Padé Approximants and the Fixed-Points of the $n_f = 3$ QCD $\beta$-Function

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Abstract

The positive zeros of [2|1], [1|2] and the most general possible [2|2] Padé approximants whose Maclaurin series reproduce the presently known terms in the three-flavour ($n_f = 3$) QCD $\beta$-function are all shown to correspond to ultraviolet fixed points.
Higher order terms of the QCD $\beta$-function

$$\mu^2 \frac{dx}{d\mu^2} \equiv \beta(x)$$

(1a)

$$\beta(x) = -\sum_{i=0}^{\infty} \beta_i x^{i+2},$$

(1b)

$[x \equiv \alpha_s(\mu)/\pi]$ are known to permit the occurrence of fixed points other than the ultraviolet fixed point at $x = 0$; e.g., the positive infrared fixed point (IRFP) which occurs for $9 \leq n_f \leq 16$ when the series for $\beta(x)$ in (1) is truncated after two terms $[\beta_0 = (11 - 2n_f/3)/4$; $\beta_1 = (102 - 38n_f/3)/16$; $x_{IRFP} = -\beta_0/\beta_1]$. However, the fixed points arising from such truncation are likely to be spurious, as the candidate-value for coefficient $\beta$ e.g. to lower order terms [3].

In the present letter, we study the zeros of Padé-approximant summations of the $n_f = 3$ $\beta$-function that are consistent with the (now fully-known [4]) coefficients $\beta_0 - 3$. In particular, we consider the explicit [2] Padé approximants, as well as the most general possible Padé approximant, whose Maclaurin expansion reproduces $\beta_0 - 3$. In each case, we find that the positive zeros of the Padé approximant correspond to ultraviolet fixed points and not to infrared fixed points.

The known coefficients of the $n_f = 3$ $\beta$-function can be expressed as follows [4]:

$$\beta(x) = -(9x^2/4)[1 + R_1x + R_2x^2 + R_3x^3 + R_4x^4 + .],$$

(2)

with $R_1 = 16/9$, $R_2 = 4.471065$, $R_3 = 20.99027$. $R_4$ and subsequent terms are not presently known. The [2] Padé approximant that successfully matches the first four terms in the series (2) is $(1 - 2.91691x - 3.87504x^2)/(1 - 4.69468x)$. The only positive numerator root is at $x = 0.2559$. This fixed point, however, is separated from the $\beta$-function’s ultraviolet fixed point (UVFP) at $x = 0$ by a smaller positive zero of the denominator $(x = 0.2130)$. Consequently, if one uses the [2] approximant to represent the $\beta$-function (2), one finds that the $\beta$-function has negative slope as one approaches $x = 0.2559$ from above, and positive slope as one approaches $x = 0.2559$ from below. This behaviour characterizes 0.2559 as an UVFP.

Such behaviour — specifically, a zero of the numerator of the UVFP that is less than the first positive zero of the numerator — characterizes the [1] Padé approximant as well: $(1 - 8.1733x)/(1 - 9.95115x + 13.2199x^2)$. The numerator zero at $x = 0.1223$ is separated from the $x = 0$ UVFP by a denominator zero at $x = 0.1194$. Thus the zero of the [1] Padé approximant that matches the known terms of the series in (2) again corresponds to a UVFP of the $\beta$-function (2).

The $R_4$ term of the series (2) can be estimated using an algorithm [5] based upon the asymptotic error formula [1,6] relating the value $R_{N+2}$ to the predicted value $R_{N+2}^{\text{Padé}}$ obtained from expanding an $[N]1$ Padé approximant into a Maclaurin series:

$$\delta_{N+2} \equiv \frac{R_{N+2}^{\text{Padé}} - R_{N+2}}{R_{N+2}} = \frac{-A}{[N + 1 + (a + b)]},$$

(3)
Using a \([0|1]\) approximant, one finds that \(\delta_2 = (R_1^2 - R_2)/R_2 = -A/[1+(a+b)]\). Using a \([1|1]\) approximant, one finds that \(\delta_3 = (R_2^3/R_2 - R_3)/R_3 = -A/[2 + (a+b)]\). Since \(R_{1,2,3}\) are known, these two relations determine the two unknowns \(A\) and \((a+b)\). One can estimate the unknown coefficient \(R_4\) by applying (3) to the \([2|1]\) approximant:

\[
R_4 = \frac{R_2^2/R_2}{1 + \delta_4} = \frac{R_2^2(R_3^2 + R_1 R_2 R_3 - 2 R_1^3 R_3)}{R_2(2 R_2^3 - R_1^3 R_2 - R_1^2 R_2^2)}.
\]

For the \(n_f = 3\) values of \(R_{1,2,3}\) given above, we find \(R_4 = -849.7\).

Using these numbers, the polynomial \(1 + R_1 x + R_2 x^2 + R_3 x^3 + R_4 x^4\) does have a positive zero which can be identified with a \(\beta\)-function IRFP at \(x = 0.2143\) \((\alpha_s = 0.673)\), provided we accept a degree-4 truncation of the series in (2). Such an IRFP [analogous to the naive IRFP \(x = -\beta_1/\beta_0\) described at the beginning of this letter] is of questionable validity because of the large magnitude of the dominant \(R_4 x^4\) term immediately preceding truncation \([7]\).

Such truncation difficulties are averted if the known coefficients \(R_{1,2,3}\) and the estimated coefficient \(R_4\) are utilized to generate a \([2|1]\) Padé approximant whose Maclaurin series reproduces \(1 + R_1 x + R_2 x^2 + R_3 x^3 + R_4 x^4\) as the first five terms of its infinite Maclaurin series. This approximant, \((1 + 94.383 x - 75.605 x^2)/(1 + 92.606 x - 244.71 x^2)\), has one positive numerator-zero \((x = 1.259)\), which is found to be larger than the only positive denominator-zero \((x = 0.3889)\). Consequently, the positive zero of the \([2|2]\) approximant generated via the estimate (4) for \(R_4\) once again corresponds to a UVFP of the \(\beta\)-function (2).

