Exact \((1+1)\)-dimensional flows of a perfect fluid

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We present a general solution of relativistic \((1+1)\)-dimensional hydrodynamics for a perfect fluid flowing along the longitudinal direction as a function of time, uniformly in transverse space. The Khalatnikov potential is expressed as a linear combination of two generating functions with polynomial coefficients of 2 variables. The polynomials, whose algebraic equations are solved, define an infinite-dimensional basis of solutions. The kinematics of the \((1+1)\)-dimensional flow are reconstructed from the potential.

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I. INTRODUCTION

A. Historical perspective

The problem of solving the \((1+1)\)-dimensional flow of a relativistic perfect fluid has a quite long history in particle physics. It has been first investigated in the pioneering work [1] where relativistic hydrodynamics has been introduced for describing high-energy multiparticle scattering. Together with the other pioneering work of Ref.[2] they are considered as the founding papers of the modern applications of hydrodynamics to heavy-ion collisions.

Ref.[1] has been followed by studies on the same guideline [3–9]. The Gaussian rapidity dependence prediction for the “Landau flow” found in [1] consistent with the observed multiplicity distributions has inspired subsequent works [10–12]. It has been revived recently [13–15] in connection with the experimental results on heavy-ion collisions at ultra-high energies [16].

The other well-known pioneering work analyzing the \((1+1)\)-dimensional flow of a relativistic perfect fluid is thus Ref.[2] (with a precursor [17]). Here, the boost-invariant solution of the \((1+1)\)-dimensional flow, the “Bjorken flow”, allows for quantitative predictions valid for the central rapidity region of heavy-ion reactions. It provided a firm theoretical basis for the prediction of the Quark-Gluon Plasma produced in subsequent heavy-ion colliders. In fact, it is now realized that the flow of relativistic particles created by the collisions can be well described by hydrodynamics, at least during some intermediate stage of the reaction where one observes the creation of a specific phase of Quantum Chromodynamics, namely the Quark-Gluon Plasma (QGP) [18].

Recent works on the hydrodynamic behavior of the QGP [19] uses numerical simulations of hydrodynamics, with the aim of solving them in a realistic way, including 4-dimensionality of space-time, initial and final conditions of the hydrodynamic regime, viscosity and other transport coefficients, realistic equation of state. However, it is useful to reconsider the initial [1, 2] problem, namely finding the exact analytic solutions of hydrodynamic equations in the simplified set-up of a perfect fluid flow in the longitudinal direction with constant speed of sound. As we shall see, this problem has not yet been solved.

There are quite a few motivations to follow this path, besides being the missing piece of a long lasting theoretical physics problem. On the phenomenological ground, it is known that in a first stage (important for later evolution, as discussed already in the seminal papers [1, 2]), the hydrodynamic flow is mainly \((1+1)\)-dimensional, \textit{i.e.} can essentially be described in the kinematic relativistic subspace defined by proper-time \(\tau\) and space-time rapidity \(\eta\).

On a more theoretical ground, the recently found Gauge/Gravity connection [20–22] between relativistic hydrodynamics and gravity in an higher-dimensional space through the AdS/CFT correspondence motivates completing the study of exact solutions of hydrodynamical equations. For instance in \((1+1)\) dimensions, the ‘Bjorken flow’ of a perfect fluid in a strongly coupled gauge theory is put in one-to-one correspondence [21] with the time-dependent 5-dimensional gravity configuration of a Black Hole escaping away in the fifth dimension. Going beyond the “Bjorken flow” is an important open question for the application of AdS/CFT correspondence to plasma physics. Hence,

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making progress in the exact solution of the hydrodynamic equations in $(1+1)$ dimensions may be quite useful in a modern perspective.

B. Position of the problem

The state of the art we have to begin with is the following. The hydrodynamic equations are a priori non-linear and as such are difficult to handle exactly through analytic methods. Only few particular exact solutions have been found. Apart the noticeable contributions of the pioneering studies, namely the analytic asymptotic solution of [1] (the “Landau flow” solution), and the boost-invariant solution of [2] (the “Bjorken flow” solution), there were only few interesting exact solutions given in the literature for specific values of the dynamical parameters (see e.g. [23–26], [27–30]). To our knowledge, a general solution for the relativistic $(1+1)$-dimensional flow of a perfect fluid is still lacking.

Recently, two developments on exact solutions of the $(1+1)$-dimensional flow appeared, which are the building blocks of the present work. On the one hand, a one-parameter family of solutions, interpolating between the “Bjorken flow” and the “Landau flow” was derived [31]. They were named harmonic flows since they are obtained assuming that the physical rapidity $y$ is an harmonic function of the light-cone kinematic variables, condition which is valid both for the “Bjorken flow” and the “Landau flow”. On the other hand, it was possible using the formalism of the Khalatnikov potential [3] to derive exact solutions of the $(1+1)$-dimensional entropy flow as a function of rapidity [32]. The Khalatnikov potential method makes use of a hodograph transformation, allowing for a substitution of the kinematic light-cone variables by the hydrodynamic ones, namely temperature and rapidity, in order to transform the initially nonlinear mathematical problem, posed by the hydrodynamic equations, into a linear one.

In the present paper we show how, by combining both approaches, i.e. the “harmonic flow” and the Khalatnikov potential approach, one generates an infinite-dimensional linear basis of exact solutions, making a sizable step towards the general solution of the relativistic $(1+1)$-dimensional flow of a perfect fluid.

Our plan is the following: In section II, we provide a reminder on the Khalatnikov potential method [3] and recall those results obtained in Refs. [31, 32] for the harmonic flow solution and its entropy flow which we will use here. In section III, we introduce the notion of regular (resp. irregular) solutions obtained by integration (resp. derivation) from the “harmonic flow” and give first generic examples of solutions. Focusing in section IV on regular solutions, we derive the more general set of solutions by solving appropriate polynomial equations in two variables. Section V is devoted to a discussion of the general solution. A final section VI provides a summary of our results and an outlook on the prospects for a complete solution of the exact $(1+1)$-dimensional flows of a perfect fluid.

