4D $\mathcal{N} = 1$ SYM supercurrent in terms of the gradient flow

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The gradient flow and its small flow-time expansion provide a very versatile method to represent renormalized composite operators in a regularization-independent manner. This technique has been utilized to construct typical Noether currents such as the energy–momentum tensor and the axial-vector current in lattice gauge theory. In this paper, we apply the same technique to the supercurrent in the four-dimensional $\mathcal{N} = 1$ super Yang–Mills theory (4D $\mathcal{N} = 1$ SYM) in the Wess–Zumino gauge. Since this approach provides a priori a representation of the properly normalized conserved supercurrent, our result should be useful, e.g., in lattice numerical simulations of the 4D $\mathcal{N} = 1$ SYM; the conservation of the so-constructed supercurrent can be used as a criterion for the supersymmetric point toward which the gluino mass is tuned.
1. Introduction

The so-called gradient flow (Refs. [1–5]) possesses a remarkable renormalization property that any local product (i.e., composite operator) composed from bare fields evolved by the flow automatically becomes a renormalized composite operator (Refs. [4, 6]). By utilizing this property, one can express physical quantities, such as a nonperturbative gauge coupling (Refs. [4, 7–9]), the topological charge (Refs. [3, 10]), the chiral condensate (Ref. [5]), and the quasi-parton distribution functions (Ref. [11] etc. in terms of bare fields evolved by the flow. Such finite representations of physical quantities are universal, i.e., independent of the regularization and are particularly useful in the context of lattice gauge theory. See Ref. [12] for a review. In general, however, to find such a representation of a desired physical quantity in terms of the flowed fields requires an ingenious argument quantity by quantity.

On the other hand, one can always employ the small flow-time expansion (Ref. [4]) to express any renormalized local composite operator in terms of the flowed fields. The combination of the gradient flow and the small flow-time expansion thus provides a very versatile method to represent a renormalized composite operator in a regularization-independent manner. This technique has been utilized to construct typical Noether currents such as the energy–momentum tensor (Refs. [13–15]) (see also Ref. [16] for a related study) and the axial-vector current (Refs. [17, 18]). The resulting representations have been numerically examined/applied in the quenched and 2 + 1 flavor QCD (Refs. [19–22]) and analytically examined in some solvable models (Refs. [23–25]).

In this paper, we apply the above technique to find a representation of the supercurrent in the four-dimensional $\mathcal{N}=1$ super Yang–Mills theory (4D $\mathcal{N}=1$ SYM) (Refs. [26–29]). A generalization of the gradient flow to this system in its off-shell supermultiplet was developed in Ref. [30] by partially aiming at identical renormalizations between the flowed gauge field and the flowed fermion (i.e., gaugino) field through the off-shell supersymmetry. Our objective here is much modest: Having the application in the conventional lattice gauge theory in mind, we take the Wess–Zumino gauge, which contains only conventional dynamical fields but preserves only the on-shell supersymmetry. Our approach provides a priori the properly normalized and conserved supercurrent, our result should be useful, e.g., in lattice numerical simulations of the 4D $\mathcal{N}=1$ SYM (Refs. [31–53]; Ref. [54] is a recent review); the conservation of the so-constructed supercurrent can be used as a criterion for the supersymmetric point toward which the gluino mass is tuned. For the 4D $\mathcal{N}=1$ SYM, one may alternatively invoke the chiral symmetry to find the supersymmetric point. A tuning of parameters to the supersymmetric point will really become demanding, however, in supersymmetric theories containing matter multiplets. In such theories, a priori knowledge of the conserved supercurrent will be quite helpful to find the supersymmetric point. Thus, the present work can be regarded as the first step toward a flow-time representation of the supercurrent in such complicated supersymmetric systems.

Now, to find a representation of the properly normalized conserved supercurrent in terms of the flowed fields, we have to know the expression of the former at least in perturbation theory. For the energy–momentum tensor (Refs. [13, 14]), the all-order expression is readily available

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1 Up to the wave-function renormalization of the flowed matter fields; see below.

2 In Refs. [17, 18], the vector current and the (pseudo-) scalar density are also studied.
in dimensional regularization (see, e.g., Refs. [55, 56]) because this regularization preserves the translational invariance exactly. For the axial-vector current, the naive expression in dimensional regularization must be corrected appropriately (Ref. [57]) so that it fulfills the corresponding Ward–Takahashi (WT) relations. It is not difficult to carry out this correction, at least in the one-loop order, because the chiral transformation acts only on the fermion fields only linearly; in Refs. [17, 18], the one-loop corrected expression in the dimensional regularization was employed.

