FRACTIONAL CAHN-HILLIARD, ALLEN-CAHN AND POROUS MEDIUM EQUATIONS

GORO AKAGI, GIULIO SCHIMPERNA, AND ANTONIO SEGATTI

Abstract. We introduce a fractional variant of the Cahn-Hilliard equation settled in a bounded domain $\Omega \subset \mathbb{R}^N$ and complemented with homogeneous Dirichlet boundary conditions of solid type (i.e., imposed in the whole of $\mathbb{R}^N \setminus \Omega$). After setting a proper functional framework, we prove existence and uniqueness of weak solutions to the related initial-boundary value problem. Then, we investigate some significant singular limits obtained as the order of either of the fractional Laplacians appearing in the equation is let tend to 0. In particular, we can rigorously prove that the fractional Allen-Cahn, fractional porous medium, and fractional fast-diffusion equations can be obtained in the limit. Finally, in the last part of the paper, we discuss existence and qualitative properties of stationary solutions of our problem and of its singular limits.

1. Introduction

Let $\Omega$ be a smooth bounded domain in $\mathbb{R}^N$. For $s, \sigma \in (0, 1)$, we consider the following class of initial and boundary value problems:

\begin{align*}
\partial_t u + (-\Delta)^s w &= 0 \quad \text{in } \Omega \times (0, +\infty), \\
w &= (-\Delta)^\sigma u + W'(u) \quad \text{in } \Omega \times (0, +\infty), \\
u(x, 0) &= u_0(x) \quad \text{in } \Omega, \\
u = w &= 0 \quad \text{in } \mathbb{R}^N \setminus \Omega.
\end{align*}

The above system constitutes a natural generalization of the well-known and extensively studied Cahn-Hilliard equation, Allen-Cahn equation, and porous medium equation. More precisely, when $s = \sigma = 1$, system (1)-(2) reduces to the Cahn-Hilliard equation (cf. [12]), when $s = 0$ and $\sigma = 1$ we have the Allen-Cahn equation (cf. [3]), and when $s = \sigma = 0$, (1)-(2) turns to an ODE. The relations between the above system and the porous medium equation will be outlined and made rigorous later on.

The function $W$ in (2) represents a configuration potential which may have two (or more) wells. The general structure of $W$ is given by

$$W(v) = \beta(v) - \frac{\lambda}{2} v^2,$$

where $\beta = \dot{\beta}'$. Here $\dot{\beta}$ is a smooth and convex function and $\lambda \geq 0$ is a constant. Hence, if $\lambda > 0$, then $W$ may be nonconvex. In the phase-transition literature, the wells of $W$ correspond to energy minima (attained at pure phases or configurations). In view of the variational structure of our system, it is convenient to keep the same interpretation also in the present case. Actually, several types of significant choices have been proposed for $W$, also including cases where $\dot{\beta}$ is nonsmooth or even singular (like the so-called logarithmic potential $\dot{\beta}(v) = (1 - v) \log(1 - v) + (1 + v) \log(1 + v)$, $v \in (-1, 1)$). For the sake of simplicity, in this paper we will just consider the case given by

$$\dot{\beta}(v) = \frac{1}{p} |v|^p, \quad p \in (1, \infty),$$

so that $W'(v) = \beta(v) - \lambda v$, where $\beta = \dot{\beta}'$.

Moreover, we will set $\lambda = 1$. This choice, up to an additive constant, includes the standard double-well potential, namely $W(v) = \frac{1}{2}(v^2 - 1)^2$, widely used in the literature. Mathematically speaking, the cases $p > 2$ and $p < 2$ enjoy rather different features. Indeed, while in the former case $W(v)$ is coercive as $|v| \to \infty$, in the latter situation $W$ is unbounded from below. Hence, we can expect (and, indeed,
we will prove) different behaviors of the solutions, especially for large values of the time variable. Note that the case $p = 2$ corresponds in fact to the linear problem and is not considered here.

A further important feature of our problem is the occurrence of solid boundary conditions of homogeneous Dirichlet type, stated by (4). Namely, the values of $u$ and $w$ are prescribed in the whole complement of $\Omega$, not only on the boundary. Of course, this assumption is strongly related to the nonlocal character of fractional Laplacians. Indeed, it is worth observing from the very beginning that, though the values of $(-\Delta)^s w$ and of $(-\Delta)^s u$ at any point $x \in \Omega$ depend also on the values of $u$ and $w$ outside $\Omega$ (which are set to be 0 by the boundary conditions), we prescribe the validity of (1) and (2) only at the points $x \in \Omega$. For $x \in \mathbb{R}^N \setminus \Omega$, (1)-(2) need not to be satisfied (and, indeed, there is no reason why they should). This observation will be further clarified in Section 2, where the appropriate concept of weak solution is introduced. Correspondingly, we will also recall the precise definition of $(-\Delta)^r$, $r \in (0, 1)$, both in the strong and in the weak (variational) form, the latter being the more appropriate one for the analysis of our problem.

The study of system (1)-(4) is motivated both from the point of applications and due to its mere mathematical interest. Under the first perspective, it is worth recalling that the Allen-Cahn and Cahn-Hilliard equations, in their standard formulation (i.e., with the usual Laplace operators), play a central role in materials science. Indeed, they commonly occur in mathematical models for phase-transition or separation, viscoelasticity, damaging, complex fluids, and whenever diffuse interfaces appear (see, e.g., [13, 28] for a comprehensive bibliography). A reason for considering a fractional version of the Cahn-Hilliard equation can be provided by observing that, in the original formulation of the physical model [12], the Laplace operator in (2) was actually replaced by a spatial convolution term, aimed at describing long-range interactions among particles. It was only in the subsequent mathematical literature that, mainly for analytical reasons, this nonlocal term has been substituted with the term $-\Delta u$. Under this perspective, the use of the fractional Laplacian $(-\Delta)^s$ (which, at least for smooth functions, may be represented exactly by a convolution integral) appears to be more adherent to the physical setting. It is worth remarking that the study of fractional (or, more generally, nonlocal) PDE’s is a lively research topic, both from the point of view of mathematical theory and in relation with the many real-world applications. Among these, we mention obstacle problems [10], finance [16], quasi-geostrophic flows [11, 15], anomalous diffusion [27, 37, 39]. A more comprehensive list of references is provided in the survey [18].

In the recent literature, a relevant number of works have been devoted to the analysis of nonlocal Allen-Cahn and Cahn-Hilliard models. Here we quote, among others, [4, 20, 29] (see also [14] for an application to complex fluids). Actually, in [4, 14, 20], the term $(-\Delta)^s u$ in (2) is replaced by

$$J[u] = a(\cdot)u - j * u, \quad a(x) = \int_\Omega j(x - y) \, dy, \quad (j * u)(x) = \int_\Omega j(x - y)u(y) \, dy,$$

where the convolution kernel $j$ enjoys suitable, and rather strong, regularity properties. For instance, in [4], $j$ is assumed to lie in $C^{2+\alpha}(\mathbb{R}^N)$ for some $\alpha > 0$, while in [14] $j$ is taken in $W^{1,1}(\mathbb{R}^N)$, and in [20] a similar condition is required. In all cases, both integrals in the right hand side of (7) are finite for (almost) every $x \in \Omega$, whenever $u$ lies, say, in $L^1(\Omega)$ (more comments on this point will be given in the Appendix).

On the other hand, the kernel $K_\gamma$ generating the fractional Laplacian $(-\Delta)^\gamma$ (cf. (10) below) is not even summable. Hence, the corresponding convolution integral needs to be intended in the principal value sense even for smooth functions $u$ (cf. (11)). In addition to that, if $u$ is not regular (for instance if it just lies in some $L^p$-space), then the “pointwise” expression (10) makes no sense at all and a variational definition of $(-\Delta)^\gamma u$ is required. In this sense, the fractional Laplacian gives rise to a much stronger singularity with respect to those considered in [4, 14, 20]. Up to our knowledge, the only papers dealing with a true fractional Allen-Cahn model are [29] and [19], where, however, the analysis is restricted to the spatial one-dimensional case and mostly concentrated on other aspects rather than weak solvability and regularity of solutions.

More recently, Abels, Bosia and Grasselli in [2] analyzed a variant of problem (1)-(2), where the diffusion operator in (the analogue of) (1) is the standard Laplacian, while the diffusion operator in (the analogue of) (2) is the so-called regional fractional Laplacian. For its definition we refer the reader to the Appendix: here it is just worth mentioning that its properties are slightly different.
compared to those of the operator considered in this paper, the main point regarding the boundary conditions. Actually, the operator of [2] can be seen as a fractional power of the Neumann Laplacian on \( \Omega \) (in particular, this implies conservation of mass, which does not hold here in view of the Dirichlet condition (4)). The authors of [2] prove existence and uniqueness of weak solutions in the case when the function \( W \) may have a singular character (as happens for the logarithmic potential mentioned before). Moreover they characterize the long-time behavior of solution trajectories proving existence of the global attractor for the dynamical process associated to the system.

The first aim of this paper is to prove existence and uniqueness of solutions to a weak formulation of problem (1)-(4). Hence, differently from [2], we will consider a fractional dynamics both in (1) and in (2). This program requires, at first, to set a proper functional framework. Our approach basically follows (and complements) the perspective given in [33], where a weak version of the fractional Laplacian can be given at least in two (equivalent) ways: one can either work with functions defined on \( \Omega \) and implicitly extend them to 0 outside \( \Omega \) when computing fractional Laplacians (which depend on values taken on the whole of \( \mathbb{R}^N \)), or one may use spaces of functions defined on the whole space but constrained to be identically equal to zero outside \( \Omega \). Generally, we shall work within the latter framework. This choice permits us to address the problem, for all fixed \( s, \sigma \in (0, 1) \), in the usual Hilbert setting, very similarly to what happens for the standard Cahn-Hilliard model. In particular, we can prove existence by means of a classical time-discretization scheme. Compactness and duality arguments are then exploited in order to pass to the limit in the discretization; in this way we can avoid any reference to finer regularity properties of solutions to fractional elliptic and parabolic problems, which may involve rather delicate issues. Uniqueness also follows from a simple contraction principle.

As anticipated, after establishing well-posedness of the model, we will turn our attention to further properties of solutions. As a first issue, we will let the “order” \( \sigma \) of the fractional Laplacian in (2) go to 0. This singular limit is motivated by noting that, as \( \sigma \searrow 0 \), \( (-\Delta)^\sigma u \to u \) in a suitable sense. As a consequence we obtain, at least formally,

\[
w = (-\Delta)^\sigma u + W'(u) = (-\Delta)^\sigma u + \beta(u) - u \xrightarrow{\sigma \searrow 0} \beta(u).
\]

Hence, on account of (1) and recalling assumption (6), one expects to get in the limit the equation

\[
\partial_t u + (-\Delta)^s (|u|^{p-1} \text{sign } u) = 0 \quad \text{in } \Omega \times (0, +\infty),
\]

\[
u = 0 \quad \text{in } \mathbb{R}^N \setminus \Omega.
\]

This corresponds, for \( p > 2 \), to the so-called \textit{fractional porous medium equation}, recently addressed and studied in a number of contributions (see, among others, [17], [5], [6] and [7]).

Actually, taking a family \( \{u_\sigma, w_\sigma\} \) of solutions to our problem, and letting \( \sigma \searrow 0 \), we can rigorously prove that, up to extraction of a subsequence, \( u_\sigma \) tends to a limit function \( u \) satisfying (8). Our result holds, under natural assumptions on the initial data, both for \( p > 2 \) and for \( p \in (1, 2) \).

The proof is, however, not straightforward and relies on some fine properties of first eigenvalues of fractional elliptic Dirichlet problems, which, to the best of our knowledge, are new (see Proposition 2.2). The case \( p \in (1, 2) \) is a bit more involved due to the lack of coercivity of the energy functional. Indeed, in that case we need to modify a bit the energy (see Theorem 3 below) in such a way to get an estimate for \( u_\sigma \) uniform as \( \sigma \searrow 0 \). However, this modification does not affect the limit equation (8), which corresponds in this case to the \textit{fractional fast-diffusion equation} studied, e.g., in [24].

On the other hand, it is also natural to investigate what happens as one lets \( s \searrow 0 \). In that case, one expects that

\[
(-\Delta)^s w \xrightarrow{s \searrow 0} \beta(w).
\]

In other words, the limit gives rise to the \textit{fractional Allen-Cahn equation}

\[
\partial_t u + (-\Delta)^s u + W'(u) = 0 \quad \text{in } \Omega \times (0, +\infty),
\]

\[
u = 0 \quad \text{in } \mathbb{R}^N \setminus \Omega,
\]

both for \( p > 2 \) and for \( 1 < p < 2 \). It is worth noting that the limit \( s \searrow 0 \) is considerably simpler than the limit \( \sigma \searrow 0 \) since keeping \( \sigma \) fixed maintains some additional space compactness, which reveals to be helpful for the purposes of obtaining a strong convergence for \( u \) and identifying the nonlinear
terms in the limit. As a result of these two procedures, we can see problem (1)-(4) as a bridge between the usual Cahn-Hilliard equation (given by $(\sigma, s) = (1, 1)$) and the fractional (or also non-fractional) porous medium and Allen-Cahn equations.

The last part of the paper is devoted to the proof of some results related to stationary solutions to problem (1)-(4) (for fixed $s, \sigma \in (0, 1)$). As expected in view of the nonconvex character of $W$, we can show that nontrivial stationary states exist if and only if the first eigenvalue $\lambda_1(\sigma)$ of $(-\Delta)^r$ is strictly smaller than 1. Indeed, whenever $\lambda_1(\sigma) \geq 1$, the coercivity given by $(-\Delta)^r$ compensates the nonconvexity of $W'(u) = \beta(u) - u$, exactly as happens for the standard Laplacian. Of course, the properties of the stationary states play an important role for what concerns the long-time behavior of solutions to the evolutionary system. We plan to address this issue, and, particularly, to investigate the properties of $\omega$-limit sets, in a forthcoming paper.

The plan of the paper is as follows. A survey on some basic definitions and tools related to the fractional Laplacian, as well as some useful lemmas, are presented in the next Section 2. In Section 3, we introduce our assumptions and state our main results. The proofs of existence and regularity properties of solutions are carried out in Section 4, while the convergence to the fractional porous medium and Allen-Cahn equations is analyzed in Section 5. Our last results regarding the properties of stationary states are presented in Section 6. Finally, in the Appendix we provide some more comments on the relations occurring between the problem analyzed here and other nonlocal models of Allen-Cahn or Cahn-Hilliard type studied in the literature.

2. Notations and Preliminaries

2.1. The fractional Laplacian. We introduce here the standard (strong) form of the fractional Laplace operator $(-\Delta)^r$ in the whole space $\mathbb{R}^N$. The reader may refer, e.g., to [18] for additional details. Given $r \in (0, 1)$, for $u$ in the Schwartz class $\mathcal{S}(\mathbb{R}^N)$ of the rapidly decaying functions at infinity, $(-\Delta)^r u$ is defined as

$$
(-\Delta)^r u(x) := C(r, N) \text{ p.v.} \int_{\mathbb{R}^N} \frac{u(x) - u(y)}{|x - y|^{N+2r}} \, dy,
$$

where the notation p.v. means that the integral is taken in the Cauchy principal value sense, namely

$$
p.v. \int_{\mathbb{R}^N} \frac{u(x) - u(y)}{|x - y|^{N+2r}} \, dy = \lim_{\epsilon \searrow 0} \int_{\mathbb{R}^N \setminus B(x, \epsilon)} \frac{u(x) - u(y)}{|x - y|^{N+2r}} \, dy.
$$

The exact value and the asymptotics with respect to $r$ of the normalizing constant $C(r, N)$ are crucial for our purposes. To this end, we recall that

$$
C(r, N) = \left( \int_{\mathbb{R}^N} \frac{1 - \cos(\zeta)}{|\zeta|^{N+2r}} \, d\zeta \right)^{-1}
$$

and that (see, e.g., [18, Corollary 4.2])

$$
\lim_{r \searrow 0} \frac{C(r, N)}{r(1 - r)} = \frac{2}{|\mathbb{S}^{N-1}|}.
$$

On the other hand, since the dependence of $C(r, N)$ with respect to the space dimension $N$ is not the major issue for this paper, we will always write $C(r)$ for $C(r, N)$ in what follows. For any $r \in (0, 1)$ and for any $x, y \in \mathbb{R}^N$ we will also use the shorthand notation

$$
K_r(x - y) = |x - y|^{-N-2r}
$$

to denote the singular kernel appearing in the definition of $(-\Delta)^r$. A second, albeit equivalent, definition can be given using the Fourier transform. Indeed, $(-\Delta)^r$ can be introduced as the pseudo-differential operator of symbol $|\xi|^{2r}$, namely

$$
(-\Delta)^r v = \mathcal{F}^{-1}(|\xi|^{2r} \hat{v}(\xi)), \quad \forall v \in \mathcal{S}(\mathbb{R}^N).
$$

We denote by $\mathcal{F}(v)$ (or by $\hat{v}$) the Fourier transform of $v$.

2.2. Fractional Sobolev spaces. In this subsection, we shall deal with fractional Sobolev spaces. We refer the reader to, e.g., [25] and [1] for further details.

For $r \in \mathbb{R}$, the fractional Sobolev space $H^r(\mathbb{R}^N)$ is defined by

$$
H^r(\mathbb{R}^N) := \left\{ v \in \mathcal{S}(\mathbb{R}^N) : (1 + |\xi|^2)^{r/2} \hat{v}(\xi) \in L^2(\mathbb{R}_\xi^N) \right\},
$$

with $\mathbb{R}_\xi^N$ being the Euclidean space $\mathbb{R}^N$.

For $r \in (0, 1)$, the fractional Sobolev space $H^r(\mathbb{R}^N)$ is a Hilbert space with inner product and norm given by

$$
\langle u, v \rangle_{H^r(\mathbb{R}^N)} := \int_{\mathbb{R}^N} \frac{u(x) \overline{v(x)}}{|x|^{N+2r}} \, dx,
$$

$$
\| u \|_{H^r(\mathbb{R}^N)} := \left( \int_{\mathbb{R}^N} \frac{|u(x)|^2}{|x|^{2N+4r}} \, dx \right)^{1/2}.
$$
where \( \mathcal{S}(\mathbb{R}^N)' \) stands for the dual space of the Schwartz class \( \mathcal{S}(\mathbb{R}^N) \) and \( L^2(\mathbb{R}^N) \) is the space of square-integrable functions with respect to the variables \( \xi \in \mathbb{R}^N \), equipped with the norm
\[
\|v\| := \left\| (1 + |\xi|^2)^{r/2} \hat{v}(\xi) \right\|_{L^2(\mathbb{R}^N)} \quad \text{for} \quad v \in H^r(\mathbb{R}^N).
\]

In particular, for \( r \in (0, 1) \), we can equivalently write
\[
H^r(\mathbb{R}^N) := \{ v \in L^2(\mathbb{R}^N) : (x, y) \mapsto K_r(|x - y|)|v(x) - v(y)|^2 \in L^1(\mathbb{R}^{2N}) \},
\]
endowed with the (equivalent) norm
\[
\|v\|^2_{H^r(\mathbb{R}^N)} := \|v\|^2_{L^2(\mathbb{R}^N)} + \frac{C(r)}{2} \|v\|^2_{H^r(\mathbb{R}^N)} \quad \text{for} \quad v \in H^r(\mathbb{R}^N).
\]

Here, \([-\cdot]_{H^r}\) denotes the so-called Gagliardo-seminorm
\[
[v]^2_{H^r(\mathbb{R}^N)} : = \iint_{\mathbb{R}^{2N}} K_r(x - y)|v(x) - v(y)|^2 \, dx \, dy.
\]
Furthermore, for \( r \in \mathbb{R} \), \( H^r(\mathbb{R}^N) \) is an intermediate space between \( H^m(\mathbb{R}^N) \) and \( L^2(\mathbb{R}^N) \), that is,
\[
H^r(\mathbb{R}^N) = [H^m(\mathbb{R}^N), L^2(\mathbb{R}^N)]_\theta
\]
for any \( \theta \in (0, 1) \) and \( m \in \mathbb{Z} \) satisfying \( r = (1 - \theta)m \).

Let \( \Omega \) be a bounded domain of \( \mathbb{R}^N \) with smooth boundary \( \partial \Omega \). For \( r \in \mathbb{R} \), the fractional Sobolev space \( H^r(\Omega) \) may be analogously defined as
\[
H^r(\Omega) := [H^m(\Omega), L^2(\Omega)]_\theta
\]
for any \( \theta \in (0, 1) \) and \( m \in \mathbb{N} \) satisfying \( (1 - \theta)m = r \). Moreover, for \( r \in (0, 1) \), one may use an alternative definition,
\[
H^r(\Omega) := \{ u \in L^2(\Omega) : (x, y) \mapsto K_r(|x - y|)|u(x) - u(y)|^2 \in L^1(\Omega \times \Omega) \}
\]
with the intrinsic norm
\[
\|v\|^2_{H^r(\Omega)} := \|v\|^2_{L^2(\Omega)} + \frac{C(r)}{2} \iint_{\Omega \times \Omega} K_r(|x - y|)|v(x) - v(y)|^2 \, dx \, dy \quad \text{for} \quad v \in H^r(\Omega).
\]

Sobolev embeddings and inequalities also hold for fractional Sobolev spaces. Thus, \( H^r(\Omega) \) is continuously (resp., compactly) embedded in \( L^q(\Omega) \), provided that \( 1 \leq q \leq 2^*_r := 2N/(N - 2r) \) (resp., \( 1 \leq q < 2^*_r \)) and \( 2r < N \). Finally, for each \( r > 0 \), \( H^r_0(\Omega) \) is defined as the closure of \( C_0^\infty(\Omega) \) in \( H^r(\Omega) \).

