On continuous variable quantum algorithms for oracle identification problems

Mark R. A. Adcock,1 Peter Høyer,1,2 and Barry C. Sanders1
1Institute for Quantum Information Science, University of Calgary,
Calgary, Alberta, Canada, T2N 1N4. Email: mkadcock@qis.ucalgary.ca
2Department of Computer Science, University of Calgary, 2500 University Drive N.W.,
Calgary, Alberta, Canada, T2N 1N4. Email: hoyer@ucalgary.ca.

We establish a framework for oracle identification problems in the continuous variable setting, where the stated problem necessarily is the same as in the discrete variable case, and continuous variables are manifested through a continuous representation in an infinite-dimensional Hilbert space. We apply this formalism to the Deutsch-Jozsa problem and show that, due to an uncertainty relation between the continuous representation and its Fourier-transform dual representation, the corresponding Deutsch-Jozsa algorithm is probabilistic hence forbids an exponential speed-up, contrary to a previous claim in the literature.

PACS numbers: 03.67.Ac

December 18, 2008

I. INTRODUCTION

Quantum information protocols have been demonstrated experimentally in both the discrete-variable (DV) and so-called continuous-variable (CV) settings. DV quantum information protocols employ qubits [1] and qudits [2], and CV quantum information protocols regard continuously parameterized canonical position states as the logical elements analogous to qubits for the DV case [3]. CV quantum information is experimentally appealing because sophisticated squeezed light experiments have led to claims of successful quantum information protocols such as teleportation [4], key distribution [5], and memory [6, 7], but the theoretical status of CV quantum information is challenged by unresolved issues concerning quantum error correction [8], non-distillability [9], no-go theorems for quantum computation [10, 11], and the absence of full security proofs for key distribution.

CV information processing has also been studied for classical models, including the now named Blum-Shub-Smale machine [12] and continuous Turing machines [13]. These models are of background relevance to the research into CV quantum information and are referenced here for contextual purposes.

In this paper, we establish a sound theoretical framework for studying quantum algorithms and apply this framework to study the CV analogue of the early DV quantum algorithm, known as the Deutsch-Jozsa (DJ) algorithm [14, 15, 16]. The problem solved by DJ algorithm is the following.

**Problem 1.** Given a function \( f : \{0,1\}^n \rightarrow \{0,1\} \) that is promised to be either constant (\( f \) takes the same value everywhere) or balanced (\( f \) takes the value 0 on exactly half the inputs), determine whether \( f \) is constant or balanced.

The best classical algorithm requires \( 2^{n-1} + 1 \) evaluations in the worst case. If error is tolerated, for any integer \( m \geq 2 \), to achieve an error of at most \( 2^{-m} \), any probabilistic algorithm requires a number of evaluations that is at least of order \( m^{1/2} [17] \). If the function is accessible on a quantum computer as a quantum oracle, then the DJ algorithm is exact and requires just one evaluation to solve the problem.

Our focus here is on the CV analogue of the DJ algorithm, and we are inspired by the Braunstein and Pati formulation [18] of the CV DJ algorithm; however, our work differs from theirs in that ours relies only on logical states that are elements in the Hilbert Space and thus provides a strict CV version of the DJ problem. We introduce a particular model for the computation of the DJ problem in a CV setting. Within the constraints of this model, our analysis shows that the CV DJ algorithm is necessarily probabilistic and its performance must therefore be compared to the classical case where bounded error is tolerated and not to the classical deterministic case.

We choose the DJ algorithm for the following reasons. Two types of quantum algorithms dominate the field, those that implement a version of the hidden subgroup problem and those that use a version of Grover’s search algorithm [1, 19]. An early example of the former is the Deutsch-Jozsa (DJ) algorithm [14], which is amongst the oracle class of problems [20] that have been important in demonstrations of quantum speed-ups. Finally the CV DJ algorithm has a head start in the work of Braunstein and Pati so our analysis can build on their concepts [18].

Our paper is presented as follows. In Sec. II we review the DJ algorithm. Although this algorithm is well known, our review serves as a foundation for careful construction of the CV version of this algorithm. Furthermore we compare the DJ algorithm’s performance against both deterministic and probabilistic strategies, especially because the CV case can only be properly compared against probabilistic strategies because the CV DJ algorithm can never be deterministic. Our description of the DV DJ algorithm comprises three steps so that these steps can be discussed separately during the construction of the CV analogue.
Our approach emphasizes a recasting of the DV DJ algorithm in that we do not need the target qubit. This approach leads to an easier adaptation to the CV case. In Sec. [II] we review the formalism of rigged Hilbert spaces (RHS) \([21]\) since our CV algorithm, as well as any other CV quantum algorithms, must work in a RHS. This will have implications when we discuss the limitations of error inherent in our CV DJ algorithm in Sec. [III] and in Sec. [IV].

In Sec. [III] we adapt the DJ problem to the CV case and develop the CV DJ algorithm through the same three fundamental steps of the algorithm. We pay particular attention to the challenge of encoding a finite \(N\)-bit string into functions over the real numbers. Overcoming this challenge enables us to recognize that perfect encoding results in the inability to determine if the encoding is of a constant string or balanced in a single execution of the algorithm. We show that this probabilistic nature of the algorithm is the result of an uncertainty relation between the continuous representation and its Fourier-transform dual representation.

In Sec. [IV] we determine an upper bound on the query complexity of the CV DJ algorithm. We note that because the CV DJ algorithm is shown to be probabilistic, its performance can only logically be compared to the classical probabilistic algorithm and not to the classical deterministic algorithm. We conclude that the formalism presented herein is applicable to a wide range of oracle identification problems in a CV setting.

II. BACKGROUND

We cast the DJ problem into the class of ‘oracle identification problems’ in Subsec. [IIA]. We then review deterministic algorithms in Subsec. [IIB] and probabilistic algorithms in Subsec. [IIC]. In Subsec. [IID] we analyze an alternative representation of the quantum DJ algorithm that uses \(n\) qubits instead of the traditional \(n + 1\) qubits. In Subsec. [IIIE] we present a primer on the rigged Hilbert space and close with a discussion of the concepts required to transition from discrete variables to continuous variables.

A. The Oracle Identification Problem

The DJ problem is an identification problem in which we are given a function from some candidate set \(S = \{f_1, f_2, \ldots, f_M\}\) of functions. The candidate set \(S = S_0 \cup S_1\) is the disjoint union of two collections of functions, and our task is to determine which of the two collections the function \(f\) is drawn from.

**Problem 2.** Consider the set of all functions from \(n\) bits to one bit, \(F = \{f \mid f: \{0, 1\}^n \rightarrow \{0, 1\}\}\). Let \(S_0\) and \(S_1\) be disjoint subsets of \(F\). Given some oracle

\[ f: \{0, 1\}^n \rightarrow \{0, 1\} \quad (2.1) \]

with the promise that either \(f \in S_0\) or \(f \in S_1\), determine the index \(b\) such that \(f \in S_b\).

For \(N = 2^n\), we impose lexicographic order on the \(N\)-bit strings of \(\{0, 1\}^n\). We can then specify any function \(f_2\) by writing all its \(N\) function values in a list \(z \in \{0, 1\}^N\) of length \(N\). The \(i^{\text{th}}\) bit \(z_i\) in the list is 1 if \(f\) takes the value 1 on the \(i^{\text{th}}\) bit-string of \(\{0, 1\}^n\). There are \(2^N\) functions from \(n\) bits to one bit, and thus our candidate set has cardinality upper bounded by \(M \leq 2^N\). In the following, we often write \(f_z\) to denote the function that corresponds to the \(N\)-bit string \(z\).

We are interested in finding an efficient strategy to identify the property of whether \(f\) belongs to set \(S_0\) or to set \(S_1\) without necessarily determining \(f\) itself. In the DJ case, the property we are interested in is whether \(f\) is balanced or constant \([14, 15, 16]\). The cost of the algorithm is the number of queries made to the oracle.

With the promise of balanced or constant functions, there are far fewer than \(2^N\) functions. The number of balanced and the number of constant functions is readily ascertained from the binomial theorem applied to power sets. The strings \(z\) of length \(N\) that correspond to the constant functions are the string consisting only of 1s and the string consisting only of 0s. There are thus just two constant functions. The strings \(z\) of length \(N\) that correspond to balanced functions are the strings in which exactly half of the bits are 0 and half are 1. There are thus precisely \(\binom{N}{N/2}\) balanced functions.

B. The Classical Deterministic Approach

On a classical Turing machine, Problem II can be solved deterministically. A deterministic algorithm corresponds to submitting queries in the form of \(n\)-bit inputs and obtain the one-bit output for each query. There are \(N\) unique input strings, but the promise of balanced versus constant functions implies that only \(N/2 + 1\) are required to determine whether the given function is balanced or constant, with certainty.

