SLOPE INEQUALITIES AND A MIYAOKA-YAU TYPE INEQUALITY

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Abstract. We prove several slope inequalities for a relative minimal surface fibration in positive characteristic. As an application, we prove a Miyaoka-Yau type inequality \( \chi(\mathcal{O}_S) \geq \frac{p^2-4p-1}{4(p+1)(p-1)}K_S^2 \) for all minimal surface \( S \) of general type in characteristic \( p \geq 5 \) and the equality holds for Raynaud’s examples. Some similar inequalities are also established for \( p = 2, 3 \), which answer completely a question of Shepherd-Barron on the positivity of \( \chi(\mathcal{O}_S) \).

1. Introduction

Let \( S \) be a smooth projective surface over an algebraically closed field \( k \) of characteristic \( p > 0 \). By a fibration of \( S \) over a smooth projective curve \( C \), we mean a flat morphism \( f : S \to C \) such that \( f_*\mathcal{O}_S = \mathcal{O}_C \). In particular, the general fibre of \( f : S \to C \) could be singular in case of positive characteristic. Let \( \omega_{S/C} := \omega_S \otimes f^*\omega_C^{-1} \) be the relative canonical sheaf of \( f \), and \( K_{S/C} := K_S - f^*K_C \) be the relative canonical divisor. Then \( K_{S/C}^2 = K_S^2 - 8(g-1)(b-1) + l \) and

\[
\chi_f := \deg(f_*\omega_{S/C}) = \chi(\mathcal{O}_S) - (g-1)(b-1) + l
\]

where \( l := \dim_k(R^1f_*\mathcal{O}_S)_{\text{tor}} \) is the torsion dimension of the direct image sheaf \( R^1f_*\mathcal{O}_S \). The fibration \( f \) is called relatively minimal if \( S \) contains no \((-1)\)-curve in fibres. Then we have the so called slope inequality.

Theorem 1. Let \( f : S \to C \) be a relatively minimal fibration, then

\[
K_{S/C}^2 \geq \frac{4(g-1)}{g}\chi_f.
\]

When \( \text{char}(k) = 0 \), this inequality was proved by Xiao (see [14]) and independently by Cornalba-Harris for semi-stable fibration (see [2]). When \( \text{char}(k) = p > 0 \), there exist a few approaches to this inequality.
We point out that all of them require the condition that the generic fibre of \( f : S \to C \) is smooth. In a recent paper [13], the authors have generalized Xiao’s approach to cover the case of characteristic \( p > 0 \). Although their theorem contains the assumption that \( f \) has smooth generic fibre, their method can be generalized to cover arbitrary relatively minimal fibration \( f \).

For a minimal smooth projective surface \( S \) of general type over an algebraically closed field of characteristic \( p > 0 \), one still has
\[
K_S^2 > 0
\]
by Bombieri-Mumford’s classification, and the Noether formula
\[
12\chi(O_S) = K_S^2 + c_2(S).
\]
But one should not expect the Miyaoka-Yau inequality
\[
c_2(S) \geq \frac{1}{3} K_S^2,
\]
or equivalently
\[
\chi(O_S) \geq \frac{1}{9} K_S^2
\]
since Raynaud’s examples show that there exist minimal smooth projective surfaces \( S \) of general type with \( c_2(S) < 0 \). It is a natural question if \( \chi(O_S) > 0 \) for all minimal smooth projective surfaces \( S \) of general type? If \( \chi(O_S) < 0 \), Shepherd-Barron has shown (see [12, Theorem 8]) that there is a fibration \( f : S \to C \) with singular general fibres of arithmetic genus \( 2 \leq g \leq 4 \) and \( p \leq 7 \). But the question if there exist surfaces \( S \) of general type with \( \chi(O_S) < 0 \) remains unsolved (see the Remark at page 268 of [12]). Shepherd-Barron also suggested that the most obvious place to look for such examples would be in the case where \( (p, g) = (2, 2) \). Indeed, it is proved in [3] that \( \chi(O_S) > 0 \) when \( p \geq 3 \). As an application of Theorem 1, we have

**Theorem 2.** Let \( S \) be a minimal smooth projective surface of general type over an algebraically closed field \( k \) of characteristic \( p > 0 \). Then

1. if \( p \geq 5 \), then we have the optimum inequality
   \[
   \chi(O_S) \geq \frac{p^2 - 4p - 1}{4(3p + 1)(p - 3)} K_S^2
   \]
   and the equality holds if \( S \) is one of Raynaud’s examples (it was conjectured in [3]);
2. if \( p = 3 \), then \( \chi(O_S) > \frac{1}{32} K_S^2 \).
3. if \( p = 2 \), we also have the optimum inequality \( \chi(O_S) \geq \frac{1}{12} K_S^2 \).
   An example is given at the end of §5 where the equality holds.

In particular, \( \chi(O_S) > 0 \) holds for all smooth projective surfaces \( S \) of general type.

The above application shows that characteristic \( p \) version of slope inequalities may be useful tools for the study of algebraic surfaces in
positive characteristic. Thus it is deserved to generalize the following result for non-hyperelliptic fibration to positive characteristic by using the characteristic \( p \) version of Xiao’s approach and some ideas similar to null characteristic.

1. \( f \) is non-hyperelliptic and \( g = 3 \): \( K_{S/C}^2 \geq 3 \chi_f \);
2. \( f \) is non-hyperelliptic and \( g = 4 \): \( K_{S/C}^2 \geq \frac{24}{7} \chi_f \);
3. \( f \) is non-hyperelliptic and \( g = 5 \): \( K_{S/C}^2 \geq \frac{40}{11} \chi_f \).

These inequalities over complex field \( \mathbb{C} \) was proved by: (1) is by Horikawa (see [4]) and Konno (see [5]). (2) & (3) was proved by Konno (see [6]) and (2) is independently proved by Chen (see [1]). We would like to remind the readers that in characteristic \( p > 0 \), both sides of all these inequalities along with Xiao’s can be negative. As a consequence and for example, in case \( f \) is non-hyperelliptic and \( g = 3 \), one can not say that (1) is better than Xiao’s.

This paper is organized as follows. Some notations are assumed and Clifford’s Theorem for Gorenstein curve is recalled in § 2. In § 3 we recall the characteristic \( p \) version of Xiao’s approach and point out that smoothness of its generic fibre is unnecessary. In § 4 we prove several slope inequalities for non-hyperelliptic fibrations in positive characteristic. In § 5 we apply slope inequality to study algebraic surfaces \( S \) of general type with \( c_2(S) < 0 \). As a consequence, we prove that \( \chi(O_S) > 0 \) for any characteristic \( p > 0 \) which answer a question of Shepherd-Barron. Finally, Raynaud’s examples are presented in § 6 and we show that if \( p \geq 5 \), the equality of \( \chi(O_S) = \frac{p^2 - 4p - 1}{4(3p + 1)(p - 3)} K_{S}^2 \) holds when \( S \) is one of Raynaud’s examples.

