FRAMES ADAPTED TO A PHASE-SPACE COVER

MONIKA DÖRFLER AND JOSÉ LUIS ROMERO

Abstract. We construct frames adapted to a given cover of the time-frequency or time-scale plane. The main feature is that we allow for quite general and possibly irregular covers. The frame members are obtained by maximizing their concentration in the respective regions of phase-space. We present applications in time-frequency, wavelet and Gabor analysis.

1. Introduction

A time-frequency representation of a distribution \( f \in \mathcal{S}'(\mathbb{R}^d) \) is a function defined on \( \mathbb{R}^d \times \mathbb{R}^d \) whose value at \( z = (x, \xi) \) represents the influence of the frequency \( \xi \) near \( x \). The short-time Fourier transform (STFT) is a standard choice for such a representation, popular in analysis and signal processing. It is defined, by means of an adequate smooth and fast-decaying window function \( \varphi \in \mathcal{S}(\mathbb{R}^d) \), as

\[
V_\varphi f(z) = \int_{\mathbb{R}^d} f(t) \overline{\varphi(t-x)} e^{-2\pi i \xi t} dt, \quad z = (x, \xi) \in \mathbb{R}^d \times \mathbb{R}^d.
\]

If the window \( \varphi \) is normalized by \( \|\varphi\|_2 = 1 \), the distribution \( f \) can be re-synthesized from its time-frequency content by

\[
f(t) = V_{\varphi}^* V_{\varphi} f(t) = \int_{\mathbb{R}^d \times \mathbb{R}^d} V_{\varphi} f(x, \xi) \varphi(t-x) e^{2\pi i \xi t} dx d\xi, \quad t \in \mathbb{R}^d.
\]

This representation is extremely redundant. One of the aims of time-frequency analysis is to provide a representation of an arbitrary signal as a linear combination of elementary time-frequency atoms, which form a less redundant dictionary. The standard choice is to let these atoms be time-frequency shifts of a single window function \( \varphi \), thus providing a uniform partition of the time-frequency plane. The resulting systems of atoms are known as Gabor frames. However, in certain applications atomic decompositions adapted to a less regular pattern may be required (see for example [32, 4, 15, 46]).

For example, a time-frequency partition may be derived from perceptual considerations. For audio signals, this means that low frequency bins are given a finer resolution than bins in high regions, where better time-resolution is often desirable, cp. [48, 52]. Such a partition is schematically depicted in the left plot of Figure 1.

More irregular partitions may be desirable whenever the frequency characteristics of an analyzed signal change over time and require adaptation in both time and frequency. For example, adaptive partitions obtained from information theoretic criteria were suggested in [39, 41]. In such a situation, partitions as irregular as shown in the right plot of Figure 1 can be appropriate.

In this article we consider the following problem. Given a - possibly irregular - cover of the time-frequency plane \( \mathbb{R}^{2d} \), we wish to construct a frame for \( L^2(\mathbb{R}^d) \) with atoms whose time-frequency concentration follows the shape of the cover members. This allows to vary the trade-off between time and frequency resolution along the time-frequency plane. The adapted frames are constructed by selecting,
for each member of a given cover, a family of functions maximizing their concentration in the corresponding region of the time-frequency domain, or phase-space. These functions can be obtained as eigenfunctions of time-frequency localization operators, as we now describe.

Given a bounded measurable set $\Omega \subseteq \mathbb{R}^{2d}$ in the time-frequency plane, the time-frequency localization operator $H_{\Omega}$ is defined by masking the coefficients in (2), cf. [16, 17],

$$H_{\Omega}f(t) = V_\varphi^* \left( 1_{\Omega} V_\varphi f \right)(t) = \int_\Omega V_\varphi f(x, \xi) \varphi(t-x) e^{2\pi i t \xi} dx d\xi.$$

$H_{\Omega}$ is self-adjoint and trace-class, so we can consider its spectral decomposition

$$H_{\Omega}f = \sum_{k=1}^{\infty} \lambda_{\Omega}^k \langle f, \phi_{\Omega}^k \rangle \phi_{\Omega}^k,$$

where the eigenvalues are indexed in descending order. Note that $\langle H_{\Omega}f, f \rangle = \langle 1_{\Omega} V_\varphi f, V_\varphi f \rangle = \int_\Omega |V_\varphi f|^2$. Hence, the first eigenfunction of $H_{\Omega}$ is optimally concentrated inside $\Omega$ in the following sense

$$\int_\Omega |V_\varphi \phi_{\Omega}^1(z)|^2 dz = \max_{\|f\|_2 = 1} \int_\Omega |V_\varphi f(z)|^2 dz.$$

More generally, it follows from the Courant minimax principle, see e.g. [45, Section 95], that the first $N$ eigenfunctions of $H_{\Omega}$ form an orthonormal set in $L^2(\mathbb{R}^d)$ that maximizes the quantity $\sum_{j=1}^N \int_\Omega |V_\varphi \phi_j^\Omega(z)|^2 dz$ among all orthonormal sets of $N$ functions in $L^2(\mathbb{R}^d)$. In this sense, their time-frequency profile is optimally adapted to $\Omega$. Figure 2 illustrates this principle by showing some time-frequency boxes $\Omega$ along with the STFT and real part of the corresponding localization operator’s first eigenfunctions.

Based on these observations, we propose the following construction of frames. Let $\{\Omega_\gamma : \gamma \in \Gamma\}$ be a cover of $\mathbb{R}^{2d}$ consisting of bounded measurable sets. In order to construct a frame adapted to the cover, we select, for each region $\Omega_\gamma$, the first $N_\gamma$ eigenfunctions $\phi_1^{\Omega_\gamma}, \ldots, \phi_{N_\gamma}^{\Omega_\gamma}$ of the operator $H_{\Omega_\gamma}$. We will prove that there is a number $\alpha > 0$ such that if $N_\gamma \geq \alpha |\Omega_\gamma|$, then the collection of all the chosen eigenfunctions spans $L^2(\mathbb{R}^d)$ in a stable fashion. (Here $|\Omega_\gamma|$ is the Lebesgue measure of $\Omega_\gamma$.) Note that the condition $N_\gamma \geq \alpha |\Omega_\gamma|$ is in accordance with the uncertainty principle, which roughly says that for each time-frequency region $\Omega_\gamma$ there are only approximately $|\Omega_\gamma|$ degrees of freedom.

We allow for covers that are arbitrary in shape as long as they satisfy the following mild admissibility condition. An indexed set $\{\Omega_\gamma : \gamma \in \Gamma\}$, is said to be an admissible cover of $\mathbb{R}^{2d}$ if the following conditions hold.
the family of functions such that for every choice of numbers 
Under this condition we prove the following.

\[ \Omega_\gamma \subseteq B_R(\gamma), \quad \gamma \in \Gamma. \]

Under this condition we prove the following.

**Theorem 1.1.** Let \( \Omega_\gamma : \gamma \in \Gamma \) be an admissible cover of \( \mathbb{R}^d \). Then there exists a constant \( \alpha > 0 \) such that for every choice of numbers \( \{ N_\gamma : \gamma \in \Gamma \} \subseteq \mathbb{N} \) satisfying 

\[ \alpha |\Omega_\gamma| \leq N_\gamma \text{ and } \sup_{\gamma \in \Gamma} N_\gamma < +\infty, \]

the family of functions \( \{ \lambda_k^{\Omega_\gamma} \phi_k^{\Omega_\gamma} : \gamma \in \Gamma, 1 \leq k \leq N_\gamma \} \), obtained from the eigenfunctions and eigenvalues of the operators \( H_{\Omega_\gamma} \) - cf. \( [3] \) and \( [11] \) - is a frame of \( L^2(\mathbb{R}^d) \). That is, for some constants \( 0 < A \leq B < +\infty \), the following frame inequality holds

\[ A \| f \|^2 \leq \sum_{\gamma \in \Gamma} \sum_{k=1}^{N_\gamma} \langle f, \lambda_k^{\Omega_\gamma} \phi_k^{\Omega_\gamma} \rangle^2 \leq B \| f \|^2, \quad f \in L^2(\mathbb{R}^d). \]

Theorem 1.1 is proved at the end of Section 5.2. We also show that if an admissible cover \( \{ \Omega_\gamma : \gamma \in \Gamma \} \) satisfies the additional inner regularity condition:

\[ \text{There exist } r > 0 \text{ such that } B_r(\gamma) \subseteq \Omega_\gamma, \quad \gamma \in \Gamma, \]

then the statement of Theorem 1.1 remains valid, if the functions \( \lambda_k^{\Omega_\gamma} \phi_k^{\Omega_\gamma} \) are replaced by their unweighted versions \( \phi_k^{\Omega_\gamma} \) (see Theorem 5.10). In this case \( \inf_{\gamma} |\Omega_\gamma| \geq |B_\gamma(0)| \), so the lower bound on \( N_\gamma \) in the hypothesis of Theorem 1.1 can be expressed as \( N_\gamma \geq \alpha \), for some constant \( \alpha > 0 \).

While Theorem 1.1 was our main motivation, it is just a sample of our results. The introduction of an abstract model for phase space provides sufficient flexibility to obtain variants of Theorem 1.1 in the context of time-scale analysis (Theorem 6.4) and of discrete time-frequency representations (Theorem 6.5).

1.1. **Technical overview.** The proofs of the main results in this paper are based on two major observations. First, the norm equivalence

\[ \| f \|^2 = \sum_{\gamma \in \Gamma} \| H_{\eta_\gamma} f \|^2, \quad f \in L^2(\mathbb{R}^d), \]

holds for a family of time-frequency localization operators

\[ H_{\eta_\gamma} f(t) = \int_{\mathbb{R}^d} \eta_\gamma(x, \xi) V_{\tilde{\varphi}} f(x, \xi) \varphi(t-x) e^{2\pi i \xi t} dx d\xi, \]

provided that the symbols \( \eta_\gamma : \mathbb{R}^d \to [0, +\infty) \) satisfy

\[ \sum_{\gamma \in \Gamma} \eta_\gamma(z) \approx 1 \]

and the enveloping condition

\[ \eta_\gamma(z) \leq g(z - \gamma), \text{ for some } g \in L^1(\mathbb{R}^{2d}) \text{ and } \gamma \in \Gamma, \text{ with } \Gamma \subseteq \mathbb{R}^{2d} \text{ a lattice.} \]

The inequalities were first proved in [19] for symbols of the form \( \eta_\gamma(z) = g(z - \gamma) \) and \( \Gamma = \mathbb{Z}^{2d} \), then for a general lattice in [20], and finally for fully irregular symbols satisfying (11) in [47]. It is interesting to note that the proofs in [19, 20] are based on the observation that under condition (10) the norm-equivalence (8) is equivalent to the fact that finitely many eigenfunctions of the operator \( H_g \)

\[ f, g : X \to (0, +\infty), \text{ the statement } f \approx g \text{ means that there exist constants } c, C \in (0, +\infty) \text{ such that } cf(x) \leq g(x) \leq Cf(x) \text{ for all } x \in X. \]
generate a multi-window Gabor frame over the lattice $\Gamma$. The proof of the general case in \cite{47} does not explicitly involve the eigenfunctions of the operators $H_{\eta}$ nor does it rely on tools specific to Gabor frames on lattices. Thus the question arises whether it is also possible in the irregular case to construct a frame consisting of finite sets of eigenfunctions of the operators $H_{\eta}$. Here, this question is given a positive answer.

Second, the observation that \cite{8} remains valid when the operators $H_{\eta_\gamma}$ are replaced by finite rank approximations $H_{\eta_\gamma}^\varepsilon$ obtained by thresholding their eigenvalues, cf. Theorem \cite{5.3}, is the core of the proof of our main results. This finite rank approximation is in turn achieved by proving that the operators $H_{\eta_\gamma}$ behave “globally” like projectors. More precisely, in Proposition \cite{5.1} we obtain the following extension of \cite{8}:

$$\|f\|_2^2 \approx \sum_{\gamma \in \Gamma} \|H_{\eta_\gamma} f\|_2^2 \approx \sum_{\gamma \in \Gamma} \|(H_{\eta_\gamma})^2 f\|_2^2.$$  

(12)

This will allow us to “localize in phase-space” the $L^2$-norm estimates relating $H_{\eta_\gamma}$ and $H_{\eta}$. Note that, in general, the operators $H_{\eta_\gamma}$ have infinite rank even if $\eta_\gamma$ is the characteristic function of a compact set (see Lemma \cite{5.8}). Consequently $\|H_{\eta_\gamma} f\|_2 \neq \|(H_{\eta_\gamma})^2 f\|_2$ and therefore the global properties of the family $\{(H_{\eta_\gamma})^2 : \gamma \in \Gamma\}$ are crucial to prove \cite{12}. While the squared operators $(H_{\eta_\gamma})^2$ are not time-frequency localization operators, their time-frequency localizing behavior is preserved under conditions as given by \cite{11} and they are the prototypical example of a family of operators that is

![Figure 2. Four different rectangular masks in time-frequency domain and the first eigenfunctions of the corresponding localization operators. Middle plots show the absolute value squared of the STFT and right plots show the real part.](image)
well-spread in the time-frequency plane. The latter notion is defined in Section 3 and we exploit the fact that the tools from \[47\] are valid for these operator families.

For clarity, we choose to accentuate the case of time-frequency analysis but all the proofs are carried out in an abstract setting that yields, for example, analogous consequences in time-scale analysis.

1.2. Organization. The article is organized as follows. Section 2 motivates the abstract model to study phase-space covers and gives the main examples to keep in mind. Section 2.3 formally introduces the abstract model for phase-space and Section 3 presents certain key technical notions, in particular the properties required for a family of localization operators to exhibit an almost-orthogonality property. In Section 4, we first prove our results in the context of time-frequency analysis, where some technical problems of the abstract setting do not arise. In addition, in the context of time-frequency analysis, we are able to extend the result on phase-space adapted frames from \(L^2(\mathbb{R}^d)\) to an entire class of Banach spaces, the modulation spaces, by exploiting spectral invariance results for pseudodifferential operators. Theorem 5.5 in Section 5 is the general version of Theorem 1.1 stated above. This latter result in proved after Theorem 5.5 as an application. Section 6 develops the results in the abstract setting. These are then applied to time-scale analysis in Section 6.1. Finally, Section 6.2 contains an additional application of the abstract results to time-frequency analysis, this time using Gabor multipliers, which are time-frequency masking operators related to a discrete time-frequency representation (Gabor frame). The atoms thus obtained maximize their time-frequency concentration with respect to a weight on a discrete time-frequency grid and the resulting frames are relevant in numerical applications.

For clarity, the presentation of the results highlights the case of time-frequency analysis, which was our main motivation. Most of the technicalities in Section 2.3 are irrelevant to that setting (although they are relevant for time-scale analysis). The reader interested mainly in time-frequency analysis is encouraged to jump directly to Section 4 and then go back to Sections 2.3 and 3 having a clear example in mind.

The article also contains two appendices providing auxiliary results related to the almost-orthogonality tools from \[47\] and spectral invariance of pseudodifferential operators.

2. Phase-space

In this section, we introduce an abstract model for phase space. We first provide some motivation and the main examples to keep in mind.

2.1. The time-frequency plane as an example of phase-space. In time-frequency analysis, a distribution \(f \in S'(\mathbb{R}^d)\) is studied by means of a time-frequency representation, e.g. the STFT \(V_\varphi f : \mathbb{R}^d \times \mathbb{R}^d \to \mathbb{C}\), cf. \[1\]. In signal processing, where \(f\) is called a signal, the domain \(\mathbb{R}^d\) is referred to as signal space while \(\mathbb{R}^d \times \mathbb{R}^d\) is referred to as phase-space. This terminology borrows some intuition from mechanics, where the position of a freely moving particle is described by a point \(x\) in the configuration space \(\mathbb{R}^d\), while \((x,\xi)\) describes a pair of position-momentum variables, belonging to the phase-space \(\mathbb{R}^d \times \mathbb{R}^d\).

Since \(f\) can be resynthesized from \(V_\varphi f\) - cf. \[2\] - all its properties can in principle be reformulated as properties of \(V_\varphi f\). However, not every function on \(\mathbb{R}^d \times \mathbb{R}^d\) is the STFT of a distribution on \(\mathbb{R}^d\). Indeed, if we let \(S := V_\varphi L^2(\mathbb{R}^d)\) be the image of \(L^2(\mathbb{R}^d)\) under \(V_\varphi\), it turns out that \(S\) is a reproducing kernel space (see \[43\] Chapters 3 and 11). This means that \(S\) is a closed subspace of \(L^2(\mathbb{R}^{2d})\) consisting of continuous functions and that for all \((x,\xi) \in \mathbb{R}^d \times \mathbb{R}^d\) the evaluation functional \(S \ni F \mapsto F(x,\xi) \in \mathbb{C}\) is continuous. In particular \(S \subseteq L^2(\mathbb{R}^d)\).

