AdS Calibrations

J.Gutowski & G.Papadopoulos

DAMTP, Silver Street, University of Cambridge, Cambridge CB3 9EW

ABSTRACT

We present a new bound for the worldvolume actions of branes with a Wess-Zumino term. For this we introduce a generalization of calibrations for which the calibration form is not closed. We then apply our construction to find the M-5-brane worldvolume solitons in an AdS background that saturate this bound. We show that these worldvolume solitons are supersymmetric and that they satisfy differential equations which generalize those of standard calibrations.
1. Introduction

The dynamics of branes in a flat background is described, in the absence of Born-Infeld type fields, by a Nambu-Goto type of action. The solutions of the field equations of such an action are minimal manifolds. A subclass of solutions of the Nambu-Goto action preserve a proportion of spacetime supersymmetry and saturate a bound. The most elegant way to formulate this bound and find the proportion of supersymmetry preserved by a solution is in terms of calibrations [1, 2, 3, 4]. Because brane worldvolume solitons have the bulk interpretation of intersecting branes, recently there has been much activity in applying calibrations to find worldvolume solitons of branes. These are then used to investigate the quantum theory of certain gauge theories [5]. Despite the many applications that calibrations have in string and M-theory, there are certain limitations. One is that there does not seem to be a generalization of the calibration bound in the presence of Born-Infeld type fields on the brane worldvolume. Another limitation is that the description of brane dynamics on curved backgrounds may also require the modification of the Nambu-Goto action by Wess-Zumino terms. In the presence of such Wess-Zumino terms, the standard argument used to establish the calibration bounds does not apply.

In this letter, we shall modify the definition of calibrations of [6, 7] to establish a new bound for the brane worldvolume actions with certain types of Wess-Zumino terms. As for standard calibrations, these new calibrations are associated with a calibration k-form \( \phi \) which satisfies the inequality

\[
\phi(\vec{\eta}) \leq 1
\]  

when evaluated at any appropriately normalized k-co-form \( \vec{\eta} \), but in this case \( \phi \) is not closed, i.e.

\[
d\phi \neq 0.
\]  

This modification in the definition of a calibration allows us to relate the form field strength \( F \) associated with the Wess-Zumino term of the brane action to the
calibration form as $\tilde{F} = d\phi$. Let $\text{Vol}(M)$ and $\text{WZ}(M)$ be the volume and the Wess-Zumino term of a manifold $M$, respectively. For static worldvolume configurations, we shall establish a bound

$$\text{Vol}(M) + \text{WZ}(M) \leq \text{Vol}(M') + \text{WZ}(M'),$$

(1.3)

where $M$ is a $k$-dimensional manifold for which $\phi(\eta_M) = 1$ and $\eta_M$ is its co-volume. We shall apply our method to find the calibrations associated with the M-5-brane in an $\text{AdS}_7 \times S^4$ background. Such backgrounds have been used to formulate a correspondence between gauge and string theories [8]. For this, we shall determine the calibration forms and relate them to the Wess-Zumino term. We shall then find that many of the differential equations associated with standard calibrated surfaces in flat spaces admit a natural generalization on $\text{AdS}_7 \times S^4$.

2. Generalized Calibrations

Let $(N, \tilde{g})$ be an $n$-dimensional oriented Riemannian manifold equipped with a metric $\tilde{g}$ and $G(k, T_pN)$ be the Grassmannian of oriented $k$-planes in the tangent space $T_pN$ at the point $p \in N$. Then for $\chi \in G(k, T_pN)$, there exists an orthonormal basis $\{e_1...e_n\}$ of $T_pN$ such that $\{e_1...e_k\}$ is a basis of $\chi$. The co-volume of $\chi$ is then

$$\chi = e_1 \wedge ... \wedge e_k.$$ 

(2.1)

We define a calibration of degree $k$ on an open subset $U \subset N$ to be a $k$-form $\phi$ with the following properties:

(i) At each $p \in U$,

$$\phi_p(\chi) \leq 1$$

(2.2)

for all $\chi \in G(k, T_pN)$, and
(ii) the contact set

\[ G(\phi) = \{ \chi \in G(k, T_p N) : \phi(\chi) = 1 \} \]  

is non-empty. Now a k-dimensional submanifold \( M \) of \( N \) is calibrated if

\[ \phi_p(\tilde{\eta}_M) = 1 \]  

for every \( p \in M \), where \( \tilde{\eta}_M \) is the co-volume of \( M \). In the conventional definition of a calibration \([6,7]\) it is further required that

\[ d\phi = 0 . \]  

To distinguish between the calibrations according to the old and new definitions, we shall refer to the former as “standard” calibrations and to the latter as “generalized” calibrations or simply calibrations. The generalized calibrations are a special cases of the \( \varphi \)-geometries of \([6]\).

