Abstract: An $n \times n$ diamond alternating sign matrix (ASM) is a $(0, +1, -1)$-matrix with $\pm 1$ entries alternating and arranged in a diamond-shaped pattern. The explicit inverse (for $n$ even) or generalized inverse (for $n$ odd) of a diamond ASM is derived. The eigenvalues of diamond ASMs are considered and when $n$ is even, the characteristic polynomial, which involves signed binomial coefficients, is determined.

Keywords: Diamond alternating sign matrix, Inverse, Generalized inverse, Eigenvalue, Binomial coefficients

MSC: 15A09, 15A18, 15B36

1 Introduction

An $n \times n$ alternating sign matrix (ASM) is a $(0, +1, -1)$-matrix with $+1$s alternating with $-1$s (ignoring $0$s) such that each row and column begins and ends with $+1$. Research on combinatorial aspects of these matrices, especially on the number of ASMs for a given $n$, is contained in Bressoud [2]. Here we are concerned with a special class of ASMs.

Definition 1.1. [4] The diamond ASM $D_n$ is the $n \times n$ ASM with the $(1, \lfloor \frac{n}{2} \rfloor + 1)$-entry equal to $+1$ and for which the number of nonzero entries in both its rows and columns is given by the vector $(1, 3, 5, \ldots, 5, 3, 1)$.

For example, $D_6$ and $D_7$ are given by

$$D_6 = \begin{bmatrix} 0 & 0 & 0 & +1 & 0 & 0 \\ 0 & 0 & +1 & -1 & +1 & 0 \\ 0 & +1 & -1 & +1 & -1 & +1 \\ +1 & -1 & +1 & -1 & +1 & 0 \\ 0 & +1 & -1 & +1 & 0 & 0 \\ 0 & 0 & +1 & 0 & 0 & 0 \end{bmatrix}, \quad D_7 = \begin{bmatrix} 0 & 0 & 0 & +1 & 0 & 0 & 0 \\ 0 & 0 & +1 & -1 & +1 & 0 & 0 \\ 0 & +1 & -1 & +1 & -1 & +1 & 0 \\ +1 & -1 & +1 & -1 & +1 & -1 & +1 \\ 0 & +1 & -1 & +1 & -1 & +1 & 0 \\ 0 & 0 & +1 & -1 & +1 & 0 & 0 \\ 0 & 0 & 0 & +1 & 0 & 0 & 0 \end{bmatrix}.$$ 

Note that the $\pm 1$ entries form a diamond-shaped pattern.
A matrix $A = [a_{ij}] \in \mathbb{R}^{n\times n}$ is called centrosymmetric if $a_{ij} = a_{n-i+1,n-j+1}$ for all $1 \leq i, j \leq n$. Note that $D_n$ is a symmetric centrosymmetric matrix for all $n \geq 1$. We denote by $P_n$ the $n \times n$ antidiagonal matrix

$$P_n = \begin{bmatrix} 0 & \cdots & 0 & 1 \\ \vdots & \ddots & 1 & 0 \\ 0 & 1 & \ddots & \vdots \\ 1 & 0 & \cdots & 0 \end{bmatrix}.$$ 

It is well known that $A \in \mathbb{R}^{n\times n}$ is centrosymmetric if and only if $P_n A P_n = A$. In [3, Theorem 2] it is proved that for a fixed $n$, $D_n$ has the maximum spectral radius over the set of $n \times n$ ASMs. This is the only result on the spectra of ASMs that we have found in the literature.

For $n$ even, $\tilde{D}_n := P_n D_n$ is also an ASM, and is also referred to as a diamond ASM in [4]. For example, $\tilde{D}_6$ is given by

$$\tilde{D}_6 = \begin{bmatrix} 0 & 0 & -1 & 0 & 0 & 0 \\ 0 & +1 & -1 & +1 & 0 & 0 \\ +1 & -1 & +1 & -1 & +1 & 0 \\ 0 & +1 & -1 & +1 & -1 & +1 \\ 0 & 0 & +1 & -1 & +1 & 0 \\ 0 & 0 & 0 & +1 & 0 & 0 \end{bmatrix}.$$ 

We focus on inverses and spectral properties of diamond ASMs $D_n$. In Section 2, we give an explicit formula for the inverse (resp. generalized inverse) for $n$ even (resp. $n$ odd). In Section 3, we show that half of the eigenvalues of $D_n$ are $\pm 1$ (resp. 0) for $n$ even (resp. $n$ odd). In the case of $n$ even, by using a recursion, we find the characteristic polynomial that specifies the remaining eigenvalues of $D_n$. Interestingly, the coefficients of this polynomial have magnitudes (but not signs) equal to the binomial coefficients.

