The Yablonskii - Vorob’ev polynomials for the second Painlevé hierarchy

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Abstract

Special polynomials associated with rational solutions of the second Painlevé equation and other equations of its hierarchy are studied. A new method, which allows one to construct each family of polynomials is presented. The structure of the polynomials is established. Formulae for their coefficients are found. The degree of every polynomial is obtained. The main achievement of the method lies in the fact that it enables one to construct the family of polynomials corresponding to any member of the second Painlevé hierarchy. Our approach can be applied for deriving the polynomials related to rational or algebraic solutions of other nonlinear differential equations.

Keywords: the Yablonskii - Vorob’ev polynomials, the second Painlevé equation, the second Painlevé hierarchy, power geometry

PACS: 02.30.Hq - Ordinary differential equations

1 Introduction

It is well known that the second Painlevé equation (P₂)

\[ w_{zz} = 2w^3 + zw + \alpha \]  

(1.1)
and all members of its hierarchy have rational solutions only at integer values of the parameter $\alpha (\alpha = n \in \mathbb{Z})$. These solutions can be written in terms of special polynomials $Q^{(N)}_n(z)$ ($N \geq 1$)

$$w^{(N)}(z; n) = \frac{d}{dz} \left\{ \ln \left[ \frac{Q^{(N-1)}_n(z)}{Q^{(N)}_n(z)} \right] \right\}, \quad n \geq 1, \quad w^{(N)}(z; -n) = -w^{(N)}(z; n),$$

(1.2)

where $w^{(N)}(z; n)$ is a solution of the $N$th equation in the hierarchy ($N=1$ is the case of $P_2$). The polynomials $Q^{(1)}_n(z)$ were suggested by A. I. Yablonskii and A. P. Vorob’ev and are called the Yablonskii – Vorob’ev polynomials [1, 2]. While their analogues were introduced by P. A. Clarkson and E. L. Mansfield [3]. All these polynomials can be regarded as nonlinear analogues of classical special polynomials [4–12].

The polynomials $Q^{(1)}_n(z)$ satisfy the differential – difference equation

$$Q^{(1)}_{n+1}Q^{(1)}_{n-1} = z(Q^{(1)}_n)^2 - 4(Q^{(1)}_n)^{\prime\prime} - (Q^{(1)}_n)^{\prime'} - (Q^{(1)}_n)^{\prime''},$$

(1.3)

where $Q^{(1)}_0(z) = 1$, $Q^{(1)}_1(z) = z$. It is not clear from the first sight that this relation defines exactly polynomials however it is so. Moreover $Q^{(1)}_n(z)$ are monic polynomials with integer coefficients. They possess a certain number of interesting properties. In particular, for every integer positive $n$ each polynomial $Q^{(1)}_n(z)$ has simple roots only and besides that, two successive polynomials $Q^{(1)}_n(z)$ and $Q^{(1)}_{n+1}(z)$ do not have a common root. The Yablonskii – Vorob’ev polynomials $Q^{(1)}_n(z)$ arise in various physical models. For example, partial solutions of the Korteweg – de Vries equation, the modified Korteweg – de Vries equation, the nonlinear Schrödinger equation, the Kadowtsev – Patviashvili equation can be expressed via the polynomials $Q^{(1)}_n(z)$ [4].

The polynomials $Q^{(2)}_n(z)$ associated with the second equation of the $P_2$ hierarchy

$$w_{zzzz} - 10 w^2 w_{zz} - 10 w w_z^2 + 6 w^5 - z w - \alpha = 0$$

(1.4)

in their turn satisfy the differential – difference equation

$$Q^{(2)}_{n+1}Q^{(2)}_{n-1} = z(Q^{(2)}_n)^2 - 4 Q^{(2)}_n(Q^{(2)}_n)^{\prime\prime\prime} - 4(Q^{(2)}_n)^{\prime\prime\prime} - (Q^{(2)}_n)^{\prime\prime} (Q^{(2)}_n)^{\prime} + 3(Q^{(2)}_n)^{\prime\prime}(Q^{(2)}_n)^{\prime\prime}$$

(1.5)

where again $Q^{(2)}_0(z) = 1$, $Q^{(2)}_1(z) = z$. As far as the polynomials $Q^{(N)}_n(z)$ ($N \geq 3$) are concerned, the differential – difference equation analogues to
yet is not obtained. In this connection an important problem is to define the polynomials in another way, i.e. without using the differential – difference equation.

In this paper we present a new method, which solves this problem. In particular, the method allows one to find the degree of each polynomial $Q_n^{(N)}(z)$, to determine its structure and to derive formulas for its coefficients. Our approach can be also applied for constructing other polynomials related to rational or algebraic solutions of nonlinear differential equations. We would like to mention that not long ago it was made an attempt to obtain formulas for coefficients of the Yablonskii – Vorob’ev polynomials $Q_n^{(1)}(z)$ [13]. As a result the coefficient of the lowest degree term was found.

The outline of this paper is as follows. In section 2 the algorithm of our method is presented and main theorems are proved. Correlations between the roots of the Yablonskii – Vorob’ev polynomials and properties of their coefficients are discussed in sections 3 and 4, accordingly. In other words sections 2 - 4 are devoted to the general case. While several examples are given in sections 5 - 7. More exactly the polynomials associated with the first, the second, and the third members of the hierarchy are studied in sections 5, 6, 7, respectively.

2 Method applied

The $P_2$ hierarchy can be obtained through the scaling reduction from the hierarchy of the modified Korteveg – de Vries equation [14–18], and is the following

$$P_2^{(N)}[w, \alpha] = \left( \frac{d}{dz} + 2w \right) L_N[w' - w^2] - zw - \alpha = 0, \quad N \geq 1,$$

(2.1)

where $L_N[w' - w^2] \equiv L_N[u]$ satisfies the Lenard recursion relation

$$d_z L_{N+1}[u] = (d_z^3 + 4ud_z + 2u_z) L_N[u], \quad L_0[u] = \frac{1}{2}.$$  

(2.2)

Every equation of the hierarchy (2.1) has a unique rational solution if and only if $\alpha$ is an integer. All these solutions are expressible via the logarithmic derivative of the polynomials $Q_n^{(N)}(z)$ $(N \geq 1)$, which will be the objects of our study. Analyzing the expression (1.2) we understand that the polynomials $Q_n^{(N)}(z)$ can be defined as monic polynomials. By $p_{n,N}$ denote the degree
of $Q^{(N)}_n(z)$. Then we can present each polynomial in the form

$$Q^{(N)}_n(z) = \sum_{k=0}^{p_{n,N}} A^{(N)}_{n,k} z^{p_{n,N} - k}, \quad A^{(N)}_{n,0} = 1 \quad (2.3)$$

Let us show that it is possible to derive the polynomials $Q^{(N)}_n(z)$ without leaning on the differential – difference recurrence formula or determinantal representation of the rational solutions. For this aim we will use power expansions at infinity for solutions of the equations (2.1).

