Triple $M$-brane configurations and preserved supersymmetries

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Abstract

We investigate all standard triple composite $M$-brane intersections defined on products of Ricci-flat manifolds for preserving supersymmetries in eleven-dimensional $\mathcal{N} = 1$ supergravity. The explicit formulae for computing the numbers of preserved supersymmetries are obtained, which generalize the relations for topologically trivial flat factor spaces presented in the classification by Bergshoeff et al. We obtain certain examples of configurations preserving some fractions of supersymmetries, e.g. containing such factor spaces as $K3$, $C^2_2/Z_2$, a four-dimensional $pp$-wave manifold and the two-dimensional pseudo-Euclidean manifold $\mathbb{R}^{1,1}/Z_2$.

1 Introduction

The developments of \cite{1, 2, 3} have led to a renewed interest in various aspects of supergravity. Classical BPS configurations of intersecting branes play an essential role in studies of non-perturbative superstring and $M$-theories as well as in establishing and proving new supergravity/gauge correspondences. Non-maximally supersymmetric solutions are important in many applications of superstring dualities. Various intersections of $M$-branes in 11-dimensional supergravity \cite{4} provide a unified viewpoint, because a large class of solutions describing brane intersections can be obtained by dimensional reduction and duality transformations.

In the basic $M2$- and $M5$-brane solutions, preserving half of the supersymmetries \cite{5, 6}, the worldvolumes have been taken to be flat pseudo-Euclidean spaces $\mathbb{R}^{1,k}$ ($k = 2, 5$) and the transverse spaces have been taken to be flat Euclidean ones $\mathbb{R}^r$ ($r = 8, 5$). Nevertheless, the brane configurations defined on the spacetimes with more complicated geometry involved are of interest \cite{7, 8}.

First examples of $M2$-brane solutions, partially preserving the supersymmetry, with Ricci-flat 8-dimensional transverse spaces and the flat brane worldvolumes $\mathbb{R}^{1,2}$ have been obtained in \cite{9} and \cite{10}. The supersymmetric $M5$-brane solutions with the flat transverse

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spaces $\mathbb{R}^8$ and Ricci-flat 6-dimensional brane worldvolumes have been found in \cite{11}, \cite{12} and \cite{13}. In \cite{19} explicit formulae for fractional numbers of supersymmetries for $M2$- and $M5$-brane solutions have been derived.

In this paper, we study triple orthogonal intersections of composite $M$-branes defined on the manifold of the form

$$M = M_0 \times M_1 \times \ldots \times M_n,$$

where all factor spaces $M_i$ are Ricci-flat manifolds. It should be noted that the study of the flat case of the factor spaces was undertaken in a variety of works \cite{14}-\cite{18}. In \cite{15} the classification of supersymmetric $M$-brane configurations on the product manifold with the factor spaces $M_i = \mathbb{R}^{k_i}$ was presented. According to this work the amount of preserved supersymmetries is given by

$$\mathcal{N} = 2^{-k} \quad \text{with} \quad k = 1, 2, 3, 4, 5. \quad (1.2)$$

However, the relation (1.2) is not well justified if $M$-brane configurations are taken into consideration on the product of Ricci-flat manifolds (1.1). In this case the fractional number of supersymmetries $\mathcal{N}$ depends upon several numbers of chiral parallel (i.e. covariantly constant) spinors on certain factor spaces $M_i$ and brane sign factors $c_s$, which define the orientations of brane worldvolumes.

For clarity, we remind that the metric for orthogonally intersecting $M$-branes is split into several parts: the common worldvolume, the relative transverse space and the totally transverse one. The classification of possible factor spaces contained in worldvolumes and transverse spaces and admitting parallel spinors can be given in terms of the holonomy groups, see \cite{11},\cite{13},\cite{20} and references therein. While for intersections of two $M$-branes, as well as for the case of single $M2$- and $M5$-branes, one can use the results obtained in works \cite{21},\cite{22}, finding non-trivial examples for triple $M$-brane configurations is complicated by increasing dominance of low-dimensional flat manifolds (of dimensions 1, 2 and 3) among factor spaces.

The purpose of this work is to find out the relations for the amounts of preserved supersymmetries for triple $M$-brane configurations defined on the product of Ricci-flat manifolds. The cases of one and two $M$-branes were considered in \cite{19} and \cite{20}, respectively. Here we deal with non-localized composite brane solutions with a vanishing contribution from the Chern-Simons term. However this may be a starting point for future considering localized brane solutions, which are of interest in view of possible applications by using the gravity/gauge correspondence \cite{26}.

The structure of the paper is as follows. In section 2 we present a set up, main definitions and notations. Here we use Propositions 1 and 2 from the previous work \cite{20} which reduce the solutions to generalized Killing equations to a search of parallel (i.e. covariantly constant) spinors on the product manifold (1.1) obeying three algebraic equations. These equations depend upon a brane configuration and brane sign factors. In section 3 we find relations for fractional numbers of preserved supersymmetries for triple $M$-brane solutions: $M5 \cap M5 \cap M5$, $M2 \cap M2 \cap M5$ and $M2 \cap M5 \cap M5$. For a completeness we start here by considering three electric branes $M2 \cap M2 \cap M2$ which was performed earlier in \cite{20}.
2 Generalized Killing spinor equations

The bosonic action in 11-dimensional supergravity is given by

$$S_{\text{act}} = \int d^{11}z \sqrt{|g|} \left( R[g] - \frac{1}{2(4!)} F^2 \right) - \frac{1}{6} \int A \wedge F \wedge F, \quad (2.1)$$

where

$$F = dA = \frac{1}{4!} F_{NPQR} dz^N \wedge dz^P \wedge dz^Q \wedge dz^R \quad (2.2)$$

is the 4-form field strength of the 3-form potential $A$.

The solutions to the equations of motion for the model (2.1) are defined on the (oriented) warped product spin manifold of the form

$$M = M_0 \times M_1 \times \ldots \times M_n \quad (2.3)$$

with the metric

$$g = e^{2\gamma(x)} g^0 + \sum_{i=1}^n e^{2\phi_i(x)} g^i, \quad (2.4)$$

where $g^0 = g^0_{\mu\nu}(x) dx^\mu \otimes dx^\nu$ is a metric on the (oriented) spin manifold $M_0$ and $g^i = g^i_{m_i n_i}(y_i) dy_{m_i} \otimes dy_{n_i}$ is a metric on the (oriented) spin manifold $M_i$, $i = 1, \ldots, n$. We denote $d_\nu = \dim M_\nu$, $\nu = 0, \ldots, n$; $\sum_{\nu=0}^n d_\nu = 11$.

The manifold (2.3) allows a frame such that the metric $g = g_{MN} dx^M \otimes dx^N$ ($M, N = 0, \ldots, 10$) can be represented in the following form

$$g_{MN} = \eta_{AB} e^A_M e^B_N, \quad \text{with} \quad \eta_{AB} = \eta^{AB} = \eta_A \delta_{AB}, \quad (2.5)$$

where $e^A = e^A_M dx^M$ is the diagonalizing 11-bein, $\eta_A = \pm 1$; $A, B = 0, \ldots, 10$. (Here $\eta_A = -1$ only for one value of the index $A$.)

The required backgrounds must admit 32-component Majorana-spinors $\epsilon$ such that the supersymmetry variation of the gravitino field $\delta \psi_M$ vanishes, i.e.

$$(D_M + B_M) \epsilon = 0. \quad (2.6)$$

Here

$$D_M = \partial_M + \frac{1}{4} \omega_{ABM} \hat{\Gamma}^A \hat{\Gamma}^B \quad (2.7)$$

is the spinorial covariant derivative, $\hat{\Gamma}^A$ are $32 \times 32$ gamma matrices in an orthonormal frame satisfying the Clifford algebra relations

$$\hat{\Gamma}_A \hat{\Gamma}_B + \hat{\Gamma}_B \hat{\Gamma}_A = 2\eta_{AB} 1_{32}, \quad (2.8)$$

$\omega^A_{BM}$ is the spin connection and $\omega_{ABM} = \eta_{AC} \omega^C_{BM} = -\omega_{BAM}$. (See also the approach of Alekseevsky et al. [23].)

In (2.6) $B_M$ is a matrix-valued covector field induced by the 4-form field strength $F$

$$B_M = \frac{1}{288} (\Gamma_M \Gamma^N \Gamma^P \Gamma^Q \Gamma^R - 12 \delta_M^N \Gamma^P \Gamma^Q \Gamma^R) F_{NPQR}, \quad (2.9)$$
where $\Gamma_M$ are world gamma matrices obeying

$$\Gamma_M \Gamma_N + \Gamma_M \Gamma_N = 2g_{MN}1_{32}, \quad \text{with} \quad \Gamma_M = e^A_M \hat{\Gamma}_A.$$  \hspace{2cm} (2.10)

It should be noted that for any manifold $M_l$ with the metric $g^l$ one considers $k_l \times k_l$ $\Gamma$-matrices with $k_l = 2^{[d/2]}$ obeying

$$\hat{\Gamma}^{a_l}_{(l)} \hat{\Gamma}^{b_l}_{(l)} + \hat{\Gamma}^{b_l}_{(l)} \hat{\Gamma}^{a_l}_{(l)} = 2\eta(\ell)_{a_l b_l}1_{k_l}.$$  \hspace{2cm} (2.11)

Here we use alternative double-number notations for indices: $a_j = 1, \ldots, (d_j)_j$, where $d_j$ is the dimension of the manifold $M_j$, $j = 0, \ldots, n$.

The local frame co-vectors are chosen in the following form

$$(e^A_M) = \text{diag} \left( e^a e^{(0)a}_\mu, e^\phi e^{(1)a_1 m_1}, \ldots, e^\phi e^{(n)a_n m_n} \right),$$ \hspace{2cm} (2.12)

where

$$g^{\mu_\nu}_0 = \eta^{(0)}_{ab} e^{(0)a}_\mu e^{(0)b}_\nu, \quad g^{i_j m_j}_m = \eta^{(i)}_{a_i b_i} e^{(i)a_i m_i} e^{(i)b_i m_i},$$

$i = 1, \ldots, n$, and the signature matrix $(\eta_{AB})$ in (2.5) can be written down in the components

$$(\eta_{AB}) = \text{diag} \left( \eta^{(0)}_{ab}, \eta^{(1)}_{a_1 b_1}, \ldots, \eta^{(n)}_{a_n b_n} \right).$$ \hspace{2cm} (2.14)

In (2.13) $\eta^{(l)}_{a_l b_l}$ is a diagonal signature matrix for the metric $g^l$, equipped by a set of (local) frame vectors with components $e^{(l)a_i m_i}$, $l = 0, \ldots, n$. We put

$$\text{det} \left( e^{(l)a_i m_i} \right) > 0,$$ \hspace{2cm} (2.15)

$l = 0, \ldots, n$, i.e. for any $l$ the oriented set of $d_l$-beins $e^{(l)a_i}$ has the orientation compatible with the orientation of the manifold $M_l$.

Henceforth the following notation for the volume $d_i$-form on the manifolds $(M_i, g^i)$ is used

$$\tau_i \equiv \sqrt{|g^i(y_i)|} dy^1_i \wedge \ldots \wedge dy^d_i,$$ \hspace{2cm} (2.16)

for $i = 1, \ldots, n$.

In this paper we continue our investigations of composite $M$-brane solutions [16] (with standard intersection rules) defined on the product of $(n + 1)$ Ricci-flat manifolds $M_l$ and governed by several harmonic functions $H_s$ on $(M_0, g^0)$.

