Order book model with herd behavior exhibiting long-range memory

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Abstract

In this work, we propose an order book model with herd behavior. The proposed model is built upon two distinct approaches: a recent empirical study of the detailed order book records by Kanazawa et al. [Phys. Rev. Lett. 120, 138301] as well as financial herd behavior model. Combining these approaches allows us to create a more plausible financial market model, which is also capable to replicate the long-range memory phenomenon of the absolute return and the trading activity as well as the other stylized facts. We compare the statistical properties of the model against the empirical statistical properties of Bitcoin exchange rates as well as New York stock exchange tickers. We also show that the fracture in the spectral density of the high frequency absolute return time series might be related to the mechanism of convergence towards the equilibrium price.

1 Introduction

In the recent decades an increasing effort by social scientists, physicists and broader interdisciplinary community has been applied to create agent-based models (ABMs) of different social systems and phenomena [1–8]. Notable part of these ABMs were created to explain various recurrent anomalous statistical patterns, collectively referred to as stylized facts, observed in the financial markets [2–4,7]. Financial ABMs vary in complexity usually trading plausibility for analytical tractability [4]. Some of the more complex financial ABMs seem to be reasonably plausible [9,10], but it seems near impossible to build upon them to further improve their agreement with empirical data.

In the recent few years we have developed a reasonably plausible yet analytically tractable financial ABM [11,12]. This financial ABM was derived from a widely recognized behavioral model [13], which emphasizes imitation (herd) behavior among socially interacting individuals, and is able to reproduce the exact probability density functions (PDFs) and power spectral densities (PSDs) of the empirical absolute return time series. While the model has a lot of desirable features, e.g., it scales well with change in the time scale, it also has some drawbacks. First of all it does not implement realistic trading strategies, though most of the financial ABMs lack this feature, unlike [10]. But this is not the main issue we currently see this model makes two assumptions, which are not fully transparently justified: the presence of the omnipotent market maker, who is able to clear the market instantaneously, (a rather common assumption in many financial ABMs [2–4]) and the presence of the exogenous noise. As in our papers we have speculated that exogenous noise most likely originates from the order book dynamics, it seems that translating this model into an order book model we could resolve these two issues at once. Additionally this would allow us to consider trading activity time series as well.

Though the empirical order book data has become available at the same time as the empirical high–frequency financial time series data, in the 1980’s, it took a longer time for detailed empirical studies of the order book dynamics to be undertaken [14–17]. Interestingly some of the observations were not as universal as the stylized facts discovered in the empirical time series. There are few recent empirical order book studies, which confirm some of the earlier findings, but fail to confirm the other findings [18]. Similarly there is a variety of simple limit order book models (e.g., [19–21]), which are not mutually compatible, but are able to reproduce some of the empirical observations in the order book data. Some of the more recent order book modeling approaches, such as [22–25], are more sophisticated and able to reproduce a variety of empirical observations. The most
recent approach by Kanazawa et al. [26,27] combines the empirical and modeling approach, they have observed the behavior of the high-frequency traders in a highly detailed order book level data set and formulated a microscopic model based on these observations by using the kinetic theory. Yet most of these models do not consider detailed reproduction of the stylized facts established for the time series data.

Recently a few order book models, which focus on the time series stylized facts, were proposed [28,29]. As the approach we plan to take here is very similar to these two approaches, let us briefly compare our approach against theirs. Biondo et al. model is inspired by the models of self-organized criticality. Information steadily reaches the agents, until an agent accumulates enough to take an action. By taking an action the agent spreads information towards its neighbors, who may also accumulate enough information to take their action. As in self-organized criticality models the avalanches of trades are observed. Navarro–Larralde model relies on heterogeneity and plausibility of the trading strategies. Both of these approaches, as well as our own, assume two types of agents, technical traders (chartists) and fundamentalists, but our approach is different as we allow our agents to switch their trading strategies. In our approach we have used highly stylized representation of technical trading strategy, while Biondo et al. model uses past reference values and Navarro–Larralde model utilizes Moving–Average–Oscillator strategy. Using Moving–Average–Oscillator strategy could improve plausibility of our approach, but would complicate by introducing additional parameters. Biondo et al. model uses multilayer networks as interaction topology for their agents, while we allow any agent to interact with any other agent. In Navarro–Larralde model agents interact only via trades, while in our and Biondo et al. approach agents also exchange additional information. Despite these differences all of the approaches produce similar results. Biondo et al. reports that the model generate heavy tailed return PDF, while Navarro–Larralde reports reproducing larger variety of stylized facts: heavy tailed return PDF, absence of raw return correlation, presence of absolute return correlation, asymmetry of returns. It also seems that Navarro–Larralde model exhibits fat tailed trading activity distribution, but this is not directly stated in the paper. Unlike in these two approaches we strive to reproduce empirical observations not qualitatively, but exactly. Namely we compare model and empirical PDFs and PSDs of absolute return and trading activity against each other.