## 2 Inverses

### 2.1 Inverses of $D_n$ and $\tilde{D}_n$, $n$ even

Let $n = 2m$. By [5, Lemma 2 (i)], $D_n$ may be written in the partitioned form

$$D_n = \begin{bmatrix} A_m & P_m B_m P_m \\ B_m & P_m A_m P_m \end{bmatrix},$$

where

$$A_m = \begin{bmatrix} 0 & \cdots & \cdots & 0 \\ \vdots & \ddots & 0 & 1 \\ \vdots & \ddots & 1 & -1 \\ 0 & \cdots & \cdots & (-1)^m \end{bmatrix}, \quad B_m = \begin{bmatrix} 1 & -1 & \cdots & (-1)^{m+1} \\ \vdots & \ddots & \ddots & \vdots \\ 0 & 1 & -1 & \cdots \\ 0 & \cdots & \cdots & 0 \\ (-1)^m \end{bmatrix}.$$ (1)

Note that

$$A_m + P_mB_m = P_m.$$ 

Furthermore, by [5, Lemma 3 (i)]

$$Q D_n Q^T = \begin{bmatrix} A_m - P_mB_m & 0 \\ 0 & A_m + P_mB_m \end{bmatrix} = \begin{bmatrix} A_m - P_mB_m & 0 \\ 0 & P_m \end{bmatrix},$$ (2)
where $Q$ is the orthogonal matrix given by

$$Q = \frac{1}{\sqrt{2}} \begin{bmatrix} I_m & -P_m \\ I_m & P_m \end{bmatrix}. \quad (3)$$

Let $S_m = \text{diag}(1, -1, \ldots, (-1)^{m-1})$. Then

$$S_m(A_m - P_mB_m)S_m = (-1)^m E_m, \quad (4)$$

where $E_m$ is the $m \times m$ matrix

$$E_m = \begin{bmatrix} 0 & \cdots & \cdots & 0 & 1 \\ \vdots & \ddots & 1 & 2 \\ \vdots & \ddots & \ddots & \ddots & \vdots \\ 0 & 1 & \ddots & \ddots & \vdots \\ 1 & 2 & \cdots & \cdots & 2 \end{bmatrix}. \quad (5)$$

A direct computation of $E_m^{-1}$ gives

$$E_m^{-1} = \begin{bmatrix} (-1)^{m+1} & (-1)^m & \cdots & -2 & 1 \\ (-1)^m & \ddots & \ddots & \ddots & \vdots \\ \vdots & -2 & \ddots & \ddots & \vdots \\ -2 & 1 & \ddots & \ddots & \vdots \\ 1 & 0 & \cdots & \cdots & 0 \end{bmatrix}. \quad (6)$$

Observe that

$$(-1)^m S_m E_m^{-1} S_m = P_mE_mP_m. $$

Using the above results, we now determine the inverse of $D_{2m}$.

**Theorem 2.1.** For $m \geq 1$, the diamond ASM $D_{2m}$ is invertible, and

$$D_{2m}^{-1} = \frac{1}{2} \begin{bmatrix} -P_mE_mP_m + P_m & P_mE_m + I_m \\ E_mE_m + I_m & -E_m + P_m \end{bmatrix}. \quad (7)$$

**Proof.** Since $A_m - P_mB_m$ and $P_m$ are both invertible, $D_{2m}$ is invertible, and using (2) and (3),

$$D_{2m}^{-1} = Q^T \begin{bmatrix} (A_m - P_mB_m)^{-1} & 0 \\ 0 & P_m^{-1} \end{bmatrix} Q$$

$$= \frac{1}{2} \begin{bmatrix} I_m & I_m \\ -P_m & P_m \end{bmatrix} \begin{bmatrix} (A_m - P_mB_m)^{-1} & 0 \\ 0 & P_m^{-1} \end{bmatrix} \begin{bmatrix} I_m & -P_m \\ -P_m & P_m \end{bmatrix}$$

$$= \frac{1}{2} \begin{bmatrix} (A_m - P_mB_m)^{-1} + P_m & -P_m(A_m - P_mB_m)^{-1}P_m + I_m \\ -P_m(A_m - P_mB_m)^{-1}P_m + I_m & P_m(A_m - P_mB_m)^{-1}P_m + P_m \end{bmatrix}. \quad (8)$$

By (4), $(A_m - P_mB_m)^{-1} = (-1)^m S_m E_m^{-1} S_m$. From (6), it follows that

$$(A_m - P_mB_m)^{-1} = -P_mE_mP_m. \quad (9)$$

Substituting (9) into (8) yields the result. 

