Topological photonic orbital angular momentum switch

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The large number of available orbital angular momentum (OAM) states of photons provides a unique resource for many important applications in quantum information and optical communications. However, conventional OAM switching devices usually rely on precise parameter control and are limited by slow switching rate and low efficiency. Here we propose a robust, fast and efficient photonic OAM switch device based on a topological process, where photons are adiabatically pumped to a target OAM state on demand. Such topological OAM pumping can be realized through manipulating photons in a few degenerate main cavities and involves only a limited number of optical elements. A large change of OAM at ~ 10^9 can be realized with only q degenerate main cavities and at most 5q pumping cycles. The topological photonic OAM switch may become a powerful device for broad applications in many different fields.

Introduction.—Discrete degrees of freedom, such as charge, spin, valleys, etc., play a crucial role in many information encoding and device applications [1–4]. In this context, a fundamental degree of freedom of photons, the orbital angular momentum (OAM), possesses a unique property that an infinite number of distinctive OAM states are available [5–7]. This unique property makes photonic OAM very attractive for various applications in optical communication [8–10], quantum simulation [11–13], quantum information [14–16], and quantum cryptography (e.g., key distribution) [17–20]. To fully utilize these applications, a tunable device that can rapidly and robustly switch between different OAM modes on demand is therefore highly desirable. However, many conventional OAM switching devices rely on precise parameter control and are usually limited by slow switching rates (~kHz) [21, 22] or low purity and efficiency [23, 24], and limited number of usable OAM modes [25, 26].

The photonic OAM is a topological degree of freedom that characterizes the winding of the azimuthal phase of photon field. Therefore a nature question is whether a topological process can be designed to create a robust photonic OAM switching device with high performance. Recently, the study of topological photonics has become one frontier direction in optical physics with the major focus on modulating photon propagation through topological edge states [27, 28], while practical topological photonic devices for on-demand switching of photon internal degrees of freedom are still largely lacked.

In this Letter, we propose a practical photonic OAM switching device through the topological adiabatic pumping of OAM states, which is robust (immune to small perturbations in system parameters), fast (~ MHz switching rate), efficient (~ 90% efficiency and ~ 100% purity in principle), and highly tunable (on demand switch for high-OAM states). Topological pumping was initially proposed by Thouless [29] for a periodically time-varying system, and has been recently realized in ultra-cold atom optical lattices [30, 31] and coupled optical waveguides [32, 33], where particles are adiabatically pumped in real space lattices. Such “Thouless pumps” along particle’s internal degrees of freedom have not been well explored, although atomic hyperfine states have been utilized as a synthetic dimension for studying quantum Hall effects [34–36], where available energy levels are very limited.

Here we show how to realize the Thouless pumps in the synthetic OAM space with large numbers of available OAM modes, which provides the basis for engineering a topological photonic OAM switch. Essentially, we propose a scheme to realize a tunable double well OAM lattice by manipulating photons in a single degenerate main cavity, where photons can be adiabatically pumped to target OAM lattice sites (i.e., the OAM states). Thanks to the topological nature of the pumping, precise switch can be realized without involving precise control of the parameters during the pumping cycle. Such robustness of the pumping against perturbations as well as its properties in the presence of photon losses are also considered. Moreover, even though the pumping process is adiabatic, the pumping cycle can be very short. With realistic optical elements, it is possible to achieve a switching rate of ~MHz, which is much faster than commonly used OAM switching devices [21, 22]. Even for a large change of OAM at ~ 10^9, only q degenerate main cavities and at most 5q pumping cycles are needed using a multi-stage setup, leading to exponential speedup of the OAM switch. The proposed topological photonic OAM switch only relies on a simple optical setup with a few degenerate main cavities, therefore it may become a powerful device for broad applications in quantum information and optical communications.

