Research Article

Sharp Bounds on the Spectral Radii of Uniform Hypergraphs concerning Diameter or Clique Number

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In this paper, we defined two classes of hypergraphs, hyperbugs and kite hypergraphs. We show that balanced hyperbugs maximize the spectral radii of hypergraphs with fixed number of vertices and diameter and kite hypergraphs minimize the spectral radii of hypergraphs with fixed number of vertices and clique number.

1. Introduction

Hypergraph theory deals extensively with hypergraph invariants, i.e., function $i_H$ represents a certain invariant of the hypergraph as a real number or integer, [1–3]. Well-known invariants are the independence and chromatic numbers, the diameter, and so on. Extremal hypergraph theory deals with the problem of characterizing the families of hypergraph $H$ for which an invariant $i_H$ is minimum or maximum. Usual hypergraph classes, such as complete hypergraphs, hyperpaths, hypercycles, and hyperstars, frequently appear as extremal hypergraphs in hypergraph spectral theory problems. Here we want to turn the readers’ attention to two novel, simply defined, hypergraph classes that appear as extremal hypergraphs in several hypergraph spectral theory problems. We call them hyperbugs and kite hypergraphs.

In recent years, a few results are known about the spectral radii of hypergraphs with property $P$, for example, Fan et al. [4] considered the maximum spectral radius of uniform hypergraphs with few edges. Xiao et al. [5] investigated the supertrees whose spectral radii attain the maximum among all uniform supertrees with given degree sequence. Xiao and Wang [6] also determined the unique hypergraph with the maximum spectral radius among all the uniform supertrees and all the connected uniform unicyclic hypergraphs with given number of pendant edges, respectively. Zhang and Li [7] characterized the hypergraph with maximum spectral radius among all connected uniform hypergraphs with given number of pendant vertices. Su et al. [8] determined the largest spectral radius of hypertrees with $r$ edges and given size of matching. Xiao et al. [9] determined the supertrees with the first two largest spectral radii among all supertrees in the set of $m$-uniform supertrees with $r$ edges and diameter $d$. Su et al. [10] determined the first $\lfloor d/2 \rfloor + 1$ largest spectral radius of $k$-uniform supertrees with size $m$ and diameter $d$. In addition, the first two smallest spectral radii of supertrees with size $m$ are also determined. For other related results, readers are referred to [11–20]. In the spirit of the general problem of Brualdi and Solheid [21], one can ask how large or how small can be the spectral radius of hypergraphs with some specific properties. For example: how large $\rho(H)$ can be if $H$ is a $k$-uniform hypergraph of order $n$ and diameter at least $r$? Similarly, how small $\rho(H)$ can be if $H$ is a $k$-uniform hypergraph of order $n$ and clique number at least 2? In fact, for $k = 2$, this question has been answered by Stevanović and Hansen in [22, 23], respectively.

Lemma 1 (see [22, 24]). Let $G$ be a graph of order $n$ with $\text{diam}(G) \geq r$. If $r = 1$, then $\rho(G) = \rho(K_n)$. If $r \geq 2$, then

$$\rho(G) \leq \rho(B_{n-r+2,\lfloor r/2 \rfloor,\lfloor r/2 \rfloor}),$$

holds if and only if $G \equiv B_{n-r+2,\lfloor r/2 \rfloor,\lfloor r/2 \rfloor}$, where $B_{p,q,r}$ denotes the graph obtained from a complete graph $K_p$ by deleting an edge and attaching paths $P_q$ and $P_r$ to its ends.
Lemma 2 (see [23, 24]). If $G$ is a connected graph of order $n$ with clique number $\omega \geq 2$, then $\rho(G) \geq \rho(\text{PK}_{n-\omega}^{n})$ holds if and only if $G \equiv \text{PK}_{n-\omega}^{n}$, where $\text{PK}_{r,q}$ denotes the graph obtained by joining an end vertex of the path $P_{r}$ to a vertex of the complete graph $K_{q}$.

In this paper, we mainly generalize the above two results to the $k$-uniform hypergraphs.

