Counting restricted orientations of random graphs

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Abstract
We count orientations of $G(n, p)$ avoiding certain classes of oriented graphs. In particular, we study $T_r(n, p)$, the number of orientations of the binomial random graph $G(n, p)$ in which every copy of $K_r$ is transitive, and $S_r(n, p)$, the number of orientations of $G(n, p)$ containing no strongly connected copy of $K_r$. We give the correct order of growth of $\log T_r(n, p)$ and $\log S_r(n, p)$ up to polylogarithmic factors; for orientations with no cyclic triangle, this significantly improves a result of Allen, Kohayakawa, Mota, and Parente. We also discuss the problem for a single forbidden oriented graph, and state a number of open problems and conjectures.

KEYWORDS
restricted orientations; random graphs; forbidden digraphs

1 | INTRODUCTION

An orientation $\vec{H}$ of a graph $H$ is an oriented graph obtained by assigning an orientation to every edge of $H$. Over 40 years ago, Erdős [8] initiated the study of $D(G, \vec{H})$, the number of $\vec{H}$-free orientations of a graph $G$, and in particular posed the problem of determining $D(n, \vec{H}) := \max \{ |D(G, \vec{H})| : |V(G)| = n \}$. For tournaments, this problem was resolved by Alon and Yuster [4], who proved that $D(n, T_k) = 2^{\text{ex}(n, K_3)}$ holds for any tournament $T_k$, and all sufficiently large $n \in \mathbb{N}$.

Recently, Allen, Kohayakawa, Mota, and Parente [2] introduced a related problem in the context of random graphs: that of determining the typical number of $C_r^\circ$-free orientations of the Erdős-Rényi random graph $G(n, p)$, where $C_r^\circ$ is the directed cycle of length $r$. The main result of [2] is as follows.
Theorem 1.1. Let \( r \geq 3 \). Then, with high probability as \( n \to \infty \),

\[
\log_2 D(G(n, p), C_r^\wedge) = \begin{cases} 
(1 + o(1)) p \binom{n}{2} & \text{if } n^{-2} \ll p \ll n^{-(r-2)/(r-1)}, \\
o(pn^2) & \text{if } p \gg n^{-(r-2)/(r-1)}. 
\end{cases}
\]

Our first theorem provides the following improvement in the case \( r = 3 \); the \( \tilde{\Theta}(\cdot) \)-notation indicates upper and lower bounds that differ by a polylogarithmic factor.

Theorem 1.2. The following bounds hold with high probability as \( n \to \infty \).

\[
\log_2 D(G(n, p), C_3^\wedge) = \begin{cases} 
(1 + o(1)) p \binom{n}{2} & \text{if } n^{-2} \ll p \ll n^{-1/2}, \\
\tilde{\Theta}(n/p) & \text{if } p \gg n^{-1/2}. 
\end{cases}
\]

We will in fact prove a similar upper bound for \( D(G(n, p), C_r^\wedge) \) for all \( r \geq 3 \) (see Theorem 6.1). However, we do not believe that our bound is sharp when \( r \geq 4 \); instead, we believe that the natural generalization of the lower bound construction in Theorem 1.2 (see Section 2) is sharp up to polylogarithmic factors.

Conjecture 1.3. Let \( r \geq 4 \). If \( p \gg n^{-(r-2)/(r-1)} \), then

\[
\log D(G(n, p), C_r^\wedge) = \tilde{\Theta}\left(\frac{n}{p^{1/(r-2)}}\right)
\]

with high probability as \( n \to \infty \).

Note that \( D(G, C_r^\wedge) \) is the number of orientations of \( G \) in which every triangle is transitive, or equivalently, in which no triangle is strongly connected. This suggests two natural generalizations of Theorem 1.2, which we discuss below.

Avoiding nontransitive tournaments. Our first generalization deals with the number of orientations of \( G(n, p) \) avoiding nontransitive tournaments of a given size.

Definition 1.4. Let \( T_r(n, p) \) denote the random variable which counts the number of orientations of the random graph \( G(n, p) \) in which every copy of \( K_r \) is transitively oriented (we will simply say that “every \( K_r \) is transitive”).

Our next main result generalizes Theorem 1.2 by determining the typical value of \( \log T_r(n, p) \) up to polylogarithmic factors for every \( r \geq 3 \).

Theorem 1.5. Let \( r \geq 3 \). The following bounds hold with high probability as \( n \to \infty \).

\[
\log_2 T_r(n, p) = \begin{cases} 
(1 + o(1)) p \binom{n}{2} & \text{if } n^{-2} \ll p \ll n^{-2/(r+1)}, \\
\tilde{\Theta}\left(p^{2-2/(r+1)} n^{4-r}\right) & \text{if } n^{-2/(r+1)} \ll p \ll n^{-2/(r+2)}, \\
\tilde{\Theta}(n/p) & \text{if } p \gg n^{-2/(r+2)}. 
\end{cases}
\]
FIGURE 1 The graph of $b = b(a)$, where $T_r(n, p) = \exp\left(n^{\Theta(1)}\right)$ and $p^\binom{n}{r} = n^a$.

Note that the functions in (2) and (3) coincide when $r = 3$. Figure 1 illustrates the typical behavior of $T_r(n, p)$ given by Theorem 1.5 when $r \geq 4$. We remark that, despite the more complicated behavior of $T_r(n, p)$ when $r \geq 4$, the proof of Theorem 1.5 is not significantly more difficult than that of Theorem 1.2.

Avoiding strongly connected tournaments. The construction used to prove the lower bounds in Theorems 1.2 and 1.5 and Conjecture 1.3 can also be used to prove the lower bound in the following conjecture.

**Conjecture 1.6.** Let $\vec{H}$ be a strongly connected tournament on $r \geq 3$ vertices, and suppose that $p \gg n^{-2/(r+1)}$. Then

$$\log D(G(n, p), \vec{H}) = \tilde{\Theta}\left(\frac{n}{p^{(r-1)/2}}\right)$$

with high probability as $n \to \infty$.