In case \( r \leq 1/2 \), the space \( H^r_0(\Omega) \) coincides with \( H^r(\Omega) \); in case \( r > 1/2 \), \( H^r_0(\Omega) \) is strictly contained in \( H^r(\Omega) \) (see, e.g., [25, p. 55, Theorem 11.1]).

2.3. The functional framework. It is apparent from (13) that, for \( v \in \mathcal{S}(\mathbb{R}^N) \), \((-\Delta)_f^r v \) does not necessarily belong to \( \mathcal{S}(\mathbb{R}^N) \) (being \( r < 1 \), the symbol \(|\xi|^{2r} \) introduces a singularity in the origin in its Fourier transform). Moreover, even for \( v \) with compact support, \((-\Delta)_f^r v \) generally does not have compact support due to the non locality of the operator. In addition to this, the above definition could make no sense when non-smooth functions are involved. Thus, it will be important for us to extend the definition of the fractional Laplacian to a more general setting. This will be accomplished by using the theory of distributions in combination with some tools of convex analysis. The framework we are going to fix will permit us to use variational and energy techniques in order to address our problem. As already observed in the introduction, although equations (1)-(2) are settled only in \( \Omega \), the behavior of \((-\Delta)_f^s u \) and \((-\Delta)_f^s w \) depends on the interplay between the values of \( u \) and of \( w \) inside and outside \( \Omega \). Proceeding along the lines of [33], we can then introduce some functional spaces.

Firstly, we set
\[
H_0 := \left\{ v \in L^2(\mathbb{R}^N) : v = 0 \text{ a.e. in } \mathbb{R}^N \setminus \Omega \right\}.
\]

The space \( L^2(\mathbb{R}^N) \) (hence its closed subspace \( H_0 \)) is endowed with its standard scalar product,
\[
(u, v) := \int_{\mathbb{R}^N} u(x)v(x) \, dx \quad \text{for} \quad u, v \in L^2(\mathbb{R}^N).
\]
Of course, in the closed subspace $H_0$ taking the scalar product of $L^2(\Omega)$ and the associated norm would make no difference. Hence, $L^2(\Omega)$ can be identified with $H_0$ by zero extension outside $\Omega$. Furthermore, we set

$$L^p_0(\mathbb{R}^N) := \{v \in L^p(\mathbb{R}^N) : v = 0 \text{ a.e. in } \mathbb{R}^N \setminus \Omega \} \quad \text{with} \quad \| \cdot \|_{L^p_0(\mathbb{R}^N)} := \| \cdot \|_{L^p(\mathbb{R}^N)},$$

$$\mathcal{D}(\Omega) := \{ \phi \in C^\infty(\mathbb{R}^N) : \phi|\Omega \in C^\infty(\Omega) \text{ and } \phi = 0 \text{ in } \mathbb{R}^N \setminus \Omega \},$$

which can be identified with $L^p(\Omega)$ and $C^\infty(\Omega)$, respectively. Then $L^p_0(\mathbb{R}^N)$ is reflexive for $p \in (1, \infty)$, since $L^p_0(\mathbb{R}^N)$ is closed in $L^p(\mathbb{R}^N)$; moreover, $L^p_0(\mathbb{R}^N)$ is separable for $p \in [1, \infty)$. Furthermore, $\mathcal{D}(\Omega)$ is dense in $L^p_0(\mathbb{R}^N)$, provided that $1 \leq p < \infty$.

As in [33], we set $Q := \mathbb{R}^N \setminus ((\mathbb{R}^N \setminus \Omega) \times (\mathbb{R}^N \setminus \Omega))$ and denote by $X_{r,0}$, $r \in (0, 1)$, the space

$$X_{r,0} := \left\{ v \in H_0 : (x, y) \mapsto (v(x) - v(y))\sqrt{K_r(x - y)} \in L^2(Q) \right\}.$$  

Actually, $X_{r,0}$ can be endowed with the scalar product

$$\langle v, z \rangle_{X_{r,0}} := \int_{Q} \frac{C(r)}{2} K_r(x - y)(v(x) - v(y))(z(x) - z(y)) \, dx \, dy \quad \text{for } v, z \in X_{r,0}$$

and the associated norm

$$\| v \|^2 := \| v \|^2_{L^2(Q)} + \frac{C(r)}{2} \int_{Q} K_r(x - y)|v(x) - v(y)|^2 \, dx \, dy \quad \text{for } v \in X_{r,0},$$

where $C(r)$ is as in (10). Then, it is easy to check that $X_{r,0}$ is a Hilbert space (i.e., the above norm is complete). Note that $X_{r,0}$ could be also presented in a more familiar form,

$$X_{r,0} = \left\{ v \in H^r(\mathbb{R}^N) \text{ such that } v = 0 \text{ a.e. in } \mathbb{R}^N \setminus \Omega \right\}.$$  

There holds the Poincaré-type inequality,

$$\| v \|^2_{L^2(\Omega)} \leq c_P(r) \frac{C(r)}{2} \| v \|^2_{H^r(\mathbb{R}^N)} \quad \text{for all } v \in X_{r,0}$$

for some constant $c_P(r)$ depending only on $r$, $N$ and the diameter of $\Omega$. Indeed, one can take $R > 0$ such that $\Omega$ is included in the open ball $B_R$ of radius $R$ centered at the origin. Then, using the definition of the Gagliardo-seminorm, we see that

$$[v]_{H^r(\mathbb{R}^N)}^2 \geq \int_{\Omega'} \int_{\Omega} K_r(|x - y|)v(x)^2 \, dx \, dy
\geq \int_{\Omega} \left( \int_{\Omega' \cap B_{R+1}} K_r(|x - y|) \, dy \right) v(x)^2 \, dx \geq \frac{|B_{R+1} \setminus \Omega|}{(2R + 2)^N} \| v \|^2_{L^2(\Omega)}$$

for all $v \in X_{r,0}$, where $\Omega'$ stands for the complement of $\Omega$ and $|B_{R+1} \setminus \Omega|$ denotes the Lebesgue measure of the set $B_{R+1} \setminus \Omega$. Note that $|B_{R+1} \setminus \Omega| \geq |B_{R+1} \setminus B_R| > 0$. Thus the quotient $[v]_{H^r(\mathbb{R}^N)}^2/\| v \|^2_{L^2(\Omega)}$ is bounded from below for $v \in X_{r,0}$, whence (20) follows. Hence, by (20), the norm on $X_{r,0}$ given by

$$\| v \|_{X_{r,0}} := \frac{C(r)}{2} \int_{Q} K_r(x - y)|v(x) - v(y)|^2 \, dx \, dy \quad \text{for } v \in X_{r,0}$$

is equivalent to that defined in (18). From now on, we will fix (21) to be the norm in $X_{r,0}$.

Now let us introduce the dual spaces $H'_0$ and $X'_{r,0}$ of $H_0$ and $X_{r,0}$, respectively. In particular, we are here concerned with the meaning of the equality in the dual space,

$$f = g \text{ in } H'_0 \quad \text{for } f, g \in H'_0,$$

which actually means

$$\langle f, v \rangle_{H'_0} = \langle g, v \rangle_{H'_0} \quad \text{for all } v \in H_0.$$  

By the Riesz representation theorem, one can uniquely take $u_f, u_g \in H_0$ such that $\langle f, v \rangle_{H'_0} = (u_f, v)$ and $\langle g, v \rangle_{H'_0} = (u_g, v)$ for all $v \in H_0$. Hence (22) yields $u_f = u_g$ in $\Omega$ (hence, over $\mathbb{R}^N$). On the other hand, one may generate $h \in H'_0$ from a function $w_h \in L^2(\mathbb{R}^N)$ which might not vanish outside $\Omega$ by setting

$$\langle h, v \rangle_{H'_0} := \int_{\mathbb{R}^N} w_h(x)v(x) \, dx \quad \text{for } v \in H_0.$$
Then \( h \) coincides with \( f \) in \( H'_0 \), provided that \( w_h = u_f \) in \( \Omega \). In other words, even if \( f \) and \( g \) have pointwise representations \( w_f, w_g \in L^2(\mathbb{R}^N) \), respectively, as in (23), the relation \( f = g \) in \( H'_0 \) ensures that \( w_f(x) = w_g(x) \) for a.e. \( x \in \Omega \) only, and it does not guarantee the coincidence of \( w_f \) and \( w_g \) outside \( \Omega \). This observation is also extended to the relation \( f = g \) in \( A'_{r,0} \).

From now on, we identify the Hilbert space \( H_0 \) with its dual space \( H'_0 \) by means of the scalar product \( (\cdot, \cdot) \). More precisely, we shall identify \( f \in H'_0 \) with its unique representation \( u_f \in H_0 \) by the Riesz representation theorem. Hence we shall write \( f = g \) in \( H_0 \) for claiming the equality of \( f, g \in H'_0 \) (i.e., (22)) as well. Here we should emphasize again that whenever are given representatives \( w_f \) and \( w_g \) of \( f, g \in H'_0 \), respectively, the identification implies that \( w_f = w_g \) only in \( \Omega \). Then since \( X_{r,0} \) can be seen as a dense subspace of \( H_0 \), one may consider the Hilbert triple,

\[
X_{r,0} \hookrightarrow H_0 \cong H'_0 \hookrightarrow X'_{r,0},
\]

with compact and densely defined canonical injections. This relation will play a crucial role throughout this paper.

On the other hand, for \( r \in (0, 1) \), the extension operator of \( u \in X_{r,0} \) to 0 outside \( \Omega \) is a continuous mapping of \( H^r(\Omega) \to H^r(\mathbb{R}^N) \). In particular (see [25, Theorem 11.4, Chapter 1]), if \( r \in (1/2, 1) \), the functions in \( X_{r,0} \) are equal to zero, in the sense of traces, on \( \partial \Omega \). Hence, \( X_{r,0} \) can be identified with \( H^r_0(\Omega) \) in that case, whereas \( X_{r,0} \cong H^r_0(\Omega) = H^r(\Omega) \) for \( r \in (0, 1/2) \). Finally, in the limit case \( r = 1/2 \), it turns out that \( X_{1/2,0} \cong H^{1/2}(\Omega) \) (again, see [25] for more details).

Based on this functional framework, we can introduce, for \( r \in (0, 1) \), the weak form \( \mathfrak{A}_r \) of the fractional Laplacian \((-\Delta)^r \). More precisely, the operator \( \mathfrak{A}_r : X_{r,0} \to X'_{r,0} \) is defined by

\[
\langle \mathfrak{A}_r v, \phi \rangle := \frac{C(r)}{2} \int_{\mathbb{R}^{2N}} K_r(x-y)(v(x) - v(y)) \phi(x) - \phi(y) \, dx \, dy,
\]

for all \( v, \phi \in X_{r,0} \),

where the integral over \( \mathbb{R}^{2N} \) can be equivalently replaced with an integral over \( \Omega \). Note that, as soon as \( v, \phi \in X_{r,0} \), the integral in the right hand side is finite. Note also that (25) can be understood as an integration by parts formula, at least when \( v, \phi \) are sufficiently regular. Indeed, to see this, we define

\[
(-\Delta)^r u(x) := C(r) \int_{\mathbb{R}^N} K_{r,\varepsilon}(x-y)(u(x) - u(y)) \, dy,
\]

where \( K_{r,\varepsilon} := K_r(1 - \chi_{B(0,\varepsilon)}) \) and \( \chi_{B(0,\varepsilon)} \) denotes the characteristic function of the ball \( B(0,\varepsilon) \) in \( \mathbb{R}^N \). Then, by symmetry of \( K_{r,\varepsilon} \),

\[
\int_{\mathbb{R}^N} (-\Delta)^r v(x) \phi(x) \, dx = C(r) \int_{\mathbb{R}^{2N}} K_{r,\varepsilon}(x-y)(v(x) - v(y)) \phi(x) \, dx \, dy
\]

\[= C(r) \int_{\mathbb{R}^{2N}} K_{r,\varepsilon}(z)(v(x) - v(x + z)) \phi(x) \, dx \, dz \]

\[= C(r) \int_{\mathbb{R}^{2N}} K_{r,\varepsilon}(z)(v(x) - v(x - z)) \phi(x) \, dx \, dz \]

\[= \frac{C(r)}{2} \int_{\mathbb{R}^{2N}} \int_{\mathbb{R}^N} K_{r,\varepsilon}(x-y)(v(x) - v(y)) \phi(x) - \phi(y) \, dx \, dy. \]

Now, letting \( \varepsilon \searrow 0 \), the right hand side converges to \( \langle \mathfrak{A}_r v, \phi \rangle \), provided that \( v, \phi \) lie in \( X_{r,0} \) (and, hence, a fortiori if \( v, \phi \) are smooth functions). On the other hand, whenever \( v \) is so smooth that \((-\Delta)^r v \) (i.e., the “strong” fractional Laplacian of \( v \) defined in (10)) is represented by, say, an \( L^2 \)-function, then the left hand side of (27) converges to \( \int_{\mathbb{R}^N} (-\Delta)^r v(x) \phi(x) \, dx \). Hence, \( \mathfrak{A}_r \) can indeed be seen as an extension of \((-\Delta)^r \) to less regular function. Moreover, formula (25) can be also expressed in terms of
Fourier transform. Actually, for $v, \phi \in \mathcal{X}_{r, 0}$, thanks also to Fubini’s theorem, we have
\[
\langle \mathfrak{A}_r v, \phi \rangle = \frac{C(r)}{2} \int_{\mathbb{R}^N} \left( \int_{\mathbb{R}^N} K_r(z)(v(y + z) - v(y))(\phi(y + z) - \phi(y)) \, dy \right) \, dz
\]
\[
= \frac{C(r)}{2} \int_{\mathbb{R}^N} \hat{v}(\xi)\hat{\phi}(\xi) \left( \int_{\mathbb{R}^N} K_r(z) \left| e^{i\xi \cdot z} - 1 \right|^2 \, dz \right) \, d\xi
\]
\[
= \int_{\mathbb{R}^N} |\xi|^{r} \hat{v}(\xi) |\xi|^{r} \hat{\phi}(\xi) \, d\xi,
\]
so that $\frac{C(r)}{2} \int_{\mathbb{R}^N} K_r(z)|e^{i\xi \cdot z} - 1|^2 \, dz = |\xi|^{2r}$ (see [18]). In particular, using (13), we get
\[
\langle \mathfrak{A}_r v, \phi \rangle = \int_{\mathbb{R}^N} |\xi|^{r} \hat{v}(\xi) |\xi|^{r} \hat{\phi}(\xi) \, d\xi = \int_{\mathbb{R}^N} (-\Delta)^{r/2} v(x) (-\Delta)^{r/2} \phi(x) \, dx = \langle \mathfrak{A}_r, v \rangle, \]
which could serve as an alternative, albeit equivalent, definition of the weak fractional Laplacian $\mathfrak{A}_r$.

A weak form of the fractional Laplacian can be introduced also for the whole space case, $\Omega = \mathbb{R}^N$. Indeed, for $r \in (0, 1)$, we can set (cf. (16))
\[
\mathcal{X}_r := \left\{ v \in L^2(\mathbb{R}^N) : (x, y) \mapsto (v(x) - v(y))\sqrt{K_r(x - y)} \in L^2(\mathbb{R}^2N) \right\} = H^r(\mathbb{R}^N).
\]
Then, the previous discussion extends, with minor modifications, to the space $\mathcal{X}_r$. Moreover, one can correspondingly consider the Hilbert triplet $(\mathcal{X}_r, H, \mathcal{X}_r^\prime)$. With a small abuse of notation we will indicate with the same symbol $\mathfrak{A}_r$ the weak form of the fractional Laplacian as an operator from $\mathcal{X}_r$ to $\mathcal{X}_r^\prime$ defined by (25) with $\mathcal{X}_{r, 0}$ replaced by $\mathcal{X}_r$.

The relations between the Gagliardo-seminorm and the Fourier-transform definition of the fractional Laplacian are also clarified by the following property (cf. [18, Propositions 3.4 & 3.6]):
\[
\frac{C(r)}{2} \left( \|v\|_{H^r(\mathbb{R}^N)}^2 \right)_{H^r(\mathbb{R}^N)} = \|\mathfrak{A}_r/v\|_{L^2(\mathbb{R}^N)}^2 + \|\mathfrak{A}_r/v\|_{L^2(\mathbb{R}^N)}^2 = \|\mathfrak{A}_r/v\|_{L^2(\mathbb{R}^N)}^2 + \left( \|\mathfrak{A}_r/v\|_{L^2(\mathbb{R}^N)}^2 \right)_{H^r(\mathbb{R}^N)}
\]

In particular, the norm in $H^r(\mathbb{R}^N)$ can be equivalently expressed as
\[
\|v\|_{H^r(\mathbb{R}^N)}^2 = \|v\|_{L^2(\mathbb{R}^N)}^2 + \|\mathfrak{A}_r/v\|_{L^2(\mathbb{R}^N)}^2 = \|v\|_{L^2(\mathbb{R}^N)}^2 + \left( \|\mathfrak{A}_r/v\|_{L^2(\mathbb{R}^N)}^2 \right)_{H^r(\mathbb{R}^N)}.
\]
It is worth noting that there is another possible approach for dealing with fractional Laplacians on bounded domains. Indeed, one may define the operator, called spectral fractional Laplacian, as
\[
(-\Delta_{\Omega})^r f(x) := \sum_{j=1}^{+\infty} \lambda_j^r \hat{f}_j \phi_j(x), \quad x \in \Omega,
\]
where $\lambda_j > 0, j = 1, 2, \ldots$, are the eigenvalues of the Dirichlet Laplacian on $\Omega$, $\phi_j$ are the corresponding normalized eigenfunctions, and
\[
\hat{f}_j := \int_{\Omega} f(x) \phi_j(x) \, dx, \quad \|\phi_j\|_{L^2(\Omega)} = 1.
\]
As observed in [34], the spectral fractional Laplacian $(-\Delta_{\Omega})^r$ is a different operator with respect to the operator considered in this paper. We refer to [9] and [7] and to the references therein for the functional framework related to (33) and for the analysis of some differential problems involving $(-\Delta_{\Omega})^r$.

2.4. An $L^2$-framework for fractional Laplacians. The weak fractional Laplacian $\mathfrak{A}_r$ can be also interpreted in the framework of convex analysis. Actually, for $r \in (0, 1)$, we can introduce the functional $G_r : H_0 \to [0, +\infty]$ given by
\[
G_r(v) := \begin{cases} \frac{C(r)}{4} \int_0^\infty K_r(x - y)|v(x) - v(y)|^2 \, dx \, dy & \text{if } v \in \mathcal{X}_{r, 0}, \\
+\infty & \text{otherwise} \end{cases}
\]
Then, it is obvious that $G_r$ is a lower semicontinuous (in $H_0$), convex functional. Moreover, given $u \in D(G_r) := \mathcal{X}_{r, 0}$ (the effective domain of $G_r$), it is clear that, for all $v \in \mathcal{X}_{r, 0}$,
\[
\lim_{t \to 0} \frac{G_r(u + tv) - G_r(u)}{t} = \langle \mathfrak{A}_r, v \rangle_{\mathcal{X}_{r, 0}}.
\]
Moreover, it is clear that, for any decreases. Hence, letting \( \varepsilon \to 0 \)
\[ (39) \]
\[ (40) \]
\[ \xi, v \]
Of course, the latter integral may well be (plus) infinity.

\[ (38) \]
\[ \|u\|_{L^q(\Omega)} \leq C\|f\|_{H^0} \quad \text{for} \quad q = \frac{2N}{N - 4r}, \]
while for \( r \in ( \frac{N}{4}, 1 ) \cap (0, 1) \) we have
\[ (40) \]
\[ \|u\|_{C^\alpha(\Omega)} \leq C\|f\|_{H^0} \quad \text{for} \quad \alpha = \min \left\{ r, 2r - \frac{N}{2} \right\}. \]
In both cases the constant \( C > 0 \) depends only on \( r, |\Omega|, \) and \( q \) (or \( \alpha \)).