**Corollary 2.2.** For $m \geq 1$, the diamond ASM $\tilde{D}_{2m}$ is invertible, and

$$\tilde{D}_{2m}^{-1} = \frac{1}{2} \begin{bmatrix} P_mE_mP_m + P_m & P_mE_m + I_m \\ -E_mE_m + I_m & E_m + P_m \end{bmatrix}. \quad (10)$$
Example 2.3. With \( m = 3 \), (7) gives
\[
D_6^{-1} = \begin{bmatrix}
-1 & -1 & 0 & 1 & 1 & 1 \\
-1 & 0 & 0 & 0 & 1 & 1 \\
0 & 0 & 0 & 0 & 0 & 1 \\
1 & 0 & 0 & 0 & 0 & 0 \\
1 & 1 & 0 & 0 & 0 & -1 \\
1 & 1 & 1 & 0 & -1 & -1
\end{bmatrix}.
\]

Note that in \( D_2^{-1} \), the 0 entries form a diamond-shaped pattern.

2.2 Generalized Inverse of \( D_n, n \) odd

Let \( n = 2m + 1 \). By [5, Lemma 2 (ii)], \( D_n \) may be written in the form
\[
D_n = \begin{bmatrix}
A_m & a & A_m P_m \\
a^T & a & a^T P_m \\
P_m A_m & P_m a & P_m A_m P_m
\end{bmatrix},
\]
where \( A_m \) is given in (1), \( a^T = \begin{bmatrix} 1 & -1 & \cdots & (-1)^{m+1} \end{bmatrix} \) and \( a = (-1)^m \). By [5, Lemma 3 (ii)]
\[
RD_n R^T = \begin{bmatrix}
O_m & 0 & 0 \\
0 & a & \sqrt{2} a^T \\
0 & \sqrt{2} a & 2A_m
\end{bmatrix},
\]
where \( O_m \) is the zero matrix and \( R \) is the orthogonal matrix
\[
R = \frac{1}{\sqrt{2}} \begin{bmatrix}
I_m & 0 & -P_m \\
0 & \sqrt{2} & 0 \\
I_m & 0 & P_m
\end{bmatrix}.
\]

Observe that \( D_{2m+1} \) has rank \( m + 1 \), so 0 is an eigenvalue of (algebraic and geometric) multiplicity \( m \). Thus the \( m + 1 \) nonzero eigenvalues of \( D_{2m+1} \) are precisely the eigenvalues of the \( (m + 1) \times (m + 1) \) matrix
\[
F_{m+1} = \begin{bmatrix}
a & \sqrt{2} a^T \\
\sqrt{2} & 2A_m
\end{bmatrix} = \begin{bmatrix}
(-1)^m & \sqrt{2} & \cdots & (-1)^m \sqrt{2} & (-1)^{m+1} \sqrt{2} \\
\sqrt{2} & 0 & \cdots & 0 & 0 \\
\vdots & \vdots & \ddots & 0 & 2 \\
\sqrt{2} & 0 & \cdots & 2 & \vdots \\
(-1)^m \sqrt{2} & 0 & 2 & \cdots & \vdots \\
(-1)^{m+1} \sqrt{2} & 0 & 2 & \cdots & (-1)^{m+2}
\end{bmatrix}.
\]

Since \( D_{2m+1} \) is symmetric, the algebraic and geometric multiplicities of the eigenvalue 0 are equal. Thus the group inverse \( D_{2m+1} \) exists and furthermore this equals the Moore-Penrose inverse \( D_{2m+1}^\dagger \), which we now determine.

Theorem 2.4. For \( m \geq 1 \), the group inverse of the diamond ASM \( D_{2m+1} \) exists (and is equal to the Moore-Penrose inverse), and
\[
D_{2m+1}^\# = D_{2m+1}^\dagger = \frac{1}{2} \begin{bmatrix}
Z_m & e_1 & Z_m P_m \\
e_1^T & 0 & e_1^T P_m \\
P_m Z_m & P_m e_1 & P_m Z_m P_m
\end{bmatrix},
\]
where
\[
Z_m = \frac{1}{2} \begin{bmatrix}
0 & \cdots & \cdots & 0 & 1 \\
\vdots & \ddots & 1 & 1 \\
\vdots & \ddots & \ddots & 0 \\
0 & 1 & \ddots & \cdots \\
1 & 1 & 0 & \cdots & 0 \\
\end{bmatrix}.
\]

**Proof.** From (10) and [1, Theorem 5, p. 140],
\[
D_{2m+1}^\theta = D_{2m+1}^{1/2} = R^T \begin{bmatrix} 0 & 0 \\ 0 & F^{-1}_{m+1} \end{bmatrix} R.
\]
(13)

