More exact predictions of SUSYM for string theory

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Abstract

We compute the coefficients of an infinite family of chiral primary operators in the local operator expansion of a circular Wilson loop in \(\mathcal{N} = 4\) supersymmetric Yang-Mills theory. The computation sums all planar rainbow Feynman graphs. We argue that radiative corrections from planar graphs with internal vertices cancel in leading orders and we conjecture that they cancel to all orders in perturbation theory. The coefficients are non-trivial functions of the 'tHooft coupling and their strong coupling limits are in exact agreement with those previously computed using the AdS/CFT correspondence. They predict the sub-leading orders in strong coupling and could in principle be compared with string theory calculations.
1 Introduction and summary of main results

The idea that a quantized gauge theory could have a dual description as a string theory has a long history. Recently one concrete realization of such a duality has emerged. It has been conjectured \([1]\) that there is an exact mapping between \(\mathcal{N} = 4\) supersymmetric Yang-Mills theory (SYM) with gauge group SU(\(N\)) on four dimensional spacetime and IIB superstring theory on background \(AdS_5 \times S_5\) with \(N\) units of RR flux.

This mapping is most useful in the low energy, weakly coupled limit of the string theory. This coincides with the large \(N\) ’tHooft limit \([2]\) of SYM theory, where \(N\) is taken to infinity holding the combination of Yang-Mills coupling constant and \(N\), defined by \(\lambda = g^2 N\), fixed and then taking the large \(\lambda\) limit. This projects onto the strong coupling limit of the sum of planar Feynman diagrams. In string theory, this coincides with the classical low energy limit where the string theory is accurately described by type IIB supergravity on the background space \(AdS_5 \times S_5\). Some explicit computations can be done there. The results can then be interpreted in terms of the gauge theory using a well-defined prescription \([3],[4]\).

Though it has been used for many computations of the strong coupling limits of gauge theory quantities (see refs. \([5]-[9]\) for reviews), it is difficult to obtain a direct check of the Maldacena conjecture. The reason for this is the fact that the correspondence with supergravity computes gauge theory in the large \(\lambda\) limit, with corrections from tree level string effects being suppressed by powers of \(1/\sqrt{\lambda}\) and sometimes computable to the next order. On the other hand, the only other analytical tool which can be used systematically in the gauge theory is perturbation theory which is an asymptotic expansion in small \(\lambda\). Generally, the only quantities for which these expansions have an overlapping range of validity is for quantities which are so protected by supersymmetry that they do not depend on the coupling constant.

There is, however, one known example of a quantity which is a non-trivial function of the coupling constant and whose large \(N\) limit is computable and is thought to be known to all orders in perturbation theory in planar diagrams. That quantity is the circular Wilson loop. Its expectation value was computed in ref. \([10]\). The contribution of a subset of all Feynman graphs, the planar rainbow diagrams, were found at each order in \(\lambda\) and the sum of all orders was taken to obtain the result

\[
\langle W[\text{circle}] \rangle = \frac{2}{\sqrt{\lambda}} I_1(\sqrt{\lambda}) \approx \sqrt{\frac{2}{\pi}} \frac{e^{\sqrt{\lambda}}}{\lambda^{3/4}} \quad \text{as} \quad \lambda \to \infty \tag{1.1}
\]

where \(I_1(x)\) is a modified Bessel function. It was also shown explicitly that the leading corrections to the sum of rainbow diagrams cancels identically. It was conjectured that this cancellation would also occur at higher orders and the result \([1.1]\) was thus the exact sum of all planar diagrams. Some support for this conjecture was developed in ref. \([11]\). They also observed that the sum over Feynman diagrams could be obtained for all orders in the \(1/N\) expansion and had a beautiful argument that, in the large \(\lambda\) limit, these higher orders produced the expected higher genus string corrections.

