NUMERICAL ANALYSIS OF A DISCONTINUOUS GALERKIN METHOD FOR CAHN–HILLIARD–NAVIER–STOKES EQUATIONS

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Abstract. In this paper, we derive a theoretical analysis of an interior penalty discontinuous Galerkin methods for solving the Cahn–Hilliard–Navier–Stokes model problem. We prove unconditional unique solvability of the discrete system, obtain unconditional discrete energy dissipation law, and derive stability bounds with a generalized chemical energy density. Convergence of the method is obtained by proving optimal a priori error estimates. Our analysis of the unique solvability is valid for both symmetric and non-symmetric versions of the discontinuous Galerkin formulation.

Key words. Cahn–Hilliard–Navier–Stokes, interior penalty discontinuous Galerkin method, unique solvability, stability analysis, error analysis

AMS subject classifications. 35G25, 65M60, 65M12, 76D05

1. Introduction. The Cahn–Hilliard–Navier–Stokes system strikes an optimal balance in terms of thermodynamical rigor and computational efficiency for modeling two-component binary flow. The model that belongs to the class of diffusive interphase or phase-field models, attracts much attention in physics, chemistry, biology, and engineering fields. In recent years, driven by the major developments of numerical algorithms and by increased availability of computational resources and capabilities, the direct numerical simulation of Cahn–Hilliard–Navier–Stokes equations has become increasingly popular. The spectrum of applications for this model involves modeling spinodal decomposition [25], transport processes in porous media [13], and wetting phenomenon [1, 15].

This paper is devoted to the numerical analysis of an interior penalty discontinuous Galerkin method for the Cahn–Hilliard–Navier–Stokes equations. We prove, in two and three dimensions, the unconditional unique solvability, obtain an unconditional energy dissipation law, and derive stability bounds with generalized chemical energy density function. Convergence of the method is obtained by proving optimal a priori error estimates for two dimensional domains. The main contributions of this paper are the unique solvability proof and the error estimates. In addition, the technique we use for proving the existence and uniqueness is valid for both symmetric and non-symmetric interior penalty discontinuous Galerkin formulations.

Over the last ten years, the convergence analysis for the Cahn–Hilliard–Navier–Stokes model has been extensively investigated, for numerical schemes based on the continuous finite element method. In [11], continuous $P_2 - P_0$ elements are used for the approximation of the velocity and pressure whereas continuous $P_r$ elements, for $r \geq 1$ are used for the approximation of the chemical potential and order parameter. Convergence of the solution is obtained via a compactness argument. Kay, Styles, and Welford in [24] analyzed semi-discrete and fully discrete finite element schemes in two-dimensions. Under a CFL-like condition, they obtained a priori error estimates for the semi-discrete method and a convergence proof based on a compactness argument for the fully discrete scheme. Diegel, Wang, Wang, and Wise in [8] analyzed a second order in time mixed finite element method, based on Crank–Nicolson method. Continuous $P_r$ elements are used for the

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chemical potential, order parameter and pressure whereas continuous $P_{r+1}$ are used for the velocity with any positive integer $r$. The work contains unconditional energy stability and optimal error estimates. In [21, 5], a projection method is used to handle the Navier–Stokes equations. Han and Wang introduce a second order in time method and show unconditional unique solvability of the algorithm. The work [21] does not contain any theoretical proof of convergence of the solution. Cai and Shen obtain unconditional unique solvability, derive error estimates and show a convergence analysis based on a compactness argument. In [5], both chemical potential and order parameter are approximated by continuous $P_2$ elements and the velocity and pressure are approximated by a stable pair of finite element spaces. In addition of using continuous finite elements in space, all the works mentioned above assume a special form of chemical energy density, namely a double-well potential, also referred to as Ginzburg–Landau potential. The coupling term in the momentum equation of the Navier–Stokes system may take several forms, that yield different numerical methods and impact their analysis. We note that in [11, 8, 5], the coupling term is the product of the chemical potential and the gradient of the order parameter. In the other works [24, 21] as well as in our present work, the coupling term is the product of the order parameter and the gradient of the chemical potential.

To the best of our knowledge, the theoretical analysis for fully discrete interior penalty discontinuous Galerkin scheme of the Cahn–Hilliard–Navier–Stokes system is not yet available in the literature. While there are few works on the numerical analysis of the coupled Cahn–Hilliard and Navier–Stokes equations, the literature on numerical methods for solving the Cahn–Hilliard equation (resp. the Navier–Stokes equations) is abundant. Finite element methods and interior penalty discontinuous Galerkin methods have been employed for each equation separately. We refer the reader to [12, 23, 2, 10] for the error analysis of Cahn–Hilliard equations and to [31, 17, 20, 19, 18] for the Navier–Stokes equations, and the references herein for a non-exhaustive list.

The outline of the paper follows. The mathematical model and related analytical properties are described in section 2. The numerical method and analysis, including the proof of unique solvability, stability analysis, and error analysis are addressed in section 3. Conclusions are given in the last section.

2. Mathematical Model. Let $\Omega \subset \mathbb{R}^d$, where $d = 2$ or $3$, be an open bounded polyhedral domain and $n$ denote the outward normal of $\Omega$. The unknown variables in Cahn–Hilliard–Navier–Stokes equations are the order parameter $c$, the chemical potential $\mu$, the velocity $v$ and the pressure $p$, satisfying:

\begin{align*}
\partial_t c - \Delta \mu + \nabla \cdot (cv) &= 0, & \text{in } (0, T) \times \Omega, \\
\mu &= \Phi'(c) - \kappa \Delta c, & \text{in } (0, T) \times \Omega, \\
\partial_t v + v \cdot \nabla v - \mu_s \Delta v &= -\nabla p - c \nabla \mu, & \text{in } (0, T) \times \Omega, \\
\nabla \cdot v &= 0, & \text{in } (0, T) \times \Omega.
\end{align*}

Equations (2.1a) and (2.1b) represent the mass conservation equations for two components. The parameter $\kappa$ is a positive constant, which is related to the thickness of the interface between the two phases. The function $\Phi$ is a scalar potential function, also called chemical energy density. Equations (2.1c) and (2.1d) are the momentum and incompressibility equations respectively. The parameter $\mu_s$ is the fluid viscosity. For our model problem, the following boundary and initial
conditions are added:

\( \nabla c \cdot n = 0 \), on \((0, T) \times \partial \Omega\), \(\nabla \cdot n \)\(= 0\), on \((0, T) \times \partial \Omega\), \(v = 0\), on \((0, T) \times \partial \Omega\), \(c = c^0\), in \(\{0\} \times \Omega\), \(v = v^0\), in \(\{0\} \times \Omega\).

The pressure \(p\) is uniquely defined up to an additive constant, to close this system, we also assume the mean pressure on \(\Omega\) is zero:

\[ \int_{\Omega} p = 0. \]

**Remark 2.1.** The order parameter \(c\) can either be a volume or a mass fraction of one of the two components \(c_1, c_2\) or the difference between mass fractions. In the former case, for instance \(c = c_1\), from the definition of the fraction it is straightforward to see \(c \in [0, 1]\). In the latter case, for instance \(c = c_1 - c_2\), due to the constraint \(c_1 + c_2 = 1\) we have \(c \in [-1, 1]\).

**Remark 2.2.** Under the assumption of the incompressibility constraint (2.1d), it is possible to consider employing advection operator \(v \cdot \nabla c\) in (2.1a) in nonconservative form instead of \(\nabla \cdot (cv)\) in conservative form. However, for the convenience of proving discrete global mass conservation property, we propose to use the conservative form here.

**Remark 2.3.** The diffusion operator \(-\Delta v\) and the convection operator \(v \cdot \nabla v\) in (2.1c) can be replaced by more general forms of \(-2 \nabla \cdot \xi(v)\) and \(\nabla \cdot (v \otimes v)\), where the deformation tensor \(\xi(v) = \frac{1}{2} (\nabla v + (\nabla v)^T)\). This equivalence directly comes from the identities

\[ 2 \nabla \cdot \xi(v) = \nabla (\nabla \cdot v) + \Delta v, \]
\[ \nabla \cdot (v \otimes v) = v \cdot \nabla v + (\nabla \cdot v)v. \]

Note since these operators are mathematically equivalent, one might consider formulating problems more generally. However, the modifications may lead unexpected behaviors of the velocity field at the outflow boundary in open boundary simulation. The details refer to [22].

**Remark 2.4.** The second term on the right-hand side of (2.1c) expresses the phase introduced force. This term appears differently in different literatures and some authors propose the form \(-\nabla \cdot (\nabla c \otimes \nabla c)\) [27], or \(\mu \nabla c\) [3], or \(-c \nabla \mu\) [9]. It can be shown that the three expressions are equivalent by redefining the pressure \(p\) [11].

In the rest of this section, we briefly summarize some well-known analytical properties of Cahn–Hilliard–Navier–Stokes model.

**Well-posedness.** A weak formulation of the Cahn–Hilliard–Navier–Stokes system (2.1) is proposed as finding the quaternion \((c, \mu, v, p)\), where

\[ c \in L^\infty(0, T; H^1(\Omega)) \cap L^4(0, T; L^\infty(\Omega)), \quad \mu \in L^2(0, T; H^1(\Omega)), \]
\[ v \in L^2(0, T; H^1(\Omega)^d) \cap L^\infty(0, T; L^2(\Omega)^d), \quad p \in L^2(0, T; L^2(\Omega)), \]
\[ \partial_t c \in L^2(0, T; H^{-1}(\Omega)), \quad \partial_t v \in L^2(0, T; H^{-1}(\Omega)^d), \]
\[ \partial_t v \in L^2(0, T; H^{-1}(\Omega)^d), \]

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such that for a.e. \( t \in (0, T) \),

\[
\begin{align*}
(2.2a) & \quad \langle \partial_t c, \chi \rangle + (\nabla \mu, \nabla \chi) - (c v, \nabla \chi) = 0, \quad \forall \chi \in H^1(\Omega), \\
(2.2b) & \quad (\mu, \varphi) - \langle \Phi'(c), \varphi \rangle - \kappa (\nabla c, \nabla \varphi) = 0, \quad \forall \varphi \in H^1(\Omega), \\
(2.2c) & \quad \langle \partial_t v, \theta \rangle + (v \cdot \nabla v, \theta) + \mu_n (\nabla v, \nabla \theta) \\
& \quad - (\nabla \cdot \theta, p) + (c \theta, \nabla \mu) = 0, \quad \forall \theta \in H^1_0(\Omega)^d, \\
(2.2d) & \quad (\nabla \cdot v, \phi) = 0, \quad \forall \phi \in L^2_0(\Omega),
\end{align*}
\]

with initial data

\[
\begin{align*}
c(0) & \in \{ c \in H^2(\Omega) : \nabla c \cdot n = 0, \text{ on } \partial \Omega \}, \\
v(0) & \in \{ v \in H^1_0(\Omega)^d : (\nabla \cdot v, \phi) = 0, \forall \phi \in L^2_0(\Omega) \}.
\end{align*}
\]

The \( L^2 \) inner-product is denoted by \( \langle \cdot, \cdot \rangle \) and the duality pairing by \( \langle \cdot, \cdot \rangle \). Standard notation is used for the Sobolev and Bochner spaces and we recall that \( L^2_0(\Omega) \) denotes the space of \( L^2 \) functions with zero average. The existence of the weak solution to (2.2) follows the argument in [8]. A generalized version of Cahn–Hilliard–Navier–Stokes model, in which the deformation tensor \( \varepsilon(v) \) was employed and the capillary stress tensor related term was expressed as \( -\nabla \cdot (\nabla c \otimes \nabla c) \), was studied in [27]. The authors there show that with periodic boundary conditions the existence of weak solution is guaranteed.

**Mass conservation.** Let \( \bar{c}_0 \) denote the mass average at time \( t_0 \). The solution of the model problem (2.1) enjoys the global mass conservation property [14].

**THEOREM 2.5.** The total amount of the order parameter \( c \) is preserved, i.e., for any \( t \in (0, T) \), we have

\[
\frac{1}{|\Omega|} \int_{\Omega} c = \frac{1}{|\Omega|} \int_{\Omega} c^0 = \bar{c}_0.
\]

**Energy dissipation.** Benefiting from the boundary conditions (2.1e-2.1g), the Cahn–Hilliard–Navier–Stokes model (2.1) is an energy dissipative system. Analysis of a similar model can be found in [30]. Define the total energy as follows

\[
(2.3) \quad F(c, v) = \int_{\Omega} \left( \frac{1}{2} |v|^2 + \Phi(c) + \frac{\kappa}{2} |\nabla c|^2 \right),
\]

Then (2.1) satisfies the following energy dissipation law and for technique details of the proof we refer to [15].

**THEOREM 2.6.** The total energy is non-increasing in time, i.e., \( \text{d}_t F(c, v)(t) \leq 0 \) for any \( t \in (0, T) \). We have the identity

\[
(2.4) \quad \frac{d}{dt} \int_{\Omega} \left( \frac{1}{2} |v|^2 + \Phi(c) + \frac{\kappa}{2} |\nabla c|^2 \right) = - \int_{\Omega} \left( \mu_n \nabla v : \nabla v + \nabla \mu \cdot \nabla v \right) \leq 0.
\]

**Remark 2.7.** The energy dissipation law still holds in case of changing the diffusion term \( -\mu_n \Delta v \) in (2.1c) to \( -2\nabla \cdot (\mu_n \varepsilon(v)) \). The modification of the proof can be easily obtained by using the tensor identity

\[
\nabla \cdot (\mu_n \varepsilon(u)v) = \mu_n \varepsilon(u) : \varepsilon(v) + \nabla \cdot (\mu_n \varepsilon(u)) \cdot v.
\]

Note here, in this general case, \( \mu_n \) is not required to be a constant any more.
Chemical energy density. The chemical energy density $\Phi$ may take several forms. Two popular expressions of $\Phi$ are the Ginzburg–Landau double well potential [28],

\[(2.5a) \quad \Phi(c) = \frac{1}{4}(1 + c)^2(1 - c)^2, \quad c \in [-1, 1], \]

and the logarithmic potential [4],

\[(2.5b) \quad \Phi(c) = \frac{\vartheta}{2} \left[ (1 + c) \log \left( \frac{1 + c}{2} \right) + (1 - c) \log \left( \frac{1 - c}{2} \right) \right] + \frac{\vartheta c}{2}(1 - c^2), \quad c \in [-1, 1]. \]

The parameters $\vartheta$ and $\vartheta_c$ are positive constants.

Convex-concave decomposition. Throughout our analysis, we assume the chemical energy density $\Phi \in C^2$, i.e., $\Phi$ is a two times continuously differentiable function with respect to $c$. Any $C^2$ function can be decomposed into the sum of a convex part and a concave part [33]. We write

\[(2.6) \quad \Phi(c) = \Phi_+(c) + \Phi_-(c), \]

where $\Phi_+$ is a convex function and $\Phi_-$ a concave function. Although the convex-concave splitting for any $C^2$ function always exists, the decomposition is not unique. We show two examples of splitting for the chemical energy density:

i. Convex-concave splitting for the Ginzburg–Landau double well potential:

\[\Phi_+(c) = \frac{1}{4}(1 + c)^4, \quad \Phi_-(c) = -\frac{1}{2}c^2.\]

ii. Convex-concave splitting for the logarithmic potential:

\[\Phi_+(c) = \frac{\vartheta}{2} \left[ (1 + c) \log \left( \frac{1 + c}{2} \right) + (1 - c) \log \left( \frac{1 - c}{2} \right) \right], \quad \Phi_-(c) = \frac{\vartheta c}{2}(1 - c^2).\]

Remark 2.8. The logarithmic potential is also called Flory–Huggins potential. For some choices of the parameters $\vartheta$ and $\vartheta_c$, the minimum of this potential may take negative value. Although the logarithmic potential is only defined on a finite interval, many authors define an extension over $\mathbb{R}$ for convenience in numerical simulations, for instance see [4, 32].

3. Numerical Analysis. In this section, we introduce an interior penalty discontinuous Galerkin method for the Cahn–Hilliard–Navier–Stokes system and analyze their numerical properties. These include uniquely solvability of the scheme, discrete mass conservation, energy dissipation, stability and error bounds. Our results are valid for any general form for the chemical energy density at the exception of the a priori error bounds that are valid in two dimensions and with additional assumptions on the chemical energy density.

