Lie-algebraic Connections Between Two Classes of Risk-sensitive Performance Criteria for Linear Quantum Stochastic Systems

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Abstract
This paper is concerned with the original risk-sensitive performance criterion for quantum stochastic systems and its recent quadratic-exponential counterpart. These functionals are of different structure because of the noncommutativity of quantum variables and have their own useful features such as tractability of evolution equations and robustness properties. We discuss a Lie algebraic connection between these two classes of cost functionals for open quantum harmonic oscillators using an apparatus of complex Hamiltonian kernels and symplectic factorizations. These results are aimed to extend useful properties from one of the classes of risk-sensitive costs to the other and develop state-space equations for computation and optimization of these criteria in quantum robust control and filtering problems.

1 Introduction
Open quantum harmonic oscillators (OQHOs) [15], governed by linear quantum stochastic differential equations (QSDEs), constitute an important application of the Hudson-Parthasarathy calculus [18] 29 to the modelling of quantum systems interacting with external bosonic fields. The class of OQHOs is closed under concatenation, and their interconnection into a quantum feedback network [16] 22 (for example, a closed-loop system formed from a plant and controller, both modelled as OQHOs) is also an OQHO whose parameters are expressed in terms of the subsystems.

Quantum control and filtering problems for such systems [3, 4, 5, 15, 20, 21, 22, 27, 44, 45, 50] aim to achieve certain dynamic properties for quantum plants by using measurement-based feedback with classical controllers and filters or coherent (measurement-free) feedback involving direct or field-mediated connection [52] with other quantum systems. The performance criteria combine qualitative requirements (such as stability) with optimality principles in the form of the minimization of cost functionals. In particular, quantum linear quadratic Gaussian (LQG) control and filtering [13, 23, 27] are concerned with minimising the mean square values of the closed-loop system variables, similarly to the classical LQG control and filtering problems [11, 24].

The quantum risk-sensitive performance criterion, originated in [20, 21] for measurement-based quantum control and filtering problems (see also [9, 51]), employs the mean square value of a time-ordered exponential (TOE) driven by a function of the system variables. This cost functional imposes an exponential penalty on the system variables and involves their multi-point quantum states at different moments of time. Since, even in the Gaussian case [8, 30], such states do not reduce to classical joint probability distributions because of the noncommutativity of quantum variables, the quantum risk-sensitive cost differs from its classical predecessors [5] 19 49. Nevertheless, this cost functional allows for tractable evolution equations and an appropriate modification of the information state techniques in application to the measurement-based quantum control settings.

The structure of the classical risk-sensitive performance criteria (as the exponential moment of a quadratic function of the system variables over a time interval) has recently been adopted in a quadratic-exponential functional (QEF) [46]. Despite a more complicated evolution (compared to the original quantum risk-sensitive cost), the QEF leads to upper bounds [46] for the tail distribution of the corresponding quadratic function of the quantum system variables in the spirit of the large deviations theory [10, 41]. Moreover, the QEF gives rise to guaranteed upper bounds [47] for the worst-case value of the quadratic cost when the actual quantum state may depart from its nominal model, with the departure being described in terms of the quantum relative entropy [29, 51]. The role of the QEF in the quantum robust performance estimates is similar to the connections between risk-sensitive control and minimax LQG control for classical stochastic systems with a relative entropy description of statistical uncertainty in the driving noise [11, 33, 36, 37].

The useful properties can be extended from one of the risk-sensitive costs to the other through bilateral links between these two classes of quantum performance criteria, which is the main theme of the present paper. To this end, we develop a continuous-time analogue of the results of [48], which leads to a Lie-algebraic correspondence between the QEF and the original TOE-based quantum risk-sensitive cost driven by a quadratic function of the system variables. An important ingredient of this connection is an isomorphism between the Lie algebra of quadratic functions of the system variables of the OQHO with complex symmetric kernels and the Lie algebra of complex Hamiltonian kernels, which are
infinitesimal generators of complex symplectic kernels (all these kernels are matrix-valued).

The paper is organised as follows. Section 2 specifies the class of linear quantum stochastic systems under consideration. Section 3 describes the original quantum risk-sensitive cost and its quadratic-exponential counterpart. Section 4 represents a class of quadratic functions of system variables using complex symmetric matrix-valued measures. Section 5 discusses the Lie-algebraic correspondence between two classes of TOE-based and quadratic-exponential functions of system variables. Section 6 presents an isomorphism of this class to a Lie algebra of complex Hamiltonian kernels. Section 7 establishes a Lie-algebraic correspondence between two classes of TOE-based and quadratic-exponential measures for the QEF and TOE-based criteria driven by quadratic functions of the current system variables. Section 8 makes concluding remarks.

2 Open quantum harmonic oscillators

We consider an OQHO with (an even number of) dynamic variables \( X_1, \ldots, X_n \) (for example, pairs of conjugate quantum mechanical positions and momenta \([39]\)). These system variables are time-varying self-adjoint operators on (a dense domain of) a complex separable Hilbert space \( \mathcal{H} \) and are assembled into a vector \( X := (X_k)_{1 \leq k \leq n} \) (vectors are organised as columns, and the time arguments are often omitted for brevity). They satisfy the canonical commutation relations (CCRs) \( [W_u, W_v] = \Theta^{uT} \Theta^{vT} \), for all \( u, v \in \mathbb{R}^n \), where \( \Theta := \sqrt{-1} \mathbb{I} \) is the imaginary unit, and \( \Theta^T = \Theta \). Here, \( \Theta \) is a nonsingular real antisymmetric matrix specifying the matrix

\[
[X, X^T] := ([X_j, X_k])_{1 \leq j, k \leq n} = 2i(\Theta \otimes \mathcal{J}_\mathcal{H})
\]