The large \(\lambda\) limit in \([1.1]\) agrees with the expectation value of the circular Wilson loop which was computed using the AdS/CFT correspondence in refs. \([12],[13]\). If \([1.1]\) is indeed an exact result, this provides a non-trivial check on the validity of the AdS/CFT correspondence. It gives the further interesting possibility of comparing corrections at sub-leading orders in \(1/\sqrt{\lambda}\) with string theory computations. Investigations
of the relevant string theory technique appears in refs. [14] - [19] but explicit calculations of the $1/\sqrt{\lambda}$ corrections have not yet been done.

In this Paper we shall report the computation of a series of expansion coefficients which are related to the circular Wilson loop and chiral primary operators in $\mathcal{N} = 4$ SYM theory. The problem that we pose is the following. When probed from a distance much larger than the size of the loop, the Wilson loop operator can be expanded in a series of local operators with some coefficients [20], [12]:

$$W[C] = \langle W[C] \rangle \sum_{\Delta} C_{A} R^{\Delta A} \mathcal{O}^{A}(0)$$  \hspace{1cm} (1.2)$$

where $\mathcal{O}^{A}(0)$ is a local operator evaluated at the center of the loop, $\Delta_{A}$ is the conformal dimension of $\mathcal{O}^{A}(x)$ and $R$ is the radius of the loop. The problem is to compute the coefficients $C_{A}$ in this operator product expansion (OPE).

We shall concentrate on computing the coefficients for a particular class of chiral primary operators (CPO). We will be able to compute the contribution of the sum of all planar rainbow graphs to the coefficients $C_{A}$ in that case. We are also able to show that the leading order corrections to this sum, which come from diagrams with internal vertices, cancels identically. This leads us to conjecture that the radiative corrections cancel to all orders and the sum of planar rainbow graphs gives the exact result.

We find that the coefficients that we compute are non-trivial functions of the coupling constant. In the limit of large $\lambda$ they coincide with results of the AdS/CFT correspondence [12]. This gives a large array of non-trivial functions of the coupling constant which could be compared with string computations of the strong coupling limit. There are various reasons why these computations could be simpler than the $1/\sqrt{\lambda}$ corrections to the expectation value of the Wilson loop itself.

The Wilson loop operator in $\mathcal{N} = 4$ SYM theory that is readily computed using the AdS/CFT correspondence and which has the right transformation properties under supersymmetry [21], [13] contains the scalar fields inside the path-ordered exponential:

$$W[C] = \frac{1}{N} \operatorname{tr} \exp \left[ \oint_{C} d\tau \left( i A_{\mu}(x) \dot{x}_{\mu} + \Phi^{i}(x) \theta_{i} |\dot{x}| \right) \right] $$  \hspace{1cm} (1.3)$$

where $x_{\mu}(\tau)$ parameterize the contour $C$ and $\theta_{i}$ are Cartesian coordinates of a point on $S^{5}$: $\theta^{2} = 1$. If the size of the contour $C$ is small, the Wilson loop can be expanded in local operators as in (1.2).

For primary operators, one can choose a basis where

$$\langle \mathcal{O}^{A}(x) \mathcal{O}^{B}(y) \rangle = \frac{\delta^{AB}}{|x - y|^{\Delta_{A} + \Delta_{B}}}$$  \hspace{1cm} (1.4)$$

Then their OPE coefficients can be extracted from the large distance behavior of connected two-point correlation functions,

$$\frac{\langle W(C) \mathcal{O}^{A}(L) \rangle}{\langle W(C) \rangle} = \mathcal{C}_{A} \frac{R^{\Delta A}}{L^{2 \Delta_{A}}} + \cdots$$  \hspace{1cm} (1.5)$$

where $L \gg R$ and the omitted terms are of higher order in $R^{2}/L^{2}$. 

