Abstract

We study the harmonic moments of Galton-Watson processes, possibly non homogeneous, with positive values. Good estimates of these are needed to compute unbiased estimators for non canonical branching Markov processes, which occur, for instance, in the modeling of the polymerase chain reaction. By convexity, the ratio of the harmonic mean to the mean is at most 1. We prove that, for every square integrable branching mechanisms, this ratio lies between $1 - A/k$ and $1 - A'/k$ for every initial population of size $k > A$. The positive constants $A$ and $A'$ such that $A \geq A'$ are explicit and depend only on the generation-by-generation branching mechanisms. In particular, we do not use the distribution of the limit of the classical martingale associated to the Galton-Watson process. Thus, emphasis is put on non asymptotic bounds and on the dependence of the harmonic mean upon the size of the initial population. In the Bernoulli case, which is relevant for the modeling of the polymerase chain reaction, we prove essentially optimal bounds that are valid for every initial population $k \geq 1$. Finally, in the general case and for large enough initial populations, similar techniques yield sharp estimates of the harmonic moments of higher degrees.
vation for this theoretical problem is the construction of unbiased estimators for samples of branching Markov processes, when the state of an individual depends on the number of its siblings. An instance, outside the realm of pure probability, where this construction is needed, arises in the modeling of the polymerase chain reaction by branching processes, see Sun (1995). In this specific case, the offspring of each individual is 1 or 2, the state of the first descendant is identical to the state of its parent and the state of the other descendant, if any, is a stochastic function of the state of its parent. One wishes to estimate, for instance, the mutation rate of the reaction from a uniform sample of a given generation. Any unbiased estimator of the state of such a sample requires to compute the harmonic mean size of the corresponding generation. But there exists no closed form of these harmonic means, except for small initial populations and for small numbers of generations. Since the mean sizes of the generations of a branching process are well known, the above problem is usually circumvented by assuming that the initial population is very large. Then, an averaging effect occurs which implies, roughly speaking, that the harmonic mean size of a generation is close to its mean size. In the context of the polymerase chain reaction, we showed in previous papers, see Piau (2004), that this approximation is accurate for surprisingly small initial populations, and we provided sharp quantitative estimates of the discrepancy between the harmonic mean and the mean, for any initial population. These results also proved useful to establish rigorous confidence intervals for the estimator of the mutation rate of the polymerase chain reaction, see Piau (2005). Our purpose in the present paper is to give the exact extent of this approximation phenomenon for general, possibly non homogeneous, Galton-Watson processes with positive values. When the approximation phenomenon indeed occurs, we quantify it through non asymptotic and essentially optimal bounds.

1 Results

In the following, $(Z_n)_{n \geq 0}$ denotes a positive Galton-Watson process, possibly non homogeneous. The distribution of this Markov process with values in $\{1, 2, \ldots\}$ is characterized by a sequence $\Xi := (\xi_n)_{n \geq 1}$ of distributions on $\{1, 2, \ldots\}$, as follows. For every $n \geq 1$, conditionally on the past of the process, $Z_n$ is the sum of $Z_{n-1}$ random variables of law $\xi_n$ which are independent of the past. Assume that each $\xi_n$ is integrable of mean $\mu_n \geq 1$. Then $Z_n$ is integrable and, if $E_k$ denotes the expectation when $Z_0 = k$, for any positive integer $k$,

$$E_k(Z_n) = k M_n, \quad \text{with} \quad M_n := n \prod_{i=1}^{n} \mu_i.$$
On the other hand, by convexity, the sequence of general term $M_n \mathbb{E}_k(1/Z_n)$ is nondecreasing for $n \geq 0$. Thus every term is at least $1/k$. Our aim is to provide explicit bounds of the harmonic moments, which imply, in particular, that $M_n \mathbb{E}_k(1/Z_n)$ is close to $1/k$ when this is so. In other words, we wish to show that the sequence of general term $M_n \mathbb{E}_k(1/Z_n)$ is nearly constant. Indeed, for every fixed $n \geq 0$ and when $k \to \infty$, the law of large numbers implies that $\mathbb{E}_k(1/Z_n)$ is equivalent to $1/(k \mathbb{E}_1(Z_n))$, whose value is $1/\mathbb{E}_k(Z_n) = 1/(k M_n)$. Much more is true, as we show below. To ease the task of the reader, we first state the consequence of our general results, in the homogeneous case.

**Theorem A** Assume that $\Xi$ is constant and square integrable. Thus $\xi_n = \xi$ and $M_n = \mu^n$ where $\xi$ is square integrable and $\mu_n =: \mu \geq 1$ for every $n \geq 1$. Then, there exists a positive constant $A$, which depends only on $\xi$, such that, for every integer $k > A$ and every $n \geq 0$,

$$1/k \leq \mu^n \mathbb{E}_k(1/Z_n) \leq 1/(k - A).$$

Assume furthermore that $\mu \neq 1$. There exists a positive constant $A'$, which depends only on $\xi$, such that $A' \leq A$ and, for every integer $k > A'$,

$$\lim_{n \to \infty} \mu^n \mathbb{E}_k(1/Z_n) \geq 1/(k - A').$$

### 1.1 Harmonic moments

Theorem A is a consequence of a general quantitative result, stated as theorem B below, which deals with non homogeneous processes. To state and prove this result, we rely on some specific families of distributions, that we define now.

**Definition 1** For every $m \geq 1$, the generating function $g_m$ of the positive, integer valued, random variable $L_m$ is such that, for any $t \in [0,1]$,

$$\mathbb{E}(t^{L_m}) := g_m(t) := t/(m - (m - 1)t).$$

For any positive $c$, the random variable $L_{c,m}$ is such that, for any $t \in [0,1]$,

$$\mathbb{E}(t^{L_{c,m}}) := g_{c,m}(t) := (g_m(t^c))^{1/c} = t/(m - (m - 1)t^c)^{1/c}.$$
Definition 2  For any positive $c$, let $A_c$ denote the set of distributions of integrable random variables $L \geq 1$ such that, for any $t \in [0, 1]$,
\[ E(t^L) \leq g_{c,m}(t), \quad m := E(L). \]

For any positive $c$, let $A'_c$ denote the set of distributions of integrable random variables $L \geq 1$ such that, for any $t \in [0, 1]$,
\[ E(t^L) \geq g_{c,m}(t), \quad m := E(L). \]

Note that one compares the distribution of $L$ to distributions of random variables, not a priori integer valued but with the same mean. We are now able to state our main result.

Theorem B (1) Let $n \geq 1$. Assume that there exists $c$ such that $\xi_i \in A_c$ for every $i \leq n$. Then, for every $k > c$,
\[ M_n E_k(1/Z_n) \leq 1/(k - c). \]

(2) Assume that $M_n \to \infty$ when $n \to \infty$ and that there exists $c$ such that $\xi_i \in A'_c$ for every $i \geq 1$. Then, for every $k > c$,
\[ \lim_{n \to \infty} M_n E_k(1/Z_n) \geq 1/(k - c). \]

Recall that, by convexity, the sequence $M_n E_k(1/Z_n)$ is nondecreasing, hence the existence of the limit when $n \to \infty$ is a general fact. Assertion (2) becomes false when $M_n$ is allowed to stay bounded, or when one replaces the limit $n \to \infty$ by a finite $n$ since, for instance, the $n = 0$ value is $1/k$.

On the other hand, in practical situations, the hypothesis that $M_n \to \infty$ is easy to check since it only involves the first moments of the generation-by-generation mechanisms.

The restriction to $k > c$ is important as well. As proposition 4 shows, the behaviours of $E_k(1/Z_n)$ and $1/M_n$ can be quite different if $k$ is not large enough. Proposition 4 deals with one generation of a branching process using random variables distributed as $L_{c,m}$, when $m \to \infty$, and corollary 5 applies this result to the $n$th generation of a branching process using random variables distributed as $L_{c,m}$ for a given $m$, when $n \to \infty$.

Definition 3 Let $Z$ denote a random variable and $\mathbb{P}_k^{c,m}$ a probability measure, such that $Z$ is distributed, with respect to $\mathbb{P}_k^{c,m}$, like the sum of $k$ i.i.d. copies of the random variable $L_{c,m}$.

Proposition 4 For any $k \leq c$, $m \mathbb{E}_k^{c,m}(1/Z) \to \infty$ when $m \to \infty$. 

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Corollary 5 Assume that $c$ is an integer and that $\xi_n$ is the distribution of $L_{c,m}$, for every $n$. Hence $M_n = Z_0 m^n$. Then the distribution of $Z_n$ coincides with the distribution of the first generation of the branching process based on $L_{c,m}$. As a consequence, $m^n \mathbb{E}^{c,m}_k(1/Z_n) \to \infty$ when $n \to \infty$.