of commutators \( [X_j, X_k] := X_j X_k - X_k X_j \) as the Heisenberg infinitesimal form (on a dense domain in \( \mathcal{H} \)) for the Weyl CCRs, with \( \otimes \) the tensor product, and \( \mathcal{J}_\mathcal{H} \) the identity operator on \( \mathcal{H} \). The matrix \( \Theta \otimes \mathcal{J}_\mathcal{H} \) will be identified with \( \Theta \). The system variables of the OQHO evolve in time according to a linear QSDE

\[
dX = AXdr + BdBW,
\]

where \( A \in \mathbb{R}^{n \times n} \), \( B \in \mathbb{R}^{n \times m} \) are constant matrices whose structure is clarified below. This QSDE is driven by the vector \( W := ([W_{jk})_{1 \leq j, k \leq m} \) of an even number \( m \) of quantum Wiener processes \( W_1, \ldots, W_m \) which are time-varying self-adjoint operators on a symmetric Fock space \( \mathcal{F} \) \([29, 31]\). These operators represent the external bosonic fields and have a complex positive semi-definite Hermitian Ito matrix \( \Omega := I_m + J \in \mathbb{C}^{m \times m} \), so that \( dW_dW^T = \Omega dt \), with \( I_m \), the identity matrix of order \( m \). Its imaginary part \( J := \begin{bmatrix} 0 & -I_m \\ I_m & 0 \end{bmatrix} \) is an orthogonal real antisymmetric matrix of order \( m \) (so that \( J^2 = -I_m \)), which specifies CCRs for the quantum Wiener processes as \( [W(s), W(t)^T] = 2i\min(s, t)J \) for all \( s, t \geq 0 \). The matrices \( A, B \) in (2.2) are not arbitrary and satisfy the physical realizability (PR) condition \([23, 40]\)

\[
A \Theta + \Theta A^T + B B^T = 0,
\]

which pertains to the preservation of the CCRs (2.1) in time. Such matrices are parameterized as \( A = 2\Theta(K + M^T/2) \), \( B = 2\Theta M^T \) in terms of the energy and coupling matrices \( K = K^T \in \mathbb{R}^{n \times n} \), \( M \in \mathbb{R}^{n \times m} \), which specify the quadratic system Hamiltonian \( \frac{1}{2}X^T K X \) and the vector \( M^T X \) of system-field coupling operators.

The relations (2.1)—(2.3), which describe the OQHO, reflect the effect of the external bosonic fields on its dynamics. Accordingly, the system-field Hilbert space is organised as the tensor product \( \mathcal{H}_d := \mathcal{H}_0 \otimes \mathcal{F} \), where \( \mathcal{H}_0 \) is a Hilbert space for the action of the initial system variables \( X_1(0), \ldots, X_n(0) \). The space \( \mathcal{H}_d \) is endowed with a filtration \( (\mathcal{H}_d_t)_{t \geq 0} \), where \( (\mathcal{H}_d_t)_{t \geq 0} \) is the Fock space filtration. At any time \( t \geq 0 \), the system variables \( X_j(t) \) act on the subspace \( \mathcal{H}_d \) for all \( j = 1, \ldots, m \), while the input field variables \( W_k(t) \) act on the subspace \( \mathcal{F}_k \) for all \( k = 1, \ldots, m \), in which sense both sets of processes (and nonanticipative functions thereof) are adapted to the filtration \( (\mathcal{H}_d_t)_{t \geq 0} \). The statistical properties of the system and field variables depend on a density operator (quantum state) \( \rho \) (a positive semi-definite self-adjoint operator on \( \mathcal{H}_d \) with unit trace \( Tr(\rho) = 1 \)) which also has a tensor-product structure: \( \rho := \rho_0 \otimes \nu \), where \( \rho_0 \) is the initial system state on \( \mathcal{H}_0 \), and the fields are in the vacuum state \( \nu \) \([18, 29]\). In particular, \( \rho \) specifies the expectation \( E_\xi := Tr(\rho_\xi) \) for quantum variables \( \xi \) on the space \( \mathcal{H}_d \).

Since the solution of the linear QSDE (2.2) satisfies \( X(t) = e^{(t-s)A}X(s) + \int_s^t e^{-(t-\tau)A}BdBW(\tau) \) for all \( s \leq t \geq 0 \), and the future Ito increments of the quantum Wiener process \( W \) commute with the past system variables (so that \( dW(\tau), X(s)^T \) = 0 for all \( \tau \geq s \geq 0 \)), then \( [X(t), X(s)^T] = e^{(s-t)A}[X(s), X(s)^T] \). Hence, the CCRs (2.1), which are concerned with one point in time, extend to different moments as

\[
[X(s), X(t)^T] = 2i\Lambda(s-t), \quad s, t \geq 0,
\]

\[
\Lambda(\tau) = -\Lambda(-\tau)^T = \begin{cases} e^{\tau A} \Theta & \text{if } \tau \geq 0 \\ \Theta e^{-\tau A} & \text{if } \tau < 0 \end{cases}
\]

where \( \Lambda \) is the two-point CCR matrix of the system variables, with \( \Lambda(0) = \Theta \). The linear structure of the QSDE (2.2) enters through the matrix \( A \), which is assumed to be Hurwitz.

