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Tome 67, n° 2 (2017), p. 863-877.

<http://aif.cedram.org/item?id=AIF_2017__67_2_863_0>
QUASICIRCLE BOUNDARIES AND EXOTIC ALMOST-ISOMETRIES

by Jean-François LAFONT, Benjamin SCHMIDT & Wouter VAN LIMBEEK (*)

Abstract. — We show that the limit set of an isometric and convex cocompact action of a surface group on a proper geodesic hyperbolic metric space, when equipped with a visual metric, is a Falconer–Marsh (weak) quasicircle. As a consequence, the Hausdorff dimension of such a limit set determines its bi-Lipschitz class. We give applications, including the existence of almost-isometries between periodic negatively curved metrics on $H^2$ that cannot be realized equivariantly.

1. Introduction

Consider a closed Riemannian manifold $(M, g)$ with sectional curvatures $\leq -1$. The metric $g$ lifts to a $\pi_1(M)$-invariant metric $\tilde{g}$ on the universal covering $X$, inducing an action of $\pi_1(M)$ on the boundary at infinity $\partial X$. This boundary, equipped with a visual metric $d_\partial$ (see Definition 2.3), exhibits fractal-like behavior as a consequence of this action. More generally, this phenomenon occurs if $\pi_1(M)$ acts properly discontinuously and convexly on a hyperbolic metric space $X$. Our first result follows this general philosophy:

Keywords: Rigidity, quasi-isometry, almost-isometry, bi-Lipschitz map, boundary at infinity, quasi-circle, limit set, Hausdorff dimension.

Math. classification: 20F67, 51F99.

(*) J.-F. L. was partially supported by the NSF, under grants DMS-1207782, DMS-1510640. B.S. was partially supported by the NSF, under grant DMS-1207655.
Theorem 1.1. — Let $\Gamma$ be a surface group acting isometrically, properly discontinuously, and convex cocompactly on a proper geodesic hyperbolic metric space $(X, d)$. Then the limit set of this action, with a visual metric $d_\partial$, is a (weak) quasicircle in the sense of Falconer–Marsh.

The following results use the special case of Theorem 1.1 where $(X, d)$ is the universal cover of a closed surface equipped with a locally $\text{CAT}(-1)$ metric, and $\Gamma$ acts on $X$ by deck transformations.

Corollary 1.2. — Let $(M_1, d_1), (M_2, d_2)$ be any pair of closed surfaces equipped with locally $\text{CAT}(-1)$ metrics, and let $(X_i, \tilde{d}_i)$ be their universal covers. Then one can find real numbers $0 < \lambda_i \leq 1$, with $\max\{\lambda_1, \lambda_2\} = 1$, having the property that $(X_1, \lambda_1 \tilde{d}_1)$ is almost-isometric to $(X_2, \lambda_2 \tilde{d}_2)$.

Recall that an almost-isometry is a quasi-isometry with multiplicative constant $= 1$. When $M_1, M_2$ are locally $\text{CAT}(-1)$ manifolds, the existence of an almost-isometry $X_1 \to X_2$ sometimes forces the universal covers to be isometric [12]. On the other hand, Corollary 1.2 yields examples of almost-isometric universal covers that are not isometric – take for example $d_1$ to be Riemannian and $d_2$ to be non-Riemannian.

Corollary 1.3. — Let $M$ be a closed surface with Riemannian metrics $g_1, g_2$ having curvatures $\leq -1$. Equip $\partial X$ with the corresponding canonical visual metrics $d_1$ and $d_2$. Then the following three statements are equivalent:

1) The topological entropies of the two geodesic flows on $T^1 M$ are equal.
2) The boundaries $(\partial X, d_1)$ and $(\partial X, d_2)$ are bi-Lipschitz equivalent.
3) The universal covers $(X, \tilde{g}_1)$ and $(X, \tilde{g}_2)$ are almost-isometric.

By Corollary 1.2, given two negatively curved Riemannian metrics $g_1$ and $g_2$ on a closed surface $M$, after possibly scaling one of the metrics (and also its area), the universal covers are almost-isometric. Our next result shows that one can arrange for equal area examples with almost-isometric universal covers.

Theorem 1.4. — Let $M$ be a closed surface of genus $\geq 2$, and $k \geq 1$ an integer. One can find a $k$-dimensional family $\mathcal{F}_k$ of Riemannian metrics on $M$, all of curvature $\leq -1$, with the following property. If $g, h$ are any two distinct metrics in $\mathcal{F}_k$, then

- $\text{Area}(M, g) = \text{Area}(M, h)$.
- the lifted metrics $\tilde{g}, \tilde{h}$ on the universal cover $X$ are almost-isometric.
• the lifted metrics \( \tilde{g}, \tilde{h} \) on the universal cover \( X \) are not isometric.

