ESSENTIAL GRADED ALGEBRA OVER POLYNOMIAL RINGS WITH REAL EXONENTS

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Abstract. The geometric and algebraic theory of monomial ideals and multigraded modules is initiated over real-exponent polynomial rings and, more generally, monoid algebras for real polyhedral cones. The main results include the generalization of Nakayama’s lemma; complete theories of minimal and dense primary, secondary, and irreducible decomposition, including associated and attached faces; socles and tops; minimality and density for downset hulls, upset covers, and fringe presentations; Matlis duality; and geometric analysis of staircases. Modules that are semialgebraic or piecewise-linear (PL) have the relevant property preserved by functorial constructions as well as by minimal primary and secondary decompositions. And when the modules in question are subquotients of the group itself, such as monomial ideals and quotients modulo them, minimal primary and secondary decompositions are canonical, as are irreducible decompositions up to the new real-exponent notion of density.

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1. INTRODUCTION

Overview. Little is known about the algebra of rings of polynomials whose exponents are allowed to be nonnegative real numbers instead of integers. The extreme failure of the noetherian condition—ideals can be uncountably generated—and the nontrivial topology on the set of exponents present daunting technical difficulties. The small amount of existing literature proceeds by restricting the study to monomial ideals that are finitely generated, in an appropriate sense [ISW13, ASW15], or to multigraded modules that are finitely presented [Les15]. Other work can be viewed as touching on the continuous nature of the exponent set via nondiscrete-monoid algebras [ACHZ07]. But the general behavior of modules over real-exponent polynomial rings remains wide open, even in the special case of monomial ideals. The issue has risen to prominence particularly because modules over the real-exponent polynomial ring emerge in applied topology [CZ09] (see also [Mil15]), where the focus is on multigraded modules.

This paper breaks ground on the earnest study of modules over real-exponent polynomial rings, including the usual setting where exponents lie in a right-angled nonnegative
orthant but also real analogues of affine semigroup rings, with arbitrary pointed polyhedral cones of allowed exponents. This first step of the investigation concerns monomial ideals and multigraded modules, which are general enough to exhibit the starkly different behavior resulting from continuous exponents and deviation from noetherianity but have enough combinatorial structure to allow complete treatment of basic theory, such as primary decomposition, Nakayama’s lemma, and minimal presentations.

The algebraic development hinges on a number of foundations whose elementary versions for finitely generated or noetherian modules fail when straightforwardly generalized to real exponents but nonetheless admit fully functioning analogues when appropriately enhanced. Most importantly, detecting

- an injective homomorphism of modules by checking at all associated primes or
- a surjective homomorphism by Nakayama’s lemma (Krull–Azumaya theorem)

falter at the outset: modules need not contain copies of quotients by prime ideals, so the notion of associated prime requires serious thought; and dually, modules—even as simple as monomial ideals—do not have minimal generators [ISW13], so considerations surrounding Nakayama’s lemma require just as much attention.

The solutions developed here to tackle these problems construct a topological framework for concepts of minimality, in the form of dense generator and cogenerator functors. For example, with definitions made properly, arbitrary real-exponent monomial ideals have canonical monomial primary decompositions that are minimal in a strong sense, generalizing the situation for polynomial and other affine semigroup rings. And these decompositions are similarly derived from canonical irreducible decompositions. But the notion of “irredundant” for irreducible decomposition must be revised: components can be omitted as long as those that remain are dense in the sense developed here.

Density engages with continuity of exponent sets in a way that gives hope of being able to lift the lessons learned here for monomial ideals and multigraded modules to arbitrary ideals and modules. In the meantime, the results here generalize classical statements about monomial ideals to the much harder and uncharted context of real exponents. And they are made more important by virtue of the central role of real multigraded algebra in the rapidly developing field of topological data analysis, where mathematical foundations for the real exponent case is sorely lacking from the theory of persistent homology with multiple real parameters [CZ09].

Acknowledgements. Justin Curry provided feedback after listening for hours about face posets infinitesimally near real persistence parameters; he enhanced the functorial viewpoint and provided references as well as insight on topics from sheaf theory to real algebraic geometry. Ashleigh Thomas played a crucial role in the genesis of this algebraic theory of real multipersistence, and continues to be a collaborator. Ville Puuska read an inchoate version of this manuscript [Mil17, §6–14] extremely carefully; among his valuable comments, he indicated a need for certain hypotheses in Matlis duality.
1.1. Real exponent issues.

Example 1.1. In the real-exponent polynomial ring \( k[\mathbb{R}^n_+] \), a monomial ideal is an ideal generated by monomials: \( I = \langle x^a \mid a \in A \rangle \) for some \( A \subseteq \mathbb{R}^n_+ \), where \( x^a = x_1^{a_1} \cdots x_n^{a_n} \).

For example, the maximal graded ideal \( m = \langle x_1^{b_1}, \ldots, x_n^{b_n} \mid b_i > 0 \text{ for } i = 1, \ldots, n \rangle \) has the exponent set

which is the nonnegative orthant with the origin missing. This ideal is not finitely generated, although it is countably generated by using any sequence of strictly positive vectors \( b = (b_1, \ldots, b_n) \) converging to \( 0 \). This ideal also has no minimal generating set, because deleting any finite subset of a sequence converging to \( 0 \) still results in a sequence converging to \( 0 \). This phenomenon was observed in [ISW13].

Example 1.2. The monomial ideal \( I = \langle x^a \mid a_1 + \cdots + a_n = 1 \rangle \subseteq k[\mathbb{R}^n_+] \) has exponent set depicted on the left:

This ideal is uncountably generated, with the given generators forming the unique minimal monomial generating set. The quotient module \( M = k[\mathbb{R}^n_+] / I \), depicted on the right, has open upper boundary. This module would appear to be primary to the maximal ideal \( m \), but as \( m^d = m \) for all positive integers \( d \), the module \( M \) is not annihilated by any power of \( m \). Worse, \( M \) has no elements annihilated by the maximal ideal: \( M \) has no simple submodules, so \( m \) is not an associated prime in the usual sense of \( M \) containing a copy of \( k = k[\mathbb{R}^n_+] / m \). In that usual sense, the set of such copies, namely the socle \( \text{Hom}(k, M) \), would be an essential submodule of \( M \), which by definition intersects every nonzero submodule of \( M \) nontrivially. But in this picture the Hom vanishes. Moreover, an essential submodule requires containing a strip of locally positive width near the upper boundary, but the intersection of all essential submodules is 0.
Question 1.3. In Example 1.1, what should Nakayama’s lemma say?

Question 1.4. In Example 1.2, what should the statement, “A homomorphism $M \to N$ is injective if and only if $M_p \hookrightarrow N_p$ is injective for all associated primes $p$ of $M$” say?

These questions are precisely dual to each other when considered from a judicious angle.

1.2. Nakayama’s lemma and primary decomposition.

The first answers to Questions 1.3 and 1.4 are two major contributions of this paper,
- Theorem 12.3 on detecting surjectivity by generator functors (tops), and the Matlis dual (Section 2.5) from which it is deduced,
- Theorem 6.7 on detecting injectivity by cogenerator functors (socles).

However, the final answers reflect the observations in Examples 1.1 and 1.2 that generating sets and essential submodules in the setting of real exponents maintain those properties when replaced with dense approximations. The precise formulation of this deeply non-discrete observation yields two additional major contributions,
- the density enhancements in Theorem 12.15 and Theorem 7.27.

Flowing from these foundational results, particularly from the socle injectivity criterion that is Theorem 6.7, are other staples of commutative algebra:
- primary decompositions, minimal in a strong sense (Theorems 9.23 and 9.27);
- canonical minimal primary decompositions of monomial ideals (Theorem 9.12);
- irreducible decomposition for monomial ideals that are canonical and irredundant up to taking dense subsets (Theorem 9.7 and Corollary 9.8);
- duals of all these for secondary decomposition and attached primes (Section 12).

