Rotational suppression of the Tayler instability in stellar radiation zones

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ABSTRACT
The study of the magnetic field in stellar radiation zones is an important topic in modern astrophysics because the magnetic field can play an important role in several transport phenomena such as mixing and angular momentum transport. We consider the influence of rotation on stability of a predominantly toroidal magnetic field in the radiation zone. We find that the effect of rotation on the stability depends on the magnetic configuration of the basic state. If the toroidal field increases sufficiently rapidly with the spherical radius, the instability cannot be suppressed entirely even by a very fast rotation although the strength of the instability can be significantly reduced. On the other hand, if the field increases slowly enough with the radius or decreases, the instability has a threshold and can be completely suppressed in rapidly rotating stars. We find that in the regions where the instability is entirely suppressed a particular type of magnetohydrodynamic waves may exist which are marginally stable.

Key words: instabilities - magnetohydrodynamics - stars: interiors - stars: magnetic field - Sun: interior

1 INTRODUCTION
In recent years, magnetic fields of various strength and topology have been detected in increasing number of stars (see, e.g., Donati et al. 2006). Despite the origin of stellar magnetic fields is usually related to the turbulent dynamo action in convection zones, it is quite possible that magnetic fields exist also in internal radiation zones as well. For instance, a uniform rotation of the solar core may be explained naturally by the presence of the magnetic field. A thin transition layer between the convection and radiation zones of the Sun, called the tachocline, cannot be understood within the frame of purely fluid-dynamical mechanisms and requires a large-scale magnetic field in the Sun’s interior (Gough & McIntyre 1998). The origin, topology, and strength of the magnetic field in internal radiation zones are the subject of debates for decades. Likely, dynamo cannot operate in stellar radiation zones. A dynamo action requires sufficiently rapid flows with \( \Re_m \gg 1 \) (\( \Re_m \) is the magnetic Reynolds number) to generate a global magnetic field but, according to the generally accepted point of view, such flows are not available in the internal radiation zones (see, e.g., Schwarzschild 1958). Sometimes, the origin of the magnetic field in radiation zones is related to the dynamo mechanism proposed by Spruit (1999) but the existence of such mechanism has not been proven yet. One might speculate that a fossil magnetic field could exist in radiation zones. Even a weak fossil field with non-vanishing poloidal component will quickly wrap up into a predominantly toroidal configuration, under the action of differential rotation. Such configurations can be generated if \( \Re_m \) of differential rotation is greater than 1, or \( |\nabla \Omega| > \eta_m / r^3 \) where \( \Omega \) and \( \eta_m \) are the angular velocity and magnetic diffusivity, respectively. Estimating \( |\nabla \Omega| \sim \Delta \Omega / r \) where \( \Delta \Omega \) is a departure from the rigid rotation and assuming that the conductivity of plasma is \( \sim 10^{16} \, \text{s}^{-1} \), one can obtain that this condition is satisfied if \( \Delta \Omega / \Omega > 10^{-18} \Omega_{\sec}^{-1} \) where \( \Omega_{\sec} \) is the angular velocity in inverse seconds. Therefore, even very weak departures from the rigid rotation lead to a generation of the toroidal field. The magnetic field with a predominant toroidal component is also typical for the liquid cores of neutron stars (Bonanno et al. 2005, 2006) where the field can be generated by the turbulent dynamo during the very early evolutionary stage. The toroidal field resulting from winding up or dynamo action cannot be arbitrarily strong because various magnetic instabilities will set in when it becomes strong enough. That is why much attention has been drawn to the instabilities that may affect the toroidal magnetic field in stably stratified radiation zones.

The magnetic field in a radiation zone can be subject to various instabilities such as the magnetic buoyancy (see, e.g., Gilman 1970, Acheson 1978) or magnetorotational in-
stability (Velikhov 1959, Balbus 1995). Likely, however, that the most efficient instabilities are caused by the electric currents maintaining the magnetic configuration (Spruit 1999). Such instabilities are well studied in cylindrical geometry in the context of laboratory fusion research (see, e.g., Friedberg 1970, Godlblośd & Hagebeuk 1972). In astrophysical conditions, the instability caused by electric currents is studied mainly in cylindrical geometry as well (see, e.g., Tayler 1973). It turns out that the properties of instability depend on the ratio of the axial and toroidal fields and, even if this ratio is small, the axial field can alter the instability substantially (see, e.g., Knobloch 1992, Bonanno & Urpin 2008a). The effect of an axial field on the Tayler instability of the toroidal field has been studied in detail in cylindrical geometry by Bonanno & Urpin (2008b, 2011). The nonlinear evolution of the Tayler instability was considered by Bonanno et al. (2012) who argued that symmetry-breaking can give rise to a saturated state with non-zero helicity even if the initial state has zero helicity. A production of non-zero helicity is likely possibility for the onset of adynamy action.