The almost-isometries between the lifted metrics in Theorem 1.4 are exotic in the sense that they cannot be realized equivariantly with respect to the \( \pi_1(M) \)-actions on \( X \). Indeed, a \( \pi_1(M) \)-equivariant almost-isometry between the two lifted metrics on \( X \) implies that the metrics on \( M \) have equal marked length spectra (see [12], for example), and are therefore isometric by [8, 16].

As a final application, we exhibit a gap phenomenon for the optimal multiplicative constant for quasi-isometries between certain periodic metrics.

**Corollary 1.5.** Let \((M_1, d_1), (M_2, d_2)\) be any pair of closed surfaces equipped with locally CAT(-1) metrics, and assume that their universal covers \((X_i, \tilde{d}_i)\) are not almost-isometric. Then there exists a constant \( \epsilon > 0 \) with the property that any \((C, K)\)-quasi-isometry from \((X_1, d_1)\) to \((X_2, d_2)\) must satisfy \( C \geq 1 + \epsilon \).

**Acknowledgments**

The authors would like to thank Marc Bourdon, Ralf Spatzier, Jeremy Tyson, and Xiangdong Xie for helpful discussions. We would also like to thank an anonymous referee for helpful comments.

The third author gratefully acknowledges support from the University of Chicago while part of this work was done. Part of this work was completed during a collaborative visit of the third author to Ohio State University (OSU), which was funded by the Mathematics Research Institute at OSU and the NSF.

**2. Background material**

This section reviews the definitions and existing results used in our proofs.

**2.1. Convex cocompact actions**

We refer the reader to [2, Section 1.8] for more details concerning this subsection. Let a finitely generated group \( \Gamma \) act properly discontinuously and isometrically on a proper geodesic hyperbolic metric space \( X \) with limit set \( \Lambda_\Gamma \subset \partial X \).
Definition 2.1. — The $\Gamma$-action on $X$ is convex cocompact if it satisfies any of the following equivalent conditions:

1. the map $\Phi : \Gamma \to X$ given by $\Phi(g) := g(x)$ is quasi-isometric.
2. the orbit of a point $\Gamma \cdot p$ is a quasi-convex subset of $X$.
3. $\Gamma$ acts cocompactly on the Gromov hull $Q(\Lambda \Gamma)$ of its limit set (consisting of all geodesics joining pairs of points in $\Lambda \Gamma$).

These conditions imply that $\Gamma$ is $\delta$-hyperbolic, and therefore has a Gromov boundary $\partial \Gamma$. An important consequence of the action being convex cocompact is that the limit set $\Lambda \Gamma$ is homeomorphic to $\partial \Gamma$. If $\Gamma$ is a surface group, then $\partial \Gamma$ is a topological circle.

2.2. Metrics on the boundary

We refer the reader to [5, Chapter III.H.3] for more details concerning this subsection. Let $X$ be a proper geodesic hyperbolic metric space with boundary $\partial X$. Fix a basepoint $w \in X$. The Gromov product $(\cdot|\cdot)_w : X \times X \to \mathbb{R}$ is defined by $(p|q)_w := \frac{1}{2} (d(w, p) + d(w, q) - d(p, q))$ for each $p, q \in X$, and extends to $\partial X \times \partial X$ by

$$(x, y)_w := \sup \lim_{n, m \to \infty} (x_n|y_m)_w$$

where the supremum is over all sequences $\{x_n\}$ and $\{y_m\}$ in $X$ such that $x_n \to x$ and $y_m \to y$. Given a basepoint $w \in X$ and $\epsilon > 0$, we set $\rho_{\epsilon}(x, y) := e^{-\epsilon(x|y)_w}$. In general $\rho_{\epsilon}$ may not define a metric on $\partial X$, so we define

$$d_{\epsilon}(x, y) := \inf \sum_{i=1}^{n} \rho_{\epsilon}(\xi_i, \xi_{i-1})$$

where the infimum is taken over all finite chains $x = \xi_0, \ldots, \xi_n = y$.

