String Theory on $AdS_3$ Revisited

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ABSTRACT

We discuss string theory on $AdS_3 \times S^3 \times M^4$ with particular emphasis on unitarity and state-operator correspondence. The AdS-CFT correspondence, in the Minkowski signature, is re-examined by taking into account the only allowed unitary representation: the principal series module of the affine current algebra $SL(2,R)$ supplemented with zero modes. Zero modes play an important role in the description of on-shell states as well as of windings in space-time at the $AdS_3$ boundary. The theory is presented as part of the supersymmetric WZW model that includes the supergroup $SU(2/1,1)$ (or $OSp(4/2)$ or $D(2,1;\alpha)$) with central extension $k$. A free field representation is given and the vertex operators are constructed in terms of free fields in $SL(2,R)$ principal series representation bases that are labeled by position space or momentum space at the boundary of $AdS_3$. The vertex operators have the correct operator products with the currents and stress tensor, all of which are constructed from free fields, including the subtle zero modes. It is shown that as $k \to \infty$, $AdS_3$ tends to flat 3D-Minkowski space and the $AdS_3$ vertex operators in momentum space tend to the vertex operators of flat 3D-string theory (furthermore the theory readjusts smoothly in the rest of the dimensions in this limit).

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1 Introduction

There has been much discussion on the topic of a string propagating on AdS$_3$ curved spacetime [1]-[16]. The early interest was due to the challenge of solving exactly the problem of a string theory in curved spacetime with Minkowski signature. At the early stages this effort revealed some problems with ghosts, despite the Virasoro constraints. The solution to the problem was presented several years later in [9] [10] where it was shown that there were two simple but essential points that were missed in earlier investigations: (i) the correct unitary representation and (ii) zero modes. Both of these are provided naturally by the explicit analysis of the non-compact SL(2, R) WZW model but it was easy to miss them in the earlier abstract current algebra approach that included wrong assumptions by analogy to the compact SU(2). While each point is an independent feature of the model they are both needed to correctly describe the string in AdS$_3$ space.

The first point is that the WZW model based on SL(2,R), in the absence of certain zero modes (in analogy to SU(2)), permits only the unitary representation called the principal series for which $j(j + 1) \leq -1/4$ (or $j = -1/2 + is$ where $s$ is real). This is similar to saying that the model $L = r \times p$ for angular momentum permits only integer quanta for angular momentum; half integer quanta cannot occur in this model. Similarly, the discrete series or the supplementary series modules of affine SL(2, R) current algebra cannot occur in the WZW model.

The ghost problems do not arise in the principal series module, they only arise in the discrete series module which was wrongly assumed to be part of the model when using abstract current algebra in the investigations prior to 1995. In recent investigations [12] - [15] the discrete series module resurfaced in connection with the AdS-CFT conjecture, however this is unsatisfactory since the relation to the underlying string theory remains obscure and the corresponding vertex operators lack unitarity or state-operator correspondence. We show that there is a subtle solution that incorporates unitarity and state-operator correspondence, consistent with the underlying unitary string theory spectrum given in [3] [11].

The second point is that a lightcone-type momentum zero mode $p^L_-$ (and a similar $p^R_-$) was also missed in the old abstract current algebra approach. The zero mode is needed to satisfy the Virasoro mass shell condition ($L_0 = a$) for left movers in the form

$$L_0 = p^L_+ p^L_- - j(j + 1)/(k - 2) + \text{integer} = a \leq 1$$

(and a similar one for right movers). In the absence of $p^L_-$ the mass shell condition cannot be satisfied with the principal series ($-j(j + 1) = \frac{1}{2} + s^2$) when the positive integer is non-zero (string excitation level). The zero mode term $p^L_+ p^L_-$ provides the only negative contribution ($p^L_+ p^L_- = -p^2_0 + p^2_1 < 0$) thereby making it possible to satisfy the on shell condition in the same manner of a string in flat spacetime (in fact an excited string in flat spacetime also cannot be put on shell if the $p^-$ zero mode is absent: $p^+ p^- + (p_2)^2 + \text{integer} = 1$). However, the presence of the $p^L_-$ zero mode in curved spacetime is non-trivial and it requires a quantization condition.
due to the periodicity of a closed string $x(\tau, \sigma + 2\pi) = x(\tau, \sigma)$. It was shown in \cite{9} \cite{10} that due to this monodromy of the SL$(2, R)$ currents, unless $p_L^-$ vanishes, one must have a negative integer for the combination $p^+ p^-$, that is

$$p_L^+ p_L^- = p_R^+ p_R^- = -r, \quad r \in \mathbb{Z}_+.$$  \hspace{1cm} (2)

For the compact SU$(2)$ (as opposed to the non-compact SL$(2, R)$) the zero modes $p_L^-, p_R^+$ are not needed, and are taken to vanish. This is why they were missed in the early investigations of SL$(2, R)$. With these conditions the complete on-shell spectrum of the theory was computed, the no-ghost theorem was proven (after the Virasoro constraints) and the exact unitary solution of the SL$(2, R)$ WZW model was established \cite{9} \cite{10}.

The recent interest in the topic is due to the AdS-CFT conjecture \cite{17}. The AdS$_3$ string provides one of the rare exact conformal field theories in which one could possibly verify the conjecture for a full superstring theory as opposed to the low energy supergravity limit. It is important to take into account the unitarity of the model as a string on AdS$_3$ if this correspondence is to be a meaningful one. Recent papers \cite{13} - \cite{15} made progress in providing an AdS$_3$-CFT map by suggesting and investigating certain vertex operators constructed in AdS$_3$ string theory that correspond to operators in the boundary CFT theory. One interesting overlap with the previous work \cite{9} \cite{10} involves the quantized zero modes $p_L^-, p_R^+$ described above: they are related to the winding numbers on the AdS boundary $\oint d\gamma/\gamma$ , $\oint d\bar{\gamma}/\bar{\gamma}$ in the language of \cite{13} and \cite{15} (this will be described more fully below). However, the recent papers did not incorporate the unitarity conditions given in \cite{9} \cite{10} and the operator-physical state correspondence that is standard and desirable in string theory is obviously lacking. For these reasons, while the CFT-AdS ideas in recent papers are very tantalizing, the connection to the underlying physical string theory remains to be established. We will show that we are not far from this when we insist on unitarity and use only the principal series of the affine current algebra supplemented with zero modes.

In this paper we briefly review the construction of \cite{9} \cite{10} and present it in the context of a superstring on $AdS_3 \times S^3 \times M^4$ partially described by the WZW model that includes the supergroup SU$(2/1, 1)$ (or OSp$(4/2)$ or D$(2, 1; \alpha)$) with central extension $k$. We emphasize the two points mentioned above: (i) only the principal series of the affine SL$(2, R)$ is allowed and (ii) it must be supplemented by the quantized zero modes $p_L^-, p_R^+$.

We give the free field representation of the model and then construct the vertex operators in terms of free fields in two SL$(2, R)$ bases. The SL$(2,R)$ labels on these representations have the interpretation of either position or momentum at the AdS boundary. The momentum space version was introduced in \cite{10} and subsequently completed in an unpublished work by two of us \cite{18}; its details will be presented here. The position space version is related to the momentum space one by ordinary Fourier transformation. It is also related to an analytic continuation of the vertex operator for a string on $H^3_+ = \text{SL}(2, C)/SU(2)$ used in the recent literature \cite{19}. 


(take $H^3_+ \to \text{SL}(2, R)$ after Euclidean to Minkowski continuation), but only in the sector in which the zero modes $p_L^-, p_R^+$ vanish.

We also go one step further by constructing the vertex operator from free fields such that the AdS string coordinates $\phi(z, \bar{z}), \gamma(z, \bar{z}), \tilde{\gamma}(z, \bar{z})$ in curved space are themselves constructed from left and right moving free fields. We show that the free field representation of the vertex operators satisfy the correct operator products with the $\text{SL}(2, R)$ currents and the stress tensor, all constructed from left/right free fields. Using the free field construction we show that the quantized zero modes $p_L^-, p_R^+$ are proportional to the winding numbers $\oint d\gamma/\gamma, \oint d\bar{\gamma}/\bar{\gamma}$ at the boundary of AdS$_3$.

We show one other desirable property of the vertex operator, namely that it becomes the flat string vertex operator $\exp(ip_{\mu}X^{\mu}(z, \bar{z}))$ when the AdS$_3$ space tends to flat space as the central extension grows $k \to \infty$.

Using the new unitary vertex operators we re-examine the AdS-CFT correspondence discussed in recent papers. We show that there is operator-physical state correspondence. Also we show that all the main arguments of the CFT-AdS correspondence can be reformulated by using only the unitary vertex operators.

There remains to compute in strictly string theoretical language the various correlation functions or operator products that would verify or refine the AdS-CFT correspondence. In principle our free field vertex operators can be used to compute any correlation function, but we leave these computations to future papers.

## 2 SU(2/1, 1) WZW Model and free fields

### 2.1 Comments on the supergroup approach

The WZW model based on the *non-compact* supergroup SU(2/1, 1) at level $k > 0$ has one timelike and five spacelike bosonic coordinates; it also has four timelike and four spacelike fermionic degrees of freedom (see next paragraph). Therefore it has ghosts that come from timelike bosonic and fermionic modes. An additional indication that it contains ghosts is that, as for any SU(N/N) current algebra [21], its Virasoro central charge is $c = -2$ (see footnote 2 of the first paper listed in [21]). Indeed as argued in [10] this is a model for a superstring on AdS$_3 \times S^3$ with Ramond-Ramond flux, but its degrees of freedom include Faddeev-Popov ghosts that arise in the quantization procedure. The physical sector of the model can be obtained by applying an elaborate set of constraints, as discussed in [10]. A simpler picture is to consider $N = 4$ local superconformal symmetry. Such a local symmetry provides one bosonic constraint (usual Virasoro) and four fermionic constraints that match the number of timelike bosonic/fermionic degrees of freedom, thus leading to a ghost-free physical sector. Assuming the sufficiency of these constraints the WZW model for the supergroup SU(2/1, 1)
would correspond to a physical superstring theory whose interpretation is obtained through the arguments of [13].

The counting of timelike degrees of freedom and constraints (gauge invariances) that remove ghosts is an essential first step for determining if WZW models (or gauged WZW models) based on supergroups can be physical superstring theories. A quick way to arrive at the counting is to consider the signature of the fields in the Lagrangian or equivalently the signature in the operator products of the currents $J^A$. The kinetic term in the Lagrangian or the double pole in the operator product $(z-w)^{-2}$ is proportional to the central extension times the supergroup Killing metric $\frac{k}{2} \text{Str}(T^A T^B)$ where $T^A$ is a graded supermatrix that represents the generators in the fundamental representation. The timelike/spacelike signature of the bosonic/fermionic string coordinate (or supergroup parameter) $X^A(z)$ that is associated with $T^A$ is directly determined by the sign of $\frac{k}{2} \text{Str}(T^A T^B)$. For $SU(2/1,1)$ with $k > 0$, the $SU(2)$ subalgebra matrices give a positive signature $\frac{k}{2} \text{Str}(T^a T^b) = k\delta^{ab}$; hence the $SU(2)$ group parameters correspond to spacelike string coordinates. For the $SU(1,1)$ subalgebra one has $\frac{k}{2} \text{Str}(T^\mu T^\nu) = -k\eta^{\mu\nu}$ where the extra minus sign comes from the definition of the supertrace, and the $SL(2,R)$ Killing metric $\eta^{\mu\nu}$ is given by $\eta^{00} = -\eta^{22} = -\eta^{11} = 1$. Therefore, the $SU(1,1)$ parameters correspond to two spacelike and one timelike coordinates. Similarly, for the fermionic parameters one finds 4 spacelike and 4 timelike string degrees of freedom.

2.1.1 Supergroups $\text{OSp}(4/2)$ and $\text{D}(2,1;\alpha)$

As another example consider the supergroup $\text{OSp}(4/2)$ with compact $\text{SO}(4)$ and non-compact $\text{Sp}(2) = \text{SL}(2,R)$. In this case there are 8 spacelike and one timelike bosonic coordinates and 4 spacelike and 4 timelike fermionic ones. The supergroup $\text{D}(2,1;\alpha)$ also has the same counting (at $\alpha = 1$ it becomes $\text{OSp}(4/2)$). Therefore, physical string theories can be constructed with the supergroups $SU(2/1,1)$, $\text{OSp}(4/2)$ and $\text{D}(2,1;\alpha)$ provided the remaining spacelike degrees of freedom added and then $N = 4$ superconformal constraints are imposed. For other supergroups one may consider gauged WZW models based on supergroups that give only one timelike coordinate (see [22] for the purely bosonic sector) supplemented by appropriate fermionic constraints that follow from gauge invariances.

\footnote{Under hermitian conjugation the $SU(2/1,1)$ matrices satisfy $(T^A)^\dagger = CT^A C^{-1}$ with

$$C = \begin{pmatrix} 1_2 & 0 \\ 0 & \tau_2 \end{pmatrix}$$

(3)

where $1_2$ is the $2 \times 2$ identity matrix and $\tau_2$ is the Pauli matrix (by taking $\tau_2$ we have chosen the $\text{SL}(2,R)$ rather than the $SU(1,1)$ basis). Then the supermatrix $T^a$ with $a = 1,2,3$ that correspond to $SU(2)$ has non-zero entries $\tau^a$ (Pauli matrices) in the upper left $2 \times 2$ block, the supermatrix $T^\mu$ with $\mu = 0,1,2$ that correspond to the $SU(1,1)=\text{SL}(2,R)$ subalgebra has non-zero entries $\sigma^\mu \equiv (\tau_2, i\tau_1, -i\tau_3)$ in the lower right $2 \times 2$ block. The supermatrices $T^a$ that correspond to the eight fermionic parameters (4 complex fermions) have non-zero entries in the off diagonal blocks.}
2.2 Free fields and reduction to bosonic AdS$_3$

Returning to SU(2/1, 1), the group element can be parametrized at the critical point in the usual form $G(z, \bar{z}) = G_L(z)G_R(\bar{z})$. Then each of the $G_{L,R}$ can most generally be parametrized in the “triangular” form

$$G_L(z) = \begin{pmatrix} 1 & 0 \\ \theta_L & 1 \end{pmatrix} \begin{pmatrix} H_L & 0 \\ 0 & g_L \end{pmatrix} \begin{pmatrix} 1 & \tilde{\theta}_L \\ 0 & 1 \end{pmatrix}$$

(4)

where $(H_L(z))_i^j$ and $(g_L(z))_i^\beta$ are group elements in the adjoint representation $(1, 0)$ and $(0, 1)$ of the SU(2)⊗SL(2) subgroup, $(\theta_L(z))_i^j$ is a 2x2 matrix of 4 complex fermions in the $(1, 1)$ representation, and $\tilde{\theta}_L(z) = \tilde{\theta}_L^\dagger \sigma_2$. In this notation canonical conjugates are determined for the WZW model, and its currents are parametrized in terms of them as in [21] [14]

$$SU(2) : \quad W^a = \theta_i^a \left( \frac{\tau^a}{2} \right)_i^j \pi_j^\alpha + J^a$$

(5)

$$SL(2, R) : \quad W^\mu = -\theta_i^\mu \left( \frac{\sigma^\mu}{2} \right)_i^j \pi_j^\beta + J^\mu$$

(6)

$$\text{coset} : \quad W_i^\alpha = \pi_i^\alpha$$

(7)

$$\tilde{W}_i^\alpha = \theta_j^i \theta_j^\alpha \pi_j^\alpha + \theta_j^i \sigma_\mu \pi_j^\beta J^\mu - J^a (\tau_a)_j^i \theta_j^\alpha + k \partial_z \theta_i^\alpha$$

(8)

The fields $(\theta_L(z))_i^j$ and its canonical conjugate $(\pi_L(z))_i^\alpha$ are just free fields akin to Wakimoto’s [23] representation ($\gamma \sim \theta_i^a$ and $\beta \sim \pi_i^\alpha(z)$), but they are fermions as well as being bosons in the present case. Also $(\partial H_L H_L^{-1})_i^j = (\tau_a)_i^j J^a(z)$ is an SU(2) current at level $k - 2$ and $(\partial g_L g_L^{-1})_i^j = (\sigma_\mu)_i^j J^\mu(z)$ is an SL(2, R) current at level $-k - 2$ (see [21] for an explanation of the shift in $k$). The Sugawara stress tensor is [21] [14]

$$T_{++}(z) =: \pi_i^\alpha \partial \theta_i^\alpha : + \frac{1}{k} : (J^a J_a - J^\mu J_\mu) :$$

(9)

This shows that the fermions $\theta, \pi$ are free fields and that the analysis of the model is reduced to solving the left/right factorized bosonic WZW model based on SU(2)×SL(2, R) at levels $k - 2$ and $-k - 2$ respectively. The SU(2) part is well known, therefore we concentrate on the purely bosonic SL(2, R) part and discuss the purely bosonic string on the AdS$_3$ background by itself. This involves the purely bosonic degrees of freedom $g_L(z)$ and the corresponding currents $J^\mu(z)$ (and their right moving counterparts $g_R(\bar{z}), \bar{J}^\mu(\bar{z})$) to which we will return in the following sections.

