Notes on the SYK model in real time

Irina Aref’eva and Igor Volovich
Steklov Mathematical Institute, Russian Academy of Sciences,
Gubkina str. 8, 119991, Moscow, Russia
E-mail: arefeva@mi.ras.ru, volovich@mi.ras.ru

ABSTRACT: Nonperturbative formulation of the Sachdev-Ye-Kitaev (SYK) model is discussed. The partition function of the model can be represented as a functional integral over the Grassmann variables in Euclidean time which is well defined but it diverges after the transformation to the fermion bilocal fields. We point out that the generating functional of the SYK model in real time is well defined even after the transformation to the bilocal fields and it can be used for nonperturbative investigations of its properties. The SYK model in zero dimensions is studied, its large $N$ expansion is evaluated and phase transitions are investigated.
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1 Introduction

The Sachdev-Ye-Kitaev (SYK) model [1–3] has been considered recently as an interesting example of the solid state system which could admit a nontrivial holographic description, [4–14]. The ordinary approach to the study of the partition function of the SYK model is based on the transformation to the fermion bilocal fields and then the using of the saddle point method to get the $1/N$ expansion. However in terms of bilocal fields one gets the functional integral that is divergent and it is difficult to use it for a nonperturbative formulation.

We point out that the generating functional of the SYK model in the real time is well defined even after the transformation to bilocal fields and it can be used for nonperturbative investigations of its properties.

A simple analogue of the Euclidean partition function of the SYK model can be described as the Gaussian integral with the random coupling constant $g$

$$
\frac{1}{\sqrt{2\pi}} \int dx dg \exp\left\{-\frac{g^2}{2}\right\} \exp\{\pm gx^2\} = \int dx \exp\left\{-\frac{1}{2}x^4\right\},
$$

which is divergent. The problem of giving meaning to the functional integral of the SYK model in the bilinear variables representation by a suitable choosing of complex contours has been discussed in [11] and in [14] with the conclusion that still there are open questions.

In the "real time" formulation discussed in this paper, we have

$$
\frac{1}{\sqrt{2\pi}} \int dx dg \exp\left\{-\frac{g^2}{2}\right\} \exp\{igx^2\} = \int dx \exp\left\{-\frac{1}{2}x^4\right\},
$$

which is well defined.

We propose to define the generating functional in the SYK model in zero dimensions with $M$ replicas in "real time" as the Grassmann (Berezin) integral over anticommuting variables

$$
Z(A, J) = \int d\mu(J) \int d\chi e^{iA[\chi, J]},
$$

where $\chi_\alpha^j$ are the Grassmann variables, $i, j, k, l = 1, \ldots N$, $\alpha, \beta = 1, \ldots M$, $J > 0$, $(A_{\alpha\beta})$ is an antisymmetric matrix with real entries and $d\mu(J)$ is a Gaussian probability measure.
and \( J = (J_{ijkl}) \) are random variables with zero mean \(<J_{ijkl}>=0\), and variance is given by \(<J_{ijkl}^2>=3!J^2/N^3>\).

Performing the integration over \( d\mu(J) \) one gets

\[
Z(A, J) = \int d\chi \exp \left\{ -\frac{1}{2} \sum_{\alpha,\beta,j} \chi^\alpha_j A_{\alpha\beta} \chi^\beta_j - \frac{NJ^2}{8} \sum_{\alpha,\beta} \left( \frac{1}{N} \sum_j \chi^\alpha_j \chi^\beta_j \right)^4 \right\}. \tag{1.5}
\]

Note that \( Z(A, J) \) is a polynomial at \( J^2 \) and \( A_{\alpha\beta} \) because there is only a finite number of the Grassmann variables. A similar natural definition can be done also for the 1-dim SYK model, which can be called the SYK model in real time, see below.

The expression (1.5) can be written also by using the bilocal variables

\[
Z(A, J) = \left( \frac{N}{2\pi} \right)^{\frac{M(M-1)}{2}} \int d\Sigma dG \left( \text{Pf}(A+i\Sigma) \right)^N \exp \left\{ N \sum_{\alpha<\beta} \left[ -\frac{J^2}{4} G^4_{\alpha\beta} + i\Sigma_{\alpha\beta} G_{\alpha\beta} \right] \right\}. \tag{1.6}
\]

Here Pf is the Pfaffian and \((G_{\alpha\beta})\) and \((\Sigma_{\alpha\beta})\) are antisymmetric matrices with real entries. Note the presence of the factor \( \exp\{-J^2 G^4_{\alpha\beta}/4\} \) which provides the convergence of the integral.

