Effect of density step on stirring properties of a strain flow

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Abstract
The influence of steep density gradient on stirring properties of a strain flow is addressed by considering the problem in which an interface separating two regions with different constant densities is stabilized within a stagnation-point flow. The existence of an analytic solution for the two-dimensional incompressible flow field allows the exact derivation of the velocity gradient tensor and of parameters describing the local flow topology. Stirring properties are affected not only by vorticity production and jump of strain intensity at the interface, but also by rotation of strain principal axes resulting from anisotropy of pressure Hessian. The strain persistence parameter, which measures the respective effects of strain and effective rotation (vorticity plus rotation rate of strain basis), reveals a complex structure. In particular, for large values of the density ratio, it indicates dominating effective rotation in a restricted area past the interface. Information on flow structure derived from the Okubo–Weiss parameter, by contrast, is less detailed. The influence of the density step on stirring properties is assessed by the Lagrangian evolution of the gradient of a passive scalar. Even for a moderate density ratio, alignment of the scalar gradient and growth rate of its norm are deeply altered. Past the interface effective rotation indeed drives the scalar gradient to align with a direction determined by the local strain persistence parameter, away from the compressional strain direction. The jump of strain intensity at the interface, however, opposes the lessening effect of the latter mechanism on the growth rate of the scalar gradient norm and promotes the rise of the gradient.
1. Introduction

In stratified environmental and engineering flows or in reacting flows where local heat release occurs, velocity gradient properties are affected by spatial variations of density. Density gradients may thus indirectly play on stirring properties of flows through the local features of the velocity gradient. The influence of a density gradient upon velocity gradient properties has been tackled in some studies. In their classification of local flow topologies, Chong et al. (1990) considered the case of compressible flows. Hua et al. (1998) have shown analytically how in geostrophic turbulence the topology of stirring undergoes the effect of a stable stratification through the velocity gradient. The interaction between strain, vorticity and density gradient in homogeneous sheared turbulence has been studied numerically by Diamessis and Nomura (2000). In combustion flows, heat release may deeply influence the mixing through scalar dissipation (Chakraborty and Swaminathan 2007, Pantano et al. 2003). This finding thus suggests a possible effect of density variations on the velocity gradient/scalar gradient interaction. In passing, it also questions the relevance of ‘cold-flow analyses’ of small-scale mixing to combustion flows. The small-scale structure of the velocity field in nonpremixed flames has been investigated in the numerical studies of Nomura and Elghobashi (1993) and Boratav et al. (1996, 1998). These authors showed how strain and vorticity properties—in particular, vorticity alignments—are changed by a variable density via dilatational and baroclinic effects. Nomura and Elghobashi (1993) also studied the local flow topology using the classification of Chong et al. (1990) for compressible flows. Boratav et al. (1998) analysed the way in which buoyancy-influenced vorticity alignment indirectly affects the dissipation rate of fuel concentration. In addition, they examined how the properties of the pressure Hessian are changed by the variable density. This is a key point, for the pressure Hessian contributes to the rotation of strain principal axes (Dresselhaus and Tabor 1991, Lapeyre et al. 1999, Nomura and Post 1998) and may thus be indirectly involved in the small-scale mixing process through alignment mechanisms.

The present study is specially focused on the effect of local steep density gradients. The latter are produced, for instance, in reacting flows with significant heat release. For the purpose of our analysis, we consider the flow field of a model problem taken from flame studies in which an interface stabilized in a pure straining velocity field divides the flow into two regions with different densities. The two-dimensional inviscid problem in which each flow region is assumed to be incompressible has an analytic solution (Kim and Matalon 1988). The main advantage of the approach lies in that the flow field is completely known and its properties in terms of strain, vorticity and pressure Hessian, etc can therefore be derived exactly. In particular, parameters relevant to local flow topology and stirring properties such as the Okubo–Weiss parameter or the strain persistence parameter can be expressed analytically. This makes it possible to examine thoroughly the way in which they are influenced by the density step. Finally, the strain persistence parameter is used to track the Lagrangian evolution of the gradient of a passive scalar in terms of orientation and norm growth rate. The effects of the density step on the stirring mechanism, then, are precisely revealed.