The fact that \(S := V_\varphi L^2(\mathbb{R}^d)\) is a “small” subspace of \(L^2(\mathbb{R}^d)\) is important to understand the problem studied in this article. In designing a frame for \(L^2(\mathbb{R}^d)\) with a prescribed phase-space profile, the challenge lies in the fact that the shapes we design in \(\mathbb{R}^{2d}\) must correspond to functions in the small subspace \(S\). The role of time-frequency localization operators is crucial and will be detailed in the rest of this section.

For \(m \in L^\infty(\mathbb{R}^d \times \mathbb{R}^d)\), the time-frequency localization operator with symbol \(m\) is

\[
H_m f(t) = \int_{\mathbb{R}^d \times \mathbb{R}^d} V_\varphi f(x,\xi) m(x,\xi) \varphi(t-x) e^{2\pi i \xi t} dx d\xi, \quad t \in \mathbb{R}^d.
\]
If \( m = 1_{\Omega} \) is the characteristic function of a set \( \Omega \) we write \( H_{\Omega} \) instead of \( H_{1_{\Omega}} \), cf. (3).

The STFT (with respect to a fixed normalized window \( \varphi \in L^2(\mathbb{R}^d) \)) defines a map \( V_{\varphi} : L^2(\mathbb{R}^d) \to L^2(\mathbb{R}^{2d}) \). Its adjoint \( V_{\varphi}^* : L^2(\mathbb{R}^{2d}) \to L^2(\mathbb{R}^d) \) is given by

\[
V_{\varphi}^* F(t) = \int_{\mathbb{R}^d \times \mathbb{R}^d} F(x, \xi) \varphi(t-x)e^{2\pi i \xi \cdot dx} d\xi, \quad t \in \mathbb{R}^d.
\]

The inversion formula in (2) says that \( V_{\varphi}^* V_{\varphi} = I_{L^2(\mathbb{R}^d)} \). Hence \( V_{\varphi} \) is an isometry on \( L^2(\mathbb{R}^d) \). Consider a symbol \( m \in L^\infty(\mathbb{R}^d \times \mathbb{R}^d) \) and the time-frequency localization operator from (13). Using \( V_{\varphi} \) and \( V_{\varphi}^* \), the definition reads

\[
H_m f = V_{\varphi}^* (mV_{\varphi} f), \quad f \in L^2(\mathbb{R}^d).
\]

The fact that \( V_{\varphi} \) is an isometry with range \( S \) implies that \( V_{\varphi} V_{\varphi}^* = P_S \) is the orthogonal projection \( L^2(\mathbb{R}^{2d}) \to S \). Explicitly, for \( F \in L^2(\mathbb{R}^d \times \mathbb{R}^d) \)

\[
P_S F(x', \xi') = \int_{\mathbb{R}^d \times \mathbb{R}^d} F(x, \xi) \left( \varphi(\cdot - x) e^{2\pi i \xi \cdot dx} \varphi(\cdot - x') e^{2\pi i \xi' \cdot dx} \right)_{L^2(\mathbb{R}^d)} d\xi.
\]

It follows from (14) that

\[
(V_{\varphi} H_m V_{\varphi}^*) V_{\varphi} f = P_S (m V_{\varphi} f), \quad f \in L^2(\mathbb{R}^d).
\]

Since \( F = V_{\varphi} f \) is the generic form of a function in \( S \), we obtain that

\[
(V_{\varphi} H_m V_{\varphi}^*) F = P_S (m F), \quad F \in S.
\]

This means that the time-frequency localization operator \( H_{\Omega} \) is unitarily equivalent to the operator \( M_m : S \to S \) given by

\[
M_m F := P_S (m F), \quad F \in S.
\]

The operator \( M_m \) consists of multiplication by \( m \) followed by projection onto \( S \). We will call these operators phase-space multipliers: they apply a mask \( m \) to a function \( F = V_{\varphi} f \in S \), typically yielding a function \( m \cdot F \notin S \), and then provide the best \( L^2 \) approximation of \( m \cdot F \) within \( S \). (These operators are sometimes also called Toeplitz operators for the STFT.)

### 2.2. Other transforms.

The interpretation of time-frequency localization operators as phase-space multipliers (multiplication followed by projection) is central to this article. We will consider a general setting where the role of the STFT can be replaced by other transforms. An important example is the wavelet transform of a function \( L^2(\mathbb{R}^d) \) with respect to an adequate window \( \psi \in \mathcal{S}(\mathbb{R}^d) \),

\[
W_\psi f(x, s) = s^{-d/2} \int_{\mathbb{R}^d} f(t) \psi \left( \frac{t-x}{s} \right) dt, \quad x \in \mathbb{R}^d, s > 0.
\]

(See Section 6.1 for details.) In analogy with time-frequency analysis, we still call \( \mathbb{R}^d \times \mathbb{R}^+ \) the phase-space associated with the wavelet transform. This terminology is justified by the fact that there is a formal analogy between the two contexts. The range of the wavelet transform \( W_\psi L^2(\mathbb{R}^d) \) is, under suitable assumptions on \( \psi \), a reproducing kernel subspace of \( L^2(\mathbb{R}^d \times \mathbb{R}^+, s^{-(d+1)} dx ds) \) and time-scale localization operators are defined in analogy to the time-frequency localization operators (see Section 6.1 for explicit formulas).

The similarity between the time-frequency and time-scale contexts stems from the fact that both are associated with representations of a locally-compact group. In the former case, the Heisenberg group acts on \( L^2(\mathbb{R}^d) \) by translations and modulations, while in the latter, the affine group acts by translations and dilations. The theory of coorbit spaces [25, 26] treats this situation in general, studying the transform associated with the representation coefficients of a group action and associating a range of function spaces to it. The model for abstract phase space to be introduced in Section 2.3 is largely inspired by [25]. It allows for the simultaneous treatment of various settings, since it makes no explicit reference to an integral transform. The main ingredients are a “big” space \( E \) called the environment and a “small” subspace called the atomic space. In the time-frequency example \( E = L^2(\mathbb{R}^{2d}) \) and \( S = V_{\varphi} L^2(\mathbb{R}^d) \). Similarly, the case of time-scale analysis (wavelets) uses \( E = L^2(\mathbb{R}^d \times \mathbb{R}^+, s^{-(d+1)} dx ds) \) and \( S = W_\psi L^2(\mathbb{R}^d) \). Building on the intuition provided by time-frequency and time-scale analysis, we think
of $S$ as a collection of phase-space representations for functions in $L^2(\mathbb{R}^d)$, while the environment $E$ is big enough for certain operations, such as pointwise multiplication by arbitrary bounded measurable functions, to be well-defined. One central assumption of the model in Section 2.3 is then the existence of a projector $P : E \to S$, so that one can consider phase-space multipliers like in [10].

The price to pay for this unified approach is a certain level of technicality. The Euclidean space is not suitable any more as a model for the domain of the functions in $S$: we need to consider a general locally-compact group. To provide an easily accessible example, the main concepts will be spelled out in the concrete case of time-frequency analysis.

2.3. Abstract phase-space.

2.3.1. Locally compact groups and function spaces. Throughout the article $G$ will be a locally compact, $\sigma$-compact, topological group with modular function $\Delta$. The left Haar measure of a set $X \subseteq G$ will be denoted by $|X|$. Integration will always be considered with respect to the left Haar measure. For $x \in G$, we denote by $L_x$ and $R_x$ the operators of left and right translation, defined by $L_x f(y) = f(xy)$ and $R_x f(y) = f(yx)$. We also consider the involution $f^\vee(x) = f(x^{-1})$.

Given two non-negative functions $f, g$ we write $f \lesssim g$ if there exists a constant $C \geq 0$ such that $f \leq Cg$. We say that $f \approx g$ if both $f \lesssim g$ and $g \lesssim f$. The characteristic function of a set $A$ will be denoted by $1_A$.

A set $\Gamma \subseteq G$ is called relatively separated if for some (or any) $V \subseteq G$, relatively compact neighborhood of the identity, the quantity - called the spreadness of $\Gamma$ -

$$\rho(\Gamma) = \rho_V(\Gamma) := \sup_{x \in G} \#(\Gamma \cap xV)$$

is finite, i.e. if the amount of elements of $\Gamma$ that lie in any left translate of $V$ is uniformly bounded.

The following definition introduces a class of function spaces on $G$. The Lebesgue spaces $L^p(G)$ are natural examples.

**Definition 2.1** (Banach function spaces). A Banach space $E$ is called a solid, translation invariant BF space if it satisfies the following.

(i) $E$ is continuously embedded into $L^1_{loc}(G)$, the space of complex-valued locally integrable functions on $G$.

(ii) Whenever $f \in E$ and $g : G \to \mathbb{C}$ is a measurable function such that $|g(x)| \leq |f(x)|$ a.e., it is true that $g \in E$ and $\|g\|_E \leq \|f\|_E$.

(iii) $E$ is closed under left and right translations (i.e. $L_x E \subseteq E$ and $R_x E \subseteq E$, for all $x \in G$) and the following relations hold with the corresponding norm estimates

$$L^1_W(G) * E \subseteq E \quad \text{and} \quad E * L^1_W(G) \subseteq E,$$

where $u(x) := \|L_x\|_{E \to E}$, $v(x) := \Delta(x^{-1})\|R_{x^{-1}}\|_{E \to E}$.

**Definition 2.2** (Admissible weights for a BF space). Given a solid, translation invariant BF space $E$, a function $w : G \to (0, +\infty)$ satisfying

$$w(x) = \Delta(x^{-1})w(x^{-1}),$$

$$w(xy) \leq w(x)w(y) \quad \text{(submultiplicativity)},$$

$$w(x) \geq C_{E,w} \max \left\{ u(x), u(x^{-1}), v(x), \Delta(x^{-1})v(x^{-1}) \right\},$$

where $u(x) := \|L_x\|_{E \to E}$, $v(x) := \Delta(x^{-1})\|R_{x^{-1}}\|_{E \to E}$,

for some constant $C_{E,w} > 0$ is called an admissible weight for $E$.

If $w$ is admissible for $E$, it follows that $w(x) \geq 1$, $L^1_w * E \subseteq E$ and $E * L^1_w \subseteq E$, and the constants in the corresponding norm estimates depend only on $C_{E,w}$, cf. [25].

For a solid translation invariant BF space $E$ and $\Gamma \subseteq G$ a relatively separated set, we construct discrete versions $E_d$ as follows. Fix $V$, a symmetric relatively compact neighborhood of the identity and let

$$E_d = E_d(\Gamma) := \left\{ c \in \mathbb{C}^\Gamma \mid \sum_{\gamma \in \Gamma} |c_\gamma| 1_{\gamma V} \in E \right\},$$

with norm $\|(c_\gamma)_{\gamma \in \Gamma}\|_{E_d} := \left\| \sum_{\gamma \in \Gamma} |c_\gamma| 1_{\gamma V} \right\|_E$. 


The definition depends on $V$, but a different choice of $V$ yields the same space with equivalent norm (this is a consequence of the right invariance of $E$, see for example [44, Lemma 2.2]). For $E = L^p_w$, the corresponding discrete space $E_d(\Gamma)$ is just $\ell^p_w(\Gamma)$, where the (admissible) weight $w$ is restricted to the set $\Gamma$.

We next define the left Wiener amalgam space with respect to a solid, translation invariant BF space $E$. Let $V$ be again a symmetric, relatively compact neighborhood of the identity. For a locally bounded function $f : \mathcal{G} \to \mathbb{C}$ consider the left local maximum function defined by

$$f^\#(x) := \text{esssup}_{y \in V} |f(xy)| = \|f \cdot (L_x 1_V)\|_\infty, \quad x \in \mathcal{G},$$

and similarly the right local maximum function is given by $f^\#(x) := \text{esssup}_{y \in V} |f(yx)| = \|f \cdot (R_x 1_V)\|_\infty$.

**Definition 2.3** (Wiener amalgam spaces).

$$W(L^\infty, E)(\mathcal{G}) := \{ f \in L^\infty_{w0}(\mathcal{G}) \mid f^\# \in E \},$$

with norm $\|f\|_{W(L^\infty, E)} := \|f^\#\|_E$. The right Wiener amalgam space $W_R(L^\infty, E)$ is defined similarly, this time using the norm $\|f\|_{W_R(L^\infty, E)} := \|f\|_E$.

A different choice of $V$ yields the same spaces with equivalent norms (see for example [22, Theorem 1] or [23]). When $E$ is a weighted $L^p$ space on $\mathbb{R}^d$, the corresponding amalgam space coincides with the classical $L^\infty - \ell^p$ amalgam space [38, 39]. In the present article we will be mainly interested in the spaces $W(L^\infty, L^1_w)$ and $W_R(L^\infty, L^1_w)$ for which we need the following facts.

**Proposition 2.4.** The spaces $W(L^\infty, L^1_w)$ and $W_R(L^\infty, L^1_w)$ are convolution algebras. That is, the relations $W(L^\infty, L^1_w) * W(L^\infty, L^1_w) \hookrightarrow W(L^\infty, L^1_w)$ and $W_R(L^\infty, L^1_w) * W_R(L^\infty, L^1_w) \hookrightarrow W_R(L^\infty, L^1_w)$ hold together with the corresponding norm estimates.

**Proof.** The left amalgam space satisfies the (translation invariance) relation $L^1_w * W(L^\infty, L^1_w) \hookrightarrow W(L^\infty, L^1_w)$. This is a particular case of [24, Theorem 7.1] and can also be readily deduced from the definitions. Since $W(L^\infty, L^1_w) \hookrightarrow L^1_w$, the statement about $W(L^\infty, L^1_w)$ follows. The involution $f^\#(x) = f(x^{-1})$ maps $W_R(L^\infty, L^1_w)$ isometrically onto $W(L^\infty, L^1_w)$ (because $V = V^{-1}$) and satisfies $(f * g)^\# = g^\# * f^\#$. Hence the statement about the right amalgam space follows from the one about the left one. \hfill \qed

**2.3.2. The model for abstract phase-space.** In the abstract model for phase-space we consider a solid BF space $E$ (called the environment) and a certain distinguished subspace $S_E$, which is the range of an idempotent operator $P$. The precise form of the model is taken from [47] and is designed to fit the theory in [25] (see also [13, 42]). We list a number of ingredients in the form of two assumptions: (A1) and (A2).

(A1) $- E$ is a solid, translation invariant BF space, called the environment.
- $w$ is an admissible weight for $E$.
- $S_E$ is a closed complemented subspace of $E$, called the atomic subspace.
- Each function in $S_E$ is continuous.

(A2) $P$ is an operator and $K$ is a non-negative function satisfying the following.
- $P : W(L^1, L^\infty_{1/w}) \to L^\infty_{1/w}$ is a (bounded) linear operator,
- $P(E) = S_E$ and $P(f) = f$, for all $f \in S_E$,
- $K \in W(L^\infty, L^1_w) \cap W_R(L^\infty, L^1_w)$,
- For $f \in W(L^1, L^\infty_{1/w})$,

$$|P(f)(x)| \leq \int_{\mathcal{G}} |f(y)| \, K(y^{-1}x) \, dy, \quad x \in \mathcal{G}. \hspace{1cm} (23)$$

When $E = L^2(\mathcal{G})$ we will additionally assume the following.

(A3) $P : L^2(\mathcal{G}) \to S_{L^2}$ is the orthogonal projection.

Note that Assumption (A2) means that the retraction $E \to S_E$ is given by an operator dominated by right convolution with a kernel in $W(L^\infty, L^1_w) \cap W_R(L^\infty, L^1_w)$. 


Example 2.5. The discussion in Section 2.1 provides the main example of the abstract model. If \( \varphi \in \mathcal{S}(\mathbb{R}^d) \) we can let \( G = \mathbb{R}^d \times \mathbb{R}^d, E = L^2(\mathbb{R}^{2d}) \) and \( S = S_E = V \varphi^* L^2(\mathbb{R}^d) \). For this choice \( w \equiv 1 \) is an admissible weight. We also let \( P = P_S : L^2(\mathbb{R}^{2d}) \rightarrow S_E \) be the orthogonal projection. To see that \( P \) satisfies (A2), note that from projection \( P \), \( E \) Phase-space multipliers.

Remark 2.8. Since \( P \) an admissible weight. We also let \( \varphi \) Example 2.5. The discussion in Section 2.1 provides the main example of the abstract model. If \( \varphi \in \mathcal{S}(\mathbb{R}^d) \) we can let \( G = \mathbb{R}^d \times \mathbb{R}^d, E = L^2(\mathbb{R}^{2d}) \) and \( S = S_E = V \varphi^* L^2(\mathbb{R}^d) \). For this choice \( w \equiv 1 \) is an admissible weight. We also let \( P = P_S : L^2(\mathbb{R}^{2d}) \rightarrow S_E \) be the orthogonal projection. To see that \( P \) satisfies (A2), note that from projection \( P \), \( E \) Phase-space multipliers.

