The relation between optical beam propagation in free space and in strongly nonlocal nonlinear media

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received 9 January 2009; accepted in final form 27 April 2009
published online 27 May 2009

PACS 42.65.Jx – Beam trapping, self-focusing and defocusing; self-phase modulation
PACS 42.65.Tg – Optical solitons; nonlinear guided waves
PACS 42.25.Fx – Diffraction and scattering

Abstract – The relation between optical beam propagation in strongly nonlocal nonlinear (SNN) media and propagation in free space is demonstrated using the technique of variable transformation. The governing equation, integral and analytical solutions, and propagation properties in free space can be directly transferred to their counterparts in SNN media through a one-to-one correspondence. The one-to-one correspondence together with the Huygens-Fresnel integral yields an efficient numerical method to describe SNN propagation. The existence conditions and possible structures of solitons and breathers in SNN media are described in a unified manner by comparing propagation properties in SNN media with those in free space. The results can be employed in other contexts in which the governing equation for the evolution of waves is equivalent to that in SNN media, such as for quadratic graded-index media, or for harmonically trapped Bose-Einstein condensates in the noninteracting limit.

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Introduction. – The propagation properties of light beams in nonlocal nonlinear media have attracted significant attention in recent years. There are various interesting properties induced by the nonlocality, such as the suppression of collapse [1] and the support of vortex solitons [2] and multi-pole solitons [3]. For the special case of the strongly nonlocal nonlinear (SNN) media in which the characteristic length of the material response function is much larger than the beam width, the propagation equation can be linearized to the well-known Snyder-Mitchell model (SMM) [4]. In fact, since Snyder and Mitchell introduced the SMM to investigate optical beam propagation in SNN media, various soliton solutions [2,3,5–10], such as Hermite-Gaussian (HG) [3], Laguerre-Gaussian (LG) [3] and Ince-Gaussian (IG) [6,9] solitons, have been predicted theoretically. Some soliton structures and their interaction have been observed experimentally in SNN materials such as nematic liquid crystal [11,12], and lead glass [13].

In addition, to the best of our knowledge, the propagation in free space has been investigated more thoroughly than the other propagation problems in the field of optics. The paraxial diffraction equation has been thoroughly investigated. Various types of beam solutions with different transverse profiles have been obtained in Cartesian, circular cylindrical, and elliptical coordinates (e.g., [14–16] and references therein). These solutions can be roughly classified into two types: i) shape-invariant beams, such as LG, HG, and IG beams; and ii) shape-variant beams, such as higher-order elegant-Hermite-Gaussian (EHG), elegant-Laguerre-Gaussian (ELG), and elegant-Ince-Gaussian (EIG) beams. The propagation of these beams has been investigated in detail and many parameters, such as width, divergence, radius of curvature, and quality factor, have been introduced to describe their propagation. In summary, the theory of free propagation has been well developed over the past decades.

Compared with that of the free propagation, the problem of SNN propagation is mathematically much more complicated [17], even for the problem of soliton which may be the simplest example of the propagation problems in SNN media. However, the structures of the HG, LG and IG solitons introduced in the literature are also the modes in free space [18], leading naturally to the question of whether any direct relation exists between free propagation and propagation in SNN media.

In fact, if there exists a one-to-one correspondence between free propagation and propagation in SNN media, the results of free-propagation theory could be applied to
the following aspects of the study of SNN propagation: i) The structures of and the existence conditions for breathers and solitons could be conveniently described with simple and intuitive physical pictures of free propagation in a unified manner. ii) Whereas the previous investigations are mainly focused on solitons and breathers that are shape invariant upon propagation, the shape-variant propagation in SNN media remains unexplored. With an established correspondence between free and SNN propagation, it would be easy to deal with the propagation of an arbitrary field in SNN media by using the well-developed theory of free propagation, avoiding complicated mathematical calculations. iii) The well-developed parameters of free propagation could be directly transplanted to the SNN case to characterize the propagation properties. iv) In experiments, beams are usually transmitted from free space into SNN media. A direct correspondence between free propagation and propagation in SNN media would be of practical interest for designing of experiments.