II. KHALATNIKOV EQUATION AND HARMONIC SOLUTIONS

A. Hydrodynamic equations

We consider a perfect fluid whose energy-momentum tensor is

$$T^\mu{}^\nu = (\epsilon + p)u^\mu u^\nu - p n^\mu n^\nu$$

(1)

where $\epsilon$ is the energy density, $p$ is the pressure and $u^\mu$ $(\mu = \{0, 1, 2, 3\})$ is the 4-velocity in the Minkowski metric $\eta^\mu{}^\nu$. It obeys the equation

$$\partial_\mu T^\mu{}^\nu = 0.$$  \hspace{1cm} (2)

We write the standard thermodynamical identities (where we have assumed for simplicity vanishing chemical potential):

$$p + \epsilon = Ts; \quad d\epsilon = Tds; \quad dp = sdT,$$

(3)

where $p, \epsilon, s$ are respectively, the pressure, energy and entropy density. The system of hydrodynamic equations closes by relating energy density and pressure through the equation of state, which, in the present study will be considered with constant speed of sound, namely

$$\frac{dp}{d\epsilon} = \frac{sdT}{Tds} = c_s^2 \equiv \text{const.}\$$

(4)

We consider now the $(1+1)$ approximation of the hydrodynamic flow, restricting it only to the longitudinal direction. Within such an approximation, the effect of the transverse dimensions is only reflected through the equation of state
(4). Note that we do not a priori assume the traceless condition $T^{\mu\mu} = 0$, and thus the fluid is considered as “perfect” (null viscosity) but not necessarily “conformal” (null trace).

Let us introduce the light-cone coordinates

$z^\pm = z^0 \pm z^1 \equiv t \pm z = \tau e^{\pm \eta} \Rightarrow \left( \frac{\partial}{\partial z^0} \pm \frac{\partial}{\partial z^1} \right) = 2 \frac{\partial}{\partial z^\pm} (\equiv 2 \partial_\pm)$,

(5)

where $\tau = \sqrt{z^+ z^-}$ is the proper time and $\eta = \frac{1}{2} \ln(z^+/z^-)$ is the space-time rapidity of the fluid. We also introduce for further use the hydrodynamical variables, namely $y$, the usual energy-momentum rapidity variable and $\theta$, the logarithm of the inverse temperature, namely (recalling $u^+ u^- = 1$)

$y = \log u^+ = - \log u^- ; \quad \theta = \log(T_0/T)$,

(6)

where $u^\pm = \log(u^0 \pm u^1)$ are the light-cone components of the fluid velocity and $T_0$ some given fixed temperature, e.g. the initial one for a cooling plasma (explaining why one choses the ratio of temperatures (6), leading to $\theta \geq 0$).

The hydrodynamic equations (2) take the form

$$
(\partial_+ + \partial_-) T^{00} + (\partial_+ - \partial_-) T^{01} = 0 \\
(\partial_+ + \partial_-) T^{01} + (\partial_+ - \partial_-) T^{11} = 0.
$$

(7)

Note that inserting the formulation of the energy-momentum tensor (1) into the system (7) using expressions (6) leads to an highly non-linear system of equations in terms of the kinematic phase-space variables (5). This explains why there happened to be so much difficulty to find exact solutions of the flow characteristics. This is our aim to find a general solution to this problem by a change of perspective.

### B. The Khalatnikov Equation

It is known [3, 4] that one can replace the non-linear problem of (1+1) hydrodynamic evolution with a linear equation for a suitably defined potential. In this section we briefly recall the results of Refs.[3, 4] (recasting the calculations in the light-cone variables, as was done in [31]).

Using the thermodynamic relations (3), one can recombine the two equations (7) into the following ones, each of them having a physical interpretation, namely

- **The flow derives from a kinematic potential**

One combination of Eqs.(7) gives

$$
\partial_+ (e^{-\theta + y}) = \partial_- (e^{-\theta - y}) \equiv \partial_+ \partial_- \Phi(z^+, z^-).
$$

(8)

Eq.(8) proves the existence of a potential $\Phi(z^+, z^-)$ such that:

$$
\partial_{\mp} \Phi(z^+, z^-) \equiv u^\pm T = T_0 e^{-\theta \mp y}.
$$

(9)

- **Conservation of entropy**

Another independent combination of equations (7) corresponds to the conservation of entropy, namely

$$
\partial_+ (u^+ s) + \partial_- (u^- s) = 0.
$$

(10)

Combining Eqs.(9) and (10), one introduces the **Khalatnikov potential**

$$
\chi(\theta, y) \equiv \Phi(z^+, z^-) - z^- u^+ T - z^+ u^- T,
$$

(11)

where $z^\pm$ are now considered as functions of $(\theta, y)$. This is called the *hodograph* transformation expressing the hydrodynamic equations as a function of the dynamical variables $(\theta, y)$ via the Legendre transformation (11). The kinematic variables are recovered from the Khalatnikov potential by the equations

$$
z^\pm(\theta, y) = \frac{1}{2T_0} e^{\theta \pm y} (\partial_\theta \chi \pm \partial_y \chi).
$$

(12)
The Khalatnikov potential has the remarkable property \[3, 4, 32\] to verify a linear partial differential equation which takes the form
\[c_s^2 \partial_y^2 \chi(\theta, y) - \left[1 - c_s^2\right] \partial_\theta \chi(\theta, y) - \partial_\theta^2 \chi(\theta, y) = 0 .\] (13)

In (13), \(c_s\) (denoted also \(1/\sqrt{g}\) for further convenience) is the speed of sound in the fluid, which will be considered as a constant in the present study.

The Khalatnikov equation has been originally derived \[3\] for the potential (11). But its range of applicability appears to be much wider. Indeed, the transformation of a nonlinear problem in terms of the kinematic variables into a linear one in terms of hydrodynamic variables has tremendous advantages, as we shall see further. It allows to obtain new solutions by arbitrary linear combinations of known ones. Furthermore, primitive integrals and derivatives of solutions are also solutions.