Proposition 2.6. \cite{[18, 53, 40]} Proposition 3
(a) \( P \) boundedly maps \( E \) into \( W(L^\infty, E) \).
(b) \( S_E \hookrightarrow W(L^\infty, E) \).
(c) If \( f \in W(L^1, L^\infty_{1/w}) \), then \( \|P(f)\|_{L^\infty_{1/w}} \lesssim \|f\|_{W(L^1, L^\infty_{1/w})} \|K\|_{W_R(L^\infty, L^1)} \).
(d) If \( f \in W(L^1, L^\infty) \), then \( \|P(f)\|_{L^\infty} \lesssim \|f\|_{W(L^1, L^\infty)} \|K\|_{W_R(L^\infty, L^1)} \).

Remark 2.7. Since \( w \gtrsim 1 \), \( L^\infty \hookrightarrow L^\infty_{1/w} \).

Remark 2.8. The estimates in Proposition 2.6 hold uniformly for all the spaces \( E \) with the same weight \( w \) and the same constant \( C_{E,w} \) (cf. \cite{[22]}). This is relevant to the applications, where the same projection \( P \) is used with different spaces \( E \) and corresponding subspaces \( S_E = P(E) \).

2.3.3. Phase-space multipliers. Recall the projector \( P : E \rightarrow S_E \) from (A2). For \( m \in L^\infty(\mathcal{G}) \), the phase-space multiplier with symbol \( m \) is the operator \( M_m : S_E \rightarrow S_E \) defined by

\[
M_m(f) := P(m f), \quad f \in S_E.
\]

It is bounded by Proposition 2.6 and the solidity of \( E \):

\[
\|M_m(f)\|_E \lesssim \|m\|_1 \|f\|_E, \quad f \in S_E.
\]

Example 2.9. As discussed in Section 2.1, time-frequency localization operators \cite{[13]} are unitarily equivalent via the STFT to phase-space multipliers, with \( E, S_E \) and \( P \) as in Example 2.5.

In the context of time-scale analysis, one can define operators in analogy to time-frequency localization operators using the wavelet transform in \cite{[17]} instead of the STFT. These are called time-scale localization operators or wavelet multipliers \cite{[18, 53, 40]}. With an adequate choice of \( E, S_E \) and \( P \), time-scale localization operators are unitarily equivalent to phase-space multipliers. This is developed in Section 6.4.

For future reference we note some Hilbert-space properties of phase-space multipliers (when \( E = L^2(\mathcal{G}) \)), which are well-known for time-frequency localization operators, cf. \cite{[7, 27, 54, 58]}.

Proposition 2.10. Let \( E = L^2(\mathcal{G}) \) and assume (A1), (A2) and (A3). Then the following hold.
(a) Let \( m_1, m_2 \in L^\infty(\mathcal{G}) \) be real-valued. If \( m_1 \leq m_2 \) a.e., then \( M_{m_1} \leq M_{m_2} \) as operators. In particular if \( m \) is non-negative and bounded, then \( M_m \) is a positive operator.
(b) Let \( m \in L^1(\mathcal{G}) \cap L^\infty(\mathcal{G}) \) be non-negative. Then \( M_m : S_{L^2} \rightarrow S_{L^2} \) is trace-class and \( \text{trace}(M_m) \lesssim \|m\|_1 \).

Proof. To prove (a), let \( f \in S_{L^2} \) and note that by (A3),

\[
\langle M_{m_1} f, f \rangle = \langle P(m_1 f), f \rangle = \langle m_1 f, f \rangle = \int_{\mathcal{G}} m_1(x) \left| f(x) \right|^2 dx 
\]

\[
\leq \int_{\mathcal{G}} m_2(x) \left| f(x) \right|^2 dx = \langle M_{m_2} f, f \rangle.
\]
Let us now prove (b). For \( f \in S_{L^2} \) and \( x \in \mathcal{G} \), by Assumption (A2) - cf. \[23\]

\[
|f(x)| \leq \int_{\mathcal{G}} |f(y)| K(y^{-1}x)dy = \int_{\mathcal{G}} |f(y)| K^\vee(x^{-1}y)dy \leq \|f\|_2\|K^\vee\|_2,
\]

which is finite because \( K^\vee \in W_R(L^\infty, L^1_w) \subseteq L^2 \). Hence \( f(x) = \langle f, E_x \rangle \), for some function \( E_x \in S_{L^2} \) with \( \|E_x\|_2 \leq \|K^\vee\|_2 \).

Let \( \{e_k\}_k \) be an orthonormal basis of \( S_{L^2} \). Since by (a) \( M_m \) is a positive operator, it suffices to check that \( \sum_k \langle M_m e_k, e_k \rangle \lesssim \|m\|_1 \) (see for example, \[49\], Theorem 2.14). To this end, note that for \( x \in \mathcal{G} \),

\[
\sum_k |e_k(x)|^2 = \sum_k |\langle e_k, E_x \rangle|^2 = \|E_x\|^2 \leq \|K^\vee\|^2_2.
\]

Hence,

\[
\sum_k \langle M_m e_k, e_k \rangle = \sum_k \int_{\mathcal{G}} m(x) |e_k(x)|^2 \leq \|K^\vee\|^2_2 \|m\|_1.
\]

This completes the proof. \( \square \)

3. Well-spread families of operators

Throughout this section, we assume (A1) and (A2) from Section 2.3.2. Recall the notion of phase-space multiplier \( M_m: S_E \to S_E \), \( M_m(f) = \hat{f}(m) \), from Section 2.3.3. If \( P: E \to S_E \) is the projector from (A2) and \( m \in L^\infty(\mathcal{G}) \) is a symbol, then (A1), (A2) intuitively imply that a phase-space multiplier spreads the mass of \( f \) in a controlled way. Indeed, using the bound in (A2) we obtain

\[
|M_m f(x)| = |P(mf)(x)| \leq \int_{\mathcal{G}} |m(y)||f(y)| K(y^{-1}x)dy, \quad x \in \mathcal{G}.
\]

Hence, if \( m \) is known to be concentrated in a certain region of \( \mathcal{G} \), so will be \( M_m(f) \).

One of the main technical insights of this article is the fact that some important tools used in the investigation of families phase-space multipliers only depend on estimates such as \[23\]. To formalize this observation, we now introduce the key concept of families of operators that are well-spread in phase-space, implying that the operators are dominated by product-convolution operators centered at suitably distributed nodes \( \gamma \).

**Definition 3.1** (Well-spread family of operators). Let \( \Gamma \subseteq \mathcal{G} \) be a relatively separated set, \( \Theta \in W(L^\infty, L^\infty_w) \cap W_R(L^\infty, L^1_w) \) and \( g \in W_R(L^\infty, L^1_w) \) be non-negative functions. A family of operators \( \{T_\gamma: S_E \to S_E: \gamma \in \Gamma\} \) is called well-spread with envelope \((\Gamma, \Theta, g)\) if the following pointwise estimate holds

\[
|T_\gamma f(x)| \leq \int_{\mathcal{G}} g(\gamma^{-1}y)|f(y)| \Theta(\gamma^{-1}x)dy, \quad \gamma \in \Gamma, x \in \mathcal{G}.
\]

If we do not want to emphasize the role of the envelope, we say that \( \{T_\gamma: \gamma \in \Gamma\} \) is a well-spread family of operators, assuming the existence of an adequate envelope.

The advantage of working with well-spread families instead of just families of phase-space multipliers is that well-spreadness is stable under various operations, e.g. finite composition. This will be essential to the proofs of our main results.

The canonical example of a well-spread family of operators is a family of phase-space multipliers \( \{M_{\eta_\gamma}: \gamma \in \Gamma\} \) associated with an adequate family of symbols.

**Definition 3.2** (Well-spread family of symbols). A family of (measurable) symbols \( \{\eta_\gamma: \mathcal{G} \to \mathbb{C}: \gamma \in \Gamma\} \) is called well-spread if

- \( \Gamma \subseteq \mathcal{G} \) is a relatively separated set and
- there is a function \( g \in W_R(L^\infty, L^1_w) \) such that \( |\eta_\gamma(x)| \leq g(\gamma^{-1}x), x \in \mathcal{G}, \gamma \in \Gamma. \)

The pair \((\Gamma, g)\) is called an envelope for \( \{\eta_\gamma: \gamma \in \Gamma\} \).

Together with the properties of the convolution kernel \( K \) dominating the projection onto \( S_E \) - cf. (A1), (A2) - we immediately obtain a family of well-spread operators.
Proposition 3.3. Let \( \{ \gamma \in \Gamma \} \) be a well-spread family of symbols. Then the corresponding family of phase-space multipliers \( \{ M_{\eta_{\gamma}} \}_{\gamma \in \Gamma} \) is well-spread.

Proof. It follows readily from the definitions and Assumptions (A1), (A2) that if \( (\Gamma, g) \) is an envelope for \( \{ \gamma \in \Gamma \} \), then \( (\Gamma, K, g) \) is an envelope for \( \{ M_{\eta_{\gamma}} : \gamma \in \Gamma \} \).

The reason why we introduce the concept of well-spread families of operators is that composition of phase-space multipliers usually fails to yield a phase-space multiplier. However, the estimate in (26) is stable under various operations. In this article we will be mainly interested in finite composition and we have the following.

Proposition 3.4. Let \( \{ \eta_{\gamma} : \gamma \in \Gamma \} \) be a well-spread family of symbols. Then the family of operators \( \{ (M_{\eta_{\gamma}})^2 : \gamma \in \Gamma \} \) is well-spread.

Proof. If \( (\Gamma, g) \) is an envelope for \( \{ \eta_{\gamma} : \gamma \in \Gamma \} \) then
\[
\|(M_{\eta_{\gamma}})^2 f(x)\| \leq \int_{\mathcal{G}} g(\gamma^{-1} y) \| M_{\eta_{\gamma}} f(y) \| K(y^{-1} x) dy \\
\leq \int_{\mathcal{G}} \int_{\mathcal{G}} g(\gamma^{-1} y) g(\gamma^{-1} z) |f(z)| K(z^{-1} y) K(y^{-1} x) dy dz \\
\leq \|g\| \int_{\mathcal{G}} g(\gamma^{-1} z) |f(z)| (K * K)(z^{-1} x) dz.
\]

Hence, if we set \( \Theta := K * K \), it follows that \( (\Gamma, \Theta, \|g\| \gamma g) \) is an envelope for \( \{ (M_{\eta_{\gamma}})^2 : \gamma \in \Gamma \} \). Since \( K \) belongs to \( W(L^\infty, L^1_w) \) and \( L^2 \) by Proposition 2.4, so does \( \Theta \).

3.1. Almost-orthogonality estimates. We now introduce one of the key estimates used in this article, that can be seen as a generalization of (5).

Theorem 3.5. Let \( \{ T_{\gamma} : \gamma \in \Gamma \} \) be a well-spread family of operators. Suppose that the operator \( \sum_{\gamma} T_{\gamma} : S_E \to S_E \) is invertible. Then the following norm equivalence holds
\[
\|f\|_E \approx \left( \|T_{\gamma} f\|_{L^2(\mathcal{G})}\right)_{\gamma \in \Gamma} \quad f \in S_E.
\]

Remark 3.6. The implicit constants depend only on \( \|K\|_{W(L^\infty, L^1_w)}, \|K\|_{W(L^\infty, L^1_w)}, \|\Theta\|_{W(L^\infty, L^1_w)}, \|g\|_{W(L^\infty, L^1_w)}, \rho(\Gamma) \) (cf. (18)) and \( C_{E,w} \) (cf. (22)), and the norms of \( \sum_{\gamma} T_{\gamma} \) and \( \gamma^{-1} \sum_{\gamma} T_{\gamma} \) of symbols, is proved in \( [47] \). However, the arguments involved work unaltered for the case of general well-spread families of operators. Indeed, Definition 3.1 is tailored to the requirements of the proof in \( [47] \). For completeness, we sketch a proof of Theorem 3.5 in Appendix A.

Theorem 3.5 is an almost orthogonality principle: because of the phase-space localization of the family \( \{ T_{\gamma} : \gamma \in \Gamma \} \), the effects of each individual operator within the sum \( \sum_{\gamma} T_{\gamma} \) decouple.

We also point out that the \( L^2 \)-norm in (27) can be replaced by any solid translation invariant norm (see [47]). However, the by far most important case is the one of \( L^2 \). Indeed, in this article we exploit (27) to extrapolate certain thresholding estimates from \( L^2 \) to other Banach spaces, cf. Theorem 4.5.

As a consequence of Theorem 3.5 we get the following fact.

Corollary 3.7. Let \( \{ T_{\gamma} : \gamma \in \Gamma \} \) be a well-spread family of operators. Suppose that \( E = L^2(\mathcal{G}) \) and that the operator \( \sum_{\gamma} T_{\gamma} : S_{L^2} \to S_{L^2} \) is invertible. Then so is the operator \( \sum_{\gamma} T_{\gamma}^* T_{\gamma} : S_{L^2} \to S_{L^2} \).

Proof. By Theorem 3.5 the invertibility of \( \sum_{\gamma} T_{\gamma} \) implies that for \( f \in S_{L^2} \),
\[
\|f\|_{L^2}^2 \approx \sum_{\gamma \in \Gamma} \|T_{\gamma} f\|_{L^2}^2 = \left( \sum_{\gamma \in \Gamma} T_{\gamma}^* T_{\gamma} f, f \right).
\]

Hence \( \sum_{\gamma} T_{\gamma}^* T_{\gamma} \) is positive definite and therefore invertible on \( S_{L^2} \).
Remark 3.8. Consider a well-spread family of self-adjoint operators \( \{ T_\gamma : \gamma \in \Gamma \} \). Since \( (\sum_\gamma T_\gamma)^2 = \sum_\gamma (T_\gamma)^2 + \sum_{\gamma \not= \gamma'} T_\gamma T_{\gamma'} \) and \( (\sum_\gamma T_\gamma)^2 \) is invertible if and only if \( \sum_\gamma T_\gamma \) is, Corollary 3.7 says that the invertibility of \( (\sum_\gamma T_\gamma)^2 \) implies that of its “diagonal part” \( \sum_\gamma (T_\gamma)^2 \). This gives further support to the interpretation of Theorem 3.7 as an almost orthogonality principle.

4. The case of Time-Frequency analysis

In this section we show how time-frequency analysis fits the abstract model from Section 2.3 and also illustrate the notions from Section 3 in this particular case. We first introduce the relevant class of function spaces.

4.1. Modulation spaces. Let \( \varphi \in \mathcal{S}(\mathbb{R}^d) \) be a non-zero function normalized by \( \| \varphi \|_2 = 1 \) and recall that the STFT of a distribution \( f \in \mathcal{S}'(\mathbb{R}^d) \) is the function \( V_\varphi f : \mathbb{R}^d \times \mathbb{R}^d \to \mathbb{C} \) given by

\[
V_\varphi f(z) = \int_{\mathbb{R}^d} f(t) \overline{\varphi(t-x)} e^{-2\pi i \xi t} dt, \quad z = (x, \xi) \in \mathbb{R}^d \times \mathbb{R}^d.
\]

Modulation spaces are defined by imposing weighted \( L^p \)-norms on the STFT, as we now describe. As weight we consider continuous functions \( v : \mathbb{R}^{2d} \to (0, +\infty) \) such that \( v(z + z') \leq C v(z)(1 + |z'|)^N \), for some constants \( N, C > 0 \) and all \( z, z' \in \mathbb{R}^{2d} \). Such functions are called polynomially moderated weights. For a polynomially moderated weight \( v \) and \( p \in [1, +\infty] \), the modulation space \( M^p_v(\mathbb{R}^d) \) is

\[
M^p_v(\mathbb{R}^d) := \{ f \in \mathcal{S}'(\mathbb{R}^d) | V_\varphi f \in L^p_v(\mathbb{R}^{2d}) \}.
\]

The space \( M^p_v(\mathbb{R}^d) \) is given the norm

\[
\|f\|_{M^p_v(\mathbb{R}^d)} := \|V_\varphi f\|_{L^p_v(\mathbb{R}^{2d})} = \left( \int_{\mathbb{R}^{2d}} |f(z)|^p v(z)^p dz \right)^{1/p},
\]

with the usual modification for \( p = +\infty \). A different choice of \( \varphi \) yields the same space with an equivalent norm, \( M^p_v(\mathbb{R}^d) \) is a Banach space, and \( V_\varphi : M^p_v(\mathbb{R}^d) \to V_\varphi(M^p_v(\mathbb{R}^d)) \) defines an isometric isomorphism, where \( V_\varphi(M^p_v(\mathbb{R}^d)) \) is considered as a subspace of \( L^p_v(\mathbb{R}^{2d}) \) (see [23], Proposition 4.3 or [33], Chapter 11). The unweighted modulation space \( M^2_v(\mathbb{R}^d) \) coincides with \( L^2(\mathbb{R}^d) \).

The assumption that \( \varphi \) is a Schwartz function can be relaxed by considering the following classes of weights.