2. Notations and Preliminaries

A hypergraph $H$ is a pair $(V, E)$ with $E \subseteq \mathcal{P}(V)$ and $\mathcal{P}(V)$ denotes the power set of $V$, $V = V(H)$ is a nonempty finite set (the vertex set), and the elements of $E = E(H)$ are called hyperedges or edges. A hypergraph $H$ is $k$-uniform if every edge $e \in E(H)$ contains precisely $k$ vertices. A sub-hypergraph $H'$ of $H$ is a $k$-uniform hypergraph such that $V(H') \subseteq V(H)$ and $E(H') \subseteq E(H)$. For a vertex $v \in V$, we denote by $e_{v}$ the set of edges containing $v$. We write $e_{uv}$ to express the edges containing the vertices $u, v$. The cardinality $|e_{v}|$ is the degree of $v$, denoted by $\deg(v)$. A vertex with degree one is called a core vertex. If any two vertices in $H$ share at most one vertex, then $H$ is said to be linear hypergraph. Let $H - e$ be obtained by deleting an edge $e$ from $H$, i.e., $H - e = (V, E \setminus \{e\})$. $H - (v)$ is obtained from $H$ by deleting $k$ vertices of edge $e$, i.e., $H - \{v_{1}, \ldots, v_{k}\}$, where $e = \{v_{1}, \ldots, v_{k}\}$.

A walk $W$ in a hypergraph $H$ is a finite alternating sequence of vertices and edges, i.e., $W = (v_{0}, e_{1}, v_{1}, e_{2}, \ldots, e_{k}, v_{k})$, satisfying that both $v_{i-1}$ and $v_{i}$ are incident to $e_{i}$ for $1 \leq i \leq k$. A walk $W$ is a path if all the vertices $v_{i}$ for $i = 0, 1, \ldots, k$ and all the edges $e_{i}$ for $j = 1, 2, \ldots, k$ in $W$ are distinct. The length of a path is the number of edges in it. A hypergraph is connected, if there is a path between any pair of vertices of $H$. In this paper, we assume that hypergraphs are $k$-uniform and connected.

Let $H = (V, E)$ be a $k$-uniform hypergraph. An edge $e$ is called a pendant edge if $e$ contains exactly $k - 1$ core vertices. If $e$ is not a pendant edge, it is called a non-pendant edge. A path $P = (v_{0}, e_{1}, v_{1}, \ldots, v_{p-1}, e_{p}, v_{p})$ of $H$ is called a pendant path (attached at $v_{0}$), if all of the vertices $v_{1}, \ldots, v_{p-1}$ are of degree two and the vertex $v_{p}$, and all the $k - 2$ vertices in the set $e_{i} \setminus \{v_{i-1}, v_{i}\}$ are core vertices in $H (i = 1, \ldots, p)$. Let $H = (V, E)$ be a $k$-uniform hypergraph of order $n$; if $E$ consists of all $k$-subsets of $V$, then $H$ is a complete $k$-uniform hypergraph, denoted by $K_{n}^{k}$. A clique of a $k$-uniform hypergraph $H$ is a complete $k$-uniform sub-hypergraph of $H$. A maximal clique is a clique that cannot be extended to a larger clique. The clique number $\omega(H)$ of a hypergraph $H$ is the number of vertices in the maximum clique of $H$.

Definition 1 (see [11]). The adjacency tensor of an $r$-uniform hypergraph $H$ on $n$ vertices is defined as the tensor $\mathcal{A}(H)$ of order $r$ and dimension $n$ whose $(i_{1}, \ldots, i_{r})$-entry is $1/(r - 1)!$ if $\{i_{1}, \ldots, i_{r}\} \in E(H)$, 0 otherwise.

The spectrum, eigenvalues, and spectral radius $\rho(H)$ of $H$ are defined to be those of its adjacency tensor $\mathcal{A}$.

The following relation between the spectral radius of a $k$-uniform hypergraph and its sub-hypergraph can be found in [11].

Lemma 3 (see [11]). Let $H$ be a $k$-uniform hypergraph, and $H'$ is a sub-hypergraph of $H$; then, $\rho(H') \leq \rho(H)$.

The first upper and lower bounds of spectral radius of a $k$-uniform hypergraph are given by Cooper et al. as follows.

Lemma 4 (see [11]). Let $H$ be a $k$-uniform hypergraph. Let $d$ be the average degree of $H$ and $\Delta$ be the maximum degree. Then,

$$d \leq \rho(H) \leq \Delta.$$  

In [14], Li et al. proposed an effective method to find a $k$-uniform hypergraph with larger spectral radius.