3.1. Preliminaries. Domain and triangulation. Let $T_h = \{E_k\}$ be a family of conforming nondegenerate (also called regular) meshes of the domain $\Omega$. The parameter $h$ denotes the maximum element diameter. Let $\Gamma_h$ denote the set of interior faces. For each interior face $e \in \Gamma_h$ shared by elements $E_{k-}$ and $E_{k+}$, we define a unit normal vector $n_e$ that points from $E_{k-}$ into $E_{k+}$. For the face $e$ on boundary $\partial \Omega$, i.e., $e = E_{k-} \cap \partial \Omega$, the normal $n_e$ is taken to be the unit outward vector to $\partial \Omega$. We also denote by $n_E$ the unit normal vector outward to the element $E$. The natural spaces to work with DG methods are the broken Sobolev spaces. For any real number $r$, we introduce

\[H^r(T_h) = \{\omega \in L^2(\Omega) : \forall E \in T_h, \omega|_E \in H^r(E)\}.\]
The average and jump of any scalar quantity $\omega$ is defined for each interior face $e \in \Gamma_h$ by

$$\{\omega\} = \frac{1}{2} \omega|_{E_{k-}} + \frac{1}{2} \omega|_{E_{k+}} , \quad \quad [\omega] = \omega|_{E_{k-}} - \omega|_{E_{k+}} .$$

If $e$ belongs to the boundary $\partial \Omega$, the jump and average of $\omega$ coincide with its trace on $e$. The related definitions of any vector quantity in $H^r(T_h)^d$ are similar [29].

**DG forms.** We introduce the forms

$$a_A : H^2(T_h) \times H^2(T_h)^d \times H^2(T_h) \to \mathbb{R},$$

$$a_C : H^2(T_h)^d \times H^2(T_h)^d \times H^2(T_h)^d \times H^2(T_h)^d \to \mathbb{R},$$

$$a_D : H^2(T_h) \times H^2(T_h) \to \mathbb{R},$$

$$a_e : H^2(T_h)^d \times H^2(T_h)^d \to \mathbb{R},$$

$$b_p : H^1(T_h) \times H^2(T_h)^d \to \mathbb{R},$$

$$b_T : H^2(T_h) \times H^2(T_h) \times H^2(T_h)^d \to \mathbb{R},$$

corresponding to DG discretization of the advection term $\nabla \cdot (cv)$, convection term $v \cdot \nabla v$, elliptic operator $-\Delta c$, diffusion term $-\Delta v$, pressure term $\nabla p$, and interface term $-c\nabla \mu$, respectively.

\begin{equation}
(3.1a) \quad a_A(c,v,\chi) = - \sum_{E \in T_h} \int_E c v \cdot \nabla \chi + \sum_{e \in \Gamma_h} \int_e \{v\} \cdot \{n_e\} [\chi],
\end{equation}

\begin{equation}
(3.1b) \quad a_C(w,v,z,\theta) = \sum_{E \in T_h} \left( \int_E (v \cdot \nabla z) \cdot \theta + \int_{\partial E} w \cdot n_E \left( z^\text{int} - z^\text{ext} \right) \cdot \theta^\text{int} \right)
+ \frac{1}{2} \sum_{E \in T_h} \int_E (\nabla \cdot v) z \cdot \theta - \frac{1}{2} \sum_{e \in \Gamma_h \cup \partial \Omega} \int_e [v \cdot n_e] \{z \cdot \theta\},
\end{equation}

\begin{equation}
(3.1c) \quad a_D(c,\chi) = \sum_{E \in T_h} \int_E \nabla c \cdot \nabla \chi - \sum_{e \in \Gamma_h} \int_e \{\nabla c \cdot n_e\} [\chi],
+ \frac{1}{h} \sum_{e \in \Gamma_h} \int_e [c] [\chi],
\end{equation}

\begin{equation}
(3.1d) \quad a_e(v,\theta) = \sum_{E \in T_h} \int_E \nabla v : \nabla \theta - \sum_{e \in \Gamma_h \cup \partial \Omega} \int_e \{\nabla v \cdot n_e\} : \theta
+ \frac{1}{h} \sum_{e \in \Gamma_h \cup \partial \Omega} \int_e [v] : [\theta],
\end{equation}

\begin{equation}
(3.1e) \quad b_p(p,\theta) = - \sum_{E \in T_h} \int_E p \nabla \cdot \theta + \sum_{e \in \Gamma_h \cup \partial \Omega} \int_e \{p\} [\theta \cdot n_e],
\end{equation}

\begin{equation}
(3.1f) \quad b_T(c,\mu,\theta) = - \sum_{E \in T_h} \int_E c \nabla \mu \cdot \theta + \sum_{e \in \Gamma_h} \int_e \{c\} [\mu] [\theta \cdot n_e].
\end{equation}

In (3.1b), the set $\partial E^\text{ext} = \{ x \in \partial E : \{w\} \cdot n_E < 0 \}$ and the superscript int (resp. ext) refers to the trace of the function on a face of $E$ coming from the interior of $E$ (resp. coming from the exterior of $E$ on that face), in addition, if the face lies on the boundary of the domain, we take the exterior trace to be zero. For more details related to (3.1b), we refer the reader to [18]. The derivation of these DG forms are given in [29]. We recall that since we use symmetric bilinear forms the penalty parameter, $\sigma$, has to be chosen large enough.
Remark 3.1. In order to ensure the discrete unconditional energy dissipation in a convenient manner, here we specially design our numerical discretization to satisfy $a_A(c, ϕ, μ) = b_T(c, ϕ, v)$ for any $c, μ$ and $v$.

3.2. Numerical scheme.

DG scheme. Uniformly partition $[0, T]$ into $N$ subintervals and let $τ$ be the time step length. For any fixed positive integer $q ∈ \mathbb{N}_+$, the set $P_q(E)$ denotes all polynomials of degree at most $q$ on an element $E$. Define the following broken polynomial spaces

$$S_h = \{ ω ∈ L^2(Ω) : ∀E ∈ T_h, ω|_E ∈ P_q(E) \}, \quad M_h = S_h ∩ L^2_0(Ω),$$
$$X_h = \{ θ ∈ L^2(Ω)^d : ∀E ∈ T_h, θ|_E ∈ P_q(E)^d \},$$
$$Q_h = \{ ω ∈ L^2_0(Ω) : ∀E ∈ T_h, ω|_E ∈ P_{q−1}(E) \},$$
$$V_h = \{ θ ∈ X_h : ∀φ ∈ Q_h, b_T(φ, θ) = 0 \}.$$  

We employ the implicit Euler method with Picard’s linearization for temporal discretization. The fully discrete mixed convex-concave splitting DG scheme reads:

for any $1 ≤ n ≤ N$, given $c_h^{n−1} ∈ S_h$ and $v_h^{n−1} ∈ X_h$ find $(c_h^n, μ_h^n, v_h^n, p_h^n) ∈ S_h × S_h × X_h × Q_h$ such that

\begin{align}
(3.2a) \quad (δ_τ c_h^n, χ) + a_D(μ_h^n, χ) + a_A(c_h^{n−1}, v_h^n, χ) &= 0, \quad ∀χ ∈ S_h, \\
(3.2b) \quad (Φ^+_ε(c_h^n) + Φ^−_ε(c_h^{n−1}), ϕ) + κ a_D(c_h^n, ϕ) − (μ_h^n, ϕ) &= 0, \quad ∀ϕ ∈ S_h, \\
(3.2c) \quad (δ_τ v_h^n, θ) + a_C(v_h^{n−1}, v_h^n, v_h^n, θ) + μ_h a_e(v_h^n, θ) &= 0, \quad ∀θ ∈ X_h, \\
&\quad + b_T(p_h^n, θ) − b_T(c_h^{n−1}, μ_h^n, θ) = 0, \quad ∀θ ∈ X_h, \\
&\quad b_T(φ, v_h^n) = 0, \quad ∀φ ∈ Q_h.
\end{align}

Here, $δ_τ$ denotes the backward finite temporal difference operator:

$$δ_τ c_h^n = \frac{c_h^n − c_h^{n−1}}{τ}.$$  

The initial data $c_h^0$ and $v_h^0$ are good approximations of $c^0$ and $v^0$ respectively. For instance, we choose $v_h^0$ as the $L^2$ projection of $v^0$ into $X_h$ and we choose $c_h^0 = P_h c^0$, where $P_h : H^2(T_h) → S_h$ is the elliptic projection operator:

\begin{equation}
(3.3) \quad a_D(P_h c − c, χ) = 0, \quad ∀χ ∈ S_h, \quad \text{with constraint} \quad (P_h c − c, 1) = 0,
\end{equation}

Operator properties. Throughout this paper, the semi-norms $|| · ||_{DG}$ for any scalar quantity $c ∈ H^1(T_h)$ and for any vector quantity $v ∈ H^1(T_h)^d$ are defined as follows, respectively

\begin{align}
&∀ c ∈ H^1(T_h), \quad || c ||^2_{DG} = \sum_{E ∈ T_h} || ∇ c ||^2_{E,L^2(e)} + \frac{σ}{h} \sum_{e ∈ Γ_h} || c ||^2_{E,L^2(τ_e)}, \\
&∀ v ∈ H^1(T_h)^d, \quad || v ||^2_{DG} = \sum_{E ∈ T_h} || ∇ v ||^2_{E,L^2(e)} + \frac{σ}{h} \sum_{e ∈ Γ_h \cup Ω} || |v| ||^2_{E,L^2(τ)}.
\end{align}

Note $|| · ||_{DG}$ is a norm on $H^1(T_h) ∩ L^2_0(Ω)$ and due to the fact that the $d − 1$ dimensional Lebesgue measure of $∂Ω$ is positive, $|| · ||_{DG}$ is a norm on $H^1(T_h)^d$ as well. Furthermore, the spaces $H^1(T_h) ∩ L^2_0(Ω)$ and $H^1(T_h)^d$ equipped with above energy norm $|| · ||_{DG}$ are reflexive Hilbert spaces. For readability, we drop the dimension in the norm notation for vector functions. Let $p_0$ be the exponent of the Sobolev embedding of $H^1(Ω)$ into $L^p(Ω)$ defined by $\frac{1}{p_0} = \frac{1}{2} − \frac{1}{d}$. Then, we have the following result
LEMMA 3.2 (Poincaré’s inequality [16]). For each \( p \leq p_0 \), there exists a constant \( C_p > 0 \) independent of mesh size \( h \) such that
\[
\| \chi - \frac{1}{|\Omega|} \int_{\Omega} \chi \|_{L^p(\Omega)} \leq C_p \| \chi \|_{DG}, \quad \forall \chi \in S_h.
\]
We also have
\[
\| \theta \|_{L^p(\Omega)} \leq C_p \| \theta \|_{DG}, \quad \forall \theta \in X_h.
\]

Many of the DG forms above satisfy important properties for the analysis of our scheme. Below, we recall several well-known results and provide a briefly proof for the boundedness of \( a_A \). We omit the other proofs for the sake of brevity – for details see [6, 19, 29].

LEMMA 3.3 (Boundedness of \( a_A \)). There exists a constant \( C_\gamma > 0 \) independent of mesh size \( h \) such that for all \( c, \chi \) in \( S_h \) and \( v \) in \( X_h \), we have the following inequalities:
\[
|a_A(c, v, \chi)| \leq C_\gamma \| c \|_{L^\infty(\Omega)} \| v \|_{L^2(\Omega)} \| \chi \|_{DG}, \quad \text{and}
\]
\[
|a_A(c, v, \chi)| \leq C_\gamma \left( \| c \|_{DG} + \left| \int_{\Omega} c \right| \right) \| v \|_{DG} \| \chi \|_{DG}.
\]

In particular, for all \( c \) in \( M_h \), \( \chi \) in \( S_h \), and \( v \) in \( X_h \), the second inequality above implies \( |a_A(c, v, \chi)| \leq C_\gamma \| c \|_{DG} \| v \|_{DG} \| \chi \|_{DG} \).

Proof. The first inequality above is a direct result after applying trace inequality and the Cauchy–Schwarz’s inequality. Now let us have a look at the second inequality. Using Hölder’s inequality and Cauchy–Schwarz’s inequality we obtain
\[
\left| \sum_{E \in T_h} \int_E c \cdot \nabla \chi \right| \leq \left( \sum_{E \in T_h} \| c \|_{L^4(E)}^4 \right)^{\frac{1}{4}} \left( \sum_{E \in T_h} \| v \|_{L^4(E)}^4 \right)^{\frac{1}{4}} \left( \sum_{E \in T_h} \| \nabla \chi \|_{L^2(E)}^2 \right)^{\frac{1}{2}}.
\]
Again, using Hölder’s inequality, Cauchy–Schwarz’s inequality, triangular inequality and trace inequality we have
\[
\left| \sum_{e \in \Gamma_h} \int_e \{ c \} \{ v \cdot n_e \} \{ \chi \} \right|
\leq \left( \sum_{e \in \Gamma_h} \| \{ c \} \|_{L^4(e)}^4 \right)^{\frac{1}{4}} \left( \sum_{e \in \Gamma_h} \| \{ v \cdot n_e \} \|_{L^4(e)} \right)^{\frac{1}{4}} \left( \sum_{e \in \Gamma_h} \| \{ \chi \} \|_{L^2(e)}^2 \right)^{\frac{1}{2}}
\leq C \left( \sum_{E \in T_h} \| c \|_{L^4(E)}^4 \right)^{\frac{1}{4}} \left( \sum_{E \in T_h} \| v \|_{L^4(E)} \right)^{\frac{1}{4}} \left( \frac{1}{h} \sum_{e \in \Gamma_h} \| \chi \|_{L^2(e)} \right)^{\frac{1}{2}}.
\]
Thus, combine above bounds, by definition of \( a_A \) and Poincaré’s inequality we conclude our proof.

LEMMA 3.4 (Continuity of \( a_C \)). The form \( a_C \) is linear with respect to its second to fourth arguments and there exists a constant \( C_\nu > 0 \) independent of mesh size \( h \) such that for all \( u_h, v_h, w_h, z_h \) in \( X_h + (H^1_0(\Omega))^d \),
\[
|a_C(z_h, u_h, v_h, w_h)| \leq C_\nu \| u_h \|_{DG} \| v_h \|_{DG} \| w_h \|_{DG}.
\]

LEMMA 3.5 (Bounds of \( a_C \)). There exists a constant \( C \) independent of \( h \) such that for any \( u \) in \((L^\infty(\Omega) \cap W^{1,3}(\Omega))^d \), any \( v_h \) in \( V_h \) and any \( w_h, z_h \) in \( X_h \), the bound holds
\[
|a_C(z_h, v_h, u, w_h)| \leq C \left( \| u \|_{L^\infty(\Omega)} + \| u \|_{W^{1,3}(\Omega)} \right) \| v_h \|_{L^2(\Omega)} \| w_h \|_{DG}.
\]
Lemma 3.6 (Positivity of $a_C$). The form $a_C$ satisfies the positivity property, i.e., for all $v, z$ in $X_h$,

$$a_C(v, v, v, v) = \frac{1}{2} \sum_{E \in T_h} \int_{\partial E} |\{v\} \cdot n_E| \|z^{\text{ext}} - z^{\text{int}}\|^2 \geq 0.$$ 

Lemma 3.7 (Continuity of $a_D$). The bilinear form $a_D$ is continuous on $S_h$ equipped with the energy norm, i.e., there exists a constant $C_\alpha > 0$ independent of mesh size $h$ such that for all $c, \chi$ in $S_h$,

$$|a_D(c, \chi)| \leq C_\alpha \|c\|_{DG} \|\chi\|_{DG}.$$ 

Lemma 3.8 (Coercivity of $a_D$). Assume that $\sigma$ is sufficiently large. Then, there exists a constant $K_\alpha > 0$ independent of mesh size $h$ such that

$$a_D(c, c) \geq K_\alpha \|c\|_{2DG}^2, \quad \forall c \in S_h.$$ 

Lemma 3.9 (Continuity of $a_\varepsilon$). The bilinear form $a_\varepsilon$ is continuous on $X_h$ equipped with the energy norm, i.e., there exists a constant $C_\varepsilon > 0$ independent of mesh size $h$ such that for all $v, \theta$ in $X_h$,

$$|a_\varepsilon(v, \theta)| \leq C_\varepsilon \|v\|_{DG} \|\theta\|_{DG}.$$ 

Lemma 3.10 (Coercivity of $a_\varepsilon$). Assume that $\sigma$ is sufficiently large. Then, there exists a constant $K_\varepsilon > 0$ independent of mesh size $h$ such that

$$a_\varepsilon(v, v) \geq K_\varepsilon \|v\|_{DG}^2, \quad \forall v \in X_h.$$ 

Lemma 3.11 (Inf-sup). There exists a constant $\beta > 0$, independent of mesh size $h$, such that

$$\inf_{\phi \in Q_h} \sup_{\theta \in X_h} \frac{b_P(\phi, \theta)}{\|\phi\|_{L^2(\Omega)} \|\theta\|_{DG}} \geq \beta.$$ 

3.3. Discrete mass conservation.

Theorem 3.12. The DG scheme (3.2) satisfies the discrete global mass conservation property, i.e., for any $1 \leq n \leq N$, we have

$$(c_h^n, 1) = (c_h^0, 1) = (c^0, 1) = (c(t^n), 1).$$

Proof. The proof for the first equality is straightforward and obtained by choosing $\chi = 1$ in (3.2a) and by using $a_D(\mu_h^n, 1) = 0$ and $a_A(c_h^n, v_h^n, 1) = 0$. Furthermore, applying (3.3) and Theorem 2.5, we obtain the second and third equalities. 

