EXPLICIT ZERO-FREE REGIONS FOR DEDEKIND ZETA FUNCTIONS

HABIBA KADIRI

Abstract. Let \( K \) be a number field, \( n_K \) its degree, and \( d_K \) the absolute value of its discriminant. We prove that, if \( d_K \) is sufficiently large, then the Dedekind zeta function \( \zeta_K(s) \) has no zeros in the region:

\[
\Re s \geq 1 - \frac{1}{\log M}, \quad |\Im s| \geq 1,
\]

where \( \log M = 12.55 \log d_K + 9.69n_K \log |\Im s| + 3.03n_K + 58.63 \). Moreover, it has at most one zero in the region:

\[
\Re s \geq 1 - \frac{1}{12.74 \log d_K}, \quad |\Im s| \leq 1.
\]

This zero if it exists is simple and is real. This argument also improves a result of Stark by a factor of 2:

\[
\zeta_K(s) \text{ has at most one zero in the region } \Re s \geq 1 - \frac{1}{2 \log d_K}, \quad |\Im s| \leq 1/2 \log d_K.
\]

1. Introduction

Let \( K \) be a number field. Its degree is denoted \( n_K = [K : \mathbb{Q}] \), the absolute value of its discriminant is \( d_K \), and the Dedekind zeta function associated to \( K \) is \( \zeta_K(s) \).

In this article, we prove an explicit classical zero-free region for \( \zeta_K(s) \).

Rosser and Schoenfeld published a series of articles devoted to obtaining improved estimates for prime counting functions (see [13], [14], [15], and [16]), enlarging de La Vallé Poussin’s classical zero-free region in [15]. By employing the global explicit formula for \( -\zeta'(s) \zeta(s) \) and building on an argument of Stechkin [18], they proved that \( \zeta(s) \) has no zeros in the region

\[
\Re s \geq 1 - \frac{1}{R_1 \log(|\Im s|/17)}
\]

where \( R_1 = 9.645908801 \). McCurley applied the same method to Dirichlet \( L \)-functions. He proved in [12] that the product \( L_q(s) = \prod_{\chi \mod q} L(s, \chi) \) has at most a single zero in the region

\[
\Re s \geq 1 - \frac{1}{R_2 \log \max (q, q|\Im s|, 10)}
\]

where \( R_2 = 9.645908801 \). The single zero, if it exists is real, simple, and corresponds to a non-principal real character. The constant is independent of the modulus \( q \), and is valid for any value of \( q \geq 3 \). Observe that the two constants agree: \( R_2 = R_1 \).

In 1992, Heath-Brown established an asymptotic result which provides a wider zero-free region for sufficiently large modulus \( q \):

\[
\Re s \geq 1 - \frac{1}{R_3 \log q}, \quad |\Im s| \leq 1
\]

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where $R_3 = 2.8735 \ldots$ is smaller than McCurley’s constant. Heath-Brown’s method is different from the one used to obtain (1.1) and (1.2). Some of the main tools in his proof are: a smooth explicit formula for zeros of Dirichlet $L$-functions, Burgess’ sub-convexity bound for Dirichlet $L$-functions, and a local Jensen type formula. The previous zero-free region is one of the main ingredients in the proof of Linnik’s theorem on the size of the smallest prime $P(a, q)$ in an arithmetic progression ($a$ modulo $q$). In his groundbreaking article [3], Heath-Brown improves drastically all previous results on Linnik’s theorem and shows that $P(a, q) \ll q^{5.5+\epsilon}$. Recently, Xylouris reduced $R_3$ to 2.2727 \ldots and Linnik’s constant to 5.2 in his Ph.D. thesis [20]. In 2000, Ford [2] applied Heath-Brown’s argument to the case of the Riemann zeta function. This allowed him to produce an explicit Korobov-Vinogradov zero-free region and to widen the region in (1.1) by replacing $R_1$ by 8.463.

In [4], the author further reduced the value of $R_1$ to 5.69693. The method used a global explicit formula applied to a smoothed version of the Riemann zeta-function, together with a generalization of Stechkin’s lemma. This method also improves McCurley’s result. In [5], the author finds that $R_2 = 6.50$ is an admissible value for any Dirichlet $L$-function.

In comparison, in the number field setting, there are no analogous theorems to (1.1), (1.2), and (1.3) with explicit constants. However, in [17] Stark established an explicit result for the Dedekind zeta function in a restricted region. He established that for any number field $K \neq \mathbb{Q}$, $\zeta_K(s)$ has at most one zero in the region

$$\Re s \geq 1 - \frac{1}{4 \log d_K}, \quad |\Im s| \leq \frac{1}{4 \log d_K}.$$\hspace{1cm} (1.4)

If such a zero exists, it is real and simple. In Lemma 2.3 of [3], Lagarias, Montgomery, and Odlyzko establish zero-free regions for Hecke $L$-functions. They prove that for all finite extensions $K$ of $\mathbb{Q}$ and Hecke characters $\chi$ on $K$, the Hecke $L$-function $L(s, \chi, K)$ has at most one zero in the region

$$\Re s > 1 - \frac{1}{R_4 \log A(\chi)}, \quad |\Im s| < \frac{1}{R_4 \log A(\chi)}, \hspace{1cm} (1.5)$$

where $A(\chi) = d_K N_{K/\mathbb{Q}} f(\chi)$, $f(\chi)$ being the conductor of $\chi$, and where $R_4$ is a positive constant, independent of $K$. They also extend the region to the whole critical strip and prove that $L(s, \chi, K)$ has no zeros in the region

$$\Re s > 1 - \frac{1}{R_4 (\log A(\chi) + n_K \log(|\gamma| + 2)).} \hspace{1cm} (1.6)$$

The classical argument of de La Vallée Poussin is used to prove the above inequalities. Moreover, (1.5) and (1.6) play an important role in their proof of a bound for the least prime ideal in the Chebotarev density theorem. However, the constant $R_4$ is not made explicit. In this article we shall apply some of the above mentioned techniques to obtain the following result.

**Theorem 1.1.** Let $d_K$ be sufficiently large. Then $\zeta_K(s)$ has no zero in the region:

$$\Re s \geq 1 - \frac{1}{12.55 \log d_K + 9.69(\log |\Im s|)n_K + 3.03n_K + 58.63}, \quad |\Im s| \geq 1. \hspace{1cm} (1.7)$$

Moreover, $\zeta_K(s)$ has at most one zero in the region:

$$\Re s \geq 1 - \frac{1}{12.74 \log d_K}, \quad |\Im s| \leq 1. \hspace{1cm} (1.8)$$
This zero if it exists is simple and is real.

An improvement of Stark’s result [1.4] follows from the method leading to (1.8):

**Corollary 1.2.** Let \( d_K \) be sufficiently large. Then \( \zeta_K(s) \) has at most one zero in the region:

\[
\Re s \geq 1 - \frac{1}{2 \log d_K}, \quad |\Im s| \leq \frac{1}{2 \log d_K}.
\]

Also, we can prove that \( \zeta_K(s) \) has at most one zero in the region:

\[
\Re s \geq 1 - \frac{1}{1.70 \log d_K}, \quad |\Im s| \leq \frac{1}{4 \log d_K}.
\]

This zero if it exists is simple and is real.

Note that the above theorems can be made completely explicit for any value of \( \log d_K \).

Our proof does not make use of Heath-Brown’s version of Jensen’s formula. We now explain why it appears difficult to apply his approach to the number field setting. He proved that for \( \chi \) a non-principal Dirichlet character modulo \( q \), \( \sigma \) close to 1, and for any \( \epsilon > 0 \), there exists a \( \delta \) such that

\[
-\Re \frac{L'(s, \chi)}{L(s, \chi)} \leq -\sum_{|1-\overline{\varphi}| \leq \delta} \Re \frac{1}{\sigma - \varphi} + \left( \frac{\phi}{2} + \epsilon \right) \log q.
\]

Here \( \phi \) is a constant associated to an upper bound for \( L(s, \chi) \). The convexity bound yields \( \phi = \frac{1}{2} \) and Burgess’ sub-convexity estimate yields \( \phi = \frac{1}{3} \). In comparison, this method applied in the context of number fields leads to the following inequality:

\[
-\Re \frac{\zeta'_K(s)}{\zeta_K(s)} \leq \frac{1}{\sigma - 1} + \left( \frac{1}{4} + \epsilon \right) \log d_K + 2n_K \log \left( \frac{\log d_K}{n_K} \right) + O(n_K).
\]

(see Lemma 4 in Li’s article [10] for a reference). The problem here is that the error terms may become larger than the main term which is of size \( \log d_K \). Note that the coefficient \( \frac{1}{4} \) follows from the convexity bound. Sub-convexity bounds for number fields only have been proven for some special cases, such as cubic extensions (see pp. 54-55 of [1] for an overview of known results). On the other hand, Stechkin’s argument leads to the inequality:

\[
-\Re \frac{\zeta'_K(s)}{\zeta_K(s)} \leq \frac{1}{\sigma - 1} - \sum_{\varphi} \Re \frac{1}{\sigma - \varphi} + \left( \frac{1 - \frac{1}{2\sqrt{5}}}{2} + \epsilon \right) \log d_K,
\]

where \( \frac{1 - \frac{1}{2\sqrt{5}}}{2} = 0.27639... \). Moreover, the above is valid for any number field \( K \).

