Singular hyperbolicity of $C^1$ generic three dimensional vector fields

Manseob Lee
Department of Marketing Big Data and Mathematics, Mokwon University, Daejeon, 302-729, Korea.

E-mail: lmsds@mokwon.ac.kr.

Abstract

In the paper, we show that for a generic $C^1$ vector field $X$ on a closed three dimensional manifold $M$, any isolated transitive set of $X$ is singular hyperbolic. It is a partial answer of the conjecture in [13].

1 Introduction

The transitivity is a symbol of chaotic property for differential dynamical systems. The $C^1$ robust transitivity for diffeomorphisms are well investigate in a series of works [2, 3, 5], and then we have a good characterization on isolated transitive sets of $C^1$ generic diffeomorphisms at the same time. From the main result of [1] we know that if every isolated transitive set of a $C^1$ generic diffeomorphism admit a nontrivial dominated splitting, then it is volume hyperbolic.

It is well known that a singularity-free flow, for an instance, a suspension of a diffeomorphism, will take similar phenomenons of diffeomorphisms. However, once the recurrent regular points can accumulates a singularity, such as the Lorenz-like systems, we will meet something new. For instance, in [14], one have to use a new notion of singular hyperbolicity to characterize the robustly transitive sets of a 3-dimensional flow. Here the singular hyperbolicity is a generalization of hyperbolicity so that we can give the Lorenz attractor and Smale’s horseshoe a unified characterization. In this article, we will show that an isolated transitive of $C^1$ generic vector field on 3-dimensional manifolds will be singular hyperbolic. That means, every isolated transitive set of a $C^1$ generic vector field looks like a Lorenz attractor[6, 9].

Let us be precise now. Denote by $M$ a compact $d(\geq 2)$-dimensional smooth Riemannian manifold without boundary and by $\mathcal{X}^1(M)$ the set of $C^1$ vector fields on $M$ endowed

2010 Mathematical Subject Classification: 37C20; 37C27; 37D50.

Key words and phrases: transitive set; generic; local star; Lyapunov stable; singular hyperbolic.
with the $C^1$ topology. Every $X \in \mathcal{X}(M)$ generates a flow $X^t : M \times \mathbb{R} \to M$ that is a $C^1$ map such that $X^t : M \to M$ is a diffeomorphism for all $t \in \mathbb{R}$ and then $X^t(0) = x$ and $X^{t+s}(x) = X^t(X^s(x))$ for all $s, t \in \mathbb{R}$ and $x \in M$. An orbit of $X$ corresponding a point $x \in M$ is the set $\text{Orb}(x) = \{X^t(x) : t \in \mathbb{R}\}$. A point $x \in M$ is called singular if $X^t(x) = x$ for all $t \in \mathbb{R}$, and $p \in M$ is called periodic if $X^T = p$ for some $T > 0$. Let $\text{Sing}(X)$ denotes the set of singularities of $X$ and $\text{Per}(X)$ is the set of periodic orbits of $X$. Denote by $\text{Crit}(X) = \text{Sing}(X) \cup \text{Per}(X)$ the set of all critical points of $X$.

Let $\Lambda \subset M$ be a closed $X^t$-invariant set. We say that $\Lambda$ is a hyperbolic set of $X$ if there are constants $C > 0, \lambda > 0$ and a $DX^t$-invariant continuous splitting $T_\Lambda M = E^s \oplus (X) \oplus E^u$ such that

$$\|DX^t|_{E^s}\| \leq Ce^{-\lambda t} \quad \text{and} \quad \|DX^{-t}|_{E^u}\| \leq Ce^{-\lambda t}$$

for $t > 0$ and $x \in \Lambda$, where $<X(x)>$ denotes the space spanned by $X(x)$, which is 0-dimensional if $x$ is a singularity or 1-dimensional if $x$ is not singularities. For any critical point $x \in \text{Crit}(X)$, if its orbit is a hyperbolic set, we denote by $\text{index}(x) = \dim E^s_x$.

Now let us recall the singular hyperbolicity firstly given by Morale, Pacifico and Pujals [14] which is an extension of hyperbolicity. We say that a compact invariant set $\Lambda$ is positively singular hyperbolic for $X$ if there are constants $K \geq 1$ and $\lambda > 0$, and a continuous invariant $T_\Lambda M = E^s \oplus E^c$ with respect to $DX^t$ such that

(i) $E^s$ is $(K, \lambda)$-dominated by $E^c$, that is,

$$\|DX^t|_{E^s}\| \cdot \|DX^{-t}|_{E^c(X^t(x))}\| \leq Ke^{-\lambda t}, \quad \forall x \in \Lambda \text{ and } t \geq 0.$$

(ii) $E^s$ is contracting, that is,

$$\|DX^t|_{E^s(x)}\| \leq Ke^{-\lambda t}, \quad \forall x \in \Lambda \text{ and } t \geq 0.$$

(iii) $E^c$ is sectional expanding, that is, for any $x \in \Lambda$ and any 2-dimensional subspace $L \subset E^c(x)$,

$$|\det(DX^t|_L)| \geq K^{-1}e^{\lambda t}, \quad \forall t \geq 0.$$

We say that $\Lambda$ is negatively singular hyperbolic for $X$ if $\Lambda$ is positively singular hyperbolic for $-X$, and then say that $\Lambda$ is singular hyperbolic for $X$ if it is either positively singular hyperbolic for $X$, or negatively singular hyperbolic for $X$. Definitely, we can see that if $\Lambda$ is singular hyperbolic for $X$ and it does not contain singularities then it is hyperbolic (see [14] Proposition 1.8 for a proof). In the paper, we consider the relation between transitivity and hyperbolicity for an isolated compact invariant set. We say that $\Lambda$ is transitive if there is $x \in \Lambda$ such that $\omega(x) = \Lambda$, where $\omega(x)$ is the omega limit set of $x$. We say that a closed $X^t$-invariant set $\Lambda$ is isolated (or locally maximal ) if there exists a neighborhood $U$ of $\Lambda$ such that

$$\Lambda = \Lambda_X(U) = \bigcap_{t \in \mathbb{R}} X^t(U).$$
Here $U$ is said to be isolated neighborhood of $\Lambda$.

For the 3-dimensional case, Morales, Pacifico and Pujals \[14\] proved that if $\Lambda$ is a robustly transitive set containing singularities then it is singular hyperbolic set for $X$. Here we will consider $C^1$ generic vector fields. We say that a subset $G \subset X^1(M)$ is residual if it contains a countable intersection of open and dense subsets of $X^1(M)$. A property is called $C^1$ generic if it holds in a residual subset of $X^1(M)$.

We give the following characterization of the isolated transitive sets of a $C^1$ generic vector field on 3-dimensional Riemannian manifold.

**Theorem A.** For $C^1$ generic $X \in X^1(M)$, an isolated transitive set $\Lambda$ is singular hyperbolic.

## 2 Transitivity and locally Star condition

Let $M$ be a three dimensional smooth Riemannian manifold and let $X \in X^1(M)$ be the set of $C^1$ vector fields on $M$ endowed with the $C^1$ topology. Here we collect some known generic properties for $C^1$ vector fields.

**Proposition 2.1** There is a residual set $G_1 \subset X^1(M)$ such that for any $X \in G_1$, $X$ satisfies the following properties:

1. $X$ is a Kupka-Smale system, that is, every periodic orbits and singularity of $X$ is hyperbolic, and the corresponding invariant manifolds intersect transversely.

2. if there is a sequence of vector fields $\{X_n\}$ with critical orbit $\{P_n\}$ of $X_n$ such that $X_n \to X$, index($P_n$) = $i$ and $P_n \to H \Lambda$ then there is a sequence of critical orbit $\{Q_n\}$ of $X$ such that index($Q_n$) = $i$ and $Q_n \to H \Lambda$, where $\to H$ is the Hausdorff limit.

The item 1 is from the famous Kupka-Smale theorem (see [15]) and item 2 is from [16, Lemma 3.5]

From item 1 of Proposition \[2.1\] we can see that if $\Lambda$ is a trivial transitive set, that is, $\Lambda$ is a periodic orbit or a singularity, then it should be hyperbolic and automatically singular hyperbolic. To prove Theorem A, we just need to consider the nontrivial case. Hereafter, we assume that $\Lambda$ is a nontrivial transitive set of $X$. One can see that if $\Lambda$ is a nontrivial transitive set, then $\Lambda$ contains no hyperbolic sinks or sources.

Let $U$ be an isolated neighborhood of $\Lambda$. Then for $Y C^1$ close to $X$, denote by

$$
\Lambda_Y(U) = \bigcap_{t \in \mathbb{R}} Y^t(U)
$$

the maximal invariant set of $Y$ in $U$.

