STRONG APPROXIMATION OF ALMOST PERIODIC FUNCTIONS

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Abstract. We consider summability methods generated by the class $GM(2\beta)$. We generalize some related results of P. Pych-Taberska [Studia Math. XCVI (1990), 91–103] on strong approximation of almost periodic functions by their Fourier series and S. M. Mazhar and V. Totik [J. Approx. Theory, 60(1990), 174–182] on approximation of periodic functions by matrix means of their Fourier series.

1. Introduction

Let $S^p$ ($1 < p \leq \infty$) be the class of all almost periodic functions in the sense of Stepanov with the norm

$$
\|f\|_{S^p} := \left\{ \begin{array}{ll}
\sup_u \left\{ \frac{1}{\pi} \int_{u}^{u+\pi} |f(t)|^p \, dt \right\}^{1/p} & \text{when } 1 < p < \infty, \\
\sup_u |f(u)| & \text{when } p = \infty.
\end{array} \right.
$$

Denote yet by $C_{2\pi}$ the class of all $2\pi$-periodic functions continuous over $Q = [-\pi, \pi]$ with the norm

$$
\|f\|_{C_{2\pi}} := \sup_{t \in Q} |f(t)|.
$$

Suppose that the Fourier series of $f \in S^p$ has the form

$$
Sf(x) = \sum_{\nu = -\infty}^{\infty} A_{\nu}(f)e^{i\lambda_{\nu}x},
$$

where $A_{\nu}(f) = \lim_{L \to \infty} \frac{1}{L} \int_{0}^{L} f(t)e^{-i\lambda_{\nu}t} \, dt$,

with the partial sums

$$
S_{N}f(x) = \sum_{|\lambda_{\nu}| \leq N} A_{\nu}(f)e^{i\lambda_{\nu}x}
$$

and that $0 = \lambda_0 < \lambda_{\nu} < \lambda_{\nu+1}$ if $\nu \in \mathbb{N} = \{1, 2, 3, \ldots\}$, \( \lim \lambda_{\nu} = \infty \), \( \lambda_{-\nu} = -\lambda_{\nu} \) \( |A_{\nu}| + |A_{-\nu}| > 0 \). Let $\Omega_{\alpha, p}$, with some fixed positive $\alpha$, be the set of functions of class $S^p$ bounded on $\mathbb{R} = (-\infty, \infty)$ whose Fourier exponents satisfy the condition

$$
\lambda_{\nu+1} - \lambda_{\nu} \geq \alpha \quad (\nu \in \mathbb{N} \cup \{0\}).
$$

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In case $f \in \Omega_{\alpha,p}$

$$S_{\lambda_k} f (x) = \int_0^\infty \{ f (x + t) + f (x - t) \} \psi_{\lambda_k, \lambda_k + \alpha} (t) \, dt,$$

where

$$\psi_{\lambda, \eta} (t) = \frac{2 \sin \left( \frac{\eta - \lambda}{2} \right) \sin \left( \frac{\eta + \lambda}{2} \right)}{\pi (\eta - \lambda) t^2} \quad (0 < \lambda < \eta, \ |t| > 0).$$

Let $A := (a_{n,k})$ be an infinite matrix of real nonnegative numbers such that

$$\sum_{k=0}^\infty a_{n,k} = 1, \text{ where } n = 0, 1, 2, \ldots, \quad (1)$$

and let the $A-$transformation of $(S_{\lambda_k} f)$ be given by

$$T_{n,A,\gamma} f (x) := \sum_{k=0}^\infty a_{n,k} S_{\lambda_k} f (x) \quad (n = 0, 1, 2, \ldots).$$

Let us consider the strong mean

$$T_{n,A,\gamma}^q f (x) = \left\{ \sum_{k=0}^\infty a_{n,k} |S_{\lambda_k} f (x) - f (x)|^q \right\}^{1/q} \quad (q > 0). \quad (2)$$

If $f \in C_{2\pi}$, then as usually

$$S f (x) = \frac{a_0 (f)}{2} + \sum_{\nu=1}^\infty (a_{\nu} (f) \cos kx + b_{\nu} (f) \sin kx)$$

and instead of $S_{\lambda_k} f$ we will consider the partial sums

$$S_k f (x) = \frac{a_0 (f)}{2} + \sum_{\nu=1}^k (a_{\nu} (f) \cos kx + b_{\nu} (f) \sin kx).$$

