The Richardson extrapolation technique for quasilinear parabolic singularly perturbed convection-diffusion equations

G I Shishkin and L P Shishkina
Institute of Mathematics and Mechanics, Russian Academy of Science, S.Kovalevskaya str., 620219 Yekaterinburg, Russia
E-mail: shishkin@imm.uran.ru, Lida@convex.ru

Abstract. The Dirichlet problem is considered for a quasilinear singularly perturbed parabolic convection-diffusion equation on a rectangular domain. For this problem classical finite difference (nonlinear) schemes on piecewise uniform meshes condensing in the boundary layer converge $\varepsilon$-uniformly at a rate that is at best first-order. Using a Richardson extrapolation technique, we construct an improved (nonlinear) scheme that is $\varepsilon$-uniformly convergent at the rate $O((N^{-1} \ln N)^2 + N_0^{-q})$, where $N$ and $N_0$ define the number of nodes in the spatial and time meshes, respectively.

This nonlinear scheme is used in the construction of a linearized scheme where at each time level the nonlinear term is evaluated using the computed solution from the previous time level. Furthermore, using the linearized and nonlinear improved schemes, we construct a linearized improved Richardson scheme that converges $\varepsilon$-uniformly at the rate $O((N^{-1} \ln N)^2 + N_0^{-q})$, where $q \geq 2$.

1. Introduction
At present, for a wide class of singularly perturbed boundary value problems, it is known that special finite difference methods allow us to obtain $\varepsilon$-uniformly convergent (in the maximum norm) computed solutions. In the case of boundary value problems for parabolic convection-diffusion equations, the order of $\varepsilon$-uniform convergence with respect to the spatial and time variables does not exceed 1 (see, e.g., [1, 4, 8, 13, 14, 17] and also their bibliographies). Thus for parabolic convection-diffusion problems it is important to construct special schemes whose $\varepsilon$-uniform convergence rate is greater than 1 with respect to both variables.

To increase the accuracy of computed approximations of the solutions of classical boundary value problems, defect correction and Richardson extrapolation (see, e.g., [2, 11, 12] and their bibliographies) are used. The same methods are used to improve the $\varepsilon$-uniform rates of convergence of computed solutions for linear singularly perturbed problems (see, e.g., [5, 6, 7, 16, 20]). Recently, using Richardson extrapolation, $\varepsilon$-uniformly convergent finite difference schemes with improved accuracy were constructed also for quasilinear singularly perturbed reaction-diffusion parabolic [21] and elliptic [19] problems.

In [9], Richardson extrapolation is considered for a convection-diffusion problem on the rectangle. Note that the technique used in [9] (which differs from the technique of the present paper) cannot be used to justify the improvement of convergence (and in particular $\varepsilon$-uniform
convergence) for Richardson extrapolation when the parameter $\varepsilon$ is not small compared to the step-size in the $x$ variable. In [9], references can be found to papers where the Richardson method is studied for ordinary differential equations.

In the present paper, we apply Richardson extrapolation to an initial-boundary value problem for a quasilinear singularly perturbed parabolic convection-diffusion equation on a rectangular domain (the problem formulation is given in Section 2). It is known that the use of meshes condensing in the boundary layer proves to be necessary (see, e.g., [3, 13, 18]) for the construction of $\varepsilon$-uniformly convergent methods for nonlinear (and in particular quasilinear) singularly perturbed problems, just as in the case of linear problems when parabolic layers are present. Therefore, it is of interest to develop special schemes of increased $\varepsilon$-uniform accuracy on nonuniform meshes.

In this connection, to construct $\varepsilon$-uniformly convergent schemes for the quasilinear problem under investigation we apply an approach based on meshes condensing in the boundary layer (for a description of this approach, see, e.g., [1, 13, 17] and their bibliographies); when constructing the schemes, monotone classical approximations [15] of the problem are used. This approach leads to a basic special scheme on a piecewise uniform rectangular mesh that is convergent $\varepsilon$-uniformly at the rate $O(N^{-1} \ln N + N_0^{-1})$, where $N + 1$ and $N_0 + 1$ are the number of nodes in the spatial and temporal meshes respectively (see Section 3). This basic nonlinear scheme is used for the construction of a nonlinear scheme with improved accuracy by means of Richardson extrapolation (see Section 4) and a linearized basic scheme where at each time level the nonlinear term is evaluated using the computed function from the previous time level (see Section 5). The improved nonlinear Richardson scheme converges $\varepsilon$-uniformly at the rate $O((N^{-1} \ln N)^2 + N_0^{-2q})$, $q \geq 2$ (see Section 6), thereby preserving the accuracy of $\varepsilon$-uniformly convergent nonlinear schemes. A preliminary version of these results appeared in [22].

