Unitals in PG(2, $q^2$) with a large 2-point stabiliser

L.Giuzzi and G.Korchmáros
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Abstract

Let $U$ be a unital embedded in the Desarguesian projective plane PG(2, $q^2$). Write $M$ for the subgroup of $\text{PGL}(3, q^2)$ which preserves $U$. We show that $U$ is classical if and only if $U$ has two distinct points $P, Q$ for which the stabiliser $G = M_{P,Q}$ has order $q^2 - 1$.

1 Introduction

In the Desarguesian projective plane PG(2, $q^2$), a unital is defined to be a set of $q^3 + 1$ points containing either 1 or $q + 1$ points from each line of PG(2, $q^2$). Observe that each unital has a unique 1-secant at each of its points. The idea of a unital arises from the combinatorial properties of the non-degenerate unitary polarity $\pi$ of PG(2, $q^2$). The set of absolute points of $\pi$ is indeed a unital, called the classical or Hermitian unital. Therefore, the projective group preserving the classical unital is isomorphic to $\text{PGU}(3, q)$ and acts on its points as $\text{PGU}(3, q)$ in its natural 2-transitive permutation representation. Using the classification of subgroups of $\text{PGL}(3, q^2)$, Hoffer [14] proved that a unital is classical if and only if it is preserved by a collineation group isomorphic to $\text{PSU}(3, q^2)$. Hoffer’s characterisation has been the starting point for several investigations of unitals in terms of the structure of their automorphism group, see [3, 6, 8, 10, 11, 12, 15, 16]; see also the survey [2, Appendix B]. In PG(2, $q^2$) with $q$ odd, L.M. Abatangelo [1] proved that a Buekenhout–Metz unital with a cyclic 2–point stabiliser of order $q^2 - 1$ is necessarily classical. In their talk at Combinatorics 2010, G. Donati e N. Durante have conjectured that Abatangelo’s characterisation holds true for any unital in PG(2, $q^2$). In this note, we provide a proof of this conjecture.

Our notation and terminology are standard, see [2], and [13]. We shall assume $q > 2$, since all unitals in PG(2, 4) are classical.

2 Some technical lemmas

Let $M$ be the subgroup of $\text{PGL}(3, q^2)$ which preserves a unital $U$ in PG(2, $q^2$). A 2-point stabiliser of $U$ is a subgroup of $M$ which fixes two distinct points of $U$.

**Lemma 2.1.** Let $U$ be a unital in PG(2, $q^2$) with a 2–point stabiliser $G$ of order $q^2 - 1$. Then, $G$ is cyclic, and there exists a projective frame in PG(2, $q^2$) such that $G$ is generated by a projectivity.
thus, we have

\[ G \times X \text{ term } = (0,0) \]

Let

\[ \lambda \text{ a primitive element of } GF(q^2) \text{ and } \mu \text{ is a primitive element of } GF(q). \]

**Proof.** Let \( O, Y_\infty \) be two distinct points of \( U \) such that the stabiliser \( G = M_{O,Y_\infty} \) has order \( q^2 - 1 \). Choose a projective frame in \( PG(2,q^2) \) so that \( O = (0,0,1) \), \( Y_\infty = (0,1,0) \) and the 1-secants of \( U \) at those points are respectively \( \ell_X : X_2 = 0 \) and \( \ell_\infty : X_3 = 0 \). Write \( X_\infty = (1,0,0) \) for the common point of \( \ell_X \) and \( \ell_\infty \). Observe that \( G \) fixes the vertices of the triangle \( OX_\infty Y_\infty \). Therefore, \( G \) consists of projectivities with diagonal matrix representation. Let now \( h \in G \) be a projectivity that fixes a further point \( P \in \ell_X \) apart from \( O, X_\infty \). Then, \( h \) fixes \( \ell_X \) point-wise; that is, \( h \) is a perspectivity with axis \( \ell_X \). Since \( h \) also fixes \( Y_\infty \), the centre of \( h \) must be \( Y_\infty \). Take any point \( R \in \ell_X \) with \( R \neq O, X_\infty \). Obviously, \( h \) preserves the line \( r = Y_\infty R \); hence, it also preserves \( r \cap U \). Since \( r \cap U \) comprises \( q \) points other than \( R \), the subgroup \( H \) generated by \( h \) has a permutation representation of degree \( q \) in which no non-trivial permutation fixes a point. As \( q = p^r \) for a prime \( p \), this implies that \( p \) divides \( |H| \). On the other hand, \( h \) is taken from a group of order \( q^2 - 1 \). Thus, \( h \) must be the trivial element in \( G \). Therefore, \( G \) has a faithful action on \( \ell_X \) as a 2-point stabiliser of \( PG(1,q^2) \). This proves that \( G \) is cyclic. Furthermore, a generator \( g \) of \( G \) has a matrix representation