Remark 3.13. One interesting property that naturally comes with the primal DG scheme is the conservation of mass on each mesh element. For instance, we fix an element $E$ that belongs to the interior of the domain, i.e., $\partial E \cap \partial \Omega = \emptyset$. It is easy to show the PDE solution of the Cahn–Hilliard–Navier–Stokes system satisfies

$$\frac{d}{dt} \int_E c - \int_{\partial E} \nabla \mu \cdot n_E + \int_{\partial E} c v \cdot n_E = 0.$$ 

The artificial numerical mass can be exactly computed by choosing $\chi = 1$ on $E$ and vanishes
corresponding \( \tilde{q} \) where the initial datum is defined to be \( a \). Since the problem is linear and finite-dimensional, it suffices to show uniqueness. This is easily recovered the discrete pressure \( \tilde{q} \) solution from the auxiliary flow problem: for any \( 1 \leq n \leq N \), given \( \mathbf{v}_h^n \in X_h \) find \( (\tilde{v}_h^n, \tilde{p}_h^n) \in X_h \times Q_h \) such that
\[
\frac{1}{\tau}(\tilde{v}_h^n - \mathbf{v}_h^n, \theta) + a_C(\mathbf{v}_h^{n-1}, \mathbf{v}_h^n, \tilde{v}_h^n, \theta) + \mu_a a_\varepsilon(\tilde{v}_h^n, \theta) + b_P(\tilde{p}_h^n, \theta) = 0, \quad \forall \theta \in X_h, \tag{3.4a}
\]
\[
\frac{1}{\tau}(\tilde{v}_h^n - \mathbf{v}_h^n, \phi) + b_P(\tilde{p}_h^n, \phi) = 0, \quad \forall \phi \in Q_h. \tag{3.4b}
\]

**Lemma 3.14.** There exists a unique solution to the auxiliary flow problem (3.4) for any mesh size \( h \) and time step size \( \tau \).

*Proof.* We first show existence and uniqueness of \( \tilde{v}_h^n \in V_h \) satisfying
\[
\frac{1}{\tau}(\tilde{v}_h^n - \mathbf{v}_h^n, \theta) + a_C(\mathbf{v}_h^{n-1}, \mathbf{v}_h^n, \tilde{v}_h^n, \theta) + \mu_a a_\varepsilon(\tilde{v}_h^n, \theta) = 0, \quad \forall \theta \in V_h.
\]
Since the problem is linear and finite-dimensional, it suffices to show uniqueness. This is easily obtained by using positivity of \( a_C \) and coercivity of \( a_\varepsilon \) (see Lemma 3.6 and Lemma 3.10). To recover the discrete pressure \( \tilde{p}_h^n \in Q_h \), we then use the inf-sup condition of Lemma 3.11. \( \square \)

Owing to the last result, we can construct the following scheme by employing the unique discrete solution from the auxiliary flow problem: for any \( 1 \leq n \leq N \), given \( (y_h^{n-1}, \mathbf{v}_h^{n-1}) \in M_h \times X_h \), and corresponding \( (\tilde{v}_h^n, \tilde{p}_h^n) \) satisfying (3.4), find \( (y_h^n, w_h^n, \tilde{v}_h^n, \tilde{p}_h^n) \in M_h \times M_h \times X_h \times Q_h \) such that
\[
\frac{1}{\tau}(y_h^n - \tilde{y}_h^{n-1}, \tilde{\chi}) + a_P(w_h^n, \tilde{\chi}) + a_\lambda(y_h^n + \tilde{\varepsilon}, \tilde{\chi}) = 0, \quad \forall \tilde{\chi} \in M_h, \tag{3.5a}
\]
\[
(\Phi^*(y_h^n + \tilde{c} \varepsilon) + \Phi^*(y_h^{n-1} + \tilde{c} \varepsilon) - \mathbf{v}_h^{n-1}, \tilde{\phi}) + \kappa a_P(y_h^n, \tilde{\phi}) = 0, \quad \forall \tilde{\phi} \in M_h, \tag{3.5b}
\]
\[
\frac{1}{\tau}(\tilde{v}_h^n, \theta) + a_C(\mathbf{v}_h^{n-1}, \mathbf{v}_h^n, \tilde{v}_h^n, \theta) + \mu_a a_\varepsilon(\tilde{v}_h^n, \theta)
+b_P(\tilde{p}_h^n, \theta) - b_\Lambda(y_h^{n-1} + \tilde{c} \varepsilon, \mathbf{v}_h^n, \theta) = 0, \quad \forall \theta \in X_h, \tag{3.5c}
\]
\[
b_P(\phi, \tilde{v}_h^n) = 0, \quad \forall \phi \in Q_h, \tag{3.5d}
\]
where the initial datum is defined to be \( y_h^0 = \mathcal{P}_h e^0 - \tilde{c} \varepsilon \) and we recall the initial velocity \( v_h^0 \) is the L2 projection of \( v^0 \) onto \( X_h \). We also denote \( y_h^{n-1} \in M_h \) the solution of
\[
(y_h^{n-1}, \tilde{\chi}) - \tau a_\lambda(y_h^{n-1} + \tilde{c} \varepsilon, \tilde{v}_h^n, \tilde{\chi}) = (\tilde{y}_h^{n-1}, \tilde{\chi}), \quad \forall \tilde{\chi} \in M_h.
\]
whose existence and uniqueness are asserted by the Riesz representation theorem. Our next goal is to prove the scheme (3.5) is equivalent to the DG scheme (3.2). Due to the translational invariance of the trilinear form $a_A$ with respect to the third argument and using the same techniques as in [26], we have

**Lemma 3.15.** The unique solvability of the DG scheme (3.2) is equivalent to the unique solvability of the problem: for any $1 \leq n \leq N$, given $(y_h^{n-1}, v_h^{n-1}) \in M_h \times X_h$ find $(y_h^n, w_h^n, v_h^n, p_h^n) \in M_h \times M_h \times X_h \times Q_h$ such that

\[(3.6a) \quad (\delta, y_h^n, \chi) + a_D(w_h^n, \chi) + a_A(y_h^{n-1} + \bar{c}_0, v_h^n, \chi) = 0, \quad \forall \chi \in M_h,\]

\[(3.6b) \quad (\Phi_+ + \epsilon_0) + \Phi_-(y_h^{n-1} + \bar{c}_0, \varphi) + \kappa a_D(y_h^n, \varphi) - (w_h^n, \varphi) = 0, \quad \forall \varphi \in M_h,\]

\[(3.6c) \quad (\delta, v_h^n, \varphi) + a_C(v_h^{n-1}, v_h^n, \varphi) + \mu_a a_E(v_h^n, \varphi) = 0, \quad \forall \varphi \in M_h,\]

\[(3.6d) \quad b_F(p_h^n, \theta) - b_I(y_h^{n-1} + \bar{c}_0, w_h^n, \theta) = 0, \quad \forall \theta \in X_h,\]

\[(3.6e) \quad b_F(\phi, v_h^n) = 0, \quad \forall \phi \in Q_h.\]

**Proof.** It is easy to check the unique solvability of DG scheme (3.2) is equivalent to the unique solvability of the problem: for any $1 \leq n \leq N$, given $y_h^{n-1} \in M_h$ and $v_h^{n-1} \in X_h$ find $(y_h^n, \mu^p_h, v_h^n, p_h^n) \in M_h \times S_h \times X_h \times Q_h$ such that

\[(3.7a) \quad (\delta, y_h^n, \chi) + a_D(\mu_h^n, \chi) + a_A(y_h^{n-1} + \bar{c}_0, v_h^n, \chi) = 0, \quad \forall \chi \in M_h,\]

\[(3.7b) \quad (\Phi_+ + \epsilon_0) + \Phi_-(y_h^{n-1} + \bar{c}_0, \varphi) + \kappa a_D(y_h^n, \varphi) - (\mu_h^n, \varphi) = 0, \quad \forall \varphi \in S_h,\]

\[(3.7c) \quad (\delta, v_h^n, \varphi) + a_C(v_h^{n-1}, v_h^n, \varphi) + \mu_a a_E(v_h^n, \varphi) = 0, \quad \forall \varphi \in M_h,\]

\[(3.7d) \quad b_F(\phi, v_h^n) = 0, \quad \forall \phi \in Q_h.\]

Thus, we only need to prove the unique solvability of (3.6) is equivalent to the unique solvability of (3.7). (Necessity) If (3.7) has a solution $(y_h^n, \mu_h^n, v_h^n, p_h^n)$. Define $w_h^n = \mu_h^n - \frac{1}{\mu^n} \mu_h^n(\mu_h^n, 1)$, then $(y_h^n, w_h^n, v_h^n, p_h^n)$ is a solution of (3.6). If the solution of (3.7) is unique. Assume $(y_h^{n,1}, w_h^{n,1}, v_h^{n,1}, p_h^{n,1})$ and $(y_h^{n,2}, w_h^{n,2}, v_h^{n,2}, p_h^{n,2})$ are two different solutions of (3.6), then

\[
\begin{align*}
(y_h^{n,1}, w_h^{n,1} & + \frac{1}{|I|} (\Phi_+ + \epsilon_0) + \Phi_-(y_h^n + \bar{c}_0) + \Phi_-(y_h^{n-1} + \bar{c}_0, 1), v_h^{n,1}, p_h^{n,1}) \quad \text{and} \\
(y_h^{n,2}, w_h^{n,2} & + \frac{1}{|I|} (\Phi_+ + \epsilon_0) + \Phi_-(y_h^n + \bar{c}_0) + \Phi_-(y_h^{n-1} + \bar{c}_0, 1), v_h^{n,2}, p_h^{n,2})
\end{align*}
\]

are two different solutions of (3.7). By contradiction argument, we know the solution of (3.6) is unique.

(Sufficiency) If (3.6) has a solution $(y_h^n, w_h^n, v_h^n, p_h^n)$. Define $\mu_h^n = w_h^n + \frac{1}{\mu^n} (\Phi_+ + \epsilon_0) + \Phi_-(y_h^n + \bar{c}_0)$, then $(y_h^n, \mu_h^n, v_h^n, p_h^n)$ is a solution of (3.7). If the solution of (3.6) is unique. Assume $(y_h^{n,1}, \mu_h^{n,1}, v_h^{n,1}, p_h^{n,1})$ and $(y_h^{n,2}, \mu_h^{n,2}, v_h^{n,2}, p_h^{n,2})$ are two different solutions of (3.7), then $(y_h^{n,1}, \mu_h^{n,1} - \frac{1}{\mu^n} (\mu_h^{n,1}, 1), v_h^{n,1}, p_h^{n,1})$ and $(y_h^{n,2}, \mu_h^{n,2} - \frac{1}{\mu^n} (\mu_h^{n,2}, 1), v_h^{n,1}, p_h^{n,1})$ are two different solutions of (3.6). This argument is valid, since if $(y_h^n, \mu_h^n, v_h^n, p_h^n)$ is a solution of (3.7), then $(y_h^n, \mu_h^n + C, v_h^n, p_h^n)$ is not a solution of (3.7), here $C$ denotes a nonzero constant.

**Theorem 3.16.** Based on the auxiliary flow problem (3.4), the unique solvability of the DG scheme (3.2) is equivalent to the unique solvability of the problem (3.5).

**Proof.** By Lemma 3.15, we only need to prove the unique solvability of (3.6) is equivalent to the unique solvability of (3.5). From Lemma 3.14, the auxiliary flow problem (3.4) is always
unconditionally unique solvable. Let \((\hat{\psi}_h^n, \hat{p}_h^n)\) be the unique solution of (3.4) and we have:

(Necessity) If (3.5) has a solution \((y_h^n, w_h^n, \hat{v}_h^n, \hat{p}_h^n)\). Then \((y_h^n, w_h^n, \hat{v}_h^n + \bar{v}_h^n, \hat{p}_h^n + \bar{p}_h^n)\) is a solution of (3.6). If the solution of (3.5) is unique. Assume \((y_h^{n,1}, w_h^{n,1}, \hat{v}_h^{n,1}, \hat{p}_h^{n,1})\) and \((y_h^{n,2}, w_h^{n,2}, \hat{v}_h^{n,2}, \hat{p}_h^{n,2})\) are two different solutions of (3.6). Then \((y_h^{n,1}, w_h^{n,1}, \hat{v}_h^{n,1} - \bar{v}_h^n, \hat{p}_h^{n,1} - \bar{p}_h^n)\) and \((y_h^{n,2}, w_h^{n,2}, \hat{v}_h^{n,2} - \bar{v}_h^n, \hat{p}_h^{n,2} + \bar{p}_h^n)\) are two different solutions of (3.5). By contradiction argument, we know the solution of (3.6) is unique.

(Sufficiency) If (3.6) has a solution \((y_h^n, w_h^n, \hat{v}_h^n, \hat{p}_h^n)\). Then \((y_h^n, w_h^n, \hat{v}_h^n - \bar{v}_h^n, \hat{p}_h^n - \bar{p}_h^n)\) is a solution of (3.5). If the solution of (3.6) is unique. Assume \((y_h^{n,1}, w_h^{n,1}, \hat{v}_h^{n,1}, \hat{p}_h^{n,1})\) and \((y_h^{n,2}, w_h^{n,2}, \hat{v}_h^{n,2}, \hat{p}_h^{n,2})\) are two different solutions of (3.5). Then \((y_h^{n,1}, w_h^{n,1}, \hat{v}_h^{n,1} + \bar{v}_h^n, \hat{p}_h^{n,1} + \bar{p}_h^n)\) and \((y_h^{n,2}, w_h^{n,2}, \hat{v}_h^{n,2} + \bar{v}_h^n, \hat{p}_h^{n,2} + \bar{p}_h^n)\) are two different solutions of (3.6). By contradiction argument, we know the solution of (3.5) is unique.

Now we are in the position to prove (3.5) is uniquely solvable. The outline is let us first express \(y_h^n\) and \((v_h^n, p_h^n)\) in terms of \(w_h^n\) by solving (3.5b) and (3.5c–3.5d) respectively, then seek the unique existence of solution \(w_h^n\).

**Lemma 3.17.** For each fixed \(w_h \in M_h\), given \(y_h^{n-1} \in M_h\) and \(\bar{c}_0 \in S_h\), there exists a unique solution \(y_h \in M_h\) satisfying

\[
(\Phi_+(y_h + \bar{c}_0) + \Phi_-'(y_h^{n-1} + \bar{c}_0), \varphi) + \kappa a_D(y_h, \varphi) - (w_h, \varphi) = 0, \quad \forall \varphi \in M_h,
\]

for any mesh size \(h\), time step size \(\tau\), and parameter \(\kappa\).