Definition 4.1 (Admissible TF weights). A continuous even function \( w : \mathbb{R}^{2d} \to (0, +\infty) \) is called an admissible TF weight if it satisfies the following.

(i) \( w \) is submultiplicative, i.e. \( w(z + z') \leq w(z)w(z') \), for all \( z, z' \in \mathbb{R}^{2d} \).

(ii) \( w(z) \leq C(1 + |z|)^N \), for some constants \( C, N > 0 \) and all \( z \in \mathbb{R}^{2d} \).

(iii) \( w \) satisfies the following condition - known as the GRS-condition after Gelfand, Raikov and Shilov [31].

\[
\lim_{n \to \infty} w(nz)^{1/n} = 1, \quad z \in \mathbb{R}^{2d}.
\]

The polynomial weights \( w(z) = (1 + |z|)^s \) with \( s \geq 0 \) are examples of admissible TF weights. A second weight \( v : \mathbb{R}^{2d} \to (0, +\infty) \) is said to be \( w \)-moderated if

\[
v(z + z') \leq C_v w(z)v(z'), \quad z, z' \in \mathbb{R}^{2d},
\]

for some constant \( C_v \). Note that in this case \( v \) is polynomially moderated because \( w \) is dominated by a polynomial. (Do not confuse the notion of admissible TF weight with the one of being a weight admissible for a certain BF space - cf. Definition 2.2)

Let \( w \) be an admissible TF weight and let \( \varphi \in M^p_w(\mathbb{R}^d) \) be non-zero. For every \( w \)-moderated weight \( v \) and \( p \in [1, +\infty] \) the modulation space \( M^p_v(\mathbb{R}^d) \) can be described as

\[
M^p_v(\mathbb{R}^d) = \{ f \in M^p_{v,w}(\mathbb{R}^d) | V_\varphi f \in L^p_v(\mathbb{R}^{2d}) \},
\]

and \( \|V_\varphi f\|_{L^p_v(\mathbb{R}^{2d})} \) is an equivalent norm on it (see [33], Chapter 11). Hence, the assumption: \( \varphi \in M^p_{v,w}(\mathbb{R}^d) \) is enough to treat the class of all modulation spaces \( M^p_v \), with \( v \) a \( w \)-moderated weight and \( p \in [1, +\infty] \). (The GRS-condition above does not play a role in what was presented so far; it is required for the technical results used in Sections 4.3 and 4.4.)
4.2. Phase-space and modulation spaces. Extending Example 2.5 in this section we explain how modulation spaces fit the model of Section 2.3.

Let \( w \) be an admissible TF weight, \( \varphi \in M^1_w(\mathbb{R}^d) \) normalized by \( \| \varphi \|_2 = 1 \), and \( v \) a \( w \)-moderated weight. To apply the model we let \( G = \mathbb{R}^{2d} \) with the usual group structure and \( E := L^p_v(\mathbb{R}^{2d}) \). Since \( v \) is \( w \)-moderated, we deduce readily that \( w \) is admissible for \( E \).

We also let \( S_E := V_\varphi M_p^v(\mathbb{R}^{2d}) \). The fact that \( S_E \) is closed within \( L^p_v(\mathbb{R}^{2d}) \) follows from the fact that \( V_\varphi : M^p_v(\mathbb{R}^d) \to S_E \) is an isometric isomorphism and \( M^p_v(\mathbb{R}^d) \) is complete. As projection \( P : L^p_v(\mathbb{R}^{2d}) \to V_\varphi M^p_v(\mathbb{R}^{2d}) \) we consider the operator composition of \( V_\varphi \) with its formal adjoint: \( P = V_\varphi V_\varphi^* \). As discussed in Section 2.1 and Example 2.5, this operator is well-defined and satisfies

\[
|PF(x', \xi')| \leq \int_{\mathbb{R}^d \times \mathbb{R}^d} |F(x, \xi)| |\varphi_\varphi(x' - x, \xi' - \xi)| \, dx d\xi, \quad F \in L^p_v(\mathbb{R}^{2d}).
\]

The fact that \( \varphi \in M^1_w(\mathbb{R}^d) \) in principle means that \( \varphi_\varphi \in L^1_w(\mathbb{R}^{2d}) \), but it also follows that \( V_\varphi \varphi \in W(L^\infty, L^1_w)(\mathbb{R}^{2d}) \) - see for example [33, Proposition 12.1.11].

Hence, (A2)-(A3) are satisfied with \( K := |V_\varphi \varphi| \). (Note that since \( G = \mathbb{R}^{2d} \) is abelian, the left and right amalgamated spaces coincide.)

Recall the time-frequency localization operator with symbol \( m \in L^\infty(\mathbb{R}^{2d}) \) formally given by \( H_m f = V_\varphi \varphi m(V_\varphi f) \), cf. [14]. In Section 2.1 we discussed the operator \( H_m : L^2(\mathbb{R}^d) \to L^2(\mathbb{R}^d) \). Analogous considerations apply to \( H_m : M^p_v(\mathbb{R}^d) \to M^p_v(\mathbb{R}^d) \) and show that \( V_\varphi H_m V_\varphi^* F = P(mF) \) for all \( F \in V_\varphi M^p_v(\mathbb{R}^d) \). This means that under the isometric isomorphism \( V_\varphi : M^p_v(\mathbb{R}^d) \to V_\varphi M^p_v(\mathbb{R}^d) \), the time-frequency localization operator \( H_m \) corresponds to phase-space multiplier \( M_m \) from Section 2.3.3.

Remark 4.2. In this article we will be concerned with time-frequency localization operators with bounded symbols \( m \). However the condition that \( m \) be bounded is by no means necessary for \( H_m : M^p_v(\mathbb{R}^d) \to M^p_v(\mathbb{R}^d) \) to be bounded. See for example [27, 12] for sharper boundedness results for time-frequency localization operators.

4.3. Spectral Invariance. We present a technical result that says that for a certain class of operators, the property of being invertible on \( L^2(\mathbb{R}^d) \) automatically implies invertibility on a range of modulation spaces. The following proposition is obtained by combining known spectral invariance results for pseudodifferential operators [50, 51, 35, 44]. For convenience of the reader, in Appendix B we provide the necessary background results and give a proof based on those results.

Proposition 4.3. Let \( w : \mathbb{R}^{2d} \to (0, +\infty) \) be an admissible TF weight - cf. Section 2.1 - and let \( \varphi \in M^1_w(\mathbb{R}^d) \) be non-zero. Let \( v \) be a \( w \)-moderated weight and let \( p \in [1, +\infty] \). Let \( T : M^p_v(\mathbb{R}^d) \to \mathbb{R}^{2d} \) be an operator satisfying the enveloping condition

\[
|V_\varphi T(f)(z)| \leq \int_{\mathbb{R}^{2d}} |V_\varphi f(z')| H(z - z') \, dz', \quad z \in \mathbb{R}^{2d},
\]

for some function \( H \in L^1_v(\mathbb{R}^{2d}) \). Assume that \( T : L^2(\mathbb{R}^d) \to L^2(\mathbb{R}^d) \) is invertible. Then \( T : M^p_v(\mathbb{R}^d) \to M^p_v(\mathbb{R}^d) \) is invertible.

4.4. Families of operators well-spread in the time-frequency plane. We now illustrate the notions and results from Section 3 in the case of time-frequency analysis.

Let \( w : \mathbb{R}^{2d} \to (0, +\infty) \) be an admissible TF weight and \( \varphi \in M^1_w(\mathbb{R}^d) \) with \( \| \varphi \|_2 = 1 \). Setting \( G = \mathbb{R}^{2d} \) in [18] we call a set \( \Gamma \subseteq \mathbb{R}^{2d} \) relatively separated if \( \#(\Gamma \cap B_1(z)) < +\infty \). In analogy to Definition 3.2 a family of symbols \( \{ \eta_\gamma : \mathbb{R}^{2d} \to \mathbb{C} | \gamma \in \Gamma \} \) is well-spread (relative to \( w \)) if \( \Gamma \) is relatively separated and there exists \( g \in W(L^\infty, L^1_w)(\mathbb{R}^{2d}) \) such that

\[
|\eta_\gamma(z)| \leq g(z - \gamma), \quad z \in \mathbb{R}^{2d}, \gamma \in \Gamma.
\]

Similarly, in analogy to Definition 3.1 a family of operators \( \{ H_\gamma : M^\infty_{1/w}(\mathbb{R}^d) \to M^\infty_{1/w}(\mathbb{R}^d) | \gamma \in \Gamma \} \) is said to be well-spread in the time-frequency plane (relative to \( (\varphi, w) \)) if there exists an envelope \( (\Gamma, \Theta, g) \), with \( \Gamma \subseteq \mathbb{R}^{2d} \) a relatively separated set and \( \Theta, g \in W(L^\infty, L^1_w)(\mathbb{R}^{2d}) \) such that

\[
|V_\varphi H_\gamma f(z)| \leq \int_{\mathbb{R}^{2d}} g(z' - \gamma) |V_\varphi f(z')| \Theta(z - z') \, dz', \quad z \in \mathbb{R}^{2d}, \gamma \in \Gamma.
\]
Note that \( \{ H_\gamma : \gamma \in \Gamma \} \) is well-spread in the TF plane if and only if the family of operators \( \{ V_\varphi H_\gamma V_\varphi^* : V_\varphi(M_1^{\infty,1}(\mathbb{R}^d)) \to V_\varphi(M_1^{\infty,1}(\mathbb{R}^d)) \mid \gamma \in \Gamma \} \) is well-spread in the sense of Section 3.

Note also that, if \( \{ H_\gamma : \gamma \in \Gamma \} \) is well-spread in the time-frequency plane, then, because of \( (33) \), for all \( w \)-moderated weights \( v \) and \( p \in [1, +\infty] \) each operator \( H_\gamma \) maps \( M^p_v(\mathbb{R}^d) \) into \( M^p_v(\mathbb{R}^d) \) with a norm bound independent of \( \gamma \),

\[
\| H_\gamma f \|_{M^p_v} \leq C_v \| g \|_\infty \| \Theta \|_{L^p_{\Delta v}} \| f \|_{M^p_v}.
\]

**Remark 4.4.** In parallel to Proposition 3.3, if \( \{ \eta_\gamma : \gamma \in \Gamma \} \) is a well-spread family of symbols, then the corresponding family of time-frequency localization operators \( \{ H_{\eta_\gamma} : \gamma \in \Gamma \} \) is well-spread in the time-frequency plane.

In the case of time-frequency analysis we can strengthen Theorem 3.5 by means of the spectral invariance result in Proposition 4.3. Due to this result, the invertibility assumption in Theorem 3.5 can be replaced by assuming invertibility on \( L^2(\mathbb{R}^d) \).

**Theorem 4.5.** Let \( w : \mathbb{R}^d \to (0, +\infty) \) be an admissible TF weight and let \( \varphi \in M^1_w(\mathbb{R}^d) \). Let \( \{ H_\gamma : \gamma \in \Gamma \} \) be well-spread in the TF plane (relative to \( (\varphi, w) \)).

Suppose that the operator \( \sum_{\gamma \in \Gamma} H_\gamma : L^2(\mathbb{R}^d) \to L^2(\mathbb{R}^d) \) is invertible. Then, for all \( w \)-moderated weights \( v \) and for all \( 1 \leq p \leq +\infty \), \( \sum_{\gamma \in \Gamma} H_\gamma : M^p_v \to M^p_v \) is invertible and the following norm equivalence holds

\[
\| f \|_{M^p_v} \approx \left( \sum_{\gamma \in \Gamma} \| H_\gamma f \|_{L^2(\mathbb{R}^d)}^p \right)^{1/p}, \quad f \in M^p_v(\mathbb{R}^d),
\]

with the usual modification for \( p = \infty \).

**Remark 4.6.** The estimates hold uniformly for \( 1 \leq p \leq +\infty \) and any family of weights having a uniform constant \( C_v \) (cf. \( (30) \)).

**Proof of Theorem 4.5.** Let \( v \) be a \( w \)-moderated weight and \( 1 \leq p \leq +\infty \). By hypothesis \( \sum_{\gamma \in \Gamma} H_\gamma : L^2(\mathbb{R}^d) \to L^2(\mathbb{R}^d) \) is invertible. In order to apply Proposition 4.3 to this operator we let \( f \in M^p_v(\mathbb{R}^d) \) and use \( (33) \) to estimate

\[
\left| V_\varphi \left( \sum_{\gamma \in \Gamma} H_\gamma f \right)(z) \right| \leq \int_{\mathbb{R}^d} |V_\varphi f(z')| \Theta(z-z') \sum_{\gamma \in \Gamma} g(z' - \gamma) dz'
\]

\[
\lesssim \| g \|_{W(L^\infty, L^1_v)} \int_{\mathbb{R}^d} |V_\varphi f(z')| \Theta(z-z') dz'.
\]

Since \( \Theta \in W(L^\infty, L^1_v)(\mathbb{R}^d) \), it follows from Proposition 4.3 that \( \sum_{\gamma \in \Gamma} H_\gamma : M^p_v(\mathbb{R}^d) \to M^p_v(\mathbb{R}^d) \) is invertible.

We use the notation \( S^p_v := V_\varphi(M^p_v(\mathbb{R}^d)) \). Let \( T_\gamma := V_\varphi H_\gamma V_\varphi^* : S^p_v \to S^p_v \). The family \( \{ T_\gamma : \gamma \in \Gamma \} \) is well-spread (in the sense of Section 3). Since \( \sum_{\gamma \in \Gamma} H_\gamma : M^p_v(\mathbb{R}^d) \to M^p_v(\mathbb{R}^d) \) is invertible, so is \( \sum_{\gamma \in \Gamma} T_\gamma : S^p_v \to S^p_v \). Therefore, we can apply Theorem 3.5 and the conclusion follows.

### 5. Frames of eigenfunctions of time-frequency localization operators

In this section we prove the main results on constructing frames with a prescribed time-frequency profile. Throughout this section let \( w : \mathbb{R}^d \to (0, +\infty) \) be an admissible TF weight and \( \varphi \in M^1_w(\mathbb{R}^d) \) be normalized by \( \| \varphi \|_2 = 1 \).

**5.1. Thresholding eigenvalues.** Let \( \{ \eta_\gamma : \gamma \in \Gamma \} \) be a family of non-negative functions on \( \mathbb{R}^d \) that is well-spread (relative to \( w \)) and consider the corresponding family of time-frequency localization operators \( H_{\eta_\gamma} f = V_\varphi^* (\eta_\gamma V_\varphi f) \). Since each \( \eta_\gamma \) is non-negative and belongs to \( L^1(\mathbb{R}^d) \), \( H_{\eta_\gamma} \) is positive
and trace class, and trace\((H_{\eta_\gamma}) = \|\eta_\gamma\|_1\), see \[\text{[27, 54, 8]}\] and Proposition \[\text{[2.10]}\]. Hence \(H_{\eta_\gamma} : L^2(\mathbb{R}^d) \rightarrow L^2(\mathbb{R}^d)\) can be diagonalized as

\[
H_{\eta_\gamma} f = \sum_{k \geq 1} \lambda_k^n(\eta, f_\gamma) \phi_k^n, \quad f \in L^2(\mathbb{R}^d)
\]

where \(\{ \phi_k^n \mid k \in \mathbb{N} \}\) is an orthonormal subset of \(L^2(\mathbb{R}^d)\) - possibly incomplete if \(\ker(H_{\eta_\gamma}) \neq \{0\}\) - and \((\lambda_k^n)_k\) is a non-increasing sequence of non-negative real numbers.

The time-frequency profile of the functions \(\{ \phi_k^n \mid k \in \mathbb{N} \}\) is optimally adapted to the mask \(\eta_\gamma\) in the following sense. For each \(N \in \mathbb{N}\), the set \(\{ \phi_1^n, \ldots, \phi_N^n \}\) is an orthonormal set maximizing the quantity,

\[
\sum_{k=1}^N \int_{\mathbb{R}^d} \eta_\gamma(z) |V_\varphi \phi_k(z)|^2 \, dz,
\]

among all orthonormal sets \(\{f_1, \ldots, f_N\} \subseteq L^2(\mathbb{R}^d)\).

Moreover, since \(\eta_\gamma \in L^1_\nu\) and \(\varphi \in M^1_\nu\), \(\phi_k^n \in M^1_\nu\) provided that \(\lambda_k^n \neq 0\) (see for example \[\text{[20] Lemma 5}\]).

