Two-Body Dirac Equation and Its Wave Function at the Origin *

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Abstract

A relativistic equation is deduced for the bound state of two particles, by assuming a proper boundary condition for the propagation of the negative-energy states. It reduces to the (one-body)Dirac equation in the infinite limit of one of the constituent mass. It also has the symmetries to assure the existence of the anti-bound-state with the same mass. The interaction kernel(pseudo-potential) is systematically constructed by diagonalizing the Hamiltonian of the background field theory, by which the retardation effects are included in the interaction. Its wave function at the origin(WFO) behaves properly in a manner suggested by the covariant field theory. The equation can well be applied in the heavy quark effective theory. It can also be used as a starting equation for the unperturbed system in the rigorous perturbation theory of QED.

1 Introduction

One of the unsatisfactory nature of the Bethe-Salpeter equation for the bound state is that it does not reduce to the Dirac equation in the infinite limit of the one of the constituent mass, even when the interaction is assumed to be instantaneous\cite{1}. We have to sum up all the crossed diagrams to recover the Dirac equation\cite{2}, which is

\footnote{A preliminary report was presented in the workshop ”Fundamental Problems in the Elementary Particle Theory” held at Nihon University in March 1995.}
impossible for the finite masses. The reducibility to the Dirac equation is essential in applications to the heavy flavored quarkonia or the other atom-like systems.

Historically, the relativistic single-time equation for the two-body system preceded the BS equation. Soon after the discovery of the Dirac equation, Breit proposed the equation of the form

\[ \{ H_1(p_1) + H_2(p_2) + V \} \psi = E \psi, \]  

(1)

where \( H_i \) is the Dirac Hamiltonian

\[ H_i(p_i) = \alpha_i \cdot p_i + m_i \beta_i \]

and \( V \) is a local potential. The Breit equation reduces to the Dirac equation in the limit mentioned above but does not have the “\( E \)-parity symmetry”, by which we mean there is symmetric negative eigenvalue \( E \) for each positive one, which is interpreted as the bound state of the antiparticles.

On the other hand, in the instantaneous BS equation we have an extra projection factor

\[ \Lambda_{++} - \Lambda_{--} \]

(2)

in front of the potential \( V \), which consists of the energy-projection operators

\[ \Lambda_{\varepsilon \eta}(p_1, p_2) = \Lambda_{\varepsilon}^1(p_1) \Lambda_{\eta}^2(p_2), \quad \varepsilon, \eta = + \text{ or } -, \]

(3)

where

\[ \Lambda_{\varepsilon}^i(p_i) = \{ E_i(p_i) + \varepsilon H_i(p_i) \} / 2E_i(p_i), \]

\[ E_i(p_i) = (p_{i2}^2 + m_i^2)^{1/2}. \]

\( E \)-parity symmetry of this equation is a result of the \( PCT \) invariance: The Dirac Hamiltonian is odd under the \( PCT \) transformation and the interaction Hamiltonian is assumed to be invariant under it. We therefore see that the symmetry is assured by a projection factor (2). We must introduce a similar factor in any attempt at the construction of an improved two-body equation.

The factor (4) comes from the Stückelberg-Feynman boundary condition for the propagation of the negative-energy state. We will construct the equation for the unequal-mass constituents by imposing similar boundary condition. Before presenting it, we restrict the framework of consideration. For definiteness, we assume Abelian gauge field interacting with the Dirac particles. We also work in the rest frame of the bound system \( (P = (E, 0)) \), since we are looking for the non-covariant approximation of the low-energy dynamics. \( p \) and \( x \), in the following, are the relative momentum and coordinate, respectively, in this frame.
2 The equation for unequal-mass constituents

In this section, we deduce a new single-time equation (Two-body Dirac equation) for the unequal-mass constituents, for which we assume that the mass \( M (= m_1) \) is larger than \( m (= m_2) \).