3 Quantum risk-sensitive cost functionals

The original quantum risk-sensitive cost functional \([20, 21]\) employs an auxiliary quantum process in the form of the (leftward) TOE

\[
R_\theta(t) := \Theta \exp \left( \frac{\theta}{2} \int_{0}^{t} \Sigma(s)ds \right), \quad t \geq 0,
\]

which is the fundamental solution of the operator differential equation (ODE)

\[
R_\theta(t) = \frac{\theta}{2} \Sigma(t) R_\theta(t), \quad R_\theta(0) = \mathcal{J}_\mathcal{H}.
\]
Here, \( \dot{} \) := \( \partial_t \) is the time derivative, \( \theta \geq 0 \) is the risk-sensitivity parameter, and \( \Sigma(t) \) is a time-dependent positive semi-definite self-adjoint quantum variable which can be a function (for example, quadratic) of the current system variables (or, more generally, their past history over the time interval \([0,t]\), so that \( \Sigma \) is an adapted quantum process. Since, in general, \( R_\theta(t) \) is a non-Hermitian operator with a complex mean value, its square is used instead as a cost functional

\[
E_\theta(t) := E(R_\theta(t)^\dagger R_\theta(t))
\]

(with \( \cdot^\dagger \) the operator adjoint), which imposes an exponential penalty on the system variables through \( \Sigma \) due to the multiplicative structure of the TOE \( R_\theta \), with \( \theta \) controlling its severity. For simplicity, we do not include an additional terminal cost (on the time interval \([0,t]\)) in (3.3), cf. [21] Eqs. (19)–(21).

If the quantum variables \( \Xi(s) \) commuted with each other for all \( 0 \leq s \leq t \), then (3.3) would reduce to

\[
E_\theta(t) = e^{\theta \int_0^t \Sigma(s) ds},
\]

which is organised as the classical exponential-of-integral-performance criteria [5] [19] [49]. In the noncommutative quantum setting, the right-hand side of (3.4) provides an alternative to the original quantum risk-sensitive cost functional in (3.1), (3.3). Its quadratic-exponential counterpart [46] is given by

\[
\Xi_\theta(t) := E e^{\theta \phi(t)} = E e^{\theta \int_0^t \psi(s) ds},
\]

where \( \phi \) is a quantum process defined for any time \( t \geq 0 \) by

\[
\phi(t) := \int_0^t \psi(s) ds, \quad \psi(s) := X(s)^T \Pi X(s).
\]

Here, \( \Pi \) is a real positive semi-definite symmetric matrix of order \( n \) (the dependence of \( \Xi_\theta(t) \) on \( \Pi \) is omitted for brevity). Accordingly, \( \phi(t), \psi(t) \) are positive semi-definite self-adjoint operators on the system-field space \( \mathcal{H} \), which follows from the representation \( \psi = \zeta^T \zeta = \sum_{k=1}^n \zeta_k^2 \) in terms of the auxiliary self-adjoint quantum variables constituting the vector \( \zeta := (\zeta_k)_{1 \leq k \leq n} := \sqrt{\Pi} \).

Although the original quantum risk-sensitive cost \( E_\theta \) in (3.1), (3.3) and its quadratic-exponential counterpart \( \Xi_\theta \) in (3.4), (3.6) are identical in the classical case if \( \Sigma = \psi \), they are different in the noncommutative quantum setting (even if \( \Sigma = \psi \)) because of the discrepancy between the TOE and the usual operator exponential. Moreover, at any given instant \( t \geq 0 \), the QEF \( \Xi_\theta(t) \) is the moment-generating function for the classical probability distribution (the averaged spectral measure [17]) of the self-adjoint quantum variable \( \phi(t) \). In contrast to \( \Xi_\theta(t) \), the quantity \( E_\theta(t) \) in (3.3) does not lend itself to a similar association with a single \( \theta \)-independent quantum variable.

Since the evolution equations for the cost functionals (3.3), (3.5) are obtained by averaging the corresponding time derivatives as \( \dot{E}_\theta = E((R_\theta^\dagger R_\theta)' \) and \( \dot{\Xi}_\theta = E((e^{\theta \phi})' \), we will be concerned mainly with the dynamics of the processes \( R_\theta^\dagger R_\theta \) and \( e^{\theta \phi} \) themselves. Also, we abandon the assumption on self-adjointness of the operator \( \Sigma(t) \) which drives (3.2). Then an appropriate modification of [47] yields

\[
\langle R^\dagger R \rangle = \frac{\theta}{2} (\Sigma R^\dagger + R^\dagger \Sigma) = \theta R^\dagger (\text{Re}\Sigma) R,
\]

where the subscript \( \theta \) in \( R_\theta \) is omitted for brevity, and the real part is extended to operators as \( \text{Re} \xi := \frac{1}{2}(\xi + \xi^T) \). Also,

\[
(e^{\theta \phi})' = \theta e^{\frac{\theta}{2} \psi} \psi e^{\frac{\theta}{2} \psi}, \quad \Psi_\theta := \text{sinhc}(\frac{\theta}{2} \text{ad}_\phi)(\psi),
\]

with \( \text{ad}_\xi(\cdot) := [\xi, \cdot] \), where the evaluation of the hyperbolic sinc function \( \text{sinhc}(z) := \text{sinh}(iz) \) at \( \text{ad}_\phi \) yields a linear superoperator acting on \( \psi \). The relations (3.8) holds regardless of the particular structure of the OQHO dynamics and the processes in (3.6) (except that \( \phi = \psi \) and follows from the identities

\[
\langle e^{\phi} \rangle = \Upsilon(\text{ad}_\phi)(\phi) = e^{\phi} \Upsilon(-\text{ad}_\phi)(\phi)
\]

(in view of the Magnus lemma [26]) for a time-varying operator \( \phi \), which reduce to the standard exponential derivative when \( [\phi, \phi] = 0 \), where

\[
\Upsilon(z) := e^{\frac{z}{e-1}} \text{sinhc} z = \begin{cases} \frac{1}{e-1} & \text{if } z = 0 \\ 1 & \text{otherwise} \end{cases}
\]

