Resolution of two fundamental issues in the dynamics of relativity and exposure of a real version of the emperor’s new clothes

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In this paper, we aim to resolve two fundamental issues in the dynamics of relativity: (i) Under what condition, the time-column space integrals of a Lorentz four-tensor constitute a Lorentz four-vector, and (ii) under what condition, the time-element space integral of a Lorentz four-vector is a Lorentz scalar; namely two “conservation laws”, which are misrepresented in traditional textbooks, and widely used in fundamental research, such as relativistic analysis of the momentum of light in a medium, and the proofs of the positive mass theorem in general relativity. To resolve issue (i), we have developed a generalized Lorentz four-vector theorem from the change of variables theorem in classical mathematical analysis. We use this four-vector theorem to verify Møller’s theorem, and surprisingly find that Møller’s theorem is fundamentally wrong. We provide a corrected version of Møller’s theorem. We also use this four-vector theorem to analyze a plane light wave in a moving uniform medium, and find that the momentum and energy of Minkowski quasi-photon constitute a Lorentz four-vector and Planck constant is a Lorentz invariant. To resolve issue (ii), we have developed a generalized Lorentz scalar theorem. We use this theorem to verify the “invariant conservation law” in relativistic electrodynamics, and unexpectedly find that it is also fundamentally wrong. Thus the two “conservation laws” in traditional textbooks, which have magically attracted several generations of most outstanding scientists, turned out to be imaginary, just like the emperor’s new clothes; creating a scientific myth in the modern theoretical and mathematical physics: Believing is seeing.

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I. INTRODUCTION

In the dynamics of relativity, the energy and momentum of a physical system is described by a Lorentz four-tensor; such a tensor is usually called energy–momentum tensor, stress tensor, stress–energy tensor, or momentum–energy stress tensor. If the tensor is divergence-less, then the system is thought to be conserved, and it is a closed system; thus the total energy and momentum can be obtained by carrying out space integration of the time-column elements of the tensor to constitute a Lorentz four-vector.

Mathematically speaking, if a tensor satisfies certain conditions, the space integrals of the tensor’s time-column elements can form a Lorentz four-vector. For the sake of convenience, we call such a mathematical statement “four-vector theorem”.

Laue set up a four-vector theorem for a tensor that is required to be time-independent. Laue’s theorem only provides a sufficient condition (instead of a sufficient and necessary condition), and it cannot be used to judge the Lorentz property of the energy and momentum of electrostatic fields. In a recent study, Laue’s theorem is improved to be a theorem that has a sufficient and necessary condition, and it is successfully used to generally resolve the electrostatic field problem.

In contrast to Laue’s theorem, Møller provided a four-vector theorem for a tensor that is required to be divergence-less, with a boundary condition imposed, but allowed to be time-dependent. Møller’s theorem only has a sufficient condition (instead of a sufficient and necessary condition), but it is more attractive because the energy–momentum tensor for electromagnetic (EM) radiation fields varies with time. It is widely recognized in the community that Møller’s theorem is absolutely rigorous so that this theorem has been widely used in quantum electrodynamics and relativistic analysis of light momentum in a dielectric medium.

In this paper, we provide a generalized Lorentz four-vector theorem for a tensor that is not required to be time-independent and divergence-less, and on which no boundary conditions are imposed. This theorem has a sufficient and necessary condition. We use this theorem to verify Møller’s theorem, surprisingly finding that Møller’s theorem is fundamentally wrong.

Like the four-vector theorem, a Lorentz scalar theorem is a mathematical statement that under what conditions, the time-element space integral of a four-vector is a Lorentz scalar. In Ref. [6], a scalar theorem for a four-vector that is required to be time-independent is set up, called “derivative von Laue’s theorem”, and it is successfully used to strictly resolve the invariance problem of total electric charge in relativistic electrodynamics.

In this paper, we also provide a generalized Lorentz scalar theorem for a four-vector that is not required to be time-independent and divergence-less, and on which no boundary conditions are imposed. This theorem has a sufficient and necessary condition. We use this scalar...
theorem to identify the validity of a well-known result in the dynamics of relativity that if a Lorentz four-vector is divergence-less, then the time-element space integral of the four-vector is a Lorentz scalar \[^{[3]}\ p.\ 168\] named “invariant conservation law” claimed by Weinberg \[^{[2]}\ p.\ 41\]. We unexpectedly find that it is also fundamentally wrong.

The paper is organized as follows. In Sec. II, proofs are given of Lorentz four-vector and scalar theorems. In Sec. III, Möller’s theorem is proved to be incorrect, and a corrected version of Möller’s theorem is provided. In Sec. IV, the validity of the well-known “invariant conservation law” claimed by Weinberg \[^{[2]}\] is identified. In Sec. V, some remarks and conclusions are given. In Appendix A the hyperplane differential-element four-vector presented in textbooks is found to be in contradiction with both the principle of classical mathematical analysis and the principle of relativity. In Appendix B as an application of Theorem 1 to Minkowski tensor for a plane light wave in a moving uniform medium, the momentum-energy four-vector of quasi-photon and the Lorentz invariance of Planck constant are naturally derived. In Appendix C physical counterexamples of Thirring’s claims are provided. In Appendix D an illustration is given of why the proofs of the positive mass theorem in general relativity are based on a flawed theoretical framework.

II. LORENTZ FOUR-VECTOR THEOREMS AND SCALAR THEOREM

In this section, proofs are given of Lorentz four-vector and scalar theorems. Four-vector theorems provide a criterion to judge under what condition the space integrals of the time-column elements of a tensor constitute a Lorentz four-vector (Theorem 1) and under what condition the space integrals of the time-row elements of a tensor constitute a Lorentz four-vector (Theorem 2), while the scalar theorem provides a criterion to judge under what condition the space integral of the time-element of a four-vector is a Lorentz scalar (Theorem 3). The proofs of Theorem 1 and Theorem 2 are very similar, and without loss of generality, only the proof of Theorem 1 is given.

Suppose that an inertial frame of \(X'Y'Z'\) moves uniformly at \(\beta c\) relatively to the laboratory frame \(XYZ\), where \(c\) is the vacuum light speed. The Lorentz transformation of time-space four-vector \(X^\mu = (x, ct)\) is given by \[^{[3]}\] and \[^{[4]}\]

\[
x' = x + \xi (\beta \cdot x) \beta - \gamma \beta ct, \quad \text{(1)}
\]

\[
c't' = \gamma (ct - \beta \cdot x), \quad \text{(2)}
\]
or conversely, given by

\[
x = x' + \xi (\beta' \cdot x') \beta' - \gamma \beta' ct', \quad \text{(3)}
\]

\[
c t = \gamma (ct' - \beta' \cdot x'), \quad \text{(4)}
\]

where \(\xi \equiv (\gamma - 1)/\beta^2 = \gamma^2/(\gamma + 1), \gamma \equiv (1 - \beta^2)^{-1/2}\), and \(\beta' = -\beta\). Note: \(X_\mu = g_{\mu\nu}X^\nu = (-x, ct)\), with \(g_{\mu\nu} = g^{\mu\nu} = \text{diag}(-1, -1, -1, +1)\) the Minkowski metric.

According to the definition of tensors \[^{[3]}\] p.108], if \(\Omega^{\mu\nu}(x, t)\) is a Lorentz four-tensor given in \(XYZ\), where \(\mu, \nu = 1, 2, 3, 4\), and \(4\), with the index 4 corresponding to time component, then in \(X'Y'Z'\) the tensor \(\Omega^{\mu\nu}(x, t) = \Omega^{\mu\nu}(x(t'))\) is obtained through “double” Lorentz transformation of \(\Omega^{\mu\nu}(x, t)\), given by

\[
\Omega^{\mu\nu}(x, t) = \frac{\partial X^\mu}{\partial X^\lambda} \frac{\partial X^\nu}{\partial X^\sigma} \Omega_{\lambda\sigma}(x, t), \quad \text{(5)}
\]

\[
\Omega^{\mu\nu}(x = x(x', t), t = t(x', t')) = \frac{\partial X^\mu}{\partial X^\lambda} \frac{\partial X^\nu}{\partial X^\sigma} \Omega_{\lambda\sigma}(x = x(x', t), t = t(x', t')), \quad \text{(6)}
\]

where \(\partial X^\mu/\partial X^\lambda\) and \(\partial X^\nu/\partial X^\sigma\) are obtained from Lorentz transformation Eqs. \[^{[1]}\] and \[^{[2]}\], while \(x = x(x', t')\) and \(t = t(x', t')\) denote Lorentz transformation Eqs. \[^{[3]}\] and \[^{[4]}\], respectively. Eq. (5) is the expression of \(\Omega^{\mu\nu}\) observed in \(XYZ\), and Eq. (6) is the expression of \(\Omega^{\mu\nu}\) observed in \(X'Y'Z'\).

Theorem 1. Suppose that \(\Theta^{\mu\nu}(x, t)\) is an integrable Lorentz four-tensor, defined in the domain \(V\) in the laboratory frame \(XYZ\), where \(\mu, \nu = 1, 2, 3, 4\), and \(4\), with the index 4 corresponding to time component, and \(V\) including its boundary is at rest in \(XYZ\), namely any \(x \in V\) is independent of \(t\). The space integrals of the time-column elements of the tensor in \(XYZ\) are defined as

\[
P^\mu = \int_{V: t = \text{const}} \Theta^{\mu4}(x, t) \, d^3x. \quad \text{(7)}
\]

The space integrals of time-column elements of the tensor in \(X'Y'Z'\) are defined as

\[
P'^\mu = \int_{V': t, t' = \text{const}} \Theta'^{\mu4}(x = x(x', t), t) \, d^3x', \quad \text{(8)}
\]

where

\[
\Theta'^{\mu4}(x = x(x', t), t) := \frac{\partial X^\mu}{\partial X^\lambda} \frac{\partial X^4}{\partial X^\sigma} \Theta_{\lambda\sigma}(x = x(x', t), t). \quad \text{(9)}
\]

The four-vector theorem states: \(P^\mu\) is a Lorentz four-vector \(\text{if and only if}\)

\[
\int_{V: t = \text{const}} \Theta^{\mu4}(x, t) \, d^3x = 0 \quad \text{for} \quad \mu = 1, 2, 3, 4 \quad \text{and} \quad j = 1, 2, 3. \quad \text{(10)}
\]
holds.

There are a few main points to understand Theorem 1 that should be noted, as follows.

(i) The importance of the definition Eq. (9) should be emphasized, otherwise the implication of $P^\mu = \int \Theta^{\mu 4} d^3 x'$ is ambiguous, and we cannot set up the transformation between $P^\mu$ and $P^\mu$. In Eq. (9), the space variables $x$ in $\Theta^{\mu \nu}(x, t)$ are replaced by $x = (x', t')$, namely the space Lorentz transformation Eq. (3), while $t$ in $\Theta^{\mu \nu}(x, t)$ is kept as it is.

(ii) Observed in XYZ, like $P^\mu$, $P^{\mu'}$ is only dependent on $t$ in general; confer Eq. (10). The quantity $t'$ in the integrand $\Theta^{\mu 4}(x = x(x', t'), t)$ of $P^\mu = P^\mu(t) = \int _{V'}: t', t' = const \Theta^{\mu 4}(x = x(x', t'), t) d^3 x'$ is introduced as a constant parameter in a change of variables in the space integrals, and thus observed in $X'Y'Z'$, the boundary of $V'$ is moving so that $P^\mu = P^\mu(t)$ does not contain $t'$. To better understand this, let us take a simple one-dimensional example, given by

$$I(t) = \int a^b \cos (x - ct) dx$$

$$= \int a^b \cos \left[ \gamma (x' + \beta ct') - ct \right] \gamma dx'$$

$$= \sin (b - ct) - \sin (a - ct)$$

which does not contain $t'$ although the integrand $\cos \left[ \gamma (x' + \beta ct') - ct \right] \gamma$ contains $t'$, and where a change of variable $x = \gamma (x' + \beta ct')$ is taken, with $dx = \gamma dx'$ and $t'$ as a constant parameter introduced, and the integral limits $x'_a = a / \gamma - | \beta | ct'$ and $x'_b = b / \gamma - | \beta | ct'$, with $a$ and $b$ being constants, are moving at a velocity of $| \beta | c$.

(iii) If $\Theta^{\lambda \sigma}(x, t)$ is independent of $t$, then both $P^\mu$ and $P^{\mu'}$ are independent of $t$, namely they are constants.

(iv) If $P^\mu = P^\mu(t)$ is set to be observed in $XY'Z'$, $t$ in $P^\mu = P^\mu(t)$ should be replaced by $t = \gamma (t' - \beta \cdot x'/c)$, namely the Lorentz transformation given by Eq. (4).

(v) The symmetry $(\Theta^{\mu \nu} = \Theta^{\nu \mu})$ and divergence-less $(\partial_a \Theta^{\mu \nu} = 0)$ are not required, and there are no boundary conditions imposed on $\Theta^{\mu \nu}(x, t)$.

Theorem 2. Suppose that $\Theta^{\mu \nu}(x, t)$ is an integrable Lorentz four-tensor, defined in the domain $V$ in the laboratory frame $XYZ$, where $\mu, \nu = 1, 2, 3, 4$, with the index 4 corresponding to time component, and $V$ including its boundary is at rest in $XY'Z'$, namely any $x \in V$ is independent of $t$. The space integrals of the time-row elements of the tensor in $XYZ$ are defined as

$$\Pi^\nu = \int _{V: t = const} \Theta^{\mu 4}(x, t) d^3 x.$$  \hspace{1cm} (11)

The space integrals of time-row elements of the tensor in $X'Y'Z'$ are defined as

$$\Pi'^\nu = \int _{V': t', t' = const} \Theta'^{\mu 4}(x = x(x', t'), t) d^3 x',$$  \hspace{1cm} (12)

where

$$\Theta'^{\mu 4}(x = x(x', t'), t)$$

:= \frac{\partial X'^4}{\partial X^4} \frac{\partial X'^\mu}{\partial X^\mu} \Theta^{\lambda \sigma}(x = x(x', t'), t).$$  \hspace{1cm} (13)

The four-vector theorem states: $\Pi'^\nu$ is a Lorentz four-vector if and only if

$$\int _{V: t = const} \Theta^{\nu}(x, t) d^3 x = 0$$

for $\nu = 1, 2, 3, 4$ and $i = 1, 2, 3$  \hspace{1cm} (14)

holds.

