GENERALIZED SUBJECTIVE LEXICOGRAPHIC EXPECTED UTILITY REPRESENTATION

HUGO CRUZ SANCHEZ

ABSTRACT. We provide foundations for decisions in face of unlikely events by extending the standard framework of Savage to include preferences indexed by a family of events. We derive a subjective lexicographic expected utility representation which allows for infinitely many lexicographically ordered levels of events and for event-dependent attitudes toward risk. Our model thus provides foundations for models in finance that rely on different attitudes toward risk (e.g. Skiadas [9]) and for off-equilibrium reasonings in infinite dynamic games, thus extending and generalizing the analysis in Blume, Brandenburger and Dekel [3].
One of the lessons we learn from the theory of refinements of Nash equilibrium in game theory is that decisions in face of unlikely events play an important role in determining how a game is to be played. In particular, the analysis of dynamic games relies heavily on off-equilibrium reasonings, that is, in determining what would have happened had players not played what they are supposed to play. We also learn from finance theory that the attitudes toward risk may depend on the kinds of events that the agent faces. For instance, it is conceivable that agents become more risk averse in face of catastrophic, unlikely events. Lexicographic Expected Utility (LEU) is a sensible approach to model decisions in face of very unlikely events, as it presumes a hierarchy of events, ordered by relative unlikeliness, and captures the idea that once the agent is faced with an unlikely event, he goes down to the level of the event in the hierarchy and performs a standard expected utility computation. Thus, a LEU model with infinitely many levels and level-dependent attitudes toward risk seems to be the right model to be used in the theory of infinite dynamic games and of financial theories with varying risk attitudes. It turns out, however, that there is no decision theoretic foundations for such a model available in the literature.

This paper fills up this gap. In particular, in a standard Savage-style framework, we consider a decision maker that is described not only by a preference relation over acts, but also by a family of preference relations over acts. Each preference in this family is indexed by some event in the state space. The idea is that a preference indexed by an event, say $\succeq_A$, where $A$ is the indexing event, represents the preferences of the agent when the agent is informed that the event $A$ has occurred. We then provide a list of axioms that such a system of preferences ought to satisfy and show that decisions that are consistent with the axioms can be represented by a Generalized Subjective Lexicographic Expected Utility (GSLEU) functional. Specifically, for a given state space $S$, a sigma-algebra $\Sigma$ on subsets of $S$ and an outcome space $O$, we consider a preference relation $\succeq$ and a family $(\succeq_A)_{A \in \Sigma}$ of preference relations over the space of acts $f : S \to O$. When these preferences satisfy our axioms, it must be that there exists a (possibly uncountable) family of events $E \subset \Sigma$ and, for each $E \in E$, a utility function $u_E : O \to \mathbb{R}$ and

\footnote{For now this family is taken as a primitive; later we will argue that each preference in the family can be inferred as a sort of conditional preference from the given preference relation over acts.}
a subjective probability measure $P_E$ such that an act $f$ is preferred to an act $g$ under $\succ$ (in short, $f \succ g$) if and only if the subjective lexicographic expected utility of $f$ is greater than that of $g$. In symbols, it must be that if
\[
\int u_E(g)dP_E > \int u_E(f)dP_E
\]
for some $E \in \mathcal{E}$, then there must exist $E' \in \mathcal{E}$ with $E \subset E'$ such that
\[
\int u_{E'}(f)dP_{E'} > \int u_{E'}(g)dP_{E'}.
\]
Moreover, for each $E \in \mathcal{E}$, $P_E$ is uniquely determined, $u_E$ is unique up to affine transformations, and $u_E$ is ordinally equivalent to $u_{E'}$, for any other $E' \in \mathcal{E}$. This last property allows the utility indices $u_E$ and $u_{E'}$ to represent different attitudes toward risk. Observe that the interpretation of the family $\mathcal{E}$ is that of a hierarchy of events, ordered by relative unlikeliness, in that higher level events are interpreted as infinitely more likely than lower level events.

To contrast with the existing literature, the most relevant contribution is that of Blume, Brandenburger and Dekel [3], which, by relaxing the Archimedian axiom in an Anscombe and Aumann [2] framework with finitely many states, provide foundations to subjective expected utility representation with finitely many levels and level-independent risk attitudes. That is, they establish the existence of one utility index $u$ and finitely many subjective probability measures $(P_\ell)_{\ell=1}^L$ such that an act $f$ is preferred to an act $g$ if and only if the existence of a level $\ell$ such that
\[
\sum u(g(s))P_\ell(s) > \sum u(f(s))P_\ell(s)
\]
implies the existence of a level $\ell' < \ell$ such that
\[
\sum u(f(s))P_{\ell'}(s) > \sum u(g(s))P_{\ell'}(s).
\]
It is apparent that the representation derived here is better suited to the analysis of the problems in game theory and finance mentioned above, as it allows for infinitely many levels and level-dependent utility indices.

Moving on to the axioms, we begin by making precise the interpretation of $\succ_A$ as the preference when the agent is informed that the event $A$ occurred. We then proceed to relativize the standard Savage axioms to each of the preferences in the family $(\succ_A)_{A \in \Sigma}$. We note that, because “preferences when informed of an event $A$” are part of our primitives, Savage’s Sure Thing Principle has an immediate formulation: if the agent prefers act $f$ to act $g$ when informed of an event $A$ and also
when informed of the complement of the event $S \setminus A$, then $f$ should indeed be preferred to $g$. We depart from Savage to allow for lexicographic introspection. What we want to capture is a decision maker who, when informed than an extremely unlikely event has occurred, performs the minimal changes in his/her world views in order to make sense of the unlikely event, and then proceeds as a standard expected utility maximizer. In lexicographic terms, this means that we are after a completely ordered hierarchy of levels events, where the order represents that higher levels are infinitely more likely than lower levels. When an unlikely event occurs, the decision maker goes down to the first level at which the event occurs and uses the level’s expected utility terms (utility index and subjective probability.) The crucial feature in our axioms that represents such a decision maker is the postulate of existence of a subfamily of events, $\mathcal{E} \subset \Sigma$, satisfying the following properties. First, it is rich enough to identify relevant events for the entire family $\Sigma$, where “relevance” of an event means that it matters for some indexed preference in the family. Second, it is not richer than what is necessary, in the sense that it avoids redundancies. Third, and more important, the family $\mathcal{E}$ connects the corresponding indexed preferences $(\succeq)_{E \in \mathcal{E}}$ with the non-indexed preference $\succsim$, in a lexicographic fashion.

It is important to note that the existence of the family $\mathcal{E}$, as postulated by our axioms, is far from enough for our representation result. In fact, our Theorem 1 shows that the other axioms already imply the existence of a hierarchy of classes of events ordered by “relative nullity”: events in a class $\alpha$ are of “comparable likelihood”, meaning that neither is null relative to the other, but an event in a class $\alpha$ is “infinitely less likely” than an event in a higher class $\beta \succ \alpha$, meaning that it is null relative to that event. In fact, such other axioms already imply the existence of a qualitative probability for each class, which is the key ingredient for our generalized lexicographic expected utility representation.\footnote{Also, as relativizations of Savage’s axioms, they also imply a SEU representation for each event $A \in \Sigma$.} The role played by the postulated family $\mathcal{E}$ is that it provides the necessary “top events” for each class $\alpha$. That is, we show that, for any class $\alpha$, there exists an event $E \in \mathcal{E}$ that can be interpreted as the “local state space” for the class $\alpha$: the expected utility representation for class $\alpha$ is determined by $E$, in that every event $A$ in the class $\alpha$ shares the same utility index $u_E$ and the subjective probability $P_A$ can be computed as the conditional of $P_E$ given $A$. 
With the axiom system in place, we are able to establish our GSLEU representation result in Theorem 3. That is, if a decision maker is represented by a preference $\succeq$ and also by a family of indexed preferences $(\succeq_A)_{A \in \Sigma}$, and this system of preferences satisfies our axioms, then choices can be represented by those that maximize the GSLEU functional. An important issue at this juncture, however, is whether the assumed “informed” preferences $(\succeq_A)_{A \in \Sigma}$ can be inferred from a given “uninformed” preference $\succeq$ over acts.

We provide a positive answer to this question, by means of a notion of conditioning the preference $\succeq$ on an event $A$ that captures what the so far primitive notion $\succeq_A$ is meant to capture. Our notion of conditioning is stronger than Savage’s notion because it also requires checking for whether “perturbations” of an act $f$ are (conditionally) preferred to another act $g$. More precisely, Savage’s notion of conditioning says that an act $f$ is preferred to an act $g$ conditional on an event $A$ if the act $fAh$ is preferred to the act $gAh$, for any other act $h$, where the notation “$fAh$” means the act that is equal to $f$ on $A$ and equal to $h$ on the complement of $A$. On top of that, we add that the act $fAh$ must be preferred to the act $gAh$ and that the act $fAh$ must be preferred to the act $gAh$, where $fAh$ represents a perturbation of the act $fAh$ that is equal to $f$ on most of $A$ but equal to some other constant act on a small part of $A$, and is equal to $h$ on the complement of $A$. We show in Theorem 4 that this stronger notion of conditioning characterizes the informed preferences $\succeq_A$, for each $A \in \Sigma$.

In words, we can in principle tease out of $\succeq$ the “strong” conditional preferences $\succeq_A$, $A \in \Sigma$, by offering choices of acts and perturbed acts. In decision theoretic jargon, this means that the primitives $\succeq_A$, $A \in \Sigma$, are observable. But Theorem 4 is interesting from other perspectives as well. For instance, a classical question in probability is the issue of probabilities conditional on zero probability events. Our strong conditioning provide insights on computing conditionals on infinitely unlikely events by also requiring that the computation be robust to small perturbations. Relatedly, refinements of equilibrium in game theory often require consideration of perturbations of strategies/payoffs. Our notion of conditioning seems to capture exactly the need for such perturbations. These are questions that we plan to address in future research.

1.1. Related Literature. As advanced above, Blume, Brandenburger and Dekel [3] provide foundations for a SLEU representation that is special in that it does not allow for infinitely many levels or for level-dependent risk attitudes. These issues are consequences of the
Anscombe-Aumann framework adopted, since it assumes a finite state space and only considers acts that are mixture-space valued. The extension to an infinite state space is not a simple matter. LaValle and Fishburn [6] provide an extension of a finite SLEU representation in a finite state space to such a representation in an infinite state space, while still only allowing for finitely many lexicographic levels. The extension is technical in nature, and does not really extend the SLEU model to an infinite version, as it is done here. Our contribution can, in fact, be viewed as a first step in providing a complete infinite extension of the analysis in Blume, Brandenburger and Dekel [3]; they do not assume, as we do here, the existence of a primitive lexicographic order; the lexicographic representation is a consequence of their relaxation of the Archimedian axiom. Because in our infinite framework we do not even have a well-ordered space of classes of events to begin with, it is not clear how a non-Archimedian approach would work. This is another question for future research. A related paper is that of Amarante [1], which does allow for infinitely many “informed” (or conditional) preferences by also considering a family of such preferences as a primitive of the model, but does not provide a lexicographic representation, as the focus of the paper is to show that a family of SEU conditional preferences does not necessarily give rise to an SEU unconditional preference.