**Theorem 2.1.** Every equation $P^{(N)}_2[w, \alpha_N]$ has a family of solutions with the expansion at infinity

$$w^{(N)}(z; \alpha) = -\frac{\alpha}{z} + \sum_{l=1}^{\infty} c_{\alpha,-(2N+1)l-1} z^{-(2N+1)l-1}, \quad z \to \infty, \quad N \geq 1. \quad (2.4)$$

**Proof.** While proving this theorem we will use the algorithms of power geometry. For more information see [19–24]. First of all it is important to mention that equations (2.1) and the operators $L_N[w' - w^2]$ can be thought of as differential sums with $z$ being an independent variable and $w$ being a dependent one. Let us show that the support $S(L_N)$ of $L_N[w' - w^2]$ satisfies the correlation

$$\{(−2N + 1, 1); (0, 2N)\} \subset S(L_N) \subseteq \{(−2N + m, m); 1 \leq m \leq 2N\}. \quad (2.5)$$

This fact can be proved by induction. For $N = 1$ it is obvious. Suppose it is true for $N = M$. Using the recurrent expression (2.2) we get

$$\{(−2(M + 1) + 1, 1)\} \subset S(L_{M+1}) \subseteq \{(−2M + m, m) + (−2, 0); 1 \leq m \leq 2M\} \cup \{(−2M + m, m) + (−1, 1); 1 \leq m \leq 2M\} \cup \{(−2M + m, m) + (0, 2); 1 \leq m \leq 2M\}.$$  

Thus we immediately obtain

$$\{(−2(M + 1) + 1, 1)\} \subset S(L_{M+1}) \subseteq \{(−2(M + 1) + m, m); 1 \leq m \leq 2M\} \cup \{(−2(M + 1) + m, m); 2 \leq m \leq 2M + 1\} \cup \{(−2(M + 1) + m, m); 3 \leq m \leq 2(M + 1)\}. \quad (2.6)$$

Deriving this expression we did not pay attention to coefficients at the monomials of $L_{M+1}[w' - w^2]$. The monomial $w^{2M}$ of $L_M[w' - w^2]$ becoming the
Figure 1: a: Polygon corresponding to the Nth equation in the \( P_2 \) hierarchy; b: The normal cones for the vertexes \( \Gamma_j^{(0)} = M_j \) and for the edges \( \Gamma_j^{(1)} \) (j=1,2,3,4).

monomial \( w^{2(M+1)} \) of \( L_{M+1}[w' - w^2] \) may disappear. However, this does not happen as the coefficient at \( w^{2M} \) in \( L_{M}[w' - w^2] \) is equal to \((-1)^M 2^{M-1}(2M-1)!/M! \neq 0, M \geq 1. \) Hence we get

\[
\{(−2(M + 1) + 1, 1); (0, 2(M + 1))\} \subset S(L_{M+1}) \subseteq \{(−2(M + 1) + m, m); 1 \leq m \leq 2(M + 1)\}. \tag{2.7}
\]

Using the expression (2.1) it can be easily proved that

\[
\{(−2N, 1); (0, 2N + 1); (1, 1); (0, 0)\} \subset S(P_2^{(N)}) \subseteq \{(−2N + m, m + 1); (1, 1); (0, 0); 0 \leq m \leq 2N\}, \tag{2.8}
\]

where \( S(P_2^{(N)}) \) stands for the support of the Nth equation in the hierarchy. From the previous expression we see that the polygon corresponding to the equation \( P_2^{(N)} \) at \( \alpha \neq 0 \) is a trapezium with the vertexes \( M_1 = (−2N, 1), M_2 = (0, 2N + 1), M_3 = (1, 1), M_4 = (0, 0) \) and with the edges \( \Gamma_1^{(1)} = [M_1, M_2], \Gamma_2^{(1)} = [M_2, M_3], \Gamma_3^{(1)} = [M_3, M_4], \Gamma_4^{(1)} = [M_1, M_4] \) (see Fig. 1). The support of \( P_2^{(N)} \) lies in the lattice \( \mathbb{Z} \) with the basis \( B_1 = (−2N, 1), B_2 = (1, 1) \)
(again $\alpha \neq 0$). Let us find the power expansion corresponding to the edge $\Gamma_3^{(1)}$. This edge is characterized by the reduced equation

\[ \hat{f}_3^{(1)} \overset{def}{=} -zw - \alpha = 0, \quad (f(z, w) \equiv P_2^{(N)}(z, w)) \quad (2.9) \]

and the normal cone $U_3^{(1)} = \{ \lambda(1, -1), \lambda > 0 \}$. Hence the power asymptotics related to the edge $\Gamma_3^{(1)}$ is the following

\[ w^{(N)}(z; \alpha) \sim -\frac{\alpha}{z}, \quad z \to \infty, \quad N \geq 1 \quad (2.10) \]

The reduced equation (2.9) is algebraic, therefore its solution does not have critical numbers. A shifted support of (2.10) is the vector $B = (-1, -1) = -B_2$. Consequently it belongs to the lattice $\mathbb{Z}$ generated by the vectors $B_1, B_2$. This lattice consists of the points $M = \{(q_1, q_2), q_1 = -2N + m, q_2 = l + m\}$, where $m$ and $l$ are whole numbers. The lattice $\mathbb{Z}$ intersects with the line $q_2 = -1$ by the points $q_1 = -(2N + 1)m - 1$. The cone of the problem is $K = \{ k < -1 \}$. Thus the exponents of the power expansion with the asymptotic behavior (2.8) belong to the set $K = \{ k = -(2N + 1)m - 1, m \in \mathbb{N} \}$. This completes the proof. \[ \square \]

The expansion (2.4) at $N = 1$ and $N = 2$ was found in [19] and [24], accordingly. All the coefficients $c_{\alpha, -(2N+1)m-1}$ ($l \geq 1$) in (2.4) can be sequently calculated. For convenience of use let us present the series (2.4) in the form

\[ w^{(N)}(z; \alpha) = \sum_{m=0}^{\infty} c_{\alpha, -(m-1)} z^{-m-1}, \quad \text{(2.11)} \]

where $c_{\alpha, -(m-1)} = 0$ unless $m$ is divisible by $(2N + 1)$.

