Radial maximal function characterizations of Hardy spaces on RD-spaces and their applications

Dachun Yang and Yuan Zhou

Abstract Let $\mathcal{X}$ be an RD-space with $\mu(\mathcal{X}) = \infty$, which means that $\mathcal{X}$ is a space of homogeneous type in the sense of Coifman and Weiss and its measure has the reverse doubling property. In this paper, we characterize the atomic Hardy spaces $H^p_{\text{at}}(\mathcal{X})$ of Coifman and Weiss for $p \in (n/(n+1), 1]$ via the radial maximal function, where $n$ is the “dimension” of $\mathcal{X}$, and the range of index $p$ is the best possible. This completely answers the question proposed by Ronald R. Coifman and Guido Weiss in 1977 in this setting, and improves on a deep result of Uchiyama in 1980 on an Ahlfors 1-regular space and a recent result of Loukas Grafakos et al in this setting. Moreover, we obtain a maximal function theory of localized Hardy spaces in the sense of Goldberg on RD-spaces by generalizing the above result to localized Hardy spaces and establishing the links between Hardy spaces and localized Hardy spaces. These results have a wide range of applications. In particular, we characterize the Hardy spaces $H^p_{\text{at}}(M)$ via the radial maximal function generated by the heat kernel of the Laplace-Beltrami operator $\Delta$ on complete noncompact connected manifolds $M$ having a doubling property and supporting a scaled Poincaré inequality for all $p \in (n/(n+\alpha), 1]$, where $\alpha$ represents the regularity of the heat kernel. This extends some recent results of Russ and Auscher-McIntosh-Russ.

Mathematics Subject Classification (2000) Primary 42B30; Secondary 42B25, 42B35

Dachun Yang was supported by the National Natural Science Foundation (Grant No. 10871025) of China.

D. Yang
School of Mathematical Sciences, Beijing Normal University,
Laboratory of Mathematics and Complex Systems, Ministry of Education,
Beijing 100875, People’s Republic of China
e-mail: dcyang@bnu.edu.cn

Y. Zhou
School of Mathematical Sciences, Beijing Normal University,
Laboratory of Mathematics and Complex Systems, Ministry of Education,
Beijing 100875, People’s Republic of China
e-mail: yuanzhou@mail.bnu.edu.cn

Y. Zhou
Department of Mathematics and Statistics, University of Jyväskylä,
P. O. Box 35 (MaD), FI-40014, Finland
e-mail: yuzhou@cc.jyu.fi
1 Introduction

The theory of Hardy spaces on the Euclidean space $\mathbb{R}^n$ plays an important role in various fields of analysis and partial differential equations; see, for examples, [33, 8, 5, 32, 9]. It is well-known that the following spaces of homogeneous type of Coifman and Weiss [6, 7] form a natural setting for the study of function spaces and singular integrals.

Definition 1.1 Let $(X, d)$ be a metric space with a Borel regular measure $\mu$ such that all balls defined by $d$ have finite and positive measure. For any $x \in X$ and $r > 0$, set the ball $B(x, r) \equiv \{ y \in X : d(x, y) < r \}$. The triple $(X, d, \mu)$ is called a space of homogeneous type if there exists a constant $C_1 \geq 1$ such that for all $x \in X$, $\lambda \geq 1$ and $r > 0$,

$$\mu(B(x, \lambda r)) \leq C_1 \lambda^n \mu(B(x, r)).$$

Here $n$, if chosen minimal, measures the “dimension” of the space $X$ in some sense.

Let $X$ be a space of homogeneous type as in Definition 1.1. In 1977, Coifman and Weiss [7] introduced the atomic Hardy spaces $H^p_{at}(X)$ for $p \in (0, 1]$ and obtained their molecular characterizations. Moreover, under certain additional geometric condition, Coifman and Weiss obtained the radial maximal function characterization of $H^1_{at}(X)$. Then they further asked the following question; see [7, pp. 641-642] or [34, p. 580].

Question 1: Is it possible to characterize $H^p_{at}(X)$ for $p \in (0, 1]$ in terms of a radial maximal function?

Recall that an Ahlfors $n$-regular metric measure space is a space of homogeneous type as in Definition 1.1 satisfying that for all $x \in X$ and $r \in (0, \text{diam } (X))$, $\mu(B(x, r)) \sim r^n$, where $\text{diam } (X) \equiv \sup_{x,y \in X} d(x, y)$. When $X$ is an Ahlfors 1-regular metric measure space, in 1980, Uchiyama [34] partially answered this question by proving the deep result that for $p \in (p_0, 1]$ and functions in $L^1(X)$, the $L^p(X)$ quasi-norms of their grand maximal functions (as in [21]) are equivalent to the $L^p(X)$ quasi-norms of their radial maximal functions defined via some kernels in [7]. However, here $p_0 > 1/2$; see [11] for the explicit value of $p_0$. Also in this setting, when $p \in (1/2, 1]$, Macías and Segovia [21] characterized $H^p_{at}(X)$ via a grand maximal function. Observe that when $X$ is an Ahlfors 1-regular metric measure space and $p \leq 1/2$, it is impossible to characterize $H^p_{at}(X)$ via the radial maximal function since atoms of $H^p_{at}(X)$ have only 0-order vanishing moment.

Recently, the following RD-spaces were introduced in [18], which are modeled on Euclidean spaces with Muckenhoupt weights, Ahlfors $n$-regular metric measure spaces (see, for example, [16]), Lie groups of polynomial growth (see, for example, [1, 38, 39]) and Carnot-Carathéodory spaces with doubling measures (see, for example, [25, 23, 32]).

Definition 1.2 The triple $(X, d, \mu)$ is called an RD-space if it is a space of homogeneous type as in Definition 1.1 and there exist constants $C_2 > 1$ and $C_3 > 1$ such that for all $x \in X$ and $r \in (0, \text{diam } (X))$,

$$\mu(B(x, C_2 r)) \geq C_3 \mu(B(x, r)).$$

We point out that the condition (1.2) can be replaced by the following geometric condition: there exists a constant $a_0 > 1$ such that for all $x \in X$ and $0 < r < \text{diam } (X)/a_0$,.
and the dyadic maximal function characterizations of $H$ also obtained in [10].

When $p$ is not far from 1, another forthcoming paper, because of the need to overcome some additional subtle technical difficulties.

Recently, by extending Uchiyama’s idea in [34], it was proved in [11] that there exists a $p_0$ close to 1 such that when $p \in (p_0, 1]$, $H^p_{at}(\mathcal{X})$ is characterized by a certain radial maximal function. However, here $p_0 > n/(n+1)$. This partially and affirmatively answers Question 1 when $\mathcal{X}$ is an RD-space. Moreover, a Littlewood-Paley theory for $H^p_{at}(\mathcal{X})$ when $p \in (n/(n+1), 1]$ was established in [17, 18] via Calderón reproducing formulae. Also, via inhomogeneous Calderón reproducing formulae, the grand, the nontangential and the dyadic maximal function characterizations of $H^p_{at}(\mathcal{X})$ for $p \in (n/(n+1), 1]$ were also obtained in [10].

On the other hand, let $M$ be a noncompact manifold having the doubling property and supporting a scaled Poincaré inequality and $\Delta$ the Laplace-Beltrami operator. There is an increasing interest in the study of the Hardy spaces and the Riesz transforms on such noncompact manifolds; see, for example, [2, 3, 26] and the references therein. In particular, let $H^p_\Delta(M)$ be the Hardy space defined by the radial maximal function generated by the heat kernel. Based on Uchiyama’s result [34], Russ [26] stated the equivalence of $H^p_\Delta(M)$ and $H^p_{at}(M)$ for $p \in (p_0, 1]$ without a proof, where $p_0$ is not far from 1; see also Theorem 8.2 of [3] for $p = 1$. Then the following question naturally appears.

**Question 2:** What is the best possible range of $p \leq 1$ which guarantees the equivalence between $H^p_{at}(M)$ and $H^p_\Delta(M)$.

This paper is devoted to Questions 1 and 2 above, and extensions of these results to localized Hardy spaces in the sense of Goldberg ([12]). Throughout the whole paper, we always assume that $\mathcal{X}$ is an RD-space and $\mu(\mathcal{X}) = \infty$. We leave the case when $\mu(\mathcal{X}) < \infty$ to another forthcoming paper, because of the need to overcome some additional subtle technical difficulties.

First, for $p \in (n/(n+1), 1]$, we characterize $H^p_{at}(\mathcal{X})$ via a certain radial maximal function; see Theorem 3.1 and Corollary 3.1 below. This completely and affirmatively answers Question 1 of Coifman and Weiss in the case when $\mathcal{X}$ is an RD-space and $\mu(\mathcal{X}) = \infty$. Observe that when $p \in (0, n/(n+1)]$, the radial maximal function cannot characterize $H^p_{at}(\mathcal{X})$ since its atoms have only 0-order vanishing moment. Thus, the range $p \in (n/(n+1), 1]$ in Theorem 3.1 and Corollary 3.1 is the best possible. Obviously, Theorem 3.1 and Corollary 3.1 generalize the result of Uchiyama [34] to RD-spaces and, moreover, improve the corresponding results in [34] and [11] by widening the range of the index $p$ to the best possible. Let $\{S_\ell\}_{\ell \in \mathbb{Z}}$ be any approximation of the identity (see Definition 2.1 below). The proof of Theorem 3.1 is based on the following two key observations: (i) Any inhomogeneous test function $\phi$ can be decomposed into $a_\ell,\phi S_\ell + b_\ell,\phi \varphi$, where $\varphi$ is a homogeneous test function, $a_\ell,\phi$ and $b_\ell,\phi$ are constants satisfying certain uniform estimates in $\ell$ and $\phi$; (ii) Any inhomogeneous distribution uniquely induces a homogeneous distribution, which enables us to use the homogeneous Calderón reproducing formula instead of the inhomogeneous one as in [10]. This combined with (i) overcomes the difficulty caused by the average term appearing in the inhomogeneous Calderón reproducing formula. Then by some simple calculations, we control the grand maximal function via the Hardy-Littlewood maximal
function of a certain power of the radial maximal function. This procedure further implies Theorem 3.1, which is much simpler than and totally different from those used by Fefferman-Stein in [8], Uchiyama in [34] and Grafakos-Liu-Yang in [11].

Secondly, via a grand maximal function and a variant of the radial maximal function, we characterize the localized Hardy space $H^p_\ell(X)$ in the sense of Goldberg with $p \in (n/(n+1), 1]$ and $\ell \in \mathbb{Z}$; see Theorems 3.2 and 3.4 below. We point out that the range of the index $p \in (n/(n+1), 1]$ here is also the best possible by a reason similar to the above and that constants appeared in Theorems 3.2 through 3.4 are uniform in $\ell \in \mathbb{Z}$.

In fact, let $H^p_{\ell, at}(X)$ be the localized Hardy space defined by the localized grand maximal function. In Theorem 3.2 (i), for $p \in (n/(n+1), 1]$, we characterize $H^p_{\ell, at}(X)$ via a variant of the localized radial maximal function, and for an element in $H^p_\ell(X)$, in Theorem 3.2 (ii), we further establish the equivalence between its $H^p_\ell(X)$-norm and the $L^p(X)$-norm of its localized radial maximal function. The proof of Theorem 3.2 (i) is also based on the key observation (i) used in the proof of Theorem 3.1 and an application of the inhomogeneous Calderón reproducing formula. Due to the inhomogeneity of such Calderón reproducing formula, we only obtain a variant of the localized radial maximal function characterization for the localized Hardy spaces; see the extra condition (3.1) in Theorem 3.2. But this is quite reasonable as the same phenomena happens in [18] for the localized Littlewood-Paley characterization of localized Hardy spaces. The proof of Theorem 3.2 (ii) requires another key observation, namely, the size of dyadic cubes $\{Q^\ell, \nu, \tau\}$ appearing in the average term of the inhomogeneous Calderón reproducing formula (see Theorem 5.2 below) can be sufficiently small, which allows us to obtain sufficiently small decay factor determined by $j$ in the estimate of $J_1$. This plays a key role in the proof of Theorem 3.2 (ii). In Theorem 3.4, we establish the equivalence between the localized Hardy space defined by the grand maximal function and the one by atoms. To prove this, we link $H^p_\ell(X)$ and $H^p(X)$ in Theorem 3.3 by using some ideas from [12] and [18] and the Calderón reproducing formula.

Finally, applying Theorems 3.1 through 3.4 to a noncompact manifold satisfying the doubling property and supporting a scaled Poincaré inequality, we obtain, in Proposition 4.1, an explicit range $p \in (n/(n+\alpha), 1]$ for Question 2, where $\alpha \in (0, 1]$ is the order of the regularity of the heat kernel. This range is the best one which can be obtained by the current approach, but it is not clear if it is optimal. However, this already extends Theorem 8.2 of Auscher, McIntosh and Russ [3] and the result stated by Russ in [26]. We also apply Theorems 3.1 through 3.4 to the Euclidean space $\mathbb{R}^m$ endowed with the measure $w(x) \, dx$, where $w \in A_2(\mathbb{R}^m)$ (the class of Muckenhoupt weights), and to the boundary of an unbounded model polynomial domain in $\mathbb{C}^2$ introduced by Nagel and Stein [24] (see also [23]); see Propositions 4.2 and 4.3 below. We point out that Theorems 3.1 through 3.4 are also valid for Lie groups of polynomial growth. On the other hand, in this setting, Saloff-Coste has already obtained certain grand and radial maximal functions and atomic characterizations of Hardy spaces; see [27, 28, 29, 30] for more details.

The paper is organized as follows. We recall, in Section 2, some notation and definitions; state, in Section 3, the main results of this paper, Theorems 3.1 through 3.4; give, in Section 4, some applications; and finally, in Section 5, prove Theorems 3.1 through 3.4 by employing Calderón reproducing formulæ (see Theorems 5.1 and 5.2) established in
[18, 11].