We will study the large \( N \) behaviour of the particular case of the 0-dim SYK model with two replicas (\( M=2 \)) which is just the integral

\[
Z_q = \frac{N}{2\pi} \int dy (A+iy)^N \int dx \exp\left\{ N(-\frac{J^2}{q} x^q + ixy) \right\} \tag{1.7}
\]

with \( q = 2 \) and \( q = 4 \). Here \( x, y \) are real variables and \( A \) is a real constant, \( J > 0 \) and \( N \) is a natural number. The similar integral as a toy model for a suitable choice of the contour in the SYK model has been considered in [11]. In the case \( q = 2 \) the expression (1.7) after rescaling of \( x \) and \( y \) coincides up to a constant with the Hermite polynomial \( H_N \left( \sqrt{2N} \frac{A}{2J} \right) \). Using the asymptotic behavior of the Hermite polynomials for \( N \to \infty \) we observe the phase transition at \( A/2J = \pm 1 \) in this model. Similarly we find the phase transition for \( q = 4 \).

The generating functional of the 1-dim SYK model with \( M \) replicas can be represented in the form of the Grassmann integral over anticommuting variables \( \chi^\alpha_j = \chi^\alpha_j(\tau), \tau \) is on real line, as

\[
<Z^M> = \int d\mu(J) \int \mathcal{D}\chi e^{iA}, \tag{1.8}
\]
where the action is
\[ A = \sum_{\alpha} \int d\tau \left( \frac{i}{2} \sum_j \chi_j^\alpha \partial_\tau \chi_j^\alpha + \sum_{i<j<k<l} \sqrt{\frac{3! J^2}{N^3}} J_{ijkl} \chi_i^\alpha \chi_j^\alpha \chi_k^\alpha \chi_l^\alpha \right). \] (1.9)

One has
\[ <Z^M> = \int \mathcal{D}\chi \exp\left( -\frac{1}{2} \sum_{\alpha,j} \int d\tau \chi_j^\alpha \partial_\tau \chi_j^\alpha - \frac{N J^2}{8} \sum_{\alpha,\beta} \int d\tau \int d\tau' \left( \frac{1}{N} \sum_j \chi_j^\alpha(\tau) \chi_j^\beta(\tau') \right)^4 \right). \]

The generating functional can be also written in terms of the bilocal variables
\[ <Z^M> = \int \mathcal{D}\Sigma \mathcal{D}G \exp(NI), \] (1.10)
where
\[ I = \ln \text{Pf}(\partial_\tau + i\Sigma) \] (1.11)
\[ + \sum_{\alpha<\beta} \int d\tau \int d\tau' \left( i\Sigma_{\alpha\beta}(\tau,\tau') G_{\alpha\beta}(\tau,\tau') - \frac{J^2}{4} G_{\alpha\beta}(\tau,\tau')^4 \right). \]

Here \( G = (G_{\alpha\beta}(\tau,\tau')) \), \( \Sigma = (\Sigma_{\alpha\beta}(\tau,\tau')) \) and \( G_{\alpha\beta}(\tau,\tau'), \Sigma_{\alpha\beta}(\tau,\tau') \) are antisymmetric real valued functions (bilocal fields). Note that the factor \( \exp\left\{ -NJ^2 G_{\alpha\beta}(\tau,\tau')^4/4 \right\} \) provides the convergence of the integral.

The paper is organized as follows. In Section 2 we obtain the representation (1.6). In Section 3 we consider the simplest case of the \( q = 2 \) model and using the relation of the model with the Hermite polynomials investigate the behaviour of the model in the large \( N \) limit. In Section 4 using the steepest descent method we investigate the behaviour of the \( q = 4 \) model in the large \( N \) limit. We conclude in Section 5 with the discussion.

## 2 The 0-dim SYK model

The generating functional for 0-dim SYK model is given by the formula (1.5). Let us demonstrate that it can be represented in the form (1.6). To this end we use the Fourier transform. If \( f(x) \) is an integrable fast decreasing function on the real axis then the following identity for the inverse Fourier transform holds
\[ f(\Theta) = \frac{1}{2\pi} \int_{-\infty}^{\infty} dy \int_{-\infty}^{\infty} dx \, f(x) e^{iy(x-\Theta)}. \] (2.1)
This formula is valued not only in the case when $\Theta$ is the real variable, but also in the case when $\Theta$ is a nilpotent element of the Grassmann algebra, in particular, for $\Theta = \Theta_{\alpha\beta}$,

$$\Theta_{\alpha\beta} = \frac{1}{N} \sum_j \chi^\alpha_j \chi^\beta_j.$$

(2.2)

It is assumed that the functions $f(\Theta)$ and $e^{-iy\Theta}$ are understood as power series in $\Theta$.

Note that we don’t use such expressions as $\delta(x - \Theta)$. In principle such an object can be defined by using superanalysis on the Banach algebras developed in [15] but at this point we don’t need it.