2. Problem formulation and aim of research

Set

$$
\overline{G} = G \cup S, \quad G = D \times (0,T),
$$

where $D = (0,d)$. On $\overline{G}$ we consider the Dirichlet problem for a quasilinear singularly perturbed parabolic convection-diffusion equation:

$$
L(u(x,t)) = L^2 u(x,t) - f(x,t, u(x,t)) = 0, \quad (x,t) \in G,
$$

$$
u(x,t) = \varphi(x,t), \quad (x,t) \in S.
$$

Here

$$
L^2 \equiv \varepsilon a(x,t) \frac{\partial^2}{\partial x^2} + b(x,t) \frac{\partial}{\partial x} - c(x,t) - p(x,t) \frac{\partial}{\partial t}, \quad (x,t) \in G,
$$

and the functions

$$
a(x,t), \quad b(x,t), \quad c(x,t), \quad p(x,t), \quad f(x,t, u) \quad \text{and} \quad \varphi(x,t)
$$

are assumed to be sufficiently smooth on $\overline{G}$, $\overline{G} \times R$ and $S$ respectively; moreover

$$
a_0 \leq a(x,t) \leq a^0, \quad b_0 \leq b(x,t) \leq b^0, \quad |c(x,t)| \leq c^0,
$$

$$
p_0 \leq p(x,t) \leq p^0, \quad (x,t) \in \overline{G};
$$

1 Here and below, $M$, $M_i$ (or $m$) denote sufficiently large (small) positive constants independent of the parameter $\varepsilon$ and of the discretization parameters.
and the parameter $\varepsilon$ takes arbitrary values in the half-interval $(0, 1]$.

Assume that the data of problem (2), (1) on the set of corner points $S^{s} = S_{0} \cap S_{L}$ satisfy compatibility conditions that ensure the required smoothness of the solution on $\overline{G}$ (see, e.g., [10]). Here $S = S_{0} \cup S_{L}$, where $S_{0}$ and $S_{L}$ are the lower and lateral parts of the boundary; $S_{0} = \overline{S}_{0}$.

For small values of $\varepsilon$, a regular boundary layer appears in a neighbourhood of the set $S_{1}^{L} = \{(x, t) : x = 0, 0 < t \leq T\}$.

Our aim is to construct a finite difference scheme that is $\varepsilon$-uniformly convergent, with order of accuracy greater than 1 with respect to both variables $x$ and $t$, for the boundary value problem (2), (1); this will be done via Richardson extrapolation.

3. Basic finite difference scheme

Let us give an $\varepsilon$-uniformly convergent finite difference scheme based on a classical approximation of problem (2), (1).

On the set $\overline{G}$, we introduce the rectangular mesh $G_{h} = \overline{\varpi} \times \overline{\varpi}_{0}$, (4)

where $\varpi$ and $\varpi_{0}$ are, in general, arbitrary nonuniform meshes on the intervals $[0, d]$ and $[0, T]$ respectively. Let

$h^{i} = x^{i+1} - x^{i}$ for $x^{i}, x^{i+1} \in \varpi$, $h = \max_{i} h^{i},$  

$h_{t}^{k} = t^{k+1} - t^{k}$ for $t^{k}, t^{k+1} \in \varpi_{0}$, $h_{t} = \max_{k} h_{t}^{k}.$

Assume that the conditions $h \leq M N^{-1}$ and $h_{t} \leq M_{0} N_{0}^{-1}$ are satisfied, where $N + 1$ and $N_{0} + 1$ are the number of nodes in the meshes $\varpi$ and $\varpi_{0}$ respectively.

