Stochastic production planning with regime switching

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Abstract
This paper considers the stochastic production planning with regime switching. There are two regimes corresponding to different economic cycles. A factory is planning its production so as to minimize production costs. We analyze this problem through the value function approach. The optimal production is characterized through the solution of an elliptic system of partial differential equations which is shown to have a solution.

1 Introduction
The purpose of this paper is to consider a stochastic production planning problem with regime switching parameters and to provide a mathematical treatment of it. Regime switching modelling is present in many areas such as financial economics and management. In finance we point the interested reader to [18], [19] and the references therein.

In the last decade, there is an extensive literature on production planning/management with regime switching. We only recall a few works. The paper [5] studies the cost minimization problem of a company within an economy characterized by two regimes. In civil engineering [12] studies the optimal stochastic control problem for home energy systems with regime switching; the two regimes are the peak and off peak energy demand. The work [13] considers the production control problem in a manufacturing system with multiple machines which are subject to breakdowns and repairs. The mathematical modelling for these problems makes it
It is possible to find solutions by simply solving stochastic control problems with regime dependent controls/value functions. The paper [11] provides the mathematical analysis and results of a fairly general class of stochastic control problems such as the ones appearing in stochastic production planning over infinite horizons and with regime dependent model parameters. Their solution approach relies on the concept of value function and the latter is characterized through a system of elliptic equations which is shown to have solutions. Among recent papers which contribute to the mathematical analysis of stochastic planning problem we mention [8], [9], and [10].

In this paper we look at production planning problem with regime switching parameters in a random environment. A factory is planning its production of several economic goods as to minimize inter temporally its production and inventory costs. A constant discount rate is used to measure on the same time scale costs which occur at different times. The stochasticity is driven by a $N$-dimensional Brownian motion and a Markov chain. The Markov chain models the different economic regimes while the multidimensional Brownian motion captures the random nature of good’s demand; the demand is also linked to economic cycles and this makes it dependent on the Markov chain as well. The constant discount rate may also depend on the Markov chain. We add a stopping criterion in evaluating the inter temporal costs, which is the time when the inventory of the goods exceeds some threshold level. We tackle this production planning problem by the value function approach. Using probabilistic techniques we derive the Hamilton Jacobi Bellman (HJB) of the value function. We employ partial differential equations (PDE) tools/techniques to analyze the HJB equation. In the end we prove a verification result, i.e., show that the HJB equation yields the optimal production.

The remainder of this paper is organized as follows. Section 2 presents the model and the objectives. In Section 3 we present the methodology.

2 Formulation of the model

We begin our presentation of the problem to be studied by considering a factory producing $N$ types of economic goods which stores them in an inventory designated place. The factory would like to tune its production of the goods in such a way to minimize production costs and inventory costs. We allow for regime switching in our model; regime switching refers to the situations when the characteristics of the state process are affected by several regimes (e.g. in finance bull and bear market with higher volatility in the bear market), or economical cycles characterized by high versus low demand of economic products (e.g. in the auto industry there is a higher demand for cars in summer time).

Next, we formulate the model mathematically. There exists a complete probability space $(\Omega, \mathcal{F}, \{\mathcal{F}_t\}_{0 \leq t \leq \infty}, P)$, on which lives a $N$-dimensional Brownian motion denoted by

$$w = (w_1, ..., w_N).$$

The regime switching is captured by a continuous time homogeneous...
Markov chain $\epsilon(t)$ adapted to $\mathcal{F}_t$ with two regimes good and bad, i.e.,

$$\epsilon(t) \in \{1, 2\}, \quad t \in [0, \infty).$$

In a specific application, $\epsilon(t) = 1$ could represent a regime of economic growth while $\epsilon(t) = 2$ could represent a regime of economic recession. In another application, $\epsilon(t) = 1$ could represent a regime in which consumer demand is high while $\epsilon(t) = 2$ could represent a regime in which consumer demand is low.