The Dedekind zeta function is similar to the Riemann zeta function as it also has a simple pole at \( s = 1 \). However, a major difference is that its zeros can lie very close to the real axis. From this perspective, it behaves similarly to a Dirichlet \( L \)-function. As a consequence, our argument is closer to McCurley’s than to Rosser and Schoenfeld’s. Note that our coefficient of \( \log d_K \) is larger than their coefficient of \( \log q \) (namely 12.55 instead of 9.645908801). This is due to the fact that the zero-free region proof compares values of \( \zeta_K(s) \) at different points \( s \) close to the 1-line. For each of these points, a contribution of \( \log d_K \) arises, even when \( s \) is real. On the other hand, this does not occur in the case of Dirichlet \( L \)-functions: there is no contribution of \( \log q \) from \( L(s, \chi) \), when \( s \) and \( \chi \) are real.
One of the interests of Theorem 1.1 is its application to the problem of finding an explicit upper bound for the least prime ideal in the Chebotarev density theorem. Given a Galois extension of number fields $E/K$ with group $G$ and a conjugacy class $C \subset G$, there exists an unramified prime ideal $p$ of degree one such that $\sigma_p = C$ and $\mathbb{N} p \leq d_K^{C_0}$ for an explicit constant $C_0 > 0$.

2. Notation and Preliminary Lemmas

Let $K$ be a number field with ring of integers $\mathcal{O}_K$. The Dedekind zeta function of $K$ is

$$\zeta_K(s) = \sum_{\substack{a \subset \mathcal{O}_K \backslash \{0\}}} \frac{1}{(\mathbb{N} a)^s}, \text{ for } \Re(s) > 1.$$ 

It possesses the Euler product

$$\zeta_K(s) = \prod_p (1 - (\mathbb{N} p)^{-s})^{-1}$$

where $p$ ranges over all prime ideals in $\mathcal{O}_K$ and $\Re(s) > 1$. It is convenient to consider the completed zeta function

$$\zeta_K(s) = \xi_K(s) = s(1-s)d_K^{-s/2} \gamma_K(s) \zeta_K(s),$$

where $r_1$ and $r_2$ are the number of real and complex places in $K$. The advantage of $\xi_K$ is that it is an entire function which satisfies the functional equation:

$$\xi_K(s) = \xi_K(1-s).$$

By the duplication formula $\Gamma(s) = \frac{2^{s-1}}{\sqrt{\pi}} \Gamma(\frac{s}{2}) \Gamma(\frac{s+1}{2})$, it follows that

$$\gamma_K(s) = \pi^{-\frac{a+b+1}{2}} \Gamma(\frac{s+a}{2}) b \pi^{-\frac{a}{2}} \Gamma(\frac{a}{2})^b,$$

(2.1)

where $a, b$ are integers which satisfy $a + b = n_K$.

Let $\sigma > 1$ and $t$ real. We shall use the following notation and assumptions throughout the rest of the article:

$$\mathcal{L} = \log d_K, \quad \kappa = \frac{1}{\sqrt{3}},$$

$$1 < \sigma < 1.15, \quad \sigma_1 = \frac{1}{2} + \frac{1}{2} \sqrt{1 + 4\sigma^2},$$

$$s = \sigma + it, \quad s_1 = \sigma_1 + it.$$ (2.2)

Let $\varrho_0 = \beta_0 + i\gamma_0$. Note that, by symmetry of the zeros of $\zeta_K(s)$, it suffices to consider $\gamma_0 \geq 0$. The classical proof of the zero-free region studies the logarithmic derivative of the zeta function. Lagarias et al. consider $-\Re\left(\frac{\zeta_K'}{\zeta_K}(s)\right)$ and follow de La Vallée Poussin’s argument. Instead, we study the differenced function

$$f(\sigma, t) = -\Re\left(\frac{\zeta_K'}{\zeta_K}(s) - \kappa \frac{\zeta_K'}{\zeta_K}(s_1)\right),$$

(2.3)

as introduced by Stechkin. Observe that, for $\sigma > 1$,

$$f(\sigma, t) = \sum_{\substack{a \subset \mathcal{O}_K \backslash \{0\}}} \frac{\Lambda(a)}{(\mathbb{N} a)^\sigma} \left(1 - \frac{\kappa}{(\mathbb{N} a)^{\sigma_1 - \sigma}}\right) \cos(\sigma \log \mathbb{N} a).$$

(2.4)
2.1. Setting up the argument.

2.1.1. First ingredient: a trigonometric inequality. Let \( P \) be a non-negative trigonometric polynomial of degree \( n \) of the form

\[
P(\theta) = \sum_{k=0}^{n} a_k \cos(k\theta),
\]

where all the \( a_k \)'s are positive. For example, de La Vallée Poussin used

\[
2(1 + \cos \theta)^2 = 3 + 4 \cos \theta + \cos(2\theta).
\]

Later, higher degree polynomials were explored. For example, Rosser and Schoenfeld in [15] for \( \zeta(s) \), and then McCurley in [11] for \( L_q(s) \) used:

\[
P(\theta) = 8(0.9126 + \cos \theta)^2(0.2766 + \cos \theta)^2 = \sum_{k=0}^{4} a_k \cos(k\theta),
\]

with \( a_0 = 11.18593553, a_1 = 19.07334401, a_2 = 11.67618784, a_3 = 4.7568, a_4 = 1. \)

Our choice of \( P \) depends on the size of the imaginary part \( \gamma_0 \). For example, for \( \gamma_0 \) real, we only consider

\[
P(\theta) = 1.
\]

Taking \( \theta = t \log \mathbb{N}a \), we combine the trigonometric polynomial with (2.4) and define

\[
S(\sigma, \gamma_0) = \sum_{k=0}^{n} a_k f(\sigma, k\gamma_0) = \sum_{a \subseteq \mathcal{O}_K, \mathfrak{a} \neq 0} \frac{\Lambda(a)}{(\mathbb{N}a)^\sigma} \left( 1 - \frac{\kappa}{(\mathbb{N}a)^{\sigma_1 - \sigma}} \right) P(\gamma_0 \log \mathbb{N}a). \tag{2.5}
\]

Thanks to the choice of \( \sigma_1 \) and \( \kappa \) as in (2.2), we have

\[
1 - \frac{\kappa}{(\mathbb{N}a)^{\sigma_1 - \sigma}} \geq 0, \text{ for all non-zero ideals.}
\]

Together with the non-negativity of \( P \), we obtain

\[
S(\sigma, \gamma_0) \geq 0. \tag{2.6}
\]

2.1.2. Second ingredient: an explicit formula. We recall the explicit formula for Dedekind Zeta functions (see equation (8.3) of [9]):

\[
-\text{Re} \frac{\zeta'_K(s)}{\zeta_K(s)} = -\sum_{\varrho} \text{Re} \left( \frac{1}{s - \varrho} + \frac{1}{2} \log d_K + \frac{1}{s} + \text{Re} \frac{1}{s - 1} + \text{Re} \frac{\gamma'_{\varrho}}{\gamma'_{\varrho}}(s) \right),
\]

where \( \varrho \) runs through the non-trivial zeros of \( \zeta_K \). It follows for \( f(\sigma, t) \) given by (2.3) that

\[
f(\sigma, t) = -\sum_{\varrho} \text{Re} \left( \frac{1}{s - \varrho} - \kappa \frac{1}{s_1 - \varrho} \right) + \frac{1 - \kappa}{2} \log d_K
\]

\[
+ \text{Re} \left( \frac{1}{s} + \frac{1}{s - 1} - \kappa \frac{1}{s_1} - \kappa \frac{1}{s_1 - 1} \right) + \text{Re} \left( \frac{\gamma'_{\varrho}}{\gamma_{\varrho}}(s) - \kappa \frac{\gamma'_{\varrho}}{\gamma_{\varrho}}(s_1) \right). \tag{2.7}
\]