The system.—As shown in Fig. 1(a), our system contains a main degenerate cavity [37], which supports a large number of OAM modes and is coupled with three auxiliary cavities by beam splitters. The beam splitters divert a small portion of the main-cavity photons towards

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Here, we focus our discussion on amplitude, whose periods are determined by other, is used to generate the modulator (EOM) [41]: the beam rotator, realized by PM contains a beam rotator [40] and an electro-optic the corresponding tunneling amplitude and phase. The lattice model) [43] with neighbor-site energy detuning ±\( \phi_s \) with respect to each s-th auxiliary cavity, and merge it back after passing through spatial light modulators (SLMs) [38, 39], which change the OAM state by \( \pm M_s \). This corresponds to tunnelings between different OAM states in the main cavity, with a tunneling rate determined by the reflectivities of the BSs. During the tunneling, the photon can also acquire a phase determined by the optical path-length difference between two arms of the auxiliary-cavity, which can be generated and tuned using high-speed phase modulators (PMs). The system is equivalent to a one-dimensional lattice model with the lattice sites represented by the OAM states [13]. The Hamiltonian is given by

\[
H = -\sum_l \sum_{s=0}^{2} J_s e^{i \phi_s} a_{l+M_s} a_l + h.c.,
\]  

where \( a_l \) is the annihilation operator of the cavity photon in the OAM state \( l \), and \( M_s \) is the step index of the SLMs in the s-th auxiliary cavity with \( J_s \) and \( \phi_s = \alpha_s + \beta_s \) the corresponding tunneling amplitude and phase. The PM contains a beam rotator [40] and an electro-optic modulator (EOM) [41]: the beam rotator, realized by two Dove prisms rotated by \( \alpha_s/2 \) with respect to each other, is used to generate the l-dependent phase \( \alpha_s \), and the EOM is used to tune the l-independent phase \( \beta_s \).

The auxiliary cavities are designed as \( \phi_2 = 0, M_0 = 0, M_1 = 2, J_1 = 2J_2 \), as shown in Fig. 1(c). Therefore the Hamiltonian describes a generalized AAH model [42] with modulations in on-site energy, tunneling phase and amplitude, whose periods are determined by \( \alpha_0 \) and \( \alpha_1 \). Here, we focus our discussion on \( \alpha_0 = \alpha_1 = 2\pi \cdot 1/2 \) in analogy to the Rice-Mele model (i.e., double-well superlattice model) [43] with neighbor-site energy detuning \( \Delta = -4J_0 \cos(\beta_0) \) and inter- and intra-cell tunnelings \( J_\pm \equiv J_1 [1 \pm e^{i \beta_1}] \) [see Fig. 2(a)]. These parameters are time dependent through the time-varying phases \( \beta_0(t) \) and \( \beta_1(t) \) [see Fig. 2(b)]. In the Bloch basis we obtain the two-band Hamiltonian \( H(k, t) = \hbar(k, t) \cdot \sigma, \) with \( h_x(k, t) + ih_y(k, t) = -[J_+ (t) + J_- (t) e^{-i k}], \) and \( h_z(k, t) = \frac{\Delta(t)}{2} \). If the parameters are modulated adiabatically and periodically, the amount of transported particles along the OAM space is quantized, and characterized by the Chern number which is defined as the change in polarization during one pump cycle (i.e., the center-of-mass displacement of the Wannier function) [29]. It can also be proven that the Chern number equals to the winding number of \( \pm \hbar(k, t) \) surrounding the origin as \( t \) varies over one period and \( k \) varies over the Brillouin zone.

The quantized transport offers a robust way to switch the OAM states of the photonic signals in three steps: (i) Input—the input photon pulse enters the cavity and the \( l_0 \) cavity-mode is excited; (ii) Topological pumping— the photon is pumped to the desired OAM state; (iii) Output— photon pulse is released out of the cavity. For the input and output, we use a tunable beam splitter (TBS) to couple the cavity with the outside world. This is crucial for improving the efficiency of the OAM switch, because the coupling between the cavity and the outside world need be turned on during input/output so that the signals can get in/out, and turned off during the pumping to avoid unwanted photon losses. The TBS is realized by sandwiching an EOM between two polarizing beam splitters (PBSs), as shown in Fig. 1(b). The EOM rotates photon’s polarizations in a tunable way as \( |H \rangle \rightarrow \sqrt{1 - r_p^2} |H\rangle + r_p |V\rangle \) and \( |V\rangle \rightarrow \sqrt{1 - r_p^2} |V\rangle - r_p |H\rangle \), with \( |H\rangle \) and \( |V\rangle \) the horizontal and vertical polarization states which will be separated by the PBS. The tunable coefficient \( r_p \) acts as the reflectivity of the TBS.