It remains to establish the bound. For this, let \( M \) be a calibrated submanifold of \( N \) and \( U \subset M \). Then let \( V \) be a deformation of \( U \) such that \( U, V \) have the same boundary \( \partial U = \partial V \). Then we have

\[ \text{Vol}(U) = \int \phi(\tilde{\eta}_U)\mu_U = \int \phi(\tilde{\eta}_V)\mu_V + \int d\phi(\tilde{\eta}_D)\mu_D \leq \text{Vol}(V) + \int d\phi(\tilde{\eta}_D)\mu_D, \]  

where \( D \) is a disc with boundary \( \partial D = U - V \) and \( \mu_U \) is the volume form of \( U \) and similarly for the rest. To establish the second equality in (2.6), we have used Stoke’s theorem. Next let \( W \) be a reference open set such that \( \partial W = \partial U = \partial V \), and \( D_1 \) and \( D_2 \) be discs such that \( \partial D_1 = U - W \), \( \partial D_2 = V - W \) and \( D_2 = D_1 + D \). Then the above inequality can be rearranged such that

\[ \text{Vol}(U) - \int d\phi(\tilde{\eta}_{D_1})\mu_{D_1} \leq \text{Vol}(V) - \int d\phi(\tilde{\eta}_{D_2})\mu_{D_2} . \]  

But using a slight variation of the definition of a Wess-Zumino term, \( WZ \), as given
in [9], we can set
\[ WZ(U) = -\int d\phi(\vec{\eta}_{D_1})\mu_{D_1} \]
\[ WZ(V) = -\int d\phi(\vec{\eta}_{D_2})\mu_{D_2} . \] (2.8)

with \((k+1)\)-form field strength \(\tilde{F} = d\phi\). Substituting the definitions of the Wess-Zumino terms above in (2.7), we derive the inequality that we have stated in the introduction*. This inequality is saturated by the calibrated submanifolds of \(N\). In the case of a standard calibration \(d\phi = 0\) and so the above inequality reduces to that of [7] for the volumes of \(U\) and \(V\).

3. Near Horizon M-5-brane Geometries

Let \(\{x^\mu; \mu = 0, \ldots, 5\}\) be the worldvolume coordinates of M-5-brane. The M-5-brane supergravity solution is
\[ ds^2 = H^{-\frac{2}{3}}\eta_{\mu\nu}dx^\mu dx^\nu + H^{\frac{2}{3}}(dr^2 + r^2ds^2(S^4)) \]
\[ G_4 = *dH \] (3.1)

where \(\eta\) is the Minkowski metric on \(\mathbb{R}^{(1,5)}\), \(r\) is a radial coordinate and
\[ H = 1 + \frac{R^3}{r^3} ; \] (3.2)

\(R\) is a constant. Near the position \(r = 0\) of the M-5-brane, the solution becomes
\[ ds^2 \equiv G_{MN}dx^Mdx^N = \frac{r}{R}\eta_{\mu\nu}dx^\mu dx^\nu + \frac{R^2}{r^2}(dr^2 + r^2ds^2(S^4)) \]
\[ G_4 = \mu_{S^4} . \] (3.3)

The near horizon geometry of the M-5-brane [10] is therefore \(AdS_7 \times S^4\).

