The Visible Universe at the Light of Modern Statistical Physics

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In the last years there has been a growing interest in the understanding of a vast variety of scale invariant and critical phenomena occurring in nature. Experiments and observations indeed suggest that many physical systems develop spontaneously correlations with power law behaviour both in space and time. Pattern formation, aggregation phenomena, biological systems, geological systems, disordered materials, clustering of matter in the universe are just some of the fields in which scale invariance has been observed as a common and basic feature. However, the fact that certain structures exhibit fractal and complex properties does not tell us why this happens. A crucial point to understand is therefore the origin of the general scale-invariance of natural phenomena. This would correspond to the understanding of the origin of fractal structures and of the properties of Self-Organized Criticality (SOC) from the knowledge of the microscopic physical processes at the basis of these phenomena. Fractal geometry can play a crucial role in extracting the correct physical properties from experimental data. In particular, the recent availability of complete three dimensional samples of galaxies and clusters permits a direct study of their spatial properties. We present a brief review of galaxy correlations based on the methods of modern Statistical Physics. These methods are able to identify self-similar and non-analytical properties, and allow us to test the usual homogeneity assumption of luminous matter distribution. The new analysis shows that all the available data are consistent with each other and show fractal correlations (with dimension $D \simeq 2$) up to the deepest scales probed until now ($1000h^{-1}\text{Mpc}$) and even more as indicated from the new interpretation of the number counts.

1 Introduction

In scale invariant phenomena, events and information spread over a wide range of length and time scales, so that no matter what is the size of the scale considered one always observes surprisingly rich structures. These systems, with very many degrees of freedom, are usually so complex that their large scale behaviour cannot be predicted from the microscopic dynamics. New types of collective behaviour arise and their understanding represents one of the most challenging areas in modern statistical physics.

Nowadays the physics of scale invariant and complex systems is a novel field which is including topics from several disciplines ranging from condensed matter physics to geology, biology, astrophysics and economics. This widespread interdisciplinary corresponds to the fact that these new ideas allow us to look at natural phenomena in a radically new and original way, eventually leading
to unifying concepts independently of the detailed structures of systems. The
interest in the field of scale free phenomena and complex systems has been
largely due to two factors. First the emerging availability of high powered
computers over the past decade has enabled to readily simulate complex and
disordered systems. Second the cross disciplinary mathematical language for
describing these phenomena evolving under conditions far from equilibrium
has only become available in the past years. The study of critical phenomena
in second order transitions introduced the concepts of scaling and power law
behavior. Fractal geometry provided the mathematical framework for the ex-
tension of these concepts to a vast variety of natural phenomena. The physics
of complex systems, however, turned out to be effectively new with respect to
critical phenomena. The theory of equilibrium statistical physics is strongly
based on the ergodic hypothesis and scale invariance develops at the critical
equilibrium between order and disorder. Reaching this equilibrium requires
the fine tuning of various parameters. On the contrary most of the scale
free phenomena observed in nature are self-organized, in the sense that they
spontaneously develop from the generating dynamical process. One is then
forced to seek the origin of the scale invariance in nature in the rich domain
of nonequilibrium systems and this requires the development of new ideas and
methods.

In order to identify phenomenologically which microscopic dynamics may
lead to fractal and SOC properties, it is necessary to define simple physical
models that should capture the essential ingredients of these phenomena. In
the past years a large number of models have been introduced and studied,
mostly by numerical simulations. Given this scenario the aim of the current
studies on fractal and self-organized systems is twofold.