The first two solutions of the equation $P_2^{(N)}$ are the following $w_0^{(N)} = 0, w_1^{(N)} = -1/z$. This fact can be verified by direct substitution into (2.4). Therefore it can be set $Q_0^{(N)} = 1, Q_1^{(N)} = z$. Suppose $a_{n,k}^{(N)} (1 \leq k \leq p_{n,N})$ are the roots of the polynomial $Q_n^{(N)}(z)$, then by $s_{n,k}^{(N)}$ we denote the symmetric functions of the roots

\[ s_{n,m}^{(N)} \overset{def}{=} \sum_{k=1}^{p_{n,N}} (a_{n,k}^{(N)})^m, \quad m \geq 1. \quad (2.12) \]

Later in this section when it does not cause any contradiction the index $N$ will be omitted. Our next step is to express $s_{n,m}$ through coefficients of the series (2.11).
Theorem 2.2. Let $c_i, -m-1$ be the coefficient in expansion (2.11) at integer $\alpha = i \in \mathbb{N}$. Then for each $n \geq 2$ the following relations hold

\[ s_{n,m} = -\sum_{i=2}^{n} c_{i, -(m+1)}, \quad m \geq 1, \quad (2.13) \]

\[ p_n = -\sum_{i=1}^{n} c_{i, -1}. \quad (2.14) \]

Proof. As far as $Q_n(z)$ is a monic polynomial, then it can be written in the form

\[ Q_n(z) = \prod_{k=1}^{p_n} (z - a_{n,k}). \quad (2.15) \]

Note that possibly $a_{n,k} = a_{n,l}, k \neq l$. This equality implies that

\[ \frac{Q'_n(z)}{Q_n(z)} = \sum_{k=1}^{p_n} \frac{1}{z - a_{n,k}}. \quad (2.16) \]

Substituting (2.16) into the expression (1.2) yields

\[ w(z; n) = \sum_{k=1}^{p_n-1} \frac{1}{z - a_{n-1,k}} - \sum_{k=1}^{p_n} \frac{1}{z - a_{n,k}}. \quad (2.17) \]

Expanding this function in a neighborhood of infinity we get

\[ w(z; n) = \frac{b_{n-1}}{z} - \frac{b_n}{z} + \sum_{m=0}^{\infty} \left[ \sum_{k=1}^{p_n-1} (a_{n-1,k})^m - \sum_{k=1+b_{n-1}}^{p_n} (a_{n,k})^m \right] \times z^{-(m+1)}, \quad |z| > \max \{\tilde{a}_{n-1}, \tilde{a}_n\}, \quad \tilde{a}_n = \max_{1 \leq k \leq p_n} \{|a_{n,k}|\}, \quad b_n = \sum_{k=1}^{p_n} \delta_{0,a_{n,k}}, \quad (2.18) \]

where $\delta_{0,a_{n,k}}$ is the Kronecker delta. The first or the second term in (2.18) are present only if the polynomials $Q_{n-1}(z), Q_n(z)$ have zero roots, accordingly. In our designations the previous expression can be rewritten as

\[ w(z; n) = -\frac{p_n - p_{n-1}}{z} + \sum_{m=1}^{\infty} \left[ s_{n-1,m} - s_{n,m} \right] z^{-(m+1)}, \quad |z| > \max \{\tilde{a}_{n-1}, \tilde{a}_n\}. \quad (2.19) \]
The absence of a zero term in sum is essential only at \( m = 0 \). Comparing expansions (2.19) and (2.11) we obtain the equalities

\[
\begin{align*}
  p_n - p_{n-1} &= -c_{n-1}, \\
  s_{n,m} - s_{n-1,m} &= -c_{n,-(m+1)}, \quad m \geq 1.
\end{align*}
\] (2.20)

Decreasing the first index by one in (2.20) and adding the result to the original one yields

\[
\begin{align*}
  p_n - p_{n-2} &= -(c_{n-1} + c_{n-1,-1}), \\
  s_{n,m} - s_{n-2,m} &= -(c_{n,-(m+1)} + c_{n-1,-(m+1)}).
\end{align*}
\] (2.21)

Note that \( c_{1,-(m+1)} = 0, \quad m \geq 1 \) and \( a_{1,1} = 0 \). Then proceeding in such a way we get the required relations (2.13) and (2.14).

**Remark 1.** It was proved that \( P_2^{(N)} \) has a unique rational solution whenever \( \alpha \) is an integer. All these solutions possess convergent series at infinity. For \( n \neq 0 \) every rational solution \( w^{(N)}(z; n) \) has the asymptotic behavior

\[
w^{(N)}(z; n) \sim -\frac{n}{z}, \quad z \to \infty,
\]
i.e. the point \( z = \infty \) is a simple root. This fact can be easily seen from the Bäcklund transformation for \( P_2^{(N)} \). Thus the formal series (2.4) at \( \alpha = n \) coincides with the expansion (2.19) and is also convergent.

**Remark 2.** Since \( c_{i,-1}^{(N)} = -i \forall N \geq 1 \), we get that the degree of each polynomial \( Q_{n}^{(N)}(z) \) is

\[
p_{n,N} = \sum_{i=1}^{n} i = \frac{n(n+1)}{2}.
\] (2.22)

Theorem (2.2) enables us to prove the following theorem.

