A SPATIAL FOOD CHAIN MODEL FOR THE BLACK SEA ANCHOVY, AND ITS OPTIMAL FISHERY

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Abstract. We present a spatial food chain model on a bounded domain coupled with optimal control theory to examine harvesting strategies. Motivated by the fishery industry in the Black Sea, the anchovy stock and two more trophic levels are modeled using nonlinear parabolic partial differential equations with logistic growth, movement by diffusion and advection, and Neumann boundary conditions. Necessary conditions for the optimal harvesting control are established. The objective for the problem is to find the spatial optimal harvesting strategy that maximizes the discounted net value of the anchovy population. Numerical simulations using data from the Black Sea are presented. We discuss spatial features of harvesting and the effects of diffusion and advection (migration speed) on the anchovy population. We also present the landing of anchovy and its net profit by applying two different harvesting strategies.

1. Introduction. In current fishery management, overfishing is one of the top problems that ecologists and bio-economists need to address with fishery management strategies. Since fish populations live in complex ecosystems and have natural predator-prey relationships with other species, the effects of interactions between species in a food chain should not be ignored ([11]). One of the well-known examples of fish populations is the Black Sea anchovy in the Black Sea, which has been facing difficulties competing with Mnemiopsis leidyi for zooplankton populations as a main source of food, and also has been experiencing predation by Mnemiopsis leidyi on its larvae and eggs ([30]). Therefore, we can not ignore the effects of predator-prey relations on this anchovy population, and we will track these effects in our model.

The commercial fishery of the anchovy population had almost stopped in the northern part of the Black Sea due to the combined effects of overfishing and the invasive jellyfish species, Mnemiopsis leidyi ([30]; [29]; [28]). Hence, the Black Sea anchovy fishery started to depend mostly on the southern part of the Black Sea in

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recent decades ([12]), and the southern part mainly consists of the Turkish coasts. Since we have seen a decrease in landings on the southern part of the Black Sea, we decided to contribute to the solution of the problem. Therefore, the essence of the study will be to find a conservative and optimal fishery management strategy for this fishery.

Most of models involving management of fish populations represented their fishery using a single equation ([16]; [34]), which ignores the predator-prey relations and the effects of the fishery on the food chains. Using a single equation may not represent the biological system well enough for modeling sustainable fishery management ([13]; [22]; [3]). The use of food chain models focusing on the key species on ecosystems becomes an increasingly useful trend in the commercial fishery to not only conserve and manage natural renewable food resources, but also to have sustainable and productive ecosystems (e.g. [17]; [36]; [1]; [19]; [37]; [4]).

The study of the management of natural renewable resources is an increasingly growing field due to its importance for sustainable ecosystems, and for economic value of the renewable resources. There have been some work investigating different techniques to protect fish populations and conserve marine ecosystems, such as no-take marine reserves (protected areas) and ecosystem-based fishery managements ([8]; [31]; [32]; [9]; [10]). Besides, these intervention methods, another way to find management policies is to use optimal control theory together with food chain models to obtain optimal management policies for the natural renewable food resources. In many cases, no-take marine reserve areas are natural results of the optimal control application in fishery management ([27], [18], [26], and [20]).

Due to the exploitation of marine systems and the lack of food chain friendly strategies for commercial fisheries, it is essential for the fishery industry to come up with new management strategies to protect fish stocks and keep the systems sustainable. That is why ecologists and economists have been investigating new management methods for ecologically sustainable commercial fisheries. In particular, the exploration of spatial strategies for improving regulatory outcomes has received much attention ([38]). Neubert built a steady state model from a diffusive partial differential equation (PDE), which is reduced to a second-order ordinary differential equation (ODE) in space at equilibrium, and he showed that depending on length of domains, marine reserves are part of optimal harvesting strategy when maximizing yields ([27]). The advantage of using a spatial model is the possibility of explicitly showing marine reserves. Ding and Lenhart extended Neubert’s work to a multidimensional spatial domain with a steady state PDE and numerically found marine reserves in the center of the domain ([5]). Furthermore, recent work showed that when spatial dynamics of a resource are ignored, then management strategies generally produce suboptimal results ([15]). Other work has included PDEs with time and space and diffusion and advection effects ([21] and [18]). These studies have showed that using spatial and temporal fishery models can offer better strategies for fishery management ([15, 21, 18]. The damage to the environment from fishing methods and equipment has also been shown to be important through PDE modeling in [20].

In our study, we formulate our model as an extension of the models presented by [21], [18]. In these studies, a single PDE is used for the stock and the models used in these studies do not represent a specific fishery scenario, but we use a system of PDEs for the Black Sea anchovy fishery in the Black Sea, which is a specific fishery. Building on the optimal harvesting results for a food chain ODE model
of the Black Sea anchovy in [4], we use three trophic levels in our PDE system. This application with spatial-temporal features for the Black Sea anchovy coupled with optimal control theory is novel. Having three trophic levels in a model will give us an opportunity to track changes in the food chain due to the effects of the commercial fishery. We also will see the effects of predator-prey interactions on this fishery. This approach can be valuable for sustainable fishery management.

In the next section, we formulate our model with biologically feasible boundary and initial conditions, and present our goal in our objective functional for an optimal fishery. In section 3, we prove existence of an optimal control and then demonstrate the necessary conditions that an optimal control must satisfy. Section 4 gives the derivation of the optimality system, which consists of the state equations, the adjoint equations, and the characterization of optimal control. In Section 5, we first obtain the parameters of our model by choosing parameters adjusted by using landing data and some rates from [4]. Then we present numerical solutions to our optimality system for some scenarios and connect these optimal harvesting results to fishery management of the Black Sea anchovy.