Theorem 1.2 proves the conjecture in the case $r = 3$. Moreover, we are able to prove that the upper bound holds if we instead forbid all strongly connected tournaments on $r$ vertices.

**Definition 1.7.** Let $S_r(n, p)$ denote the random variable which counts the number of orientations of the random graph $G(n, p)$ in which no copy of $K_r$ is strongly connected.

Note that $T_r(n, p) \leq S_r(n, p)$, and that $D(G(n, p), C_r^\bigcirc) \leq S_r(n, p)$, since every strongly connected orientation of $K_r$ contains a Hamiltonian cycle. The following theorem determines the typical value of $\log S_r(n, p)$ up to polylogarithmic factors for every $r \geq 3$.

**Theorem 1.8.** Let $r \geq 3$. The following bounds hold with high probability as $n \to \infty$.

$$\log_2 S_r(n, p) = \begin{cases} (1 + o(1))p\binom{n}{2} & \text{if } n^{-2} \ll p \ll n^{-2/(r+1)}, \\ \tilde{\Theta}\left(\frac{n}{p^{(r-1)/2}}\right) & \text{if } p \gg n^{-2/(r+1)}. \end{cases}$$
We consider this result to be reasonably strong evidence in favor of Conjecture 1.6. In Section 7 we will discuss other forbidden oriented graphs; we do not know of any oriented graph $\vec{H}$ containing a cycle for which our lower bound construction fails to give sharp bounds on $D(G(n,p),\vec{H})$.

The rest of the paper is organized as follows: in Section 2 we prove the various lower bounds; in Sections 3-5 we prove the upper bounds in Theorems 1.2, 1.5, and 1.8 respectively; in Section 6 we prove an upper bound for arbitrary cycles, and in Section 7 we discuss some further open problems and conjectures.

## 2 | LOWER BOUNDS

Each of the bounds proved in this section (and the general lower bound given in Section 7.2) follows from the same simple construction. Roughly speaking, we fix a linear order on the vertex set (let us identify it with the set $[n] = \{1, \ldots, n\}$), choose a “critical length” $a$, and orient all edges of length at least $a$ in the same direction (“forward”). The edges shorter than $a$ may be oriented in either direction, as long as they are not at risk of creating a forbidden substructure; by choosing $a$ carefully, we can guarantee that (with high probability) there are many such “free” edges.

To illustrate this construction with a simple example, let us begin by proving the lower bound in Equation (1) in the case $p = o(1)$. (The case $p = \Theta(1)$ follows from Proposition 2.2.)

**Proposition 2.1.** If $\omega \gg 1$ and $n^{-1/2} \leq p = o(1)$, then

$$\log D(G(n,p), C_3^{\odot}) \geq \frac{n}{\omega \cdot p}$$

with high probability as $n \to \infty$.

**Proof.** We may assume that $\omega \to \infty$ sufficiently slowly. We will show that with high probability there exists a set of at least $n/\omega p$ edges which can be oriented freely without creating a nontransitive triangle. To do so, set $a = 2 \cdot p^{-2}/\omega$, and note that $a = o(n)$, since $p \gg n^{-1/2}$ and $\omega \to \infty$, and that $a \gg 1$, since $p = o(1)$ and $\omega \to \infty$ sufficiently slowly. Let us say that an edge $uv$ is $a$-short if its length $|u - v|$ is less than $a$, and $a$-long otherwise. Note that if we orient all $a$-long edges forward, then any nontransitive triangle must contain at least two $a$-short edges (a “dangerous triangle”), at least one of which must be oriented backward.

Now, observe (e.g., by Chernoff’s inequality) that with high probability the number of $a$-short edges in $G(n,p)$ is $(1 + o(1)) p a n$, and that the expected number of triangles in $G(n,p)$ containing at least two $a$-short edges is $O(p^3 a^2 n) \ll p a n$. By Markov’s inequality, it follows that with high probability there exists a set of at least $p a n/2 = n/\omega p$ edges that are not in any such triangle and therefore can be oriented freely, as required.

We conjecture (see Conjecture 7.5) that the lower bound on $D(G(n,p), C_3^{\odot})$ given by Proposition 2.1 is sharp up to a constant factor in the exponent when $p$ is not too large. On the other hand, when $p \gg 1/\log n$ we can obtain a stronger lower bound by varying the linear order used in the construction. The following proposition also provides a suitable lower bound for all nonacyclic $\vec{H}$ when $p = \Theta(1)$.

**Proposition 2.2.** If $\vec{H}$ contains a cycle and $p \gg (\log n)/n$, then

$$D(G(n,p), \vec{H}) \geq p^n \cdot n! / e^{o(n)}$$

with high probability as $n \to \infty$. 
Proposition 2.2 is a straightforward consequence of a result of Goddard, Kenyon, King, and Schulman [10, Theorem 2.5] which gives a lower bound on the number of acyclic orientations of an arbitrary graph $G$ in terms of its degree sequence (see also [11]). However, since the proof requires some tedious calculation using Stirling’s formula, we provide the details of the proof in the appendix of [7].

It is also straightforward to replace the factor of $1/\omega$ in (4) by a small fixed constant using the second moment method. To illustrate this, we will prove the following bound on $T_r(n, p)$.

**Proposition 2.3.** Let $r \geq 3$. There exists $c > 0$ such that if $n^{-2/(r-1)} \ll p \ll n^{-2/(r+2)}$, then

$$\log T_r(n, p) \geq c \cdot \frac{n^{4-r}}{p^{(r-2)}}$$

with high probability as $n \to \infty$.

Observe that Propositions 2.1 and 2.2 imply the lower bound in Equation (3), and Proposition 2.3 implies the lower bound in (2). Indeed, the lower bound in (3) follows simply by noticing that in a $C_3^\circ$-free oriented graph, all copies of $K_r$ are transitive.