In this letter, we describe our research into the relation between the SNN propagation and the free propagation, and describe the propagation in SNN media using the well-developed theory and clear physical pictures of free propagation. The relation between propagation in SNN media and that in free space is studied using the variable transformation technique. This shows that the governing equation, beam solutions, and propagation properties for free propagation can be directly transplanted with a one-to-one correspondence to the case of propagation in SNN media. On the basis of the one-to-one correspondence and the Huygens-Fresnel integral, we develop an efficient numerical method to describe SNN propagation. We describe the existence conditions and possible structures of solitons and breathers in SNN media in a unified manner by comparing the propagation properties in SNN media with those in free space. As an example, the theoretical predictions are illustrated for the case of EHG beams.

One-to-one correspondence. – We begin by using the technique of variable transformation to connect propagation in SNN media to free propagation. This technique is frequently used to study similaritons in nonlinear wave guides (e.g., [19,20]). The variable transformation technique allows the governing equation (e.g., the inhomogeneous nonlinear Schrödinger equation (NLSE)) to be reduced to a simpler mathematical form (e.g., the standard NLSE), and the solution of the former can be obtained from the solution of the latter by exploiting a one-to-one correspondence. Because the structures of the HG, LG and IG solitons in SNN media are also the modes in free space [18], it is reasonable to expect that a transformation exists that can reduce the governing equation of SNN propagation to that of free propagation.

In the laboratory reference frame, beam propagation in nonlocal nonlinear media is governed by the nonlocal NLSE

\[ 2ikn_0 \partial_z A + n_0(\partial_{xx} + \partial_{yy})A + 2k^2 \Delta n A = 0, \]  

where \( k \) represents the wave number in the media with a linear refractive index \( n_0 \), \( \Delta n = n_2 \int R(\mathbf{r} - r_m) |\Phi|^2 d^2 r_m \) represents the nonlinear perturbation of refractive index, where \( n_2 \) is the nonlinear index coefficient, \( R \) is the normalized symmetric real spatial response function of the media, and \( r = (x,y) \) represents the 2-dimensional transverse coordinate vector.

In the case of SNN media the nonlocal NLSE can be simplified to the modified SMM [10]

\[ 2ik\partial_z \Phi + (\partial_{x'x'} + \partial_{y'y'})\Phi - k^2 \gamma^2 P_0 r^2 \Phi = 0 \]  
in a new reference frame that moves with the center of mass,

\[ z' = z, \quad r' = r - r_\epsilon(z). \]

The adoption of this reference frame is important for the input fields whose transverse spatial momentum is nonzero [10]. In this reference frame the field becomes

\[ \Phi(r', z') = A(r' + r_\epsilon, z') \times \exp \left[ -\frac{ikM \cdot (r' + r_\epsilon) + ikM^2 z'}{P_0} \right]. \]

In eqs. (2)–(4), \( \gamma \) is a material constant, \( P_0 = \int |\Phi|^2 d^2 r' \) is the input power, \( r_\epsilon(z') = r_\epsilon(0) + Mz'/P_0 \) is the center of mass of the beam, \( M = (i/2k) \int (A \nabla_\bot A^* - A^* \nabla_\bot A) dz dy \) is the transverse spatial momentum, \( r' = (x', y') \), and \( r_\epsilon = (x_c, y_c) \).

To connect the governing equation of SNN propagation with that of free propagation, we adopt the transformations

\[ \begin{cases} r' = (-1)^a \frac{w_{c0}}{w_c(\zeta)} s, \\ z' = z_{c0} \arctan \left( \frac{\zeta}{z_{c0}} \right) + a \pi, \\ \Phi(r', z') = (-1)^a \frac{w_c(\zeta)}{w_{c0}} \exp \left[ -\frac{ik^2 s^2}{2R_c(\zeta)} \right] \Psi(s, \zeta), \end{cases} \]

where \( w_c(\zeta) = w_{c0}[1 + (\zeta/z_{c0})^2]^{1/2}, R_c(\zeta) = \zeta[1 + (z_{c0}/\zeta)^2], z_{c0} = kw_{c0}^2, w_{c0} = (k^2 \gamma^2 P_0)^{-1/4}, a = 0, 1, -1, 2, -2, \cdots \), and \( s = (\mu, \nu) \). Then eq. (2) reduces to

\[ (\partial_{\mu \mu} + \partial_{\nu \nu}) \Psi + 2ik \partial_\zeta \Psi = 0. \]  