Proposition 2.2 (See [5, III.H.3.21]). — Fix $w \in X$. Then for $\epsilon > 0$ sufficiently small, $d_{\epsilon}$ is a metric on $\partial X$. Further, $d_{\epsilon}$ is bi-Lipschitz equivalent to $\rho_{\epsilon}$, i.e. there exists $C$ such that for every $x, y$, we have

$$\frac{1}{C} \rho_{\epsilon}(x, y) \leq d_{\epsilon}(x, y) \leq C \rho_{\epsilon}(x, y).$$

Further the bi-Lipschitz class of $d_{\epsilon}$ is independent of $w$.

Definition 2.3. — With a slight abuse of language, we refer to the family of bi-Lipschitz equivalent metrics $d_{\epsilon}$ obtained in Proposition 2.2 (with $w$ varying in $X$) as the visual metric on $\partial X$ with parameter $\epsilon$.

If $X$ is CAT(-1), then Bourdon showed that we can choose $\epsilon = 1$ [3]. In this case we refer to $d_1$ as the canonical visual metric on $\partial X$. 

Ann. Inst. Fourier (Grenoble) 66 (2016) 1579–1637.
Let $\Gamma$ be a properly discontinuous group of isometries of $X$ acting convex cocompactly with limit set $\Lambda \subseteq \partial X$. Let $s = \text{Hdim}(\partial X)$ be the Hausdorff dimension of a visual metric $d_\text{v}$ on $\partial X$, and $H^s$ the corresponding Hausdorff measure. Then $H^s$ is a finite measure, fully supported on $\Lambda$. The Hausdorff dimension and measure can be estimated using the following result (see Patterson–Sullivan [19] and its verbatim extension to the Gromov hyperbolic setting by Coornaert [7, Section 7]).

**Theorem 2.4** (Patterson–Sullivan, Coornaert).

(i) $s = \epsilon \lim_{n \to \infty} \frac{1}{n} \log \# \{ \gamma \in \Gamma \mid d(w, \gamma w) \leq n \}$

(ii) There exists a constant $C \geq 1$ such that for each metric ball $B(x, r)$ in $\partial X$ (with center $x$ and radius $r$), $C^{-1}r^s \leq H^s(B(x, r)) \leq Cr^s$.

**Remark 2.5.** — When $(M, g)$ is a closed Riemannian manifold with sectional curvatures $\leq -1$, its universal Riemannian covering $X$ is a CAT(-1) space equipped with a geometric action of $\Gamma = \pi_1(M)$ by deck transformations. Hence we can equip $\partial X$ with the canonical visual metric (i.e. $\epsilon = 1$). In this case, Sullivan (for Kleinian groups) and Otal and Peigné [17] proved that $s = h(g)$, where the latter denotes the topological entropy of the geodesic flow on the unit tangent bundle $T^1(M)$.

2.3. (Weak) Quasicircles according to Falconer–Marsh

Next let us briefly review some notions and results of Falconer and Marsh [9].

**Definition 2.6** (Falconer–Marsh). — A metric space $(C, d)$ is a quasicircle if

(i) $C$ is homeomorphic to $S^1$,

(ii) (expanding similarities) There exist $a, b, r_0 > 0$ with the following property. For any $r < r_0$ and $N \subseteq C$ with $\text{diam}(N) = r$, there exists an expanding map $f : N \to C$ with expansion coefficient between $\frac{a}{r}$ and $\frac{b}{r}$, i.e. for all distinct $x, y \in N$, we have the estimate

$$\frac{a}{r} \leq \frac{d(f(x), f(y))}{d(x, y)} \leq \frac{b}{r}$$

Note that $a, b$ are independent of the size of $N$, and of the choice of points in $N$. 

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(iii) (contracting similarities) There exist $c, r_1 > 0$ with the following property. For any $r < r_1$ and ball $B \subseteq C$ with radius $r$, there exists a map $f : C \to B \cap C$ contracting no more than a factor $cr$.

Let $s = \text{Hdim}(C)$ be the Hausdorff dimension of $C$, and $H^s$ the corresponding Hausdorff measure. The following alternate property to (iii) is implied by (ii) and (iii):

(iiiia) For any open $U \subseteq C$ one has $0 < H^s(U) < \infty$, and $H^s(U) \to 0$ as $\text{diam}(U) \to 0$.

The main result of Falconer–Marsh [9] is that two quasicircles $C_1$ and $C_2$ are bi-Lipschitz equivalent if and only if their Hausdorff dimensions are equal. While stated this way, their proof only uses conditions (i), (ii), and (iiiia). For this reason, we will say that a metric space $(C, d)$ is a weak quasicircle when conditions (i), (ii), and (iiiia) are satisfied.