The Markov chain’s rate matrix is

$$A = \begin{pmatrix} -a_1 & a_1 \\ a_2 & -a_2 \end{pmatrix},$$

(1)

for some $a_1 > 0$, $a_2 > 0$. Diagonal elements $A_{ii}$ are defined such that

$$A_{ii} = -\sum_{j \neq i} A_{ij},$$

(2)

where

$$A_{11} = -a_1, A_{12} = a_1, A_{21} = a_2, A_{22} = -a_2.$$  

In this case, if $p_t = \mathbb{E}[\epsilon(t)] \in \mathbb{R}^2$, then

$$\frac{d\epsilon(t)}{dt} = A\epsilon(t).$$

(3)

Moreover

$$\epsilon(t) = \epsilon(0) + \int_0^t A\epsilon(u) \, du + M(t),$$

(4)

where $M(t)$ is a martingale with respect to $\mathcal{F}_t$. The filtration $\mathcal{F}_t$ is generated by the $N$-dimensional Brownian motion and the Markov chain.

Next, let us introduce the control variables in our model. Let

$$p(t) = (p_1(t, \epsilon(t)), ..., p_N(t, \epsilon(t))),$$

represent the production rate at time $t$ (control variable) adjusted for the demand rate. That means we subtract the demand rate so that we obtain net production rate. Next, let $y_{t,0,0}^i$ denote the initial inventory level of good $i$, and $y_t(t, \epsilon(t))$ the inventory level of good $i$ at time $t$, adjusted for demand. Again, we look at the net inventory since it is this quantity which incurs inventory costs. These adjusted for demand inventory levels are modelled by the following system of stochastic differential equations

$$dy_t(t, \epsilon(t)) = p_t \, dt + \sigma_{\epsilon(t)} dw_t, \quad y_t(0, \epsilon(0)) = y_{t,0,0}^i, \quad i = 1, ..., N,$$

(5)

where $\sigma_{\epsilon(t)}$ is a regime dependent constant (non-zero) diffusion coefficient taking on two values, $\sigma_1$ and $\sigma_2$. The stochasticity here is due to demand adjustment which is random in nature and dependent on the regime. Another source of randomness our model can accommodate are inventory spoilages. One can think of examples when the demand is more volatile in some periods (e.g. some states of the Markov chain) and less volatile in other periods.
We impose a stopping production criterion; that is when the (net) inventory exceeds an exogenous threshold level then the production stops (this is often the case in auto industry when the storage capacity of newly produced cars is exhausted). Let us formalize this mathematically; \( \tau \) denotes the stopping time representing the moment when the (net) inventory level reaches some positive threshold \( R \), i.e.,

\[
\tau = \inf_{t > 0} \{ |y(t, \epsilon(t))| \geq R \}.
\]

Here, \(| \cdot |\) stands for the Euclidian norm. At this point we are ready to state our objective.

2.1 The Objective

The performance over time of a demand adjusted production rate(s)

\[
p(t, \epsilon(t)) = (p_1(t, \epsilon(t)), \ldots, p_N(t, \epsilon(t)))
\]

is measured by means of its production costs and inventory costs. At this point we introduce the cost functional which measures the quadratic loss:

\[
J(p_1, \ldots, p_N) := \mathbb{E} \int_0^\tau (|p(t, \epsilon(t))|^2 + |y(t, \epsilon(t))|^2) e^{-\alpha_\epsilon(t) t} dt,
\]

where \( |p(t, \epsilon(t))|^2 \) and \( |y(t, \epsilon(t))|^2 \) denote the quadratic holding cost and the quadratic production cost functions, respectively. Again let us recall that we measure deviations from the demand, whence the loss. Here \( \alpha_\epsilon(t) \) is a regime dependent (taking on two values \( \alpha_1 \) and \( \alpha_2 \)), constant psychological rate of time discount, whence the exponential discounting. The constant psychological rate of time discount is employed to measure on the same time scale outcomes which occur at different times.