Stability of the spherical magnetic configurations is much less studied because of mathematical problems. With numerical simulations Braithwaite & Nordlund (2006) studied the stability of a random initial field and argued that this field relaxes on a stable mixed magnetic configuration with both poloidal and toroidal components. Stability of magnetic configurations with a predominantly toroidal field draws particular attention. Numerical modeling by Braithwaite (2006) confirmed that the toroidal field with \( B_\theta \propto s \) or \( s^2 \) (where \( s \) is the cylindrical radius) is unstable to the \( m = 1 \) mode as it was predicted by Tayler (1973). It is widely believed that rotation and stratification can play an important role in stellar radiation zones providing a stabilizing influence on the Tayler instability (see, e.g., Pitts & Tayler 1985, Spruit 1999). Recently, the effect of stratification and thermal conductivity on the Tayler instability of the toroidal field has been considered by Bonanno & Urpin (2012). The authors studied a linear stability and used a local approximation in latitude and global in the radial direction. They calculated the growth rate of instability and argued that the combined influence of gravity and thermal conductivity can never suppress the instability entirely. Stratification suppresses the instability at the pole more efficiently than at the equator. The growth rate of instability can be essentially reduced by a stable stratification. A decrease of the growth rate caused by stratification is inversely proportional to the Brunt-Väisälä frequency. Therefore, if gravity is strong, the instability develops very slowly. A simple fitting expression has been obtained for the growth rate of instability in a stratified radiation zone. Bonanno & Urpin (2012) argued that the instability of modes with a large number of nodes in the radial direction is significantly more suppressed than the fundamental eigenmode. The reason of this is qualitatively clear. As it was pointed out by Spruit (1999), the stabilizing influence of stratification is less pronounced for perturbations with short radial lengthscales. Therefore, it seems at the first glance that the instability should operate most efficiently on very short radial lengthscales. However, this conclusion is incorrect because a destabilizing effect of electric currents (that is the reason of the Tayler instability) also decreases if the radial lengthscale decreases. This occurs because the instability becomes almost two-dimensional if the radial lengthscale goes to 0 but, as it was shown by Kitchatinov & Rüdiger (2008), the Tayler instability does not exist in 2D. Therefore, the instability cannot arise for perturbations with very short radial lengthscales despite the stabilizing influence of stratification is minimal for them.

Rotation can also suppress the Tayler instability and stabilize the magnetic configurations. The effect of rotation has been considered by a number of authors. Spruit (1999) found that the growth rate of the Tayler instability near the rotation axis should be of the order of \( \sim \omega_A(\omega_A/\Omega) \) if \( \Omega \gg \omega_A \) in a particular case of the magnetic field dependent on \( s \) alone, where \( \omega_A \) and \( \Omega \) are the Alfvén frequency and angular velocity of the star, respectively. In numerical modelling by Braithwaite (2006), the Tayler instability was suppressed if \( \Omega \) is above a certain value of the order of \( \omega_A \). Above this value a distinct oscillatory behaviour sets in with marginal stability. Stability of the toroidal field in rotating radiation zones has been considered also by Kitchatinov (2008) and Kitchatinov & Rüdiger (2008) who clarified that the Tayler’s instability recovers only in 3D. Radial displacements are essential for this instability. It does not exist in the 2D case of strictly horizontal (perpendicular to gravity) disturbances and only stable modes exist in this case. These authors argued that the magnetic instability is characterized by the threshold field strength. The instability arises if the magnetic field is stronger than this threshold but it does not occur if the field is weaker than the threshold. Estimating this threshold in the solar radiation zone, Kitchatinov & Rüdiger (2008) impose the upper limit on the magnetic field \( \approx 600 \) G. The stability of the toroidal field in a rotating radiation zone has been studied also by Zahn et al. (2007) in the particular case \( B_\theta \propto s \). The particular type of oscillatory modes found by these authors is relevant to rotation and is stable in the non-dissipative limit. However, instability of the considered modes can occur in a form of an oscillatory diffusive instability if dissipation is provided by radiative or Ohmic diffusion. Unfortunately, the authors of these studies did not compare the results of their calculations and the reason of such qualitatively different behaviours is unclear.

