Abstract—Time and fuel balance is an important topic in the dynamic tracking problem of unmanned systems. In this paper, we propose a quasi time-fuel optimal control strategy (QTFOC) to solve the dynamic tracking problem of a double integrator system, which is capable of multi-target switching tracking tasks, such as the multi-target strike of weapons and the rapid multi-target grabbing of robots on industrial assembly lines. Compared with the previous optimal control algorithms, the proposed controller retains the time-fuel optimal characteristic when switching among multiple dynamic targets, and overcomes the high-frequency oscillation problem by incorporating local linear control region and two nonlinear buffer areas. That is, when switching among multiple dynamic targets, the proposed control strategy enables the corresponding system to achieve desired transient performance and satisfactory steady-state performance simultaneously. In addition, the asymmetry of friction load is further explored, which affects the dynamic performance of the actual system. Extensive experiments based on the visual tracking turntable demonstrate the superiority and feasibility of the proposed method.

Note to Practitioners—This paper was motivated by the problem of multi-target switching tracking, such as the multi-target strike of weapons with vision sensors and the rapid multi-target grabbing of robots on industrial assembly lines. In recent years, various algorithms have been developed for trajectory planning and tracking. However, it is still challenging for unmanned servos to capture dynamic targets quickly due to the constraints of motion performance, energy, and computational power. In order to reduce the computational burden and obtain ideal response under various physical constraints, a quasi time-fuel optimal control strategy (QTFOC) with analytical solutions is proposed in this paper. First, the static target is extended to the dynamic target by further investigating the traditional time-fuel optimal control theory for double integrator system. Second, to deal with the oscillation problem caused by system disturbances, we incorporate buffer areas and local linear control region into the control algorithm, which makes the system more robust. Third, the system performance is further improved by analyzing the frictional load asymmetry. This algorithm requires small computational resources and can be implemented directly on microcontrollers such as STM32. The experimental results demonstrate the superiority of our proposed method in handling the multi-target switching tracking problem, which can also adjust the response speed weight, making the unmanned system perform better under different operating conditions. Furthermore, the proposed QTFOC can be extended to other unmanned systems, such as trajectory planning and motion control for unmanned vehicles and bionic robots.

Index Terms—Optimal control method, constrained unmanned system, tracking control algorithm, local linear control, multiple targets.

I. INTRODUCTION

Motion control and motion planning are closely related in the dynamic tracking problem of unmanned systems. In some studies, motion control and motion planning were considered together, which is one of the core research topics for unmanned control systems represented by robots [1], [2], [3]. It becomes particularly challenging when the ultimate performance of unmanned control systems is limited by practical factors such as mechanical structure, economic cost, and equipment size. Many researchers have contributed to the problem of motion planning under control performance constraints for unmanned systems [4], [5], [6], which enable unmanned systems to avoid the adverse effects of control saturation. However, the limitations of energy and controller processing power are also need to be taken into consideration. Therefore, in addition to ensuring the transient performance of the system under constraints, it is also necessary to adjust the control strategy according to the operating conditions. In other words, the control strategy designed for an unmanned system needs to find a balance among several performance indices.

For problems with performance indices requirements and control constraints, optimal control theory provides a promising research direction. There have been numerous efforts for exploring energy or fuel saving control strategies, which have been successfully applied to unmanned systems [7], [8], [9], [10]. Cao et al. [8] constructed an optimal mixed depletion mode control strategy by reasonably balancing the battery power and the engine power. Shen et al. [9] presented an optimization algorithm for energy-efficient deriving of electric vehicles, which translated perturbations and constraints (e.g., slope changes, speed limits, safety headway, and traffic lights) to interior-point conditions. Zhang et al. [10] proposed an in-vehicle energy optimal control (EOC) method to solve the problem of short driving distance per charge for electric vehicles, which improves the efficiency of the motor drive...
system. Similarly, an optimal energy management strategy was developed based on the Pontryagin’s Minimum Principle (PMP) in [11], which distributes the required propulsion power and the regenerative braking energy to the energy storage systems. Thus, the energy management strategy can minimize the electricity usage of the electric vehicles. In addition to exploring energy or fuel saving control strategies, it is also important to investigate the optimal transient response of unmanned systems. Bobrow et al. [12] pioneered the time-optimal trajectory planning strategy for industrial manipulators. Based on this, Kim et al. [13] investigated the near-minimum-time control of asymmetric rigid spacecraft and developed analytic closed-form solutions for the switch times and final time. In [14], the global time optimal control law for triple integrator with input saturation and full state constraints is considered and successfully applied to a robot control system. In order to address the challenges of human-computer interaction systems due to their complicated structure and the need of satisfying many constraints during task execution, a numerical solution of time-optimal control problems for variable-stiffness-actuated systems was proposed in [15]. Meanwhile, time-optimal control was also applied to power systems with model uncertainty [16]. These investigations focused on a single performance index, such as energy-optimal or time-optimal.

However, unmanned systems are usually supposed to balance rapidity and economy in the control process. For example, an unmanned platform with limited fuel requires a longer standby time when cruising and a faster response speed when tracking a target. This trade-off among the minimum-time control and the fuel saving control is named as the time-fuel optimal control problem [17]. In the 1960s, the time-fuel optimal feedback control law of the ideal double integrator was systematically derived by Athans and Falb [18]. Subsequent researchers found that some undesirable phenomena (i.e., chattering and overshoot [19]) may occur if the control strategy proposed in [17] was applied to a perturbed double-integrator system. This is due to the fact that the time-fuel optimal control strategy is sensitive to disturbance, unmodeled dynamics, and parameter variations [20].

Based on the above discussion, our work aims to investigate the trade-off between time and fuel consumption when a double integrator system tracks dynamic targets. A quasi-time-fuel optimal control strategy (QTFOC) is proposed based on Pontryagin’s minimum principle. It can not only adjust the time-fuel weight according to different operation conditions and extend the working time of the system, but also have the time-optimal characteristic at the position far away from the target. Furthermore, local linear control region and two buffer areas are incorporated to address the highly oscillatory problem caused by system perturbations. In this way, the proposed algorithm is capable of multi-target switching tracking tasks, such as the multi-target strike of weapons and the rapid multi-target grabbing of robots on industrial assembly lines.

