DESIGNING COILS TO PRODUCE USER-SPECIFIED MAGNETIC FIELD PROFILES INSIDE A CLOSED FINITE-LENGTH HIGH-PERMEABILITY CYLINDER

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ABSTRACT

Commercially available passive magnetic shields with active components that allow the generation of a tailored low-field environment are required for many applications in science, engineering, and medical imaging. Until now, accurate field nulling, or field generation, has only been possible over a small fraction of the overall volume of the shield. This is due to the interaction between the active field-generating components and the multiple layers of surrounding high-permeability passive shielding material. Such material distorts the field generated by the active parts, making it hard to optimize the spatial variation of the field ab initio. Here, we show that this design problem can be overcome for cylindrical shields with planar end-caps by explicitly including the interactions with the passive shielding layers in the optimization procedure for the active field-generating components. Specifically, we consider the interaction between a finite closed cylindrical passive shield and an arbitrary cylindrical coaxial static current source inside the shield. We modify the Green’s function for the magnetic vector potential so that it satisfies the boundary conditions of the passive shield, thereby incorporating the response of the high-permeability material, and then apply a harmonic minimization method. We illustrate the validity of our method, and its applicability to generating a range of user-specified magnetic field profiles, by using it to design two hybrid active–passive systems inside a closed cylindrical perfect magnetic conductor, with a length-to-diameter ratio of 2, which generate a constant transverse field, $B_x$, across the cylinder, and a linear transverse field gradient, $B = (z \hat{x} + x \hat{z})$, along the axis of the cylinder. We then analyze our constant transverse field-generating system, $B_x$, in a simulated passive magnetic shield of finite magnetic permeability while varying the shielding thickness and introducing axial entry holes of varying sizes to find the physical limitations in our model. In the limiting cases, the fields generated by both the constant transverse and linear gradient systems are within 0.1\% and 0.5\% of the desired target field, respectively, over 40\% of the central radial and axial extent of a simulated passive magnetic shield with magnetic permeability, thickness, and axial entry holes that recreate an example experimental system. Our optimization procedure can be adapted to design active–passive magnetic field shaping systems that accurately generate any physical user-specified static magnetic field in the interior of a closed cylindrical shield of any length, enabling the development and miniaturization of systems that require accurate magnetic shielding and control.

1 Introduction

Regions of space that require precisely-controlled magnetic field environments are essential for a wide range of experiments, devices, and applications. These include quantum sensing of gravity and gravity gradients for geophysical survey and mapping \cite{1, 2, 3, 4}; atomic magnetometry \cite{5, 6, 7} for applications including material characterization \cite{8}.
and magnetoencephalography\textsuperscript{[9,10]}, noise reduction in fundamental physics experiments\textsuperscript{[11,12]} such as timing using high precision atomic clocks\textsuperscript{[13,14]}, measuring the electric dipole moment of fundamental systems\textsuperscript{[15,16,17,18]}, and testing Lorentz-invariance by observing the limits of spin precession\textsuperscript{[19,20,21]}. Typically, these systems are enclosed within high-permeability materials, such as mumetal, to shield magnetically sensitive components from undesired magnetic fields. Cylindrical geometries, in particular, are widely used due to their comparatively high shielding factor relative to their low manufacturing cost\textsuperscript{[22,23,24]}. However, the fields produced by active coil systems needed either for field generation or further cancellation, are distorted by the presence of a high-permeability magnetic shield, hindering the accurate generation of specified target magnetic field profiles\textsuperscript{[23]}. Consequently, optimization of magnetic field cancellation, or other field-shaping, systems in the presence of a material with high magnetic permeability is a long-standing challenge in electromagnetism.

Finite element methods (FEMs) can be used to develop models of hybridized active and passive shielding systems. One method of optimizing active–passive systems using FEMs would be to use genetic algorithms\textsuperscript{[26,27]} to evolve the coil parameters iteratively, including the effect of the passive material on the magnetic field generated actively at each iteration. Due to their computational complexity, however, FEMs can be slow, and, if coupled with a forward optimization procedure, are limited to locally optimal designs as the magnetic field profile depends highly non-linearly on the system parameters. Analytical formulations of the magnetic field generated by hybrid active–passive systems have the distinct advantage that optimal solutions can be calculated at a range of target points with minimal computation\textsuperscript{[28]}. Thus far, analytical solutions for coils in high-permeability cylindrical magnetic shields have only been formulated in the specific cases of magnetically-shielded solenoids and discrete current loops\textsuperscript{[29,30,31]}. The geometric parameters of the active structure in these systems, such as the separation distances of wire loops, are adapted to account for the interaction between the active and passive system. Previously, the method of mirror images\textsuperscript{[32]} has been used to formulate the response of a planar material with high magnetic permeability to a magnetic field generated by a current source\textsuperscript{[33,34,35,36]}. In these formulations, Maxwell’s equations are solved explicitly by including the reflections of the current source generated by the high-permeability material in order to match the required boundary conditions. More generally, Green’s function solutions to boundary value problems have been calculated for an extensive range of electromagnetic systems\textsuperscript{[37]}, but have not previously been found generally for the case of a finite closed cylindrical high-permeability shield in the presence of an arbitrary cylindrical coaxial static current source.

To address this, here, we incorporate \textit{ab initio} the effect of a finite length high-permeability cylinder on the magnetic field generated by an arbitrary static current flow on a conducting cylinder, and use our results to determine the flow required to generate specified static target fields optimally. This enables the geometry of the active components on the surface of a cylinder to be determined entirely from the required magnetic field profile. Guided by\textsuperscript{[38]}, we first derive a Green’s function for a hybrid active–passive field-generating system described in cylindrical coordinates. We then utilize a modified Fourier basis to define an adjusted current density distribution, which accounts for the effect of the high-permeability material. From this, we determine globally-optimal current density maps using a quadratic optimization procedure, akin to magnetic field design methodologies used previously in Magnetic Resonance Imaging (MRI)\textsuperscript{[39,40]}. To enable the construction of a practical current source, we then use the streamfunction of the cylindrical current continuum to approximate it by a set of closed-loop current-carrying wire geometries. Using this formulation, we present two example coil designs optimized in the interior of a closed cylindrical magnetic shield of finite length and high permeability, $\mu_r \gg 1$, and show that our analytical model agrees well with FEM simulations performed for the optimized current flow patterns. We then show how the design methodology can be used in a real-world situation where the cylindrical magnetic shield has finite thickness and permeability as well as axial entry holes in the end caps to allow experimental access. Using this formulation, we transform the performance of systems designed to generate user-specified magnetic field profiles in the magnetostatic regime, reducing the amount of high-permeability material required, and find globally-optimal solutions required for practical, low Size, Weight, Power, and Cost (SWaP-C) technologies for the applications listed above.

