Discrete Symmetry, Non-Commutative Geometry and Gravity

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Abstract

We describe the geometry of a set of scalar fields coupled to gravity. We consider the formalism of a differential \( \mathbb{Z}_2 \)-graded algebra of \( 2 \times 2 \) matrices whose elements are differential forms on space-time. The connection and the vierbeins are extended to incorporate additional scalar and vector fields. The resulting action describes two universes coupled in a non-minimal way to a set of scalar fields. This picture is slightly different from the description of general relativity in the framework of non-commutative geometry.

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1 Introduction

Although the standard model of electroweak interactions [1] has successfully passed all the experimental tests at the presently available energies, we still lack a convincing understanding of the phenomena of spontaneous symmetry breaking. The appearance of the Higgs fields, their self interactions and their Yukawa couplings to the other fields in the standard model is quite artificial. Many attempts, however, have been made to give the Higgs sector a geometrical origin. These attempts are usually in the form of some Kaluza-Klein theories [2] or compactified string models. In these descriptions of the Higgs fields, space-time is taken to be locally written as $M_4 \times F$. When $M_4$ and $F$ are smooth manifolds, one needs only the usual mathematical tools of differential geometry. However, when the internal space $F$ is a discrete set of points, non-commutative geometry does apply.

Recently, a geometrical picture unifying the gauge fields and the Higgs fields of the standard $SU(2) \times U(1)$ model was put forward by A. Connes [3,4]. The proposed space-time is a product of a continuous Euclidean manifold and a discrete space consisting of two points. The vector potential defined on this space has the usual $SU(2)$ gauge fields along the continuous directions and the $U(1)$ component in the form of a Higgs field along the discrete direction [5-7]. Furthermore, by enlarging the discrete space to three points, one obtains other models of particle physics such as grand unification models [8].

There is however another approach to geometrically describe the Higgs sector and spontaneous symmetry breaking without using the machinery of non-commutative geometry. This method is mathematically simpler and uses the algebra of $2 \times 2$ matrices whose entries are functions or $p$-forms on space-time [9]. The basic mathematical object here is a “generalized connection” whose diagonal elements give the Yang-Mills fields and whose off-diagonal elements characterize the Higgs fields (see also refs.[10-16] for related topics on discrete symmetry and non-commutative geometry).

We will, in this note, apply this last approach to the theory of gravity. Our fundamental mathematical objects are $2 \times 2$ “generalized spin connection” together with a $2 \times 2$ “generalized vierbeins”. We then construct out of these objects a gauge invariant action and obtain, in the most general case, a theory describing two distinct universes (the left and right universes). These two universes, however, are coupled to each other in a non-
trivial way through the presence of an action for a set of scalar fields. We note here that in the framework of non-commutative geometry one obtains a model of only one single scalar field coupled in a minimal manner to Einstein-Hilbert gravity [17]. As it is well-known, scalar theories coupled to gravity are of crucial importance in cosmology and account, for instance, for the inflation of the universe. Hence giving a geometrical origin to these theories might be relevant to cosmology.

We start this note by reviewing the mathematical notions used to describe the algebra of $2 \times 2$ matrices whose elements are differential forms. The setting for analysing gravity in the context of this algebra is then presented and the gauge invariant action is proposed.

2 An Algebra of $2 \times 2$ Matrices

We would like to construct in this section a differential $\mathbb{Z}_2$-graded algebra of $2 \times 2$ matrices whose elements are functions or $p$-forms. The most general element in this algebra is written as

$$X = \begin{pmatrix} A & C \\ D & B \end{pmatrix},$$

where $A$, $B$, $C$, and $D$ can be complex numbers, functions or differential forms. This algebra is also going to be constructed as the tensor product of two graded associative and differential algebras. The first consists of the algebra of $2 \times 2$ complex matrices, where the $\mathbb{Z}_2$-grading is defined by associating a degree 0 to diagonal matrices (even) and $-1$ to the off-diagonal ones (odd). The second differential graded algebra is the algebra of differential forms (the algebra for the addition and wedge product of forms).