The presentation of our approach is structured as follows. In Section 2 we will briefly introduce herd behavior model proposed by Kirman [13]. In Section 3 we will discuss two different financial market models: the previously proposed herd behavior model with instantaneous clearing [11,12] and the newly proposed order book model with herd behavior. We will show that under certain parameter values both of the models produce mostly identical statistical properties. Further in Section 4 we will compare the order book model with herd behavior against the empirical Bitcoin and NYSE data. We will examine the sensitivity of the model to parameter value changes in Section 5. While we will provide concluding remarks and future outlooks in Section 6.

2 Kirman’s herd behavior model

Let us start with discussion about Kirman’s herd behavior model [13], which is the base upon we build financial market models in the following sections. In the seminal paper Kirman shared an observation that social scientists and behavioral biologists observe remarkably similar patterns in rather distinct systems. In the experiments involving ants described by Kirman, entomologists observed the emergence of asymmetry in a symmetric experimental setup: despite having two identical food sources available, majority of the ants in the ant colony preferred to forage from a single food source at a time. Numerous references in Kirman’s paper suggest that humans also seem to prefer the more popular product over the less popular despite both being of a similar quality.

To account for these empirical observations Kirman proposed a simple probabilistic herd behavior model in which the probability for an agent to switch to another state is proportional to the fraction of agents in that
state:

\[ p(X \rightarrow X + 1) = (N - X) \left( \sigma_1 + h \frac{X}{N} \right) \Delta t, \]  

\[ p(X \rightarrow X - 1) = X \left[ \sigma_2 + h \frac{N - X}{N} \right] \Delta t, \]  

where \( X \) is a number of agents in the first state, \( N \) – a total number of agents, \( \sigma_i \) – an idiosyncratic behavior parameter (encodes preferences for the states), \( h \) herd behavior parameter, \( \Delta t \) – arbitrarily small time step.

This formulation of the herd behavior model is often referred to as “local” or “extensive”, because the fluctuations of \( X \) quickly disappear as \( N \) becomes larger \[30–33\]. In other words \( X \) rapidly converges to a certain value and remains almost constant afterwards.

What we described above in the literature is often referred to as the \( N \)-dependence problem \[30–32\]. This problem can be circumvented by assuming that the probability to switch is proportional to a total number of agents \( X \):

\[ p(X \rightarrow X + 1) = (N - X) \left( \sigma_1 + h \frac{X}{N} \right) \Delta t, \]  

\[ p(X \rightarrow X - 1) = X \left[ \sigma_2 + h \frac{N - X}{N} \right] \Delta t, \]  

To contrast the previous formulation of the model, this formulation of the herd behavior model is often referred to as the “non-extensive” or “global” formulation. In this formulation \( X \) no longer converges to a fixed value even in the limit of infinite \( N \). This is desirable feature to have in the financial market models as it is well-known that the stylized facts hold for small and large markets alike \[34\]. Hence there is a variety of the agent-based financial market models, which were inspired by the non-extensive formulation of the Kirman’s model \[9,11,12,35–38\]. There are also papers in the opinion dynamics, which claim that the fluctuating nature of opinion change can be explained by assuming the presence of collective peer-pressure instead of inter-personal communication \[31–33,39,40\].

Let us take the infinite \( N \) limit and introduce an almost continuous state variable \( x = \frac{X}{N} \). This allows us to rewrite the model driven by Eqs. (3) and (4) as a stochastic differential equation \[11,35,36\]:

\[ d x = h [\varepsilon_1 (1 - x) - \varepsilon_2 x] \Delta t + \sqrt{2h \varepsilon_1 (1 - x)} \Delta W, \]  

where \( \varepsilon_i = \frac{\sigma_i h}{N} \). From the Eq. (5) it is straightforward to conclude that \( x \) is Beta distributed, \( x \sim Beta(\varepsilon_1, \varepsilon_2) \).

If we would set \( \varepsilon_1 = \varepsilon_2 \) and \( \varepsilon_1 < 1 \) then we would observe the same pattern entomologists did as the Beta distribution is multimodal in this case.