Note that the combined contribution of the bosonic currents $J^a, J^\mu$ to the Virasoro central charge $c_{\text{bosons}}$ is independent of $k$

$$c_{SU(2)} = \frac{3(k - 2)}{(k - 2) + 2} = 3 - \frac{6}{k}, \quad c_{SL(2, R)} = \frac{3(-k - 2)}{(-k - 2) + 2} = 3 + \frac{6}{k}$$

(10)

$$c_{\text{bosons}} = c_{SU(2)} + c_{SL(2, R)} = 6, \quad c_{\text{fermions}} = -8, \quad c_{\text{total}} = -2.$$
Therefore the value of the central extension $k$ can be changed arbitrarily without changing the central charge $c_{bosons}$. This feature will allow us to consider the limit $k \to \infty$ for which AdS$_3$ tends to flat 3D Minkowski space while the rest of the theory adjusts to this limit smoothly. In this limit we will show that our AdS$_3$ vertex operator tends to the vertex operator in flat 3D space $\exp (i k \mu X^\mu (z, \bar{z}))$: which is obviously a desired property for the correct vertex operator.

The same procedure can be applied to OSp(4/2) and D(2, 1; $\alpha$) for which again the fermions are free fields [21] and the non-trivial part is the string on AdS$_3$. These supergroups can be used to describe superstrings on AdS$_3 \times S^3 \times S^3$. In this case the bosonic part corresponds to SL(2, $\mathbb{R}$) $\times$ SU(2)$_1 \times$ SU(2)$_2$ at levels $-k - 2$, $(1 + \alpha) k - 2$ and $(1 + 1/\alpha) k - 2$ respectively ($\alpha = 1$ is for OSp(4/2)). One can then see that $c_{bosons} = 9$, $c_{fermi} = -8$, $c_{tot} = 1$ are again independent of $k$ or $\alpha$ and therefore these parameters can be taken to various limits to better understand the structure of the model.

2.3 Bosonic string on AdS$_3$

We now discuss the purely bosonic string in an AdS$_3$ background. This problem was already solved satisfactorily in [9] [10] but the lessons learned there remain to be incorporated in the study of the AdS-CFT conjecture, which we will address in later sections. In this section we outline the main important features that relate to unitarity and to zero modes (the notation is slightly different here as compared to [9] [10]).

To parametrize the group element in the $j = 1/2$ representation we use the matrix representation $t^\mu = \sigma^\mu / 2$ given in footnote 1. Using the lightcone type combinations

$$
t^+ = t^1 + t^0 = i \frac{\tau^1}{2} + \frac{\tau^2}{2} = i \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}$$

$$
t^- = t^1 - t^0 = i \frac{\tau^1}{2} - \frac{\tau^2}{2} = i \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}$$

$$
t^2 = -i \frac{\tau^3}{2} = -i \begin{pmatrix} \frac{1}{2} & 0 \\ 0 & -\frac{1}{2} \end{pmatrix}$$

we can always parametrize the SL(2, $\mathbb{R}$) group element in the triangular form

$$
g(z, \bar{z}) = e^{it^+ \gamma^-(z, \bar{z})} e^{it^2 \phi(z, \bar{z})} e^{it^- \gamma^+(z, \bar{z})}$$

$$
= \begin{pmatrix} 1 & 0 \\ -\gamma^- (z, \bar{z}) & 1 \end{pmatrix} \begin{pmatrix} e^{\frac{i}{2} \phi(z, \bar{z})} & 0 \\ 0 & e^{-\frac{i}{2} \phi(z, \bar{z})} \end{pmatrix} \begin{pmatrix} 1 & -\gamma^+ (z, \bar{z}) \\ 0 & 1 \end{pmatrix}
$$

$$
= \begin{pmatrix} e^{\frac{i}{2} \phi} & -\gamma^+ e^{\frac{i}{2} \phi} \\ -\gamma^- e^{-\frac{i}{2} \phi} + e^{\frac{i}{2} \phi} \gamma^+ \gamma^- \end{pmatrix}
$$

Note that all the fields $\phi(z, \bar{z})$, $\gamma^-(z, \bar{z})$, $\gamma^+(z, \bar{z})$ are real. $\gamma^\pm (z, \bar{z})$ are lightcone combinations
$\gamma^{\pm} = \gamma^1 \pm \gamma^0$. The SL(2, R) WZW Lagrangian takes the form

$$L = (k + 2) \left[ \partial \phi \bar{\partial} \phi + e^\phi \partial \gamma^+ \bar{\partial} \gamma^- \right]$$

This shows that it describes a string with one time-like coordinate and two space-like coordinates propagating on the AdS$_3$ background. The boundary of AdS$_3$ is defined by $\phi \to \infty$.

The classical solution for the fields $\phi(z, \bar{z}), \gamma^-(z, \bar{z}), \gamma^+(z, \bar{z})$ are extracted from the general classical solution for the group element in the form

$$g(z, \bar{z}) = g_L(z) g_R(\bar{z})$$

where $g_{L,R}$ are arbitrary SL(2, R) group elements. Note that the $g_L(z)$ that appears here is identified with the $g_L(z)$ in the parametrization of the SU(2/1, 1) group element $G_L(z)$ in eq. (4) so that the discussion given in this section directly applies to the SU(2/1, 1) model. The most general $g_{L,R}$ can always be parametrized in the triangular form

$$g_L(z) = e^{it^+X^-} e^{it_2X_2} e^{it^-X^+} = \begin{pmatrix} e^{\frac{i}{2}X_2} & -X^+ e^{\frac{i}{2}X_2} \\ -X^- e^{\frac{i}{2}X_2} & e^{-\frac{i}{2}X_2} + e^{\frac{i}{2}X_2} X^+ X^- \end{pmatrix}$$

$$g_R(\bar{z}) = e^{it^+\bar{X}^-} e^{it_2\bar{X}_2} e^{it^-\bar{X}^+} = \begin{pmatrix} e^{\frac{i}{2}\bar{X}_2} & -\bar{X}^+ e^{\frac{i}{2}\bar{X}_2} \\ -\bar{X}^- e^{\frac{i}{2}\bar{X}_2} & e^{-\frac{i}{2}\bar{X}_2} + e^{\frac{i}{2}\bar{X}_2} \bar{X}^+ \bar{X}^- \end{pmatrix}$$

By comparing (16) to (18) and (19,20) one obtains the general classical solution for the fields $\phi(z, \bar{z}), \gamma^-(z, \bar{z}), \gamma^+(z, \bar{z})$ in terms of the left/right moving fields $X^\pm(z), X_2(z)$ and $\bar{X}^\pm(\bar{z}), \bar{X}_2(\bar{z})$ (see the explicit formulas (43-45) below).

The quantum theory at the conformal critical point also takes the form (18). The left/right currents are $J(z) = ik (\partial g) g^{-1} = ik (\partial g_L) g_L^{-1}$ and $\bar{J}(\bar{z}) = -ikg^{-1} (\bar{\partial} g) = -ikg_R^{-1} (\bar{\partial} g_R)$. It was shown in [9] [10] that the quantum theory parametrized in this form reduces to a free field theory. The currents and the energy-momentum tensor are then expressed in terms of canonical sets of free fields: The left moving sets are $(X^-(z), P^+(z))$ and $(X_2(z), P_2(z))$ in terms of which the left currents $J^\mu(z)$ take the form (after taking into account normal ordering which shifts the overall $k$ to $k + 2$ in the second equation below)

$$J^1(z) + J^0(z) = P^+(z)$$
$$J^1(z) - J^0(z) = -: X^- P^+ X^- : -2P_2 X^- + i (k + 2) \partial_z X^-$$
$$J_2(z) = : X^- P^+ : + P_2(z)$$

where the canonical momenta $P^+$ and $P_2$ are identified as follows

$$P^+(z) = i k \partial_z X^+ e^{X_2}$$
$$P_2(z) = -\frac{1}{2} i k \partial_z X_2$$

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Therefore $X^+(z)$ must be expressed in terms of $P^+(z)$ as follows

$$X^+(z) = q^+ - \frac{i}{k} \int^z dz' P^+(z') e^{-X_2(z')}(25)$$

The Sugawara energy momentum tensor takes the form (after careful ordering of operators, including zero modes, to insure hermiticity)

$$T_{++}(z) =: P^+ i \partial X^- : + \frac{1}{k} \left( : P^2_2 : - \frac{i}{z} \partial (z P^2) + \frac{1}{4 z^2} \right)$$

(26)

(and similarly for $\bar{T}_{--}(z)$). The $L_0$ Virasoro operator that follows from this is

$$L_0 = p^+ p^- + \frac{1}{k} \left( \frac{1}{4} + p^2_2 \right) + \text{oscillators}$$

(27)

where the zero modes $p^+, p^-, p_2$ come from

$$X^-(z) = q^- - ip^- \ln z + \text{oscillators}, \quad P^+(z) = \frac{p^+}{z} + \text{oscillators},$$

$$P_2(z) = \frac{p_2}{z} + \text{oscillators}$$

(28)

(29)

This free field representation is similar to Wakimoto’s [23] (with $\gamma \sim X^-$ and $\beta \sim P^+$) but there are two important differences:

- The fields $(X^-, P^+)$ and $(X_2, P_2)$ are all hermitian, and similarly all the currents $J^\mu$ are hermitian, e.g. the hermitian conjugate of $J^1(z) + J^0(z)$ is itself (see details in [9]). This feature is different in SU(2) and this is why we can build a unitary representation with free fields in this parametrization for SL(2, R). By contrast for SU(2), hermiticity and hence unitarity is not straightforward in the Wakimoto free field representation and it requires certain screening charges and singular states etc. to identify the unitary subset of states in the Hilbert space. None of these occur in SL(2, R) and hermiticity (and unitarity) is manifest at every step in the formulation given above. Note that $T_{++}(z)$ and the Virasoro operators $L_n$ that follow from it are also hermitian (i.e. $L^\dagger_n = -L_{-n}$) thanks to the hermiticity of the structure $-\frac{i}{z} \partial (z P^2) + \frac{1}{4z^2}$.

- The momentum zero mode $p^-$ (and similarly $\tilde{p}^+$ for right movers) contributes logarithmic terms to the currents given above (24) [23]. In SU(2) these zero modes are dropped to insure that the currents are periodic for a string when $\sigma \rightarrow \sigma + 2\pi$. However for SL(2, R) it is essential to retain the $p^-$ (and $\tilde{p}^+$) zero mode because otherwise the string cannot be put on shell when it is excited as will be discussed now.

If the zero mode $p^-$ is dropped then the currents are holomorphic as in usual affine current algebra (usual affine currents do not have logarithmic terms $\ln z$ and $(\ln z)^2$ contributed through the $X^-$ in (28)). We define the affine currents as $j^\mu(z) = [J^\mu(z)]_{p^- = 0}$. These currents obey
the standard operator products expected from SL(2, R) currents. Then the spectrum is given
in the form $L_0|_{p^-=0} = -\frac{1}{2} j (j + 1) + \text{oscillators}$, a formula that is valid only for affine currents. Comparing to the expression of $L_0$ above we learn that the affine current algebra is realized
only in the principal series

$$j (j + 1) = -\frac{1}{4} - p_2^2, \quad \rightarrow \quad j = -\frac{1}{2} + ip_2. \quad (30)$$

The discrete series or supplementary series do not occur in this realization of the affine current algebra since for them $j (j + 1) > -1/4$.

However there is a physical problem if $p^- = 0$: since every term is positive in $L_0|_{p^-=0}$ it is impossible to put the theory on shell for any excited level by requiring $L_0|_{p^-=0} = a \leq 1$ (the oscillators contribute an integer $\geq 1$ for an excited level, and any additional spacelike dimensions contribute only positive terms). This is why the $p^-$ zero mode must be included since $p^+ p^- = -p_0^2 + p_1^2$ is the only negative term in the correct $L_0$, and its presence enables the theory to be put on-shell, as is also the case in flat string theory with only one timelike coordinate. The presence of $p^-$ is completely natural as part of $X^-$ in the WZW model, and there would be no issue if one treated the model directly in terms of the free field representation given above. However a non-zero $p^-$ modifies certain standard affine current algebra results and therefore one must be careful in trying to apply results obtained in affine current algebra to the SL(2, R) WZW string theory as explained in the next paragraph and in the later sections of this paper. The early papers until 1995 and some recent papers have overlooked this point. As we will see below it has consequences for and adds refinements to the AdS-CFT correspondence.

With a non-zero $p^-$ the current algebra is not the usual affine current algebra since it includes the logarithmic terms $\ln z$ and $(\ln z)^2$. In terms of the affine currents $j^\mu (z)$ the correct SL(2, R) WZW currents $J^\mu (z)$ are given by

\begin{align*}
J^0 (z) + J^1 (z) &= [j^0 (z) + j^1 (z)], \\
J^0 (z) - J^1 (z) &= [j^0 (z) - j^1 (z)] - 2ip^- \ln z \ j^2 (z) \\
&- \frac{k+2}{z} p^- + (ip^- \ln z)^2 [j^0 (z) + j^1 (z)], \\
J^2 (z) &= j^2 (z) - ip^- \ln z [j^0 (z) + j^1 (z)].
\end{align*}

Despite the logarithms, The Sugawara form $T \sim JJ$ gives the correct energy momentum tensor in (26), and the WZW currents $J^\mu (z)$ have the correct operator products among themselves and with the energy-momentum tensor as shown in detail in [9] [10].

The remaining problem is the periodicity of the currents $J^\mu (ze^{in2\pi}) = J^\mu (z)$, which is not true in general, but must be true at least in the physical sector of the theory. It was shown that this condition imposes a quantization condition on the zero modes [9] [10]

$$p^+ p^- = -r, \quad r \in Z_+ \quad (31)$$

where $r$ is a positive integer or zero. Therefore the physical states must come in sectors labeled
by \( r \). The spectrum of the SL(2, \( R \)) WZW model may now be written as

\[
L_0 = -r - \frac{j(j+1)}{k} + \text{integer} = a \leq 1
\]  

(32)

where \( j = -1/2 + is \) (\( s \) is the eigenvalue of \( p_2 \)) labels the representation of the affine current algebra \( j^\mu(z) \) and \(-r\) comes purely through the extra zero mode \( p^- \). As argued below, \( r \) is related to winding modes at the AdS boundary and therefore represents winding strings.