It was shown in [20] that the solutions to eqs. (2.6), corresponding to composite $M$-brane backgrounds, admit a representation in the following form

$$\varepsilon = \left( \prod_{s \in S_s} H_s \right)^{-1/6} \left( \prod_{s \in S_m} H_s \right)^{-1/12} \eta,$$ \hspace{2cm} (2.17)

where $\eta$ is a covariantly constant spinor on the spin manifold (2.3) with the metric

$$\tilde{g} = g^0 + \sum_{i=1}^n g^i,$$ \hspace{2cm} (2.18)
satisfying projection conditions
\[ \hat{\Gamma}_s \eta = c_s \eta, \quad (2.20) \]
s \in S, where the brane set S contains two subsets S_e and S_m denoting electric and magnetic brane sets, respectively. Under this convention in (2.17) harmonic functions \( H_s \) with \( s \in S_e \) correspond to electric M2-branes and \( H_s \) with \( s \in S_m \) correspond to magnetic M5-branes.

We remind that in (2.20) \( c_s = \pm 1 \) are brane sign factors and \( \hat{\Gamma}_s \) are brane operators, \( s \in S \), which are defined as follows: for the electric case
\[ \hat{\Gamma}_s = \hat{\Gamma}^{A_1} \hat{\Gamma}^{A_2} \hat{\Gamma}^{A_3}, \quad s \in S_e, \quad (2.21) \]
where three indices describe the position of the \( s \)-th M2-brane worldvolume and for the magnetic case
\[ \hat{\Gamma}_s = \hat{\Gamma}^{B_1} \hat{\Gamma}^{B_2} \hat{\Gamma}^{B_3} \hat{\Gamma}^{B_4} \hat{\Gamma}^{B_5}, \quad s \in S_m, \quad (2.22) \]
where five indices \( B_1, B_2, B_3, B_4, B_5 \) describe the position not occupied by the \( s \)-th M5-brane worldvolume. These operators are idempotent, i.e., \( (\hat{\Gamma}_s)^2 = 1 \) for all \( s \).

Eqs. (2.19) are equivalent to the following set of equations \[ \bar{D}_m \eta = 0, \quad (2.23) \]
l = 0, ..., \( n \). Here
\[ \bar{D}_m = \partial_m + 1/4 \omega^{(l)}_{ablm} \hat{\Gamma}^{a_l} \hat{\Gamma}^{b_l}, \quad (2.24) \]
where \( \omega^{(l)}_{ablm} = \eta^{(l)}_{ac} \omega^{(l)}_{bcm} \) and \( \omega^{(l)}_{ablm} \) are components of the spin connection corresponding to the metric \( g^l \) equipped with diagonalizing \( d_l \)-bein vectors \( e^{(l)a_l}, l = 0, ..., n \).

In what follows operators \( (2.24) \) will generate the covariant spinorial derivatives corresponding the manifolds \( M_l \)
\[ D^{(l)}_{m_l} = \partial_{m_l} + 1/4 \omega^{(l)}_{ablm} \hat{\Gamma}^{a_l} \hat{\Gamma}^{b_l}, \quad (2.25) \]
l = 0, ..., \( n \).

In the next section we calculate the fractional numbers \( N \) of unbroken supersymmetries (SUSY) for all triple \( M \)-brane configurations. Here we consider generic solutions to Eqs. (2.6) in the form (2.17). For any configuration we have
\[ N = N/32, \quad (2.26) \]
where \( N \) is the dimension of the linear space of solutions to algebraic equations (2.19) and (2.20). (For special non-generic choices of harmonic functions \( H_s \) the real fractional numbers \( N \) of unbroken SUSY may be higher than those given by (2.26)).

Remark. In this paper as in [20] we put for simplicity \( \varepsilon(z) \in \mathbb{C}^{32} \). The imposing of the Majorana condition will give the same number \( N \) for the dimension of the real linear space of parallel Majorana spinors obeying (2.19) and (2.20).

\[^{1}\text{In [20] } \bar{D}_{m_l} \text{ was denoted by } \bar{D}^{(l)}_{m_l}.\]
3 Triple $M$-brane configurations

In this section we deduce relations for fractional numbers of supersymmetries preserved by triple $M$-brane configurations defined on the product manifold (3.1).

3.1 $M2 \cap M2 \cap M2$

Let us consider the configuration of three electric 2-branes intersecting over a point. The configuration which is defined on the manifold

$$M_0 \times M_1 \times M_2 \times M_3 \times M_4,$$

with $d_0 = 4$, $d_1 = d_2 = d_3 = 2$ and $d_4 = 1$, can be presented as in Fig.1.

$$g = \begin{cases} - - - - \times \times - - - - \times \times - - - - \times : H_1 \\ - - - - - \times - - - - \times - - - - \times \times : H_2 \\ - - - - - - - - - - \times - - - - - - - - : H_3. \end{cases}$$

Figure 1: $M2 \cap M2 \cap M2$-intersection over a point. $M_0$ is the totally transverse space, $M_1, M_2, M_3$ are the relative transverse spaces and $M_4$ is the common worldvolume.

For each figure we denote by $\times$ a coordinate corresponding to a worldvolume direction and every direction transverse to the brane by $-$. The solution is given by

$$g = H_1^{1/3} H_2^{1/3} H_3^{1/3} \{ g^0 + H_1^{-1} g^1 + H_2^{-1} g^2 + H_3^{-1} g^3 + H_1^{-1} H_2^{-1} H_3^{-1} g^4 \},$$

$$F = c_1 dH_1^{-1} \wedge \tau_1 \wedge \tau_4 + c_2 dH_2^{-1} \wedge \tau_2 \wedge \tau_4 + c_3 dH_3^{-1} \wedge \tau_3 \wedge \tau_4,$$

where $c_1^2 = c_2^2 = c_3^2 = 1$; $H_1$, $H_2$, $H_3$ are harmonic functions on $(M_0, g^0)$, metrics $g^i$, $i = 0, 1, 2, 3, 4$, are Ricci-flat (the last four metrics are flat). The metrics $g^i$, $i = 0, 1, 2, 3$, have Euclidean signatures and the metric $g^4$ has the signature $(-)$. Here we put $M_4 = \mathbb{R}$, $g^4 = -dt \otimes dt$ ($\tau_4 = dt$). The brane sets are $I_1 = \{1, 4\}$, $I_2 = \{2, 4\}$ and $I_3 = \{3, 4\}$.

Using the rules of decomposition for $\Gamma$-matrices on product spaces from [25] the set of $\Gamma$-matrices can be represented in the following form

$$\hat{\Gamma}^{a_0} = \hat{\Gamma}^{(0)} \otimes \hat{\Gamma}^{a_1} \otimes \hat{\Gamma}^{(1)} \otimes \hat{\Gamma}^{a_2} \otimes \hat{\Gamma}^{(2)} \otimes \hat{\Gamma}^{a_3} \otimes \hat{\Gamma}^{(3)} \otimes 1,$$

where $\hat{\Gamma}^{a_0}$ are $4 \times 4$ gamma matrices corresponding to $M_0$:

$$\hat{\Gamma}^{a_0} \hat{\Gamma}^{b_0} + \hat{\Gamma}^{b_0} \hat{\Gamma}^{a_0} = 2\delta_{a_0 b_0} 1_4,$$
\[ \hat{\Gamma}^{a_i}_{(i)} \] are \( 2 \times 2 \) gamma matrices corresponding to \( M_i, i = 1, 2, 3 \):

\[
\hat{\Gamma}^{a_i}_{(i)} \hat{\Gamma}^{b_i}_{(i)} + \hat{\Gamma}^{b_i}_{(i)} \hat{\Gamma}^{a_i}_{(i)} = 2\delta_{a_ib_i}\mathbf{1}_2. \tag{3.6}
\]

and

\[
\hat{\Gamma}(0) = \hat{\Gamma}^{10}_{(0)} \ldots \hat{\Gamma}^{40}_{(0)}, \quad \hat{\Gamma}(i) = \hat{\Gamma}^{1i}_{(i)} \hat{\Gamma}^{2i}_{(i)}, \tag{3.7}
\]

obey

\[
(\hat{\Gamma}(0))^2 = \mathbf{1}_4, \quad (\hat{\Gamma}(i))^2 = -\mathbf{1}_2. \tag{3.8}
\]

The spinor monomial can be written as follows

\[
\eta = \eta_0(x) \otimes \eta_1(y_1) \otimes \eta_2(y_2) \otimes \eta_3(y_3) \otimes \eta_4(y_4), \tag{3.9}
\]

where \( \eta_0 = \eta_0(x) \) is a 4-component spinor on \( M_0 \), \( \eta_i = \eta_i(y_i) \) is a 2-component spinor on \( M_i, i = 1, 2, 3 \), and \( \eta_4 = \eta_4(y_4) \) is a 1-component spinor on \( M_4 \).

Due to (3.4) and (3.8) the covariant derivative acts on the spinor \( \eta \) as follows

\[
\begin{align*}
\hat{D}_{m_0} \eta &= (D^{(0)}_{m_0} \eta_0) \otimes \eta_1 \otimes \eta_2 \otimes \eta_3 \otimes \eta_4, \\
\hat{D}_{m_1} \eta &= \eta_0 \otimes (D^{(1)}_{m_1} \eta_1) \otimes \eta_2 \otimes \eta_3 \otimes \eta_4, \\
\hat{D}_{m_2} \eta &= \eta_0 \otimes \eta_1 \otimes (D^{(2)}_{m_2} \eta_2) \otimes \eta_3 \otimes \eta_4, \\
\hat{D}_{m_3} \eta &= \eta_0 \otimes \eta_1 \otimes \eta_2 \otimes (D^{(3)}_{m_3} \eta_3) \otimes \eta_4, \\
\hat{D}_{m_4} \eta &= \eta_0 \otimes \eta_1 \otimes \eta_2 \otimes \eta_3 \otimes (D^{(4)}_{m_4} \eta_4),
\end{align*}
\tag{3.10}
\]

where \( D^{(i)}_{m_i} \) is a covariant derivative corresponding to \( M_i, i = 0, 1, 2, 3 \). Here \( D^{(4)}_{m_4} = \partial_{m_4} \).

The operators corresponding to \( M_2 \)-branes are given by

\[
\hat{\Gamma}_{[s]} = \hat{\Gamma}^{11}_{(0)} \hat{\Gamma}^{22}_{(0)} \hat{\Gamma}^{14}_{(0)} = -\hat{\Gamma}(0) \otimes 1_2 \otimes 1_2 \otimes \hat{\Gamma}(3) \otimes 1 \tag{3.11}
\]

for \( s = I_1 \),

\[
\hat{\Gamma}_{[s]} = \hat{\Gamma}^{12}_{(0)} \hat{\Gamma}^{22}_{(0)} \hat{\Gamma}^{14}_{(0)} = -\hat{\Gamma}(0) \otimes \hat{\Gamma}(1) \otimes 1_2 \otimes 1 \tag{3.12}
\]

for \( s = I_2 \) and

\[
\hat{\Gamma}_{[s]} = \hat{\Gamma}^{13}_{(0)} \hat{\Gamma}^{23}_{(0)} \hat{\Gamma}^{14}_{(0)} = -\hat{\Gamma}(0) \otimes \hat{\Gamma}(1) \otimes 1_2 \otimes 1 \tag{3.13}
\]

for \( s = I_3 \).