Definition 2 (see [14]) (general edge-moving operation). Suppose that $H = (V, E)$ is a hypergraph with $u \in V$ and $e_{1}, e_{2}, \ldots, e_{r} \in E$, such that $u \notin e_{i}, i = 1, 2, \ldots, r (r \geq 1)$. Let $v_{i} \in e_{i}$ and $e_{i}' = (e_{i} \setminus \{v_{i}\}) \cup \{u\}$. Let $H^{*} = (V, E^{*})$ be the hypergraph with $E^{*} = (E \setminus \{e_{1}, \ldots, e_{r}\}) \cup \{e_{1}', \ldots, e_{r}'\}$. Then, we say that $H^{*}$ is obtained from $H$ by moving edges $(e_{1}, \ldots, e_{r})$ from $(v_{1}, \ldots, v_{r})$ to $u$.

According to the definition of general edge-moving operation, Li et al. obtained the following relation of spectral radius.

Lemma 5 (see [14]). Suppose that $H$ is a $k$-uniform hypergraph and $H^{*}$ is the hypergraph obtained from $H$ by moving edge $(e_{1}, \ldots, e_{r})$ from $(v_{1}, \ldots, v_{r})$ to $u$, where $H^{*}$ contains no multiple edges. If $x \in \mathbb{R}^{n}$ is the Perron eigenvector of $\mathcal{A}(H)$ corresponding to $\rho(H)$ and $x_{u} \geq \max_{1 \leq i \leq r} x_{v_{i}}$, then $\rho(H^{*}) > \rho(H)$.

In [14], Li et al. also gave two extremal results about upper and lower bounds of the $k$th power of an ordinarly tree $T$.

Lemma 6 (see [14]). Let $T^{k}$ be the $k$th power of an ordinary tree $T$. Suppose that $T^{k}$ has $n$ vertices. Then, we have...
\[ \rho(\mathcal{A}(P_n^k)) < \rho(\mathcal{A}(T_k^n)) < \rho(\mathcal{A}(S_n^k)), \]  
\[ \rho(H'_{p,q}) < \rho(H_{p,q}) < \rho(H'_{p,q+1}), \]  
where the former equalities hold if and only if \( T_k^n \cong P_n^k \), and the latter equalities hold if and only if \( T_k^n \cong S_n^k \).

Let \( H \) be a \( k \)-uniform hypergraph. Let \( H_{p,q}(u) \) be the hypergraph obtained by attaching the paths \( P_p \) and \( P_q \) to \( u \in H \). Similarly, let \( H_{p,q}(u,v) \) be the hypergraph obtained by attaching the paths \( P_p \) to \( u \in H \) and \( P_q \) to \( v \in H \).

In [17], Shan et al. gave an operation to find a \( k \)-uniform hypergraph with larger spectral radius.

**Lemma 7** (see [17]). Let \( u, v \) be two non- pendant vertices of hypergraph \( H \). If there exists an internal path \( P \) with \( s \) length in hypergraph \( H_{p,q}(u,v) \) for any \( p \geq q \geq 1 \), then we have

\[ \rho(H_{p+1,q-1}(u,v)) < \rho(H_{p,q}(u,v)), \quad \text{for } p - q + 1 \geq s \geq 0. \]  

In [12], Guo and Zhou gave another operation to find a \( k \)-uniform hypergraph with larger spectral radius.

**Lemma 8** (see [12]). For \( k \geq 2 \), let \( H \) be a \( k \)-uniform hypergraph with \( |E(G)| \geq 1 \) and \( u \in V(G) \). For \( p \geq q \geq 1 \), we have

\[ \rho(H_{p,q}(u)) > \rho(H_{p+1,q-1}(u)). \]  

A hypergraph \( H \) is isomorphic to a hypergraph \( H' \), if there is a bijection \( \sigma : V(H) \rightarrow V(H') \) such that \( \{v_1, v_2, \ldots, v_k\} \in E(H) \) if and only if \( \{\sigma(v_1), \sigma(v_2), \ldots, \sigma(v_k)\} \in E(H') \). The bijection \( \sigma \) is called an isomorphism of \( H \). If \( H \cong H' \), then \( \sigma \) is called an automorphism of \( H \). Let \( \mathbf{x} \) be a vector defined on \( V(H) \) and \( \sigma \) be an automorphism of \( H \). Two vertices \( u \) and \( v \) are equivalent in \( H \), if there exists an automorphism \( \sigma \) of \( H \) such that \( \sigma(u) = v \), \( u \sim v \). Denote \( \mathbf{x}_\sigma \) to be the vector such that \( x_\sigma(u) = x_{\sigma(u)} \) for each \( u \in V(H) \).