• Although fractal growth models and SOC have provided a useful insight
  into a vast array of problems, many important questions are still open.
  In fact it is particularly important to reach a more complete and predic-
tive theoretical understanding of fractal growth and the SOC mechanism.
  These issues are still unresolved, the present picture being based on the
  analysis of particular models. In this sense, it is still missing a gen-
  eral and precise definition of the circumstances leading to fractals and
  SOC and the identification of common features in different systems. For
  instance a crucial issue is the role of universality in fractal and SOC phe-
nomena. In usual critical phenomena, the same exponents that define
  the onset of magnetisation, also describe the liquid vapor transition in
  water. This strong universality appears to be a characteristic of equilib-
  rium systems. Self-organized systems, on the other hand, do not seem
  to exhibit the same degree of universality as the fractal dimension can
be easily altered by simple changes in the growth process. This lack of universality is sometimes viewed as a negative element because one is forced to describe specific systems instead of a single universal model. The truth is probably the opposite. Some theoretical concepts can be considered as general or universal, but the inherent diversity of the various models that have been studied, adds another fascinating dimension in the intellectual search. This is a difficult and deep question because, in addition to its intrinsic interest, it has deep implications on the validity of the simplified computer models with respect to physical reality. Clearly such a problem can only be investigated by a comprehensive effort that involves computer simulations, analytical tools and suitably designed experiments.

• While the theoretical activity is focused mainly on simple cellular automata or toy models, it is strikingly important to understand the relations between theory and real experiments. Therefore, inspired from the early generation of prototype models one intends to formulate more realistic models for a detailed interpretation of specific phenomena in various fields. These models should be concrete tools for the understanding of the phenomenology and the characterization of several experimental problems. On the other hand, there is a need of real experiments which can discriminate among the various theoretical framework. One also expects that experiments could point out new properties and new models that may enlighten specific theoretical questions.

The introduction of new ideas, inspired by fractal geometry and scaling, irreversible and non ergodic dynamical systems leading to self-organization and stochastic processes of various types, give rise to a considerable enrichment of the traditional framework of Statistical physics and provides efficient methods for characterizing and understanding complex systems.

The impact of fractals in physics can be assessed along three different lines of increasing complexity:

(a) Fractal geometry merely as a mathematical framework which leads to a re-analysis of known data that results in a revamping of long standing points of view.

(b) The development of physical models for systems that exhibit fractal and SOC behaviour.

(c) The construction of physical theories that allow us to understand the origin of fractal structures and SOC properties in various systems. An
additional question which refers to all the previous points, is the study of the physical properties of fractal and SOC systems such as transport, vibrational, electronic properties etc.

The first physical model of fractal growth was the Diffusion Limited Aggregation (DLA) introduced in 1981 by Witten and Sander (see for a review [1]). This model was generalized by Niemeyer, Pietronero and Wiesmann with the Dielectric Breakdown Model (DBM), inspired by discharges in gases, that also clarified the underlying mathematical properties based on Laplace equation. In this respect, it may be surprising to see that the Laplace operator, which usually leads to smooth properties, in the case of these models drives spontaneously its boundaries into strongly irregular fractal structures. These Laplacian fractals are believed to capture the essential properties of a variety of phenomena such as electrochemical deposition, dielectric and mechanical breakdown, viscous fingering in fluids, propagation of fractures and various properties of colloids.

More recently, in order to put in a broader framework the self-organization properties of the above models, Bak, Tang and Wiesenfeld invoked the concept of Self-Organized Criticality (SOC) as a unifying framework to describe a vast class of dynamically driven systems which evolve spontaneously in a stationary state with a broad power law distribution of discrete energy dissipating events. In these models criticality seems to emerge automatically without the fine tuning of parameters. Because of the enormous conceptual power, SOC ideas have invaded rapidly throughout the sciences, from physics and geophysics to biology and economics, as a prototype mechanism to understand the manifestation of scale invariance and complexity in natural phenomena.

An important area in which fractal geometry can play a crucial role consists in extracting the correct physical properties from experimental data of the galaxy distribution in space. The usual analysis measures the deviations of the conditional density at a given distance from the average density. Note however that this procedure implicitly assumes homogeneity and thus cannot objectively test if the considered portion of the universe is or not homogeneous. Some years ago we proposed a more general method of analysis based on the concepts and methods of modern statistical physics. This led to the surprising result that galaxy correlations are fractal and not homogeneous up to the limit of the available data. All the structures observed in the universe appear to be characterized by strongly irregular, scale invariant fluctuations. These results led to a large debate in the field and the new data, expected to appear in the near future, should provide a definitive test of these properties. These results may lead to fascinating conceptual implications about our knowledge of the universe and to a new scenario about the theoretical challenge in this field.
2 Self-similarity and power law correlations