The right moving currents \( J^R_R = -ikg^{-1}_R(\partial g^R) \) are quantized in a similar way. In this case the independent canonical degrees of freedom are \( (\tilde{\mathcal{X}}^+, \tilde{\mathcal{P}}^-) \) and \( (\tilde{\mathcal{X}}_2^+, \tilde{\mathcal{P}}_2^-) \), and \( \tilde{\mathcal{X}}^- (\bar{z}) \) is expressed in terms of \( \tilde{\mathcal{P}}^- \)

\[
\tilde{\mathcal{X}}^- (\bar{z}) = \tilde{q}^- - \frac{i}{k} \int d\bar{z}' \tilde{\mathcal{P}}^- (\bar{z}') e^{-\tilde{\mathcal{X}}_4(\bar{z}')}
\]  

(33)

while the currents are

\[
\tilde{J}^1(\bar{z}) - \tilde{J}^0(\bar{z}) = -\tilde{\mathcal{P}}^- (\bar{z})
\]  

(34)

\[
\tilde{J}^1(\bar{z}) + \tilde{J}^0(\bar{z}) = \tilde{\mathcal{X}}^+ \tilde{\mathcal{P}}^- \tilde{\mathcal{X}}^+ + 2\tilde{\mathcal{P}}^2 \tilde{\mathcal{X}}^+ - i(k+2) \partial_2 \tilde{\mathcal{X}}^+ (\bar{z})
\]  

(35)

\[
\tilde{J}^2(\bar{z}) = -\tilde{\mathcal{X}}^+ \tilde{\mathcal{P}}^- \tilde{\mathcal{X}}^+ - \tilde{\mathcal{P}}^2 (\bar{z})
\]  

(36)

The spectrum of right movers is given by

\[
\tilde{L}_0 = \tilde{\mathcal{P}}^+ \tilde{\mathcal{P}}^- + \frac{1}{k} \left( \frac{1}{4} + \tilde{p}^2 \right) + \text{oscillators}
\]  

(37)

Combining the left and right movers and using the conditions \( j = \tilde{j} \) and \( L_0 = \tilde{L}_0 \) we find \( p_2 = \tilde{p}_2 \) and

\[
\tilde{p}^+ \tilde{p}^- p^+ p^- = -r, \quad r \in Z_+.
\]  

(38)

Therefore physical states are constructed by applying oscillators to the base that is labeled by \( |\text{base} > = |p^+, p^-, p_2; \tilde{p}^+ , \tilde{p}^- , \tilde{p}_2 > \)

(39)

with the relations and quantization conditions given above. The Virasoro constraints are applied to single out the physical states. The no ghost theorem was proven in \[9\] \[10\].

The procedure above establishes the physical spectrum of the string on AdS\(_3\). The next step is to construct vertex operators that correspond to these states.

### 3 Vertex operator

Vertex operators are constructed by starting with the “tachyon” vertex operator, which is the group element in a representation as specified by the generators \( \hat{t}^\mu \) (the relevant representation will be specified below)

\[
V(g) = e^{i\gamma^- \hat{t}^\mu} e^{i\phi^2} e^{i\gamma^+ \hat{t}^-}
\]  

(40)

\[
= V(g_L) V(g_R)
\]  

(41)
The second line $V(g) = V(g_L) V(g_R)$ follows from the group property in any representation. Furthermore, $V(g_L)$ and $V(g_R)$ are constructed from free fields $X^\mu (z)$ and $\bar{X}^\mu (\bar{z})$ given above

$$V(g_L) = e^{iX^{-}i} e^{iX^2 e^{iX^{+}i}}, \quad V(g_R) = e^{i\bar{X}^{-}i} e^{i\bar{X}^{2} e^{i\bar{X}^{+}i}}$$

(42)

The representation of the generators $\hat{t}^\mu$ should correspond to the physical states determined above (as opposed to the non-unitary $j = 1/2$ representation of footnote 1). We need to take into account the effect of the zero mode $p^-$ in the representation $\hat{t}^\mu$ so that there is operator-state correspondence. The relevant representation will be discussed in the next subsection.

In the quantum theory the expressions for $V(g_{L,R})$ must be normal ordered appropriately so that their operator products with the currents and stress tensor give the correct results for the single and double poles. An additional desirable property is that the AdS vertex operators should tend to the flat 3D string vertex operators $\exp(i k_\mu X^\mu (z, \bar{z}))$ since the large $k$ limit is smooth for the complete theory as we have already discussed. We have accomplished all of these properties as described below. Using this construction in principle one can perform computations of correlation functions using free fields.

For semi-classical arguments used in the interpretation of the model, it is also useful to express the AdS coordinates $\phi (z, \bar{z}), \gamma^+ (z, \bar{z}), \gamma^- (z, \bar{z})$ themselves in terms of the free fields. If we ignore orders of operators in a semi-classical approach, we can obtain the result by computing $V(g_{L,R})$ in the $j = 1/2$ representation. Thus, using \([18, 16]\) and \([13, 20]\) we obtain

$$\phi (z, \bar{z}) = X_2 (z) + \bar{X}_2 (\bar{z}) + 2 \ln \left( 1 + X^+ (z) \bar{X}^- (\bar{z}) \right)$$

(43)

$$\gamma^- (z, \bar{z}) = X^- (z) + \frac{e^{-X_2(z)} \bar{X}^-(\bar{z})}{1 + X^+ (z) \bar{X}^- (\bar{z})}$$

(44)

$$\gamma^+ (z, \bar{z}) = \bar{X}^+ (\bar{z}) + \frac{e^{-\bar{X}_2(\bar{z})} X^+ (z)}{1 + X^+ (z) \bar{X}^- (\bar{z})}$$

(45)

where $X^+ (z), \bar{X}^- (\bar{z})$ are given by \([23]\) and \([33]\)

$$X^+ (z) = q^+ - \frac{i}{k} \int^z dz' P^+(z') e^{-X_2(z')},$$

(46)

$$\bar{X}^- (\bar{z}) = \bar{q}^- - \frac{i}{k} \int^\bar{z} d\bar{z}' \bar{P}^- (\bar{z}') e^{-\bar{X}_2(\bar{z}')}. $$

(47)

We will use these expressions below to discuss windings at the AdS boundary $\phi \rightarrow \infty$.

### 3.1 Position-momentum basis and state-operator correspondence

The following is an operator representation of the SL(2, R) generators $\hat{t}^\mu$ that correspond to the physical states determined in the previous section\(^2\). They should be inserted into the expression

\(^2\)This unitary representation is obtained by considering the quantum theory of a particle in AdS3 space. In momentum space its complete normalizable wavefunctions can be related to those of the Hydrogen atom in an appropriate basis. For an informal discussion see \([24]\).
of the vertex operators given in (40-42)

\[
\hat{t}^+ \equiv \hat{p}^+ \equiv -\hat{x}^+ \hat{p}^- \hat{x}^+ + 2s \hat{x}^+ - \frac{kr}{\hat{p}^-} \\
\hat{t}_2 \equiv \frac{1}{2} (\hat{x}^- \hat{p}^+ + \hat{p}^+ \hat{x}^-) + s \equiv -\frac{1}{2} (\hat{x}^+ \hat{p}^- + \hat{p}^- \hat{x}^+) + s \\
\hat{t}^- \equiv -\hat{x}^- \hat{p}^+ \hat{x}^- - 2s \hat{x}^- - \frac{kr}{\hat{p}^+} \equiv \hat{p}^-
\]

These forms may look familiar except for the terms that contain \(kr\). The integer \(r\) corresponds to the quantum label of the state (39) as determined by the monodromy argument (38), and \(k\) is the central extension of the current algebra. The operators \((\hat{x}^-, \hat{p}^+)\) form a canonical pair. These \(\hat{x}^-, \hat{p}^+\) operators are just a convenient device, they are not the zero modes of the fields \(X^-, P^+\) etc. Similarly the pair \((\hat{x}^+, \hat{p}^-)\) is canonical, but it is not independent of the pair \((\hat{x}^-, \hat{p}^+)\). The relationship between these pairs is non-linear and is given by the two forms of the generators above. The structure of \(\hat{t}^\mu\) given above is obtained by studying the particle moving in an AdS space and it is interesting that it can represent all possible representations of \(SL(2, R)\) by taking various values of \(s\) and \(kr\).

By using the canonical commutation rules \([\hat{x}^-, \hat{p}^+] = [\hat{x}^-, \hat{p}^+] = i\) it is easy to see that either form of the operators \(\hat{t}^\mu\) satisfy the same commutation rules as the \(2 \times 2\) matrix representation of footnote 1. Furthermore the \(\hat{t}^\mu\) are manifestly hermitian thus insuring that they correspond to a unitary representation (unlike the \(t^\mu\) of footnote 1). The Casimir operator for either form of \(\hat{t}^\mu\) is

\[
\hat{C}_2 = -\frac{1}{2} (\hat{t}^+ \hat{t}^- + \hat{t}^- \hat{t}^+) - \hat{t}_2^2 \\
= kr - \frac{1}{4} - s^2
\]

(50)

(the corresponding Casimir eigenvalue for the \(t^\mu\) of footnote 1 is \(C_2 = 3/4\)). Observe that \(-\frac{1}{k} \hat{C}_2 = -r + \frac{1}{k} \left(\frac{1}{4} + s^2\right)\) is the same as the formula for the spectrum of a physical state as given in (27). We will see that the operator product of the energy-momentum tensor with the vertex operator gives

\[
T(z) V^{r,s}(w) \sim -r + \frac{1}{k} \left(\frac{1}{4} + s^2\right) V^{r,s}(z) + \frac{\partial V^{r,s}(z)}{(z-w)}
\]

\[
\Delta^{r,s} = -r + \frac{1}{k} \left(\frac{1}{4} + s^2\right)
\]

(52)

where \(\Delta^{r,s}\) is the conformal dimension. This identifies \(\hat{t}^\mu\) given above as the representation that provides the desired state-operator correspondence. Note that the contribution of the zero modes are taken into account through the term \(kr\). Without the \(kr\) term one cannot construct primary vertex operators of the form : \(J^\mu V^{r,s}\) (needed for the AdS-CFT correspondence) with total conformal weight 1, since this requires \(\Delta^{r,s} = 0\).

There is here the possibility for some confusion about this representation and we would like
to comment on it. Of course we may define a \( \hat{j} \) through
\[
\hat{C}_2 = kr - \frac{1}{4} - s^2 = \hat{j}(\hat{j} + 1)
\]
and note that we must have \( \hat{j}(\hat{j} + 1) > 0 \) to be able to satisfy the mass shell condition. This identifies the \( \hat{i}^\mu \) in the discrete series representation. How is this possible since we made the point that only the principal series representation for the affine currents is allowed? The answer is that the \( \hat{i}^\mu \) takes into account the contribution of the zero mode while the affine currents continue to be in the principal series. That is, the full current algebra module, including the string excitations, is in the principal series of the affine currents \( j^\mu(z) \), which is a very different module than the discrete series module. When this is combined with the zero mode, the excitation spectrum of the current algebra module remains the same, but the mass shell condition changes, and this is taken into account in the WZW vertex operators constructed with \( \hat{i}^\mu \). The affine current algebra module is still in the principal series, but the physics is supplemented by the zero modes. The zero mode is related to winding strings at the AdS boundary. This shows that the effects of the zero modes are subtle, as we will continue to witness further in the following sections.

The group theoretical states for \( SL(2, \mathbb{R}) \) are usually labeled as \( |jm > \) where \( m \) is the eigenvalue of the compact generator \( \hat{p}_0 \). One may consider states in which other operators are diagonal. For our purposes it is useful to diagonalize \( \hat{t}^+ \) or \( \hat{t}^- \). When \( \hat{t}^+ \) is diagonal in the basis \( < s, p^+ > \), it corresponds to diagonalizing the operator \( \hat{p}^+ \) and when \( \hat{t}^- \) is diagonal in the basis \( |s, p^- > \), it corresponds to diagonalizing the operator \( \hat{p}^- \). The Fourier transform of these states correspond to diagonalizing the operators \( \hat{x}^- \) or \( \hat{x}^+ \) in the basis \( < s, x^- | \) or \( |s, x^+ > \), respectively. In position space we have \( < s, x^- | \hat{p}^+ = -i \frac{\partial}{\partial x} < s, x^- | \) and \( \hat{p}^- |s, x^+ > = i \frac{\partial}{\partial x^+} |s, x^+ > \) consistent with the commutation rules.

One may evaluate the matrix elements of the vertex operators (40-42) in the position or momentum basis in the same way that one would compute the group elements in the \( |jm > \) basis \( D^j_{mm'}(g) \). In particular we find it convenient to use the momentum basis or the position basis as follows
\[
V^{r,s}_{p^+,p^-} (g) = < p^+ | e^{i\gamma^- \hat{t}^+} e^{i\phi_2 \hat{t}^+} | p^- >
\]
\[
V^{r,s}_{x^-,x^+} (g) = < x^- | e^{i\gamma^- \hat{t}^-} e^{i\phi_2 \hat{t}^-} | x^+ >
\]
and similarly for the left/right vertex operators. Note that in momentum basis \( \hat{t}^+ \) is diagonal on the bra and the \( \hat{t}^- \) is diagonal on the ket, showing that this form is already fairly close to the vertex operator of a flat string. We will see that these labels \( p^\pm \) or \( x^\pm \) have the interpretation of momenta or positions at the boundary of AdS\(_3\). The position basis will be useful to connect to the discussions in the recent literature and show the refinements that need to be made. The momentum basis will be useful for establishing the operator product (51) at the fully quantum level (ordering of operators taken into account) and for discussing the flat limit as \( k \to \infty \). Our new vertex operator has the desirable properties.
3.2 Vertex operator in position basis

The vertex operator in position space is defined by

\[
V_{x^-x^+}^{r,s}(g) = \langle x^- | e^{i\gamma^-t^+} e^{i\phi t_2^2} e^{i\gamma^+t^-} | x^+ \rangle
\]

(56)

\[
= \langle x^- + \gamma^- | e^{i\phi t_2} | x^+ - \gamma^+ \rangle
\]

(57)

\[
= e^{i\phi} (x^- + \gamma^-) | x^+ - \gamma^+ > e^{i(s-i/2)\phi}
\]

(58)

\[
= \langle x^- + \gamma^- | e^{\phi} (x^+ - \gamma^+) > e^{i(s-i/2)\phi}
\]

(59)

where we have used that \( \hat{t}^+ = \hat{p}^+ \) and \( \hat{t}^- = \hat{p}^- \) are translation operators on \( x^\pm \) space, and \( \hat{t}_2 \) is a dilation operator on \( x^\pm \) space (except for the extra factor \( e^{i(s-i/2)\phi} \)). If we define \( f_{r,s}(x^-x^+) \equiv \langle x^- | x^+ \rangle \) then we must have a function of only the product \( x^-x^+ \) on account of its properties under dilations. Then the full vertex is

\[
V_{x^-x^+}^{r,s}(g) = e^{i(s-i/2)\phi} f_{r,s} \left( e^{\phi} \left( x^+ - \gamma^+ \right) \left( x^- + \gamma^- \right) \right)
\]

(60)

To find \( f_{r,s}(x^-x^+) \) sandwich \( \hat{t}^- \) and use its action on the right and left states to obtain the differential equation

\[
\langle x^- | \hat{t}^- | x^+ \rangle = i\partial_+ f_{r,s} \tag{61}
\]

\[
= ix^- \partial_- (x^- f_{r,s}) - 2sx^- f_{r,s} - \frac{kr}{-i\partial_-} f_{r,s} \tag{62}
\]

At \( r = 0 \) this first order differential equation has the unique solution \( f_{0,s} = \sqrt{\frac{1}{\pi}} (x^+x^- + 1)^{-1-2is} \), and the full vertex operator at \( r = 0 \) becomes

\[
V_{x^-x^+}^{0,s}(g) = \sqrt{\frac{1}{\pi}} \left( e^{\phi/2} \left( x^+ + \gamma^+ \right) \left( x^- + \gamma^- \right) + e^{-\phi/2} \right)^{-1-2is} \tag{63}
\]

At non-zero values of \( r \) we apply \(-i\partial_-\) on both sides to obtain the second order differential equation

\[
- \partial_- (x^- \partial_- (x^- f_{r,s})) - 2is\partial_- (x^- f_{r,s}) + kr f_{r,s} + \partial_- \partial_+ f_{r,s} = 0 \tag{64}
\]

This becomes the hypergeometric equation in one variable. The exact solution that is well behaved at infinity is

\[
f_{r,s}(x^-x^+) = \frac{(1+x^-x^+)^{-1-2is}}{\sqrt{\pi} \left( x^-x^+ \right)^{\sigma-is}} 2F_1 \left( \sigma - is, \sigma - is; 1 + 2\sigma; \frac{-1}{x^-x^+} \right), \tag{65}
\]

\[
\sigma \equiv \sqrt{kr - s^2} \tag{66}
\]

When \( r = 0 \), we get \( \sigma = is \) and \( 2F_1 \left( 0,0; 1 + 2is; \frac{-1}{x^-x^+} \right) = 1 \), and the solution reduces to \( V_{x^-x^+}^{0,s}(g) \) given above. For general \( r \), with \( kr > s^2 \) that satisfies the mass shell condition for
excited strings, the vertex operator becomes

$$V_{x^-,x^+}^{r,s}(g) = \sqrt{\frac{1}{\pi}} \left( e^{\phi/2} (x^+ - \gamma^+) \left( x^- + \gamma^- \right) + e^{-\phi/2} \right)^{-1-2is} \times$$

$$\times 2F1 \left( \sigma - is, \sigma - is; 1 + 2\sigma; \frac{1}{(x^+ - \gamma^+) (x^- + \gamma^-) e^{\phi}} \right)$$

We see that the zero mode modifies the vertex operator by the factor in the second line. The conformal weight of this vertex operator is

$$\Delta_{r,s} = -r + \frac{1}{k} \left( \frac{1}{4} + s^2 \right)$$

as in (68) which will be proven below at the full quantum level.

