Hermite wavelet method for the numerical solution of nonlinear singular initial value problems

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Abstract
In this article, we are going to use Hermite wavelet method and here we are proposed scheme for the numerical results of nonlinear singular initial value problems. presently, series of derivatives is come into picture by using Hermite wavelets. With the help of these derivatives, proposed method is developed. Properties of Hermite wavelets are used to convert nonlinear singular initial value problems with initial conditions and collocation points to systems of nonlinear algebraic equations. And then this system of equations can be solved by using suitable methods. Some of the numerical test problems are given to express the validity and the accuracy of the proposed method.

Keywords
Hermite wavelets; nonlinear singular initial value problems; collocation technique.

AMS Subject Classification
65T60, 97N40, 49K20.

1. Introduction
Wavelets are simply defined as a localized waves or also called it as a small wave. Instead of oscillating rapidly, they drop down to zero. Wavelet theory is a recently emerging field in the mathematical research area. It has so many applications or features in science as well as engineering disciplines; such as signal analysis for wave form representation and also segmentations, harmonic analysis and also time frequency analysis etc. Wavelets gives the accurate representation of a variety of functions and operators. Wavelets are assumed as basis functions \( \psi_{j,k}(t) \) in continuous time. Special feature of the wavelets basis is that all functions \( \psi_{j,k}(t) \) are constructed from a single mother wavelet \( \psi(t) \) which is a small pulse. Usually set of linearly independent functions created by translation and dilation of mother wavelet.

There is plenty of work has been done in the field of wavelet area. and we are formulated as initial value problems. The nonlinear singular initial value problems are very helpful in so many fields, particularly in chemical reactions, fluid dynamics, gas dynamics, simply in physics [3]. In many cases, getting the exact results for this type of problems from analytical methods are not gives solution in all cases. For avoiding that type of situations we switch over to the numerical techniques are: collocation method, Galerkin method, and there are so many methods. Now presently going to use Series of derivatives collocation technique is one of the best schemes to get solution for differential equations. This idea is based on replacing differential equation into algebraic equations with the aid of series of derivatives and replacing the unknowns in order to reduce the problem into system of algebraic equations.

The main work for this paper is to develop the new Hermite wavelets series of derivatives collocation method as an alternative method to solve nonlinear singular initial value problems is as follows: (1.1). for a given set of conditions. Here we have to bring solutions of (1.1) under the Hermite space and Hermite wavelet collocation method will gives us a good accuracy for linear singular problems. Therefore, we mainly concentrate on the nonlinear problems i.e., we are expressing
solutions of these nonlinear singular initial value problems in to Hermite wavelets basis. Let 
\( p(t), q(t), r(t, u) : (a, b) \rightarrow R \) 
be continuous real valued functions. Now we are considering the arbitrary nonlinear singular second order equations given by

\[
\{ u''(t) + p(t)u'(t) + q(t)u(t) = r(t), \quad a < t < b, \tag{1.1} \]

subjected to following initial conditions:

\[
\{ u(0) = A, u'(0) = B, \tag{1.2} \]

where \( A, B \) are constants and are available. There is some work has been done on this type of problems and are available on singular initial value problems. Adomian decomposition method (ADM) [6], Laguerre wavelets method [8], Hermite wavelets method for boundary value problem [1], operational matrix of integration for Chebyshev wavelet [2], and also, some another methods are there to solving partial differential equations [4]. We are concentrating on converting differential equations into algebraic equations then omitting the unknowns by Hermite Wavelet collocation Method. There is a very huge amount of the source is available on Series of derivatives wavelets such as: Legendre wavelet, Laguerre wavelet but there is very least source is available on Hermite wavelet series of derivatives. Therefore this point is attracts us to generate series of derivatives with the aid of Hermite wavelet.