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The coefficient corresponding to the CPO of lowest conformal dimension, which in this case is $\Delta = 2$, is important as it determines the correlator of two Wilson loops with large separation

$$\langle W[C_1]W[C_2]\rangle_c = C_2^2 \left( \frac{R}{L} \right)^4 + \ldots$$  \hspace{1cm} (1.6)$$

The coefficients of various CPOs in the expansion of the circular Wilson loop were calculated in ref. [12] both perturbatively at $\lambda \sim 0$ and at strong coupling, $\lambda \sim \infty$, using the AdS/CFT correspondence. Evaluation of the correlators that define coefficients in the strong-coupling regime involves a hybrid of the supergravity and the string calculations.

In AdS/CFT, the Wilson loop operator (1.3) naturally couples to stringy degrees of freedom. It creates a classical string world-sheet which is embedded in $AdS_5 \times S_5$ and whose boundary is the contour of the Wilson loop [21],[22].

On the other hand, a local operator $O^A(x)$ emits one of the supergravity fields at point $x$. When it contributes to a correlator of $O^A(x)$ with the Wilson loop, this supergravity mode propagates on the background $AdS_5 \times S_5$ and is then absorbed by a vertex operator which must be integrated over the string world-sheet.

1.1 Dimension two operators

Let us begin by considering the CPO with smallest conformal dimension, $\Delta = 2$. It is the symmetric traceless part of a gauge invariant product of scalar fields,

$$O^{ij} = \frac{8\pi}{\sqrt{2}} \frac{1}{\lambda} \text{tr} \left( \Phi^i \Phi^j - \frac{1}{6} \delta^{ij} \Phi^2 \right). \hspace{1cm} (1.7)$$

This operator is the lowest weight component of a short multiplet of $\mathcal{N} = 4$ superconformal algebra. Such chiral primary operators have very special properties. The super-conformal algebra guarantees that their conformal dimensions do not receive radiative corrections, so in this case the conformal dimension is exactly two. Furthermore, it is known that their two and three-point correlation functions are given by the free field values, that is, that they are independent of the coupling constant, $g$. It is known that their four-point functions are non-trivial, so they are not free fields in disguise [23]-[26].

In (1.7), the overall coefficient is chosen to give a canonical normalization of the two-point function:

$$\langle O^{ij}(x)O^{kl}(y) \rangle = \frac{1}{2} \left( \delta^{ik} \delta^{jl} + \delta^{il} \delta^{jk} - \frac{1}{3} \delta^{ij} \delta^{kl} \right) \frac{1}{|x-y|^4}. \hspace{1cm} (1.8)$$

The small coupling limit of the correlator of $O^{ij}$ with the Wilson loop is straightforward to obtain. To leading order in perturbation theory:

$$\langle W(C)O^{ij} \rangle \left( \frac{W(C)}{W(C)} \right) = \frac{1}{N} \frac{1}{2\sqrt{2}} \lambda \left( \theta^i \theta^j - \frac{1}{6} \delta^{ij} \right) \frac{R^2}{L^4} \hspace{1cm} (\lambda \to 0). \hspace{1cm} (1.9)$$

The linear dependence on $\lambda$ is an obvious consequence of the fact that the correlator contains two propagators and one power of $\lambda$ is cancelled by the normalization.
The AdS dual of the dimension two operator is the negative mass scalar which is a linear combination of the trace of the metric and the Ramond-Ramond four-form field. Its contribution to the OPE of the circular Wilson loop was calculated in [12]:

\[ \frac{\langle W(C) O^{ij} \rangle}{\langle W(C) \rangle} = \frac{1}{N} \sqrt{\frac{2\lambda}{\lambda}} \left( \theta^i \theta^j - \frac{1}{6} \delta^{ij} \right) \frac{R^2}{L^4} \quad (\lambda \to \infty). \]  