At this point we are ready to frame our objective, which is to minimize the cost functional. i.e.,

\[
\inf_{(p_1, \ldots, p_N) \in \mathbb{R}^N} \{ J(p_1, \ldots, p_N) \},
\]

subject to the stochastic differential equation system (5).

3 The Methodology

Having presented the problem we want to solve, now we provide our means to tackle it. Our approach is based on the value function and dynamic programming which leads to an HJB system of equations.

We apply probabilistic techniques to characterize the value function; that is we search for functions \( z_i : \mathbb{R} \rightarrow \mathbb{R}, \ i = 1, 2 \) such that the stochastic process \( Z^\epsilon(t) \) defined below

\[
Z^\epsilon(t) = -e^{-\alpha_\epsilon(t) t} z_i(t) (y(t, \epsilon(t))) - \int_0^t \left[ |p(s, \epsilon(s))|^2 + |y(s, \epsilon(s))|^2 e^{-\alpha_\epsilon(s) s} ds,
\]

(8)
is supermartingale for all
\[ p(t, \epsilon(t)) = (p_1(t, \epsilon(t)), ..., p_N(t, \epsilon(t))) \]
and martingale for the optimal control
\[ p^*(t, \epsilon(t)) = (p^*_1(t, \epsilon(t)), ..., p^*_N(t, \epsilon(t))) \]
We search for \( z_1, z_2 \) functions in \( C^2([0, R]) \), and the supermartingale/martingale requirement yields by means of Itô’s Lemma for Markov modulated diffusions the HJB system of equations which characterizes the value function
\[
- a_1 z_2 + (a_1 + \alpha_1) z_1 - \frac{\sigma^2}{2} \Delta z_1 - |x|^2 = \inf_{p \in \mathbb{R}^N} \left\{ p \nabla z_1 + |p|^2 \right\},
\]
and
\[
- a_2 z_1 + (a_2 + \alpha_2) z_2 - \frac{\sigma^2}{2} \Delta z_2 - |x|^2 = \inf_{p \in \mathbb{R}^N} \left\{ p \nabla z_2 + |p|^2 \right\}.
\]
This HJB system can be turned into a partial differential equation system (PDE system) since a simple calculation yields
\[
\inf_{p \in \mathbb{R}^N} \left\{ p \nabla z_j + |p|^2 \right\} = -\frac{1}{4} |\nabla z_j|^2, \ j = 1, 2.
\]
Thus, the HJB system becomes the PDE system
\[
\begin{align*}
- a_1 z_2 + (a_1 + \alpha_1) z_1 - \frac{\sigma^2}{2} \Delta z_1 - |x|^2 &= -\frac{1}{4} |\nabla z_1|^2 \quad \text{for} \ x \in B_R, \\
- a_2 z_1 + (a_2 + \alpha_2) z_2 - \frac{\sigma^2}{2} \Delta z_2 - |x|^2 &= -\frac{1}{4} |\nabla z_2|^2 \quad \text{for} \ x \in B_R,
\end{align*}
\]
where
\[
B_R = \left\{ x \in \mathbb{R}^N \ | \ |x| < R \right\},
\]
denotes the open ball of radius \( R > 0 \) centered at the origin. In order to perform the verification, i.e., show that the HJB system gives the solution of the optimization problem, one needs to impose the following boundary condition
\[
z_1(x) = z_2(x) = 0 \quad \text{for} \ x \in \partial B_R.
\]
The gradient term in the above PDE system can be removed by the change of variable
\[
u_j (x) = e^{-\frac{x_j}{2\sigma_j}}, \ j = 1, 2,
\]
to get a simpler PDE system
\[
\begin{align*}
\Delta u_1 (x) &= u_1 (x) \left[ \frac{1}{\sigma_j^2} |x|^2 + \frac{2(a_1 + \alpha_1)}{\sigma_j^2} \ln u_1 (x) - 2a_1 \frac{\sigma^2}{\sigma_j^2} \ln u_2 (x) \right] \quad \text{for} \ x \in B_R, \\
\Delta u_2 (x) &= u_2 (x) \left[ \frac{1}{\sigma_j^2} |x|^2 + \frac{2(a_2 + \alpha_2)}{\sigma_j^2} \ln u_2 (x) - 2a_2 \frac{\sigma^2}{\sigma_j^2} \ln u_1 (x) \right] \quad \text{for} \ x \in B_R, \\
u_j (x) > 0, \ u_1 (x) > 0 \quad \text{for} \ x \in B_R,
\end{align*}
\]
with the corresponding boundary condition
\[
u_1 (x) = u_2 (x) = 1 \quad \text{for} \ x \in \partial B_R.
\]
The value function will give us in turn the candidate optimal control. The first order optimality conditions on the lefthand side of (11) are sufficient for optimality since we deal with a quadratic function and they produce the candidate optimal control as follows:

\[ p_i(t, \epsilon(t)) = \overline{p}_i(y_1(t, \epsilon(t)), \ldots, y_N(t, \epsilon(t))), \text{ } i = 1, \ldots, N, \]

and

\[ \overline{p}_i(x_1, \ldots, x_N, j) = -\frac{1}{2} \frac{\partial^2}{\partial x_i^2} (x_1, \ldots, x_N), \text{ for } i = 1, \ldots, n, \text{ } j = 1, 2. \]  

(16)

The system (12)-(13) is key in solving our problem so we need to analyze it. We prove the following result:

**Theorem 3.1.** The system of equations (12)-(13) has a unique positive classical solution \((z_1, z_2)\). Moreover,

\[ z_i(x) \leq -2\sigma_i^2 K_i (R^2 - |x|^2), \text{ for some } K_i < 0, \text{ } i = 1, 2. \]  

(17)

**Proof** Our approach, being constructive, will be useful for a computational scheme for numerical approximations of the solution. Since the system (12)-(13) is equivalent to (14)-(15) we will work with the later. We proceed in three steps: step 1) establishes a sub-solution and a supersolution; step 2) provides an approximating sequence of functions which converges to the solution; step 3) established the uniqueness.

**Step 1** The main problem is reduced to the construction of the function \((\overline{u}_1, \overline{u}_2)\) called sub-solution and a function \((\overline{\nu}_1, \overline{\nu}_2)\) named super-solution with order (i.e., \(\overline{u}_1(x) \leq \overline{\nu}_1(x)\) and \(\overline{u}_2(x) \leq \overline{\nu}_2(x)\), for all \(x \in \overline{B}_R\)) to the system (14), which satisfy the inequalities

\[
\begin{align*}
\Delta \overline{u}_1(x) &\geq \overline{u}_1(x) \left[ \frac{1}{\sigma_1^2} |x|^2 + \frac{2(a_1 + a_1)}{\sigma_1^2} \ln \overline{u}_1(x) - \frac{2a_1^2}{\sigma_1^4} \ln \overline{u}_2(x) \right], \text{ } x \in \overline{B}_R, \\
\Delta \overline{u}_2(x) &\geq \overline{u}_2(x) \left[ \frac{1}{\sigma_2^2} |x|^2 + \frac{2(a_2 + a_2)}{\sigma_2^2} \ln \overline{u}_1(x) - \frac{2a_2^2}{\sigma_2^4} \ln \overline{u}_2(x) \right], \text{ } x \in \overline{B}_R, \\
\Delta \overline{\nu}_1(x) &\leq \overline{\nu}_1(x) \left[ \frac{1}{\sigma_1^2} |x|^2 + \frac{2(a_1 + a_1)}{\sigma_1^2} \ln \overline{\nu}_1(x) - \frac{2a_1^2}{\sigma_1^4} \ln \overline{\nu}_2(x) \right], \text{ } x \in \overline{B}_R, \\
\Delta \overline{\nu}_2(x) &\leq \overline{\nu}_2(x) \left[ \frac{1}{\sigma_2^2} |x|^2 + \frac{2(a_2 + a_2)}{\sigma_2^2} \ln \overline{\nu}_1(x) - \frac{2a_2^2}{\sigma_2^4} \ln \overline{\nu}_2(x) \right], \text{ } x \in \overline{B}_R.
\end{align*}
\]  