2. Model formulation. In this study, we focus on optimal control of harvesting anchovy on the southern part of the Black Sea by using a food chain model in one space dimension. We consider a parabolic PDE system with Neumann boundary conditions in a bounded domain \( Q = (0, L) \times (0, T) \). We define \( A(x, t) \), \( P(x, t) \), and \( Z(x, t) \) as biomass of the Black Sea anchovy, the jellyfish (Mnemiopsis Leidyi, predator on anchovy), and the zooplankton at location \( x \), and time \( t \), respectively. Each biomass has logistic growth with an intrinsic growth rate, directed movement (advection), and simple diffusion (movement from regions of high concentration to regions of low concentration). The Black Sea anchovy is harvested with the term, \( h(x, t)A(x, t) \), proportional to the biomass of the anchovy and the effort, \( h(x, t) \). Given a control \( h \), the corresponding state variables, \( A(h) \), \( P(h) \), and \( Z(h) \) satisfy the state system:

\[
\begin{align*}
L_1 A &= f_1(A) + m_0 AZ - m_1 PA - hA \\
L_2 P &= f_2(P) + m_2 PA + m_3 PZ - m_6 P \\
L_3 Z &= f_3(Z) - m_4 AZ - m_5 PZ
\end{align*}
\]

with homogeneous Neumann boundary conditions:

\[
\frac{\partial A}{\partial \eta} = 0, \quad \frac{\partial P}{\partial \eta} = 0, \quad \frac{\partial Z}{\partial \eta} = 0 \quad \text{on } \partial(0, L) \times (0, T)
\]

and the initial conditions in \( L^\infty(0, L) \):

\[
A(x, 0) = A_0(x), \quad P(x, 0) = P_0(x), \quad Z(x, 0) = Z_0(x) \quad \text{for } x \in (0, L) \subset \mathbb{R}
\]

where \( f_1(A) \), \( f_2(P) \), and \( f_3(Z) \) represent the logistic growth terms of \( A \), \( P \), and \( Z \) respectively with intrinsic growth rates \( r_1, r_2, \) and \( r_3 \) as follows:

\[
\begin{align*}
f_1(A) &= r_1 A (1 - \frac{A}{K_1}), \\
f_2(P) &= r_2 P (1 - \frac{P}{K_2}), \\
f_3(Z) &= r_3 Z (1 - \frac{Z}{K_3}),
\end{align*}
\]
and the terms $m_0AZ, m_1PA, m_2PA, m_3AZ, m_4AZ,$ and $m_5PZ$ represent interaction (predation) terms between species. For example, $-m_1PA$ is a decay term for the anchovy population and $+m_2PA$ is a growth term for the Jellyfish population. Note that the decay term $m_0P$ comes from the predator–prey relation between the jellyfish $M. leidyi$ and its predators, like $B. ovata$, in the food chain. The following notation is used for the operators in the left hand side of system (1)

\[
\begin{align*}
L_1A &= A_1 + b_1(x,t)A_x - (D_1(x,t)A_x)_x \\
L_2P &= P_1 + b_2(x,t)P_x - (D_2(x,t)P_x)_x \\
L_3Z &= Z_1 + b_3(x,t)Z_x - (D_3(x,t)Z_x)_x
\end{align*}
\]

where $b_1$, $b_2$, and $b_3$ are advection coefficients of anchovy, jellyfish, and zooplankton, respectively. The advection coefficients are positive since the system moves from west to east during the fishery season. We have $D_1$, $D_2$, and $D_3$ as the corresponding diffusion coefficients.

We define the class of admissible controls as

\[ A = \{h \in L^\infty(Q) : 0 \leq h(x,t) \leq M \text{ and } h = 0 \text{ on } Q \setminus Q^* \} \]

where $M \in \mathbb{R}$, $Q^* = (x_1, x_2) \times (a, b) \subset Q$, the interval $(x_1, x_2)$ represents the area with harvesting, and the interval $(a, b)$ represents time interval in which commercial harvesting is allowed. This season takes about 3 months (November - January) in the commercial fishery of the Black Sea anchovy.

Our objective functional for control $h \in A$ is

\[
J(h) = \int_Q e^{-\alpha t}(phA - \mu_1h - \mu_2h^2)dxdt = \int_{Q^*} e^{-\alpha t}(phA - \mu_1h - \mu_2h^2)dxdt \quad (4)
\]

subject to the PDE system (1), where $Q^*$, shows the allowable location and timing of the harvest. In this discounted net profit $J$, the factor $e^{-\alpha t}$ represents the discount term with interest rate $\alpha$. The term $e^{-\alpha t}phA$ represents the revenue of the fishery (with price $p$), and $e^{-\alpha t}(\mu_1h + \mu_2h^2)$ represents the cost of the fishery. The cost has two terms, where the first term involves the workers’ wages and vessel expenses and the second term is nonlinear due to congestion that accounts for crowding when the harvest is concentrated in space. Our purpose is to maximize the objective functional over the admissible class of controls, such that

\[ J(h^*) = \sup_{h \in A} J(h). \]

We will consider a weak solution of the system (1)-(3) with the space $V = L^2(0,T;H^1(0,L))$ for the solution, and the space $V^* = L^2(0,T;(H^1(0,L))^*)$ for the time derivative of the weak solution, and then we define the bilinear forms as

\[
B^i(t,u,\phi) = \int_{(0,L)} D_i(x,t)u_x\phi_x dx + \int_{(0,L)} b_i(x,t)u_x\phi dx
\]

for $u, \phi \in V$, and $i = 1, 2, 3$.

**Definition 2.1.** For the system (1) with BCs (2) and initial conditions (3), a weak solution $(A, P, Z) \in V^3 \cap (L^\infty(Q))^3$ with $A_t, P_t, Z_t \in V^*$ satisfies

\[
\int_0^T \langle A_t, \phi_1 \rangle dt + \int_0^T B^1(t,A,\phi_1)dt = \int_Q [f_1(A) + A(m_0Z - m_1P - h)]\phi_1 dxdt
\]
Theorem 2.1. Given for our system.

\[ \int_0^T \langle P, \phi_2 \rangle dt + \int_0^T B^2(t, P, \phi_2) dt = \int_Q [f_2(P) + P(m_2A + m_3Z - m_6)]\phi_2 dx \, dt \]

\[ \int_0^T \langle Z, \phi_3 \rangle dt + \int_0^T B^3(t, Z, \phi_3) dt = \int_Q [f_3(Z) - Z(m_4A + m_5P)]\phi_3 dx \, dt \]

(5)

for any test functions \( \phi_1, \phi_2, \phi_3 \in V \), where \( \langle \ , \ \rangle \) is the duality between \( H^1(0, L) \) and \( (H^1(0, L))^* \), and satisfies initial conditions (3).

Remark 2.1. When \( A, P, Z \in V \) and \( A, P, Z \in V^* \), then \( A, P, Z \in C([0, T]; L^2(0, L)) \) by using results of [7]. Thus the initial conditions, \( A(x, 0), P(x, 0), \) and \( Z(x, 0) \), make sense in \( L^2(0, L) \).

We need to have \( L^\infty(Q) \) bounds on the state variables due to having non-linear terms.