Equation (6) is the well-known paraxial diffraction equation that governs the paraxial propagation of monochromatic beams in free space. Thus, eqs. (2)–(6) establish the one-to-one correspondence between the beam solution in SNN media and that in free space:

\[ \Phi(r', z') = F_1 F_2 \times \Psi(F_1 r', F_2), \]
where

\[
\begin{align*}
F_1(z') &= (-1)^s \left[ 1 + \tan^2 \left( \frac{z'}{z_{c0}} \right) \right]^{1/2}, \\
F_2(r', z') &= \exp \left\{ -\frac{ikF_1(z')^2 r'^2}{2z_{c0}[\tan(z'/z_{c0}) + 1/\tan(z'/z_{c0})]} \right\}, \\
F_3(z') &= z_{c0} \tan \left( \frac{z'}{z_{c0}} \right), \\
a(z') &= \pi \left\{ \frac{z'}{z_{c0}} - \arctan \left[ \tan \left( \frac{z'}{z_{c0}} \right) \right] \right\}, \\
z_{c0}(P_0) &= \frac{1}{\sqrt{P_0}}.
\end{align*}
\]

(8)

Note that \( z_{c0} \) is not a constant, but varies with the input power \( P_0 \). Equation (7) connects the propagation in SNN media with free propagation. The numerous free-space monochromatic beam solutions and propagation properties can be conveniently transplanted to their counterparts in SNN media using eq. (7).

Since a one-to-one correspondence exists between the beam solutions in SNN media and those in free space, a general comparison between the propagation properties in SNN media and those in free space would be constructive. We therefore compare three beam propagation properties.

Beginning with beam patterns, we see from eq. (7) that the beam in SNN media evolves periodically with a period \( \Delta z = 2\pi z_{c0} \). For convenience of discussion, we divide each period (from \( z' = 2(2a-1/2)\pi z_{c0} \) to \( z' = 2(2a+3/2)\pi z_{c0} \)) into two half-periods. In the leading half-period (from \( z' = 2(2a-1/2)\pi z_{c0} \) to \( z' = 2(2a+1/2)\pi z_{c0} \)), the evolution of the pattern is a condensed configuration of that in free space from \(-\infty\) to \(+\infty\). There is a one-to-one correspondence between patterns in SNN media and those in free space, i.e., the pattern shape at the cross-section \( z' \) in SNN media is the same as that at the cross-section \( z_{c0} \) tan \( z' / z_{c0} \) in free space. At special cross-sections where \( z'/z_{c0} - 2a\pi = -\pi/2, -\pi/4, 0, \pi/4, \pi/2 \), the beam pattern shapes in SNN media are, respectively, the same as at \( z = -\infty, -z_{c0}, 0, z_{c0}, +\infty \) in free space. In the trailing half-period (from \( z' = (2a+1/2)\pi z_{c0} \) to \( z' = (2a+3/2)\pi z_{c0} \)), the beam patterns are the reverse of those in the leading half-period, and the evolution of the pattern is corresponding to that of a inverse field \( (\Psi(-s, \zeta)) \) in free space. In fact, patterns of most beams are symmetrical. For these beams, the beam patterns in the trailing half-period are the same as those in the leading half-period, therefore the period of pattern evolution reduces to \( \Delta z = \pi z_{c0} \).

The second beam property we compare is the beam width. Although the pattern shape at the cross-section \( z' \) in SNN media is the same as that at the cross-section \( z_{c0} \) tan \( z' / z_{c0} \) in free space, the beam width in SNN media decreases by a factor of \( |F_1(z')| \) compared with that in free space. Explicitly,

\[
w^{(s)}(z') \bigl|_{z'=z'_{s}} = \frac{w^{(f)}(\zeta)_{|z=z_{c0} \tan(z'/z_{c0})}}{F_1'(z')} \quad (9)
\]

where the superscripts \((f)\) and \((s)\) refer to the cases of propagation in free space and in SNN media, respectively, and \( w^{(s)} \) and \( w^{(f)} \) are the corresponding beam widths. Equation (9) results from the self-focusing effect of SNN media. Accordingly the amplitude increases by a factor of \( |F_1(z')| \), in accordance with energy conservation.

