Antilinear superoperator and quantum geometric invariance for higher-dimensional quantum systems

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We present an investigation of the antilinear superoperators and their applications in studying higher-dimensional quantum systems. The antilinear superoperators are introduced and various properties are discussed. We study several crucial classes of antilinear superoperators, including antilinear quantum channels, antilinearly unital superoperators, antiunitary superoperators and the generalized Θ-conjugation. Then using the Bloch representation, we present a systematic investigation of the quantum geometric transformations of higher-dimensional quantum systems. By choosing different generalized Θ-conjugation, different metrics for the space of Bloch space-time vectors are obtained, including the Euclidean metric and Minkowskian metric. Then using these geometric structures, we investigate the entanglement distribution over a multipartite system restricted by quantum geometric invariance.

I. INTRODUCTION

With the development of quantum information and quantum computation theory, the investigation of preparation, transformation, and measurement of quantum states becomes more and more important [1–3]. According to Wigner theorem [4, 5], symmetry operations for a closed quantum system can only be unitary or antiunitary. It’s well-known that for an open quantum system, the unitary operations reduce to completely positive trace-preserving (CPTP) maps. Therefore it’s natural to consider the case corresponding to antiunitary operations. To our knowledge, this has not been systematically investigated. By tracing the environment, the antiunitary operations of closed system reduce to antilinear CPTP maps, which are special cases of more general antilinear superoperators. Though, some special classes of antilinear operators, like time-reversal symmetry operation and Hill-Wootters conjugation have been studied [6]. A more systematic investigation is given by Ulhmann [7, 8], he proposed an antilinear transformation called Θ-conjugation, which plays a crucial role in studying quantum entanglement [6, 7, 9]. However, in an open quantum system, it’s more natural to consider the antilinear superoperators and this will be one of the main focuses of this work.

For the simplest case, qubit system, Bloch representation provides an extremely convenient description of the quantum system [1, 2], where the Θ-conjugation has a concise representation using the Bloch vectors. However, the generalization of Bloch representation to the higher-dimensional case remains open [10–12]. Two reasons behind this are that in higher-dimensional case, the set of Bloch vector for quantum states is no longer a ball and the higher-dimensional vectors are more complicated to visualize and manipulate.

In this work, we will discuss the Bloch representation for a higher-dimensional quantum system based on the Hilbert-Schmidt basis of the real vector space consisting of all Hermitian operators. This provides us a good framework to study the generalized Θ-conjugation, which, by definition, is a class of antilinear superoperators. To this end, we present a systematical investigation of the antilinear superoperators, including antilinear quantum channel, antilinear unital superoperators, antiunitary superoperator and generalized Θ-conjugation. The generalized Θ-conjugation turns out to be closely related to the geometric transformations of a quantum state. For qubit case, these geometric transformations and geometric invariance have been extensively explored [13–17]. The Lorentz transformation corresponds to stochastic local operation and classical communication (SLOCC) (see, e.g. [18] and references therein). For the qubit case, using the generalized Θ-conjugation, we present a concise description of the Lorentzian and Euclidean invariance of a quantum system. The Lorentzian invariance of the quantum states can be used to study the entanglement distribution over a pure multipartite system [17].

The work is organized as follows. In Sec. II, we briefly discussed the Bloch representation of higher-dimensional quantum states based on Hilbert-Schmidt basis. Then in Sec. III, we systematically study the antilinear superoperators, including antilinearly CPTP maps, antilinear unital superoperators, antiunitary superoperators, and generalized Θ-conjugation. The geometric representation of generalized Θ-conjugation and its relationship with quantum geometric invariance are discussed in Sec. IV. Using the quantum geometric invariance of the quan-

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For semidefinite trace-one operators: d-dimensional 
\begin{equation}
\Lambda = 2 \frac{d(d-1)}{2} \text{symmetric GGM which corresponds to Pauli X-matrix}
\end{equation}
\begin{equation}
\Lambda_{x}^{j} = \sqrt{\frac{d}{2}} |j\rangle \langle k| + |k\rangle \langle j|, \quad 0 \leq j < k \leq d-1;
\end{equation}
\begin{equation}
(2) \frac{d(d-1)}{2} \text{antisymmetric GGM which corresponds to Pauli Y-matrix}
\end{equation}
\begin{equation}
\Lambda_{y}^{j} = \sqrt{\frac{d}{2}} (-i) |j\rangle \langle k| + i |k\rangle \langle j|, \quad 0 \leq j < k \leq d-1;
\end{equation}
\begin{equation}
(3) \text{(d-1) diagonal GGM which corresponds to Pauli Z-matrix}
\end{equation}
\begin{equation}
\Lambda_{z} = \sqrt{\frac{d}{(l+1)(l+2)}} \left( \sum_{j=0}^{l} |j\rangle \langle j| - (l+1)l \langle l+1| l+1\right), \quad 0 \leq l \leq d-2;\)
\begin{equation}
(4) \text{The identity matrix I.}
\end{equation}
There are in total \(\frac{d(d-1)}{2} + \frac{d(d-1)}{2} + (d-1) + 1 = d^2 - 1\) matrices. Notice that GGM matrices are generators of \(su(d)\) and they also serve as a basis of complex vector space \(B(H)\).

Since \(H(H) \simeq \mathbb{R}^{d^2}\), a density operator can be uniquely represented in Hilbert-Schmidt basis as
\begin{equation}
\rho = \frac{1}{d} \sum_{\mu=0}^{d^2-1} x_\mu \sigma_\mu = \frac{1}{d} (\sigma_0 + \vec{x} \cdot \vec{\sigma}),
\end{equation}
where \(x_\mu \in \mathbb{R}\) and \(x_0 = 1\) since all \(\sigma_\mu\) are traceless except \(\sigma_0\) and the density operator is trace-one. The \((d^2-1)\)-dimensional vector \(\vec{x}\) is called a Bloch vector (or coherence vector). Notice all GGM matrices are Hermitian, thus, they can be regarded as observables. One of the advantage for this kind of representation is that
\begin{equation}
\langle \sigma_\mu \rangle = \text{Tr}(\sigma_\mu \rho) = x_\mu.
\end{equation}
By measuring the expectation value of these \(d^2 - 1\) observables, we can determine the state \([10]\).