We make the following assumptions:

1. The initial conditions satisfy \( A_0(x), P_0(x), Z_0(x) \in L^\infty(0, L) \), and \( 0 \leq A_0(x) < B \),
   \[ 0 \leq P_0(x) < B, \ 0 \leq Z_0(x) < B \] for some \( B \in \mathbb{R} \).
2. \( r_i, K_i \) (for \( i = 1, 2, 3 \)), \( m_i \) (for \( i = 0, 1, 2, 3, 4, 5, 6 \)), and \( h \in L^\infty(Q) \) and are non-negative constants.
3. Uniform ellipticity on the diffusion coefficients:
   \[ 0 < \theta \leq D_i(x, t) \text{ for all } (x, t) \in Q. \] (6)

For a control \( h \in \mathcal{A} \), we have existence of a solution for our state system as in [25] and [2], and then in the next section, we show existence of an optimal control for our system.

Theorem 2.1. Given \( h \in \mathcal{A} \), and sufficiently small \( T \), there exists a unique non-negative weak solution \( (A, P, Z) \in \mathcal{L}^3 \cap (L^\infty(0, L))^3 \) satisfying (2), (3), and (5). Moreover, \( 0 \leq A(x, t) \leq C, \ 0 \leq P(x, t) \leq C, \) and \( 0 \leq Z(x, t) \leq C \) a.e. \( (x, t) \in Q \) for some constant \( C \).

3. Existence of optimal control. For a control \( h \in \mathcal{A} \), there exists a unique weak state solution \( (A, P, Z) = (A^n, P^n, Z^n)(h) \). Now, we will show that there exist an optimal control \( h^* \) and the corresponding state solutions \( (A^*, P^*, Z^*) = (A, P, Z)(h^*) \) to our optimal control problem.

Theorem 3.1. There exists an optimal control \( h^* \) in \( \mathcal{A} \), which maximizes the objective functional \( J(h) \) subject to our PDE system (1)-(3).

Proof. The state variables, \( A, P, Z \), and the control variable \( h \) are uniformly \( L^\infty(Q) \) bounded, and thus there exists a maximizing sequence \( h^n \) in \( \mathcal{A} \) such that

\[ \lim_{n \to \infty} J(h^n) = \sup_{h \in \mathcal{A}} J(h). \]

By using the results of Theorem 2.1, we define

\[ (A^n, P^n, Z^n) = (A, P, Z)(h^n) \quad \text{for all } n \in \mathbb{N}. \]
Now, we use the weak formulation of the state system (1) satisfied by \( A^n, P^n, \) and \( Z^n \). By Gronwall’s Inequality, we can conclude that

\[
\sup_{s \in (0,T)} \left( \int_{(0,L)} (A^n)^2(x,s) + (P^n)^2(x,s) + (Z^n)^2(x,s)dx \right) + \theta \int_{Q_s} |A^n|^2 + |P^n|^2 + |Z^n|^2 dxdt \\
\leq C_5 \int_{(0,L)} (A^n)^2(x) + (P^n)^2(x) + (Z^n)^2(x)dx
\]

where \( Q_s = (0,L) \times (0,s) \) with \( s \leq T \) and \( C_5 \) depends on \( T, \theta, \) and \( L^\infty \) bounds on the coefficients, the states, and the control. These bounds together with the PDE system (1) imply that \( ||A^n||_{V^*}, ||P^n||_{V^*}, ||Z^n||_{V^*} \) are uniformly bounded in \( V^* = L^2(0,T;H^1(0,L))^3 \) for any \( n \in \mathbb{N} \).

Thus, we can obtain that there exist subsequences \( A^n, P^n, Z^n, A^n_t, P^n_t, Z^n_t, h^n, \) and \( (A^*, P^*, Z^*) \in V^3 \cap (L^\infty(Q))^3 \) with \( (A^n_t, P^n_t, Z^n_t) \in (V^*)^3 \) such that

\[
A^n \rightarrow A^*, \quad P^n \rightarrow P^*, \quad Z^n \rightarrow Z^* \quad \text{weakly in } V \quad (7)
\]

\[
A^n_t \rightarrow A^*_t, \quad P^n_t \rightarrow P^*_t, \quad Z^n_t \rightarrow Z^*_t \quad \text{weakly in } V^* \quad (8)
\]

\[
h^n \rightarrow h^* \quad \text{weakly in } L^2(Q) \quad (9)
\]

Also, by using the compactness result from [33], we can obtain the following:

\[
A^n \longrightarrow A^*, \quad P^n \longrightarrow P^*, \quad Z^n \longrightarrow Z^* \quad \text{strongly in } L^2(Q). \quad (10)
\]

We need to show that \( (A^*, P^*, Z^*) = (A, P, Z)(h^*) \) in weak sense. By these weak and strong convergences, we obtain convergences directly of terms like:

\[
\int_0^T (A^n_t - A^*_t)\phi_1 dxdt \rightarrow 0
\]

\[
\int_Q (f_1(A^n) - f_1(A^*))\phi_1 dxdt \rightarrow 0
\]

\[
\int_Q D_3(x,t)(Z^n - Z^*)(\phi_3)_{x} dxdt \rightarrow 0
\]

\[
\int_Q b_1(x,t)(A^n - A^*)(\phi_1) dxdt \rightarrow 0
\]

The nonlinear terms also converge. Since \( h^n \) is bounded in \( L^\infty(Q) \), and \( A^*, P^*, \) and \( Z^* \) are uniformly bounded in \( V \), we obtain

\[
\left| \int_Q (h^n A^n \phi_1 - h^* A^* \phi_1) dxdt \right| \leq \left| \int_Q h^n (A^n - A^*) \phi_1 dxdt \right| + \left| \int_Q (h^n - h^*) A^* \phi_1 dxdt \right| \rightarrow 0,
\]

since \( A^n \rightarrow A^* \) strongly in \( L^2(Q) \). Therefore, we can conclude that

\[
(A^*, P^*, Z^*) = (A, P, Z)(h^*)
\]

To complete the proof, we will use the lower semi-continuity argument for our objective functional, which says every lower semi-continuous convex function of
a real vector space remains lower semi-continuous when supplied with the weak topology ([6]). Since \( h^n \to h^* \) weakly in \( L^2(Q) \),
\[
\int_{Q^*} (h^*)^2 dxdt \leq \liminf_{n \to \infty} \int_{Q^n} (h^n)^2 dxdt.
\]
This gives us
\[
\sup_{h \in \mathcal{A}} J(h) = \lim_{n \to \infty} J(h^n) = \liminf_{n \to \infty} J(h^n) 
\leq \int_{Q^*} e^{-\alpha t} (h^* A^* - \mu h^* - \mu(h^*)^2) dxdt = J(h^*)
\]
Therefore, \( h^* \) is an optimal control which maximizes the objective functional. \( \square \)

4. **Derivation of the optimality system.** We now derive the optimality system which consists of the state system coupled with the adjoint system and the optimal control characterization. We differentiate the objective functional with respect to the control to obtain necessary conditions for the optimality system. Since the objective functional contains the anchovy population \( A \), we also have to differentiate the map \( h \mapsto (A, P, Z)(h) \) with respect to the control, \( h \). These derivatives are called the sensitivity functions and will be used to find our adjoint system.