The reason that fewer than \(N/2 + 1\) queries is insufficient is that only \(N/2\) queries may reveal all output bits being the same, suggesting a constant function, whereas the remaining \(N/2\) outputs could all be the opposite of the first \(N/2\) queries.

C. The Classical Probabilistic Approach

In Subsec. [IIIB] we saw that fewer than \(N/2 + 1\) queries is insufficient for a deterministic algorithm, but that case seems highly unlikely. More formally, fewer than \(N/2 + 1\) queries will identify most of the balanced functions as non-constant in much fewer than \(N/2 + 1\) queries. Here we ask the question about how many queries are required if we are prepared to tolerate a small number of errors.
In fact a probabilistic algorithm achieves an exponentially small error of \(2^{-m}\) with a number of queries that is only linear in \(m\) \([17]\). To understand how a probabilistic algorithm can help, consider that, although a single query with a random input provides no information, two queries with two random inputs can be highly informative. If the output from the second query differs from the first output, then the function is proved not to be constant and therefore must be balanced. If, on the other hand, the second output is the same as the first, then the outcome is not certain, but the more times the outputs are the same, the more confident one can be about the function being constant.

We calculate the probability of successfully determining whether the given function \(f_x\) is balanced or constant. A lower bound on the success probability \(\Pr_\forall\) for \(m\) queries can be achieved by examining a sampling-without-replacement strategy, which is expressed as

\[
\Pr_\forall = 1 - \prod_{j=1}^m \frac{N/2 - (j - 1)}{N - (j - 1)} \geq 1 - \left(\frac{1}{2}\right)^m. \tag{2.2}
\]

Here the equality is calculated assuming sampling without replacement and shows dependency on \(N\), whereas the inequality in Eq. (2.2) is based on sampling with replacement and is independent of \(N\). The failure probability \(1 - \Pr_\forall\) declines exponentially in \(m\), the number of queries.

In Subsec. II D we study the quantum DJ algorithm next where we show that the problem can be solved with a single query independent of \(N\). Although this exponential speed-up is impressive when compared to the classical deterministic approach, it is much less so when compared to the classical probabilistic approach.

D. The Quantum DJ Algorithm

The quantum DJ algorithm has been shown to solve Problem 1 in a single query \([14]\). The quantum DJ algorithm is usually studied via its corresponding quantum circuit. We present a standard circuit version \([16]\) in Fig. 1.

The state represented by the lower line in Fig. 1 is referred to as the target qubit. In order for easier adaption of this circuit to the CV setting, we choose an alternative, and equivalent, circuit formulation — one without the target state. We take this approach to avoid some of the difficulties the target state introduces in \([18]\). The unitary operator associated with the oracle function changes slightly in this alternative circuit. We discuss these differences before proceeding with analysis of the circuit.

This simpler algorithm without the target qubit is given in Fig. 2. Oracle application is the critical part of the algorithm. The oracle construct originally proposed by DJ is expressed, for \(x \in \{0, 1\}^n\) and \(y \in \{0, 1\}\), as

\[
U_f : |x\rangle|y\rangle \mapsto |x\rangle|y \oplus f(x)\rangle. \tag{2.3}
\]

This construction yields a matrix representation for the \(U_f\) as a permutation matrix, hence always unitary \([1]\).

With respect to the ordered basis

\[
B = \{|0 \cdots 0\rangle, |0 \cdots 0\rangle|1\rangle, \ldots, |1 \cdots 1\rangle|0\rangle, |1 \cdots 1\rangle|1\rangle\},
\]

the unitary matrix \(U_f\) can be expressed in the following insightful form

\[
U_f = \begin{pmatrix}
X^{f(0\cdots0)} & 0 & \cdots & 0 \\
0 & X^{f(0\cdots1)} & \cdots & 0 \\
\vdots & \vdots & \ddots & \vdots \\
0 & 0 & \cdots & X^{f(1\cdots1)}
\end{pmatrix}, \tag{2.4}
\]

with \(X\) the \(2 \times 2\) NOT operator in this case. Here \(U_f\) is a \(2^{(n+1)} \times 2^{(n+1)}\) matrix, which results from there being \(2^n\) strings (the arguments of \(f\)) and an additional target qubit.

The operator \(U_f\) can also be expressed in the alternative ordered basis

\[
B' = \{|0 \cdots 0\rangle\rangle, |0 \cdots 1\rangle\rangle, \ldots, |1 \cdots 0\rangle\rangle, |1 \cdots 1\rangle\rangle\},
\]

FIG. 1: Quantum circuit implementing the Discrete Variable DJ Algorithm. The upper line represents the \(n\)-qubit “control” state, and the lower line represents the 1-qubit “target” state.

FIG. 2: Alternative quantum circuit implementing the discrete variable DJ Algorithm. Note the absence of the target qubit and the use of the operator \(\hat{U}_f\) defined in Eq. (2.4).
as

\[ U_f = \left( \begin{array}{cc} -1^{f(0\cdots0)} & 0 \\ 0 & -1^{f(0\cdots1)} \\ \vdots & \vdots \\ 0 & 0 \end{array} \right), \]

with \( I \) the \( 2^n \times 2^n \) identity operator. Furthermore the operator \( \hat{U}_f \) is expressed as the \( 2^n \times 2^n \) matrix

\[
\hat{U}_f = \left( \begin{array}{cccc} -1^{f(0\cdots0)} & 0 & \cdots & 0 \\ 0 & -1^{f(0\cdots1)} & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & -1^{f(1\cdots1)} \end{array} \right),
\]

and thus provides a reduced representation for \( U_f \). It is apparent that the operator \( \hat{U}_f \) acts on a \( 2^n \times 2^n \) subspace of \( U_f \) since

\[ U_f = \left( \hat{U}_f \otimes |\rangle\langle \cdot| \right) \oplus \left( I \otimes_n \otimes |\rangle\langle +| \right). \]

We make the assumption that if we have the oracle \( U_f \), we we also have the oracle \( \hat{U}_f \). We thus conclude that the construction employing both control and target qubits is not strictly necessary. That is, one could construct this algorithm employing the \( n \)-qubit control state only. Apparently the choice of representation simply depends on the nature of the actual physical implementation.

We now present a step-by-step analysis of the alternative circuit presented in Fig. 2. We shall analyze the CV circuit in the same steps for cross reference and comparison.

1. State preparation

We use the hat notation \( |\hat{\Psi} \rangle \) in order to emphasize that this analysis is of the algorithm presented in Fig. 4 which employs \( n \)-qubit states and not of that presented in Fig. 1 which employs \( (n+1) \)-qubit states. The \( n \)-qubit input state of the circuit in Fig. 2 is a string of qubits prepared in \(|0\cdots0\rangle\). The next step in state preparation is to place the state \( |\hat{\Psi}_0 \rangle \) into an equal superposition of all computational basis states

\[ H^\otimes n |\hat{\Psi}_0 \rangle \mapsto |\hat{\Psi}_1 \rangle = 2^{-n/2} \sum_{x \in \{0,1\}^n} |x\rangle \]

for \( H \) the single qubit Hadamard operator.

2. Oracle application

Given the definition of the reduced operator \( \hat{U}_f \) defined in Eq. 2.5 its effect on the equal superposition of basis states expressed in the state \(|\hat{\Psi}_1 \rangle\) is to effectively encode the \( N \)-bit string \( z \) unitarily into the state \(|\hat{\Psi}_2 \rangle\).

We express this as

\[ \hat{U}_f |\hat{\Psi}_1 \rangle \mapsto |\hat{\Psi}_2 \rangle = 2^{-n/2} \begin{pmatrix} (-1)^{f(0\cdots0)} \\ (-1)^{f(0\cdots1)} \\ \vdots \\ (-1)^{f(1\cdots1)} \end{pmatrix}, \]

which is a convenient representation. We shall show that this representation naturally extends to the CV setting.

3. Measurement

Measurement proceeds by first undoing the superposition created during the state preparation step. This is achieved through the application of the operator \( \hat{U}_3 = H^\otimes n \), which modifies the state after oracle application

\[ \hat{U}_3 |\hat{\Psi}_2 \rangle \mapsto |\hat{\Psi}_3 \rangle. \]

The resultant state is

\[ |\hat{\Psi}_3 \rangle = 2^{-n/2} H^\otimes n \sum_{x \in \{0,1\}^n} (-1)^{f(x)} |x\rangle. \]

We rewrite Eq. (2.8) with the operator \( H^\otimes n \) expressed in terms of a recursive definition as follows

\[ |\hat{\Psi}_3 \rangle = 2^{-(n+1)/2} \begin{pmatrix} H^\otimes (n-1) \\ H^\otimes (n-1) \end{pmatrix} \begin{pmatrix} (-1)^{f(0\cdots0)} \\ (-1)^{f(0\cdots1)} \\ \vdots \\ (-1)^{f(1\cdots1)} \end{pmatrix}. \]

Given

\[ H = \frac{1}{\sqrt{2}} \begin{pmatrix} 1 & 1 \\ 1 & -1 \end{pmatrix}, \]

the combination of Eq. (2.9) and Eq. (2.10) allows us to see that all of the rows (and columns) of the operator \( H^\otimes n \) have an equal number of positive and negative ones except for the first row, which consists entirely of plus ones. It is this feature that permits the constant and balanced functions to be distinguished in a single measurement.