We finally make some conventions. Throughout this paper, we always use $C$ to denote a positive constant which is independent of the main parameters involved but whose value may differ from line to line. Constants with subscripts do not change throughout the whole paper. Set $a \wedge b \equiv \min\{a, b\}$ for $a, b \in \mathbb{R}$.

## 2 Preliminaries

The following notion of approximations of the identity on RD-spaces were first introduced in [18]. In what follows, we set $V_r(x) \equiv \mu(B(x, r))$ and $V(x, y) \equiv \mu(B(x, d(x, y)))$ for $x, y \in \mathcal{X}$ and $r \in (0, \infty)$.

**Definition 2.1** Let $\epsilon_1 \in (0, 1)$, $\epsilon_2 > 0$ and $\epsilon_3 > 0$. A sequence $\{S_k\}_{k \in \mathbb{Z}}$ of bounded linear integral operators on $L^2(\mathcal{X})$ is called an approximation of the identity of order $(\epsilon_1, \epsilon_2, \epsilon_3)$ (for short, $(\epsilon_1, \epsilon_2, \epsilon_3)$-AOTI), if there exists a positive constant $C_4$ such that for all $k \in \mathbb{Z}$ and $x, x', y, y' \in \mathcal{X}$, $S_k(x, y)$, the integral kernel of $S_k$, is a measurable function from $\mathcal{X} \times \mathcal{X}$ into $\mathbb{C}$ satisfying

(i) $|S_k(x, y)| \leq C_4 \frac{1}{V_{2-k}(x) + V(x, y)} \left[ \frac{2^{-k}}{d(x, y)+d(x, y)} \right]^{\epsilon_2}$;

(ii) $|S_k(x, y) - S_k(x', y)| \leq C_4 \frac{1}{V_{2-k}(x) + V(x, y)} \left[ \frac{2^{-k}}{d(x, y)+d(x, y)} \right]^{\epsilon_2}$ for $d(x, x') \leq [2^{-k} + d(x, y)]/2$;

(iii) Property (ii) also holds with $x$ and $y$ interchanged;

(iv) $|[S_k(x, y) - S_k(x', y)] - [S_k(x', y) - S_k(x', y')]| \leq C_4 \frac{1}{V_{2-k}(x) + V(x, y)} \left[ \frac{2^{-k}}{d(x, y)+d(x, y)} \right]^{\epsilon_3}$ for $d(x, x') \leq [2^{-k} + d(x, y)]/3$ and $d(y, y') \leq [2^{-k} + d(x, y)]/3$;

(v) $\int_{\mathcal{X}} S_k(x, z) d\mu(z) = 1 = \int_{\mathcal{X}} S_k(z, y) d\mu(z)$ for all $x, y \in \mathcal{X}$.

**Remark 2.1** (i) In [18], for any $N > 0$, it was proved that there exists a $(1, N, N)$-AOTI with bounded support in the sense that $S_k(x, y) = 0$ when $d(x, y) > C2^{-k}$, where $C$ is a fixed positive constant independent of $k$. In this case, $\{S_k\}_{k \in \mathbb{Z}}$ is called a 1-AOTI with bounded support; see [18].

(ii) If a sequence $\{S_{\tilde{t}}\}_{t>0}$ of bounded linear integral operators on $L^2(\mathcal{X})$ satisfies (i) through (v) of Definition 2.1 with $2^{-k}$ replaced by $t$, then we call $\{S_{\tilde{t}}\}_{t>0}$ a continuous approximation of the identity of order $(\epsilon_1, \epsilon_2, \epsilon_3)$ (for short, continuous $(\epsilon_1, \epsilon_2, \epsilon_3)$-AOTI). For example, if $\{S_k\}_{k \in \mathbb{Z}}$ is an $(\epsilon_1, \epsilon_2, \epsilon_3)$-AOTI and if we set $\tilde{S}_t(x, y) \equiv S_k(x, y)$ for $t \in (2^{-k-1}, 2^{-k}]$ with $k \in \mathbb{Z}$, then $\{\tilde{S}_t\}_{t>0}$ is a continuous $(\epsilon_1, \epsilon_2, \epsilon_3)$-AOTI.

(iii) If $S_k$ (resp. $\tilde{S}_t$) satisfies (i), (ii), (iii) and (v) of Definition 2.1, then $S_k S_k$ (resp. $\tilde{S}_t \tilde{S}_t$) satisfies the conditions (i) through (v) of Definition 2.1; see [18].

(iv) For any RD-space $(\mathcal{X}, d, \mu)$, if we relax $d$ to be a quasi-metric, then there exist constants $\theta \in (0, 1)$ and $C > 0$ and quasi-metric $\tilde{d}$ which is equivalent to $d$ such that $|d(x, y) - \tilde{d}(z, y)| \leq C[d(z, x)]^\theta [d(x, y) + \tilde{d}(z, y)]$ for all $x, y, z \in \mathcal{X}$; see [20]. By this and an argument similar to that used in [18], we know the existence of the approximation of the identity $(\theta, N, N)$-AOTI with bounded support, where $N > 0$. 

---

**Hardy spaces**
The following spaces of test functions play an important role in the theory of function spaces on spaces of homogeneous type; see [17, 18].

**Definition 2.2** Let $x \in X$, $r > 0$, $\beta \in (0, 1]$ and $\gamma > 0$. A function $f$ on $X$ is said to belong to the space of test functions, $\mathcal{G}(x, r, \beta, \gamma)$, if there exists a nonnegative constant $C$ such that

(i) $|f(y)| \leq C \frac{1}{V(x,y)} \left(\frac{r}{r+d(x,y)}\right)^\gamma$ for all $y \in X$;

(ii) $|f(z)-f(y)| \leq C \left(\frac{d(y,z)}{r+d(x,y)}\right)^\beta \frac{1}{V(x,y)} \left(\frac{r}{r+d(x,y)}\right)^\gamma$ for all $y, z \in X$ satisfying that $d(y, z) \leq [r + d(x, y)]/2$.

Moreover, for any $f \in \mathcal{G}(x, r, \beta, \gamma)$, we define its norm by

$$\|f\|_{\mathcal{G}(x, r, \beta, \gamma)} \equiv \inf \{C : (i) \text{ and } (ii) \text{ hold}\}.$$

The space $\mathcal{G}(x, r, \beta, \gamma)$ is defined to be the set of all functions $f \in \mathcal{G}(x, r, \beta, \gamma)$ satisfying that $\int_X f(y) \, d\mu(y) = 0$. Moreover, we endow the space $\mathcal{G}(x, r, \beta, \gamma)$ with the same norm as the space $\mathcal{G}(x, \beta, \gamma)$.

It is easy to see that $\mathcal{G}(x, r, \beta, \gamma)$ is a Banach space. Let $\epsilon \in (0, 1]$ and $\beta, \gamma \in (0, \epsilon]$. For applications, we further define the space $\mathcal{G}_0^\epsilon(x, r, \beta, \gamma)$ to be the completion of the set $\mathcal{G}(x, r, \epsilon, \beta)$ in $\mathcal{G}(x, r, \beta, \gamma)$. For $f \in \mathcal{G}_0^\epsilon(x, r, \beta, \gamma)$, define $\|f\|_{\mathcal{G}_0^\epsilon(x, r, \beta, \gamma)} \equiv \|f\|_{\mathcal{G}(x, r, \beta, \gamma)}$.

Let $(\mathcal{G}_0^\epsilon(x, r, \beta, \gamma))'$ be the set of all continuous linear functionals on $\mathcal{G}_0^\epsilon(x, r, \beta, \gamma)$, and as usual, endow $(\mathcal{G}_0^\epsilon(x, r, \beta, \gamma))'$ with the weak* topology. Throughout the whole paper, we fix $x_1 \in X$ and write $\mathcal{G}(\beta, \gamma) = \mathcal{G}(x_1, 1, \beta, \gamma)$, and $(\mathcal{G}_0^\epsilon(\beta, \gamma))' \equiv (\mathcal{G}_0^\epsilon(x_1, 1, \beta, \gamma))'$.

Observe that for any $x \in X$ and $r > 0$, $\mathcal{G}_0^\epsilon(x, r, \beta, \gamma) = \mathcal{G}_0^\epsilon(\beta, \gamma)$ with equivalent norms.

Similarly, define the space $\mathcal{G}_0^\epsilon(x, r, \beta, \gamma)$ to be the completion of the set $\mathcal{G}(x, r, \epsilon, \beta)$ in $\mathcal{G}(x, r, \beta, \gamma)$. For $f \in \mathcal{G}_0^\epsilon(x, r, \beta, \gamma)$, define $\|f\|_{\mathcal{G}_0^\epsilon(x, r, \beta, \gamma)} \equiv \|f\|_{\mathcal{G}(x, r, \beta, \gamma)}$. Denote by $(\mathcal{G}_0^\epsilon(x, r, \beta, \gamma))'$ the set of all continuous linear functionals from $\mathcal{G}_0^\epsilon(x, r, \beta, \gamma)$ to $C$, and endow $(\mathcal{G}_0^\epsilon(x, r, \beta, \gamma))'$ with the weak* topology. Write $\mathcal{G}_0^\epsilon(\beta, \gamma) \equiv \mathcal{G}(x_1, 1, \beta, \gamma)$. For any $x \in X$ and $r > 0$, we also have $\mathcal{G}_0^\epsilon(x, r, \beta, \gamma) = \mathcal{G}_0^\epsilon(\beta, \gamma)$ with equivalent norms.

Now we recall the following maximal functions.

**Definition 2.3** (i) Let $\epsilon \in (0, 1]$, $\beta, \gamma \in (0, \epsilon)$ and $\ell \in \mathbb{Z}$. For any $f \in (\mathcal{G}_0^\epsilon(\beta, \gamma))'$, the grand maximal function $G^{(\epsilon, \beta, \gamma)}(f) \equiv \sup \{ (f, \varphi) : \varphi \in C_0^\infty(\beta, \gamma), \|\varphi\|_{\mathcal{G}(x, r, \beta, \gamma)} \leq 1 \text{ for some } r > 0 \}$, and the localized grand maximal function $G^{(\epsilon, \beta, \gamma)}_\ell(f)$ by setting, for all $x \in X$,

$$G^{(\epsilon, \beta, \gamma)}_\ell(f)(x) \equiv \sup \left\{ (f, \varphi) : \varphi \in C_0^\infty(\beta, \gamma), \|\varphi\|_{\mathcal{G}(x, r, \beta, \gamma)} \leq 1 \text{ for some } r \in (0, 2^{-\ell}) \right\}.$$

(ii) Let $\epsilon_1 \in (0, 1]$, $\epsilon_2$, $\epsilon_3 > 0$, $\epsilon \in (0, \epsilon_1 \wedge \epsilon_2)$ and $\{S_k\}_{k \in \mathbb{Z}}$ be an $(\epsilon_1, \epsilon_2, \epsilon_3)$-AOTI. Let $\ell \in \mathbb{Z}$. For any $\beta, \gamma \in (0, \epsilon)$ and $f \in (\mathcal{G}_0^\epsilon(\beta, \gamma))'$, the radial maximal function $S^+(f)$ is defined by setting, for all $x \in X$,

$$S^+(f)(x) \equiv \sup_{k \in \mathbb{Z}} |S_k(f)(x)|,$$
and the localized radial maximal function \( S_{\ell}^{+}(f) \) by setting, for all \( x \in \mathcal{X} \),

\[
S_{\ell}^{+}(f)(x) \equiv \sup_{k \geq \ell} |S_k(f)(x)|.
\]

When there exists no ambiguity, we write \( G^{(r, \beta, \gamma)}(f) \) and \( G_{\ell}^{(r, \beta, \gamma)}(f) \) simply by \( G(f) \) and \( G_{\ell}(f) \), respectively. It is easy to see that for all \( x \in \mathcal{X} \), \( S^{+}(f)(x) \leq G(f)(x) \), \( S_{\ell}^{+}(f)(x) \leq G_{\ell}(f)(x) \) and for any \( \ell \geq k \), \( G_{\ell}(f)(x) \leq G_k(f)(x) \leq CG_{\ell}(f)(x) \), where \( C \) is a positive constant depending on \( k \) and \( \ell \), but not on \( f \) and \( x \).

**Definition 2.4** Let \( p \in (n/(n + 1), 1] \) and \( n(1/p - 1) < \beta, \gamma < \epsilon < 1 \).

(i) The Hardy space \( H^p(\mathcal{X}) \) is defined by

\[
H^p(\mathcal{X}) \equiv \{ F \in (G_{0}(\beta, \gamma))^\prime : \| F \|_{H^p(\mathcal{X})} \equiv \| G(F) \|_{L^p(\mathcal{X})} < \infty \}.
\]

(ii) Let \( \ell \in \mathbb{Z} \). The localized Hardy space \( H^p_{\ell}(\mathcal{X}) \) is defined by

\[
H^p_{\ell}(\mathcal{X}) \equiv \{ F \in (G_{0}(\beta, \gamma))^\prime : \| F \|_{H^p_{\ell}(\mathcal{X})} \equiv \| G_{\ell}(F) \|_{L^p(\mathcal{X})} < \infty \}.
\]

It was proved in [10] that if \( p \in (n/(n + 1), 1] \), then the definition of \( H^p(\mathcal{X}) \) is independent of the choices of \( \epsilon \in (n(1/p - 1), 1) \) and \( \beta, \gamma \in (n(1/p - 1), \epsilon) \). This also holds for \( H^p_{\ell}(\mathcal{X}) \) by a similar argument. Here we omit the details.

Now we recall the notion of the atomic Hardy space of Coifman and Weiss [7].