The operator \( \mathfrak{A}_r \) also enjoys a useful monotonicity property. Indeed, let \( \beta : \mathbb{R} \to \mathbb{R} \) be any smooth monotone function such that \( \beta(0) = 0 \) and let \( \hat{\beta} : \mathbb{R} \to [0, +\infty] \) be the (convex) function such that \( \hat{\beta}(0) = 0 \) and \( \hat{\beta}' = \beta \). Then, setting for instance
\[ (41) \]
\[ \beta_\varepsilon(r) := \begin{cases} 
\beta(r) & \text{if } |r| \leq \varepsilon^{-1}, \\
\beta(\varepsilon^{-1}) & \text{if } r > \varepsilon^{-1}, \\
\beta(-\varepsilon^{-1}) & \text{if } r < -\varepsilon^{-1},
\end{cases} \]

then, for fixed \( \varepsilon > 0 \), \( \beta_\varepsilon \) is bounded, Lipschitz continuous, and monotone. Hence, it is immediate to check that, if \( v \in X_{r,0} \), then \( \beta_\varepsilon(v) \in X_{r,0} \) for each \( \varepsilon > 0 \). Hence, by monotonicity,
\[ (42) \]
\[ (43) \]
Of course, the latter integral may well be (plus) infinity.
2.5. **Asymptotics of \((-\Delta)^r\) and principal eigenvalues as \(r \searrow 0\).** Let us start with the following lemma on the behavior of \((-\Delta)^r\) as \(r \searrow 0\), which will play an important role in the sequel:

**Lemma 2.1.** For any \(\phi \in S(\mathbb{R}^N)\), there holds
\[
(-\Delta)^r \phi \xrightarrow{r \searrow 0} \phi \quad \text{strongly in } H^{\alpha}(\mathbb{R}^N) \quad \text{for any } \alpha \geq 0.
\]

*Proof.* The Plancherel identity and the definition of \((-\Delta)^r\) by Fourier transform imply
\[
\|(-\Delta)^r \phi - \phi\|_{H^\alpha(\mathbb{R}^N)}^2 = \|(1 + |\xi|^2)^{\alpha/2}(|\xi|^{2r} - 1)^2\hat{\phi}\|_{L^2(\mathbb{R}^N)}^2.
\]
Consequently, since \(\hat{\phi}\) belongs to the Schwartz class \(S(\mathbb{R}^N)\) of rapidly decreasing functions, we can find a positive \(L^1(\mathbb{R}^N)\)-function \(g\) such that
\[
\|(1 + |\xi|^2)^{\alpha/2}(|\xi|^{2r} - 1)^2\hat{\phi}(\xi)| \leq g(\xi) \quad \text{for all } \xi \in \mathbb{R}^N.
\]
Then, noting that \((1 + |\xi|^2)^{\alpha/2}(|\xi|^{2r} - 1)^2\hat{\phi}(\xi) \xrightarrow{r \searrow 0} 0\) for any \(\xi \in \mathbb{R}^N\), we obtain (44) via the dominated convergence theorem and (45).

It is also important to recall the following (asymptotic) relation between the \(H^r\)- and the \(L^2\)-norm (see [18] and [26]),
\[
\lim_{r \searrow 0} \frac{C(r)}{2} \|v\|_{H^r(\mathbb{R}^N)}^2 = \|v\|_{L^2(\mathbb{R}^N)}^2 \quad \text{for any } v \in \bigcup_{\tau \in (0,1)} \mathcal{X}_{\tau,0},
\]
which is clearly related to Lemma 2.1.

Next, we shall characterize the behavior of the (weak) fractional Laplacian \(\mathcal{A}_r\) as the index \(r\) goes to 0 (cf. Lemma 2.1). Indeed, we can prove that \(\mathcal{A}_r\) tends to the identity operator in a suitable way.

**Lemma 2.2.** Let \(\{r_k\} \subset (0,1)\) be a sequence with \(r_k \searrow 0\) as \(k \nearrow \infty\). Let \(\{v_k\} \subset H_0\) be a sequence such that \(v_k \in D(\mathcal{A}_{r_k})\) for all \(k \in \mathbb{N}\). Moreover, let us assume both \(\{v_k\}\) and \(\{\mathcal{A}_{r_k} v_k\}\) to be uniformly bounded in \(H_0\), respectively. Then, denoting by \(v\) the weak limit of \(v_k\) in \(H_0\) (up to a non-relabeled subsequence of \(k\)), we have
\[
\mathcal{A}_{r_k} v_k \xrightarrow{k \nearrow \infty} v \quad \text{weakly in } H_0.
\]

*Proof.* The boundedness of \(\mathcal{A}_{r_k} v_k\) in \(H_0\) entails the existence of \(w \in H_0\) such that
\[
\mathcal{A}_{r_k} v_k \xrightarrow{k \nearrow \infty} w \quad \text{weakly in } H_0,
\]
up to a non-relabeled subsequence of \(k\) (of course, we can assume that this subsequence is extracted from the subsequence along which \(v_k\) weakly converges to \(v\) in \(H_0\)). To conclude, we have to prove that \(w \equiv v\) a.e. in \(\Omega\). We test the weak convergence \(\mathcal{A}_{r_k} v_k \rightharpoonup v\) by \(\varphi \in C^\infty(\mathbb{R}^N)\) with support in \(\Omega\). Recalling Lemma 2.1 and (29), we then have
\[
\int_{\mathbb{R}^N} w \varphi \, dx = \lim_{k \nearrow \infty} \int_{\mathbb{R}^N} \mathcal{A}_{r_k} v_k \varphi \, dx = \lim_{k \nearrow \infty} \int_{\mathbb{R}^N} v_k \mathcal{A}_{r_k} \varphi \, dx = \int_{\mathbb{R}^N} v \varphi \, dx,
\]
which implies \(w \equiv v\) almost everywhere in \(\mathbb{R}^N\).

The optimal constant of the Poincaré-type inequality (20) depends on the first eigenvalue \(\lambda_1(r)\) of the fractional eigenvalue problem
\[
v \in \mathcal{X}_{r,0}, \quad \mathcal{A}_r v = \lambda v \quad \text{in } H_0,
\]
which also does not mean that \(\mathcal{A}_r v\) vanishes outside \(\Omega\) (see §2.3). More precisely, the optimal value of \(c_{\mathcal{A}}(r)\) is given as \(1/\lambda_1(r)\), that is,
\[
\|v\|_{H_0}^2 \leq \frac{1}{\lambda_1(r)} \|v\|_{\mathcal{X}_{r,0}}^2 \quad \text{for all } v \in \mathcal{X}_{r,0}.
\]
The first eigenvalue \(\lambda_1(r)\) is characterized in the next lemma, which comprises results from [33] and [34] and clarifies the spectral properties of the fractional Laplacian.
Lemma 2.3 ([33], [34]). Let \( r \in (0, 1) \). Then the fractional eigenvalue problem (48) admits a first eigenvalue \( \lambda_1(r) \) which is strictly positive, simple, isolated, and can be characterized as

\[
\lambda_1(r) = \min_{u \in \mathcal{X}_{r,0}, \|u\|_{H_0} = 1} \frac{C(r)}{2} \frac{\|u\|_{H^r(\mathbb{R}^N)}^2}{\|u\|_{H_0}^2}.
\]

Moreover, there exists a unique positive first eigenfunction \( e_1 \in \mathcal{X}_{r,0} \) which satisfies \( \|e_1\|_{H_0} = 1 \), and attains the minimum in (50).

Finally, denoting with \( \lambda_1 \) the first eigenvalue of the (standard) Dirichlet problem

\[
-\Delta v = \lambda v \quad \text{in } \Omega, \quad v = 0 \quad \text{on } \partial\Omega,
\]

then for any \( r \in (0, 1) \) there holds

\[
\lambda_1(r) < \lambda'_1.
\]

Proof. To simplify the notation, for any \( k \in \mathbb{N} \) we denote by \( \lambda_k \) and \( v_k \) the first eigenvalue and the (normalized in \( H_0 \)) first eigenfunction of (48), respectively. Then, the couple \( (\lambda_k, v_k) \) solves the eigenvalue problem

\[
\mathcal{A}_r v_k = \lambda_k v_k \quad \text{in } H_0.
\]

As a consequence, the only uniform (in \( k \)) estimate on which we can rely is \( \|v_k\|_{L^2(\mathbb{R}^N)} = 1 \). Hence, the only convergence we expect for \( v_k \) is a weak \( L^2 \)-convergence. Indeed, this is enough to pass to the limit in (54) (recall Lemma 2.2). On the other hand, since a priori we cannot exclude that the weak limit of \( v_k \) is zero, we are not able, at this level, to conclude that \( \lambda_k \to 1 \). For this reason, we have to use a different approach. First of all, thanks to (52) of Lemma 2.3, we only need to prove a lower bound for the limit in (53), namely

\[
\lim_{k \nearrow + \infty} \lambda_k \geq 1.
\]

To this aim, we test (54) by \( v_k \). Using that \( \|v_k\|_{H_0} = 1 \) and recalling (29) and (31), we get

\[
\lambda_k = \lambda_k \|v_k\|_{H_0}^2 = (\mathcal{A}_r v_k, v_k)_{\mathcal{X}_{r,0}} = \int_{\mathbb{R}^N} |(-\Delta)^{r/2} v_k|^2 \, dx = \int_{\mathbb{R}_\xi^N} |\xi|^{2r} \hat{v}_k(\xi)^2 \, d\xi.
\]

For each \( k \in \mathbb{N} \), set \( f_k(\xi) = |\hat{v}_k(\xi)|^2 \). Then \( f_k \geq 0 \) a.e. in \( \mathbb{R}_\xi^N \) and, by Plancherel’s identity, \( f_k \in L^1(\mathbb{R}_\xi^N) \) with \( \int_{\mathbb{R}_\xi^N} f_k(\xi) \, d\xi = 1 \) for any \( k \in \mathbb{N} \). Moreover, by (56) and (52), the sequence of the \( 2rk \)-moments of \( f_k \) is uniformly bounded with respect to \( k \). Namely, we have

\[
\int_{\mathbb{R}_\xi^N} |\xi|^{2rk} f_k(\xi) \, d\xi = \lambda_k < (\lambda_1)^{rk}.
\]

Now, combining this information with the fact that \( v_k \) is zero outside \( \Omega \), we can easily deduce that \( f_k \) lies in \( L^\infty(\mathbb{R}_\xi^N) \). Indeed, using \( |e^{-ix\cdot\xi}| \leq 1 \) together with the Hölder inequality, we obtain

\[
f_k(\xi) = \frac{1}{(2\pi)^N} \left| \int_{\mathbb{R}^N} e^{-ix\cdot\xi} v_k(x) \, dx \right|^2 \leq \frac{1}{(2\pi)^N} \left( \int_{\Omega} \left| e^{-ix\cdot\xi} v_k(x) \right| \, dx \right)^2 \frac{|\Omega|}{(2\pi)^N} \frac{\|v_k\|_{H_0}^2}{\|v_k\|_{H_0}^2}, \quad \text{for all } \xi \in \mathbb{R}_\xi^N.
\]
At this point, we need to estimate the $L^1$-norm of $f_k$ in a quantitative way. To this purpose, we use the following interpolation lemma, whose proof is based on a technique widely used in the kinetic theory community (alternatively, one could refer to [38, Lemma 2.1], where a similar inequality is shown in a different way):

**Lemma 2.4.** Given $f \in L^1(\mathbb{R}^N) \cap L^{\infty}(\mathbb{R}^N)$ with $\int_{\mathbb{R}^N} |x|^\alpha |f(x)| \, dx < +\infty$, where $\alpha > 0$, there holds

$$\|f\|_{L^1(\mathbb{R}^N)} \leq \kappa(N, \alpha) \left( \|x|^\alpha f\|_{L^N(\mathbb{R}^N)} \right) \left( \|f\|_{L^{\infty}(\mathbb{R}^N)} \right),$$

where

$$\kappa(N, \alpha) = \alpha^{-\frac{\alpha}{d}} (\alpha + N)^{-\frac{N}{d}} d^\frac{N}{\alpha}$$

and $d = d(N)$ is the volume of the unit ball in $\mathbb{R}^N$.

**Proof.** To simplify the presentation, we set $L := \|f\|_{L^{\infty}(\mathbb{R}^N)}$ and $M := \int_{\mathbb{R}^N} |x|^\alpha |f(x)| \, dx$. We compute, for $R > 0$ to be chosen later,

$$\int_{\mathbb{R}^N} |f| \, dx = \int_{|x| \leq R} |f| \, dx + \int_{|x| > R} |f| \, dx \leq dR^N L + R^{-\alpha} M.$$

Now, optimizing with respect to $R$ the function $g(R) := dR^N L + R^{-\alpha} M$, we find

$$\|f\|_{L^1(\mathbb{R}^N)} \leq \inf_{R > 0} g(R) = g(\bar{R}), \quad \text{with} \quad \bar{R} = \left( \frac{\alpha M}{dN L} \right)^{1/(N + \alpha)}.$$

Hence, computing $g(\bar{R})$, we readily get (59). \qed

**Continuation of Proof of Proposition 2.2.** Applying the lemma, with $f = f_k$, $\alpha = 2r_k$, $\|\lambda_k^N f_k\|_{L^1(\mathbb{R}^N)} = \lambda_k$ and $\|f_k\|_{L^{\infty}(\mathbb{R}^N)} \leq \frac{|\Omega|}{(2\pi)^N}$, we then get

$$\|f_k\|_{L^1(\mathbb{R}^N)} \leq \kappa(N, 2r_k) \lambda_k^N \left( \frac{|\Omega|}{(2\pi)^N} \right)^{\frac{2N}{N + 2r_k}},$$

which, together with the fact that $\|f_k\|_{L^1(\mathbb{R}^N)} = 1$ by construction, implies

$$\lambda_k \geq \kappa(N, 2r_k) \frac{N + 2r_k}{2} \left( \frac{(2\pi)^N}{|\Omega|} \right)^{\frac{2r_k}{N}}.$$

Hence, letting $k \searrow +\infty$ and noting that $\lim_{k \searrow +\infty} \kappa(N, 2r_k) = 1$, we get (55), which implies the thesis. \qed

3. Assumptions and statement of main results

Let us start with listing our hypotheses on the data of the problem. First of all, we make precise the choice of the confining potential by choosing

$$W(v) = \frac{1}{p} |v|^p - \frac{1}{2} v^2, \quad p \in (1, \infty) \setminus \{2\}.$$

We also set $\hat{\beta}(v) := \frac{1}{p} |v|^p$ and $\beta(v) := \beta'(v) = |v|^{p-1} \text{sign} v$, for brevity. In general, we note the case $p > 2$ as the coercive case, while for $p \in (1, 2)$ we speak of a non-coercive potential. Indeed, $W$ is unbounded from below in the latter situation. As noted in Introduction, the coercive case includes, up to an additive constant, the standard double well potential

$$W(v) = \frac{1}{4} (v^2 - 1)^2.$$

Next, for $v \in H_0$, we introduce the energy functional

$$\mathcal{E}_\sigma(v) := \frac{1}{2} \|v\|_{X_{\sigma, 0}}^2 + \int_\Omega W(v) \, dx,$$

whenever it makes sense. In particular, in the case $p > 2$, the domain of $W$, i.e., the set where it takes finite values, is given by

$$\mathcal{E}_\sigma = \{ v \in H_0 \text{ such that } \mathcal{E}_\sigma(v) < +\infty \} = X_{\sigma, 0} \cap L^p(\mathbb{R}^N).$$
Of course, $E_\sigma$ is a Banach space with the natural norm
\begin{equation}
\|v\|_{E_\sigma} := \|v\|_{X_{\sigma,0}} + \|v\|_{L^p(\mathbb{R}^N)}.
\end{equation}

If $p \leq \frac{2N}{N-2\sigma}$, then by Sobolev’s embeddings $E_\sigma$ coincides in fact with $X_{\sigma,0}$ (hence we can use the norm of $X_{\sigma,0}$ instead of (68)). On the other hand, in the non-coercive case $p \in (1, 2)$, the functional $E_\sigma$ may be unbounded from below. In that situation, existence and uniqueness of (global in time) solutions still hold. However, when dealing with the singular limits, some difficulties arise from the lack of coercivity.

In what follows, we will look for solutions taking values in the energy space $E_\sigma$. Correspondingly, we ask that the same regularity is satisfied by the initial datum:
\begin{equation}
u_0 \in E_\sigma.
\end{equation}

Next, we introduce a weak (energy) formulation of the system (1)-(2). Here and henceforth, the notation $C_w([0,T];X)$ will represent the class of continuous functions on $[0,T]$ in the weak topology of a normed space $X$.

**Definition 3.1.** We say that $(u,w)$ is a weak solution to the Cauchy-Dirichlet problem (1)-(4) for the fractional Cahn-Hilliard system if, for all $T > 0$, we have
\begin{align}
&u \in C_w([0,T];E_\sigma) \cap C([0,T];H_0) \cap W^{1,2}(0,T;X'_{\sigma,0}), \\
&w \in L^2(0,T;X_{\sigma,0});
\end{align}
moreover, the couple $(u,w)$ satisfies, a.e. in $(0,T)$, the following weak formulation of (1)-(2):
\begin{align}
&\partial_t u + \mathcal{A}_s w = 0 \quad \text{in } X'_{\sigma,0}, \\
&w = \mathcal{A}_s u + B(u) - u \quad \text{in } E_\sigma,
\end{align}
where $B(u)$ denotes the bounded linear functional on $L^p(\mathbb{R}^N)$ defined by
\begin{equation}
(B(u),v)_{L^p(\mathbb{R}^N)} = \int_{\Omega} \beta(u(x))v(x) \, dx \quad \text{for } v \in L^p(\mathbb{R}^N),
\end{equation}
and, finally, the initial condition (3) holds in the following sense:
\begin{equation}
u(t) \to u_0 \quad \text{strongly in } H_0 \text{ and weakly in } E_\sigma \text{ as } t \searrow 0.
\end{equation}

Correspondingly, we can prove the following existence and uniqueness result:

**Theorem 1.** Let $s,\sigma \in (0,1)$, $p \in (1, \infty) \setminus \{2\}$, and assume (64) and (69). Then, the fractional Cahn-Hilliard system (1)-(4) admits a unique weak solution $(u,w)$ in the sense of Definition 3.1, which additionally satisfies
\begin{align}
&\|\beta(u(\cdot,t))\|_{H_0}^2 \leq 2 \left( \|w(t)\|_{H_0}^2 + \|u(t)\|_{H_0}^2 \right) \quad \text{for a.e. } t \in (0,T), \\
&\beta(u) \in L^2(0,T;H_0).
\end{align}
Moreover, $u(t) := u(\cdot,t)$ and $E_\sigma(u(t))$ are right-continuous on $[0,T]$ with respect to the strong topologies of $E_\sigma$ and $\mathbb{R}$, respectively, and the following energy inequality holds true:
\begin{equation}
\|w(t)\|_{X_{\sigma,0}}^2 + \frac{d}{dt} E_\sigma(u(t)) \leq 0 \quad \text{for a.e. } t \in (0,T).
\end{equation}
In particular, if $\sigma \geq s$, then $u \in C([0,T];E_\sigma)$ and $E_\sigma(u(t))$ is absolutely continuous on $[0,T]$; moreover, the inequality (77) can be replaced by an equality, namely we have
\begin{equation}
\|w(t)\|_{X_{\sigma,0}}^2 + \frac{d}{dt} E_\sigma(u(t)) = 0 \quad \text{for a.e. } t \in (0,T).
\end{equation}

A proof of the above result will be carried out in Section 4 by means of time-discretization, a-priori estimates, and compactness arguments. It is worth noting that, in addition to (70)-(76), one could show that weak solutions satisfy parabolic time-smoothing properties. We will analyze this issue in a forthcoming paper, where we also plan to consider a more general class of potentials $W$.