Unfortunately, the situation is much more complicated for the supercurrent in the Wess–Zumino gauge. First, there is no regularization that manifestly preserves supersymmetry; we thus adopt dimensional regularization in what follows for computational convenience. Second, the super transformation acts on both the gauge field and the fermion field. Then the WT relations necessarily contain contributions from the gauge-fixing term and the Faddeev–Popov ghost term, which are neither gauge invariant nor supersymmetric in the Wess–Zumino gauge. Finally, the super transformation is nonlinear in the Wess–Zumino gauge and thus the WT relations necessarily contain composite operators. Even though the gauge field and the fermion field are related by supersymmetry, the wave-function renormalization factors for those fields differ in the Wess–Zumino gauge. The validity of supersymmetry WT relations in the Wess-Zumino gauge thus crucially depends on the renormalization of composite operators appearing in the WT relations. The fact that gauge invariance is broken by the gauge-fixing term and the Faddeev–Popov term further complicate the situation, because one has also to take into account the operator mixing with gauge noninvariant operators. In short, to find the correct expression of the supercurrent, one has to fully understand the renormalization/mixing structure of composite operators appearing in supersymmetry WT relations.

Somewhat surprisingly, to our knowledge, this program to find an explicit form of the supercurrent in the 4D $\mathcal{N} = 1$ SYM in the Wess–Zumino gauge (e.g., in dimensional regularization) has not been carried out thoroughly in the literature. The only exception we could find is Ref. [61] but in this reference only the case of the abelian gauge theory is studied. Thus, we had to carry out this program by ourselves; Sect. 2 is devoted to this complicated analysis in the one-loop order. Our conclusion is that in dimensional regularization, the properly normalized supercurrent in correlation functions of gauge invariant operators

3 One might think that the use of the dimensional reduction (Ref. [58]) rather simplifies our task. However, since the Fierz identity must be given up with the dimensional reduction (Ref. [59, 60]), we could not find that the dimensional reduction is particularly useful for our purpose.
(which do not contain the ghost field and the anti-ghost field) is given by

\[ S_{\mu R}(x) = \frac{1}{2g_0}g_0\rho_\sigma \gamma_\mu \psi^\sigma(x)F^{a\sigma}_\rho(x) + \mathcal{O}(g_0^3), \tag{1.2} \]

where \( g_0 \) is the bare gauge coupling, \( \psi^\sigma(x) \) is the bare gaugino field and \( F^{a\sigma}_\rho(x) \equiv \partial_\rho A^a_\sigma(x) - \partial_\sigma A^a_\rho(x) + f^{abc}A^b_\rho(x)A^c_\sigma(x) \) is the bare field strength. The expression to this order is thus identical to the naive classical expression of the supercurrent; we emphasize however that Eq. (1.2) is the result of a lengthy analysis of the renormalization/mixing of composite operators in supersymmetry WT relations.

Once Eq. (1.2) is obtained, it is basically straightforward to express the supercurrent in terms of the small flow-time limit of flowed fields. This is done in Sect. 3 after some calculation, we have

\[ S_{\mu R}(x) = \lim_{t \to 0} \left( -\frac{1}{2g_0(1/\sqrt{8t})} \left\{ 1 + \frac{g(1/\sqrt{8t})}{(4\pi)^2} C_2(G) \left[ -\frac{7}{2} + \frac{3}{2} \ln \pi + \frac{1}{2} \ln(432) \right] \right\} \right. \]

\[ \times \sigma_\rho \gamma_\mu \chi^a(t, x)G^a_\rho(t, x) \]

\[ \left. -\frac{g(1/\sqrt{8t})}{(4\pi)^2} C_2(G)3\gamma_\mu \chi^a(t, x)G^a_\rho(t, x) \right). \tag{1.3} \]

In this expression, \( \bar{g}(\mu) \), \( \chi^a(t, x) \), and \( G^a_\rho(t, x) \) are the running gauge coupling in the minimal subtraction (MS) scheme at the renormalization scale \( \mu \), the flowed gaugino field, and the flowed field strength, respectively; the precise definitions of these quantities will be given in Sect. 3. At this point, it is interesting to note that the above expression reproduces the gamma-trace anomaly (superconformal anomaly) \([62, 70]\) \( \gamma_\mu S_{\mu R}(x) = -g/(4\pi)^2 C_2(G)3\sigma_\mu \psi^\sigma(x)F^{a\mu}_\rho(x) + \mathcal{O}(g^3) \) at least in the one-loop order, because \( \gamma_\mu \sigma_\rho \gamma_\mu = 0 \) for \( D = 4 \) and flowed fields simply reduce to the corresponding un-flowed fields in the \( t \to 0 \) limit in the lowest order of perturbation theory. This is reassuring, because the properly normalized conserved supercurrent must possess this gamma-trace anomaly.\(^5\)

Section 4 is devoted to the conclusion. In Appendix A we clarify how the charge conjugation matrix should be treated in dimensional regularization, because a description of this issue is also hard to find in the literature.