The problem under study and the relating flow field are described in section 2. The flow structure derived from the velocity gradient tensor is studied in section 3. In particular, the way in which the level of the density step affects the local flow topology is closely analysed through the behaviour of the strain persistence parameter. In section 4, we show how the local alignment and growth rate of the gradient of a passive scalar and thus stirring properties are influenced by the density step. Conclusions are drawn in section 5.
2. Flow conditions and dynamic field equations

We consider the stationary problem analysed by Kim and Matalon (1988) which consists of a plane premixed flame stabilized in the stagnation-point flow of a body (figure 1). The flame thickness is assumed to be much smaller than its standoff distance. At the flow scale, the flame front may thus be regarded as a discontinuity separating the unburnt from lighter burnt gases. In addition, the flow is considered to be inviscid and the flowfield in each region of constant density ($\rho_u$ and $\rho_b$, respectively, in the unburnt and burnt gases) is described by the two-dimensional incompressible Euler equations. Stability analysis has been carried out by Kim and Matalon (1990) and Jackson and Matalon (1993). In the present study, the stability is ensured according to the criteria derived by Kim and Matalon (1990) for the two-dimensional case, in particular, unity Lewis number is prescribed.

The equations are made dimensionless by using the adiabatic flame speed and the length scale characterizing the standoff distance as velocity and length units; density, temperature and pressure are normalized by their respective values in the unburnt gas. Kim and Matalon (1988) derived the velocity field from the Euler and continuity equations combined with Rankine–Hugoniot relations for the jumps at the interface and boundary conditions for velocity upstream ($x \to \infty$) and at the surface of the body ($x = 0$). The flame is found to stabilize at distance $d$ from the body; for unity Lewis number and plane geometry, $d$ is given by:

$$d = \frac{2\gamma \arctan[(\gamma - 1)^{1/2}]}{\sigma_0(\gamma - 1)^{1/2}}. \quad (1)$$

In the case of nonunity Lewis number, $d$ is given by a transcendental equation that precludes a complete analytic study. Parameters $\gamma = \rho_u/\rho_b$ and $\sigma_0$ are, respectively, the thermal expansion coefficient (or density ratio) and the intensity of the imposed strain. Strain intensity is defined...
as $\sigma = (\sigma_n^2 + \sigma_s^2)^{1/2}$, where $\sigma_n = \partial u / \partial x - \partial v / \partial y$ and $\sigma_s = \partial v / \partial x + \partial u / \partial y$ are the normal and shear components of strain, respectively. In the unburnt region $\sigma = \sigma_0$. The coordinate system is defined in figure 1 and $u$ and $v$ are the velocity components normal and parallel to the interface, respectively.

The velocity and pressure fields are described by the following equations (Kim and Matalon 1988):

For $x > d$:

$$\rho = 1,$$
$$u = -\frac{1}{2} \sigma_0 (x - a),$$
$$v = \frac{1}{2} \sigma_0 y,$$
$$p = p_a - \frac{1}{8} \sigma_0^2 \left[ (x - a)^2 + y^2 \right],$$

where $p_a$ is the pressure at the virtual stagnation point, $(x, y) = (a, 0)$, and constant $a$ is given by

$$a = \left[ 1 - \frac{(\gamma - 1)^{1/2}}{\gamma \arctan [(\gamma - 1)^{1/2}]} \right] d.$$