A. Related Work

Time-fuel optimal control problem refers to the optimal control problem penalizing both time and fuel consumption. This kind of problem plays an important role in control field owing to its ability to reduce fuel consumption and deliver desired response speed in finite time [21]. Initially, similar to fuel-optimal control, time-fuel-optimal control was applied to spaceflight. The pioneer Athans [22] suggested that fuel minimization studies for spacecraft are imperfect since these methods do not impose a limit on the response time, which may allow formulations of this nature to admit very large response times, making it impossible to achieve the desired operation of the spacecraft. In order to optimally compromise the response time of the spacecraft with the fuel consumption, Sarles [17] proposed a time-fuel optimal controller, which solves the problem of repeatedly pointing an orbiting spacecraft to allow the exposure of high resolution weather photographs. The minimization of the control function (fuel) integral extends the useful life of spacecrafts in orbit, while the minimization of the response time allows the device to expose pictures faster. Due to the excellent performance of time-fuel optimal control, the method is increasingly applied in other domains. In [23], the concept of averaged dynamics was introduced for robotic arm control to design the proposed NMTF controller, in which the dynamics is continuously updated at each sampling interval using the available dynamic information of the current and final states. Zhang et al. [24] applied a time-fuel optimal control approach to model the human control strategies in a simplified rendezvous and docking (RVD) task, which provides a useful guide for cognitive modeling of RVD task. In [25], the authors extended the applicability of time-fuel optimal control and proposed a method to determine the switching instances of the time-fuel optimal control for a second order linear time invariant (LTI) system, which computes the optimal time instances by solving a hierarchy of semidefinite relaxations of an equivalent generalized moment problem. Furthermore, based on the findings of [25], the time-fuel optimal control was also combined with the consensus tracking problem for multi-agent system to derive a Nash equilibrium feedback strategy, which solves the two-player time-fuel pursuit evasion game problem [21]. The results show that by applying these local pursuer policies, the multi-agent system can consensus in finite time with reduced fuel consumption.

However, Pontryagin’s minimum principle states that the time-fuel optimal control strategy is necessarily “bang-off-bang” in nature, i.e., the control input switches among maximal, zero, and minimal values. Therefore, the system often generates highly oscillatory behavior in the neighborhood of the switching function and the target state due to the switching delay, which degrades the performance and reduces the serviceable lifetime of the control system [19]. To tackle with this issue, [26] proposed to introduce commutation memory into the time-fuel optimal switching curve to eliminate the system chattering, which relies on using the controller to store past control inputs. More broadly, there are some approaches dedicated to solving the highly oscillatory behavior caused by system uncertainty under the purely time optimal constraint, which provide new insights.

In the time optimal control framework, a popular improvement method to suppress highly oscillatory behavior is the...
proximate time-optimal control strategy [27]. This approach starts with a time-optimal controller, and then switches to a local controller when the system output is close to a given target [28]. Choi et al. [29] further proposed a novel damping scheduling PTOS (DSPTOS), which can accelerate the system dynamic response while ensuring smaller overshoot and oscillation by setting different damping in the deceleration and stabilization phases. In [30], the local linear control strategy of PTOS was replaced with a nonlinear control and stabilization phases. In [31], the local linear control region and two nonlinear buffer areas are introduced to address the oscillatory issue caused by high frequency switching of control input. The essence of this improvement is to increase a small amount of cost in exchange for better robustness of the control strategy.

3) The ultimate performance of the actual system is further explored by considering the asymmetry of the friction load.

4) Extensive experiments using a visual tracking system demonstrate the effectiveness and feasibility of the proposed method.

The remainder of this paper is organized as follows. In Section II, the preliminary and problem formulation are provided. In Section III, we present the quasi time-fuel optimal control strategy. In Section IV, the influence of the asymmetry on the control algorithm when tracking dynamic targets is further explored.

The main contributions and novelties can be summarized as follows.

1) We propose a quasi time-fuel optimal control strategy for tracking dynamic targets. It extends the previously studied time-fuel optimal control strategy from approaching static targets to tracking dynamic targets, which provides an important basis for unmanned double integrator systems to balance working periods and response speed.

2) Local linear control region and two nonlinear buffer areas are introduced to address the oscillatory issue caused by high frequency switching of control input. The essence of this improvement is to increase a small amount of cost in exchange for better robustness of the control strategy.

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The remainder of this paper is organized as follows.

II. PRELIMINARIES AND PROBLEM FORMULATION

We investigate a visual tracking system in the form of a double integrator, which includes smart sensors, a fuel judgment mechanism and a quasi time-fuel optimal controller (see Fig. 1). The red dashed box in Fig. 1 is the focus of our work. For this constrained and fuel-limited double integrator system, the system constrains, transient response, and fuel consumption are all need to be taken into consideration for better control performance.

B. Original Contributions

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The model of the system is given by

\[ \dot{x}_1 = x_2, \]
\[ x_2 = u, \quad |u| \leq M. \]  

(1)

The performance index can be defined as

\[ J(u) = \int_{t_0}^{t_f} [\lambda + |u(t)|]dt, \]

(2)

where \( t_0 \) and \( t_f \) represent the initial time and terminal time respectively. \( \lambda \) indicates the weight of time and \( |\lambda| \in \mathbb{R} \) (\( \lambda > 0 \)).

According to Pontryagin’s minimum principle, the time-fuel optimal control for the double integrator system is as follows [18]:

\[ u(t) = \begin{cases} 
+M, & p_2(t) < -1, \\
0, & -1 < p_2(t) < 1, \\
-M, & p_2(t) > 1, \\
-v(t) \cdot \text{sign}[p_2(t)], & |p_2(t)| = 1, 
\end{cases} \]

(3)

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where \( v(t) \in [0, M] \) is an integrable function and \( p^T(t) = [p_1(t), p_2(t)] \) is the costate of the system. The optimality condition stipulates that the time derivative of costate \( p^T(t) \) satisfies:

\[
\begin{align*}
\dot{p}_1(t) &= 0, \\
\dot{p}_2(t) &= -p_1(t),
\end{align*}
\]

and

\[
\begin{align*}
p_1(t) &= c_1, \\
p_2(t) &= -c_1 t + c_2.
\end{align*}
\]

**Definition 1:** If the tracking error between the system and the dynamic target is bounded under the control strategy, and the tracking error will reduce to 0 when the dynamic target velocity \( x_{2d}(t) = 0 \), then the system is said to be able to keep up with the dynamic target.

**Remark 1:** Since \( p_2(t) \) is a linear function of time \( t \), the possible optimal control input sequences include: \( \{0, [+M], [-M], [0, +M], [0, -M], [+M, 0], [-M, 0], [0, +M, 0], [-M, 0, +M] \) \. Moreover, when \( u(t) \) switches from \( +M \) to 0 or from 0 to \( +M \), \( p_2(t) = 1 \); when \( u(t) \) switches from \( -M \) to 0 or from 0 to \( -M \), \( p_2(t) = -1 \). According to Definition 1, the system can maintain its current state to save fuel when the error between the system and the target is bounded and constant. Therefore, the control sequences \( \{+M, 0\} \) and \( \{-M, 0\} \) are feasible when \( x_{2d}(t) \neq 0 \).

Note that \( x_{1d}(t) \) and \( x_{2d}(t) \) represent the position and velocity of the target, respectively. The system cannot accurately predict the future motion state of the target because the target state will change suddenly when the target is switched. In order to track dynamic targets, the system will update the current motion state of the target in real time by additional sensors ([35], [36]).