Thus, instead of $T_{n,A,\gamma} f$ and $T_{n,A,\gamma}^q f$ we will consider the quantities $T_{n,A} f$ and $T_{n,A}^q f$ defined by the formulas

$$T_{n,A} f (x) := \sum_{k=0}^\infty a_{n,k} S_k f (x) \quad (n = 0, 1, 2, \ldots) \quad (3)$$

and

$$T_{n,A}^q f (x) = \left\{ \sum_{k=0}^\infty a_{n,k} |S_k f (x) - f (x)|^q \right\}^{1/q} \quad (q > 0), \quad (4)$$

respectively. As measures of approximation by the quantities (2), (3) and (4) we use the best approximation of $f$ by trigonometric polynomials $t_k$ of order at most $k$ or by
entire functions $g_\sigma$ of exponential type $\sigma$ bounded on the real axis, shortly $g_\sigma \in B_\sigma$ and the modulus of continuity of $f$, defined by the formulas

$$E_k(f)_{C_2\pi} = \inf_t \|f - t_k\|_{C_2\pi}$$

or

$$E_\sigma(f)_{Sp} = \inf_{g_\sigma} \|f - g_\sigma\|_{Sp}$$

and

$$\omega f(\delta)_X = \sup_{|t| \leq \delta} \|f(\cdot + t) - f(\cdot)\|_X,$$  

$X = C_2\pi$ or $X = Sp$,

respectively.

In [10] S. M. Mazhar and V. Totik proved the following theorem:

**Theorem 1.** Let $f \in C_2\pi$. Suppose $A := (a_{n,k})$ satisfies (1), $\lim_{n \to \infty} a_{n,0} = 0$ and

$$a_{n,k} \geq a_{n,k+1} \quad k = 0, 1, 2, ..., \quad n = 0, 1, 2, ...,$$

then

$$\|T_{n,A} f(x) - f\|_{C_2\pi} \leq K \sum_{k=0}^{\infty} a_{n,k} \omega f \left( \frac{1}{k+1} \right)_{C_2\pi}.$$

Recently, L. Leindler [5] defined a new class of sequences named as sequences of rest bounded variation, briefly denoted by $RBVS$, i.e.,

$$RBVS = \left\{ a := (a_n) \in \mathbb{C} : \sum_{k=m}^{\infty} |a_k - a_{k+1}| \leq K(a) |a_m| \text{ for all } m \in \mathbb{N} \right\}, \quad (5)$$

where here and throughout the paper $K(a)$ always indicates a constant only depending on $a$.

Denote by $MS$ the class of nonincreasing sequences. Then it is obvious that

$$MS \subset RBVS.$$

In [6] L. Leindler considered the class of mean rest bounded variation sequences $MRBVS$, where

$$MRBVS = \left\{ a := (a_n) \in \mathbb{C} : \sum_{k=m}^{\infty} |a_k - a_{k+1}| \leq K(a) \frac{1}{m} \sum_{k\geq m/2} |a_k| \text{ for all } m \in \mathbb{N} \right\}. \quad (6)$$

It is clear that

$$RBVS \subset MRBVS.$$ 

In [13] the second author proved that $RBVS \neq MRBVS$. Moreover, the above theorem was generalized for the class $MRBVS$ in [12].
Further, the class of general monotone coefficients, $GM$, is defined as follows (see [14]):

$$GM = \left\{ a := (a_n) \in \mathbb{C} : \sum_{k=m}^{2m-1} |a_k - a_{k+1}| \leq K(a) |a_m| \text{ for all } m \in \mathbb{N} \right\}. \quad (7)$$

It is clear

$$RBVS \subset GM.$$  

In [7, 14, 15, 16] was defined the class of $\beta$–general monotone sequences as follows:

**DEFINITION 1.** Let $\beta := (\beta_n)$ be a nonnegative sequence. The sequence of complex numbers $a := (a_n)$ is said to be $\beta$–general monotone, or $a \in GM(\beta)$, if the relation

$$\sum_{k=m}^{2m-1} |a_k - a_{k+1}| \leq K(a) \beta_m \quad (8)$$

holds for all $m$.

In the paper [16] Tikhonov considered, among others, the following examples of the sequences $\beta_n$:

1. $1\beta_n = |a_n|$, 
2. $2\beta_n = \sum_{k=\lfloor n/c \rfloor}^{\lfloor cn \rfloor} |a_k|/k$ for some $c > 1$.