In this paper, we consider in detail the effect of rotational suppression of the Tayler instability in the case of a predominantly toroidal field. We show that rotation can influence the instability in different ways depending on the background magnetic configuration. The paper is organized as follows. The basic equations and mathematical formulation of the problem are presented in Sec. 2. This is followed by results of numerical calculations of the growth rate and frequency of unstable modes in Sec. 3. The paper closes with a summary of the main results and some remarks in Sec. 4.

2 BASIC EQUATIONS

We assume that the magnetic field in a radiation zone is sub-thermal and the magnetic pressure is smaller than the gas pressure. In this case, the Boussinesq approximation is applied for a consideration of low-frequency modes. The ideal MHD equations read in this approximation

\[
\frac{\partial \vec{v}}{\partial t} + (\vec{v} \cdot \nabla)\vec{v} = \frac{-\nabla p}{\rho} + \frac{1}{4\pi \rho} (\nabla \times \vec{B}) \times \vec{B},
\]  

(1)
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\[
\frac{\partial \vec{B}}{\partial t} - \nabla \times (\vec{v} \times \vec{B}) = 0, \tag{2}
\]

\[
\nabla \cdot \vec{v} = 0, \quad \nabla \cdot \vec{B} = 0, \tag{3}
\]

where \( \vec{g} \) is gravity, \( p \) and \( \rho \) are the gas pressure and density, respectively. The equation of thermal balance reads in the Boussinesq approximation

\[
\frac{\partial T}{\partial t} + \vec{v} \cdot (\nabla T - \nabla_{ad} T) = 0, \tag{4}
\]

where \( \nabla_{ad} T \) is the adiabatic temperature gradient.

Consider the stability of an axisymmetric toroidal magnetic field using spherical coordinates \((r, \theta, \varphi)\). We assume that the radiation zone rotates with the angular velocity \( \Omega = \text{const} \) and that the toroidal field depends on \( r \) and \( \theta \), \( B_\theta(B_\theta(r, \theta)) \). In the unperturbed state, the radiation zone is assumed to be in hydrostatic equilibrium, then

\[
\nabla \cdot \vec{p} = \vec{g} + \frac{1}{4\pi \rho} (\nabla \times \vec{B}) \times \vec{B} + \vec{e}_r \Omega^2 r \sin \theta, \tag{5}
\]

where \( \vec{e}_r \) is the unit vector in the cylindrical radial direction. The rotational energy is assumed to be much smaller than the gravitational one, \( g \gg r \vec{B}^2 \). Since the magnetic energy is subthermal, \( \vec{g} \) is approximately radial in our basic state. Therefore, longitudinal variations of the unperturbed density and pressure are small.

We consider a linear stability. Small perturbations will be indicated by subscript 1, while unperturbed quantities will have no subscript. Linearizing Eqs.(1)-(4), we take into account that small perturbations of the density and temperature in the Boussinesq approximation are related by \( \rho_1/\rho = -\beta(T_1/T) \) where \( \beta \) is the thermal expansion coefficient. We use a local approximation in the \( \theta \)-direction and assume that small perturbations depend on \( \theta \) as \( \exp(-i l \theta) \), where \( l \geq 1 \) is the polar wavenumber. Since the basic state is stationary and axisymmetric, the dependence of perturbations on \( t \) and \( \varphi \) can be taken in the exponential form as well. Then, perturbations are proportional to \( \exp(\sigma t - i l \theta - i m \varphi) \) where \( m \) is the azimuthal wavenumber and \( \sigma \) is the growth rate. The dependence of perturbations on \( r \) should be determined from Eqs.(1)-(4).

