MARKOV AND ARTIN NORMAL FORM THEOREM FOR BRAID GROUPS

L. A. Bokut
Sobolev Institute of Mathematics, Novosibirsk, Russia

V. V. Chaynikov
Faculty of Mathematics and Mechanics, Novosibirsk State University, Novosibirsk, Russia

K. P. Shum
Faculty of Science, The Chinese University of Hong Kong, Hong Kong, China

In this article, we will present the results of Artin–Markov on braid groups by using the Gröbner–Shirshov basis. As a consequence, we can reobtain the normal form of Artin–Markov–Ivanovsky as an easy corollary.

Key Words: Braid group; Diamond lemma; Gröbner–Shirshov basis; Normal form; Rewriting system.

Mathematics Subject Classification: Primary 20F36; Secondary 16S15.

1. INTRODUCTION

The theory of braid groups was first studied by Artin, dated back to 1925 (see Artin, 1925). Artin established generators and defining relations of the braid group and gave a faithful representation theorem for the braid group as a subgroup of the automorphism group of a free group. Later on, Markov (1945) and Artin (1947) enriched the Artin generators by the Burau elements (Burau, 1932) and find a presentation of the braid group in the Artin–Burau generators. We call it the Artin–Markov presentation of the braid group in the Artin–Burau generators. The main result of Markov (1945) and Artin (1947) is the normal form theorem for the braid group. We call it the Artin–Markov–Ivanovsky normal form theorem for Markov credited the result to Ivanovsky; it seems that there is no article published by Ivanovskii himself. However, the proof of the normal form theorem of the braid group given by Markov and Artin are rather complicated because the automorphism of the free groups are involved. In this note, we will try to escape the technical details of the proof given by Markov (1945) and Artin (1947). We will show, by direct calculations (of compositions), that the Artin–Markov presentation of the braid
group is the minimal Gröbner–Shirshov basis of the braid group in the Artin–Burau generators and under an appropriate ordering of group words, namely, the inverse tower ordering. As a consequence, by using the Composition–Diamond Lemma, the Artin–Markov–Ivanovsky normal form in the braid group follows as an immediate corollary of our result.

2. BASIC NOTATIONS AND RESULTS

In this section, we give some basic notations and cite some useful results in the literature. We first let $X$ be a linearly ordered set and $k$ a field. Let $k\langle X \rangle$ the free associative algebra over $X$ and $k$. On the set $X^*$ of words generated by $X$, we impose a well order $\leq$ which is compatible with the multiplication of words. We call this kind of order a monomial order.

Now, let $f \in k\langle X \rangle$ be a polynomial with leading word $\bar{f}$. We say that $f$ is monic if $\bar{f}$ occurs in $f$ with coefficient 1. We now formulate the following definitions.

**Definition 2.1.** Let $f$ and $g$ be two monic polynomials.

(i) Let $w$ be a word such that $w = \bar{f}b = a\bar{g}$, with $\circ(\bar{f}) + \circ(\bar{g}) > \circ(w)$. Then we call the polynomial $(f, g)_w$ the *intersection composition* of $f$ and $g$ with respect to $w$ if $(f, g)_w = fb - ag$.

(ii) Let $w = \bar{f} = a\bar{g}b$. Then we call the polynomial $(f, g)_w = f - agb$ the *inclusion composition* of $f$ and $g$ with respect to $w$. In the above case, the transformation $f \mapsto (f, g)_w = f - agb$ is called the *elimination of the leading word* (ELW) of $g$ in $f$.

(iii) Let $S \subset k\langle X \rangle$. We call a composition $(f, g)_w$ *trivial* relative to $S$ (and $w$) if

$$(f, g)_w = \sum \alpha_i a_i t_i b_i,$$

where $t_i \in S$, $a_i, b_i \in X^*$, and $a_i \bar{t}_i b_i < w$. In a notation, $(f, g)_w \equiv 0 \mod (S, w)$. In particular, if $(f, g)_w$ goes to zero by using ELW of polynomials from $S$, then $(f, g)_w$ is trivial relative to $S$. We assume that $f_1$ and $f_2$ are some polynomials satisfying the condition $f_1 \equiv f_2 \mod (S, w)$ if $f_1 - f_2 \equiv 0 \mod (S, w)$.