It is clear that $GM(1\beta) = GM$. Moreover (see [16, Remark 2.1])

$$GM(1\beta + 2\beta) \equiv GM(2\beta).$$

Consequently, we assume that the sequence $(K(\alpha_n))_{n=0}^\infty$ is bounded, that is, that there exists a constant $K$ such that

$$0 \leq K(\alpha_n) \leq K$$

holds for all $n$, where $K(\alpha_n)$ denote the sequence of constants appearing in the inequalities (5)-(8) for the sequences $\alpha_n := (a_{nk})_{k=0}^\infty$.

Now we can give the conditions to be used later on. We assume that for all $n$

$$\sum_{k=m}^{2m-1} |a_{n,k} - a_{n,k+1}| \leq K \sum_{k=\lfloor m/c \rfloor}^{\lfloor cn \rfloor} \frac{a_{n,k}}{k} \quad (9)$$

holds if $\alpha_n = (a_{n,k})_{k=0}^\infty$ belongs to $GM(2\beta)$, for $n = 1, 2, ...$

In this paper we consider the class $GM(2\beta)$ in estimate of the quantity $\left\| T_{n,A,\gamma}^q f \right\|_{Sp}$. Precisely, we extend the result of S. M. Mazhar and V. Totik (see [10, Theorem 1]) and generalize the following result of P. Pych-Taberska (see [11, Theorem 5]):
THEOREM 2. If \( f \in \Omega_{\alpha, \infty} \), \( \alpha > 0 \) and \( q \geq 2 \), then

\[
\left\| T_{q}^{n} \alpha_{n} f \right\|_{S_{\infty}^{q}} \ll \left\{ \frac{1}{n+1} \sum_{k=0}^{n} \left[ \phi f \left( \frac{\pi}{k+1} \right) \right]^{q} \right\}^{1/q} + \frac{\| f \|_{S_{\infty}^{q}}}{(n+1)^{1/q}},
\]

for \( n = 0, 1, 2, \ldots \), where \( \gamma = (\gamma_{k}) \) is a sequence with \( \gamma_{k} = \frac{\alpha k}{2} \), \( a_{n,k} = \frac{1}{n+1} \) when \( k \leq n \) and \( a_{n,k} = 0 \) otherwise.

We shall write \( I_{1} \ll I_{2} \) if there exists a positive constant \( K \), sometimes depended on some parameters, such that \( I_{1} \leq K I_{2} \).

2. Statement of the results

We start with two propositions.

PROPOSITION 1. If \( f \in \Omega_{\alpha,p} \), \( \alpha > 0 \), \( n = O(r_{n}) \) and \( q > 0 \), then

\[
\left\| \left\{ \frac{1}{r_{n}} \sum_{k=n-r_{n}}^{n-1} \left| S_{\frac{\alpha k}{2}} f - f \right|^{q} \right\} \right\|_{S_{p}}^{1/q} \ll \| f \|_{S_{p}},
\]

for \( n = 0, 1, 2, \ldots \).

PROPOSITION 2. If \( f \in \Omega_{\alpha,p} \), \( \alpha > 0 \), \( n = O(r_{n}) \) and \( q > 0 \), then

\[
\left\| \left\{ \frac{1}{r_{n}} \sum_{k=n-r_{n}}^{n-1} \left| S_{\frac{\alpha k}{2}} f - f \right|^{q} \right\} \right\|_{S_{p}}^{1/q} \ll E_{\alpha(n-r_{n})} (f)_{S_{p}},
\]

for \( n = 0, 1, 2, \ldots \).

In the special case \( p = \infty \) and \( f \in C_{2\pi} \) Proposition 2 reduce to the fundamental result of L. Leindler (see [8, Theorem 1]).