A fractal consists of a system in which more and more structures appear at smaller and smaller scales and the structures at small scales are similar to the one at large scales. Starting from a point occupied by an object we count how many objects are present within a volume characterized by a certain length scale in order to establish a generalized "mass-length" relation from which one can define the fractal dimension. We can then write a relation between $N$ ("mass") and $r$ ("length") of type:

$$N(r) = B \cdot r^D$$

(1)

where the fractal dimension is $D$ and the prefactor $B$ is instead related to the lower cut-offs. It should be noted that Eq.1 corresponds to a smooth convolution of a strongly fluctuating function. Therefore a fractal structure is always connected with large fluctuations and clustering at all scales. From Eq.1 we can readily compute the average density $\langle n \rangle$ for a spherical sample of radius $R_s$ which contains a portion of the fractal structure:

$$\langle n \rangle = \frac{N(R_s)}{V(R_s)} = \frac{3}{4\pi} BR_s^{-(3-D)}$$

(2)

From Eq.2 we see that the average density is not a meaningful concept in a fractal because it depends explicitly on the sample size $R_s$. We can also see that for $R_s \to \infty$ the average density $\langle n \rangle \to 0$, therefore a fractal structure is asymptotically dominated by voids.

It is useful to introduce the conditional density from an point occupied as:

$$\Gamma(r) = S^{-1} \frac{dN(r)}{dr} = \frac{D}{4\pi} Br^{-(3-D)}$$

(3)

where $S(r)$ is the area of a spherical shell of radius $r$. Usually the exponent that defines the decay of the conditional density $(3-D)$ is called the codimension and it corresponds to the exponent $\gamma$ of the galaxy distribution.

We can now describe how to perform the correct correlation analysis which can be applied in the case of an irregular distribution as well as of a regular one. We may start recalling the concept of correlation. If the presence of an object at the point $r_1$ influences the probability of finding another object at $r_2$, these two points are correlated. Therefore there is a correlation at $r$ if, on average

$$G(r) = \langle n(0)n(r) \rangle \neq \langle n \rangle^2.$$

(4)
where we average over all occupied points chosen as origin. On the other hand there is no correlation if
\[ G(r) \approx \langle n \rangle^2. \]  
(5)
The physically meaningful definition of \( \lambda_0 \) is therefore the length scale which separates correlated regimes from uncorrelated ones.

In practice, it is useful to normalize the correlation function (CF) of Eq.4 to the size of the sample under analysis. Then we use, following Coleman & Pietronero
\[ \Gamma(r) = \frac{\langle n(r)n(0) \rangle}{\langle n \rangle} = \frac{G(r)}{\langle n \rangle} \]  
(6)
where \( \langle n \rangle \) is the average density of the sample. We stress that this normalization does not introduce any bias even if the average density is sample-depth dependent, as in the case of fractal distributions, because it represents only an overall normalizing factor. In order to compare results from different catalogs it is however more useful to use \( \Gamma(r) \), in which the size of a catalog only appears via the combination \( N^{-1} \sum_{i=1}^{N} \), so that a larger sample volume only enlarges the statistical sample over which averages are taken. \( G(r) \) instead has an amplitude that is an explicit function of the sample’s size scale.

The CF of Eq.4 can be computed by the following expression
\[ \Gamma(r) = \frac{1}{N} \sum_{i=1}^{N} \frac{1}{4\pi r^2 \Delta r} \int_{r}^{r+\Delta r} n(\vec{r} + \vec{r}' \mid \vec{r}) d\vec{r}' = \frac{BD}{4\pi} r^{D-3} \]  
(7)
where the last equality follows from Eq.3. This function measures the average density at distance \( r \) from an occupied point at \( \vec{r}_i \) and it is called the conditional density. If the distribution is fractal up to a certain distance \( \lambda_0 \), and then it becomes homogeneous, we have that
\[ \Gamma(r) = \frac{BD}{4\pi} r^{D-3} \quad r < \lambda_0 \]
\[ \Gamma(r) = \frac{BD}{4\pi} \lambda_0^{D-3} \quad r \geq \lambda_0 \]  
(8)
It is also very useful to use the conditional average density in the 3-d space
\[ \Gamma^*(r) = \frac{3}{4\pi r^3} \int_{0}^{r} 4\pi r'^2 \Gamma(r') dr' = \frac{3B}{4\pi} r^{D-3}. \]  
(9)
This function would produce an artificial smoothing of rapidly varying fluctuations, but it correctly reproduces global properties.4,6
For a fractal structure, $\Gamma(r)$ has a power law behaviour and the conditional average density $\Gamma^*(r)$ has the form

$$\Gamma^*(r) = \frac{3}{D} \Gamma(r).$$

(10)