* We do not consider here the case for which \(\tilde{F}\) is closed but not exact. For our argument, it suffices to assume that in the deformation region \(D\) of \(U\), \(\tilde{F} = d\phi\), where \(\phi\) is the calibration form.
The near horizon M-5-brane above can be used as a background for an M-5-brane probe with transverse scalars \( \{X^a; a = 1, \ldots, 5\} \). The worldvolume action of the probe which is parallel to the background M-5-brane in the static gauge \([11, 12]\) is

\[
S = \int d^6x \sqrt{-\text{det}(g_{\mu\nu})} - \int F. \tag{3.4}
\]

where \( g_{\mu\nu} \) is the pull-back to the worldvolume of the background \( AdS_7 \times S^4 \) metric, i.e.

\[
g_{\mu\nu} = \frac{r}{R} \eta_{\mu\nu} + \frac{R^2}{r^2} \delta_{ab} \partial_\mu X^a \partial_\nu X^b, \tag{3.5}
\]

\[
r^2 = \delta_{ab} X^a X^b, \tag{3.6}
\]

and

\[
F = d\left(\frac{r^3}{R^3}\right) \wedge dt \wedge dx^1 \wedge dx^2 \wedge dx^3 \wedge dx^4 \wedge dx^5. \tag{3.7}
\]

In the above action we have not included the (self-dual) 3-form worldvolume field which we have set to zero.

The field equations for the transverse scalars \( X \) are

\[
-\partial_\mu \left( \frac{R^2}{r^2} \sqrt{-g} g^{\mu\nu} \partial_\nu X^a \right) + \Omega X^a = 0 \tag{3.8}
\]

where

\[
\Omega = \frac{3}{2r} \sqrt{-g} \left( \frac{1}{R} g^{\mu\nu} \eta_{\mu\nu} - \frac{4}{r} \right) - \frac{3r}{R^3}. \tag{3.9}
\]

We shall be considering static solutions to these field equations which have a p-brane interpretation, i.e. solutions that are invariant under the action of the Poincaré group on \( E^{(1,p)} \). For such solution, the transverse scalars \( X \) will depend
on \( \{x^i; i = 1, \ldots, 5-p\} \) worldvolume coordinates. Placing the worldvolume soliton along the first \( p \) worldvolume coordinates of the M-5-brane probe, we find that

\[
\sqrt{-\det(g_{\mu\nu})} = \left( \frac{r}{R} \right)^{\frac{p+1}{2}} \sqrt{\det(g_{ij})} = \sqrt{\det(\tilde{g}_{ij})},
\]

where

\[
g_{ij} = \frac{r}{R} \delta_{ij} + \frac{R^2}{r^2} \delta_{ab} \partial_i X^a \partial_j X^b,
\]

and

\[
\tilde{g}_{ij} = \left( \frac{r}{R} \right)^{\frac{5-p}{2}} g_{ij}.
\]

We remark that the metric \( g \) is the pull back of the metric on \( H^{6-p} \times S^4 \) with respect to \( X \), where \( H^{6-p} \) is a \((6-p)\)-dimensional hyperbolic space. Using all these redefinitions, the non-trivial part of the field equations for the \( p \)-brane worldvolume soliton can be derived by varying the action

\[
S_p = \lambda \left( \int d^{5-p}x \sqrt{\det(\tilde{g}_{ij})} - \int_D \tilde{F} \right),
\]

where

\[
\tilde{F} = d \left( \frac{r^3}{R^3} \right) \wedge dx^1 \wedge \ldots \wedge dx^{5-p}
\]

and \( \lambda \) is a constant. In what follows, we shall apply our calibrations to find solutions to field equations of the action (3.13). We shall find that that there are generalized analogues to well known standard K"ahler, Special Lagrangian (SLAG) and exceptional calibrations in \( \mathbb{R}^n \).

The supersymmetry projection operator is

\[
\Gamma = \frac{1}{6! \sqrt{-g}} \epsilon_{\mu \nu \rho \sigma \tau \lambda} \hat{\gamma}_\mu \hat{\gamma}_\nu \hat{\gamma}_\rho \hat{\gamma}_\sigma \hat{\gamma}_\tau \hat{\gamma}_\lambda
\]

where \( \hat{\gamma}_\mu \) are the pull-backs of the bulk \( \hat{\Gamma} \) matrices. The proportion of supersymmetry preserved by a solution of the field equations of (3.13) is determined by the
number of linearly independent solutions to [13]
\[(1 - \Gamma)\epsilon = 0, \quad (3.16)\]
where $\epsilon$ satisfy the Killing spinor equations for the $AdS_7 \times S^4$ background.