Based on the above statement, the following parts will explain how to get the switch plane of time-fuel optimal control for the dynamic target step by step, and further improvements are proposed to solve the problems of asymmetrical friction load and system oscillation. Finally, a practical control strategy is obtained.

**III. QUASI TIME-FUEL OPTIMAL CONTROL STRATEGY**

In this section, we explore the analytical solution of quasi time-fuel optimal control strategy based on dynamic terminal target set. Analysis and practical experiments show that time-fuel optimal control (TFOC) [24], [25], [26] is prone to chattering when the system is located in the target neighborhood or converges to the target in the vicinity of the switching curves. Therefore, we design a local linear control strategy near the target, which avoids the saturation of the control input in the local linear control region and keeps the system state from exceeding the boundary due to the switching of the control law. That is, it prevents the system state from oscillating back and forth between the time-fuel optimal region and the local linear control region. In the vicinity of the switching curves, we design buffer areas in which the control input is gradually changed from 0 to \( M \) or \( -M \) depending on the distance the system state from switching curves. This allows the system to approach the target smoothly along the switching curves without frequently switching of control inputs due to system perturbations.

We will then show the structure of the QTOC which can be used to improve the above problem, and then analyze how to obtain this structure in the subsequent content. The QTOC for the double integrator system (1)-(3) is:

1) \( u(t) = M \), iff \( (x_1, x_2) \in \Omega_1 \), where

\[
\Omega_1 = \{ (x_1, x_2) | Q_9 > 0 \cap \{ \Omega_{11} \cup \{ \Omega_{12} \cap \Omega_{13} \} \} \}.
\]

The sets \( \Omega_{11}, \Omega_{12} \) and \( \Omega_{13} \) are defined as follows:

\[
\Omega_{11} = \{ (x_1, x_2) | Q_5 \geq 0 \cap Q_5 \leq 0 \},
\]

\[
\Omega_{12} = \{ (x_1, x_2) | Q_6 < 0 \cap Q_6 \geq 0 \},
\]

\[
\Omega_{13} = \{ (x_1, x_2) | (Q_2 + Q_3 \geq 0) \cap Q_6 \geq 0 \cup x_2 \geq 0 \}.
\]

2) \( u(t) = M \min \{ 1, Q_6 \} \), iff \( (x_1, x_2) \in \Omega_2 \), where

\[
\Omega_2 = \{ (x_1, x_2) | Q_9 > 0 \cap \{ \Omega_{21} \cup \Omega_{22} \} \}.
\]

The sets \( \Omega_{21} \) and \( \Omega_{22} \) are defined as follows:

\[
\Omega_{21} = \{ (x_1, x_2) | Q_5 > 0 \cap Q_5 \leq 0 \},
\]

\[
\Omega_{22} = \{ (x_1, x_2) | Q_2 + Q_3 \geq 0 \cap x_2 < 0 \}.
\]

3) \( u(t) = -M \), iff \( (x_1, x_2) \in \Omega_3 \), where

\[
\Omega_3 = \{ (x_1, x_2) | Q_9 < 0 \cap \{ \Omega_{31} \cup \{ \Omega_{32} \cap \Omega_{33} \} \} \}.
\]

The sets \( \Omega_{31}, \Omega_{32} \) and \( \Omega_{33} \) are defined as follows:

\[
\Omega_{31} = \{ (x_1, x_2) | Q_5 > 0 \cap Q_5 \geq 0 \},
\]

\[
\Omega_{32} = \{ (x_1, x_2) | Q_2 < 0 \cap Q_2 \geq 0 \},
\]

\[
\Omega_{33} = \{ (x_1, x_2) | (Q_1 - Q_2 < 0) \cup Q_2 < 0 \cup x_2 < 0 \}.
\]

4) \( u(t) = -M \min \{ 1, Q_6 \} \), iff \( (x_1, x_2) \in \Omega_4 \), where

\[
\Omega_4 = \{ (x_1, x_2) | Q_9 < 0 \cap \{ \Omega_{41} \cup \Omega_{42} \} \}.
\]

The sets \( \Omega_{41} \) and \( \Omega_{42} \) are defined as follows:

\[
\Omega_{41} = \{ (x_1, x_2) | Q_5 \geq 0 \cap Q_5 > 0 \},
\]

\[
\Omega_{42} = \{ (x_1, x_2) | (Q_1 - Q_2 > 0) \cup x_2 < 0 \}.
\]

5) \( u(t) = 0 \), iff \( (x_1, x_2) \in \Omega_5 \), where

\[
\Omega_5 = \{ (x_1, x_2) | Q_9 < 0 \cap \{ \Omega_{51} \cup \Omega_{52} \} \}.
\]

The sets \( \Omega_{51} \) and \( \Omega_{52} \) are defined as follows:

\[
\Omega_{51} = \{ (x_1, x_2) | Q_5 < 0 \cap Q_5 > 0 \},
\]

\[
\Omega_{52} = \{ (x_1, x_2) | Q_2 = 0 \cap Q_2 < 0 \cup Q_6 < 0 \}.
\]

6) \( u(t) = -M \left( x_1(t) - x_{1d}(t) \right) + \frac{\delta}{2M} (x_2(t) - x_{2d}(t)) \right) + a_d(t) \), iff \( (x_1, x_2) \in \Omega_6 \), where

\[
\Omega_6 = \{ (x_1, x_2) | Q_6 \leq 0 \}.
\]

Note that \( \delta \geq 0 \) and \( \xi \) is the width of the buffer areas which are used to eliminate the highly oscillatory behavior of the system. \( a_d(t) \) denotes the acceleration of the target. The different regions \( \Omega_1 \to \Omega_6 \) of \( \mathbb{R}^2 \) labeled in Fig. 2 are divided by switch curves \( (Q_1 \to Q_6) \) and the switch curves are described mathematically in Table I. Without loss
of generality, the value of $\delta$ is set to 0 in Fig. 2 and the light blue line with arrow indicates the direction of the system state change.

The local linear control region $\Omega_6$ is determined by $Q_9 \leq 0$. The region of $Q_9 \leq 0$ is an ellipse and the expression of $Q_9$ can be shown as follows:

$$Q_9 = a_1(x_1(t) - x_{1d}(t))^2 + a_2(x_2(t) - x_{2d}(t))^2 + 2a_2(x_1(t) - x_{1d}(t))(x_2(t) - x_{2d}(t)) - k_F,$$

where

$$k_F = \frac{Ma_1(1 - \sqrt{M}a_2 + \frac{Ma_2}{4} - (\sqrt{M}a_2 - \frac{Ma_2}{4})^2}{4(a_1 - \sqrt{M}a_2 + \frac{Ma_2}{4})^2},$$

$$a_1 = \frac{M + 1}{4\sqrt{M}}, \quad a_2 = \frac{1}{2M}, \quad a_4 = \frac{1 + M}{4M},$$

and $M$ is the upper limit of the control constraint as shown in (1).