The supersymmetry constraints (2.20) are satisfied if

\[
\hat{\Gamma}(0) \eta_0 = c_0(0) \eta_0, \quad c_0^2(0) = 1, \tag{3.14}
\]

\[
\hat{\Gamma}(j) \eta_j = c_j(0) \eta_j, \quad c_j^2(0) = -1, \tag{3.15}
\]

\( j = 1, 2, 3 \), and

\[
-c_0(0)c_1(2)c_3(3) = c_1, \quad -c_0(0)c_1(1)c_3(3) = c_2, \quad -c_0(0)c_1(1)c_2 = c_3. \tag{3.16}
\]

Then one obtains the following solution to SUSY equations (2.6) corresponding to the field configuration from (3.2), (3.3)

\[
\varepsilon = H_1^{-1/6} H_2^{-1/6} H_3^{-1/6} \eta_0(x) \otimes \eta_1(y_1) \otimes \eta_2(y_2) \otimes \eta_3(y_3) \otimes \eta_4. \tag{3.17}
\]

Here \( \eta_i, i = 0, 1, 2, 3 \), are chiral parallel spinors defined on \( M_i \), respectively \( D^{(i)}_{m_i} \eta_i = 0 \), obeying (3.14), (3.15) and (3.16); \( \eta_4 \) is constant.
Eqs. (3.16) have the following solutions
\[ c_{(0)} = c_1 c_2 c_3, \quad c_{(j)} = \pm i c_j, \quad (3.18) \]
\[ j = 1, 2, 3. \]

Thus, the number of linear independent solutions given by (3.17)-(3.18) reads
\[ N = 32 \mathcal{N} = n_0(c_1 c_2 c_3) \sum_{c=\pm 1} n_1(i c c_1) n_2(i c c_2) n_3(i c c_3), \quad (3.19) \]

where \( n_j(c_{(j)}) \) is the number of chiral parallel spinors on \( M_j, \ j = 0, 1, 2, 3 \); see (3.14) and (3.15).

**Examples.**

Let \( M_1 = M_2 = M_3 = \mathbb{R}^2 \). The manifold \( \mathbb{R}^2 \) has one parallel spinor of chirality \( (+i) \) and one — of chirality \( (-i) \), hence all \( n_j(c) = 1, j = 1, 2, 3, c = \pm 1 \), and one gets from (3.19)
\[ \mathcal{N} = \frac{1}{16} n_0(c_1 c_2 c_3). \quad (3.20) \]

It is worth noting that the chirality of the spinor \( \eta_0 \) on the manifold \( M_0 \) is defined by the product of the brane sign constants \( c_1, c_2, c_3 \).

{a} For \( M_0 = \mathbb{R}^4 \) we have \( n_0(c) = 2 \) and hence \( \mathcal{N} = 1/8 \) for all values of \( c_i, i = 1, 2, 3 \).

{b} Consider the case of the curved transverse space. Let \( M_0 = K3, K3 = CY_2 \), which is a 4-dimensional Ricci-flat Kähler manifold with the holonomy group \( SU(2) = Sp(1) \). The \( K3 \) surface has two parallel spinors of the same chirality. We put \( n_0(1) = 2 \) and \( n_0(-1) = 0 \). Then we get \( \mathcal{N} = 1/8 \) if \( c_1 c_2 c_3 = 1 \) and \( \mathcal{N} = 0 \) if \( c_1 c_2 c_3 = -1 \).

{c} One obtains the same result for the conic space \( M_0 = \mathbb{C}^2/Z_2 \), where \( \mathbb{C}^2 = \mathbb{C}^2 \setminus \{0\} \). It also has parallel spinors of the same chirality: the pair \( (n_0(1), n_0(-1)) \) is either \( (2, 0) \) or \( (0, 2) \) depending on the choice of the spin structure. The completion of \( \mathbb{C}^2/Z_2 \) is the orbifold \( \mathbb{C}^2/Z_2 \).

### 3.2 \( M5 \cap M5 \cap M5 \)

According to the classification of \( M \)-brane configurations which is presented in [15] there are three possible intersections of three magnetic branes depending on the position of the branes in the bulk space.

**(i)**

The first case of the solution describing three intersecting \( M5 \)-branes is defined on the manifold of the form
\[ M_0 \times M_1 \times M_2 \times M_3 \times M_4, \quad (3.21) \]

where \( d_0 = 1, d_1 = d_2 = d_3 = 2, d_4 = 4 \). The configuration is given in Figure 2.

The solution reads
\[ g = H_1^{2/3} H_2^{2/3} H_3^{2/3} \left\{ g^0 + H_1^{-1} g^1 + H_2^{-1} g^2 + H_3^{-1} g^3 + H_1^{-1} H_2^{-1} H_3^{-1} g^4 \right\}, \quad (3.22) \]
Figure 2: $M5 \cap M5 \cap M5$-intersection over a 3-brane. $M_0$ is the totally transverse space, $M_1, M_2, M_3$ are the relative transverse spaces and $M_4$ is the common worldvolume.

\[ F = c_1(*_0dH_1) \wedge \tau_2 \wedge \tau_3 + c_2(*_0dH_2) \wedge \tau_1 \wedge \tau_3 + c_3(*_0dH_3) \wedge \tau_1 \wedge \tau_2, \]  

(3.23)

where $c_1^2 = c_2^2 = c_3^2 = 1$; $H_1, H_2, H_3$ are harmonic functions on $(M_0, g^0)$. The metrics $g^i, i = 0, 1, 2, 3$, have Euclidean signatures and the metric $g^4$ has the signature $(-, +, +, +)$. The branes sets are $I_1 = \{1, 4\}, I_2 = \{2, 4\}, I_3 = \{3, 4\}$. Here and in what follows $*_0$ is the Hodge operator on $(M_0, g^0)$.

Using the rules of decomposition from [25] one can write $\Gamma$-matrices in the following form

\[
(\hat{\Gamma}^A) = \begin{pmatrix}
1 \\ i \hat{\Gamma}^{a_1}_{(1)} \\ \hat{\Gamma}^{a_2}_{(2)} \\ \hat{\Gamma}^{a_3}_{(3)} \\ \hat{\Gamma}^{a_4}_{(4)}
\end{pmatrix}.
\]

(3.24)

Here the operators $\hat{\Gamma}^{(i)}, i = 1, 2, 3, 4$, are given by

\[
\hat{\Gamma}^{(1)} = \hat{\Gamma}^{1}_1 \hat{\Gamma}^{2}_1, \quad \hat{\Gamma}^{(2)} = \hat{\Gamma}^{1}_2 \hat{\Gamma}^{2}_2, \quad \hat{\Gamma}^{(3)} = \hat{\Gamma}^{1}_3 \hat{\Gamma}^{2}_3, \quad \hat{\Gamma}^{(4)} = \hat{\Gamma}^{1}_4 \hat{\Gamma}^{2}_4 \hat{\Gamma}^{3}_4 \hat{\Gamma}^{4}_4
\]

(3.25)

obey

\[
(\hat{\Gamma}^{(i)})^2 = -1, \quad (\hat{\Gamma}^{(4)})^2 = -1
\]

(3.26)

with $i = 1, 2, 3$. $\hat{\Gamma}^{(1)}, \hat{\Gamma}^{(2)}, \hat{\Gamma}^{(3)}$ are $2 \times 2$ $\Gamma$-matrices corresponding to $M_1, M_2, M_3$ respectively, $\hat{\Gamma}^{(4)}$ is a set of gamma matrices corresponding to $M_4$.

The covariant derivatives can be written down as

\[
\bar{D}_{m_1} = \partial_{m_1} + \frac{1}{4} \omega^{(1)}_{a_1 b_1 m_1} \left(1 \otimes \hat{\Gamma}^{a_1}_{(1)} \hat{\Gamma}^{b_1}_{(1)} \otimes \mathbf{1}_2 \otimes \mathbf{1}_2 \otimes \mathbf{1}_4\right),
\]

\[
\bar{D}_{m_2} = \partial_{m_2} + \frac{1}{4} \omega^{(2)}_{a_2 b_2 m_2} \left(1 \otimes \hat{\Gamma}^{a_2}_{(2)} \hat{\Gamma}^{b_2}_{(2)} \otimes \mathbf{1}_2 \otimes \mathbf{1}_2 \otimes \mathbf{1}_4\right),
\]

\[
\bar{D}_{m_3} = \partial_{m_3} + \frac{1}{4} \omega^{(3)}_{a_3 b_3 m_3} \left(1 \otimes \mathbf{1}_2 \otimes \mathbf{1}_2 \otimes \hat{\Gamma}^{a_3}_{(3)} \hat{\Gamma}^{b_3}_{(3)} \otimes \mathbf{1}_4\right),
\]

\[
\bar{D}_{m_4} = \partial_{m_4} + \frac{1}{4} \omega^{(4)}_{a_4 b_4 m_4} \left(1 \otimes \mathbf{1}_2 \otimes \mathbf{1}_2 \otimes \mathbf{1}_2 \otimes \hat{\Gamma}^{a_4}_{(4)} \hat{\Gamma}^{b_4}_{(4)}\right),
\]

(3.27)

where $\omega^{(i)}_{a_i b_i c_i}$ are components of the spin connection corresponding to the manifold $M_i$, $\bar{D}^{(i)}_{m_i}$ is a covariant derivatives corresponding to $M_i, i = 1, 2, 3, 4$, $\bar{D}_{m_0} = \partial_{m_0}$ and $D_{m_0}^{(0)} = \partial_{m_0}$.
Let $\eta$ be represented in the following form

$$\eta = \eta_0(x) \otimes \eta_1(y_1) \otimes \eta_2(y_2) \otimes \eta_3(y_3) \otimes \eta_4(y_4), \quad (3.28)$$

where $\eta_0(x)$ is a 1-component spinor on $M_0$, $\eta_i = \eta_i(y_i)$ is a 2-component spinor on $M_i$, $i = 1, 2, 3$, $\eta_4 = \eta_4(y_4)$ is a 4-component spinor on $M_4$. Then for spinorial covariant derivatives we get relations (3.10).

The number of linear independent solutions given by (3.34), (3.35) and (3.36) reads

$$n = 1,$$  

and hence we get from (3.37)

$$\epsilon = \left( \prod_{s=1}^{3} H_s^\perp \right) \eta_0 \otimes \eta_1(y_1) \otimes \eta_2(y_2) \otimes \eta_3(y_3) \otimes \eta_4(y_4), \quad (3.34)$$

where $\eta_i$, $i = 1, 2, 3, 4$, are parallel spinors defined on $M_i$, respectively $\left( D_m^{(i)} \eta_i = 0 \right)$, obeying (3.32) and (3.33), and $\eta_0$ is a constant one-dimensional spinor on $M_0$.

Eqs. (3.33) have the following solutions

$$c(1) = -ic_1, \quad c(2) = -ic_2, \quad c(3) = -ic_3, \quad c(4) = i, \quad (3.35)$$

or

$$c(1) = ic_1, \quad c(2) = ic_2, \quad c(3) = ic_3, \quad c(4) = -i. \quad (3.36)$$

The number of linear independent solutions given by (3.34), (3.35) and (3.36) reads

$$N = 32N = n_1(-ic_1)n_2(-ic_2)n_3(-ic_3)n_4(i) + n_1(ic_1)n_2(ic_2)n_3(ic_3)n_4(-i), \quad (3.37)$$

where $n_j(c_j)$ is the number of chiral parallel spinors on $M_j$, $j = 1, 2, 3, 4$.

Examples.

Let $M_0 = \mathbb{R}$ and $M_1 = M_2 = M_3 = \mathbb{R}^2$. Then all $n_j(c) = 1$, $j = 1, 2, 3$, with $c = \pm i$, and hence we get from (3.37)

$$N = 32N = n_4(i) + n_4(-i). \quad (3.38)$$
\{a\} If we take the common worldvolume to be Minkowski space \( M_4 = \mathbb{R}^{1,3} \), then we have \( n_4(c) = 2 \) and hence \( \mathcal{N} = 1/8 \) for all values of \( c_i \), \( i = 1, 2, 3 \).

\{b\} Now consider the case of \( M5 \)-branes intersecting by the manifold \( M_4 = (\mathbb{R}^{1,1}_s / Z_2) \times \mathbb{R}^2 \), where \( \mathbb{R}^{1,1}_s = \mathbb{R}^{1,1} \setminus \{0\} \). Since \( \mathbb{R}^{1,1}_s \) has only one parallel spinor (left or right) depending of the choice of the spin structure, one obtains \( n_4(c) = 1 \) for any \( c \) and hence \( \mathcal{N} = 1/16 \).