The factor in the second line of (67) has been missed in recent discussions of the AdS-CFT correspondence. The vertex operator used in recent literature either is not unitary [13]-[15], or as in [19] corresponds to only the first line of (67), which is the vertex operator at $r = 0$. As already emphasized several times, with $r = 0$ the mass shell condition, or the primary operator condition (with $\Delta_{r,s} \leq 0$) could not be satisfied and therefore the state-operator correspondence and/or unitarity would be lost. Our new vertex operator has the desirable properties of unitarity and state-operator correspondence.

### 3.2.1 Vertex operator near the AdS$_3$ boundary

The behavior of the correct vertex operator near the AdS boundary $\phi \to \infty$ can now be examined. As it turns out the important quantity is not the vertex operator, which is the wavefunction, but rather the probability, that involves the absolute value square of the wavefunction

$$\Omega_{x^-,x^+}^{r,s}(g) = \left| V_{x^-,x^+}^{r,s}(g) \right|^2.$$  

Noting that $2F1(a,b;c;0) = 1$, we see that as long as $(x^+ - \gamma^+) (x^- + \gamma^-) \neq 0$, the vertex operator falls off as $\exp \left( -\phi \left( \frac{1}{2} + \sqrt{kr - s^2} \right) \right)$, or $\Omega_{x^-,x^+}^{r,s}(g) \to \exp \left( -\phi \left( 1 + 2\sqrt{kr - s^2} \right) \right)$, indicating that this wavefunction is normalizable. When $(x^+ - \gamma^+) (x^- + \gamma^-) \sim 0$ the first factor in (67) behaves like a delta function as $\phi \to \infty$, while the second factor modifies it mildly with logarithms

$$\Omega_{x^-,x^+}^{r,s} \to \delta \left( x^+ + \gamma^+ \right) \delta \left( x^- - \gamma^- \right) \times \left[ c_1 \ln \left( x^+ - \gamma^+ \right) \left( x^- + \gamma^- \right) + c_1 \phi + c_2 \right]^2$$

where $c_{1,2}$ are independent of $(x^+ - \gamma^+), (x^- + \gamma^-)$. In arriving at this result we have used the small $z \to 0$ behavior of the hypergeometric function

$$\frac{2F1 \left( \sigma - is, \sigma - is; 1 + 2\sigma; \frac{1}{z} \right)}{z^{\sigma - is}} \to \frac{\Gamma \left( 1 + 2\sigma \right) \times \ln z}{\Gamma \left( \sigma - is \right) \Gamma \left( 1 + \sigma + is \right)} + c_2 \left( \sigma, s \right)$$

15
Note that at \( r = 0 \) we get \( \sigma = is \) and the coefficients \( c_1 = 0 \) and \( c_2 = 1 \). When \( r \neq 0 \) the term \( c_1 \phi + c_2 \) is neglected as compared to the first term in \( c_1 \ln (x^+ - \gamma^+) \) \((x^- + \gamma^-) + (c_1 \phi + c_2)\) since \((x^+ - \gamma^+)\) \((x^- + \gamma^-) \sim 0 \) before taking the limit \( \phi \to \infty \). So we see that the support of the probability at the boundary of AdS \( \phi \to \infty \) is precisely at

\[
\gamma^+ (z, \bar{z}) = x^+, \quad \gamma^- (z, \bar{z}) = -x^-.
\]  

(72)

Therefore, the labels \( x^\pm \) must be interpreted as the coordinates at the boundary of the AdS space, in agreement with \[14\], \[19\], for any \( r \).

### 3.2.2 Windings at the AdS boundary

To examine further the properties of the string theory near the boundary at \( \phi \to \infty \), in particular its zero modes, we take the center of mass positions of \( X_2, \bar{X}_2 \) to infinity. That is, use \( X_2 (z) = q_2 + \cdots \) and \( \bar{X}_2 (\bar{z}) = \bar{q}_2 + \cdots \), and let \( q_2 = Q + q \) and \( \bar{q}_2 = Q - q \) and then let \( Q \to \infty \). Then note that \( \gamma^\pm (z, \bar{z}) \) given in \((44, 45)\) become purely left or right moving in this limit

\[
\gamma^- (z, \bar{z}) \to X^- (z), \quad \gamma^+ (z, \bar{z}) \to \bar{X}^+ (\bar{z}).
\]  

(73)

Then using the Minkowski signature on the worldsheet replace \( z \to \exp (i (\tau + \sigma)) \) and \( \bar{z} \to \exp (i (\tau - \sigma)) \)

\[
X^- (\tau + \sigma) = q^- + p^- (\tau + \sigma) + \text{oscillators}, \quad P^+ (z) = p^+ + \text{oscil.},
\]  

(74)

\[
\bar{X}^+ (\tau - \sigma) = \bar{q}^+ + \bar{p}^+ (\tau - \sigma) + \text{oscillators}, \quad \bar{P}^- (\bar{z}) = \bar{p}^- + \text{oscil.},
\]  

(75)

and examine the periodicity of the AdS string as \( \sigma \to \sigma + 2\pi \). The oscillator part is periodic; for full periodicity we must make a periodic lattice in \( q^- \) space with lattice size \( R \) and identify points on this lattice

\[
q^- \sim q^- + 2\pi p^-, \quad p^- = nR
\]  

(76)

Then the canonical conjugate \( p^+ = m/R \) must be quantized in units of \( 1/R \). Thus, we learn that the string winds \( n \) times and that \( p^- \) is quantized in terms of winding number \( n \) while \( p^+ \) is quantized in terms of the Kaluza-Klein quantum number \( m \), and similarly for \( \bar{p}^-, \bar{p}^+ \)

\[
p^- = nR, \quad p^+ = \frac{m}{R}, \quad \bar{p}^- = \bar{n}R, \quad \bar{p}^+ = \frac{\bar{m}}{R}.
\]  

(77)

This observation is related to the winding of the long strings discussed in \[13\], \[15\], \[25\], so we can identify the winding numbers

\[
n = \oint \frac{d\gamma^-}{\gamma^-}, \quad \bar{n} = \oint \frac{d\gamma^+}{\gamma^+}.
\]  

(78)

Now we can compute the products \( p^+ p^-, \bar{p}^+ \bar{p}^- \) that appear in the spectrum \((38)\) and establish the following relation between these integers

\[
mn = \bar{m}\bar{n} = -r, \quad r \in Z_+
\]  

(79)
Thus, the winding of the long strings on the boundary as discussed in [13], [15], [25] demands that \( p^- = nR \) be non-zero. Of course, this is in agreement with the requirements of on shell and monodromy [9] [10] we emphasized above. We now see that the effect of the zero mode \( p^- \) must be included in the vertex operator if it is to describe a string that has non-trivial windings in AdS space.

### 3.3 Vertex operator in momentum basis

We will now take advantage of the factorized form of the vertex operator. The full vertex operator is

\[
V_{r,s}(z, \bar{z}) = \langle s, p_+ | e^{iX^- \hat{t}^+} e^{iX_2 \hat{t}^2} e^{iX^+ \hat{t}^-} | s, p_- \rangle
\]

\[
= \int d\tilde{p}^- \langle s, p_+ | e^{iX^- \hat{t}^+} e^{iX_2 \hat{t}^2} e^{iX^+ \hat{t}^-} | s, p_- \rangle
\]

\[
= \int d\tilde{p}^- \overline{V}_{r,s}(\tilde{z}) \overline{V}_{r,s}(\tilde{z})
\]

In this section we are going to describe only the left moving part of the factorized vertex operator in the momentum basis \( V_{r,s}(z, \bar{z}) \) and verify that it has the correct operator product properties with the left moving currents and the stress tensor. The right moving factor of the factorized vertex operator \( \overline{V}_{r,s}(\tilde{z}) \) is insensitive to these operator products and therefore we do not include it in the discussion. To discuss operator products for right movers we insert the intermediate states \( 1 = \int d\tilde{p}^+ |\tilde{p}^+ > < \tilde{p}^+| \) and then the discussion for right movers parallels the discussion for left movers. After defining the left moving vertex operator \( V_{r,s}(z) \) at the fully quantum level (ordering of operators taken into account) we are going to show that it has correct quantum operator products with the currents and that it has the desired conformal dimension

\[
\Delta_{r,s} = -r + \frac{1}{k} \left( \frac{1}{4} + s^2 \right).
\]

#### 3.3.1 Classical expression

The left moving part of the vertex operator in momentum space is defined by

\[
V_{r,s}(g_L)(z) = \langle s, p_+ | e^{iX^- \hat{t}^+} e^{iX_2 \hat{t}^2} e^{iX^+ \hat{t}^-} | s, p_- \rangle
\]

\[
= e^{iX^- p^+} \langle s, p_+ | e^{iX_2 \hat{t}^2} | s, p^- \rangle e^{iX^+ p^-}
\]

\[
= e^{iX^- p^+} e^{-\frac{1}{2}X_2 (1-2is)} \langle s, e^{-X_2 p^+} | s, p^- \rangle e^{iX^+ p^-}
\]

\[
= e^{iX^- p^+} e^{-\frac{1}{2}X_2 (1-2is)} \langle s, p^+ | s, e^{-X_2 p^-} \rangle e^{iX^+ p^-}
\]

where we have used the fact that \( \hat{t}^+ \) is diagonal on \( < s, p^+ | \) and \( \hat{t}^- \) is diagonal on \( | s, p^- \rangle \) and that the operator \( \hat{t}^2 \) is the dilation operator on functions of \( p^\pm \) except for the additional overall phase. If we define the function \( F_{r,s}(p^+ p^-) = < s, p^+ | s, p^- \rangle \) then we must have a function of
the single variable $p^+p^-$ on account of its properties under dilations. Then the vertex operator is

$$V_{p^+p^-}^{r,s}(z) = e^{iX^+p^-} e^{-\frac{k}{2}X_2(1-2i\alpha)} F_{r,s} (e^{-X_2p^-}p^+) e^{iX^+p^-}$$

(87)

To find the function $F_{r,s}$ we sandwich $<s,p^+|\hat{t}^\mp|s,p^->$ and derive a differential equation by operating on both sides

$$\left(p^\mp - \partial_\pm p^\mp \partial_\pm + 2is\partial_\pm + \frac{kr}{p^\pm}\right) F_{r,s}(p^+,p^-) = 0.$$  

(88)

By multiplying through with $p^\pm$ this becomes a single equation in one variable $\kappa = p^+p^-$

$$\left(\kappa - (\kappa\partial_\kappa)^2 + 2is(\kappa\partial_\kappa) + kr\right) F_{r,s}(\kappa) = 0$$  

(89)

The solution is given in terms of a Bessel function

$$F_{r,s}(\kappa) = \kappa^{is} J_{2\sigma}\left(2\sqrt{-\kappa}\right), \quad \sigma = \sqrt{kr - s^2}.$$  

(90)

Therefore the middle factor in the vertex operator is obtained by rescaling $\kappa$ with the factor $e^{-X^2}$. The result is

$$V_{p^+p^-}^{r,s}(z) = Ce^{iX^-p^+} e^{-\frac{k}{2}X_2} J_{2\sigma}\left(2\sqrt{-p^+p^-}e^{-\frac{1}{2}X_2(z)}\right) e^{iX^+p^-}$$

(91)

where $s, p^\pm$ have been absorbed into an overall factor $C(r,s,p^\pm) = C(s,r) \times (p^+p^-)^{is}$.

### 3.3.2 Quantum ordering and operator products

At the full quantum level (with an ordering of the operators that will be given below) the correct operator products with the currents are given as

$$J^\mu(z) \times V_{p^+p^-}^{r,s}(w) = \frac{1}{z-w} <s,p^+|\hat{t}^\mu V_{p^+p^-}^{r,s}(w)|s,p^->$$  

(92)

where $\mu$ is $+, -, $ or $2$. The action of $\hat{t}^\mu$ on the bra is a differential operator that follows from the left side of (48). Thus, $\hat{t}^+ = p^+$, $\hat{t}_2=$dilations, etc. In addition to getting the correct operator products with the currents we will also see that the dimension of the vertex operator $V_{p^+p^-}^{r,s}(w)$ is $\Delta^{r,s} = -r + \frac{1}{k}\left(\frac{1}{4} + s^2\right)$.

To define the vertex operator at the quantum level, we begin by preserving the order of the factors due to the group theoretical origin of their order, and then apply normal ordering within each factor as follows

$$V_{p^+p^-}^{r,s}(z) = e^{iX^+(z)p^+} <s,p^+| \left( e^{iX_2(z)\hat{\partial}_z} \right) |s,p^-> \left( e^{ip^-(z)\hat{\partial}_z} e^{-\frac{1}{2}X_2(z')} \right)$$  

(93)

where $X^+$ has been written in terms of canonical variables. The leftmost factor does not need normal ordering (see definition of canonical variables). This already provides the order of operators. The next step is a reordering of operators (and picking up factors due to the reordering)
for the purpose of performing computations. A methodical approach for the computation of operator products is the use of Wick’s theorem for free fields. This requires fully normal ordered expressions. For this purpose we need to reorder the operators above to rewrite the vertex operator in fully normal ordered form. This reordering gives the following expression

\[ V_{p^+ p^-}^{r, s}(z) = : e^{iX^-(z) p^+} < s, p^+ | e^{iX_2(z) \hat{t}_2} e^{ip^- q^+} \]

\[ \times e^{- \int^z dz' \frac{1}{2} \left( p^+ (z') - \frac{p^+}{z' - z} - p^+ \right) e^{-x_2(z')} \left( \frac{z - z_1}{\sqrt{zz_1}} \right)^{\frac{2n}{k}} | s, p^- : \]  

(94)

Note that now the whole expression is sandwiched within the normal ordering columns, and this is what produces the complicated additional factors. In computations it will be convenient to further rewrite it by expanding the last exponential as a series and then evaluating the matrix elements at the end

\[ V_{p^+ p^-}^{r, s}(z) = : \exp (iX^-(z) p^+) < s, p^+ | \exp (iX_2(z) \hat{t}_2) e^{ip^- q^+} \sum_{n=0}^{\infty} \frac{1}{n!} \left( \frac{p^-}{k} \right)^n \]

\[ \times \prod_{i=1}^{n} \int^z dz_i \left( P^+(z_i) - \frac{p^+}{2z_i} - \frac{p^+}{z - z_i} \right) e^{-x_2(z_i)} \left( \frac{z - z_1}{\sqrt{zz_1}} \right)^{\frac{2n}{k}} | s, p^- : \]  

(95)

\[ = : C e^{iX^-(z) p^+} e^{-\frac{1}{2} X_2(z)} e^{ip^- q^+} \sum_{n=0}^{\infty} \frac{1}{n!} \left( \frac{p^-}{k} \right)^n \]

\[ \times \prod_{i=1}^{n} \int^z dz_i \left( P^+(z_i) - \frac{p^+}{2z_i} - \frac{p^+}{z - z_i} \right) e^{-x_2(z_i)} \left( \frac{z - z_1}{\sqrt{zz_1}} \right)^{\frac{1}{k}} \]

\[ \times J_{2\sigma} \left( 2 \sqrt{-p^+ p^-} e^{-\frac{1}{2} X_2(z)} \left( \frac{z - z_1}{\sqrt{zz_1}} \right)^{\frac{1}{k}} \right) : \]  

(96)

Since the first definition of the vertex operator is not fully normal ordered, we should show that its vacuum expectation value is finite. For this we use the last form and find

\[ < 0| V_{p^+ p^-}^{r, s}(z)| 0 > = : C \sum_{n=0}^{\infty} \frac{1}{n!} \left( \frac{p^-}{k} \right)^n \prod_{i=1}^{n} \int^z dz_i \left( - \frac{p^+}{2z_i} - \frac{p^+}{z - z_i} \right) \]

\[ \times \left( \frac{z - z_1}{\sqrt{zz_1}} \right)^{-\frac{1}{k}} J_{2\sigma} \left( 2 \sqrt{-p^+ p^-} \left( \frac{z - z_1}{\sqrt{zz_1}} \right)^{-\frac{1}{k}} \right) : \]  

(97)

Changing the integration variable to \( y_i = 2 \sqrt{-p^+ p^-} (\frac{z - z_1}{\sqrt{zz_1}})^{-\frac{1}{k}} \), we see

\[ < 0| V_{p^+ p^-}^{r, s}(z)| 0 >= C \exp \left( \frac{1}{2} \sqrt{-p^+ p^-} \int_0^\infty dy J_{2\sigma}(y) \right) \]  

(98)

which is finite.