To make the description clearer, we will explain how to obtain the boundaries of other regions ($\Omega_6$-$\Omega_4$) without considering local linear regions ($\Omega_6$). The $\Omega_5$ is very critical in this paper, so we start the analysis of QTFOC from this region where $u(t) = 0$.

1. The control input sequence in Remark 1 will be discussed separately to obtain the switching boundary of $\Omega_5$. The initial state of the system is set to $x_0 = [x_{10}, x_{20}]$.

**Definition 2:** Suppose there is a control input sequence $u_n(t) = \phi_1, \phi_2, \cdots, \phi_i, \phi_{i+1}, \cdots, \phi_n$, $n \in \mathbb{N}^+$, $i \in (1, 2, \cdots, n)$, then $u(t) = \{\phi_1, \phi_{i+1}, \cdots, \phi_{n-1}, \phi_n\}$ is called a control input sub-sequence of $u_n(t)$.

Note that the control input of the time-fuel optimal control problem is only related to the current state of the system and the target. Therefore, $u_1(t)$ is a control input sub-sequence of $u_2(t)$, which means that $u_1(t)$ can be regarded as a special case of $u_2(t)$. According to the possible control input sequences of the system discussed in Remark 1, it is obvious that $\{0\}$ is the control input sub-sequence of $\{+M, 0\}$ and $\{-M, 0\}$, $\{+M\}$ and $\{0, +M\}$ are the control input sub-sequences of $\{-M, 0, +M\}$, $\{-M\}$ and $\{0, -M\}$ are the control input sub-sequences of $\{+M, 0, -M\}$. So for the optimal control input sequences in Remark 1, we only need to analyze $\{+M, 0\}$, $\{-M, 0\}$, $\{-M, 0, +M\}$, $\{+M, 0, -M\}$.

The current state of the dynamic target is $[x_{1d}(t), x_{2d}(t)]$. When $u(t) = \{+M, 0, -M\}$, we assume that the switching instants are $t_{1i}$ and $t_{2i}$. According to (18), the system state at $t_{2i}$ will satisfy the following equation:

$$x_1(t_{2i}) = -\frac{1}{2M}x_2^2(t_{2i}) + x_{1d}(t_{2i}) + \frac{1}{2M}x_2^2(t_{2i})^2.$$  (14)

Since the control strategy is updated through real-time feedback, we can rewrite $t_{2i}$ to $t$. Therefore, the curve $Q_1$, and (14) have the same form, which is called the switching boundary function from $u(t) = 0$ to $u(t) = -M$ in this situation. Meanwhile, during $[t_{1i}, t_{2i}]$, $u(t) = 0$ so that

$$x_1(t_{2i}) = x_1(t_{1i}) + x_1(t_{1i})[t_{2i} - t_{1i}].$$  (15)

Substituting (15) into (14) yields:

$$x_1(t_{1i}) + x_1(t_{1i})[t_{2i} - t_{1i}] + \frac{1}{2M}x_2^2(t_{1i}) = x_{1d}(t_{2i}) + \frac{1}{2M}x_2^2(t_{2i}).$$  (16)

In order to obtain the relationship between $c_1$ and $c_2$, we consider the singular values of $p_2(t)$, i.e., $p_2(t) = \pm 1$, which implies that

$$-c_1t_{1i} + c_2 = -1,$$

$$-c_1t_{2i} + c_2 = 1.$$  (17)

Moreover, the following formula can be obtained:

$$[t_{2i} - t_{1i}] = -\frac{2}{c_1}. $$  (18)

In this case $u(t) = 0$ and the Hamiltonian is equal to zero for the steady state system, which means that

$$\lambda + c_1x_2(t_{1i}) = 0.$$  (19)

Further, $x_2(t_{2i}) \neq 0$ so that

$$[t_{2i} - t_{1i}] = \frac{2x_2(t_{1i})}{\lambda}.$$  (20)
Substituting (20) into (16) yields:
\[
x_1(t_1^-) = -\frac{\lambda + 4M}{2M}\phi_2\hat{t}_1^- + x_{1d}(t_1^-) + \frac{1}{2M}\phi_2^2\hat{t}_1^-.
\] (21)

Since the control strategy is updated through real-time feedback, we can rewrite \( t_1^- \) to \( t \). Therefore, the curve \( Q_1 \) and (21) have the same form, which is called the switching boundary function from \( u(t) = +M \) to \( u(t) = 0 \).

In addition, when the control input sequence of the system is \( \{+M, 0, -M\} \), the control process \( u(t) = -M \) can decelerate the system after \( t_1^- \). Thus, it is obvious that \( x_2(t_1^-) > x_{2d}(t_1^-) \). Refer to (21), the constraint on \( x_1(t_1^-) \) is given by
\[
x_1(t_1^-) < -\frac{\lambda + 4M}{2M}\phi_2\hat{t}_1^- + x_{1d}(t_1^-) + \frac{1}{2M}\phi_2^2\hat{t}_1^- = x_{1d}(t_1^-) - \frac{2\phi_2^2\hat{t}_1^-}{\lambda}.
\] (22)

When \( u(t) = \{-M, 0, +M\} \), we assume that the switching instants are \( t_1^+ \) and \( t_2^+ \). It is completely similar to \( u(t) = \{+M, 0, -M\} \) so that we can make the following analysis concisely. The switching boundary function from \( u(t) = 0 \) to \( u(t) = +M \) in this situation is that
\[
x_1(t_2^+) = \frac{1}{2M}\phi_2^2\hat{t}_2^+ + x_{1d}(t_2^+) - \frac{1}{2M}\phi_2^2\hat{t}_2^+.
\] (23)

Since the control strategy is updated through real-time feedback, we can rewrite \( t_2^+ \) to \( t \). Therefore, the curve \( Q_2 \) and (23) have the same form in this case. The switching boundary function from \( u(t) = -M \) to \( u(t) = 0 \) is that
\[
x_1(t_1^+) = \frac{\lambda + 4M}{2M}\phi_2^2\hat{t}_1^+ + x_{1d}(t_1^+) - \frac{1}{2M}\phi_2^2\hat{t}_1^+ = \frac{\lambda + 4M}{2M}\phi_2^2\hat{t}_1^+ + x_{1d}(t_1^+) - \frac{1}{2M}\phi_2^2\hat{t}_1^+.
\] (24)

which is the switch curve \( Q_4 \). Moreover, similar to (22),
\[
x_1(t_1^+) > x_{1d}(t_1^+) + \frac{2\phi_2^2\hat{t}_1^+}{\lambda}.
\] (25)

Next, we will analyze the situation where the control input sequence is \( \{+M, 0\} \) and \( \{-M, 0\} \).