For $x < d$:

$$\rho = \frac{1}{\gamma},$$
$$u = -\frac{1}{2} \sigma_0 \delta \sin(kx),$$
$$v = \frac{1}{2} \sigma_0 k \delta y \cos(kx),$$
$$p = p_s - \frac{1}{8} \sigma_0^2 \left[ \frac{\delta^2}{\gamma} \sin^2(kx) + y^2 \right],$$

with pressure at the actual stagnation point, $p_s$, given by

$$p_s = p_a - \frac{1}{8} \sigma_0^2 (\gamma - 1)(d - a)^2 \sigma_0^2,$$

and constants $\delta$ and $k$ by

$$\delta = \gamma \left( \frac{\gamma}{\gamma - 1} \right)^{1/2} (d - a)$$

and

$$k = \frac{(\gamma - 1)^{1/2}}{\gamma (d - a)}.$$

The dynamic field is completely defined by (1)–(13) and parameters $\gamma$ and $\sigma_0$. Note that from (1) and (6) $d - a$ only depends on the imposed strain as $d - a = 2/\sigma_0$. In the following, we use the above model for deriving the change in flow structure brought about by the density step. We are more specifically interested in stirring properties resulting from the flow topology determined by the velocity gradient tensor.
3. Influence of the density step on flow structure

The velocity gradient tensor can be derived analytically from (1) to (13). In particular, calculation of its symmetric and antisymmetric parts, namely strain and rotation is straightforward.

The properties of the prescribed strain flow \((x > d)\) are, obviously, those of a pure straining motion with \(\sigma_n = -\sigma_0\) and \(\sigma_s = 0\); the Okubo–Weiss parameter, \(Q = \sigma^2 - \omega^2\), where \(\omega = \partial v/\partial x - \partial u/\partial y\) is the vorticity, amounts to \(Q = \sigma_0^2\). The pressure Hessian, \(H\), is isotropic; from (5):

\[
H_{ij} = \frac{1}{\rho} \frac{\partial^2 p}{\partial x_i \partial x_j} = -\frac{1}{4} \sigma_0^2 \delta_{ij}.
\]

The strain persistence parameter, \(r\), is defined as (Lapeyre et al 1999)

\[
r = \frac{\omega + \Omega}{\sigma},
\]

where \(\Omega\) is the rotation rate of strain principal axes (Lapeyre et al 1999):

\[
\Omega = \frac{1}{\sigma^2} \left[ \sigma_s \left( \frac{d \sigma_n}{dt} \right) - \sigma_n \left( \frac{d \sigma_s}{dt} \right) \right].
\]

Because it includes the acceleration gradient tensor via the strain derivatives, the strain persistence parameter is more general than the Okubo–Weiss parameter. In particular, accounting for the Lagrangian variation of strain allows a better estimate of stirring properties (Hua and Klein 1998, Lapeyre et al 1999). Dominating strain corresponds to \(r^2 < 1\), while effective rotation prevails if \(r^2 > 1\) and balance occurs for \(r^2 = 1\). Analysis of stirring through the behaviour of a passive scalar gradient in terms of orientation and norm based on the latter different regimes has been performed by Lapeyre et al (1999). They assumed \(r\) varying ‘slowly’ along the Lagrangian trajectories, that is, on a timescale much larger than the gradient response timescale. The effect of fluctuations of \(r\) faster than the response of the scalar gradient has been studied by Garcia et al (2005, 2008). In an inviscid incompressible flow, equations for \(\sigma_n\) and \(\sigma_s\) only include the pressure Hessian components (Lapeyre et al 1999):

\[
\frac{d \sigma_n}{dt} = \frac{1}{\rho} \frac{\partial^2 p}{\partial y^2} - \frac{1}{\rho} \frac{\partial^2 p}{\partial x^2},
\]

\[
\frac{d \sigma_s}{dt} = -\frac{2}{\rho} \frac{\partial^2 p}{\partial x \partial y}
\]

and reveal that the rotation rate of strain axes, \(\Omega\), results from the anisotropic part of \(H\). Equations (14) and (16)–(18) thus show that in the imposed straining flow \(\Omega = 0\) and, as a consequence, \(r = 0\) that indicates a perfectly persistent strain. The Okubo–Weiss and strain persistence parameters obviously agree in this simple case. However, depending on the density ratio, they may differ from one another in their diagnoses in the flow region past the interface as shown in the following.