2 Theory

The matching conditions for a magnetic field along a boundary, $S$, between two materials are that the normal component of the magnetic field, $\mathbf{B}$, and tangential component of the magnetic field strength, $\mathbf{H}$, are continuous. Considering the interface between air and a material, working in the magnetostatic regime, where no eddy currents are induced, and in the case where no surface currents are present, these matching conditions are

\[
\langle \mathbf{B}_{\text{mat.}} - \mathbf{B}_{\text{air}} \rangle \cdot \hat{n} = 0 \quad \text{on } S,  \tag{1}
\]

and

\[
\left( \frac{1}{\mu_r} \mathbf{B}_{\text{mat.}} - \mathbf{B}_{\text{air}} \right) \wedge \hat{n} = 0 \quad \text{on } S,  \tag{2}
\]
where \( \hat{n} \) is the unit vector normal to the boundary, and \( \mu_r \) is the relative permeability of the material. In free space, the magnetic field is related to the magnetic field strength and the magnetization, \( \mathbf{M} \), by

\[
\mathbf{B} = \mu_0 (\mathbf{H} + \mathbf{M}).
\]

Physically, the magnetization of the sub-domains of a material change in response to an applied magnetic field to satisfy the boundary condition, (2), at the material’s surface. Here, we choose to formulate this response in terms of a pseudo-current density, \( \mathbf{\tilde{J}} \), confined to the surface of the material, which relates to the curl of the magnetization,

\[
\nabla \wedge \mathbf{M} = \mathbf{\tilde{J}}.
\]

The magnetic field strength resulting from a current source, \( \mathbf{J}_c \), can be expressed using Ampère’s law

\[
\nabla \wedge \mathbf{H} = \mathbf{J}_c.
\]

Substituting (4) and (5) into the curl of (3), noting \( \mathbf{B} = \nabla \wedge \mathbf{A} \) where \( \mathbf{A} \) is the vector potential, results in the Poisson equation in free space,

\[
\nabla^2 \mathbf{A} = -\mu_0 (\mathbf{J}_c + \mathbf{\tilde{J}}).
\]

As shown in many papers and textbooks \cite{32}, the solution to the Poisson equation, for an arbitrary current density, \( \mathbf{J} \), is given by

\[
\mathbf{A}(\mathbf{r}) = \mu_0 \int_{\mathbf{r}'} d^3 \mathbf{r}' G(\mathbf{r}, \mathbf{r}') \mathbf{J}(\mathbf{r}'),
\]

where \( G(\mathbf{r}, \mathbf{r}') \) is the associated Green’s function.

Let us now use the boundary condition on the magnetic field, (2), to examine the effect of the high-permeability material on the magnetic field generated by a specific current distribution. Consider a hollow high-permeability, \( \mu_r \rightarrow \infty \), cylinder of radius, \( \rho_s \), length, \( L_s \), thickness, \( d \), with high-permeability planar end caps located at \( z = \pm L_s/2 \) that is surrounded by free space (Fig. 1). An arbitrary static current flows over an internal open coaxial cylinder of radius, \( \rho_c < \rho_s \), and length, \( L_1 - L_2 = L_c < L_s \), where \( -L_s/2 < L_2 < L_1 < L_s/2 \), as shown in Fig. 1. Since high-permeability materials, such as mumetal, can have \( \mu_r \) values \( >100,000 \) times that of air, the magnetic field must abruptly change direction at the boundary between its surface and air to satisfy the boundary conditions (1)-(2). When the shield is of sufficient thickness, the tangential components of the magnetic field at its boundary must be approximately zero. The boundary condition, (2), on the interior surface of the exterior closed cylinder for the example depicted in Fig. 1, requires that

\[
B_r \bigg|_{z=\pm L_s/2} \approx 0, \quad B_z \bigg|_{\rho=\rho_c} \approx 0, \quad B_\phi \bigg|_{z=\pm L_s/2, \rho=\rho_s} \approx 0.
\]
where the magnetic field, \( \mathbf{B} = B_\rho \hat{\rho} + B_\phi \hat{\phi} + B_z \hat{z} \), is expressed in cylindrical polar coordinates. The response of the high-permeability material to the magnetic field generated by current flow over the inner cylinder deviates from that of a perfect magnetic conductor on the scale of \( O(\mu^{-1}) \) \[41, 42\]. Therefore, we may assume the high-permeability material is a perfect magnetic conductor without introducing significant errors.

Due to the symmetries of the system it is natural to work in cylindrical coordinates. Following the formulation of Turner [38], we may express the vector potential (7) due to a current distribution, \( \mathbf{J} = J_{\rho}(r') \hat{\rho} + J_{\phi}(r') \hat{\phi} + J_z(r') \hat{z} \), on an arbitrary cylinder as

\[
A_\rho(r) = \mu_0 \int_{r'} d^3r' G(r, r') (J_\rho(r') \cos(\phi - \phi') + J_\phi(r') \sin(\phi - \phi')) ,
\]

\[
A_\phi(r) = -\mu_0 \int_{r'} d^3r' G(r, r') (J_\rho(r') \sin(\phi - \phi') - J_\phi(r') \cos(\phi - \phi')) ,
\]

\[
A_z(r) = \mu_0 \int_{r'} d^3r' G(r, r') J_z(r') .
\]

Since the current has been restricted to flow over an interior cylindrical shell centred radially about the origin, its radial components may be set to zero, resulting in the simplified vector potentials

\[
A_\rho(r) = \mu_0 \int_{r'} d^3r' G(r, r') J_\phi(r') \sin(\phi - \phi') ,
\]

\[
A_\phi(r) = \mu_0 \int_{r'} d^3r' G(r, r') J_\phi(r') \cos(\phi - \phi') .
\]

Substituting the Green’s function solution from (7) in cylindrical coordinates [32],

\[
G(r, r') = \frac{1}{4\pi^2} \sum_{m=-\infty}^{\infty} \int_{-\infty}^{\infty} dk e^{ik(z-z')} I_m(|k|\rho_<) K_m(|k|\rho_>) ,
\]

into (11)-(13), the components of the vector potential may be expressed as

\[
A_\rho(\rho, \phi, z) = -i\frac{\mu_0 \rho'}{4\pi} \sum_{m=-\infty}^{\infty} \int_{-\infty}^{\infty} dk e^{im\phi} e^{ikz} I_{m-1}(|k|\rho_<) K_{m-1}(|k|\rho_>) \left[ I_{m+1}(|k|\rho_<) K_{m+1}(|k|\rho_>) - I_{m+1}(|k|\rho_<) K_{m+1}(|k|\rho_>) \right] J_m^m(k) ,
\]

\[
A_\phi(\rho, \phi, z) = \frac{\mu_0 \rho'}{4\pi} \sum_{m=-\infty}^{\infty} \int_{-\infty}^{\infty} dk e^{im\phi} e^{ikz} I_{m-1}(|k|\rho_<) K_{m-1}(|k|\rho_>) \left[ I_{m+1}(|k|\rho_<) K_{m+1}(|k|\rho_>) + I_{m+1}(|k|\rho_<) K_{m+1}(|k|\rho_>) \right] J_m^m(k) ,
\]