For an homogenous distribution ($D = 3$) these two functions are exactly the same and equal to the average density.

Pietronero and collaborators have clarified some crucial points of the standard correlations analysis, and in particular they have discussed the physical meaning of the so-called "correlation length" $r_0$ found with the standard approach and defined by the relation:

$$\xi(r_0) = 1$$

(11)

where

$$\xi(r) = \frac{\langle n(n_0^n n_0^n + n) \rangle}{\langle n \rangle^2} - 1$$

(12)

is the two-point correlation function used in the standard analysis. The basic point that stressed, is that the mean density, $\langle n \rangle$, used in the normalization of $\xi(r)$, is not a well defined quantity in the case of self-similar distribution and it is a direct function of the sample size. Hence only in the case that the homogeneity has been reached well within the sample limits the $\xi(r)$-analysis is meaningful, otherwise the a priori assumption of homogeneity is incorrect and the characteristic lengths, like $r_0$, became spurious.

For example the expression of the $\xi(r)$ in the case of fractal distributions is:

$$\xi(r) = ((3 - \gamma)/3)(r/R_s)^{-\gamma} - 1$$

(13)

where $R_s$ is the depth of the spherical volume where one computes the average density from Eq. 2. From Eq. 13 it follows that

i.) the so-called correlation length $r_0$ (defined as $\xi(r_0) = 1$) is a linear function of the sample size $R_s$

$$r_0 = ((3 - \gamma)/6)^{1/\gamma} R_s$$

(14)

and hence it is a spurious quantity without physical meaning but it is simply related to the sample finite size.

ii.) $\xi(r)$ is power law only for

$$((3 - \gamma)/3)(r/R_s)^{-\gamma} >> 1$$

(15)

hence for $r \ll r_0$: for larger distances there is a clear deviation from the power law behaviour due to the definition of $\xi(r)$. This deviation, however, is just
due to the size of the observational sample and does not correspond to any real change of the correlation properties. It is clear that if one estimates the exponent of $\xi(r)$ at distances $r \lesssim r_0$, one systematically obtains a higher value of the correlation exponent due to the break of $\xi(r)$ in the log-log plot.

The analysis performed by $\xi(r)$ is therefore mathematically inconsistent, if a clear cut-off towards homogeneity has not been reached, because it gives an information that is not related to the real physical features of the distribution in the sample, but to the size of the sample itself.

### 3 Correlation properties of galaxy distribution

The main data of our correlation analysis are collected in Fig.1 (left part) in which we report the conditional density as a function of scale for the various catalogues. The relative position of the various behaviours of the conditional density in different catalogs, is not arbitrary but it is fixed by the luminosity function (a part for the cases of IRAS and SSRS1 for which this is not possible). The properties derived from different catalogues are compatible with each other and show a power law decay for the conditional density from $1h^{-1}Mpc$ to $150h^{-1}Mpc$ without any tendency towards homogenization (flattening). This implies necessarily that the value of $r_0$ (derived from the $\xi(r)$ approach) will scale with the sample size $R_s$ as shown also from the specific data about $r_0$ of the various catalogues. The behaviour observed corresponds to a fractal structure with dimension $D \approx 2$. The smaller value of CfA1 was due to its limited size. An homogeneous distribution would correspond to a flattening of the conditional density which is never observed. It is remarkable to stress that the amplitudes and the slopes of the different surveys match quite well. From this figure we conclude that galaxy correlations show very well defined fractal properties in the entire range $0.5 \div 1000h^{-1}Mpc$ with dimension $D = 2 \pm 0.2$. Moreover all the surveys are in agreement with each other.

