A STOCHASTIC OPTIMAL CONTROL PROBLEM GOVERNED BY SPDES VIA A SPATIAL-TEMPORAL INTERACTION OPERATOR

ZHUN GOU AND NAN-JING HUANG*

Department of Mathematics, Sichuan University
Chengdu 610064, China

MING-HUI WANG

School of Economic Mathematics, Southwestern University of Finance and Economics
Chengdu 610074, China
and
Department of Mathematics, Sichuan University
Chengdu 610064, China

YAO-JIA ZHANG

Department of Mathematics, Sichuan University
Chengdu 610064, China

(Communicated by Qi Li)

ABSTRACT. In this paper, we first introduce a new spatial-temporal interaction operator to describe the space-time dependent phenomena. Then we consider the stochastic optimal control of a new system governed by a stochastic partial differential equation with the spatial-temporal interaction operator. To solve such a stochastic optimal control problem, we derive an adjoint backward stochastic partial differential equation with spatial-temporal dependence by defining a Hamiltonian functional, and give both the sufficient and necessary (Pontryagin-Bismut-Bensoussan type) maximum principles. Moreover, the existence and uniqueness of solutions are proved for the corresponding adjoint backward stochastic partial differential equations. Finally, our results are applied to study the population growth problems with the space-time dependent phenomena.

1. Introduction. In last decades, many scholars have focused on the topic of stochastic partial differential equations (SPDEs) [15, 20, 24, 26]. As far as applications are concerned, the study of SPDEs have been motivated much from biological sciences, in particular from population biology [7, 32]. Recently, Lenhart et al. [18, 19] investigated the stochastic optimal control problem for a system of SPDEs, where
the system is motivated by modeling a population that inhabits an environment containing resources. For more works concerned with the stochastic optimal control problem for SPDEs, we refer the reader to [5, 6, 9, 12, 16, 21, 35].

In this paper, we consider a stochastic optimal control problem governed by new type SPDEs with a spatial-temporal interaction operator which describes the space-time dependent phenomena in population growth problems. To explain the motivations of our work, we first recall some other recent works concerning on the stochastic optimal control problem governed by SPDEs.

In 2005, Øksendal [29] studied the stochastic optimal control problem mentioned above, proved a sufficient maximum principle for such a problem, and applied to the optimal harvesting problem described by SPDEs without time delays. However, in realistic world, it is necessary to study the optimal control problem governed by some dynamic system with time delays. For example, for biological reasons, time delays occur naturally in population dynamic models [27, 30]. When dealing with optimal harvesting problems of biological systems, one can be led to the optimal control problem of the system with time delays. Motivated by this fact, Øksendal et al. [31] investigated the stochastic optimal control problem governed by the delay stochastic partial differential equation (DSPDE), established both sufficient and necessary stochastic maximum principles for this problem, and illustrated their results by an application to the optimal harvesting problem from a biological system. Another area of applications for the optimal control of time delay systems is mathematical finance (see, for example, [3]). For more applications, we refer the reader to [4, 13, 17, 25, 28] and the references therein.

On the other hand, it is equally important to study the stochastic optimal control problem governed by dynamic systems with the spatial dependence because it also has many applications in real problems such as the harvesting problems of biological systems [14, 33]. To deal with the problems, Agram et al. [1] introduced the space-averaging operator and considered a system of the SPDE with this type operator. Then they proved both sufficient and necessary stochastic maximum principles for the problem governed by such an SPDE and applied the results to solve the optimal harvesting problem for a population growth system in an environment with space-mean interactions. Following [1], Agram et al. [2] also solved a singular control problem of optimal harvesting from a fish population, of which the density is driven by the SPDE with the space-averaging operator.

Now, a natural question arises: can we capture the desired features of both the past dependence and the space-mean dependence within the same framework? To this end, we construct the spatial-temporal interaction operator, and then consider the stochastic optimal control problem modeled by a system of SPDEs with this operator in the filtered probability space \((\Omega, \mathcal{F}, \mathcal{F}_t, \mathbb{P})\) satisfying the usual hypothesis. This system takes the following form:

\[
\begin{aligned}
    dX(t, x) &= (A_x X(t, x) + b(t, x) - u(t, x)) dt + \sigma(t, x) dB_t \\
    &\quad + \int_{\mathbb{R}_0} \gamma(t, x, \zeta) \tilde{N}(dt, d\zeta), \quad (t, x) \in (0, T] \times D; \\
    X(t, x) &= \xi(t, x), \quad (t, x) \in (0, T] \times \partial D; \\
    X(t, x) &= \eta(t, x), \quad (t, x) \in [-\delta, 0] \times \overline{D}; \\
    u(t, x) &= \beta(t, x), \quad (t, x) \in [-\delta, 0] \times \overline{D},
\end{aligned}
\]

where \(X(t, x)\) is the state, \(dX(t, x)\) is the differential with respect to \(t\), \(u(t, x)\) is the
control process, $D \subset \mathbb{R}^d$ is an open set with $C^1$ boundary $\partial D$, and $\overline{D} = D \cup \partial D$. Moreover, some simplified notations are used in equation (1):

$$
\begin{align*}
&b(t, x) = b(t, x, X(t, x), \overline{X}(t, x), u(t, x), \overline{u}(t, x)), \\
&\sigma(t, x) = \sigma(t, x, X(t, x), \overline{X}(t, x), u(t, x), \overline{u}(t, x)), \\
&\gamma(t, x, \zeta) = b(t, x, \zeta, X(t, x), \overline{X}(t, x), u(t, x), \overline{u}(t, x)).
\end{align*}
$$

Here $\overline{X}(t, x)$ denotes the space-time dependent density.

Consequently, the main purpose of this paper is to study the following stochastic optimal control problem.

**Problem.** Suppose that the performance functional associated to the control $u \in \mathcal{U}^{ad}$ takes the form

$$
J(u) = \mathbb{E} \left[ \int_0^T \int_D f(t, x, X(t, x), \overline{X}(t, x), u(t, x), \overline{u}(t, x))dxdt + \int_D g(x, X(T, x))dx \right],
$$

where $X(t, x)$ is described by (1), $f$ and $g$ are two given functions satisfying some mild conditions, and $\mathcal{U}^{ad}$ is the set of all admissible control processes. The problem is to find the optimal control $\hat{u} = \hat{u}(t, x) \in \mathcal{U}^{ad}$ such that

$$
J(\hat{u}) = \sup_{u \in \mathcal{U}^{ad}} J(u). \tag{2}
$$

The rest of this paper is structured as follows. The next section introduces some necessary preliminaries including the definition of the spatial-temporal interaction operator, and derives an adjoint backward stochastic partial differential equation (BSPDE) with spatial-temporal dependence by defining a Hamiltonian functional. In Section 3, the sufficient and necessary maximum principles of the related control problem are derived, respectively. In Section 4, the existence and uniqueness of solutions are obtained for the related BSPDE of the control problem with the spatial-temporal interaction operator. Before concluding this paper in Section 6, two examples are presented in Section 5 as applications of our main results.

### 2. Preliminaries

In this section, some necessary definitions and propositions are given to state (1) in detail. We also give several examples to show that all these definitions are well-posed.

Now, in (1), the terms $B_t$ and $\widetilde{N}(dt, d\zeta)$ denote a one-dimensional $\mathcal{F}_t$-adapted Brownian motion and a compensated Poisson random measure, respectively, such that

$$
\widetilde{N}(dt, d\zeta) = N(dt, d\zeta) - \nu(dt, d\zeta),
$$

where $N(dt, d\zeta)$ is a Poisson random measure associated with the one-dimensional $\mathcal{M}_t$-adapted Poisson process $P_N(t)$ defined on $\mathbb{R}_0 = \mathbb{R} \setminus \{0\}$ with the characteristic measure $\nu(dt, d\zeta)$. Here, $B_t$ and $P_N(t)$ are mutually independent. Moreover, $\sigma$-algebras $\mathcal{F} = (\mathcal{F}_t)_{t \geq 0}$ and $\mathcal{M} = (\mathcal{M}_t)_{t \geq 0}$ are right-continuous and increasing. The augmented $\sigma$-algebra $\mathcal{F}_t$ is generated by

$$
\mathcal{F}_t = \sigma(\mathcal{F}_t \vee \mathcal{M}_t).
$$

We extend $X(t, x)$ to the process on $[0, T] \times \mathbb{R}^d$ by setting

$$
X(t, x) = 0, \quad (t, x) \in [-\delta, T] \times \mathbb{R}^d \setminus \overline{D}.
$$

Next, we recall some useful sets and spaces which will be used throughout this paper.
Definition 2.1.

- $H = L^2(D)$ is the set of all Lebesgue measurable functions $f : D \to \mathbb{R}$ such that
  \[ \|f\|_H := \left( \int_D |f(x)|^2 \, dx \right)^{1/2} < \infty, \quad x \in D. \]
  In addition, $(f(x), g(x))_H = \int_D f(x)g(x) \, dx$ denotes the inner product in $H$.

- $\mathcal{R}$ denotes the set of Lebesgue measurable functions $r : \mathbb{R}_0 \times D \to \mathbb{R}$. $L^r_0(H)$ is the set of all Lebesgue measurable functions $\gamma \in \mathcal{R}$ such that
  \[ \|\gamma\|_{L^2_0(H)} := \left( \int_D \int_{\mathbb{R}_0} |\gamma(x, \zeta)|^2 \nu(d\zeta) \, dx \right)^{1/2} < \infty, \quad x \in D. \]

- $H_T = L^2_3([0, T] \times \Omega, H)$ is the set of all $\mathcal{F}$-adapted processes $X(t, x)$ such that
  \[ \|X(t, x)\|_{H_T} := \mathbb{E} \left( \int_D \int_0^T |X(t, x)|^2 \, dt \, dx \right)^{1/2} < \infty. \]

- $H_T^{-\delta} = L^2_3([-\delta, T] \times \Omega, H)$ is the set of all $\mathcal{F}$-adapted processes $X(t, x)$ such that
  \[ \|X(t, x)\|_{H_T^{-\delta}} := \mathbb{E} \left( \int_D \int_{-\delta}^T |X(t, x)|^2 \, dt \, dx \right)^{1/2} < \infty. \]

- $V = W^{1,2}(D)$ is a separable Hilbert space (the Sobolev space of order 1) which is continuously, densely imbedded in $H$. Consider the topological dual of $V$ as follows:
  \[ V \subset H \cong H^* \subset V^*. \]
  In addition, let $\langle A_x u, u \rangle_*$ be the duality product between $V$ and $V^*$, and $\|\cdot\|_V$ the norm in the Hilbert space $V$.