### 2. Properties of Hermite Wavelets

We have the following family of discrete wavelets, \( \psi_{n,m}(t) = |a|^{-1/2} \varphi (a^{1/2} - nb/a) \) for all \( a, b \in R a \neq 0 \), where \( \psi_{n,m} \) form a wavelet basis for \( L^2(R) \). In particular, when \( a_0 = 2 \) and \( b_0 = 1 \) then \( \psi_{n,m}(t) \) forms an orthonormal basis. Hermite wavelets are defined as [1]

\[
\psi_{n,m}(t) = \begin{cases} \frac{2^{1/2}}{\sqrt{\pi}} 2^{k+1/2} H_m (2^k t - 2n + 1), & 0 \leq t \leq \frac{n}{2^k+1} \\ 0, & \text{Otherwise} \end{cases} \tag{2.1} \]

where \( m = 0, 1, 2, ..., M - 1 \) Here \( H_m \) is Hermite polynomials of degree \( m \) with respect to weight function \( W(t) = \sqrt{1 - t^2} \), on the real line \( R \) and satisfies the following recurrence formula, \( H_0(t) = 1, H_1(t) = 2t \),

\[
H_{m+2}(t) = 2tH_{m+1}(t) - 2(m + 1)H_m(t), m = 0, 1, 2, ... \tag{2.2} \]

Function Approximation:

\[
u(t) \sim \sum_{n=1}^{\infty} \sum_{m=0}^{\infty} B_{n,m} \psi_{n,m}(t) \tag{2.3} \]

where \( \psi_{n,m}(t) \) is given in eq (2.1). We approximate \( u(t) \) by truncating the series represented in eq (2.3) as

\[
u(t) \approx \sum_{n=1}^{2k-1} \sum_{m=0}^{M-1} B_{n,m} \psi_{n,m}(t) = B^T \psi(t) \tag{2.4} \]

where \( B \) and \( \psi(t) \) are \( 2^{k-1} \times 1 \) matrices,

\[
B^T = [\psi_{1,0}, \psi_{1,1}, \psi_{1,2}, ..., \psi_{1,2^{k-1}M-1}], \tag{2.5} \]

\[
\psi(x) = [\psi_{1,0}, \psi_{1,1}, \psi_{1,2}, ..., \psi_{1,2^{k-1}M-1}] \tag{2.6} \]

### 3. Method of solution for Hermite Wavelet

Solution for the given nonlinear singular second order equation (1.1) and (1.2) can be expressed as Hermite wavelets:

\[
u(t) \sim \sum_{n=1}^{\infty} \sum_{m=0}^{\infty} B_{n,m} \psi_{n,m}(t) \tag{3.1} \]

where \( \psi_{n,m}(t) \) is wavelets in eq (2.1) Then we approximating it \( u(t) \) by truncated series,

\[
u(t) \approx \sum_{n=1}^{2k-1} \sum_{m=0}^{M-1} B_{n,m} \psi_{n,m}(t) = B^T \psi(t) \tag{3.2} \]

Then should be total number of conditions are \( 2^{k-1} \). Now we are going to determine the \( 2^{k-1} \) co-efficients.

Let, the given differential equation is of the form non-linear second order and from problem itself have two conditions, namely

\[
u(0) \approx \sum_{n=1}^{2k-1} \sum_{m=0}^{M-1} B_{n,m} \psi_{n,m}(0), \quad \frac{d\nu(0)}{dt} \approx \sum_{n=1}^{2k-1} \sum_{m=0}^{M-1} B_{n,m} \psi_{n,m}(0) \tag{3.2} \]