2.1 Two-body Dirac equation

Let us first consider the 2-body propagator for which we impose the boundary condition that the negative-energy states propagate backward in time. If we keep individuality of the constituents and use the usual Feynman propagator in the instantaneous BS equation, the equation (1) modified by the factor (2) results. However, we can choose the other possibility in which we incorporate the idea that the bound two bodies should be treated as a quantum-mechanical unity. Since \( M \) is larger than \( m \), the free part of the Hamiltonian has the same sign as that of the particle 1 in the CM system. Let us then modify the boundary condition as follows: A bound two-particle state propagates backward in time if their net energy is negative. By assuming it, we have the propagator

\[
S_F^{(2)}(P, p) = \sum_{\varepsilon\eta} \frac{1}{\lambda_1 E - p_0 - H_1(-p) + i\varepsilon\delta \lambda_2 E + p_0 - H_2(p) + i\varepsilon\delta \Lambda_{\varepsilon\eta}\gamma_0^1\gamma_0^2}, \tag{4}
\]

where \( \lambda_1 + \lambda_2 = 1 \) and the limit \( \delta \to +0 \) is assumed.

By assuming (4) we obtain the single-time equation. The pseudo-4-dimensional form of it, in the momentum space, is

\[
\psi(p) = iS_F^{(2)}(P, p) \int V(p, q)\psi(q)d^4q/(2\pi)^4, \tag{5}
\]

where \( V(p, q) \) represents the interaction, which does not depend on the relative energies but is not necessarily an instantaneous local potential. After integrating out the redundant degree of the freedom, (5) becomes

\[
\{H_1(-p) + H_2(p) + \sum_{\varepsilon\eta} \varepsilon\Lambda_{\varepsilon\eta}V\}\psi = E\psi, \tag{6}
\]

which reduces to the Dirac equation in the infinite limit of \( M \). It is easy to see the \( E \)-parity symmetry of this equation.

\footnote{We can establish the concept of individuality in the quantum mechanics only through observation, which brings about a subtle point to the bound system: To detect an individual one in the bound constituents, we need to separate them by applying the 3rd interaction, which inevitably destroys the original state. So there is no a priori reason to apply the free propagator individually to each constituent. Note that “the free propagator” itself for the bound state is merely a convention of approximation.}
If we would apply our equation to the scattering state, we shall have, from the
time-dependent equation, the conserved probability density
\[
\rho(x, t) = \psi(x, t)\dagger \sum_{\varepsilon \eta} \varepsilon \Lambda_{\varepsilon \eta} \psi(x, t),
\]
which is in accord with the boundary condition that the negative-energy state prop-
agates backward in time carrying the negative probability density. However, it
does not necessarily provide the normalization condition for the bound state: For
the scattering processes, the projected wave function \(\Lambda_{\varepsilon \eta} \psi\) corresponds to the physi-
cal state of the (free)particles with the positive or negative energy \(E\). And the above
interpretation of the probability current actually says that the state with the negative
\(E\) is carrying the negative probability density. However, for the bound state with the
positive eigenvalue \(E\), \(\Lambda_{-+} \psi\) or \(\Lambda_{--} \psi\) is merely a negative-energy component in the
representation in which the energy of the free particle is diagonal.

Taking the above consideration into account, we restore the probability interpre-
tation of the wave function and normalize it by assuming the probability density
\[
\rho(x) = \psi(x, t)\dagger \psi(x, t),
\]
for the stationary state. Observables except the Hamiltonian are self-adjoint under
this metric:
\[
(\phi, \hat{O}\psi) = (\hat{O}\phi, \psi).
\]
The Hamiltonian is the operator ruling the time development of the system and its
interaction part is modified by the factor \(\sum_{\varepsilon \eta} \varepsilon \Lambda_{\varepsilon \eta}\). Though it is not a self-adjoint
operator, its eigenvalue is proved to be real if the inner product (13) below exists.