2. Notations

Let us assume the following notions:
- \( k \): an algebraically closed field of characteristic \( p > 0 \);
- \( C \): a proper nonsingular algebraic curve of genus \( b := g(C) \) over \( k \);
- \( S \): a proper nonsingular algebraic surface over \( k \);
- \( f : S \to C \) a relatively minimal fibration with \( f_* O_S = O_C \), in particular, all geometric fibres are connected;
- \( K := K(C) \) the function field of \( C \);
- \( c \): a general closed point of \( C \);
- \( F := S_c \) a general closed fibre of \( f \), in particular, it is integral;
- \( \omega_{S/C} \): the relative dualizing sheaf and \( K_{S/C} \) the relative canonical divisor;
- \( g := p_a(F) \) the arithmetic genus of fibre \( F \).
\( \chi_f := \deg(f_*\omega_{S/C}) \);

- \( l := \dim_k(R^1f_*\mathcal{O}_S)_{\text{tor}} \) the torsion dimension of the higher direct image sheaf \( R^1f_*\mathcal{O}_S \).

**Definition 1.** An irreducible, reduced and Gorenstein curve over \( k \) is called hyperelliptic if it admits a double cover to \( \mathbb{P}_k^1 \).

The above fibration \( f : S \to C \) is called hyperelliptic if the general fibre \( F \) is so.

**Theorem 3** (Clifford’s Theorem, see [8]). Let \( D \) be a special Cartier divisor on an irreducible, reduced and Gorenstein curve \( V \), then

\[
h^0(D) \leq \frac{1}{2} \deg D + 1
\]

and the equality holds only if either \( D = 0 \) or \( D = K_V \), or \( V \) is hyperelliptic.

**Remark 1.**

1. We have \( l = 0 \) unless there is some multiple fibre of multiplicity divided by \( p \).
2. We do not assume the generic fibre \( F \) is smooth unless otherwise stated in this paper.

3. Xiao’s approach of slope inequality in positive characteristic

In the paper [14], Xiao introduces an approach of slope inequality for surface fibrations by studying the Harder-Narasimhan filtration of the direct image \( f_*\omega_{S/C} \). For readers’ convenience, we briefly recall the idea.

For a vector bundle \( E \) on a smooth projective curve \( C \), let

\[
\mu(E) = \frac{\deg(E)}{\rk(E)}
\]

where \( \rk(E) \) and \( \deg(E) \) denote the rank and degree of \( E \) respectively. \( E \) is called semi-stable (resp., stable) if for any nontrivial subbundle \( E' \subset E \), one has \( \mu(E') \leq \mu(E) \) (resp., \( < \)). If \( E \) is not semi-stable, one has the following well-known theorem.

**Theorem 4.** (Harder-Narasimhan filtration) For any vector \( E \) on \( C \), there exists a unique filtration

\[
0 := E_0 \subset E_1 \subset \cdots \subset E_n = E
\]

which is the so-called Harder-Narasimhan filtration, such that

1. each subquotient \( E_i/E_{i-1} \) is semi-stable for \( 1 \leq i \leq n \),
2. \( \mu_1 > \cdots > \mu_n \), where \( \mu_i := \mu(E_i/E_{i-1}) \) for \( 1 \leq i \leq n \).
By using Harder-Narasimhan filtration
\[ 0 := E_0 \subset E_1 \subset \cdots \subset E_n = E = f^* \omega_{S/C} \]
of \( f^* \omega_{S/C} \), Xiao constructs a sequence of effective divisors
\[ Z_1 \geq Z_2 \geq \cdots \geq Z_n \geq 0 \]
such that \( N_i = K_{S/C} - Z_i - \mu_i F \ (1 \leq i \leq n) \) are nef \( \mathbb{Q} \)-divisors. Then he uses the following elementary lemma to get a lower bound of \( K_{S/C}^2 \).

**Lemma 1.** ([14, Lemma 2]) Let \( f : S \to C \) be a relatively minimal fibration, with a general fibre \( F \). Let \( D \) be a divisor on \( S \), and suppose that there are a sequence of effective divisors
\[ Z_1 \geq Z_2 \geq \cdots \geq Z_n \geq Z_{n+1} = 0 \]
and a sequence of rational numbers
\[ \mu_1 > \mu_2 > \cdots > \mu_n, \quad \mu_{n+1} = 0 \]
such that for every \( i \), \( N_i = D - Z_i - \mu_i F \) is a nef \( \mathbb{Q} \)-divisor. Then
\[ D^2 \geq \sum_{i=1}^{n} (d_i + d_{i+1})(\mu_i - \mu_{i+1}), \]
where \( d_i = N_i \cdot F \).

It is important that semi-stability of \( E_i/E_{i-1} \) implies nefness of
\[ \mathcal{O}_{\mathbb{P}(E_i)}(1) - \mu_i \Gamma_i \]
where \( \Gamma_i \) is a fibre of \( \mathbb{P}(E_i) \to C \) when \( \text{char.}(k) = 0 \). A key observation of [13] is that strongly semi-stability of \( E_i/E_{i-1} \) implies nefness of
\[ \mathcal{O}_{\mathbb{P}(E_i)}(1) - \mu_i \Gamma_i \]
when \( \text{char.}(k) = p > 0 \). Recall, let \( F_C : C \to C \) be the (absolute) Frobenius morphism, a bundle \( E \) on \( C \) is called strongly semi-stable (resp., stable) if its pullback by \( k \)-th power \( F_C^k \) is semi-stable (resp., stable) for any integer \( k \geq 0 \).

**Lemma 2.** ([7, Theorem 3.1]) For any bundle \( E \) on \( C \), there exists an integer \( k_0 \) such that all quotients \( E_i/E_{i-1} \ (1 \leq i \leq n) \) appearing in the Harder-Narasimhan filtration
\[ 0 := E_0 \subset E_1 \subset \cdots \subset E_n = F_C^{k_0} f^* \omega_{S/C} \]
are strongly semi-stable whenever \( k \geq k_0 \).
Lemma 3. For each sub-bundle $E_i$ in the Harder-Narasimhan filtration

$$0 := E_0 \subset E_1 \subset \cdots \subset E_n = F_C^{k*} f_* \omega_{S/C}$$

when $k \geq k_0$, the divisor $O_{\mathbb{P}(E_i)}(1) - \mu_i \Gamma_i$ is a nef $\mathbb{Q}$-divisor, where $\Gamma_i$ is a fibre of $\mathbb{P}(E_i) \to C$ and $\mu_i = \mu(E_i/E_{i-1})$.