Since the proof of Proposition 2.3 requires some slightly tedious calculations, we will give here just a sketch of the proof; for the full details, see the appendix of [7].

**Sketch proof of Proposition 2.3.** We repeat the proof strategy of Proposition 2.1, using the second moment method instead of Markov’s inequality. Choose $a = c \cdot n^{3-r} p^{1/(r)}$, and observe that the lower bound on $p$ implies that $a = o(n)$, and the upper bound implies that $a = \Omega(p^{-2})$. As before, say that an edge is $a$-short if its length is less than $a$, and $a$-long otherwise. Note that any nontransitive copy of $K_r$ contains a cyclic triangle, so if we orient all $a$-long edges forward, then any nontransitive $K_r$ must contain at least two $a$-short edges sharing a vertex, at least one of which must be oriented backward.

Let $X$ denote the number of copies of $K_r$ in $G(n, p)$ that are “dangerous,” in the sense that they contain two $a$-short edges incident to a single vertex. A straightforward calculation gives

$$\mathbb{E}[X] = \Theta(n^{r-2} a^2 p^{(r-1)}) \leq \frac{pan}{r^2}$$

if $c > 0$ is sufficiently small, and

$$\text{Var}(X) = O\left(n^{2r-5} a^2 p^{2(1)} + n^{2r-5} a^2 p^{2(2)} + n^{r-1} a^2 p^{(r-1)} \right) \ll (pan)^2,$$

since $pan \gg 1$ and $a = \Omega(p^{-2})$. It follows, by Chebyshev’s inequality, that $X \leq \frac{pan}{r^2}$ with high probability, and hence there exists a set of at least $pan / 2$ edges that can be oriented freely, as required.

It is straightforward to generalize this idea to prove the lower bounds in Theorem 1.8 and Conjectures 1.3 and 1.6, so we will be somewhat brief with the details. To avoid tedious calculations, we will prove the slightly weaker bounds given by Markov’s inequality, rather than the slightly stronger bounds given by the second moment method.

**Proposition 2.4.** Let $r \geq 3$ and $\omega \gg 1$. If $p \geq n^{-(r-2)/(r-1)}$, then

$$\log D\left(G(n, p), C_r^\circ\right) \geq \frac{n}{\omega \cdot p^{1/(r-2)}}$$

with high probability as $n \to \infty$. 
Theorem 3.1. Let whose proof contains the key idea introduced in this paper. In this section we prove the following theorem, which implies the upper bound in Theorem 1.2, and therefore the expected number of such copies of \( C \) could form a \( C^3 \) -free graph is sometimes enough to uniquely determine \( C \). Given any \( C^3 \) -free orientation \( \vec{G} \) of \( G(n, p) \), we will use this fact to find a small set of “nonredundant” edges \( S \subseteq E(\vec{G}) \) such that the prescribed orientation \( \vec{G} \) is uniquely determined by \( S \). This is formalized in the deterministic lemma 3.2. We remark that the next sections will use similar deterministic statements (in Claims 4.4 and 5.4, and Lemma 6.4).

Proof. The lower bound in Proposition 2.2 is stronger when \( p = \Omega(1) \), so we may assume \( p = o(1) \). Set \( a = p^{-1/(r-2)}/\omega = o(n) \), and consider any orientation \( \vec{G} \) of \( G(n, p) \) in which all \( a \)-long edges are oriented forward. We claim that if a copy \( C = (c_1, \ldots, c_r) \) of \( C_r \) in \( G(n, p) \) is oriented cyclically in \( \vec{G} \), then \( |c_i - c_j| \leq (r-1)(a-1) \) for every \( i, j \in [r] \). To see this, simply note that every backwards edge has length at most \( a - 1 \), and at most \( r - 1 \) of the edges can be directed backwards.

In particular, if we orient only the \( a \)-long edges of \( G(n, p) \) forward, the number of choices for \( V(C) \) of size \( r \) such that \( C \) could form a \( C^3 \) -free orientation of the \( a \)-short edges is at most \( n(ra)^{r-1} \). Therefore, the expected number of such copies of \( C_r \) in \( G(n, p) \) is \( O(p^r a^{r-1}) \ll p n \), so the claimed bound follows as in the proof of Proposition 2.1.

The next proposition implies the lower bounds in Conjecture 1.6 and Theorem 1.8.

Proposition 2.5. Let \( r \geq 3 \) and \( \omega \gg 1 \). If \( p \geq n^{-2/(r+1)} \), then

\[
\log S_r(n, p) \geq \frac{n}{\omega \cdot p^{r-1/2}}
\]

with high probability as \( n \to \infty \).

Proof. By Proposition 2.2, we may again assume that \( p = o(1) \). Set \( a = p^{-1/(r+1)}/\omega = o(n) \). Denoting by \( v \) and \( w \) the leftmost and rightmost vertices of a strongly connected copy of \( K_r \), as in Proposition 2.4, we see that \( |w - v| \leq (r-1)(a-1) \). Therefore, all vertices in a “dangerous” copy of \( K_r \) are within distance \( O(a) \) of each other. The expected number of such copies of \( K_r \) in \( G(n, p) \) is \( O(p^{r-1}a^{r-1}n) \ll p n \), so the claimed bound again follows as before.

Finally, the lower bounds of the form \((1 + o(1))p^n\) in Theorems 1.5 and 1.8 follow easily from the observation that, by Markov’s inequality, if \( p \ll n^{-2/(r+1)} \) then with high probability \( G(n, p) \) contains \( o(p n^2) \) copies of \( K_r \). Indeed, if we orient the edges that are contained in a copy of \( K_r \) according to a fixed linear order of the vertex set, then we may orient the remaining edges arbitrarily.

3 UPPER BOUND FOR TRIANGLES

In this section we prove the following theorem, which implies the upper bound in Theorem 1.2, and whose proof contains the key idea introduced in this paper.