**Theorem 4.1 (Sensitivity PDEs).** Let \( h \) be an optimal control for problem (4) with corresponding state solution \( (A, P, Z) = (A, P, Z)(h) \), and let \( h' \) be another control with the corresponding state solutions \( (A', P', Z') = (A(h+\epsilon), P(h+\epsilon), Z(h+\epsilon)) \) such that \( h' = h + \epsilon \), is in \( \mathcal{A} \) for all sufficiently small \( \epsilon > 0 \) with \( l \in L^\infty(Q) \). Then, for \( h \in \mathcal{A} \) the mapping, \( h \mapsto (A(h), P(h), Z(h)) \) is weakly differentiable in the directional derivative sense and there exists \( \Psi_1, \Psi_2, \Psi_3 \in V \) with \( (\Psi_1)_t, (\Psi_2)_t, (\Psi_3)_t \in V^* \) such that
\[
\lim_{\epsilon \to 0} \frac{A(h+\epsilon) - A(h)}{\epsilon} = \Psi_1 \quad \text{weakly in } L^2(Q) \\
\lim_{\epsilon \to 0} \frac{P(h+\epsilon) - P(h)}{\epsilon} = \Psi_2 \quad \text{weakly in } L^2(Q) \\
\lim_{\epsilon \to 0} \frac{Z(h+\epsilon) - Z(h)}{\epsilon} = \Psi_3 \quad \text{weakly in } L^2(Q)
\]
Furthermore, the sensitivity functions \( \Psi_1, \Psi_2, \text{ and } \Psi_3 \) satisfy the linearized sensitivity system:
\[
\begin{align*}
L_1 \Psi_1 - f_1'(A) \Psi_1 - m_0(A \Psi_3 + Z \Psi_1) + m_1(P \Psi_1 + A \Psi_2) + h \Psi_1 &= -lA \\
L_2 \Psi_2 - f_2'(P) \Psi_2 - m_2(P \Psi_1 + A \Psi_2) - m_3(P \Psi_3 + Z \Psi_2) + m_6 \Psi_2 &= 0 \\
L_3 \Psi_3 - f_3'(Z) \Psi_3 + m_4(A \Psi_3 + Z \Psi_1) + m_5(P \Psi_3 + Z \Psi_2) &= 0 \quad \text{in } (0,L) \times (0,T)
\end{align*}
\]
with boundary conditions:
\[
\frac{\partial \Psi_1}{\partial \eta} = 0, \quad \frac{\partial \Psi_2}{\partial \eta} = 0, \quad \frac{\partial \Psi_3}{\partial \eta} = 0 \quad \text{on } \partial(0,L) \times (0,T)
\]
and initial conditions: \( \Psi_1(x,0) = \Psi_2(x,0) = \Psi_3(x,0) = 0 \) for \( x \in (0,L) \).
Proof. Let us choose \( l \in L^\infty(Q) \) and \( h \in \mathcal{A} \) such that \((h + \epsilon l) \in \mathcal{A}\) for small \( \epsilon > 0 \). Using the equations satisfied by the quotients \((\frac{A'}{\epsilon} - A), (\frac{P'}{\epsilon} - P), \) and \((\frac{Z'}{\epsilon} - Z)\), we obtain that \( ||(\frac{A'}{\epsilon} - A)||_V, ||(\frac{P'}{\epsilon} - P)||_V, ||(\frac{Z'}{\epsilon} - Z)||_V \) are uniformly bounded for any \( 0 < \epsilon < 1 \). These bounds together with the sensitivity system imply that \( ||(\frac{A'}{\epsilon} - A)||_V, ||(\frac{P'}{\epsilon} - P)||_V, ||(\frac{Z'}{\epsilon} - Z)||_V \) are uniformly bounded in \( V^* \) for any \( 0 < \epsilon < 1 \). Using a result from [33] to get strong convergence in \( L^2 \) of these quotients, these bounds justify the existence of \( \Psi \) such that as \( \epsilon \to 0 \)

\[
\frac{A'}{\epsilon} - A \to \Psi_1, \quad \frac{P'}{\epsilon} - P \to \Psi_2, \quad \frac{Z'}{\epsilon} - Z \to \Psi_3,
\]

weakly in \( V \) and strongly in \( L^2(Q) \),

\[
(\frac{A'}{\epsilon})_t \to \Psi_1, \quad (\frac{P'}{\epsilon})_t \to \Psi_2, \quad (\frac{Z'}{\epsilon})_t \to \Psi_3,
\]

weakly in \( V^* \).

Thus, we obtain that \( \Psi_1, \Psi_2, \) and \( \Psi_3 \) satisfy the sensitivity system (12)-(13). \( \Box \)

The sensitivity system (12)-(13) can be written in terms of the linear operator \( \mathcal{L} \) as follows:

\[
\mathcal{L} \begin{pmatrix} \Psi_1 \\ \Psi_2 \\ \Psi_3 \end{pmatrix} = \begin{pmatrix} L_1 \Psi_1 \\ L_2 \Psi_2 \\ L_3 \Psi_3 \end{pmatrix} + M \begin{pmatrix} \Psi_1 \\ \Psi_2 \\ \Psi_3 \end{pmatrix} = \begin{pmatrix} -lA \\ 0 \\ 0 \end{pmatrix}
\]

with

\[
M = \begin{pmatrix}
-f'_1(A) - m_0Z + m_1P + h & m_1A \\
-m_2P & f'_2(P) - m_2A - m_3Z + m_6 \\
 m_3Z & -f'_3(Z) + m_4A + m_5P
\end{pmatrix}.
\]