Later we will be interested in taking the large \( k \) limit to show that we can recover the vertex operator in flat 3D. In this limit our finite normal ordered vertex operator will be the same as the finite normal ordered vertex operator in flat Minkowski space-time.
The laborious technical details of the calculation of the operator products of the currents and stress tensor with the vertex operator are given in the appendix. With the ordering prescription described above, we find the correct operator products \((92)\). We also show that the operator product of the vertex operator with the energy-momentum tensor gives the correct conformal dimension as in \([19,51]\)

\[
\Delta_{r,s} = \frac{1}{k} \left[ (t_2)^2 - \hat{t}_2 + \hat{t}^+ \hat{t}^- \right] = -r + \frac{1}{k} \left( \frac{1}{4} + s^2 \right).
\]

### 3.4 Zero curvature limit \((k \to \infty)\) and flat vertex operator

#### 3.4.1 \(k \to \infty\) limit for currents

If we rescale the affine currents \(\alpha_n^\mu \equiv J_n^\mu / \sqrt{k+2}\) and then send \(k \to \infty\) we find

\[
k \neq \infty : \quad [J_n^\mu, J_m^\nu] = i \varepsilon^{\mu\nu\lambda} \eta_{\lambda\rho} J_n^{\rho} - (k + 2) n \delta_{n+m} \eta^{\mu\nu}
\]

\[
\frac{\L_n}{\sqrt{k+2}} \cdot \frac{\L_n}{\sqrt{k+2}} = \frac{1}{\sqrt{k+2}} [\varepsilon^{\mu\nu\lambda} \eta_{\lambda\rho} J_n^{\rho} - n \delta_{n+m} \eta^{\mu\nu}]
\]

\[
k \to \infty : \quad [\alpha_n^\mu, \alpha_m^\nu] = -n \delta_{n+m} \eta^{\mu\nu}, \quad \eta^{\mu\nu} = \text{diag} (1, -1, -1)
\]

Therefore, we get the flat theory in this limit. We do the same with the free field formulation of the currents, rescale the free fields, and then identify the flat free fields as the limit of the rescaled free fields.

\[
\left( \frac{J_1 + J_0}{\sqrt{k+2}} \right) = \left( \frac{P^+}{\sqrt{k+2}} \right)
\]

\[
\left( \frac{J_2}{\sqrt{k+2}} \right) = \frac{1}{\sqrt{k+2}} : (\sqrt{k+2} X^-) \left( \frac{P^+}{\sqrt{k+2}} \right) : + \left( \frac{P_2}{\sqrt{k+2}} \right)
\]

\[
\left( \frac{J_1 - J_0}{\sqrt{k+2}} \right) = \frac{1}{k+2} : (\sqrt{k+2} X^-) \left( \frac{P^+}{\sqrt{k+2}} \right) \left( \sqrt{k+2} X^- \right) :
\]

\[
- \frac{2}{\sqrt{k+2}} \left( \frac{P_2}{\sqrt{k+2}} \right) \left( \sqrt{k+2} X^- \right) + i \partial_z \left( \sqrt{k+2} X^- \right)
\]

As \(k \to \infty\) the expressions in parentheses on both sides are kept fixed. Therefore, due to the extra factors of \(1/\sqrt{k+2}\) some terms vanish. Now we identify the flat string coordinates \(\tilde{X}^\mu\) and their conjugate momenta \(\tilde{P}^\mu\) in the large \(k\) limit

\[
\left( \frac{J_\mu}{\sqrt{k+2}} \right) = \tilde{P}^\mu
\]

as the expressions inside parentheses that remain fixed (using \(\tilde{P}^\pm = (\tilde{P}^1 \pm \tilde{P}^0)\) and \(\tilde{X}^\pm = \frac{1}{2} (\tilde{X}^1 \pm \tilde{X}^0)\)), then in the large \(k\) limit we identify

\[
\left( \frac{P^\pm}{\sqrt{k+2}} \right) = \tilde{P}^\pm, \quad \left( \frac{P_2}{\sqrt{k+2}} \right) = \tilde{P}^2, \quad i \partial_z \left( \sqrt{k+2} X^- \right) = \tilde{P}^-
\]

\[
\left( \sqrt{k+2} X^+ \right) = \tilde{X}^+, \quad \left( \sqrt{k+2} X_2 \right) = \tilde{X}_2, \quad \left( \sqrt{k+2} X^- \right) = \tilde{X}^-.
\]
The fundamental canonical pairs are \((\hat{X}^-, \hat{P}^+)\) and \((\hat{X}^2, \hat{P}^2 = i\partial \hat{X}^2)\), which are written in terms of the elementary oscillators (rescaled ones \(\alpha_{n,\pm}^{-}\)). Note that \(\hat{X}^+\) is derived in terms of the oscillators in \(\hat{P}^+\), since \(\hat{P}^+ = i\partial \hat{X}^+\) in the large \(k\) limit according to (23).

In terms of the rescaled variables \(\hat{X}^\mu, \hat{P}^\mu = i\partial_\hat{X}^\mu\), the operator products in the large \(k\) limit become simple for \(\hat{X}^+\) as well, so that we have the operator products of the usual flat string fields

\[
k \to \infty : \quad \hat{X}^\pm (z) \hat{P}^\mp (w) \sim \hat{X}^2 (z) \hat{P}^2 (w) \sim \frac{i}{z-w} \quad (104)
\]

### 3.4.2 Vertex operator in the flat limit

We want to show that the asymptotic form of the vertex operator as \(k \to \infty\) matches the flat form. For this we need to replace \(p^+ = \sqrt{k}\hat{p}^+\) and \(p^- = \sqrt{k}\hat{p}^-\) so that in the limit the factors

\[
\exp \left( i p^\pm X^\pm \right) \to \exp \left( i \hat{p}^\pm \hat{X}^\pm \right)
\]

come out right. Similarly we define \(s = \sqrt{k}\hat{p}_2\) so that \(\exp (isX_2) \to \exp (i\hat{p}_2\hat{X}_2)\). Replacing the rescaled variables in (87) and taking the large \(k\) limit we find that the flat vertex operator emerges

\[
V^{r,s}_{\hat{p}^+, \hat{p}^-} (z) \to e^{i\hat{X}^-\hat{p}^+} e^{i\hat{X}^2\hat{p}_2^2} e^{i\hat{X}^+\hat{p}^-} F^{r,s} \left( k\hat{p}^+ \hat{p}^- \right) \quad (106)
\]

The overall field independent constant \(F^{r,s} (k\hat{p}^+ \hat{p}^-) = (k\hat{p}^+ \hat{p}^-)^{(\sqrt{k}\hat{p}_2}) J_{2m\sqrt{k}} \left( 2\sqrt{-\hat{p}^+ \hat{p}^-} \right)\), is given in terms of the Bessel function where we have defined the mass \(m\) through \(\sigma = \sqrt{k}m\), or

\[
m = \sqrt{r - \hat{p}_2^2} = \sqrt{-\hat{p}^+ \hat{p}^- - \hat{p}_2^2} \quad (107)
\]

To evaluate the large \(k\) limit of \(F^{r,s} (k\hat{p}^+ \hat{p}^-)\) we find it useful to examine the large \(k\) limit of the differential equation in (89). After rescaling \(s = \sqrt{k}\hat{p}_2\) and \(\kappa = k\hat{\kappa}\), and taking the large \(k\) limit we learn from the leading term that \(F^{r,s}\) behaves like a delta function. After using the definition of mass given in (107) we obtain

\[
F^{r,s} \to C\delta \left( \hat{p}^+ \hat{p}^- + \hat{p}_2^2 + m^2 \right) \quad (108)
\]

where \(C\) is a constant. Note that this is the mass shell condition for a flat string. If we substitute this in the vertex operator, then in the \(k \to \infty\) limit we get (up to a constant phase)

\[
\left( V^{r,s}_{\hat{p}^+, \hat{p}^-} (z) \right)_{k \to \infty} = e^{i\hat{X}^-\hat{p}^+} e^{i\hat{X}^2\hat{p}_2^2} e^{i\hat{X}^+\hat{p}^-} \delta \left( \hat{p}^+ \hat{p}^- + \hat{p}_2^2 + m^2 \right) \quad (109)
\]

which is the correct vertex operator in flat 3D-Minkowski space-time, with the mass shell condition imposed.
4 AdS-CFT correspondence and unitarity

4.1 Boundary operators and principal series

The results presented in the previous sections establish properties and vertex operators of the bulk theory in AdS(3). We now discuss the implications of our results for the proposed structure of the boundary conformal field theory (CFT) which is claimed to provide a non-perturbative and second-quantized formulation of string theory on $AdS_3 \times S^3 \times M^4$ \cite{13}, \cite{14}, \cite{15}. This CFT part of our discussion is much less complete as compared to our discussion of the bulk theory, but we feel compelled to include it here because of the considerable confusion and controversy that exists in the literature.

We are motivated by the proposal of \cite{13}, as discussed in more detail in \cite{14} and as recently presented in \cite{15}. The puzzle we want to address is how to implement this proposal for constructing the boundary CFT theory from the bulk string theory without neglecting the information about the unitarity and the physical state spectrum of the bulk string theory that we have presented in the previous sections. There is a puzzle since the vertex operators used in \cite{13} \cite{14} \cite{15} have no relation to the physical states of the theory, and therefore it is not evident how the bulk and boundary theories are related. The proposals we make in this section are aimed at trying to close this gap by connecting our results to those of \cite{13} \cite{14} \cite{15}. This will involve some conceptual issues and some generalizations. In particular, our proposals for boundary operators include strings that wind at the boundary. There will still remain unsolved puzzles as will be seen in the discussion below.

We have particularly emphasized the issues of unitarity and operator-state correspondence, and therefore we are forced to re-examine the proposal of \cite{13} in view of the only allowed unitary representation in the bulk theory discussed above, that is - the principal series representation of the affine current algebra supplemented by the zero modes. We show that the proposal of \cite{13} has to be generalized and modified to include the effect of the zero modes, while working in the unitary principal series representation of the affine current algebra. The vertex operator $V_{x^- , x^+}(g)$ discussed in the previous section of this paper plays a crucial role in the following construction.

First we briefly summarize the proposal of \cite{13} in order to make our presentation more convenient for the reader. We follow the notation of \cite{13}. Let the world-sheet currents $k^a(z)$ of some world-sheet current algebra satisfy the following OPEs

$$k^a(z)k^b(w) \sim \frac{1}{2} \frac{k_G \delta^{ab}}{(z-w)^2} + \frac{f^{abc}k^c(w)}{(z-w)},$$

(110)

where $k_G$ denotes the level of the world-sheet affine Lie algebra $\hat{G}$ and $f^{abc}$ are the structure constants of the corresponding Lie group G. Let the corresponding space-time currents be
denoted by $K^a(x)$. They are expected to satisfy the analogous OPEs

$$K^a(x)K^b(y) \sim \frac{1}{2} \frac{k_G^{(st)} \delta^{ab}}{(x-y)^2} + \frac{f^{abc}K^c(y)}{(x-y)},$$

(111)

where $k_G^{(st)}$ stands for the level of the space-time current algebra.

The proposed form for $K^a(x)$ is dictated by symmetry and the fact that $K^a(x)$ is a $(1,0)$ operator in space-time. Following the proposal of [13] [15] we present a generalized form of the space-time current $K^a$ while insisting on the only unitary representation and taking into account the contribution of the zero modes

$$K^a(x) = -\frac{1}{k} \int d^2z \; k^a(z) \; J(x^+; \bar{z}) \; \Omega^r_{x^-x^+}(z, \bar{z}),$$

(112)

where, the probability distribution $\Omega^r_{x^-x^+}(z, \bar{z})$ is defined as in subsection (3.2.1), which we repeat here for convenience

$$\Omega^r_{x^-x^+}(z, \bar{z}) = \left| V^r_{x^-x^+}(g) \right|^2.$$

(113)

Furthermore, $J$ is defined to be, as in [13] [14]

$$J(x^-; z) \equiv 2x^- J^0(z) - (J^1(z) + J^0(z)) - \left( x^- \right)^2 (J^1(z) - J^0(z)).$$

(114)

and similarly for $\bar{J}(x^+; \bar{z})$. Note that the expression for the space-time current $K^a(x)$ given in [13] formally corresponds to the form of the probability distribution at $r = 0$ since it then coincides with the same expression

$$\Omega^0_{x^-x^+}(g) = \frac{1}{\pi} \left( e^{\phi/2} \left( x^+ - \gamma^+ \right) \left( x^- + \gamma^- \right) + e^{-\phi/2} \right)^{-2}$$

(115)

donated as $\Phi_1(x; \bar{x}; z, \bar{z})$ in [13]. However, for $r \neq 0$ our proposal includes winding strings.

The form of $K^a(x)$ may be interpreted as “dressing” each worldsheet operator with a vertex operator that corresponds to a physical state in the theory. Thus $k^a(z) V^r(z, \bar{z})$ and $\bar{J}(x^+; \bar{z}) (V^r(z, \bar{z}))^*$ are both “dressed”, resulting in the factor $\Omega^r_{x^-x^+}(z, \bar{z})$. Our proposal here differs from [13] [15] only by inserting $\Omega^r_{x^-x^+}(z, \bar{z})$ which is of the form $V V^*$ instead of a single power of the vertex operator $V$. A single $V$ represents a wavefunction related to a physical state. Since a wavefunction is generally complex it cannot result in a real $K^a(x)$. On the other hand, the $\Omega^r_{x^-x^+}(z, \bar{z}) \sim V V^*$ is a probability density that is real and carries the relevant information about the physical state.

We note that we must take the worldsheet conformal dimension $\Delta^r = 0$, so that the operators have the correct dimensions. This is possible provided the zero modes $p^-, \bar{p}^+$ contribute a non-zero $r$. The explicit form of the vertex operator $V^r_{x^-x^+}(g(z, \bar{z}))$ and its dimension $\Delta^r$ was derived and discussed in detail in the previous section. In arriving at this expression, in our case we only used the principal series representation at $r = 0$, and we never had to refer to the non-existent discrete series representation that is not consistent with unitarity. This
approach maintains operator-state correspondence as well as unitarity, since we have used only the positive norm physical states that correspond to the probability $\Omega_{x^-}^{r,s}(g)$. In the next section we will briefly discuss the OPEs of the space-time currents $K^a$ and compare to the results of [15].