Let us denote by $\odot$ the product which defines this $\mathbb{Z}_2$-graded associative algebra. Let us also denote by $\partial A$, $\partial B$, $\partial C$ and $\partial D$ the degrees of the differential forms $A$, $B$, $C$ and $D$ respectively. The associative product of two elements in this algebra, $X$ and $X'$, is defined as

$$X \odot X' = \begin{pmatrix} A & C \\ D & B \end{pmatrix} \odot \begin{pmatrix} A' & C' \\ D' & B' \end{pmatrix} = \begin{pmatrix} A \wedge A' + (-1)^{BC} C \wedge D' & C \wedge B' + (-1)^{BD} A \wedge C' \\ D \wedge A' + (-1)^{DB} B \wedge D' & B \wedge B' + (-1)^{DD} B \wedge C' \end{pmatrix}. \tag{2.2}$$

1In this section, we will present only the necessary mathematical definitions and refer the reader to ref.[9] for more details.
The next step would be to define a differential operator \( \hat{d} \) acting on elements like \( X \), satisfying graded Leibniz rule and being nilpotent \( \hat{d}^2 = 0 \). This would be a generalization of the notion of the usual exterior derivative \( d \) in ordinary differential geometry. The action of \( \hat{d} \) on \( X \) is given by [9]

\[
\hat{d}X = \begin{pmatrix}
  dA + C + D & -dC - (A - B) \\
  -dD + (A - B) & dB + C + D
\end{pmatrix}.
\] (2.3)

Notice that \( \hat{d}X \neq 0 \) even when \( A, B, C \) and \( D \) are just complex numbers. It is also easy to see that \( \hat{d}^2 X = 0 \). Furthermore, \( \hat{d} \) obeys graded Leibniz rule

\[
\hat{d} (X \circ X') = \hat{d}X \circ X' + (-1)^{\partial X} X \circ \hat{d}X' .
\] (2.4)

Here \( \partial X \) is the total \( Z_2 \)-grading of \( X \). That is, \( \partial X \) is the sum of the \( Z_2 \)-grading of \( X \) as a matrix (diagonal or off-diagonal) and as a differential form (even or odd degree).

To compute the second term of the right-hand side one has, therefore, to write \( X \) as a sum of four matrices each having only one non-zero element. The previous mathematical ingredients are just what we need to construct the gravitational theory that we have in mind. We will work with a “generalized spin connection”, \( \Omega^a_b \), and a “generalized vierbein”, \( E^a \), in the form of 2 \( \times \) 2 matrices

\[
\Omega^a_b = \begin{pmatrix}
  \omega^a_b & \varphi^a_b \\
  \bar{\omega}^a_b & \bar{\varphi}^a_b
\end{pmatrix} , \quad E^a = \begin{pmatrix}
  e^a & s^a \\
  \bar{s}^a & \bar{e}^a
\end{pmatrix} .
\] (2.5)

Here \( \omega^a_b, \bar{\omega}^a_b, e^a, \bar{e}^a \) are all real one-forms and we have \( \omega^a_b = \omega^a_{\mu b} dx^\mu \), \( \bar{\omega}^a_b = \bar{\omega}^a_{\mu b} dx^\mu \), \( e^a = e^a_\mu dx^\mu \), \( \bar{e}^a = \bar{e}^a_\mu dx^\mu \), while \( \varphi^a_b, \bar{\varphi}^a_b, s^a, \bar{s}^a \) are real functions.

The generalized curvature is defined as

\[
\mathcal{R}^a_b = \hat{d}\Omega^a_b + \Omega^a_c \circ \Omega^c_b .
\] (2.6)

In analogy with ordinary differential geometry, we define the generalized torsion to be given by

\[
\mathcal{T}^a = \hat{d}E^a + \Omega^a_b \circ E^b .
\] (2.7)

The last two equations are the generalization of Cartan’s structure equations in differential geometry. Furthermore, by acting on both sides of (2.6) with \( \hat{d} \), we find the generalized Bianchi identities

\[
\hat{d}\mathcal{R}^a_b + \Omega^a_c \circ \mathcal{R}^a_c - \mathcal{R}^a_c \circ \Omega^c_b \equiv DR^a_b = 0 .
\] (2.8)
The action of \( \hat{d} \) on (2.7) gives the following consistency conditions

\[
\hat{d} T^a + \Omega^a_c \otimes T^c = R^a_c \otimes E^c .
\] (2.9)

In deriving the above two consistency equations, we have used the fact that the total \( \mathbb{Z}_2 \)-grading of \( \Omega^a_b \) is +1. This is because the diagonal elements of \( \Omega^a_b \) are one-forms and its off-diagonal terms are zero-forms.