For the two constant cases, Eq. (2.9) may be expressed as

\[ |\hat{\Psi}_{3C} \rangle = \pm \frac{1}{\sqrt{2^n}} \begin{pmatrix} 1 & 1 & \cdots & 1 \\ 1 & -1 & \cdots & -1 \\ \vdots & \vdots & \ddots & \vdots \\ 1 & -1 & \cdots & 1 \end{pmatrix} \begin{pmatrix} 1 \\ 1 \\ \vdots \\ 1 \end{pmatrix} = \pm \begin{pmatrix} 1 \\ 0 \\ \vdots \\ 0 \end{pmatrix} \]

as only the first row does not result in amplitude cancellation of the \( 2^n \) constant amplitude components of the state \(|\hat{\Psi}_2 \rangle\). Each of the balanced functions result in the amplitudes of the state \(|\hat{\Psi}_2 \rangle\) having an equal number of
positive and negative ones. This feature coupled with action of the operator $H^\otimes n$ results in the first component of the state $|\Psi_3\rangle$ having zero amplitude for all the balanced functions. We express this result as

$$|\Psi_{3B}\rangle = \pm \begin{pmatrix} 0 \\ x \\ \vdots \\ x \end{pmatrix}, \quad (2.12)$$

where we use the symbol $x$ to represent that the non-zero value(s) will land on the other $N-1$ components depending on which of the $(N/2)$ balanced functions the oracle is set to. It is interesting to note that the number of rows in the state $|\Psi_{3B}\rangle$ potentially having a non-zero value is $N-1$ whereas the number of balanced functions is exponential in $N$. This means that many of the balanced states can be expressed as real-valued mixtures of the computational basis states with the condition that the amplitude of the first component is always zero.

For the final measurement step, we employ the projection operator $|m\rangle\langle m|$ defined for $m \in \{0,1\}^N$ as follows

$$M_m = |m\rangle\langle m|. \quad (2.13)$$

We are only concerned with the first component as discussed above, so for the constant cases we have

$$\Pr[|m = 0\cdots 0\rangle] = \langle \Psi_{3C}|M_{0\cdots 0}\rangle |\Psi_{3C}\rangle = 1, \quad (2.14)$$

and for all balanced cases we have

$$\Pr[|m = 0\cdots 0\rangle] = \langle \Psi_{3B}|M_{0\cdots 0}\rangle |\Psi_{3B}\rangle = 0 \quad (2.15)$$

as required.

We have completed the study of the quantum DJ algorithm in a form that allows us to adapt readily to the CV setting. Our strategy will be to construct a CV algorithm analogous to that shown in Fig. 2 and whose operator representation is given by

$$|\Psi_3\rangle = H^\otimes n \tilde{U}_f H^\otimes n |0\cdots 0\rangle. \quad (2.16)$$

This approach is simpler, and we can worry about whether or not an implementation will require target states when a particular implementation is considered. Before delving into the CV algorithm, we present some background CV information.

### E. CV Background

The transition from DV to CV quantum information requires an extension of Hilbert spaces to rigged Hilbert spaces [21], which allows the use of position states $|x\rangle$ with $x \in \mathbb{R}$ but restricts dual states to so-called ‘test functions’. An inner product between position states and test functions is meaningful, but the inner product between two position states leads to the Dirac relation $\langle x'|x\rangle = \delta(x-x')$, which must be treated carefully. For $n$ the size of Problem I the target-less quantum DJ algorithm requires $n$ qubits, which requires a Hilbert space of size $N = 2^n$ [22]. The Hilbert space for CV problems seems quite generous in this respect as it is infinite-dimensional.

In fact the CV Hilbert space is congruent to the space of square-integrable complex functions over the real field $L^2(\mathbb{R})$ [22]. A function $f : [a, b] \to \mathbb{C}$, for $[a, b] \subset \mathbb{R}$, is in $L^2(\mathbb{R})$ if

$$\int_b^a |f(x)|^2 dx < \infty. \quad (2.17)$$

The inner product of two functions $f, f'$ is

$$\langle f'|f \rangle = \int_a^b f^n(x)f(x)dx, \quad (2.18)$$

with positive definite norm and distance metric defined by

$$\|f\| = \sqrt{\langle f'|f \rangle}, \quad d(f, f') = \|f - f'\|, \quad (2.19)$$

respectively.

Typically in CV quantum information discourse, the position states $|x\rangle$ are introduced as a basis set of the Hilbert space with each $|x\rangle$ an eigenstate of a position operator $\hat{x}$, with $x \in \mathbb{R}$. Unfortunately the state $|x\rangle$ does not exist in the Hilbert space; this problem is evident in the standard inner product

$$\langle x'|x \rangle = \delta(x-x'). \quad (2.20)$$

As $\delta$ is not a proper function, position states are not proper states. Fortunately the position states are correct as a representation; for example $f(x) = \langle x|f \rangle$ is the position representation of test function $f$ within the context of the rigged Hilbert space. Also Eq. (2.20) is meaningful in the context of distribution theory.

A rigged Hilbert space is a pair $(\mathcal{H}, \Phi)$ such that $\mathcal{H}$ is a Hilbert space and $\Phi$ is a vector space that is included by a continuous mapping into $\mathcal{H}$: $\Phi \subseteq \mathcal{H}$. Elements of $\Phi$ are referred to as ‘test functions’, and the dual to $\Phi$ is $\Phi^* \supseteq \mathcal{H}^*$, for $\mathcal{H}^*$ dual to $\mathcal{H}$ and $\Phi^*$ comprising generalized functions, or ‘distributions’. The inner product $\langle f'|f \rangle$ is in $[0, 1]$ for any $f' \in \Phi^*$ and for any $f \in \Phi$ [21].

Note that the adaptation of the DV DJ algorithm to the CV regime needs to be done in the context of a computational problem. Here the relevant problem is still Problem I and the notion of the oracle remains unchanged. Thus, in the CV case, our task is still to determine whether the function $f_x$ belongs to the set of constant functions or to the set of balanced functions.

### III. CV REPRESENTATION OF THE DJ PROBLEM

We begin by giving a strategic overview in order to convey the key concepts of our approach to developing a
CV computation model. We follow this with a subsection giving some preliminary definitions allowing us to set the stage for detailed analysis. We then proceed with a step-by-step analysis of our CV DJ algorithm.

A. Strategy Overview

Although we are now working with CV, instead of DV, quantum information, the computational problem to be solved remains Problem \[1\]. In other words, we want to learn whether the function \(f_z\) is constant or balanced with as few oracle queries as possible. Another way to think of this is that we wish to determine the index \(b \in \{0, 1\}\) such that \(f_z \in \mathcal{S}_b\). We now give a conceptual overview of our model for CV quantum computation of the DJ problem, which we follow later with a rigorous treatment.

In our model of CV quantum computation, we will use the continuous position and momentum variables of a particle. For \(x, p \in \mathbb{R}\), we use the particle’s position wave function, \(\phi(x)\), to describe where the particle is concentrated and the particle’s momentum wave function, \(\tilde{\phi}(p)\), to describe its momentum distribution. The position and momentum wave functions are Fourier transform pairs, and the relationship between the particle’s position and its momentum is governed by Heisenberg’s uncertainty principle.

There are many position and momentum wave function pairs on which we could base our computational model. We select our particular pair as follows. First, we wish to encode the unknown \(N\)-bit string, \(z\), in the momentum domain. We do so because encoding in the momentum domain is the continuous analogue of the discrete case, where encoding is performed on an equal superposition of computational basis states. Second in order to fix one of the degrees of freedom of the problem, we want each of the bits comprising the string \(z\) to be unambiguously represented in the momentum space. By unambiguous we mean that each of the bits are represented by equal-sized, non-overlapping, contiguous regions in the momentum space.

Since we want each of the \(N\)-bits comprising the string to be represented unambiguously, we naturally think of each bit as being manifested by a finite-width square pulse whose position in momentum space represents the bit position in the string \(z\) and whose magnitude represents the bit value. Continuing along this line of reasoning to the representation of the entire string \(z\), we can imagine we have a region of momentum extending from \(-P\) to \(+P\). All the contiguous momentum pulses within this region thus have “width” \(\delta_p = 2P/N\), and for \(j \in \{0, N - 1\}\), the \(j\)th momentum pulse is centred at position \(-P + (j + 1/2)\delta_p\) and takes on value \((-1)^j\). We illustrate this concept in Fig. 3 for a particular \(N = 4\) case.