**Theorem 2.7** (Falconer–Marsh). — Two weak quasicircles $C_1$ and $C_2$ are bi-Lipschitz equivalent if and only if their Hausdorff dimensions are equal.

**Remark 2.8.** — In geometric function theory, the term quasicircle is used for the image of $S^1$ under a quasisymmetric map $f$. We will call these quasisymmetric circles. Falconer–Marsh quasicircles form a strict subset of the quasisymmetric circles. Indeed, a Falconer–Marsh quasicircle $X$ has the same Hausdorff dimension as the visual boundary $Y$ of some negatively curved surface. By Theorems 1.1 and 2.7, $X$ and $Y$ are bi-Lipschitz equivalent. Since $Y$ is a quasisymmetric circle, so is $X$. On the other hand, Property (ii) of Falconer–Marsh quasicircles implies that any nonempty open subset $U$ of a Falconer–Marsh quasicircle $X$ has the same Hausdorff dimension as $X$, a property that quasisymmetric circles need not have (see [18]).

### 2.4. Almost-isometries and the work of Bonk–Schramm

Now let us recall some results of Bonk and Schramm [1] that we will need.

**Definition 2.9.** — A map $f : X \to Y$ between metric spaces $(X, d_X)$ and $(Y, d_Y)$ is quasi-isometric if there exists constants $C, K$ such that

$$\frac{1}{C} d_X(x, y) - K \leq d_Y(f(x), f(y)) \leq C d_X(x, y) + K.$$


A map $f : X \to Y$ is coarsely onto if $Y$ lies in a bounded neighborhood of $f(X)$. If the quasi-isometric map $f$ is coarsely onto, then we call it a quasi-isometry, and we say that $X,Y$ are quasi-isometric. A map is almost-isometric if it is quasi-isometric with multiplicative constant $C = 1$. An almost-isometric map $f : X \to Y$ which is coarsely onto is called an almost-isometry, in which case we say that $X,Y$ are almost-isometric.

Special cases of the results in [1] relate the existence of almost-isometries with metric properties of $\partial X$, as follows.

**Theorem 2.10** (Bonk–Schramm). — Let $X,Y$ be a pair of CAT(-1) spaces. Then $X,Y$ are almost-isometric if and only if the canonical visual metrics on the boundaries $\partial X, \partial Y$ are bi-Lipschitz homeomorphic to each other.

The fact that an almost-isometry between $X$ and $Y$ induces a bi-Lipschitz homeomorphism between the boundaries $\partial X$ and $\partial Y$ appears in [1, proof of Theorem 6.5] – where it should be noted that, in the notation of their proof, our more restrictive context corresponds to $\epsilon = \epsilon' = 1$ and $\lambda = 1$. As for the converse, the interested reader should consult [1, Theorem 7.4] to see that a bi-Lipschitz map between boundaries $\partial X$ and $\partial Y$ induces an almost-isometry between the metric spaces $\text{Con}(X)$ and $\text{Con}(Y)$. The comment following [1, Theorem 8.2] applies in our context where $a = 1$, so that $\text{Con}(X)$ and $\text{Con}(Y)$ are almost isometric to $X$ and $Y$, respectively, whence $X$ and $Y$ are almost-isometric.

### 3. Surface boundaries are (weak) quasicircles

Having covered the preliminaries, let us now give a proof of Theorem 1.1, as well as the various corollaries.

**Proof of Theorem 1.1.** — Looking at Definition 2.6, there are three properties to establish. Property (i) is obvious – see the discussion in Section 2.1. If $\Gamma$ acts cocompactly on $X$, then Property (ii) is a consequence of the more general result of Bourdon and Kleiner – see [4, Section 3.1]. In the convex cocompact case, let $Y$ be the Gromov hull (in $X$) of the limit set of $\Gamma$. Then the Bourdon–Kleiner argument with minor modifications (using quasi-convexity of $Y$ in $X$) applies to the $\Gamma$ action on $Y$, and this proves Property (ii) in general (an earlier version of this paper, available on the arXiv [15], contains a more detailed discussion of this point).
Finally, property (iii) is an immediate consequence of Theorem 2.4. We conclude that the limit set is indeed a (weak) quasicircle in the sense of Falconer–Marsh.

The proof of all three corollaries are now completely straightforward.

Proof of Corollary 1.3. — Theorem 1.1 gives us that \((\partial X, d_1)\) and \((\partial X, d_2)\) are weak quasicircles. Then Falconer and Marsh’s Theorem 2.7 and Remark 2.5 gives the equivalence of statements (1) and (2), while Bonk and Schramm’s Theorem 2.10 gives the equivalence of statements (2) and (3).