\[
\begin{pmatrix}
\lambda & 0 & 0 \\
0 & \mu & 0 \\
0 & 0 & 1
\end{pmatrix}
\]

with \( \lambda \) a primitive element of \( GF(q^2) \).

As \( G \) preserves the set \( \Delta = U \cap OY_\infty \), it also induces a permutation group \( \bar{G} \) on \( \Delta \). Since any projectivity fixing three points of \( OY_\infty \) must fix \( OY_\infty \) point-wise, \( \bar{G} \) is semiregular on \( \Delta \). Therefore, \( |\bar{G}| \) divides \( q - 1 \). Let now \( F \) be the subgroup of \( G \) fixing \( \Delta \) point-wise. Then, \( F \) is a perspectivity group with centre \( X_\infty \) and axis \( \ell_Y : X_1 = 0 \). Take any point \( R \in \ell_Y \) such that the line \( r = RX_\infty \) is a \((q+1)\)-secant of \( U \). Then, \( r \cap U \) is disjoint from \( \ell_Y \). Hence, \( F \) has a permutation representation on \( r \cap U \) in which no non-trivial permutation fixes a point. Thus, \( |F| \) divides \( q + 1 \). Since \( |G| = q^2 - 1 \), we have \( |\bar{G}| \leq q - 1 \) and \( |G| = |\bar{G}| |F| \). This implies \( |\bar{G}| = q - 1 \) and \( |F| = q + 1 \). From the former condition, \( \mu \) must be a primitive element of \( GF(q) \). □

**Lemma 2.2.** In \( PG(2,q^2) \), let \( \mathcal{H}_1 \) and \( \mathcal{H}_2 \) be two non-degenerate Hermitian curves which have the same tangent at a common point \( P \). Denote by \( I(P, \mathcal{H}_1 \cap \mathcal{H}_2) \) the intersection multiplicity of \( \mathcal{H}_1 \) and \( \mathcal{H}_2 \) at \( P \). Then,

\[
I(P, \mathcal{H}_1 \cap \mathcal{H}_2) = q + 1.
\]

**Proof.** Since, up to projectivities, there is a unique class of Hermitian curves in \( PG(2,q^2) \), we may assume \( \mathcal{H}_1 \) to have equation \(-X_1^{q+1} + X_2^q X_3 + X_2 X_3^q = 0 \). Furthermore, as the projectivity group \( PGU(3,q) \) preserving \( \mathcal{H}_1 \) acts transitively on the points of \( \mathcal{H}_1 \) in \( PG(2,q^2) \), we may also suppose \( P = (0,0,1) \). Within this setting, the tangent \( r \) of \( \mathcal{H}_1 \) at \( P \) coincides with the line \( X_2 = 0 \). As no term \( X_1^j \) with \( 0 < j \leq q \) occurs in the equation of \( \mathcal{H}_1 \), the intersection multiplicity \( I(P, \mathcal{H}_1 \cap r) \) is equal to \( q + 1 \).