One can derive a similar SDE, by taking finitely large \( N \) limit under extensive formulation of the model. Yet in this case SDE becomes ordinary differential equation as with larger \( N \) the diffusion term becomes negligible and only the drift term remains:

\[ d x = h [\varepsilon_1 (1 - x) - \varepsilon_2 x] dt + \sqrt{\frac{2h \varepsilon_1 (1 - x)}{N}} dW \approx h [\varepsilon_1 (1 - x) - \varepsilon_2 x] dt. \]  

In a special case, \( p(X \rightarrow X - 1) = 0 \), one can also derive the well-known Bass diffusion equation \[41\]:

\[ d x = (1 - x) (\sigma_1 + hx) dt. \]  

\[ 3 \]
3 Herd behavior model in the context of the financial markets

Kirman’s herd behavior model is in a sense a generic model. In the financial market context it would be natural for these states to represent different trading strategies. In the agent-based modeling literature one can find the most common is to consider interaction between agents using fundamentalist and chartist trading strategies [4,6,10]. It is worth to note a couple approaches which consider modeling market sentiments instead [36,42,43]. In our previous works [11,12] we have used both of these approaches to build a highly sophisticated agent-based model which was able to replicate the empirical absolute return PDF and PSD exactly. Here we will start with the fundamentalist-chartist model under instantaneous clearing and later use it to build order book model.

3.1 Herd behavior model with instantaneous clearing

Let us define the trading strategies in a rather stylized manner. One could in general define more sophisticated strategies (as discussed in [10]), but we would like to keep the model compatible with our earlier works. This will allow us to use some of the earlier analytically obtained results which would be impossible while using a more realistic trading strategies.

In [11] we have assumed that excess demand by chartist traders is conditioned on their mood $\xi(t)$:

$$D_c(t) = r_0 N_c(t) \xi(t),$$

where $r_0$ is the relative impact of chartists’ trading activity and $N_c(t)$ is the number of chartists. In contrast fundamentalists’ demand is conditioned on their knowledge about the market fundamentals, which is quantified as the fundamental price $P_f$:

$$D_f(t) = N_f(t) \ln \frac{P_f}{P(t)},$$

where $N_f(t)$ is the number of fundamentalists and $P(t)$ is the current price. We have also assumed that a market maker instantaneously clears the market by setting the price to the Walras equilibrium price, which is obtained in the following manner:

$$D_f(t) + D_c(t) = 0 \Rightarrow P(t) = P_f \exp \left( r_0 \cdot \frac{N_c(t)}{N_f(t)} \xi(t) \right).$$

If $\xi(t)$ fluctuates significantly faster than $N_i(t)$, then the return

$$r(t) = \ln \frac{P(t)}{P(t-T)} \propto \frac{N_c(t)}{N_f(t)} = \frac{N_c(t)}{N - N_c(t)} = y.$$ 

Through out our papers we have referred to $y$ as modulating return as it describes longer-term fluctuations of the return, while $\xi(t)$ dictates the rapid fluctuations. In some of the earlier works $\xi(t)$ was even modeled as a noise [35].

Previously [11] we have also extended the original herd behavior model by introducing the feedback of the modulating return $y$ on the switching dynamics:

$$p(N_c \rightarrow N_c + 1) = (N - N_c) [\sigma_{fc} + h N_c] \frac{\Delta t}{\tau(N_c)},$$

$$p(N_c \rightarrow N_c - 1) = N_c [\sigma_{cf} + h (N - N_c)] \frac{\Delta t}{\tau(N_c)},$$

where

$$\tau(N_c) = \left( \frac{N_c}{N - N_c} \right)^{-\alpha} \equiv y^{-\alpha}.$$ 

That is, $\tau(N_c)$ adjusts the characteristic time scale of microscopic switching events according to the current global value of return. Such feedback scenario implements the coupling between returns and trading activity,
which is well established empirical fact \cite{44}.

Introduction of the feedback scenario enables us to obtain a more general form of the SDE for $y$, which has tunable noise multiplicativity exponent:

$$
d y = h [\varepsilon_{fc} + (2 - \varepsilon_{cf}) y] \frac{1 + y}{\tau(y)} \, dt + \sqrt{2 h y^{1+\alpha}} (1 + y) \, d W =
$$

$$
= h [\varepsilon_{fc} + (2 - \varepsilon_{cf}) y] (1 + y)^{\alpha} \, dt + \sqrt{2 h y^{1+\alpha}} (1 + y) \, d W \approx
$$

$$
\approx h (2 - \varepsilon_{cf}) y^{2+\alpha} \, dt + \sqrt{2 h y^{3+\alpha}} \, d W. \quad (15)
$$

This SDE, assuming $y \gg 1$, belongs to a class of SDEs exhibiting power–law statistics described in \cite{45}. Thus it the $y$ time series should exhibit power–law statistics:

$$
P(y) \sim y^{\varepsilon_{cf} - \alpha - 1}, \quad S(f) \sim f^{-1 - \frac{\varepsilon_{cf} + \alpha - 2}{1 + \alpha}}. \quad (16)
$$

This simple model already reproduces two main stylized facts. In the later papers, e.g., \cite{12}, we have extended this model by describing the mood dynamics using the same herd behavior model. Though in order to reproduce the empirical absolute return PDF and PSD exactly we had to introduce exogenous noise, which we assumed to represent additional randomness arising from the order book dynamics and possibly an exogenous information inflow. In the next section we build the order book model to address this assumption.

