Indiscernible pairs of countable sets of reals at a given projective level

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Abstract

Using an invariant modification of Jensen’s “minimal \( \Pi^1_2 \) singleton” forcing, we define a model of ZFC, in which, for a given \( n \geq 2 \), there exists an \( \Pi^1_n \) unordered pair of non-OD (hence, OD-indiscernible) countable sets of reals, but there is no \( \Sigma^1_n \) unordered pairs of this kind.

Any two reals \( x_1 \neq x_2 \) are discernible by a simple formula \( \varphi(x) := x < r \) for a suitable rational \( r \). Therefore, the lowest (type-theoretic) level of sets where one may hope to find indiscernible elements, is the level of sets of reals. And indeed, identifying the informal notion of definability with the ordinal definability (OD), one finds indiscernible sets of reals in appropriate generic models.

Example 1. If reals \( a \neq b \) in \( 2^\omega \) form a Cohen-generic pair over \( L \), then the constructibility degrees \( [a]_L = \{ x \in 2^\omega : L[x] = L[a] \} \) and \( [b]_L \) are OD-indiscernible disjoint sets of reals in \( L[a,b] \), by rather straightforward forcing arguments, see [2, Theorem 3.1] and a similar argument in [3, Theorem 2.5].

Example 2. As observed in [5], if reals \( a \neq b \) in \( 2^\omega \) form a Sacks-generic pair over \( L \), then the constructibility degrees \( [a]_L \) and \( [b]_L \) still are OD-indiscernible disjoint sets in \( L[a,b] \), with the additional advantage that the unordered pair \( \{ [a]_L, [b]_L \} \) is an OD set in \( L[a,b] \) because \( [a]_L, [b]_L \) are the only two minimal degrees in \( L[a,b] \). (This argument is also presented in [3, Theorem 4.6].) In other words, it is true in such a generic model \( L[a,b] \) that \( P = \{ [a]_L, [b]_L \} \) is an OD pair of non-OD (hence OD-indiscernible in this case) sets of reals.

Unordered OD pairs of non-OD sets of reals were called Groszek – Laver pairs in [4], while in the notation of [3, 6] the sets \( [a]_L, [b]_L \) are ordinal-algebraic (meaning that they belong to a finite OD set) in \( L[a,b] \), but neither of the two

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sets is straightforwardly OD in L[a, b]. From the other angle of view, any (OD or not) pair of OD-indiscernible sets x ≠ y is a special violation of the Leibniz – Mycielski axiom LM of Enayat [2] (see also [11]).

Given an unordered pair of disjoint sets A, B ⊆ 2^ω, to measure its descriptive complexity, define the equivalence relation E_{AB} on the set A ∪ B by x E_{AB} y iff x, y ∈ A or x, y ∈ B. It holds in the Sacks × Sacks generic model L[a, b] that E_{a,b}L is the restriction of the Σ^1_2 relation L[x] = L[y] to the Δ^1_3 set

\[ [a]_L \cup [b]_L = \{ x \in 2^\omega : x \notin L \land \exists z \in 2^\omega (z \notin L[x]) \} \]
\[ = \{ x \in 2^\omega : x \notin L \land \forall y \in 2^\omega \cap L[x] (y \in L \lor x \in L[y]) \}. \]

Thus the Groszek – Laver (unordered) pair \{[a]_L, [b]_L\} of Example 2 can be said to be a Σ^1_3 pair in L[a, b] because so is the equivalence relation E_{a,b}L.

**Example 3.** A somewhat better result was obtained in [3]: a generic model L[a, b] in which the E_0-equivalence classes \{[a]_{E_0}, [b]_{E_0}\} form a Π^1_2 Groszek – Laver pair of countable sets.

Thus Δ^1_3, and even Π^1_2 Groszek – Laver pairs of countable sets in 2^ω exist in suitable extensions of L. This is the best possible existence result since Σ^1_2 Groszek – Laver pairs do not exist by the Shoenfield absoluteness.

The main result of this paper is the following theorem. It extends the research line of our recent papers [12, 13, 14], based on some key methods and approaches outlined in Harrington’s handwritten notes [7] and aimed at the construction of generic models in which this or another property of reals or pointsets holds at a given projective level.