Now, we come to the behavior of (families of) solutions as $\sigma$ tends to $0$. Then, in the coercive case we can prove the following
Theorem 2 (from Cahn-Hilliard to porous medium). Let \( p \in (2, \infty) \), \( s \in (0, 1) \) and let \( \{ \sigma_k \} \subset (0, 1) \) be such that \( \sigma_k \searrow 0 \) as \( k \uparrow +\infty \). Moreover, let us given a sequence of initial data \( \{ u_{0,k} \} \) and \( u_0 \in \mathcal{X}_{s,0}^\prime \) satisfying
\[
\sup_{k \in \mathbb{N}} \mathbb{E}_{\sigma_k}(u_{0,k}) < +\infty, \quad u_{0,k} \rightharpoonup u_0 \text{ in } \mathcal{X}_{s,0}^\prime.
\]
Let \( (u_k, w_k) \) denote the corresponding sequence of unique weak solutions to (1)-(4), with \( \sigma = \sigma_k \) and initial datum \( u_{0,k} \). Then, there exist a (non-relabeled) subsequence of \( \{ k \} \) and a pair of limit functions \( (u, w) \) such that
\[
\begin{align*}
&u_k \rightharpoonup u \quad \text{weakly star in } L^\infty(0, T; L^p_0(\mathbb{R}^N)), \\
&\text{strongly in } L^p(0, T; L^p_0(\mathbb{R}^N)) \cap C([0, T]; \mathcal{X}_{s,0}^\prime), \\
&w_k \rightharpoonup w \quad \text{weakly in } W^{1,2}(0, T; \mathcal{X}_{s,0}^\prime),
\end{align*}
\]
Moreover,
\[
\begin{align*}
&u \in C_w([0, T]; L^0_0(\mathbb{R}^N)) \cap W^{1,2}(0, T; \mathcal{X}_{s,0}^\prime), \\
&\beta(u) = w \in L^2(0, T; \mathcal{X}_{s,0}^\prime).
\end{align*}
\]
Furthermore, \( u \) is a (weak) solution to the fractional porous medium equation
\[
\partial_t u + \mathfrak{A}_s \beta(u) = 0 \quad \text{in } \mathcal{X}_{s,0}^\prime, \quad \text{a.e. in } (0, T),
\]
with the initial condition \( u|_{t=0} = u_0 \), i.e.,
\[
\begin{align*}
&u(t) \rightharpoonup u_0 \quad \text{strongly in } \mathcal{X}_{s,0}^\prime \text{ as } t \searrow 0.
\end{align*}
\]

Now we address the non-coercive case \( p \in (1, 2) \). In this situation (which corresponds to the fast diffusion range when dealing with equations of the type (84)) we can prove similar results but the analysis requires some extra assumption together with a small modification of the system. First of all, we need a compatibility condition between \( p \) and the fractional order \( s \) in (72). To be precise, we ask that
\[
p > \frac{2N}{N + 2s} =: 2_*.\]

Note that the above condition implies that \( H^r(\Omega) \hookrightarrow L^{p'}(\Omega) \) with compact embedding. Hence \( \mathcal{X}_{s,0} \) is compactly embedded in \( L^{p'}(\mathbb{R}^N) \). Moreover, we need to modify a bit the energy functional. Namely, we replace \( \mathbb{E}_\sigma \) with the following
\[
\tilde{\mathbb{E}}_\sigma(v) := \frac{1}{2} ||v||_{\mathcal{X}_{s,0}}^2 + \frac{1}{p} ||v||_{L^p(\Omega)}^p - \frac{\lambda_1(\sigma)}{2} ||v||_{L^2(\Omega)}^2 \quad \text{for } v \in \mathcal{E}_\sigma = \mathcal{X}_{s,0},
\]
where \( \lambda_1(\sigma) \) is the first eigenvalue of (48) with \( r = \sigma \). Note that, in view of \( \lambda_1(\sigma) \overset{\sigma \searrow 0}{\to} 1 \) (cf. Prop. 2.2), one expects that the contribution of \( \lambda_1(\sigma) \) in the limit is idle. The reason why we need to replace \( \mathbb{E}_\sigma \) with \( \tilde{\mathbb{E}}_\sigma \) lies in the fact that, thanks to Poincaré’s inequality (49), one has
\[
\tilde{\mathbb{E}}_\sigma(v) \geq \frac{1}{p} ||v||_{L^p(\mathbb{R}^N)}^p \quad \text{for all } v \in \mathcal{X}_{s,0},
\]
and independently of the value of \( \sigma \). Of course, modifying the energy through the choice (87) leads correspondingly to a modification of the “original” system (72)-(73), which is now replaced by
\[
\begin{align*}
&\partial_t u + \mathfrak{A}_s w = 0 \quad \text{in } \mathcal{X}_{s,0}^\prime, \\
&w = \mathfrak{A}_s u + B(u) - \lambda_1(\sigma) u \quad \text{in } \mathcal{X}_{s,0}^\prime,
\end{align*}
\]
still complemented with the initial condition (74). At fixed \( \sigma \), existence and uniqueness of energy solutions for the modified system (89)-(90) follow from the very same argument given for Theorem 1. Moreover, the uniform coercivity provided by (88) permits us to prove a counterpart of Theorem 2 for the fast-diffusion case in the following theorem:
Theorem 3 (from Cahn-Hilliard to fast-diffusion). Let $s \in (0, 1)$, $p \in (2_*, 2)$, $2_*$ being given by (86), and let $\{\sigma_k\}$ be as before. Moreover, let us given a sequence of initial data $\{u_{0,k}\}$ and $u_0 \in X_{s,0}'$ satisfying

$$
\sup_{k \in \mathbb{N}} \mathbb{E}_{\sigma_k}(u_{0,k}) < +\infty, \quad u_{0,k} \to u_0 \quad \text{in} \quad X_{s,0}'.
$$

Let $(u_k, w_k)$ denote the corresponding sequence of unique weak solutions of (1)-(4), with $\sigma = \sigma_k$, $W'(v) = \beta(v) - \lambda_1(\sigma_k)v$ and initial datum $u_{0,k}$. Then, there exist a (non-relabelled) subsequence of $\{k\}$ and a pair of limit functions $(u, w)$ satisfying (79)-(83). Moreover, $u$ is a (weak) solution to the fractional fast-diffusion equation

$$
\partial_t u + \mathbb{A}_\sigma \beta(u) = 0 \quad \text{in} \quad X_{s,0}', \quad \text{a.e. in} \quad (0, T),
$$

with the initial condition (85).

Finally, we investigate the behavior of weak solutions to (1)-(4) when the order $s$ of the fractional Laplacian in (72) is let tend to 0. In this case, we can prove the following

Theorem 4 (from Cahn-Hilliard to Allen-Cahn). Let $p \in (1, \infty) \setminus \{2\}$, $\sigma \in (0, 1)$ and let $\{s_k\} \subset (0, 1)$ be such that $s_k \searrow 0$ as $k \not\to +\infty$. Let also $u_{0,k} \in X_{s_k,0}$ and $u_0 \in H_0$ satisfy

$$
\sup_{k} \left( \mathbb{E}_{\sigma}(u_{0,k}) + \|u_{0,k}\|_{X_{s_k,0}'}^2 \right) < \infty, \quad u_{0,k} \to u_0 \quad \text{strongly in} \quad H_0.
$$

Let us denote as $(u_k, w_k)$ the corresponding sequence of unique weak solutions to (1)-(4) with $s = s_k$ and initial datum $u_{0,k}$. Then, there exist a (non-relabelled) subsequence of $\{k\}$ and a pair of limit functions $(u, w)$ such that

$$
u_k \to u \quad \text{weakly star in} \quad L^\infty(0, T; \mathcal{E}_\sigma),
$$

$$
\text{strongly in} \quad C([0, T]; H_0),
$$

$$
\text{weakly in} \quad W^{1,2}(0, T; \mathcal{E}_\sigma'),
$$

$$
w_k \to w \quad \text{weakly in} \quad L^2(0, T; H_0),
$$

$$
\mathbb{A}_{s_k} w_k \to w \quad \text{weakly in} \quad L^2(0, T; \mathcal{E}_\sigma').
$$

Moreover,

$$
u \in C_w([0, T]; \mathcal{E}_\sigma) \cap W^{1,2}(0, T; \mathcal{E}_\sigma'), \quad w \in L^2(0, T; H_0),
$$

$$
w = \mathbb{A}_{s} u + B(u) - u, \quad \text{and} \quad u \text{ is a (weak) solution to the fractional Allen-Cahn equation}
$$

$$
\partial_t u + \mathbb{A}_\sigma \beta(u) + B(u) - u = 0 \quad \text{in} \quad \mathcal{E}_\sigma', \quad \text{a.e. in} \quad (0, T),
$$

with the initial condition $u|_{t=0} = u_0$, i.e.,

$$
u(t) \to u_0 \quad \text{strongly in} \quad H_0 \quad \text{and weakly in} \quad \mathcal{E}_\sigma \quad \text{as} \quad t \not\to 0.
$$

Theorems 2, 3 and 4 will be proved in Section 5 below. Actually, while the proof of Theorem 4 is almost straightforward, taking the limit $\sigma \not\to 0$ will require a careful analysis of the behavior of the eigenvalues of $\mathbb{A}_{s_k}$ as $\sigma_k \not\to 0$. The proof of Theorem 3 is simplified by the fact that we will not start from the original system (72)-(73), but rather from its modified version (89)-(90), whose energy is bounded from below, uniformly w.r.t. $\sigma$, thanks to (88). As will be clarified from the proof, this fact makes the analysis simpler. Indeed, we will not need to use the results in Prop. 2.2 regarding the asymptotics of the first eigenvalue of the fractional Laplacian, which, instead, are essential for proving Theorem 2. Actually, we expect that the convergence to the fast diffusion equation could hold, still for $p \in (2_*, 2)$, also for the non-modified fractional Cahn-Hilliard (72)-(73). However, for the moment, this remains as an open question.

4. Existence and uniqueness

This section is devoted to giving a proof of Theorem 1, i.e., existence and uniqueness of weak solutions along with some regularity properties and energy inequalities. In the existence part, the basic strategy of a proof is more or less standard, and it consists of time-discretization, a priori estimates, compactness arguments to obtain convergence, and Minty’s trick for the identification of the limit. On the other hand, we will face some difficulty in deriving the energy inequality (77),
particularly, the differentiability of the energy $t \mapsto \mathcal{E}_\sigma(u(t))$, and for proving the right-continuity of $u(t)$ in the strong topology of $\mathcal{E}_\sigma$. These difficulties are due to the simultaneous presence of two fractional Laplacians of (possibly) different order. However, the energy inequality seems to be one of the minimum requirements for the analysis of singular limits in latter sections and also for the investigation of the long-time behavior of global (in time) solutions. Moreover, the right-continuity of solutions plays a crucial role to prove the convergence of each global solution as $t$ goes to infinity. More details on the long-time behavior of solutions will be discussed in a forthcoming paper. Here, it is also worth emphasizing that, in contrast with usual studies on the Cahn-Hilliard equation, our argument does not rely on the coercivity of the double-well potential $W$. In other words, we can treat the cases $p > 2$ and $p \in (1, 2)$, simultaneously. This fact potentially means that our proof could also be extended to fractional Cahn-Hilliard equations with more general classes of potentials.

4.1. Uniqueness of solution. We first note that $\mathfrak{A}_s : \mathcal{X}_{s,0} \to \mathcal{X}_{s,0}'$ is invertible, i.e., there exists the inverse mapping $\mathfrak{A}_s^{-1} : \mathcal{X}_{s,0}' \to \mathcal{X}_{s,0}$ of $\mathfrak{A}_s$. Indeed, the functional on $\mathcal{X}_{s,0}$ defined by

$$v \mapsto \frac{1}{2} \|v\|_{\mathcal{X}_{s,0}}^2$$

is smooth, convex and coercive in $\mathcal{X}_{s,0}$. Hence its Fréchet derivative $\mathfrak{A}_s$ is invertible. Furthermore, one observes by the definition of norms in dual spaces that

$$\langle \mathfrak{A}_s u, w \rangle_{\mathcal{X}_{s,0}} = \|u\|_{\mathcal{X}_{s,0}}^2 = \|\mathfrak{A}_s u\|_{\mathcal{X}_{s,0}'}^2 \quad \forall u \in \mathcal{X}_{s,0}.$$

Hence $\mathfrak{A}_s$ is a duality mapping between $\mathcal{X}_{s,0}$ and $\mathcal{X}_{s,0}'$.

Now, let $(u_1, w_1)$ and $(u_2, w_2)$ be weak solutions of the fractional Cahn-Hilliard equation for the same initial data $u_0$. Let $\hat{u} := u_1 - u_2$ and $\hat{w} := w_1 - w_2$. Then we have

$$\partial_t \hat{u} + \mathfrak{A}_s \hat{w} = 0 \quad \text{in} \ \mathcal{X}_{s,0}', \quad \hat{w} = \mathfrak{A}_s \hat{w} + B(u_1) - B(u_2) - \hat{u} \quad \text{in} \ \mathcal{E}_\sigma'.$$

Applying $\mathfrak{A}_s^{-1} : \mathcal{X}_{s,0}' \to \mathcal{X}_{s,0}$ to both sides of the first equation of (96), we have

$$\frac{d}{dt} \mathfrak{A}_s^{-1} \hat{u} + \hat{w} = 0 \quad \text{in} \ \mathcal{X}_{s,0}.$$

Test it by $\hat{u} \in \mathcal{X}_{s,0} \mapsto \mathcal{X}_{s,0}'$ (see (24)) to get

$$\frac{1}{2} \frac{d}{dt} \|\hat{u}\|_{\mathcal{X}_{s,0}'}^2 + \langle \hat{u}, \hat{w} \rangle_{\mathcal{X}_{s,0},0} = 0.$$

Here we used the fact that

$$\langle \hat{u}, \frac{d}{dt} \mathfrak{A}_s^{-1} \hat{u} \rangle_{\mathcal{X}_{s,0}} = \langle \mathfrak{A}_s \circ \mathfrak{A}_s^{-1} \hat{u}, \frac{d}{dt} \mathfrak{A}_s^{-1} \hat{u} \rangle_{\mathcal{X}_{s,0}} = \frac{1}{2} \frac{d}{dt} \|\mathfrak{A}_s^{-1} \hat{u}\|_{\mathcal{X}_{s,0}}^2 = \frac{1}{2} \frac{d}{dt} \|\hat{u}\|_{\mathcal{X}_{s,0}'}^2,$$

since $\mathfrak{A}_s^{-1}$ is a duality mapping between $\mathcal{X}_{s,0}'$ and $\mathcal{X}_{s,0}$. Test now the second equation of (96) by $\hat{u}$. Then, by the monotonicity of $\beta$, it follows that

$$\langle \hat{w}, \hat{u} \rangle_{\mathcal{E}_\sigma} = \|\hat{u}\|_{\mathcal{X}_{s,0}}^2 + \int_\Omega (\beta(u_1) - \beta(u_2)) \hat{u} \, dx - \|\hat{u}\|_{L^2(\Omega)}^2 \geq \|\hat{u}\|_{\mathcal{X}_{s,0}}^2 - \|\hat{u}\|_{L^2(\Omega)}^2.$$

We also note that, for any functions $u \in \mathcal{E}_\sigma$ ($\mapsto \mathcal{X}_{s,0}'$ by (24)) and $w \in \mathcal{X}_{s,0}$ ($\mapsto \mathcal{E}_\sigma$),

$$\langle u, w \rangle_{\mathcal{X}_{s,0}} = \int_{\mathbb{R}^N} u(x) w(x) \, dx = \langle w, u \rangle_{\mathcal{E}_\sigma}.$$

Combining (97) and (99) with (100), we deduce that

$$\frac{1}{2} \frac{d}{dt} \|\hat{u}\|_{\mathcal{X}_{s,0}'}^2 + \|\hat{u}\|_{\mathcal{X}_{s,0}}^2 \leq \|\hat{u}\|_{L^2(\Omega)}^2 \leq \frac{1}{2} \|\hat{u}\|_{\mathcal{X}_{s,0}}^2 + C \|\hat{u}\|_{\mathcal{X}_{s,0}'}^2,$$

for some constant $C \geq 0$. Here we used Ehrling’s lemma (see [36, Lemma 8]), i.e., for each $\varepsilon > 0$, one can take a constant $C_\varepsilon \geq 0$ such that

$$\|v\|_{L^2(\Omega)} = \|v\|_{H_0} \leq \varepsilon \|v\|_{\mathcal{X}_{s,0}} + C_\varepsilon \|v\|_{\mathcal{X}_{s,0}'} \quad \forall v \in \mathcal{X}_{s,0}.$$
Thus we get
\[
\frac{1}{2} \frac{d}{dt} \| \hat{u} \|_{X_{t,0}}^2 + \frac{1}{2} \| \hat{u} \|_{X_{t,0}}^2 \leq C \| \hat{u} \|_{X_{t,0}}^2,
\]
which along with the fact that \( \hat{u}(0) = 0 \) in \( X_{t,0} \) implies \( \hat{u}(t) = 0 \) in \( X_{t,0} \) for all \( t \geq 0 \). Consequently, since \( u_1(t) \) and \( u_2(t) \) belong to \( H_0 \), one can assure that \( u_1(t) = u_2(t) \) a.e. in \( \mathbb{R}^N \) for all \( t \in [0, T] \), that is, \( u_1 = u_2 \) as desired.

4.2. **Time-discretization.** Let \( N \in \mathbb{N} \) and let \( \tau = T/N \) be a time step. In order to construct discretization of (72)-(73), we first carry out the following discretization of (72)-(73):

\[
\frac{u_n - u_{n-1}}{\tau} + \mathcal{A}_s w_n = 0 \quad \text{in } X_{s,0},
\]

\[
w_n = \mathcal{A}_s u_n + B(u_n) - u_{n-1} \quad \text{in } E_{\sigma}
\]
for \( n = 1, 2, \ldots, N \). In order to show existence of \((u_n, w_n)\) satisfying (102)-(103), let us introduce the functional \( F_n : E_\sigma \to \mathbb{R} \) given by
\[
F_n(u) := \frac{\tau}{2} \left\| \frac{u - u_{n-1}}{\tau} \right\|_{X_{s,0}}^2 + \frac{C(\sigma)}{4} \int \int_{\mathbb{R}^N} \frac{|u(x) - u(y)|^2}{|x - y|^{N+2\sigma}} \, dx \, dy + \int_\Omega \hat{\beta}(u) \, dx - \int_\Omega u_{n-1} \, dx
\]
for \( u \in E_\sigma \), where \( \hat{\beta} \) denotes the primitive function of \( \beta \) such that \( \hat{\beta}(0) = 0 \). Then \( F_n \) is strictly convex, coercive and of class \( C^1 \) in \( E_\sigma \). Therefore one can take a unique minimizer \( u_n \in E_\sigma \) of \( F_n \). Let us recall that \( \mathcal{A}_s^{-1} : X_{s,0}^* \to X_{s,0} \) is the inverse duality mapping, whence \( \mathcal{A}_s^{-1} \) coincides with the Fréchet derivative of the functional
\[
K(v) := \frac{1}{2} \| v \|_{X_{s,0}}^2 \quad \text{for } v \in X_{s,0}^*.
\]
Correspondingly, \( K(\cdot) \) is of class \( C^1 \) in \( X_{s,0}^* \) moreover,
\[
K'(v) = \mathcal{A}_s^{-1} v \quad \text{for } v \in X_{s,0}.
\]
Indeed, for each \( u \in E_\sigma \), one has
\[
K(u) - K(v) \leq \langle u - v, \mathcal{A}_s^{-1} u \rangle_{X_{s,0}} \quad \text{for all } v \in E_\sigma,
\]
which implies (104). Thus we have
\[
\mathcal{A}_s^{-1} \left( \frac{u_n - u_{n-1}}{\tau} \right) + \mathcal{A}_s u_n + B(u_n) - u_{n-1} = 0 \quad \text{in } E_{\sigma},
\]
Setting
\[
w_n := -\mathcal{A}_s^{-1} \left( \frac{u_n - u_{n-1}}{\tau} \right) \in X_{s,0}
\]
and applying \( \mathcal{A}_s \) to both sides, we obtain (102) along with (103).

4.3. **A priori estimates.** Test (105) by \( u_n \in E_\sigma \) \((\mapsto X_{t,0}^*)\). Then as in (98) it follows that
\[
\| u_n \|_{X_{t,0}}^2 - \| u_{n-1} \|_{X_{t,0}}^2 + 2\tau \left( \| u_n \|_{X_{s,0}}^2 + \| u_n \|_{L^p(\Omega)}^p \right) \leq 2\tau \langle u_n, u_{n-1} \rangle_{L^2(\Omega)}
\]
\[
\leq \tau \left( \| u_n \|_{L^2(\Omega)}^2 + \| u_{n-1} \|_{L^2(\Omega)}^2 \right).
\]
Summing up, we deduce that
\[
\| u_n \|_{X_{t,0}}^2 + 2 \sum_{j=1}^n \tau \left( \| u_j \|_{X_{s,0}}^2 + \| u_j \|_{L^p(\Omega)}^p \right) \leq \| u_0 \|_{X_{t,0}}^2 + 2 \sum_{j=0}^{n} \tau \| u_j \|_{L^2(\Omega)}^2.
\]
Recall (101) again to obtain
\[
\| u_n \|_{X_{t,0}}^2 \leq C \| u_0 \|_{X_{t,0}}^2 + C \sum_{j=1}^n \tau \| u_j \|_{X_{t,0}}^2.
\]
Hence, due to the discrete Gronwall inequality, we obtain
\begin{equation}
\max_n \|u_n\|_{X_{s,0}}^2 + \sum_{n=0}^N \tau \left( \|u_n\|^2_{X_{s,0}} + \|u_n\|^p_{L^p(\Omega)} \right) \leq C. \tag{106}
\end{equation}

Test (102) by \(w_{n} \in X_{s,0}\) and (103) by \((u_{n} - u_{n-1})/\tau \in \mathcal{E}_{\sigma}\) and employ (100) to get
\begin{equation}
\frac{\|u_{n}\|^2_{X_{s,0}} - \|u_{n-1}\|^2_{X_{s,0}}}{2\tau} + \frac{\|u_{n}\|^p_{L^p(\Omega)} - \|u_{n-1}\|^p_{L^p(\Omega)}}{p\tau} \leq \frac{\|u_{n}\|^2_{L^2(\Omega)} - \|u_{n-1}\|^2_{L^2(\Omega)}}{2\tau}. \tag{107}
\end{equation}

By summing up, we have
\begin{equation}
\sum_{j=1}^n \tau \|w_j\|^2_{X_{s,0}} + \frac{1}{2} \|u_{n}\|^2_{X_{s,0}} + \frac{1}{p} \|u_{n}\|^p_{L^p(\Omega)} \leq \frac{1}{2} \|u_0\|^2_{X_{s,0}} + \frac{1}{p} \|u_0\|^p_{L^p(\Omega)} + \frac{1}{2} \|u_n\|^2_{L^2(\Omega)}. \tag{108}
\end{equation}

Using (101) and (106), assumption (69), and the the boundedness of \(\mathfrak{A}_s : X_{s,0} \to X'_{s,0}\) along with (102), we then deduce
\begin{equation}
\sum_{n=1}^N \tau \|w_n\|^2_{X_{s,0}} + \max_n \left( \|u_{n}\|^2_{X_{s,0}} + \|u_{n}\|^p_{L^p(\Omega)} \right) \leq C, \quad \sum_{n=1}^N \frac{\|u_{n} - u_{n-1}\|_{X_{s,0}}}{\tau} \leq C. \tag{109}
\end{equation}