Example 1.5. The upper boundary of the interval in $\mathbb{R}^2$ at the left end of the display

```
\begin{align*}
\begin{bmatrix}
    1 & 0 \\
    0 & 1
\end{bmatrix} \oplus \begin{bmatrix}
    0 & 1 \\
    1 & 0
\end{bmatrix}
\end{align*}
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has the vertical axis as an asymptote, whereas the horizontal axis is exactly parallel to the positive horizontal end of the upper boundary. The corresponding interval module has the indicated canonical minimal primary decomposition by Theorem 9.12.

The minimality in Theorem 9.27 is a requirement that the socle of the module should map isomorphically to the direct sum of the socles of the quotients modulo its primary components (Definition 9.25). This isomorphism is stronger than usually proposed in noetherian commutative algebra. When applied to an injective hull or irreducible
decomposition in noetherian situations, socle-minimality as in Definition 9.25 is equivalent to there being a minimal number of indecomposable summands. In contrast, minimal primary decompositions in noetherian commutative algebra do not require socle-minimality in any sense; they stipulate only minimal numbers of summands, with no conditions on socles. This has unfortunate consequences: even in noetherian settings, different choices of primary components for an embedded (i.e., nonminimal) prime can strictly contain one another, for example. Requiring socle-minimality as in Definition 9.25 recovers a modicum of uniqueness over arbitrary noetherian rings, as socles are functorial even if primary components themselves need not be. This tack is more commonly taken in combinatorial commutative algebra, typically involving objects such as monomial or binomial ideals. In particular, the “witnessed” forms of minimality for mesoprimary decomposition [KM14, Definition 13.1 and Theorem 13.2] and irreducible decomposition of binomial ideals [KMO16] serve as models for the type of minimality in primary decompositions considered here.

In ordinary noetherian commutative algebra socle-minimal primary decompositions are anyway automatically produced by the usual existence proof, which leverages the noetherian hypothesis to create an irreducible decomposition. Indeed, a noetherian primary decomposition is socle-minimal if and only if each primary component is obtained by gathering some of the components in a minimal irreducible decomposition. When real exponents enter, truly minimal irreducible decompositions are impossible by Theorem 9.7 and Corollary 9.8, which force us to settle for irredundancy up to taking dense subsets. Nonetheless, the primary component formed by gathering all irreducible components with a given associated face is well defined, regardless of which dense subset of irreducible components was present. That is how uniqueness of the minimal primary decomposition in Theorem 9.12 arises even from nonunique irreducible decomposition.

Secondary decomposition is lesser known, even to algebraists, than its Matlis dual, primary decomposition, but secondary decomposition has been in the literature for decades [Kir73, Mac73, Nor72] (see [Sha76, Section 1] for a brief summary of the main concepts). The unfamiliarity of secondary decomposition and its related functors is a primary reason why the bulk of the technical development over real exponents is carried out in terms of cogenerators and socles instead of generators and tops.

Minimal primary and dense irreducible decomposition owe their existence to definitions tailored to real exponents. These include especially

- a definition of associated prime by socle nonvanishing (Definition 9.1) that yields
- characterizations of coprimary modules as those with only one associated prime (Proposition 2.29 and Theorem 9.2).

These, in turn, rely on the heart of the matter regarding density, which draws, at the most fundamental level, on the topological algebra surrounding real exponents:

- the characterization of essential submodules by socle inclusion (Theorem 8.5).
1.3. **Socles, cogenerators, and staircases.**

The results discussed thus far all rest on the main socle injectivity criterion in Theorem 6.7. As such, the entire edifice is built on socles. Identifying the right definition of socle in Section 4 to account for the departure from discrete exponents is the most subtle and difficult aspect of the theory. But the answer turns out to be pretty and, as luck would have it, finite.

The problem to be overcome is seen in Example 1.2: the socle of the quotient module \( M \) there should lie along its upper boundary, but the upper boundary is missing: \( M \) is zero in the corresponding \( \mathbb{R}^n \)-graded degrees. The solution is to keep track of the directions in which limits must be taken to reach the missing boundary points. The finiteness of the answer comes down to the fact that it matters only which of the finitely many faces of the exponent cone the limits are taken along. The main product of Section 4 is not a theorem, but nonetheless a major contribution, namely

- the notion of socle in Definition 4.29.1 as well as
- cogenerator and nadir in Definition 4.29.3.

Readers from persistent homology should view these as functorializing the notion of “closed or open right endpoint of an interval” and generalizations to more parameters.

**Example 1.6.** Consider the module \( k[y, x] \) at the left-hand end of Example 1.5. Any point along the curved portion of its upper boundary—that is, along the upper boundary of the middle illustration—represents a usual (“closed”) socle element of \( k[y, x] \) (Definition 4.1), because such an element is annihilated by moving up in any direction, including straight vertically or horizontally, so it yields an injection \( k \rightarrow k[y, x] \) in the relevant multigraded degree. In contrast, the horizontal ray in the upper boundary represents a closed socle element of \( k[y, x] \) along the \( x \)-axis (Definition 4.15), because it (i) extends infinitely far to the right, so it yields an injection \( k[x, \text{axis}_+] \rightarrow k[y, x] \) but (ii) is annihilated upon moving upward in any direction, notably the vertical direction.

**Example 1.7.** In the right-hand illustration from Example 1.2, the module is 0 at any point along the antidiagonal upper boundary line segment. However, any such point can be approached within the interior of the triangle from below or from the left:

This open boundary point represents an element in the socle (Definition 4.5). Limits can be taken along any nonzero face of the cone \( \mathbb{R}^2_+ \) to reach this point, but the two minimal such faces, namely the positive \( x \)-axis and the positive \( y \)-axis, are the two nadirs of this cogenerator (Definition 4.29.3).

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\(^1\)This is appropriate to the English definition of *socle*: the base of a column.

\(^2\)It is an accident of history that in illustrations, socles lie along upper boundaries instead of along lower boundaries at the bottoms of pictures, where tops quite unfortunately reside.
Example 1.8. As the upper boundary in Example 1.5 is closed, its has no open cogenerators: its socle is closed. So modify the interval there by omitting the horizontal ray and leaving the rest of the points as they are.

The missing horizontal boundary ray represents an element in the socle along the $x$-axis. In more detail, the ray yields a summand of the upper boundary (Definition 3.15) that contains a copy of $\mathbb{k}[x\text{-axis}_+]$. Localizing along the $x$-axis yields the translation-invariant version in the middle picture, which can be reached by vertical limits from the interior of the lower half-plane. The quotient-restriction (Definition 2.22), thought of either as modding out by horizontal translation or by restricting to a vertical slice, yields a situation analogous to Example 1.7. The nadir is the face of $\mathbb{R}^2_+$ along the $y$-axis.

The socle along a face $\tau$ of the exponent cone is a module over the quotient $Q/\mathbb{R}\tau$ of the ambient real vector space $Q$ modulo its subspace generated by $\tau$, via quotient-restriction (Definition 2.22). That aspect is not novel to real exponents; it is the analogue of the socle $\text{Hom}_Q(R/p, M)_p$ at a prime $p$ of positive dimension in a ring $R$ being a module over the local ring $R_p$. What is novel, however, is the set of nadirs, as in the examples. A nadir of positive dimension indicates an “open” cogenerator of $M$, which is not an element of $M$ but an element in its upper boundary $\delta M$ (Definition 3.15). The upper boundary is a module not merely over a real-exponent polynomial ring, but also over the face poset of the exponent cone. That is the crucial ingredient entailed by real exponents: every point of the grading group $Q$ gets replaced by an infinitesimal copy of the face poset of the exponent cone.

This treatment of socles highlights the price to pay for real exponents. First, generalizing standard constructions from noetherian commutative algebra demands care. Notably, for instance, localization fails to commute with Hom, materially complicating proofs; see Remark 4.22, which explains how this failure to commute is not an artifact of the proofs but rather an intrinsic facet of the real-exponent theory. Second, socles of modules over real exponents are not submodules, but instead are functorially manufactured from auxiliary modules derived from $M$, namely upper boundary modules $\delta M$. Honest submodules must be reconstructed from cogenerators (this is done in Section 8). Finally, it would have been nice to develop module theory over real exponents entirely within the language of monoid algebras, but the infinitesimal structure of real-exponent polynomial rings is unavoidably poset-theoretic in nature, the poset being the face lattice of the positive cone of exponents. Hence this paper is phrased in terms of modules over posets [Mil20a, Mil20b], whose theory is reviewed in Section 2.
The geometry and combinatorics that rules the construction of socles, especially the entrance of face posets, is that of downsets and their boundaries, otherwise known as staircases [Mil02, §2]. These are rich objects at the interface of geometry, algebra, and combinatorics with connections to other areas of mathematics and science; see [Oko16], for example, where they are Ising crystals at zero temperature, or [BP19], where the modules called “ephemeral” in topological data analysis are those with no upper boundary along the interior of the exponent cone. The functorial viewpoint on staircases in Section 3, particularly

- the upper boundary functors (Definition 3.15),
- what it means to divide an upper boundary element (Definition 3.19), and
- computations of downset upper boundaries (Lemma 3.20 and Proposition 3.21)

translate topological limits in partially ordered real vector spaces into algebraic colimits on modules. Taking ordinary socles of the upper boundary of a module $M$, to get a module over the real exponent ring and the face poset of the exponent cone, completes the detection of missing boundary points that the ordinary socle of $M$ itself misses.

Section 5 assures that functorial constructions surrounding socles preserve additional semialgebraic or piecewise linear structure when they are present in the input. To wit,

- Theorem 5.2 says that left-exact functors with predictable actions on quotients modulo monomial ideals preserve additional geometric structure, and
- Theorem 5.13 verifies the hypotheses to draw this conclusion for the cogenerator functors, which take socles.

This conclusion is reasonable, because socles take each downset to a well behaved subset of its boundary (Lemma 3.20 and Proposition 3.21). Preserving additional geometric structure is particularly crucial for algorithms: it is hopeless for a computer to manipulate an arbitrary real-exponent monomial ideal, for its staircase could be missing a Cantor set or some more arbitrary, unfathomable antichain. Multigraded modules that arise in practice—such as from persistent homology—come from finite, computable procedures. Constraints from linear inequalities or comparisons among (squared) distances between points or other simple geometric objects yield PL or semialgebraic modules.

The entire theory for tame modules over real exponents has a simpler analogue over affine semigroup rings. It is barely new, being based on more elementary foundations, but it is worth phrasing precisely and collecting the results for the record (Section 10).

The dual discrete theory surrounding generator functors is interspersed with the corresponding real-exponent theory of tops in Sections 11–12, which is Matlis duality applied to earlier sections; see especially

- Theorem 11.31 on socle and top duality over real-exponent polynomials, and
- Theorem 11.22 on closed socle and top duality over any partially ordered group.

To locate results for affine semigroup rings, look for results over arbitrary partially ordered groups, such as Theorem 11.22, or look for the keywords “discrete polyhedral
group”; these indicate the same context as “affine semigroup” (Example 2.3) but refer to modules over posets, which is the language adopted (of necessity) for real exponents.

The paper closes with its final major results, in a discussion (Section 13) of

- minimal presentations, including downset, upset, and fringe presentations; and
- resolutions, including conjectures about minimal lengths of resolutions as well as lengths of minimal resolutions along the lines of the Hilbert syzygy theorem.

2. Algebra over partially ordered groups

The algebra surrounding localization, support, primary decomposition, and Matlis duality works over a broad class of partially ordered abelian groups with finitely many faces, appropriately defined [Mil20b], and sometimes in more generality. The main settings of this paper restrict primarily to the continuous case of real vector spaces but also secondarily the discrete case of finitely generated free abelian groups. Nonetheless, since some of the surrounding algebra works for all partially ordered abelian groups, this section reviews the basic setup, always indicating the allowed generality. For reference, the definitions and claims in Sections 2.1, 2.2, and the start of 2.3 are taken from [Mil20a, §2, §3.1, §4.1, §4.5] and [Mil20b, §5]; those in Section 2.4 are taken from [Mil20b, §4, §5]. The expositions in Section 2.5 and the remainder of Section 2.3 do not review specific sources, as their levels of generality are likely new, though they build on indicated well known material in straightforward ways.

2.1. Real and discrete polyhedral groups.

Definition 2.1. Let $Q$ be a partially ordered set (poset) and $\preceq$ its partial order. A module over $Q$ (or a $Q$-module) is

- a $Q$-graded vector space $M = \bigoplus_{q \in Q} M_q$ with
- a homomorphism $M_q \to M_{q'}$ whenever $q \preceq q'$ in $Q$ such that
- $M_q \to M_{q''}$ equals the composite $M_q \to M_{q'} \to M_{q''}$ whenever $q \preceq q' \preceq q''$.

A homomorphism $M \to N$ of $Q$-modules is a degree-preserving linear map, or equivalently a collection of vector space homomorphisms $M_q \to N_q$, that commute with the structure homomorphisms $M_q \to M_{q'}$ and $N_q \to N_{q'}$.

The posets of interest in the paper are the following, primarily the real case in Example 2.3, although some results are naturally stated in the generality of Definition 2.2.

Definition 2.2. An abelian group $Q$ is partially ordered if it is generated by a submonoid $Q_+$, called the positive cone, that has trivial unit group. The partial order is:

$q \preceq q' \iff q' - q \in Q_+$.

Example 2.3. A real polyhedral group $Q$ is a real vector space of finite dimension partially ordered so that its positive cone $Q_+$ is an intersection of finitely many closed half-spaces. The notation $\mathbb{R}^n$ and especially $\mathbb{R}^n_+$ is reserved for the case where the positive cone is the nonnegative orthant, so the partial order is componentwise comparison.
Example 2.4. A discrete polyhedral group is a finitely generated free abelian group partially ordered so that its positive cone is a finitely generated submonoid. (Equivalently a discrete polyhedral group is the Grothendieck group of an affine semigroup with trivial unit group.) The notation $\mathbb{Z}^n$ is reserved for the special case where the positive cone is the nonnegative orthant $\mathbb{N}^n$, so the partial order is componentwise comparison.

Remark 2.5. Examples 2.3 and 2.4 are instances of the class of polyhedral partially ordered groups, introduced in [Mil20b, Definition], which have finitely many faces: submonoids of the positive cone whose complements are ideals of the positive cone.

Example 2.6. Fix a poset $Q$. The vector space $k[Q] = \bigoplus_{q \in Q} k$ that assigns $k$ to every point of $Q$ is a $Q$-module with identity maps on $k$. More generally,
1. an upset (also called a dual order ideal) $U \subseteq Q$, meaning a subset closed under going upward in $Q$ (so $U + Q_+ = U$, when $Q$ is a partially ordered group) determines an indicator submodule or upset module $k[U] \subseteq k[Q]$;
2. dually, a downset (also called an order ideal) $D \subseteq Q$, meaning a subset closed under going downward in $Q$ (so $D - Q_+ = D$, when $Q$ is a partially ordered group) determines an indicator quotient module or downset module $k[Q] \twoheadrightarrow k[D]$; and
3. an interval $I \subseteq Q$, meaning the intersection of an upset and a downset, determines an interval module $k[I]$ that is a subquotient of $k[Q]$: if $I = U \cap D$ then $k[I] \hookrightarrow k[D]$ and $k[U] \twoheadrightarrow k[I]$, since $I$ is an upset in $D$ and a downset in $U$.

Remark 2.7. For polyhedral groups the language of $Q$-modules is equivalent to that of $Q$-graded $k[Q_+]$-modules. Indeed, a module over any partially ordered abelian group $Q$ is the same thing as a $Q$-graded module over the monoid algebra $k[Q_+]$ of the positive cone [Mil20b, Lemma 2.6]. When $Q = \mathbb{Z}^n$ and $Q_+ = \mathbb{N}^n$, the relevant monoid algebra is the polynomial ring $k[\mathbb{N}^n] = k[\mathbf{x}]$, where $\mathbf{x} = x_1, \ldots, x_n$ is a sequence of $n$ variables. This is the classical case; see [MS05, §8.1], for instance. When $Q = \mathbb{R}^n$ and $Q_+ = \mathbb{R}_+^n$, the relevant monoid algebra is the real-exponent polynomial ring $k[\mathbb{R}_+^n]$, whose elements are polynomials in $\mathbf{x} = x_1, \ldots, x_n$ with exponents that are nonnegative real numbers.

2.2. Tame modules and morphisms.

Definition 2.8. A constant subdivision of a poset $Q$ subordinate to a $Q$-module $M$ is a partition of $Q$ into constant regions such that for each constant region $I$ there is a single vector space $M_I$ with an isomorphism $M_I \rightarrow M_i$ for all $i \in I$ that has no monodromy: if $J$ is some (perhaps different) constant region, then all comparable pairs $i \preceq j$ with $i \in I$ and $j \in J$ induce the same composite homomorphism $M_I \rightarrow M_i \rightarrow M_j \rightarrow M_J$.

Definition 2.9. Fix a poset $Q$ and a $Q$-module $M$.
1. A constant subdivision of $Q$ is finite if it has finitely many constant regions.
2. The $Q$-module $M$ is $Q$-finite if its components $M_q$ have finite dimension over $k$.
3. The $Q$-module $M$ is tame if it is $Q$-finite and $Q$ admits a finite constant subdivision subordinate to $M$. 

Definition 2.10. Fix a subposet $Q$ of a partially ordered real vector space (e.g., a real polyhedral group). A partition of $Q$ into subsets is

1. semialgebraic if the subsets are real semialgebraic varieties;
2. piecewise linear (PL) if the subsets are finite unions of convex polyhedra, where a convex polyhedron is an intersection of finitely many closed or open half-spaces.

A module over $Q$ is semialgebraic or PL if $Q_+$ is and the module is tamed by a subordinate finite constant subdivision of the corresponding type.

Definition 2.11. Fix a poset $Q$. An encoding of a $Q$-module $M$ by a poset $P$ is a poset morphism $\pi : Q \to P$ together with a $P$-module $H$ such that $M \cong \pi^*H = \bigoplus_{q \in Q} H_{\pi(q)}$, the pullback of $H$ along $\pi$, which is naturally a $Q$-module. The encoding is finite if

1. the poset $P$ is finite, and
2. the vector space $H_p$ has finite dimension for all $p \in P$.

Definition 2.12. Fix a poset $Q$ and a $Q$-module $M$.

1. A poset morphism $\pi : Q \to P$ or an encoding of a $Q$-module (perhaps different from $M$) is subordinate to $M$ if there is a $P$-module $H$ such that $M \cong \pi^*H$.
2. When $Q$ is a subposet of a partially ordered real vector space, an encoding of $M$ is semialgebraic or PL if the constant subdivision of $Q$ formed by the fibers of $\pi$ [Mil20a, Theorem 4.22] is of the corresponding type (Definition 2.10).

Definition 2.13. A homomorphism $\varphi : M \to N$ of $Q$-modules is tame if $Q$ admits a finite constant subdivision subordinate to both $M$ and $N$ such that for each constant region $I$ the composite isomorphism $M_I \to M_i \to N_i \to N_I$ does not depend on $i \in I$.

1. This constant subdivision is subordinate to the morphism $\varphi$.
2. The morphism $\varphi$ dominates a constant subdivision or poset encoding if the subdivision or encoding is subordinate to $\varphi$.
3. The morphism $\varphi$ is semialgebraic or PL if it dominates a constant subdivision of the corresponding type.

Definition 2.14. The category of tame modules is the subcategory of $Q$-modules whose objects are the tame modules and whose morphisms are the tame homomorphisms. The subcategories of semialgebraic modules and PL modules have the correspondingly restricted objects and tame morphisms.

Proposition 2.15 ([Mil20a, Proposition 4.31]). Over any poset $Q$, the category of tame $Q$-modules is abelian. If $Q$ is a subposet of a partially ordered real vector space, then the categories of semialgebraic and PL modules are abelian.

Since socles involve essential submodules, which are divorced from generators, the theory can often get by with less than tameness, which requires finite upset covers and downset hulls. In the pictures, only phenomena near upper boundaries matter for socles, not anything near lower boundaries; see [Mil20b, Remark 5.6] for discussion.
Definition 2.16. A downset hull of a module $M$ over an arbitrary poset is an injection $M \hookrightarrow \bigoplus_{j \in J} E_j$ with each $E_j$ being a downset module. The hull is finite if $J$ is finite. The module $M$ is downset-finite if it admits a finite downset hull.

2.3. Localization and restriction.

Definition 2.17. Fix a face $\tau$ of a partially ordered group $Q$. The localization of a $Q$-module $M$ along $\tau$ is the tensor product $M_{\tau} = M \otimes_{k[Q+]} k[Q_+/\mathbb{Z}\tau]$, viewing $M$ as a $Q$-graded $k[Q_+]$-module.

Example 2.18. The localization $D_{\tau}$ of a downset $D$ is the downset with $k[D]_{\tau} = k[D_{\tau}]$.

The remainder of Section 2.3 is new in this generality, although $\mathbb{Z}^n$-graded versions date back to [Mil98, §4] and [Mil00, §3.6]

Definition 2.19. For a partially ordered abelian group $Q$ and a face $\tau$ of $Q_+$, write $Q/\mathbb{Z}\tau$ for the quotient of $Q$ modulo the subgroup generated by $\tau$. If $Q$ is a real polyhedral group then write $Q/\mathbb{R}\tau = Q/\mathbb{Z}\tau$.

Remark 2.20. The image $Q_+/\mathbb{Z}\tau$ of $Q_+$ in $Q/\mathbb{Z}\tau$ is a submonoid that generates $Q/\mathbb{Z}\tau$, but $Q_+/\mathbb{Z}\tau$ can have units, so $Q/\mathbb{Z}\tau$ need not be partially ordered in a natural way. However, if $Q$ is a real polyhedral group then the group of units (lineality space) of the cone $Q_+ + \mathbb{R}\tau$ is just $\mathbb{R}\tau$ itself, because $Q_+$ is pointed, so $Q/\mathbb{R}\tau$ is a real polyhedral group whose positive cone $(Q/\mathbb{R}\tau)_+ = Q_+/\mathbb{R}\tau$ is the image of $Q_+$. Similar reasoning applies to the intersection of the real polyhedral situation with any subgroup of $Q$; this includes the case of normal affine semigroups, where the subgroup of $Q$ is discrete.

Lemma 2.21. The subgroup $\mathbb{Z}\tau \subseteq Q$ of a partially ordered group $Q$ acts freely on the localization $M_{\tau}$ of any $Q$-module $M$ along a face $\tau$. Consequently, if $I_{\tau} \subseteq k[Q_+]$ is the augmentation ideal $\langle m - 1 \mid m \in k[\tau] \text{ is a monomial} \rangle$, then the $Q/\mathbb{Z}\tau$-graded module $M/\tau = M/I_{\tau}M$ over the monoid algebra $k[Q_+/\mathbb{Z}\tau]$ satisfies

$$M_{\tau} \cong \bigoplus_{a} (M/\tau)_{\hat{a}}.$$

Proof. The monomials of $k[Q_+]$ corresponding to elements of $\tau$ are units on $M_{\tau}$ acting as translations along $\tau$. Since the augmentation ideal sets every monomial equal to 1, the quotient $M \rightarrow M/\tau$ factors through $M_{\tau}$. □

Definition 2.22. The $k[Q/\mathbb{Z}\tau]$-module $M/\tau$ in Lemma 2.21 is the quotient-restriction of $M$ along $\tau$. 
Remark 2.23. Over (any subgroup of) a real polyhedral group $Q$, the functor $M_{\tau} \mapsto M/\tau$ has a “section” $M/\tau \mapsto M_{\tau}|_{\tau^\perp}$, where $N_{\tau^\perp} = \bigoplus_{a \in \tau^\perp} N_a$ is the restriction of $N$ to any linear subspace $\tau^\perp$ complementary to $\mathbb{R}\tau$. (When $Q_+ = \mathbb{R}_+^n$, a complement is canonically spanned by the face orthogonal to $\tau$, or equivalently, the unique maximal face of $\mathbb{R}_+^n$ intersecting $\tau$ trivially.) The restriction is a module over the real polyhedral group $\tau^\perp$ with positive cone $(Q_+ + \mathbb{R}\tau) \cap \mathbb{R}\tau^\perp$, which projects isomorphically to the positive cone of $Q/\mathbb{R}\tau$. Thus the quotient-restriction is both a quotient and a restriction of $M_{\tau}$. While a section can exist over polyhedral partially ordered groups that are not real, it need not. For example, when $Q = \mathbb{Z}_2$ and the columns of $\begin{bmatrix} 2 & 1 & 0 \\ 0 & 1 & 1 \end{bmatrix}$ generate $Q_+$, taking $\tau = \langle \begin{bmatrix} 0 \\ 2 \end{bmatrix} \rangle$ to be the face along the $x$-axis yields a quotient monoid $Q_+/\mathbb{Z}_2 \cong \mathbb{Z}/2\mathbb{Z} \oplus \mathbb{N}$ with torsion, preventing $k[Q_+]_{\tau} \mapsto k[Q_+]_{\tau}/\tau$ from having a section to any category of modules over a subgroup of $Q$.

Lemma 2.24. The quotient-restriction functors $M \mapsto M/\tau$ are exact.

Proof. Localizing along $\tau$ is exact because the localization $k[Q_+ + \mathbb{Z}\tau]$ of $k[Q_+]$ is flat as a $k[Q_+]$-module. The exactness of the functor that takes each $k[Q_+/\mathbb{Z}\tau]$-module $M_{\tau}$ to $M/\tau$ can be checked on each $Q/\mathbb{Z}\tau$-degree individually. □

2.4. Support and primary decomposition.

Definition 2.25. Fix a face $\tau$ of a partially ordered group $Q$. The submodule of $M$ globally supported on $\tau$ is

$$\Gamma_\tau M = \bigcap_{\tau' \notin \tau} (\ker(M \to M_{\tau'})) = \ker(M \to \prod_{\tau' \notin \tau} M_{\tau'}).$$

Fix a $Q$-module $M$ for a polyhedral partially ordered group $Q$. The local $\tau$-support of $M$ is the module $\Gamma_\tau M_{\tau}$ of elements globally supported on $\tau$ in the localization $M_{\tau}$, or equivalently [Mil20b, Proposition 4.6] the localization along $\tau$ of the submodule of $M$ globally supported on $\tau$.

Example 2.26. The global supports of the interval module for the interval in $\mathbb{R}^2$ on the left-hand side of this display (the same interval as in Example 1.5) are the interval modules for the intervals on the right-hand side, each labeled by the face $\tau$ on which the support is taken. Extending the first two of these global supports modules downward,
so they become quotients instead of submodules of the original interval module, yields the canonical primary components in Example 1.5.

**Definition 2.27.** A module $M$ over a polyhedral partially ordered group is *coprimary* if for some face $\tau$, the localization map $M \hookrightarrow M_\tau$ is injective and $\Gamma_\tau M_\tau$ is an essential submodule of $M_\tau$, meaning every nonzero submodule of $M_\tau$ intersects $\Gamma_\tau M_\tau$ nontrivially.

**Definition 2.28.** Fix a face $\tau$ of the positive cone $Q_+$ in a partially ordered group $Q$. A homogeneous element $y \in M_q$ in a $Q$-module $M$ is

1. $\tau$-*persistent* if it has nonzero image in $M_{q'}$ for all $q' \in q + \tau$;
2. $\overline{\tau}$-*transient* if, for each $f \in Q_+ \setminus \tau$, the image of $y$ vanishes in $M_{q'}$ whenever $q' = q + \lambda f$ for $\lambda \gg 0$;
3. $\tau$-*coprimary* if it is $\tau$-persistent and $\overline{\tau}$-transient.

**Proposition 2.29** ([Mil20b, Theorem 4.13]). Fix a face $\tau$ of the positive cone $Q_+$ in a real or discrete polyhedral partially ordered group $Q$. A $Q$-module $M$ is $\tau$-coprimary if and only if every homogeneous element divides a $\tau$-coprimary element, where $y \in M_q$ divides $y' \in M_{q'}$ if $q \preceq q'$ and $y$ has image $y'$ under the structure morphism $M_q \to M_{q'}$.

**Definition 2.30.** Fix a $Q$-module $M$ over a polyhedral partially ordered group $Q$. A primary decomposition of $M$ is an injection $M \hookrightarrow \bigoplus_{i=1}^r M/M_i$ into a direct sum of coprimary quotients $M/M_i$, called components of the decomposition.

**Example 2.31.** When $M = k[I]$ in Definition 2.30 is an interval module, a primary decomposition $k[I] \hookrightarrow \bigoplus_{i=1}^r k[I_i]$ may also be expressed as a primary decomposition of $I$ itself: $I = \bigcup I_i$, where each $I_i$ is a coprimary interval in $I$. That is, $I_i$ is an interval in $Q$ but a downset in the subposet $I$. See Example 1.5, where the interval on the left side is the union of the two intervals on the right side.

2.5. Matlis duality.

**Definition 2.32.** A poset $Q$ is *self-dual* if it is given a poset isomorphism $Q \cong Q^{op}$ with its opposite poset. On elements denote this isomorphism by $q \mapsto -q$.

**Example 2.33.** Inversion makes partially ordered abelian groups self-dual as posets.

**Definition 2.34.** Fix a poset $Q$ with opposite poset $Q^{op}$. The Matlis dual of a $Q$-module $M$ is the $Q^{op}$-module $M^\vee$ defined by $(M^\vee)_q = \text{Hom}_k(M_q, k)$. When $Q \cong Q^{op}$ is a self-duality, then

$$(M^\vee)_q = \text{Hom}_k(M_{-q}, k),$$

so the homomorphism $(M^\vee)_q \to (M^\vee)_{q'}$ is transpose to $M_{-q'} \to M_{-q}$. 

Example 2.35. The Matlis dual over a partially ordered group $Q$ is equivalently
\[ M^\vee = \text{Hom}_Q(M, k[Q+]^\vee) \]