We move now to Section 2 where the basic framework and axioms are presented. The results are in Section 3 and the conclusion in Section 4. The Appendix contains the results not proved in the main text.

2. Setting and Axioms

Following Savage [8] and Machina and Schmeidler [7] (including the naming of axioms), our setting is as follows: $S$ is the infinite set of states, which capture the uncertainty present in the decision-making process, $\Sigma \subseteq 2^S$ is the $\sigma$–algebra of events, $O$ is the set of outcomes, $F$ is the set of $\Sigma$–measurable functions from $S$ to $O$ with finite range. This is the set of acts that represents the possible choices in the decision-making process. A weak order (complete and transitive binary relation) represents the ranking of the decision-maker over $F$. In addition to this classical setting, $\succsim_A$ is a weak preference on $F$ for each event $A$ in $\Sigma$.

**Notation 1.** For an event $A$, and a pair of acts $f$ and $g$, $fAg$ represents the composed act that is equal to $f$ on $A$ and $g$ on $S\setminus A$. 
Notation 2. For a weak order $\succeq$ on a non empty set $X$, and a pair of elements of $X$, $f$ and $g$,
\[ f > g \Leftrightarrow (f \succeq g) \land (\neg (g \succeq f)) \]
and
\[ f \sim g \Leftrightarrow (f \succeq g) \land (g \succeq f). \]

Definition 1 (agree). $\succeq_A$ agree with $\succeq_B$ iff, for each pair of acts $f$ and $g$,
\[ f \succeq_A g \Leftrightarrow f \succeq_B g. \]

Our first axiom on the primitives makes precise the idea that $\succeq_A$ is indeed a “preference when informed of event $A$”. Such definition meets Savage’s idea of ordering if $A$ were known to obtain (page 22, Savage [8], “What technical interpretation can be attached to the idea that $f$ would be preferred to $g$, if $B$ were known to obtain? Under any reasonable interpretation, the matter would seem not to depend on the values $f$ and $g$ assume at states outside of $B$”) In particular, the preference indexed by the empty event is degenerate (i.e. each act is weakly preferred to any act). However, axioms $P3_1^1$ and $P5_2^1$ below ensure that no weak preference indexed by a non empty event is degenerate.

Axiom 1 ($P1_2^1$). For each pair of acts $f$ and $g$,
\[ f \succeq_A g \Rightarrow fA h \succeq_A gA h \text{ for each act } h \]
and
\[ fA h \succeq_A gA h \text{ for some act } h \Rightarrow f \succeq_A g. \]

In Savage [8], a null event is an event that is irrelevant for the decision-making process. However, in our framework, the decision-making process has multiple levels, and an event can be irrelevant at some levels, but relevant for others. Thus, our concept of null event is relative to each indexed preference.

Definition 2 (null event). For each pair of events $A$ and $B$ such that $B \subseteq A$, $B$ is a null event at $A$ if and only if $\succeq_{A \setminus B}$ agrees with $\succeq_A$.

Note that the concept of null event at $A$ refers only to events (i.e. belonging to $\Sigma$) contained in $A$. The empty event is null at every $A$, even when $A$ is the empty event.

The next axiom is the sure thing principle, and it relates different indexed preferences. It says that the act $f$ is weakly preferred to the act $g$ for the preference indexed by an event $A$ if $f$ is weakly preferred
to $g$ for the preferences indexed by each event of a bipartition of $A$, and
the order between $f$ and $g$ is strict for the preference indexed by $A$ if it
is strict for the preference indexed by an event of the bipartition which
is non null at $A$. This is consistent with Savage’s first formulation of
the sure thing principle (page 21, Savage [8]), “Having suggested what
I shall tentatively call the sure-thing principle, let me give it relatively
formal formal statement thus: If the person would not prefer $f$ to $g$,
either knowing that the event $B$ obtained, or knowing that the event
$S\setminus B$ obtained, then he does not prefer $f$ to $g$. Moreover (provided he
does not regard $B$ as virtually impossible) if he would definitely prefer
$g$ to $f$, knowing that $B$ obtained, and, if he would not prefer $f$ to $g$,
knowing that $B$ did not obtain, then he definitely prefers $g$ to $f$.”)

**Axiom 2** ($P_{2\frac{1}{2}}$). For each pair of events $A$ and $B$ such that $B \subseteq A$, and each pair of acts $f$ and $g$,

$$f \succ_B g \text{ and } f \succ_{A\setminus B} g \iff f \succ_A g$$

and

$$B \text{ is non null at } A \Rightarrow (f \succ_B g \text{ and } f \succ_{A\setminus B} g \Rightarrow f \succ_A g).$$

The next axiom says that constant acts are equally ordered by the
preferences indexed by non empty events, and it is called eventwise
monotonicity.

**Axiom 3** ($P_{3\frac{1}{2}}$). For each non empty event $A$, and constant acts $f$ and $g$,

$$f \succeq_A g \iff f \succeq_S g.$$  

The next axiom says that an event is more likely than a second event,
if a prize resulting from the first event is preferred to the same prize
resulting from the second event, independently of the prize. It is called
weak comparative probability - prize independence.

**Axiom 4** ($P_{4\frac{1}{2}}$). For each triple of events $A, B$ and $C$ such that $B, C \subseteq A$, and constant acts $f, f', g$ and $g'$ such that $f \succeq_S f'$ and $g \succeq_S g'$,

$$fBf' \succeq_A fCf' \Rightarrow gBg' \succeq_A gCg'.$$

The next axiom implies nondegeneracy for each preference indexed
by a non empty event, and it is called nondegeneracy.

**Axiom 5** ($P_{5\frac{1}{2}}$). There is at least two constant acts $f$ and $g$ such that $f \succ_S g$. 

The axioms above are readily seen as translations of the usual Savage axioms to our setting with an additional family of informed preferences. The next two axioms are particular of our framework. The first restricts the allowed families of indexed preferences. It says that any allowed family of indexed preferences must contain an essential subfamily that is sufficient to determine if an event is relevant for the family (i.e. an event whose subtraction matters for some indexed preference of the family). It is called separability axiom (SE) because the subfamily separates events (by symmetric difference on \( \Sigma \)), which means that each event of \( \Sigma \) is used in the comparison between a pair of acts, if preferences indexed by non empty events are non degenerate (which is true by \( P5\frac{1}{2} \)).

**Axiom 6 (SE).** There exists a subfamily \( \succsim_E \), with \( E \in \mathcal{E} \subseteq \Sigma \setminus \{\emptyset\} \), of the family of indexed preferences satisfies the following:

\[
\forall B \in \Sigma \left( \forall E \in \mathcal{E} \,(B \subseteq E \Rightarrow \succsim_E \text{ agrees with } \succsim_{E \setminus B}) \Rightarrow \forall A \in \Sigma \,(B \subseteq A \Rightarrow \succsim_A \text{ agrees with } \succsim_{A \setminus B}) \right),
\]

and,

\[
\forall A \in \Sigma, \forall E \in \mathcal{E} \left( E \subseteq A \Rightarrow \begin{array}{l}
\succsim_A \text{ agrees with } \succsim_{A \setminus E} \\
\text{or} \\
\succsim_A \text{ agrees with } \succsim_E
\end{array} \right).
\]

In order to better understand what is implied by Axiom SE, we argue that it is a necessary condition for any subjective lexicographic utility representation with at least one event of probability one in each class of the hierarchy of classes. In fact, if we have a subjective lexicographic utility representation with at least one event of probability one in each class of the hierarchy of classes, then for any class, we have a preference induced by the subjective expected utility representation of the class. Also, for each event of the class (so, a non null event), we have a preference indexed by the event corresponding to the preference induced by the subjective expected utility representation conditioned on the event. The preference indexed by the event of probability one of the class (by assumption) agrees with the preference induced by the subjective expected utility representation of the class, and this is true for any other event of probability one of the class. Any event of the class containing an event of probability one of the class is an event of probability one of the class, in particular, the union of an event of the
class with an event of probability one of the class. Thus, the class contains a cofinal subset of events of probability one, which is a singleton only when the class contains an event containing every event of the class. A preference indexed by an event $A$ containing an event $E$ of this subset does not agree with the preference indexed by $E$ if and only if $A$ is of a higher class of the hierarchy (in this case, the preference indexed by $A \setminus E$ agree with the preference indexed by $A$, because $E$ is irrelevant for higher classes). And finally, an event $B$ irrelevant for an event $E \supset B$ of the subset is irrelevant for the class, so for any event $A \supset B$ of the class.

On the other hand, when combined with the other axioms above, SE is a sufficient condition for a subjective lexicographic utility representation with at least one event of probability one in each class of the hierarchy of classes.

The next axiom says that the ordering between acts for the preference $\succsim$ coincides with the ordering between acts resulting from using the lexicographic rule of comparison on preferences indexed by chains of events in $\mathcal{E}$ ordered by the set inclusion $\supseteq$.

**Axiom 7** ($P0\frac{1}{2}$). For each pair of acts $f$ and $g$,

\[ f \succsim g \]

iff

\[ g \succ_E f \text{ for some } E \in \mathcal{E} \Rightarrow f \succ_{E'} g \text{ for some } E' \in \mathcal{E} \text{ st } E' \supseteq E. \]

The lexicographic rule of comparison on a (maybe uncountable) infinite and order-dense hierarchy implies a transitive strict binary relation, but the indifference binary relation might violate transitivity. Axiom $P0\frac{1}{2}$ allows us to bypass this intransitivity issue, by working as a criterion on the set of families of indexed preferences. Similarly as above, note that $P0\frac{1}{2}$ is a necessary condition for any subjective lexicographic utility representation of a weak order in general.

Our final axiom is the usual technical requirement to obtain quantitative probabilities out of qualitative ones. It is called small-event continuity.

**Axiom 8** ($P6\frac{1}{2}$). For each non empty event $A$, and each triple of acts $f, g$ and $h$ such that $f \succ_A g$ and $h$ constant, there exists a finite $\Sigma$-measurable partition of $A$, $A_i$, such that, for each $i$,

\[ f \succ_A hA_ig \]
and

$$hA_i f \succ_A g.$$  

3. Results

We begin with a result that follows from $P1\frac{1}{2}$ to $P5\frac{1}{2}$, asserting the existence of a hierarchy of disjoint families of events, and describing its main properties. The proof is a series of results in the appendix A.

