Safe Distance Mathematical Calculation and Quantitative Analysis on Covid-19 Using Einstein Equation and Gaussian Distribution

Hanwen Miao*
Princeton International School of Mathematics and Science
*Corresponding author: Hanwen_Miao2021@foxmail.com

Abstract. Most of the diseases caused by virus mainly spread through droplets in the air. The pathogen bearing droplets go deep into people’s lungs and cause infection. In this paper, we analyze the safe distance, the minimum range to keep droplets containing virus particles from entering lungs, and thereby carrying the virus inside the lung. Einstein equation for diffusivity of a particle and the wide of the Gaussian distribution of the particles in Brownian movement are used in the calculation of the range a virus-containing mucosal vary droplet can reach. Moreover, we used datas recorded in a previous paper named “Visualization of sneeze ejecta: steps of fluid fragmentation leading to respiratory droplets” by B. E. Scharfman et. all to generate our results.

Keywords: Safe Distance, Mathematical Calculation, Einstein Equation, Gaussian Distribution.

1. Introduction
Coronavirus disease 2019 (COVID-19) is an infectious disease caused by severe acute respiratory syndrome coronavirus 2 (SARS-CoV-2) (1). It was first identified in December 2019 in Wuhan, Hubei, China, and has resulted in an ongoing pandemic (1). Common symptoms include cough, sore throat, congestion or runny nose, nausea or vomiting, and diarrhea (2). If not treated properly, it may lead to death. Unfortunately, ejecta caused by coughing and sneezing feature turbulent, multiphase flows that may contain pathogen-bearing droplets of mucosal vary fluid (3).

Influenza (Flu) and COVID-19 are both contagious respiratory illnesses (2). COVID-19 transmits easily between people, more easily than influenza but not as easily as measles, and are mainly spread by people who show symptoms (4). People will be easily infected by COVID-19 when they are in close contact and inhale small droplets produced by an infected person coughing, sneezing, or talking (5). These droplets can land in the mouths or noses of people who are in close proximity with one another (within about 6 feet) and possibly be inhaled into the lungs (3). Therefore, keeping a safe distance more than approximately 6 feet is important to avoid being infected by COVID-19. This essay is focusing on the calculation of that safe distance.

Smaller droplets that are able to stay suspended in the air for longer periods of time are more dangerous (6). The radii of the coronavirus vary from 60~140 nm (7), so the coronavirus with radius of 60 nm is the most dangerous type of coronavirus. It stays the longest in air and goes the deepest into human body, increasing its possibility to reach the lung and cause infection. Thus, when calculating the
safe distance, it is reasonable to use the calculation of the coronavirus whose radius is 60 nm as the final result.

In section 3 “Materials and Methods”, the safe distance of coronavirus is calculated based on known constants using two different methods. The first method combines equations and data from the “Visualization of sneeze ejecta: steps of fluid fragmentation leading to respiratory droplets” (6). The second method uses pure equations simulating the process theoretically. In section 4 “Discussion”, the solutions to prevent infections are provided, the potential error sources are analysed, and the literary discussion is included. In section 5 “Conclusion”, the discovery of this experiment is restated and declared, and the further improvements are made.

2. Materials and Methods
Considering the process of droplets spreading in the air as a motion that is similar to the horizontal projectile motion, we separate this motion into two directions, vertical and horizontal. In the vertical motion, the droplets are free falling (considering air resistance since the droplets have small radii). In the horizontal motion, the droplets are doing linear motion with constant acceleration. The following graph demonstrates the situation.

In order to calculate the maximum spreading distance of the droplets, we first need to calculate the time it takes the droplets to fall to the ground. After that, we use this time to calculate the safe distance, the maximum range the droplets can reach in horizontal direction.