- $\mathcal{U}^ad$ is the set of all stochastic processes which take values in a convex subset $\mathcal{U}$ of $\mathbb{R}^d$ and are adapted to a given subfiltration $\mathcal{G} = (\mathcal{G}_t)_{t \geq 0}$. Here, $\mathcal{G}_t \subseteq \mathcal{F}_t$ for all $t \geq 0$. Moreover, $\mathcal{U}^ad$ is called the set of admissible control processes $u$.

Definition 2.2. The adjoint operator $A^*_x$ of a linear operator $A_x$ on $C_0^\infty(\mathbb{R}^d)$ is defined by
\[ \langle A_x \phi, \psi \rangle_{L^2(\mathbb{R}^d)} = \langle \phi, A^*_x \psi \rangle_{L^2(\mathbb{R}^d)}, \quad \forall \phi, \psi \in C_0^\infty(\mathbb{R}^d). \]
Here, $\langle \phi_1, \phi_2 \rangle_{L^2(\mathbb{R}^d)} = \int_{\mathbb{R}^d} \phi_1(x) \phi_2(x) \, dx$ is the inner product in $L^2(\mathbb{R}^d)$. If $A_x$ is the second order partial differential operator acting on $x$ given by
\[ A_x \phi = \sum_{i,j=1}^n \alpha_{ij}(x) \frac{\partial^2 \phi}{\partial x_i \partial x_j} + \sum_{i=1}^n \beta_i(x) \frac{\partial \phi}{\partial x_i}, \quad \forall \phi \in C^2(\mathbb{R}^d), \]
where $(\alpha_{ij}(x))_{1 \leq i, j \leq n}$ is a given nonnegative definite $n \times n$ matrix with entries $\alpha_{ij}(x) \in C^2(D) \cap C(\overline{D})$ for all $i, j = 1, 2, \ldots, n$ and $\beta_i(x) \in C^2(D) \cap C(\overline{D})$ for all $i = 1, 2, \ldots, n$, then it is easy to show that
\[ A^*_x \phi = \sum_{i,j=1}^n \frac{\partial^2 \phi}{\partial x_i \partial x_j}(\alpha_{ij}(x) \phi(x)) - \sum_{i=1}^n \frac{\partial}{\partial x_i}(\beta_i(x) \phi(x)), \quad \forall \phi \in C^2(\mathbb{R}^d). \]
We interpret \( X(t, x) \) as a weak (variational) solution to (1), if for \( t \in [0, T] \) and all \( \phi \in C^\infty_0(D) \), the following equation holds.

\[
(X(t, x), \phi)_H = (\beta(0, x), \phi)_H + \int_0^t (X(s, x), A_t^x \phi)_ds + \int_0^t (b(s, X(s, x)), \phi)_H ds
+ \int_0^t (\sigma(s, X(s, x)), \phi)_H dB_s + \int_0^t \int_{\mathbb{R}_0} (\gamma(s, X(s, x), \zeta), \phi)_H d\tilde{N}(s, \zeta). \tag{3}
\]

In equation (3), these coefficients \( b, \sigma \) and \( \gamma \) are all the simplified notations.

Now, we give the definition of the spatial-temporal interaction operator.

**Definition 2.3.** \( S \) is said to be a spatial-temporal interaction operator if it takes the following form

\[
S(X(t, x)) = \int_{R_\theta} \int_{t-\delta}^t Q(t, s, x, y)X(s, x + y)dsdy, \quad (X(t, x) \in H_T^{-\delta}), \tag{4}
\]

where \( Q(t, s, x, y) \) denotes the density function such that

\[
\int_{y-R_\theta} \int_{s\in\mathbb{R}} |Q(t, s, x, y - x)|^2 dt dx \leq M. \tag{5}
\]

Here the set

\[
R_\theta = \{ y \in \mathbb{R}^d; \|y\|_2 < \theta \}
\]

is an open ball of radius \( \theta > 0 \) centered at \( 0 \), where \( \| \cdot \|_2 \) represents the Euclid norm in \( \mathbb{R}^d \).

**Proposition 1.** For any \( X(t, x) \in H_T^{-\delta} \), one has

\[
\|S(X(t, x))\|_{H_T} \leq \sqrt{M}\|X(t, x)\|_{H_T^{-\delta}}. \tag{6}
\]

This implies that \( S : H_T^{-\delta} \to H_T \) is a bounded linear operator.

**Proof.** Applying Cauchy-Schwartz’s inequality and Fubini’s theorem, we have

\[
\|S(X(t, x))\|_{H_T}^2 = E \int_{D} \int_0^T \left[ \int_{R_\theta} \int_{t-\delta}^t Q(t, s, x, y)X(s, x + y)dsdy \right]^2 dx dt
\leq E \int_{D} \int_0^T \int_{R_\theta} \int_{t-\delta}^t |Q(t, s, x, y)|^2 |X(s, x + y)|^2 dsdydx dt
\leq E \int_D \int_0^T \left( \int_{s\in\mathbb{R}} |Q(t, s, x, y)|^2 dt \right) |X(s, x + y)|^2 dy dx dt
\leq E \int_D \int_0^T \left( \int_{s\in\mathbb{R}} |Q(t, s, x, y)|^2 dy \right) |X(s, x + y)|^2 ds dx dt
\leq E \int_D \int_{t-\delta}^T \left( \int_{z\in\mathbb{R}} |Q(t, s, x, z - x)|^2 dz \right) |X(s, z)|^2 ds dx
\leq E \int_D \int_{t-\delta}^T \left( \int_{z\in\mathbb{R}} |Q(t, s, x, z - x)|^2 dz dx \right) |X(s, z)|^2 ds dz
\leq ME \int_D \int_{t-\delta}^T |X(s, z)|^2 dz ds = M\|X(t, x)\|_{H_T^{-\delta}}^2
\]

This completes the proof. \( \square \)
Example. We give examples for spatial-temporal interaction operators in the following three cases, respectively.

(i) If we set

\[ Q_0(t, s, x, y - x) = e^{-\rho_1(t-s)}e^{-\rho_2\|y\|^2}, \]

where \( \rho_1, \rho_2 \) are two positive constants, then \( Q_0(t, s, x, y - x) \) clearly satisfies condition (5) and \( S_0 : H_T^{-\delta} \to H_T \),

\[ S_0(X(t, x)) = \int_{R^d} \int_{t-\delta}^t e^{-\rho_1(t-s)}e^{-\rho_2\|y\|^2}X(s, x + y)dyds \quad (\forall X(t, x) \in H_T^{-\delta}) \]

becomes the spatial-temporal interaction operator. It shows that an increase in distance \( \|y\|^2 \) or time interval \( t - s \) results in a decreasing effect for local population density.

(ii) When there is no temporal dependence, we set \( S_1 : H \to H : \)

\[ S_1(X(t, x)) = \int_{R^d} Q_1(x, y)X(t, x + y)dy \quad (\forall X(t, x) \in H), \]

where the density function \( Q_1(x, y) \) satisfies

\[ \int_{y - R^d} |Q_1(x, y - x)|^2dy \leq M. \]

For \( Q_1(x, y) = \frac{1}{V(R^d)} \), where \( V(\cdot) \) is the Lebesgue volume in \( \mathbb{R}^d \), \( S_1 \) reduces to the space-averaging operator proposed in [1].

(iii) When there is no spatial dependence, we set \( S_2 : H_T^{-\delta} \to H_T \),

\[ S_2(X(t, x)) = \int_{t-\delta}^t Q_2(t, s)X(s, x)ds, \quad \forall X(t, x) \in H, \]

where the density function \( Q_2(x, y) \) satisfies

\[ \int_{t\in[0,T]} |Q_2(t, s)|^2dt \leq M. \]

For \( Q_2(t, s) = 1 \), \( S_2 \) reduces to the well-known moving average operator.

In the sequel, we illustrate the Fréchet derivative for spatial-temporal interaction operators.

Definition 2.4. The Fréchet derivative \( \nabla_S F \) of a map \( F : H_T^{-\delta} \to H_T \) has a dual function if for \( \forall X(t, x) \in H_T^{-\delta} \)

\[ E \left[ \int_D \int_T \langle \nabla_S F, X \rangle(t, x)dxdt \right] = E \left[ \int_D \int_{t-\delta}^T \nabla_S^* F(t, x)X(t, x)dxdt \right]. \]

Example. Let \( F : H_T^{-\delta} \to H_T \) be a given map by setting

\[ F(X)(t, x) = \langle F, X \rangle(t, x) = S(X(t, x)) = \int_{R^d} \int_{t-\delta}^t Q(t, s, x, y)X(s, x + y)dyds \]

for \( t \in [0, T] \), where \( X(t, x) \in H_T^{-\delta} \). Since \( F \) is linear, we have

\[ \langle \nabla_S F, \psi \rangle(t, x) = \langle F, \psi \rangle(t, x) = \int_{R^d} \int_{t-\delta}^t Q(t, s, x, y)X(s, x + y)dyds. \]
Thus for $\forall \psi \in H$, we have
\[
E \left[ \int_D \int_0^T (\nabla_S F, \psi) dx dt \right] \\
= E \left[ \int_D \int_0^T \int_{R_0} \int_{t-\delta}^{t} Q(t, s, x, y) X(s, x + y) ds dy dx dt \right] \\
= E \left[ \int_D \int_{-\delta}^T \left( \int_{R_0} \int_{s \wedge 0}^{(s+\delta) \wedge T} Q(t, s, x, y) dy ds \right) X(s, x + y) ds dx \right] \\
= E \left[ \int_D \int_{-\delta}^T \left( \int_{D \cap (z-R_0)} \int_{s \vee 0}^{(s+\delta) \vee T} Q(t, s, x, z - x) ds dx \right) X(s, z) ds dz \right].
\]
This implies that
\[
\nabla^*_S F(s, z) = \int_{D \cap (z-R_0)} \int_{s \wedge 0}^{(s+\delta) \wedge T} Q(t, s, x, z - x) dx dt.
\]
Therefore for $t \in [-\delta, T],
\[
\nabla^*_S F(t, x) = \int_{D \cap (x-R_0)} \int_{t \vee 0}^{(t+\delta) \vee T} Q(s, t, y, x - y) dy ds \\
= \int_D \int_T Q(s, t, y, x - y) 1_{x-R_0}(y) 1_{[0,T-\delta]}(t) dy ds.
\]