The matrix for the curvature is found to be

\[
R^a_b = \left( \begin{array}{cc}
(R_{11})^a_b & (R_{12})^a_b \\
(R_{21})^a_b & (R_{22})^a_b
\end{array} \right) ,
\] (2.10)

with

\[
(R_{11})^a_b = R^a_b + \varphi^a_b + \varphi^a_c \varphi^c_b + \tilde{\varphi}^a_b + \varphi^a_c \tilde{\varphi}^c_b,
\]

\[
(R_{12})^a_b = -\nabla \varphi^a_b - (\omega^a_b - \tilde{\omega}^a_b),
\]

\[
(R_{21})^a_b = -\nabla \tilde{\varphi}^a_b + (\omega^a_b - \tilde{\omega}^a_b),
\]

\[
(R_{22})^a_b = \tilde{R}^a_b + \varphi^a_b + \varphi^a_c \varphi^c_b + \tilde{\varphi}^a_b + \varphi^a_c \tilde{\varphi}^c_b .
\] (2.11)

where

\[
R^a_b = d\omega^a_b + \omega^a_c \wedge \omega^c_b = \frac{1}{2} R^a_{b \mu \nu} dx^\mu \wedge dx^\nu,
\]

\[
\tilde{R}^a_b = d\tilde{\omega}^a_b + \tilde{\omega}^a_c \wedge \tilde{\omega}^c_b = \frac{1}{2} \tilde{R}^a_{b \mu \nu} dx^\mu \wedge dx^\nu,
\]

\[
\nabla \varphi^a_b = d\varphi^a_b - \varphi^a_b \varphi^c_b + \omega^a_c \varphi^c_b,
\]

\[
\nabla \tilde{\varphi}^a_b = d\tilde{\varphi}^a_b - \varphi^a_b \tilde{\varphi}^c_b + \tilde{\omega}^a_b \tilde{\varphi}^c_b .
\] (2.12)

Similarly, the torsion matrix is written as

\[
T^a = \left( \begin{array}{cc}
T^a_{11} & T^a_{12} \\
T^a_{21} & T^a_{22}
\end{array} \right) ,
\] (2.13)

where

\[
T^a_{11} = de^a + \omega^a_b \wedge e^b + s^a + \tilde{s}^a + \varphi^a_b \tilde{e}^b,
\]

\[
T^a_{12} = -\nabla s^a - (e^a - \tilde{e}^a) + \varphi^a_b \tilde{e}^b,
\]

\[
T^a_{21} = -\nabla \tilde{s}^a + (e^a - \tilde{e}^a) + \tilde{\varphi}^a_b \tilde{e}^b,
\]

\[
T^a_{22} = d\tilde{e}^a + \tilde{\omega}^a_b \wedge \tilde{e}^b + \tilde{s}^a + \tilde{s}^a + \tilde{\varphi}^a_b s^b .
\] (2.14)
with

\[
\nabla s^a = ds^a + \omega^a_{\ b} s^b \\
\nabla s^a = d\tilde{s}^a + \tilde{\omega}^a_{\ b} s^b .
\] (2.15)

We would like now to examine the issue of gauge transformations. In analogy with differential geometry, we consider a generalized orthogonal rotation of the generalized orthonormal frame

\[
E^a \rightarrow E'^a = \mathcal{H}^a_{\ b} \circ E^b .
\] (2.16)

here \(\mathcal{H}^a_{\ b}\) is a \(2 \times 2\) matrix whose entries are all zero-forms. This takes the general form

\[
\mathcal{H}^a_{\ b} = \left( \begin{array}{cc}
 h^a_{\ b} & f^a_{\ b} \\
 f^a_{\ b} & \tilde{h}^a_{\ b}
\end{array} \right)
\] (2.17)

We define in what follows the generalized Cartesian flat metric, \(\Sigma_{ab}\), and the generalized Kronecker delta-function, \(\Upsilon^a_{\ b}\), as

\[
\Sigma_{ab} = \left( \begin{array}{cc}
 \eta_{ab} & 0 \\
 0 & \eta_{ab}
\end{array} \right), \quad \Upsilon^a_{\ b} = \left( \begin{array}{cc}
 \delta^a_{\ b} & 0 \\
 0 & \delta^a_{\ b}
\end{array} \right) .
\] (2.18)

