The $U(1)$ Topological Gauge Field Theory for Topological Defects in Liquid Crystals

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A novel $U(1)$ topological gauge field theory for topological defects in liquid crystals is constructed by considering the $U(1)$ gauge field is invariant under the director inversion. Via the $U(1)$ gauge potential decomposition theory and the $\phi$-mapping topological current theory, the decomposition expression of $U(1)$ gauge field and the unified topological current for monopoles and strings in liquid crystals are obtained. It is revealed that monopoles and strings are located in different spatial dimensions and their topological charges are just the winding numbers of $\phi$-mapping.

§1. Introduction

Topological defects have attracted a lot of interest in liquid crystals$^{1,3}$ and play an important role in the static structures and dynamical behavior of liquid crystals.$^{4,5}$ Because of the soft-liquid character and optic transparency of liquid crystals, a rich variety of topological defects are exhibited, such as dislocations, disclinations, point defects, global defects. Theoretically, these topological defects have been classified by the homotopy group, which provides a natural language for the classification of defects in ordered media. Work in this field was originated by Finkelstein,$^6$ Toulouse and Klémàn$^7$ et al and was summarized by Mermin,$^8$ Anderson$^9$ and Bray .$^{10}$ Consider the symmetry breakdown of Lie group $G \rightarrow H$, taking for $G$ the simply connected cover of the initial Lie symmetry and $H \subset G$ its unbroken subgroup. Monopoles lie on the vacuum manifold with nontrivial $\pi_2(G/H)$. Strings occur in theories with nontrivial $\pi_1(G/H)$, domain walls occur with nontrivial $\pi_0(G/H)$ and textures in theories with nontrivial $\pi_3(G/H)$. In the case of biaxial liquid crystals in three-dimensional space, the vacuum manifold is a sphere, where diametrically opposite points have to be identified because of the equivalence of director $\mathbf{d}$ and $-\mathbf{d}$. $^{11}$ This is the projection plane $P_2$, whose group element is $G/H \cong SO(3)/D_2$. $^{12}$ In addition to their importance in ordered media, these classifications are also relevant to problems in cosmological structure formation. However, this earlier work is mainly concerned with the topological classification of individual defects, and the topological gauge field theory for liquid crystals is not established yet.

In the present paper, we will construct a $U(1)$ topological gauge field theory for topological defects in liquid crystals, in which the $U(1)$ gauge field is invariant...
under the director inversion $d \rightarrow -d$. We will derive a unified topological current for monopoles and strings in liquid crystals from the $U(1)$ gauge field tensor and show that monopoles and strings are generated from the singularities of the director field in different spatial dimensions. This work is based on the so-called gauge potential decomposition theory and $\phi$-mapping topological current theory.\textsuperscript{[13],[14]}

This paper is arranged as follows. In Sec. 2 we investigate the decomposition expression of the $U(1)$ gauge potential in liquid crystals. In Sec. 3 the topological current for monopoles and strings in liquid crystals is presented. In Sec. 4 we show that the topological charges of monopoles and strings are just the winging numbers. In the final section, we provide a brief conclusion and outlook for the further work.

§2. The $U(1)$ gauge potential decomposition in terms of the director

As is well known, the director $d$ has unit length by definition and is to be identified with $-d$ in liquid crystals. In this sense the director becomes a unit double-valued vector\textsuperscript{[15]} and generalizes the former concept of a single-valued vector. Generally, the topological characteristics of a manifold are represented by the properties of a smooth vector field on it, or, in other words, the smooth vector fields carry the topological information of a manifold. In the single-valued vector field system, the gauge potential decomposition theory is established by means of the unit vector field $n$.\textsuperscript{[16]} In liquid crystals, its inner structure should be based on the director field $d$. In the case of the 2-dimensional director, the director inversion corresponds to the $U(1)$ gauge transformation, which inspires us to study the $U(1)$ gauge potential decomposition in terms of the unit director field on the vacuum manifold.