2.2 Green’s function and the vertex equation
We define the Green’s function \(G\) for (6) by the operator equation
\[
\{E - H_1 - H_2 - \sum_{\varepsilon \eta} \varepsilon \Lambda_{\varepsilon \eta} V\}G = \sum_{\varepsilon \eta} \varepsilon \Lambda_{\varepsilon \eta},
\]
and
\[
G\{E - H_1 - H_2 - V \sum_{\varepsilon \eta} \varepsilon \Lambda_{\varepsilon \eta}\} = \sum_{\varepsilon \eta} \varepsilon \Lambda_{\varepsilon \eta}.
\]
In the momentum representation, it can be written, by using the eigen-function \(\chi_\eta(p)\)
of (6), as
\[
G(p, p') = \sum_n \frac{1}{N_n(E - E_n)} \chi_n(p)\chi_n(p')\dagger + \text{continuum},
\]
where \(E_n\) is an eigenvalue and \(N_n\) is a normalization factor defined by
\[
N_n = (\chi_n, \sum_{\varepsilon \eta} \varepsilon \Lambda_{\varepsilon \eta} \chi_n).
\]
When one of the constituents (labeled with 2) belongs to the same class of the antiparticle of the other, there can be an annihilation process through the weak interaction for which the unamputated-decay-vertex $\Phi$ is given by

$$\Phi = C\gamma G,$$

where $\gamma$ is the lowest vertex and $C$ is the charge-conjugation matrix of the particle 2.

$\Phi$ satisfies the vertex equation

$$\Phi(E - H_1 - H_2 - V \sum_{\varepsilon\eta} \varepsilon \Lambda_{\varepsilon\eta}) = C\gamma \sum_{\varepsilon\eta} \varepsilon \Lambda_{\varepsilon\eta}$$

and the amputated vertex is

$$\Gamma = \gamma + C\Phi V.$$ 

We can determine the renormalization constant $Z_1$ for the wave function at the origin (WFO) from (15) and (14), if we need it[6].

### 2.3 Interaction Hamiltonian

We have, so far, not specified the interaction Hamiltonian (quasipotential). In this section, we investigate it for the one-(Abelian)gauge-boson exchange in the Coulomb gauge as an example. For the instantaneous part of the interaction, $V$ is obvious. For the remaining part, we can specify the quasipotential in a clear way from the background field theory. We have already shown, for the Salpeter equation, that the quasipotential from the one-boson exchange is given through the diagonalization of Fukuda, Sawada, Taketani[7] and Okubo[8] (FSTO)[9]. We also apply this method to the present equation.

We introduce the generalized Fock subspace of the free constituents, the bases of which are denoted by $|\varepsilon, \eta, p\rangle$, where $\varepsilon$ and $\eta$ are the signs of the energies of the particles 1 and 2 respectively. We then diagonalize the Hamiltonian in the Schrödinger picture by using the FSTO method. The second-order boson-exchange potential in this subspace is given by

$$\langle \varepsilon, \eta, p | V(1b) | \varepsilon', \eta', p' \rangle = \frac{g^2}{(2\pi)^3} \sum_{ij} \alpha_{1i}(\delta_{ij} - \frac{1}{q^2} q_i q_j) \alpha_{2j}$$

$$\times \frac{1}{2} \left[ q^2 - \{ \varepsilon E_1(p) - \varepsilon' E_1(p') \}^2 \right] + \frac{1}{q^2 - \{ \eta E_2(p) - \eta' E_2(p') \}^2},$$

There was some error concerning the Fock space in Ref.[9]. Namely, we employed the usual (positive-energy) Fock space and reinterpreted the matrix elements of the interaction Hamiltonian including the negative-energy indices as the ones in this space according to the hole theory. It, however, brings the procedure into confusion, since we have revived the negative energy in Eq.(6). However, the error is only conceptual for the Salpeter equation which has only projection factors $\Lambda_{++}$ and $\Lambda_{--}$. The result of Ref.[9] is correct.
where \( q = p - p' \). The retardation effects are included in this equation.