From the condition that purity \(\text{Tr}(\rho^2) \leq 1\), we see that
\begin{equation}
\|\vec{x}\|^2 \leq d - 1.
\end{equation}
For pure states, \(\|\vec{x}\|^2 = d - 1\), and for mixed states, \(\|\vec{x}\|^2 < d - 1\). Notice that Eq. (6) is not sufficient for \(\rho\) in Eq. (4) to be a density operator, the condition that \(\rho \geq 0\) still need to be imposed \([10-12]\). It has been shown that the angle between any two pure-state Bloch vectors \(\vec{x}\) and \(\vec{y}\) must satisfy \([11]\)
\begin{equation}
- \frac{1}{d-1} \leq \cos(\theta_{\vec{x},\vec{y}}) \leq 1.
\end{equation}
This implies that the set of all Bloch vectors for qudit states forms a convex subset of \((d^2-1)\)-dimensional ball \(B^{d^2-1}(0; \sqrt{\langle d - 1\rangle})\) with radius \(\sqrt{\langle d - 1\rangle}\). To impose the condition that \(\rho \geq 0\) (all eigenvalues are nonnegative), we consider the characteristic polynomial
\begin{equation}
\det(\lambda I - \rho) = \sum_{j=0}^{d} (-1)^j a_j \lambda^{d-j}.
\end{equation}
Using Vieta’s theorem \(a_j = \sum_{1 \leq k_1 < \cdots < k_j \leq d} \lambda_{k_1} \cdots \lambda_{k_j}\), it can be proved that \(\rho \geq 0\) is equivalent to \(a_0, \ldots, a_d \geq 0\) \([12]\). With this, each Bloch vector \(\vec{x}\) corresponds to a set of real coefficients of the characteristic polynomial, \(a_j(\vec{x})\). Thus the Bloch convex body corresponding to the set of all density matrices can be defined as
\begin{equation}
B(d^2 - 1) = \{ \vec{x} \in \mathbb{R}^{d^2-1} | a_j(\vec{x}) \geq 0, \forall j \}.
\end{equation}
For \(d = 2\), the Bloch body is exactly a ball, for higher-dimensional case, the shapes are very complicated.

Example 1 (3-dimensional Bloch convex body). For 3-dimensional case, the 9 GGM matrices are
The efficient of characteristic polynomial in Eq. (8) can be where the nonzero structure constants are:

\[ f = \begin{pmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}, \]

\[ g = \begin{pmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \\ 1 & 0 & 0 \end{pmatrix}, \]

\[ h = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 1 & 0 \\ 1 & 0 & 0 \end{pmatrix}. \]

3 antisymmetric matrices:

\[ \sigma_2 = \sqrt{\frac{3}{2}} \begin{pmatrix} 0 & -i & 0 \\ i & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}, \]

\[ \sigma_5 = \sqrt{\frac{3}{2}} \begin{pmatrix} 0 & 0 & -i \\ 0 & 0 & 0 \\ -i & 0 & 0 \end{pmatrix}, \]

\[ \sigma_7 = \sqrt{\frac{3}{2}} \begin{pmatrix} 0 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & -2 \end{pmatrix}. \]

2 diagonal matrices:

\[ \sigma_3 = \sqrt{\frac{3}{2}} \begin{pmatrix} 1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & 0 \end{pmatrix}, \]

\[ \sigma_8 = \frac{1}{\sqrt{2}} \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 2 \end{pmatrix}. \]

1 identity operator \( \sigma_0 = I \).

These are \( \mathfrak{su}(3) \) generators and satisfy the commutation relation

\[ [\sigma_j, \sigma_k] = i2\sqrt{\frac{3}{2}} \sum_l f_{jkl}\sigma_l, \]

where the nonzero structure constants are: \( f_{123} = 1, f_{147} = f_{246} = f_{257} = f_{345} = 1/2, f_{156} = f_{367} = -1/2, f_{246} = 1/2 \) and \( f_{458} = f_{678} = \sqrt{3}/2. \) The anticommutation relation is

\[ \{\sigma_j, \sigma_k\} = 2\delta_{jk}I + \sqrt{6} \sum_l g_{jkl}\sigma_l, \]

where nonzero structure constants are \( g_{118} = g_{228} = g_{338} = -g_{888} = 1/\sqrt{3}, g_{448} = g_{558} = g_{668} = g_{778} = -1/2\sqrt{3}, g_{146} = g_{157} = g_{256} = g_{344} = g_{355} = -g_{247} = -g_{366} = -g_{377} = 1/2. \)

For a density operator \( \rho = \frac{1}{2}(I + \sum_{i=1}^{8} x_i\sigma_i) \), the coefficient of characteristic polynomial in Eq. (8) can be explicitly calculated using the Newton identities,

\[ ka_k = \sum_{j=1}^{k}(-1)^{k-j}N_ja_{k-j}, k = 1, 2, 3 \]

where \( N_j = \sum_{i=1}^{d}(\lambda_i)^j \) is the \( j \)-th power sum of all eigenvalues of \( \rho \). In this way, we see that

\[ 1\alpha_1 = N_1 = \text{Tr} \rho = 1, \]

\[ 2\alpha_2 = N_1^2 - N_2 = 1 - \text{Tr}(\rho^2), \]

\[ 3\alpha_3 = N_1^3 - N_1N_2 = 2N_2 + 2N_3 = 1 - 3\text{Tr}(\rho^2) + 2\text{Tr}(\rho^3). \]

The condition that \( \rho \geq 0 \) now becomes \( j\alpha_j \geq 0 \) for all \( j \), which can be explicitly calculated by using the structure constants \( f_{jkl} \) and \( g_{jkl} \) of communication and anti-communication relations \( \sigma_j\sigma_k = \frac{1}{2}(\{\sigma_j, \sigma_k\} + \{\sigma_j, \sigma_k\}) = \frac{1}{2}(i2\sqrt{\frac{3}{2}} \sum_l f_{jkl}\sigma_l + 2\delta_{jk}I + \sqrt{6} \sum_l g_{jkl}\sigma_l). \) See Ref. [11] for more details.

III. ANTIKERNEL SUPEROPERATORS

The antilinear operators have many applications in physics since Wigner observed that time-reversal operation in quantum mechanics is characterized by antunitary operators [4, 5]. In quantum information theory, Hill-Wootters conjugation [6] and Uhlmann’s generalization of \( \Theta \)-conjugation [7] are crucial examples of the antilinear operator which play a key role in investigating quantum correlations. However, a systematic investigation of antilinear operators is still lacking [8, 21]. In this section, we will study these more general case, the antilinear superoperators, for which antilinear operators are special cases. Various special classes of the antilinear superoperators will be discussed, and these antilinear superoperators may have more potential applications in investigating quantum information theory. Particular applications in the study of geometric invariance and distribution of quantum correlations for higher-dimensional quantum systems will be discussed later in this work.

A. Representations of antilinear superoperators

To begin with, let’s introduce the basic definitions related to the antilinear (or conjugate linear) superoperators, which are natural generalizations of their linear counterparts.

Definition 2. Let \( \mathcal{M} : B(X) \rightarrow B(Y) \) be a mapping between Banach spaces \( B(X) \) and \( B(Y) \). It is called an antilinear superoperator if

\[ \mathcal{M}(\alpha \rho_1 + \beta \rho_2) = \alpha^* \mathcal{M}(\rho_1) + \beta^* \mathcal{M}(\rho_2) \]

for all \( \alpha, \beta \in \mathbb{C} \) and \( \rho_1, \rho_2 \in B(X) \). Since \( B(X) \) and \( B(Y) \) are both inner-product spaces with Hilbert-Schmidt inner product, we can introduce the Hermitian adjoint superoperator \( \mathcal{M}^\dagger \) by

\[ \langle \mathcal{M}^\dagger(\sigma), \rho \rangle_{B(X)} = \langle \sigma, \mathcal{M}(\rho) \rangle_{B(Y)}, \]
and similarly, we can introduce the antilinear Hermitian adjoint superoperator $\mathcal{M}^\dagger$ by
\begin{equation}
\langle \mathcal{M}^\dagger(\sigma), \rho \rangle_{\mathcal{B}(\mathcal{X})} = \langle \sigma, \mathcal{M}(\rho) \rangle_{\mathcal{B}(\mathcal{Y})},
\end{equation}
for all $\rho \in \mathcal{B}(\mathcal{Y})$, and $\sigma \in \mathcal{B}(\mathcal{X})$. The set of antilinear superoperators forms a linear space, which we denote as $\mathcal{B}^{(2)}_{\text{anti}}(\mathcal{X}, \mathcal{Y})$; when $\mathcal{X} = \mathcal{Y}$, we will simply denote it as $\mathcal{B}^{(2)}_{\text{anti}}(\mathcal{X})$.  