Thus $\mathbb{E}_k(1/Z_n)$ when $k \leq c$ may decay on a different scale than $1/M_n$, see more on this case in section 8. On the other hand, theorem 11 describes every square integrable Galton-Watson process if $k$ is large enough, as the following theorem shows.

Theorem C Any square integrable distribution on $[1, +\infty[$ belongs to $A_c$ for $c$ large enough, respectively to $A'_c$ for $c$ small enough. Conversely, any distribution on $[1, +\infty[$ which belongs to $A_c$ is square integrable and its variance is at most $cm(m-1)$, where $m$ denotes its mean. Likewise, the variance of any distribution on $[1, +\infty[$ which belongs to $A'_c$ is, either finite and at least $cm(m-1)$, or infinite.

We shall precise the optimal values of $c$ for some usual distributions. Finally, theorem 11 indeed describes the behaviour of $\mathbb{E}_k(1/Z_n)$ when $k$ is large enough, for any square integrable branching process.

1.2 Bernoulli case

We apply theorem 11 to the Bernoulli case when the offspring is always 1 or 2. This case is relevant in the context of the polymerase chain reaction. Our techniques yield accurate bounds of $\mathbb{E}_k(1/Z_n)$ for every positive $k$, that is, even when $k = 1$, for instance in the homogeneous case, see theorem 11 below. We first state uniform bounds that are simple consequences of the results of section 11.

Theorem D Let $n \geq 0$. Assume that $\xi_i = (1 - x_i) \delta_1 + x_i \delta_2$ with $x_i$ in $[0,1]$ for every $i \leq n$. Then, for any $k \geq 2$,

$$1/k \leq M_n \mathbb{E}_k(1/Z_n) \leq 1/(k-1).$$

In the homogeneous case, one can prove better bounds. We write $\mathbb{E}_k^x$ for $\mathbb{E}_k$ when $\xi_i = (1 - x) \delta_1 + x \delta_2$ for every $i \geq 1$. For every $x$ in $(0,1)$, define

$$\alpha''(x) := 1 - x, \quad \alpha'(x) := (1 - x)/(1 + x).$$

Then $0 \leq \alpha' \leq \alpha'' \leq 1$ and $\alpha''$ and $\alpha'$ decrease from $\alpha''(0) = \alpha'(0) = 1$ to $\alpha''(1) = \alpha'(1) = 0$. 


**Theorem E** (1) For any $k \geq 1$ and $n \geq 0$,
\[
1/k \leq (1 + x)^n \mathbb{E}_k^x(1/Z_n) \leq 1/(k - \alpha''(x)).
\]

(2) For any $k \geq 1$,
\[
\lim_{n \to \infty} (1 + x)^n \mathbb{E}_k^x(1/Z_n) \geq 1/(k - \alpha'(x)).
\]

These estimates are precise enough to imply the following side result about the case $k = 1$.

**Proposition 6** There exists no uniform upper bound of $(1 + x)^n \mathbb{E}_1^x(1/Z_n)$ for $n \geq 0$ and $x \in (0, 1)$, since
\[
\lim_{x \to 0} \lim_{n \to \infty} (1 + x)^n \mathbb{E}_1^x(1/Z_n) \text{ is infinite.}
\]

More precisely, for every $x$ in $(0, 1)$ and $n \geq 1$,
\[
c(x)/x < \lim_{n \to \infty} (1 + x)^n \mathbb{E}_1^x(1/Z_n) < 1/x,
\]
where $c(x) := 1 - x(1-x)/(1+3x)$ is such that $8/9 \leq c(x) < 1$.

In theorem E the value of $\alpha'(x)$ stems from the general construction of section 1.1, but the value of $\alpha''(x)$ does not. In other words, a direct application of section 1.1 to the Bernoulli case yields $\alpha(x)$ instead of $\alpha''(x)$, with
\[
\alpha(x) := -\log(1+x)/\log(1-x).
\]

For every $x$ in $(0, 1)$, $\alpha'(x) < \alpha''(x) < \alpha(x)$.

Theorem E follows from the more general case below.

**Theorem F** Let $\xi_i = (1 - x_i) \delta_1 + x_i \delta_2$ for every $i$.

(1) If $x_i \geq x$ for every $i \leq n$, then, for any $k \geq 1$,
\[
1/k \leq \mathbb{E}_k(1/Z_n) \prod_{i=1}^n (1 + x_i) \leq 1/(k - \alpha''(x)).
\]

(2) If $x_i \leq x$ for every $i$ and $\sum_{i \geq 1} x_i$ diverges, then, for any $k \geq 1$,
\[
\lim_{n \to \infty} \mathbb{E}_k(1/Z_n) \prod_{i=1}^n (1 + x_i) \geq 1/(k - \alpha'(x)).
\]
1.3 A discontinuity result

In the Bernoulli case, the functions $\alpha''(x)$ and $\alpha'(x)$ have a nonzero limit at $x \to 0^+$, hence the second part of theorem E above shows that the limit of the normalized harmonic moments does not always depend continuously on the parameters of the model. We show in this section that the phenomenon is general. For the sake of simplicity, we deal with the homogeneous case.

Let $\mathcal{M}$ denote a given subset of $(1, +\infty)$ such that $1$ is a limit point of $\mathcal{M}$. Below, the limits when $\mu \to 1$ are implicitly restricted to $\mu \in \mathcal{M}$. For each $\mu \in \mathcal{M}$, let $\xi_{\mu}$ denote a distribution of mean $\mu$. If $\xi_i = \xi_{\mu}$ for every $i \geq 1$, define a function $h_k$ on $\mathcal{M}$ by

$$h_k(\mu) := \lim_{n \to \infty} \mu^n E_k(1/Z_n).$$

**Proposition 7** Assume that, for each $\mu \in \mathcal{M}$, there exists $a(\mu)$ and $a'(\mu)$ such that $\xi_{\mu}$ belongs to $A_{a(\mu)}$ and to $A'_{a'(\mu)}$. Then

$$1/(k - a'_*) \leq \liminf_{\mu \to 1} h_k(\mu) \leq \limsup_{\mu \to 1} h_k(\mu) \leq 1/(k - a_*),$$

where $a_* := \limsup_{\mu \to 1} a(\mu)$ and $a'_* := \liminf_{\mu \to 1} a'(\mu)$. Thus, if $a'_*$ is positive, the function $h_k$ is not continuous at $\mu = 1^+$.

**Theorem G** In the homogeneous case, assume that each $\xi^\mu$ is the law of $1 + X$, where the law of $X$ is either binomial or Poisson or geometric. Then, for every $k \geq 1$, $h_k$ is discontinuous at $\mu = 1$, since $h_k(1) = 1/k$ and

$$\lim_{\mu \to 1} h_k(\mu) = 1/(k - 1).$$

If the law of $X$ is geometric, then $h_k(\mu) = 1/(k - 1)$ for every $\mu > 1$ and $h_k(1) = 1/k$.

Likewise, assume that each $\xi^\mu$ is the law of $L_{c,\mu}$ for a given positive integer $c$. Then, $h_k(1) = 1/k$ for every $k$. Furthermore, for every $\mu > 1$, $h_k(\mu) = 1/(k - c)$ if $k > c$ and $h_k(\mu) = +\infty$ if $k \leq c$.

1.4 Higher harmonic moments

We now state an extension of theorem H to higher harmonic moments. Theorem H is but a special case of proposition 31 in section 5.

**Theorem H (1)** Let $n \geq 1$. Assume that there exists $c$ such that $\xi_i \in A_c$ for every $i \leq n$. Then, for every positive integer $r$ and every integer $k > rc$,

$$M_n^r E_k(1/Z_n^r) \leq 1/[(k - c)(k - 2c) \cdots (k - rc)].$$
(2) Assume that \( M_n \to \infty \) when \( n \to \infty \) and that there exists \( c \) such that \( \xi_i \in A'_c \) for every \( i \). Then, for every positive integer \( r \) and every integer \( k > rc \),

\[
\lim_{n \to \infty} M_n^r \mathbb{E}_k(1/Z_n^r) \geq 1/[(k - c)(k - 2c) \cdots (k - rc)].
\]

**Corollary 8** Let \( n \geq 1 \). Assume that there exists \( c \) such that \( \xi_i \in A'_c \) for every \( i \leq n \) and write \( \sigma^2_k(1/Z_n) \) for the variance of \( 1/Z_n \) when \( Z_0 = k \). Then, for every integer \( k > 2c \),

\[
M_n^2 \sigma^2_k(1/Z_n) \leq (3c)/[k(k - c)(k - 2c)].
\]

If, additionally, there exists \( c' \) such that \( \xi_i \in A'_c \) for every \( i \leq n \), then the sequence

\[
k^3 M_n^2 \sigma^2_k(1/Z_n)
\]

is bounded above and below by finite positive constants, independently of \( n \) and \( k \), for large enough values of \( k \).