Comparing OPE coefficients at strong and at weak coupling, we see that the scaling with \( \lambda \) is different. The OPE coefficients are clearly renormalized by radiative corrections. We shall conjecture that, in the large N limit, this renormalization is entirely due to planar rainbow diagrams. We shall also obtain the sum of planar rainbows as

\[ \frac{\langle W(C) O^{ij} \rangle}{\langle W(C) \rangle} = \frac{1}{N} \sqrt{\frac{2\lambda}{\lambda}} \frac{I_2(\sqrt{\lambda})}{I_1(\sqrt{\lambda})} \left( \theta^i \theta^j - \frac{1}{6} \delta^{ij} \right) \frac{R^2}{L^4}, \]  

where \( I_2 \) and \( I_1 \) are modified Bessel functions. By construction, this expression reduces to (1.9) at small \( \lambda \). Since

\[ \lim_{\lambda \to \infty} \frac{I_k(\sqrt{\lambda})}{I_1(\sqrt{\lambda})} = 1 \]  

for any \( k \), the AdS/CFT prediction (1.10) is also exactly reproduced at large \( \lambda \). The sum of rainbow diagrams thus interpolates between perturbative and strong coupling limits of the OPE coefficient.

### 1.2 Chiral primary operators

The \( O^{ij} \) is the first in an infinite sequence of CPOs. The operator of dimension \( k \) in this sequence is a symmetrized trace of \( k \) scalar fields:

\[ O^I_k = \frac{(8\pi^2)^{k/2}}{\sqrt{k\lambda^{k/2}}} C^I_{i_1 \ldots i_k} \text{tr} \Phi^{i_1} \ldots \Phi^{i_k}, \]  

where \( C^I_{i_1 \ldots i_k} \) are totally symmetric traceless tensors normalized as

\[ C^I_{i_1 \ldots i_k} C^J_{i_1 \ldots i_k} = \delta^I_J. \]  

The AdS duals of CPOs are Kaluza Klein modes of the \( AdS_5 \) tachyonic scalar on \( S^5 \) and each CPO is associated with a spherical harmonic:

\[ Y^I(\theta) = C^I_{i_1 \ldots i_k} \theta^{i_1} \ldots \theta^{i_k}. \]  

Here, we are following the notation of refs. [23], [12].

The OPE coefficients depend on how the operators are normalized. When comparing perturbative calculations with the AdS/CFT predictions, we need to use the same normalization. For operators in short multiplets of \( N = 4 \) supersymmetry, this is easy to achieve, since the two point correlation functions of such operators do not receive radiative corrections and can be used to fix normalization. The coefficient in (1.13) is chosen to unit normalize the two point function:

\[ \langle O^I_k(x) O^J_k(y) \rangle = \frac{\delta^I_J}{|x - y|^{2k}}. \]  

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The same conventions were used in the supergravity calculations of ref. [12].

At weak coupling, the OPE coefficient of the circular Wilson loop is proportional to \( \lambda^{k/2} \), where \( k \) is the dimension of the CPO:

\[
\frac{\langle W(C) O_k \rangle}{\langle W(C) \rangle} = \frac{1}{N} 2^{-k/2} \frac{\sqrt{k}}{k!} \lambda^{k/2} \frac{R^k}{L^{2k}} Y^I(\theta) \quad (\lambda \to 0). \tag{1.17}
\]

It turns out that AdS/CFT correspondence predicts a universal scaling of the OPE coefficients with \( \lambda \) at strong coupling: all of them are proportional to \( \sqrt{\lambda} \) independently of \( k \). This can be easily understood by considering a pair correlator of the Wilson loops, which is quadratic in OPE coefficients. The Wilson loop correlator is described by an annulus string amplitude and therefore is proportional to the string coupling \( g_s \). According to the AdS/CFT dictionary,

\[
g_s = \frac{g^2}{4\pi} = \frac{\lambda}{4\pi N}.
\]

Hence, OPE coefficients must scale as \( \sqrt{\lambda} \). An explicit calculation gives [12]:

\[
\frac{\langle W(C) O_k \rangle}{\langle W(C) \rangle} = \frac{1}{N} 2^{k/2-1} \sqrt{k\lambda} \frac{R^k}{L^{2k}} Y^I(\theta) \quad (\lambda \to \infty). \tag{1.18}
\]