In order to illustrate the powerfulness of this method, let us give two already known \[31\] examples. If \(\chi(\theta, y)\) is solution of (13), then the potential \(\Phi\), defined through (9), but now expressed in terms of the hydrodynamic variables through (12), reads
\[\Phi(\theta, y) \equiv \Phi\{z + (\theta, y), z - (\theta, y)\} = \chi(\theta, y) + \partial_\theta \{e^\theta \chi(\theta, y)\},\] (14)
and thus it verifies also the Khalatnikov equation (13). As a direct consequence, the physical entropy flow as a function of rapidity also verifies (13). Indeed, one has \[32\] (see also \[9\])
\[\frac{dS}{dy}(\theta, y) = \frac{s_0}{2gT_0} e^{-(g-1)\theta} \partial_\theta \Phi(\theta, y),\] (15)
where we use the thermodynamic relation \(s = s_0 e^{-g\theta}\) for the overall, temperature-dependent, entropy density of a perfect fluid, recalling that by definition \(c_s^2 \equiv 1/g\).

Before going further, it proves useful to use new variables, which allow to put the Khalatnikov equation (13) in a simple and symmetric form. Introducing
\[a = \frac{1}{2} \sqrt{(g-1)(\theta + c_s y)}, \quad b = \frac{1}{2} \sqrt{(g-1)(\theta - c_s y)}\] (16)
and redefining \(\chi(a, b)\) as a function of the reduced variables of (16), and introducing
\[Z(a, b) = e^{-(a^2 + b^2)} \chi(a, b),\] (17)
then the Khalatnikov equation (13) takes one or the other simple forms
\[\partial_a^2 \partial_b^2 \chi(a, b) = \{\partial_a^2 + \partial_b^2\} \chi(a, b) \quad (18)\]
\[\partial_a^2 \partial_b^2 Z(a, b) = Z(a, b) .\] (19)

C. Harmonic Flow

As noticed in \[31\], a specific combination of the hydrodynamic set of equations (3,7) allows one to eliminate the temperature and write a consistency condition on the rapidity. It reads
\[4 \partial_+ \partial_- y = \frac{g-1}{g+1} \{\partial_- \partial_- [e^{-2y}] - \partial_+ \partial_+ [e^{+2y}]\} .\] (20)

Eq. (20) explicitly exhibits the highly nonlinear character of the hydrodynamic equations (7) written in terms of kinematic differentials.

Despite this nonlinear feature, an analytic one-parameter family of solutions interpolating between the Landau and Bjorken flows has been obtained \[31\] imposing the harmonic condition (with \{+, -\} signature)
\[\partial_+ \partial_- y \equiv \{((\partial_1)^2 - (\partial_2)^2\} y = 0 .\] (21)

\footnote{Note that in this relation there is a sign difference comparing to that of \[31, 32\], due to the sign-difference in the \(\theta\)-definition that we use in this work.}
Harmonicity and the parity symmetry by $z_{\pm}$ interchange can be realized by writing
\[ y(z_+, z_-) = \frac{1}{2} \left[ l^2(z_+) - l^2(z_-) \right], \quad (22) \]
with
\[ z_{\pm} = \int_{l_{\pm}} \, dl \, e^{l^2} \quad (23) \]
as an implicit equation defining $l(z_{\pm})$ as a function of the kinematics (up to constants).

The relation between thermodynamic and kinematic variables can then be explicitly written
\[ \theta = \frac{g + 1}{4g} (l_+^2 - l_-^2) - \frac{g - 1}{2g} l_+ l_- \]
and reversely by simple algebraic manipulation
\[ l_+ + l_- = \sqrt{2g} \left( \theta + \sqrt{\theta^2 - y^2/g} \right)^{1/2} \quad (24) \]
\[ l_+ - l_- = \sqrt{2} \left( \theta - \sqrt{\theta^2 - y^2/g} \right)^{1/2} = \sqrt{\frac{4}{g - 1}} (a - b), \]
where we introduce the reduced variables (16). Note that each of the variables $a, b$ is a function of both $z_{\pm}$, contrary to the “harmonic variables” $l_+(z_+), l_-(z_-)$.

Using the property (9) of the potential $\Phi$ one writes
\[ \frac{\partial \Phi}{\partial l_{\pm}} = \frac{dz_{\pm}}{dl_{\pm}} \partial_{l_{\pm}} \Phi = \frac{dz_{\pm}}{dl_{\pm}} \, T_0 \, e^{-\theta \mp y}. \quad (26) \]
Now, inserting (22) and (24), one obtains
\[ \frac{\partial \Phi}{\partial l_{\pm}} = e^{l_{\pm}^2} e^{-\theta \mp y} = e^{\frac{g - 1}{4g} (l_+ + l_-)^2}. \quad (27) \]
The expression (27) is symmetric in $l_{\pm}$ and thus, by mere integration and using (25), one gets for the kinematic potential
\[ \Phi(a, b) \propto \int_{c}^{a+b} \, dt \, e^{t^2}, \quad (28) \]
where the initial integration value $c$ is matter of convention.

Hence the harmonic flow derives from a simple potential which corresponds to is the “Imaginary Error Function” defined as
\[ \text{erfi}[z] = \frac{2}{\sqrt{\pi}} \int_{0}^{z} e^{-t^2} dt = \frac{2}{\sqrt{\pi}} \int_{0}^{\frac{z}{2}} e^{t^2} dt. \quad (29) \]
Using relation (15) and the definitions (16) one easily obtains the entropy distribution corresponding to the harmonic flow
\[ \frac{dS}{dy} = \frac{s_0}{2g T_0} e^{-(g - 1) \theta} \, \partial_{\theta} \Phi(\theta, y) \propto \frac{a + b}{ab} \, e^{-(a-b)^2}. \quad (30) \]
As noticed in [31], this entropy distribution, considered for freeze-out at a fixed proper-time, leads to a density which interpolates between the Landau Gaussian solution and the Bjorken boost-invariant one. However the distribution contains a singularity at $a, b = 0$, i.e. when $c, y \to \pm \theta$, which causes a phenomenological problem. We will see that besides irregular solutions for the entropy flow generalizing the one obtained for the harmonic flow (30), a full set of regular solutions will be found in the set of 1+1 flow solutions, thus avoiding the phenomenological problems of the harmonic flow.
III. REGULAR AND IRREGULAR SOLUTIONS

A. Derivatives of the harmonic flow

Thanks to the linear form of Khalatnikov equation (13), it is obvious that any derivative of $\chi$ with respect to $y$ or/and $\theta$ will also provide a solution. In terms of the reduced variables (16), the derivatives of $\chi$ with respect to $a^2$ and $b^2$ (which are linear in $y$ and $\theta$) will be also solutions of the Khalatnikov equation. Hence, this is also valid for the potential $\Phi(a, b)$, thanks to the general linear relation (14). Note that symmetric derivatives in $a^2$ and $b^2$ will correspond to symmetric solutions in rapidity which we keep studying in the present paper.