4. AdS Hermitian Calibrations

The AdS Hermitian calibrations are the analogue to the Kähler calibrations on $\mathbb{R}^n$. To define the degree $2\ell$ AdS Hermitian calibrations, we shall take \( N \) to be conformal to the space \( H^{2\ell+1} \times S^{2m-1} \) equipped with the metric
\[
d^2 = \left( \frac{r}{R} \right)^{3-\ell} (ds^2(H^{2\ell+1}) + R^2 ds^2(S^{2m-1})) = \left( \frac{r}{R} \right)^{3-\ell} \left[ \frac{r}{R} ds^2(S^{2\ell}) + \frac{R^2}{r^2} ds^2(S^{2m}) \right], \quad (4.1)\]
where
\[
d^2(H^{2\ell+1}) = \frac{r}{R} ds^2(S^{2\ell}) + \frac{R^2}{r^2} dr^2 \quad (4.2)\]
is the metric on the $(2\ell+1)$-dimensional hyperbolic space. We then use the orthonormal basis
\[
e^A = \left( \frac{r}{R} \right)^{3-\ell} dx^A, \quad 1 \leq A \leq 2\ell, \quad (4.3)\]
\[
e^A = \left( \frac{r}{R} \right)^{3-\ell} dx^A, \quad 2\ell + 1 \leq A \leq 2(\ell + m)\]
of the above metric and define an almost complex structure $I$ with Kähler form
\[
\omega_I = \sum_{q=1}^{\ell+m} e^{2q-1} \wedge e^{2q}. \quad (4.4)\]
In the coordinate basis $\omega_I$ can be written as
\[
\omega_I = \left( \frac{r}{R} \right)^{3-\ell} \sum_{i=1}^{\ell} dx^{2i-1} \wedge dx^{2i} + \left( \frac{r}{R} \right)^{3-\ell} \sum_{a=1}^{m} dx^{2i-1+2\ell} \wedge dx^{2i+2\ell}. \quad (4.5)\]
Using this, we observe that $I$ is a complex structure because it is a constant tensor
in the coordinate basis. We then proceed to define the calibration form

\[ \phi = \frac{1}{\ell!} \wedge^\ell \omega_f. \]  

(4.6)

The form \( \phi \) satisfies Wirtinger’s inequality. This is because in the orthonormal basis the calibration form in the same as that of the Kähler calibration. The subspaces of \( Gr(2\ell, T_pN) \) that saturate the bound are those that are complex with respect to the complex structure \( I \) defined above. So \( \phi \) defines a calibration in the generalized sense with contact set

\[ G(\phi) = U(\ell + m)/U(\ell) \times U(m). \]  

(4.7)

The calibrated submanifolds are the complex submanifolds of \( N \) with respect to the complex structure \( I \).

Our AdS Hermitian calibrated manifolds correspond to M-5-brane worldvolume p-brane solitons for which \( p = 5 - 2\ell \) and with \( 2m \) non-vanishing transverse scalars \( X \). To establish this, it remains to show that \( \tilde{F} = d\phi \), where \( \tilde{F} \) is given in (3.14). This can be verified by a straightforward computation. The argument for computing the supersymmetry preserved by brane worldvolume solitons associated with standard calibrations of \([2, 3]\) can be easily generalized in this case. In particular, we find that the above worldvolume solitons preserve \( 1/2^{\ell+m} \) of spacetime supersymmetry. In an orthonormal frame of the metric \( G \), the projection operators are precisely those that have been found for the standard Kähler case.

4.1. AdS SAS Calibrations

Special Almost Symplectic (SAS) calibrations are the analogues of the standard Special Lagrangian (SLAG) calibrations. To define a degree \( \ell \) AdS SAS calibrations, we shall take \( N \) to be conformal to the space \( H^{\ell+1} \times S^{\ell-1} \) equipped with
metric
\[ ds^2 = \left( \frac{r}{R} \right)^{\frac{4}{\ell}} (ds^2(H^{(\ell+1)}) + R^2 ds^2(S^{\ell-1})) \]
\[ = \left( \frac{r}{R} \right)^{\frac{4}{\ell}} \left[ \frac{r}{R} ds^2(\mathbb{H}^\ell) + \frac{R^2}{r^2} ds^2(\mathbb{S}^\ell) \right], \]  
(4.8)