3 Inclusion of crossed diagrams and the counter correction

We have modified the boundary condition in (4), which amounts to inclusion of some parts of the crossed diagram. To estimate this effect, we take up the 4th-order scattering amplitude in the Coulomb potential for the mass-shell particles. We are interested in the contribution of the intermediate state with the \( \Lambda^- \) projection.

Now, let us compare the two amplitudes restricted to this intermediate state: the ‘ladder diagram’(l) with the propagator (4) and the crossed diagram(c) with the usual Feynman propagator. We see that they are different by the following factors:

\[
I_l = \frac{\sqrt{(p_2 + q)^2 + m^2 - \alpha^{(2)} \cdot (p_2 + q) - \beta^{(2)} m}}{\sqrt{(p_2 + q)^2 + m^2 (A + \sqrt{p_2^2 + m^2 + (p_2 + q)^2 + m^2})}}
\]  

(18)

for the ‘ladder diagram’ and

\[
I_c = \frac{\sqrt{(p'_2 - q)^2 + m^2 - \alpha^{(2)} \cdot (p'_2 - q) - \beta^{(2)} m}}{\sqrt{(p'_2 - q)^2 + m^2 (-A + \sqrt{p'_2^2 + m^2 + (p'_2 - q)^2 + m^2})}}
\]  

(19)

for the crossed diagram, where \( p_2 \) and \( p'_2 \) are the initial and final momentum of the lighter particle respectively and \( q \) is the loop momentum to be integrated. The common element \( A \) is given by

\[
A = \sqrt{p_1^2 + M^2} - \sqrt{(p_1 - q)^2 + M^2}.
\]

We see that \( I_l \) and \( I_c \) have the same sign and the order of magnitude for momenta much smaller than \( M \). Therefore, the term with the factor \( I_l \) can be interpreted as a substitute for the crossed diagram.

The (pseudo)potential model is an effective theory in the strong coupling QED, for which the effective inclusion of the crossed diagrams is desirable. In contrast to it, we are able to get corrections of any desired accuracy by using the perturbation theory in the weak coupling QED. We could start with the Schrödinger equation for the unperturbed system for this purpose. However, more suitable choice is the relativistic equation with the good features. The two-body equation of the present paper is a candidate for it.

If we intend to proceed to the next order approximation to the interaction \( V \), we shall calculate the 4th-order crossed diagram by assuming ordinary Feynman propagator. We should, further, add counter correction terms to compensate the part of the crossed diagram already included effectively in the iteration of the lowest kernel.
mentioned above. The counter correction to the two-body propagator is given by
taking the difference of the equation \((1)\) and the Feynman's two-body propagator. 
The counter corrections should be added also in the higher order approximations.

4 Some properties

4.1 WFO of the \(^1S_0\) state

We can use our equation as the basic equation of the effective theory of QCD. When it
is applied to the system in which the pair annihilation of the constituents can occur,
an important physical quantity is the wave function at the origin. For example, the
decay amplitude of the pseudo-scalar \(Q\bar{q}\) meson via a weak boson is proportional to
the average WFO \(\text{Tr}\{\gamma_5\gamma_0\psi(0)\}\), where \(\psi(0)\) is the charge conjugated(with respect
to the particle 2(\(\bar{q}\))) WFO. We investigate, in Appendix, the asymptotic behavior
of the momentum-space wave function by using the method given in Ref.\([6]\). We
assume instantaneous exchange of a gauge boson\([3]\). The average WFO thus obtained
is finite. This result is consistent with consideration on the covariant field theory.
We note that the average WFO becomes divergent in the limit of the one-body Dirac
equation, for which we have the renormalization procedure\([10]\).

There are many “two-body Dirac equation” proposed. An interesting one from
the point of view of the present paper is the one by Mandelzweig and Wallace\([11]\).
They intended to include the effects of the higher-order interaction (the crossed
Feynman diagram) and got an equation which has the proper one-body limit and
the \(E\)-parity symmetry. An important difference from our equation is in the average
WFO considered above. It is divergent in their equation if the transverse part of the
gauge-boson exchange is included\([12]\).