To simplify notation, we set

\[
F(s, z) = \text{Re} \left( \frac{1}{s - z} + \frac{1}{s - 1 + z} \right).
\]
Using the symmetry of the zeros with respect to the \(1/2\)-line, we have

\[
- \sum_\varepsilon \Re \left( \frac{1}{s - \varepsilon} - \kappa \frac{1}{s_1 - \varepsilon} \right) = - \sum'_{\beta \geq \frac{1}{2}} (F(s, \varepsilon) - \kappa F(s_1, \varepsilon)),
\]

(2.8)

where \(\sum'_{\beta \geq \frac{1}{2}} = \frac{1}{2} \sum_{\Re \varepsilon = 1/2} + \sum_{1/2 < \Re \varepsilon \leq 1} \). It follows that

\[
f(\sigma, k\gamma_0) = - \sum'_{\beta \geq \frac{1}{2}} (F(\sigma + ik\gamma_0, \varepsilon) - \kappa F(\sigma_1 + ik\gamma_0, \varepsilon)) + \frac{1 - \kappa}{2} \log d_K
\]

\[
+ F(\sigma + ik\gamma_0, 1) - \kappa F(\sigma_1 + ik\gamma_0, 1) + \Re \left( \frac{\gamma_K}{\gamma_K} (\sigma + ik\gamma_0) - \kappa \frac{\gamma_K}{\gamma_K} (\sigma_1 + ik\gamma_0) \right).
\]

(2.9)

Now (2.6) becomes

\[
S_1(\sigma, \gamma_0) + S_2 + S_3(\sigma, \gamma_0) + S_4(\sigma, \gamma_0) \geq 0
\]

(2.10)

where

\[
S_1(\sigma, \gamma_0) = - \sum_{k=0}^{n} a_k \sum'_{\beta \geq \frac{1}{2}} (F(\sigma + ik\gamma_0, \varepsilon) - \kappa F(\sigma_1 + ik\gamma_0, \varepsilon)),
\]

(2.11)

\[
S_2 = 1 - \frac{\kappa}{2} \left( \sum_{k=0}^{n} a_k \right) \log d_K,
\]

(2.12)

\[
S_3(\sigma, \gamma_0) = \sum_{k=0}^{n} a_k (F(\sigma + ik\gamma_0, 1) - \kappa F(\sigma_1 + ik\gamma_0, 1)),
\]

(2.13)

\[
S_4(\sigma, \gamma_0) = \sum_{k=0}^{n} a_k \Re \left( \frac{\gamma_K}{\gamma_K} (\sigma + ik\gamma_0) - \kappa \frac{\gamma_K}{\gamma_K} (\sigma_1 + ik\gamma_0) \right).
\]

(2.14)

It remains to bound each of the \(S_i\)'s so as to exhibit \(\beta_0\) and deduce from (2.10) an upper bound for it. Observe that for \(\Re z \leq 1\), \(\Re (\sigma_1 + ik\gamma_0 - z)\) is large enough, implying that the terms \(F(\sigma_1 + ik\gamma_0, z)\) are insignificant. On the other hand, Stechkin’s trick reduces the coefficient of \(\log d_K\) from \(\frac{1}{2}\) to \(\frac{1 - \kappa}{2}\). We choose the parameter \(\sigma\) such that \(\sigma - 1\) and \(\sigma - \beta_0\) are both of size \(\frac{1}{L}\). We split our argument and evaluate each of the \(S_i\)’s for:

- **Case 1**: \(\gamma_0 > 1\),
- **Case 2**: \(\frac{d_1}{2} < \gamma_0 \leq 1\),
- **Case 3**: \(\frac{d_2}{2} < \gamma_0 \leq \frac{d_2}{2}\),
- **Case 4**: \(0 < \gamma_0 \leq \frac{d_1}{2}\),
- **Case 5**: \(\gamma_0 = 0\). In this case, we consider \(\beta_1\) and \(\beta_2\) two real zeros satisfying \(\beta_1 \leq \beta_2\).

It is possible to prove an upper bound for \(\beta_1\), and thus establish a region free of zeros, with the exception of \(\beta_2\).

Here \(d_1\) and \(d_2\) are positive constants chosen to make the zero-free regions as wide as possible. For each case, we make a specific choice for the trigonometric coefficients \(a_k\).
2.1.3. Bounding the sum over the zeros \( S_1 \). In de La Vallée Poussin’s argument, he makes use of the positivity condition
\[
\Re \frac{1}{\sigma + ik\gamma_0 - \varrho} \geq 0, \text{ for all non-trivial zeros } \varrho \text{ and for all } \sigma > 1.
\]
Later, Stechkin showed (see Lemma 2.1 below) that
\[
F(\sigma + ik\gamma_0, \varrho) - \kappa F(\sigma_1 + ik\gamma_0, \varrho) \geq 0, \text{ for all non-trivial zeros } \varrho.
\]
Moreover, \( \kappa = 1/\sqrt{5} \) is the largest value such that the inequality holds. Observe that for the zeros \( \varrho \) where \(|\sigma + ik\gamma_0 - \varrho|\) is small, then \( \Re \frac{1}{\sigma + ik\gamma_0 - \varrho} \) is a large positive term. We retain these zeros in the sum (2.11) and we discard the other ones by using Stechkin’s Lemma.

Cases 1, 2, and 3: We have \( \gamma_0 \gg \sigma - \beta_0 \). We isolate \( \varrho_0 = \beta_0 + i\gamma_0 \) only for the \( k = 1 \) term:
\[
S_1(\sigma, \gamma_0) \leq -\frac{a_1}{\sigma - \beta_0} + O(1).
\]
Case 4: We isolate both zeros \( \varrho_0 \) and \( \varrho_0' \):
\[
S_1(\sigma, 0) \leq -\frac{2(1 + o(1)) \sum_{k=0}^n a_k}{\sigma - \beta_0} + O(1).
\]
Case 5: We isolate both zeros \( \beta_1 \) and \( \beta_2 \):
\[
S_1(\sigma, 0) \leq -\frac{2(1 + o(1)) \sum_{k=0}^n a_k}{\sigma - \beta_1} + O(1).
\]

2.1.4. Bounding the polar terms \( S_3 \). Note that \( F(\sigma + ik\gamma_0, 1) - \kappa F(\sigma_1 + ik\gamma_0, 1) \) is essentially of size \( \Re \frac{1}{\sigma - 1 + ik\gamma_0} \).

Cases 1 and 2: We have \( \gamma_0 \gg \sigma - 1 \). Thus \( F(\sigma + ik\gamma_0, 1) - \kappa F(\sigma + ik\gamma_0, 1) \ll 1 \) for all \( k \neq 0 \) and
\[
S_3(\sigma, \gamma_0) \leq \frac{a_0}{\sigma - 1} + O(1).
\]
Cases 3, 4, and 5: Since \( \gamma_0 \ll \sigma - 1 \), \( F(\sigma + ik\gamma_0, 1) - \kappa F(\sigma + ik\gamma_0, 1) \ll \frac{1}{\sigma - 1} \) for all \( k \) and
\[
S_3(\sigma, \gamma_0) \leq \frac{(1 + o(1)) \sum_{k=0}^n a_k}{\sigma - 1} + O(1).
\]

2.1.5. Bounding the \( \gamma K \) terms \( S_4 \). An analysis of \( \psi(x) = \frac{\psi'(x)}{\Gamma(x)} \) together with the definition (2.1) of \( \frac{\gamma K}{\gamma_k} \) gives
\[
\sum_{k=0}^n a_k \Re \frac{\gamma K}{\gamma_k}(\sigma + ik\gamma_0) \leq \begin{cases} (1+o(1))(\log \gamma_0)n_K \text{ in Case 1,} \\ O(n_K) \text{ in Case 2, 3, 4, 5.} \end{cases}
\]

2.1.6. Conclusion. We deduce from the above bounds an inequality depending on \( \beta_0 \) (respectively \( \beta_1 \)), \( \gamma_0, d_K, n_K \), and \( \sigma \). We choose \( \sigma \) so as to obtain the smallest upper bound possible for \( \beta_0 \) and \( \beta_1 \).

The following sections establish in complete detail the results mentioned in section 2.1.
2.2. Preliminary Lemma about the zero terms. We define $s_1(\sigma, \gamma_0, k)$ to be the $k$-th summand of $S(\sigma, \gamma_0)$:

$$s_1(\sigma, \gamma_0, k) = -\sum_{\beta \geq 4}^\prime (F(\sigma + ik\gamma_0, \beta) - \kappa F(\sigma + ik\gamma_0, \theta)).$$

We employ the following lemma to establish a bound for it.

