Implications of Efimov physics for the description of three and four nucleons in chiral effective field theory

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In chiral effective field theory the leading order (LO) nucleon-nucleon potential includes two contact terms, in the two spin channels $S = 0, 1$, and the one-pion-exchange potential. When the pion degrees of freedom are integrated out, as in the pionless effective field theory, the LO potential includes two contact terms only. In the three-nucleon system, the pionless theory includes a three-nucleon contact term interaction at LO whereas the chiral effective theory does not. Accordingly arbitrary differences could be observed in the LO description of three- and four-nucleon binding energies. We analyze the two theories at LO and conclude that a three-nucleon contact term is necessary at this order in both theories. In turn this implies that subleading three-nucleon contact terms should be promoted to lower orders. Furthermore this analysis shows that one single low energy constant might be sufficient to explain the large values of the singlet and triplet scattering lengths.

I. INTRODUCTION

Strong efforts have been done in recent years to determine the nuclear interaction from first principles. The starting point is an effective Lagrangian in terms of the relevant degrees of freedom, nucleons and pions, maintaining the symmetries of the strong interaction, and a perturbative framework named chiral perturbation theory (ChPT) (see Refs. 1,2 and references therein). This setting allowed to construct two and three-body interactions up to next-to-next-to-next-leading order (N4LO). Though potentials up to N5LO have been studied 3,4, at present most of the calculations in finite nuclei and infinite nuclear matter have been done with the nucleon-nucleon (NN) interaction up to N3LO and the three-nucleon interaction up to N2LO. At LO the chiral perturbative expansion includes two contact terms, characterized by the low energy constants (LECs) $C_S$ and $C_T$ governing the (spin) singlet and triplet s-waves, and the (regularized) one-pion-exchange potential (OPEP). At very low energies the pion degrees of freedom can be integrated out resulting in what is called pionless effective theory 5. At LO this theory includes only two contact terms without the inclusion of the pionic tail. In the low energy regime the chiral and pionless effective theories should be equivalent. In fact both theories have two LECs to be determined by two observables, such as e.g. the singlet and triplet scattering lengths, implying an equivalent description of the low energy s-wave scattering. Since the OPEP tail includes the tensor interaction, small differences between the two theories arise from those observables depending on the d-state component of the deuteron as for example the quadrupole moment. However these quantities are small and can be considered to be inside the theoretical error produced by the truncation of the perturbative expansion at LO. Accordingly both theories give an equivalent description of the two nucleon system at LO.

The extension to the three-nucleon sector produces a drastic difference between the two theories and this is the motivation of the present investigation. In chiral perturbation theory there is no new term at LO when the three-nucleon system is considered 6. Accordingly the LO NN potential used in the two-body system has to be used to describe the three-body system without the inclusion of any new LEC. Conversely, and it is well known from Efimov physics, the pionless theory includes a contact three-body force at LO with a new LEC that can be used to determine the energy of the three-nucleon bound state 7. Assuming that there are no other many-body forces at LO, systems with $A \geq 3$ are described in the LO pionless theory with a sum of a two- plus a three-body term whereas the LO chiral potential consists only in a two-body term. Since the new LEC can be used to fix the triton binding energy, we expect a better description of the $^3$H, $^3$He and $^4$He binding energies using the LO pionless potential than using the LO chiral one. As a result, the pionless expansion can be expected to converge better than the chiral one. In principle there is no contradiction: ChPT, being a more microscopic theory, yields a more economic description, with less unknowns than the pionless theory; differences in the convergence pattern are also expected, as the perturbative expansions are different in the two theories (e.g. the expansion parameters are different). Nevertheless we consider this situation as undesirable from a practical point of view, and think that
this difference in the description of the $s$-wave nuclei at LO needs a deeper analysis.