**Lemma 9** (see [13]). Let \( H \) be a \( k \)-uniform hypergraph and \( \sigma \) be an automorphism of \( H \). Let \( \mathbf{x} \) be an eigenvector of \( \mathcal{A}(H) \); then, \( \mathbf{x}_\sigma = \mathbf{x} \). Further, for two vertices \( u \) and \( v \) in \( V(H) \), if \( u \sim v \), there must be \( x_u = x_v \).

Next we discuss two novel hypergraph families, i.e., hyperbugs and kite hypergraphs, which are defined as follows.

**Definition 3.** A hyperbug \( \mathcal{B}_{p,q,s} \) is a \( k \)-uniform hypergraph obtained from a complete \( k \)-uniform hypergraph \( K^n_k \) by deleting an edge \( \{u, u_1, \ldots, u_{k-2}, v\} \) attaching paths \( P_p \) and \( P_s \) at \( u \) and \( v \). A hyperbug is balanced if \(|q - s| \leq 1\) (see Figure 1 for an example).

**Definition 4.** A kite hypergraph \( \mathcal{P}K_{p,q,s} \) is a \( k \)-uniform hypergraph obtained by joining an end vertex of the path \( P_p \) to a vertex of the complete \( k \)-uniform hypergraph \( K^n_q \) (see Figure 2 for an example).

In this paper, we obtain that if \( H \) is a \( k \)-uniform hypergraph of order \( n \) and diameter at least \( r \), then

\[ \rho(H) \leq \rho(\mathcal{B}_{n-(r-2)(k-1),\lfloor n/r \rfloor,\lceil n/r \rceil}). \]  

holds if and only if \( H \equiv \mathcal{B}_{n-(r-2)(k-1),\lfloor n/r \rfloor,\lceil n/r \rceil} \). Furthermore, we also obtain that if \( H \) is a \( k \)-uniform hypergraph of order \( n \) with clique number \( \omega \geq 2 \), then \( \rho(H) \geq \rho(\mathcal{P}K_{n-\omega,\omega}) \) holds if and only if \( H \equiv \mathcal{P}K_{n-\omega,\omega} \). These generalize some related results of Nikiforov and Rojo [24] and Hansen and Stevanović [22].

### 3. Main Results

#### 3.1. The Spectral Radii of Uniform Hypergraphs with Fixed Number of Vertices and Diameter

Let \( H \) be a \( k \)-uniform hypergraph containing a path \( P \) as a sub-hypergraph (see Figure 3). We say that \( P \) is a pendant path if one of its ends is a cut vertex of \( H \); we call this vertex the root of \( P \). Note that a hypergraph can have multiple pendant paths, which may share roots; e.g., the hypergraph \( \mathcal{B}_{p,q,s} \) has two pendant paths.

From Figure 3, we see that if \( H' \) is a \( k \)-uniform hypergraph and \( H \) is a \( k \)-uniform hypergraph with a pendant path \( P \) and \( \rho(H) = \rho \geq 2 \), then the distribution of the entries of an eigenvector to \( \rho \) along \( P \) is well determined.

In fact, let \( P = (u_1, \ldots, u_i, \ldots, u_{i+k-1}) \) be a pendant path in \( H \) with root \( u_i \). Let \( x_1, \ldots, x_i, \ldots, x_{i+k-1} \) \((i \in \mathbb{N})\) be the entries of a positive unit eigenvector to \( \rho(H) \) corresponding to \( u_1, \ldots, u_i, \ldots, u_{i+k-1} \) \((i = n(k-1) + 1, n \in \mathbb{N})\). The eigenvalue of \( \mathcal{A}(H) \) for \( x_j \) \((1 \leq j \leq i + k - 1)\) is

\[ \rho x_j^{i+k-1} = x_{j-k+1} x_{j-k+2} \ldots x_{j-1} + x_{j+1} x_{j+2} \ldots x_{j+k-1}, \]  

which implies that
\[ x_{i-k+1} x_{i-k+2}, \ldots, x_{i-1} - \rho x_i^{k-1} + x_{i+1} x_{i+2}, \ldots, x_{i+k-1} = 0, \]  
\[ x_{i-k+1} x_{i-k+2}, \ldots, x_{i-1} \] 
\[ X_i^{k-1} x_{i+1} x_{i+2}, \ldots, x_{i+k-1}^{-1} \] 
\[ X_i^{k-1} x_{i+1} x_{i+2}, \ldots, x_{i+k-1}^{-1} \]