**Theorem 2.3.** All the coefficients \( A_{n,m} \) of the polynomial \( Q_n(z) \) can be obtained with a help of \( n(n+1)/2 + 1 \) first coefficients of the expansion (2.11) for the solutions of \( P_2^{(N)} \).
Proof. For every polynomial there exists a connection between its coefficients and the symmetric functions of its roots $s_{n,m}$. This connection is the following

$$mA_{n,m} + s_{n,1}A_{n,m-1} + \ldots + s_{n,m}A_{n,0} = 0, \quad 1 \leq m \leq p_n. \quad (2.23)$$

Taking into account that in our case $A_{n,0} = 1$ and $p_n = n(n + 1)/2$ we get

$$A_{n,m} = -s_{n,m} + \frac{s_{n,m-1}A_{n,1} + \ldots + s_{n,1}A_{n,m-1}}{m}, \quad 1 \leq m \leq n(n + 1)/2. \quad (2.24)$$

The function $s_{n,m}$ can be derived using the expression (2.13). Hence recalling the fact that (2.11) is exactly (2.4) we obtain

$$s_{n,m} = 0, \quad m \in \mathbb{N} / \{(2N + 1)l, \quad l \in \mathbb{N}\},$$

$$s_{n,(2N+1)l} = -\sum_{i=2}^{n} c_{i-1(2N+1)l-1}, \quad l \in \mathbb{N}. \quad (2.25)$$

Substituting this into (2.24) yields

$$A_{n,m} = 0, \quad m \in \{1, 2, \ldots, n(n + 1)/2\} / \{(2N + 1)l, \quad l \in \mathbb{N}\};$$

$$A_{n,(2N+1)l} = -\frac{1}{(2N + 1)l}\left\{s_{n,(2N+1)l} + s_{n,(2N+1)l-(2N+1)}A_{n,(2N+1)} + \ldots + s_{n,(2N+1)l-(2N+1)}\right\}, \quad l \in \mathbb{N}, \quad (2N + 1)l \leq n(n + 1)/2. \quad (2.26)$$

Thus we see that the coefficients $A_{n,k}$ of the polynomial $Q_n(z)$ are uniquely defined by coefficients $c_{n-1(2N+1)l-1}$ of the expansion (2.4). This completes the proof. \qed

Remark 3. Expression (2.26) defines the structure of the polynomial $Q_n(z)$. Namely if $n(n + 1)/2$ is divisible by $(2N + 1)$, i.e. $n \equiv 0 \ mod \ (2N+1)$ or $n \equiv 2N \ mod \ (2N + 1)$, then $Q_n(z)$ is a polynomial in $z^{2N+1}$. Otherwise (if $n(n + 1)/2$ is not divisible by $(2N + 1)$) $Q_n(z)/z^r$ is a polynomial in $z^{2N+1}$, where $r = n(n + 1)/2 \ mod \ (2N + 1)$. In other words

$$q^{(N)}_n(z) \overset{\text{def}}{=} \frac{Q^{(N)}_n\left(\xi^{1/(2N+1)}\right)}{\xi^{r/(2N+1)}}, \quad \xi = z^{2N+1} \quad (2.27)$$

is a polynomial of degree $\lfloor n(n + 1)/(4N + 2) \rfloor$ with $\lfloor x \rfloor$ denoting the integer part of $x$. 

9
3 Symmetric functions of the roots

In this section we are discussing properties of the symmetric functions. It is important to note that the functions \( s^{(N)}_{n,m} \) can be regarded as relations between the roots \( a^{(N)}_{n,k} \) of the polynomials \( Q^{(N)}_n(z) \). Recently the location of the roots in the complex plane was investigated. It was shown that the structure of the roots is very regular. In order to establish our main results we need a lemma.

**Lemma 3.1.** The coefficient \( c_{\alpha,-(2N+1)l-1} \) in the expansion (2.4) is a polynomial in \( \alpha \) of degree \( 2Nl + 1 \).

**Proof.** The proof is by induction on \( l \). For \( l = 0 \) there is nothing to prove as \( c_{\alpha,-1} = -\alpha \). Other coefficients can be obtained from the recursion relation, which for \( N = 1 \) is

\[
c_{\alpha,-3(l+1)-1} = (3l + 2)(3l + 1)c_{\alpha,-3l-1} - 2 \sum_{m=0}^{l} \sum_{n=0}^{m} c_{\alpha,-3n-1} \tag{3.1}
\]

\[
c_{\alpha,-3(m-n)-1}c_{\alpha,-3(l-m)-1}, \quad l \geq 1.
\]

In the general case the recursion relation has the similar structure (see (2.1) and (2.8)):

\[
c_{\alpha,-(2N+1)(l+1)-1} = ((2N+1)l + 2N) \ldots ((2N+1)l + 1)c_{\alpha,-(2N+1)l-1}
\]

\[+ \ldots + \frac{(-1)^N 2^N (2N-1)!!}{N!} \sum_{k_1=0}^{l} \sum_{k_2=0}^{k_1} \ldots \sum_{k_{2N}=0}^{k_{2N-1}} c_{\alpha,-(2N+1)k_{2N}-1} \tag{3.2}
\]

\[
c_{\alpha,-(2N+1)(k_{2N-1}-k_{2N})-1} \ldots c_{\alpha,-(2N+1)(l-k_1)-1}, \quad l \geq 1.
\]

Suppose that \( c_{\alpha,-(2N+1)m-1} \) is a polynomial in \( \alpha \) of degree \( (2Nm+1) \) \((0 < m \leq l)\). Then from (3.2) we see that \( c_{\alpha,-(2N+1)(l+1)-1} \) is also a polynomial in \( \alpha \) and \( \deg(c_{\alpha,-(2N+1)(l+1)-1}) = 2k_{2N}N + 1 + 2(k_{2N-1} - k_{2N})N + 1 + \ldots + 2(l - k_1)N + 1 = 2Nl + 2N + 1 = 2N(l + 1) + 1 \). Q.E.D.

**Theorem 3.1.** The following statements are true:

1. at given \( n \geq 2 \) the functions \( s^{(N)}_{n,m} \) \((m > p_{n,N} = n(n+1)/2)\) do not contain any new information about the roots of \( Q^{(N)}_n(z) \);
2. \( s^{(N)}_{n,(2N+1)l} \) is a polynomial in \( n \) of degree \( 2(Nl + 1) \).
Proof. The first statement of the theorem immediately follows from the correlation

\[ s_{n,m} + s_{n,m-1}A_{n,1} + \ldots + s_{n,m-p_n}A_{n,p_n} = 0, \quad m > p_n, \quad m \in \mathbb{N} \quad (3.3) \]

and the expression (2.24) (the index \( N \) is omitted). Now let us prove the second statement. From (2.13) and Lemma (3.1) we see that in order to find \( s_n^{(N)}(2N+1)_l \) one should calculate finite amount of sums \( \sum_{i=1}^{n} i^m, \quad m \in \mathbb{N}, \max m = 2Nl + 1 \). Such sum is computable. And the result is a polynomial in \( n \) of degree \( m + 1 \). This completes the proof.