The equation of the other Hermitian curve \( \mathcal{H}_2 \) might be written as

\[
F(X_1, X_2, X_3) = a_0 X_3^q X_2 + a_1 X_3^{q-1} G_1(X_1, X_2) + \ldots + a_q G_q(X_1, X_2) = 0,
\]
where \( a_0 \neq 0 \) and \( \deg G_i(X_1, X_2) = i + 1 \). Since the tangent of \( \mathcal{H}_2 \) at \( P \) has no other common point with \( \mathcal{H}_2 \), even over the algebraic closure of \( \text{GF}(q^2) \), no terms \( X_1^j \) with \( 0 < j \leq q \) can occur in the polynomials \( G_i(X_1, X_2) \). In other words, \( I(P, \mathcal{H}_2 \cap r) = q + 1 \).

A primitive representation of the unique branch of \( \mathcal{H}_1 \) centred at \( P \) has components

\[
x(t) = t, \quad y(t) = ct + \ldots, \quad x_3(t) = 1
\]

where \( i \) is a positive integer and \( y(t) \in \text{GF}(q^2)[[t]] \), that is, \( y(t) \) stands for a formal power series with coefficients in \( \text{GF}(q^2) \).

From \( I(P, \mathcal{H}_1 \cap r) = q + 1 \),

\[
y(t)^q + y(t) - t^{q+1} = 0,
\]

whence \( y(t) = t^{q+1} + H(t) \), where \( H(t) \) is a formal power series of order at least \( q + 2 \). That is, the exponent \( j \) in the leading term \( ct^j \) of \( H(t) \) is larger than \( q + 1 \).

It is now possible to compute the intersection multiplicity \( I(P, \mathcal{H}_1 \cap \mathcal{H}_2) \) using [13, Theorem 4.36]:

\[
I(P, \mathcal{H}_1 \cap \mathcal{H}_2) = \text{ord}_t F(t, y(t), 1) = \text{ord}_t (a_0 t^{q+1} + G(t)),
\]

with \( G(t) \in \text{GF}(q^2)[[t]] \) of order at least \( q + 2 \). From this, the assertion follows. \( \square \)

**Lemma 2.3.** In \( \text{PG}(2, q^2) \), let \( \mathcal{H} \) be a non-degenerate Hermitian curve and let \( \mathcal{C} \) be a Hermitian cone whose centre does not lie on \( \mathcal{H} \). Assume that there exist two points \( P_i \in \mathcal{H} \cap \mathcal{C} \), with \( i = 1, 2 \), such that the tangent line of \( \mathcal{H} \) at \( P_i \) is a linear component of \( \mathcal{C} \). Then

\[
I(P_i, \mathcal{H} \cap \mathcal{C}) = q + 1.
\] (2)

**Proof.** We use the same setting as in the proof of Lemma 2.2 with \( P = P_1 \). Since the action of \( \text{PGU}(3, q) \) is \( 2 \)-transitive on the points of \( \mathcal{H} \), we may also suppose that \( P_2 = (0, 1, 0) \). Then the centre of \( \mathcal{C} \) is the point \( X_\infty = (1, 0, 0) \), and \( \mathcal{C} \) has equation \( c^2 X_2^3 X_3 + c X_2 X_3^q = 0 \) with \( c \neq 0 \). Therefore,

\[
I(P, \mathcal{H} \cap \mathcal{C}) = \text{ord}_t (c^2 y(t)^q + c y(t)) = \text{ord}_t (c^2 t^{q+1} + K(t))
\]

with \( K(t) \in \text{GF}(q^2)[[t]] \) of order at least \( q + 2 \), whence the assertion follows. \( \square \)

## 3 Main result

**Theorem 3.1.** In \( \text{PG}(2, q^2) \), let \( \mathcal{U} \) be a unital and write \( M \) for the group of projectivities which preserves \( \mathcal{U} \). If \( \mathcal{U} \) has two distinct points \( P, Q \) such that the stabiliser \( G = M_{P,Q} \) has order \( q^2 - 1 \), then \( \mathcal{U} \) is classical.