At the base of the difference between the two theories is the fact that the singlet and triplet scattering lengths $a_0 \approx -24 \text{ fm}$ and $a_1 \approx 5.4 \text{ fm}$ are large with respect to the range of the nuclear interaction $r_N \approx 1.4 \text{ fm}$. A direct consequence is the shallow characteristic of the deuteron binding energy $E_D \approx h^2/m_N^2$ with corrections in powers of the small quantity $r_{\text{eff}}^1/a_1$, with $r_{\text{eff}}^1 \approx r_N$ the effective range in the triplet state. The existence of large scattering lengths puts the two-, three- and four-nucleon systems inside a window in which the concepts of the Efimov physics can be applied (see Refs. [8, 9] for a recent discussion). This physics describes systems close to the unitary limit, $a_0, a_1 \to \infty$, using a zero-range theory. In this limit the three-body system is characterized by an infinite series of excited states, called Efimov effect [10]. Close to the unitary limit there is no length scale and the three-body contact term is needed to stabilize the system against variations of the cutoff introduced at the two-body level to regularize the theory. It could be thought that since the OPEP tail introduces a length scale there is no need for a three-body counter term to stabilize the theory even if the system is close to the unitary limit. However, as we are dealing with shallow states, the inclusion of the potential tail is in competition with the zero-range description suggested by the large values of the scattering lengths. Accordingly there is a noticeable sensitivity to variations of the cutoff, though smaller compared to the pionless case, not completely dampened by the inclusion of the OPEP tail.

### II. NUMERICAL ANALYSIS

We perform a numerical analysis of both theories at LO in order to assess the necessity of including a three-body force at LO in both theories. We start analyzing the two-body sector. The LO effective Hamiltonian for two nucleons can be put in the form

$$H_{LO} = -\frac{\hbar^2 \nabla^2}{m_N} + V_{sr} + V_\pi$$

where $V_{sr}$ is the (regularized) short-range contact interaction and $V_\pi$ is the OPEP. The short-range interaction, considered to act in $s$-waves, has a spin dependence and can be written as

$$V_{sr} = C_S V_c + C_T V_\sigma \sigma_1 \cdot \sigma_2 .$$

Using a local gaussian regulator, the two potentials $V_c$ and $V_\sigma$ have the following form

$$V_c = V_\sigma = V_0(r) = e^{-r^2/\Lambda_0^2}$$

with $\Lambda_0 = 1/r_0$ the cutoff parameter. $V_\sigma$ reads

$$V_\sigma(r) = \tau_1 \cdot \tau_2 \left[ \sigma_1 \cdot \sigma_2 Y_\beta(r) + S_{12} T_\beta(r) \right]$$

with the regularized factors $(x = m_\pi r)$

$$Y_\beta(x) = \frac{g_A^2 m_\pi^2}{12\pi F_\pi^2} \frac{e^{-x}}{x} \left(1 - e^{-r^2/\beta^2} \right)$$

$$T_\beta(x) = \frac{g_A^2 m_\pi^2}{12\pi F_\pi^2} \frac{e^{-x}}{x} \left(1 + \frac{3}{x} + \frac{3}{x^2} \right) (1 - e^{-r^2/\beta^2})^2 .$$

Here $m_\pi = 138.03 \text{ MeV}$ is the average pion mass, $g_A = 1.29$ is the nucleon coupling constant and $F_\pi = 2 F_\pi = 184.80 \text{ MeV}$ is the pion decay amplitude.

Using $H_{LO}$, we solve the two-nucleon Schrödinger equation for different values of the cutoff $\Lambda_0$ and the regulator parameter $\beta$. For each case the two LECs $C_S$ and $C_T$ are selected in order to reproduce the two scattering lengths, $a_0$ and $a_1$. It should be noticed that in the case $\beta \to \infty$ the chiral Hamiltonian $H_{LO}$ tends to the pionless one. In this case the coupling constants $C_0 = C_S - 3 C_T$ and $C_1 = C_S + C_T$, corresponding to spin channels $S = 0, 1$ respectively, can be expanded in powers of the small parameters $r_0/a_0$ as

$$C_\beta = \frac{\sqrt{\pi} m_\pi^2}{2} C_\lambda = C_\infty \left(1 + \alpha_1 \frac{r_0}{a_\lambda} + \alpha_2 \left(\frac{r_0}{a_\lambda}\right)^2 + \ldots \right)$$