**Proof.** We first prove the existence of a solution. For each fixed \(w_h \in M_h\), define the mapping \(\mathcal{F} : M_h \to M_h\) by

\[
(\mathcal{F}(y_h), \varphi) = (\Phi_+(y_h + \bar{c}_0) + \Phi_-'(y_h^{n-1} + \bar{c}_0), \varphi) + \kappa a_D(y_h, \varphi) - (w_h, \varphi), \quad \forall y_h, \varphi \in M_h.
\]

The fact that \(\mathcal{F}\) is well defined is guaranteed by the Riesz representation theorem. Taking the Taylor expansion of \(\Phi_+(y_h + \bar{c}_0)\) at \(\bar{c}_0\) to first order, there exists \(\xi_h\) between \(\bar{c}_0\) and \(y_h + \bar{c}_0\), such that

\[
\Phi_+(y_h + \bar{c}_0) = \Phi_+(\bar{c}_0) + \Phi_+'(\xi_h) y_h.
\]

Considering \(y_h \in M_h\), and the fact that \(\Phi_+\), the convex part of \(\Phi\), satisfies \(\Phi_+'' \geq 0\), we obtain the following inequality

\[
(\Phi_+(y_h + \bar{c}_0), y_h) = (\Phi_+(\bar{c}_0), y_h) + (\Phi_+''(\xi_h), y_h^2) = (\Phi_+''(\xi_h), y_h^2) \geq 0.
\]

We next turn to derive a lower bound of \((\mathcal{F}(y_h), y_h)\). Applying Cauchy–Schwarz’s inequality, Young’s inequality, and Poincaré’s inequality, we have

\[
- (\Phi_-'(y_h^{n-1} + \bar{c}_0), y_h) + (w_h, y_h) \\
\leq \|\Phi_-'(y_h^{n-1} + \bar{c}_0)\|_L^2(\Omega)\|y_h\|_L^2(\Omega) + \|w_h\|_L^2(\Omega)\|y_h\|_L^2(\Omega) \\
\leq \frac{C_p^2}{K_\alpha}\|\Phi_-'(y_h^{n-1} + \bar{c}_0)\|_L^2(\Omega) + \frac{K_\alpha K}{4C_p^2}\|y_h\|_L^2(\Omega) + \frac{C_p^2}{K_\alpha}\|w_h\|_L^2(\Omega) + \frac{K_\alpha K}{4C_p^2}\|y_h\|_L^2(\Omega) \\
\leq \frac{K_\alpha K}{2}\|y_h\|_{DG}^2 + \frac{C_p^2}{K_\alpha}(\|\Phi_-'(y_h^{n-1} + \bar{c}_0)\|_L^2(\Omega) + \|w_h\|_L^2(\Omega)).
\]
Combining this result with (3.9) and using the coercivity of \( a_D \), we obtain
\[
(F(y_h), y_h) \geq \frac{K_a\kappa}{2} \| y_h \|^2_{DG} - \frac{C_P^2}{K_a\kappa^2} \left( \| \Phi_-(y_h^{n-1} + \bar{c}_0) \|^2_{L^2(\Omega)} + \| w_h \|^2_{L^2(\Omega)} \right).
\]
Define the sphere \( \Xi \) in \( M_h \) as follows
\[
\Xi = \left\{ y_h \in M_h : \| y_h \|^2_{DG} = \frac{2C_P^2}{K_a\kappa^2} \left( \| \Phi_-(y_h^{n-1} + \bar{c}_0) \|^2_{L^2(\Omega)} + \| w_h \|^2_{L^2(\Omega)} \right) \right\}.
\]
We have \((F(y_h), y_h) \geq 0\) for any \( y_h \in \Xi \). By Brouwer’s fixed point theorem, there exists a function \( y_h \in M_h \) such that \( F(y_h) = 0 \). In particular \((F(y_h), \varphi) = 0\) for all \( \varphi \in M_h \), i.e., the function \( y_h \) is a solution of (3.8). Next, let us prove the solution of (3.8) is unique. Assume \( y_h \in M_h \) and \( \tilde{y}_h \in M_h \) are two solutions of (3.8), then
\[
(\Phi_+(y_h + \bar{c}_0) + \Phi_-'(y_h^{n-1} + \bar{c}_0), \varphi) + \kappa a_D(y_h, \varphi) - (w_h, \varphi) = 0,
\]
\[
(\Phi_+(\tilde{y}_h + \bar{c}_0) + \Phi_-'(y_h^{n-1} + \bar{c}_0), \varphi) + \kappa a_D(\tilde{y}_h, \varphi) - (w_h, \varphi) = 0.
\]
Subtracting above two equations, taking \( \varphi = y_h - \tilde{y}_h \in M_h \), by the coercivity of \( a_D \), we have
\[
K_a\kappa \| y_h - \tilde{y}_h \|^2_{DG} \leq \kappa a_D(y_h - \tilde{y}_h, y_h - \tilde{y}_h) = -(\Phi_+(y_h + \bar{c}_0) - \Phi_+(\tilde{y}_h + \bar{c}_0), y_h - \tilde{y}_h).
\]
Since \( \Phi_+ \) is convex, then \( \Phi_+ \) is non-decreasing, hence we have
\[
(\Phi_+(y_h + \bar{c}_0) - \Phi_+(\tilde{y}_h + \bar{c}_0))(y_h - \tilde{y}_h) \geq 0.
\]
Therefore, we have \( \| y_h - \tilde{y}_h \|^2_{DG} \leq 0 \), which means \( \| y_h - \tilde{y}_h \|^2_{DG} = 0 \). Due to the fact that \( \| \cdot \|_{DG} \) is a norm in \( M_h \), we obtain \( y_h = \tilde{y}_h \), i.e., the solution of (3.8) is unique.

**Lemma 3.18.** For each fixed \( w_h \in M_h \), given \((y_h^{n-1}, \nu_h^{n-1}) \in M_h \times X_h \) and \( \bar{c}_0 \in S_h \), there exists a unique solution \((\nu_h, p_h) \in X_h \times Q_h \) satisfying

\[
\begin{align}
\frac{1}{\tau}(\nu_h, \theta) + a_C(v_h^{n-1}, v_h^{n-1}, \nu_h, \theta) + \mu_s a_s(v_h, \theta) &+ b_\tau(p_h, \theta) - b_\tau(y_h^{n-1} + \bar{c}_0, w_h, \theta) = 0, \quad \forall \theta \in X_h, \\
&b_\tau(\phi, v_h) = 0, \quad \forall \phi \in Q_h,
\end{align}
\]

for any mesh size \( h \), time step size \( \tau \), and parameter \( \mu_s \).

The technique of proving Lemma 3.18 is similar to deal with the unique solvability of the auxiliary flow problem (3.4). With the help of Lemma 3.17 and Lemma 3.18, we can establish the unconditionally unique solvability of our DG scheme (3.2) by invoking the Minty–Browder theorem [7].

**Lemma 3.19.** The scheme (3.5) is uniquely solvable for any mesh size \( h \), time step size \( \tau \), parameter \( \kappa \), and parameter \( \mu_s \).

**Proof.** For any \( w_h \in M_h \), let \( y_h \) and \((\nu_h, p_h) \) be the unique solutions of (3.5b) and (3.5c–3.5d) which are defined in Lemma 3.17 and Lemma 3.18, respectively. Define an operator \( \mathcal{G} : M_h \rightarrow M_h' \) (the dual space of \( M_h \)) as follows

\[
\langle \mathcal{G}(w_h), \chi \rangle = (y_h - y_h^{n-1}, \chi) + \tau a_D(w_h, \chi) + \tau a_A(y_h^{n-1} + \bar{c}_0, \nu_h, \chi), \quad \forall \chi \in M_h.
\]
Let us first check the boundedness of $G$. By triangle inequality, Cauchy–Schwarz’s inequality, Poincaré’s inequality, and the continuity of $a_D$, we have

$$
|G(w_h), \hat{\chi}| \leq \|y_h\|_{L^2(\Omega)} \|\hat{\chi}\|_{L^2(\Omega)} + \|\hat{\chi}\|_{L^2(\Omega)} \|\hat{\chi}\|_{L^2(\Omega)} \\
+ \tau |a_D(w_h, \hat{\chi})| + \tau |a_A(y_h^{n-1} + \bar{c}_0, v_h, \hat{\chi})| \\
\leq C_B \|y_h\|_{DG} \|\hat{\chi}\|_{DG} + C_P \|\hat{\chi}\|_{DG} + \tau |a_A(y_h^{n-1} + \bar{c}_0, v_h, \hat{\chi})|.
$$

For the last term in above inequality, considering $y_h^{n-1} \in M_h$, the boundedness of $a_A$ implies

$$
|a_A(y_h^{n-1} + \bar{c}_0, v_h, \hat{\chi})| \\
\leq C_\gamma \left( \|y_h^{n-1} + \bar{c}_0\|_{DG} + \int_{\Omega} (y_h^{n-1} + \bar{c}_0) \right) \|v_h\|_{DG} \|\hat{\chi}\|_{DG} \\
= C_\gamma \|y_h^{n-1} + \bar{c}_0\|_{DG} + \|\hat{\chi}\|_{DG} \|v_h\|_{DG} \|\hat{\chi}\|_{DG},
$$

which means, for any $\hat{\chi} \in M_h$ with $\|\hat{\chi}\|_{DG} = 1$, we have

$$
|G(w_h), \hat{\chi}| \leq C_\alpha \tau \|w_h\|_{DG} + C_B \|y_h\|_{DG} \\
+ C_P \|\hat{\chi}\|_{DG} + \|\hat{\chi}\|_{DG} \|v_h\|_{DG} + C_P \|y_h^{n-1}\|_{L^2(\Omega)}.
$$

Our next step is to bound $\|y_h\|_{DG}$ and $\|v_h\|_{DG}$ by $\|w_h\|_{DG}$. Since $y_h = y_h(w_h) \in M_h$ is the unique solution of (3.8) which is defined in Lemma 3.17, take $\hat{\varphi} = y_h$ then

$$
(\Phi_+(y_h + \bar{c}_0) + \Phi_-(y_h^{n-1} + \bar{c}_0), y_h) + \kappa a_D(y_h, y_h) - (w_h, y_h) = 0.
$$

Recall the nonnegativity of $(\Phi_+(y_h + \bar{c}_0), y_h)$ in (3.9). By the coercivity of $a_D$, Cauchy–Schwarz’s inequality and Poincaré’s inequality, we have

$$
K_\alpha \|y_h\|_{DG}^2 \leq (\Phi_+(y_h + \bar{c}_0), y_h) + \kappa a_D(y_h, y_h) \\
= (w_h, y_h) - (\Phi_-(y_h^{n-1} + \bar{c}_0), y_h) \\
\leq \|w_h\|_{L^2(\Omega)} \|y_h\|_{L^2(\Omega)} + \|\Phi_-(y_h^{n-1} + \bar{c}_0)\|_{L^2(\Omega)} \|y_h\|_{L^2(\Omega)} \\
\leq C_B \|w_h\|_{DG} \|y_h\|_{DG} + C_P \|\Phi_-(y_h^{n-1} + \bar{c}_0)\|_{L^2(\Omega)} \|y_h\|_{DG}.
$$

Therefore, we obtain the following bound

$$
\|y_h\|_{DG} \leq \frac{C^2_B}{K_\alpha \kappa} \|w_h\|_{DG} + \frac{C_P}{K_\alpha \kappa} \|\Phi_-(y_h^{n-1} + \bar{c}_0)\|_{L^2(\Omega)}.
$$

Since $(v_h, p_h) = (v_h(w_h), p_h(w_h))$ is the unique solution of (3.10) which is defined in Lemma 3.18, take $\theta = v_h$ we have

$$
\frac{1}{\tau} (v_h, v_h) + a_C(v_h^{n-1}, v_h^{n-1}, v_h, v_h) + \mu_a a_\varepsilon(v_h, v_h) - b_\varepsilon (y_h^{n-1} + \bar{c}_0, w_h, v_h) = 0.
$$

Recall the definition of DG forms $a_A$ and $b_\varepsilon$ in (3.1). By the positivity of $a_C$, the coercivity of $a_\varepsilon$, and considering $v_h$ is discrete divergence-free, we obtain

$$
a_A(y_h^{n-1} + \bar{c}_0, v_h, w_h) = b_\varepsilon (y_h^{n-1} + \bar{c}_0, w_h, v_h) \\
= \frac{1}{\tau} (v_h, v_h) + a_C(v_h^{n-1}, v_h^{n-1}, v_h, v_h) + \mu_a a_\varepsilon(v_h, v_h) \\
\geq K_\varepsilon \mu a_{DG} \|v_h\|_{DG}^2 \geq 0.
$$
Taking \( \hat{\chi} = w_h \) in (3.11) and combining the result with (3.15), we obtain the following bound

\[
\lVert w_h \rVert_{DG} \leq \frac{C \gamma}{K \epsilon} (\lVert y^{n-1}_h + \bar{c}_0 \rVert_{DG} + \lVert \Omega \rVert_{DG}) \lVert w_h \rVert_{DG}.
\]

Substituting (3.14) and (3.16) into (3.12), we have

\[
\lVert G(w_h) \rVert_{M'_h} = \sup_{\hat{\chi} \in M_h} |\langle G(w_h), \hat{\chi} \rangle| \\
\leq \left( C_\alpha + \frac{C_p}{K \alpha} \right) \frac{C_\gamma}{K \epsilon} (\lVert y^{n-1}_h + \bar{c}_0 \rVert_{DG} + \lVert \Omega \rVert_{DG}) \lVert w_h \rVert_{DG} + \frac{C_3}{K \alpha} \lVert \Phi^\prime(y^{n-1}_h + \bar{c}_0) \rVert_{L^2(\Omega)} + C_p \lVert \hat{y}_h^{n-1} \rVert_{L^2(\Omega)}.
\]

Due to the fact that \( y^{n-1}_h, \hat{y}_h^{n-1} \in M_h \) and \( \bar{c}_0 \in S_h \) are given quantities, the above inequality shows that the operator \( G \) maps bounded sets in \( M_h \) to bounded sets in \( M'_h \), i.e., we have proved boundedness of the operator. Second, we show the coercivity of \( G \). By Cauchy–Schwarz’s inequality and Poincaré’s inequality, we have

\[
(y^{n-1}_h, w_h) \leq \lVert y^{n-1}_h \rVert_{L^2(\Omega)} \lVert w_h \rVert_{L^2(\Omega)} \leq C_p \lVert y^{n-1}_h \rVert_{L^2(\Omega)} \lVert w_h \rVert_{DG}.
\]

We again use (3.13) and (3.9). By the coercivity of \( a_D \), Cauchy–Schwarz’s inequality, Young’s inequality, and Poincaré’s inequality, we have

\[
-(w_h, y_h) \leq -\langle \Phi^\prime(y^{n-1}_h + \bar{c}_0), y_h \rangle - \alpha a_D(y_h, y_h) \\
\leq \lVert \Phi^\prime(y^{n-1}_h + \bar{c}_0) \rVert_{L^2(\Omega)} \lVert y_h \rVert_{L^2(\Omega)} - K_a \alpha \lVert y_h \rVert_{DG}^2 \\
\leq \frac{C_p}{4 K_a \alpha} \lVert \Phi^\prime(y^{n-1}_h + \bar{c}_0) \rVert_{L^2(\Omega)}^2 - \frac{K_a \alpha}{C_p} \lVert y_h \rVert_{L^2(\Omega)}^2 - K_a \alpha \lVert y_h \rVert_{DG}^2 \\
\leq \frac{C_p}{4 K_a \alpha} \lVert \Phi^\prime(y^{n-1}_h + \bar{c}_0) \rVert_{L^2(\Omega)}^2.
\]

Using the definition of \( G \), the coercivity of \( a_D \), the bounds (3.15), (3.17), and (3.18), we obtain

\[
\langle G(w_h), w_h \rangle = (y_h - \hat{y}_h^{n-1}, w_h) + \tau a_D(w_h, w_h) + \tau a_A(y^{n-1}_h + \bar{c}_0, w_h, w_h) \\
\geq K_\alpha \tau \lVert w_h \rVert_{DG}^2 - C_p \lVert y^{n-1}_h \rVert_{L^2(\Omega)} \lVert w_h \rVert_{DG} - \frac{C_p}{4 K_a \alpha} \lVert \Phi^\prime(y^{n-1}_h + \bar{c}_0) \rVert_{L^2(\Omega)}^2.
\]

Since \( y^{n-1}_h, \hat{y}_h^{n-1} \in M_h \) and \( \bar{c}_0 \in S_h \) are given quantities, it is obvious that

\[
\lim_{\lVert w_h \rVert_{DG} \to +\infty} \frac{\langle G(w_h), w_h \rangle}{\lVert w_h \rVert_{DG}} = +\infty.
\]

Therefore we proved the coercivity of \( G \). Third, let us check the monotonicity of \( G \). For any \( w_h \) and \( s_h \) in \( M_h \), we have

\[
\langle G(w_h) - G(s_h), w_h - s_h \rangle \\
= \langle G(w_h) - G(s_h), s_h \rangle - \langle G(s_h), w_h \rangle + \langle G(s_h), s_h \rangle \\
= (y_h(w_h) - y_h(s_h), w_h - s_h) + \tau a_D(w_h - s_h, w_h - s_h) \\
+ \tau a_A(y^{n-1}_h + \bar{c}_0, v_h(w_h) - v_h(s_h), w_h - s_h).
\]
Due to the coercivity of $a_D$, the second term above is always nonnegative, which means we only need to check the sign of the first and the third terms. From Lemma 3.17, for any $\varphi \in M_h$, we obtain

\[
(w_h, \varphi) = (\Phi'_+(y_h(w_h) + \bar{c}_0) + \Phi'_-(y_h^{n-1} + \bar{c}_0), \varphi) + \kappa a_D(y_h(w_h), \varphi),
\]

\[
(s_h, \varphi) = (\Phi'_+(y_h(s_h) + \bar{c}_0) + \Phi'_-(y_h^{n-1} + \bar{c}_0), \varphi) + \kappa a_D(y_h(s_h), \varphi).
\]