where
\[ ds^2(H^{(\ell+1)}) = \frac{r}{R} ds^2(\mathbb{H}^\ell) + \frac{R^2}{r^2} dr^2 \]  
(4.9)
is the metric on the \((\ell + 1)\)-dimensional hyperbolic space. We then use the orthonormal basis
\[ e^A = \left( \frac{r}{R} \right)^{\frac{4}{\ell}} dx^A, \quad 1 \leq A \leq \ell, \]
\[ e^A = \left( \frac{r}{R} \right)^{\frac{4}{\ell}} d\ell dx^A, \quad \ell + 1 \leq A \leq 2\ell \]  
(4.10)
of the above metric and define an almost symplectic form
\[ \omega = \sum_{q=1}^{\ell} e^q \wedge e^{q+\ell}. \]  
(4.11)

Let \( I \) be the almost complex structure with Kähler form \( \omega \). The SAS calibration form is
\[ \phi = \text{Re}(e^1 + ie^{\ell+1}) \wedge \ldots \wedge (e^\ell + ie^{2\ell}) \]  
(4.12)

Since the form \( \phi \) coincides in an orthonormal basis with that used as a calibration form in the SLAG case, it satisfies the same inequalities as those that can be found in [6]. These inequalities are saturated by the special Lagrangian \( \ell \)-planes in \( Gr(\ell, \mathbb{R}^{2\ell}) \). So \( \phi \) defines a calibration in the generalized sense with contact set
\[ G(\phi) = SU(\ell)/SO(\ell). \]  
(4.13)

A consequence of this is that
\[ \text{Im}(e^1 + ie^{\ell+1}) \wedge \ldots \wedge (e^\ell + ie^{2\ell}) = 0, \]  
(4.14)
and conversely, any plane that satisfies (4.14) is special Lagrangian.
To derive the equations that our calibrated manifolds satisfy, we first rewrite the almost symplectic form in a coordinate basis as

$$\omega = \left(\frac{r}{R}\right)^{\frac{12-\ell}{2\ell}} \sum_{i=1}^{\ell} dx^i \wedge dy_i$$

(4.15)

where $y_i = x^{i+\ell}$. We expect that the calibrated manifolds $M$ can be locally expressed as $y_i = y_i(x^j)$. Moreover, the pull-back of $\omega$ on $M$ must vanish. This implies that there is a function $f = f(x^i)$ such that

$$y_i = \partial_i f.$$  

(4.16)

Substituting this into (4.14), we find that the function $f$ satisfies

$$\Im \left[ \det \left( \delta_{ij} + i \frac{R^3}{r} \partial_i \partial_j f \right) \right] = 0,$$

(4.17)

where

$$r^2 = \delta^{ij} \partial_i f \partial_j f.$$ 

(4.18)

Degree two SAS calibrations are the same as the degree two AdS Hermitian calibrations investigated in the previous section. For degree three SAS calibrations the equation (4.17) reduces to

$$\frac{(\delta^{mn} \partial_m f \partial_n f)^2}{R^3} \delta^{ij} \partial_i f \partial_j f = \det(\partial_i \partial_j f),$$

(4.19)

where $i, j, m, n = 1, 2, 3$. This equation corresponds to the Monge–Ampère equation of the degree three SLAG calibrations. To find solutions to this equation, we assume that $f$ is spherically symmetric, i.e. $f = f(u)$ where $u^2 = \delta_{ij} x^i x^j$. Then
the equation (4.19) becomes
\[
\left( \frac{u^2 f'}{R^3} - 1 \right) f'' + 2u \frac{f'}{R^3} f^2 = 0
\]  
(4.20)
where \( f' = \frac{d}{du} f \). A solution is
\[
f(u) = -R^3 \int_{u_0}^{u} \frac{1}{v^2} L(\beta v^2) dv ,
\]  
(4.21)
where \( u_0, \beta \) are constants and \( L(v) \) is the principal branch of the Lambert function which is analytic at \( v = 0 \) and satisfies
\[
L(v)e^{L(v)} = v .
\]  
(4.22)
For degree four SAS calibrations, the equation (4.17) reduces to
\[
\frac{(\delta^{mn}\partial_m f \partial_n f)^2}{R^3} \delta^{ij} \partial_i \partial_j f = \sum_k \text{Det}_{k|k}(\partial_i \partial_j f) 
\]  
(4.23)
where \( i, j, m, n = 1, 2, 3, 4 \) and \( \text{Det}_{k|k}(\partial_i \partial_j f) \) is the determinant of the matrix \( (\partial_i \partial_j f) \) with the \( k \)-column and \( k \)-row missing. A spherically symmetric solution is
\[
f(u) = \frac{5R^3}{u}
\]  
(4.24)
where \( u^2 = \delta_{ij} x^i x^j \).