Theorem 3.1. Let \( 0 < p \leq 1 \). Then, with high probability as \( n \to \infty \),

\[
D(G(n, p), C^3) \leq \exp \left( \frac{6n(\log n)^2}{p} \right).
\]

In order to prove Theorem 3.1, we start with the simple (but key) observation that knowing the orientation of two edges of a triangle in a \( C^3 \) -free graph is sometimes enough to uniquely determine the orientation of the third edge. Given any \( C^3 \) -free orientation \( \vec{G} \) of \( G(n, p) \), we will use this fact to find a small set of “nonredundant” edges \( S \subseteq E(\vec{G}) \) such that the prescribed orientation \( \vec{G} \) is uniquely determined by \( S \). This is formalized in the deterministic lemma 3.2. We remark that the next sections will use similar deterministic statements (in Claims 4.4 and 5.4, and Lemma 6.4).
Lemma 3.2. Let $\tilde{G}$ be a $C_3^\cap$-free orientation of a graph $G$ on $n$ vertices. There exists a set $S \subseteq E(\tilde{G})$ with

$$|S| \leq 2n \cdot \alpha(G)$$

such that $\tilde{G}$ is the unique $C_3^\cap$-free orientation of $G$ containing $S$.

Proof. We prove the lemma by induction on $n$. For $n = 1$ it is trivial, so let $n \geq 2$ and assume that the statement is true for $n - 1$. Pick a vertex $v \in V(G)$, and let $S'$ be the set given by the lemma applied to $\tilde{G}' = \tilde{G} - v$. Note that $|S'| \leq 2(n - 1)\alpha(G)$, since $\alpha(G') \leq \alpha(G)$, and that $\tilde{G}'$ is the unique $C_3^\cap$-free orientation of $G'$ containing $S'$.

Now, let $T \subseteq E(\tilde{G}) \setminus E(\tilde{G}')$ be minimal such that $\tilde{G}$ is the unique $C_3^\cap$-free orientation of $G$ containing $S' \cup T$. (Note that such a set exists, since $E(\tilde{G}) \setminus E(\tilde{G}')$ has this property.) Set $T^+ := \{(v, w) : (v, w) \in T\}$ be the edges of $T$ oriented away from $v$; we claim that

$$|T^+| \leq \alpha(G).$$

Indeed, suppose that there exists an edge $(u, w) \in E(\tilde{G}')$ such that $(v, u), (v, w) \in T^+$. Recall that only one orientation of the edge $uw$ can appear together with $S'$ in a $C_3^\cap$-free graph, and therefore every $C_3^\cap$-free graph containing $T \setminus \{(v, w)\}$ contains the edge $(v, w)$, as we can deduce the orientation of $vw$ from those of $uv$ and $uw$. Thus, setting $T' := T \setminus \{(v, w)\}$, it follows that $\tilde{G}$ is the unique $C_3^\cap$-free orientation of $G$ containing $S' \cup T'$, contradicting the minimality of $T$. Hence the set $\{w : (v, w) \in T^+\}$ must in fact be independent, and therefore we have $|T^+| \leq \alpha(G)$, as claimed.

To complete the proof, simply note that the same bound holds for $T^- := T \setminus T^+$, by symmetry, and hence, setting $S := S' \cup T$, we have $|S| \leq 2n \cdot \alpha(G)$, as required.

We remark that Lemma 3.2 can be stated as an extremal result about a deterministic process that resembles graph bootstrap percolation (also known as weak saturation), see for example [5, 6].

The following corollary is an immediate consequence of Lemma 3.2.

Corollary 3.3. A graph $G$ on $n$ vertices admits at most

$$\sum_{i=0}^{2n\alpha(G)} \binom{e(G)}{i} 2^i$$

$C_3^\cap$-free orientations.

We can now prove Theorem 3.1.

Proof of Theorem 3.1. Note that $p^2n \ll (\log n)^2$ then the trivial bound $D(G, C_3^\cap) \leq 2^{\alpha(G)}$ is sufficient, so we may assume this is not the case. It follows that, with high probability, we have $e(G(n, p)) = (1 + o(1))p(n)^2$ and $\alpha(G(n, p)) \leq \frac{3\log n}{p}$ (by the first moment method, since the expected number of independent sets of this size is $o(1)$). Therefore, by Corollary 3.3,

$$D(G(n, p), C_3^\cap) \leq \sum_{i=0}^{2n\cdot \alpha(G(n, p))} \binom{e(G(n, p))}{i} 2^i$$

$$\leq \frac{6n \log n}{p} \cdot \left( \frac{pn^2}{6n \log n} \right) \cdot n^{6n/p} \leq \exp \left( \frac{6n(\log n)^2}{p} \right)$$

with high probability, as required.
4 | AVOIDING NONTRANSITIVE TOURNAMENTS

Recall that \( T_r(n, p) \) denotes the number of orientations of \( G(n, p) \) in which every copy of \( K_r \) is transitive. In this section we prove the following two theorems, which (together with Theorem 3.1) imply the upper bounds in Theorem 1.5.

**Theorem 4.1.** Let \( r \geq 4 \). If \( p > n^{-2/(r+2)} \), then

\[
T_r(n, p) \leq \exp \left( \frac{O(n(\log n)^2)}{p} \right)
\]

with high probability as \( n \to \infty \).

**Theorem 4.2.** Let \( r \geq 4 \). If \( p \leq n^{-2/(r+2)} \), then

\[
T_r(n, p) \leq \exp \left( \frac{O(n^{4-r}(\log n)^2)}{p^{(r)}-2} \right)
\]

with high probability as \( n \to \infty \).

Before proving these theorems, let us first note that a slightly weaker version of Theorem 4.1 follows easily from Theorem 3.1. Indeed, if \( p \gg n^{-2/(r+1)}(\log n)^{2/(r+2)(r-3)} \), then with high probability every triangle in \( G(n, p) \) is contained in a copy of \( K_r \) (see [14]), and hence every orientation of \( G(n, p) \) in which every \( K_r \) is transitive is also \( C^r_3 \)-free.