Letting
\[
F^{-1}_{m+1} = \begin{bmatrix} \omega & Y_m^T \\ y & Y_m \end{bmatrix},
\]
we now determine \(Y_m, y\) and \(\omega\). The expression in (13) becomes
\[
D_{2m+1}^\theta = D_{2m+1}^{1/2} = \frac{1}{2} \begin{bmatrix} I_m & 0 & I_m \\ 0 & \sqrt{2} & 0 \\ -P_m & 0 & P_m \end{bmatrix} \begin{bmatrix} I_m & 0 & -P_m \\ 0 & \omega & Y_m^T \\ -P_m & 0 & P_m \end{bmatrix}
\]
\[
= \frac{1}{2} \begin{bmatrix} Y_m & \sqrt{2}y & Y_mP_m \\ \sqrt{2}y & 2\omega & \sqrt{2}yP_m \\ P_mY_m & \sqrt{2}P_my & P_mY_mP_m \end{bmatrix}.
\]
(14)

To find \(Y_m, y\) and \(\omega\), we use (11) and a well-known formula (see, e.g., [6, page 42]) for the inverse of a partitioned matrix to obtain
\[
Y_m = G_m^{-1},
\]
\[
y^T = \frac{1}{a} + \sqrt{2}a^{-1}G_m^{-1}
\]
\[
\omega = \frac{1}{a} + \sqrt{2}a^{-1}G_m^{-1}\sqrt{2}a^{-1} = \frac{1}{a} + 2aG_m^{-1}a,
\]
where
\[
G_m = 2A_m - \frac{\sqrt{2}a\sqrt{2}a^T}{a} = 2 \left( A_m - \frac{aa^T}{a} \right).
\]

Now
\[
A_m - \frac{aa^T}{a} = A_m + (-1)^{m+1}aa^T =
\]
\[
\begin{bmatrix}
(-1)^{m+1} & (-1)^{m} & \cdots & -1 & 1 \\
(-1)^{m} & \ddots & \ddots & \vdots & 0 \\
\vdots & \ddots & \ddots & \ddots & \vdots \\
-1 & 1 & \ddots & \cdots & \vdots \\
1 & 0 & \cdots & 0 \\
\end{bmatrix},
\]

thus
\[
Y_m = G_m^{-1} = \frac{1}{2} \begin{bmatrix}
0 & \cdots & \cdots & 0 & 1 \\
\vdots & \ddots & 1 & 1 \\
\vdots & \ddots & \ddots & \cdots & 0 \\
0 & 1 & \ddots & \cdots & \vdots \\
1 & 1 & 0 & \cdots & 0 \\
\end{bmatrix} = Z_m.
\]
With $e^T_1 = [1, 0, \cdots, 0]$, 

$$y^T = -\frac{1}{\alpha} \sqrt{2} a^T G_m^{-1} = (-1)^{m+1} \frac{\sqrt{2}}{2} \begin{bmatrix} 1 & -1 & \cdots & (-1)^{m+1} \end{bmatrix} \begin{bmatrix} 0 & \cdots & \cdots & 0 & 1 \\ \vdots & \ddots & \ddots & \vdots & 1 \\ \vdots & \ddots & \ddots & \vdots & \vdots \\ 0 & 1 & \ddots & \ddots & \vdots \\ 1 & 1 & 0 & \cdots & 0 \end{bmatrix} = \frac{1}{\sqrt{2}} e^T_1,$$

and

$$\omega = \frac{1}{\alpha} + 2 a^T G_m^{-1} a = (-1)^m + \begin{bmatrix} 1 & -1 & \cdots & (-1)^{m+1} \end{bmatrix} \begin{bmatrix} 0 & \cdots & \cdots & 0 & 1 \\ \vdots & \ddots & \ddots & \vdots & 1 \\ \vdots & \ddots & \ddots & \vdots & \vdots \\ 0 & 1 & \ddots & \ddots & \vdots \\ 1 & 1 & 0 & \cdots & 0 \end{bmatrix} \begin{bmatrix} 1 \\ \vdots \\ \vdots \\ 1 \\ \vdots \\ (-1)^{m+1} \end{bmatrix} = 0.$$

Using (14), this completes the proof. \hfill \Box

**Example 2.5.** With $m = 3$, (12) gives

$$D^T_7 = D^T_1 = \frac{1}{q} \begin{bmatrix} 0 & 0 & 1 & 2 & 1 & 0 & 0 \\ 0 & 1 & 1 & 0 & 1 & 1 & 0 \\ 1 & 1 & 0 & 0 & 0 & 1 & 1 \\ 2 & 0 & 0 & 0 & 0 & 0 & 2 \\ 1 & 1 & 0 & 0 & 0 & 1 & 1 \\ 0 & 1 & 1 & 0 & 1 & 1 & 0 \\ 0 & 0 & 1 & 2 & 1 & 0 & 0 \end{bmatrix}.$$ 

### 3 Eigenvalues

For all $n \geq 1$, $D_n$ and $\overline{D}_n$ have 1 as an eigenvalue with eigenvector $[1, 1, \cdots, 1]^T$ since all row sums of ASMs are 1.