with $\lambda = 0, 1$ and $C_\infty = 2.379 \ [12, 13]$. The pure number 2.379 is universal and gives the value of the coupling constant at the unitary limit. It also gives the ratio $\sqrt{\pi} m_\pi^2 C_\lambda/(2h^2)$ at the scaling limit, $r_0 \to 0$. Eq. (6) maps the renormalization group (RG) trajectories as the interaction approaches the scaling limit. In Fig. 4 the different trajectories are shown as a fit of the results, given by the solid circles ($S = 0$ state) and empty circles ($S = 1$ state), as a function of $r_0/a_0$ for different values of $\beta$. In all cases $C_\infty$ is independent of the spin state and of the regulator $\beta$ and represents the fixed point to which all trajectories flow. Moreover, in the case of $\beta \to \infty$ the calculations of the spin $S = 0, 1$ states lie on a single trajectory. The fact that all trajectories converge to the same fixed point can be seen as an indication that both theories, pionless and chiral, need a three-body contact interaction to stabilize the three-body energy against variations in the cutoff $\Lambda_0$ (see below).

We extend our analysis to the three- and four-nucleon systems. The effective potential is

$$\sum_{i\neq j} [V_{sr}(i,j) + V_\pi(i,j)] + \sum_{i\neq j\neq k} W(i,j,k)$$

where we have added a (regularized) contact three-body term of the form

$$W(i,j,k) = W_0 e^{-r_i^2/r_0^2} e^{-r_j^2/r_0^2} e^{-r_k^2/r_0^2} .$$

In first place we calculate the $^3\text{H}$ energy, $B(^3\text{H})$, considering only the two-body potential terms for different values of $r_0$ and $\beta$. The results are shown in Fig. 4 as a function of the regulator of the OPEP $\beta$. The limit $\beta \to \infty$ corresponds to the pionless theory whereas the lowest value, $\beta = 1.0 \text{ fm}$, is well inside the region compatible with the
formation of the OPEP tail. We observe a noticeable spread in \( B(^3\text{H}) \) for all the \( r_0 \) values considered, more pronounced as \( \beta \to \infty \). In the figure, the lowest result \((\approx -18 \text{ MeV})\) corresponds to the shortest range considered as expected due to the Thomas collapse. For the lower values of \( \beta \) the spread in \( B(^3\text{H}) \) is reduced, but not enough to judge the inclusion of the OPEP tail sufficient for a stable description of \( B(^3\text{H}) \) at LO. It should be noticed that extending the analysis to shorter values of \( r_0 \) the spread increases more and more. So we can conclude that the reduction of the spread in the three-nucleon energy when the OPEP tail is included is not significant, a three-body contact term is needed in both cases, pionless and chiral, to stabilize the results.

In Fig. 2 the (red) solid squares indicates those values of the LECs verifying \( C_1 = C_0 \) which implies that the short-range potentials includes only the central term, \( V_{sr} = C_0 V_c \). This case corresponds to consider the Wigner spin-flavor symmetry \( SU(4)_W \), which in turn is justified in the context of the large \( N_c \) limit of QCD \cite{14, 15}. Thus, for the corresponding values of the cutoff \( \beta \), the difference between the singlet and triplet scattering length is entirely ascribed to the presence of the OPEP tail. Finer cutoff effects change the conclusion, but they may be considered to lie beyond the LO picture. It is reassuring to see that the fine-tuning which brings the NN system close to the unitary limit in both spin channels is only due to one single LEC, \( C_S \).

As suggested from Efimov physics and pionless theory, we include now the (regularized) three-body contact term, given in Eq. 8. We solve the Schrödinger equation with two and three-body forces for the different values of \( r_0 \) and \( \beta \), with the strength \( W_0 \) fixed in such a way that in all cases \( B(^3\text{H}) = -8.48 \text{ MeV} \). At this point predic-

\[ \text{FIG. 1. Color online. RG trajectories of the coupling constant } \hat{C}_\lambda \text{ as a function of the inverse cutoff parameter } r_0 \text{ in units of the singlet } (\lambda = 0, \text{ negative values}) \text{ and triplet } (\lambda = 1, \text{ positive values}) \text{ scattering length } a_\lambda \text{ for different values of the regulator } \beta. \]

