Density of rational points on a family of del Pezzo surfaces of degree one

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(With an appendix by Jean-Louis Colliot-Thélène)

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Abstract

Let $k$ be an infinite field of characteristic 0, and $X$ a del Pezzo surface of degree $d$ with at least one $k$-rational point. Various methods from algebraic geometry and arithmetic statistics have shown the Zariski density of the set $X(k)$ of $k$-rational points in $X$ for $d \geq 2$ (under an extra condition for $d = 2$), but fail to work in generality when the degree of $X$ is 1, leaving a large class of del Pezzo surfaces for which the question of density of rational points is still open. In this paper, we prove the Zariski density of $X(k)$ when $X$ has degree 1 and is represented in the weighted projective space $\mathbb{P}(2,3,1,1)$ with coordinates $x, y, z, w$ by an equation of the form $y^2 = x^3 + az^6 + bz^4w^3 + cw^6$ for $a, b, c \in k$ with $a, c$ non-zero, under the condition that the elliptic surface obtained by blowing up the base point of the anticanonical linear system $|−K_X|$ contains a smooth fiber above a point in $\mathbb{P}^1 \setminus \{(1 : 0), (0 : 1)\}$ with positive rank over $k$. When $k$ is of finite type over $\mathbb{Q}$, this condition is sufficient and necessary.

1 Introduction

A del Pezzo surface over a field $k$ is a smooth, projective, geometrically integral surface over $k$ with ample anticanonical divisor. The degree of a del Pezzo surface is the self-intersection number of the canonical divisor, and this is an integer between 1 and 9. Over an algebraically closed field, a del Pezzo surface of degree $d$ is isomorphic to either $\mathbb{P}^1 \times \mathbb{P}^1$ (for $d = 8$), or to $\mathbb{P}^2$ blown up in $9 − d$ points in general position. Over a non-algebraically closed field, this is not true in general. A variety $X$ over a field $k$ is $k$-unirational if there is a dominant rational map $\mathbb{P}^n_k \dashrightarrow X$ for some $n$. Del Pezzo surfaces of degree at least 2 over a field $k$ with a $k$-rational point are known to be $k$-unirational under the extra condition for degree 2 that the $k$-rational point lies outside the ramification curve of the anticanonical map, and is not contained in the intersection of 4 exceptional curves. This is proved consecutively in [Seg43] and [Seg51] for degree 3 and $k = \mathbb{Q}$, in [Man86] Theorems 29.4, 30.1 for $d \geq 5$, as well as for $d = 3, 4$ for large enough cardinality of $k$, in [Kol02] for the complete case $d = 3$, in [Pie12] Proposition 5.19 for the complete case $d = 4$, and in [STVA14] for $d = 2$. A del Pezzo surface is minimal if and only if there exists no birational map over its groundfield to a del Pezzo surface of higher degree. Therefore, if a del Pezzo surface $X$ of degree 1 over a field $k$ is not minimal, it is birationally equivalent to a del Pezzo surface $X'$ of higher degree, and $X$ is unirational if and only if $X'$ is. A minimal del Pezzo surface of degree 1 has Picard rank 1 or 2.

For a long time, nothing about unirationality for minimal del Pezzo surfaces of degree 1 was known, even though they always contain a rational point. In 2017, Kollár and Mella proved that minimal del Pezzo surfaces of degree 1 over a field $k$ with char $k \neq 2$ that have Picard rank 2 are $k$-unirational [KM17]. Outside this case, the question of $k$-unirationality for minimal del Pezzo surfaces of degree 1 is wide open: we do not have any example of a minimal del Pezzo surface of degree 1 with Picard rank 1 that is known to be $k$-unirational, nor of one that is known not to be $k$-unirational.

If $k$ is infinite, then $k$-unirationality implies density of the set of $k$-rational points. While unirationality for del Pezzo surfaces of degree 1 is still out of reach, there are several partial results on density of their set of rational points; see Remark 1.3. Moreover, if $k$ is a number field, density of the set of $k$-rational points for del Pezzo surfaces of degree 1 is implied by a conjecture of Colliot-Thélène and Sansuc, stating that for a geometrically rational variety over a number field, its set of rational points is dense in the Brauer–Manin set for the adelic topology [CTS80] Question (j1)].

In this paper we give sufficient conditions for del Pezzo surfaces of degree 1 in a certain family...
over an infinite field of characteristic 0 to have a dense set of rational points. The conditions are necessary if the field is of finite type over \( \mathbb{Q} \).

### 1.1 Main result

A del Pezzo surface of degree 1 over a field \( k \) can be described by a smooth sextic in the weighted projective space \( \mathbb{P}(2, 3, 1, 1) \) with coordinates \( (x : y : z : w) \). For char \( k \neq 2, 3 \), this sextic can be written as

\[
g^2 = x^3 + x \cdot f(z, w) + g(z, w),
\]

where \( f, g \in k[z, w] \) are homogeneous of degree 4 and 6, respectively. For a del Pezzo surface \( X \) of degree 1, the anticanonical linear system \( |-K_X| \) has a unique base point given by \( \mathcal{O} = (1 : 1 : 0 : 0) \). Blowing up this basepoint gives a surface \( \mathcal{E} \) with elliptic fibration \( \mathcal{E} \to \mathbb{P}^1 \), which, when restricted to \( S \), is given by the projection to \( (z : w) \). The fibration admits a section \( \tilde{O} \) given by the exceptional curve above \( \mathcal{O} \).

In this paper we prove the following theorem.

**Theorem 1.1.** Let \( k \) be a field of characteristic 0, and \( a, b, c \in k \) with \( a, c \) non-zero. Let \( S \) be the del Pezzo surface given by

\[
y^2 = x^3 + az^6 + bw^6 + cw^6 \tag{2}
\]

in the weighted projective space \( \mathbb{P}(2, 3, 1, 1) \) with coordinates \( (x, y, z, w) \). Let \( \mathcal{E} \) be the elliptic surface obtained by blowing up the base point of the linear system \( |-K_S| \). If \( S \) contains a rational point with non-zero \( z, w \)-coordinates, such that the corresponding point on \( \mathcal{E} \) is non-torsion on its fiber, then \( S(k) \) is dense in \( S \) with respect to the Zariski topology. If \( k \) is of finite type over \( \mathbb{Q} \), the converse holds as well.

**Remark 1.2.** Theorem [1.1] is the first result that gives sufficient and necessary conditions for the \( k \)-rational points on the family given by (2) to be dense, even when \( b = 0 \), where \( k \) is any field of finite type over \( \mathbb{Q} \); see also Remark [1.3]. We require \( k \) to be of finite type over \( \mathbb{Q} \) in order to bound the torsion in a family of elliptic curves over \( k \); see also Theorem 4.1.