### 1.5 Related studies

As mentioned above, Piau (2004) uses preliminary versions of our results, especially in the Bernoulli case, which is relevant for the study of the polymerase chain reaction. In this specific case, we are now able to deal directly with every initial population \( k \), even \( k = 1 \).

Ney and Vidyashankar (2003) give asymptotics of the harmonic moments of every integrable homogeneous branching process starting from \( k = 1 \) particle. When furthermore \( L \log L \) is integrable, their results specialize as follows, see also Bingham (1988) for some classical facts that are recalled below.

Let \( p_1 := \mathbb{P}(L = 1) \), \( \mu := \mathbb{E}(L) \), and let \( \gamma \) denote the Karlin–McGregor exponent of the distribution of \( L \) (\( \gamma \) is also called the Schröder constant), defined by the equality

\[
p_1 \mu^\gamma = 1.
\]

Let \( W \) denote the almost sure limit of the nonnegative martingale \( Z_n/\mu^n \). The Poincaré function is the Laplace transform \( P(s) := \mathbb{E}_1(\exp(-sW)) \) of the distribution of \( W \) when \( k = 1 \), and solves Poincaré’s functional equation

\[
P(\mu s) = f(P(s)).
\]

Three cases may arise. First, when \( r > \gamma \), \( \mathbb{E}_1(1/Z_n^r)/p_1^r \) converges to a finite positive limit, whose expression is an integral which involves the Schröder function \( S \), defined for any \( t \) in \([0, 1]\), by

\[
S(t) := \lim_{n \to \infty} \mathbb{E}_1(t^{Z_n})/p_1^n.
\]
Up to a multiplicative constant, \( S \) is the unique finite solution on \([0,1)\) of Schröder’s functional equation

\[
S(f(t)) = p_1 S(t).
\]

Second, when \( r = \gamma \), \( \mathbb{E}_1(1/Z_n^\gamma)/(n p_1^\gamma) \) converges to a finite positive limit, whose expression involves Poincaré and Schröder functions. Third, when \( r < \gamma \), \( \mu^{nr} \mathbb{E}_1(1/Z_n^\gamma) \) converges to a finite positive limit. Ney and Vidyashankar provide an expression of the limit in terms of an integral of Poincaré function. One can readily check that this limit is in fact \( \mathbb{E}_1(1/W^r) \) and that the limit is also an upper bound.

When \( L \log L \) is not integrable, the results are similar but one must replace the normalizations \( n p_1^\gamma = n/\mu^\gamma \) when \( r = \gamma \) and \( 1/\mu^{nr} \) when \( r < \gamma \), by similar expressions which involve the Seneta-Heyde constants.

Coming back definitely to the \( L \log L \) case, we recall that the distribution of \( W \) has a density \( w \) on the nonnegative real line, such that \( w(x)/x^{\gamma-1} \) is bounded between two finite positive constants, when \( x \to 0 \), see Dubuc (1971).

The comparison of our results with those recalled above is based on two elementary lemmas.

**Lemma 9** For any distribution \( \xi \) in \( A_c \), \( \gamma(\xi) \geq 1/c \). For any distribution \( \xi \) in \( A'_c \), \( \gamma(\xi) \leq 1/c \).

In other words (see definition \( \text{[25]} \) in section \( \text{[4]} \)),

\[
A'_\xi(\xi) \leq 1/\gamma(\xi) \leq A(\xi).
\]

**Lemma 10** For any branching process, \( k \geq 1, n \geq 0 \), and positive \( r \),

\[
k^r \mathbb{E}_k(1/Z_n^r) \leq \mathbb{E}_1(1/Z_n^{r/k})^k.
\]

Corollary \( \text{[11]} \) is not stated as such in the papers that we mentioned above but it follows from results that we recalled.

**Corollary 11** For any homogeneous branching process of Schröder exponent \( \gamma \) and any \( k > r/\gamma \), the sequence \( \mu^{nr} \mathbb{E}_k(1/Z_n^\gamma) \) is bounded as \( n \) varies, by the finite constant \( \mathbb{E}_k(1/W^r) \).

An interesting feature of corollary \( \text{[11]} \) is that it deals with the entire regime where such a control of \( \mu^{nr} \mathbb{E}_k(1/Z_n^\gamma) \) may hold, namely, with every initial population \( k > r/\gamma(\xi) \). In other words, when \( k \leq r/\gamma(\xi) \), \( \mu^{nr} \mathbb{E}_k(1/Z_n^\gamma) \) is
not bounded. Our upper bounds are restricted to higher values of $k$, namely, to the regime $k > rA(\xi)$.

One could think of recovering the dependence with respect to $k$ from the results of Ney and Vidyashankar even when $k \geq 2$, starting from the inequality

$$\mathbb{E}_k(1/W^r) \leq \mathbb{E}_1(1/W^{r/k})^k/k^r.$$  

However, the bounds one gets cannot be optimal for $k \geq 2$, since

$$\mathbb{E}_c^k(1/W) = 1/(k - c) < \mathbb{E}_1^c(1/W^{1/k})^k/k.$$  

Furthermore, as stressed by Bingham (1988), the law of $W$, hence the value of $\mathbb{E}(1/W^r)$, may be explicitly computed only in very specific cases. In contrast with every other paper we are aware of, the bounds we provide are explicit. The assumptions involve only elementary, step-by-step, mechanisms of the branching process, that is, the distributions of the number of descendants at each generation. Also, we allow for inhomogeneous processes, as long as the reproducing laws belong uniformly to a given space $\mathcal{A}_c$, respectively $\mathcal{A}_c'$, and we make explicit the dependence of the bounds on the initial population.

The introduction of the family of distributions described by $g_{c,m}$ for integer values of $c$ is hardly new, see Harris (1948) for instance. A key point is that we use them for noninteger values of $c$ and as a reference scale of any square integrable distribution. For instance, the $k = 1$ case of Bernoulli distributions requires to make use of values of $c$ in $(0, 1)$. Although these distributions do not correspond to a branching process for noninteger values of $c$, they still satisfy a semigroup property, and this property is sufficient for our purposes. Finally, our methods do not determine the behaviour of the harmonic moments of homogeneous processes whose reproducing law is not square integrable.

1.6 Plan

The remainder of the paper is organised as follows. In section 2 we reduce the case of branching processes in $\mathcal{A}_c$ and $\mathcal{A}_c'$ to the case of well-chosen distributions $g_{c,m}$, which we solve in section 3. In section 4 we show that the $\mathcal{A}_c$ and $\mathcal{A}_c'$ cases imply the result for every square integrable branching process. In section 5 we deal with harmonic moments of higher degrees. In section 6 we study thoroughly the Bernoulli case, that is, the case when the offspring is 1 or 2, sharpening our previous results on this subject. We provide an algorithm to compute the asymptotic harmonic moments, up to any accuracy, and we present some simulations and conjectures about this specific case. Section 7 is a remark about size-biased offsprings. Finally, in section 8 we briefly explain how to deal with cases when the asymptotic behaviours of the harmonic mean and the mean do not coincide.
2 From $g_{c,m}$ to $A_c$ and $A'_c$

We show that every branching process whose branching mechanism uses only laws in $A_c$ can be reduced to the case of $L_{c,m}$ for a suitable $m$, and we solve this case. Similar results hold, as regards the comparison with $A'_c$.

2.1 Results

Lemma 12 describes the semi-group structure of each family $(g_{c,m})_m$. This is the starting point of our computations. Corollary 14 is a special case of corollary 13 and corollary 13 is a consequence of lemma 12. Corollary 13 uses definition 3 in the introduction.

**Lemma 12** For any positive $c$ and any $m \geq 1$ and $m' \geq 1$, $g_{c,m} \circ g_{c,m'} = g_{c,m''}$ with $m'' := mm'$.

**Corollary 13** Let $\varphi$ denote a nonnegative completely monotone function. For every branching process in $A_c$, every $k \geq 1$ and $n \geq 0$,

$$
\mathbb{E}_k(\varphi(Z_n)) \leq \mathbb{E}^{cm}_k(\varphi(Z)), \text{ where } m := M_n.
$$

For every branching process in $A'_c$, every $k \geq 1$ and $n \geq 0$,

$$
\mathbb{E}_k(\varphi(Z_n)) \geq \mathbb{E}^{cm}_k(\varphi(Z)), \text{ where } m := M_n.
$$

Recall that $\varphi$ is completely monotone if and only if its derivatives are such that $(-1)^i \varphi^{(i)}$ is nonnegative for every positive integer $i$. Nonnegative completely monotone functions are Laplace transforms of nonnegative measures on $[0, +\infty)$, see chapter IV of Widder (1948).