The main idea of the proof is to build up a projective plane of order \( q \) using, for the definition of points, non-trivial \( G \)-orbits in the affine plane \( \text{AG}(2, q^2) \) which arise from \( \text{PG}(2, q^2) \) by removing the line \( \ell_\infty : X_3 = 0 \) with all its points. To this purpose, take \( \mathcal{U} \) and \( G \) as in Lemma 2.1 with \( \mu = \lambda^{q+1} \), and define an incidence structure \( \Pi = (\mathcal{P}, \mathcal{L}) \) as follows:

1. Points are all non-trivial \( G \)-orbits in \( \text{AG}(2, q^2) \). 

2. Lines are \(\ell_Y\), and the non-degenerate Hermitian curves of equation

\[ \mathcal{H}_b : -X_1^{q+1} + bX_2X_3^q + b^qX_3X_2 = 0, \]  

with \(b\) ranging over \(\text{GF}(q^2)^*\), together with the Hermitian cones of equation

\[ \mathcal{C}_c : c^qX_2^3X_3 + cX_2X_3^q = 0, \]  

with \(c\) ranging over a representative system of cosets of \((\text{GF}(q), *)\) in \((\text{GF}(q^2), *)\).

3. Incidence is the natural inclusion.

**Lemma 3.2.** The incidence structure \(\Pi = (\mathcal{P}, \mathcal{L})\) is a projective plane of order \(q\).

**Proof.** In \(\text{AG}(2, q^2)\), the group \(G\) has \(q^2 + q + 1\) non-trivial orbits, namely its \(q^2\) orbits disjoint from \(\ell_Y\), each of length \(q^2 - 1\), and its \(q + 1\) orbits on \(\ell_Y\), these of length \(q - 1\). Therefore, the total number of points in \(\mathcal{P}\) is equal to \(q^2 + q + 1\). By construction of \(\Pi\), the number of lines in \(\mathcal{L}\) is also \(q^2 + q + 1\). Incidence is well defined as \(G\) preserves \(\ell_Y\) and each Hermitian curve and cone representing lines of \(\mathcal{L}\).

We now count the points incident with a line in \(\Pi\). Each \(G\)-orbit on \(\ell_Y\) distinct from \(O\) and \(Y_\infty\) has length \(q - 1\). Hence there are exactly \(q + 1\) such \(G\)-orbits; in terms of \(\Pi\), the line represented by \(\ell_Y\) is incident with \(q + 1\) points. A Hermitian curve \(\mathcal{H}_b\) of Equation (3) has \(q^3\) points in \(\text{AG}(2, q^2)\) and meets \(\ell_Y\) in a \(G\)-orbit, while it contains no point from the line \(\ell_X\). As \(q^3 - q = q(q^2 - 1)\), the line represented by \(\mathcal{H}_b\) is incident with \(q + 1\) points in \(\mathcal{P}\). Finally, a Hermitian cone \(\mathcal{C}_c\) of Equation (4) has \(q^3\) points in \(\text{AG}(2, q^2)\) and contains \(q\) points from \(\ell_Y\). One of these \(q\) points is \(O\), the other \(q - 1\) forming a non-trivial \(G\)-orbit. The remaining \(q^3 - q\) points of \(\mathcal{C}_c\) are partitioned into \(q\) distinct \(G\)-orbits. Hence, the line represented by \(\mathcal{C}_c\) is also incident with \(q + 1\) points. This shows that each line in \(\Pi\) is incident with exactly \(q + 1\) points.

Therefore, it is enough to show that two any two distinct lines of \(\mathcal{L}\) have exactly one common point. Obviously, this is true when one of these lines is represented by \(\ell_Y\). Furthermore, the point of \(\mathcal{P}\) represented by \(\ell_X\) is incident with each line of \(\mathcal{L}\) represented by a Hermitian cone of equation (1). We are led to investigate the case where one of the lines of \(\mathcal{L}\) is represented by a Hermitian curve \(\mathcal{H}_b\) of equation (1), and the other line of \(\mathcal{L}\) is represented by a Hermitian curve \(\mathcal{H}\) which is either another Hermitian curve \(\mathcal{H}_d\) of the same type of Equation (3), or a Hermitian cone \(\mathcal{C}_c\) of Equation (4).