Consider a man whose height is $h$. He sneezes and coughs, producing a spherical droplet whose radius is $R$. Suppose the drag force is $F$, the drag coefficient is $ζ$, the viscosity of the air is $η$, the gravitational acceleration is $g$, the density of the droplet is $ρ$ (assume it is the same density of water), and the mass of the droplet is $m = \frac{4}{3}πR^3ρ$. The following equations show the relationship between these constants:

$$F = ζv$$  \hspace{2cm} [1]

$$ζ = 6πηR$$  \hspace{2cm} [2]

$$g = 9.806 m/s^2$$  \hspace{2cm} [3]

$$ρ = 1 \cdot 10^{3} kg \cdot m^3$$  \hspace{2cm} [4]

$$R = 5.0 \cdot 10^{-6} m$$  \hspace{2cm} [5]
\[ \eta = 2.98 \cdot 10^{-3} Pa \cdot S \]  \[6\]

**Method 1: Combination of Theoretical Simulation and Data from Previous Paper.**  
First, we consider the vertical motion of the droplets. Suppose the droplets moves a distance of \( x \) in vertical direction, and it takes the droplets a time \( t \) to fall to the ground.  
Based on Newton’s Second Law,  
\[ mg - \zeta v = ma \]  \[7\]

By replacing velocity \( v \) and acceleration \( a \) with the differs coal relationship between displacement \( x \) and time \( t \), Equation 7 is equivalent to  
\[ mg - \zeta \left( \frac{dx}{dt} \right) = m \left( \frac{d^2x}{dt^2} \right) \]  \[8\]

The initial conditions are:  
\[ x_0 = 0 \ (x = 0 \text{ at } t = 0) \]  \[9\]
\[ \frac{dx}{dt} = v_0 = 0 \ (x = 0 \text{ at } t = 0) \]  \[10\]

Therefore, we can get the following equations of \( x_t \) and \( v_t \) (The calculation process is shown in the “Attachment” section):  
\[ x_t = \frac{mv \cdot ln[cosh(\sqrt{\frac{mg\zeta}{v^3}}) \ t]}{\zeta} = h \]  \[11\]
\[ v_t = \frac{dx}{dt} = \sqrt{\frac{mgv}{\zeta}} \ tanh(\sqrt{\frac{g\zeta}{mv}} t) \]  \[12\]

In which:  
\[ \cosh x = \frac{e^x + e^{-x}}{2} \]  \[13\]
\[ \tanh x = \frac{e^x - e^{-x}}{e^x + e^{-x}} \]  \[14\]

Therefore, we can get that the maximum value of the velocity:  
\[ v_{t_{\text{max}}} = \frac{mgv}{\zeta} \ t = \frac{2\rho g R}{9\eta} \]  \[15\]

Based on Equation 12, we need to solve the following equation to get the value of \( t \):  
\[ \frac{mv \cdot ln[cosh(\sqrt{\frac{mg\zeta}{v^3}}) \ t]}{\zeta} = h \]  \[16\]
Known that the radius of droplets $R = 5.0 \times 10^{-6} m$ (6), the viscosity of water $\eta = 2.98 \times 10^{-3} Pa \cdot s$ (70), the density of water $\rho = 1.0 \cdot 10^3 kg/m^3$ and the gravitational acceleration $g = 9.806 m/s^2$ (8). Therefore, it can be derived from Equation 11 that

$$v_{t_{\text{max}}} = \frac{2 \rho g R}{\eta} = \frac{2(1.0 \cdot 10^3 kg/m^3)(9.806 m/kg)(5.0 \cdot 10^{-6}m)}{9.298 \cdot 10^{-3} Pa \cdot s} = 3.66 m/s$$ \[17\]

The density of mucosalivary liquid is similar to the density of water, which is $\rho = 1.0 \cdot 10^3 kg/m^3$ (9). Therefore, its mass can be calculated:

$$m_{ms} = \rho_{ms} \cdot V_{ms} = \rho_{ms} \cdot \frac{4}{3} \pi r^3 = (1.0 \cdot 10^3 m/s) \cdot \frac{4}{3} \pi (5 \cdot 10^{-6})^3 m^3 = 5.23 \cdot 10^{-13} kg$$ \[18\]

Based on Equation 16, we can get the value of time $t$.

$$t = 2e^\frac{\chi h}{m v} \ln(e^{\sqrt{\frac{mg\chi}{v}}} + e^{-\sqrt{\frac{mg\chi}{v}}})$$ \[19\]

By plugging in the average height of human beings $h$ and the drag coefficient of air $\chi$ shown in Equation 20 and Equation 21, we can calculate the exact value of time $t$ based on Equation 19.