**Remark 1.** For any $X = X(t, x) \in H$, we set
\[
\overline{X}(t, x) = S(X(t, x)), \quad \min(t, x) = S(u(t, x)).
\]

Now, we introduce these coefficients of SPDE (1) and the functions in Problem 1 in detail. We assume that all of these are functions in $C^1(H)$ and take the following forms:
\[
\begin{align*}
    b(t, x, X, S_\xi, u, S_u) &= b(t, x, X, S_\xi, u, S_u, \omega) : E \times \Omega \rightarrow \mathbb{R}; \\
    \sigma(t, x, X, S_\xi, u, S_u) &= \sigma(t, x, X, S_\xi, u, S_u, \omega) : E \times \Omega \rightarrow \mathbb{R}; \\
    \gamma(t, x, X, S_\xi, u, S_u, \zeta) &= \gamma(t, x, X, S_\xi, u, S_u, \omega, \zeta) : E \times \mathbb{R}_0 \times \Omega \rightarrow \mathbb{R}; \\
    f(t, x, X, S_\xi, u, S_u) &= f(t, x, X, S_\xi, u, S_u, \omega) : E \times \Omega \rightarrow \mathbb{R}; \\
    g(x, X(T)) &= g(x, X(t, x), \omega) : D \times \mathbb{R} \times \Omega \rightarrow \mathbb{R},
\end{align*}
\]
where $E = [-\delta, T] \times D \times \mathbb{R} \times \mathbb{R} \times \mathcal{U}^{ad} \times \mathbb{R}$.

Next, we define the related Hamiltonian functional.

**Definition 2.5.** Define the Hamiltonian functional with respect to the optimal control problem (2) by $H : [0, T + \delta] \times D \times \mathbb{R} \times \mathbb{R} \times \mathcal{U}^{ad} \times \mathbb{R} \times \mathbb{R} \times \mathbb{R} \times \mathbb{R} \times \Omega \rightarrow \mathbb{R}$ as follows:
\[
H(t, x) = H(t, x, X, S_\xi, u, S_u, p, q, r) \\
= H(t, x, X, S_\xi, u, S_u, p, q, r(\cdot), \omega) \\
= f(t, x, X, S_\xi, u, S_u) + b(t, x, X, S_\xi, u, S_u)p + \sigma(t, x, X, S_\xi, u, S_u)q \\
+ \int_{R_0} \gamma(t, x, X, S_\xi, u, S_u, \zeta) r d\zeta.
\]
Moreover, we suppose that functions $b, \sigma, \gamma$ and $f$ all admit bounded Fréchet derivatives with respect to $X, S_\xi, u$ and $S_u$, respectively.
We associate the following adjoint BSPDE to the Hamiltonian (7) in the unknown processes \( p(t, x), q(t, x), r(t, x, \cdot) \).

\[
\begin{align*}
\frac{dp(t, x)}{dt} &= - \left( \frac{\partial H}{\partial x} (t, x) + A^*_x p(t, x) + \mathbb{E} \left[ \nabla_{S_x}^* H(t, x) \big| \mathcal{F}_t \right] \right) dt + q(t, x) dB_t \\
&\quad + \int_{\mathbb{R}_0} r(t, x, \zeta) \tilde{N}(dt, d\zeta), \quad (t, x) \in [0, T) \times D; \\
p(t, x) &= \frac{\partial g}{\partial x} (T, x), \quad (t, x) \in [T, T + \delta] \times D; \\
q(t, x) &= 0, \quad (t, x) \in [0, T) \times \partial D; \\
r(t, x, \cdot) &= 0, \quad (t, x) \in [T, T + \delta] \times \mathcal{D}.
\end{align*}
\]

\[ (8) \]

3. **Maximum principles.** We are now able to derive the sufficient version of the maximum principle.

3.1. **A sufficient maximum principle.**

**Assumption.** Let \( \hat{u} \in \mathcal{U}^{ad} \) be a control with corresponding solutions \( \hat{X}(t, x) \) to (1) and \( (\hat{p}(t, x), \hat{q}(t, x), \hat{r}(t, x, \cdot)) \) to (8), respectively. Furthermore, the control and the solutions satisfy

(i) \( \hat{X} \big|_{t \in [0, T]} \in \mathcal{H}_T; \)

(ii) \( (\hat{p}, \hat{q}, \hat{r}(.)) \big|_{(t, x) \in [0, T] \times \mathcal{D}} \in V \times H \times L^2_v(H); \)

(iii) \( \mathbb{E} \left[ \int_0^T \|\hat{p}(t, x)\|_V^2 + \|\hat{q}(t, x)\|_H^2 + \|\hat{r}(t, x, \cdot)\|_{L^2_v(H)}^2 \right] dt < \infty. \)

**Theorem 3.1.** Suppose that Assumption 3.1 holds. For arbitrary \( u \in \mathcal{U} \), put

\[
H(t, x) = H(t, x, \hat{X}, \hat{S}_X, u, S_u, \hat{p}, \hat{q}, \hat{r}(.)), \quad \tilde{H}(t, x) = H(t, x, \tilde{X}, \tilde{S}_X, \tilde{u}, \tilde{S}_u, \hat{p}, \hat{q}, \hat{r}(.)).
\]

Assume that

- (Concavity) For each \( t \in [0, T] \), the functions
  \[
  (X, S_X, u, S_u) \rightarrow H(t, x, X, S_X, u, S_u, \hat{p}, \hat{q}, \hat{r})
  \]
  \[
  X(T) \rightarrow g(x, X(T))
  \]
  are concave a.s..

- (Maximum condition) For each \( t \in [0, T] \),
  \[
  \mathbb{E} \left[ \tilde{H}(t, x) \big| \mathcal{G}_t \right] = \sup_{u \in \mathcal{U}} \mathbb{E} \left[ H(t, x) \big| \mathcal{G}_t \right], \quad a.s.
  \]

Then, \( \hat{u} \) is an optimal control.

**Proof.** Consider

\[ J(u) - J(\hat{u}) = I_1 + I_2. \]

Here,

\[ I_1 = \mathbb{E} \left[ \int_0^T \int_D f(t, x, X(t, x), \tilde{X}(t, x), u(t, x), \pi(t, x)) \right. \\
- \left. f(t, x, \hat{X}(t, x), \tilde{X}(t, x), \tilde{u}(t, x), \tilde{\pi}(t, x)) dxdt \right]. \]
By the First Green formula [34], there exist first order boundary differential operators $A_1$ and $A_2$ such that

\[
I_2 = \int_{D} \mathbb{E} \left[ g(x, X(T, x)) - g(x, \tilde{X}(T, x)) \right] dx.
\]

Setting $\tilde{X}(t, x) = X(t, x) - \tilde{X}(t, x)$ and applying Itô’s formula, one has

\[
I_2 \leq \int_{D} \mathbb{E} \left[ \frac{\partial \tilde{g}}{\partial x}(T, \tilde{X}(T, x) - \tilde{X}(t, x)) \right] dx + \int_{D} \mathbb{E} \left[ \hat{p}(T, x) \tilde{X}(T, x) \right] dx
\]

\[
= \int_{D} \mathbb{E} \left[ \int_{0}^{T} \hat{p}(t, x)dX(t, x) + \tilde{X}(t, x) + \tilde{g}(t, x)\tilde{\sigma}(t, x)dt + d\tilde{p}(t, x)d\tilde{X}(t, x) \right]
\]

\[
= \int_{D} \mathbb{E} \left\{ \int_{0}^{T} \hat{p}(t, x) \left[ b(t, x) + A_x \tilde{X}(t, x) \right] - \tilde{X}(t, x) \left[ \frac{\partial H}{\partial X}(t, x) + A_x^* \tilde{p}(t, x) \right]
\]

\[
+ \mathbb{E} \left[ \nabla^*_S H(t, x) \left| \mathfrak{F}_t \right. \right] + \tilde{g}(t, x)\tilde{\sigma}(t, x)dt + \int_{\mathfrak{F}_0} \hat{r}(t, x, \zeta)\tilde{\gamma}(t, x, \zeta)\nu(d\zeta, dt) \right\} dx.
\]

(10)

By the First Green formula [34], there exist first order boundary differential operators $A_1$ and $A_2$ such that

\[
\int_{D} \hat{p}(t, x) A_x \tilde{X}(t, x) - \tilde{X}(t, x) A_x^* \tilde{p}(t, x) dx \]

\[
= \int_{\partial D} \hat{p}(t, x) A_1 \tilde{X}(t, x) - \tilde{X}(t, x) A_2 \tilde{p}(t, x) dS = 0. \tag{11}
\]