Let $S_n = \text{diag}(1, -1, \ldots, (-1)^{n+1})$ and $|D_n|$ be the $(0, 1)$ matrix obtained by replacing each entry of $D_n$ by its absolute value. Then $S_n D_n S_n = (-1)^p |D_n|$, where $p = 0$ if $n = 4k$ or $4k + 1$; $p = 1$ if $n = 4k + 2$ or $4k + 3$. This leads to the following observation for the spectral radius $\rho(D_n)$.

**Observation 3.1.** For all $n \geq 1$,

1. $\rho(D_n)$ is an eigenvalue of $D_n$ if $n = 4k$ or $4k + 1$; and
2. $-\rho(D_n)$ is an eigenvalue of $D_n$ if $n = 4k + 2$ or $4k + 3$.

### 3.1 Eigenvalues of $D_n$ and $\overline{D}_n$, $n$ even

From (2) and (4), the eigenvalues of $D_n$ for $n = 2m$ are the union of the eigenvalues of $P_m$ and $(-1)^m E_m$, where $E_m$ is the matrix given in (5). Since $\overline{D}_n = P_m D_n$,

$$\overline{D}_n = \begin{bmatrix} 0 & P_m \\ P_m & 0 \end{bmatrix} \begin{bmatrix} A_m & P_m B_m P_m \\ B_m & P_m A_m P_m \end{bmatrix} = \begin{bmatrix} P_m B_m & A_m P_m \\ P_m A_m & B_m P_m \end{bmatrix},$$
where $A_m, B_m$ are as in (1). Furthermore, with $Q$ as in (3),

$$Q\tilde{D}_nQ^T = \begin{bmatrix} (-1)^{m+1}S_mE_mS_m & 0 \\ 0 & P_m \end{bmatrix}.$$ 

Thus, the eigenvalues of $\tilde{D}_{2m}$ are the union of the eigenvalues of $P_m$, which are 1 with multiplicity $\left\lceil \frac{m}{2} \right\rceil$ and -1 with multiplicity $\left\lfloor \frac{m}{2} \right\rfloor$, and of $(-1)^{m+1}E_m$.

We now give some observations about the eigenvalues of $E_m$, and since numerical computations indicate that some eigenvalues are irrational, we then determine the characteristic polynomial of $E_m$. The following observation is a consequence of (6).

**Observation 3.2.** For all $m \geq 1$, the following statements hold:

1. If $m$ is even, then $E_m^1$ and $-E_m$ have exactly the same eigenvalues.
2. If $m$ is odd, then $E_m^1$ and $E_m$ have exactly the same eigenvalues.

Note that $\det P_mE_m = 1$, so $\det E_m = \det P_m = (-1)^{m(m-1)/2} = (-1)^{\left\lfloor \frac{m}{2} \right\rfloor}$, i.e.,

$$\det E_m = \det P_m = \begin{cases} 1 & \text{if } m = 4k \text{ or } 4k + 1 \\ -1 & \text{if } m = 4k + 2 \text{ or } 4k + 3. \end{cases} \quad (15)$$

Thus, from (2) and (4),

$$\det D_{2m} = \det((-1)^m E_m) \det P_m = (-1)^m (\det P_m)^2 = (-1)^m,$$

and

$$\det \tilde{D}_{2m} = \det P_{2m} \det D_{2m} = 1.$$

Observation 3.2 gives a nice relationship between eigenvalues of $D_n$ and $D_n^{-1}$ for $n$ even.

**Observation 3.3.** For all $m \geq 1$, the following statements hold:

1. If $m$ is even, then the eigenvalues of $D_{2m}^{-1}$ are the negative of the eigenvalues of $D_{2m}$.
2. If $m$ is odd, then the eigenvalues of $D_{2m}^{-1}$ are precisely the eigenvalues of $D_{2m}$.

**Example 3.4.** With $m = 3$, the eigenvalues of $D_3$ are the union of the eigenvalues of $P_3$ (namely $-1, 1, 1$) and those of $-E_3$. The latter are $1, -2 \pm \sqrt{3}$ since the characteristic polynomial of $E_3$ is $x^3 - 3x^2 - 3x + 1$.