$$h = 1.71 m$$ \[20\]

$$\chi = 1.8 \cdot 10^{-5} Pa/s$$ \[21\]

$$t = 1.308 s$$ \[22\]

We can then consider about the horizontal motion using the result above. According to the estimation of the data from *Visualization of sneeze ejecta: steps of fluid fragmentation leading to respiratory droplets* by B. E. Scharman, A. H. Techet, J. W. M. Bush, and L. Bourouiba (will be discussed in Section IV) (1), we get:

![Figure 2. Picture from “Drop Sneeze MIT”](image-url)
Figure 3. Picture from “Drop Sneeze MIT”

The following table shows the analysis to the data shown in this graph: the horizontal distances of the particle with different time $t$, from which we get the acceleration $a$ of the particle.

Table 1. The data estimated according to “Drop Sneeze MIT”

|   | D / T                  | 1   | 2   | 3    | 4    | 5    | 6    |
|---|------------------------|-----|-----|------|------|------|------|
|   | Distance measured in picture | 0.5cm | 0.8cm | 1.5cm | 3cm  | 10.5cm |
|   | Distance in reality calculated | 3.35cm | 5.34cm | 10.05cm | 20.01cm | 70cm |
|   | Time                   | 0.005s | 0.008s | 0.015s | 0.032s | 0.150s |
|   | Distance measured in picture | 1cm | 2cm | 3.7cm | 6.4cm | 8.9cm | 10.1cm |
| Picture 2 | Distance in reality calculated | 6.93cm | 13.86cm | 25.64cm | 44.35cm | 61.68cm | 70cm |
|   | Time                   | 0.007s | 0.030s | 0.107s | 0.162s | 0.251s | 0.340s |

Therefore, we can get the following data of the relationship between the velocity and the distance of the droplets created by sneezing and coughing ejecta travels horizontally.

\[
\therefore x_r = v_0t - \frac{1}{2}at^2
\] [23]
∴ \( x_{h1} = v_0 t_1 - \frac{1}{2} a t_1^2 \), \( x_{h2} = v_0 t_2 - \frac{1}{2} a t_2^2 \) [24]

∴ \( a = \frac{2(t_2 x_{h2} - t_1 x_{h1})}{t_1 t_2 (t_1 + t_2)} \) [25]

The result is shown in the following table.

**Table 2.** The change of distance, the change of time, and the acceleration from “Drop Sneeze MIT”

| D / T       | 1  | 2  | 3  | 4  | 5  |
|-------------|----|----|----|----|----|
| Picture 1   |    |    |    |    |    |
| \( \Delta D \) | 1.99cm | 4.71cm | 9.96cm | 40.99cm |
| \( \Delta T \) | 0.003s | 0.007s | 0.017s | 0.118s |
| Acceleration|    |    |    |    |    |
| \( \Delta D \) | 19.05 \( \frac{m}{s^2} \) | 72.48 \( \frac{m}{s^2} \) | 35.33 \( \frac{m}{s^2} \) |
| Picture 2   |    |    |    |    |    |
| \( \Delta T \) | 0.023s | 0.077s | 0.055s | 0.089s | 0.089s |
| Acceleration|    |    |    |    |    |
| \( \Delta D \) | 29.66 \( \frac{m}{s^2} \) | 28.36 \( \frac{m}{s^2} \) | 20.20 \( \frac{m}{s^2} \) | 11.37 \( \frac{m}{s^2} \) |

Therefore, we can get the average acceleration \( a \) and the initial velocity \( v_0 \) in the horizontal direction is:

\[
a = 32.345 \frac{m}{s^2} \quad [26]
\]
\[
v_0 = 6.795 \frac{m}{s} \quad [27]
\]

Therefore, we can calculate the distance in horizontal direction \( x_h \):

\[
x_h = v_0 t + \frac{1}{2} a t^2 \quad [28]
\]

Finally, we get that the value of the safe distance \( x_h \):

\[
x_h = (6.795 \frac{m}{s}) \times (1.308s) + \frac{1}{2} \left(32.345 \frac{m}{s^2}\right) \times (1.308s)^2 = 36.557m \quad [29]
\]