\{c\} The same result takes place when \( (M_4, g^4) \) is a 4-dimensional Ricci-flat \( pp \)-wave space from \cite{13} with the holonomy group \( H = \mathbb{R}^2 \) (see \cite{24}). In this case \( (n_4(i), n_4(-i)) = (1, 1) \) and \( \mathcal{N} = 1/16 \).

\[(ii)\] The second possible configuration of three \( M5 \)-branes is the pairwise intersection over 3-branes defined on the manifold of the form

\[
M_0 \times M_1 \times M_2 \times M_3 \times M_4,
\]

where \( d_0 = 3 \), \( d_1 = d_2 = d_3 = d_4 = 2 \). In Figure 3 one presents the intersection of three magnetic branes.

\[
g = \begin{cases} 
- - - - - - \times \times \times \times \times \times & : H_1 \\
- - - - \times \times - - \times \times \times \times & : H_2 \\
- - - - \times \times \times \times - - \times \times \times & : H_3.
\end{cases}
\]

Figure 3: The pairwise intersection of three \( M5 \)-branes over 3-branes. \( M_0 \) is the totally transverse space, \( M_1, M_2, M_3 \) are the relative transverse spaces and \( M_4 \) is the common worldvolume.

The metric and the 4-form field strength can be represented in the following form

\[
g = H_1^{2/3} H_2^{2/3} H_3^{2/3} \frac{1}{2} \{ g^0 + H_2^{-1} H_3^{-1} g^1 + H_2^{-1} H_3^{-1} g^2 + H_1^{-1} H_2^{-1} g^3 + H_1^{-1} H_2^{-1} H_3^{-1} g^4 \},
\]

\[
F = c_1 (*_0 dH_1) \wedge \tau_1 + c_2 (*_0 dH_2) \wedge \tau_2 + c_3 (*_0 dH_3) \wedge \tau_3,
\]

where \( c_1^2 = c_2^2 = c_3^2 = 1 \); \( H_1, H_2, H_3 \) are harmonic functions on \( (M_0, g^0) \). The metrics \( g^i \), \( i = 0, 1, 2, 3 \), have Euclidean signatures and the metric \( g^4 \) has the signature \((-+, +)\). The branes sets are \( I_1 = \{2, 3, 4\} \), \( I_2 = \{1, 3, 4\} \), \( I_3 = \{1, 2, 4\} \).

The gamma matrices can be split in the following form

\[
(\hat{\Gamma}^A) = \begin{pmatrix}
(i \hat{\Gamma}_0^{a_0} & \hat{\Gamma}_1 & \hat{\Gamma}_2 & \hat{\Gamma}_3 & \hat{\Gamma}_4, \\
\hat{\Gamma}_1^{a_1} & \hat{\Gamma}_0 & \hat{\Gamma}_2 & \hat{\Gamma}_3 & \hat{\Gamma}_4, \\
\hat{\Gamma}_2^{a_2} & \hat{\Gamma}_1 & \hat{\Gamma}_0 & \hat{\Gamma}_3 & \hat{\Gamma}_4, \\
\hat{\Gamma}_3^{a_3} & \hat{\Gamma}_2 & \hat{\Gamma}_1 & \hat{\Gamma}_0 & \hat{\Gamma}_4, \\
\hat{\Gamma}_4^{a_4} & \hat{\Gamma}_3 & \hat{\Gamma}_2 & \hat{\Gamma}_1 & \hat{\Gamma}_0
\end{pmatrix}
\]

(3.42)
where

\[ \hat{\Gamma}_{(0)} = \hat{\Gamma}_{(0)}^{10} \hat{\Gamma}_{(0)}^{20} \hat{\Gamma}_{(0)}^{30}, \quad \hat{\Gamma}_{(1)} = \hat{\Gamma}_{(1)}^{11} \hat{\Gamma}_{(1)}^{21}, \quad \hat{\Gamma}_{(2)} = \hat{\Gamma}_{(2)}^{12} \hat{\Gamma}_{(2)}^{22}, \quad \hat{\Gamma}_{(3)} = \hat{\Gamma}_{(3)}^{13} \hat{\Gamma}_{(3)}^{23}, \quad \hat{\Gamma}_{(4)} = \hat{\Gamma}_{(4)}^{14} \hat{\Gamma}_{(4)}^{24}. \]  

(3.43)

satisfy

\[ (\hat{\Gamma}_{(0)})^2 = (\hat{\Gamma}_{(1)})^2 = (\hat{\Gamma}_{(2)})^2 = (\hat{\Gamma}_{(3)})^2 = -1_2, \quad (\hat{\Gamma}_{(4)})^2 = 1_2. \]  

(3.44)

Here \( \hat{\Gamma}_{(i)}^{a} \) are 2 \times 2 gamma matrices corresponding to \( M_i, \ i = 0, 1, 2, 3, 4 \). One can write down the gamma matrices corresponding to \( M_0 \) as \( (\hat{\Gamma}_{(0)}^{a}) = (a_1, a_2, a_3) \) and hence

\[ \hat{\Gamma}_{(0)} = i 1_2. \]  

(3.45)

Here we put

\[ \eta = \eta_0(x) \otimes \eta_1(y_1) \otimes \eta_2(y_2) \otimes \eta_3(y_3) \otimes \eta_4(y_4), \]  

(3.46)

where \( \eta_0 = \eta_0(x) \) is a 2-component spinor on \( M_0 \), \( \eta_i = \eta_i(y_i) \) is a 2-component spinor on \( M_i, i = 1, 2, 3, 4 \).

The factorization relations (3.10) are valid for spinorial covariant derivatives where

\[ D_{m_0} = \partial_{m_0} + \frac{1}{4} \omega_{a_0 b_0 m_0}^{(0)} \left( \hat{\Gamma}_{(0)}^{a_0} \hat{\Gamma}_{(0)}^{b_0} \otimes 1_2 \otimes 1_2 \otimes 1_2 \right), \]

\[ D_{m_4} = \partial_{m_4} + \frac{1}{4} \omega_{a_4 b_4 m_4}^{(4)} \left( 1_2 \otimes 1_2 \otimes 1_2 \otimes 1_2 \otimes \hat{\Gamma}_{(4)}^{a_4} \hat{\Gamma}_{(4)}^{b_4} \right), \]  

(3.47)

\( D_{m_i} \) is a covariant derivative corresponding to \( M_i, i = 0, 1, 2, 3, 4 \).

Due to (3.42) and (3.43)-(3.45) the brane operators corresponding to the magnetic branes can be written down as

\[ \hat{\Gamma}_{[s]} = \hat{\Gamma}_{(0)}^{10} \hat{\Gamma}_{(0)}^{20} \hat{\Gamma}_{(0)}^{30} \hat{\Gamma}_{(1)}^{11} \hat{\Gamma}_{(1)}^{21} = 1_2 \otimes 1_2 \otimes 1_2 \otimes 1_2 \otimes \hat{\Gamma}_{(2)} \otimes \hat{\Gamma}_{(3)} \otimes \hat{\Gamma}_{(4)} \]  

(3.48)

for \( s = I_1 \),

\[ \hat{\Gamma}_{[s]} = \hat{\Gamma}_{(0)}^{10} \hat{\Gamma}_{(0)}^{20} \hat{\Gamma}_{(0)}^{30} \hat{\Gamma}_{(2)}^{12} \hat{\Gamma}_{(2)}^{22} = 1_2 \otimes 1_2 \otimes 1_2 \otimes 1_2 \otimes \hat{\Gamma}_{(3)} \otimes \hat{\Gamma}_{(4)} \]  

(3.49)

for \( s = I_2 \),

\[ \hat{\Gamma}_{[s]} = \hat{\Gamma}_{(0)}^{10} \hat{\Gamma}_{(0)}^{20} \hat{\Gamma}_{(0)}^{30} \hat{\Gamma}_{(3)}^{13} \hat{\Gamma}_{(3)}^{23} = 1_2 \otimes 1_2 \otimes 1_2 \otimes 1_2 \otimes \hat{\Gamma}_{(4)} \]  

(3.50)

for \( s = I_3 \).

The supersymmetry restrictions (2.20) are satisfied if

\[ \hat{\Gamma}_{(1)} \eta_1 = c_1 \eta_1, \quad c_1^2 = 1, \]  

(3.51)

\[ \hat{\Gamma}_{(j)} \eta_j = c_{(j)} \eta_j, \quad c_{(j)}^2 = -1, \]  

(3.52)

with \( j = 0, 2, 3, 4 \). Then the conditions for the chirality constants are given by

\[ c_{(1)} c_{(3)} c_{(4)} = c_1, \quad c_{(1)} c_{(2)} c_{(3)} = c_2, \quad c_{(1)} c_{(2)} c_{(4)} = c_3. \]  

(3.53)
We get the following solution to Eqs. (2.6) corresponding to the field configuration from (3.40)-(3.41)

\[ \varepsilon = H_1^{-1/12} H_2^{-1/12} H_3^{-1/12} \eta_0(x) \otimes \eta_1(y_1) \otimes \eta_2(y_2) \otimes \eta_3(y_3) \otimes \eta_4(y_4). \]  

(3.54)

Here \( \eta_i, \ i = 0, 1, 2, 3, 4, \) are chiral parallel spinors defined on \( M_i \) \( \left(D^{(i)}_m \eta_i = 0\right) \), obeying (3.51)-(3.53). Eqs. (3.53) have the following solutions

\[ c^{(1)} = \pm ic_2 c_3, \quad c^{(2)} = \pm ic_1 c_2, \quad c^{(3)} = \pm i, \quad c^{(4)} = -c_1 c_2 c_3. \]  

(3.55)

The number of linear independent solutions is

\[ N = 32 \mathcal{N} = n_0 n_4 (-c_1 c_2 c_3) \sum_{c=\pm 1} n_1 (icc_2 c_3) n_2 (icc_1 c_2) n_3 (ic), \]  

(3.56)

where \( n_j(c_j) \) is the number of chiral parallel spinors on \( M_j, \ j = 1, 2, 3, 4, \) \( n_0 \) is the number of parallel spinors on \( M_0 \).

**Examples.**

Consider the case when all factor spaces are flat: \( M_0 = \mathbb{R}^3, \ M_1 = M_2 = M_3 = \mathbb{R}^2 \).

Then all \( n_j(ic) = 1, \ j = 1, 2, 3, \ c = \pm 1. \) The amount of preserved supersymmetries is given by

\[ \mathcal{N} = \frac{1}{8} n_4 (-c_1 c_2 c_3). \]  

(3.57)

\{a\} Let \( M_4 = \mathbb{R}^{1,1} \), then we obtain \( \mathcal{N} = \frac{1}{8} \) for any choice of brane sign factors.

\{b\} If \( M_4 = \mathbb{R}^{1,1}/\mathbb{Z}_2 \) with \( n_4(1) = 1 \) and \( n_4(-1) = 0 \), we get \( \mathcal{N} = \frac{1}{8} \) for \( c_1 c_2 c_3 = -1 \) and \( \mathcal{N} = 0 \) otherwise.