We first study the $U(1)$ gauge potential decomposition in the single-valued vector field system, which reveals the inner structure of the $U(1)$ gauge potential in terms of $n$. It is known that the $U(1)$ gauge covariant derivative and its complex conjugate are given by

$$D_\mu \psi = \partial_\mu \psi - iA_\mu \psi, \quad (D_\mu \psi)^* = \partial_\mu \psi^* + iA_\mu \psi^*,$$

where $A_\mu$ is the $U(1)$ gauge potential (connection) and $\psi$ is the complex wave function in the single-valued vector field. And the $U(1)$ gauge field tensor (curvature) is defined by

$$F_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu,$$

which is invariant under the $U(1)$ gauge transformation: $A'_\mu(x) = A_\mu(x) + \partial_\mu \theta(x)$, where $\theta(x) \in \mathbb{R}$ is a phase factor denoting the $U(1)$ gauge transformation. Multiplying $\psi^*$ with Eq. (2.1) and Eq. (2.2) with $\psi$, it is easy to find the decomposition expression of the $U(1)$ gauge potential

$$A_\mu = \frac{1}{2i\psi^* \psi} (\psi^* \partial_\mu \psi - \partial_\mu \psi^* \psi) - \frac{1}{2i\psi^* \psi} (\psi^* D_\mu \psi - (D_\mu \psi)^* \psi).$$

To study the inner structure of the above expression. The complex wave function $\psi(x)$ can be regarded as a complex representation of a 2-dimensional vector field
$\vec{\psi} = (\varphi^1, \varphi^2)$ over the vacuum manifold, i.e.,

$$\psi(x) = \varphi^1(x) + i\varphi^2(x), \quad (2.5)$$

where $\varphi^1(\vec{x})$ and $\varphi^2(\vec{x})$ are two real functions. The 2-dimensional unit vector $\mathbf{n}$ is naturally expressed as

$$n^a = \frac{\varphi^a}{\|\varphi\|}, \quad (a = 1, 2), \quad (2.6)$$

where $\|\varphi\|^2 = \varphi^a \varphi^a = \psi^* \psi$. Then Eq. (2.4) can be simplified as

$$A_\mu = \varepsilon^{ab} n^a \partial_\mu n^b - \varepsilon^{ab} n^a D_\mu n^b.$$  \quad (2.7)

Let $k^a = \varepsilon^{ab} n^b$ be another 2-dimensional unit vector which is orthogonal to $n^a$: $k^an^a = 0$, $k^ak^a = 1$, then using $n^a$ and $k^a$, Eq. (2.7) is rewritten as

$$A_\mu = k^a \partial_\mu n^a - k^a D_\mu n^a.$$  \quad (2.8)

Suppose $u^a$ be another unit vector field satisfying: $D_i u^a = 0$ ($u^au^a = 1$) and be expressed as $u^a = \cos \theta n^a + \sin \theta k^a$, it can be proved that $-k^a D_i n^a = \partial_i \theta$. We see that the second term of (2.7), $\partial_i \theta$, behaves as a $U(1)$ gauge transformation of $A_\mu$, which contributes nothing to the gauge field tensor $F_{\mu\nu}$ defined by Eq. (2.3) and can be ignored in $U(1)$ decomposition theory. Therefore the decomposition of $U(1)$ gauge potential is simplified as

$$A_\mu = \frac{1}{2i} \frac{1}{\psi^*\psi} (\psi^* \partial_i \psi - \partial_i \psi^* \psi) = \varepsilon^{ab} n^a \partial_\mu n^b.$$  \quad (2.9)

which means that the $U(1)$ gauge potential $A_\mu$ possesses an inner structure in terms of the 2-dimensional unit vector $\mathbf{n}$. In superconductivity theory, the form of (2.9) actually corresponds to the London relation and is a fundamental expression in $U(1)$ topological quantum mechanics. Thus, the $U(1)$ gauge field tensor $F_{\mu\nu}$ in Eq. (2.3) is changed into

$$F_{\mu\nu} = 2\varepsilon^{ab} \partial_\mu n^a \partial_\nu n^b.$$  \quad (2.10)

From Eqs. (2.9) and (2.10), it is easy to see that the $U(1)$ gauge field is invariant when the unit vector field $\mathbf{n}$ transforms into $-\mathbf{n}$.