The above equation is equivalent to the following crucial equation:

\[ X^2 - \rho X + 1 = 0. \]  

(10)

We can write \( \gamma \) for the root of (10):

\[ \gamma = \frac{\rho + \sqrt{\rho^2 - 4}}{2}, \]  

(11)

and note that \( \gamma \) is real since \( \rho \geq 2 \); moreover, \( \gamma \geq 1 \), with strict inequality if \( \rho > 2 \). Note also that the other root (10) is equal to \( \gamma^{-1} \).

By equations (10) and (11), we can obtain the following theorem.

**Theorem 1.** Let \( H \) be a \( k \)-uniform hypergraph with \( \rho = \rho(H) \geq 2 \). Let \( P = (u_1, \ldots, u_i, \ldots, u_{i+k-1}) \) be a pendant path in \( H \) with root \( u_i \). Let \( x_1, x_2, \ldots, x_{i+k-1}, i = nk (n \in \mathbb{N}) \) be the entries of a positive unit eigenvector to \( \rho(H) \) corresponding to \( u_1, \ldots, u_i, \ldots, u_{i+k-1} \), \( i = n(k-1) + 1, n \in \mathbb{N} \). If \( \gamma \) is defined by (11), then for every \( x_j (1 \leq j \leq i + k - 1) \), we have

\[ x_{i-k+1} > \gamma x_{i-k+2}, x_{i-k+1} > \gamma x_i, (i = nk(k - 1) + 1, n \in \mathbb{N}). \]  

(12)

**Proof.** The eigenvalue of \( \omega(H) \) for \( x_j (i + 1 \leq j \leq i + k - 1) \) is

\[ \rho x_i^{k-1} = x_i x_{i+1}, \ldots, x_{i+k-2}, \]  

\[ \rho x_{i+k-2} = x_i x_{i+1}, \ldots, x_{i+k-3} x_{i+k-1}, \]  

\[ \vdots \]  

\[ \rho x_{i+k-1} = x_i x_{i+2}, \ldots, x_{i+k-1}. \]  

(13)

Since \( \rho > \gamma \), then

\[ x_i x_{i+1}, \ldots, x_{i+k-2} > \gamma x_i^{k-1}, \]  

\[ x_i x_{i+1}, \ldots, x_{i+k-3} x_{i+k-1} > \gamma x_i^{k-2}, \]  

\[ \vdots \]  

\[ x_i x_{i+2} \cdots x_{i+k-1} > \gamma x_i^{k-1}. \]  

(14)

By multiplying the above \( k - 1 \) inequalities, we have

\[ x_{i-k+1} x_{i-k+2}, \ldots, x_{i-1} \] 
\[ x_{i-k+1} x_{i-k+2}, \ldots, x_{i+k-1} \] 
\[ x_{i-k+1} x_{i-k+2}, \ldots, x_{i+k-1} \] 
\[ x_{i-k+1} x_{i-k+2}, \ldots, x_{i+k-1} \]

So, we have

\[ x_{i-k+1} > \gamma x_{i-k+2}, \ldots, x_{i-k+1} > \gamma x_{i+k-2}, \ldots, x_{i-k+1} > \gamma x_{i+k-1}. \]  

(15)
we have

\begin{align}
\text{Lemma 6, we have}
\rho(H) \leq \rho(H_r) \leq \rho(\mathcal{B}_{n-(r-2)(k-1),[r/2],[r/2]}),
\end{align}

holds if and only if \( H = \mathcal{B}_{n-(r-2)(k-1),[r/2],[r/2]} \).