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4 Notation: Without noting otherwise, repeated indices are understood to be summed over. The generators \( T^a \) of the gauge group \( G \) are anti-Hermitian and the structure constants are defined by \([T^a, T^b] = f^{abc}T^c\). Quadratic Casimirs are defined by \( f^{acd}f^{bcd} = C_2(G)\delta^{ab} \) and, for a gauge representation \( R \), \( \text{tr}_R(T^aT^b) = -T(R)\delta^{ab} \) and \( T^aT^a = -C_2(R)1 \). We also denote \( \text{tr}_R(1) = \dim(R) \). For the fundamental \( N \) representation of \( SU(N) \) for which \( \dim(N) = N \), our choice is

\[ C_2(SU(N)) = N, \quad T(N) = \frac{1}{2}, \quad C_2(N) = \frac{N^2 - 1}{2N}. \tag{1.1} \]

Our Dirac matrices \( \gamma_\mu \) are all Hermitian and for the trace over the spinor index we set \( \text{tr}(1) = 4 \) for any spacetime dimension \( D \). The chiral matrix is defined by \( \gamma_5 = \gamma_0\gamma_1\gamma_2\gamma_3 \) for any \( D \). We also use the symbol \( \sigma_{\mu\nu} = (1/2)[\gamma_\mu, \gamma_\nu] \).

5 Although \( g(1/\sqrt{8t}) \to 0 \) as \( t \to 0 \) due to the asymptotic freedom, \( \mathcal{O}(g(1/\sqrt{8t})) \) terms still provide useful information because they tell us how the representation approaches the real supercurrent as \( t \to 0 \); we cannot simply set \( t \to 0 \) in lattice numerical simulations with finite lattice spacings. See Refs. [19, 21] for the situation for a similar representation of the energy–momentum tensor.\(^6\)

6 In Appendix B by using the information of the superconformal anomaly to the two-loop order, we further improve the small flow-time representation \( \mathcal{I}_3 \).
2. Properly normalized conserved supercurrent in dimensional regularization

2.1. Actions, the super transformation and the BRS transformation

In what follows, we denote the spacetime dimension as $D \equiv 4 - 2\epsilon$, assuming dimensional regularization. The classical action of the $\mathcal{N} = 1$ SYM in the Wess–Zumino gauge is given by

$$S = \frac{1}{4g_0^2} \int d^D x \, F^a_{\mu \nu}(x) F^a_{\mu \nu}(x) + \frac{1}{2} \int d^D x \, \bar{\psi}^a(x) \gamma^{ab} \psi^b(x), \quad (2.1)$$

where the covariant derivative is defined in the adjoint representation,

$$D^a_{\mu} \equiv \delta^a_{\mu} \partial_{\mu} + f^{acb} A^c_{\mu}(x),$$

and $\gamma^{ab} \equiv \gamma_{\mu} D_{\mu}^{ab}$.

The gaugino field $\psi(x)$ is the Majorana spinor. This implies that $\bar{\psi}(x)$ is not an independent dynamical variable and it is given from $\psi(x)$ by

$$\bar{\psi}(x) = \psi^T(x)(-C^{-1}), \quad (2.3)$$

where $T$ denotes the transpose on the Dirac index and $C$ is the charge conjugation matrix.

The properties of $C$ in dimensional regularization are summarized in Appendix A. In particular, the matrix $C^{-1} \gamma_{\mu}$ is symmetric in its Dirac indices for any $D$; this is crucial for the action (2.1) to be meaningful for any $D$.

The super transformation in the Wess-Zumino gauge is given by

$$\delta_{\xi_{\mu}} A^a_{\mu}(x) = g_0 \bar{\xi} \gamma_{\mu} \psi(x), \quad (2.4)$$

$$\delta_{\xi} \psi(x) = -\frac{1}{2g_0} \sigma_{\mu \nu} \xi F_{\mu \nu}(x), \quad \delta_{\xi} \bar{\psi}(x) = \frac{1}{2g_0} \bar{\xi} \sigma_{\mu \nu} F_{\mu \nu}(x), \quad (2.5)$$

where the parameter $\xi$ is also the Majorana spinor:

$$\bar{\xi} = \xi^T(-C^{-1}). \quad (2.6)$$

The classical action (2.1) with $D = 4$ is invariant under this transformation (see below).