Beyond the interface \((x < d)\) the velocity gradient has a rotational part due to vorticity production at the discontinuity. Strain and vorticity are derived from (8), (9), (12) and (13):

\[
\sigma_n = -\sigma_0 y^{1/2} \cos(kx),
\]

\[
\sigma_s = -\frac{1}{2} \sigma_0 y^{1/2} ky \sin(kx),
\]

\[
\omega = -\frac{1}{2} \sigma_0 y^{1/2} ky \sin(kx).
\]
At the interface, the normal component of strain is conserved, $\sigma_n(d^-) = \sigma_n(d^+) = -\sigma_0$, whereas the shear component undergoes a step from $\sigma_s(d^-) = 0$ to $\sigma_s(d^+) = -\sigma_0 (\gamma - 1)^{1/2} \arctan((\gamma - 1)^{1/2})y/2d$ resulting in a step in strain intensity expressed by

$$\sigma (d^-) = \sigma (d^+) \left[ 1 + (\gamma - 1) \arctan \left[ \frac{y^2}{4d^2} \right] \right]^{1/2}. \quad (22)$$

The Okubo–Weiss parameter depends on both imposed strain and density ratio and varies with $x$ as

$$Q = \sigma_0^2 \gamma \cos^2(kx).$$

Because from (1) and (13) $kx \leq kd = \arctan((\gamma - 1)^{1/2}) < \pi/2$, parameter $Q$ never takes a zero value. It grows with increasing density ratio and is strictly positive, thus indicating a prevailing strain.

The pressure Hessian derived from (7) and (10) (using also (12) and (13)) is anisotropic:

$$\frac{1}{\rho} \frac{\partial^2 p}{\partial x^2} = - \frac{1}{4} \sigma_0^2 \gamma \cos(2kx), \quad (23)$$

$$\frac{1}{\rho} \frac{\partial^2 p}{\partial y^2} = - \frac{1}{4} \sigma_0^2 \gamma, \quad (24)$$

$$\frac{1}{\rho} \frac{\partial^2 p}{\partial x \partial y} = \frac{1}{\rho} \frac{\partial^2 p}{\partial y \partial x} = 0. \quad (25)$$

From (13) and (23), it is clear that anisotropy is caused by the density step in the $x$-direction (which was foreseeable) and that isotropy of the pressure Hessian is retrieved for $\gamma = 1$. The level of anisotropy of $H$ is given by the anisotropy tensor, $A$, defined by

$$A_{ij} = \frac{2H_{ij} - H_{aa} \delta_{ij}}{H_{aa}}. \quad (26)$$

From (23)–(26):

$$A_{11} = - A_{22} = - \tan^2(kx)$$

and

$$A_{12} = A_{21} = 0.$$ 

The norm of tensor $A$, $|A| = \sqrt{2} \tan^2(kx)$, is plotted in figure 2 for different values of $\gamma$. As $\gamma$ is increased, the level of anisotropy grows and the largest anisotropy is restricted to a region closer to the interface. Defining $x_{1/2}^*$ as the location in $x/d$ where $|A|$ reaches half its maximum value, it is straightforward to show that

$$x_{1/2}^* = \frac{\arctan((\gamma - 1)/2)^{1/2}}{\arctan((\gamma - 1)^{1/2})} \frac{y}{d},$$

and, then, $x_{1/2}^*$ tends to unity with increasing $\gamma$. Note that plotting results (possibly normalized by the imposed strain, $\sigma_0$, when needed) in function of $x/d$, $y/d$ or $\sigma_0 t$ (for Lagrangian tracking) allow to get rid of explicit dependence on $\sigma_0$ and to restrict the analysis to the effects of density ratio.
The anisotropy of the pressure Hessian results in rotation of the strain principal axes. Using (16) with (17)–(20) and the components of the pressure Hessian given by (23)–(25):

$$\Omega = \frac{\sigma_0 \gamma^{1/2} ky \sin(kx)\tan^2(kx)}{4 + k^2 y^2 \tan^2(kx)}.$$  \hspace{1cm} (27)

It is worth noticing that from their respective signs the rotation rate of strain basis, \(\Omega\), and vorticity, \(\omega\) (given by (21)), oppose each other.