The composition of an antilinear superoperator with a linear superoperator is antilinear, the composition of an antilinear superoperator with an antilinear superoperator is linear. This means that different from the space of all linear superoperators $\mathcal{B}^{(2)}(\mathcal{X})$ which is an algebra, $\mathcal{B}^{(2)}_{\text{anti}}(\mathcal{X})$ is not an algebra, because that the composition is not closed. However, we can show that $\mathcal{B}^{(2)}_{\text{anti}}(\mathcal{X})$ is a bimodule over $\mathcal{B}^{(2)}(\mathcal{X})$. For antilinear $\mathcal{M}$, it’s easy to show that the Hermitian adjoint $\mathcal{M}^\dagger$ becomes linear, but the antilinear Hermitian adjoint $\mathcal{M}^\dagger$ is still antilinear. We also have $(\mathcal{M}^\dagger)^\dagger = \mathcal{M}$, and $(\mathcal{M} \circ \mathcal{N})^\dagger = \mathcal{N}^\dagger \circ \mathcal{M}^\dagger$.

Tensor product between two antilinear superoperators $\mathcal{M}$ and $\mathcal{N}$ is well-defined in the way that $\mathcal{M} \otimes \mathcal{N}(\rho \otimes \sigma) = \mathcal{M}(\rho) \otimes \mathcal{N}(\sigma)$. However, the tensor product between linear and antilinear superoperators cannot be consistently defined in this way. In particular, we cannot define the tensor product between antilinear $\mathcal{M}$ and (linear) identity superoperator $\mathcal{I}$.

One of the most crucial examples of an antilinear superoperator is the complex conjugation $\mathcal{K}$ in a particular basis, $\mathcal{K}(\rho) = \rho^*$. It can be checked that $\mathcal{K}^{-1} = \mathcal{K}^\dagger = \mathcal{K}$. As we will show later, the complex conjugation superoperator plays a key role in characterizing and studying antilinear superoperators. The following result is the main tool that we will utilize to investigate the antilinear superoperators.

**Theorem 3.** Any antilinear superoperator $\mathcal{M}$ can be decomposed as $\mathcal{M} = \mathcal{M}_L \circ \mathcal{K} = \mathcal{K} \circ \mathcal{M}_L^*$ with $\mathcal{M}_L$ and $\mathcal{M}_L^*$ linear superoperators called left and right linearizations of $\mathcal{M}$, respectively. The decomposition will be called the fundamental decomposition of an antilinear superoperator.

**Proof.** By choosing a basis $\{ E_{ij} = |i\rangle \langle j| \}$, notice that the complex conjugation of an operator under this basis has the property: $\mathcal{K}(E_{ij}) = E_{ij}$ and $\mathcal{K}(\rho) = \mathcal{K}(\sum_{ij} \rho_{ij} E_{ij}) = \sum_{ij} \rho_{ij}^* E_{ij} = \rho^*$. We can define a linear superoperator $\mathcal{M}_L$ such that $\mathcal{M}_L(E_{ij}) := \mathcal{M}(E_{ij})$. Then we see that $\mathcal{M}(\rho) = \sum_{ij} \rho_{ij} \mathcal{M}_L(E_{ij}) = \mathcal{M}_L \circ \mathcal{K}(\rho)$ for all $\rho$, which shows that $\mathcal{M} = \mathcal{M}_L \circ \mathcal{K}$. We also define a linear operator $\mathcal{M}_L^*$ such that $\mathcal{M}_L^*(E_{ij}) = \mathcal{M}_L(E_{ij})^*$. In this way, $\mathcal{M}(\rho) = \sum_{ij} \rho_{ij} \mathcal{M}_L(E_{ij}) = \sum_{ij} \rho_{ij}^* \mathcal{M}_L^*(E_{ij})^* = \mathcal{K}(\sum_{ij} \rho_{ij} \mathcal{M}_L^*(E_{ij})) = \mathcal{K} \circ \mathcal{M}_L^*(\rho)$ for all $\rho$. This completes the proof. 

In the same spirit as in the above proof, we can show that, if $\mathcal{A} : \mathcal{X} \rightarrow \mathcal{Y}$ is an antilinear operator, then there exist corresponding linear operators $A_L$ and $A_L^*$ such that $\mathcal{A} = A_L \mathcal{K} = \mathcal{K} A_L^*$ with $\mathcal{K}$ the complex conjugate operator in the standard basis. There is corresponding antilinear superoperator $\mathcal{M}(\rho) = A_L \mathcal{K}(\rho) A_L^*$.

Now let’s discuss the representations of antilinear superoperators. The simplest one is natural representation. By introducing the vector map
\begin{equation}
v : |i\rangle \langle j| \mapsto |i\rangle \langle j|,
\end{equation}

it’s clear that the mapping
\begin{equation}
\mathcal{A}(\mathcal{M}) : v(\rho) \mapsto v(\mathcal{M}(\rho))
\end{equation}
is antilinear, thus we obtain an antilinear operator representation of $\mathcal{M}$. Using theorem 3 we see that $\mathcal{M}_L$ corresponds to linearization $\mathcal{A}(\mathcal{M})_L$ of $\mathcal{M}(\rho)$, and the explicit form reads
\begin{equation}
\mathcal{A}(\mathcal{M})_L = \sum_{i,j} \sum_{k,l} (\mathcal{E}_{k,l}, \mathcal{M}_L(E_{i,j})) \mathcal{E}_{k,i} \otimes \mathcal{E}_{l,j}.
\end{equation}
The antilinear operator $\mathcal{A}(\mathcal{M})_L \mathcal{K}$ will be called the natural representation of $\mathcal{M}$. The natural representation for the antilinear Hermitian conjugate $\mathcal{M}^\dagger$ is of the form
\begin{equation}
\mathcal{A}(\mathcal{M})^\dagger_\mathcal{L} = \sum_{i,j} \sum_{k,l} (\mathcal{E}_{i,j}, \mathcal{M}_L^*(E_{k,l})) \mathcal{E}_{k,i} \otimes \mathcal{E}_{l,j},
\end{equation}

which means that $\mathcal{A}(\mathcal{M}^\dagger)_\mathcal{L} = \mathcal{A}(\mathcal{M})_\mathcal{L}^\dagger$.

Using theorem 3, we see that $\mathcal{M}$ has a Kraus decomposition
\begin{equation}
\mathcal{M}(\rho) = \sum_j A_j \rho^* B^T_j,
\end{equation}

where $\{ A_j \}$ and $\{ B_j \}$ are Kraus operators for $\mathcal{M}_L$. The Kraus representation for $\mathcal{M}^\dagger$ is thus
\begin{equation}
\mathcal{M}^\dagger(\rho) = \sum_j A_j^T \rho^* B^*_j.
\end{equation}
The Kraus representations exist for all antilinear superoperators but they are not unique in general.