We approximate problem (2), (1) by the finite difference scheme [15]

$$\Lambda(z(x, t)) \equiv \Lambda^{2} z(x, t) - f(x, t, z(x, t)) = 0, \quad (x, t) \in G_{h},$$

$$z(x, t) = \varphi(x, t), \quad (x, t) \in S_{h}. \quad (5)$$

Here, $G_{h} = G \cap \overline{G}_{h}$, $S_{h} = S \cap \overline{G}$, $\Lambda^{2} \equiv \varepsilon a(x, t) \delta_{x}^{2} + b(x, t) \delta_{x} - c(x, t) - p(x, t) \delta_{T}$, $(x, t) \in G_{h},$

$\delta_{x}^{2} z(x, t)$ is the second-order central difference derivative on the nonuniform mesh, i.e.,

$$\delta_{x}^{2} z(x, t) = 2(h^{i} + h^{i-1})^{-1}[\delta_{x} z(x, t) - \delta_{T} z(x, t)], \quad (x, t) = (x^{i}, t) \in G_{h},$$

while $\delta_{x} z(x, t)$ and $\delta_{T} z(x, t)$ are the first-order forward and backward difference derivatives.

The scheme (5), (4) is $\varepsilon$-uniformly monotone [15]. The following version of the comparison theorem holds.
**Theorem 1** Let the functions \( z^1(x,t) \) and \( z^2(x,t) \) satisfy the conditions
\[
\Lambda(z^1(x,t)) < \Lambda(z^2(x,t)), \quad (x,t) \in G_h, \quad z^1(x,t) > z^2(x,t), \quad (x,t) \in S_h.
\]
Then \( z^1(x,t) > z^2(x,t) \) for all \( (x,t) \in \overline{G}_h \).

In the case where the mesh \( G_h = \omega \times \omega_0 \) (6) is uniform in both variables, by using the maximum principle we can show that
\[
|u(x,t) - z(x,t)| \leq M \left[ (\varepsilon + N^{-1})^{-1} N^{-1} + N_0^{-1} \right], \quad (x,t) \in \overline{G}_h.
\]
(7)

Let us construct the basic \( \varepsilon \)-uniformly convergent scheme (see, e.g., [13, 17]). On the set \( \overline{G} \), we introduce the mesh \( \overline{G}_h = \omega^* \times \omega_0 \), (8a)
where \( \omega_0 = \omega_0(6) \) and \( \omega^* \) is a special piecewise uniform mesh constructed as follows. The interval \([0,d]\) is divided into two parts \([0,\sigma]\) and \([\sigma,d]\), while the step size in each part is constant and equal to
\[
h^{(1)} = 2\sigma N^{-1} \quad \text{and} \quad h^{(2)} = 2(d - \sigma)N^{-1},
\]
respectively. The parameter \( \sigma \) is defined by
\[
\sigma = \sigma(\varepsilon, N, l) = \min \left[ 2^{-1}d, l m^{-1} \varepsilon \ln N \right], \quad (8b)
\]
where \( m \) is an arbitrary number in the interval \((0,m_0)\),
\[
m_0 = \min_{\overline{G}} [a^{-1}(x,t) b(x,t)].
\]
(8c)
Here,
\[
l = 1;
\]
for later meshes, this parameter will be chosen differently. The construction of the meshes \( \omega^* \) and \( \overline{G}_h = \overline{G}_h(l = 1) \) is now complete.

For solutions of the difference scheme (5), (8), we obtain the estimate
\[
|u(x,t) - z(x,t)| \leq M \left[ N^{-1} \min \left[ \varepsilon^{-1}, \ln N \right] + N_0^{-1} \right], \quad (x,t) \in \overline{G}_h,
\]
(9)
and also the \( \varepsilon \)-uniform estimate
\[
|u(x,t) - z(x,t)| \leq M \left[ N^{-1} \ln N + N_0^{-1} \right], \quad (x,t) \in \overline{G}_h.
\]
(10)

**Theorem 2** Let the solution \( u(x,t) \) of problem (2), (1) satisfy the estimates in Theorem 6, where \( K = 4 \). Then the difference scheme (5), (6) converges for fixed values of the parameter \( \varepsilon \), while the scheme (5), (8) converges \( \varepsilon \)-uniformly. Their discrete solutions satisfy the respective estimates (7), (9) and (10).
4. Richardson extrapolation for (2), (1)

Let us describe the Richardson extrapolation method used to increase the accuracy of computed solutions of the special basic scheme.