**Theorem 1.** There is a partition of $\Sigma$ in classes of events ordered by an irreflexive, transitive and total order $\gg$. The lowest class is the class containing the empty event (a singleton called trivial class), and the highest class is the class containing $S$. An event of a non trivial class $\alpha$ is non null relatively to other event of the class (i.e. each of the two events is non null at the union of both events), but it is null relatively to an event of a higher class $\beta$ (i.e. $\beta \gg \alpha$). Subevents (superevents) of an event of a non trivial class $\alpha$ belong to a class no higher (lower) than $\alpha$. Each class $\alpha$ is closed for the union of an event of $\alpha$ and an event of a class no higher than $\alpha$.

From $P1\frac{1}{2}$ to $P6\frac{1}{2}$, we have the following theorem asserting the existence of a subjective expected utility representation for the indexed preferences, and describing the relations intraclass and interclass for these representations. The proof is a series of results in appendices A and B.

**Theorem 2.** For a non trivial class $\alpha$, there exists a unique (up to a positive affine transformations) function $u_\alpha : O \to \mathbb{R}$, and for each event $A$ of the class, there exists a unique finitely additive, convex-valued probability measure $P_A$ on $\Sigma$, such that, for each pair of acts $f$ and $g$,

$$f \succeq_A g \iff \int u_\alpha(f) \, dP_A \geq \int u_\alpha(g) \, dP_A.$$  

Moreover, for another event $B \subseteq A$ of the class, $P_A(C) = P_B(C) P_A(B)$ for an event $C \subseteq B$. In addition, different non trivial classes admit different attitudes toward risk when $O$ is not a mixture space.

We are now ready to establish our main representation result. It follows from $P0\frac{1}{2}$ to $P6\frac{1}{2}$, and $SE$, and it asserts the existence of a generalized lexicographic subjective expected utility representation.

\footnote{Indeed, $P_A(\cdot)$ is defined on $\Sigma|_A$, but it can be extended to a probability on $\Sigma$ as $P_A(A \cap \cdot)$. We will keep the simpler notation in the paper.}
Theorem 3 (Representation Result). There exists a family of real functions on $O$, $\{u_E\}_{E \in \mathcal{E}}$, with each $u_E$ unique (up to positive affine transformations), and a unique family of finitely additive, convex-valued probability measures on $\Sigma$, $\{P_E\}_{E \in \mathcal{E}}$, such that, for each pair of acts $f$ and $g$,

$$f \succeq g \quad \text{iff} \quad \int u_E(g) \, dP_E > \int u_E(f) \, dP_E \quad \text{for some } E \in \mathcal{E} \Rightarrow \int u_{E'}(f) \, dP_{E'} > \int u_{E'}(g) \, dP_{E'} \quad \text{for some } E' \in \mathcal{E} \text{ st } E' \supseteq E.$$ 

Proof. First, note that $P3\frac{1}{2}$ and $P5\frac{1}{2}$ implies that preferences indexed by non empty events are non degenerated, so no non empty event is null at itself, and, by $SE$, for each non empty event $A$, there is an event $E$ in $\mathcal{E}$, containing $A$, and belonging to same class than $A$. Thus, $\mathcal{E}$ contains a cofinal subset of each non trivial class. By $SE$, for each pair of events of $\mathcal{E}$ in a class, $E$ and $E'$, $\succeq_E$ agrees with $\succeq_{E'}$, so $E \setminus E'$, $E' \setminus E$ and $E \Delta E'$ are null events at some event of $\mathcal{E}$ in the class, containing $E \cup E'$ (the class is closed for finite unions, and $\mathcal{E}$ contains a cofinal subset of each non trivial class). By (2), each event of $\mathcal{E}$ in the class has a subjective expected utility representation, and they are equivalent representations. More specifically, the Bernoulli indexes of those representations are equal to the Bernoulli index of the class, and the beliefs (extended to $\Sigma$) assign the same probability to the events. As every event in the class is contained in some event in that subset of the class, the belief (extended to $\Sigma$) of the subjective utility representation of every event in the class which contains an event in that subset of the class assign probability one to each event in that subset of the class. In other words, they are essential top events. And, given the derived properties of $\mathcal{E}$, the representation for $\succeq$ follows from $P0\frac{1}{2}$. \hfill \Box

Each essential “top events” in $\mathcal{E}$ corresponds to a class, so the description of the relations intraclass and interclass in Theorem 2 holds for the representation in Theorem 3.

We now move to the issue of observability of $\succeq_A$, that is, that $\succeq_A$ can be inferred from choices that respect $\succeq$ over acts that “strongly” reveal the subjective assessments of the decision maker as to the event $A$ relative to its complement $S \setminus A$.

Theorem 4 (Observability of informed preferences). For each non empty event $A$, and each triple of acts $f$, $g$ and $h$, $f \succ_A g$ if, and only if, $fAh \succ gAh$ and, for each constant act $k$, there exists a finite
constant act with outcome belonging to $f$.

Besides, the constant act with outcome $-\Sigma$ act with outcome belonging to $fAh$ and $fAh \succ kA_i(gAh)$.

Proof. Note that $fAh \sim gAh$ implies $fAh \sim_E gAh$ for any top event $E \supseteq A$ of the class of $A$, and, by the lemma order-preserving (see appendix Proofs), this implies $fAh \sim_A gAh$, which implies $f \sim_A g$ (eq. $f \succ_A g$ implies $fAh \succ_A gAh$, and using the same steps and lemma, we have $fAh \succ gAh$), but the converse is not necessarily true. We can have $fAh \succ gAh$ and $fAh \sim_A gAh$, if $fAh$ is (strictly) preferred to $gAh$ for some class $\alpha$ lower than the class of $A$, and for any class higher than $\alpha$, $gAh$ is not (strictly) preferred to $fAh$. In what follows we discuss the effects of perturbations on the case $fAh \sim_A gAh$.

By $P3_1^1$ and $P5_2^1$, we can choose a pair of constant acts $k \succ_S k'$. Besides, $k$ and $k'$ can satisfy the following: $k$ is preferred (by $P3_2^1$, we do not need to specify an indexed preference) to any constant act with outcome belonging to $f(A) \cup g(A)$ (a finite set), and any constant act with outcome belonging to $f(A) \cup g(A)$ is preferred to $k'$. A finite $\Sigma$-measurable partition of $A$, $A_i$, defines a finer finite $\Sigma$-measurable partition of $A$, $P_f = \{A_i \cap f^{-1}(o) : i$ and $o \in f(A)\}$ (for $g$, the discussion that follows is analogous). For $P_f$, we have $\cup_i A_i \cap f^{-1}(o) = A \cap f^{-1}(o)$, so, if $A \cap f^{-1}(o)$ is non null at $A$, $A_i \cap f^{-1}(o)$ is non null at $A$ for some $i$ (name it $i_o$). Thus, if for each $o' \in f(A)$ such that $A \cap f^{-1}(o')$ is non null at $A$, $k$ is preferred to the constant act with outcome $o'$, and this ordering is strict at least once (name it $o$), then $kA_i(o)(fAh) \succ_A fAh$. And, if for each $o' \in f(A)$ such that $A \cap f^{-1}(o')$ is non null at $A$, the constant act with outcome $o'$ is preferred to $k'$, and this ordering is strict at least once (name it $o$), then $fAh \succ_A k'A_i(o)(fAh)$. Summing up, for each $o \in f(A)$ such that

If for each $o' \in f(A)$ such that $A \cap f^{-1}(o')$ is non null at $A$, $k$ is preferred to the constant act with outcome $o'$, and this ordering is strict at least once (name it $o$), then

\[ k\left(A_i \cap f^{-1}(o)\right)(fAh) \succ_{A_i \cap f^{-1}(o)} fAh \]

and

\[ k\left(A_i \cap f^{-1}(o)\right)(fAh) \sim_{A \cap A_i \cap f^{-1}(o)} fAh, \]

so, by $P2_1^1$,

\[ k\left(A_i \cap f^{-1}(o)\right)(fAh) \succ_A fAh. \]

This one element substitution can be repeated for each $o'' \in f(A)$ such that $A_i \cap f^{-1}(o'')$ is non empty, and, by $P2_1^1$ at each step, after a finite number of one element substitutions, we obtain

\[ kA_i(fAh) \succ_A fAh. \]
A \cap f^{-1}(o) is non null at A, either k is strictly preferred to the constant act with outcome o, or the constant act with outcome o is strictly preferred to k', or both, so, for each o ∈ f(A) such that A \cap f^{-1}(o) is non null at A, and for each finite Σ-measurable partition of A, Ai,

\[ kA_i(o) \succ_A fAh \text{ or } fAh \succ_A kA_i(o) \text{.} \]

In addition (for f and g shifted, the discussion that follows is analogous), if, for each o ∈ f(A) such that A \cap f^{-1}(o) is non null at A, the constant act with outcome o is indifferent to k', and, for each o' ∈ g(A) such that A \cap g^{-1}(o') is non null at A, k is indifferent to the constant act with outcome o', then gAh \sim_A kAh \succ_A k'Ah \sim_A fAh, an absurd. So, if k'Ah \sim_A fAh, then for at least one o' ∈ g(A) such that A \cap g^{-1}(o') is non null at A, k is strictly preferred to the constant act with outcome o', and kA_i(o) \succ_A gAh. And, if kAh \sim_A fAh, then for at least one o' ∈ g(A) such that A \cap g^{-1}(o') is non null at A, the constant act with outcome o' is strictly preferred to k', and gAh \succ_A k'A_i(o)\sim_A (gAh).

Summing up, for each finite Σ-measurable partition of A, Ai,

\[ k'Ah \sim_A fAh \Rightarrow \exists i \, (kA_i(gAh) \succ_A gAh) , \]

\[ kAh \sim_A fAh \Rightarrow \exists i \, (gAh \succ_A k'A_i(gAh)) , \]

and,

\[ kAh \succ_A fAh \succ_A k'Ah \Rightarrow \exists i, j \, (kA_i(fAh) \succ_A fAh \succ_A k'A_j(fAh)) . \]

Concluding, the set of rules above shows that, for fAh \sim_A gAh, and using k and k', any finite Σ-measurable partition of A, Ai, generates a set of perturbed versions of fAh, and a set of perturbed versions of gAh, which meet at some side (strict upper/lower set) of fAh \sim_A gAh.

On the other hand, for fAh \succ_A gAh, P6_{\frac{1}{2}} ensures that, for each constant act k, there exists a finite Σ-measurable partition of A, Ai, such that, for each i, kA_i(fAh) \succ_A gAh and fAh \succ_A kA_i(gAh).

As, for each strict ordering for the preference indexed by A corresponds a strict ordering for \succ (Savage-like) conditioned by A, we can conclude that fAh \succ_A gAh (gAh \succ_A fAh) with fAh \sim_A gAh is not preserved by perturbations in the sense of P6_{\frac{1}{2}}.

As we advanced in the introduction, our concept of conditioning is more intricate than Savage’s conditioning. It involves Savage’s conditioning, and also conditioning of perturbations of acts. Using this

\[ k'Ah \sim_A fAh \text{ means that perturbations of } fAh \text{ are in the upper set of } fAh, \]

with some in the strict upper set.