Proof of Corollary 1.2. — When two metrics on \(X\) are related by a scale factor \(\lambda\), it easily follows from the formula for the canonical visual metric that the Hausdorff dimension of the boundary at infinity scales by \(1/\lambda\). Note also that if \((X, d)\) is CAT(-1) and \(0 < \lambda \leq 1\), then the rescaled metric \((X, \lambda d)\) is also CAT(-1). Corollary 1.2 then immediately follows by combining our Theorem 1.1, Falconer and Marsh’s Theorem 2.7, and Bonk and Schramm’s Theorem 2.10.

Proof of Corollary 1.5. — Combining our Theorem 1.1, Falconer and Marsh’s Theorem 2.7, and Bonk and Schramm’s Theorem 2.10, we see that the boundaries at infinity \((\partial X_i, d_i)\) must have distinct Hausdorff dimensions. Without loss of generality, let us assume \(\Hdim(\partial X_1, d_1) < \Hdim(\partial X_2, d_2)\). If \(\phi : X_1 \to X_2\) is a \((C, K)\)-quasi-isometry, we want to obtain a lower bound on \(C\). But the quasi-isometry induces a homeomorphism \(\partial \phi : \partial X_1 \to \partial X_2\) which, from the definition of the Gromov product, has the property that

\[
\rho_2(\partial \phi(x), \partial \phi(y)) \leq e^K \cdot \rho_1(x, y)^C
\]

for all \(x, y \in \partial X_1\). Because \(d_i\) and \(\rho_i\) are bi-Lipschitz equivalent, we obtain the desired inequality

\[
1 < \frac{\Hdim(\partial X_2, d_2)}{\Hdim(\partial X_1, d_1)} \leq C.
\]

4. Constructing exotic almost-isometries

This section is devoted to the proof of Theorem 1.4. In view of Corollary 1.3, we want to produce a \(k\)-dimensional family \(\mathcal{F}_k\) of equal area metrics on a higher genus surface \(M\), which all have the same topological entropy, but whose lifts to the universal cover are not isometric to each other.
4.1. Perturbations of metrics

Start with a fixed reference hyperbolic metric $g_0$ on $M$, normalized to have constant curvature $-2$. Pick $(k + 2)$ distinct points $p_1, \ldots, p_{k+2} \in M$, and choose $r_2$ smaller than the injectivity radius of $M$ and satisfying $2r_2 < \inf_{i \neq j} \{d(p_i, p_j)\}$. Let $U_i$ denote the open metric ball of radius $r_2$ centered at $p_i$ — note that the $U_i$ are all isometric to each other, and are pairwise disjoint. Now choose $r_1 < r_2$ so that the area of the ball of radius $r_1$ is at least $4/5$ the area of the ball of radius $r_2$. Denote by $V_i \subset U_i$ the ball of radius $r_1$ centered at each $p_i$.

We will vary the metric $g_0$ by introducing a perturbation on each of the $U_i$ in the following manner. Let us choose a smooth bump function $\rho : [0, \infty) \to [0, 1]$ with the property that $\rho|_{[0, r_1]} = 1$ and $\rho|_{[r_2, \infty]} = 0$. Next define $u_i : M \to [0, 1]$ via $u_i(x) := \rho(d(x, p_i))$. Given a parameter $\vec{t} := (t_1, \ldots, t_{k+2}) \in \mathbb{R}^{k+2}$, define the function $u_\vec{t} : M \to [0, \infty)$ by setting $u_\vec{t} := t_1 u_1 + \cdots + t_{k+2} u_{k+2}$. Finally, we define the metric $g_\vec{t} := e^{2u_\vec{t}} g_0$ (and note the identification $g_{\vec{0}} = g_0$). The family $\mathcal{F}_k$ will be obtained by choosing suitable values of $\vec{t}$ close to $\vec{0}$.