The final propagation property we compare involves the cophaseal surfaces. For convenience of discussion, we assume the radius of the cophaseal surface at the cross-section \( \zeta = z_{c0} \) tan \( z'/z_{c0} \) in free space is \( R^{(f)}(s, \zeta) \), so that in free space the phase variation across the transverse plane can be written as exp \( [ikr^2/2R^{(f)}] \). In SNN media, according to eq. (7), the phase variation across the transverse plane at the corresponding cross-section \( z' = z'_{s} \) would be exp \( [ikr^2/2R^{(s)}(z')] \), where \( R^{(s)}(z') \) is the radius of the cophaseal surface in SNN media, and is given by

\[
R^{(s)}(z') = \frac{1}{\sqrt{R^{(f)(s, \zeta)}} - F_1'(z')} \quad (10)
\]

In the special case where \( R^{(f)}(s, \zeta) = R_{c0}(\zeta) \), \( R^{(s)}(z') \) approaches infinity, the cophaseal surface remains planar upon propagation.

The evolution of the beam width and of the cophaseal surfaces is interdependent for SNN propagation. For example, when a HG beam is input at the waist and the relation \( z_R = z_{c0} \) (where \( z_R \) is the Rayleigh distance) is satisfied, the free propagation increases the beam width by a factor of \( F_1(z') \) (i.e., \( w^{(f)}(\zeta) = F_1(z')w^{(f)}(0) \)), so that in SNN media the beam width remains invariant during propagation (i.e., \( w^{(s)}(z') = w^{(f)} / F_1'(z') = w^{(f)}(0) \)). Simultaneously, the cophaseal surface remains planar during propagation, because \( R^{(f)}(s, \zeta) = R_{c0}(\zeta) \) is satisfied. If the beam is not input at the waist and/or the relation \( z_R = z_{c0} \) is not satisfied, the beam width and the cophaseal surface both evolve periodically during propagation. This property is important for the existence of solitons and breathers, as will be discussed below.

**Numerical description of SNN propagation.** In real optical systems, there exist many beams with irregular amplitude and phase profiles. Numerical methods play an important role in studying these situations. The most popular numerical method is the split-step Fourier method (SSFM). The SSFM is based on dividing the propagation length into a large number of segments and assuming that in each segment the diffractive and nonlinear effects are independent and may be calculated separately. Although the SSFM is much faster than the finite-difference method, it is still time consuming, because of the large number of segments which is required to ensure the accuracy. Since in the SNN case the nonlocal NLSE can be simplified to the SMM model (connected to free propagation), we develop a simple numerical method that is more efficient than the SSFM.

In the case of free propagation, an efficient numerical method exists based on the integral solution of eq. (6),
i.e., the Huygens-Fresnel integral [18]

\[
\Psi(s, \zeta) = \frac{-ik}{2\pi \zeta} \int \Psi(s_0, 0) \exp \left[ \frac{ik}{2\zeta} (s - s_0)^2 \right] d^2s_0. \tag{11}
\]

Because eq. (11) represents a convolution of the input field with a spherical wave function, the field at any plane can be obtained easily from the input plane by using the fast Fourier transform algorithm.

Since there is a one-to-one correspondence between the beam solution in SNN media and that in free space, we get the integral solution in SNN media using eqs. (7) and (11), as follows:

\[
\Phi(r', z') = \int \varphi(r', r_0') \Phi(r_0', 0) d^2r_0', \tag{12}
\]

where

\[
\varphi(r', r_0') = \frac{-i}{2\pi w_0^2 \sin(\frac{\zeta}{z_0})} \times \exp \left[ \frac{ir'^2 + ir_0'^2 - 2ir' \cdot r_0' \sec(\frac{\zeta}{z_0})}{2w_0^2 \tan(\frac{\zeta}{z_0})} \right]. \tag{13}
\]

Equation (12) is equivalent to \( \Phi(r', z') = F_{\alpha} \{ \Phi(r_0', 0) \} e^{-i\alpha} \), where \( F_{\alpha} \) represents the fractional Fourier transform of order \( \alpha = z'/z_0 \). For this reason, in a separate publication [21] we refer to the propagation in SNN media as the self-induced fractional Fourier transform.