### 3.2 Order book model with herd behavior

Most of the agent-based models, which consider statistical properties of the various financial time series, directly or indirectly assume presence of the market maker \cite{3}. While the real financial markets are not cleared by an idealized market maker, most of the contemporary financial markets implement trading by using the order books. Similarly to the market makers order books record and execute orders that the traders submit. The difference is that the orders in the order book are executed only if there is an overlap between the buy (bid) and the ask (sell) sides of the order book or if a market order is submitted. While there is a significant body of literature considering order book modeling \cite{20–28}, most of these models consider reproducing patterns observed at the order book level and usually neglect stylized facts related to the financial time series. It is worth to note that there are numerous papers in economics, which suggest that the prices in various auctions converge towards Walras equilibrium and that this convergence might be comparatively fast \cite{46–51}. Yet this convergence is not instantaneous and we might observe some interesting effects in the high frequency financial time series. Here, while building our order book model, we will partly rely on an empirically motivated order book model proposed by Kanazawa et al. \cite{26,27}.

As in \cite{26,27} we assume that chartists as high-frequency traders submit unit volume limit orders to the both sides of the order book. The submitted quotes, $Q_{ask}^{i}$ and $Q_{bid}^{i}$, are placed by the $i$-th agent the same distance, $S_{i}$, away from the current valuation of the stock, $V_{i}$:

$$
Q_{ask}^{i} (t) = V_{i} (t) + S_{i}, \quad (17)
$$

$$
Q_{bid}^{i} (t) = V_{i} (t) - S_{i}, \quad (18)
$$

$$
S_{i} \sim \text{Gamma} (k, \theta), \quad (19)
$$

where $k$ is the shape parameter of the Gamma distribution and $\theta$ is the scale parameter of the Gamma distribution. Further in this paper we will use the empirically determined values $k = 4$ and $\theta = 15.5$ (see \cite{26,27}) unless specified otherwise.

In \cite{26,27} it is assumed that high-frequency traders could potentially execute market orders due to the trend following. Let us replace the trend following mechanism with a simpler one. Namely, let the chartists submit
We use the Gillespie algorithm \cite{52,53} to implement the order book model. The main idea behind the Gillespie
algorithm is that events are triggered according to a Poisson process with event rate \( \lambda \). As the number of agents in this model will always be finite, the probability of \( N \) switching does not include parameter \( h \), which is because it is equivalent to \( \lambda_c \). As soon as a chartist becomes fundamentalist his limit orders are canceled, also if fundamentalist becomes chartist, then he immediately submits his limit orders.

As the number of agents in this model will always be finite, the probability of \( N_c(t) = 0 \) (or alternatively \( N_c(t) = N \)) and \( \tau(N_f) = \infty \) (or alternatively \( N_c(t) = 0 \)) will be non-zero, which would lead to “over-heating” or “freezing” of the strategy switching dynamics. To avoid these edge cases let us redefine the feedback scenario as:

\[
\tau^{-1}(N_c(t)) = \lambda_0 + \begin{cases} 2N_c(t) \alpha N_c(t) \\ \frac{N_c(t)}{N-N_c(t)} \alpha \end{cases},
\]

where \( \lambda_0 \) is the relative minimum switching rate. In the above we have multiplied \( N_c \) by 2 when taking the \( N_c = N \) edge case into account, because the previous increase in \( y \), number of chartists increasing from \( N_c = N - 2 \) to \( N_c = N - 1 \), is approximately double given large \( N \).

We use the Gillespie algorithm \cite{52,53} to implement the order book model. The main idea behind the Gillespie
algorithm is that we can sum all of the event rates to obtain the total event rate:

$$\lambda^T = \lambda_{cf} + \lambda_{fc} + \lambda_M + \lambda_tF + \lambda_tC,$$

which enables us to generate random inter-event times, which are distributed exponentially:

$$\Delta t_i \sim \text{Exp}(\lambda^T).$$

After each $\Delta t_i$ one of the possible events happens. The probability for any of the possible events to happen is proportional to its rate:

$$p_{cf} = \frac{\lambda_{cf}}{\lambda^T}, \quad p_{fc} = \frac{\lambda_{fc}}{\lambda^T}, \quad p_M = \frac{\lambda_M}{\lambda^T}, \quad p_F = \frac{\lambda_{tF}}{\lambda^T}, \quad p_C = \frac{\lambda_tC}{\lambda^T}.$$  

As these probabilities sum to 1, one of the five possible events is bound to happen: either randomly selected chartist switches to fundamentalist trading strategy (with probability $p_{cf}$) or randomly selected fundamentalist switches to chartist trading strategy (with probability $p_{fc}$) or the mood flips its sign (with probability $p_M$) or the randomly selected fundamentalist submits market order (with probability $p_F$) and randomly selected chartist submits market order (with probability $p_C$).