Proof of Theorem 1. From Eqs. (8) and (9) we have

$$P^\mu = \int _{V': t', t' = const} \Theta^{\mu 4}(x = x(x', t'), t) d^3 x' = \frac{\partial X'^\mu}{\partial X^\mu} \frac{\partial X'^4}{\partial X^4} \int _{V': t', t' = const} \Theta^{\lambda \sigma}(x = x(x', t'), t) d^3 x'.$$  \hspace{1cm} (15)

Note that $x = x(x', t')$ in $\Theta^{\lambda \sigma}(x = x(x', t'), t)$ denotes Eq. (3). By the change of variables $x(x', t') = x$ or $(x', y', z', t) \rightarrow (x, y, z)$ with $t'$ as a constant parameter, from above Eq. (15) we obtain

$$P^\mu = \frac{\partial X'^\mu}{\partial X^\mu} \frac{\partial X'^4}{\partial X^4} \gamma \int _{V: t = const} \Theta^{\lambda \sigma}(x, t) d^3 x,$$  \hspace{1cm} (16)

where $d^3 x = |\partial (x, y, z)/\partial (x', y', z')| d^3 x'$ is employed, with the Jacobian determinant $\partial (x, y, z)/\partial (x', y', z') = \gamma$ being explained as the effect of Lorentz contraction physically (confer Appendix A). Since $t'$ is introduced as a constant parameter in the change of variables, $P^\mu$ is independent of $t'$.

From Eq. (16), with $\partial X'^4/\partial X^4 = \gamma$ and the definition given by Eq. (7), $P^\mu = \int _{V: t = const} \Theta^{\mu 4}(x, t) d^3 x$, taken into account, we have
If \( P^\mu \) is a Lorentz four-vector, then

\[
P^\mu = \frac{\partial X^\mu}{\partial X^a} P^\lambda
\]

must hold. Inserting Eq. [18] into Eq. [17], we have

\[
\frac{\partial X^\mu}{\partial X^a} \frac{\partial X^a}{\partial X^\lambda} \gamma \int_{V: t=const} \Theta^{\lambda j}(x, t) d^3x = 0,
\]

namely,

\[
\begin{pmatrix}
1 + \xi \beta^2 \\
\xi \beta_x \beta_y \\
\xi \beta_y \beta_x \\
-\gamma \beta_x
\end{pmatrix} \begin{pmatrix}
\xi \beta^2 \\
1 + \xi \beta^2 \\
\xi \beta_x \beta_y \\
-\gamma \beta_x
\end{pmatrix} \begin{pmatrix}
\xi \beta^2 \\
\xi \beta_y \beta_x \\
1 + \xi \beta^2 \\
-\gamma \beta_x
\end{pmatrix} \begin{pmatrix}
1 + \xi \beta^2 \\
\xi \beta_x \beta_y \\
\xi \beta_y \beta_x \\
-\gamma \beta_x
\end{pmatrix} = \begin{pmatrix}
0 \\
0 \\
0 \\
0
\end{pmatrix},
\]

where \( a^{\lambda j} = \int_{V: t=const} \Theta^{\lambda j}(x, t) d^3x \), with \( \lambda = 1, 2, 3, 4 \) and \( j = 1, 2, 3 \).

From above it is seen that Eq. [20] \( \Rightarrow \) Eq. [18] through Eq. [17] is valid. Thus for \( P^\mu \) to be a Lorentz four-vector, the sufficient and necessary condition is given by

\[
a^{\lambda j} = \int_{V: t=const} \Theta^{\lambda j}(x, t) d^3x = 0
\]

for \( \lambda = 1, 2, 3, 4 \) and \( j = 1, 2, 3 \). (21)

The sufficiency of Eq. [21] is apparent because we directly have Eq. [21] \( \Rightarrow \) Eq. [20] \( \Rightarrow \) Eq. [19] \( \Rightarrow \) Eq. [18] from Eq. [17]. The necessity is based on the fact that a four-vector must follow Lorentz rule between any two inertial frames, namely \( \beta c \) is arbitrary, and thus \( a^{\lambda j} = 0 \) must hold for all \( \lambda \) and \( j \), because \( (\beta_x \neq 0, \beta_y = 0, \beta_z = 0) \Rightarrow a^{\lambda 1} = 0, (\beta_x = 0, \beta_y \neq 0, \beta_z = 0) \Rightarrow a^{\lambda 2} = 0, \) and \( (\beta_x = 0, \beta_y = 0, \beta_z \neq 0) \Rightarrow a^{\lambda 3} = 0 \). Thus we finish the proof of the sufficiency and necessity.

**Theorem 3.** Suppose that \( \Lambda^\mu(x, t) = (\Lambda, \Lambda^4) \) is an integrable Lorentz four-vector, defined in the domain \( V \) in the laboratory frame \( XYZ \), where \( \mu = 1, 2, 3, \) and 4, with the index 4 corresponding to time component, and \( V \) including its boundary is at rest in \( XYZ \), namely any \( x \in V \) is independent of \( t \). The Lorentz scalar theorem states: The time-element space integral

\[
\Phi = \int_{V: t=const} \Lambda^4(x, t) d^3x
\]

is a Lorentz scalar if and only if

\[
\int_{V: t=const} \Lambda^i(x, t) d^3x = 0 \quad \text{for} \quad i = 1, 2, 3 \quad \text{or}
\]

\[
\int_{V: t=const} \Lambda(x, t) d^3x = 0
\]

holds.

**Proof.** Corresponding to \( \Phi = \int_{V: t=const} \Lambda^4(x, t) d^3x \) given by Eq. [22], we first have to define \( \Phi' = \int \Lambda^4 d^3x' \) in \( X'Y'Z' \), because the implication of \( \Phi' = \int \Lambda^4 d^3x' \) itself is ambiguous before the dependence of \( \Lambda^4 \) on \( x', t' \), and \( t \) is defined. For this end, from Lorentz transformation we have

\[
\Lambda^4(x, t) = \frac{\partial X^4}{\partial X^\lambda} \Lambda^\lambda(x, t)
\]

\[
\Rightarrow \Lambda^4(x = x(x', t'), t) = \frac{\partial X^4}{\partial X^\lambda} \Lambda^\lambda(x = x(x', t'), t)
\]

(24)

where the space variables \( x \) in \( \Lambda^\lambda(x, t) \) are replaced by \( x = x(x', t') \), namely the space Lorentz transformation Eq. [4], but \( t \) in \( \Lambda^\lambda(x, t) \) is kept as it is.

Making integration in Eq. [24] with respect to \( (x, y, z) \) over \( V \) in the laboratory frame, we have

\[
\int_{V: t=const} \Lambda^4(x = x(x', t'), t) d^3x
\]

\[
= \frac{\partial X^4}{\partial X^\lambda} \int_{V: t=const} \Lambda^\lambda(x = x(x', t'), t) d^3x.
\]

(25)
By the change of variables \((x, y, z) \rightarrow (x', y', z'; t')\) with \(t'\) as a constant parameter in the left-hand side of Eq. (25), while keeping the integrals of the right-hand side to be computed in \(XYZ\) frame, we obtain

\[
\int_{V': t', t=\text{const}} A^4(x = x(x', t'), t) \gamma d^3x' = \frac{\partial X^4}{\partial X^3} \int_{V: t=\text{const}} A^4(x, t) d^3x. \tag{26}
\]

where \(d^3x = |\partial(x, y, z)/\partial(x', y', z')| d^3x' = \gamma d^3x'\) is taken into account, with \(\partial(x, y, z)/\partial(x', y', z') = \gamma\) the Jacobian determinant.

We define

\[
\Phi' = \int_{V': t', t=\text{const}} A^4(x = x(x', t'), t) d^3x', \tag{27}
\]

where \(A^4(x = x(x', t'), t)\) is defined in Eq. (24). Since \(t'\) is introduced as a constant parameter in the change of variables in the space integral, \(\Phi'\) does not contain \(t'\) although the integrand \(A^4(x = x(x', t'), t)\) in Eq. (27) contains \(t'\). Thus with the both sides of Eq. (26) divided by \(\gamma\), we have

\[
\Phi = \int_{V': t', t=\text{const}} A^4(x = x(x', t'), t) d^3x',
\]

\[
= \frac{\partial X^4}{\partial X^3} \frac{1}{\gamma} \int_{V: t=\text{const}} A^4(x, t) d^3x
\]

\[
= \frac{\partial X^4}{\partial X^4} \frac{1}{\gamma} \int_{V: t=\text{const}} A^4(x, t) d^3x
\]

\[
+ \frac{\partial X^4}{\partial t} \frac{1}{\gamma} \int_{V: t=\text{const}} A^4(x, t) d^3x
\]

\[
= \Phi - \beta \cdot \int_{V: t=\text{const}} \Lambda(x, t) d^3x, \tag{28}
\]

where \(\partial X^4/\partial X^4 = \gamma, (\partial X^4/\partial X^t) A^t = -\gamma \beta \cdot \Lambda,\) and the definition \(\Phi = \int_{V: t=\text{const}} A^4(x, t) d^3x\) given by Eq. (22) are employed.

From Eq. (28) we obtain the sufficient and necessary condition for \(\Phi = \Phi'\) (Lorentz scalar), given by

\[
\int_{V: t=\text{const}} A^i(x, t) d^3x = 0 \quad \text{for } i = 1, 2, 3 \quad \text{or}
\]

\[
\int_{V: t=\text{const}} \Lambda(x, t) d^3x = 0. \tag{29}
\]

The sufficiency is apparent, while the necessity comes from the fact that \(\beta\) is arbitrary. Thus we complete the proof.

There are some main points to understand Theorem 3 that should be noted:

(i) If \(A^\mu(x, t)\) is independent of \(t\), namely \(\partial A^\mu/\partial t = 0\), then both \(\Phi\) and \(\Phi'\) are constants.

(ii) The divergence-less \((\partial_\mu A^\mu = 0)\) is not required, and there are no boundary conditions imposed on \(A^\mu(x, t)\).

(iii) Asymmetry arising from resting \(V\) and moving \(V'\). Directly from Eq. (2), we have

\[
A^4 = \gamma(A^4 - \beta \cdot \Lambda)
\]

\[
\Rightarrow \int_V A^4 d^3x' = \int_V \gamma(A^4 - \beta \cdot \Lambda) d^3x'
\]

\[
\Rightarrow \Phi' = \Phi - \beta \cdot \int_V \Lambda d^3x
\]

with \(d^3x' = d^3x/\gamma\) used, namely Eq. (28). Conversely, from Eq. (4) we have

\[
A^4 = \gamma(A^4 - \beta' \cdot \Lambda')
\]

\[
\Rightarrow \int_V A^4 d^3x = \int_V \gamma(A^4 - \beta' \cdot \Lambda') d^3x
\]

\[
\Rightarrow \Phi = \gamma^2 \Phi' - \gamma^2 \beta' \cdot \int_V \Lambda' d^3x
\]

with \(d^3x = \gamma d^3x'\) used. We find that

\[
\Phi = \Phi - \beta \cdot \int_V \Lambda d^3x \quad \text{and}
\]

\[
\Phi = \gamma^2 \Phi' - \gamma^2 \beta' \cdot \int_V \Lambda' d^3x
\]

are not symmetric, although

\[
A^4 = \gamma(A^4 - \beta \cdot \Lambda) \quad \text{and}
\]

\[
A^4 = \gamma(A^4 - \beta' \cdot \Lambda')
\]

are symmetric. This asymmetry comes from the fact that \(V\) is fixed in \(XYZ\), while \(V'\) is moving in \(X'Y'Z'\).
III. INVALIDITY OF MÖLLER’S THEOREM

In this section, (i) Møller’s theorem is proved to be incorrect; (ii) based on Theorem 1, a counterexample of Møller’s theorem is given; and (iii) a corrected version of Møller’s theorem is provided, with a detailed elucidation of why the corrected Møller’s theorem only defines a trivial zero four-vector for EM stress–energy tensor.

Møller’s theorem. Suppose that \( \Theta^{\mu\nu}(x, t) \) is an integrable Lorentz four-tensor, defined in the domain \( V \) in the laboratory frame \( XYZ \), where \( \mu, \nu = 1, 2, 3, \) and 4, with the index 4 corresponding to time component, \( X \) including its boundary is at rest in \( XYZ \), namely any \( x \in V \) is independent of \( t \). All the elements of the tensor have first-order partial derivatives with respect to time-space coordinates \( X^\mu = (x, ct) \). Møller’s theorem states: If \( \Theta^{\mu\nu}(x, t) \) is divergence-less (\( \partial_\nu \Theta^{\mu\nu}(x, t) = 0 \)), and \( \Theta^{\mu\nu}(x, t) = 0 \) holds on the boundary of \( V \) for any time \( (-\infty < t < +\infty) \) — zero boundary condition, then the time-column space integrals

\[
P^\mu = \int_{V: t=const} \Theta^{\mu\nu}(x, t) d^3x \tag{30}
\]

constitute a Lorentz four-vector [3, pp.166-169].

Proof. From Møller’s sufficient condition, we first demonstrate that the time-column space integrals, given by Eq. (30), are time-independent (\( \partial P^\mu / \partial t = 0 \)), then we prove that the sufficient condition is not enough to make Eq. (30) be a four-vector, and we conclude that Møller’s theorem is incorrect.

Since \( \Theta^{\mu\nu}(x, t) = 0 \) holds on the boundary of \( V \), using 3-dimensional Gauss’s divergence theorem we obtain

\[
\int_{V: t=const} \partial_\iota \Theta^{\mu\iota}(x, t) d^3x = 0, \quad \text{with} \quad i = 1, 2, 3. \tag{31}
\]

Because the boundary of \( V \) is at rest in the laboratory frame, we have

\[
\int_{V: t=const} \frac{\partial}{\partial t} (\cdot \cdot \cdot) d^3x = \int_{V: t=const} (\cdot \cdot \cdot) d^3x. \tag{32}
\]

From \( \partial_\iota \Theta^{\mu\iota}(x, t) = 0 \), with Eq. (31), Eq. (32), and \( X^4 = ct \) taken into account, we have

\[
0 = \int_{V: t=const} \partial_\iota \Theta^{\mu\iota}(x, t) d^3x
= \int_{V: t=const} \partial_\iota \Theta^{\mu\iota}(x, t) d^3x + \int_{V: t=const} \partial_4 \Theta^{\mu4}(x, t) d^3x
= \int_{V: t=const} \partial_4 \Theta^{\mu4}(x, t) d^3x
= \frac{\partial}{\partial (ct)} \int_{V: t=const} \Theta^{\mu4}(x, t) d^3x. \tag{33}
\]

Inserting Eq. (30) into above Eq. (33) yields

\[
\frac{\partial P^\mu}{\partial t} = \int_{V: t=const} \Theta^{\mu\nu}(x, t) d^3x \equiv 0. \tag{34}
\]

Thus \( P^\mu = \int_{V: t=const} \Theta^{\mu\nu}(x, t) d^3x \) is constant although the integrand \( \Theta^{\mu\nu}(x, t) \) may depend on \( t \). However it should be emphasized that

\[
\frac{\partial}{\partial t} \int_{V: t=const} \Theta^{\mu\nu}(x, t) d^3x = 0
\]

for \( j = 1, 2, 3 \) may not hold. (35)

From the divergence-less (\( \partial_\nu \Theta^{\mu\nu} = 0 \)) and the zero-boundary condition (\( \Theta^{\mu\nu} = 0 \) on boundary), we have achieved a conclusion that the time-column space integrals \( P^\mu = \int_{V: t=const} \Theta^{\mu\nu}(x, t) d^3x \) are time-independent constants. In what follows, we will show that the divergence-less and the zero-boundary condition is not sufficient to make \( P^\mu \) be a four-vector. In other words, Møller’s sufficient condition is not sufficient.