(18)

The construction of the sub-solution requires some work. More exactly, by direct calculations we observe that there exist

\((u_1(x), u_2(x)) = \left( e^{K_1 (R^2 - |x|^2)}, e^{K_2 (R^2 - |x|^2)} \right), \text{ with } K_1, K_2 \in (-\infty, 0), \)  

(19)
satisfying (18). We are trying to prove that

\[
\begin{align*}
4K_1^2 |x|^2 - 2K_1 N &\geq \frac{1}{\sigma_1^2} |x|^2 + \frac{2(a_1 + a_1)K_1}{\sigma_1^2} (R^2 - |x|^2) - \frac{2a_1^2 K_1}{\sigma_1^4} (R^2 - |x|^2), \\
4K_2^2 |x|^2 - 2K_2 N &\geq \frac{1}{\sigma_2^2} |x|^2 + \frac{2(a_2 + a_2)K_2}{\sigma_2^2} (R^2 - |x|^2) - \frac{2a_2^2 K_2}{\sigma_2^4} (R^2 - |x|^2),
\end{align*}
\]

or, equivalently

\[
\begin{align*}
[K_1^2 - \frac{2a_1^2 K_2}{\sigma_1^2} K_1 + 1 - 2(a_1 + a_1)K_1 \sigma_1^2] |x|^2 - \frac{2(a_1 + a_1)K_1}{\sigma_1^2} R^2 - 2K_1 N + \frac{2a_1^2 K_2}{\sigma_1^4} R^2 &\geq 0, \\
[K_2^2 - \frac{2a_2^2 K_1}{\sigma_2^2} K_2 + 1 - 2(a_2 + a_2)K_2 \sigma_2^2] |x|^2 - \frac{2(a_2 + a_2)K_2}{\sigma_2^2} R^2 - 2K_2 N + \frac{2a_2^2 K_1}{\sigma_2^4} R^2 &\geq 0.
\end{align*}
\]
Since, a simple calculation shows that there exist \( K_1, K_2 \in (-\infty, 0) \) such that
\[
\begin{align*}
4K_1^2 - \frac{1}{\sigma_1} - 2a_1 \frac{\sigma_1^2}{\sigma_1^2} K_1 + \frac{2(a_{11} + a_{12})}{\sigma_1^2} K_1 &\geq 0 \\
4K_2^2 - \frac{1}{\sigma_2} - 2a_2 \frac{\sigma_2^2}{\sigma_2^2} K_2 + \frac{2(a_{21} + a_{22})}{\sigma_2^2} K_2 &\geq 0 \\
-\left(\frac{2(a_{11} + a_{12})}{\sigma_1^2} K_1 + 2a_1 \frac{\sigma_1^2}{\sigma_1^2} R^2 K_1 \right) &\geq 0 \\
-\left(\frac{2(a_{21} + a_{22})}{\sigma_2^2} K_2 + 2a_2 \frac{\sigma_2^2}{\sigma_2^2} R^2 K_2 \right) &\geq 0
\end{align*}
\]
one only has to notice that \( \overline{u} \) is a sub-solution for the system \( \text{[13]} \). Constructing a super-solution is easier. It turns out that
\[
\left( \overline{u}_1 (x), \overline{u}_2 (x) \right) = (1, 1),
\]
is a super-solution of \( \text{[14]} \).