2.1 Bilocal variables in the 0-dim model

We start from the generating functional in the real time formulation given by representation (1.5). Using the representation (2.1) for $f(\Theta_{\alpha\beta}) = \exp\{-NJ^2\Theta^4_{\alpha\beta}/4\}$, where

$$\Theta_{\alpha\beta} = \frac{1}{N} \sum_j \chi^\alpha_j \chi^\beta_j,$$

we get

$$e^{-NJ^2\Theta^4_{\alpha\beta}} = \int \prod_{\alpha<\beta} \frac{d\Sigma_{\alpha\beta}}{2\pi} \int \prod_{\alpha<\beta} dG_{\alpha\beta} \exp\{-\frac{NJ^2}{4}G^4_{\alpha\beta} + i\Sigma_{\alpha\beta}(G_{\alpha\beta} - \Theta_{\alpha\beta})\}$$

and

$$Z(A, J) = \int \prod_{j,\alpha} d\chi^\alpha_j \int \prod_{\alpha<\beta} \frac{Nd\Sigma_{\alpha\beta}}{2\pi} \int \prod_{\alpha<\beta} dG_{\alpha\beta} \exp\{N\sum_{\alpha<\beta} \left[ -\frac{J^2}{4}G^4_{\alpha\beta} + i\Sigma_{\alpha\beta}G_{\alpha\beta} \right]\} \cdot \exp\left\{-\frac{1}{2} \sum_{\alpha,\beta,j} A_{\alpha\beta} \chi^\alpha_j \chi^\beta_j - \frac{i}{2} \sum_{\alpha,\beta,j} \Sigma_{\alpha\beta} \chi^\alpha_j \chi^\beta_j \right\}.$$  

(2.3)

Integrating over the Grassmann variables we get

$$Z(A, J) = \int \prod_{\alpha<\beta} \frac{Nd\Sigma_{\alpha\beta}}{2\pi} \int \prod_{\alpha<\beta} dG_{\alpha\beta} (\text{Pf} (i\Sigma_{\alpha\beta} + A_{\alpha\beta}))^N \cdot \exp\left\{N\sum_{\alpha<\beta} \left[ -\frac{J^2}{4}G^4_{\alpha\beta} + i\Sigma_{\alpha\beta}G_{\alpha\beta} \right]\right\}.$$  

(2.4)

3 Zero-dim SYK model with 2 replicas

3.1 Real time for zero-dim SYK model with 2 replicas

Let us consider the 0-dim SYK model with 2 replicas. In this case we have only the following variables $G_{12} = x$, $\Sigma_{12} = y$ and $A_{12} = A$ and the generating functional has
the form

\[ Z_q(A, J) = \frac{N}{2\pi} \int dy \int dx \left( iy + A \right)^N \exp\left\{ N\left( -\frac{J^2}{q} x^q + ixy \right) \right\}, \quad q \geq 2. \quad (3.1) \]

We will study its asymptotic behaviour as \( N \to \infty \). The case \( A = 0 \) corresponds to the vacuum functional, \( A \neq 0 \) stands for the generating functional.

### 3.2 Quadratic model and Hermite polynomials

Let us now consider the generating functional for the quadratic case

\[ Z_2(A, J) = \frac{N}{2\pi} \int dy \int dx \left( iy + A \right)^N \exp\left\{ N\left( -\frac{J^2}{2} x^2 + ixy \right) \right\}, \quad J > 0. \quad (3.2) \]

The following formula holds

\[ Z_2(A, J) = 2 \left( \frac{J}{\sqrt{2N}} \right)^N H_N(\sqrt{2N} \cdot \frac{A}{2J}), \quad (3.3) \]

where \( H_N(x) \) are the Hermite polynomials, \( N = 1, 2, ... \). Indeed,

\[ Z_2(A, J) = \frac{1}{J} \sqrt{\frac{N}{2\pi}} \int dy \left( iy + A \right)^N e^{-\frac{Ny^2}{2}} \quad (3.4) \]

and by using the representation

\[ H_N(x) = \frac{2^N}{\sqrt{\pi}} \int_{-\infty}^{\infty} dt \left( x + it \right)^N e^{-t^2} \quad (3.5) \]

after the change of variables \( y = Jt \sqrt{\frac{N}{2}} \) in (3.4) we obtain (3.3).

### 3.3 Phase transition

It was found by Plancherel and Rotach that there are two regions on the half-line \( x > 0 \) where the Hermite polynomials \( H_N((2N + 1)^{\frac{1}{2}} x) \) have different asymptotic behaviour as \( N \to \infty \) for \( 0 < x < 1 \) and for \( 1 < x < \infty \), see [16]. Recall that the Hermite polynomials satisfy the symmetry condition \( H_N(-x) = (-1)^N H_N(x) \). In our case we have to use a slightly modified asymptotic, that has been found later in [17] and more recent paper [18]. In this case also there is different asymptotical behavior of the Hermite polynomials \( H_N(\sqrt{2N} \cdot x) \) for \( |x| > 1 \) and \( |x| < 1 \). The critical points for the Hermite polynomials \( H_N(\sqrt{2N} \cdot \frac{A}{2J}) \) are \( A/2J = \pm 1 \). We have:
For $A/2J > 1$, using parametrization
\[
\frac{A}{2J} = \cosh \Psi, \tag{3.6}
\]
we have for $\psi > 0$ asymptotically for $N \to \infty$
\[
Z_2(A, J) \sim P e^{NF}
\equiv 2 \exp \left[ \frac{N}{2} (e^{-2\Psi} + 2\Psi + 2 \ln J) + \frac{1}{2} \Psi \right] \sqrt{\frac{1}{2 \sinh \Psi}}, \tag{3.7}
\]
where
\[
F = \frac{1}{2} (e^{-2\Psi} + 2\Psi + 2 \ln J). \tag{3.8}
\]
Note that $F$ could take positive and also negative values.