From Eqs. (15)-17) in the proof of Theorem 1, we have

\[
P^\mu = \int_{V': t',t'=const} \Theta^{\mu\nu}(x(x',t'), t) d^3x'
= \int_{V': t',t'=const} \frac{\partial X^\nu}{\partial X^\lambda} \frac{\partial X^\lambda}{\partial \gamma} \Theta^{\nu\gamma}(x, t) d^3x
= \int_{V: t=const} \Theta^{\lambda\gamma}(x, t) d^3x \tag{34}
\]

(allowed to be \( t \)-dependent)

\[
+ \frac{\partial X^\nu}{\partial X^\lambda} P^\lambda, \quad \text{with} \quad j = 1, 2, 3. \tag{36}
\]

(\( t \)-independent)

Thus like Eq. (17), we obtain a sufficient and necessary condition for constant \( P^\mu \) to be a Lorentz four-vector, given below

\[
\Theta^{\lambda\gamma}(x, t) d^3x = 0
\]

for \( \lambda = 1, 2, 3, 4 \) and \( j = 1, 2, 3, \) (37)

which is the same as Eq. (10). However Møller’s sufficient condition does not include this sufficient and necessary condition, and accordingly, Møller’s theorem is fundamentally wrong. Thus we finish the proof.

Counterexample of Møller’s theorem. To further convince readers, given below is a pure mathematical counterexample to disprove Møller’s theorem based on Theorem 1. As indicated in Sec. [Møller’s theorem is also the counterexample of Landau-Lifshitz and Weinberg’s versions of Laue’s theorem [6].]
Suppose that there is a symmetric Lorentz four-tensor

\[ A^{\mu\nu} = \begin{pmatrix} 0 & 0 & 0 & f(x) \\ 0 & 0 & 0 & f(x) \\ f(x) & f(x) & f(x) & (ct)(f_x + f_y + f_z) \end{pmatrix}, \]

(38)

defined in the cubic domain \( V (-\pi \leq x, y, z \leq \pi) \), where \( f(x) = (\sin x)^2 (\sin y)^2 (\sin z)^2 \) is independent of time, with \( \int_V f(x) \, d^3x = \pi^3 \), and \( f_x = \partial f/\partial x \), \( f_y = \partial f/\partial y \), and \( f_z = \partial f/\partial z \). \( A^{\mu\nu} \) is divergence-less (\( \partial_\mu A^{\mu\nu} = 0 \) \( \Leftrightarrow \partial_\nu A^{\mu\nu} = 0 \)) because of \( A^{\mu\nu} = A^{\nu\mu} \), and satisfies the Møller’s zero boundary condition: \( A^{\mu\nu} = 0 \) holds on the boundary \( x, y, z = \pm \pi \) for \(-\infty < t < +\infty \). Thus \( A^{\mu\nu} \) satisfies the sufficient condition of Møller’s theorem.

\[ M^\mu = \int_{V: \ t=\text{const}} A^{4\mu} \, d^3x = (\pi^3, \pi^3, \pi^3, 0) \]

(39)
is supposed to be a Lorentz four-vector. However because

\[ \int_{V: \ t=\text{const}} A^{41} \, d^3x = \int_{V: \ t=\text{const}} A^{42} \, d^3x \]

\[ = \int_{V: \ t=\text{const}} A^{43} \, d^3x \]

\[ = \pi^3 \neq 0, \]

(40)

\( A^{\mu\nu} \) does not satisfy the sufficient and necessary condition Eq. (10) of Theorem 1, and accordingly, \( M^\mu = \int_{V: \ t=\text{const}} A^{\mu\nu} \, d^3x \) is not a four-vector. Thus Møller’s theorem is disproved by this counterexample based on Theorem 1.

The above counterexample shows that the sufficient condition of Møller’s theorem indeed does not include the sufficient and necessary condition Eq. (10) of Theorem 1. Obviously, Møller’s theorem can be easily corrected by adding the condition Eq. (10) as follows.

Corrected Møller’s theorem. Suppose that \( \Theta^{\mu\nu}(x, t) \) is an integrable Lorentz four-tensor, defined in the domain \( V \) in the laboratory frame XYZ, where \( \mu, \nu = 1, 2, 3 \), and 4, with the index 4 corresponding to time component, and \( V \) including its boundary is at rest in XYZ, namely any \( x \in V \) is independent of \( t \). It is assumed that \( \Theta^{\mu\nu}(x, t) \) is divergence-less (\( \partial_\mu \Theta^{\mu\nu}(x, t) = 0 \)), and \( \Theta^{\mu\nu}(x, t) = 0 \) holds on the boundary of \( V \) for any time \((-\infty < t < +\infty) \) — zero boundary condition. The corrected Møller’s theorem states: The time-column space integrals

\[ P^\mu = \int_{V: \ t=\text{const}} \Theta^{\mu4}(x, t) \, d^3x \]

(41)

constitute a Lorentz four-vector if and only if

\[ \int_{V: \ t=\text{const}} \Theta^{\mu j}(x, t) d^3x = 0 \]

for \( \mu, j = 1, 2, 3, 4 \) \( \text{and} \) \( j = 1, 2, 3 \). (42)

However we would like to indicate, by enumerating specific examples as follows, that the corrected Møller’s theorem has a limited application.

Example 1 for corrected Møller’s theorem. Consider Minkowski EM stress-energy tensor for “a pure radiation field in matter” [10], given by

\[ \tilde{T}^{\mu\nu} = (T^{\mu\nu})^T, \quad \text{with} \quad T^{\mu\nu} = \begin{pmatrix} T_M \ c g_A \\ c g_M \ W_{em} \end{pmatrix}, \]

(43)

where \( \tilde{T}^{\mu\nu} \) is the transpose of \( T^{\mu\nu} \), with \( \partial_\nu \tilde{T}^{\mu\nu} = \partial_\nu T^{\mu\nu} = (\nabla \cdot T_M + \partial g_M/\partial t, \nabla \cdot (c g_A) + \partial W_{em}/\partial (ct)) \); \( g_A = E \times H/c^2 \) is the Abraham momentum; \( g_M = D \times B \) is the Minkowski momentum; \( W_{em} = 0.5 (D \cdot E + B \cdot H) \) is the EM energy density; and \( T_M = -D E - B H + 0.5(D \cdot E + B \cdot H) \) is the Minkowski stress tensor, with \( I \) the unit tensor [6]. We first assume that the corrected Møller’s theorem is applicable for this EM tensor. Then let us see what conclusion we can get.

The pre-assumption of corrected Møller’s theorem is the tensor’s divergence-less plus a zero-boundary condition. The zero-boundary condition requires that all the tensor elements be equal to zero on the boundary for any time \((-\infty < t < +\infty) \). Thus for the EM stress-energy tensor given by Eq. (43), the pre-assumption requires \( \partial_\nu \tilde{T}^{\mu\nu} = 0 \) holding within the finite domain \( V \) of a physical system, and Poynting vector \( E \times H = 0 \) and Minkowski momentum \( D \times B = 0 \) holding on the boundary of \( V \) for any time \((-\infty < t < +\infty) \).

Physically, the pre-assumption is extremely strong and severe, because it requires that (i) within the domain \( V \), there are no any sources (\( \partial_\nu \tilde{T}^{\mu\nu} = 0 \)), and (ii) the EM energy and Minkowski momentum never flow through the closed boundary of \( V \) for any time \((-\infty < t < +\infty) \).

Example 2 for corrected Møller’s theorem. Nevertheless, the corrected Møller’s theorem may define a non-zero four-vector in general. As an example, consider the tensor given by
\[ B^{\mu\nu}(x, t) = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & f(x) \end{pmatrix}, \quad (44) \]
defined in the cubic domain \( V (-\pi \leq x, y, z \leq \pi) \), where \( f(x) = (\sin x)^2(\sin y)^2(\sin z)^2 \), with \( \int_V f(x) d^3x = \pi^3 \). \( B^{\mu\nu}(x, t) \) is divergence-less (\( \partial_\nu B^{\mu\nu} = 0 \)), and satisfies the zero boundary condition: \( B^{\mu\nu}(x, t) = 0 \) on the boundary \((x, y, z = \pm \pi)\) for \(-\infty < t < +\infty\); thus the pre-assumption of corrected Møller’s theorem is satisfied. On the other hand, \( \int_{V: t=\text{const}} B^{\mu\nu}(x, t) d^3x = 0 \) holds for \( \mu = 1, 2, 3, 4 \) and \( j = 1, 2, 3 \); thus \( B^{\mu\nu}(x, t) \) also satisfies the sufficient and necessary condition Eq. \([12] \) for the corrected Møller’s theorem. Accordingly, \( \int_{V: t=\text{const}} B^{\mu\nu}(x, t) d^3x = (0, 0, 0, \pi^3) \neq 0 \) is a four-vector — the corrected Møller’s theorem may define a non-zero four-vector in general.

**Conclusion for corrected Møller’s theorem.** In conclusion, the corrected Møller’s theorem may define a non-zero four-vector in general; however, it only defines a trivial zero four-vector for an EM stress–energy tensor of a finite closed physical system. Thus the application of the theorem is limited.

**Differences between three four-vector theorems.** We have three four-vector theorems: Theorem 1 and corrected Møller’s theorem (both presented in the present paper), and generalized von Laue’s theorem (presented in Ref. \[6\]). For the convenience to compare, we write down the generalized von Laue’s theorem from Ref. \[6\] as follows.

**Generalized von Laue’s theorem.** Assume that \( \Theta^{\mu\nu}(x) \) is an integrable Lorentz four-tensor, defined in the domain \( V \) in the laboratory frame XYZ, where \( \mu, \nu = 1, 2, 3, 4 \), with the index 4 corresponding to time component, \( V \) including its boundary is at rest in XYZ, and \( \Theta^{\mu\nu} \) is independent of time (\( \partial \Theta^{\mu\nu}/\partial t = 0 \)). The generalized von Laue’s theorem states: The time-cum-moment-space integrals \( P^\mu = \int_V \Theta^{\mu\nu}(x) d^3x \) constitute a Lorentz four-vector if and only if \( \int_V \Theta^{\mu\nu}(x) d^3x = 0 \) holds for all \( \mu = 1, 2, 3, 4 \) and \( j = 1, 2, 3 \).

Between the corrected Møller’s theorem and the above generalized von Laue’s theorem, the difference is that in the corrected Møller’s theorem, the divergence-less (\( \partial_\nu \Theta^{\mu\nu} = 0 \)) plus a zero boundary condition (\( \Theta^{\mu\nu} = 0 \) on boundary) is taken as a pre-assumption, and \( \Theta^{\mu\nu}(x, t) \) is allowed to be time-dependent, while in the generalized von Laue’s theorem, \( \partial \Theta^{\mu\nu}/\partial t = 0 \) is taken as a pre-assumption, and \( \Theta^{\mu\nu}(x, t) = \Theta^{\mu\nu}(x) \) is not allowed to be time-dependent, but no boundary condition is required. Compared with the corrected Møller’s theorem and the generalized von Laue’s theorem, Theorem 1 does not have any pre-assumption; however, the three theorems have the same definition \( P^\mu \), as shown below.

From Eq. \([36] \), we know that the same definition of \( P^\mu \) is used in both Theorem 1 and the corrected Møller’s theorem, given by

\[
P^\mu = \int_{V': t', t=\text{const}} \Theta^{\mu\nu}(x = x(x', t'), t) d^3x' = \frac{\partial X^\mu}{\partial X^i} \frac{\partial X^i}{\partial X^\nu} \Theta^\nu(\pi) \int_{V: t=\text{const}} \int_{V: t=\text{const}} \Theta^\nu(\pi) d^3x. \quad (45)
\]

If \( \Theta^{\mu\nu}(x, t) \) is independent of \( t \), namely \( \Theta^{\mu\nu}(x, t) = \Theta^{\mu\nu}(x) \), then the above Eq. \([45] \) becomes

\[
P^\mu = \int_{V': t', t=\text{const}} \Theta^{\mu\nu}(x = x(x', t'), t) d^3x' = \frac{\partial X^\mu}{\partial X^i} \frac{\partial X^i}{\partial X^\nu} \Theta^\nu(\pi) \int_{V: t=\text{const}} \Theta^\nu(\pi) d^3x. \quad (46)
\]

This is exactly the case of von Laue’s theorem presented in Ref. \[6\], where \( \Theta^{\mu\nu}(x = x(x', t')) \) is written as \( \Theta^{\mu\nu}(x', c't') \), and \( t \) does not show up.

**Adaptability of Theorem 1.** Since Theorem 1 does not have a pre-assumption, it may have a better adaptability. To show this, a specific example is given below.

Suppose that there is a symmetric Lorentz four-tensor defined in the cubic domain \( V (-\pi \leq x, y, z \leq \pi) \) with its boundary \( S (x, y, z = \pm \pi) \).

From Eq. \([47] \), we know that

(i) \( \partial R^{\mu\nu}/\partial t = 0 \) holds but \( R^{\mu\nu}(x, t) \) does not have a zero-boundary condition (\( R^{44}(x, t) = \pi \neq 0 \) on the boundary: \( x = \pi \) and \( -\pi \leq y, z \leq \pi \), for example). Thus the corrected Møller’s theorem does not apply.

(ii) \( \partial R^{\mu\nu}/\partial t = 0 \) does not hold, because of \( \partial R^{44}/\partial t = -c \neq 0 \). Thus the generalized von Laue’s theorem does not apply either.

(iii) \( \int_{V: t=\text{const}} R^{33}(x, t) d^3x = 0 \) for \( \mu = 1, 2, 3, 4 \) and \( j = 1, 2, 3 \) holds, satisfying the sufficient and necessary condition Eq. \([10] \) of Theorem 1. Thus \( N^\mu = \int_{V: t=\text{const}} R^{33}(x, t) d^3x = \int_{V: t=\text{const}} (x, 0, 0, -ct) d^3x = (0, 0, 0, -\pi^3 ct) \) is a Lorentz four-vector.