These two conditions were obtained by substituting in eq (3.1) and with the help of collocation points \( t_i(2^{k-1}M - 2), t_j^s \) are limit points of the sequence:

\[
t_i = \frac{1}{2} \left( 1 + \cos \left( \frac{(i-1)\pi}{2^{k-1}M - 2} \right) \right), \quad i = 2, 3, ... \]

which will gives us a system of equations and on combining these system of equations with the eq (3.2) to gives us \( 2^{k-1}M \) system of equations from which we can generate the values for the unknown coefficients \( Z_{n,m} \). Same procedure is repeated for differential equations of higher order also. Now, substituting eq (3.1) in eq (1.1):

\[
d^2 \sum_{n=1}^{2k-1} \sum_{m=0}^{M-1} Z_{n,m} \psi_{n,m}(t) + d \sum_{n=1}^{2k-1} \sum_{m=0}^{M-1} Z_{n,m} \psi_{n,m}(t) \]

\[
+ q(t) \sum_{n=1}^{2k-1} \sum_{m=0}^{M-1} Z_{n,m} \psi_{n,m}(t) = r(t, u_i) \quad a < t < b \tag{3.3} \]
Substituting the \((2^k M - 2)\) collocation points \(t'_i\) as follows:

\[
\frac{d^2}{dt^2} \sum_{n=1}^{2^k M - 1} \sum_{m=0}^{M-1} Z_{n,m} \psi_{n,m}(t_i) + \frac{d}{dt} \sum_{n=1}^{2^k M - 1} \sum_{m=0}^{M-1} Z_{n,m} \psi_{n,m}(t_i)
\]

\[+ q(t) \sum_{n=1}^{2^k M - 1} \sum_{m=0}^{M-1} Z_{n,m} \psi_{n,m}(t) = r(t, u), \quad \alpha < t < b. (3.4)\]

Joining above equations eq 3.4 and eq (3.2) then we get equations from which we can calculate values for the unknown coefficients. Lastly, substituting obtained unknown coefficients in to eq 3.1 will be the final our Hermite Wavelets solution.

4. Examples

Test Problem 1.

First, Here we are considering the Non-linear singular initial value problem [6],

\[u'' + \frac{1}{t} u' - u^3 + 3u^5 = 0, \quad 0 < t < 1, \quad (4.1)\]

with the initial conditions are \(u(0) = 1, u'(0) = 0\). This problem has the Analytical solution as \(u(t) = (1 + t^2)^{-1/2}\). We have been solved this problem by our Hermite wavelet collocation method and we were presented in the section 3. Here, Tables and figures are as follows. Table 1 and fig 1 shows the absolute error of Hermite wavelet method based solution with analytical solution for different values of \(M\). Graphical representation of Exact and Hermite wavelet method solution is drawn in Figure 2., of the test problem 1.

Table 1 Examine the absolute errors (AE) of test problem 1 for fixed value of \(k = 1\).

| \(t\) | \(HWM \ AE\) at \(M = 5\) | \(HWM \ AE\) at \(M = 7\) | \(HWM \ AE\) at \(M = 9\) |
|------|----------------|----------------|----------------|
| 0.1  | 4.1628 \times 10^{-6} | 1.1349 \times 10^{-5} | 7.30393 \times 10^{-8} |
| 0.2  | 8.5851 \times 10^{-6} | 3.41711 \times 10^{-5} | 8.12089 \times 10^{-8} |
| 0.3  | 7.4690 \times 10^{-6} | 9.0696 \times 10^{-5} | 1.26332 \times 10^{-7} |
| 0.4  | 1.1798 \times 10^{-5} | 1.68623 \times 10^{-6} | 9.65309 \times 10^{-9} |
| 0.5  | 6.8992 \times 10^{-6} | 1.78929 \times 10^{-6} | 2.99931 \times 10^{-9} |
| 0.6  | 1.3075 \times 10^{-5} | 1.6670 \times 10^{-7} | 1.42449 \times 10^{-9} |
| 0.7  | 1.5157 \times 10^{-5} | 9.2475 \times 10^{-6} | 5.19913 \times 10^{-9} |
| 0.8  | 1.3174 \times 10^{-5} | 2.3562 \times 10^{-5} | 7.3647 \times 10^{-7} |
| 0.9  | 9.4053 \times 10^{-6} | 3.2636 \times 10^{-6} | 5.84180 \times 10^{-7} |

Figure 1. Relation of Analytical and Hermite wavelet collocation method solution for the test problem 1.