For $\psi < 0$ with the same parametrization we have
\[
Z_2(A, J) \sim 2 \exp \left[ \frac{N}{2} (e^{2\Psi} - 2\Psi + 2 \ln J) - \frac{1}{2} \Psi \right] \sqrt{-\frac{1}{2 \sinh \Psi}}. \tag{3.9}
\]

For $0 < A/2J < 1$, using parametrization
\[
\frac{A}{2J} = \cos \psi, \tag{3.10}
\]
we have for $\psi > 0$
\[
Z_2(A, J) \sim P e^{NF} \equiv 2 \exp \left\{ \frac{N}{2} [2 \ln J + \cos 2\psi] \right\}
\times \cos \left\{ N \left[ \frac{1}{2} (\sin 2\psi - 2\psi) \right] + \frac{\pi}{4} - \frac{\psi}{2} \right\} \sqrt{\frac{2}{\sin \psi}}, \tag{3.11}
\]
where
\[
F = \frac{1}{2} (2 \ln J + \cos 2\psi). \tag{3.12}
\]
For $\psi < 0$ with the same parametrization we have the similar asymptotics with $\psi \to -\psi$ in (3.11), i.e.
\[
Z_2(A, J) \sim 2 \exp \left\{ \frac{N}{2} [2 \ln (J) + \cos (2\psi)] \right\}
\times \cos \left\{ N \left[ \frac{1}{2} (\sin 2\psi - 2\psi) \right] - \frac{\pi}{4} + \frac{\psi}{2} \right\} \sqrt{-\frac{2}{\sin \psi}}. \tag{3.13}
\]
• For $A \approx 2J$ one uses the parametrization $A = 2J = 1 - \frac{z}{2^{2/3} 3^{2/3} N^{2/3} \sqrt{2N+1}}$ with $z$ complex and bounded, one has

$$Z_2(A, J) \sim (2J \sqrt{1/e})^N (2\pi)^{1/2} N^{1/6} e^{\frac{z^2}{2}} \left( \text{Ai} \left(-3^{-\frac{3}{2}} z\right) + O \left(N^{-\frac{5}{2}}\right) \right). \quad (3.14)$$

Note that the same asymptotics (3.9) and (3.11) can be found by using the steepest descent method.

The half-plane $(A, J > 0)$ is divided by the curves $F(J, A) = 0$ and $A = 2J$ into regions with different asymptotic behaviour of $Z_2(A, J)$ at large $N$, see Fig. 1.
4 Quartic case

4.1 Partition function

Let us first consider the partition function for the quartic interaction

$$Z_4(J) = \frac{N}{2\pi} \int dy \int dx \exp\{N \mathcal{S}_4\}, \quad J > 0,$$

where

$$\mathcal{S}_4 = -\frac{J^2}{4} x^4 + ixy + \log y + \log i.$$ 

(4.2)

The stationary points are derived from equations

$$-J^2 x^3 + iy = 0,$$
$$\frac{1}{y} + ix = 0.$$ 

(4.3) (4.4)

The solutions are

$$\left( x_{(1)}^0, y_{(1)}^0 \right) = \left( -\frac{e^{\frac{3\pi}{4}}}{\sqrt{2}J}, -\sqrt{2}Je^{\frac{3\pi}{4}} \right),$$

$$\left( x_{(2)}^0, y_{(2)}^0 \right) = \left( \frac{e^{\frac{3\pi}{4}}}{\sqrt{2}J}, \sqrt{2}Je^{\frac{3\pi}{4}} \right),$$

$$\left( x_{(3)}^0, y_{(3)}^0 \right) = \left( -\frac{e^{\frac{3\pi}{4}}}{\sqrt{2}J}, \sqrt{2}Je^{\frac{3\pi}{4}} \right),$$

$$\left( x_{(4)}^0, y_{(4)}^0 \right) = \left( \frac{e^{\frac{3\pi}{4}}}{\sqrt{2}J}, -\sqrt{2}Je^{\frac{3\pi}{4}} \right).$$ 

(4.5) (4.6) (4.7) (4.8)