The exact algorithm behind this model is summarized as a flowchart in Fig. 1. The code implementing this model is publicly available on [https://github.com/akononovicius/herding-OB-model](https://github.com/akononovicius/herding-OB-model).

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Figure 1: (color online) Flowchart illustrating the order book model with herd behavior.
3.3 Comparison between the models

It is possible to approximately estimate equilibrium prices for the order book model. From the discussion in the previous section we know that chartists submit unit volume market orders at rate $\lambda_{tC}$, with probability $p_{\text{bid}}$ they buy the stock and with probability $1 - p_{\text{bid}}$ they sell the stock. Hence their excess demand is given by:

$$D_c = \lambda_{tC} p_{\text{bid}} - \lambda_{tC} (1 - p_{\text{bid}}) = \frac{\lambda_{tC}}{\tau(N_c(t))} \lambda_{tC} N_c(t) \xi(t).$$  \hspace{1cm} (30)

Note that the final result is similar to Eq. (8). Fundamentalists on the other hand submit unit volume market orders at rate $\lambda_{tF}$, they submit ask orders if there is such $j$ for which $Q_{\text{bid}}^j(t) > P_f$ and bid orders if there is such $j$ for which $Q_{\text{ask}}^j(t) < P_f$. Assuming that $P_f$ is not in the spread, the excess demand of fundamentalists will be given by:

$$D_f = \frac{\lambda_{tc} \xi(t)}{\tau(N_c(t))} \lambda_{tf} [N - N_c(t)] \ln \left( \frac{P(t)}{P_f} \right).$$  \hspace{1cm} (31)

Note that the final result is similar to Eq. (9). Assuming that order book is non-empty and almost uniformly filled we can obtain the equilibrium price:

$$D_c + D_f = 0 \quad \Rightarrow \quad P(t) = P_f \exp \left( \frac{\lambda_{tc} N_c(t)}{\lambda_{tf} N_c(t)} \cdot \xi(t) \right).$$  \hspace{1cm} (32)

As the expression for the equilibrium price has the same form as in Eq. (10), we can expect that $y$ and return will have similar statistical properties. The main condition we have to ensure for the similarity to be observable is that enough trades happen between $N_c$ changes, so that the equilibrium price could be reached. This means that $\lambda_{tc}$ and $\lambda_{tf}$ have to be rather large. As you can see in Fig. 2, the agreement between the statistical properties of $y$ and absolute return improves as $\lambda_{tc}$ and $\lambda_{tf}$ grow larger.

![Figure 2](image-url)  \hspace{1cm} Figure 2: (color online) Comparison between the statistical properties, (a) PDFs and (b) PSDs, of $y$ time series (gray curves) and absolute return time series (red, blue and green curves), black curves show the expected slopes of the statistical properties, as per Eq. (16). For the best comparison all of the time series were normalized to unit standard deviation. The following parameter values were used in numerical simulations: $N = 500$, $\lambda_c = 10^{-7}$, $\varepsilon_{cf} = \varepsilon_{fc} = 1$, $\xi_0 = 0.2$, $\lambda_m = 10^3$, $\lambda_0 = 0.1$, $\alpha = 1$, $k = 4$, $\theta = 15.5$, $P_f = 3 \cdot 10^4$ (all cases), $\lambda_{tc} = \lambda_{tf} = 3 \cdot 10^4$ (red curves), 300 (blue and gray curves), 3 (green curves).

Similar intuition can be obtained from Fig. 3. As you can see in (a) and (c), for large $\lambda_{tc}$ and $\lambda_{tf}$ (parameters the same as for the red curve in Fig. 2) the price tends to follow the equilibrium price. Though the following is far from being perfect as can be seen by zooming in on the series, (c). The correlation between the price and the equilibrium price time series is mild $\rho \approx 0.5$. While for small $\lambda_{tc}$ and $\lambda_{tf}$, (b) and (d), (parameters the same as for the green curve in Fig. 2) it is evident that the price does not follow the equilibrium price. As expected, no correlation is detected between the time series, $\rho \approx 0$. 