(iii)

The third intersection of magnetic branes is defined on the manifold

\[ M_0 \times M_1 \times M_2 \times M_3 \times M_4 \times M_5 \times M_6 \times M_7, \]  

(3.58)

where \( d_0 = 2, \ d_1 = d_2 = d_3 = d_4 = d_5 = d_6 = 1, \ d_7 = 3. \) For this configuration we have Fig.4.

\[
 g = \begin{cases} 
 \begin{array}{cccccccc}
 - & - & \times & - & - & \times & \times & \times & \times \\
 - & - & - & \times & - & \times & \times & \times & \times \\
 \end{array} & : H_1 \\
 \begin{array}{cccccccc}
 - & - & \times & - & - & \times & \times & \times & \times \\
 \end{array} & : H_2 \\
 \begin{array}{cccccccc}
 \end{array} & : H_3. \\
 \end{cases}
\]

Figure 4: The pairwise intersection of three \( M5 \)-branes over 3-branes. \( M_0 \) is the totally transverse space, \( M_i, \ i = 1, 2, 3, 4, 5, 6, \) are the relative transverse spaces and \( M_7 \) is the common worldvolume.
The metric of three intersecting $M5$-branes now reads

$$g = H_1^{2/3}H_2^{2/3}H_3^{2/3}\left\{g^0 + H_1^{-1}g^1 + H_2^{-1}g^2 + H_3^{-1}g^3 + H_2^{-1}H_3^{-1}g^4 + H_1^{-1}H_3^{-1}g^5 + H_1^{-1}H_2^{-1}H_3^{-1}g^6 + H_1^{-1}H_2^{-1}H_3^{-1}g^7\right\}. \quad (3.59)$$

The corresponding field strength is

$$F = c_1(*_0dH_1) \wedge \tau_2 \wedge \tau_3 \wedge \tau_4 + c_2(*_0dH_2) \wedge \tau_1 \wedge \tau_3 \wedge \tau_5 + c_3(*_0dH_3) \wedge \tau_1 \wedge \tau_2 \wedge \tau_6, \quad (3.60)$$

where $c_1^2 = c_2^2 = c_3^2 = 1$; $H_1, H_2, H_3$ are harmonic functions on $(M_0, g^0)$. The metrics $g^i, i = 0, 1, 2, 3, 4, 5, 6$, have Euclidean signatures and the metric $g^7$ has the signature $(-, +, +)$. The branes sets are $I_1 = \{1, 5, 6, 7\}, I_2 = \{2, 4, 6, 7\}, I_3 = \{3, 4, 5, 7\}$.

Under the decomposition rules the set of gamma matrices can be presented in the form

$$\hat{\Gamma}^A = \begin{pmatrix}
\hat{\Gamma}^{(0)}_0 & \hat{\Gamma}^{(0)}_1 & \hat{\Gamma}^{(0)}_2 & \hat{\Gamma}^{(0)}_3 & \hat{\Gamma}^{(0)}_4 & \hat{\Gamma}^{(0)}_5 & \hat{\Gamma}^{(0)}_6 & \hat{\Gamma}^{(0)}_7 \\
\hat{\Gamma}^{(7)}_0 & \hat{\Gamma}^{(7)}_1 & \hat{\Gamma}^{(7)}_2 & \hat{\Gamma}^{(7)}_3 & \hat{\Gamma}^{(7)}_4 & \hat{\Gamma}^{(7)}_5 & \hat{\Gamma}^{(7)}_6 & \hat{\Gamma}^{(7)}_7
\end{pmatrix},$$

Here the operators

$$\hat{\Gamma}^{(0)}_0 = \hat{\Gamma}^{(7)}_0 = \hat{\Gamma}^{(0)}_7 = \hat{\Gamma}^{(7)}_7, \quad \hat{\Gamma}^{(7)}_1 = \hat{\Gamma}^{(0)}_1 = \hat{\Gamma}^{(7)}_2 = \hat{\Gamma}^{(0)}_2 = \hat{\Gamma}^{(7)}_3 = \hat{\Gamma}^{(0)}_3$$

obey

$$(\hat{\Gamma}^{(0)}_0)^2 = -1_2, \quad (\hat{\Gamma}^{(7)}_0)^2 = 1_2. \quad (3.62)$$

The gamma matrices corresponding to $M_0$ and $M_7$ manifolds can be written down in the form

$$\left(\hat{\Gamma}^{(0)}_0, \hat{\Gamma}^{(0)}_7\right) = (\sigma_1, \sigma_2), \hat{\Gamma}^{(7)}_0 = i\sigma_3, \left(\hat{\Gamma}^{(7)}_0\right) = (i\sigma_1, \sigma_2, \sigma_3), \hat{\Gamma}^{(7)}_7 = -1_2,$$

respectively.

We put the following relation for the 32-component spinor

$$\eta = \eta_0(x) \otimes \eta_1(y_1) \otimes \eta_2(y_2) \otimes \eta_3(y_3) \otimes \eta_4(y_4) \otimes \eta_5(y_5) \otimes \eta_6(y_6) \otimes \eta_7(y_7) \otimes \chi, \quad (3.64)$$

where $\eta_i = \eta_i(y_i)$ is a 1-component spinor on $M_i, i = 1, 2, 3, 4, 5, 6$, $\eta_0 = \eta_0(x)$ is a 2-component spinor on $M_0$, $\eta_7 = \eta_7(y_7)$ is a 2-component spinor on $M_7$ and

$$\chi = \sum_{r=1}^{8} \chi_1^r \otimes \chi_2^r \otimes \chi_3^r, \quad (3.65)$$

belongs to $V = \mathbb{C}^2 \otimes \mathbb{C}^2 \otimes \mathbb{C}^2$; $\chi_1^r, \chi_2^r, \chi_3^r$ are “auxiliary” 2-dimensional spinors.

We remind that auxiliary spinors appear when the dimension of the spinorial space $2^{[D/2]}$ corresponding to the product manifold $M$ is not equal to the product of dimensions of spinorial spaces $2^{[d_i/2]}$ corresponding to factor spaces $M_i$ (or, equivalently, when the integer part $[D/2]$ is not equal to the sum of integer parts $[d_i/2]$). The simplest example of
the product manifold with two factor spaces of odd dimensions $d_1$ and $d_2$ was considered in [25]. In this case the auxiliary spinor is two-dimensional one.

Here the covariant derivatives act on $\eta$ as

$$
\bar{D}_{m_0} \eta = \left( D_{m_0}^{(0)} \eta_0 \right) \otimes \eta_1 \otimes \eta_2 \otimes \eta_3 \otimes \eta_4 \otimes \eta_5 \otimes \eta_6 \otimes \eta_7 \otimes \chi,
$$

$$
\bar{D}_{m_7} \eta = \eta_0 \otimes \eta_1 \otimes \eta_2 \otimes \eta_3 \otimes \eta_4 \otimes \eta_5 \otimes \eta_6 \otimes \left( D_{m_7}^{(7)} \eta_7 \right) \otimes \chi,
$$

(3.66)

where

$$
\bar{D}_{m_0} = \partial_{m_0} + \frac{1}{4} w_{a_0b_0m_0}^{(0)} \left( \hat{\Gamma}_{a_0}^{b_0} \hat{\Gamma}_{b_0} \otimes 1 \otimes 1 \otimes 1 \otimes 1 \otimes 1 \otimes 1 \otimes 1 \otimes \chi \right),
$$

$$
\bar{D}_{m_7} = \partial_{m_7} + \frac{1}{4} w_{a_7b_7m_7}^{(7)} \left( 1 \otimes 1 \otimes 1 \otimes 1 \otimes 1 \otimes 1 \otimes 1 \otimes \hat{\Gamma}_{a_7}^{b_7} \hat{\Gamma}_{b_7} \otimes 1 \otimes 1 \otimes 1 \otimes \chi \right),
$$

(3.67)

$D_{m_i} = D_{m_i}^{(i)} = \partial_{m_i}$, $i = 1, 2, 3, 4, 5, 6$. $D_{m_i}^{(i)}$ is a covariant derivative corresponding to $M_i$, $i = 0, 7$. Using (3.61) one can write down the operators for $M5$-branes as

$$
\hat{\Gamma}_{[s]} = \hat{\Gamma}_{10} \hat{\Gamma}_{20} \hat{\Gamma}_{12} \hat{\Gamma}_{13} \hat{\Gamma}_{14} = -1_2 \otimes 1 \otimes 1 \otimes 1 \otimes 1 \otimes 1 \otimes 1 \otimes \hat{\Gamma}_{(7)} \otimes B_1
$$

(3.68)

for $s = I_1$, 

$$
\hat{\Gamma}_{[s]} = \hat{\Gamma}_{10} \hat{\Gamma}_{20} \hat{\Gamma}_{11} \hat{\Gamma}_{13} \hat{\Gamma}_{15} = -1_2 \otimes 1 \otimes 1 \otimes 1 \otimes 1 \otimes 1 \otimes 1 \otimes \hat{\Gamma}_{(7)} \otimes B_2
$$

(3.69)

for $s = I_2$, 

$$
\hat{\Gamma}_{[s]} = \hat{\Gamma}_{10} \hat{\Gamma}_{20} \hat{\Gamma}_{12} \hat{\Gamma}_{16} = -1_2 \otimes 1 \otimes 1 \otimes 1 \otimes 1 \otimes 1 \otimes 1 \otimes \hat{\Gamma}_{(7)} \otimes B_3
$$

(3.70)

for $s = I_3$.

Here we denote by $B_s$ the following self-adjoint commuting idempotent (i.e. $B_s^2 = 1_2 \otimes 1_2 \otimes 1_2$) operators acting on the 8-dimensional Hilbert space $V = \mathbb{C}^2 \otimes \mathbb{C}^2 \otimes \mathbb{C}^2$

$$
B_1 = -\sigma_1 \otimes \sigma_2 \otimes 1_2, \quad B_2 = \sigma_3 \otimes \sigma_1 \otimes 1_2, \quad B_3 = -1_2 \otimes \sigma_2 \otimes \sigma_1.
$$

(3.71)

Due to the proposition from Appendix A there exists a basis of eigenvectors $\psi_{\varepsilon_1, \varepsilon_2, \varepsilon_3}$ in $V$ with $\varepsilon_1 = \pm 1, \varepsilon_2 = \pm 1, \varepsilon_3 = \pm 1$, obeying

$$
B_s \psi_{\varepsilon_1, \varepsilon_2, \varepsilon_3} = \varepsilon_s \psi_{\varepsilon_1, \varepsilon_2, \varepsilon_3},
$$

(3.72)

$s = 1, 2, 3$.

Let the gamma matrices corresponding to the 3-dimensional manifold $M_7$ be chosen as follows $\hat{\Gamma}_{(7)} = \left( i \sigma_1, \sigma_2, \sigma_3 \right)$, and hence $\hat{\Gamma}_{(7)} = -1_2$.

Then the solutions to SUSY equations (2.6) corresponding to the field configuration from (3.59), (3.60) are generated by the following set of monomial solutions

$$
\varepsilon = H_1^{-1/12} H_2^{-1/12} H_3^{-1/12} \eta_0(x) \otimes \eta_1 \otimes \eta_2 \otimes \eta_3 \otimes \eta_4 \otimes \eta_5 \otimes \eta_6 \otimes \eta_7(y_7) \otimes \psi_{\varepsilon_1, \varepsilon_2, \varepsilon_3},
$$

(3.73)

where $\eta_0(x)$ and $\eta_7(y_7)$ are parallel spinors defined on $M_0$ and $M_7$, respectively, $\eta_i$ are constant 1-dimensional spinors, $i = 1, 2, 3, 4, 5, 6$. Here $\varepsilon_s$ parameters obey the relations

$$
\varepsilon_s = c_s,
$$

(3.74)
relative transverse spaces, $M$-branes and one

$\{ \ldots \}$

where $c_1, c_2, c_3$ are harmonic functions on $(M_0, g^0)$. The metrics $g^i, i = 0, 1, 2, 3, 4, 5$, have Euclidean signatures and we put the metric $g^0 = -dt \otimes dt$. The brane sets are $I_1 = \{1, 5, 6\}, I_2 = \{2, 4, 6\}, I_3 = \{3, 4, 5, 6\}$. 

\[ \sum_{s=1}^{3, \ldots, 3} \text{ where } \}$

Figure 5: $M2 \cap M2 \cap M5$: $M2$-branes intersect over a point, each $M2$-brane intersects $M5$-brane over a string, $M0$ is the totally transverse space, $M_i, i = 1, 2, 3, 4, 5$, are the relative transverse spaces, $M_6$ is the common worldvolume.