Subtracting the two equations above, for any $\varphi \in M_h$, we have

\[
(w_h - s_h, \varphi) = (\Phi'_+(y_h(w_h) + \bar{c}_0) - \Phi'_+(y_h(s_h) + \bar{c}_0), \varphi) + \kappa a_D(y_h(w_h), \varphi) - y_h(s_h), \varphi).
\]

By Lemma 3.17, we know that $y_h(w_h)$ and $y_h(s_h)$ belong to $M_h$. We may then choose $\varphi = y_h(w_h) - y_h(s_h) \in M_h$ in the equation above. Using the fact that $\Phi'_+$ is non-decreasing and the coercivity of $a_D$, we obtain

\[
\begin{aligned}
(y_h(w_h) - y_h(s_h), w_h - s_h) &= (\Phi'_+(y_h(w_h) + \bar{c}_0) - \Phi'_+(y_h(s_h) + \bar{c}_0), y_h(w_h) - y_h(s_h)) + \kappa a_D(y_h(w_h) - y_h(s_h), \varphi) \\
&\geq K_\alpha \kappa \|y_h(w_h) - y_h(s_h)\|_{D^G}^2 \geq 0.
\end{aligned}
\]

From Lemma 3.18, for any $\theta \in X_h$, we obtain

\[
\begin{aligned}
b_I(y_h^{n-1} + \bar{c}_0, w_h, \theta) &= \frac{1}{\tau}(v_h(w_h), \theta) + a_C(v_h^{n-1}, v_h^{n-1}, v_h(w_h), \theta) \\
&\quad + \mu_s a_\varepsilon(v_h(w_h), \theta) + b_D(p_h(w_h), \theta),
\end{aligned}
\]

\[
\begin{aligned}
b_I(y_h^{n-1} + \bar{c}_0, s_h, \theta) &= \frac{1}{\tau}(v_h(s_h), \theta) + a_C(v_h^{n-1}, v_h^{n-1}, v_h(s_h), \theta) \\
&\quad + \mu_s a_\varepsilon(v_h(s_h), \theta) + b_D(p_h(s_h), \theta).
\end{aligned}
\]

Subtracting the two equations above, for any $\theta \in X_h$, we have

\[
\begin{aligned}
b_I(y_h^{n-1} + \bar{c}_0, w_h - s_h, \theta) &= \frac{1}{\tau}(v_h(w_h) - v_h(s_h), \theta) \\
&\quad + a_C(v_h^{n-1}, v_h^{n-1}, v_h(w_h) - v_h(s_h), \theta) \\
&\quad + \mu_s a_\varepsilon(v_h(w_h) - v_h(s_h), \theta) + b_D(p_h(w_h) - p_h(s_h), \theta).
\end{aligned}
\]

We may then choose $\theta = v_h(w_h) - v_h(s_h) \in X_h$ in the equation above. Using the positivity of $a_C$, the coercivity of $a_\varepsilon$, considering $v_h(w_h)$ and $v_h(s_h)$ are discretely divergence-free, and by Remark 3.1, we obtain

\[
\begin{aligned}
a_A(y_h^{n-1} + \bar{c}_0, v_h(w_h) - v_h(s_h), w_h - s_h) &= b_I(y_h^{n-1} + \bar{c}_0, w_h - s_h, v_h(w_h) - v_h(s_h)) \\
&= \frac{1}{\tau}(v_h(w_h) - v_h(s_h), v_h(w_h) - v_h(s_h)) \\
&\quad + a_C(v_h^{n-1}, v_h^{n-1}, v_h(w_h) - v_h(s_h), v_h(w_h) - v_h(s_h)) \\
&\quad + \mu_s a_\varepsilon(v_h(w_h) - v_h(s_h), v_h(w_h) - v_h(s_h)) \\
&\quad + b_D(p_h(w_h) - p_h(s_h), v_h(w_h) - v_h(s_h)) \\
&\geq K_\epsilon \mu_s \|v_h(w_h) - v_h(s_h)\|_{D^G}^2 \geq 0.
\end{aligned}
\]
Substituting (3.20) and (3.21) into (3.19), considering \( \| \cdot \|_{DG} \) is a norm in \( M_h \), the following inequality is strict whenever \( w_h \neq s_h \), i.e.,

\[
(\mathcal{G}(w_h) - \mathcal{G}(s_h), w_h - s_h) \geq K_\alpha \tau \| w_h - s_h \|^2_{DG} \geq 0.
\]

Thus we establish the strict monotonicity of \( \mathcal{G} \). Finally, let us show the continuity of \( \mathcal{G} \). For any \( \bar{\chi} \in M_h \) with \( \| \bar{\chi} \|_{DG} = 1 \), by triangle inequality, Cauchy–Schwarz’s inequality, the continuity of \( a_D \), the boundedness of \( a_A \), and Poincaré’s inequality, we have

\[
|\langle \mathcal{G}(w_h) - \mathcal{G}(s_h), \bar{\chi} \rangle| \\
\leq (y_h(w_h) - y_h(s_h), \bar{\chi}) + \tau |a_D(w_h - s_h, \bar{\chi})| \\
+ \tau |a_A(y_h^{n-1} + \bar{\epsilon}_0, v_h(w_h) - v_h(s_h), \bar{\chi})| \\
\leq \| y_h(w_h) - y_h(s_h) \|_{L^2(\Omega)} \| \bar{\chi} \|_{L^2(\Omega)} + C_\alpha \tau \| w_h - s_h \|_{DG} \| \bar{\chi} \|_{DG} \\
+ C_\gamma \tau (\| y_h^{n-1} + \bar{\epsilon}_0 \|_{DG} + \| \bar{\epsilon}_0 \|_{DG}) \| v_h(w_h) - v_h(s_h) \|_{DG} \| \bar{\chi} \|_{DG} \\
\leq C_\alpha \tau \| w_h - s_h \|_{DG} + C_p^2 \| y_h(w_h) - y_h(s_h) \|_{DG} \\
+ C_\gamma \tau (\| y_h^{n-1} + \bar{\epsilon}_0 \|_{DG} + \| \bar{\epsilon}_0 \|_{DG}) \| v_h(w_h) - v_h(s_h) \|_{DG}.
\]

We now estimate the second term above. By (3.20), Cauchy–Schwarz’s inequality, and Poincaré’s inequality, we obtain

\[
K_\alpha \kappa \| y_h(w_h) - y_h(s_h) \|^2_{DG} \leq (y_h(w_h) - y_h(s_h), w_h - s_h) \\
\leq \| y_h(w_h) - y_h(s_h) \|_{L^2(\Omega)} \| w_h - s_h \|_{L^2(\Omega)} \\
\leq C_p^2 \| y_h(w_h) - y_h(s_h) \|_{DG} \| w_h - s_h \|_{DG},
\]

which implies the following bound

\[
(3.23) \quad \| y_h(w_h) - y_h(s_h) \|_{DG} \leq \frac{C_p^2}{K_\alpha \kappa} \| w_h - s_h \|_{DG}.
\]

Similarly, by (3.21) and the boundedness of \( a_A \), we have

\[
K_\varepsilon \mu \| v_h(w_h) - v_h(s_h) \|^2_{DG} \leq a_A(y_h^{n-1} + \bar{\epsilon}_0, v_h(w_h) - v_h(s_h), w_h - s_h) \\
\leq C_\gamma (\| y_h^{n-1} + \bar{\epsilon}_0 \|_{DG} + \| \bar{\epsilon}_0 \|_{DG}) \| v_h(w_h) - v_h(s_h) \|_{DG} \| w_h - s_h \|_{DG},
\]

which implies the following bound

\[
(3.24) \quad \| v_h(w_h) - v_h(s_h) \|_{DG} \leq \frac{C_\gamma}{K_\varepsilon \mu} (\| y_h^{n-1} + \bar{\epsilon}_0 \|_{DG} + \| \bar{\epsilon}_0 \|_{DG}) \| w_h - s_h \|_{DG}.
\]

Combining (3.23), (3.24), and (3.22), we obtain

\[
\| \mathcal{G}(w_h) - \mathcal{G}(s_h) \|_{M_h} = \sup_{\forall \bar{\chi} \in M_h, \| \bar{\chi} \|_{DG} = 1} |\langle \mathcal{G}(w_h) - \mathcal{G}(s_h), \bar{\chi} \rangle| \\
\leq \left( C_\alpha \tau + \frac{C_p^2}{K_\alpha \kappa} + \frac{C_\gamma^2}{K_\varepsilon \mu} (\| y_h^{n-1} + \bar{\epsilon}_0 \|_{DG} + \| \bar{\epsilon}_0 \|_{DG})^2 \right) \| w_h - s_h \|_{DG},
\]

which means \( \| \mathcal{G}(w_h) - \mathcal{G}(s_h) \|_{M_h} \) tends to zero whenever \( \| w_h - s_h \|_{DG} \) tends to zero, i.e., we proved the continuity of the operator \( \hat{\mathcal{G}} \). All conditions of the Minty–Browder theorem are satisfied. We conclude that there exists a unique solution \( w_h^n \) such that \( \langle \mathcal{G}(w_h^n), \bar{\chi} \rangle = 0 \) for all \( \bar{\chi} \in M_h \). Recall Lemma 3.17 and Lemma 3.18, this implies that \( (y_h(w_h^n), w_h^n, v_h(w_h^n), p_h(w_h^n)) \) is the unique solution of scheme (3.5).
Therefore we have proved the following result.

**Theorem 3.20.** The DG scheme (3.2) is uniquely solvable for any mesh size $h$, time step size $\tau$, parameter $\kappa$, and parameter $\mu_s$.

**Remark 3.21.** It is easy to check that Theorem 3.20 is valid for non-symmetric version of the discontinuous Galerkin formulation as well.

### 3.5. Stability Analysis

In this section, we show the discrete solution of (3.2) satisfies the energy dissipation property and we derive stability bounds valid for any chemical energy density $\Phi$. Analogously to the energy (2.3) at the continuous level, we define the discrete energy:

$$
(3.25) \quad F_h(c_h, v_h) = \frac{1}{2} (v_h, v_h) + \left( \Phi(c_h), 1 \right) + \frac{\kappa}{2} a_D(c_h, c_h).
$$

The next statement, the discrete energy dissipation law, stems directly from the positivity in Lemma 3.6 and the convex-concave splitting.

**Theorem 3.22.** Let $(c^n_h, \mu^n_h, v^n_h, p^n_h) \in S_h \times S_h \times X_h \times Q_h$ be the unique solution of the DG scheme (3.2). Then for any mesh size $h$, time step size $\tau$, parameter $\kappa$, and parameter $\mu_s$, the discrete energy (3.25) is non-increasing in time.

$$
\forall 1 \leq n \leq N, \quad F_h(c^n_h, v^n_h) \leq F_h(c^{n-1}_h, v^{n-1}_h).
$$

**Proof.** Take $\chi = \mu^n_h$ in (3.2a), $\varphi = \delta_{c^n_h}$ in (3.2b), $\theta = v^n_h$ in (3.2c), and $\phi = -p^n_h$ in (3.2d):

$$
(\delta_{r} c^n_h, \mu^n_h) + a_D(\mu^n_h, \mu^n_h) + a_A(c^n_h, v^n_h, \mu^n_h) = 0,
$$

$$
(\Phi_{+}^\prime(c^n_h) + \Phi_{-}\,\prime(c^{n-1}_h), \delta_{c^n_h}) + \kappa a_D(\mu^n_h, \delta_{c^n_h}) = 0,
$$

$$
(\delta, v^n_h, v^n_h) + a_C(v^n_h, v^{n-1}_h, v^n_h, \mu^n_h) + \mu_s a_\varepsilon(v^n_h, v^n_h)
$$

$$
+b_D(p^n_h, v^n_h) - b_I(c^n_h, \mu^n_h, v^n_h) = 0,
$$

$$
-b_D(p^n_h, v^n_h) = 0.
$$

Adding the equations above, considering Remark 3.1, and benefitting from the positivity of $a_C$, the coercivity of $a_D$ and $a_\varepsilon$, we have

$$
(\delta_{r} v^n_h, v^n_h) + (\Phi_{+}^\prime(c^n_h) + \Phi_{-}\,\prime(c^{n-1}_h), \delta_{c^n_h}) + \kappa a_D(\mu^n_h, \delta_{c^n_h})
$$

$$
= - a_D(\mu^n_h, \mu^n_h) - a_C(v^n_h, v^{n-1}_h, v^n_h, \mu^n_h) - \mu_s a_\varepsilon(v^n_h, v^n_h)
$$

$$
- a_A(c^{n-1}_h, v^n_h, \mu^n_h) + b_I(c^n_h, \mu^n_h, v^n_h)
$$

$$
\leq - a_D(\mu^n_h, \mu^n_h) - \mu_s a_\varepsilon(v^n_h, v^n_h) \leq - K_\alpha \| \mu^n_h \|_D^2 - K_\varepsilon \mu_s \| v^n_h \|_{DG}^2 \leq 0.
$$

For the term $(\Phi_{+}^\prime(c^n_h) + \Phi_{-}\,\prime(c^{n-1}_h), \delta_{c^n_h})$, we utilize Taylor expansions up to the second order. There exist $\xi_h$ and $\eta_h$ between $c^{n-1}_h$ and $c^n_h$ such that

$$
\Phi_+^\prime(c^n_h)(c^n_h - c^{n-1}_h) = \Phi_+^\prime(c^n_h) - \Phi_+^\prime(c^{n-1}_h) = \frac{1}{2} \Phi_+^\prime\prime(\xi_h)(c^n_h - c^{n-1}_h)^2,
$$

$$
\Phi_-^\prime(c^{n-1}_h)(c^n_h - c^{n-1}_h) = \Phi_-^\prime(c^n_h) - \Phi_-^\prime(c^{n-1}_h) = \frac{1}{2} \Phi_-^\prime\prime(\eta_h)(c^n_h - c^{n-1}_h)^2.
$$

Adding above two equations and using the fact that $\Phi_+$ is convex and $\Phi_-$ is concave, we have

$$
(\Phi_+^\prime(c^n_h) + \Phi_-^\prime(c^{n-1}_h), \delta_{c^n_h})
$$

$$
= (\delta, \Phi(c^n_h), 1) + \frac{1}{2\tau} (\Phi_+^\prime\prime(\xi_h), (c^n_h - c^{n-1}_h)^2) - \frac{1}{2\tau} (\Phi_-^\prime\prime(\eta_h), (c^n_h - c^{n-1}_h)^2)
$$

$$
\geq (\delta, \Phi(c^n_h), 1).
$$
For the terms \( \langle \delta \nu^n_h, v_h^n \rangle \) and \( \kappa a_T(c_h^n, \delta \nu^n_h) \), since the inner product and \( a_T \) are both symmetric bilinear forms, we immediately have

\[
\begin{align*}
(\delta \nu^n_h, v_h^n) &\geq \frac{1}{2\tau} (v_h^n, v_h^n) - \frac{1}{2\tau} (v_h^{n-1}, v_h^{n-1}), \\
\kappa a_T(c_h^n, \delta \nu^n_h) &\geq \frac{1}{2\tau} \kappa a_T(c_h^n, c_h^n) - \frac{1}{2\tau} \kappa a_T(c_h^{n-1}, c_h^{n-1}).
\end{align*}
\]

(3.28)

(3.29)

Combine (3.26) – (3.29) together and recall the definition of the discrete energy (3.25), then

\[
0 \geq - K_a \| \mu_h^n \|^2_{DG} - K_\varepsilon \mu_s \| v_h^n \|^2_{DG}
\]

\[
\geq - \frac{1}{2\tau} (v_h^n, v_h^n) - \frac{1}{2\tau} (v_h^{n-1}, v_h^{n-1}) + \frac{1}{\tau} (\phi(c_h^n), 1)
\]

\[
- \frac{1}{\tau} (\phi(c_h^{n-1}), 1) + \frac{\kappa}{2\tau} a_T(c_h^n, c_h^n) - \frac{\kappa}{2\tau} a_T(c_h^{n-1}, c_h^{n-1})
\]

\[
= \frac{1}{\tau} F_h(c_h^n, v_h^n) - \frac{1}{\tau} F_h(c_h^{n-1}, v_h^{n-1}),
\]

which means the discrete energy \( F_h(c_h, v_h) \) is non-increasing in time.