Proof. The proof follows a slight modification of its null characteristic counterpart in [9, Theorem 3.1] with the notification that pull-backs of strongly semi-stable bundles under a finite morphism are still strongly semi-stable. □

Then Xiao’s approach applies in positive characteristic, the assumption that $f : S \to C$ has smooth general fibre in the previous work [13, Theorem 6] is unnecessary. Indeed, we have

Theorem 5. Let $f : S \to C$ be a relatively minimal fibration over an algebraically closed field of positive characteristic, then

$$K_S^2 \geq \frac{4(g - 1)}{g} \chi_f.$$ 

Proof. One can prove it by following the same argument in the proof of [13, Theorem 6], and we give a sketch here. Take the following commutative diagram (for any integer $k \geq k_0$):

$$
\begin{array}{ccc}
S & \xrightarrow{F_C^k} & S \\
f \downarrow & & f \\
C & \xrightarrow{F_C^k} & C
\end{array}
$$

Let $0 := E_0 \subset E_1 \subset \cdots \subset E_n = F_C^{k*} E$ be the Harder-Narasimhan filtration of $F_C^{k*} E = F_C^{k*} f_* \omega_{S/C}$ with $r_i = \text{rk}(E_i)$ and $\mu_i = \mu(E_i/E_{i-1})$ for $1 \leq i \leq n$. By Lemma 2 all quotients $E_i/E_{i-1}$ appears in above filtration are strongly semi-stable.

Let $L_i \subset F_C^{k*} \mathcal{O}_S(K_{S/C})$ be the image of $f^* E_i$ under homomorphism

$$f^* E_i \hookrightarrow f^* F_C^{k*} E = F_C^{k*} f^* \mathcal{O}_S(K_{S/C}) \to F_S^{k*} \mathcal{O}_S(K_{S/C}) = \mathcal{O}_S(p^k K_{S/C}),$$

which is a torsion-free sheaf of rank 1 and is locally free on an open set $U_i \subset S$ of codimension at least 2. Thus there is a morphism (over $C$)

$$\phi_i : U_i \to \mathbb{P}(E_i)$$

such that $\phi_i^* \mathcal{O}_{E_i}(1) = L_i|_{U_i}$, which implies that $c_1(L_i) - \mu_i F$ is nef by Lemma 3 where $F$ is a general fibre of $f : S \to C$. Let

$$p^k K_{S/C} = c_1(L_i) + Z_i, \quad 1 \leq i \leq n.$$
Then we get a sequence of effective divisors $Z_1 \geq Z_2 \geq \cdots \geq Z_n \geq 0$ and a sequence of rational numbers $\mu_1 > \mu_2 > \cdots > \mu_n$ such that

$$N_i = p^k K_{S/C} - Z_i - \mu_i F \quad (1 \leq i \leq n)$$

are nef $\mathbb{Q}$-divisors. Let $d_i = N_i \cdot F = \text{deg} (\mathcal{L}_i|_F)$, then

$$d_n = p^k K_{S/C} \cdot F := d_{n+1}$$

since $\omega_{S/C}|_F$ is generated by global sections and $Z_n$ is supported on fibres of $f : S \to C$. For $1 \leq i < n$, there are $r_i = \text{rk} (E_i)$ sections

$$\{s_{1i}, \ldots, s_{ri}\} \in H^0(\mathcal{O}_S(K_{S/C})|_F)$$

such that $\mathcal{L}_i|_F \subset \mathcal{O}_S(p^k K_{S/C})|_F$ is generated by the global sections $s_{1i}^p, \ldots, s_{ri}^p$. Since $\mathcal{O}_S(K_{S/C})|_F$ is special, the sub-sheaf $L_i \subset \mathcal{O}_S(K_{S/C})|_F$ generated by $\{s_{1i}, \ldots, s_{ri}\} \in H^0(\mathcal{O}_S(K_{S/C})|_F)$ is special. Although we do not assume that general fibre $F$ is smooth, Clifford theorem (see Theorem 3) holds and $\text{deg}(L_i) \geq 2r_i - 2$. Then we have

$$d_i = N_i \cdot F = \text{deg}(\mathcal{L}_i|_F) = p^k \text{deg}(L_i) \geq p^k(2r_i - 2) \quad (1 \leq i \leq n),$$

which and Lemma 1 imply the slope inequality.

**Remark 2.** From above proof, we see that the equality holds only if

- either the general fibre $F$ is hyperelliptic;
- or $f^* \omega_{S/C}$ is strongly semistable.

**Proposition 1.** Assume $\chi_f > 0$, then the slope inequality

$$K_{S/C}^2 \geq \frac{4g - 4}{g} \chi_f$$

holds without equality unless $f$ is hyperelliptic.

**Proof.** From above remark, we see that $K_{S/C}^2 = \frac{4g - 4}{g} \chi_f$ unless either $f$ is hyperelliptic or $f^* \omega_{S/C}$ is strongly semi-stable. It follows from [13, Proposition 1] that if $f^* \omega_{S/C}$ is strongly semi-stable, then

$$K_{S/C}^2 \geq \frac{5g - 6}{g} \chi_f.$$

Thus there is no equality when both $\chi_f > 0$ and $f$ is non-hyperelliptic.

**Remark 3.** The RHS of (3.2) in [13, Proposition 1] should be replaced by $K_{S/B}^2 + \text{deg}(f^* \omega_{S/B}) - l$, where $l$ is the torsion dimension of $R^1 f_* \mathcal{O}_S$. However this mistake does not affect the proof and its conclusion.
4. Slope inequalities for non-hyperelliptic fibrations

Over $\mathbb{C}$, after Xiao’s approach, another technique named counting relative hyperquadrics is used in developing the slope inequality. In the paper [6], such a method is systematically studied and the following slope inequalities for non-hyperelliptic fibrations are proved.

1. If $g = 3$, then $K^2_{S/C} \geq 3\chi_f$ (see also [4], [5]);
2. If $g = 4$, then $K^2_{S/C} \geq \frac{24}{7} \chi_f$ (see also [1]);
3. If $g = 5$, then $K^2_{S/C} \geq \frac{40}{11} \chi_f$.

The method in [6] can however be modified to work in positive characteristic to obtain the same slope inequalities. For the restriction of context, we shall not go through all the detailed arguments for above three inequalities. Instead, we mainly take the case $g = 3, 4$ as an example to show how Konno’s method in [6] is modified and applied in positive characteristic.

In the following we assume that $f : S \to C$ is a non-hyperelliptic fibration of genus $g$ defined over an algebraically closed field $k$ of characteristic $p > 0$. The main result in this section is the following generalization of slope inequalities.