**Proof.** The statement is clear if \( r = 1 \), for \( K_{n}^{r} \) is the only hypergraph of order \( n \) and diameter 1. Suppose that \( r \geq 2 \); let \( H \) be a hypergraph with maximal spectral radius among all \( k \)-uniform hypergraphs of order \( n \) and \( \text{diam}(H) \geq r \). This choice implies that \( H \) is edge-maximal, that is, no edge can be added to \( H \) without diminishing its diameter. According to Lemma 9, we only need to show that \( H = \mathcal{B}_{n-(r-2)(k-1),[r/2],[r/2]} \) for some \( r \), satisfying \( 1 \leq r \leq r-1 \).

Let \( \rho = \rho(H) \) and \( u, v \) be vertices of \( H \) at distance exactly \( r \), for every \( i = 0, \ldots, r \). Let \( V_i \) be the set of the vertices at distance \( i \) from \( u \), and degrees of these vertices are greater than 2. Since \( H \) is edge-maximal, the set \( V_i \cup V_{i+1} \) induces a linear complete hypergraph, for every \( i = 0, \ldots, r-1 \). It is also clear that \( |V_0| = 1 \); moreover, it is not hard to see that \( |V_r| = 1 \). Indeed, assume for a contradiction that \( |V_i| \geq 2 \) and add all edges between \( V_{r-2} \) and \( V_\{v\} \). These additional edges do not diminish the distance between \( u \) and \( v \); hence, \( H \) is not edge-maximal, contradicting its choice; therefore, \( |V_r| = 1 \).

Furthermore, by Lemma 3, we have

\begin{align}
\Delta(H) \geq \rho(H) > \rho(\mathcal{B}_{n-(r-2)(k-1),[r/2],[r/2]}) > \rho(K_{n-(r-2)(k-1)}) - 2\epsilon,
\end{align}

and so \( \Delta(H) \geq C_{n-1-(r-2)(k-1)}^{k-1} \). Suppose that \( w \) is vertex of maximum degree in \( H \), and let \( w \in V_j \). Clearly, \( 0 < i < r \), and in view of

\begin{align}
d(w) \leq C_{n-1-(r-2)(k-1)}^{1} \leq C_{n-1-(r-2)(k-1)}^{1},
\end{align}

we find that

\begin{align}
C_{n-1-(r-2)(k-1)}^{1} = C_{n-1-(r-2)(k-1)}^{1}.
\end{align}

Hence, if \( j < i - 1 \) or \( j > i + 1 \), then \( |V_i| = 1 \); furthermore,

\begin{align}
|V_{i-1}| + |V_i| + |V_{i+1}| (k-1) = n - (r-2)(k-1).
\end{align}

If \( |V_{i-1}| = |V_{i+1}| = 1 \), then obviously \( H = \mathcal{B}_{n-(r-2)(k-1),[r/2],[r/2]} \), so Theorem 2 is proved in this case. Next we will show that all other cases lead to contradictions, by constructing a hypergraph \( H' \) of order \( n \) and \( \text{diam}(H') = r \) with \( \rho(H') > \rho \). Suppose that \( x_1, \ldots, x_n \) is a positive unit vector to \( \rho(H) \).

First, consider the case \( |V_{i-1}| = 1 \) and \( |V_{i+1}| \geq 2 \). If \( |V_i| = 1 \), the proof is completed. So, we suppose that \( |V_i| \geq 2 \). Let \( V_{i-1} = \{a\}, V_{i+1} = \{b\} \), and suppose by symmetry that \( x_k \geq x_a \). Choose a vertex \( w \in V_i \), obtain \( H' \) from \( H \), delete the edge \( e_{aw} \), and add the edge \( e_{wb} \). In other words, \( H' \) is
obtained by moving the vertex \( w \) from \( V_j \) into \( V_{i+1} \). By symmetry and Lemma 8, \( x_w = x_w \) for any \( w \in V_i \); thus, the choice of \( H \) implies that

\[
0 \geq \rho(H') - \rho \geq x^T A(H') x - x^T A(H) x,
\]

\[
= k \sum_{e \in E(H')} x_e - k \sum_{e \in E(H)} x_e
\]

\[
= 2k(x_{e_{ab}} - x_{e_{wa}})
\]

\[
\geq 0,
\]

implies that \( \rho(H') = \rho(H) \) and that \( x \) is an eigenvector to \( \rho(H) \). However, the neighborhood of \( H \) is a proper subset of the neighborhood of \( a \) in \( H \), so the eigenequations for \( \rho(H') \) and \( \rho(H) \) for the vertex \( a \) are contradictory.