Throughout the paper, the constant \( C \) denotes a generic constant that takes different values at different places and that is independent of \( h \) and \( \tau \). It is reasonable to assume the initial energy \( F_h(c_h^0, v_h^0) \) is finite. The following a priori bounds for the order parameter, chemical potential and velocity are a direct result of the discrete energy dissipation law (Theorem 3.22).

**Theorem 3.23.** Let \( (c_h^n, \mu_h^n, v_h^n, p_h^n) \in S_h \times S_h \times X_h \times Q_h \) be the unique solution of the DG scheme (3.2). Then for any mesh size \( h \), time step size \( \tau \), parameter \( \kappa \), and parameter \( \mu_s \), and for any \( 1 \leq \ell \leq N \) we have

\[
\begin{align*}
0 &\geq - K_a \| \mu_h^n \|^2_{DG} - K_\varepsilon \mu_s \| v_h^n \|^2_{DG}
\]

\[
\geq - \frac{1}{2\tau} (v_h^n, v_h^n) - \frac{1}{2\tau} (v_h^{n-1}, v_h^{n-1}) + \frac{1}{\tau} (\phi(c_h^n), 1)
\]

\[
- \frac{1}{\tau} (\phi(c_h^{n-1}), 1) + \frac{\kappa}{2\tau} a_T(c_h^n, c_h^n) - \frac{\kappa}{2\tau} a_T(c_h^{n-1}, c_h^{n-1})
\]

\[
= \frac{1}{\tau} F_h(c_h^n, v_h^n) - \frac{1}{\tau} F_h(c_h^{n-1}, v_h^{n-1}),
\]

In addition, if the chemical energy density \( \phi \) is bounded from below by a constant (not necessarily positive), as it is the case for the Ginzburg–Landau double well potential or the logarithmic potential in (2.5), then there is a positive constant \( C \) independent of \( h \) and \( \tau \) such that

\[
\begin{align*}
\frac{1}{\tau} \sum_{n=1}^{\ell} \| \mu_h^n \|^2_{DG} + \frac{1}{\tau} \sum_{n=1}^{\ell} \| v_h^n \|^2_{DG} &\leq F_h(c_h^0, v_h^0),
\end{align*}
\]

(3.30)

In addition, if the chemical energy density \( \phi \) is bounded from below by a constant (not necessarily positive), as it is the case for the Ginzburg–Landau double well potential or the logarithmic potential in (2.5), then there is a positive constant \( C \) independent of \( h \) and \( \tau \) such that

\[
\begin{align*}
\max_{1 \leq n \leq \ell} \| c_h^n \|^2_{DG} + \max_{1 \leq n \leq \ell} \| v_h^n \|^2_{L^2(\Omega)} &\leq C,
\end{align*}
\]

(3.31a)

\[
\tau \sum_{n=1}^{\ell} \| \mu_h^n \|^2_{DG} + \tau \sum_{n=1}^{\ell} \| v_h^n \|^2_{DG} &\leq C.
\]

(3.31b)

**Proof.** From the proof of Theorem 3.22, we know that

\[
\tau K_a \| \mu_h^n \|^2_{DG} + \tau K_\varepsilon \mu_s \| v_h^n \|^2_{DG} \leq F_h(c_h^n, v_h^n) - F_h(c_h^0, v_h^0).
\]

For any \( 1 \leq \ell \leq N \), take the summation of \( n \) from 1 to \( \ell \), then

\[
\tau K_a \sum_{n=1}^{\ell} \| \mu_h^n \|^2_{DG} + \tau K_\varepsilon \mu_s \sum_{n=1}^{\ell} \| v_h^n \|^2_{DG} \leq F_h(c_h^0, v_h^0) - F_h(c_h^{\ell}, v_h^{\ell}).
\]
Finally (3.30) is obtained by moving $F_h(c_h^n, v_h^m)$ to the left-hand side and using the coercivity of $a_D$. In case $\Phi$ is bounded from below by a constant, since the parameters $\kappa, \mu, \alpha$ and constants $K, K^\varepsilon$ are all positive, it is straightforward to show (3.31) holds.

3.6. Error analysis. In this section, we derive an optimal error estimate for the fully discrete scheme (3.2) in terms of time and space discretization parameters. We restrict ourselves to two-dimensional domains as we have in this case the following a priori bound.

\[
\|c_h^p\|_{L^\infty(T)} \leq C, \quad \forall 0 \leq n \leq N.
\]

Indeed, this bound is a direct consequence of the stability bound (3.31) and an inverse inequality, if we assume that the chemical energy density is bounded from below by a constant, not necessarily positive. We also assume that the weak solutions are regular enough. More precisely, we have

\[
\begin{align*}
(3.33a) \quad c, \partial_t c & \in L^\infty(0, T; H^s(\Omega)), \quad \partial_t \mu \in L^2(0, T; L^2(\Omega)), \\
(3.33b) \quad \mu & \in L^\infty(0, T; H^{s+1}(\Omega)) \cap L^\infty(0, T; W^{1,4}(\Omega)), \\
(3.33c) \quad v & \in L^\infty(0, T; H^{s+1}(\Omega))^2, \quad \partial_t v \in L^2(0, T; H^q(\Omega))^2, \\
(3.33d) \quad \partial_t \mathbf{v} & \in L^2(0, T; L^2(\Omega)^2), \quad p \in L^\infty(0, T; H^q(\Omega)).
\end{align*}
\]

For simplicity, we denote by $c^n, \mu^n, v^n,$ and $p^n$ the functions $c, \mu, v,$ and $p$ evaluated at $t^n$. With regularities (3.33), it is straightforward to check that, for any $1 \leq n \leq N$, the weak solution $(c, \mu, v, p)$ to model problem (2.1) satisfies

\[
\begin{align*}
(3.34a) \quad (\partial_t c(t^n), \chi) + a_D(\mu^n, \chi) + a_A(c^n, v^n, \chi) & = 0, \quad \forall \chi \in S_h, \\
(3.34b) \quad (\Phi(c^n) + \Phi'(c^n), \varphi) + \kappa a_D(c^n, \varphi) - (\mu^n, \varphi) & = 0, \quad \forall \varphi \in S_h, \\
(3.34c) \quad (\partial_t v(t^n), \theta) + a(c(v^n, v^n, \theta) + \mu a_x(v^n, \theta) + b_D(p^n, \theta) - b_T(c^n, \mu^n, \theta) & = 0, \quad \forall \theta \in X_h, \\
(3.34d) \quad b_D(c, v^n) & = 0, \quad \forall \phi \in Q_h.
\end{align*}
\]

Before starting error analysis, let us briefly review several useful definitions and properties. Let $\Pi_h : L^2(\Omega) \to Q_h$ be the $L^2$ projection operator onto $Q_h$:

\[
(\Pi_h \omega - \omega, \phi) = 0, \quad \forall \phi \in Q_h, \quad \forall \omega \in L^2(\Omega).
\]

Lax–Milgram theorem allows us to define an invertible operator $J : M_h \to M_h$ via the following variational problem: given $\lambda \in M_h$, for any $\phi \in M_h$, find $J(\lambda) \in M_h$ such that

\[
\begin{align*}
(3.35) \quad a_D(\phi, J(\lambda)) & = (\lambda, \phi).
\end{align*}
\]

**Lemma 3.24.** The operator $J$ is linear and the identity (3.35) still holds for any $\phi \in S_h$ and any $\lambda \in M_h$. In addition, there exists a constant $C_1 > 0$ independent of mesh size $h$, such that

\[
|\langle \lambda, \phi \rangle| \leq C_1 \|\phi\|_{DG} \|J(\lambda)\|_{DG}, \quad \forall \phi \in H^1(T_h), \quad \forall \lambda \in M_h.
\]

**Proof.** The linearity of the operator $J$ is easy to check. For any $\phi \in S_h$ and any $\lambda \in M_h$, due to the fact $\phi - \frac{1}{|\Omega|} \int_{\Omega} \phi$ belongs to $M_h$, we have

\[
\begin{align*}
\begin{aligned}
\quad a_D(\phi, J(\lambda)) & = a_D(\phi - \frac{1}{|\Omega|} \int_{\Omega} \phi, J(\lambda)) + a_D\left(\frac{1}{|\Omega|} \int_{\Omega} \phi, J(\lambda)\right) \\
& = \langle \lambda, \phi \rangle - \frac{1}{|\Omega|} \int_{\Omega} \phi = \langle \lambda, \phi \rangle - \langle \lambda, \phi \rangle = \langle \lambda, \phi \rangle.
\end{aligned}
\end{align*}
\]
Let $\tilde{\Pi}_h : H^1(\mathcal{T}_h) \to S_h$ denote the $L^2$ projection operator onto $S_h$. It is easy to show that $\tilde{\Pi}_h$ is stable with respect to the DG norm, i.e., we have the inequality $\|\tilde{\Pi}_h \phi\|_{DG} \leq C \|\phi\|_{DG}$. Therefore, by triangular inequality, the definition of operator $J$, and the continuity of $a_D$, we obtain for any $\lambda$ in $M_h$:

\[(\lambda, \phi) = ((\phi - \tilde{\Pi}_h \phi, \lambda) + (\tilde{\Pi}_h \phi, \lambda) = (\tilde{\Pi}_h \phi, \lambda) = a_D(\tilde{\Pi}_h \phi, J(\lambda)) \leq C_0 \|\tilde{\Pi}_h \phi\|_{DG} \|J(\lambda)\|_{DG},\]

which concludes our proof.

We recall the following approximation operator (see Lemma 6.1 in [6]).

**Lemma 3.25.** There is an approximation operator $\mathcal{R}_h : H^1_0(\mathcal{T}_h)^2 \to X_h$ satisfying

\[
(3.36) \quad b_P(q, \mathcal{R}_h(v) - v) = 0, \quad \forall v \in H^1_0(\mathcal{T}_h)^2, \quad \forall q \in Q_h,
\]

and for all $E$ in $\mathcal{T}_h$, for all $v$ in $H^1_0(\mathcal{T}_h)^2 \cap W^{s,r}(E)^2$, $1 \leq r \leq \infty$, $1 \leq s \leq q + 1$,

\[
(3.37) \quad \|\mathcal{R}_h(v) - v\|_{L^r(E)} \leq C h^s |v|_{W^{s,r}(\Delta_E)},
\]

\[
\|\nabla(\mathcal{R}_h(v) - v)\|_{L^r(E)} \leq C h^{s-1} |v|_{W^{s,r}(\Delta_E)},
\]

with a constant $C$ independent of $h$ and $E$, where $\Delta_E \subset \Omega$ is a macro-element. We also have for all $s$, $1 \leq s \leq q + 1$,

\[
(3.38) \quad \forall v \in H^1_0(\mathcal{T}_h)^2 \cap H^s(\Omega)^2, \quad \|\mathcal{R}_h(v) - v\|_{DG} \leq C h^{s-1} |v|_{H^s(\Omega)}.
\]

With the operator $\mathcal{R}_h$, we have a bound for the form $a_C$ (see Proposition 6.2 in [6]).

**Lemma 3.26** (Bounds of $a_C$). There exists a constant $C$ independent of $h$ such that for any $u$ in $(L^{\infty}(\Omega) \cap W^{1,3}(\Omega) \cap H^{3/2}(\Omega))^2$, any $v_h$ in $V_h$ and any $w_h, z_h$ in $X_h$, the bound holds

\[
|a_C(z_h, v_h, u - \mathcal{R}_h u, w_h)| \leq C \left( \|u - \mathcal{R}_h u\|_{L^\infty(\Omega)} + \|u - \mathcal{R}_h u\|_{W^{1,3}(\Omega)} + \|u\|_{H^{3/2}(\Omega)} \right) \times \|v_h\|_{L^2(\Omega)} \|w_h\|_{DG}.
\]

Recall that $\mathcal{P}_h c^n$ and $\mathcal{P}_h \mu^n$ are the elliptic projections of $c^n$ and $\mu^n$, which are defined in (3.3). The DG error analysis for elliptic problems yields the following error bounds [29].

**Lemma 3.27.** There exist a constant $C$, independent of mesh size $h$ and time step size $\tau$, such that for all $0 \leq n \leq N$

\[
\|c^n - \mathcal{P}_h c^n\|_{DG} \leq C h^q \|c\|_{L^\infty(0,T; H^{q+1}(\Omega))},
\]

\[
\|\mu^n - \mathcal{P}_h \mu^n\|_{DG} \leq C h^q \|\mu\|_{L^\infty(0,T; H^{q+1}(\Omega))},
\]

\[
\|\delta e^n - \mathcal{P}_h e^n\|_{L^2(\Omega)} \leq C h^q \|\partial e\|_{L^\infty(0,T; H^{q+1}(\Omega))}.
\]

We define the projection errors and the discretization errors as follows

\[
\xi_c^n = c^n - \mathcal{P}_h c^n, \quad \xi_c^n = \mathcal{P}_h c^n - c_h^n,
\]

\[
\xi_{\mu}^n = \mu^n - \mathcal{P}_h \mu^n, \quad \xi_{\mu}^n = \mathcal{P}_h \mu^n - \mu_h^n,
\]

\[
\xi_{v}^n = v^n - \mathcal{R}_h v^n, \quad \xi_{v}^n = \mathcal{R}_h v^n - v_h^n,
\]

\[
\xi_p^n = p^n - \Pi_h p^n, \quad \xi_p^n = \Pi_h p^n - p_h^n.
\]
Now we are in the position of stating the error equation. We note that for all \( n \geq 1 \)
\[
a_D(\xi^n_\mu, \chi) = 0, \quad b_P(\phi, \xi^n_\nu) = 0, \quad \forall \chi \in S_h, \quad \forall \phi \in Q_h.
\]
Therefore, from (3.2) and (3.34), the error equation becomes, for any \( \chi \in S_h \), \( \phi \in S_h \), \( \theta \in X_h \), and \( \phi \in Q_h \):
\[
(\delta_\tau \xi^n_\mu, \chi) + a_D(\xi^n_\mu, \chi) = (\delta_\tau c^n - (\partial_t c)^n - \delta_\tau \xi^n_\mu, \chi) - a_A(c^n, v^n, \chi) + a_A(c_h^{-1}, v^n_h, \chi),
\]
(3.39a)
\[
\kappa a_D(\xi^n_\epsilon, \phi) - (\xi^n_\mu, \phi) = (\xi^n_\mu, \phi)
\]
(3.39b)
\[
(\delta_\tau \xi^n_\nu, \theta) + \mu_\alpha a_\epsilon(\xi^n_\nu, \theta) + b_P(\xi^n_\nu, \theta) = (\delta_\tau v^n - (\partial_t v)^n - \delta_\tau \xi^n_\nu, \theta)
\]
(3.39c)
\[
-a_\Gamma(c^{-1}, v^{-1}_h, \theta) + b_P(\xi^n_\nu, \theta) - a_\Gamma(v^n, v^n, \theta)
\]
(3.39d)
We assume the chemical energy density satisfies the following Lipschitz condition. **Assumption A:**
There is a constant \( C_2 > 0 \) independent of \( h \) and \( \tau \) such that for all \( n \geq 0 \)
\[
\| \Phi_+('c^n) - \Phi_+'(c^n) \|_{DG} \leq C_2 c^n_h - c^n_{DG},
\]
\[
\| \Phi_-('c^n) - \Phi_'(c^n) \|_{DG} \leq C_2 c^n_h - c^n_{DG}.
\]
In addition, we also assume that both \( \Phi_+'' \) and \( \Phi_-'' \) are bounded. In two-dimensions, for the Ginzburg–Landau potential, **Assumption A** is automatically satisfied because of (3.32). We now state the main theorem.