Therefore, the processes \( R_\theta^\dagger R_\theta \) and \( e^{\theta \phi} \) reproduce each other (in which case, \( R_\theta \) is a non-Hermitian operator square root of \( e^{\theta \phi} \)) if \( \text{Re} \Sigma \) in (3.7) is appropriately matched (and becomes unitarily equivalent to \( \Psi_\theta \) in (3.8), similarly to [47] Theorem 3). This suggests a link between the TOE-based quantum risk-sensitive functionals (3.3) and the QEFs (3.5), which requires a more explicit representation of the process \( \Psi_\theta \). To this end, the two-point CCRs (2.4) and the specific quadratic dependence of \( \psi \) on the past history of the system variables lead to

\[
\Psi_\theta(t) := \psi(t) + \frac{\theta}{2} \int_0^t \text{Re}(X(s)^T X(s)) ds
\]

(3.11)

\[
+ \int_{[0,t]^2} X(s)^T \beta_{\theta,j} X(t) ds dt,
\]

which is a quadratic function of the past history of the system variables over the time interval \([0,t]\), with the functions \( \alpha_{\theta,j} : [0,t] \rightarrow \mathbb{R}^{n \times n} \), \( \beta_{\theta,j} : [0,t]^2 \rightarrow \mathbb{R}^{n \times n} \), being related to the two-point CCR matrix \( \Lambda \) in (2.5), with \( \beta_{\theta,j} \) being symmetric: \( \beta_{\theta,j}(\sigma, \tau) = \beta_{\theta,j}(\tau, \sigma)^T \). These kernel functions are obtained in [46] Theorem 1, Lemma 2] using the fact that quadratic forms in quantum variables with CCRs form a Lie algebra with respect to the commutator (see, for example, [46] Appendix A) and references therein).

4 A class of quadratic functions of system variables

In view of the structure of the right-hand side of (3.11), consider the following unified representation for a class of quadratic functions of the system variables of the OQHO. Let
$Q : \mathcal{B}_+^2 \to C^{n \times n}$ be a countably additive measure of bounded total variation on the $\sigma$-algebra $\mathcal{B}_+^2$ of Borel subsets of the orthant $\mathbb{R}_+^2$ (with $\mathbb{R}_+ := [0, +\infty)$ the set of nonnegative real numbers). With any such $Q$, we associate a quantum variable

\begin{equation}
\phi_Q := \int_{\mathbb{R}_+^2} X(\sigma)^T Q(d\sigma \times d\tau) X(\tau),
\end{equation}

which is a quadratic function of the system variables. For example, $\phi(t), \psi(t)$ in (4.6) and $\phi \theta(t)$ in (4.11) are particular cases of (4.1), as discussed below. Since we will be concerned with commutators of the quantum variables (4.1), then, due to the two-point CCRs (2.4), the kernel measure $Q$ can be assumed to be symmetric in the sense that $Q(A \times B) = Q(B \times A)^T$ for any $A, B \in \mathcal{B}_+$, (such measures form a complex linear space, which we denote by $\mathcal{E}_n$).

Indeed, in view of the two-point CCRs (2.4), for any antisymmetric $C^{n \times n}$-valued measure $Q := (q_{jk})_{1 \leq j, k \leq n}$ on $\mathbb{R}_+^2$ (with $Q(A \times B) = -Q(B \times A)^T$ for any $A, B \in \mathcal{B}_+$), the quantum variable (4.1) is a scalar: $\phi_Q = \int_{\mathbb{R}_+^2} \sum_{j, k = 1}^n X_j(s)X_k(t)q_{jk}(ds \times dt) = -i \int_{\mathbb{R}_+^2} \sum_{j, k = 1}^n X_j(t)X_k(s)q_{jk}(ds \times dt) = -i \int_{\mathbb{R}_+^2} \sum_{j, k = 1}^n [X_j(s)X_k(t) - X_j(t)X_k(s)]q_{jk}(ds \times dt) = 0$ for all $s, t \geq 0$.

This integral operator corresponds to complex Hamiltonian matrices. In order to emphasize this analogy, $\Lambda Q$ will be referred to as a complex Hamiltonian kernel (CHK) (in the sense of the symplectic structure specified by $\Lambda$). CHKs are infinitesimal generators of complex symplectic kernels (CSKs) $S : \mathbb{R}_+ \times \mathcal{B}_+ \to C^{n \times n}$ (which are also measures over the second argument) satisfying

\begin{equation}
\int_{\mathbb{R}_+^2} (s, ds) \Lambda(s, t, \sigma) S(s, dt) = \Lambda(t-s, s, t).
\end{equation}

Such kernels $S$ form a semigroup, which preserves the two-point CCRs in the sense that the latter are inherited by the quantum process $X(t) := \int_{\mathbb{R}_+^2} S(t, ds)X(\sigma)$ as \[ \int_{\mathbb{R}_+^2} (s, ds) \Lambda(s, t, \sigma) S(t, dt) = 2i \int_{\mathbb{R}_+^2} (s, ds) \Lambda(s, t, \sigma) S(t, dt) = 2i \int_{\mathbb{R}_+^2} (s, ds) \Lambda(s, t, \sigma) S(t, dt) \]

5 Lie-algebraic isomorphism to complex Hamiltonian kernels

The significance of the CHK $\Lambda Q$ in (4.4), (4.5) for commutation relations is clarified by

\[ [\phi_Q, X(t)] = \int_{\mathbb{R}_+^2} [X(\sigma)^T Q(d\sigma \times d\tau) X(\tau), X(t)] \]

\begin{equation}
= -i \int_{\mathbb{R}_+^2} [X(t), X(\sigma)^T] Q(d\sigma \times d\tau) X(\tau) + \int_{\mathbb{R}_+^2} (X(\sigma)^T Q(d\sigma \times d\tau) X(\tau), X(t)^T) ^T \]

\begin{equation}
= -2i \int_{\mathbb{R}_+^2} \Lambda(t-s) Q(d\sigma \times d\tau) X(\tau) + 2i \int_{\mathbb{R}_+^2} (X(\sigma)^T Q(d\sigma \times d\tau) \Lambda(t-s, t)^T \]

\begin{equation}
= -4i \int_{\mathbb{R}_+^2} \Lambda(t-s) Q(d\sigma \times d\tau) X(\tau), t \geq 0,
\end{equation}

so that $[\phi_Q, X] = -4i(\Lambda Q)(X)$. Here, the derivation and antisymmetry properties of the commutator have been combined with the antisymmetry of $\Lambda$ in (2.4), (2.5) and the symmetry of $Q$.