**Remark 1.3.** Several partial results on density of rational points on a del Pezzo surface of degree 1 are known. In [VA11], Várilly-Alvarado proves Zariski density of the set of \( \mathbb{Q} \)-rational points on all surfaces of the form (1) with \( f = 0 \) and \( g = az^6 + bw^6 \) for non-zero \( a, b \in \mathbb{Z} \), such that either \( 3a/b \) is not a square, or gcd(\( a, b \)) = 1 and \( 9 \nmid ab \), under the condition that the Tate–Shafarevich group of elliptic curves with \( j \)-invariant 0 is finite. Ulas and Togbé prove Zariski density of the set of \( \mathbb{Q} \)-rational points of surfaces of the form (1) in the following cases. (i) either \( g = 0 \) and \( \deg(f(z, 1)) \leq 3 \), or \( g = 0 \) and \( \deg(f(z, 1)) = 4 \) with \( f \) not even, or \( f = 0 \) and \( g(z, 1) \) is monic of degree 6 and not even [Ula07] Theorems 2.1 (1), 2.2, and 3.1, (ii) \( g = 0 \) and \( \deg(f(z, 1)) = 4 \), or \( f = 0 \) and \( g(z, 1) \) is even and monic of degree 6, both cases under the condition that there is a fiber of \( \mathcal{E} \) with infinitely many rational points [Ula07] Theorems 2.1 (2) and 3.2, (iii) The surface can be defined by \( y^2 = x^3 - h(z, w) \), with \( h(z, 1) = z^3 + az^2 + bw^2 + cz + d \in \mathbb{Z}[z] \), and the set of rational points on the elliptic curve \( Y^2 = X^3 + 135(2a - 15)X - 1350(5a + 2b - 26) \) is infinite [Ula08] Theorem 2.1. (iv) \( f(z, 1) \) and \( g(z, 1) \) are both even of degree 4 and there is a fiber of \( \mathcal{E} \) with infinitely many rational points [UT10] Theorem 2.1. Jabara generalized the results from [Ula07] mentioned above in [Jab12] Theorems C and D. Though the proofs of these two theorems are incomplete (see [SvL14] Remark 2.7), they hold for sufficiently general cases. Finally, in [SvL14], Saltina and van Luijk generalize some of the previous results, proving Zariski density of the set of \( k \)-rational points of surfaces of the form (1) for any infinite field \( k \) with char \( k \neq 2, 3 \), assuming that there exists a point \( Q \) on a smooth fiber of \( \mathcal{E} \) satisfying several conditions, among which that a multisection that they construct from \( Q \) has infinitely many \( k \)-rational points.

**Remark 1.4.** For an elliptic surface, the Zariski density of the set of rational points is equivalent to having infinitely many fibers with non-zero rank. Given the difficulty in calculating the rank in general, a reasonable substitute when \( k = \mathbb{Q} \) is the root number \( W(E) \), defined as the parity of the analytic rank. A consequence of the Birch and Swinnerton-Dyer conjecture (known as the Parity conjecture) relates the root number to the parity of the geometric rank by \( W(E) = (-1)^{r(E)} \), reducing the question of Zariski density on elliptic surfaces to finding an infinite set of fibers with root number \(-1\), which would imply odd and thus non-zero rank. Work in this direction includes papers from Manduchi [Man95], Helfgott [Hel] and Várilly-Alvarado (mentioned in the previous remark). The latest result in the literature [Des18], by the first author, proves that for non-isotrivial elliptic surfaces, the set of fibers with negative root number is infinite, assuming the Chowla conjecture for some other reasons.
on the product of polynomials corresponding to places of multiplicative reduction, and the Square-
free conjecture on certain\footnote{To be precise, the polynomials on which we need the Squarefree conjecture are the non-insipid ones, according to the vocabulary introduced in \cite{Des21}.} polynomials associated to places of bad reduction. This proves Zariski
density conditionally on these two conjectures and on the Parity conjecture for all non-isotrivial
elliptic surfaces. The Chowla and Squarefree conjectures are known to hold for polynomials of small
degree, rendering Desjardins’s work on finding the root number unconditional for a lot of rational
elliptic surfaces.

However, on isotrivial elliptic surfaces it can happen that every fiber has a positive root number,
in which case one cannot use this quantity to predict whether the geometric rank is non-zero on
infinitely many fibers. When the $j$-invariant of the generic fiber is non-zero and the elliptic surface
is rational, the first author proves in \cite{Des19} Theorem 1.2 \( \mathbb{Q} \)-unirationality when \( j \neq 1728 \), and
Zariski density otherwise, both unconditionally.

A rational elliptic surface with $j$-invariant zero and only irreducible singular fibers is birationally
equivalent to a del Pezzo surface of degree 1, by blowing down the zero section. Combining \cite{VA11}
Theorem 2.1 and \cite[Proposition 2.2]{DN}, we know that the elliptic surfaces with \( j = 0 \) and root
number $+1$ on every fiber are of the form $y^2 = x^3 + c(f(t)^2 + 3g(t)^2)$, for some $c \in \mathbb{Q}$, and \( f, g \in \mathbb{Q}(t) \)
non-zero and with no common factors. In particular, all examples in Section 5 are in this family.
Sufficient conditions for such surfaces to have root number $+1$ on all fibers are given in \cite{Des19} in
the case $f(t) = At^3$, $g(t) = B$ for some $A, B \in \mathbb{Z}$ non-zero. Finally, in the paper \cite{DN} by Naskr\'{e}cki
and the first author, the generic rank of all surfaces given by an equation $y^2 = x^3 + at^6 + c$ for $a, c \in \mathbb{Z}$
non-zero is computed, proving \( \mathbb{Q} \)-unirationality for those with non-zero generic rank. However,
it is found that most of these surfaces have rank zero.

1.2 Set-up and idea of the proof

We set up some terminology that we will use throughout the paper. Let $C$ be a smooth, projective,
geometrically integral curve over a field $k$. An elliptic surface with base $C$ is a smooth, projective,
geometrically integral surface $E$ endowed with an elliptic fibration $\pi : E \rightarrow C$, that is, a surjective
morphism such that for almost all points $v \in C$, the fiber $E_v = \pi^{-1}(v)$ above $v$ is a smooth
genus 1 curve, and moreover, the morphism $\pi$ admits a section: a morphism $s : C \rightarrow E$ such that
$\pi \circ s = \text{id}_C$. The existence of a section implies that all smooth genus 1 fibers of $\pi$ are elliptic curves.
If $E$ is an elliptic surface with base $C$, then its generic fiber is an elliptic curve $E$ over the function
field $k(C)$ of $C$. The set of sections of $\pi : E \rightarrow C$ form a group which is in natural correspondence
with the group of $k(C)$-rational points of $E$ \cite[Proposition 5.4]{SS19}, also called the Mordell–Weil
group of $E$. A multisection of degree $d$ or $d$-section of $E$ is an irreducible curve $D$ contained in $E$
such that the projection $\pi_D : D \rightarrow C$ is non-constant and of degree $d$. Note that, when $d = 1$, this
is simply a section. We often switch between viewing a (multi-)section as a map between schemes
and a curve on $E$.

Remark 1.5. Let $k$, $S$, and $E$ be as in Theorem 1.1. If $E$ admits a $k$-rational section $s$, then, since
the Mordell-Weil group of $E$ is torsion-free \cite[Theorem 7.4]{SS19}, there are infinitely many distinct
multiples of $s$ contained in $E$, all of which have a dense set of rational points. This implies that $E(k)$
contains an infinite union of distinct one-dimensional irreducible closed subsets, hence $E(k) = E$, and
thus the Zariski density of $E(k)$ follows, implying the density of $S(k)$ in $S$. Theorem 1.1 is
therefore especially interesting in the cases where the Mordell-Weil group of $E$ has rank zero over $k$.

We prove Theorem 1.1 by constructing, for $k, S, \text{ and } E$ as in the theorem, a family of multisections
of $E$, and using these multisections to show that the set $E(k)$ is dense in $E$.

1.3 Contents of the paper

The paper is organized as follows. In Section 2 we construct a family of multisections on $E$, and show
that there is a member of this family that contains a section over $k$, or there is a member
of this family that is geometrically integral of genus 0, or there are infinitely many members of this
family that are geometrically integral of genus 1. In Section 3 we show that, in the latter case,
this family gives rise to an elliptic fibration over a smooth fiber of $E$, and that this elliptic fibration
has a non-torsion section (see also Figure 2). In Section 4 we use all this to prove Theorem 1.1.
Finally, in Section 5 we give examples of specific surfaces for which Theorem 1.1 shows that the set
of rational points is dense.
Acknowledgements. We thank Ronald van Luijk for giving us the idea to look for certain multisections, and for many useful ideas and discussions afterwards. We thank Jean-Louis Colliot-Thélène for showing us how to extend the necessary condition over a number field to any field of finite type over \( \mathbb{Q} \). We are grateful to Martin Bright, János Kollár, Bartosz Naskręcki, Marta Pieropan, and Anthony Várilly-Alvarado for helpful discussions and remarks. We thank the anonymous referee for useful comments.