Let $\Phi^{(h)}_n$ denote the $n$-th derivative of the harmonic potential (28) with respect to $a^2$ and $b^2$. After some algebra, one realizes that the general solution for $n \geq 1$ is the product of an exponential with a rational fraction of symmetric polynomials in $a$ and $b$

$$
\Phi^{(h)}_n(a, b) \equiv (\partial_{a^2})^n (\partial_{b^2})^n \Phi^{(h)}(a, b) = e^{(a+b)^2} \frac{Q_n(a, b)}{R_n(a, b)} ,
$$

(31)

where the denominator takes the form

$$
R_n(a, b) = a^n b^n ,
$$

(32)

and $Q_n(a, b)$ is a polynomial which can be straightforwardly determined through the iteration of derivatives. In fact, unless very particular cases (we did not find a counter-example) the general derivatives of the harmonic solution possess multiple poles at $a = 0$ and $b = 0$, i.e when $y = \pm \sqrt{g} \theta$ which appear as singularities in the entropy distribution (15). We call them irregular solutions since they lead to singularities in a physical distribution, the first example being the single poles of the harmonic flow itself, see (30).

Finally, note that all derivative solutions depend only on one non-meromorphic function $e^{(a+b)^2}$, which we call the seed function since all derivatives come from and factor out this function.

B. Integrals of the harmonic flow

Integrals of the harmonic flow potential (28) verify the Khalatnikov equation (13), provided one takes care of the boundary conditions (this amounts to keep $c = 0$ in (28)). Let $\Phi^{(h)n}$ denote the $n$-th integral with respect to $a^2$ and $b^2$. Thanks to the mathematical property of the error function

$$
\int \text{erfi}[z]dz = z \text{erfi}[z] - e^{-z^2}/\sqrt{\pi} .
$$

(33)

One realizes the interesting novelties of the integral solutions with respect to the derivative solutions, namely :

- The solution contains two transcendental seed functions, namely $\text{erfi}[a + b]$ and $e^{(a+b)^2}$, instead of only the last one.
- By iteration of formula (33) and appropriate integrations by parts, the solutions are always combinations of the two seed functions with polynomials.
- The solutions are regular at $a = 0$ and $b = 0$.

Indeed, using the reduced variables (16) for a more economic notation, one obtains the general form:

$$
\Phi^{(h)n}(a, b) = \left[ \Pi_n^{(0)}(a, b) \Phi_0(a, b) + \Pi_n^{(1)}(a, b) \Phi_1(a, b) \right] ,
$$

(34)

where

$$
\Phi_0(a, b) = e^{(a+b)^2} ; \quad \Phi_1(a, b) = \frac{\sqrt{\pi}}{2} \text{erfi}(a + b) \equiv \int_0^{a+b} e^{t^2} dt ,
$$

(35)

$^2$ However higher-order derivatives may be singular.
and $\Pi^0_n(a, b), \Pi^{(1)}_n(a, b)$ are symmetric polynomials in $(a, b)$. Hence $\Phi^{(h)\mu}$ does not possess poles.

As we shall see in the next section, there is a general derivation of these regular solutions which will give practical access to the infinite set of polynomials $\Pi^0_n(a, b), \Pi^{(1)}_n(a, b)$. However as a first example we provide the first polynomials, explicitly after 1 and 2 integrations symmetrically in $a$ and $b$:

$$\Pi^0_1 = 2a^3 - 2ba^2 - a + (a \leftrightarrow b)$$
$$\Pi^{(1)}_1 = -4a^4 + 4a^2(1 + b^2) + \frac{1}{2} + (a \leftrightarrow b),$$  \hspace{1cm} (36)

$$\Pi^0_2 = -2[a^7 - ba^6 - a^5(22 + 3b^2) + a^4(18b + 3b^2) + a^3(36 + 4b^2) + 12ba^2 + 72a] + (a \leftrightarrow b)$$
$$\Pi^{(1)}_2 = 2\{a^8 - a^6(24 + 4b^2) + a^4(72 + 24b^2 + 3b^4) + a^2(96 + 24b^2) + 72 + (a \leftrightarrow b)\}. \hspace{1cm} (37)$$

Note that, if $\Pi^{(i)}_n$ is solution also is $\left(\lambda \Pi^{(i)}_n + \mu\right)$ for any $\lambda, \mu$ being constants, that is independent of $i, n$.

IV. GENERAL REGULAR SOLUTION

Let us now present a systematic way to find the general solutions for the integral case. Using the form (19) of the Khalatnikov equation, let us propose a general Ansatz for the solution having the form

$$Z(a, b) = \left[P^0(a, b) Z_0(a, b) + P^{(1)}(a, b) Z_1(a, b)\right],$$  \hspace{1cm} (38)

where

$$Z_0(a, b) = e^{2ab}$$
$$Z_1(a, b) = e^{-a^2-b^2} \int_0^{a+b} e^{t^2} dt,$$  \hspace{1cm} (39)

and $P^0, P^{(1)}$ are functions determined in such a way that (19) be satisfied.