**Lemma 5.1.** The quantum variables $\phi_Q$ in (4.1), associated with measures $Q \in \mathcal{E}_n$, form a Lie algebra, in which

\begin{equation}
[\phi_{Q_1}, \phi_{Q_2}] = \phi_{\Lambda Q_1 Q_2 - Q_2 \Lambda Q_1},
\end{equation}

where

\begin{equation}
Q = 4i(Q_1 Q_2 - Q_2 Q_1).
\end{equation}
is also such a measure given by
\[ Q(A \times B) = 4i \int_{\mathbb{R}_+} (Q_1(A \times ds)\Lambda(s-t)Q_2(ds \times B)) \]
\[ (5.4) \]
for all \( A, B \in \mathcal{B}_+ \), where \( \Lambda \) is the two-point CCR function from (2.2).

**Proof.** By a reasoning, similar to that in (5.1), (4.1) implies
\[ [\phi_{Q_1}, \phi_{Q_2}] = \int_{\mathbb{R}_+} [\phi_{Q_1}, X(\sigma)]^T Q_2(d\sigma \times d\tau)X(\tau) \]
\[ = \int_{\mathbb{R}_+} [\phi_{Q_1}, X(\sigma)]^T Q_2(d\sigma \times d\tau)X(\tau) \]
\[ + \int_{\mathbb{R}_+} X(\sigma)^T Q_2(d\sigma \times d\tau)[\phi_{Q_1}, X(\tau)] \]
\[ = -4i \int_{\mathbb{R}_+} [\phi_{Q_1}, X(\sigma)]^T Q_2(d\sigma \times d\tau)(\Lambda Q_1)(X(\tau)) \]
\[ -4i \int_{\mathbb{R}_+} X(\sigma)^T Q_2(d\sigma \times d\tau)(\Lambda Q_1)(X(\tau)) \]
\[ (5.5) \]
where \( Q \in \mathcal{B}_n \) is given by (5.4), or, equivalently, (5.1), thus establishing (5.2). The symmetry of \( Q \) follows from that of the measures \( Q_1, Q_2 \) and the antisymmetry of \( \Lambda \). In (5.3), use is also made of the relation
\[ -\int_{\mathbb{R}_+} [\phi_{Q_1}, X(\sigma)]^T Q_2(d\sigma \times B) = -\int_{\mathbb{R}_+} X(\sigma)^T Q_1(d\sigma \times B) = \int_{\mathbb{R}_+} X(\sigma)^T Q_2(d\sigma \times B) - \int_{\mathbb{R}_+} X(\sigma)^T Q_2(d\sigma \times B) \]
\[ = \int_{\mathbb{R}_+} X(\sigma)^T (Q_1 \Lambda Q_2)(d\sigma \times B), \]
where \( Q_1 \Lambda Q_2 \) is a \( C^{\infty} \)-valued measure (not necessarily symmetric) given by
\[ (Q_1 \Lambda Q_2)(A \times B) = \int_{\mathbb{R}_+} Q_1(A \times d\sigma)\Lambda(\sigma - \tau)Q_2(d\sigma \times B) \]
for all \( A, B \in \mathcal{B}_+ \).

In accordance with (4.4), the multiplications of measures in \( Q_1 \Lambda Q_2 \) is associative. From Lemma (5.1) it follows that the Lie algebra of quantum variables \( \phi_Q \) in (4.1), considered for measures \( Q \in \mathcal{B}_n \), is isomorphic to the Lie algebra of CHKs. Indeed, since (5.3) implies that \( 4iQ_1 \Lambda Q_2 = (4i)^2(\Lambda Q_1 \Lambda Q_2 - \Lambda Q_2 \Lambda Q_1) = [4i\Lambda Q_1, 4i\Lambda Q_2] \), the Lie-algebraic isomorphism is described by the correspondence
\[ (5.6) \]
\[ \phi_Q \leftrightarrow 4i\Lambda Q. \]

Note that \( Q \in \mathcal{B}_n \) can be recovered from the two-sided Laplace transform of \( \Lambda Q \) given by
\[ \int_{\mathbb{R}} e^{-st} \int_{\mathbb{R}_+} \Lambda(t - \sigma)Q(ds \times B) \]
\[ = \Lambda(s) \int_{\mathbb{R}_+} e^{-st} Q(ds \times B) \]
\[ (5.7) \]
in the strip \( \{ s \in \mathbb{C} : 0 < Re s < |\ln r(e^A)| \} \) for any \( B \in \mathcal{B}_+ \), where \( r(\cdot) \) is the spectral radius of a square matrix, so that
\[ \ln r(e^A) = \max_{1 \leq k \leq n} \text{Re} \lambda_k, \text{ with } \lambda_1, \ldots, \lambda_n \text{ the eigenvalues of the Hurwitz matrix } A. \]

Here, the two-sided Laplace transform
\[ \Lambda(s) := \int_{\mathbb{R}} e^{-st} \Lambda(t) \]
\[ = (sI_n - A)^{-1}(\Theta(sI_n + A^T) - (sI_n - A)) \]
\[ = (sI_n - A)^{-1}((\Theta A^T + \Theta^T A)(sI_n + A^T) - (sI_n - A)) \]
\[ (5.8) \]
is a rational function, which is obtained by using the matrix exponential structure of \( A \) in (2.3) and the PR property (2.3) of the matrices \( A, B \). Since \( A \) is assumed to be Hurwitz, the integrals in (5.8) are convergent over the strip \( 0 < |\ln r(A)| \). A sufficient condition for unique recoverability of \( Q \) from \( \Lambda Q \) using (5.7) is \( det(BJ^T) \neq 0 \), for which it is necessary that \( n \leq m \).