\[ \text{FIG. 2. Color online. The triton energy } B(^3\text{H}) \text{ as a function of the regulator } \beta \text{ for different values of the potential range } r_0. \text{ The (red) squares indicate the results obtained with } C_1 = C_0 \text{ corresponding to the } SU(4)_W \text{ symmetry.} \]
motting the three-body contact term to LO in the chiral expansion of the nuclear interaction. In the normal counting the three-body contact term considered here at LO, and usually called $c_E$-term, appears at N2LO together with the $c_D$-term, a two-body contact term plus a pion exchange (see Ref. [1]). At N2LO two-pion exchange terms appear governed by the constants $c_1$, $c_3$ and $c_4$. At this level of the chiral expansion the new LECs are $c_E$ and $c_D$. Numerical values were determined in the literature from a fit to observables as for example $B(^3\text{H})$ and $^\alpha_\text{nd}$). In these fits usually the NN two-body potential was considered at N3LO. Consistency between the two- and three-body forces requires to construct the three-body interaction at N3LO [17, 18]. At this level however no new LECs appear. First applications of the complete N3LO nuclear interaction have addressed the description of the three-nucleon scattering, however some problems still remain in those observables governed by $P$-waves [19]. This is an indication that spin structures appearing at higher orders could be important parts of the three-nucleon interaction.

Parallel to this analysis the contact three-body force at N4LO has been discussed in Ref. [20]. In this study it was shown that the N4LO subleading contact interaction has 10 new LECs with the explicit expression given in that reference. The arguments given here to promote the $c_E$ term at LO entails a corresponding promotion of the N4LO subleading contact terms at N2LO, since they merely specify finer details of the short-distance interaction, of relative order $O(p^2)$ compared to the LO one. Accordingly the three-body force at N2LO will have, in addition to the $c_E$ appearing at leading order, the contact-one-pion LEC $c_D$ plus the 10 LECs of the subleading terms. This mechanism will improve the convergence of the ChPT since it increases the accuracy of the nuclear potential at N2LO which, considering only the two-body part, has sufficient flexibility to describe NN phases with a $\chi^2$ per datum $\approx 1$ up to 125 MeV lab energy [21]. It is thus sensible to investigate the three- and four-nucleon continuum using a N2LO interaction with two- and three-body forces having in the three-body interaction 12 independent LECs. The capability of the new terms to improve the theoretical description of spin observables has been recently investigated [22]. A more extended analysis is underway.

### III. LEADING ORDER FORMULATION FOR SHALLOW STATES

In the following we analyze the consequences for the power counting of the additional scale introduced by the shallow characteristic of the deuteron binding energy. It is worth noticing that, while the underlying (approximate) chiral symmetry of strong interactions allows to explain the hierarchy of many-body forces, it has little to say about the large cancellation between the kinetic and potential energy produced in shallow states. On the ba-

![Image of graph](image.png)

**FIG. 3.** Color online. The $B(^4\text{He})$ binding energy as a function of the regulator $\beta$ for different values of the potential range $r_0$ with and without the three-body interaction. In the first case the $r_0$ values of the different curves are given in that reference. The curves are very close to each other and are almost indistinguishable. In the second case the curves are well separated and the $r_0$ values are explicitly shown on the curves. The (red) squares indicate the results obtained with $C_1 = C_0$ corresponding to the $SU(4)_W$ symmetry.

sis of chiral counting both kinetic and potential energies are expected to be of order $O(p^2/\Lambda) \sim m_\pi^2/\Lambda \sim 20$ MeV, while the sum of the two is smaller by one order of magnitude in the deuteron (here $m_\pi$ is the pion mass and $\Lambda \approx 1$ GeV is the high momentum scale, see below). Whether this fine tuning is an accident or depends on some underlying physics is a question that need not concern us for the present discussion. In order to provide a rationale for the promotion of a three-body contact term to LO, we have to incorporate the small scale $\epsilon \sim 1/a$ associated with the large scattering length $a$ into the effective theory. It is well known that the Goldstone boson character of pions allows to estimate the size of interaction Lagrangians as