Another representation is Choi-Jamiolkowski representation, which is a useful representation for characterizing positivity of the superoperators. For $\mathcal{M} \in \mathcal{B}^{(2)}_{\text{anti}}(\mathcal{X}, \mathcal{Y})$, we have
\begin{equation}
\mathcal{J}(\mathcal{M}) = (\mathcal{M}_L \otimes \mathcal{I}_\mathcal{X})(|\Omega\rangle \langle \Omega|)
\end{equation}
\begin{equation}
= \sum_{i,j} \mathcal{M}_L(E_{i,j}) \otimes E_{i,j},
\end{equation}

where $|\Omega\rangle = \sum_i |i\rangle |i\rangle$. The operator $\mathcal{J}(\mathcal{M})$ is called Choi-Jamiolkowski representation of $\mathcal{M}$. It’s easy to verify that
\begin{equation}
\mathcal{M}(\rho) = \text{Tr}_{\mathcal{X}}(\mathcal{J}(\mathcal{M})(|\Omega\rangle \langle \Omega|)).
\end{equation}
From the open-system point of view, the superoperator is a quantum operation obtained by partially tracing the environment of a closed system. This also works for antilinear superoperator $\mathcal{M}$, the resulting representation is called Stinespring representation. Suppose that $U, V \in \mathcal{B}(\mathcal{X}, \mathcal{Y} \otimes \mathcal{Z})$ are Stinespring representation operators of $\mathcal{M}_L$, then from theorem 3 we have

$$\mathcal{M}(\rho) = \text{Tr}_Z(U \rho^* V^\dagger). \quad (33)$$

The Kraus representation is usually regarded as the most fundamental representation. In the following result, we discuss how to translate the Kraus representation into other representations.

**Lemma 4.** Suppose $\mathcal{M} \in \mathcal{B}^{(2)}_{\text{anti}}(\mathcal{X}, \mathcal{Y})$ is an antilinear superoperator and its Kraus representation is given by Eq. (29), then we have

1. The natural representation of $\mathcal{M}$ is $A(\mathcal{M})_L = \sum_k A_j \otimes B_j^\dagger$.
2. The Choi-Jamiołkowski representation is $J(\mathcal{M}) = \sum_j v(A_j) \otimes v(B_j)^\dagger$.
3. The Stinespring dilation is given by Eq. (33) with $U = \sum_j A_j \otimes e_j$ and $V = \sum_j B_j \otimes e_j$, here $e_j$ is orthonormal basis of $\mathcal{Z}$.

Proof. These claims can be verified by straightforward calculation. 

**B. Antilinear quantum channel**

Let’s now consider the antilinear channel, which is a natural generalization of the notion of the (linear) quantum channel.

**Definition 5.** The following are some crucial classes of antilinear superoperators:

- $\mathcal{M}$ is called antilinearly trace-preserving (TP) if $\text{Tr}(\mathcal{M}(\rho)) = (\text{Tr}(\rho))^*$ for all $\rho \in \mathcal{B}(\mathcal{X})$.
- $\mathcal{M}$ is called antilinearly completely positive (CP) if, it’s positive, i.e., it maps positive semidefinite operators to positive semidefinite operators, and $\mathcal{K} \otimes \mathcal{M}$ is positive.
- The antilinearly CPTP superoperators are called antilinear channels.

Using theorem 3, we have the following characterizations of the antilinear quantum channel.

**Theorem 6.** Suppose antilinear superoperator $\mathcal{M}$ has the decomposition $\mathcal{M} = \mathcal{M}_L \circ \mathcal{K}$, then

1. $\mathcal{M}$ is antilinearly TP if and only if $\mathcal{M}_L$ is TP.
2. $\mathcal{M}$ is antilinearly CP if and only if $\mathcal{M}_L$ is CP.
3. $\mathcal{M}$ is antilinearly channel if and only if $\mathcal{M}_L$ is a linear channel.

Proof. (1) For sufficiency, suppose that $\mathcal{M}_L$ is a TP superoperator, it’s easily checked that $\mathcal{M} = \mathcal{M}_L \circ \mathcal{K}$ is an antilinear TP superoperator. For necessity, using the Kraus decomposition $\mathcal{M}(\rho) = \sum_j A_j \rho^* B_j^\dagger$, since $\text{Tr}(\mathcal{M}(\rho)) = \text{Tr}(\rho^*)$ for all $\rho^*$, this further implies that $\mathcal{M}_L$ is CP. More concisely, from the fact $\mathcal{K} \otimes \mathcal{M} = (\mathcal{I} \otimes \mathcal{M}_L) \circ (\mathcal{K} \otimes \mathcal{K})$ is positive, we obtain that $\mathcal{I} \otimes \mathcal{M}_L$ is positive.

(2) Notice that $\rho^*$ is positive semidefinite if $\rho$ is positive semidefinite. Then $\mathcal{M}_L$ is CP directly implies that $\mathcal{M}$ is CP. For the other direction, consider the Kraus decomposition $\mathcal{M}(\rho) = \sum_j A_j \rho^* B_j^\dagger$ for all $\rho^*$, this further implies that $\mathcal{M}_L$ is CP. More concisely, from the fact $\mathcal{K} \otimes \mathcal{M} = (\mathcal{I} \otimes \mathcal{M}_L) \circ (\mathcal{K} \otimes \mathcal{K})$ is positive, we obtain that $\mathcal{I} \otimes \mathcal{M}_L$ is positive.

(3) is a direct result of (1) and (2). 

As for (linear) superoperators, various representations we have discussed before can be used to characterize the antilinearly CP and TP superoperators. Using theorem 6, this becomes a straightforward generalization.

**Lemma 7.** For $\mathcal{M} \in \mathcal{B}^{(2)}_{\text{anti}}(\mathcal{X}, \mathcal{Y})$, the following statements are equivalent:

1. $\mathcal{M}$ is antilinearly CP superoperator.
2. In the Kraus representation Eq. (29) of $\mathcal{M}$, $A_j = B_j$, thus

$$\mathcal{M}(\rho) = \sum_j A_j \rho^* A_j^\dagger. \quad (34)$$

3. The Choi-Jamiołkowski representation $J(\mathcal{M})$ is a positive semidefinite operator.
4. In the Stinespring representation Eq. (33) of $\mathcal{M}$, $U = V$, thus

$$\mathcal{M}(\rho) = \text{Tr}_Z(U \rho^* U^\dagger). \quad (35)$$

**Lemma 8.** For $\mathcal{M} \in \mathcal{B}^{(2)}_{\text{anti}}(\mathcal{X}, \mathcal{Y})$, the following statements are equivalent:

1. $\mathcal{M}$ is antilinearly TP superoperator.
2. The Kraus representation in Eq. (29) satisfies

$$\sum_j A_j^\dagger B_j = \mathcal{I}_X. \quad (36)$$

3. The Choi-Jamiołkowski representation $J(\mathcal{M})$ satisfies

$$\text{Tr}_Y J(\mathcal{M}) = \mathcal{I}_X. \quad (37)$$

4. For the Stinespring representation as in Eq. (33), the operators $U, V$ satisfy $U^\dagger V = \mathcal{I}_X$. 

Using the relations between different representations of $\mathcal{M}$ presented in Lemma 4, the proofs of the above two lemmas are straightforward. Combining the lemma 7 and lemma 8, we obtain a complete characterization of antilinear quantum channels.