where $\text{Hom}_Q(M, N) = \bigoplus_{q \in Q} \text{Hom}_Q(M, N(q))$ is the direct sum of all degree-preserving homomorphisms from $M$ to $Q$-graded translates of $N$, i.e., $N(q)_a = N_{a+q}$. This is proved using the adjunction between $\text{Hom}$ and $\otimes$; see [MS05, Lemma 11.16], noting that the nature of the grading group is immaterial. And as in [MS05, Lemma 11.16],
\[ \text{Hom}_Q(M, N^\vee) = (M \otimes_Q N)^\vee. \]

Example 2.36. It is instructive to compute the Matlis dual of localization along a face $\tau$ over a partially ordered abelian group: the Matlis dual of $M_{\tau}$ is
\[ (M_{\tau})^\vee = \text{Hom}(k[Q_+]_{\tau} \otimes M, k) = \text{Hom}(k[Q_+]_{\tau}, \text{Hom}(M, k)) = \text{Hom}(k[Q_+]_{\tau}, M^\vee), \]
the module of homomorphisms from a localization of $k[Q+]$ into $M^\vee$. The unfamiliarity of this functor is one of the reasons for developing most of the theory in this paper in terms of socles and cogenerators instead of tops and generators.

Lemma 2.37. $(M^\vee)^\vee$ is canonically isomorphic to $M$ if $M$ is $Q$-finite (Def. 2.9.2). \hfill \Box

Remark 2.38. Every $Q$-finite injective module over a discrete polyhedral group $Q$ is, by [MS05, Theorem 11.30], isomorphic to a direct sum of downset modules $k[D]$ for downsets of the form $D = a + \tau - Q_+$, said to be cogenerated by $a$ along the face $\tau$. Taking Matlis duals, every $Q$-finite flat module over a discrete polyhedral group $Q$ is isomorphic to a direct sum of upset modules $k[U]$ for upsets of the form $U = b + Z\tau + Q_+$. These upset modules are the graded translates of localizations of $k[Q+]$ along faces.

Lemma 2.39. $\text{Hom}(k[Q_+]_{\tau}, (-)^\vee)$ is exact for all faces $\tau$ of any partially ordered group $Q$. Consequently, $\text{Hom}(k[Q_+]_{\tau}, -)$ is exact on the category of $Q$-finite modules.

Proof. Localization is exact and so is Matlis duality, so the first sentence follows from Example 2.36. The consequence comes from Lemma 2.37: for $Q$-finite modules, $\text{Hom}(k[Q_+]_{\tau}, -)$ is the composite $(-)^\vee$ followed by $\text{Hom}(k[Q_+]_{\tau}, (-)^\vee)$. \hfill \Box

Remark 2.40. What really drives the lemma is the observation that while the opposite notion to injective is projective (reverse all of the arrows in the definition), the adjoint notion to injective is flat. That is, a module is flat if and only if its Matlis dual is injective. This is an instance of a rather general phenomenon that can be phrased in terms of a monoidal abelian category $\mathcal{C}$ possessing a Matlis object $E$ for a Matlis dual pair of subcategories $\mathcal{A}$ and $\mathcal{B}$ such that $\text{Hom}(-, E)$ restricts to exact contravariant functors $\mathcal{A} \to \mathcal{B}$ and $\mathcal{B} \to \mathcal{A}$ that are inverse to one another. The idea is to set $M^\vee = \text{Hom}(M, E)$, the Matlis dual of any object $M$ of $\mathcal{C}$, and define an object of $\mathcal{C}$ to be $\mathcal{B}$-flat if $F \otimes -$ is an exact functor on $\mathcal{B}$. Then an object $F$ of $\mathcal{A}$ is $\mathcal{B}$-flat if and only if $\text{Hom}(F, -)$ is an exact functor on $\mathcal{A}$. Examples of this situation include artinian and
noetherian modules over a complete local ring; modules of finite length over any local ring (in both cases $E = E(R/m)$ is the injective hull of the residue field); and of course $Q$-finite modules over a partially ordered abelian group $Q$. The latter two examples feature a single Matlis self-dual subcategory.

**Example 2.41.** It is important to note that $\text{Hom}(k[Q_+], -)$ is not exact on the category of all—that is, not necessarily $Q$-finite—modules over a partially ordered group $Q$. Indeed, let $F \to k[Q]$ be any free cover of the localization of $k[Q_+]$ along the maximal face. (When $Q_+ = \mathbb{N}$, this writes the module $k[\mathbb{Z}]$ of Laurent polynomials as a quotient of a graded free module $F$ over the ordinary polynomial ring $k[\mathbb{N}]$ in one variable.) Then $k[Q] = k[Q_+]_\tau$ for $\tau = Q$ itself, and applying $\text{Hom}(k[Q], -)$ to the surjection $F \to k[Q]$ yields the homomorphism $0 \to k[Q]$, which is not surjective.

### 3. Geometry of real staircases

The difference between ordinary noetherian commutative algebra and algebra over real polyhedral groups begins with the local geometry of downsets near their boundaries (Section 3.1) and the functorial view of this geometry (Section 3.2).

#### 3.1. Tangent cones of downsets.

**Definition 3.1.** The tangent cone $T_aD$ of a downset $D$ in a real polyhedral group $Q$ (Example 2.3) at a point $a \in Q$ is the set of vectors $v \in -Q_+$ such that $a + \varepsilon v \in D$ for all sufficiently small (hence all) $\varepsilon > 0$.

**Remark 3.2.** Since the real number $\varepsilon$ in Definition 3.1 is strictly positive, the vector $v = 0$ lies in $T_aD$ if and only if $a$ itself lies in $D$, and in that case $T_aD = -Q_+$.

**Example 3.3.** The tangent cone defined here is not the tangent cone of $D$ as a stratified space, because the cone here only considers vectors in $-Q_+$. A specific simple example to see the difference is the closed half-plane $D$ beneath the line $y = -x$ in $\mathbb{R}^2$, where the usual tangent cone at any point along the boundary line is the half-plane, whereas $T_aD = -\mathbb{R}^2_+$. Furthermore, $T_aD$ can be nonempty for a point $a$ in the boundary of $D$ even if $a$ does not lie in $D$ itself. For an example of that, take $D^o$ to be the interior of $D$; then $T_aD^o = -\mathbb{R}^2_+ \setminus \{0\}$ for any $a$ on the boundary line.

The most important conclusion concerning tangent cones at points of downsets, Proposition 3.10, says that such cones are certain unions of relative interiors of faces. Some definitions and preliminary results are required.

**Definition 3.4.** Fix a real polyhedral group $Q$.

1. For any face $\sigma$ of the positive cone $Q_+$, write $\sigma^o$ for the relative interior of $\sigma$.
2. For any set $\nabla$ of faces of $Q_+$, write $Q_\nabla = \bigcup_{\sigma \in \nabla} \sigma^o$, the cone of shape $\nabla$.
3. A cocomplex in $Q_+$ is an upset in the face poset $\mathcal{F}_Q$ of $Q_+$, where $\sigma \preceq \tau$ if $\sigma \subseteq \tau$. 


Example 3.5. The cocomplex $\nabla \sigma = \{ \text{faces } \sigma' \text{ of } Q_+ \mid \sigma' \supseteq \sigma \}$ is the open star of the face $\sigma$. It determines the cone $Q_{\nabla \sigma}$ of shape $\nabla \sigma$, which plays an important role.

Remark 3.6. The next proposition is the reason for specializing this section to real polyhedral groups instead of arbitrary polyhedral partially ordered groups, where limits might not be meaningful. For example, although limits make formal sense in the integer lattice $\mathbb{Z}^n$ with the usual discrete topology, it is impossible for a sequence of points in the relative interior of a face to converge to the origin of the face. This quantum separation has genuine finiteness consequences for the algebra of $\mathbb{Z}^n$-modules that usually do not hold for $\mathbb{R}^n$-modules.

Proposition 3.7. If $\{a_k\}_{k \in \mathbb{N}}$ is any sequence converging to $a$, then $\bigcup_{k=0}^{\infty} (a_k - Q_+) \supseteq a - Q_+$. If the sequence is contained in $a - \sigma^0$, then the union equals $a - Q_{\nabla \sigma}$.

Proof. For each point $b \in a - Q_+$, every linear function $\ell : \mathbb{R}^n \to \mathbb{R}$ that is nonnegative on $Q_+$ eventually takes values on the sequence that are bigger than $\ell(b)$; thus $b$ lies in the union as claimed. When the sequence is contained in $a - \sigma^0$, the union is contained in $a - \sigma^0 - Q_+$ by hypothesis, but the union contains $a - \sigma^0$ by the first claim applied with $\sigma$ in place of $Q_+$. The union therefore equals $a - \sigma^0 - Q_+$ because it is a downset.

The next lemma completes the proof.

Lemma 3.8. If $\sigma$ is any face of the positive cone $Q_+$ then $\sigma^0 + Q_+ = Q_{\nabla \sigma}$.

Proof. Fix $f + b \in \sigma^0 + Q_+$. If $\ell(f + b) = 0$ for some linear function $\ell : \mathbb{R}^n \to \mathbb{R}$ that is nonnegative on $Q_+$, then $\ell(f) = 0$, too. Therefore the support face of $f + b$ (the smallest face in which it lies) contains $\sigma$. On the other hand, suppose $b$ lies interior to some face of $Q_+$ that contains $\sigma$. Then pick any $f \in \sigma^0$. If $\ell(b) = 0$ then also $\ell(f) = 0$, because the support face of $b$ contains $\sigma$. But if $\ell(b) > 0$, then $\ell(f) > \ell(\varepsilon f)$ for any sufficiently small positive $\varepsilon$. As $Q_+$ is an intersection of only finitely many closed half-spaces, a single $\varepsilon$ works for all relevant $\ell$, and then $b = \varepsilon f + (b - \varepsilon f) \in \sigma^0 + Q_+$.

For $Q_+ = \mathbb{R}^n_+$ the following is essentially [MMc15, Lemma 5.1].

Corollary 3.9. If $D \subseteq Q$ is a downset in a real polyhedral group, then $a$ lies in the closure $\overline{D}$ if and only if $D$ contains the interior $a - Q_+^0$ of the negative cone with origin $a$.

Proposition 3.10. If $a \in \overline{D}$ for a downset $D$, then $T_a D = -Q_+$ is the negative cone of shape $\nabla$ for some cocomplex $\nabla$ in $Q_+$. In this case $\nabla' = \nabla_a^a$ is the shape of $D$ at $a$.

Proof. The result is true when $n = 1$ because there are only three possibilities for $a \in \mathbb{R}$: either $a \in D$, in which case $T_a D = -\mathbb{R} = Q_+$ for $\nabla = F_Q$ (Remark 3.2); or $a \notin D$ but $a$ lies in the closure of $D$, in which case $T_a D = Q_+$ for $\nabla = \{Q_+^0\} \subseteq F_Q$; or $a$ is separated from $D$ by a nonzero distance, in which case $T_a D = Q_+^0$ is empty.

Write $D_\sigma$ for the intersection of $D$ with the $a$-translate of the linear span of $\sigma$:

$$D_\sigma = D \cap (a + \mathbb{R} \sigma).$$
If \( \sigma \subseteq Q_+ \), then \( T_\sigma D_\sigma = \sigma \nabla \) for some upset \( \nabla \subseteq \mathcal{F}_\sigma \) by induction on the dimension of \( \sigma \). In actuality, only the case \( \dim \sigma = n - 1 \) is needed, as the face posets \( \mathcal{F}_\sigma \) for \( \dim \sigma = n - 1 \) almost cover \( \mathcal{F}_Q \): only the open maximal face \( Q^\circ \) itself lies outside of their union, and that case is dealt with by Corollary 3.9.

3.2. Upper boundary functors.

**Definition 3.11.** For a module \( M \) over a real polyhedral group \( Q \), a face \( \sigma \) of \( Q_+ \), and a degree \( a \in Q \), the upper boundary atop \( \sigma \) at \( a \) in \( M \) is the vector space

\[
(\delta^\sigma M)_a = M_{a-\sigma} = \lim_{a' \in a-\sigma^\circ} M_{a'}.
\]

**Lemma 3.12.** The functor \( M \mapsto \delta^\sigma M = \bigoplus_{a \in Q} (\delta^\sigma M)_a \) is exact.

**Proof.** Direct limits are exact in categories of vector spaces (or modules over rings). \( \square \)

**Lemma 3.13.** The structure homomorphisms of \( M \) as a \( Q \)-module induce natural homomorphisms \( M_{a-\sigma} \rightarrow M_{b-\tau} \) for \( a \preceq b \) in \( Q \) and faces \( \sigma \supseteq \tau \) of \( Q_+ \).

**Proof.** The natural homomorphisms come from the universal property of colimits. First a natural homomorphism \( M_{a-\sigma} \rightarrow M_{b-\tau} \) is induced by the composite homomorphisms \( M_c \rightarrow M_{c+b-a} \rightarrow M_{b-\tau} \) for \( c \in a - \sigma^\circ \) because adding \( b - a \) takes \( a - \sigma^\circ \) to \( b - \sigma^\circ \). For \( M_{b-\tau} \rightarrow M_{b-\tau} \) the argument is similar, except that existence of natural homomorphisms \( M_c \rightarrow M_{b-\tau} \) for \( c \in b - \sigma^\circ \) requires Proposition 3.7 and Lemma 3.8. \( \square \)

**Remark 3.14.** The face poset \( \mathcal{F}_Q \) of the positive cone \( Q_+ \) can be made into a commutative monoid in which faces \( \sigma \) and \( \tau \) of \( Q_+ \) have sum

\[
\sigma + \tau = \sigma \cap \tau.
\]

Indeed, these monoid axioms use only that \( (\mathcal{F}_Q, \cap) \) is a bounded meet semilattice, the monoid unit element being the maximal semilattice element—in this case, \( Q_+ \) itself. When \( \mathcal{F}_Q \) is considered as a monoid in this way, the partial order on it has \( \sigma \preceq \tau \) if \( \sigma \supseteq \tau \), which is the opposite of the default partial order on the faces of a polyhedral cone. For utmost clarity, and because both monoid partial orders are relevant (see Remark 11.3), \( \mathcal{F}_Q^{\text{op}} \) is written when this monoid partial order is intended.

**Definition 3.15.** Fix a module \( M \) over a real polyhedral group \( Q \) and a degree \( a \in Q \). The upper boundary functor takes \( M \) to the \( Q \times \mathcal{F}_Q^{\text{op}} \)-module \( \delta M \) whose fiber over \( a \in Q \) is the \( \mathcal{F}_Q^{\text{op}} \)-module

\[
(\delta M)_a = \bigoplus_{\sigma \in \mathcal{F}_Q} M_{a-\sigma} = \bigoplus_{\sigma \in \mathcal{F}_Q} (\delta^\sigma M)_a.
\]

The fiber of \( \delta M \) over \( \sigma \in \mathcal{F}_Q^{\text{op}} \) is the upper boundary \( \delta^\sigma M \) of \( M \) atop \( \sigma \).
**Example 3.16.** The upper boundary of the triangular module $M$ in Examples 1.2 and 1.7 has vector spaces of dimension either 0 or 1 in every graded component indexed by $\mathbb{R}_+^2 \times \mathcal{F}_{\mathbb{R}^2}^{\text{op}}$. For each $a \in \mathbb{R}^2$, depict the corresponding $\mathcal{F}_{\mathbb{R}^2}^{\text{op}}$-module using a solid dot for a vector space of dimension 1 and no solid dot for a vector space of dimension 0. Then $\delta M$ is drawn at left and $\mathcal{F}_{\mathbb{R}^2}^{\text{op}}$ is drawn at right. The $\mathcal{F}_{\mathbb{R}^2}^{\text{op}}$-module at $a$ is drawn with $a$ at the upper-right corner, to convey the idea that one should stand at $a$ and see what direct limits result as $a$ is approached along the various faces. When $M_a = 0$, the point $a$ is drawn as an empty dot.

**Remark 3.17.** Upper boundaries contain the later\(^3\) notion of ephemeral modules [BP19]: an $\mathbb{R}^n$ module $M$ is ephemeral if its upper boundary $\delta^\sigma M$ atop the interior $\sigma$ of $\mathbb{R}^n_+$ vanishes. This notion is key to the difference between poset module theory [Mil20a] and the formulation of persistent homology via constructible sheaves [KS18], as detailed in [Mil20c].

**Remark 3.18.** The face of $Q_+$ that contains only the origin 0 is an absorbing element: it acts like infinity, in the sense that $\sigma + \{0\} = \{0\}$ in the monoid $\mathcal{F}_{Q}^{\text{op}}$ for all faces $\sigma$. Adding the absorbing element 0 in the $\mathcal{F}_{Q}^{\text{op}}$ component therefore induces a natural $Q \times \mathcal{F}_{Q}^{\text{op}}$-module projection from the upper boundary $\delta M$ to $M$. At a degree $a \in Q$, this projection is $M_{a-\sigma} \to M_{a-0} = M_a$. Interestingly, the frontier of a downset $D$—those points in the topological closure but outside of $D$—is the set of nonzero degrees of a functor, namely $\ker(\delta^\sigma M \to M)$ for $\sigma = Q_+^\circ$. The proof is by Corollary 3.9.

There is no natural map $M \to \delta^\sigma M$ when $\sigma \neq \{0\}$ has positive dimension, because an element of degree $a$ in $M$ comes from elements of $\delta^\sigma M$ in degrees less than $a$. However, that leaves a way for Lemma 3.13 to afford a notion of divisibility of upper boundary elements by elements of $M$.

**Definition 3.19.** An element $y \in M_b$ divides $x \in (\delta^\sigma M)_a$ if $b \in a - Q_{\chi_\sigma} = a - \sigma^\circ - Q_+$ (Lemma 3.8) and $y \mapsto x$ under the natural map $M_b \to M_{a-\sigma}$ (Lemma 3.13). The element $y$ is said to $\sigma$-divide $x$ if, more restrictively, $b \in a - \sigma^\circ$.

---

\(^3\)Upper boundary functors were introduced in [Mil17].
Now come the fundamental calculations of upper boundary functors, in the next lemma and proposition, that drive all of the results in the rest of the paper.

**Lemma 3.20.** If \( \sigma \in \mathcal{F}_Q \) and \( D \) is a downset in \( Q \) then \( \delta^\circ k[D] = k[\delta^\circ D] \), where

\[
\delta^\circ D = \bigcup_{x \in Q} D \cap (x + R\sigma) = D \cup \bigcup_{x \in \partial D} D \cap (x + R\sigma)
\]

It suffices to take the middle union over \( x \) in any subspace complement to \( R\sigma \).

**Proof.** For the second displayed equality, observe that the middle union contains the right-hand union because the middle one contains \( D \). For the other containment, if \( x + R\sigma \) contains no boundary point of \( D \), then \( D \cap (x + R\sigma) = \overline{D} \cap (x + R\sigma) \) is already closed, so the contribution of \( D \cap (x + R\sigma) \) to the middle union is contained in \( D \).

For the other equality, \( \delta^\circ k[D] \) is nonzero in degree \( a \) if and only if \( a - \sigma^\circ \subseteq D \). That condition is equivalent to \( a - \sigma^\circ \subseteq D \cap (a + R\sigma) \) because \( a - \sigma^\circ \subseteq a + R\sigma \). Translating \( D \cap (a + R\sigma) \) back by \( a \) yields a downset in the real polyhedral group \( R\sigma \), with \((R\sigma)_+ = \sigma\), thereby reducing to Corollary 3.9. \( \square \)

**Proposition 3.21.** If \( \sigma \in \mathcal{F}_Q \) and \( D \) is a downset in a real polyhedral group then \( \delta^\circ k[D] = k[\delta^\circ D] \) is the indicator quotient for a downset \( \delta^\circ D \) that satisfies

1. \( D \subseteq \delta^\circ D \subseteq \overline{D} \) and
2. \( \delta^\circ D = \{ a \in \overline{D} \mid \sigma \in \nabla^a_D \} \).

**Proof.** Item 1 follows from item 2. What remains to show is that \( \delta^\circ D \) is a downset in \( \overline{D} \) characterized by item 2 and that it is semialgebraic if \( D \) is.

First, \( \sigma \in \nabla^a_D \) means that \( a - \sigma^\circ \subseteq D \), which immediately implies that \( a \in \delta^\circ D \) by Lemma 3.20. Conversely, suppose \( a \in \delta^\circ D \). Lemma 3.20 and Corollary 3.9, the latter applied to the downset \(-a + D \cap (a + R\sigma) \) in \( R\sigma \), imply that \( a - \sigma^\circ \subseteq D \), and hence \( \sigma \in \nabla^a_D \) by definition, proving item 2. Given that \( a - \sigma^\circ \subseteq D \), Proposition 3.7 yields \( D \cap (a - Q_+) \supseteq a - Q_{\nabla^a} \). Consequently, if \( b \in a - Q_+ \) then \( D \cap (b - \sigma) \supseteq b - \sigma^\circ \), whence \( b \in \delta^\circ D \). Thus \( \delta^\circ D \) is a downset. \( \square \)

### 4. Socles and cogenerators

The main contribution of this section is Definition 4.29, which introduces the notions of cogenerator functor and socle along a face with a given nadir. Its concomitant foundations include ways to decompose the cogenerator functors into continuous and discrete parts (Proposition 4.38), interactions with localization (Proposition 4.40), and left-exactness (Proposition 4.44), along with the crucial calculation of socles in the simplest case, namely the indicator module of a single face (Example 4.43). The theory is built step by step, starting with ordinary (closed) socles over arbitrary posets in Section 4.1 and proceeding through Section 4.4 to cogenerator functors and socles with increasing levels of complexity and (necessarily) decreasing freedom regarding the poset.
4.1. Closed socles and closed cogenerator functors.

In commutative algebra, the socle of a module over a local ring is the set of elements annihilated by the maximal ideal. These elements form a vector space over the residue field \( k \) that can alternately be characterized by taking homomorphisms from \( k \) into the module. Either characterization works for modules over partially ordered groups, but only the latter generalizes readily to modules over arbitrary posets. Note that socles over face posets of polyhedra occur naturally in the theory over real polyhedral groups.

**Definition 4.1.** Fix an arbitrary poset \( P \). The *skyscraper* \( P \)-module \( k_p \) at \( p \in P \) has \( k \) in degree \( p \) and 0 in all other degrees. The *closed cogenerator functor* \( \text{Hom}_P(k, -) \) takes each \( P \)-module \( M \) to its *closed socle*: the \( P \)-submodule

\[
\text{soc} M = \text{Hom}_P(k, M) = \bigoplus_{p \in P} \text{Hom}_P(k_p, M).
\]

When it is important to specify the poset, the notation \( P \text{-soc} \) is used instead of \( \text{soc} \). A *closed cogenerator* of degree \( p \in P \) is a nonzero element in \( (\text{soc} M)_p \).

**Remark 4.2.** The bar over “soc” is meant to evoke the notion of closure or “closed”.

The bar under Hom is the usual one in multigraded commutative algebra for the direct sum of homogeneous homomorphisms of all degrees (see [GW78, Section I.2] or [MS05, Definition 11.14], for example).

**Example 4.3.** The closed socle of \( M \) consists of the elements that are annihilated by moving up in any direction, or that have maximal degree. In particular, the interval module \( k[I] \) for any interval \( I \subseteq P \) (Example 2.6.3) has closed socle

\[
\text{soc} k[I] = k[\text{max} I],
\]

the interval module supported on the set of elements of \( I \) that are maximal in \( I \).

**Lemma 4.4.** The closed cogenerator functor over any poset is left-exact.

*Proof.* A \( P \)-module is the same thing as a module over the path algebra of (the Hasse diagram of) \( P \) with relations to impose commutativity, namely equality of the morphism induced by \( p < p'' \) and the composite morphism induced by \( p < p' < p'' \). A \( P \)-module is also the same as a sheaf on \( P \) in which the topology comprises the upsets of \( P \). (See [Yuz81] as well as [Cur14, §4.2] and [Cur19] for discussions of these and references.) The purpose of viewing things this way is merely to demonstrate that the category of \( P \)-modules is abelian, so Hom from a fixed source is automatically left-exact. \( \square \)
4.2. Socles and cogenerator functors.

**Definition 4.5.** The cogenerator functor takes a module over a real polyhedral group to its socle:

\[ \text{soc}_0 M = \overline{\text{soc}} \delta M, \]

which is the closed socle, computed over the poset \( Q \times \mathcal{F}\text{op}_Q \) (see Remark 3.14) of the upper boundary module of \( M \) (Definition 3.15).

**Remark 4.6.** Notationally, the lack of a bar over “soc_0” serves as a visual cue that the functor is over a real polyhedral group, as the upper boundary \( \delta \) is not defined in more generality. This visual cue persists throughout the more general notions of socle: bar means arbitrary poset, and no bar means real polyhedral group. The subscript “0” refers to the minimal face of the positive cone of the real polyhedral group; to be precise, it is the \( \tau = 0 \) special case of Definition 4.29.

**Lemma 4.7.** The cogenerator functor \( M \mapsto \text{soc}_0 M \) is left-exact.

**Proof.** Use exactness of upper boundaries atop \( \sigma \) (Lemma 3.12), exactness of the direct sums forming \( \delta M \) from \( \delta^0 M \), and left-exactness of closed socles (Lemma 4.4). \( \square \)

Sometimes is it useful to apply the closed socle functor to \( \delta M \) over \( Q \times \mathcal{F}\text{op}_Q \) in two steps, first over one poset and then over the other. These yield the same result.

**Lemma 4.8.** The functors \( Q^{-}\text{soc} \) and \( \mathcal{F}\text{op}_Q^{-}\text{soc} \) commute. In particular,

\[ \mathcal{F}\text{op}_Q^{-}\text{soc} (Q^{-}\text{soc} \delta M) \cong \text{soc}_0 M \cong Q^{-}\text{soc} (\mathcal{F}\text{op}_Q^{-}\text{soc} \delta M). \]

**Proof.** By taking direct sums over \( a \) and \( \sigma \), this follows from the natural isomorphisms \( \text{Hom}_{\mathcal{F}_Q}\text{op} (k_\sigma, \text{Hom}_Q (k_a, -)) \cong \text{Hom}_{Q\times\mathcal{F}_Q}\text{op} (k_\sigma k_a, -) \cong \text{Hom}_Q (k_a, \text{Hom}_{\mathcal{F}_Q}\text{op} (k_\sigma, -)). \) \( \square \)

The fundamental examples—indicator quotients for downsets—require a no(ta)tion.

**Definition 4.9.** In the situation of Definition 3.4, write \( \partial \nabla \) for the antichain of faces of \( Q_+ \) that are minimal under inclusion in \( \nabla \).

**Remark 4.10.** The reason for writing \( \partial \nabla \) instead of max or min is that it would be ambiguous either way, since both \( \mathcal{F}_Q \) and \( \mathcal{F}\text{op}_Q \) are natural here. Taking the “op” perspective (Remark 3.14), the \( \mathcal{F}\text{op}_Q \)-module \( k[\partial \nabla] \) with basis \( \partial \nabla \) resulting from Definition 4.9 is really just a \( \mathcal{F}\text{op}_Q \)-graded vector space: the antichain condition ensures that every non-identity element of \( \mathcal{F}\text{op}_Q \) acts by 0, unless \( \partial \nabla = \emptyset \), in which case all of \( \mathcal{F}\text{op}_Q \) acts by 1.

The case of most interest here is \( \nabla = \nabla_\partial D \), the shape of \( D \) at \( a \) (Proposition 3.10).
Example 4.11. For a downset $D$ in a real polyhedral group, $\mathcal{F}^{\text{op}} - \text{soc} \delta k[D]$ has $k[\partial D] \cap \delta k[D]$ in each degree $a$, because $\delta k[D]$ itself has $k[\partial D] \cap \delta k[D]$ in each degree $a$ by Definition 3.15 and Proposition 3.21. Note that $\delta k[D]$ and $\mathcal{F}^{\text{op}} - \text{soc} \delta k[D]$ are direct sums over faces $\sigma$, since they are $\mathcal{F}^{\text{op}}$-modules. What $Q - \text{soc}$ then does, for each $\sigma$, is find the degrees $a \in Q$ maximal among those where $\sigma \in \partial D$, by Proposition 3.21.2 and Example 4.3.

Taking socles in the other order, first $Q - \text{soc} k[\partial D]$ asks whether $\sigma \in \partial D$ but $\sigma \notin \partial D$ for any $b \succ a$ in $Q$. That can happen even if $\sigma$ contains a smaller face where it still happens. What $\mathcal{F}^{\text{op}} - \text{soc}$ then does is return the smallest faces at $a$ where it happens.

Corollary 4.12. The socle of the indicator quotient $k[D]$ for any downset $D$ in a real polyhedral group $Q$ is nonzero only in degrees lying in the topological boundary $\partial D$.

Proof. By Proposition 3.21.1, $\delta k[D]$ is a direct sum of indicator quotients. Example 4.3 and Proposition 3.10 show that the socle of an indicator quotient over a real polyhedral group lies along the boundary of the downset in question. $\square$

Remark 4.13. Corollary 4.12 is false for intervals with no interior.

4.3. Closed socles along faces of positive dimension.

The definition of closed socle and closed cogenerator are expressed in terms of Hom functors analogous to those in Definition 4.1. They are more general in that they occur along faces of $Q$, but more restrictive in that $Q$ needs to be a partially ordered group instead of an arbitrary poset for the notion of face to make sense.

Definition 4.14. Fix a face $\tau$ of a partially ordered group $Q$. The skyscraper $Q$-module at $a \in Q$ along $\tau$ is $k[a + \tau]$, the subquotient $k[a + Q_+]/k[a + m_\tau]$ of $k[Q]$, where $m_\tau = Q_+ \setminus \tau$. Set

$$\text{Hom}_Q(k[\tau], -) = \bigoplus_{a \in Q} \text{Hom}_Q(k[a + \tau], -).$$

Definition 4.15. Fix a partially ordered group $Q$, a face $\tau$, and a $Q$-module $M$.

1. The global closed cogenerator functor along $\tau$: the $k[Q_+/\mathbb{Z}_\tau]$-module

$$\overline{\text{soc}}_\tau M = \overline{\text{Hom}}_Q(k[\tau], M)/\tau.$$

2. If $Q/\mathbb{Z}_\tau$ is partially ordered, then the local closed cogenerator functor along $\tau$: the $Q/\mathbb{Z}_\tau$-module

$$\overline{\text{soc}}(M/\tau) = \overline{\text{Hom}}_{Q/\mathbb{Z}_\tau}(k, M/\tau).$$

Elements of $\overline{\text{soc}}(M/\tau)$ are identified with elements of $M/\tau$ via $\varphi \mapsto \varphi(1)$.

3. Regard the $Q$-module $\overline{\text{Hom}}_Q(k[\tau], M)$ naturally as contained in $M$ via $\varphi \mapsto \varphi(1)$.

A homogeneous element in this $Q$-submodule that maps to a nonzero element of $\overline{\text{soc}}_\tau M$ is a global closed cogenerator of $M$ along $\tau$. If $I \subseteq Q$ is an interval, then a closed cogenerator of $I$ is the degree in $Q$ of a closed cogenerator of $k[I]$. 

4. Regard $\text{soc}(M/\tau)$ naturally as contained in $M/\tau$ via $\varphi \mapsto \varphi(1)$. A nonzero homogeneous element in $\text{soc}(M/\tau)$ is a local closed cogenerator of $M$ along $\tau$. Assume the default modifier “global” when neither “local” nor “global” is written.

**Remark 4.16.** Notationally, a subscript on “soc” serves as a visual cue that the functor is over a partially ordered group, as faces of posets are not defined in more generality. This visual cue persists throughout the more general notions of socle.

**Remark 4.17.** The closed cogenerator functor over a partially ordered group is the global closed cogenerator functor along the trivial face: $\text{soc} = \text{soc}_{\{0\}}$ and it equals the local cogenerator functor along $\{0\}$.

**Remark 4.18.** In looser language, a closed cogenerator of $M$ along $\tau$ is an element

- annihilated by moving up in any direction outside of $\tau$ but that
- remains nonzero when pushed up arbitrarily along $\tau$.

Equivalently, a closed cogenerator along $\tau$ is an element whose annihilator under the action of $Q_+$ on $M$ equals the prime ideal $m_\tau = Q_+ \setminus \tau$ of the positive cone $Q_+$. Elements like this are sometimes known as “witnesses” in commutative algebra.

**Example 4.19.** The closed socle along a face $\tau$ of the indicator quotient $k[I]$ for any interval $I$ in a partially ordered group $Q$ with partially ordered quotient $Q/\mathbb{Z}\tau$ is

$$\text{soc}_\tau k[I] = k[\max_\tau I],$$

where $\max_\tau I$ is the image in $Q/\mathbb{Z}\tau$ of the set of closed cogenerators of $I$ along $\tau$:

$$\max_\tau I = \{a \in I \mid (a + Q_+) \cap I = a + \tau\} / \mathbb{Z}\tau.$$

The set of closed cogenerators of a downset $D$ along $\tau$ can also be characterized as the elements of $D$ that become maximal in the localization $D_\tau$ of $D$ (Example 2.18).

Every global closed cogenerator yields a local one.

**Proposition 4.20.** Fix a partially ordered group $Q$. There is a natural injection

$$\text{soc}_\tau M \hookrightarrow \text{soc}(M/\tau)$$

for any $Q$-module $M$ if $\tau$ is a face with partially ordered quotient $Q/\mathbb{Z}\tau$. 

**Proof.** Localizing any homomorphism $k[a + \tau] \to M$ along $\tau$ yields a homomorphism $k[a + \mathbb{Z}\tau] \to M_\tau$, so $\text{Hom}_Q(k[\tau], M)_\tau$ is naturally a submodule of $\text{Hom}_Q(k[\mathbb{Z}\tau], M_\tau)$. The claim now follows from Lemma 2.24 and the next result. \qed

**Lemma 4.21.** If $Q$ and $Q/\mathbb{Z}\tau$ are partially ordered, there is a canonical isomorphism

$$\text{Hom}_Q(k[\mathbb{Z}\tau], M_\tau) / \tau \cong \text{Hom}_{Q/\mathbb{Z}\tau}(k, M/\tau).$$

