LONG-TIME BEHAVIOR OF SCALAR CONSERVATION LAWS WITH CRITICAL DISSIPATION

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ABSTRACT. The critical Burgers equation \( \partial_t u + u\partial_x u + \Lambda u = 0 \) is a toy model for the competition between transport and diffusion with regard to shock formation in fluids. It is well known that smooth initial data does not generate shocks in finite time. Less is known about the long-time behavior for ‘shock-like’ initial data: \( u_0 \to \pm a \) as \( x \to \mp \infty \). We describe this long-time behavior in the general setting of multidimensional critical scalar conservation laws \( \partial_t u + \text{div} f(u) + \Lambda u = 0 \) when the initial data has limits at infinity. The asymptotics are given by certain self-similar solutions, whose stability we demonstrate with the optimal diffusive rates.

1. INTRODUCTION

Our motivating example is the Burgers equation with critical non-local dissipation

\[ \partial_t u + u\partial_x u + \Lambda u = 0 \]  

(1.1)

and ‘shock-like’ initial data:

\[ u_0(x) \to \pm a \text{ as } x \to \mp \infty, \]  

(1.2)

where \( \Lambda = (-\Delta)^{1/2} \) and \( a > 0 \). This equation arises as a toy model in fluid mechanics. It models the competition between the transport non-linearity \( u\partial_x u \), which drives the solution towards a shock, and the dissipation term \( \Lambda u \), whose smoothing effects counteract the tendency of the non-linearity to form shocks. The equation is critical in the sense that these two terms are in balance. In PDE terms, the strongest known monotone quantities, the \( L^\infty \) norm and total variation, are invariant under the scaling symmetry

\[ u \to u(\lambda x, \lambda t), \]  

(1.3)

which preserves the equation (1.1).

By now, it is well known that solutions of (1.1) evolving from smooth initial data do not form shocks in finite time. What happens in infinite time? We answer this question for the critical Burgers equation (1.1) and in the more general context of scalar conservation laws with critical non-local dissipation in \( \mathbb{R}^n \):

\[ \partial_t u + \text{div} f(u) + \Lambda u = 0, \]  

(1.4)

where the initial data has ‘limits at infinity’. The long-time behavior is described to leading order by certain self-similar solutions, that is, solutions invariant under the scaling symmetry (1.3).

Let \( h \in C^\infty(S^{n-1}) \) and \( u_0^{ss}(x) = h(|x|/|x|) \). Let \( f : \mathbb{R} \to \mathbb{R}^n \) belong to \( C^\infty_{\text{loc}}(\mathbb{R}; \mathbb{R}^n) \). Let \( v_0 \in L^\infty(\mathbb{R}^n) \) with \( |v_0| \to 0 \) as \( |x| \to +\infty \), specifically,

\[ \|v\|_{L^\infty(\mathbb{R}^n \setminus B_R)} \to 0 \text{ as } R \to +\infty. \]  

(1.5)

Let \( u_0 = u_0^{ss} + v_0 \) and \( \|u_0^{ss}\|_{L^\infty(\mathbb{R}^n)}, \|u_0\|_{L^\infty(\mathbb{R}^n)} \leq m \). Let \( u^{ss} \), \( u \) be the unique entropy solutions to (1.4) with initial data \( u_0^{ss}, u_0 \), respectively. In the context of (1.4), the notion of entropy solution was introduced by Alibaud in [3], and we review it below. Notice that, by virtue of its uniqueness, \( u^{ss} \) must be self similar.
Here is our main theorem:

**Theorem 1.1** (Long-time behavior). The above entropy solution \( u \) converges to the self-similar solution \( u^{ss} \) with the following (diffusive) rates:

\[
\| u(\cdot, t) - u^{ss}(\cdot, t) \|_{L^q(\mathbb{R}^n)} \lesssim_{m,n} o_t \rightarrow +\infty (1) t \frac{n}{q} \| u_0 - u^{ss}_0 \|_{L^p(\mathbb{R}^n)} \tag{1.6}
\]

for all \( 1 < p \leq q \leq +\infty \). If \( p = 1 \), then (1.6) holds with \( O(1) \) instead of \( o(1) \) on the RHS.

When \( f \equiv 0 \), (1.4) reduces to the fractional heat equation, and the above diffusive rates are easily seen to be sharp.

In dimension \( n = 1 \), we also have stability in BV:

**Theorem 1.2** (BV convergence). If also \( u_0 \in BV(\mathbb{R}) \), then

\[
\| u(\cdot, t) - u^{ss}(\cdot, t) \|_{TV(\mathbb{R}^n)} \rightarrow 0 \text{ as } t \rightarrow +\infty. \tag{1.7}
\]

Additionally, \( u^{ss} \) is monotone and satisfies the following spatial asymptotics:

\[
C^{-1} |x|^{-1} \leq |u^{ss}(x, 1) - u^{ss}_0| \leq C |x|^{-1}, \quad |x| \geq C, \tag{1.8}
\]

provided that \( u^{ss}_0 \) is not identically constant.\(^1\) Notice that, when \( n \geq 2 \), \( u^{ss}_0 \) does not generally belong to \( BV(\mathbb{R}^n) \), and the total variation is no longer scaling invariant.

**Comparison with existing literature.** The critical Burgers equation (1.1) belongs to the following family of Burgers equations with fractional diffusion:

\[
\partial_t u + u \partial_x u + \Lambda^s u = 0, \tag{1.9}
\]

where \( s \in (0, 2] \). These models were considered by Biler et al. in [7], where they are known as *fractal Burgers equations*. One may consider also the analogous conservation laws with fractional diffusion \( \Lambda^s \). The relevant literature is fairly extensive:

*Regularity theory*. A detailed picture of the regularity theory of (1.9) was shown by Kiselev, Nazarov, and Shterenberg in [33] in the periodic setting. When \( s \geq 1 \), smooth initial data gives global smooth solutions, whereas when \( s < 1 \), solutions may develop shocks in finite time. In that case, solutions may be continued uniquely within the class of entropy solutions. The shock formation was also shown in [4].

The proof of global regularity in [33] in the critical case follows the method of ‘moduli of continuity’.\(^2\) This method was introduced in [31] by Kiselev, Nazarov, and Volberg in the context of the critical SQG equation:

\[
\partial_t \theta + \vec{R} \cdot \nabla \theta + \Lambda \theta = 0. \tag{SQG}
\]

Other proofs of the regularity of (SQG) are contained in [11] (De Giorgi’s method), [30], [34] (Nash’s method), [18] (‘nonlinear maximum principle’), and [17]. The above proofs can be categorized as proofs of *smoothing* [11, 34] or *propagation of regularity* [31, 30, 18, 17]. The smoothing proofs notably ‘forget’ that the equation is nonlinear. Alternative proofs of regularity for (1.1), based on smoothing, were given in [13] (De Giorgi’s method) and [38, 39] (non-divergence form techniques). We rely on these smoothing estimates, particularly those of Silvestre, in an essential way below.\(^3\)

\(^1\)It may be possible to obtain more precise spatial asymptotics for \( u^{ss} \) and its derivatives by analyzing the similarity profile \( u^{ss}(\cdot, 1) \), which satisfies a quasilinear nonlocal elliptic equation.