2 Constructing a family of multisections

Let \( k \) be an infinite field with \( \text{char } k \neq 2,3 \), take \( a, b, c \in k \) with \( a, c \) non-zero, let \( S \) be the del Pezzo surface of degree 1 given by \( \mathbb{P}_k(2,3,1,1) \) with canonical divisor \( K_S \), and assume that \( S \) contains a rational point as in Theorem [1.1]. Let \( E \) be the elliptic surface obtained by blowing up the base point of the linear system \( |-K_S| \). In this section we construct a family of multisections on \( E \); see Proposition 2.5. We introduce some notation.

**Notation 2.1.** Let \( \pi : E \rightarrow S \) be the blow-up of \( S \) in \( \mathcal{O} = \{ 1 : 1 : 0 : 0 \} \) with exceptional divisor \( \tilde{O} \).

Since \( \pi \) gives an isomorphism between \( E \setminus \tilde{O} \) and \( S \setminus \{ \mathcal{O} \} \), we denote a point \( \bar{R} \in E \setminus \tilde{O} \) by the coordinates of \( \pi(\bar{R}) \) in \( \mathbb{P}_k(2,3,1,1) \). Let \( \nu : E \rightarrow \mathbb{P}^1 \) be the elliptic fibration on \( E \), which is given on \( S \) by the projection onto \((z:w)\). For \( R = (x_R : y_R : z_R : w_R) \in S \setminus \{ \mathcal{O} \} \), we denote by \( R_\nu \) the inverse image \( \nu^{-1}(R) \) on \( E \), which is a point on the fiber \( \nu^{-1}(w) \).

**Definition 2.2.** For any point \( R = (x_R : y_R : z_R : 1) \) in \( E \) with \( y_R, z_R \neq 0 \), we define the curve \( C_R \subset E \) as the strict transform of the intersection of \( S \) with the surface given by

\[
3x^2z^2xz - 2yRz^2y - (x^3R - 2az^6R - bz^3R)c^3 + (2cz^3R + bcz^6R)w^3 = 0.
\]

**Remark 2.3.** The curve \( C_R \) in (3) was found by finding generators for the subsystem of \( |-3K_S| \) of curves that contain \( R \) as a double point. A general such curve has arithmetic genus 3, and by forcing two extra singularities we hoped to find curves of genus at most 1. When computing the generators in magma, the curve \( C_R \) was one of them. Later, the following was pointed out to us by János Kollár. By setting \( z = 1, x = \frac{x}{x}, y' = \frac{-y}{x}, w' = \frac{w}{x} \), we obtain the affine model of \( S \) in \( \mathbb{A}^3 \) given by \( y'^2 = x'^3 + a + bw^3 + cw^6 \). Setting \( \omega = w^3 \), we obtain a map \( f \) from this affine model to the cubic surface \( T \) given by \( y'^2 = x'^3 + a + b\omega + c\omega^2 \) in \( \mathbb{A}^3 \) with coordinates \( x', y', \omega \). For a point \( R \in S \) with non-zero \( y, z, w \)-coordinates, the tangent plane to \( T \) at \( Q = f(R) \) intersects \( T \) in a cubic surface \( C_Q \), and the curve \( C_R \) as in Definition 2.2 is the strict transform on \( E \) of the image of \( C_Q \) under \( f \).

**Remark 2.4.** For \( R = (x_R : y_R : z_R : 1) \) in \( E \) with \( y_R, z_R \neq 0 \), the curve \( \pi(C_R) \) does not contain the point \( \mathcal{O} \), so we identify the curve \( C_R \) with \( \pi(C_R) \subset \mathbb{P}(2,3,1,1) \); see Notation 2.1.

The main result in this section is the following. We prove this at the end of the section.

**Proposition 2.5.** Let \( \mathcal{F} \) be a smooth fiber on \( E \) above a point \((z_0 : 1) \in \mathbb{P}^1 \) with \( z_0 \in k \) non-zero, such that \( \mathcal{F} \) has non-zero rank over \( k \). Then there is a point \( R \in \mathcal{F}(k) \) such that \( C_R \) contains a section defined over \( k \), or there is a point \( R \in \mathcal{F}(k) \) such that \( C_R \) is geometrically integral of genus 0, or there is a non-empty open subset \( \mathcal{F}_0 \) of \( \mathcal{F} \) such that for every point \( R \in \mathcal{F}_0(k) \), the curve \( C_R \) is geometrically integral of genus 1.

**Remark 2.6.** Let \( R = (x_R : y_R : z_R : 1) \) be a point in \( E \), with \( y_R, z_R \neq 0 \), and let \( C_R \) be the corresponding curve as in Definition 2.2. Let \( \mathbb{A}^3 \) be the affine open subset of \( \mathbb{P}(2,3,1,1) \) given by \( w \neq 0 \), with coordinates \( X = \frac{x}{w}, Y = \frac{y}{w^3}, \) and \( T = \frac{z}{w} \). We describe the intersection \( C_R \cap \mathbb{A}^3 \). Write

\[
F = Y^2 - X^3 - aT^6 - bT^3 - c, \quad G = 3x^2Rz^2RXT - 2yRz^3Y - (x^3R - 2aRz^6 - bRz^3R)c^3 + 2cz^3R + bcz^6R.
\]

We have \( C_R \cap \mathbb{A}^3 = Z(F) \cap Z(G) \), where \( Z(F) \) and \( Z(G) \) are the zero loci of \( F \) and \( G \), respectively. Since \( y_R, z_R \neq 0 \), the projection \( p: \mathbb{A}^3 \rightarrow \mathbb{A}^2 \) to the \( X, T \)-coordinates has a section given by

\[
r: (X, T) \mapsto \left( X, \frac{3x^2Rz^2RXT - (x^3R - 2aRz^6 - bRz^3R)c^3 + 2cz^3R + bcz^6R}{2yRz^3R}, T \right).
\]
Note that $p$ induces an isomorphism $Z(G) \to \mathbb{A}^2$ with inverse $r$. It follows that $C_R \cap \mathbb{A}^3$ is isomorphic to $p(Z(F) \cap Z(G))$, and the latter is defined by $H_R = 0$, where

$$
H_R = 4y_R^4z^6x^3 - 9x_R^4z_R^4x^2T^2 + (6x_R^4z_R^4 - 12x_R^3z_R^5 - 6x_R^3z_R^5)xT^4
$$

\[ - (12x_R^3z_R^5 + 6x_R^4z_R^5)XT + (4acz_R^6 + 8ax_R^3z_R^5 - b^2z_R^5 + 2bx_R^3z_R^5 - x_R^6)T^6
\]

\[ - 2(4acz_R^6 - 2cx_R^4z_R^2 - 3bx_R^4z_R^2 - b^2z_R^2)T^3 + 4acz_R^6 + 4cx_R^3z_R^2 + b^2z_R^2. \tag{5}
\]

We denote by $K_E$ the canonical divisor of $E$. Let $\overline{k}$ be an algebraic closure of $k$, and write $C_R$ for the base change $C_R \times_k \overline{k}$. Recall that $\nu : E \to \mathbb{P}^1$ is the elliptic fibration on $E$ (Notation 2.1).

**Lemma 2.7.** Let $R = (x_R : y_R : z_R : 1)$ be a point in $E$ with $y_R, z_R \neq 0$, and let $C_R$ be the curve in Definition 2.2. The following hold.

(i) The curve $C_R$ does not contain a fiber of $E$.

(ii) The curve $C_R$ is contained in the linear system $|-3K_E + 3\mathcal{O}|$, and intersects every fiber of $\nu$ in three points counted with multiplicity.