The picture that thus emerges is that each of the \(2^N\) possible strings may be represented by a uniquely shaped “square wave” having extent \(\pm P\) comprising \(N\) pulses each of width \(2P/N\) and having magnitude \(\pm 1\). With the encoding concept clear, we conceptually illustrate the four key stages of the algorithm in Fig. 4. We begin with a position wave function centered at \(x = x_0\) and illustrated in Fig. 4(a). Note that this position wave function is a sinc function since sinc/pulse functions are Fourier transform pairs. In Fig. 4(b), we present the momentum wave function, a pulse function, which acts as the “substrate” into which the \(N\)-bit strings are encoded. In Fig. 4(c), the pulse function is encoded with the particular \(N\)-bit string \(z = 0 \cdots 01 \cdots 1\). Finally, the inverse Fourier transform of this “square wave” is presented in

---

FIG. 3: Illustration of the concept for encoding an \(N\)-bit string in a region of momentum extending from \(-P\) to \(+P\) using the \(N = 4\), \(z = 0101\) example. Note that each of the bits \(z_j\) are uniquely represented.

FIG. 4: An illustrative overview of the four stages of our conceptual CV DJ algorithm: (a) The probability distribution of the input state wave function positioned at \(x = x_0\) is that of a sinc function. (b) The Fourier transform of the position state wave function is a “pulse” function in the momentum domain, which acts as the encoding “substrate”. (c) The \(N\)-bit string \(z = 0 \cdots 01 \cdots 1\) modulates this momentum “substrate”. (d) The inverse Fourier transform of the encoded “square wave” produces a “generalized” sinc function whose infinite position extent necessitates an optimal measurement “window” parameterized by \(\pm \delta\).
Fig. 1(d). Since the inverse Fourier transforms of finite pulses in the momentum domain have infinite extent in the position domain, we need to limit the extent of our measurement to \( \pm \delta \).

In summary, we see that our algorithm will need the parameters \( N, P \), and \( \delta \). We note that as \( N \) gets large, the individual pulse width associated with a single bit becomes small, appearing to pose a limit on the maximum value of \( N \). We will return to this issue once we have determined the relationship between \( P \) and \( \delta \).

There are many potential models for quantum computation in a CV setting. We have chosen to study one where we unambiguously encode an \( N \)-bit string into the continuous momentum variable of a particle. Within the constraints of this model, we will show that the CV DJ problem is necessarily probabilistic and prove an upper bound on the query complexity of the CV DJ problem.

We speculate that we can’t do better than this. For example if the momentum/position pair are described by Gaussian/Gaussian functions, as would be the case for the physically meaningful states of quantum optics, imperfect encoding of the \( N \)-bit string in the momentum domain will result in increased position error. Whether or not this will in turn impact the “big Oh” representation of the query complexity requires further research as does a general proof of a lower bound. The challenge will be to show that another strategy can do better than the model described herein.

**B. Algorithm Preliminaries**

We now proceed to formalize some of the concepts presented in the previous subsection. Here we describe a ‘natural’ way of encoding a finite-dimension, \( N \)-bit string in a continuous domain. We define the following function, along with its Fourier dual, to help us achieve this end.

For \( P > 0 \), the ‘top hat’ function

\[
\nabla(p; P, P_0) = \langle p \rangle \cap \langle P, P_0 \rangle = \begin{cases} 1, & \text{if } p \in [P_0 - P, P_0 + P] \\ 0, & \text{if } p \notin [P_0 - P, P_0 + P] \end{cases}
\]

will be especially useful in bridging the gap between DV and CV quantum information because

\[
\lim_{P \to 0} \nabla(p; P, P_0) = \delta(p - P_0),
\]

so the state \( |\nabla\rangle \) is, in some sense, a momentum eigenstate \( |p = P_0\rangle \) in the limit \( P \to 0 \). The inverse Fourier transform of the function \( e^{ix_0} \nabla(p; P, P_0) \) is

\[
\phi(x) = \langle x | \phi \rangle = \pi^{-1/2} \sin(P(x - x_0); P_0) / \sqrt{P\pi(x - x_0)},
\]

where \( x_0 \) defines the position of the sinc function. The limit of \( \phi(x) \) as \( P \) goes to \( 0 \) yields \( \delta(x - x_0) \). The position eigenstate \( |x = x_0\rangle \) is likewise formed in the limit \( P \to \infty \).

Now imagine we want to sum a contiguous string of “pulses” described by the top hat function (3.1) with all pulses having width \( \delta_p \) and the \( j^{th} \) pulse having complex amplitude \( \psi_j \). This results in the composite function

\[
\psi(p) = \sum_j \nabla_j \cap (p, -P + j\delta_p, -P + (j + 1)\delta_p).
\]

This function can also be used as a basis of CV kets in Dirac notation as

\[
|\psi\rangle = \int_{-\infty}^{\infty} dp |\psi(p)\rangle |p\rangle,
\]

thus allowing us to encode quantum information in the CV domain. Note that \( \psi(p) \) is the complex amplitude for real-valued \( p \). This affords a consistent way of encoding a discrete wave function over a continuous domain.

Before proceeding with a formal analysis of the algorithm, we give an overview of our proof strategy. The oracle is either set to one of two constant strings or to one of \( \binom{N}{N/2} \) balanced strings. A string and its complement have indistinguishable probability distributions, so there are a total of one constant probability distribution plus \( \frac{1}{2} \binom{N}{N/2} \) balanced probability distributions representing the possible oracle settings. In order to simplify the analysis, we wish to replace this exponential number of balanced probability distributions with a single “worst-case” balanced probability distribution. Thus we seek a particular balanced string (and its complement) whose probability distribution is most likely to “fool” us into concluding it is a constant string.

Intuitively, the balanced strings that have the fewest number of changes between adjacent bits in the interval \([-P, P] \) will be the most “constant like” of the balanced strings. There are no balanced strings with zero changes - this is the key feature that separates the constant strings from the balanced strings. There is, however, a single pair of balanced strings having only one change. These strings exhibit the feature that the first \( N/2 \) bits are constant and the second \( N/2 \) bits are the complement of the first. We call these strings the anti-symmetric balanced (ASB) strings. One of these two strings is illustrated in Fig. 1(c). Note that all other balanced strings have more than one change.

Our proof strategy begins by making the assumption that the ASB case is the “worst case” of all balanced cases. We use this assumption to determine the optimum value of \( \delta \), which is the extent of our measurement in the position domain and is illustrated conceptually in Fig. 1(d). Given this optimum value of \( \delta \), we then prove by induction that the worse balanced case is indeed the ASB case.
C. The CV Quantum DJ Algorithm

Our strategy is to create a CV analogue of the alternative formulation of the discrete DJ algorithm presented in Fig. 2. The CV extension of this is presented in Fig. 3. Our construction of a CV DJ algorithm employs some of Braunstein and Pati’s techniques [18] and avoids the pitfalls. In particular, we employ position states as a logical representation (states in $\Phi^*$) analogous to the discrete computational basis states. Encoding is not, however, into the position states but rather into test functions $f_z \in \Phi$ with $z \in \{0, 1\}^N$. Furthermore we employ a Fourier transform to operate as a CV version of the DV Hadamard transform (extending the Hadamard transformation to the CV case is not unique [2, 25]).

For $x$ the canonical position and $p$ the canonical momentum, the Fourier transform maps a function $\phi(x)$ to its dual $\tilde{\phi}(p)$ according to [20]

$$F : \phi(x) \mapsto \tilde{\phi}(p),$$

such that

$$\tilde{\phi}(p) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} dx \, e^{ipx} \phi(x),$$

and

$$\phi(x) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} dp \, e^{-ipx} \tilde{\phi}(p).$$

Note that we make use of the momentum variable $p$ as the Fourier dual of the position variable $x$. The function $\phi$ can be a test function in $\Phi$ and $\phi(x)$ is the inner product of $\phi$ with the position in $\Phi^*$: $\phi(x) = \langle x|\phi \rangle$. The momentum state $|p\rangle$ is the Fourier transform of $|x\rangle$ and $\tilde{\phi}(p) = \langle p|\phi \rangle$.

With these concepts in order, we now proceed through the CV DJ algorithm analogous to the three steps in the DV DJ algorithm. The function notation $\phi(x)$ and $\phi(p)$ is more convenient here rather than the Dirac notation in the previous section.

1. State preparation

We have argued previously that we need the Fourier transform of the input state to be the top hat function defined in Eq. (3.1). We add several conditions that do not take away from the generality of the solution. First, we want the top hat to have zero phase, which gives $x_0 = 0$ and to be centred at $p_0 = 0$. Second, we want the pulse to have extent $\pm P$. This gives us the simplest form of the sinc function for the initial state

$$\phi_0(x) = \frac{\sin(Px)}{\sqrt{\piPx}}$$

We note that the limit of $\phi_0(x)$ as $P \to \infty$ gives a $\delta(x)$. Thus we can think of the quantity $P$ as playing the role of the standard deviation in a Gaussian distribution.