Let \(\bar{u}_\tau\) and \(\bar{w}_\tau\) be the piecewise constant interpolants of \(\{u_n\}\) and \(\{w_n\}\), respectively, and let \(u_\tau\) be the piecewise linear interpolant of \(\{u_n\}\). More precisely, we define \(\bar{u}_\tau, u_\tau\) by
\begin{equation}
\bar{u}_\tau(t) \equiv u_n, \quad u_\tau(t) = \frac{t - t_{n-1}}{\tau} u_n + \frac{t_n - t}{\tau} u_{n-1} \quad \text{for } t \in [t_{n-1}, t_n),
\end{equation}
and \(\bar{w}_\tau\) analogously. Then they satisfy
\begin{align}
\partial_t u_\tau + \mathfrak{A}_s \bar{w}_\tau &= 0 \quad \text{in } X'_{s,0}, \tag{110} \\
\bar{w}_\tau &= \mathfrak{A}_\sigma \bar{u}_\tau + B(\bar{u}_\tau) - \bar{u}_\tau(\cdot - \tau) \quad \text{in } \mathcal{E}'_{\sigma}. \tag{111}
\end{align}

Moreover, the previous estimates (109) can be rewritten in the form
\begin{equation}
\int_0^T \|\bar{w}_\tau(t)\|^2_{X_{s,0}} \, dt + \sup_{t \in [0, T]} \left( \|\bar{u}_\tau(t)\|^2_{X_{s,0}} + \|u_\tau(t)\|^p_{L^p(\mathbb{R}^N)} \right) \leq C, \quad \int_0^T \|\partial_t u_\tau(t)\|^2_{X'_{s,0}} \, dt \leq C, \tag{112}
\end{equation}
whence we easily get also
\begin{equation}
\sup_{t \in [0, T]} \left( \|u_\tau(t)\|^2_{X_{s,0}} + \|u_\tau(t)\|^p_{L^p(\mathbb{R}^N)} \right) \leq C. \tag{113}
\end{equation}

Note that
\begin{equation}
\langle B(\bar{u}_\tau(t)), \phi \rangle_{L^p(\mathbb{R}^N)} = \int_\Omega \beta(\bar{u}_\tau(x, t)) \phi(x) \, dx \leq \|\bar{u}_\tau(t)\|_{L^p(\Omega)}^{p-1} \|\phi\|_{L^p(\mathbb{R}^N)} \quad \text{for all } \phi \in L^p(\mathbb{R}^N),
\end{equation}
which implies
\begin{equation}
\sup_{t \in [0, T]} \|B(\bar{u}_\tau(t))\|_{L^{p'}(\mathbb{R}^N)} \leq \sup_{t \in [0, T]} \|\bar{u}_\tau(t)\|_{L^p(\Omega)}^{p-1} \leq C. \tag{114}
\end{equation}

4.4. **Convergence as \(\tau \to 0\).** From the estimates established so far, one can take a (non-labeled) subsequence of \(\tau \to 0\) (equivalently, \(N \to \infty\)) such that
\begin{align}
\bar{w}_\tau &\to w \quad \text{weakly in } L^2(0, T; X_{s,0}), \tag{115} \\
\mathfrak{A}_s \bar{w}_\tau &\to \mathfrak{A}_s w \quad \text{weakly in } L^2(0, T; X'_{s,0}), \tag{116} \\
\bar{u}_\tau &\to u \quad \text{weakly star in } L^\infty(0, T; \mathcal{E}_{\sigma}), \tag{117} \\
\mathfrak{A}_s \bar{u}_\tau &\to \mathfrak{A}_s u \quad \text{weakly in } L^2(0, T; X'_{\sigma,0}), \tag{118} \\
B(\bar{u}_\tau(\cdot)) &\to \chi \quad \text{weakly star in } L^\infty(0, T; L^p(\mathbb{R}^N)), \tag{119} \\
u_\tau &\to u \quad \text{weakly star in } L^\infty(0, T; \mathcal{E}_{\sigma}), \tag{120} \\
\partial_t u_\tau \to \partial_t u \quad \text{weakly in } L^2(0, T; X'_{s,0}). \tag{121}
\end{align}
Combining these facts with the compact embeddings $\mathcal{E}_\sigma \hookrightarrow H_0 \hookrightarrow \mathcal{X}^{\prime}_{\sigma,0}$, and using the Aubin-Lions-Simon compactness lemma (see [36, Theorem 5]), one can verify that

$$
(122) \quad u_t \to u \quad \text{strongly in } C([0,T];H_0).
$$

Then $u$ belongs to $C([0,T];H_0) \cap C_w([0,T];\mathcal{E}_\sigma)$ as well. Moreover, we observe by (109) that

$$
(123) \quad \|\bar{u}_\tau(t) - u_\tau(t)\|_{\mathcal{X}^{\prime}_{\sigma,0}} = \left\| u_n - \left( t - t_{n-1} \right) \frac{t_n - t}{\tau} u_n + \frac{t_n - t}{\tau} u_{n-1} \right\|_{\mathcal{X}^{\prime}_{\sigma,0}}
$$

$$
= \frac{t_n - t}{\tau} \|u_n - u_{n-1}\|_{\mathcal{X}^{\prime}_{\sigma,0}} \leq C \sqrt{T} \quad \text{for all } t \in [t_{n-1}, t_n),
$$

which along with (122) yields

$$
\sup_{t \in [0,T]} \|\bar{u}_\tau(t) - u(t)\|_{\mathcal{X}^{\prime}_{\sigma,0}} \leq \sup_{t \in [0,T]} \|\bar{u}_\tau(t) - u_\tau(t)\|_{\mathcal{X}^{\prime}_{\sigma,0}} + \sup_{t \in [0,T]} \|u_\tau(t) - u(t)\|_{\mathcal{X}^{\prime}_{\sigma,0}} \to 0.
$$

Moreover, exploiting (101) and (112), for any $\varepsilon > 0$, one can take $C_\varepsilon \geq 0$ such that

$$
\|\bar{u}_\tau(t) - u(t)\|_{H_0} \leq \varepsilon \|\bar{u}_\tau(t) - u(t)\|_{\mathcal{X}^{\prime}_{\sigma,0}} + C_\varepsilon \|u_\tau(t) - u(t)\|_{\mathcal{X}^{\prime}_{\sigma,0}}
$$

$$
\leq \varepsilon C + C_\varepsilon \sup_{r \in [0,T]} \|\bar{u}_\tau(r) - u(r)\|_{\mathcal{X}^{\prime}_{\sigma,0}} \quad \text{for any } t \in [0,T],
$$

whence there follows that

$$
(124) \quad \bar{u}_\tau \to u \quad \text{strongly in } L^\infty(0,T;H_0),
$$

$$
(125) \quad \bar{u}_\tau(t) \to u(t) \quad \text{strongly in } H_0 \quad \text{for all } t \in [0,T],
$$

$$
(126) \quad \bar{u}_\tau(t) \to u(t) \quad \text{weakly in } \mathcal{E}_\sigma \quad \text{for all } t \in [0,T].
$$

Furthermore, noting that

$$
\|\bar{u}_\tau(t - \tau) - u(t)\|_{\mathcal{X}^{\prime}_{\sigma,0}} \leq \|\bar{u}_\tau(t - \tau) - u_\tau(t - \tau)\|_{\mathcal{X}^{\prime}_{\sigma,0}} + \|u_\tau(t - \tau) - u_\tau(t)\|_{\mathcal{X}^{\prime}_{\sigma,0}} + \|u_\tau(t) - u(t)\|_{\mathcal{X}^{\prime}_{\sigma,0}}
$$

$$
\leq C_\varepsilon \sqrt{T} + \int_{t - \tau}^t \|\partial_r u_\tau(r)\|_{\mathcal{X}^{\prime}_{\sigma,0}} \, dr + \|u_\tau(t) - u(t)\|_{\mathcal{X}^{\prime}_{\sigma,0}}
$$

for all $t \in [\tau, T]$, we also deduce from (112) and (122) that

$$
\sup_{t \in [\tau,T]} \|\bar{u}_\tau(t - \tau) - u(t)\|_{\mathcal{X}^{\prime}_{\sigma,0}} \leq C \sqrt{T} + \sup_{t \in [0,T]} \|u_\tau(t) - u(t)\|_{\mathcal{X}^{\prime}_{\sigma,0}} \to 0 \quad \text{as } \tau \to 0.
$$

As in (124), one can further obtain

$$
(127) \quad \bar{u}_\tau(-\tau) \to u \quad \text{strongly in } L^\infty(0,T;H_0),
$$

$$
(128) \quad \bar{u}_\tau(t - \tau) \to u(t) \quad \text{strongly in } H_0 \quad \text{for all } t > 0.
$$

In particular, (128) also implies $w = \mathcal{A}_\sigma u + \chi - u \in \mathcal{E}^{\prime}_\sigma$.

We next verify that $\chi = B(u)$ by using Minty's trick. To this end, we observe that

$$
\langle B(\bar{u}_\tau(t)), \bar{u}_\tau(t) \rangle_{L^p(R^N)} = \langle \bar{w}(t), \bar{u}_\tau(t) \rangle - \|\bar{u}_\tau(t)\|_{\mathcal{X}^{\prime}_{\sigma,0}}^2 + \langle \bar{u}_\tau(t - \tau), \bar{u}_\tau(t) \rangle.
$$

Hence, using (115), (124) and (128), we obtain

$$
\lim_{\tau \to 0} \sup_{t \in [0,T]} \int_0^T \langle B(\bar{u}_\tau(t)), \bar{u}_\tau(t) \rangle_{L^p(R^N)} \, dt
$$

$$
= \lim_{\tau \to 0} \int_0^T \langle \bar{w}(t), \bar{u}_\tau(t) \rangle \, dt - \liminf_{\tau \to 0} \int_0^T \|\bar{u}_\tau(t)\|_{\mathcal{X}^{\prime}_{\sigma,0}}^2 \, dt
$$

$$
+ \lim_{\tau \to 0} \int_0^T \langle \bar{u}_\tau(t - \tau), \bar{u}_\tau(t) \rangle \, dt
$$

$$
\leq \int_0^T (w(t), u(t)) \, dt - \int_0^T \|u(t)\|_{\mathcal{X}^{\prime}_{\sigma,0}}^2 \, dt + \int_0^T (u(t), u(t)) \, dt
$$

$$
= \int_0^T \langle \chi(t), u(t) \rangle_{L^p(R^N)} \, dt,
$$
which along with (117), (119) and the maximal monotonicity of the operator $u \mapsto B(u)$ from $L^p(\mathbb{R}^N)$ to $L^p' (\mathbb{R}^N)$ yields $\chi = B(u)$ in $L^p(\mathbb{R}^N)$. Thus $(u, w)$ is a weak solution of (1)-(4).

4.5. Energy inequalities. In order to derive (75), let us fix $t \in (0, T)$ at which (73) holds and define the Yosida approximation $\beta_\varepsilon: \mathbb{R} \to \mathbb{R}$ of $\beta$, i.e., $\beta_\varepsilon(r) := (r - j_\varepsilon(r))/\varepsilon = \beta(j_\varepsilon(r))$ for $r \in \mathbb{R}$, where $j_\varepsilon$ stands for the resolvent of $\beta$ defined by $j_\varepsilon(r) := (1 + \varepsilon \beta)^{-1}(r)$. Then, since $\beta_\varepsilon$ is Lipschitz continuous and $\beta_\varepsilon(0) = 0$, one can observe that $\beta_\varepsilon(u(\cdot)) \in \mathcal{E}_\sigma$ if $u \in \mathcal{E}_\sigma$. Hence, we can test (73) by $\beta_\varepsilon(u(\cdot, t))$ to get

$$\langle \mathfrak{A}_\sigma u(t), \beta_\varepsilon(u(\cdot, t)) \rangle_{X_{\sigma, 0}} + \langle B(u(t)), \beta_\varepsilon(u(\cdot, t)) \rangle_{\mathcal{E}_\sigma} = (w(t) + u(t), \beta_\varepsilon(u(\cdot, t))).$$

Here we note as in (42) that

$$\langle \mathfrak{A}_\sigma u(t), \beta_\varepsilon(u(\cdot, t)) \rangle_{X_{\sigma, 0}} \geq 0.$$ 

Moreover, by the definition of Yosida approximation and the monotonicity of $\beta$, we infer that

$$\langle B(u(t)), \beta_\varepsilon(u(\cdot, t)) \rangle_{\mathcal{E}_\sigma} = \int_\Omega \beta(u(x, t)) \beta_\varepsilon(u(x, t)) \, dx \geq \|\beta_\varepsilon(u(\cdot, t))\|_{L^2(\Omega)}^2.$$ 

Thus we obtain

$$\|\beta_\varepsilon(u(\cdot, t))\|_{H_0} \leq \|w(t)\|_{H_0} + \|u(t)\|_{H_0} \quad \text{for a.e. } t \in (0, T),$$

which implies

$$\beta_\varepsilon(u(\cdot, t)) \to b_t \quad \text{weakly in } H_0 \text{ as } \varepsilon \to 0$$

for some $b_t \in H_0$. On the other hand, let us recall that $\beta_\varepsilon(r) = \beta(j_\varepsilon(r))$. Moreover, since $j_\varepsilon$ is non-expansive (i.e., Lipschitz continuous with the Lipschitz constant 1) and $j_\varepsilon(0) = 0$, one can easily check that

$$\|j_\varepsilon(u(\cdot, t))\|_{\mathcal{E}_\sigma} \leq \|u(t)\|_{\mathcal{E}_\sigma},$$

which yields

$$j_\varepsilon(u(\cdot, t)) \to u(t) \quad \text{weakly in } \mathcal{E}_\sigma \text{ and strongly in } H_0,$$

as $\varepsilon \to 0$. Here we also used the fact that

$$|j_\varepsilon(u(x, t)) - u(x, t)| \leq \varepsilon |\beta_\varepsilon(u(x, t))| \leq \varepsilon |\beta(u(x, t))| \quad \text{for a.e. } x \in \Omega.$$ 

Therefore, by virtue of the demiclosedness of the maximal monotone operator $u \mapsto \beta(u(\cdot))$ in $H_0 \times H_0$, we conclude that $b_t = \beta(u(\cdot, t))$ a.e. in $\Omega$. Moreover, (75) follows from the weak lower-semicontinuity of the norm $\|\cdot\|_{H_0}$ in $H_0$. Furthermore, integrating (75) in time, we obtain $\beta(u) \in L^2(0, T; H_0)$, as $u$ and $w$ belong to $L^2(0, T; H_0)$.

We shall finally prove that $t \mapsto \mathcal{E}_\sigma(u(t))$ is differentiable a.e. in $(0, T)$ and derive the energy inequality, namely

$$\|w(t)\|^2_{X_{\sigma, 0}} + \frac{d}{dt} \mathcal{E}_\sigma(u(t)) \leq 0 \quad \text{for a.e. } t \in (0, T),$$

which can be also rewritten as

$$\langle \partial_t u(t), w(t) \rangle_{X_{\sigma, 0}} \geq \frac{d}{dt} \mathcal{E}_\sigma(u(t)) \quad \text{for a.e. } t \in (0, T).$$

Moreover, the right-continuity of the function $t \mapsto u(t)$ in the strong topology of $\mathcal{E}_\sigma$ will also follow as a by-product of our argument.

**Remark 4.1.** Before proceeding with a proof, it is worth stressing that, differently from what happens in the non-fractional case, the differentiability of $\mathcal{E}_\sigma(u(t))$ and the energy inequality (131) are not straightforward. Indeed, the energy functional $\mathcal{E}_\sigma$ is smooth in $\mathcal{E}_\sigma$ but non-convex. Hence, if one attempts to apply a standard chain-rule to $\mathcal{E}_\sigma$ and $u(t)$, the differentiability of $u(t)$ in the strong topology of $\mathcal{E}_\sigma$ is needed. However, $t \mapsto u(t)$ turns out to be differentiable only in the weaker space $X'_{\sigma, 0}$. When dealing with the standard Cahn-Hilliard equation (i.e., for $s = \sigma = 1$), this problem may be overcome by rewriting the energy functional corresponding to $\mathcal{E}_\sigma$ as the sum of a convex and of a concave part and by applying a generalized chain-rule for convex but (possibly) non-smooth functionals (see, e.g., [8]). However, in the present case, this kind of procedure seems to work only when $\sigma \geq s$. We shall give the highlights of a proof of this fact in Subsec. 4.6 below.
In order to show (131), we start with noting that, from (108) and interpolation, there follows
\[
\int_0^t \| \tilde{w}_t(r) \|^2_{X_{\sigma},0} \, dr + C_\sigma(\tilde{u}_t(t)) \leq C_\sigma(u_0) - \frac{1}{2} \| u_0 \|^2_{L^2(\Omega)} \quad \text{for} \quad 0 \leq t \leq T,
\]
where \( C_\sigma(\cdot) \) denotes the convex functional of class \( C^1 \) on \( E_\sigma \) given by
\[
C_\sigma(v) = \frac{1}{2} \| v \|^2_{X_{\sigma,0}} + \frac{1}{p} \| v \|^p_{L^p(\Omega)} \quad \text{for} \quad v \in E_\sigma.
\]
Using the convergence relations obtained so far and the weak lower semicontinuity of \( C_\sigma(\cdot) \) in \( E_\sigma \), we deduce that
\[
\int_0^t \| w(r) \|^2_{X_{\sigma,0}} \, dr + E_\sigma(u(t)) \leq E_\sigma(u(0)) \quad \text{for all} \quad t \in [0, T].
\]
From the uniqueness of the solution and the fact that \( u(t) \in E_\sigma \) for all \( t \in [0, T] \), one can also derive
\[
\int_t^T \| w(r) \|^2_{X_{\sigma,0}} \, dr + E_\sigma(u(\tau)) - E_\sigma(u(t)) \leq 0 \quad \text{for all} \quad 0 \leq t \leq \tau \leq T,
\]
which also implies that \( E_\sigma(u(\cdot)) \) is nonincreasing on \( [0, T] \), whence it is differentiable a.e. in \( (0, T) \). Since \( u \in C([0, T]; H_0) \cap C_w([0, T]; E_\sigma) \) and \( C_\sigma(\cdot) \) is weakly lower semicontinuous in \( E_\sigma \), \( E_\sigma(u(\cdot)) \) is right-continuous on \( [0, T) \), i.e., \( E_\sigma(u(\tau)) \to E_\sigma(u(t)) \) as \( \tau \searrow t \). Then, the same property holds for \( \| u(\cdot) \|^2_{X_{\sigma,0}} \) and \( \| u(\cdot) \|^p_{L^p(\Omega)} \) by the weak lower semicontinuity of the norms. Therefore due to the uniform convexity of \( X_{\sigma,0} \) and \( L^p(\mathbb{R}^N) \), we can also verify that \( u \) is right-continuous on \( [0, T) \) in the strong topology of \( E_\sigma \).

Furthermore, let \( t \) belong to the set
\[
\mathcal{I} := \{ t \in [0, T] : E_\sigma(u(\cdot)) \text{ is differentiable at } t \text{ and } t \text{ is a Lebesgue point of } \| w(\cdot) \|^2_{X_{\sigma,0}} \}.
\]
Then \( (0, T) \setminus \mathcal{I} \) has zero Lebesgue measure. Dividing both sides of (134) by \( \tau - t > 0 \) and passing to the limit as \( \tau \searrow t \), we obtain (131).

4.6. Energy equality. We prove here that, under the condition \( \sigma \geq s \), \( u \) belongs to \( C([0, T]; E_\sigma) \), the energy \( E_\sigma(u(t)) \) is absolutely continuous on \( [0, T] \), and the inequality (77) can be replaced by the following energy identity:
\[
\| w(t) \|^2_{X_{\sigma,0}} + \frac{d}{dt} E_\sigma(u(t)) = 0 \quad \text{for a.e. } t \in (0, T).
\]
The key tool in order to get (135) is the following chain-rule formula, which can be proved by adapting the argument given in [30, Lemma 4.1]:

**Lemma 4.1.** Let \( (\mathcal{V}, \mathcal{H}, \mathcal{V}') \) be a Hilbert triple and let \( \Psi : \mathcal{H} \to (-\infty, +\infty] \) be a convex, proper and lower semicontinuous functional. Moreover, let us assume that, for some \( k_1 > 0, k_2 \geq 0 \), there holds
\[
\Psi(v) \geq k_1 \| v \|^2_{\mathcal{H}} - k_2 \quad \forall v \in \mathcal{H}.
\]
Denote with \( \mathcal{A} \) the subdifferential of \( \Psi \) with respect to the scalar product of \( \mathcal{H} \), and consider, for \( T > 0 \), \( v \in W^{1,2}(0, T; \mathcal{V}) \cap L^2(0, T; \mathcal{V}) \) and \( \eta \in L^2(0, T; \mathcal{V}) \) with \( \eta(t) \in \mathcal{A}v(t) \) for a.e. \( t \in (0, T) \). Then, the function \( t \mapsto \Psi(v(t)) \) is absolutely continuous in \( [0, T] \). Moreover,
\[
\int_0^T \langle \partial_v \Psi(\tau), \eta(\tau) \rangle_{\mathcal{V}, \mathcal{V}'} \, d\tau = \Psi(v(T)) - \Psi(v(0)) \quad \text{for all} \quad 0 \leq r \leq t \leq T.
\]
We apply the above Lemma with the following choices:
- \( \mathcal{H} = H_0, \mathcal{V} = X_{\sigma,0} \), see (24).
- \( \Psi(v) = \frac{1}{2} \| v \|^2_{X_{\sigma,0}} + \int_\Omega \beta(v(x)) \, dx \).