**Lemma 2.2** Let $G_1 \subset X^1(M)$ be the residual set given in Proposition \[2.1\]. For any $X \in G_1$, if $\Lambda$ is an isolated nontrivial transitive set of $X$, then there are a $C^1$ neighborhood $U(X)$ of $X$ and a neighborhood $U$ of $\Lambda$ such that for any $Y \in U(X)$, we have every $\gamma \in \Lambda_Y(U) \cap \text{Per}(Y)$ is hyperbolic and index($\gamma$) = 1.
Proof. Let $G_1$ be the residual set in Proposition 2.1 and let $\Lambda$ be an isolated transitive set of $X \in G_1$. Arguing by contradiction, we assume that for any $C^1$ neighborhood $U(X)$ of $X$ and any neighborhood $U$ of $\Lambda$, there is $Y \in U(X)$ such that $Y$ has a periodic orbit $Q$ whose index is not 1. Then we have three cases: (i) $Q$ is not hyperbolic, (ii) $Q$ is hyperbolic but $\text{index}(Q) = 0$ or (iii) $\text{index}(Q) = 2$. Note that if the periodic orbit $Q$ is not hyperbolic for $Y$ then we can take a vector field $Z \in C^1$ arbitrary close to $Y$ such that either $Q$ is a sink for $Z$ or $Q$ is a source for $Z$. Then we also have the case cases (ii) or (iii) happening. Thus we can take sequences $Y_n \to X$ and a periodic orbit $P_n$ of $Y_n$ such that $\lim_{n \to \infty} P_n = \Gamma \subset \Lambda$. Without loss of generality, we can assume that all $Q_n$ have the same index 0 or 2 once we take a subsequence. By the item 2 of Proposition 2.1, we know that there is a sequence $P_n$ of periodic orbit of $X$ with index 0 or 2 converging into $\Lambda$. Since $\Lambda$ is isolated, for sufficiently large $n$, we have $P_n \subset \Lambda$. This is a contradiction since $\Lambda$ is a nontrivial transitive set.

Let $\Lambda$ be a closed $X^t$-invariant set. We say $\Lambda$ is locally star if there are a $C^1$ neighborhood $U(X)$ of $X \in X^1(M)$ and a neighborhood $U$ of $\Lambda$ such that for any $Y \in U(X)$, every periodic orbit of $Y$ in $\Lambda_Y(U) = \bigcap_{t \in \mathbb{R}} Y^t(U)$ is hyperbolic and has same indices.

Corollary 2.3 There is a residual set $R \subset X^1(M)$ such that for any $X \in R$, if $\Lambda$ is an isolated transitive set of $X$ which is not an orbit then $\Lambda$ is a local star.

Proof. Let $X \in R = G_1$ and let $\Lambda$ be an isolated transitive set. By Lemma 2.2, there are a $C^1$ neighborhood $U(X)$ of $X$ and a neighborhood $U$ of $\Lambda$ such that for any $Y \in U(X)$, every periodic orbit $\gamma \in \Lambda_Y(U) \cap \text{Per}(Y)$ is hyperbolic and $\text{index}(\gamma) = 1$. Thus $\Lambda$ is a local star.

3 Transitivity and Lyapunov stability

Suppose $\sigma \in \text{Sing}(X)$ is hyperbolic. Then we denote by

$W^s(\sigma) = W^s(\sigma, X) = \{ y \in M : d(X^t(\sigma), X^t(y)) \to 0 \text{ as } t \to \infty \}$

$W^u(\sigma) = W^u(\sigma, X) = \{ y \in M : d(X^t(\sigma), X^t(y)) \to 0 \text{ as } t \to -\infty \}$,

where $W^s(\sigma, X)$ is said to be the stable manifold of $\sigma$ and $W^u(\sigma, X)$ is said to be the unstable manifold of $\sigma$. It is known that $\text{index}(\sigma) = \dim W^s(\sigma)$.
If $X$ is a Kupka-Smale vector field, then $X$ contains finitely many singularities and every singularity is hyperbolic. Thus by the structurally stability of hyperbolic singularity we know that there are a $C^1$ neighborhood $\mathcal{U}(X)$ of $X$ and a neighborhood $U$ of $\Lambda$ such that for any $Y \in \mathcal{U}(X)$, every $\sigma \in \Lambda_Y(U) \cap \text{Sing}(Y) \subset U$ is hyperbolic.

**Lemma 3.3** Let $\mathcal{G}_1 \subset \mathcal{X}^1(M)$ be the residual set given in Proposition 2.4. For any $X \in \mathcal{G}_1$, if $\Lambda$ is an isolated nontrivial transitive set of $X$, then there are a $C^1$ neighborhood $\mathcal{U}(X)$ of $X$ and a neighborhood $U$ of $\Lambda$ such that for any $Y \in \mathcal{U}(X)$, every singularities in $\Lambda_Y(U)$ is saddles.

**Proof.** We prove it by contradiction. Assume the contrary of the lemma, then we can find a sequence of vector fields $X_n$ tends to $X$ and a sequence of singularity $\sigma_n$ of $X_n$ such that $\sigma_n$ tends to a point $\sigma$ such that the index of $\sigma_n$ equals to 0 or 2. Without loss of generality, we assume that every $\sigma_n$ has index 0, then we can see that $\sigma$ is a singularity. Since $X \in \mathcal{G}_1$, we have $\sigma$ is hyperbolic. By the structurally stability of $\sigma$ we know $\sigma$ have index 0 too. This contradicts with $\Lambda$ is a nontrivial transitive set. □

**Lemma 3.2** Let $\Lambda$ be a transitive set of a $C^1$ vector field $X$. If $\sigma \in \Lambda \cap \text{Sing}(X)$ is hyperbolic then $(W^s(\sigma) \setminus \{\sigma\}) \cap \Lambda \neq \emptyset$ and $(W^u(\sigma) \setminus \{\sigma\}) \cap \Lambda \neq \emptyset$.

**Proof.** We consider the case of $(W^s(\sigma) \setminus \{\sigma\}) \cap \Lambda \neq \emptyset$ (Other case is similar). Since $\sigma \in \Lambda = \omega(x)$ for some $x \in \Lambda$, there is $t_n \in \mathbb{R}^+$ with $t_n \to \infty$ such that $X^{t_n}(x) \to \sigma$. Since $\sigma$ is hyperbolic, we can take $\epsilon > 0$ such that

$$\{x : X^t(x) \in B_\epsilon(\sigma), \text{ for all } t > 0\} \subset W^s(\sigma).$$

Denote by $x_n = X^{t_n}(x)$. For $n$ large enough, $x_n \in B_\epsilon(\sigma)$. Let $\tau_n = \sup\{t : X^{(t,0)}(x_n) \subset B_\epsilon(\sigma)\}$. Then we have $X^{-\tau_n}(x_n) \in \partial B_\epsilon(\sigma)$. Let $y_n = X^{-\tau_n}(x_n)$. We can see that $\tau_n \to +\infty$ as $n \to \infty$. Take a subsequence if necessary, we can assume that $y_n \to y$ as $n \to \infty$. It is easy to see that $y \neq \sigma$. For every $y_n$, we have $X^{(0,\tau_n)}(y_n) \in \partial B_\epsilon(\sigma)$. By the continuity of the flow $X^t$, we have $X^{(0,+\infty)}(y) \subset B_\epsilon(\sigma)$, then $y \in W^s(\sigma) \setminus \{\sigma\}$. □

The following is the connecting lemma for $C^1$ vector fields.

**Lemma 3.3** Let $X \in \mathcal{X}^1(M)$ and $z \in M$ be neither periodic nor singular of $X$. For any $C^1$ neighborhood $\mathcal{U}(X) \subset \mathcal{X}^1(M)$ of $X$, there exist three numbers $\rho > 1, L > 1$ and $\delta_0 > 0$ such that for any $0 < \delta \leq \delta_0$ and any two points $x, y$ outside the tube $\Delta = B_{\delta}(X^{[0,L]}(z))$ (or $\Delta = B_{\delta}(X^{-[L,0]}(z))$), if the positive $X$-orbit of $x$ and the negative $X$-orbit of $y$ both hit $B_{\delta/\rho}(z)$, then there exists $Y \in \mathcal{U}(X)$ with $Y = X$ outside $\Delta$ such that $y$ is on the positive $Y$-orbit of $x$.