To carry out perturbation theory, we also introduce the gauge-fixing term

$$S_{\text{gf}} = \frac{\lambda_0}{2g_0} \int d^D x \, \partial_{\mu} A^a_{\mu}(x) \partial_{\nu} A^a_{\nu}(x), \quad (2.7)$$

where $\lambda_0$ is the bare gauge-fixing parameter and the corresponding Faddeev–Popov ghost term

$$S_{\bar{c}c} = -\frac{1}{g_0} \int d^D x \, \bar{c}^a(x) \partial_{\mu} D_{\mu}^{ab} c^b(x). \quad (2.8)$$

c^b(x) is the ghost field and $\bar{c}^a(x)$ is the anti-ghost field. Then the whole action $S + S_{\text{gf}} + S_{\bar{c}c}$ is invariant under the following BRS transformation:

$$\delta_B A^a_{\mu}(x) = D^{ab}_{\mu} c^b(x), \quad \delta_B c^a(x) = -\frac{1}{2} f^{abc} c^b(x) c^c(x), \quad (2.9)$$

$$\delta_B \bar{c}^a(x) = \lambda_0 \partial_{\mu} A^a_{\mu}(x),$$

$$\delta_B \psi^a(x) = -f^{abc} c^b(x) \psi^c(x), \quad \delta_B \bar{\psi}^a(x) = -f^{abc} c^b(x) \bar{\psi}^c(x). \quad (2.10)$$
2.2. Bare Ward–Takahashi relations in the gauge-fixed theory

Now, the classical expression of the supercurrent can be found by making the parameter $\xi$ in Eqs. (2.4) and (2.5) local, $\xi \rightarrow \xi(x)$. Then, the variation of the action (2.1) is given by

$$\delta \xi S = \int d^D x \left[ \partial_\mu \xi(x) S_\mu(x) - \xi(x) X_{\text{Fierz}}(x) \right],$$  \hspace{1cm} (2.12)

where the classical supercurrent is given by

$$S_\mu(x) \equiv -\frac{1}{2g_0} \sigma_\rho \gamma_\mu \bar{\psi}^a(x) F^a_{\rho\sigma}(x),$$  \hspace{1cm} (2.13)

and

$$X_{\text{Fierz}}(x) \equiv \frac{1}{2g_0} f^{abc} \partial_\mu \bar{c}^a(x) c^b(x) \gamma_\mu \psi^c(x).$$  \hspace{1cm} (2.14)

This $X_{\text{Fierz}}(x)$ identically vanishes for $D = 4$ because of the Fierz identity. For $D \neq 4$, however, $X_{\text{Fierz}}(x)$ does not vanish and we will see that in quantum theory, because of the UV divergences, this breaking term gives rise to a nonzero contribution even for $D \rightarrow 4$.

We assume that the Faddeev–Popov ghost and anti-ghost are not transformed under supersymmetry. Then we have the following breaking terms from the gauge-fixing term and the ghost term:

$$\delta \xi (S_{\text{gf}} + S_{\text{gc}}) = -\int d^D x \xi(x) [X_{\text{gf}}(x) + X_{\text{gc}}(x)],$$  \hspace{1cm} (2.15)

where

$$X_{\text{gf}}(x) \equiv \frac{\lambda_0}{g_0} \partial_\mu \partial_\nu A^a_{\nu}(x) \gamma_\mu \bar{\psi}^a(x),$$  \hspace{1cm} (2.16)

and

$$X_{\text{gc}}(x) \equiv \frac{1}{g_0} f^{abc} \partial_\mu \bar{c}^a(x) c^b(x) \gamma_\mu \psi^c(x).$$  \hspace{1cm} (2.17)

We note that $X_{\text{gf}}(x) + X_{\text{gc}}(x)$ is BRS exact:

$$X_{\text{gf}}(x) + X_{\text{gc}}(x) = \hat{\delta}_B \frac{1}{g_0} \partial_\mu \bar{c}^a(x) \gamma_\mu \psi^a(x)$$  \hspace{1cm} (2.18)

and thus $X_{\text{gf}}(x) + X_{\text{gc}}(x)$ does not contribute in correlation functions of gauge invariant operators.

Now, taking the functional integrals (here $d\mu$ denotes the functional measure of all field variables),

$$\int d\mu e^{-S - S_{\text{gf}} - S_{\text{gc}}} A^b_{\alpha}(y) \bar{\psi}^c(z),$$  \hspace{1cm} (2.19)

and

$$\int d\mu e^{-S - S_{\text{gf}} - S_{\text{gc}}} \bar{\psi}^b(y) c^e(z) \bar{c}^d(w),$$  \hspace{1cm} (2.20)

and considering the change of integration variables of the above form, we have the following Ward–Takahashi relations:

$$\left\langle [\partial_\mu S_\mu(x) + X_{\text{Fierz}}(x) + X_{\text{gf}}(x) + X_{\text{gc}}(x)] A^b_{\alpha}(y) \bar{\psi}^c(z) \right\rangle$$

$$= -\delta(x - y) \left\langle g_0 \gamma_\alpha \bar{\psi}^b(y) \bar{\psi}^c(z) \right\rangle - \delta(x - z) \left\langle A^b_{\alpha}(y) \frac{1}{2g_0} \sigma_\beta \gamma F^c_{\beta\gamma}(z) \right\rangle,$$  \hspace{1cm} (2.21)
and
\[
\left\langle \left[ \partial_\mu S_\mu(x) + X_{\text{Fierz}}(x) + X_g(x) + X_{\bar{c}c}(x) \right] \bar{\psi}^b(y)c^c(z)c^d(w) \right\rangle \\
= -\delta(x - y) \left\langle \frac{1}{2g_0} \sigma_{\beta\gamma} F^b_{\beta\gamma}(y)c^c(z)c^d(w) \right\rangle ,
\]
(2.22)
where we have used the fact that the ghost and the anti-ghost are not transformed under the super transformation. These are identities holding under dimensional regularization.