Now we generalize the above $U(1)$ gauge potential decomposition to the case of the double-valued vector system—liquid crystals. We will show that the $U(1)$ gauge field in liquid crystals possesses an inner structure in terms of the 2-dimensional director $\mathbf{d}$. Note that, in nematic liquid crystals, the director inversion $\mathbf{d} \rightarrow -\mathbf{d}$ corresponds to the $U(1)$ gauge transformation,

$$d' = e^{i\alpha(\vec{x})} d, \quad A'_\mu = A_\mu + \partial_\mu \alpha(\vec{x})$$  \quad (2.11)

with the phase factor $\alpha = \pi$. Then the $U(1)$ gauge potential $A'_\mu$ equals to $A_\mu$ under $\mathbf{d} \rightarrow -\mathbf{d}$, which means the director inversion leaves the gauge field unchanged and does not influence the physics of the system. Because of the invariance of the above
decomposition expressions in Eqs. (2.9) and (2.10) under \( n \to -n \), the single-valued vector \( n \) can be substituted by the double-valued director \( d \), i.e.,

\[
A_\mu = \epsilon^{ab} d^a \partial_\mu d^b, \quad F_{\mu\nu} = 2\epsilon^{ab} \partial_\mu d^a \partial_\nu d^b, \quad \tag{2.12}
\]

which are naturally invariant under the director inversion \( d \to -d \). As a result, in liquid crystals the \( U(1) \) gauge potential \( A_\mu \) and gauge field tensor \( F_{\mu\nu} \) can be decomposed in terms of the 2-dimensional director \( d^a (a = 1, 2) \).

In the following section, using the \( \phi \)-mapping topological current theory, we show that there is a topological current which can be derived from \( F_{\mu\nu} \), and topological defects structure is just inhering in this topological current.

§3. The topological structure of monopoles and strings in liquid crystals

Although a rich array of topological defects may appear in liquid crystals, the exotic types of defects, i.e., domain walls and textures, are more energetic and therefore rare in appearance. Here we focus on the monopole and string configurations with the defects’ dimensionality 0 and 1, respectively. The study of the topological defects is often involved in a \( n \)-dimensional order parameter associated with the \( d \)-dimensional space. For the 2-dimensional director case, the defects are point defects (monopoles) for the spatial dimensionality \( d = 2 \), line defects (strings) for \( d = 3 \). More generally, for \( n = d \), one has point defects; for \( n = d - 1 \), one generates line defects.

Let us consider the \( \phi \)-mapping \( \phi^a : x \to \phi^a(x) (a = 1, 2) \) with \( x = (x^1, x^2, \cdots, x^i) \) the local coordinate in real physical space, which is called the order parameter of topological defects in liquid crystals. Since the unit director \( d \) is parallel to the order parameter \( \vec{\phi} = (\phi^1, \phi^2) \), it can be further expressed as direction field of order parameter at the places \( \phi^a(x) \neq 0 \), i.e.,

\[
d^a = \frac{\phi^a}{||\phi||}, \quad ||\phi||^2 = \phi^a\phi^a, \quad d^a d^a = 1, \quad a = 1, 2. \quad \tag{3.1}
\]

From Eq. (3.1), one can see that the director inversion \( d \to -d \) corresponds to the transformation of \( \vec{\phi} \to -\vec{\phi} \) and the zeros of \( \vec{\phi} \) are just the singularities of \( d \) at which the director field is indefinite. In liquid crystals, it is known that topological defects locate at the singularities of the director field \( d \). This property will be reflected mathematically in the following.

From the expression of the gauge field tensor \( F_{\mu\nu} \) in Eq. (2.12), the topological current for monopoles and strings in liquid crystals is introduced by

\[
j^i = \frac{1}{4\pi} \epsilon^{ijk} F_{jk} = \frac{1}{2\pi} \epsilon^{ijk} \epsilon_{ab} \partial_j d^a \partial_k d^b, \quad i, j, k = 1, 2, 3. \quad \tag{3.2}
\]

where for monopoles \( i, j, k \) denote the \((2 + 1)\)-dimensional space-time coordinates with \( x^1 \) the time parameter, and for strings, \( i, j, k \) denote the 3-dimensional space coordinates. The topological current in the above equation is the special case of the \( \phi \)-mapping topological current theory. Then using \( \partial_i d^a = \partial_i (\phi^a/||\phi||) = \)
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\[ \partial_t \phi^a / ||\phi|| + \phi^a \partial_t (1 / ||\phi||) \] and the Green function equation in \( \phi \) space, \( \partial_t \partial_a ln ||\phi|| = 2\pi \delta^2(\vec{\phi})(\partial_a = \frac{\partial}{\partial \phi^a}) \), we obtain the \( \delta \)-function like topological current \(^{20}\)

\[ j^i = \delta^2(\vec{\phi}) D^i(\frac{\partial}{\partial x}), \quad (3.3) \]

where \( D^i(\vec{\phi}) = \frac{1}{2} \epsilon^{ijk} \epsilon^{ab} \partial_j \phi^a \partial_k \phi^b \) is the Jacobian vector. Since the \( \delta \)-function and the Jacobian vector \( D^i(\vec{\phi}) \) satisfy

\[ \delta(\vec{\phi}) = \delta(-\vec{\phi}) \quad \text{and} \quad D^i(\vec{\phi}) = D^i(-\vec{\phi}), \quad (3.4) \]

the topological current in Eq. (3.3) is invariant under the order parameter transformation \( \vec{\phi} \rightarrow -\vec{\phi} \), which means the topological current is also applicable for liquid crystals.