The strain persistence parameter, then, is derived from (15) together with (19)–(21) and (27):

$$r = -\frac{[4 + (k^2 y^2 - 2) \tan^2(kx)]ky \tan(kx)}{[4 + k^2 y^2 \tan^2(kx)]^{3/2}}.$$  \hspace{1cm} (27)

Function \(r\) depends on the density ratio through \(k\) which is given by (13). It is antisymmetric in \(y\) and tends to \(-1\) from above as \(y\) tends toward \(+\infty\) and to 1 from below as \(y\) tends toward \(-\infty\). A closer analytic examination reveals a rather rich behaviour that depends on the value of the density ratio. For \(\gamma \leq 3\), \(r\) is a monotonically decreasing function as shown in figure 3. For \(3 < \gamma \leq 9\), extrema appear near the interface, in a region defined by \((x/d)_{1} < x/d \leq 1\) (figure 4). Finally, if \(\gamma > 9\), \(r\) has a three-layered structure in \(x; x/d \leq (x/d)_{1}\); \(r\) is monotonic and decreasing; \((x/d)_{1} < x/d \leq (x/d)_{2}\); \(r\) has extrema within the bounds \(-1\) and 1; \((x/d)_{2} < x/d\): the extrema exceed \(-1\) and 1. This behaviour is displayed in figure 5. The threshold values \((x/d)_{1}\) and \((x/d)_{2}\) are given by

$$\left(\frac{x}{d}\right)_{1} = \frac{\arctan \sqrt{2}}{\arctan \left[\left(\gamma - 1\right)^{1/2}\right]}$$
and

$$\left(\frac{x}{d}\right)^2 = \frac{\arctan(2\sqrt{2})}{\arctan\left[(\gamma - 1)^{1/2}\right]}.$$

and, as $\gamma$ is increased, tend to constant values, namely $(2\arctan\sqrt{2})/\pi \simeq 0.608$ and $[2\arctan(2\sqrt{2})]/\pi \simeq 0.784$. Figure 6 shows the two-dimensional field of $r^2$ derived for a
Figure 5. Profiles of strain persistence parameter for density ratio $\gamma = 10$. Solid: $x/d = 0.25$; dashed: $x/d = 0.50$; dashdot: $x/d = 0.75$; dotted: $x/d = 0.90$ and longdash: $x/d = 1.0$.

Figure 6. Field of strain persistence parameter—squared—for density ratio $\gamma = 10$.

large value of the density ratio, $\gamma = 10$. Beyond the interface, strain significantly dominates only in the vicinity of the plane of symmetry and of the slip boundary located at $x/d = 0$.

The evolution of $r$ at large density ratio is explained by the rotation of strain principal axes. The vorticity-to-strain ratio defined as $R = \omega/\sigma$ discarding the rotation rate of strain...
basis, $\Omega$, has a simpler behaviour. From (19) to (21):

$$R = -\frac{ky \tan(kx)}{\left[4 + k^2 y^2 \tan^2(kx)\right]^{1/2}},$$

which shows that $R$ is monotonic and bounded by $-1$ and $1$, no matter what the value of the density ratio. This is in no way surprising, for the criterion based on $R$ only includes vorticity and strain and therefore amounts to the Okubo–Weiss criterion. Figures 7 and 8 display profiles of $R$ for $\gamma = 2$ and 10, respectively. Better still, from (15) and the fact that from (21) and (27) $\omega$ and $\Omega$ have opposed signs, $r$ starts developing a positive (resp. negative) extremum for $y > 0$ (resp. $y < 0$), near the origin, provided that $|\Omega/\omega| > 1$. From the exact expression for $\Omega/\omega$ derived using (21) and (27), it can be checked that $|\Omega/\omega| > 1$ for $\gamma > 3$. The region where $|\Omega/\omega| > 1$ is found to coincide with the one where $r$ displays extrema. It is also found that in the present case, $|r| > 1$ for $|\Omega/\omega| > 4$.