Expanding around the stationary points we get

$$\mathcal{S}_4^{(1)} = \mathcal{S}_{01} - \frac{3}{2} iJ \left( x + \frac{e^{\frac{3\pi}{4}}}{\sqrt{2}J} \right)^2 + \frac{i}{2} \left( y + \sqrt{2}Je^{\frac{3\pi}{4}} \right)^2 + \frac{1}{2} \left( x + \frac{e^{\frac{3\pi}{4}}}{\sqrt{2}J} \right) \left( y + \sqrt{2}Je^{\frac{3\pi}{4}} \right),$$

$$\mathcal{S}_{01} = \log \left( -\sqrt{2}Je^{\frac{3\pi}{4}} \right) - \frac{3}{4},$$ 

(4.9)
\( S_0^{(2)} = S_{02} - \frac{3}{2} i J \left( x - \frac{e^{i\pi}}{\sqrt{2} J} \right)^2 + \frac{i \left( y - \sqrt{2} J e^{i\pi/4} \right)^2}{2 J} + i \left( x - \frac{e^{i\pi}}{\sqrt{2} J} \right) \left( y - \sqrt{2} J e^{i\pi/4} \right), \)

\( S_{02} = \log \left( \sqrt{2} J e^{i\pi/4} \right) - \frac{3}{4}, \)

(4.10)

\( S_0^{(3)} = S_{02} + \frac{3}{2} i J \left( x + \frac{e^{i3\pi/4}}{\sqrt{2} J} \right)^2 - \frac{i \left( y - \sqrt{2} J e^{i3\pi/4} \right)^2}{2 J} + i \left( x + \frac{e^{i3\pi/4}}{\sqrt{2} J} \right) \left( y - \sqrt{2} J e^{i3\pi/4} \right), \)

\( S_{03} = \log \left( \sqrt{2} J e^{i3\pi/4} \right) - \frac{3}{4}, \)

(4.11)

\( S_0^{(4)} = S_{02} + \frac{3}{2} i J \left( x - \frac{e^{i3\pi/4}}{\sqrt{2} J} \right)^2 - \frac{i \left( y + \sqrt{2} J e^{i3\pi/4} \right)^2}{2 J} + i \left( x - \frac{e^{i3\pi/4}}{\sqrt{2} J} \right) \left( y + \sqrt{2} J e^{i3\pi/4} \right), \)

\( S_{04} = \log \left( -\sqrt{2} J e^{i3\pi/4} \right) - \frac{3}{4}. \)

(4.12)

We note that the real parts of \( S_0^{(i)}, i = 1, 2, 3, 4 \) are equal and all four points contribute to the partition function. These contributions are the following. Integrating near the first, second, third and fourth points

\[ Z_4^{(i)}(J) = \frac{N}{2\pi} \int dy \int dx \exp \{ N S_0^{(i)} \}, \]

(4.13)

we get

\[ Z_4^{(1)}(J) = \frac{\pi e^{-3N/4} \left( -1 - i \right) \sqrt{J/2}^N}{N}, \]

(4.14)

\[ Z_4^{(2)}(J) = \frac{\pi e^{-3N/4} \left( 1 + i \right) \sqrt{J/2}^N}{N}, \]

(4.15)

\[ Z_4^{(3)}(J) = \frac{\pi e^{-3N/4} \left( -1 + i \right) \sqrt{J/2}^N}{N}, \]

(4.16)

\[ Z_4^{(4)}(\tilde{J}) = \frac{\pi e^{-3N/4} \left( 1 - i \right) \sqrt{J/2}^N}{N}. \]

(4.17)
Summing up over all critical points we obtain

\[ Z_4(\tilde{J}) = \sum_{i=1}^{4} Z_4^{(i)}(\tilde{J}) = e^{-3N/4}(\sqrt{\frac{J}{2}})^N((-1-i)^N + (1+i)^N + (-1+i)^N + (1-i)^N) \]

\[ = e^{-3N/4}J^{N/2} \begin{cases} 
4, & N = 8n \\
0, & N = 8n + 1 \\
0, & N = 8n + 2 \\
0, & N = 8n + 3 \\
-4, & N = 8n + 4 \\
0, & N = 8n + 5 \\
0, & N = 8n + 6 \\
0, & N = 8n + 7 
\end{cases} \text{(4.18)} \]

### 4.2 Generating function

The generating function depending on \(A\) and \(J\) is defined as

\[ Z_4(A,J) = \frac{N}{2\pi} \int dy \int dx \exp\{NS_4(A,J)\}, \quad J > 0 \text{(4.19)} \]

where

\[ S_4(A,J) = -J^2x^4 + ixy + \log(iy + A). \text{(4.20)} \]

We can rewrite this expression as

\[ Z_4(A,J) = \frac{N}{2\pi} J^{N/2} \int dy \int dx (iy + \frac{A}{\sqrt{J}})^N \exp\{-\frac{x^4}{4} + ixy\} \text{(4.21)} \]

\[ = \frac{N}{2\pi} J^{N/2} \int dy \int dx \exp\{NS_4(\tilde{A})\}, \quad J > 0, \]

where

\[ S_4(\tilde{A}) = -\frac{x^4}{4} + ixy + \log(iy + \tilde{A}) \text{(4.22)} \]

and \(\tilde{A} = A/\sqrt{J}\).

