Evolution of Cosmological Perturbation in Reheating Phase of the Universe

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Abstract

The evolution of the cosmological perturbation during the oscillatory stage of the scalar field is investigated. For the power law potential of the inflaton field, the evolution equation of the Mukhanov’s gauge invariant variable is reduced to the Mathieu equation and the density perturbation grows by the parametric resonance.

Recently, the importance of the parametric resonance is recognized in the reheating phase of the inflationary model [1,2]. Due to the coherent oscillation of the background inflaton field, the fluctuation of the non-linearly coupled boson field or the fluctuation of self interacting inflaton field itself are amplified and the catastrophic particle creation occurs. This effect drastically changes the scenario of the reheating considered so far. When one pay attention to the evolution of the metric perturbation during the coherent oscillatory phase of the scalar field dominated universe, a naive question arises: does the metric perturbation undergo a influence of the parametric resonance by the background oscillation of the scalar field?

It is well known that the cosmological perturbation of the scalar field in the oscillatory stage has the problematic aspect [3]. If one try to write down the evolution equation for the density contrast or the Newtonian potential, the coefficient of the equation becomes singular periodically in time due to the background oscillation of the scalar field. This
behavior is not real. It appears through the reduction of the constrained system to the second order differential equation. But it makes difficult to understand the behavior of the metric perturbation in the oscillatory phase.

In this paper, we consider the evolution of the metric perturbation using the gauge invariant variable introduced by Mukhanov. The evolution equation for this variable has no singular behavior and is suitable to apply the oscillatory phase of the scalar field. We treat a spatially flat FRW universe with a minimally coupled scalar field with a potential

\[ V(\phi) = \frac{\lambda}{4} \phi_0^4 \left( \frac{\phi}{\phi_0} \right)^{2n}, \quad (n = 1, 2, \ldots) \]  

(1)

The background equations are

\[ H^2 = \frac{\kappa}{3} \left( \frac{1}{2} \dot{\phi}^2 + V(\phi) \right), \]

(2)

\[ \dot{H} = -\frac{\kappa}{2} \ddot{\phi}, \]

(3)

\[ \ddot{\phi} + 3H \dot{\phi} + V_{\phi} = 0, \]

(4)

where \( \kappa = 8\pi G \). The scalar field oscillates if the condition \( \phi \ll m_{pl} \) is satisfied. In such a situation, we cannot have the exact solution, but using the time averaged potential energy and the kinetic energy have the relation

\[ < \dot{\phi}^2 > = 2n < V(\phi) >, \]

(5)

the scale factor and the Hubble parameter can be approximately expressed as

\[ a(t) \approx \left( \frac{t}{t_0} \right)^{\frac{n+1}{3n}}, \quad H \approx \frac{n+1}{3n} \frac{1}{t}. \]

(6)

We define new variables for the scalar field and the time:

\[ \eta = n t_0 a^{-\frac{3}{n+1}} = nt_0 \left( \frac{t}{t_0} \right)^{1/n}, \]

(7)

\[ \phi(t) = \phi_0 a^{-\frac{1}{n+1}} \tilde{\phi}, \]

(8)

where \( \tilde{\phi} = 1 \) at \( t = t_0 \) and \( \phi_0 \) is a initial value of the scalar field\( (\phi_0 \ll m_{pl}) \). Using the new variables, the evolution equation of the scalar field becomes
\[ \ddot{\phi}_{\eta} + m^2 n \phi^{2n-1} = 0, \quad m^2 = \frac{\lambda}{2} \phi_0^2. \]  

(9)

For \( n = 1 \) (massive scalar field), \( \tilde{\phi} = \cos(m\eta) \). For \( n = 2 \), \( \tilde{\phi} = cn(\sqrt{\lambda} \phi_0 \eta; \frac{1}{\sqrt{2}}) \) where \( cn \) is an elliptic function.