Our main result is an expression for correlators of the circular Wilson loop with CPOs:

\[
\frac{\langle W(C) O_k \rangle}{\langle W(C) \rangle} = \frac{1}{N} 2^{k/2-1} \sqrt{k\lambda} \frac{I_k(\sqrt{\lambda})}{I_1(\sqrt{\lambda})} \frac{R^k}{L^{2k}} Y^I(\theta) \tag{1.19}
\]

which we expect is exact in the large \( N \) limit. Its expansion in \( \lambda \) reproduces (1.17). The strong-coupling limit exactly coincides with the AdS/CFT prediction (using eq. (1.12)).

### 2 Re-summation of rainbow diagrams

The Euclidean action of \( \mathcal{N} = 4 \) supersymmetric Yang-Mills theory is

\[
S = \int d^4x \frac{N}{\lambda} \text{Tr} \left\{ \frac{1}{2} F_{\mu \nu}^2 + (D_\mu \Phi^i)^2 - \sum_{i<j} [\Phi^i, \Phi^j]^2 + \psi^T \Gamma^\mu D_\mu \psi - i\psi^T \Gamma [\Phi^i, \psi] \right\} \tag{2.1}
\]

where \((\Gamma^\mu, \Gamma^i)\) are ten dimensional Dirac matrices in the Majorana-Weyl representation.

We will work in the Feynman gauge where the gauge field propagator has the form

\[
\left\langle A_{\mu}^{ab}(x) A_{\nu}^{cd}(y) \right\rangle_0 = \lambda^2 \frac{\delta_{\mu \nu}}{8\pi^2 (x-y)^2} \frac{\delta^{ad}\delta^{bc}}{N}. \tag{2.2}
\]

Our calculation of the OPE coefficients begins with summing all planar rainbow diagrams of the kind shown in fig. [1]. They contain \( k \) scalar propagators connecting the point \( L \) to the Wilson loop and propagators of scalars and gauge fields connecting points in segments of the loop.
If $L$ lies on the axis of symmetry of the circle, the scalar propagators are constants equal to

$$\frac{1}{8\pi^2(L^2 + R^2)}.$$  

If the origin is displaced from the axis of symmetry, the propagators will depend on positions of their endpoints on the circle. In any case, we will be interested in the large-distance asymptotics, and the propagators can be set to $1/(8\pi^2 L^2)$, up to corrections of higher order in $1/L$.

The problem thus reduces to re-summation of rainbow diagrams for each of the segments of the circle. This problem was solved in ref. [27]. In the following we will review the salient points involved in finding the solution. We start with the dimension two operator (1.7), when there are only two segments.

### 2.1 Dimension two operators

In this case we have:

$$\langle W(C) O^{ij} \rangle = \frac{1}{N} \frac{8\pi}{\sqrt{2}\lambda} \frac{\lambda^2}{(8\pi^2 L^2)^2} \left( \theta^i \theta^j - \frac{1}{6} \delta^{ij} \right) R^2 2\pi \int_0^{2\pi} d\varphi \ W(\varphi) W(2\pi - \varphi),$$

(2.3)

where $W(\varphi' - \varphi)$ denotes the sum of rainbow graphs for a segment of the circle between polar angles $\varphi$ and $\varphi'$. To compute this sum, we notice that the sum of the scalar and the gluon propagators between any two points on a circle does not depend on the positions of these points:

$$\left\langle \left( iA_\mu(x) x_\mu + \Phi^i(x) \theta^i \right)_{ab} \left( iA_\mu(y) y_\mu + \Phi^i(y) \theta^i \right)_{cd} \right\rangle_{ab} = \frac{\lambda}{N} \delta_{ab} \delta_{cd} \frac{|x||y| - \hat{x} \cdot \hat{y}}{8\pi^2|x - y|^2} \delta_{ad} \delta_{be}.$$