**Proof.** Follows from the definitions, using that $k[\mathbb{Z}\tau] / \tau = k$ in $(Q/\mathbb{Z}\tau)$-degree 0. \qed
The following crucial remark highlights the difference between real-graded algebra and integer-graded algebra. It is the source of much of the subtlety in the theory developed in this paper, particularly Sections 4–9.

**Remark 4.22.** In contrast with taking support on a face [Mil20b, Proposition 4.6] and also with socles in commutative algebra over noetherian local or graded rings localization need not commute with taking closed socles along faces of positive dimension in real polyhedral groups. In other words, the injection in Proposition 4.20 need not be surjective: there can be local closed cogenerators that do not lift to global ones. The problem comes down to the homogeneous prime ideals of the monoid algebra \( k[Q_+] \) not being finitely generated, so the quotient \( k[\tau] \) fails to be finitely presented; it means that \( \text{Hom}_{k[Q_+]}(k[\tau], -) \) need not commute with \( A \otimes_{k[Q_+]} - \), even when \( A \) is a flat \( k[Q_+] \)-algebra such as a localization of \( k[Q_+] \). The context of \( \mathbb{R}^n \)-modules complicates the relation between support on \( \tau \) and closed cogenerators along \( \tau \) because the “thickness” of the support can approach 0 without ever quantum jumping all the way there and, importantly, remaining there along an entire translate of \( \tau \), as it would be forced to for a discrete group like \( \mathbb{Z}^n \). See Example 2.26, for instance, where the support on the \( y \)-axis does not contribute any closed socle along the \( y \)-axis to the ambient module. This issue is independent of the density phenomenon explored in Section 7; indeed, the interval in Example 2.26 is closed, so its socle equals its closed socle and is closed.

**Proposition 4.23.** The global closed cogenerator functor \( \text{soc}_\tau \) along any face \( \tau \) of a partially ordered group is left-exact, as is the local version if \( Q/\mathbb{Z}\tau \) is partially ordered.

**Proof.** For the global case, \( \text{Hom}_Q(k[\tau], -) \) is exact because it occurs in the category of graded modules over the monoid algebra \( k[Q_+] \), and quotient-restriction is exact by Lemma 2.24. For the local case, use exactness of \( M \mapsto M/\tau \) again (Lemma 2.24) and left-exactness of closed socles (Lemma 4.4), the latter applied over \( Q/\mathbb{Z}\tau \). \( \square \)

**Remark 4.24.** Closed socles, without reference to faces, work over arbitrary posets and are actually used that way in this work (over \( \mathcal{P}_{Q}^{\text{op}} \), for instance, in Section 4.2). That explains why this separate section on closed socles along faces of positive dimension is required, instead of simply doing Section 4.1 in this specificity in the first place.

### 4.4. Socles along faces of positive dimension.

**Lemma 4.25.** If \( \tau \) is a face of a real polyhedral group \( Q \) then the face poset of the quotient real polyhedral group \( Q/\mathbb{R}\tau \) is isomorphic to the open star \( \nabla\tau \) from Example 3.5 by the map \( \nabla\tau \to (Q/\mathbb{R}\tau)_+ \) sending \( \sigma \in \nabla\tau \) to its image \( \sigma/\tau \) in \( Q/\mathbb{R}\tau \).

**Proof.** See Remark 2.20. \( \square \)

**Definition 4.26.** In the situation of Lemma 4.25, endow \( \nabla\tau \) with the monoid and poset structures from Remark 3.14, so \( \sigma \preceq \sigma' \) in \( \nabla\tau \) if \( \sigma \supseteq \sigma' \). The upper boundary functor along \( \tau \) takes \( M \) to the \( Q \times \nabla\tau \)-module \( \delta_\tau M = \bigoplus_{\sigma \in \nabla\tau} \delta^\sigma M = \delta M/\bigoplus_{\sigma \not\in \nabla\tau} \delta^\sigma M \).
The notation is such that $\delta^\sigma \neq 0 \iff \sigma \supseteq \tau$.

**Definition 4.27.** Fix a partially ordered group $Q$, a face $\tau$, and an arbitrary commutative monoid $P$. The skyscraper $(Q \times P)$-module at $(a, \sigma) \in Q \times P$ along $\tau$ is

$$k_\sigma[a + \tau] = k[a + \tau] \otimes_k k_\sigma,$$

the right-hand side being a module over the ring $\mathbb{k}[Q_+] \otimes_k \mathbb{k}[P] = \mathbb{k}[Q_+ \times P]$ with tensor factors as in Definitions 4.1 and 4.14. Set

$$\text{Hom}_{Q \times P}(k[\tau], -) = \bigoplus_{(a, \sigma) \in Q \times P} \text{Hom}_{Q \times P}(k_\sigma[a + \tau], -).$$

**Remark 4.28.** When $P$ is trivial, this notation agrees with Definition 4.14, because $Q \times \{0\} \cong Q$ canonically, so $\text{Hom}_{Q \times \{0\}}(k[\tau], -) = \text{Hom}_Q(k[\tau], -)$.

**Definition 4.29.** Fix a real polyhedral group $Q$, a face $\tau$, and a $Q$-module $M$.

1. The **global cogenerator functor** along $\tau$ takes $M$ to its **global socle** along $\tau$:

$$\text{soc}_\tau M = \overline{\text{Hom}}_{Q \times \nabla \tau}(k[\tau], \delta_\tau M) / \tau.$$

The $\nabla \tau$-graded components of $\text{soc}_\tau M$ are denoted by $\text{soc}_\sigma^\tau M$ for $\sigma \in \nabla \tau$.

2. The **local cogenerator functor** along $\tau$ takes $M$ to its **local socle** along $\tau$:

$$\text{soc}_0(M/\tau) = \overline{\text{soc}} \delta(M/\tau) = \overline{\text{Hom}}_{Q/\mathbb{R} \tau \times \nabla \tau}(k, \delta(M/\tau)),$$

where the upper boundary is over $Q/\mathbb{R} \tau$ and the closed socle is over $Q/\mathbb{R} \tau \times \nabla \tau$. Elements of $\text{soc}_0(M/\tau)$ are identified with elements of $\delta(M/\tau)$ via $\varphi \mapsto \varphi(1)$.

3. Regard $\overline{\text{Hom}}_{Q \times \nabla \tau}(k[\tau], \delta_\tau M)$ as a $(Q \times \nabla \tau)$-submodule of $\delta_\tau M$ via $\varphi \mapsto \varphi(1)$. A homogeneous element $s$ in this submodule that maps to a nonzero element of $\text{soc}_\tau M$ is a **global cogenerator** of $M$ along $\tau$, and if $s \in \delta^\sigma_\tau M$ then it has nadir $\sigma$. If $I \subseteq Q$ is an interval, then a cogenerator of $I$ along $\tau$ with nadir $\sigma$ is the degree in $Q$ of a cogenerator of $k[I]$ with nadir $\sigma$ along $\tau$.

4. Regard $\text{soc}_0(M/\tau)$ as contained in $\delta(M/\tau)$ via $\varphi \mapsto \varphi(1)$. A nonzero homogeneous element in $\text{soc}_0(M/\tau)$ is a **local cogenerator** of $M$ along $\tau$.

Assume the default modifier “global” when neither “local” nor “global” is written.

**Example 4.30.** The boundary $\delta M$ in Example 3.16 explains the nadirs in Example 1.7. The other $\mathcal{F}_{\mathbb{R}^2}^{\text{op}}$-modules in Example 3.16, which have solid points in their upper-right corners, do not yield socle elements of $\delta M$ because the $\mathbb{R}_+^2$-component can be moved up without annihilation. That is not a universal statement, though: it only holds in this example because all of the closed socles of $M$ (Definition 4.15) vanish.

**Example 4.31.** See Example 1.8 for a socle along a face of positive dimension.
Remark 4.32. The reason to quotient by $\tau$ in Definition 4.29.1 is to lump together all cogenerators with nadir $\sigma$ along the same translate of $\mathbb{R}\tau$. This lumping makes it possible for a socle basis to produce a downset hull that is (i) as minimal as possible and (ii) finite. The lumping also creates a difference between the notion of socle element and that of cogenerator: a socle element is a class of cogenerators, these classes being indexed by elements in the quotient-restriction. In contrast, a local cogenerator is a cogenerator of the quotient-restriction itself, so a local cogenerator is already an element in the socle of the quotient-restriction. This difference between socle element and cogenerator already arises for closed socles along faces (Definition 4.15) but disappears in the context of socles not along faces (see Remark 4.17), be they over real polyhedral groups (Definition 4.5) or closed over posets (Definition 4.1).

Remark 4.33. If localization commuted with cogenerator functors, then the restriction from $\mathcal{F}_Q^{\text{op}}$ to $\nabla\tau$ in Definition 4.29.1 would happen automatically, because localizing $M$ along $\tau$ would yield a module over $Q^+ + \mathbb{R}\tau$, whose face poset is naturally $\nabla\tau$. But in this real polyhedral setting, the restriction from $\mathcal{F}_Q^{\text{op}}$ to $\nabla\tau$ must be imposed manually because the Hom must be taken before localizing (Remark 4.22), when the default face poset is still $\mathcal{F}_Q^{\text{op}}$.

Remark 4.34. If $a$ is a cogenerator of a downset $D$ along $\tau$, then the topology of $D$ at $a$ is induced by downsets of the form $a' - \sigma^\circ$ for faces $\sigma \in \nabla\tau$ and elements $a' \in a + \tau^\circ$. This subtle issue regarding shapes of cogenerators along $\tau$ is a vital reason for using $\nabla\tau$ instead of $\mathcal{F}_Q^{\text{op}}$. It is tempting to expect that if a face $\sigma$ is minimal in the shape $\nabla_D^a$, then any expression of $D$ as an intersection of downsets must induce the topology of $D$ at $a$ by explicitly taking $a - \sigma^\circ$ into account in one of the intersectands. One way to accomplish that would be for an intersectand to be a union of downsets of the form $b - Q\nabla_D^a$ (see Definition 3.4) in which one of the elements $b$ is $a$. But if $\sigma \in \nabla_D^a$ for all $a' \in a + \tau$, or even merely for a single element $a' \in a + \tau^\circ$, then

$$a - \sigma^\circ = a' - (a' - a - \sigma^\circ) \in a' - (\tau^\circ + \sigma^\circ) \subseteq a' - (\tau \lor \sigma)^\circ.$$

As the purpose of cogenerators is to construct downset decompositions as minimally as possible, it is counterproductive to think of $\sigma$ as being a valid $\mathcal{F}_Q^{\text{op}}$-socle degree unless $\sigma \in \nabla\tau$, because otherwise it fails to give rise to an essential cogenerator. See Theorem 7.19 for the most general possible view of considerations in this Remark.

Remark 4.35. In terms of persistent homology, cogenerators are deaths of classes. In that context, the need for upper boundary functors and socle theory beyond closed socles is particularly crucial, because the modules most pertinent to applied topology are precisely those whose closed socles vanish [KS18, BP19, Mil20c]. That is, the upper boundaries of these modules are as far from closed as possible.

Remark 4.36. Although $\text{soc}_\tau M$ is a module over $Q/\mathbb{R}\tau \times \nabla\tau$ by construction, the actions of $Q/\mathbb{R}\tau$ and $\nabla\tau$ on it are trivial, in the sense that attempting to move a
nonzero homogeneous element up in one of the posets either takes the element to 0 or leaves it unchanged. (The latter only happens if the degree is unchanged, which occurs only when acting by the identity $0 \in Q/\mathbb{R}\tau$ or when acting by $\sigma \in \nabla\tau$ on an element of $\nabla\tau$-degree $\sigma' \subseteq \sigma$.) That is what it means to be a direct sum of skyscraper modules. It implies that any direct sum decomposition of $\text{soc}_\tau M$ as a vector space graded by $Q/\mathbb{R}\tau \times \nabla\tau$ is also a decomposition of $\text{soc}_\tau M$ as a $Q/\mathbb{R}\tau$-module or as a $\nabla\tau$-module.

**Lemma 4.37.** If $\tau$ is a face of $a$ is a real polyhedral group $Q$ and $N = \bigoplus_{\sigma \in \nabla\tau} N_\sigma$ is a module over $Q \times \nabla\tau$, then $\text{Hom}_{\nabla\tau}(k_\sigma, N) / \tau \cong \text{Hom}_{\nabla\tau}(k_\sigma, N/\tau)$, and hence

$$(\nabla\tau-\text{soc} N) / \tau \cong \nabla\tau-\text{soc}(N/\tau).$$

**Proof.** $\text{Hom}_{\nabla\tau}(k_\sigma, N)$ is the intersection of the kernels of the $Q$-module homomorphisms $N_\sigma \to N_{\sigma'}$ for faces $\sigma \supseteq \sigma'$, so the isomorphism of Hom modules follows from Lemma 2.24. The socle isomorphism follows by taking the direct sum over $\sigma \in \nabla\tau$. □

**Proposition 4.38.** The functors $\text{soc}_\tau$ and $\nabla\tau-\text{soc}$ commute. In particular,

$$\nabla\tau-\text{soc}(\text{soc}_\tau \delta_\tau M) \cong \text{soc}_\tau M \cong \text{soc}_\tau(\nabla\tau-\text{soc} \delta_\tau M).$$

**Proof.** By taking direct sums over $a$ and $\sigma$, this is mostly the natural isomorphisms

$$\text{Hom}_{\nabla\tau}(k_\sigma, \text{Hom}_{Q}(k[a + \tau], -)) \cong \text{Hom}_{Q \times \nabla\tau}(k[a + \tau], -) \cong \text{Hom}_{Q}(k[a + \tau], \text{Hom}_{\nabla\tau}(k_\sigma, -))$$

that result from the adjunction between $\text{Hom}$ and $\otimes$. Taking the quotient-restriction along $\tau$ (Definition 2.22) almost yields the desired result; the only issue is that the left-hand side requires Lemma 4.37. □

**Example 4.39.** If $a$ is a cogenerator of a downset $D \subseteq Q$ along $\tau$ with nadir $\sigma$, then reasoning as in Proposition 4.11 and using Definition 4.9, computing $\nabla\tau-\text{soc}$ first in Proposition 4.38 shows that $\sigma \in \partial(\nabla^a \cap \nabla\tau)$. What $\text{soc}_\tau$ then does is verify that the image $\bar{a}$ of $a$ in $Q/\mathbb{R}\tau$ is maximal with this property, by Example 4.19.

**Proposition 4.40.** There is a natural injection

$$\text{soc}_\tau M \hookrightarrow \text{soc}_0(M/\tau)$$

for any module $M$ over a real polyhedral group $Q$ and any face $\tau$ of $Q$.

**Proof.** By Proposition 4.20 $\text{soc}_\tau N \hookrightarrow \text{soc}(N/\tau)$ for $N = \nabla\tau-\text{soc} \delta_\tau M$ viewed as a $Q/\mathbb{R}\tau$-module. Proposition 4.38 yields $\text{soc}_\tau N = \text{soc}_\tau M$. It remains to show that $(Q/\mathbb{R}\tau)-\text{soc}(N/\tau) = \text{soc}_0(M/\tau)$. To that end, first note that

$$(\nabla\tau-\text{soc} \delta_\tau M) / \tau \cong \nabla\tau-\text{soc}(\delta_\tau M / \tau) \cong \nabla\tau-\text{soc} \delta(M / \tau),$$

the first isomorphism by Lemma 4.37 and the second by Lemma 4.41, which shows that the modules acted on by $\nabla\tau-\text{soc}$ are isomorphic. Now apply the last isomorphism in Lemma 4.8, with $Q$ replaced by $Q/\mathbb{R}\tau$ so that automatically $\mathcal{F}_Q^{\text{op}}$ must be replaced by $\nabla\tau$ via Lemma 4.25. □
Lemma 4.41. If $\sigma \supseteq \tau$ then $(\delta^\sigma M)/\tau \cong \delta^{\sigma/\tau}(M/\tau)$.

Proof. Explicit calculations from the definitions show that in degree $a/\tau$ both sides equal

$$\lim_{a' \in a - \sigma^\circ, v \in \tau} M_{a' + v},$$

although they take the colimits in different orders: $v$ first or $a'$ first. The hypothesis that $\sigma \supseteq \tau$ enters to show that any direct limit over $\{a' \in Q \mid a'/\tau \in a/\tau - (\sigma/\tau)^\circ\}$ can equivalently be expressed as a direct limit over $a' \in a - \sigma^\circ$. \hfill $\square$

Corollary 4.42. An indicator quotient for a downset in a real polyhedral group has at most one linearly independent socle element along each face with given nadir and degree. In fact, the degrees of independent socle elements along $\tau$ with fixed nadir are incomparable in $Q/\mathbb{R},$ and nadirs of socle elements with fixed degree are incomparable in $\nabla \tau$.

Proof. A socle element of an indicator quotient $E$ along a face $\tau$ of $Q$ is a local socle element of $E$ along $\tau$ by Proposition 4.40. Local socle elements along $\tau$ are socle elements (along the minimal face $\{0\}$) of the quotient-restriction along $\tau$ by Definition 4.29.2. But $E/\tau$ is an indicator quotient of $\k[Q/\mathbb{R},]$, so its socle degrees with fixed nadir $\sigma$ are incomparable, as are its nadirs with fixed socle degrees, by Example 4.11. \hfill $\square$

Example 4.43. Propositions 4.38 and 4.40 ease some socle computations. To see how, consider the indicator $Q$-module $\k[\rho]$ for a face $\rho$ of $Q$. Proposition 4.40 immediately implies that $\text{soc}_\tau \k[\rho] = 0$ unless $\rho \supseteq \tau$, because localizing along $\tau$ yields $\k[\rho]/\tau = 0$ unless $\rho \supseteq \tau$.

Next compute $\delta^\sigma \k[\rho]$. When either $a \notin \rho$ or $\sigma \nsubseteq \rho$, the direct limit in Definition 3.11 is over a set $a - \sigma^\circ$ of degrees in which $\k[\rho] = 0$ in a neighborhood of $a$. Hence the only faces that can appear in $\delta^\tau \k[\rho]$ lie in the interval between $\tau$ and $\rho$, so assume $\tau \subseteq \sigma \subseteq \rho$. If $(\delta^\sigma \k[\rho])_a \neq 0$ then it equals $\k$ because $\k[\rho]$ is an indicator module for a subset of $Q$. Moreover, if $(\delta^\sigma \k[\rho])_a = \k$ then the same is true in any degree $b \in a + \rho$ because $(b-a)+(a-\sigma^\circ) \cap \rho \subseteq (b-\sigma^\circ) \cap \rho$. Thus $\delta^\sigma \k[\rho]$ is torsion-free as a $\k[\rho]$-module.

The $\text{soc}_\tau$ on the left side of Proposition 4.38, which by Definition 4.15.1 is a quotient-restriction of a module $\text{Hom}_Q(\k[\tau], \delta_\tau \k[\rho])$, can only be nonzero if $\tau = \rho$, as any nonzero image of $\k[\tau]$ is a torsion $\k[\rho]$-module. Hence the socle of $\k[\rho]$ along $\tau$ equals the closed socle along $\tau = \rho$, which is computed directly from Definition 4.29.1 and Definition 2.22 to be $\text{Hom}_Q(\k[\tau], \k[\tau])/\tau = \k[\tau]/\tau$. In summary,

$$\text{soc}_\tau \k[\rho] = \begin{cases} \k_0 \text{ for } 0 \in Q/\mathbb{R}, & \text{if } \tau = \rho \\ 0 & \text{otherwise}. \end{cases}$$

Proposition 4.44. The global cogenerator functor $\text{soc}_\tau$ along any face $\tau$ of a real polyhedral group is left-exact, as is the local cogenerator functor along $\tau$.

Proof. Proposition 4.23 and Lemma 3.12. \hfill $\square$
5. Tame, semialgebraic, and PL socles

The tame, semialgebraic, and PL conditions are preserved under taking socles. That is the goal of this section, Theorem 5.13, which states such a result for the most general form of socle over any real polyhedral group. But because the various forms of socles in Section 4 occur in contexts more general than real polyhedral groups, it is necessary to record the statements separately for each form of socle. The order in which they are covered here is the same as in Section 4: closed socles \( \text{soc} \) over an arbitrary poset \( Q \) (Proposition 5.7); socles \( \text{soc}_0 \) over real polyhedral groups (Corollary 5.10); closed socles \( \text{soc}_{\tau} \) along faces of polyhedral groups (Proposition 5.12); and finally socles \( \text{soc}_{\tau} \) along faces of real polyhedral groups (Theorem 5.13). The proofs are based on the observation that everything reduces to the effects of cogenerator functors on downset modules. Verifying the hypotheses for the criterion in Theorem 5.2 for cogenerator functors in the tame case is relatively straightforward. The semialgebraic and PL cases require more power (Lemma 5.4 and onward). First, here is a handy concept [Mil20a, Definition 3.14].

**Definition 5.1.** Let each of \( S \) and \( S' \) be a nonempty interval in \( Q \) (Example 2.6.3). A homomorphism \( \varphi : k[S] \to k[S'] \) of interval modules is *connected* if there is a scalar \( \lambda \in k \) such that \( \varphi \) acts as multiplication by \( \lambda \) on the copy of \( k \) in degree \( q \) for all \( q \in S \cap S' \).

**Theorem 5.2.** Fix posets \( Q \) and \( Q' \). Suppose a left-exact functor \( S \) from the category of \( Q \)-modules to the category of \( Q' \)-modules takes each
- downset module \( k[D] \) to a subquotient \( S(k[D]) = k[SD] \) of \( k[Q'] \), and
- connected morphism \( k[D] \to k[D'] \) of downset modules to a connected morphism \( k[SD] \to k[SD'] \) of interval modules.

Then
1. the restriction of \( S \) to the category of tame \( Q \)-modules (see Proposition 2.15) yields a functor to the category of tame \( Q' \)-modules; and
2. if \( Q \) and \( Q' \) are partially ordered real vector spaces, and \( SD \) is semialgebraic in \( Q' \) for all semialgebraic downsets \( D \subseteq Q \), then \( S \) restricts to a functor from the category of semialgebraic \( Q \)-modules to the category of semialgebraic \( Q' \)-modules.

The previous claim remains true with “PL” in place of “semialgebraic”.

**Proof.** Assume \( M \) is a tame \( Q \)-module. Then \( M = \ker(E^0 \to E^1) \) is the kernel of a tame morphism of finite direct sums of downset modules by the syzygy theorem for poset modules [Mil20a, Theorem 6.12.5]. Left-exactness implies that \( SM = \ker(SE^0 \to SE^1) \). The first goal is to show that \( SM \) is a tame module. For that it suffices by Proposition 2.15 to show that \( SE^0 \to SE^1 \) is a tame morphism. But that also follows from Proposition 2.15 because each component of \( SE^0 \to SE^1 \) has the form \( k[SD] \to k[SD'] \) and is hence a tame morphism by hypothesis. The argument works mutatis mutandis in the semialgebraic and PL cases.

Now suppose that a morphism \( M \to M' \) is given. By the syzygy theorem again [Mil20a, Theorem 6.12], there is a copresentation \( M' = \ker(E^{0'} \to E^{1'}) \) such that
the map $M \to M'$ is induced by a morphism of their copresentations. The composite morphism $SM \to SE^0 \to SE^{0'}$ has image in $SM'$. The morphism $SM \to SM'$ is tame, semialgebraic, or PL by any common refinement of two encodings (Definition 2.11) of $SE^{0'}$ subordinate to the morphisms (Definition 2.13) from $SM$ and $SM'$.

Example 5.3. The upper boundary functor $S = \delta^\sigma$ atop $\sigma$ in Definition 3.15 satisfies the hypotheses of Theorem 5.2.1 with $Q' = Q$ by Proposition 3.21. Therefore $\delta^\sigma M$ is a tame $Q$-module if $M$ is tame, and $\delta^\sigma M \to \delta^\sigma M'$ is a tame morphism if $M \to M'$ is. Hence the same is true with $\delta$ in place of $\delta^\sigma$. The corresponding semialgebraic conclusions are true as well, this time using Theorem 5.2.2, but checking that uses Proposition 5.5, which requires a bit more power.

Lemma 5.4. If $X \subseteq \mathbb{R}^n$ and $X \to Y$ is a morphism of semialgebraic varieties, then the family $\overline{X_Y}$ obtained by taking the closure in $\mathbb{R}^n$ of every fiber of $X$ is semialgebraic.

Proof. This is a consequence of Hardt’s theorem [Har80, Theorem 4] (see also [Shi97, Remark II.3.13]), which says that over a subset of $Y$ whose complement in $Y$ has dimension less than $\dim Y$, the family $X \to Y$ is trivial.

Proposition 5.5. If $D$ is a semialgebraic or PL downset in a real polyhedral group $Q$ and $\sigma \in F_Q$ is a face then $\delta^\sigma D$ is similarly semialgebraic or PL.

Proof. Semialgebraic case: Lemma 5.4 with $Y = Q / \mathbb{R}\sigma$ and $X = D$ by Lemma 3.20.

PL case: as in Lemma 3.20 the union can be broken over finitely many relatively open polyhedral cells comprising $D$. So assume $D$ is a single relatively open polyhedral cell. The union in Lemma 3.20 is plainly a subset of $\overline{D}$. Indeed, if $\pi : \overline{D} \to Q / \mathbb{R}\sigma$, then the union is $\pi^{-1}(\pi(D))$. That is, the union is the complement in $\overline{D}$ of the (closed) faces of $\overline{D}$ whose projections mod $\mathbb{R}\sigma$ are contained in the boundary of $\pi(\overline{D})$.

Proposition 5.7 covers the case of closed socles over an arbitrary poset, with the next lemma needed for the semialgebraic and PL cases.

Lemma 5.6. If $D$ is a semialgebraic or PL downset in a real polyhedral group then $\max D$ is similarly semialgebraic or PL, as well.

Proof. The semialgebraic proof relies on standard operations on subsets that preserve the semialgebraic property; see [Shi97, Chapter II], for instance. As it happens, the proof works verbatim for the PL case because the relevant (in)equalities are linear.

Inside of $\mathbb{R}^n \times \mathbb{R}^n$, consider the subset $X$ whose fiber over each point $a \in D$ is $a + m$, where $m = Q \setminus \{0\}$ is the maximal monoid ideal of $Q_+$. Note that $m$ is semialgebraic because it is defined by linear inequalities and a single linear inequaion. The subset $X \subseteq \mathbb{R}^n \times \mathbb{R}^n$ is semialgebraic because it is the image of the algebraic morphism $D \times m \to D \times \mathbb{R}^n$ sending $(a, q) \mapsto (a, a + q)$. The intersection of $X$ with
the semialgebraic subset $D \times D$ remains semialgebraic, as does the projection of this intersection to $D$. The image of the projection is $D \setminus \max D$ because $(a + m) \cap D = \emptyset$ precisely when $a \in \max D$. Therefore $\max D = D \setminus (D \setminus \max D)$ is semialgebraic. □

**Proposition 5.7.** If a module $M$ over any poset is tame then so is its closed socle $\overline{\text{soc}} M$. If $M$ is semialgebraic or PL over a real polyhedral group then so is $\overline{\text{soc}} M$. If $M \to M'$ is a tame, semialgebraic, or PL morphism, then so is $\overline{\text{soc}} M \to \overline{\text{soc}} M'$.

**Proof.** Apply Theorem 5.2: left-exactness is Lemma 4.4, the criteria on downset modules and connected morphisms between them both follow from Example 4.3, and the semialgebraic or PL criterion is Lemma 5.6. □

The next three results cover the case of socles over real polyhedral groups.

**Lemma 5.8.** The homomorphisms $\delta^\sigma M \to \delta^{\sigma'} M$ for faces $\sigma \supseteq \sigma'$ afforded by Proposition 3.13 are tame, semialgebraic, or PL if $M$ is.

**Proof.** Use [Mil20a, Theorem 6.12.5] to express $M = \ker(E^0 \to E^1)$ as the kernel of a downset copresentation that is tame, semialgebraic, or PL as the case may be. For a single downset $D$, observe that $\delta^\sigma D \subseteq \delta^{\sigma'} D$ whenever $\sigma \supseteq \sigma'$ by Proposition 3.21.2. Therefore, by Proposition 3.21, the natural map $\delta^\sigma k[D] \to \delta^{\sigma'} k[D]$ is a quotient of downset modules, which is a connected homomorphism (Definition 5.1) and hence tame, semialgebraic, or PL, as the case may be. The homomorphism $\delta^\sigma M \to \delta^{\sigma'} M$ is induced by the morphism $\delta^\sigma E^* \to \delta^{\sigma'} E^*$ of copresentations. The argument in the final two sentences of the proof of Theorem 5.2 therefore works here. □

The next result is stated in the generality of $\nabla_{\tau-\overline{\text{soc}} \delta_{\tau}}$ (see Section 4.4) for the eventual purpose of Theorem 5.13, even though for the time being all that is needed is the case $\tau = \{0\}$, where $\nabla_{\tau-\overline{\text{soc}} \delta_{\tau}} = F_{Q^0-\overline{\text{soc}} \delta}$ (see Section 4.1).

**Proposition 5.9.** Fix a real polyhedral group $Q$ and a face $\tau \in F_Q$. The endofunctor on the category of $Q$-modules that takes $M$ to $\nabla_{\tau-\overline{\text{soc}} \delta_{\tau}}$ restricts to endofunctors on the categories of tame $Q$-modules, semialgebraic $Q$-modules, and PL $Q$-modules.

**Proof.** The same proof works, mutatis mutandis, for the semialgebraic and PL cases.

The $\nabla_{\tau}$-graded component of $\nabla_{\tau-\overline{\text{soc}} \delta_{\tau}} M$ in $\nabla_{\tau}$-degree $\sigma$ is the intersection of the kernels of the $Q$-module homomorphisms $\delta^\sigma M \to \delta^{\sigma'} M$ for $\sigma \supseteq \sigma'$. These are tame morphisms, if $M$ is tame, by Lemma 5.8. The intersection of their kernels is tame by Proposition 2.15 because any intersection of kernels of morphisms from a single object to finitely many objects in any abelian category is the kernel of the morphism to the direct sum. So $\nabla_{\tau-\overline{\text{soc}} \delta_{\tau}} M \to \delta_{\tau} M$ is tame.

Any given tame morphism $M \to N$ induces a tame morphism $\delta M \to \delta N$ by Example 5.3. Hence the composite $\nabla_{\tau-\overline{\text{soc}} \delta_{\tau}} M \to \delta_{\tau} M \to \delta_{\tau} N$ is a tame morphism that happens to have its image in $\nabla_{\tau-\overline{\text{soc}} \delta_{\tau}} N$. On the other hand, $\nabla_{\tau-\overline{\text{soc}} \delta_{\tau}} N \to \delta_{\tau} N$ is also a tame morphism. The morphism $\nabla_{\tau-\overline{\text{soc}} \delta_{\tau}} M \to \nabla_{\tau-\overline{\text{soc}} \delta_{\tau}} N$ is tame by any
common refinement of two poset encodings of \( \delta, N \) subordinate to the morphisms from \( \nabla_\tau \text{-soc} \delta, M \) and \( \nabla_\tau \text{-soc} \delta, N \).

**Corollary 5.10.** If a module \( M \) over a real polyhedral group is tame, semialgebraic, or PL then so is its socle \( \text{soc}_0 M \). If \( M \to M' \) is a tame, semialgebraic, or PL morphism, then the natural map \( \text{soc}_0 M \to \text{soc}_0 M' \) is, as well.

**Proof.** By Lemma 4.8, \( \text{soc}_0 M \) is the composite of the functors \( Q \text{-soc} \) and \( F_{Q_+}^{\text{op}} \text{-soc} \delta \), which preserve the tame, semialgebraic, and PL categories by Propositions 5.7 and 5.9, the latter of which has \( \nabla_\tau \text{-soc} \delta = F_{Q_+}^{\text{op}} \text{-soc} \delta \) when \( \tau = \{0\} \).

The next two results cover closed socles along faces of arbitrary polyhedral groups.

**Lemma 5.11.** If \( D \) is a semialgebraic or PL downset in a real polyhedral group \( Q \) then \( \max_\tau D \) is similarly semialgebraic or PL in \( Q/\mathbb{R} \tau \) for any face \( \tau \) of \( Q_+ \).

**Proof.** The projection of a semialgebraic set is semialgebraic, so by Example 4.19 it suffices to prove that the set of degrees of closed cogenerators of \( k[D] \) along \( \tau \) is semialgebraic. The argument comes in two halves, both following the framework of the proof of Lemma 5.6. For the first half, simply replace \( m \) by \( m_\tau = Q_+ \setminus \tau \) to find that \( \{a \in D \mid (a + Q_+) \cap D \subseteq a + \tau \} \) is semialgebraic. The second half uses \( \tau \) instead of \( m \), and this time it intersects the subset \( X \) with \( D \times (Q \setminus D) \) to find that \( \{a \in D \mid (a + Q_+) \cap D \supseteq a + \tau \} \) is semialgebraic. The desired set of degrees is the intersection of these two semialgebraic sets. Replacing “semialgebraic” with “PL” works because, again, the relevant (in)equalities are linear.

**Proposition 5.12.** If a module \( M \) over a partially ordered group is tame then so is its closed socle \( \text{soc}_\tau M \) along any face \( \tau \). If \( M \) is semialgebraic or PL over a real polyhedral group then so is \( \text{soc}_\tau M \). If \( M \to M' \) is a tame, semialgebraic, or PL morphism, then the natural map \( \text{soc}_\tau M \to \text{soc}_\tau M' \) is, as well.

**Proof.** Apply Theorem 5.2: left-exactness is Proposition 4.23; the criteria on downset modules and connected morphisms between them follow from Example 4.19, noting that the set \( \max_\tau D \) is an upset in the downset \( D/\mathbb{Z} \tau \) because it is contained in the set of maximal elements of \( D/\mathbb{Z} \tau \); and the semialgebraic or PL criterion is Lemma 5.11.

Finally, here is the version covering total socles over real polyhedral groups.