Then, clearly, \( \Psi \) is proper, lower semicontinuous and convex. Moreover, thanks to the fractional Poincaré inequality (20), it satisfies the coercivity assumption (136). Now, let \( u, w \) be the solution given by Theorem 1. Then, by Definition 3.1, we have
\[
u \in W^{1,2}(0, T; \mathcal{V}) \quad \text{and} \quad \theta \in L^2(0, T; \mathcal{V}).
\]
On the other hand, being \( \sigma \geq s \), we also have
\[u \in L^2(0, T; \mathcal{V}).\]
Thus, setting \( \eta := w + u \), it follows that \( \eta \in L^2(0,T; V) \); moreover, thanks to equation (73), \( \eta(t) \in \partial \Psi(u(t)) \) for a.e. \( t \in (0,T) \). Hence, (135) follows from Lemma 4.1.

**Remark 4.2.** When \( \| \cdot \| \) is the fractional Cahn-Hilliard system (1)-(4). To this end, let us recall the inequality (75) of Theorem 1, first establish uniform estimates with respect to \( X \) and uniform estimates.

Then, integrating both sides in \( \Omega \) we get

\[
\frac{d}{dt} \mathcal{E}_\sigma(u(t)) + \| u(t) \|_{X_\sigma,0}^2 \leq 0 \quad \text{for a.e. } t \in (0,T).
\]

Moreover, we also recall the energy inequality (77),

\[
\mathcal{E}_\sigma(u(t)) + \| u(t) \|_{X_\sigma,0}^2 \leq Q(\mathcal{E}_\sigma(u_0)).
\]

Here and henceforth, \( Q(\cdot) \) denotes a computable nonnegative valued function which is monotone increasing in its argument(s) and may vary from line to line. In particular, the expression of \( Q \) may depend on \( p, T \) and \( |\Omega| \); however, it is always independent both of \( \sigma \) and of \( s \).

In case \( p > 2 \) since \( W \) is coercive, it follows immediately from (139) that

\[
\| u \|_{L^\infty(0,T;X_\sigma,0)}^p + \| u \|_{L^p(0,T;L^p(\Omega))} + \| u \|_{L^2(0,T;X_\sigma,0)}^2 \leq Q(\mathcal{E}_\sigma(u_0)),
\]

which along with the boundedness of \( \mathfrak{A}_\sigma : X_\sigma,0 \to X'_{\sigma,0} \) and a comparison of terms in (72) gives

\[
\| \partial_t u \|_{L^2(0,T;X'_{\sigma,0})} \leq Q(\mathcal{E}_\sigma(u_0)).
\]

Moreover, by (114),

\[
\| \mathcal{A}_\sigma u \|_{L^2(0,T;X'_{\sigma,0})} \leq Q(\mathcal{E}_\sigma(u_0)).
\]

On the other hand, combining (140) with (138), we get

\[
\| \beta(u) \|_{L^\infty(0,T;L^p(\mathbb{R}^N))} \leq \| u \|_{L^p(0,T;L^p(\Omega))}^{p-1} \leq Q(\mathcal{E}_\sigma(u_0)).
\]

In turn, this estimate clearly implies that

\[
\| W(u) \|_{L^2(0,T;H_0)} \leq Q(\mathcal{E}_\sigma(u_0)).
\]

Hence by (73), we find that

\[
\langle \mathfrak{A}_\sigma u(t), \phi \rangle_{E_\sigma} \leq \langle u(t) - B(u(t)) + u(t), \phi \rangle_{E_\sigma} = \int_\Omega (u(x,t) - \beta(u(x,t)) + u(x,t)) \phi(x) \, dx \leq \| u \|_{H_0} + \| \beta(u) \|_{H_0} + \| u(t) \|_{H_0} \| \phi \|_{H_0} \quad \text{for all } \phi \in \mathfrak{D}_H(\mathbb{R}^N).
\]

Since \( \mathfrak{D}_H(\mathbb{R}^N) \) is dense in \( H_0 \), one has

\[
\| \mathfrak{A}_\sigma u \|_{L^2(0,T;H_0)} \leq Q(\mathcal{E}_\sigma(u_0)),
\]

where \( \mathfrak{A}_\sigma : H_0 \to H_0 \) stands for the \( H_0 \)-fractional Laplacian with domain \( D(\mathfrak{A}_\sigma) \subsetneq H_0 \) (see §2.4) and \( \mathfrak{A}_\sigma u(t) : H_0 \to \mathbb{R} \) is the unique bounded linear extension onto \( H_0 \) of the functional \( \mathfrak{A}_\sigma u(t) : X_{\sigma,0} \to \mathbb{R} \).

**In case \( 1 < p < 2 \) and \( \sigma \in (0,1) \) is fixed:** (the argument below is still available for \( p > 2 \) as well) by applying \( \mathfrak{A}_\sigma^{-1} \) to both sides of (72) and by utilizing (73), we have

\[
\mathfrak{A}_\sigma^{-1} (\partial_t u(t)) + \mathfrak{A}_\sigma u(t) + B(u(t)) - u(t) = 0 \quad \text{in } E_\sigma', \quad 0 < t < T.
\]
Test it by \( u(t) \in \mathcal{E}_\sigma \). It follows that
\[
\frac{1}{2} \frac{d}{dt} \| u(t) \|^2_{X_{1,0}^\prime} + \| u(t) \|^2_{X_{2,0}} + \| u(t) \|^p_{L^p(\Omega)} = \| u(t) \|^2_{L^2(\Omega)} \quad \text{for a.e. } 0 < t < T.
\]

Set \( X = X_{r,0} \) or \( X = \mathcal{E}_r := X_{r,0} \cap L^p(\mathbb{R}^N) \) for a fixed constant \( r \in (s, 1) \) or \( X = H^1_0(\Omega) \). Then \( X \) is continuously embedded in \( X_{r,0} \) uniformly for \( s \to 0 \). More precisely, there exists a constant \( C_0 > 0 \) independent of \( s \to 0 \) such that
\[
\| u \|_{X_{r,0}} \leq C_0 \| v \|_{X} \quad \text{for all } v \in X.
\]

Indeed, as in [18, Proof of Proposition 2.1], one can verify that, for all \( v \in X \),
\[
\frac{C(s)}{2} \iint_{\mathbb{R}^{2N}} \frac{|v(x) - v(y)|^2}{|x - y|^{N+2s}} \, dx \, dy
\]
\[
= \frac{C(s)}{2} \iint_{\mathbb{R}^N} \int_{\{x \in \mathbb{R}^N: |x - y| > 1\}} \frac{|v(x) - v(y)|^2}{|x - y|^{N+2s}} \, dx \, dy
\]
\[
+ \frac{C(s)}{2} \iint_{\mathbb{R}^N} \int_{\{x \in \mathbb{R}^N: |x - y| \leq 1\}} \frac{|v(x) - v(y)|^2}{|x - y|^{N+2s}} \, dx \, dy
\]
\[
\leq \frac{C(s)}{s} |S^{N-1}| \| v \|^2_{L^2(\Omega)} + \begin{cases} \frac{C(s)}{C(r)} \| v \|^2_{X_{r,0}} & \text{for } X = X_{r,0} \text{ or } \mathcal{E}_r, \\ \frac{|G^{N-1}|}{4} \frac{C(s)}{r^2} \| \nabla v \|^2_{L^2(\Omega)} & \text{for } X = H^1_0(\Omega). \end{cases}
\]

Here \( |S^{N-1}| \) stands for the surface area of a unit sphere in \( \mathbb{R}^N \). Finally, exploit the asymptotics (12) of \( C(r) \) as \( r \downarrow 0 \) to obtain (146). Moreover, (146) yields
\[
\| \zeta \|_{X'} \leq C_0 \| \zeta \|_{X_{s,0}'}, \quad \text{for all } \zeta \in X_{s,0}'.
\]

which particularly gives
\[
\| u \|_{X'} \leq C_0 \| u \|_{X_{s,0}'} \quad \text{for all } u \in H_0(\simeq H_0')
\]

Indeed, for any \( \phi \in X \subset X_{s,0} \), one finds that
\[
(\zeta, \phi) = (\zeta, \phi)_{X_{r,0}} \leq \| \zeta \|_{X_{r,0}} \| \phi \|_{X_{s,0}} \leq C_0 \| \zeta \|_{X_{s,0}} \| \phi \|_{X} \quad \text{for } \zeta \in X_{s,0}',
\]

which gives (147). On the other hand, from the dense and compact embeddings \( X_{s,0} \hookrightarrow H_0(\simeq H_0') \hookrightarrow X' \) along with Ehrlich’s compactness lemma [36, Lemma 8], for any \( \varepsilon > 0 \) there exists a positive constant \( C_{\varepsilon, \sigma} \), which is independent of \( s \) but may depend on \( \sigma \), such that
\[
\| u \|^2_{L^2(\Omega)} = \| u \|^2_{H_0} \leq \varepsilon \| v \|^2_{X_{s,0}} + C_{\varepsilon, \sigma} \| v \|^2_{X'} \quad \text{for all } v \in X_{s,0}.
\]

Therefore we deduce that
\[
\frac{1}{2} \frac{d}{dt} \| u(t) \|^2_{X_{1,0}'} + \frac{1}{2} \| u(t) \|^2_{X_{2,0}} + \| u(t) \|^p_{L^p(\Omega)} \leq C_{\varepsilon, \sigma} \| u(t) \|^2_{X'} \quad \text{for a.e. } 0 < t < T.
\]

The integration of both sides over \((0, t)\) along with (148) yields
\[
\| u(t) \|_{X_{1,0}'}^2 + \int_0^t \left( \| u(\tau) \|_{X_{2,0}}^2 + \| u(\tau) \|^p_{L^p(\Omega)} \right) \, d\tau \leq C_{\varepsilon, \sigma} \left( \| u_0 \|_{X_{1,0}'}^2 + \int_0^t \| u(\tau) \|_{X'}^2, \, d\tau \right)
\]

for some constant \( C_{\varepsilon, \sigma} > 0 \) (depending on \( \sigma \)). Hence, exploiting Gronwall’s inequality, we obtain
\[
\sup_{t \in [0, T]} \| u(t) \|_{X_{1,0}'}^2 + \int_0^T \left( \| u(t) \|_{X_{2,0}}^2 + \| u(t) \|^p_{L^p(\Omega)} \right) \, dt \leq Q(C_{\varepsilon, \sigma}, \| u_0 \|_{X_{1,0}'}).\]

Apply (149) to (139) and employ (150). Then we obtain (140) with a bound depending on \( C_{\varepsilon, \sigma}, \| u_0 \|_{X_{1,0}'} \) and \( \mathcal{E}_\sigma(u_0) \). Furthermore, relations analogous to (141)-(145) also follow with similar bounds. More precisely, one deduces that
\[
\| u \|^2_{L^\infty(0,T;X_{s,0})} + \| u \|^p_{L^\infty(0,T;L^p(\Omega))} + \| \beta(u) \|^2_{L^2(0,T;H_0)} + \| W'(u) \|^2_{L^2(0,T;H_0)}
\]
\[
+ \| \mathfrak{A}_u u \|^2_{L^\infty(0,T;X_{s,0}')} \leq Q(C_{\varepsilon, \sigma}, \| u_0 \|_{X_{1,0}'}^2, \mathcal{E}_\sigma(u_0)).
\]

Moreover, we also have
\[
\| \partial_t u \|^2_{L^2(0,T;X_{s,0}')}, \| \mathfrak{A}_u u \|^2_{L^2(0,T;X_{s,0}')} + \| u \|^2_{L^2(0,T;X_{s,0})} \leq Q(C_{\varepsilon, \sigma}, \| u_0 \|_{X_{1,0}'}^2, \mathcal{E}_\sigma(u_0)) \].
Hence by virtue of (147) and Poincaré’s inequality (49) along with Proposition 2.2, it follows that
\[
\|\partial_t u\|_{L^2(0,T;X')} + \|A_u w\|_{L^2(0,T;X')} + \|w\|_{L^2(0,T;H_0)}^2 \leq Q(C_\sigma, \|u_0\|_{X'_{\beta,0}}, E_\sigma(u_0)).
\]
Finally, by (114), it holds that
\[
\|B(u)\|_{L^\infty(0,T;L^{p'}(\mathbb{R}^N))} \leq Q(C_\sigma, \|u_0\|_{X'_{\beta,0}}, E_\sigma(u_0)).
\]

5.2. Limit of fractional Laplacian in Bochner spaces. In this section, we shall generalize Lemma 2.2 for later use of proving the convergence of $A_k$ for $k \to \infty$ in an appropriate Bochner space. Throughout this subsection, we use the notation $X_\beta$ and $X_{\beta,0}$ even for $\beta \geq 1$ in the following sense
\[
X_\beta := H^\beta(\mathbb{R}^N) = \left\{ u \in S(\mathbb{R}^N)' : (1 + |\xi|^2)^{\beta/2} \hat{u}(\xi) \in L^2(\mathbb{R}^N) \right\},
\]
\[
X_{\beta,0} := \left\{ u \in X_\beta : u = 0 \text{ in } \mathbb{R}^N \setminus \Omega \right\} \text{ for } \beta \geq 1.
\]
Then one finds that
\[
X_\beta \hookrightarrow X_\gamma \hookrightarrow X_0 = L^2(\mathbb{R}^N), \quad X_{\beta,0} \hookrightarrow X_{\gamma,0} \hookrightarrow X_{0,0} = H_0 \quad \text{if } \beta \geq \gamma > 0
\]
with continuous densely defined canonical injections. Hence we also have dual relations,
\[
L^2(\mathbb{R}^N)' \hookrightarrow X_\gamma' \hookrightarrow X'_{\beta}, \quad H_0' \hookrightarrow X_{\gamma,0}' \hookrightarrow X_{\beta,0}' \quad \text{if } \beta \geq \gamma > 0
\]
densely and continuously. For any $u \in X_{\beta,0}$ and $\beta, \gamma \geq 0$, one can define $T(u) \in X_\gamma$ by
\[
\langle T(u), \phi \rangle_{X_\gamma} := \int_{\Omega} u(x)\phi(x) \, dx \quad \text{for } \phi \in X_\gamma.
\]
Then $T : X_{\beta,0} \to X'_\gamma$ is continuous due to the continuous embeddings described above. Hence $X_{\beta,0}$ is continuously embedded in $X'_\gamma$ by $T$. From now on, we simply write $u$ instead of $T(u)$ if no confusion may arise.

Lemma 5.1. Let $u$ and $\xi$ be integrable functions of $(0,T)$ with values in $X_{\beta,0}'$ for some constant $\beta > 0$ satisfying $\beta \neq n - 1/2$ with $n \in \mathbb{N}$. Let $\{r_k\}$ be a sequence in $(0,\beta)$ such that $r_k \searrow 0$ as $k \to \infty$ and consider a sequence $\{u_k\}$ of strongly measurable functions in $(0,T)$ with values in $X_{r_k,0}$. In addition, assume that
\[
\int_0^T \langle u_k(t), \varphi_k \rangle_{X_{\beta,0}} \phi(t) \, dt \to \int_0^T \langle u(t), \varphi \rangle_{X_{\beta,0}} \phi(t) \, dt,
\]
\[
\int_0^T \langle A_{r_k} u_k(t), \varphi \rangle_{X_{r_k,0}} \phi(t) \, dt \to \int_0^T \langle \xi(t), \varphi \rangle_{X_{\beta,0}} \phi(t) \, dt
\]
for any $\varphi \in \mathcal{D}_{\Omega}(\mathbb{R}^N)$, $\phi \in C_\infty(0,T)$ and $\varphi_k \in X_\beta$ satisfying $\varphi_k \to \varphi$ strongly in $X_\beta$. Then it holds that $u(t) = \xi(t)$ in $X_{\beta,0}'$ for a.e. $t \in (0,T)$.

Proof. From the continuous embeddings $X_{r_k,0} \hookrightarrow L^2(\mathbb{R}^N) \simeq (L^2(\mathbb{R}^N))' \hookrightarrow X'_{\beta}$, we note that the function $u_k$ is strongly measurable with values in $X'_{\beta}$ as well. For any $\varphi \in \mathcal{D}_{\Omega}(\mathbb{R}^N)$ and $\phi \in C_\infty(0,T)$, we observe by (154) that
\[
\int_0^T \langle A_{r_k} u_k(t), \varphi \rangle_{X_{r_k,0}} \phi(t) \, dt = C(r_k) \int_0^T \left( \int_{\mathbb{R}^N} \frac{(u_k(x,t) - u_k(y,t))(\varphi(x) - \varphi(y))}{|x-y|^{N+2r_k}} \, dx \, dy \right) \phi(t) \, dt
\]
\[
= \int_0^T (-\Delta)^{r_k} \varphi, u_k(t) \, \phi(t) \, dt
\]
\[
= \int_0^T \langle u_k(t), (-\Delta)^{r_k} \varphi \rangle_{X_{\beta,0}} \phi(t) \, dt \to \int_0^T \langle u(t), \varphi \rangle_{X_{\beta,0}} \phi(t) \, dt.
\]
Here we used the fact that $(-\Delta)^{r_k} \varphi \in X_\beta$ and $(-\Delta)^{r_k} \varphi \to \varphi$ strongly in $X_\beta$ (uniformly in $t$) as $k \to \infty$ by Lemma 2.1. By virtue of (155), the left-hand side of (156) converges as follows:
\[
\int_0^T \langle A_{r_k} u_k(t), \varphi \rangle_{X_{r_k,0}} \phi(t) \, dt \to \int_0^T \langle \xi(t), \varphi \rangle_{X_{\beta,0}} \phi(t) \, dt.
\]
Thus we obtain
\[ \int_0^T \langle \xi(t), \varphi \rangle_{X'_{\beta,0}} \phi(t) \, dt = \int_0^T \langle u(t), \varphi \rangle_{X'_{\beta,0}} \phi(t) \, dt. \]

Recall that \( \mathcal{D}_\Omega(\mathbb{R}^N) \) is dense in \( X'_{\beta,0} \) if \( \beta \neq n - 1/2 \) for \( n \in \mathbb{N} \) (see [25, Theorems 11.4 and 11.1, Chap. I]). Hence, from the arbitrariness of \( \varphi \in \mathcal{D}_\Omega(\mathbb{R}^N) \), we have
\[ \int_0^T (\xi(t) - u(t)) \phi(t) \, dt = 0 \text{ in } X'_{\beta,0} \quad \text{for all } \phi \in C_0^\infty(0, T). \]

Finally, applying du Bois-Reymond’ lemma for Bochner integrals, we conclude that \( \xi(t) = u(t) \) in \( X'_{\beta,0} \) for a.e. \( t \in (0, T) \). □

**Remark 5.1.**

(i) All the assumptions of Lemma 5.1 can be proved to hold whenever \( u_k \to u \) weakly in \( L^1(0, T; X'_{\beta,t}) \) and \( \mathfrak{A}_{r_k} u_k(t) \to \xi \) weakly in \( L^1(0, T; X'_{\beta,0}) \) as \( k \to \infty \) for some \( u \in L^1(0, T; X'_{\beta}) \) and \( \xi \in L^1(0, T; X'_{\beta,0}) \). Indeed, the product of test functions \( \varepsilon \varphi \phi \) converges to \( \varphi \phi \) strongly in \( L^\infty(0, T; X'_{\beta}) \) and \( \varepsilon \varphi \phi \) belongs to \( L^\infty(0, T; X'_{\beta,0}) \), since \( \varphi_k \to \varphi \) strongly in \( X'_{\beta} \) with \( \varphi \in \mathcal{D}_\Omega(\mathbb{R}^N) \subset X'_{\beta,0} \) and \( \phi \in C_0^\infty(0, T) \). Moreover, since \( X_{\beta,0} \subset X'_{\beta} \), one observes that \( X'_{\beta} \to X'_{\beta,0} \), whence \( u \) belongs to \( L^1(0, T; X'_{\beta,0}) \).

(ii) One can also derive a similar result for sequences independent of \( t \). More precisely, if \( u_k \in X'_{r_k,0} \), \( u_k \to u \) weakly in \( X'_{r_k,0} \), \( \mathfrak{A}_{r_k} u_k \to \xi \) weakly in \( X'_{\beta,0} \), then \( \xi = u \) in \( X'_{\beta,0} \). Indeed, set \( u_k(t) \equiv u_k \). Then \( u_k \to v(\cdot) \equiv u \) weakly in \( L^p(0, T; X'_{\beta}) \) and \( \mathfrak{A}_{r_k} u_k(\cdot) \equiv \mathfrak{A}_{r_k} u_k \)
converges to \( \eta(\cdot) \equiv \xi \) weakly in \( L^p(0, T; X'_{\beta,t}) \) for any \( p \in [1, \infty) \); hence, all the assumptions of Lemma 5.1 hold true by (i) above.