Our main results are following

THEOREM 3. If \( f \in \Omega_{\alpha,p} \), \( \alpha > 0 \), \( p \geq q \), \( (a_{n,k})_{k=0}^{\infty} \in GM(2\beta) \) for all \( n \), (1) and \( \lim_{n \to \infty} a_{n,0} = 0 \) hold, then

\[
\left\| T_{n}^{q} \alpha_{n,0} f \right\|_{S_{p}} \ll \left\{ \sum_{k=0}^{\infty} a_{n,k} E_{\frac{\alpha k}{2^{(c+1)}}}^{q} \frac{(f)_{S_{p}}^{q}}{2^{(c+1)}} \right\}^{1/q},
\]

for some \( c > 1 \) and \( n = 0, 1, 2, \ldots \), where \( \gamma = (\gamma_{k}) \) is a sequence with \( \gamma_{k} = \frac{\alpha k}{2} \).
Theorem 4. If \( f \in \Omega_{\alpha, p}, \alpha > 0, p \geq q, (a_{n,k})_{k=0}^{\infty} \in GM(2\beta) \) for all \( n, (1) \) and \( \lim_{n \to \infty} a_{n,0} = 0 \) hold, then

\[
\left\| T_{n,A, \gamma}^q f \right\|_{S^p} \ll \left\{ \sum_{k=0}^{\infty} a_{n,k} \omega^q \left( \frac{\pi}{k+1} \right)^{\frac{1}{s_p}} \right\}^{1/q},
\]

for \( n = 0, 1, 2, \ldots \), where \( \gamma = (\gamma_k) \) is a sequence with \( \gamma_k = \frac{\alpha k}{2} \).

Theorem 5. If we additionally suppose that \( (a_{n,k})_{k=0}^{\infty} \in MS \) then

\[
\left\| T_{n,A, \gamma}^q f \right\|_{S^p} \ll \left\{ \sum_{k=0}^{\infty} a_{n,k} E_{\frac{ak}{2}}^p (f) \right\}^{1/q},
\]

for \( n = 0, 1, 2, \ldots \), where \( \gamma = (\gamma_k) \) is a sequence with \( \gamma_k = \frac{\alpha k}{2} \).

Remark 1. Taking \( a_{n,k} = \frac{1}{n+1} \) when \( k \leq n \) and \( a_{n,k} = 0 \) otherwise, in the case \( p = \infty \) we obtain the better estimate than this one from [11, Theorem 5].

3. Proofs of the results

3.1. Proof of Proposition 1

Denote by \( S_k^x f \) the sums of the form

\[
S_{\frac{ak}{2}}^x f (x) = \sum_{|\lambda| \leq \frac{ak}{2}} A_\lambda (f) e^{i\lambda \cdot x}
\]

such that the interval \( \left( \frac{ak}{2}, \frac{a(k+1)}{2} \right) \) does not contain any \( \lambda_\nu \). Applying Lemma 1.10.2 of [9] we easily verify that

\[
S_k^x f (x) - f (x) = \int_0^{\infty} \varphi_\lambda (t) \Psi_k (t) dt,
\]

where \( \varphi_\lambda (t) := f (x+t) + f (x-t) - 2 f (x) \) and \( \Psi_k (t) = \Psi_{\frac{ak}{2}, \frac{a(k+1)}{2}} (t) \) i.e.,

\[
\Psi_k (t) = \frac{4 \sin \frac{\alpha t}{4} \sin \frac{\alpha (2k+1)t}{4}}{\pi \alpha t^2}
\]

(see also [3], p. 41). Evidently, if the interval \( \left( \frac{ak}{2}, \frac{a(k+1)}{2} \right) \) contains a Fourier exponent \( \lambda_\nu \), then

\[
S_{\frac{ak}{2}}^x f (x) = S_{k+1}^x f (x) - \left( A_\lambda (f) e^{i\lambda \cdot x} + A_{-\lambda} (f) e^{-i\lambda \cdot x} \right).
\]

Since

\[
\left\{ \sum_{\nu = -\infty}^{\infty} |A_\nu (f)|^q \right\}^{1/q} \leq \| f \|_{B^p} \quad \text{for } 1 < p \leq 2 \text{ and } q = \frac{p}{p-1} \left( [1, \text{p. 78}] \right)
\]
and
\[ \|f\|_{B^p} \leq \|f\|_{S^p} \quad \text{for } p \geq 1 \text{ ([2, p. 7])}, \]
where \( \| \cdot \|_{B^p} \) is the Besicovitch norm, we have
\[ |A_{\pm \nu}(f)| \leq \|f\|_{S^p} \quad \text{for } p > 1, \]
whence the deviation
\[ \frac{1}{r_n} \sum_{k=n-r_n}^{n-1} |S_{g_k}(f) - f(x)|^q \]
can be estimated from above by
\[ \frac{1}{r_n} \sum_{k=n-r_n}^{n-1} \left( \int_0^\infty |\varphi_x(t)\Psi_{k+\kappa}(t)| \, dt \right)^q + \frac{1}{r_n} \sum_{k=n-r_n}^{n-1} (\|f\|_{S^p})^q, \]
where \( \kappa \) equals 0 or 1. Putting \( h = \frac{2\pi}{\alpha n} \) we obtain
\[ \int_0^\infty |\varphi_x(t)\Psi_{k+\kappa}(t)| \, dt = \left( \int_0^h + \int_h^{nh} + \int_{nh}^\infty \right) |\varphi_x(t)\Psi_{k+\kappa}(t)| \, dt = I_1(k) + I_2(k) + I_3(k). \]