The stationary points are derived from equations

\[ -x^3 + i y = 0 \quad \text{(4.23)} \]

\[ i \frac{x}{\tilde{A} + iy} + ix = 0, \quad \text{(4.24)} \]
that give

\[ x_0^{(1)} = \frac{1}{2} M - \frac{1}{2} K_1, \quad x_0^{(2)} = \frac{1}{2} M + \frac{1}{2} K_1, \]
\[ x_0^{(3)} = -\frac{1}{2} M - \frac{1}{2} K_2, \quad x_0^{(4)} = -\frac{1}{2} M + \frac{1}{2} K_2, \]

\( (4.25) \)

\[
y_0^{(1)} = i \left( K_2 \frac{3}{4} \bar{A} + \frac{1}{8} K_1 + \sqrt{\frac{9}{4} K_1} - \frac{1}{8} M^3 \right) + \sqrt{\frac{9}{4} M} + \frac{1}{8} \sqrt{\frac{3}{2} K_1 P} + \frac{1}{8} \sqrt{\frac{3}{2} M P} \right),
\]
\[
y_0^{(2)} = i \left( \frac{3}{4} \bar{A} - \frac{1}{8} K_1 - \sqrt{\frac{9}{4} K_1} - \frac{1}{8} M^3 \right) + \sqrt{\frac{9}{4} M} + \frac{1}{8} \sqrt{\frac{3}{2} K_1 P} + \frac{1}{8} \sqrt{\frac{3}{2} M P} \right),
\]
\[
y_0^{(3)} = i \left( \frac{3}{4} \bar{A} + \frac{1}{8} K_2 + \sqrt{\frac{9}{4} K_2} + \frac{1}{8} M^3 \right) - \sqrt{\frac{9}{4} M} + \frac{1}{8} \sqrt{\frac{3}{2} K_2 P} - \frac{1}{8} \sqrt{\frac{3}{2} M P} \right),
\]
\[
y_0^{(4)} = i \left( \frac{3}{4} \bar{A} - \frac{1}{8} K_2 - \sqrt{\frac{9}{4} K_2} - \frac{1}{8} M^2 \right) - \sqrt{\frac{9}{4} M} - \frac{1}{8} \sqrt{\frac{3}{2} K_2 P} - \frac{1}{8} \sqrt{\frac{3}{2} M P} \right),
\]

\( (4.26) \)

where

\[ P = \sqrt{\sqrt{3} \sqrt{27 \bar{A}^4 - 256 + 9 \bar{A}^2}}, \quad M = \sqrt{\frac{P}{3 \bar{A}^2}} + \frac{2}{3} \sqrt{\frac{4}{3} P}, \]

\( (4.27) \)

\[ K_1 = \sqrt{-\frac{2 \bar{A}}{M} - M^2}, \quad K_2 = \sqrt{\frac{2 \bar{A}}{M} - M^2}. \]

\( (4.28) \)

In Fig. 2 we present the location of these roots.

- We see that for \( |\bar{A}| < \frac{4}{3^{1/2}} \) there are two pairs of complex conjugated x-roots: \( x_0^{(2)} = \bar{x}_0^{(1)} \) and \( x_0^{(4)} = \bar{x}_0^{(3)} \) and two pairs of y-roots related as: \( y_0^{(2)} = -\bar{y}_0^{(1)} \), \( y_0^{(4)} = -\bar{y}_0^{(3)} \). We call this domain the \( \mathcal{D} \) domain (the trigonometrical domain).
- For \( \bar{A} = \pm \frac{4}{3^{1/2}} \) we have for x-roots: one pair of complex conjugated values \( x_0^{(1)} = \bar{x}_0^{(2)} \) and one pair of equal real \( x_0^{(3)} = x_0^{(4)} \), and for y-roots: one pair of y-roots related as \( y_0^{(2)} = -\bar{y}_0^{(1)} \) and one pair of equal pure imaginary \( y_0^{(3)} = y_0^{(4)} \). This is the critical domain \( \mathcal{C} \).
- For \( \bar{A} > \frac{4}{3^{1/2}} \) we have for x-roots: one pair of complex conjugated values \( x_0^{(1)} = \bar{x}_0^{(2)} \) and one pair of non-equal real \( x_0^{(3)} < x_0^{(4)} < 0 \), and for y-roots: one pair of y-roots related as : \( y_0^{(2)} = -\bar{y}_0^{(1)} \) and one pair of non-equal pure imaginary \( 0 < \text{Im} \bar{y}_0^{(4)} < \text{Im} y_0^{(3)} \). This is the hyperbolic domain \( \mathcal{H}_+ \).
For $A < -\frac{4}{3\sqrt{4}}$ we have for x-roots: one pair of complex conjugated values $x^{(3)}_0 = \bar{x}^{(4)}_0$ and one pair of non-equal reals $0 < x^{(1)}_0 < x^{(2)}_0$, and for y-roots: one pair of y-roots related as $y^{(3)}_0 = -\bar{y}^{(3)}_0$ and one pair of non-equal pure imaginary $\text{Im} y^{(2)}_0 < \text{Im} y^{(1)}_0 < 0$. This is the hyperbolic domain $\mathcal{H}_-$. 