The matrix \(\mathcal{H}^a_{\ b}\) satisfies

\[
\Sigma_{cd} = \Sigma_{ab} \circ \mathcal{H}^a_{\ c} \circ \mathcal{H}^b_{\ d} \\
\Upsilon^a_{\ b} = \mathcal{H}^a_{\ c} \circ \left( \mathcal{H}^{-1} \right)^c_{\ b} \\
\left( d\mathcal{H}^a_{\ c} \right) \circ \left( \mathcal{H}^{-1} \right)^c_{\ b} = \left[ \left( \mathcal{H}^a_{\ c} \right)_{\ a} - \left( \mathcal{H}^a_{\ c} \right)_{\ e} \right] \circ \left( d\mathcal{H}^{-1} \right)^c_{\ b} ,
\] (2.19)

where the subscripts \(e\) and \(o\) stand, respectively, for the diagonal (even) and off-diagonal (odd) parts of \(2 \times 2\) matrices.

We require the torsion to transform like

\[
\mathcal{T}'^a = dE'^a + \Omega'^a_{\ b} \circ E^b = \mathcal{H}^a_{\ b} \circ \mathcal{T}^b ,
\] (2.20)

and the connection as

\[
\Omega'^a_{\ b} = \mathcal{H}^a_{\ c} \circ \Omega'^c_{\ d} \circ \left( \mathcal{H}^{-1} \right)^d_{\ b} + \mathcal{H}^a_{\ c} \circ \left( d\mathcal{H}^{-1} \right)^c_{\ b} .
\] (2.21)
By computing $T^a = \mathcal{H}^a_b \odot T^b$ and looking at the resulting transformations for the elements of $\Omega^a_b$ and $E^a$, we deduce that $\mathcal{H}^a_b$ must be such that $f^a_b = f^a_b = 0$ and $h^a_b = \tilde{h}^a_b$. Therefore, $(\mathcal{H}^{-1})^a_b$ has $(h^{-1})^a_{b}$ along the diagonal and 0 along the off-diagonal. Consequently, the curvature transforms as expected

$$
\mathcal{R}^a_b = \hat{d}\Omega^a_b + \Omega^a_c \odot \Omega^c_b = \mathcal{H}^a_c \odot \mathcal{R}^c_d \odot (\mathcal{H}^{-1})^d_b \ .
$$

The elements of $\Omega^a_b$ and $E^a$ transform in the following way

$$
c^a = h^a_b s^b , \quad \tilde{c}^a = h^a_b \tilde{s}^b , \quad s^a = h^a_b s^b , \quad \tilde{s}^a = h^a_b \tilde{s}^b
$$

$$
\omega^a_b = h^a_c \omega^d_c (h^{-1})^d_b + h^a_c (dh^{-1})_c^b , \quad \tilde{\omega}^a_b = h^a_c \tilde{\omega}^d_c (h^{-1})_c^b + h^a_c (dh^{-1})^c_b
$$

$$
\varphi^a_b = h^a_c \varphi^d_c (h^{-1})_b^d , \quad \tilde{\varphi}^a_b = h^a_c \tilde{\varphi}^d_c (h^{-1})_b^d .
$$

Notice that $s^a$ and $\tilde{s}^a$ transform like vectors while $\varphi^a_b$ and $\tilde{\varphi}^a_b$ transform like scalars.

## 3 The Action

We turn now to the construction of an action which is gauge invariant under the above gauge transformations. Using the scalar product of differential forms, where the space-time metric is $g_{\mu\nu}$, and the trace on the space of $2 \times 2$ matrices, we define the following gauge invariant Lagrangian

$$
S = \int Tr \left[ \left( E^a \odot E^b \right) \odot \left( \Sigma_{bc} \odot \mathcal{R}^c_a \right) \right]
$$

$$
= \int \left[ \left( E^a \odot E^b \right)_{11} \wedge \left( \Sigma_{bc} \odot \mathcal{R}^c_a \right)_{11} - \left( E^a \odot E^b \right)_{12} \wedge \left( \Sigma_{bc} \odot \mathcal{R}^c_a \right)_{21} 
$$

$$
- \left( E^a \odot E^b \right)_{21} \wedge \left( \Sigma_{bc} \odot \mathcal{R}^c_a \right)_{12} 
$$

$$
+ \left( E^a \odot E^b \right)_{22} \wedge \left( \Sigma_{bc} \odot \mathcal{R}^c_a \right)_{22} \right] .
$$