Since the influence of gravity on instability has been studied in detail by Bonanno & Uspin (2012), we concentrate in this paper mainly on the effect of rotation. We consider a simplified problem assuming that stratification is neutral and \( \nabla T = \nabla_{ad} T \). In this case, the stabilizing effect of stratification is excluded and we can study only the influence of rotation alone. The combined influence of rotation and stratification will be considered elsewhere. For the sake of simplicity, we also assume that the unperturbed density is approximately homogeneous in the radiation zone.

Eliminating all variables in favor of \( v_{ir} \), we obtain with the accuracy in terms of the lowest order in \((k_B)^{-1}\)

\[
\left( \sigma_0^2 + \omega_A^2 + D \Omega^2 \right) v'_{ir} + \left( \frac{4}{\rho} \sigma_0^2 + \frac{2}{H} \omega_A^2 \right) v'_{ir} + \left( \frac{2}{r^2} \sigma_0^2 - k_\perp^2 (\omega_0^2 + \omega_A^2) - D \Omega^2 k_\parallel^2 \right) v'_{ir} + \frac{2 k_\parallel^2}{r} D = \sigma_0 \Omega e \left( \frac{k_\parallel^2}{r} + 4 D \frac{k_\parallel}{r} \sigma_0 \omega_A^2 \right) v'_{ir} = 0, \tag{6}
\]

where the prime denotes a derivative with respect to \( r \) and

\[
\sigma_0 = \sigma - i m \Omega, \quad \omega_A^2 = \frac{k_\parallel^2 B_\alpha^2}{4 \pi \rho}, \quad D = \frac{\sigma_0^2}{\sigma_0^2 + \omega_A^2}, \quad \Omega = 2 \Omega \cos \theta, \quad \Omega_e = 2 \Omega \sin \theta, \quad k_\perp^2 = k_\theta^2 + k_\varphi^2, \quad \frac{1}{H} = \frac{\partial}{\partial r} \ln(r B_\varphi). \tag{7}
\]

The polar and azimuthal wavevectors are \( k_\theta = l/r, k_\varphi = m/r \sin \theta \), respectively.

This equation with the corresponding boundary conditions describes the stability problem as a non-linear eigenvalue problem. Fortunately, the main qualitative features of this problem are not sensitive to the choice of boundary conditions. That is why we choose the simplest boundary conditions and assume that \( v_{ir} = 0 \) at the inner and outer boundaries, \( r = R_i \) and \( r = R \), respectively.

3 NUMERICAL RESULTS

We assume that the radiation zone is located at \( R_i \leq r \leq R \), introducing the dimensionless radius \( x = r/R \), at \( x_i = x \leq 1 \) where \( x_i = R_i/R \). We choose the internal radius of the radiation zone, \( x_i \), to be small but finite for computational convenience. In particular, \( x_i = 0.1 \) in this investigation. Our results do not depend qualitatively on the precise value of \( x_i \), as we have explicitly verified by performing calculations with \( x_i \) in the range \( 0.1 \leq x_i \leq 0.4 \) (less than 1% difference in the growth rate for \( x_i = 0.1 \) and \( x_i = 0.3 \) for \( \alpha = 2 \) for instance).

We represent the toroidal field as

\[
B_\varphi = B_0 (x/x_i)^\alpha \sin \theta, \tag{8}
\]

where \( B_0 \) is the field strength at \( x = x_i \) at the equator. The dependence of \( B_\varphi \) on \( r \) is uncertain in the radiation zone. Therefore, we consider different possibilities, varying the parameter \( \alpha \). The radial dependence in Eq.(8) is the simplest one and convenient from the computational point of view. A power law dependence on the cylindrical radius was also used, for example, in the pioneering paper by Tayler (1973) and, therefore, it is more convenient to compare the results for spherical and cylindrical geometries, choosing the dependence (8). Eq.(8) can mimic various physical situations. The case \( \alpha > 0 \), for instance, can be a representative of the radiation zone in a star with the outer convective zone. On the contrary, the case \( \alpha < 0 \) can mimic a radiation zone with the magnetic field generated in the inner convective core. Certainly, the magnetic field in a radiation zone can have a more complicated dependence than Eq.(8) but the main qualitative features can be understood from this simple model.

It is convenient to introduce the dimensionless quantities

\[
\Gamma = \frac{\sigma_0}{\omega_{A0}}, \quad \eta = \frac{2 \Omega}{\omega_{A0}}, \tag{9}
\]

where \( \omega_{A0} = B_0/R \sqrt{4 \pi \rho} \). We will represent the results in terms of \( \Gamma \) and \( \eta \).