The final step in state preparation is to perform the Fourier transform, which yields the top hat function with extent $\pm P$

$$\tilde{\phi}(p) = \frac{1}{\sqrt{2P}} \left\{ \begin{array}{ll} 1, & \text{if } p \in [-P,P] \\ 0, & \text{if } p \notin [-P,P]. \end{array} \right.$$  

This function forms the raw substrate, which will be ‘modulated’ by the individual $N$-bit strings $z$.

2.Oracle application

We perform encoding by partitioning the real numbers representing momentum into non-overlapping, contiguous and equal-sized bins. In this digital-to-analogue strategy, the width of each $p$-bin is $2P/N$, and

$$\land_i^{(N)}(p) = \left\{ \begin{array}{ll} 1, & p \in \left[ -\left(1 - \frac{2N_{i-1}}{N} \right), -\left(1 - \frac{2N_i}{N} \right) \right] \\ 0, & \text{otherwise.} \end{array} \right.$$  

The oracle encodes the index $i$ into the function $f_z$ as follows:

$$f_z^{(N)}(p) = \sum_{i=0}^{N-1} (-1)^{z_i} \land_i^{(N)}(p),$$

where the factor $(-1)^{z_i}$ serves to modulate the phase of the top hat function according to the bit value.

Example 1. Consider the case $n = 2$; hence $N = 2^2 = 4$. As one case, the function corresponding to the four-bit string $0011$ is

$$f_{0011}^{(N)}(p) = \land_0^{(N)}(p) + \land_1^{(N)}(p) - \land_2^{(N)}(p) - \land_3^{(N)}(p).$$

The only two four-bit strings yielding constant functions would be 0000, for which the function is identically unity over the whole domain $[-P,P]$, and 1111, for which the function is identically $-1$ over $[-P,P]$. Four cases are presented in Fig. 6. We refer to the function $f_{0011}(p)$ as the “lowest-order” antisymmetric balanced wave as it has just one zero crossing in $[-P,P]$.
In the limit that \( N \to \infty \) with \( P \) fixed, \( \Gamma_i(p) \to \delta(p-p_i) \) for \( p_i \) the midpoint of the \( i \)th bin. The limit \( N \to \infty \) thus gives a prescription for approaching a continuous variable representation where the \( z \) index seems to approach a continuum; however this limit yields a countable, rather than uncountable, set \( \{z\} \), and the finite domain \([-P,P]\) has important ramifications on the nature of the functions corresponding to Fourier transforms of \( \Gamma_i(p) \). We express the state after encoding as

\[
\hat{\phi}_z^{(N)}(p) = f_z^{(N)}(p)\hat{\phi}(p),
\]

(3.10)

where we observe the “modulating” effect of the encoded string \( f_z \) on the momentum “substrate” \( \hat{\phi}(p) \).

In the context of the digital-to-analogue strategy, the constant functions are analogous to direct current (DC) signals and the balanced functions to alternating current (AC) signals. The number of zero-crossings corresponds to frequency information, and the question of whether the output is balanced or constant is essentially a problem of querying whether there is a non-zero frequency component of the output signal. As noted previously, the ASB function has the lowest frequency component. We now proceed to analyze the measurement stage.

3. Measurement

We have the strings \( z \in \{0, 1\}^N \) encoded into the momentum state \( \hat{\phi}(p) \). The next step prior to the final measurement is to take the inverse Fourier transform of this pulse train. For \( z_j \) the \( j \)th bit of \( z \), this is expressed as

\[
\phi_z^{(N)}(x) = F^{-1}\left(\hat{\phi}_z^{(N)}(p)\right) = \frac{1}{2\sqrt{P\pi x}} \sum_{j=1}^{N} (-1)^{z_j} \times \left(e^{i\left(\frac{N-2j-1}{N}\right)Px} - e^{i\left(\frac{N-2j}{N}\right)Px}\right).
\]

(3.11)

The expression given in Eq. (3.11) can be simplified to yield

\[
\phi_z^{(N)}(x) = \frac{\sin(Px/N)}{\sqrt{P\pi x}} \sum_{j=1}^{N} (-1)^{z_j} e^{i\varphi_j(x)}
\]

(3.12)

where we have defined

\[
\varphi_j(x) = \left(\frac{N - (2j - 1)}{N}\right)Px.
\]

(3.13)

We see that the magnitude of an individual generalized sinc function, \( \phi_z^{(N)}(x) \), is determined by a vector sum of \( N \) phasors, which is modulated by a particular \( N \)-bit string \( z \).

Note that the phasors, \( e^{i\varphi_j(x)} \), are equiangular divisions of the angular interval

\[\pm(N - 1)Px/N, -(N - 1)Px/N,\]

and they exhibit the pairwise complex conjugate property \( \varphi_j(x) = -\varphi_{N+1-j}(x) \). In Fig. 7 we present the phasors for \( N = 8 \) with \( x = \pi/2 \) and \( x = \pi/4 \) to illustrate these features. Note that the phasors are added constructively or de-constructively depending on the phase of the angles, which results from the term \((-1)^{z_j}\). This effect defines the magnitude of the resulting sinc function.

We note that the only functions with \( \phi_z^{(N)}(0) \neq 0 \) are the two constant sinc functions. This is clear given that \( \sum_{j=1}^{N} (-1)^{z_j} = \pm N \) for the two constant cases, and \( \sum_{j=1}^{N} (-1)^{z_j} = 0 \) since for all balanced cases, the latter

FIG. 6: Encoded functions \( f_z^{(4)}(p) \) (a) \( z = 0000 \), (b) \( z = 0011 \), (c) \( z = 0101 \), and (d) \( z = 0110 \).

FIG. 7: The phasors \( e^{i\varphi_j(x)} \) defined in Eq. (3.12) for \( N = 8 \) (a) \( x = \pi/2 \) phasors range between \( \pm 7\pi/16 \) in steps of \( \pi/8 \), and (b) \( x = \pi/4 \) phasors range between \( \pm 7\pi/32 \) in steps of \( \pi/16 \).
sum always resolves to $N/2 - N/2 = 0$. This feature of the set of \( \binom{N}{N/2} + 2 \) sinc functions represented by \( \phi_z^{(N)}(x) \) defined by Eq. (3.12) implies the strategy for measurement that will distinguish between the constant and balanced cases.

In order to refine this strategy, we focus on two cases. The first of these cases is for the two constant functions for which Eq. (3.12) gives the probability distribution

\[
\mathcal{P}_c(x) = |\phi_c^{(N)}(x)|^2 = \frac{\sin^2(Px)}{P\pi x^2}, \tag{3.14}
\]

where we have

\[
C \in \left\{ \underbrace{0 \cdots 0}_{N/2}, \underbrace{1 \cdots 1}_{N/2} \right\}.
\]

The second case deals with the two balanced functions having the lowest ‘frequency’ content, which occurs when the first \( N/2 \) bits and the last \( N/2 \) bits have opposite values.

We think of these two balanced functions as having the lowest frequency content that will maximize our ability to distinguish between the constant and balanced cases. We will use these cases to bound the success probability of distinguishing between the constant and all balanced cases.

**Example 2.** Again consider the case \( P = 1 \), \( n = 2 \); hence \( N = 2^2 = 4 \). As one case, the function corresponding to the four-bit string 0011 is

\[
\phi_{0011}^{(4)}(x) = \frac{\sin(x/4)}{\sqrt{\pi x}} \left( e^{ix/4} + e^{ix/4} - e^{-ix/4} - e^{-ix/4} \right) = -i \frac{(\cos x - 1)}{\sqrt{\pi x}}, \tag{3.17}
\]

This function corresponds to the \( N = 4 \) ASB function. The probability distributions for the four distinct \( N = 4 \) cases are presented in Fig. 8. We clearly see that of the three balanced cases, the \( N = 4 \) ASB function has probability peaks closest to \( x_0 = 0 \).

![FIG. 8: The probability distributions \( |\phi_z^{(N)}(x)|^2 \) for (a) \( z = 0000 \), (b) \( z = 0011 \), (c) \( z = 0101 \), and (d) \( z = 0110 \). We clearly see that of the three balanced cases (b) through (d), the ASB function (b) has probability peaks closest to \( x_0 = 0 \).](image_url)

Our measurement strategy is to measure the probability distribution in a small band around the position \( x_0 = 0 \) parameterized by \( \pm \delta \). The CV analog of the projection operator given in Eq. (2.13) is defined as

\[
E_a = \int_{-\infty}^{\infty} D_a^b(x) |x| dx, \tag{3.18}
\]

where

\[
D_a^b(x) = \begin{cases} 
1, & \text{if } a \leq x \leq b \\
0, & \text{otherwise}.
\end{cases}
\tag{3.19}
\]

Due to the symmetry of the sinc functions about \( x_0 \), we set \( a = -\delta \) and \( b = +\delta \). We now need to determine the optimal value of \( \delta \) that will maximize our ability to distinguish between the constant and balanced cases. We will determine the optimum value of \( \delta \) by first assuming that the probability distribution \( \mathcal{P}_{\text{ASB}}(x) \) given by Eq. (3.15) dominates all other balanced probability distributions in the region \([-\delta, \delta]\). After using this assumption to determine a value for the optimal delta, we will state and prove a theorem justifying our assumption. As an illustration that our assumption is true for the \( N = 4 \) case, we plot the four distinct cases in Fig. [x].