To derive the adjoint equations, denoted by \( \lambda_i, i = 1, 2, 3 \), let \( \mathcal{L}^* \) be our adjoint operator of the operator \( \mathcal{L} \) associated with the sensitivity PDEs, such that

\[
\int_Q e^{-\alpha t}(\Psi_1, \Psi_2, \Psi_3)\mathcal{L}^* \begin{pmatrix} \lambda_1 \\ \lambda_2 \\ \lambda_3 \end{pmatrix} \, dx \, dt = \int_Q e^{-\alpha t}(\lambda_1, \lambda_2, \lambda_3)\mathcal{L} \begin{pmatrix} \Psi_1 \\ \Psi_2 \\ \Psi_3 \end{pmatrix} \, dx \, dt. \quad (14)
\]

We write the adjoint system as

\[
\mathcal{L}^* \begin{pmatrix} \lambda_1 \\ \lambda_2 \\ \lambda_3 \end{pmatrix} = \begin{pmatrix} L_1^* \lambda_1 \\ L_2^* \lambda_2 \\ L_3^* \lambda_3 \end{pmatrix} + M^T \begin{pmatrix} \lambda_1 \\ \lambda_2 \\ \lambda_3 \end{pmatrix} + \alpha \begin{pmatrix} \lambda_1 \\ \lambda_2 \\ \lambda_3 \end{pmatrix} = \begin{pmatrix} ph \\ 0 \\ 0 \end{pmatrix} \quad \text{on } Q. \quad (15)
\]

where

\[
\begin{align*}
L_1^* \lambda_1 &= -(\lambda_1)_t - (b_1(x, t) \lambda_1)_x - (D_1(x, t) (\lambda_1)_x)_x, \\
L_2^* \lambda_2 &= -(\lambda_2)_t - (b_2(x, t) \lambda_2)_x - (D_2(x, t) (\lambda_2)_x)_x, \\
L_3^* \lambda_3 &= -(\lambda_3)_t - (b_3(x, t) \lambda_3)_x - (D_3(x, t) (\lambda_3)_x)_x.
\end{align*}
\]

The last equality in RHS of the adjoint system (15) is obtained by differentiating the integrand of the objective functional (without \( e^{-\alpha t} \)) with respect to the states. The \( \alpha \) term, comes from integrating by parts in the terms with \( e^{-\alpha t}(\Psi_i)_t \lambda_i \) for \( i = 1, 2, 3 \).
Theorem 4.2. Given an optimal control $h^*$ and corresponding state solution $(A^*, P^*, Z^*)$, there exists a weak solution $(\lambda_1, \lambda_2, \lambda_3)$ satisfying the adjoint system,

\[
\begin{align*}
L_1^* \lambda_1 &= ph + \left[ f_1'(A) + m_0Z - m_1P - h - \alpha \right] \lambda_1 + m_2P\lambda_2 - m_4Z\lambda_3 \\
L_2^* \lambda_2 &= \left[ f_2'(P) + m_2A + m_3Z - m_6 - \alpha \right] \lambda_2 - m_1A\lambda_1 - m_5A\lambda_3 \quad \text{in } (0,L) \times (0,T) \\
L_3^* \lambda_3 &= \left[ f_3'(Z) - m_3A - m_5P - \alpha \right] \lambda_3 + m_0A\lambda_1 + m_3P\lambda_2
\end{align*}
\] (16)

with boundary conditions:

\[
\begin{align*}
D_1(x,t) \frac{\partial \lambda_1}{\partial x} + b_1(x,t)\lambda_1 &= 0 \\
D_2(x,t) \frac{\partial \lambda_2}{\partial x} + b_2(x,t)\lambda_2 &= 0 \\
D_3(x,t) \frac{\partial \lambda_3}{\partial x} + b_3(x,t)\lambda_3 &= 0
\end{align*}
\] (17) on $\partial(0,L)$ ($\partial(0,L) \times (0,T)$

and transversality conditions:

\[
\lambda_1(x,T) = 0, \quad \lambda_2(x,T) = 0, \quad \lambda_3(x,T) = 0 \quad \text{for } x \in (0,L) \subset \mathbb{R} \quad (18)
\]

Furthermore, the optimal control is characterized by

\[
h^* = \max\left(0, \min\left(\frac{(p - \lambda_1)A - \mu_1}{2\mu_2}, M\right)\right) \quad \text{on } Q^*
\] (19)

Proof. Since the adjoint PDE system is linear, there exist $\lambda_1$, $\lambda_2$, and $\lambda_3$ satisfying the adjoint PDE system in a weak sense. To obtain the characterization of an optimal control, assume $h^*$ is an optimal control, and $A(h)$, $P(h)$, and $Z(h)$ are its corresponding solutions. For $\epsilon > 0$, consider $h^* + \epsilon l \in \mathcal{A}$ with associated solutions $A^*$, $P^*$, and $Z^*$. Since the maximum of the objective functional is attained at $h^*$, we obtain the following:

\[
0 \geq \lim_{\epsilon \to 0^+} \frac{J(h^* + \epsilon l) - J(h^*)}{\epsilon}
= \int_{Q^*} (e^{-at}ph^*\Psi_1 + pe^{-at}lA)dxdt - \int_{Q^*} e^{-at}\left(2\mu_2h^*l + \mu_1l\right)dxdt
= \int_{Q^*} (e^{-at}(\Psi_1, \Psi_2, \Psi_3) \begin{pmatrix} ph^* \\ 0 \\ 0 \end{pmatrix} + pe^{-at}lA)dxdt - \int_{Q^*} e^{-at}\left(2\mu_2h^*l + \mu_1l\right)dxdt
= \int_{Q^*} e^{-at}(\lambda_1, \lambda_2, \lambda_3) \begin{pmatrix} -lA \\ 0 \\ 0 \end{pmatrix} dxdt + \int_{Q^*} e^{-at}l(pA - 2\mu_2h^* - \mu_1)dxdt
\]
\[
\int_{Q^*} e^{-\alpha t} (-\lambda_1 lA) dx dt + \int_{Q^*} e^{-\alpha t} \left( pA - 2\mu_2 h^* - \mu_1 \right) dx dt
\]
\[
= \int_{Q^*} e^{-\alpha t} \left( (p - \lambda_1)A - 2\mu_2 h^* - \mu_1 \right) dx dt
\]

We use standard optimality techniques on the above inequality to obtain our characterization of \( h^* \).

Thus, our optimality system consist of the equations (1)-(3), (16)-(18), and (19). See the work by [24] for a technique to obtain uniqueness of the optimality system, which implies uniqueness of the optimal control.