Using the relations:

$$\partial_{a^2}Z_0 = Z_0 \frac{b}{a} \quad \partial_{b^2}Z_0 = Z_0 \frac{a}{b}$$
$$\partial_{a^2}Z_1 = -Z_1 + \frac{Z_0}{2a} \quad \partial_{b^2}Z_1 = -Z_1 + \frac{Z_0}{2b}$$
$$\partial_{a^2}\partial_{b^2}Z_1 = Z_1 \quad \partial_{a^2}\partial_{b^2}Z_1 = Z_1 + \frac{Z_0}{2ab},$$  \hspace{1cm} (40)

we find that the Khalatnikov equation (19) breaks into two coupled equations, namely:

$$\{\partial_{a^2} + \partial_{b^2} - \partial_{a^2}\partial_{b^2}\} P^{(1)} = 0$$
$$\{a\partial_a + b\partial_b + 1 + \frac{1}{2}\partial_a\partial_b\} P^{(0)} = -\frac{1}{2}\{\partial_a + \partial_b\} P^{(1)}. \hspace{1cm} (42)$$

In the following we will consider only “symmetric” solutions in the interchange $a \leftrightarrow b$, but the method can be of more general validity.

Let us now for convenience use the variables:

$$u = a^2 + b^2 \equiv \frac{g-1}{2} \theta \quad v = a^2 - b^2 \equiv \frac{g-1}{2\sqrt{g}} y,$$  \hspace{1cm} (43)

thus $\partial_{a^2} = \partial_u + \partial_v$ and $\partial_{b^2} = \partial_u - \partial_v$. In terms of these variables, equation (42) becomes:

$$\left[\partial_v^2 - \partial_u^2 - 2\partial_u\right] P^{(1)}(u, v) = 0. \hspace{1cm} (44)$$

As a trial, since we expect $P^{(1)}$ to be a polynomial in two variables $(u, v)$, which in terms of $(a, b)$ is symmetric (thus containing only even powers of $v$), we consider the expansion:

$$P^{(1)}(u, v) = \sum_{k=0}^K v^{2k} P_k^{(1)}(u),$$  \hspace{1cm} (45)
where, for an arbitrarily chosen maximal value of the index $k_{\text{max}} \equiv K$, $P^{(1)}_K(u)$ are functions of $u$. We shall verify later on that they are indeed well-defined polynomials.

At this stage we introduce the Laplace transform of the functions $P^{(1)}_k(u)$ as:

$$\tilde{P}^{(1)}_k(\lambda) = \int_0^\infty du \, e^{-\lambda u} \, P^{(1)}_k(u)$$
$$P^{(1)}_k(u) = \int_{\lambda_0-i\infty}^{\lambda_0+i\infty} \frac{d\lambda}{2\pi i} \, e^{\lambda u} \, \tilde{P}^{(1)}_k(\lambda), \tag{46}$$

where $\lambda_0$ is some positive real constant at the right of all singularities of the integrand. Therefore, insertion of (46) into (45) and then into differential equation (44) gives:

$$\int_{\lambda_0-i\infty}^{\lambda_0+i\infty} \frac{d\lambda}{2\pi i} \, e^{\lambda u} \lambda (\lambda - 2) \tilde{P}^{(1)}_k(\lambda) = 0. \tag{47}$$

Starting with the highest power $v^{2K}$, all coefficients of the lower powers of $v^2$ must be zero. As we observe, the highest power contains only one term, namely the coefficient of $v^{2K}$, which amounts to impose

$$\int_{\lambda_0-i\infty}^{\lambda_0+i\infty} \frac{d\lambda}{2\pi i} \, e^{\lambda u} (\lambda - 2) \tilde{P}^{(1)}_K(\lambda) \equiv 0. \tag{48}$$

In order to avoid any singularity in the $\lambda$ complex plane, one gets two (non-trivial) possibilities, namely

$$\tilde{P}^{(1)}_K(\lambda) \propto \frac{1}{\lambda} \quad \Rightarrow \quad P^{(1)}_K(u) = \text{const.} \tag{49}$$
$$\tilde{P}^{(1)}_K(\lambda) \propto \frac{1}{\lambda - 2} \quad \Rightarrow \quad P^{(1)}_K(u) = \text{const.} \times e^{2u}. \tag{50}$$

This leads a priori to two families of solutions, depending on the chosen highest degree $K$. As we shall see further on, only the first family solution of (49) will survive the system of equations (42). The second family (50) will meet an obstruction when trying to solve the second equation of (42). Hence only polynomial solutions for $P^{(0,1)}$ are allowed.

For all smaller powers of $v^2$ we always have two terms, and the condition for the coefficients to be identically zero read iteratively as

$$\tilde{P}^{(1)}_{K-1}(\lambda) = \frac{2K(2K-1)}{\lambda(\lambda - 2)} \tilde{P}^{(1)}_K(\lambda) = \frac{\Gamma(2K+1)}{\Gamma(2K-1)\lambda(\lambda - 2)} \tilde{P}^{(1)}_K(\lambda)$$
$$\tilde{P}^{(1)}_{K-2}(\lambda) = \frac{(2K-2)(2K-3)}{\lambda(\lambda - 2)} \tilde{P}^{(1)}_{K-1}(\lambda) = \frac{\Gamma(2K+1)}{\Gamma(2K-3)\lambda(\lambda - 2)} \tilde{P}^{(1)}_K(\lambda)$$
$$\ldots \tag{51}$$

which straightforwardly leads (up to a common constant) to

$$\tilde{P}^{(1)}_K(\lambda) = \frac{\Gamma(2K+1)}{\Gamma(2K+1)} \times \frac{\tilde{P}^{(1)}_K(\lambda)}{[\lambda(\lambda - 2)]^{K-K}}, \tag{52}$$

with $1/\lambda$ (for the family of Eq. (49)) or $1/(\lambda - 2)$ (for the family of Eq. (50)). Finally, the inverse Laplace transform (46) gives

$$P^{(1)}_k(u) = \int_{\gamma} \frac{d\lambda}{2\pi i} \, e^{\lambda u} \frac{\Gamma(2K+1)}{\Gamma(2K+1)[\lambda(\lambda - 2)]^{K-K}} \tilde{P}^{(1)}_K(\lambda), \tag{53}$$

where the complex integration contour $\gamma$ encircles\(^3\) either $\lambda = 0$ or $\lambda = 2$. From (53), it is clear enough that first family of solutions are polynomials, the second family being made of polynomials factors of $e^{2u}$.