**Theorem 6.** Let $f : S \to C$ be a non-hyperelliptic fibration over an algebraically closed field $k$ of positive characteristic. We have

1. If $g = 3$, then $K^2_{S/C} \geq 3\chi_f$;
2. If $g = 4$, then $K^2_{S/C} \geq \frac{24}{7} \chi_f$;
3. If $g = 5$, then $K^2_{S/C} \geq \frac{40}{11} \chi_f$.

**Remark 4.** In the original statement of the three slope inequalities above, the gonality of $F$ is considered. Here for simplicity, we shall not go that much further.

Recall, for non-hyperelliptic fibration $f : S \to C$, the second multiplication map: $S^2(f_*\omega_{S/C}) \to f_*(\omega_{S/C}^\otimes 2)$ is generically surjective by Max Noether’s theorem. On the other hand, we have

$$\deg(f_*\omega_{S/C}^\otimes 2) = \chi_f + K^2_f - l$$

where $l$ is the torsion dimension of $R^1 f_*\mathcal{O}_S$.

4.1. **Genus 3.** When $g = 3$, the second multiplication map

$$S^2(f_*\omega_{S/C}) \to f_*(\omega_{S/C}^\otimes 2)$$
is an isomorphism outside finitely many points on $C$, so we have:
\[
\deg(f_*\omega_{S/C}^{\otimes 2}) \geq \deg S^2(f_*\omega_{S/C}) = 4\chi_f.
\] (2)

Combining this inequality with (1), we immediately have
\[
K_{S/C}^2 \geq 3\chi_f.
\]

4.2. Genus 4. The inequality $K_{S/C}^2 \geq \frac{24}{7}\chi_f$ in this case is proved for complex field $\mathbb{C}$ in both [1] and [6]. We shall here mainly follow the idea of [6] which can shed more lights on $g = 5$.

Given a vector bundle $E$ on $C$, we shall use the following notations:
- $\pi : \mathbb{P}(E) \to C$ is the projective bundle associated to $E$;
- $D$ is a general fibre of $\pi$, it is a projective variety of dimension $\text{rank}(E) - 1$;
- a relative hyperquadric $Q$ associated to $\delta \in \text{Pic}(C)$ is an effective divisor in the linear system $|O(2) \otimes \pi^*\delta|$ on $\mathbb{P}(E)$.
- for a relative hyperquadric $Q$, its rank is defined to be rank of the smallest subbundle $E' \subseteq E$ such that $Q$ is defined over $E'$.

In the paper [6], the author firstly gives several key lemmas (Lemma 1.1–1.6) on relative hyperquadrics in characteristic zero. By using of the existence of Harder-Narasimhan filtrations with strongly semi-stable sub-quotients (see Lemma 2), we can generalize these lemmas to the case of positive characteristic.

Let $0 = E_0 \subsetneq E_1 \subsetneq \ldots \subsetneq E_n = (F^k)^*E$ be the Harder-Narasimhan filtration of $(F^k)^*E$ for some $k \gg 0$ such that the subquotient bundle $E_i/E_{i-1}, i = 1, \ldots, n$ are all strongly semi-stable. We write
\[
\mu_i = \frac{1}{p^k}\mu(E_i/E_{i-1}), \ r_i = \text{rank}(E_i);
\]
and define $\nu_j := \mu_i$ for $r_{i-1} < j \leq r_i$ and $i = 1, \ldots, n$. With these $\mu_i$, all Lemma 1.1–1.6 in [6] hold in positive characteristic. Here we include Lemma 1.5 & 1.6 as an example.

**Lemma 4.** ([6] Lemma 1.5 & 1.6) Suppose $Q$ is a relative hyperquadric associated to $\delta \in \text{Pic}(C)$ and $x = -\deg \delta$. We have

1. if the rank of $Q$ is at least 3, then $x \leq \min\{2\nu_2, \nu_1 + \nu_3\}$;
2. if the rank of $Q$ is at least 4, then $x \leq \min\{\nu_2 + \nu_3, \nu_1 + \nu_4\}$.

**Proof of Theorem 2 when $g = 4$.** We may firstly assume $\chi_f > 0$, otherwise, Xiao’s inequality (see Theorem 5) $K_{S/C}^2 \geq 3\chi_f$ is already stronger than the desired one.
Take $E = f_*\omega_{S/C}$ for short. Consider the second multiplication map, we have the following exact sequence:

$$0 \to K \to S^2(E) \xrightarrow{\tau} f_*\omega_{S/C}^\otimes \to C \to 0 \quad (3)$$

where $K$ is an invertible sheaf and $C$ is a skyscraper sheaf. It therefore gives a relative hyperquadric $Q \in |O(2) \otimes \pi^*\mathcal{K}^{-1}|$. Let $x := \deg \mathcal{K}$, recall that $\deg f_*\omega_{S/C}^\otimes = K_{S/C}^2 + \chi_f - l$ with $l = \dim (R^1 f_*\mathcal{O}_S)_{\text{tor}}$. Thus $K_{S/C}^2 - 4\chi_f - l + x = \dim \mathcal{C}$ by (3). In particular,

$$K_{S/C}^2 \geq 4\chi_f - x. \quad (4)$$

We can next assume that $E$ is not strongly semi-stable, otherwise

$$K_{S/C}^2 \geq \frac{7}{2}\chi_f > \frac{24}{7}\chi_f$$

by [13, Proposition 1]. Using $\mu_i, \nu_j$ introduced above, we can still apply the trick of Xiao as in §3. Indeed, we have

$$K_{S/C}^2 \geq \sum_{i=1}^{n-1} (d_i + d_{i+1})(\mu_i - \mu_{i+1}) + 12\mu_n \quad (5)$$

and

$$K_{S/C}^2 \geq (d_1 + 6)(\mu_1 - \mu_n) + 12\mu_n \quad (6)$$

where $d_i = N_i \cdot F$ is defined as in §3. Here we remark that the inequality (6) is obtained using the similar trick of Xiao with filtration $0 \subseteq E_1 \subseteq E$. Now, combining inequalities (4), (5) and (6), we can prove the desired inequality after a case-by-case discussion as below.

**Case: $Q$ has rank 4**

In this case, we have $x \leq \min\{\nu_2 + \nu_3, \nu_1 + \nu_4\}$ by Lemma 4. So

$$2x \leq \sum_{j=1}^{4} \nu_j = \chi_f.$$

It then follows from (4) that $K_{S/C}^2 \geq \frac{7}{2}\chi_f \geq \frac{24}{7}\chi_f$.

**Case: $Q$ has rank 3**

Following the idea in [6], we firstly discuss the case $x \leq \mu_1 + \mu_n$.