2.3. The effect of \( X_{\text{Fierz}}(x) \)

Let us first examine the effect of the supersymmetry breaking term \( X_{\text{Fierz}}(x) \) (2.14). Since it vanishes for \( \epsilon \to 0 \) (where \( D = 4 - 2\epsilon \)) in the classical theory, the term can contribute only if it is multiplied by \( 1/\epsilon \), i.e., only through UV divergences in loop diagrams. The unique one-loop 1PI diagram that contributes to the left-hand side of the WT relation (2.21) is diagram i in Fig. 2.

Also for the left-hand side of Eq. (2.22) only diagram i in Fig. 1 contributes in the one-loop level because the gaugino fields should make a loop; in Eq. (2.22), the ghost fields are simply spectators with respect to \( X_{\text{Fierz}}(x) \). The computation of diagram i yields

\[
X_{\text{Fierz}}(x) \overset{D \to 4}{\longrightarrow} \frac{g_0}{(4\pi)^2} C_2(G) \frac{2}{3} \partial_\mu F^a_{\mu\nu}(x) \gamma_\nu \psi^a(x).
\]
(2.23)

This tells that we have to add the following finite counterterm \( S' \) to the original action \( S \),
\[
S' = -\frac{1}{(4\pi)^2} C_2(G) \frac{1}{6} \int d^D x F^a_{\mu\nu}(x) F^a_{\mu\nu}(x) ,
\]
(2.24)
so that the supervariation of this term compensates the effect of \( X_{\text{Fierz}}(x) \) (up to \( \mathcal{O}(A_\mu^2) \) terms). We thus have modified WT relations,
\[
\left\langle \left[ \partial_\mu S_\mu(x) + X_g(x) + X_{\bar{c}c}(x) \right] A^b_{\alpha}(y)\bar{\psi}^c(z) \right\rangle' \\
= -\delta(x - y) \left\langle A^b_{\alpha}(y) \frac{1}{2g_0} \sigma_{\beta\gamma} F^b_{\beta\gamma}(z) \right\rangle' - \delta(x - z) \left\langle A^b_{\alpha}(y) \frac{1}{2g_0} \sigma_{\beta\gamma} F^b_{\beta\gamma}(z) \right\rangle' ,
\]
(2.25)
and
\[
\left\langle \left[ \partial_\mu S_\mu(x) + X_g(x) + X_{\bar{c}c}(x) \right] \bar{\psi}^b(y)c^c(z)c^d(w) \right\rangle' \\
= -\delta(x - y) \left\langle \frac{1}{2g_0} \sigma_{\beta\gamma} F^b_{\beta\gamma}(y)c^c(z)c^d(w) \right\rangle' .
\]
(2.26)

In what follows, the wavy line stands for the gauge boson propagator, the solid line the gaugino propagator, and the broken line the ghost propagator; the blob denotes the composite operator under consideration.
In these expressions, the prime (') implies that expectation values are evaluated with respect to the modified action $S + S_{gf} + S_{c\bar{c}} + S'$, where $S'$ is given by Eq. (2.24). The effect of $X_{\text{Fierz}}(x)$ in dimensional regularization is absorbed in this way.

2.4. Renormalization of composite operators

Next, we study the renormalization/mixing of the composite operators appearing in Eqs. (2.25) and (2.26). In what follows, we adopt the MS scheme and set

$$\Delta \equiv \frac{g^2}{(4\pi)^2} C_2(G) \frac{1}{\epsilon}, \quad \text{(2.27)}$$

where $g$ is the renormalized gauge coupling. In the Feynman gauge $\lambda_0 = 1$, in the one-loop order, bare and renormalized quantities in the $\mathcal{N} = 1$ SYM in the Wess–Zumino gauge are related as (in what follows, quantities without the subscript 0 and quantities with the subscript $R$ denote renormalized quantities)

$$g_0 = \mu^\epsilon \left(1 - \frac{3}{2} \Delta\right) g, \quad \text{(2.28)}$$
$$\lambda_0 = (1 - \Delta) \lambda, \quad \text{(2.29)}$$
$$A^a_\mu(x) = (1 - \Delta) A^a_{\mu R}(x), \quad \text{(2.30)}$$
$$\{\psi^a(x)\} = \left(1 - \frac{1}{2} \Delta\right) \left\{\psi^a_{\mu R}(x), \psi^a_{\nu R}(x)\right\}, \quad \text{(2.31)}$$
$$\{c^a(x)\} = \left(1 - \frac{5}{4} \Delta\right) \left\{c^a_{\mu R}(x), c^a_{\nu R}(x)\right\}, \quad \text{(2.32)}$$

where $\mu$ is the renormalization scale and

$$F^a_{\mu \nu}(x) = \left(1 - \frac{5}{2} \Delta\right) \left[\partial_\mu A^a_{\nu R}(x) - \partial_\nu A^a_{\mu R}(x)\right] + \left(1 - \frac{11}{4} \Delta\right) \left\{f^{abc} A^b_{\mu R} A^c_{\nu R}\right\}(x). \quad \text{(2.33)}$$