**Theorem 5.13.** Over a real polyhedral group \( Q \), the cogenerator functor \( \text{soc}_\tau \) along any face \( \tau \) restricts to endofunctors on the categories of tame, semialgebraic, or PL modules over \( Q \). For any face \( \sigma \supseteq \tau \), this statement remains true for the cogenerator functor \( \text{soc}_\tau \) along \( \tau \) with nadir \( \sigma \).

**Proof.** The cogenerator functor \( \text{soc}_\tau M \) is the composite of \( \text{soc}_\tau \) and \( \nabla_\tau \text{-soc} \delta \), by Proposition 4.38. These functors preserve the tame, semialgebraic, and PL categories by Propositions 5.12 and 5.9. The \( \text{soc}_\tau \) claim follows by taking \( \nabla_\tau \)-graded pieces.
6. Essential property of socles

In this section, \( Q \) is a real polyhedral group unless otherwise stated.

The culmination of the foundations developed in Section 4 says that socles and co-
generators detect injectivity of homomorphisms between tame modules over real poly-
hedral groups (Theorem 6.7), as they do for noetherian rings in ordinary commutative
algebra. The theory is complicated by there being no actual submodule containing a
given non-closed socle element; that is why socles are functors that yield submodules
of localizations of auxiliary modules rather than submodules of localizations of the
given module itself. Nonetheless, it comes down to the fact that, when \( D \subseteq Q \) is a
downset, every element can be pushed up to a cogenerator. Theorem 6.5 contains a
precise statement that suffices for the purpose of Theorem 6.7, although the definitive
version of Theorem 6.5 occurs in Section 7, namely Theorem 7.19.

The proof of Theorem 6.5 requires a definition—essentially the notion dual to that of
shape (Proposition 3.10). Informally, it is the set of faces \( \sigma \) such that a neighborhood
of \( a \) in \( a + \sigma^\circ \) is contained in the downset \( D \). The formal definition reduces by negation
to the discussion surrounding tangent cones of downsets (Section 3.1), noting that the
negative of an upset is a downset.

**Definition 6.1.** The upshape of a downset \( D \) in a real polyhedral group \( Q \) at \( a \) is
\[
\Delta^D_a = \mathcal{F}_Q \setminus \nabla_{-U},
\]
where \( U = Q \setminus D \) is the upset complementary to \( D \).

**Lemma 6.2.** The upshape \( \Delta^D_a \) is a polyhedral complex (a downset) in \( \mathcal{F}_Q \). As a function
of \( a \), for fixed \( D \) the upshape \( \Delta^D_a \) is decreasing, meaning \( a \preceq b \Rightarrow \Delta^D_a \supseteq \Delta^D_b \).

**Proof.** These claims are immediate from the discussion in Section 3.1. \( \square \)

**Remark 6.3.** The upshape \( \Delta^D_a \) is a rather tight analogue of the Stanley–Reisner com-
plex of a simplicial complex, or more generally the lower Koszul simplicial complex
[MS05, Definition 5.9] of a monomial ideal in a degree from \( \mathbb{Z}^n \). (The complex \( K_b(I) \)
would need to be indexed by \( b - \text{supp}(b) \) to make the analogy even tighter.) Similarly,
the shape of a downset at an element of \( Q \) is analogous to the upper Koszul simplicial
complex of a monomial ideal [MS05, Definition 1.33].

The general statement about pushing up to cogenerators relies on the special case
of closed cogenerators for closed downsets.

**Lemma 6.4.** If \( D \subseteq Q \) is a downset and the part of \( D \) above \( b \in D \) is closed, so
\((b + Q_+) \cap D = (b + Q_+) \cap D\), then \( b \preceq a \) for some closed cogenerator \( a \) of \( D \).
Proof. It is possible that $b + Q_+ \subseteq D$, in which case $D = Q$ and $b$ is by definition a closed cogenerator along $\tau = Q_+$. Barring that case, the intersection $(b + Q_+) \cap \partial D$ of the principal upset at $b$ with the boundary of $D$ is nonempty. Among the points in this intersection, choose $a$ with minimal upshape. Observe that $\{0\} \in \Delta^D_a$ because $a \in D$, so $\Delta^D_a$ is nonempty.

Let $\tau \in \Delta^D_a$ be a facet. The goal is to conclude that $\Delta^D_a = F_\tau$ has no facet other than $\tau$, for then $\Delta^D_a = F_\tau$ for all $a' \geq a$ in $D$ by upshape minimality and Lemma 6.2, and hence $a$ is a cogenerator of $D$ along $\tau$ by Definition 4.15 (see also Remark 4.18).

Suppose that $\rho \in F_Q$ is a ray that lies outside of $\tau$. If $\rho \in \Delta^D_a$ then upshape minimality implies $\rho \in \Delta^D_a$ for any $a' \in (a + \tau^o) \cap D$, and such an $a'$ exists by definition of upshape. Consequently, some face containing both $\rho$ and $\tau$ lies in $\Delta^D_a$: if $v$ is any sufficiently small vector along $\rho$, then $a' + v = a + (a' - a) + v \in D$, and the smallest face containing $(a' - a) + v$ contains both the interior of $\tau$ (because it contains $a' - a$) and $\rho$ (because it contains $v$). But this is impossible, so in fact $\Delta^D_a = F_\tau$. \hfill $\square$

Theorem 6.5. If $D$ is a downset in a real polyhedral group $Q$ and $b \in D$, then there are faces $\tau \subseteq \sigma$ of $Q_+$ and a cogenerator $a$ of $D$ along $\tau$ with nadir $\sigma$ such that $b \preceq a$.

Proof. It is possible that $b + Q_+ \subseteq D$, in which case $D = Q$ and $b$ is by definition a closed cogenerator along $\tau = Q_+$ which is the same as a cogenerator along $Q_+$ with nadir $Q_+$. Barring that case, the intersection $(b + Q_+) \cap \partial D$ of the principal upset at $b$ with the boundary of $D$ is nonempty. Among the points in this intersection, there is one with minimal shape, and it suffices to treat the case where this point is $b$ itself.

Minimality of $\nabla^b_D$ implies that the shape does not change upon going up from $b$ while staying in the closure $\overline{D}$. Consequently, given any face $\sigma \in \nabla^a_D$, the shape of $D$ at every point in $b + Q_+$ that lies in $\overline{D}$ also contains $\sigma$. Equivalently by Proposition 3.21.2, $(b + Q_+) \cap \delta^o D = (b + Q_+) \cap \overline{D}$. Lemma 6.4 produces a closed cogenerator $a$ of $\delta^o D$, along some face $\tau$, satisfying $b \preceq a$. Since $\nabla^a_D$ is a nonempty cocomplex, its intersection with $\nabla^b_D$ is nonempty, so assume $\sigma \in \nabla^a_D \cap \nabla^b_D$. The closed cogenerator $a$ of $\delta^o D$ need not be a cogenerator of $D$, but if $\sigma$ is minimal under inclusion in $\nabla^a_D \cap \nabla^b_D$, then $a$ is indeed a cogenerator of $D$ along $\tau$ with nadir $\sigma$ by Proposition 4.38—specifically the first displayed isomorphism—applied to Example 4.3. \hfill $\square$

Remark 6.6. The arguments in the preceding two proofs are essential to the whole theory of socles, which hinges upon them. The structure of the arguments dictate the forms of all of the notions of socle, particularly those involving cogenerators along faces.

Theorem 6.7 is intended for tame modules, but because it has no cause to deal with generators, in actuality it only requires half of a fringe presentation (or a little less; see Definition 2.16). The statement uses divisibility (Definition 3.19), which works verbatim for $\delta \cdot M$, by Definition 4.26, because it refers only to upper boundaries atop a single face $\sigma$. 
Theorem 6.7 (Essentiality of socles). Fix a homomorphism $\varphi : M \rightarrow N$ of modules over a real polyhedral group $Q$.

1. If $\varphi$ is injective then $\text{soc}_\tau \varphi : \text{soc}_\tau M \rightarrow \text{soc}_\tau N$ is injective for all faces $\tau$ of $Q_+$.
2. If $\text{soc}_\tau \varphi : \text{soc}_\tau M \rightarrow \text{soc}_\tau N$ is injective for all faces $\tau$ of $Q_+$ and $M$ is downset-finite, then $\varphi$ is injective.

If $M$ is downset-finite then each homogeneous element of $M$ divides a cogenerator of $M$.

Proof. Item 1 is a special case of Proposition 4.44. Item 2 follows from the divisibility claim, for if $y$ divides a cogenerator $s$ along $\tau$ then $\varphi(y) \neq 0$ whenever $\text{soc}_\tau \varphi(\tilde{s}) \neq 0$, where $\tilde{s}$ is the image of $s$ in $\text{soc}_\tau M$.

For the divisibility claim, fix a downset hull $M \hookrightarrow \bigoplus_{j=1}^k E_j$ and a nonzero $y \in M_b$. For some $j$ the projection $y_j \in E_j$ of $y$ divides a cogenerator of $E_j$ along some face $\tau$ with some nadir $\sigma$ by Theorem 6.5. Choose one such cogenerator $s_j$, and suppose it has degree $a \in Q$. There can be other indices $i$ such that $(\text{soc}_\tau^\sigma E_i)\tilde{a} \neq 0$, where $\tilde{a}$ is the image of $a$ in $Q/\mathbb{R}\tau$. For any such index $i$, as long as $y_i \neq 0$ it divides a unique cogenerator in $s_i \in \delta_\tau^\sigma E_i$ by Corollary 4.42. Therefore the image of $y$ in $E = \bigoplus_{j=1}^k E_j$ divides the sum of these cogenerators $s_j$. But that sum is itself another cogenerator of $E$ along $\tau$ with nadir $\sigma$ in degree $a$, and the fact that $y$ divides it places the sum in the image of the injection (Lemma 3.12) $\delta_\tau^\sigma M \hookrightarrow \delta_\tau^\sigma E$. \hfill $\square$

Remark 6.8. In terms of persistent homology, Theorem 6.7 says that a homomorphism of real multipersistence modules is injective if and only if it takes the “right endpoints” of the source injectively to a subset of the “right endpoints” of the target.

Corollary 6.9. Fix a downset-finite module $M$ over a real polyhedral group.

1. $M = 0$ if and only if $\text{soc}_\tau M = 0$ for all faces $\tau$.
2. $\text{soc}_\tau M' \cap \text{soc}_\tau M'' = \text{soc}_\tau (M' \cap M'')$ in $\text{soc}_\tau M$ for submodules $M'$ and $M''$ of $M$.

Proof. That $M = 0 \Rightarrow M = 0$ is trivial. On the other hand, if $\text{soc}_\tau M = 0$ for all $\tau$ then $M$ is a submodule of $0$ by Theorem 6.7.2.

The second equality follows from left-exactness (Proposition 4.44):

\[
\text{soc}_\tau (M' \cap M'') = \text{soc}_\tau \ker(M' \rightarrow M/M''')
= \ker(\text{soc}_\tau M' \rightarrow \text{soc}_\tau (M/M'''))
= \ker(\text{soc}_\tau M' \rightarrow \text{soc}_\tau M/\text{soc}_\tau M'')
= \text{soc}_\tau M' \cap \text{soc}_\tau M'' ,
\]

where the penultimate equality is because $\text{soc}_\tau M''$ is the kernel of the homomorphism $\text{soc}_\tau M \rightarrow \text{soc}_\tau (M/M'')$, so that $\text{soc}_\tau M/\text{soc}_\tau M'' \hookrightarrow \text{soc}_\tau (M/M'')$. \hfill $\square$

There is a much stronger statement connecting socles to essential submodules (Theorem 8.5), but it requires language to speak of density in socles as well as tools to produce submodules from socle elements, which are the main themes of Section 7.
7. Minimality of socle functors

Socles capture the entirety of a downset by maximal elements in closures along faces; that is in some sense the main content of socle essentiality (Theorem 6.7), or more precisely Theorem 6.5. But since closures are involved, it is reasonable to ask if anything smaller still captures the entirety of every downset. Algebraically, for arbitrary modules, this asks for subfunctors of cogenerator functors. The particular subfunctors here concern the graded degrees of socle elements, for which notation is needed.

Definition 7.1. The degree set of any module $N$ over a poset $P$ is

$$\deg N = \{ a \in P \mid N_a \neq 0 \}.$$  

Write $\deg_P = \deg$ if more than one poset could be intended.

7.1. Neighborhoods of group elements.

The topological condition characterizing when enough cogenerators are present is a sort of density in the set of all cogenerators. Lemma 3.20 has a related closure notion. Recall the statement and context of Lemma 3.8, which says that $Q_{\nabla^\sigma} = \sigma^\circ + Q_+.$

Definition 7.2. Fix faces $\sigma \supseteq \tau$ of a real polyhedral group $Q$. A $\sigma$-vicinity of a point $\bar{a} \in Q/\mathbb{R}\tau$ is a subset of $Q/\mathbb{R}\tau$ of the form $(u + Q_{\nabla^\sigma})/\mathbb{R}\tau$ such that $a - u \in \sigma^\circ$ and $a$ is a representative for the the coset $\bar{a} = a + \mathbb{R}\tau \in Q/\mathbb{R}\tau$.

Lemma 7.3. The $\sigma$-vicinities of points in $Q/\mathbb{R}\tau$ form a base for a topology on $Q/\mathbb{R}\tau$. More strongly, the intersection of any finite set of $\sigma$-vicinities (for perhaps different points in $Q/\mathbb{R}\tau$) contains a $\sigma$-vicinity of each point in their intersection.

Proof. The $\sigma$-vicinities of the points of $Q/\mathbb{R}\tau$ cover $Q/\mathbb{R}\tau$ because $\bar{a}$ lies in any of its $\sigma$-vicinities. So it suffices to prove the stronger claim, which by induction reduces to: the intersection of any $\sigma$-vicinity $(u + Q_{\nabla^\sigma})/\mathbb{R}\tau$ of $\bar{a}$ and any $\sigma$-vicinity $(v + Q_{\nabla^\sigma})/\mathbb{R}\tau$ of $\bar{b}$ contains a $\sigma$-vicinity of each point $\bar{c}$ in their intersection. Pick a coset representative $c \in Q$ for $c + \mathbb{R}\tau = \bar{c}$. Translating $c$ by a vector far inside of $\tau^\circ$, if necessary, assume that $c \in u + Q_{\nabla^\sigma}$. Perhaps translating along $\tau^\circ$ further, assume that $c \in v + Q_{\nabla^\sigma}$ also. Then $u + Q_{\nabla^\sigma}$ contains the intersection with $c - \sigma^\circ$ of an open neighborhood of $c$, as does $v + Q_{\nabla^\sigma}$. Any vector $w \in c - \sigma^\circ$ in the intersection of these neighborhoods yields a $\sigma$-vicinity $(w + Q_{\nabla^\sigma})/\mathbb{R}\tau$ of $\bar{c}$ contained in both of the given $\sigma$-vicinities. \qed

Definition 7.4. The topology in Lemma 7.3 is called the $\nabla^\sigma$-topology on $Q/\mathbb{R}\tau$.

Remark 7.5. In this paper, all topological notions in real vector spaces—limit, closure, neighborhood, and so on—refer to the usual topology unless explicitly otherwise stated. For example, Remark 7.9 refers to $\sigma$-closure, $\nabla^\sigma$-closure, and $\nabla^\sigma$-open neighborhoods.
Remark 7.6. The $\nabla\sigma$-topologies for various $\sigma$ are more general than the $\gamma$-topologies from [KS18] because the cone $\nabla\sigma$ is not necessarily closed. Its non-closedness reflects directions that ought to be thought of as inverted, and the image of $\nabla\sigma$ in the collapse modulo the inverted directions is closed, but the $\nabla\sigma$-topology is needed on the vector space before this collapse.

Remark 7.7. The strength of Lemma 7.3 beyond providing a base for a topology rests on a $\sigma$-vicinity of $\tilde{a}$ not being the same as a basic $\nabla\sigma$-open set containing $\tilde{a}$. Indeed, a $\sigma$-vicinity $u + Q_{\nabla\sigma}$ is required to contain a representative for $\tilde{a}$ that lies in the face $u + \sigma^o$, not merely somewhere arbitrary in $u + Q_{\nabla\sigma}$.

Definition 7.8. Fix faces $\sigma \supseteq \tau$ of a real polyhedral group $Q$.

1. A $\sigma$-limit point of a subset $X \subseteq Q/\mathbb{R}\tau$ is a point $\tilde{a} \in Q/\mathbb{R}\tau$ that is a limit (in the usual topology) of points in $X$ each of which lies in a $\sigma$-vicinity of $\tilde{a}$.

2. The $\sigma$-closure of $X \subseteq Q/\mathbb{R}\tau$ is the set of points $\tilde{a} \in Q/\mathbb{R}\tau$ such that $X$ has at least one point in every $\sigma$-vicinity of $\tilde{a}$.

Remark 7.9. The $\sigma$-closure of $X$ equals its $\nabla\sigma$-closure, by which is meant the closure of $X$ in the $\nabla\sigma$-topology. The reason: every basic $\nabla\sigma$-open neighborhood of a point contains a $\sigma$-vicinity of that point by Lemma 7.3.

Remark 7.10. The sets $X$ to which Definition 7.8 is applied are typically decomposed as finite unions of antichains (but see Proposition 7.12 for an instance where this is not the case). Such sets “cut across” subsets of the form $(u + Q_{\nabla\sigma})/\mathbb{R}\tau$, rather than being swallowed by them, so $\sigma$-vicinities have a fighting chance of reflecting some concept of closeness in antichains. If $\sigma = Q_+$ and $\tau = \{0\}$, for example, and $X$ is an antichain in $Q$, then a $\sigma$-vicinity of a point $a \in X$ is the same thing as a usual open neighborhood of $a$ in $X$, so $\sigma$-closure is the usual topological closure. If, at the other extreme, $\sigma = \tau$, then every antichain in $Q/\mathbb{R}\tau$ is $\tau$-closed.

Example 7.11. Let $Q = \mathbb{R}^2$ and $\tau = \{0\}$. Take for $X \subset \mathbb{R}^2$ the convex hull of $0, e_1, e_2$ but with the first standard basis vector $e_1$ removed. If $\sigma$ is the $x$-axis of $\mathbb{R}^2_+$, then in

$$X = \text{the blue points of}$$

constitute a $\sigma$-vicinity of $e_1$ in $X$. In addition, the bold blue segment in the half-open hypotenuse $H$ is a $\sigma$-vicinity of $e_1$ in $H$. The point $e_1$ itself is a $\sigma$-limit point of $H$.

The concept of $\sigma$-vicinity provides a means to connect socles (Definition 4.29) with support (Definition 2.25) and primary decomposition (Definition 2.30).
Proposition 7.12. In a real polyhedral group \( Q \), every cogenerator of a downset \( D \) along \( \tau \) with nadir \( \sigma \) has a \( \sigma \)-vicinity \( O \) in \( D \subseteq Q \) (so \( \sigma \supseteq \{0\} \) are the faces in Definition 7.8) such that \( k[O] \subseteq k[D] \) is \( \tau \)-coprimary and globally supported on \( \tau \).

Proof. Let \( a \) be such a cogenerator of \( D \). Suppose \( \{a_k\}_{k \in \mathbb{N}} \subseteq a - \sigma^0 \subseteq D \) is any sequence converging to \( a \). If \( a_k \) is supported on a face \( \tau' \), then \( \tau' \supseteq \tau \) because \( a \succeq a_k \) and \( a \) remains a cogenerator of the localization of \( D \) along \( \tau \) by Proposition 4.40. The same argument shows that the \( \sigma \)-vicinity \( O = a_k + Q\sigma \) yields a submodule \( k[O] \subseteq k[D] \) such that \( k[O] \hookrightarrow k[D] \) is \( \tau \)-coprimary and globally supported on \( \tau \).

If each \( a_k \) is supported on a face properly containing \( \tau \), then, restricting to a subsequence if necessary, assume that it is the same face \( \tau' \) for all \( k \). (This uses the finiteness of the number of faces.) But then \( a + \tau' = \lim_k(a_k + \tau') \) is contained in \( \delta^\sigma D \) by Definition 3.11, contradicting the fact that \( a \) is supported on \( \tau \) in \( \delta^\sigma D \). \( \square \)

Example 7.13. All three of the downsets

\[
\begin{align*}
D_1 & \quad D_2 & \quad D_3
\end{align*}
\]

in \( \mathbb{R}^2 \) have a cogenerator at the open corner \( a \) along the face \( \tau = \{0\} \), but their behaviors near \( a \) differ in character. Write \( \sigma_x \) and \( \sigma_y \) for the faces of \( \mathbb{R}^2_+ \) that are its horizontal and vertical axes, respectively.

1. Here \( a \) has two nadirs: it is a cogenerator along \( \tau = \{0\} \) for both \( \sigma_x \) and \( \sigma_y \) by Proposition 3.21 and Example 4.3. The blue set in Example 7.11 constitutes a \( \tau \)-coprimary \( \sigma_x \)-vicinity of \( a \) globally supported on \( \tau \), as in Proposition 7.12, if the open point there is also \( a \).

2. Here \( a \) has only the nadir \( \sigma_x \), because the downset has no points in \( a + \mathbb{R}\sigma_y \) to take the closure of in Lemma 3.20. Again, the blue set in Example 7.11 constitutes the desired \( \sigma_x \)-vicinity of \( a \).

3. Here \( a \) has only the nadir \( \sigma_y \). It is possible to compute this directly, but it is more apropos to note that Proposition 7.12 rules out \( \sigma_x \) as a nadir. Indeed, every \( \sigma_x \)-vicinity of \( a \) in \( D_3 \) is an infinite vertical strip. None of these \( \sigma_x \)-vicinities are supported on \( \tau = \{0\} \), since elements therein persist forever along \( \sigma_y \). In contrast, every choice of \( v \in \sigma_y^0 \) yields a \( \sigma_y \)-vicinity \( (-v + \mathbb{R}\sigma_y) \cap D_3 \) supported on \( \{0\} \); that is, the entire negative \( y \)-axis is supported on \( \{0\} \).

Compare the following with Example 4.43; it is the decisive more or less explicit calculation that justifies the general theory of socles and provides its foundation.
**Corollary 7.14.** Fix a face $\tau$ of a real polyhedral group $Q$ and a subquotient $M$ of $k[Q]$ that is $\tau$-coprimary and globally supported on $\tau$. Then $soc_{\tau'} M = 0$ unless $\tau' = \tau$.

*Proof.* Proposition 4.40 implies that $soc_{\tau'} M = 0$ unless $\tau' \supseteq \tau$ by definition of global support: localizing along $\tau'$ yields $M_{\tau'} = 0$ unless $\tau' \subseteq \tau$. On the other hand, $M$ being a subquotient of $k[Q]$ means that $M \subseteq k[D]$ for some downset $D$. By left-exactness of socles (Proposition 4.44), every cogenerator of $M$ is a cogenerator of $k[D]$. Applying Proposition 7.12 to any such cogenerator along $\tau'$ implies that $\tau = \tau'$, because no $\tau$-coprimary module has a submodule supported on a face strictly contained in $\tau$. \(\square\)

**Remark 7.15.** Remember that being $\tau$-coprimary does not require the whole module to be globally supported on $\tau$; only an essential submodule need be globally supported on $\tau$. This occurs for the global support on $\tau = \{0\}$ in Example 2.26, which is strictly contained in the corresponding $\tau$-primary component from Example 1.5. Dually, being globally supported on $\tau$ allows for elements with support strictly contained in $\tau$.

**Remark 7.16.** Definition 7.8.1 stipulates no condition the generators of the relevant $\sigma$-limitivities—the vectors $u$ in Definition 7.2. The a priori difference between being a $\sigma$-limit point and lying in the $\sigma$-closure is hence that for $\sigma$-closure, the convergence is stipulated on the generators of the $\sigma$-vicinities rather than on the points of $X$. That said, the a priori weaker (that is, more inclusive) notion of $\sigma$-limit point is equivalent: the generators can be forced to converge.

**Proposition 7.17.** If a sequence $\{a'_k\}_{k \in \mathbb{N}}$ in a real polyhedral group $Q$ has $a'_k \to a$ and $a'_k \in a_k + Q_{\nu_\sigma}$ for some $a_k \in a - \sigma^0$, where $\sigma$ is a fixed face, then it is possible to choose the elements $a'_k$ so that $a_k \to a$. Consequently, if $\sigma \supseteq \tau$ then the $\sigma$-closure of any set $X \subseteq Q/R\tau$ equals the set of its $\sigma$-limit points.

*Proof.* Writing $a'_k = a - v_k + z_k$ with $v_k \in \mathbb{R}\sigma$ and $z_k \in \sigma^\perp$, the only relevant consequence of the hypothesis $a'_k \in a_k + Q_{\nu_\sigma}$ is to force $z_k$ to land in $Q_+ / \mathbb{R}\sigma$ when projected to $Q / \mathbb{R}\sigma$. Consider the set $Z \subseteq \sigma^\perp$ of vectors in $\sigma^\perp$ whose images in $Q / \mathbb{R}\sigma$ lie in $Q_+ / \mathbb{R}\sigma$ and have magnitude $\leq 1$. Let $V \subseteq \mathbb{R}\sigma$ be the ball of radius $1$. Find $s \in \sigma^0$ so that $s + V + Z \subseteq Q_+$. To see that such an $s$ exists, first find $s' \in \sigma^0$ so that $s' + Z \subseteq Q_+$. To construct $s'$, rescale any element $s'' \in \sigma^0$; this works because the projection of $(s'' + \sigma^\perp) \cap Q_+$ to $Q / \mathbb{R}\sigma$ contains a neighborhood of $0$ in $Q_+ / \mathbb{R}\sigma$. Then observe that the condition $s' + Z \subseteq Q_+$ remains true after adding any element of $\sigma$ to $s'$. In particular, construct $s$ by adding the center of any ball in $\sigma^0$ of radius $1$, which exists because $\sigma^0$ is nonempty, open in $\mathbb{R}\sigma$, and closed under positive scaling.

Having fixed $s$ with $s + V + Z \subseteq Q_+$, set $a_k = a - 2\varepsilon_k s$, where $\varepsilon_k = |a'_k - a|$. The reason for this choice of $\varepsilon_k$ is that $2\varepsilon_k \to 0$ (because $a'_k \to a$) and $\varepsilon_k$ bounds the magnitudes of $v_k$ and $z_k$. This latter condition implies $a'_k = a - v_k + z_k \in a + \varepsilon_k V + \varepsilon_k Z = (a - 2\varepsilon_k s) + \varepsilon_k s + (\varepsilon_k s + \varepsilon_k V + \varepsilon_k Z) \subseteq a_k + \sigma^0 + Q_+ = a_k + Q_{\nu_\sigma}$ by Lemma 3.8.
The claim involving $\tau$ follows, when $\tau = \{0\}$, from Proposition 3.7: it implies that each element $b \in a - \sigma^o$ precedes some $a_k$, and hence the $\sigma$-vicinity generated by $b$ contains $a_k'$. The case of arbitrary $\tau$ reduces to $\tau = \{0\}$ by working modulo $R\tau$. □

7.2. Dense cogeneration of downsets.

The subfunctor version of density in socles for modules requires first a geometric version for downsets. For geometric intuition, it is useful to again recall Lemma 3.8, which says that $Q_{\nabla^o} = \sigma^o + Q_+$. Thus $a - Q_{\nabla^o}$ is the “coprincipal” downset with apex $a$ and shape $\nabla^o$. Adding $\tau$ to get $a + \tau - Q_{\nabla^o} = a' - Q_{\nabla^o}$ per coset $\tilde{a} = a + R\tau$.

**Lemma 7.18.** Let $Q$ be a real polyhedral group with faces $\sigma \supseteq \tau$. Write $\tilde{a} \in Q/R\tau$ for the coset $a + R\tau$ containing $a \in Q$. Then $a + \tau - Q_{\nabla^o} = a' + \tau - Q_{\nabla^o}$ for all $a' \in \tilde{a}$. Hence there is only one downset $a + \tau - Q_{\nabla^o} = \tilde{a} - Q_{\nabla^o}$ per coset $\tilde{a} = a + R\tau$.

**Proof.** Using Lemma 3.8 to write $a + \tau - Q_{\nabla^o} = a + \tau - \sigma^o - Q_+$, this set is translation-invariant along $R\tau$ because $-Q_+$ contains $-\tau$. □

The question is which of these downsets must appear in any decomposition.

**Theorem 7.19.** Let $A^o \subseteq Q$ be a set of cogenerators of a downset $D$ in a real polyhedral group $Q$ along a face $\tau$ with nadir $\sigma$ for each $\sigma \in \nabla^o$, and let $A_\tau = \bigcup_{\sigma \supseteq \tau} A^o_{\sigma\tau}$. Then

$$D = \bigcup_{\text{faces } \sigma, \tau} \bigcup_{a \in A^o_{\sigma\tau} \text{ with } \sigma \supseteq \tau} a + \tau - Q_{\nabla^o}$$

as long as the $\sigma$-closure of the image of $A_\tau$ in $Q/R\tau$ contains the projection modulo $R\tau$ of every cogenerator of $D$ along $\tau$ with nadir $\sigma$.

**Proof.** Theorem 6.5 is equivalent to the desired result in the case that every $A^o_{\sigma\tau}$ is the set of all cogenerators of $D$ along $\tau$ with nadir $\sigma$, by Example 4.39 and Remark 4.34. Hence it suffices to show that

$$\bigcup_{\sigma' \supseteq \tau} \bigcup_{a' \in A^o_{\sigma'}} a' + \tau - Q_{\nabla^o} \supseteq a + \tau - Q_{\nabla^o}$$

for any fixed cogenerator $a$ of $D$ along $\tau$ with nadir $\sigma$. In fact, by definition of $\sigma$-limit point, it is enough to show that

$$\bigcup_{k=1}^\infty a_k' + \tau - Q_{\nabla^o} \supseteq a + \tau - Q_{\nabla^o}$$

where $\{a_k\}_{k \in \mathbb{N}}$ is a sequence of elements of $A_\tau$ such that

- $a_k$ lands in a $\sigma$-vicinity of the image $\tilde{a}$ of $a$ when projected to $Q/R\tau$, and
- these images $\tilde{a}_k'$ converge to $\tilde{a}$ in $Q/R\tau$.
and \( \sigma_k \) is a nadir of the cogenerator \( \mathbf{a}'_k \) along \( \tau \).

Note that there is something to prove even when \( \sigma = \tau \) (see the end of Remark 7.10) because \( A'_\tau \) only needs to have at least one closed cogenerator in \( Q \) for each closed socle degree in \( Q/\mathbb{R}\tau \), whereas the set of all closed cogenerators along \( \tau \) mapping to a given socle degree might not be a single translate of \( \tau \). On the other hand, \( \tau - Q_{\nabla \tau} = \tau - \tau^\circ - Q_+ \) by Lemma 3.8, and this is just \( \mathbb{R}\tau - Q_+ \). Therefore \( \mathbf{a} + \tau - Q_{\nabla \tau} \) contains the translate of the negative cone \( -Q_+ \) at every point mapping to \( \tilde{\mathbf{a}} \), cogenerator or otherwise, completing the case \( \sigma = \tau \).

For general \( \sigma \supseteq \tau \), Lemma 7.18 reduces the question to the quotient \( Q/\mathbb{R}\tau \), where it becomes

\[
\bigcup_{k=1}^{\infty} \tilde{\mathbf{a}}'_k - Q_{\nabla \sigma_k}/\tau \supseteq \tilde{\mathbf{a}} - Q_{\nabla \sigma}/\tau.
\]

But as \( Q_{\nabla \sigma_k}/\tau = \sigma_k^\circ/\tau + (Q/\mathbb{R}\tau)_+ \) by Lemma 3.8, it does no harm (and helps the notation) to assume that \( \tau = \{0\} \). The desired statement is now

\[
\bigcup_{k=1}^{\infty} \mathbf{a}'_k - Q_{\nabla \sigma_k} \supseteq \mathbf{a} - Q_{\nabla \sigma},
\]

the hypotheses being those of Proposition 7.17. The proof is completed by applying Proposition 3.7 to the sequence \( \{\mathbf{a}_k\}_{k \in \mathbb{N}} \) produced by Proposition 7.17, noting that \( \mathbf{a}_k - Q_+ \subseteq \mathbf{a}'_k - Q_{\nabla \sigma_k} \) as soon as \( \mathbf{a}_k \in \mathbf{a}'_k - Q_{\nabla \sigma_k} \), because \( \mathbf{a}'_k - Q_{\nabla \sigma_k} \) is a downset. \( \square \)

**Example 7.20.** Consider the downsets in Example 7.13. The question is whether \( \mathbf{a} \) is forced to appear in the union from Theorem 7.19 or not.

1. The point \( \mathbf{a} \) is a \( \sigma_x \)-limit point of \( D_1 \) by Example 7.11, which shares its geometry with \( D_1 \) on the relevant set, namely the \( x \)-axis and above. Thus Theorem 7.19 wants to force \( \mathbf{a} \) to appear. And indeed it appears, but only because of the \( y \)-axis: every \( \sigma_y \)-vicinity of \( \mathbf{a} \) in \( D_1 \) contains exactly one cogenerator, namely \( \mathbf{a} \) itself.

2. In contrast, \( \mathbf{a} \) is not needed for \( D_2 \). Abstractly, this is because \( D_2 \) is missing precisely the negative \( y \)-axis that caused \( \mathbf{a} \) to be forced in \( D_1 \). But geometrically it is evident that \( D_2 \) equals the union of the closed negative quadrants hanging from the open diagonal ray.

3. Here \( \mathbf{a} \) is the sole cogenerator of \( D_3 \) along \( \tau = \{0\} \), so it is forced to appear.

**Example 7.21.** The curve atop each of the following two downsets in \( \mathbb{R}^2 \) is a hyperbola.
1. Plucking out a single point from the hyperbola has an odd effect. At the frontier point Definition 4.29 detects two open cogenerators, with different nadirs as in Example 1.7, but they are redundant: $D_1$ equals the union of the closed negative orthants cogenerated by the points along the rest of the hyperbola.