## 6 A Lie-algebraic correspondence between TOE-based and quadratic-exponential functions of system variables

Similarly to the case (4.3) of products of quadratic-exponential functions of a finite number of quantum variables with CCRs, a combination of Dynkin’s lemma (12) with the Lie-algebraic isomorphism (5.6) leads to
\[ e^{\partial G_t} e^{\partial G_t} = e^{\partial G_t}, \]
\[ (6.1) \]
where \( Q_1, Q_2, Q \in \mathcal{B}_n \) are related by the complex symplectic factorization:
\[ e^{4iQ_1 \Lambda} e^{4iQ_2 \Lambda} = e^{4iQ \Lambda}. \]

All three exponentials in (6.2) are integral operators with CSKs in the sense of (4.6). A continuous-product version of this representation formula is
\[ \hat{\text{exp}} \left( \int_0^t \phi_f ds \right) = e^{\partial G_t}. \]

Here, \( F_t, G_t \in \mathcal{B}_n \) are time-dependent measures satisfying
\[ \hat{\text{exp}} \left( 4i \int_0^t \Lambda F_t ds \right) = e^{4iQ \Lambda G_t}, \]
\[ (6.4) \]
for all \( t \geq 0 \), which is equivalent to the ODE
\[ (6.5) \]
\[ (e^{4iQ \Lambda G_t}) = 4iQ e^{4iQ \Lambda G_t}, \quad G_0 = 0. \]

A similar representation holds for the rightward TOEs
\[ \hat{\text{exp}} \left( \int_0^t \phi_f ds \right) = e^{\partial G_t}, \quad \hat{\text{exp}} \left( 4i \int_0^t \Lambda F_t ds \right) = e^{4iQ \Lambda G_t}, \]
\[ (6.6) \]
in which case, (6.3) is replaced with
\[ (e^{4iQ \Lambda G_t}) = 4e^{4iQ \Lambda G_t}, \quad G_0 = 0. \]

The following theorem employs (6.1)-(6.7) in order to relate two extended classes of functions of the OQHO variables whose averaging leads to the TOE-based and QEF costs in (3.3). (3.5).
Theorem 6.1. Suppose the quantum process $R$ in (6.7), (6.2)\footnote{the parameter $\theta$ is incorporated in the measures below, and the dependence on $\theta$ is omitted for brevity, or, equivalently, $\theta = 1$} is driven as

$$R(t) := \exp \left( \int_0^t \phi_f ds \right)$$

by the quantum variable (4.1) with a time-dependent measure $F_t \in \mathscr{F}_n$. Then

$$R(t)^\dagger R(t) = e^{\phi_{N_t}},$$

where $N_t \in \mathscr{B}_n$ is a time-dependent measure, evolving as

$$\Lambda N_t = 4\tanh \Lambda G_t \Lambda \left( \Lambda e^{\Lambda G_t} \right) e^{\Lambda G_t}, \quad N_0 = 0,$$

and $G_t \in \mathscr{G}_n$ is a time-dependent measure governed by

$$\text{(6.11)} \quad \Lambda G_t = 2i\Phi_t e^{\Lambda G_t}, \quad G_0 = 0.$$ 

Proof. By applying (6.1)–(6.5), it follows that the process $R$ in (6.3) can be represented as

$$R(t) = e^{\phi_{G_t}},$$

where the time-dependent measure $G_t \in \mathscr{G}_n$ satisfies

$$e^{i\Lambda G_t} = \Lambda G_t,$$

which is equivalent to (6.11). In view of (4.3), the adjoint (6.12) takes the form

$$R(t)^\dagger = e^{\phi_{G_t}} = e^{\phi_{N_t}}.$$

By combining (6.12) with (6.14) and using (6.1), (6.2), it follows that $R(t)^\dagger R(t) = e^{\phi_{N_t}} e^{\phi_{G_t}} = e^{\phi_{N_t}}$, thus establishing (6.9), where $\phi_{N_t}$ is self-adjoint, and $N_t \in \mathscr{B}_n$ satisfies the complex symplectic factorization

$$\text{(6.15)} \quad e^{\Lambda N_t} = e^{\Lambda G_t} e^{\Lambda G_t}.$$

On the other hand, (6.8), (4.3) imply that

$$R(t)^\dagger = \exp \left( \int_0^t \phi_f ds \right).$$

Hence, application of (6.6), (6.7) to (6.14), (6.16) leads to

$$e^{i\Lambda G_t} = \Lambda G_t,$$

by substituting (6.13), (6.17) into (6.15) and differentiating, it follows that $\Lambda G_t = (\Lambda G_t) e^{\Lambda G_t} + e^{\Lambda G_t} (\Lambda G_t) = 2i e^{\Lambda G_t} \Lambda (F_t + T) e^{\Lambda G_t} = 4i e^{\Lambda G_t} \Lambda (\Lambda e^{\Lambda G_t}) e^{\Lambda G_t}$, which proves (6.10), with $N_0 = 0$ due to $R(0) = \mathbb{I}$.

7 Moving along the Lie-algebraic bridge

Theorem 6.1 allows $N_t$ on the right-hand side of (6.9) to be found for a given measure $F_t$ in (6.3), and the other way around, $F_t$ can be found for a given $N_t$.