**Proof.** (i). From (3) it is clear that $C_R$ does not contain the fiber $w = 0$. Moreover, since the coefficient of $X^3$ of $H_R$ in (6) as a polynomial in $k[T]$ is constant and non-zero, $C_R$ does not contain any fiber with $w \neq 0$, either.

(ii). The linear system $|-3K_S|$ induces the 3-uple embedding of $S$ into $\mathbb{P}^6$ [CO98, page 1200]. Under this embedding, the curve $\pi(C_R)$ is given by the intersection of $S$ with a hyperplane, hence we have $\pi(C_R) \sim -3K_S$. Since $y_R, z_R \neq 0$, the image $\pi(C_R)$ does not contain the point $\mathcal{O}$, so this implies

$$
C_R = \pi^*(\pi(C_R)) \in \pi^*(-3K_S) = |-3K_E + 3\mathcal{O}|.
$$

Since a fiber $F$ of $\nu$ is linearly equivalent to $-K_E$, which has self-intersection $(\pi^*(K_S) - \mathcal{O})^2 = 0$, and $\mathcal{O}$ is a section of $\nu$, we have $F \cdot C_R = F \cdot (\mathcal{-3K_E + 3\mathcal{O}}) = 0 + 3 = 3$. Since $F$ is irreducible, it follows that, since $F$ is not contained in $C_R$, the number of intersection points of $F$ and $C_R$ is 3, counted with multiplicity. \qed

In Proposition 2.10 we describe for which choices of $R \in E$ the curve $C_R$ has genus at most 1. In the proof we use several known results on the exceptional curves on $S$, which we state in the following remark.

**Remark 2.8.** The surface $S$ contains 240 exceptional curves, which are defined over the separable closure $k^{\text{sep}}$ of $k$ in $\overline{k}$; this follows from [CO98, Propositions 5 and 7], see for example [VA09, Theorem 2.1.1]. Therefore, from [VA08, Theorem 1.2] it follows that the exceptional curves on $S$ are exactly the curves given by

$$
x = p(z, w), \quad y = q(z, w),
$$

where $p, q \in k[z, w]$ are homogeneous of degrees 2 and 3. Note that this implies that an exceptional curve never contains $\mathcal{O} = (1 : 1 : 0 : 0)$. Therefore, for an exceptional curve $C$ on $S$, its strict transform $\pi^*(C)$ on $E$ satisfies

$$
\pi^*(C)^2 = -1, \quad \pi^*(C) \cdot -K_E = \pi^*(C) \cdot (\pi^*(K_S) + \mathcal{O}) = 1 + 0 = 1,
$$

so $\pi^*(C)$ is an exceptional curve on $E$ as well. Moreover, since a fiber of $\nu$ is linearly equivalent to $-K_E$, the curve $\pi^*(C)$ intersects every fiber once. This gives a section of $\nu$. From [SS90, Lemma 7.11] it follows that the sections on $E$ that come from exceptional curves on $S$ are exactly those that are disjoint from $\mathcal{O}$.

Let $\zeta_3 \in \overline{k}$ be a primitive third root of unity. Note that, for a curve $C_R$ as in Definition 2.2, the morphism of $\mathbb{P}(2, 3, 1, 1)$ given by multiplying the $w$-coordinate with $\zeta_3^2$ restricts to an automorphism of $C_R$.

**Definition 2.9.** Let $R = (x_R : y_R : z_R : 1)$ be a point in $E$, with $y_R, z_R \neq 0$, and let $C_R$ be the corresponding curve as in Definition 2.2. By $\sigma$ we denote the automorphism of $C_R$ given by

$$
\sigma : (x : y : z : w) \mapsto (x : y : z : \zeta_3^2w) = (\zeta_3^2x : y : \zeta_3z : w) \tag{6}
$$

Recall that $\pi : E \to S$ is the blow-up of $S$ in $\mathcal{O}$, and $\nu : E \to \mathbb{P}^1$ is the elliptic fibration on $E$.

**Proposition 2.10.** Let $R = (x_R : y_R : z_R : 1)$ be a point in $E$, with $x_R \in k$, $y_R, z_R \in k$ non-zero, and let $C_R$ be the curve in Definition 2.2. The following hold.

(i) The curve $C_R$ is singular in $R$, $\sigma(R)$, and $\sigma^2(R)$.

(ii) If $\pi(R)$ is not contained in an exceptional curve on $\mathbb{F} = S \times_k \overline{k}$, then $C_R$ either contains a section that is defined over $k$, or it is geometrically integral and has geometric genus at most 1, in which case $R$, $\sigma(R)$, $\sigma^2(R)$ are all double points.

Proof. (i). It is an easy check that $R$ is contained in $C_R$. Let $m_R$ be the maximal ideal in the local ring of $R$ on $\mathcal{E}$. The point $R$ lies in $\mathcal{A}^3 \subset \mathbb{P}(2,3,1)$ defined by $w \neq 0$ as in Remark 2.3. The ideal $m_R$ is generated by $X - x_R$, $Y - y_R$, and $T - z_R$. Let $F$, $G$ be as in (4). We have $\mathcal{E} \cap \mathcal{A}^3 = Z(F)$, and using the identity $e = y_R^2 - x_R - az_R^2 - bz_R$, we can write $F$ as

$$F = 2y_R(Y - y_R) - 3x_R^2(X - x_R) - (6a z_R^2 + 3b z_R^3)(T - z_R) + (Y - y_R)^2 - (X - x_R)^3 - 3x_R(X - x_R)^2 - a(T - z_R)^6 - 6a z_R(T - z_R)^5 - 15a z_R(T - z_R)^4 - (20a z_R^3 + b)(T - z_R)^3 - (15a z_R^4 + 3b z_R)(T - z_R)^2.$$ 

Set $\alpha = 2y_R(Y - y_R) - 3x_R^2(X - x_R) - (6a z_R^2 + 3b z_R^3)(T - z_R)$, then it follows that $\alpha$ is contained in $m_R^2$, and the tangent line to $\mathcal{E}$ at $R$ is given by $\alpha = 0$.

Similarly, we can rewrite $G$ as

$$G = -x_R^2 \alpha + 3x_R^2 z_R^2(X - x_R)(T - z_R) - (x_R^2 - 2a z_R - b z_R^3)(T - z_R)^3 - (3x_R z_R^2 - 6a z_R^2 - 3b z_R^4)(T - z_R)^2,$$

so we conclude that $G$ is contained in $m_R^2$, hence $C_R$ is singular in $R$. Since $\sigma$ is an automorphism of $C_R$, this implies that $C_R$ is singular in $\sigma(R)$ and $\sigma^2(R)$, as well.