**Definition 2.5** Let \( p \in (0, 1], q \in [1, \infty] \cap (p, \infty] \) and \( \ell \in \mathbb{Z} \).

(i) A measurable function \( a \) is called a \((p, q)\)-atom associated to the ball \( B(x, r) \) if

(A1) \( \text{supp} \ a \subset B(x, r) \) for certain \( x \in \mathcal{X} \) and \( r > 0 \),

(A2) \( \| a \|_{L^q(\mathcal{X})} \leq [\mu(B(x, r))]^{1/q - 1/p} \),

(A3) \( \int_{\mathcal{X}} a(x) \, d\mu(x) = 0 \).

(ii) A measurable function \( a \) is called a \((p, q)\)-atom associated to the ball \( B(x, r) \) if \( r \leq 2^{-\ell} \) and \( a \) satisfies (A1) and (A2), and when \( r < 2^{-\ell} \), \( a \) also satisfies (A3).

**Definition 2.6** Let \( p \in (0, 1] \).

(i) The space \( \text{Lip}(1/p - 1, \mathcal{X}) \) is defined to be the collection of all functions \( f \) satisfying

\[
\| f \|_{\text{Lip}(1/p - 1, \mathcal{X})} \equiv \sup_{x, y \in \mathcal{X}, B \ni x, y} \left[ \mu(B) \right]^{1 - 1/p} |f(x) - f(y)| < \infty,
\]

where the supremum is taken over all \( x, y \in \mathcal{X} \) and all balls containing \( x \) and \( y \).

(ii) The space \( \text{Lip}_{\ell}(1/p - 1, \mathcal{X}) \) is defined to be the collection of all functions \( f \) satisfying

\[
\| f \|_{\text{Lip}_{\ell}(1/p - 1, \mathcal{X})} \equiv \sup_{x, y \in \mathcal{X}, B \ni x, y} \left[ \mu(B) \right]^{1/p - 1} \frac{|f(x) - f(y)|}{B} + \sup_{B, r < 2^{-\ell}} \frac{1}{\mu(B)} \int_{B} |f(z)| \, d\mu(z) < \infty,
\]

where \( I_{\ell}(x, y) \) denotes all balls containing \( x \) and \( y \) with radius no more than \( 2^{-\ell} \), the second supremum is taken over all balls with radius more than \( 2^{-\ell} \).
Definition 2.7 Let $p \in (0, 1]$ and $q \in [1, \infty] \cap (p, \infty]$.

(i) The space $H^{p, q}(\mathcal{X})$ is defined to be the set of all $f = \sum_{j \in \mathbb{N}} \lambda_j a_j$ in $(\text{Lip}(1/p - 1, \mathcal{X}))'$ when $p < 1$ and in $L^1(\mathcal{X})$ when $p = 1$, where $\{a_j\}_{j \in \mathbb{N}}$ are $(p, q)$-atoms and $\{\lambda_j\}_{j \in \mathbb{N}} \subset \mathbb{C}$ such that $\sum_{j \in \mathbb{N}} |\lambda_j|^p < \infty$. For any $f \in H^{p, q}(\mathcal{X})$, define $\|f\|_{H^{p, q}(\mathcal{X})} = \inf(\{\sum_{j \in \mathbb{N}} |\lambda_j|^p)^{1/p}\}$, where the infimum is taken over all the above decompositions of $f$.

(ii) The space $H^{p, q}_\ell(\mathcal{X})$ is defined as in (i) with $(p, q)$-atoms replaced by $(p, q)\ell$-atoms and $(\text{Lip}(1/p - 1, \mathcal{X}))'$ replaced by $(\text{Lip}_\ell(1/p - 1, \mathcal{X}))'$.

Since $H^{p, q}(\mathcal{X}) = H^{p, \infty}(\mathcal{X})$ and $H^{p, q}_\ell(\mathcal{X}) = H^{p, \infty}_\ell(\mathcal{X})$ (see [7] and also [35]), we always write $H^{p, q}(\mathcal{X})$ and $H^{p, q}_\ell(\mathcal{X})$ as $H^{p, \infty}(\mathcal{X})$ and $H^{p, \infty}_\ell(\mathcal{X})$, respectively. Moreover, the dual spaces of $H^{p, \infty}(\mathcal{X})$ and $H^{p, \infty}_\ell(\mathcal{X})$ are, respectively, $\text{Lip}(1/p - 1, \mathcal{X})$ and $\text{Lip}_\ell(1/p - 1, \mathcal{X})$ when $p < 1$, and $\text{BMO}(\mathcal{X})$ and $\text{BMO}_\ell(\mathcal{X})$ when $p = 1$; see [7] and also [35] for the details.

3 Main results

The first result is on the characterization of the radial maximal function of the Hardy space $H^p(\mathcal{X})$.

Theorem 3.1 Let $\epsilon_1 \in (0, 1], \epsilon_2, \epsilon_3 > 0$, $\epsilon \in (0, \epsilon_1 \wedge \epsilon_2)$ and $\{S_k\}_{k \in \mathbb{Z}}$ be an $(\epsilon_1, \epsilon_2, \epsilon_3)$-AOTI. Let $p \in (n/(n + \epsilon), 1]$ and $\beta, \gamma \in (n(1/p - 1), \epsilon)$. Then for any $f \in (\mathcal{G}_0(\beta, \gamma))'$, $f \in H^p(\mathcal{X})$ if and only if $\|S^+(f)\|_{L^p(\mathcal{X})} < \infty$; moreover, for all $f \in H^p(\mathcal{X})$, $\|f\|_{H^p(\mathcal{X})} \sim \|S^+(f)\|_{L^p(\mathcal{X})}$.

The proof of Theorem 3.1 is given in Section 5. Moreover, by Theorem 4.16 in [10] and Theorem 3.1, we have the following result.

Corollary 3.1 Let $\epsilon_1 \in (0, 1], \epsilon_2, \epsilon_3 > 0$, $\epsilon \in (0, \epsilon_1 \wedge \epsilon_2)$ and $\{S_k\}_{k \in \mathbb{Z}}$ be an $(\epsilon_1, \epsilon_2, \epsilon_3)$-AOTI. Let $p \in (n/(n + \epsilon), 1]$ and $\beta, \gamma \in (n(1/p - 1), \epsilon)$. Then $f \in H^p(\mathcal{X})$ if and only if $f \in H^p_{\text{at}}(\mathcal{X})$ or if and only if $f \in (\mathcal{G}_0(\beta, \gamma))'$ and $S^+(f) \in L^p(\mathcal{X})$; moreover, for all $f \in H^p_{\text{at}}(\mathcal{X})$, $\|S^+(f)\|_{L^p(\mathcal{X})} \sim \|f\|_{H^p(\mathcal{X})} \sim \|f\|_{H^p_{\text{at}}(\mathcal{X})}$.

Remark 3.1 (i) If $\{S_k\}_{k \in \mathbb{Z}}$ is replaced by a continuous $(\epsilon_1, \epsilon_2, \epsilon_3)$-AOTI as in Remark 2.1 (ii), then Theorem 3.1 and Corollary 3.1 still hold for all $p \in (n/(n + \epsilon_1), 1]$.

(ii) If we relax $d$ to be a quasi-metric, then Theorem 3.1 and Corollary 3.1 still hold by replacing $p \in (n/(n + 1), 1]$ with $p \in (n/(n + \theta), 1]$ for any $(\theta, 1, 1)$-AOTI $\{S_k\}_{k \in \mathbb{Z}}$, where $\theta \in (0, 1)$ is the same as in Remark 2.1 (iv).

(iii) Corollary 3.1 tells us that for $p \in (n/(n + 1), 1]$, $H^p_{\text{at}}(\mathcal{X})$ is characterized by the radial maximal function in Definition 2.3 (ii), which completely answers Question 1 when $\mu(\mathcal{X}) = \infty$ asked by Coifman and Weiss. We also remark that Theorem 3.1 and Corollary 3.1 improve the deep result of Uchiyama [34] and that of [11] to the best range $p \in (n/(n + 1), 1]$.

(iv) Notice that atoms of $H^p_{\text{at}}(\mathcal{X})$ have only 0-order vanishing moment for $p \in (0, n/(n + 1)]$ here. It is easy to see that the Poisson kernel is just a $(1, 1, 1)$-AOTI as in Definition 2.1, and the radial Poisson maximal function characterizes certain atomic Hardy spaces on $\mathbb{R}^n$. 
which asks the vanishing moment of atoms no less than 1-order, when $p \in (0, n/(n+1)]$; see, for example, [32, pp.91, 107, 133]. This atomic Hardy space is essentially different from $H^p_{at}(\mathcal{X})$ considered here. So it is impossible to use the Poisson maximal function to characterize $H^p_{at}(\mathcal{X})$ when $p \in (0, n/(n+1)]$ here. In this sense, the range $p \in (n/(n+1), 1]$ is the best possible for which $H^p_{at}(\mathcal{X})$ can be characterized by the radial maximal function.

Now we turn to the localized Hardy spaces.

**Theorem 3.2** Let $\epsilon_1 \in (0, 1], \epsilon_2, \epsilon_3 > 0$, $\epsilon \in (0, \epsilon_1 \wedge \epsilon_2)$ and $\{S_k\}_{k \in \mathbb{Z}}$ be an $(\epsilon_1, \epsilon_2, \epsilon_3)$-AOTI. Let $p \in \mathbb{Z}$, $p \in (n/(n+\epsilon), 1]$ and $\beta, \gamma \in (n(1/p - 1), \epsilon)$.

(i) Then $f \in H^p_{\mathcal{X}}(\mathcal{X})$ if and only if $f \in (\mathcal{G}_0^{(\beta, \gamma)})', S_{k+1}^+(f) \in L^p(\mathcal{X})$, and for any $k \in \mathbb{Z}$ and $a > 0$,

$$S_k^{(a)}(f)(x) = \frac{1}{V_{a^{-k}}(x)} \int_{B(x, a^{-k})} |S_k(f)(y)| \, d\mu(y) \in L^p(\mathcal{X}),$$

where $x \in \mathcal{X}$. Moreover, for any $a > 0$, there exists a positive constant $C$ depending on a such that for all $\ell \in \mathbb{Z}$ and $f \in (\mathcal{G}_0^{(\beta, \gamma)})'$,

$$C^{-1} \|f\|_{H^p_{\mathcal{X}}(\mathcal{X})} \leq \|S_{k+1}^+(f)\|_{L^p(\mathcal{X})} + \|S_k^{(a)}(f)\|_{L^p(\mathcal{X})} \leq C \|f\|_{H^p_{\mathcal{X}}(\mathcal{X})}.$$ 

(ii) If $f \in (\mathcal{G}_0^{(\beta, \gamma)})'$ and (3.1) holds for any $k \in \mathbb{Z}$, then $f \in H^p_{\mathcal{X}}(\mathcal{X})$ if and only if $S_k^+(f) \in L^p(\mathcal{X})$; moreover, there exists a positive constant $C$, independent of $k$ and $\ell$, such that for all $f \in H^p_{\mathcal{X}}(\mathcal{X})$, $C^{-1} \|f\|_{H^p_{\mathcal{X}}(\mathcal{X})} \leq \|S_k^+(f)\|_{L^p(\mathcal{X})} \leq C \|f\|_{H^p_{\mathcal{X}}(\mathcal{X})}.$

The basic idea of the proof of Theorem 3.2 (i) is similar to that used in the proof of Theorem 3.1. To prove Theorem 3.2 (ii), observe that (3.1) and $S_k^+(f) \in L^p(\mathcal{X})$ imply that $G_{t}(f) \in L^p(\mathcal{X})$ by Theorem 3.2 (i). Based on the observation that the constants appearing in Calderón reproducing formulae are uniform in $j$, where $j$ measures the size of dyadic cubes $\{Q^{(\beta, \gamma)}_{k, \nu}\}$ appearing in the average term of the inhomogeneous Calderón reproducing formula (see Theorem 5.2 below). Then we prove that for certain $r \in (n/(n+\epsilon_1), p)$, and all $j$ large enough and $x \in \mathcal{X}$,

$$G_t(f)(x) \leq C 2^{j(n(1-r-1)}[HL([S_{k}^+(f)])^r](x)]^{1/r} + C 2^{j(\epsilon_1+n-n/r)}[HL([G_t(f)])^r](x)]^{1/r},$$

where $C$ is a positive constant independent of $j$, $f$ and $x$ and HL denotes the Hardy-Littlewood maximal function. Taking $j$ such that $C 2^{j(\epsilon_1+n-n/r)} \leq 1/2$, we then obtain Theorem 3.2 (ii).

By first establishing a connection between $H^p(\mathcal{X})$ and $H^p_{\mathcal{X}}(\mathcal{X})$, we then obtain the equivalence between $H^p_{\mathcal{X}}(\mathcal{X})$ and $H^p_{\mathcal{X}, at}(\mathcal{X})$ via Corollary 3.1.