2. The ability to omit cogenerators is even more striking upon deleting an interval from the hyperbola, instead of a single point, to get the downset $D_2$. Along the deleted curve, Definition 4.29 detects cogenerators of the same shape as those for $D_1$. Hence $D_2$ is the union of coprincipal downsets of these shapes along the deleted curve together with closed coprincipal downsets along the rest of the hyperbola. However, any finite number of the coprincipal downsets along the deleted curve can be omitted, as can be checked directly. In fact, any subset of them that is dense in the deleted curve can be omitted by Theorem 7.19. Note that a closed negative orthant is required at the lower endpoint of the deleted curve, because the endpoint has not been deleted, whereas the cogenerator at the upper endpoint of the deleted curve can always be omitted because of open negative orthants hanging from points along the hyperbola just below it.

Here is a restatement of Theorem 6.5, phrased as a special case of Theorem 7.19, in terms of coprincipal downsets via Lemma 7.18.

**Corollary 7.22.** Every downset $D$ in a real polyhedral group $Q$ is the union of the coprincipal downsets $\tilde{a} - Q_{\sigma}$ indexed by the degrees $\tilde{a} \in Q/\mathbb{R}\tau$ of socle elements of $k[D]$ along all faces $\tau$ with all nadirs $\sigma$:

$$D = \bigcup_{\text{faces } \sigma, \tau} \bigcup_{\tilde{a} \in \deg_{Q/\mathbb{R}\tau} \soc_{\tau}^0 k[D]} \tilde{a} - Q_{\sigma}.$$

**Remark 7.23.** Theorem 7.19 is an analogue for real polyhedral groups of the fact that monomial ideals in affine semigroup rings admit unique irredundant irreducible decompositions [MS05, Corollary 11.5]. To see the analogy, note that expressing a downset as a union is the same as expressing its complementary upset as an intersection. In Theorem 7.19 the union is neither unique nor irredundant, but only in the sense that a topological space can have many dense subsets, each of which can usually be made smaller by omitting some points. The union in Corollary 7.22 is still canonical, though redundant in a predictable manner, namely that of Theorem 7.19. The analogy in this remark is not the tightest possible; see Section 9.2 for the true analogy.

**Remark 7.24.** In the case of $Q = \mathbb{R}^n$ with componentwise partial order, Ingebretson and Sather-Wagstaff characterized the downsets in $Q_+$ that admit decompositions as in Theorem 7.19 with finitely many terms [ISW13, Theorem 4.12]: they are the ones whose complementary upsets have finitely many generators in the sense of [Mil20d]. The new aspects here, for arbitrary downsets, are the related notions of minimality and density.
of cogenerating sets. Theorem 7.19 implies the characterization of irreducible downsets [ISW13, Theorem 3.9] because irreducible decompositions are assumed finite there.

The final result in this subsection is applied in the proof of Theorem 7.27.

**Corollary 7.25.** Fix a cogenerator \( \mathbf{a} \) of a downset \( D \) along a face \( \tau \) with nadir \( \sigma \) in a real polyhedral group. If \( \mathbf{b} \in D \) and \( \mathbf{b} \preceq \mathbf{a} \), then the image \( \tilde{\mathbf{a}} \) of \( \mathbf{a} \) in \( \mathbb{Q}/\mathbb{R} \tau \) has a \( \sigma \)-vicinity \( \mathcal{O} \) in \( \deg_{\mathbb{Q}/\mathbb{R} \tau} \soc_{\tau} k[D] \) such that \( \tilde{\mathbf{b}} \preceq \tilde{\mathbf{a}} \) for all \( \tilde{\mathbf{a}} \in \mathcal{O} \).

**Proof.** Assume \( \mathbf{b} \in D \) and \( \mathbf{b} \preceq \mathbf{a} \). Theorem 7.19 implies that \( \mathbf{b} \in \mathbf{a} + \mathbf{R} \tau - \sigma - \mathbf{Q} = \mathbf{a} + \mathbf{R} \tau - \sigma - \mathbf{Q}_+ \). Therefore \( \mathbf{a} + \mathbf{R} \tau = \mathbf{b} + \mathbf{R} \tau + \mathbf{s} + \mathbf{q} \) for some \( \mathbf{s} \in \sigma \) and \( \mathbf{q} \in \mathbf{Q}_+ \). The \( \sigma \)-vicinity in question is \( (\tilde{\mathbf{b}} + \tilde{\mathbf{q}} + \mathbf{Q} \sigma) \cap \deg_{\mathbb{Q}/\mathbb{R} \tau} \soc_{\tau} k[D] \). \( \square \)

7.3. Dense subfunctors of socles.

In general, a subfunctor \( \Phi : A \to B \) of a covariant functor \( \Psi : A \to B \) is a natural transformation \( \Phi \to \Psi \) such that \( \Phi(A) \subseteq \Psi(A) \) for all objects \( A \in A \) [EM45, Chapter III]; denote this by \( \Phi \subseteq \Psi \). (This notation assumes that the objects of \( B \) are sets, which they are here; in general, \( \Phi(A) \to \Psi(A) \) should be monic.)

**Definition 7.26.** A subfunctor \( S_\tau = \bigoplus_{\sigma \in \nabla \tau} S_\sigma \subseteq \soc_{\tau} \) from modules over \( \mathbb{Q} \) to modules over \( \mathbb{Q}/\mathbb{R} \tau \times \nabla \tau \) is dense if the \( \sigma \)-closure of \( \deg_{\mathbb{Q}/\mathbb{R} \tau} S_\tau k[D] \) contains \( \deg \soc_{\tau} k[D] \) for all faces \( \sigma \supseteq \tau \) and downsets \( D \subseteq \mathbb{Q} \) in the proof of Theorem 7.27. Fix subfunctors \( S_\tau \subseteq \soc_{\tau} \) for all faces \( \tau \) of a real polyhedral group.

**Theorem 7.27.** Fix subfunctors \( S_\tau \subseteq \soc_{\tau} \) for all faces \( \tau \) of a real polyhedral group. Theorem 6.7 holds with \( S \) in place of \( \soc \) if and only if \( S_\tau \) is dense in \( \soc_{\tau} \) for all \( \tau \).

**Proof.** Every subfunctor of any left-exact functor takes injections to injections; therefore Theorem 6.7.1 holds for any subfunctor of \( \soc_{\tau} \) by Proposition 4.44. The content is that Theorem 6.7.2 is equivalent to density of \( S_\tau \) in \( \soc_{\tau} \) for all \( \tau \).

First suppose that \( S_\tau \) is dense in \( \soc_{\tau} \) for all \( \tau \). It suffices to show that each homogeneous element \( y \in M \) divides some \( S \)-cogenerator \( s \), for then \( \varphi(y) \neq 0 \) whenever \( S \varphi(s) \neq 0 \), where \( s \) is the image of \( s \) in \( S_\tau M \subseteq \soc_{\tau} M \). There is no harm in assuming that \( M \) is a submodule of its downset hull: \( M \subseteq E = \bigoplus_{j=1}^k E_j \). Theorem 6.7 produces a cogenerator \( x \in E \) that is divisible by \( y \), and \( x \) is automatically a cogenerator of \( M \) — say \( x \in \delta_{\tau} E \). For any index \( j \) such that \( x_j \neq 0 \), Corollary 7.25 and the density hypothesis yields a \( \sigma \)-vicinity of \( \tilde{\mathbf{a}} \) containing a socle element \( \tilde{s}_j \) mapped to by an \( S \)-cogenerator \( s_j \) that is divisible by \( y_j \). An \( S \)-cogenerator \( s \) of \( M \) divisible by \( y \) is constructed from \( s_j \) just as an ordinary cogenerator is constructed from \( s_j \) in the proof of Theorem 6.7.

Now suppose that \( S_\tau \) is not dense in \( \soc_{\tau} \) for some face \( \tau \), so some downset \( D \subseteq \mathbb{Q} \) has a cogenerator \( \mathbf{a} \in \mathbb{Q} \) whose image \( \tilde{\mathbf{a}} \in \deg \soc_{\tau} k[D] \subseteq \mathbb{Q}/\mathbb{R} \tau \) has a \( \sigma \)-vicinity \( \deg_{\mathbb{Q}/\mathbb{R} \tau} \soc_{\tau} k[D] \cap (\mathbf{u} + \mathbf{Q} \sigma)/\mathbb{R} \tau \) devoid of images of \( S \)-cogenerators along \( \tau \). Appealing to Lemma 7.3, the intersection of \( \mathbf{u} + \mathbf{Q} \sigma \) with a \( \sigma \)-vicinity \( \mathcal{O} \) of \( \mathbf{a} \) in \( D \)
from Proposition 7.12 contains another \( \sigma \)-vicinity \( \mathcal{O}' \) of \( \mathfrak{a} \) that still satisfies the conclusion of Proposition 7.12 because every submodule of any \( \tau \)-coprimary module globally supported on \( \tau \) is also \( \tau \)-coprimary and globally supported on \( \tau \). The injection \( k[\mathcal{O}'] \hookrightarrow k[D] \) yields an injection \( \mathcal{S}_\tau k[\mathcal{O}'] \hookrightarrow \mathcal{S}_\tau k[D] \), but by construction \( \mathcal{S}_\tau k[D] \) vanishes in all degrees from \( \deg_{\mathbb{Q}/\mathbb{R}_\tau} \text{soc}_\tau k[\mathcal{O}'] \), so \( \mathcal{S}_\tau k[\mathcal{O}'] = 0 \). On the other hand, \( \text{soc}_{\tau'} k[\mathcal{O}'] = 0 \) for \( \tau' \neq \tau \) by Corollary 7.14, so the subfunctor \( \mathcal{S}_{\tau'} \) vanishes on \( k[\mathcal{O}'] \) for all faces \( \tau' \). Consequently, applying \( \mathcal{S}_{\tau'} \) to the homomorphism \( \varphi : k[\mathcal{O}'] \to 0 \) yields an injection \( 0 \hookrightarrow 0 \) for all faces \( \tau' \) even though \( \varphi \) is not injective. \( \Box \)

8. ESSENTIAL SUBMODULES VIA DENSITY IN SOCLES

Given a closed cogenerator of \( M \), there is an obvious submodule of \( M \) containing the socle element, namely the submodule generated by the cogenerator itself. In contrast, open cogenerators are not elements of \( M \) itself. How, then, do cogenerators detect injectivity in Theorem 6.7? Each cogenerator must still yield a submodule to witness the injectivity, because injectivity means there is no actual submodule of \( M \) that goes to 0. The cogenerator merely indicates the presence of such a submodule, rather than being an element of it. This section reconstructs an honest submodule around each cogenerator. It requires much of the theory in earlier sections. As a consequence, a submodule \( M' \subseteq M \) is an essential submodule precisely when the socle of \( M' \) is dense in that of \( M \) (Theorem 8.5).

More precisely, the \( \sigma \)-vicinities in Proposition 7.12 transfer cogenerators back into honest submodules; they are, in that sense, the reverse of Definition 3.11. In fact this transference of cogenerators into submodules works not merely for indicator quotients but for arbitrary modules with finite downset hulls, as in Theorem 8.5. The key is the generalization of \( \sigma \)-vicinities to arbitrary downset-finite modules.

**Definition 8.1.** Fix a module \( M \) over a real polyhedral group \( \mathcal{Q} \) and a face \( \tau \).

1. A \( \sigma \)-divisor (Definition 3.19) \( y \in M \) of a cogenerator of \( M \) along \( \tau \) with nadir \( \sigma \) (Definition 4.29) is nearby if \( y \) is globally supported on \( \tau \) (Definition 2.25).
2. A \( \sigma \)-vicinity in \( M \) of a cogenerator \( s \in \delta^\sigma \mathcal{Q} M \) is a submodule of \( M \) generated by a nearby \( \sigma \)-divisor of \( s \).
3. A neighborhood in \( \text{soc}_\tau M \) of a homogeneous socle element \( \bar{s} \in \text{soc}_\tau^\sigma M \) is \( \text{soc}_\tau N \) for a \( \sigma \)-vicinity \( N \) in \( M \) of a cogenerator in \( \delta^\sigma \mathcal{Q} M \) that maps to \( \bar{s} \).
4. An inclusion \( \mathcal{S}_\tau \subseteq \text{soc}_\tau M \) of \( (\mathcal{Q}/\mathbb{R}_\tau \times \nabla_\tau) \)-modules is dense if for all \( \sigma \supseteq \tau \), every neighborhood of every homogeneous element of \( \text{soc}_\tau^\sigma M \) intersects \( \mathcal{S}_\tau \) nontrivially.

**Lemma 8.2.** Every neighborhood in \( M \) of every homogeneous element in \( \text{soc}_\tau^\sigma M \) is a \( \tau \)-coprimary submodule of \( M \) globally supported on \( \tau \).

**Proof.** Let \( y \) be a nearby \( \sigma \)-divisor of a cogenerator \( s \in \delta^\sigma \mathcal{Q} M \). Let \( x \) be a homogeneous multiple of \( y \). That \( x \) is supported on \( \tau \) is automatic from the hypothesis that \( y \) is supported on \( \tau \). To say that \( \langle y \rangle \) is \( \tau \)-coprimary means, given that it is supported
on \( \tau \), that \( \langle y \rangle \) is a submodule of its localization along \( \tau \). But \( s \) remains a cogenerator after localizing along \( \tau \) by Proposition 4.40, so \( x \) must remain nonzero because it still divides \( s \) after localizing.

**Proposition 8.3.** Fix a downset-finite module \( M \) over a real polyhedral group with faces \( \sigma \supseteq \tau \). Every cogenerator in \( \delta_\tau^\sigma M \) has a \( \sigma \)-vicinity in \( M \).

*Proof.* Let \( s \in \delta_\tau^\sigma M \) be the cogenerator, and let its degree be \( \deg_Q(s) = a \in Q \). Choose a downset hull \( M \hookrightarrow E = \bigoplus_{j=1}^k E_j \), so \( E_j = k[D_j] \) for a downset \( D_j \). Express \( s = s_1 + \cdots + s_k \in \delta_\tau E = \bigoplus_{j=1}^k \delta_\tau^\sigma E_j \). Proposition 7.12 produces a \( \sigma \)-vicinity \( O_j \) of \( a \) in \( Q \), for each index \( j \), such that \( k[O_j \cap D_j] \) is a \( \sigma \)-vicinity in \( E_j \) of the image \( \tilde{s}_j \in \text{soc}_\tau^\sigma E_j \). Lemma 7.3 then yields a single \( \sigma \)-vicinity \( O = a - v + Q_+ \) of \( a \) in \( Q \) that lies in the intersection \( \bigcap_{j=1}^k O_j \). The cogenerator \( s \in \delta_\sigma^\tau \) is a direct limit over \( a - \sigma^\circ \); since \( O \) contains a neighborhood (in the usual topology) of \( a \) in \( \sigma^\circ \), some element \( y \in M \) with degree in \( O \) is a \( \sigma \)-divisor of \( s \). This element \( y \) is nearby \( s \) by construction. \( \square \)

The following generalization of Corollary 7.14 to modules with finite downset hulls is again the decisive computation.

**Corollary 8.4.** Fix a downset-finite \( \tau \)-coprimary \( Q \)-module \( M \) globally supported on a face \( \tau \) of a real polyhedral group \( Q \). Then \( \text{soc}_\tau^\tau M = 0 \) unless \( \tau' = \tau \).

*Proof.* Proposition 4.40 implies that \( \text{soc}_\tau^\tau M = 0 \) unless \( \tau' \supseteq \tau \) by definition of global support: localizing along \( \tau' \) yields \( M_{\tau'} = 0 \) unless \( \tau' \subseteq \tau \). On the other hand, applying Proposition 8.3 to any cogenerator of \( M \) along a face \( \tau' \) implies that \( \tau = \tau' \), because no \( \tau \)-coprimary module has a submodule supported on a face strictly contained in \( \tau \). \( \square \)

**Theorem 8.5.** In a downset-finite module \( M \) over a real polyhedral group, \( M' \) is an essential submodule if and only if \( \text{soc}_\tau^\tau M' \subseteq \text{soc}_\tau^\tau M \) is dense for all faces \( \tau \).

*Proof.* First assume that \( M' \) is not an essential submodule, so \( N \cap M' = 0 \) for some nonzero submodule \( N \subseteq M \). Let \( s \in \delta_\tau^\sigma N \) be a cogenerator. Any \( \sigma \)-vicinity of \( s \) in \( N \), afforded by Proposition 8.3, has a socle along \( \tau \) that is a neighborhood of \( \tilde{s} \) in \( \text{soc}_\tau^\tau M \) whose intersection with \( \text{soc}_\tau^\tau M' \) is 0. Therefore \( \text{soc}_\tau^\tau M' \subseteq \text{soc}_\tau^\tau M \) is not dense.

Now assume that \( \text{soc}_\tau^\tau M' \subseteq \text{soc}_\tau^\tau M \) is not dense for some \( \tau \). That means \( \text{soc}_\tau^\sigma M \) for some nadir \( \sigma \) has an element \( \tilde{s} \) with a neighborhood \( \text{soc}_\tau N \) that intersects \( \text{soc}_\tau^\tau M' \) in 0. But \( \text{soc}_\tau N \cap \text{soc}_\tau^\sigma M' = \text{soc}_\tau(N \cap M') \) by Corollary 6.9.2. The vanishing of this socle along \( \tau \) means that \( \text{soc}_\tau(N \cap M') = 0 \) for all faces \( \tau' \) by Corollary 8.4, and thus \( N \cap M' = 0 \) by Corollary 6.9.1. Therefore \( M' \) is not an essential submodule of \( M \). \( \square \)

**Example 8.6.** The convex hull of \( 0, e_1, e_2 \) in \( \mathbb{R}^2 \) but with the first standard basis vector \( e_1 \) removed defines a subquotient \( M \) of \( k[\mathbb{R}^2] \). It has submodule \( M' \) that is the indicator module for the same triangle but with the entire \( x \)-axis removed. All of the cogenerators of both modules occur along the face \( \tau = \{0\} \) because both modules are
globally supported on \{0\}. However the ambient module—but not the submodule—has a cogenerator \( y \in \delta_\sigma^M \) with nadir \( \sigma = x\)-axis of degree \( e_1 \):

\[
\begin{array}{c}
\subseteq \\
\end{array}
\quad
\begin{array}{c}
\text{with } \sigma\text{-vicinities} \\
\end{array}
\quad
\begin{array}{c}
of e_1.
\end{array}
\]

A typical \( \sigma\)-vicinity of \( y \) in \( M \) is shaded in light blue (in fact, \( M \) itself is also a \( \sigma\)-vicinity of \( y \), with the corresponding vicinity in \( \text{soc}_\sigma^M \) in bold blue. Every such vicinity contains socle elements in \( \text{soc}_\sigma^M \), so \( M' \subseteq M \) is an essential submodule by Theorem 8.5. Trying to mimic this example in a finitely generated context is instructive: pixelated rastering of the horizontal lines either isolates the socle element at the right-hand endpoint of the bottom edge or prevents it from existing in the first place by aligning with the right-hand end of the line above it.

9. Primary decomposition over real polyhedral groups

This section takes the join of [Mil20b], which develops primary decomposition as far as possible over arbitrary polyhedral partially ordered groups, and Section 4, which develops socles over real polyhedral groups. That is, it investigates how socles interact with primary decomposition in real polyhedral groups.

9.1. Associated faces and coprimary modules.

What makes the theory for real polyhedral groups stronger than for arbitrary polyhedral partially ordered groups is the following notion familiar from commutative algebra, except that (as noted in Section 8) socle elements do not lie in the original module.

**Definition 9.1.** A face \( \tau \) of a real polyhedral group \( Q \) is associated to a downset-finite \( Q \)-module \( M \) if \( \text{soc}_\tau M \neq 0 \). If \( M = \mathbb{k}[D] \) for a downset \( D \) then \( \tau \) is associated to \( D \). The set of associated faces of \( M \) or \( D \) is denoted by \( \text{Ass}(M) \) or \( \text{Ass}(D) \).

**Theorem 9.2.** A downset-finite module \( M \) over a real polyhedral group is \( \tau \)-coprimary (Definition 2.27) if and only if \( \text{soc}_\tau M = 0 \) whenever \( \tau' \neq \tau \) or equivalently \( \text{Ass}(M) = \{\tau\} \).

**Proof.** If \( M \) is not \( \tau \)-coprimary then either \( M \to M_\tau \) has nonzero kernel \( N \), or \( M \to M_\tau \) is injective while \( M_\tau \) has a submodule \( N_\tau \) supported on a face strictly containing \( \tau \). In the latter case, moving up by an element of \( \tau \) shows that \( N = N_\tau \cap M \) is nonzero. In either case, any cogenerator of \( N \) lies along a face \( \tau' \neq \tau \), so \( 0 \neq \text{soc}_{\tau'} N \subseteq \text{soc}_{\tau'} M \).

On the other hand, if \( M \) is \( \tau \)-coprimary then \( \Gamma_\tau M \) is an essential submodule of \( M \) because every nonzero submodule of \( M \subseteq M_\tau \) has nonzero intersection with \( \Gamma_\tau M_\tau \), and hence with \( M \cap \Gamma_\tau M_\tau = \Gamma_\tau M \), inside of the ambient module \( M_\tau \) by Definition 2.27. Theorem 8.5 says that \( \text{soc}_{\tau'} \Gamma_\tau M \subseteq \text{soc}_{\tau'} M \) is dense for all \( \tau' \). But \( \text{soc}_{\tau'} \Gamma_\tau M = 0 \) for \( \tau' \neq \tau \) by Corollary 8.4, so density implies \( \text{soc}_{\tau'} M = 0 \) for \( \tau' \neq \tau \). \( \square \)
Lemma 9.3. A downset \( D \) in a real polyhedral group is \( \tau \)-coprimary if and only if
\[
D = \bigcup_{\text{faces } \sigma} \bigcup_{a \in A^\sigma_\tau} a + \tau - Q_{\nabla \sigma}
\]
for sets \( A^\sigma_\tau \subseteq Q \) such that the image in \( Q/\mathbb{R} \tau \times \nabla \tau \) of \( \bigcup_{\sigma \supseteq \tau} A^\sigma_\tau \times \{\sigma\} \) is an antichain, and in that case \( A^\sigma_\tau \) projects to a subset of \( \text{deg}_{Q/\mathbb{R} \tau} \soc^\sigma_\tau k[D] \subseteq Q/\mathbb{R} \tau \) for each \( \sigma \).

Proof. If \( D \) is \( \tau \)-coprimary, then it is such a union by Theorem 9.2 and Theorem 7.19, keeping in mind the antichain consequences of Example 4.11.
On the other hand, if \( D \) is such a union, then first of all it is stable under translation by \( \mathbb{R} \tau \) by Lemma 7.18. Working in \( Q/\mathbb{R} \tau \), therefore, assume that \( \tau = \{0\} \).
Example 4.11 implies that every element of \( A^\sigma_\tau \) is a cogenerator of \( D \) with nadir \( \sigma \).
Proposition 7.12 produces a \( \sigma \)-vicinity \( O^\sigma_\sigma \) of \( a \) in \( D \) that is globally supported on \( \{0\} \) (and hence \( \{0\} \)-coprimary). But every element \( b \in D \) that precedes \( a \) also precedes some element in \( O^\sigma_\sigma \); that is, \( b \leq a \Rightarrow (b + Q_\tau) \cap O^\sigma_\sigma \neq \emptyset \). The union of the \( \sigma \)-vicinities \( O^\sigma_\sigma \) over all faces \( \sigma \) and elements \( a \in A^\sigma_\tau \) therefore cogenerates \( D \), so \( D \) is coprimary by Proposition 2.29. \( \square \)

Remark 9.4. The antichain condition in Lemma 9.3 is necessary: \( Q \) itself is the union of all translates of \( -Q \), but \( Q \) is \( Q_+ \)-coprimary, whereas \( -Q \) is \( \{0\} \)-coprimary. Moreover, the \( \nabla \tau \) component of the antichain condition is important; that is, the nadirs also come into play. For a specific example, take \( D \subseteq \mathbb{R}^2 \) to be the union of the open negative quadrant cogenerated by \( 0 \) and the closed negative quadrant cogenerated by any point on the strictly negative \( x \)-axis. The \( Q \)-components of the two cogenerated are comparable in \( Q \), but the nadirs are comparable the other way (it is crucial to remember that the ordering on the nadirs is by \( F^Q_\mathbb{Q} \), not \( F_Q \), so smaller faces are higher in the poset). Of course, no claim can be made that \( \text{deg}_{Q/\mathbb{R} \tau} \soc^\sigma_\tau k[D] \) equals the image in \( Q/\mathbb{R} \tau \) of \( A^\sigma_\tau \); only the density claim in Theorem 7.19 can be made.

Lemma 9.5. Over any real polyhedral group, every submodule of a coprimary module is coprimary.

Proof. Theorem 9.2, Theorem 6.7, and Definition 9.1. \( \square \)

9.2. Canonical decompositions of intervals.
The true real polyhedral generalizations of unique monomial minimal irreducible and primary decomposition for monomial ideals in ordinary polynomial rings are Theorem 9.7 and Theorem 9.12, respectively. These concern intervals rather than downsets because while the set of (exponents of) monomials outside of an ideal in \( k[N^n] \) is a downset in \( N^n \), as a subset of \( \mathbb{Z}^n \) this set of exponents is merely an interval. These real polyhedral decompositions make full use of the notions of socle, cogenerator, and density introduced in earlier sections: the topological graded algebra is essential.
Theorem 6.5, on the existence of enough downset cogenerators, holds verbatim for intervals instead of downsets. The proof is a telling application of the dense-socle characterization of essential submodules in Theorem 8.5, combined with its precursor, Theorem 7.19, that dense subsets of cogenerators suffice to express a downset as a union. With no additional effort, enough cogenerators of an interval can be located in any dense subset of the full set of cogenerators, generalizing Theorem 7.19 directly. To state this version it is useful to make a final definition concerning density.

**Definition 9.6.** Fix a real polyhedral group \( Q \). Let \( S \) be a family of sets \( S_{\tau}^\sigma \), indexed by pairs of faces \( \sigma \) and \( \tau \) of \( Q_+ \) with \( \sigma \supseteq \tau \), such that \( S_{\tau}^\sigma \subseteq Q/\mathbb{R} \tau \). This family \( S \) is dense in another such family \( \hat{S} \) if the \( \sigma \)-closure (Definition 7.8) of \( S_{\tau}^\sigma = \bigcup_{\sigma \supseteq \tau} S_{\tau}^\sigma \) in \( Q/\mathbb{R} \tau \) contains \( \hat{S}_{\tau}^\sigma \) for all \( \sigma \supseteq \tau \).

**Theorem 9.7.** Fix an interval \( I \) in a real polyhedral group \( Q \) and \( b \in I \). There are faces \( \tau \subseteq \sigma \) of \( Q_+ \) and a cogenerator \( a \) of \( I \) along \( \tau \) with nadir \( \sigma \) such that \( b \preceq a \). In fact, the cogenerator \( a \) can be selected from any given family \( A = \{ A^\sigma_{\tau} \}_{\sigma \supseteq \tau} \), where \( A^\sigma_{\tau} \subseteq Q \) is a set of cogenerators of \( I \) along \( \tau \) with nadir \( \sigma \), as long as the family \( \{ A^\sigma_{\tau} / \mathbb{R} \tau \}_{\sigma \supseteq \tau} \) of sets of cosets in \( Q/\mathbb{R} \tau \) is dense in \( \text{deg soc}_k[I] = \{ \text{deg} Q/\mathbb{R} \tau \text{ soc}_k^\sigma k[I] \}_{\sigma \supseteq \tau} \). In this case, \( I = \bigcup_{\text{faces } \sigma, \tau \text{ with } \sigma \supseteq \tau} \bigcup_{a \in A^\sigma_{\tau}} (a + \tau - Q_{\nabla \sigma}) \cap I \).

**Proof.** Let \( D \) be the downset cogenerated by \( I \). The interval module \( k[I] \) is an essential submodule of \( k[D] \) by construction. Theorem 8.5 implies that \( \text{soc} k[I] \) is dense in \( \text{soc} k[D] \), and hence any family that is dense in \( \text{soc} k[I] \) is dense in \( \text{soc} k[D] \). Theorem 7.19 therefore expresses \( D \) as the union of downsets cogenerated by the given cogenerators of \( I \). This means that every element of \( D \) precedes one of the given cogenerators of \( I \), so certainly every element of \( I \) precedes such a cogenerator of \( I \). \( \square \)

**Corollary 9.8.** Every interval \( I \) in a real polyhedral group has a canonical irreducible decomposition as a union of intervals in \( I \) cogenerated by the family of global cogenerators of \( I \). This decomposition is minimal in the sense that the only subfamilies yielding a union that still equals \( I \) are dense in the canonical family.

**Proof.** Let \( A^\sigma_{\tau} \) in Theorem 7.19 comprise all cogenerators of \( I \) along \( \tau \) with nadir \( \sigma \). \( \square \)

**Remark 9.9.** The case of Theorem 9.7 corresponding to finitely generated monomial ideals in real-exponent polynomial rings is treated in [ASW15].

**Example 9.10.** Looking at the proof of Theorem 9.7, it is tempting to posit that if \( D \) is the downset cogenerated by an interval \( I \) in a real polyhedral group \( Q \), then the inclusion \( I \subseteq D \) induces a socle isomorphism \( \text{soc} k[I] \cong \text{soc} k[D] \). But alas, it fails: the module \( M' \) in Example 8.6 cogenerates a downset that shares with the ambient module \( M \) its cogenerator \( y \), which is not a cogenerator of \( M' \) itself. Consequently, socle density as in Theorem 8.5 is the best one can hope for. (See Remark 9.14.)
Definition 9.11. A primary decomposition (Example 2.31) $I = \bigcup_{j=1}^{k} I_j$ of an interval in a real polyhedral group is minimal if
1. each face associated to $I$ is associated to precisely one of the downsets $I_j$, and
2. the natural map $\text{soc}_\tau k[I] \rightarrow \bigoplus_{j=1}^{k} \text{soc}_\tau k[I_j]$ is an isomorphism for all faces $\tau$.

Theorem 9.12. Every interval $I \subseteq Q$ in a real polyhedral group $Q$ has a canonical primary decomposition (Example 2.31) whose corresponding primary decomposition of the interval module $k[I]$ is minimal. Explicitly, this decomposition expresses $I$ as a union

$$I = \bigcup_{\tau \in \text{Ass} I} \bigcup_{\sigma \supseteq \tau, a \in A_\sigma} (a + \tau - Q) \cap I$$

of coprimary intervals, where $A_\tau \subseteq Q$ is the set of cogenerators of $I_\tau$ along $\tau$ with nadir $\sigma$.

Proof. That $I$ equals the union is a special case of Theorem 9.7. The inner union for fixed $\tau$ is $\tau$-coprimary by Lemma 9.5 because it is coprimary when the intersection with $I$ is omitted, by Lemma 9.3. To see that the socle maps are isomorphisms, let $I_\tau$ be the inner union for fixed $\tau \in \text{Ass} I$. Applying $\text{soc}_\tau$ to the primary decomposition $k[I] \hookrightarrow \bigoplus_{\sigma \supseteq \tau} k[I_\sigma]$ yields $\text{soc}_\tau k[I] \hookrightarrow \text{soc}_\tau k[I_\sigma]$ by Theorem 6.7. The question is whether every cogenerator of $I_\tau$ is indeed a cogenerator of $I$. Each cogenerator of $I_\tau$ has a nadir $\sigma$ for some face $\sigma \supseteq \tau$, and Lemma 8.2 produces a corresponding $\sigma$-vicinity (Definition 8.1) in $k[I_\tau]$. But $I_\tau$ is contained in $I$, explicitly by the way it is defined as a union, given Theorem 9.7, so this vicinity is contained in $I$. Therefore the cogenerator of $I_\tau$ in question must also be a cogenerator of $I$. \hfill \Box

Example 9.13. The global support at the right-hand end of Example 2.26 yields a redundant primary component, and hence is not part of the minimal primary decomposition in Example 1.5 that comes from Theorem 9.12, because the interval $I = \mathbb{R}_+ \subseteq Q$ has no cogenerators along the $y$-axis, in the functorial sense of Definition 4.29. The illustrations in Example 2.26 show that $k[I]$ has elements supported on the face of $\mathbb{R}^2$ that is the (positive) $y$-axis, but the socle of $k[I]$ along the $y$-axis is 0, as the top half of the boundary curve of $k[I]$ is supported on the origin.

Remark 9.14. The sole reason why Example 9.10 happens is the penultimate sentence of the proof of Theorem 9.12: the downset cogenerated by $I_\tau$ need not be contained in $I$, so a vicinity of a cogenerator of this downset can have empty intersection with $I$.

Example 9.15. The canonical $\tau$-primary component in Theorem 9.12 can differ from the $\tau$-primary component $P_\tau(D)$ in [Mil20b, Definition 3.2.6 and Corollary 3.11], namely the downset $\Gamma_\tau(D_\tau) - Q_+$ cogenerated by the local $\tau$-support of $D$. However, it takes dimension at least 3 to force a difference. For a specific case, let $\tau$ be the $z$-axis in $\mathbb{R}^3$, and let $D_1$ be the $\{0\}$-coprimary (Lemma 9.3) downset in $\mathbb{R}^3$ cogenerated by the nonnegative points on the surface $z = 1/(x^2 + y^2)$. Then every point on the positive $z$-axis is supported on $\tau$ in $D_1$. That would suffice, for the present purpose,
but for the fact that $\tau$ fails to be associated to $D_1$. The remedy is to force $\tau$ to be associated by taking the union of $D_1$ with any downset $D_2 = a + \tau - \mathbb{R}^3_+$ with $a = (x, y, z)$ satisfying $xy < 0$, the goal being for $D_2 \not\subseteq D_1$ to be $\tau$-coprimary but not contain the $z$-axis itself. The canonical $\tau$-primary component of $D = D_1 \cup D_2$ is just $D_2$ itself, but by construction $\Gamma_\tau D$ also contains the positive $z$-axis. (Note: $D = D_1 \cup D_2$ is not the canonical primary decomposition of $D$ because $D_2$ swallows an open set of cogenerators of $D_1$, so these cogenerators must be omitted from the $\{0\}$-primary component to induce an isomorphism on socles.) The reason why three dimensions are needed is that $\tau$ must have positive dimension, because elements supported on $\tau$ must be cloaked by those supported on a smaller face; but $\tau$ must have codimension more than 1, because there must be enough room modulo $\mathbb{R} \tau$ to have incomparable elements.