In Fig. 1, we plot the growth rate and frequency of unstable modes as functions of \( \eta \) for different \( \theta \) and \( \alpha = 3 \). Such distribution of \( B_\varphi \) can mimic, for example, the radiation zone of a star with a convective envelope. In this case,
the bottom of a convection zone is likely the location of the
toroidal field generated by a dynamo action. However, this
case can be also the representative of a star with relic mag-
netic fields. Details of the formation of such fields are very
uncertain but it is quite possible that differential rotation
is stronger in the outer layers at the early stage of stellar
evolution. Then, the toroidal field generated by differential
rotation should be stronger in the outer layers as well.

Fig. 1 shows that the instability is most efficient at the
equator and does not occur around the pole. There always
exist a range of \( \theta \) around the pole where the instability does
not occur and this range depends on \( \alpha \). Note that the insta-
bility exhibits a similar behaviour also in the non-rotating
case (see Bonanno & Urpin 2012). Our result is at variance
with a widely accepted opinion that toroidal magnetic con-
figurations are always unstable at the axis (see, e.g., Spruit
1999). This opinion is based on similarity of the spherical
magnetic configuration near the axis and the axisymmetric
cylindrical configuration. However, this analogy is gener-
ally incorrect because, in spherical geometry, the toroidal field
near the axis depends also on distance along the magnetic
axis and not only on the distance from it. This dependence
plays the crucial role in stability. At small \( \eta \), the maximum
growth rate is of the order of \( \omega_{A0} \) and is reached at the equa-
tor. However, the growth rate clearly shows some suppress-
ion for a faster rotation. Suppression becomes important
already at relatively small values of \( \eta \sim 1-2 \). We can distin-
guish two substantially different regimes of suppression. If \( \theta \)
is greater than some characteristic value, \( \theta_0 \), the growth rate
decreases with an increase of \( \eta \) approximately as \( 1/\eta \) and it
does not vanish even at very large \( \eta \). It turns out, there-
fore, that rotation can never suppress the Tayler instabil-
y in the region around the equator, where \( \theta > \theta_0 \), but can
only decrease its growth rate. Note that the Tayler’s modes
are complex in this region in contrast to the non-rotating
case. The frequency \((\sim \text{Im } \Gamma)\) is basically comparable to the
growth rate and also decreases if rotation becomes faster.
Note also that similar dependence of the growth rate on \( \eta \)
was obtained by Spruit (1999) who considered the instabil-
ity in cylindrical geometry with the toroidal field depend-
on the cylindrical radius \( \alpha \), \( B_z = B_\alpha(s) \) where \( s \) is the
cylindrical radius. The exact value of \( \theta_0 \) is rather difficult
to calculate because of computational problems but, in the
case \( \alpha = 3 \), it is equal approximately to \( 26^\circ \). For \( \theta < \theta_0 \),
the effect of rotation is substantially stronger. In contrast to the
instability near the equator, the instability at \( \theta < \theta_0 \) is
characterized by the threshold, \( \eta_{cr} \). The instability occurs if
\( \eta < \eta_{cr} \) (Re \( \Gamma > 0 \) in this region) but it does not occur if
\( \eta > \eta_{cr} \) since \( \Gamma \) is imaginary at such \( \eta \). The threshold, \( \eta_{cr} \),
depends on \( \theta \) and is \( \sim 1 \) if \( \theta = 24^\circ \). The threshold value
\( \eta_{cr} \) is lower for smaller \( \theta \). Therefore, even a relatively slow
rotation with \( \eta < \eta_{cr} \) can suppress the instability entirely in
some region around the pole. Note that the Tayler’s modes
become oscillatory beyond the threshold (at \( \eta > \eta_{cr} \)): their
growth rate is equal to zero but the frequency is always non-
vanishing. In the region around the pole (at \( \theta < \theta_{cr} \)), the
instability does not arise even if rotation is very slow (\( \eta \ll 1 \)).
The critical value of the polar angle, \( \theta_{cr} \), that restricts the
stable region for a non-rotating star, is \( \sim 21^\circ \).