**Lemma 3.4** Let $\Lambda$ be a transitive set for $X$ and $\sigma \in \Lambda \cap \text{Sing}(X)$ be hyperbolic. Then for any $C^1$ neighborhood $\mathcal{U}(X)$ of $X$, any point $y \in \Lambda$ and any neighborhood $U$ of $y$, there is $Y \in \mathcal{U}(X)$ such that $W^s(\sigma, Y) \cap U \neq \emptyset$, where $W^s(\sigma, Y)$ is the stable manifold of $\sigma$ with respect to $Y.
Proof. Let $U(X)$ be fixed. By Lemma 3.2, there is a point $x \in (W^s(\sigma) \setminus \{\sigma\}) \cap \Lambda$. Then $x$ is neither a singularity nor a periodic point. Let $L, \rho$ and $\delta_0$ be the constant given by Lemma 3.3. Take a point $X^T(x)$ with $T > L$ and $\delta > 0$ such that the tube 

$$\Delta = B_\delta(X^{[0,L]}(x)) \cap X^{[T,+\infty)}(x) = \emptyset.$$ 

Since $\Lambda$ is transitive, there is $z \in \Lambda$ such that $\omega(z) = \Lambda$. For any small neighborhood $U$ of $y$, we can find $0 < s < t$ such that $X^s(z) \in U$ and $X^t(z) \in B_{\delta/\rho}(x)$. Let $q = X^T(x)$ and $p = X^s(z)$. Then by Lemma 3.3, there is $Y \in U(X)$ such that $Y^t(p) = q$ for some $t > 0$. Since $q = X^T(x) \in W^s(\sigma)$, we have $p \in W^s(\sigma, Y)$. □

From Lemma 3.1 we know that if $X \in G_1$, and $\Lambda$ is an isolated nontrivial transitive set of $X$, then every $\sigma \in \Lambda \cap \text{Sing}(X)$ has index 1 or 2.

**Lemma 3.5** There is a residual set $G_2 \subset \mathcal{X}^1(M)$ with the following property. For any $X \in G_2$ and any isolated nontrivial transitive set $\Lambda$ of $X$, if there is $\sigma \in \Lambda \cap \text{Sing}(X)$ with $\text{index}(\sigma) = 2$ then $\Lambda \subset W^u(\sigma)$. Symmetrically, if there is $\sigma \in \Lambda \cap \text{Sing}(X)$ with $\text{index}(\sigma) = 1$ then $\Lambda \subset W^s(\sigma)$.

**Proof.** Let $\mathcal{O} = \{O_1, O_2, \ldots, O_n, \ldots\}$ be a countable basis of $M$. For each $m, k \in \mathbb{N}$, let

$$\mathcal{H}_{m,k} = \{X \in \mathcal{X}^1(M) : \text{there is a $C^1$ neighborhood $U(X)$ of $X$ such that for any $Y \in U(X)$, $Y$ has a singularity $\sigma \in O_m$ with $\text{index}(\sigma) = 2$ such that $W^u(\sigma, Y) \cap O_k \neq \emptyset\}.$$$$

Then $\mathcal{H}_{m,k}$ is an open in $\mathcal{X}^1(M)$. Let

$$\mathcal{N}_{m,k} = \mathcal{X}^1(M) \setminus \overline{\mathcal{H}_{m,k}}.$$ 

Then $\mathcal{H}_{m,k} \cup \mathcal{N}_{m,k}$ is open and dense in $\mathcal{X}^1(M)$. Let

$$G_2 = \bigcap_{m,k \in \mathbb{N}} (\mathcal{H}_{m,k} \cup \mathcal{N}_{m,k}).$$

We will show that the residual set $G_2$ satisfies the request of lemma. Let $X \in G_2$ and $\Lambda$ be an isolated transitive set and let $\sigma \in \Lambda \cap \text{Sing}(X)$ with $\text{index}(\sigma) = 2$. Since $\sigma$ is hyperbolic, we can take $O_m$ such that $O_m$ is an isolated neighborhood of $\sigma$. By the structurally stability of hyperbolic singularity, there is a $C^1$ neighborhood $U(X)$ of $X$ such that for any $Y \in U(X)$, $Y$ has a unique hyperbolic singularity in $O_m$. For any $y \in \Lambda$ and any neighborhood $U$ of $y$, we can choose $O_k \in \mathcal{O}$ such that $y \in O_k \subset U$.

**Claim** $X \notin \mathcal{N}_{m,k}$
Proof of Claim. For any neighborhood $V(X) \subset U(X)$, by Lemma 3.3 there is $Y \in V(X)$ such that $Y$ has a singularity $\sigma \in O_m$ with $\text{index}(\sigma) = 2$ and $W^u(\sigma, Y) \cap O_k \neq \emptyset$. Note that if $W^u(\sigma, Y) \cap O_k \neq \emptyset$ then there is a $C^1$ neighborhood $U(Y)$ of $Y$ such that for any $Z \in U(Y)$, we know that $W^u(\sigma, Z) \cap O_k \neq \emptyset$ by the continuity of the unstable manifold. Thus we have $Y \in H_{m,k}$. Hence $X \in \overline{H_{m,k}}$. This ends the proof of claim.

Then by claim, since $X \in \mathcal{G}_2$, we have $X \in H_{m,k}$. Note that $O_m$ is an isolated neighborhood of $\sigma$, by the definition of $H_{m,k}$, we know that $W^u(\sigma) \cap O_k \neq \emptyset$. This prove that for every neighborhood $U$ of $\sigma$, we know that $W^u(\sigma) \cap U \neq \emptyset$. This means that $\Lambda \subset W^u(\sigma)$. 

We say that a closed $X^t$-invariant set $\Lambda$ is Lyapunov stable for $X$ if for every neighborhood $U$ of $\Lambda$ there is a neighborhood $V \subset U$ of $\Lambda$ such that $X^t(V) \subset U$ for every $t \geq 0$.

Let $\sigma$ be a hyperbolic singularity of $X$ with $\dim W^u(\sigma) = 1$. Then $W^u(\sigma) \setminus \{\sigma\}$ can be divided into two connected branches $\Gamma_1, \Gamma_2$, that is, $W^u(\sigma) = \{\sigma\} \cup \Gamma_1 \cup \Gamma_2$.

Lemma 3.6 Let $X \in X^1(M)$ and $\Lambda$ be a transitive set of $X$. Assume $\sigma \in \Lambda$ is a hyperbolic singularity of $X$ with $\dim W^u(\sigma) = 1$. Let $\Gamma_1 = \text{Orb}(x_1)$ and $\Gamma_2 = \text{Orb}(x_2)$ be the two branches of $W^u(\sigma) \setminus \{\sigma\}$. If $x_1 \in \Lambda$, then for any neighborhood $U(X)$ of $X$, and any neighborhood $V$ of $x_2$, there is $Y \in U(X)$ such that $x_1$ is still in the unstable manifold of $\sigma$ and the positive orbit of $x_1$ will cross $V$ with respect to $Y$.

Proof. We prove this lemma by a standard application of the connecting lemma. By Lemma 3.2, we know that there is a point $z \in (W^s(\sigma) \setminus \{\sigma\}) \cap \Lambda$. Then we have two triple of $\rho > 1, L > 1$ and $\delta_0$ with the properties stated as in Lemma 3.3 with respect to the point $x_1$ and $z$ and the neighborhood $U(X)$ of $X$. By taking the larger $\rho, L$, and smaller $\delta_0$, we get a triple, still denoted by $\rho, L$ and $\delta_0$, works both for $x_1$ and $z$.

Now we can take $\delta > 0$ small enough such that the two tubes $\Delta_1 = B_\delta(X[0,L](x_1))$ and $\Delta_2 = B_\delta(X[-L,0](z))$ are disjoint. For any neighborhood $V$ of $x_2$ and any neighborhood $V'$ of $z$, by the inclination lemma we know that there are a point $y \in V$ and $T > 0$ such that $X^{-T}(y) \in V'$. If $\delta > 0$ is choosing small enough, we can take $y$ and $T$ such that $X^{[-T,0]}(y)$ does not touch $\Delta_1$.

Since $\Lambda$ is transitive, we can find a point $x \in \Lambda$ such that $\Lambda = \omega(x)$. Then we can find $t_1 < t_2$ such that $X^{t_1}(x) \in B_{\delta/\rho}(x_1)$ and $X^{t_2}(x) \in B_{\delta/\rho}(z)$ and a point $y \in V$ with $X^{-T}(y) \in B_{\delta/\rho}(z)$. Then apply Lemma 3.3 we can find a vector filed $Y \in U(X)$ differs from $X$ at tubes $\Delta_1$ and $\Delta_2$ such that the negative orbit of $x_1$ is not changed and $y$ is contained in the positive orbit of $x_1$. It is easy to see that $Y$ satisfies the request of lemma.

Lemma 3.7 Let $\mathcal{G}_2 \subset X^1(M)$ be the residual set chosen as in Lemma 3.5. Then for any $X \in \mathcal{G}_2$ and any isolated nontrivial transitive set $\Lambda$ of $X$, if there is a singularity $\sigma \in \Lambda$ with $\text{index}(\sigma) = 2$, then we have $\overline{W^u(\sigma)} \subset \Lambda$. 