From above example we see that Theorem 1 has a better adaptability. It is interesting to indicate that Theorem 1 can be used to analyze the EM stress–energy tensor for a plane light wave in a dielectric medium, as shown in Appendix \[B\].
IV. INVALIDITY OF WEINBERG’S CLAIM

In relativistic electrodynamics, there are two mainstream arguments for the Lorentz invariance of total electric charge. One of them comes from an assumption that the total electric charge is an experimental invariant, as presented in the textbook by Jackson [3, p.555]; the other comes from a well-accepted “invariant conservation law” that the divergence-less of current density four-vector makes the total charge be a Lorentz scalar, as claimed in the textbook by Weinberg [2] p.41. In this section, by enumerating a counterexample we use Theorem 3 to disprove Weinberg’s claim.

For a physical system defined in the domain $V$ with $S$ as its closed boundary, we will show that the divergence-less ($\partial_\mu J^\mu = 0$) of current density four-vector $J^\mu = (J, c\rho)$ plus a boundary zero-integral given by $\oint_S J(x,t) \cdot dS = 0$ makes the total charge $Q$ in $V$ be a time-independent constant; however, it is not enough to make the constant be a Lorentz scalar.

**Constant of total electric charge.** From $\partial_\mu J^\mu = 0 \Rightarrow \nabla \cdot J + \partial \rho / \partial t = 0$, with $Q = \int_V \rho d^3x$ taken into account we have $\oint_S J(x,t) \cdot dS + dQ / dt = 0$. If $\oint_S J(x,t) \cdot dS = 0$ holds, then we have $dQ / dt = 0 \Rightarrow Q = \text{const}$. Physically, the current density $J = u \rho$ is a charge density flow, where $u$ is the charge moving velocity, and $\oint_S J(x,t) \cdot dS = 0$ means that there is no net charge flowing into or out of $V$. Thus the total electric charge $Q$ in $V$ is constant; however, it never means that this constant is a Lorentz invariant. (It should be emphasized that only from $\partial_\mu J^\mu = 0$ without $\oint_S J(x,t) \cdot dS = 0$ considered, one cannot derive $Q = \text{const}$ in $V$.)

Why is the constant total charge $Q$ in $V$ in above, resulting from $\partial_\mu J^\mu = 0 \Rightarrow \nabla \cdot J = -\partial \rho / \partial t$, not a Lorentz invariant? That is because $Q$ in $V$ in frame $XYZ$ and $V'$ in $V'$ in frame $X'Y'Z'$ do not refer to the same total charge in the same volume physically, where $X'Y'Z'$ moves at $\beta c \neq 0$ relatively to $XYZ$. This conclusion directly comes from the fact that, the volume $V$ in $\int_V (\partial \rho / \partial t) d^3x$ is fixed in $XYZ$, and the volume $V'$ in $\int_{V'} (\partial \rho' / \partial t') d^3x'$ is fixed in $X'Y'Z'$ so that $\partial \rho / \partial t$ and $\partial \rho' / \partial t'$ and $\partial \rho / \partial t'$ are exchangeable, respectively, as shown in Eq. (32), in order to make both $\int_V (\partial \rho / \partial t) d^3x = dQ / dt$ and $\int_{V'} (\partial \rho' / \partial t') d^3x' = dQ' / dt'$ hold in general. However according to Einstein’s relativity [20], $V$ fixed in $XYZ$ is moving observed in $X'Y'Z'$. Now $V'$ in $\int_{V'} (\partial \rho' / \partial t') d^3x'$ is at rest in $X'Y'Z'$, and thus $V$ and $V'$ do not denote the same volume, and $Q$ and $Q'$ do not denote the same total charge. Therefore, the continuity equation $\partial_\mu J^\mu = 0$ cannot be taken as the “invariant [charge] conservation law” in relativity [2] p.40, which can be better seen from the counterexample below.

**Counterexample of Weinberg’s claim.** Why is $\partial_\mu J^\mu = 0$ not a sufficient condition to make $Q = \text{const}$ be a Lorentz scalar, even additionally plus a zero-boundary condition $J^\mu = (J, c\rho) = 0$ on $S$ ? To understand this, consider a mathematical four-vector, given by

$$W^\mu(x,t) = (W, W^4) = (f(x), 0, 0, -(ct) f_x)$$

defined in the cubic domain $V (\pi \leq x, y, z \leq \pi)$, where $W = (f(x), 0, 0)$, $W^4 = -(ct) f_x$, and $f(x) = (\sin x)^2 (\sin y)^2 (\sin z)^2$, with $\int_V f(x) d^3x = \pi^3$. $W^\mu(x,t)$ satisfies the zero-boundary condition, namely $W^\mu = (W, W^4) = 0$ holds on the boundary $S (x, y, z = \pm \pi)$.

$W^\mu$ is divergence-less, namely $\partial_\mu W^\mu = 0$, and in addition, $W^\mu$ has a zero-boundary condition $\Rightarrow \oint_S W \cdot dS = 0$ holds. According to Weinberg’s claim, $\partial_\mu W^\mu = 0$ makes $\int W^4 d^3x$ a Lorentz scalar.

However because of $\int_{V': t = \text{const}} W d^3x = (\pi^3, 0, 0) \neq 0$, $W^\mu$ does not satisfy the sufficient and necessary condition Eq. (23) of Theorem 3. Thus according to Theorem 3, $\Phi = \int_{V': t = \text{const}} W d^3x = \int_{V': t = \text{const}} - (ct) f_x d^3x (x = 0)$ is not a Lorentz scalar. To better understand this, from Eq. (28) we have

$$\Phi' = \Phi - \beta \cdot \int_{V': t = \text{const}} W d^3x = \Phi - \beta x \cdot \pi^3,$$

and $\Phi' = \Phi$ cannot hold for any $\beta x \neq 0$. Thus Weinberg’s claim is disproved, namely $\partial_\mu W^\mu = 0$ is not a sufficient condition to make $\int W^4 d^3x$ be a scalar.

In the above counterexample, $\partial_\mu W^\mu = 0$ holds but $\int W^4 d^3x = 0$ does not hold. Thus in general, the sufficient condition $\partial_\mu N^\mu = 0$ of Weinberg’s claim does not includes the sufficient and necessary condition $\int A d^3x = 0$ of Theorem 3. The current density four-vector $J^\mu = (J, c\rho)$ and the above counterexample $W^\mu = (W, W^4)$ are all divergence-less, while the Lorentz property of their time-element space integrals does not depends on the divergence-less. Now let us take a look of four-vectors that are not divergence-less, and see what difference they may have.

**Example 1.** Consider a four-vector, given by $\Gamma^\mu = (\Gamma, \Gamma^4) = (\sin x \sin y \sin z, 0, 0, 0)$ defined in the cubic domain $V (\pi \leq x, y, z \leq \pi)$, with $\partial_\mu \Gamma^\mu = \cos x \sin y \sin z \neq 0$ holding except for some individual discrete points, and $\int_{V': t = \text{const}} \Gamma d^3x = (0, 0, 0) \neq 0$ holding. According to Theorem 3, $\Phi = \int_{V': t = \text{const}} \Gamma d^3x = (0, 0, 0)$ is a Lorentz scalar, because $\Phi' = \Phi - \beta \cdot \int_{V': t = \text{const}} \Gamma d^3x = \Phi$. Put it simply, for $\partial_\mu \Gamma^\mu \neq 0$, $\int \Gamma d^3x$ is a Lorentz scalar.

**Example 2.** Consider a four-vector, given by $U^\mu = (U, U^4) = (\sin^2 x \sin^2 y \sin^2 z, 0, 0, 0)$ defined in the cubic domain $V (\pi \leq x, y, z \leq \pi)$, with $\partial_\mu U^\mu = 2 \cos x \sin x \sin^2 y \sin^2 z \neq 0$ holding except for some individual discrete points, and $\int_{V: t = \text{const}} U d^3x = (\pi^3, 0, 0) \neq 0$ holding. According to Theorem 3, $\Phi = \Phi'$.


\[ \int_{V; \, t = \text{const}} U^4 \, d^3 x \, (\equiv 0) \] is not a Lorentz scalar, because \( \Phi' = \Phi - \beta \cdot \int_{V; \, t = \text{const}} U^3 d^3 x = \Phi - \beta_x x^3 \Rightarrow \Phi' \neq \Phi \) for any \( \beta_x \neq 0 \). Put it simply, for \( \partial_\mu U^\mu \neq 0 \), \( \int U^4 d^3 x \) is not a Lorentz scalar.

From above Example 1 and Example 2, we know that \( \int \Gamma^4 d^3 x \) is a Lorentz scalar for \( \partial_\mu \Gamma^\mu \neq 0 \), and \( \int U^4 d^3 x \) is not a Lorentz scalar for \( \partial_\mu U^\mu \neq 0 \). We have known that \( \int J^4 d^3 x \) is a Lorentz scalar for \( \partial_\mu J^\mu = 0 \) \[6\], while the counterexample of Weinberg’s claim tells us that \( \int W^4 d^3 x \) is not a Lorentz scalar for \( \partial_\mu W^\mu = 0 \). Thus we can generally conclude that whether the time-element space integral of a four-vector is a Lorentz scalar has nothing to do with the divergence-less property of the four-vector.

The invariance problem of total electric charge has been resolved by using “derivative von Laue’s theorem” in Ref. \[6\], which indicates that the invariance comes from two facts: (a) \( J^\mu \) is a four-vector and (b) the moving velocity of any charged particles is less than vacuum light speed. This strict theoretical result removes the assumption that the total charge is an experimental invariant \[4, p.555\].

The difference between the derivative von Laue’s theorem \[6\] and Theorem 3 is that the derivative von Laue’s theorem has a pre-assumption of \( \partial A^\mu / \partial t \equiv 0 \), namely \( A^\mu = (A, A^t) \) is not allowed to be time-dependent, while Theorem 3 does not. For example, we also can use the derivative von Laue’s theorem \[6\] to analyze the four-vector \( \Gamma^\mu = (\Gamma, \Gamma^t) = (\sin x \sin y \sin z, 0, 0, 0) \) discussed above because \( \partial \Gamma^\mu / \partial t \equiv 0 \) holds, but we cannot use it to analyze \( W^\mu (x, t) \) given by Eq. (38), because \( \partial W^\mu / \partial t \equiv 0 \) does not hold. Thus Theorem 3 has a better adaptability.

V. REMARKS AND CONCLUSIONS

In this paper, we have developed Lorentz four-vector theorems (Theorem 1 for column four-vector and Theorem 2 for row four-vector; they are essentially the same) and Lorentz scalar theorem (Theorem 3). Based on Theorem 1, we find that the well-established Møller’s theorem is fundamentally wrong, and we provided a corrected version of Møller’s theorem (see Sec. III). Based on Theorem 3, we disproved Weinberg’s claim, and obtained a general conclusion for the Lorentz property of a four-vector’s time-element space integral (see Sec. IV).

We have shown that the sufficient condition of Møller’s theorem makes the time-column space integrals of a tensor be time-independent constants; however, it is not a sufficient condition to make the integrals constitute a Lorentz four-vector. The corrected Møller’s theorem has a limited application; especially for Minkowski EM stress–energy tensor, the corrected Møller’s theorem only defines a trivial zero four-vector.

We have shown that there are three four-vector theorems: (a) generalized von Laue’s theorem; (b) corrected Møller’s theorem; and (c) Theorem 1. The generalized von Laue’s theorem, presented in Ref. \[6\], has a pre-assumption that tensor \( \Theta^{\mu \nu} \) is required to be time-independent \( (\partial \Theta^{\mu \nu} / \partial t \equiv 0) \). The corrected Møller’s theorem, provided in the present paper, also has a pre-assumption that tensor \( \Theta^{\mu \nu} \) is required to be divergence-less \( (\partial_\nu \Theta^{\mu \nu} = 0) \) and required to satisfy a zero boundary condition \( (\Theta^{\mu \nu} = 0 \text{ on boundary}) \) but \( \Theta^{\mu \nu} \) is allowed to be time-dependent. Compared with the generalized von Laue’s theorem and corrected Møller’s theorem, Theorem 1 does not have any pre-assumption, while the three theorems have the same sufficient and necessary condition. Thus Theorem 1 has a better adaptability, as shown by a specific example described by Eq. (47) in Sec. III.

However it should be noted that, just because the generalized von Laue’s theorem has a pre-assumption of \( \partial \Theta^{\mu \nu} (x, t) / \partial t \equiv 0 \) (but no boundary condition required) and the corrected Møller’s theorem has a pre-assumption of \( \partial_\nu \Theta^{\mu \nu} (x, t) = 0 \) plus \( \Theta (x, t) = 0 \) on boundary (zero boundary condition), the four-vector \( P^\mu = \int_{V; \, t = \text{const}} \Theta^{\mu \nu} (x, t) d^3 x \) defined by the two theorems is time-independent \( (\partial P^\mu / \partial t \equiv 0) \). Thus the generalized von Laue’s theorem and the corrected Møller’s theorem can be taken as “conservation laws” in a traditional sense.

We also have shown that there are three incorrect four-vector theorems: (i) Møller’s theorem, which is also called “Møller’s version of Laue’s theorem” in Ref. \[6\]; (ii) Landau-Lifshitz version of Laue’s theorem; and (iii) Weinberg’s version of Laue’s theorem. Møller’s version is disproved in the present paper by taking the mathematical tensor Eq. (38) as a counterexample, while Landau-Lifshitz and Weinberg’s versions are disproved in Ref. \[6\] by taking the EM tensor of a charged metal sphere in free space as a counterexample. All the sufficient conditions of the three disproved versions of Laue’s theorem include the divergence-less of a tensor \( (\partial_\nu \Theta^{\mu \nu} (x, t) = 0) \), but they do not include or derive the sufficient and necessary condition Eq. (10). Thus it is not appropriate that \( \partial_\nu \Theta^{\mu \nu} (x, t) = 0 \) is recognized to be “conservation Law” \[18\, p.310\] or to “guarantee conservation of the total 4-momentum” \[21\, p.443\] in traditional textbooks.