The last term denotes the renormalized composite operator corresponding to $f^{abc} A^b_{\mu}(x) A^c_{\nu}(x)$.

Let us next consider the renormalization of the composite operators $X_{gf}(x)$ in Eq. (2.16) and $X_{c\bar{c}}(x)$ in Eq. (2.17). By substituting bare quantities in Eqs. (2.16) and (2.17) by Eqs. (2.28)–(2.32), in the one-loop order, we have

$$X_{gf}(x) = (1 - \Delta) \frac{\lambda}{g} \partial_\mu \partial_\nu A^a_{\nu R}(x) \gamma_\mu \psi^a_{\nu R}(x), \quad \text{(2.34)}$$
$$X_{c\bar{c}}(x) = \left(1 - \frac{3}{2} \Delta\right) \frac{1}{g} f^{abc} \partial_\mu c^b_{\nu R}(x) c^a_{\nu R}(x) \gamma_\mu \bar{\psi}_{R}(x). \quad \text{(2.35)}$$

The UV divergences (being proportional to $\Delta$) in these expressions arise from self-energy corrections in external lines of the composite operators and the renormalization of involved parameters. Besides these divergences, the composite operators produce further divergences associated with the vertex part (i.e., 1PI part) containing the composite operators. Relevant one-loop diagrams are depicted in Figs. 2–7.

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8 The counterterm $S'$ in Eq. (2.24) does not influence the following one-loop analysis of the renormalization of composite operators, because $S'$ is a one-loop order quantity and it is UV finite.
In the Feynman gauge $\lambda_0 = 1$, we find that the sum of divergent parts of these diagrams is given by

\[
\left[ \frac{\lambda}{g} \partial_\mu \partial_\nu A^a_{\nu R}(x) \gamma^\mu \psi^a_R(x) + \frac{1}{g} f^{abc} \partial_\mu c^a_{R}(x)c^b_R(x) \gamma_\mu \psi^c_R(x) \right]_{1PI, \text{divergent part}}
\]

\[= 2\Delta \frac{\lambda}{g} \partial_\mu \partial_\nu A^a_{\nu R}(x) \gamma^\mu \psi^a_R(x) + \frac{1}{2} \Delta \frac{1}{g} f^{abc} \partial_\mu c^a_{R}(x)c^b_R(x) \gamma_\mu \psi^c_R(x)\]

\[+ \Delta \partial_\mu \left\{ -\frac{1}{2g} \sigma_{\rho \sigma} \gamma^\mu \psi^a_R(x) \left[ \partial_\rho A^a_{\sigma R}(x) - \partial_\sigma A^a_{\rho R}(x) \right] \right\} \]

\[+ 2\Delta \left( -\frac{1}{g^2} \right) \partial_\mu \partial_\sigma A^a_{\nu R}(x) g^{\gamma \nu} \psi^a_R(x)\]

\[+ \frac{3}{2} \frac{\Delta}{2g} \left[ \partial_\mu A^a_{\nu R}(x) - \partial_\nu A^a_{\mu R}(x) \right] \sigma_{\mu \nu} \psi^a_R(x)\]

\[+ \Delta \frac{1}{4g} \partial_\mu \left\{ \left[ A^a_{\nu R}(x) \gamma_\nu \gamma^\mu + 2A^a_{\mu R}(x) \right] \psi^a_R(x) \right\} + \Delta O(A^2_{\mu R}). \quad (2.36)\]

Since here we are considering only correlation functions with at most one $A_\mu$-line, we cannot determine the last $\Delta O(A^2_{\mu R})$ term; we will fix the associated ambiguity in the supercurrent by imposing gauge invariance in the very final stage of our analysis.
Then, from the sum of Eqs. (2.34), (2.35), and (2.36), we see that to one-loop order all the divergences are canceled out in $X_{gIR}(x)$ and in $X_{ceR}(x)$ defined by

$$X_{gR}(x) + X_{ce}(x)$$

$$= (1 + \Delta)X_{gIR}(x) + (1 - \Delta)X_{ceR}(x)$$

$$+ \Delta \partial_\mu \left\{ -\frac{1}{2g} \sigma_{\rho\sigma} \gamma_\mu \psi_R^a(x) \left[ \partial_\rho A_{\sigma R}^a(x) - \partial_\sigma A_{\rho R}^a(x) \right] \right\}$$