\[
A_z(\rho, \phi, z) = \frac{\mu_0 \rho'}{2\pi} \sum_{m=-\infty}^{\infty} \int_{-\infty}^{\infty} dk e^{im\phi} e^{ikz} I_m(|k|\rho_<) K_m(|k|\rho_>) J_z^m(k) ,
\]

where \( \rho' \) is the radius of the cylinder and \( \rho_>, \rho_< \) is either \( \rho, \rho' \) if \( \rho > \rho' \) or \( \rho', \rho \) if \( \rho < \rho' \), respectively. Equations (15)-(17) contain Fourier transforms of the current densities,

\[
J_{\phi}^m(k) = \frac{1}{2\pi} \int_{0}^{2\pi} d\phi' e^{-im\phi'} \int_{-\infty}^{\infty} dz' e^{-ikz'} J_{\phi}(\phi', z') ,
\]

\[
J_z^m(k) = \frac{1}{2\pi} \int_{0}^{2\pi} d\phi' e^{-im\phi'} \int_{-\infty}^{\infty} dz' e^{-ikz'} J_z(\phi', z') ,
\]

with their corresponding inverse transforms given by

\[
J_{\phi}(\phi', z') = \frac{1}{2\pi} \sum_{m=-\infty}^{\infty} \int_{-\infty}^{\infty} dk e^{im\phi'} e^{ikz'} J_{\phi}^m(k) ,
\]
Writing the magnetic field in cylindrical coordinates, as a result of this formulation, we can now combine methods for matching the boundary conditions at the radial interface, akin to the formulation of passive screening of magnetic field gradients for MRI \[43\], with the method of mirror images, accounting for the effect of the planar end caps, to determine the effect of the high-permeability material on the overall magnetic field profile. Due to the cylindrical symmetry of the system, the radial boundary condition may be satisfied by matching the azimuthal Fourier modes in the magnetic field, generated by the cylindrical current source, through the use of a pseudo-current density induced on an infinite cylinder. As the planar end caps are spatially orthogonal to the pseudo-current generated by the infinite cylinder, any image current formed by applying the method of mirror images continues to satisfy the radial boundary condition. Consequently, we can decouple the boundary conditions on the high-permeability cylinder and at the planar end caps boundaries. This must be done sequentially to generate mirror images of the pseudo-current density induced on the high-permeability cylindrical shell and, hence, satisfy the boundary conditions over the entire domain of the closed finite high-permeability cylinder. As a result, using \(15\)-\(17\) and the usual formulation of the method of mirror images with two infinite parallel planes, depicted in Fig. 2, the vector potential in the region \(\rho_{c} \leq \rho \leq \rho_{s}\), including the effect of the high-permeability cylinder, may be written in terms of an infinite summation over the reflected image currents,

\[
A_{\rho}(\rho, \phi, z) = -\frac{i\mu_{0}}{4\pi} \sum_{m=-\infty}^{\infty} \sum_{p=-\infty}^{\infty} \int_{-\infty}^{\infty} dk \ e^{im\phi} e^{ikz} \left\{ \rho_{c} I_{m-1}(|k|\rho_{c}) K_{m-1}(|k|\rho) - I_{m+1}(|k|\rho_{c}) K_{m+1}(|k|\rho) J_{\phi}^{mp}(k) + \rho_{s} I_{m-1}(|k|\rho_{s}) K_{m-1}(|k|\rho_{s}) \right\} \right. ,
\]

\[
A_{\phi}(\rho, \phi, z) = \frac{\mu_{0}}{4\pi} \sum_{m=-\infty}^{\infty} \sum_{p=-\infty}^{\infty} \int_{-\infty}^{\infty} dk \ e^{im\phi} e^{ikz} \left\{ \rho_{c} I_{m-1}(|k|\rho_{c}) K_{m-1}(|k|\rho) + I_{m+1}(|k|\rho_{c}) K_{m+1}(|k|\rho) J_{\phi}^{mp}(k) + \rho_{s} I_{m-1}(|k|\rho_{s}) K_{m-1}(|k|\rho_{s}) \right\} \right. ,
\]

\[
A_{z}(\rho, \phi, z) = \frac{\mu_{0}}{2\pi} \sum_{m=-\infty}^{\infty} \sum_{p=-\infty}^{\infty} \int_{-\infty}^{\infty} dk \ e^{im\phi} e^{ikz} \left[ \rho_{c} I_{m}(|k|\rho_{c}) K_{m}(|k|\rho) J_{\phi}^{mp}(k) + \rho_{s} I_{m}(|k|\rho_{s}) K_{m}(|k|\rho_{s}) \right] J_{z}^{mp}(k) ,
\]

where

\[
J_{\phi}^{mp}(k) = \frac{1}{2\pi} \int_{0}^{2\pi} d\phi' \ e^{-im\phi'} \int_{-\infty}^{\infty} dz' \ e^{-ikz'} J_{\phi}^{p}(\phi', z') ,
\]

\[
J_{z}^{mp}(k) = \frac{1}{2\pi} \int_{0}^{2\pi} d\phi' \ e^{-im\phi'} \int_{-\infty}^{\infty} dz' \ e^{-ikz'} J_{z}^{p}(\phi', z') ,
\]

are the Fourier transforms of the \(p^{th}\) reflected image current and \(J_{\phi}^{\hat{z}}(k)\) is the Fourier transform of the \(p^{th}\) reflected pseudo-current induced on the high-permeability cylinder. Fig. 3 depicts how azimuthal, \(J_{\phi}^{p}(\phi', z')\), and axial, \(J_{z}^{p}(\phi', z')\), image currents are generated by two parallel planar perfect magnetic conductors.

Writing the magnetic field in cylindrical coordinates,

\[
B = \nabla \times A = \left( \frac{1}{\rho} \frac{\partial A_{z}}{\partial \phi} - \frac{\partial A_{\phi}}{\partial z} \right) \hat{\rho} + \left( \frac{\partial A_{\rho}}{\partial z} - \frac{\partial A_{z}}{\partial \rho} \right) \hat{\phi} + \frac{1}{\rho} \left( \frac{\partial}{\partial \rho} (\rho A_{\phi}) - \frac{\partial A_{\rho}}{\partial \phi} \right) \hat{z} ,
\]
Azimuthal currents located in the $\rho\phi$ equation, \(20)-(24), and (30), the magnetic field interior to the conducting cylinder, for all points.