By elementary calculations we get
\[ |I_1(k)| \leq \frac{(2k+3)\alpha}{4\pi} \int_0^h |\varphi_x(t)| \, dt \ll \frac{1}{h} \int_0^h |\varphi_x(t)| \, dt \]
and
\[ |I_3(k)| \leq \int_{nh}^\infty |\varphi_x(t)\Psi_{k+\kappa}(t)| \, dt \ll \int_{nh}^\infty \frac{|\varphi_x(t)|}{t^2} \, dt. \]

Therefore
\[ \frac{1}{r_n} \sum_{k=n-r_n}^{n-1} \left[ |I_1(k)| + |I_3(k)| \right]^q \ll \left[ \frac{1}{h} \int_0^h |\varphi_x(t)| \, dt + \int_{nh}^\infty \frac{|\varphi_x(t)|}{t^2} \, dt \right]^q. \]

Consequently, we have to estimate the quantity \( \frac{1}{r_n} \sum_{k=n-r_n}^{n-1} |I_2(k)|^q \). The inequality of Hausdorff-Young [17, Chap. XII, Th. 3.3 II] yields (cf. [11, p. 102])
\[ \frac{1}{r_n} \sum_{k=n-r_n}^{n-1} |I_2(k)|^q \ll \frac{1}{n} \sum_{k=1}^n |I_2(k)|^q \ll \frac{1}{n} \left[ \int_{nh}^{nh} \frac{|\varphi_x(t)|^q}{t^{q'}} \, dt \right] \frac{q'}{q}, \]
where \( q' = \frac{q}{q-1} \) and \( q \geq 2 \).
By monotonicity of \( \left\{ \frac{1}{r_n} \sum_{k=n-r_n}^{n-1} \left| S_{\frac{ak}{2}} f - f \right| \right\}^{1/v} \) with respect to \( v > 0 \),

\[
\left\| \left\{ \frac{1}{r_n} \sum_{k=n-r_n}^{n-1} \left| S_{\frac{ak}{2}} f - f \right| \right\}^{1/v} \right\|_{S^p} \leq \left\| \left\{ \frac{1}{r_n} \sum_{k=n-r_n}^{n-1} \left| S_{\frac{ak}{2}} f - f \right|^q \right\}^{1/q} \right\|_{S^p} 
\]

\[
\leq \frac{1}{h} \int_0^h \| \phi (t) \|_{S^p} dt + \int_{nh}^\infty \frac{\| \phi (t) \|_{S^p} t^2}{t} dt + \left\{ \frac{1}{n} \int_{nh}^{nh} t^{q'} \right\}^{1/q'} + \| f \|_{S^p} 
\]

\[
\leq \left\| f \right\|_{S^p} \left[ 2 + \int_{nh}^\infty \frac{1}{t^2} dt \right] + \left\{ \frac{1}{n} \int_{nh}^{nh} t^{q'} dt \right\}^{1/q'} \leq \| f \|_{S^p},
\]

for any \( v \in (0, q) \) such that \( q' \leq p \).