\(^2\)This method has since been generalized to other models, including the 1d critical Keller-Segel equations [10], the 1d fractional Euler alignment system [20], and the 2d Muskat problem [12].

\(^3\)For supercritical SQG, global regularity remains open, though it is possible to show eventual regularity [37, 19, 32, 14]. We mention also the recent extension of [11] to bounded domains in [41].
Long-time behavior. The long-time behavior of (1.9) is perhaps less well studied than its regularity. When \( s \in (0, 2) \) and the initial data is well localized, the non-linearity of (1.9) is ‘irrelevant’, in the sense of [9], for the long-time dynamics. When \( s = 1 \), Iwabuchi [24, 25] demonstrated that all solutions with \( u_0 \in L^1 \cap B^0_{\infty, 1} \) converge to the Poisson kernel. When \( s = 2 \), the spaces \( L^1 \) and \( M \) (finite measures) are critical, and it is classical that the long-time behavior is given by a self-similar solution, sometimes called a diffusion wave. This case and its precise asymptotic behavior can be illuminated by the Cole-Hopf transformation [15, 35, 29, 6].

What about non-decaying solutions? The current best results in this direction concern ‘rarefaction-like’ initial data, that is, \( a < 0 \) in (1.2). In [28], it was shown that such solutions converge to an inviscid rarefaction wave when \( s > 1 \). In [5], it was shown that when \( s = 1 \), the solutions converge to a certain self-similar solution, and when \( s < 1 \), the non-linearity is ‘irrelevant’ in the long-time asymptotic expansion. Notably, in the rarefaction case, the potential term in the energy estimates for the linearized equation appears with a good sign.

In this paper, we analyze the case of ‘shock-like’ initial data, which is less clear. Initially, one might wonder whether (i) solutions converge to a smooth traveling or standing wave, known as a ‘viscous shock’, or perhaps (ii) solutions form a shock in infinite time. Regarding (i), it was already shown in [7] that traveling wave solutions satisfying reasonable regularity conditions do not exist when \( s \in (0, 1] \). Regarding (ii), one might additionally wonder whether the standing waves constructed in the subcritical case \( s > 1 \) in [16] converge to a shock as \( s \to 1^+ \). This is apparently also not the case, as we show in Theorem 1.1.

It is tempting to conjecture that, in the subcritical case \( s > 1 \), shock-like solutions of (1.9) behave as in the classical case \( s = 2 \), where there is a unique viscous shock, whose global asymptotic stability was shown by Il’in and Oleǐnik in [23]. See [36, 27, 44] and many others for further developments and precise asymptotics. For \( s \in (1, 2) \), the uniqueness, spatial asymptotics, and global asymptotic stability of the monotone viscous shocks constructed in [16] seem to be unknown, although local asymptotic stability was recently demonstrated in [1].

Self-similarity and (non-)uniqueness. The two-dimensional Navier-Stokes equations

\[
\partial_t \omega + u \cdot \nabla \omega - \Delta \omega = 0, \quad u = \nabla^\perp \Delta^{-1} \omega
\]

(NS)

exhibit a family of self-similar solutions known as the Oseen vortices: \( \omega(x, t) = \alpha \Gamma(x, t/\nu) \), where \( \Gamma \) is the heat kernel and \( \alpha = \int \omega_0 \, dx \) is the circulation. In [21], Gallay and Wayne famously showed that all localized solutions converge to Oseen vortices as \( t \to +\infty \), and, moreover, the vortex solutions are unique within a natural solution class. Our situation is analogous, with the circulation \( \alpha \) corresponding to the jump parameter \( a \) in (1.2). By contrast, self-similar solutions of the three-dimensional Navier-Stokes equations are expected to be non-unique [26, 22]. For (SQG), this is investigated in forthcoming work of Bradshaw and the first author. While the entropy solutions of (1.1) are unique, there may be a different class of self-similar solutions with potential non-uniqueness, for example, with \( u_0 \sim \log x \), so that \( \nabla u_0 \) is \(-1\)-homogeneous.

Main idea. Our starting point is the existence and uniqueness of \( L^\infty \) entropy solutions to (1.4), due to Alibaud [3], which immediately gives the existence and uniqueness of a self-similar solution \( u^{ss} \). Let \( v = u - u^{ss} \) be the difference between an entropy solution \( u \) and the self-similar solution. Consider a sequence \( (v^{(k)})_{k \in \mathbb{N}} \) obtained by ‘zooming out’ on \( v \) using the scaling symmetry (1.3). Then establishing \( v(\cdot, t) \to 0 \) as \( t \to +\infty \) is the same as establishing \( v^{(k)}(\cdot, t) \to 0 \) on \( \mathbb{R}^n \times (1/2, 1) \)

It is also interesting to examine the convergence of viscous solutions to shocks as \( \nu \to 0^+ \). In the case \( s > 1 \), this was done in [2].
as \( k \to +\infty \). To analyze the new problem, we exploit a key (standard) observation about viscous scalar conservation laws, namely, that \( v \) satisfies the viscous continuity equation

\[
\partial_t v + \text{div}(gv) + \Lambda v = 0,
\]

where

\[
g = \frac{f(u) - f(u^{ss})}{u - u^{ss}} \in L^\infty(\mathbb{R}^n \times (0, +\infty)).
\]

In our setting, \textit{the main difficulty is that at the initial time,} \( g \) \textit{is no better than bounded, since} \( u^{ss} \) \textit{is not continuous.} This is an essential feature of the problem, and we handle it using two tools:

1. \textit{smoothing for drift-diffusion equations.} By the known regularity theory, the solution, which is initially merely bounded, instantaneously becomes \( C^\alpha \)-in-\( x \). This may be bootstrapped to higher regularity. The key point is then to move the problem past \( t = 0 \), which is done by the

2. \textit{controlled speed of propagation.} Solutions of (1.10) have finite propagation speed \textit{up to the effect of the diffusion.} This allows us to keep the initial spatial decay of the solution for small positive times and exploit (1.10) with smooth coefficients and smooth, decaying initial data.