The first of these problems pertains to representing the TOE-based original quantum risk-sensitive cost functional as a QEF. An intermediate step of this procedure is concerned with finding the measure $G_t$ in (6.12) for the given $F_t$. A comparison of the ODE (6.11) with the general exponential derivative $(e^{\Lambda G_t})^\dagger = 4i\Gamma(4i\Lambda G_t) (\Lambda G_t) e^{\Lambda G_t}$ (following from (3.9)) leads to

$$\text{(7.1)} \quad \Upsilon(4i\Lambda G_t) (\Lambda G_t) = \frac{1}{2} \Lambda F_t,$$

and hence,

$$\text{(7.2)} \quad \Lambda G_t = \frac{1}{2} \Upsilon(4i\Lambda G_t) (\Lambda F_t),$$

where the function $\Upsilon$ is given by (3.10), and its reciprocal $\Upsilon(z) := \frac{1}{\Upsilon(z)} = \sum_{k=0}^{\infty} \frac{\Lambda^k}{k!} z^k$ is the generating function of the Bernoulli numbers $b_0, b_1, b_2, \ldots$. The relation (7.2) is a nonlinear ODE whose linearisation version takes the form $\Lambda G_t = \frac{1}{2} \Lambda F_t + i\Lambda F_t (\Lambda F_t) + \ast$ in view of $b_0 = 1, b_1 = -\frac{1}{2}$, where $\ast$ contains the higher-order terms, nonlinear with respect to $G_t$. However, finding $N_t$ from (6.15) requires the CSK $S_t : \mathbb{R}_+ \times \mathbb{B}_+ \to \mathbb{C}^{n \times n}$ of the integral operator $e^{\Lambda G_t}$ in (6.13) rather than $G_t$ itself. In contrast to (7.2), $S_t$ satisfies a linear integro-differential equation (IDE)

$$\text{(7.3)} \quad \partial_t S_t(v, B) = 2i \int_{\mathbb{R}_+} \Lambda (v - \sigma) F_t (d\sigma \times d\tau) S_t(\tau, B)$$

for all $t, v \geq 0, B \in \mathbb{B}_+$, with the initial condition $S_0(v, B) = \chi_B(v) I_n$, where $\chi_B$ is the indicator function of the set $B$. Then the measure $N_t$ is recovered from the CSK $T_t : \mathbb{R}_+ \times \mathbb{B}_+ \to \mathbb{C}^{n \times n}$ of the integral operator $e^{\Lambda G_t}$ satisfying the complex symplectic factorization

$$\text{(7.4)} \quad \int_{\mathbb{R}_+} \Upsilon(v, d\sigma) T_t(\sigma, B) = S_t(v, B).$$

The latter is a linear equation (of Fredholm first kind) obtained from (6.15) due to the property that $\Upsilon$ is the CSK of the integral operator $e^{\Lambda G_t} = (e^{\Lambda G_t})^{-1}$. By a similar reasoning, the following IDE form of (7.4) for finding $T_t$ (after the IDE (7.3) is solved for $S_t$) is obtained from (6.10):

$$\int_{\mathbb{R}_+} \Upsilon(v, d\sigma) \partial_t T_t(\sigma, B)$$

$$= 4i \int_{\mathbb{R}_+} \Lambda (v - \sigma) F_t (d\sigma \times d\tau) S_t(\tau, B)$$

(with the same initial condition $T_0 = S_0$). Therefore, the representation of the TOE-based left-hand side of (6.9) as a quadratic-exponential function of the OQHO variables on the right-hand side can be carried out by consecutive solution of the IDEs (7.3), (7.5).
The inverse problem (to be described below) is to represent the QEF, specified by a given measure \( N_t \in \mathcal{M}_n \), in the form of the original quantum risk-sensitive functional driven by \( \mathbf{F}_t \), where \( \mathbf{F}_t \) is to be found for \( N_t \). To this end, the measure \( \mathbf{F}_t \in \mathcal{M}_n \) in Theorem 6.1 can be organised so that the TOE \( R(t) \) remains a positive definite self-adjoint square root of \( e^{\theta N_t} \) over the course of time: \( R(t) = e^{\frac{i}{2} \theta N_t} \) for all \( t \geq 0 \). Then the corresponding measure \( G_t \) in (6.12) is given by \( G_t = \frac{1}{2} N_t \), and its substitution into (7.1) relates \( F_t \) to \( N_t \) as

\[
\Lambda F_t = \mathbf{Y}(2i \text{ad}_{\Lambda N_t})(\Lambda N_t) = \frac{1}{2} \int_0^1 L_{\lambda, t} d\lambda.
\]

Here, \( N_t \) is assumed to have an appropriate distributional time derivative \( L \), and

\[
L_{\lambda, t} := e^{2\lambda \text{ad}_{\Lambda N_t}}(\Lambda N_t)
\]

is a CHK satisfying

\[
\partial^2_t L_{\lambda, t} = 2i[\Lambda N_t, L_{\lambda, t}], \quad 0 \leq \lambda \leq 1,
\]

with the initial condition (in the sense of the parameter \( \lambda \)) \( L_{\lambda, 0} = \Lambda N_t \). Therefore, the quadratic-exponential function of the OQHO variables on the right-hand side of (6.9) can be represented in the TOE-based form on the left-hand side of (6.9) by solving the IDE (7.8) and performing the integration in (7.6).