**Lemma 5.** If $x \leq \mu_1 + \mu_n$, then the inequality $K_{S/C}^2 \geq \frac{24}{7}\chi_f$ holds.

**Proof.** If $\mu_1 + \mu_n \geq \frac{4}{7}\chi_f$, then it follows from (6) that

$$K_{S/C}^2 \geq 6(\mu_1 + \mu_n) \geq \frac{24}{7}\chi_f.$$
If \( \mu_1 + \mu_n < \frac{4}{7} \chi_f \), then it follows from (4) that
\[
K_{S/C}^2 \geq 4 \chi_f - x \geq 4 \chi_f - (\mu_1 + \mu_n) > \frac{24}{7} \chi_f.
\]
\[\square\]

**Lemma 6.** If \( x > \mu_1 + \mu_n \), then \( r_{n-1} = 3 \) and \( d_{n-1} = 6 \).

**Proof.** By [6, Lemma 1.4], we have \( r_{n-1} \geq \text{rank}(Q) = 3 \), hence \( r_{n-1} = 3 \). In this case \( d_{n-1} < 6 \) only if the general fibre \( F \) pass through the vertex of the cone \( Q \cap D \). This is however not the case. \[\square\]

**Lemma 7.** If \( x > \mu_1 + \mu_n \), we still have \( K_{S/C}^2 \geq \frac{24}{7} \chi_f \).

**Proof.**

**Case** \( n = 2 \). We have \( r_1 = 3, d_1 = 6 \) by the previous lemma, and \( x \leq 2 \mu_1 \) by Lemma 4. It follows from (5) that
\[
K_{S/C}^2 \geq 12 \mu_1.
\]

Note that \( \chi_f > 0 \) implies \( \mu_1 > 0 \), so this gives \( x \leq \frac{1}{6} K_{S/C}^2 \). It then follows from (4) that \( K_{S/C}^2 \geq \frac{24}{7} \chi_f \).

**Case** \( n = 3 \). Since \( r_2 = 3, d_2 = 6 \) by the previous lemma, we have
\[
K_{S/C}^2 \geq (d_1 + 6)(\mu_1 - \mu_2) + 12(\mu_2 - \mu_3) + 12 \mu_3 \geq 6(\mu_1 + \mu_2).
\]

On the other hand we have \( x \leq \mu_1 + \mu_2 \) by Lemma 4. Hence \( x \leq \frac{1}{6} K_{S/C}^2 \) again holds. The same trick then gives \( K_{S/C}^2 \geq \frac{24}{7} \chi_f \).

**Case** \( n = 4 \). In this case, we have \( d_1 = 0, d_2 \geq 3 \) (see Clifford’s Theorem) and \( d_3 = 6 \) by the previous lemma. Hence we have
\[
K_{S/C}^2 \geq 3(\mu_1 - \mu_2) + 9(\mu_2 - \mu_3) + 12(\mu_3 - \mu_4) + 12 \mu_4 = 3(\mu_1 + 2\mu_2 + \mu_3).
\]

Again, we have Lemma 4 that \( x \leq \min(\mu_1 + \mu_3, 2 \mu_2) \). In particular \( x \leq \frac{1}{6} K_{S/C}^2 \) holds again and the same trick then applies. \[\square\]

**4.3. Genus 5.** One can again follow the argument of Konno (see [6]) to obtain \( K_{S/C}^2 \geq \frac{40}{11} \chi_f \) with his key lemmas modified. For the restriction of context, we shall not go through the details here.
5. Application of Xiao’s slope inequality: Miyaoka-Yau type inequality in positive characteristic

Over the field of complex numbers $\mathbb{C}$, the celebrated Miyaoka-Yau inequality asserts

$$c_1^2(S) \leq 3c_2(S)$$  \hspace{1cm} (7)

holding for all complex algebraic surface $S$ of general type. Using Noether’s formula:

$$12\chi(O_S) = c_1^2(S) + c_2(S),$$

Miyaoka-Yau inequality can be rewritten as:

$$\chi(O_S) \geq \frac{1}{9}K_S^2$$  \hspace{1cm} (8)

in terms of $\chi(O_S)$ and $K_S^2$ for all complex algebraic surface $S$ of general type.

In positive characteristic, there are minimal algebraic surface $S$ of general type with $c_2(S) < 0$ (see Raynaud’s examples in §6), while $c_1^2(S) > 0$ by Bombieri-Mumford’s classification of algebraic surfaces. Henceforth, the original form (7) of Miyaoka-Yau inequality does not hold in positive characteristic. As Miyaoka-Yau inequality is such a powerful tool over $\mathbb{C}$, one would naturally like to find a certain modified inequality of similar form as (8) in positive characteristic.

In the paper [12], Shepherd-Barron also asked whether $\chi(O_S) > 0$ holds true for all algebraic surface $S$ of general type in any characteristic $p > 0$ and later the first author of the present paper (see [3]) raised the question whether we can find an inequality of the same form as Miyaoka-Yau inequality (8):

$$\chi(O_S) \geq \kappa_p K_S^2, \kappa_p > 0.$$  \hspace{1cm} (9)

in each characteristic. Once such an inequality exists, Shepherd-Barron’s question is solved. Following the work of Shepherd-Barron (see [12]), such inequalities are given for each $p \geq 3$ in [3] and meanwhile a conjecture of the optimum inequality for $p \geq 5$ is given:

**Conjecture 1.** The inequality

$$\chi(O_S) \geq \frac{p^2 - 4p - 1}{4(3p + 1)(p - 3)} K_S^2$$  \hspace{1cm} (9)

holds for any algebraic surface $S$ over an algebraically closed field of characteristic $p \geq 5$.

In this section, we solve Shepherd-Barron’s question and prove above conjecture completely by applying the characteristic-$p$ version of Xiao’s
inequality (see Theorem 5). From now on, we always assume that \( S \) is a minimal algebraic surface of general type over an algebraically closed field \( k \) of characteristic \( p > 0 \).

To proceed, we firstly recall a fundamental theorem on the structure of algebraic surfaces of general type with negative \( c_2 \) due to Shepherd-Barron.

**Theorem 7.** (Shepherd-Barron, see [12, Theorem 8]) If \( c_2(S) < 0 \), then the Albanese map of \( S \) induces a fibration: \( f : S \to C \) with

- \( C \) is a nonsingular projective curve of genus \( b := g(C) \geq 2 \) and \( f_*O_S \cong O_C \);
- the fibre (arithmetic) genus \( g := p_a(F) \geq 2 \);
- the geometric generic fibre is a singular rational curve with cusp singularity.

Let \( \Delta \) be the divisorial part of the nonsmooth locus of \( f : S \to C \), which has be studied in [3]. An important fact is

- the horizontal part \( \Delta_h \) of \( \Delta \) is nonempty and consisting of divisors inseparable over the base \( C \).