7
Proof. Let \( O = \{O_1, O_2, \ldots, O_n, \ldots\} \) be a countable basis of \( M \). Recall that for each \( m, k \in \mathbb{N} \), we take
\[
H_{m,k} = \{X \in X^1(M) : \text{there is a } C^1 \text{ neighborhood } U(X) \text{ of } X
\]
such that for any \( Y \in U(X), Y \) has a singularity \( \sigma \in O_m \) with
\[
\text{index}(\sigma) = 2 \text{ such that } W^u(\sigma, Y) \cap O_k \neq \emptyset\}.
\]
Then take \( N_{m,k} = X^1(M) \setminus \overline{H_{m,k}} \) and
\[
G_2 = \bigcap_{m,k \in \mathbb{N}} (H_{m,k} \cup N_{m,k}).
\]
We will see that this \( G_2 \) satisfies the request of lemma.

Let \( X \in G_2 \) and \( \Lambda \) be an isolated transitive set of \( X \). Assume there is singularity \( \sigma \in \Lambda \) with index 2. Let \( \Gamma_1 = \text{Orb}(x_1) \) and \( \Gamma_2 = \text{Orb}(x_2) \) be the two branches of \( W^u(\sigma) \setminus \sigma \). By Lemma 3.2, we know that either \( x_1 \) or \( x_2 \) is contained in \( \Lambda \). Without loss of generality, we assume that \( x_1 \in \Lambda \). To prove \( W^u(\sigma) \subset \Lambda \), we just need to prove that \( x_2 \) is also contained in \( \Lambda \). By the compactness of \( \Lambda \), we just need to prove that for any neighborhood \( U \) of \( x_2 \), \( U \cap \Lambda \neq \emptyset \). For a given arbitrarily small neighborhood \( U \) of \( x \), we can find \( k \) such that \( O_k \subset U \). Let \( O_m \) be an isolated neighborhood of \( \sigma \). Then we have
\[
\text{Claim } X \notin N_{m,k}
\]
Proof of Claim: For any neighborhood \( V(X) \subset U(X) \), by Lemma 3.6, there is \( Y \in V(X) \) such that \( Y \) has a singularity \( \sigma \in O_m \) with \( \text{index}(\sigma) = 2 \) and \( W^u(\sigma, Y) \cap O_k \neq \emptyset \). By the continuity of the unstable manifold we know that there is a \( C^1 \) neighborhood \( U(Y) \) of \( Y \) such that for any \( Z \in U(Y), W^u(\sigma, Z) \cap O_k \neq \emptyset \) Thus we have \( Y \in H_{m,k} \). Hence \( X \in \overline{H_{m,k}} \). This ends the proof of claim.

Since \( X \in G_2 \) and \( X \notin N_{m,k} \), we have \( X \in H_{m,k} \). Since \( \sigma \) is the only singularity of \( X \) in \( O_m \), by the definition of \( H_{m,k} \) we can see that \( W^u(\sigma) \cap O_k \neq \emptyset \). Hence for any neighborhood \( U \) of \( x_2 \), there is a point contained in \( W^u(\sigma) \). This ends the proof of Lemma 3.7. \( \square \)

The following lemma is collected from [4].

Lemma 3.8 [4] Proposition 4.1 There is a residual set \( G_3 \subset X^1(M) \) such that for any \( X \in G_3, W^u(\sigma) \) is Lyapunov stable for \( X \) and \( W^s(\sigma) \) is Lyapunov stable for \( -X \) for all \( \sigma \in \text{Sing}(X) \).

Proposition 3.9 There is a residual set \( S \subset X^1(M) \) such that for any \( X \in S \), and any isolated nontrivial transitive set \( \Lambda \) of \( X \), if there is a singularity \( \sigma \in \Lambda \cap \text{Sing}(X) \) with \( \text{index}(\sigma) = 2 \) then \( \Lambda \) is Lyapunov stable for \( X \). Symmetrically, if there is \( \sigma \in \Lambda \cap \text{Sing}(X) \) with \( \text{index}(\sigma) = 1 \) then \( \Lambda \) is Lyapunov stable for \( -X \).
Proof. Let \( X \in S = \mathcal{G}_2 \cap \mathcal{G}_3 \) and \( \Lambda \) be an isolated transitive set of \( X \). Suppose that \( \sigma \in \Lambda \cap \text{Sing}(X) \) with \( \text{index}(\sigma) = 2 \). Then by Proposition 3.5 and Lemma 3.7, we have \( W^u(\sigma) = \Lambda \). By Lemma 3.8, \( \Lambda \) is Lyapunov stable for \( X \).

A point \( \sigma \in \text{Sing}(X) \) of \( X \) is called Lorenz-like if \( DX(\sigma) \) has three real eigenvalues \( \lambda_1, \lambda_2, \lambda_3 \) such that \( \lambda_2 < \lambda_3 < 0 < -\lambda_3 < \lambda_1 \). Let \( \sigma \in \text{Sing}(X) \) be a Lorenz-like singularity, then we use \( E^{sss}_X(\sigma), E^{cs}_X(\sigma), E^{su}_X(\sigma) \) to denote the eigenspaces of \( DX(\sigma) \) corresponding the eigenvalues \( \lambda_2, \lambda_3, \lambda_1 \) respectively. Denoted by \( W^{ss}_X(\sigma) \) the one-dimensional invariant manifold of \( X \) associated to the eigenvalue \( \lambda_2 \). We have the following lemma was proved in [13].

Lemma 3.10 [13, Lemma A. 4] There is a residual set \( G_4 \subset X^1(M) \) such that for any \( X \in \mathcal{R} \), if \( \Lambda \) is a Lyapunov stable nontrivial transitive set of \( X \), then every singularity \( \sigma \in \Lambda \) is Lorenz-like and one has \( W^{ss}_X(\sigma) \cap \Lambda = \{ \sigma \} \).

Here is the main conclusion in this section.

Proposition 3.11 There is a residual set \( T \subset X^1(M) \) with the following properties. Let \( X \in \mathcal{T} \) and \( \Lambda \) be an isolated transitive set of \( X \). If there is a singularity with index 2, then for all singularity \( \sigma \in \Lambda \), one has (1) \( \text{index}(\sigma) = 2 \), (2) \( \sigma \) is Lorenz-like, and (3) \( W^{ss}_X(\sigma) \cap \Lambda = \{ \sigma \} \). Symmetrically, if there is singularity with index 1 then for all singularity \( \sigma \in \Lambda \), one has (1) \( \text{index}(\sigma) = 1 \), (2) \( \sigma \) is Lorenz-like for \( -X \), and (3) \( W^{uu}_X(\sigma) \cap \Lambda = \{ \sigma \} \).

Proof. Let \( X \in \mathcal{T} = S \cap \mathcal{G}_4 \) and \( \Lambda \) be an isolated transitive set of \( X \). Suppose that there is \( \eta \in \Lambda \cap \text{Sing}(X) \) such that \( \text{index}(\eta) = 2 \). By Proposition 3.9, \( \Lambda \) is Lyapunov stable for \( X \). On the other hand, since \( X \in \mathcal{G}_4 \), according to Lemma 3.11, \( \sigma \) is Lorenz-like, and \( W^{ss}_X(\sigma) \cap \Lambda = \{ \sigma \} \). We directly obtained \( \text{index}(\sigma) = 2 \), for all \( \sigma \in \Lambda \cap \text{Sing}(X) \).

4 Proof of Theorem A

To prove Theorem A, we prepare two techniques here. One is the extended linear Poincaré flow given by Li, Gan and Wen [10], and another one is the ergodic closing lemma given by Mañé [11, 17].

Firstly we recall the notion of linear Poincaré flow firstly given by Liao [7, 8]. For any regular point \( x \in M \setminus \text{Sing}(X) \), we can put a normal space

\[
N_x = \{ v \in T_x M : v \perp X(x) \}.
\]

Then we have a normal bundle

\[
N = N(X) = \bigcup_{x \in M \setminus \text{Sing}(X)} N_x.
\]

Denote by \( \pi_x \) the orthogonal projection from \( T_x M \) to \( N_x \) for any \( x \in M \setminus \text{Sing}(X) \). From the tangent flow, we can define the linear Poincaré flow

\[
P_t^X : N(X) \to N(X)
\]
\[
P_t^X(v) = \pi_{X'(x)}(DX^t(v)), \text{ for all } v \in N_x, \text{ and } x \in M \setminus Sing(X).
\]
Note that the linear Poincaré flow is defined on the normal bundle over a non compact set. We consider a compactification for \(P_t^X\) as following.

Let

\[G^1 = \{ L : L \text{ is a one dimensional subspaces in } T_xM, x \in M \}\]

be the Grassmannian manifold of \(M\). Then for any \(L \in G^1\), assuming \(L \subset T_xM\) for some \(x \in M\), we can define a normal space associated to \(L\) as in following,

\[N_L = \{ v \in T_xM : v \perp L \}.
\]

Now can take a normal bundle

\[N = N_{G^1} = \bigcup_{L \in G^1} N_L.
\]

Note that \(G^1\) is a compact manifold, so \(N_{G^1}\) is a bundle over a compact space.