We can easily extend our discussion and consider the generators of the space-time Virasoro symmetry. The form of the space-time stress tensor $T^{st}(x)$ is also dictated by symmetry and the fact that $T^{st}(x)$ should be a $(2,0)$ operator in space-time. Again, we generalize the proposal presented in [13] [15], and present formulae valid for the unitary principal series representation of the affine current algebra supplemented with zero modes

$$T^{st}(x) = \frac{1}{2k} \int d^2 z \left( \partial_x - J(x^-; z) \partial_x - \Omega_{x^-}^{r,s}(g) + 2\partial_x - J(x^-; z) \Omega_{x^-}^{r,s}(g) \right) \tilde{J}(x^+; \bar{z}). \quad (116)$$

As before, the expression given in [15] is formally the special case of this formula at $r = 0$, however in our case we do not refer to the discrete series representation which is not consistent with unitarity. In our version unitarity and operator-state correspondence is preserved by using only the probability $\Omega_{x^-}^{r,s}(g)$ associated with physical states. The corresponding OPE

$$T^{st}(x)T^{st}(y) \sim \frac{1}{2} \frac{c^{st}}{(x-y)^4} + \frac{2T^{st}(y)}{(x-y)^2} + \frac{\partial_y T^{st}(y)}{(x-y)}, \quad (117)$$

in principle determines the value of the central charge of the boundary conformal field theory $c^{st}$. This value should be compared to the classical expression for the central charge of the Virasoro algebra at the boundary of $AdS_3$ found in the framework of pure three-dimensional quantum gravity [27]. In the next section we will briefly discuss the OPEs of the space-time stress tensor $T(x)$ and compare to the results of [15].

Note that besides coinciding with the expressions in [15] at $r = 0$, our proposal takes into account the winding strings when $r \neq 0$.

4.2 Operator products

In this section we want to discuss the OPEs of the space-time currents $K^a(x)$ in the space-time current algebra and the OPEs of the space-time stress tensor $T^{st}(x)$ in the space-time Virasoro algebra, given the general expressions for $K^a(x)$ and $T^{st}(x)$ presented above.

In principle, the relevant OPEs should be evaluated for all values of the field $\phi(z, \bar{z})$. To accomplish this it seems necessary to use the general OPEs of two vertex operators $V_{x^-}^{r,s}(g)$. These expressions can be evaluated in principle by using the free field representation discussed in this paper. The corresponding analytically continued expression (relevant for the $SL(2,C)/SU(2)$ WZW model) have been presented by Teschner [19], [20], however only at $r = 0$. The general form of the $V_{x^-}^{r,s}(g) V_{x^-}^{r,s}(g)$ turns out to be rather complicated, even at $r = 0$, as in can be seen from [19], [20]. However, provided one takes the limit $\phi \to \infty$, which
corresponds to the boundary of $AdS_3$, the corresponding semiclassical expression for the OPEs of two vertex operators $V^{r,s}_{x^-} (g)$ simplify considerably.

In particular, we have already seen that for $\phi \to \infty$

$$\Omega^{r,s}_{x^-} (g) \to \delta (x^+ - \gamma^+) \delta (x^- - \gamma^-) \left( [c_1 \log ((x^+ - \gamma^+) (x^- - \gamma^-)) + c_1 \phi + c_2] \right)^2,$$

where $c_1 = 0$, $c_2 = 1$ if $r = 0$, and $c_1 \phi + c_2$ is dropped if $r \neq 0$. When $r = 0$ only the delta function survives in this limit. From this expression it follows that $\Omega_{j_1} \equiv \Omega^{0,s_1}_{x_1^-} (g)$ and $V_{j_2} \equiv V^{0,s_2}_{x_2^-} (g)$ satisfy (with $j_{1,2} = -1/2 + is_{1,2}$, as implied by the unitary principal series representation)

$$\lim_{z_1 \to z_2} \Omega_{j_1} (x_1, \bar{x}_1; z_1, \bar{z}_1) \Omega_{j_2} (x_2, \bar{x}_2; z_2, \bar{z}_2) = \delta (x_1^+ - x_2^+) \delta (x_1^- - x_2^-) \Omega_{j_1+j_2-1} (x_2, \bar{x}_2; z_2, \bar{z}_2)$$

$$\lim_{z_1 \to z_2} \Omega_{j_1} (x_1, \bar{x}_1; z_1, \bar{z}_1) V_{j_2} (x_2, \bar{x}_2; z_2, \bar{z}_2) = \delta (x_1^+ - x_2^+) \delta (x_1^- - x_2^-) V_{j_1+j_2-1} (x_2, \bar{x}_2; z_2, \bar{z}_2)$$

$$\lim_{z_1 \to z_2} V_{j_1} (x_1, \bar{x}_1; z_1, \bar{z}_1) V_{j_2} (x_2, \bar{x}_2; z_2, \bar{z}_2) = \sqrt{\pi \delta (x_1^+ - x_2^+) \delta (x_1^- - x_2^-)} V_{j_1+j_2-1/2} (x_2, \bar{x}_2; z_2, \bar{z}_2)$$

For the last case we used (with $\varepsilon = \exp (-\phi/2) \to 0$)

$$\lim_{\varepsilon \to 0} \frac{1}{\varepsilon^{-1}u^2 + \varepsilon} = \frac{1}{2i} \left( \frac{1}{u - i\varepsilon} - \frac{1}{u + i\varepsilon} \right) = \pi \delta (u),$$

which is different than [15]. Note that these expressions are obviously consistent with the fact that we are working in the principal series representation $j = -1/2 + is$ which closes under operator products. These expressions can be used as a semiclassical limit of the complicated OPE involving probability distributions $\Omega^{0,s}_{x^-} (g)$ or vertex operators $V^{0,s}_{x^-} (g)$ at $r = 0$. Notice that the analogous expressions used in [15] (eqs. (2.34) and (2.35) of [15]) are not consistent with the closure in the principal series representation.

Nevertheless, we use the semiclassical reasoning of [15], to argue that the space-time currents $K^a (x)$ and the space-time stress tensor $T^{st} (x)$ defined above satisfy the OPEs expected from the space-time current algebra and the Virasoro algebra respectively. However, it should be noted that the $\phi \to \infty$ asymptotics of the general vertex operator $V^{r,s}_{x^-} (g)$ involves, apart from the delta function, a log piece as well, so that the general formula for the semiclassical limit of the OPE of two vertex operators is not as simple as the $r = 0$ case, and the semiclassical analysis along the lines of [15] becomes more involved. These refinements are still to be worked out (in principle, by using our free fields), but it is likely that the exact formulas at $r \neq 0$ and general $\phi$ will be similar.

With the caviats above in mind, following the same steps presented in [15] one sees that

$$K^a (x) K^b (y) \sim \frac{1}{2} \frac{I_{[G]}^{(st)} \delta^{ab}}{(x-y)^2} + \frac{f^{abc} K^c (y)}{(x-y)},$$

where in our case the operator $I$ (in the notation of [15]) is defined as

$$I \equiv \frac{1}{k^2} \int d^2 z J (x; z) \tilde{J} (\bar{x}; \bar{z}) \Omega^{r,s}_{x^-} (g).$$

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The analogous formula given in [15] is formally once again a special case of this expression for \( r = 0 \). However, we emphasize that in general (i.e. when \( r \neq 0 \)) it is much harder to argue (as done in [15]) that the value of the expectation value of the operator \( I \) corresponds precisely to the value of the central charge expected from the classical analysis of [27]. Similar observations are valid for the OPEs of the space-time stress tensor \( T^{st}(x) \).

We conclude this section with the following remark: In many papers on \( AdS_3/CFT_2 \) correspondence [13] [14] [15] it is claimed that the Casimir label \( j \) of the bulk \( SL(2,R) \) current algebra is related to the spacetime Virasoro highest weight \( h \) (because the global \( SL(2,R) \) symmetry of the bulk \( AdS_3 \) corresponds to the global conformal symmetry of the boundary \( CFT_2 \) generated by \( L_0 \) and \( L_\pm [26] \)), and since \( h \) is a conformal weight of the boundary \( CFT_2 \), \( j \) has to be real, forcing one to consider the non-unitary discrete series representation. We comment on this argument as follows.

The Virasoro heighest weight \( h \) is undoubtedly real since it is associated with the hermitian spacetime conformal operator \( T^{st}(x) \). However, the relation between \( h \) and the complex \( j \) is not necessarily a simple one. The spacetime \( h \) corresponds to an operator that forms a representation of the space-time theory \( K^a, T^{st} \), and such an operator is constructed from bulk vertex operators \( V^{r,s} \) that carry the label \( j \), or more precisely \( s, r \). As we have seen in the previous section, our construction differs from previous ones and the computation of \( h \) is thereby altered. For example, the probability density \( \Omega^{r,s}_{x-,x+}(g) \) at \( r = 0 \) has the space-time weight \( h = -(j + j^*) = 1 \) which is what was needed to construct the \( \Phi_1(x, \bar{x}; z, \bar{z}) \) in [15] at \( r = 0 \), yet the value of \( j \) is complex.

This example serves to illustrate that one should distinguish the representation space of physical states in the space-time theory from the representation space of physical states in the bulk theory, although one expects some relationship between them, and therefore that \( j \) in the bulk theory is quite happy to be complex while \( h \) in the boundary theory is real. As we have insisted in this paper, the bulk theory is unitary with its physical states appearing in the principal series representation of the affine currents \( j^\mu(z) \), with \( j = -1/2 + is \), and distorted by the effects of zero modes that describe winding strings, represented by sectors labelled with a quantum number \( r \). The dependence of \( h \) on \( s, r \) (or \( j \)) for the representation space of the space-time theory is currently an unsolved problem. Likewise the relation of the physical states of the bulk to the physical states of the boundary theory remains to be clarified.

In this paper we presented what we believe to be the correct description of the Hilbert space of the bulk theory, but our work remains incomplete with regards to the Hilbert space of the boundary theory, and its relationship to the bulk vertex operators. Thus we think that the right interpretation of \( j \) consistent with the unitary principal series representation, supplemented with zero modes, is the one already given in the second part of this paper. The “problems” associated with alleged non-unitarity and failure of operator-state correspondence for the case of string theory on \( AdS_3 \), as discussed in many papers in the literature, simply do
not exist in our construction. To fully establish the AdS$_3$-CFT conjecture, the physical vertex operators need to be used as building blocks, as we have shown partially in this paper, and the refinements discussed in this paper for the vertex operator need to be implemented more rigorously.

5 Conclusions

To summarize, in this paper we have discussed string theory on $AdS_3 \times S^3 \times M^4$ while emphasizing the issues of unitarity and state-operator correspondence. In particular we have re-examined the $AdS_3$-$CFT_2$ correspondence in the Minkowski signature, by taking into account the only allowed unitary representation, the principal series of SL(2,R) supplemented with the zero modes. The zero modes play an important role in the description of on-shell physical states or vertex operators. Without them a unitary formulation of the on-shell theory is not possible. Also, the zero modes describe the winding of long strings around the $AdS_3$ boundary.

The theory is presented as part of the supersymmetric WZW model that includes the supergroup SU(2/1,1) (or OSp(4/2) or $D(2,1;\alpha)$) with central extension $k$. A free field representation is given and the vertex operators are constructed in terms of free fields in SL(2,R) principal series representation bases that are labeled by position space or momentum space at the boundary of $AdS_3$. Our vertex operator includes factors that have been missed in the recent literature on the $AdS_3$-$CFT_2$ correspondence. These factors are related to winding strings. We have shown explicitly that our vertex operator has the correct operator products with the currents and stress tensor, all of which are constructed from free fields. We have also shown that in the limit when $AdS_3$ tends to flat 3D-Minkowski space ($k \rightarrow \infty$), the $AdS_3$ vertex operators in momentum space tend to the vertex operators of flat 3D-string theory (and furthermore the theory readjusts smoothly in the rest of the dimensions in the same limit).

There remains to compute in strictly string theoretical language the various correlation functions or operator products that would verify or refine the AdS-CFT correspondence. In principle our free field vertex operators can be used to compute any correlation function. We intend to discuss these computations in the future.

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A Appendix: Operator products

In this appendix we are going to describe in detail the full quantum treatment of the operator products of the vertex operator with currents and energy-momentum tensor. This calculation is done using the same notation as in [9] which we keep here so that we can use previous results. Therefore one should be careful about the following relations between the notation in this appendix and the notation in the previous sections of this paper:

\begin{equation}
\text{Previous sections : } X^-(z), \quad X_2(z), \quad X^+(z) \\
\text{Appendix : } -X^-(z), \quad -\frac{2}{k} u(z), \quad -X^+(z)
\end{equation}

(124)

In the first subsection of the appendix we are going to describe the quantum ordered expression for the $2 \times 2$ group element ($j = 1/2$ representation) and discuss its operator product with the energy-momentum tensor. Besides serving to double check the result that follows from products with currents $J^\mu(z) g(\omega) \sim t^\mu g(\omega) / (z - w)$ [9], this calculation is also useful to clarify some delicate points of the calculation in more general representations $j$. In the second subsection we are going to discuss operator products of the full vertex operator with the currents and also find the conformal dimension of the vertex operator as stated in (51).

The method of computation is based on Wick’s theorem for free fields. The expressions for contractions between free fields which will be used in the following computations are as follows

\begin{align}
<X^-(z) P^+(w) > & \equiv \frac{i}{2w} + \frac{i}{z-w}, \quad <u(z) S(w) > \equiv k \left( \frac{i}{2w} + \frac{i}{z-w} \right) \\
<P^+(z) X^-(w) > & \equiv \frac{i}{2z} + \frac{i}{z-w}, \quad <u(z) u(w) > \equiv -\frac{k}{2} \ln \left( \frac{z-w}{\sqrt{zw}} \right).
\end{align}

(125)

(126)

The terms of the form $\frac{i}{2z}$ arise from the careful ordering of zero modes such that hermiticity is respected at every step [9]. The definition of the contraction includes this effect of the zero modes.

A.1 Operator products with the group element

We have the quantum operator version of the left moving group element $g_{mn}$, as in [9] modulo the map (124)

\begin{align}
g_{11} &= : e^{-\frac{u(w)}{k}} : , \quad g_{12} = : e^{-\frac{u(w)}{k}} : X^+(w) \\
g_{21} &= X^-(w) : e^{-\frac{u(w)}{k}} : , \quad g_{22} = e^{\frac{u(w)}{k}} + e^{-\frac{u(w)}{k}} X^{-} X^+.
\end{align}

(127)

(128)

The orders of the operators are important in these expressions. This order of operators is dictated by the orders of the matrices that make up $g$. This order will also be respected in more general representations. Then these $g_{mn}$ are not fully normal ordered (recall that the dependent operator $X^+$ given in (25) contains $P^+$ and $e^{2u(w)/k} :$ ). In order to do the calculations of operator products by using Wick’s theorem, first we should rewrite each $g_{mn}$ in normal ordered form, by keeping all the terms that arise from re-ordering the operators, as follows
\[ g_{11} = : e^{-\frac{u(w)}{k}} : \]  
\[ g_{12} = -i \int \frac{dw'}{k} P^+(w') \left( : e^{-\frac{2a(w') - u(w)}{k}} : \right) \left( \frac{w - w'}{\sqrt{ww'}} \right) \frac{1}{k} \]  
\[ g_{21} = X^-(w) : e^{-\frac{u(w)}{k}} : \]  
\[ g_{22} = -i \int \frac{dw'}{k} \left( : X^-(w) P^+(w') e^{-\frac{2a(w') - u(w)}{k}} : \right) \left( \frac{w - w'}{\sqrt{ww'}} \right) \frac{1}{k} \]  
\[ + \int \frac{dw'}{k} \left( \frac{1}{2w'} + \frac{1}{w - w'} \right) \left( \frac{w - w'}{\sqrt{ww'}} \right) \frac{1}{k} \left( : e^{-\frac{2a(w') - u(w)}{k}} : \right) \]  
\[ + \left( : e^{-\frac{u(w)}{k}} : \right) \left( : e^{-\frac{u(w)}{k}} : \right) \]  
\[ (129) \]
\[ (130) \]
\[ (131) \]
\[ (132) \]
\[ (133) \]
\[ (134) \]