**Corollary 14** For every branching process in $A_c$ and every positive $r$,

$$
\mathbb{E}_k(1/Z_n^r) \leq \mathbb{E}^{cm}_k(1/Z^r), \text{ where } m := M_n.
$$

For every branching process in $A'_c$ and every positive $r$,

$$
\mathbb{E}_k(1/Z_n^r) \geq \mathbb{E}^{cm}_k(1/Z^r), \text{ where } m := M_n.
$$

2.2 Proofs

**Proof of lemma 12** Since each $g_{c,m}$ is the conjugate of $g_m$ by the bijection $t \mapsto t^c$, the case $c = 1$ implies the general case. When $c = 1$, $1/(1 - g_m(t))$ is an affine function of $1/(1 - t)$. By composition, $g_m \circ g_{m'}$ is also an affine function of $1/(1 - t)$ and it remains to compute its coefficients to prove the semigroup property. □
Proof of corollary 13 The representation of completely monotone functions which we recalled after the statement of the corollary shows that
\[ \varphi(z) = \int_0^1 t^z \, d\pi(t), \]
for a given measure \( \pi \) on \([0, 1]\). Thus \( \mathbb{E}_k(\varphi(Z_n)) \) is a positive linear functional of the generating function \( \mathbb{E}_k(t^{Z_n}) \) of \( Z_n \). The function \( \mathbb{E}_k(t^{Z_n}) \) is the \( k \)th power of the composition from \( i = 1 \) to \( n \) of the generating function of \( \xi_i \). When the branching process belongs to \( \mathcal{A}_c \), the generating function of \( \xi_i \) is bounded above by \( g_{c,\mu_i} \), thus the composition is bounded above by the composition of the functions \( g_{c,\mu_i} \), which equals \( g_{c,m} \). Finally,
\[ \mathbb{E}_k(\varphi(Z_n)) \leq \int_0^1 g_{c,m}(t)^k \, d\pi(t) = \mathbb{E}^{c,m}_k(\varphi(Z)). \]
The proof of the result for branching processes in \( \mathcal{A}_c' \) is similar. \( \square \)

Proof of corollary 14 For every positive \( r \), \( \varphi(z) := 1/z^r \) is completely monotone. To see this, choose \( d\pi(t) = (\log 1/t)^{r-1} \, dt / (\Gamma(r) t) \) in the representation of \( \varphi \) which we used to prove corollary 13. \( \square \)

3 The case \( g_{c,m} \)

Our task in this section is to evaluate the moments of \( 1/Z \) under the measure \( \mathbb{P}_k^{c,m} \). The cases \( k > c \) and \( k \leq c \) yield different asymptotic behaviours of the first moment of \( 1/Z \). We begin with the direct way to deal with \( \mathbb{P}_k^{c,m} \) when \( k \) is large enough, namely, the computation of factorial moments of \( Z \) instead of the usual moments, see proposition 15. Starting with lemma 18 which gives a representation formula valid for every \( k \), we study in depth the first harmonic moment, both in the small \( k \) and large \( k \) regimes. Corollary 13 in section 2 and lemma 19 below then imply theorem 13. Lemma 20 deals with the case \( k = c \). Lemma 21 provides an alternative formulation of the integral of lemma 18, a formulation that lemma 22 uses to settle the case \( k < c \). Proposition 4, which concludes the case \( k \leq c \), is then an easy consequence.

3.1 Results

We begin with exact formulas. Theorem 11 in section 1.4 is a consequence of corollary 16 below.
Proposition 15 (1) For every nonnegative integer \( r \),
\[
\mathbb{E}_k^{c,m}(Z(Z + c) \cdots (Z + rc)) = m^{r+1}k(k + c) \cdots (k + rc).
\]

(2) For every real number \( r \) such that \( k > rc \),
\[
m^r \mathbb{E}_k^{c,m}(\Gamma((Z/c) - r)/\Gamma(Z/c)) = \Gamma((k/c) - r)/\Gamma(k/c).
\]

(3) For instance, for every nonnegative integer \( r \) such that \( k > rc \),
\[
m^r \mathbb{E}_k^{c,m}(1/[(Z - c) \cdots (Z - rc)]) = 1/[(k - c) \cdots (k - rc)].
\]

Corollary 16 For every nonnegative integer \( r \) such that \( k > rc \),
\[
1/k^r \leq m^r \mathbb{E}_k^{c,m}(1/Z^r) \leq 1/[(k - c) \cdots (k - rc)].
\]
For instance, for every \( k > c \),
\[
1/k \leq m \mathbb{E}_k^{c,m}(1/Z) < m \mathbb{E}_k^{c,m}(1/(Z - c)) = 1/(k - c).
\]

Here is a slight generalization of the \( r = 1 \) assertion in corollary 16.

Proposition 17 For every \( c > 0, m \geq 1, u \geq 0 \) and positive integer \( k > u \),
\[
1/k \leq m \mathbb{E}_k^{c,m}(1/(Z - u)) \leq 1/(k - \sup\{c, u\}).
\]

Proposition 15 and corollary 16 are the results that we use to settle the case \( k > c \) in the rest of the paper. We turn to the evaluation of the exact harmonic moment of \( Z \) with respect to \( \mathbb{P}_k^{c,m} \). The results below are mostly used to deal with the case \( k \leq c \).

Lemma 18 For every positive integer \( k \), positive \( c \) and \( m > 1 \),
\[
\mathbb{E}_k^{c,m}(1/Z) = G(k/c, m)/c, \quad G(u, m) := \int_0^1 t^{u-1}dt/(1 + (m - 1)t).
\]
Alternatively,
\[
G(u, m) = B_{u, 1-u}(1 - 1/m)/(m - 1)^u,
\]
where \( B_{u,v} \) denotes the incomplete Beta function of parameters \( u \) and \( v \), that is, for every \( x \in [0, 1) \),
\[
B_{u,v}(x) := \int_0^x t^{u-1}(1 - t)^{v-1}dt.
\]
Lemma 19. Assume that $u > 1$. Then,

$$m G(u, m) \leq 1/(u - 1).$$

The order of this upper bound is exact when $m$ is large, since the function $(m-1)G(u, m)$ increases when $m$ increases and converges to $1/(u-1)$ when $m \to \infty$.

Lemma 20. $G(1, m) = (\log m)/(m - 1)$.

Lemma 21. For any $u$, $(m-1)^u G(u, m)$ is an increasing function of $m \geq 1$.

Lemma 22. Assume that $u < 1$. When $m \to \infty$, $(m-1)^u G(u, m)$ converges to $c_u := \pi/\sin(\pi u)$. Thus, for any $m > 1$,

$$(m-1)^u G(u, m) \leq c_u.$$

On the other hand, for any $m \geq 2$, $(m-1)^u G(u, m) \geq 1/(2u)$. Bounds of $c_u$ are, for every $u < 1$, $c_u \geq \pi$ and $1/(2u(1-u)) \leq c_u \leq 1/(u(1-u))$.

Corollary 23. Let $k < c$ and $\ell(k, c) := c/(k(c-k))$. For every $m$,

$$(m-1)^{k/c} E_k^c(m, 1/Z) \leq \ell(k, c).$$

The order of this upper bound is exact, since

$$\lim_{m \to \infty} (m-1)^{k/c} E_k^c(m, 1/Z) \geq \ell(k, c)/2.$$

3.2 Proofs

Lemmas 20, 21 and 22 stem from the definitions.

Proof of proposition 15 (1) For any $|x| < 1$ and any positive $y$,

$$1/(1 - x)^y = \sum_{r \geq 0} x^r \Gamma(y + r)/[\Gamma(y) \Gamma(r + 1)].$$

Setting $y = Z/c$ and integrating yields

$$E_k^c(m, 1/(1 - x)^{Z/c}) = \sum_{r \geq 0} E_k^c(m, [\Gamma(r + (Z/c))/\Gamma(Z/c)] x^r /\Gamma(r + 1)).$$

On the other hand,

$$g_{c,m}(1/(1 - x)^{1/c})^k = 1/(1 - mx)^{k/c}. $$
Using the expansion of $1/(1 - mx)^{k/c}$ given above and equating the coefficients of the two series yield the result for any nonnegative integer $r$.

