WARPED PRODUCTS AND REISSNER-NORDSTROM METRIC

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Abstract. We study a multiply warped products manifold associated with the Reissner-Nordstrom metric to investigate the physical properties inside the black hole event horizons. It is shown that, different from the uncharged Schwarzschild metric, the Ricci curvature components inside the Reissner-Nordstrom black hole horizons are nonvanishing, while the Einstein scalar curvature vanishes even in the interior of the charged metric. Introducing a perfect fluid inside the Reissner-Nordstrom black hole, it is also shown that the charge plays effective roles of decreasing the mass-energy density and the pressure of the fluid inside the black hole.

I. Introduction

Since the pioneering work in 1976, thermal Hawking effects on a curved manifold have been studied as an Unruh effect in a higher flat dimensional spacetime. Following the global embedding Minkowski space approach, several authors recently have shown that this approach could yield a unified derivation of temperature for various curved manifolds in (2+1) dimensions and in (3+1) dimensions. However all these higher dimensional embedding solutions have been constructed outside the event horizons of the metrics.

On the other hand, the concept of a warped products manifold was introduced by Bishop and O’Neill, where it served to provide a class of complete Riemannian manifolds with everywhere negative curvature. The connection with general relativity was first made by Beem, Ehrich, and Powell, who pointed out that several of the well-known exact solutions to Einstein field equations are pseudo-Riemannian warped products. Beem and Ehrich further explored the extent to which certain causal and completeness properties of a spacetime maybe determined by the presence of a warped products structure. After first developing the general theory of warped products to spaces, O’Neill then applied the theory to discuss, in turn, the special cases of Robertson-Walker and Schwarzshild spacetime. The role of warped products in the study of exact solutions to Einstein’s equations is now firmly established, and it appears that these structures are generating interest in other areas of geometry. Very recently, Choi has represented the interior Schwarzschild spacetime as a multiply warped products spacetime with warping functions to yield the Ricci curvature in terms of $f_1$, $f_2$ for the multiply warped products of the form $M = R^1 \times f_1 R^1 \times f_2 S^2$.

In this paper we will further analyze the multiply warped products manifold associated with the Schwarzschild metric by including the charge degrees of freedom so that we can investigate the physical properties inside the event horizons of the Reissner-Nordstrom black hole.

Key words and phrases. multiply warped products, Reissner-Nordstrom metric.
In section II, we will briefly recapitulate the multiply warped products manifold and the corresponding Ricci curvature components. In section III, we will apply the multiply warped products manifold scheme to the Reissner-Nordstrom metric to explicitly obtain the Ricci and Einstein curvatures inside the event horizons of the metric and to study the corresponding charge effects. Introducing a perfect fluid inside the Reissner-Nordstrom black hole, we will discuss the charge effects on the mass-energy density and the pressure of the fluid inside the black hole.

II. Multiply warped products and Ricci curvature

In this section, we briefly review a multiply warped products manifold to investigate Ricci curvature inside the black hole horizons.

A Lorentzian manifold \((M, g)\) is a connected smooth manifold of \(d\)-dimension \((d \geq 2)\) with a countable basis together with a Lorentzian metric \(g\) of signature \((-,+,+,...,+).\) Let \((F_i, g_i)\) be Riemannian manifolds, and let \((B, g_B)\) be either a spacetime, or let \(B = \mathbb{R}_1\) with \(g_B = -dt^2.\) Let \(f_i > 0, i = 1,..., n\) be smooth functions on \(B.\) A multiply warped products spacetime with base \((B, g_B),\) fibers \((F_i, g_i) i = 1,..., n\) and warping functions \(f_i > 0\) is the product manifold \((B \times F_1 \times ... \times F_n, g)\) endowed with the Lorentzian metric:

\[
g = \pi_B^* g_B + \sum_{i=1}^{n} (f_i \circ \pi_B)^2 \pi_i^* g_i \equiv -dt^2 + \sum_{i=1}^{n} f_i^2 g_i
\]

where \(\pi_B, \pi_i, i = 1,..., n\) are the natural projections of \(B \times F_1 \times ... \times F_n\) onto \(B\) and \(F_1,...,F_n,\) respectively.

Thus, warped product spaces are extended to richer class of spaces involving multiply products. Multiply warped products spaces were studied by Flores and Sánchez\(^{22}\). The conditions of spacelike boundaries in the multiply warped products spacetimes were studied by Harris\(^{23}\). The Kasner metric was studied as a cosmological model by Schücking and Heckmann\(^{24}\). Choi has investigated the curvature of a multiply warped product with \(C^0\)-warping functions\(^{21}\).

From a physical point of view, these spacetimes are interesting, first, because they include classical examples of spacetimes: when \(n = 1\) they are generalized Robertson-Walker spacetimes, standard models of cosmology; when \(n = 2\) the intermediate zone of Reissner-Norsdtröm spacetime and interior of Schwarzschild spacetime appear as particular cases.