We use the gauge invariant variables to treat the perturbation whose wavelength is larger than the horizon scale. We found that the most convenient variable is

\[ Q = \delta \phi - \frac{\dot{\phi}}{H} \psi = \delta \phi^{(g)} - \frac{\dot{\phi}}{H} \Psi, \]

(10)

where \( \psi \) is the perturbation of the three curvature, \( \delta \phi^{(g)} \) is the gauge invariant variable for the scalar field perturbation and \( \Psi \) is the gauge invariant Newtonian potential. For the zero curvature slice, \( Q \) represents the fluctuation of the scalar field. This variable \( Q \) was first introduced by Mukhanov [4]. As already mentioned, the coefficient of the evolution equation for the Newtonian potential or the gauge invariant density contrast becomes singular because of the background oscillation of the scalar field. But the evolution equation for \( Q \) does not have such a singular behavior:

\[ \ddot{Q} + 3H \dot{Q} + \left[ V_{\phi \phi} + \left( \frac{k}{a} \right)^2 + 2 \left( \frac{\dot{H}}{H} + 3H \right) \right] Q = 0. \]  

(11)

The Newtonian potential and the variable \( Q \) is connected by the relation

\[ -\frac{k^2}{a^2} \Psi = \frac{\kappa \dot{\phi}^2}{2H} \left( \frac{H}{\phi} Q \right), \]  

(12)

and the gauge invariant density contrast \( \Delta \) which is equal to \( \left( \frac{\delta \rho}{\rho} \right) \) on the co-moving time slice is

\[ \Delta = \frac{\kappa \dot{\phi}^2}{3H^2} \left( \frac{H}{\phi} Q \right). \]  

(13)

To treat the eq.(11) more tractable, we change the variable

\[ Q = a^{-\frac{n+1}{n+2}} \tilde{Q}, \quad \eta = nt_0 \left( \frac{t}{t_0} \right)^{1/n}, \]

(14)

Then
\[
\tilde{Q}_{\eta\eta} + a \dfrac{\eta(n-1)}{n+1} \left[ V_{\phi\phi} + \left( \dfrac{k}{a} \right)^2 + 2 \left( \dfrac{\dot{H}}{H} + 3H \right) - \dfrac{9n}{(n+1)^2} H^2 - \dfrac{3}{n+1} \dot{H} \right] \tilde{Q} = 0. \quad (15)
\]

Using the background equation, we can estimate the time dependence of the each terms in this equation:

\[
V_{\phi\phi} = n(2n-1)m^2 a \dfrac{\eta^{2n-2}}{\eta^{n+1}} \tilde{\phi}^2 \sim O \left( \left( \dfrac{\eta}{t_0} \right)^{2-2n} \right),
\]

\[
\left( \dfrac{\dot{H}}{H} + 3H \right) = 6H n^n \left( \dfrac{\eta}{t_0} \right)^{1-n} \tilde{\phi}^{2n-1} \tilde{\phi}_\eta \sim O \left( \left( \dfrac{\eta}{t_0} \right)^{1-2n} \right),
\]

\[
H^2, \dot{H} \sim O \left( \left( \dfrac{\eta}{t_0} \right)^{-2n} \right).
\]

As we are considering the situation \( \eta \geq t_0 \), we can neglect the \( H^2, \dot{H} \) terms in eq.(15) because they are higher order in powers of \( (1/\eta) \) compared to other terms. Our basic equation for the gauge invariant variable becomes

\[
\tilde{Q}_{\eta\eta} + \left[ n(2n-1)m^2 \tilde{\phi}^{2n-2} + k^2 a \dfrac{4(n-2)}{n+1} \tilde{\phi}^2 + 4n(n+1) \dfrac{1}{\eta} \tilde{\phi}^{2n-1} \tilde{\phi}_\eta \right] \tilde{Q} = 0. \quad (16)
\]