Substituting \((22)-(24)\) into \((28)-(29)\), the Fourier transformed pseudo-current density on the cylindrical wall of the high-permeability cylinder is found to be

\[ B\phi (\rho, \phi, z) = \frac{i \mu_0 \rho_c}{2\pi} \sum_{m=-\infty}^{\infty} \sum_{p=-\infty}^{\infty} \int_{-\infty}^{\infty} d k k e^{i m \phi} e^{i k z} I'_m (|k| \rho_c) R_m (|k| \rho, \rho_c, \rho_s) J^{mp}_\phi (k), \]

where $I'_m(z)$ and $K'_m(z)$ are the derivatives with respect to $z$ of $I_m(z)$ and $K_m(z)$, respectively. Using the continuity equation, \((20)-(24), and (30)\), the magnetic field interior to the conducting cylinder, for all points $\rho < \rho_c$, is given by

\[ B\phi (\rho, \phi, z) = -\frac{\mu_0 \rho_c}{2\pi} \sum_{m=-\infty}^{\infty} \sum_{p=-\infty}^{\infty} \int_{-\infty}^{\infty} d k m \frac{|k|}{k} e^{i m \phi} e^{i k z} I_m (|k| \rho) R_m (|k| \rho, \rho_c, \rho_s) J^{mp}_\phi (k), \]
\[ B_z (\rho, \phi, z) = -\frac{\mu_0 \rho c}{2\pi} \sum_{m=-\infty}^{\infty} \sum_{p=-\infty}^{\infty} \int_{-\infty}^{\infty} dk \, |k| e^{im\phi} e^{ikz} I_m(|k|\rho) R_m(k, \rho_c, \rho_s) J_q^m(k), \]  

(33)

where \( R_m(k, \rho_c, \rho_s) = K'_m(|k|\rho_c) - \frac{I'_m(|k|\rho_c) K_m(|k|\rho_s)}{I_m(|k|\rho_s)}. \) In order to determine the magnetic field generated by an arbitrary cylindrical current source, we must construct an orthogonal basis defined on a finite cylinder, which accounts for the mirror images. To do this we use a modified Fourier basis, defining the \( p^\text{th} \) reflected azimuthal current to be

\[
J^p_\phi (\phi', z') = (T^p_\rho (z' \mid L_1, L_2, L_s) + T^p_\sigma (z' \mid L_1, L_2, L_s)) \left[ \sum_{n=1}^{N} W_{n0} \sin \left( \frac{n\pi ((-1)^p (z' - pL_s) - L_2)}{L_c} \right) \right. \\
\left. + \sum_{n=1}^{N} \sum_{m=1}^{M} (W_{nm} \cos(m\phi') + Q_{nm} \sin(m\phi')) \cos \left( \frac{n\pi ((-1)^p (z' - pL_s) - L_2)}{L_c} \right) \right],
\]

(34)

in which

\[
T^p_\rho (z' \mid L_1, L_2, L_s) = (H (z' - L_2 - pL_s) - H (z' - L_1 - pL_s)) \left( \frac{1 + (-1)^p}{2} \right),
\]

(35)

\[
T^p_\sigma (z' \mid L_1, L_2, L_s) = (H (z' + L_1 - pL_s) - H (z' + L_2 - pL_s)) \left( \frac{1 - (-1)^p}{2} \right),
\]

(36)

where \( H(x) \) is the Heaviside function and \( (W_{n0}, W_{nm}, Q_{nm}) \) are Fourier coefficients to be determined. Substituting (34) into (31)-(33), we derive a set of governing equations which relate the magnetic field to the set of Fourier coefficients,

\[
B_\rho (\rho, \phi, z) = \sum_{n=1}^{N} W_{n0} F_n (\rho, z) + \sum_{n=1}^{N} \sum_{m=1}^{M} (W_{nm} G^w_{nm} (\rho, \phi, z) + Q_{nm} G^q_{nm} (\rho, \phi, z)), 
\]

(37)

\[
B_\phi (\rho, \phi, z) = \sum_{n=1}^{N} \sum_{m=1}^{M} (W_{nm} H^w_{nm} (\rho, \phi, z) + Q_{nm} H^q_{nm} (\rho, \phi, z)), 
\]

(38)

\[
B_z (\rho, \phi, z) = \sum_{n=1}^{N} W_{n0} D_n (\rho, z) + \sum_{n=1}^{N} \sum_{m=1}^{M} (W_{nm} S^w_{nm} (\rho, \phi, z) + Q_{nm} S^q_{nm} (\rho, \phi, z)), 
\]

(39)

where the functions \( F_n (\rho, z), G^w_{nm} (\rho, \phi, z), H^w_{nm} (\rho, \phi, z), D_n (\rho, z), \) and \( S^w_{nm} (\rho, \phi, z) \) are defined in Appendix A.

Having determined the magnetic field produced by an arbitrary cylindrical current, we may now use an inverse method to solve the system of governing equations, (37)-(39), to determine the unknown Fourier coefficients \( (W_{n0}, W_{nm}, Q_{nm}) \) for a specified target magnetic field. Following work done by Carlson et al. [29], this may be done by a least squares minimization with the addition of a penalty term to regularize the problem. This regularization term may take many forms, with individual contributions to it representing, for example, the curvature of a given wire geometry, the power consumption, or any other physical parameter that depends quadratically on the geometry of the coil. In this work we focus on the overall power dissipated in the cylindrical current flow, but this choice is somewhat arbitrary since all of the regularization parameters act to achieve the same general goal. If the regularization term is large, the result is a well-conditioned inverse problem that yields a simple current flow, but reduced field fidelity. On the other hand, if the regularization term is small, then the result is a less well-conditioned inverse problem that yields a more intricate pattern of current flow but a higher-fidelity magnetic field. The power dissipation in the conducting cylinder of thickness, \( t \), and resistivity, \( \rho \), is given by

\[
P = \frac{\rho_c \rho}{t} \int_{-L_c/2}^{L_c/2} dz' \int_{0}^{2\pi} d\phi' |J_z(\phi', z')|^2 + |J_\phi(\phi', z')|^2,
\]

(40)

which, when integrated over the surface of the cylinder, using the continuity equation and (34), gives

\[
P = \frac{\rho_c \rho}{t} \left[ \sum_{n=1}^{N} \sum_{m=1}^{M} W_{n0}^2 \pi L_c + \sum_{n=1}^{N} \sum_{m=1}^{M} (W_{nm}^2 + Q_{nm}^2) \left( \frac{\pi L_c}{2} + \frac{m^2 \pi^3 L_c^3}{2\pi^2 n^2 \rho_c^2} \right) \right].
\]

(41)

The least squares optimization is applied to a cost function,

\[
\Phi = \alpha \sum_k \left[ \mathbf{B}_{\text{desired}} (\mathbf{r}_k) - \mathbf{B} (\mathbf{r}_k) \right]^2 + \beta P,
\]