1. On the set $\mathcal{G}$, we construct the meshes

$$\mathcal{G}_{h}^{i} = \mathcal{W}^{*i} \times \mathcal{W}_{0}^{*i}, \quad i = 1, 2,$$

which are uniform in the $t$ direction and piecewise uniform in the $x$ direction. Here $\mathcal{G}_{h}^2$ is $\mathcal{G}_{h(8a)}$, where

$$\omega = \sigma(8b)(\varepsilon, N, l) \text{ with } l = 2;$$

$\mathcal{G}_{h}^{0}$ is a coarsened mesh. For the parameters $\sigma^i$ that define piecewise uniform meshes $\mathcal{W}^{*i} = \mathcal{W}^{*i}(\sigma^i)$, we impose the condition $\sigma^1 = \sigma^2$, where $\sigma^2 = \sigma(11b)$; that is, the intervals on which the meshes $\mathcal{W}^{*1}$ and $\mathcal{W}^{*2}$ have a constant step-size are identical. The step-sizes in the mesh $\mathcal{W}^{*1}$ are $k$ times larger than the step-sizes in the mesh $\mathcal{W}^{*2}$, and the step-size in the mesh $\mathcal{W}_{0}^{*1}$ (on the interval $[0, T]$) is $k$ times larger than the step-size in the mesh $\mathcal{W}_{0}^{*2}$; here $k^{-1}N + 1$ and $k^{-1}N_{0} + 1$ are the number of nodes in the meshes $\mathcal{W}^{*1}$ and $\mathcal{W}^{*1}$ respectively. Let

$$\mathcal{G}_{h}^{0} = \mathcal{G}_{h}^{1} \cap \mathcal{G}_{h}^{2}.$$  

We have $\mathcal{G}_{h}^{0} = \mathcal{G}_{h}^{1}$ for integer $k \geq 2$, and $\mathcal{G}_{h}^{0} \neq \mathcal{G}_{h}^{1}$ for noninteger $k$.

Let $z^{i}(x, t)$, $(x, t) \in \mathcal{G}_{h}^{i}$, for $i = 1, 2$, be the solutions of the difference schemes

$$\Lambda_{(5)} \left( z^{i}(x, t) \right) = 0, \quad (x, t) \in \mathcal{G}^{i}_{h}, \quad \text{ (12a)}$$

$$z^{i}(x, t) = \phi(x, t), \quad (x, t) \in S^{i}_{h}, \quad i = 1, 2.$$  \hspace{1cm} \text{ (12b)}$$

We set

$$z^{0}(x, t) = \gamma z^{1}(x, t) + (1 - \gamma) z^{2}(x, t), \quad (x, t) \in \mathcal{G}_{h}^{0}, \quad \text{ (12c)}$$

where

$$\gamma = \gamma(k) = -(k - 1)^{-1}.$$ 

The difference scheme (12), (11) is constructed on the basis of the nonlinear scheme (5), (8) using a Richardson extrapolation technique on two embedded meshes. We call this scheme the Richardson scheme (12), (11). We call the function $z^{0}_{(12)}(x, t)$, $(x, t) \in \mathcal{G}_{h}^{0}$, the solution of the Richardson scheme (12), (11); we call the functions $z^{1}_{(12)}(x, t)$, $(x, t) \in \mathcal{G}_{h}^{1}$ and $z^{2}_{(12)}(x, t)$, $(x, t) \in \mathcal{G}_{h}^{2}$ the components that generate the solution of the Richardson scheme (12), (11).

2. To justify the convergence of the Richardson scheme (12), (11), we apply a technique similar to one used in [7, 19, 21]. It is convenient to consider the expansions of the functions $z^{i}(x, t)$, $(x, t) \in \mathcal{G}^{i}_{h}$, $i = 1, 2$, with respect to the values $N^{-1}$ and $N_{0}^{-1}$:

$$z^{i}(x, t) = u(x, t) + k^{i} \left[ N^{-1} u_{1}(x, t) + N_{0}^{-1} u_{0}(x, t) \right] + v^{i}(x, t), \quad (x, t) \in \mathcal{G}^{i}_{h}, \quad i = 1, 2,$$

\[
\text{ (13)}
\]

\[\text{---}
\]

2 Throughout the paper, the notation $L_{(j,k)}$ ($M_{(j,k)}$, $G_{h(j,k)}$) means that these operators (constants, grids) are introduced in formula $(j,k)$.
where \( v^i(x, t) \) is the remainder term. Here the function \( u_0(x, t) \) is the solution of the problem

\[
L_{(14)} u_0(x, t) = \left\{ L_2 f(x, t, u(x, t)) \right\} u_0(x, t)
\]

\[
= - 2^{-1} T p(x, t) \frac{\partial^2}{\partial t^2} u(x, t), \quad (x, t) \in G,
\]

\[
u_0(x, t) = 0, \quad (x, t) \in S.
\]
The function \( u_1(x, t) \) can be written as the sum of functions

\[
u_1(x, t) = u_{11}(x, t) + u_{12}(x, t) + u_{13}(x, t), \quad (x, t) \in \overline{G}.
\]