**Theorem 4.** Let n ≥ 3. There is a generic extension L[a] of L, the constructible universe, by a real a ∈ 2^ω, such that the following is true in L[a]:

(i) there exists a Π^1_0 Groszek – Laver pair of countable sets in 2^ω;

(ii) every countable Σ^1_0 set consists of OD elements, and hence there is no Σ^1_0 Groszek – Laver pairs of countable sets.

The proof of Theorem 4 makes use of a forcing notion P = P_n ∈ L, defined in [12] for a given number n ≥ 2, which satisfies the following key requirements.

1°. P ∈ L and P consists of Silver trees in 2^{<ω}. A perfect tree T ⊆ 2^{<ω} is a Silver tree, in symbol T ∈ ST, whenever there exists an infinite sequence

1. LM claims that if x ≠ y then there exists an ordinal α and a (parameter-free) ∈-formula \( \varphi(\cdot) \) such that x, y ∈ V_α and \( \varphi(x) \) holds in V_α but \( \varphi(x) \) fails in V_α — in this case x, y are OD-discernible (with α ∈ Ord as a parameter), of course.
2. The first line says that x is nonconstructible and not ≤_L-maximal, the second line says that x is nonconstructible and ≤_L-minimal; this happens to be equivalent in that model.
3. E_0 is defined on the Cantor space 2^ω so that x E_0 y iff the set \{ n : x(n) ≠ y(n) \} is finite.
of strings $u_k = u_k(T) \in 2^{<\omega}$ such that $T$ consists of all strings of the form $s = u_0 \concat i_0 \concat u_1 \concat i_1 \concat u_2 \concat i_2 \cdots \concat u_m \concat i_m$, and their substrings (including $\Lambda$, the empty string), where $m < \omega$ and $i_k = 0,1$.

2°. If $s \in T \in \mathbb{P}$ then the subtree $T|_s = \{ t \in T : s \subset t \}$ belongs to $\mathbb{P}$ as well — then clearly the forcing $\mathbb{P}$ adjoins a new generic real $a \in 2^\omega$.

3°. If $x \in 2^\omega$, then the tree $T = \{ s \cdot T : s \in 2^\omega \}$ belongs to $\mathbb{P}$ as well.[4] It follows that if $a \in 2^\omega$ is $\mathbb{P}$-generic over $L$ then any real $b \in [a]_{\mathbb{E}_0}$ is $\mathbb{P}$-generic over $L$ too.

In other words, $\mathbb{P}$ adjoins a whole $\mathbb{E}_0$-class $[a]_{\mathbb{E}_0}$ of $\mathbb{P}$-generic reals.

4°. Conversely, if $a \in 2^\omega$ is $\mathbb{P}$-generic over $L$ and a real $b \in 2^\omega \cap L[a]$ is $\mathbb{P}$-generic over $L$, then $b \in [a]_{\mathbb{E}_0}$.

5°. The property of “being a $\mathbb{P}$-generic real in $2^\omega$ over $L$” is (lightface) $\Pi^1_1$ in any generic extension of $L$.

6°. If $a \in 2^\omega$ is $\mathbb{P}$-generic over $L$, then it is true in $L[a]$ that

(1) (by [3] [4] [5]) $[a]_{\mathbb{E}_0}$ is a $\Pi^1_1$ set containing no OD elements, but

(2) every countable $\Sigma^1_0$ set contains OD elements.[6]

Proof (Theorem 4). Let $\mathbb{P} \in L$ be a forcing satisfying conditions [1'] - [6']. Let $a_0 \in 2^\omega$ be a real $\mathbb{P}$-generic over $L$. Then, in $L[a_0]$, the $\mathbb{E}_0$-class $[a_0]_{\mathbb{E}_0}$ is a $\Pi^1_1$ set containing no OD elements, by [6] (1).