(42)
We now analyze our model by designing and testing hybrid active–passive magnetic field-generating systems. Regarding the validation of our calculations, we first note that, as expected from previous work \cite{29, 30, 31} and shown in the Supplementary Material, our calculations confirm that the optimal coil design for generating a constant axial field inside a closed cylindrical perfect magnetic shield is a perfect solenoid that runs along the full length of the cylindrical shield. This current distribution is then related, through the continuity equation, to the streamfunction

$$\varphi(x', z') = (H(z' - L_1) - H(z' - L_2)) \left[ \sum_{n=1}^{N} \frac{L_c}{n\pi} W_{n0} \cos \left( \frac{n\pi (z' - L_2)}{L_c} \right) \right. $$

$$- \sum_{n=1}^{N} \sum_{m=1}^{M} \frac{L_c}{n\pi} (W_{nm} \cos(m\varphi') + Q_{nm} \sin(m\varphi')) \sin \left( \frac{n\pi (z' - L_2)}{L_c} \right) \right],$$  

allowing the optimal Fourier coefficients to be found by matrix inversion for any physically attainable target magnetic field. The current density is then related, through the continuity equation, to the streamfunction

$$\varphi(x', z') = (H(z' - L_1) - H(z' - L_2)) \left[ \sum_{n=1}^{N} \frac{L_c}{n\pi} W_{n0} \cos \left( \frac{n\pi (z' - L_2)}{L_c} \right) \right. $$

$$- \sum_{n=1}^{N} \sum_{m=1}^{M} \frac{L_c}{n\pi} (W_{nm} \cos(m\varphi') + Q_{nm} \sin(m\varphi')) \sin \left( \frac{n\pi (z' - L_2)}{L_c} \right) \right].$$  

3 Results

We now analyze our model by designing and testing hybrid active–passive magnetic field-generating systems. Regarding the validation of our calculations, we first note that, as expected from previous work \cite{29, 30, 31} and shown in the Supplementary Material, our calculations confirm that the optimal coil design for generating a constant axial field inside a closed cylindrical perfect magnetic shield is a perfect solenoid that runs along the full length of the cylindrical shield.

In Fig.3 and Fig.4, respectively, we show active–passive systems for generating a constant transverse field, $B_x$, normal to the axis of the cylinder, and a linear transverse field gradient, $B$, along the axis of the cylinder. Each of these systems has a cylindrical surface of length $L_c = 0.95$ m and radius $\rho_c = 0.245$ m, which carries the coil current distribution. This current distribution is interior to and centred about the origin of a closed perfect magnetically conducting cylinder of length $L_s = 1$ m and radius $\rho_s = 0.25$ m. The field is optimized over a central cylindrical region spanning half the radius and length of the coil cylinder. Fig.3a and Fig.4a show the respective streamfunctions on the surface of the cylinder and their discretization into wire patterns. The magnetic fields shown in Fig.3a–c and Fig.4a are calculated in three ways: analytically, using our theoretical model in (37)–(29); numerically, using COMSOL Multiphysics\textsuperscript{\textregistered} Version 5.3a with the shield treated as a perfect magnetic conductor; and numerically in free space, i.e. excluding the high-permeability material and calculating the magnetic fields by applying the elemental Biot–Savart law directly to the discrete coil geometries in Fig.3a and Fig.4a. It is clear from Figs.3–4 that our design methodology is capable of generating highly accurate user-specified target magnetic fields inside the optimized field region with good agreement between the theoretical model and numerical simulations.

We quantify this agreement by analyzing the deviation from the target fields in the optimization region, $\Delta B_x$ and $\Delta dB_x/dz$, for the constant transverse field and linear transverse field gradient systems, respectively. Specifically, along the $z$-axis of the optimized region the maximum absolute deviations from the target fields are 0.11 % and 0.24 %, respectively. Over the same region, the maximum absolute deviations between the numerically-simulated and analytically-calculated field profiles are 0.002 % and 0.003 %, respectively. We can also see the hybrid nature of our optimization via the improved performance of the active systems when inside, and coupled to, the passive shield. For example, the strength of the $B_x$ field is nearly doubled, and its uniformity is improved by a factor of 20, when the high-permeability cylinder is added to the constant transverse field-generating system (see Fig.3d).

In Fig.5 we show the numerically-calculated color maps of the $B_z$ field in the $y$-$z$ plane generated by the constant transverse field (Fig.3a) and linear transverse field gradient (Fig.4a) systems, using COMSOL Multiphysics\textsuperscript{\textregistered} Version 5.3a with the shield treated as a perfect magnetic conductor. We summarize the performance of both systems in Table 1 with the shield treated as a perfect magnetic conductor. The minimum deviations from the target fields are less than 0.01 %, 0.05 %, 0.1 %, 0.5 %, 1 %, and 5 %, respectively.

When physically constructing active–passive structures for real-world experiments, additional limitations must be taken into consideration in order to generate accurate magnetic fields using our design methodology. These limitations...
Figure 3: Wire layouts (a) and performance (b–c) of an optimized hybrid active–passive constant transverse field-generating system in which current flows on a cylinder of length $L_c = 0.95$ m and radius $\rho_c = 0.245$ m. The wire layouts are optimized to generate a constant transverse field, $B_x$, across the cylinder and normal to its axis of symmetry. The current-carrying cylinder is placed symmetrically inside a perfect closed magnetic shield of radius $\rho_s = 0.25$ m and length $L_s = 1$ m and the magnetic field is optimized between $\rho = \rho_c/2$ and $z = \pm L_c/4$; dashed black lines in (b) and (c). (a) Color map of the optimal current streamfunction on the cylinder [blue and red shaded regions correspond to the flow of current in opposite senses respectively and their intensity shows the streamfunction magnitude from low (white) to high (intense color)]. Solid and dashed black curves represent discrete wires with opposite senses of current flow, approximating the current continuum. (b) Transverse magnetic field, $B_x$, versus axial position, $z$, calculated from the current continuum in (a) in three ways: analytically using (37)-(39) (solid red curve); numerically using COMSOL Multiphysics® Version 5.3a and modelling the high-permeability cylinder as a perfect magnetic conductor (blue dotted curve); numerically without the high-permeability cylinder and using the Biot–Savart law with $N_{\phi} = 100$ contours (dashed green). (c) Enlarged section of (b) emphasizing the high level of field uniformity and the agreement between the numerical and analytical results over the optimization region.

originate either from the theoretical model itself or from experimental practicalities. The limitations in the theoretical model are primarily associated with how accurately the high-permeability cylinder approximates a perfect magnetic conductor. This depends on the value of the finite permeability, the thickness of the shielding material, and the required experimental access holes in the shielding system. The errors introduced by these parameters depend on the lengths, radii, and positions of the conducting and high-permeability cylinders relative to the location of the optimization region. The experimental limitations on the field fidelity relate to the stability of the experimental equipment and errors in manufacturing an accurate representation of the current continuum. These errors include coarse discretization of the current continuum, inexact wire placement and construction, and imprecise positioning of the active structure inside the high-permeability shield. In practice, highly stable experimental equipment is available, particularly power
Figure 4: Wire layouts (a) and performance (b) of an optimized hybrid active–passive linear transverse gradient field-generating system in which current flows on a cylinder of length $L_c = 0.95$ m and radius $\rho_c = 0.245$ m. The wire layouts are optimized to generate a linear transverse field gradient, $dB_x/dz$, along the $z$-axis of the cylinder. The current-carrying cylinder is placed symmetrically inside a perfect closed magnetic shield of radius $\rho_s = 0.25$ m and length $L_s = 1$ m and the magnetic field is optimized between $\rho = \rho_c/2$ and $z = \pm L_c/4$; dashed black lines in (b). (a) Color map of the optimal current streamfunction on the cylinder [blue and red shaded regions correspond to the flow of current in opposite senses respectively and their intensity shows the streamfunction magnitude from low (white) to high (intense color)]. Solid and dashed black curves represent discrete wires with opposite senses of current flow, approximating the current continuum. (b) Transverse magnetic field, $B_x$, versus axial position, $z$, calculated from the current continuum in (a) in three ways: analytically using (37)-(39) (solid red curve); numerically using COMSOL Multiphysics® Version 5.3a and modelling the high-permeability cylinder as a perfect magnetic conductor (blue dotted curve); numerically without the high-permeability cylinder and using the Biot–Savart law with $N_x = 100$ contours (dashed green).