\(^3\) Initially the straight imaginary line contour of (48) can be deformed and leads to encircle the two multipole singularities at $\lambda = 0$ and $\lambda = 2$. However, as we shall see, at each step $k$ of the iteration (53), the choice $\lambda = 0$ will be selected by the second equation (42).
Let us first examine the first family generated at all steps by the multipole at $\lambda = 0$. The method will be to obtain the explicit solution (53) of the first of equations (42) and then plough the solutions as an input in the second member of the second equation (42).

From (53) with $P_{K}^{(1)}(\lambda) \equiv 1/\lambda$, i.e. $P_{K}^{(1)}(u) \equiv 1$, one obtains

$$P_{K}^{(1)}(u) = \int_{0}^{\infty} \frac{d\lambda}{2\pi i} e^{\lambda u} \frac{\Gamma(2K+1)}{\Gamma(2K+1)[\lambda(\lambda-2)]^{K-k}} \times 1 = \frac{\Gamma(2K+1)}{\Gamma(2K+1)\Gamma(K-k+1)} \left[ \frac{e^{\lambda u}}{(\lambda-2)^{K-k}} \right]_{\lambda=0}. \hspace{1cm} (54)$$

In order to now introduce the second equation (42), it is convenient to expand the polynomials $P^{(0,1)}$ in terms of their homogeneity components of degree $d$ in the variables $(a,b)$, namely

$$P^{(1)}(a,b) = \sum_{p=0}^{2K} P_{d}^{(1)}(a,b) \hspace{0.5cm} ; \hspace{0.5cm} d = 2p$$

$$P^{(0)}(a,b) = \sum_{p=0}^{2K-1} P_{d}^{(0)}(a,b) \hspace{0.5cm} ; \hspace{0.5cm} d = 2p + 1, \hspace{1cm} (55)$$

where the maximal degrees of homogeneity are dictated by the expansion (45) and the expression (54) for $P^{(1)}(a,b)$ and then by the second member of the second equation (42) for $P^{(0)}$. Indeed, from (54) it is straightforward to realize that the maximal degree at level $k$ for $P^{(1)}$ is $d = 4k + 2(K-k) = 2(k + K)$. Note also that $P^{(1)}(a,b)$ has only even degrees while $P^{(0)}(a,b)$ only odd ones.

Inserting the expansions (55) into the inhomogeneous second equation of (42), one finds the following nested recurrence

$$(4K) \hspace{0.5cm} P_{4K-1}^{(0)}(a,b) = -\frac{1}{2} (\partial_{a} + \partial_{b}) P_{4K}^{(1)}(a,b)$$

$$(4K-2) \hspace{0.5cm} P_{4K-3}^{(0)}(a,b) = -\frac{1}{2} \left\{ \partial_{a} \partial_{b} P_{4K-1}^{(0)}(a,b) + (\partial_{a} + \partial_{b}) P_{4K-2}^{(1)}(a,b) \right\}$$

$$(4K-4) \hspace{0.5cm} P_{4K-5}^{(0)}(a,b) = -\frac{1}{2} \left\{ \partial_{a} \partial_{b} P_{4K-3}^{(0)}(a,b) + (\partial_{a} + \partial_{b}) P_{4K-4}^{(1)}(a,b) \right\}$$

$$(\cdots) \hspace{1cm} (56)$$

Hence, degree by degree, all homogeneity components of $P_{4K-3}^{(0)}(a,b)$ are determined from those of $P^{(1)}(a,b)$.

2. Second family (forbidden)

Let us now consider the second family defined by the Ansatz (50) corresponding to the multipole at $\lambda = 2$. The first equation of (42) would give

$$Q_{K}^{(1)}(u) = \int_{0}^{\infty} \frac{d\lambda}{2\pi i} e^{\lambda u} \frac{\Gamma(2K+1)}{\Gamma(2K+1)[\lambda(\lambda-2)]^{K-k}} \times 1 = \frac{\Gamma(2K+1)}{\Gamma(2K+1)\Gamma(K-k+1)} e^{2u} \left[ \frac{e^{\lambda u}}{(\lambda-2)^{K-k}} \right]_{\lambda=0}. \hspace{1cm} (57)$$

where the second equality is obtained by the change of variable $\lambda \to \lambda + 2$. Hence now $Q_{K}^{(0,1)}(u) \equiv e^{2u} \hat{P}^{(0,1)}$ are to be taken as products of $e^{2u}$ by a polynomial.

Now the second equation of (42) reads

$$\left\{ a\partial_{a} + b\partial_{b} + 1 + \frac{1}{2} \partial_{a}\partial_{b} \right\} [e^{2u} \hat{P}^{(0)}] = -\frac{1}{2} (\partial_{a} + \partial_{b}) [e^{2u} \hat{P}^{(1)}]. \hspace{1cm} (58)$$

After differentiating both sides of (58) and simplifying by the common exponential factor, one gets

$$\left\{ (a + 2b)\partial_{a} + (b + 2a)\partial_{b} + 1 + \frac{1}{2} \partial_{a}\partial_{b} + 4(a + b)^{2} \right\} \hat{P}^{(0)} = -\frac{1}{2} (4(a + b) + \partial_{a} + \partial_{b}) \hat{P}^{(1)}, \hspace{1cm} (59)$$

where the additional terms with respect to the initial second equation (42) come from the exponential.
Again, as in Eqs. (56) but with more terms, there exists a priori a hierarchy of equations allowing to determine, term by term, the homogeneous components $\tilde{P}^{(1)}_d$ from those of $P^{(1)}$. One obtains

\begin{align*}
4(a+b)^2 \tilde{P}^{(0)}_{4K-1} &= -2(a+b)\tilde{P}^{(1)}_{4K} \\
4(a+b)^2 \tilde{P}^{(0)}_{4K-3} &= -(4K + 2b\partial_a + 2a\partial_b)\tilde{P}^{(0)}_{4K-1} - 2(a+b)\tilde{P}^{(1)}_{4K-2} - \frac{1}{2}(\partial_a + \partial_b)\tilde{P}^{(1)}_{4K} \\
4(a+b)^2 \tilde{P}^{(0)}_{4K-5} &= -(4K-2+2b\partial_a + 2a\partial_b)\tilde{P}^{(0)}_{4K-3} - 2(a+b)\tilde{P}^{(1)}_{4K-4} - \frac{1}{2}(\partial_a + \partial_b)\tilde{P}^{(1)}_{4K-2} \\
4(a+b)^2 \tilde{P}^{(0)}_{4K-7} &= -(4K-4+2b\partial_a + 2a\partial_b)\tilde{P}^{(0)}_{4K-5} - 2(a+b)\tilde{P}^{(1)}_{4K-4} - \frac{1}{2}(\partial_a + \partial_b)\tilde{P}^{(1)}_{4K-2}
\end{align*}