**Definition 2.2.** Then we call $S$ a *Gröbner–Shirshov set* (basis) in $k\langle X \rangle$ if any composition of polynomials from $S$ is trivial relative to $S$. From now on, we denote the algebra with set of generators $X$ and set of defining relations $S$ by $\langle X | S \rangle$.

The following lemma and its applications to Gröbner–Shirshov bases was due to Newman (1942), Shirshov (1962/1999), Buchberger (1965, 1970) and Bergman (1978). The lemma given below was formulated by (see Bokut, 1972, 1976).

**Lemma 2.3** (Composition–Diamond Lemma). *Let $R = \langle X | S \rangle$. The set of defining relations $S$ is a Gröbner–Shirshov set if and only if the set*

$$\text{Irr}(S) = \{u \in X^* | u \neq \bar{a}fb, \text{ for any } f \in S\}$$

*of $S$-irreducible words consists of a linear basis of $R$.**
Definition 2.4. Let $S$ be a Gröbner–Shirshov basis in $k\langle X \rangle$. Then $S$ is called a minimal Gröbner–Shirshov basis if for any $s \in S$, $s$ is a linear combination of $S \setminus \{s\}$-irreducible words. Any ideal of $k\langle X \rangle$ has a unique minimal Gröbner–Shirshov basis (i.e., a set of generators of the ideal).

Definition 2.5. A polynomial $f$ is called a semigroup relation if $f$ is of the form $u - v$, where $u, v \in X^*$. If $S$ is a set of semigroup relations, then any nontrivial composition of $S$ has the same form. Consequently, the set $S^\text{comp}$ also consists of semigroup relations.

Remark 2.6. Let $A = \text{sgp}\langle X | S \rangle$ be a semigroup presentation. Then $S \subset k\langle S \rangle$ and we can find a Gröbner–Shirshov basis $S^\text{comp}$. This set does not depend on $k$ and it consists of semigroup relations. In this case, we use $S^\text{comp}$ to denote a Gröbner–Shirshov basis of $A$. It is the same as a Gröbner–Shirshov basis of the semigroup algebra $kA = \langle X | S \rangle$.

We now introduce the concept of inverse tower ordering of words.

Definition 2.7. Let $X = Y \cup Z$, words $Y^*$ and the letters $Z$ are well ordered. Suppose that the order on $Y^*$ is monomial. Then, any word in $X$ has the form $u = u_0z_1 \ldots u_{k-1}z_ku_k$, where $k \geq 0$, $z_i \in Z$, $u_i \in Y^*$. Define the inverse weight of the word $u \in X$ by

$$\text{inwt}(u) = (k, u_k, z_k, \ldots, z_1, u_0).$$

Now we order the inverse weights lexicographically and define

$$u > v \iff \text{inwt}(u) > \text{inwt}(v).$$

Then we call the above order the inverse tower order. Clearly, the above order is a monomial order.

In case if $Y = T \cup U$ and $Y^*$ are endowed with the inverse tower order, we call the order of words on $X$ the inverse tower order of word relative to the presentation

$$X = (T \cup U) \cup Z.$$

In general, we can define the inverse tower order of $X$-words relative to the presentation

$$X = \left( \ldots \left( X^{(n)} \cup X^{(n-1)} \right) \cup \ldots \right) \cup X^{(0)},$$

where $X^{(n)}$-words are endowed by a monomial order.

Definition 2.8. Let $\Sigma = \{\sigma_1, \ldots, \sigma_{n-1}\}$ be a finite alphabet. Then, the following group presentation defines the $n$-strand braid group:

$$B_n = \langle \Sigma | \sigma_i \sigma_{i+1} \sigma_i = \sigma_{i+1} \sigma_i \sigma_i, \sigma_i \sigma_j = \sigma_j \sigma_i, i - j > 1 \rangle.$$  

Here any index falls into the interval $[1, n - 1]$. 
In the braid group $B_n$, we now introduce a new set of generators. We call them the Artin–Burau generators.

In the braid group $B_n$, we let

$$s_{i,i+1} = \sigma_i^2, \quad s_{i,i+1} = \sigma_j \ldots \sigma_{i+1} \sigma_i^{-1} \sigma_{i+1} \ldots \sigma_j^{-1},$$

where $1 \leq i < j \leq n - 1$.