**Lemma 2.1** (Stechkin - [18]). Let $s = \sigma + it$ with $\sigma > 1$. If $0 < \Re z < 1$, then

$$F(s, z) - \kappa F(s_1, z) \geq 0. \quad (2.15)$$

If $\Im z = t$ and $1/2 \leq \Re z < 1$, then

$$\Re \left(\frac{1}{s - 1 + \beta} - \kappa F(s_1, z)\right) \geq 0. \quad (2.16)$$

For the rest of the article, we consider $\theta_0 = \beta_0 + i\gamma_0$ a non-trivial zero of $\zeta_K$. We assume

$$\beta_0 \geq 0.85 \quad \text{and} \quad \gamma_0 \geq 0. \quad (2.17)$$

Note that otherwise, the zero-free region $\Re \Omega > 0.85$ is established. When $k = 1$, we isolate $\theta_0$ from the sum in (2.8). Together with (2.15) and (2.16), we obtain

$$s_1(\sigma, \gamma_0, 1) \leq -(F(\sigma + i\gamma_0, \theta_0) - \kappa F(\sigma_1 + i\gamma_0, \theta_0))$$

$$= \left(\Re \frac{1}{\sigma + i\gamma_0 - \theta_0} + \Re \frac{1}{\sigma + i\gamma_0 - 1 + \theta_0} - \kappa F(\sigma + i\gamma_0, \theta_0)\right) \leq -\frac{1}{\sigma - \beta_0}. \quad (2.18)$$

When $k = 0, 2, 3, 4$, we consider various cases.

If $\gamma_0 > 1$ (Case 1), we use (2.15) for all zeros:

$$s_1(\sigma, \gamma_0, k) \leq 0. \quad (2.19)$$

For $\gamma_0$ as in Case 2, we apply (2.15) except for $\theta_0$:

$$s_1(\sigma, \gamma_0, k) \leq -(F(\sigma + ik\gamma_0, \beta_0 + i\gamma_0) - \kappa F(\sigma_1 + ik\gamma_0, \beta_0 + i\gamma_0)). \quad (2.20)$$

For $\gamma_0$ as in Case 3, we apply (2.15) except for $\theta_0$ and $\bar{\theta}_0$:

$$s_1(\sigma, \gamma_0, k) \leq -(F(\sigma + i\gamma_0, \beta_0 + i\gamma_0) - \kappa F(\sigma_1 + i\gamma_0, \beta_0 + i\gamma_0))$$

$$- (F(\sigma + i\gamma_0, \beta_0 - i\gamma_0) - \kappa F(\sigma_1 + i\gamma_0, \beta_0 - i\gamma_0)). \quad (2.21)$$

We observe that, for $x$ real, $F(x, \theta_0) = F(x, \bar{\theta}_0)$. For Cases 4 and 5, (2.19) becomes

$$s_1(\sigma, 0, k) \leq -2(F(\sigma_0 + i\gamma_0) - \kappa F(\sigma_1 + i\gamma_0)). \quad (2.22)$$

It remains to bound $-(F(\sigma + ik\gamma_0, \beta_0 \pm i\gamma_0) - \kappa F(\sigma_1 + ik\gamma_0, \beta_0 \pm i\gamma_0))$ in (2.18), (2.19), and (2.20) for $k = 0, 1, 2, 3, 4$ and $\gamma_0 \leq 1$. The following elementary lemma may be used for this.

**Lemma 2.2.** For $a, b, c > 0$, we define

$$g(a, b, c; x) = \kappa \left(\frac{a}{a^2 + x^2} + \frac{b}{b^2 + x^2}\right) - \frac{c}{c^2 + x^2}.$$

Let $a_0 = \frac{\sqrt{5} - 1}{2}, b_0 = \frac{1 + \sqrt{5}}{2}$, and $c_0 = 1$.

(i) Let $g_0 = -0.121585107$. Then the inequality

$$g_0 \leq g(a_0, b_0, c_0; x) \leq 0$$
is valid for all \( x \in \mathbb{R} \).

(ii) Let \( a, b, c > 0 \) and let \( 0 < \epsilon < \epsilon_0 \). If there exist constants \( m_1, m_2, m_3 \) such that
\[
|a - a_0| < m_1 \epsilon, \quad |b - b_0| < m_2 \epsilon, \quad \text{and} \quad |c - c_0| < m_3 \epsilon,
\]
then
\[
go - m_0 \epsilon \leq g(a, b, c; x) \leq m_0 \epsilon,
\]
where
\[
m_0 = \frac{\kappa m_1}{(a_0 - m_1 \epsilon_0)^2} + \frac{\kappa m_2}{(b_0 - m_2 \epsilon_0)^2} + \frac{m_3}{(c_0 - m_3 \epsilon_0)^2}.
\]

Proof. (i) Differentiating we find that
\[
g(a_0, b_0, c_0; x) = 2x \left( \kappa \left( \frac{a}{a_0^2 + x^2} - \frac{a_0}{a_0^2 + x^2} + \frac{b}{b_0^2 + x^2} - \frac{b_0}{b_0^2 + x^2} \right) \right) + \frac{c_0}{c_0^2 + x^2} + \frac{c}{c_0^2 + x^2}.
\]
For positive real numbers \( u \) and \( u_0 \) we have that
\[
\left| \frac{u}{u^2 + x^2} - \frac{u_0}{u_0^2 + x^2} \right| = \left| \frac{(u - u_0)(x^2 - u_0)}{(u^2 + x^2)(u_0^2 + x^2)} \right| \leq \frac{|u - u_0| (x^2 + \max(u, u_0)^2)}{(x^2 + u^2)(x^2 + u_0^2)} \leq \frac{|u - u_0|}{\min(u, u_0)^2}.
\]
Using this bound, the triangle inequality implies that
\[
|g(a, b, c; x) - g(a_0, b_0, c_0; x)| \leq \epsilon \left( \frac{\kappa m_1}{(a_0 - m_1 \epsilon_0)^2} + \frac{\kappa m_2}{(b_0 - m_2 \epsilon_0)^2} + \frac{m_3}{(c_0 - m_3 \epsilon_0)^2} \right) = m_0 \epsilon.
\]
Hence \( go - m_0 \epsilon \leq g(a, b, c; x) \leq m_0 \epsilon \). \( \square \)

Observe that
\[
- \left( F(\sigma + ik\gamma_0, \beta_0 + i\gamma_0) - \kappa F(\sigma + ik\gamma_0, \beta_0 + i\gamma_0) \right)
- \frac{\sigma - \beta_0}{(\sigma - \beta_0)^2 + ((k - 1)\gamma_0)^2} - \frac{\sigma - \beta_0}{(\sigma - \beta_0)^2 + ((k - 1)\gamma_0)^2}
+ \kappa \frac{\sigma_1 - \beta_0}{(\sigma_1 - \beta_0)^2 + ((k - 1)\gamma_0)^2}
+ \kappa \frac{\sigma_1 - \beta_0}{(\sigma_1 - \beta_0)^2 + ((k - 1)\gamma_0)^2}
- \frac{\sigma_1 - \beta_0}{(\sigma_1 - \beta_0)^2 + ((k - 1)\gamma_0)^2}
- \frac{\sigma_1 - \beta_0}{(\sigma_1 - \beta_0)^2 + ((k - 1)\gamma_0)^2}
- g(a, b, c; x),
\]
where \( a = \sigma_1 - \beta, b = \sigma_1 - 1 + \beta, c = \sigma - 1 + \beta, \text{and} \ x = |k - 1|\gamma_0 \). From \( 2.17 \), it follows that \( 1 - \epsilon \leq \beta < 1 \) and \( 1 < \sigma \leq 1 + \epsilon, \) with \( \epsilon = 0.15 \). Thus
\[
|a - a_0| \leq 1.9064 \epsilon, |b - b_0| \leq 0.9064 \epsilon, \text{and} \ |c - c_0| \leq \epsilon.
\]
Moreover, for \( \gamma_0 > 1 \) as in Case 1, we have
\[
- \sum_{\beta \geq \frac{1}{2}} (F(\sigma + ik\gamma_0, \varrho) - \kappa F(\sigma_1 + ik\gamma_0, \varrho)) \leq -\frac{1}{\sigma - \beta_0}.
\] (2.21)

For \( k \neq 1 \) and \( \gamma_0 > 1 \) as in Case 1, we have
\[
- \sum_{\beta \geq \frac{1}{2}} (F(\sigma + ik\gamma_0, \varrho) - \kappa F(\sigma_1 + ik\gamma_0, \varrho)) \leq \frac{\sigma - \beta_0}{(\sigma - \beta_0)^2 + (k - 1)^2\gamma_0^2} + 2\alpha_1.
\] (2.23)