Based on these results we would like to argue that the fracture in the PSD of the high-frequency absolute return time series happens due to order book dynamics. Namely, the absolute return PSD in the high frequency range is less sloped, because the markets are unable to discover the equilibrium price that fast. It seems that it could take a day or two (as the fracture is usually between $10^{-5}$ and $10^{-4}$ Hz) for the markets to discover the new equilibrium price.

### 4 Comparison against the empirical data

In this paper we use publicly available tick by tick trading data from 12 different Bitcoin exchanges. We have downloaded the data from bitcoincharts.com website on July 5, 2018. List of the considered Bitcoin time series is given in Table 1. Note that Coinbase and Kraken exchanges appear twice in the table, because they contribute more than one exchange pair. These time series were selected, because their data files were among top 5% of the largest. Fisco’s BTC/JPY, Zaif’s BTC/JPY and Zyado’s BTC/EUR were also among top 5% of the largest, but these time series were excluded, because their statistical properties were too different from the rest of the time series. Note that for the same reason we have truncated some of the Bitcoin time series which remained under our consideration.
Table 1: List of the considered Bitcoin time series

| exchange | exchange pair | period available | period used       | trades       |
|----------|---------------|------------------|-------------------|--------------|
| bitfinex| BTC/USD       | 2013-03-31/2016-12-22 | from 2013-05-01 | 10121872 (99.4%) |
| bitflyer| BTC/JPY       | 2017-07-04/2018-07-04 | whole            | 30650659     |
| bitstamp| BTC/USD       | 2011-09-13/2018-07-04 | except 2016-06-23 | 26065456 (99.9%) |
| btcbtc  | BTC/JPY       | 2014-04-09/2018-07-04 | whole            | 8900784      |
| bttc    | BTC/USD       | 2011-08-14/2017-07-25 | whole            | 32904793     |
| btcn    | BTC/CNY       | 2011-06-13/2017-09-30 | from 2013-04-01 | 1114077056 (99.9%) |
| btcod   | BTC/IDR      | 2014-02-09/2018-07-04 | whole            | 8037858      |
| btctrade| BTC/CNY      | 2013-05-19/2017-09-30 | whole            | 19672943     |
| coinbase| BTC/EUR       | 2015-04-23/2018-07-04 | whole            | 15012224     |
| coinbase| BTC/USD       | 2014-12-01/2018-07-04 | from 2017-05-01 | 30553110 (67.5%) |
| coincheck| BTC/JPY     | 2014-10-31/2018-07-04 | except 2017-08-07 | 102124064 (99.9%) |
| kraken  | BTC/EUR       | 2014-01-08/2018-07-04 | whole            | 21392263     |
| kraken  | BTC/USD       | 2014-01-07/2018-07-04 | whole            | 10462050     |
| okcoin  | BTC/CNY       | 2013-06-12/2015-04-05 | whole            | 99999546     |

For each of the considered Bitcoin time series we have produced one minute absolute return (normalized to standard deviation) and trading activity (trades per window normalized to mean) time series. For each of the produced one minute time series we have calculated PDF and PSD. The obtained statistical properties were averaged to produce average profile for each of the statistical properties. To select the model parameters we have used simulated annealing technique with a goal to reproduce these averaged statistical properties. As you can see in Fig. 4 the obtained agreement is rather good.

Figure 4: (color online) Comparison between the empirical Bitcoin statistical properties (gray curves) and statistical properties generated by the model (red curves): (a) one minute absolute return PDF, (b) one minute absolute return PSD, (c) trading activity per one minute PDF, (d) trading activity per one minute PSD. The following parameter set was used in numerical simulations: $N = 500$, $\lambda_c = 10^{-7}$, $\varepsilon_{fc} = 5$, $\varepsilon_{cf} = 2$, $\xi_0 = 0.2$, $\lambda_{mc} = 10$, $\lambda_{tc} = 25$, $\lambda_{tf} = 75$, $\lambda_0 = 0.4$, $\alpha = 2$, $k = 4$, $\theta = 15.5$, $P_f = 3 \cdot 10^4$.

We have also considered the statistical properties of 26 tickers from NYSE. The considered tickers include: ABT, ADM, BA, BMY, C, CVX, DOW, FON, FNM, GE, GM, HD, IBM, JNJ, JPM, KO, LLY, MMM, MO, MOT, MRK, SLE, PFE, T, WMT and XOM. All their time frames are from January, 2005 to March, 2007. As with
the Bitcoin time series, we have obtained averaged statistical properties for NYSE data set. Using simulated annealing technique we have obtained another best fit parameter set for our model. To obtain a better fit we had to divide absolute return time series generated by the model by factor of $3$. This indicates that the model still lacks something, though it seems to reproduce correct behavior for the tail of the distribution. As we can see in Fig. 5 after this correction the agreement between the model and the data appears to be rather good. The obtained parameter set is similar to the one obtained for the Bitcoin case. Though there are some differences. The mood swings seem to be larger in NYSE case (the respective $\xi_0$ is larger). While chartists have smaller impact on prices for NYSE ($\lambda_{tc}$ is smaller). Base trading activity seems to be higher, $\lambda_0$ is larger, for NYSE.