In order to remove this polylogarithmic factor, and to prove Theorem 4.2, we will use the following slightly technical lemma, which follows easily from the Janson inequalities (see, e.g., [3, 12]). For completeness, we provide a proof in the appendix of [7].

**Lemma 4.3.** For each \( r \geq 4 \), there exists \( C > 0 \) such that if

\[
p \gg n^{-2/(r+1)}(\log n)^{4/(r+1)(r-2)},
\]

then the following holds with high probability as \( n \to \infty \). Set

\[
t_r(n, p) := \begin{cases} 
    C p^{2-(r)} n^{3-r} \log n & \text{if } p \leq n^{-2/(r+2)} \\
    C \log n / p & \text{if } p > n^{-2/(r+2)}.
\end{cases}
\]

(6)

For every \( v \in V(G(n, p)) \) and every \( T \subset N(v) \) of size at least \( t_r(n, p) \), there exists a copy of \( K_r \) in \( G(n, p) \) containing \( v \) and at least two vertices of \( T \).

To prove Theorems 4.1 and 4.2, we now simply repeat the proof of Theorem 3.1, replacing \( a(G) \) by \( t_r(n, p) \).

**Proof of Theorems 4.1 and 4.2.** We begin with a deterministic claim corresponding to Lemma 3.2. Given \( n \in \mathbb{N} \) and \( p \in (0, 1) \), let \( G \) be a graph on \( n \) vertices that satisfies the conclusion of Lemma 4.3,
that is, for every \( v \in V(G) \) and every \( T \subset N(v) \) of size at least \( t_r(n,p) \), there exists a copy of \( K_r \) in \( G \) containing \( v \) and at least two vertices of \( T \).

**Claim 4.4.** Let \( \tilde{G} \) be an orientation of \( G \) in which every \( K_r \) is transitive. There exists a set \( S \subset E(\tilde{G}) \) with

\[
|S| \leq 2n \cdot t_r(n,p)
\]

such that \( \tilde{G} \) is the unique orientation of \( G \) containing \( S \) in which every \( K_r \) is transitive.

**Proof of Claim 4.4.** Observe first that, since every copy of \( K_r \) in \( G \) is transitive in \( \tilde{G} \), every triangle in \( G \) that is contained in a copy of \( K_r \) is also transitive. Fix an ordering \( v_1, \ldots, v_n \) of \( V(G) \), let \( k \in [n] \), and suppose that we have already found a set \( S_k \subset E(\tilde{G}[\{v_1, \ldots, v_k\}]) \) such that

\[
|S_k| \leq 2k \cdot t_r(n,p),
\]

and \( \tilde{G}[\{v_1, \ldots, v_k\}] \) is the unique orientation of \( G[\{v_1, \ldots, v_k\}] \) containing \( S_k \) in which every triangle that is contained in a copy of \( K_r \) in \( G \) is transitive.

Now, let \( S_{k+1} \subset E(\tilde{G}[\{v_1, \ldots, v_{k+1}\}]) \) be minimal such that \( S_{k+1} \supset S_k \), and such that \( \tilde{G}[\{v_1, \ldots, v_{k+1}\}] \) is the unique orientation of \( G[\{v_1, \ldots, v_{k+1}\}] \) containing \( S_{k+1} \) in which every triangle that is contained in a copy of \( K_r \) in \( G \) is transitive. Setting \( T^+ := \{w : (v_{k+1}, w) \in S_{k+1}\} \), we claim that

\[
|T^+| \leq t_r(n,p).
\]  

Indeed, if \( |T^+| > t_r(n,p) \) then there exists a copy of \( K_r \) in \( G \) containing \( v_{k+1} \) and at least two vertices \( u, w \in T^+ \), and hence the triangle \( uwv_{k+1} \) must be transitive in \( \tilde{G} \). But this means that (as in the proof of Lemma 3.2) we can deduce the orientation of either \( uv_{k+1} \) or \( wv_{k+1} \) from that of the other, together with that of \( uw \), and hence \( S_{k+1} \) is not minimal. This contradiction proves (7). Similarly, defining \( T^- := \{w : (w, v_{k+1}) \in S_{k+1}\} \), an analogous argument shows that \( |T^-| \leq t_r(n,p) \). Proceeding inductively, we obtain a set \( S = S_n \) as claimed. \( \blacksquare \)

We now prove Theorem 4.1. Suppose \( p > n^{-2/(r+2)} \). It follows from Claim 4.4 (and Lemma 4.3) that, with high probability,

\[
T_r(n,p) \leq \sum_{i=0}^{2n \cdot t_r(n,p)} \binom{pn^2}{i} 2^i \leq \binom{pn^2}{2n^2 \log n^2} \leq \exp \left( \frac{2Cn(\log n)^2}{p} \right),
\]

as required.

To prove Theorem 4.2, we suppose from now on that \( p \leq n^{-2/(r+2)} \). Note that if \( p^{(r-1)n^{-r-2}} \leq (\log n)^2 \) then the trivial bound \( 2^{t_r(G)} \) is sufficient, so we may assume this is not the case. It follows that, with high probability, we have \( e(G(n,p)) = (1 + o(1)) \binom{n}{2} \) and \( G(n,p) \) satisfies the conclusion of Lemma 4.3. Hence, by Claim 4.4,

\[
T_r(n,p) \leq \sum_{i=0}^{2n \cdot t_r(n,p)} \binom{pn^2}{i} 2^i \leq \exp \left( O(t_r(n,p) \cdot n \log n) \right)
\]

with high probability, as required. \( \blacksquare \)
5 | AVOIDING STRONGLY CONNECTED TOURNAMENTS

Recall that $S_r(n, p)$ denotes the number of orientations of $G(n, p)$ in which no copy of $K_r$ is strongly connected. In this section we prove the following theorems, which (together with Theorem 3.1) imply the upper bound in Theorem 1.8.