Let us split the $\mathbb{E}_0$-class $[a_0]_{\mathbb{E}_0}$ into two equivalence classes of the subrelation $\mathbb{E}^{\text{even}}_0$ defined on $2^\omega$ so that $x \mathbb{E}^{\text{even}}_0 y$ iff the set $x \triangle y = \{ k : x(k) \neq y(k) \}$ contains a finite even number of elements. Thus $[a_0]_{\mathbb{E}_0} = [a_0]_{\mathbb{E}^{\text{even}}_0}$ is the partition, where $[a]_{\mathbb{E}^{\text{even}}_0}$ is the $\mathbb{E}^{\text{even}}_0$-class of any $x \in 2^\omega$, and $b \in [a_0]_{\mathbb{E}_0} \setminus [a_0]_{\mathbb{E}^{\text{even}}_0}$ is any real $\mathbb{E}_0$-equivalent but not $\mathbb{E}^{\text{even}}_0$-equivalent to $a_0$. We claim that, in $L[a_0]$, these two $\mathbb{E}^{\text{even}}_0$-subclasses of $[a_0]_{\mathbb{E}_0}$ form a $\Pi^1_3$ Groszek – Laver pair required.

Basically, we have to prove that $[a_0]_{\mathbb{E}^{\text{even}}_0}$ is not OD in $L[a_0]$. Suppose to the contrary that $[a_0]_{\mathbb{E}^{\text{even}}_0}$ is OD in $L[a_0]$, say $[a_0]_{\mathbb{E}^{\text{even}}_0} = \{ x \in 2^\omega : \varphi(x) \}$, where $\varphi(x)$

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4 Here $s \cdot t \in 2^{<\omega}$, $\text{dom}(a \cdot t) = \text{dom} t$, if $k < \min\{\text{dom}s, \text{dom} t\}$ then $(a \cdot t)(k) = t(k) +_2 s(k)$ (and $+_2$ is the addition mod 2), while if $\text{dom} s \leq k < \text{dom} t$ then $(a \cdot t)(k) = t(k)$.

5 Earlier results in this direction include a model in [11] with a $\Pi^1_3$ $\mathbb{E}_0$-class in $2^\omega$, containing no OD elements — which is equivalent to case $n = 2$ in [6]. The forcing employed in [11] is an invariant, as in [7] “Silver tree” version $F = \mathbb{P}_2$, of a forcing notion, call it $\check{J}$, introduced by Jensen [9] to define a model with a nonconstructible minimal $\Pi^1_3$ singleton. See also 28A in [8] on Jensen’s original forcing. The invariance implies that instead of a single generic real, as in [7], $\mathbb{P}_2$ adjoins a whole $\mathbb{E}_0$-equivalence class $[a]_{\mathbb{E}_0}$ of $\mathbb{P}_2$-generic reals in [11]. Another version of a countable lightface $\Pi^1_3$ non-empty set of non-OD reals was obtained in [10] [15] by means of the finite-support product $J''$ of Jensen’s forcing $J$, following the idea of Ali Enayat [2]. See [12] Introduction] on a more detailed account of the problem of the existence of countable OD sets of non-OD elements.
is a $\in$-formula with ordinals as parameters. This is forced by a condition $T \in \mathbb{P}$, so that if $a \in [T]$ is $\mathbb{P}$-generic over $L$ then $[a]_{E_{0}^{\text{even}}} = \{ x \in 2^\omega : \varphi(x) \}$ in $L[a]$.

Representing $T$ in the form of $1^c$, let $m = \text{dom}(u_0)$ and let $s = 0^m \cdot 1$, so that $s \in 2^{<\omega}$ is the string of $m$ 0s, followed by 1 as the rightmost term; $\text{dom} s = m + 1$. Then $s \cdot T = T$, so that the real $b = s \cdot a$ still belongs to $[T]$, and hence we have $[b]_{E_{0}^{\text{even}}} = \{ x \in 2^\omega : \varphi(x) \}$ in $L[b] = L[a]$ by the choice of $T$. We conclude that $[a]_{E_{0}^{\text{even}}} = [b]_{E_{0}^{\text{even}}}$. However, on the other hand, $a \in E_{0}^{\text{even}}$ fails by construction since the set $a \Delta b = \{ m \}$ contains one (an odd number) element. The contradiction ends the proof of (i) of Theorem 4.

To prove (ii) apply (i).

**A problem.** Can (ii) of Theorem 4 be improved to the nonexistence of $\Sigma^1_n$ Groszek – Laver pairs of not-necessarily-countable sets in the model considered?

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