(2.4)
This observation allows us to replace the field-theory Wick contraction by the matrix-model average defined by the partition function

\[ Z = \int dM \exp \left( -\frac{8\pi^2}{\lambda} N \text{tr} M^2 \right). \] (2.5)

Upon the replacement of \( iA_\mu(x)\dot{x}_\mu + \Phi^i(x)|\dot{x}| \) by \( M \), the sum of rainbow diagrams in the segment of the length \( \varphi \) reduces to the matrix-model counterpart of the Wilson loop:

\[ W(\varphi) = \left\langle \frac{1}{N} \text{tr} \ e^{\varphi M} \right\rangle. \] (2.6)

The matrix model can be viewed as a combinatorial tool which simply counts the number of planar graphs [10].

The Wilson loop in the matrix model satisfies Schwinger-Dyson identity in the large-\( N \) limit (the loop equation [28]):

\[ W'(\vartheta) = \frac{\lambda}{16\pi^2} \int_0^\vartheta d\varphi W(\varphi)W(\vartheta - \varphi). \] (2.7)

The solution [10], [29], [27] of the loop equation is

\[ W(\varphi) = \frac{4\pi}{\sqrt{\lambda \varphi}} I_1 \left( \frac{\sqrt{\lambda \varphi}}{2\pi} \right). \] (2.8)

The integral we need to compute in order to calculate the correlation function (2.3) is the right-hand-side of the loop equation at \( \vartheta = 2\pi \). Using properties of the modified Bessel functions,

\[ I_k(z) = \frac{1}{2} (I_{k-1}(z) + I_{k+1}(z)), \]

\[ kI_k(z) = \frac{z}{2} (I_{k-1}(z) - I_{k+1}(z)), \] (2.9)

we get:

\[ \int_0^\vartheta d\varphi W(\varphi)W(\vartheta - \varphi) = \frac{32\pi^2}{\lambda \vartheta} I_2 \left( \frac{\sqrt{\lambda \vartheta}}{2\pi} \right). \] (2.10)

Setting \( \vartheta = 2\pi \) and substituting into (2.3), we obtain:

\[ \langle W(C) O^{ij} \rangle = \frac{1}{N} 2\sqrt{2} I_2 \left( \sqrt{\lambda} \right) \left( \vartheta^i \vartheta^j - \frac{1}{6} \delta^{ij} \right) \frac{R^2}{L^4}. \] (2.11)

Dividing by the vacuum expectation value of the Wilson loop,

\[ \langle W(C) \rangle = W(2\pi) = \frac{2}{\sqrt{\lambda}} I_1 \left( \sqrt{\lambda} \right), \] (2.12)

we arrive at the result (1.11) which we quoted earlier.
2.2 Chiral primary operators

The correlator of the Wilson loop with the CPO of dimension $k$ contains an integral over $k-1$ endpoints of the scalar propagators (one integration yields an overall factor of $2\pi$):

\[ \langle W(C) O_k \rangle = \frac{1}{N} \frac{(8\pi^2)^{k/2}}{\sqrt{k} \lambda^{k/2}} \frac{\lambda^k}{(8\pi^2 L^2)^k} \left| C_{i_1 \cdots i_k} \theta^{i_1} \cdots \theta^{i_k} \right| R_k \]

\[
\times 2\pi \int_0^{2\pi} d\phi_1 \cdots \int_0^{\phi_{k-2}} d\phi_{k-1} W(\phi_{k-1}) W(\phi_{k-2} - \phi_{k-1}) \cdots W(2\pi - \phi_1). \quad (2.13)
\]

It is useful to introduce

\[ F_k(\varphi) = \int_0^\varphi d\varphi_1 \cdots \int_0^{\phi_{k-2}} d\phi_{k-1} W(\phi_{k-1}) W(\phi_{k-2} - \phi_{k-1}) \cdots W(\varphi - \varphi_1). \quad (2.14) \]