**Step 2** By the above construction one gets
\[
\underline{u}_1 (x) \leq \overline{u}_1 (x) \quad \text{and} \quad \underline{u}_2 (x) \leq \overline{u}_2 (x) \quad \text{for all} \quad x \in \overline{B}_R.
\]
We are showing that, the problem \( \text{[13]} \) admits a unique solution
\[
(u_1, u_2) \in \left[ C^2(B_R) \cap C(\overline{B}_R) \right]^2 = \left[ C^2(B_R) \cap C(\overline{B}_R) \right] \times \left[ C^2(B_R) \cap C(\overline{B}_R) \right],
\]
such that
\[
\underline{u}_1 (x) \leq u_1 (x) \leq \overline{u}_1 (x) \quad \text{and} \quad \underline{u}_2 (x) \leq u_2 (x) \leq \overline{u}_2 (x), \quad \text{for} \quad x \in \overline{B}_R.
\]
Denote
\[
M_1 = e^{K_1R^2}, \quad M_2 = e^{K_2R^2} \quad \text{and} \quad M = 1.
\]
Let \( g_1 : \overline{B}_R \times [M_1, M] \times [M_2, M] \to \mathbb{R} \) and \( g_2 : \overline{B}_R \times [M_2, M] \times [M_1, M] \to \mathbb{R} \) defined by
\[
g_1(x, t, s) = \frac{1}{\sigma_1} |x|^2 t + \frac{2(a_{11} + a_{12})}{\sigma_1^2} t \ln t - \frac{2a_1 \sigma_1^2}{\sigma_1^2} t \ln s,
\]
\[
g_2(x, t, s) = \frac{1}{\sigma_2^2} |x|^2 s + \frac{2(a_{21} + a_{22})}{\sigma_2^2} s \ln s - \frac{2a_2 \sigma_2^2}{\sigma_2^2} s \ln t.
\]
Since \( g_1 \) is a continuous function with respect to the first variable in \( \overline{B}_R \)
and continuously differentiable with respect to the second and third in \([M_1, M] \times [M_2, M], \) it allows to choose \( \Lambda_1 \in (0, \infty) \) such that
\[
-\Lambda_1 \geq \frac{g_1(x, t_1, s) - g_1(x, t_2, s)}{t_2 - t_1},
\]
for every \( t_1, t_2 \) with \( \underline{u}_1 \leq t_2 < t_1 \leq \overline{u}_1 \) and \( x \in B_R. \) Similarly for \( g_2, \) we can choose \( \Lambda_2 \in (0, \infty) \) such that
\[
-\Lambda_2 \geq \frac{g_2(x, t, s_1) - g_2(x, t, s_2)}{s_2 - s_1},
\]
for every \( s_1, s_2 \) with \( \underline{u}_2 \leq s_2 < s_1 \leq \overline{u}_2 \) and \( x \in B_R. \)

We develop a sequence of approximations of solution. The sub- and super- solution will be used as the initial iteration in the Picard type of
monotone iteration process. Namely, with the starting point \((u_1^0, u_2^0) = (u_1, u_2)\) we inductively define a sequence \(\{(u_1^k, u_2^k)\}_{k \in \mathbb{N}}\) such that
\[
\begin{align*}
\Delta u_1^k + \Lambda_1 u_1^k &= g_1(x, u_1^{k-1}, u_2^{k-1}) + \Lambda_1 u_1^{k-1} \quad \text{for } x \in B_R, \\
\Delta u_2^k + \Lambda_2 u_2^k &= g_2(x, u_1^{k-1}, u_2^{k-1}) + \Lambda_2 u_2^{k-1} \quad \text{for } x \in B_R, \\
u_1^k(x) &= u_2^k(x) = 1 \quad \text{for } x \in \partial B_R.
\end{align*}
\]
Clearly, the sequence \(\{(u_1^k, u_2^k)\}_{k \in \mathbb{N}}\) is well defined. Next, assuming that
\[
u_1^{k-1} \leq u_1^k \text{ and } u_2^{k-1} \leq u_2^k \text{ on } \overline{B}_R,
\]
we prove that
\[
u_1^k \leq u_1^{k+1} \text{ and } u_2^k \leq u_2^{k+1} \text{ on } \overline{B}_R.
\]
The constants \(\Lambda_1\) and \(\Lambda_2\) was chosen so that
\[
\begin{align*}
(\Delta + \Lambda_1) (u_1^{k+1}(x) - u_1^k(x)) &\leq 0 \quad \text{in } B_R, \\
(\Delta + \Lambda_2) (u_2^{k+1}(x) - u_2^k(x)) &\leq 0 \quad \text{in } B_R,
\end{align*}
\]
if
\[
u_1^{k-1} \leq u_1^k \text{ and } u_2^{k-1} \leq u_2^k \text{ on } B_R,
\]
for \(k = 1, 2, \ldots\), which is true for \(k = 1\) and thus for every larger \(k\) by (20) and the maximum principle.