Thus the desired result follows. \( \square \)

### 3.2. Proof of Proposition 2

The proof is standard and the estimate follows from that of Proposition 1. Namely, taking \( g \sigma \in B \sigma \) such that \( E_\sigma (f)_{S^p} = \| f - g \sigma \|_{S^p} \) we obtain

\[
\left\| \left\{ \frac{1}{r_n} \sum_{k=n-r_n}^{n-1} \left| S_{\frac{ak}{2}} f - f \right|^q \right\}^{1/q} \right\|_{S^p} = \left\| \left\{ \frac{1}{r_n} \sum_{k=n-r_n}^{n-1} \left| S_{\frac{ak}{2}} f - g \sigma - (f - g \sigma) \right|^q \right\}^{1/q} \right\|_{S^p} 
\]

\[
\leq \left\| f - g \sigma \right\|_{S^p},
\]

with \( \sigma = \frac{\alpha (n-r_n)}{2} \), and thus our result follows. \( \square \)

### 3.3. Proof of Theorem 3

Let

\[
\left\| T^q_{nA, Y} f \right\|_{S^p} = \left\| \left\{ 2^{c_n} \sum_{k=0}^{a_n k} \left| S_{\frac{ak}{2}} f - f \right|^q + \sum_{k=2^{c_n}}^{\infty} a_n k \left| S_{\frac{ak}{2}} f - f \right|^q \right\}^{1/q} \right\|_{S^p}
\]
for some \( c > 1 \). Using Proposition 2 we obtain, for \( p \geq q \),

\[
I_1 \leq \left\| \left\{ \sum_{k=0}^{2^c-1} a_{n,k} \left| S_{\frac{k}{2}} f - f \right|^q \right\}_{Sp}^{1/q} \right\|
\]

\[
\leq \left\{ \sum_{k=0}^{2^c-1} a_{n,k} \left( \sum_{l=k/2}^{k} \left| S_{\frac{l}{2}} f - f \right|^q \right)^{1/q} \right\}_{Sp}^{1/q}
\]

\[
= \left\{ \sum_{k=0}^{2^c-1} a_{n,k} E_{\frac{q}{2}}^q (f)_{Sp} \right\}^{1/q}.
\]

By partial summation, our Proposition 2 gives

\[
I_2 = \left\| \left\{ \sum_{m=[c]}^{\infty} \sum_{k=2^m}^{2^{m+1}-2} \left( a_{n,k} - a_{n,k+1} \right) \left| S_{\frac{k}{2}} f - f \right|^q \right\}_{Sp}^{1/q} \right\|
\]

\[
+ a_{n,2^{m+1}-1} \left( \sum_{l=2^m}^{2^{m+1}-2} \left| S_{\frac{l}{2}} f - f \right|^q \right)^{1/q} \right\}_{Sp}^{1/q}
\]

\[
= \left\{ \sum_{m=[c]}^{\infty} \sum_{k=2^m}^{2^{m+1}-2} \left| a_{n,k} - a_{n,k+1} \right| \left( \sum_{l=2^m}^{2^{m+1}-2} \left| S_{\frac{l}{2}} f - f \right|^q \right)^{1/q} \right\}_{Sp}^{1/q}
\]

\[
+ a_{n,2^{m+1}-1} \left( \sum_{l=2^m}^{2^{m+1}-2} \left| S_{\frac{l}{2}} f - f \right|^q \right)^{1/q} \right\}_{Sp}^{1/q}
\]

\[
= \left\{ \sum_{m=[c]}^{\infty} \sum_{k=2^m}^{2^{m+1}-2} \left| a_{n,k} - a_{n,k+1} \right| \left( E_{\frac{q}{2}}^q (f)_{Sp} \right)_{Sp}^{1/q}
\]

\[
+ 2^m a_{n,2^{m+1}-1} E_{\frac{q}{2}}^q (f)_{Sp} \right\}^{1/q}
\]

\[
= \left\{ \sum_{m=[c]}^{\infty} \sum_{k=2^m}^{2^{m+1}-2} \left| a_{n,k} - a_{n,k+1} \right| \left( E_{\frac{q}{2}}^q (f)_{Sp} \right)_{Sp}^{1/q}
\]

for \( p \geq q \).
Since (9) holds, we have
\[ a_{n,s+1} - a_{n,r} \]
\[ \leq |a_{n,r} - a_{n,s+1}| \leq \sum_{k=r}^{s} |a_{n,k} - a_{n,k+1}| \]
\[ \leq 2^{m+1} - 2 \sum_{k=2^m} \left| a_{n,k} - a_{n,k+1} \right| \leq \sum_{k=[2^m/c]}^{[2^{m+1}]} \frac{a_{n,k}}{k} \quad (2 \leq 2^m \leq r \leq s \leq 2^{m+1} - 2), \]
whence
\[ a_{n,s+1} \ll a_{n,r} + \sum_{k=[2^m/c]}^{[2^{m+1}]} \frac{a_{n,k}}{k} \quad (2 \leq 2^m \leq r \leq s \leq 2^{m+1} - 2). \]