8 Specific nonanticipative time-varying measures

The above problems in Section 7 of finding \( N_t \) for \( F_t \), and \( F_t \) for \( N_t \) are particularly important for nonanticipative time-varying measures \( Q_t \in \mathcal{M}_n \) satisfying

\[
\supp Q_t \subset [0, t]^2, \quad t \geq 0.
\]

Then \( \phi_Q = \int_{[0, t]^2} X(\tau) Q_t(d\sigma \times d\tau)X(\tau) \) in (4.1) depends only on the past history of the system variables over the time interval \([0, t]\) (and is, therefore, \( \partial_t \)-adapted). Furthermore, in view of (8.1) and in accordance with (4.4), the corresponding CHK \( \Lambda Q_0 \) takes the form \( \Lambda Q_0(v, B) = \int_{\mathbb{R}} \Lambda(\tau - \sigma) Q_t(d\sigma \times \{0, t\}) + \int_{\mathbb{R}} \Lambda(\tau - \sigma) Q_t(d\sigma \times \{t, 1\}) \) for any \( v, t \geq 0, B \in \mathcal{B}_+ \), and hence, its support (over the second argument) satisfies \( \supp(\Lambda Q_0)(v, \cdot) \subset [0, t] \) for any \( t \geq 0 \).

Nonanticipative measures specify the quantum processes \( \varphi, \psi \) in (3.5) and also play a role when the process \( \Sigma \), which drives the TOE in (3.2), is a quadratic function of the current system variables. More precisely, the operator \( \varphi(t) \) in (3.6), which gives rise to the QEF in (3.5), is a particular case of (4.1) obtained as \( \varphi(t) = \phi_{N_t} \) by using a nonanticipative measure \( N_t \) given by

\[
N_t(C) := \mu(\{ \sigma \in [0, t] : (\sigma, \sigma) \in C \}) \Pi, \quad C \in \mathcal{B}_+^2,
\]

where \( \mu \) is the one-dimensional Lebesgue measure. The distributional time derivative of (8.2) is an atomic nonanticipative measure concentrated at the singleton \( \{(t, t)\} \) as

\[
N_t(C) = \chi_C((t, t)) \Pi,
\]

which allows the quantum variable \( \psi(t) \) in (3.6) to be represented in the form (4.1) as \( \psi(t) = \phi_{N_t} \). Substitution of (8.2) into (7.8) (which pertains to the problem of finding the TOE-based representation for the QEF) leads to the following IDE for the CHK \( L_{\lambda, t} \) in (7.7):

\[
\partial^2_t L_{\lambda, t}(v, B) = 2i\left\{ \int_0^t \Lambda(\tau - \sigma) \Pi L_{\lambda, \tau}(\sigma, B) d\sigma \right\} - \int_0^t L_{\lambda, \tau}(v, d\sigma) \left( \int_0^t \Lambda(\tau - \sigma) \Pi d\tau \right).
\]

with the initial condition \( L_{\lambda, 0}(v, B) = \chi_B(t) \Lambda(\tau - v) \Pi \) for all \( v \geq 0, B \in \mathcal{B}_+ \). Here, use is also made of the fact that the measure \( N_t \) in (8.2) satisfies \( N_t(A \times B) = \mu(\{0, t\} \cap A \cap \{\sigma \geq 0 \}) \Pi \) for all \( A, B \in \mathcal{B}_+ \).

Furthermore, the quantum process \( \Psi_1 \) in (3.8) (we let \( \theta = 1 \) as mentioned above) can also be represented in the form (4.1) as

\[
\Psi_1(t) = \sinh(\frac{1}{2} \text{ad}_{\theta N_t})(\phi_{N_t}) = \phi_{M_t}
\]

with a nonanticipative measure \( M_t \in \mathcal{M}_n \). It follows from (3.11) that \( M_t \) consists of an absolutely continuous part over the square \([0, t]^2\), a singular part concentrated at the two edges \([0, t] \times \{0\}\) and \(\{0\} \times [0, t]\) of the square, including an atomic part concentrated at the corner \(\{(t, t)\}\). Due to the Lie-algebraic isomorphism of Section 5, the measure \( M_t \) in (8.5) satisfies \( \sinh(\frac{1}{2} \text{ad}_{\theta N_t})(\Lambda N_t) = \Lambda M_t \).

We will now return to the first of the problems in Section 7 on the Lie-algebraic correspondence of Theorem 6.1 in application to representing the TOE-based criterion as a QEF. Suppose the quantum process \( R \) in (3.1), (3.2) (with \( \theta = 1 \) for simplicity) in (3.8) is driven by the atomic measure (8.6) and reduces to a PDE:

\[
\partial_t S_t(v, B) = 2i\Lambda(v - t) \Pi S_t(t, B), \quad t, v \geq 0, B \in \mathcal{B}_+ \,
\]

with the same initial condition \( S_0(v, B) = \chi_B(v) \Pi \). The transformation \( S_t(u, B) := S_t(t + u, B) \) allows the PDE (8.7) to be represented as

\[
\partial_t S_t(u, B) = 2i\Lambda(u) \Pi S_t(0, B) + \partial_u S_t(u, B).
\]

The PDE (8.7) (or its equivalent form (8.8)) can be solved by the method of characteristics or the Laplace transform techniques. The latter employ the two-sided Laplace transform (8.8) of the two-point CCR function (2.5) for the system variables and are also applicable to the IDE (8.4).
9 Conclusion

For linear quantum stochastic systems, we have established a Lie-algebraic link between two classes of quantum risk-sensitive cost functionals, which pertain to the original TOE-based performance criterion and its recent QEF version. We have used a unified representation for the quadratic functions of system variables in these criteria in terms of complex symmetric matrix-valued measures. The Lie-algebraic correspondence has been reduced to IDEs for related complex Hamiltonian and symplectic kernels which involve the two-point CCR matrix of the system variables. These relations will be employed in subsequent publications for extending useful features, such as robustness properties, simplicity of evolution, and applicability of information state techniques, from one of the classes of risk-sensitive costs to the other. The results of the paper will also be used in order to develop state-space equations for computation and minimization of these functionals in quantum robust control and filtering problems.

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