Consequently, by induction we get a monotone increasing sequence \(\{(u_1^k, u_2^k)\}_{k \in \mathbb{N}}\) of iterates
\[
\begin{align*}
u_1^1 &\leq u_1^1 \leq u_1^2 \leq \cdots \leq u_1^{k-1} \leq u_1^k \leq \cdots \leq \overline{u}_1 \text{ on } \overline{B}_R, \\
u_2^1 &\leq u_2^1 \leq u_2^2 \leq \cdots \leq u_2^{k-1} \leq u_2^k \leq \cdots \leq \overline{u}_2 \text{ on } \overline{B}_R.
\end{align*}
\]
To sum up, we have constructed a monotonic and bounded sequence \(\{(u_1^k, u_2^k)\}_{k \in \mathbb{N}}\) that converge
\[
\lim_{k \to \infty} \left( u_1^k(x), u_2^k(x) \right) = (u_1(x), u_2(x)), \text{ for all } x \in \overline{B}_R.
\]
Clearly, the limit function \((u_1(x), u_2(x))\) exists as a continuous function on \(\overline{B}_R\). Via standard, see [20] p. 26], bootstrap arguments
\[
\left( u_1^k, u_2^k \right) \overset{k \to \infty}{\to} (u_1, u_2) \text{ in } [C^2(B_R) \cap C(\overline{B}_R)]^2,
\]
and \((u_1, u_2)\) is a solution of problem (13) satisfying
\[
u_1(x) \leq u_1(x) \leq \overline{u}_1(x) \text{ and } u_2(x) \leq u_2(x) \leq \overline{u}_2(x) \text{ for all } x \in \overline{B}_R.
\]
Then
\[
(z_1(x), z_2(x)) = (-2\sigma_1 \ln u_1(x), -2\sigma_2 \ln u_2(x)) \in [C^2(B_R) \cap C(\overline{B}_R)]^2,
\]
is the positive solution of (12) with quadratic growth.

**Step 3** Let us point that using a maximum principle coupled with the scalar case in (11), for any classical solution \((u_1, u_2)\) of the system (14)-(15) there exist constants \(c_1, c_2 \in (0, 1)\) such that
\[
1 \geq u_1(x) \geq c_1 \text{ and } 1 \geq u_2(x) \geq c_2, \text{ for all } x \in B_R.
\]
This coupled with (16) Section 3, [4], [6] guarantee the uniqueness of a classical solution of the system (12)-(15). This completes our proof.
3.1 Verification

In this subsection we show that the control of \([16]\) is indeed optimal. This is formalized in the following Theorem.