The most convincing way to disprove a mathematical conjecture is to provide its counterexample. The counterexample Eq. (38) for Møller’s version of Laue’s theorem, \( A^{\mu \nu} \), is also a counterexample of Landau-Lifshitz and Weinberg’s versions of Laue’s theorem, because \( A^{\mu \nu} \) is both divergence-less \( (\partial_\nu A^{\mu \nu} = 0) \) and symmetric \( (A^{\mu \nu} = A^{\nu \mu}) \), while the Landau-Lifshitz version takes the divergence-less of a tensor as a sufficient condition, and the Weinberg’s version takes the divergence-less plus a symmetry of a tensor as a sufficient condition. In other words, \( A^{\mu \nu} \) given by Eq. (38) is a common mathematical counterexample to disprove Møller’s, Landau-Lifshitz, and Weinberg’s versions of Laue’s theorem which are all mathematical conjectures, independent of physics although originating from physics, because all physical implications have already been turned into mathematical descriptions.
There is a well-known result in the dynamics of relativity that if a Lorentz four-vector is divergence-less, then the time-element space integral of the four-vector is a Lorentz scalar; for example, Weinberg claims that “for any conserved four-vector”, namely for any four-vector \( J^\mu = (J, \epsilon p) \) that satisfies equation \( \partial_\mu J^\mu = 0 \),
\[
e Q = \int J^4 d^3x = \int \epsilon c p d^3x \text{ “defines a time-independent scalar”} \ [2, p.41], and Møller also claims a proof of such a result in his textbook \ [3, p.168]. However in the present paper, we have shown based on Theorem 3 in Sec. IV that this well-known result is not correct. We have found a general conclusion for the Lorentz property of a four-vector’s time-element space integral, stating that whether the time-element space integral of a four-vector is a Lorentz scalar has nothing to do with the four-vector’s divergence-less property. Accordingly, it is not appropriate for \( \partial_\mu J^\mu (x, t) = 0 \) to be called “invariant conservation Law” \ [2, p.40] or “the law of charge conservation” \ [21, p.559, p.443] in textbooks.

As presented in traditional textbooks, the local conservation law of energy–momentum in general relativity, given by \( T^{\mu\nu} = 0 \) \ [22, p.298], is covariantly generalized from the conservation law \( T^{\mu\nu} = \partial_\mu T^{\mu\nu} = 0 \) in special relativity \ [1, p.83] \ [2, p.45] \ [24]. Thus the validity of the latter is a necessary condition for the validity of the former. This law is often used for relativistic analysis of the Abraham-Minkowski debate on the momentum of light in a medium \ [9–11], and it is also thought to play an important role in gravitation theory \ [21, p.132], because “the GR [general relativity] field equations should be consistent with energy and momentum conservation, \( T^{\mu\nu} = 0 \)” \ [22, p.299]. This law is so well-established that often no citations are given for its origin in research articles \ [27–30]. However in fact, the traditional conservation laws including both \( \partial_\mu A^\mu = 0 \) and \( \partial_\nu T^{\mu\nu} = 0 \) \ [21, p.443] can be directly disproved by simple physical counterexamples, as shown in Appendix C, although they are claimed to have been already proved with the use of the modern language of exterior calculus \ [19, p.318]. Thus clarifying the two conservation laws in the present paper has a general significance.

In conclusion, we have generally resolved two fundamental issues in the dynamics of relativity: (a) Under what condition, the time-column space integrals of a Lorentz four-tensor constitute a Lorentz four-vector (Theorem 1), and (b) under what condition, the time-element space integral of a Lorentz four-vector is a Lorentz scalar (Theorem 3). Both Theorem 1 and Theorem 3 have their own sufficient and necessary conditions, which have nothing to do with the divergence-less (\( \partial_\mu \Theta^{\mu\nu} = 0 \) or \( \partial_\mu A^\mu = 0 \)) symmetry (\( \Theta^{\mu\nu} = \Theta^{\nu\mu} \)), and boundary conditions. This point is especially important. For example, from Theorem 1 we can directly judge that Møller’s theorem is incorrect, because the sufficient condition of Møller’s theorem does not include the sufficient and necessary condition of Theorem 1, as shown in counterexample Eq. 35. A similar argument is applicable to the “invariant conservation law” claimed by Weinberg \ [2, p.40], as shown in counterexample Eq. (48).

As a practical application, we have used Theorem 1 to analyze Minkowski EM tensor for a plane light wave in a moving medium with Einstein’s light–quantum hypothesis taken into account, and we find that the four-momentum of quasi-photon and the Lorentz invariance of Planck constant can be naturally derived (see Appendix D). Einstein’s light–quantum hypothesis is the basis of Bohr frequency condition of radiation from atomic transitions, the invariance of Planck constant is an implicit postulate in Dirac relativistic quantum mechanics \ [7], and the existence of four-momentum of the quasi-photon is required by momentum–energy conservation law in Einstein-box thought experiment \ [23]. Thus this natural derivation is compatible with the quantum theory and the momentum–energy conservation law, and the result obtained strongly supports the Minkowski tensor as being the most qualified momentum–energy tensor for descriptions of light–matter interactions in the frame of the principle of relativity.

In relativity, charge conservation law refers to that the total charge is a time- and frame-independent constant \ [2, p.41], as demonstrated in Eq. (A6) of Appendix A, while energy-momentum conservation law refers to that the total energy and momentum constitute a covariant and time-independent four-vector \ [2, p.46], as demonstrated for a plane light wave in Appendix D. The problems of \( \partial_\mu A^\mu = 0 \) and \( \partial_\nu T^{\mu\nu} = 0 \) as being conservation laws in relativity were first discovered in \ [1,7], and they are explicitly illustrated and generally resolved in the present paper. One might argue that in the establishment of the positive mass theorems in general relativity, rigorous mathematics has been used and loopholes possibly left in earlier works do not exist. Unfortunately, this is not true. These problems never got any attention in the proofs of the positive mass theorem by Schoen and Yau in 1979 \ [32] and 1981 \ [39], Witten \ [33] and Nester \ [24] in 1981, Parker and Taubes in 1982 \ [35], and Gibbons and coworkers in 1983 \ [36]. All the proofs are based on a flawed theoretical framework set up by Arnovitt, Deser, and Misner \ [37,38], where the total energy–momentum \( P^\mu \) in an asymptotically flat spacetime is required to “obey the differential conservation law \( T^{\mu\nu} = 0 \) [of the energy–momentum tensor \( T^{\mu\nu} \)] and required to “transform as a four-vector” \ [38]. However in fact, the conservation law \( T^{\mu\nu} = 0 \) cannot guarantee that \( P^\mu \) is a four-vector, as shown in Appendix D. Apparently, this problem is ignored in all the proofs \ [32,33,39]. Moreover, the equations \( T^{\mu\nu} = 0, T^{\mu\nu} = 0, T^{\mu\nu} = 0 \), or \( T^{\mu\nu} = 0 \) have been always claimed as conservation law in later literature, such as in Ref. \ [28] by Ratra and Peebles in 1988, Ref. \ [27] by Witten in 1991, Ref. \ [22, p.299] by Rindler in 2006, and Ref. \ [29] by Bčák and Schmidt in 2016. From this we can see that the community in the field of relativity has never become aware of these problems before the publications of \ [6,7].
Appendix A: What is the correct technique for change of variables in triple integrals?

In this Appendix, we provide a proof of why the hyperplane differential-element four-vector contradicts both the principle of classical mathematical analysis and the principle of relativity, although it is Lorentz covariant and widely accepted in the community [4, p. 757].

There is no ambiguity technically for how to calculate a triple integral over a volume that is at rest in an inertial frame. However when the volume is moving, the important ambiguity that has to be solved is how to define the moving volume. In Einstein’s special relativity [20], a moving volume is defined as the set of all space points, observed at the same time, contained in the volume. This definition is essentially the same as that of the length of a moving ruler. The length of a moving ruler is defined in terms of the coordinates, measured at the same time, of the two ends of the ruler. In the ruler-rest frame, the coordinates of the ends of the ruler are not dependent on the times when to measure the ends respectively. Thus this definition has no contradiction against the relativity of simultaneity.

In relativistic electrodynamics, there are two techniques for a change of variables in space (triple) integrals. The first technique is based on the change of variables theorem, as presented in mathematical analysis [14], and used in Laue’s original paper to derive Laue’s theorem [12], and also used to develop my theory in the present paper and the previous work [6]; called Technique-I for convenience. In this technique, the change of variable formula from the laboratory frame $XYZ$ to the moving frame $X’Y’Z’$ is given by

$$\text{d}^3x = \frac{\partial(x, y, z)}{\partial(x’, y’, z’)} \frac{\partial(x’, y’, z’)}{\partial(x, y, z)} \text{d}^3x’$$  \hspace{1cm} (A1)

where $\partial(x, y, z)/\partial(x’, y’, z’)$ is Jacobian determinant; $\text{d}^3x$ is the proper differential element fixed in $XYZ$; $\text{d}^3x’$ is the corresponding moving element in $X’Y’Z’$. In obtaining $\partial(x, y, z)/\partial(x’, y’, z’)$ from $x = x(x’, t’)$ given in Eq. (3), the time $t’$ is taken to be a constant, because the volume element $\text{d}^3x’$ is measured at the same time in $X’Y’Z’$, as mentioned above.

The second technique is based on the hyperplane differential-element four-vector, as presented in some respected textbooks, such as the book by Jackson [4, p. 757]; called Technique-II for convenience. In Technique-II, the integral domain $V$ is assumed to be at rest in the laboratory frame, and then similar to the classical particle’s four-momentum $= \text{proper mass multiplied by its four-velocity}$, a differential-element four-vector (= proper element multiplied by its four-velocity normalized to light speed $c$) is constructed to define the change of variables in space integrals, given by

$$\text{d}S_{\mu} = (-\text{d}S, \text{d}S) = (0, 0, 0, 1) \text{d}^3x$$  \hspace{1cm} (A2)

in the laboratory frame $XYZ$, and

$$\text{d}S’_{\mu} = (-\text{d}S’, \text{d}S’) = (\gamma\beta, \gamma) \text{d}^3x$$  \hspace{1cm} (A3)

in the frame $X’Y’Z’$ moving at $\beta c$ with respect to $XYZ$, where $\gamma = (1 - \beta^2)^{-1/2}$; thus leading to

$$\text{d}S’_{\mu} = \gamma \text{d}^3x’ = \gamma \text{d}S_4$$  \hspace{1cm} (A4)

—the change of variable formula for Technique-II. Note: $\text{d}S’_{\mu} = (\text{d}S’, \text{d}S’)$ means that $\text{d}^3x$ is fixed in $XYZ$, while $\text{d}S’_{\mu} = (\text{d}S’, \text{d}S’)$ = $((\gamma\beta’, \gamma)) \text{d}^3x$ means that $\text{d}^3x’ = \gamma \text{d}^3x$ moves at $\beta’ c = -\beta c$ in $X’Y’Z’$, and $(\gamma\beta’, \gamma)$ is its normalized four-velocity.

Technique-II is widely accepted in the community as a strong basis to define EM momentum-energy four-vector, [4, p. 757]. Unfortunately, Technique-II is fundamentally flawed, as shown below.

Because $\text{d}^3x$ is fixed in $XYZ$ while $\text{d}^3x’$ is moving in $X’Y’Z’$, the size of $\text{d}^3x$ is independent of what times to measure the boundary points of $\text{d}^3x$ respectively, while the size of $\text{d}^3x’$ has to be determined by measuring the boundary points of $\text{d}^3x’$ at the same time $t’$, according to Einstein [20]. Thus we have $\partial(x, y, z)/\partial(x’, y’, z’)|_{t’=\text{const}} = \gamma$. Then inserting it into Eq. (A1) we have

$$\text{d}^3x’ = \frac{\text{d}^3x}{\gamma}.$$  \hspace{1cm} (A5)

Obviously, Lorentz contraction is imposed on $\text{d}^3x’$, just like a moving ruler.

However according to Eq. (A4), Technique-II requires $\text{d}^3x’ = \gamma \text{d}^3x$; thus Technique-II contradicts both Technique-I and the effect of Lorentz contraction in Einstein’s special relativity. More seriously, Technique-II directly contradicts Lorentz invariance of total charge; in other words, if Technique-II were used, then a non-zero total charge would not be a Lorentz invariant, which is shown below.

Total charge $Q$ = invariant in Technique-I. First we show that Lorentz invariance of total charge is always valid in Technique-I. Without loss of generality, we assume that a charge distribution is created by charged particles which move at the same velocity, otherwise it can be treated discretely, as shown in Ref. [4]. According to special relativity, there must exist an inertial frame $XYZ$ where the charged particles are at rest. Thus in the charge-rest frame $XYZ$ (taken as the laboratory frame), the total charge can be formulated as $Q = \int_V \rho(x) \text{d}^3x$, where $\rho(x)$ with $\partial\rho/\partial t = 0$ is the charge distribution, $V$ is the volume at rest, and $Q$ is the (time-independent) total charge in $V$. In such a case, all charged particles are stationary and frozen in $V$ so that no current exists ($J = 0$). Observed in a frame $X’Y’Z’$ moving at $\beta c \neq 0$ with respect to the charge-rest frame $XYZ$, the volume $V’$ is moving at $\beta’ c = -\beta c$, but there are no charged particles crossing through the boundary of $V’$ although the current $J’ = -\gamma \beta \rho \neq 0$ holds. In Technique-I, as shown in Eq. (A5), the change of variable formula is given by $\text{d}^3x’ = \text{d}^3x/\gamma$. From this, with $\rho’ = \gamma(\rho - \beta \cdot J)$ and


J = 0 taken into account, we have
\[
cQ' := \int_V (cp')d^3x' = \int_V \gamma (cp - \beta \cdot J)(d^3x/\gamma) = \int_V (cp)d^3x = cQ
\]

= invariant. \hspace{1cm} (A6)

Thus we finish the proof that the total charge is always a Lorentz invariant in Technique-I.

Total charge Q is non-invariant in Technique-II. Now we show why Technique-II contradicts the Lorentz invariance of total charge. In the laboratory frame XYZ (charge-rest frame), the four-current density is given by
\[
J^\mu = (J, J^4)
\]

with J = 0 and J^4 = c\rho, and the total charge Q is defined by cQ = \int_V (cp)d^3x = \int_V J^4dS_4.

Observed in the frame X'Y'Z' moving at \beta c with respect to the laboratory frame XYZ, we have
\[
J'^\mu = (J', J'^4)
\]

with J' = -\gamma \beta J^4 and J'^4 = c\rho' = \gamma J^4, and the total charge Q' is defined by cQ' = \int_{V'} (cp')d^3x' = \int_{V'} J'^4dS_4'.

According to Eq. (A2) and Eq. (A3) of Technique-II, we have
\[
\begin{align*}
\int_V J^4dS_4 &= \int_{V'} J'^4dS'_4 = \int_{V'} J' \cdot dS' \\
&= \int_{V'} J'^4dS'_4 - \int_{V'} (-\gamma \beta J^4) \cdot (-\gamma \beta dS_4) \\
&= \int_{V'} J'^4dS'_4 - (-\gamma \gamma')^2 \int_{V'} J^4dS_4 \\
&\Rightarrow (1 + (\gamma \beta)^2) \int_{V'} J'^4dS_4 = \int_{V'} J^4dS_4' \\
&\Rightarrow \gamma^2 cQ = cQ' \Rightarrow Q \neq Q'
\end{align*}
\]

if Q \neq 0 holds, where \beta \neq 0 is assumed \Rightarrow \gamma > 1. Thus we finish the proof that a non-zero total charge (Q \neq 0) is not a Lorentz invariant in Technique-II.