The numerical simulation of the SNN propagation in the laboratory reference frame can therefore be accomplished in three steps: First, we calculate the initial center of mass \( r_0, 0 \) as well as the transverse momentum \( \mathbf{M} \) and use eq. (4) to transform the field in the laboratory reference frame (i.e. \( A(r, z) \)) to the field in the reference frame \( z' = z, r' = r - r_c \) (i.e. \( \Phi(r', z') \)). Next, we propagate the field in the reference frame \( z' = z, r' = r - r_c \) from the input plane \( z' = 0 \) to the later plane \( z' = z' \) using eq. (12). Finally, we transform the field at the plane \( z' = z' \) in the reference frame \( z' = z, r' = r - r_c \) to that in the laboratory reference frame using the inverse transformation

\[
A(r, z) = \Phi(r - r_c, z) \exp \left[ \frac{-ikM \cdot r}{P_0} + \frac{ikM^2 z}{2P_0^2} \right]. \tag{14}
\]

This approach provides a straightforward way to numerically propagate any field from the input plane to an arbitrary later plane in SNN media. Since the propagation distance is not required to be divided into a large number of segments, only a single fractional Fourier transform is required, making this approach much more efficient than the SSFM for the SNN case. We believe the fast fractional Fourier transform algorithm which has been developed in recent years (e.g. [22] and references therein) would make this approach even more efficient.

Breathers and solitons in SNN media. – A special feature of SNN media is that the nonlocality supports (2+1)D solitons and breathers [1,3,6,9], preventing the catastrophic collapse that occur in local nonlinear media. Here, using eq. (7) and comparing propagation properties in SNN media to those in free space, the existence conditions of breathers and solitons in SNN media can be described conveniently in a unified manner. If a beam keeps its shape during free propagation, it would be a breather in SNN media, because the beam shape would remain invariant and the beam width as well as the cophasal surface would evolve periodically with the period \( \Delta \zeta = \pi z_0 \) in SNN media. Furthermore, if the input power and the entrance plane are designed appropriately so that the beam width and the beam shape simultaneously remain invariant during SNN propagation, the breather would reduce to a soliton.

This explains why the HG, LG, and IG solitons exist in the SNN media. They retain the beam shape in free space, therefore they generally evolve as breathers in SNN media. In the special case that the field is input at the beam waist and the input power \( P_0 \) equals the critical power \( P_c \) (so that \( z_0 = z_R \), where \( P_c = 1/(k^2\gamma^2w_0^4) \), and \( z_R \) is the Rayleigh distance of the input field), then diffraction increases and self-focusing decreases the beam width by the same factor of \( F_1(z') \), and the deform of the cophasal surfaces caused by self-focusing exactly balances that caused by diffraction. Therefore the beam width in addition to the beam shape remains invariant, and the breather reduces to a soliton.

Furthermore, based on the analysis above we can extend the range of breathers and solitons in SNN media to the input fields that are linear superpositions of the degenerate solutions of HG, LG, and IG beams with the same Rayleigh distance, beam waist location, and Gouy phase shift in free space. These superposed fields are shape invariant in free propagation, thus they propagate as solitons in SNN media when the entrance plane is located at the beam waist and \( P_0 = P_c \), otherwise they propagate as breathers. The various structures of these superposed fields would greatly enrich the family of solitons and breathers.

Example. – To illustrate the predictive capacity of our calculations, we take the EHG beams as an example. In free space, the field of the \((m, n)\) mode EHG beam can be written as [18]

\[
\Psi_{mn}(s, \zeta) = \psi \left( \frac{q_0}{q(\zeta)\mu} \right)^{m+n+1} H_m(\sqrt{c(\zeta)}\mu) \times H_n(\sqrt{c(\zeta)}\nu) \exp[-c(\zeta)s^2], \tag{15}
\]

where \( c(\zeta) = -ik/2q(\zeta), \quad q(\zeta) = \zeta - iz_R, \quad z_R = kw_0^2 \) is the Rayleigh distance, and the coefficient \( \psi \) is determined by the input power through \( P_0 = \int |\Psi|^2 d^2s \). In free propagation, the \((m, n)\) mode EHG beam is shape invariant when \( m, n \leq 1 \), otherwise it is shape variant.