**Theorem 3.28.** Suppose \((c, \mu, v, p)\) is a weak solution of (3.34) with regularity (3.33). Then, under **Assumption A** and sufficiently small time step size \( \tau \), there exists a constant \( C \) independent of mesh size \( h \) and time step size \( \tau \) such that for any \( m \geq 1 \)
\[
\max_{1 \leq n \leq m} \left( \| \xi^n_\mu \|_{DG}^2 + \| \xi^n_\nu \|_{L^2(\Omega)}^2 \right) + \tau \sum_{n=1}^m \| \xi^n_\nu \|_{DG}^2 \leq C (\tau^2 + h^{2q}),
\]
\[
\tau \sum_{n=1}^m \| \xi^n_\mu \|_{DG}^2 \leq C (\tau^2 + h^{2q}).
\]
**Proof.** From Theorem 3.12 and (3.3), it is obvious that \( \delta_\tau \xi^n_\mu \) belongs to \( M_h \), which means that the function \( \mathcal{J}(\delta_\tau \xi^n_\mu) \) is well defined in \( M_h \). Choosing \( \chi = \mathcal{J}(\delta_\tau \xi^n_\mu) \) in (3.39a) and using Lemma 3.24, we have
\[
a_D(\mathcal{J}(\delta_\tau \xi^n_\mu), \mathcal{J}(\delta_\tau \xi^n_\nu)) + (\delta_\tau \xi^n_\mu, \mathcal{J}(\delta_\tau \xi^n_\mu)) = (\delta_\tau c^n - (\partial_t c)^n - \delta_\tau \xi^n_\mu, \mathcal{J}(\delta_\tau \xi^n_\mu)) - a_A(c^n, v^n, \mathcal{J}(\delta_\tau \xi^n_\mu)) + a_A(c_h^{-1}, v^n_h, \mathcal{J}(\delta_\tau \xi^n_\mu)).
\]
(3.41a)
Choosing \( \varphi = \delta_\tau \xi^n_\nu \) in (3.39b) and adding and subtracting the appropriate terms, we obtain
\[
\kappa a_D(\xi^n_\epsilon, \delta_\tau \xi^n_\nu) - (\xi^n_\mu, \delta_\tau \xi^n_\nu) = (\xi^n_\mu, \delta_\tau \xi^n_\nu) + (\Phi_+'(c^n_h) - \Phi_+'(c^n), \delta_\tau \xi^n_\nu) + (\Phi_+'(c^n_h) - \Phi_+'(c^n), \delta_\tau \xi^n_\nu) + (\Phi_-'(c^n_h) - \Phi_-'(c^n_h), \delta_\tau \xi^n_\nu) + (\Phi_-'(c^n_h) - \Phi_-'(c^n_h), \delta_\tau \xi^n_\nu).
\]
(3.41b)
Choosing $\theta = \xi^0$ in (3.39c), $\phi = -\xi^0$ in (3.39d) and combining the resulting equations, we have

\[ (\delta_r \xi^0, \xi^0_n) + \mu a_{\varepsilon}(\xi^0_n, \xi^0_n) = (\delta_r v^n - (\partial_t v)^n - \delta_r \xi^0_n, \xi^0_n) \]

\[ -\mu a_{\varepsilon}(\xi^0_n, \xi^0_n) - b_{\varepsilon}(v^n, v^n, v^n, \xi^0_n) + a_{\varepsilon}(v^n_{h-1}, v^n_{h-1}, v^n_h, \xi^0_n) + b_{\varepsilon}(c^n, \mu^n, \xi^0_n) - b_{\varepsilon}(c^n_{h-1}, \mu^n_{h-1}, \xi^0_n). \]

Summing (3.41a) – (3.41c), we obtain the following equation

\[ a_{\varepsilon}(J_{\delta_r \xi^0_n}, J_{\delta_r \xi^0_n}) + \kappa a_{\varepsilon}(\xi^0_n, \delta_r \xi^0_n) + \mu a_{\varepsilon}(\xi^0_n, \xi^0_n) + (\delta_r \xi^0_n, \xi^0_n) \]

\[ -\delta_r(c^n_{h-1}, \xi^0_n) + (\Phi^'_+(c^n_{h-1}) - \Phi^'_+(c^n), \delta_r \xi^0_n) + (\Phi^'_-(c^n_{h-1}) - \Phi^'_-(c^n-1), \delta_r \xi^0_n) \]

\[ + (\Phi^'_-(c^n-1) - \Phi^'_-(c^n), \delta_r \xi^0_n) + (\xi^0_n, \delta_r \xi^0_n) - \mu a_{\varepsilon}(\xi^0_n, \xi^0_n) - b_{\varepsilon}(c^n, \mu^n, \xi^0_n) - b_{\varepsilon}(c^n_{h-1}, \mu^n_{h-1}, \xi^0_n) = T_1 + \cdots + T_{16}. \]

The remainder of the proof consists of finding lower bounds for the terms in the left-hand side and upper bounds for the terms in the right-hand side of the equation above. We will then utilize Gronwall’s lemma. For the left-hand side of (3.42), since $a_{\varepsilon}$ and the inner product are both symmetric bilinear forms, using the formula $a(a - b) \geq \frac{1}{2}a^2 - \frac{1}{2}b^2$, and the coercivity of $a_{\varepsilon}$ and $a_{\varepsilon}$, we have

\[ a_{\varepsilon}(J_{\delta_r \xi^0_n}, J_{\delta_r \xi^0_n}) + \kappa a_{\varepsilon}(\xi^0_n, \delta_r \xi^0_n) + \mu a_{\varepsilon}(\xi^0_n, \xi^0_n) + (\delta_r \xi^0_n, \xi^0_n) \]

\[ \geq K_{\varepsilon} ||J_{\delta_r \xi^0_n}||_{DG}^2 + \frac{\kappa}{2\varepsilon} a_{\varepsilon}(\xi^0_n, \xi^0_n) - \frac{\kappa}{2\varepsilon} a_{\varepsilon}(\xi^0_n, \xi^0_{n-1}) \]

\[ + \mu K_{\varepsilon} ||\xi^0_n||_{DG}^2 + \frac{1}{2\varepsilon} ||\xi^0_n||_{L^2(\Omega)}^2 - \frac{1}{2\varepsilon} ||\xi^0_{n-1}||_{L^2(\Omega)}^2. \]

Now, let us proceed to estimate the right-hand side of (3.42) term by term. At several places, we will use Young’s inequality with positive real numbers $r_1$ and $r_2$ to be chosen later. By Cauchy–Schwarz’s inequality, Poincaré’s inequality, Young’s inequality, and using a Taylor expansion, we have

\[ T_1 \leq ||\delta_r c^n - (\partial_t c)^n||_{L^2(\Omega)} ||J_{\delta_r \xi^0_n}||_{L^2(\Omega)} \]

\[ \leq C_P ||\delta_r c^n - (\partial_t c)^n||_{L^2(\Omega)} ||J_{\delta_r \xi^0_n}||_{DG} \]

\[ \leq \frac{r_1}{2} ||J_{\delta_r \xi^0_n}||_{DG}^2 + \frac{C_P^2}{2r_1} ||\delta_r c^n - (\partial_t c)^n||_{L^2(\Omega)}^2 \]

\[ \leq \frac{r_1}{2} ||J_{\delta_r \xi^0_n}||_{DG}^2 + \frac{C_P^2}{6r_1} \int_{t_{n-1}}^{t_n} ||\partial_t c||_{L^2(\Omega)}^2. \]

By Cauchy–Schwarz’s inequality, Poincaré’s inequality, and Young’s inequality, the term $T_2$ is simply bounded

\[ T_2 \leq ||\delta_r \xi^0_n||_{L^2(\Omega)} ||J_{\delta_r \xi^0_n}||_{L^2(\Omega)} \]

\[ \leq C_P ||\delta_r \xi^0_n||_{L^2(\Omega)} ||J_{\delta_r \xi^0_n}||_{DG} \]

\[ \leq \frac{r_1}{2} ||J_{\delta_r \xi^0_n}||_{DG}^2 + \frac{C_P^2}{2r_1} ||\delta_r \xi^0_n||_{L^2(\Omega)}^2. \]
Next, we remark that $\mathcal{P}_h(\delta_r c^n) = \delta_r(\mathcal{P}_h c^n)$ and with the approximation result of Lemma 3.27, we have
\[
\|\delta_r \xi^n_v\|_{L^2(\Omega)} \leq C h^q \|\partial_t v\|_{L^\infty(0,T;H^{s+1}(\Omega))}.
\]
Therefore, we have
\[
T_2 \leq \frac{r_1}{2} \|\mathcal{J}(\delta_r \xi^n_v)\|_{DG}^2 + \frac{C}{r_1} h^{2q}.
\]
The terms $T_3$ and $T_4$ are bounded by employing a similar technique as for $T_1$ and $T_2$.
\[
T_3 \leq \|\xi^n_v\|_{L^2(\Omega)}^2 + \frac{r}{12} \int_{t_n-1}^{t_n} \|\partial_t v\|_{L^2(\Omega)}^2,
\]
\[
T_4 \leq \|\xi^n_v\|_{L^2(\Omega)}^2 + \frac{r}{4} \|\delta_r \xi_v^n\|_{L^2(\Omega)}^2.
\]
We write
\[
\delta_r \xi^n_v = \frac{1}{r} \int_{t_n-1}^{t_n} \partial_t (v - R_h v).
\]
This implies with Lemma 3.25:
\[
(3.44) \quad \|\delta_r \xi^n_v\|_{L^2(\Omega)} \leq \frac{C}{\sqrt{r}} h^q \|\partial_t v\|_{L^2((t_n-1,t_n;H^s(\Omega)))}.
\]
Therefore we have for $T_4$:
\[
T_4 \leq \|\xi^n_v\|_{L^2(\Omega)}^2 + \frac{C}{r} h^{2q} \|\partial_t v\|_{L^2((t_n-1,t_n;H^s(\Omega)))}^2.
\]
For the term $T_5$, using Lemma 3.24, Young’s inequality, assumption (3.40), and triangular inequality, we have
\[
T_5 \leq C_1 \|\Phi_+'(c^n_r) - \Phi_+'(c^n_v)\|_{DG} \|\mathcal{J}(\delta_r \xi^n_v)\|_{DG}
\leq \frac{r_1}{2} \|\mathcal{J}(\delta_r \xi^n_v)\|_{DG}^2 + \frac{C_1^2}{2r_1} \|\Phi_+'(c^n_r) - \Phi_+'(c^n_v)\|_{DG}^2
\leq \frac{r_1}{2} \|\mathcal{J}(\delta_r \xi^n_v)\|_{DG}^2 + \frac{C_1^2}{2r_1} \|\xi^n_v - c^n\|_{DG}^2
\leq \frac{r_1}{2} \|\mathcal{J}(\delta_r \xi^n_v)\|_{DG}^2 + \frac{C_1^2}{2r_1} \|\xi^n_v\|_{DG}^2 + C h^{2q} \|c\|_{L^\infty(0,T;H^{s+1}(\Omega))}^2.
\]
Repeating exactly the same steps of bounding the term $T_5$ as above, we have the following bound for the term $T_6$
\[
T_6 \leq \frac{r_1}{2} \|\mathcal{J}(\delta_r \xi^n_v)\|_{DG}^2 + \frac{C_1^2}{2r_1} \|\xi^n_v - c^n\|_{DG}^2 + C h^{2q} \|c\|_{L^\infty(0,T;H^{s+1}(\Omega))}^2.
\]
For the term $T_7$, we use Lemma 3.24, Young’s inequality, and a Taylor expansion of first order and obtain
\[
T_7 \leq \frac{r_1}{2} \|\mathcal{J}(\delta_r \xi^n_v)\|_{DG}^2 + \frac{C_1^2}{2r_1} \|\Phi_-'(c^{n-1}) - \Phi_-'(c^n)\|_{DG}^2
\leq \frac{r_1}{2} \|\mathcal{J}(\delta_r \xi^n_v)\|_{DG}^2 + \frac{C_1^2}{2r_1} \|\partial_t c\|_{L^\infty(0,T;H^s(\Omega))}^2.
\]
For the term $T_8$, since $\zeta^n_1$ belongs to $H^1(\mathcal{T}_h)$, with Lemma 3.24, Young’s inequality and the approximation bound in Lemma 3.27, we have

$$T_8 \leq C \|\zeta^n_1\|_{DG} \|\mathcal{J}(\delta_r \zeta^n_1)\|_{DG} \leq \frac{r_1}{2} \|\mathcal{J}(\delta_r \zeta^n_1)\|_{DG} + \frac{C h^{2q}}{r_1} \|\mu\|_{L^\infty(0,T; H^{r+1}(\Omega))}^2.$$  

The way of processing the terms $T_9$ and $T_{10}$ follows the argument in [29] (page 127), we have

$$T_9 \leq r_2 \mu_s K \|\zeta^n_1\|_{DG}^2 + \frac{C h^{2q}}{r_2} \|\nu^n\|_{H^{r+1}(\Omega)}^2,$$

$$T_{10} \leq r_2 \mu_s K \|\zeta^n_1\|_{DG}^2 + \frac{C h^{2q}}{r_2} \|\nu^n\|_{H^{r}(\Omega)}^2.$$  

The handling of the terms $T_{11}$ and $T_{12}$ is complicated; however these terms have been analyzed in papers for the Navier–Stokes equations, for instance in [6]. We give an outline of the proof for completeness.

$$T_{11} + T_{12} = -a_C(v^{n-1}_h, v^n, \zeta^n_1) + a_C(v^{n-1}_h, v^n, \zeta^n_1)$$

$$= -a_C(v^{n-1}_h, \zeta^n_1, \zeta^n_1) - a_C(v^{n-1}_h, \zeta^n_1, R_h \nu^n, \zeta^n_1)$$

$$- a_C(v^{n-1}_h, \zeta^n_1, R_h \nu^n, \zeta^n_1) + a_C(v^{n-1}_h, \nu^n - \nu^n, R_h \nu^n, \zeta^n_1)$$

$$= T_1^3 + \cdots + T_6^3.$$  

We know from Lemma 3.6 that the first term $T_1^3$ is negative. We rewrite the second term as:

$$T_2^3 = -a_C(v^{n-1}_h, \zeta^n_1, \nu^n, \zeta^n_1) + a_C(v^{n-1}_h, \zeta^n_1, \zeta^n_1).$$

Note that $\zeta^n_1$ belongs to $V_h$ and we apply Lemma 3.5 to the first term

$$|a_C(v^{n-1}_h, \zeta^n_1, \nu^n, \zeta^n_1)| \leq C (\|\nu^n\|_{L^\infty(\Omega)} + |\nu^n|_{W^{1,3}(\Omega)}) \|\zeta^n_1\|_{L^2(\Omega)} \|\zeta^n_1\|_{DG}.$$  

We apply Lemma 3.26 to the second term

$$|a_C(v^{n-1}_h, \zeta^n_1, \zeta^n_1, \zeta^n_1)| \leq C (\|\zeta^n_1\|_{L^\infty(\Omega)} + |\zeta^n_1|_{W^{1,3}(\Omega)} + |\nu^n|_{H^{3/2}(\Omega)})$$

$$\times \|\zeta^n_1\|_{L^2(\Omega)} \|\zeta^n_1\|_{DG}.$$  

Combining both terms, we obtain

$$T_2^3 \leq r_2 \mu_s K \|\zeta^n_1\|_{DG}^2 + C h^{2q} \|\nu^n\|_{L^\infty(0,T; H^{r+1}(\Omega))}^2.$$  

We apply Lemma 3.4 to the term $T_3^3$ and obtain using the regularity of the weak solution:

$$T_3^3 \leq r_2 \mu_s K \|\zeta^n_1\|_{DG}^2 + \frac{C h^{2q}}{r_2} \|\nu^n\|_{L^\infty(0,T; H^{r+1}(\Omega))}^2.$$

For the term $T_4^3$, we note that $\nu^n$ is divergence free and has no jumps. The term simplifies

$$T_4^3 = \sum_{E \in \mathcal{T}_h} \int_E ((\nu^{n-1} - \nu^n) \cdot \nabla R_h \nu^n) \cdot \zeta^n_1$$

$$+ \sum_{E \in \mathcal{T}_h} \int_{\partial E \setminus \partial \Omega} |(\nu^{n-1} - \nu^n) \cdot n_E| \left((R_h \nu^n)^{\text{int}} - (R_h \nu^n)^{\text{ext}}\right) \cdot \zeta^n_1.$$  