(ii). Assume that $\pi(R)$ is not contained in an exceptional curve on $\mathbb{S}$. We distinguish two cases. First assume that $C_R$ is not irreducible or not reduced. Since $C_R$ does not contain a fiber and intersects every fiber with multiplicity 3 (Lemma 2.7), this implies that there is a curve that intersects every fiber with multiplicity one (hence is a section), say $H_1$, such that $C_R$ contains $H_1$ as irreducible component, or $C_R$ is a multiple of $H_1$. Since $C_R$ is disjoint from the zero section, it follows that $\pi(H_1)$ is an exceptional curve on $\mathbb{S}$ (Remark 2.8). Therefore, by our assumption, $R$ is not contained in $H_1$, so $C_R$ is not a multiple of $H_1$, and $H_1$ is an irreducible component of $C_R$. Let $H_2$ be the other (not necessarily irreducible or reduced) component of $C_R$, which contains $R$. If $H_2$ were not irreducible or not reduced, it would either be a double section or two sections intersecting in $R$. In both cases, $\pi(R)$ lies on an exceptional curve, contradicting our first assumption. We conclude that $H_2$ is irreducible and reduced. Since $C_R$ is defined over $k$, it is fixed by the action of the absolute Galois group of $k$ on $\text{Pic} S$. The exceptional curves of $\mathbb{S}$ are all defined over the separable closure $k^\text{sep}$ of $k$ by Remark 2.8 so the Galois group $\text{Gal}(k^\text{sep}/k)$ acts on them. Since $C_R$ contains only one exceptional curve of $S$, which is $H_1$, it follows that this component is invariant under the Galois action, hence it is defined over $k$. This finishes the first case. Now assume that $C_R$ is irreducible and reduced. Since $C_R$ is contained in the line $y = x^3 + 3x T$ by Lemma 2.7 from the adjunction formula it follows that its arithmetic genus is $\frac{1}{2} \cdot (9 - 3) + 1 = 4$. Since the three distinct points $R$, $\sigma(R)$, $\sigma^2(R)$ are all singular on $C_R$ with the same multiplicity, we conclude that they all have multiplicity 2, and the geometric genus of $C_R$ is at most 1. \qed

Remark 2.11. Proposition 2.10 (i) also follows from the description of $C_R$ in Remark 2.3. The curve $C_Q$ defined there is clearly singular in $Q$, as it is contained in the tangent plane to $T$ at $Q$. The inverse image of $C_Q$ under $f$ contains the 3 singular points $(x', y', w')$ for which $w'^3 = w_R$, which are exactly the points $R$, $\sigma(R)$, and $\sigma^2(R)$.

Remark 2.12. In the last proof, we concluded that in the case where $C_R$ is geometrically integral, the geometric genus of $C_R$ is at most 1. If it were 0, then $C_R$ would contain exactly one more singular point besides $R$, $\sigma(R)$, $\sigma^2(R)$, say $Q$. Then $\sigma(Q)$ and $\sigma^2(Q)$ would be singular points of $C_R$ as well, so $Q$ would be a fixed point of $\sigma$. Note that the points on the intersection of $C_R$ with the fiber above $(1:0)$ are fixed points of $\sigma$. Assume that $\sigma$ has a fixed point $Q = (x_Q : y_Q : z_Q : 1)$ in $C_R \setminus (C_R \cap \mathcal{E}(1,0))$. From [6] it follows that there is a $\lambda \in k$ such that $\lambda^3y_Q = y_Q$, $\lambda^2x_Q = x_Q$, $\lambda z_Q = z_Q$, and $\lambda^3 = 1$. The last equation implies $\lambda = \zeta_3^{n - 2}$ for some $n > 0$, and it follows that $x_Q = z_Q = 0$. From the fact that $Q$ lies in $\mathcal{E}$ it follows that we have $y_Q^3 = c$. We conclude that if $C_R$ is geometrically integral, then it has genus 0 if and only if it has a singular point which is a fixed point of $\sigma$ and which either lies on the fiber $(1:0)$, or is the point $(0: \sqrt[3]{c} : 0 : 1)$. In our experiments with different surfaces and different points we have not found an example where this happens.
Proof of Proposition 2.5. Let $F$ be as in the proposition. There are finitely many points on $F$ with $y$-coordinate 0, and from Remark 2.8 it follows that there are at most 240 points $R$ on $F$ with $\pi(R)$ contained in an exceptional curve on $S$; let $V_1$ be the set of points in $F$ for which either of these two conditions holds. Since $F$ has positive rank over $k$ by assumption, the set of $k$-rational points on $F_0 = F \setminus V_1$ is infinite. For each point $R \in F_0(k)$ we can define the curve $C_R$ as in Definition 2.2, and $C_R$ contains either a section defined over $k$, or it is geometrically integral and has geometric genus at most 1 by Proposition 2.10. Therefore, if there is no $R \in F(k)$ such that $C_R$ either contains a section over $k$ or is geometrically integral of genus 0, then for all $R \in F_0(k)$, the curve $C_R$ is geometrically integral of genus 1.

3 An elliptic fibration with non-torsion section

Let $F$ be a smooth fiber on $E$ above a point $(z_0 : 1) \in \mathbb{P}^1$ with $z_0 \neq 0$, such that $F$ has positive rank over $k$. Assume that there is an open subset of $F$, which we denote by $F_0$, such that for every point $R \in F_0$, the curve $C_R$ is well-defined and geometrically integral of genus 1. In this section we show that this implies that there is an elliptic fibration over $F$ that admits a non-torsion section.

Remark 3.1. Let $R$ be a point in $F_0$. Since $C_R$ intersects every fiber of $\nu$ in three points counted with multiplicity (Lemma 2.7), it is a 3-section. Moreover, since $R$ is a double point on $C_R$ by Proposition 2.10, there is a unique third point of intersection of $C_R$ with $F$, say $Q$ (see Figure 1). Hence $E_R = (\tilde{C}_R, Q)$ is an elliptic curve, where $\tilde{C}_R$ is the normalization of $C_R$. The curve $E_R$ contains a rational point, which we denote by $D_R$, which is the sum of the points corresponding to $\sigma(Q)$ and $\sigma^2(Q)$ on $C_R$.

![Figure 1: The multisection $C_R$ on $E$, when $C_R$ is geometrically integral of genus 1.](image)

Notation 3.2. For a point $R \in F_0$, we denote by $E_R$ the corresponding elliptic curve and by $D_R$ the point on it, both as defined in Remark 3.1.

Let $\eta$ be the generic point of $F$, that is, $\eta$ is the point given by $(\tilde{x} : \tilde{y} : z_0 : 1)$ over the function field $k(F) = k(\tilde{x}, \tilde{y}) = \text{Frac}(k[x, y]/(y^2 - x^3 - az_0^6 - bz_0^3 - c))$ of $F$. Let $C_\eta \subseteq \mathbb{P}_{k(F)}(2, 3, 1, 1)$ be the corresponding curve given by (4). From Proposition 2.10 and Remark 2.12 it follows that $C_\eta$ is geometrically integral of genus 1. Let $E_\eta$ be the corresponding elliptic curve with point $D_\eta$ as in Notation 3.2.

In Lemma 3.3 we give a Weierstrass model for the curve $E_\eta$, which we use in Proposition 3.5. Recall that $a, b, c,$ and $z_0$ are fixed elements in $k$, and $a, c, z_0$ are non-zero. We define the polynomial

$$q = q_1 q_2 q_3 q_4$$

(7)
L2

\[ q_1 = \tilde{x}; \]
\[ q_2 = -\tilde{x}^6 + 2z_0^3 (4a^2z_0^3 + b) \tilde{x}^3 + (4ac - b^2)z_0^6; \]
\[ q_3 = \tilde{x}^6 + 8 (a\tilde{z}_0^6 - c) \tilde{x}^3 + 8 (2a^2z_0^{12} + 3ab\tilde{z}_0^9 + (2ac + b^2)z_0^6 + bc\tilde{z}_0^3); \]
\[ q_4 = 29\tilde{x}^{12} + (40c + 24a\tilde{z}_0^3)\tilde{x}^9 + 8(12a^2z_0^{12} + 9ab\tilde{z}_0^9 + (18ac - 5b^2)z_0^6 - 5bc\tilde{z}_0^3 - 2c^2\tilde{z}_0^6) + 32(4a^2z_0^{18} + 9a^2b\tilde{z}_0^6) + (5ab^2 + 12a^2c)\tilde{z}_0^9 + 14abc\tilde{z}_0^6 + (b^2c + 8ac^2)z_0^6 + be^2z_0^3)\tilde{x}^3 + 16((4a^2c - a^2b^2)\tilde{z}_0^6 + (8a^2bc - 2ab^3)z_0^6 + (8a^2c^2 + 2ab^2c - b^4)z_0^6 + (8abc^2 - 2b^3c)z_0^6 + (4ac^3 - b^2c^2)z_0^6). \]

**Lemma 3.3.** There exists a unique polynomial \( \delta \in k[\tilde{x}] \) with leading term \(-27c\tilde{x}_0^{18}\tilde{z}_0^{81}\) such that the following holds. There is an isomorphism \( \omega \) between the elliptic curve \( E_\eta \) and the curve with Weierstrass equation given by

\[ \gamma^2 = \xi^3 + \delta, \]

where the denominators in the defining equations of \( \omega \) and \( \omega^{-1} \) are all of the form \( 2^p3^r(q_2q_4)^s \) for positive integers \( p, r, s \).