For \( k \neq 1 \) and \( \gamma_0 \leq 1 \) as in Case 3, we have
\[
- \sum_{\beta \geq \frac{1}{2}} (F(\sigma + ik\gamma_0, \varrho) - \kappa F(\sigma_1 + ik\gamma_0, \varrho)) \leq \frac{1}{(\sigma - \beta_0)^2 + (k - 1)^2\gamma_0^2} + 2\alpha_1.
\] (2.24)

Moreover, for \( \gamma_0 \leq 1 \) as in Cases 4 and 5, we have
\[
- \sum_{\beta \geq \frac{1}{2}} (F(\sigma, \varrho) - \kappa F(\sigma_1, \varrho)) \leq -\frac{\sigma - \beta_0}{(\sigma - \beta_0)^2 + \gamma_0^2} + 2\alpha_1.
\] (2.25)

2.3. Preliminary Lemma about the polar terms.

Lemma 2.4. Assume (2.2). We define
\[
\alpha_{20} = 0.0215, \quad \alpha_{21} = 1.5166, \quad \alpha_{22} = 1.6666.
\]

If \( k = 0 \) or \( \gamma_0 = 0 \), then
\[
F(\sigma + ik\gamma_0, 1) - \kappa F(\sigma_1 + ik\gamma_0, 1) \leq \frac{1}{\sigma - 1} + \alpha_{20}.
\] (2.26)

If \( k = 1, 2, 3, 4 \) and \( 0 < \gamma_0 < 1 \), then
\[
F(\sigma + ik\gamma_0, 1) - \kappa F(\sigma_1 + ik\gamma_0, 1) \leq \frac{1}{(\sigma - 1)^2 + (k\gamma_0)^2} + \alpha_{21}.
\] (2.27)

If \( k = 1, 2, 3, 4 \) and \( \gamma_0 \geq 1 \), then
\[
F(\sigma + ik\gamma_0, 1) - \kappa F(\sigma_1 + ik\gamma_0, 1) \leq \alpha_{22}.
\] (2.28)

Proof. When \( k\gamma_0 = 0 \), we have
\[
F(\sigma, 1) - \kappa F(\sigma_1, 1) = \frac{1}{\sigma - 1} + \frac{1}{\sigma} - \frac{\kappa}{\sigma_1} - \frac{\kappa}{\sigma_1 - 1}.
\]

A Maple computation shows that the maximum of \( \frac{1}{\sigma} - \frac{\kappa}{\sigma_1} - \frac{\kappa}{\sigma_1 - 1} \) for \( \sigma \in [1, 1.15] \) occurs at \( \sigma = 1.15 \), and is 0.02146... Thus
\[
F(\sigma, 1) - \kappa F(\sigma_1, 1) \leq \frac{1}{\sigma - 1} + \alpha_{20}.
\] (2.29)
Observe that

\[ F(\sigma + ik\gamma_0, 1) - \kappa F(\sigma_1 + ik\gamma_0, 1) = \frac{\sigma}{\sigma^2 + (k\gamma_0)^2} + \frac{\sigma - 1}{(\sigma - 1)^2 + (k\gamma_0)^2} - \frac{\kappa\sigma_1}{\sigma_1^2 + (k\gamma_0)^2} - \frac{\kappa(\sigma_1 - 1)}{(\sigma_1 - 1)^2 + (k\gamma_0)^2} \]

Taking \( \epsilon = 0.15 \), it follows from Lemma 2.2 that

\[ g(\sigma_1 - 1, \sigma_1, \sigma; k\gamma_0) \geq g_0 - m_0\epsilon \geq -1.5166. \]

Thus

\[ F(\sigma + ik\gamma_0, 1) - \kappa F(\sigma_1 + ik\gamma_0, 1) \leq \frac{\sigma - 1}{(\sigma - 1)^2 + (k\gamma_0)^2} + \alpha_{21}. \quad (2.30) \]

Moreover, when \( \gamma_0 \geq 1 \), the above becomes

\[ F(\sigma + ik\gamma_0, 1) - \kappa F(\sigma_1 + ik\gamma_0, 1) \leq 0.15 + \alpha_{21}. \quad (2.31) \]

The announced inequalities follow from (2.29), (2.30), and (2.31).

2.4. Preliminary Lemma about the \( \gamma_K \) terms. We now bound the expression

\[ \Re \left( \frac{\gamma_K}{\gamma_K}(\sigma + ik\gamma_0) - \kappa \frac{\gamma_K}{\gamma_K}(\sigma_1 + ik\gamma_0) \right) = -\frac{(1 - \kappa)\log \pi}{2} n_K \]

\[ + \frac{a}{2} \Re \left( \frac{\Gamma'}{\Gamma} \left( \frac{\sigma + ik\gamma_0 + c}{2} \right) - \kappa \frac{\Gamma'}{\Gamma} \left( \frac{\sigma_1 + ik\gamma_0 + c}{2} \right) \right) + \frac{b}{2} \Re \left( \frac{\Gamma'}{\Gamma} \left( \frac{\sigma_1 + ik\gamma_0 + 1}{2} \right) - \kappa \frac{\Gamma'}{\Gamma} \left( \frac{\sigma_1 + ik\gamma_0 + 1}{2} \right) \right), \]

where \( a + b = n_K \) as in (2.1). We have:

Lemma 2.5. Assume (2.2). Let \( k = 0, 1, 2, 3, 4 \) and \( c = 0 \) or 1. Then

\[ \frac{1}{2} \Re \left( \frac{\Gamma'}{\Gamma} \left( \frac{\sigma + ik\gamma_0 + c}{2} \right) - \kappa \frac{\Gamma'}{\Gamma} \left( \frac{\sigma_1 + ik\gamma_0 + c}{2} \right) \right) \]

\[ \leq \begin{cases} 
\begin{align*}
    d(0) &= D(0) &\text{if } k\gamma_0 = 0, \\
    d(k) &= 0 &\text{if } 0 < \gamma_0 \leq 1 \text{ and } k = 1, 2, 3, 4, \\
    \frac{1 - \kappa}{2} \log \gamma_0 + D(k) &= 0 &\text{if } \gamma_0 > 1 \text{ and } k = 1, 2, 3, 4,
\end{align*}
\end{cases} \]

where admissible values of \( D(k) \) and \( d(k) \) are given in the following chart.

| \( k \) | 0 | 1 | 2 | 3 | 4 |
|---|---|---|---|---|---|
| \( d(k) \) | -0.0512 | -0.0390 | 0.2469 | 0.4452 | 0.5842 |
| \( D(k) \) | -0.0512 | 0.3918 | 0.3915 | 0.4062 | 0.4266 |

The cases \( k = 1, 2, 3, 4 \) are a direct consequence of Lemmas 1 and 2 of [11]. The case \( k = 0 \) can be obtained by a Maple computation. Thus we deduce

Lemma 2.6. Assume (2.2). If \( \kappa \gamma_0 = 0 \), then

\[ \Re \left( \frac{\gamma_K}{\gamma_K}(\sigma) - \kappa \frac{\gamma_K}{\gamma_K}(\sigma_1) \right) \leq \left( -\frac{(1 - \kappa)\log \pi}{2} + d(0) \right) n_K. \quad (2.32) \]

If \( 0 < \gamma_0 \leq 1 \) and \( k = 1, 2, 3, 4 \), then

\[ \Re \left( \frac{\gamma_K}{\gamma_K}(\sigma + ik\gamma_0) - \kappa \frac{\gamma_K}{\gamma_K}(\sigma_1 + ik\gamma_0) \right) \leq \left( -\frac{(1 - \kappa)\log \pi}{2} + d(k) \right) n_K. \quad (2.33) \]
If \( \gamma_0 > 1 \) and \( k = 1, 2, 3, 4 \), then

\[
\Re \left( \frac{\gamma_K'}{\gamma_K} (\sigma + ik\gamma_0) - \frac{\gamma_K'}{\gamma_K} (\sigma_1 + ik\gamma_0) \right) \leq \left( \frac{1 - \kappa}{2} \log \gamma_0 + \frac{1 - \kappa}{2} \log \frac{k}{\pi} + D(k) \right) n_K.
\]

### 3. Zero-free regions

In this section, we continue to assume conditions (2.2) and (2.17) for the parameters \( \sigma, L \), and for the non-trivial zero \( \gamma_0 = \beta_0 + i\gamma_0 \). We use lemmas 2.3, 2.4, and 2.6 from Section 2 to provide upper bounds for the \( S_j \)'s and thus derive zero-free regions.