The metric and the 4-form field strength corresponding to the intersection of two $M2$-branes and one $M5$-brane can be represented in the following form

\[ g = H_1^{1/3} H_2^{1/3} H_3^{2/3} \left\{ g^0 + H_1^{-1} g^1 + H_2^{-1} g^2 + H_3^{-1} g^3 + H_2^{-1} H_3^{-1} g^4 + H_1^{-1} H_3^{-1} g^5 + H_1^{-1} H_2^{-1} H_3^{-1} g^6 \right\}, \]  

\[ F = c_1 dH_1^{-1} \wedge \tau_1 \wedge \tau_5 \wedge \tau_6 + c_2 dH_2^{-1} \wedge \tau_2 \wedge \tau_4 \wedge \tau_6 + c_3 (\ast_0 dH_3) \wedge \tau_1 \wedge \tau_2, \]
The gamma matrices may be chosen in the following form

\[
\begin{align*}
\hat{\Gamma}^A &= (\hat{\Gamma}_{a0}^{(0)} \otimes 1 \otimes 1 \otimes 1_2 \otimes 1 \otimes 1 \otimes 1 \otimes \sigma_3 \otimes 1_2 \otimes 1_2, \\
&\quad 1_2 \otimes 1 \otimes 1 \otimes 1_2 \otimes 1 \otimes 1 \otimes 1 \otimes \sigma_1 \otimes 1_2 \otimes 1_2, \\
&\quad 1_2 \otimes 1 \otimes 1 \otimes 1_2 \otimes 1 \otimes 1 \otimes 1 \otimes \sigma_2 \otimes \sigma_3 \otimes 1_2, \\
&\quad 1_2 \otimes 1 \otimes 1 \otimes \hat{\Gamma}_{a3}^{(3)} \otimes 1 \otimes 1 \otimes 1 \otimes \sigma_2 \otimes \sigma_1 \otimes 1_2, \\
&\quad 1_2 \otimes 1 \otimes 1 \otimes 1_2 \otimes 1 \otimes 1 \otimes 1 \otimes \sigma_2 \otimes \sigma_2 \otimes \sigma_3, \\
&\quad 1_2 \otimes 1 \otimes 1 \otimes 1_2 \otimes 1 \otimes 1 \otimes 1 \otimes \sigma_2 \otimes \sigma_2 \otimes \sigma_1, \\
&\quad 1_2 \otimes 1 \otimes 1 \otimes 1_2 \otimes 1 \otimes 1 \otimes i \otimes \sigma_2 \otimes \sigma_2 \otimes \sigma_2).
\end{align*}
\]  

(3.79)

Here the operators

\[
\hat{\Gamma}_0 = \hat{\Gamma}_{10}^{(0)} \hat{\Gamma}_{20}^{(0)} \hat{\Gamma}_{30}^{(0)}, \quad \hat{\Gamma}_3 = \hat{\Gamma}_{13}^{(3)} \hat{\Gamma}_{23}^{(3)} \hat{\Gamma}_{33}^{(3)}
\]  

(3.80)

obey

\[
(\hat{\Gamma}_0)^2 = (\hat{\Gamma}_3)^2 = -1_2.
\]  

(3.81)

The spinor monomial reads

\[
\eta = \eta_0(x) \otimes \eta_1(y_1) \otimes \eta_2(y_2) \otimes \eta_3(y_3) \otimes \eta_4(y_4) \otimes \eta_5(y_5) \otimes \eta_6(y_6) \otimes \chi,
\]  

(3.82)

where \(\eta_i = \eta_i(y_i)\) is a 1-component spinor on \(M_i\), \(i = 1, 2, 4, 5, 6\), \(\eta_0 = \eta_0(x)\) is a 2-component spinor on \(M_0\), \(\eta_3 = \eta_3(y_3)\) is a 2-component spinor on \(M_3\) and \(\chi\) belongs to \(V = \mathbb{C}^2 \otimes \mathbb{C}^2 \otimes \mathbb{C}^2\).

The covariant derivatives \(\bar{D}_{m_i}\) act on \(\eta\) as follows

\[
\begin{align*}
\bar{D}_{m_0} \eta &= (\bar{D}_{m_0}^{(0)} \eta_0) \otimes \eta_1 \otimes \eta_2 \otimes \eta_3 \otimes \eta_4 \otimes \eta_5 \otimes \eta_6 \otimes \chi, \\
\cdots \\
\bar{D}_{m_3} \eta &= \eta_0 \otimes \eta_1 \otimes \eta_2 \otimes (\bar{D}_{m_3}^{(3)} \eta_3) \otimes \eta_4 \otimes \eta_5 \otimes \eta_6 \otimes \chi, \\
\cdots \\
\bar{D}_{m_6} \eta &= \eta_0 \otimes \eta_1 \otimes \eta_2 \otimes \eta_3 \otimes \eta_4 \otimes \eta_5 \otimes (\bar{D}_{m_6}^{(6)} \eta_6) \otimes \chi,
\end{align*}
\]  

(3.83)

where

\[
\begin{align*}
\bar{D}_{m_0} &= \partial_{m_0} + \frac{1}{4} \omega_{a_0 b_0 m_0}^{(0)} (\hat{\Gamma}_{a0}^{(0)} \hat{\Gamma}_{b0}^{(0)} \otimes 1 \otimes 1 \otimes 1_2 \otimes 1 \otimes 1 \otimes 1_2 \otimes 1_2 \otimes 1_2), \\
\bar{D}_{m_3} &= \partial_{m_3} + \frac{1}{4} \omega_{a_3 b_3 m_3}^{(3)} (1_2 \otimes 1 \otimes 1 \otimes \hat{\Gamma}_{a3}^{(3)} \hat{\Gamma}_{b3}^{(3)} \otimes 1 \otimes 1 \otimes 1_2 \otimes 1_2 \otimes 1_2,
\end{align*}
\]  

(3.84)

(3.85)

\[
\bar{D}_{m_i} = D_{m_i}^{(i)} = \partial_{m_i} \text{ for } i = 1, 2, 4, 5, 6 \text{ and } D_{m_i}^{(i)} \text{ is a covariant derivative corresponding to } M_i, \ i = 0, 3.
\]

Under (3.79) the operators corresponding to the \(M2\)-branes read

\[
\hat{\Gamma}_{[s]} = \hat{\Gamma}^{11} \hat{\Gamma}^{15} \hat{\Gamma}^{16} = -1_2 \otimes 1 \otimes 1 \otimes 1_2 \otimes 1 \otimes 1 \otimes B_1
\]  

(3.85)

for \(s = I_1\) and

\[
\hat{\Gamma}_{[s]} = \hat{\Gamma}^{12} \hat{\Gamma}^{14} \hat{\Gamma}^{16} = -1_2 \otimes 1 \otimes 1 \otimes 1_2 \otimes 1 \otimes 1 \otimes B_2
\]  

(3.86)
for \( s = I_2 \). The operator for the \( M5 \)-brane can be written in the form
\[
\hat{\Gamma}_{[s]} = \hat{\Gamma}^{10} \hat{\Gamma}^{20} \hat{\Gamma}^{30} \hat{\Gamma}^{11} \hat{\Gamma}^{12} = i \hat{\Gamma}_{(0)} \otimes 1 \otimes 1 \otimes 1_2 \otimes 1 \otimes 1 \otimes 1 \otimes B_3 \tag{3.87}
\]
for \( s = I_3 \).

In (3.85)-(3.87) \( B_s \) are self-adjoint commuting idempotent operators acting on \( V = \mathbb{C}^2 \otimes \mathbb{C}^2 \otimes \mathbb{C}^2 \)
\[
B_1 = \sigma_1 \otimes 1_2 \otimes \sigma_3, \quad B_2 = \sigma_2 \otimes \sigma_3 \otimes \sigma_1, \quad B_3 = 1_2 \otimes \sigma_3 \otimes 1_2. \tag{3.88}
\]

The gamma matrices corresponding to \( M_0 \) can be chosen in the form (3.88) and hence \( \hat{\Gamma}_{(0)} = i1_2 \).

Under the proposition from Appendix A there exists a basis of eigenvectors \( \psi_{\varepsilon_1 \varepsilon_2 \varepsilon_3} \) in \( V \) with \( \varepsilon_1 = \pm 1, \varepsilon_2 = \pm 1, \varepsilon_3 = \pm 1 \), satisfying (3.72).

The solutions to generalized Killing equations (2.6) corresponding to the field configuration from (3.77), (3.78) are given by the following monomial solutions
\[
\varepsilon = H_1^{-1/6} H_2^{-1/6} H_3^{-1/12} \eta_0(x) \otimes \eta_1 \otimes \eta_2 \otimes \eta_3(y_3) \otimes \eta_4 \otimes \eta_5 \otimes \eta_6 \otimes \psi_{\varepsilon_1 \varepsilon_2 \varepsilon_3}, \tag{3.89}
\]
where \( \eta_0(x) \) and \( \eta_3(y_3) \) are parallel spinors defined on \( M_0 \) and \( M_3 \), respectively, \( \eta_k \) is a constant 1-dimensional spinor on \( M_i, i = 1, 2, 4, 5, 6 \).

Using the relations (2.20), (3.82) with \( \chi = \psi_{\varepsilon_1 \varepsilon_2 \varepsilon_3}, (3.85)-(3.87) \) one can obtain the restrictions for the parameters \( \varepsilon_s \)
\[
- \varepsilon_s = c_s. \tag{3.90}
\]
Thus the number of linear independent solutions given by (3.89) and (3.90)
\[
N = 32\mathcal{N} = n_0 n_3, \tag{3.91}
\]
where \( n_j \) is the number of parallel spinors on the 3-dimensional manifolds \( M_j, j = 0, 3 \).

**Example.**

Let the totally transverse space \( M_0 \) and relatively transverse space \( M_6 \) be coinciding with the 3-dimensional Euclidean space: \( M_0 = M_6 = \mathbb{R}^3 \). Then one obtains \( \mathcal{N} = 1/8 \) in agreement with [15].

### 3.4 \( M2 \cap M5 \cap M5 \)

According to the classification from [15] there are two possible configurations for the intersection of one \( M2 \)-brane and two \( M5 \)-branes.

**i**

The first configuration \( M2 \cap M5 \cap M5 \) is defined on the manifold
\[
M_0 \times M_1 \times M_2 \times M_3 \times M_4 \times M_5, \tag{3.92}
\]
where \( d_0 = d_2 = d_3 = d_4 = d_5 = 2 \) and \( d_1 = 1 \) and describes a \( M2 \)-brane intersecting each of the two \( M5 \)-branes over a string with the \( M5 \)-branes intersecting over a 3-brane (see Fig. 6).
\[ g = \begin{cases} 
\begin{array}{cccccccc}
\times & \times & \times & \times & \times & \times & \times & \times \\
M_0 & M_1 & M_2 & M_3 & M_4 & M_5 
\end{array} : H_1 \\
\begin{array}{cccccccc}
\times & \times & \times & \times & \times & \times & \times & \times \\
M_0 & M_1 & M_2 & M_3 & M_4 & M_5 
\end{array} : H_2 \\
\begin{array}{cccccccc}
\times & \times & \times & \times & \times & \times & \times & \times \\
M_0 & M_1 & M_2 & M_3 & M_4 & M_5 
\end{array} : H_3.
\end{cases} \]

Figure 6: \( M_2 \cap M_5 \cap M_5 \)-intersection. \( M_0 \) is the totally transverse space, \( M_1, M_2, M_3, M_4 \) are the relative transverse spaces and \( M_5 \) is the common worldvolume.