![Figure 5: (color online) Comparison between the empirical NYSE stocks’ statistical properties (gray curves) and statistical properties generated by the model (red curves). Parameter values are identical to the ones used in Fig. 4 except: $\xi_0 = 1$, $\lambda_{tc} = 2$, $\lambda_0 = 1.5$.]

5 Impact of the model parameters

In this section we check to see how changing the model’s parameter values impact the statistical properties of absolute return and trading activity generated by the model. In all figures in this section we will show three curves. Usually one will be generated with a larger parameter value than used to produce Fig. 4 (blue curve), one smaller (green curve) and one identical (red curve).

Models built on the non-extensive formulation of the herd behavior model are known to avoid the $N$-dependence problem \[11,12,30,33,35,36\], but as we can see in Fig. 6 this model has some kind of $N$ dependence. Yet the fluctuations do not disappear with larger $N$, it seems that the model starts to exhibit even fatter tails. This is most likely occurs due to the implemented mood mechanism: the more agents and the more chartist agents, the more mood is reflected in the time series.
Figure 6: (color online) Influence of $N$ parameter on the model’s statistical properties. Parameter values are identical to the ones used in Fig. 4 except: $N = 5000$ (blue curve), $N = 50$ (green curve).

We can partly eliminate this dependence by recalling that while $y$ dynamics are not influenced by $N$, but the number of trades per time window does depend on $N$. By requiring that $N\lambda_{tc} = \text{const}$ and $N\lambda_{tf} = \text{const}$ we eliminate this dependence. And as we can see in Fig. 7 then changing $N$ does not influence the statistical properties of absolute return. Though $N$ retains the impact on the trading activity.

Figure 7: (color online) Influence of $N$ parameter on the model’s statistical properties, when $N\lambda_{tc} = \text{const}$ and $N\lambda_{tf} = \text{const}$. Parameter values are identical to the ones used in Fig. 4 except: $N = 5000$, $\lambda_{tc} = 2.5$, $\lambda_{tf} = 7.5$ (blue curve), $N = 50$, $\lambda_{tc} = 250$, $\lambda_{tf} = 750$ (green curve).
Changing $\lambda_e$ parameter value seems to have a similar impact as changing $h$ in the original herd behavior model (see Fig. 8). Namely the PSD of the absolute return shifts to the right as we increase $\lambda_e$. Though due to the absolute return formula comparing log-prices at two different points in time, the PDF of the absolute return might also be impacted: it seem that the PDF might obtain heavier tails as $\lambda_e$ becomes larger. Interestingly changing $\lambda_e$ does not seem to have any qualitative effect on the statistical properties of the trading activity. Most likely increasing $\lambda_e$ simply increase the mean of trades per time window without changing anything else.

Parameter $\varepsilon_{fc}$ does not have significant impact on the statistical properties of the model with instantaneous clearing, Eq. 16 but it seems that it is able to impact the statistical properties of the absolute return in the order book model (see Fig. 9). As $\varepsilon_{fc}$ increases the tails of the PDF becomes heavier and the PSD becomes flatter. The statistical properties of the trading activity do not seem to change qualitatively, the tail of the PDF remains the same as well as the steepness of the PSD. Most likely larger $\varepsilon_{fc}$ simply increases mean trading activity.
Parameter $\varepsilon_{cf}$ seems to have the opposite effect on the statistical properties of absolute return (see Fig. 10). As $\varepsilon_{cf}$ increases the tail of the PDF become lighter, while the slope of the PSD becomes steeper. The impact on the statistical properties of the trading activity seems to be both quantitative, the mean number of trades per time window decreases as $\varepsilon_{cf}$ increases, and qualitative, the tail of the PDF becomes lighter and the slope of the PSD becomes steeper. These effects are most likely caused by the dynamics reflected by the model with instantaneous clearing as such dependence is predicted by Eq. 16.
The mood dynamics, $\xi_0$ and $\lambda_{mc}$, doesn’t seem to have a significant impact on the statistical properties of both absolute return and trading activity (see Figs. 11 and 12). Though small effect of $\xi_0$ on the absolute return PDF and PSD are visible. Larger $\xi_0$ makes the tail of the PDF less fat and the PSD flatter. Changing $\lambda_{mc}$ would have a larger if the base value of $\xi_0$ was larger. Increasing $\lambda_{mc}$ would have similar effect as decreasing $\xi_0$ as with larger $\lambda_{mc}$ the effective mood (the average trend) would be smaller than than the true value of $\xi_0$.