$$+ 2\Delta \left( -\frac{1}{g^2} \right) \partial_\mu \partial_\nu A_{\nu R}^a(x) g \gamma_\nu \psi_R^a(x)$$

$$+ \frac{3}{2} \Delta \frac{1}{2g} \left[ \partial_\mu A_{\nu R}^a(x) - \partial_\nu A_{\mu R}^a(x) \right] \sigma_{\mu\nu} \phi^a_R(x)$$

$$+ \Delta \frac{1}{4g} \partial_\mu \left\{ [A_{\nu R}^a(x) \gamma_\nu \gamma_\mu + 2A_{\mu R}^a(x)] \phi^a_R(x) \right\} + \Delta \mathcal{O}(A_{\mu R}^2). \quad (2.37)$$

Here, we have assumed that $X_{gIR}(x)$ and $X_{ceR}(x)$ in the tree level are given by bare ones, $X_{gR}(x)$ and $X_{ce}(x)$, respectively. Thus, $X_{gIR}(x)$ and $X_{ceR}(x)$ in the above expression are renormalized finite composite operators corresponding to $X_{gR}(x)$ and $X_{ce}(x)$, respectively.

Next, we analyze the supercurrent $S_\mu(x)$ in Eq. (2.13). The substitutions (2.28), (2.30), and (2.31) give rise to

$$S_\mu(x) = -\frac{1}{2g} \sigma_{\rho\sigma} \gamma_\mu \psi_R^a(x) \left[ \partial_\rho A_{\sigma R}^a(x) - \partial_\sigma A_{\rho R}^a(x) + f^{abc} A_{\rho R}^b(x) A_{\sigma R}^c(x) \right] + \Delta \mathcal{O}(A_{\mu R}^2). \quad (2.38)$$

Here, note that the coefficient of the $f^{abc} A_{\rho R}^b(x) A_{\sigma R}^c(x)$ term is uniquely fixed because the $\Delta \mathcal{O}(A_{\mu R}^2)$ term should be proportional to $\Delta$. On the other hand, a somewhat tedious calculation of diagrams in Figs. 2, 3, 8, and 9 shows that

$$-\frac{1}{2g} \sigma_{\rho\sigma} \gamma_\mu \psi_R^a(x) \left[ \partial_\rho A_{\sigma R}^a(x) - \partial_\sigma A_{\rho R}^a(x) + f^{abc} A_{\rho R}^b(x) A_{\sigma R}^c(x) \right]_{\text{IPI, divergent part}}$$

$$= -\Delta \frac{1}{4g} \left[ A_{\nu R}^a(x) \gamma_\nu \gamma_\mu + 2A_{\mu R}^a(x) \right] \phi^a_R(x) + \Delta \mathcal{O}(A_{\mu R}^2). \quad (2.39)$$

Then from the sum of Eqs. (2.38) and (2.39), we see that all divergences are canceled out in the combination $S_{\mu R}(x)$, defined by

$$S_\mu(x) = S_{\mu R}(x) - \Delta \frac{1}{4g} \left[ A_{\nu R}^a(x) \gamma_\nu \gamma_\mu + 2A_{\mu R}^a(x) \right] \phi^a_R(x) + \Delta \mathcal{O}(A_{\mu R}^2). \quad (2.40)$$

Thus, $S_{\mu R}(x)$ is the renormalized supercurrent up to a one-loop $\mathcal{O}(A_{\mu R}^2)$ term.

---

\[9\] One-loop diagrams with external ghost lines turn out to be UV finite.
In summary, the renormalization of the composite operators, $X_{gf}(x)$, $X_{ce}(x)$, and $S_\mu(x)$, is accomplished by Eqs. (2.37) and (2.40); $X_{gfR}(x)$, $X_{ceR}(x)$, and $S_{\mu R}(x)$ are corresponding renormalized composite operators.

2.5. Supersymmetry WT relations in terms of renormalized operators

Now, we substitute Eqs. (2.28)–(2.33), (2.37), and (2.40) in the WT identities Eqs. (2.25) and (2.26). For Eq. (2.25), we note the following relations hold in the tree-level approximation (in the Feynman gauge $\lambda_0 = 1$):

\[
\left\langle \left( -\frac{1}{g^2} \right) \partial_\mu \partial_\nu A^a_{\nu R}(x) A^b_{\mu R}(y) \right\rangle' = \delta^{ab} \delta_{\nu \mu} \delta(x - y),
\]

(2.41)

\[
\left\langle \bar{\psi}_R^a(x) \bar{\psi}_R^b(y) \right\rangle' = \delta^{ab} \delta(x - y),
\]

(2.42)

\[
\left\langle X_{ceR}(x) A_{\alpha R}(y) \psi^c_R(z) \right\rangle' = 0.
\]