In Fig. 2, we plot the growth rate and frequency for the
case \( \alpha = 2 \). Qualitatively, the behaviour is same but
suppression turns out to be more essential. Again, the in-
stability is most efficient at the equator and does not occur
around the pole. This is qualitatively clear because, on the
equator, with a neutral entropy gradient, we have an equiv-
lent situation to the cylindrical geometry. The growth rate
is maximal for small \( \eta \) and is \( \approx \omega_{A0} \) at the equator that is
a bit lower than in the case \( \alpha = 3 \). Rotational suppres-
sion becomes important already at \( \eta \sim 1 \) and can operate
in two different regimes, depending on \( \theta \). The characteris-
tic value \( \theta_0 \) that distinguishes these two regimes is \( \sim 37^\circ \)
in this case. Unfortunately, we cannot calculate the growth
rate when both \( \theta \) and \( \eta \) are close to their critical values, \( \theta_0 \)
and \( \eta_{cr} \), respectively. This is caused by computational prob-
lems. At \( \theta > \theta_0 \), the growth rate decreases with an increase
of \( \eta \) approximately as \( 1/\eta \) and it does not vanish at large
\( \eta \). Rotation can never suppress the instability in this re-
gion of \( \theta \) but can only reduce its growth rate. Like the case
\( \alpha = 3 \), unstable modes are oscillatory in this region with
the frequency being comparable to the growth rate. Closer
to the pole, at \( \eta < \eta_{cr} \), the instability is characterized by the
threshold, \( \eta_{cr} \): the instability can occur if \( \eta < \eta_{cr} \) but it is
suppressed if \( \eta > \eta_{cr} \). For example, at \( \theta = 37^\circ \), the critical
value \( \eta_{cr} \) is \( \approx 3.5 \) but it is lower for smaller \( \theta \). In the region
near the pole, \( \theta < \theta_{cr} = 24^\circ \), the instability is completely
suppressed and does not occur at any \( \eta \). However, there exist
stable oscillatory modes with a low frequency.

Fig. 3 plots the growth rate and frequency for the
toroidal field with \( \alpha = 1 \). Like the previous cases, the in-
stability is most efficient at the equator and its growth rate
decreases if \( \theta \) decreases. Generally, the instability at \( \alpha = 1 \)
is suppressed stronger than in the cases \( \alpha = 3 \) and \( \alpha = 2 \).
However, there is a qualitative difference in comparison to
these cases: there is no regime near the equator in which the
growth rate is non-vanishing at large \( \eta \) or, in other words,
\( \theta_0 = \pi/2 \). It turns out that such regime of instability can
occur only at a relatively large \( \alpha \). The instability at \( \alpha = 1 \)
is characterized by the threshold, \( \eta_{cr} \), even in the region near
the equator. The threshold is lower than for \( \alpha = 2 \) and is
equal to \( \approx 2 \) at the equator. As usual, the threshold is lower
for smaller \( \theta \). The field near the magnetic axis turns out to

Figure 1. The growth rate (left panel) and frequency (right panel) of the Tayler’s modes as functions of
the rotational parameter \( \eta \) for \( \alpha = 3 \) and \( \alpha = 90^\circ \) (solid), \( 36^\circ \) (dashed), \( 46^\circ \)
(dash-and-dotted), \( 34^\circ \) (dash-dot-dotted), and \( 24^\circ \) (dotted). The
longitudinal wavenumber is \( l = 10 \).
be stable. The instability does not occur at any \( \eta \) in the region with \( \theta < \theta_{cr} \approx 31^\circ \).

The condition of stability \( \eta > \eta_{cr} \) can easily be reformulated in terms of the angular velocity and magnetic field. Rotation suppresses completely the Tayler instability if the star rotates with the angular velocity

\[
\Omega > \frac{\eta_{cr}}{2} \frac{B_0}{R\sqrt{4\pi \rho}}.
\]

We can also rewrite this inequality as the condition for the magnetic field,

\[
B_0 < \frac{2}{\eta_{cr}}\Omega R\sqrt{4\pi \rho}.
\]

The Tayler modes become oscillatory (\( \text{Im} \Gamma = 0 \)) if condition (10) or (11) are satisfied. Note that such behaviour was also seen in numerical modelling of the Tayler instability by Braithwaite (2006). The frequency of marginally stable waves is determined by the Coriolis and Lorentz force and waves is of the order of \( \omega_{A0}(\Omega_i/\Omega) \). The solution of this equation is

\[
\sigma_0^2 + \omega_A^2 + D \Omega_i^2 = 0,
\]

or

\[
\sigma_1^4 + \sigma_1^2(2\omega_A^2 + \Omega_i^2) + \omega_A^2 = 0.
\]