The solution for the intersection of an electric \( M_2 \)-brane and two magnetic \( M_5 \)-branes is given by
\[ g = H_1^{1/3} H_2^{2/3} H_3^{2/3} \{ g^0 + H_1^{-1} g^1 + H_2^{-1} g^2 + H_3^{-1} g^3 + H_2^{-1} H_3^{-1} g^4 + H_1^{-1} H_2^{-1} H_3^{-1} g^5 \}. \] (3.93)

\[ F = c_1 dH_1^{-1} \wedge \tau_1 \wedge \tau_5 + c_2 (\ast_0 dH_2) \wedge \tau_1 \wedge \tau_3 + c_3 (\ast_0 dH_3) \wedge \tau_1 \wedge \tau_2, \] (3.94)

where \( c_1^2 = c_2^2 = c_3^2 = 1 \); \( H_1, H_2, H_3 \) are harmonic functions defined on \((M_0, g^0)\). The metrics \( g^i, i = 0, 1, 2, 3, 4 \), have Euclidean signatures and the metric \( g^5 \) has the signature \((-+, +)\). The branes sets are \( I_1 = \{1, 5\}, I_2 = \{2, 4, 5\} \) and \( I_3 = \{3, 4, 5\}\).

We introduce the following set of \( \hat{\Gamma} \)-matrices
\[
(\hat{\Gamma}^A) = \begin{cases}
\hat{\Gamma}_{(0)}^{a_0} \otimes 1 \otimes \hat{1}_2 \otimes \hat{1}_2 \otimes \hat{1}_2 \otimes \hat{1}_2, \\
\hat{\Gamma}_{(0)}^{a_2} \otimes \hat{\Gamma}_{(2)} \otimes \hat{\Gamma}_{(3)} \otimes \hat{\Gamma}_{(4)} \otimes \hat{\Gamma}_{(5)}, \\
\hat{\Gamma}_{(0)}^{a_3} \otimes \hat{\Gamma}_{(2)} \otimes \hat{\Gamma}_{(3)} \otimes \hat{\Gamma}_{(4)} \otimes \hat{\Gamma}_{(5)}.
\end{cases}
\] (3.95)

where \( \hat{\Gamma}_{(i)}^{a_i} \) are \( 2 \times 2 \) \( \Gamma \)-matrices, \( a_i = 1, 2, \), \( i = 0, 2, 3, 4, 5 \), corresponding to \( M_i \), respectively, and the operators
\[ \hat{\Gamma}_{(i)} = \hat{\Gamma}_{(i)}^{a_1} \hat{\Gamma}_{(i)}^{a_2}, \quad \hat{\Gamma}_{(5)} = \hat{\Gamma}_{(5)}^{a_3} \hat{\Gamma}_{(5)}^{a_4} \] (3.96)
satisfy
\[ (\hat{\Gamma}_{(i)})^2 = -1_2, \quad (\hat{\Gamma}_{(5)})^2 = 1_2, \] (3.97)
\( i = 0, 2, 3, 4 \).

Consider \( \eta \) in the form \( \eta = \eta_0(x) \otimes \eta_1(y_1) \otimes \eta_2(y_2) \otimes \eta_3(y_3) \otimes \eta_4(y_4) \otimes \eta_5(y_5), \) where \( \eta_i = \eta_i(y_i) \) is a 2-component spinor on \( M_i, i = 0, 2, 3, 4, 5 \), \( \eta_1(y_1) \) is a 1-component spinor on \( M_1 \).

Due to (3.95) and (3.97), the operator \( \tilde{D}_{m_i} \) acts on \( \eta \) as
\[ \tilde{D}_{m_i} \eta = \ldots \otimes \eta_{i-1} \otimes \left( D_{m_i}^{(i)} \eta_i \right) \otimes \eta_{i+1} \otimes \ldots, \] (3.98)
where \( D_{m_i}^{(i)} \) is the spinorial covariant derivative corresponding to \( M_i, i = 0, 2, 3, 4, 5 \), and \( D_{m_1}^{(1)} = \partial_{m_1} \). Thus the relations (2.23) are satisfied if \( \eta_i \) is a parallel spinor on \( M_i, i = 0, 1, 2, 3, 4, 5 \), and \( \eta_1(y_1) = \eta_1 \) is constant.
The operator corresponding to the $M2$-brane is
\[
\hat{\Gamma}_{[s]} = \hat{\Gamma}^{11} \hat{\Gamma}^{15} \hat{\Gamma}^{25} = \hat{\Gamma}(0) \otimes 1 \otimes \hat{\Gamma}(2) \otimes \hat{\Gamma}(3) \otimes \hat{\Gamma}(4) \otimes \mathbf{1}_2
\] (3.99)
for $s = I_1$, the operators corresponding to the $M5$-branes are
\[
\hat{\Gamma}_{[s]} = \hat{\Gamma}^{10} \hat{\Gamma}^{20} \hat{\Gamma}^{11} \hat{\Gamma}^{21} \hat{\Gamma}^{22} = \mathbf{1}_2 \otimes 1 \otimes \hat{\Gamma}(2) \otimes \hat{\Gamma}(3) \otimes \hat{\Gamma}(4) \otimes \hat{\Gamma}(5)
\] (3.100)
for $s = I_2$ and
\[
\hat{\Gamma}_{[s]} = \hat{\Gamma}^{10} \hat{\Gamma}^{20} \hat{\Gamma}^{11} \hat{\Gamma}^{12} \hat{\Gamma}^{22} = \mathbf{1}_2 \otimes 1 \otimes \hat{\Gamma}(3) \otimes \hat{\Gamma}(4) \otimes \hat{\Gamma}(5)
\] (3.101)
for $s = I_3$.

The projections (2.20) are satisfied if
\[
\hat{\Gamma}(j) \eta_j = c(j) \eta_j, \quad c^2_{(j)} = -1,
\]
\[
\hat{\Gamma}(5) \eta_5 = c(5) \eta_5, \quad c^2_{(5)} = 1,
\] (3.102)
for $j = 0, 3, 2, 4$, and
\[
c(0)c(2)c(3)c(4) = c_1, \quad c(2)c(4)c(5) = c_2, \quad c(3)c(4)c(5) = c_3.
\] (3.103)

Eqs. (3.103) have the following solutions
\[
c(0) = i c_1 c_2 c_3 \varepsilon_4, \quad c(2) = -i c_2 \varepsilon_4 \varepsilon_5,
\]
\[
c(3) = -i c_3 \varepsilon_4 \varepsilon_5, \quad c(4) = i \varepsilon_4, \quad c(5) = \varepsilon_5,
\] (3.104)
where $\varepsilon_4 = \pm 1$, $\varepsilon_5 = \pm 1$.

For the field configuration (3.93) and (3.94) we obtain the following solution to SUSY equations
\[
\varepsilon = H_1^{-1/6} H_2^{-1/12} H_3^{-1/12} \eta_0(x) \otimes \eta_1 \otimes \eta_2(y_2) \otimes \eta_2(y_3) \otimes \eta_4(y_4) \otimes \eta_5(y_5),
\] (3.106)
where $\eta_i, \ i = 0, 2, 3, 4, 5$, are chiral parallel spinors defined on $M_i \left( D_{m(i)} \eta_i = 0 \right)$, obeying (3.102) and (3.103). $\eta_1$ is constant.

Thus, the number of linear independent solutions is
\[
N = 32 \mathcal{N} = \sum_{\varepsilon_4 = \pm 1, \varepsilon_5 = \pm 1} n_0(i \varepsilon_4 c_1 c_2 c_3) n_2(-i c_2 \varepsilon_4 \varepsilon_5) n_3(-i c_3 \varepsilon_4 \varepsilon_5) n_4(i \varepsilon_4) n_5(\varepsilon_5),
\] (3.107)
where $n_j(c_j)$ is the number of chiral parallel spinors on $M_j, \ j = 0, 2, 3, 4, 5$.

**Examples.**

Let $M_0 = M_2 = M_3 = M_4 = \mathbb{R}^2$ and $M_1 = \mathbb{R}$.

**a** Then for $M_5 = \mathbb{R}^{1,1}$ one gets $\mathcal{N} = 1/8$.

**b** While in the case of $M_5 = \mathbb{R}_{0+}^{1,1}/\mathbb{Z}_2$, we find $\mathcal{N} = 1/16$ for any choice of brane sign factors.
(ii) The second possible intersection of M2-brane and two M5-branes is defined on the manifold

\[ M_0 \times M_1 \times M_2 \times M_3 \times M_4 \times M_5 \times M_6, \quad (3.108) \]

where \( d_0 = 3 \), \( d_1 = d_2 = d_4 = d_5 = d_6 = 1 \), \( d_3 = 3 \).

The configuration is given in Fig. 7.

\[
g = \begin{pmatrix}
- & - & - & - & - & - & - & - & \chi & \chi & \chi & : H_1 \\
- & - & - & - & - & - & - & - & \chi & \chi & \chi & : H_2 \\
- & - & - & - & - & - & - & - & \chi & \chi & \chi & : H_3 \\
M_0 & M_1 & M_2 & M_3 & M_4 & M_5 & M_6
\end{pmatrix}
\]

Figure 7: \( M_2 \cap M_5 \cap M_5 \)-intersection: the M2-brane intersect each of the two M5-branes over a string, the M5-branes intersect over a 3-brane. \( M_0 \) is the totally transverse space, \( M_i, i = 1, 2, 3, 4, 5 \), are the relatively transverse spaces, \( M_6 \) is the common worldvolume.

The solution describing the intersection of one electric brane and two magnetic ones reads now

\[
g = H_1^{1/3} H_2^{2/3} H_3^{2/3} \left\{ g^0 + H_2^{-1} g^1 + H_3^{-1} g^2 + H_2^{-1} H_5^{-1} g^3 + H_1^{-1} H_3^{-1} g^4 + \right. \\
\left. H_1^{-1} H_2^{-1} g^5 + H_1^{-1} H_2^{-1} H_3^{-1} g^6 \right\}, \quad (3.109)
\]

\[
F = c_1 dH_1^{-1} \wedge \tau_4 \wedge \tau_5 \wedge \tau_6 + c_2 (\ast_0 dH_2) \wedge \tau_2 \wedge \tau_4 + c_3 (\ast_0 dH_3) \wedge \tau_1 \wedge \tau_5, \quad (3.110)
\]

where \( c_1^2 = c_2^2 = c_3^2 = 1 \); \( H_1, H_2, H_3 \) are harmonic functions on \( (M_0, g^0) \). The metrics \( g^i, i = 0, 1, 2, 3, 4, 5 \), have Euclidean signatures and we put the metric \( g^0 = -dt \otimes dt \). The branes sets are \( I_1 = \{4, 5, 6\}, I_2 = \{1, 3, 5, 6\}, I_3 = \{2, 3, 4, 6\} \).

The corresponding set of gamma matrices matches with \( (3.79) \) for two electric and one magnetic branes due to the same space configuration. The expressions for \( \hat{\Gamma}^{(i)} \), \( (\hat{\Gamma}^{(i)})^2 \), \( i = 0, 3 \), coincide with \( (3.80) \) and \( (3.81) \) as well.

The 32-component spinor \( \eta \) can be represented in the form \( (3.82) \); \( \eta = \eta_0(x) \otimes \eta_1(y_1) \otimes \eta_2(y_2) \otimes \eta_3(y_3) \otimes \eta_4(y_4) \otimes \eta_5 \otimes \eta_6 \otimes \chi \), where \( \eta_i = \eta_i(y_i) \) is a 1-component spinor on \( M_i \), \( i = 1, 2, 4, 5, 6 \), \( \eta_0 = \eta_0(x) \) is a 2-component spinor on \( M_0 \) and \( \eta_3 = \eta_3(y_3) \) is a 2-component spinor on \( M_3 \) and \( \chi \) is an element of \( V = \mathbb{C}^2 \otimes \mathbb{C}^2 \otimes \mathbb{C}^2 \).