The expression of Eq. (3.3) provides an important conclusion

\[ j^i \begin{cases} \neq 0, & \text{if and only if } \vec{\phi} \neq 0, \\ \neq 0, & \text{if and only if } \vec{\phi} = 0, \end{cases} \quad (3.5) \]

so it is necessary to study the zero points of \( \vec{\phi} \) to determine the non-zero solutions of \( j^i \). Suppose that the order parameter \( \phi^a(x) \) processes \( N \) isolated zeros, according to the implicit function theory,\(^{18}\) under the regular condition

\[ D^i(\phi/x) \neq 0, \quad (3.6) \]

the general solutions of

\[ \begin{aligned} \phi^1(x^1, x^2, x^3) &= 0, \\
\phi^2(x^1, x^2, x^3) &= 0, \end{aligned} \quad (3.7) \]

can be point-like form: \( z_k = z_k(t) \) \((k = 1, 2, \ldots, N)\) located in the \((2 + 1)\)-dimensional space-time or string-like form \( L_k: z_k = z_k(s) \) \((s \text{ is a line parameter})\) located in the 3-dimensional space. These singular solutions are just monopoles or strings generated from the zeros of the order parameter field \( \vec{\phi} \), i.e., the singularities of the director field \( \mathbf{d} \).

We notice that \( \delta(\vec{\phi}) = \infty \) for \( x = z_k(t) \) or \( z_k(s) \), and vanishes outside these zeros. According to the \( \delta \)-function theory,\(^{19}\) \( \delta(\vec{\phi}) \) can be expanded as

\[ \delta(\vec{\phi}) = \begin{cases} \sum_{k=1}^{N} \frac{\beta_k}{D(\vec{\phi})|_{z_k}} \delta(x - z_k(t)) & \text{for monopoles,} \\ \sum_{k=1}^{N} \int_{L_k} \left| \frac{\partial \delta}{\partial \phi} \right|_{\Sigma_k} \delta(x - z_k(s)) ds & \text{for strings,} \end{cases} \quad (3.8) \]

where \( D(\vec{\phi})|_{z_k} = \frac{1}{2} \epsilon^{ijk} \epsilon^{ab} \frac{\partial \phi^m}{\partial x^j} \frac{\partial \phi^n}{\partial x^k} \); \( D(\vec{\phi})|_{\Sigma_k} = \frac{1}{2} \epsilon^{ijk} \epsilon^{ab} \frac{\partial \phi^m}{\partial u^j} \frac{\partial \phi^n}{\partial u^k} \) and \( \Sigma_k \) is the \( k \)-th planer element transversal to the string \( L_k \) with local coordinates \((u^1, u^2)\). The positive \( \beta_k \) is the Hopf index of \( \phi \)-mapping. In liquid crystals, due to the equivalence of the director \( \mathbf{d} \) and \(-\mathbf{d}\), the Hopf index can be integers and half-integers. Meanwhile the general velocity \( V^i \) of the \( k \)-th monopole and string is respectively given by,\(^{20}\)

\[ \begin{aligned} \frac{dz_k^i}{dt} \bigg|_{z_k(t)} &= \frac{D^i(\phi/x)}{D(\phi/x)} \bigg|_{z_k}, \\
\frac{dz_k^i}{ds} \bigg|_{z_k(s)} &= \frac{D^i(\phi/x)}{D(\phi/u)} \bigg|_{z_k}. \end{aligned} \quad (3.9) \]
which determines the motion of the $k$th topological defect $z_k$. Then applying Eqs. (3.3), (3.8) and (3.9), the topological current becomes

$$ j^i = \left\{ \begin{array}{ll} \sum_{k=1}^{N} \beta_k \eta_k \frac{dz_k^i}{dt} \delta(x - z_k(t)) & \text{for monopoles}, \\
\sum_{k=1}^{N} \beta_k \eta_k \int_{L_k} \frac{dz_k}{ds} \delta(x - z_k(s)) ds & \text{for strings}, \end{array} \right. $$