Using the remark at the end of the previous section we obtain that the roots of \( Q_n^{(N)}(z) \) lie on circles with center the origin. The radii of the circles are equal to \((2N+1)\)th roots of the absolute values of the non-zero roots of \( q_n^{(N)}(\xi) \). Furthermore there are \( 2N + 1 \) equally spaced roots of \( Q_n^{(N)}(z) \) on a circle and for the real roots of \( q_n^{(N)}(\xi) \) and \( 2(2N + 1) \) roots of \( Q_n^{(N)}(z) \) (\( 2N + 1 \) complex conjugate pairs) are located on a circle for the complex roots of \( q_n^{(N)}(\xi) \).

4 Coefficients of the Yablonskii - Vorob’ev polynomials

In the next sections we will find first several coefficients of the Yablonskii - Vorob’ev polynomials for some members of the \( P_2 \) hierarchy. While now let us study the general case and prove a theorem.

**Theorem 4.1.** The coefficient \( A_{n,(2N+1)}^{(N)} \) is a polynomial in \( n \) of degree \( 2(N+1)^l \). Moreover \( A_{n,(2N+1)}^{(N)}/T_{n,l}^{(N)} \) is also a polynomial in \( n \), where \( T_{n,l}^{(N)} = n(n-1)(n-2)\ldots(n-(n_0-1)) \) and \( n_0 = \lfloor \sqrt{1/4 + 2(2N+1)l} - 1/2 \rfloor \) with \( [x] \) denoting \( x \) if \( x \) is an integer and \( [x]+1 \) otherwise.

**Proof.** The first part of the theorem can be proved by induction on \( l \). Indeed in the case \( l = 1 \) we see that \( A_{n,(2N+1)} = -s_{n,(2N+1)}/(2N + 1) \). (Here and up to the end of the proof the upper index \( N \) is omitted.) Consequently using the second statement of theorem (3.1) we get the correlation \( \deg A_{n,(2N+1)} = \deg s_{n,(2N+1)} = 2(N+1) \). Let \( A_{n,(2N+1)m} \) be a polynomial in \( n \) of degree \( 2(N+1)m \) \( (m < l) \). The coefficient \( A_{n,(2N+1)l} \) can be found with a help of (2.26). Analyzing this expression we understand that the term
\[ s_{n,(2N+1)} A_{n,(2N+1)} \] gives the greatest contribution into the degree of \( A_{n,(2N+1)} \). Hence \( \deg A_{n,(2N+1)} = 2(N + 1) + 2(N + 1)(l - 1) = 2(N + 1)l \).

In order to prove the second part of the theorem first of all we should note that \( \sum_{i=1}^{n} i^m / n \) is a polynomial in \( n \) for all \( m \in \mathbb{N} \cup \{0\} \). Next let us regard the left-hand side of (3.3) also as a polynomial in \( n \)

\[ R_{n,(2N+1)} \overset{\text{def}}{=} s_{n,(2N+1)} l + s_{n,(2N+1)} l - 1 A_{n,2N+1} + \ldots + s_{n,(2N+1)} (l-k) A_{n,(2N+1)k}, \quad k = \max_{(2N+1)j \leq p_n} j, \quad (2N+1)l > p_n, \quad l \in \mathbb{N}. \] (4.1)

As far as (3.3) holds, then \( R_{n_0-1,(2N+1)} l = 0 \), where \( n_0 = \min_{n(n+1)/2 \geq (2N+1)l} n \). At the same time \( A_{n,(2N+1)} \) is equal to \( R_{n,(2N+1)} \) accurate to a numerical parameter \( (\tilde{l} = \min_{(2N+1)j > n_0(n_0-1)/2} j) \). Continuing in such a way we complete the proof.

5 The Yablonskii - Vorob’ev polynomials associated with rational solutions of \( P_2 \)

In this section our interest is in the polynomials \( Q_n^{(1)}(z) \) associated with the equation (1.1). Later in this section the upper index will be omitted. The power expansion at infinity corresponding to the solutions of \( P_2 \equiv P_2^{(1)} \) is the following [21]

\[ w(z; \alpha) = -\frac{\alpha}{z} + \sum_{l=1}^{\infty} c_{\alpha,-3l-1} z^{-3l-1}, \quad z \to \infty, \] (5.1)

where \( c_{\alpha,-3l-1} \) (\( l > 1 \)) satisfy the recurrence relation (3.1). Using theorems (3.1) and (4.1) we obtain

\[ s_{n,m} = \begin{cases} 0, & m \in \mathbb{N} / \{3l, \quad l \in \mathbb{N}\}; \\
S_{2(l+1)}(n), & m = 3l, \quad l \in \mathbb{N}. \end{cases} \] (5.2)

\[ A_{n,m} = \begin{cases} 0, & m \in \{1, 2, \ldots, n(n+1)/2\} / \{3l, \quad l \in \mathbb{N}\}; \\
A_{d^l}(n), & m = 3l, \quad l \in \mathbb{N}, \quad 3l \leq n(n+1)/2. \end{cases} \] (5.3)
Here $S_{2(l+1)}(n)$, $A_{4l}(n)$ are polynomials in $n$ of degree $2(l + 1)$ and $4l$, accordingly. Each polynomial $Q_n(z)$ is a monic polynomial of degree $n(n + 1)/2$ and can be written as

$$Q_n(z) = \sum_{l=0}^{[n(n+1)/6]} A_{n,3l} z^{n(n+1)/2 - 3l}, \quad A_{n,0} = 1. \tag{5.4}$$

Hence the polynomial $Q_n(z)$ is divisible by $z$ if and only if $n \mod 3 = 1$. Besides that $Q_n(z)$ is a polynomial in $z^3$ if $n \mod 3 \neq 1$ and $Q_n(z)/z$ is a polynomial in $z^3$ if $n \mod 3 = 1$. Since every polynomial $Q_n(z)$ has simple roots and two successive polynomials do not have a common root the rational solution $w(z; n)$ of $P_2$ has $n(n-1)/2$ poles with residue 1, which are the roots of $Q_{n-1}(z)$ and $n(n+1)/2$ poles with residue $-1$, which in their turn are the roots of $Q_n(z)$.