For every \(\varepsilon > 0\), we define the operator \(H_{\eta_\gamma}^\varepsilon\) by applying a threshold to the eigenvalues of \(H_{\eta_\gamma}\),

\[
H_{\eta_\gamma}^\varepsilon f := \sum_{k: \lambda_k^n > \varepsilon} \lambda_k^n(\eta, f_\gamma) \phi_k^n, \quad f \in L^2(\mathbb{R}^d).
\]

Hence,

\[
\|H_{\eta_\gamma}^\varepsilon f\|_2 \leq \|H_{\eta_\gamma} f\|_2 \leq \|H_{\eta_\gamma}^\varepsilon f\|_2 + \varepsilon \|f\|_2, \quad f \in L^2(\mathbb{R}^d).
\]

As a first step to analyze the effect of the thresholding operation \(H_{\eta_\gamma} \mapsto H_{\eta_\gamma}^\varepsilon\), we show that the operators \(\{H_{\eta_\gamma}: \gamma \in \Gamma\}\) behave “globally” like projectors. Since these operators may have infinite rank, in general \(\|(H_{\eta_\gamma})^2 f\|_2 \neq \|H_{\eta_\gamma} f\|_2\) holds. However, we prove the following.

**Proposition 5.1.** Let \(w : \mathbb{R}^{2d} \rightarrow (0, +\infty)\) be an admissible TF weight and let \(\varphi \in M^1_\nu(\mathbb{R}^d)\). Let \(\{\eta_\gamma: \gamma \in \Gamma\}\) be a well-spread family of non-negative symbols on \(\mathbb{R}^{2d}\) (relative to \(w\)) with \(\sum_{\gamma} \eta_\gamma \approx 1\). Then for all \(w\)-moderated weight \(v\) and all \(p \in [1, +\infty]\)

\[
\|f\|_{M_p^w} \approx \left( \sum_{\gamma \in \Gamma} \|H_{\eta_\gamma} f\|_{L^2(\mathbb{R}^d)}^p v(\gamma)^p \right)^{1/p} \approx \left( \sum_{\gamma \in \Gamma} \|(H_{\eta_\gamma})^2 f\|_{L^2(\mathbb{R}^d)}^p v(\gamma)^p \right)^{1/p}, \quad f \in M_p^w(\mathbb{R}^d).
\]

**Remark 5.2.** The estimate \(\|f\|_{M_p^w} \approx \|(H_{\eta_\gamma} f)\|_{L^2(\mathbb{R}^d)}\) is contained in \[\text{[19, 20]}\] for the case of families of symbols consisting of lattice translates of a single function, and in \[\text{[17]}\] in the present generality. The norm equivalence involving \((H_{\eta_\gamma})^2 f\) is new.

**Proof of Proposition 5.1.** Since the symbols \(\eta_\gamma\) satisfy \(m := \sum_{\gamma} \eta_\gamma \geq A\) for some constant \(A > 0\) it follows that

\[
\left\langle \sum_{\gamma \in \Gamma} H_{\eta_\gamma} f, f \right\rangle = \left\langle H_m f, f \right\rangle = \left\langle V_\varphi^* (m V_\varphi f), f \right\rangle
\]

\[
= \left\langle m V_\varphi f, V_\varphi f \right\rangle = \int_{\mathbb{R}^{2d}} m(z) |V_\varphi f(z)|^2 \, dz \geq A \|V_\varphi f\|_2^2 = A \|f\|_2^2.
\]

Hence, \(\sum_{\gamma} H_{\eta_\gamma} : L^2(\mathbb{R}^d) \rightarrow L^2(\mathbb{R}^d)\) is invertible. Now Proposition \[\text{[3.3]}\] and Theorem \[\text{[4.5]}\] yield the estimate involving \(\|f\|_{M_p^w}\) and \(\|(H_{\eta_\gamma} f)\|_{L^2(\mathbb{R}^d)}\).

For the estimate involving the squared operators, note that Corollary \[\text{[3.7]}\] implies that \(\sum_{\gamma} (H_{\eta_\gamma})^2 : S_{L^2}^2 \rightarrow S_{L^2}^2\) is also invertible. According to Proposition \[\text{[3.4]}\] the family \(\{H_{\eta_\gamma}: \gamma \in \Gamma\}\) is also well-spread. Hence, a second application of Theorem \[\text{[4.5]}\] concludes the proof.

We now state and prove that a sufficiently fine thresholding of the eigenvalues of the operators \(\{H_{\eta_\gamma}: \gamma \in \Gamma\}\) still decomposes the family of modulation spaces.
Theorem 5.3. Let \( w : \mathbb{R}^d \rightarrow (0, +\infty) \) be an admissible TF weight and let \( \varphi \in M^1_w(\mathbb{R}^d) \). Let \( \{\eta_\gamma : \gamma \in \Gamma\} \) be a family of non-negative symbols on \( \mathbb{R}^d \) that is well-spread (relative to \( w \)) and such that \( \sum_{\gamma} \eta_\gamma \approx 1 \). Let \( v \) be a \( w \)-moderated weight. Then there exist constants \( 0 < c \leq C < +\infty \) such that for all sufficiently small \( \varepsilon > 0 \) and all \( p \in [1, +\infty] \)

\[
\| f \|_{M^p_w} \leq \left( \sum_{\gamma \in \Gamma} \| H_{\eta_\gamma}^e f \|_{L^2(\mathbb{R}^d)}^p v(\gamma) \right)^{1/p} \leq C \| f \|_{M^p_w}, \quad f \in M^p_w(\mathbb{R}^d),
\]

with the usual modification for \( p = \infty \).

The choice of \( \varepsilon \) and the estimates are uniform for \( 1 \leq p \leq +\infty \) and any family of weights having a uniform constant \( C_v \) (cf. (30)).

Proof. Given \( \varepsilon > 0 \) and \( f \in M^p_w(\mathbb{R}^d) \) we apply (37) to \( H_{\eta_\gamma}(f) \) to obtain

\[
\| (H_{\eta_\gamma})^2 f \|_2 \leq \| H_{\eta_\gamma} \|_2 + \varepsilon \| H_{\eta_\gamma} \|_2.
\]

Since \( H_{\eta_\gamma} \) and \( H_{\eta_\gamma} \) commute, \( \| H_{\eta_\gamma} H_{\eta_\gamma} f \|_2 = \| H_{\eta_\gamma} H_{\eta_\gamma} f \|_2 \lesssim \| H_{\eta_\gamma} f \|_2 \). Hence,

\[
\| (H_{\eta_\gamma})^2 f \|_2 \lesssim \| H_{\eta_\gamma} f \|_2 + \varepsilon \| H_{\eta_\gamma} f \|_2.
\]

Taking \( \ell^p_v \) norm on \( \gamma \) yields

\[
\| (H_{\eta_\gamma}^e f) \|_{L^2(\mathbb{R}^d)} \|_{\ell^p_v} \lesssim \| (H_{\eta_\gamma} f) \|_{L^2(\mathbb{R}^d)} \|_{\ell^p_v} + \varepsilon \| (H_{\eta_\gamma} f) \|_{L^2(\mathbb{R}^d)} \|_{\ell^p_v}.
\]

Using the estimates in Proposition 5.1 we get

\[
\| f \|_{M^p_w} \leq C \| (H_{\eta_\gamma}^e f) \|_{L^2(\mathbb{R}^d)} \|_{\ell^p_v} + \varepsilon \| f \|_{M^p_w},
\]

for some constants \( c, C \). Hence,

\[
(1 - \varepsilon) \| f \|_{M^p_w} \leq C \| (H_{\eta_\gamma}^e f) \|_{L^2(\mathbb{R}^d)} \|_{\ell^p_v}.
\]

This gives the desired lower bound (for all \( 0 < \varepsilon < 1/c \)). The upper bound follows from the first inequality in (37) and Proposition 5.1. \( \square \)

Remark 5.4. Note that the proof of Theorem 5.3 only uses the estimate in (37) and the fact that \( H_{\eta_\gamma} \) and \( H_{\eta_\gamma}^e \) commute. Hence, more general thresholding rules besides the one in (36) can be used.

5.2. Frames of eigenfunctions. Finally, we obtain the desired result on frames of eigenfunctions.

Theorem 5.5. Let \( w : \mathbb{R}^d \rightarrow (0, +\infty) \) be an admissible TF weight and let \( \varphi \in M^1_w(\mathbb{R}^d) \). Let \( \{\eta_\gamma : \gamma \in \Gamma\} \) be a family of non-negative symbols on \( \mathbb{R}^d \) that is well-spread (relative to \( w \)) and such that \( \sum_{\gamma} \eta_\gamma \approx 1 \). Let \( v \) be a \( w \)-moderated weight. Then there exists a constant \( \alpha > 0 \) such that, for every choice of finite subsets of eigenfunctions of \( H_{\eta_\gamma} \) \( \{\phi^\gamma_k : \gamma \in \Gamma, 1 \leq k \leq N_\gamma\} \) with

\[
\alpha \| \eta_\gamma \|_1 \leq N_\gamma \quad \text{and} \quad \sup_{\gamma \in \Gamma} N_\gamma < +\infty,
\]

the following frame estimates hold simultaneously for all \( 1 \leq p \leq +\infty \), with the usual modification for \( p = \infty \):

\[
\| f \|_{M^p_w} \approx \left( \sum_{\gamma \in \Gamma} \sum_{k=1}^{N_\gamma} \langle f, \lambda_k^\gamma \phi^\gamma_k \rangle^p v(\gamma) \right)^{1/p}, \quad f \in M^p_w(\mathbb{R}^d).
\]

Moreover, \( \alpha \) can be chosen uniformly for any class of weights \( v \) having a uniform constant \( C_v \) (cf. (30)).

Before proving Theorem 5.5 we make some remarks.

Remark 5.6. As opposed to other constructions that partition either the time or frequency domain (see e.g. [24, 4, 9, 10]), the symbols \( \eta_\gamma \) partition the time-frequency plane simultaneously.

Remark 5.7. Note that, if \( (\Gamma, \Theta, g) \) is an envelope for \( \{\eta_\gamma : \gamma \in \Gamma\} \), since \( \| \eta_\gamma \|_1 \leq ||g||_1 \), in Theorem 5.5 it is always possible to make a uniform choice \( N_\gamma = N_0 \).
Proof of Theorem 5.5. For every $\gamma \in \Gamma$ and $\varepsilon > 0$, let $I_\varepsilon^\gamma := \{k \in \mathbb{N} | \lambda_k^\gamma > \varepsilon\}$, which is a finite set. Using Theorem 5.5 and Proposition 5.1 and the orthonormality of the eigenfunctions, we can find a value of $\varepsilon > 0$ such that

$$\|f\|_{M^p_\varepsilon} \approx \left( \sum_{\gamma \in \Gamma} \left( \sum_{k \in \mathbb{N}} |(f, \lambda_k^\gamma \phi_k^\gamma)|^2 \right)^{p/2} v(\gamma)^{p} \right)^{1/p} \approx \left( \sum_{\gamma \in \Gamma} \left( \sum_{k \in I_\varepsilon^\gamma} |(f, \lambda_k^\gamma \phi_k^\gamma)|^2 \right)^{p/2} v(\gamma)^{p} \right)^{1/p},$$

with the usual modification for $p = +\infty$. This implies that for any choice of subsets of indices $J_\varepsilon^\gamma \supseteq I_\varepsilon^\gamma$ we also have,

$$\|f\|_{M^p_\varepsilon} \approx \left( \sum_{\gamma \in \Gamma} \left( \sum_{k \in J_\varepsilon^\gamma} |(f, \lambda_k^\gamma \phi_k^\gamma)|^2 \right)^{p/2} v(\gamma)^{p} \right)^{1/p}. \quad (39)$$

Furthermore, since $\sum_k \lambda_k^\gamma = \text{trace}(H_\gamma) = \|\eta\|_1$, we have $\# I_\varepsilon^\gamma \leq \varepsilon^{-1} \sum_k \lambda_k^\gamma = \varepsilon^{-1} \|\eta\|_1$. Hence, setting $\alpha := \varepsilon^{-1}$, we ensure that for $N_\gamma \geq \alpha \|\eta\|_1$, the set $J_\varepsilon^\gamma := \{k \in \mathbb{N} | 1 \leq k \leq N_\gamma \}$ contains $I_\varepsilon^\gamma$ and therefore satisfies $\varepsilon$. Let $N := \sup_\gamma N_\gamma$. By hypothesis, $N < \infty$. Since $\# J_\varepsilon^\gamma = N_\gamma \leq N$, it follows that

$$\left( \sum_{k \in J_\varepsilon^\gamma} |(f, \lambda_k^\gamma \phi_k^\gamma)|^2 \right)^{1/2} \approx \left( \sum_{k \in J_\varepsilon^\gamma} |(f, \lambda_k^\gamma \phi_k^\gamma)|^p \right)^{1/p},$$

with a constant that depends on $N$ (and the usual modification for $p = +\infty$). Combining the last estimate with (39) we obtain the desired conclusion. \qed

As an application of Theorem 5.5 we can now prove Theorem 1.1 (stated in the Introduction).

Proof of Theorem 1.1. Let $\{\Omega, \gamma \in \Gamma\}$ be an admissible cover of $\mathbb{R}^d$, as defined in the Introduction - cf. [5]. Let $\eta_\gamma := 1_{\Omega_\gamma}$ be the characteristic function of $\Omega_\gamma$. The fact that $\{\Omega, \gamma \in \Gamma\}$ covers $\mathbb{R}^d$ implies that $\sum_\gamma \eta_\gamma = \sum_\gamma 1_{\Omega_\gamma} \geq 1$. In addition, if we let $g := 1_{B_{R}(0)}$, it follows from [5] that $\{\eta_\gamma : \gamma \in \Gamma\}$ is well-spread with envelope $(\Gamma, g)$ and constant weight $v \equiv 1$. We note that $|\Omega_\gamma| = \|\eta_\gamma\|_1$ and apply Theorem 5.5 with $p = 2$ and $v \equiv 1$. (Recall that $M^2(\mathbb{R}^d) = L^2(\mathbb{R}^d).$) \qed

5.3. Inner regularity. Finally we derive a variant of Theorem 5.5 where, under an inner regularity assumption on the family of symbols, we renormalize the frame of eigenfunctions so that each frame element has norm 1. To this end we first prove the following lemma which may be of independent interest.

Lemma 5.8. Let $\varphi \in L^2(\mathbb{R}^d) \setminus \{0\}$ and let $\Omega \subseteq \mathbb{R}^d$ be a measurable set with non-empty interior. Then the time-frequency localization operator $H_\Omega : L^2(\mathbb{R}^d) \to L^2(\mathbb{R}^d)$,

$$H_\Omega f(t) = \int_\Omega V_\varphi f(x, \xi) \varphi(t-x)e^{2\pi i\xi t} dx d\xi, \quad t \in \mathbb{R}^d,$$

has infinite rank.

Proof. The proof is based on the fact that the STFT of Hermite functions are weighted polyanalytic functions (cf. [1],[2],[3]) and therefore cannot vanish on a ball.

Suppose, for the sake of contradiction, that $H_\Omega$ has rank $n - 1 < +\infty$. Let $h_1, \ldots, h_n \in L^2(\mathbb{R}^d)$ be multi-dimensional Hermite functions of order $\leq n$. For example, if $g_1, \ldots, g_n \subseteq L^2(\mathbb{R})$ are the first one-dimensional Hermite functions, we can let $h_k \in L^2(\mathbb{R}^d)$ be the tensor product $h_k(x_1, \ldots, x_d) := g_k(x_1)g_1(x_2) \ldots g_1(x_d)$. Let $V_n$ be the subspace of $L^2(\mathbb{R}^d)$ spanned by $h_1, \ldots, h_n$.

Since $V_n$ has dimension $n$, it follows that there exists some nonzero $f = \sum_k c_k h_k \in V_n$ such that $H_\Omega f = 0$. Consequently, $0 = \langle H_\Omega f, f \rangle = \int_\Omega |V_\varphi f(z)|^2 dz$,
and therefore $V_\varphi f \equiv 0$ on $\Omega$. With the notation $(x, \xi) =: z \in \mathbb{C}^n$ and $m(z) = e^{-x\xi + \pi|z|^2}$, let $F(z) := m(z)V_\varphi \varphi(z)$. Since $V_\varphi f(z) = V_\varphi \varphi(-z)$, it follows that $F$ vanishes on $\Omega' := -\Omega$. We will show that $F \equiv 0$. Since $m$ never vanishes, this will imply that $f \equiv 0$, thus yielding a contradiction.

The function $F(z) = \sum_k c_k m(z)V_{h_k} \varphi(-z)$ is a polyanalytic function of order (at most) $n$, i.e. $(\partial/\partial x)^d F \equiv 0$, for every multi-index $|\beta| \leq n$, see [11][12]. A polyanalytic function that vanishes on a set of non-empty interior must vanish identically. For $d = 1$ this can be proved directly by induction on $n$ or deduced from much sharper uniqueness results (see [14]). The case of general dimension $d$ reduces to $d = 1$ by fixing $d - 1$ variables of $F$ and applying the one-dimensional result. \hfill \Box

We also quote the following intertwining property.