**Theorem 3.3** Let $\epsilon_1 \in (0, 1], \epsilon_2, \epsilon_3 > 0$, $\epsilon \in (0, \epsilon_1 \wedge \epsilon_2)$ and $\{S_k\}_{k \in \mathbb{Z}}$ be an $(\epsilon_1, \epsilon_2, \epsilon_3)$-AOTI. Let $p \in (n/(n+\epsilon), 1]$ and $\beta, \gamma \in (n(1/p - 1), \epsilon)$. Then there exists a positive constant $C$ such that for all $\ell \in \mathbb{Z}$ and $f \in (\mathcal{G}_0^{(\beta, \gamma)})'$, $\|f-S_{k}(f)\|_{H^p(\mathcal{X})} \leq C \|G_{\ell}(f)\|_{L^p(\mathcal{X})}.$
**Theorem 3.4** Let \( \epsilon \in (0, 1], p \in (n/(n + \epsilon), 1] \) and \( \beta, \gamma \in (n(1/p - 1), \epsilon) \). Then for each \( \ell \in \mathbb{Z}, f \in H^{p}_{\ell, at}(\mathcal{X}) \) if and only if \( f \in (\mathcal{G}_0(\beta, \gamma), \gamma)' \) and \( \|G_\ell(f)\|_{L^p(\mathcal{X})} < \infty \); moreover, there exists a positive constant \( C \), independent of \( \ell \), such that for all \( f \in H^{p}_{\ell, at}(\mathcal{X}) \),

\[
C^{-1}\|f\|_{H^{p}_{\ell, at}(\mathcal{X})} \leq \|G_\ell(f)\|_{L^p(\mathcal{X})} \leq C\|f\|_{H^{p}_{\ell, at}(\mathcal{X})}.
\]

**Remark 3.2**

(i) If \( p = 1 \), then Theorem 3.2 (i) gives the radial maximal function characterization of \( H^{p}_{\ell, at}(\mathcal{X}) \) since (3.1) is just \( S_\ell(f) \in L^1(\mathcal{X}) \). This result when \( p = 1 \) was also obtained in [36]. When \( p < 1 \), it is still unclear if one can remove the extra assumption (3.1). But this is quite reasonable since we use the inhomogeneous Calderón reproducing formula and the same phenomena happens in the Littlewood-Paley characterization of the localized Hardy space established in [18]. We also notice that when \( \mathcal{X} \) is an RD-space and \( \mu(\mathcal{X}) < \infty \), Theorem 3.2 also holds; moreover, in this setting, (3.1) holds automatically.

(ii) Combining Theorems 3.2 with 3.4, we characterize \( H^{p}_{\ell, at}(\mathcal{X}) \) with \( p \in (n/(n + 1), 1] \) via the radial maximal function and the grand maximal function. By a reason similar to that of Remark 3.1, the range \( p \in (n/(n + 1), 1] \) is the best possible for which \( H^{p}_{\ell, at}(\mathcal{X}) \) can be characterized by the radial maximal function.

(iii) If we relax \( d \) to be a quasi-metric, then Theorems 3.2 through 3.4 still hold by replacing \( p \in (n/(n + 1), 1] \) with \( p \in (n/(n + \theta), 1] \) for any \( (\theta, 1, 1) \) - AOTI \( \{S_k\}_{k \in \mathbb{Z}} \), where \( \theta \in (0, 1) \) is the same as in Remark 2.1 (iv).

(iv) Theorems 3.2 through 3.4 are also true if we replace the discrete approximation of the identity by the continuous one as in Remark 2.1 (ii).

## 4 Applications to some differential operators

Beyond the Ahlfors \( n \)-regular metric measure space and Lie groups of polynomial growth, we list several other specific settings where Theorems 3.1 through 3.4 work.

(I) **Hardy spaces associated to a certain Laplace-Beltrami operator**

Let \( M \) be a complete noncompact connected Riemannian manifold, \( d \) the geodesic distance, \( \mu \) the Riemannian measure and \( \nabla \) the Riemannian gradient. Denote by \( | \cdot | \) the length in the tangent space. One defines \( \Delta \), the Laplace-Beltrami operator, as the self-adjoint positive operator on \( L^2(M) \) by the formal integration by parts \( \langle \Delta f, f \rangle = \|\nabla f\|_{L^2(M)}^2 \) for all \( f \in C_0^\infty(M) \). Denote by \( T_t(x, y) \) with \( t > 0 \) and \( x, y \in M \) the heat kernel of \( M \), namely, the kernel of the heat semigroup \( \{e^{-t\Delta}\}_{t \geq 0} \). One says \( T_t \) satisfies the Li-Yau type estimates if there exist some positive constants \( C_5, C_5 \) and \( C \) such that

\[
C^{-1}\frac{1}{V_\sqrt{t}(x)} \exp\left\{-\frac{d(x, y)^2}{C_5 t}\right\} \leq T_t(x, y) \leq C\frac{1}{V_\sqrt{t}(x)} \exp\left\{-\frac{d(x, y)^2}{C_5 t}\right\} \quad (4.1)
\]

for all \( x, y \in M \) and \( t > 0 \). It is well-known [19] that such estimates hold on manifolds with non-negative Ricci curvature. Later it has been proved in [31] that the Li-Yau type estimates are equivalent to the conjunction of the doubling property (1.1) and the scaled Poincaré inequality that for every ball \( B \equiv B(x, r) \) and every \( f \) with \( f, \nabla f \in L^1_{loc}(M) \),

\[
\int_B |f(y) - f_B|^2 \, d\mu(y) \leq C r^2 \int_B |\nabla f(y)|^2 \, d\mu(y), \quad (4.2)
\]
where $f_B$ denotes the average of $f$ on $B$, namely, $f_B = \frac{1}{\mu(B)} \int_B f(x) \, d\mu(x)$.

Assume that $M$ satisfies the doubling property (1.1) and supports a scaled Poincaré inequality (4.2). Then $(M, d, \mu)$ is a connected space of homogeneous type and hence, an RD-space. Observe that the Li-Yau type estimates do already imply some regularity estimates for the heat kernel: there exist positive constants $C$ and $\alpha \in (0, 1)$ such that

$$|T_t(x, y) - T_t(z, y)| \leq \left( \frac{d(x, z)}{\sqrt{t}} \right)^\alpha \frac{C}{V_{\sqrt{t}}(y)} \quad (4.3)$$

for all $x, y, z \in M$ and $t > 0$; see, for example, [31, 13]. See also [2] for more discussions on the regularity of the heat kernels and connections with the boundedness of Riesz transforms in this setting.

Notice that $e^{-t\Delta} 1 = 1$; see [14]. From this, (4.1), (4.3), the semigroup property and Remark 2.1, it follows that $\{T_t\}_{t > 0}$ is just a continuous $(\alpha', N, N)$-AOTI for each $N \in \mathbb{N}$ and $\alpha' \in (0, \alpha)$ as in Definition 2.1. Define the semigroup maximal function and the localized one by $T^+(f)(x) \equiv \sup_{t > 0} |e^{-t\Delta}(f)(x)|$ and

$$T_{2^{-\ell}}^+(f)(x) \equiv \sup_{0 < \epsilon < 2^{-\ell}} |e^{-t\Delta}(f)(x)| + \frac{1}{V_{2^{-\ell}}(x)} \int_{B(x, 2^{-\ell})} |T_{2^{-\ell}}(f)(y)| \, d\mu(y)$$

for all $\ell \in \mathbb{Z}$, suitable distributions $f$ and $x \in \mathcal{X}$.

Let $n$ be the same as in (1.1). For $p \in (n/(n+\alpha), 1]$, the Hardy space $H^p_\Delta(M)$ and the localized one $H^p_{\ell, \Delta}(M)$ with $\ell \in \mathbb{Z}$ on $M$ associated to the Laplace-Beltrami operator $\Delta$ are defined, respectively, by

$$H^p_\Delta(M) \equiv \left\{ f \in (G^0_\beta(\mathcal{X}))' : \|f\|_{H^p_\Delta(M)} = \|T^+(f)\|_{L^p(\mathcal{X})} < \infty \right\}, \quad (4.4)$$

and

$$H^p_{\ell, \Delta}(M) \equiv \left\{ f \in (G^0_\beta(\mathcal{X}))' : \|f\|_{H^p_{\ell, \Delta}(M)} = \|T^+_{2^{-\ell}}(f)\|_{L^p(\mathcal{X})} < \infty \right\}, \quad (4.5)$$

where $\epsilon \in (n(1/p - 1), \alpha)$ and $\beta, \gamma \in (n(1/p - 1), \epsilon)$. Applying Theorems 3.1, 3.2 and 3.4, we have the following result.

**Proposition 4.1** Let $p \in (n/(n+\alpha), 1]$ and $\ell \in \mathbb{Z}$.

(i) Then $H^p_\Delta(M) = H^p_{\ell, \Delta}(M)$ with equivalent norms.

(ii) Then $H^p_{\ell, \Delta}(M) = H^p_{\ell, at}(M)$ with equivalent norms uniformly in $\ell$.

**Remark 4.1**

(i) Proposition 4.1 also implies that the definition of the (localized) Hardy spaces are independent of the choices of $\epsilon \in (n(1/p - 1), \alpha)$ and $\beta, \gamma \in (n(1/p - 1), \epsilon)$.

(ii) Proposition 4.1 improves the result stated by Russ [26] and Theorem 8.2 of Auscher, McIntosh and Russ [3].

(iii) According to the approach here, the range $p \in (n/(n+\alpha), 1]$ is the best possible to define $H^p_\Delta(\mathcal{X})$ and $H^p_{\ell, \Delta}(M)$, and to establish their equivalences with the corresponding atomic Hardy spaces, respectively.
Let $m \geq 3$ and $\mathbb{R}^m$ be the $m$-dimensional Euclidean space endowed with the Euclidean norm $| \cdot |$ and the Lebesgue measure $dx$. Recall that a nonnegative locally integrable function $w$ is called an $A_2(\mathbb{R}^m)$ weight in the sense of Muckenhoupt if

$$\sup_B \left\{ \frac{1}{|B|} \int_B w(x) \, dx \right\} \left\{ \frac{1}{|B|} \int_B |w(x)|^{-1} \, dx \right\} < \infty,$$

where the supremum is taken over all the balls in $\mathbb{R}^m$. Observe that if we set $w(E) \equiv \int_E w(x) \, dx$ for any measurable set $E$, then there exist positive constants $C, n$ and $\kappa$ such that for all $x, \xi \in \mathbb{R}^m$ of the heat kernel and Remark 2.1 implies that

$$\left\{ \text{Theorems 2.1, 2.3, 2.4 and 2.7, and Corollary 3.4 of [15]. This together with the symmetry} \right\},$$

namely, the measure $w(x) \, dx$ satisfies (1.1) and (1.2). Thus $(\mathbb{R}^m, | \cdot |, w(x) \, dx)$ is an RD-space.

Let $w \in A_2(\mathbb{R}^m)$ and $\{a_{i,j}\}_{1 \leq i, j \leq m}$ be a real symmetric matrix function satisfying that for all $x, \xi \in \mathbb{R}^m$,

$$C^{-1} w(x)|\xi|^2 \leq \sum_{1 \leq i, j \leq m} a_{i,j}(x)\xi_i\xi_j \leq C w(x)|\xi|^2.$$

Then the (possibly) degenerate elliptic operator $\mathcal{L}_0$ is defined by

$$\mathcal{L}_0 f(x) \equiv -\frac{1}{w(x)} \sum_{1 \leq i, j \leq m} \partial_i (a_{i,j}(\cdot) \partial_j f)(x),$$

where $x \in \mathbb{R}^m$. Denote by $\{T_t\}_{t \geq 0} \equiv \{e^{-t\mathcal{L}_0}\}_{t \geq 0}$ the semigroup generated by $\mathcal{L}_0$. We also denote the kernel of $T_t$ by $T_t(x, y)$ for all $x, y \in \mathbb{R}^m$ and $t > 0$. Then it is known that there exist positive constants $C, C_0, \tilde{C}_0$ and $\alpha \in (0, 1]$ such that for all $t > 0$ and $x, y \in \mathbb{R}^m$,

$$C^{-1} \frac{1}{V_{\sqrt{t}}(x)} \exp \left\{ -\frac{|x-y|^2}{C_0 t} \right\} \leq T_t(x, y) \leq C \frac{1}{V_{\sqrt{t}}(x)} \exp \left\{ -\frac{|x-y|^2}{C_0 t} \right\};$$

that for all $t > 0$ and $x, y, y' \in \mathbb{R}^m$ with $|y - y'| < |x - y|/4$,

$$|T_t(x, y) - T_t(x, y')| \leq C \frac{1}{V_{\sqrt{t}}(x)} \left( \frac{|y - y'|}{\sqrt{t}} \right)^\alpha \exp \left\{ -\frac{|x-y|^2}{C_0 t} \right\};$$

and, moreover, that for all $t > 0$ and $x, y \in \mathbb{R}^n$, $\int_{\mathbb{R}^m} T_t(x, y) w(y) \, dy = 1$; see, for example, Theorems 2.1, 2.3, 2.4 and 2.7, and Corollary 3.4 of [15]. This together with the symmetry of the heat kernel and Remark 2.1 implies that $\{T_t\}_{t \geq 0}$ is a continuous $(\alpha, N, N)$-AOTI for any $N \in \mathbb{N}$. Thus, Theorems 3.1 through 3.4 in Section 3 also work here for $p \in (n/(n+\alpha), 1]$. Define the Hardy spaces $H^p_{\mathcal{L}_0}(\mathbb{R}^m, w)$ and the localized one $H^p_{\ell, \mathcal{L}_0}(\mathbb{R}^m, w)$ with $\ell \in \mathbb{Z}$ associated to the (possibly) degenerate Laplace as in (4.4) and (4.5) of (1). We then have the following conclusions.
Proposition 4.2 Let $p \in (n/(n + \alpha), 1]$ and $\ell \in \mathbb{Z}$.

(i) Then $H^p_{\mathcal{C}}(\mathbb{R}^m, w) = H^p_{\mathcal{A}}(\mathbb{R}^m, w)$ with equivalent norms.

(ii) Then $H^p_{\ell, \mathcal{C}_0}(\mathbb{R}^m, w) = H^p_{\ell, \mathcal{A}}(\mathbb{R}^m, w)$ with equivalent norms uniformly in $\ell$.

(III) Hardy spaces associated to a certain sub-Laplace operator

The following example deals with the differential operators on the noncompact $C^\infty$-manifold $M$ arising as the boundary of an unbounded model polynomial domain in $\mathbb{C}^2$ introduced by Nagel and Stein [24] (see also [23]).