Combining (10), (11) and the fact that $u(t)$ is $\mathcal{G}_t$-measurable gives

\[
I_2 \leq \int_{D} \mathbb{E} \left\{ \int_{0}^{T} \hat{p}(t, x) b(t, x) - \tilde{X}(t, x) \left[ \frac{\partial H}{\partial X}(t, x) + \mathbb{E} \left[ \nabla^*_S H(t, x) \left| \mathfrak{F}_t \right. \right] \right] \right\} dx
\]

\[
+ \int_{0}^{T} \tilde{g}(t, x)\tilde{\sigma}(t, x)dt + \int_{\mathfrak{F}_0} \hat{r}(t, x, \zeta)\tilde{\gamma}(t, x, \zeta)\nu(d\zeta, dt) \right\} dx
\]

\[
= - I_1 + \int_{D} \mathbb{E} \left[ \int_{0}^{T} H(t, x) - \tilde{X}(t, x) \left[ \frac{\partial H}{\partial X}(t, x) + \mathbb{E} \left[ \nabla^*_S H(t, x) \left| \mathfrak{F}_t \right. \right] \right] dt \right] dx
\]

\[
\leq - I_1 + \int_{D} \mathbb{E} \left[ \int_{0}^{T} \bar{u}(t, x) \left[ \frac{\partial H}{\partial u}(t, x) + \mathbb{E} \left[ \nabla^*_S H(t, x) \left| \mathfrak{F}_t \right. \right] \right] dt \right] dx
\]

\[
= - I_1 + \int_{D} \mathbb{E} \left[ \int_{0}^{T} \bar{u}(t, x) \frac{\partial H}{\partial u}(t, x) + \bar{u}(t, x) \nabla^*_S H(t, x) dt \right] dx
\]

\[
= - I_1 + \int_{D} \mathbb{E} \left[ \int_{0}^{T} \bar{u}(t, x) \left[ \frac{\partial H}{\partial u}(t, x) + \nabla^*_S H(t, x) \right] \mathfrak{G}_t dt \right] dx
\]

\[
\leq - I_1,
\]

where the last inequality is derived by the maximum condition imposed on $H(t, x)$. This implies that

\[
J(u) - J(\bar{u}) = I_1 + I_2 \leq 0.
\]

Therefore $\bar{u}$ becomes the optimal control. □
3.2. A necessary maximum principle. We now proceed to study the necessary version of maximum principle.

**Assumption.** For each \( t_0 \in [0, T] \) and all bounded \( \mathcal{G}_{t_0} \)-measurable random variable \( \pi(x) \), the process \( \vartheta(t, x) = \pi(x) \mathbb{I}_{(t_0, T]}(t) \) belongs to \( \mathcal{U}^{ad} \).

**Remark 2.** Thanking to the convex condition imposed on \( \mathcal{U}^{ad} \), one has

\[
u^{\epsilon} = (1 - \epsilon)\hat{u} + \epsilon u \in \mathcal{U}^{ad}, \quad \epsilon \in [0, 1]
\]

for any \( u, \hat{u} \in \mathcal{U}^{ad} \).

Consider the process \( Z(t, x) \) obtained by differentiating \( X^{\epsilon}(t, x) \) with respect to \( \epsilon \) at \( \epsilon = 0 \). Clearly, \( Z(t, x) \) satisfies the following equation:

\[
dZ(t, x) = \left[ (\nabla b(t, x))^T (Z(t, x), u(t, x)) + (\nabla \sigma(t, x))^T (Z(t, x), u(t, x)) + A_x Z(t, x) \right] dt
\]

\[
+ \int_{\mathcal{F}_x} (\nabla \gamma(t, x, \xi))^T (Z(t, x), u(t, x)) N(dt, d\xi), \quad (t, x) \in (0, T] \times D,
\]

\[
Z(t, x) = 0, \quad (t, x) \in [-\delta, 0] \times D.
\]

(12)

where

\[
(\nabla b(t, x))^T = \left( \frac{\partial b}{\partial X}(t, x) + \mathbb{E} \left[ \nabla_{\mathcal{S}_x} b(t, x) \bigg| \mathcal{G}_t \right] \right)^T,
\]

\[
(\nabla \sigma(t, x))^T = \left( \frac{\partial \sigma}{\partial X}(t, x) + \mathbb{E} \left[ \nabla_{\mathcal{S}_x} \sigma(t, x) \bigg| \mathcal{G}_t \right] \right)^T,
\]

\[
(\nabla \gamma(t, x, \xi))^T = \left( \frac{\partial \gamma}{\partial X}(t, x, \xi) + \mathbb{E} \left[ \nabla_{\mathcal{S}_x} \gamma(t, x, \xi) \bigg| \mathcal{G}_t \right] \right)^T.
\]

Theorem 3.2. Suppose that Assumptions 3.1 and 3.2 hold. Then the following equalities are equivalent.

(i) For all bounded \( u \in \mathcal{U}^{ad} \),

\[
0 = \left. \frac{d}{dx} J(\hat{u} + \epsilon u) \right|_{\epsilon = 0}.
\]

(ii)

\[
0 = \left. \int_D \mathbb{E} \left[ \frac{\partial H}{\partial u}(t, x) + \nabla_{\mathcal{S}_x}^* H(t, x) \bigg| \mathcal{G}_t \right] dx \right|_{u = \hat{u}} = \forall t \in [0, T].
\]

Proof. Assume that (13) holds. Then

\[
0 = \left. \frac{d}{dx} J(\hat{u} + \epsilon u) \right|_{\epsilon = 0}
\]

\[
= \mathbb{E} \left[ \int_0^T \left( (\nabla f(t, x))^T (Z(t, x), u(t, x)) dt + \int_D \frac{\partial \tilde{Z}}{\partial X}(T, x) \tilde{Z}(T, x) dx \right) \right],
\]

where \( \tilde{Z}(t, x) \) is the solution to (12), and

\[
(\nabla f(t, x))^T = \left( \frac{\partial f}{\partial X}(t, x) + \mathbb{E} \left[ \nabla_{\mathcal{S}_x}^* f(t, x) \bigg| \mathcal{G}_t \right] \right)^T.
\]
By Itô’s formula,

\[
E \left[ \int_D \frac{\partial^2}{\partial x^2}(T, x) \tilde{Z}(T, x) dx \right] = E \left[ \int_D \tilde{p}(T, x) \tilde{Z}(T, x) dx \right]
\]

\[
= E \left[ \int_D dx \int_0^T \tilde{p}(t, x) d\tilde{Z}(t, x) + \tilde{Z}(t, x) d\tilde{p}(t, x) + d\tilde{Z}(t, x) d\tilde{p}(t, x) \right]
\]

\[
= E \left[ \int_D dx \int_0^T \left( \tilde{p}(t, x) \left( \nabla b(t, x) \right)^T \tilde{Z}(t, x), u(t, x) \right) + \tilde{p}(t, x) \cdot A \tilde{Z}(t, x) - \tilde{Z}(t, x) A^*_\sigma \tilde{p}(t, x) \right.
\]

\[
- \left( \frac{\partial H}{\partial x}(t, x) + E \left[ \nabla S^* H(t, x) \left| \tilde{S}_t \right. \right] \right) \tilde{Z}(t, x) + \tilde{q}(t, x) (\nabla \sigma(t, x))^T (\tilde{Z}(t, x), u(t, x))
\]

\[
+ \int_{\tilde{S}_0} \tilde{r}(t, x, \zeta) \int_{\tilde{S}_0} (\nabla \gamma(t, x, \zeta))^T (\tilde{Z}(t, x), u(t, x)) N(d\zeta) \right] dt
\]

\[
= E \left[ \int_0^T \int_D \frac{\partial H}{\partial u}(t, x) + E \left[ \nabla S^* H(t, x) \left| \tilde{S}_t \right. \right] u(t, x) - (\nabla f(t, x))^T (\tilde{Z}(t, x), u(t, x)) dx dt \right].
\]

where the last step follows from the first Green formula [34].

Combining (15) and (16), one has

\[
0 = E \left[ \int_0^T \int_D \frac{\partial H}{\partial u}(t, x) + E \left[ \nabla S^* H(t, x) \left| \tilde{S}_t \right. \right] u(t, x) dx dt \right].
\]

Now we set \( u(t, x) = \pi(x) \mathbb{I}_{(t_0, T]}(t) \), where \( \pi(x) \) is a bounded \( \mathcal{G}_{t_0} \)-measurable random variable. Then, we have

\[
0 = \int_0^T E \left[ \int_D \frac{\partial H}{\partial u}(t, x) \pi(x) \mathbb{I}_{(t_0, T]}(t) + E \left[ \nabla S^* H(t, x) \left| \tilde{S}_t \right. \right] \pi(x) \mathbb{I}_{(t_0, T]}(t) \right] dt
\]

\[
= \int_0^T E \left[ \int_D \frac{\partial H}{\partial u}(t, x) \pi(x) + E \left[ \nabla S^* H(t, x) \left| \tilde{S}_t \right. \right] \pi(x) \right] dt
\]

\[
= \int_{t_0}^T E \left[ \int_D \frac{\partial H}{\partial u}(t, x) \pi(x) + \nabla S^* H(t, x) \pi(x) \right] dt.
\]

Differentiating with respect to \( t_0 \), it follows that

\[
0 = E \left[ \int_D \frac{\partial H}{\partial u}(t_0, x) \pi(x) + \nabla S^* H(t_0, x) \pi(x) dx \right], \quad \forall t_0 \in [0, T].
\]

Since this holds for all such \( \pi(x) \), we have

\[
0 = \int_D E \left[ \frac{\partial H}{\partial u}(t_0, x) + \nabla S^* H(t_0, x) \bigg| \mathcal{G}_t \right] dx, \quad \forall t_0 \in [0, T].
\]

The argument above is reversible. Thus (13) and (14) are equivalent.
4. Existence and uniqueness. In this section, we prove the existence and uniqueness of the solution to the following general BSPDE (8) with spatial-temporal dependence:

\[
\begin{align*}
    dp(t,x) &= E[F(t,\tilde{\xi})]dt + q(t,x)dB_t + \int_{\mathbb{R}^n} r(t,x,\zeta)\tilde{N}(dt,d\zeta) \\
    - A_2 p(t,x)dt, & (t,x) \in [0,T] \times D; \\
    p(t,x) &= \theta(t,x), & (t,x) \in [T,T+\delta] \times \partial D; \\
    q(t,x) &= \chi(t,x), & (t,x) \in [0,T) \times \partial D; \\
    r(t,x,\cdot) &= 0, & (t,x) \in [T,T+\delta] \times \partial D; \\
    \end{align*}
\]