**Theorem 4.3.** When \( T \) is small enough, then the solution of the optimality system is unique.

5. **Numerical simulations and parameters.** For our numerical simulations, the rates are obtained from using the data of Black Sea landing in [35] and from appropriate adaption of the parameters in [4]. The initial total biomass for zooplankton was found from [4] when anchovy population is near its lowest biomass and placed spatially due to the known location at that time (coming in from the north part of the Black Sea). For the advection rate of 0.18 in Table 1, we used the known distance of the migration of the anchovy over the months in our simulation. We considered the spread of population to choose the diffusion rate as 0.05. By using these baseline parameters given in Table 1, the optimality system, consisting of the state (1)-(3) and adjoint system (16)-(18) coupled with the optimal control characterization is solved using the forward-backward sweep method discussed in [23]; the convergence of one type of implementation of this iterative method can be found in [14]. In the forward sweeps of the state system and the backward sweeps of the adjoint system, an explicit finite difference method is used to solve the PDEs.

In our numerical results, the length of our space domain is 4, and we assume that the southern part of Black Sea is between 0.5 and 3.5 in space. But, we will consider just the Turkish coast since our data of anchovy fishery is from Turkey ([35]). Thus, for Turkish coast, we will consider harvesting between the space 0.5 and 3.0, which corresponds 1700km. We also assume that the space between 3.0 and 3.5 belongs to the coast of Georgia. Thus, the space between 0 and 0.5, and space between 3.5 and 4.0 will be considered as a part of north coast of the Black Sea. In our numerical results, the spatial area for harvest is the interval (0, 3.0).

Furthermore, for the final time, we use \( T = 9 \), which corresponds to 9 months. Note that we will investigate this fishery just for one season.

In our numerical setup, our space and time steps are \( \Delta x = 0.05 \) and \( \Delta t = \frac{(\Delta x)^2}{4} \), respectively. Our initial guess for the harvest control \( h \) was zero. The convergence criterion was achieved with relative errors between controls, states, and adjoints of successive iterates were less than 0.001.

5.1. **Benefits of using optimal control in anchovy fishery.** In fishery management, there are many benefits of applying optimal control. For example, investigating the marine protected areas, finding spatial harvesting strategies that depend on
A with harvesting strategy. For this subsection only, we used constant initial conditions optimal control strategy, we see that this strategy actually offers us a seasonal decreasing the value of $\mu$ small initial anchovy population. Thus, we first reduce the cost of the fishery by seasonal harvest result may be due to the high cost of the fishery, or due to having mainly recommend for the harvest to occur in the last few months. Obtaining this left graph in Figure 1, in which we allowed controls for 9 months and the results possible time for harvest is the interval $0 \leq t \leq 9$ months. When we apply an optimal control strategy, we see that this strategy actually offers us a seasonal harvesting strategy. For this subsection only, we used constant initial conditions with $A_0 = 180,000$, $P_0 = 15,000$, and $Z_0 = 39,000$ tonnes of biomass. See the left graph in Figure 1, in which we allowed controls for 9 months and the results mainly recommend for the harvest to occur in the last few months. Obtaining this seasonal harvest result may be due to the high cost of the fishery, or due to having small initial anchovy population. Thus, we first reduce the cost of the fishery by decreasing the value of $\mu_1$ from baseline $44,750,000$ to $22,375,000$ to see the effect on the optimal harvesting strategy. We see that the fishery season enlarges from 2 months to 3 months in the optimal control harvesting strategy, but it still offers a seasonal harvesting. (See the right plot in Figure 1).

In Figure 2, we increased the initial biomass of anchovy population from 180,000 tonnes to 350,000 tonnes and we still obtain a seasonal harvesting (with baseline cost). These results show that optimal control tools capture the seasonal fishery depending on the dynamics of food chain. This simple example also shows the benefits of optimal control tools in fishery management.

5.2. Comparison of strategies in anchovy fishery. We compare two strategies, the optimal harvesting strategy and the constant harvesting strategy. We harvest the system for only 3 months (November-January). Note that fishing schools are becoming larger and more appropriate for commercial fishing around November ([12]). Initial biomass of populations are concentrated in left side of the space domain (see Figure 3). Due to the seasonal behavior of anchovy population, it migrates from the western part to the eastern part during the fishery season in the southern part of the Black Sea. Hence, we will have a positive advection to capture the migration in our model. We will then apply the optimal control (harvesting) strategy and compare it with a constant harvesting strategy in the anchovy fishery.

| Parameters | Descriptions | Unit | Value |
|------------|--------------|------|-------|
| $r_1$ | Intrinsic growth rate of anchovy, $A$ | days$^{-1}$ | 3.3 |
| $r_2$ | Intrinsic growth rate of jelly fish, $P$ | days$^{-1}$ | 0.75 |
| $r_3$ | Intrinsic growth rate of zoo-plankton, $Z$ | days$^{-1}$ | 0.9 |
| $K_1$ | Carrying capacity of anchovy, $A$ | Tonnes | 3.5e+5 |
| $K_2$ | Carrying capacity of jelly fish, $P$ | Tonnes | 7.5e+3 |
| $K_3$ | Carrying capacity of zoo-plankton, $Z$ | Tonnes | 4e+4 |
| $m_0$ | Growth rate of $A$ due to predation of $Z$ | (days x Tonnes)$^{-1}$ | 1.4e-6 |
| $m_1$ | Consumption rate of $A$ by its predator, $P$ | (days x Tonnes)$^{-1}$ | 0.66e-5 |
| $m_2$ | Growth rate of $P$ due to predation of $A$ | (days x Tonnes)$^{-1}$ | 4.95e-6 |
| $m_3$ | Growth rate of $P$ due to predation of $Z$ | (days x Tonnes)$^{-1}$ | 5.7e-6 |
| $m_4$ | Consumption rate of $Z$ due to its predator $A$ | (days x Tonnes)$^{-1}$ | 0.2e-5 |
| $m_5$ | Consumption rate of $Z$ due to its predator $P$ | (days x Tonnes)$^{-1}$ | 1e-5 |
| $m_6$ | Consumption rate of $P$ due to its predators | days$^{-1}$ | 0.2 |
| $\mu_1$ | Coefficient of linear part of the cost function | US Dollar | 44,750,000 |
| $\mu_2$ | Coefficient of quadratic part of the cost function | US Dollar x (Tonnes)$^{-1}$ | $10000$ |
| $p$ | Price of anchovy per tonne | US Dollar x (Tonnes)$^{-1}$ | $1000$ |
| $\alpha$ | The interest rate of the discount rate | year$^{-1}$ | 0.01 |
| $b_i$ | Advection Coefficient, $(b_i > 0$ for $i = 1, 2, 3)$ | km x days$^{-1}$ | 0.18 |
| $D_i$ | Diffusion Coefficient, $(D_i > 0$ for $i = 1, 2, 3)$ | km$^2$ x days$^{-1}$ | 0.05 |
Note that some optimal control (harvesting) strategies and constant harvesting strategies were tested by [4] by using ODEs, which ignores spatial features, and in that work, constant harvesting strategies were the second best option. Using the maximum harvest rate as 0.35, we obtain a yearly landing close to the average yearly landing in 2003-2016 ([35]). Also, note that using a smaller harvest rate will make a smaller net profit and maintain seasonality. But using a too high harvest rate will be cause the population to decrease too much as noted in a previous work [4].