Lemma 5.9. For $z = (x, \xi) \in \mathbb{R}^d \times \mathbb{R}^d$, consider the time-frequency shift
\[
\pi(z)f(t) = e^{2\pi i \xi t}f(t-x), \quad f \in L^2(\mathbb{R}^d).
\]
For $m \in L^\infty(\mathbb{R}^2d)$, let $H_m : L^2(\mathbb{R}^d) \to L^2(\mathbb{R}^d)$ be the time-frequency localization operator with symbol $m$, i.e. $H_m f := V_\varphi^* (mV_\varphi f)$. Then
\[
\pi(z)H_m \pi(z)^* = H_m(-z), \quad z = (x, \xi) \in \mathbb{R}^d \times \mathbb{R}^d, m \in L^\infty(\mathbb{R}^2d).
\]

The proof of Lemma 5.9 is a straightforward calculation, see [19, Lemma 2.6]. Using Lemma 5.9 we obtain a variant of Theorem 5.10.

Theorem 5.10. Let $w : \mathbb{R}^{2d} \to (0, +\infty)$ be an admissible TF weight and let $\varphi \in M_{\infty}(\mathbb{R}^d)$. Let $\{\eta_\gamma : \gamma \in \Gamma\}$ be a family of non-negative symbols on $\mathbb{R}^{2d}$ that is well-spread (relative to $w$) and such that $\sum_\gamma \eta_\gamma \approx 1$. Let $v$ be a $w$-moderated weight. Assume in addition that there exists a ball $B_r$ and a constant $c > 0$ such that
\[
\eta_\gamma(z) \geq c1_{B_r}(z-\gamma), \quad z \in \mathbb{R}^{2d}, \gamma \in \Gamma.
\]
Then there exists a constant $\alpha > 0$ such that, for every choice of finite subsets of eigenfunctions $\{\phi_k^\gamma \mid \gamma \in \Gamma, 1 \leq k \leq N_\gamma\}$ with $\inf_\gamma N_\gamma \geq \alpha$ and $\sup_\gamma N_\gamma < +\infty$, the following frame estimates hold simultaneously for all $1 \leq p \leq +\infty$, with the usual modification for $p = \infty$:
\[
\|f\|_{M^p_{\varphi}} \approx \left( \sum_{\gamma \in \Gamma} \sum_{k=1}^{N_\gamma} |\langle f, \phi_k^\gamma \rangle|^p v(\gamma)^p \right)^{1/p}, \quad f \in M^p_{\varphi}(\mathbb{R}^d).
\]
Moreover, $\alpha$ can be chosen uniformly for any class of weights $v$ having a uniform constant $C_v$ (cf. (30)).

Before proving Theorem 5.10 we make some remarks.

Remark 5.11. When $\eta_\gamma$ is the characteristic function of a set $\Omega_\gamma$, the condition in (40) holds whenever the sets satisfy: $B_r(\gamma) \subseteq \Omega_\gamma \subseteq B_R(\gamma)$, with $R > r > 0$ and $\Gamma$ a relatively separated set.

Remark 5.12. The frame in Theorem 5.10 comprises the first $N_\gamma$ elements of each of the orthonormal sets $\{\phi_k^\gamma : k \geq 1\}$. These first $N_\gamma$ functions are the ones that are best concentrated, according to the weight $\eta_\gamma$. This resembles the problem studied in [10]. However the results there require very precise information on the frames being pieced together and hence do not apply here.

Remark 5.13. In the language of [33][34], Theorem 5.10 shows that the subspaces spanned by the finite families of eigenfunctions form a stable splitting or fusion frame. From an application point of view, it is useful to have orthogonal projections onto subspaces with time-frequency concentration in a prescribed area of the time-frequency plane.

Remark 5.14. When $\eta_\gamma$ is the characteristic function of a set $\Omega + \gamma$ and $\Gamma$ is a lattice, then Theorem 5.10 reduces to the main technical result in [20]. The proof there does not adapt to the irregular context, since it relies on the use of rotation algebras (non-commutative tori). The proof we give here resorts instead to the almost-orthogonality techniques from [17] (cf. Theorem 3.3) together with spectral invariance results for pseudodifferential operators with symbols in the Sjöstrand class [50][51][53][54].
Proof of Theorem 5.10. First note that (40), together with the well-spreadness condition, implies that \( \|\eta_\gamma\|_1 \approx 1 \), (the constants, of course, depend on \( r, R \) and \( c \)). Hence the condition \( \alpha \|\eta_\gamma\|_1 \leq N_\gamma \) required by Theorem 5.5 can be granted by simply requiring \( \alpha \lesssim N_\gamma \) to hold with a different constant \( \tilde{\alpha} \).

By Theorem 5.5 we have that \( \|f\|_{M^p} \approx \left( \sum_{\gamma \in \Gamma} \sum_{k=1}^{N_\gamma} |\langle f, \lambda_k^\gamma \phi_k^\gamma \rangle|^p v(\gamma)^p \right)^{1/p} \). Hence, it suffices to show that \( \lambda_k^\gamma \approx 1 \), for \( 1 \leq k \leq N_\gamma \).

The upper bound follows from the well-spreadness condition, because if \( (\Gamma, \Theta, g) \) is an envelope for \( \{H_{\eta_\gamma} : \gamma \in \Gamma\} \), then all the singular values of \( H_{\eta_\gamma} \) are bounded by \( \|H_{\eta_\gamma}\|_{2 \rightarrow 2} \leq \|g\|_\infty \), cf. (24).

Let \( N := \sup_\gamma N_\gamma \). By Lemma 5.8, the localization operator \( H_{B_r} \) has infinite rank. Hence, the non-zero eigenvalues of \( H_{B_r} \) form an infinite non-increasing sequence \( \lambda_k^R > 0, \; k \geq 1 \). From (40) it follows that \( H_{\eta_\gamma} \approx cH_{B_{R,\gamma}} \) (cf. Proposition 2.10). By Lemma 5.9, the sequence of eigenvalues of \( H_{B_{R,\gamma}} \) coincides with the one of \( H_{B_{r,\gamma}} \). Consequently, for \( 1 \leq k \leq N_\gamma \),

\[
\lambda_k^\gamma \geq c\lambda_k^R \geq c\lambda_k^N \geq c\lambda_k^R > 0,
\]
as desired. \( \square \)

Remark 5.15. For the results in this section, the abstract setting of Section 2.3 allows for the replacement of \( L_p^\alpha \) by more general normed spaces. Indeed, the results derived in the abstract setting cover modulation spaces defined with respect to general translation-invariant solid spaces, cf. [25].

6. Frames of eigenfunctions: general \( L^2 \) estimates

In this section we prove results similar to the ones in Section 5, but this time in the abstract setting of Section 3. We work only with the space \( L^2 \) instead of treating a class of Banach spaces. The reason for this restriction is that in Section 5 we used tools from the theory of pseudodifferential operators to extend certain results from \( L^2(\mathbb{R}^d) \) to a range of modulation spaces, and those tools are not available in the abstract setting. The proofs in this section are, mutatis mutandis, the same as in Section 4 and will be just sketched.

Let \( \mathcal{G} \) be a locally compact, \( \sigma \)-compact group. Let \( E = L^2(\mathcal{G}) \) and let us assume that (A1), (A2) and (A3) from Section 2.3.2 hold. Let a well-spread family \( \{\eta_\gamma : \gamma \in \Gamma\} \) of non-negative functions on \( \mathcal{G} \) be given. We consider the corresponding family of phase-space multipliers,

\[
M_{\eta_\gamma} f = P(\eta_\gamma f), \quad f \in S_{L^2}.
\]

Since each \( \eta_\gamma \) is non-negative and belongs to \( L^1(\mathcal{G}) \), according to Proposition 2.10, the corresponding operator \( M_{\eta_\gamma} : S_{L^2} \to S_{L^2} \) is positive and trace-class, and trace\( (M_{\eta_\gamma}) \lesssim \|\eta_\gamma\|_1 \). Let \( M_{\eta_\gamma} : S_{L^2} \to S_{L^2} \) be diagonalized as

\[
M_{\eta_\gamma} f = \sum_{k \geq 1} \lambda_k^\gamma \langle f, \phi_k^\gamma \rangle \phi_k^\gamma, \quad f \in S_{L^2},
\]

where \( \{ \phi_k^\gamma \mid k \geq 1 \} \) is an orthonormal subset of \( S_{L^2} \) and \( \{ \lambda_k^\gamma \mid k \geq 1 \} \) is decreasing. Let us define

\[
M_{\eta_\gamma}^\varepsilon f = \sum_{k : \lambda_k^\gamma > \varepsilon} \lambda_k^\gamma \langle f, \phi_k^\gamma \rangle \phi_k^\gamma, \quad f \in S_{L^2}.
\]

If \( M_{\eta_\gamma} \) has finite rank, then \( \lambda_k^\gamma = 0 \), for \( k \gg 1 \) and the choice of the corresponding eigenfunctions is arbitrary.

We now derive results similar to Theorems 5.3 and 5.5 but this time in the current abstract setting.

Theorem 6.1. Under Assumptions (A1), (A2) and (A3), let \( \{\eta_\gamma : \gamma \in \Gamma\} \) be a well-spread family of non-negative symbols such that \( \sum_\gamma \eta_\gamma \approx 1 \). Then there exist constants \( 0 < c \leq C < +\infty \) such that for all sufficiently small \( \varepsilon > 0 \),

\[
c\|f\|_2^2 \leq \sum_{\gamma \in \Gamma} \|M_{\eta_\gamma}^\varepsilon f\|_2^2 \leq C\|f\|_2^2, \quad f \in S_{L^2}.
\]
Remark 6.2. Note again that when $\eta_\gamma$ is the characteristic function of a set $\Omega_\gamma$, we are picking $\approx |\Omega_\gamma|$ eigenfunctions from each phase-space multiplier. Here, $|\Omega_\gamma|$ is the Haar measure of $\Omega_\gamma$.

Remark 6.3. The operator $M_{\eta_\gamma}$ may have finite rank (for example if $G$ is a discrete group and $\eta_\gamma$ is the characteristic function of a finite set). In this case the choice of the eigenfunctions associated to the singular value zero is irrelevant, since in (43) these are multiplied by zero.

Proof of Theorem 6.1. We parallel the proofs in Section 5. Since $\sum_{\gamma} M_{\eta_\gamma} = M_{\sum_{\gamma} \eta_\gamma}$ and $\sum_{\gamma} \eta_\gamma \geq A > 0$, it follows from Proposition 2.10 that $\sum_{\gamma} M_{\eta_\gamma}$ is positive definite and therefore invertible. Theorem 3.5 consequently yields,

$$\sum_{\gamma \in \Gamma} \|M_{\eta_\gamma} f\|_{L^2(G)}^2 \approx \|f\|_{L^2}^2, \quad f \in S_{L^2}.\tag{44}$$

In addition, Proposition 3.4, Corollary 3.7 and a second application of Theorem 3.5 yield

$$\sum_{\gamma \in \Gamma} \|M_{\eta_\gamma}^2 f\|_{L^2(G)}^2 \approx \|f\|_{L^2}^2, \quad f \in S_{L^2}.\tag{44}$$

The thresholded operators $M_{\eta_\gamma}^\varepsilon$ satisfy

$$\|M_{\eta_\gamma}^\varepsilon f\|_{L^2} \leq \|M_{\eta_\gamma} f\|_{L^2} \leq \|M_{\eta_\gamma}^\varepsilon f\|_{L^2} + \varepsilon \|f\|_{L^2}, \quad f \in S_{L^2}.\tag{44}$$

Applying this to $M_{\eta_\gamma} f$ and noting that $M_{\eta_\gamma}$ and $M_{\eta_\gamma}^\varepsilon$ commute gives

$$\|M_{\eta_\gamma}^2 f\|_{L^2} \leq \|M_{\eta_\gamma}^\varepsilon M_{\eta_\gamma} f\|_{L^2} + \varepsilon \|M_{\eta_\gamma} f\|_{L^2},$$

$$\leq \|M_{\eta_\gamma} f\|_{L^2} + \varepsilon \|M_{\eta_\gamma} f\|_{L^2},$$

$$\approx \|M_{\eta_\gamma} f\|_{L^2} + \varepsilon \|M_{\eta_\gamma} f\|_{L^2}, \quad f \in S_{L^2.}$$

Putting all these inequalities together gives

$$\left(\sum_{\gamma \in \Gamma} \|M_{\eta_\gamma}^\varepsilon f\|_{L^2}^2\right)^{1/2} \approx \|f\|_{L^2} \approx \left(\sum_{\gamma \in \Gamma} \|M_{\eta_\gamma}^\varepsilon f\|_{L^2}^2\right)^{1/2} + \varepsilon \|f\|_{L^2}, \quad f \in S_{L^2}.\tag{44}$$

This implies that, for $0 < \varepsilon \ll 1$, $\sum_{\gamma \in \Gamma} \|M_{\eta_\gamma}^\varepsilon f\|_{L^2}^{1/2} \approx \|f\|_{L^2}$, as claimed. The fact that the system in (43) is a frame of $S_{L^2}$ now follows like in Theorem 5.5, this time using Proposition 2.10 to estimate:

$$\# \{\lambda_k : \lambda_k^2 > \varepsilon\} \leq \varepsilon^{-1} \text{trace}(M_{\eta_\gamma}) \leq \varepsilon^{-1} \|\eta_\gamma\|_1.\tag{44}$$

6.1. Application to time-scale analysis. We now show how to apply Theorem 6.1 to time-scale analysis. Let $\psi : \mathbb{R}^d \to \mathbb{C}$ be a Schwartz-class radial function with several vanishing moments. The wavelet transform of a function $f \in L^2(\mathbb{R}^d)$ with respect to $\psi$ is defined by

$$W_\psi f(x, s) = s^{-d/2} \int_{\mathbb{R}^d} f(t) \psi\left(\frac{t-x}{s}\right) dt, \quad x \in \mathbb{R}^d, s > 0.\tag{45}$$

If $\psi$ is properly normalized (and we assume so thereof), $W_\psi$ maps $L^2(\mathbb{R}^d)$ isometrically into $L^2(\mathbb{R}^d \times \mathbb{R}^+, s^{-(d+1)} dx ds)$. For a measurable bounded symbol $m : \mathbb{R}^d \times \mathbb{R}^+ \to \mathbb{C}$, the wavelet multiplier $\text{WM}_m : L^2(\mathbb{R}^d) \to L^2(\mathbb{R}^d)$ is defined as

$$\text{WM}_m f(t) = \int_0^{+\infty} \int_{\mathbb{R}^d} m(x, s) W_\psi f(x, s) \pi(x, s) \psi(t) dx \frac{ds}{s^{d+1}}, \quad t \in \mathbb{R}^d,\tag{46}$$

where $\pi(x, s) \psi(t) := s^{-d/2} \psi\left(\frac{t-x}{s}\right)$. (Here, the integral converges in the weak sense.) Note that $\text{WM}_m = W_\psi^* (m W_\psi F)$. The operator $\text{WM}_m$ is also known as wavelet localization operator [16, 18, 17].
In order to apply the model from Section 2.3.2 we consider the affine group \( G = \mathbb{R}^d \times \mathbb{R}_+ \), where multiplication is given by \((x, s) \cdot (x', s') = (x + a x', s')\). The Haar measure in \( G \) is given by
\[
|X| = \int_X s^{-(d+1)} \, dx ds,
\]
and the modular function is given by \( \Delta(x, s) = s^{-d} \).

We let \( E := L^2(G) \) and \( S_E := W_0L^2(\mathbb{R}^d) \). In complete analogy to the time-frequency analysis case, we let \( P := W_0W_0^* : L^2(G) \to W_0L^2(\mathbb{R}^d) \) be the orthogonal projection and \( K := |W_0\psi| \). We further let \( u(x, s) := \max \{1, s^d\} \). The kernel \( K \) belongs to \( W(L_\infty, L^1_\infty) \cap W_R(L_\infty, L^1_\infty) \) if \( \psi \) has sufficiently many vanishing moments (see [37, Section 4.2]).

As an example of a well-spread family of symbols we consider the characteristic functions of a cover \( G \) by irregular boxes. Let us take as centers the points \( \{ \gamma_{j,l} \} \) with corresponding eigenvalues \( \{ \lambda_{j,l} \} \) ordered decreasingly.

We let \( U_{j,l} := \gamma_{j,l}V_{j,l} \), \( j \in \mathbb{Z}, l \in \mathbb{Z}^d \).

The family of characteristic functions \( \{ U_{j,l} : j \in \mathbb{Z}, l \in \mathbb{Z}^d \} \) is well-spread, with envelope \((\Gamma, g)\), where
\[
\Gamma := \{ \gamma_{j,l} \} := (2^j, 2^l) \mid j \in \mathbb{Z}, l \in \mathbb{Z}^d \}
\]
and \( g := 1_{[a/2, a/2]} \times [b^{-1}, b] \).

We let \( \|1_{U_{j,l}}\|_{L^1(\mathbb{R}^d)} = \|U_{j,l}\| = |V_{j,l}| = \frac{1}{2} \prod_{i=1}^d a_{j,l} \cdot [(b_{j,l})^d - (b_{j,l})^{-d}] \). Theorem 6.1 yields the following.