Form the set

$$S_j = \{s_{i,j}, s_{i,j}^{-1}, 2 < i < j < n\}$$

and

$$\Sigma^{-1} = \{\sigma_1^{-1}, \ldots, \sigma_{n-1}^{-1}\}.$$

Then the set

$$S = S_n \cup S_{n-1} \cup \cdots \cup S_2 \cup \Sigma^{-1}$$

generates $B_n$ as a semigroup. We call the elements of $S$ the Artin–Burau generators of $B_n$. Observe that generators $\sigma_i$ are omitted as well as the trivial group relations on them. With the above notation, Markov (1945) used $s_{i,i+1} \sigma_i^{-1}$ to replace $\sigma_i$, and used $\sigma_i^{-2} = s_{i,i+1}$ to replace $\sigma_i^{-1} \sigma_i = 1$.

Now, we order the set $S$ in the following way:

$$S_n < S_{n-1} < \cdots < S_2 < \Sigma^{-1},$$

Clearly, in the above chain, any letter of $S_n$ is less than any letter of $S_{n-1}$ and so on. Also we define for each $j$

$$s_{1,j}^{-1} < s_{1,j} < s_{2,j}^{-1} < \cdots < s_{j-1,j}, \quad \text{and} \quad \sigma_i^{-1} < \sigma_2^{-1} < \cdots < \sigma_{n-1}^{-1}.$$

With the above notation, we are now able to order the $S$-words by using the inverse tower order, according to the fixed presentation of $S$ as the union of $S_j$ and $\Sigma^{-1}$. We order the $S_n$-words by the $\text{deg–inlex}$ order, i.e., we first compare the words by length and then by inverse lexicographical order, starting from their last letters.

The following abbreviations are taken from Markov (1945):

$$\sigma_{i,j+1} = \sigma_i^{-1} \ldots \sigma_j^{-1}, \quad 1 \leq i \leq j \leq n - 1, \quad \sigma_n = 1.$$

Also we denote $\{a, b\} = b^{-1}ab$.

### 3. MAIN RESULTS

We first cite some crucial results from Markov (1945) and Artin (1947). The first lemma is fairly easy.
Lemma 3.1 (Artin, 1947, p. 119; Markov, 1945, Lemma 3). The following relations hold in the braid group $B_n$ for $(\delta = \pm 1)$:

\[ \sigma_k^{-1} s_{i,j}^\delta = s_{i,j}^\delta \sigma_k^{-1}, \quad k \neq i - 1, i, j - 1, j; \]  
\[ \sigma_i^{-1} s_{i+1,i}^\delta = s_{i+1,i}^\delta \sigma_i^{-1}; \]  
\[ \sigma_{i-1}^{-1} s_{i,j}^\delta = s_{i,j}^\delta \sigma_{i-1}^{-1}; \]  
\[ \sigma_i^{-1} s_{i,j}^\delta = \{s_{i+1,j}, s_{i+1}^\delta \} \sigma_i^{-1}; \]  
\[ \sigma_{j-1}^{-1} s_{i,j}^\delta = s_{i,j-1}^\delta \sigma_{j-1}^{-1}; \]  
\[ \sigma_j^{-1} s_{i,j}^\delta = \{s_{i,j+1}, s_{j+1}^\delta \} \sigma_j^{-1}. \]

Also we have

\[ \sigma_{i-1} s_{i,j}^\delta = \{s_{i-1,j}, s_{i-1,j}^{-1} \} \sigma_{i-1}; \]  
\[ \sigma_i s_{i,j}^\delta = s_{i+1,j}^\delta \sigma_i; \]  
\[ \sigma_{j-1} s_{i,j}^\delta = \{s_{i,j-1}, s_{j-1,j}^{-1} \} \sigma_{j-1}; \]  
\[ \sigma_j s_{i,j}^\delta = s_{i,j+1}^\delta \sigma_j. \]

Remark 3.2. We note that the last relations from the above lemma will not be in Gröbner–Shirshov basis of the braid group $B_n$ (see below). We shall use them for sketch of a proof of Lemma 3.3.