Let \( \omega(D_\eta) \) be the point on (8) corresponding to the point \( D_\eta \) on \( E_\eta \) given by

\[ \omega(D_\eta) = (\xi_D, \gamma_D), \]

where

\[ \xi_D = \frac{\alpha}{(q_1q_4)^2}, \quad \gamma_D = \frac{\beta}{(q_1q_4)^2} \]

are rational functions, and \( \alpha, \beta \) are polynomials in \( k[\tilde{x}] \), with leading terms \( \frac{1}{4}\tilde{x}_0^{16}x^{42} \) and \( \frac{1}{8}\tilde{x}_0^{24}x^{63} \), respectively.

**Proof.** The magma code that is used in this proof can be found in [Code]. Let \( Q \) be the third point of intersection of \( C_\eta \) with the fiber of \( \eta \) on the base change \( \mathcal{E} \times_k k(\mathcal{F}) \) over \( \mathbb{P}^1 \times_k k(\mathcal{F}) \). Write \( Q = (x_Q, y_Q: z_0 : 1) \), with \( x_Q, y_Q \in k(\mathcal{F}) \). Then \( Q \) lies in \( C_\eta \cap (k^3 \times_k k(\mathcal{F})) \), which is isomorphic to the curve \( C_\eta^1 \) defined by \( H_0 = 0 \), where \( H_0 \) is given in (5) after substituting \( R \) by \( \eta \). We find \( x_Q \) by substituting \( T = z_0, c = \tilde{x}^2 - \tilde{x}^3 - az_0^6 - b\tilde{z}_0^6 \) in (5) and factorizing in \( k(\mathcal{F})[X] \), which yields

\[ x_Q = \frac{9\tilde{x}^4 - 8\tilde{x}^2y^2}{4y^2}. \]

We conclude that the elliptic curve \( E_\eta \) as defined in Remark 3.1 is isomorphic to the curve \( \left( \tilde{C}_\eta^1, \left( \frac{9\tilde{x}^4 - 8\tilde{x}^2y^2}{4y^2}, z_0 \right) \right) \), where \( \tilde{C}_\eta^1 \) is the normalization of \( C_\eta^1 \). With magma we compute a Weierstrass model for \( E_\eta \), which is given by

\[ \gamma' = \xi^3 + \left( \frac{3 \cdot 2^5}{(q_2q_4)^2} \delta \right), \]

where \( \delta \) is a polynomial in \( k[\tilde{x}] \) with leading term \(-27c\tilde{x}_0^{18}\tilde{z}_0^{81}\). We verify with magma that the denominators in the defining equations of the isomorphism \( \omega_1 \) between \( E_\eta \) and the curve (10), as well as those of \( \omega_1^{-1} \), are all of the form \( 2^p3^r(q_2q_4)^s \) for positive integers \( p', r', s' \). The change of coordinates

\[ \xi' = \left( \frac{3 \cdot 2^5}{(q_2q_4)^2} \right) \xi, \quad \gamma' = \left( \frac{3 \cdot 2^5}{(q_2q_4)^2} \right) \gamma, \]

induces an isomorphism \( \omega_2 \) between the curve (11) and the curve defined by

\[ \gamma^2 = \xi^3 + \delta. \]

We conclude that \( \omega = \omega_2 \circ \omega_1 \) is an isomorphism between \( E_\eta \) and the curve (11), and the denominators in the defining equations of \( \omega \) and \( \omega^{-1} \) are all of the form \( 2^p3^r(q_2q_4)^s \) for positive integers \( p, r, s \).

If \( \delta' \) was another polynomial in \( k[\tilde{x}] \) such that \( E_\eta \) were isomorphic to the curve given by \( \gamma^2 = \xi^3 + \delta' \), then we would have \( \delta' = \epsilon^\delta \delta \) for some \( \epsilon \in k(\mathcal{F}) \), hence \( \delta' \) would not have leading term \(-27c\tilde{x}_0^{18}\tilde{z}_0^{81}\). We conclude that \( \delta \) is the unique polynomial with leading term \(-27c\tilde{x}_0^{18}\tilde{z}_0^{81}\) such that \( E_\eta \) is isomorphic to the curve with Weierstrass model (11). With magma we compute the sum \( D \) on the curve (11) of the points corresponding to \( \left( \xi^3_{\tilde{C}_\eta^1}, \tilde{z}_0 \right) \) and \( \left( \xi^3_{\tilde{C}_\eta^1} - 8\tilde{x}^2y^2, \tilde{z}_0 \right) \) on \( C_\eta \). We find \( D = (\xi_D, \gamma_D) \) with \( \xi_D = \frac{\alpha}{(q_1q_4)^2}, \gamma_D = \frac{\beta}{(q_1q_4)^2} \), where \( \alpha, \beta \) are elements in \( k[\tilde{x}] \) with leading terms given by \( \frac{1}{4}\tilde{x}_0^{16}x^{42} \) and \( \frac{1}{8}\tilde{x}_0^{24}x^{63} \), respectively. \( \square \)
Remark 3.4. The curve in \([8]\) over the function field \(k(F)\) of \(F\) gives rise to a unique relatively minimal elliptic surface \(\rho: C \rightarrow F\) over \(F\) \(\text{[SS19, Theorem 5.19]}\), such that the generic fiber of \(\rho\) is isomorphic to \(E_\eta\). Recall the polynomial \(q\) in \([7]\). From Lemma 3.3 it follows that for every \(R = (x_R : y_R : z_0 : 1) \in F_0\) with \(q(x_R) \neq 0\), the fiber of \(\rho\) above \(R\) is isomorphic to the curve \(E_R\) as in Notation 3.2. Moreover, the point \(D_\eta\) on \(E_\eta\) gives rise to a section \(D\) on \(C\). See Figure 2. In Proposition 3.5 we show that \(D\) is a non-torsion section.

![Diagram of \(E\) and \(C\)](image)

**Figure 2:** Left: two points \(R, R' \in F_0(k)\), with corresponding curves \(C_R, C_{R'}\). Right: the fibration on \(F\) with two fibers that are the normalizations of the multisections \(C_R, C_{R'}\), and the section \(D\).

Proposition 3.5. If \(k\) has characteristic 0, then the point \(D_\eta\) is non-torsion on \(E_\eta\).

**Proof.** It suffices to show that the section \(D\) intersects a fiber of \(\rho: C \rightarrow F\) in a non-torsion point. We use the Weierstrass equation for the elliptic surface \(C\) given in Lemma 3.3, and look at the fiber above the point at infinity on \(F\) by setting \(\psi = \frac{1}{2}\), multiplying by factors of \(\psi\) to obtain polynomials in \(k[\psi]\), and evaluating at \(\psi = 0\).

After setting \(\psi = \frac{1}{2}\) and applying the change of coordinates \(\xi' = \psi^{28}\xi\), \(\gamma' = \psi^{42}\gamma\) in \([8]\), we obtain \(\gamma'^2 = \xi'^3 + \delta'\), where \(\delta' = \psi^{84}\delta\). Since the leading term of \(\delta\) in \(k[x]\) has degree 81, the rational function \(\delta'\) is in fact a polynomial in \(k[\psi]\), and it is divisible by \(\psi^{3}\). Therefore, evaluating \(\delta'\) at \(\psi = 0\) gives 0, and the fiber above infinity on \(F\) is given by \(F_\infty : \gamma'^2 = \xi'^3\).