Here the \( u_{ij}(x, t), (x, t) \in G \), are the solutions of the problems

\[
L_{(14)} u_{11}(x, t) = - \sigma b(x, t) \frac{\partial^2}{\partial x^2} V(x, t), \quad (x, t) \in G,
\]

\[
u_{11}(x, t) = 0, \quad (x, t) \in S;
\]

\[
L_{(14)} u_{12}(x, t) = - \sigma b(x, t) \frac{\partial^2}{\partial x^2} U(x, t), \quad (x, t) \in G,
\]

\[
u_{12}(x, t) = 0, \quad (x, t) \in S;
\]

\[
L_{(14)} u_{13}(x, t) = \left\{ \begin{array}{ll}
0, & x < \sigma \\
-d - 2 \sigma b(x, t) \frac{\partial^2}{\partial x^2} U(x, t), & x > \sigma
\end{array} \right\}, \quad (x, t) \in G,
\]

\[
u_{13}(x, t) = 0, \quad (x, t) \in S,
\]

with

\[
U(x, t) = U_{(27)}(x, t), \quad V(x, t) = V_{(27)}(x, t), \quad (x, t) \in \overline{G}, \quad \sigma = \sigma_{(11)}.
\]
The functions \( u_0(x, t), u_{11}(x, t), u_{12}(x, t), u_{13}(x, t) \) satisfy the estimates

\[
|u_0(x, t)|, |u_{12}(x, t)| \leq M,
\]

\[
|u_{11}(x, t)| \leq M \ln N,
\]

\[
|u_{13}(x, t)| \leq M \sigma, \quad (x, t) \in \overline{G}.
\]
The components \( u_0(x, t), u_{11}(x, t), u_{12}(x, t) \) are sufficiently smooth on \( \overline{G} \), and the component \( u_{13}(x, t) \) is sufficiently smooth on \( \overline{G} \) in the sets \( x \leq \sigma \) and \( x \geq \sigma \). Taking into account estimates for the components \( u_1(x, t) \) and \( u_0(x, t) \), we obtain the estimate for the function \( v^i(x, t) \):

\[
|v^i(x, t)| \leq M \left[ (N^{-1} \ln N)^2 + N_0^{-2} \right], \quad (x, t) \in \overline{G}_h, \quad i = 1, 2.
\]

Then the estimate (19) yields

\[
|u(x, t) - z^0(x, t)| \leq M \left[ (N^{-1} \ln N)^2 + N_0^{-2} \right], \quad (x, t) \in \overline{G}_h.
\]

**Theorem 3** In the boundary value problem (2), (1), assume that \( a, b, c, p \in C^6+\alpha(\overline{G}), f \in C^{6+\alpha}(\overline{G} \times R), \varphi \in C^{6+\alpha}(S), \alpha > 0 \), that the conditions (3) and (31) are satisfied for \( l = 6 \), and that for the components in the representation (27) of solutions to the problem (2), (1), the a priori estimates (33) are satisfied for \( K = 6 \). Then the solution \( z^0(x, t) \) of the Richardson scheme (12), (11) converges to the solution \( u(x, t) \) of the boundary value problem at an \( \varepsilon \)-uniform rate of \( O((N^{-1} \ln N)^2 + N_0^{-2}) \): the computed solution satisfies the estimate (20).
5. Linearized basic scheme

On the mesh (4), we consider a difference scheme in which the nonlinear term of the differential equation is evaluated at each time level by using the computed solution at the previous time level. Thus to the boundary value problem (2), (1) we assign the linearized difference scheme (see [15])

\[
\Lambda_{(21)}(z(x, t)) \equiv \Lambda_0^2 z(x, t) - f(x, t, \tilde{z}(x, t)) = 0, \quad (x, t) \in G_h,
\]

\[
z(x, t) = \varphi(x, t), \quad (x, t) \in S_h.
\]  

(21)

Here

\[
\tilde{z}(x, t) = z(x, t - h_t), \quad (x, t) \in G_h, \quad t > 0
\]

Under the condition

\[
\frac{\partial}{\partial u} f(x, t, u) \leq c(x, t), \quad (x, t, u) \in G \times R,
\]  

(22)

the difference scheme (21), (4) is monotone.