In order to simplify future computations, we record the following elementary lemma:

**Lemma 3.1.** Let \( a, b, q, t > 0 \) be fixed.

(i) If \( 2a - b > 0 \), then \( f_1(x) = -\frac{a}{x} - \frac{bx}{x^2 + t^2} \) is increasing.

(ii) \( f_2(a, b; x) = \frac{a}{a^2 + x^2} - \frac{b}{b^2 + x^2} \) has opposite sign of \( (b - a)(x^2 - ab) \).

(iii) If \( qb^3 \geq a^3, qb \geq a, \) and \( qa \geq b \), then \( f_3(a, b, q; x) = q \frac{a}{a^2 + x^2} - \frac{b}{b^2 + x^2} \) is decreasing with \( x \).

**Proof.** We have

\[
f_1'(x) = \frac{(a + b)x^4 + (2a - b)x^2 + at^4}{x^2(x^2 + t^2)^2},
\]

\[
f_2(a, b; x) = -\frac{(b - a)(x^2 - ab)}{(a^2 + x^2)(b^2 + x^2)},
\]

\[
\frac{\partial}{\partial x} f_3(a, b, q; x) = (-2x) \left( \frac{ab(qb^3 - a^3) + 2ab(qb - a)x^2 + (qa - b)x^4}{(a^2 + x^2)(b^2 + x^2)^2} \right).
\]

The lemma follows from the above three formulae. \( \square \)

### 3.1. Case 1: Zero-free region when \( \gamma_0 > 1 \). Let \( r > 0 \). We choose \( \sigma \) such that

\( \sigma - 1 = r(1 - \beta_0). \)

We define the trigonometric polynomial

\[
P(\theta) = 8(0.8924 + \cos \theta)^2(0.1768 + \cos \theta)^2 = \sum_{k=0}^{4} a_k \cos(k\theta),
\]

where \( a_0 = 9.034112058, a_1 = 15.52951106, a_2 = 9.834965120, a_3 = 4.2768, a_4 = 1. \)

We apply Lemma 2.3 using equations (2.21) when \( k = 1 \), and (2.22) when \( k = 0, 2, 3, 4 \):

\[
S_1(\sigma, \gamma_0) \leq -\frac{a_1}{\sigma - \beta_0}.
\]

We apply Lemma 2.4 using equations (2.20) when \( k = 0 \), and (2.28) otherwise:

\[
S_3(\sigma, \gamma_0) \leq \frac{a_0}{\sigma - 1} + a_0\alpha_20 + \alpha_22 \sum_{k=1}^{4} a_k.
\]
We apply Lemma 2.5, using equations (2.32) for \( k = 0 \) and (2.34) otherwise:

\[
S_4(\sigma, \gamma_0) \leq a_0 \left( -\frac{(1 - \kappa) \log \pi}{2} + d(0) \right) n_K + \sum_{k=1}^{4} a_k \left( -\frac{(1 - \kappa) \log \gamma_0}{2} + \frac{1 - \kappa}{2} \log \frac{k}{\pi} + D(k) \right) n_K.
\]

Together with (2.10), (2.9), and the above inequalities, we deduce

\[
0 \leq \left( -\frac{a_1}{1 + r} + \frac{a_0}{r} \right) \frac{1}{1 - \beta_0} + c_1 \log d_K + c_2 (\log \gamma_0) n_K + c_3 n_K + c_4,
\]

where

\[
c_1 = \frac{1 - \kappa}{2} \sum_{k=0}^{4} a_k, \quad c_2 = \frac{1 - \kappa}{2} \left( \sum_{k=1}^{4} a_k \right),
\]

\[
c_3 = a_0 \left( -\frac{(1 - \kappa) \log \pi}{2} + d(0) \right) + \sum_{k=1}^{4} a_k \left( -\frac{(1 - \kappa)}{2} \log \frac{k}{\pi} + D(k) \right),
\]

\[
c_4 = a_0 a_2 + a_{22} \sum_{k=1}^{4} a_k.
\]

Thus

\[
\beta_0 \leq 1 - \frac{1}{c_1 \log d_K + c_2 (\log \gamma_0) n_K + c_3 n_K + c_4}.
\]

The largest value for \( \frac{a_1}{1 + r} - \frac{a_0}{r} \) occurs for \( r = \frac{\sqrt{\pi}}{\sqrt{a_1} - \sqrt{a_0}} = 3.21438 \ldots \) and hence

\[
\beta_0 \leq 1 - \frac{1}{12.5419 \log d_K + 9.6861 (\log \gamma_0) n_K + 3.0297 n_K + 58.6265}.
\]

This proves the zero free region (1.7) of Theorem 1.1.

**Remark 3.2.** We ran a Maple computation to determine the \( a_k \)'s which minimized

\[
\frac{c_1}{1 + r} - \frac{a_0}{r} = \frac{\frac{1}{r} n_k a_k}{\sqrt{a_1} - \sqrt{a_0}}.
\]

Let \( 0 < r, c < 1 \). For the remainder of the article, we consider

\[
\sigma - 1 = \frac{r}{\sqrt{\beta_0}} \quad \text{and} \quad 1 - \beta_0 = \frac{c}{\sqrt{\beta_0}}.
\]

### 3.2. Case 2: Zero-free region when \( \frac{d}{2} < \gamma_0 \leq 1 \), with \( d_2 = 2.374 \).

For the trigonometric polynomial, we choose

\[
P(\theta) = 8(0.8918 + \cos \theta)^2 (0.1732 + \cos \theta)^2 = \sum_{k=0}^{4} a_k \cos(k\theta),
\]

\[
a_0 = 8.96344062, \quad a_1 = 15.41199431, \quad a_2 = 9.77257808, \quad a_3 = 4.26, \quad a_4 = 1.
\]

In addition to our conditions on \( \sigma \) and \( \bar{\gamma}_0 \), we impose the conditions

\[
0 < \frac{a_0}{a_1 - a_0} c < r < 1, \quad \text{for} \quad d_2 > \frac{\sqrt{r(r+c)}}{2}, \quad (3.2)
\]
We apply Lemma 2.3 using (2.21) for $k = 1$, and (2.23) for $k = 0, 2, 3, 4$:

$$S_1(\sigma, \gamma_0) \leq -\frac{a_1}{\sigma - \beta_0} - \sum_{k=0,2,3,4} \frac{a_k(\sigma - \beta_0)}{(\sigma - \beta_0)^2 + (k-1)^2\gamma_0^2} + \frac{1}{2} \log \frac{1 - \kappa}{2} d_k \sum_{k=0}^{4} a_k.$$

We apply Lemma 2.4 using (2.26) for $k = 0$, and (2.27) otherwise:

$$S_3(\sigma, \gamma_0) \leq \frac{a_0}{\sigma - 1} + a_0 \alpha_2 + \sum_{k=1}^{4} \frac{a_k(\sigma - 1)}{(\sigma - 1)^2 + k^2\gamma_0^2} + \alpha_2 \sum_{k=1}^{4} a_k.$$

We apply Lemma 2.5 using equations (2.32) for $k = 0$, and (2.33) otherwise:

$$S_4(\sigma, \gamma_0) \leq \sum_{k=0}^{4} a_k \left( -\frac{1 - \kappa}{2} \log \pi + d(k) \right) n_K.$$

Note that the coefficient of $n_K$ is negative and may be dispensed. Together with (2.10), (2.9), and the above inequalities, we deduce

$$0 \leq \frac{a_0}{\sigma - 1} - \frac{a_1}{\sigma - \beta_0} + \frac{a_1(\sigma - 1)}{(\sigma - 1)^2 + \gamma_0^2} - \frac{a_0(\sigma - \beta_0)}{(\sigma - \beta_0)^2 + \gamma_0^2} + \frac{1 - \kappa}{2} \log d_k \sum_{k=0}^{4} a_k$$

$$+ \sum_{k=2,3,4} a_k \left( \frac{\sigma - 1}{(\sigma - 1)^2 + k^2\gamma_0^2} - \frac{\sigma - \beta_0}{(\sigma - \beta_0)^2 + (k-1)^2\gamma_0^2} \right)$$

$$+ \alpha_1 \sum_{k=0,2,3,4} a_k + \alpha_2 a_0 + \alpha_2 \sum_{k=1}^{4} a_k.$$  \hspace{1cm} (3.3)