With a help of the expression (2.25) we can obtain the symmetric functions of the roots $s_{n,3l}$. The first few of them are the following

$$s_{n,3} = -\frac{1}{2} n (n^2 - 1) (n + 2), \tag{5.5}$$

$$s_{n,6} = 2 n (n^2 - 1) (n + 2) (n^2 + n - 5), \tag{5.6}$$

$$s_{n,9} = -4 n (n^2 - 1) (n + 2) (n^2 + n - 7) (3 n^2 + 3 n - 20), \tag{5.7}$$

$$s_{n,12} = 8 n (n^2 - 1) (n + 2) (11 n^6 + 33 n^5 - 259 n^4 - 573 n^3$$

$$+ 2348 n^2 + 2640 n - 7700), \tag{5.8}$$

$$s_{n,15} = -8 n (n^2 - 1) (n + 2) (91 n^8 + 364 n^7 - 3468 n^6 - 11678 n^5 + 57138 n^4 + 134164 n^3 - 454161 n^2 - 523250 n + 1401400); \tag{5.9}$$

The coefficients $A_{n,3l}$ of the polynomials $Q_n(z)$ can be found using the expression (2.25) and the symmetric functions $s_{n,3l}$. The first few of them are written out below

$$A_{n,3} = \frac{n}{6} (n^2 - 1) (n + 2). \tag{5.10}$$
\[ A_{n,6} = \frac{n}{72} \left( n^2 - 1 \right) \left( n^2 - 4 \right) (n - 4)(n + 5)(n + 3), \quad (5.11) \]

\[ A_{n,9} = \frac{n}{1296} \left( n^2 - 1 \right) \left( n^2 - 4 \right) \left( n^2 - 9 \right) (n + 4) \]
\[ \quad \quad \left( n^4 + 2n^3 - 57n^2 - 58n + 1120 \right), \quad (5.12) \]

\[ A_{n,12} = \frac{n}{31104} \left( n^2 - 1 \right) \left( n^2 - 4 \right) \left( n^2 - 9 \right) \left( n^2 - 16 \right) (n + 5) \]
\[ \quad \quad \left( n^6 + 3n^5 - 109n^4 - 223n^3 + 5148n^2 + 5260n - 110880 \right). \quad (5.13) \]

\[ A_{n,15} = \frac{n}{933120} \left( n^2 - 1 \right) \left( n^2 - 4 \right) \left( n^2 - 9 \right) \left( n^2 - 16 \right) (n + 5) \left( n^{10} + \right. \]
\[ \quad \quad \begin{aligned} & 5n^9 - 200n^8 - 830n^7 + 18917n^6 + 59677n^5 - 1072550n^4 - \end{aligned} \]
\[ \quad \quad \begin{aligned} & 2245540n^3 + 35648392n^2 + 36781248n - 484323840, \end{aligned} \quad (5.14) \]

Besides that the following estimations can be derived

\[ A_{n,3l} \approx \frac{n^4}{6(l + 1)} A_{n,3l}, \quad n \to \infty, \quad l \geq 0; \quad (5.15) \]

\[ A_{n,3(l+1)} \approx \frac{l}{(l + 1)} \frac{A_{n,3l}}{A_{n,3(l-1)}}, \quad n \to \infty, \quad l \geq 1. \quad (5.16) \]

Analysing the expressions (5.10) – (5.14) we get

\[ A_{n,3l} = \frac{n(n^2 - 1)(n + 2)}{6l!} \left[ n^{4l-4} + 2(l - 1)n^{4l-5} - 5(l - 1)(2l + 1) \right. \]
\[ \quad \quad \begin{aligned} & \times n^{4l-6} + \ldots + 2^{l-1}(3l - 1)! \left], \quad (l > 1) \end{aligned} \quad (5.17) \]

Consequently calculating the coefficients \( c_{\alpha,3l-1} \) of the expansion (5.1) one can construct each polynomial \( Q_n(z) \). Several polynomials \( Q_n(z) \) are given in Table 5.1.

Substituting the polynomials \( Q_{n-1}(z) \) and \( Q_n(z) \) into (1.2) we obtain the rational solution \( w(z; n) \) of \( P_2 \). Several of them are
Table 5.1: The Yablonskii - Vorob'ev polynomials for $P_2$

\begin{align*}
Q_0(z) &= 1, \\
Q_1(z) &= z, \\
Q_2(z) &= z^3 + 4, \\
Q_3(z) &= z^6 + 20z^3 - 80, \\
Q_4(z) &= (z^9 + 60z^6 + 11200)z, \\
Q_5(z) &= z^{15} + 140z^{12} + 2800z^9 + 78400z^6 - 313600z^3 - 6272000, \\
Q_6(z) &= z^{21} + 280z^{18} + 18480z^{15} + 627200z^{12} - 17248000z^9 + 1448832000z^6 \\
&\quad + 193177600000z^3 - 38635520000z, \\
Q_7(z) &= (z^{27} + 504z^{24} + 75600z^{21} + 5174400z^{18} + 62092800z^{15} + 13039488000z^{12} \\
&\quad - 828731904000z^9 - 49723914240000z^6 - 3093932441600000)z
\end{align*}

6 The Yablonskii - Vorob'ev polynomials associated with rational solutions of $P_2^{(2)}$

In this section we will deal with the polynomials $Q_n^{(2)}(z)$ associated with the forth-order analogue to $P_2^{(1)}$. Again the upper index will be omitted. The power expansion at infinity corresponding to the solutions of $P_2^{(1)}$ is the following [24, 25]

\begin{equation}
w(z; \alpha) = -\frac{\alpha}{z} + \sum_{t=1}^{\infty} c_{\alpha, -5t-1} z^{-5t-1}, \quad z \to \infty, \tag{6.1}
\end{equation}
where \( c_{\alpha,-5l-1} \) \( (l > 1) \) can be sequently found. Using the results of the previous sections we get

\[
s_{n,m} = \begin{cases} 
0, & m \in \mathbb{N} / \{5l, \ l \in \mathbb{N}\}; \\
S_{2(2l+1)}(n), & m = 5l, l \in \mathbb{N}. 
\end{cases} \tag{6.2}
\]

\[
A_{n,m} = \begin{cases} 
0, & m \in \{1, 2, \ldots, n(n+1)/2\} / \{5l, \ l \in \mathbb{N}\}; \\
A_{6l}(n), & m = 5l, \ l \in \mathbb{N}, \ 5l \leq n(n+1)/2. 
\end{cases} \tag{6.3}
\]

Here \( S_{2(2l+1)}(n), A_{6l}(n) \) are polynomials in \( n \) of degree \( 2(2l+1) \) and \( 6l \), accordingly. Being of degree \( n(n+1)/2 \) the polynomial \( Q_n(z) \) can be written as

\[
Q_n(z) = \sum_{l=0}^{[n(n+1)/10]} A_{n,5l} z^{n(n+1)/2 - 5l}, \quad A_{n,0} = 1. \tag{6.4}
\]

Therefore the polynomial \( Q_n(z) \) is a polynomial in \( z^5 \) if \( n \mod 5 = 0 \) or \( n \mod 5 = 4 \), \( Q_n(z)/z \) is a polynomial in \( z^5 \) if \( n \mod 5 = 1 \) or \( n \mod 5 = 3 \), and \( Q_n(z)/z^3 \) is a polynomial in \( z^5 \) if \( n \mod 5 = 2 \).