For covariant derivatives we have relations \( (3.83) \) and \( (3.84) \). Using the representation of gamma matrices \( (3.79) \) the operators corresponding to \( M \)-branes are given by

\[
\hat{\Gamma}_{[s]} = \hat{\Gamma}^{14} \hat{\Gamma}^{15} \hat{\Gamma}^{16} = -1_2 \otimes 1 \otimes 1 \otimes 1_2 \otimes 1 \otimes 1 \otimes 1 \otimes B_1 \quad (3.111)
\]

for \( s = I_1 \),

\[
\hat{\Gamma}_{[s]} = \hat{\Gamma}^{16} \hat{\Gamma}^{20} \hat{\Gamma}^{30} \hat{\Gamma}^{12} \hat{\Gamma}^{14} = i \hat{\Gamma}_0 \otimes 1 \otimes 1 \otimes 1_2 \otimes 1 \otimes 1 \otimes 1 \otimes B_2 \quad (3.112)
\]

for \( s = I_2 \),

\[
\hat{\Gamma}_{[s]} = \hat{\Gamma}^{16} \hat{\Gamma}^{20} \hat{\Gamma}^{30} \hat{\Gamma}^{11} \hat{\Gamma}^{15} = i \hat{\Gamma}_{(0)} \otimes 1 \otimes 1 \otimes 1_2 \otimes 1 \otimes 1 \otimes 1 \otimes B_3 \quad (3.113)
\]
for $s = I_3$.

Here
\[ B_1 = \sigma_2 \otimes \sigma_2 \otimes 1_2, \quad B_2 = \sigma_3 \otimes \sigma_1 \otimes \sigma_3, \quad B_3 = 1_2 \otimes \sigma_2 \otimes \sigma_1 \quad (3.114) \]
are self-adjoint commuting operators.

As in the case with two $M2$-branes and one $M5$-brane we put $(\hat{\Gamma}^{(0)}) = (\sigma_1, \sigma_2, \sigma_3)$ and hence $\hat{\Gamma}_{(0)} = i1_2$.

Due to the proposition from Appendix A there exists a basis of eigenvectors $\psi_{\varepsilon_1,\varepsilon_2,\varepsilon_3} \in V$ satisfying the relations
\[ B_s \psi_{\varepsilon_1,\varepsilon_2,\varepsilon_3} = \varepsilon_s \psi_{\varepsilon_1,\varepsilon_2,\varepsilon_3}, \quad (3.115) \]
where $\varepsilon_1 = \pm 1, \quad \varepsilon_2 = \pm 1, \quad \varepsilon_3 = \pm 1, \quad s = 1, 2, 3$.

The solutions to eqs. (2.6) corresponding to the field configuration from (3.109), (3.110) are given by
\[ \varepsilon = H_1^{-1/6} H_2^{-1/12} H_3^{-1/12} \eta_0(x) \otimes \eta_1 \otimes \eta_2 \otimes \eta_3(y_3) \otimes \eta_4 \otimes \eta_5 \otimes \eta_6 \otimes \psi_{\varepsilon_1,\varepsilon_2,\varepsilon_3}, \quad (3.116) \]
where $\eta_0(x)$ and $\eta_3(y_3)$ are parallel spinors on $M_0$ and $M_3$, $\eta_i$ are constant 1-dimensional spinors, $i = 1, 2, 4, 5, 6$.

The parameters $\varepsilon_s$ obey the chirality restrictions
\[ -\varepsilon_s = c_s, \quad (3.117) \]
$s = 1, 2, 3$, following from relations (2.20), (3.65) with $\chi = \psi_{\varepsilon_1,\varepsilon_2,\varepsilon_3}$, (3.111)-(3.113), (3.115).

The number of linear independent solutions given by (3.116)-(3.117) can be computed as follows
\[ N = 32N = n_0 n_3, \quad (3.118) \]
where $n_0$ and $n_3$ are numbers of parallel spinors on the manifolds $M_0$ and $M_3$, respectively.

**Example.**

Here the only example we have is the trivial one: $M_0 = M_2 = \mathbb{R}^3$ with $N = 1/8$ in agreement with [15].

### 4 Conclusions

In this paper we have considered the generalized Killing equations in $D = 11$ supergravity for triple $M$-brane solutions defined on the product of Ricci-flat manifolds. The first configuration with three electric branes $M2 \cap M2 \cap M2$ has been studied earlier in [20], while six others ones: $M2 \cap M2 \cap M5, M2 \cap M5 \cap M5$ (two configurations) and $M5 \cap M5 \cap M5$ (three configurations) have been considered here for the first time. Using the approach of [19, 20] we have obtained explicit formulae for computing the amounts of preserved supersymmetries for all triple $M$-brane configurations. These formulae have generalized the relations obtained earlier by several authors for flat factor spaces $\mathbb{R}^{k_i}$ [14, 15]. The deduced fractional numbers of preserved SUSY $N$ depend upon certain numbers of (chiral or all) parallel spinors on some factor spaces and in several cases upon brane sign factors $c_s$.

We have presented examples of partially supersymmetric configurations which do not belong to the classification of Bergshoeff et al. [15]. These examples use the following factor
spaces: $K3$, $\mathbb{C}^2/Z_2$ (for $M2 \cap M2 \cap M2$), $\mathbb{R}^{1,1}_+ / \mathbb{Z}_2$ (for case (i) of $M2 \cap M5 \cap M5$ and case (ii) of $M5 \cap M5 \cap M5$) and $(\mathbb{R}^{1,1}_+ / \mathbb{Z}_2) \times \mathbb{R}$ (for case (iii) of $M5 \cap M5 \cap M5$), a 4-dimensional $pp$-wave manifold from [13] and $(\mathbb{R}^{1,1}_+ / \mathbb{Z}_2) \times \mathbb{R}^2$ (for case (i) of $M5 \cap M5 \cap M5$). This list of factor spaces contains only few 4-dimensional Ricci-flat factor spaces ($K3$, $pp$-wave) which are not flat. All other two- and three-dimensional factor spaces are flat. (Any $d = 2, 3$ Ricci-flat space is flat.)

In three cases: $M2 \cap M2 \cap M2$ and $M5 \cap M5 \cap M5$ ((i) and (ii)) we have presented examples where $N$ depend upon brane sign factors $c_s = \pm 1$.

An open problem here is to analyze special solutions with certain “near-horizon” harmonic functions $H_s$ for which the unbroken numbers of supersymmetries might be larger then the numbers $N$ obtained here for generic $H_s$-functions. In this case one should deal with Freund-Rubin-type solutions with composite $M$-branes, see [28] and references therein. Such partially supersymmetric solutions will lead to certain relations which contain numbers of (chiral) Killing spinors on certain Einstein factor spaces. This may be of interest in a context of the AdS/CFT approach, its generalizations and applications [26].

It is also a straightforward task to use the obtained results for studying partially supersymmetric solutions (with Ricci-flat or non-trivial flat factor spaces) in $IIA$-, $IIB$- and other ($d < 10$) supergravitational models using dimensional reductions and duality transformations. Another problem of interest may be related to a search of ”pseudo-supersymmetric” brane solutions [27] defined on a product of Ricci-flat manifolds by using a possible generalization of the approach from [19, 20].

**Acknowledgments**

The authors are grateful to D.V. Alekseevsky and H. Baum for helpful comments on related issues. A.G. was supported in part by The Ministry of education and science of Russian Federation, project 14.B37.21.2035.

**A Appendix**

Here we outline a proposition on simultaneous diagonalization of a set of linear idempotent operators arising in the decomposition of gamma matrices for some cases of product spaces. This proposition is a special case of the so-called ”$2^{-k}$-splitting” theorem from [19].

**Proposition.**

Let $B_1, \ldots, B_k : V \to V$ be a set of linear operators defined on the vector space $V = \mathbb{C}^2 \otimes \ldots \otimes \mathbb{C}^2$ ($k$-times) which are idempotent

$$B_i^2 = 1_V$$

and commute with each other

$$B_i B_j = B_j B_i,$$

$i, j = 1, \ldots, k$.

Let $A_1, \ldots, A_k : V \to V$ be bijective operators obeying the following relations

\[ A_i A_j = A_j A_i, \]

\[ i, j = 1, \ldots, k. \]
\[ A_i B_i + B_i A_i = 0, \]  
\[ A_i B_j = B_j A_i, \quad i \neq j, \]  
\[ i, j = 1, \ldots, k. \]

Then there exists a basis of \( 2^k \) vectors in \( V \), \( \psi_{\varepsilon_1, \ldots, \varepsilon_k}, \varepsilon_1, \ldots, \varepsilon_k = \pm 1 \), which are eigenvectors of \( B_i \):

\[ B_i \psi_{\varepsilon_1, \ldots, \varepsilon_k} = \varepsilon_i \psi_{\varepsilon_1, \ldots, \varepsilon_k}, \]

\[ i = 1, \ldots, k. \]

According to the proof of the theorem in [19], the operator \( A_i \) defines an isomorphism between vector eigen-spaces \( V_{\varepsilon_1, \ldots, \varepsilon_i} \) and \( V_{\varepsilon_1, \ldots, -\varepsilon_i} \) (all other indices \( \varepsilon_j, j \neq i \), are coinciding). Hence the basis may be found as follows. First we find a non-zero "ground-state" vector \( \psi_{-1, \ldots, -1} = \psi \) satisfying

\[ B_i \psi = -\psi \]

for all \( i \) and afterwards we put

\[ \psi_{\varepsilon_1, \ldots, \varepsilon_k} \equiv A_{1+\varepsilon_1}^{(1+\varepsilon_1)/2} \cdots A_{1+\varepsilon_k}^{(1+\varepsilon_k)/2} \psi, \]

where \( \varepsilon_1, \ldots, \varepsilon_k = \pm 1 \). Here \( A_i^0 \equiv 1_V \) is identity operator on \( V \).

Let us consider three examples of the operators \( B_1, B_2, B_3 \), which appeared in Section 3 (in the cases of intersections \( M5 \cap M5 \cap M5 \) (iii), \( M2 \cap M2 \cap M5 \) and \( M2 \cap M5 \cap M5 \) (ii)).

**Example 1.**

Let

\[ B_1 = -\sigma_1 \otimes \sigma_2 \otimes 1_2, \quad B_2 = \sigma_3 \otimes \sigma_1 \otimes \sigma_3, \quad B_3 = -1_2 \otimes \sigma_2 \otimes \sigma_1. \]

A set of the operators \( A_s, s = 1, 2, 3 \), obeying (3)-(4) may be chosen as

\[ A_1 = \sigma_3 \otimes 1_2 \otimes 1_2, \quad A_2 = \sigma_1 \otimes 1_2 \otimes 1_2, \quad A_3 = 1_2 \otimes 1_2 \otimes \sigma_3. \]

**Example 2.**

For

\[ B_1 = \sigma_1 \otimes 1_2 \otimes \sigma_3, \quad B_2 = \sigma_2 \otimes \sigma_3 \otimes 1_1, \quad B_3 = 1_2 \otimes \sigma_3 \otimes 1_2, \]

the operators \( A_s \) read

\[ A_1 = \sigma_2 \otimes 1_2 \otimes \sigma_3, \quad A_2 = \sigma_1 \otimes 1_2 \otimes 1_2, \quad A_3 = \sigma_1 \otimes \sigma_1 \otimes 1_2. \]

**Example 3.**

Let

\[ B_1 = \sigma_2 \otimes \sigma_2 \otimes 1_2, \quad B_2 = \sigma_3 \otimes \sigma_2 \otimes \sigma_3, \quad B_3 = 1_2 \otimes \sigma_2 \otimes \sigma_1. \]

Then \( A_s \)-operators can be presented in the following form

\[ A_1 = \sigma_3 \otimes 1_2 \otimes 1_2, \quad A_2 = \sigma_2 \otimes 1_2 \otimes 1_2, \quad A_3 = 1_2 \otimes 1_2 \otimes \sigma_3. \]
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