where it clearly appears that at each step the left hand side is determined by all previously obtained coefficients. However, the key difference with the set of equations (56) is that one has, at each step to factor out the $(a+b)^2$ factors, in order to ensure the polynomial nature of $P^{(0)}(a,b)$. Our conjecture, based on various trials, is that Eq. (60), contrary to the previous case, leads to an obstruction when reaching the lower homogeneity degrees.

As a simple but significant example, let us consider the solution of (57) with $K = 1$, namely

$$
\tilde{P}^{(1)}(a,b) \equiv \sum_{p=0}^{2} \tilde{P}^{(1)}_{2p}(a,b) = (a^2 - b^2)^2 + a^2 + b^2 - 1/2 .
$$

The system (60) leads to

\begin{align*}
4(a+b)^2 \tilde{P}^{(0)}_{3}(a,b) &= -2(a+b)^3(a-b)^2 \\
4(a+b)^2 \tilde{P}^{(0)}_{1}(a,b) &= -(a+b)(3a^2 + 3b^2 - 2ab).
\end{align*}

We see that, while the first equation can be satisfied by polynomials set, the second cannot. Unless exceptional cases, which we did not encounter in our tests, there exists an obstruction to realize the system (60). A more general study of this mathematical conjecture would be interesting.

V. SOLUTION AND PROPERTIES

Let us discuss in detail our results. As we saw, our resulting basis of regular solutions of the Khalatnikov equation (13), under the form (19), reads

$$
F_K(a,b) = \left[ P^{(0)}(a,b|K) F^{(0)}(a,b) + P^{(1)}(a,b|K) F^{(1)}(a,b) \right],
$$

where the change $P^{(0,1)} \rightarrow P^{(0,1)}(a,b|K)$ identifies the obtained solutions and

\begin{align*}
F^{(0)}(a,b) &= e^{(a+b)^2} \\
F^{(1)}(a,b) &= \int_0^{a+b} e^t dt .
\end{align*}

$P^{(0)}(a,b|K), \ (resp. P^{(1)}(a,b|K))$ span a family of polynomials, each one indexed by a different integer $K \in \mathbb{N}$, its higher degree being $d = 4K \ (d = 4K - 1)$. Note That $P^{(0)}(a,b|0) = 1, \ P^{(1)}(a,b|0) = 0$.

The following questions are in order:

- **How to reconstruct the flow from the solutions of the Khalatnikov equation?**

The guiding line for constructing the hydrodynamic flow solutions in the hodographic method is to obtain the kinematic variables from the solution of the Khalatnikov potential. The corresponding relation (12) expressed using the variables $a, b$ writes

$$
z^{\pm} = \frac{1 - c_s^2}{8T_0 c_s^2} \exp \left\{ \frac{2c_s}{1-c_s^2} \left[ c_s (a^2 + b^2) \pm (a^2 - b^2) \right] \right\} \times \left\{ \left( \frac{\partial}{\partial a^2} + \frac{\partial}{\partial b^2} \right) \pm c_s \left( \frac{\partial}{\partial a^2} - \frac{\partial}{\partial b^2} \right) \right\} \chi(a,b) .
$$

\[\tag{66}\]
Now, the physical requirement we are using is the regularity of the entropy distribution. Its expression is obtained through rewriting the relations (14,15) as the hierarchy of relations starting from the Khalatnikov potential $\chi$.

$$\Phi = \chi(a, b) + \frac{g-1}{2} \{\partial_{a^2}\partial_{b^2}\} \chi(a, b)$$
$$\frac{dS}{dy} = \frac{s_0(g-1)}{8T_0g} e^{-2(a^2+b^2)} \{\partial_{a^2}\partial_{b^2}\} \Phi(a, b) ,$$

(67)

where we make use of the relation

$$\{\partial_{a^2} + \partial_{b^2}\} F(a, b) = \partial_{a^2}\partial_{b^2} F ,$$

(68)

valid for any solution of the Khalatnikov equation $F(a, b)$ by inserting the definition $Z(a, b) = e^{-(a^2+b^2)} F(a, b)$ into (19). Note that the operator $\{\partial_{a^2} + \partial_{b^2}\}$ corresponds exactly to the shift $K \rightarrow K-1$ acting on our basis of solutions (63).

From Eqns.(67) we see that the entropy distribution $dS/dy$ is obtained from up to two successive actions of $\{\partial_{a^2}\partial_{b^2}\}$ on the Khalatnikov potential $\chi$. Hence, in order to obtain a regular $dS/dy$ it is required to start with a Khalatnikov potential $\chi$ at level $K \geq 2$. One obtains the full solution given by

$$\chi = \sum_{K_i = 2} \lambda_{K_i} F_{K_i} ,$$
$$\Phi = \sum_{K_i = 2} \lambda_{K_i} \left( F_{K_i} + \frac{g-1}{2} F_{K_{i-1}} \right) ,$$
$$\frac{dS}{dy} = \frac{s_0(g-1)}{8T_0g} e^{-2(a^2+b^2)} \sum_{K_i = 2} \lambda_{K_i} \left( F_{K_{i-1}} + \frac{g-1}{2} F_{K_{i-2}} \right) .$$

(69)

In order to illustrate this discussion, in Fig. 1 we present the entropy rapidity-distribution $dS/dy$ for the first components of the new basis of solutions. In particular, we depict $dS/dy$ for the first two irregular components (starting with the harmonic potential), and for the first two regular components. Additionally, for completeness we present also the well-known “Landau flow” solutions, namely the asymptotic Gaussian one [1] and the exact solution of Ref.[4].