The term in the second row may be dropped since, for $k = 2, 3, 4$,

$$\frac{\sigma - 1}{(\sigma - 1)^2 + k^2\gamma_0^2} - \frac{\sigma - \beta_0}{(\sigma - \beta_0)^2 + (k-1)^2\gamma_0^2} \leq 0.$$  \hspace{1cm} (3.4)

This is established as follows:

$$\frac{\sigma - 1}{(\sigma - 1)^2 + k^2\gamma_0^2} - \frac{\sigma - \beta_0}{(\sigma - \beta_0)^2 + (k-1)^2\gamma_0^2} \leq f_2(a, b; x)$$

where $a = \sigma - 1$, $b = \sigma - \beta_0$, and $x = k\gamma_0$. We have $b - a = (\sigma - \beta_0) - (\sigma - 1) \geq 0$ and $x^2 - ab = (k\gamma_0)^2 - (\sigma - \beta_0)(\sigma - 1) \geq \frac{2(\sigma - \beta_0)}{\sigma - 1} \geq 0$ by condition (3.2). Hence Lemma 3.1 gives that $f_2(a, b; x) \leq 0$, and (3.4) is established. Next, we have

$$\frac{a_1(\sigma - 1)}{(\sigma - 1)^2 + \gamma_0^2} - \frac{a_0(\sigma - \beta_0)}{(\sigma - \beta_0)^2 + \gamma_0^2} \leq a_0 f_3(a, b, q; x)$$  \hspace{1cm} (3.5)

where $a = \sigma - 1 = \frac{1}{\sigma}$, $b = \sigma - \beta_0 = \frac{1}{\sigma}$, $q = \frac{a_0}{a} = 1.71942\ldots$, and $x = \gamma_0$. The conditions of Lemma 3.1 (iii) are satisfied. Hence $f_3(a, b, q; x)$ decreases with $x$ on $(d_2, \mathbb{L}, 1)$ and

$$f_3(a, b, q; x) \leq f_3(a, b, q, d_2; \mathbb{L}) = \left( q - \frac{r}{r^2 + d_2^2} - \frac{r + c}{(r + c)^2 + d_2^2} \right) \mathbb{L}.$$
This proves (3.5). Together with (3.3) and (3.5), we obtain
\[
0 \leq \frac{a_0}{r} - \frac{a_1}{r+c} + \frac{a_1 r}{r^2+d^2} - \frac{a_0(r+c)}{(r+c)^2+d^2} + \frac{1-\kappa}{2} \sum_{k=0}^{4} a_k
\]
\[+ \left( a_1 \sum_{k=0,2,3,4} a_k + a_0 \alpha_{20} + a_{21} \sum_{k=1}^{4} a_k \right) \frac{1}{\mathcal{L}}, \]
which becomes, for \( \mathcal{L} \) asymptotically large, \( 0 \leq \mathcal{E}(d_2, r, c) \), where
\[
\mathcal{E}(d_2, r, c) = \frac{a_0}{r} - \frac{a_1}{r+c} + \frac{a_1 r}{r^2+d^2} - \frac{a_0(r+c)}{(r+c)^2+d^2} + \frac{1-\kappa}{2} \sum_{k=0}^{4} a_k. \quad (3.6)
\]
Observe that since \( 2a_1 > a_0 \), (1) of Lemma 3.1 implies that \( \mathcal{E}(d_2, r, c) \) increases with \( c \). Thus the smallest value for \( c = c(d_2, r) \) satisfying the above inequality is the root of \( \mathcal{E}(d_2, r, c) \). We now choose the parameters \( d_2 \) and \( r \) such that \( c(d_2, r) \) is as small as possible. A GP-Pari computation gives
\[
\begin{array}{ccc}
d_2 & r & \frac{1}{c} \\
2.374 & 0.248 & 12.7305
\end{array}
\]

**Remark 3.3.** We now give some motivation for the choice of the trigonometric polynomial. Numerically, we expect \( d_2 \) to be close to 2.5. Thus it will be much larger than the expected values for \( r \) and \( c \). To simplify the analysis of (3.6), we drop the terms depending on \( d_2 \). We expect the values for \( r \) and \( c \) to be very close to \( \tilde{r} \) and \( \tilde{c} \) respectively, where \( \tilde{r} \) and \( \tilde{c} \) are numbers which satisfy
\[
a_0 \frac{1}{\tilde{r}} - a_1 \frac{1}{\tilde{r}+\tilde{c}} + \frac{1-\kappa}{2} \sum_{k=0}^{4} a_k \geq 0.
\]
This occurs as long as
\[
\tilde{c} \geq \frac{(a_1 - a_0)\tilde{r} - \left( \frac{1-\kappa}{2} \sum_{k=0}^{4} a_k \right) \tilde{r}^2}{a_0 + \left( \frac{1-\kappa}{2} \sum_{k=0}^{4} a_k \right) \tilde{r}}.
\]
By calculus, the expression on the right is minimized for
\[
\tilde{r} = a_0 \left( \sqrt{\frac{a_0}{a_1}} - 1 \right)
\]
\[= \frac{1-\kappa}{2} \sum_{k=0}^{4} a_k.
\]
We set \( d_2 = 2.5 \), and run a Maple computation to determine which \( a_k \)’s make the root \( c \) of \( \mathcal{E}(2.5, \tilde{r}, \tilde{c}) = 0 \) as small as possible.

### 3.3. Case 3: Zero-free region when \( \frac{d_1}{r} < \gamma_0 = \frac{d_2}{r} \), with \( d_1 = 1.021, d_2 = 2.374 \).

We choose for our trigonometrical polynomial:
\[
P(\theta) = 8(0.8924 + \cos \theta)^2(0.1771 + \cos \theta)^2 = \sum_{k=0}^{4} a_k \cos(k\theta),
\]
\[
a_0 = 9.039496, a_1 = 15.538449, a_2 = 9.839673, a_3 = 4.278, a_4 = 1.
\]
We impose the condition
\[
\frac{a_0}{a_1 - a_0} c < r < 1. \quad (3.7)
\]
We apply Lemma 2.3 using (2.21) for \( k = 1 \), and (2.24) otherwise:

\[
S_1(\sigma, \gamma_0) \leq -\frac{a_1}{\sigma - \beta_0} + 2a_1 \sum_{k=0,2,3,4} a_k \\
- \sum_{k=0,2,3,4} a_k \left( \frac{\sigma - \beta_0}{(\sigma - \beta_0)^2 + (k - 1)^2 \gamma_0} + \frac{\sigma - \beta_0}{(\sigma - \beta_0)^2 + (k + 1)^2 \gamma_0} \right).
\]

We apply Lemma 2.4 using (2.26) for \( k = 0 \), and (2.27) otherwise:

\[
S_3(\sigma, \gamma_0) \leq a_0 + a_0 \alpha_{20} + \sum_{k=1}^4 a_k (\sigma - 1) + \alpha_{21} \sum_{k=1}^4 a_k.
\]

We apply Lemma 2.5 using equations (2.28) for \( k = 0 \), and (2.29) otherwise:

\[
S_4(\sigma, \gamma_0) \leq \sum_{k=0}^4 a_k \left( -\frac{1}{2} \log \pi + d(k) \right) n_K \leq 0.
\]

Together with (2.10), (2.3), and the above inequalities, we deduce

\[
0 \leq \frac{a_0}{\sigma - 1} - \frac{a_1}{\sigma - \beta_0} + \frac{a_1 (\sigma - 1)}{(\sigma - 1)^2 + \gamma_0} - \frac{2a_0 (\sigma - \beta_0)}{(\sigma - \beta_0)^2 + 2 \gamma_0} - \frac{a_1 (\sigma - \beta_0)}{(\sigma - \beta_0)^2 + 4 \gamma_0} \\
+ \sum_{k=2,3,4} a_k \left( \frac{\sigma - 1}{(\sigma - 1)^2 + k^2 \gamma_0} - \frac{\sigma - \beta_0}{(\sigma - \beta_0)^2 + (k - 1)^2 \gamma_0} - \frac{\sigma - \beta_0}{(\sigma - \beta_0)^2 + (k + 1)^2 \gamma_0} \right) \\
+ \frac{1}{2} \left( \sum_{k=0}^4 a_k \right) \log d_K + 2a_1 \sum_{k=0,2,3,4} a_k + a_0 \alpha_{20} + \alpha_{21} \sum_{k=1}^4 a_k. \tag{3.8}
\]