---

\[ 5k'Ah \sim_A fAh \]
concept of conditioning, we can define indexed preferences from \( \succsim \). In this approach, the axioms are properties of \( \succsim \). In particular, \( P_0 \frac{1}{2} \) says that \( \succsim \) has an internal consistency rule that says that more inclusive events are more decisive for the ordering between acts.

4. Conclusion

We provided a first step into a fully general foundation to subjective lexicographic expected utility. For the applied literature, we provide foundations for source-dependent risk attitudes (e.g. Skiadas \[9\].) For off-equilibrium reasonings in dynamic games, we provide a general theory supporting standard arguments that invoke “infinitely less likely events” in dynamic games with infinite horizon. Although we do show that there is no need to use a family of indexed preferences as a primitive of the model, we still need to impose an a priori lexicographic order to obtain our representation. For the future, we plan to dispense with such assumption, and obtain a lexicographic order directly from the implied hierarchy of classes of events that follow from our other axioms. We also plan to explore the implications of our “perturbed” conditionals to the foundations of refinements of equilibria in game theory.
Appendix A. Proofs

**Lemma 1.** For each $A \in \Sigma$, $\succsim_A$ is a weak preference on $F$ such that, for each acts $f$ and $g$,

$$f \succ_A g \Rightarrow fAh \succ_A gAh \text{ for each act } h$$

and

$$fAh \succ_A gAh \text{ for some act } h \Rightarrow f \succ_A g.$$ 

**Proof.** Observe that $gAh \succsim_A fAh$ for some act $h$ implies $g \succsim_A f$, thus, if $f \succ_A g$ then $fAh \succ_A gAh$ for each act $h$. Besides, observe that $g \succsim_A f$ implies that $gAh \succsim_A fAh$ for each act $h$, thus, if $fAh \succ_A gAh$ for some act $h$, then $f \succ_A g$. □

**Lemma 2.** For an event $B$, $f \sim_A fBh$ for each pair of acts $f$ and $h$, iff, for each event $C$ disjoint of $B$, $fCh \sim_A gCh$ for each triple of acts $f$, $g$ and $h$.

**Proof.** Note that $fCh$ and $gCh$ are equal at $B$. □

**Lemma 3.** If $B \subseteq A$ is a null event at $A$, then, for each event $C$ disjoint of $A \setminus B$, $fCh \sim_A gCh$ for each triple of acts $f$, $g$ and $h$.

**Proof.** By $P1\frac{1}{2}$, $f \sim_{A \setminus B} f( A \setminus B)h$ for each pair of acts $f$ and $h$. Thus, the result follows from [2] and the definition of null event. □

In (3), if, for each event $C$ disjoint of $A \setminus B$, $fCh \sim_A gCh$ for each triple of acts $f$, $g$ and $h$, then [2] implies that $f \sim_A f( A \setminus B)h$ for each pair of acts $f$ and $h$. However, $\succsim_{A \setminus B}$ does not need to agree with $\succsim_A$, so $B$ does not need to be a null event at $A$. Nevertheless, $P1\frac{1}{2}$ to $P3\frac{1}{2}$, and $P5\frac{1}{2}$, ensure that $B \subseteq A$ is a null event at $A$.

**Lemma 4** (order preserving - new version). If $B \subseteq A$ is a non null event at $A$, then, for each pair of acts $f$ and $g$,

$$f \succsim_B g \iff fBh \succsim_A gBh \text{ for each act } h.$$ 

**Proof.** By $P1\frac{1}{2}$. (i) $f \succsim_B g$ iff $fBh \succsim_B gBh$ for each act $h$, and (ii) $fBh \sim_{A \setminus B} gBh$ for each act $h$.

By the first part of $P2\frac{1}{2}$, if $fBh \succsim_B gBh$ for each act $h$, then $fBh \succsim_A gBh$ for each act $h$. As $B$ is non null at $A$, by the second part of $P2\frac{1}{2}$, if $fBh \succ_B gBh$ for each act $h$, then $fBh \succ_A gBh$ for each act $h$. □
**Lemma 5** (order-preserving - old version). For each $A, B \in \Sigma$ such that $B \subseteq A$, $B$ is non null at $A$, and acts $f$ and $g$,

$$f \succsim_B g \iff fBh \succsim_A g Bh \text{ for each act } h.$$  

**Proof.** By definition, $f \succsim_B g$ implies $fBh \succsim_B g Bh$ for each act $h$. By sure-thing consistency, and $h \not\sim_A h$ for each act $h$, $fBh \succsim_A g Bh$ for each act $h$. Analogously, by lemma above, $f \succ Bg$ implies $fBh \succ_A g Bh$ for each act $h$. By sure-thing consistency, $B$ is non null at $A$, and $h \not\sim_A h$ for each act $h$. □

Next lemma says that, a Savage-like null event with relation to an indexed preference, and contained in the indexing event, is a null event for the indexed preference.

**Lemma 6.** If, for an event $B \subseteq A$, $fBh \sim_A g Bh$, for each triple of acts $f, g$ and $h$, then $B$ is a null event at $A$ (a partial converse for (3)).

**Proof.** If $A$ is the empty event, the result is trivial.

Assume that $A$ is a non empty event. By $P1\frac{1}{2}$, $fBh \sim_B g Bh$, for each triple of acts $f, g$ and $h$. If $B$ is non null at $A$, then, by the second part of $P2\frac{1}{2}$, $fBh \sim_B g Bh$, for each triple of acts $f, g$ and $h$. By $P1\frac{1}{2}$, $P3\frac{1}{2}$ and $P5\frac{1}{2}$, $B$ is the empty event, a contradiction ($B$ is non null at $A$). Thus, $B$ is null at $A$. □

**Theorem 5** (Nullity). For each triple of events $A, B$ and $C$ such that $C \subseteq B \subseteq A$,

$$B \text{ is null at } A \implies C \text{ is null at } A,$$

$$C \text{ and } B \setminus C \text{ are null at } A \implies B \text{ is null at } A,$$

and

$$C \text{ is null at } B \implies C \text{ is null at } A.$$

**Proof.** By (3) and (6), $C$ is null at $A$ if $B$ is null at $A$.

If, $fCh \sim_A gCh$ and $f(B\setminus C)h \sim_A g(B\setminus C)h$, for each triple of acts $f, g$ and $h$, then $fCf(B\setminus C)h \sim_A gCf(B\setminus C)h$ and $f(B\setminus C)gCh \sim_A g(B\setminus C)gCh$, for each triple of acts $f, g$ and $h$. By transitivity, $fBh \sim_A g Bh$, for each triple of acts $f, g$ and $h$. So, by (3) and (6), $B$ is null at $A$ if $C$ and $B\setminus C$ are null at $A$.

If $B$ is null at $A$, the third case above is a particular case of the first case above. So, assuming that $B$ is non null at $A$, if $fCh \sim_B gCh$ for each triple of acts $f, g$ and $h$, then, by (4), $fCh \sim_A gCh$ for each triple
of acts \(f\), \(g\) and \(h\). So, by (3) and (6), \(C\) is null at \(A\) if \(C\) is null at \(B\).

**Definition 3** \((\geq_A)\). For each triple of events \(A, B\) and \(C\) such that \(B, C \subseteq A\), and constant acts \(f\) and \(g\) such that \(f \succeq_S g\), \(B\) is at least as probable as \(C\) at \(A\), and denoted by \(B \geq_A C\) (for the definitions of \(>_A\) and \(=_A\) see (2)), when

\[
fBg \succeq_A fCg.
\]

**Lemma 7.** If \(B, C \subseteq D\) are null at \(D\), then \(B =_D C\).

**Proof.** By nullity, \(B \cup C\) is null at \(D\), consequently, \(\succeq_{D \setminus (B \cup C)}\) and \(\succeq_D\) agree. I.e., prizes at \(B\) or \(C\) are negligible. \(\square\)

**Lemma 8.** If \(B \subseteq D\) is null at \(D\), then \(B =_D \emptyset\).

**Proof.** By definition, \(\emptyset\) is null at \(D\), thus, by lemma above, \(B =_D \emptyset\). \(\square\)

**Lemma 9.** If \(A \subseteq B \subseteq C\) then \(B \geq_C A\). Besides, \(B >_C A\) iff \(B \setminus A\) is not null at \(C\).

**Proof.** First, observe that for each pair of acts \(f\) and \(g\), by the definition of \(\succeq_{C \setminus B}\) and \(\succeq_A\), \(fBg \sim_{C \setminus B} fAg\) and \(fBg \sim_A fAg\).

Given constant acts \(f\) and \(g\) such that \(f \succeq_S g\), by eventwise monotonicity or by the definition of \(\succeq_{\emptyset}\), \(f \succeq_{B \setminus A} g\). By nondegeneracy, \(f\) and \(g\) can satisfy \(f \succ_S g\), thus, if \(B \setminus A\) is non null at \(C\), then \(f \succ_{B \setminus A} g\) and, by nullity, \(B\) is non null at \(C\) and \(B \setminus A\) is non null at \(B\). If \(B \setminus A\) is null at \(C\), then \(\succeq_{(C \setminus B) \cup A}\) agree with \(\succeq_C\), consequently, \(fBg \sim_C fAg\); otherwise, by the discussion above and the sure-thing consistency, \(fBg \succ_C fAg\). \(\square\)

**Lemma 10.** If \(A, B \subseteq C\), \(A\) is null at \(C\) and \(B\) is non null at \(C\), then \(A \cup B =_C B\).

**Proof.** By nullity, as \(B\) is non null at \(C\), then \(A \cup B\) is non null at \(C\). Now, take \(B \subseteq A \cup B \subseteq C\) and use the lemma above. \(\square\)

**Lemma 11.** If \(C \subseteq D\) and \(C\) is non null at \(D\), then \(C >_D \emptyset\).

**Proof.** Given constant acts \(f\) and \(g\) such that \(f \succ_S g\) (nondegeneracy), by eventwise monotonicity, \(f \succ_C g\), consequently, by order-preserving lemma,

\[
fCg \succ_D gCg = g = f\emptyset g.
\]

\(\square\)
Lemma 12. If $B, C \subseteq D$, $B$ is null at $D$ and $C$ is non null at $D$, then $C >_D B$.

Proof. Both lemmas above imply that, given constant acts $f$ and $g$ such that $f \succ_S g$ (nondegeneracy),

$$fCg >_D f\emptyset g \sim_D fBg.$$ 

\[\square\]

Lemma 13. $C \neq \emptyset$ is non null at $C$.

Proof. By nondegeneracy and eventwise monotonicity, $\succ_\emptyset$ and $\succ_C$ do not agree. \[\square\]

Lemma 14. If $A \subseteq C \neq \emptyset$ is null at $C$ then $C \setminus A$ is non null at $C$.

Proof. By nullity and lemma above. \[\square\]

Lemma 15. If $A \subseteq B \subseteq C$, $B$ is non null at $C$ and $A$ is null at $C$, then $A$ is null at $B$.