Since in the problem under study the solution given by (3)–(6) and (8)–(13) depends on Lewis number through the distance $d$ (Kim and Matalon 1988), the above threshold values for $\gamma$ and $x/d$ derived in the case of unity Lewis number are not universal. Nevertheless, the analysis leads to the conclusion that past the density interface parameter $r$ indicates dominating strain except when the density ratio is large enough; in this case, an effective rotation prevails near the plane of symmetry, in a region close to the interface.

4. Effect of density ratio on alignment and stirring properties

Stirring and mixing properties of a flow field can be analysed through the evolution of the gradient of an advected passive scalar. In this view, alignment of the scalar gradient with respect to the local strain principal axes is a key mechanism (Brethouwer et al 2003, Pumir 1994, Vedula et al 2001). The increase of the gradient norm brought about by an alignment
with the local compressional direction causes enhancement of mixing through acceleration of molecular diffusion.

Lapeyre et al (1999) have shown that in two-dimensional flows the opposed effects of strain and effective rotation result in the existence of directions that are mostly different from those of strain principal axes. Their analysis is based on the equation in the strain basis for the orientation of the gradient of a nondiffusive passive scalar:

\[
\frac{d\zeta}{d\tau} = r - \cos \zeta. 
\]

In the fixed frame of reference the scalar gradient is defined by vector \( \mathbf{G} = |\mathbf{G}|(\cos \theta, \sin \theta) \) with \( \zeta = 2(\theta + \Phi) \); \( \Phi \) gives the orientation of the strain principal axes through \( \tan(2\Phi) = \sigma_n/\sigma_s \). Note that the rotation rate, \( \Omega \), of the strain principal axes defined by (16) coincides with \( 2d\Phi/dt \). Time \( \tau \) is a strain normalized time:

\[
\tau = \int_0^t \sigma(t')dt',
\]

where \( t \) stands for the Lagrangian time and \( \sigma \) for the strain intensity. Assuming slow variations of \( r \) along Lagrangian trajectories, Lapeyre et al (1999) showed how the local flow topology determines the orientation of the scalar gradient. If strain prevails over effective rotation \( (r^2 < 1) \), the orientation equation (28) has a stable fixed point,

\[
\zeta_{eq}(r) = -\arccos r,
\]

corresponding to an equilibrium orientation. It is only in the special case \( r = 0 \)—i.e. in the pure hyperbolic regime—that the equilibrium orientation, \( \zeta_{eq} \), coincides with the local compressional direction, \( \zeta_c = -\pi/2 \), which corresponds to \( \theta = -\Phi - \pi/4 \) in the fixed frame of reference. When strain and effective rotation balance each other \( (r^2 = 1) \), the equilibrium
Figure 9. Normalized difference between the local compressional direction and the equilibrium orientation of the gradient of a passive scalar for density ratio γ = 6.

orientation is a bisector of strain principal axes. If effective rotation dominates ($r^2 > 1$), the scalar has no equilibrium orientation, but a most probable one coinciding with a bisector of the strain basis. From the numerical simulations of Lapeyre et al (1999) in two-dimensional turbulence, it appears that in strain-dominated regions the scalar gradient statistically aligns better with the local equilibrium orientation than that with the compressional direction. In fact, Garcia et al (2005, 2008) have shown that the scalar gradient aligns with the equilibrium direction rather than with the compressional one provided that its response timescale to $r$ fluctuations is short enough compared with the Lagrangian timescale of $r$.

In the above approach, the equation for the norm of the scalar gradient is (Lapeyre et al 1999)

$$2 \frac{d|G|}{dt} = -\sigma \sin \zeta$$

and obviously defines the local growth rate of the gradient norm as $-\sigma \sin \zeta$. Clearly, the alignment with the compressional direction, $\zeta = \zeta_c = -\pi/2$, corresponds to the maximum growth rate. The growth rate in the unaltered region, $x > d$, obviously amounts to $\sigma_0$.