For \( k = 2, 3, 4 \), since \( \gamma_0 \in (d_1/\mathcal{L}, d_2/\mathcal{L}) \) we have

\[
\frac{\sigma - 1}{(\sigma - 1)^2 + k^2 \gamma_0} - \frac{\sigma - \beta_0}{(\sigma - \beta_0)^2 + (k - 1)^2 \gamma_0} - \frac{\sigma - \beta_0}{(\sigma - \beta_0)^2 + (k + 1)^2 \gamma_0} \\
\leq \left( \frac{r}{r^2 + k^2 d_1^2} - \frac{r + c}{(r + c)^2 + (k - 1)^2 d_2^2} - \frac{r + c}{(r + c)^2 + (k + 1)^2 d_2^2} \right) \frac{1}{\mathcal{L}}.
\]

Since \( r \) and \( c \) satisfy (3.7), the same argument that gave (3.5) applies. Thus

\[
\frac{a_1 (\sigma - 1)}{(\sigma - 1)^2 + \gamma_0} - \frac{a_0 (\sigma - \beta_0)}{(\sigma - \beta_0)^2 + \gamma_0}
\]

decreases with \( \gamma_0 \in (d_1/\mathcal{L}, d_2/\mathcal{L}) \). We obtain

\[
\frac{a_1 (\sigma - 1)}{(\sigma - 1)^2 + \gamma_0} - \frac{a_0 (\sigma - \beta_0)}{(\sigma - \beta_0)^2 + \gamma_0} \leq \left( \frac{a_1 r}{r^2 + d_1^2} - \frac{a_0 (r + c)}{(r + c)^2 + d_2^2} \right) \mathcal{L}.
\]

We use the trivial bound for \( \gamma_0 \in (d_1/\mathcal{L}, d_2/\mathcal{L}):

\[
- \frac{a_0 (\sigma - \beta_0)}{(\sigma - \beta_0)^2 + \gamma_0} - \frac{a_1 (\sigma - \beta_0)}{(\sigma - \beta_0)^2 + 4 \gamma_0} \leq - \left( \frac{a_0 (r + c)}{(r + c)^2 + d_2^2} + \frac{a_1 (r + c)}{(r + c)^2 + 4 d_2^2} \right) \mathcal{L}.
\]
We deduce that (3.8) becomes \(0 \leq \mathcal{E}(d_1, d_2, r, c)\). Here

\[
\mathcal{E}(d_1, d_2, r, c) = \frac{a_0}{r} \frac{a_1}{r+c} + \frac{a_1 r}{r^2 + d_1^2} - \frac{a_0(r+c)}{(r+c)^2 + d_1^2} - \frac{a_0(r+c)}{(r+c)^2 + d_2^2} - \frac{a_1(r+c)}{(r+c)^2 + 4d_2^2} + \sum_{k=2,3,4} a_k \left( \frac{r}{r^2 + k^2d_1^2} - \frac{r+c}{(r+c)^2 + (k-1)^2d_2^2} \right)
\]

\[
- \frac{r+c}{(r+c)^2 + (k+1)^2d_2^2} + \frac{1-\kappa}{2} \sum_{k=0}^{4} a_k.
\]

(3.9)

Calculus gives that the above increases with \(c\). Thus the smallest value of \(c\) satisfying the inequality (3.9) is the root of \(\mathcal{E}(d_1, d_2, r, c) = 0\). We obtain

| \(d_1\) | \(d_2\) | \(r\) | \(1/c\) |
|-------|-------|------|-------|
| 1.021 | 2.374 | 0.236 | 12.7301 |

Remark 3.4. We explain our choice of \(P\) in this section. To simplify the analysis, we drop the terms depending on \(d_1\) and \(d_2\), and consider

\[
0 = \frac{a_0}{\tilde{r}} - \frac{a_1}{\tilde{r} + \tilde{c}} + \sum_{k=0}^{4} \frac{a_k(1 - \kappa)}{2}.
\]

As in remark 3.3 a similar analysis leads to

\[
\tilde{c} = \frac{(a_1 - a_0)\tilde{r} - \left(\frac{1+\kappa}{2} \sum_{k=0}^{4} a_k \right) \tilde{r}^2}{a_0 + \left(\frac{1+\kappa}{2} \sum_{k=0}^{4} a_k \right) \tilde{r}}
\]

with

\[
\tilde{r} = \frac{a_0 \left(\sqrt{\frac{a_1}{a_0}} - 1\right)}{\frac{1+\kappa}{2} \sum_{k=0}^{4} a_k}.
\]

We set \(d_1 = 1\), \(d_2 = 2.5\), and run a Maple computation to determine which \(a_k\)'s make the root \(c\) of \(\mathcal{E}(1, 2.5, \tilde{r}, c) = 0\) as small as possible.

3.4. Case 4: Zero-free region when \(0 < \gamma_0 \leq \frac{d_2}{2}\) with \(d_1 = 1.021\). In this case we consider

\[
S(\sigma, 0) = f(\sigma, 0),
\]

where

\[
f(\sigma, 0) = -\sum_{\beta \geq \frac{1}{2}} (F(\sigma, \beta) - \kappa F(\sigma_1, \beta)) + \frac{1-\kappa}{2} \log d_K + F(\sigma, 1) - \kappa F(\sigma_1, 1) + \Re \left( \frac{\gamma_K}{\gamma_K} - \kappa \frac{\gamma_K}{\gamma_K} \right).
\]

Using (2.25), (2.26), and (2.32), we obtain

\[
f(\sigma, 0) \leq -2 \frac{\sigma - \beta_0}{(\sigma - \beta_0)^2 + \gamma_0} + 2\alpha_1 + \frac{1-\kappa}{2} \log d_K + \frac{1}{\sigma - 1} + \alpha_2 + \left( -\frac{1-\kappa}{2} \log \pi + d(0) \right) n_K.
\]
The coefficient of $n_K$ is negative and may be dropped. Together with the fact that $f(\sigma, 0) \geq 0$, we deduce that, for $\gamma_0 < \frac{r}{2}$,

$$0 \leq \frac{1}{r} - 2 \frac{r + c}{(r + c)^2 + d_1^2} + \frac{1 - \kappa}{2} + \frac{2\alpha_1 + \alpha_20}{L},$$

which for $L$ asymptotically large gives

$$0 \leq \frac{1}{r} - 2 \frac{r + c}{(r + c)^2 + d_1^2} + \frac{1 - \kappa}{2}.$$

We solve:

$$c \geq -\frac{1-\kappa}{2} r^2 + \sqrt{r^2 - d_1^2 (1 + \frac{1-\kappa}{2} r)^2 \frac{1}{1+\frac{1-\kappa}{2} r}},$$

and find

| $d_1/4$ | $r$ | $1/c$ |
|--------|-----|------|
| 1.021  | 2.1426... | 12.5494 |
| 1/4    | 1.5344... | 1.6918 |
| 1/1.9996... | 1.644 | 1.9997 |

The two last rows justify the regions announced in (1.9) and (1.10).

3.5. Case 5: The case of real zeros. Consider $\beta_1$ and $\beta_2$, two real zeros with $\beta_1 \leq \beta_2$. We isolate both of them from the sum over the zeros, and use the trivial inequality:

$$-\frac{1}{\sigma - \beta_1} - \frac{1}{\sigma - \beta_2} \leq -\frac{2}{\sigma - \beta_1}.$$

It follows from $f(\sigma, 0) \geq 0$ that

$$0 \leq f(\sigma, 0) \leq -\frac{2}{\sigma - \beta_1} + 2\alpha_1 + \frac{1 - \kappa}{2} \log d_K + \frac{1}{\sigma - 1} + \alpha_20.$$

We write $1 - \beta_1 = \frac{c_1}{L}$ and obtain for $L$ sufficiently large

$$0 \leq \frac{1}{r} - 2 \frac{r + c_1}{r + c} + \frac{1 - \kappa}{2}.$$

The largest value for $c_1$ is given by $\frac{r-\frac{1}{2\kappa}\kappa}{1+\frac{1}{2\kappa}r}$ and this expression is maximized for $r = 2\sqrt{2-\frac{1}{\kappa}}$.

| $r$ | $1/c_1$ |
|-----|--------|
| 1.4986... | 1.6110 |

This proves that there is at most one zero in the region

$$\Re s \geq 1 - \frac{1}{1.62 L} \text{ and } \Im s = 0.$$

3.6. Conclusion. Observe that $\max(12.7305, 12.7301, 12.5494) = 12.7305$. Combining the results proven in Sections 3.2, 3.3, 3.4, and 3.5 we deduce that $\zeta_K(s)$ has at most one zero in the region:

$$\Re s \geq 1 - \frac{1}{12.74 \log d_K}, \vert \Im s \vert \leq 1.$$

Moreover, it follows from Section 3.5 that this zero, if it exists, is real and simple. This completes the proof of (1.8) of Theorem 1.1.
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