9.3. Downset and interval hulls of modules.

**Definition 9.16.** A downset hull $M \hookrightarrow E = \bigoplus_{j=1}^k E_j$ (Definition 2.16) of a module over a real polyhedral group is

1. **coprimary** if $E_j = \mathbb{k}[D_j]$ is coprimary for all $j$, so $D_j$ is a coprimary downset, and
2. **dense** if the induced map $\text{soc}_\tau M \hookrightarrow \text{soc}_\tau E$ is dense (Definition 8.1.4) for all $\tau$.

**Remark 9.17.** The density condition in Definition 9.16 is equivalent to $M$ being an essential submodule of the downset hull $E$, by Theorem 8.5.

**Theorem 9.18.** Every downset-finite module $M$ over a real polyhedral group admits a dense coprimary downset hull.

**Proof.** Suppose that $M \rightarrow \bigoplus_{j=1}^k E_j$ is any finite downset hull. Replacing each $E_j$ by a primary decomposition of $E_j$, using Theorem 9.12, assume that this downset hull is coprimary. Let $E^*$ be the direct sum of the $\tau$-coprimary summands of $E$. Then $\text{soc}_\tau E = \text{soc}_\tau E^*$ by Theorem 9.2. Replacing $M$ with its image in $E^*$, it therefore suffices to treat the case where $M$ is $\tau$-coprimary and $E = E^*$.

The proof is by induction on the number $k$ of summands of $E$. If $k = 1$ then $M = \mathbb{k}[I] \subseteq \mathbb{k}[D] = \mathbb{k}[D^\tau]$ is an interval submodule of a $\tau$-coprimary downset module by hypothesis. If $D'$ is the downset cogenerated by $I$, then $\mathbb{k}[I] \subseteq \mathbb{k}[D']$ is an essential submodule by construction and coprimary by Lemma 9.5. Thus $M = \mathbb{k}[I] \subseteq \mathbb{k}[D'] = E$ is a dense coprimary downset hull by Theorem 8.5.

When $k > 1$, let $M' = \ker(M \rightarrow E_k)$. Then $M' \hookrightarrow \bigoplus_{j=1}^{k-1} E_j$, so it has a dense coprimary downset hull $M' \hookrightarrow E'$ by induction. The $k = 1$ case proves that the quotient $M'' = M/M'$ has a dense coprimary downset hull $M'' \hookrightarrow E''$. The exact sequence $0 \rightarrow M' \rightarrow M \rightarrow M'' \rightarrow 0$ yields an exact sequence

$$0 \rightarrow \text{soc}_\tau M' \rightarrow \text{soc}_\tau M \rightarrow \text{soc}_\tau M'' \rightarrow 0 \rightarrow$$

which, if exact, automatically splits by Remark 4.36. Hence it suffices to prove that $\text{soc}_\tau M \rightarrow \text{soc}_\tau M''$ is surjective. For that, note that the image of $\text{soc}_\tau M$ in $\text{soc}_\tau E$
surjects onto its projection to $\text{soc}_\tau E_k$, but the image of $\text{soc}_\tau M \rightarrow \text{soc}_\tau E_k$ is the image of the injection $\text{soc}_\tau M'' \hookrightarrow \text{soc}_\tau E_k$ by construction. \hfill \Box

**Remark 9.19.** Theorem 9.18 is the analogue of existence of minimal injective hulls for finitely generated modules over noetherian rings [BH98, Section 3.2] (see also [Mil20a, Proposition 5.7 or Theorem 5.19] for finitely determined $\mathbb{Z}^n$-modules, which need not be finitely generated). The difference here is that a direct sum—as opposed to direct product—can only be attained by gathering cogenerators into finitely many bunches.

**Example 9.20.** The indicator module for the disjoint union of the strictly negative axes in the plane injects in an appropriate way into one downset module (the punctured negative quadrant) or a direct sum of two (negative quadrants missing one boundary axis each). Thus the “required number” of downsets for a downset hull of a given module is not necessarily obvious and might not be a functorial invariant. This may sound bad, but it should not be unexpected: the quotient by an artinian monomial ideal in an ordinary polynomial ring can have socle of arbitrary finite dimension, so the number of coprincipal downsets required is well defined, but if downsets that are not necessarily coprincipal are desired, then any number between 1 and the socle dimension would suffice. This phenomenon is related to Remark 4.36: breaking the socle of a downset into two reasonable pieces expresses the original downset as a union of the two downsets cogenerated by the pieces.

**Definition 9.21.** An **interval hull** of a module $M$ over an arbitrary poset is an injection $M \hookrightarrow H = \bigoplus_{j \in J} H_j$ with each $H_j$ being an interval module (Example 2.6.3). The hull is **finite** if $J$ is finite. Over a real polyhedral group a finite interval hull is

1. **coprimary** if $H_j = k[I_j]$ is coprimary for all $j$, so $I_j$ is a coprimary interval, and
2. **minimal** if the induced map $\text{soc}_\tau M \hookrightarrow \text{soc}_\tau H$ is an isomorphism for all faces $\tau$.

**Remark 9.22.** Minimality of interval hulls is stronger than density of downset hulls: the induced socle inclusion is required to be an isomorphism rather than merely dense.

**Theorem 9.23.** Every downset-finite module $M$ over a real polyhedral group admits a minimal coprimary interval hull.

**Proof.** The transition from downsets to intervals alters one key aspect of the proof of Theorem 9.18, namely the base of the induction: when $k = 1$ the module is already an interval module, so the identity map is a minimal interval hull, as the socle inclusion is the identity isomorphism. The rest of the proof goes through mutatis mutandis, changing “downset” to “interval”, “dense” to “minimal”, and all instances of “$E$” to “$H$”.\hfill \Box

**Remark 9.24.** The proof of Theorem 9.23 shows more than its statement: any coprimary interval hull $M \hookrightarrow H = H_1 \oplus \cdots \oplus H_k$ of a coprimary module $M$ induces a filtration $0 = M_0 \subset M_1 \subset \cdots \subset M_k = M$ such that $\text{soc}_\tau M = \bigoplus_{j=1}^k \text{soc}_\tau (M_j/M_{j-1})$, and furthermore $M \hookrightarrow H$ can be “minimalized”, in the sense that a minimal hull $H'$ can be constructed inside of $H$ so that $\text{soc}_\tau M \cong \text{soc}_\tau H'$ decomposes as direct sum.
of factors $\text{soc}_\tau(M_j/M_{j-1}) \cong \text{soc}_\tau H_j$. Reordering the summands $H_j$ yields another filtration of $M$ with the same property. That $\text{soc}_\tau M$ breaks up as a direct sum in so many ways should not be shocking, in view of Remark 4.36. The main content is that all of the socle elements of $M/M_{k-1}$ are inherited from $M$, essentially because $M_{k-1}$ is the kernel of a homomorphism to a direct sum of downset modules, so $M_{k-1}$ has no cogenerators that are not inherited from $M$.

9.4. Minimal primary decomposition of modules.

**Definition 9.25.** A primary decomposition $M \hookrightarrow \bigoplus_{i=1}^{r} M/M_i$ (Definition 2.30) of a module over a real polyhedral group is minimal if $\text{soc}_\tau M \to \text{soc}_\tau \bigoplus_{i=1}^{r} M/M_i$ is an isomorphism for all faces $\tau$.

**Definition 9.26.** Given a coprimary interval hull $M \hookrightarrow H$ of an arbitrary downset-finite module $M$ over a real polyhedral group, write $H^\tau$ for the direct sum of all summands of $H$ that are $\tau$-coprimary. The kernel $M^\tau$ of the composite homomorphism $M \to H \to H^\tau$ is the $\tau$-primary component of 0 for this particular interval hull of $M$.

**Theorem 9.27.** Every downset-finite module $M$ over a real polyhedral group admits a minimal primary decomposition. In fact, if $M \hookrightarrow H$ is a coprimary interval hull then $M \hookrightarrow \bigoplus_{\tau} M/M^\tau$ is a primary decomposition that is minimal if $M \hookrightarrow H$ is minimal.

**Proof.** Fix a coprimary interval hull $M \hookrightarrow H$. The quotient $M/M^\tau$ is $\tau$-coprimary by Lemma 9.5 since it is a submodule of the coprimary module $H^\tau$, and $M \to \bigoplus_{\tau} M/M^\tau$ is injective because the injection $M \hookrightarrow \bigoplus_{\tau} H^\tau = H$ factors through $\bigoplus_{\tau} M/M^\tau \subseteq H$.

Theorem 9.2 implies that $\text{soc}_{\tau'}(M/M^\tau) = 0$ unless $\tau = \tau'$, regardless of whether $M \hookrightarrow H$ is minimal. And if the hull is minimal, then $\text{soc}_{\tau} M \to \text{soc}_{\tau} H^\tau$ is an isomorphism (by hypothesis) that factors through the injection $\text{soc}_{\tau}(M/M^\tau) \hookrightarrow \text{soc}_{\tau} H^\tau$ (by construction), forcing $\text{soc}_{\tau} M \cong \text{soc}_{\tau}(M/M^\tau)$ to be an isomorphism for all $\tau$. $\square$

**Remark 9.28.** Theorem 9.27 enables full access to interpretations of primary decomposition in persistent homology, now with a notion of minimality for multiple real parameters instead of versions without minimality for partially ordered groups [Mil20b] or with minimality in the discrete case [HOST19]. Primary decomposition has important statistical implications for applications of multipersistence [MT20].

10. Socles and essentiality over discrete polyhedral groups

The theory developed for real polyhedral groups in Sections 4–11 applies as well to discrete polyhedral groups (Example 2.4). The theory is easier in the discrete case, in the sense that only closed cogenerator functors are needed, and none of the density considerations in Sections 7–8 are relevant. The deduction of the discrete case is elementary, but it is worthwhile to record the results, both because they are useful and for comparison with the real polyhedral case.
For the analogue of Theorem 6.7, the notion of divisibility in Definition 3.19 makes sense, when \( \sigma = \{0\} \), verbatim in the discrete polyhedral setting: an element \( y \in M_b \) divides \( x \in M_a \) if \( b \in a - Q_+ \) and \( y \mapsto x \) under the natural map \( M_b \to M_a \).

**Theorem 10.1** (Discrete essentiality of socles). Fix a homomorphism \( \varphi : M \to N \) of modules over a discrete polyhedral group \( Q_+ \).

1. If \( \varphi \) is injective then \( \text{soc}_\tau \varphi : \text{soc}_\tau M \to \text{soc}_\tau N \) is injective for all faces \( \tau \) of \( Q_+ \).
2. If \( \text{soc}_\tau \varphi : \text{soc}_\tau M \to \text{soc}_\tau N \) is injective for all faces \( \tau \) of \( Q_+ \) and \( M \) is downset-finite, then \( \varphi \) is injective.

Each homogeneous element of \( M \) divides some closed cogenerator of \( M \).

**Proof.** Item 1 is a special case of Proposition 4.23. Item 2 follows from the divisibility claim, for if \( y \) divides a closed cogenerator \( s \) along \( \tau \) then \( \varphi(y) \neq 0 \) whenever \( \text{soc}_\tau \varphi(s) \neq 0 \), where \( \tilde{s} \) is the image of \( s \) in \( \text{soc}_\tau M \). The divisibility claim follows from the case where \( M \) is generated by \( y \in M_b \). But \( \langle y \rangle \) is a noetherian \( k[Q_+] \)-module and hence has an associated prime. This prime equals the annihilator of some homogeneous element of \( \langle y \rangle \), and the quotient of \( k[Q_+] \) modulo this prime is \( k[\tau] \) for some face \( \tau \) [MS05, Section 7.2]. That means, by definition, that the homogeneous element is a closed cogenerator along \( \tau \) divisible by \( y \).

The discrete analogue of Theorem 7.27 is simpler in both statement and proof.

**Theorem 10.2.** Fix subfunctors \( \overline{\mathbf{S}}_\tau \subseteq \text{soc}_\tau \) for all faces \( \tau \) of a discrete polyhedral group. Theorem 10.1 holds with \( \mathbf{S} \) in place of \( \text{soc}_\tau \) if and only if \( \overline{\mathbf{S}}_\tau = \text{soc}_\tau \) for all \( \tau \).

**Proof.** Every subfunctor of any left-exact functor takes injections to injections; therefore Theorem 10.1.1 holds for any subfunctor of \( \text{soc}_\tau \) by Proposition 4.23. The content is that Theorem 10.1.2 fails as soon as \( \overline{\mathbf{S}}_\tau M \not\subseteq \text{soc}_\tau M \) for some module \( M \) and some face \( \tau \). To prove that failure, suppose \( \tilde{s} \in \text{soc}_\tau M \setminus \overline{\mathbf{S}}_\tau M \) for some closed cogenerator \( s \) of \( M \) along \( \tau \). Then \( \langle s \rangle \subseteq M \) induces an injection \( \overline{\mathbf{S}}_\tau \langle s \rangle \to \overline{\mathbf{S}}_\tau M \), but by construction the image of this homomorphism is 0, so \( \overline{\mathbf{S}}_\tau \langle s \rangle = 0 \). But \( \text{soc}_\tau \langle s \rangle = 0 \) for all \( \tau' \neq \tau \) because \( \langle s \rangle \) is abstractly isomorphic to \( k[\tau] \), which has no associated primes other than the kernel of \( k[Q_+] \to k[\tau] \). Consequently, applying \( \overline{\mathbf{S}}_{\tau'} \) to the homomorphism \( \varphi : \langle s \rangle \to 0 \) yields an injection \( 0 \to 0 \) for all faces \( \tau' \) even though \( \varphi \) is not injective.

The analogue of Theorem 8.5 is similarly simpler.

**Theorem 10.3.** In any module \( M \) over a discrete polyhedral group, \( M' \) is an essential submodule if and only if \( \text{soc}_\tau M' = \text{soc}_\tau M \) for all faces \( \tau \).

**Proof.** First assume that \( M' \) is not an essential submodule, so \( N \cap M' = 0 \) for some nonzero submodule \( N \subseteq M \). Any closed cogenerator \( s \) of \( N \) along any face \( \tau \) maps to a nonzero element of \( \text{soc}_\tau M \) that lies outside of \( \text{soc}_\tau M' \). Conversely, if \( \text{soc}_\tau M' \neq \text{soc}_\tau M \), then any closed cogenerator of \( M \) that maps to an element \( \text{soc}_\tau M \setminus \text{soc}_\tau M' \) generates a nonzero submodule of \( M \) whose intersection with \( M' \) is 0.
The analogue of Theorem 9.12 uses slightly modified definitions but its proof is easier.

**Definition 10.4.** A primary decomposition (Definition 2.31) \( I = \bigcup_{j=1}^{k} I_j \) of an interval in a discrete polyhedral group is **minimal** if

1. the intervals \( I_j \) are coprimary for distinct associated faces of \( I \), and
2. the natural map \( \text{soc}_\tau k[I] \to \bigoplus_{j=1}^{k} k[I_j] \) is an isomorphism for all faces \( \tau \), where \( \tau \) is **associated** if some element generates an upset in \( I \) that is a translate of \( \tau \).

**Theorem 10.5.** Every interval \( I \) in a discrete polyhedral group has a canonical minimal primary decomposition \( I = \bigcup_{\tau} I_\tau \) as a union of coprimary intervals

\[
I_\tau = \bigcup_{a_\tau \in \text{deg} \text{soc}_\tau k[I]} (a_\tau - Q_+) \cap I,
\]

where \( a_\tau \) is viewed as an element in \( Q/\mathbb{Z} \) to write \( a_\tau \in \text{deg} \text{soc}_\tau k[I] \) but \( a_\tau \subseteq Q \) is viewed as a coset of \( \mathbb{Z} \tau \) to write \( a_\tau - Q_+ \).

**Proof.** The interval \( I \) is contained in the union by the final line of Theorem 10.1, but the union is contained in \( I \) because every closed cogenerator of \( I \) is an element of \( I \). It remains to show that \( I_\tau \) is coprimary and that the socle maps are isomorphisms.

Each nonzero homogeneous element \( y \in k[I_\tau] \) divides an element \( s_y \) whose degree lies in some coset \( a_\tau \in \text{deg} \text{soc}_\tau k[I] \) by construction. As \( I_\tau \subseteq I \), each such element \( s_y \) is a closed cogenerator of \( I_\tau \) along \( \tau \). Therefore \( k[I_\tau] \) is coprimary, inasmuch as no prime other than the one associated to \( k[\tau] \) can be associated to \( I_\tau \). The same argument shows that these elements \( s_y \) generate an essential submodule of \( I_\tau \), and then Theorem 10.3 yields the isomorphism on socles. \( \square \)

**Corollary 10.6.** Every interval \( I \) in a discrete polyhedral group \( Q \) has a unique irredundant irreducible decomposition as a union of its irreducible components, namely the coprincipal intervals \( (a_\tau - Q_+) \cap I \) in Theorem 10.5.

**Proof.** The irredundant condition is a consequence of the socle isomorphisms. \( \square \)

**Definition 10.7.** A downset hull \( M \hookrightarrow E = \bigoplus_{j=1}^{k} E_j \) (Definition 2.16) of a module over a discrete polyhedral group is

1. **coprimary** if \( E_j = k[D_j] \) is coprimary for all \( j \), so \( D_j \) is a coprimary downset, and
2. **minimal** if the induced map \( \text{soc}_\tau M \to \text{soc}_\tau E \) is an isomorphism for all faces \( \tau \).

The discrete analogue of Theorem 9.18 appears to be new.

**Theorem 10.8.** Every downset-finite module \( M \) over a discrete polyhedral group admits a minimal coprimary downset hull.
Proof. The argument follows that of Theorem 9.18, using Theorem 10.5 instead of Theorem 9.12 and Theorem 10.3 instead of Theorem 8.5. In the course of the proof, note that the discrete analogue of Theorem 9.2 is the definition of associated prime, making the analogue of Lemma 9.5 trivial, and that the analogue of Remark 4.36 holds (more easily) in the discrete polyhedral setting. □

Remark 10.9. Remark 9.24 holds verbatim over discrete polyhedral groups.

Finally, here is the discrete version of minimal primary decomposition.

Definition 10.10. A primary decomposition $M \hookrightarrow \bigoplus_{i=1}^{r} M/M_{i}$ (Definition 2.30) of a module over a discrete polyhedral group is minimal if $\text{soc}_{\tau} M \to \bigoplus_{i=1}^{r} M/M_{i}$ is an isomorphism for all faces $\tau$.

Definition 10.11. Given a coprimary downset hull $M \hookrightarrow E$ of an arbitrary downset-finite module $M$ over a discrete polyhedral group, write $E^{\tau}$ for the direct sum of all summands of $E$ that are $\tau$-coprimary. The kernel $M^{\tau}$ of the composite homomorphism $M \to E \to E^{\tau}$ is the $\tau$-primary component of 0 for this particular downset hull of $M$.

Theorem 10.12. Every downset-finite module $M$ over a discrete polyhedral group admits a minimal primary decomposition. If $M \hookrightarrow E$ is a coprimary downset hull then $M \hookrightarrow \bigoplus_{\tau} M/M^{\tau}$ is a primary decomposition that is minimal if $M \hookrightarrow E$ is minimal.

Proof. Follow the proof of Theorem 9.27, using downset hulls instead of interval hulls because, in contrast with the real polyhedral case (Theorems 9.18 and 9.23), downset hulls are minimal—not merely dense—in the discrete context (Theorem 10.8). □

11. Generator functors and tops

The theory of generators is Matlis dual (Section 2.5) to the theory of cogenerators. Every result for socles, downsets, and cogenerators therefore has a dual. All of these dual statements can be formulated so as to be straightforward, but sometimes they are less natural (see Remarks 11.19 and 11.20, for example), sometimes they are weaker (see Remark 11.5), and sometimes there are natural formulations that must be proved equivalent to the straightforward dual (see Definition 11.28 and Theorem 11.31, for example). This section presents those Matlis dual notions that are used in later sections.

11.1. Lower boundary functors.

The following are Matlis dual to Definition 3.11. Lemma 3.13, and Definition 3.15.

Definition 11.1. For a module $M$ over a real polyhedral group $Q$, a face $\xi$ of $Q_{+}$, and a degree $b \in Q$, the lower boundary beneath $\xi$ at $b$ in $M$ is the vector space

$$(\partial^{\xi} M)_{b} = M_{b+\xi} = \lim_{\substack{b' \to b+\xi \circ \xi \circ \xi}} M_{b'}.$$
Lemma 11.2. The structure homomorphisms of $M$ as a $Q$-module induce natural homomorphisms $M_{b+\xi} \to M_{c+\eta}$ for $b \preceq c$ in $Q$ and faces $\xi \subseteq \eta$ of $Q_+$. □

Remark 11.3. In contrast with Remark 3.14, the relevant monoid structure here on the face poset $F_Q$ of the positive cone $Q_+$ is opposite to the monoid denoted $F_Q^\text{op}$. In this case the monoid axioms use that $F_Q$ is a bounded join semilattice, the monoid unit being $\{0\}$. The induced partial order on $F_Q$ is the usual one, with $\xi \preceq \eta$ if $\xi \subseteq \eta$.

Definition 11.4. Fix a module $M$ over a real polyhedral group $Q$ and a degree $b \in Q$. The lower boundary functor takes $M$ to the $Q \times F_Q$-module $\partial M$ whose fiber over $b \in Q$ is the $F_Q$-module $(\partial M)_b = \bigoplus_{\xi \in F_Q} M_{b+\xi} = \bigoplus_{\xi \in F_Q} (\delta^\xi M)_b$.

The fiber of $\partial M$ over $\xi \in F_Q$ is the lower boundary $\partial^\xi M$ of $M$ beneath $\xi$.

Remark 11.5. Direct and inverse limits play differently with vector space duality. Consequently, although the notion of lower boundary functor is categorically dual to the notion of upper boundary functor, the duality only coincides unfettered with vector space duality in one direction, and some results involving tops are necessarily weaker than the corresponding results for socles; compare Theorem 6.7 with 12.3 and Example 12.7, for instance. To make precise statements throughout this section on generator functors, starting with Lemma 11.7, it is necessary to impose a finiteness condition that is somewhat stronger than $Q$-finiteness (Definition 2.9.2).

Definition 11.6. A module $M$ over a real polyhedral group $Q$ is infinitesimally $Q$-finite if its lower boundary module $\partial M$ is $Q$-finite.

Lemma 11.7. If $\xi$ is a face of a real polyhedral group $Q$, then
1. $\partial^\xi (M^\vee) = (\delta^\xi M)^\vee$ for all $Q$-modules $M$, and
2. $(\partial^\xi M)^\vee = \delta^\xi (M^\vee)$ if $M$ is infinitesimally $Q$-finite.

Proof. Degree by degree $b \in Q$, the first of these is because the vector space dual of a direct limit is the inverse limit of the vector space duals. Swapping “direct” and “inverse” only works with additional hypotheses, and one way to ensure these is to assume infinitesimal $Q$-finiteness of $M$. Indeed, then $M = \partial^{[0]} M$ is $Q$-finite, so replacing $M$ with $M^\vee$ in the first item yields $\partial^\xi M = (\delta^\xi (M^\vee))^\vee$ by Lemma 2.37. Thus $(\partial^\xi M)^\vee = \delta^\xi (M^\vee)$, as $\partial^\xi M$—and hence $(\delta^\xi (M^\vee))^\vee$ and $\delta^\xi (M^\vee)$—is also $Q$-finite. □

Example 11.8. Any module $M$ that is a quotient of a finite direct sum of upset modules (“upset-finite” in Definition 12.2) over a real polyhedral group is infinitesimally $Q$-finite. Indeed, the Matlis dual of such a quotient is a downset hull demonstrating that $M^\vee$ is downset-finite and hence $Q$-finite. Proposition 3.21 and exactness of upper boundary functors (Lemma 3.12) implies that $\delta (M^\vee)$ remains downset-finite and hence $Q$-finite. Applying Lemma 11.7.1 to $M^\vee$ and using that $(M^\vee)^\vee = M$ (Lemma 2.37) on
the left-hand side yields that $\partial M$ is $Q$-finite. This example includes all tame modules by the syzygy theorem [Mil20a, Theorem 6.12.4].

**Proposition 11.9.** The category of infinitesimally $Q$-finite modules over a real polyhedral group $Q$ is a full abelian subcategory of the category of $Q$-modules. Moreover, the lower boundary functor is exact on this subcategory.

**Proof.** Use Matlis duality, in the form of Lemma 11.7, along with Lemma 3.12. □

11.2. **Closed generator functors.**

Here is the Matlis dual of Definition 4.1. Recall the skyscraper $P$-module $k_p$ there.

**Definition 11.10.** Fix an arbitrary poset $P$. The closed generator functor $k \otimes_P -$ takes each $P$-module $N$ to its closed top: the quotient $P$-module

$$\text{top } N = k \otimes_P N = \bigoplus_{p \in P} k_p \otimes_P N.$$

When it is important to specify the poset, the notation $P\text{-top}$ is used instead of top. A closed generator of degree $p \in P$ is a nonzero element in $\text{top } N_p$.

**Example 11.11.** Elements of $N_p$ that persist from lower in the poset die in the tensor product $k \otimes_P N$. Consequently, $\mathbb{R}$-modules like the maximal monomial ideal $m \subseteq k[\mathbb{R}_+]$ have vanishing closed top, because every monomial with nonzero positive degree is divisible by a monomial of smaller positive degree (its square root, for instance). This is merely the statement that $m$ is not minimally generated.

**Remark 11.12.** $P\text{-soc } N \hookrightarrow N$ is the universal $P$-module monomorphism that is 0 when composed with all nonidentity maps induced by going up in $P$. The Matlis dual is $N \twoheadrightarrow P\text{-top } N$, the universal $P$-module epimorphism that is 0 when composed with all nonidentity maps induced by going up in $P$. This is the essence of Proposition 11.15.

**Remark 11.13.** Matlis duality has an intrinsic asymmetry regarding the behavior of tops and socles. In the presence of sufficient finiteness, the asymmetry disappears, but in general it requires care to insert some finiteness appropriately. The following definition, proposition, and proof are presented in (perhaps too much) detail to highlight how finiteness enters. Local finiteness (Definition 11.14) can fail for modules over a partially ordered group, but it is useful for the discrete (face lattice) half of the poset used to compute tops and socles over real polyhedral groups; see Proposition 11.25. The other finiteness restriction, namely $P$-finiteness (Definition 2.9.2) has already appeared, with consequences (see Example 2.41).

**Definition 11.14.** A module $N$ over a poset $P$ is locally finite if, for each poset element $p \in P$, there is a finite subset $P'(p, N) \subseteq P$ such that, if $N_p \rightarrow N_{p''}$ is nonzero for some $p'' \in P$, there is some $p' \in P'(p, N)$ with $p' \prec p''$.

Loosely, $P'(p, N)$ sits between $N_p$ and any of its nonzero images higher in $P$. 

Proposition 11.15. Fix a poset $P$ with opposite poset $P^{\text{op}}$. For a $P$-module $N$, Matlis duality interacts with tops and socles as follows:

1. $P\cdot\text{top}(N)^\vee = P^{\text{op}}\cdot\text{soc}(N^\vee)$ for any $P$-module $N$, and
2. $P\cdot\text{top}(N^\vee) = (P^{\text{op}}\cdot\text{soc} N)^\vee$ for any $P^{\text{op}}$-finite or locally finite $P^{\text{op}}$-module $N$.

All of these hold with $P$ and $P^{\text{op}}$ swapped.

Proof. View the $P$-module $N$ as a diagram of vector spaces indexed by $P$. Tensoring with $k$ in Definition 11.10 takes each vector space $N_p$ to the cokernel of the homomorphism $\bigoplus_{p' \prec p} N_{p'} \to N_p$ induced by the maps going up in $P$ from $p'$ to $p$. Let $L_p$ be the image in $N_p$ of $\bigoplus_{p' \prec p} N_{p'}$. The vector space dual $(\cdot)^*$ of the surjection $N_p \twoheadrightarrow N_p/L_p$ is the kernel of the homomorphism $\prod_{p' \succ p} N_{p'}^* \hookrightarrow N_p^*$, where $p' \succ p$ in the partial order on $P^{\text{op}}$ here. This kernel is $\text{Hom}_{P^{\text{op}}}(k, N_p^\vee)$, proving the first equation by Definition 4.1.

The second equation is similar: $\text{Hom}_{P^{\text{op}}}(k, \cdot)$ takes each vector space $N_p$ in the diagram $N$ indexed by $P^{\text{op}}$ to the kernel of the homomorphism $\prod_{p' \succ p} N_{p'}^* \hookrightarrow N_p^*$. If a finite sub-product—over a subset $P(p, N)$, say—suffices to compute the kernel, as the $P^{\text{op}}$-finite or locally finite conditions guarantee, then the vector space dual of the kernel is the cokernel of the homomorphism $\bigoplus_{p' \in P(p, N)} (N_{p'})^* \to (N_p)^*$ induced by the maps going up in $P$ from $p'$ to $p$ in $N^\vee$. \hfill \square

11.3. Closed generator functors along faces.

Generators along faces of partially ordered groups make sense just as cogenerators along faces do; however, they are detected not by localization but by the Matlis dual operation in Example 2.36, which is likely unfamiliar (and is surely less familiar than localization). An element in the following can be thought of as an inverse limit of elements of $M$ taken along the negative of the face $\rho$. This is Matlis dual to the construction of the localization $M_\rho$ as a direct limit.

Definition 11.16. Fix a face $\rho$ of a partially ordered group $Q$ and a $Q$-module $M$. Set

$$M^\rho = \text{Hom}_Q(k[Q_+], M).$$

The following is Matlis dual to Definition 4.15.1; see Theorem 11.22. Duals for the rest of Definition 4.15 are omitted for reasons detailed in Remarks 11.19 and 11.20.

Definition 11.17. Fix a partially ordered group $Q$, a face $\rho$, and a $Q$-module $M$. The closed generator functor along $\rho$ takes $M$ to its closed top along $\rho$:

$$\overline{\text{top}}_\rho M = (k[\rho] \otimes_Q M)^\rho/\rho.$$
Example 11.18. When $Q = \mathbb{R}^2$ and $\rho$ is the face of $\mathbb{R}^2_+$ along the $x$-axis, the module $M^\rho = \text{Hom}_{\mathbb{R}^2}(k[\mathbb{R}^2_+]_\rho, M)$ for the depicted module $M$ is

$$\text{Hom} \left( \begin{bmatrix} k & \square \end{bmatrix}, \begin{bmatrix} k & \triangle \end{bmatrix} \right) = \begin{bmatrix} k & \square \end{bmatrix}.$$  

The effect is push the jagged right-hand boundary off to $+\infty$. It is interesting to note that the bottom edge of $M$ does become homogenized in $M^\rho$, because there is no way to place the closed purple upper half plane onto the blue module (either one, actually, though $M$ with the jagged upper boundary is meant here). Indeed, the dotted left-pointing horizontal ray has no location to place the solid left-pointing boundary ray of the half plane. In contrast, in any given homomorphism $k[\mathbb{R}^2_+]_\rho \to M$, the right-pointing solid purple ray goes to 0 once it exits the blue region.

The tensor product $k[\rho] \otimes_{\mathbb{R}^2} M$ is nonzero only on the solid blue segment along the bottom edge of $M$. Therefore $\left( k[\rho] \otimes_{\mathbb{R}^2} M \right)^\rho = 0$. Thus, while a nonzero homomorphism $k[\mathbb{R}^2_+]_\rho \to M$, the right-pointing solid purple ray goes to 0 once it exits the blue region.

Remark 11.19. The notion of global closed cogenerator has a Matlis dual, but since the dual of an element—equivalently, a homomorphism from $k[Q_+]$—is not an element, the notion of closed generator is not Matlis dual to a standard notion related to socles. A generator along a face $\rho$ can be defined as an element of $M^\rho$ that is not a multiple of any generator of lesser degree, but making this precise requires care regarding what “degree of generator” and “lesser” mean; see [Mil20d].

Remark 11.20. The notion of local socle has a Matlis dual, but it is not in any sense a local top, because localization does not Matlis dualize to localization (Example 2.36). Instead, Matlis dualizing the local socle yields a functor $k \otimes_{Q/\mathbb{Z}_\rho} M^\rho$ that surjects onto $\text{top}^\rho M$ by the Matlis dual of Proposition 4.40. Local socles found uses in proofs here and there, such as Corollary 4.42, Proposition 7.12, Corollary 7.14, Lemma 8.2, and Corollary 8.4, via Proposition 4.40. But since Matlis duals of statements hold regardless of their proofs, given appropriate finiteness conditions (Definition 11.6), local socles and their Matlis duals have no further use in this paper.