For any \(L \in G^1\) contained in \(T_xM\), denoted by \(\pi_L\) the orthogonal projection from \(T_xM\) to \(N_L\) along \(L\). Let \(X\) be a \(C^1\) vector field. Similar to the linear Poincaré flow, we can define the extended linear Poincaré flow

\[
\tilde{P}_t^X : N_{G^1} \rightarrow N_{G^1}
\]

\[
\tilde{P}_t^X(v) = \pi_{DX'(L)}(DX^t(v)),
\]

for all \(L \in G^1\) and \(v \in N_L\).

One can check that for any \(x \in M \setminus Sing(X)\), then we have \(N_x = N_{(X(x))}\) and \(P_t^X|_{N_x} = \tilde{P}_t^X|_{N_{(X(x))}}\). Here, \(\tilde{P}_t^X\) is said to be the extended linear Poincaré flow.

For any compact invariant set \(\Lambda\) of the vector fields \(X\), we use \(\hat{\Lambda}\) to denote the closure of

\[\{ (X(x)) : x \in \Lambda \setminus Sing(X) \}\]

in the space of \(G^1\). Let \(\sigma \in \Lambda\) be a singularity, denote by

\[\hat{\Lambda}_\sigma = \{ L \in \hat{\Lambda} : L \subset T_{\sigma}M \}.
\]

From the facts we got from Proposition 3.11, we have the following characterization of \(\hat{\Lambda}_\sigma\).

**Lemma 4.1** Let \(X \in \mathcal{T}\) and \(\Lambda\) be an isolated transitive set of \(X\). Suppose there is a singularity with index 2. Then for all singularity \(\sigma \in \Lambda\), we have \(L \subset E_{\sigma}^{cs} \oplus E_{\sigma}^{u}\) for all \(L \in \hat{\Lambda}_\sigma\).
Proof. Let $X \in T$ and $\Lambda$ be an isolated transitive set of $X$. Suppose on the contrary, that is, there is $L \in \Lambda_\sigma$ such that $L$ is not a subspace in $E^{cs}_\sigma \oplus E^u_\sigma$. Note that $DX^t(L)$ is contained in $\Lambda_\sigma$ for all $t \in \mathbb{R}$ and $\Lambda_\sigma$ is a closed set. By taking a limit line of $DX^t(L)$ as $t \to -\infty$, we know that there is $L \in \Lambda_\sigma$ such that $L \subset E^{ss}_\sigma$. From now on, we assume that $L \in \Lambda$ and $L \subset E^{ss}_\sigma$. By the definition of $\Lambda$, we know that there exist $x_n \in \Lambda \setminus Sing(X)$ such that $<X(x_n)> \to L \subset E^{ss}_\sigma$.

For the simplicity of notations, we assume everything happens in a local chart containing $\sigma$. For any $0 < \eta \leq 1$, denote by $E^{cu}_\sigma = E^{cs}_\sigma \oplus E^u_\sigma$ and $$C^{cu}_\eta(\sigma) = \{v = v^{ss} + v^{cu} \in T_\sigma M : |v^{ss}| < \eta|v^{cu}|, v^{ss} \in E^{ss}_\sigma, v^{cu} \in E^{cu}_\sigma\}$$
the $cu$-cone at the singularity $\sigma$. These cones can be parallel translated to $x$ who is close to $\sigma$. Since $E^{ss}_\sigma \oplus E^{cu}_\sigma$ is a dominated splitting for the tangent flow $DX^t$, there are two constants $T > 0$ and $0 < \lambda < 1$ such that

$$DX^t(C^{cu}_1(\sigma)) \subset C^{cu}_\lambda(\sigma),$$
for any $t \in [T, 2T]$. By the continuous property of the cone to a cone field in a small neighborhood $U_\sigma$ of $\sigma$, for any $t \in [T, 2T]$, $X^{[t,0]}(x) \subset U_\sigma$ then we have $DX^t(C^{cu}_1(x)) \subset C^{cu}_1(X^t(x))$.

Now let $t_n = \sup\{t > 0 : X^{[-t,0]}(x_n) \subset U_\sigma\}$. We know that $t_n \to +\infty$ as $n \to \infty$ because $x_n \to \sigma$ as $n \to \infty$. Denote by $y_n = X^{-t_n}(x_n)$. Then we can take $q = \lim_{n \to \infty} y_n \in \partial U_\sigma$ by taking the subsequence if necessary. We know that for $t > 0$, $X^t(q) \in U_\sigma$ and so, $q \in W^s(\sigma)$. Since $y_n \in \Lambda$ we know $q \in \Lambda$. If $q \in W^{ss}(\sigma) \cap \Lambda$, because we have already $q \in \partial U_\sigma$, hence $q \neq \sigma$, then from the fact that $X \in T_1$ and $\Lambda$ is an isolated nontrivial transitive set, this is a contradiction by Proposition 3.11. Now we assume that $q \in W^s(\sigma) \setminus W^{ss}(\sigma)$. We have $<X(X^t(q))> \to E^{ss}_\sigma$ as $t \to +\infty$. Thus there is $T_1 > 0$ big enough such that $X(X^T_1(q)) \subset C^{cu}_1(X^T_1(q))$. For $n$ big enough we have $X(X^T_1(y_n)) \subset C^{cu}_1(X^T_1(y_n))$. Since $t_n \to \infty$, we assume that $t_n - T_1 > T$. Since $X^{[T_1,t_n]}(y_n) \subset U_\sigma$, we know that

$$X(x_n) = X(X^{t_n}(y_n)) = DX^{t_n-T_1}(X(X^T_1(y_n))) 
\in DX^{t_n-T_1}(C^{cu}_1(X^T_1(y_n))) 
\subset C^{cu}_1(X^{t_n}(y_n)) = C^{cu}_1(x_n).$$

This is a contradiction with the assumption $<X(x_n)> \to L \subset E^{ss}_\sigma$. □

It is proved in section 2 that generically, if $\Lambda$ is an isolated transitive set, then it is locally star. By some well know results from the proof of stability conjecture, we have the following proposition.

Proposition 4.2 [8, 11] Let $\Lambda$ be a locally star set for $X \in \mathcal{X}^1(M)$ and let $\mathcal{U}(X), U$ be the neighborhoods in the definition of local star. Then there are constants $0 < \lambda_0 < 1, T_0 > 0$ such that for any $Y \in \mathcal{U}(X)$ and any $p \in \Lambda_Y(U) \cap Per(Y)$, the following properties hold:
(a) $\Delta^s \oplus \Delta^u$ is a dominated splitting with respect to the linear Poincaré flow. Precisely, for any $t \geq T_0$ and any $x \in \text{Orb}(p)$,

$$\|P_t^Y|_{\Delta^s(x)}\| \cdot \|P_{-t}^Y|_{\Delta^u(y^\tau(x))}\| \leq e^{-2\lambda_0 t};$$

(b) if $\tau$ is the period of $p$ and $m$ is any positive integer, and if $0 = t_0 < t_1 < \cdots < t_k = m\tau$ is any partition of the time interval $[0, m\tau]$ with $t_{i+1} - t_i \geq T_0$, then

$$\frac{1}{m\tau} \sum_{i=0}^{k-1} \log \|P_{t_{i+1}-t_i}^Y|_{\Delta^s(y^{t_i}(p))}\| < -\lambda_0,$$

and

$$\frac{1}{m\tau} \sum_{i=0}^{k-1} \log \|P_{-t_{i+1}+t_i}^Y|_{\Delta^u(y^{t_i+1}(p))}\| < -\lambda_0,$$

where $\Delta^s \oplus \Delta^u$ is the hyperbolic splitting with respect to $P_t^X|_{N_{\text{Orb}(p)}}$.

Now we assume that $\Lambda$ is an isolated transitive set of a $C^1$-generic vector field $X$. By the closing lemma we know that for any $x \in \Lambda \setminus \text{Sing}(X)$, one can find a sequence of periodic points $p_n$ of $X$ such that $p_n \to x$ as $n \to \infty$. Consequently, for any $L \in \hat{\Lambda}$, we can find a sequence of periodic points $p_n$ of $X$, such that $L$ is the limit of $<X(p_n)>$. Since $\Lambda$ is locally star, from item (a) of Proposition 4.2 we can see that for any $L \in \hat{\Lambda}$, we can get two one dimensional subspaces $\Delta^1(L) = \lim_{n \to \infty} \Delta^s(p_n)$ and $\Delta^2(L) = \lim_{n \to \infty} \Delta^u(p_n)$ with the property: for any $t \geq T_0$,

$$\|\tilde{P}_t^Y|_{\Delta^1(L)}\| \cdot \|\tilde{P}_t^Y|_{\Delta^2(DX^1(L))}\| \leq e^{-2\lambda_0 t}.$$  

This implies that there is a dominated splitting $N_{\hat{\Lambda}} = \Delta^1 \oplus \Delta^2$ for the extended linear Poincaré flow $\tilde{P}_t^X$. For any $x \in \Lambda \setminus \text{Sing}(X)$, we can put $\Delta^i(x) = \Delta^i(<X(x)>)$ for $i = 1, 2$, then we can get a dominated splitting $N_{\Lambda \setminus \text{Sing}(X)} = \Delta^1 \oplus \Delta^2$ for the linear Poincaré flow $P_t^X$.