The above tools, due to [38, 39] and [3], respectively, are key to our arguments. We encounter a further, technical difficulty in that the controlled speed of propagation only allows us to propagate \( L^1 \)-based quantities. This requires the use of special norms \( \| v \|_{k, L^p_x(\mathbb{R}^n)} \), for example,

\[
\| v \|_{k, L^p_x(\mathbb{R}^n)} = \sup_{k \in \mathbb{Z}^n} \int_{k+(-1/2,1/2)^n} |v(x)| \, dx.
\]

After the initial time, we use the smoothing effect to estimate more standard quantities, such as \( \| v \|_{L^q(\mathbb{R}^n)} \), in terms of these special norms.\(^5\) For this, we use pointwise estimates for fundamental solutions of non-local parabolic equations with subcritical lower order terms, due to Xie and Zhang [43]. When \( f \) is merely Lipschitz, we offer less precise asymptotics, see Remark 3.2.

2. Preliminaries

In the sequel, constants in the \( C, \lesssim \) notation may implicitly depend on \( n \geq 1, f \in W^{1,\infty}_{\text{loc}}(\mathbb{R}; \mathbb{R}^n) \).

Recall that the Poisson kernel \( P \) is given by

\[
P(x, t) = c_n \frac{t}{(|x|^2 + t^2)^{n+1}},
\]

where \( c_n > 0 \) is chosen to satisfy \( \int P(x, t) \, dx = 1 \) for all \( t > 0 \).

In [3, Definition 2.3], Alibaud introduced the notion of \textit{entropy solution} to the critical scalar conservation law (1.4). We summarize only the facts we need about entropy solutions. See Section 3 of [3]. For each \( u_0 \in L^\infty(\mathbb{R}^n) \), there exists a unique entropy solution \( u \) of (1.4). This solution exists globally and satisfies the maximum principle

\[
\| u \|_{L^\infty_{t,x}(\mathbb{R}^n \times (0, +\infty))} \leq \| u_0 \|_{L^\infty(\mathbb{R}^n)}.
\]

The PDE (1.4) is satisfied in the distributional sense. Finally, \( u \) belongs to \( C([0, T]; L^1(K)) \) for each \( T > 0 \) and compact \( K \subset \mathbb{R}^n \).

The following proposition is contained in [3, Theorem 3.2]:

\(^5\)Similar norms appear in the Navier-Stokes literature. See [8] and the references therein. Apparently, these spaces are known as \textit{Wiener amalgam spaces.}
Proposition 2.1 (Controlled speed of propagation). Let $u_0, \tilde{u}_0 \in L^\infty(\mathbb{R}^n)$. Consider $u, \tilde{u}$ entropy solutions to (1.4) with initial conditions $u_0$ and $\tilde{u}_0$, respectively. Then for all $x_0 \in \mathbb{R}^n$, all $t > 0$ and all $R > 0$,
\[
\int_{B(x_0,R)} |u(x,t) - \tilde{u}(x,t)| \, dx \leq \int_{B(x_0,R+Lt)} P(\cdot,t) * |u_0 - \tilde{u}_0| \, dx \tag{2.3}
\]
where $L$ is the Lipschitz constant of $f$ on $[-m,m]$, with $m = \max(\|u_0\|_{L^\infty(\mathbb{R}^n)}, \|\tilde{u}_0\|_{L^\infty(\mathbb{R}^n)})$.

We use Proposition 2.1 to establish

Proposition 2.2 (Controlled BV). If $u_0 \in \text{BV}(\mathbb{R}^n)$, then $u(\cdot,t) \in \text{BV}(\mathbb{R}^n)$ with $\|u(\cdot,t)\|_{\text{TV}(\mathbb{R}^n)} \leq \|u_0\|_{\text{TV}(\mathbb{R}^n)}$ for all $t > 0$. Let $\psi \in C_0^\infty(\mathbb{R}^n)$ be non-negative and radial with $\psi \equiv 1$ in a neighborhood of the origin. Let $\psi(x,t) = \psi(x - xLt/|x|)$ when $|x| \geq Lt$ and $\psi \equiv 1$ otherwise. Then, for all $x_0 \in \mathbb{R}^n$, all $t > 0$, and $k = 1, \ldots, n$,
\[
\int_{\mathbb{R}^n} \psi(x - x_0)|\omega_k(x,t)| \, dx \leq \int_{\mathbb{R}^n} \psi(x - x_0,t)P(\cdot,t) * |\omega_k| \, dx, \tag{2.4}
\]
where $\omega_k = \partial_k u$ and $\omega_{k,0} = \partial_k \omega_0$ are finite measures.\(^6\)

By approximation, if also $\nabla u(\cdot,t) \in L^1(\mathbb{R}^n)$ for a given $t > 0$, then $\psi = 1_{B_R}$ with $R > 0$ is an admissible weight function.

Proof. The global BV bound is directly from [3, Proposition 3.4]. Let us justify (2.4) with $x_0 = 0$ when $\nabla u_0 \in L^1(\mathbb{R}^n)$ is compactly supported. First, Alibaud’s formula (2.3) holds with integration against weight $\psi$ as in (2.4). This is shown by integrating (2.3) according to the principle $\int_{\mathbb{R}^n} \psi F \, dx = \int_0^\infty \int_{\{\psi > \lambda\}} F \, dx \, d\lambda$. Let $D_k^\varepsilon$ be the different quotient operator $D_k^\varepsilon u = (u(x) - u(x - \varepsilon \tilde{c}_k))/\varepsilon$. Letting $\tilde{u} = u(\cdot - \varepsilon \tilde{c}_k, t)$ in (2.3) with weight $\psi$, and dividing by $\varepsilon$, we have
\[
\int_{\mathbb{R}^n} \psi(x)|D_k^\varepsilon u(x,t)| \, dx \leq \int_{\mathbb{R}^n} \psi(x,t)P(\cdot,t) * |D_k^\varepsilon u_0| \, dx. \tag{2.5}
\]
We have $D_k^\varepsilon u_0 \to \omega_{k,0}$ strongly in $L^1(\mathbb{R}^n)$. Then $P(\cdot,t) * |D_k^\varepsilon u_0| \to P(\cdot,t) * |\omega_{k,0}|$ in $L^1(\mathbb{R}^n)$ also. This implies that the LHS remains bounded as $\varepsilon \to 0^+$. Hence, $\nabla u(\cdot,t)$ actually belongs to $L^1(\mathbb{R}^n)$, and the LHS converges to $\int_{\mathbb{R}^n} \psi(x)|\omega_k(x,t)| \, dx$ as $\varepsilon \to 0^+$. To complete the proof for general $u_0 \in \text{BV}(\mathbb{R}^n)$, we approximate $u_0$ in $L^\infty(\mathbb{R}^n)$ by $u_0(i), i \in \mathbb{N}$, and we approximate $\nabla u_0$ compactly supported, belonging to $L^1(\mathbb{R}^n)$, and satisfying $|\omega_{k,0}| \to |\omega_{k,0}|$ in the sense of measures. Then one may verify, using the Lebesgue dominated convergence theorem and kernel estimates, that $P(\cdot,t) * |\omega_{k,0}(i)| \to P(\cdot,t) * |\omega_{k,0}|$ strongly in $L^1(\mathbb{R}^n)$. The LHS is handled by lower semicontinuity. This completes the proof. \(\square\)