As a consequence, we see that

**Proposition 2.** If \( f \) has a multiple fibre, then its multiplicity is divided by \( p \).

By abuse of language, we call such \( f : S \to C \) as the Albanese fibration of \( S \). As an application of Theorem 5 we have

**Theorem 8.** Let \( S \) be a minimal algebraic surface of general type over an algebraically closed field \( k \) of characteristic \( p > 0 \) with \( c_2(S) < 0 \). Then we have

\[
\chi(O_S) \geq \frac{g^2 - g - 1}{(12g + 8)(g - 1)} K_S^2 > 0
\]

where \( g \) is defined as in Theorem 7.

Before our proof, we recall the following numerical relations.

\[
K_{S/C}^2 = K_S^2 - 8(g - 1)(b - 1); \tag{10}
\]

\[
\chi_f = \chi(O_S) - (g - 1)(b - 1) + l, \tag{11}
\]

where \( l := \dim_k(R^1 f_* O_S)_{\text{tor}} \) is the torsion dimension of the direct image sheaf \( R^1 f_* O_S \).

**Proof.** Applying Theorem 5 to the Albanese fibration \( f \), one has

\[
K_{S/C}^2 - \frac{4g - 4}{g} \deg \chi_f \geq 0 \tag{12}
\]
which is equivalent to
\[ K_S^2 - 8(b-1)(g-1) - \frac{4g - 4}{g} \left[ \frac{1}{12} (K_S^2 + c_2(S)) - (b-1)(g-1) + l \right] \geq 0. \tag{13} \]

After simplification, the inequality (13) turns into
\[ \frac{2g + 1}{3g} K_S^2 \geq \frac{(12g + 8)(g-1)}{3g} (b-1) + \frac{g - 1}{3g} (4(b-1) + c_2(S)) + \frac{(4g - 4)l}{g}; \]
and it implies that
\[ K_S^2 \geq \frac{(12g + 8)(g-1)}{2g + 1} (b-1). \]
by the inequality (see \[3, (3.3)\]):
\[ c_2(S) \geq -4(b-1). \tag{14} \]

Now, it is clear that
\[ \frac{12\chi(O_S)}{K_S^2} = 1 - \frac{-c_2(S)}{K_S^2} \geq 1 - \frac{4(b-1)}{(12g + 8)(g-1)} = \frac{12(g^2 - g - 1)}{(12g + 8)(g-1)} \]
or equivalently,
\[ \frac{\chi(O_S)}{K_S^2} \geq \frac{g^2 - g - 1}{(12g + 8)(g-1)} > 0. \]

\[ \square \]

**Colloary 1.** Let \( S \) be a minimal algebraic surface of general type over an algebraically closed field \( k \) of characteristic \( p > 0 \). Then

1. if \( p \geq 5 \),
\[ \chi(O_S) \geq \frac{p^2 - 4p - 1}{4(3p + 1)(p - 3)} K_S^2. \tag{15} \]
   The equality holds for Raynaud’s examples (see § 7).

2. if \( p = 2, 3 \), \( \chi(O_S) \geq \frac{1}{32} K_S^2. \]

In particular, \( \chi(O_S) > 0 \) holds for all algebraic surface of general type in positive characteristic and Conjecture \( \square \) holds true.

**Proof.** In order to prove our results, note that if \( c_2(S) \geq 0 \) Noether’s formula implies \( \chi(O_S) \geq \frac{1}{12} K_S^2 \), it is enough to consider that \( c_2(S) < 0 \).
Since we always have \( g \geq \frac{p - 1}{2} \) by genus change formula (see \[3, § 2.1\]), then (15) is a direct consequence of Theorem \( \square \). When \( S \) is one of
Raynaud’s examples, the equality of (15) holds according to a direct computation (see Remark 5 in § 6).

When \( p = 2, 3 \), the inequality follows immediately from Theorem 8 by that \( g \geq 2 \). We are done. \( \square \)

In the rest of this section, we concentrate into the case \( p = 2, 3 \).

**Proposition 3.** When \( p = 3 \), we have \( \chi(O_S) > \frac{1}{32} K^2_S \).

**Proof.** According to the previous corollary and theorem, the equality

\[
\chi(O_S) = \frac{1}{32} K^2_S
\]

holds only if:

1. \( c_2(S) < 0 \);
2. the Albanese fibration (see Theorem 7) \( f : S \to C \) is a genus 2 fibration;
3. when \( p = 3 \), each fibre of \( f \) is irreducible and reduced (by equality (14) and Proposition 2).

We shall show that above three conditions lead to a contradiction. Note that the relative canonical map gives a morphism: \( \pi : S \to \mathbb{P}(f_* \omega_{S/C}) \) since each fibre of \( f \) is irreducible and reduced, and such morphism \( \pi \) is necessarily a separable flat double cover (see [3, § 2]). Let \( M \subseteq \mathbb{P}(f_* \omega_{S/C}) \) be the branch divisor of \( \pi \), which satisfies:

- \( M \) is a smooth, horizontal divisor and \([M : C] = 6\);
- each component of \( M \) is inseparable over \( C \);
- for each point \( c \in C \), its inverse image in \( M \) has exactly two points as a set. In fact, otherwise there is some \( c \) has one inverse image. Then the fibre of \( f \) at \( c \) by construction is a flat double cover of \( \mathbb{P}^1_k \) branching at a single point of multiplicity 6. Such fibre is clearly not irreducible.

Then there are two possibilities:

A) \( M = M_1 + M_2 \) with \( M_1 \cdot M_2 = 0 \), and the projections \( M_i \to C \) \((i = 1, 2)\) are both isomorphic to the Frobenius morphism;

B) \( M \) is irreducible and the projection \( u : M \to C \) factors as

\[
\begin{array}{ccc}
M & \xrightarrow{F_M} & M' \\
\downarrow u & & \downarrow v \\
C & & \end{array}
\]

where \( F_M \) is the frobenius morphism and \( v \) is an étale double cover.
Indeed we only need to consider the case A), since replacing $C$ by the base change $v$ above which is an étale double cover, the case B) can be turned into A).

Finally we go to exclude case A). Let $\Sigma$ be the divisor class $O(1)$ of $\mathbb{P}(f_*\omega_{S/C})$, and $M_i \sim_{\text{num}} 3\Sigma + u_i F$ for $i = 1, 2$. Recall that $\Sigma^2 = \deg f_*\omega_{S/C} = \chi_f$, and we have

\begin{align*}
2b - 2 &= (3\Sigma + u_1 F)^2 + (3\Sigma + u_1 F)(-2\Sigma + (\chi_f + 2b - 2)F) \quad (16) \\
0 &= (3\Sigma + u_1 F)(3\Sigma + u_2 F) \quad (17)
\end{align*}

Thus $u_1 = u_2$ and $b = 1$, which is a contradiction. \hfill \Box

**Question 1.** Is it true that

$$\inf \{\frac{\chi(O_S)}{K_S^2} \mid \text{minimal algebraic surface of general type} \} = \frac{1}{32}$$

when $p = 3$? If not, it may be interesting to work out the value of LHS above.