**Topological pumping.**—We start by considering the pumping process in step (ii), as shown in Fig. 2(c), where the pumping cycle corresponds to a loop in the 2D parameter space spanned by \( \beta_0 \) and \( \beta_1 \). The two bands of the system are gapped in the parameter space except at the critical point \( \beta_0 = \beta_1 = \pi/2 \) with \( \Delta = \delta J = 0 \) and \( \delta J \equiv |J_+| - |J_-| \). A loop which encloses this critical point is topological non-trivial, and the topology of the pump is invariant under deformation of the loop without cutting through the critical point. Therefore it is more convenient to consider the pump in the \( \Delta - \delta J \) plane, with the corresponding pump loop and photon movement illustrated in Fig. 2(d).

The band structures \( E_{\pm} = \pm |\hbar(k, t)| \) along the pumping loop are shown in Fig. 3(a), with a smallest bandgap \( 4J_1 \). The two gapped bands have different transport properties due to their different topologies, characterized by Chern number \( C = \pm 1 \) respectively [44]. We start the pumping at \( \delta J = 2J_1 \) and \( \Delta = -4J_0 \) with

\[
\Delta = -4J_0 \cos(\beta_0) \text{ and } \delta J = 0.
\]
Each unit cell (enclosed by the dashed square) contains two sites with detuning \( \Delta = -4J_0 \cos[\beta_0(t)] \). \( J_{\pm} = J_1[1 \pm e^{i\beta_1(t)}] \) are the inter- and intra-cell tunnelings respectively, whose dependence on the tunneling phase \( \beta_1 \) is shown in (b). The pumping loop in the \( \beta_0-\beta_1 \) plane. Since the tunneling phase has a period of \( 2\pi \), their choice is not unique, and the two loops (red dashed and blue solid) have the same topology. Illustration of the pumping process. The loop in the \( \delta J-\Delta \) plane can be realized by either loop shown in (c). Red solid circle represents a site occupied by a photon, which moves to the right by two sites (one unit cell) during one pumping cycle.

For on-demand switch of the OAM states of photonic signal pulses, the cavity needs to be coupled with the input/output channel. The dynamics are characterized by [44, 45]

\[
\dot{a}_1 = \frac{1}{i} [a_1, H] - \frac{\kappa}{2} a_1 + \delta_{l,0} \sqrt{\kappa_c} \mathcal{E}_{in}(t),
\]

where \( \mathcal{E}_{in}(t) \) is the input photonic field in \( l_0 \)-OAM state, \( \kappa \) is the total photon loss, and the tunable coupling strength between the cavity and the input/output fields is \( \sqrt{\kappa_c} \simeq |r_p| \sqrt{\frac{2\kappa}{\omega_0}} \) [46] with \( r_p^2 \ll 1 \) and \( \Omega_0 \) the free spectral range (FSR) of the main cavity.

In an ideal case, all optical elements are perfectly designed, and the only photon loss channel is the TBS, so we have \( \kappa = \kappa_c \). The TBS is tuned such that the
FIG. 4: (a) Normalized cavity photon distribution and (b) Input/output photon pulses during the OAM switching. The normalized input pulse is $E_{\text{in}}(t) = e^{-iE_{\text{in}}t - 0.2J_1 t^5}$ with $b_0 = 0$ (i.e., only the zero-OAM Wannier orbital is excited). $E_{\text{in}}$ is the lower band energy. After the input pulse enters the cavity, we pump the state to higher OAM modes, and then release the signal by increasing the cavity loss $\kappa$ [see the inset in (a)]. Other parameters are the same as in Fig. 3(b).

range of our system is $l \in [-l_{\max}, l_{\max}]$. To obtain high OAM states, large numbers of pump cycles may be required, which slow down the switching rate. To accelerate the switching rate, we consider a multistage setup with several cascaded degenerate-cavity systems as shown in Fig. 5. In the $n$-th stage, we choose the tunneling step of the SLMs as $M_1 = M_2 = N^n$ with $N$ an integer, and $\alpha_0 = \alpha_1$ satisfying $\text{mod}(\alpha_0 N^n, 2\pi) = \pi$. For an arbitrary switching distance $\Delta l = \sum_n c_n N^n$ (with $c_n$ the expansion coefficients), the corresponding pump pulse of the $n$-th stage is $c_n/2$, which gives the total switching time $T \sum_n c_n/2$. For example, consider $\Delta l = 512$ and $N = 10$, the total switching time is only $4T$. Both the maximum number of stages $\leq \log N l_{\max}$ and the maximum switching time $\leq T N^{\log N} l_{\max}$ are logarithmic, yielding exponential speedup for the OAM switch.