Let $\Omega \equiv \{(z, \omega) \in \mathbb{C}^2 : \text{Im}[\omega] > P(z)\}$, where $P$ is a real, subharmonic polynomial of degree $m$. Then $M \equiv \partial \Omega$ can be identified with $\mathbb{C} \times \mathbb{R} \equiv \{(z, t) : z \in \mathbb{C}, t \in \mathbb{R}\}$. The basic $(0, 1)$ Levi vector field is then $\mathbf{Z} = \partial_t - i \frac{\partial P}{\partial \overline{t}}\partial_t$, and rewrite as $\mathbf{Z} = X_1 + iX_2$. The real vector fields $\{X_1, X_2\}$ and their commutators of orders $\leq m$ span the tangent space at each point of $M$. Thus $M$ is of finite type $m$.

We denote by $d$ the control distance in $M$ generated by $X_1$ and $X_2$; see [25] for the definition of the control distance. Let $\mu$ be the Lebesgue measure on $M$. Then, Nagel and Stein showed that there exist positive constants $Q \geq 4$ and $C$ such that for all $s \geq 1$, $x \in \mathcal{X}$ and $r > 0$,

$$C^{-1}s^4V(x, r) \leq V(x, sr) \leq Cs^QV(x, r),$$

which implies that $(M, d, \mu)$ is an RD-space; see [23, 24].

Moreover, denote by $X_j^*$ the formal adjoint of $X_j$, namely, $\langle X_j^* \varphi, \psi \rangle = \langle \varphi, X_j \psi \rangle$ for $\varphi, \psi \in C^\infty_c(M)$, where $\langle \varphi, \psi \rangle = \int_M \varphi(x)\psi(x)dx$. In general, $X_j^* = -X_j + a_j$, where $a_j \in C^\infty(M)$. The sub-Laplacian $\mathcal{L}$ on $M$ is formally given by $\mathcal{L} \equiv X_1^*X_1 + X_2X_2X_2$. Set the heat operator $T_t \equiv e^{-t\mathcal{L}}$ for $t > 0$.

Nagel and Stein further established the non-Gaussian upper bound estimate and regularity of $\{T_t\}_{t>0}$; see [23, 22]. These properties further imply that $\{T_t\}_{t>0}$ forms a continuous $(1, 1, 1)$-AOTI as in Definition 2.1. Thus Theorems 3.1 through 3.4 in Section 3 also work here. In particular, we define the Hardy space $H^p_{\mathcal{L}_t}(M)$ and the localized one $H^p_{\ell, \mathcal{L}_t}(M)$ with $\ell \in \mathbb{Z}$ associated to the sub-Laplace as in (4.4) and (4.5) of (I). We then have the following conclusions.

Proposition 4.3 Let $p \in (n/(n + 1), 1]$ and $\ell \in \mathbb{Z}$.

(i) Then $H^p_{\mathcal{L}_t}(M) = H^p_{\mathcal{A}_t}(M)$ with equivalent norms.

(ii) Then $H^p_{\ell, \mathcal{L}_t}(M) = H^p_{\ell, \mathcal{A}_t}(M)$ with equivalent norms uniformly in $\ell$.

5 Proofs of main theorems

We need the Calderón reproducing formula to prove Theorems 3.1 through 3.4. First we recall the dyadic cubes on spaces of homogeneous type constructed by Christ [4].

Lemma 5.1 Let $\mathcal{X}$ be a space of homogeneous type. Then there exists a collection $\{Q^k_\alpha \subset \mathcal{X} : k \in \mathbb{Z}, \alpha \in I_k\}$ of open subsets, where $I_k$ is certain index set, and positive constants $\delta \in (0, 1)$, $C_7$ and $C_8$ such that

(i) $\mu(\mathcal{X} \setminus \bigcup_{\alpha} Q^k_\alpha) = 0$ for each fixed $k$ and $Q^k_\alpha \cap Q^k_\beta = \emptyset$ if $\alpha \neq \beta$;
(ii) for any $\alpha, \beta, k, \ell$ with $\ell \geq k$, either $Q^\ell_\beta \subset Q^k_\alpha$ or $Q^\ell_\beta \cap Q^k_\alpha = \emptyset$;

(iii) for each $(k, \alpha)$ and $\ell < k$, there exists a unique $\beta$ such that $Q^k_\alpha \subset Q^\ell_\beta$;

(iv) $\text{diam}(Q^k_\alpha) \leq C\delta^k$;

(v) each $Q^k_\alpha$ contains certain ball $B(z^k_\alpha, C\delta^k)$, where $z^k_\alpha \in \mathcal{X}$.

In fact, we can think of $Q^k_\alpha$ as being a dyadic cube with diameter roughly $\delta^k$ and centered at $z^k_\alpha$. In what follows, for simplicity, we may assume that $\delta = 1/2$; see [18, p. 25] as to how to remove this restriction.

Fix $j_0 \in \mathbb{N}$ such that $2^{-j_0}C_7 < 1/3$. For any given $j \geq j_0$, and for any $k \in \mathbb{Z}$ and $\tau \in I_k$, we denote by $Q^k_\tau$, $\nu = 1, 2, \ldots, N(k, \tau)$, the set of all cubes $Q^k_\tau \subset Q^\ell_\tau$. Denote by $z^k_\tau$ the center of $Q^k_\tau$ and let $y^k_\tau$ be a point in $Q^k_\tau$.

For any given $\ell \in \mathbb{Z} \cup \{-\infty\}$ and $j \geq j_0$, we pick a point $y^k_\ell$ in the cube $Q^k_\ell$ for each integer $k \geq \ell$ when $\ell \in \mathbb{Z}$ or $k \in \mathbb{Z}$ when $\ell = -\infty$, $\tau \in I_k$ and $\nu = 1, \ldots, N(k, \tau)$, and denote by $D(\ell, j)$ the set of all these points; namely, $D(\ell, j) = \{y^k_\ell : k \in \mathbb{Z}, \tau \in I_k, \nu = 1, \ldots, N(k, \tau)\}$.

The following Calderón reproducing formula was established in Theorem 4.1 of [18].

**Theorem 5.1** Let $\epsilon_1 \in (0, 1], \epsilon_2, \epsilon_3 > 0$, $\epsilon \in (0, 0, \epsilon_1 \wedge \epsilon_2)$ and $\{S_k\}_{k \in \mathbb{Z}}$ be an $(\epsilon_1, \epsilon_2, \epsilon_3)$-AOTI. Set $D_k \equiv S_k - S_{k-1}$ for $k \in \mathbb{Z}$. Then there exists $j > j_0$ such that for any choice of $D(-\infty, j)$, there exists a sequence of operators $\{\tilde{D}_k\}_{k \in \mathbb{Z}}$ such that for all $f \in (G_0^0(\beta, \gamma))^\ell$ with $\beta, \gamma \in (0, \epsilon)$, and $x \in \mathcal{X}$,

$$f(x) = \sum_{k=\infty}^{\infty} \sum_{\tau \in I_k}^{\tau = \infty} \sum_{\nu = 1}^{N(k, \tau)} \mu(Q^k_\tau) \tilde{D}_k(x, y^k_\nu) D_k(y^k_\nu),$$

where the series converge in $(G_0^0(\beta, \gamma))^\ell$. Moreover, for any $\epsilon' \in (\epsilon, \epsilon_1 \wedge \epsilon_2)$, there exists a positive constant $C$ depending on $\epsilon'$, but not on $j$ and $D(-\infty, j)$, such that $\{\tilde{D}_k(x, y)\}_{k \in \mathbb{Z}}$, the kernels of $\{\tilde{D}_k\}_{k \in \mathbb{Z}}$, satisfy (i) and (ii) of Definition 2.1 with $\epsilon_1$ and $\epsilon_2$ replaced by $\epsilon'$ and the constant $C_4$ replaced by $C$, and $\int_X \tilde{D}_k(x, y) d\mu(x) = 0 = \int_X D_k(x, y) d\mu(y)$.

The following inhomogeneous Calderón reproducing formula was established in [18] and [11]. For a measurable set $E$, we set $m_E(f) \equiv \frac{1}{\mu(E)} \int_E f(y) d\mu(y)$.

**Theorem 5.2** Let $\epsilon_1 \in (0, 1], \epsilon_2, \epsilon_3 > 0$, $\epsilon \in (0, 0, \epsilon_1 \wedge \epsilon_2)$ and $\{S_k\}_{k \in \mathbb{Z}}$ be an $(\epsilon_1, \epsilon_2, \epsilon_3)$-AOTI. Set $D_k \equiv S_k - S_{k-1}$ for $k \in \mathbb{Z}$. Then there exist $N \in \mathbb{N}$ and $j_1 > j_0$ such that for all $j > j_1$, $\ell \in \mathbb{Z}$ and $D(\ell + N + 1, j)$, there exist operators $\{\tilde{D}_k(x, y)\}_{k=\ell}^\infty$ such that for any $f \in (G_0^0(\beta, \gamma))^\ell$ with $\beta, \gamma \in (0, \epsilon)$, and all $x \in \mathcal{X}$,

$$f(x) = \sum_{\tau \in I_\ell} \sum_{\nu = 1}^{N(\ell, \tau)} \int_{Q^\ell_\nu} \tilde{D}_\ell(x, y) d\mu(y) m_{Q^\ell_\nu}(S_\ell(f)) + \sum_{k=\ell+1}^{\ell+N} \sum_{\tau \in I_k}^{\tau = \infty} \sum_{\nu = 1}^{N(k, \tau)} \int_{Q^k_\nu} \tilde{D}_k(x, y) d\mu(y) m_{Q^k_\nu}(D_k(f))$$
To this end, it suffices to prove that for any \( \phi \) and \( \nu \) constant \( C \),
\[
\sum_{k=\ell+1}^{\infty} \sum_{\tau \in I_k} \sum_{\nu=1}^{N(k,\tau)} \mu(Q^{k,\nu}_\tau) \tilde{D}_k(x, y^{k,\nu}_\tau) D_k(f)(y^{k,\nu}_\tau),
\]
where the series converge in \( (G^0_0(\beta, \gamma))^r \). Moreover, for any \( \epsilon' \in [\epsilon, \epsilon_1 \wedge \epsilon_2] \), there exists a positive constant \( C \) depending on \( N \) and \( \epsilon' \), but not on \( \ell, j \) and \( D(\ell + N + 1, j) \), such that \( \tilde{D}_k \) for \( k \geq \ell \) satisfies (i) and (ii) of Definition 2.1 with \( \epsilon_1 \) and \( \epsilon_2 \) replaced by \( \epsilon' \) and the constant \( C_4 \) replaced by \( C \); and \( \int X \tilde{D}_k(x, y) d\mu(x) = \int X \tilde{D}_k(x, y) d\mu(y) = 1 \) when \( \ell \leq k \leq \ell + N \); \( = 0 \) when \( k \geq \ell + N + 1 \).

**Remark 5.1** (i) We remark that by checking the proofs of Theorem 4.1 and Theorem 4.3 in [18], it is easy to see that the constants \( C \) in Theorems 5.1 and 5.2 are independent of \( j \). This observation plays a key role in the proof of Theorem 3.2 (ii).

(ii) For simplicity, in Theorem 5.2, we always assume that \( N = 0 \) in what follows.

The following technical lemma is also used; see [18] for its proof.

**Lemma 5.2** Let \( \epsilon \in (0, 1) \), \( r \in (n/(n + \epsilon), 1] \) and \( j \geq j_0 \). Then there exists a positive constant \( C \) independent of \( j \) such that for all \( k, k' \in \mathbb{Z} \), \( a^{k,\nu}_{\tau} \in \mathbb{C} \), \( y^{k,\nu}_\tau \in Q^{k,\nu}_\tau \) with \( \tau \in I_k \) and \( \nu = 1, \ldots, N(k, \tau) \) and \( x \in X \),
\[
\sum_{\tau \in I_k} \sum_{\nu=1}^{N(k,\tau)} \frac{\mu(Q^{k,\nu}_\tau)}{V^{-2(\nu\wedge k)}(x)} \left| \frac{2^{-(k'\wedge k)}}{2^{-(k'\wedge k)} + d(x, y^{k,\nu}_\tau)} \right|^\epsilon \leq C 2^{n(1/r-1)} 2^{|(k'\wedge k)-k|n(1-1/r)} \left\{ \text{HL} \left( \sum_{\tau \in I_k} \sum_{\nu=1}^{N(k,\tau)} |a^{k,\nu}_{\tau}|^r \chi_{Q^{k,\nu}_\tau} \right)(x) \right\}^{1/r}.
\]

Now we prove Theorem 3.1 by using the homogeneous Calderón formula given in Theorem 5.1.