4.2 On application to the heavy quark effective theory

One of the research fields in which we can utilize the two-body Dirac equation is the
physics of the heavy flavored quarkonia. The recent trends in this field are led by
the heavy quark effective theory(HQET). The \(Q\bar{q}\) system is well described by using
the two-body Dirac equation with some phenomenological (pseudo)potential. If we
take the heavy quark limit(\(M \to \infty\)) of the quark \(Q\), the equation itself becomes the
one-body Dirac equation, which is the basic equation of HQET.

There are, however, divergent(as \(M \to \infty\)) portions in the perturbative correction
to the WFO. For example, the matrix element of the current for the annihilation
decay of a pseudoscalar quarkonium is given by

\[^3\]The analysis in the Appendix cannot be applied to the retarded interaction.
\[ <0|\bar{q}\gamma_{\nu}\gamma_5Q|Q\bar{q}> = -C(\mu)\text{tr}\{\gamma_{\nu}\gamma_5\Psi(0)\}, \quad (20) \]

where \(\Psi(0)\) is the WFO of the bound state and we have separated the correction \(C(\mu)\) coming from the range of momenta \(\mu \sim M\) in the integral over the loop momenta. \(C(\mu)\) is determined from the perturbative loop corrections summed up by using the renormalization group equation \([13][14]\).

\[ C(\mu) = \left(\frac{\alpha_s(M)}{\alpha_s(\mu)}\right)^d, \quad d = -6/(33 - 2N_f). \quad (21) \]

To improve the approximation, we can calculate \(1/M\) corrections to HQET by using the two-body Dirac equation.

4.3 Another approach to the heavy quark phenomenology

An alternative way of investigating the heavy quark systems is due to direct application of the two-body Dirac equation. The binding interaction, in this formulation, consists of the Coulomb and the confining potentials. We further add the gluonic correction term to the potential, for which we should attach the high momentum cutoff\([15]\). The correction factor \(C(\mu)\) depends on this cutoff. If we take the cutoff of the QCD scale, we should include all the corrections from the high momenta, except for the Coulomb contribution, to the vertex correction. By subtracting it in the equation \(\text{21}\), we get \(d = 10/(33 - 2N_f)\) for the exponent.\(^4\)

4.4 On the equal-mass limit

The unequal-mass equation \((\text{6})\) is well applied to the system with \(M \gg m\). For \(M \simeq m\), the reason justifying \((\text{4})\) becomes obscure and we will have two different equations in the limit \(M = m\). Though we concern ourselves in the case \(M > m\), it is meaningful to investigate the degree of ambiguity near the equal mass limit. For \(M = m\), the projection factor in front of the interaction term of \((\text{6})\) includes a part which violates the exchange symmetry: It is shown that

\[ \Lambda^{(V)} = \Lambda_{++} - \Lambda_{--} \quad (22) \]

and the Heisenberg’s exchange operator

\[ P_H = \frac{1}{4}(1 + \sigma_1 \cdot \sigma_2)(1 + \rho_1 \cdot \rho_2)P_M \quad (23) \]

\(^4\) We have assumed the counter correction which compensates the modification of the two-body propagator, since it cannot be regarded as a substitute for crossed diagram in QCD. If we do not include the counter correction we shall get \(d = 2/(33 - 2N_f)\).\([16]\).
anticommute, where the operator $P_M$ exchanges the momenta (or coordinates). $\Lambda^{(V)}$ violates the symmetry since the remaining part of the Hamiltonian and $P_H$ commute.

For equal-mass limit, we have two equations. One is the equation (6) with $M = m$ and another is obtained by assigning a minus sign in front of $\Lambda^{(V)}$, which is the equal-mass limit of the equation with $m > M$. It is the conjugate equation of (6) in the sense that (6) is converted into it by the transformation $P_H$. It is easy to show that these equations have the common eigenvalue spectrum: If an eigenfunction $\chi_n$ of (6) belongs to some eigenvalue $E_n$, $P_H\chi_n$ is the solution of the conjugate equation with the same eigenvalue. However, $\chi_n$ does not have the definite $P_H$-parity.