### 3.1.1 Characteristic Polynomials for $E_m$

Let $p_m(x) = x^m + a_{m-1}x^{m-1} + \cdots + a_1x + a_0$ denote the characteristic polynomial of $E_m$, $m \geq 1$. By (15),

$$a_0 = \det E_m = 1, \quad \text{if } m = 4k$$

$$a_0 = -\det E_m = -1, \quad \text{if } m = 4k + 1$$

$$a_0 = \det E_m = -1, \quad \text{if } m = 4k + 2$$

$$a_0 = -\det E_m = 1, \quad \text{if } m = 4k + 3. \quad (16)$$

By Observation 3.2 and (16),

$$x^m p_m(-\frac{1}{2}) = p_m(x) \quad \text{if } m = 4k$$

$$x^m p_m(-\frac{1}{2}) = -p_m(x) \quad \text{if } m = 4k + 2$$

$$x^m p_m(\frac{1}{2}) = -p_m(x) \quad \text{if } m = 4k + 1$$

$$x^m p_m(\frac{1}{2}) = p_m(x) \quad \text{if } m = 4k + 3.$$
Thus the coefficients \( a_i, i = 0, \ldots, m, \) of \( p_m(x) \) satisfy

\[
\begin{align*}
 a_{m-i} &= (-1)^i a_i & \text{if } m = 4k \\
 a_{m-i} &= (-1)^{i+1} a_i & \text{if } m = 4k + 2 \\
 a_{m-i} &= -a_i & \text{if } m = 4k + 1 \\
 a_{m-i} &= a_i & \text{if } m = 4k + 3.
\end{align*}
\]

To illustrate these relations, we list \( p_m(x) \) below for small values of \( m \) (calculated by Maple):

\[
\begin{align*}
p_1(x) &= x - 1 \\
p_2(x) &= x^2 - 2x - 1 \\
p_3(x) &= x^3 - 3x^2 - 3x + 1 \\
p_4(x) &= x^4 - 4x^3 - 6x^2 + 4x + 1 \\
p_5(x) &= x^5 - 5x^4 - 10x^3 + 10x^2 + 5x - 1 \\
p_6(x) &= x^6 - 6x^5 - 15x^4 + 20x^3 + 15x^2 - 6x - 1 \\
p_7(x) &= x^7 - 7x^6 - 21x^5 + 35x^4 + 35x^3 - 21x^2 - 7x + 1 \\
p_8(x) &= x^8 - 8x^7 - 28x^6 + 56x^5 + 70x^4 - 56x^3 - 28x^2 + 8x + 1.
\end{align*}
\]

We note that these coefficients coincide with the binomial coefficients up to signs; this motivates Theorem 3.9, for the proof of which we use the following lemmas. The matrix \( f_m \) is the \( m \times m \) matrix with all entries 1, and \( 1_m \) is the \( m \)-vector with all entries 1.

**Lemma 3.5.** [6, Theorem 2.2] Let \( M = \begin{bmatrix} A & B \\ C & D \end{bmatrix}, \) where \( A \) and \( D \) are square matrices. Then, taking a Schur complement,

\[
\det M = \det A \det(D - CA^{-1}B), \text{ if } A \text{ is invertible,}
\]

and

\[
\det M = \det(AD - CB), \text{ if } AC = CA.
\]

The following identities can be easily verified.

**Lemma 3.6.** For \( r \geq 2, \)

\[
\binom{2r + 2}{k} = 2 \left( \binom{2r}{k} + \binom{2r}{k-2} - \binom{2r-2}{k} - 2 \binom{2r-2}{k-2} + \binom{2r-2}{k-4} \right)
\]

and

\[
\binom{2r + 3}{k} = 2 \left( \binom{2r + 1}{k} + \binom{2r + 1}{k-2} - \binom{2r - 1}{k} - 2 \binom{2r - 1}{k-2} + \binom{2r - 1}{k-4} \right).
\]

**Lemma 3.7.** For \( m \geq 1, \) \( E_m^2 \) is the leading \( m \times m \) principal submatrix of \( E_{m+1}^2. \)

**Proof.** Observe that \( E_{m+1}^2 = (E_{m+1}P_{m+1})(P_{m+1}E_{m+1}). \) In block form this becomes

\[
E_{m+1}^2 = \begin{bmatrix} E_m P_m & 0 \\ 2 \cdot 1_m & 1 \end{bmatrix} \begin{bmatrix} P_m E_m & 2 \cdot 1_m \\ 0 & 1 \end{bmatrix} = \begin{bmatrix} E_m^2 & 2 E_m P_m 1_m \\ 2 \cdot 1_m P_m E_m & 4m + 1 \end{bmatrix}.
\]

**Observation 3.8.** The \( m \)-th, \((m + 1)\)-th columns (or rows) of \( E_{m+1}^2 \) are \([2, 6, 10, \ldots, 4m - 6, 4m - 3, 4m - 2]\) and \([2, 6, 10, \ldots, 4m - 6, 4m - 2, 4m + 1]\), respectively.
Theorem 3.9. Let \( p_m(x) = x^m + a_{m-1}x^{m-1} + \cdots + a_1x + a_0 \) be the characteristic polynomial of the \( m \times m \) nonsingular matrix \( E_m \) given in (5). Then for \( q = 0, 1, \ldots \),

\[
a_{m-k} = \begin{cases} 
\binom{m}{k} & \text{if } k = 4q \text{ or } k = 4q + 3 \\
-\binom{m}{k} & \text{if } k = 4q + 1 \text{ or } k = 4q + 2
\end{cases}
\]

with the convention that \( \binom{m}{k} = 0 \) if \( k > m \) or \( k < 0 \).