The local stirring properties of the flow under study can be derived from the field of the strain persistence parameter, $r$. In particular, the local equilibrium orientation is known through (29). Figure 9 displays the field of variable $\Delta\zeta = |2[\zeta_{eq}(r) + \pi/2]/\pi|$ in the case of density ratio $\gamma = 6$. For the latter value of $\gamma$, the whole flow is dominated by strain and $\zeta_{eq}$ is defined everywhere. Variable $\Delta\zeta$ ranges from 0 to 1 and gives the normalized difference between the local equilibrium and compressional directions. It takes the maximum value, $\Delta\zeta = 1$, for $\zeta_{eq}$ corresponding to a bisector of strain principal axes, namely $\zeta_{eq} = 0$ or $\zeta_{eq} = -\pi$ which coincide, respectively, with directions $-\Phi$ and $-\Phi - \pi/2$ in the fixed frame of reference. From figure 9 it is clear that past the interface $\zeta_{eq}$ significantly departs from $\zeta_c$ over most of the flow field except near the plane of symmetry and the slip boundary where the strain significantly prevails over the effective rotation. For $x/d > 1$, pure strain makes the
Theoretical growth rate of the norm of a passive scalar gradient normalized by local strain; density ratio $\gamma = 6$.

Theoretical growth rate of the norm of a passive scalar gradient normalized by the imposed strain; density ratio $\gamma = 6$.

Equilibrium orientation coincide with the compressional direction. According to the analysis of function $r$ we made in section 3, the difference $\Delta \zeta$ obviously tends to its maximum value with increasing $y/d$.

The limitation of stirring properties that would result from the alignment with the local equilibrium orientation instead of the compressional direction is given by the theoretical growth rate, normalized by local strain, $\eta_{\text{eq}}/\sigma = -\sin[\zeta_{\text{eq}}(r)] = \eta^*_{\text{eq}}$ (figure 10). As expected, $\eta^*_{\text{eq}}$ is close to its maximum value near the plane of symmetry and the slip boundary where the
equilibrium and compressional directions almost coincide. However, $\eta_{eq}^{*}$ displays lower values over the rest of the flow field and even tends to 0 for large values of $y/d$. This trend results from $\zeta_{eq}$ drawing near values 0 or $-\pi$ as $r$ tends towards 1 or $-1$, respectively. But the step in strain intensity given by (22) overcomes the effect of possible misalignment of the scalar gradient with respect to the compressional direction. Figure 11 shows the field of theoretical growth rate normalized by the imposed strain, $\eta_{eq}/\sigma_{0} = \sigma_{0}\sin[\zeta_{eq}(r)]/\sigma_{0}$, and reveals the gain in growth rate at the interface caused by the step in strain intensity.

The response of a passive scalar gradient to the variations of strain persistence along the Lagrangian trajectories brings further information. Equation (28) has been solved starting from initial conditions at the interface ($x/d = 1$) for different values of $y/d$, namely $y(0)/d = 1, 2, 5$ and 10 and density ratio $\gamma = 6$. Figure 12 shows the corresponding trajectories (cut at $y/d = 100$). Initial conditions for velocity are given by (8) and (9) with $x = d$. Initially, the value of strain intensity is the imposed strain, $\sigma_{0}$, strain persistence, $r$, is zero and scalar gradient orientation is $\zeta = \zeta_{c} = -\pi/2$.

Figure 13 displays the trend of strain persistence towards zero, a pure hyperbolic value after the sharp change at the interface. The larger $y(0)/d$, the deeper the variation caused by the interface, and the slower the evolution of $r$; at $\sigma_{0}t = 100 |r|$ is close to 0.1 if $y(0)/d = 1$, but is around 0.7 in the case $y(0)/d = 10$. As expected, the local equilibrium orientation displays a similar behaviour (figure 14): for $y(0)/d = 1 \zeta_{eq}$ almost retrieves its initial value, $\zeta_{c}$, at $\sigma_{0}t = 100$; for $y(0)/d = 10 \zeta_{c} - \zeta_{eq}$ is still greater than $0.2\pi$. Now, the local, instantaneous orientation of the scalar gradient coincides with $\zeta_{eq}$ only if its response timescale is short enough as compared with the timescale of $r$ variations (Garcia et al 2005, 2008). Figure 15 shows that after $\sigma_{0}t = 2$ this condition is realized and $\zeta$ varies as $\zeta_{eq}$, no matter the value of $y(0)/d$. The actual growth rate of the scalar gradient norm thus coincides with the theoretical growth rate (figure 11) over most of the field past the interface and, as shown in figure 16, reveals a significant change of stirring properties with increasing $y(0)/d$. However,
\[ \sigma_0 t \]

\[ \zeta_{eq} / \pi \]

\[ \zeta_c / \pi \]

\[ \eta / \sigma_0 \]