Clearly, both \(O\) and \(Y_\infty\) are common points of \(\mathcal{H}_b\) and \(\mathcal{H}\). From Kestenband’s classification [17], see also [2] Theorem 6.7, \(\mathcal{H}_b \cap \mathcal{H}\) cannot consist of exactly two points. Therefore, there exists another point, say \(P \in \mathcal{H}_b \cap \mathcal{H}\). Since \(\ell_X\) and \(\ell_0\) are 1-secants of \(\mathcal{H}_b\) at the points \(O\) and \(Y_\infty\), respectively, either \(P\) is on \(\ell_Y\) or \(P\) lies outside the fundamental triangle. In the latter case, the \(G\)-orbit \(\Delta_1\) of \(P\) has size \(q^2 - 1\) and represents a point in \(\mathcal{P}\). Assume that \(\mathcal{H}_b \cap \mathcal{H}\) contains a further point, not lying in \(\Delta_1\). If the \(G\)-orbit of \(Q\) is \(\Delta_2\), then

\[ |\mathcal{H}_b \cap \mathcal{H}| \geq |\Delta_1| + |\Delta_2| = 2(q^2 - 1) + 2 = 2q^2. \]

However, from Bézout’s theorem, see [13] Theorem 3.14,

\[ |\mathcal{H}_b \cap \mathcal{H}| \leq (q + 1)^2. \]

Therefore, \(Q \in \ell_Y\), and the \(G\)-orbit \(\Delta_3\) of \(Q\) has length \(q - 1\). Hence, \(\mathcal{H}_b\) and \(\mathcal{H}\) shear \(q + 1\) points on \(\ell_Y\). If \(\mathcal{H} = \mathcal{H}_d\) is a Hermitian curve of Equation (3), each of these \(q + 1\) points is the tangency point
of a common inflection tangent with multiplicity $q + 1$ of the Hermitian curves $H_b$ and $H$. Write $R_1, \ldots, R_{q+1}$ for these points. Then, by Lemma 2.8, the intersection multiplicity is $I(R_i, H_b \cap H) = (q + 1)^2$. This holds true also when $H$ is a Hermitian cone $C_\nu$ of Equation (4); see Lemma 2.8. Therefore, in any case,

\[ \sum_{i=1}^{q+1} I(R_i, H_b \cap H) = (q + 1)^2. \]

From Bézout’s theorem, $H_b \cap H = \{R_1, \ldots, R_{q+1}\}$. Therefore, $H_b \cap H = \Delta_2 \cup \{O, Y_\infty\}$. This shows that if $Q \notin \ell_Y$, the lines represented by $H_b$ and $H$ have exactly one point in common. The above argument can also be adapted to prove this assertion in the case where $Q \in \ell_Y$. Therefore, any two distinct lines of $L$ have exactly one common point.

**Proof of Theorem 3.1** Assume first $\mu = \lambda^{q+1}$. Construct a projective plane $\Pi$ as in Lemma 3.2. Since $U \setminus \{O, Y_\infty\}$ is the union of $G$-orbits, $U$ represents a set $\Gamma$ of $q + 1$ points in $\Pi$. From [7], $N \equiv 1 \pmod{p}$ where $N$ is the number of common points of $U$ with any Hermitian curve $H_b$. In terms of $\Pi$, $\Gamma$ contains some point from every line $\Lambda$ in $L_b$ represented by a Hermitian curve of Equation (4). Actually, this holds true when the line $\Lambda$ in $L_b$ is represented by a Hermitian curve of Equation (4). To prove it, observe that $\mathcal{C}$ contains a line $\ell$ distinct from both lines $\ell_X$ and $\ell_0$. Then $\ell \cap U$ is non-empty, and contains neither $O$ nor $Y_\infty$. If $P$ is point in $\ell \cap U$, then the $G$-orbit of $P$ contains a common point of $\Gamma$ and $\Lambda$. Since the line in $L$ represented by $\ell_Y$ meets $\Gamma$, it turns out that $\Gamma$ contains some point from every line in $L$.