**Theorem 6.4.** Suppose that the sets \( \{ U_{j,l} : j \in \mathbb{Z}, l \in \mathbb{Z}^d \} \) from (47) and (48) cover \( \mathbb{R}^d \times \mathbb{R}_+ \).

For each \( j \in \mathbb{Z}, l \in \mathbb{Z}^d \) let the wavelet multiplier \( W_{M_{1_{U_{j,l}}}} \) - cf. (46) - have eigenfunctions \( \{ \phi_{j,l} : k \geq 1 \} \) with corresponding eigenvalues \( \{ \lambda_{j,l} \} \) ordered decreasingly.

Then there exists a constant \( \alpha > 0 \) such for every choice of numbers \( N_{j,l} : j \in \mathbb{Z}, l \in \mathbb{Z}^d \) satisfying
\[
\alpha \|U_{j,l}\| \leq N_{j,l} \text{ and } \sup_{j,l} N_{j,l} < +\infty,
\]
the family \( \{ \lambda_{j,l} \phi_{j,l} : j \in \mathbb{Z}, l \in \mathbb{Z}^d, 1 \leq k \leq N_{j,l} \} \) is a frame of \( L^2(\mathbb{R}^d) \).

**6.2. Application to Gabor analysis.** Let us consider a window \( \varphi \in M^1(\mathbb{R}^d) \) with \( \|\varphi\|_2 = 1 \) and a (full rank) lattice \( \Lambda \subseteq \mathbb{R}^{2d} \), i.e. \( \Lambda = P \mathbb{Z}^{2d} \), where \( P \in \mathbb{R}^{2d \times 2d} \) is an invertible matrix. The Gabor system associated with \( \varphi \) and \( \Lambda \) is the collection of functions
\[
\mathcal{G}_{\varphi, \Lambda} := \{ \varphi_{\lambda}(t) := e^{i \pi \mathbf{t}^t \varphi(t - x) \mid \lambda = (x, \xi) \in \Lambda \} \}
\]
We assume that this collection of functions is a tight-frame. This means that for some constant \( A > 0 \), every function \( f \in L^2(\mathbb{R}^d) \) admits the expansion
\[
f = A \sum_{\lambda \in \Lambda} \langle f, \pi(\lambda) \varphi \rangle \pi(\lambda) \varphi.
\]

(See [37] for a survey on Gabor multipliers; see also [21, 33]...) If \( m \geq 0 \), the first \( N \) eigenfunctions of \( GM_m \) form an orthonormal set in \( L^2(\mathbb{R}^d) \) that maximizes the quantity
\[
\sum_{j=1}^N \sum_{\lambda \in \Lambda} m(\lambda) |\langle f_j, \pi(\lambda) \varphi \rangle|^2
\]
among all orthonormal sets of $W$.

Let us show how the abstract setting of Section 2.3.2 can be applied. The discussion is analogous to Example 2.5. We get $\mathcal{G} = \Lambda$, considered as a group and $E := \ell^2(\Lambda)$. Consider the analysis operator $C_{\Lambda,\varphi} : L^2(\mathbb{R}^d) \to \ell^2(\Lambda)$ given by $C_{\Lambda,\varphi}(\lambda) = \sqrt{A} \langle f, \pi(\lambda) \varphi \rangle$. Let $S_{\ell^2} := C_{\Lambda,\varphi}(L^2(\mathbb{R}^d))$. Since we assume that $\mathcal{G}_{\varphi,\lambda}$ is a tight frame, the operator $C_{\Lambda,\varphi} : L^2(\mathbb{R}^d) \to \ell^2(\Lambda)$ is an isometry - cf. (49). The orthogonal projection $P : \ell^2(\Lambda) \to S_{\ell^2}$ is then $P = C_{\Lambda,\varphi} C_{\Lambda,\varphi}^*$. Thus $\mathcal{G}_{\varphi,\lambda}$ is therefore represented by the matrix $\kappa(\lambda, \varphi) = A(\varphi, \varphi)\approx A(|\varphi|^2) - \lambda A(|\varphi|^2)$,

Consequently, $|\kappa(\mu, \lambda)| = A(|\varphi, \varphi\rangle) = A(|\varphi, \varphi\rangle) - \lambda A(|\varphi, \varphi\rangle)\approx A(|\varphi|^2) - \lambda A(|\varphi|^2)\approx \lambda(|\varphi|^2)\approx |\varphi|^2$.

Since $\varphi \in M^1(\mathbb{R}^d)$, $C_{\Lambda,\varphi}$ maps $M^1(\mathbb{R}^d)$ into $\ell^1(\Lambda)$ (see for example (28)) and we conclude that $K := A(|\varphi, \varphi\rangle) = \sqrt{A} C_{\Lambda,\varphi}\varphi \in \ell^1(\Lambda) = W(\ell^\infty, \ell^1)(\Lambda)$. Hence, (A1), (A2) and (A3) from Section 2.3 are satisfied with $\mathcal{G} = \Lambda$, $E = \ell^2(\Lambda)$ and $w \equiv 1$.

In addition, note that the Gabor multiplier in (50) satisfies $GM_f = C_{\Lambda,\varphi}(mC_{\Lambda,\varphi}f)$. Hence $C_{\Lambda,\varphi}GM_mC_{\Lambda,\varphi} : S_{\ell^2} \to S_{\ell^2}$ is a phase-space multiplier with symbol $m$ - cf. Section 2.3.3.

As an example of a well-spread family of symbols on $\Lambda$ we may now consider a well-spread family $\{\eta_\gamma : \gamma \in \Gamma\}$ of non-negative symbols defined on $\mathbb{R}^{2d}$, where $\Gamma \subseteq \Lambda$ is a relatively separated subset of $\mathbb{R}^d$ and $\sum_\gamma \eta_\gamma \approx 1$.

Then there exists a constant $\alpha > 0$ such for every choice of numbers $\{N_\gamma : \gamma \in \Gamma\} \subseteq \mathbb{N}$ satisfying

$$\alpha \parallel \gamma \parallel_{\ell^1(\Lambda)} \leq N_\gamma$$

the family $\{\lambda_\gamma^k \eta_\gamma \in \Gamma, 1 \leq k \leq N_\gamma\}$ is a frame of $L^2(\mathbb{R}^d)$.

**Remark 6.6.** While Theorem 5.5 provides frames for $L^2(\mathbb{R}^d)$ consisting of functions having a spectrum that is optimally adapted to a given weight on $\mathbb{R}^{2d}$ - cf. (35), Theorem 6.5 provides frame elements where the profile of the functions associated with the discrete expansion in (49) is optimized with respect to a weight on $\Lambda$.

**APPENDIX A: PROOF OF THEOREM 5.5**

In this appendix we prove Theorem 5.5. The proof is essentially contained in [57], but is not explicitly stated in the required generality. We therefore show how to derive Theorem 5.5 from some technical lemmas in [57].

**Remark 6.7.** We quote simplified versions of some statements in [57]. The article [57] considers a technical variant of the amalgam space $W_R(L^\infty, L^1_w)(\mathcal{G})$, called the weak amalgam space $W^\text{weak}_R(L^\infty, L^1_w)$ (see [57] Section 2.4), which we do not wish to introduce here. By [57] Proposition 1, $L^1(\mathcal{G}) \Rightarrow W^\text{weak}_R(L^\infty, L^1_w)(\mathcal{G}) \Rightarrow W_R(L^\infty, L^1_w)(\mathcal{G})$. Some results from [57] that we quote assume that a certain function $g$ belongs to $W_R(L^\infty, L^1_w)(\mathcal{G})$ and are proved in [57] under the weaker assumption: $g \in W^\text{weak}_R(L^\infty, L^1_w)$.

We quote the following estimate.

**Lemma 6.8.** Let $E$ be a solid, translation invariant BF space, let $w$ be an admissible weight for it and let $\Gamma \subseteq \mathcal{G}$ be a relatively separated set. Then for every $f \in E$ and $g \in W_R(C_0, L^1_w)$, the sequence $(\langle f, \mathcal{L}_A g \rangle)_{A \in \mathcal{G}}$ belongs to $E_d(\Lambda)$ and

$$\|\langle f, \mathcal{L}_A g \rangle\|_{E_d} \lesssim \|f\|_E \|g\|_{W_R(L^\infty, L^1_w)}.$$

The implicit constants depend on the spreadness $\rho(\Gamma)$ - cf. (18).
Suppose that Assumptions (A1) and (A2) from Section 2.3.2 hold. For a solid, translation invariant BF space \(E\) we consider an \(L^2\)-valued version of \(E_d(\Gamma)\),

\[
E_{d,L^2} = E_{d,L^2}(\Gamma) := \{ (f_\gamma)_{\gamma \in \Gamma} \in (L^2(\mathcal{G}))^\Gamma \mid (\|f_\gamma\|_{L^2})_{\gamma \in \Gamma} \in E_d(\Gamma) \},
\]
and endow it with the norm \(\|(f_\gamma)_{\gamma \in \Gamma}\|_{E_{d,L^2}} := \|(\|f_\gamma\|_{L^2})_{\gamma \in \Gamma}\|_{E_d} \).

Let \(\{ T_\gamma \mid \gamma \in \Gamma \} \) be a well-spread family of operators - cf. Section 3. Let \(U \subseteq \mathcal{G}\) be a relatively compact neighborhood of the identity. Consider the operators \(C_T\) and \(S_U\) formally defined by

\[
C_T(f) := (T_\gamma(f))_{\gamma \in \Gamma}, \quad f \in S_E,
\]

\[
S_U((f_\gamma)_{\gamma \in \Gamma}) := \sum_{\gamma \in \Gamma} P(f_\gamma)1_{\gamma U}, \quad f_\gamma \in L^2(\mathcal{G}),
\]
where \(1_{\gamma U}\) denotes the characteristic function of the set \(\gamma U\). These operators satisfy the following mapping properties.

**Proposition 6.9.** Assume (A1) and (A2) and let \(\{ T_\gamma \mid \gamma \in \Gamma \} \) be a well-spread family of operators. Then the operators \(C_T\) and \(S_U\) in (51) and (52) satisfy the following.

(a) The analysis operator \(C_T\) maps \(S_E\) boundedly into \(E_{d,L^2}(\Gamma)\).
(b) For every relatively compact neighborhood of the identity \(U\), and every sequence \(F \equiv (f_\gamma)_{\gamma \in E_{d,L^2}}\), the series defining \(S_U(F)\) converge absolutely in \(L^2(\mathcal{G})\) at every point. Moreover, the operator \(S_U\) maps \(E_{d,L^2}(\Gamma)\) boundedly into \(E\) (with a bound that depends on \(U\)).

**Proof.** Part (b) is proved in [47, Proposition 4 (b)] under weaker hypothesis. Part (a) is a slight variant of Proposition 6.11. For completeness we give a full argument. Let \((\Gamma, \Theta, g)\) be an envelope for \(\{ T_\gamma \mid \gamma \in \Gamma \} \).

Let \(f \in S_E\). Since \(\eta_\gamma\) is bounded, \(f\eta_\gamma \in E\). By the definitition of well-spread family - cf. (26)

\[
|T_\gamma f(x)| \leq \int_G |f(y)| g(\gamma^{-1}y) \Theta(y^{-1}x) dy = |f| L_\gamma g \ast \Theta(x).
\]

By Young’s inequality \(L^1 \ast L^2 \hookrightarrow L^2\) we have

\[
\|T_\gamma f\|_2 \leq \|\Theta\|_2 \int_G |f(y)| g(\gamma^{-1}y) dy \lesssim \|\Theta\|_{W(L^\infty, L^2_\gamma)} \int_G |f(y)| g(\gamma^{-1}y) dy.
\]

Now Lemma 6.8 yields \(\|C_T(f)\|_{E_{d,L^2}} \lesssim \|f\|_E \|g\|_{W(R(L^\infty, L^2_\gamma))}\), as desired. \(\square\)

**Remark 6.10.** Note that in the last proof the use of the \(L^2\) norm is somewhat arbitrary; a number of other function norms could have been used instead (cf. [47 Proposition 4]).

Now we prove the key approximation result (cf. [47 Theorem 1]).

**Theorem 6.11.** Assume (A1) and (A2) and let \(\{ T_\gamma \mid \gamma \in \Gamma \} \) be a well-spread family of operators. Given \(\varepsilon > 0\), there exists \(U_0\), a relatively compact neighborhood of \(e\) such that for all \(U \supseteq U_0\)

\[
\|\sum_{\gamma \in \Gamma} T_\gamma f - S_U C_T(f)\|_E \leq \varepsilon \|f\|_E, \quad f \in S_E.
\]

**Remark 6.12.** The neighborhood \(U_0\) can be chosen uniformly for any class of spaces \(E\) having the same weight \(w\) and the same constant \(C_{E,w}\) (cf. (22)).

Concerning the parameters in Assumptions (A1) and (A2) and (26), the choice of \(U_0\) only depends on \(\|K\|_{W(L^\infty, L^\infty_\gamma)}, \|K\|_{W(R(L^\infty, L^\infty_\gamma))}, \|\Theta\|_{W(L^\infty, L^\infty_\gamma)}, \|\Theta\|_{W(R(L^\infty, L^\infty_\gamma))}, \|g\|_{W(R(L^\infty, L^\infty_\gamma))}\) and \(p(\Gamma)\) (cf. (18)).

**Proof of Theorem 6.11.** Let \(f \in S_E\) and let \(U\) be a relatively compact neighborhood of \(e\). Because of the inclusion \(S_E \hookrightarrow W(L^\infty, E)\) in Proposition 2.6, it suffices to dominate the left-hand side of (53) by \(\varepsilon \|f\|_{W(L^\infty, E)}\).
Note that since $T_\gamma f \in S_E$, $S_U C_T f(x) = \sum_{\gamma \in \Gamma} T_\gamma f(x) 1_{\gamma U}(x)$. Hence, using (26) let us estimate
\[
\left| \sum_{\gamma \in \Gamma} T_\gamma f(x) - S_U C_T f(x) \right| = \left| \sum_{\gamma \in \Gamma} 1_{(\gamma \in U)}(x) T_\gamma(f(x)) \right| 
\leq \sum_{\gamma \in \Gamma} \int |f(y)| g(\gamma^{-1}y) \Theta(y^{-1}x) 1_{(\gamma \in U)}(x) dy.
\]
The rest of the proof is carried out exactly as in [47, Theorem 1]. Indeed, the proof there only depends on the estimate just derived\(^2\) (The definition of well-spread family of operators was tailored so that the proof in [47, Theorem 1] would still work.)

Finally we can prove Theorem 3.5.

**Proof of Theorem 3.5**. Let \( \{T_\gamma : \gamma \in \Gamma\} \) be a well-spread family of operators and suppose that the operator \( \sum_\Gamma T_\gamma : S_E \to S_E \) is invertible. We have to show that for \( f \in S_E \), \( \|f\|_E \approx \|C_T(f)\|_{E_{d, L^2}(\Gamma)} \). The estimate \( \|C_T(f)\|_{E_{d, L^2}(\Gamma)} \lesssim \|f\|_E \) is proved in Proposition 6.11 (a). To establish the second inequality, consider the operator \( PS_U C_T : S_E \to S_E \). Then for \( f \in S_E \)
\[
\left\| \sum_{\gamma \in \Gamma} T_\gamma f - PS_U C_T f \right\|_E = \left\| P \sum_{\gamma \in \Gamma} T_\gamma f - PS_U C_T f \right\|_E \lesssim \left\| \sum_{\gamma \in \Gamma} T_\gamma f - S_U C_T f \right\|_E.
\]
This estimate, together with Theorem 6.11 implies that \( \|\sum_\Gamma T_\gamma - PS_U C_T\|_{S_E \to S_E} \to 0 \) as \( U \) grows to \( \mathcal{G} \). Hence, there exists \( U \) such that \( PS_U C_T \) is invertible on \( S_E \). Consequently, for \( f \in S_E \), \( \|f\|_E \approx \|PS_U C_T f\|_E \lesssim \|C_T(f)\|_{E_{d, L^2}(\Gamma)} \). Here we have used the boundedness of \( S_U \) - contained in Proposition 6.9 (b) - and the boundedness of \( P : E \to W(L^\infty, E) \hookrightarrow E \) - contained in Proposition 2.6.

**Appendix B: Pseudodifferential operators and proof of Proposition 4.3**

The Weyl transform of a distribution \( \sigma \in S'(\mathbb{R}^d \times \mathbb{R}^d) \) is an operator \( \sigma^w \) that is formally defined on functions \( f : \mathbb{R}^d \to \mathbb{C} \) as
\[
(54) \quad \sigma^w(f)(x) := \int_{\mathbb{R}^d \times \mathbb{R}^d} \sigma \left( \frac{x+y}{2}, \xi \right) e^{2\pi i (x-y) \xi} f(y) dy d\xi, \quad x \in \mathbb{R}^d.
\]
The fundamental results in the theory of pseudodifferential operators provide conditions on \( \sigma \) for the operator \( \sigma^w \) to be well-defined and bounded on various function spaces. We now quote some results about pseudodifferential operators acting on modulation spaces - cf. Section 4.1.