**Method 2: Pure Theoretical Simulation**

Considering the condition of the pathogen-containing particle as soon as it leaves the naval path of the infected patient, we can get the following analyzation:
At time $t = 0$:

$v = v_0 = (v_{x0}, v_{y0})$

$$\frac{d}{dt} mv = F = (F_x, F_y)$$ [31]

The force on the x-direction acting on the particle is the drag force, which has a value of:

$$F_D = 6\pi \eta R v_x$$ [32]

The forces on the y-direction acting on the particle are drag force and gravitational force, so the net force acting on the y-direction is:

$$F_y = mg - 6\pi \eta R v_x$$ [33]

Therefore, on the x-direction, combining Equation 31 and Equation 32, we get that:

$$\frac{d}{dt} mv_x = 6\pi \eta R v_x$$ [34]

Known that:

$$m = \frac{4\pi}{3} R^3 \Delta \rho$$ [35]

In which

$$\Delta \rho = \rho_{\text{liquid}} - \rho_{\text{air}}$$ [36]

Combining Equation 35 and Equation 36 to solve Equation 34, we get:

$$\frac{dv_x}{dt} = -\frac{9\eta v_x}{2RA\Delta \rho}$$ [37]

Let
Then based on Equation 37, we get
\[
\frac{dv_x}{dt} = -\beta v_x
\]  

[39]

Since we know that \( v_x(t = 0) = v_{x0} \), we can get:
\[
v_x = v_{x0} e^{-\beta t}
\]  

[40]

On the other hand, for the y-direction, combining Equation 31 and Equation 33, we can get:
\[
\frac{dv_y}{dt} m v_y = mg - 6\pi \eta R v_x
\]  

[41]

Applying the \( \beta \) notation to Equation 41,
\[
\frac{dv_y}{dt} = g - \frac{6\pi \eta R v_y - g - \beta v_y}{\frac{2\pi^2 R^3}{\eta}} = g - \beta v_y
\]  

[42]

Since we know that \( v_y(t = 0) = v_{y0} \), we can get:
\[
v_y e^{\beta t} = g \int_0^t e^{\beta t'} dt' + v_{y0} \rightarrow v_y = \frac{g}{\beta} (1 - e^{-\beta t}) + V_{y0} e^{-\beta t}
\]  

[43]

Then, we start investigating the displacement on both the x-direction and y-direction based on the velocity on them derived from Equation 40 and Equation 43.

Since it is also known that
\[
v_x = v_{x0} e^{-\beta t} = \frac{dx}{dt}
\]  

[44]

\( x(t = 0) = 0 \)

[45]

we can get that the relationship between the horizontal displacement \( x \) and time \( t \) by integration:
\[
x(t) = -\frac{v_{x0}}{\beta} e^{-\beta t} t |_0^{-1} = -\frac{2\pi^2 R^3}{\eta} \frac{\rho}{\beta} v_{x0} (1 - e^{-\beta t})
\]  

[46]

Similarly, since
\[
\frac{dy}{dt} = \frac{g}{\beta} (1 - e^{-\beta t}) + v_{y0} e^{-\beta t}
\]  

[47]

\( y(t = 0) = h \)

[48]

we can get the displacement on y-direction by integration:
\[
y = \frac{gt}{\beta} + \frac{g}{\beta^2} e^{-\beta t} t |_0^{-1} + \frac{v_{y0}}{\beta} (1 - e^{-\beta t}) + h = \frac{gt}{\beta} + \frac{g}{\beta^2} (e^{-\beta t} - 1) + \frac{v_{y0}}{\beta} (1 - e^{-\beta t}) + h
\]  

[49]
Then we express $y$ in terms of $x$ based on Equation 46 and Equation 49, deriving:

$$1 - e^{-\beta t} = \frac{\beta}{v_0} x; e^{-\beta t} = 1 - \frac{\beta}{v_0} x$$  \[50\]

$$-\beta t = \ln(1 - \frac{\beta}{v_0} x); t = \frac{1}{\beta} (\frac{1}{1 - \frac{\beta}{v_0} x})$$  \[51\]