For simplicity, we assume that the condition (22) is satisfied.

Taking into account estimates of the solution to problem (2), (1) for the linearized difference scheme (21) on the special mesh (8), we obtain an estimate similar to estimate (10):

\[
|u(x, t) - z(x, t)| \leq M \left[ N^{-1} \ln N + N_0^{-1} \right], \quad (x, t) \in \overline{G}_h.
\]  

(23)

If the condition (22) is not satisfied, then in problem (21), (4), we pass from the function \( z(x, t) \) to the function \( z^*(x, t) \) defined by

\[
z(x, t) = z^*(x, t) \exp(\alpha t),
\]

and we choose the constant \( \alpha \) sufficiently large so that

\[
\frac{\partial}{\partial u} f(x, t, u) \leq c(x, t) + \delta_T \left[ \exp(\alpha t) \right] p(x, t), \quad (x, t, u) \in G \times R;
\]

this will ensure the monotonicity of the discrete problem obtained. Furthermore, we can then establish the convergence of the function \( z^*(x, t) \) to the function \( u^*(x, t) \) defined by

\[
u(x, t) = u^*(x, t) \exp(\alpha t).
\]

Returning to the function \( z(x, t) \), we finally obtain the estimate (23).

**Theorem 4** Let the hypotheses of Theorem 2 be fulfilled. Then the solution of the linearized difference scheme (21), (8) converges \( \varepsilon \)-uniformly: the estimate (23) holds for the computed solution.

6. Linearized scheme of improved accuracy

Let us present a Richardson scheme of improved accuracy whose construction is based on the linearized scheme (21), (8).

Let \( z^i(x, t), (x, t) \in \overline{G}_h^i, \ i = 1, 2 \), be the solutions of the difference schemes

\[
\Lambda_{(21)} \left( z^i(x, t) \right) = 0, \quad (x, t) \in G_h^i,
\]  

(24a)

\[
z^i(x, t) = \varphi(x, t), \quad (x, t) \in S_h^i, \quad i = 1, 2,
\]
where $G_{h}^{i} = \overline{G}_{h}^{i(1)}$. On the set $G_{h}^{0} = \overline{G}_{h}^{0(1)}$, we introduce the function

$$z^{0}(x, t) = \gamma z^{1}(x, t) + (1 - \gamma) z^{2}(x, t), \quad (x, t) \in G_{h}^{0},$$

(24b)

where $\gamma = \gamma(12)$; here the functions

$$z^{i}(x, t), \quad (x, t) \in G_{h}^{i}, \quad i = 1, 2,$$

are the solutions of the problem (24), (11).

We call the linearized difference scheme (24), (11) the linearized Richardson scheme (24), (11). We call the function $z^{0}(24)(x, t), (x, t) \in G_{h}^{0}$, the solution of the linearized Richardson scheme (24), (11).

The analysis of the scheme (24), (11) is similar to that for the scheme (12), (11). In the representation (13) of the solution of scheme (24), (11), the component $u_{0}(x, t)$ is the solution of a problem similar to problem (14):

$$L_{(14)}u_{0}(x, t) = -T \left[ 2^{-1} p(x, t) \frac{\partial}{\partial t} u(x, t) + \frac{\partial}{\partial u} f(x, t, u(x, t)) \frac{\partial}{\partial t} u(x, t) \right],$$

$$u_{0}(x, t) = 0, \quad (x, t) \in S.$$

In the same representation (13), the component $u_{i}(x, t)$ has the form (15). In (15) the functions $u_{1j}(x, t), (x, t) \in G$, are the solutions of problems (16)–(18).

For the solution of problem (24), (11) we obtain an estimate similar to (20):

$$|u(x, t) - z^{0}(x, t)| \leq M \left[ (N^{-1} \ln N)^{2} + N_{0}^{-2} \right], \quad (x, t) \in G_{h}^{0}. \quad (25)$$

**Theorem 5** Let the hypotheses of Theorem 3 be fulfilled. Then the solution of the linearized Richardson scheme (24), (11) converges to the solution of the boundary value problem (2), (1) $\varepsilon$-uniformly at the rate $O((N^{-1} \ln N)^{2} + N_{0}^{-q})$: the estimate (25) is valid.