Proposition 2.1 only allows us to propagate $L^1$-based quantities, which then smooth to $L^q$-based quantities, $q \geq 1$, after the initial time:

Let $\ell > 0$ and $\ell \Box(k)$ be the open cube with center at $k$ and side length $\ell$. That is, $\ell \Box(k) = k + (\ell/2, \ell/2)^n$. We write $\Box(k) = 1\Box(k)$. Define
\[
\|f\|_{\ell^q_k L^2(\mathbb{R}^n)} = \left\| \|f\|_{L^2_k(\Box(k))} \right\|_{\ell^q_k(\mathbb{R}^n)}. \tag{2.6}
\]

\(^6\)This is why we require integration against continuous $\psi$ on the LHS.
When \( p = +\infty \), the space \( \ell^\infty_k L^2_k(\mathbb{R}^n) \) is known in the literature as \( L^0_{\text{uloc}}(\mathbb{R}^n) \). We have \( L^p(\mathbb{R}^n) = \ell^p_k L^p_k(\mathbb{R}^n) \) with equality of norms. We also have the obvious embeddings

\[
\| f \|_{L^q_k(\mathbb{R}^n)} \leq \| f \|_{L^p_k(\mathbb{R}^n)}
\]

(2.7)  
when \( q_1 \leq q_2 \), and

\[
\| f \|_{L^p_k(\mathbb{R}^n)} \leq \| f \|_{L^q_k(\mathbb{R}^n)}
\]

(2.8)  
when \( p_1 \leq p_2 \). The short-time and small-distance behavior of these spaces is akin to that of \( L^q(\mathbb{R}^n) \), whereas the large-distance behavior is more closely akin to that of \( L^p(\mathbb{R}^n) \).

We will require the following smoothing estimates when \( q = 1 \) or \( p = q \). However, it is no more effort to prove the general estimates:

**Lemma 2.3** (Smoothing for the heat equation). Let \( p, q_1, q_2 \in [1, +\infty] \) with \( q_1 \leq q_2 \). Let \( w_0 \in \ell^p_k L^{q_1}_k(\mathbb{R}^n) \). Define

\[
w(\cdot, t) = P(\cdot, t) * w_0.
\]

(2.9)  
Then for \( t \leq 1 \),

\[
\| w(\cdot, t) \|_{L^p_k(\mathbb{R}^n)} \lesssim \frac{t^{\frac{n}{2q_2}}}{n} \| w_0 \|_{L^{q_1}_k(\mathbb{R}^n)}.
\]

(2.10)  

**Proof.** Let \( k \in \mathbb{Z}^n \). We decompose \( \mathbb{R}^n \) into near-to-\( k \) and far-from-\( k \) regions:

\[
\| w(x, t) \|_{L^2_k(\mathbb{R}^n)} \leq \left\| \int_{y \in 3\mathbb{Z}(k)} P(x-y, t) |w_0|(y) \, dy \right\|_{L^2_k(\mathbb{R}^n)} = I_1(k)
\]

\[
+ \left\| \int_{y \in \mathbb{R}^n \setminus 3\mathbb{Z}(k)} P(x-y, t) |w_0|(y) \, dy \right\|_{L^2_k(\mathbb{R}^n)} = I_2(k).
\]

(2.11)  
Eventually, we will sum in \( \ell^p_k(\mathbb{Z}^n) \). First, we estimate \( I_1(k) \):

\[
I_1(k) \leq \left\| \sum_{|j| \leq 1} P(\cdot, t) * (1_{\mathbb{Z}(k+j)} |w_0|) \right\|_{L^2_k(\mathbb{R}^n)} \lesssim t^{\frac{n}{2q_2}} \sum_{|j| \leq 1} \| w_0 \|_{L^{q_1}_k(\mathbb{R}^n)}.
\]

(2.12)  
by Young’s convolution inequality. We now sum in \( \ell^p_k(\mathbb{Z}^n) \). By the triangle inequality, and since there are only a finite number of boxes (specifically, \( 3^n \)) in the \( j \) sum, we have

\[
\| I_1(k) \|_{\ell^p_k(\mathbb{Z}^n)} \lesssim t^{\frac{n}{2q_2}} \| w_0 \|_{\ell^q_k(\mathbb{R}^n)}.
\]

(2.13)  
Now we estimate \( I_2(k) \):

\[
I_2(k) \leq \sum_{|j| > 1} \int_{\mathbb{Z}(k-j)} |w_0(y)| \| P(x-y, t) \|_{L^2_k(\mathbb{R}^n)} \, dy
\]

\[
\leq \sum_{|j| > 1} \left[ \sup_{x \in \mathbb{Z}(k)} \sup_{y \in \mathbb{Z}(k-j)} P(x-y, t) \right] \int_{\mathbb{Z}(k-j)} |w_0(y)| \, dy,
\]

(2.14)  
where we used Hölder’s inequality in \( x \) and \( |\mathbb{Z}(k)| = 1 \). When \( x \in \mathbb{Z}(k) \) and \( y \in \mathbb{Z}(k-j) \), we have \( |x-y| \geq |j| - 1 \). Recall that \( P(z, t) \lesssim t/|z|^{n+1} \lesssim 1/|z|^{n+1} \) for \( t \leq 1 \). Hence,

\[
\sup_{x \in \mathbb{Z}(k)} \sup_{y \in \mathbb{Z}(k-j)} P(x-y, t) \lesssim \frac{1}{(|j| - 1)^{n+1}},
\]

(2.15)
2.16

\[ I_2(k) \lesssim \sum_{|j| > 1} \frac{1}{(|j| - 1)^{n+1}} \int_{B(k-j)} |w_0(y)| \, dy. \]  

(2.16)

One may recognize (2.16) as a discrete convolution with a summable-in-\(j\) kernel. Applying \(\|\cdot\|_{L^p_\ell(Z^n)}\), we have

\[ \|I_2(k)\|_{L^p_\ell(Z^n)} \lesssim \sum_{|j| > 1} \frac{1}{(|j| - 1)^{n+1}} \|w_0\|_{L^p_\ell(\mathbb{R}^n)} \lesssim \|w_0\|_{L^p_\ell(\mathbb{R}^n)}, \]  

(2.17)

where we use the trivial embedding (2.7). This completes the proof.