Surprisingly, this correspondence holds even if we discard (4) entirely and develop a general \([2|2]\) Padé approximant whose \(R_4\) dependence is explicit \([8]\). Using \(R_{1-3}\) appropriate for the \(n_f = 3\) \(\beta\)-function, one finds the most general \([2|2]\) approximant whose Maclaurin series reproduces \(1 + R_1 x + R_2 x^2 + R_3 x^3 + R_4 x^4\) with \(R_4\) arbitrary to be \((1 + a_1 x + a_2 x^2)/(1 + b_1 x + b_2 x^2)\), such that \(a_1 = 7.1945 - 0.10261 R_4\), \(b_2 = -11.329 + 0.075643 R_4\), \(b_1 = 5.4168 - 0.10261 R_4\), \(b_2 = -25.430 + 0.25806 R_4\). The numerator and denominator zeros are (respectively) denoted by \(x_\pm = (-a_1 \pm \sqrt{a_1^2 - 4a_2})/2a_2\), \(y_\pm = (-b_1 \pm \sqrt{b_1^2 - 4b_2})/2b_2\). For \(R_4 < 98.54\), both \(a_2\) and \(b_2\) are negative, in which case \(x_-\) and \(y_-\) are positive, and \(x_+\) and \(y_+\) are negative. Fig. 1 shows that \(0 < y_- < x_-\) through this range, in which case the positive root \(x_-\) necessarily corresponds to a UVFP. For \(R_4\) between 98.54 and 149.76, \(x_-\), \(y_-\) and \(y_+\) are all positive \((x_+\) is negative). Noting that \(y_- > y_- > x_- > 0\), \(x_- > y_- > y_- > 0\). Neither of these sets of inequalities is upheld over this range of \(R_4\). Instead \(y_- > x_- > y_- > 0\) [Fig. 1], consistent with \(x_-\) corresponding to a UVFP of the \(\beta\)-function. Finally, if \(R_4 > 149.76\), we see that \(x_+ > x_- > 0\) and \(y_+ > y_- > 0\). However, these four positive roots are seen to satisfy \(x_+ > y_+ > x_- > y_- > 0\) [Fig. 1], consistent with identifying both zeros of the Padé-approximant numerator with UVFP’s of the \(n_f = 3\) \(\beta\)-function. Corresponding behaviour of the coupling constant is heuristically presented in Fig 2. This same behaviour has already been shown to characterize the exact \(\beta\) function for SUSY gluodynamics \([10]\).

Thus, no matter what \(\beta_4\) \((= 3 \beta_0 R_4)\) is eventually found to be, the \([2|2]\) Padé approximant whose Maclaurin expansion matches the \(\beta_{0-4}\) terms of the \(\beta\)-function will not support the existence of any positive IRFPs; zeros of this approximant all correspond to UVFPs. As is evident from Fig. 2, the structure of the \([2|2]\) approximant to the \(\beta\)-function (2) decouples the IR-region entirely from coupling-constant evolution between UVFPs — i.e. if \(x\) is between zero and \(x_-\), \(\mu\) cannot be smaller than \(\mu(y_-)\). Finally we note that the existence of a UVFP different from zero [necessarily leading to a double-valued function for \(\alpha_s(\mu)\)] could indicate an additional strong-coupling phase of QCD at short distances \([9]\), with possible implications for dynamical electroweak symmetry breaking. QCD may conceivably furnish its own technicolour.
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[6] J. Ellis, E. Gardi, M. Karliner and M. A. Samuel, Phys. Lett. B 366, 268 (1996) and Phys. Rev. D 54, 6986 (1996).
[7] The large negative value of $R_4$ is a consequence of the near cancellation of $2R_3^2 - R_1^3R_3 - R_2^3R_2^2$ in the denominator of (4), which could change sign under relatively small variations in $R_{1-3}$. Such a sign change would be expected to lead to a large positive value for $R_4$, thereby eliminating the positive IRFP zero. A large positive value for $R_4$ can be extracted from ref. 4’s fit of the explicit $n_f$ dependence of $\beta_4$.
[8] As noted in ref. 4, the algorithm culminating in eq. (4) cannot anticipate quadratic-casimir contributions to $R_4$.
[9] We have verified for $n_f = 5$ (and $n_f = 0$) that all zeros of the most general $[2|2]$ Padé approximation of $\beta(x)$ [matching $\beta_0$ and $\beta_{1-3} = \beta_0 R_{1-3}$ in (2), with $R_4$ arbitrary] also correspond to UVFPs.
[10] See Fig. 1 of I.I. Kogan and M. Shifman, Phys. Rev. Lett. 75, 2085 (1995).

Figure Captions:

Figure 1: Relative size of eq. (4)’s numerator zeros $x_{\pm}$ and denominator zeros $y_{\pm}$, expressed as functions of the horizontal-axis variable $R_4$. $x_-$ approaches $y_-$ from above for large positive values of $R_4$.

Figure 2: Schematic behaviour of $x(\mu)$ obtained from use of the $[2|2]$ Padé approximant for $x_+ > y_+ > x_- > y_- > 0$. $x_+$ and $x_-$ are numerator zeros corresponding to UVFP’s. Corresponding behaviour of (positive) $x(\mu)$ when $x_- > y_- > 0$ with $x_+, y_+$ both negative (see Fig. 1) is obtained by excising the middle branch of the above figure. The value of $\mu$ when $x = y_-$, the first zero of the Padé-denominator, is denoted by $\mu(y_-)$.