It is interesting to compare the analysis of Fig.1 with the usual one, made with the function $\xi(r)$, for the same galaxy catalogs. This is reported in Fig.2 and, from this point of view, the various data appear to be in strong disagreement with each other. This is due to the fact that the usual analysis looks at the data from the prospective of analyticity and large scale homogeneity (within each sample). These properties have never been tested, and they are not present in the real galaxy distribution, so the result is rather confusing (Fig.2). Once the same data are analyzed with a broader perspective, the situation becomes clear (Fig.1) and the data of different catalogs result in agreement with each other. It is important to remark that analyses like those of Fig.3 have had a deep influence in the field in various ways: first, in the
Figure 1: Full correlation analysis for the various available redshift surveys in the range of distance $0.5 \div 1000h^{-1} Mpc$. A reference line with slope $-1$ is also shown, that corresponds to fractal dimension $D = 2$. 
Figure 2: Traditional analyses based on the function $\xi(r)$ of the same galaxy catalogs of the previous figure. The usual analysis is based on the a priori untested assumptions of analyticity and homogeneity. These properties are not present in the real galaxy distribution and the results appear therefore rather confusing. This lead to the impression that galaxy catalogs are not good enough and to a variety of theoretical problems like the galaxy-cluster mismatch, luminosity segregation, linear and non-linear evolution, etc.. This situation changes completely and becomes quite clear if one adopts the more general conceptual framework that is at the basis the previous figure.
standard analysis, the different catalogues appear in conflict with each other. This has generated the concept of *not fair samples* and a strong mutual criticism about the validity of the data between different authors. In the other cases the discrepancies observed in Fig. 2 have been considered real physical problems for which various technical approaches have been proposed. These problems are, for example, the galaxy-cluster mismatch, luminosity segregation, the richness-clustering relation and the linear non-linear evolution of the perturbations corresponding to the "small" or "large" amplitudes of fluctuations. We can now see that all this problematic situation is not real and it arises only from a statistical analysis based on inappropriate and too restrictive assumptions that do not find any correspondence in the physical reality. It is also important to note that, even if the galaxy distribution would eventually became homogeneous at larger scales, the use of the above statistical concepts is anyhow inappropriate for the range of scales in which the system shows fractal correlations as those shown in Fig. 3.

### 4 Conclusions and theoretical implications

Most of theoretical physics is based on analytical functions and differential equations. This implies that structures should be essentially smooth and irregularities are treated as single fluctuations or isolated singularities. The study of critical phenomena and the development of the Renormalization Group (RG) theory in the seventies was therefore a major breakthrough. One could observe and describe phenomena in which *intrinsic self-similar irregularities develop at all scales* and fluctuations cannot be described in terms of analytical functions. The theoretical methods to describe this situation could not be based on ordinary differential equations because self-similarity implies the absence of analyticity and the familiar mathematical physics becomes inapplicable. In some sense the RG corresponds to the search of a space in which the problem becomes again analytical. This is the space of scale transformations but not the real space in which fluctuations are extremely irregular. For a while this peculiar situation seemed to be restricted to the specific critical point corresponding to the competition between order and disorder. In the past years instead, the development of Fractal Geometry has allowed us to realize that a large variety of structures in nature are intrinsically irregular and self-similar (Fig. 3).

Mathematically this situation corresponds to the fact that these structures are singular in every point. This property can be now characterized in a quantitative mathematical way by the fractal dimension and other suitable concepts. However, given these subtle properties, it is clear that making a
Figure 3: Example of analytical and nonanalytic structures. Top panels (Left) A cluster in a homogenous distribution. (Right) Density profile. In this case the fluctuation corresponds to an enhancement of a factor 3 with respect to the average density. Bottom panels (Left) Fractal distribution in the two dimensional Euclidean space. (Right) Density profile. In this case the fluctuations are non-analytical and there is no reference value, i.e. the average density. The average density scales as a power law from any occupied point of the structure.
theory for the physical origin of these structures is going to be a rather challenging task. This is actually the objective of the present activity in the field [1]. The main difference between the popular fractals like coastlines, mountains, trees, clouds, lightnings etc. and the self-similarity of critical phenomena is that criticality at phase transitions occurs only with an extremely accurate fine tuning of the critical parameters involved. In the more familiar structures observed in nature, instead, the fractal properties are self-organized, i.e. they develop spontaneously from the dynamical process. It is probably in view of this important difference that the two fields of critical phenomena and Fractal Geometry have proceeded somewhat independently, at least at the beginning.