(3.1)

Here the integration is over a $n$-dimensional space and $*$ is the usual Hodge star which acts on the individual elements of the matrix $(\Sigma_{ac} \odot \mathcal{R}^c_b)$. An explicit computation gives

$$
S = \int d^nx \sqrt{g} \left[ \left( e^a_{\mu} e^b_{\nu} \eta_{bc} R^c_{\alpha \rho \sigma} + e^a_{\mu} e^b_{\nu} \eta_{bc} R^c_{\alpha \rho \sigma} \right) g^{\mu \nu} g^{\rho \sigma} 
$$

$$
+ \left( s^a e^b_{\mu} - e^a s^b_{\mu} \right) \eta_{bc} D_{\nu} \tilde{\varphi}^c_a g^{\mu \nu} + \left( \tilde{s}^a e^b_{\mu} - e^a \tilde{s}^b_{\mu} \right) \eta_{bc} D_{\nu} \varphi^c_a g^{\mu \nu} 
$$

$$
+ s^a e^b_{\mu} \eta_{bc} \left( \varphi^c_a + \tilde{\varphi}^c_a + \varphi^c_d \tilde{s}^d_a \right) 
$$

$$
+ \tilde{s}^a e^b_{\mu} \eta_{bc} \left( \varphi^c_a + \tilde{\varphi}^c_a + \tilde{\varphi}^c_d \tilde{s}^d_a \right) \right] ,
$$

(3.2)
where

\[
D_\mu \varphi^a_b = \nabla_\mu \varphi^a_b + (\omega^a_b - \tilde{\omega}^a_b),
\]

\[
D_\mu \tilde{\varphi}^a_b = \nabla_\mu \tilde{\varphi}^a_b - (\omega^a_b - \tilde{\omega}^a_b).
\]  

(3.3)

At this stage all the fields entering in the construction of the above Lagrangian are independent of each other. One can, therefore, simply take \( S \) as a starting point and eliminate all the non-propagating fields by their equations of motion. Indeed, the equations of motion for \( s^a \) and \( \tilde{s}^a \) can be easily solved and we find

\[
s^a = (M^{-1})^{ab} \left( \epsilon^{cd}_{\mu \nu} \eta_{bc} D_\nu \varphi^c_d - \epsilon^{cd}_{\mu \nu} \eta_{dc} D_\nu \varphi^c_b \right) g^{\mu \nu},
\]

\[
\tilde{s}^a = (N^{-1})^{ab} \left( \epsilon^{cd}_{\mu \nu} \eta_{bc} D_\nu \tilde{\varphi}^c_d - \epsilon^{cd}_{\mu \nu} \eta_{dc} D_\nu \tilde{\varphi}^c_b \right) g^{\mu \nu},
\]  

(3.4)

where

\[
M_{ab} = \eta_{ac} \left( \varphi^c_b + \tilde{\varphi}^c_b + \varphi^c_d \tilde{\varphi}^d_b \right) + \eta_{bc} \left( \varphi^c_a + \tilde{\varphi}^c_a + \varphi^c_d \tilde{\varphi}^d_a \right),
\]

\[
N_{ab} = \eta_{ac} \left( \varphi^c_b + \tilde{\varphi}^c_b + \varphi^c_d \tilde{\varphi}^d_b \right) + \eta_{bc} \left( \varphi^c_a + \tilde{\varphi}^c_a + \varphi^c_d \tilde{\varphi}^d_a \right).
\]  

(3.5)

By substituting for \( s^a \) and \( \tilde{s}^a \) in the above action we get

\[
S = \int d^3x \sqrt{\mathcal{g}} \left[ (e^{a}_{\mu \nu} \eta_{bc} R^c_{\mu \nu} + \tilde{e}^{a}_{\mu \nu} \tilde{R}^c_{\mu \nu}) g^{\mu \nu} g^{\nu \sigma} + \mathcal{F}^{da, \sigma \nu}_{cs} D_\sigma \varphi^c_d D_\nu \varphi^s_a \right],
\]  