\( \tilde{\phi} \) is the solution of eq.(9). We first consider \( n = 1 \) case(massive scalar). The background scalar field is sinusoidal and given by

\[
\tilde{\phi} = \cos(m \eta). \quad (17)
\]

Eq.(16) becomes

\[
\tilde{Q}_{\eta\eta} + \left[ m^2 + \left( \dfrac{k}{a} \right)^2 - \dfrac{8}{\eta} \sin(m \eta) \cos(m \eta) \right] \tilde{Q} = 0. \quad (18)
\]

We introduce a dimensionless time variable \( \tau = m \eta \),

\[
\tilde{Q}_{\tau\tau} + \left[ 1 + \left( \dfrac{k}{ma} \right)^2 - \dfrac{4}{\tau} \sin(2\tau) \right] \tilde{Q} = 0, \quad (19)
\]

where \( a \propto \tau^{2/3} \). This equation has the same form as the Mathieu equation:

\[
Y_{\tau\tau} + [A - 2q \sin(2\tau)] Y = 0. \quad (20)
\]

In our case the coefficient \( A, q \) are time dependent functions and the relation between \( A \) and \( q \) is
\[ A = 1 + \left( \frac{mt_0}{2} \right)^{4/3} \left( \frac{k}{m} \right)^2 q^{4/3}. \]  

(21)

Using the stability/instability chart of the Mathieu equation, we can know that the perturbation will have the effect of parametric resonance of the first unstable band of the Mathieu function and grows in time (see figure). We can derive its time evolution by solving eq. (19) using multi-time scale method [5]. We introduce a parameter

\[ \epsilon = \frac{4}{\tau_0}. \]

(22)

As the condition \( \tau_0 \gg 1 \) is equivalent to the condition of coherent oscillation, \( \epsilon \) is a small parameter. Rewrite the eq. (19) as

\[ \tilde{Q}_{\tau \tau} + \left[ 1 + 2\epsilon \omega_1 - \epsilon \frac{\tau_0}{\tau} \sin(2\tau) \right] \tilde{Q} = 0, \quad (\tau \geq \tau_0) \]

(23)

where \( \omega_1 = \frac{1}{2\epsilon} \left( \frac{k}{ma} \right)^2 \). We assume the condition \( \left( \frac{k}{ma} \right)^2 < 1 \) to be the term \( 2\epsilon \omega_1 \) smaller. This means we consider the wavelength larger than the Compton length. We define slow time scale \( \tau_n = \epsilon^n \tau \). The time derivative with respect to \( \tau \) is replaced by

\[ \frac{d}{d\tau} = D_0 + \epsilon D_1 + \cdots, \]

(24)

where \( D_n = \frac{\partial}{\partial \tau_n} \). We expand

\[ \tilde{Q} = Q^{(0)} + \epsilon Q^{(1)} + \cdots. \]

(25)

By substituting these expression to eq. (19) and collect the terms of each order of \( \epsilon \). From the \( O(\epsilon^0) \) and \( O(\epsilon^1) \), we have

\[ O(\epsilon^0): \quad D_0^2 Q^{(0)} + Q^{(0)} = 0, \]

(26)

\[ O(\epsilon^1): \quad D_0^2 Q^{(1)} + Q^{(1)} = -\left( 2D_0 D_1 Q^{(0)} + 2\omega_1 Q^{(0)} - \frac{\tau_0}{\tau} \sin(2\tau) Q^{(0)} \right). \]

(27)

The solution of eq. (26) is

\[ Q^{(0)} = A(\tau_1) e^{i\tau} + A^*(\tau_1) e^{-i\tau}. \]

(28)
We substitute this to the right hand side of eq.(27) and demand that the secular term which is proportional to $e^{i\tau}$ vanishes. This gives the equation for the amplitude $\mathcal{A}$:

$$i\frac{\partial \mathcal{A}}{\partial \tau_1} + \omega_1 A + \frac{\tau_0}{4\tau} A^* = 0.$$  \hspace{1cm} (29)