Since the metric $g_\vec{t}$ is obtained by making a conformal change on each $U_i$, and since the $U_i$ are pairwise disjoint, we first analyze the behavior of such a change on an individual $U_i$. To simplify notation, denote by $V \subset U$ open balls of radius $r_1 < r_2$ centered at a point $p$ in the hyperbolic plane $\mathbb{H}^2_{-2}$ of curvature $-2$, and set $g_t := e^{2tu} g_0$ where $g_0$ is the hyperbolic metric of curvature $-2$, and $u : \mathbb{H}^2_{-2} \to [0, \infty)$ is given by $u(x) := \rho(d(p, x))$. We start with the easy:

**Lemma 4.1.** As $t \to 0$, we have the following estimates:

1. the curvatures $K(g_t)$ tend uniformly to $-2$.
2. the area $\text{Area}(U; g_t)$ of the ball $U$ tends to $\text{Area}(U; g_0)$.
3. the area $\text{Area}(V; g_t)$ of the ball $V$ tends to $\text{Area}(V; g_0)$.

**Proof.** This is straightforward from the formulas expressing how curvature and area change when one makes a conformal change of metric. We have that the new curvature $K(g_t)$ is related to the old curvature $K(g_0)$ via the formula

$$K(g_t) = (e^{-2u})^t K(g_0) - t(e^{-2u})^t \Delta u$$

where $\Delta u$ denotes the Laplacian of the function $u$ in the hyperbolic metric $g_0$. As $t$ tends to zero, it is clear that the expression on the right converges to $K(g_0)$ uniformly, giving (1). Similarly, the area form $dg_t$ for the new
metric is related to the area form \( dg_0 \) for the original metric via the formula
\[
dg_t = (e^{2u})^t dg_0
\]
giving us (2) and (3).

4.2. Lifted metrics are almost-isometric

Next we establish that, for suitable choices of the parameter \( \vec{t} \), we can arrange for the lifted metrics to be almost-isometric. By Lemma 4.1, we can take the parameters \( \vec{t} \) close enough to \( \vec{0} \) to ensure that all the metrics we consider have sectional curvatures \( \leq -1 \). Then from Corollary 1.3, it suffices to consider values of the parameter \( \vec{t} \) for which the corresponding metrics have the same topological entropy for the geodesic flow on \( T^1 \mathbb{M} \).

Notice that varying \( \vec{t} \) near \( \vec{0} \) gives a \( C^\infty \) family of perturbations of the metric \( g_0 \). Work of Katok, Knieper, Pollicott and Weiss [13, Theorem 2] then implies that the topological entropy map \( h \), when restricted to any line \( l(s) \) through the origin \( \vec{0} \) in the \( \vec{t} \)-space, is a \( C^\infty \) map. Moreover the derivative of \( h \) along the line is given by (see [14, Theorem 3])
\[
\frac{\partial}{\partial s}
\bigg|_{s=0}

h \left( g_{l(s)} \right) = -\frac{h(g_0)}{2} \int_{T^1 \mathbb{M}} \frac{\partial}{\partial s}
\bigg|_{s=0}
g_{l(s)}(v,v) d\mu_0
\]
where \( T^1 \mathbb{M} \) denotes the unit tangent bundle of \( \mathbb{M} \) with respect to the \( g_0 \)-metric, and \( \mu_0 \) denotes the Margulis measure of \( g_0 \) (the unique measure of maximal entropy for the \( g_0 \)-geodesic flow on \( T^1 \mathbb{M} \)).

Consider the map \( F : \mathbb{R}^{k+2} \to \mathbb{R} \) given by \( F(t_1, \ldots, t_{k+2}) = h(g_{(t_1, \ldots, t_{k+2})}) \), where \( h \) denotes the topological entropy of (the geodesic flow associated to) a metric. Let us compute the directional derivative in the direction \( \frac{\partial}{\partial t_1} \):
\[
\frac{\partial F}{\partial t_1}(0, \ldots, 0) = \frac{d}{dt}
\bigg|_{t=0}

h \left( g_{(t,0,\ldots,0)} \right) = -\frac{h(g_0)}{2} \int_{T^1 \mathbb{M}} \frac{d}{dt} \bigg|_{t=0}
g_{(t,0,\ldots,0)}(v,v) d\mu_0
= -\frac{h(g_0)}{2} \int_{T^1 \mathbb{M}} \frac{d}{dt} \bigg|_{t=0} e^{2t u_1(\pi(v))} d\mu_0
= -\frac{h(g_0)}{2} \int_{T^1 \mathbb{M}} 2u_1(\pi(v)) d\mu_0
\]
where \( \pi : T^1 \mathbb{M} \to \mathbb{M} \) is the projection from the unit tangent bundle onto the surface \( \mathbb{M} \). Finally, we observe that by construction \( u_1 \) is a non-negative function, which is identically zero on the complement of \( U_1 \), and identically one on the set \( V_1 \). Hence the integral above is positive, and we obtain \( \frac{\partial F}{\partial t_1}(0) < 0 \).
Now, a similar calculation applied to each of the other coordinates gives us the general formula for the directional derivative of $F$. The gradient of $F$ is given by the non-vanishing vector:

$$\nabla F = -h(g_0) \int_{T^1M} \langle u_1(\pi(v)), \ldots, u_{k+2}(\pi(v)) \rangle d\mu_0.$$ 