III. Reissner-Nordstrom black hole as a multiply warped product manifold

In this section, to investigate a multiply warped product manifold for the Reissner-Nordstrom interior solution, we start with the four-metric inside the horizon

\[
ds^2 = N^2 dt^2 - N^{-2} dr^2 + r^2 d\Omega^2
\]

where the lapse function for the interior solution is given by

\[
N^2 = -1 + \frac{2m}{r} - \frac{Q^2}{r^2},
\]

where the lapse function for the interior solution is given by
with a mass $m$ and a charge $Q$, and $d\Omega^2 = d\theta^2 + \sin^2 \theta d\phi^2$. Note that, for the nonextremal case, there exist two event horizons $r_{\pm}(Q)$ satisfying the equations

$$0 = -1 + 2m/r_{\pm} - Q^2/r_{\pm}^2$$

such that

$$r_{\pm} = m \pm (m^2 - Q^2)^{1/2}.$$  \hspace{1cm} (3.3)

Furthermore the lapse function can be rewritten in terms of these outer and inner horizons as follows

$$N^2 = \frac{(r_+ - r)(r - r_-)}{r^2}$$

which, for the interior solution, is well defined in the region $r_- < r < r_+$.

**Proposition 3.1** Let $M$ be a manifold with the Reissner-Nordstrom metric solution $ds^2 = N^2dt^2 - N^{-2}dr^2 + r^2d\Omega^2$ (where $d\Omega^2 = d\theta^2 + \sin^2 \theta d\phi^2$), then we have $M$ as a multiply warped manifolds with warping functions $f_1, f_2$

$$ds^2 = -d\mu^2 + f_1^2(\mu)d\nu^2 + f_2^2(\mu)d\Omega^2$$  \hspace{1cm} (3.5)

where

$$\mu = 2m \cos^{-1} \left( \frac{r_+ - r}{r_+ - r_-} \right) - (r_+ - r)^{1/2}(r - r_-)^{1/2} = F(r),$$

$$f_1(\mu) = \left( -1 + \frac{2m}{F^{-1}(\mu)} - \frac{Q^2}{F^{-2}(\mu)} \right)^{1/2},$$

$$f_2(\mu) = F^{-1}(\mu).$$  \hspace{1cm} (3.7)\hspace{1cm} (3.8)

**Proof:** Define a new coordinate $\mu$ by

$$d\mu^2 = N^{-2}dr^2,$$  \hspace{1cm} (3.9)

which can be integrated to yield

$$\mu = \int_{r_-}^r \frac{dx}{(r_+ - x)^{1/2}(x - r_-)^{1/2}} + \mu(r_-).$$  \hspace{1cm} (3.10)

We establish $\mu(r_-) = 0$ to yield the analytic solution (3.6). Note that in (3.6) $dr/d\mu > 0$ implies $F^{-1}(\mu)$ is well-defined function. Exploiting the above new coordinate (3.6) and redefinition $\nu = t$, we rewrite the metric (3.1) as a multiply warped product $M$ as in the form of (2.1) to yield (3.5) with $f_1$ and $f_2$ in (3.7) and (3.8).

Note that in the vanishing $Q$ limit the above solution is reduced to that of the uncharged Schwarzschild case\textsuperscript{21}. Moreover, we have the following boundary conditions for $F(r)$

$$\lim_{r \to r_+} F(r) = m\pi, \quad \lim_{r \to r_-} F(r) = 0.$$  \hspace{1cm} (3.11)
After some algebra, we obtain the following nonvanishing Ricci curvature components

\[ R_{\mu\mu} = -\frac{f_1''}{f_1} - \frac{2f_2''}{f_2}, \]
\[ R_{\nu\nu} = \frac{2f_1f_2'}{f_2} + f_1f_1'', \]
\[ R_{\theta\theta} = \frac{f_1f_2'}{f_1} + f_2f_2'' + f_2^2 + 1, \]
\[ R_{\phi\phi} = \left( \frac{f_1f_2'}{f_1} + f_2f_2'' + f_2^2 + 1 \right) \sin^2 \theta, \quad (3.12) \]

which hold also in the Schwarzschild metric case. Note that, as shown in (3.6)-(3.8), the argument \( \mu \) of \( f_1 \) and \( f_2 \) in (3.12) and \( f_1 \) itself are however different from those of the Schwarzschild black hole, since \( \mu \) is described in terms of \( r_\pm \) possessing the charge and \( f_1 \) has the additional charge term.