The same argument disposes also of the case \(|V_{i-1}| \geq 2 \) and \(|V_{i+1}| = 1 \); thus, to complete the proof, it remains to consider the case \(|V_{i-1}| \geq 2 \) and \(|V_{i+1}| \geq 2 \).

Let \( V_{i-2} = \{a\} \), \( c \in V_{i-1}, d \in V_{i+1} \), and \( V_{i+2} = \{b\} \). Our first step is to show that

\[
x_e \geq x_a.
\]

Note that if \( i \geq 3 \) and \( V_{i-3} = \{z\} \), then Theorem 1 gives \( x_z < x_a \). Hence, setting \( I = |V_{i-1}| \), the eigenequation for the vertex \( a \) implies that

\[
\rho x_a^{k-1} = x_{a|a} + I x_{a|a},
\]

According to \( x_z < x_a \) and Theorem 1, we have

\[
\rho x_a^{k-1} < x_a^{k-1} + I x_a^{k-1},
\]

yielding in turn

\[
\left( \frac{x_a}{x_a} \right)^{k-1} > \frac{\rho - 1}{I}.
\]

Since

\[
\rho - 1 \geq C^{k-1}_{n-1-(r-2)(k-1)} - 1 - 1
\]

\[
= C^{k-1}_{|V_{i-1}|+|V_{i+1}|+|V_{i-1}|}(k-1) - 1 - 2
\]

\[
> C^{k-1}_{|V_{i-1}|+2}(k-1) - 1 - 2
\]

\[
\geq l + 2
\]

\[
= l,
\]

inequality (39) is proved. By symmetry, we also see that \( x_d > x_a \). Suppose, again by symmetry, that \( x_d \geq x_a \), which yields \( x_z \leq x_d \). Choose a vertex \( w \in V_{i-1} \), obtain \( H' \) from \( H \), delete the edges \( e_{a|a} \), and add the edges \( e_{a|s} \) for all \( s \in V_{i+1} \). In other words, \( H' \) is obtained by moving the vertex \( w \) from \( V_{i-1} \) into \( V_i \). The choice of \( H \) implies that

\[
0 \geq \rho(H') - \rho \geq x^T A(H') x - x^T A(H) x,
\]

\[
= k \sum_{e \in E(H')} x_e - k \sum_{e \in E(H)} x_e
\]

\[
= 2k(x_{e_{ab}} - x_{e_{wa}})
\]

\[
\geq 0.
\]

This contradiction completes the proof of Theorem 2. □

Remark 1. From the above result, when \( k = 2 \), the result of Lemma 1 is obvious [22].

3.2. The Spectral Radii of Uniform Hypergraphs with Fixed Number of Vertices and Clique Number

Lemma 11. For \( k \geq 2 \), let \( H \) be a \( k \)-uniform hypergraph with \(|E(H)| = 1 \) and \( u \in V(H) \). For \( p \geq q \geq 1 \), if \( H_T(u) \) is the \( k \)-uniform hypergraph obtained by identifying \( u \) with a vertex of a hypertree \( T \) of order \( n \), then

\[
\rho(H_T(u)) > \rho(H_{n,1}(u)).
\]

Proof. This can be carried out along well-known lines by applying Lemma 7 to recursively flatten \( T \) until it becomes a path. □

Theorem 3. If \( H \) is a \( k \)-uniform hypergraph of order \( n \) with clique number \( \omega \geq 2 \), then \( \rho(H) \geq \rho(\mathcal{H}'_{n,\omega}) \) holds if and only if \( H \equiv \mathcal{H}'_{n,\omega} \).

Proof. Let \( H \) be a \( k \)-uniform hypergraph with minimal spectral among all connected \( k \)-uniform hypergraphs of order \( n \) and clique number \( \omega \). If \( \omega = 2 \), by Lemma 5, \( H \) must be a path, as the path is the hypergraph with smallest spectral radius among connected \( k \)-uniform hypergraphs of given order. Thus, we suppose that \( \omega \geq 3 \) and let \( H' \) be a complete \( k \)-uniform sub-hypergraph of \( H \) of order \( \omega \).