**Theorem 5.1.** Let $r \geq 4$. If $n^{-2/(r+1)} \ll p \leq (\log n)^{-2/(r-2)}$, then

$$S_r(n, p) \leq \exp\left(\frac{O(n(\log n)^{1+1/(r-2)})}{p^{r-1/2}}\right)$$

with high probability as $n \to \infty$.

**Theorem 5.2.** If $p > (\log n)^{-2/(r-2)}$, then

$$S_r(n, p) \leq \exp\left(\frac{O(n(\log n)^{2})}{p}\right)$$

with high probability as $n \to \infty$.

The proofs of Theorems 5.1 and 5.2 are similar to the proofs of Theorems 4.1 and 4.2. Instead of Lemma 4.3, we will use the following straightforward fact, which also follows easily from the Janson inequalities (see the appendix of [7]).

**Lemma 5.3.** For every $r \geq 3$, there exists $C > 0$ such that the following holds with high probability as $n \to \infty$. Set

$$s_r(n, p) = \begin{cases} \frac{C(\log n)^{r/(r-1)}}{p^{r/2}} & \text{if } p \leq (\log n)^{-2/(r-1)} \\ \frac{C \log n}{p} & \text{if } p > (\log n)^{-2/(r-1)} \end{cases}$$

Then every set $S \subset V(G(n, p))$ with $|S| \geq s_r(n, p)$ contains a copy of $K_r$.

We can now easily deduce Theorem 5.1, using the method of the previous two sections.

**Proof of Theorems 5.1 and 5.2.** Once again, we begin with a deterministic claim (cf. Lemma 3.2 and Claim 4.4). Given $n \in \mathbb{N}$ and $p \in (0, 1)$, let $G$ be a graph on $n$ vertices that satisfies the conclusion of Lemma 5.3 for $r-1$, that is, every set of vertices of $G$ of size at least $s_{r-1}(n, p)$ contains a copy of $K_{r-1}$.

**Claim 5.4.** Let $\vec{G}$ be an orientation of $G$ in which no copy of $K_r$ is strongly connected. There exists a set $S \subset E(\vec{G})$ with

$$|S| \leq 2n \cdot s_{r-1}(n, p)$$

such that $\vec{G}$ is the unique orientation of $G$ containing $S$ with no strongly connected $K_r$.

**Proof of Claim 5.4.** We will use the simple observation that every orientation of $K_{r-1}$ contains a Hamiltonian path. Now, we can simply choose the set $S$ greedily, vertex by vertex, as before. To be...
In this section, we show an upper bound for the number of orientations avoiding oriented cycles of length $r$.

To prove Lemma 6.4, we start with the following simple observation.

Indeed, if $T^+$ were larger than this, then there would exist a copy of $K_{r-1}$ in $G[T^+]$, and this copy of $K_{r-1}$ contains a Hamiltonian path in $\tilde{G}$, from $a$ to $b$, say. Moreover, the orientations of the edges in this path are determined by $S_k$, and we can therefore deduce the orientation of the edge $bv_{k+1}$ from $S_{k+1} \setminus \{(v_{k+1}, b)\}$. This contradicts the minimality of $S_{k+1}$, and hence proves (10).

Suppose now that $n^{-2/(r+1)} < p \leq (\log n)^{-2/(r-2)}$. It follows from Claim 5.4 (and the first case of Lemma 5.3) that, with high probability,

$$S_r(n, p) \leq \sum_{i=0}^{2n \cdot s_{r-1}(n, p)} \binom{pn^2}{i} 2^i \leq \exp \left( \frac{O(n(\log n)^{1+1/(r-2)})}{p^{(r-1)/2}} \right),$$

and Theorem 5.1 is proved. To prove Theorem 5.2, note that if $p > (\log n)^{-2/(r-2)}$, then a similar calculation using Lemma 5.3 shows that

$$S_r(n, p) \leq \sum_{i=0}^{2n \cdot s_{r-1}(n, p)} \binom{pn^2}{i} 2^i \leq \exp \left( \frac{O(n(\log n)^2)}{p} \right)$$

with high probability, as claimed.

### 6 Avoiding Longer Cycles

In this section, we show an upper bound for the number of orientations avoiding oriented cycles of length $r$ (denoted by $C_r^\circ$). As stated in Conjecture 1.3, we believe substantially better upper bounds are possible for $r \geq 4$.

**Theorem 6.1.** Let $r \geq 3$. Then

$$\log D(G(n, p), C_r^\circ) = \tilde{O} \left( \frac{n}{p} \right)$$

with high probability as $n \to \infty$.

The proof of Theorem 6.1 follows directly from Lemma 6.4, which is a generalization of Lemma 3.2 to longer cycles. To prove Lemma 6.4, we start with the following simple observation.
Lemma 6.2. Let \( \tilde{G} \) be a \( C_r^\circ \)-free graph, \( v \) a vertex of \( \tilde{G} \) and \( P = (w_1, \ldots, w_k) \) a directed path in \( \tilde{G}[N(v)] \) such that \( w_i \in N^+(v) \) for \( 1 \leq i \leq r-2 \). Then \( P \subseteq N^+(v) \).

Proof. If the conclusion were not true, there would exist a minimal \( i \) such that \( w_i \in N^-(v) \). By hypothesis, we would have \( i > r-2 \). But then \( (v, w_{i-(r-2)}, \ldots, w_i, v) \) would be a directed cycle of length \( r \), a contradiction.

By reversing the orientation of all edges of \( G \), we can deduce from Lemma 6.2 that if the last \( r-2 \) vertices of a directed path contained in \( N(v) \) are in \( N^-(v) \), then the whole path is in \( N^-(v) \).