### 5.3. Proof of Theorem 2.

Let \( \{\sigma_k\} \) be a sequence in \((0, 1)\) such that \( \sigma_k \searrow 0 \) and let \((u_k, w_k)\) be the family of weak solutions to
\[
\begin{align*}
\partial_t u_k + \mathfrak{A}_s w_k &= 0 \quad \text{in } X'_{s,0}, \\
 w_k &= \mathfrak{A}_{\sigma_k} u_k + B(u_k) - u_k \quad \text{in } X'_{s,0},
\end{align*}
\]
with \( u_k(0) = u_{0,k} \). Then recalling uniform estimates (140)-(145) in §5.1 along with hypothesis (78), one can take weak limits \( u \in L^\infty(0, T; L^p_0(\mathbb{R}^N)) \cap W^{1,2}(0, T; X'_{s,0}) \), \( w \in L^2(0, T; X'_{s,0}) \), \( \beta \in L^2(0, T; H_0) \)
and \( \xi \in L^2(0, T; H_0) \) such that, up to a (non-relabeled) subsequence of \( \{k\} \),
\[
\begin{align*}
 u_k &\to u \quad \text{weakly star in } L^\infty(0, T; L^p_0(\mathbb{R}^N)), \\
 w_k &\to w \quad \text{weakly in } W^{1,2}(0, T; X'_{s,0}), \\
 w_k &\to w \quad \text{weakly in } L^2(0, T; X'_{s,0}), \\
 \mathfrak{A}_s w_k &\to \mathfrak{A}_s w \quad \text{weakly in } L^2(0, T; X'_{s,0}), \\
 \beta(u_k) &\to \beta \quad \text{weakly in } L^2(0, T; H_0), \\
 \mathfrak{A}_{\sigma_k} u_k &\to \xi \quad \text{weakly in } L^2(0, T; H_0).
\end{align*}
\]

It follows immediately that \( \partial_t u + \mathfrak{A}_s w = 0 \) in \( X'_{s,0} \). Applying Lemma 5.1 with any \( \beta > 0 \) to \( \mathfrak{A}_{\sigma_k} u_k(t) \) and \( u_k(t) \) along with the weak convergence relations (159) and (164), we obtain \( \xi(t) = u(t) \) in \( X'_{\beta,0} \) for a.e. \( t \in (0, T) \). Moreover, since \( \xi(t) \) and \( u(t) \) lie in \( H_0 \), which is dense in \( X'_{\beta,0} \), we see that \( \xi = u \) a.e. in \( \Omega \times (0, T) \). For all \( \phi \in \mathcal{D}_\Omega(\mathbb{R}^N) \) and \( \phi \in C_0^\infty(0, T) \), it follows from (158) that
\[
\begin{align*}
\int_0^T (w_k(t), \varphi) \phi(t) \, dt &= \int_0^T (w_k(t), \varphi) \phi(t) \, dt \\
&\quad + \int_0^T (\beta(u_k(\cdot,t)), \phi) \phi(t) \, dt - \int_0^T (u_k(t), \phi) \phi(t) \, dt.
\end{align*}
\]

Passing to the limit as \( k \searrow \infty \), we obtain
\[
\begin{align*}
\int_0^T (w(t), \varphi) \phi(t) \, dt &= \int_0^T (\xi(t), \varphi) \phi(t) \, dt \\
&\quad + \int_0^T (\beta(t), \phi) \phi(t) \, dt - \int_0^T (u(t), \phi) \phi(t) \, dt.
\end{align*}
\]
which together with the density of $\mathcal{D}_0(\mathbb{R}^N)$ in $H_0$ and the arbitrariness of $\phi \in C_0^\infty(0, T)$ implies that

$$w = \xi + \bar{\beta} - u \text{ in } H_0, \text{ a.e. in } (0, T).$$

Thus we obtain $w(t) = \bar{\beta}(t)$ in $H_0$ for a.e. $t \in (0, T)$, or, in other words,

$$w = \bar{\beta} \text{ a.e. in } \Omega \times (0, T).$$

(165)

For each $t \in [0, T]$, since $\{u_k(t)\}$ is bounded in $H_0$ and $H_0$ is compactly embedded in $\mathcal{X}_{s,0}$, the sequence $\{u_k(t)\}$ is precompact in $\mathcal{X}_{s,0}$. Moreover, $t \mapsto u_k(t)$ is equicontinuous on $[0, T]$ with values in $\mathcal{X}_{s,0}$ for $k \in \mathbb{N}$. Therefore, thanks to Ascoli’s lemma, we infer that

(166)

$$u_k \to u \quad \text{strongly in } \mathcal{C}([0, T]; \mathcal{X}_{s,0}).$$

Since $u_{0,k} \to u_0$ strongly in $\mathcal{X}_{s,0}$ by assumption, one can check that $u(t) \to u_0$ strongly in $\mathcal{X}_{s,0}$ as $t \searrow 0$. In particular, $u(0) = u_0$.

Now, the major task is to identify the limit $\bar{\beta}$ as $\beta(u)$, namely proving that $\bar{\beta} = \beta(u)$ a.e. in $\mathbb{R}^N \times (0, T)$. To this end, we shall use Minty’s trick, i.e., we claim that

(167)

$$\limsup_{k, r \to +\infty} \int_0^T \int_{\mathbb{R}^N} \beta(u_k)u_k \, dx \, dt \leq \int_0^T \int_{\mathbb{R}^N} \bar{\beta} u \, dx \, dt.$$

Actually, testing (158) by $u_k$, we find that

(168)

$$\int_0^T \int_{\mathbb{R}^N} \beta(u_k)u_k \, dx \, dt = -\int_0^T \int_{\mathbb{R}^N} \|u_k(t)\|^2_{\mathcal{X}_{s,0}} \, dt + \int_0^T \int_{\mathbb{R}^N} \|u_k(t)\|^2_{L^2(\mathbb{R}^N)} \, dt + \int_0^T \int_{\mathbb{R}^N} w_k u_k \, dx \, dt.$$

Taking the $\limsup_{k, r \to +\infty}$ of both sides, we have

(169)

$$\limsup_{k, r \to +\infty} \int_0^T \int_{\mathbb{R}^N} \beta(u_k)u_k \, dx \, dt \leq \limsup_{k, r \to +\infty} \int_0^T \left(\|u_k(t)\|^2_{L^2(\mathbb{R}^N)} - \|u_k(t)\|^2_{\mathcal{X}_{s,0}}\right) \, dt + \int_0^T \int_{\mathbb{R}^N} w u \, dx \, dt.$$

In particular, in order to take the limit of the last integral, we used (161) together with (166), and observed that

(170)

$$\int_0^T \int_{\mathbb{R}^N} w_k u_k \, dx \, dt = \int_0^T \int_{\mathbb{R}^N} \langle u_k(t), w_k(t) \rangle_{\mathcal{X}_{s,0}} \, dt \to \int_0^T \int_{\mathbb{R}^N} \langle u(t), w(t) \rangle_{\mathcal{X}_{s,0}} \, dt = \int_0^T \int_{\mathbb{R}^N} w u \, dx \, dt.$$

The Poincaré inequality (49) gives

(171)

$$D_k(t) := \|u_k(t)\|^2_{L^2(\mathbb{R}^N)} - \|u_k(t)\|^2_{\mathcal{X}_{s,0}} \leq \left(\frac{1}{\lambda_1(\sigma_k)} - 1\right) \|u_k(t)\|^2_{\mathcal{X}_{s,0}} \text{ for a.e. } t \in (0, T).$$

Thus, recalling Proposition 2.2 and the energy estimate (139), we conclude that

$$\limsup_{k, r \to +\infty} \int_0^T D_k(t) \, dt \leq 0.$$

Then (167) follows from the above along with (165), (169) and the fact that $u = 0$ outside $\Omega$. Therefore thanks to (159), (163), and the maximal monotonicity of the mapping $u \mapsto \beta(u)$ in $L^2(\mathbb{R}^N) \times L^2(\mathbb{R}^N)$, one deduces that $\beta(u) = \bar{\beta}$ a.e. in $\mathbb{R}^N \times (0, T)$. In particular, $\bar{\beta} = \beta(u)$ vanishes outside $\Omega$. Hence by (165) together with the fact that $w = 0$ in $\mathbb{R}^N \setminus \Omega$, one obtains

(172)

$$\beta(u) = \bar{\beta} = w \quad \text{a.e. in } \mathbb{R}^N \times (0, T),$$

which also implies $\beta(u) = w \in L^2(0, T; \mathcal{X}_{s,0})$. Consequently, $u$ solves for almost any $t \in (0, T)$

$$\partial_t u + \mathcal{A}_s \beta(u) = 0 \quad \text{in } \mathcal{X}_{s,0}'.$$

Note that, as a consequence of the procedure, recalling (159) again, we also get

(173)

$$u_k \to u \quad \text{strongly in } L^p(0, T; L^p_0(\mathbb{R}^N))$$

by utilizing the uniform convexity of $L^p_0(\mathbb{R}^N)$. 

26 GORO AKAGI, GIULIO SCHIMPERNA, AND ANTONIO SEGATTI
5.4. **Proof of Theorem 3.** We first remark that, as in the Riesz representation theorem for standard Lebesgue spaces, one can also identify the dual space \((L^0_0(\mathbb{R}^N))'\) of \(L^0_0(\mathbb{R}^N)\) with \(L^0_0(\mathbb{R}^N)\), where \(q \in (1, \infty)\) and \(q' := q/(q - 1)\).

Let \(\sigma_k \searrow 0\) and let \((u_k, w_k)\) be the family of weak solutions to

\[
\begin{align*}
\partial_t u_k + \mathfrak{A}_s w_k &= 0 \quad \text{in } X'_{s,0}, \\
w_k &= \mathfrak{A}_s u_k + B(u_k) - \lambda_k u_k \quad \text{in } X'_{\sigma_k,0},
\end{align*}
\]

with \(u_k(0) = u_{0,k}\), where \(\lambda_k := \lambda_1(\sigma_k)\) denotes the first eigenvalue of (48) with \(r\) replaced by \(\sigma_k\). As in \S 5.1, (formally) test (174) by \(w_k\) and (175) by \(\partial_t u_k\) to get

\[
\|w_k(t)\|^2_{X'_{s,0}} + \frac{d}{dt} \tilde{E}_{\sigma_k}(u_k(t)) \leq 0 \quad \text{for a.e. } t \in (0, T),
\]

where \(\tilde{E}_{\sigma_k} : X_{\sigma_k,0} \to \mathbb{R}\) is defined as in (87). Indeed, the energy inequality above can be rigorously derived as in the proof of Theorem 1. Integrate both sides over \((0, t)\). It follows that

\[
\|w_k\|^2_{L^2(0,T;X'_{s,0})} + \tilde{E}_{\sigma_k}(u_k(t)) \leq \tilde{E}_{\sigma_k}(u_{0,k}).
\]

By (88), we have

\[
\|w_k\|^2_{L^2(0,T;X'_{s,0})} + \|u_k\|^p_{L^\infty(0,T;L^p_0(\mathbb{R}^N))} \leq Q(\tilde{E}_{\sigma_k}(u_{0,k})),
\]

which also implies

\[
\|\partial_t u_k\|^2_{L^2(0,T;X'_{s,0})} + \|\mathfrak{A}_s w_k\|^2_{L^2(0,T;X'_{s,0})} + \|\beta(u_k)\|^p_{L^\infty(0,T;L^p_0(\mathbb{R}^N))} \leq Q(\tilde{E}_{\sigma_k}(u_{0,k})).
\]

As in (145), by (175) and estimates above along with the fact that \(1 < p < 2\) (i.e., \(p' > 2\)), we can take the unique bounded linear extension \(\overline{\mathfrak{A}_s u_k(t)} : L^p_0(\mathbb{R}^N) \to \mathbb{R}\) onto \(L^p_0(\mathbb{R}^N)\) of the functional \(\mathfrak{A}_s u_k(t)|_{L^p_0(\mathbb{R}^N)} : X_{\sigma_k,0} \cap L^p_0(\mathbb{R}^N) \to \mathbb{R}\) such that, by the identification \((L^p_0(\mathbb{R}^N))' \simeq L^0_0(\mathbb{R}^N)\),

\[
\left\|\overline{\mathfrak{A}_s u_k(t)}\right\|_{L^p_0(\mathbb{R}^N)} \leq \|w_k(t)\|_{L^p_0(\mathbb{R}^N)} + \lambda_k \|u_k(t)\|_{L^0_0(\mathbb{R}^N)} + \|\beta(u_k(\cdot, t))\|_{L^p_0(\mathbb{R}^N)}
\]

\[
\leq C \left(\|w_k(t)\|_{H_0} + \lambda_k \|u_k(t)\|_{L^0_0(\mathbb{R}^N)} + \|u_k(t)\|_{L^0_0(\mathbb{R}^N)}^{p-1}\right)
\]

for some constant \(C \geq 0\) independent of \(k\) and \(t\). We shall simply write \(\mathfrak{A}_s u_k\) instead of \(\overline{\mathfrak{A}_s u_k(\cdot)}\) below. Thus we obtain

\[
\|\mathfrak{A}_s u_k\|_{L^2(0,T;L^0_0(\mathbb{R}^N))} \leq Q(\tilde{E}_{\sigma_k}(u_{0,k})).
\]

Therefore, there exist weak limits \(u, w, \beta, \xi\) such that, up to a (non-relabeled) subsequence,

\[
\begin{align*}
&u_k \to u \quad \text{weakly star in } L^\infty(0,T;L^p_0(\mathbb{R}^N)), \\
&w_k \to w \quad \text{weakly in } W^{1,2}(0,T;X'_{s,0}), \\
&\mathfrak{A}_s u_k \to \beta \quad \text{weakly star in } L^\infty(0,T;L^p_0(\mathbb{R}^N)), \\
&\mathfrak{A}_s u_k \to \xi \quad \text{weakly in } L^2(0,T;L^p_0(\mathbb{R}^N)).
\end{align*}
\]

Moreover, apply Ascoli’s compactness lemma along with the compact embedding \((L^p_0(\mathbb{R}^N))' \hookrightarrow X'_{\beta} (\hookrightarrow X'_{\beta,0})\) (see (86)) to get

\[
\partial_t u + \mathfrak{A}_s w = 0 \quad \text{in } X'_{s,0}, \quad w = \beta \quad \text{in } X'_{\beta,0}.
\]

To prove \(\xi = u\), we use Lemma 5.1. Indeed, choose \(\beta\) sufficiently large so that \(L^p_0(\mathbb{R}^N)\) is densely and continuously embedded in \(X'_{\beta} (\hookrightarrow X'_{\beta,0})\). Then the weak (star) convergence of \(u_k\) (cf. (176)) and that of \(\mathfrak{A}_s u_k\) (cf. (180)) suffice to apply the lemma (see (i) of Remark 5.1) and obtain the conclusion. As \(\lambda_k \to 1\) by Prop. 2.2, we then arrive at

\[
\partial_t u + \mathfrak{A}_s w = 0 \quad \text{in } X'_{s,0}, \quad w = \beta \quad \text{in } X'_{\beta,0}.
\]
Hence $w = \bar{\beta}$ in $\Omega \times (0, T)$. It remains to prove that $\bar{\beta} = \beta(u)$ in $\mathbb{R}^N \times (0, T)$. By Poincaré’s inequality (49) and (153), we note that
\[
\int_0^T \int_{\mathbb{R}^N} \beta(u_k) u_k \, dx \, dt \leq \int_0^T \int_{\mathbb{R}^N} w_k u_k \, dx \, dt \to \int_0^T \int_{\mathbb{R}^N} w u \, dx \, dt = \int_0^T \int_{\mathbb{R}^N} \beta u \, dx \, dt
\]
by (178) and (181). Therefore by (176) and (179), the maximal monotonicity of $u \mapsto \beta(u)$ in $L^p(\mathbb{R}^N) \times L^p(\mathbb{R}^N)$ yields $\bar{\beta} = \beta(u)$ in $\mathbb{R}^N \times (0, T)$. Moreover, $\beta(u)$ coincides with $w$ on $\mathbb{R}^N \times (0, T)$, as $w(\cdot, t)$ vanishes outside $\Omega$. The rest of proof runs as in the proof of Theorem 2.

5.5. Proof of Theorem 4. Let $s_k \in (0, 1)$, $s_k \searrow 0$ and let $(u_k, w_k)$ be the family of weak solutions to the problem
\[
\begin{align*}
\partial_t u_k + A_{s_k} w_k &= 0 \quad \text{in } \mathcal{L}^{p'}_{\sigma} \times (0, T), \\
w_k &= A_{\sigma} u_k + B(u_k) - u_k \quad \text{in } \mathcal{L}^{p'}_{\sigma},
\end{align*}
\]
with $u_k(0) = u_{0,k}$. Compared to the proofs of Theorems 2 and 3, this proof is definitely easier. Actually, since $\sigma$ is kept fixed, the sequence $u_k$ retains some space compactness.

Put $X = \mathcal{E}_\sigma$ and suppose that $s_k < \sigma$ for all $k \in \mathbb{N}$ without any loss of generality. Thanks to the uniform estimates (151)-(153) obtained in §5.1 along with (93), one has, up to a (non-relabeled) subsequence,
\[
\begin{align*}
u_k &\to u \quad \text{weakly star in } L^\infty(0, T; \mathcal{E}_\sigma), \\
A_{\sigma} u_k &\to A_{\sigma} u \quad \text{weakly star in } L^{1,2}(0, T; X'), \\
w_k &\to w \quad \text{weakly in } L^2(0, T; H_0), \\
A_{s_k} w_k &\to w \quad \text{weakly in } L^2(0, T; X'), \\
\beta(u_k) &\to \bar{\beta} \quad \text{weakly in } L^2(0, T; H_0),
\end{align*}
\]
which immediately gives $w = A_{\sigma} u + \bar{\beta} - u \in \mathcal{E}_\sigma'$. Here we used Lemma 5.1 with $\beta > 0$ sufficiently large so that $X' \hookrightarrow \mathcal{L}^{p'}_{\sigma,0}$ (as in §5.3) to identify the limit of $A_{s_k} w_k$ as $w$. By [36, Theorem 5], one can obtain
\[
u_k \to u \quad \text{strongly in } C([0, T]; H_0).
\]
By assumption, $u_{0,k} \to u_0$ strongly in $H_0$. Hence we obtain $u(t) \to u_0$ strongly in $H_0$ as $t \searrow 0$. Moreover, by applying Minty’s trick to the maximal monotone operator $u \mapsto \beta(u(\cdot))$ in $H_0$, one concludes that $\bar{\beta} = \beta(u)$. For all $\varphi \in \mathcal{D}_\Omega(\mathbb{R}^N) \subset X$ and $\phi \in C_0^\infty(0, T)$, one can derive
\[
\int_0^T \langle \partial_t u_k(t), \varphi \rangle_X \phi(t) \, dt + \int_0^T \langle A_{s_k} w_k(t), \varphi \rangle_X \phi(t) \, dt = 0.
\]
Passing to the limit as $k \nearrow \infty$, we obtain
\[
\int_0^T \langle \partial_t u(t) + w(t), \varphi \rangle_X \phi(t) \, dt = 0.
\]
Since $\mathcal{D}_\Omega(\mathbb{R}^N)$ is dense in $X$, we conclude that
\[
\partial_t u + w = 0 \quad \text{in } X', \quad 0 < t < T.
\]
Recalling that $X = \mathcal{E}_\sigma$, we conclude that $u$ solves
\[
\partial_t u + A_{\sigma} u + \beta(u) - u = 0 \quad \text{in } \mathcal{E}_\sigma', \quad 0 < t < T.
\]

6. Stationary states

In this section we analyze the behavior of stationary states of system (1)-(4) in the coercive case $p > 2$. We will put a particular emphasis on the asymptotic behavior of the stationary states when $\sigma \searrow 0$.

The function $u$ is called a stationary state of (1)-(4) when $\partial_t u = 0$ for a.e. $(x, t) \in \Omega \times (0, +\infty)$. Then by (1) we have
\[
(-\Delta)^s w = 0 \quad \text{in } \Omega, \quad w = 0 \quad \text{in } \mathbb{R}^N \setminus \Omega,
\]
Indeed, let for all \( \varepsilon \) is nontrivial. Provided that \( w \equiv 0 \) by Poincaré’s inequality (20). Hence, by (2), \( u \) solves the problem

\[
\begin{align*}
\left\{( -\Delta )^\sigma u + \beta (u) - u &= 0 \quad \text{in } \Omega, \\
u &= 0 \quad \text{in } \mathbb{R}^N \setminus \Omega.
\end{align*}
\]

First of all, let us provide a weak formulation of (184).