The correlator is expressed in terms of $F_k(2\pi)$ as

\[ \langle W(C) O_k \rangle = \frac{1}{N} \frac{2\pi}{\sqrt{k} (8\pi^2)^{k/2}} \lambda^{k/2} F_k(2\pi) \frac{R_k}{L^{2k}} Y^I(\theta). \quad (2.15) \]

To find the functions $F_k(\varphi)$, we again use the loop equation. Differentiating $F_k(\varphi)$ and using (2.7), we get the recurrence relations:

\[ F_k'(\varphi) = F_{k-1}(\varphi) + \frac{\lambda}{16\pi^2} F_{k+1}(\varphi), \quad (2.16) \]

which are supplemented by initial conditions

\[ F_1(\varphi) = W(\varphi), \quad F_0(\varphi) = 0. \quad (2.17) \]

These unambiguously determine all $F_k$. A systematic way to solve these recurrence relations is to introduce a generating function and then use a Laplace transform to convert differential equations into algebraic equations. The result is

\[ F_k(\varphi) = \frac{k}{\varphi} \left( \frac{4\pi}{\sqrt{\lambda}} \right)^k I_k \left( \frac{\sqrt{\lambda} \varphi}{2\pi} \right). \quad (2.18) \]

It is straightforward to check that this expression solves the recurrence relations with the help of (2.3).

Substituting (2.18) into (2.17), we obtain

\[ \langle W(C) O_k \rangle = \frac{1}{N} 2^{k/2} \sqrt{k} I_k \left( \sqrt{\lambda} \right) \frac{R_k}{L^{2k}} Y^I(\theta). \quad (2.19) \]

Normalizing by the Wilson loop expectation value (2.12), we get (1.19).

3 Radiative Corrections

The leading radiative corrections come from the Feynman diagrams which are shown in fig.2.*

*The diagram similar to e, but with scalar lines replaced by gluon propagators does not contribute because of R-charge conservation.
Each of these diagrams is separately divergent and regularization is required to define them properly. We use a regularization by dimensional reduction which was previously used in ref. [10]. The essential observation is that $\mathcal{N} = 4$ SYM is obtained by dimensional reduction of ten dimensional SYM. This dimensional reduction retains sixteen supersymmetries in any dimension. Thus, a supersymmetric dimensional regularization of $\mathcal{N} = 4$ SYM theory is obtained by dimensionally reducing ten dimensions to $4 - \epsilon$ dimensions.

In this dimensional regularization, the diagram in fig. 2a is of higher order than the relevant leading power, $R^2/L^4$, and therefore does not contribute to $C_2$.

In the limit $L \gg R$, using dimensional regularization, the leading, $R^2/L^4$, contributions of the remaining diagrams in fig. 2 can be seen to be identical to results of computing the diagrams which are displayed in fig. 3. This sum of diagrams is known
to vanish when the dimension is exactly four, due to the non-renormalization theorem for the two-point function of the CPO. This non-renormalization results from super-conformal invariance.
Similar arguments apply to the higher CPO’s for which similar non-renormalization theorems can be applied [30].

This is by no means a proof that all radiative corrections vanish. But the excellent agreement with strong coupling AdS/CFT results gives optimism that it is indeed the case.

4 Remarks

Our main results (1.11) and (1.19) are valid when the distance from operator insertion to the loop is much larger than the loop’s radius. However, it is not hard to restore the dependence on the radius and on the orientation of the circle. Consider first the case when the operator is inserted at the symmetry axis of the circle. As follows from the discussion in the beginning of Sec. 2 we can find the correlators of the Wilson loop with CPOs at any $R$ and $L$, not only at large distances, simply by replacing $L^2$ with $L^2 + R^2$. As expected, this is perfectly consistent with the supergravity prediction, which actually is known in complete generality. If we denote the displacement of the operator insertion from the axis of symmetry by $r$ and, as before, $L$ is the distance from the plane of the circle, the correlation function is obtained from the large-distance asymptotics by replacing

$$\frac{R^k}{L^{2k}} \rightarrow \frac{R^k}{(L^2 + r^2 - R^2)^2 + 4L^2R^2}.$$  (4.1)

At $r = 0$, the denominator on the right hand side indeed coincides with $(L^2 + R^2)^k$.