For equal masses, it is reasonable to use the Salpeter equation which includes the projection factor (4). The eigenvalue of this equation is different from the above $E_n$. The difference is, however, small since it is of the 4th order in the symmetry-breaking part of the Hamiltonian.

5 Summary and discussion

By assuming a proper boundary condition for the two-body propagation in the negative-energy state, we have modified the propagator and proposed a new bound-state equation for the unequal-mass constituents. It has the symmetrical energy eigenvalues $E_n$ and $-E_n$ and reduces to the (one-body) Dirac equation in the infinite limit of one of the constituents masses. Secondly, we have discussed the normalization of the wave function and pointed out that the positive-definite probability density should be assumed. We can consistently calculate the observables of the bound state by assuming this normalization.

The interaction Hamiltonian of the equation is constructed by diagonalizing the field theoretic Hamiltonian in the generalized Fock subspace of the two particles. Relativistic effects such as the retardation are taken into account in systematic way in the framework of the perturbation theory.

We can use this two-body equation as the foundation of the rigorous perturbation theory of the weak-coupling QED. In this case, we should correct back the modification of the propagator.

We have further investigated the WFO’s of the proposed equation in some detail for the instantaneous interaction and shown that the average WFO in the $^1S_0$ state which determines the leptonic decay rate of a pseudo-scalar meson is finite, which is in accord with the expectation from the field theory.

One of the system for which we may utilize the two-body Dirac equation is the heavy flavored quarkonium. There are various possibilities of choosing the framework of the approximation according to the treatment for the high-momentum interaction. We have briefly discussed the correction factor to the leptonic decay width caused by the high momenta. The equation affords a good foundation of the heavy quark effective theory.
Finally, we discuss some feature of the spectrum of the equation (6). There are physical eigenvalues $E_n$ which reduce to $M + m$ in the weak coupling limit and the corresponding negative ones which are interpreted as the bound states of the antiparticles 1 and 2. Besides those, there may be a series of eigenvalues which reduce to $M - m$ in the weak-coupling limit and its negative counterpart, which are unphysical. Fortunately, we can identify and discard these unphysical spectra by inspecting the weak-coupling limit.

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**Appendix**

We examine the asymptotic ($p \to \infty$) behavior of the momentum-space wave function and show that the average WFO $\frac{1}{\sqrt{2}} \text{Tr}\{\gamma_5 \gamma_0 \psi(0)\}$ is finite.$^5$

There are 4 partial amplitudes $h_{\varepsilon\eta}(p)$ in the $^1S_0$ state, with which the wave function is expanded as

$$\chi(p) = \sum_{\varepsilon\eta} \sum_r c_r u_r^\varepsilon(-p)v_r^\eta(p)h_{\varepsilon\eta}(p)\left(\frac{1}{16\pi E_1 E_2}\right)^{1/2}, \quad (24)$$

where $c_{1/2} = -c_{-1/2} = 1/\sqrt{2}$ and the spinors $u$ and $v$, for the particle 1 and 2 respectively, are defined in [3].

The average WFO for the annihilation decay through the axial-vector current is given by

$$\frac{1}{\sqrt{2}} \text{Tr}\{\gamma_5 \gamma_0 \psi(0)\} = \frac{1}{\sqrt{8\pi}} \int \left(\frac{1}{E_1 E_2}\right)^{1/2}$$

$$\times \sum_{\varepsilon\eta} \{\sqrt{(E_1 + \varepsilon M)(E_2 + \eta m)} - \varepsilon\eta\sqrt{(E_1 - \varepsilon M)(E_2 - \eta m)}\} h_{\varepsilon\eta}(p)p^2 dp. \quad (25)$$

We first assume the Coulomb potential. The partial-wave equation for the $^1S_0$ state is given by

$$\{E - \varepsilon E_1(p) - \eta E_2(p)\}h_{\varepsilon\eta}(p) = -\varepsilon \frac{\alpha}{4\pi} \sum_{\varepsilon'\eta'} \int dq$$