Lemma 3.3 (Artin, 1947, Theorem 18; Markov, 1945, Lemmas 6 and 7). The following relations hold in the braid group $B_n$ for all $i < j < k < l$, $\varepsilon = \pm 1$:

\[ s_{j,k}^{-1} s_{j,l}^\varepsilon = \{s_{j,l}^\varepsilon, s_{j,l}^{-1}\} s_{j,k}^{-1}; \]  
\[ s_{j,k} s_{j,l}^\varepsilon = \{s_{j,l}^\varepsilon, s_{j,l} s_{j,k}\} s_{j,k}; \]  
\[ s_{j,k}^{-1} s_{j,l}^\varepsilon = \{s_{j,l}^\varepsilon, s_{j,l}^{-1}s_{j,l}^{-1}\} s_{j,k}^{-1}; \]  
\[ s_{j,k} s_{j,l}^\varepsilon = \{s_{j,l}^\varepsilon, s_{j,l} s_{j,k}\} s_{j,k}; \]  
\[ s_{i,k}^{-1} s_{i,l}^\varepsilon = \{s_{i,l}^\varepsilon, s_{i,l} s_{i,k}^{-1}s_{i,l}^{-1}\} s_{i,k}^{-1}; \]  
\[ s_{i,k} s_{i,l}^\varepsilon = \{s_{i,l}^\varepsilon, s_{i,l}^{-1}s_{i,l}s_{i,k}\} s_{i,k}. \]

Also for $j < i < k < l$ or $i < k < j < l$, and $\varepsilon, \delta = \pm 1$

\[ s_{i,k}^\delta s_{i,l}^\varepsilon = s_{i,l}^\varepsilon s_{i,k}^\delta. \]

**Proof.** We provide here a proof of (3.7) for $\varepsilon = 1$ as a typical example. We use arguments given by Markov (1945). At first, we can easily see that the relation holds for $j = k - 1, l = k + 1$. We assume that (3.7) holds for $j < k < l$. Then we prove for
By using Lemma 3.1. We deduce the following equalities by direct computation:

\[
\begin{align*}
    s_{j,k}^{-1}s_{k,l+1} &= s_{j,k}^{-1}\{s_{k,l}, \sigma_i^{-1}\} = \{(s_{k,l}, s_{j,k}^{-1})\}, \sigma_i^{-1}s_{l,k}^{-1} = \{s_{k,l+1}, s_{j,l+1}^{-1}\} s_{j,k}^{-1}, \\
    s_{j-1,k}^{-1}s_{j,l}^{-1} &= s_{j,k}^{-1}, s_{j-1} = \sigma_{j-1,j}^{-1}s_{j,k}^{-1}\sigma_{j-1,j} = \sigma_{j-1,j}^{-1}s_{j,k}^{-1}\sigma_{j-1,j} = \{(s_{k,l}, s_{j,l}^{-1})\} s_{j,k}^{-1}, \sigma_{j-1,j} = \{s_{k,l}, s_{j-1,l}^{-1}\} s_{j-1,k}^{-1}.
\end{align*}
\]

This shows that (3.7) holds.

Finally, we have the following relations in the braid group \(B_n\) (see Markov, 1945, Lemma 5). A proof is fairly simple.

**Lemma 3.4.** The following relations hold in the braid group \(B_n\):

\[
\begin{align*}
    \sigma_j^{-1}\sigma_k^{-1} &= \sigma_k^{-1}\sigma_j^{-1}, & j < k - 1, \\
    \sigma_{j,j+1}\sigma_{k,j+1} &= \sigma_{k,j+1}\sigma_{j,j+1}, & k < j, \\
    \sigma_i^{-2} &= s_i^{-1}.
\end{align*}
\]

We now call the relations (3.1)–(3.16) together with the trivial relations

\[
    s_{i,j}^\pm s_{i,j}^{-1} = 1
\]

the Artin–Markov’s relations \(S\) for the braid group \(B_n\), in terms of Artin–Burau generators.

Using the above relations \(S\) together with the definition \(\sigma_i = s_{i+1}\sigma_i^{-1}\), we can deduce Artin’s relations for \(B_n\).