**Proposition 3.2** Let \( M \) be a multiply warped product manifold \( M = R^1 \times f_1 R^1 \times f_2 S^2 \) for the Reissner-Nordstrom metric solution \( ds^2 = -d\mu^2 + f_1^2(\mu)d\nu^2 + f_2^2(\mu)d\Omega^2 \) (where \( d\Omega^2 = d\theta^2 + \sin^2 \theta d\phi^2 \)) with warping functions \( f_1, f_2 \), then we have Ricci curvature components as follows,

\[ R_{\mu\mu} = \frac{Q^2}{f_2^2}, \quad \text{(1)} \]
\[ R_{\nu\nu} = -\frac{Q^2f_1^2}{f_2^2}, \quad \text{(2)} \]
\[ R_{\theta\theta} = \frac{Q^2}{f_2^2}, \quad \text{(3)} \]
\[ R_{\phi\phi} = \frac{Q^2}{f_2^2} \sin^2 \theta. \quad \text{(3.13)} \]

**Proof:** Using the explicit expressions for \( f_1 \) and \( f_2 \) in (3.7) and (3.8), we can obtain identities for \( f_1, f_1', f_1'' \) in terms of \( f_1, f_2 \) and their derivatives

\[ f_1 = f_2, \]
\[ f_1' = -\frac{m}{f_2^2} + \frac{Q^2}{f_2^2}, \]
\[ f_1'' = -\frac{2f_1f_1'}{f_2} - \frac{Q^2f_1}{f_2^2}. \quad (3.14) \]

Substituting (3.14) into (3.12), we evaluate \( R_{\mu\mu} \) as follows,

\[ R_{\mu\mu} = 2\left( \frac{f_1'}{f_2} + \frac{Q^2}{2f_2^2} \right) - \frac{2f_1''}{f_2} = \frac{Q^2}{f_2^2}. \]

Similarly, we establish the other components \( R_{\nu\nu}, R_{\theta\theta} \) and \( R_{\phi\phi} \). 

Here one notes that, differently from the Schwarzschild case where we have the flat Ricci curvature components as shown with \( Q = 0 \), we have the nonvanishing Ricci tensor components for \( r_- < r < r_+ \). Moreover, it is amusing to see that, exploiting the metric (3.5), the Einstein scalar curvature is given as follows

\[ R = 0, \quad (3.15) \]
even in the interior of the charged Reissner-Nordstrom black hole horizons. However, due to the charge of the black hole, the vacuum solution in the uncharged Schwarzschild case does not hold any more as shown in (3.13).

Next, in order to investigate the charge effects inside the Reissner-Nordstrom black hole, we assume a perfect fluid which is a continuous distribution of matter with stress-energy tensor $T_{ab}$ of the form,

$$T_{ab} = \rho u_a u_b + P (g_{ab} + u_a u_b),$$

where $u^a$ is a unit timelike 4-velocity vector field of the fluid, $\rho$ is the mass-energy density and $P$ is the pressure of the fluid as measured in its rest frame.

**Proposition 3.3** In the interior ($r_- < r < r_+$) of charged Reissner-Nordstrom metric solution, we obtain the following equations.

1. $R_{\mu\mu} - 8\pi T_{\mu\mu} = \frac{Q^2}{f^2} - 8\pi P f_1^2 = 0$,
2. $R_{\nu\nu} - 8\pi T_{\nu\nu} = -\frac{Q^2 f_1^2}{f^2} + 8\pi \rho = 0$,
3. $R_{\theta\theta} - 8\pi T_{\theta\theta} = \frac{Q^2}{f^2} - 8\pi P f_2^2 = 0$,
4. $R_{\phi\phi} - 8\pi T_{\phi\phi} = \frac{Q^2}{f^2} \sin^2 \theta - 8\pi P f_2^2 \sin^2 \theta = 0$.

**Proof:** Substituting into the Einstein equations of motion

$$R_{ab} - \frac{1}{2} R g_{ab} = 8\pi T_{ab},$$

the Ricci components given by (3.13), the Einstein scalar curvature (3.15) and the stress-energy tensor (3.16) given in terms of the warped products, we obtain the above results.

Here one notes that in the case of the vanishing $Q$ limit which is the Schwarzschild case, one has contributions from the mass-energy density and the pressure of the fluid in the interior of the black hole. On the other hand, in the charged Reissner-Nordstrom case, the existence of nonvanishing charge $Q$ plays effective roles of decreasing the mass-energy density and the pressure of the fluid inside the black hole, since the terms associated with the charge are positive definite in the Einstein equations of motion. From a physical viewpoint, the above result is quite consistent with the phenomenology that the charge generates a repulsive force inside the charged black hole to dilute the density in the perfect fluid itself.

**IV. Conclusions**

We have studied a multiply warped product manifold associated with the Reissner-Nordstrom metric to evaluate the Ricci curvature components inside the charged black hole horizons. Differently from the uncharged Schwarzschild metric where
both the Ricci and Einstein curvatures vanish inside the horizon, the Ricci curvature components inside the Reissner-Nordstrom black hole horizons are nonvanishing, even though the Einstein scalar curvature vanishes in the interior of the charged metric. Introducing a perfect fluid inside the Reissner-Nordstrom black hole, we have also explicitly evaluated the Einstein equations of motion inside the horizons to investigate the charge effects that the charge decreases the mass-energy density and the pressure of the fluid inside the black hole.

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