**Lemma 1:** The optimal control sequence of plant (1) is \( \{+M, 0\} \) only if \( x_{2d}(t) > 0 \). The optimal control sequence of plant (1) is \( \{-M, 0\} \) only if \( x_{2d}(t) < 0 \).

**Proof:** For a steady state system, when \( u(t) = 0 \), the Hamiltonian of plant (1) with a performance index of (2) will satisfies
\[
H = \lambda + c_1x_2(t) = 0,
\] (26)
because the system reaches the target state \( [x_{1d}(t) \ x_{2d}(t)]^T \) under the control of \( u(t) = 0 \). Therefore, since \( \lambda > 0 \), \( c_1 > 0 \) \( (c_1 < 0) \) when \( x_{2d}(t) < 0 \) \( (x_{2d}(t) > 0) \). Moreover, from (3) and (5), the control input sequence \( u(t) = \{+M, 0\} \) requires \( p_2(t) \) to increase with time, which means \( c_1 < 0 \). The control input sequence \( u(t) = \{-M, 0\} \) requires \( p_2(t) \) to decrease with time, which means \( c_1 > 0 \). It indicates that \( u(t) = \{+M, 0\} \) only if \( x_{2d}(t) > 0 \) and \( u(t) = \{-M, 0\} \) only if \( x_{2d}(t) < 0 \). This completes the proof. □

**Remark 2:** According to Definition 1, the control input sequence \( u(t) = [\phi_1, \phi_{i+1}, \ldots, \phi_n] \) \( (i.e. \ \phi_n = 0) \) allows the system to keep up with the dynamic target only if \( x_{2d}(t) \neq 0 \) and
\[
\begin{align*}
x_2(t) &= x_{2d}(t) > 0, & \text{or} & \quad x_2(t) &= x_{2d}(t) < 0, \\
x_{1d}(t) &> x_{1n-1}, & & x_{1d}(t) < x_{1n-1}.
\end{align*}
\]

where \( x_{1n-1} \) represents the position of the system at the end of \( u(t) = \phi_{n-1} \).

**Theorem 1:** If the control input sequence \( \{+M, 0\} \) is applied to plant (1), there will be a switching instant \( t_1^- \) in control process and the following conditions will be satisfied:
\[
\begin{align*}
x_1(t_1^-) &\geq x_{1d}(t_1^-) - \frac{2\phi_2^2\hat{t}_1^-}{\lambda}, \\
x_2(t_1^-) &= x_{2d}(t_1^-).
\end{align*}
\] (27)

**Proof:** Referring to Lemma 1, it is obvious that the control input sequence \( u(t) = \{+M, 0\} \) means \( x_{2d}(t) > 0 \). Therefore,
\[
u(t) = \begin{cases} +M, & t \in [0, t_1^-], \\
0, & t \in [t_1^-, x_{1d}(t_1^-) - x_{1d}(t_1^-)] + t_1^- \end{cases}
\] (28)

Since the control input \( u(t) \) of the system is equal to 0 before reaching the target state, refer to (4) and (26), the costate \( p(t) = [p_1(t) \ p_2(t)] \) will satisfy
\[
\begin{align*}
p_1(t) &= \frac{-\lambda}{x_2(t)}, \\
p_2(t) &= \frac{\lambda}{x_2(t)}t + c_2,
\end{align*}
\] (29)

According to Remark 1, \( p_2(t_1^-) \) will satisfy
\[
p_2(t_1^-) = \frac{\lambda}{x_2(t_1^-)}t_1^- + c_2 = -1.
\] (30)

Note that, according to Remark 2, \( x_2(t_1^-) = x_{2d}(t_1^-) > 0 \), thus we can get the value of \( c_2 \):
\[
c_2 = -1 - \frac{\lambda}{x_{2d}(t_1^-)}t_1^- < -1.
\] (31)

Furthermore, when \( t \in [t_1^-, \frac{x_{2d}(t_1^-)-x_{1d}(t_1^-)}{x_{2d}(t_1^-)} + t_1^-] \), \( p_2(t) \leq 1 \) always holds. And the equality \( p_2(t) = 1 \) holds if \( t = \frac{x_{2d}(t_1^-)-x_{1d}(t_1^-)}{x_{2d}(t_1^-)} + t_1^- \). It means that the following equation needs to be satisfied:
\[
\frac{\lambda}{x_{2d}(t_1^-)}t_1^- + \frac{\lambda}{x_{2d}(t_1^-)}x_{1d}(t_1^-) - x_{1d}(t_1^-) + 1 - \frac{\lambda}{x_{2d}(t_1^-)}t_1^- \leq 1.
\] (32)

Evidently,
\[
x_1(t_1^-) \geq x_{1d}(t_1^-) - \frac{2\phi_2^2\hat{t}_1^-}{\lambda},
\] (33)
which completes the proof. □

**Theorem 2:** If the control input sequence \( \{-M, 0\} \) is applied to plant (1), there will be a switching instant \( t_1^- \) in control process and the following conditions will be satisfied:
\[
\begin{align*}
x_1(t_1^+) &\leq x_{1d}(t_1^+) + \frac{2\phi_2^2\hat{t}_1^+}{\lambda}, \\
x_2(t_1^+) &= x_{2d}(t_1^+).
\end{align*}
\] (34)
Proof: Referring to Lemma 1, it is obvious that the control input sequence \( u(t) = [-M, 0] \) means \( x_{2d}(t_1^+) < 0 \). Similar to the proof of Theorem 1, we have

\[
u(t) = \begin{cases}  
-M, & t \in [0, t_1^+], \\
0, & t \in [t_1^+, \frac{x_{1d}(t_1^+) - x_1(t_1^+)}{x_{2d}(t_1^+)} + t_1^+].
\end{cases}
\] (35)

According to (29) and Remark 1, \( p_2(t_1^+) \) will satisfy

\[ p_2(t_1^+) = \frac{\lambda}{x_2(t_1^+)} + c_2 = 1. \] (36)

Note that, according to Remark 2, \( x_2(t_1^+) = x_{2d}(t_1^+) < 0 \), thus we can get the value of \( c_2 \):

\[ c_2 = 1 - \frac{\lambda}{x_{2d}(t_1^+)} > 1. \] (37)

Furthermore, when \( t \in [t_1^+, \frac{x_{1d}(t_1^+) - x_1(t_1^+)}{x_{2d}(t_1^+)}, t_1^+] \), \( p_2(t) \) always holds if \( t = \frac{x_{1d}(t_1^+) - x_1(t_1^+)}{x_{2d}(t_1^+)} + t_1^+ \). It means that the following equation needs to be satisfied:

\[ \frac{\dot{x}_1}{x_{2d}(t_1^+)} + \frac{2(x_{1d}(t_1^+) - x_1(t_1^+))}{x_{2d}(t_1^+)} + 1 - \frac{\lambda}{x_{2d}(t_1^+)} \geq -1. \] (38)