- The boundaries of the $\mathcal{H}_\pm$ and $\mathcal{T}_\pm$ come from solutions of the equation

$$K_2 = 0 \Rightarrow 2\tilde{A} - M^3 = 0$$

or explicitly

$$144\tilde{A}^2 \left( \sqrt{81\tilde{A}^4 - 768 + 9\tilde{A}^2} \right) = \left( \sqrt{2} \left( \sqrt{81\tilde{A}^4 - 768 + 9\tilde{A}^2} \right)^{2/3} + 8\sqrt{3} \right)^3.$$  

(4.30)

The solutions of (4.30) are

$$\tilde{A}_0 = \pm \frac{4}{3\sqrt{4}} \approx \pm 1.755.$$  

(4.31)

Expanding the action (4.20) around these stationary points we get

$$S_4(A, J) \bigg|_{x \sim x^{(i)}_0} \approx S_4^{(i)}(A, J) = S_{0i}(A, J) + Q_{\alpha \beta}^{(i)}(\tilde{A}) X^{(i)\alpha} X^{(i)\beta}, \quad X^\alpha = (X, Y), \quad i = 1, 2, 3, 4$$

$$X = x - x^{(i)}_0, \quad Y = y - y^{(i)}_0,$$  

(4.32)

where

$$S_{0i}(A, J) = S_{0i}(\tilde{A}) + \log \sqrt{J},$$

$$S_{0i}(\tilde{A}) = -\frac{x^{(i)}_0^4}{2} + ix^{(i)}_0 y^{(i)}_0 + \log(iy^{(i)}_0 + \tilde{A}).$$

(4.33)

(4.34)

We have two types of solutions: trigonometric in the region $\mathcal{T}$ and hyperbolic in the region $\mathcal{H}$ of the real line. In the region $\mathcal{T}$ there are two pairs of complex conjugated roots and in the region $\mathcal{H}$ there is one pair of the complex conjugated and two reals roots of (4.23).

In the domain $\mathcal{T}$ the real parts of $S_{01}(\tilde{A})$ and $S_{02}(\tilde{A})$, and the real parts of $S_{03}(\tilde{A})$ and $S_{04}(\tilde{A})$ coincide,

$$\text{Re}[S_{01}(\tilde{A})] = \text{Re}[S_{02}(\tilde{A})],$$

$$\text{Re}[S_{03}(\tilde{A})] = \text{Re}[S_{04}(\tilde{A})].$$

(4.35)

(4.36)
Figure 2. Location of stationary points \((x_0(i), y_0(i))\), \(i = 1, 2, 3, 4\), depending on \(\tilde{A}\).

It turns out that

\[
\text{Re}[S_{01}(\tilde{A})] > \text{Re}[S_{03}(\tilde{A})], \quad \text{for } \tilde{A} \in \mathcal{T}_-,
\]

(4.37)

\[
\text{Re}[S_{01}(A)] < \text{Re}[S_{03}(A)], \quad \text{for } A \in \mathcal{T}_+,
\]

(4.38)
\( \mathfrak{T}_\pm \) are parts of \( \mathfrak{T} \) where \( \tilde{A} < 0 \) and \( \tilde{A} > 0 \), respectively. In \( \mathfrak{H} \) we have

\[
\text{Re}[S_02(\tilde{A})] > \text{Re}[S_{01}(\tilde{A})] > \text{Re}[S_{03}(\tilde{A})] = \text{Re}[S_{04}(\tilde{A})], \quad \text{for } \tilde{A} \in \mathfrak{H}_-,
\]

\[
\text{Re}[S_{03}(\tilde{A})] > \text{Re}[S_{04}(\tilde{A})] > \text{Re}[S_{01}(\tilde{A})] = \text{Re}[S_{02}(\tilde{A})], \quad \text{for } \tilde{A} \in \mathfrak{H}_+,
\]

\( \mathfrak{H}_\pm \) are parts of \( \mathfrak{H} \) where \( \tilde{A} < 0 \) and \( \tilde{A} > 0 \), respectively, see Fig.3.

**Figure 3.** Values of \( \text{Re}[S_{0i}(\tilde{A})], i = 1, 2, 3, 4 \) for arbitrary \( \tilde{A} \)

In Fig.3 we present the dependence of the \( \text{Re}[S_{0i}(\tilde{A})], i = 1, 2, 3, 4 \), on \( \tilde{A} \).