(17)

Here \( F = F(t) : [0,T+\delta] \times \mathbb{R} \times \mathbb{R} \times \mathbb{R} \times \mathbb{R} \times \mathbb{R} \rightarrow \mathbb{R} \) is a functional on \( C^1(H) \) as follows:

\[ F(t) = F(t,p(t,x),\bar{p}(t,\delta,x),q(t,x),\bar{q}(t,\delta,x),r(t,x,\cdot),\tau(t+\delta,x,\cdot)). \]

**Assumption.** Assume that \( A_2 : V \rightarrow V^* \) is a bounded and linear operator. Moreover, there exists two constants \( \alpha_1 > 0 \) and \( \alpha_2 \geq 0 \) such that

\[ 2\langle A_2 u, u \rangle_\ast + \alpha_1 \| u \|^2_V \leq \alpha_2 \| u \|^2_H, \quad \forall u \in V. \]

(18)

**Assumption.** Suppose that the following assumptions hold:

(i) \( \theta(t,x) \) is a given \( \mathcal{F}_t \)-measurable process such that

\[ \mathbb{E}\left[ \sup_{t \in [T,T+\delta]} \| \theta(t,x) \|^2_H \right] < \infty; \]

(ii) \( F(t,0,0,0,0,0) \in H_T; \)

(iii) There is a constant \( C > 0 \) such that, for any \( t,p_1,q_1,r_1,p_2,q_2,r_2, \)

\[
\begin{align*}
    |F(t,p_1,\bar{p}_1,q_1,\bar{q}_1,r_1) - F(t,p_2,\bar{p}_2,q_2,\bar{q}_2,r_2)||^2 \\
    \leq C\left(|p_1 - p_2|^2 + |q_1 - q_2|^2 + \int_{\mathbb{R}^n} |r_1 - r_2|^2 \nu(d\zeta) + |\bar{p}_1 - \bar{p}_2|^2 \right. \\
    \left. + |\bar{q}_1 - \bar{q}_2|^2 + \int_{\mathbb{R}^n} |\bar{r}_1 - \bar{r}_2|^2 \nu(d\zeta). \right)
\end{align*}
\]

In the sequel, we use \( C \) to represent the constant large enough such that all the inequalities are satisfied.

**Theorem 4.1.** Under Assumptions 4 and 4, BSPDEs (17) has a unique solution \((p,q,r(\cdot))\) such that the restriction on \((t,x) \in [0,T] \times \overline{D}\) satisfies

\[
\begin{align*}
    (i) \quad & \left( p,q,r(\cdot) \right) \big|_{(t,x) \in [0,T] \times \overline{D}} \in V \times H \times L^2(H); \\
    (ii) \quad & \mathbb{E}\left[ \int_0^T \| p(t,x) \|^2_V + \| q(t,x) \|^2_H + \| r(t,x,\cdot) \|^2_{L^2(H)} dt \right] < \infty.
\end{align*}
\]

**Proof:** We decompose the proof into five steps.

**Step 1:** Assume that the driver \( F(t) \) is independent of \( p \) and \( \bar{p} \) such that \((p,q,r(\cdot)) \in V \times H \times L^2(H)\) satisfies...
\[
\begin{align*}
\frac{dp(t,x)}{dt} &= -(A_x p(t,x) - \mathbb{E}[F(t, q(t,x), \eta(t+\delta,x), r(t,x,\cdot), \tau(t+\delta,x,\cdot))]|\mathcal{F}_t]) dt \\
&+ q(t,x)dB_t + \int_{\mathbb{R}_o} r(t,x,\zeta)\tilde{N}(dt,d\zeta), \quad (t,x) \in [0,T) \times D;
\end{align*}
\]

\[
\begin{align*}
p(t,x) = \zeta(t,x), \quad (t,x) \in [T,T+\delta] \times \overline{D}; \\
p(t,x) = \theta(t,x), \quad (t,x) \in [0,T) \times \partial D; \\
q(t,x) = 0, \quad (t,x) \in [T,T+\delta] \times \overline{D}; \\
r(t,x,\cdot) = 0, \quad (t,x) \in [T,T+\delta] \times \overline{D}.
\end{align*}
\]

(19)

We first prove the uniqueness and existence of solutions to (19). By Theorem 4.2 in [31], it is easy to show that for each fixed \( n \in \mathbb{N} \), there exists a unique solution to the following BSPDE

\[
\begin{align*}
\frac{dp^{n+1}(t,x)}{dt} &= \mathbb{E}[F(t, q^n(t,x), \eta^n(t+\delta,x), r^n(t,x,\cdot), \tau^n(t+\delta,x,\cdot))]|\mathcal{F}_t] dt \\
&- A_x p^{n+1}(t,x) dt + q^{n+1}(t,x) dB_t + \int_{\mathbb{R}_o} r^{n+1}(t,x,\zeta)\tilde{N}(dt,d\zeta), \\
&\quad (t,x) \in [0,T) \times D;
\end{align*}
\]

\[
\begin{align*}
p^{n+1}(t,x) = \zeta(t,x), \quad (t,x) \in [T,T+\delta] \times \overline{D}; \\
p^{n+1}(t,x) = \theta(t,x), \quad (t,x) \in [0,T) \times \partial D; \\
q^{n+1}(t,x) = 0, \quad (t,x) \in [T,T+\delta] \times \overline{D}; \\
r^{n+1}(t,x,\cdot) = 0, \quad (t,x) \in [T,T+\delta] \times \overline{D}.
\end{align*}
\]

(20)

such that

\[
(p^n, q^n, r^n(\cdot)) \big|_{(t,x) \in [0,T] \times \overline{D}} \in V \times H \times L^2(H).
\]

Here, \( q^0(t,x) = r^0(t,x,\cdot) = 0 \) for all \( (t,x) \in [0,T+\delta] \times \overline{D} \).

We now aim to show that \( (p^n, q^n, r^n(\cdot)) \) forms a Cauchy sequence. By similar arguments in Proposition 1, we have

\[
\begin{align*}
\mathbb{E} \left[ \int_t^T \| p^{n+1}(s+\delta,x) - p^n(s+\delta,x) \|^2_{H} ds \right] \\
&= \mathbb{E} \left[ \int_D \int_{\mathbb{R}_o} \int_s^{s+\delta} |Q(s+\delta,\varsigma,x,y)|^2 |p^{n+1}(\varsigma,x+y) - p^n(\varsigma,x+y)|^2 d\varsigma dxdyds \right] \\
&\leq CE \left[ \int_D \int_{\mathbb{R}_o} \left( \int_{(s-R_o) \cap \{z: |z| = \delta\}} |Q(s+\delta,\varsigma,x,z-x)|^2 dsdz \right) \| p^{n+1}(\varsigma,z) - p^n(\varsigma,z) \|^2_{L^2} d\varsigma dx \right] \\
&= CE \left[ \int_{D} \int_{\mathbb{R}_o} \| p^{n+1}(\varsigma,z) - p^n(\varsigma,z) \|^2_{L^2} d\varsigma dz \right].
\end{align*}
\]

Similarly, one has

\[
\begin{align*}
\mathbb{E} \left[ \int_t^T \| p^{n+1}(s+\delta,x) - \eta^n(s+\delta,x) \|^2_{H} ds \right] &\leq CE \left[ \int_t^T \| q^{n+1}(s,x) - q^n(s,x) \|^2_{H} ds \right], \\
\mathbb{E} \left[ \int_t^T \| \tau^{n+1}(s+\delta,x,\cdot) - \tau^n(s+\delta,x,\cdot) \|^2_{H} ds \right] &\leq CE \left[ \int_t^T \| \gamma^{n+1}(s,x,\cdot) - \gamma^n(s,x,\cdot) \|^2_{L^2(H)} ds \right].
\end{align*}
\]

(21)
Step 2: For simplicity, we write
\[ F^n_t = F(t, q^n_t(x), \eta^n_t(x, \cdot), n^n_t(x + \delta, x, \cdot)). \]

Applying Itô’s formula to \( \|p^{n+1}(t, x) - p^n(t, x)\|_H^2 \), one has
\[
- \|p^{n+1}(t, x) - p^n(t, x)\|_H^2 = 2 \int_t^T \langle p^{n+1}(s, x) - p^n(s, x), A_x(p^{n+1}(s, x) - p^n(s, x)) \rangle_t ds \\
- 2 \int_t^T \langle p^{n+1}(s, x) - p^n(s, x), E[F^n(s) - F^{n-1}(s) | \mathcal{F}_s] \rangle_t H ds \\
+ 2 \int_t^T \langle p^{n+1}(s, x) - p^n(s, x), q^n(s, x) - q^n(s, x) \rangle_t dB_s \\
+ \int_t^T \|q^n(s, x) - q^n(s, x)\|_{L^n_2(H)}^2 ds + \int_t^T \|r^{n+1}(s, x, \cdot) - r^n(s, x, \cdot)\|_{L^n_2(H)}^2 ds \\
+ 2 \int_t^T \int_{\Omega_0} \langle p^{n+1}(s, x) - p^n(s, x), r^{n+1}(s, x, \zeta) - r^n(s, x, \zeta) \rangle_t N(ds, d\zeta). \tag{22}
\]