Evidently,

\[ x_1(t_1^+) \leq x_{1d}(t_1^+) + \frac{2x_2^2(t_1^+)}{\lambda}, \] (39)

which completes the proof. \( \square \)

On the basis of (14)-(22) and Theorem 1, we can find that there are two situations in which \( u \) switches from \( +M \) to 0. When the control input sequence is \( [+M, 0, -M] \), the system state at the switching instant \( t_1^+ \) satisfies the formula (21) and (22), thus the switch curve is \( Q_3 \) in Fig. 2. When the control input sequence is \( [+M, 0] \), the switching time \( t_1^+ \) satisfies the formula (27), thus the switch curve is \( Q_5 \) which intersects \( Q_3 \) at the point \( \left( x_{1d}(t_1^+) - \frac{2x_2^2(t_1^+)}{\lambda}, x_{2d}(t_1^+) \right) \) and intersects \( Q_1 \) at the point of target state. On the basis of (23)-(25) and Theorem 2, we can also find that there are two situations in which \( u \) switches from \( -M \) to 0 and the switch curve will be \( Q_4 \) and \( Q_6 \) as shown in Fig. 2. In summary, the region \( Q_5 \) where \( u = 0 \) can be expressed as the form in (10). Note that increasing the value of \( \delta \) will move the switching curves \( Q_5 \) and \( Q_6 \), so that the system can reach the target faster.

Another difference between QTFOC and TFOC for static targets is that there exist a new plane area which can be named as transient loop. It is equivalent to \( \{Q_1 < 0 \cap Q_2 > 0\} \). The system cannot reach the target state in the transient loop because \( |x_2(t)| < |x_{2d}(t)| \) so that we can choose \( x_1(t) \) as the first control target and \( x_2(t) \) as the auxiliary control target, hence the control input in transient loop can be chosen as \( u(t) = +M \) if \( Q_S < 0 \) and \( u(t) = -M \) if \( Q_S > 0 \). Combining the analysis on \( Q_5 \), we can naturally obtain \( Q_1 \) and \( Q_3 \).

To solve the problem of chattering and overshoot when applying the time-fuel optimal control strategy to a perturbed double integrator [19], we design the local linear control region (11). The parameters in (13) are chosen to ensure that \( u(t) \) does not saturate in \( \Omega_6 \) and that the system state converges to the target point after entering \( \Omega_6 \). Specific analysis is as follows:

In accordance with the form of local linear controller (11) and (12), \( \Omega_6 \) is an ellipse centered at point \( (x_{1d}(t), x_{2d}(t)) \). If the acceleration of the target is exceeds the performance constraint of the system for a period of time, the state of the system will escape from the local linear control region \( (\Omega_6) \) and enter the peripheral control region \( (\Omega_1 - \Omega_3) \), which causes the system to chase the target with the extreme performance. In this case, the error between the system and the target will gradually increase, which does not need to be considered in the local linear control region. Therefore, we will investigate how to keep the system from switching repeatedly between the \( \Omega_6 \) and other control regions when the target acceleration does not change drastically and is smaller than the control constraint of the system. Two conditions need to be met to achieve this goal:

1) The control input never saturates for \( (x_1, x_2) \in \Omega_6 \).
2) There exists a Lyapunov function for the system when \( (x_1, x_2) \in \Omega_6 \) which has a negative definite derivative when the system state is on the boundary of \( \Omega_6 \).

Notice that the \( \Omega_6 \) designed in this paper consists of \( Q_9 \leq 0 \), where the formula of \( Q_9 \) is shown in (12). Based on the properties of ellipses, in order to make \( x_2(t) - x_{2d}(t) \) in \( Q_9 \) = 0 have nonzero real solutions when \( x_1(t) = x_{1d}(t) \), we should let \( a_4 \) and \( k_F \) have the same sign. For the sake of discussion, we stipulate that \( a_4 > 0 \) which is also required in the next step when selecting the Lyapunov function, then we have \( k_F > 0 \). Rewrite the local linear control law in (11) as

\[ u(t) = -M \left[ x_{1e}(t) + \frac{2}{\sqrt{M}} v_2(t) \right] + a_4(t), \] (40)

where \( x_{1e}(t) = x_1(t) - x_{1d}(t) \) and \( x_{2e}(t) = x_2(t) - x_{2d}(t) \). To ensure that the control input does not saturate on the boundary of the ellipse and within the ellipse, \( \Omega_6 \) must be contained within the region of \( |u(t)| \leq M \), i.e., neither of the two systems of equations represented by (41) has distinct real roots.

\[
\begin{bmatrix}
-M & \frac{2}{\sqrt{M}} v_2(t) \\
a_1 x_{1e}(t) + a_4 v_2(t) + 2 a_2 x_{1e}(t) v_2(t) - k_F = 0
\end{bmatrix}
\] (41)

Thus, when \( \Delta \leq 0 \) and \( M > 0 \), \( k_F \) needs to satisfy the following constraint: If \( a_4(t) < 0 \),

\[
k_F \leq \frac{\varphi_2}{4} \left[ M a_4 (a_1 - \sqrt{M} a_2 + \frac{M a_4}{4}) - (\sqrt{M} a_2 - \frac{M a_4}{2})^2 \right],
\] (42)

otherwise

\[
k_F \leq \frac{\varphi_2}{4} \left[ M a_4 (a_1 - \sqrt{M} a_2 + \frac{M a_4}{4}) - (\sqrt{M} a_2 - \frac{M a_4}{2})^2 \right],
\] (43)

where \( \varphi_1 = 1 - \frac{a_4(t)}{M} \) and \( \varphi_2 = 1 + \frac{a_4(t)}{M} \).
From the properties of ellipses, it follows that the lengths of the long and short axes of the ellipse described by $Q_0$ are positively related to $k_F$, and the area $S$ of $Q_0$ is also positively related to $k_F$. Furthermore, when $a_d(t) \to 0$, $k_F \to 0$ and $S \to 0$. That is, when the acceleration of the target tends to the ultimate performance of the system, it is almost impossible to eliminate the error if the initial states of the target and the system are different. In this case, the system needs to operate in the time-fuel optimal control region rather than in the local linear control region.

Note that the parameter $k_F$ designed by the local linear controller in (13) satisfies the condition of (42) and (43), so the control input never saturates for $x \in Q_0$.