Figure 11: (color online) Influence of $\xi_0$ parameter on the model’s statistical properties. Parameter values are identical to the ones used in Fig. 4 except: $\xi_0 = 0.6$ (blue curve), $\xi_0 = 0.1$ (green curve).

Figure 12: (color online) Influence of $\lambda_{mc}$ parameter on the model’s statistical properties. Parameter values are identical to the ones used in Fig. 4 except: $\lambda_{mc} = 300$ (blue curve), $\lambda_{mc} = 0.3$ (green curve).
As we have seen in the previous section larger $\lambda_{tc}$ and $\lambda_{tf}$ values force the realized prices to more closely follow the equilibrium prices. While looking at the statistical properties of the model we see that $\lambda_{tf}$ does not seem to have a noticeable effect (see Fig. 14). This is because the fundamentalists activate only if the current price deviates far from the the fundamental price and their trades rapidly push the price back to the fundamental price. On the other hand $\lambda_{tc}$ seems to have a significant effect (see Fig. 13): larger values lead to the fatter tails of the PDFs, while the PSDs flatten.

Figure 13: (color online) Influence of $\lambda_{tc}$ parameter on the model's statistical properties. Parameter values are identical to the ones used in Fig. 4 except: $\lambda_{tc} = 250$ (blue curve), $\lambda_{tc} = 2.5$ (green curve).
Figure 14: (color online) Influence of $\lambda_{tf}$ parameter on the model’s statistical properties. Parameter values are identical to the ones used in Fig. 4 except: $\lambda_{tf} = 750$ (blue curve), $\lambda_{tc} = 7.5$ (green curve).

As we can see in Fig. 15 changing $\lambda_0$ values does not have a significant effect besides increasing the overall level of trading activity.

Figure 15: (color online) Influence of $\lambda_0$ parameter on the model’s statistical properties. Parameter values are identical to the ones used in Fig. 4 except: $\lambda_0 = 4$ (blue curve), $\lambda_0 = 0.04$ (green curve).

Changing the power of the feedback scenario $\alpha$ seems to have an adverse effect (see Fig. 16): with smaller values the tails of the PDFs become lighter and the slopes of the PSDs become steeper.
Figure 16: (color online) Influence of $\alpha$ parameter on the model’s statistical properties. Parameter values are identical to the ones used in Fig. 4 except: $\alpha = 1$ (blue curve), $\alpha = 0$ (green curve).

Interestingly, as can be seen in Figs. 17 and 18, the parameters influencing the overall shape of the order book itself, $k$ and $\theta$, do not seem to have any effect on the statistical properties obtained from the normalized time series. This would potentially imply that a different underlying model for the order book dynamics could have been chosen instead and similar results would still be obtained.

Figure 17: (color online) Influence of $k$ parameter on the model’s statistical properties. Parameter values are identical to the ones used in Fig. 4 except: $k = 16$ (blue curve), $k = 1$ (green curve).
6 Conclusions

Here we have proposed an order book model with herd behavior, which is able to reproduce the stylized facts of the financial markets. The order book model with herd behavior was built upon empirical insights by Kanazawa et al. [26,27], who have studied a very detailed records of the order book level events, and our previously proposed theoretical agent-based model [11,12], which successfully reproduces statistical properties of the high-frequency absolute return. Incorporating order book dynamics improves upon our previous work in numerous ways. First of all, we were able to scrap two not very realistic, but still common in the literature, assumptions: we no longer need to introduce an efficient market maker to define the market price (introduced in [11]), as well as we no longer the exogenous noise introduced in [12]. Another key improvement is that now we are able to consider statistical properties of the trading activity alongside the statistical properties of absolute return. Using simulated annealing we were able to calibrate the model parameters to match the Bitcoin’s statistical properties. Calibrating the model to match the statistical properties observed in NYSE was not as successful, which indicates that the model still lacks something. We believe that introducing heterogeneity into chartist and fundamentalist valuation of the stock might be the key, but this would further complicate the model introducing additional parameters. Another possibility would be to complicate the modeling of the chartist mood swings. Finally, the model structure itself suggests that some of the parameter values could be gleaned from the order book level data. Currently we are gathering publicly available Bitcoin order book level data in hopes to use the data for the better calibration of the model.

Another possible future extension of the model would be to describe its dynamics analytically. This might be possible, because one of the underlying models can be alternatively described using stochastic differential equations, but appropriately describing the order book part of the model will be tricky. There could be a couple of possible approaches: a superstatistical or a coupled SDE approach as used in [54,55] or a coarse–grained approximation of the model as discussed in [56].
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