We now justify that the entropy solutions immediately become Hölder continuous and better:

**Proposition 2.4** (Regularity). Let \(u\) be the unique entropy solution of (1.4) with initial data satisfying \(\|u_0\|_{L^\infty(\mathbb{R}^n)} \leq m\). Suppose also that \(f \in C^\infty_{\text{loc}}(\mathbb{R}; \mathbb{R}^n)\). There exists \(\alpha = \alpha(m, n) \in (0, 1)\) such that \(u \in L^\infty_{t, \text{loc}} C^\alpha_{x}(\mathbb{R}^n \times (0, +\infty))\) and

\[ \text{ess sup}_{t > 0} t\|\nabla u(\cdot, t)\|_{L^\infty(\mathbb{R}^n)} + t^2\|\nabla^2 u(\cdot, t)\|_{L^\infty(\mathbb{R}^n)} + t^{2+\alpha}\|\nabla^2 u(\cdot, t)\|_{C^0(\mathbb{R}^n)} \lesssim_m 1. \]  

(2.18)

**Proof.** The estimate

\[ t^\alpha [u(\cdot, t)]_{C^\alpha(\mathbb{R}^n)} \lesssim_m 1 \]  

(2.19)

follows from a direct application of the \(L^\infty_{t, \text{loc}} \to C^\alpha_{x}\) smoothing estimates developed by Silvestre in [39, Theorem 1.1] and [38] for bounded ‘solutions’ of non-local drift-diffusion equations

\[ \partial_t u + b \cdot \nabla u + \Lambda u = g, \]  

(2.20)

where \(b, g\) are bounded. Notably, \(b\) may be large and not necessarily divergence free. In our situation, \(b(x, t) = f'(u(x, t))\) and \(g = 0\). The notion of ‘solution’ is in quotation marks because, here, \(b\) is allowed to be discontinuous, so the notion of viscosity solution may not be directly applicable.\(^7\) To employ Silvestre’s estimates rigorously, one may mollify the initial data, argue at the level of classical solutions, and pass to the limit.

To bootstrap \(C^\alpha_{x} \to C^{1,\alpha}_{x}\), we apply linear estimates due to Silvestre in [40, Theorem 1.1] for the drift-diffusion equation (2.20). It is also possible to proceed more directly, as in Appendix B of [11] or in [18, 17]. Since \(b = f'(u)\) is \(\alpha\)-Hölder continuous in \(\mathbb{R}^n \times (1/2, 1)\) with bounds depending only on \(m\), Theorem 1.1 in [40] gives

\[ \|u\|_{L^\infty_{t} C^{1,\alpha}_{x}(\mathbb{R}^n \times (1/2, 1))} \lesssim_m 1. \]  

(2.22)

Hence, \(b = f'(u)\) satisfies the same bounds. Next, we apply \(\partial_k\), \(1 \leq k \leq n\), to the PDE. This gives

\[ \partial_t \partial_k u + \Lambda \partial_k u + b \cdot \nabla \partial_k u = -\partial_k b \cdot \nabla u. \]  

(2.23)

We regard \(g = -\partial_k b \cdot \nabla u\) as a forcing term belonging to \(L^\infty_{t} C^{\alpha}_{x}(\mathbb{R}^n \times (1/2, 1))\) Finally, Theorem 1.1 in [40] gives

\[ \|\partial_k u\|_{L^\infty_{t} C^{1,\alpha}_{x}(\mathbb{R}^n \times (3/4, 1))} \lesssim_m 1. \]  

(2.24)

Scaling invariance gives the sharp dependence on \(t\). One could also proceed to higher derivatives.\(^8\)

\(^7\)This is discussed in Section 5 of Silvestre’s paper [39], see also Section 3 of [40]. Silvestre mentions that, if viscosity solutions are unavailable, then one may justify the estimates at the level of the vanishing viscosity approximation

\[ \partial_t u^\varepsilon + b^\varepsilon \cdot \nabla u^\varepsilon + \Lambda u^\varepsilon = \varepsilon \Delta u^\varepsilon \]  

(2.21)

with \(\varepsilon \to 0^+\). In principle, this is possible. However, in our setting, the construction in [3] was by an operator splitting method, rather than regularization by \(\varepsilon \Delta u^\varepsilon\), so we argue differently.

\(^8\)
Consider the linear PDE
\[ \partial_t u + \Lambda u + b \cdot \nabla u + cu = 0 \]  
(2.25)
where \( b \in L^\infty_t C^1_{x} (Q_1) \) and \( c \in L^\infty_t C^\alpha_x (Q_1) \) with \( \|b\|_{L^\infty_t C^1_{x}(Q_1)} + \|c\|_{L^\infty_t C^\alpha_x(Q_1)} \leq M \). Here, \( Q_T = \mathbb{R}^n \times (0, T) \).

**Proposition 2.5** (Fundamental solution estimates). There exists a continuous function \( \Gamma = \Gamma(x, t; y, s) \), \( x, y \in \mathbb{R}^n \) and \( 0 \leq s < t \leq 1 \), satisfying the following properties:

- **(Pointwise upper and lower bounds)** For all \( 0 \leq S \leq s < t \leq T \leq 1 \),
  \[ C_0^{-1} P(x, t; y, s) \leq \Gamma(x, t; y, s) \leq C_0 P(x, t; y, s), \]  
  \[ \text{where } C_0 = C_0(T - S, M) > 0 \text{ and } P \text{ is the Poisson kernel.} \]

- **(Maximum principle)** If \( c \equiv 0 \), then \( \int_{\mathbb{R}^n} P(x, t; y, s) \, dy = 1 \).

- **(Representation formula)** If \( w \in L^{t,x}_1(Q_1) \), with \( w \in L^{t,loc}_1 C^1_{x} (Q_1) \) for some \( \beta \in (0, 1) \), is a solution of (2.25) on \( Q_1 \) and \( w(\cdot, t) \rightharpoonup w_0 \) in \( L^\infty(\mathbb{R}^n) \) as \( t \to 0^+ \), then
  \[ w(x, t) = \int_{\mathbb{R}^n} \Gamma(x, t; y, 0)w_0(y) \, dy. \]  
  \[ \text{(2.27)} \]

Solutions given by the representation formula belong to the above class.

Proposition 2.5 was obtained in the paper [43] of Xie and Zhang by E. E. Levi’s parametrix method, except for uniqueness, which we sketch below. In [43], the authors work with more general assumptions: \( b \) in the subcritical space \( L^{t,loc}_1 C^1_{x} (Q_1) \) and \( c \) in a critical Kato space.