Obtaining the field of EHG beams in SNN media using SMM directly is mathematically complicated. However,
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Fig. 1: (Colour on-line) Evolution of the pattern of the (2, 0) mode EHG beam in free space (row 1) and in SNN media (rows 2–4). The input power of the SNN propagation is $P_0 = 0.5P_c$ (row 2), $P_c$ (row 3), and $2P_c$ (row 4), respectively, where $P_c = 1/(k^2R^2w_0^2)$. In row 1 the transverse dimension is scaled by a factor of $1/[(\zeta/z_{c0})^{1/2}]$, whereas in other rows it is not scaled.

Using eq. (7) it is easily obtained as

$$\Phi_{mn}(r', z') = F_1(z')F_2(r', z')\psi\left[\frac{q_0}{Q(z')}\right]^{n+m+1} \times H_m[\sqrt{C(z')}F_1(z')x']H_n[\sqrt{C(z')}F_1(z')y'] \exp\{-C(z')|F_1(z')r'|^2\},$$

(16)

where $C(z') = -ik/2Q(z')$, $Q(z') = z_{c0}\tan(z'/z_{c0}) - iz_R$, $z_{c0} = 1/\gamma\sqrt{P_0}$.

Figure 1 shows the evolution of the pattern of the (2, 0) mode EHG beam in a SNN medium under varying input power compared with the same situation for free propagation. We find that if the pattern varies upon propagation in free space, it does so in the SNN medium as well. Because the transverse pattern is distributed symmetrically, the pattern evolves periodically with the period $\Delta z = z_{c0}$.

Furthermore, at the cross-section $z' = (a + 1/2)\pi z_{c0}$, the pattern is the same as that at the far field in free space and does not vary with the input power, and the beam width $w$ is inversely proportional to $\sqrt{P_0}$, because the field at the cross-section may be regarded as the conventional Fourier transform of the input field. At an arbitrary cross-section where $z' \neq a\pi z_{c0}/2$, the pattern shape is the same as that at the free-space cross-section $\zeta = z_{c0}\tan(z'/z_{c0})$. Since $\zeta$ varies with the input power, the pattern is different for different input power.

Because the (1, 1) mode EHG beam is shape invariant in free space, its pattern remains invariant in SNN media (fig. 2). Generally, the (1, 1) mode EHG in SNN media propagates as a breather, i.e., the beam width as well as the radius of the cophasal surfaces varies periodically with the period $\Delta z = \pi z_{c0}$. At special cross-sections where $z'/z_{c0} = a\pi/2$ the width is either maximum or minimum (in fact, the width reaches its maximum and minimum alternately during propagation), and the radius of the cophasal surfaces approaches infinity (i.e., $\tan^{-1}(R) = \pi/2$). For the special case where $P_0 = P_c$, the relation $z_{c0} = z_R$ is ensured and diffraction is exactly balanced by self-focusing. Therefore the beam width and the radius of the cophasal surfaces are invariant during propagation, and the breather reduces to a soliton.

Conclusion. – In conclusion, optical beam propagation in SNN media is connected with free propagation using the technique of variable transformation. The fact that the solutions as well as the propagation properties in free space can be mapped to their counterparts in SNN media through a one-to-one correspondence makes this technique useful for investigating the propagation problems in SNN media. The efficient numerical method provided in this letter is of interest for the investigation of beams with irregular amplitude and phase profiles in SNN media. The technique provides a unified description of the existence conditions and gives possible structures of solitons and breathers in SNN media. The various soliton and breather structures predicted herein would greatly enrich the family of solitons and breathers.

Mathematically, the modified SMM is equivalent to the famous equation for the linear harmonic oscillator, which is widely used in many branches of physics. Therefore, the relation proposed in this letter can connect the free propagation not only with propagation in SNN media, but also with the evolution of waves in other systems in which the governing equations reflect the equation for the linear harmonic oscillator. The technique described in this letter can be readily employed in other contexts with equivalent governing equation, such as for quadratic graded-index media [23], or for harmonically trapped Bose-Einstein condensates in the noninteracting limit [24,25].

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This research was supported by the National Natural Science Foundation of China (Grants No. 10674050 and
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