(2) For any positive $y$ and $r$ with $y > r$,

$$\Gamma(r) \frac{\Gamma(y - r)}{\Gamma(y)} = \int_0^1 t^{y-r-1} (1 - t)^{r-1} \, dt.$$ 

Setting $y = Z/c$ and performing the integration yields

$$\Gamma(r) \mathbb{E}^{c, m}_k \Gamma((Z/c) - r) / \Gamma(Z/c) = \int_0^1 g_{c, m}(t'^{1/c})^k (1 - t')^{r-1} \, dt'/t^{r+1}.$$ 

The change of variables $s := g_{c, m}(t'^{1/c}) = g_m(t)$ yields

$$\Gamma(r) \mathbb{E}^{c, m}_k \Gamma((Z/c) - r) / \Gamma(Z/c) = \int_0^1 s^{k/c-r-1} (1 - s)^{r-1} \, ds/m^r,$$

that is, the desired formula. □

**Proof of lemma 18** Write $\mathbb{E}^{c, m}_k (1/Z)$ as the integral of $g_{c, m}(t)^k / t$ on $(0, 1)$. Use the change of variable $t' := g_{c, m}(t)^c$. This yields the first expression of $G$ in the lemma. To get the expression of $G$ in terms of incomplete Beta function, use the change of variable $t' := (m - 1)t/(1 + (m - 1)t)$ in the first expression of $G$. □

**Proof of lemma 19** In the first expression of $G$ in lemma 18 use the fact that $1 + (m - 1)t$ lies between $mt$ and $m$. Thus, $G(u, m)$ lies between the integral of $t^{u-2}/m$ and the integral of $t^{u-1}/m$, that is, between $1/((u-1)m)$ and $1/(um)$. □

**Proof of corollary 23** Use the bound of lemma 22 by $c_u/c$ for $u := k/c$, then the bound of $c_u$ by $1/(u(1-u))$. This yields the bound for every finite value of $m$. The limit when $m \to \infty$ is $c_u/c \geq 1/(2uc(1-u)) = \ell(k, c)/2$. □

4 From $\mathcal{A}_c$ and $\mathcal{A}'_c$ to the general case

In this section, we show that every square integrable branching process belongs to the set $\mathcal{A}_c$, respectively to the set $\mathcal{A}'_c$, for a suitable value of $c$, we prove theorem C and we describe the best possible constants $c$ of theorem B in some specific examples.
4.1 Comparisons

Our next proposition is related to theorem C and justifies definition 25 below.

**Proposition 24** If $\frac{c_1}{c_2} \leq m \geq 1$, $g_{c_1,m} \leq g_{c_2,m}$. If $\frac{c_1}{c_2} < m > 1$, the distribution of $L_{c_2,m}$ belongs to $\mathcal{A}_{c_2}$ but not to $\mathcal{A}_{c_1}$ and the distribution of $L_{c_1,m}$ belongs to $\mathcal{A}'_{c_1}$ but not to $\mathcal{A}'_{c_2}$. Thus, $(\mathcal{A}_c)_c$ is a strictly increasing sequence and $(\mathcal{A}'_c)_c$ is a strictly decreasing sequence.

**Definition 25** For any square integrable distribution $\xi$ on $[1, +\infty]$, let

$$A(\xi) := \inf\{c > 0 : \xi \in \mathcal{A}_c\}, \quad A'(\xi) := \sup\{c > 0 : \xi \in \mathcal{A}'_c\}.$$  

4.2 Examples

We now study some specific transformations and examples. Proposition 26 follows from the definitions.

**Proposition 26** For every $\xi$, $A'(\xi) \leq A(\xi)$. The inequality is strict except in two cases: either $A(\xi) = A'(\xi) = 0$, and in that case, $\xi$ is a Dirac measure at $m \geq 1$; or $A(\xi) = A'(\xi) = c$ is positive, and in that case, $\xi$ is the distribution of a random variable $L_{c,m}$.

**Proposition 27** (1) If the laws of the independent $1 + X$ and $1 + X'$ belong to $\mathcal{A}'_c$, the law of $1 + X + X'$ belongs to $\mathcal{A}'_c$ as well. This statement is false when one replaces $\mathcal{A}'_c$ by $\mathcal{A}_c$.

(2) If the law of $1 + X$ belongs to $\mathcal{A}_c$ and if $b$ is positive, the law of $1 + bX$ belongs to $\mathcal{A}_{cb}$. A similar statement holds if one replaces $\mathcal{A}_c$ and $\mathcal{A}_{cb}$ by $\mathcal{A}'_c$ and $\mathcal{A}'_{cb}$.

(3) If the law of $L$ belongs to $\mathcal{A}_c$ and if $L'$ dominates stochastically $L$, then the law of $L'$ belongs to $\mathcal{A}_c$ as well. For instance, if $b$ is nonnegative, the law of $L + b$ belongs to $\mathcal{A}_c$. A similar statement holds if one replaces $\mathcal{A}_c$ by $\mathcal{A}'_c$.

If $\xi$ is a Dirac measure, $A'(\xi) = A(\xi) = 0$. Other usual cases are as follows.

**Proposition 28** (1) If $\xi$ is uniform on $\{1, \ldots, n\}$, $A(\xi) < 1$. More precisely, $2^{nA(\xi)} = n + 1$.

(2) If $\xi$ is uniform on $\{1, n\}$, $A'(\xi) = (n - 1)/(n + 1)$ and $2^{A'(\xi) + 1} = n + 1$.

(3) If $\xi$ is the law of $1 + X$ where $X$ is binomial $(n, x)$, $A(\xi) < 1$. More precisely,

$$(1 + x n) (1 - x)^{nA(\xi)} = 1.$$
(4) If \( \xi \) is the law of \( 1 + X \) where \( X \) is Poisson of mean \( x \), \( A(\xi) < 1 \). More precisely,
\[
e^{xA(\xi)} = 1 + x.
\]

With the notations of section 1.5, cases (1) to (4) of proposition 28 are such that \( A(\xi) = 1/\gamma(\xi) > A'(\xi) \).

To check that the three values \( A(\xi), 1/\gamma(\xi) \) and \( A'(\xi) \) can indeed be different, assume that \( \xi := (1 - p)\delta_1 + (\delta_2 + \delta_3)p/2 \) with \( p \in (0,1) \).

Then \( \gamma(\xi) := -\log(1 - p)/\log(1 + 3p) \). For \( p = 1/5 \), one can check that the function \( t \mapsto E(tL)/g_{1/\gamma,m}(t) \) has positive derivatives at \( t = 0 \) and \( t = 1 \). Thus some values of this function are greater than 1 and some are smaller than 1. This implies that \( A(\xi) > 1/\gamma(\xi) > A'(\xi) \).

4.3 Bernoulli case

Definition 29 If \( \xi = (1 - x)\delta_1 + x\delta_2 \), write \( \alpha(x) \) for \( A(\xi) \) and \( \alpha'(x) \) for \( A'(\xi) \).

Proposition 30 For any \( x \in (0,1) \), \( \alpha'(x) < \alpha(x) < 1 \), since
\[
\alpha'(x) = \frac{(1 - x)}{(1 + x)}, \quad (1 - x)^{\alpha(x)}(1 + x) = 1.
\]

Thus, \( \alpha \) and \( \alpha' \) decrease on \( (0,1] \) from \( \alpha(0^+) = \alpha'(0^+) = 1 \) to \( \alpha(1) = \alpha'(1) = 0 \). Both are discontinuous at 0 since \( \alpha(0) = \alpha'(0) = 0 \).

One can note that \( \alpha'(x) \leq 1 - x \leq \alpha(x) \).