(3.6)

where the quantity \( \mathcal{F}^{da, \sigma \nu}_{cs} \) is given by

\[
\mathcal{F}^{da, \sigma \nu}_{cs} = g^{\mu \rho} g^{\rho \sigma} \eta_{rc} \eta_{bs} \left[ (M^{-1})^{ar} e^{d}_{\rho \mu} + (M^{-1})^{bd} e^{a}_{\mu \rho} \right] - (M^{-1})^{ad} e^{d}_{\rho \mu} - (M^{-1})^{br} e^{a}_{\mu \rho} \\
+ g^{\mu \nu} g^{\sigma \rho} \eta_{r s} \eta_{b c} \left[ (N^{-1})^{d r} e^{a}_{\rho \mu} + (N^{-1})^{b a} e^{d}_{\mu \rho} \right] - (N^{-1})^{d a} e^{d}_{\rho \mu} - (N^{-1})^{b r} e^{a}_{\mu \rho} \\
+ g^{\mu \sigma} g^{\rho \nu} \eta_{r c} \eta_{s b} \left[ (\varphi^l_t + \tilde{\varphi}^l_t + \varphi^l_k \tilde{\varphi}^k_t) \right] \left[ (M^{-1})^{l r} (N^{-1})^{b f} e^{d}_{\mu \rho} \right] - (M^{-1})^{l a} e^{d}_{\mu \rho} - (N^{-1})^{b f} e^{a}_{\mu \rho} \\
+ (M^{-1})^{l f} e^{r}_{\mu \rho} \right].
\]
\[ g^{\mu\nu} g^{\rho\sigma} \eta_{rs} \eta_{ij} \left( \bar{\varphi}_i^t + \bar{\varphi}_i^l + \bar{\varphi}_k^t \right) \left( (N^{-1})^r_s (M^{-1})^b_f e^a_\mu e^d_\rho \right. \\
- \left. \left( (N^{-1})^r_s (M^{-1})^b_d e^a_\mu e^f_\rho - (N^{-1})^a_t (M^{-1})^b_f e^a_\mu e^d_\rho \right) \right. \\
+ \left. \left( (N^{-1})^a_t (M^{-1})^b_d e^a_\mu e^f_\rho \right) \right]. \tag{3.7} \]

The resulting action describes two distinct universes (the “tilded” universe and the “untilded” one) which are coupled through the presence of the \( \mathcal{F}_{\alpha\beta\gamma} \) term. In order to obtain Einstein-Hilbert gravity, we restrict ourselves to the case when

\[ \tilde{e}^a_\mu = e^a_\mu, \quad \tilde{\omega}^a_{\mu b} = \omega^a_{\mu b}, \quad g_{\mu\nu} = e^a_\mu e^b_\nu \eta_{ab} \tag{3.8} \]

and \( \omega^a_{\mu b} \) is the Levi-Civita spin connection satisfying the metricity condition \( (\omega^a_{\mu b} = \eta_{ac} \omega^c_{\mu b} = -\omega^c_{\mu a}) \) and the torsion-free constraints \( (de^a + \omega^a_{\mu b} \wedge e^b = 0) \). The first term in the action \( (3.6) \) is then the Einstein-Hilbert action while the second term describes a non-minimal coupling of a set of scalar fields to gravity.

The other possibility in dealing with our starting action \( (3.2) \) would be to impose the generalized torsion-free conditions, \( \mathcal{T}^a = 0 \). It turns out, however, that these constraints are very restrictive and, apart from some trivial solutions, no general solutions were found.

In what follows we will propose another definition for the curvature and the torsion in such a way that the torsion-free constraints can be solved.

We define the new generalized curvature as

\[ \hat{\mathcal{K}}^a_b = d\Omega^a_b + \Omega^a_c \odot \Omega^c_b - (\Omega^a_c)_o \odot (\Omega^c_b)_o, \tag{3.9} \]

where \( (\Omega^a_b)_o \) is the off-diagonal part of \( \Omega^a_b \). It is easy to see that under the gauge transformations \( \mathcal{H}^a_b \) (for which \( f^a_b = \tilde{f}^a_b = 0 \) and \( h^a_b = \tilde{h}^a_b \) ) the new curvature tensor transforms as

\[ \hat{\mathcal{K}}^a_b = \mathcal{H}^a_c \odot \hat{\mathcal{K}}^c_d \odot \left( \mathcal{H}^{-1} \right)^d_b. \tag{3.10} \]