The solution of this equation is

\[
\sigma_1^2 = \frac{1}{2} \left( \Omega_i^2 + 2\omega_A^2 \right) \pm \frac{1}{2} \Omega_i^2 \sqrt{1 + 4\omega_A^2/\Omega_i^2}.
\]

If \( \Omega_i \gg \omega_A \) then we can expand a square root in a power series of \( (\omega_A/\Omega_i)^2 \). Then, choosing the upper sign in Eq.(14), we obtain with accuracy in the lowest order in \( (\omega_A/\Omega_i)^2 \)

\[
\sigma_1^2 \approx -\omega_A^2 (\omega_A/\Omega_i)^2.
\]

This dispersion relation describes a new type of oscillatory modes that can exist in rapidly rotating stars. These modes can be called the “magneto-inertial” waves because they can exist only in a magnetized and rotating plasma. In the considered case, these waves are stable but, likely, they can be unstable under certain conditions. We consider the magneto-inertial waves in more detail elsewhere.

Fig. 4 shows the growth rate and frequency for \( \alpha = -0.4 \). The instability is very much suppressed by rotation in this case. It can occur only in a narrow region around the equator, \( 90^\circ \gg \theta > \theta_{cr} \approx 65^\circ \), and does not occur in the extended region around the rotation axis, \( \theta <\theta_{cr} \approx 65^\circ \). The threshold value of \( \eta \) at the equator is \( \approx 2 \). Therefore, the Tayler instability is entirely suppressed everywhere in the radiation zone with \( \alpha = -0.4 \) if \( \eta > 2 \), or \( \Omega > \omega_{A0} \). However, stable oscillating modes can exist even at much higher \( \eta \).

Magnetic configurations with a rapidly decreasing toroidal field are stable in a cylindrical geometry. For example, the field can be unstable to non-axisymmetric perturbations only if \( d\ln B_\theta(s)/d\ln s > -1/2 \) (Tayler 1973a). Our calculations do not show the presence of instability in the spherical geometry if \( \alpha < -1/2 \).

Fig. 5 plots two examples of eigenfunctions for \( \alpha = 2 \) and \( \eta = 2 \). The eigenfunction at \( \theta = 90^\circ \) corresponds to the unstable mode that cannot be suppressed by rotation. Motions in this mode take place basically in the outer part of the radiation zone. Note that, at larger \( \eta \), this mode (corresponding to \( \theta = 90^\circ \)) turns out to be localized in a more narrow region near the outer boundary. On the contrary, the eigenfunction at \( \theta = 26^\circ \) corresponds to a stable oscillating mode that exists beyond the instability threshold, \( \eta > \eta_{cr} \).

In this case, the mode tends to be located near the inner boundary. This type of modes also becomes sharper with increasing \( \eta \). Note that this seems to be a rather general rule (at least, we did not find an exception): modes tend...
the toroidal field in stellar radiation zones is generally misleading. In order to apply the cylindrical geometry near the axis of a spherical star, one needs to make sure that i) the length scale of perturbations in the axial direction is small compared to the radius and ii) the radial (cylindrical) length scale of perturbations is small compared to the radius. It is clear that perturbations should be affected by the spherical geometry if their radial length scale is comparable to the stellar radius. Therefore, the behaviour of such modes with a large radial length scale may be different in the spherical and cylindrical geometries. As far as perturbations with a small radial length scale are concerned, they are always stable (see Eq.(14)-(15)). Therefore, the analogy with an infinitely long cylinder is generally incorrect and the Tayler instability can be suppressed near the magnetic axis in the spherical geometry. It seems that a dependence on the axial coordinate always makes perturbations more stable.