Therefore, $\Gamma$ is itself a line in $L$. Note that $U$ contains no line. In terms of $PG(2, q^2)$, this yields that $U$ coincides with a Hermitian curve of Equation (3). In particular, $U$ is a classical unital.

To investigate the case $\mu \neq \lambda^{q+1}$, we still work in the above plane $\Pi$. By a straightforward computation, the projectivity $g$ given in Lemma 2.1 induces a non-trivial collineation on $\Pi$. Also, $g$ preserves every Hermitian cone of Equation (4) and the common line $\ell_X$ of these Hermitian cones. In terms of $\Pi$, $g$ is a perspectivity with centre at the point represented by $\ell_X$. Since $g$ also preserves the line $\ell_Y$, the axis of $g$ is $\ell_Y$, regarded as a line in $\Pi$. Therefore, every point of $\Pi$ lying on $\ell_Y$ is fixed by $g$. Consequently, $g^{q+1}$ is the identity collineation. As $g$ has order $q^2 - 1$, this yields that $g^{q+1}$ preserves every Hermitian curve of Equation (3). Thus, $g^{q+1} = (\lambda^{q+1})^{q+1}$, whence $\mu = -\lambda^{q+1}$. In particular, $p \neq 2$.

Consider now the $q + 1$ non-trivial $G$-orbits in $U$ with $G = \langle g \rangle$. For any point $P \in \Pi$, let $n_P$ the number of the non-trivial $G$-orbits in $U$ intersecting the set $\rho(P)$ representing $P$ in $PG(2, q^2)$. Then $n_P = 1$ when $\rho(P)$ is the unique $G$-orbit in $U$ which lies on $\ell_Y$. Otherwise, $0 \leq n_P \leq 2$, with $n_P = 2$ if and only if $\rho(P)$ is a $G$-orbit but the union of two $H$-orbits with $H = \langle g^2 \rangle$.

Let $\Gamma$ be the multiset consisting of all points with $n_P > 0$ and define the weight $\nu_P$ of $P$ to be either 1 or 2, according as $n_P = 2$ or $n_P = 1$. Then, $\sum_{P \in \Gamma} \nu_P = 2q + 2$. We show that $\Gamma$ is a 2-fold blocking multiset of $\Pi$. For this purpose, let $H$ be either a Hermitian curve of Equation (4) or a Hermitian cone of Equation (4). Write $m$ for the number of common points of $H_b$ and $H$, different from $O$ and $Y_\infty$; thus, the total number of common points is $N = m + 2$. As $N \equiv 1 \pmod{p}$, we have $m \geq 1$. Take $P \in H \cap U$. If $\nu_P = 2$, then the line representing $H$ meets $\Gamma$ in a point with weight 2. If $\nu_P = 1$, then the $H$-orbit of $P$ has size $(q^2 - 1)/2$ and lies on both $H$ and $U$. Since $(q^2 - 1)/2 \neq 1 \pmod{p}$, $H$ and $U$ must share a further point $Q$ other than $O$ and $Y_\infty$. Therefore, the points $P'$ and $Q'$ of $\Pi$ which represent the subsets containing $P$ and $Q$ are distinct. This shows that $\Gamma$ meets the line represented by $H$ in two distinct points. Therefore, $\Gamma$ is a 2-fold blocking multiset.
Since $\Gamma$ has at least one point with weight 2, this yields that $\Gamma$ comprises of all points of a line, each with weight 2. Hence, $U$ coincides with the Hermitian curve representing that line. This is to say that $U$ is a classical unital.

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