Similarly, the new torsion is given by

\[ \hat{\mathcal{T}}^a = dE^a + \Omega^a_c \odot E^c - (\Omega^a_b)_o \odot (E^b)_o, \tag{3.11} \]

and it transforms as

\[ \hat{\mathcal{T}}^a \propto \mathcal{H}^a_b \odot \hat{\mathcal{T}}^b. \tag{3.12} \]
The elements of $\hat{\mathcal{R}}_a^b$ are simply given by the expressions in (2.12) but without the terms $\varphi^a_c \tilde{\varphi}^c_b$ and $\tilde{\varphi}^a_c \varphi^c_b$ in $(\mathcal{R}^{11})_b^a$ and $(\mathcal{R}^{22})_b^a$ respectively. Also the elements of $\hat{T}_a^a$ are given by the expressions in (2.14) but without the terms $\varphi^a_b \tilde{s}^b$ and $\tilde{\varphi}^a_b s^b$ in $(\mathcal{T}_a)_1^1$ and $(\mathcal{T}_a)_2^2$ respectively.

The new gauge invariant action is given by

$$S = \int Tr \left[ (E^a \circ E^b) \circ \ast \left( \Sigma_{bc} \circ \hat{\mathcal{R}}_a^c \right) \right]$$

$$= \int d^n x \sqrt{g} \left[ (\tilde{e}_a^\mu \tilde{e}_\nu^b \eta_{bc} R^c_{a\rho\sigma} + \tilde{\tilde{e}}^b_{\mu} \tilde{e}_\nu^b \eta_{bc} R^c_{a\rho\sigma}) g^{\mu\rho} g^{\nu\sigma} \right.$$

$$+ \left. \left[ (s^a \tilde{e}_\mu^b - e^a_\mu s^b) \eta_{bc} D_\nu \tilde{\varphi}^c_a + (\tilde{s}^a \tilde{e}_\mu^b - \tilde{e}^a_\mu \tilde{s}^b) \eta_{bc} D_\nu \varphi^c_a \right] g^{\mu\nu} \right.$$  

$$+ \left. \left( s^a \tilde{s}^b + \tilde{s}^a s^b \right) \eta_{bc} \left( \varphi^c_a + \tilde{\varphi}^c_a \right) \right] . \quad (3.13)$$

The torsion-free constraints, $\hat{T}_a^a = 0$, are solved by

$$g_{\mu\nu} = \eta_{ab} e^a_\mu e^b_\nu , \quad e^a = \tilde{e}^a , \quad \omega^a_b = \tilde{\omega}^a_b$$

$$\tilde{s}^a = - s^a , \quad \tilde{\varphi}^a_b = - \varphi^a_b = \lambda_\mu^a \nabla_\mu s^a , \quad (3.14)$$

where $\omega^a_b$ is the Levi-Civita spin connection and $\lambda^a_\mu$ is the inverse of $e_\mu^a$, such that $\lambda^a_\mu e^b_\mu = \delta^b_a$, $g^{\mu\nu} = \lambda^a_\mu \lambda^b_\nu \eta^{ab}$ and $\eta_{ab} = g_{\mu\nu} \lambda^a_\mu \lambda^b_\nu$.

Substituting these expressions in the action (3.13) and integrating by parts, we find

$$S = 2 \int d^n x \sqrt{g} \left\{ R - \left[ g^{\mu\nu} \eta_{ab} - \frac{1}{2} \left( \lambda^a_\mu \lambda^b_\nu + \lambda^a_\nu \lambda^b_\mu \right) \right] \nabla_\mu s^a \nabla_\nu s^b \right\} . \quad (3.15)$$

This is the action for a vector field coupled in a non-conventional manner to Einstein-Hilbert gravity.

To summarize, we have considered gravity in the framework of a differential $Z_2$-graded and associative algebra of $2 \times 2$ matrices with $p$-forms entries. We have given an extension of Cartan’s structure equations and constructed a gauge invariant action. The extended torsion-free conditions are very restrictive and would get rid of all the extra scalar fields in the theory. Instead, we eliminate the non-dynamical fields by their equations of motion and obtain an action describing a non-minimal coupling of a set of scalar fields to two distinct universes. It is also possible to modify the definitions of the extended curvature and the extended torsion in such a way that the extended torsion-free constraints can be solved. We obtain, in this last case, a theory characterizing the coupling of a vector field to Einstein-Hilbert gravity.
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