Proof. By contradiction, assume that $A$ is non null at $B$. By nondegeneracy and eventwise monotonicity, there exist constant acts $f$ and $g$ such that $f \succ_A g$. By order-preserving lemma, as $A$ is non null at $B$, $fAh \succ_B gAh$ for each act $h$. Again, by order-preserving lemma, as $B$ is non null at $C$, $fAh \succ_C gAh$ for each act $h$. Thus, $A$ is non null at $C$, an absurd. \[\square\]

If $B$ was null at $C$ in the lemma above, $A$ could be null at $B$ (e.g. $A = \emptyset$) or $A$ could be non null at $B$ (e.g. $A = B$).

Definition 4 (qualitative probability). A relation $\succeq_A$ between events is a qualitative probability on $A \neq \emptyset$, iff, for each triple of events $B$, $C$ and $D$ contained in $A$, $D$ disjoint of $B \cup C$,

$\succeq_A$ is a weak ordering,

$$B \succeq_A \emptyset, \ A >_A \emptyset$$

and

$$B \succeq_A C \iff B \cup D \succeq_A C \cup D.$$ 

Theorem 6. Given $C \in \Sigma$, $C \neq \emptyset$, the relation $\succeq_C$ is a qualitative probability on $(C, \{A \in \Sigma : A \subseteq C\})$. 


Proof. (1). $\geq_C$ is a weak preference follows from the fact that $\gtrsim_C$ is a weak preference.

(2). For each $B \subseteq C$, and constant acts $f$ and $g$ such that $f \nvl s g$; by eventwise monotonicity ($B \neq \emptyset$), or by definition of $\gtrsim_{\emptyset} (B = \emptyset)$, $f \gtrsim_B g$. Using the definition of $\gtrsim_B$, $fBg \gtrsim_B g$ is true. As $fBg \sim_{C \setminus B} g$, by sure-thing principle,

$$fBg \gtrsim_C g = f\emptyset g,$$

i.e., $B \geq_C \emptyset$.

(3). By nondegeneracy, there are constant acts $f$ and $g$ such that $f \succ_S g$, so, by eventwise monotonicity, $f \succ_C g$. Using the lemma after the definition of $\gtrsim_C$, $fCg \succ_C g = f\emptyset g$ is true. I.e.$C \succ_C \emptyset$.

(4). For $A, B, D \in \Sigma$ such that $A, B \subseteq C \subseteq D$ and $D \cap A = D \cap B = \emptyset$, and constant acts $f$ and $g$; if $C \setminus D$ is null at $C$ then $\gtrsim_D$ agree with $\gtrsim_C$ and, using the definition of $\gtrsim_D$,

$$fBg \sim_D fAg \quad f (B \cup D) g \sim_D f (A \cup D) g,$$

so, $B =_C A =_C \emptyset$ and $B \cup D =_C A \cup D$; but if $C \setminus D$ is non null at $C$, as, by the definition of $\gtrsim_{C \setminus D}$,

$$fBg \sim_{C \setminus D} f (B \cup D) g \quad fAg \sim_{C \setminus D} f (A \cup D) g$$

and, by sure-thing consistency and $C \setminus D$ is non null at $C$,

$$fBg \gtrsim_{C \setminus D} fAg \iff fBg \gtrsim_C fAg \quad f (B \cup D) g \gtrsim_{C \setminus D} f (A \cup D) g \iff f (B \cup D) g \gtrsim_C f (A \cup D) g,$$

then

$$fBg \gtrsim_C fAg \iff f (B \cup D) g \gtrsim_C f (A \cup D) g.$$  

\[\square\]

The lemma below defines the relation $\geq_D$ as the unique qualitative probability ($C, D \neq \emptyset$, by assumption) at any $C \subseteq D$ non null at $D$.

**Lemma 16** (weak comparative probability - event independence). *Given $A, B, C, D \in \Sigma$ such that $A, B \subseteq C \subseteq D$, $C$ non null at $D$,

$$B \geq_C A \iff B \geq_D A.$$*

*Proof. Given constant acts $f$ and $g$, by order-preserving lemma and $C$ non null at $D$,

$$fAg \gtrsim_C fBg \iff fAg \gtrsim_D fBg.$$*
Definition 5 (fine). A qualitative probability $\geq_A (A \neq \emptyset)$ is fine, iff, for each event $B$ contained in $A$ such that $B >_A \emptyset$, there exits a finite $\Sigma-$ measurable partition of $A$, $A_i$, satisfying $B >_A A_i$ for each $i$.

Lemma 17 (fineness). For each $C \neq \emptyset$, $\geq_C$ is fine.

Proof. Let $B \subseteq C$ be a non null event at $C$. By nondegeneracy, there exist constant acts $f$ and $g$ such that, $f \succ_S g$, and, by eventwise monotonicity, $f \succ_B g$. By order-preserving lemma and $B$ is non null at $C$, $fBg \succ_C fAg$ for each $i = 1, \ldots, n$. In other words, $B >_C A_i$ for each $i = 1, \ldots, n$ (i.e., $\geq_C$ is fine).

Definition 6 (tight). A qualitative probability $\geq_A (A \neq \emptyset)$ is tight, iff, $B =_A C$, for each pair of events $B$ and $C$ contained in $A$, satisfying:

$$B \cup D >_A C \quad \text{and} \quad C \cup E >_A B,$$

for each pair of events $D$ and $E$ contained in $A$, and such that,

$$D, E >_A \emptyset \quad \text{and} \quad B \cap D = \emptyset = C \cap E.$$

Lemma 18 (fineness and tightness). Given $A, B \subseteq C$, if $B >_C A$ then there exists a finite partition of $C$, $\{A_i\}_{i=1}^n \subseteq \Sigma$, such that $B >_C A \cup A_i$ for each $i = 1, \ldots, n$.

Proof. $B >_C A$ implies $fBg \succ_C fAg$ for some constant acts $f$ and $g$ such that $f \succ_S g$ (nondegeneracy), $C \neq \emptyset$ and $B$ non null at $C$. By small event continuity, there is a finite partition of $C$, $\{A_i\}_{i=1}^n \subseteq \Sigma$, such that $fBg \succ_C fA_i(fAg) = f(A \cup A_i)g$ for each $i = 1, \ldots, n$. In other words, $B >_C A \cup A_i$ for each $i = 1, \ldots, n$.

Several conclusions can be obtained from facts above (see theorem 3, page 37, [8], and page 195, [4]). Besides, properties provided above imply that, for each qualitative probability at each non empty event, there exists a unique finitely additive probability (fap) representing it (see theorem 14.2, page 195, and its proof in 198-199, [4]), with the caveat that, for any $C \subseteq D$ non null at $D$, the fap at $C$ is the fap at $D$ conditioned at $C$. Furthermore, these properties imply the existence of a SEU representation for each non empty event (see next appendix).

From the theorem proved above, for each $C \neq \emptyset$, $\geq_C$ is a qualitative probability. Using the lemma [HS] and theorem 4 (page 38, [8]), $\geq_C$ is fine and tight. By the corollary 1 (page 38, [8]), the only probability measure that almost agrees with $\geq_C$ (if an event is at least as probable
as another event, then the probability of the first event is greater than or equal to the probability of the second event), strictly agrees (an event is at least as probable as another event, iff, the probability of the first event is greater than or equal to the probability of the second event) with it. By the theorem 3 (page 37, [8]), there exists one and only one \( P_C \) on \( (C, \{A \in \Sigma : A \subseteq C\}) \) that almost agrees with \( \geq C \). Thus, there exists one and only one \( P_C \) on \( (C, \{A \in \Sigma : A \subseteq C\}) \) that strictly agrees with \( \geq C \).

In those lemmas and theorems other properties of \( P_C \) are provided. The most important is convex-valuedness, that is, for each \( B \subseteq C \) and \( \lambda \in [0, 1] \) there exists \( A \subseteq B \) such that \( P_C(A) = \lambda P_C(B) \). This property is used for defining specific partitions of an event with pre-established probabilities and outcomes.

Let \( \{P_A\}_{A \in \Sigma \setminus \{\emptyset\}} \) be the family of finitely additive probabilities from the theorem 14.2, [4], where \( P_A \) represents \( \geq A \), and if \( A \) is non null at \( B \supseteq A \), by weak comparative probability - event independence, and uniqueness, \( P_A \) is \( P_B \) conditioned on \( A \). Of course, if \( C \supseteq B \) and \( A \) is non null at \( C \), then \( B \) is non null at \( C \), \( P_A \) is \( P_C \) conditioned on \( A \) and \( P_B \) is \( P_C \) conditioned on \( B \). As it is known, for \( D \subseteq A \), \( P_B(D) = \frac{P_C(D)}{P_C(B)} \) and \( P_B(A) = \frac{P_C(A)}{P_C(B)} \), so, \( P_A(D) = \frac{P_C(D)}{P_C(A)} \). In the general case, qualitative probabilities are not the same necessarily, but for each \( A \subseteq B \subseteq C \), \( B \neq \emptyset \), \( P_C(A) = P_B(A) P_C(B) \).

The notion of relative null events defines an ordering in the events space with elements boundlessly smaller than other elements. When \( A \) is null and \( B \) is non null at \( C \), for any finite partitioning \( \{B_k\}_{k=1}^n \) of \( B \), by nullity, for some \( k \), \( B_k \) is non null at \( C \). I.e., \( A \) is “infinitely” smaller than \( B \), violating the Archimedean property (see [5]). As it is shown below, this non-Archimedean ordering defines an equivalence relation in the events space such that the quotient space is a linearly ordered set with the straightforward extension of the non-Archimedean ordering to this quotient space.

**Definition 7 (\(\triangleright\)).** Given events \( A \) and \( B \), \( A \triangleright B \) iff there exists a event \( C \supseteq A, B \) such that \( A \) is non null at \( C \), but \( B \) is null at \( C \).

**Lemma 19.** \(\triangleright\) is well defined.

**Proof.** Let \( A \) and \( B \) be events such that \( A \triangleright B \). Suppose, by absurd, there is a event \( D \supseteq A, B \) such that \( B \) is non null at \( D \). Then, for
some event $C \supseteq A, B$, $A$ is non null at $C$ and $B$ is null at $C$, but for some event $D \supseteq A, B$, $B$ is non null at $D$.

By nullity, $A \cup B$ is non null at $C$ because $A \subseteq A \cup B$. Besides, as $B$ is null at $C$, by lemma above, $B$ is null at $A \cup B$. Moreover, as $B$ is non null at $D$, by nullity, $B$ is non null at $A \cup B$, a contradiction. □

By definition, $A \gg \emptyset$ for each event $A \neq \emptyset$.