In the next lemma, a prerequisite to duality of closed socles and tops, a localization of $N$ along $\rho$ on the left-hand side is hiding in the quotient-restriction (Definition 2.22).

Lemma 11.21. For any module $N$ over a partially ordered group $Q$ and any face $\rho$,

$$(N/\rho)^\vee = (N^\vee)^\rho/\rho.$$  

If $N$ is $Q$-finite, then

$$(N^\vee)/\rho = (N^\rho/\rho)^\vee.$$
Proof. This is Example 2.36 plus the observation that quotient-restriction along $\rho$ commutes with Matlis duality on modules that are already localized along $\rho$ as can be seen directly from Definitions 2.34 and 2.22. The detailed calculation goes like this:

$$(N/\rho)^\vee = (N_\rho/\rho)^\vee = (N_\rho)^\vee/\rho = (N^\vee)^\rho/\rho.$$ 

To derive the second displayed equation, use Lemma 2.37 to replace $N$ by $N^\vee$ in the first equation, and then use Lemma 2.37 again to take the Matlis duals of both sides. □

**Theorem 11.22.** For a module $M$ over a partially ordered group $Q$,

1. $(\text{top}_\rho M)^\vee = \text{soc}_\rho (M^\vee)$ if $k[\rho] \otimes_Q M$ is $Q$-finite, and
2. $\text{top}_\rho (M^\vee) = (\text{soc}_\rho M)^\vee$ if $M$ is $Q$-finite.

Proof. The two are similar, but to indicate why the finitenesses must be assumed, both are written out. The first and last lines of each half are by Definitions 11.17 and 4.15.1:

$$
(\text{top}_\rho M)^\vee = ((k[\rho] \otimes_Q M)^\rho/\rho)^\vee
= (k[\rho] \otimes_Q M)^\vee/\rho \quad \text{by Lemma 11.21 and } Q\text{-finiteness of } k[\rho] \otimes_Q M
= \text{Hom}_Q (k[\rho], M^\vee)^\rho/\rho \quad \text{by Example 2.35}
= \text{soc}_\rho (M^\vee),
$$

and

$$
(\text{soc}_\rho M)^\vee = (\text{Hom}_Q (k[\rho], M)^\rho/\rho)^\vee
= (\text{Hom}_Q (k[\rho], M^\vee)^\rho/\rho \quad \text{by Lemma 11.21}
= (k[\rho] \otimes_Q M^\vee)^\rho/\rho \quad \text{by Example 2.35 and } Q\text{-finiteness of } M
= \text{top}_\rho (M^\vee).
$$

Remark 11.23. A blanket hypothesis that $M$ be $Q$-finite would suffice for both parts of Theorem 11.22, because tensoring the surjection $k[Q] \twoheadrightarrow k[\rho]$ with $M$ yields a surjection $M \twoheadrightarrow k[\rho] \otimes_Q M$, but the additional generality may be useful.

11.4. Generator functors over real polyhedral groups.

Here is the Matlis dual to Definition 4.26, using Definition 11.4.

**Definition 11.24.** For a face $\rho$ of real polyhedral group, set $\Delta \rho = (\nabla \rho)^{\text{op}}$, the open star of $\rho$ (Example 3.5) with the partial order opposite to Definition 4.26, so

$$
\xi \preceq \eta \text{ in } \Delta \rho \text{ if } \xi \subseteq \eta.
$$

The lower boundary functor along $\rho$ takes $M$ to the $Q \times \Delta \rho$-module $\partial_\rho M = \bigoplus_{\xi \in \Delta \rho} \partial_\xi M$.

**Proposition 11.25.** Fix a face $\rho$ of a real polyhedral group $Q$. The Matlis dual $N^\vee$ over $Q$ of any module $N$ over $Q \times \Delta \rho$ is naturally a module over $Q \times \nabla \rho$ (without altering degrees in $\Delta \rho = \nabla \rho$). Moreover,

1. $(\Delta \rho \text{-top} N)^\vee = \nabla \rho \cdot \text{soc} (N^\vee)$ for any module $N$ over $Q \times \Delta \rho$, and
2. $\Delta \rho \text{-top} (N^\vee) = (\nabla \rho \cdot \text{soc} N)^\vee$ for any module $N$ over $Q \times \nabla \rho$,
where the Matlis duals are taken over $Q$. All of these hold with $\nabla \rho$ and $\Delta \rho$ swapped.

**Proof.** Matlis duality over $Q$ reverses the arrows in the $\Delta \rho$-module structure on $N$, making $N^\vee$ into a module over $Q \times \nabla \rho$. An adjointness calculation then yields

$$
(\Delta \rho \text{-}\text{top} N)^\vee = (k \otimes_{\Delta \rho} N)^\vee = \text{Hom}_{\nabla \rho}(k, N^\vee) = \nabla \rho \text{-}\text{soc} (N^\vee).
$$

The other adjointness is similar, but it uses finiteness of $\Delta \rho$ via Proposition 11.15:

$$
\Delta \rho \text{-}\text{top} (N^\vee) = k \otimes_{\Delta \rho} (N^\vee) = \text{Hom}_{\nabla \rho}(k, N^\vee) = (\nabla \rho \text{-}\text{soc} N)^\vee.
$$

**Corollary 11.26.** For a face $\rho$ over a real polyhedral group $Q$, the lower boundary is Matlis dual over $Q$ to the upper boundary: as modules over $Q \times \Delta \rho$,  

1. $\partial \rho (M^\vee) = (\delta \rho M)^\vee$ for all $Q$-modules $M$ (see Definitions 11.24 and 4.26), and 
2. $(\partial \rho M)^\vee = \delta \rho (M^\vee)$ if $M$ is infinitesimally $Q$-finite.

**Proof.** Lemma 11.7 plus the first part of Proposition 11.25. □

Next is the Matlis dual of Definition 4.27, using the skyscraper modules $k_\xi [b + \rho]$ there, followed by the Matlis dual of Definition 4.29.

**Definition 11.27.** Fix a partially ordered group $Q$, a face $\rho$, and an arbitrary commutative monoid $P$. Define a functor $k[\rho] \otimes_{Q \times P}$ on modules $N$ over $Q \times P$ by

$$
k[\rho] \otimes_{Q \times P} N = \bigoplus_{(b, \xi) \in Q \times P} k_\xi [b + \rho] \otimes_{Q \times P} N.
$$

**Definition 11.28.** Fix a real polyhedral group $Q$, a face $\rho$, and a $Q$-module $M$. The generator functor along $\rho$ takes $M$ to its top along $\rho$: the $(Q/\mathbb{R} \rho \times \Delta \rho)$-module

$$
\text{top}_\rho M = (k[\rho] \otimes_{Q \times \Delta \rho} \partial \rho M)^\rho/\rho.
$$

The $\Delta \rho$-graded components of $\text{top}_\rho M$ are denoted by $\text{top}_\rho^\xi M$ for $\xi \in \Delta \rho$.

**Proposition 11.29.** Fix a real polyhedral group $Q$ and a module $N$ such that $k[\rho] \otimes_Q N$ is $Q$-finite. The functors $\text{top}_\rho$ and $\Delta \rho \text{-}\text{top}$ commute on $N$. In particular, if $M$ is a $Q$-module such that $k[\rho] \otimes_Q \partial \rho M$ is $Q$-finite (e.g., if $M$ is $Q$-finite), then

$$
\Delta \rho \text{-}\text{top} (\text{top}_\rho \partial \rho M) \cong \text{top}_\rho \Delta \rho \text{-}\text{top} \partial \rho M.
$$
Proof. This is Matlis dual to Proposition 4.38, but to prove it without an infinitesimal $Q$-finiteness restriction requires a direct argument:

$$
\Delta^\rho \top_{\rho}(\top_{\rho} N) = k \otimes_{\Delta^\rho} (k[\rho] \otimes_{Q} N)^\rho / \rho 
$$

by Definitions 11.10 and 11.17

$$= k \otimes_{\Delta^\rho} \left( \Hom(k[Q_+], k[\rho] \otimes_{Q} N) / \rho \right) \quad \text{by Definition 11.16}
$$

$$= \left( k \otimes_{\Delta^\rho} \Hom(k[Q_+], k[\rho] \otimes_{Q} N) \right) / \rho \quad \text{by Lemma 2.24}
$$

$$= \left( k[\rho] \otimes_{Q} N \otimes_{\Delta^\rho} \kappa \right) / \rho \quad \text{by Lemma 2.39}
$$

$$= \top_{\rho}(\Delta^\rho \top N).
$$

The penultimate line is equal to $(k[\rho] \otimes_{Q \times \Delta^\rho} N)^\rho / \rho$, and $N = \partial_{\rho} M$ yields $\top_{\rho} M$. \qed

Remark 11.30. The condition in Proposition 11.29 that $k[\rho] \otimes_{Q} \partial_{\rho} M$ be $Q$-finite is weaker than $M$ being $Q$-finite. Roughly, in each degree the former counts generators supported on $\rho$ while the latter takes into account all elements, generator or otherwise. The difference is visible when $Q = R$ and $\rho = 0$, in which case $k[\rho] = \kappa$. The module $M = \bigoplus_{\alpha \in \mathbb{R}} \kappa(\alpha + \mathbb{R}_+)$ with one generator in each real degree $\alpha$ yields a module $k \otimes_{R} M$ that is $\mathbb{R}$-finite, since it has dimension 1 in every graded degree, but $M$ itself has uncountable dimension in each graded degree.

Theorem 11.31. Over a real polyhedral group, the generator functor along a face $\rho$ is Matlis dual to the cogenerator functor along $\rho$: if $M$ is infinitesimally $Q$-finite, then

1. $\top_{\rho} (M^\vee) = (\soc_{\rho} M)^\vee$ and
2. $(\top_{\rho} M)^\vee = \soc_{\rho} (M^\vee)$.

Proof. $(\top_{\rho} M)^\vee = (\Delta^\rho \top (\top_{\rho} \partial_{\rho} M))^\vee$ \quad by Proposition 11.29

$$= \nabla^\rho \soc \left( (\top_{\rho} \partial_{\rho} M)^\vee \right) \quad \text{by Proposition 11.25.1}
$$

$$= \nabla^\rho \soc \left( \soc_{\rho} (\partial_{\rho} M)^\vee \right) \quad \text{by Theorem 11.22.1}
$$

$$= \nabla^\rho \soc \left( \soc_{\rho} \delta_{\rho} (M^\vee) \right) \quad \text{by Corollary 11.26.2}
$$

$$= \soc_{\rho} (M^\vee) \quad \text{by Proposition 4.38},
$$

and $(\soc_{\rho} M)^\vee = (\nabla^\rho \soc (\soc_{\rho} \delta_{\rho} M))^\vee$ \quad by Proposition 4.38

$$= \Delta^\rho \top \left( (\soc_{\rho} \delta_{\rho} M)^\vee \right) \quad \text{by Proposition 11.25.2}
$$

$$= \Delta^\rho \top \left( (\top_{\rho} ((\delta_{\rho} M)^\vee)) \right) \quad \text{by Theorem 11.22.2}
$$

$$= \Delta^\rho \top \left( (\top_{\rho} (\top_{\rho} (\delta_{\rho} M)^\vee)) \right) \quad \text{by Corollary 11.26.1}
$$

$$= \top_{\rho} (M^\vee) \quad \text{by Proposition 11.29}. \quad \Box
$$

12. Essential properties of tops

Secondary decomposition and attached primes are Matlis dual to primary decomposition and associated primes. To start, here is the dual to Definition 9.1 and Theorem 9.2.
Definition 12.1. Fix a face $\rho$ and a module $M$ over a real or discrete polyhedral group.  
1. The face $\rho$ is attached to $M$ if $\text{top}_\rho M \neq 0$ (Definition 11.28).
2. If $M = \mathbb{k}[I]$ for an interval $I$ then $\rho$ is attached to $I$.
3. The set of attached faces of $M$ or $I$ is denoted by $\text{Att} M$ or $\text{Att} I$.
4. The module $M$ is $\rho$-secondary if $\text{Att}(M) = \{\rho\}$.

Next comes the Matlis dual of Definitions 2.16 and 9.21.

Definition 12.2. An interval cover of a module $M$ over an arbitrary poset is a surjection $\bigoplus_{j \in J} F^j \twoheadrightarrow M$ with each $F^j$ an interval module. The cover is finite if $J$ is finite. It is an upset cover if the intervals are all upsets. The module $M$ is upset-finite if it admits a finite upset cover. If the poset is a real or discrete polyhedral group, the cover is
1. secondary if $F^j = \mathbb{k}[I_j]$ is secondary for all $j$, so $I_j$ is a secondary upset, and
2. minimal if the induced map $\text{top}_\rho F \rightarrow \text{top}_\rho M$ is an isomorphism for all faces $\rho$.

The Matlis dual to Theorems 6.7 has an infinitesimal $\mathbb{Q}$-finiteness hypotheses because duality between tops and socles in Theorem 11.31 requires it (see also Example 12.7), but the dual to Theorem 10.1 has only the upset-finiteness dual to downset-finiteness.

Theorem 12.3 (Essentiality of real tops). Fix a homomorphism $\varphi : N \rightarrow M$ of modules over a real polyhedral group $Q$.
1. If $\varphi$ is surjective with $M$ and $N$ both being infinitesimally $\mathbb{Q}$-finite modules, then $\text{top}_\rho \varphi : \text{top}_\rho N \rightarrow \text{top}_\rho M$ is surjective for all faces $\rho$ of $Q_+$.
2. If $\text{top}_\rho \varphi : \text{top}_\rho N \rightarrow \text{top}_\rho M$ is surjective for all faces $\rho$ of $Q_+$ and $M$ is upset-finite, then $\varphi$ is surjective. □

Remark 12.4. One of the versions of Nakayama’s lemma says that a homomorphism $M \rightarrow N$ of finitely generated modules over a local ring $R$ is surjective if and only if it becomes surjective upon tensoring with the residue field $\mathbb{k}$. In the language of tops and socles, $M \otimes_R \mathbb{k} = \text{top} M$. Therefore Theorem 12.3 is the direct generalization of Nakayama’s lemma to multigraded modules over real-exponent polynomial rings. Some finiteness is still required, but it is vastly weaker than finitely generated, rather requiring roughly that the generators can be gathered into finitely many coherent clumps. There is, in addition, a quintessentially real-exponent further weakening that allows the top to be replaced by a dense image (Theorem 12.15).

Theorem 12.5 (Essentiality of discrete tops). Fix a homomorphism $\varphi : N \rightarrow M$ of modules over a discrete polyhedral group $Q$.
1. If $\varphi$ is surjective then $\overline{\text{top}}_\rho \varphi : \overline{\text{top}}_\rho N \rightarrow \overline{\text{top}}_\rho M$ is surjective for all faces $\rho$ of $Q_+$.
2. If $\overline{\text{top}}_\rho \varphi : \overline{\text{top}}_\rho N \rightarrow \overline{\text{top}}_\rho M$ is surjective for all faces $\rho$ of $Q_+$ and $M$ is upset-finite, then $\varphi$ is surjective. □

Remark 12.6. In terms of persistent homology, Theorems 12.3 and 12.5 say that a homomorphism of multipersistence modules is surjective if and only if it maps the “left endpoints” of the source surjectively onto the “left endpoints” of the target.
Example 12.7. Some hypothesis is needed in Theorem 12.3.1, in contrast to Theorem 6.7.1 or indeed Theorem 12.5.1. Let $M = \mathbb{k}[U]$ for the open half-plane $U \subset \mathbb{R}^2$ above the antidiagonal line $y = -x$. Then $M$ is $\{0\}$-secondary, with $(\top^\epsilon_0 M)_b \neq 0$ precisely when $b$ lies on the antidiagonal and $\xi$ is the $x$-axis or $y$-axis. The direct sum $\bigoplus_{b \neq 0} (\mathbb{k}[b + Q_{\nabla y}] \oplus \mathbb{k}[b + Q_{\nabla x}])$ surjects onto $M$, but the map on tops fails to hit any element in $\mathbb{R}^2$-degree $0$. This kind of behavior might lead one to wonder: why is its Matlis dual not a counterexample to Theorem 6.7.1? Because $M^\vee$ does not possess a well defined map to a direct sum indexed by $a \neq 0$ along the antidiagonal line, only to a direct product. Any sequence of points $v_k \in -U$ converging to $0$ yields a sequence of elements $z_k \in M$. The image of the sequence $\{z_k\}_{k=1}^\infty$ in any particular one (or finite direct sum) of the downset modules of the form $\mathbb{k}[a - Q_{\nabla x}]$ with $a \neq 0$ is eventually $0$, but in the direct product the sequence $\{z_k\}_{k=1}^\infty$ survives forever. The direct limit of the image sequence witnesses the nonzero socle of the direct product at the missing point $0$.

Theorem 12.8. Every upset-finite module $M$ over a real or discrete polyhedral group admits a minimal secondary interval hull. When the polyhedral group is discrete, it is possible to use upsets for all of the intervals.

Proof. This is the Matlis dual of Theorems 9.23 and 10.8, using Example 11.8 to allow the results of Section 11 to be applied at will, as the strongest hypothesis there is infinitesimal $Q$-finiteness. □

Remark 12.9. Matlis duality in persistent homology might appear to indicate that generators (births) are in adamantine antisymmetry with cogenerators (deaths), but when it comes to interactions between the two, the symmetry is broken by the partial order on $Q$: elements in $Q$-modules move from birth inexorably toward death. Definition 12.10 treats elements functorially, as homomorphisms from the monoid algebra of the positive cone. Doing so makes it clear that the dual of an element is not an element. It is instead a homomorphism to the injective hull of the residue field, as in Definition 12.11. This complication in dealing with generators rather than cogenerators cements the choice to develop the theory in terms of cogenerators in Sections 3–10.

Density considerations are important for use in connection with resolutions (Section 13). They dualize directly, but phrasing them accurately is touchy because of issues like those in Remarks 11.19 and 11.20. Clearer duality comes from a functorial recasting of Definition 3.19; the following is precisely equivalent to that definition.

Definition 12.10. An element $\mathbb{k}[b + Q] \xrightarrow{\beta} M$ is said to divide a boundary element $\mathbb{k}[a + Q] \xrightarrow{\alpha} \delta^\sigma M$ if $b \in a - Q_{\nabla \sigma} = a - \sigma^\epsilon - Q_+$ (Lemma 3.8) and $\alpha$ equals the composite

$$
\mathbb{k}[a + Q] \xrightarrow{\beta} \mathbb{k}[b + Q] \xrightarrow{\alpha} M_b \xrightarrow{M_{a-\sigma}}
$$

of the inclusion of principal upset $Q$-modules induced by $a + Q_+ \subseteq b + Q_+$ with $\beta$ and the natural map from Lemma 3.13. The element $\beta$ is said to $\sigma$-divide $\alpha$ if, more restrictively, $b \in a - \sigma^\epsilon$. 


**Definition 12.11.** Fix a module $M$ over a real polyhedral group $Q$.

1. A basin $M \xrightarrow{\beta} k[b - Q_+]$ is said to **attract** a boundary basin $\partial^\xi M \xrightarrow{\alpha} k[a - Q_+]$ if $b \in a + Q_+ \xi = a + \xi^+ + Q_+$ (Lemma 3.8) and $\alpha$ equals the composite

$$M_{a+\xi} \xrightarrow{\beta} M \xrightarrow{\delta} k[b - Q_+] \xrightarrow{\alpha} k[a - Q_+]$$

of the natural map from Lemma 11.2 with $\beta$ and the surjection of coprincipal downset $Q$-modules induced by $b - Q_+ \supseteq a - Q_+$.

2. The basin $\beta$ is said to **$\xi$-attract** $\alpha$ if, more restrictively, $b \in a + \xi^+$.

3. The basin $\beta$ is **$\xi$-secondary** if its image in $k[b - Q_+]$ is a $\xi$-secondary module.

**Example 12.12.** Because of quotients modulo faces in Definition 11.28, a basin $\tilde{t} : \text{top}_\rho M \to k[\tilde{a} - Q_+ / \mathbb{R}\rho]$ takes values modulo $\rho$. This basin lifts canonically to a homomorphism

$$(k[\rho] \otimes_{Q \times \Delta \rho} \partial^\xi M)^\rho \to k[a - Q_+ + \mathbb{R}\rho]$$

that is not itself a basin but is induced by (perhaps many) basins

$$k[\rho] \otimes_{Q \times \Delta \rho} \partial^\xi M \to k[a - Q_+]$$

under applying the Matlis dual $(-)^\rho$ of localization (Definition 11.16). Note that

$$\partial^\xi M \to k[\rho] \otimes_{Q \times \Delta \rho} \partial^\xi M$$

by right-exactness of colimits. (For the current purpose, surjectivity of this last map is irrelevant, but it might be handy to keep in mind for intuition.) Composing these various lifts, the basin $\tilde{t}$ lifts to (perhaps many) boundary basins $t : \partial^\xi \to k[a - Q_]$.

**Definition 12.13.** Fix a module $M$ over a real polyhedral group $Q$.

1. A **neighborhood in $\text{top}_\rho M$** of a basin $\tilde{t}$ of $\text{top}_\rho M$ is $\text{top}_\rho(\beta M)$ for a $\xi$-secondary basin $\beta$ that $\xi$-attracts a lifted boundary basin $t$ of $\partial_\rho M$ (Example 12.12).

2. A surjection $\text{top}_\rho M \to T_\rho$ of $(Q / \mathbb{R}\rho \times \Delta \rho)$-modules is **dense** if for all $\xi \supseteq \rho$, every neighborhood of every basin of $\text{top}_\rho^\xi M$ has nonzero image in $T_\rho$.

3. A quotient functor $\text{top}_\rho \to T_\rho$ from modules over $Q$ to modules over $Q / \mathbb{R}\rho \times \Delta \rho$ is **dense** if $\text{top}_\rho k[U] \to T_\rho k[U]$ is dense for all faces $\rho$ and upsets $U \subseteq Q$.

**Remark 12.14.** Definition 12.13 skips the duals to notions of “nearby” and “vicinity” from Definition 8.1 because the work of defining “coprimary”—and hence “secondary”, by taking duals—has already been done in Section 9. The contrast between Definitions 12.13 and 8.1 is simple: principal primary submodules become coprincipal secondary quotients. Definition 12.13.3 is Matlis dual to Definition 7.26.

**Theorem 12.15.** Fix quotient functors $\text{top}_\rho \to T_\rho$ for all faces $\rho$ of a real polyhedral group. Theorem 12.3 holds with $T$ instead of top if and only if $\text{top}_\rho \to T_\rho$ is dense for all $\rho$. 
Proof. Apply the exact Matlis duality functor to the statement of Theorem 7.27 in the presence of the finiteness hypothesis in Theorem 12.3.

The straightforward dualization of primary decomposition in Sections 9.4 and 10 to secondary decomposition is omitted.

13. Minimal presentations over discrete or real polyhedral groups

Algebra of modules over arbitrary posets [Mil20a] and primary decomposition over partially ordered groups [Mil20b] lack a crucial aspect of noetherian commutative algebra, namely minimality. Much of the edifice of modern commutative algebra is built on numerical, homological, combinatorial, or geometric behavior whose quantification rests firmly on notions of minimality: Betti numbers, Castelnuovo–Mumford regularity, primary and irreducible decomposition, homological dimension, computational complexity bounds—all of these depend on minimal resolutions, or minimal decompositions, or minimal degrees of some nature. When the partially ordered group is a real vector space, earlier sections rescue notions of minimality, perhaps with density amendments, for generators and decompositions. This section explores to what extent minimality applies to presentations and resolutions.

Definition 13.1 ([Mil20a, Definitions 3.16, 6.1, 6.4]). Fix a module $M$ over a partially ordered abelian group $Q$.

1. An upset presentation of $M$ is an expression of $M$ as the cokernel of a homomorphism $F_1 \to F_0$ such that each $F_i$ is a direct sum of upset modules.
2. A downset copresentation of $M$ is an expression of $M$ as the kernel of a homomorphism $E^0 \to E^1$ such that each $E^i$ is a direct sum of downset modules.
3. A fringe presentation of $M$ is a direct sum $F$ of upset modules $k[U]$, a direct sum $E$ of downset modules $k[D]$, and a homomorphism $F \to E$ of $Q$-modules with
   - image isomorphic to $M$ and
   - components $k[U] \to k[D]$ that are connected (Definition 5.1).
4. An upset resolution of $M$ is a complex $F \cdot$ of $Q$-modules, each a direct sum of upset modules, whose differential $F_i \to F_{i-1}$ decreases homological degrees and has only one nonzero homology $H_0(F \cdot) \cong M$.
5. A downset resolution of $M$ is a complex $E^\cdot$ of $Q$-modules, each a direct sum of downset modules, whose differential $E^i \to E^{i+1}$ increases cohomological degrees and has only one nonzero homology $H^0(E^\cdot) \cong M$.

Definition 13.2. Each indicator presentation or indicator resolution in Definition 13.1

1. is finite if it has only finitely many summands in total;
2. dominates a constant subdivision (Definition 2.8) or poset encoding (Definition 2.11) of $M$ if the morphism does (Definition 2.13);
3. is semialgebraic or PL if the morphism has that type (Definition 2.13).
Definition 13.3. Over a real polyhedral group, a module morphism \( \varphi : M \to N \) is

1. \textit{injectively minimal} or \textit{injectively dense} if the canonical inclusion \( \text{im} \varphi \hookrightarrow N \) induces an isomorphism \( \text{soc}(\text{im} \varphi) \cong \text{soc} N \) or dense inclusion \( \text{soc}(\text{im} \varphi) \rightarrow \text{soc} N \);

2. \textit{surjectively minimal} or \textit{surjectively dense} if the canonical surjection \( M \twoheadrightarrow \text{im} \varphi \) induces an isomorphism \( \text{top} M \cong \text{top}(\text{im} \varphi) \) or dense surjection \( \text{top} M \twoheadrightarrow \text{top}(\text{im} \varphi) \).

Over a discrete polyhedral group the definition of \textit{injectively minimal} and \textit{surjectively minimal} are unchanged. In either the real or discrete polyhedral setting, a complex of modules is \textit{injectively or surjectively minimal} or \textit{dense} if all of its differentials are.

Remark 13.4. Category-theoretically, injective minimality or density should naturally be phrased in terms of the image morphism of \( \varphi \), while surjective minimality and density should be phrased in terms of the coimage morphism of \( \varphi \).

Remark 13.5. The notion of minimal morphism makes sense in ordinary commutative algebra much more generally: minimal resolutions and essential submodules are transparently special cases. Irredundant irreducible decompositions \( 0 = \bigcap W_j \) in a module \( M \) also correspond to also injectively minimal morphisms \( M \hookrightarrow \bigoplus M/W_j \). In contrast, for historical reasons, a minimal primary decomposition \( 0 = \bigcap P_j \) in a module \( M \) is usually defined to have a minimal number of intersectands, a condition that need not induce an injectively minimal morphism \( M \hookrightarrow \bigoplus M/P_j \). Consequently, minimal primary decompositions by this definition suffer from annoying non-uniqueness. For example, the \( p \)-primary component in one minimal primary decomposition can strictly contain the \( p \)-primary component in another. Defining a primary decomposition to be \textit{minimal} precisely when it induces an injectively minimal morphism would rectify this containment problem and other defects.

Definition 13.6. Fix a module \( M \) over a real or discrete polyedral group \( Q \).

1. A downset copresentation or resolution \( E^\bullet \) of \( M \) is \textit{minimal} or \textit{dense} if the exact augmented complex \( 0 \to M \to E^\bullet \) is correspondingly injectively minimal or dense.

2. An upset presentation or resolution \( F^\bullet \) of \( M \) is \textit{minimal} or \textit{dense} if the exact augmented complex \( 0 \leftarrow M \leftarrow F^\bullet \) is correspondingly top-minimal or top-dense.

3. A fringe presentation \( F \to E \) of \( M \) is \textit{minimal} or \textit{dense} if it is the composite of a correspondingly minimal or dense upset cover and downset hull of \( M \).

Theorem 13.7. A module over a real polyhedral group \( Q \) is tame if and only if it admits

1. a dense finite fringe presentation; or

2. a dense finite upset presentation; or

3. a dense finite downset copresentation.

Over a discrete polyedral group these presentations can be chosen minimal instead of dense. When the module is semialgebraic or PL these presentations can all be chosen semialgebraic or PL, respectively.
Proof. In both the real and discrete cases, any one of these presentations is, in particular, finite, so the existence of any of them implies that the module is tame by the syzygy theorem [Mil20a, Theorem 6.12]. It is the other direction that requires the theory in this paper.

In the real polyhedral case, any finite downset hull can be densitized by Theorem 9.18 and Remark 9.24. The Matlis dual of this statement says that any finite upset cover can be densitized, as well. Composing these from a given finite fringe presentations yields a dense finite fringe presentation. In addition, the cokernel of any downset hull (dense or otherwise) of a tame module is tame by Proposition 2.15, so the cokernel has a dense finite downset hull by Theorem 9.18 again. That yields a dense finite downset copresentation. The Matlis dual of a dense finite downset copresentation of the Matlis dual $M^\vee$ is a dense upset presentation of $M$ by Theorem 11.31 (which applies unfettered to tame modules by Example 11.8).

The minimal discrete polyhedral case follows the parallel proof, using Theorem 10.8 and Remark 10.9 instead of Theorem 9.18 and Remark 9.24.

If $M$ is semialgebraic, then the densitization procedure in Theorem 9.18 and Remark 9.24 is semialgebraic by induction on the number $k$ of summands there, the base case being the canonical primary decomposition of a semialgebraic interval in Theorem 9.12, which is semialgebraic by Theorem 5.13. □

Remark 13.8. If minimal instead of dense presentations are desired in the real polyhedral setting, then they can be achieved by combining Definitions 9.21 and 13.1 to form interval copresentations instead of downset copresentations, or the Matlis dual interval presentations instead of upset presentations. Splicing these yields interval fringe presentations instead of fringe presentations. Theorem 9.23 and the interval version of Remark 9.24 forms the basis for a minimalizing version of the proof of Theorem 13.7.

Remark 13.9. Comparing Theorem 13.7 to the syzygy theorem for tame modules over arbitrary posets [Mil20a, Theorem 6.12], various items are missing.

1. Theorem 13.7 makes no claim concerning whether the presentations can be densitized if a poset encoding $\pi : Q \to P$ (Definition 2.11) has been specified beforehand. It is a priori possible that deleting redundant generators of upsets and cogenerators of downsets could prevent an indicator summand from being constant on fibers of $\pi$.

2. Theorem 13.7 makes no claim concerning finite poset encodings dominating any one of the three presentations there, but as each of these presentations is finite, existence is already implied by the syzygy theorem for tame modules [Mil20a, Theorem 6.12], including semialgebraic and PL considerations.

3. Theorem 13.7 makes no claim concerning finiteness of minimal or dense indicator resolutions. Dense resolutions of tame modules over real polyhedral groups (or minimal ones in the discrete polyhedral setting) can be constructed from scratch
by Theorem 9.18, Theorem 10.8, and their Matlis duals, but there is no a priori guarantee that such resolutions must terminate after finitely many steps.

**Conjecture 13.10.** Every tame, semialgebraic, or PL module $M$ over a real polyhedral group $Q$ has finite dense downset and upset resolutions of the corresponding type.

**Conjecture 13.11.** Every tame module $M$ over a discrete polyhedral group $Q$ has finite minimal downset and upset resolutions of the corresponding type.

**Remark 13.12.** Remark 13.9.3 raises an intriguing point about indicator resolutions: the bound on the length in the syzygy theorem over arbitrary posets [Mil20a, Theorem 6.12] comes from the order dimension of an encoding poset, which is more or less unrelated to the dimension of the real or discrete polyhedral group. It seems plausible that the geometry of the polyhedral group asserts control to prevent the lengths from going too high, just as it does to prevent the cohomological dimension of an affine semigroup ring from going too high via Ishida complexes to compute local cohomology [MS05, Section 13.3.1]. This points to potential value of developing a derived functor side of the top-socle / birth-death / generator-cogenerator story for indicator resolutions to solve Conjecture 13.14, which would be an even tighter indicator analogue of the Hilbert Syzygy Theorem.

**Definition 13.13.** Fix a module $M$ over a poset $Q$.

1. The **downset-dimension** of $M$ is the smallest length of a downset resolution of $M$.
2. The **upset-dimension** of $M$ is the smallest length of an upset resolution of $M$.
3. The **indicator-dimension** of $M$ is maximum of its downset- and upset-dimensions.
4. The **indicator-dimension** of $Q$ is the maximum of the indicator-dimensions of its tame modules.

**Conjecture 13.14.** The indicator-dimension of any real or discrete polyhedral group $Q$ equals the rank of $Q$ as a free module (over the field $\mathbb{R}$ or group $\mathbb{Z}$, respectively).

**Remark 13.15.** No uniform bound on the lengths of finite upset and downset resolutions over an arbitrary posets $Q$ is known when $Q$ has quotients with unbounded order dimension. It is already open to find a module over $\mathbb{R}^2$ whose indicator-dimension is provably as high as 2. It would not be shocking if the rank of $Q$ were an upper bound instead of an equality for the indicator-dimension in Conjecture 13.14: the use of upset modules instead of free modules could prevent the final syzygies that, in finitely generated situations, come from elements supported at the origin by local duality.
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