4.4 Proofs

Proof of proposition 24 Compare the logarithmic derivatives. \( \square \)

Proof of theorem C Both results stem from the expansion of \( g_{c,m} \) near 1, which reads as follows, when \( t = o(1) \),
\[
g_{c,m}(1 - t) = 1 - mt + c(c + 1)m(m - 1)t^2/2 + o(t^2).
\]

On the other hand,
\[
E((1 - t)^L) = 1 - E(L)t + E(L(L - 1))t^2/2 + o(t^2).
\]

A comparison of the second order terms of these expansions yield the condition on the variance of \( L \) for \( L \) to belong to \( \mathcal{A}_c \), respectively to \( \mathcal{A}'_c \).
To show that any square integrable distribution belongs to $A_c$ for suitable values of $c$, we first choose values of $d$ and $s < 1$, both large enough to make sure that $\mathbb{E}(t^L) \leq g_{d,m}(t)$ for every $t \geq s$. Thanks to the expansion above, this is possible for any $d$ such that
\[
d(d + 1)m(m - 1) > \mathbb{E}(L(L - 1)), \quad m := \mathbb{E}(L).
\]
Then we choose a value of $c > d$ large enough such that $1/m^{1/c} \geq g_{d,m}(s)/s$. Thus $\mathbb{E}(t^L) \leq g_{d,m}(t) \leq g_{c,m}(t)$ for every $t \geq s$, and, since $\mathbb{E}(t^L)/t$ is a nondecreasing function of $t$, for any $t \leq s$,
\[
\mathbb{E}(t^L) \leq t \mathbb{E}(s^L)/s \leq t g_{d,m}(s)/s \leq t/m^{1/c} \leq g_{c,m}(t).
\]
The proof for the comparison with distributions in $A'_c$ is similar. □

**Proof of proposition 27** Part (1) stems from the property
\[
g_{c,m}(t)g_{c,m'}(t) \geq t g_{c,mm'}(t),
\]
which we leave as an exercise for the reader. Parts (2) and (3) are direct. □

5 Higher moments

Assume that $\xi_i \in A_c$ for every $i \leq n$ and let $m := M_n$. Then,
\[
M^r_n \mathbb{E}_k(1/Z^r_n) \leq m^r \mathbb{E}^c_k(1/Z^r).
\]
Expansions of $g_{c,m}(t)$ when $m \to \infty$ show that the distribution of $Z/m$ with respect to $\mathbb{P}^c_1$ converges to the distribution of $W$ with respect to a measure $\mathbb{P}^c_1$ such that
\[
\mathbb{E}^c_1(e^{-tW}) = (1 + ct)^{-1/c}.
\]
The distribution of $W$ is Gamma $(c, 1/c)$, that is, its density with respect to the Lebesgue measure $dw$ is
\[
w^{c-1}e^{-w/c} 1_{w \geq 0}/(c^c \Gamma(c)).
\]
Furthermore, $g_{c,m}(t) \leq g_c(t) := \mathbb{E}^c_1(t^W) = (1 - c \log t)^{-1/c}$. Hence,
\[
M^r_n \mathbb{E}_k(1/Z^r_n) \leq \mathbb{E}^c_k(1/W^r) = \Gamma((k/c) - r)/(c^r \Gamma(k/c)).
\]
This inequality holds for every positive values of $r$ and $c$ and every positive integer $k$ such that $k > cr$. The lines above prove the following result.

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Proposition 31 (1) Let $c$ such that $\xi_i \in A_c$ for every $i \leq n$. For every positive $r$ and $k$ such that $k > rc$,

$$M_n^r \mathbb{E}_k(1/Z_n^r) \leq 1/[k, c]_r,$$

where $[k, c]_r := c^r \Gamma(k/c)/\Gamma((k/c) - r)$.

When $r$ is an integer,

$$[k, c]_r = \prod_{i=1}^r (k - ic).$$

(2) Conversely, let $c$ such that $\xi_i \in A'_c$ for every $i$. Assume that $M_n \to \infty$. Then, for every $k$ and every positive real number $r \geq k/c$,

$$\lim_{n \to \infty} M_n^r \mathbb{E}_k(1/Z_n^r) = +\infty.$$

6 Bernoulli branching processes

6.1 Preliminaries

We first set some notations, to be able to deal with non homogeneous processes.

Definition 32 The efficiency of a Bernoulli branching process is the sequence $X := (x_i)_{i \geq 1}$ such that $\xi_i = (1 - x_i) \delta_1 + x_i \delta_2$. Let $L$, respectively $L^*$, denote the set of efficiencies $X$ such that $x_i \in [0, 1]$, respectively $x_i \in (0, 1]$, for every $i \geq 1$. For any $X \in L$, let $s(X) := (x_{i+1})_{i \geq 1}$ denote the shifted sequence.

Definition 33 For any $k \geq 1$ and any efficiency $X$, let

$$B_k(X) := \lim_{n \to \infty} \mathbb{E}_k(1/Z_n) \prod_{i=1}^n (1 + x_i).$$

In the homogeneous case $x_i = x$ for every $i \geq 1$, we write $B_k(x)$ for $B_k(X)$.

By convexity, the limit which defines $B_k(X)$ is also a supremum over $n \geq 0$, thus $B_k(X) \geq 1/k$. The functional $B_k$ describes $\mathbb{E}_k(1/Z_n)$ for finite values of $n$ as well, since replacing every $x_i$ with $i \geq n + 1$ by 0 freezes the branching process at its value $Z_n$. Thus, uniform upper bounds of $B_k$ on $L$ yield upper bounds of $\mathbb{E}_k(1/Z_n)$ for finite values of $n$.

6.2 Results

The following uniform result is a consequence of the fact that $A(\xi) < 1$ for every Bernoulli $\xi$, see proposition 30.
Proposition 34  For every efficiency \( X \in L \) and every \( k \geq 1 \),

\[
1/k \leq B_k(x) \leq 1/(k-1).
\]

Thus, for every \( n \geq 0 \),

\[
1/k \leq \mathbb{E}_k(1/Z_n) \prod_{i=1}^n (1 + x_i) \leq 1/(k-1).
\]

The sequence \((B_k)_{k \geq 1}\) satisfies recursion relations, which we state in proposition 35, and which characterize it fully, see proposition 36.

Proposition 35  For every \( k \geq 1 \), the function \( B_k \) is measurable on \( L \).

Furthermore, for every \( k \geq 1 \) and \( X \in L \),

\[
B_k(X) = (1 + x_1) \sum_i \binom{k}{i} x_1^i (1 - x_1)^{k-i} B_{k+i}(s(X)).
\]

(\( \star \))

Proposition 36  Let \((F_k)_{k \geq 1}\) denote a sequence of functionals defined on \( L^* \). Assume that \( k F_k(X) \to 1 \) when \( k \to \infty \), uniformly over \( X \in L^* \), and that \((F_k)_{k \geq 1}\) solves \( \star \) on \( L^* \) for every \( k \geq 1 \). These conditions define a unique sequence \((F_k)_{k \geq 1}\), such that \( F_k = B_k \) on \( L^* \) for every \( k \geq 1 \).

Thus, the sequence \((B_k)_{k \geq 1}\) is entirely determined on \( L^* \) by the recursion \( \star \) and by the bounds \( 1/k \leq B_k \leq 1/(k-1) \).

Note finally that the recursion \( \star \) is but a special case of the following result.

For any branching process of reproducing law \( \Xi = (\xi_i)_{i \geq 1} \) and any \( k \geq 1 \), introduce

\[
H_k(\Xi) := \lim_{n \to \infty} M_n \mathbb{E}_k(1/Z_n),
\]

and the shifted mechanism \( s(\Xi) := (\xi_{i+1})_{i \geq 1} \). Let \( \xi_1^k \) denote the convolution of the measure \( \xi_1 \) with itself \( k \) times. Then,

\[
H_k(\Xi) = \mu_1 \sum_{i \geq k} \xi_1^k(i) H_i(s(\Xi)).
\]

6.3 Homogeneous case

We start with a version of the relation \( \star \) in the homogeneous case.

Proposition 37  For every \( x \) in \((0,1)\),

\[
B_k(x) = (1 + x) \sum_i \binom{k}{i} x^i (1 - x)^{k-i} B_{k+i}(x).
\]
The recursion whose left hand side is $B_k(x)$ involves the whole set of values $B_k(x), B_{k+1}(x), \ldots, B_{2k}(x)$. Thus, this system of equations does not yield directly the value of each $B_k(x)$. The exception is the case $k = 1$.

**Corollary 38** For every $x \neq 0$, $B_1(x) = B_2(x)(1 + x)/x$.

Our main result in this section is proposition 39.

**Proposition 39** Let $\alpha'(x) := (1 - x)/(1 + x)$ and $\alpha''(x) := 1 - x$. For every $k \geq 1$, 

$$1/(k - \alpha'(x)) \leq B_k(x) \leq 1/(k - \alpha''(x)).$$

Thus, $B_k(0^+) = 1/(k - 1)$ and $B_k(1^-) = 1/k$.

For $k = 1$, proposition 39 states that $B_1(x)$ is at least $1/(1 - \alpha'(x))$. A better bound obtains if one uses corollary 38 and then proposition 39, namely

$$(1 + x)^2/(1 + 3x) \leq x B_1(x) \leq 1.$$

The lower bound is always greater than $8/9 = .889^-$. From our numerical simulations in section 6.9 below, some values of $\lambda B_1(\lambda)$ are as small as $B_* = .9274 \pm .0002$, to be compared to $8/9 = .8889^-$. One could iterate the procedure, getting yet tighter upper and lower bounds of $\mathbb{E}_x(1/Z_n)$, or of any $\mathbb{E}_k^x(1/Z_n)$ with $k \geq 1$, with any prescribed accuracy. We develop this idea in section 6.7 below.