Figure 2. Relation between the of Exact and Hermite wavelet method solutions considered Test problem 2.

Test Problem 2.

Next, We are considering the singular nonlinear Lane-Emden type equation [7],

\[u'' + \frac{2}{t} u' + u^5 = 0, \quad 0 < t < 1, \quad (4.2)\]

with the initial conditions \(u(0) = 1, u'(0) = 0\) This problem has the Analytical solution as \(u(t) = (1 + t^2)^{-1/2}\). We were solved this test problem eq (4.2) by using the Hermite wavelet collocation method and we were presented in section 3. Table 2 and fig 3 explain us difference between the Exact and numerical solution of the test problem 2. Graphs shows us Exact and Hermite wavelet method solution is drawn in Fig. 4., of the eq (4.2).

Table 1 Relation between Exact and Hermite wavelet method of the test problem 2 for fixed value of \(k = 1\).

| \(t\) | \(HWM \ AE\) at \(M = 5\) | \(HWM \ AE\) at \(M = 7\) | \(HWM \ AE\) at \(M = 9\) |
|------|----------------|----------------|----------------|
| 0.1  | 2.6398 \times 10^{-6} | 4.9737 \times 10^{-7} | 1.53595 \times 10^{-7} |
| 0.2  | 4.821363 \times 10^{-6} | 5.984305 \times 10^{-7} | 1.86322 \times 10^{-7} |
| 0.3  | 2.21691 \times 10^{-5} | 1.910199 \times 10^{-6} | 7.80514 \times 10^{-10} |
| 0.4  | 4.799125 \times 10^{-5} | 1.4709916 \times 10^{-5} | 1.36485 \times 10^{-8} |
| 0.5  | 1.28565 \times 10^{-5} | 2.7631 \times 10^{-7} | 3.86880 \times 10^{-7} |
| 0.6  | 1.761980 \times 10^{-5} | 1.382300 \times 10^{-6} | 1.48085 \times 10^{-8} |
| 0.7  | 1.589676 \times 10^{-5} | 6.840884 \times 10^{-7} | 8.04376 \times 10^{-7} |
| 0.8  | 7.92488 \times 10^{-5} | 7.644986 \times 10^{-7} | 5.88231 \times 10^{-6} |
| 0.9  | 6.839527 \times 10^{-5} | 9.525524 \times 10^{-7} | 5.88831 \times 10^{-6} |

Figure 3. Relation between absolute error analysis for considered problem 2 for Hermite wavelet method solution and Wazwaz Solution [7].

Table 3 Relation between Hermite wavelet method and other existing methods[7,8] of solution for the considered test problem 2.

| \(t\) | \(Wazwaz \ AE\) at \(M = 10\) | \(AE\) for \(HWM\) at \(M = 9\) |
|------|----------------|----------------|
| 0.1  | 0.9980428441 \times 10^{-10} | 2.9464705080 \times 10^{-10} |
| 0.2  | 0.922189498 \times 10^{-10} | 1.2098329099 \times 10^{-10} |
| 0.5  | 0.9519611019 \times 10^{-10} | 8.8078209299 \times 10^{-10} |
| 1.0  | 0.8182516669 \times 10^{-10} | 1.5601033675 \times 10^{-10} |
| 1.5  | 0.92553184 \times 10^{-10} | 3.2684643135 \times 10^{-10} |
5. Conclusions

In this current work, our proposed algorithm is tested for some test problems and those problems were gives us results are quite satisfactory in relation with the already available numerical results. Finally, we summarize the output of this analysis is: The current work gives us better accuracy in relating with the other numerical techniques [7, 8] available in sources. This scheme is very easy to implement in the computer programs and we can extend this scheme for higher order also with slight changes in the current method. This method is applicable for all type of differential equations problems like a singular nonlinear initial value problems.

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