**Proof of uniqueness.** Let \( u_0 \in L^2(\mathbb{R}^n) \). Define
\[ v(x, t) = \int_{\mathbb{R}^n} \Gamma(x, t; y, 0)u_0(y) \, dy. \]  
\[ \text{(2.28)} \]
Let \( L = \Lambda + b \cdot \nabla + c \) and \( L^* = \Lambda - b \cdot \nabla + (c - \text{div } b) \). Under our additional regularity assumptions, it is possible to show that \( v \) is a weak solution\(^8\) of the PDE in the sense that
\[ \iint v(x, t)(-\partial_t + L^*) \varphi \, dx \, dt = 0 \]  
\[ \text{(2.29)} \]
for all \( \varphi \in C^\infty_0(\mathbb{R}^n \times (0, 1)) \). Additionally, we have \( v \in L^\infty_t L^2_x(Q_1) \) and \( v \in L^{t,loc}_t H^{1/2}_x(\mathbb{R}^n \times (0, 1]) \), among many other spaces, and \( v(\cdot, t) \rightharpoonup u_0 \) in \( L^2(\mathbb{R}^n) \) as \( t \to 0^+ \). This follows from the pointwise upper bounds of the fundamental solution and its first derivatives (see [43, Theorem 1.1 (v)]) and the convergence result in [43, Theorem 1.1 (ii)]). One may show, via energy estimates, that the above solution is unique in its class and, additionally, belongs to \( C([0, 1]; L^2(\mathbb{R}^n)) \cap L^2_t H^{1/2}_x(Q_1) \).\(^9\)

Assume now that \( u_0 \in L^1 \cap L^\infty(\mathbb{R}^n) \). Then the above solution \( v \) also belongs to \( L^\infty_t L^1_x \cap L^\infty_t L^\infty_x(Q_1) \).

By uniqueness within the energy class, the solution \( v \) may be obtained by vanishing viscosity:
\[ \partial_t u^\varepsilon + \Lambda u^\varepsilon + b \cdot \nabla u^\varepsilon + cu^\varepsilon = \varepsilon \Delta u^\varepsilon. \]  
\[ \text{(2.30)} \]
According to Silvestre’s estimates, we have that \( v \in L^{t,loc}_t C^{1,\alpha}_{x} (\mathbb{R}^n \times (0, 1]) \) for some \( \alpha \in (0, 1) \) with estimates depending only on \( \|u_0\|_{L^\infty(\mathbb{R}^n)} \) and the coefficients. By approximation, we have that when \( u_0 \in L^\infty(\mathbb{R}^n) \), \( v \) satisfies the same a priori estimates.

\(^8\)Due to the quite general conditions in [43], the authors avoided classical solutions and space-time distributional solutions. Instead, they connect the fundamental solution to the PDE via the ‘generator’ notion.

\(^9\)It is important for the energy estimates that \( b \in L^{t,loc}_1 C^{1/2}_{x}(Q_1) \).
We now demonstrate the following uniqueness theorem by duality: If \( u \in L^\infty_{t,\text{loc}} C^{1,\alpha}_x(\mathbb{R}^n \times (0, 1]) \) is a solution of the linear PDE (2.25) with \( u(\cdot, t) \to 0^* \) in \( L^\infty(\mathbb{R}^n) \) as \( t \to 0^+ \), then \( u \equiv 0.\)

Let \( T \in (0, 1) \) and \( \psi_0 \in L^1 \cap L^\infty(\mathbb{R}^n) \). The above analysis demonstrated that there exists \( \psi \in L^\infty_T L^1_x \cap L^\infty_t (Q_T) \) with \( \psi \in L^\infty_{t,\text{loc}} C^{1,\alpha}_x(\mathbb{R}^n \times [0, T)) \) and satisfying the adjoint problem

\[
- \partial_t \psi + L^* \psi = 0
\]

with \( \psi(T) = \psi_0 \). Let \( 0 < t_0 < t_1 < T \) and \( R, \varepsilon > 0 \). Let \( \chi \in C^\infty_0(B_2) \) with \( \chi \equiv 1 \) on \( B_1 \) and \( \chi_R = \chi(x/R) \). Let \( \varphi_{t_0,t_1}^{R,\varepsilon} \) be a mollification of the indicator function \( 1_{(t_0,t_1)} \) at scale \( \varepsilon \ll 1 \). We test (2.25) against \( \psi R \varphi_{t_0,t_1}^{R,\varepsilon} \psi \) and omit \( t_0, t_1, R, \varepsilon \) from the notation as convenient:

\[
\iint_0^\infty \partial_t u + Lu \psi \varphi \, dx \, dt = \iint_0^\infty -\partial_t \psi + L^* \psi \varphi \, dx \, dt \\
+ \iint_0^\infty ( -\partial_t \varphi ) \psi u \varphi + \varphi (-\nabla \chi) u \varphi + \varphi u [\Lambda, \chi] \psi \, dx \, dt.
\]

Upon sending \( \varepsilon \to 0^+ \), we have

\[
\int \chi_R u(x, t_1) \psi(x, t_1) \, dx - \int \chi_R u(x, t_0) \psi(x, t_0) \, dx = \int_{t_0}^{t_1} \int \nabla \chi_R u \psi + u [\Lambda, \chi] \psi \, dx \, dt
\]

for a.e. \( t_0, t_1 \in (0, 1) \). Moreover, (2.33) is valid for all \( t_0, t_1 \in [0, T] \), since \( u : [0, 1] \to L^\infty(\mathbb{R}^n) \) is weak-\(*\) continuous and \( \psi \in C([0, T]; L^2(\mathbb{R}^n)) \). We focus on \( t_0 = 0 \) and \( t_1 = T \). Upon sending \( R \to +\infty \), we have

\[
\iint \psi u [\Lambda, \chi] \psi \, dx \, dt \lesssim R^{-1} R^{n/p} \| u \|_{L^\infty_T L^p_x(Q_1)} \| \psi \|_{L^\infty_T L^p_x(Q_1)} \to 0
\]

when \( p > n \), since \( \| [\Lambda, \chi] \|_{L^p(\mathbb{R}^n)} \to L^p(\mathbb{R}^n) \lesssim_p R^{-1} \) for all \( p \in (1, +\infty) \). The term containing \( b \cdot \nabla \chi_R \) is \( O(R^{-1}) \), since \( b, u \in L^\infty_T L^p_x(Q_1) \) and \( \psi \in L^\infty_T L^1_x(Q_T) \). Hence, (2.33) becomes

\[
\int u(x, T) \psi_0 \, dx = 0,
\]

for all \( T \in (0, 1) \) and \( \psi_0 \in L^1 \cap L^\infty(\mathbb{R}^n) \). Therefore, \( u \equiv 0 \) on \( Q_1 \).

\[ \square \]

3. PROOF OF MAIN RESULTS

**Proof of Theorem 1.1.** Let \( u_0 \in L^\infty \) and \( u \) be the corresponding entropy solution. For each \( v_0 \), we consider the solution \( \widetilde{u} = u + v \) with initial data \( \widetilde{u}_0 = u_0 + v_0 \in L^\infty \).

Let \( m > 0 \) and \( \| u_0 \|_{L^\infty}, \| \widetilde{u}_0 \|_{L^\infty} \leq m \).

We prove continuity with respect to \( v_0 \).