We end this section with a conjecture.

**Problem 40** We conjecture that every function $x \mapsto B_k(x)$ is decreasing on $x \in (0, 1]$. Prove this and find a natural explanation of the fact that $B_k(0^+)$ and $B_k(0)$ are not equal.

### 6.4 Proofs

**Proof of proposition 39** Since each $\mathbb{E}_k(1/Z_n)$ is measurable with respect to $(x_i)_{i \leq n}$, $B_k(x)$ is the limit of a measurable nondecreasing sequence, hence $B_k$ is measurable. As regards the recursion relation, we consider the conditioning by $Z_1$ of the Bernoulli branching process starting from $Z_0 = k$. On the event $\{Z_1 = k + i\}$, $(Z_{n+1})_{n \geq 0}$ follows the law of a Galton-Watson branching process of efficiency $s(x)$, starting from $k + i$. Hence it follows from the fact that the distribution of $Z_1 - k$ is binomial $(k, x_1)$. \qed
Proof of proposition 36

The existence follows from the construction of each $B_k$. A proof of the uniqueness is as follows. Assume that the sequences of functionals $(B'_k)$ and $(B''_k)$ are solutions. In particular, $B'_k/B''_k \to 1$ uniformly on $L^*$, when $k \to \infty$. Fix $\varepsilon$. For every $k$ large enough and for every $x \in L^*$,

$$B'_k(x) \leq (1 + \varepsilon) B''_k(x).$$

Since $(B'_k)$ and $(B''_k)$ both solve the recursion relations (34), a recursion over the decreasing values of $k$ shows that $B'_k(x) \leq (1 + \varepsilon) B''_k(x)$ for every $k \geq 1$ and every $x \in L^*$. This recursion uses as a crucial tool the fact that no $x_k$ is zero. Now, since $\varepsilon$ is arbitrary, $B'_k \leq B''_k$ on $L^*$ for every $k \geq 1$. Exchanging the roles of the two sequences, one sees that $B'_k = B''_k$ on $L^*$, for every $k \geq 1$. □

6.5 Outline of the proof of proposition 39

We start from relations between the functions $B_k$ in proposition 37, which read, for every $k \geq 1$,

$$B_k(x) = (1 + x) \mathbb{E}_k^x(B_{Z_1}(x)).$$

With respect to the probability $\mathbb{P}_k^x$, $Z_1$ is distributed like the sum of $k$ i.i.d. random variables of distribution $(1 - x) \delta_1 + x \delta_2$. Lemma 41 below follows from the fact that $k B_k(x) \to 1$ when $k \to \infty$.

Lemma 41 Assume that $\lim \inf k \varphi(k) \geq 1$ and that, for every $k \geq 1$,

$$(1 + x) \mathbb{E}_k^x(\varphi(Z_1)) \leq \varphi(k).$$

Then $B_k(x) \leq \varphi(k)$ for every $k \geq 1$. Conversely, if $\lim \sup k \psi(k) \leq 1$ and if, for every $k \geq 1$,

$$(1 + x) \mathbb{E}_k^x(\psi(Z_1)) \geq \psi(k),$$

then $B_k(x) \geq \psi(k)$ for every $k \geq 1$.

Definition 42 For every $k \geq 1$, let $c_k$ denote the unique solution in $(0,1)$ of the equation

$$(1 + x) \mathbb{E}_k^x(1/(Z_1 - c_k)) = 1/(k - c_k).$$

Lemma 43 becomes obvious when one uses an equivalent definition of $c_k$, given below in part (ii) of lemma 16.

Lemma 43 If $c_k \leq c$ for every $k \geq 1$, then $B_k(x) \leq 1/(k - c)$ for every $k \geq 1$. Conversely, if $c_k \geq c$ for every $k \geq 1$, then $B_k(x) \geq 1/(k - c)$ for every $k \geq 1$. 22
Lemma 43 asserts that \( B_k(x) \leq \frac{1}{(k - c)} \) for \( c = \sup \{ c_k : k \geq 1 \} \) and, by lemma 44, this supremum is \( c_1 = 1 - x \), thus proposition 39 follows.

**Lemma 44** For every \( k \geq 1 \), \( c_k \leq c_1 = 1 - x \).

The following result shows that the technique above cannot yield a better value of \( \alpha'(x) \) than \( \alpha'(x) = (1-x)/(1+x) \).

**Lemma 45** When \( k \to \infty \), \( c_k \to (1-x)/(1+x) \). Furthermore, for \( k \geq 2 \),
\[
c_k + x c_{k-1} \geq 1 - x.
\]

Finally, we use the characterizations below to evaluate \( c_k \).

**Lemma 46** For every \( k \geq 2 \), the following inequalities are equivalent and equivalent to the fact that \( c \geq c_k \).

(i) \( (1 + x) \mathbb{E}_k^x (1/(Z_1 - c)) \leq 1/(k - c) \).

(ii) \( k (1 + x) \mathbb{E}_{k-1}^x (1/(Z_1 + 2 - c)) \geq 1 \).

(iii) \( k (k - 1) x (1 + x) \mathbb{E}_{k-2}^x (1/(Z_1 + 4 - c)) \leq x k - 1 + c \).

The reversed inequalities (i'), (iii') and (iii'') are equivalent and equivalent to the fact that \( c \leq c_k \).

6.6 Technical steps of the proof of proposition 39

We prove lemmas 44 and 45 assuming lemma 46 for the moment. By Jensen’s inequality, the expectation of the inverse is greater than the inverse of the expectation. Thus (ii'') implies
\[
k(1 + x) \leq (k - 1)(1 + x) + 2 - c.
\]
This reads \( c \leq 1 - x \). Since \( c_1 = 1 - x \), we are done with lemma 43. Furthermore, we can and we will restrict the reasoning below to \( c \leq 1 - x \).

To prove lemma 45, we first note that (ii) involves the expected value of a concave function of \( u := 1/(Z_1 - c_{k-1}) \), namely the function \( u \mapsto u/(1 + bu) \) with \( b := c_{k-1} + 2 - c \). The expected value of a concave function is at most the function of the expected value. From the definition of \( c_{k-1} \), (ii) implies
\[
k(1 + x) \geq (k - 1 - c_{k-1})(1 + x) + c_{k-1} + 2 - c.
\]
This is equivalent to \( c \geq 1 - x - x c_{k-1} \). Hence, for any \( k \geq 2 \),
\[
1 - x - x c_{k-1} \leq c_k \leq 1 - x. \tag{†}
\]

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This is enough to show that \( c_k \geq (1 - x)^2 \) for every \( k \geq 1 \). Thus we can and we will further restrict the reasoning below to \( c \geq (1 - x)^2 \).

In the second step of the proof of lemma \( \text{45} \) we use (iii') like we used (ii). Namely, we note that (iii') involves the expected value of a concave function of \( 1/(Z_1 - c_{k-2}) \) and we apply Jensen's inequality once again. From the definition of \( c_{k-2} \), (iii') implies

\[
k (k - 1) x (1 + x) \geq [(1 + x)(k - 2 - c_{k-2}) + c_{k-2} + 4 - c] (x k - 1 + c).
\]

After some simplifications, this reads

\[
A_1 k + A_0 \geq 0, \quad A_1 := (1 - x)^2 - c + x^2 c_{k-2} - (1 - c)(2(1 - x) - c - x c_{k-2}).
\]

Since \( c \geq (1 - x)^2 \) and \( c_{k-2} \geq (1 - x)^2 \), simple bounds show that \( A_0 \leq 1 \).

Hence (iii') implies that \( A_1 \geq -1/k \). Finally, for every \( k \geq 3 \),

\[
(1 - x)^2 \leq c_k \leq (1 - x)^2 + x^2 c_{k-2} + 1/k.
\]

One uses the a priori bounds of (†) and (‡) as follows. On the one hand, the upper bound of \( c_k \) in (‡) implies

\[
\limsup c_k \leq (1 - x)^2 + x^2 \limsup c_k.
\]

On the other hand, the lower bound of \( c_k \) in (†) implies

\[
1 - x - x \limsup c_k \leq \liminf c_k.
\]

Hence \( \limsup c_k = \liminf c_k = (1 - x)/(1 + x) \). This proves lemma \( \text{45} \).