Further, \( H \) should be edge-minimal, that is, the removal of any edge of \( H \) either makes \( H \) disconnected or its clique number diminishes. In particular, if \( H'' \) is the hypergraph obtained by removing the edges of \( H' \), then the components of \( H'' \) are superhypergraphs, and each component has exactly one vertex in common with \( H' \). It follows that \( H \) is isomorphic to a complete \( k \)-uniform hypergraph of order \( \omega \) with superhyperedges attached to some of its common vertices of edges. Moreover, Lemma 10 implies that each of those superhyperedges must be a pendant path. To complete the proof, we show that there is only one such path.

Let \( S = \{v: \text{d}(v) \geq 2\} \subseteq V(H') \), and \( u, v \in S \). Suppose that a path \( P_{uv} = (v_1 = v, \ldots, v_p) \) is attached to \( v \) and \( P_{u_i} = (u_1 = u, \ldots, u_{p_i}) \) is attached to \( u \). Let \( H^* \) be the hypergraph obtained by deleting the edge \( e_{u_1} \) and adding the edge \( e_{u_1} \), that is, \( H^* \) is obtained by removing \( P_{uv} \) and extending \( P_{u_i} \) to \( P_{u_i+1} \). To complete the proof, we need to show that \( \rho(H) > \rho(H^*) \).
Let $\rho = \rho(H^*)$ and $x$ be a positive eigenvector of $H^*$ to $\rho$. Write $x_1, \ldots, x_{|V^*|}$ for the entries of $x$ corresponding to $v_1, \ldots, v_r, u_2, \ldots, u_r$, and let $\gamma$ be defined by (2). Now, if $x_u \geq \gamma^{-1}x_v$, then Theorem 1 implies that $x_u \geq \gamma^{-1}x_1 > x_2 \geq x_v$, and so Lemma 4 implies that $\rho(H) > \rho(H^*)$. Thus, we focus on showing that $x_u \geq \gamma^{-1}x_v$.

Since the eigenequation for $u$ is

$$\rho u_x^{k-1} = \sum_{i \in S \setminus \{u\}} x_{x_u^{i \in S \setminus \{u\}}}$$

$$< -x_u^{k-1} + \sum_{i \in S} x_{x_u^{i \in S \setminus \{u\}}}.$$

Hence,

$$\rho + 1)x_u^{k-1} < \sum_{i \in S} x_{x_u^{i \in S \setminus \{u\}}}.$$  \hspace{1cm} (47)

Likewise, for any $w \in S \setminus \{u, v\}$, the eigenequation for $w$ gives

$$(\rho + 1)x_w^{k-1} > \sum_{i \in S} x_{x_u^{i \in S \setminus \{w\}}}.$$  \hspace{1cm} (48)

In particular, we see that $x_u > x_u \geq$ for any $w \in S \setminus \{u, v\}$, since $\rho + 1 > 0$.

Returning to the eigenequation for $u$, we find that

$$\rho u_x^{k-1} = \sum_{i \in S \setminus \{u\}} x_{x_u^{i \in S \setminus \{u\}}}$$

$$\geq x_{x_u^{i \in S \setminus \{u\}}} + (\omega - 2)x_u^{k-1}.$$  \hspace{1cm} (49)

According to Theorem 1,

$$(\rho - \omega + 2)x_u \geq x_1. \hspace{1cm} (50)$$

Assuming for a contradiction that $x_u < \gamma^{-1}x_1$, after some algebra, we get

$$\rho - \omega + 2 > \gamma = \sqrt{\rho^2 - 4},$$

and therefore,

$$\rho - 2\omega + 4 > \sqrt{\rho^2 - 4}. \hspace{1cm} (52)$$

It is known that $\rho < \Delta(H^*) = \omega$, since $H^*$ is not a $\omega$-regular hypergraph. Hence,

$$\omega - 2\omega + 4 > 0, \hspace{1cm} (53)$$

which is a contradiction if $\omega \geq 4$. If $\omega = 3$, inequality (51) becomes

$$\rho - 2 > \sqrt{\rho^2 - 4}. \hspace{1cm} (54)$$

Squaring both sides of this inequality, we get

$$\rho^2 - 4\rho + 4 > \rho^2 - 4,$$

and so

$$8 > 4\rho. \hspace{1cm} (56)$$

Therefore, $\rho < 2$, an obvious contradiction, and this completes the proof of Theorem 3.

Remark 2. From the above result, when $k = 2$, the result of Lemma 2 is obvious [23].

Data Availability

No data were used to support this study.

Conflicts of Interest

The authors declare that they have no conflicts of interest.

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