Namely, in relation (3.15), we can let \(k = j - 1\). Then we have

\[
    \sigma_j^{-1}\sigma_{j-1}^{-1}\sigma_j^{-1} = \sigma_{j-1,j}^{-1}\sigma_j^{-1}\sigma_{j-1}^{-1}.
\]

Also, we have

\[
    \sigma_i^{-1}\sigma_i = \sigma_i^{-1}s_{i,i+1}\sigma_i^{-1} = s_{i,i+1}\sigma_i^{-2} = s_{i,i+1}s_{i,i+1}^{-1} = 1,
\]

and the same for \(\sigma_i^2 = 1\).

**Corollary 3.5.** The following relations can be deduced by using the ELW of \(S\) (to be more precise, by the ELW of (3.7)–(3.16)):

\[
\begin{align*}
    \sigma_{i,j}\sigma_{k,j} &= \sigma_{k,j}\sigma_{i-1,j-1}, & k < i, \\
    \sigma_{i,j}\sigma_{k,j} &= s_{i,k+1}\sigma_{k+j,1}\sigma_{i,j-1}, & i \leq k.
\end{align*}
\]

4. MAIN THEOREM

Using the Artin–Markov relations given in the Section 3, we establish the following theorem.
**Theorem 4.1.** The Artin–Markov relations form a minimal Gröbner–Shirshov basis of the braid group $B_n$ in term of the Artin–Burau generators relative to the inverse tower order of words.

**Proof.** There are no inclusion compositions of defining relations. We only need to check all possible intersection compositions. Let us do some for examples.

Let us check a composition of intersection of two relations $f, g$ of the form (3.8) relative to the ambiguity

$$w = (ij)(jk)(kl), \quad i < j < k < l,$$

where $(ij) = s_{i,j}$. We have

$$f = (ij)(jk) - ((jk), (ik)(jk))((ij), \quad g = (jk)(kl) - ((kl), (jl)(kl))(jk).$$

We need to prove that

$$(jk), (ik)(jk))((ij)(kl) \equiv (ij)((kl), (jl)(kl))(jk) \mod(S, w). \quad (4.1)$$

In fact, by computation, we deduce that

$$(ij)((kl), (jl)(kl))(jk) \equiv ((kl), (il)(jl)(kl))(ij)(jk) \equiv ((kl), (il)(jl)(kl))(jk), (ik)(jk))((ij)). \quad (4.2)$$

For the left-hand side of (4.1), we have

$$(jk), (ik)(jk))((ij)(kl) \equiv ((jk), (ik)(jk))(k)(ij) \equiv (jk)^{-1}(ik)^{-1}(jk)(ik)(jk)(ij) \equiv (jk)^{-1}(ik)^{-1}(jk)((kl), (jl)(kl))(jk)(ij) \equiv (jk)^{-1}(ik)^{-1}(jk)\{(kl), (il)(kl)\}(ik)(jk)(ij) \equiv (jk)^{-1}(ik)^{-1}(jk)\{(kl), (il)(kl-1)\}(il)(kl)(ik)(jk)(ij) \equiv (X = \{(jl), (kl)(il)(kl^{-1})\}((kl), (il^{-1})\} = ((kl), (il^{-1})\}((jl), (kl))) \times (jk)^{-1}\{((kl), (il^{-1})\}, (jl)(kl))\}(ik^{-1}(jk)(ik)(jk)(ij) \equiv \{((kl), (il^{-1})\}, (jl)(kl))\}(ik^{-1}(jk)(ik)(jk)(ij) \times (il)(jl)(kl^{-1})\}((kl), (jl^{-1})\}((jl^{-1})\}((kl), (jl^{-1})\}((jl^{-1})\}\}((kl), (jl^{-1})\}\}((jl^{-1})\}\}((jl^{-1})\}\}((kl), (jl^{-1})\}\}((jl^{-1})\}\}((kl), (jl^{-1})\}\}((jl^{-1})\}\}((kl), (jl^{-1})\}\}((jl^{-1})\}\}((kl), (jl^{-1})\}\}((jl^{-1})\}\}((kl), (jl^{-1})\}\}((jl^{-1})\}\}((kl), (jl^{-1})\}\}((jl^{-1})\}\}((kl), (jl^{-1})\}\}((jl^{-1})\}\}((kl), (jl^{-1})\}\}((jl^{-1})\}\}((kl), (jl^{-1})\}\}((jl^{-1})\}\}((kl), (jl^{-1})\}\}((jl^{-1})\}\}((kl), (jl^{-1})\}\}((jl^{-1})\}\}((kl), (jl^{-1})\}\}((jl^{-1})\}\}((kl), (jl^{-1})\}\}((jl^{-1})\}\}((kl), (jl^{-1})\}\}((jl^{-1})\}\}((kl), (jl^{-1})\}\}((jl^{-1})\}\}((kl), (jl^{-1})\}\}((jl^{-1})\}\}((kl), (jl^{-1})\}\}((jl^{-1})\}\}((kl), (jl^{-1})\}\) \equiv \{((kl), (jl^{-1})\}, (il)(jl)(kl))((jk), (ik)(jk))((ij)). \quad (4.3)
Words (4.2) and (4.3) are the same for
\[
\{ (kl), (jl), (il)(jl) \} (kl) = \{ (kl), (jl)^{-1} (il)^{-1}, (jl), (il)(jl)(kl) \}.
\]
Thus, (4.1) is verified and the composition is checked.
To check the compositions of relations \( f, g \) of the forms (3.12), (3.2), respectively, we let \( w = \sigma^{-1}_q (jk)(kl) \) and let
\[
f = \sigma^{-1}_q s_{jk} - s_{jk} \sigma^{-1}_q, \quad q \neq j - 1, j, k - 1, k,
g = (jk)(kl) - \{ (kl), (jl)(kl) \} (jk).
\]
Then we have
\[
(f, g)_w = \sigma^{-1}_q (kl), (jl)(kl) \} (jk) - (jk) \sigma^{-1}_q (kl).
\]
If \( q \neq l - 1, l \) then it is clear that the composition is trivial. So, we need to consider the following cases:

a) \( q = l \). We have
\[
\sigma^{-1}_l \{ (kl), (jl) (kl) \} (jk) = \{ (k, l + 1), (l, l + 1) \},
\]
\[
\times \{ (k, l + 1), (l, l + 1) \} (jk) \sigma^{-1}_l
\]
\[
\equiv \{ (k, l + 1), (j + 1), (k, l + 1), (l, l + 1) \} (jk) \sigma^{-1}_l,
\]
\[
(jk) \sigma^{-1}_l (kl) \equiv (jk) \{ (k, l + 1), (l, l + 1) \} \sigma^{-1}_l
\]
\[
\equiv \{ (k, l + 1), (j, l + 1), (k, l + 1), (l, l + 1) \} (jk) \sigma^{-1}_l.
\]
Hence, the case is verified.

b) \( q = l - 1 \). We have
\[
\sigma^{-1}_{l-1} \{ (kl), (jl) (kl) \} (jk) = \{ (k, l - 1), (j, l - 1) \} (jk) \sigma^{-1}_{l-1},
\]
\[
(jk) \sigma^{-1}_{l-1} (kl) \equiv (jk) (kl) \sigma^{-1}_{l-1} \equiv \{ (k, l - 1), (j, l - 1) \} (jk) \sigma^{-1}_{l-1}.
\]
Hence, the case is also verified.

Finally, we need to check the composition of relations (3.16). We first let
\[
f = \sigma^{-1}_j \sigma^{-1}_k \ldots \sigma^{-1}_j - \sigma^{-1}_k \ldots \sigma^{-1}_j \sigma^{-1}_{j-1}, \quad k < j,
g = \sigma^{-1}_j \sigma^{-1}_i \ldots \sigma^{-1}_j - \sigma^{-1}_i \ldots \sigma^{-1}_j \sigma^{-1}_{j-1}, \quad l < j,
\]
and let \( w = \sigma^{-1}_j \sigma^{-1}_k \ldots \sigma^{-1}_j \sigma^{-1}_i \ldots \sigma^{-1}_j \).