The fact that we are traditionally accustomed to think in terms of analytical structures has a crucial effect of the type of questions we ask and on the methods we use to answer them. If one has never been exposed to the subtileness on nonanalytic structures, it is natural that analyticity is not even questioned. It is only after the above developments that we could realize that the property of analyticity can be tested experimentally and that it may or may not be present in a given physical system.

These results have important consequences from a theoretical point of view. In fact, when one deals with self-similar structures the relevant physical phenomenon that leads to the scale-invariant structures is characterized by the exponent and not the amplitude of the physical quantities that characterizes such distributions.

Indeed, the only relevant and meaningful quantity is the exponent of the power law correlation function (or of the space density), while the amplitude of the correlation function, or of the space density and of the luminosity function, is just related to the sample size and to the lower cut-offs of the distribution. The geometric self-similarity has deep implications for the non-analyticity of these structures. In fact, analyticity or regularity would imply that at some small scale, the profile becomes smooth and one can define a unique tangent. Clearly this is impossible in a self-similar structure because, at any small scale, a new structure appears and the distribution is never smooth. Self-similar structures are therefore intrinsically irregular at all scales and correspondingly one has to change the theoretical framework into one which is capable of dealing with non-analytical fluctuations. This means going from differential equations to something like the Renormalization Group to study the exponents. For example the so-called ”Biased theory of galaxy formation” [14] is implemented considering the evolution of density fluctuations within an analytic Gaussian framework, while the non-analyticity of fractal fluctuations implies a breakdown of the central limit theorem which is the cornerstone of Gaussian processes [15].
The highly irregular galaxy distributions with large structures and voids strongly point to a new statistical approach in which the existence of a well defined average density is not assumed a priori and the possibility of non analytical properties should be addressed specifically. The new approach for the study of galaxy correlations in all the available catalogues shows that their properties are actually compatible with each other and they are statistically valid samples. The severe discrepancies between different catalogues that have led various authors to consider these catalogues as *not fair*, were due to the inappropriate methods of analysis.

The correct two point correlation analysis shows well defined fractal correlations up to the present observational limits, from 1 to $1000 h^{-1} Mpc$ with fractal dimension $D \simeq 2$. Of course the statistical quality and solidity of the results is stronger up to $100 \div 200 h^{-1} Mpc$ and weaker for larger scales due to the limited data. It is remarkable, however, that at these larger scales one observes exactly the continuation of the correlation properties of the small and intermediate scales. These results are currently at the center of acute debates in the field and we refer to \cite{16} and \cite{17} for a review of the two different points of view on this subject.

From the theoretical point of view the fact that we have a situation characterized by *self-similar structures*, implies that we should not use concept like $\xi(r)$, $r_0$, $\delta N/N$ and certain properties of the power spectrum, because they are not suitable to represent the real properties of the observed structures. In this respect also the N-body simulations should be considered from a new perspective. One cannot talk about ”small” or ”large” amplitudes for a self-similar structure because of the lack of a reference value like the average density. The Physics should shift from ”amplitudes” towards ”exponent” and the methods of modern statistical Physics should be adopted. This requires the development of constructive interactions between two fields.

As we have already mentioned, the correct reanalysis of the experimental data is just the first step in the understanding of the properties of the galaxy large scale structures. We refer the reader to Sylos Labini *et al.*\cite{5} for a review of the theoretical problem. It is worth to mention that Sanchez *et al.*\cite{18,19} have proposed a field theory approach to the fractal structure of the universe. In such a model the dominant dynamical mechanism responsible for the scale invariant distribution is self-gravity itself. Although there are several open problems, as for example the assumption of quasi-isothermal equilibrium of galaxy distribution, this model represents an interesting approach and a first attempt to focus the theoretical investigation on the behaviour of the exponents rather than on the amplitudes of correlations.
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