Consequently,
\[ 2^m a_{n,2^{m+1}-1} = \frac{2^m}{2^m - 1} \sum_{r=2^m}^{2^{m+1} - 2} a_{n,2^{m+1}-1} \]
\[ \ll \sum_{r=2^m}^{2^{m+1} - 2} \left( a_{n,r} + \sum_{k=[2^m/c]}^{[2^{m+1}]} \frac{a_{n,k}}{k} \right) \]
\[ \ll \sum_{r=2^m}^{2^{m+1} - 1} a_{n,r} + 2^m \sum_{k=[2^m/c]}^{[2^{m+1}]} \frac{a_{n,k}}{k} \]

and therefore
\[ I_2 \ll \left\{ \sum_{m=[c]}^{\infty} \frac{2^m E_{\alpha 2^m}^q (f)_{Sp}}{2} \sum_{k=[2^m/c]}^{[2^{m+1}]} \frac{a_{n,k}}{k} + E_{\alpha 2^m}^q (f)_{Sp} \sum_{k=2^m}^{2^{m+1} - 1} a_{n,k} \right\}^{1/q}. \]

Using typical transformations we get
\[ I_2 \ll \left\{ \sum_{m=[c]}^{\infty} \frac{2^m E_{\alpha 2^m}^q (f)_{Sp}}{2} \sum_{k=2^{m-1}}^{2^{m+1}-1} \frac{a_{n,k}}{k} + E_{\alpha 2^m}^q (f)_{Sp} \sum_{k=2^m}^{2^{m+1}} a_{n,k} \right\}^{1/q}, \]
\[ \ll \left\{ \sum_{m=[c]}^{\infty} \frac{2^m E_{\alpha 2^m}^q (f)_{Sp}}{2} \sum_{k=2^{m-1}}^{2^{m+1}-1} a_{n,k} \right\}^{1/q}, \]
\[ = \left\{ \sum_{m=[c]}^{\infty} \frac{2^m E_{\alpha 2^m}^q (f)_{Sp}}{2} \sum_{k=2^{m-1}}^{2^{m-1}} a_{n,k} + \sum_{m=[c]}^{\infty} \frac{2^m E_{\alpha 2^m}^q (f)_{Sp}}{2} \sum_{k=2^m}^{2^{m+1}} a_{n,k} \right\}^{1/q}, \]
\[ \ll \left\{ \sum_{m=[c]}^{\infty} \frac{2^{m-1}}{2^{m-1}} a_{n,k} E_{\alpha 2^m}^q (f)_{Sp} + \sum_{m=[c]}^{\infty} \frac{2^m E_{\alpha 2^m}^q (f)_{Sp}}{2} \sum_{m=[c]}^{2^{m+1}} a_{n,k} \right\}^{1/q}. \]
and basic properties of the modulus of continuity

3.4. Proof of Theorem 4

The proof follows by the Jackson type theorem

\[ E_\sigma(f)_{Sp} \ll \omega_f \left( \frac{1}{\sigma} \right)_{Sp} \]

and basic properties of the modulus of continuity \( \omega_f (\cdot)_{Sp} \). □

3.5. Proof of Theorem 5

If \((a_{n,k})_{k=0}^\infty \in MS\), then \((a_{n,k})_{k=0}^\infty \in GM(2\beta)\) and using Theorem 4 we obtain

\[
\left\| T_{n,A,\gamma}^q f \right\| \ll \left\{ \sum_{k=0}^{\infty} a_{n,k} E_{\alpha k^{2}}^q (f)_{Sp} \right\}^{1/q} = \left\{ \sum_{k=0}^{(k+1)2^{c+1}-1} a_{n,m} E_{\alpha k^{2}}^q (f)_{Sp} \right\}^{1/q}
\]

\[
\ll \left\{ \sum_{k=0}^{\infty} E_{\alpha k^{2}}^q (f)_{Sp} \sum_{m=2^c}^{2^{c+1}-1} a_{n,m} \right\}^{1/q} \ll \left\{ \sum_{k=0}^{2^c} E_{\alpha k^{2}}^q (f)_{Sp} a_{n,k} \right\}^{1/q}
\]

This ends our proof. □
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