**Lemma 20.** Given events $A$ and $B$, $A \gg B$ iff $A$ is non null at $A \cup B$, but $B$ is null at $A \cup B$.

**Proof.** Let $A$ and $B$ be events such that $A \gg B$. Then, for some event $C \supseteq A, B$, $A$ is non null at $C$ and $B$ is null at $C$. By nullity, $A \cup B$ is non null at $C$ because $A \subseteq A \cup B$. Besides, as $B$ is null at $C$, by lemma above, $B$ is null at $A \cup B$. Moreover, as $A$ is non null at $C$, by nullity, $A$ is non null at $A \cup B$. □

**Lemma 21** (dominance - [5]). $\gg$ is irreflexive and transitive.

**Proof.** $A \gg A$ iff $A$ is non null at $A$, so $A \neq \emptyset$, but $A$ is null at $A$, so $A = \emptyset$, a contradiction.

$A \gg B \gg C$ iff $A$ is non null at $A \cup B$, but $B$ is null at $A \cup B$, and $B$ is non null at $B \cup C$, but $C$ is null at $B \cup C$. Then, $A, B \neq \emptyset$; and $A \cup B$ is non null at $A \cup B \cup C$, otherwise, by nullity, $C \gg B$. Besides, if $B \cup C$ is non null at $A \cup B \cup C$, by lemma above (contrapositive), $B$ is non null at $A \cup B \cup C$, and by nullity, $B$ is non null at $A \cup B$, a contradiction. Thus, $A \cup B$ is non null at $A \cup B \cup C$, and $B \cup C$ is null at $A \cup B \cup C$, implying, by nullity, $A \gg C$. □

From the non-Archimedean ordering, an equivalence relation on the events space can be derived, and from this equivalence relation, a partitioning of this space in equivalence classes of Archimedean-orderable events, resulting in a non-Archimedean linear ordering on the quocient space.

**Definition 8** ($\approx$). Given events $A$ and $B$, $A \approx B$ iff $(A \gg B)$ and $¬(B \gg A)$.

By definition, $A \approx \emptyset$ iff $A = \emptyset$. The class of the empty set is the trivial equivalence class.

**Lemma 22.** Given non trivial events $A$ and $B$, $A \approx B$ iff $A$ and $B$ are non null at $A \cup B$. 

Proof. By lemma above and definition of $\approx$, $A$ and $B$ are non null at $A \cup B$, or both are null at $A \cup B$. As $A$ and $B$ are non trivial, both are non null at $A \cup B$. \hfill \Box

**Lemma 23.** $\approx$ is an equivalence relation on $\Sigma$.

*Proof.* See [5]. \hfill \Box

**Lemma 24.** On $\Sigma/\approx$, $\gg$ is irreflexive, transitive and total.

*Proof.* Trivial. \hfill \Box

**Lemma 25** (weak comparative probability - extended event independence). Given $A, B, C, D \in \Sigma$ such that $A, B \subseteq C, D$, $C \approx D \gg \emptyset$,

\[
B \geq_C A \iff B \geq_{C \cup D} A \iff B \geq_D A.
\]

*Proof.* By weak comparative probability - event independence, using $A, B \subseteq C, D \subseteq C \cup D$. \hfill \Box

The lemma above defines the relation $\geq_{C \cup D}$ as the unique qualitative probability at $C$ and $D$ such that $C \approx D \gg \emptyset$. I.e., there is a unique (in the sense above) qualitative probability on each non trivial equivalence class in $\Sigma/\approx$.

The discussion above shows that an agent would consider non null, at least, each event of the equivalence class of a non empty event $A$, if $A$ was relevant for his/her decision making.
Definition 9 \((L_A^f)\). For each \(A \in \Sigma \setminus \{\emptyset\}\) and simple act \(f\), define a simple lottery on \(O\) as \(L_A^f = P_A \circ f|_A^{-1}\).

In short, it is proved that \(f \sim_A g\) if \(L_A^f = L_A^g\) for each couple of simple acts \(f\) and \(g\). The space of simple lotteries on \(O\) is endowed with a weak order \(\geq_A\) defined as \(L_A^f \geq_A L_A^g\) iff \(f \succ_A g\), for each couple of simple acts on \(A\), \(f\) and \(g\). It is proved that this lottery space endowed with \(\geq_A\) satisfies independence and Archimedean properties. The theorem 8.2 (page 107, [4]) provides the representation. The Bernoulli index is the same for every \(A \in \Sigma \setminus \{\emptyset\}\) if \(O\) is a mixture space satisfying weak forms of the independence and Archimedean properties, otherwise, each class has a Bernoulli index that preserves ordering for constant acts only.

Definition 10 (non redundancy). Given a simple lottery \(\sum_{k=1}^{m} q_k 1_{o_k}\), for some finite \(m \geq 1\), where \(1_{o_k}(o) = \begin{cases} 1, & o = o_k \\ 0, & o \neq o_k \end{cases}\), \(\sum_{k=1}^{m} q_k = 1, q_k \geq 0\) for \(k = 1, \ldots, m\), its non redundant representation is \(\sum_{k=1}^{n} p_k 1_{o_k}\), for some finite \(n \geq 1\), where \(n \leq m\), \(\sum_{k=1}^{n} p_k = 1, p_k > 0\) for \(k = 1, \ldots, n\), and \(o_k = o_l \implies k = l\).

Lemma 26. For each \(A \in \Sigma \setminus \{\emptyset\}\) and simple lottery \(L = \sum_{k=1}^{n} p_k 1_{o_k}\) (non redundancy), there exist a simple act \(f\) such that \(L_A^f = L\).

Proof. For each \(A \in \Sigma \setminus \{\emptyset\}\) there is a \(P_A\) such that, for each \(B \subseteq A\) and \(\lambda \in [0, 1]\), there is an event \(C \subseteq B\) satifying \(P_A(C) = \lambda P_B(B)\). Given that, choose \(A_1 \subseteq A\) such that \(P_A(A_1) = p_1 P_A(A)\), \(A_2 \subseteq A \setminus A_1\) such that \(P_A(A_2) = \frac{p_2}{1-p_1} P_A(A \setminus A_1)\), \(A_3 \subseteq A \setminus (A_1 \cup A_2)\) such that \(P_A(A_3) = \frac{p_3}{1-p_1-p_2} P_A(A \setminus (A_1 \cup A_2))\), and so forth. Defining \(f = o_1 A_1 \ldots o_n A_n h\), where \(h\) is an arbitrary simple act, a simple act satisfying \(L_A^f = L\) is obtained. \(\square\)

Lemma 27. For each \(A, B \in \Sigma\), \(B \subseteq A\), \(B\) non null at \(A\), simple lotteries \(L = \sum_{k=1}^{n} p_k 1_{o_k}\) and \(L' = \sum_{k=1}^{n'} p'_k 1_{o'_k}\) (non redundancy), simple acts \(f\) and \(g\) such that \(L_A^f = L\) and \(L_B^f = L'\), and \(\lambda \in (0, 1]\), there is \(C \subseteq B\) such that \(P_A(C) = \lambda P_B(B)\), \(L_C^f = L\) and \(L_C^g = L'\).
Proof. Take $C = \bigcup_{i=1,\ldots,n} C_{i,j}$ such that $C_{i,j} \subseteq f^{-1}(a_i) \cap g^{-1}(a'_j) \cap B$ and $P_A(C_{i,j}) = \lambda P_A(f^{-1}(a_i) \cap g^{-1}(a'_j) \cap B)$. It is straightforward that $P_A(C) = \lambda P_A(B)$, $L'_{i} = L$ and $L''_{i} = L'$. \hfill $\Box$

**Lemma 28.** For each $A \in \Sigma \setminus \{\emptyset\}$ and simple acts $f$ and $g$, $f \sim_{A} g$ if $L'_{A} = L''_{A}$.

**Proof.** For each $A \in \Sigma \setminus \{\emptyset\}$ and simple act $f$, define a simple lottery as $L'_{A} = P_A \circ f^{-1}$, which can be represented by $\sum_{k=1}^{n} p_k \lambda_{o_k}(\text{non redundancy})$.

**Claim 8.** For each $A \in \Sigma \setminus \{\emptyset\}$ and each couple of simple acts $f$ and $g$, $f \sim_{A} g$ if $L'_{A} = L''_{A} = 1$.

**Proof.** Observe that $f = oBf$ and $g = oCg$, where $B, C \subseteq A$ and $P_A(B) = P_A(C) = 1$. It is straightforward that $P_A(B \cap C) = 1$, so, $A \setminus (B \cap C)$ is null at $A$, consequently, $\succsim_{A}$ and $\succsim_{B \cap C}$ agree. By definition of $\succsim_{B \cap C}$, $f \sim_{B \cap C} g$, thus, for $n = 1$ the result follows. \hfill $\Box$

As induction hypothesis, assume that, for each $A \in \Sigma \setminus \{\emptyset\}$, each couple of simple acts $f$ and $g$, and each $m < n$, $f \sim_{A} g$ if $L'_{A} = L''_{A} = \sum_{k=1}^{m} p_k \lambda_{o_k}$.

If $L'_{A} = L''_{A} = \sum_{k=1}^{n} p_k \lambda_{o_k}$ for two simple acts $f$ and $g$, by the definition of $\succsim_{A}$, there are $A_i, B_i \subseteq A$ and $P_A(B_i) = P_A(C_i) = p_i$ for each $i = 1, \ldots, n$, and a simple act $h$, such that

$$f \sim_{A} A_0 B_1 \ldots A_n B_n h$$

$$g \sim_{A} A_0 C_1 \ldots A_n C_n h.$$ 

Now, take $D = C_1 \cap B_n$ and $E \subseteq C_n \setminus B_n$ such that $P_A(E) = P_A(D)$, and define the simple act

$$k = o_1 E o_n D g.$$ 

As $g \sim_{A \setminus (D \cup E)} k$, if $D$ is null at $A$ then, by nullity, $D \cup E$ is null at $A$, $\succsim_{A}$ agree with $\succsim_{A \setminus (D \cup E)}$, consequently, $g \sim_{A} k$.

However, if $D$ is non null at $A$ then, there are three possibilities:

1. $o_n \sim_{S} o_1$, so, by eventwise monotonicity and the definition of $\succsim_{E}$ and $\succsim_{D}$,

$$o_n E o_1 \sim_{E} o_n \sim_{E} o_1 \sim_{E} o_n D o_1$$

$$o_n D o_1 \sim_{D} o_n \sim_{D} o_1 \sim_{D} o_n E o_1$$
so, by sure-thing consistency,

\[ o_n E o_1 \sim_{D \cup E} o_n D o_1, \]

and, given that \( D \) is non null at \( A \), by order-preserving lemma

\[ g = (o_n E o_1) (D \cup E) g \sim_A (o_n D o_1) (D \cup E) g = k. \]

(2). \( o_n \succ_S o_1 \), so,

\[ o_n E o_1 \sim_{D \cup E} g \succ_{D \cup E} k \sim_{D \cup E} o_n D o_1 \Rightarrow E \succ_{D \cup E} D, \]

and, given that \( D \) is non null at \( A \), by weak comparative probability - event independence,

\[ E \succ_A D, \]

an absurd. Analogously for \( k \succ_{D \cup E} g \), consequently, \( k \sim_{D \cup E} g \), so, given that \( D \) is non null at \( A \), by order-preserving lemma

\[ k = k (D \cup E) g \sim_A g (D \cup E) g = g. \]

(3). \( o_1 \succ_S o_n \), it is analogous to (2).