(3.10)

where $\eta_k = \text{sgn} D(\phi/x)_{z_k} = \pm 1$ (for monopoles) or $\eta_k = \text{sgn} D(\phi/u)_{\Sigma_k} = \pm 1$ (for strings) is the Brouwer degree of $\phi$-mapping: $\eta_k = +1$ corresponds to the defects solutions, while $\eta_k = -1$ corresponds to anti-defects solutions. Between the topological current and topological current density, there is an important relation: $j^i = \rho V^i$. From Eq. (3.10), the topological current density $\rho$ becomes

$$ \rho = \left\{ \begin{array}{ll} \sum_{k=1}^{N} \beta_k \eta_k \delta(x - z_k(t)) & \text{for monopoles}, \\
\sum_{k=1}^{N} \beta_k \eta_k \int_{L_k} \delta(x - z_k(s)) ds & \text{for strings}, \end{array} \right. $$

(3.11)

This quantized current in Eq. (3.10) and current density in Eq. (3.11) are the unified form of the defect densities done by Liu and Mazenko. In their work the topological index $\beta_k \eta_k$ is absent and it requires additional homotopic analysis to classify defects.

From Eq. (3.11), one can obtain the topological charge of the $k$th monopole and string

$$ Q_k = \int_{\Sigma_k} \rho d\sigma = \beta_k \eta_k, $$

(3.12)

where the surface $\Sigma_k$ is a neighborhood of $z_k$ with $z_k \neq \partial \Sigma_k$, $\Sigma_k \cap \Sigma_i = \emptyset$. For monopoles, the surface element $d\sigma$ satisfies: $\epsilon^{ij} d\sigma = dx^i \wedge dx^j$ ($i, j = 1, 2$); while for strings, $\epsilon^{ij} d\sigma = du^i \wedge du^j$ ($i, j = 1, 2$). Then the total topological charge carried by $N$ topological defects on $\Sigma = \cup \Sigma_k$ is

$$ Q = \sum_{k=1}^{N} Q_k = \sum_{k=1}^{N} \beta_k \eta_k. $$

(3.13)

It is obvious to see that the topological current in Eq. (3.10) represents $N$ topological defects of which the $k$th topological defect is charged with the topological charge $\beta_k \eta_k$. Therefore, this current describes how the defects move and how the topological charges are distributed for topological defects in a topologically quantized way.

Recent experiments with nematic liquid crystals have revealed interesting knot-like vortex lines, which include the ring-like and knot-like configurations. For the topology of the closed lines $L_{\text{ring}}$ and $L_{\text{knot}}$, we obtain the topological current

$$ j^i = \sum_{k=1}^{N} \beta_{r_k} \eta_{r_k} \oint_{L_{\text{ring}}} \frac{D^i(\phi/x)}{D(\phi/u)} \delta(x - z_{r_k}(s)) ds + \sum_{i=1}^{N} \beta_{k} \eta_{k} \oint_{L_{\text{knot}}} \frac{D^i(\phi/x)}{D(\phi/u)} \delta(x - z_{k}(s)) ds, $$

(3.14)

where $\beta_{r_k} \eta_{r_k}$ and $\beta_{k} \eta_{k}$ are the $i$th topological number of $\tilde{\phi}$ surrounding the ring and knot-like lines $L_{\text{ring}}$ and $L_{\text{knot}}$, respectively.
§ 4. The winding numbers of monopoles and strings in liquid crystals

Defects are classified according to their topological strength $S$, and a more general definition of the topological strength is the winding number $W$ of topological defects. In liquid crystals, half-integer values of winding numbers are also allowed ($W = \pm \frac{1}{2}, \pm 1, \pm \frac{3}{2}, \ldots$) because of the “head to tail” invariance of the director. Defects of opposite winding number may annihilate each other.