So far we have shown that total charge Q \neq 0 is never a Lorentz invariant in Technique-II, while total charge Q is always a Lorentz invariant in Technique-I; both cases have nothing to do with the boundary conditions of J^\mu = (J, c\rho).

Conclusion. From above analysis we can conclude that Technique-II is based on the hyperplane differential-element four-vector, as presented in [4, p. 757], and it has three flaws: (i) contradicting the effect of Lorentz contraction in special relativity; (ii) contradicting the change of variables theorem in mathematical analysis [13]; and (iii) contradicting the Lorentz invariance of total charge.

To put it simply, Technique-II follows neither the principle of mathematical analysis nor the principle of relativity. Thus the Lorentz covariant hyperplane differential-element four-vector Eq. (A3) is only a mathematical four-vector, instead of a physical four-vector [24].

Two subtle issues for checking Lorentz invariance of total charge. In analysis of the Lorentz invariance of total charge in a volume in specific cases, a subtle issue is about how to define the volume. If there are charged particles crossing through the boundary of the volume, the total charge in the volume may not be Lorentz invariant [13], possibly leading to a doubt of the completeness of Maxwell EM theory [16]. Thus the correct volume is supposed to be moving at the same velocity as that of the charge, as argued above. Another subtle issue is how to correctly understand the definition of total charge. For example, by analysis of an infinite straight charged wire, Bilić puzzled that the standard definition Q = \int_V \rho d^3x and the so-called covariant definition Q = \int J^\mu dS_\mu (in units with c = 1) are not equivalent in general [13]: now we know that the problem turned out to be here: the transformation of triple integral \int_V \rho d^3x = \int J^4dS_4 = \int J^\mu dS_\mu contradicts the change of variables theorem in mathematical analysis [13], as shown above.

Appendix B: Natural derivation of four-momentum of quasi-photon and invariance of Planck constant

Quasi-photon carries momentum and energy and it is a macroscopic description of light-matter microscopic interactions [17]. Lorentz invariance of Planck constant is an implicit postulate in Dirac relativistic quantum theory [7]. In this appendix, we will apply Theorem 1 to Minkowski EM tensor for a plane light wave in a moving uniform medium, and find that the momentum–energy four-vector (four-momentum) of the quasi-photon and the Lorentz invariance of Planck constant can be naturally derived.

For a plane light wave propagating in a moving non-dispersive, isotropic, uniform medium, observed in the medium-rest frame, Minkowski quasi-photons characterizing the light-matter interactions move along the wave vector \mathbf{k}_\nu at the speed c/n, with n (> 1) the refractive index. In such a case, there is a photon-rest frame moving at the quasi-photon velocity relatively to the medium-rest frame [4]. Observed in the photon-rest frame XYZ (taken as the laboratory frame here), there are some fantastic EM phenomena to take place.

1. The EM fields \mathbf{E} = 0, \mathbf{H} = 0, \mathbf{D} = D_0 \cos \Psi, and \mathbf{B} = B_0 \cos \Psi hold, where D_0 \neq 0 and B_0 \neq 0 with \mathbf{D}_0 \perp \mathbf{B}_0 are the constant amplitudes, leading to EM energy density \( W_{\text{em}} = 0.5(\mathbf{D} \cdot \mathbf{E} + \mathbf{B} \cdot \mathbf{H}) = 0 \) and Abraham momentum \( \mathbf{g}_A = \mathbf{E} \times \mathbf{H}/c^2 = 0 \), but the Minkowski momentum \( \mathbf{g}_M = \mathbf{D} \times \mathbf{B} \neq 0 \).
2. The wave angular frequency \( \omega = 0 \) and the wave phase \( \Psi = (\omega t - k_x \cdot x) = (-k_x \cdot x) \) hold, with the wave vector \( k_x \neq 0 \), leading to all the EM fields behaving as static fields \([17]\).

The Minkowski EM stress-energy tensor \( \tilde{T}^{\mu \nu} \) is given by Eq. (43), with \( T_M = -DE - BH + IW_{\text{em}} = 0 \) and \( c g_{\alpha \beta} = 0 \) holding \( \Rightarrow \) the holding of \( \int_V: t = \text{const}, T_M d^3x = 0 \) and \( \int_V: t = \text{const}, g_{\alpha \beta} d^3x = 0 \) \( \Rightarrow \) the holding of \( \int_V: t = \text{const}, T_{\mu \nu} d^3x = 0 \) for \( \mu, j = 1, 2, 3, 4 \) and \( j = 1, 2, 3, 4 \). According to the sufficient and necessary conditions Eq. \([10]\) of Theorem 1, \( \tilde{T}^{\mu \nu} \)-time-column \( (= T^{\mu \nu} \)-time-row) space integrals

\[
P^\mu = c \int_{V': t' = \text{const}} \left( g_M(\Psi), \frac{W_{\text{em}}(\Psi)}{c} \right) d^3x \quad \text{(B1)}
\]

constitute a Lorentz four-vector, which is time-independent because of all the EM fields behaving as being static in the photon-rest frame \( \Rightarrow \partial T^{\mu \nu} / \partial t = 0 \), including \( \partial (T_M, W_{\text{em}} / c) / \partial t = 0 \).

Observed in the inertial frame \( X'Y'Z' \) moving relatively to the photon-rest frame \( XYZ \) at a velocity of \( \beta c \), we have

\[
P^\mu = c \int_{V': t' = \text{const}} \left( g_M'(\Psi'), \frac{W_{\text{em}}'(\Psi')}{c} \right) d^3x' \quad \text{(B2)}
\]

where the phase is given by \( \Psi' = (\omega' t' - k_x' \cdot x') \), and \( V' \) is moving at \( \beta' c = -\beta c \).

From the Lorentz transformation of EM field-strength tensors \([17]\) see Eqs. (3) and (4) there], with \( E = 0 \) and \( H = 0 \) taken into account we have

\[
E' = \gamma \beta \times B_c, \quad \text{(B3)}
\]
\[
H' = -\gamma \beta \times D_c, \quad \text{(B4)}
\]
\[
D' = \gamma D - \xi (\beta \cdot D) \beta, \quad \text{(B5)}
\]
\[
B' = \gamma B - \xi (\beta \cdot B) \beta, \quad \text{(B6)}
\]

where \( \gamma = (1 - \beta^2)^{-1/2} \) and \( \xi = (\gamma - 1) / \beta^2 \). From above Eqs. \( \text{(B3)-(B6)} \), we obtain

\[
\frac{E' \times H'}{W_{\text{em}}} = -\beta c = \beta' c. \quad \text{(B7)}
\]

called the energy velocity traditionally, where \( W_{\text{em}} = 0.5(D' \cdot E' + B' \cdot H') = D' \cdot E' = B' \cdot H' \) is the EM energy density. Because \( (E' \times H') / W_{\text{em}} = \beta' c \) given by above Eq. \( \text{(B7)} \) is the moving velocity of the volume \( V' \), seemingly it indeed looks like the velocity of energy transport. However in fact, it is not the real energy velocity in general. The real energy velocity is the phase velocity, required by both Fermat’s principle and energy conservation law which are physical postulates independent of Maxwell equations \([23]\). That is why \( \beta' c = (E' \times H') / W_{\text{em}} \) is called “photon apparent velocity” in Ref. \([17]\).

According to the Lorentz transformation of the wave four-vector \( K^\mu = (k_x, \omega / c) \), with \( \omega = 0 \) in \( XYZ \) taken into account, we have

\[
\omega = \gamma (\omega' - k_x' \cdot \beta' c) \Rightarrow \omega' = k_x' \cdot \beta' c. \quad \text{(B8)}
\]

As mentioned above, both Fermat’s principle and energy conservation law require the EM energy to propagate at the phase velocity \([7]\), given by

\[
\beta' c = \frac{\omega'}{|k_x'| |k_x'|} \Rightarrow \omega' = k_x' \cdot \beta' c. \quad \text{(B9)}
\]

From Eq. \( \text{(B8)} \) and Eq. \( \text{(B9)} \), we have

\[
k_x' \cdot \beta' c = k_x' \cdot \beta' c. \quad \text{(B10)}
\]

As indicated, the volume \( V' \) in Eq. \( \text{(B2)} \) moves at \( \beta' c \). If the same volume \( V' \) moves at \( \beta' c \), we label it as \( V_{\text{light}}' \), called “light volume” \([7]\). Observed in the photon-rest frame \( XYZ \), the volume \( V \) is at rest, and the light volume \( V_{\text{light}} \) is also at rest due to \( \omega = 0 \Rightarrow \beta' c = 0 \); namely \( V_{\text{light}} = V \).

Observed in \( X'Y'Z' \), from Eq. \( \text{(B10)} \) we find that \( \Psi' = (\omega' t' - k_x' \cdot x') \) takes the same value for \( x' = x_0' + \beta' ct' \) with \( x' \in V' \) and for \( x' = x_0 + \beta' phct' \) with \( x' \in V_{\text{light}}' \). Thus with \( V' \) in Eq. \( \text{(B2)} \), replaced by \( V_{\text{light}}' \), we obtain an equivalent expression, given by

\[
P^\mu = c \int_{V_{\text{light}}'}: t' = \text{const} \left( g_M'(\Psi'), \frac{W_{\text{em}}'(\Psi')}{c} \right) d^3x'. \quad \text{(B11)}
\]

The above Eq. \( \text{(B11)} \) is a time-independent and Lorentz covariant four-vector.

So far, with Theorem 1 applied to Minkowski EM tensor we have arrived at a conclusion that for a plane light wave in a medium, the total Minkowski EM momentum and energy contained in a light volume constitute a Lorentz four-vector.

The above result implies that observed in any inertial frame, all photons in a light volume are moving at the same velocity as that of the light volume and no photons cross the boundary of the volume; in other words, observed at the same time in all inertial frames, respectively, the photons in the light volume are the same photons. This Lorentz property of light volume was first formulated in Ref. \([7]\) see Eq. \( \text{(50)} \) there], directly based on the Lorentz covariance of EM field-strength tensors (instead of Minkowski EM tensor). This makes sense, because Minkowski tensor is a covariant combination of the EM field-strength tensors \([17]\) footnote \([7]\), and in principle, all physical results obtained from Minkowski tensor are already embodied in the EM field-strength tensors \([24]\).
Especially, if there is only one photon contained in the light volume \( V_{\text{light}} \), and Einstein light-quantum hypothesis is taken into account, namely

\[
\int_{V_{\text{light}}} W_{\text{em}} d^3x' = \hbar \omega', \quad (B12)
\]
then

\[
\int_{V_{\text{light}}} \mathbf{g}'_{\text{M}} d^3x' = \int_{V_{\text{light}}} (\mathbf{D}' \times \mathbf{B}') d^3x' = \int_{V_{\text{light}}} (W_{\text{em}}' / \omega') \mathbf{k}_o' d^3x' = \hbar k_w' \quad (B13)
\]
is the momentum of the photon in \( V_{\text{light}}' \), where the Lorentz invariant expression \( \mathbf{D}' \times \mathbf{B}' = (W_{\text{em}}' / \omega') \mathbf{k}_o' \) is used [7, see Eq. (37) there], and \( \hbar \) is the reduced Planck constant. Inserting Eq. (B12) and Eq. (B13) into Eq. (B11), we find that

\[
(\hbar \mathbf{k}_o', \mathbf{h}_o/c) = \hbar K'^\mu \quad (= P'^\mu/c) \quad (B14)
\]
is the photon’s four-momentum, where \( K'^\mu = (\mathbf{k}_o', \omega'/c) \) is a known (wave) four-vector. Since \( \hbar K'^\mu \) and \( K'^\mu \) are both four-vectors, \( h(g_{\mu \nu} K'^\mu X'^\nu) = \text{scalar and } (g_{\mu \nu} K'^\mu X'^\nu) = \text{scalar} \) must hold, where \( g_{\mu \nu} \) is the Minkowski metric and \( X'^\nu \) is the time-space four-vector. Note that \( (g_{\mu \nu} K'^\mu X'^\nu) \) is the scalar of phase, and it can be any real number. Thus the Planck constant \( \hbar \) must be a Lorentz invariant. This conclusion was first obtained in Ref. [7] directly from two EM field-strength tensors.

**Conclusion.** From above it is seen that the four-momentum of quasi-photon in a medium and the invariance of Planck constant are naturally obtained by applying Theorem 1 to Minkowski tensor, with taken into account Einstein light-quantum hypothesis that is the basis of Bohr frequency condition of atomic transitions in quantum theory. On the other hand, as shown in Einstein-box thought experiment [25], momentum–energy conservation law requires the quasi-photon to have a four-momentum. Thus the Minkowski tensor is compatible with the quantum theory of atomic light radiation and the momentum–energy conservation law, and it is the correct momentum–energy tensor for descriptions of light–matter interactions.

**Question 1:** Is the generalized von Laue’s theorem [6] applicable for identifying the Lorentz property of light momentum and energy for a plane light wave in a medium? The answer is yes, because its pre-assumption \( \partial T'^\mu / \partial t \equiv 0 \) is satisfied, as shown above.

**Question 2:** Is the corrected Møller’s theorem applicable for identifying the Lorentz property of light momentum and energy for a plane light wave in a medium?
\[ N_p = \frac{W_{em}}{\hbar \omega} = \left| c_0 E_0^2 / (\hbar \omega) \right| \cos^2 \Psi \] is a wave, dependent on time and space locations.) On the other hand, because of the relativity of simultaneity, photons may cross through the boundary of \( V \) at the same time in one frame, but these photons cannot cross through the boundary of \( V \) at the same time in other frames; thus leading to a result that the photons in \( V \) are not the same photons observed in different frames. That is why the total momentum and energy of the photons contained in \( V \) cannot constitute a four-vector in such a case.

Therefore, Theorem 1 only can be used to identify the Lorentz property of the total momentum and energy of materials or particles, which are moving at a velocity less than the vacuum light speed \( c \) so that there is a material-rest or particle-rest inertial frame, such as in the case for a plane light wave in a dielectric medium shown above, where Minkowski quasi-photon propagates at a velocity of \( c/n < c \)\(^{17}\). (Note: If a momentum–energy tensor is contributed by materials or particles which move at different velocities individually, then the tensor should be discretized so that each of the discretized tensors is contributed by the materials or particles which move at the same velocity, just like in the proof of the Lorentz invariance of total charge given in Ref. [6].)

It should be noted in free space, the invariance of Planck constant \( h \) and the covariance of wave four-vector \( K^\mu \) have been already proved in [7] and [20], respectively. Thus the four-momentum of photon in free space, equal to \( hK^\mu \), is a solved problem theoretically.