Thus, \( g \sim_A k \) in each possible case. However, repeating this process for \( D' = C_2 \cap B_n \) and \( E' \subseteq (C_n \backslash B_n) \backslash E \) such that \( P_A (E') = P_A (D') \), and defining the simple act

\[ k' = o_2 E' o_n D' k, \]

it is obtained that \( k \sim_A k' \). So, after finite steps, it is obtained the simple act

\[ r \sim_A o_1 D_1 ... o_{n-1} D_{n-1} o_n B_n h, \]

which, by construction, is equivalent to \( g \) at \( A \), and, by induction hypothesis and the definition of \( \succ_A \backslash B_n \) and \( \succ_B \backslash n \), satisfies

\[ r \sim_A f, \]

\[ r \sim_B f, \]

so, by sure-thing consistency,

\[ r \sim_A f. \]

The lemma above has proved that, for each simple lottery \( L \), each \( A \in \Sigma \backslash \{\emptyset\} \) and each couple of simple acts \( f \) and \( g \),

\[ L^L_A = L^g_A = L \Rightarrow f \sim_A g. \]

**Definition 11 \((\succeq_A)\).** For each each \( A \in \Sigma \backslash \{\emptyset\} \) and lotteries \( L \) and \( L' \), \( L \succeq_A L' \) if there are acts \( f \) and \( g \), such that, \( L^f_A = L, L^g_A = L' \) and \( f \succeq_A g. \) (\( \equiv_A \) and \( \succeq_A \) are defined as usual).
As it is shown above, each simple lottery is generated by a simple act. Besides, it is shown above that two simple acts that generate the same lottery are equivalent. Thus, if a couple of simple acts \( f \) and \( g \) satisfy \( f \sim_A g \), \( L_f^A = L \) and \( L_g^A = L' \), then each couple of simple acts \( h \) and \( k \) satisfying \( L_h^A = L \) and \( L_k^A = L' \) must satisfy \( h \sim_A k \). I.e. \( \geq_A \) is well defined. Moreover, \( \geq_A \) is a weak preference.

**Lemma 29** (sure-thing consistency for lotteries). For each \( A, B \in \Sigma \) such that \( B \subseteq A \), and lotteries \( L_1 \) and \( L_2 \),

\[
L_1 \geq_B L_2 \quad \text{and} \quad L_1 \geq_{A\setminus B} L_2 \implies L_1 \geq_A L_2
\]

and

\[
B \text{ is non null at } A \implies (L_1 \geq_B L_2 \quad \text{and} \quad L_1 \geq_{A\setminus B} L_2 \implies L_1 \geq_A L_2).
\]

**Proof.** By lemmas 26, 27, 28 and sure-thing consistency, the result follows.

**Lemma 30** (order-preserving for lotteries). For each \( A, B \in \Sigma \) such that \( B \subseteq A \), \( B \) is non null at \( A \), and lotteries \( L \) and \( L' \),

\[
L \geq_B L'
\]

\[
\Leftrightarrow P_A(B) L + (1 - P_A(B)) L'' \geq_A P_A(B) L' + (1 - P_A(B)) L'' \quad \text{for each } L''
\]

**Proof.** Take acts \( f \) and \( g \) such that \( L_f^B = L \) and \( L_g^B = L' \). Observe that, for each act \( h \), \( L_f^{Bh} = P_A(B) L + (1 - P_A(B)) L_h^A \) and \( L_g^{Bh} = P_A(B) L' + (1 - P_A(B)) L_h^{A\setminus B} \). Now, by the order-preserving lemma, and using lemmas 26 and 28, the result follows.

Observe that, by order-preserving for lotteries, for each \( A \in \Sigma \), if, for some \( B \subseteq A \) such that \( B \) non null at \( A \), \( L \geq_B L' \), then, for each \( C \subseteq A \) such that \( C =_A B \), \( L \geq_C L' \). I.e., the ordering between lotteries at some non null event at \( A \) depend of the probability of this event at \( A \) and nothing else.

**Lemma 31.** For each \( A, B, C \in \Sigma \) such that \( B, C \subseteq A \), \( B =_A C >_A \emptyset \), and lotteries \( L_1 \) and \( L_2 \), \( L_1 \geq_B L_2 \iff L_1 \geq_C L_2 \).

**Proof.** By order-preserving for lotteries lemma,

\[
L_1 \geq_B L_2
\]

\[
\Leftrightarrow P_A(B) L_1 + (1 - P_A(B)) L_3 \geq_A P_A(B) L_2 + (1 - P_A(B)) L_3 \quad \text{for each } L_3
\]
\[ P_A(C) L_1 + (1 - P_A(C)) L_3 \geq_A P_A(C) L_2 + (1 - P_A(C)) L_3 \] for each \( L_3 \)

\[ L_1 \geq_C L_2. \]

**Lemma 32** (small event continuity for lotteries). For each each \( A \in \Sigma \setminus \{\emptyset\} \), and lotteries \( L_1, L_2 \) and \( L_3 \) such that \( L_1 \geq_A L_2 \) and \( L_3 \) constant, there exists a finite partition of \( A \), \( \{A_k\}_{k=1}^n \subseteq \Sigma \), such that

\[ L_1 \geq_A (1 - P_A(A_k)) L_2 + P_A(A_k) L_3 \]

and

\[ (1 - P_A(A_k)) L_1 + P_A(A_k) L_3 \geq_A L_2, \]

for each \( k = 1, \ldots, n \).

**Proof.** By lemma 26 and small event continuity, the result follows. □

Observe that versions of sure-thing consistency and small event continuity for lotteries are simpler than the original versions.

**Lemma 33** (Independence). For each each \( A \in \Sigma \setminus \{\emptyset\} \), each \( \rho \in (0, 1] \) and each pair of lotteries \( L_1 \) and \( L_2 \),

\[ L_1 \geq_A L_2 \iff \rho L_1 + (1 - \rho) L_3 \geq_A \rho L_2 + (1 - \rho) L_3 \] for each \( L_3 \).

**Proof.** First, observe that the lemma can be rewritten as: for each \( A, B \in \mathcal{S} \) such that \( B \subseteq A \) and \( B \) is non null at \( A \), \( \geq_A \) and \( \geq_B \) agree.

Given \( A, B \in \Sigma \) such that \( B \subseteq A \) and \( B \) is non null at \( A \), and lotteries \( L_1 \) and \( L_2 \), by lemma 31 whether \( L_1 \geq_A L_2 \iff L_1 \geq_B L_2 \) is true is independent of \( B \), although it is dependent on \( P_A(B) \). Thus, henceforth, the notation \( B(\alpha) \) is used for an incognito event in \( A \) with \( P_A(B(\alpha)) = \alpha \).

Given the discussion above, it shall be proved that, for each \( \alpha > 0 \), \( L_1 \) and \( L_2 \), \( L_1 \geq_A L_2 \iff L_1 \geq_{B(\alpha)} L_2 \).

If, for each \( \alpha > 0 \), \( L_1 \equiv_{B(\alpha)} L_2 \), there is nothing to prove, because \( \geq_A \) and \( \geq_{B(1)} \) agree, consequently, for each acts \( f \) and \( g \) such that \( L_A^f = L_1 \) and \( L_A^g = L_2 \), it follows that \( L_{B(1)}^f = L_1 \), \( L_{B(1)}^g = L_2 \) and \( f \sim_{B(1)} g \), so, \( f \sim_A g \) and \( L_1 \equiv_A L_2 \). Thus, henceforth, it is assumed that there exists \( \alpha_0 > 0 \) such that \( L_1 \geq_{B(\alpha_0)} L_2 \). (\( \geq \) is analogous)

If \( \alpha + \beta \leq 1 \), and \( B(\alpha) \) and \( B(\beta) \) are such that \( L_1 \geq_{B(\alpha)} L_2 \) and \( L_1 \geq_{B(\beta)} L_2 \) then, by lemma 31 they can be taken disjoint and, by
lemma \(26\), there are acts \(f\) and \(g\) such that \(L^f_{B(\alpha)} = L^f_{B(\beta)} = L_1\) and \(L^g_{B(\alpha)} = L^g_{B(\beta)} = L_2\), which imply that \(f \gtrdot_{B(\alpha)} g\) and \(f \gtrdot_{B(\beta)} g\), and by sure-thing consistency, \(f \gtrdot_{B(\alpha) \cup B(\beta)} g\). As, by construction, \(L^f_{B(\alpha) \cup B(\beta)} = L_1\), \(L^g_{B(\alpha) \cup B(\beta)} = L_2\) and \(B(\alpha) \cup B(\beta)\) is an event \(B(\alpha + \beta)\), \(L_1 \succeq_{B(\alpha + \beta)} L_2\).

If \(B(\alpha)\) is such that \(L_1 \preceq_{B(\alpha)} L_2\) then, taken a \(n\)-fold uniform partition of \(B(\alpha)\), \(\{B_j(\frac{m}{n})\}_{j=1}^n\), by lemma \(26\) there are acts \(f\) and \(g\) such that \(L^f_{B_j(\frac{m}{n})} = L_1\) and \(L^g_{B_j(\frac{m}{n})} = L_2\) for each \(j = 1, \ldots, n\), consequently, \(L^f_{B(\alpha)} = L_1\), \(L^g_{B(\alpha)} = L_2\), \(f \gtrdot_{B(\alpha)} g\) and, by sure-thing consistency, \(f \gtrdot_{B_j(\frac{m}{n})} g\) for some \(j\), i.e. \(L_1 \succeq_{B(\frac{m}{n})} L_2\).

Both paragraphs above and \(L_1 \preceq_{B(\alpha)} L_2\) imply that,

\[
L_1 \preceq_{B(q\alpha)} L_2 \quad \text{for each} \quad q \in \mathbb{Q} \cap \left(0, \frac{1}{\alpha_0}\right).
\]

As \(L_1\) and \(L_2\) are simple lotteries and \(L_1 \preceq_{B(\alpha)} L_2\), taking the most preferred outcome in \(\{w : L_1(w) + L_2(w) > 0\}\), \(v\), by lemma \(26\) there are acts \(f\) and \(g\) such that \(L^f_{B(\alpha)} = L_1\) and \(L^g_{B(\alpha)} = L_2\), and by small event continuity, given the constant act \(v\), there is a finite partition \(\{B_j(\alpha_{0j})\}_{j=1}^n\) of \(B(\alpha_0)\) such that

\[
vB_j(\alpha_{0j}) f \prec_{B(\alpha_0)} g
\]

for each \(j = 1, \ldots, n\).