We were not able to compute the sum of rainbow diagrams for $r \neq 0$, but the validity of the prescription (4.1) can be proved by an indirect argument. The point is that, once we know the correlators at $r = 0$, the dependence on $r$ is unambiguously fixed by conformal invariance, because $r$ can be always set to zero by a special conformal transformation, which maps a circle onto a circle. This transformation is not anomalous and therefore does not affect the dependence of the correlator on $\lambda$.

Another remark concerns the dependence of the OPE coefficients on $k$. It was argued on general grounds that coupling of a Wilson loop to states of very large spin should be factorially suppressed [31]. In a theory with an exponential density of states, unsuppressed coupling to such states would lead to a catastrophe in the pair correlator of Wilson loops similar to the Hagedorn transition. The CPOs we consider in this paper carry the spin of $SO(6)$ R-symmetry which for $O^I_k$ is equal to $k$. At weak coupling, the OPE coefficients are indeed suppressed at large $k$, as follows from (1.17), but the supergravity result (1.18) seems to suggest that this suppression disappears at strong coupling. Careful inspection of the exact OPE coefficients (1.19) shows that limits $\lambda \rightarrow \infty$ and $k \rightarrow \infty$ do not commute and coupling of the Wilson loop to operators of very high spin is always suppressed, but the suppression begins with operators of parametrically large spin $k \sim \lambda$. At $k \gg \lambda$:

$$C_k \propto 2^{-k/2-1} \frac{\sqrt{k}}{k!} \frac{\lambda^{(k+1)/2}}{I_1(\sqrt{\lambda})}.$$  (4.2)
5 Discussion

We have calculated the OPE coefficients of the circular Wilson loops by re-summation of the planar Feynman graphs without internal vertices. It is likely that other diagrams cancel to all orders of perturbation theory. We have successfully checked this conjecture up to two loops. Complete agreement of an infinite set of OPE coefficients with the supergravity predictions at strong coupling strongly suggests that our results are indeed exact in the 't Hooft limit. It would be interesting to see if the arguments of Ref. [11] based on diagram-by-diagram conformal transformations can be invoked to prove that only rainbow diagrams contribute to the OPE coefficients of the circular Wilson loop with CPOs.

Our result for the OPE coefficients, along with the known exact expression for the vacuum expectation value of the circular loop, can be regarded as a prediction for the string theory in $AdS_5 \times S^5$. The usual $\alpha'$ expansion of the world-sheet sigma-model then coincides with the expansion in $1/\sqrt{\lambda}$:

$$\frac{\langle W(C) \mathcal{O}_k^I \rangle}{\langle W(C) \rangle} = \frac{1}{N} 2^{k/2-1} \sqrt{k\lambda} \left( 1 - \frac{k^2 - 1}{2\sqrt{\lambda}} + \frac{k^4 - 4k^2 + 3}{8\lambda} + O\left(\frac{1}{\lambda^{3/2}}\right) \right) \frac{R^k}{L^{2k}} Y^I(\theta).$$

(5.1)

The calculation of the stringy correction to the expectation of the Wilson loop is a hard problem analogous to instanton calculations in field theory (see [19], for details). As usual, such problems require delicate treatment of various normalization factors associated with zero modes and with regularization of fluctuation determinants. However, in the ratio of the expectation values (5.1), these normalization factors cancel. For this reason, a calculation of stringy corrections to the OPE coefficients seems less complicated and perhaps can be accomplished without tremendous effort.

It would also be interesting to consider similar correlators of Wilson loops with different contours [32] or with other operators, where some preliminary results are contained in ref. [33], [34].

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