**Lemma 8.** When $p = 2$, the equality $\chi(O_S) = \frac{1}{32} K_S^2$ holds if and only if there exists a fibration $f : S \to C$ satisfying:

1. $f : S \to C$ is a genus 2 fibration;
2. any fibre of $f$ is irreducible, singular and rational.

**Proof.** If $\chi(O_S) = \frac{1}{32} K_S^2$, then one can directly check that its Albanese fibration of $S$ is what we want. Conversely, we return to the proof of Theorem 8, the inequality $\chi(O_S) = \frac{1}{32} K_S^2$ holds if and only

- $c_2(S) = -4(b - 1)$ (satisfied if and only if each fibre of $f$ is irreducible);
- $l = 0$ (satisfied since no genus 2 fibre has multiplicities); and
- Xiao’s inequality: $K_f^2 = 2 \deg f_*\omega_{S/C}$ holds.

It suffices to show under above two conditions, Xiao’s inequality holds. In fact, when all fibres of $f$ are integral, the relative canonical map

$$v : S \to P = \mathbb{P}(f_*\omega_{S/C})$$

is a morphism without base point. In particular, we have $\omega_{S/C} = v^* O(1)$. Therefore

$$K_f^2 = 2c_1^2(O(1)) = 2 \deg (f_*\omega_{S/C}).$$

\hfill \Box
Finally, we go to construct an explicit surface fibration $f : S \to C$ satisfying conditions (1) and (2) of Lemma 8. Define $C$ to be the quintic plane curve defined by homogeneous equation:

$$Y^4Z + YZ^4 = X^5$$

over an algebraically closed field $k$ of characteristic $p = 2$. One can easily check that $C$ is a smooth curve of genus $b := g(C) = 6$. There are two affine subset $C_i$ ($i = 0, 1$) of $C$ as below:

$C_0$ ($Z = 1$) : $y^4 + y = x^5$, $x = \frac{X}{Z}$, $y = \frac{Y}{Z}$, with $C \setminus C_0 = \{(0, 1, 0)\}$;

$C_1$ ($Y = 1$) : $z^4 + z' = x^5$, $x' = \frac{X}{Y}$, $z' = \frac{Z}{Y} = \frac{1}{y}$, with $C \setminus C_1 = \{(0, 0, 1)\}$.

For simplicity, we introduce the following notations:

- $\infty$ is the point $(0, 1, 0)$ which is the complement of $C_0$ in $C$;
- $\Lambda := \{(0, 1, \lambda)|\lambda \in \mathbb{F}^*_1\} = \{(0, \tau, 1)|\tau \in \mathbb{F}^*_1\} \subset C$.
- $C'_1 := C_1 \setminus \Lambda$;
- $C_{10} = C_0 \cap C'_1$.

One can check immediately that

- $dx$ (resp., $dx'$) is a generator of $\Omega_C$ over $C_0$ (resp., $C_1$);
- $C_{10}$ is the open subset where $x$ and $x'$ is invertible;
- $C'_1$ is the open subset where $1 + z'^3$ is invertible.

Over $C_0$: $S$ is defined as

$$Y_0^2 = S_0T_0^5 + xS_0^6;$$

in the weighted projective space $\text{Proj}(\mathcal{O}_{C_0}[S_0^1, T_0^1, Y_0^3])$. Here the superscript on each element is its homogeneous degree.

Over $C'_1$: $S$ is defined as

$$Y_1^2 = S_1T_1^5 + \frac{x'}{1 + z'^6}S_1^6;$$

in the weighted projective space $\text{Proj}(\mathcal{O}_{C'_1}[S_1^1, T_1^1, Y_1^3])$.

The homogeneous translation relation is given by

$$\begin{cases} S_1 &= x'^3S_0 \\ T_1 &= x'T_0 \\ Y_1 &= x'^4Y_0 + (1 + z'^3)T_0^3 \end{cases}$$

and this construction makes sense because that

$$x'^8(Y_0^2 - S_0T_0^5 + xS_0^6) = Y_1^2 = S_1T_1^5 + \frac{x'}{1 + z'^6}S_1^6.$$

Moreover, one can check that $S$ is a non-singular surface as following:
(Over $C_0$) : When $T_0 = 1$, take $s = \frac{S_0}{T_0}$, then the function defining $S$ is

$$y^2 = s + xs^6.$$  

It is even smooth over $C_0$ by Jacobian criterion. When $S_0 = 1$, take $t = \frac{T_0}{S_0}$, then the function defining $S$ is

$$y'^2 = t^5 + x.$$  

Since $dx$ is a generator of $\Omega_C$ on $C_0$, $S$ is smooth over $k$ by Jacobian criterion.

(Over $C'_1$) : When $T_1 = 1$, $s_1 = \frac{S_1}{T_1}$, then the function defining $S$ is

$$y_1^2 = s_1 + \frac{x'}{1 + z^6}s_1^6.$$  

It is even smooth over $C'_1$ by Jacobian criterion. When $S_1 = 1$, $t_1 = \frac{T_1}{S_1}$, the function defining $S$ is

$$y_1'^2 = t_1^5 + \frac{x'}{1 + z^6}.$$  

As $\frac{1}{1 + z^6}$ is invertible on $C'_1$ and $dx'$ is a generator of $\Omega_C$, $S$ is smooth over $k$ by Jacobian criterion.

On each fibre, the unique singularity lies on the infinity $T_0 = 0$ or $T_1 = 0$, and up to étale equivalence the singularity is as the following cusp

$$y^2 = t^5.$$  

One can also find out that each fibre $F$ of the fibration $f : S \to C$ is irreducible, rational and $p_a(F) = 2$. Thus, by Lemma 8, we have

**Proposition 4.** When $p = 2$, the inequality $\chi(O_S) \geq \frac{1}{32}K_S^2$ is optimum.

In fact, we have $\chi(O_S) = 1$ and $K_S^2 = 32$ in above example. We would also like to indicate that $S$ is given from $C \times \mathbb{P}^1$ by taking quotient of the foliation $D = s^6 \frac{\partial}{\partial s} + \frac{\partial}{\partial x}$, where $s$ is the parameter of $\mathbb{P}^1$. 
6. Raynaud's examples

In the paper [11], Raynaud constructed a class of pairs \((S, \mathcal{L})\), where \(S\) is a smooth projective algebraic surface in positive characteristic and \(\mathcal{L}\) is an ample line bundle on \(S\) such that \(H^1(S, \mathcal{L}) \neq 0\). These pairs then give counterexamples to Kodaira's vanishing theorem. Honestly, Raynaud's example is so special that it does not only violates Kodaira's vanishing theorem, but also leads to many pathologies in positive characteristic.