The symmetric functions of the roots \( s_{n,5l} \) can be obtained with a help of the expression (2.5). The first few of them are the following

\[
s_{n,5} = n \left( n^2 - 1 \right) \left( n^2 - 4 \right) (n + 3), \tag{6.5}
\]

\[
s_{n,10} = 6n \left( n^2 - 1 \right) \left( n^2 - 4 \right) (n + 3) \left( 3n^4 + 6n^3 - 73n^2 - 76n + 504 \right), \tag{6.6}
\]

\[
s_{n,15} = 36n \left( n^2 - 1 \right) \left( n^2 - 4 \right) (n + 3) \left( 15n^8 + 60n^7 - 1010n^6 - 3240n^5 + 28759n^4 + 62988n^3 - 388124n^2 - 420168n + 2018016 \right). \tag{6.7}
\]

Using the expression (2.25) and the symmetric functions \( s_{n,5l} \) we find the coefficients \( A_{n,5l} \) of the polynomials \( Q_n(z) \). The first few of them are

\[
A_{n,5} = -\frac{n}{5} \left( n^2 - 1 \right) \left( n^2 - 4 \right) (n + 3), \tag{6.8}
\]

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\[ A_{n,10} = \frac{n}{50} \left( n^2 - 1 \right) \left( n^2 - 4 \right) \left( n^2 - 9 \right) (n+4) \]
\[ \left( n^4 + 2n^3 - 85n^2 - 86n + 1260 \right) , \] (6.9)

\[ A_{n,15} = -\frac{n}{750} \left( n^2 - 1 \right) \left( n^2 - 4 \right) \left( n^2 - 9 \right) \left( n^2 - 16 \right) (n+5) \]
\[ \left( n^8 + 4n^7 - 248n^6 - 758n^5 + 26959n^4 + 55186n^3 - 1107792n^2 - \right.
\[ \left. -1135512n + 15135120 \right) . \] (6.10)

Besides that we obtain the following estimations

\[ A_{n,5(l+1)} \simeq -\frac{n^6}{5(l+1)} A_{n,5l}, \quad n \to \infty, \quad l \geq 0; \] (6.11)

\[ A_{n,5(l+1)} \simeq \frac{l}{(l+1)} \frac{A_{n,5l}^2}{A_{n,5l-1}}, \quad n \to \infty, \quad l \geq 1. \] (6.12)

Thus calculating the coefficients \( c_{\alpha, -5l-1} \) of the expansion (6.1) each polynomial \( Q_\alpha(n) \) can be constructed. Several polynomials \( Q_\alpha(n) \) are gathered in Table 6.1

**Table 6.1: The Yablonskii - Vorob’ev polynomials for \( P_2^{(2)} \)**

\[
\begin{align*}
Q_0(z) &= 1, \\
Q_1(z) &= z, \\
Q_2(z) &= z^3, \\
Q_3(z) &= z(z^5 - 144) \\
Q_4(z) &= z^{10} - 1008z^5 - 48384, \\
Q_5(z) &= z^{15} - 4032z^{10} - 3048192z^5 + 146313216, \\
Q_6(z) &= z(z^{20} - 12096z^{15} - 21337344z^{10} - 33798352896z^5 - 4866962817024), \\
Q_7(z) &= z^3(z^{25} - 30240z^{20} - 55883520z^{15} - 1182942351360z^{10} + 701543488297107456) \\
\end{align*}
\]

Substituting the polynomials \( Q_{n-1}(z) \) and \( Q_n(z) \) into (1.2) yields the rational solution \( w(z; n) \) of \( P_2^{(2)} \). Several of them are
\[
\begin{align*}
  w(z; 1) &= -\frac{1}{z} \\
  w(z; 2) &= -\frac{2}{z} \\
  w(z; 3) &= -\frac{3(z^5 + 96)}{z(z^5 - 144)} \\
  w(z; 4) &= -\frac{4(z^{15} - 72z^{10} + 217728z^5 - 1741824)}{z(z^5 - 144)(z^{10} - 1008z^5 - 48384)} \\
  w(z; 5) &= -\frac{5z^4(z^{20} - 2016z^{15} + 6967296z^{10} + 97542144z^5 + 294967443456)}{(z^{15} - 4032z^{10} - 3048192z^5 + 146313216)(z^{10} - 1008z^5 - 48384)}
\end{align*}
\]

7 The Yablonskii - Vorob’ev polynomials associated with rational solutions of \( P_2^{(3)} \)

In this section we will briefly review the case of the polynomials \( Q_n^{(3)}(z) \) associated with the rational solutions of the sixth-order analogue to \( P_2 \)

\[
\begin{align*}
  w_{zzzzzz} - 14 w^2 w_{zzzz} - 56 w w_z w_{zzz} - 42 w (w_z) w_{zz} - 70 (w_z)^2 w_{zz} \\
  + 70 w^4 w_{zz} + 140 w^3 (w_z)^2 - 20 w^7 - z w - \alpha &= 0.
\end{align*}
\] (7.1)

Later the upper index will be omitted. The power expansion at infinity corresponding to the solutions of (7.1) can be written as

\[
w(z; \alpha) = -\frac{\alpha}{z} + \sum_{l=1}^{\infty} c_{\alpha,-7l-1} z^{-7l-1}, \quad z \to \infty,
\] (7.2)

where \( c_{\alpha,-7l-1} (l > 1) \) are sequentially found. Again using the results of the previous sections we obtain

\[
s_{n,m} = \begin{cases} 
0, & m \in \mathbb{N} / \{ 7l, \quad l \in \mathbb{N} \}; \\
S_{2(3l+1)}(n), & m = 7l, \quad l \in \mathbb{N}.
\end{cases}
\] (7.3)

\[
A_{n,m} = \begin{cases} 
0, & m \in \{ 1, 2, \ldots, n(n+1)/2 \} / \{ 7l, \quad l \in \mathbb{N} \}; \\
\mathcal{A}_{8l}(n), & m = 7l, \quad l \in \mathbb{N}, \quad 7l \leq n(n+1)/2.
\end{cases}
\] (7.4)
Here $S_{2(3l+1)}(n)$, $A_{8l}(n)$ are polynomials in $n$ of degree $2(3l + 1)$ and $8l$, accordingly. Since the polynomial $Q_n(z)$ is of degree $n(n + 1)/2$, then it can be presented in the form

$$Q_n(z) = \sum_{l=0}^{[n(n+1)/14]} A_{n, 7l} z^{n(n+1)/2 - 7l}, \quad A_{n, 0} = 1. \tag{7.5}$$