Using the definition of $\epsilon$ and $\tau_1$, we have

$$i\frac{\partial \mathcal{A}}{\partial \tau} + \frac{1}{2} \left( \frac{k}{ma} \right)^2 \mathcal{A} + \frac{i}{\tau} \mathcal{A}^* = 0.$$  \hspace{1cm} (30)

Writing $\mathcal{A} = u + iv (u, v$ are real), $u$ satisfies the following second order differential equation:

$$u_{\tau\tau} + \frac{4}{3\tau} u_{\tau} + \left[ \frac{1}{4} \left( \frac{k}{ma} \right)^4 - \frac{2}{3\tau} \right] u = 0.$$  \hspace{1cm} (31)

Using the eq.(13) and (28),

$$\Delta \propto \tilde{Q}_{\eta} \tilde{\phi}_{\eta} - \tilde{Q}_{\eta} \tilde{\phi}_{\eta\eta} - \frac{1}{\eta} \tilde{Q}_{\eta} \tilde{\phi} = u + O\left(\frac{u}{\eta}\right).$$  \hspace{1cm} (32)

Therefore the function $u$ is equal to the gauge invariant density contrast $\Delta$ within the approximation we are using here. The solution of eq.(31) is

$$u = \tau^{-1/6} Z_{\pm 5/2} \left( \left( \frac{k}{a} \right)^2 \frac{1}{mH} \right),$$  \hspace{1cm} (33)

where $Z_{\nu}$ is a Bessel function of order $\nu$. We have the critical wavelength $\lambda_J = (mH)^{-1/2}$. The mode whose wavelength is larger than $\lambda_J$ can grow. If the wavelength is shorter than $\lambda_J$ initially, the wavelength is stretched by the cosmic expansion and its exceeds the critical length. We can see this behavior by using the chart of Mathieu function. The trajectory which started from the stable region moves to the unstable region. For $k \to 0$ limit,

$$\Delta \propto \tau^{2/3} = a, \quad \tau^{-1} = a^{-3/2},$$

$$\Psi \propto \text{constant}, \quad \tau^{-5/3} = \frac{H}{a}.$$  \hspace{1cm} (34)

This behavior is the same as the perturbation in the dust dominated universe.

For $n \geq 2$, the scalar filed oscillation is not sinusoidal. We start searching the solution of the equation for $y = \tilde{\phi}_{\eta}$:
$$y_{\eta\eta} + n(2n - 1)m^2 \tilde{\phi}^{2n-2}y = 0. \quad (35)$$

Eq. (16) reduces to this equation if the third and the forth term do not exist. We approximate the solution of eq. (35) by sinusoidal function: $y = m \sin(cm\eta)$. $c$ is a some numerical constant which defines the period of scalar field oscillation. Using the equation for $\tilde{\phi}$, we have

$$n(2n - 1)m^2 \tilde{\phi}^{2n-2} = -\frac{y_{\eta\eta}}{y},$$

$$2n(n + 1)m^2 \tilde{\phi}^{2n-1} \tilde{\eta} = (\tilde{\phi}_{\eta} \tilde{\phi})_{\eta\eta},$$

and using $y = \tilde{\phi}_{\eta} = m \sin(cm\eta)$, the equation of $\tilde{Q}$ becomes

$$\tilde{Q}_{\eta\eta} + \left[ c^2 m^2 - \frac{4cm}{\eta} \sin(2cm\eta) + k^2 a^{\frac{4(n-2)}{n+1}} \right] \tilde{Q} = 0. \quad (36)$$

By introducing the dimensionless time variable $\tau = cm\eta$,

$$\tilde{Q}_{\tau\tau} + \left[ 1 - \frac{4}{\tau} \sin(2\tau) + \left( \frac{k}{cm} \right)^2 a^{\frac{4(n-2)}{n+1}} \right] \tilde{Q} = 0. \quad (37)$$