Antilinear superoperator $\mathcal{M}$ is called antilinearly unitary (antiunitary) if $\mathcal{M}(\rho) = U^* \rho^* U$ for some unitary operator $U$. From Stinespring representation of the antilinear quantum channel, every antilinear channel can be implemented by performing an antunitary transformation on system and some ancilla and tracing over the ancilla.

### C. Antilinearly unital superoperator

The unital superoperators have broad applications in quantum information theory [3, 22]. The unital channels are also called doubly stochastic quantum channels. In this subsection, we study their antilinear counterparts.

**Definition 9.** Let $\mathcal{M} \in \mathcal{B}^{(2)}(\mathcal{X}, \mathcal{Y})$ be an antilinear superoperator, it’s called antilinearly unital (stochastic) if $\mathcal{M}(I_{\mathcal{X}}) = I_{\mathcal{Y}}$; an antilinearly unital and TP superoperator is called antilinearly doubly stochastic superoperator.

**Theorem 10.** $\mathcal{M}$ is antilinearly unital if and only if $\mathcal{M}_{\mathcal{L}}$ is unital; $\mathcal{M}$ is antiunitary if and only if $\mathcal{M}_{\mathcal{L}}$ is unitary.

The antilinearly unital superoperators are closely related to the antilinearly TP superoperator.

**Theorem 11.** $\mathcal{M}$ is antilinearly TP if and only if $\mathcal{M}_{\mathcal{L}}$ is unital.

**Proof.** From theorem 6, $\mathcal{M}$ is antilinearly TP if and only if $\mathcal{M}_{\mathcal{L}}$ is TP, and this is further equivalent to that $\mathcal{M}_{\mathcal{L}}$ is unital. \hfill \blacksquare

It’s clear that the antiunitary superoperator is antilinearly unital. We can also introduce the mixture of antunitaries

$$\mathcal{U}(\rho) = \sum_j p(j) U_j^* \rho U_j^\dagger,$$

where $p(j)$ is a probability distribution and $U_j$’s are a collection of unitary operators. Antilinearly Weyl-covariant channel is also a crucial example of unitary superoperator,

$$\mathcal{W}(\rho) = \sum_{i,j \in \mathbb{Z}^N} p(i,j) W_{ij} \rho W_{ij}^\dagger,$$

where $W_{ij} = X_{i} Z_{j}$ and $X_N, Z_N$ are generalized Pauli operators. It can be proved that antilinearly Weyl-covariant channel is mixed antilinearly unitary channel. Another crucial class of antilinearly unitary superoperator is generalized $\Theta$-conjugation, which is useful for us to investigate the geometric properties of higher-dimensional quantum systems.

![FIG. 1: A depiction of relations between different classes of antilinear superoperators. Here $\mathcal{B}^{(2)}_{\text{anti}}$ represents the set of all antilinear superoperators; $\text{CPTP}_{\text{anti}}$ represents the set of all antilinear CPTP superoperators; $\text{Unital}_{\text{anti}}$ represents the set of all antilinearly unital superoperators; $\text{MixedUnitary}_{\text{anti}}$ represents the set of all antilinearly mixed-unitary superoperators; $\text{Unitary}_{\text{anti}}$ represents the set of all antunitary superoperators; $\mathcal{B}^{(2)}_{\text{anti},G\Theta}$ represents the set of all generalized $\Theta$-conjugations; $\mathcal{B}^{(2)}_{\text{anti},\Theta}$ represents the set of all $\Theta$-conjugations;](#)

#### D. Generalized $\Theta$-conjugation

We now introduce the notion of the generalized $\Theta$-conjugation. This is a useful definition for investigating quantum fidelity, quantum concurrence, and quantum geometric invariance.

**Definition 12.** An antilinear superoperator is called a generalized $\Theta$-conjugation if $\Theta$ is unital and it satisfies $\Theta^\dagger = \Theta^{-1}$.

The motivation is like this. For maps between $\mathcal{B}(\mathcal{X})$ and $\mathcal{B}(\mathcal{Y})$, we consider the the maps that preserve that norm of inner, i.e.,

$$|\langle \mathcal{M}(\sigma), \mathcal{M}(\rho) \rangle| = |\langle \sigma, \rho \rangle|.$$

(40)

Inspired by the Wigner theorem, we can consider two classes of such maps: (i) for linear case $\mathcal{M}^\dagger = \mathcal{M}^{-1}$; (ii) for antilinear case $\mathcal{M}^\dagger = \mathcal{M}^{-1}$. Here the generalized $\Theta$-conjugation is just the generalization of antilinear unitary operators that preserve the inner product between quantum states. The relations between different classes of antilinear superoperators are shown in Fig 1.

The $\Theta$-conjugation [7] is a crucial example of generalized $\Theta$-conjugation, other examples are time-reversal operation [4], Hill-Wootters conjugation [6]. The $\Theta$-conjugated quantum fidelity and quantum concurrence play key roles in studying quantum information and quantum correlations. Here we can also introduce their counterparts for generalized $\Theta$-conjugation.

Fidelity measures how close two states are. For completeness, we can introduce the $p$-norm fidelity between quantum states $\rho$ and $\sigma$,

$$F_p(\rho, \sigma) = \| \sqrt{\rho} \sqrt{\sigma} \|_p.$$  

(41)
When taking \( p = 1 \), we obtain the normal fidelity \( F(\rho, \sigma) = \text{Tr}(\sqrt{\rho \sigma}) \). Following the definition of \( \Theta \)-fidelity [7], we introduce the definition below:

**Definition 13.** For a given generalized \( \Theta \)-conjugation, the \( p \)-norm generalized \( \Theta \)-fidelity between two states is defined as

\[
F_{\Theta,p}(\rho, \sigma) := F_{p}(\rho, \Theta(\sigma)) = \| \sqrt{\rho} \sqrt{\Theta(\sigma)} \|_p. \tag{42}
\]

By abbreviating \( \tilde{\rho} := \Theta(\rho) \), the \( p \)-norm generalized \( \Theta \)-fidelity of \( \rho \) is defined as

\[
F_{\Theta,p}(\rho) := F_{p}(\rho, \tilde{\rho}). \tag{43}
\]

The most commonly used one is 1-norm fidelity.

Notice that generalized \( \Theta \)-conjugation may not be an antilinear channel, thus, it may maps a positive semidefinite operators to an operator which is not positive semidefinite. To remedy this problem, we can shrink the Bloch vector correspondind to \( \Theta(\rho) \), such that it becomes a vector in the Bloch body (see Sec. IV for a detail discussion).