The landing of anchovy is 221,850 tonnes with the net profit as $J = \$122,920,000$ in the optimal harvest strategy, and 245,500 tonnes anchovy landing with the net profit as $J = \$117,020,000$ in the constant harvest strategy for one fishery season on the southern part of the Black Sea (See Figure 4). We obtain 5% more net profit in the optimal control strategy than in the constant harvest strategy, but with 10% less landing. In the optimal control strategy, we harvest mostly in the eastern part of the region because of the direction of the advection (See Figures 4 and 5), and this result of harvested areas in the optimal control strategy matches with literature ([12] and [35]).
5.3. Effect of migration speed of anchovy in its fishery. In the southern part of the Black Sea, we have migration of anchovy from the western part to the eastern part of the Black Sea during the fishery seasons, and the speed of this migration can change season to season ([12]). That is why we would like to quantify the effect of migration speed on the Black Sea Anchovy fishery by applying different advection rates (see Figure 6). When we increase the constant advection rate (speed of the migration) from 0.18 to 0.23, the harvesting area decreases and shifts to the right side of the harvesting region in the optimal harvesting. Any change in the speed of
anchovy migration will directly affect the harvesting regions. Moreover, when the speed of anchovy migration is high, then the anchovy population hits the eastern boundary of harvesting area faster in the southern part of the Black Sea, which results in less total landing and discounted net profit of the anchovy fishery (see the right plot in Figure 6). As the advection rate of anchovy increases (decreases), the landing and net profit of the anchovy fishery decreases (increases) in the southern part of the Black Sea.

5.4. Effect of diffusion in anchovy fishery. We assume that when anchovy enter the harvesting regions in southern part of Black Sea, its population aggregated since it is kind of a initial pulse of anchovy population and densely concentrated. The value of the diffusion rate is difficult to estimate. Depending on the diffusion rate, the aggregation will stay strong or get weaker. If the aggregation is lower due to a high diffusion, then the landing and total net profit will be reduced. Conversely, we see some increase in landing and total net profit when this aggregation stay strong with a low diffusion rate (see Figure 7). Thus, diffusion in anchovy population/schools has an important effect in anchovy fishery on the southern part of the Black Sea. The landing and net profit results are quite different depending
on the advection and diffusion rates, which illustrates the importance of being able to estimate those rates.

![Optimal harvesting strategies with different speeds of anchovy migration](image)

**Figure 7.** Seasonal optimal harvesting rate applied for 3 months with different diffusion rates, $D_i$ for $i = 1, 2, 3$.

6. **Conclusion.** In the study, we modeled the spatial features of the Black Sea anchovy. We showed the advantages of using food chain models coupled with optimal control theory. We showed that seasonal harvesting is the best option for the Black Sea anchovy using optimal control theory. Note that when we got the optimal harvesting strategy that advised a seasonal harvesting, we just used the knowledge of the Black Sea anchovy population and applied an optimal control strategy for 9 months, not just for 3 months. This shows the value of optimal control theory in finding an appropriate harvesting strategy for a given aquatic system.

Including advection and diffusion to our dynamical system, we obtain extra information about the Black Sea anchovy fishery in terms of spatial features. For example, when the speed of advective movement is increased, then the landing and the net profit of the fishery are decreased on the southern part of the Black Sea. Increasing the diffusion rate has a similar effect. Thus, it is important to include the effects of advection and diffusion in estimate of a spatial harvesting strategy. Furthermore, the speed of migration (the advection rate) plays a significant role in harvesting areas presented by optimal harvesting strategy. When the speed of anchovy migration is high, the harvesting regions shift to the east with the shift of the anchovy population and the harvest level is lower, due to some anchovy leaving the region. (See the right plot in Figure 6).

The spatial optimal control strategy gives better profit even if it offers less landing when we compare with the constant harvest rate, 0.35. The spatial optimal control strategy promises a landing that is close to the annual average of anchovy landing data ([35]). The spatial harvesting strategy obtained in the study is also very realistic in terms of the recommended harvesting areas ([12]). In the spatial optimal harvesting strategy, we harvest the stocks mostly on the eastern part of the Turkish coast due to the advection movement.

We made our analysis just for one fishing season, and we are planning to construct a model for more than one season. One needs to model carefully the migration across the northern coast and the return to the southern coast. One also needs to understand other features with seasonality like birth rates.
REFERENCES

[1] K. S. Chaudhuri and S. S. Ray, On the combined harvesting of a prey predator system, *Biol. Syst.*, **4** (1996), 373–389.

[2] K. R. De Silva, T. V. Phan, T. and S. Lenhart, Advection control in parabolic PDE systems for competitive populations, *Discrete Contin. Dyn. Syst. Ser. B*, **22** (2017), 1049–1072.

[3] M. Demir, *Optimal Control Strategies in Ecosystem-Based Fishery Models*, Ph.D dissertation, University of Tennessee in Knoxville, 2019. Available from: [https://trace.tennessee.edu/utk_graddiss/5421](https://trace.tennessee.edu/utk_graddiss/5421).

[4] M. Demir and S. Lenhart, Optimal sustainable fishery management of the Black Sea anchovy with food chain modeling framework, *Nat. Resour. Model.*, **33** (2020), 29pp.

[5] W. Ding and S. Lenhart, Optimal harvesting of a spatially explicit fishery model, *Nat. Resour. Model.*, **22** (2009), 173–211.

[6] I. Ekeland and R. Temam, *Convex Analysis and Variational Problems*, Classics in Applied Mathematics, **28**, Society for Industrial and Applied Mathematics (SIAM), Philadelphia, PA, 1999.