Discussion and Conclusion.—Conventional spatial light modulator [21] and digital micro-mirror device [22] are limited by the switching rates of $\sim$kHz. Higher switching rates can be achieved by combing acousto-optic (electro-optic) modulator with SLMs (q-plate) [23, 26], or using on-chip resonators [24]. However, the acousto-optic modulator would induce unwanted change in wavelength, and the on-chip switching has a very low efficiency. Moreover, all these approaches require precise control of experimental parameters and the number of usable OAM modes is usually very limited. In contrast, our scheme is robust against perturbations due to its topological feature, and also able to rapidly switch to high OAM modes with high efficiency. Our results of single-photon pumping can be generalized to multi-photon states or even classic coherent states. Since the system is linear with no interaction, every photon is pumped independently.

In summary, we proposed a topological photonic OAM switch which is fast, robust, efficient and accessible to exponentially large OAM states. The proposed topological pumping in the OAM-based synthetic dimension offers a powerful platform to study 1D topological physics of the generalized AAH model (e.g., the effects of disorder with $\alpha_0$ and $\alpha_1$ incommensurate, or long-range tunneling with $M_\alpha > 1$). The simple optical setup for the topological photonic OAM switch opens a wide range of experimental opportunities and may find important applications in quantum information processing and optical communications.

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Supplementary Materials

The band topology

Our system shown in Fig. 1 in the main text is equivalent to a 1D lattice, with lattice sites represented by the OAM states. Using three properly designed auxiliary cavities, we obtain a super-lattice Hamiltonian. For a double-well lattice we have

\[
H = -\sum_j J_+ b_{j,2}^\dagger b_{j,1} + J_- b_{j+1,1}^\dagger b_{j,2} + h.c. \\
+ \sum_j \frac{\Delta}{2} (b_{j,1}^\dagger b_{j,1} - b_{j,2}^\dagger b_{j,2}).
\]  

(S1)

We have introduced the unit cell index \(j\) with \(b_{j,1} = a_{2j}\) and \(b_{j,2} = a_{2j+1}\). After a Fourier transformation \(b_{k,1(2)} = \sum_j e^{ikj} b_{j,1(2)}\), the Hamiltonian becomes

\[
H = -\sum_k \begin{bmatrix} \Delta e^{-ik} & -J_+ - J_- e^{ik} \\ -J_+ - J_- e^{ik} & \Delta e^{-ik} \end{bmatrix} H_k \begin{bmatrix} b_{k,1} \\ b_{k,2} \end{bmatrix}^T.
\]  

(S2)

with the Bloch Hamiltonian

\[
H_k = \begin{bmatrix} -J_+ - J_- e^{ik} & -J_+ - J_- e^{ik} \\ -J_+ - J_- e^{ik} & -J_+ - J_- e^{ik} \end{bmatrix}.
\]  

(S3)

The band structure can be obtained by solving \(H_k u_n(k) = E_n u_n(k)\), with band spectrum \(E_\pm = \pm \sqrt{\Delta^2 + |J_+|^2 + |J_-|^2}\) and the Bloch wave function \(e^{ikj} u_n(k)\).

The parameters \(\Delta = -4J_0 \cos(\beta_0)\) and \(J_\pm = J_1 (1 \pm e^{i\theta})\) depend on the time-varying phases \(\beta_{0,1}(t)\), so does the Hamiltonian \(H(t)\). If the phases are modulated adiabatically and periodically, the amount of transported particles for a filled band is characterized by the Chern number

\[
\mathcal{C}_n = \frac{1}{2\pi} \int_0^T dt \int_0^{2\pi} d\Omega_{kt}
\]

(S4)

with

\[
\Omega_{kt} = \langle \partial_t u_n(k,t) \partial_k u_n(k,t) \rangle - \langle \partial_k u_n(k,t) \partial_t u_n(k,t) \rangle.
\]

The two band is gapped in the parameter space except the critical point when \(\Delta = |J_+| - |J_-| = 0\). It can be proven that a loop which encloses the critical point is topological non-trivial. For a clockwise loop enclosed the critical point,
we find that \( C_{\pm} = \mp 1 \). This can also be seen by writing \( H_k(t) \) as
\[
H_k(t) = h_x(k,t) \sigma_x + h_y(k,t) \sigma_y + h_z(k,t) \sigma_z,
\]
with \( h_x(k,t) = -\text{Re}[J^*_+ + J_- e^{ik}] \), \( h_y(k,t) = \text{Im}[J^*_+ + J_- e^{ik}] \) and \( h_z(k,t) = \frac{i}{2} \). The Chern number is equal to the winding number of \( \pm h(k,t) \) surrounding the origin as \( t \) varies over one period and \( k \) varies over the Brillouin zone.