- Is the family identical to the one obtained in section IIIB by multiple integrations?

The uniqueness of our solutions, up to overall multiplicative and additive constants, for a given maximal degree $4K$ gives a strong argument that it corresponds exactly to the solutions obtained by successive integrations over $a^2, b^2$. In order to give a simple example, one gets for $K = 1$

$$P^{(1)}(a, b|1) = (a^2 - b^2)^2 - a^2 - b^2 - \frac{1}{4} = -\frac{1}{4} \Pi_1^{(1)}$$
$$\hat{P}^{(0)}(a, b|1) = -\frac{1}{2} (a + b)(a^2 - b^2) + \frac{1}{4} (a + b) \equiv -\frac{1}{4} \Pi_1^{(0)} ,$$

(70)

where the polynomials $\Pi_1^{(0,1)}$ were obtained in section IIIB from first integration, see (36). So, the $K$-indexed basis of solutions is eventually identical to the multiple integration of the “harmonic family”. A rigorous mathematical proof deserves some more further effort.

- Is the $K$-indexed family forming a complete basis?

This question appears to be more involved. A well-known example of solution to (19) (see e.g. [32] for a complete discussion) is given by

$$Z_0(a, b) = I_0(2ab) = \partial_{a^2}\partial_{b^2} I_0(2ab) = \partial_{a^2}\{a/b \times I_1(2ab)\} ,$$

(71)

where $I_{0,1}$ are the well-known Modified Bessel Functions. It is clear that those functions cannot be put into the form (63) for a combination of solutions with maximal finite $K_{max}$. The only possibility would be that $I_{0,1}(a, b)$ belong to the closure of the basis (63), i.e. expressed as a convergent expansion over the basis with $K \rightarrow \infty$. This example, and more generally the proof of the completeness of the basis (63) appears to be a non-trivial mathematical problem which deserves to be studied on its own.
FIG. 1: Entropy distribution. Various solutions for $dS/dy$ arising from the initial harmonic potential $\Phi^{(h)}$ (see (28)) are represented for the speed of sound $c_s = 1/\sqrt{3} \ (g = 3)$ and $\theta = \log T_0/T = 2$. Regular solutions: 1) $\{\Phi^{(h)}_1\}$, solid (black) line: first integral of $\Phi^{(h)}$. 2) $\{\Phi^{(h)}_2\}$, dashed (blue) line: second integral of $\Phi^{(h)}$. Singular solutions: 1) $\{\Phi^{(h)}_1\}$, dotted (red) line: harmonic potential; 2) $\{\Phi^{(h)}_1\}$, dashed-dotted (magenta) line: first derivative. “Landau flow” solutions: 1) $\{\text{Gaussian}\}$, short-dashed (green): “Landau flow” asymptotics; 2) $\{I_0\}$ thin (green) line: Landau-Belenkij exact solution [4].

- How to introduce the initial conditions?

The equation (63) appears to propose a rich possibility of solutions to the flow equations. However, one would be interested to modulate these solutions as a function of the boundary conditions, at least the initial ones. For this sake, one would have to solve the Green functions [32], namely the solutions of the equation

$$
c_s^2 \partial^2_y G(\theta, y) + \left[1 - c_s^2\right] \partial_\theta G(\theta, y) - \partial^2_y G(\theta, y) = \delta(\theta)\delta(y). \tag{72}
$$

We hope our method could be useful to solve in the near future this equation.

VI. CONCLUSION AND OUTLOOK

Let us summarize our results:

i) We have derived a basis of solutions for the (1+1)-dimensional flows of a perfect fluid with arbitrary constant speed of sound. It spans an infinite-dimensional linear vectorial space of solutions.

ii) The basis elements can be indexed by a positive or negative integer number $K \in \mathbb{Z}$, where the negative indices $K < 0$ correspond to singular solutions while $K \in \mathbb{N}$ correspond to regular ones.

iii) The singular basis can be obtained by successive differentiation of the “harmonic flow” solution, while the regular one arises from successive integration.

iv) The general regular solution is characterized by a Khalatnikov potential which is an arbitrary combination of components of a linear basis

$$
\{\chi(a, b|K)\} = \left\{P^{(0)}(a, b|K) e^{(a+b)^2} + P^{(1)}(a, b|K) \int_0^{a+b} e^{t^2} dt \mid K \in [2, \infty) \right\}, \tag{73}
$$
where \( P^{(1,0)}(a, b|K) \) are polynomials of maximal homogeneity degree \( d = (4K, 4K-1) \) defined by recurrence relations, see formulas (54,56). They are uniquely defined (up to an overall constant) by the value of \( K \).

v) The variables describing the flow

\[
a = \frac{\sqrt{\theta} + \frac{g}{2}(y-1)}{2}, \quad b = \frac{\sqrt{\theta} - \frac{g}{2}(y-1)}{2}
\]

(74)

are functions of the dynamical variables \( \theta = \log T_0/T, \ y = \frac{1}{2}\log u^+/u_- \) depending on the flow temperature \( T \) and longitudinal velocity \( u_\pm = u_0 \pm u_1 \). The kinematics of the flow are recovered from the Khalatnikov potential through the inverse hodograph transformation \( \theta, y \rightarrow z_\pm = z_0 \pm z_1 \).

As an outlook, let us quote a few interesting problems following our present study:

- Can we prove (or disprove) the completion of our vectorial space of regular solutions or, equivalently, can we prove (or disprove) that any regular Khalatnikov potential solution arises from a convergent expansion \( \sum_{K=2}^{\infty} \lambda_K \chi(a, b|K) \)?
- Can we solve the solution with given boundary conditions or, equivalently, solve the Green function associated with the Khalatnikov equation?
- Can we strictly satisfy the energy conservation inside the forward light-cone? This arises from the remark [33] that the harmonic flow does not verify this constraint.

We hope that the presented solution for the rather old problem of \((1+1)\)-dimensional flows of a perfect fluid can serve for the modern and stimulating hydrodynamic investigations both on the phenomenological (through heavy-ion experiments) and theoretical (through Gauge/Gravity duality) points of view. In any case, it was quite pleasant to deal with this problem.

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