We now briefly recall his example, and one can also refer to [3, § 4]. Fixing an algebraically closed field \(k\) of characteristic \(p > 2\). Let \((C, P = \mathbb{P}(\mathcal{E}), \Sigma)\) be a triple satisfying:

- \(C\) is a projective nonsingular curve over \(k\) of genus \(b\);
- \(\pi : P = \mathbb{P}(\mathcal{E}) \to C\) is the \(\mathbb{P}^1\)-bundle over \(C\) associated to a rank-2 locally free sheaf \(\mathcal{E}\);
- \(\Sigma \subseteq P\) is a reduced horizontal divisor consisting of two irreducible components \(\Sigma_i\) \((i = 1, 2)\) such that
  - \(X_1\) \(\pi : \Sigma_1 \to C\) is an isomorphism, or equivalently \(\Sigma_1\) is a rational section of \(\pi\);
  - \(X_2\) \(\pi : \Sigma_2 \to C\) is isomorphic to the Frobenius morphism;
  - \(X_3\) \(\Sigma_1 \cap \Sigma_2 = \emptyset\).

According to the computations in [11], one can easily check that such data \((C, P, \Sigma)\) is equivalent to the pair \((C, f)\) with the same curve \(C\) and a rational function \(f \in K(C) \setminus K(C)^p\) with

\[
\text{div}(df) = pD
\]

for some divisor \(D\) on \(C\). As a (non-direct) consequence, one can also prove that \(\Sigma\) is an even divisor on \(P\).

**Example 1** (Artin-Schreier curve). *Let \(C\) be the complete normal curve associated to the following plane equation:

\[
y^p - y = x^{p-1}, \varphi(x) \in k[x].
\]

Then

- \(b = g(C) = \frac{(p - 1)(p - 2)}{2}\);
- \(\text{div}(dy) = (2q - 2)\infty = p((p - 3)\infty)\),

where \(\infty \in C\) is the unique point at infinity. In particular, such example of \((C, f)\) or equivalently \((C, P, \Sigma)\) satisfying above properties does exist in any characteristic \(p > 2\).*
Definition 2. Let $S$ be a smooth projective surface over $k$ and $\sigma : S \to P$ be any flat double cover with branch divisor $\Sigma$

\[
\begin{array}{c}
S \xrightarrow{\sigma} P \\
\downarrow f \quad \quad \quad \downarrow \pi \\
C
\end{array}
\]

where $(C, P, \Sigma)$ satisfies above properties. We call such a fibration $f : S \to C$ as one of Raynaud's examples.

One can directly check that such a fibration $f$ satisfies the following properties:

a) the surface $S$ itself is nonsingular over $k$;

b) every geometric fibre of $f$ is a singular rational curve of arithmetic genus

$$p_a = \frac{p - 1}{2};$$

c) every geometric fibre is hyperelliptic and integral (irreducible & reduced).

Lemma 9. Suppose $f : S \to C$ is any surface fibration with above properties a), b) and c). Then $S$ must be one of Raynaud’s examples.

Proof. Let $\rho$ be the hyperelliptic involution, and $\sigma : S \to P := S/\rho$ be the quotient map. Then condition c) implies that the canonical homomorphism $\pi : P \to C$ has integral fibres. Note that $\pi : P \to C$ is birational to a ruled surface (recall that $K(C)$ is $C1$ by Tsen’s Theorem, hence a smooth plane conic over $K(C)$ is $\mathbb{P}^1_{K(C)}$) and $P$ is normal with integral $\pi$-fibres, we see that $P$ is exactly a smooth minimal ruled surface over $C$. Thus the quotient map $\sigma : S \to P$ is a flat double cover with some branch divisor $\Sigma \subseteq P$, and $\Sigma$ itself is smooth over $k$ (see [3, § 2.2]). on the other hand, it can be deduced from [3, § 2.1 & 2.2] that $\Sigma_{K(C)} := \Sigma \times_{\pi,C} K(C)$ is a divisor of $P_{K(C)} := P \times_{\pi,C} K(C) \simeq \mathbb{P}^1_{K(C)}$ with

- $\deg_{K(C)} \Sigma_{K(C)} = p + 1$ (by $p_a = \frac{p - 1}{2}$);
- $\Sigma_{K(C)}$ contains a point inseparable over $K(C)$ (by the fact all fibres are singular).

By degree counting, it in all concludes that $\Sigma$ consists of two components with property $\{X_1, X_2\} \& X_3$) as above. We are done.  

According to the proof of Corollary 1, when $p \geq 5$ the equality of (15) holds only if $c_2(S) < 0$. Let $f : S \to C$ be the Albanese fibration, then
moreover we have that $g = \frac{p - 1}{2}$ and both of the inequalities (12) and (14) are equalities, thus we have

- the fibration $f$ is of arithmetic genus $\frac{p - 1}{2}$;
- each geometric fibre of $f$ is irreducible and reduced. In fact, the equality (14) gives that each fibre of $f$ is irreducible. On the other hand, since the genus is $\frac{p - 1}{2}$, we can have no multiple fibre by Proposition 2.

As a direct consequence of Lemma 9, we have

**Colloary 2.** Let $S$ be a minimal algebraic surface of general type over an algebraically closed filed $k$ of characteristic $p \geq 5$. Assume that the equality of (15) holds and its Albanese fibration is hyperelliptic, then $S$ is one of Raynaud’s examples.

We end this section with a remark about some numerical facts of Raynaud’s examples. Let $f : S \to C$ be one of Raynaud’s examples associated to the triple $(C, P, \Sigma)$, then we have (see [3, § 4])

- $2b - 2 = pl'$ for some integer $l' \in \mathbb{N}_+$;
- $2g - 2 = p - 3$;
- $K_S^2 = \frac{(3p^2 - 8p - 3)}{2}l'$;
- $\chi(O_S) = \frac{(p^2 - 4p - 1)}{8}l'$.

Thus

$$K_f^2 = K_S^2 - 8(g - 1)(b - 1) = -\frac{(p - 1)(p - 3)}{2}l'$$  \hspace{1cm} (21)

$$\chi_f = \chi(O_S) - (g - 1)(b - 1) = -\frac{(p - 1)^2}{8}l'$$  \hspace{1cm} (22)

**Remark 5.** Let $f : S \to C$ be one of Raynaud’s examples, then the equality of (15) holds and

$$K_f^2 = \frac{4g - 4}{g} \chi_f$$

while both sides of the equality are negative.

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