**Definition 6.1.** A function \( u \in \mathcal{E}_\sigma \) is called a weak solution of (184), if

\[
\frac{C(\sigma)}{2} \iint_{\mathbb{R}^{2N}} \frac{(u(x) - u(y))(\varphi(x) - \varphi(y))}{|x - y|^{N + 2\sigma}} \, dx \, dy + \int_\Omega \beta(u) \varphi \, dx - \int_\Omega u \varphi \, dx = 0
\]

for all \( \varphi \in \mathcal{E}_\sigma \). The weak form (185) can be equivalently rewritten as

\[
\mathcal{A}_\sigma u + B(u) - u = 0 \quad \text{in } \mathcal{E}_\sigma'.
\]

We next prove existence of a solution to (184). To this end, we use the direct method of calculus of variations. Recall that the energy functional \( E_\sigma : \mathcal{E}_\sigma \to \mathbb{R} \) is defined by

\[
E_\sigma (v) := \frac{C(\sigma)}{4} \iint_{\mathbb{R}^{2N}} \frac{|v(x) - v(y)|^2}{|x - y|^{N + 2\sigma}} \, dx \, dy + \int_\Omega \beta(u) \, dx - \frac{1}{2} \int_\Omega |v|^2 \, dx \quad \text{for } v \in \mathcal{E}_\sigma.
\]

It is easy to check that \( E_\sigma \) is coercive in \( \mathcal{E}_\sigma \). Indeed, by Hölder’s and Young’s inequalities,

\[
E_\sigma (v) \geq \frac{1}{2} \|v\|^2_{\mathcal{E}_\sigma,0} + \frac{1}{2p} \int_\Omega |v|^p \, dx - C \quad \text{for all } v \in \mathcal{E}_\sigma.
\]

Hence, the existence of a (global) minimizer \( u \in \mathcal{E}_\sigma \) follows from the compactness of the embedding \( \mathcal{X}_{\sigma,0} \hookrightarrow H_0 \). Note that \( u \) actually solves equation (184) in the sense of Definition 6.1. Now, let \( u \) be a global minimizer of \( E_\sigma \). Since \( |u(x)| - |u(y)| \leq |u(x) - u(y)| \) for \((x, y) \in \mathbb{R}^N \times \mathbb{R}^N, |u| \) has the same energy of \( u \), namely \( E_\sigma (u) = E_\sigma (|u|) \). Hence \( |u| \) also minimizes \( E_\sigma \) and solves (184). Consequently, by applying maximum principle for the fractional Laplacian (see, e.g., [31, 21]) to the nonnegative solution \( |u| \), we infer that \( |u| > 0 \) or \( u \equiv 0 \) in \( \Omega \). Therefore every minimizer \( u \) of \( E_\sigma \) turns out to be sign-definite or identically equal to zero over \( \Omega \). Now, we are going to give conditions implying that there exist nontrivial solutions to (184). We can prove the following

**Proposition 6.1.** Let \( \lambda_1 (\sigma) \) be the first eigenvalue of (48) with \( r \) replaced by \( \sigma \). Then, if \( \lambda_1 (\sigma) < 1 \), problem (184) admits a nontrivial weak solution.

**Proof.** First of all, we claim that

\[
\inf_{\mathcal{E}_\sigma} E_\sigma < 0 \quad \text{if } \lambda_1 (\sigma) < 1.
\]

Indeed, let \( v_\sigma \) be the eigenfunction of \( \mathcal{A}_\sigma \) corresponding to the first eigenvalue \( \lambda_1 (\sigma) \), normalized with respect to the \( H_0 \)-norm. Then it is proved that \( \lambda_1 (\sigma) \) is simple in [33, Prop. 9 (c)]. Moreover, \( v_\sigma \) is Hölder continuous up to the boundary by [32, Theorems 1.1 and 1.3] and [35, Proposition 4]. By a simple calculation, we have for \( \varepsilon > 0 \)

\[
E_\sigma (\varepsilon v_\sigma) = \frac{\varepsilon^2}{2} |v_\sigma|_{\mathcal{X}_{\sigma,0}}^2 + \frac{\varepsilon^p}{p} \int_\Omega |v_\sigma|^p \, dx - \frac{\varepsilon^2}{2} \int_\Omega |v_\sigma|^2 \, dx
\]

\[
= \frac{\varepsilon^2}{2} \lambda_1 (\sigma) + \frac{\varepsilon^p}{p} \int_\Omega |v_\sigma|^p \, dx - \frac{\varepsilon^2}{2}
\]

\[
= \varepsilon^2 \left[ \frac{1}{2} \lambda_1 (\sigma) - 1 \right] + \frac{\varepsilon^{p-2}}{p} \int_\Omega |v_\sigma|^p \, dx < 0,
\]

provided that \( \varepsilon \) is chosen so that

\[
\frac{1}{2} \lambda_1 (\sigma) - 1 + \frac{\varepsilon^{p-2}}{p} \int_\Omega |v_\sigma|^p \, dx < 0, \quad \text{i.e. } \varepsilon^{p-2} < \frac{p(1 - \lambda_1 (\sigma))}{2 \| v_\sigma \|^p_{\mathcal{X}_{\sigma,0}}}.
\]

Hence the infimum of \( E_\sigma \) over \( \mathcal{E}_\sigma \) is negative whenever \( \lambda_1 (\sigma) < 1 \). Therefore every global minimizer is nontrivial.

**Remark 6.1.** If the first eigenvalue \( \lambda_1 \) of \(-\Delta\) equipped with the homogeneous Dirichlet condition is not greater than one, then by the upper estimate \( \lambda_1 (\sigma) < \lambda_1^\sigma \) (see (52)), we have \( \lambda_1 (\sigma) < 1 \). Hence (184) possesses a nontrivial weak solution.
Lemma 6.1. Let $u$ be a weak solution of (184). Then its energy is nonpositive, i.e. $E_{\sigma}(u) \leq 0$. Moreover, if $E_{\sigma}(u) = 0$, then $u \equiv 0$.

Proof. Let $u$ be a weak solution of (184). Test (184) by $u$ (i.e., substitute $\varphi = u$ in (185)) to get

$$\|u\|_{X_{\sigma,0}}^2 + \int_\Omega |u|^p \, dx - \int_\Omega |u|^2 \, dx = 0,$$

which yields

$$E_{\sigma}(u) = -\left(\frac{1}{2} - \frac{1}{p}\right) \int_\Omega |u|^p \, dx \leq 0.$$

In particular, if $E_{\sigma}(u) = 0$, then $u \equiv 0$. □

As a consequence, we have the following criterion for non-existence of nontrivial solutions:

Corollary 6.1. If $\lambda_1(\sigma) \geq 1$, then (184) admits only the trivial solution.

Proof. We observe that, by (49),

$$E_{\sigma}(u) = \frac{1}{2}\|u\|_{X_{\sigma,0}}^2 + \frac{1}{p} \int_\Omega |u|^p \, dx - \frac{1}{2} \int_\Omega |u|^2 \, dx
\geq \frac{1}{2} (\lambda_1(\sigma) - 1) \int_\Omega |u|^2 \, dx + \frac{1}{p} \int_\Omega |u|^p \, dx \quad \text{for all } u \in E_{\sigma}.$$

(187)

Hence, if $\lambda_1(\sigma) \geq 1$, then $E_{\sigma}(u) \geq 0$. Therefore, due to Lemma 6.1, problem (184) has no nontrivial weak solution. □

We are now in position to state a result on the asymptotic behavior of nontrivial weak solutions as $\sigma \searrow 0$.

Proposition 6.2. Suppose that $\lambda_1(\sigma) < 1$ for all $\sigma \in (0,1)$. Let, for $\sigma \in (0,1)$, $u_\sigma$ be a nontrivial weak solution of (184). Then $u_\sigma$ converges to zero strongly in $H_0$ as $\sigma \searrow 0$.

Proof. Let $\sigma$ converge to 0 along a sequence $\sigma_k$ and denote by $u_k$ the corresponding nontrivial weak solution to (184). Let $r > 0$ and $v \in E_{\sigma_k}$ be such that $\|v\|_{H_0} = r$. Let us first note that

$$\int_\Omega |v|^2 \, dx \leq |\Omega|^{(p-2)/p} \left(\int_\Omega |v|^p \, dx\right)^{2/p}.$$

Hence, combining this with (187), we infer

$$E_{\sigma}(v) \geq \frac{r^2}{2} (\lambda_1(\sigma_k) - 1) + \frac{r^p}{p|\Omega|^{(p-2)/2}} = \frac{r^2}{2} \left(\lambda_1(\sigma_k) - 1 + \frac{2r^{p-2}}{p|\Omega|^{(p-2)/2}}\right) \geq 0,$$

provided that $\lambda_1(\sigma_k) - 1 + 2r^{p-2}/(p|\Omega|^{(p-2)/2}) \geq 0$, which corresponds to $r^{p-2} \geq (p/2)|\Omega|^{(p-2)/2}(1 - \lambda_1(\sigma_k))$. Set

$$r_k := \left(\frac{p}{2}|\Omega|^{(p-2)/2}(1 - \lambda_1(\sigma_k))\right)^{1/(p-2)}.$$

Then one has

$$\inf \{ E_{\sigma}(v) : v \in E_{\sigma_k}, \|v\|_{H_0} \geq r_k \} \geq 0.$$

Therefore, by Lemma 6.1, being $u_k$ non trivial, we have

$$\|u_k\|_{H_0} < r_k.$$

Since $\lambda_1(\sigma_k) \to 1$ as $k \uparrow +\infty$ (cf. Proposition 2.2), we conclude that $u_k \to 0$ strongly in $H_0$ as $k \uparrow +\infty$. □
7. Appendix

Here we discuss in some more detail the relations between our problem and other nonlocal Allen-Cahn or Cahn-Hilliard models. We just comment on the assumptions on the fractional diffusion operators, neglecting the differences occurring in the choice of the potential.

Let us start describing the relations between our problem and the Cahn-Hilliard equation analyzed in [4, 14, 20]. Actually, these problems share a common variational structure as the variable \( w \) is introduced as the first variation (in \( L^2 \)) of some functional: in our case, of \( E_\sigma \) (cf. (66)), while the nonlocal operator \( J[u] \) of [4, 14, 20] (recall (7)) corresponds to the gradient of

\[
E_J[u] = \frac{1}{4} \int_{\Omega \times \Omega} j(x - y) (u(x) - u(y))^2 \, dx \, dy,
\]

as a direct computation shows.

Apart from the lower regularity of the kernel \( K_r \) associated to \((-\Delta)^r\), one major difference is that the integration domain in (188) is \( \Omega \times \Omega \) in place of \( \mathbb{R}^{2N} \). In a sense, this is similar to what happens in the case of the so-called regional Laplacian (cf., e.g., [23]), defined, for smooth functions, as (compare with (10))

\[
(-\Delta)^{r, \text{reg}} u(x) := C(r,N) \text{p.v.} \int_{\Omega} \frac{u(x) - u(y)}{|x - y|^{N + 2r}} \, dy.
\]

In other words, the above position corresponds to assuming that, as \( x \in \Omega \), the value of \( u(y) \) influences that of \( u(x) \) only for \( y \in \Omega \). Hence, in fact, no boundary conditions are taken in that case. In our setting, instead, also the outer (Dirichlet) value \( u(y) = 0 \), \( y \in \mathbb{R}^N \setminus \Omega \), carries some influence on the value of \((-\Delta)^r u(x)\).

This difference is also reflected when one looks at the first variation \( J \) of \( E_J \) (cf. (7)), where the function \( a(\cdot) \) depends in fact on the variable \( x \in \Omega \). Indeed, taking \( W \equiv 0 \) and neglecting the principal value for simplicity, also in our case it would be possible to write (at least formally)

\[
(\mathcal{E}_\sigma)(u) = au - K_\sigma * u, \quad a = \int_{\mathbb{R}^N} K_\sigma(x - y) \, dy.
\]

However, \( a \), representing the “total mass” of \( K_\sigma \), is now independent of \( x \) (note, instead, that the convolution term is the same in both cases as we assume \( u \) be identically 0 out of \( \Omega \)).

A further difference between the two models is related to the regularity of the kernels. In [4, 14, 20], \( j \) satisfies some kind of summability property. On the other hand, in the case of the fractional Laplacian, the kernel \( K_r \) is somehow “less than \( L^1 \)”. This is a trivial remark, of course. However, some consequences from the point of view of regularity analysis deserve to be discussed. Indeed, in the present case the term \( \mathcal{L}_r u \) is less regular than \( u \) and this fact implies that the equation enjoys some smoothing effect, as happens in the standard parabolic case (i.e., for the usual Laplacian). This fact permits us to “embed” compactness and density tools in the Hilbert formulation by working in Hilbert triplets like \( (\mathcal{X}_{r,0}, H_0, \mathcal{X}_{r,0}') \).

Instead, in the models analyzed in [4, 14, 20], the term \( J[u] \) is strictly more regular than the function \( u \) on which \( J \) acts. This means that the PDE system has limited regularization effects (actually, this is especially true for Allen-Cahn based models, see, e.g., [22], in the Cahn-Hilliard case the Laplacian in the equation \( u_t = \Delta u \) corresponding to our (1) partially compensates this). In other words, a singularity in the initial datum tends to propagate with time without smoothing out (at most it decreases in amplitude like in a dissipative ODE).

The regional Laplace operator (189) characterizes also the model studied more recently in [2]. To be precise, in [2] the standard Laplacian is taken in the analogue of (1), while the regional operator \((-\Delta)^{r, \text{reg}}\) appears in the analogue of (2). Moreover, no-flux conditions are assumed for \( u \). This, in particular, entails a mass-conservation property; namely, the spatial mean value of \( u \) is constant with respect to time, as one expects to occur in models describing phase separation, on account of the underlying physics. Using that \((-\Delta)^{r, \text{reg}}\) can be seen as a fractional power of the Neumann Laplacian (restricted to the class of zero-mean functions), the authors of [2] can show that, at least for sufficiently smooth solutions, also the component \( u \) turns out to satisfy a no-flux condition on \( \partial \Omega \) (though no explicit boundary condition is required in the mathematical formulation of their problem). It is also worth mentioning that the analysis given in [2] admits a much wider class of potentials \( W \), including
in particular singular functions like the “logarithmic potential” mentioned in the Introduction. In principle, this would be possible also for our model and we plan to address this issue in a forthcoming work. We also observe that, for the model considered in [2], one expects that the solution, at least asymptotically in time and for non-singular functions $W$, could satisfy strong regularization properties. This is due to the fact that the regional Laplacian, as a fractional power of the Neumann Laplacian, satisfies the property $(-\Delta)^{r/2} \circ (-\Delta)^{t/2} = (-\Delta)^{r+t}$, which may allow use of bootstrap regularity methods. On the other hand, an analogue property fails for our operators $\mathfrak{A}$, due to the occurrence of the “solid” Dirichlet condition. For this reason, we expect that the analysis of asymptotic regularity properties of weak solutions could be more challenging in our case.

References

[1] R.A. Adams and J.J.F. Fournier, *Sobolev spaces, second edition*, Elsevier, Amsterdam, 2003.
[2] H. Abels, S. Bosia, and M. Grasselli, *Cahn-Hilliard equation with nonlocal singular free energies*, Ann. Mat. Pura Appl. (4), to appear.
[3] S.M. Allen and J.W. Cahn, *A macroscopic theory for antiphase boundary motion and its application to antiphase domain coarsening*, Acta Metallurgica, 27 (1979), 1085–1095.
[4] P.W. Bates and J. Han, *The Dirichlet boundary problem for a nonlocal Cahn-Hilliard equation*, J. Math. Anal. Appl., 311 (2005), 289–312.
[5] M. Bonforte and J.L. Vázquez, *Quantitative local and global a priori estimates for fractional nonlinear diffusion equations*, Adv. Math., 250 (2014), 242–284.
[6] M. Bonforte and J.L. Vázquez, *A priori estimates for fractional nonlinear degenerate diffusion equations on bounded domains*, Arch. Ration. Mech. Anal., to appear (2015).
[7] M. Bonforte, Y. Sire, and J.L. Vázquez, *Existence, uniqueness and asymptotic behaviour for fractional porous medium equations on bounded domains*, Discrete Contin. Dyn. Syst., to appear (2015).
[8] H. Brézis, “Opérateurs Maximaux Monotones et Sémi-groupes de Contractions dans les Espaces de Hilbert”, North-Holland Mathematics Studies, No. 5, North-Holland Publishing Co., 1973.
[9] X. Cabré and J. Tan, *Positive solutions of nonlinear problems involving the square root of the Laplacian*, Adv. Math, 5 (2010), 2052–2093.
[10] L. Caffarelli, S. Sandro, and L. Silvestre, *Regularity estimates for the solution and the free boundary of the obstacle problem for the fractional Laplacian*, Invent. Math., 171 (2008), 425–461.
[11] L. Caffarelli and A. Vasseur *Drift diffusion equations with fractional diffusion and the quasi-geostrophic equation*, Ann. of Math. (2), 3 (2010), 1903–1930.
[12] J.W. Cahn and J.E. Hilliard, *Free energy of a nonuniform system I. Interfacial free energy*, J. Chem. Phys., 28 (1958), 258–267.
[13] L. Cherfils, A. Miranville, and S. Zelik, *The Cahn-Hilliard equation with logarithmic potentials*, Milan J. Math., 79 (2011), 561–596.
[14] F. Colli, S. Frigeri, and M. Grasselli, *Global existence of weak solutions to a nonlocal Cahn-Hilliard-Navier-Stokes system*, J. Math. Anal. Appl., 386 (2012), 428–444.
[15] P. Constantin and J. Wu, *Behavior of solutions of 2D quasi-geostrophic equations*, SIAM J. Math. Anal., 30 (1999), 937–948.
[16] R. Cont and P. Tankov, *Financial modelling with jump processes*, Chapman & Hall/CRC Financial Mathematics Series, Chapman & Hall/CRC, Boca Raton, FL, 2004.
[17] A. de Pablo, F. Quirós, A. Rodríguez, and J.L. Vázquez, *A general fractional porous medium equation*, Comm. Pure Appl. Math., 65 (2012), 1242–1284.
[18] E. Di Nezza, G. Palatucci, and E. Valdinoci, *Hitchiker’s guide to the fractional Sobolev spaces*, Bull. Sci. Math., 136 (2012), 521–573.
[19] S. Dipierro, G. Palatucci, and E. Valdinoci, *Dislocation dynamics in crystals: A macroscopic theory in a fractional Laplace setting*, Comm. Math. Phys., to appear.
[20] H. Gajewski and K. Zacharias, *On a nonlocal phase separation model*, J. Math. Anal. Appl., 286 (2003), 11–31.
[21] A. Greco and R. Servadei, *Hopf’s lemma and constrained radial symmetry for the fractional Laplacian*, preprint, available online at http://www.ma.utexas.edu/mp_arc-bin/mpa?yn=14-69
[22] M. Grasselli and G. Schimperna, *Nonlocal phase-field systems with general potentials*, Discrete Contin. Dyn. Syst., 33 (2013), 5089–5106.
[23] Q.-Y. Guan, *Integration by parts formula for regional fractional Laplacian*, Comm. Math. Phys., 266 (2006), 289–329.
[24] S. Kim and K.A. Lee, *Hölder estimates for singular non-local parabolic equations*, J. Funct. Anal., 261 (2011), 3482–3518.
[25] J.-L. Lions and E. Magenes, “Problèmes aux Limites non Homogènes et Applications”, Vol. 1. Travaux et Recherches Mathématiques, No. 17, Dunod, Paris, 1968.
[26] V. Maz’ya and T. Shaposhnikova, *On the Bourgain, Brezis, and Mironescu theorem concerning limiting embeddings of fractional Sobolev spaces*, J. Funct. Anal., 195 (2002), 230–238.
[27] R. Metzler and J. Klafter, *The random walk’s guide to anomalous diffusion: a fractional dynamics approach*, Phys. Rep., 339 (2000), 77 pp.
[28] A. Novick-Cohen, The Cahn-Hilliard equation: mathematical and modeling perspectives, Adv. Math. Sci. Appl., 8 (1998), 965–985.
[29] Y. Nec, A.A. Nepomnyashchy, and A.A. Golovin, Front-type solutions of fractional Allen-Cahn equation, Phys. D, 237 (2008), 3237–3251.
[30] E. Rocca and G. Schimperna, Universal attractor for some singular phase transition systems, Phys. D, 192 (2004), 279–307.
[31] X. Ros-Oton and J. Serra, The extremal solution for the fractional Laplacian, Calc. Var. Partial Differential Equations, 50 (2014), 723–750.
[32] X. Ros-Oton and J. Serra, The Pohozaev identity for the fractional Laplacian, Arch. Ration. Mech. Anal., 213 (2014), 587–628.
[33] R. Servadei and E. Valdinoci, Variational methods for non-local operators of elliptic type, Discrete Contin. Dyn. Syst., 33 (2013), 2105–2137.
[34] R. Servadei and E. Valdinoci, On the spectrum of two different fractional operators, Proc. Roy. Soc. Edinburgh Sect. A, 144 (2014), 831–855.
[35] R. Servadei and E. Valdinoci, A Brezis-Nirenberg result for non-local critical equations in low dimension, Commun. Pure Appl. Anal., 12 (2013), 2445–2464.
[36] J. Simon, Compact sets in the space $L^p(0,T;B)$, Ann. Math. Pura. Appl. (4), 146 (1987), 65–96.
[37] W. Woyczyński, Lévy processes in the physical sciences, Lévy processes, 241–266, Birkhäuser Boston, Boston, MA, 2001.
[38] S.Y. Yolcu and T. Yolcu, Estimates for the sums of eigenvalues of the fractional Laplacian on a bounded domain, Commun. Contemp. Math., 15 (2013), 1250048, 15 pp.
[39] G.M. Zaslavsky, Chaos, fractional kinetics, and anomalous transport, Phys. Rep., 371 (2002), 461–580.

Graduate School of System Informatics, Kobe University

Dipartimento di Matematica “F. Casorati”, via Ferrata 1, I–27100 Pavia, Italy

E-mail address: akagi@port.kobe-u.ac.jp
E-mail address: giusch04@unipv.it
E-mail address: antonio.segatti@unipv.it
URL: http://www2.kobe-u.ac.jp/~akagi56/index.html
URL: http://www-dimat.unipv.it/giulio
URL: http://www-dimat.unipv.it/segatti