By the Lipschitz condition imposed on \( F \) and (21), for any fixed constant \( \rho > 0 \), one has
\[
- 2E \left[ \int_t^T \langle p^{n+1}(s, x) - p^n(s, x), E[F^n(s) - F^{n-1}(s) | \mathcal{F}_s] \rangle_t H ds \right] \\
= -2E \left[ \int_t^T \langle p^{n+1}(s, x) - p^n(s, x), F^n(s) - F^{n-1}(s) \rangle_t H ds \right] \\
\leq 2E \left[ \int_t^T \|p^{n+1}(s, x) - p^n(s, x)\|_H \|F^n(s) - F^{n-1}(s)\|_H ds \right] \\
\leq \frac{1}{\rho} E \left[ \int_t^T \|p^{n+1}(s, x) - p^n(s, x)\|_H^2 ds \right] + \rho E \left[ \int_t^T \|F^n(s) - F^{n-1}(s)\|_H^2 ds \right] \\
\leq \frac{1}{\rho} E \left[ \int_t^T \|p^{n+1}(s, x) - p^n(s, x)\|_H^2 ds \right] + \rho CE \left[ \int_t^T \|q^n(s, x) - q^{n-1}(s, x)\|_{L^n_2(H)}^2 ds \right] \\
+ \|r^n(s, x, \cdot) - r^{n-1}(s, x, \cdot)\|_{L^n_2(H)}^2 + \|\eta^{n+1}(s + \delta, x) - \eta^n(s + \delta, x)\|_H^2 \\
+ \|\gamma^{n+1}(s + \delta, x, \cdot) - \gamma^n(s + \delta, x, \cdot)\|_{L^n_2(H)}^2 ds \right] \\
\leq \frac{1}{\rho} E \left[ \int_t^T \|p^{n+1}(s, x) - p^n(s, x)\|_H^2 ds \right] + \rho CE \left[ \int_t^T L^n(s) ds \right], \tag{23}
\]

where
\[ L^n(s) = \|q^n(s, x) - q^{n-1}(s, x)\|_{L^n_2(H)}^2 + \|r^n(s, x, \cdot) - r^{n-1}(s, x, \cdot)\|_{L^n_2(H)}^2. \]

Taking expectation on both sides of (22), and applying both (18) and (23), we have
\[
E \|p^{n+1}(t, x) - p^n(t, x)\|_H^2 \\
\leq E \left[ \int_t^T \alpha_2 \|p^{n+1}(s, x) - p^n(s, x)\|_H^2 ds + \alpha_1 \|p^{n+1}(s, x) - p^n(s, x)\|_V^2 ds \right] \\
+ \frac{1}{\rho} E \left[ \int_t^T \|p^{n+1}(s, x) - p^n(s, x)\|_H^2 ds \right] - E \left[ \int_t^T \|q^{n+1}(s, x) - q^n(s, x)\|_H^2 ds \right].
\]
\( \int_{t}^{T} L^{n}(s) ds \) - \( \mathbb{E} \left[ \int_{t}^{T} \| p^{n+1}(s, x, \cdot) - p^{n}(s, x, \cdot) \|_{L_{v}(H)}^{2} ds \right] \) 
\leq \left( \frac{1}{\rho} + \alpha_{2} \right) \mathbb{E} \left[ \int_{t}^{T} \| p^{n+1}(s, x) - p^{n}(s, x) \|_{H}^{2} ds \right] + \rho \mathbb{C} \mathbb{E} \left[ \int_{t}^{T} L^{n}(s) ds \right]
\) 
- \( \mathbb{E} \left[ \int_{t}^{T} L^{n+1}(s) ds \right] - \alpha_{1} \mathbb{E} \left[ \int_{t}^{T} \| p^{n+1}(s, x) - p^{n}(s, x) \|_{V}^{2} ds \right] \). \tag{24}

One can choose a \( \rho > 0 \) such that
\( \mathbb{E} \| p^{n+1}(t, x) - p^{n}(t, x) \|_{H}^{2} - \alpha_{3} \mathbb{E} \left[ \int_{t}^{T} \| p^{n+1}(s, x) - p^{n}(s, x) \|_{H}^{2} ds \right] \)
\leq \frac{1}{2} \mathbb{E} \left[ \int_{t}^{T} L^{n}(s) ds \right] - \alpha_{1} \mathbb{E} \left[ \int_{t}^{T} \| p^{n+1}(s, x) - p^{n}(s, x) \|_{V}^{2} ds \right] \mathbb{E} \left[ \int_{t}^{T} L^{n+1}(s) ds \right]. \tag{25}

where \( \alpha_{3} = \alpha_{2} + \frac{1}{\rho} \). Multiplying by \( e^{\alpha_{3}t} \) and integrating both sides in \([0, T]\), we have
\( \int_{0}^{T} e^{\alpha_{3}t} \mathbb{E} \left[ \| p^{n+1}(t, x) - p^{n}(t, x) \|_{H}^{2} \right] dt \)
\leq \frac{1}{2} \int_{0}^{T} e^{\alpha_{3}t} \mathbb{E} \left[ \int_{t}^{T} L^{n}(s) ds \right] dt - \int_{0}^{T} e^{\alpha_{3}t} \mathbb{E} \left[ \int_{t}^{T} L^{n+1}(s) ds \right] dt
and so
\( \int_{0}^{T} e^{\alpha_{3}t} \mathbb{E} \left[ \int_{t}^{T} L^{n+1}(s) ds \right] dt \)
\leq \frac{1}{2} \int_{0}^{T} e^{\alpha_{3}t} \mathbb{E} \left[ \int_{t}^{T} L^{n}(s) ds \right] dt. \tag{26}

In particular,
\( \int_{0}^{T} e^{\alpha_{3}t} \mathbb{E} \left[ \int_{t}^{T} L^{n+1}(s) ds \right] dt \leq \frac{1}{2} \int_{0}^{T} e^{\alpha_{3}t} \mathbb{E} \left[ \int_{t}^{T} L^{n}(s) ds \right] dt \leq C \left( \frac{1}{2} \right)^{n}. \tag{27} \)

**Step 3: Existence**

Substituting (27) into (26), one has
\( \| p^{n+1}(t, x) - p^{n}(t, x) \|_{H_{T}}^{2} \leq C \left( \frac{1}{2} \right)^{n} \).

It follows from (25) that
\( \mathbb{E} \left[ \int_{t}^{T} L^{n+1}(s) ds \right] \leq C \left( \frac{1}{2} \right)^{n} + \frac{1}{2} \mathbb{E} \left[ \int_{t}^{T} L^{n}(s) ds \right] \)
\leq C \left( \frac{1}{2} \right)^{n} + C \left( \frac{1}{2} \right)^{n} + \frac{1}{2^{2n}} \mathbb{E} \left[ \int_{t}^{T} L^{n-1}(s) ds \right]
\leq \frac{nC}{2^{n}} + \frac{1}{2^{n}} \mathbb{E} \left[ \int_{t}^{T} L^{1}(s) ds \right]
\leq \frac{nC}{2^{n}} + \frac{C}{2^{n}} = \frac{n(C + 1)}{2^{n}}.
Letting $n \to \infty$, we have

$$0 = \lim_{n \to \infty} \mathbb{E} \left[ \int_{t}^{T} L^{n+1}(s) \, ds \right].$$

(28)

Substituting (28) into (25) yields

$$\lim_{n \to \infty} \mathbb{E} \left[ \int_{t}^{T} \| p^{n+1}(s, x) - p^n(s, x) \|_V^2 \, ds \right] = 0.$$

Therefore, for $(t, x) \in [0, T] \times \mathcal{D}$, the sequence $\{(p^n, q^n, r^n(\cdot)) : (p^n, q^n, r^n(\cdot)) \in V \times H \times L^2_\nu(H)\}$ converges to $(p, q, r(\cdot)) \in V \times H \times L^2_\nu(H)$. Letting $n \to \infty$ in (20), we can see that the limit $(p, q, r(\cdot)) = \lim_{n \to \infty} (p^n, q^n, r^n(\cdot))$ is indeed the solution to (19) on $(t, x) \in [0, T] \times \mathcal{D}$.

**Step 4: Uniqueness**

Suppose that $(p, q, r(\cdot)), (p^{(0)}, q^{(0)}, r^{(0)}(\cdot))$ are two solutions to (19). As the same arguments in Step 2, we see that

$$\mathbb{E}\|p(t, x) - p^{(0)}(t, x)\|_H^2$$

$$\leq \alpha_3 \mathbb{E} \left[ \int_{t}^{T} \| p(s, x) - p^{(0)}(s, x) \|_H^2 \, ds \right] - \alpha_1 \mathbb{E} \left[ \int_{t}^{T} \alpha_1 \| p(s, x) - p^{(0)}(s, x) \|_V^2 \, ds \right]$$

$$- \frac{1}{2} \mathbb{E} \left[ \int_{t}^{T} \| p(t, x) - p^{(0)}(t, x) \|_H^2 + \| q(s, x) - q^{(0)}(s, x) \|_H^2 \right.$$        

$$+ \| r(s, x, \cdot) - r^{(0)}(s, x, \cdot) \|_{L^\infty_\nu(H)}^2 \, ds \right].$$

It follows that

$$\mathbb{E}\|p(t, x) - p^{(0)}(t, x)\|_H^2 \leq \alpha_3 \mathbb{E} \left[ \int_{t}^{T} \| p(s, x) - p^{(0)}(s, x) \|_H^2 \, ds \right].$$

By Gronwall’s Lemma, we know that $\mathbb{E}\|p(t, x) - p^{(0)}(t, x)\|_H^2 = 0$ and $p(t, x) = p^{(0)}(t, x)$ a.s. and so

$$\mathbb{E} \left[ \int_{t}^{T} \alpha_1 \| p(s, x) - p^{(0)}(s, x) \|_V^2 \, ds \right] + \frac{1}{2} \mathbb{E} \left[ \int_{t}^{T} \| q(s, x) - q^{(0)}(s, x) \|_H^2 \, ds \right]$$

$$+ \frac{1}{2} \mathbb{E} \left[ \int_{t}^{T} \| r(s, x, \cdot) - r^{(0)}(s, x, \cdot) \|_{L^\infty_\nu(H)}^2 \, ds \right] \leq 0,$$

which implies $q(t, x) = q^{(0)}(t, x)$ and $r(t, x, \cdot) = r^{(0)}(t, x, \cdot)$ a.s.