Figure 13. Evolution of strain persistence parameter along trajectories starting from \( x/d = 1 \) and \( y/d = 1, 2, 5 \) and 10; density ratio \( \gamma = 6 \).

Figure 14. Evolution of equilibrium orientation along trajectories starting from \( x/d = 1 \) and \( y/d = 1, 2, 5 \) and 10; density ratio \( \gamma = 6 \).

\( \eta / \sigma_0 \) rapidly tends to its asymptotic value, \( \gamma^{1/2} \), whatever the value of \( y(0)/d \), whereas strain persistence (figure 13) and strain (not shown) do not. This can be understood by noticing that \( \eta / \sigma_0 = \sigma (1 - r^2)^{1/2} / \sigma_0 \simeq \gamma^{1/2} \cos kx \) and that \( x \) rapidly tends to 0; for \( \gamma = 6 \), \( x/d \simeq 0.1 \) at \( \sigma_0 t = 2 \).
Figure 15. Departure of scalar gradient orientation to equilibrium orientation along trajectories starting from $x/d = 1$ and $y/d = 1, 2, 5$ and $10$; density ratio $\gamma = 6$.

Figure 16. Evolution of the actual growth rate of scalar gradient norm normalized by the imposed strain along trajectories starting from $x/d = 1$ and $y/d = 1, 2, 5$ and $10$; density ratio $\gamma = 6$.

5. Conclusion

The flow field resulting from stabilization of an infinitely thin flame within a stagnation-point flow has been used for studying analytically the effect of a density step upon the structure and
stirring properties of a straining flow. The analysis rests on the velocity gradient tensor and subsequent parameters indicating the local flow topology.

In addition to vorticity production and rise of strain intensity, the density step causes rotation of strain principal axes through anisotropy of the pressure Hessian. Strain basis rotation and vorticity—the sum of which corresponds to effective rotation—do not combine, but oppose each other. The Okubo–Weiss parameter, which is based only on strain and vorticity, is sensitive to the density step and indicates prevailing strain whatever the value of the density ratio. Because it accounts for the acceleration gradient through the rotation rate of the strain basis, the strain persistence parameter betrays a more subtle behaviour. More specifically, it reveals the existence of a restricted area past the interface where the strain is reduced relatively to the effective rotation as the density ratio is increased. For large values of the density ratio, strain basis rotation overcomes vorticity and the effective rotation prevails over the strain in this region of the flow.

Moreover, alignment properties of the gradient of a passive scalar and relating stirring properties of the flow field can be derived from the strain persistence parameter. For a moderate value of the density ratio, the difference between the equilibrium orientation of the scalar gradient resulting from the opposed effects of strain and effective rotation and the compressional direction is significant over most of the flow field beyond the interface. This misalignment with respect to compressional direction tends to lessen the growth rate of the gradient norm, but is overcome by the rise of strain intensity. Lagrangian analysis shows that past the interface the local, instantaneous orientation of the gradient of a passive scalar rapidly coincides with the equilibrium orientation and confirms the change in gradient growth rate (hence in stirring properties) brought about by the density step.

In this two-dimensional inviscid flow, rotation of the strain principal axes is only caused by anisotropy of the pressure Hessian, whereas in the three-dimensional case it also results from local vorticity. It would therefore be worth defining a three-dimensional flow in which the question could be re-examined. The strain/vorticity interaction also makes the three-dimensional case more complex. Another extension of the work should consist in relaxing the assumption of an infinitely thin density interface by defining a finite length scale of the density gradient. The analytic approach would certainly be harder, but numerical computation could be used for this purpose.

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