**Remark 1** Using Richardson extrapolation on $q > 2$ embedded meshes in the time variable $t$ allows us to obtain schemes that are $\varepsilon$-uniformly convergent at the rate $O((N^{-1} \ln N)^{2} + N_{0}^{-q})$ (see, e.g., [16]). The consideration of examples—even for linear problems—shows however that increasing the number of embedded meshes in the spatial variable $x$ does not allow us to construct schemes that are $\varepsilon$-uniformly convergent with accuracy in $N^{-1}$ greater than 2.

7. **A priori estimates of the solution and its derivatives**

Let us give a priori estimates for the solution of the boundary value problem (2), (1) and its derivatives. The derivation of these estimates is similar to [17].

Using the majorant function technique (see, e.g., [10]), we obtain

$$|u(x, t)| \leq M, \quad (x, t) \in G. \quad (26)$$

Write the solution of the problem as the sum of functions

$$u(x, t) = U(x, t) + V(x, t), \quad (x, t) \in G, \quad (27a)$$
where \( U(x, t) \) and \( V(x, t) \) are the regular and singular parts of the solution. The function \( U(x, t) \), 
\((x, t) \in \overline{G} \), is the restriction to \( \overline{G} \) of the function \( U^0(x, t) \), 
\((x, t) \in \overline{G}^0 \). Here the function \( U^0(x, t) \), 
\((x, t) \in \overline{G}^0 \), is the solution of the boundary value problem

\[
L^0 \left( U^0(x, t) \right) = L^{20} U^0(x, t) - f^0 \left( x, t, U^0(x, t) \right) = 0, \quad (x, t) \in \overline{G}^0, \quad (28)
\]

\[
U^0(x, t) = \varphi^0(x, t), \quad (x, t) \in S^0,
\]

and \( \overline{G}^0 \) is the half-strip \((x \leq d)\), which is an extension of \( \overline{G} \) beyond the side \( S^1_1 \); the data of problem (28) are smooth continuations of the data of problem (2), (1) that preserve properties (3) on \( \overline{G}^0 \). Assume that the functions

\[
f^0(x, t, u), \quad (x, t, u) \in \overline{G} \times R, \quad \text{and} \quad \varphi^0(x, t), \quad (x, t) \in S^0,
\]

are equal to zero outside the \( m_1 \)-neighbourhood of the set \( \overline{G} \); finally, the operator \( L^{20} \) is defined by

\[L^{20} \equiv \varepsilon \frac{\partial^2}{\partial x^2} + \frac{\partial}{\partial x} - 1 - \frac{\partial}{\partial t} \]

The function \( V(x, t) \), \((x, t) \in \overline{G} \), is the solution of the problem

\[
L^2_{(2)} V(x, t) - \left[ f(x, t, U(x, t) + V(x, t)) - f(x, t, U(x, t)) \right] = 0, \quad (x, t) \in G,
\]

\[
V(x, t) = \varphi(x, t) - U(x, t), \quad (x, t) \in S.
\]

We write the function \( U(x, t) \) as a sum of functions:

\[
U(x, t) = U_0(x, t) + \varepsilon U_1(x, t) + \ldots + \varepsilon^n U_n(x, t) + v_\Sigma(x, t), \quad (x, t) \in \overline{G}. \quad (27b)
\]

This is simply the restriction to \( \overline{G} \) of the following representation of the function \( U^0(x, t) \):

\[
U^0(x, t) = U^0_0(x, t) + \varepsilon U^0_1(x, t) + \ldots + \varepsilon^n U^0_n(x, t) + v^0_\Sigma(x, t), \quad (x, t) \in \overline{G}^0, \quad (30a)
\]

which is the solution of the boundary value problem (28). Here \( v^0_\Sigma(x, t) \) is the remainder term and

\[
U(x, t) = U^0(x, t), \ldots, v_\Sigma(x, t) = v^0_\Sigma(x, t), \quad (x, t) \in \overline{G}.
\]