**Proposition 3.1 (Continuity estimate).** Let \( 1 \leq p \leq q \leq +\infty \). If \( v_0 \in L^p(\mathbb{R}^n) \), we have

\[
\| v(\cdot, t) \|_{L^q(\mathbb{R}^n)} \lesssim_{m, p, q} \| v_0 \|_{L^p(\mathbb{R}^n)}.
\]

**Proof of Proposition 3.1.** By scaling invariance, it is enough to demonstrate (3.1) with \( t = 1 \).

**Step 1. Propagation of localization.** First, we demonstrate that, for all \( t \in (0, 1/2] \), we have

\[
\| v(\cdot, t) \|_{L^q_x(\mathbb{R}^n)} \lesssim_{m, p} \| v_0 \|_{L^p_x(\mathbb{R}^n)}.
\]

---

10This argument is modeled off a similar argument in a forthcoming work by the first author and Zachary Bradshaw.
Using Proposition 2.1 (Controlled speed of propagation), we have
\[ \int \nabla(k)(x,t)dx \leq \int B(k,\sqrt{2n/2}) |v(x,t)|dx \]
\[ \leq \int B(k,\sqrt{2n/2+L}) P * |v_0|(x)dx \]
\[ = \sum_{|j| \leq R} \int B(k+j) P * |v_0|(x)dx \]
where \( j \in \mathbb{Z}^n \) and \( R = R(n, L) > 0 \). We apply \( \| \cdot \|_{\mathbb{L}^q_k(\mathbb{Z}^n)} \) to each side of (3.3). By the triangle inequality, and since the sum in \( j \) has only finitely many boxes, we have
\[ \|v(\cdot, t)\|_{\mathbb{L}^q_k(\mathbb{Z}^n)} \lesssim_R \|P * |v_0|\|_{\mathbb{L}^q_k(\mathbb{Z}^n)}. \]
(3.4)

Now Lemma 2.3 (Smoothing for the heat equation) with \( q_1 = q_2 = 1 \) gives (3.2).

**Step 2. Smoothing.** Second, we demonstrate that, for all \( t \in (3/4, 1] \), we have
\[ \|v(\cdot, t)\|_{\mathbb{L}^q_k(\mathbb{Z}^n)} \lesssim_{m, p, q} \|v(\cdot, 1/2)\|_{\mathbb{L}^q_k(\mathbb{Z}^n)}. \]
(3.5)

By Proposition 2.4 (Regularity), \( u \) and \( \tilde{u} \) belong to \( L^\infty \cap C^2_\alpha(x(\mathbb{R}^n \times (1/2, 1))) \) with bounds depending only on \( m \). Hence, \( v = \tilde{u} - u \) belongs to the same space. Let \( w(\cdot, t) = v(\cdot, t + 1/2) \) when \( t \in (0, 1/2] \). Let \( w_0 = v(\cdot, 1/2) \). Then
\[ \partial_t w + \text{div}(g(x,t)w) + \Lambda w = 0, \]
(3.6)
where
\[ g(x,t) = \frac{f(\tilde{u}) - f(u)}{\tilde{u} - u} \]
(3.7)
and
\[ \|g\|_{L^\infty C^2_\alpha(\mathbb{R}^n \times (0, 1/2))} \lesssim m 1. \]
(3.8)

Therefore, we may use the representation formula from Proposition 2.5 (Fundamental solution estimates):
\[ w(x,t) = \int_{\mathbb{R}^n} \Gamma(x,t; y, 0) w_0(y) dy. \]
(3.9)

In particular, the pointwise upper bound in Proposition 2.5 gives
\[ |w(x,t)| \lesssim_m \int_{\mathbb{R}^n} P(x-y,t)|w_0|(y) dy. \]
(3.10)

Then Lemma 2.3 (Smoothing for the heat equation) with \( q_1 = 1 \) and \( q_2 = q \), along with the embedding \( \mathbb{L}^q_k L^q(\mathbb{R}^n) \hookrightarrow L^q(\mathbb{R}^n) \), gives (3.5).

Finally, we combine (3.2) and (3.5) to complete the proof.

When \( v_0 \in L^1(\mathbb{R}^n) \), the propagation of localization comes ‘for free’ from the \( L^1 \)-contraction property, which was shown in [3].

**Proof of Theorem 1.1.** Our goal is to demonstrate the \( o_{t \to +\infty}(1) \) improvement over the conclusion of Proposition 3.1 (Continuity estimate) when \( p > 1 \). We approximate \( v_0 \) strongly in \( L^p(\mathbb{R}^n) \) by \( v_0^{(k)} \) belonging to \( L^1 \cap L^\infty(\mathbb{R}^n) \) and satisfying the decay condition (1.5), \( |v_0^{(k)}| \leq |v_0| \), and \( \|v_0^{(k)}\|_{L^\infty(\mathbb{R}^n)} \leq 2m \). Let \( v^{(k)} \) be the solution corresponding to the initial data \( v_0^{(k)} \). The \( o_{t \to +\infty}(1) \)
improvement is obvious for \( v^{(k)} \), which satisfies a faster decay rate because its initial data belongs to \( L^1(\mathbb{R}^n) \). Next, the triangle inequality and Proposition 3.1 yield
\[
\|v(\cdot, t)\|_{L^q(\mathbb{R}^n)} \leq \|v^{(k)}(\cdot, t)\|_{L^q(\mathbb{R}^n)} + \|v(\cdot, t) - v^{(k)}(\cdot, t)\|_{L^q(\mathbb{R}^n)}
\]
\[
\lesssim \frac{t^{\frac{2}{q}} - t^{\frac{2}{q_0}}}{o(t \to +\infty)}(1) + t^{\frac{2}{q} - \frac{2}{q_0}} \|v_0 - v^{(k)}(0)\|_{L^p(\mathbb{R}^n)}. \tag{3.11}
\]
This completes the proof.

**BV convergence and spatial asymptotics.**

**Proof of Theorem 1.2.** Let \( t_k \to +\infty \) with \( t_k \geq 1 \). It will be convenient to work with the rescaled solutions
\[
u^{(k)}(y, s) = u(t_k y, t_k s),
\]
with \( \omega^{(k)} = \partial_y u^{(k)} \). Then
\[
\|
\omega^{(k)}(\cdot, 1) - \omega^{ss}(\cdot, 1)\|_{L^1(\mathbb{R})} = \|
\omega(\cdot, t_k) - \omega^{ss}(\cdot, t_k)\|_{L^1(\mathbb{R})}. \tag{3.13}
\]

By Proposition 2.4 (Regularity) we can bootstrap the decay of \( \|u^{(k)}(\cdot, 1) - u^{ss}(\cdot, 1)\|_{L^\infty(\mathbb{R})} \) given by Theorem 1.1 to get
\[
\|\omega^{(k)}(\cdot, 1) - \omega^{ss}(\cdot, 1)\|_{L^\infty(B(\mathbb{R}))} \to 0 \text{ as } k \to +\infty \tag{3.14}
\]
for all \( R \geq 1 \). Therefore, it suffices to show that there is no mass of \( \omega^{(k)} \) escaping to infinity. Let \( R \geq L + 10 \). By Alibaud’s BV formula (2.4), and covering \( \mathbb{R} \setminus B(R) \) by an appropriate sequence of balls \( B(x_0, 1) \), we have
\[
\int_{\mathbb{R} \setminus B(R)} |\omega^{(k)}(x, 1)| \, dx \lesssim \int_{\mathbb{R} \setminus B(R-L)} P(\cdot, 1) * |\omega^{(k)}_0| \, dx. \tag{3.15}
\]
It is not difficult to show that the quantity on the RHS is \( o_{R \to +\infty}(1) \) uniformly in \( k \).