**Theorem 3.2.** The production rate(s) \(p_i^*(t, \epsilon(t)), i = 1, 2, \ldots N\) defined in \([16]\) is optimal. That is for every production rate(s) \((p_1(t, \epsilon(t)), \ldots, p_N(t, \epsilon(t)))\)

\[
J(p_1, \ldots, p_N) \geq J(p_1^*, \ldots, p_N^*).
\]

**Proof** Let us denote by \(y^*(t, \epsilon(t))\) the vector of inventory levels associated with \(p_i^*(t, \epsilon(t)), i = 1, 2, \ldots N\). Recall that

\[
\tau^* = \inf_{t \geq 0} \{|y^*(t, \epsilon(t))| \geq R\},
\]

and

\[
\tau = \inf_{t \geq 0} \{|y(t, \epsilon(t))| \geq R\}.
\]

We proceed in two steps: step 1) shows that the stochastic process \(Z^p(t)\) defined in \([3]\) is supermartingale for all

\[
p(t, \epsilon(t)) = (p_1(t, \epsilon(t)), \ldots, p_N(t, \epsilon(t))),
\]

on \(0 \leq t \leq \tau\), and martingale for

\[
p^*(t, \epsilon(t)) = (p_1^*(t, \epsilon(t)), \ldots, p_N^*(t, \epsilon(t))),
\]

on \(0 \leq t \leq \tau^*.\) In step 2) we establish optimality of \(p_i^*(t, \epsilon(t)), i = 1, 2, \ldots N\) defined in \([16]\).

**Step 1** Itô Lemma for Markov modulated diffusion (see \([23]\) for more on this) yields

\[
dZ^p(t) = e^{-\alpha(t)z} \left[ -\frac{\sigma_t^2}{2} \Delta z_t (y(s, \epsilon(s))) + |(y(s, \epsilon(s)))|^2 - p(s, \epsilon(s)) \nabla z_t (y(s, \epsilon(s))) \right]
\]

\[
+ |(p(s, \epsilon(s)))|^2 \left( \alpha(s) + \sum_{i=1}^{\infty} a_{i} \mathbb{1}_{\{i = \epsilon(s)\}} \right) z_t (y(s, \epsilon(s)))
\]

\[
+ \sum_{i=1}^{\infty} a_{i} \mathbb{1}_{\{i \neq \epsilon(s)\}} z_t (y(s, i)) ds - e^{-\alpha(s)z} \sigma_t \mathbb{1}_{\{i = \epsilon(s)\}} p(s, \epsilon(s)) \nabla z_t (y(s, \epsilon(s))) dw(s).
\]

Then, the claim yields in light of HJB equation \([9]\) and \([10]\).

**Step 2** In a second step let us establish the optimality of \((p_1^*, \ldots, p_N^*)\).

The martingale/supermartingale principle yields

\[
E e^{-\alpha(\tau^*)} z_\tau (y^*(\tau^*, \epsilon(\tau^*))) + \int_0^{\tau^*} e^{-\alpha(u)} \left[ |p^*(u, \epsilon(u))|^2 + |y^*(u, \epsilon(u))|^2 \right] du
\]

\[
= z_\tau (y^*(0, \epsilon(0)));
\]

and

\[
E e^{-\alpha(\tau)} z_\tau (y(\tau, \epsilon(\tau))) + \int_0^{\tau} e^{-\alpha(u)} \left[ |p(u, \epsilon(u))|^2 + |y(u, \epsilon(u))|^2 \right] du
\]
\[ \geq z_{\epsilon(0)}(y(0, \epsilon(0))). \]

Moreover
\[
E e^{-\alpha_{\epsilon(\tau^*)} z_{\epsilon(\tau^*)}} (y^\tau, \epsilon(\tau^*)) = E e^{-\alpha_{\epsilon(\tau)} z_{\epsilon(\tau)}} (y(\tau, \epsilon(\tau))) = z_{\epsilon(\tau)}(R) = z_{\epsilon(\tau)}(R) = 0,
\]
since
\[ z_i(R) = 0, \quad i = 1, 2. \]

This together with \( y^\tau(0, \epsilon(0)) = y(0, \epsilon(0)) \) finishes the proof.

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