Similar as fidelity, we can also introduce the generalized \( \Theta \)-concurrence. Consider the operator \( \sqrt{\rho \sqrt{\sigma}} \), its singular values are some nonnegative numbers: \( \lambda_1 \geq \lambda_2 \geq \cdots \geq \lambda_d \). Consider the \( p \)-th powers of these singular values, viz., the spectrum of \( (\sqrt{\rho \sqrt{\sigma}})^p/2 \), the concurrence of order \( p \) between \( \rho \) and \( \sigma \) are defined as

\[
C_p(\rho, \sigma) = \max\{0, \lambda_1^p - \sum_{j=2}^{d} \lambda_j^p\}. \tag{44}
\]

When taking \( p = 1 \), we obtain the normal definition of concurrence between \( \rho \) and \( \sigma \), \( C(\rho, \sigma) = \max\{0, \lambda_1 - \sum_{j=2}^{d} \lambda_j\} \). Following the definition of \( \Theta \)-concurrence [7], we introduce the definition below:

**Definition 14.** For a fixed generalized \( \Theta \)-conjugation, consider the spectrum of the operator \( (\sqrt{\rho \sqrt{\sigma}})^{p/2} \), by ordering its eigenvalues in a descending order \( \lambda_1^p \geq \lambda_2^p \geq \cdots \geq \lambda_d^p \), the generalized \( \Theta \)-concurrence of order \( p \) is defined as

\[
C_{\Theta,p}(\rho, \sigma) = \max\{0, \lambda_1^p - \sum_{j=2}^{d} \lambda_j^p\}. \tag{45}
\]

By abbreviating \( \tilde{\rho} := \Theta(\rho) \), the generalized \( \Theta \)-concurrence of order \( p \) of \( \rho \) is defined as \( C_{\Theta,p}(\rho) = C_{\Theta,p}(\rho, \tilde{\rho}) \).

As we will see later, for the qubit system, quantum concurrence is related to the Bloch vectors in a simple way. Similar to the generalized \( \Theta \)-conjugated fidelity, there is also a problem of positivity, and the problem will be discussed in the next section.

**IV. QUANTUM GEOMETRIC INVARIANCE FOR QUDIT SYSTEM**

With the above preparation, we are now in a position to discuss the quantum geometric invariance of qudit systems. Consider the linear isomorphism \( \Phi : \mathbb{H}(\mathbb{C}^d) \rightarrow \mathbb{R}^{d^2} \) given by Bloch representation. By mapping \( \sigma_\mu \) to \( e_\mu = (0, \cdots, 0, 1, 0, \cdots, 0) \), each Hermitian matrix has a corresponding vector \( x(\rho) = (x_0, \cdots, x_{d^2-1}) \). Conversely, for every \( x \in \mathbb{R}^{d^2} \), we obtain a Hermitian matrix of the form

\[
\rho(x) = \frac{1}{2} \sum_{\mu=0}^{d^2-1} x_\mu \sigma_\mu.
\]

We aim to study the transformations of states corresponding to the geometric transformations of Bloch space-time vector \( x_\mu \).

For a given superoperator \( \mathcal{E} \), it maps \( \rho \rightarrow \mathcal{E}(\rho) = \frac{1}{d}(\sum_\mu x'_\mu \sigma_\mu) \), there is a corresponding geometric transformation \( T_\mathcal{E} : x_\mu \rightarrow x'_\mu(x) \). For a given geometric transformation \( T \), there also exists a corresponding superoperator \( \mathcal{E}_T \). See Fig. 2 for a depiction. For these geometric transformations, like Lorentz transformation, investigating the corresponding transformations of states plays a crucial role in studying entanglement, monogamy relations, Jones vector transformation in quantum optics, and so on [13–17].

We regard the \( x \in \mathbb{R}^{d^2} \) as a space-time vector, by introducing generalized \( \Theta \)-conjugation, different geometric structures on \( \mathbb{R}^{d^2} \) are obtained. In this different geometric space-time, different geometric transformations and their properties will be discussed.

**A. Generalized \( \Theta \)-conjugation and space-time metric**

One of the well-known example of \( \Theta \)-conjugation is the Hill-Wootters spin-flip operation \( \rho \rightarrow \sigma_y K(\rho) \sigma_y [6, 23] \). The corresponding geometric transformation is the parity transformation: \( x_0 \rightarrow x_0, x \rightarrow -x \). This can be naturally generalized to the qudit case. Consider a special generalized \( \Theta \)-conjugation, which, when acting on GGM matrices, has the form

\[
\Theta(\sigma_0) = \sigma_0, \Theta(\sigma_j) = -\sigma_j, 1 \leq j \leq d^2 - 1. \tag{46}
\]

**FIG. 2:** An illustration of the geometric transformation of quantum states for Bloch space-time vector.
It’s easily checked that $\Theta^\dagger = \Theta^{-1}$. Notice that under this generalized $\Theta$-conjugation, a density operator $\rho$ may be mapped to an Hermitian operator $\Theta(\rho)$ with negative eigenvalues. This is because that the Bloch convex body for $d \geq 3$ case do not have rotational symmetry.

To remedy this problem, we can shrink the Bloch vector $\vec{x}' = f_\Theta(\vec{x})$ such that $\vec{x}'' = \vec{x}'/\lambda$ gives a density operator. The shrinking process works as follows: suppose that $a_{\vec{x}'}$ is the Bloch vector in $\vec{x}'$ direction with the maximum length $\alpha$, then $\vec{x}'' := a_{\vec{x}'}/\sqrt{d-1}$. This shrinking process may break the linearity in general, so for given convex combination $\rho_1 + (1 - p)\rho_2$, we must first calculate corresponding overall Bloch vector, then maps it to a new Bloch vector. This can remedy the problems when we define generalized $\Theta$-conjugated fidelity and concurrence.

We can similarly consider the generalized $\Theta$-conjugation corresponding to the partial parity transformation,

$$\Theta(\sigma_{jk}) = -\sigma_{jk}, \forall k = 1, \cdots, q,$$  \hspace{1cm} (47)

while leaving all other $\sigma_j$ unchanged. The corresponding Bloch vector transformation is

$$x_{jk} \mapsto -x_{jk}, \forall k = 1, \cdots, q,$$

$$x_{jk} \mapsto x_{jk}, \forall k = q + 1, \cdots, d^2 - 1.$$  \hspace{1cm} (48)

This kind of generalized $\Theta$-conjugation is crucial for us to study the Lorentzian invariance of the qudit state.

B. Quantum Euclidean invariance for qudit system

Let us now consider Euclidean invariance of single qudit state. We assume that the space $\mathbb{R}^{d^2}$ is equipped with the Euclidean norm $\|x\|_E = \sum_{\mu \nu} \delta_{\mu \nu} x_\mu x_\nu = \sum_{\mu=0}^{d^2-1} x_\mu^2$. In $\mathcal{H}(\mathbb{C}^d)$, the norm corresponds to

$$\langle \rho, \rho \rangle_{HS} = \text{Tr}(\rho^2) = \frac{1}{d^2} \sum_{\mu \nu} x_{\mu \nu}^2 = \frac{1}{d} \sum_{j,k} \|x\|^2,$$  \hspace{1cm} (49)

which is nothing but the purity of the state. Thus the Euclidean invariance for single qudit state is the quantum transformations which preserve the purity of states. In Bloch representation, this corresponds to the orthogonal group $O(d^2 - 1)$. Since the qudit Bloch representation does not have the rotational symmetry, we still need to do some shrinking if the rotated vector is not a Bloch vector that corresponds to the positive semidefinite operator.