In fact, since each $u_i$ is supported solely on $U_i$, and each $u_i$ is defined as $\rho(d(p_i, x))$ on the $U_i$, each of the integrals in the expression for $\nabla F$ has the same value. So $\nabla F$ is just a nonzero multiple of the vector $\langle 1, \ldots, 1 \rangle$.

The implicit function theorem locally gives us an embedded codimension one submanifold $\sigma(z)$ (where $z \in \mathbb{R}^{k+1}, ||z|| < \epsilon$) in the $(t_1, \ldots, t_{k+2})$-space, with normal vector $\langle 1, \ldots, 1 \rangle$ at the point $\sigma(\vec{0}) = \vec{0}$, on which the topological entropy functional is constant. From Corollary 1.3, we see that the lifts of these metrics to the universal cover are all pairwise almost-isometric.

### 4.3. Lifted metrics are not isometric

**Lemma 4.2.** — There is an $\epsilon > 0$ so that if the parameters $\vec{s} = (s_1, \ldots, s_{k+2})$ and $\vec{t} = (t_1, \ldots, t_{k+2})$ satisfy $0 < |s_i| < \epsilon$ and $0 < |t_i| < \epsilon$ and the lifted metrics $(\tilde{M}, \tilde{g}_{\vec{s}})$ and $(\tilde{M}, \tilde{g}_{\vec{t}})$ are isometric to each other, then we must have an equality of sets $\{s_1, \ldots, s_{k+2}\} = \{t_1, \ldots, t_{k+2}\}$.

**Proof.** — By Lemma 4.1, it is possible to pick $\epsilon$ small enough so that, for all parameters $\vec{s}, \vec{t}$ within the $\epsilon$-ball around $\vec{0}$, we have that

$$\text{Area}(V_i; g_{\vec{t}}) \geq \frac{3}{4} \text{Area}(U_j; g_{\vec{s}})$$

for every $1 \leq i, j \leq k+2$.

Now let us assume that there is an isometry $\Phi : (\tilde{M}, \tilde{g}_{\vec{s}}) \to (\tilde{M}, \tilde{g}_{\vec{t}})$. Observe that the lifted metrics have the following properties:

(i) on the complement of the lifts of the $U_i$, both metrics have curvature identically $-2$.

(ii) on any lift of the set $V_1$, the metric $\tilde{g}_{\vec{s}}$ has curvature identically $-2e^{-2s_1}$.

(iii) on any lift of the set $V_i$, the metric $\tilde{g}_{\vec{t}}$ has curvature identically $-2e^{-2t_i}$.

Take a lift $\tilde{V}_1$ of $V_1$ in the source, and consider its image under $\Phi$. The metric in the source has curvature identically $-2e^{-2s_1}$ on this lift $\tilde{V}_1$, and since $\Phi$ is an isometry, the image set $\Phi(\tilde{V}_1)$ must have the same curvature. From property (i), we see that $\Phi(\tilde{V}_1)$ must lie, as a set, inside the union
of lifts of the $U_i$. Since $\Phi(\tilde{V}_1)$ is path-connected, it must lie inside a single connected lift $\tilde{U}_i$ of one of the $U_i$. But from the area estimate, we see that for the $\tilde{V}_i \subset \tilde{U}_i$ inside the lift, one has that the intersection $\Phi(\tilde{V}_1) \cap \tilde{V}_i$ is non-empty. Looking at the curvature of a point in the intersection, we see that

$$-2e^{-2s_i} = -2e^{-2t_i}$$

and hence that $s_1 = t_i$ for some $i$. Applying the same argument to each of $s_i, t_i$ completes the proof. \hfill \Box

Now pick a vector $\vec{v} = (v_1, \ldots, v_{k+2})$ with the property that $v_1 + \cdots + v_{k+2} = 0$, and such that $v_i \neq v_j$ for each $i \neq j$. Notice that the first constraint just means that $\vec{v} \cdot \nabla F = 0$, and hence that $\vec{v}$ is tangent to the $(k+1)$-dimensional submanifold $\sigma$. So there exists a curve $\gamma \subset \sigma$ satisfying $\gamma(0) = \vec{0}$, and $\gamma'(0) = \vec{v}$. Notice that, from our second condition, when $t \approx 0$ we have $\gamma(t) \approx (v_1t, \ldots, v_{k+2}t)$, and hence the point $\gamma(t)$ has all coordinates distinct. It follows from Lemma 4.2 that, for any $t \approx 0$ ($t \neq 0$), one can find a small enough connected neighborhood $W_t$ of $\gamma(t)$ with the property that all the metrics in that neighborhood have lifts to the universal cover that are pairwise non-isometric.