**Further specific explanation:** Why is
\[
\int_{V: t = \text{const}} c g_{\alpha\beta} d^3 x = \int_{V: t = \text{const}} (E \times H/c) d^3 x = 0
\]
never valid for a finite \( V \neq 0 \) for a plane wave in free space?

For a non-trivial plane wave in free space, observed in any inertial frames, the power flow or Poynting vector
\[
E \times H = E_0 \times H_0 \cos^2 \Psi \neq 0
\]
holds; otherwise, there are no energy flowing and no wave.

On the other hand, we have \( \int_{V: t = \text{const}} c g_{\alpha\beta} d^3 x > 0 \) holding; thus leading to the holding of \( \int_{V: t = \text{const}} c g_{\alpha\beta} d^3 x \neq 0 \) for a plane wave in free space.

Note: \( \int_{V: t = \text{const}} c g_{\alpha\beta} d^3 x > 0 \) comes from the fact that \( V \neq 0 \) is a finite 3D volume, and \( \cos^2 \Psi \geq 0 \) holds with the zero points only appearing on discrete planes, and thus there must exist a smaller volume \( V^* \subset V \), where \( \cos^2 \Psi > 0 \) exactly holds \( \Rightarrow \) the holding of \( \int_{V: t = \text{const}} c g_{\alpha\beta} d^3 x \geq \int_{V^*: t = \text{const}} c g_{\alpha\beta} d^3 x > 0 \).

### Appendix C: Physical counterexamples of Thirring’s claims

The most effective and convincing way to disprove a claim is to give its counterexample. In this appendix, we will provide two physical counterexamples of the claims made by Thirring.

In his book [19], with the help of exterior calculus Thirring claims:

(i) \( \partial_{\alpha} A^\alpha = 0 \) makes \( \int A^\mu d^3 x \) be a Lorentz scalar (namely “invariant conservation Law” claimed by Weinberg [21 p. 40]);

(ii) \( \partial_{\nu} T^{\mu\nu} = 0 \) makes \( \int T^{\mu\nu} d^3 x \) be a Lorentz four-vector (namely, Landau-Lifshitz version of Laue’s theorem [6]).

In both claims (i) and (ii), no boundary conditions are required. However, claim (i) can be directly disproved by a simple counterexample \( A^\mu = K^\mu \), and claim (ii) can be directly disproved by a simple counterexample \( T^{\mu\nu} = K^\mu K^\nu \), where \( K^\mu = (k_\omega, \omega/c) \) is the wave four-vector for a plane wave in free space, first shown by Einstein [20], with \( k_\omega \) the wave vector, \( \omega (\neq 0) \) the angular frequency, and \( |k_\omega| = \omega/c \) holding. This is illustrated as follows.

Apparently, \( \partial_{\mu} K^\mu = 0 \) and \( \partial_{\mu}(K^\mu K^\nu) = 0 \) are both valid because \( K^\mu \) is independent of space and time variables \((x, t)\). Suppose that \( X'Y'Z' \) frame moves with respect to the laboratory frame \( XYZ \) at a velocity of \( \beta c \) along the wave vector \( k_\omega \). Observed in \( XYZ \), we have
\[
K^\mu = \omega/c
\]
and
\[
\int_{V} K^\mu d^3 x = \frac{\omega}{c} \int_{V} d^3 x,
\]
where the integral domain \( V \) is fixed in \( XYZ \). Observed in \( X'Y'Z' \), we have \( K'^\mu = \omega'/c \) and
\[
\int_{V'} K'^\mu d^3 x' = \int_{V'} \frac{\omega'}{c} d^3 x',
\]
where the integral domain \( V' \) is moving at \( \beta' c = -\beta c \).

From the Lorentz transformation of \( K^\mu \), we obtain the Doppler frequency shift [20], given by
\[
\frac{\omega'}{c} = \gamma \left( \frac{\omega}{c} - \beta \cdot k_\omega \right) = \gamma (1 - |\beta|) \frac{\omega}{c},
\]
where \( \gamma = (1 - \beta^2)^{-1/2} \) is the time dilation factor, and \( |k_\omega| \Rightarrow |k_\omega| = \beta \cdot k_\omega = |\beta| \times |k_\omega| = |\beta|\omega/c \) is employed.

As shown in Eq. (A.7) of Appendix A, the change of variable formula in such a case is given by \( d^3 x' = d^3 x / \gamma \).

Inserting Eq. (C.3) into Eq. (C.2), with \( d^3 x' = d^3 x / \gamma \) and...
Eq. (C1) taken into account, we have
\[
\int_{V'} K'^4 d^3x' = \int_{V} \frac{\omega'}{c} d^3x' \\
= \gamma (1 - |\beta|) \int_{V} \frac{d^3x}{\gamma} \\
= (1 - |\beta|) \int_{V} K^4 d^3x.
\] (C4)

Note that in above Eq. (C4), the integrand \(K'^4\) satisfies the definition Eq. (24) in the proof of Theorem 3.

From above Eq. (C4) we have
\[
\int_{V'} K'^4 d^3x' \neq \int_{V} K^4 d^3x \text{ for } \beta c \neq 0.
\] (C5)

Thus we conclude that \(\int_{V} K^4 d^3x\) is not a Lorentz scalar, and Thirring’s claim (i) is disproved.

On the other hand, we have
\[
\int_{V} T^\mu_4 d^3x = \int_{V} K^\mu K^4 d^3x = K^\mu \int_{V} K^4 d^3x.
\] (C6)

In above Eq. (C6), \(K^\mu\) is a four-vector, but \(\int_{V} K^4 d^3x\) is not a Lorentz scalar; thus their product
\[
(K^\mu) \times (\int_{V} K^4 d^3x) = \int_{V} T^\mu_4 d^3x
\] (C7)

must not be a four-vector, and Thirring’s claim (ii) is disproved as well.

**Proof by contradiction that Eq. (C7) is not a four-vector.** If Eq. (C7) were a four-vector, then
\[
(g_{\mu\nu} K^\mu X^\nu) \left( \int_{V} K^4 d^3x \right) = \text{scalar}
\] (C8)

would hold, where \(X^\nu = (x, ct)\) is the time-space four-vector, and
\[
(g_{\mu\nu} K^\mu X^\nu) = (\omega t - \mathbf{k}_w \cdot \mathbf{x}) = \text{scalar}
\] (C9)

is the scalar of phase, with \(g_{\mu\nu} = \text{diag}(-1, -1, -1, +1)\) the Minkowski metric. Thus from Eq. (C8) and Eq. (C9) it follows that \((\int_{V} K^4 d^3x)\) must be a scalar, but \((\int_{V} K^4 d^3x)\) is not a scalar according to Eq. (C5). This contradiction indicates that Eq. (C7), \(\int_{V} T^\mu_4 d^3x = K^\mu (\int_{V} K^4 d^3x)\), cannot be a Lorentz four-vector.

An interesting question: Why is \(\int K^4 d^3x\) not a scalar for the wave four-vector \(K^\mu = (k_\omega, \omega/c)\) while \(\int J^4 d^3x\) is a scalar for the current density four-vector \(J^\mu = (J, c\rho)\)?

That is because the moving velocity of any charged particle is less than the vacuum light speed \(c\), and there is a particle-rest frame where the particle current \(J = 0 \Rightarrow \int J d^3x = 0\) holds, with the sufficient and necessary condition Eq. (23) of Theorem 3 satisfied, \(\Rightarrow \int J^4 d^3x = \text{scalar},\) as shown in Eq. (A6) of Appendix A. However for \(K^\mu = (k_\omega, \omega/c)\), there is no such a frame where \(k_\omega = 0 \Rightarrow \int k_\omega d^3x = 0\) holds; thus \(\int K^4 d^3x\) is not a scalar. Why is there no such a frame for \(k_\omega = 0\)? As we know, the photon momentum–energy four-vector is given by \(h K^\mu = (h k_\omega, h\omega/c)\), with \(h\) the Planck constant. According to Einstein’s hypothesis of constancy of light speed, there is no photon-rest frame in free space, and the photon momentum \(h k_\omega \neq 0 \Rightarrow k_\omega \neq 0\) holds in any frames.

**Appendix D: Why is the theoretical framework for the positive mass theorem flawed?**

In general relativity, the metric \(g_{\mu\nu}\) are solutions of Einstein’s field equations for a given energy–momentum tensor \(T^\mu_\nu\) that causes space to curve [21, p. 5]. According to Arnowitt, Deser and Misner [37], the definition of the total energy–momentum \(P^\mu\) is given by the volume integral of the components of \(T^0_\mu\), namely
\[
P^\mu := \int T^{0\mu} d^3x,
\] (D1)

which can be expressed as surface integrals through Einstein’s field equations and Gauss’s theorem, and where \(P^0 = E\) is the total energy [38].

In the proofs of the positive mass theorem [33, 35, 39], the total energy follows the definition in [37, 38, 21, p. 462], given by
\[
E = \frac{1}{16\pi} \int (g_{jk,k} - g_{kk,j}) d^2S^j,
\] (D2)

where the integral is evaluated over a closed surface in the asymptotically flat region surrounding the source of gravitation.

Arnowitt, Deser and Misner claim that because of the conservation law \(T_{\mu\nu}^{\text{matter}} = 0\), “\(P^\mu\) should transform as a four-vector” [38], which is clearly endorsed by Nester [34]. In the textbook by Misner, Thorne and Wheeler [21, p. 462], it is also emphasized that the conservation law \(T_{\mu\nu}^{\text{matter}} = 0\) \((T_{\mu\nu}^{\text{matter}} = 0)\) makes \(P^\mu = \int T^{0\mu} d^3x\) \((P^\mu = \int T^{0\mu} d^3x)\) be a four-vector.

Unfortunately, as shown by the counterexample in Appendix C \(T_{\mu\nu}^{\text{matter}} = 0\) cannot guarantee that \(P^\mu\) is a four-vector. Thus the Arnowitt-Deser-Misner theoretical framework used for the proofs of the positive mass theorem is flawed.

Nevertheless, one might argue that in the framework of the positive mass theorem, in addition to \(T_{\mu\nu}^{\text{matter}} = 0\) there is another important requirement, called *dominant energy condition* [39], reading:
\[
T^{00} \geq |T^{\mu\nu}|
\] (D3)
for each $\mu, \nu$ [30][35]; and both $T^{\mu\nu}$, $0$ and $T^{00} \geq |T^{\mu\nu}|$
together make $P^\mu = \int T^{\mu\nu} \mathrm{d}^3x$ be a four-vector (in above
Arnol’it-Deser-Misner practice $\mu, \nu = 0, 1, 2, 3$ used).
However this is not true, because the counterexample
\[ T^{\mu\nu} = K^\mu K^\nu \quad \text{(D4)} \]
in Appendix C itself satisfies the dominant energy condition
\[ T^{44} \geq |T^{\mu\nu}|, \quad \text{(D5)} \]
namely
\[ K^4 K^4 \geq |K^\mu K^\nu| \quad \text{(D6)} \]
or
\[ (\omega/c)^2 \geq |K^\mu K^\nu|, \quad \text{(D7)} \]
where $K^\mu = (k_w, \omega/c)$ with $|k_w| = \omega/c \geq |K^\mu|$ (here
Rindler’s practice $\mu, \nu = 1, 2, 3, 4$ [22] p. 138) used).

That is to say, although the conservation law $T^{\mu\nu} = 0$
or $\partial_\nu T^{\mu\nu} = \partial_\nu (K^\mu K^\nu) = 0$ and the dominant energy
condition $T^{44} \geq |T^{\mu\nu}|$ or $K^4 K^4 \geq |K^\mu K^\nu|$ are both satisfied,
\[ P^\mu = \int T^{\mu\nu} \mathrm{d}^3x = \int K^\mu K^4 \mathrm{d}^3x = \left( \int K^4 \mathrm{d}^3x \right) K^\nu \quad \text{(D8)} \]
is not a four-vector, because $\left( \int K^4 \mathrm{d}^3x \right)$ is not a scalar,
as shown in Eq. (C5) of Appendix C.
From above we can see that all the proofs of the
positive mass theorem [32][36][39] are based on a flawed
theoretical framework, and thus the validity of the
theorem itself could be called into question.

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logical* (Oxford, NY, 2006), 2nd edition. In his book
Rindler argues that in special relativity, four-current
density $J^\mu = (J, cp)$ satisfies charge conservation equation
$J^\mu = \partial_\nu (K^\mu K^\nu) = 0$ (on p. 142 of the book), and charge
$Q = \int (|J^\mu/c|^2 \mathrm{d}^3x)$ is “Lorentz invariant from frame
to frame” (p. 140). However as shown in Appendix C
$\partial_\nu J^\mu = 0$ is not a sufficient condition for $Q$ to be a
Lorentz invariant. Rindler also argues that $T^{\mu\nu} = 0$ is the energy–momentum conservation equa-
tions for a closed physical system in special relativity (p. 298); however, there is no definition that is provided for what is the total four-momentum of the system. Thus the invariance of total charge and the covariance of total four-momentum, namely the conservation laws of total charge and total energy-momentum, are not appropriately formulated in this excellent textbook.

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Dear Readers,

In my paper, I think that I have generally resolved two fundamental issues in the dynamics of relativity: (a) Under what condition, the time-column space integrals of a Lorentz four-tensor constitute a Lorentz four-vector (Theorem 1), and (b) under what condition, the time-element space integral of a Lorentz four-vector is a Lorentz scalar (Theorem 3). Both Theorem 1 and Theorem 3 have their own sufficient and necessary conditions, which have nothing to do with the divergence-less, symmetry, and boundary conditions. This point is especially important. For example, from Theorem 1 we can directly judge that Møller’s theorem is incorrect, because the sufficient condition of Møller’s theorem does not include the sufficient and necessary condition of Theorem 1, as shown in counterexample Eq. (38). A similar argument is applicable to the “invariant conservation law” claimed by Weinberg [2, p.40], as shown in counterexample Eq. (48).

Why is the constant total charge $Q$ in $V$, resulting from $\partial_{\mu}J^{\mu} = 0 \Rightarrow \nabla \cdot \mathbf{J} = -\partial \rho / \partial t$, not a Lorentz invariant? As shown in Sec. IV, that is because $Q$ in $V$ in frame XYZ and $Q'$ in $V'$ in frame $XYZ'$ do not refer to the same total charge in the same volume physically, where $XYZ'$ moves at $\beta c \neq 0$ relatively to $XYZ$. This conclusion directly comes from the fact that the volume $V$ in $\int (\partial \rho / \partial t) d^3x$ is fixed in $XYZ$, and the volume $V'$ in $\int (\partial \rho / \partial t') d^3x'$ is fixed in $XYZ'$ so that $\partial / \partial t$ and $\int$, and $\partial / \partial t'$ and $\int$ are exchangeable, respectively, in order to make both $\int (\partial \rho / \partial t) d^3x = dQ / dt$ and $\int (\partial \rho / \partial t') d^3x' = dQ' / dt'$ hold in general. However according to Einstein’s relativity [20], $V$ fixed in $XYZ$ is moving observed in $XYZ'$. Now $V'$ in $\int (\partial \rho / \partial t') d^3x'$ is at rest in $XYZ'$, and thus $V$ and $V'$ do not denote the same volume, and $Q$ and $Q'$ do not denote the same total charge. Therefore, the current continuity equation $\partial_{\mu}J^{\mu} = 0$ cannot be taken as the “invariant [charge] conservation law” in relativity [2, p.40].