Figure 5: Color maps showing the magnitude of the transverse magnetic field, $B_x$, in the $y$-$z$ plane inside a closed finite length perfect magnetic conductor generated by two active–passive systems: (a) the constant transverse field-generating system depicted in Fig. 3; (b) the linear transverse gradient field-generating system depicted in Fig. 4. The field profiles were calculated numerically using COMSOL Multiphysics® Version 5.3a. Contours show where the field deviates from the target field by $1\%$ (solid curves), $0.1\%$ (dashed curves), $0.01\%$ (dot-dashed curves), and $0.001\%$ (dotted curves; in (a) only).
We use the root mean square field deviation, $\Delta B_{x}$, when there is no end cap, at this point, regarding the accuracy of our model, there is little advantage to increasing which reduce these system errors, such as screen-printed foldable PCBs [45] and 3D-printing technologies [46], it may. Version 5.3a working in the magnetostatic regime, to determine how the uniformity of the $B_x$ field generated by the constant transverse, $B_x$, and linear transverse gradient, $dB_x/dz$, systems, depicted in Fig. 3a and Fig. 4, respectively. The magnetic field deviations were calculated numerically using COMSOL Multiphysics® Version 5.3a in two ways: inside a perfect magnetic conductor (Perfect case); inside a magnetic shield with finite permeability $\mu_r = 20000$, thickness $d = 1$ mm, and a circular entry hole of normalized radius, $\rho_h = 0.25\rho_s$ in both end-caps (Imperfect case).

| Max. Field Deviation (%) | $B_x$ | $dB_x/dz$ | $B_x$ | $dB_x/dz$ |
|--------------------------|-------|-----------|-------|-----------|
| 0.01                     | 0.320 | 0.237     | 0.190 | 0.168     |
| 0.05                     | 0.398 | 0.315     | 0.350 | 0.268     |
| 0.1                      | 0.437 | 0.348     | 0.442 | 0.320     |
| 0.5                      | 0.472 | 0.448     | 0.474 | 0.440     |
| 1                        | 0.496 | 0.470     | 0.499 | 0.470     |
| 5                        | 0.598 | 0.553     | 0.598 | 0.551     |

Table 1: Cylindrical shield fractions – defined as the ratio of the radius and length of the central region to those of the passive shield – where the maximum magnetic field deviations are within $0.01\%$, $0.05\%$, $0.1\%$, $0.5\%$, $1\%$, and $5\%$ of the target fields generated by the constant transverse, $B_x$, and linear transverse gradient, $dB_x/dz$, systems, respectively. The analytical value of $\mu_r = 20000$, thickness $d = 1$ mm, converging to the material limit where the thickness is assumed to be infinite. The horizontal dashed red line shows the analytical value of $\Delta B_{x}^{RMS} = 0.0232\%$ calculated using (37)-(39). The numerical $\Delta B_{x}^{RMS}$ values decrease asymptotically below this analytical limit and approach the difference $O(\mu_r^{-1}) \approx 0.005\%$ [41][42] that we predicted for our model in Section 2. This intrinsic error, resulting from the small difference between thick high-permeability materials and a perfect magnetic conductor, sets the hard limit on the accuracy of any magnetic field that can be designed using our methodology. In reality, however, this limit is so small that for a thick material with a high permeability, such as mumetal, the errors in manufacturing and construction will be much more significant. As technologies advance which reduce these system errors, such as screen-printed foldable PCBs [45] and 3D-printing technologies [46], it may become more relevant to develop a model which accounts $ab initio$ for magnetic shields of finite permeability and thickness.

We see from Fig. 6a that the asymptotic limit is reached at approximately $d = 1$ mm, where $\Delta B_{x}^{RMS} = 0.019\%$. At this point, regarding the accuracy of our model, there is little advantage to increasing $d$ further. Consequently, in Fig. 6b we take $d = 1$ mm and examine the effect of introducing a circular axial entry hole in both the end caps of the high-permeability cylinder. Although $\Delta B_{x}^{RMS}$ increases as the hole radius, $\rho_h$, increases from no hole, $\rho_h = 0$, to when there is no end cap, $\rho_h = \rho_s$, we see that a small hole in both end caps can give experimental access without significantly worsening the accuracy of the field generated. In particular, for our system, the hole radius can be made.
as large as $\rho_h = 0.25\rho_s$ while only increasing $\Delta B_x^{\text{RMS}}$ by 0.0012% when compared to the no hole case (horizontal dashed light blue line).

In Fig. [7] we show the numerically-calculated color maps of the $B_x$ field in the $y$-$z$ plane generated by the constant transverse field (Fig. 3b) and linear transverse field gradient (Fig. 4a) systems. Both of these systems are simulated with the same imperfect high-permeability cylindrical magnetic shield that has had its properties determined in the above analysis: $\mu_r = 20000$, $d = 1 \text{ mm}$, and $\rho_h = 0.25\rho_s$. The performance of both systems in terms of the cylindrical shield fraction is summarized in Table 1.

Finally, we see from Table 1 that this imperfect magnetic shield does not introduce significant magnetic field deviations above 0.1% when compared to a perfect shield with the same geometry. In particular, the maximum difference between the perfect and imperfect cylindrical shield fractions for deviations above 0.1% is only 0.028. Large deviations in the field accuracy below 0.1% can be graphically seen when comparing the color maps for the perfect (Fig. 5) and imperfect (Fig. 7) cases. The contours showing field deviations of 0.01% and 0.001% are strongly perturbed, as expected from the analysis in Fig. 6, demonstrating the hard intrinsic limit on our model when generating target field profiles inside real-world magnetic shields. Similar analysis should be applied when designing active–passive systems using this methodology in order to quantify its accuracy for a specific experimental setup. Further analysis could also be performed to determine the low-frequency limit in which a time-dependent current source could be included. However, if the magnitudes of any induced eddy currents are much less than the magnitude of the coil current, such effects will be negligible.
Figure 7: Color maps showing the magnitude of the transverse magnetic field, $B_x$, in the $y$-$z$ plane inside a magnetic shield with permeability $\mu_r = 20000$, thickness $d = 1$ mm, and a circular axial entry hole of normalized radius, $\rho_h = 0.25\rho_s$, around the centre of each end cap generated by two active–passive systems: (a) the constant transverse field-generating system depicted in Fig. 3; (b) the linear transverse gradient field-generating system depicted in Fig. 4. The field profiles were calculated numerically using COMSOL Multiphysics® Version 5.3a. Contours show where the field deviates from the target field by $1\%$ (solid curves), $0.1\%$ (dashed curves), $0.01\%$ (dot-dashed curves), and $0.001\%$ (dotted curves; in (a) only).