In these expressions the ordering of the zero modes of \( : e^{\gamma u(w)/k} : \), where \( \gamma \) is a constant, is defined as
\[ [ : e^{\gamma u(w)/k} : ]_{\text{zero modes}} = e^{-i\gamma \ln z/k} e^{\gamma u_0/k} z(\gamma^2 - 2\gamma)/4k. \]  
\[ (135) \]

Operator products of the currents with the group element were already presented in [4]. The operator product of the energy-momentum tensor
\[ T(z) = : P^+ i\partial X^- : + T_{P_2}(z) \]  
\[ T_{P_2}(z) = \frac{1}{k} \left( : P_2^2 : - \frac{i}{z} \partial(z P_2) + \frac{1}{4z^2} \right) \]  
with the group element is given as
\[ T(z) \times g(w) \rightarrow \frac{hq(w)}{(z - w)^2} + \frac{\partial_w g(w)}{(z - w)}, \]  
\[ h = -\frac{j(j + 1)}{k} = -\frac{3/4}{k} \]  
\[ (136) \]
\[ (137) \]
\[ (138) \]
\[ (139) \]

In order to calculate this expression firstly we need the result of the following two operator products: the operator products of \( T(z) \) with \( : e^{-u(w)/k} : \) and \( : e^{2u(w')/k} : \)
\[ T(z) \times : e^{-u(w)/k} : = -\frac{3/4k}{(z - w)^2} : e^{-u(w)/k} : + \frac{1}{z - w} : \partial_w e^{-u(w)/k} : \]  
\[ (140) \]
\[ T(z) \times : e^{2u(w')/k} : = \frac{1}{z - w} : \partial_w e^{2u(w')/k} : \]  
\[ (141) \]

In this calculations one should be careful about the normal ordering of the zero modes as described above. Using (140), we find
\[ T \times g_{11} \rightarrow \frac{-3/4k}{(z - w)^2} g_{11} + \frac{1}{(z - w)} \partial_w g_{11} \]  
\[ (142) \]
\[ T \times g_{21} \rightarrow : X^- T_{P_2} e^{-u(w)/k} : + : i \partial_z X^- < P^+ X^- > e^{-u(w)/k} : \]  
\[ = \frac{-3/4k}{(z - w)^2} g_{21} + \frac{1}{(z - w)} \partial_w g_{21} \]  
\[ (143) \]
\[ (144) \]
The operator product of $T(z)$ with $g_{12}$ is expected to be

$$T \times g_{12} \to \left( -\frac{3}{4k} \right) \frac{g_{12}(w)}{(z-w)^2} + \frac{\partial_w g_{12}}{(z-w)} \quad (145)$$

where

$$i\partial_w g_{12} = \int^w dw' \frac{P^+(w')}{k} \partial_w \left( : e^{\frac{2u(w')}{k} - \frac{u(w)}{k}} : \right) \left( \frac{w-w'}{\sqrt{ww'}} \right)^\frac{1}{2} + \frac{P^+(w)}{k} \left( : e^{\frac{2u(w)}{k} - \frac{u(w)}{k}} : \right) \left( \frac{w-w'}{\sqrt{ww'}} \right)^\frac{1}{2} \quad (146)$$

$$+ \int^w dw' \frac{P^+(w')}{k} \left( e^{\frac{2u(w')}{k} - \frac{u(w)}{k}} : \right) \frac{1}{w-w'} \left( \frac{w-w'}{\sqrt{ww'}} \right)^\frac{1}{2} \quad (147)$$

$$- \int^w dw' \frac{P^+(w')}{k} \left( e^{\frac{2u(w')}{k} - \frac{u(w)}{k}} : \right) \frac{1}{2k} \left( \frac{w-w'}{\sqrt{ww'}} \right)^\frac{1}{2} \quad (148)$$

To see how this arises through operator products we list the contractions to be calculated

$$T \times g_{12} \to \ : iP^+ \left( -i \int^w dw' \frac{1}{k} < \partial_z X^- P^+(w') > : e^{\frac{2u(w')}{k} - \frac{u(w)}{k}} : \left( \frac{w-w'}{\sqrt{ww'}} \right)^\frac{1}{2} \right) \quad (150)$$

$$-i \int^w dw' \frac{P^+(w')}{k} \left( < T_p e^{\frac{2u(w')}{k} - \frac{u(w)}{k}} : \right) \left( \frac{w-w'}{\sqrt{ww'}} \right)^\frac{1}{2} \quad (151)$$

$$-i \int^w dw' \frac{P^+(w')}{k} \left( : e^{\frac{2u(w')}{k} - \frac{u(w)}{k}} > T_p e^{\frac{2u(w)}{k} - \frac{u(w)}{k}} : \right) \left( \frac{w-w'}{\sqrt{ww'}} \right)^\frac{1}{2} \quad (152)$$

$$-\frac{2i}{k} \int^w dw' \frac{P^+(w')}{k} \left( < P_2 e^{\frac{2u(w')}{k} - \frac{u(w)}{k}} > P_2 e^{\frac{2u(w)}{k} - \frac{u(w)}{k}} : \right) \left( \frac{w-w'}{\sqrt{ww'}} \right)^\frac{1}{2} \quad (153)$$

where the factor of 2 in the last term comes from the permutations of $P_2$'s. The first term in (143) and the term in (146) are obtained from contraction in (152). The contraction in (153) is equal to

$$-\frac{i}{k} \int^w dw' \frac{P^+(w')}{k} \left( \frac{1}{z-w}(z-w') \left( e^{\frac{2u(w')}{k} - \frac{u(w)}{k}} : \right) \left( \frac{w-w'}{\sqrt{ww'}} \right)^\frac{1}{2} \right) \quad (154)$$

$$+ \frac{i}{k} \int^w dw' \frac{P^+(w')}{k} \left( \frac{1}{2z(z-w)} \left( e^{\frac{2u(w')}{k} - \frac{u(w)}{k}} : \right) \left( \frac{w-w'}{\sqrt{ww'}} \right)^\frac{1}{2} \right) \quad (155)$$

$$+ \frac{i}{k} \int^w dw' \frac{P^+(w')}{k} \left( \frac{1}{2z(z-w)} \left( e^{\frac{2u(w')}{k} - \frac{u(w)}{k}} : \right) \left( \frac{w-w'}{\sqrt{ww'}} \right)^\frac{1}{2} \right) \quad (156)$$

The term (153) is just the term in (149). Writing

$$\frac{1}{(z-w')} = \frac{1}{(w-w')} - (z-w) \frac{1}{(z-w')(w-w')} \quad (157)$$

in the first term (154) one gets

$$-\frac{i}{z-w} \int^w dw' \frac{P^+(w')}{k} \left( e^{\frac{2u(w')}{k} - \frac{u(w)}{k}} : \right) \frac{1}{w-w'} \left( \frac{w-w'}{\sqrt{ww'}} \right)^\frac{1}{2} \quad (158)$$

$$-i \int^w dw' \frac{P^+(w')}{k} \left( e^{\frac{2u(w')}{k} - \frac{u(w)}{k}} : \right) \partial_w' (w-w') \left( \frac{1}{\sqrt{ww'}} \right)^\frac{1}{2} \quad (159)$$

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where the first term is equal to the term in (148). The partial integration of the second term (159) gives

\[-\frac{i}{k}\left\langle e^{\frac{2u(w)}{k}} : \frac{w - w'}{\sqrt{ww'}} \right\rangle \right|_{w'=w} \]

\[+i \int^w dw' \frac{1}{k} \left( e^{\frac{2u(w')}{k}} - e^{\frac{u(w')}{k}} \right) \frac{w - w'}{\sqrt{ww'}} \]

Finally the last term (164) gives a finite result combined with the term in (156).

In this expression the first term (160) gives the term in (147), the terms (161,162) combined cancel the result of the term in (150), likewise the fourth term (163) cancels the result of the term in (151). Finally the last term (164) gives a finite result combined with the term in (150).

The calculation of the operator product \( T(z) \times g_{22}(w) \) is similar to the above calculation. Since in the integrand there are extra \( w \) dependent functions, one will see their \( w \)-derivatives appear in the operator product expansion. The double contraction \( i < P^+(z)X^-(w) > \partial_zX^+(z)P^+(w') > \) gives the term coming from the \( w \)-derivative of \( 1/(w-w') \). Other than this, the contraction \( < P^+(z)X^-(w) > i\partial_zX^+(z)P^+(w') > \) gives the term that contains \( \partial_wX^-(w) \).

If one calculates the conformal weight of \( e_{\frac{u(w)}{k}} \) : one finds that it is equal to \( 1/(4k) \), but not \(-3/(4k)\). However, one should be careful about a special term coming from the double contraction of \( P_z^2 \) : with \( g_{22} \). Following the same steps as in the previous calculation we found a term similar to the term in (159):

\[-i \int^w dw' \frac{1}{k} \left( X^-(w)P^+(w') + \frac{i}{2w'} + \frac{i}{w - w'} \right) \left( e^{\frac{2u(w')}{k}} - e^{\frac{u(w')}{k}} \right) \frac{1}{(z - w')} \]

Here the \( 1/(w-w') \) part is new compared to the previous calculation. The other parts work as before. The partial integration of this extra term contains in particular

\[-\frac{i}{k} \left[ \left( e^{\frac{2u(w')}{k}} - e^{\frac{u(w')}{k}} \right) \frac{w - w'}{\sqrt{ww'}} \right]_{w'=w} \]

\[= -\frac{i}{k} \left[ \left( e^{\frac{2u(w')}{k}} - e^{\frac{u(w')}{k}} \right) \frac{w - w'}{\sqrt{ww'}} \right] \]

\[-\frac{1}{k} \frac{1}{(z - w)^2} \left( e^{\frac{2u(w)}{k}} - e^{\frac{u(w)}{k}} \right) \]

\[-\frac{1}{k} \frac{1}{(z - w)^2} \left( e^{\frac{2u(w)}{k}} - e^{\frac{u(w)}{k}} \right) \]

\[-\frac{1}{k} \frac{1}{(z - w)^2} \left( e^{\frac{2u(w)}{k}} - e^{\frac{u(w)}{k}} \right) \]

\[-\frac{1}{k} \frac{1}{(z - w)^2} \left( e^{\frac{2u(w)}{k}} - e^{\frac{u(w)}{k}} \right) \]

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The term (169) comes from the derivative of $g_{22}(w)$. The term (170) is the term that we were looking for to obtain the correct conformal weight $-3/(4k)$. Changing $:e^{\frac{u(w)}{k}}:$ to $:e^{\frac{2u(w)}{k}}:$ times $:e^{-\frac{u(w)}{k}}:$ does not change its conformal dimension. Adding its conformal dimension, $1/4k$, to the coefficient of the term in (170) we again find $-3/4k$. Therefore the operator product of $g_{22}(w)$ with $T(z)$ dictates that the quantum theory be written in the normal ordered form $:e^{\frac{u(w)}{k}}:$ instead of $:e^{\frac{2u(w)}{k}}:$. This calculation shows that the conformal dimension of each entry in the group element $g_{mn}$ is $-3/4k$.

### A.2 Operator products with the vertex operator

In order to simplify the expressions of the operator products we are going to use the expression (94) for the vertex operator, which we write again for convenience in the form

$$V_{r,s}^{p,+}(w) =: f(w) < j p^+ | g(w) h(w) | j p^- > : (171)$$

where

$$f(w) = e^{-iX^-(w)p^+}$$

$$g(w) = :e^{-\frac{2u(w) t_2}{k}}:$$

$$h(w) = \left( e^{-p^- \int w \, dw'} \left( p^+(w') + \frac{p^+}{2} + \frac{p^+}{w - w'} \right) e^{-\frac{2u(w')}{k}} : \left( \frac{w - w'}{\sqrt{ww'}} \right)^{\frac{2i}{k}} : \right) : (172, 173, 174)$$

In the following expressions we will often have $\partial_p^+ g(w)$. Even though in the above expression $g(w)$ seem not to contain $p^+$ explicitly, one sees $p^+$ dependence after calculating its matrix element between the states $< s, p^+ |$ and $| s, p^- > (91)$. Keeping this in mind, we will also often omit the states $< s, p^+ |$ and $| s, p^- >$ in most of the following expressions for simplicity, but their presence is implied.

In the case of operator product of $J^+$ with the vertex operator there is only one contraction

$$J^+(z) \times V_{r,s}^{p,+p^-}(w) \rightarrow : P^+ f > gh := -\frac{p^+}{z - w} V_{r,s}^{p,+p^-}(w)$$

$$= -\frac{1}{z - w} < s, p^+ | t^+ V_j(w) | s, p^- >$$

For the operator product of $J_2$ with the vertex operator we need to calculate the following contractions

$$J_2(z) \times V_{r,s}^{p,+p^-}(w) \rightarrow : X^- : P^+ f > gh : + : P^+ f > g < X^- h :$$

$$+ : f < P_2 g > h : + : f g < J_2(z) h :$$

These contractions give the following results:

$$: X^- : P^+ f > gh : \rightarrow -\frac{1}{z - w} : (ip^+ \partial_p^+ f) gh :$$

(178)
and
\[
<P^+ f > g < X^- h > : \rightarrow - \frac{1}{z - w} : f g (ip^+ \partial_{p^+} h) :
\]
\[
+ : f g \left( -ip^+ p^- \int^w \frac{dw'}{k} \frac{1}{z - w'} \frac{1}{w - w'} : e^{\frac{2u(w')}{k}} : \left( \frac{w - w'}{\sqrt{ww'}} \right)^{2it_2} k \right) h :
\]
(179)

and
\[
: f < P_2 g > h : \rightarrow - \frac{1}{z - w} : f ((ip^+ \partial_{p^+} + \frac{i}{2} + s) g) h :
\]
(180)

and
\[
: fg < J_2(z) h > : \rightarrow
\]
\[
+ : fg \left( ip^+ p^- \int^w \frac{dw'}{k} \frac{1}{z - w'} \frac{1}{w - w'} : e^{\frac{2u(w')}{k}} : \left( \frac{w - w'}{\sqrt{ww'}} \right)^{2it_2} k \right) h :
\]
(181)

In the second and the last contractions we have used the relation
\[
\frac{1}{(z - w)(w - w')} = \frac{1}{z - w} \left( \frac{1}{w - w'} - \frac{1}{z - w'} \right)
\]
(182)

Combining these terms we get
\[
J_2(z) \times V^{r,s}_{p^+ p^-} (w) \rightarrow \frac{-1}{z - w} (ip^+ \partial_{p^+} + \frac{i}{2} + s) : f (w) g(w) h(w) :
\]
\[
= \frac{-1}{z - w} < s, p^+ | t_2 V_2 (w) | s, p^- >
\]
(183)

The most involved operator product is the operator product of \(J^-\) with the vertex operator. It contains the following contractions:
\[
J^- (z) \times V^{r,s}_{p^+ p^-} (w) \rightarrow - : (X^-)^2 < P^+ f > gh :
\]
(184)
\[
-2 : X^- < P^+ f > g < X^- h >:
\]
(185)
\[
- : < P^+ f > g < (X^-)^2 h >:
\]
(186)
\[
-2 : X^- f < P_2 g > h : -2 : f < P_2 g > < X^- h >:
\]
(187)
\[
+ : fg < J^- (z) h > :
\]
(188)

where in the second contraction the factor of 2 comes from the permutations of \(X^-\)'s. The last contraction is equal to
\[
: fg < J^- (z) h > : \rightarrow
\]
\[
+ : fg \left( -p^- \int^w \frac{dw'}{k} \frac{1}{z - w'} : \partial_{w'} e^{\frac{2u(w')}{k}} : \left( \frac{w - w'}{\sqrt{ww'}} \right)^{2it_2} k \right) h :
\]
(189)
\[
-2p^+ : (\partial_{p^+} f) \left( -p^- \int^w \frac{dw'}{k} \frac{1}{z - w'} \frac{1}{w - w'} : e^{\frac{2u(w')}{k}} : \left( \frac{w - w'}{\sqrt{ww'}} \right)^{2it_2} k \right) h :
\]
(190)
\[
+ : fg \left( -p^- \int^w \frac{dw'}{(z - w')^2} : e^{\frac{2u(w')}{k}} : \left( \frac{w - w'}{\sqrt{ww'}} \right)^{2it_2} k \right) h :
\]
(191)
Partial integration of the third term gives

\[- : fg \left[p^{- \frac{1}{z-w'}} e^{\frac{2u(w')}{k}} : \left(\frac{w-w'}{\sqrt{ww'}} \right)^{2it_k} \right]_{w'=w} h : \] (192)

\[- : fg \left(-p^{- \int^w dw' \frac{1}{z-w'} : \partial_{w'} e^{\frac{2u(w')}{k}} : \left(\frac{w-w'}{\sqrt{ww'}} \right)^{2it_k} \right) h : \] (193)

\[- : fg \left(-p^{- \int^w dw' \frac{1}{z-w'} : e^{\frac{2u(w')}{k}} : \partial_{w'} \left(\frac{w-w'}{\sqrt{ww'}} \right)^{2it_k} \right) h : \] (194)

where the term (193) cancels the term in (189). The term (192) can be written as (writing also the end states explicitly)

\[- \frac{1}{z-w} f(w) < s, p^+ | e^{-i\frac{2u(w)}{k}t_z} : e^{\frac{2u(w)}{k}} : p^- (e^{-it^+X^+(w)}) | s, p^- > \] (195)