The quadratic forms have the following coefficients

\[
Q^{(i)}_{11} = -\frac{3}{2} x^{(i)}_0, \quad Q^{(i)}_{12} = i, \quad Q^{(i)}_{22} = \frac{1}{2 \left( \tilde{A} + iy^{(i)}_0 \right)^2},
\]

and integrating over \( x \) and \( y \) we get the prefactor

\[
\Phi^{(k)} = \frac{2\pi}{N \sqrt{1 - \frac{3x^{(k)}_0}{(\tilde{A} + iy^{(k)}_0)^2}}},
\]

that can be written in the form

\[
\Phi^{(k)} = R^{(k)} e^{i\phi^{(k)}}, \quad R^{(k)} > 0, \quad 0 \leq \phi^{(k)} \leq 2\pi,
\]

\( k = 1, 2, 3, 4 \).

We can approximate the integral by the contributions of these four points,

\[
Z_4(A, J) = \sum_{i=1}^{4} \Phi^{(i)} e^{NS_{0i}(A, J)},
\]
but according to the inequalities (4.37) (see Fig.3) in the region $\mathcal{H}_-$ the contribution from the second point $(x_0^{(2)}, y_0^{(2)})$ dominates, in the $\mathcal{H}_+$ the contribution from the third point $(x_0^{(3)}, y_0^{(3)})$ dominates. In $\mathcal{T}_-$ the contributions from the first and the second points, $(x_0^{(1)}, y_0^{(1)})$ and $(x_0^{(2)}, y_0^{(2)})$ dominate, and in $\mathcal{T}_+$ the contributions from the third and the forth points, $(x_0^{(3)}, y_0^{(3)})$ and $(x_0^{(4)}, y_0^{(4)})$ dominate. The final result is

$$Z_4(A, J) \approx P\left(\frac{A}{\sqrt{J}}\right) e^{NF(A, J)}, \quad (4.44)$$

where

$$F(A, J) \approx \begin{cases} 
\text{Re}S_{01}(\frac{A}{\sqrt{J}}) + \frac{1}{2} \log J, & \tilde{A} \in \mathcal{T}_- \\
\text{Re}S_{03}(\frac{A}{\sqrt{J}}) + \frac{1}{2} \log J, & \tilde{A} \in \mathcal{T}_+ \\
\text{Re}S_{01}(\frac{A}{\sqrt{J}}) + \frac{1}{2} \log J, & \tilde{A} \in \mathcal{H}_- \\
\text{Re}S_{03}(\frac{A}{\sqrt{J}}) + \frac{1}{2} \log J, & \tilde{A} \in \mathcal{H}_+ 
\end{cases}, \quad (4.45)$$

and

$$P(\tilde{A}) = \begin{cases} 
2R^{(1)}(\tilde{A}) \cos(\phi^{(1)}(\tilde{A}) + N\text{Im}S_{01}(\tilde{A})), & \tilde{A} \in \mathcal{T}_- \\
2R^{(3)}(\tilde{A}) \cos(\phi^{(3)}(\tilde{A}) + N\text{Im}S_{03}(\tilde{A})), & \tilde{A} \in \mathcal{T}_+ \\
2R^{(1)}(\tilde{A}), & \tilde{A} \in \mathcal{H}_- \\
2R^{(3)}(\tilde{A}), & \tilde{A} \in \mathcal{H}_+ 
\end{cases}. \quad (4.46)$$

**Figure 4.** The equal level plot for free energy $F(A, J)$ given by (4.45). The magenta line shows the phase transition line and the red one shows the change of sign of $F(A, J)$
In Fig.4 we plot the dependence of free energy $F(A, J)$ on parameters $A$ and $J$. We see that the half plane $\{(A, J) : J > 0\}$ is divided into 8 regions by critical lines.

5 Conclusion

The formulation of the SYK model in real time is considered. The large $N$ asymptotic behavior of the generating functional $Z_q(A, J)$ in the zero dimensional SYK model with $M = 2$ replicas has been studied and the phase transitions were investigated. There are 8 regions in the upper half plane $\{(A, J) : J > 0\}$ with different behavior of the free energy $F(A, J)$. For $q = 2$ the critical curves are $A^2/J^2 = 4$, $F(A, J) = 0$ and for $q = 4$ the critical curves are $A^4/J^2 = 4^4/3^3$ as well as $F(A, J) = 0$ and $A = 0$. For the arbitrary even $q$ the critical curve is given by the equation $A^q/J^2 = q^q/(q-1)^{(q-1)}$. It would be interesting to study the behavior in the model in this transition layer, i.e. an analogue of (3.14). The large $N$ behavior and phase transitions in the 0-dim SYK model in real time with arbitrary $M > 2$ also deserves to study.

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