4 Conclusion

In this paper, we have developed an analytical model of the total magnetic field generated by an arbitrary current flow on a cylinder that is coaxially nested within a finite closed high-permeability cylinder. We modified the Green’s function for the magnetic vector potential, matched the radial and planar boundary conditions of the magnetic field through the introduction of a pseudo-current density, and incorporated a harmonic minimization procedure to design optimal user-specified magnetic fields by using a modified cylindrical Fourier basis. We then verified this optimization procedure by designing coils to generate a constant transverse field, $B_x$, across the cylinder, and a linear transverse field gradient, $dB_x/dz$, along the length of the cylinder. Our analytical calculations of these field profiles agreed well with numerical simulations. The optimization procedure generated highly uniform $B_x$ and $dB_x/dz$ field profiles, inside a high-permeability cylinder, with peak-to-peak deviations from the target profiles below $0.11\%$ and $0.24\%$, respectively. The analytically-predicted deviations agreed with the numerical simulations to within $0.002\%$ and $0.003\%$ for the constant and linear gradient systems, respectively.

We further investigated the validity of our model by analyzing the behavior of the constant transverse field-generating system inside a magnetic shield of permeability $\mu_r = 20000$, finite thickness, and with circular axial entry holes in the end caps. We found a range of parameters where the analytical predictions for a perfect cylindrical magnetic conductor remain close to numerical simulations for a cylindrical shield with finite permeability and thickness and including entry holes; showing that the designs generated by our model are applicable to real-world magnetic shields. Notably, when the active field-generating systems were enclosed by a passive magnetic shield with realistic experimental parameters ($\mu_r = 20000$, thickness 1 mm, and entry holes of radius equal to 25% of the shield’s radius), the deviation from the desired constant and linear gradient field profiles was less than $0.1\%$ and $0.5\%$, respectively, over more than 40% of the central radial and axial extent of this simulated real-world magnetic shield.

Our flexible optimization procedure enables the design of new active–passive magnetic field shaping systems that accurately generate any physical magnetic field in the interior of a finite closed magnetic shield. This facilitates the development and miniaturization of systems and technologies which require such control, including quantum sensors, fundamental physics experiments, and medical technologies. Further investigation could consider an analytical treatment of finite magnetic shield thickness and permeability and interactions with an open magnetic shield topology.
Addendum

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Competing Interests The authors M. Packer, P. J. Hobson, T. M. Fromhold, M. J. Brookes, and R. Bowtell declare that they have a patent pending to the UK Government Intellectual Property Application office (Application Number 1913549.0) regarding the magnetic field optimization techniques described in this work. The authors have no other competing financial interests.

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we can simplify the expressions in (45), (46), (48), (50), and (51), to find that

\[ F_n(\rho, z) = \sum_{p=-\infty}^{\infty} \int_{-\infty}^{\infty} dk \, I_0^p(|k|\rho) \left( K'_0(|k|\rho_c) - \frac{I'_0(|k|\rho_c)}{I_0(|k|\rho_s)} \right) C_{np}^1(k, z), \]  

(45)

\[ G_{nm}(\rho, z) = \sum_{p=-\infty}^{\infty} \int_{-\infty}^{\infty} dk \, I'_m(|k|\rho) \left( K'_m(|k|\rho_c) - \frac{I'_m(|k|\rho_c)K_m(|k|\rho_s)}{I_m(|k|\rho_s)} \right) C_{np}^2(k, z), \]  

(46)

\[ G^w_{nm}(\rho, \phi, z) = \cos(m\phi) G_{nm}(\rho, z), \quad G^q_{nm}(\rho, \phi, z) = \sin(m\phi) G_{nm}(\rho, z), \]  

(47)

\[ H_{nm}(\rho, z) = \sum_{p=-\infty}^{\infty} \int_{-\infty}^{\infty} dk \, mI_m(|k|\rho) \left( K'_m(|k|\rho_c) - \frac{I'_m(|k|\rho_c)K_m(|k|\rho_s)}{I_m(|k|\rho_s)} \right) C_{np}^3(k, z), \]  

(48)

\[ H^w_{nm}(\rho, \phi, z) = \sin(m\phi) H_{nm}(\rho, z), \quad H^q_{nm}(\rho, \phi, z) = -\cos(m\phi) H_{nm}(\rho, z), \]  

(49)

\[ D_n(\rho, z) = \sum_{p=-\infty}^{\infty} \int_{-\infty}^{\infty} dk \, I_0(|k|\rho) \left( K'_0(|k|\rho_c) - \frac{I'_0(|k|\rho_c)K_0(|k|\rho_s)}{I_0(|k|\rho_s)} \right) C_{np}^4(k, z), \]  

(50)

\[ S_{nm}(\rho, z) = \sum_{p=-\infty}^{\infty} \int_{-\infty}^{\infty} dk \, I_m(|k|\rho) \left( K'_m(|k|\rho_c) - \frac{I'_m(|k|\rho_c)K_m(|k|\rho_s)}{I_m(|k|\rho_s)} \right) C_{np}^5(k, z), \]  

(51)

\[ S^w_{nm}(\rho, \phi, z) = \cos(m\phi) S_{nm}(\rho, z), \quad S^q_{nm}(\rho, \phi, z) = \sin(m\phi) S_{nm}(\rho, z), \]  

(52)

where,

\[ C_{np}^1(k, z) = i\mu_0\rho c L_c k e^{i(k-z-pL_s)} \left( \frac{e^{-(1)^pikL_s} + (-1)^{n+1}e^{-(1)^plL_s}}{n^2\pi^2 - L_s^2k^2} \right), \]  

(53)

\[ C_{np}^2(k, z) = \frac{i(-1)^pL_c k}{n\pi} C_{np}^1(k, z), \]  

(54)

\[ C_{np}^3(k, z) = -\frac{i(-1)^pL_c |k|}{n\pi k \rho} C_{np}^1(k, z), \]  

(55)

\[ C_{np}^4(k, z) = \frac{i|k|}{k} C_{np}^3(k, z), \]  

(56)

\[ C_{np}^5(k, z) = -\frac{(1)^pL_c |k|}{n\pi} C_{np}^3(k, z). \]  

(57)

Performing the integrals over \( k \) by splitting up the odd and even terms in \( p \) while expressing the summation over the infinite pseudo-current densities through a Fourier series expansion

\[ \sum_{p=-\infty}^{\infty} e^{2ipkL_s} = \frac{\pi}{L_s} \sum_{p=-\infty}^{\infty} \delta \left( k - \frac{\pi p}{L_s} \right), \]  