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We then have
\[
\left| \int_E (v^{n-1} - v^n) \cdot \nabla R_h v^n \cdot \xi^n \right| \leq \int_{\tau_{n-1}}^{\tau_n} \| \partial_t v \|_{L^2(E)} \| \xi^n \|_{L^2(E)} \| R_h v^n \|_{W^{1,3}(E)}.
\]
Using the stability of $R_h$ and summing over the elements yields:
\[
\left| \sum_{E \in T_h} \int_E (v^{n-1} - v^n) \cdot \nabla R_h v^n \cdot \xi^n \right| 
\leq C \sqrt{\tau} \| \partial_t v \|_{L^2(t_{n-1}, t_n; L^2(\Omega))} \| \xi^n \|_{DG} \| v \|_{L^\infty(0, T; W^{1,3}(\Omega))}.
\]
For the term on the faces, we rewrite
\[
\left| (R_h v^n)^{\text{int}} - (R_h v^n)^{\text{ext}} \right| = \| R_h v^n - v^n \|
\]
and employ trace inequalities to obtain a similar bound:
\[
\left| \sum_{E \in T_h} \int_{\partial E \setminus \partial \Omega} | (v^{n-1} - v^n) \cdot n_E | | (R_h v^n)^{\text{int}} - (R_h v^n)^{\text{ext}} | \cdot (\xi^n)^{\text{int}} \right| 
\leq C \sqrt{\tau} \| \partial_t v \|_{L^2(t_{n-1}, t_n; L^\infty(\Omega))} \| \xi^n \|_{DG} \| v \|_{L^\infty(0, T; H^1(\Omega))}.
\]
Therefore we finally obtain for the term $T_{\varepsilon}^4$:
\[
T_{\varepsilon}^4 \leq r_2 \mu_3 K_{\varepsilon} \| \xi^n \|^2_{DG} + \frac{C}{r_2} \tau \| \partial_t v \|^2_{L^2(t_{n-1}, t_n; L^\infty(\Omega))}.
\]
The bound for $T_{\varepsilon}^5$ is similar but simpler:
\[
T_{\varepsilon}^5 \leq 4 r_2 \mu_3 K_{\varepsilon} \| \xi^n \|^2_{DG} + \frac{C}{r_2} h^{2q} \| v \|^2_{L^2(t_{n-1}, t_n; H^{q+1}(\Omega))}.
\]
Combining the bounds above, we obtain:
\[
T_{11} + T_{12} \leq 4 r_2 \mu_3 K_{\varepsilon} \| \xi^n \|^2_{DG} + \frac{C}{r_2} \left( \| \xi^n \|_{L^2(\Omega)}^2 + h^{2q} + \tau \| \partial_t v \|_{L^2(t_{n-1}, t_n; L^\infty(\Omega))}^2 \right).
\]
We can rewrite the terms $T_{13} + T_{14}$ as follows
\[
T_{13} + T_{14} = - a_{\mathcal{A}}(c^{n-1}, v^n, \mathcal{J}(\delta_r \xi^n)) - a_{\mathcal{A}}(c^n, v^n, \mathcal{J}(\delta_r \xi^n))
- a_{\mathcal{A}}(c^n - c^{n-1}, v^n, \mathcal{J}(\delta_r \xi^n)) - a_{\mathcal{A}}(c^{n-1}, \xi^n, \mathcal{J}(\delta_r \xi^n))
- a_{\mathcal{A}}(c^{n-1}, \xi^n, \mathcal{J}(\delta_r \xi^n)) = T_{\mathcal{A}}^1 + \cdots + T_{\mathcal{A}}^5.
\]
Expanding $T_{\mathcal{A}}^1$ by definition in (3.1a), by Cauchy–Schwarz’s inequality, trace inequality, Poincaré’s inequality, and Lemma 3.27, we have
\[
| T_{\mathcal{A}}^1 | \leq C h^q c \| v \|_{L^\infty(0, T; H^{q+1}(\Omega))} \| \xi^n \|_{L^\infty(\Omega)} \| \mathcal{J}(\delta_r \xi^n) \|_{DG}.
\]
Using a similar technique as above, we obtain
\[
| T_{\mathcal{A}}^2 | \leq C \| \xi^n \|_{L^2(\Omega)} \| v^n \|_{L^\infty(\Omega)} \| \mathcal{J}(\delta_r \xi^n) \|_{DG} \leq C \| \xi^n \|_{DG} \| v^n \|_{L^\infty(\Omega)} \| \mathcal{J}(\delta_r \xi^n) \|_{DG}.
\]
Taking a Taylor expansion of $c$ at $t^{n-1}$ and using similar techniques as above, we have

$$|T_{A}^{3}| \leq C\tau\|\partial_{t}c\|_{L^{\infty}(\Omega)}\|v^{n}\|_{L^{\infty}(\Omega)}\|J(\delta_{\tau}\xi_{\tau}^{n})\|_{DG}.$$  

Using similar techniques as above, using the bound (3.32) and the approximation in Lemma 3.25, we have

$$|T_{A}^{4}| \leq Ch^{q}\|v^{n}\|_{H^{q+1}(\Omega)}\|a_{A}c^{n-1}\|_{L^{\infty}(\Omega)}\|J(\delta_{\tau}\xi_{\tau}^{n})\|_{DG}.$$  

Using the boundedness of $a_{A}$, the stability bound of $\|c^{n-1}\|_{L^{\infty}(\Omega)}$, we have

$$|T_{A}^{5}| \leq C\gamma\|c^{n-1}\|_{L^{\infty}(\Omega)}\|\xi_{\tau}^{n}\|_{L^{2}(\Omega)}\|J(\delta_{\tau}\xi_{\tau}^{n})\|_{DG}.$$  

Therefore, combining the bounds above and using Young’s inequality, we obtain

$$|T_{13} + T_{14}| \leq \frac{C}{r_{1}}\|\xi_{\tau}^{n}\|_{L^{2}(\Omega)}^{2} + r_{1}\|J(\delta_{\tau}\xi_{\tau}^{n})\|_{DG}^{2} + \frac{C}{r_{1}}\|c^{n-1}\|_{DG}^{2} + \frac{C}{r_{1}}(r^{2} + h^{2q}).$$  

For the terms $T_{15}$ and $T_{16}$, by Remark 3.1, we may write

$$T_{15} + T_{16} = a_{A}(c^{n}, \xi_{\tau}^{n}, \mu^{n}) - a_{A}(c_{h}^{n-1}, \xi_{\tau}^{n}, \mu_{h})$$

$$= a_{A}(c_{h}^{n-1}, \xi_{\tau}^{n}, \mu^{n}) + a_{A}(c_{h}^{n-1}, \xi_{\tau}^{n}, \mu_{h})$$

$$+ a_{A}(c^{n} - c_{h}^{n-1}, \xi_{\tau}^{n}, \mu^{n}) + a_{A}(c_{h}^{n-1}, \xi_{\tau}^{n}, \mu_{h}) + a_{A}(c_{h}^{n-1}, \xi_{\tau}^{n}, \mu_{h}).$$

The first three terms simplify since $\mu^{n}$ does not have any jump. Using Poincaré’s inequality, we obtain:

$$|a_{A}(c_{h}^{n-1}, \xi_{\tau}^{n}, \mu^{n})| \leq \|c_{h}^{n-1}\|_{L^{2}(\Omega)}\|\xi_{\tau}^{n}\|_{L^{2}(\Omega)}\|\mu^{n}\|_{W^{1,4}(\Omega)} \leq Ch^{q}\|c^{n-1}\|_{H^{q+1}(\Omega)}\|\xi_{\tau}^{n}\|_{DG},$$

$$|a_{A}(c_{h}^{n-1}, \xi_{\tau}^{n}, \mu_{h})| \leq \|c_{h}^{n-1}\|_{L^{2}(\Omega)}\|\xi_{\tau}^{n}\|_{L^{2}(\Omega)}\|\mu_{h}\|_{W^{1,4}(\Omega)} \leq C\|c_{h}^{n-1}\|_{L^{2}(\Omega)}\|\xi_{\tau}^{n}\|_{DG},$$

and with Taylor expansion

$$|a_{A}(c^{n} - c_{h}^{n-1}, \xi_{\tau}^{n}, \mu^{n})| \leq \|c^{n} - c_{h}^{n-1}\|_{L^{2}(\Omega)}\|\xi_{\tau}^{n}\|_{L^{2}(\Omega)}\|\mu^{n}\|_{W^{1,4}(\Omega)} \leq C\tau\|\xi_{\tau}^{n}\|_{DG}.$$  

For the other terms, we use the bound (3.32) and the approximation bound in Lemma 3.27:

$$|a_{A}(c_{h}^{n-1}, \xi_{\tau}^{n}, \mu_{h})| \leq Ch^{q}\|\mu_{h}\|_{H^{q+1}(\Omega)}\|c_{h}^{n-1}\|_{L^{\infty}(\Omega)}\|\xi_{\tau}^{n}\|_{L^{2}(\Omega)},$$

$$|a_{A}(c_{h}^{n-1}, \xi_{\tau}^{n}, \mu_{h})| \leq C\|\mu_{h}\|_{DG}\|c_{h}^{n-1}\|_{L^{\infty}(\Omega)}\|\xi_{\tau}^{n}\|_{L^{2}(\Omega)}.$$  

Therefore we obtain:

$$T_{15} + T_{16} \leq r_{2}\mu_{h}K_{\alpha}\|\xi_{\tau}^{n}\|_{DG}^{2} + \frac{C}{r_{2}}(h^{2q} + \tau^{2})$$

$$+ \frac{C}{r_{2}}\|c_{h}^{n-1}\|_{L^{2}(\Omega)}^{2} + C\|\xi_{\tau}^{n}\|_{DG}\|\xi_{\tau}^{n}\|_{L^{2}(\Omega)}.$$  

It remains to find a bound for $\|\xi_{\tau}^{n}\|_{DG}$. We choose $\chi = \xi_{\mu}^{n}$ in (3.39a), use coercivity of $a_{D}$ and (3.35) to obtain:

$$K_{\alpha}\|\xi_{\tau}^{n}\|_{DG}^{2} \leq |a_{D}(\xi_{\tau}^{n}, J(\delta_{\tau}\xi_{\tau}^{n}))| + (\|\delta_{\tau}c^{n} - (\partial_{t}c)^{n}\|_{L^{2}(\Omega)} + \|\delta_{\tau}\xi_{\tau}^{n}\|_{L^{2}(\Omega)})\|\xi_{\tau}^{n}\|_{L^{2}(\Omega)}$$

$$+ |a_{A}(c_{h}^{n-1}, \xi_{\tau}^{n}, \mu_{h}) + a_{A}(c^{n}, \xi_{\tau}^{n}, \mu_{h}) - a_{A}(c^{n}, \xi_{\tau}^{n}, \mu_{h})|.$$
The last two terms are handled similarly than the terms $T_{13}$ and $T_{14}$. Using continuity of $a_D$, we have
\[
\|\xi^n_{\mu}\|_{DG} \leq C\left(\|\xi^n_{\nu}\|_{L^2(\Omega)}^2 + \|\xi^{n-1}_{\nu}\|_{DG}^2 + \tau^2 + h^{2q}\right)
\]
\[
+ \|J(\delta_t \xi^n_{\nu})\|_{DG}^2 + \|\delta_t \xi^n_{\nu}\|_{L^2(\Omega)}^2 + \tau \int_{t_{n-1}}^{t_n} \|\partial_t c\|_{L^2(\Omega)}^2 \right)^{1/2}.
\]

Next, we remark that $P_t(\delta_t c^n) = \delta_t(P_t c^n)$ and with the approximation result in Lemma 3.27, we have
\[
\|\delta_t \xi^n_{\nu}\|_{L^2(\Omega)} \leq C h^{q/9}\|\partial_t c\|_{L^\infty(0,T;H^{s+1}(\Omega))}.
\]

With this bound, the bound for $T_{15}$ and $T_{16}$ becomes:
\[
T_{15} + T_{16} \leq 2\mu_t K_\epsilon \|\xi^n_{\nu}\|_{DG}^2 + \frac{\tau}{2} \|J(\delta_t \xi^n_{\nu})\|_{DG}^2 + C(1 + \mu_t h^{2q} + \tau^2) + C \int_{t_{n-1}}^{t_n} \|\partial_t c\|_{L^2(\Omega)}^2.
\]

Combining (3.43) with all the bounds for $T_1$ to $T_{16}$, and choosing the values $r_1 = K_\alpha/9$ and $r_2 = 1/4$ yields
\[
\frac{K_\alpha}{2} \|J(\delta_t \xi^n_{\nu})\|_{DG}^2 + \frac{\kappa}{2\tau} a_D(\xi^n_{\nu}, \xi^n_{\nu}) - \frac{\kappa}{2\tau} a_D(\xi^{n-1}_{\nu}, \xi^{n-1}_{\nu})
\]
\[
+ \frac{\mu_t K_\epsilon}{2} \|\xi^n_{\nu}\|_{DG}^2 + \frac{1}{2\tau} \|\xi^n_{\nu}\|_{L^2(\Omega)}^2 - \frac{1}{2\tau} \|\xi^{n-1}_{\nu}\|_{L^2(\Omega)}^2\leq C\left(\|\xi^n_{\nu}\|_{DG}^2 + \|\xi^{n-1}_{\nu}\|_{DG}^2 + \|\xi^n_{\nu}\|_{L^2(\Omega)}^2 + \|\xi^{n-1}_{\nu}\|_{L^2(\Omega)}^2\right)
\]
\[
+ C \|\partial_t c\|_{L^2(\nu^n_{0}, \nu_{L^n_{0}}; H^s(\Omega))}^2 + C \int_{t_{n-1}}^{t_n} \|\partial_t c\|_{L^2(\Omega)}^2 + \|\partial_t \nu\|_{L^2(\Omega)}^2.
\]

Multiply by $2\tau$ and sum from $n = 1$ to $n = m$, use the coercivity of $a_D$, the fact that $\xi^0_{\nu}$ is optimally bounded and $\xi^0_{\nu} = 0$:
\[
\tau \sum_{n=1}^{m} K_\alpha \|J(\delta_t \xi^n_{\nu})\|_{DG}^2 + \kappa K_\alpha \|\xi^m_{\nu}\|_{DG}^2
\]
\[
+ \tau \sum_{n=1}^{m} \mu_t K_\epsilon \|\xi^n_{\nu}\|_{DG}^2 + \|\xi^n_{\nu}\|_{L^2(\Omega)}^2\leq C \tau \sum_{n=1}^{m} \|\xi^n_{\nu}\|_{DG}^2 + \|\xi^n_{\nu}\|_{L^2(\Omega)}^2 + C \|\partial_t c\|_{L^2(\nu^n_{0}, \nu_{L^n_{0}}; H^s(\Omega))}^2
\]
\[
+ C \tau^2 \left(\|\partial_t c\|_{L^2(0,T; L^2(\Omega))}^2 + \|\partial_t \nu\|_{L^2(0,T; L^\infty(\Omega))}^2\right).
\]

Then, for sufficiently small time step size $\tau$, we can conclude using discrete Gronwall inequality
\[
\tau \sum_{n=1}^{m} \|J(\delta_t \xi^n_{\nu})\|_{DG}^2 + \|\xi^m_{\nu}\|_{DG}^2 + \tau \sum_{n=1}^{m} \|\xi^n_{\nu}\|_{DG}^2 + \|\xi^n_{\nu}\|_{L^2(\Omega)}^2 \leq C(\tau^2 + \tau^{2q}).
\]

Furthermore it is easy to obtain the following error estimate result
\[
\tau \sum_{n=1}^{m} \|\xi^m_{\nu}\|_{DG}^2 \leq C(\tau^2 + \tau^{2q}).
\]
Corollary 3.29. Suppose \((c, \mu, v, p)\) is a weak solution of (3.34) with regularity (3.33). Then, under Assumption A and sufficiently small time step size \(\tau\), there exists a constant \(C\) independent of mesh size \(h\) and time step size \(\tau\) such that for any \(m \geq 1\)

\[
\max_{1 \leq n \leq m} \left( \|c(t^n) - c_h^n\|_{DG}^2 + \|v(t^n) - v_h^n\|_{L^2(\Omega)}^2 \right) + \tau \sum_{n=1}^{m} \|v(t^n) - v_h^n\|_{DG}^2 \\
+ \tau \sum_{n=1}^{m} \|\mu(t^n) - \mu_h^n\|_{DG}^2 \leq C(\tau^2 + h^{2q}).
\]

4. Conclusions. In this paper, we have formulated an interior penalty discontinuous Galerkin method for solving the Cahn–Hilliard–Navier–Stokes equations. The time discretization utilizes a convex-concave splitting of the chemical energy density and a Picard’s linearization for the convection term. Existence and uniqueness of the numerical solution is proved for any general chemical energy density. We show that the discrete total free energy is always dissipative at any time and we obtain stability bounds with any generalized chemical energy density. Under the assumption of a global Lipschitz bound for the chemical energy density, which is automatically satisfied by Ginzburg–Landau double well potential in two dimensions, we derive optimal error estimates in time and space. Our analysis of the unique solvability is also valid for non-symmetric versions of the discontinuous Galerkin formulation.

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