In [33, 35] it was shown that modulation spaces on \( \mathbb{R}^{2d} \) serve as symbol classes to study pseudodifferential operators acting on modulation spaces on \( \mathbb{R}^d \), recovering and extending classical results from Sjöstrand [50, 51]. We quote the following simplified version of [35] Theorems 4.1 and 4.6, and Corollaries 3.3 and 4.7. (The GRS condition for admissible weights in Section 4.1 is important here.)

**Theorem 6.13**. Let \( w \) be an admissible TF weight - cf. Definition 4.4 - and let \( \varphi \in M_w^1(\mathbb{R}^d) \) be non-zero. Let us denote \( \tilde{w}(z_1, z_2) = w(-z_2, z_1) \). Then the following statements hold true.

- (i) If \( \sigma \in M_w^{\infty, -1}(\mathbb{R}^{2d}) \), then \( \sigma^w \) is bounded on \( M_w^p(\mathbb{R}^d) \), for all \( w \)-moderated weights \( v \) and all \( p \in [1, +\infty] \).
- (ii) If \( \sigma \in M_w^{\infty, -1}(\mathbb{R}^{2d}) \) and \( \sigma^w \) is invertible as an operator on \( L^2(\mathbb{R}^d) \), then \( \sigma^w \) is invertible as an operator on \( M_w^p(\mathbb{R}^d) \), for all \( w \)-moderated weights \( v \) and all \( p \in [1, +\infty] \).
- (iii) Let \( T : S(\mathbb{R}^d) \to S'(\mathbb{R}^d) \) be a linear and continuous operator. For \( (x, \xi) \in \mathbb{R}^d \times \mathbb{R}^d \) let us denote \( \varphi(x, \xi)(t) := e^{2\pi i t \xi} \varphi(t-x) \). If there exists a function \( H \in L_w^1(\mathbb{R}^{2d}) \) such that
\[
|\langle T(\varphi(x, \xi)), \varphi'(x', \xi') \rangle| \leq H(x' - x, \xi' - \xi), \quad (x, \xi), (x', \xi') \in \mathbb{R}^d \times \mathbb{R}^d,
\]
then the exists \( \sigma \in M_w^{\infty, -1}(\mathbb{R}^{2d}) \) such that \( T = \sigma^w \) on \( S(\mathbb{R}^d) \).

As an application of Theorem 6.13 we now prove Proposition 4.3.

\(^2\)The function \( \Theta \) is called \( H \) in the proof [47, Theorem 1].
Proof of Proposition 4.3. With the notation of Theorem 6.13 we use (32) to estimate
\[
|\langle T(\varphi(x,\xi),\varphi(x',\xi')\rangle| = |V_\varphi T(\varphi(x,\xi))(x',\xi')|
\leq \int_{\mathbb{R}^d} |V_\varphi \varphi(x,\xi)(z'')| H((x',\xi') - z'')dz''
\leq \int_{\mathbb{R}^d} |V_\varphi \varphi(z'' - (x,\xi))| H((x',\xi') - z'')dz'' = (H * |V_\varphi \varphi|)((x',\xi') - (x,\xi)).
\]
Since \(\varphi \in M^1_0(\mathbb{R}^d)\) and \(H \in L^1_v(\mathbb{R}^{2d})\), we deduce that \(H * |V_\varphi \varphi| \in L^1_v(\mathbb{R}^{2d})\).

Hence, Theorem 6.13 implies that there exists \(\sigma \in M^\infty_0(\mathbb{R}^{2d})\) such that \(T = \sigma^w\) on \(S(\mathbb{R}^d)\). Since both operators are bounded on \(L^2(\mathbb{R}^d)\) it follows that \(T = \sigma^w\) on \(L^2(\mathbb{R}^d)\). By hypothesis \(T = \sigma^w: L^2(\mathbb{R}^d) \to L^2(\mathbb{R}^d)\) is invertible. A new application of Theorem 6.13 implies that \(\sigma^w: M^p_v(\mathbb{R}^d) \to M^p_v(\mathbb{R}^d)\) is invertible. It is tempting to conclude that then \(T: M^p_v(\mathbb{R}^d) \to M^p_v(\mathbb{R}^d)\) is invertible because it “is” \(\sigma^w\). If \(p = +\infty\) that conclusion is indeed correct because both operators coincide on the dense space \(S\). The case \(p = +\infty\) requires some carefulness. We now discuss this in detail.

We note that \(M^p_v(\mathbb{R}^d) \to M^\infty_1(\mathbb{R}^d)\) and use the facts that \(M^\infty_1(\mathbb{R}^d)\) can be identified with the dual space of the (separable) Banach space \(M^1_v(\mathbb{R}^d)\) and that \(S\) is dense in \(M^\infty_1(\mathbb{R}^d)\) with respect to the weak* topology (see [33] Chapter 11). To conclude that \(T = \sigma^w\) on \(M^p_v(\mathbb{R}^d)\) we show that both operators are continuous with respect to the weak* topology of \(M^\infty_1(\mathbb{R}^d)\).

Let \(f \in M^p_v(\mathbb{R}^d)\) and let us show that \(T(f) = \sigma^w(f)\). Let \(\{f_k: k \in \mathbb{N}\} \subseteq S(\mathbb{R}^d)\) be a sequence such that \(f_k \to f\) in the weak*-topology of \(M^\infty_1(\mathbb{R}^d)\). The operator \(\sigma^w: M^\infty_1(\mathbb{R}^d) \to M^\infty_1(\mathbb{R}^d)\) is weak* continuous because it is the adjoint of the operator \(\sigma^w: M^1_v(\mathbb{R}^d) \to M^1_v(\mathbb{R}^d)\). Hence \(T(f_k) = \sigma^w(f_k) \to \sigma^w(f)\) in the weak*-topology of \(M^\infty_1(\mathbb{R}^d)\). Let us note that this implies that
\[
(56) \quad V_\varphi T(f_k)(z) \to V_\varphi \sigma^w(f)(z), \text{ as } k \to +\infty, \text{ for all } z \in \mathbb{R}^{2d}.
\]
Indeed, if \(z = (x,\xi) \in \mathbb{R}^d \times \mathbb{R}^d\) the function \(\varphi_{x,\xi} := e^{2\pi i \xi} \varphi(-x)\) belongs to \(M^1_v(\mathbb{R}^d)\) and consequently \(V_\varphi T(f_k)(z) = \langle T(f_k), \varphi_{x,\xi} \rangle \to \langle \sigma^w(f), \varphi_{x,\xi} \rangle = V_\varphi \sigma^w(f)(z)\). Similarly, since \(f_k \to f\) in the weak* topology of \(M^\infty_1(\mathbb{R}^d)\) we know that \(V_\varphi f_k(z) \to V_\varphi f(z)\), for all \(z \in \mathbb{R}^{2d}\).

Using the enveloping condition in (32), we estimate for \(z \in \mathbb{R}^{2d}\)
\[
|V_\varphi T(f)(z) - V_\varphi T(f_k)(z)| \leq \int_{\mathbb{R}^d} |V_\varphi(f - f_k)(z')| H(z - z')dz'.
\]
The integrand in the last expression tends to 0 pointwise as \(k \to +\infty\). In order to apply Lebesgue’s dominated convergence theorem we show that the integrand is dominated by an integrable function. Since \(H \in L^1_v(\mathbb{R}^{2d})\) it suffices to show that \(\sup_k \|V_\varphi(f - f_k)\|_{L^1_v} < +\infty\). This is true because \(\|V_\varphi(f - f_k)\|_{L^1_v} = \|f - f_k\|_{M^\infty_1} \text{ and weak*-convergent sequences are bounded. Hence, Lebesgue’s dominated convergence theorem can be applied and we conclude that } V_\varphi T(f_k)(z) \to V_\varphi T(f)(z), \text{ for all } z \in \mathbb{R}^{2d}.\) Combining this with (56) we conclude that \(V_\varphi T(f) \equiv V_\varphi \sigma^w(f)\). Hence \(T(f) = \sigma^w(f)\), as desired.

References

[1] L. D. Abreu. Sampling and interpolation in Bargmann-Fock spaces of polyanalytic functions. Appl. Comput. Harmon. Anal., 29(3):287–302, 2010.

[2] L. D. Abreu. On the structure of Gabor and super Gabor spaces. Monatsh. Math., 161(3):237–253, 2010.

[3] L. D. Abreu and K. Gröchenig. Banach Gabor frames with Hermite functions: polyanalytic spaces from the Heisenberg group. Appl. Anal., 91:1981–1997, 2012.

[4] A. Aldroubi, C. Cabrelli, and U. Molter. Wavelets on irregular grids with arbitrary dilation matrices, and frame atoms for \(L^2(\mathbb{R}^d)\). Appl. Comput. Harmon. Anal., Special Issue on Frames II.:119–140, 2004.

[5] M. Balk. Polyanalytic functions and their generalizations. Complex analysis I. Encycl. Math. Sci. 85, 195-253 (1997); translation from Itogi Nauki Tekh., Ser. Sovrem. Probl. Math., Fundam Napravlenniya 85, 187-246 (1991)., 1991.

[6] P. Boggio. Localization operators with \(L^p\) symbols on modulation spaces. In Advances in Pseudo-differential Operators, volume 155 of Oper. Theory Adv. Appl., pages 149–163. Birkhäuser, Basel, 2004.

[7] P. Boggio and E. Cordero. Anti-Wick quantization with symbols in \(L^p\) spaces. Proc. Amer. Math. Soc., 130(9):2679–2685 (electronic), 2002.
[8] P. Boggiatto and J. Toft. Schatten classes for Toeplitz operators with Hilbert space windows on modulation spaces. *Adv. Math.*, 217(1):305–333, 2008.

[9] M. Bownik and K. P. Ho. Atomic and molecular decompositions of anisotropic Triebel-Lizorkin spaces. *Trans. Amer. Math. Soc.*, 358(4):1497–1510, 2006.

[10] C. Cabrelli, U. Molter, and J. L. Romero. Non-uniform painless decompositions for anisotropic Besov and Triebel-Lizorkin spaces. *Adv. Math.*, 232(1):98–120, 2013.

[11] P. G. Casazza and G. Kutyniok. Frames of subspaces. In *Wavelets, Frames and Operator Theory*, volume 345 of *Contemp. Math.*, pages 87–113. Amer. Math. Soc., Providence, RI, 2004.

[12] E. Cordero and K. Gröchenig. Time-frequency analysis of localization operators. *J. Funct. Anal.*, 205(1):107–131, 2003.

[13] E. Cordero and K. Gröchenig. Symbolic calculus and Fredholm property for localization operators. *J. Fourier Anal. Appl.*, 12(4):371–392, 2006.

[14] S. Dahlke, G. Steidl, and G. Teschke. Weighted coorbit spaces and Banach frames on homogeneous spaces. *J. Fourier Anal. Appl.*, 10(5):507–539, 2004.

[15] W. Dahmen, S. Dekel, and P. Petrushev. Two-level-split decomposition of anisotropic Besov spaces. *Constr. Approx.*, 31(2):149–194, 2010.

[16] I. Daubechies. Time-frequency localization operators: a geometric phase space approach. *IEEE Trans. Inform. Theory*, 36(5):961–1005, 1990.

[17] I. Daubechies. The wavelet transform, time-frequency localization and signal analysis. *IEEE Trans. Inform. Theory*, 34(4):605–612, July 1988.

[18] I. Daubechies and T. Paul. Time-frequency localisation operators - a geometric phase space approach: II. The use of dilations. *Inverse Probl.*, 4(3):661–680, 1988.

[19] M. Dörfler, H. G. Feichtinger, and K. Gröchenig. Time-frequency partitions for the Gelfand triple $(S_0, L^2, S_0')$. *Math. Scand.*, 98(1):81–96, 2006.

[20] M. Dörfler and K. Gröchenig. Time-frequency partitions and characterizations of modulations spaces with localization operators. *J. Funct. Anal.*, 260(7):1903 – 1924, 2011.

[21] M. Dörfler and B. Torrésani. Representation of operators in the time-frequency domain and generalized Gabor multipliers. *J. Fourier Anal. Appl.*, 16(2):261–293, 2010.

[22] H. G. Feichtinger. Banach convolution algebras of Wiener type. In *Proc. Conf. on Functions, Series, Operators, Budapest 1980*, volume 35 of *Colloq. Math. Soc. Janos Bolyai*, pages 509–524. North-Holland, Amsterdam, Eds. B. Sz.-Nagy and J. Szabados, edition, 1983.

[23] H. G. Feichtinger. Generalized amalgams, with applications to Fourier transform. *Canad. J. Math.*, 42(3):395–409, 1990.

[24] H. G. Feichtinger and P. Gröbner. Banach spaces of distributions defined by decomposition methods. *I. Math. Nachr.*, 123:97–120, 1985.

[25] H. G. Feichtinger and K. Gröchenig. Banach spaces related to integral group representations and their atomic decompositions, *I. J. Funct. Anal.*, 86(2):307–340, 1989.

[26] H. G. Feichtinger and K. Gröchenig. Banach spaces related to integral group representations and their atomic decompositions, *II. Monatsh. Math.*, 108(2–3):129–148, 1989.

[27] H. G. Feichtinger and K. Nowak. A first survey of Gabor multipliers. In H. G. Feichtinger and T. Strohmer, editors, *Advances in Gabor Analysis*, Appl. Numer. Harmon. Anal., pages 99–128. Birkhäuser, 2003.

[28] H. G. Feichtinger and G. Zimmermann. A Banach space of test functions for Gabor analysis. In H. G. Feichtinger and T. Strohmer, editors, *Gabor Analysis and Algorithms: Theory and Applications*, Applied and Numerical Harmonic Analysis, pages 123–170. Birkhäuser Boston, Boston, MA, 1998.

[29] Y. Liu, A. Mohammed, and M. Wong. Wavelet multipliers on $L^p(R^n)$. *Proc. Amer. Math. Soc.*, 136(3):1009–1018, 2008.
[41] M. Liuni, A. Röbel, M. Romito, and X. Rodet. Rényi information measures for spectral change detection. In Proceedings of the IEEE International Conference on Acoustics, Speech and Signal Processing (ICASSP), 2011, pages 3824 – 3827, May 2011.

[42] M. Nashed and Q. Sun. Sampling and reconstruction of signals in a reproducing kernel subspace of $L^p(\mathbb{R}^d)$. J. Funct. Anal., 258(7):2422–2452, 2010.

[43] P. Oswald. Stable subspace splittings for Sobolev spaces and domain decomposition algorithms. In Keyes, David E (ed) et al, Domain Decomposition Methods in Scientific and Engineering Computing Proceedings of the 7th International Conference on Domain Decomposition, October 27-30, 1993, Pennsylvania State University, PA, USA Providence, RI: Ameri.

[44] H. Rauhut. Wiener amalgam spaces with respect to quasi-Banach spaces. Colloq. Math., 109(2):345–362, 2007.

[45] F. Riesz and B. Sz.-Nagy. Functional analysis. Frederick Unger Publishing Co., New York, 1955.

[46] J. L. Romero. Surgery of spline-type and molecular frames. J. Fourier Anal. Appl., 17:135 – 174, 2011.

[47] J. L. Romero. Characterization of coorbit spaces with phase-space covers. J. Funct. Anal., 262(1):59–93, 2012.

[48] C. Schörkhuber and A. Klapuri. Constant-Q toolbox for music processing. In Proceedings of the 7th Sound and Music Computing Conference (SMC), 2010, 2010.

[49] B. Simon. Trace Ideals and their Applications. Cambridge University Press, Cambridge, 1979.

[50] J. Sjöstrand. An algebra of pseudodifferential operators. Math. Res. Lett., 1(2):185–192, 1994.

[51] J. Sjöstrand. Wiener type algebras of pseudodifferential operators. Séminaire sur les équations aux Dérivées Partielles, 1994-1995, École Polytech, Palaiseau, Exp. No. IV, 21, 1995.

[52] G. A. Velasco, N. Holighaus, M. Dörfler, and T. Grill. Constructing an invertible constant-Q transform with non-stationary Gabor frames. Proceedings of DAFX11, Paris, 2011.

[53] M. Wong. Localization Operators. Seoul National University, Seoul, 1999.

[54] M.-W. Wong. Wavelet Transforms and Localization Operators. Operator Theory: Advances and Applications. 136. Basel: Birkhäuser, 2002.

E-mail address: monika.doerfler@univie.ac.at

E-mail address: jose.luis.romero@univie.ac.at

Faculty of Mathematics, University of Vienna, Oskar-Morgenstern-Platz 1, A-1090 Wien, Austria