### 4.4. Metrics with equal area

Now consider the smooth function

$$A : \sigma \rightarrow \mathbb{R}$$

defined by $A(z) := \text{Area}(g_{\sigma(z)})$ for each $z \in \sigma$. The change of area formula for a conformal change of metric (see the proof of Lemma 4.1) implies that $A$ is nonconstant on $W_t$. By Sard’s theorem, there is a regular value $r$ of $A$ in the interval $A(W_t)$. Then $\tau := A^{-1}(r)$ is a smooth $k$-dimensional submanifold of the $(k+1)$-dimensional manifold $\sigma$ consisting of parameters for area $r$ metrics. A connected component $F_k$ of $W_t \cap A^{-1}(r)$ satisfies all of the constraints of Theorem 1.4.

### 5. Concluding remarks

As the reader undoubtedly noticed, our results rely heavily on the surprising result of Falconer and Marsh. As such, it is very specific to the case of circle boundaries – which essentially restricts us to surface groups (see
Gabai [11]). In higher dimensions, we would not expect the bi-Lipschitz class of a self-similar metric on a sphere to be classified by its Hausdorff dimension. Thus, the following problem seems substantially more difficult.

**Conjecture 5.1.** — Let $M$ be a smooth closed manifold of dimension $\geq 3$, and assume that $M$ supports a negatively curved Riemannian metric. Then $M$ supports a pair of equal volume Riemannian metrics $g_1, g_2$ with curvatures $\leq -1$, and having the property that the Riemannian universal covers $(\hat{M}, \hat{g}_i)$ are almost-isometric, but are not isometric.

In another direction, if one were to drop the dimension, then there are many examples of 0-dimensional spaces having analogous self-similarity properties (i.e. properties (ii), (iii) in Definition 2.6). The metrics on these boundaries turn them into Cantor sets — and the classification of (metric) Cantor sets up to bi-Lipschitz equivalence seems much more complex than in the circle case (for some foundational results on this problem, see for instance Falconer and Marsh [10] and Cooper and Pignaturo [6]). Of course, from the viewpoint of boundaries, such spaces would typically arise as the boundary at infinity of a metric tree $T$. This suggests the following:

**Problem.** — Study periodic metrics on trees up to the relation of almost-isometry.

In particular, invariance of the metric under a cocompact group action translates to additional constraints on the canonical visual metric on $\partial T$, e.g. the existence of a large (convergence) group action via conformal automorphisms (compare with the main theorem in [6]). It would be interesting to see if this makes the bi-Lipschitz classification problem any easier.

Finally, given a pair of quasi-isometric spaces, we can consider the collection of all quasi-isometries between them, and try to find the quasi-isometry which has smallest multiplicative constant. More precisely, given a pair of quasi-isometric metric spaces $X_1, X_2$, define the real number $\mu(X_1, X_2)$ to be the infimum of the real numbers $C$ with the property that there exists some $(C, K)$-quasi-isometry from $X_1$ to $X_2$. We can now formulate the:

**Problem.** — Given a pair of quasi-isometric metric spaces $X_1, X_2$, can one estimate $\mu(X_1, X_2)$? Can one find a $(C, K)$-quasi-isometry from $X_1$ to $X_2$, where $C = \mu(X_1, X_2)$? In particular, can one find a pair of quasi-isometric spaces $X_1, X_2$ which are not almost-isometric, but which nevertheless satisfy $\mu(X_1, X_2) = 1$?

Our Corollary 1.5 gives a complete answer in the case where the $X_i$ are universal covers of locally CAT(-1) metrics on surfaces — the real number
$\mu(X_1, X_2)$ is exactly the ratio of the Hausdorff dimensions of the canonical visual metrics on the boundary, and one can always find a quasi-isometry with multiplicative constant $\mu(X_1, X_2)$. It is unclear what to expect in the more general setting.

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Manuscrit reçu le 4 octobre 2014,
révisé le 16 février 2015,
accepté le 26 mars 2015.

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