**Step 5: General case**

For the general driver

$$F_p^n(t) = F \left( t, p^n(t, x), p^n(t + \delta, x), q^{n+1}(t, x), \rho^{n+1}(t + \delta, x), r^{n+1}(t, x, \cdot), \rho^{n+1}(t + \delta, x, \cdot) \right),$$

consider the following iteration
\[ \begin{align*}
&\begin{cases}
    dp^{n+1}(t, x) = \mathbb{E}\left[ F^n_p(t) \left| \tilde{\mathcal{F}}_t \right. \right] dt + q^{n+1}(t, x) dB_t + \int_{\mathbb{R}_0} r^{n+1}(t, x, \xi) \tilde{N}(dt, d\xi) \\
    \quad - Ax^{n+1}(t, x) dt, \quad (t, x) \in [0, T) \times \mathcal{D};
    \\
    p^{n+1}(t, x) = \zeta(t, x), \quad (t, x) \in [T, T + \delta) \times \mathcal{D};
    \\
    q^{n+1}(t, x) = \theta(t, x), \quad (t, x) \in [0, T) \times \partial \mathcal{D};
    \\
    r^{n+1}(t, x, \cdot) = 0, \quad (t, x) \in [T, T + \delta) \times \mathcal{D},
\end{cases} \\
\end{align*} \]

where \( p^0(t, x) = 0 \). Similar to the proofs of Steps 1-2, we can easily obtain the following inequality:

\[
\begin{align*}
&\mathbb{E}\|p^{n+1}(t, x) - p^n(t, x)\|^2_H - \alpha_3 \mathbb{E}\left[ \int_t^T \|p^{n+1}(s, x) - p^n(s, x)\|^2_H ds \right] \\
&\leq \rho C \mathbb{E}\left[ \int_t^T \|p^n(s, x) - p^{n-1}(s, x)\|^2_H ds \right] - (1 - \rho C) \mathbb{E}\left[ \int_t^T L^{n+1}(s) ds \right] \\
&\quad - \alpha_1 \mathbb{E}\left[ \int_t^T \|p^{n+1}(s, x) - p^n(s, x)\|^2_H ds \right].
\end{align*}
\]

Choosing \( \rho = \frac{1}{2\alpha_3} \), we have

\[
\begin{align*}
&\mathbb{E}\|p^{n+1}(t, x) - p^n(t, x)\|^2_H - \alpha_3 \mathbb{E}\left[ \int_t^T \|p^{n+1}(s, x) - p^n(s, x)\|^2_H ds \right] \\
&\leq \frac{1}{2} \mathbb{E}\left[ \int_t^T \|p^n(s, x) - p^{n-1}(s, x)\|^2_H ds \right] - \alpha_1 \mathbb{E}\left[ \int_t^T \|p^{n+1}(s, x) - p^n(s, x)\|^2_H ds \right] \\
&\quad - \frac{1}{2} \mathbb{E}\left[ \int_t^T L^{n+1}(s) ds \right] \leq \frac{1}{2} \mathbb{E}\left[ \int_t^T \|p^n(s, x) - p^{n-1}(s, x)\|^2_H ds \right].
\end{align*}
\]

Multiplying by \( e^{\alpha_3 t} \) and integrating both sides in \([0, T]\), one has

\[
\begin{align*}
&\mathbb{E}\left[ \int_t^T \|p^{n+1}(s, x) - p^n(s, x)\|^2_H ds \right] \leq \frac{1}{2} \int_t^T e^{\alpha_3 t} \mathbb{E}\left[ \int_t^T \|p^n(s, x) - p^{n-1}(s, x)\|^2_H ds dt \right] \\
&\leq C \mathbb{E}\left[ \int_t^T \int_t^T \|p^n(s, x) - p^{n-1}(s, x)\|^2_H ds dt \right] \\
&\leq C \int_t^T \mathbb{E}\left[ \int_t^T \|p^n(s, x) - p^{n-1}(s, x)\|^2_H ds \right] dt.
\end{align*}
\]

Note that for any \( \tau \in [0, T] \),

\[
\begin{align*}
&\mathbb{E}\left[ \int_\tau^T \|p^2(s, x) - p^1(s, x)\|^2_H ds \right] \leq C \int_\tau^T \mathbb{E}\left[ \int_0^T \|p^1(s, x) - p^0(s, x)\|^2_H ds \right] \\
&\leq C^2 (T - \tau).
\end{align*}
\]

Iterating the above inequality shows that

\[
\begin{align*}
&\mathbb{E}\left[ \int_\tau^T \|p^{n+1}(s, x) - p^n(s, x)\|^2_H ds \right] \leq \frac{C^{n+1} (T - \tau)^n}{n!}.
\end{align*}
\]
Using (30) and a similar argument as in Steps 3-4, it can see that the limit
\( (p,q,r(\cdot)) = \lim_{n\to\infty} (p^n,q^n,r^n(\cdot)) \) is indeed the solution to (8) on \((t,x) \in [0,T] \times \mathcal{D} \). This ends the proof.

5. Applications. In this section, our results obtained in the previous sections are applied to the stochastic population dynamic models with spatial-temporal dependence.

As discussed in [29], the following SPDE is a natural model for population growth:

\[
\begin{align*}
\frac{dX(t,x)}{dt} &= \left[ \frac{1}{2} \Delta X(t,x) + \alpha X(t,x) - u(t,x) \right] dt + \beta X(t,x) dB_t, \\
(t,x) &\in (0,T] \times \mathcal{D}; \\
X(0,x) &= \xi(x), \quad x \in \mathcal{D}; \\
X(t,x) &= \eta(t,x) \geq 0, \quad (t,x) \in (0,T] \times \partial \mathcal{D}.
\end{align*}
\]

(31)

Here, \( X(t,x) \) is the density of a population (e.g. fish) at \((t,x)\), \( u(t,x) \) is the harvesting rate at \((t,x) \in [0,T] \times \mathcal{D} \), and

\[
\Delta = \sum_{i=1}^{n} \frac{\partial^2}{\partial x_i^2}
\]

is a Laplacian operator.

Taking account of the spatial-temporal dependence in (31), we have the following system:

\[
\begin{align*}
\frac{dX(t,x)}{dt} &= \left[ \frac{1}{2} \Delta X(t,x) - u(t,x) \right] dt + (\gamma_1(t,x)X(t,x) + \gamma_2(t,x)\bar{X}(t,x)) \gamma_3(t,x) dt \\
&\quad + \gamma_4(t,x) dB_t + \int_{\mathbb{R}_0} \gamma_5(t,x,\zeta) \bar{N}(dt,d\zeta), \quad (t,x) \in (0,T] \times \mathcal{D}; \\
X(t,x) &= \xi(t,x), \quad (t,x) \in [-\delta,0] \times \mathcal{D}; \\
X(t,x) &= \eta(t,x) \geq 0, \quad (t,x) \in (0,T] \times \partial \mathcal{D}; \\
u(t,x) &= \beta(t,x) \geq 0, \quad (t,x) \in [-\delta,0] \times \partial \mathcal{D},
\end{align*}
\]

(32)

where \( \gamma_i(t,x) \in H_T \) \((i = 1,2,3,4)\), \( \int_{\mathbb{R}_0} \gamma_5(t,x,\zeta) \bar{N}(dt,d\zeta) \in H_T \) are all given.

Now, we give two examples concerned with the stochastic optimal control problems governed by (32) under different performance functionals.

Example. We consider the following performance functional:

\[
J_0(u) = \mathbb{E} \left[ \int_{\mathcal{D}} \int_{0}^{T} \log \left( u(t,x) \right) dt dx + \int_{\mathcal{D}} k(x) \log \left( X(T,x) \right) dx \right].
\]

(33)

Here, \( \beta \in (0,1) \) is a constant and \( k(x) \in H \) is a nonnegative, \( \mathcal{F}_T \)-measurable process.

Now, we aim to find \( \tilde{u}(t,x) \in \mathcal{U}^{ad} \) such that

\[
J_0(\tilde{u}) = \sup_{u \in \mathcal{U}^{ad}} J_0(u).
\]

Since the Laplacian operator \( \Delta \) is self-adjoint, the Hamiltonian functional associated to this problem takes the following form

\[
H(t,x,S,z,u,p,q,r(\cdot)) = \log u(t,x) + [\gamma_1(t,x)\gamma_3(t,x)X(t,x) + \gamma_2(t,x)\bar{X}(t,x) - u(t,x)]p(t,x)
\]

\[
+ \gamma_4(t,x)[\gamma_1(t,x)X(t,x) + \gamma_2(t,x)\bar{X}(t,x)]q(t,x)
\]
where \((p, q, r(\cdot))\) is the unique solution to the following BSPDE:
\[
\begin{align*}
dp(t, x) = \left[ -\frac{1}{2} \Delta p(t, x) + (\gamma_1(t, x) + \gamma_2(t, x) \nabla_S^* (t, x)) \right] (\gamma_3(t, x)p(t, x) \\
+ \gamma_4(t, x)q(t, x) + \int_{\mathbb{R}_0} \gamma_5(t, x, \zeta)r(t, x, \zeta) \nu(d\zeta)), \quad (t, x) \in [0, T) \times D;
\end{align*}
\]
\[
\begin{align*}
p(t, x) = \frac{k(x)}{X(T, x)}, \quad (t, x) \in [T, T + \delta] \times \partial D;
p(t, x) = 0, \quad (t, x) \in [0, T) \times \partial D;
q(t, x) = 0, \quad (t, x) \in [T, T + \delta] \times \partial D;
r(t, x, \cdot) = 0, \quad (t, x) \in [T, T + \delta] \times \partial D.
\end{align*}
\]
Here \(\nabla_S^*\) has been given in Example 2. By Theorems 3.1 and 3.2, the optimal control \(\hat{u}\) satisfies
\[
\hat{u}(t, x) = \frac{1}{\bar{p}(t, x)},
\]
where \((\bar{p}, \bar{q}, \bar{r}(\cdot))\) is the unique solution to (34) for \(u = \hat{u}\) and \(X = \hat{X}\).