On the basis of (12), the region of the local linear controller $Q_0$ can be rewritten as follows:

$$x_e^T P x_e \leq k_F,$$

where

$$x_e = \begin{bmatrix} x_{1e} \\ x_{2e} \end{bmatrix}, \quad P = \begin{bmatrix} a_1 & a_2 \\ a_2 & a_4 \end{bmatrix}.$$ (45)

Let $a_1 > 0$, $a_2 > 0$ and $a_1 a_4 - a_2^2 > 0$, then $P$ is a positive definite matrix. A Lyapunov function that can show the stability of the system in $Q_0$ is

$$V = x_e^T P x_e > 0.$$ (46)

Combining (40) and differentiating (46) gives

$$\dot{V} = x_e^T (A_{cl}^T P + P A_{cl}) x_e,$$ (47)

where

$$A_{cl} = \begin{bmatrix} 0 & 1 \\ -M & -2\sqrt{M} \end{bmatrix}.$$ (48)

Choosing $\Psi$ as the unit matrix $I_{2 \times 2}$ and letting $\Psi = -(A_{cl}^T P + P A_{cl})$, we can obtain

$$a_1 = \frac{M + S}{4\sqrt{M}}, \quad a_2 = \frac{1}{2M}, \quad a_4 = \frac{1 + M}{4M\sqrt{M}}.$$ (49)

which are the parameters selected in (13). At this point, the Lyapunov function $V$ has a negative definite derivative when the state is on the boundary of $Q_0$, so that the system is stable to the region of $Q_0$.

However, although the local linear controller (11) can eliminate the highly oscillatory behavior of the system near the target, it can also be triggered by the feedback delay of the target state near the switch curves. In other words, the system will oscillate due to the changes of control input near the switch curves $Q_1$ and $Q_2$ when $x_{2d}(t) < 0$ and $x_{2d}(t) < 0$ respectively. Therefore, we design a buffer area near $Q_1$ with a width of $\zeta$. When the system state leaves $Q_1$ to the right plane, the value of $u(t)$ gradually increases, and when it exceeds this buffer, the value of $u(t)$ becomes $M$. This is the region $Q_2$ in (7) and in the same way we can obtain $Q_3$ in (9).

So far, all the regions in Fig. 2 have been analyzed. Next, we will consider the actual servo system which is named as visual tracking system, and Section IV will focus on the impact of Coulomb friction on the control strategy.

IV. THE INFLUENCE OF COULOMB FRICTION LOAD ASYMMETRY ON QTOC

This section mainly studies how to maximize the limit performance of the system under the condition of asymmetric load. The following model is not only applicable to the visual tracking system driven by a servo motor, but also to a variety of actuators.

$$J\alpha = J\dot{\theta} = T_d - B\dot{\theta} - K\theta - T_f,$$ (50)

where $J$ denotes the moment of inertia of the visual tracking system, $\theta$ and $\alpha$ represent the rotation angle and rotation angular acceleration, respectively. $B$ is the damping coefficient (including mechanical damping, electromagnetic damping, etc.). $K$ denotes the elastic coefficient of the system. $T_f$ indicates the Coulomb friction moment, and $T_d$ indicates the electromagnetic torque generated by the servo motor.

We focus on a visual tracking system (see Fig. 3) for tracking dynamic targets, which has a small rotation angle range (i.e., $[-120^\circ, +120^\circ]$) and a low tracking speed, but requires a high angular acceleration and often works at the maximum output torque. Therefore, compared with the Coulomb friction load and inertial load, the damping load proportional to the rotation speed and the elastic load proportional to the rotation angle are small enough to be ignored in the following analysis. Note that the Coulomb friction loads present asymmetric characteristics relative to the motion process, which will adversely affect the acceleration performance of the system but benefit the deceleration performance.

On the basis of the above analysis, we can transform (50) into a form similar to a double integrator, the model of visual servo system is rewritten as $\dot{x}_1 = x_2$, $\dot{x}_2 = \alpha$, where $x_1$ and $x_2$ are equivalent to $\theta$ and $\dot{\theta}$, respectively. $T_d$ ranges in $[-JM - T_f, JM + T_f]$. Therefore, $\alpha \in [-M, M]$ when the system is accelerating and $\alpha \in [-M - T_f, M + T_f]$ when the system is slowing down. That is, when the system is decelerating and deviates from the switch curves $Q_1$ or $Q_2$, $T_f$ contributes to deceleration. In order to be consistent with the previous analysis, we rewrite $\alpha$ to $u(t)$ here. When $Q_1 - \zeta \leq 0$, the allowable maximum value of $u(t)$ can be modified to $u_{\text{max}}(t) = M + K_1$. When $Q_1 - \zeta > 0$, the allowable minimum value of $u(t)$ can be modified to $u_{\text{min}}(t) = -M - K_1$. The value of $K_1$ depends on the asymmetric $T_f$ in (50). This is an engineering
method, but it often plays a satisfactory role in the actual application process. In addition, this method only works when the system is decelerating, so a buffer area with width $K_\xi$ is set, similar to that in (7) and (9).

According to the above discussion, the value of $u(t)$ proposed in (7) and (9) is redefined as follows:

$$1) \quad u(t) = M' \min \left\{ 1, \frac{\Omega}{\xi} \right\}, \text{iff } (x_1, x_2) \in \Omega_3,$$

$$2) \quad u(t) = -M' \min \left\{ 1, \frac{\Omega}{\xi} \right\}, \text{iff } (x_1, x_2) \in \Omega_4,$$

where $M' = M + K_1$ and $\xi^* = \frac{M + K_1 u}{M'}$.

V. EXPERIMENTAL VALIDATION

In this section, the performance of QTFOC is evaluated using an experimental setup (see Fig. 3), which consists of a vision sensor, a LiDAR, and an execution motor, to show the efficacy and superiority of the control strategy and verify the theoretical results. Considering the vertical scanning field of LiDAR is limited, when the vision camera and LiDAR perform fusion detection, the vision camera and LiDAR can obtain high-altitude target information through the rotation of the turntable. Therefore, the turntable needs to perform large-angle adjustments based on the given information to aim at targets beyond the vertical scanning field of perception system, and then track the movement of targets in a small range.

On the basis of Section IV, the model of the visual tracking system is shown below:

$$\begin{align*}
\dot{\theta}(t) &= \omega(t), \\
\dot{\omega}(t) &= \frac{1}{J} (T_d(t) - T_f).
\end{align*}$$

$T_f$ will exhibit the load asymmetry. The relevant parameters of the experimental setup are presented in Table II. From the Section IV and Table II, we can know that $\omega \in [-359.2, 359.2]$ when the system is accelerating and $\omega \in [-410.9, 410.9]$ when the system is slowing down.