As a practical application, I have used Theorem 1 to analyze Minkowski EM tensor for a plane light wave in a moving medium with Einstein’s light-quantum hypothesis taken into account, finding that the four-momentum of quasi-photon and the Lorentz invariance of Planck constant can be naturally derived (see Appendix B).

In relativity, charge conservation law refers to that the total charge is a time- and frame-independent constant [2, p. 41], while energy-momentum conservation law refers to that the total energy and momentum constitute a covariant and time-independent four-vector [2, p. 46]. The problems of $\partial_{\nu}A^{\nu} = 0$ and $\partial_{\nu}\Theta^{\nu} = 0$ as being conservation laws in relativity were first discovered in [6,7], and they are explicitly illustrated and generally resolved in the present paper. These problems never got any attention in literature before the publications of [6,7], such as in the proofs of the positive mass theorem by Schoen and Yau [39] and by Witten [33], and in the textbook of Rindler [22].

My manuscript has been greatly and carefully revised according to the referee reports from Physical Review D.
Referee A did not provide any specific criticisms and comments about novelty and originality, and whether analysis and calculations support claimed conclusions.

Referee B has two main points which constitute thought-provoking challenges over the novelty and originality of my research work, and took me a lot of time to understand.

Point (1). Referee B argues that Rindler’s book has already solved the problems about the traditional conservations laws.

Point (2). Referee B argues that in the proofs of the positive mass theorem, “rigorous mathematics has been used and loopholes possibly left in earlier works do not exist.”

By careful analysis, I find that Rindler did not realize the problems of conservation laws, not to speak of solving them, while the positive mass theorem itself is based on the incorrect conservation law $T^{\mu}_{\mu} = 0$, which is not just a “loophole” but an error that is not fixable.

I think that I have clarified the above two points in the revised manuscript, and all detailed explanations are attached in Response to Referee B. Thank you for your precious time to review my paper.

Sincerely,
Changbiao Wang

Attached:
Response to Referee B
Response to Referee B

Response to Point (1). Referee B argues that Rindler’s book has already solved the problems about the traditional conservations laws.

I am sorry, I don’t think so, which is shown below.

(i) Invariance of total charge in a closed physical system is the charge conservation law, which means that the total charge is a time-independent and frame-independent constant. It is a fundamental issue in the dynamics of relativity.

In his book, Prof. Rindler argues that in special relativity, four-current density \( J^\mu = (J, c\rho) \) satisfies “conservation of charge” equation \( \partial_\mu J^\mu = 0 \), copied below

\[
\text{and recall that the conservation of charge is expressed by the following equation of continuity:} \\
\frac{\partial \rho}{\partial t} + \text{div} \ j = 0. \\
\text{(7.37)}
\]

Since \( \text{div} \ j \) measures the outflux of charge from a (small) unit volume in unit time, this equation simply states that to the precise extent that charge leaves a small region, the total charge inside that region must decrease.

We next define the four-current density \( J^\mu \) by the first of the following equations (provisionally—hence the brackets),

\[
J^\mu = [\rho_0 U^\mu = \rho_0 \gamma(u)(u, c)] = (J, c\rho), \\
\text{(7.38)}
\]

and note that it allows us to express the equation of continuity (7.37) in the following tensorial form [cf. (7.7)]:

\[
J^\mu_{,\mu} = 0, \\
\text{(7.39)}
\]

W. Rindler, Relativity (Oxford, NY, 2006), p. 142

and the charge \( q = \int (J^+ / c) d^3 x \) is “[Lorentz] invariant from frame to frame”, copied below

(Note that any charged particle takes up a finite space.) However the “conservation of charge” equation \( \partial_\mu J^\mu = 0 \) is not a sufficient condition for \( q \) to be “invariant from frame to frame”. Apparently, Prof. Rindler did not realize this problem.

In Appendix C of my paper, a counterexample is given to show why \( \partial_\mu J^\mu = 0 \) is not a sufficient condition for \( q \) to be “invariant from frame to frame”. The counterexample is the wave four-vector

\[
K^\mu = (k^+, \omega/c)
\]

for a plane wave in free space, first shown by Einstein, with \( k^+ \) the wave vector, \( \omega \neq 0 \) the angular frequency, and \( |k^+| = \omega/c \) holding. Like the four-current density \( J^\mu = (J, c\rho) \) satisfying \( \partial_\mu J^\mu = 0 \), the wave four-vector \( K^\mu \) also satisfies

\[
\partial_\mu K^\mu = 0, \\
\text{but } \int K^+ d^3 x = (\omega/c)^{\frac{1}{2}} d^3 x \text{ is not a Lorentz invariant. Why? This is physically explained in Appendix C, copied below.}
From above, we can see that Prof. Rindler did not realize the problem of $J^\mu = 0$ as being “conservation of charge”, not to speak of solving this problem. Therefore, in his book Prof. Rindler did not provide a proper formulation of the charge conservation law in the frame of relativity.

(ii) The covariance of total four-momentum in a closed physical system is the energy-momentum conservation law, which means that the total energy and momentum constitute a covariant and time-independent four-vector for a closed physical system. It is also a fundamental issue in the dynamics of relativity.

In his book, Prof. Rindler argues that $T^\mu_{\rho}(-\partial^\rho, T^\nu_\nu) = 0$ is the energy-momentum four conservation equations for a closed physical system in special relativity, copied below

\[ T^\mu_{\rho}(-\partial^\rho, T^\nu_\nu) = 0 \]

however, there is no definition that is provided for what is the total four-momentum of the system.

Since Prof. Rindler did not provide the definition of total four-momentum for the energy-momentum tensor $T^\nu_\rho$ that satisfies the conservation equation $\partial^\rho T^\nu_\rho = 0$, Prof. Rindler did not provide a proper formulation of the covariance of total four-momentum for a closed system — the energy-momentum conservation law in the frame of relativity.
**Conclusion to Point (1)**

In his book,

(i) Prof. Rindler did not realize that the “conservation of charge” equation \( \partial_{\mu} J^{\mu} = 0 \) is not a sufficient condition for \( q \int = \frac{1}{(J^4/c)d^4x} \) to be “invariant from frame to frame”;

(ii) Prof. Rindler did not provide the definition of total four-momentum based on the energy-momentum tensor \( T^{\mu \nu} \) for a closed system.

Thus the issues about the invariance of total charge and the covariance of total four-momentum for a closed system are not appropriately formulated. In other words, Prof. Rindler did not solve the problems of \( \partial_{\mu} J^{\mu} = 0 \) and \( \partial_{\nu} T^{\mu \nu} = 0 \) as being conservation laws.

In my revised manuscript, I added Ref. [22] to answer this comment by Referee B.

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**Response to Point (2).** Referee B argues that in the proofs of the positive mass theorem, “rigorous mathematics has been used and loopholes possibly left in earlier works do not exist.”

I am sorry, I don’t think so, because in the proofs of the positive mass theorem, the traditional conservation law \( \partial_{\nu} T^{\mu \nu} = 0 \) is employed. In other words, the proofs are based on a flawed theoretical framework, which is shown below.

The theoretical framework used for the proofs are shown in

- Arnowitt-Deser-Misner framework (part 1)
- Arnowitt-Deser-Misner framework (part 2)
- Arnowitt-Deser-Misner framework (part 3)
- Arnowitt-Deser-Misner framework (part 4)

The problem of the conservation law \( \partial_{\nu} T^{\mu \nu} = 0 \) never got any attention in the proofs of the positive mass theorem by Schoen and Yau in 1979 and in 1981, Witten and Nester in 1981, Parker and Taubes in 1982, and Gibbons and coworkers in 1983. All the proofs are based on a flawed theoretical framework set up by Arnowitt, Deser, and Misner, where the total energy-momentum \( P^\nu \) in an asymptotically flat spacetime is required to “obey the differential conservation law \( T^{\mu \nu} = 0 \) [of the energy-momentum tensor \( T^{\mu \nu} \)]” and required to “transform as a four-vector”; see Arnowitt-Deser-Misner framework (part 2). However in fact, the conservation law \( T^{\mu \nu} = 0 \) cannot guarantee that \( P^\nu \) is a four-vector, as shown in Appendix D. Apparently, this problem is ignored in all the proofs.

**Conclusion to Point (2)**

The proofs of the positive mass theorem are based on a flawed theoretical framework where \( \partial_{\nu} T^{\mu \nu} = 0 \) is taken to be the energy-momentum conservation law, and this problem has never got any attention in the community.

To answer this comment by Referee B, I greatly revised Sec. V and added Appendix D.
Arnowitt-Deser-Misner framework (part 1)

The energy-momentum of the field is just the volume integral of the components of $T^{\alpha\beta}$ when a solution of the field equations is substituted in for $g^{\alpha\beta}$ and $\pi^i$. In

II. DEFINITION AND PROPERTIES OF ENERGY AND MOMENTUM

From Eq. (1.6) we see that the total energy may be written as

$$ E = - \int g^{\alpha\beta} d\Phi = - \int g^{\alpha\beta} dS_{\alpha} $$

(2.1)

where $dS_{\alpha} = dx^\alpha dx^\beta$, etc., are the rectangular surface elements at spatial infinity. Using Eq. (1.3a), the energy then becomes

$$ E = \int (\epsilon_{\alpha\beta} - \epsilon_{\beta\alpha}) dS_{\alpha} $$

(2.2a)

Similarly, the momentum $P_{\alpha}$ is given by

$$ P_{\alpha} = - \int 2(\sigma_{\alpha j} + \sigma_{j\alpha}) dS_{\alpha} = - \int 2\pi_{\alpha} dS_{\alpha} $$

(2.2b)

R. Arnowitt, S. Deser, and C. W. Misner, Phys. Rev. 118, 1100 (1960)

Arnowitt-Deser-Misner framework (part 2): According to Arnowitt, Deser and Misner, because of the conservation law

$$ 0 = \sum_{\alpha=0}^{3} \mu_{\alpha} T^{\alpha\beta} $$

$\mu_{\alpha}$ should transform as a four-vector. Our boundary conditions have not stated that $g_{\alpha\beta,\alpha}$ should behave as $1/r^{2}$ at infinity; in fact, it is most natural simply to require that $g_{\alpha\beta,\alpha}$ also vanish as $1/r$. For example, we thus allow the metric to decrease as $(r^{2})/r$.

Let us now examine the various proposed expressions for $P_{\alpha}$ within the above framework. A common characteristic of all these is that they can be cast into the

$P_{\alpha}$ should transform as a four-vector. Our boundary conditions have not stated that $g_{\alpha\beta,\alpha}$ should behave as $1/r^{2}$ at infinity; in fact, it is most natural simply to require that $g_{\alpha\beta,\alpha}$ also vanish as $1/r$. For example, we thus allow the metric to decrease as $(r^{2})/r$.

Let us now examine the various proposed expressions for $P_{\alpha}$ within the above framework. A common characteristic of all these is that they can be cast into the

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Arnowitt-Deser-Misner framework (part 2): According to Arnowitt, Deser and Misner, because of the conservation law $\sum_{\alpha=0}^{3} \mu_{\alpha} T^{\alpha\beta} = 0$, $P_{\alpha}$ should transform as a four-vector, namely it is a four-vector.
Arnowitt-Deser-Misner framework (part 3)

In this textbook, it is claimed that \( \partial_\mu J^\mu = 0 \) makes \( \int_{x_0}^x dJ^\mu Q = \text{const} \) be an invariant, and \( \partial_\mu T^{\mu\nu} = 0 \) makes \( \int_{\text{body}} dT_{\mu\nu}(t, x) dx dy dz \equiv P^\mu = \text{const} \).

Arnowitt-Deser-Misner framework (part 4): Here in this textbook, \( P^\mu = \int T^{\mu\nu} d^4x \) is, once again, clearly claimed as a four-vector because of the source conservation laws \( \partial_\mu T^{\mu\nu} = 0 \). The total energy expression \( P^\mu = E \) is exactly the one used for the proofs of the positive mass theorem by Schoen and Yau, Witten, and Parker and Taubes.
**Schoen-Yau proof**

Abstract. The positive mass theorem states that for a nontrivial isolated physical system, the total energy, which includes contributions from both matter and gravitation is positive. This assertion was demonstrated in our previous paper in the important case when the space-time admits a maximal slice. Here this assumption is removed and the general theorem is demonstrated. Abstracts of the results of this paper appeared in [11] and [13].

With each $N_k$ we associate a total mass $M_k$ defined by the flux integral

$$M_k = \frac{1}{16\pi} \sum_{i,j} (g_{ij} - \eta_{ij}) d\sigma_i$$

which is the limit of surface integrals taken over large two spheres in $N_k$.

This number $M_k$ is called the ADM mass of $N_k$ (see Arnowitt, Deser, and Misner [1]). Classically it was assumed that the first term in the asymptotic expansion of $g_{ij}$ is spherical. It was pointed out by York [11] that physically it is

Schoen and Yau, Commun. Math. Phys. 79, 231-260 (1981)

**Schoen-Yau proof**: The total mass definition is taken from the one in Arnowitt-Deser-Misner framework (part 4). This theorem has been proved in their “previous paper in the important case” [Commun. Math. Phys. 65, 45 (1979)]. The two papers takes the same Arnowitt-Deser-Misner framework.

**Witten’s proof**

The total energy of this system is defined as a surface integral over the asymptotic behavior of the gravitational field,

$$E = \frac{1}{16\pi} \int d^2S \left( \frac{\partial}{\partial x^k} g_{ik} - \frac{\partial}{\partial x^k} g_{ik} \right)$$

(21)

where the integral is evaluated over a bounding surface in the asymptotically flat region of the initial value surface. The problem is to prove that this total energy $E$ is always positive or zero, and zero only for flat Minkowski space.

Witten, Commun. Math. Phys. 80, 381 (1981)

**Witten’s proof**: The total mass definition is taken from the one in Arnowitt-Deser-Misner framework (part 4). Nester [Phys. Lett. A 83, 241 (1981)], Parker and Taubes [Commun. Math. Phys. 84, 223 (1982)], and Gibbons, Hawking, Horowitz, and Perry [Commun. Math. Phys. 88, 295 (1983)] follow Witten’s work, using Arnowitt-Deser-Misner framework.

--------------------- End of Response to Referee B ---------------------