Then, we have
\[
(f, g)_w = -\sigma^{-1}_k \ldots \sigma^{-1}_j \sigma^{-1}_j \sigma^{-1}_i \ldots \sigma^{-1}_j + \sigma^{-1}_j \sigma^{-1}_k \ldots \sigma^{-1}_j \sigma^{-1}_j \sigma^{-1}_i \ldots \sigma^{-1}_j \sigma^{-1}_{j-1}.
\]
We consider the following cases:

a) \( l = j - 1 \). In this case, by Corollary 3.5, we have

\[
(f, g)_w = -\sigma_k^{-1} \ldots \sigma_j^{-1} s_{j-l} \sigma_j^{-1} + \sigma_j^{-1} \sigma_k^{-1} \ldots \sigma_{j-2} s_{j-l} \sigma_{j-1};
\]

\[
\sigma_k^{-1} \ldots \sigma_j^{-1} s_{j-l} \sigma_j^{-1} \equiv s_{j-l} \sigma_k^{-1} \ldots \sigma_j^{-1} s_{j-l} \sigma_j^{-1} \equiv s_{j-l} \sigma_k^{-1} \ldots \sigma_{j-2} s_{j-l} \sigma_{j-1};
\]

\[
\sigma_j^{-1} \sigma_k^{-1} \ldots \sigma_{j-2} s_{j-l} \sigma_j^{-1} \sigma_{j-1}^{-1} \equiv \sigma_j^{-1} s_{j-l} \sigma_k^{-1} \ldots \sigma_{j-2} s_{j-l} \sigma_{j-1}^{-1} \equiv \{s_{j-l}^{-1}, s_{j-l+1} \} s_{j-l} \sigma_k^{-1} \ldots \sigma_{j-1}^{-1} \equiv s_{j-l}^{-1} \sigma_{l+j+1} \sigma_k^{-1} \ldots \sigma_{j-2}^{-1},
\]

and the case is done.

b) \( l < j - 1 \). We use again Corollary 3.5 to obtain

\[
\sigma_k^{-1} \ldots \sigma_j^{-1} \sigma_l^{-1} \ldots \sigma_j^{-1} \sigma_{j-2}^{-1} \equiv \sigma_k^{-1} \ldots \sigma_j^{-1} \sigma_l^{-1} \ldots \sigma_j^{-1} \sigma_{j-2}^{-1} \equiv \sigma_{l+j+1} \sigma_{l+j} \sigma_{l+j+1} \sigma_{j-2}^{-1} \equiv \sigma_{l+j+1} \sigma_{l+j+1} \sigma_k^{-1} \sigma_{j-2}^{-1}
\]

If \( l < k \), then

\[
\sigma_j^{-1} \sigma_k^{-1} \ldots \sigma_{j-1}^{-1} \sigma_l^{-1} \ldots \sigma_j^{-1} \sigma_{j-1}^{-1} \equiv \sigma_j^{-1} \sigma_k^{-1} \sigma_l^{-1} \sigma_j^{-1} \sigma_{j-1}^{-1} \equiv \sigma_j^{-1} \sigma_l^{-1} \sigma_k^{-1} \sigma_{j-1}^{-1} \sigma_{j-1}^{-1} \equiv \sigma_j^{-1} \sigma_{l+j+1} \sigma_{k-1} \sigma_{j-1}^{-1} \sigma_{j-1}^{-1} \equiv \sigma_j^{-1} \sigma_{l+j+1} \sigma_{k-1} \sigma_{j-1}^{-1} \sigma_{j-1}^{-1}
\]

and we are done. If \( l \geq k \), then \( k \leq l < j - 1 \) and so we have

\[
\sigma_j^{-1} \sigma_k^{-1} \ldots \sigma_{j-1}^{-1} \sigma_l^{-1} \ldots \sigma_j^{-1} \sigma_{j-1}^{-1} \equiv \sigma_j^{-1} s_{k-l+1} \sigma_{l+j+1} \sigma_{k-1} \sigma_{j-1}^{-1} \sigma_{j-1}^{-1} \equiv \sigma_j^{-1} s_{k-l+1} \sigma_{l+j+1} \sigma_{k-1} \sigma_{j-1}^{-1} \sigma_{j-1}^{-1}
\]

and

\[
\sigma_j^{-1} s_{k-l+1} \sigma_{l+j+1} \sigma_k^{-1} \equiv s_{k-l+1} \sigma_{l+j+1} \sigma_k^{-1}
\]

as desired. \( \square \)

Applying the Composition-Diamond Lemma, we obtain the following corollary.