**Proof of (1.8).** First, we remark that \( u^{ss} \) is monotone because the evolution of \( \omega \) preserves its sign. This is true at the level of entropy solutions, as can be seen from their construction by the splitting argument in Alibaud’s paper.

In the following, we allow the constant \( C \) to depend on \( u_0 = u_0^{ss} \) and \( \tilde{u}_0 \). Let \( a, b \in \mathbb{R} \) represent the limits of \( u_0 \) as \( x \to \mp\infty \).

**Step 1. Asymptotics for smooth approximation \( \tilde{u} \).** Let \( \tilde{u}_0 \in C^\infty(\mathbb{R}) \) with \( \tilde{u}_0 \equiv u_0 \) outside of \( B_1 \). Let \( \tilde{u} \) be the corresponding entropy solution, which may be shown to belong to \( L^\infty C^{2,\alpha}_x(\mathbb{R}^n \times (0, 1)) \) by combining local-in-time well-posedness\(^{11}\) with the estimates in Proposition 2.4 (Regularity).

By Proposition 2.5 (Fundamental solution estimates), we have
\[
\tilde{u}(x, t) - u_0(x) = \int_{\mathbb{R}^n} P(x, t; y, 0)[\tilde{u}_0(y) - u_0(x)] \, dy,
\]
\[
since \int P(x, t; y, 0) \, dy = 1 \text{ when } c \equiv 0. \text{ Let } x \leq -1. \text{ Hence,}
\]
\[
\tilde{u}(x, t) - u_0(x) = \int_{B_1} P(x, t; y, 0)[\tilde{u}_0(y) - u_0(x)] \, dy + (a - b) \int_{y \geq 1} P(x, t; y, 0) \, dy. \tag{3.17}
\]

\(^{11}\)See the expository blog post [42] of Tao on quasilinear well-posedness.
Since \([\bar{u}_0 - u_0(x)]1_{B_1}\) is compactly supported, we have that \(|I_1(x)| \lesssim \langle x \rangle^{-2}\). On the other hand, when \(a \neq b\), we have
\[
C^{-1}\langle x \rangle^{-1} \leq I_2(x)/(a - b) \leq C\langle x \rangle^{-1}.
\] (3.18)
A similar argument holds for \(x \geq 1\). The \(I_2\) term will dominate when \(|x| \geq C\).

**Step 2. Faster decay for the difference \(v\).** Let \(v = u^{ss} - \bar{u}\). We will exploit that \(v_0 = v(\cdot, 1)\) is supported in \(B_1\) to demonstrate
\[
|v(x, 1)| \leq C\langle x \rangle^{-2}.
\] (3.19)
This will complete the proof, since the \(I_2\) term will dominate \(|v|\) when \(|x| \geq C\). We follow the scheme of propagation of localization and smoothing as in the proof of Proposition 3.1. Let \(k \in \mathbb{Z}\) with \(|k| \geq 10\). By Alibaud’s formula and the decay of the Poisson kernel, we have
\[
\int_{\Box(k)} |v(\cdot, 1/2)| \, dx \leq C \int_{B(k, \sqrt{2}/2 + L)} P(\cdot, 1/2) * |v_0| \, dx \leq C\langle k \rangle^{-2}.
\] (3.20)
The difference \(v\) also satisfies this estimate when \(|k| < 10\). Next, we consider \(w(\cdot, t) = v(\cdot, t + 1/2)\) and analyze its representation formula when \(t \in (1/4, 1/2]\):
\[
|w(x, t)| \leq \int_{\mathbb{R}} \Gamma(x, t; y, 0)|w_0(y)| \, dy \leq C \sum_{k \in \mathbb{Z}} \langle k \rangle^{-2} \|\Gamma(x, t; \cdot, 0)\|_{L^\infty_{\nu}(\Box(k))}
\]
\[
\leq C \sum_{k \in \mathbb{Z}} \langle k \rangle^{-2} \langle x - k \rangle^{-2}
\]
\[
\leq C\langle x \rangle^{-2}.
\] (3.21)
The proof is complete. \(\square\)

**Remark 3.2 (Rough \(f\)).** Suppose that \(f\) is locally Lipschitz and \(n \geq 1\). It is possible to show that, for each \(R > 0\), we have
\[
\|u - u^{ss}\|_{L^\infty(B(Rt))} \to 0 \text{ as } t \to +\infty.
\] (3.22)
That is, \(u\) converges to \(u^{ss}\) locally uniformly in self-similar coordinates \(y = x/t, s = \log(t/t_0)\) where \(t_0 > 0\) is a reference time. Indeed, consider any sequence of rescaled solutions \(u^{(k)}\) as above. Since \(u^{(k)}_0 \to u_0^{ss}\) in \(L^1_{uloc}(\mathbb{R}^n)\), Alibaud’s formula (2.3) gives that \(u^{(k)}(\cdot, 1)\) converges in \(L^1_{uloc}(\mathbb{R}^n)\) to \(u^{ss}(\cdot, 1)\). By the \(a priori\) Hölder estimates (2.19) and the Ascoli-Arzelá theorem,\(^{12}\) \(u^{(k)}(\cdot, 1)\) converges in \(L^\infty_{uloc}(\mathbb{R}^n)\), and its limit must be \(u^{ss}(\cdot, 1)\).

If \(n = 1\) and \(u_0 \in BV(\mathbb{R})\), then we may choose \(R = +\infty\) in (3.22), since the \(BV(\mathbb{R})\) norm manages the behavior in \(L^\infty(\mathbb{R} \setminus B_R)\) for \(R \gg 1\) according to Alibaud’s BV formula (2.4). If \(f \in C^{1,\alpha}_{loc}(\mathbb{R})\), then it is possible to upgrade to \(BV(\mathbb{R})\) convergence, since Silvestre’s estimates in [40] allow the solution to be bootstrapped from \(C^{\alpha}_{x}(\mathbb{R})\) to \(C^{1,\alpha}_{x}(\mathbb{R})\).

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\(^{12}\)To justify (2.19) with Lipschitz \(f\), one could mollify \(f\) or apply a parabolic regularization \(\varepsilon \Delta u^\varepsilon\), justify the estimates at the regularized level, and pass to the limit.
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