For bipartite case, it is convenient to introduce the joint observables $x_{\mu \nu} = \text{Tr}(\rho_{\mu \nu} \otimes \sigma_\nu)$ and express the state as

$$\rho = \frac{1}{d^2} \sum_{\mu \nu} x_{\mu \nu} \sigma_\mu \otimes \sigma_\nu$$

$$= \frac{1}{d^2} \left( \sum_{j,k} x_{jk} \sigma_j \otimes \sigma_k + \sum_{j} x_{j0} \sigma_0 \otimes \sigma_j + \sum_{k} x_{0k} \sigma_0 \otimes \sigma_k \right)$$

$$+ \sum_{j,k} x_{jk} \sigma_j \otimes \sigma_k.$$  \hspace{1cm} (50)

The tensor product of Hilbert-Schmidt matrices $\sigma_\mu \otimes \sigma_\nu$ provides a basis for the $\mathcal{H}(\mathbb{C}^d) \otimes \mathcal{H}(\mathbb{C}^d)$. In this case, the Euclidean norm is

$$\langle \rho, \rho \rangle_{HS} = \text{Tr}(\rho^2) = \frac{1}{d^2} \sum_{\mu \nu} x_{\mu \nu}^2.$$  \hspace{1cm} (51)

Since $x_{00} = 1$, we see that the operation preserving the Euclidean norm is the group $O(d^2 - 1)$. For the $n$-qudit case, the generalization is straightforward,

$$\rho = \frac{1}{d^n} \left( \sum_{\mu_1 \cdots \mu_n} x_{\mu_1 \cdots \mu_n} \sigma_{\mu_1} \otimes \cdots \otimes \sigma_{\mu_n} \right).$$  \hspace{1cm} (52)

The Euclidean norm is

$$\langle \rho, \rho \rangle_{HS} = \text{Tr}(\rho^2) = \frac{1}{d^n} \sum_{\mu_1 \cdots \mu_n} x_{\mu_1 \cdots \mu_n}^2.$$  \hspace{1cm} (53)

The symmetry group is $O(d^{2n} - 1)$. Thus we see that the Euclidean norm corresponds to the purity of the state, and it can be expressed as the Hilbert-Schmidt inner product of $\rho$ with itself. The Euclidean invariance of quantum states is the equivalent class of states which is invariant under the group $O(d^{2n} - 1)$. To consider the more general case, i.e., the Euclidean rotation over the $d^{2n}$ space, we need do some shrinking of the Bloch tensors we obtained first, then the Euclidean-invariant class of state will have more general meaning and the symmetry group will be $O(d^{2n}, \mathbb{R})$.

C. Quantum Lorentzian invariance for qudit system

To investigated the Lorentzian invariance, we need to consider the space-time $\mathbb{R}^{p,q}$ with the Minkowskian metric defined as $\eta_{\mu \nu} = \text{diag}(+ \cdots +, - \cdots -)$, here $p + q = d^2$. The Lorentzian norm is thus $\|x\|^2 = \sum_{\mu, \nu} \eta_{\mu \nu} x_\mu x_\nu = \sum_{\mu=0}^{p-1} x_\mu^2 - \sum_{\mu=p}^{p+q-1} x_\mu^2$. The Lorentz transformation is a linear transformation

$$x'_{\mu} = \sum_{\nu} A_{\mu \nu} x_\nu$$  \hspace{1cm} (54)

such that $\|x\|^2 = \|x'\|^2$. The set of all Lorentz transformations forms the Lorentz group $O(p, q) = \{ \Lambda \in GL(p + q; \mathbb{R}) | \Lambda^T \eta \Lambda = \eta \}$.
For single-qubit state in Bloch representation \( \rho = \frac{1}{2}(I + \vec{x} \cdot \vec{\sigma}) \), the Hill-Wootters conjugation gives \( \tilde{\rho} = \sigma_y \rho^* \sigma = \frac{1}{2}(I - \vec{x} \cdot \vec{\sigma}) \), then we see that

\[
4 \det \rho = 2 \text{Tr}(\rho \tilde{\rho}) = x_0^2 - x_1^2 - x_2^2 - x_3^2
\]

\[
= \sum_{\mu, \nu} \eta_{\mu \nu} x_\mu x_\nu.
\] (55)

For higher-dimensional situation, we can utilize the generalized \( \Theta \)-conjugation which maps \( \sigma_j \) to \( -\sigma_j \) for \( j = 1, \ldots, p \) and leaves all other Hilbert-Schmidt basis matrices unchanged. We introduce the generalized \( \Theta \)-conjugated Hilbert-Schmidt inner product \( \langle \rho, \chi \rangle_\Theta = \langle \rho, \Theta(\chi) \rangle_{HS} = \text{Tr}(\rho \Theta(\chi)) \). Then for Hilbert-Schmidt operator we have \( \langle \sigma_\mu, \sigma_\nu \rangle_\Theta = \eta_{\mu \nu} \), hence the Lorentzian norm for a density operator can be expressed as

\[
\langle \rho, \rho \rangle_\Theta = \text{Tr}(\rho \Theta(\rho)) = \frac{1}{d} \sum_{\mu, \nu} \eta_{\mu \nu} x_\mu x_\nu.
\] (56)

We usually take \( p = 1 \), namely, \( \Theta(\sigma_0) = \sigma_0 \) and \( \Theta(\sigma_j) = -\sigma_j \) for all \( j \geq 1 \).

For bipartite state \( \rho \) in Eq. (50), we introduce \( R_\rho = \rho \Theta^{\otimes 2}(\rho) \), the Lorentzian norm is therefore

\[
\langle \rho, \rho \rangle_\Theta = \text{Tr}(R_\rho) = x_0^2 - \sum_{j=1}^{d^2-1} (x_{0j}^2 + x_{j0}^2) + \sum_{j,k=1}^{d^2-1} x_{jk}^2
\]

\[
= L_0 - L_1 + L_2.
\] (57)

Hereafter, \( L_k \) denotes the sum of all terms \( x_{j_1 \cdots j_n}^2 \) with \( k \) space-like indices.

Similar as Euclidean case, the Lorentzian invariance is characterized by equivalence class which is invariant under Lorentz transformation. One of the main problems here is that the Lorentz boost may map a Bloch space-time vector \( x_\mu \) to the one with time component \( x_0 \neq 1 \). This can also be remedied by shrinking or dilating the Bloch space-time vectors.

V. QUANTUM GEOMETRIC INVARiance AND ENTANGLEMENT DISTRIBUTION

One of the characteristic features of quantum correlations is that they cannot be shared freely in a many-body system. The phenomenon is now known as monogamy of quantum correlations. It’s shown that there exist monogamy relations for Bell nonlocality [24–28], quantum steering [29], and entanglement [17, 30, 31]. In this section, we study the monogamy equalities of entanglement restricted by quantum geometric invariance.