If $X \in \mathcal{T}$ and $\Lambda$ be an isolated transitive set of $X$, then we have only finitely many singularity in $\Lambda$. Without loss of generality, after a change of equivalent Riemannian structure, we can assume that for any $\sigma \in \hat{\Lambda}$ with index 2, the subspaces $E^s_\sigma, E^c_\sigma, E^u_\sigma$ are mutually orthogonal. From Lemma 4.1 we know that every $L \in \hat{\Lambda}_\sigma$ is orthogonal to $E^s_\sigma$, this fact derives the following lemma.

**Lemma 4.3** Let $X \in \mathcal{T}$ and $\Lambda$ be an isolated transitive set of $X$. Suppose there is a singularity with index 2. Then for all singularity $\sigma \in \Lambda$ with mutually orthogonal $E^s_\sigma, E^c_\sigma, E^u_\sigma$, we have $\Delta^1(L) = E^s_\sigma$ and $\tilde{P}_S^X|_{\Delta^1(L)} = DX^1|_{E^s_\sigma}$ for any $L \in \hat{\Lambda}_\sigma$.  

12
Proof. We denote by $E^c_{\sigma} \triangleq E^s_{\sigma} \oplus E^u_{\sigma}$ for any given singularity $\sigma \in \Lambda$. For any $L \in \tilde{\Lambda}_\sigma$, we set $N^1(L) = E^s_{\sigma}$ and $N^2(L) = E^c_{\sigma} \cap N_L$. By the fact that $L$ is orthogonal to $E^s_{\sigma}$ we know that $N^1(L) \subset N_L$ for any $L \in \tilde{\Lambda}_\sigma$. Now we have two subbundles

$$N^1_{\tilde{\Lambda}_\sigma} = \bigcup_{L \in \tilde{\Lambda}_\sigma} N^1(L), \quad N^2_{\tilde{\Lambda}_\sigma} = \bigcup_{L \in \tilde{\Lambda}_\sigma} N^2(L).$$

These two subbundles are $\tilde{P}_t^X$-invariant by the fact that $L \subset E^c_{\sigma}$ for any $L \in \tilde{\Lambda}_\sigma$ and both $E^s_{\sigma}$ and $E^c_{\sigma}$ are $DX^t$-invariant.

Since $E^s_{\sigma} \oplus E^c_{\sigma}$ is a dominated splitting for $DX^t$, we know that there are contants $C > 1, \lambda > 0$ such that

$$\frac{\|DX^t(u)\|}{\|DX^t(v)\|} \leq Ce^{-\lambda t}$$

for any unit vectors $u \in E^c_{\sigma}$ and $v \in E^s_{\sigma}$ and any $t > 0$. Then for any $L \in \tilde{\Lambda}_\sigma$ and any unit vectors $u \in N^2(L), v \in N^1(L)$, we have

$$\frac{\|\tilde{P}_t^X(u)\|}{\|\tilde{P}_t^X(v)\|} \leq \frac{\|DX^t(u)\|}{\|DX^t(v)\|} \leq Ce^{-\lambda t}.$$ 

This says that $N_{\tilde{\Lambda}_\sigma} = N^1_{\tilde{\Lambda}_\sigma} \oplus N^2_{\tilde{\Lambda}_\sigma}$ is a dominated splitting on $\tilde{\Lambda}_\sigma$ with respect to the extended linear Poincaré flow $\tilde{P}_t^X$. By the uniqueness of dominated splitting we know that $N^1_{\tilde{\Lambda}_\sigma} = \Delta^1_{\tilde{\Lambda}_\sigma}$. Thus we have $\Delta^1(L) = E^s_{\sigma}$ for all $L \in \tilde{\Lambda}_\sigma$. By the definition of extended linear Poincaré flow, we directly have the fact that $\tilde{P}_t^X|_{\Delta^1(L)} = DX^t|_{E^s_{\sigma}}$ for all $L \in \tilde{\Lambda}_\sigma$. 

Now let us recall the ergodic closing lemma. A point $x \in M \setminus Sing(X)$ is called a well closable point of $X$ if for any $C^1$ neighborhood $U(X)$ of $X$ and any $\delta > 0$, there are $Y \in U(X), z \in M, \tau > 0$ and $T > 0$ such that the following conditions are hold:

(a) $Y^\tau(z) = z$,

(b) $d(X^t(x), Y^t(z)) < \delta$ for any $0 \leq t \leq \tau$, and

(c) $X = Y$ on $M \setminus B(X^{[-T,0]}(x), \delta)$. 

Denote by $\Sigma(X)$ the set of all well closable points of $X$. Here we will use the flow version of the ergodic closing lemma which was proved in [17].

Lemma 4.4 [17] For any $X \in \mathcal{X}^1(M), \mu(\Sigma(X) \cup Sing(X)) = 1$ for every $T > 0$ and every $X^T$-invariant Borel probability measure $\mu$.

Assume $X \in \mathcal{T}$ and $\Lambda$ be an isolated transitive set of $X$. From Proposition 4.2 we have already known that there is a dominated splitting $N_{\Lambda \setminus Sing(X)} = \Delta^1 \oplus \Delta^2$ with $\dim(\Delta^1) = \dim(\Delta^2) = 1$ with respect to the linear Poincaré flow $P_t^X$. By applying the ergodic closing lemma, we have the following lemma.

13
Lemma 4.5  Let $X \in \mathcal{T}$ and $\Lambda$ be an isolated transitive set of $X$. Suppose there is a singularity with index 2. Then there are constant $C > 1$ and $\lambda > 0$ such that

$\|DX^t_{|\langle X(x)\rangle}\|^{-1} \cdot \|P^X_t|\Delta^1(x)\| < C e^{-\lambda t},$

$\|DX^{-t}_{|\langle X(x)\rangle}\| \cdot \|P^X_{-t}|\Delta^2(x)\| < C e^{-\lambda t}$

for all $x \in \Lambda \setminus \text{Sing}(X)$ and $t \geq 0$.

Proof. Let $X \in \mathcal{T}$ and $\Lambda$ be an isolated transitive set of $X$. Then there is a $\tilde{P}^X_t$ invariant splitting $N_{\tilde{\Lambda}} = \Delta^1 \oplus \Delta^2$ with constant $T_0 > 0$ and $\lambda_0 > 0$ such that the followings are satisfied:

1. if $L = \langle X(x)\rangle$ for some $x \in \Lambda \setminus \text{Sing}(X)$, then $\Delta^i(\langle X(x)\rangle) = \Delta^i(x)$ for $i = 1, 2$,

2. $\|\tilde{P}^Y_t|\Delta^1(L)\| \cdot \|\tilde{P}^Y_{-t}|\Delta^2(DX^t(L))\| \leq e^{-2\lambda_0 t}$ for any $t > T_0$, and

3. $L \in \tilde{\Lambda}$.

To prove the lemma, we just need to prove that there is $C > 1$ and $\lambda > 0$ such that for any $L \in \tilde{\Lambda}$ and any $t > 0$, we have

$\|DX^t_{|L}\|^{-1} \cdot \|\tilde{P}^X_t|\Delta^1(L)\| < C e^{-\lambda t},$

$\|DX^{-t}_{|L}\| \cdot \|\tilde{P}^X_{-t}|\Delta^2(L)\| < C e^{-\lambda t}.$

Since $\tilde{\Lambda}$ is compact, we just need to show that for any $L \in \tilde{\Lambda}$, there is a $T > 0$ such that

$log \|\tilde{P}^X_T|\Delta^1(L)\| - log \|DX^T_{|L}\| < 0,$

$log \|\tilde{P}^X_{-T}|\Delta^2(L)\| + log \|DX^{-T}_{|L}\| < 0.$

Now let us prove these properties of $\Delta^1 \oplus \Delta^2$ by contradiction. Firstly we prove the first half part. Assume that for any $L \in \tilde{\Lambda}$ and any $t > 0$

$log \|\tilde{P}^X_T|\Delta^1(L)\| - log \|DX^t_{|L}\| \geq 0.$

Similar to [12] Lemma I.5, by a typical application of Birkhoff ergodic theorem, for any $S > 0$ there is an ergodic $DX^T$-invariant measure $\mu \in \mathcal{M}(G^1)$ with supp$(\mu) \subset \tilde{\Lambda}$ such that

$\int (log \|\tilde{P}^X_S|\Delta^1(L)\| - log \|DX^S_{|L}\|) d\mu(L) \geq 0.$

In the following, we will always choose $S$ is big enough.