In the aspect of feedback communication, all the necessary data are sent to controller every 3 ms by the serial communication port of the STM32 with the baud rate being 115200. For the actual systems, the sensor signal is usually accompanied by noise and sometimes there is no available sensor to measure the output derivatives of the system. Therefore, kalman filter [37] and tracking differentiator [38] are needed in this case. This is only a signal processing mechanism independent of the algorithm proposed in this paper, so it will not be described here. Readers interested in this aspect can refer to the above-mentioned papers. On the other side, for the STM32 microcontroller, the computational complexity should be taken into consideration. Unlike classical process control applications, the sampling time of the fast dynamic systems (embedded systems) is typically of millisecond order. The research results in [39] showed that the classical Dynamic Matrix Control (DMC) and Generalized Predictive Control (GPC) algorithms is unsuitable for STM32, since calculations last longer than the sampling period. The proposed QTFOC can obtain an analytical solution, and the calculation time required for this algorithm is very short. Therefore, this is beneficial to reduce the computational burden of STM32.

In addition, the system shown in Fig. 3 is powered by lithium batteries. Significantly, the capacity of a lithium battery refers to the limited amount of charge released when the battery is fully discharged, that is, the amount of charge that can be released is fixed when a battery is fully charged. Since the intensity of the current is the quantity of charge which passes in a conductor per unit of time, for an servo motor powered by a lithium battery, the integral of $|u|$ with respect to time, which is proportional to the current intensity, can reflect the power loss of the lithium battery during the control process. That is, for the experimental setup shown in Fig. 3, it is feasible to use the method proposed in this paper to trade off the “fuel” and the response speed. Similar researches on the systems powered by lithium batteries have been introduced in [40].

After completing the above preparations, we conduct following experiments. We will let the experimental setup switch and track among three dynamic targets, which are named “Object A”, “Object B”, and “Object C”. The trajectories of “Object A”, “Object B”, and “Object C” are $5\sin(\pi t) + 10\sin(2t)$, $50 + 5\sin(\pi t) + 10\sin(2t)$, and $-10 + 5\sin(\pi t) + 10\sin(2t)$ respectively. For the convenience of description, we establish an experimental scene named “CASE-F”. It is described as following: In 0 to 5 seconds, the visual tracking system is required to track “Object A”. In 5-10 seconds, the visual tracking system is required to track “Object B”. After 10 seconds, the visual tracking system is required to track “Object C”. Therefore, “CASE-F” can evaluate the speed of the system switching among multiple dynamic targets and the dynamic tracking performance of the system under the corresponding control strategy.

In order to better demonstrate the effect of the control strategy, we compare the performance of QTFOC for dynamic targets with TFOC and proportional-integral-derivative (PID) control law. The PID control law is improved by compensator, which can resist the effects of differential noise. The transfer function is as follows:

$$G(s) = P + \frac{1}{s} + \frac{D}{1 + N\frac{s}{\tau}}.$$  (52)
The control input of the PID controller should satisfy $|u| \leq M$, where $M = 359.2$. After actual experimental testing, two sets of representative PID parameters are selected:

1. **PID1**: $K_p = 45.0$, $K_i = 3.51$, $K_d = 9.45$, $N_1 = 10$.
2. **PID2**: $K_p = 24.3$, $K_i = 2.04$, $K_d = 11.51$, $N_2 = 10$.

The initial states of the visual tracking system are given as $\theta(0) = 45^\circ$, $\omega(0) = 0^\circ/s$ and the experimental setup is required to track “CASE-F”. The time weight $\lambda$ is set to
\( \lambda = 10000. \) The experimental performance of QTFOC, TFOC, PID1, and PID2 are shown in Fig. 4-7, which demonstrate the superiority of QTFOC in dynamic target tracking. Moreover, as shown in Fig. 6, the control input of TFOC will switch frequently because of the velocity mismatch and feedback delays. Fig. 7 verifies that the frequent switching of the TFOC control input (see Fig. 6) results in a larger accumulation of \(|x|\), that is, TFOC will cause the system’s lithium battery to consume more power with the same \( \lambda \) value. In summary, TFOC is inapplicable when tracking dynamic targets, and QTFOC is feasible for this. On the other side, compared with QTFOC, when two sets of PID controllers with different parameters are used to track “CASE-F”, the switching and tracking performance will be worse due to input saturation constraints (see Fig. 4-6). In addition, from Fig. 4 and Fig. 5, we can observe that in the process of target switching, QTFOC eliminates the large position error caused by target switching faster than PID1 and PID2, and thus quickly recovers the tracking of the new target.

The influence of different \( \lambda \) on system tracking performance and fuel consumption are shown in Fig. 8-11. The initial states are given as \( \theta(0) = 45^\circ \), \( \omega(0) = 0^\circ/s \), and the experimental setup are also required to track “CASE-F”. It can be seen from Fig. 8 and Fig. 9 that increasing the value of \( \lambda \) will increase the response speed of the system, which is beneficial to quickly reduce the tracking error when switching targets. Combining the above results in Fig. 10 and Fig. 11, we can conclude that when increasing the value of \( \lambda \) to obtain a faster response speed, QTFOC will make the system’s lithium battery consume more power, which will reduce the endurance of the unmanned system to a certain extent. In other words, the continuous operating time of the unmanned system can be increased by reducing the value of \( \lambda \).

Actually, the actuator executes the delayed signal from the controller instead of the real-time signal. The experiments in Fig. 4-11 were all performed with a 5 ms control period, which means that there was at least a 5 ms interval between the change of the actuator state and the output of the corresponding control signal from the controller. Since the maximum sampling period of the sensor is 3 ms, we choose 5 ms as the control period after extensive practical tests, which make the system operate stably. In order to illustrate the effect of the controller’s signal delay on the controller performance, we tested the controller performance at different control periods. In Fig. 12-15, the control period of the controller is set to 5 ms, 10 ms, 20 ms, and 50 ms respectively. The system is also required to track “CASE-F”.

Fig. 12 and Fig. 13 illustrate that when the system control period is increased from 5 ms to 10 ms, the system still tracks the target quite well, but the overshoot increases slightly. When the system control period is further increased to 20 ms, it can be observed from Fig. 14 that control the input contains chattering, while Fig. 13 shows that the tracking performance of the system becomes worse. When the control period of the system is set to 50 ms, Fig. 12 and Fig. 13 indicate that the system is no longer able to track the target effectively because of the excessive delay, and the system fluctuates back and forth around the target. Fig. 14 and Fig. 15 also demonstrate that the control input to the system switches repeatedly between maximum and minimum values in this case, thus the cumulative cost of the control input increases sharply.

VI. Conclusion

In this work, we studied the dynamic target tracking problem in the time-fuel optimal control framework. The proposed QTFOC was implemented with a visual tracking system. The experimental results shown in Fig. 4-9 demonstrate that our method achieves superior performance in dynamic target tracking problems, especially the the multi-target switching tracking problems. What’s more, QTFOC can also adjust the weight of response speed, boosting the performance of the unmanned system under different operation conditions. A further step is to investigate how to exploit the funnel control strategy to improve the local linear control strategy, which is a non-trivial extension of the current work. Future studies will also include the compensation of controller signal delay and trajectory prediction of dynamic targets.

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