Observe that, by definition of \(v\), using sure-thing consistency if it is needed, \(v \gtrdot_{B_j(\alpha_{0j})} g\) for each \(j = 1, \ldots, n\), and, by sure-thing consistency again, \(v \gtrdot_{B(\alpha_0)} g\). Besides, for each \(j\), \(\alpha_{0j} < \alpha_0\), otherwise, \(P_{B(\alpha_0)}(B_j(\alpha_{0j})) = 1\), consequently, \(\gtrdot_{B(\alpha_0)}\) and \(\gtrdot_{B_j(\alpha_{0j})}\) agree, and \(v \prec_{B_j(\alpha_{0j})} g\), an absurd. Moreover, for some \(j\), \(\alpha_{0j} > 0\), otherwise, \(\alpha_0 = 0\), an absurd. Thus, given a \(j\) such that \(0 < \alpha_{0j} < \alpha_0\), given the definition of \(v\), if \(f \gtrdot_{B(\alpha_0) \setminus B_j(\alpha_{0j})} g\) then \(vB_j(\alpha_{0j}) f \gtrdot_{B(\alpha_0) \setminus B_j(\alpha_{0j})} g\). As \(vB_j(\alpha_{0j}) f \gtrdot_{B(\alpha_0) \setminus B_j(\alpha_{0j})} g\), by sure-thing consistency, \(vB_j(\alpha_{0j}) f \gtrdot_{B(\alpha_0)} g\), an absurd, i.e.

\[
f \prec_{B(\alpha_0) \setminus B_j(\alpha_{0j})} g.
\]

Now, take any \(\beta \in (0, \alpha_{0j})\). So, for each \(B(\beta) \subseteq B_j(\alpha_{0j})\), given the definition of \(v\), using sure-thing consistency if it is needed, \(v \gtrdot_{B(\beta)} g\) and \(v \gtrdot_{B_j(\alpha_{0j}) \setminus B(\beta)} f\), so, \(vB(\beta) f \gtrdot_{B(\beta)} g\) and \(vB_j(\alpha_{0j}) f \gtrdot_{B(\alpha_0) \setminus B_j(\alpha_{0j}) \setminus B(\beta)} vB(\beta) f\), consequently, as \(vB_j(\alpha_{0j}) f \sim_{B(\alpha_0) \setminus B_j(\alpha_{0j})} vB(\beta) f\), by sure-thing consistency, \(vB_j(\alpha_{0j}) f \gtrdot_{B(\alpha_0) \setminus B(\beta)} vB(\beta) f\). Now, if \(vB_j(\alpha_{0j}) f\)
\[ \succsim_{B(\alpha_0) \setminus B(\beta)} g \text{ then, as } vB_j(\alpha_{0j}) f \succsim_{B(\beta)} g, \text{ by sure-thing consistency,} \\
vB_j(\alpha_{0j}) f \succsim_{B(\alpha_0)} g, \text{ an absurd, consequently,} \\
vB(\beta) f \prec_{B(\alpha_0) \setminus B(\beta)} vB_j(\alpha_{0j}) f \prec_{B(\alpha_0) \setminus B(\beta)} g, \]

\[ f \prec_{B(\alpha_0) \setminus B(\beta)} g. \]

From what was shown above, if \(0 < \beta < \alpha_{0j} < \alpha_0\) and \(L_1 \preceq_{B(\alpha_0)} L_2\), then, by lemma 26 and lemma 27,

\[ L_1 \preceq_{B(\alpha_0 - \beta)} L_2 \]

The steps above imply that, for each \(\beta \in (0, \alpha_{0j})\),

\[ L_1 \preceq_{B(q(\alpha_0 - \beta))} L_2 \text{ for each } q \in \mathbb{Q} \cap \left(0, \frac{1}{\alpha_0 - \beta}\right), \]

consequently,

\[ L_1 \preceq_{B(x)} L_2 \text{ for each } x \in (0, 1], \]

implying that \(L_1 \preceq_A L_2\). \(\Box\)

**Corollary 1.** For each \(A, B \in \Sigma \setminus \{\emptyset\}\) such that \(B \approx A, \preceq_A \text{ and } \preceq_B\) agree.

**Proof.** Take \(A \cup B\) and apply the independence lemma. \(\Box\)

**Lemma 34.** For each \(A \in \Sigma \setminus \{\emptyset\}\), if \(L_2 \succeq_A L_1\) and \(0 \leq \rho < \sigma \leq 1\), then \(\rho L_1 + (1 - \rho) L_2 \succeq_A \sigma L_1 + (1 - \sigma) L_2\).

**Proof.** It follows straightforwardly from the independence lemma (see page 72, [8]). It is just algebra. \(\Box\)

**Lemma 35** (Archimedean). For each \(A \in \Sigma \setminus \{\emptyset\}\), if \(L_2 \succeq_A L_1\) and \(L_2 \succeq_A L_3 \succeq_A L_1\) for some fixed lottery \(L_3\), then there is one and only one \(\rho \in [0, 1]\) such that \(L_3 \equiv_A \rho L_1 + (1 - \rho) L_2\).

**Proof.** It follows straightforwardly from lemmas 26 and 34 Dedekind cut and small event continuity (see page 73, [8]). The Fishburn’s version is more complete (see page 205, [4]). \(\Box\)

**Theorem 9.** For each \(A \in \Sigma \setminus \{\emptyset\}\), there is a real-valued function \(u_A\) on \(O\) satisfying

\[ L \succeq_A L' \iff \sum_{\alpha : L(\alpha) > 0} u_A(\alpha) L(\alpha) \geq \sum_{\alpha : L'(\alpha) > 0} u_A(\alpha) L'(\alpha) \]

for all simple lotteries \(L\) and \(L'\) on \(O\). Besides, \(u_A\) is unique up to a positive affine transformation.
Proof. For each $A \in \Sigma \setminus \{\emptyset\}$, as $\succeq_A$ is a weak preference satisfying independence and Archimedean properties, using the theorem 8.2, page 107, \[4\], the result follows. □

**Lemma 36.** For each $A, B \in \Sigma \setminus \{\emptyset\}$ such that $B \approx A$, $u_A$ and $u_B$ are related by a positive affine transformation.

Proof. It is straightforward from corollary \[1\] □

**Lemma 37.** For the set of constant lotteries, $\succeq_A$ and $\succeq_B$ agree for each $A, B \in \Sigma \setminus \{\emptyset\}$.

Proof. It is a straightforward consequence of eventwise monotonicity and the definition of $\succeq_A$ and $\succeq_B$. □

**Definition 12** (mixture space). A mixture space is a 2-uple $(X, m)$ with $X$ a non empty set and $m : [0, 1] \times X^2 \to X$ such that $m(1, x, y) = x$, $m(\alpha, x, y) = m(1 - \alpha, y, x)$ and $m(\alpha, m(\beta, x, y), y) = m(\alpha \beta, x, y)$.

**Theorem 10.** For a mixture space $(X, m)$ and a weak preference $\succeq$ on $X$, there exist a unique, up to affine transformations, real-valued function $v : X \to \mathbb{R}$ such that

$$\forall x, y \in X \ (x \succeq y \iff v(x) \geq v(y))$$

and

$$\forall x, y \in X \forall \alpha \in [0, 1] \ (v(m(\alpha, x, y)) = \alpha v(x) + (1 - \alpha) v(y)),$$

if, and only if,

(1) $\forall x, y, z \in X \forall \alpha \in (0, 1) \ (x \succ y \Rightarrow m(\alpha, x, z) \succeq m(\alpha, y, z))$

and

(2) $\forall x, y, z \in X \ (x \succ y \succ z \Rightarrow \exists \alpha, \beta \in (0, 1) \ (m(\alpha, x, z) \succ y \succ m(\beta, x, z)))$.

Proof. See theorem 8.4, page 112, \[4\]. □

**Lemma 38.** If $O$ is convex then the set of constant acts with $m(\alpha, o, o') = \alpha o + (1 - \alpha) o' \in O$ is a mixture space. Besides, if $\succ_S$ on this mixture space satisfies \[1\] and \[2\], then $\succeq_A$ and $\succeq_B$ agree for each $A, B \in \Sigma \setminus \{\emptyset\}$.

Proof. A convex combination of two constant acts $o$ and $o'$ is a constant act $\alpha o + (1 - \alpha) o'$ for some $\alpha \in [0, 1]$, so, the first part follows straightforwardly.
If $≿_S$ on this mixture space satisfies 1 and 2, then, by theorem 10, there exist a unique, up to affine transformations, real function $v : O \rightarrow \mathbb{R}$ such that
\[
\forall o, o' \in O (o \equiv_S o' \iff v(o) \geq v(o'))
\]
and
\[
\forall o, o' \in O \forall \alpha \in [0, 1] (v(\alpha o + (1 - \alpha) o') = \alpha v(o) + (1 - \alpha) v(o')).
\]

By lemma 37 and theorem 9, for each $A \in \Sigma \setminus \{\emptyset\}$, $u_A$ and $v$ are related by a positive affine transformation. Thus, the second part follows.

**Corollary 2.** Under the conditions of lemma 38, there is a real-valued function $v$ on $O$ such that, for each $A \in \Sigma \setminus \{\emptyset\}$,
\[
L \equiv_A L' \iff \sum_{o : L(o) > 0} v(o) L(o) \geq \sum_{o : L'(o) > 0} v(o) L'(o)
\]
for all simple lotteries $L$ and $L'$ on $O$. Besides, $v$ is unique up to a positive affine transformation.

**Proof.** It is straightforward from lemma 38.

It is shown above that, under the conditions of lemma 38, there exists a unique, up to affine transformations, event-independent weak preference for simple lotteries on $O$. I.e., an agent has a unique well defined ordering on simple lotteries and discrepancies between different classes are implied by different hypothetical conditional beliefs only. This idea is formalized below.

**Definition 13 ($\equiv$).** Under the conditions of lemma 38, for each pair of simple lotteries $L$ and $L'$ on $O$,
\[
L \equiv L' \iff L \equiv_A L',
\]
for some $A \in \Sigma \setminus \{\emptyset\}$.

**Lemma 39.** Under the conditions of lemma 38, for each pair of simple acts $f$ and $g$, and each $A \in \Sigma \setminus \{\emptyset\}$,
\[
f \equiv_A g
\]
\[
L_f^A \equiv L_g^A
\]
\[
\sum_{o : L_f^A(o) > 0} v(o) L_f^A(o) \geq \sum_{o : L_g^A(o) > 0} v(o) L_g^A(o)
\]
\[
\int_S v(f(s)) dP_A(s) \geq \int_S v(g(s)) dP_A(s)
\]
Proof. By definition of $\geq$ and corollary 2.

Anscombe and Aumann [2] assumes properties similar to lemma 38 but in this work there is not an explicit mixture space structure on non constant acts (i.e. Savage’s axioms are needed) and those properties on constant acts are assumed because they imply the uniqueness mentioned above.

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