**Proof of Theorem 3.1** To prove Theorem 3.1, it suffices to prove that for any fixed \( \beta, \gamma \in (n(1/p - 1), \epsilon) \), there exist positive constants \( r \in (n/(n + \beta \wedge \gamma), p) \) and \( C \) such that for all \( f \in (G^0_0(\beta, \gamma))^r \) and \( x \in X \),
\[
G(f)(x) \leq C \left\{ \text{HL}([S^+(f)]^r)(x) \right\}^{1/r}. \tag{5.1}
\]

To this end, it suffices to prove that for any \( \varphi \in G^0_0(\beta, \gamma) \) with \( \|\varphi\|_{G^0_0(\beta, \gamma)} \leq 1 \),
\[
|\langle f, \varphi \rangle| \leq C \left\{ \text{HL}([S^+(f)]^r)(x) \right\}^{1/r}. \tag{5.2}
\]

Assume that (5.2) holds for the moment. Then for any \( \phi \in G^0_0(\beta, \gamma) \) with
\[
\|\phi\|_{G^0_0(\beta, \gamma)} \leq 1
\]
and \( \sigma \equiv \int_{X} \phi(y) \, d\mu(y) \), we have
\[
|\sigma| \leq \int_{X} \frac{1}{V_{2^{-\ell}}(x) + V(x, y)} \left( \frac{2^{-\ell}}{2^{-\ell} + d(x, y)} \right)^{\gamma} \, d\mu(y) \lesssim 1.
\]
Set \( \varphi(y) \equiv \frac{1}{1+|\sigma|\tau\gamma} [\phi(y) - \sigma S_{\ell}(x, y)] \). Then \( \int_{X} \varphi(y) \, d\mu(y) = 0 \) and hence, \( \varphi \in \mathcal{G}_{0}^{\beta, \gamma} \) with \( \|\varphi\|_{\mathcal{G}_{0}^{\beta, \gamma}} \lesssim 1 \). Since
\[
|(f, \varphi)| \leq |\sigma||S_{\ell}(f)(x)| + (1 + |\sigma|\gamma)|\langle f, \varphi \rangle|,
\]
by (5.2) and taking the supremum over all \( \ell \in \mathbb{Z} \) and \( \phi \in \mathcal{G}_{0}^{\beta, \gamma} \) with \( \|\phi\|_{\mathcal{G}_{0}^{\beta, \gamma}} \lesssim 1 \), we further obtain that for all \( x \in X \),
\[
G(f)(x) \leq |\sigma|S^+(f)(x) + C(1 + \sigma\gamma) \left\{ \text{HL}([S^+(f)]')(x) \right\}^{1/r},
\]
which yields (5.1).

To prove (5.2), for each \( Q^{k, \nu}_{\tau} \), we pick a point \( y^{k, \nu}_{\tau} \in Q^{k, \nu}_{\tau} \) such that
\[
|D_{k}(f)(y^{k, \nu}_{\tau})| \leq 2 \inf_{z \in Q^{k, \nu}_{\tau}} |D_{k}(f)(z)|,
\]
which further implies that
\[
|D_{k}(f)(y^{k, \nu}_{\tau})| \leq 2 \inf_{z \in Q^{k, \nu}_{\tau}} (|S_{k}(f)(z)| + |S_{k-1}(f)(z)|) \leq 4 \inf_{z \in Q^{k, \nu}_{\tau}} S^+(f)(z).
\]

Denote by \( \mathcal{D}(-\infty, j) \) the collection of all such points \( y^{k, \nu}_{\tau} \). Observe that \( \mathcal{G}_{0}^{\beta, \gamma} \) is a subspace of \( \mathcal{G}_{0}^{\beta, \gamma} \) and for any \( \varphi \in \mathcal{G}_{0}^{\beta, \gamma} \), \( \|\varphi\|_{\mathcal{G}_{0}^{\beta, \gamma}} = \|\varphi\|_{\mathcal{G}_{0}^{\beta, \gamma}} \). So for any \( f \in (\mathcal{G}_{0}^{\beta, \gamma})' \), we know that \( f \) uniquely induces a bounded linear functional \( g \) on \( \mathcal{G}_{0}^{\beta, \gamma} \), namely, \( g \in (\mathcal{G}_{0}^{\beta, \gamma})' \); moreover, \( \|g\|_{(\mathcal{G}_{0}^{\beta, \gamma})'} \leq \|f\|_{(\mathcal{G}_{0}^{\beta, \gamma})'} \). By \( \varphi \in \mathcal{G}_{0}^{\beta, \gamma} \), \( D_{k}(y, \cdot) \in \mathcal{G}_{0}^{\beta, \gamma} \) for all \( y \in X \) and Theorem 5.1, we have
\[
\langle f, \varphi \rangle \equiv \langle g, \varphi \rangle = \sum_{k=-\infty}^{\infty} \sum_{\tau \in I_{k}} \sum_{\nu=1}^{N(k, \tau)} \mu(Q^{k, \nu}_{\tau}) \tilde{D}^*_k(\varphi)(y^{k, \nu}_{\tau})D_k(g)(y^{k, \nu}_{\tau})
\]
\[
= \sum_{k=-\infty}^{\infty} \sum_{\tau \in I_{k}} \sum_{\nu=1}^{N(k, \tau)} \mu(Q^{k, \nu}_{\tau}) \tilde{D}^*_k(\varphi)(y^{k, \nu}_{\tau})D_k(f)(y^{k, \nu}_{\tau}),
\]
where and in what follows, for all \( y \in X \),
\[
\tilde{D}^*_k(\varphi)(y) = \int_{X} \tilde{D}_k(z, y) \varphi(z) \, d\mu(z).
\]

We recall that there exist positive constants \( \beta' \in (n(1/p - 1), \beta) \) and \( C \) such that for all \( y \in X \),
\[
|\tilde{D}^*_k(\varphi)(y)| \leq C 2^{-|k-\ell| \beta'} \frac{1}{V_{2^{-|k-\ell| \beta'}}(x) + V(x, y)} \left( \frac{2^{-(|k-\ell| \beta')}}{2^{-(|k-\ell| \beta')} + d(x, y)} \right)^{\gamma}; \quad (5.3)
\]
see the proof of Proposition 5.7 of [18] for the details. Recall that \( \ell \wedge k \equiv \min\{\ell, k\} \). We point out that, to obtain the decay factor \( 2^{-(k-\ell)|\beta'} \) in (5.3), we need to use the fact that \( \int_X \varphi(z) \, d\mu(z) = 0 \) and \( \int_X D_k(z, y) \, d\mu(z) = 0 \) for all \( y \in X \).

Then by Lemma 5.2 with \( r \in (n/(n + \beta' \wedge \gamma), p) \), we have

\[
|\langle f, \varphi \rangle| \lesssim \sum_{k=-\infty}^{\infty} 2^{-k-\ell|\beta'} \sum_{\tau \in I_k} \sum_{\nu=1}^{N(k, \tau)} \mu(Q_{\tau}^{k, \nu}) \inf_{z \in Q_{\tau}^{k, \nu}} S^+(f)(z) \\
\times \frac{1}{V_{2^{-\ell}k}(x) + V(x, y_{\tau}^{k, \nu})} \left( 2^{-k(\ell + \gamma)} + d(x, y_{\tau}^{k, \nu}) \right)^{\gamma} \\
\lesssim \sum_{k=-\infty}^{\infty} 2^{-k-\ell|\beta'} 2[(\ell + k)n(1-1/r) \\
\times \left\{ \text{HL} \left( \sum_{\tau \in I_k} \sum_{\nu=1}^{N(k, \tau)} \inf_{z \in Q_{\tau}^{k, \nu}} S^+(f)(z) \left| x_{Q_{\tau}^{k, \nu}} \right| \right)^{r}(x) \right\}^{1/r} \\
\lesssim \left\{ \text{HL} \left( \left| S^+(f) \right|^r \right)(x) \right\}^{1/r},
\]

which gives (5.2). This finishes the proof of Theorem 3.1.

To prove Theorem 3.2, we first introduce a variant of the localized radial maximal function.

**Definition 5.1** Let \( \epsilon_1 \in (0, 1], \epsilon_2, \epsilon_3 > 0, \epsilon \in (0, \epsilon_1 \wedge \epsilon_2) \) and \( \{S_k\}_{k \in \mathbb{Z}} \) be an \((\epsilon_1, \epsilon_2, \epsilon_3)\)-AOTI. Let \( p \in (n/(n + \epsilon), 1] \) and \( \beta, \gamma \in (n(1/p - 1), \epsilon) \). For \( j \geq j_0 \) and \( f \in (G_0^\beta(\beta, \gamma))' \), define

\[
S_{\ell}^{+j}(f) \equiv S_\ell^+(f) + \left\{ \sum_{\tau \in I_\ell} \sum_{\nu=1}^{N(\ell, \tau)} [m_{Q_{\tau}^{\ell, \nu}}(|S_{\ell}(f)|)]^p x_{Q_{\tau}^{\ell, \nu}} \right\}^{1/p},
\]

where \( \{Q_{\tau}^{\ell, \nu} \}_{\nu=1}^{N(\ell, \tau)} \) is the collection of all dyadic cubes \( Q_{\tau}^{\ell, j} \) contained in \( Q_{\tau}^\ell \).

**Proof of Theorem 3.2** To prove (i), we first claim that for \( p \in (n/(n + \epsilon), 1] \), all \( \ell \in \mathbb{Z} \) and \( f \in (G_0^\beta(\beta, \gamma))' \),

\[
\|S_{\ell}^{(a)}(f)\|_{L^p(X)} \sim \sum_{\tau \in I_\ell} \sum_{\nu=1}^{N(\ell, \tau)} [m_{Q_{\tau}^{\ell, \nu}}(|S_{\ell}(f)|)]^p \mu(Q_{\tau}^{\ell, \nu}),
\]

(5.4)

where the constants depend on \( j, p \) and \( a \), but not on \( \ell \) and \( f \).
To see this, we observe that for each given \( j \geq 0 \), there exists a positive constant \( C(j) \) depending on \( j \) such that for all \( \ell \in \mathbb{Z} \) and \( \tau \in I_\ell \), \( N(\ell, \tau) \leq C(j) \). In fact, by Lemma 5.1, (1.1) and (1.2), we have

\[
\mu(Q^\ell_\tau) = \sum_{\nu=1}^{N(\ell, \tau)} \mu(Q^\ell_\tau, \nu) \geq \frac{\sum_{\nu=1}^{N(\ell, \tau)} V_{C_\ell 2^{-\ell-j}}(z_\tau^\ell, \nu)}{\sum_{\nu=1}^{N(\ell, \tau)} V_{2C_\ell 2^{-\ell-j}}(z_\tau^\ell, \nu)} \geq N(\ell, \tau) \mu(Q^\ell_\tau),
\]

which implies our claim. Then it is easy to see that for all \( j \geq 0 \),

\[
\sum_{\tau \in I_\ell} \sum_{\nu=1}^{N(\ell, \tau)} [m_{Q^\ell_\tau, \nu}(|S_\ell(f)|)]^p \mu(Q^\ell_\tau, \nu) \sim \sum_{\tau \in I_\ell} [m_{Q^\ell_\tau}(|S_\ell(f)|)]^p \mu(Q^\ell_\tau), \tag{5.5}
\]

where the equivalent constants are independent of \( f \) and \( \ell \). For any given \( a > 0 \), let \( j \) be large enough such that \( a > C_\tau 2^{-\ell-j} \geq \text{diam}(Q^\ell_\tau, \nu) \). Since for any \( x \in Q^\ell_\tau, \nu \),

\[
m_{Q^\ell_\tau, \nu}(|S_\ell(f)|) \leq \frac{V_{a2^{-\ell}}(x)}{\mu(Q^\ell_\tau, \nu)} S^a_\ell(f)(x) \leq \frac{V_{a2^{-\ell+1}}(x)}{V_{C_\ell 2^{-\ell-j}}(x)} S^a_\ell(f)(x) \leq S^a_\ell(f)(x),
\]

we know that the left hand side of (5.4) controls its right hand side for this \( j \) and thus for all \( j \geq 0 \) by (5.5). For \( a \geq 0 \) and each \( Q^\ell_\tau \), by (1.2) and a similar argument, we have

\[
\int_{Q^\ell_\tau} [S^a_\ell(f)(x)]^p \, d\mu(x) \lesssim [\mu(Q^\ell_\tau)]^{1-p} \left( \int_{B(x^\ell_\tau, (a+C_\ell)2^{-\ell})} S_\ell(f)(y) \, d\mu(y) \right)^p \lesssim \sum_{Q^\ell_\tau \cap B(x^\ell_\tau, (a+C_\ell)2^{-\ell}) \neq \emptyset} [m_{Q^\ell_\tau, \nu}(|S_\ell(f)|)]^p \mu(Q^\ell_\tau, \nu).
\]

Similarly to above, using Lemma 5.5, (1.1) and (1.2), we have that \( \sharp \{ \tau' : Q^\ell_\tau \cap B(x^\ell_\tau, (a+C_\ell)2^{-\ell}) \neq \emptyset \} \) is bounded uniformly in \( \ell \) and \( \tau \), from which it follows that the right hand side of (5.4) controls its left hand side for \( j = 0 \) and thus for all \( j \) by (5.5). Here, \( \sharp E \) denotes the cardinality of the set \( E \).

By Lemma 5.1 and the H"older inequality, we have

\[
\|S_\ell(f)\|^p_{L^p(X)} = \sum_{\tau \in I_\ell} \sum_{\nu=1}^{N(\ell, \tau)} m_{Q^\ell_\tau, \nu}(|S_\ell(f)|)^p \mu(Q^\ell_\tau, \nu) \leq \sum_{\tau \in I_\ell} \sum_{\nu=1}^{N(\ell, \tau)} [m_{Q^\ell_\tau, \nu}(|S_\ell(f)|)]^p \mu(Q^\ell_\tau, \nu),
\]

which together with (5.4) leads to

\[
\|S_\ell^{+j}(f)\|_{L^p(X)} \sim \|S_\ell^{+1}(f)\|_{L^p(X)} + \|S_\ell^{a}(f)\|_{L^p(X)}
\]

with constants independent of \( \ell \) and \( f \). Then the proof of Theorem 3.2 (i) is reduced to showing that

\[
C^{-1}\|S_\ell^{+j}(f)\|_{L^p(X)} \leq \|G_\ell(f)\|_{L^p(X)} \leq C2^{jn(1/p-1)}\|S_\ell^{+j}(f)\|_{L^p(X)}, \tag{5.6}
\]
where $C$ is a positive constant independent of $j$, $\ell$ and $f$.