Claim If $S$ is big enough, then for any singularity $\sigma \in \Lambda \cap \text{Sing}(X)$, one has $\mu(\tilde{\Lambda}_\sigma) = 0.$
Proof of Claim: According to Lemma 4.1, for every $L \in \tilde{\Lambda}_\sigma$, $L \subset E_{cs}^s \oplus E_{cu}^s \triangleq E_{cu}^s$. Without loss of generality, we assume that $E_{ss}^s$ is orthogonal to $E_{cu}^s$. Then by Lemma 4.3, we have $\tilde{P}_S^{X_{\Delta^1(L)}} = DX^S_{<X(x)>}$ for any $L \in \tilde{\Lambda}_\sigma$. Since $E_{ss}^s$ is dominated by $E_{cu}^s$, we can take $S$ big enough such that

$$\log \| \tilde{P}_S^{X_{\Delta^1(L)}} \| - \log \| DX^S_{<X(x)>} \| < 0$$

for any $L \in \tilde{\Lambda}_\sigma$. If $\tilde{\mu}(\tilde{\Lambda}_\sigma) \neq 0$, then we have $\tilde{\mu}(\tilde{\Lambda}_\sigma) = 1$ by the invariant of $\tilde{\Lambda}_\sigma$ and the ergodicity of $\tilde{\mu}$, thus we have,

$$\int (\log \| P_S^{X_{\Delta^1(L)}} \| - \log \| DX^S_{<X(x)>} \|) d\tilde{\mu}(L) < 0.$$

This is a contradiction. This ends the proof of claim.

In the following, we will take $S$ is a multiple of $T_0$ which is big enough such that the above claim is satisfied. One can see $S$ have also the properties of $T_0$.

For any Borel set $A \subset \Lambda$, we denote by $\tilde{A} = \{ L : L = <X(x)> \text{ for some } x \in A \}$. Then we define $\mu(A) = \tilde{\mu}(\tilde{A})$. By the fact that $\tilde{\mu}(\tilde{\Lambda}_\sigma) = 0$ for any $\sigma \in \Lambda \cap Sing(X)$, we know that $\mu$ is an ergodic measure support in $\Lambda$ with $\mu(\Lambda \setminus Sing(X)) = 1$. From the inequality

$$\int (\log \| P_S^{X_{\Delta^1(x)}} \| - \log \| DX^S_{<X(x)>} \|) d\mu(x) \geq 0,$$

we have

$$\int_{\Lambda \setminus Sing(X)} (\log \| P_S^{X_{\Delta^1(x)}} \| - \log \| DX^S_{<X(x)>} \|) d\mu(x) \geq 0.$$

By Lemma 4.3,

$$\int_{\Lambda \cap \Sigma(X)} (\log \| P_S^{X_{\Delta^1(x)}} \| - \log \| DX^S_{<X(x)>} \|) d\mu(x) \geq 0.$$

By the ergodic theorem of Birkhoff, there is a point $y \in \Lambda \cap \Sigma(X)$ such that

$$\lim_{n \to \infty} \frac{1}{nS} \sum_{j=0}^{n-1} (\log \| P_S^{X_{\Delta^1(X^{jS}(y))}} \| - \log \| DX^S_{<X(X^{jS}(y)>)} \|) \geq 0. \tag{1}$$

Claim $y$ is not a periodic point of $X$.

Proof of Claim: By the fact that $\| DX^S_{<X(x)>} \| = \frac{|X(X^S(x))|}{|X(x)|}$, we have

$$\sum_{j=0}^{n-1} \log \| DX^S_{<X(X^{jS}(y)>)} \| = \sum_{j=0}^{n-1} \log \frac{|X(X^{j+1S}(y))|}{|X(X^{jS}(y))|} = \log |X(X^{nS}(y))| - \log |X(y)|.$$
If $y \in Per(X)$ then by Proposition 4.2, we have

$$\limsup_{n \to \infty} \frac{1}{nS} \sum_{j=0}^{n-1} \log \| P^X_S | \Delta^s_{X^jS(y)} \| \leq -\lambda_0.$$ 

Since $\sup |\log(X(x))|$ is bounded for $x \in Orb(y)$, we have

$$\limsup_{n \to \infty} \frac{1}{nS} \left( \sum_{j=0}^{n-1} \log \| P^X_S | \Delta^s_{X^jS(y)} \| - \log |X(X^nS(y))| - \log |X(y)|| \right) \leq -\lambda.$$

This is contradiction by (1). Thus $y$ is not periodic. $\square$

Since $y$ is a well closable point, for any $n > 0$, there are $X_n \in X^1(M), z_n \in M$, and $\tau_n > 0$ such that

(i) $Y^\tau_n(z_n) = z_n$ and $\tau_n$ is the prime period of $z_n$,
(ii) $d(X^t(y), Y^\tau_n(z_n)) \leq 1/n$, for any $0 \leq t \leq \tau_n$, and
(iii) $\| Y_n - X \| \leq 1/n$.

Since $y$ is not a periodic point, we have $\tau_n \to +\infty$ as $n \to \infty$. We also have the following uniformly continuity for $P^Y_t | \Delta^1$.

Claim For any $\epsilon > 0$ there is $\delta > 0$ and a $C^1$ neighborhood $U(X)$ of $X$ such that for any $x, y \in M$, if (i) $x \in \Lambda \setminus Sing(X)$, (ii) there is $Y \in U(X)$ such that $y \in Per(Y)$, $Orb(y) \subset U$, and $d(x,y) < \delta$, then

$$| \log \| P^X_t | \Delta^1(x) \| - \log \| P^Y_t | \Delta^s(y) \| | < \epsilon,$$ (2)

for any $t \in [0,2S]$. Here $\Delta^s(y)$ denotes the stable subspace of $y$ with respect to the vector field $Y$.

Proof of Claim : We prove this by deriving a contradiction. Assume the contrary. Then there is $\eta > 0$ such that for any $n > 0$ there exists $t_n \in [0,2S], X_n \to X$ and two sequences $\{x_n\}, \{y_n\}$ such that (i) $x_n \in \Lambda \setminus Sing(X)$, (ii) $y_n \in Per(X_n)$ and $Orb(y_n) \subset U$, (iii) $d(x_n, y_n) < 1/n$, and

$$| \log \| P^{X_n}_{t_n} | \Delta^1(x_n) \| - \log \| P^{X_n}_{t_n} | \Delta^s(y_n) \| | \geq \eta.$$ 

Since $[0,2S]$ and $\Lambda$ are compact, we can take sequences $\{t_n\} \subset [0,2S]$ and $\{x_n\} \subset \Lambda$ (take subsequences if necessary) such that $t_n \to t_0$ and $x_n \to x_0$. Then we have $y_n \to y_0$ by the above item (iii).

If $x_0 \not\in Sing(X)$ then by the continuity of dominated splitting, we know $\Delta^1(x_n) \to \Delta^1(x_0)$ and $\Delta^s(y_n) \to \Delta^1(x_0)$ as $n \to \infty$, then we have

$$| \log \| P^X_{t_0} | \Delta^1(x_0) \| - \log \| P^X_{t_0} | \Delta^1(x_0) \| | \geq \eta.$$
This is a contradiction.

If \( x_0 \in Sing(X) \) then we can take sequence \( \{<X(x_n)>\}, \{<X_n(y_n)>\} \) (take subsequences if necessary) such that \( <X(x_n)> \to L \in \tilde{A}_{x_0} \) and \( <X_n(y_n)> \to L_1 \in \tilde{A}_{x_0} \). Since both \( L, L_1 \in \tilde{A}_{x_0} \), we have \( \tilde{P}^X_t|\Delta^1(L) = \tilde{P}^X_t|\Delta^1(L_1) = DX^t|_{E_{x_0}} \) by Lemma 4.3. But on the other hand, we have

\[
|\log \|\tilde{P}^X_t|\Delta^1(L)\| - \log \|\tilde{P}^X_t|\Delta^1(L_1)\| \geq \eta.
\]

This is also a contradiction. This ends the proof of Claim. \( \square \)

By (2), there is \( n_0 \) such that for any \( k > n_0, t \in [0,2S] \) and \( t_0 \in [0,\tau_0] \), one has

\[
|\log \|P^X_t|\Delta^1_{X^\tau(y)}\| - \log \|P^X_t|\Delta^1_{X^{\tau_0}(y)}\| < S\lambda_0/3,
\]

where \( \lambda_0 \) as in Proposition 4.2. Let \( \tau_n = m_nS + s_n \) \( (m_n \in \mathbb{Z} \) and \( s_n \in [0,S]) \). Then we consider the partition

\[
0 = t_0 < t_1 = S < \cdots < t_{m_n-1} = (m_n - 1)S < t_{m_n} = \tau_n.
\]

According to Proposition 4.2, we know

\[
\sum_{j=0}^{m_n-2} \log \|P^X_S|\Delta^1_{X^j(y)}\| + \log \|P^X_{S+s_n}|\Delta^1_{X^{(m_n-1)j}(y)}\| \leq -\tau_n\lambda_0.
\]

Then by (3) we have

\[
\sum_{j=0}^{m_n-2} \log \|P^X_S|\Delta^1_{X^j(y)}\| + \log \|P^X_{S+s_n}|\Delta^1_{X^{(m_n-1)j}(y)}\|
\leq m_nS\lambda_0/3 - \tau_n\lambda_0 = -2m_nS\lambda_0/3 - s_n\lambda_0 \leq -2m_nS\lambda_0/3.
\]

For sufficiently small \( r > 0 \), let \( B_r(y) \) be a neighborhood of \( X^{[-2S,0]}(y) \) such that \( B_r(y) \cap Sing(X) = \emptyset \).