**Example.** We modify (33) to the following performance functional:
\[
J_1(u) = \mathbb{E} \left[ \frac{1}{\beta} \int_D \int_0^T u^\beta(t, x) dtdx + \int_D k(x)X(T, x) dx \right].
\]
Here, \(\beta \in (0, 1)\) is a constant and \(k(x) \in H_T\) is a nonnegative, \(\mathcal{F}_T\)-measurable process. Now, we aim to find \(\hat{u}(t, x) \in \mathcal{U}^{ad}\) such that
\[
J_1(\hat{u}) = \sup_{u \in \mathcal{U}^{ad}} J_1(u).
\]
The associated Hamiltonian functional in this example becomes the following form
\[
H(t, x, S, z, u, p, q, r(\cdot)) = \frac{1}{\beta} u^\beta(t, x) + [\gamma_1(t, x) \gamma_3(t, x) X(t, x) + \gamma_2(t, x) \gamma_3(t, x) \hat{X}(t, x) - u(t, x)] p(t, x)
\]
\[
+ \gamma_4(t, x) [\gamma_1(t, x) X(t, x) + \gamma_2(t, x) \hat{X}(t, x)] q(t, x)
\]
\[
+ \int_{\mathbb{R}_0} \gamma_5(t, x, \zeta) [\gamma_1(t, x) X(t, x) + \gamma_2(t, x) \hat{X}(t, x)] r(t, x, \zeta) \nu(d\zeta),
\]
where \((p, q, r(\cdot))\) is the unique solution to the following BSPDE:
\[
\begin{align*}
\left\{ \begin{array}{l}
dp(t, x) = \left[ -\frac{1}{2} \Delta p(t, x) + (\gamma_1(t, x) + \gamma_2(t, x) \nabla_S^* (t, x)) \right] (\gamma_3(t, x)p(t, x)
\end{align*}
\]
\[
+ \gamma_4(t, x)q(t, x) + \int_{\mathbb{R}_0} \gamma_5(t, x, \zeta)r(t, x, \zeta) \nu(d\zeta)), \quad (t, x) \in [0, T) \times D;
p(t, x) = k(x), \quad (t, x) \in [T, T + \delta] \times \partial D;
p(t, x) = 0, \quad (t, x) \in [0, T) \times \partial D;
q(t, x) = 0, \quad (t, x) \in [T, T + \delta] \times \partial D;
r(t, x, \cdot) = 0, \quad (t, x) \in [T, T + \delta] \times \partial D.
\]
Here $\nabla_S^2$ has been given in Example 2. By Theorems 3.1 and 3.2, the optimal control $\hat{u}$ satisfies

$$\hat{u}(t, x) = (\hat{p}(t, x))^\frac{1}{\alpha},$$

where $(\hat{p}, \hat{q}, \hat{r}(\cdot))$ is the unique solution to (36) for $u = \hat{u}$ and $X = \hat{X}$.

**Remark 3.**

(i) If we take $\gamma_1(t, x) = 1$, $\gamma_2(t, x) = \gamma_5(t, x, \zeta) = 0$ $\gamma_3(t, x) = \gamma_3$, $\gamma_4(t, x) = \gamma_4$ and $k(x) = k > 0$ in (32), Example 5 reduces to Example 3.1 in [29];

(ii) If we take $\delta = 0$, $\gamma_1(t, x) = 0$, $\gamma_4(t, x) = \gamma_4$ ($i = 2, 3, 4$), $\gamma_5(t, x, \zeta) = \gamma_5(\zeta)$, $X(t, x) = S_1(X(t, x))$ and $Q_1(x, y) = \frac{1}{V(R_0)}$ in (32), where $S_1, Q_1$ are represented in Example 2, Example 5 reduces to Optimal Harvesting (II) in [1]. In addition, if $k(x) = 1$, then Example 5 reduces to Optimal Harvesting (I) in [1].

6. **Concluding remarks.** This paper is devoted to study the stochastic optimal control problem governed by the system of SPDEs with the spatial-temporal interaction operator. The main contributions of this paper are as follows: (i) The spatial-temporal interaction operator is constructed to capture the space-time dependent phenomena in realistic world; (ii) Sufficient and necessary maximum principles for the optimal control problem are obtained under the partial information case; (iii) The existence and uniqueness of solutions are proved for BSPDEs with the spatial-temporal interaction operator.

We would like to mention that the Pontryagin-Bismut-Bensoussan type maximum principle is indeed a first order necessary condition for optimal controls. It is well known that the first order necessary condition is not always effective to determine the optimal controls, especially when the optimal controls are singular, i.e., the optimal control trivially satisfies the first order necessary condition. For the singular control problem, higher order necessary conditions should be established to find the optimal controls. On the other hand, the convexity condition of the control domain is really hard to be satisfied in practical control systems. As pointed out by Frankowska and Lü [1], it had been a longstanding problem to obtain the stochastic maximum principle for general infinite dimensional nonlinear stochastic systems in which the diffusion coefficient is control dependent and the control domain is non-convex till some recent papers [8, 12, 23, 22]. We note that some integral type second order necessary conditions are obtained in [11] while the control domain is non-convex. Therefore, it would be important and interesting to study the following problems in the controlled system of SPDEs with the spatial-temporal interaction operator: (i) Find the second or higher order necessary condition for optimal controls; (ii) Find the necessary condition for optimal controls when the control domain is non-convex. We leave these problems for future work.

**Acknowledgments.** The authors are grateful to the editors and reviewers for their constructive comments, which helps us to improve the paper.

**REFERENCES**

[1] N. Agram, A. Hilbert and B. Øksendal, SPDEs with space-mean dynamics, preprint, arXiv:1807.07303, 2019.

[2] N. Agram, A. Hilbert and B. Øksendal, Singular control of SPDEs with space-mean dynamics, Mathematical Control and Related Fields, 10 (2020), 425–441.

[3] N. Agram and B. Øksendal, Stochastic control of memory mean-field processes, Applied Mathematics & Optimization, 79 (2019), 181–204.
[4] A. Basse-O’Connor, M. S. Nielsen, J. Pedersen and V. Rohde, Multivariate stochastic delay differential equations and CAR representations of CARMA processes, *Stochastic Processes and Their Applications*, 129 (2019), 4119–4143.

[5] A. Bensoussan, *Stochastic Control of Partially Observable Systems*, Cambridge University Press, Cambridge, 1992.

[6] G. Da Prato and J. Zabczyk, *Stochastic Equations in Infinite Dimensions*, 2nd edition, Cambridge University Press, Cambridge, 2014.

[7] K. Du and Q. Meng, A maximum principle for optimal control of stochastic evolution equations, *SIAM Journal on Control and Optimization*, 51 (2013), 4343–4362.

[8] R. Dumitrescu, B. Øksendal and A. Sulem, Stochastic control for mean-field stochastic partial differential equations with jumps, *Journal of Optimization Theory and Applications*, 176 (2018), 559–584.

[9] S. N. Evans, P. L. Ralph, S. J. Schreibe and A. Sen, Stochastic population growth in spatially heterogeneous environments, *Journal of Mathematical Biology*, 66 (2013), 423–476.

[10] H. Frankowska and Q. Lü, First and second order necessary optimality conditions for controlled stochastic evolution equations with control and state constraints, *Journal of Differential Equations*, 268 (2020), 2949–3015.

[11] M. Fuhrman and C. Orrieri, Stochastic maximum principle for optimal control of a class of nonlinear SPDEs with dissipative drift, *SIAM Journal on Control and Optimization*, 54 (2016), 341–371.

[12] S. Lenhart, J. Xiong and J. Yong, Optimal controls for stochastic partial differential equations with an application in population modeling, *SIAM Journal on Control and Optimization*, 54 (2016), 495–535.

[13] J. Liu and C. A. Tudor, Analysis of the density of the solution to a semilinear SPDE with fractional noise, *Stochastics*, 88 (2016), 959–979.

[14] Q. Lü and X. Zhang, Transposition method for backward stochastic evolution equations revisited, and its application, *Mathematical Control and Related Fields*, 8 (2018), 337–381.

[15] Q. Meng and Y. Shen, Optimal control of mean-field jump-diffusion systems with delay: A stochastic maximum principle approach, *Journal of Computational and Applied Mathematics*, 279 (2015), 13–30.

[16] J. B. Mijena and E. Nane, Intermittence and space-time fractional stochastic partial differential equations, *Potential Analysis*, 44 (2016), 295–312.

[17] S. E. A. Mohammed, Stochastic differential systems with memory: Theory, examples and applications, in *Stochastic Analysis and Related Topics VI* (eds. Decreusefond L., Øksendal B., Gjerde J. and Üstünel A. S.), Birkhäuser, Boston, (1998), 1–77.
[28] F. Z. Mokkedem and X. Fu, Optimal control problems for a semilinear evolution system with infinite delay, *Applied Mathematics & Optimization*, 79 (2019), 41–67.

[29] B. Øksendal, Optimal control of stochastic partial differential equations, *Stochastic Analysis and Applications*, 23 (2005), 165–179.

[30] B. Øksendal, A. Sulem and T. Zhang, Optimal control of stochastic delay equations and time-advanced backward stochastic differential equations, *Advances in Applied Probability*, 43 (2011), 572–596.

[31] B. Øksendal, A. Sulem and T. Zhang, Optimal partial information control of SPDEs with delay and time-advanced backward SPDEs, in *Stochastic Analysis and Applications to Finance: Essays in Honour of Jia-an Yan* (eds. T. Zhang and X. Zhou), World Scientific Publishing, Hackensack, (2012), 355–383.

[32] T. Reichenbach, M. Mobilia and E. Frey, Noise and correlations in a spatial population model with cyclic competition, *Physical Review Letters*, 99 (2007), 238105.

[33] S. J. Schreiber and J. O. Lloyd-Smith, Invasion dynamics in spatially heterogeneous environments, *Applied Mathematics & Optimization*, 174 (2009), 490–505.

[34] J. Wloka, *Partial Differential Equations*, Cambridge University Press, Cambridge, 1987.

[35] H.-N. Wu and X.-M. Zhang, Boundary static output feedback control for nonlinear stochastic parabolic partial differential systems via fuzzy-model-based approach, *IEEE Transactions on Fuzzy Systems*, DOI: 10.1109/TFUZZ.2019.2941698.

Received March 2020; revised June 2020.

E-mail address: 2019322010030@stu.scu.edu.cn
E-mail address: nanjinghuang@hotmail.com
E-mail address: wmhfranklin@foxmail.com
E-mail address: 906542324@qq.com