The ability to effectively distinguish between two random events is proportional to the separation of the individual probabilities of occurrence. Thus we need to select \( \delta \) such that we get as much separation between the constant distribution and the ASB distribution as possible. Given this concept we can think that when we make a measurement we are distinguishing between two events, the probabilities for which we define as follows

\[
\Pr_{\text{Const}}(\delta) = \Pr \left[ |\phi_z^{(N)}|^2 = \mathcal{P}_c(x) \right] = E_{\pm \delta}(\mathcal{P}_c(x)), \tag{3.20}
\]

and

\[
\Pr_{\text{ASB}}(\delta) = \Pr \left[ |\phi_z^{(N)}|^2 = \mathcal{P}_{\text{ASB}}(x) \right] = E_{\pm \delta}(\mathcal{P}_{\text{ASB}}(x)). \tag{3.21}
\]
Theorem 1. Max subject to the balanced condition \( \phi \) note the pairwise conjugate property \( \delta \).

Now consider \( S \) conditions gives a global maximum at \( \delta \).

We shall return to this concept in our discussion in the conclusion. We have determined the optimum value for \( \delta \) based on our assumption that for \( -\delta \leq x \leq \delta \) the balanced probability distribution \( P_{\text{ASB}}(x) \) domi- nates all other balanced probability distributions. We now proceed to prove this assumption.

In order to proceed with the proof, we define a set \( \Phi \) of \( m \) pairwise conjugate angles with \( 2m = N \). Note that \( N \) is not restricted to being equal to \( 2^n \) for the purpose of this proof. Also for the purpose of this proof, we set \( P = 1 \) and incorporate \( x \) into the definition of \( \varphi_j = \frac{N-(2j-1)}{N} \) for \( -\pi/2 \leq x \leq \pi/2 \). We let \( \Phi = \{ \varphi_1, \varphi_2, \ldots, \varphi_m, \varphi_{m+1}, \ldots, \varphi_{2m} \} \) with \( j = 1, \ldots, m \) and note the pairwise conjugate property \( \varphi_j = -\varphi_{2m+1-j} \).

Now consider \( S = \sum_{j=0}^{2m} g(j) e^{i\varphi_j} \) where \( g : [2m] \mapsto \pm 1 \) subject to the balanced condition \( \sum_j g(j) = 0 \), then

\[
g(j) = \begin{cases} 1 & \text{if } 1 \leq j \leq m \\ -1 & \text{if } m+1 \leq j \leq 2m, \end{cases}
\]

\[ (3.24) \]

We can determine the optimum value of \( \delta \) by maximizing the expression \( |Pr_{\text{Conc}}(\delta) - Pr_{\text{ASB}}(\delta)| \). It suffices to find the value of \( \delta \) for which \( \frac{d}{d\delta} |Pr_{\text{Conc}}(\delta) - Pr_{\text{ASB}}(\delta)| = 0 \), which may be expressed as

\[
\begin{align*}
\frac{d}{d\delta} |Pr_{\text{Conc}}(\delta) - Pr_{\text{ASB}}(\delta)| &= \frac{d}{d\delta} \left| \int_{-\delta}^{\delta} \left( \frac{\sin^2(Px)}{P\pi x^2} - \frac{(\cos(Px) - 1)^2}{P\pi x^2} \right) dx \right| \\
&= \frac{\sin^2(P\delta)}{P\pi \delta^2} - \frac{(\cos(P\delta) - 1)^2}{P\pi \delta^2} = 0. \quad (3.22)
\end{align*}
\]

This occurs where \( \cos(P\delta) = \cos(P\delta)^2 \) for \( \delta \neq 0 \), which gives a global maximum at \( \delta = \frac{\pi}{2P} \). It is interesting to think of this result as an uncertainty relationship

\[
P\delta = \frac{\pi}{2} \quad (3.23)
\]

Theorem 1. Max \( |S| \) occurs under the specific balanced conditions

\[
g(j) = \begin{cases} 1 & \text{if } 1 \leq j \leq m \\ -1 & \text{if } m+1 \leq j \leq 2m, \end{cases}
\]

\[ (3.24) \]

FIG. 9: For \( P = 1 \), the optimal value of \( \delta = \frac{\pi}{P} \). This graph shows that only the Constant and the Antisymmetric Balanced Functions significantly contribute to probability between \( \pm \delta \).

\[
\begin{align*}
\text{FIG. 10: } & \text{Definition of the phasor angles for the } N = 4 \text{ base } \\
& \text{base. Note that the effect of varying } x \text{ over } [-\pi/2, \pi/2] \text{ simply focuses or expands the double angles } 2\varphi_1 \text{ and } 2\varphi_2 \text{ proportionally.}
\end{align*}
\]

and

\[
g(j) = \begin{cases} -1 & \text{if } 1 \leq j \leq m \\ 1 & \text{if } m+1 \leq j \leq 2m, \end{cases}
\]

\[ (3.25) \]

which we refer to as the asymmetric balanced functions (ASB).

\textbf{Proof.} Proof is done by induction on \( m \). We begin with the base case \( m = 1, N = 2 \). This case is trivial since the only balanced cases are the two ASB cases represented by the strings \{01, 10\}. We proceed with the base case for \( m = 2, N = 4 \). This case is a little more involved. We begin by labelling the angles and phasors as shown in Fig. 10.

There are \( \binom{4}{2} = 6 \) balanced cases, and we have to consider the strings \{0011, 0101, 0110, 1100, 1010, 1001\}. Since the latter three are complements of the first three, we have to consider only three vector sums.

With reference to Fig. 10 we have \( S_{\{0011\}} = e^{i\varphi_1} + e^{i\varphi_2} - e^{-i\varphi_1} - e^{-i\varphi_2} \). We simplify and express the resultant along with the three other cases as

1. \( S_1 = S_{\{0011\},\{1100\}} = \pm 2i(\sin(\varphi_1) + \sin(\varphi_2)) \)
2. \( S_2 = S_{\{0011\},\{1010\}} = \pm 2(\cos(\varphi_1) - \cos(\varphi_2)) \)
3. \( S_3 = S_{\{0110\},\{1001\}} = \pm 2i(\sin(\varphi_1) - \sin(\varphi_2)) \).

Clearly \( |S_1| > |S_3| \). We use the trigonometric identities,

\[
|S_1| = 2 \sin \left( \frac{\varphi_1 + \varphi_2}{2} \right) \cos \left( \frac{\varphi_1 - \varphi_2}{2} \right) \\
|S_2| = 2 \sin \left( \frac{\varphi_1 + \varphi_2}{2} \right) \sin \left( \frac{\varphi_1 - \varphi_2}{2} \right),
\]

to establish the relationship between \( |S_1| \) and \( |S_2| \). We note that max \((\frac{2\varphi_1 - \varphi_2}{2}) = (\frac{3}{2} - \frac{1}{N}) x \) and min \((\frac{2\varphi_1 - \varphi_2}{2}) = \]
By inspection, this gives the same result as $\cos \left(\frac{\varphi_1 - \varphi_2}{2}\right) > \sin \left(\frac{\varphi_1 - \varphi_2}{2}\right)$.

and thus $|S_1| > |S_2|$ for $0 \leq x \leq \pi/2$. This proves that the theorem is true for the $m = 2, N = 4$ base case. We are now ready to prove the inductive step.

We consider two cases. Case (i) assumes every pair is balanced. By this we mean that $g(j) = -g(2m + 1 - j)$. By inspection, this gives the same result as $|S_1|$ and $|S_3|$ for the $m = 2, N = 4$ case. Case(ii) assumes that Case(i) is not true and is proved by induction. Since Case(i) is not true, there must exist two non-balanced pairs for which $g(j) = g(2m + 1 - j) = +1$ and $g(k) = g(2m + 1 - k) = -1$. As an illustration in the $m = 4, N = 8$ case, the balanced string $\{01000111\}$ has this property. The inductive step is

$$\sum_{j=1}^{2m} g(j)e^{i\phi_j} \leq |S| \{\{l, 2m + 1 - l, k, 2m + 1 - k\}\} | + |S(u)|,$$  

(3.27)

where $S(u)$ is maximized for the $m = 2, N = 4$ base case. Only when $|S| \{\{l, 2m + 1 - l, k, 2m + 1 - k\}\}$ itself is maximized is equality achieved and the total sum maximized. This occurs for the ASB strings.