[7] L. C. Evans, *Partial Differential Equations*, Graduate Studies in Mathematics, **19**, American Mathematical Society, Providence, RI, 1998.

[8] FAO, *Fisheries Management, Marine Protected Areas and Fisheries*, Food and Agriculture Organization (FAO) of the United Nations, Rome, (2011).

[9] W. J. Fletcher, J. Shaw, S. J. Metcalf and D. J. Gaughan, An ecosystem based fisheries management framework: The efficient, regional-level planning tool for management agencies, *Marine Policy*, **34** (2010), 1226–1238.

[10] E. A. Fulton, A. D. M. Smith, D. C. Smith and P. Johnson, An integrated approach is needed for ecosystem based fisheries management: Insights from ecosystem-level management strategy evaluation, *PLoS One*, **9** (2014).

[11] A. Grishin, G. Daskalov, V. Shlyakhov and V. Mihneva, Influence of gelatinous zoo-plankton on fish stocks in the Black Sea: Analysis of biological time-series, *Marine Ecological J.*, **6** (2007), 5–24.

[12] A. C. Gücü, Y. Genç, M. Dağtekin, S. Sakman and O. Ak, et al., On Black Sea anchovy and its fishery, *Rev. Fisheries Sci. Aquaculture*, **25** (2017), 230–244.

[13] C. R. Gwaltney, M. P. Styczynski and M. A. Stadtherr, Reliable computation of equilibrium states and bifurcations in food chain models, *Comput. Chemical Engrg.*, **28** (2004), 1981–1996.

[14] W. Hackbusch, A numerical method for solving parabolic equations with opposite orientations, *Computing*, **20** (1978), 229–240.

[15] G. E. Herrera and S. Lenhart, Spatial optimal control of renewable resource stocks, in *Spatial Ecology*, CRC Press, 2009, 343–357.

[16] R. Hilborn and C. J. Walters, *Quantitative Fisheries Stock Assessment: Choice, Dynamics and Uncertainty*, Springer, 1992.

[17] J. Hoekstra and J. C. J. M. van den Bergh, Harvesting and conservation in a predator-prey system, *J. Econom. Dynam. Control*, **29** (2005), 1097–1120.

[18] H. R. Joshi, G. E. Herrera, S. Lenhart and M. G. Neubert, Optimal dynamic harvest of a mobile renewable resource, *Nat. Resour. Model.*, **22** (2009), 322–343.

[19] T. K. Kar and K. S. Chaudhuri, Harvesting in a two-prey one-predator fishery: A bioeconomic model, *ANZIAM J.*, **45** (2004), 443–456.

[20] M. R. Kelly Jr., M. G. Neubert and S. Lenhart, Marine reserves and optimal dynamic harvesting when fishing damages habitat, *Theoretical Ecology*, **12** (2019), 131–144.

[21] M. R. Kelly Jr., Y. Xing and S. Lenhart, Optimal fish harvesting for a population modeled by a nonlinear parabolic partial differential equation, *Nat. Resour. Model.*, **29** (2016), 36–70.

[22] T. Lauck, C. W. Clark, M. Mangel and G. R. Munro, Implementing the precautionary principle in fisheries management through marine reserves, *Ecological Appl.*, **8** (1998), S72–S78.

[23] S. Lenhart and J. T. Workman, *Optimal Control Applied to Biological Models*, Chapman & Hall/CRC Mathematical and Computational Biology Series, Chapman & Hall/CRC, Boca Raton, FL, 2007.

[24] R. Miller Neilan, *Optimal Control Applied to Population and Disease Models*, Ph.D. dissertation, University of Tennessee in Knoxville, 2009. Available from: [https://trace.tennessee.edu/utk_graddiss/74/](https://trace.tennessee.edu/utk_graddiss/74/).

[25] R. Miller Neilan and S. Lenhart, Optimal vaccine distribution in a spatiotemporal epidemic model with an application to rabies and raccoons, *J. Math. Anal. Appl.*, **378** (2011), 603–619.
E. A. Moberg, E. Shyu, G. E. Herrera, S. Lenhart, Y. Lou and M. G. Neubert, On the bioeconomics of marine reserves when dispersal evolves, Nat. Resour. Model., 28 (2015), 456–474.

M. G. Neubert, Marine reserves and optimal harvesting, Ecology Lett., 6 (2003), 843–849.

T. Oguz, Controls of multiple stressors on the Black Sea fishery, Frontiers in Marine Sci., 4 (2017).

T. Oguz, E. Akoglu and B. Salihoglu, Current state of overfishing and its regional differences in the Black Sea, Ocean & Coastal Mgmt., 58 (2012), 47–56.

T. Oguz, B. Fach and B. Salihoglu, Invasion dynamics of the alien ctenophore Mnemiopsis leidyi and its impact on anchovy collapse in the Black Sea, J. Plankton Res., 30 (2008), 1385–1397.

B. Öztürk, B. A. Fach, Ç. Keskin, S. Arkin and B. Topaloğlu, et al., Prospects for marine protected areas in the Turkish Black Sea, in Management of Marine Protected Areas: A Network Perspective, John Wiley & Sons Ltd., 2017, 247–262.

E. K. Pikitch, C. Santora, E. A. Babcock, A. Bakun and R. Bonfil, et al., Ecosystem-based fishery management, Science, 305 (2004), 346–347.

J. Simon, Compact sets in the space $L^p(0, T, B)$, Ann. Mat. Pura Appl. (4), 146 (1987), 65–96.

M. Skern-Mauritzen, G. Ottersen, N. O. Handegard, G. Huse and G. E. Dingsor, et al., Ecosystem processes are rarely included in tactical fisheries management, Fish and Fisheries, 17 (2016), 165–175.

STECF, Scientific, Technical and Economic Committee for Fisheries (STECF) Black Sea Assessments, Publications Office of the European Union, Luxembourg, EU, 2017, 284pp.

W. J. Ströbele and H. Wacker, The economics of harvesting predator-prey systems, Zeitschr. f. Nationalökonomie, 61 (1995), 65–81.

A. W. Trites, V. Christensen and D. Pauly, Competition between fisheries and marine mammals for prey and primary production in the pacific ocean, J. Northwest Atlantic Fishery Sci., 22 (1997), 173–187.

J. E. Wilen, Spatial management of fisheries, Marine Resource Economics, 19 (2004), 7–19.