This is also Mathieu equation. It is surprising that this equation contains the $n = 1$ case if we set $c = 1$. The function $A$ and $q$ are

$$A = 1 + \alpha \tau^{\frac{4(2-n)}{3}}, \quad q = \frac{2}{\tau}, \quad (38)$$

where $\alpha = \left( \frac{k}{cm} \right)^2 (cnt_0m)^{4(n-2)/3}$ and we have the relation

$$A(q) = 1 + \alpha \left( \frac{q}{2} \right)^{\frac{4(2-n)}{3}}. \quad (39)$$

Using the chart of Mathieu function, we find that the perturbation also get the effect of the parametric resonance of the first unstable band and grows. But as the time goes on, the trajectory moves from the unstable region to the stable region and the perturbation will oscillate with a constant amplitude. To investigate these behavior, we introduce slow time variable and derive the equation for slowly changing amplitude of $\tilde{Q}$. The procedure is the completely same as $n = 1$ case. The result is

$$\tilde{Q} = Ae^{i\tau} + A^*e^{-i\tau}, \quad (40)$$

$$i\frac{\partial A}{\partial \tau} + \frac{1}{2} \left( \frac{k}{cm} \right)^2 a^{\frac{4(n-2)}{n+1}} A + i\frac{1}{\tau} A^* = 0. \quad (41)$$
The real part of $\mathcal{A}$ obeys

$$u_{\tau\tau} - \frac{4}{3}(n - 2)\frac{1}{\tau}u_{\tau} + \left[\frac{\alpha^2}{4}\tau^{8(n-2)} - \left(\frac{4}{3}n - 2\right)\frac{1}{\tau^2}\right]u = 0. \quad (42)$$

The solution of this equation is

$$u = \tau^{\frac{1}{6}(4n-5)}Z_\nu\left(\frac{3\alpha}{2(4n-5)}\tau^{\frac{1}{3}(4n-5)}\right), \quad \nu = \pm \frac{4n + 1}{2(4n - 5)}. \quad (43)$$

For $k \to 0$ limit, we have

$$\Delta \propto \tau^{\frac{2}{3}(2n-1)} = a^{\frac{2(2n-1)}{n+1}}, \quad \tau^{-1} = a^{-\frac{3}{n+1}}, \quad \Psi \propto \text{constant}, \quad \tau^{-\frac{1}{3}(4n+1)} = \frac{H}{a}. \quad (44)$$

In summary, we found that the evolution equation of the Mukhanov’s gauge invariant variable in the oscillatory phase of the scalar field can be reduced to the Mathieu equation and the evolution of this variable undergoes the effect of the parametric resonance. We can interpret the growth of the density perturbation in this phase is caused by the parametric resonance. Now we comment on previous works. In paper [6], the analysis is done by using the Newtonian approximation which means the wavelength of the perturbation is smaller than the horizon length. But the obtained equation for $(\delta \rho/\rho)$ coincides with the result of this paper(eq.(31)). In paper [7], the long wave approximation is used. As the eq.(11) has the exact solution for $k = 0$, they take in the effect of small $k$ perturbatively and derive the evolution of the gauge invariant variables whose wavelength is larger than the horizon length. The assumption we used in this paper is the wavelength is larger than the Compton length which is smaller than the horizon scale in the oscillatory phase of the scalar field. So our treatment is more general. Extension to the non-linearly interacting two scalar field system which is a realistic model of the reheating is straightforward and the analysis is now going on. We will show the result in a separate publication.
The stability/instability chart of the Mathieu equation (20). The grey is stable and the white is unstable region. Three curves show the time evolution of the parameter $A, q$ for the power index $n$ of the scalar field potential $V = \frac{A}{4} \phi_0^4 \left( \frac{\phi}{\phi_0} \right)^2 n$. $A = 1$ line corresponds to $k = 0$. 
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