In (30a), the functions

\[
U^0_0(x, t), \quad U^0_i(x, t) \quad \text{for} \quad i = 1, \ldots, n,
\]

are the solutions of the problems

\[
L^0_{(30a)} U^0_0(x, t) - f^0(x, t, U^0_0(x, t)) = 0, \quad (x, t) \in G^0, \quad (30b)
\]

\[
U^0_i(x, t) = \varphi^0(x, t), \quad (x, t) \in S^0;
\]

\[
L^0_{(30a)} U^0_i(x, t) - \varepsilon^{-i} \left[ f^0 \left( x, t, \sum_{j=0}^{i-1} \varepsilon^j U^0_j(x, t) \right) - f \left( x, t, \sum_{j=0}^{i-1} \varepsilon^j U^0_j(x, t) \right) \right] = -a^0(x, t) \frac{\partial^2}{\partial x^2} U^0_{i-1}(x, t), \quad (x, t) \in G^0,
\]

\[
U^0_i(x, t) = 0, \quad (x, t) \in S^0, \quad i = 1, \ldots, n.
\]
Here \( L^0_{(30)} \) is the operator \( L^{20}_{(28)} \) for \( \varepsilon = 0 \), i.e.,
\[
L^0_{(30)} = b^0(x, t) \frac{\partial}{\partial x} - c^0(x, t) - p^0(x, t) \frac{\partial}{\partial t}, \quad (x, t) \in \mathcal{G}^0,
\]
and the function \( a^0(x, t) \) is the coefficient of the derivative \( \partial^2/\partial x^2 \) in the operator \( L^{20}_{(28)} \); it is an extension of \( a(x, t) \) to \( \mathcal{G}^0 \) that preserves the properties (3).

Assume that the data of problem (2), (1) satisfy additional conditions on the set \( S_2^* = S_0 \cap \overline{S}_2^L \) that ensure sufficient smoothness of the functions \( U_i^0(x, t) \) and \( U_i^0(x, t) \) for \( i = 1, \ldots, n \). It is not difficult to write down such conditions, for example in the case when the boundary function \( \varphi(x, 0) \) and its derivatives vanish on the set \( S^* \).

For simplicity, we assume that the following conditions are valid:
\[
\frac{\partial^k}{\partial x^k} \varphi(x, t), \quad \frac{\partial^{k_0}}{\partial t^{k_0}} \varphi(x, t) = 0, \quad (x, t) \in S^*, \tag{31}
\]
where \( l = 3n + 2, \ n \geq 1 \).

Assuming sufficient smoothness of the coefficients of the operator \( L^2_{(2)} \) on \( \mathcal{G} \), and of the functions \( \varphi(x, t) \) on \( S_0, S^L \) and \( f(x, t, u) \) on \( \mathcal{G} \times R \), one has
\[
u, U \in C^{l_1, l_1}(\mathcal{G}), \quad U_i \in C^{l_2, l_2}(\mathcal{G}), \quad i = 0, 1, \ldots, n, \tag{32}
\]
where
\[
l_1 = n + 1 + \alpha, \quad l_2 = 3n + 2 - 2i, \quad n \geq 1, \quad \alpha > 0.
\]
In this case, \( V \in C^{l_1, l_1}(\mathcal{G}) \).

For \( U(x, t) \) and \( V(x, t) \) we obtain the estimates
\[
\left| \frac{\partial^{k+k_0}}{\partial x^k \partial t^{k_0}} U(x, t) \right| \leq M [1 + \varepsilon^{n+1-k-k_0}], \tag{33a}
\]
\[
\left| \frac{\partial^{k+k_0}}{\partial x^k \partial t^{k_0}} V(x, t) \right| \leq M \varepsilon^{-k} \exp(-m^{-1}x), \quad (x, t) \in \mathcal{G}, \quad k + k_0 \leq K, \tag{33b}
\]
where \( m \) is an arbitrary number in the interval \( (0, m_0) \) and \( m_0 = \min \{a^{-1}(x, t) b(x, t) \} \); here \( K = n + 1 \) when the data of the boundary value problem (2), (1) are sufficiently smooth.

The function \( u(x, t) \) satisfies the estimate
\[
\left| \frac{\partial^{k+k_0}}{\partial x^k \partial t^{k_0}} u(x, t) \right| \leq M \varepsilon^{-k-k_0}, \quad (x, t) \in \mathcal{G}, \quad k + k_0 \leq K. \tag{34}
\]

**Theorem 6** In the boundary value problem (2), (1) assume that \( a, b, c, p \in C^{l, l_1}(\mathcal{G}), f \in C^{l, l_1}(\mathcal{G} \times R), \varphi \in C^{l_1}(\mathcal{G}), l = 3n + 2, n \geq 1 \), and also that condition (31) is satisfied. Then the estimates (26), (33) and (34), where \( K = n + 1 \), are satisfied for the solution of the boundary value problem and its components in the representation (27).
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