We have established that we can bound the probabilities of determining whether an unknown function is balanced or constant in a single query in a CV setting. In the next section will determine an upper bound for the query complexity of a CV algorithm in terms of success probability in terms of the number of queries.

**IV. BOUNDING THE QUERY COMPLEXITY OF THE CONTINUOUS VARIABLE DJ ALGORITHM**

Before we bound the query complexity, we make some important observations regarding the comparison between the discrete DJ algorithm and the CV DJ algorithm.

First, we note that probability distributions $P_c(x)$ and $P_{\text{ASB}}(x)$ defined by Eqs. (4.1) and (4.5) respectively, are in $H_2$, the Hilbert space of $L^2(\mathbb{R})$ functions over the interval $[-\infty, \infty]$. This implies that since we are measuring over a finite interval, the CV DJ algorithm is necessarily probabilistic. Furthermore, we noted that $P$ and $\delta$ are related by the uncertainty relation given in Eq. (4.29). This leads to the conclusion that even in the limit of the improper delta function $\delta(x - x_0)$, the CV DJ algorithm remains probabilistic. This conclusion is contrary to that made in [3].

Second, we compare the operator descriptions of the DV DJ and the CV DJ, which we express as

$$|\tilde{\Psi}_3\rangle = H^{\otimes n} \tilde{U}_f H^{\otimes n}|0\cdots0\rangle$$

$$\phi_z^{(N)}(x) = F_{\hat{z}}^{(N)}(p) F_{\hat{x}} \phi_0(x).$$  

(4.1)

The first equation represents the quantum DJ algorithm operator expression given in Eq. (5.29). The second equation is the analogous CV operator expression determined by concatenating the steps of the previous section. There is a high degree of similarity between these two expressions, but there are mathematical subtleties.

1. The CV position state $\phi_0(x)$ is not a perfect analog to the computational basis state $|0\cdots0\rangle$ except in the limit. However, this limit creates a state that is not in the RHS we argued is necessary for consistency [21].

2. The continuous Fourier transform is not equal to the CV extension of the Hadamard operator in a CV parameterized system with a finite Hilbert space. It is however, a convenient extension when the Hilbert space is infinite.

3. Finally, the diagonal operator $\tilde{U}_f$ given by Eq. (4.6) has each entry taking on the value $\pm 1$ dependent on the value of $f_z$. The CV analogue to this operator is the function $f_z^{(N)}(p)$, where each of the $N$ partitions of the real interval $[-P, P]$ similarly take on the value $\pm 1$.

We now determine numerical values of the probabilities determined in Eqs. (5.20) and (5.21). We can readily calculate the probability of detecting if the function is constant

$$P_{\text{Const}} = \int_{-\delta}^{\delta} \frac{\sin^2(Px)}{P\pi x^2} \, dx$$

$$= \frac{\cos(2\delta P) + 2\delta \text{Si}(2\delta P) - 1}{\delta P\pi},$$  

(4.2)

where the sine integral is given by

$$\text{Si}(z) = \int_0^z \frac{\sin t}{t} \, dt.$$  

(4.3)

Note this probability depends only on the product $P\delta$. If the function is the lowest-order antisymmetric balanced (ASB)

$$P_{\text{ASB}} = \int_{-\delta}^{\delta} \frac{(\cos(Px) - 1)^2}{P\pi x^2} \, dx$$

$$= -8 \sin^4 \left(\frac{\delta P}{4}\right) + 4\delta \text{Si}(\delta P) - 2\delta \text{Si}(2\delta P) \frac{\delta P\pi}{\delta P\pi},$$  

(4.4)

For $P\delta = \pi/2$, the numerical values of these two probabilities are

$$P_{\text{Const}} = \frac{2(\pi \text{Si}(\pi) - 2)}{\pi^2} \approx 0.77.$$  

(4.5)
and
\[ \Pr_{\text{ASB}} = \frac{4\pi \Si(\pi/2) - 2\pi \Si(\pi) - 4}{\pi^2} \approx 0.16. \]  (4.6)

Given this probabilistic nature of the CV DJ algorithm, we need to develop a strategy to bound the error probability. We will employ the technique sometimes called probability amplification [27, 28].

Our strategy will be to make \( m \) repetitions of the CV DJ algorithm where we assume that the oracle is set to the same function for each of the repetitions. Each repetition ends with a measurement. From this sequence of measurements we want to determine whether the unknown function is balanced or constant with high probability.

**Theorem 2.** An error of \( O(e^{-m}) \) can be achieved by making \( O(m) \) repetitions of the CV DJ algorithm.

**Proof.** We will adopt the convention that when we make a query to the CV DJ algorithm we either detect something (algorithm returns a 1), or we do not (algorithm returns a 0). We can thus treat multiple queries as a sequence of Bernoulli trials [29]. We assume that we have set our measurement limits to the optimal \( \pm \delta \). The two events we are trying to uncover are the constant cases where, for ease of calculation we set the probability of detecting something is \( \Pr_\text{C} \geq 3/4 \), and the balanced cases where the probability detecting something is \( \Pr_\text{B} \leq 1/4 \). Note that we have set the probabilities to these rational numbers for illustrative purposes and to simplify the calculation. We can make this arbitrary setting, and we will get the same result as long as the probabilities are bound from 1/2 by a constant.

If each measurement is based on an independent preparation of the state \( \phi_0(\alpha) \), then each of the queries are independent. After a series of \( m \) queries, we can use the Chernoff bounds of the binomial distribution to amplify the success probability [28, 29]. The simplest (but somewhat weak) Chernoff bound on the lower tail is given by [28] as
\[ \Pr[X < (1 - \epsilon)\mu] < e^{-\mu \epsilon^2 / 2}, \]  (4.7)
and on the upper tail as
\[ \Pr[X > (1 + \epsilon)\mu] < e^{-\mu \epsilon^2 / 2}, \]  (4.8)
where \( \mu \) is the expected mean of the resulting binomial distributions after \( m \) queries, and \( \epsilon \) is the relative distance from the respective means.

First, we bound the lower tail corresponding to the distribution of the constant case for which we have \( \mu = m \Pr_\text{c} \). Here we set \( \epsilon = 1/4 \), which expresses the probability for the value being less than half way between the two means as \( \Pr[X < (m/2)] < e^{-\frac{m}{16}} \). Clearly the success is worse for the lower tail allowing us to bound the success probability of the CV DJ algorithm after \( m \) queries as
\[ \Pr[\text{Success}] \geq 1 - e^{-\frac{m}{16}}. \]  (4.9)

This gives an error probability that is \( O(e^{-m}) \) as required.

\[ \square \]

We note that this is of the same order as the exponentially good success probability we have for the classical probabilistic approach given by Eq. (2.2). Also note that this query complexity is independent of the value of \( N \). We have made no attempt to obtain a tighter bound preferring to show only that the success probability of the CV DJ algorithm is of the same order of that of the classical probabilistic approach to solving the DJ problem.

\[ \text{V. CONCLUSIONS} \]

In this paper we have presented a rigorous framework for the analysis of the DJ oracle identification problem in a CV setting. The rigged Hilbert space (RHS) affords a consistent transition from the traditional discrete Hilbert space to the CV setting. Our framework allows us to define a consistent way of encoding \( N \)-bit strings into functions over the real numbers.

We have used this framework, and the selection of the sinc/pulse Fourier transform pair, to prove that a CV implementation of the DJ algorithm cannot provide the exponential speed-up of its discrete quantum counterpart. Additionally, we have presented a bounded-error, upper bound on the query complexity of the DJ problem within the constraints of our model. The lack of speed-up results from an uncertainty principle between the ability to encode perfectly in a continuous representation and the subsequent inability to measure perfectly in the Fourier-dual representation. This uncertainty relationship is manifest in Eq. (4.23) which relates \( P \), the encoding extent, to \( \delta \), the measurement extent. A natural extension of this work would be a lower bound perhaps exploring the techniques along the lines of [30] from the perspective of different Fourier transform pairs.

This uncertainty relationship appears to be a natural feature of the CV setting, but it could also be used to advantage. There is likely to be oracle function symmetries that are particularly suited to different CV settings. For example in Sec. III, we showed that balanced functions with a higher number of zero crossings create sinc functions with frequency components further away from \( x_0 \). It appears that an oracle identification problem designed to separate balanced functions according to frequency separation could be implemented in a CV setting and possibly provide advantage over classical or discrete quantum settings.

Furthermore, it would be interesting to classify the balanced functions from the perspective of different coherent states [31] in CV parameterized settings of both finite and
infinite dimensions. The former would naturally involve the study of the coherent spin systems [32]. Furthermore, the use of squeezed spin states should be studied [33]. Infinite dimension systems would naturally involve the study of implementations involving the coherent states of quantum optics [32, 34].