Proof. Note that \( p_m(x) = \det(xI_m - E_m) \). We partition \( xI_m - E_m \) into a \( 2 \times 2 \) block form, and in doing so we distinguish two cases and use mathematical induction.

Suppose first that \( m = 2r \) with \( r \geq 1 \). For \( r = 1 \) and \( 2 \), the result holds with \( p_2(x) \) and \( p_6(x) \) given by (17). Using Lemma 3.5,

\[
p_{2r}(x) = \det(xI_{2r} - E_{2r}) = \det \begin{pmatrix} xI_r & -E_r \\ -E_r & xI_r - 2I_r \end{pmatrix} = \det \left( x^2I_r - 2xI_r - E^2_r \right).
\]

Using induction, suppose that the result holds for \( p_{2r}(x) \) and \( p_{2r-2}(x) \). We proceed to find \( p_{2r+2}(x) \). Let

\[
M_{r+1} = \begin{pmatrix}
1 & 0 & \cdots & \cdots & 0 \\
0 & 1 & \ddots & & \\
\vdots & \ddots & \ddots & \ddots & \vdots \\
\vdots & \ddots & 1 & -1 & \\
0 & \cdots & \cdots & 0 & 1
\end{pmatrix}
\]

Then

\[
p_{2r+2}(x) = \det \left( x^2I_{r+1} - 2xI_{r+1} - E^2_{r+1} \right) = \det M^T_{r+1} \left( x^2I_{r+1} - 2xI_{r+1} - E^2_{r+1} \right) M_{r+1},
\]

where the multiplication by \( M^T_{r+1} \) subtracts the second last row from the last row, and the multiplication by \( M_{r+1} \) subtracts the second last column from the last column. Now expansion about the last column of this determinant gives the recursion

\[
p_{2r+2}(x) = 2(x^2 - 1)p_{2r}(x) - (x^2 + 1)^2p_{2r-2}(x).
\]

If \( C[p, x^k] \) denotes the coefficient of \( x^k \) from the polynomial \( p \), then for \( 0 \leq k \leq 2r + 2 \), this gives

\[
C[p_{2r+2}, x^{2r+2-k}] = 2C[p_{2r}, x^{2r-k}] - 2C[p_{2r}, x^{2r-(k-2)]} - (C[p_{2r-2}, x^{2r-2-k}] + 2C[p_{2r-2}, x^{2r-2-(k-2)]} + C[p_{2r-2}, x^{2r-2-(k-4)]}).
\]

By the induction hypothesis, for even \( m \leq 2r \) and \( q = 0, 1, \ldots \),

\[
C[p_m, x^{m-k}] = a_{m-k} = \begin{cases} 
\binom{m}{k} & \text{if } k = 4q \text{ or } k = 4q + 3 \\
-\binom{m}{k} & \text{if } k = 4q + 1 \text{ or } k = 4q + 2.
\end{cases}
\]
When $k = 4q$, the righthand side of (19) is
\[
2 \left( \frac{2r}{k} \right) + 2 \left( \frac{2r}{k-2} \right) - \left[ \left( \frac{2r-2}{k} \right) - 2 \left( \frac{2r-2}{k-2} \right) + \left( \frac{2r-2}{k-4} \right) \right],
\]
which by the first identity in Lemma 3.6 is equal to $\left( \frac{2r^2}{k} \right)$. Thus Theorem 3.9 is proved for $k = 4q$.

Similarly, for $k = 4q + 3$, the righthand side of (19) is equal to $\left( \frac{2r^2}{k} \right)$ by Lemma 3.6; whereas for $k = 4q + 1$ and $4q + 2$, the righthand side of (19) is equal to $\left( \frac{2r^2}{k} \right)$ by Lemma 3.6.