Denote by \( C = \sup \{|\log |X(x)|| : x \in B_r(y)\} + \sup \{|\log \|P^X_t|\Delta^1(x)\| : x \in B_r(y), t \in [0,2S]\} \) \( < \infty \).

Since \( d(y,z_n) < 1/n \) and \( d(X^{\tau_n}(y),z_n) = d(X^{\tau_n}(y),X^{\tau_n}(z_n)) < 1/n \), we know \( d(X^{\tau_n}(y),y) < 2/n \). Thus there is \( n_1 > n_0 \) such that for any \( n > n_1 \) and \( t \in [0,2S] \) we have \( X^{\tau_n-t}(y) \in B_r(y) \). Since \( \tau_n - (m_n - 1)S = S + s_n < 2S \), we know

\[
|\log |X(X^{(m_n-1)S}(y))|| + |\log \|P^X_{S+s_n}|\Delta^1_{X^{(m_n-1)S}(y)}\|| \leq C.
\]

(4)

By (1) and \( m_n \to +\infty \) as \( n \to +\infty \), there is \( n_2 \geq n_1 \) such that for any \( n > n_2 \)

\[
\sum_{j=0}^{m_n-2} \log \|P^X_S|\Delta^1_{X^j(y)}\| - (|\log |X(X^{(m_n-1)S}(y))|| - |\log |X(y)|| \geq -(m_n - 1)S\lambda_0/3.
\]

17
Then by
\[ \sum_{j=0}^{m_n-2} \log \| P^X_S \|_{\Delta^j(XJ_S(y))} + \log \| P^X_{S+s_n} \|_{\Delta^j(X^{(m_n-1)}S(y))} \leq -2m_nS\lambda_0/3, \]
and (4), we have
\[ -(m_n - 1)S\lambda_0/3 \leq -2m_nS\lambda_0/3 + C + \log |X(y)|. \]
If \( n \) is big enough then it is not happen, and so, it is a contradiction. This proves that for any \( L \in \tilde{\Lambda} \), there is a \( T > 0 \) such that
\[ \log \| \tilde{P}^X_T \|_{\Delta^1(L)} - \log \| DX^T \|_L < 0. \]
And then by the compactness of \( \tilde{\Lambda} \), we can find \( C > 1 \) and \( \lambda > 0 \) such that for any \( L \in \tilde{\Lambda} \) and any \( t > 0 \), we have
\[ \| DX^t \|_L \cdot \| \tilde{P}^X_T \|_{\Delta^2(L)} < Ce^{-\lambda t}. \]
By a similar argument we can prove that for any \( L \in \tilde{\Lambda} \), there is a \( T > 0 \) such that
\[ \log \| \tilde{P}^{-X}_T \|_{\Delta^2(L)} + \log \| DX^{-T} \|_L < 0, \]
and then there exist \( C > 1 \) and \( \lambda > 0 \) such that for any \( L \in \tilde{\Lambda} \) and any \( t > 0 \), we have
\[ \| DX^{-t} \|_L \cdot \| \tilde{P}^{-X}_T \|_{\Delta^2(L)} < Ce^{-\lambda t}. \]
This ends the proof of the lemma. \( \square \)

Theorem A is a direct corollary of Lemma 4.5 and the following lemma in [18].

**Lemma 4.6** [18 Theorem A] Assume \( \Lambda \) is a non-trivial transitive set such that all singularity in \( \Lambda \) is hyperbolic. If there is a dominated splitting \( N_{\Lambda \setminus \text{Sing}(X)} = \Delta^1 \oplus \Delta^2 \) on \( \Lambda \setminus \text{Sing}(X) \) with respect to \( P^X_t \) and there are constant \( C > 1 \) and \( \lambda > 0 \) such that
\[ \| DX^t \|_{\Delta^1(x)} \cdot \| P^X_t \|_{\Delta^1(x)} < Ce^{-\lambda t}, \]
\[ \| DX^{-t} \|_{\Delta^2(x)} \cdot \| P^X_{-t} \|_{\Delta^2(x)} < Ce^{-\lambda t} \]
for all \( x \in \Lambda \setminus \text{Sing}(X) \) and \( t \geq 0 \), then \( \Lambda \) is positively singular hyperbolic.

**Proof of Theorem A.** Let \( X \in \mathcal{T} \) and \( \Lambda \) be an isolated transitive set of \( X \). If there is singularity \( \sigma \in \Lambda \) with index 2, then \( \Lambda \) is positively singular hyperbolic by Lemma 4.5 and Lemma 1.6. If there is a singularity \( \sigma \in \Lambda \) with index 1, then by reversing the vector fields, we know that \( \Lambda \) is negatively singular hyperbolic. This ends the proof of Theorem A. \( \square \)

**Acknowledgement.** The authors wish to express grateful to Xiao Wen for the hospitality at Beihang University in China. This work is supported by Basic Science Research Program through the National Research Foundation of Korea(NRF) funded by the Ministry of Science, ICT & Future Planning (No. 2017R1A2B4001892 and -2020R1F1A1A01051370).
References

[1] F. Abdenur, C. Bonatti, and S. Crovisier, *Global dominated splittings and the $C^1$ Newhouse phenomenon*. Proc. Amer. Math. Soc. **134** (2006), no. 8, 2229–2237.

[2] C. Bonatti and L. Díaz, *Persistent nonhyperbolic transitive diffeomorphisms*, Ann. of Math. 143 (1996), 357–396.

[3] C. Bonatti, L. Díaz and E. Pujal, *A $C^1$-generic dichotomy for diffeomorphisms: Weak forms of hyperbolicity or infinitely many sinks or sources*, Ann. of Math., **158** (2003), 355–418.

[4] C. M. Carballo, C. A. Morales and M. J. Pacifico, *Maximal transitive sets with singularities for generic $C^1$ vector fields*, Bol. Soc. Bras. Mat. **31** (2000), 287–303.

[5] L. Díaz, E. Pujals, and R. Ures, *Partial hyperbolicity and robust transitivity*, Acta Math. **183** (1999), 1–43.

[6] J. Guckenheimer, *A strange, strange attractor*, The Hopf Bifurcation Theorem and its Applications, Springer-Verlag, New York (1976).

[7] S. Liao, *Obstruction sets (I)*, Acta Math. Sinica **23** (1980), 411–453.

[8] S. Liao, *Obstruction sets (II)*, Acta Sci. Natur. Univ. Pekinensis 2 (1981), 1–36.

[9] E. N. Lorenz, *Deterministic nonperiodic flow*. J. Atmosph. Sci. **20** (1963), 130–141.

[10] M. Li, S. Gan and L. Wen, *Robustly transitive singular sets via approach of extended linear Poincaré flow*, Discrete Contin. Dyn. Syst., **13** (2005), 239–269.

[11] R. Mañé, *An erodic closing lemma*, Ann. of Math. **116** (1982), 503–540.

[12] R. Mañé, *A proof of the $C^1$ stability conjecture*, Publ. Math. IHES **66** (1988), 161–210.

[13] C. A. Morales and M. J. Pacifico, *A dichotomy for three-dimensional vector fields*, Ergod. Th. & Dynam. Syst. **23** (2003), 1575–1600.

[14] C. A. Morales, M. J. Pacifico and E. R. Pujals, *Robust transitive singular sets for 3-flows are partially hyperbolic attractors or repellers*, Ann. of Math. **160**(2004), 375–432.

[15] J. Palis and de Melo, *Geometric theory of dynamical system: an introduction*, Springer-Verlag, Nwe-York (1982).

[16] L. Wen, *Generic diffeomorphisms away from homoclinic tangencies and heterodimensional cycles*, Bull. Braz. Math. Soc. New Ser. **35**(2004), 419–452.

[17] L. Wen, *On the $C^1$ stability conjecture for flows*, J. Differ. Equat., **129**(1996), 334-357.
[18] X. Wen, L. Wen and D. Yang, *A characterization of singular hyperbolicity via the linear Poincaré flow*, J. Diff. Equat., 268 (2020), 4256–4275.

[19] L. Wen and Z. Xia, *$C^1$-connecting lemma*, Trans. Amer. Math. Soc., 352 (2000), 5213–5230.

[20] S. Zhu, S. Gan and L. Wen, *Indices of singularities of robustly transitive sets*, Discrete Contin. Dyn. Syst., 21 (2008), 945–957.