Additionally, we have set up this framework in a manner that should allow any oracle identification problem to be analyzed in a similar manner in the CV setting. A implementation of a discrete quantum oracle, for example [35, 36], requires a unitary operator representing the oracle. Provided we can create a diagonal representation of this oracle along the lines of $\hat{U}_f$ given in Eq. (2.10), our framework will naturally extend to it. Of course we need to be able to create an implementation of these oracles and that remains an important open question.

Other avenues of the extension of this framework include CV implementations of other hidden subgroup problems. The solution of Simon’s problem [37] in this setting would be an obvious starting point as would the exploration of a CV implementation of Shor’s algorithm [38]. Additionally, the CV framework could be extended to include analysis of noisy oracles along the lines of [24, 39].

In closing we note that the transition from a discrete quantum information setting to a CV setting has many subtleties. In particular the improper delta functions must not be used. Limiting behaviour can be explored, but only if the limits are taken from the perspective of functions defined in the rigged Hilbert space.

Acknowledgements

We appreciate financial support from the Alberta Ingenuity Fund (AIF), Alberta’s Informatics Circle of Research Excellence (iCORE), Canada’s Natural Sciences and Engineering Research Council (NSERC), the Canadian Network Centres of Excellence for Mathematics of Information Technology and Complex Systems (M-ITACS), and General Dynamics Canada. PH is a Scholar and BCS is an Associate of the Canadian Institute for Advanced Research (CIFAR).

[1] M. A. Nielsen and I. L. Chuang, Quantum Computation and Quantum Information (Cambridge University Press, Cambridge, UK, 2000).
[2] D. Gottesman, A. Kitaev, and J. Preskill, “Encoding a qubit in an oscillator”, Phys. Rev. A 64, 012310 (2001).
[3] S. L. Braunstein and A. K. Pati, eds. Quantum Information with Continuous Variables, (Kluwer, Dordrecht, 2003); arXiv: quant-ph/0207108.
[4] A. Furusawa, J. L. Sørensen, S. L. Braunstein, C. A. Fuchs, H. J. Kimble, and E. S. Polzik, “Unconditional quantum teleportation”, Science 23, Vol. 282. no. 5389, pp. 706-709, (1998)
[5] F. Grosshans and P. Grangier, “Continuous variable quantum cryptography using coherent states”, Phys. Rev. Lett. 88, 057902-1-057902-4 (2002).
[6] J. Appel, E. Figueroa, D. Korystov, M. Lobino, and A. I. Lvovsky, “Quantum memory for squeezed light”, Phys. Rev. Lett. 100, 093602 (2008).
[7] D. Akamatsu, Y. Yokoi, M. Arikawa, S. Nagatsuka, T. Tanimura, A. Furusawa, and M. Kozuma, “Ultraslow propagation of squeezed vacuum pulses with electromagnetically induced transparency”, Phys. Rev. Lett. 99, 153602 (2007).
[8] S. L. Braunstein, “Error correction for continuous quantum variables”, Phys. Rev. Lett. 80, 4084 (1998).
[9] J. Eisert and M. B. Plenio, “Distilling Gaussian states with Gaussian operations is impossible”, Phys. Rev. Lett. 89, 137903 (2002).
[10] S. D. Bartlett and B. C. Sanders, “Efficient classical simulation of optical quantum information circuits”, Phys. Rev. Lett. 89, 207903 (2002).
[11] S. D. Bartlett, B. C. Sanders, S. L. Braunstein, and K. Nemoto, “Efficient classical simulation of continuous variable quantum information processes”, Phys. Rev. Lett. 88, 097904 (2002).
[12] L. Blum, M. Shub, and S. Smale, “On a theory of computation and complexity over the real numbers: NP-completeness, recursive functions and universal machines”, Bull. Am. Math. Soc. 21, 1 (1989).
[13] “Continuous turing machines”, Manuscript, not dated. (Available at http://www.mathpages.com/home/kmath135.htm Reference downloaded 03 Dec. 2008.)
[14] D. Deutsch and R. Jozsa, “Rapid solution of problems by quantum computation”, Proc. Royal Soc. (Lond.) A 439, 553 (1992).
[15] D. Deutsch, “Quantum theory, the Church-Turing principle and the universal quantum computer”, Proc. of the Royal Society (London) A 400, 97 (1985).
[16] R. Cleve, A. Ekert, C. Macchiavello, and M. Mosca, “Quantum algorithms revisited”, Proc. Royal Soc. (Lond.) A 454, 339-354 (1998).
[17] D. Bacon, “Lecture 4: Quantum algorithms from sum- mer school in Siena, Italy, August 30-September 2, 2005”, Manuscript, 2005. (Available at http://www.cs.washington.edu/homes /dabacon/teaching/siena. Reference downloaded 20 Oct. 2008.)
[18] S. L. Braunstein and A. K. Pati, “Deutsch-Jozsa algorithm for continuous variables”, in [3]
[19] L. K. Grover, “A fast quantum mechanical algorithm for database search”, Proc. 28th Ann. ACM Symp. on Theory of Computing (STOC ’96), pp. 212–219 (1996).
[20] A. Ambainis, K. Iwama, A. Kawauchi, R. Raymond, and S. Yamashita, “Robust quantum algorithms for oracle identification”, arXiv: quant-ph/0411204.
[21] R. de la Madrid, “The role of the rigged Hilbert space in quantum mechanics”, Eur. J. Phys. 26, 287 (2005).
[22] R. Blume-Kohout, C. M. Caves, and I. H. Deutsch, “Climbing mount scalable: Physical resource require-
ments for a scalable quantum computer", Found. Phys. 32, 1641 (2002).
[23] C. M. Caves, “Physical resources, entanglement, and the power of quantum computation”, Lecture given in SQuInT Summer Retreat University of Southern California, 2005 July 7. Reference downloaded 03 Dec. 2008.
[24] C. I. Tan, “Notes on Hilbert Space”, Manuscript, not dated. (Available at http://jcbmac.chem.brown.edu/bairyl/QuantumPDF Reference downloaded 20 Oct. 2008.)
[25] S. D. Bartlett, H. de Guise, and B. C. Sanders, “Quantum encodings in spin systems and harmonic oscillators”, Phys. Rev. A 65, 052316 (2002).
[26] R. N. Bracewell, The Fourier Transform and Its Applications, 2nd Ed. (McGraw-Hill, New York, 1986).
[27] M. Adcock “The classical and quantum complexity of the Goldreich-Levin problem with applications to bit commitment”, Master of Science Thesis, Department of Computer Science, University of Calgary (2004).
[28] J. F. Canny, “Chernoff bounds”, Manuscript, 2001. (Available at http://www.cs.berkeley.edu/~jfc/cs174/lec10/lec10.pdf. Reference downloaded 20 Oct. 2008.)
[29] T. H. Cormen, C. E. Leiserson, and R. L. Rivest, Introduction to Algorithms, MIT Press, Cambridge, MA 1990
[30] P. Høyer and R. Špalek, “Lower Bounds on quantum query complexity”, Bull. Euro. Assoc. for Theoretical Computer Science Vol. 87, 2005.
[31] A. Perelomov, Generalized Coherent States and Their Applications (Springer-Verlag, New York, 1986).
[32] F. T. Arrechi, E. Courtens, R. Gilmore, and H. Thomas, “Atomic coherent states in quantum optics”, Phys. Rev. A 6, 2211 (1972).
[33] M. Kitagawa and M. Ueda, “Squeezed spin states”, Phys. Rev. A 47, 5138 (1993).
[34] U. Leonhardt, Measuring the Quantum State of Light (Cambridge University Press, Cambridge UK, 1997).
[35] E. Bernstein and U. V. Vazirani, “Quantum complexity theory”, SIAM J. on Comp. 26, No. 5, pp. 1411–1473 (1997).
[36] R. Cleve, W. van Dam, M. Nielsen, and A. Tapp, “Quantum entanglement and the communication complexity of the inner product function”, Lecture Notes in Computer Science 1509 (Springer-Verlag), pp. 61-74 (1999).
[37] D. R. Simon, “On the power of quantum computing”, SIAM J. on Comp. 26, No. 5, pp. 1474–1483 (1997).
[38] P. W. Shor, “Polynomial-time algorithms for prime factorization and discrete logarithms on a quantum computer”, SIAM J. on Comp. 26, No. 5, pp. 1484–1509 (1997).
[39] M. Adcock and R. Cleve, “A quantum Goldreich-Levin theorem with cryptographic applications” Proc. 19th International Symp. Theor. Aspects Comp. Sci. (STACS 2002), H. Alt and A. Ferreira, eds., Lecture Notes in Computer Science 2285 (Springer-Verlag), pp. 323-334 (2002).