Observe that for all $x, y \in Q^k_{\ell, \nu}$ and $k \geq \ell$, $S_\ell(f)(x) \lesssim G_\ell(f)(y)$, which implies the first inequality in (5.6). To prove the second inequality in (5.6), it suffices to prove that for any fixed $\beta, \gamma \in (n(1/p - 1), e)$, there exist positive constants $r \in (n/(n + \beta \wedge \gamma), p)$ and $C$ such that for all $j \geq j_0$, $\ell \in \mathbb{Z}$, $f \in (G^s_0(\beta, \gamma))'$ and $x \in \mathcal{X}$,

$$G_\ell(f)(x) \leq C 2^{jn(1/p - 1)} \left\{ \text{HL}([|S_\ell^{+, j}(f)|^r](x)) \right\}^{1/r}. \tag{5.7}$$

To prove (5.7), we only need to prove that for all $k' \geq \ell$ and $\varphi \in G^s_0(\beta, \gamma)$ with $\|\varphi\|_{G^s_0(2^{-k'}, \beta, \gamma)} \leq 1$,

$$\langle f, \varphi \rangle \leq C 2^{jn(1/p - 1)} \left\{ \text{HL}([|S_\ell^{+, j}(f)|^r](x)) \right\}^{1/r}. \tag{5.8}$$

In fact, assume that $\phi \in G^s_0(\beta, \gamma)$ with $\|\phi\|_{G^s_0(2^{-k'}, \beta, \gamma)} \leq 1$ for certain $k \geq \ell$. Set $\sigma = \int_X \phi(y) \, d\mu(y)$ and $\varphi(y) = \frac{1}{1 + |\sigma|_C} \{ \phi(y) - \sigma S_{k'}(x, y) \}$. Then we have $\int_X \varphi(y) \, d\mu(y) = 0$ and hence, $\varphi \in G^s_0(\beta, \gamma)$ with $\|\varphi\|_{G^s_0(2^{-k'}, \beta, \gamma)} \leq 1$. Moreover, we have

$$\langle f, \varphi \rangle \leq \|\varphi\|_{G^s_0(2^{-k'}, \beta, \gamma)} \leq 1.$$ 

By taking the supremum over all $k' \geq \ell$ and $\phi \in G^s_0(\beta, \gamma)$ with $\|\phi\|_{G^s_0(2^{-k'}, \beta, \gamma)} \leq 1$ together with (5.8), we have

$$G_\ell(f)(x) \leq \sigma S_\ell^{+, j}(f)(x) + C(1 + |\sigma|_C) 2^{jn(1/p - 1)} \left\{ \text{HL}([|S_\ell^{+, j}(f)|^r](x)) \right\}^{1/r},$$

which combined with the uniform boundedness of $\sigma$ yields (5.7).

To prove (5.8), for each $Q^k_{\ell, \nu}$ with $k \geq \ell + 1$, we choose a point $y^{k, \nu}_{\ell, \nu}$ such that

$$|D_k(f)(y^{k, \nu}_{\ell, \nu})| \leq 2 \inf_{z \in Q^k_{\ell, \nu}} |D_k(f)(z)| \leq 4 \inf_{z \in Q^k_{\ell, \nu}} S_\ell^{+, j}(f)(z)$$

and denote by $\mathcal{D}(\ell + 1, j)$ the collection of all such points $y^{k, \nu}_{\ell, \nu}$. Then by Theorem 5.2, we write

$$\langle f, \varphi \rangle = \sum_{\tau \in I_\ell} \sum_{\nu=1}^{N(\ell, \tau)} \int_{Q^k_{\ell, \nu}} \overline{D}_k^{*}(\varphi)(z) \, d\mu(z) m_{Q^k_{\ell, \nu}}(S_{\ell}(f))$$

$$+ \sum_{k=\ell + 1}^{\infty} \sum_{\tau \in I_\ell} \sum_{\nu=1}^{N(k, \tau)} \mu(Q^k_{\ell, \nu}) \overline{D}_k^{*}(\varphi)(y^{k, \nu}_{\ell, \nu}) D_k(f)(y^{k, \nu}_{\ell, \nu}).$$

Similarly to the proof of (5.3), there exist positive constants $\beta' \in (n(1/p - 1), \beta)$ and $C$, independent of $j$, $\ell$ and $\mathcal{D}(\ell + 1, j)$, such that for all $k \in \mathbb{Z}$ and $y \in \mathcal{X}$,

$$|\overline{D}_k^{*}(\varphi)(y)| \leq C 2^{-|k-\ell|/\beta'} \frac{1}{V_{2^{-|k-\ell|}}(x) + V(x, y)} \left( \frac{2^{-2(k' \wedge k)}}{2^{-2(k' \wedge k)} + d(x, y)} \right)^\gamma.$$
Then by Lemma 5.2 with \( r \in (n/(n + \beta' \land \gamma), p) \), we have

\[
\langle f, \varphi \rangle \lesssim \sum_{k=\ell}^\infty 2^{-|k-k'|\beta'} \sum_{\tau \in I_k} \sum_{\nu=1}^{N(k, \tau)} \mu(Q_{\tau}^k, \nu) \inf_{z \in Q_{\tau}^k, \nu} S_{\ell}^+ (f)(z) \times \frac{1}{V_{2^{-\nu}}(x)} \left( \frac{2^{-\beta' k}}{2^{-\beta' k^1} + d(x, y_{\tau}^k)} \right)^{\gamma} \lesssim \sum_{k=\ell}^\infty 2^{-|k-k'|\beta'} 2^{(k' \land \gamma) - k} n(1-1/r) 2^{jn(1/r-1)} \times \left\{ \text{HL} \left( \sum_{\tau \in I_k} \sum_{\nu=1}^{N(k, \tau)} \inf_{z \in Q_{\tau}^k, \nu} S_{\ell}^+ (f)(z) \left| x_{Q_{\tau}^k, \nu} \right| \right) \right\}^{1/r} \lesssim 2^{jn(1/r-1)} \left\{ \text{HL} \left( \left| S_{\ell}^+ (f) \right|^{r} \right) \right\}^{1/r}.
\]

Thus (5.8) holds, which gives (i).

To prove (ii), by the fact that \( S_{\ell}^+ (f)(y) \leq G_\ell (f)(y) \) for all \( y \in X \), we only need to prove that for any \( \beta, \gamma \in (n(1/p - 1), \epsilon) \), there exist \( r \in (n/(n + \epsilon), p) \) and a positive constant \( C \) such that for all \( \ell \in \mathbb{Z}, j > j_0 \) and \( f \in (G_0^\epsilon(\beta, \gamma))' \),

\[
G_\ell (f) \leq C 2^{jn(1/r-1)} \left\{ \text{HL} \left( \left| S_{\ell}^+ (f) \right|^{r} \right) \right\}^{1/r} + C 2^{-j(\epsilon_1 + n - n/r)} \left\{ \text{HL} \left( \left| G_\ell (f) \right|^{r} \right) \right\}^{1/r}. \tag{5.9}
\]

Assume that (5.9) holds for the moment. Then, by the \( L^{p/r}(X) \)-boundedness of HL, we have

\[
\|G_\ell (f)\|_{L^p(X)} \leq C 2^{jn(1/r-1)} \|S_{\ell}^+ (f)\|_{L^p(X)} + C 2^{-j(\epsilon_1 + n - n/r)} \|G_\ell (f)\|_{L^{p/r}(X)}.
\]

Notice that from the assumption (3.1) and Theorem 3.2 (i), we deduce that \( \|G_\ell (f)\|_{L^p(X)} < \infty \). Thus, choosing \( j \) large enough such that \( C 2^{-j(\epsilon_1 + n - n/r)} \leq 1/2 \), we obtain

\[
\|G_\ell (f)\|_{L^p(X)} \leq C 2^{jn(1/r-1)} \|S_{\ell}^+ (f)\|_{L^p(X)}.
\]

To prove (5.9), for each \( Q_{\tau}^k, \nu \) with \( k \geq \ell \), we pick a point \( y_{\tau}^k, \nu \in Q_{\tau}^k, \nu \) such that

\[
|S_{\ell} (f)(y_{\tau}^k, \nu)| \leq 2 \inf_{z \in Q_{\tau}^k, \nu} S_{\ell}(f)(z)
\]

and for \( k \geq \ell + 1 \),

\[
|D_k (f)(y_{\tau}^k, \nu)| \leq 2 \inf_{z \in Q_{\tau}^k, \nu} |D_k (f)(z)| \leq 4 \inf_{z \in Q_{\tau}^k, \nu} S_{\ell}^+ (f)(z).
\]

Denote by \( D(\ell, j) \) the collection of all such points \( y_{\tau}^k, \nu \). For any \( k' \geq \ell \) and any \( \varphi \in \bar{G}_0(\beta, \gamma) \) with \( \|\varphi\|_{\bar{G}_0(2^{-k'}, \beta', \gamma)} \leq 1 \), by Theorem 5.2, we have

\[
\langle f, \varphi \rangle = \sum_{\tau \in I_\ell} \sum_{\nu=1}^{N(\ell, \tau)} \int_{Q_{\tau}^k} D_\tau^+ (\varphi)(y) d\mu(y) \left[ m_{Q_{\tau}^k, \nu}(S_{\ell}(f)) - S_{\ell}(f)(y_{\tau}^k, \nu) \right]
\]
which implies that
\[
y
\]
Proof of Theorem 3.3
This gives (5.9) and hence, finishes the proof of Theorem 3.2.

AOTI. By Theorem 3.1, it suffices to prove that
\[
\|\sum_{k=\ell+1}^{\infty} \sum_{\nu=1}^{N(k, \tau)} \mu(Q_{k, \tau}^k, \nu) D_k(f)(y_{k, \nu}^k)\| \equiv J_1 + J_2 + J_3.
\]

Similarly to the proofs of (5.8) and (5.2), we have
\[
|J_2| + |J_3| \lesssim 2^{j_1(1,r)} \left\{ \text{HL} \left( \|S_{\ell}^+ (f)\|^r \right)(x) \right\}^{1/r}.
\]

To estimate $J_1$, observe that there exists a positive constant $C$, independent of $j$ and $\ell$, such that for all $y, z, y_{\ell, \nu}^k \in Q_{k, \tau}^k$,
\[
\|S_{\ell}(y, \cdot) - S_{\ell}(y_{\ell, \nu}^k, \cdot)\|_{g_0^k(z, 2^{-\ell}, \beta, \gamma)} \lesssim 2^{-j_1},
\]
which implies that
\[
|m_{Q_{k, \tau}^k, \nu}(S_{\ell}(f)) - S_{\ell}(f)(y_{k, \nu}^k)| \lesssim 2^{-j_1} \inf_{z \in Q_{k, \tau}^k} G_{\ell}(f)(z).
\]

Therefore, by Lemma 5.2 and the fact that $2^{-\ell} + d(x, y) \sim 2^{-\ell} + d(x, y_{\ell, \nu}^k)$ for all $y \in Q_{k, \tau}^k$, we have
\[
J_1 \lesssim 2^{-j_1} \sum_{\tau \in I_{\ell}} \sum_{\nu=1}^{N(\ell, \tau)} \mu(Q_{k, \tau}^k, \nu) \inf_{z \in Q_{k, \tau}^k} G_{\ell}(f)(z)
\]
\[
\times \frac{1}{V_{2^{-\ell}}(x) + V(x, y_{\ell, \nu}^k)} \left( \frac{2^{-\ell}}{2^{-\ell} + d(x, y_{\ell, \nu}^k)} \right)^{\gamma}
\]
\[
\lesssim 2^{-j_1(\ell + n - n/r)} \left\{ \text{HL} \left( \|G_{\ell}(f)\|^r \right)(x) \right\}^{1/r}
\]
\[
\times \left\{ \text{HL} \left( \sum_{\tau \in I_{\ell}} \sum_{\nu=1}^{N(\ell, \tau)} \left[ \inf_{z \in Q_{k, \tau}^k} G_{\ell}(f)(z) \right]^{r} \chi_{Q_{k, \tau}^k} \right)(x) \right\}^{1/r}
\]

This gives (5.9) and hence, finishes the proof of Theorem 3.2.

Now we give the proof of Theorem 3.3.

Proof of Theorem 3.3 Assume that $\|G_{\ell}(f)\|_{L^p(\mathcal{X})} < \infty$ and $\{S_k\}_{k \in \mathbb{Z}}$ is an $(\epsilon_1, \epsilon_2, \epsilon_3)$-AOTI. By Theorem 3.1, it suffices to prove that
\[
\|S^+(f - S_{\ell}(f))\|_{L^p(\mathcal{X})} \lesssim \|G_{\ell}(f)\|_{L^p(\mathcal{X})}
\]
with a constant independent of $f$ and $\ell$. Write
\[
\|S^+(f - S_{\ell}(f))\|_{L^p(\mathcal{X})} \sim \|S_{\ell}^+(f)\|_{L^p(\mathcal{X})} + \sup_{k \geq \ell + 1} |S_k S_{\ell}(f)|_{L^p(\mathcal{X})}
\]
Moreover, for any \( k \) or any \( \geq 1 \), we have that \( \approx \) with a constant independent of \( k \).

Case 1 of the proof of [17, Proposition 5.11], we have that \( \approx \) and

\[ \| S_k S_\ell (x, \cdot) \|_\mathcal{G}(x, 2^{-\ell}, 2^\epsilon) \leq 1 \]

with a constant independent of \( \ell, k \) and \( f \). From this, it follows that \( \approx \| G_\ell(f) \|_{\mathcal{L}^p(X)} \).