The main additional ingredient in the proof of Lemma 6.4 is the Gallai-Milgram theorem [9]; the proof of Theorem 6.1 was inspired by a similar application in [1].

Theorem 6.3 (Gallai-Milgram [9]). The vertex set of every directed graph \( \tilde{G} \) can be partitioned into at most \( \alpha(G) \) vertex-disjoint directed paths.

We are now ready to prove the main lemma of this section.

Lemma 6.4. Let \( \tilde{G} \) be a \( C_r^\circ \)-free orientation of a graph \( G \) on \( n \) vertices. There exists a set \( S \subseteq E(\tilde{G}) \) with

\[
|S| \leq 2n \cdot (r-2) \alpha(G)
\]

such that \( \tilde{G} \) is the unique \( C_r^\circ \)-free orientation of \( G \) containing \( S \).

Proof. We proceed by induction on \( n \). Let \( v \in V(G) \) be any vertex of \( G \), and let \( G' = G \setminus v \). By induction, there exists a set \( S' \subseteq E(\tilde{G}') \) of size \( 2(n-1) \cdot (r-2) \alpha(G) \) such that \( \tilde{G}' = \tilde{G} \setminus v \) is the unique \( C_r^\circ \)-free orientation of \( G' \) containing \( S' \). Our aim is to find a set of edges \( T \) of size \( 2(r-2) \alpha(G) \) such that \( \tilde{G} \) is the unique \( C_r^\circ \)-free orientation of \( G \) containing \( S' \cup T \).

We start by applying Theorem 6.3 to partition the graph \( \tilde{G}[N^+(v)] \) into a collection \( P^+ \) of at most \( \alpha(G) \) oriented paths, and define \( T^+ \) to be the set of edges given by

\[
T^+ = \{(v, w) : w \text{ is one of the first } r-2 \text{ vertices in some } P \in P^+\}.
\]

We define \( T^- \) similarly by decomposing \( \tilde{G}[N^-(v)] \) into at most \( \alpha(G) \) oriented paths and taking the last \( r-2 \) vertices of each path. We claim that \( T = T^+ \cup T^- \) has the desired property.

To check the claim, we must show that any \( C_r^\circ \)-free orientation \( \tilde{H} \) of \( G \) containing \( S' \cup T \) equals \( \tilde{G} \). By the induction hypothesis, \( \tilde{G}[V(G')] = \tilde{H}[V(G')] \), so it suffices to show that \( N^+_G(v) \subseteq N^+_H(v) \) and \( N^-_G(v) \subseteq N^-_H(v) \). Since \( \tilde{H} \) contains \( T^+ \subseteq T \), we know \( N^+_H(v) \) contains the first \( r-2 \) vertices in each path of \( P^+ \). By Lemma 6.2, \( N^+_H(v) \) contains all paths in \( P^+ \), and since \( P^+ \) was a partition of \( N^+_G(v) \), we conclude that \( N^+_G(v) \subseteq N^+_H(v) \). A similar argument works for \( T^- \), and therefore we have checked the claim. Taking \( S = S' \cup T \) finishes the proof.

Proof of Theorem 6.1. This proof closely mirrors that of Theorem 3.1, with Lemma 6.4 replacing Lemma 3.2. We obtain

\[
D(G(n, p), C_r^\circ) \leq \sum_{i=1}^{2n(r-2) \cdot \alpha(G(n, p))} \binom{e(G(n, p))}{i} 2^i \leq \exp \left( \frac{6n(r-2)(\log n)^2}{p} \right)
\]

with high probability, as required.
7 | OPEN PROBLEMS

In this section we will mention some further open problems and possible directions for future research; in particular, we will discuss the problem of removing the polylogarithmic factors that separate the upper and lower bounds in Theorems 1.2, 1.5, and 1.8, and the problem of determining the behavior of $D(G(n,p), \vec{H})$, the number of $\vec{H}$-free orientations of $G(n,p)$, for an arbitrary oriented graph $\vec{H}$.

First, we remark that if $\vec{H}$ is contained in a transitive tournament then the situation is different; more precisely, the following theorem is an easy consequence of a well-known theorem of Rödl and Ruciński [13].

**Theorem 7.1.** Let $r \geq 3$ and $p \gg n^{-2/(r+1)}$. With high probability, every orientation of $G(n,p)$ contains a transitive copy of $K_r$.

To deduce Theorem 7.1 from the main theorem of [13], simply fix a linear order of the vertices, and define an edge-coloring from an orientation by coloring forward-pointing edges blue and backwards edges red. In this setting, a monochromatic copy of $K_r$ corresponds to a transitively oriented $K_r$ (but not vice-versa). If $p \gg n^{-2/(r+1)}$, then [13, Theorem 1] ensures (with high probability) the existence of a monochromatic copy of $K_r$ in any 2-coloring of $G(n,p)$, and therefore for this range of $p$ it is impossible to avoid a transitively oriented copy of $K_r$.

We remark that Theorem 7.1 does not give the correct threshold for the event “every orientation of $G(n,p)$ contains a transitive triangle,” since every orientation of $K_4$ contains a transitive triangle, and the event $\{K_4 \subseteq G(n,p)\}$ has a threshold at $\Theta(n^{-2/3})$. Nevertheless, we suspect that $n^{-2/(r+1)}$ is the correct threshold for the event “every orientation of $G(n,p)$ contains a transitive copy of $K_r$” for every $r \geq 4$.

Therefore, we will assume throughout this section that $\vec{H}$ contains a cycle. For this case, we state (somewhat imprecisely) the central question that is suggested by the work in this paper.

**Question 7.2.** Is the lower bound construction described in Section 2 always sharp?

The results proved in this paper provide some evidence in favor of a positive answer to this question (at least in a weak sense). It is moreover plausible that it is true in a much stronger sense: that (with high probability) almost all $\vec{H}$-free orientations of $G(n,p)$ are “close” to one of the orientations given by the construction described in Section 2.