Therefore,

$$y = \frac{g}{\beta^2} \ln\left(\frac{1}{1 - \frac{\beta}{v_0} x}\right) - \frac{g}{\beta^2} \frac{\beta x}{v_0} + \frac{v_y}{\beta} \frac{\beta}{v_0} x + h$$  \[52\]

$$y = \frac{g}{\beta^2} \ln\left(\frac{1}{1 - \frac{\beta}{v_0} x}\right) - \frac{g}{\beta v_0} x + \frac{v_y}{v_0} x + h$$  \[53\]

Combine Equation 53 with Equation and the definition of $\beta$, we get the relationship between the vertical displacement $y$ and time $t$:

$$y(t) = \frac{8gR^3\Delta \rho^3}{31\eta^2 t} - \frac{4gR^2\Delta \rho^2}{31\eta^2} (1 - e^{\frac{9\eta}{2K2\rho} t}) + \frac{2KR^2\rho}{9\eta} (1 - e^{\frac{9\eta}{2K2\rho} t}) + h$$  \[54\]

It is known that $g = 9.8 \text{ m/s}^2$, $R = 5.0 \cdot 10^{-6} \text{ m}$, $\eta = 2.98 \cdot 10^{-3} \text{ Pa} \cdot \text{s}$, $\rho = 1 \cdot 10^3 \text{ kg/m}^3$, $h = 1.71 \text{ m}$. By bringing the constants into Equation 28 and Equation 34, we get the equation for the horizontal and vertical displacement relative to the initial velocity and the time taken:

$$x(t) = -0.37v_{x_0}(1 - e^{2.68t})$$  \[55\]

$$y(t) = \frac{0.014}{t} - (1.36 - 0.37v_{y_0})(1 - e^{2.68t}) + 1.71$$  \[56\]

By plugging in the appropriate value of $v_{x_0}$ and $v_{y_0}$ of different types of particles containing virus for the corresponding diseases, the safe distance of a certain pandemic is identifiable by using Equation 55.

3. Discussion

"Visualization of sneeze ejecta: steps of fluid fragmentation leading to respiratory droplets" indicates that coughs and sneezes feature turbulent, multiphase flows that may contain pathogen-bearing droplets of mucosal vary fluid contribute to the spread of numerous infectious diseases, including COVID-19 pandemic (5). Larger droplets spread in smaller range, while smaller droplets spread in larger range. Direct observation of the physical mechanisms of droplet formation at the exit of the mouth during sneezing is reported in the essay. The researchers use high-speed imaging to study the trajectory of the sneeze ejecta after it is split out from mouths. They reveal for the first time that the breakup of the fluid into droplets outside of the respiratory tract during violent exhalations is a continuous process (5). Based on the high-speed image they presented in their paper, the velocity of the sneeze ejecta is solvable, and therefore the safe distance can be indicated.

In this experiment, the main error sources are contributed by the estimation of the constant $h, R, \rho, k, \eta$, and the ignorance of the movement of the pathogen containing particle inside the nasal path of the infected patient. The second method gives a specific equation calculating the path of a particle with respect to air resistance and drag force. However, in the first method, we used the result of
“Visualization of sneeze ejecta: steps of fluid fragmentation leading to respiratory droplets” to estimate the initial condition of the particle. If there is a chance to directly get the exact motion of particle with calibrated measurement equipments.

The most obvious drawback of this research is that there is no any type of experimental data supporting the theoretical calculation mentioned above. However, since all of the process are statistically correct, the result of this study of the safe distance of COVID 19 is still reliable.

4. Conclusion
Through this experiment, we roughly estimated the safe distance of the coronavirus to be 36.557m. Also, we get the exact equations for the range of coronavirus relative to the time of the motion and the initial velocity, which is \( x(t) = -0.37v_x(1 - e^{2.68t}) \) and \( y(t) = \frac{0.014}{t} - (1.36 - 0.37v_y)(1 - e^{2.68t}) + 1.71 \). As for the further improvements, we could do experiment measuring all of the constants used in this experiment under the environment of the experiment being proceed to improve the accuracy. Also, we can redo the experiment proceeds in “Visualization of sneeze ejecta: steps of fluid fragmentation leading to respiratory droplets” to generate a precise description of the motion of the particle when it does not exit human body to calculate the calibrated initial velocity.

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