### Single particle pumping

The pumping for a single particle is slightly different from the pumping of a filled band. The modulation is adiabatic so that the state will follow its initial band, and \( k \) is a good quantum number during the pumping. Considering an arbitrary single particle initial state in the \( n \)-th band
\[
|\Psi(j,0)\rangle = \sum_k \psi_k e^{ijk} |u_n(k,0)\rangle.
\]
The final state can be written as
\[
|\Psi(j,t)\rangle = \sum_k \psi_k e^{ijk} e^{-i \int_0^t dt' E_n(k,t')} e^{i \gamma_n(k,t)} |u_n(k,t)\rangle,
\]
with \( \gamma_n(t) \) the Berry phase given by
\[
\gamma_n(k,t) = i \int_0^t \langle u_n(k,t') | \partial_{t'} u_n(k,t') \rangle dt'.
\]
The center of mass of the particle is
\[
\bar{j}(t) = \sum_j \langle \Psi(j,t) | j | \Psi(j,t) \rangle = \sum_k \langle \bar{\Psi}(k,t) | i \partial_k | \bar{\Psi}(k,t) \rangle
\]
with
\[
|\bar{\Psi}(k,t)\rangle = \psi_k e^{-i \int_0^t dt' E_n(k,t')} e^{i \gamma_n(k,t)} |u_n(k,t)\rangle.
\]
So we have
\[
\bar{j}(t) = \sum_k i \psi_k \partial_k \psi_k + \int_0^t I(t') dt'
\]
with the average current given by
\[
I(t) = \sum_k |\psi_k|^2 [\partial_k E(k,t) + \Omega_{kt}].
\]
So the displacement after one pumping cycle would be
\[
\Delta \bar{j} = \bar{j}(T) - \bar{j}(0) = \int_0^T I(t) dt.
\]

For the simple case with \( \psi_k = \frac{1}{\sqrt{N}} \), \( N \) is the total number of lattice sites, \( |\Psi(j,0)\rangle \) is reduced to the Wannier function \( |W_n(j,0)\rangle \),
\[
|\Psi(j,0)\rangle = \sum_k \frac{1}{\sqrt{N}} e^{ijk} |u_n(k,0)\rangle \equiv |W_n(j,0)\rangle,
\]
\[\text{(S12)}\]
and we have

$$\Delta j = \frac{1}{N} \sum_k \int_0^T \Omega_{kl} dt = \int_0^T dt \int_{-\pi}^{2\pi} \frac{dk}{2\pi} \Omega_{kl} = C_n.$$  \hspace{1cm} (S13)$$

We can see that the average displacement is exactly quantized even for a single particle state. However, the final state is not a simple displacement of the initial state because there exists diffusion during the pumping, which can be seen from the dynamical phase \( e^{-i \int_0^T dt E_n(k,t)} \) in the final state. The diffusion effects can be reduced if the band is flat in momentum space, thus the dynamical phase becomes a constant phase independent of \( k \).

For the pumping process considered in this paper, we start at \( \beta_{0,1} = 0 \), that is \( \Delta = -4J_0, J_+ = 2J_1 \) and \( J_- = 0 \), so we obtain two flat bands with \( E_\pm = \pm \sqrt{4J_0^2 + 4J_1^2} \), and the eigenvectors \( |u_- (k,0)\rangle = |\cos(\theta), -\sin(\theta)\rangle \) and \( |u_+ (k,0)\rangle = |\sin(\theta), \cos(\theta)\rangle \), with \( \tan(2\theta) = J_1/J_0 \) independent of \( k \). The Wannier functions of the lower band is \( |W_- (0)\rangle = \cos(\theta)|l = 0\rangle - \sin(\theta)|l = 1\rangle \), which is well localized on \( l = 0 \) for \( J_1/J_0 \ll 1 \). For the initial state \( |l = 0\rangle \), the pumping is characterized by the lower band which gives a quantized transport of \( C_- = 1 \).