$$L = c_{\ell mn} \left( \frac{\bar{N}_\ell N}{f_\pi^2 \Lambda} \right)^\ell \left( \frac{\pi}{f_\pi} \right)^m \left( \frac{\partial_{\mu} m_\pi}{\Lambda} \right)^n \Lambda^2 f_\pi^2, \quad (9)$$

with dimensionless rescaled LECs $c_{\ell mn} \sim O(1)$ under the hypothesis of naturalness. This is called naive dimensional analysis [23] (cfr. also [24]). $f_\pi$ is the scale of Goldstone bosons’ fields, and sets the chiral symmetry breaking scale as $\Lambda \sim 4\pi f_\pi$, with the factor $4\pi$ coming from momenta integrations. $\Lambda$ represents the mass scale of hadrons unprotected by chiral symmetry, e.g. $\Lambda \sim m_N$, the nucleon mass. The scaling of each factor in the above formula can be simply obtained by applying it to the LO Lagrangian of interacting pions and nucleons [24]. The same can be done in a regime in which pions play no role, only nucleons are present, and a relevant scale is identified in $\epsilon$. The free nucleon Lagrangian is
not enough to constrain the scaling of nucleon fields and derivatives. We also need an interaction term,

\[ L_0^F = \tilde{N} (i\partial_{\mu} \gamma^\mu - m_N)N - \frac{D_0}{2} (\tilde{N}N)^2, \]

with the contact LEC \( D_0 \) related to the scattering length as \( D_0 \approx 4\pi a/m_N \sim 4\pi/m_N \epsilon \sim 1/(f_\pi \epsilon) \), and \( f_\pi \) serving here only a mnemonic purpose. This fixes uniquely the \( \epsilon \) in a combined small momenta and small \( D \) as
\[ \epsilon \sim \frac{\pi a}{m_N} \sim \frac{\pi}{f_\pi \Lambda} \sim \frac{1}{f_\pi \Lambda}. \]

We then see that contact terms are naturally enhanced by the small scale \( \epsilon \), e.g. a three-nucleon contact term would have a natural LEC \( c_E \) of order \( c_E \sim 1/(f_\pi^2 \Lambda^2) \); in a combined small momenta and small \( \epsilon \) expansion, it would have to be promoted to lower orders. The naive dimensional analysis is recovered imposing that \( \epsilon \) is natural, i.e. \( \epsilon \sim O(p) \sim f_\pi \). The two forms are not in contradiction since, in a matching between the two regimes, there could be dimensionless ratios of the two scales \( \epsilon/f_\pi \) restoring the proper scaling. Thus we see that, allowing \( \epsilon \) to be smaller than natural, pion-range interactions should actually be regarded as small compared to contact interactions, in agreement with the so-called KSW counting \[25\]. Therefore, if a three-body parameter is needed at LO in the pionless theory, the same is true in the chiral effective theory. The increased importance of contact interactions in ChPT would also explain the crucial role they play for the construction of phenomenologically realistic interactions, both in the two- and three-nucleon sector, as well as in the nuclear interactions with external electroweak probes. The proposed mechanism of promotion of contact terms, contrary to arguments based on the non-perturbative renormalization of the one-pion-exchange potential \[26\] doesn’t rely in any way on the role of pions. It just reflects the fact that the scattering length is larger than natural. In particular it is conceivable that, in a theory with no spontaneous chiral symmetry breaking, as in the case of QCD with one single flavor, gauge interactions (which are flavor-blind) could lead in principle to a similar scenario of quasi-universality.

IV. CONCLUSIONS

In the present work we have discussed two aspects of the two-, three- and four-nucleon systems in a LO description. The main conclusion is that the \( c_E \) term appearing in ChPT at N2LO has to be considered at LO due to the same situation already analyzed in pionless theory (or Efimov physics) and related to the Thomas collapse. This promotion has the consequence of moving to lower orders subleading contact terms, increasing the number of LEC’s and allowing for a faster convergence of the perturbative series. A second point regards the unusual large values of the two-nucleon scattering lengths in singlet and triplet states. It was shown that, considering the Wigner spin-flavor symmetry, the single LEC associated to the central short-range potential, in combination with the OPEP is responsible for these large values. This observation reduces, to some extent, the amount of fine-tuning required to explain the closeness of the nuclear interaction to the unitary limit.

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