**Corollary 4.2.** The set of \( S \)-irreducible words of \( B_n \) corresponding to the above Gröbner–Shirshov basis \( S \) consists of the words

\[
f_n f_{n-1} \ldots f_2 \sigma_{i_2} \sigma_{i_1} \sigma_{i_2} \sigma_{i_1} \ldots \sigma_{i_2}, \quad (4.4)
\]

where \( f_j \) are free irreducible words in \( \{s_{ij} \mid i < j\} \), \( 2 \leq j \leq n \).
Corollary 4.3 (Artin-Markov-Ivanovskii, 1947, Theorem 17 and Remark of Theorem 18; Markov, 1945, Theorem 6). Every word of $B_n$ has a unique presentation in the form (4.4).

Let $\Sigma_n$ be the symmetric group, i.e.,

$$\Sigma_n = \langle s_1, \ldots, s_{n-1} \mid s_i^2 = 1, s_i s_{i-1} s_i = s_i s_{i+1} s_i, s_i s_j = s_j s_i, i-j > 1 \rangle,$$

and let

$$S_{i,j} = 1 \quad \text{and} \quad S_{i,j+1} = s_i s_{i+1} \ldots s_j, \quad i < j.$$

The following lemma was proved by Markov (1945, Theorem 4, Corollary 6). It also follows from the fact that

$$\{s_i^2 = 1, \ s_i s_j = s_j s_i, i-j > 1, S_{j,j+1} S_{k,j+1} = S_k S_{j,k}, k < j\}$$

is a Gröbner–Shirshov basis of $\Sigma_n$ under the deg-inlex order of words in $\{s_i\}$ (see Bokut and Shiao, 2001).

Lemma 4.4. Every element of $\Sigma_n$ has a unique presentation in a form

$$S_{i,n} S_{i,n-1} \ldots S_{i,2},$$

where $i,j \leq j$ and $2 \leq j \leq n$.

Let $P_n$ be the group of pure braids. This is the kernel of the natural homomorphism of $B_n$ onto $\Sigma_n$. From Theorem 4.1, Corollary 4.3, and Lemma 4.4, the following corollary results.

Corollary 4.5 (Artin-Markov-Ivanovskii, 1947, Theorem 18; Markov, 1945, Theorem 8). $P_n$ is a group with generators $\{s_{ij}\}$ and defining relations (3.7)–(3.13) (which, together with the trivial relations, form a minimal Gröbner–Shirshov basis of $P_n$ relative the inverse tower order of words in the generators).

ACKNOWLEDGMENT

Bokut is supported in part by the Russia’s Fund for Fundamental Research and the Leading Scientific Schools Fund. The research of the third author is partially supported by a RGC grant (CUHK) #2060297 (05-07).

REFERENCES

Artin, E. (1925). Theory der Zöpfe. Abh. Math. Seminar., Hamburg Univ. 4:47–72.
Artin, E. (1947). Theory of braids. Ann. of Math. 48:101–126.
Bergman, G. M. (1978). The diamond lemma for ring theory. Adv. in Math. 29:178–218.
Bokut, L. A. (1972). Unsolvability of the word problem, and subalgebras of finitely presented Lie algebras. Izv. Akad. Nauk SSSR Ser. Mat. 36:1173–1219.
Bokut, L. A. (1976). Imbeddings into simple associative algebras. Algebra i Logika 15:117–142, 245.

Bokut, L. A., Shiao, L.-S. (2001). On Gröbner–Shirshov basis of Coxeter groups. Commun. in Algebra 29(9):4305–4319.

Buchberger, B. (1965). An Algorithm for Finding a Basis for the Residue Class Ring of a Zero-Dimensional Polynomial Ideal. (German). Ph.D. thesis, University of Innsbruck, Austria.

Buchberger, B. (1970). An algorithmical criteria for the solvability of algebraic systems of equations. (German). Aequationes Math. 4:374–383.

Burau, W. (1932). Über Zopfinvarianten. Abh. Math. Sem. Hamburg Univ. 9(2):117–124.

Markov, A. A. (1945). Foundations of the algebraic theory of Braids. In: Proceedings of the Steklov Mathematical Institute. 16. Moscow-Leningrad.

Newman, M. H. A. (1942). On theories with a combinatorial definition of “equivalence.” Ann. of Math. 43:223–243.

Shirshov, A. I. (1962/1999). Some algorithm problems for Lie algebras (Russian). Translation in ACM SIGSAM Bull. 33. Original Sibirsk. Mat. Z. 3(2):292–296.