**Problem 7.3.** Determine the typical structure of a $\vec{H}$-free orientation of $G(n,p)$.

For example, in the case $\vec{H} = C_3^O$ one might hope to prove that if $p \gg n^{-1/2}$, then the following holds with high probability: for almost all $C_3^O$-free orientations of $G(n,p)$, there exists an ordering of the vertices such that $\Theta(n/p)$ edges are oriented backwards, and all but $o(n/p)$ of those edges have length $O(1/p^2)$.

Of course, one can ask the same questions for a family $\vec{H}$ of forbidden oriented graphs, such as the family of nontransitive tournaments (or the family of strongly connected tournaments) of a given size.

### 7.1 | Removing the polylogarithmic terms

An important (and probably very challenging) step in the direction of Problem 7.3 would be to remove the polylogarithmic factor between our upper and lower bounds on $\log T_r(n,p)$ and $\log S_r(n,p)$.

**Problem 7.4.** Determine the typical values of $\log T_r(n,p)$ and $\log S_r(n,p)$ up to a constant factor for each $r \geq 3$ and every function $p \gg n^{-2/(r+1)}$. 
Note that some polylogarithmic factor is necessary, at least when $p$ is large, since

$$T_r(n, 1) = S_r(n, 1) = n!$$

for every $r \geq 3$. In the case $r = 3$, we conjecture that a combination of the lower bounds given by Propositions 2.1 and 2.2 is sharp up to the implicit constant factor in the exponent.

**Conjecture 7.5.** $D(G(n, p), C^G_3) \leq 2^{O(n/p)} \cdot n!$ for every $p \geq n^{-1/2}$.

### 7.2 General forbidden structures

In this section we will discuss the general lower bound on $D(G(n, p), \vec{H})$ given by the construction described in Section 2, where $\vec{H}$ is an arbitrary oriented graph that contains a cycle. Let $m_2(\vec{H})$ denote the 2-density of the underlying graph of $\vec{H}$,

$$m_2(\vec{H}) = \max \left\{ \frac{e(\vec{F}) - 1}{\nu(\vec{F}) - 2} : \vec{F} \subset \vec{H}, \nu(\vec{F}) \geq 3 \right\}. $$

Note that any oriented graph can be decomposed into strongly connected components in a unique way, and let $s(\vec{H})$ denote the number of strongly connected components of $\vec{H}$.

**Proposition 7.6.** Suppose that $\vec{H}$ contains a cycle. If $p \gg n^{-1/m_2(\vec{H})}$ and $\omega \gg 1$, then

$$\log D(G(n, p), \vec{H}) \geq \frac{pn}{\omega} \cdot \max \left\{ \left( p^{e(\vec{F})-1}n^{s(\vec{F})-1} \right)^{\frac{1}{\nu(\vec{F})-s(\vec{F})-1}} : \vec{F} \subset \vec{H}, \nu(\vec{F}) \geq s(\vec{F}) + 2 \right\}$$

with high probability as $n \to \infty$.

**Proof.** When $p = \Omega(1)$, the claimed lower bound is of order $o(n)$, and therefore follows from Proposition 2.2. When $p = o(1)$, we first need to observe that, since $\vec{H}$ contains a cycle, there is a strongly connected component $\vec{F} \subset \vec{H}$ with $\nu(\vec{F}) \geq 3$, and this implies that there exists $\vec{F} \subset \vec{H}$ with $\nu(\vec{F}) \geq s(\vec{F}) + 2$ and $p^{e(\vec{F})-1}n^{s(\vec{F})-1} = o(1)$. Given any such $\vec{F}$, consider the construction of Section 2 with

$$a = \frac{1}{\omega} \cdot \left( p^{e(\vec{F})-1}n^{s(\vec{F})-1} \right)^{\frac{1}{\nu(\vec{F})-s(\vec{F})-1}},$$

that is, orient all edges of length at least $a$ from left to right. Note that $a \gg 1$, by our choice of $\vec{F}$, and that $p \gg n^{-1/m_2(\vec{H})}$ implies that $a = o(n)$. Now, observe that the expected number of potential copies of $\vec{F}$ in $G(n, p)$ is

$$O(n^{s(\vec{F})}a^{e(\vec{F})-s(\vec{F})}p^{s(\vec{F})}) \ll \text{pan},$$

since any two vertices in the same strongly connected component of a copy of $\vec{F}$ must lie within distance $O(a)$ of one another, and since $\nu(\vec{F}) \geq s(\vec{F}) + 2$. By Markov’s inequality, it follows that with high probability there exists a set of at least $\text{pan}/2$ edges that can be oriented freely without creating a copy of $\vec{F}$, and hence of $\vec{H}$. Since this holds for each $\vec{F} \subset \vec{H}$ with $\nu(\vec{F}) \geq s(\vec{F}) + 2$ and $p^{e(\vec{F})-1}n^{s(\vec{F})-1} = o(1)$, the claimed bound follows.

We can now rephrase Question 7.2 more precisely in this setting.
Question 7.7. Suppose that $\vec{H}$ contains a cycle, and let $p \gg n^{-1/m_2(\vec{H})}$. Is it true that

$$\log D(G(n,p), \vec{H}) = \tilde{\Theta}\left( \max_{\vec{F} \in H, v(\vec{F}) \geq 2} \left\{ pn \cdot \left( p^{v(\vec{F})-1} n^{v(\vec{F})-1} \right)^{1/v(\vec{F})-1} \right\} \right)$$

with high probability as $n \to \infty$?

Observe that the oriented subgraph $\vec{F} \subset \vec{H}$ which corresponds to the maximum in Proposition 7.6 depends on $p$, and in general it can change arbitrarily many times as $p$ increases. A positive answer to Question 7.7 would therefore imply the existence of $\vec{H}$ for which $D(G(n,p), \vec{H})$ exhibits arbitrarily many thresholds between $n^{-1/m_2(\vec{H})}$ and 1.

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