If one inserts \( t^- \) on the right instead of \( p^- \) and moves it to the other side of \( e^{-i\frac{2u(w)}{k}t_z} \), one gets

\[ (e^{-i\frac{2u(w)}{k}t_z} : e^{\frac{2u(w)}{k}} : t^- = t^- (e^{-i\frac{2u(w)}{k}t_z} : \) (196)

Therefore the contraction of \( J^- (z) \) with only the last exponential piece of \( V^r, s_{p^+ p^-} (w) \) gives us

\[ - \frac{1}{z-w} : (p^+ f(\partial^2_{p^+} g)h + (1-2is)f(\partial_{p^+} g)h + \left( -\frac{kr}{p^+} \right) fgh) : \] (197)

\[ -2p^+ : (\partial_{p^+} f) g \left(-p^{- \int^w dw' \frac{1}{k} \frac{1}{z-w'} \frac{1}{w-w'} : e^{\frac{2u(w')}{k}} : \left(\frac{w-w'}{\sqrt{ww'}} \right)^{2it_k} \right) h : \] (198)

\[ - : fg \left(-p^{- \int^w dw' \frac{1}{z-w'} : e^{\frac{2u(w')}{k}} : \partial_{w'} \left(\frac{w-w'}{\sqrt{ww'}} \right)^{2it_k} \right) h : \] (199)

where in the first line the representation of \( t^- \) on \( < s, p^+ | \) (18) is used.

The other contractions are as follows

\[ : (X^-)^2 < P^+ f > gh \rightarrow \frac{1}{z-w} : p^+ (\partial^2_{p^+} f) gh : \] (200)

and

\[ 2 : X^- < P^+ f > g < X^- h > \rightarrow \frac{1}{z-w} : 2p^+ (\partial_{p^+} f) g(\partial_{p^+} h) : \] (201)

\[ -2p^+ : (\partial_{p^+} f) g \left(-p^{- \int^w dw' \frac{1}{k} \frac{1}{z-w'} \frac{1}{w-w'} : e^{\frac{2u(w')}{k}} : \left(\frac{w-w'}{\sqrt{ww'}} \right)^{2it_k} \right) h : \] (202)
and
\[ 2: X^- f < P_2 g > h : \rightarrow \frac{1}{z-w} : 2p^+(\partial_{p^+} f)(\partial_{p^+} g)h + (1-2is)(\partial_{p^+} f)gh : \]  \hspace{1cm} (203)

and
\[ 2: f < P_2 g >> X^- h : \rightarrow \]
\[ \frac{1}{z-w} : 2p^+ f(\partial_{p^+} g)(\partial_{p^+} h) + (1-2is)f g(\partial_{p^+} h) : \]
\[ + : fg \left( -p^- f^w dw' \frac{1}{z-w'} : e^{2u(w')} - \frac{1}{w-w'} + \frac{1}{2z} \left( \frac{w-w'}{\sqrt{w w'}} \right)^{2it_2/k} \right) h : \]
\hspace{1cm} (204)

In writing (201) and (204) we again used the relation (182).

Combining these terms we get the desired result
\[ J^-(z) \times V_{p^+p^-}^{r,s}(w) \rightarrow -\frac{1}{z-w} (p^+ \partial_{p^+}^2 + (1-2is)\partial_{p^+} - \frac{k r}{p^+}) : f g h : \]
\[ = -\frac{1}{z-w} < s, p^+ | t^- V_j(w) | s, p^- > \]  \hspace{1cm} (205)

Next, we calculate the operator product of the vertex operator with the energy-momentum tensor (136) and determine its conformal weight. The expected result of this operator product is
\[ T(z) \times V_{p^+p^-}^{r,s}(w) \rightarrow \frac{-r + \frac{1}{k} \left( \frac{1}{4} + s^2 \right)}{(z-w)^2} V_{p^+p^-}^{r,s}(w) + \frac{1}{z-w} \partial_w V_{p^+p^-}^{r,s}(w) \]
\hspace{1cm} (207)

where the right hand side may be written as
\[ \frac{1/k}{(z-w)^2} < j p^+ | \left( (t_2)^2 - it_2 + t^+ t^- \right) V_j(w) | j p^- > \]  \hspace{1cm} (208)
\[ + \frac{1}{z-w} : (\partial_w f) g h : + : f(\partial_w g) h : + f g (\partial_w h) \]
\hspace{1cm} (209)

and the last term has the form
\[ \frac{fg (\partial_w h)}{z-w} = : \frac{fg}{z-w} \left[ -\frac{p^-}{k} \left( P^+(w') + \frac{p^+}{2w'} + \frac{p^+}{w-w'} \right) : e^{2u(w')} - \frac{1}{\sqrt{w w'}} \right] h : \]
\[ - : \frac{fg}{z-w} \left( -p^- \int w dw' \frac{p^+}{k (w-w')} : e^{2u(w')} - \frac{1}{\sqrt{w w'}} \right) h : \]  \hspace{1cm} (210)
\[ + : \frac{fg}{z-w} \left( -p^- \int w dw' (\cdots) : e^{2u(w')} - \frac{1}{\sqrt{w w'}} \right) h : \]  \hspace{1cm} (211)
\[ - : \frac{fg}{z-w} \left( -p^- \int w dw' (\cdots) : e^{2u(w')} - \frac{1}{\sqrt{w w'}} \right) h : \]  \hspace{1cm} (212)

where \((\cdots)\) in the last two terms stand for \((P^+(w') + \frac{p^+}{2w'} + \frac{p^+}{w-w'})\).
In this operator product there are the following contractions to be calculated and then compared to the above result

\[ T(z) \times V^{e_1}_{p_1^+,p_2^-}(w) \rightarrow :i\partial_z X^- < P^+ f > gh : + : iP^+ fg < \partial_z X^- h : \]

\[ + : i < P^+ f > g < \partial_z X^- h : \]

\[ + : f < T_{P_2}(z) g > h : + : fg < T_{P_2}(z) h : \]

\[ + \frac{2}{k} : f < P_2 g > < P_2 h : \]

where the factor of 2 in the last term comes from the permutations of \( P_2 \)'s. The result of these contractions are

\[ : i\partial_z X^- < P^+ f > gh : \rightarrow -\frac{ip^+}{z - w} : \partial_w X^- f gh : = \frac{1}{z - w} : (\partial_w f) gh : \]

and

\[ : iP^+ fg < \partial_z X^- h : \rightarrow : fg \left(-p^- \int^w dw' \frac{P^+(z)}{k} \frac{1}{(z - w')^2} \right) e^{\frac{2u(w')}{k}} \left( \frac{w - w'}{\sqrt{ww'}} \right)^{\frac{2it_2}{k}} : \]

and

\[ : i < P^+ f > g < \partial_z X^- h : \rightarrow i \left( \frac{u_k}{z - w} - \frac{u^k}{z - w} \right) : fg \left(-p^- \int^w dw' \frac{1}{k} \frac{1}{(z - w')^2} \right) e^{\frac{2u(w')}{k}} \left( \frac{w - w'}{\sqrt{ww'}} \right)^{\frac{2it_2}{k}} : \]

and

\[ : f < T_{P_2} g > h : \rightarrow \frac{1/k}{(z - w)^2} : (i t_2)^2 : fgh : + \frac{1}{z - w} : f(\partial_w g)h : \]

and

\[ : fg < T_{P_2} h : \rightarrow : fg \left(-p^- \int^w dw' \frac{1}{k} \frac{1}{z - w'} \right) e^{\frac{2u(w')}{k}} \left( \frac{w - w'}{\sqrt{ww'}} \right)^{\frac{2it_2}{k}} : \]

and

\[ \frac{2}{k} : f < P_2 g > < P_2 h : \rightarrow \]

\[ \frac{2it_2}{k} \left( \frac{1}{2z} - \frac{1}{z - w} \right) : fg \left(-p^- \int^w dw' \frac{1}{k} \frac{1}{2z} \frac{1}{z - w'} \right) e^{\frac{2u(w')}{k}} \left( \frac{w - w'}{\sqrt{ww'}} \right)^{\frac{2it_2}{k}} : \]

\[ = \frac{2it_2}{k} : fg \left(-p^- \int^w dw' \frac{1}{k} \frac{1}{z - w}(z - w') \right) e^{\frac{2u(w')}{k}} \left( \frac{w - w'}{\sqrt{ww'}} \right)^{\frac{2it_2}{k}} : \]

\[ - \frac{2it_2}{k} : fg \left(-p^- \int^w dw' \frac{1}{2z(z - w')} \frac{1}{2z} e^{\frac{2u(w')}{k}} \left( \frac{w - w'}{\sqrt{ww'}} \right)^{\frac{2it_2}{k}} : \]
\[-\frac{2it_2}{k} : f g \left(-p^- \int \frac{dw'}{k} (\cdots) \frac{1}{2z(z-w)} : e^{\frac{2u(w')}{k}} : \left(\frac{w-w'}{\sqrt{ww'}}\right)^{2it_2} \right) h : \quad (226)\]

where \((\cdots)\) in the last two contractions stand for \((P^+(w') + \frac{p^+}{2w'} + \frac{p^+}{w'-w})\).

As one notices the first three terms in \(T \times V_{p^+p^-}^{r,s} (218-219)\), except \(\frac{1}{k(z-w)^2} < t^+V_j(w) >\) piece, is given by the contractions (218 and (221). The term in (224) is the last term in \(T \times V_{p^+p^-}^{r,s} (213)\). Adding the contractions (219) and (220) one gets

\[
: f g \left(-p^- \int \frac{dw'}{k} \left(P^+(z) + \frac{p^+}{2z} - \frac{p^+}{z-w} \right) \frac{1}{(z-w')^2} : e^{\frac{2u(w')}{k}} : \left(\frac{w-w'}{\sqrt{ww'}}\right)^{2it_2} \right) h : \quad (227)
\]

\[
= : f g \left(-p^- \int \frac{dw'}{k} \left(P^+(z) + \frac{p^+}{2z} \right) \frac{1}{(z-w')^2} : e^{\frac{2u(w')}{k}} : \left(\frac{w-w'}{\sqrt{ww'}}\right)^{2it_2} \right) h : \quad (228)
\]

\[
- : f g \left(-p^- \int \frac{dw'}{k} \frac{p^+}{z-w(w-w')^2} : e^{\frac{2u(w')}{k}} : \left(\frac{w-w'}{\sqrt{ww'}}\right)^{2it_2} \right) h : \quad (229)
\]

\[
+ : f g \left(-p^- \int \frac{dw'}{k} \left(\frac{p^+}{z-w'} + \frac{1}{w-w'} \right) \frac{1}{w-w'} : e^{\frac{2u(w')}{k}} : \left(\frac{w-w'}{\sqrt{ww'}}\right)^{2it_2} \right) h : \quad (230)
\]

The second term in the above expression is the term in \(T \times V_{p^+p^-}^{r,s} (211)\).

The term in (224) can be written as

\[
: \frac{2it_2}{k(z-w)} f g \left(-p^- \int \frac{dw'}{k} (\cdots) \frac{1}{z-w'} : e^{\frac{2u(w')}{k}} : \left(\frac{w-w'}{\sqrt{ww'}}\right)^{2it_2} \right) h : \quad (231)
\]

\[
+ : f g \left(-p^- \int \frac{dw'}{k} (\cdots) \frac{1}{z-w} : e^{\frac{2u(w')}{k}} : \left(\frac{1}{\sqrt{ww'}}\right)^{2it_2} \partial_w (w-w')^{2it_2} \right) h : \quad (232)
\]

by using the relation (237). The term in (231) gives the term in (212). Partial integration of the second term above gives

\[
: f g \left[ -p^- \int \left(P^+(w') + \frac{p^+}{2w'} + \frac{p^+}{w'-w} \right) \frac{1}{z-w'} : e^{\frac{2u(w')}{k}} : \left(\frac{w-w'}{\sqrt{ww'}}\right)^{2it_2} \right] \bigg|_{w'=w} - \frac{iit_2}{w} : e^{\frac{2u(w')}{k}} : \left(\frac{w-w'}{\sqrt{ww'}}\right)^{2it_2} \right) h : \quad (233)
\]

\[
- : f g \left(-p^- \int \frac{dw'}{k} \left(\partial_w P^+(w') - \frac{p^+}{2(w-w')^2} + \frac{p^+}{(w-w')^2} \right) \frac{1}{z-w} : e^{\frac{2u(w')}{k}} : \left(\frac{w-w'}{\sqrt{ww'}}\right)^{2it_2} \right) h : \quad (234)
\]

\[
- : f g \left(-p^- \int \frac{dw'}{k} \left(P^+(w') + \frac{p^+}{2w'} + \frac{p^+}{w'-w} \right) \frac{1}{z-w'} : \partial_w e^{\frac{2u(w')}{k}} : \left(\frac{w-w'}{\sqrt{ww'}}\right)^{2it_2} \right) h :
\]
The second term above cancels the term in (223) and the last term cancels (222). Combining the third and the fourth terms with the terms of (228,230) one gets finite terms. Therefore, after all the calculations done so far we obtained all the terms in $T \times V^{r,s}_{p^+p^-}$ (208-213), except the term $\frac{1}{(z-w)^2} < t^+ t^- V_j(w) >$ piece and the term (210) and the only remaining term coming from the contractions is the first term in (233). This term can be written as

$$\frac{1}{z-w} : fg \left[ \frac{-p^+}{k} \left( P^+(w') + \frac{p^+}{2w'} \right) : e^{\frac{2u(w')}{k}} : \left( \frac{w-w'}{\sqrt{ww'}} \right)^{2i\tau} \right] \frac{1}{w' = w} h :$$

$$+ \frac{1}{z-w} : fg \left[ \frac{-p^-}{k} \frac{p^+}{w-w'} : e^{\frac{2u(w')}{k}} : \left( \frac{w-w'}{\sqrt{ww'}} \right)^{2i\tau} \right] \frac{1}{w' = w} h :$$

$$- \frac{1}{z-w} : fg \left[ \frac{-p^-}{k} \frac{p^+}{z-w} : e^{\frac{2u(w')}{k}} : \left( \frac{w-w'}{\sqrt{ww'}} \right)^{2i\tau} \right] \frac{1}{w' = w} h :$$

In writing this expression we used the relation (182). The combination of first two terms in (234) gives the term (210). Whereas the last term in (234) can be written as

$$\frac{1}{(z-w)^2} < s, p^+ | p^+ f(w) \left( e^{-i\frac{2u(w)}{k}} t_2 \right) \left( e^{\frac{2u(w)}{k}} \right) : e^{-i t^+ X^+} : | p^- s, p^- >$$

Changing $p^+$ to $t^+$ and $p^-$ to $t^-$, and then moving $t^-$ past $t_2$ (196) one gets

$$\frac{1}{(z-w)^2} < s, p^+ | \frac{t^+ t^-}{k} V^{r,s}(w) | s, p^- >$$

This is the last piece needed in the operator product of the vertex operator with the energy-momentum tensor (208-213).

This computation proves that we have constructed the correct vertex operator at the quantum level.

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