In the single-valued vector field system, the winding number $W_k$ is defined by the Gauss map $n$. As to monopoles, we definitely consider the 2-dimensional space. Moreover, as to strings in the system, for simplicity and without loss of generality, we also suppose that the director field $d$ lies locally in the plane which is vertical to strings with the local coordinates $(u^1, u^2)$. The Gauss map is $n : \partial \Sigma_k \rightarrow S^2$ and the winding number $W_k$ is defined by

$$W_k = \frac{1}{2\pi} \int_{\partial \Sigma_k} n^* (\epsilon_{abc} n^a \wedge dn^b), \quad (4.1)$$

where $n^*$ is the pull back of the map $n$ and $\partial \Sigma_k$ is the boundary of $\Sigma_k$. The winding number is a topological invariant and is called the degree of Gauss map. Using the Stokes' theorem, the above formula can be written as

$$W_k = \frac{1}{2\pi} \int_{\Sigma_k} \epsilon_{ab} d n^a \wedge d n^b = \frac{1}{2\pi} \int_{\Sigma_k} \epsilon^{ij} \epsilon_{ab} \partial_i n^a \partial_j n^b d\sigma. \quad (4.2)$$

From the above equation, the winding number $W_k$ is an integral invariant under the transformation $n \rightarrow -n$. Therefore, the winding number $W_k$ for topological defects in liquid crystals can be expressed as

$$W_k = \frac{1}{2\pi} \int_{\Sigma_k} \epsilon_{ab} d(d^a) \wedge d(d^b) = \frac{1}{2\pi} \int_{\Sigma_k} \epsilon^{ij} \epsilon_{ab} \partial_i d^a \partial_j d^b d\sigma. \quad (4.3)$$

In topology, the winding number $W_k$ means that when the topological defect $z_k$ covers $\Sigma_k$ in the real space once, the unit vector will cover the unit sphere $S^2$ for $W_k$ times. In the single-valued vector field system, to return the same, the vector field $n$ rotates by $2k\pi$ around the topological defects, where $k$ is an integer. While in liquid crystals, due to the equivalence of $d$ and $-d$, the director field $d$ only needs to rotate by $k\pi$. By the topological meaning of the winding number, $W_k$ in liquid crystals is half of that in the single-valued vector field system, that is to say, $W_k$ can be integers and half-integers, which is a generalization of our previous integer winding number.

On the other hand, $W_k$ is also the topological charge $Q_k$ of the $k$th topological defect. Substituting Eq. (3.14) into Eq. (4.3), one can find the relation between the winding number $W_k$, the Hopf index $\beta_k$ and the Brouwer degree $\eta_k$

$$W_k = \int_{\Sigma_k} \rho d\sigma = \beta_k \eta_k, \quad (4.4)$$
which shows that the winding number $W_k$ is just the topological charge $Q_k$ in Eq. (3.12) of the $k$th topological defect. For liquid crystals with a set of topological defects, the total winding number is

$$W = \sum_{k=1}^{\beta_k \eta_k} W_k = \sum_{k=1}^{\beta_k \eta_k} \beta_k \eta_k = W_+ - W_-,$$

(4.5)

where $W_+$ and $W_-$ are the total winding number of defects and anti-defects. The above expression in Eq. (4.5) naturally arrives at the conclusion that, in liquid crystals, each isolated topological defect created from the singularities of the director is characterized by the topological number $W_k$. As some corollaries of Eq. (4.5), the following two points are obtained. (i) For a certain surface, the total winding number of topological defects is not arbitrary but a topological invariant—the Euler characteristic on the vacuum manifold obtained from the Gauss-Bonnet-Chern theorem and Hopf index theorem. (ii) In liquid crystals, since the free energy is proportional to the square of winding number, there exists in general the monopoles of winding number $\pm 1$ and strings of winding number $\pm \frac{1}{2}$ with the principle of least free energy.

§5. Conclusion

In summary, in the light of the gauge potential decomposition theory and the $\phi$-mapping topological current theory, we construct a $U(1)$ topological gauge field theory for monopoles and strings in liquid crystals, in which the $U(1)$ gauge field is invariant under the director inversion. It is revealed that in liquid crystals the $U(1)$ gauge potential and $U(1)$ gauge field tensor can be decomposed in terms of the 2-dimensional director $d$, which is essential to study the topological properties of monopoles and strings located at the singularities of the director field. We derive a unified topological current for monopoles and strings from the $U(1)$ gauge field tensor and show that the topological current takes the form of $\delta$-function. Furthermore, the topological charge carried by monopole and string is just the winding number labelled by the Hopf index and Brouwer degree. The theory formulated in this paper is a new concept for topological defects in liquid crystals.

At last there are two points which should be stressed. Firstly, when the regular condition Eq. (3.6) fails caused by the external field (e.g. the magnetic field, the electric field), the bifurcation processes of topological defects will occur. Secondly, the topological current of the knotlike vortex lines in liquid crystals is given in Eq. (3.14), which will give us a insight into the topological essence of the knotlike vortex lines and be detailed in our further work.

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