Next, suppose that $m$ is odd, that is $m = 2r + 1$ for some $r \geq 0$. The base case for induction is true from $p_1(x)$ and $p_3(x)$ given in (17). In order to find a recursion, partition $xI_{2r+3} - E_{2r+3}$ into a $2 \times 2$ block form with the leading block of order $r + 1$. Then using Lemma 3.5,
\[
p_{2r+3}(x) = \det(xI_{2r+3} - E_{2r+3})
\]
\[
= x^{r+1} \det \begin{bmatrix} x - 1 & -2 & -2 \\ -2 & x - 2 & \vdots \\ \vdots & \ddots & \ddots \\ -2 & \cdots & x - 2 \end{bmatrix} - \frac{1}{x} \begin{bmatrix} 0 & 0 \\ E_{r+1} & E_{r+1} \end{bmatrix}^T
\]
\[
= \frac{1}{x} \begin{bmatrix} x(x-1) & -2x \cdots -2x \\ -2x & x^2I_{r+1} - 2xI_{r+1} - E_{r+1}^2 \end{bmatrix}.
\]

Let $M_{r+2}$ be as defined in (18) and $\hat{p}_{2r+3}(x) = xp_{2r+3}(x)$. From
\[
\hat{p}_{2r+3}(x) = \det M_{r+2}^T \begin{bmatrix} x(x-1) & -2x \cdots -2x \\ -2x & x^2I_{r+1} - 2xI_{r+1} - E_{r+1}^2 \end{bmatrix} M_{r+2},
\]
the recursion
\[
\hat{p}_{2r+3}(x) = 2(x^2 - 1)\hat{p}_{2r+1}(x) - (x^3 + 1)^2 \hat{p}_{2r-1}(x)
\]
is obtained. That is,
\[
p_{2r+3}(x) = 2(x^2 - 1)p_{2r+1}(x) - (x^3 + 1)^2 p_{2r-1}(x).
\]

For $0 \leq k \leq 2r + 3$, this recursion gives
\[
C[p_{2r+3}, x^{2r+3-k}] = 2C[p_{2r+1}, x^{2r+1-k}] - 2C[p_{2r+1}, x^{2r+1-(k-2)}] - \left( C[p_{2r-1}, x^{2r-1-k}] + 2C[p_{2r-1}, x^{2r-1-(k-2)}] + C[p_{2r-1}, x^{2r-1-(k-4)}] \right).
\]
The remaining steps are similar to the case in which $m$ is even using the second identity in Lemma 3.6. □

Letting $E[a]$ denote a principal minor of order $|a|$ of $E$, the above theorem and (15) lead to a simple expression for the sum of these principal minors.
Corollary 3.10. If $\alpha \subseteq \{1, 2, \ldots, m\}$ and $E_m = P_m U_m$ where $U_m$ is upper triangular with +1 in each diagonal entry, then for $k = 1, \ldots, m$,

$$\sum_{|a|=k} \det E_m[a] = \binom{m}{k} \det P_k = \det P_k \sum_{|a|=k} \det U_m[a].$$

3.2 Eigenvalues of $D_n$, $n$ odd

From the discussion in Section 2.2, $D_{2m+1}$ has an eigenvalue 0 of multiplicity $m$, and the $m+1$ nonzero eigenvalues of $D_{2m+1}$ are precisely the eigenvalues of the $(m+1) \times (m+1)$ matrix $F_{m+1}$ that is given in (11). Note that $F_{m+1}$ has 1 as an eigenvalue.

Let $q_{i}(x)$ denote the characteristic polynomial of $F_i$ for $\ell \geq 2$. Calculations using Maple show, for example, that

\[
\begin{align*}
q_2(x) &= x^2 + x - 2 \\
q_3(x) &= x^3 - 3x^2 - 2x + 4 \\
q_4(x) &= x^4 + 3x^3 - 8x^2 - 4x + 8 \\
q_5(x) &= x^5 - 5x^4 - 8x^3 + 20x^2 + 8x - 16 \\
q_6(x) &= x^6 + 5x^5 - 18x^4 - 20x^3 + 48x^2 + 16x - 32 \\
q_7(x) &= x^7 - 7x^6 - 18x^5 + 56x^4 + 48x^3 - 112x^2 - 32x + 64.
\end{align*}
\]

If $V_{m+1} = \begin{bmatrix} 0 & I_m \\ 1 & 0 \end{bmatrix}$, then

$$V_{m+1} F_{m+1} V_{m+1}^T = \begin{bmatrix} 2A_m & \sqrt{2}a \\ \sqrt{2}a^T & a \end{bmatrix},$$

where $A_m$ is given by (1) and $a$, $a$ are given at the beginning of Section 2. The Schur complement on the first \left\lceil \frac{m}{2} \right\rceil$ rows and columns can be used to find an expression for the characteristic polynomial of $F_{m+1}$, but we have been unable to find a recursion (as we used for $E_m$), and thus do not have a general formula for the characteristic polynomial of $D_n$ for $n$ odd.

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