(58)

we can simplify the expressions in (45), (46), (48), (50), and (51), to find that

\[ F_n(\rho, z) = \sum_{p=-\infty}^{\infty} I'_0\left(\frac{\pi p}{L_s}\rho \right) K_0\left(\frac{\pi p}{L_s}\rho_c \right) - \frac{I'_0\left(\frac{\pi p}{L_s}\rho_c \right) K_0\left(\frac{\pi p}{L_s}\rho_s \right)}{I_0\left(\frac{\pi p}{L_s}\rho_s \right)} \right) C_{np}^1(k, z), \]  

(59)

Appendices

A Mathematical Definitions
\[ G_{nm}(\rho, z) = \sum_{p=-\infty}^{\infty} I'_m \left( \frac{\pi p}{L_s} \rho \right) \left( K'_m \left( \frac{\pi p}{L_s} \rho_c \right) \frac{K_m \left( \frac{\pi p}{L_s} \rho_s \right)}{I_m \left( \frac{\pi p}{L_s} \rho_s \right)} \right) C_{np}^2(z), \] 

\[ H_{nm}(\rho, z) = \sum_{p=-\infty}^{\infty} m I_m \left( \frac{\pi p}{L_s} \rho \right) \left( K'_m \left( \frac{\pi p}{L_s} \rho_c \right) \frac{K_m \left( \frac{\pi p}{L_s} \rho_s \right)}{I_m \left( \frac{\pi p}{L_s} \rho_s \right)} \right) C_{np}^3(z), \] 

\[ D_n(\rho, z) = \sum_{p=-\infty}^{\infty} I_0 \left( \frac{\pi p}{L_s} \rho \right) \left( K'_0 \left( \frac{\pi p}{L_s} \rho_c \right) \frac{K_0 \left( \frac{\pi p}{L_s} \rho_s \right)}{I_0 \left( \frac{\pi p}{L_s} \rho_s \right)} \right) C_{np}^4(z), \] 

\[ S_{nm}(\rho, z) = \sum_{p=-\infty}^{\infty} I_m \left( \frac{\pi p}{L_s} \rho \right) \left( K'_m \left( \frac{\pi p}{L_s} \rho_c \right) \frac{K_m \left( \frac{\pi p}{L_s} \rho_s \right)}{I_m \left( \frac{\pi p}{L_s} \rho_s \right)} \right) C_{np}^5(z), \]

where,

\[ C_{np}^1(z) = i\mu_0 \rho_c n L_c \frac{pe^{i\frac{\pi p}{L_s}}}{n^2 L_s^2 - L_c^2 p^2} \left( (-1)^p e^{i\frac{\pi p L_s}{L_c}} + e^{-i\frac{\pi p L_s}{L_c}} \right) \] 

\[ + (-1)^{n+1} \left( (-1)^p e^{i\frac{\pi p L_s}{L_c}} + e^{-i\frac{\pi p L_s}{L_c}} \right), \] 

\[ C_{np}^2(z) = \mu_0 \rho_c L_c^2 \frac{p^2 e^{i\frac{\pi p}{L_s}}}{n^2 L_s^2 - L_c^2 p^2} \left( (-1)^p e^{i\frac{\pi p L_s}{L_c}} - e^{-i\frac{\pi p L_s}{L_c}} \right) \] 

\[ + (-1)^{n+1} \left( (-1)^p e^{i\frac{\pi p L_s}{L_c}} - e^{-i\frac{\pi p L_s}{L_c}} \right), \] 

\[ C_{np}^3(z) = -\frac{|p| L_s}{p^2 \pi \rho} C_{np}^2(z), \]

\[ C_{np}^4(z) = \frac{i |p|}{p} C_{np}^1(z), \]

\[ C_{np}^5(z) = \frac{i |p|}{p} C_{np}^2(z). \]
Supplementary Material

A solenoidal coil

As shown in previous work \[29, 30, 31\], a solenoid of the same length as the high-permeability cylinder provides the most optimal solution to generate a constant axial field. Due to the image currents, the finite solenoid effectively acts as one of infinite extension, resulting in the most homogeneous possible magnetic field in the \(z\)-direction. A perfect finite solenoid of length \(L_c\) carries a total current of

\[ I_t = \int_{-L_c/2}^{L_c/2} dz' I, \]  

where \(I\) is the current density, i.e. current per unit solenoid length, resulting in the azimuthal current \(I_t/L_c\). Using (25) the Fourier transform of a finite solenoid given by

\[ J_{mp}^\phi(k) = I_t \delta_{m0} e^{-ikpL_c} \text{Sinc}(kL_c/2). \]  

Substituting the above equation into (33) gives the magnetic field in the \(z\)-direction

\[ B_z(\rho, \phi, z) = \mu_0 I_t \rho_c L_c \sum_{p=-\infty}^{\infty} \int_{-\infty}^{\infty} dk |k| e^{ikz} e^{-ikpL_c} \text{Sinc}(kL_c/2) I_0(|k|\rho) \]

\[ \times \left[ K_1(|k|\rho_c) + \frac{I_1(|k|\rho_c) K_0(|k|\rho_s)}{I_0(|k|\rho_s)} \right]. \]  

Performing the integral over \(k\) by expressing the summation over the infinite pseudo-current reflections through a Fourier series expansion

\[ \sum_{p=-\infty}^{\infty} e^{ikpL_s} = \frac{2\pi}{L_s} \sum_{p=-\infty}^{\infty} \delta \left( k - \frac{2\pi p}{L_s} \right), \]  

we find

\[ B_z(\rho, \phi, z) = \frac{\mu_0 I_t \rho_c}{L_c} \sum_{p=-\infty}^{\infty} \left| \frac{2\pi p}{L_c} \right| \cos \left( \frac{2\pi p z}{L_c} \right) \text{Sinc}(\pi p) I_0 \left( \frac{2\pi p}{L_c} \rho \right) \]

\[ \times \left[ K_1 \left( \frac{2\pi p}{L_c} \rho_c \right) + \frac{I_1 \left( \frac{2\pi p}{L_c} \rho_c \right) K_0 \left( \frac{2\pi p}{L_c} \rho_s \right)}{I_0 \left( \frac{2\pi p}{L_c} \rho_s \right)} \right]. \]  

This summation can be simplified as the only contributing term is \(p = 0\), which, when evaluated results in

\[ B_z(\rho, \phi, z) = \frac{\mu_0 I_t}{L_c}. \]  

This result might seem counter intuitive because the magnetic field is identical to a long solenoid in free space with \(N\) turns, i.e. \(B_z(\rho, \phi, z) = \mu_0 I N/L_c\), with no field created by the passive shield. This is, however, entirely physical for the following reason. An infinite solenoid generates a uniform magnetic field in the \(z\)-direction inside and zero field outside of the solenoid. Consequently, there is no field parallel to the surface of the cylindrical wall and, therefore, no effect on the cylindrical walls of the perfect magnetic conductor, i.e. the high-permeability cylinder.