We first modulate \( \beta_0 \) from 0 to \( \pi \) during which the bands are always flat, and \( \Delta \) changes from \(-4J_0 \) to \( 4J_0 \). Then we modulate \( \beta_1 \) from 0 to \( \pi \), such that \( J_+ \) changes from \( 2J_1 \) to 0 while \( J_- \) changes from 0 to \( 2J_1 \). During this modulation, the bands become

$$E_\pm = \pm \sqrt{4J_0^2 + J_1^2} e^{i\beta_1}$$

$$\simeq \pm \left\{ 2J_0 + \frac{J_1^2}{J_0} + \frac{J_1^2}{4J_0^2} \left[ \cos(k) - \cos(k + 2\beta_1) \right] \right\}.$$ 

which are no longer flat and such band dispersion would lead to diffusion. In the limit \( J_1/J_0 \ll 1 \), the bands are approximately flat, and the diffusion effect is negligible. The band gap is also very large during this modulation, and the diffusion effect can be reduced further by increasing the modulating speed properly. Finally we modulate \( \beta_0 \) back to 0, followed by modulation of \( \beta_1 \) back to 0. The loop encloses the critical point in a clockwise manner, thus corresponds to a quantized particle transport of +1.

**Effects of photon losses**

Typically, the optical path length of the cavity is about tens of centimeters which leads to a FSR \( \Omega_F = 2\pi c/L \sim 2\pi \times 1\text{GHz} \), where \( L \) is the optical path length of the cavity and \( c \) the speed of light. The reflectivity \( r_F \) of the TBS can be tuned by the EOM. If the tuning rate is much smaller than the FSR \( \Omega_F \), its effect is well characterized by a time-dependent coupling \( \sqrt{\kappa_o(t)} \simeq |r_F(t)| \sqrt{\frac{2\pi}{2\pi}} \), where \( \kappa_o \) can be tuned from 0 to a few MHz within a few \( \mu s \) and vice versa. In realistic experiments, there are other losses due to factors such as the finite Q-factor of the cavity, the intrinsic loss of the SLMs and phase modulators. Such photon loss can be made as low as tens of kHz (much smaller than the switching rate) using high performance optical elements. For extremely high OAM states, the beam size becomes comparable with the aperture of the optical elements, which gives a upper limit that the OAM can be
switched to. We consider the pumping between OAM states smaller than the upper limit, the dynamics of the lossy system is characterized by the master equation

$$ \dot{\rho} = \frac{1}{i}[H, \rho] + \kappa_0 \sum_l (a_l \rho a_l^\dagger - \frac{1}{2} \rho a_l^\dagger a_l - \frac{1}{2} a_l^\dagger a_l \rho) $$

(S14)

with photon loss rate $\kappa_0$. For the pumping of a single photon state, the solution is simply given by

$$ \rho = (1 - e^{-\kappa_0 t})|\text{vac}\rangle\langle\text{vac}| + e^{-\kappa_0 t} |\Psi(t)\rangle\langle\Psi(t)|, $$

(S15)

with $|\Psi(t)\rangle$ the solution of non-dissipative case. The photon density distribution

$$ N(l, t) = \frac{\text{Tr}[\rho(t)a_l^\dagger a_l]}{\sum_l \text{Tr}[\rho(t)a_l^\dagger a_l]} = \langle \Psi(t)|a_l^\dagger a_l|\Psi(t)\rangle $$

(S16)

is the same as the non-dissipative case, except that the probability to find the photon inside the cavity is reduced to $e^{-\kappa_0 t_p}$ with $t_p$ the total pumping time. Our system is linear with no interactions, thus the results of single photon pumping can also be generalized to multi-photon state and even classic coherent states.

When the cavity is coupled with the input/output signal, the dynamics are characterized by the Langevin equation

$$ \dot{a}_l = \frac{1}{i} [a_l, H] - \frac{\kappa}{2} a_l + \delta_{l, l_0} \sqrt{\kappa e} \mathcal{E}_{\text{in}}(t), $$

(S17)

with $\kappa = \kappa_0 + \kappa_c$ and $\mathcal{E}_{\text{in}}(t)$ the input optical field in $l_0$-OAM state which can be either a single-photon pulse or a classic coherent pulse. The dynamics of both single-photon and coherent input signal are described by Eq. (S17), with $a_l$ being the coherent (single-photon) amplitude of OAM state $l$. Typically, the intrinsic photon loss $\kappa_0$ is of the order of tens of kHz (which can be made even smaller by improving the performance of the optical elements), and it only reduces the switching efficiency without affecting the quantized transport (even for a strong loss), as shown in Fig. S1 with a large $\kappa_0$ ($\sim 2\pi \times 100$kHz).