Our AdS SAS calibrated manifolds correspond to p-brane worldvolume M-5-brane solitons for which \( p = 5 - \ell \) and with \( \ell \) non-vanishing transverse scalars \( X \). To establish this, it remains to show that \( \tilde{F} = d\phi \) for \( \ell = 5 - p \), where \( \tilde{F} \) is given in (3.14). This can be verified by a straightforward computation for the cases \( \ell = 3, 4 \) but it is not the case for \( \ell = 5 \). The argument for computing the supersymmetry preserved by brane worldvolume solitons associated with standard calibrations of [2, 3] can be easily generalized in this case. In particular, we find that the SAS worldvolume M-5-brane solitons preserve \( 1/2^\ell \) of spacetime supersymmetry. In the orthonormal frame of the metric \( G \), the projection operators are precisely those that have been found for the standard SLAG case.
5. AdS Exceptional Calibrations

The construction of AdS exceptional calibrations proceeds in the same way as for Hermitian and for SAS calibrations. One begins with the standard constant exceptional calibration form in each case and constructs the calibration form of the AdS calibration by contracting it with an orthonormal basis of the associated curved metric. It is clear that all the necessary inequalities follow by those presented in [6] for the exceptional cases. The contact sets of the AdS exceptional calibrations are those of the standard exceptional ones. In all these cases $d\phi = \tilde{F}$ and so the calibrated manifolds have a p-brane worldvolume soliton interpretation. The supersymmetry preserved by each such solution is the same as that for p-brane worldvolume solitons associated with the standard exceptional calibrations. In what follows we shall simply state the metric on the manifold $N$ and the equations which the calibrated manifolds satisfy.

5.1. AdS Cayley Calibrations

The AdS Cayley calibration is a degree four calibration in $N$ equipped with metric

$$d\tilde{s}^2 = \frac{\delta^2}{R^2} \delta_{ij} dx^i dx^j + \frac{R^3}{r^2} \delta_{ij} dy^i dy^j,$$

where $r^2 = \delta_{ij} y^i y^j$ and $i, j = 1, \ldots, 4$. The calibration four-form is constructed from the Spin(7) invariant self-dual form on $\mathbb{R}^8$. The calibrated manifolds can be locally expressed as $y^i = y^i(x)$ and $y^i$ satisfy the modified Cayley equations as follows:

$$\left(\frac{R}{r}\right)^3 (\partial_1 \Theta - \partial_2 \Theta i - \partial_3 \Theta j - \partial_4 \Theta k) = \partial_2 \Theta \times \partial_3 \Theta \times \partial_4 \Theta + \partial_1 \Theta \times \partial_3 \Theta \times \partial_4 \Theta i$$
$$- \partial_1 \Theta \times \partial_2 \Theta \times \partial_4 \Theta j + \partial_1 \Theta \times \partial_2 \Theta \times \partial_3 \Theta k$$

and

$$\text{Im}[ (\partial_1 \Theta \times \partial_2 \Theta - \partial_3 \Theta \times \partial_4 \Theta) i + (\partial_1 \Theta \times \partial_3 \Theta + \partial_2 \Theta \times \partial_4 \Theta) j$$
$$+ (\partial_1 \Theta \times \partial_4 \Theta - \partial_2 \Theta \times \partial_3 \Theta) k] = 0$$
where $\Theta = y^1 + iy^2 + jy^3 + ky^4$, $a \times b \times c = \frac{1}{2}(\bar{a}bc - c\bar{b}a)$, $a \times b = -\frac{1}{2}(\bar{a}b - ba)$ and $i, j, k$ are the imaginary unit quaternions.