Lemma \( \text{46} \) is a consequence of the following trick. Part (i) involves

\[
(k - c)/(Z_1 - c) = 1 - (Z_1 - k)/(Z_1 - c) =: 1 - v.
\]

By exchangeability, \( \mathbb{E}_k^x(v) \) is \( k \) times the expected value of \( (L_1 - 1)/(Z_1 - c) \), where \( L_1 \) denotes the number of descendants of the first individual in the initial population. The event \( \{L_1 - 1 \neq 0\} \) is \( \{L_1 = 2\} \) and has probability \( x \). Thus, for every \( k \geq 2 \),

\[
(k - c) \mathbb{E}_k^x(1/(Z_1 - c)) = 1 - k x \mathbb{E}_{k-1}^x(1/(Z_1 + 2 - c)).
\]

With the convention that \( Z_1 = 0, \mathbb{P}_0^x \) almost surely, this relation holds for \( k = 1 \) as well. This translates (i) or (i') into (ii) or (ii'). The translation of (ii) or (ii') into (iii) or (iii') uses the same trick, starting from \( 1/(Z_1 + 2 - c) \). This concludes the proof of proposition \( \text{39} \).
6.7 Algorithm

The following algorithm yields approximate values of $B_k$ on $L^*$, with any prescribed accuracy.

- Fix $n \geq 1$.
- For every $k \geq n + 1$ and $x$, let
  \[ B^0_{k,n}(x) := 1/k, \quad B^1_{k,n}(x) := 1/(k - 1). \]
- Find the unique sequence $(B^1_{k,n})_{k \leq n}$ that solves the system of equations (1) for $k \leq n$, when one replaces every $B_k(s(x))$ such that $k \geq n + 1$, by $B^1_{k,n}(s(x))$, that is, by the value $1/(k - 1)$.
- Likewise, find the unique sequence $(B^0_{k,n}(x))_{k \leq n}$ that solves the system of equations (2) for $k \leq n$, when one replaces every $B_k(s(x))$ such that $k \geq n + 1$, by $B^1_{k,n}(x)$, that is, by the value $1/k$.
- Then, for every $k \geq 1$ and every $x$,
  \[ B^0_{k,n}(x) \leq B_k(x) \leq B^1_{k,n}(x) \leq (1 + 1/n) B^0_{k,n}(x). \]

6.8 Comments on the algorithm

Neither $(B^0_{k,n})_{k \geq 1}$ nor $(B^1_{k,n})_{k \geq 1}$ solve the full system of equations (2). For any fixed values of $k$ and $x$, $(B^0_{k,n}(x))_{n \geq 1}$ is a nondecreasing sequence that converges to $B_k(x)$ when $n \to \infty$. Likewise, $(B^1_{k,n}(x))_{n \geq 1}$ is a nonincreasing sequence that converges to $B_k(x)$ when $n \to \infty$.

Increasing values of $n$ yield more and more accurate approximations of each $B_k(x)$ and the relative error is of order at most $1/n$.

In the Bernoulli case, one can use some initial values, better than $1/k$, respectively $1/(k - 1)$, namely, for every $k \geq n + 1$ and $x$,
\[ b^0_{k,n}(x) := 1/(k - \alpha'(x)), \quad b^1_{k,n}(x) := 1/(k - \alpha''(x)). \]

The relative error that was at most $1 + 1/n$ in the first version of the algorithm becomes at most
\[ 1 + (\alpha'' - \alpha')(x)/(n + 1 - \alpha''(x)) \leq 1 + (3 - 2\sqrt{2})/(n + x). \]

Numerically, this is at most $1 + 0.172/n$, for every $x$. 
6.9 Simulations in the homogeneous case

The algorithm above with \( n := 1000 \) suggest the following refinements. Define

\[
B(x) := B_1(x) x = B_2(x) (1 + x).
\]

Simulations show that \( B \) decreases on \((0, x_*)\) from \( B(0^+) = 1 \) to \( B(x_*) := B_* \) and increases on \((x_*, 1]\) from \( B_* \) to \( B(1) = 1 \), with

\[
x_* = .38 \pm .01, \quad B_* = .9274 \pm .0002.
\]

This would imply that, for every positive \( x \),

\[
B_* / x \leq B_1(x) \leq 1 / x.
\]

Simulations show that \( B_2 \), hence \( B_1 \) as well, decreases on \((0, 1]\).

7 Size-biased offspring

When computing harmonic means, it may prove convenient to use size-biased distributions, as follows. Assume that \( L \) and \( L_i \) are i.i.d. positive integrable random variables and that \( L' \) is an independent size-biased copy of \( L \), that is, for every \( t \in [0, 1] \),

\[
E(tL') := E(L t^L) / E(L).
\]

Then, for any nonnegative integer \( k \),

\[
E(L) E(1/(L_1 + \cdots + L_k + L')) = 1/(k + 1).
\]

Can one use this in our branching setting? Assume first that \( 1 \leq L \leq c + 1 \) almost surely, for a given integer \( c \). Since \( L' \leq c + 1 \leq L_{k+1} + \cdots + L_{k+c+1} \) almost surely, this proves that

\[
E(L) E_k(1/Z_1) \leq 1/(k - c),
\]

for every \( k \geq c + 1 \). More generally, the last inequality above holds as soon as \( E(tL') \geq E(t^L)^{c+1} \) for every \( t \in [0, 1] \) and for a given positive \( c \).

In our setting, this line of reasoning suffers from two drawbacks. First, unless we miss something, to be able to iterate this inequality during \( n \) generations, one must assume that \( k \geq nc \). Second, the inequality \( E(tL') \geq E(t^L)^{c+1} \) implies that \( c \geq A(\xi) \), where \( \xi \) denotes the law of \( L \) (the proof is easy and omitted). In other words, \( k \geq c \) implies that \( \xi \in A_c \).
8 Case $k \leq c$

This section is a brief description of the behaviour of $E_k(1/Z_n)$ when the hypotheses of theorem are not met. Consider, for the sake of simplicity, a homogeneous branching process and let

$$p_i := \xi(i) = \mathbb{P}(L = i) = \mathbb{P}(Z_1 = i \mid Z_0 = 1).$$

Our first remark is that, for every $n \geq 0$ and $k \geq 1$,

$$E_k(1/Z_n) \geq r_k^n/k, \quad r_k := \max\{p_k^1, 1/\mu\}.$$ 

The $1/\mu$ bound is due to the convexity. The $p_k^1$ bound is due to the fact that the probability of the event $\{Z_n = k\}$ is $p_k^1$. The parameters $\mu$ and $p_1$, through $r_k$, indeed describe the asymptotics of $E_k(1/Z_n)$, as follows. For the sake of simplicity, we exclude the degenerate case $p_k^1 \mu = 1$, where polynomial corrections appear. For a given $k \geq 1$, there exists a finite positive $h_k$ such that

$$\lim_{n \to \infty} E_k(1/Z_n)/r_k^n = h_k.$$

In the Bernoulli case, $p_1 < 1/\mu$ hence $r_k = 1/\mu$ for every $k \geq 1$ and the $r_k = p_k^1$ regime is nonexistent.

The limits $h_k$ satisfy the following relations. Assume for instance that one wishes to compute $h_1$ and that $\mu p_1 > 1$, hence $r_1 = p_1$. Conditioning on the value of $Z_1$, one gets a relation between $E_1(1/Z_{n+1})$ and the sequence $(E_k(1/Z_n))_{k \geq 1}$. Letting $n$ go to infinity yields

$$h_1 = 1 + \sum_{k \geq 2} H_k p_k/p_1, \quad H_k := \sum_{n \geq 0} E_k(1/Z_n)/p_1^n.$$

The term $E_2(1/Z_n)/p_1^n$ behaves like $(r_2/p_1)^n$, that is, like $1/(\mu p_1)^n$ if $\mu p_1^2 > 1$, or like $p_1^n$ if $\mu p_1^2 < 1$. Since both quantities are summable, $H_2$ is finite. Since $H_k \leq H_2$ for every $k \geq 2$, $h_1$ is finite as well.

When $\mu p_1 < 1$, $r_k = 1/\mu$ for every $k \geq 1$ and the same reasoning as above yields

$$h_1 = \mu \sum_{k \geq 1} p_k h_k.$$ 

Since $\mu p_1 < 1$, this gives $h_1$ as a linear combination of $(h_k)_{k \geq 2}$. In turn, for every $k \geq 2$, a one-step recursion, similar to the one we used before, shows that $h_k$ is such that

$$h_k = \mu \sum_{i \geq k} h_i \xi^k(i).$$ 

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Since the coefficient of $h_k$ in the right hand side is $\mu p_1^k < 1$, this relation implies that $h_k$ is a linear combination of the sequence $(h_i)_{i \geq k+1}$ and that this linear combination uses nonnegative coefficients. Since $h_i \leq h_2$ for every $i \geq k+1$, this series converges. However, it does not seem easy to get information about the coefficients $h_1$ or $h_k$ with $k \geq 2$ from these relations.

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