Rotation provides an additional stabilizing influence on the Tayler instability. The effect of rotation is characterized by the parameter \( \eta = 2\Omega/\omega_{A0} \) that can be large in radiation zones \((\sim 10^2 - 10^3 \text{ if } B \sim 10^4 \text{ G})\). A reduction of the growth rate becomes significant already at a relatively low angular velocity, \( \Omega \sim \omega_{A0} \), that corresponds to \( \eta \sim 1 \). It turns out that the effect of rotation can be twofold and depends crucially on the radial dependence of the toroidal field. If the toroidal field increases with the spherical radius sufficiently rapidly \((\alpha = 2-3)\) then rotation cannot suppress completely the Tayler instability in the region around the equator, even if \( \eta \) is very large. This region is more extended for larger \( \alpha \) but shrinks around the equator and disappear for smaller \( \alpha \).

The growth rate of instability in this region is non-vanishing for any angular velocity but it can be substantially reduced at large \( \eta \). At large \( \eta \), the growth rate decreases \( \propto 1/\eta \) and is approximately equal to \( \omega_{A0}(\omega/\omega_{A0}/\Omega) \). This expression was also obtained by Spruit (1999) for the growth rate near the rotation axis in a particular case \( B_\psi = B_{\psi}(s) \). Note, however, that the Alfvén timescale, \( \omega_{A0}^{-1} \), is typically short \((\sim 1 - 3 \text{ yrs if } B \sim 10^3 \text{ G})\) compared to the stellar lifetime, therefore even a suppressed instability with a reduced growth rate can be important in radiation zones.

Such region near the equator exists only if \( \alpha \) is sufficiently large \((\gtrsim 1.5 - 2)\) but it shrinks and disappears if \( \alpha \) decreases. At smaller \( \theta \), the region is located where the rotational suppression is qualitatively different and where the instability is determined by the threshold value of \( \eta \). The Tayler instability turns out to be suppressed completely in this region if \( \eta > \eta_{cr, \psi} \gtrsim 1 - 2 \). Therefore, modes are stable everywhere in this region for such rapidly rotating stars. The threshold is not very high and corresponds approximately to \( \Omega \sim \omega_{A0} \). Higher eigenmodes are suppressed stronger than the fundamental one and perturbations with a short radial wavelength are always stable. Since the instability is suppressed at \( \eta > \eta_{cr, \psi} \) but not suppressed at \( \eta < \eta_{cr, \psi} \), this implies that the magnetic field should satisfy the condition

\[
B_0 \gtrsim \frac{2}{\eta_{cr}} \Omega R \sqrt{4\pi \rho}
\]  

in order the instability could occur. Estimating \( \Omega R \sim 2 \times 10^5 \text{ cm/s} \) and \( \rho \sim 0.1 \text{ g/cm}^3 \), we obtain that instability can arise in the radiation zone of the Sun if \( B_0 \gtrsim 10^5 \text{ G} \). This estimate of the critical field is more than two orders of magnitude higher than that obtained by Kichatinov & Rüdiger (2008).

Figure 4. The same as in Fig.2 but for \( \alpha = -0.4 \) and \( \theta = 90^\circ \) (solid), and \( 69^\circ \) (dashed).

Figure 5. The eigenfunctions \( v_{1r} \), \( \alpha = 2, \eta = 2, \) and \( \theta = 90^\circ \) (solid) and \( 26^\circ \) (dashed).
The hydrodynamic motions generated by instability can be important for the transport processes in radiation zones. Motions in the unstable modes depend generally on the polar and azimuthal wavenumbers as well as on the parameter $\eta$. In real conditions, $\eta$ is likely large ($\sim 10^3 - 10^4$) and the radial length scale of eigenmodes is rather short. If it is shorter than $r/l$ or $l/m$, then one can easily estimate from the continuity equation that the radial component of velocity is small compared to the polar and azimuthal ones. However, the character of motions can be different at the non-linear stage when the interaction between modes becomes essential.

It turns out that a new type of magneto-inertial waves might exist in the radiation zones of rapidly rotating stars in the regions where $\eta > \eta_{cr}$ and the Tayler instability is suppressed. These waves are marginally stable and their dispersion relation is given by Eq.(15) at large $\eta$. The frequency of this oscillation decreases with an increase of $\Omega$. These waves are stable in our simplified model but they can be unstable in more realistic conditions (the presence of a poloidal field, differential rotation, etc.). The magneto-inertial waves can play an important in various processes in the radiation zone (mixing, transport of the angular momentum, formation of the tachocline, etc.). These waves will be considered in more detail in a subsequent publication.

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