5.2. AdS Associative Calibrations

The AdS associative calibration is a degree three calibration in $N$ with metric

$$d\tilde{s}^2 = \frac{r^2}{R^2}\delta_{ij}dx^i dx^j + \frac{R}{r}\delta_{ab}dy^a dy^b,$$  \hspace{1cm} (5.4)

where $r^2 = \delta_{ab}y^a y^b$, $a, b = 1, \ldots, 4$ and $i, j = 1, \ldots, 3$. The calibration form can be constructed from the $G_2$ invariant three-form on $\mathbb{R}^7$ in the way that we have explained above. The calibrated manifolds can be locally expressed as $y^a = y^a(x)$ and $y^a$ satisfy the following equations:

$$-(\frac{r}{R})^3(\partial_1 \Theta i + \partial_2 \Theta j + \partial_3 \Theta k) = \partial_1 \Theta \times \partial_2 \Theta \times \partial_3 \Theta$$  \hspace{1cm} (5.5)

where $\Theta = y^1 + iy^2 + jy^3 + ky^4$ and $\times$ are defined as for the Cayley case above.

5.3. AdS Coassociative Calibrations

The AdS coassociative calibration is a degree four calibration in $N$ equipped with metric

$$d\tilde{s}^2 = \frac{r^3}{R^2}\delta_{ij}dx^i dx^j + \frac{R^3}{r^2}\delta_{ab}dy^a dy^b,$$  \hspace{1cm} (5.6)

where $r^2 = \delta_{ab}y^a y^b$, $a, b = 1, \ldots, 3$ and $i, j = 1, \ldots, 4$. The calibration form can be constructed from the $G_2$ invariant four-form on $\mathbb{R}^7$ in the way that we have explained above. The calibrated manifolds can be locally expressed as $y^a = y^a(x)$ and $y^a$ satisfy the following equations:

$$-(\frac{r}{R})^3(\partial X i + \partial Y j + \partial Z k) = \partial X \times \partial Y \times \partial Z$$  \hspace{1cm} (5.7)

where $X = y^1$, $Y = y^2$, $Z = y^3$ and $\partial = \partial_1 + i\partial_2 + j\partial_3 + k\partial_4$. A well-known example of a standard co-associative calibrated manifold is the Lawson-Osserman
co-associative cone. We may generalize this solution by considering the ansatz

\[ X_i + Y_j + Z_k = \text{Im}(f(u)\bar{x}i) , \]  \hspace{1cm} (5.8)

where \( x = x^1 + ix^2 + jx^3 + kx^4 \) and \( u^2 = \delta_{ij}x^i x^j \). This solves the equation (5.7) for

\[ f(u) = -\frac{4R^3}{u^4} + \frac{R^3e^\alpha}{2u^6}(e^\alpha \pm \sqrt{e^\alpha - 16u^2}) , \]  \hspace{1cm} (5.9)

where \( \alpha \) is a constant.

6. Conclusions

We have found that the investigation of M-5-brane supersymmetric worldvolume solitons on \( AdS_7 \times S^4 \) background requires the generalization of calibrations due to the presence of Wess-Zumino terms in the M-5-brane worldvolume action. We have presented such a generalization of calibrations and we have established a bound that is saturated by the calibrated manifolds. Our construction is general but when it is applied to the case of M-5-brane, the functional that is minimized is related to the worldvolume action without Born-Infeld type fields. We have also found that there is a correspondence between calibrations on \( AdS_7 \times S^4 \) and calibrations on \( \mathbb{R}^{(1,10)} \) which induces a correspondence between the equations satisfied by these calibrations.

There are many other cases to consider. For example solutions to brane probe actions in the near horizon geometries of the M-2-brane and the D-3-brane; some of the AdS Hermitian calibrations for the D-3-brane have been investigated in [14, 15]. Our construction can be generalized to treat both these cases. Since brane probes in generic string and M-theory backgrounds which have non-vanishing form field strengths couple with terms that include the Wess-Zumino ones, the investigation of supersymmetric worldvolume solutions may require a further generalization of calibrations. One such example is the case of D-brane probes in NS\( \otimes \)NS backgrounds. Such backgrounds have been investigated from the conformal field theory.
point of view in [16]. These may have applications in strings, M-theory and differential geometry.

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