The section \(D\) intersects \(F_\infty\) in a point \((\xi_\infty, \gamma_\infty)\). The leading coefficient of \((q_1q_2)^2\) equals \(\bar{z}^{14}\), so we have

\[
\xi_D = \frac{\alpha(\frac{1}{\psi})}{(q_1(\frac{1}{\psi})q_3(\frac{1}{\psi}))^2} = \frac{\psi^{14}\alpha(\frac{1}{\psi})}{\psi^{14}(q_1(\frac{1}{\psi})q_3(\frac{1}{\psi}))^2},
\]

where the denominator of the latter is a polynomial \(q'\) in \(k[\psi]\) with constant coefficient 1, and the numerator, as polynomial in \(\bar{z}\), has leading term \(\frac{1}{4}z_{0}^{16}\bar{z}^{28}\). We conclude that we have

\[
\xi_\infty = (\psi^{28}\xi_D)|_{\psi=0} = \frac{1}{4}z_{0}^{16}q'(0) = \frac{1}{4}z_{0}^{16}.
\]

Similarly, we have \(\gamma_\infty = \frac{1}{4}z_{0}^{24}\). So \(D\) intersects \(F_\infty\) in the non-zero point \(\left(\frac{1}{4}z_{0}^{16}, \frac{1}{4}z_{0}^{24}\right)\). Since the group \(F_\infty(k)\) of non-singular \(k\)-rational points on \(F_\infty\) is isomorphic to the additive group of \(k\), and we have \(\text{char } k = 0\), we conclude that \(\left(\frac{1}{4}z_{0}^{16}, \frac{1}{4}z_{0}^{24}\right)\) is non-torsion on \(F_\infty\).

**4 Proof of Theorem 1.1**

In this section we prove Theorem 1.1. Let \(a, b, c, k, S, \) and \(E\) be as in the theorem; in particular, \(k\) is now a field of characteristic 0. Recall Notation \(2.3\).

We use the following theorem of Colliot-Thélène, which gives us a stronger version of Merel’s theorem for bounding the torsion in a family of elliptic curves. The proof can be found in the Appendix at the end of this paper. The same result is also mentioned in Footnote 1 of the paper \(\text{[CT12]}\) by Cadoret and Tamagawa.

**Proposition 4.1**
Theorem 4.1. [Théorème in the Appendix] Let $l$ be a field that is finitely generated over $\mathbb{Q}$. There exists an integer $N = N(l)$ such that if $C/l$ is a geometrically integral $l$-variety, and $E/C$ a smooth family of elliptic curves, then for all points $P \in C(l)$, the order of a torsion point on the fiber $E_P(l)$ is at most $N$.

Proof of Theorem 1.1. By assumption, there is a point $P \in S(k)$ such that the corresponding point $P_\nu$ on $E$ lies on a smooth fiber $F_P$ above $(z_\nu : 1) \in \mathbb{P}^1$ for some $z_\nu \in k$ non-zero, and $P_\nu$ is non-torsion on $F_P$. From Proposition 2.5 it follows that at least one of the following holds.

(i) There is a point $R \in F_P(k)$ such that the curve $C_R$ as in [3] contains a section defined over $k$;

(ii) there is a point $R \in F_P(k)$ such that the curve $C_R$ as in [3] is geometrically integral of genus 0;

(iii) there is an open subset $F_{P,0}$ of $F_P$ such that for every point $R \in F_{P,0}(k)$, the curve $C_R$ is geometrically integral of genus 1.

Note that in case (i) we are done by Remark 1.5. In case (ii), the normalization $n: \tilde{C}_R \rightarrow C_R$ gives a smooth curve of genus 0. Since $R$ is not a triple point on $C_R$ (Proposition 2.10), the latter contains a rational point given by the unique other point in the intersection of $C_R$ with $F_P$, hence $C_R$ contains infinitely many $k$-rational points. Consider the base change $\nu_R: E \times_k \tilde{C}_R \rightarrow \tilde{C}_R$ of $\nu$, which is an elliptic fibration with section $\tilde{C}_R \rightarrow E \times_{\mathbb{P}^1} \tilde{C}_R$, $p \mapsto (\nu(p), p)$. The latter has infinite order [Sin14, Theorem 6.4], and is defined over $k$, so the set $(E \times_{\mathbb{P}^1} \tilde{C}_R)(k)$ is dense in $E \times_{\mathbb{P}^1} \tilde{C}_R$ by Remark 1.5. Since $E \times_{\mathbb{P}^1} \tilde{C}_R$ maps dominantly to $E$, and hence to $S$, the density of $S(k)$ in $S$ follows. If we are in case (iii), it follows from Remark 3.4 and Proposition 3.5 that there is an elliptic fibration $\rho: C_P \rightarrow F_P$ such that almost all fibers are the normalizations of a 3-section of $E$, and such that $\rho$ admits a section defined over $k$ of finite order. From Remark 1.5 it follows that $C_P(k)$ is Zariski dense in $C_P$. Since almost all fibers of $\rho$ are normalizations of $3$-sections of $E$, the surface $C_P$ maps dominantly to $E$, and hence to $S$. It follows that $S(k)$ is dense in $S$ as well. This proves the first statement of the theorem.

Now assume that $k$ is finitely generated over $\mathbb{Q}$, and that $S(k)$ is dense in $S$. Since every smooth fiber of $E$ is an elliptic curve over $k$, there is an upper bound $N = N(k)$ such that on all the fibers, all the torsion points have order at most $N$ (Theorem 4.1). Let $m \leq N$ be an integer, and let $T_m$ be the zero locus of the $m$-th division polynomial $\psi_m \in k[x,t]$ of the generic fiber $E$ of $\mathcal{E}$, which is an elliptic curve over the function field $k(t)$ of $\mathbb{P}^1$. We have $\psi_m \in k[x,t]$, and for any $\tau \in k$, the polynomial $\psi_m(x,\tau) \in k[x]$ has degree $m^2$. So $T_m$ is an $m^2$-section of $E$. If $S$ contains a point $P$ as in the theorem, then $S(k)$ would be contained in the union of the torsion locus $\cup_{m \leq N} T_m$, with the two fibers $(1:0)$ and $(0:1)$ and the singular fibers, which is a strict closed subset of $S$, contradicting the assumption that $S(k)$ is dense in $S$. This finishes the proof.

5 Examples

We conclude this paper by giving examples where we prove the density of rational points on specific surfaces. The rank of the Mordell–Weil group over $\mathbb{Q}$ of the surfaces in Examples 5.1 and 5.2 is 0 by [DN, Corollary 2.4], so in these cases the density of the $\mathbb{Q}$-rational points can not be proven by the existence of a section over $\mathbb{Q}$ (see also Remark 1.5). The surface in Example 5.3 has Mordell–Weil rank 2 over $\mathbb{Q}$ [DN, Corollary 2.4], so the density of the set of rational points is implied by Remark 1.5. For this surface, we show how our method can construct a rational section as a component of the curve $C_R$ for a certain point $R$ (this is one of the cases in Proposition 2.5).

Example 5.1. Let $k$ be field of characteristic 0 and let $S$ be the surface given by

$$y^2 = x^3 + 6(27z^6 + w^6).$$

Note that $S$ does not satisfy the conditions of [VA11, Theorem 1.1] since $3 \cdot 27$ is a square and $\gcd(6, 27, 6) \neq 1$, hence the density of $\mathbb{Q}$-rational points could not be proven by Várilly-Alvarado [VA11, Example 7.2]. However, the fiber $\mathcal{E}_{(1:1)}$ of the anticanonical elliptic surface $\mathcal{E}$ above $(1 : 1)$ is smooth, and with magma we find that this fiber has rank 2. So $S$ contains a point that lies on a smooth fiber of $\mathcal{E}$ and has infinite order, hence $S(k)$ is dense in $S$ by Theorem 1.1.