$^5$See Ref.[17] and references therein, for the Salpeter equation.
\[
\times \frac{1}{p^4 E_1(p) E_2(q) E_2(q)} \left[ \{ A^{1\varepsilon}_{\varepsilon}, A^{2}_{\eta\eta'} + \varepsilon' \eta' A^{1}_{\varepsilon', \varepsilon} A^{2}_{\eta, \eta'} \} Q_0(z) + \{ \varepsilon' A^{1}_{\varepsilon, \varepsilon'} A^{2}_{\eta, \eta'} + \eta' A^{1}_{\varepsilon', \varepsilon} A^{2}_{\eta', \eta} \} Q_1(z) \right] h_{\varepsilon, \eta}(q),
\]

where \( z = (p^2 + q^2)/2pq \) and \( Q_\ell(z) \) is the Legendre’s function. \( A^i_{\varepsilon, \eta} \) is defined by

\[
A^i_{\varepsilon, \eta} = \sqrt{(E_i(p) + \varepsilon m_i)(E_i(q) + \varepsilon' m_i)}.
\]

The asymptotic behavior of the wave function is determined from the integral region near the infinity. We then expand the both sides of (26) into the series of \( \varepsilon \) and \( \frac{1}{p} \) where the terms of the higher power in \( \frac{1}{p} \) and \( \varepsilon \) are chosen to be

\[
C(i, j, k) = \{ \varepsilon^{i-1} \eta^{j-1} \varepsilon'^{k-1} \} \frac{1}{p^i} \quad \text{terms of the higher power in } \frac{1}{p}.
\]

The asymptotic amplitudes

\[
A^i_{\varepsilon, \eta} = \sum_{n} C^n_{\varepsilon, \eta} \frac{1}{\pi X^2} p^{-2n-1}.
\]

Integrals on the right-hand side can be done if we neglect infrared-divergent terms which are irrelevant to the leading asymptotic behavior. Now, we can determine the asymptotic indices \( \beta \)'s from consistency\[6\]: We get, for \( h_A \) and \( h_B \)

\[
2C_A^0 (1 - \beta_A) p^{-\beta_A} = \frac{\alpha}{\pi} C_A^0 \frac{\pi}{1 - \beta_A} \cot \left( \frac{\pi}{2} \beta_A \right) p^{-\beta_A},
\]

and

\[
2C_B^0 (1 - \beta_B) p^{-\beta_B} = \frac{\alpha}{\pi} C_B^0 \frac{\pi}{\beta_B (2 - \beta_B)} \tan \left( \frac{\pi}{2} \beta_B \right) p^{-\beta_B},
\]

where the terms of the higher power in \( \frac{1}{p} \) are neglected. If we neglect the second term in the left-hand sides of (27), we find \( \beta_A \) in the range \( 1 < \beta_A < 2 \) and get \( \beta_B = \beta_A + 1 \) from (28). We obtain another series by neglecting the second term in (28). For this, \( \beta_B \) is found to be in the range \( 2 < \beta_0 < 3 \), where the lower bound \( \beta_0 \) corresponds to the upper bound \( 4/\pi \) of \( \alpha \) above which the index \( \beta_A \) from (27) becomes complex. \( \beta_A \) of the second series is given by \( \beta_A = \beta_B + 1 \).

The asymptotic amplitudes \( h_C \) and \( h_D \) are determined dependently on \( h_A \) and \( h_B \). We get, for the minimum indices

\[
\beta_C = \min(\beta_A + 2, \beta_B + 1) \quad \beta_D = \beta_A + 1.
\]

We see that the average WFO (23) is finite, because

\[
\beta_B > 1 \quad \text{and} \quad \beta_C > 2
\]

hold for the asymptotic amplitudes. This conclusion is valid even if the instantaneous exchange(transverse part) of the gauge boson is added.

\[6\text{See Ref.}[6]\ for the details.
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