We see that the polynomial $Q_n(z)$ is a polynomial in $z^7$ if $n \mod 7 = 0$ or $n \mod 7 = 6$, $Q_n(z)/z$ is a polynomial in $z^7$ if $n \mod 7 = 1$ or $n \mod 7 = 5$, $Q_n(z)/z^3$ is a polynomial in $z^7$ if $n \mod 7 = 2$ or $n \mod 7 = 4$ and $Q_n(z)/z^6$ is a polynomial in $z^7$ if $n \mod 7 = 3$.

With a help of the expressions (2.5) the symmetric functions of the roots $s_{n, 7l}$ can be obtained. The first few of them are the following

$$s_{n, 7} = -\frac{5n}{2} \left( n^2 - 1 \right) \left( n^2 - 4 \right) \left( n^2 - 9 \right) \left( n + 4 \right), \tag{7.6}$$

$$s_{n, 14} = 40 n \left( n^2 - 1 \right) \left( n^2 - 4 \right) \left( n^2 - 9 \right) \left( n + 4 \right) \left( 5n^6 + 15n^5 ight. \right.

$$-340 n^4 - 705 n^3 + 8651 n^2 + 9006 n - 77220), \tag{7.7}$$

$$s_{n, 21} = -800 n \left( n^2 - 1 \right) \left( n^2 - 4 \right) \left( n^2 - 9 \right) \left( n + 4 \right) \left( 35n^{12} + 210n^{11} ight. \right.

$$-6860 n^{10} - 36225 n^9 + 629265 n^8 + 2736720 n^7 - 32792630 n^6 

$$-108110865 n^5 + 989372966 n^4 + 2162197152 n^3 

$$-16042160664 n^2 - 17141744880 n + 107749699200). \tag{7.8}$$

Using the expression (2.25) and the symmetric functions $s_{n, 7l}$ we find the coefficients $A_{n, 7l}$ of the polynomials $Q_n(z)$

$$A_{n, 7} = \frac{5n}{14} \left( n^2 - 1 \right) \left( n^2 - 4 \right) \left( n^2 - 9 \right) \left( n + 4 \right), \tag{7.9}$$

$$A_{n, 14} = \frac{5n}{392} \left( n^2 - 1 \right) \left( n^2 - 4 \right) \left( n^2 - 9 \right) \left( n^2 - 16 \right) \left( n + 5 \right) 

\left( 5n^6 + 15n^5 - 1105 n^4 - 2235 n^3 + 56540 n^2 + 57660 n - 864864 \right), \tag{7.10}$$

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\[ A_{n,21} = \frac{25n}{16464} (n^2 - 1) (n^2 - 4) (n^2 - 9) (n^2 - 16) (n^2 - 25) (n + 6) \\
(5n^{12} + 30n^{11} - 3235n^{10} - 16450n^9 + 985395n^8 + 4040610n^7 \\
-137483057n^6 - 426660726n^5 + 9606229564n^4 + 19928307448n^3 \\
-331282163616n^2 - 341318105856n + 4505374089216). (7.11) \]

Besides that the following estimations can be derived

\[ A_{n,7(l+1)} \approx \frac{5n^8}{14(l+1)} A_{n,7l}, \quad n \to \infty, \quad l \geq 0; \quad (7.12) \]

\[ A_{n,7(l+1)} \approx \frac{l}{(l+1)} A_{n,7l}^2, \quad n \to \infty, \quad l \geq 1. \quad (7.13) \]

Thus calculating the coefficients \( c_{\alpha,-7l-1} \) of the expansion (7.2) each polynomial \( Q_n(z) \) can be constructed. Several polynomials \( Q_n(z) \) are gathered in Table \( \ref{tab:7.1} \).

**Table 7.1: The Yablonskii - Vorob’ev polynomials for \( P_2^{(3)} \)**

| \( n \) | \( Q_n(z) \) |
|---|---|
| 0 | \( Q_0(z) = 1 \), |
| 1 | \( Q_1(z) = z \), |
| 2 | \( Q_2(z) = z^3 \), |
| 3 | \( Q_3(z) = z^6 \), |
| 4 | \( Q_4(z) = z^3(z^7 + 14400) \), |
| 5 | \( Q_5(z) = z(z^{14} + 129600z^7 - 373248000) \), |
| 6 | \( Q_6(z) = z^{21} + 648000z^{14} - 24634368000z^7 - 35473489920000 \), |
| 7 | \( Q_7(z) = z^{28} + 2376000z^{21} - 825251328000z^{14} - 30436254351360000z^7 + 43828206265958400000 \) |

Substituting the polynomials \( Q_{n-1}(z) \) and \( Q_n(z) \) into (1.12) yields the rational solution \( w(z; n) \) of \( P_2^{(3)} \). Several of them are
\[ w(z; 1) = -\frac{1}{z} \]
\[ w(z; 2) = -\frac{2}{z} \]
\[ w(z; 3) = -\frac{3}{z} \]
\[ w(z; 4) = -\frac{4(z^7 - 5400)}{z(z^7 + 7200)} \]
\[ w(z; 5) = -\frac{5(z^{21} - 8640z^{14} + 634521600z^7 + 268738560000)}{z(z^{14} + 64800z^7 - 93312000)(z^7 + 7200)} \]

8 Conclusion

In this paper a method for constructing the polynomials associated with rational solutions of the \( P_2 \) hierarchy has been presented. The basic idea of the method is to use power expansions for solutions of the equations studied. These power expansions can be obtained with a help of algorithms of power geometry [19, 20]. Using our approach we have found: the degree of each polynomial, formulas for its coefficients, correlations between its roots. Besides that we have established the structure of the polynomials. The main computations have been done simultaneously for all members of the hierarchy. Our method is sufficiently general and can be applied for constructing polynomials associated with rational or algebraic solutions of other nonlinear differential equations [6–10].

9 Acknowledgments

This work was supported by the International Science and Technology Center under Project B 1213.

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