To estimate \( I_3 \), using Theorem 5.2, for \( k \leq \ell \), we have

\[ \| S_k (I - S_\ell) f(x) \|
= \sum_{\tau \in I_\ell} \sum_{\nu=1}^{N(\ell, \tau)} \int_{Q_\ell^{\ell, \nu}} [S_k (I - S_\ell) \hat{D}_\ell](x, u) d\mu(u) m_{Q_\ell^{\ell, \nu}}(S_\ell(f))
+ \sum_{k' = \ell + 1} \sum_{\tau \in I_{k'}} \sum_{\nu=1}^{N(k', \tau)} \mu(Q_{k'}^{k', \nu}) [S_k (I - S_\ell) \hat{D}_{k'}](x, y_{k'}^{k', \nu}) D_{k'}(f)(y_{k'}^{k', \nu}). \]

For any \( y_{k'}^{k', \nu} \in Q_{k'}^{k', \nu} \), since \( |S_\ell(f)(z)| \leq G_\ell(f)(y_{k'}^{k', \nu}) \) for any \( z \in Q_{k'}^{k', \nu} \), we further obtain

\[ |m_{Q_\ell^{k', \nu}}(S_\ell(f))| \leq G_\ell(f)(y_{k'}^{k', \nu}). \]

Moreover, for any \( k' \geq \ell \geq k \) and \( y_{k'}^{k', \nu} \in Q_{k'}^{k', \nu} \), by an argument similar to that used in Case 1 of the proof of [17, Proposition 5.11], we have

\[ \| [S_k (I - S_\ell) \hat{D}_{k'}](x, y_{k'}^{k', \nu}) \| \leq 2^{-(k' - \ell)} \frac{1}{V_{2-k}(x) + V(x, y_{k'}^{k', \nu})} \left( \frac{2^{-k}}{2^{-k} + d(x, y_{k'}^{k', \nu})} \right)^\epsilon. \]

We omit the details here. Thus, for any \( r \in (n/(n + \epsilon), p) \), we have

\[ |S_k (I - S_\ell) f(x)|\]
\[ \lesssim \sum_{k' = \ell} \sum_{\tau \in I_{k'}} \sum_{\nu=1}^{N(k', \tau)} \mu(Q_{k'}^{k', \nu}) 2^{-(k' - \ell)} \frac{1}{V_{2-k}(x) + V(x, y_{k'}^{k', \nu})} \times \left( \frac{2^{-k}}{2^{-k} + d(x, y_{k'}^{k', \nu})} \right)^\epsilon \inf_{z \in Q_{k'}^{k', \nu}} G_\ell(f)(z) \]
\[ \lesssim \sum_{k' = \ell} 2^{-(k' - \ell)(\epsilon + n - n/r)} \left\{ \text{HL} \left( \sum_{\tau \in I_{k'}} \sum_{\nu=1}^{N(k', \tau)} [G_\ell(f)]^{\tau} \chi_{Q_{k'}^{k', \nu}} \right) \right\}^{1/r} \]
\[ \lesssim \sum_{k' = \ell} 2^{-(k' - \ell)(\epsilon + n - n/r)} \left\{ \text{HL} \left( [G_\ell(f)]^{\tau} \right) \right\}^{1/r} \lesssim \left\{ \text{HL} \left( [G_\ell(f)]^{\tau} \right) \right\}^{1/r}. \]
This implies that
\[ I_3 \lesssim \left\| \{\text{HL}([G_\ell(f)])^r \}^{1/r} \right\|_{L^p(X)} \lesssim \|G_\ell(f)\|_{L^p(X)}, \]
which completes the proof of Theorem 3.3.

Now we turn to the proof of Theorem 3.4.

**Proof of Theorem 3.4** Let \( \ell \in \mathbb{Z} \). Then it is quite standard (see, for example, the proof of [17, Theorem 2.21]) to prove that for any \( f \in H^p_{\ell, \infty}(\mathcal{X}) \),
\[ \|G_\ell(f)\|_{L^p(X)} \lesssim \|f\|_{H^p_{\ell, \infty}(\mathcal{X})} \]
with a constant independent of \( \ell \) and \( f \). Observe that if \( f \in (\text{Lip}_\ell(1/p - 1, \mathcal{X}))' \), then \( f \in G_0^0(\beta, \gamma)' \) for \( \epsilon \in (n(1/p - 1), 1] \) and \( \beta, \gamma \in (n(1/p - 1), \epsilon) \). Then by Corollary 3.1, we have \( H^p_{\ell, \infty}(\mathcal{X}) \subset H^p_{\ell, \infty}(\mathcal{X}) \). We omit the details.

On the other hand, if \( f \in (G_0^0(\beta, \gamma))' \) and \( \|G_\ell(f)\|_{L^p(X)} < \infty \), then by Theorem 3.3, we have \( f - S_\ell f \in H^p_{\ell, \infty}(\mathcal{X}) \) and
\[ \|f - S_\ell f\|_{H^p_{\ell, \infty}(\mathcal{X})} \sim \|S_\ell^+(f - S_\ell f)\|_{L^p(X)} \lesssim \|G_\ell(f)\|_{L^p(X)}. \]

Write
\[ S_\ell(f) = \sum_{\tau \in I_\ell} \sum_{\nu = 1}^{N(\ell, \nu)} S_\ell(f) \chi_{Q^\ell_{\tau, \nu}} = \sum_{\tau \in I_\ell} \sum_{\nu = 1}^{N(\ell, \nu)} \lambda^\ell, \nu G_{\tau, \nu}, \]
where \( \lambda^\ell, \nu = [\mu(Q^\ell_{\tau, \nu})]^{1/p} \|S_\ell(f)\|_{L^\infty(Q^\ell_{\tau, \nu})} \) and \( a^\ell, \nu = (\lambda^\ell, \nu)^{-1} S_\ell(f) \chi_{Q^\ell_{\tau, \nu}} \). It is easy to see that \( \{a^\ell, \nu\}_{\tau \in I_\ell, \nu = 1, \ldots, N(\ell, \nu)} \) are \((p, \infty)\)-atoms. Since for any \( z \in Q^\ell_{\tau, \nu} \),
\[ \|S_\ell(f)\|_{L^\infty(Q^\ell_{\tau, \nu})} \lesssim G_\ell(f)(z), \]
we have
\[ \left( \lambda^\ell, \nu \right)^p \lesssim \int_{Q^\ell_{\tau, \nu}} |G_\ell(f)(z)|^p d\mu(z), \]
which implies that
\[ \sum_{\tau \in I_\ell} \sum_{\nu = 1}^{N(\ell, \nu)} \left( \lambda^\ell, \nu \right)^p \lesssim \sum_{\tau \in I_\ell} \sum_{\nu = 1}^{N(\ell, \nu)} \int_{Q^\ell_{\tau, \nu}} |G_\ell(f)(z)|^p d\mu(z) \lesssim \|G_\ell(f)\|_{L^p(X)}^p. \]
Thus \( S_\ell^+ f \in H^p_{\ell}(\mathcal{X}) \) and \( \|S_\ell^+ f\| \lesssim \|G_\ell(f)\|_{L^p(X)} \), which implies that \( f \in H^p_{\ell, \infty}(\mathcal{X}) \) and
\[ \|f\|_{H^p_{\ell, \infty}(\mathcal{X})} \lesssim \|G_\ell(f)\|_{L^p(X)}. \] This finishes the proof of Theorem 3.4.

**Acknowledgements** Dachun Yang would like to thank Professor Pascal Auscher for some useful discussions on the subject of this paper. Part of this paper was written during the stay of Yuan Zhou in University of Jyväskylä and he would like to thank Professor Pekka Koskela and Professor Xiao Zhong for their support and hospitality. Both authors would also like to thank the referee for his/her many valuable remarks which improve the presentation of the paper.
References

1. Alexopoulos, G.: Spectral multipliers on Lie groups of polynomial growth. Proc. Amer. Math. Soc. **120**, 973-979 (1994)
2. Auscher, P., Coulhon, T., Duong, X.T., Hofmann, S.: Riesz transform on manifolds and heat kernel regularity. Ann. Sci. École Norm. Sup. (4) **37**, 911-957 (2004)
3. Auscher, P., McIntosh, A., Russ, E.: Hardy spaces of differential forms on Riemannian manifolds. J. Geom. Anal. **18**, 192-248 (2008)
4. Christ, M.: A $T(b)$ theorem with remarks on analytic capacity and the Cauchy integral. Colloq. Math. **60/61**, 601-628 (1990)
5. Coifman, R.R., Lions, P.-L., Meyer, Y., Semmes, S.: Compensated compactness and Hardy spaces. J. Math. Pures Appl. (9) **72**, 247-286 (1993)
6. Coifman, R.R., Weiss, G.: Analyse Harmonique Non-commutative sur Certain E-spaces Homogènes. Lecture Notes in Math. 242, Springer, Berlin (1971)
7. Coifman, R.R., Weiss, G.: Extensions of Hardy spaces and their use in analysis. Bull. Amer. Math. Soc. **83**, 569-645 (1977)
8. Fefferman, C., Stein, E.M.: $H^p$ spaces of several variables. Acta Math. **129**, 137-193 (1972)
9. Grafakos, L.: Modern Fourier Analysis. Second Edition, Graduate Texts in Math., No. 250, Springer, New York (2008)
10. Grafakos, L., Liu, L., Yang, D.: Maximal function characterizations of Hardy spaces on RD-spaces and their applications. Sci. China Ser. A **51**, 2253-2284 (2008)
11. Grafakos, L., Liu, L., Yang, D.: Radial maximal function characterizations for Hardy spaces on RD-spaces. Bull. Soc. Math. France **137**, 225-251 (2009)
12. Goldberg, D.: A local version of real Hardy spaces. Duke Math. J. **46**, 27-42 (1979)
13. Grigor'yan, A.A.: Stochastically complete manifolds. Dokl. Akad. Nauk SSSR **290**, 534-537 (1986) (in Russian); English translation in Soviet Math. Dokl. **34**, 310-314 (1987)
14. Grigor'yan, A.A.: The heat equation on noncompact Riemannian manifolds. Mat. Sb. **182**, 55-87 (1991) (in Russian); English translation in Math. USSR-Sb. **72**, 47-77 (1992)
15. Hebisch, W., Saloff-Coste, L.: On the relation between elliptic and parabolic Harnack inequalities. Ann. Inst. Fourier (Grenoble) **51**, 1437-1481 (2001)
16. Heinonen, J.: Lectures on Analysis on Metric Spaces. Universitext, Springer-Verlag, New York (2001)
17. Han, Y., Müller, D., Yang, D.: Littlewood-Paley characterizations for Hardy spaces on spaces of homogeneous type. Math. Nachr. **279**, 1505-1537 (2006)
18. Han, Y., Müller, D., Yang, D.: A theory of Besov and Triebel-Lizorkin spaces on metric measure spaces modeled on Carnot-Carathéodory spaces. Abstr. Appl. Anal. Art. ID 893409, 250 pp. (2008)
19. Li, P., Yau, S.-T.: On the parabolic kernel of the Schrödinger operator. Acta Math. **156**, 153-201 (1986)
20. Macías, R.A., Segovia, C.: Lipschitz functions on spaces of homogeneous type. Adv. in Math. **33**, 257-270 (1979)
21. Macías, R.A., Segovia, C.: A decomposition into atoms of distributions on spaces of homogeneous type. Adv. in Math. 33, 271-309 (1979)
22. Nagel, A., Stein, E.M.: The $\Box_b$-heat equation on pseudoconvex manifolds of finite type in $\mathbb{C}^2$. Math. Z. 238, 37-88 (2001)
23. Nagel, A., Stein, E.M.: On the product theory of singular integrals. Rev. Mat. Ibero. 20, 531-561 (2004)
24. Nagel, A., Stein, E.M.: The $\overline{\partial}_b$-complex on decoupled boundaries in $\mathbb{C}^n$. Ann. of Math. (2) 164, 649-713 (2006)
25. Nagel, A., Stein, E.M., Wainger, S.: Balls and metrics defined by vector fields I. Basic properties. Acta Math. 155, 103-147 (1985)
26. Russ, E.: $H^1$-BMO duality on graphs. Colloq. Math. 86, 67-91 (2000)
27. Saloff-Coste, L.: Analyse sur les groupes de Lie nilpotents. C. R. Acad. Sci. Paris Sér. I Math. 302, 499-502 (1986)
28. Saloff-Coste, L.: Fonctions maximales sur certains groupes de Lie. C. R. Acad. Sci. Paris Sér. I Math. 305, 457-459 (1987)
29. Saloff-Coste, L.: Analyse réelle sur les groupes à croissance polynomiale. C. R. Acad. Sci. Paris Sér. I Math. 309, 149-151 (1989)
30. Saloff-Coste, L.: Analyse sur les groupes de Lie à croissance polynomiale. Ark. Mat. 28, 315-331 (1990)
31. Saloff-Coste, L.: Parabolic Harnack inequality for divergence-form second-order differential operators, Potential Anal. 4, 429-467 (1995)
32. Stein, E.M.: Harmonic Analysis: Real-variable Methods, Orthogonality, and Oscillatory Integrals. Princeton University Press, Princeton, N. J. (1993)
33. Stein, E.M., Weiss G.: On the theory of harmonic functions of several variables. I. The theory of $H^p$-spaces. Acta Math. 103, 25-62 (1960)
34. Uchiyama, A.: A maximal function characterization of $H^p$ on the space of homogeneous type. Trans. Amer. Math. Soc. 262, 579-592 (1980)
35. Yang, Da., Yang, Do., Zhou, Y.: Localized Morrey-Campanato spaces related to admissible functions on metric measure spaces and applications to Schrödinger operators. arXiv: 0903.4576
36. Yang, D., Zhou, Y.: Localized Hardy spaces $H^1$ related to admissible functions on RD-spaces and applications to Schrödinger operators. arXiv: 0903.4581
37. Yang, D., Zhou, Y.: New properties of Besov and Triebel-Lizorkin spaces on RD-spaces. arXiv: 0903.4583
38. Varopoulos, N.Th.: Analysis on Lie groups. J. Funct. Anal. 76, 346-410 (1988)
39. Varopoulos, N.Th., Saloff-Coste, L., Coulhon, T.: Analysis and Geometry on Groups. Cambridge University Press, Cambridge (1992)