Consider von Neumann entropy \( S(\rho) = -\text{Tr}(\rho \ln \rho) \), using the Mercator series for \( \ln \rho = \ln(I + (\rho - I)) \approx (\rho - I) - (\rho - I)^2/2 + \cdots \) and keeping only linear term, we obtain a quantity called linear entropy

\[
S_L(\rho) = \text{Tr}(\rho) - \text{Tr}(\rho^2).
\] (58)

Note that many authors made the convention for the linear entropy differing from the one we give here with a multiple two. For two-qubit state \( \psi_{AB} \), the linear entropy of the reduced state are related to the concurrence of the state via

\[
2S_L(\rho_A) = 2S_L(\rho_B) = C_{A,B}^2(\psi_{AB}).
\] (59)

From this, we see that linear entropy can be used to measure the quantum correlation of the state. Using this observation, Eichtsha and Siewert derive the distribution of quantum correlations from the quantum Lorentz invariance of qubit states [17]. Here we generalize this result to the qudit case.

For \( n \)-qudit pure state \( \rho_\psi \), the Euclidean norm is

\[
d^n \text{Tr}(\rho_\psi^2) = L_0 + L_1 + \cdots + L_n.
\] (60)

If we set \( R_\rho = \rho \Theta^{\otimes n}(\rho) \), the Lorentzian norm is given by

\[
d^n \text{Tr}(R_\rho) = L_0 - L_1 + \cdots + (-1)^n L_n.
\] (61)

It’s easy to see that

\[
(-1)^n d^n \text{Tr}(R_\rho) = d^n \text{Tr}(\rho_\psi^2) - 2 \sum_{k=\delta_n}^{[n/2]} L_{2k-\delta_n},
\] (62)

where \( \delta_n = (1 + (-1)^n)/2 \). Let us denote the bipartition of \( n \)-particle system as \( A | A^c \), the linear entropy \( S_L(A) \) for this bipartition measures the entanglement between \( A \) and \( A^c \). It’s straightforward to check that

\[
\text{Tr}(R_\rho) = \sum_{A|A^c} \alpha_{A|A^c} S_L(A),
\] (63)

where we have denoted the linear entropy for empty partition as \( S_L(\emptyset) = 1 \). This in fact characterizes the distribution of the entanglement for a multipartite state, since \( \text{Tr}(R_\rho) \) measures the overall entanglement of the state \( \psi \).

Example 15. Consider an \( n \)-qubit state \( \psi \), the coefficient in Eq. (63) has been carefully calculated in Ref. [17]:

\[
\text{Tr}(R_\rho) = (-1)^{n+1} S_L(\rho) + \sum_{A|A^c \neq \emptyset} (-1)^{|A^c|+n+1} S_L(A).
\] (64)

This characterize how entanglement is distributed over these \( n \) particles. For \( n = 3 \) case, the constraint becomes trivial \( S(A) = S(A^c) \). But for \( n = 4 \), we see that

\[
E(ABCD) + E(AB|CD) + E(AC|BD) + E(AD|BC) + E(BC|AD) + E(BD|AC) + E(CD|AB)
\]

\[
= E(A|BCD) + E(B|ACD) + E(C|ABD) + E(D|ABC) + E(ABC|D) + E(ABD|C) + E(ACD|B) + E(BCD|A).
\] (65)

Here we use \( E(A|A^c) \) to denote the entanglement between \( A \) and \( A^c \). This provides a constraint of the pattern for the entanglement of the state.
Example 16. Let us now consider an example of 3-qutrit state $\rho_{ABC} = |\psi\rangle\langle\psi|$, the Bloch representation can be given explicitly using matrices in example 1.

$$\rho_{ABC} = \sum_{\mu,\nu,\gamma=0}^{8} x_{\mu\nu\gamma} \sigma_{\mu} \otimes \sigma_{\nu} \otimes \sigma_{\gamma}. \quad (66)$$

From Eq. (62), we see that given explicitly using matrices in example 1.

$$L \rho = \sum_{\mu,\nu,\gamma=0}^{8} x_{\mu\nu\gamma} \sigma_{\mu} \otimes \sigma_{\nu} \otimes \sigma_{\gamma}. \quad (67)$$

$L_0 = 1$ is a constant term. For $L_2 = L^{AB}_2 + L^{AC}_2 + L^{BC}_2$, consider $L^{AB}_2 = \sum_{\mu,\nu=1}^{8} x^{2}_{\mu\nu}$, it can be derived from the reduced density matrix $\rho_{AB}$, $\rho_{A}$ and $\rho_{B}$,

$$L^{AB}_2 = 3^2 Tr \rho_{AB}^2 - 3 Tr \rho_{A}^2 - 3 Tr \rho_{B}^2. \quad (68)$$

This further implies that

$$L^{AB}_2 = 3^2(1 - S_{L}(\rho_{AB})) - 3(2 - S_{L}(\rho_{A}) - S_{L}(\rho_{B})). \quad (69)$$

All other terms can be calculated similarly. In this way, we obtain all coefficients in Eq. (63), and thus the distribution formula of entanglement over the state $\psi$.

Notice that for more complicated cases, we could also use Lorentzian norms $\|\rho\|_{p,q}$ with a metric $\eta_{p,q} = (+,\cdots,+,-,\cdots,-)$. By assuming the invariance of these norms, we can obtain a formula that characterizes the distribution of entanglement over a multipartite state. This means that these monogamy relations can be regarded as a result of the quantum geometric invariance.

VI. CONCLUSION

In this work, we systematically study the Bloch representation of a higher-dimensional quantum system under the Hilbert-Schmidt basis and the geometric properties of the corresponding Bloch vectors. We investigate the antilinear superoperators, including various representation and properties of antilinear quantum channels, antilinear unital superoperators, antiunitary superoperators, and generalized $\Theta$-conjugation. The generalized $\Theta$-conjugation plays an important role in studying the geometric properties of the quantum states. By using the different generalized $\Theta$-conjugation, the Lorentzian and Euclidean invariance of the quantum states are investigated. The invariant class is just the set of states which have the same norms in the corresponding space. Using these geometric properties, we derive the monogamy equalities of entanglement from these geometric invariance for many-body quantum states. This means that the distribution of entanglement over a multipartite state can be regarded as a result of geometric invariance.

The generalized $\Theta$-conjugated fidelity and concurrence are also introduced. Though we did not discuss more details about the $\Theta$-conjugated fidelity in this work, we would like to point out that the quality is closely related to the antilinear superoperator. Notice that for pure state $F_{1}(\psi,\varphi) = |\langle\psi|\varphi\rangle|$, Wigner’s theorem claims that the quantum operations that preserve this fidelity are unitary and antiunitary ones. It’s natural to ask what is the quantum operations that preserve the generalized $p$-norm $\Theta$-conjugated fidelity. This interesting open problem will be left for our future study. For qubit case, the generalized $\Theta$-conjugated concurrence is related to the norm of Bloch vectors, however, for higher-dimensional case, there is no such correspondence.

The framework is also useful for investigating various quantum correlations, including Bell nonlocality, quantum steering and quantum entanglement. Especially for two-particle case, using the Bloch vectors, we can obtain the geometric bodies corresponding to these correlations and study their relations with antilinear superoperators. This part is also left for our future study.

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[32] The notation here has a categorical meaning, the morphisms between two Hilbert spaces are called 1-morphisms (operators), the morphisms between 1-morphism spaces are called 2-morphisms (superoperators). In this way, we can introduce n-morphisms between the spaces of (n-1)-morphisms and denote the space of n-morphisms as $B^{(n)}(X, Y)$ and we denote $B^{(1)}(X, Y)$ simply as $B(X, Y)$. 