We illustrate this by constructing a 3-section as in [3]. With magma we find two generators for $\mathcal{E}_{(1:1)}(\mathbb{Q})$, given by $P_1 = (1 : 13 : 1 : 1)$ and $P_2 = (22 : 104 : 1 : 1)$. The curve $C_{P_1}$ is cut out from $S$ by $3xz - 26y + 323z^3 + 12w^3$, and it has geometric genus 1. We find $C_{P_1} \cap \mathcal{E}_{(1:1)} = \{P_1, Q_1\}$ with
Let \( k \) be a field of characteristic 0 and consider the surface \( S \) given by
\[
y^2 = x^3 + 243z^6 + 16w^6.
\]
Note that this surface does not satisfy the conditions of [VA11 Theorem 1.1], so the method there failed in this case [VA11 Remark 7.4]. Salgado and van Luijk made the observation that this surface contains the point \( P = (0 : 4 : 0 : 1) \), which is 3-torsion on its fiber on the corresponding elliptic surface \( E \) (more generally, for \( \beta \in k^* \), the elliptic curve of the form \( y^2 = x^3 + \beta^2 \) has the 3-torsion point \((0, \beta)\)). However, this point is contained in 9 exceptional curves, so their method does not work with \( P \). They did not find another point for which the computations were doable to show density of \( S(k) \) [SV14 Examples 7.3 and 4.4 (iii)]. Finally, Elkies showed that the set \( S(\mathbb{Q}) \) is Zariski dense in \( S \), by constructing a multisection with infinitely many rational points in the linear system \( -3K_S \) that contains \( P \) as a point of multiplicity 3 (this idea was generalized to any surface with a torsion point in the master thesis [Bul18], though under the assumption that at least one of the infinitely many multisections constructed there has infinitely many rational points).

We prove the density of \( S(k) \) in \( S \) using Theorem 1.1 with magma we find that the fiber \( E_{(1,5)} \) above \((1 : 5)\) is smooth and has rank 2, so \( S \) contains a point that lies on a smooth fiber of \( E \) and has infinite order (for example \( P = (-63 : -14 : 1 : 5) \)), hence \( S(k) \) is dense in \( S \).

Example 5.3. Let \( k \) be a field of characteristic 0, and let \( S \) be the del Pezzo surface of degree 1 over \( k \) given in \( \mathbb{P}(2,3,1,1) \) by the equation
\[
y^2 = x^3 + 27z^6 + 16w^6.
\]
For \( k = \mathbb{Q} \), the rank of the Mordell–Weil group of the corresponding elliptic surface \( E \) is 2 [DN Corollary 2.4]. In this example we illustrate that different cases in Proposition 2.5 can happen on the same surface; we give a point \( P \) on \( S \) such that \( C_P \) contains a section defined over \( k \), and we give a point \( Q \) on \( S \) such that \( C_Q \) is geometrically integral of genus 1. We have not found a point for which the corresponding curve is geometrically integral of genus 0 (see also Remark 2.12).

The point \( P = (-3 : -4 : 1 : 1) \) on \( S \) corresponds to a non-torsion point on a smooth fiber of \( E \). The curve \( C_P \) is cut out from \( S \) by the hypersurface \( 27xz + 8y + 81z^3 + 32w^3 = 0 \). It is the union of a 2-section of genus 1 and a section of genus 0, both containing \( P \). For \( k = \mathbb{Q} \) we compute with magma that the 2-section is an elliptic curve of rank 3. The section is given by the curve \( x + 3z^2 = y + 4w^3 = 0 \) in \( \mathbb{P}(2,3,1,1) \), and it corresponds to the point \((-3t^2, -4)\) on the generic fiber of \( E \) over the function field \( k(t) \) of \( \mathbb{P}^1 \), where we set \( t = z/w \). This is the first case in Proposition 2.5.

By starting with the point \( Q = (36 : -220 : 2 : 1) \) on \( S \), which also corresponds to a point on a smooth fiber of \( E \), we obtain the curve \( C_Q \) cut out on \( S \) by the surface \( 243xz + 55y - 675z^3 + 4w^3 \). The curve \( C_Q \) is now geometrically integral of genus 1. For \( k = \mathbb{Q} \) we compute with magma, under the condition SetClassGroupBounds("GRH"), that its normalization is an elliptic curve of rank 4.

Appendix. Un corollaire d’un théorème de Merel (par Jean-Louis Colliot-Thélène)

Un théorème bien connu de L. Merel [Mer96] borne la torsion des courbes elliptiques sur un corps de nombres \( k \), et ce de façon uniforme en fonction uniquement du degré du corps de \( k \) sur \( \mathbb{Q} \). Je remarque qu’on en déduit facilement une extension au cas des corps de type fini sur \( \mathbb{Q} \).

On utilise le lemme bien connu suivant.
Lemme. Soient $k$ un corps de caractéristique zéro, $Y$ une $k$-variété intègre et $f : X \rightarrow Y$ une famille lisse de variétés abéliennes. Si la fibre générique de $f$ possède un point exactement de $n$-torsion, alors pour tout point (schématique) $P$ du schéma $Y$, la fibre $\mathcal{X}_P/\mathcal{X}(P)$ possède un point exactement de $n$-torsion.

Démonstration. Pour tout entier $n$, le schéma des points de $n$-torsion est fini étale sur $Y$. En particulier le sous-schéma formé des sections d’ordre exactement $n$ est une union disjointe d’images de sections de $f$. CQFD

Voici l’extension du théorème de Merel.

Théorème. Soit $k$ un corps de type fini sur $\mathbb{Q}$, soit $C$ une $k$-variété intègre, et soit $E/C$ une famille lisse de courbes elliptiques. Alors il existe un entier $N$ (dépendant de $k$) tel que, pour tout point $P \in C(k)$, l’ordre d’un point $k$-rationnel de torsion sur la fibre $E_P$ est au plus $N$.

Démonstration. Le corps $k$ s’écrit comme le corps des fractions d’une $\mathbb{Q}$-variété intègre $U = \text{Spec}(A)$, qu’on peut choisir finie étale, d’un certain degré $d$, sur un ouvert d’un espace affine $\mathbb{A}^r_d$.

Quitter à restreindre $U$ on peut étendre la situation $E/C/k/\mathbb{Q}$ à $F/D/U/\mathbb{Q}$ avec $F/D$ famille de courbes elliptiques sur $D$. Un point $k$-rationnel $P$ de $C$ s’étend en une section $\tau_P : V \rightarrow D$ de la projection $D \rightarrow U$ sur un ouvert $V \subset U$ (ouvert dépendant de $P$) non vide. L’image réciproque de $F \rightarrow D$ au-dessus de $V$ via $\tau_P$ est une famille de courbes elliptiques dont la fibre générique est $E_P$. L’ensemble des points fermés de $V$ de degré au plus $d$ est Zariski dense dans $V$ (considérer les images réciproques des points de $\mathbb{A}^r_1(\mathbb{Q}))$, en particulier est non vide. Le théorème de Merel assure que l’ordre des points de torsion des courbes elliptiques sur un corps de nombres de degré au plus $d$ est borné par un entier $N(d)$. Le lemme permet alors de conclure. CQFD

Si l’on note $\phi(d)$ la borne sur l’ordre d’un point de torsion donnée par le théorème de Merel sur les corps de nombres de degré au plus $d$ et si, pour $k$ de type fini sur $\mathbb{Q}$, on note $d_{\min}(k)$ le degré minimal de la présentation de $k$ comme extension finie $k/E$ d’une extension transcendantale pure $E$ de $\mathbb{Q}$, alors on peut borner $N$ dans le théorème par $\phi(d_{\min}(k))$.

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