(2.43)

We can use these tree-level relations in the terms proportional to $\Delta$, because $\Delta$ is already a one-loop-order quantity. Then we find, to the one-loop order,

\[
\left\langle \left[ \partial_\mu S_{\mu R}(x) + X_{gfR}(x) + X_{ceR}(x) + \Delta O(A^2_{\mu R}) \right] A^b_{\alpha R}(y) \bar{\psi}_R^c(z) \right\rangle' = -\delta(x - y) \left\langle g\gamma_\alpha \bar{\psi}_R^b(y) \bar{\psi}_R^c(z) \right\rangle'.
\]

(2.44)

For Eq. (2.26), on the other hand, by using the tree-level relation

\[
\left\langle \left( -\frac{1}{g^2} \right) \partial_\mu \partial_\nu A^a_{\nu R}(x) g\gamma_\nu \psi^a_R(x) \bar{\psi}_R^b(y) c^c_R(z) \bar{c}^d_R(w) \right\rangle' = \left\langle X_{ce}(x) \bar{\psi}_R^b(y) c^c_R(z) \bar{c}^d_R(w) \right\rangle',
\]

(2.45)

in the term proportional to $\Delta$, we have in the one-loop order,

\[
\left\langle \left[ \partial_\mu S_{\mu R}(x) + X_{gfR}(x) + X_{ceR}(x) + \Delta O(A^2_{\mu R}) \right] \bar{\psi}_R^b(y) c^c_R(z) \bar{c}^d_R(w) \right\rangle' = -\delta(x - y) \left\langle \frac{1}{2g} \sigma_{\beta \gamma} \left[ \partial_\beta A^b_{\gamma R}(y) - \partial_\gamma A^b_{\beta R}(y) + \{ f^{cde} A^c_{\beta} A^d_{\gamma} \} R(y) \right] c^c_R(z) \bar{c}^d_R(w) \right\rangle'.
\]

(2.46)

Remarkably, Eqs. (2.44) and (2.46) show that in the one-loop order, the following combination of renormalized finite composite operators,

\[
\partial_\mu S_{\mu R}(x) + X_{gfR}(x) + X_{ceR}(x),
\]

(2.47)

up to possible $\Delta O(A^2_{\mu R})$ terms that cannot be read off from our present calculation, generates the properly normalized super transformation on renormalized elementary fields. The existence of such a finite operator would be expected on general grounds (i.e., supersymmetry should be free from quantum anomaly). Nevertheless, the validity of supersymmetry WT relations in the Wess–Zumino gauge that we have observed above still appears miraculous, because it resulted from nontrivial renormalization/mixing of various composite operators.

---

\textsuperscript{10} We also note that we can neglect the effect of the counterterm $S'$ in Eq. (2.24) to Eqs. (2.41)–(2.43), because Eq. (2.24) is already a one-loop-order quantity and we use Eqs. (2.41)–(2.43) in the terms proportional to $\Delta$. 

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2.6. Final step

We have observed that the combination (2.47) generates the correct super transformation on renormalized elementary fields. Whether the same combination generates the correct super transformation on composite operators is far from obvious and to answer this requires further complicated analyses. However, if we consider only “on-shell” correlation functions in which all composite operators are separated from the combination (2.47) in position space, we can still regard the combination (2.47) as properly normalized because new UV divergences associated with composite operators at an equal point do not arise.

Within such on-shell correlation functions, in the one-loop order we can neglect the term proportional to \( \Delta \partial_\mu \bar{\psi}_R(x) \) in the supercurrent (2.40), because one can use the tree-level equation of motion of the gaugino field. Moreover, since we are practically interested only in correlation functions of gauge-invariant operators, requiring that the supercurrent is gauge invariant, we can eliminate the possibility of the \( \Delta \mathcal{O}(A^2_{\mu R}) \) term in Eq. (2.40) (note that there is no dimension 7/2 gauge-invariant fermionic combination of \( \mathcal{O}(A^2_{\mu}) \)). Thus, in the one-loop order, we may set

\[
S_{\mu R}(x) \to S_\mu(x) = -\frac{1}{2g_0} \sigma_{\rho \sigma} \gamma_\mu \bar{\psi}^a(x) F^a_{\rho \sigma}(x),
\]  

in the on-shell correlation functions of gauge-invariant operators.

For the combination \( X_{gfR}(x) + X_{ceR}(x) \) in Eq. (2.47), by using the identity at \( D = 4 \),

\[
\sigma_{\rho \sigma} \gamma_\mu = \gamma_\rho \delta_{\sigma \mu} - \gamma_\sigma \delta_{\rho \mu} - \gamma_5 \gamma_\alpha \epsilon_{\alpha \rho \sigma \mu},
\]

we can rewrite Eq. (2.37) as

\[
X_{gfR}(x) + X_{ceR}(x) = X_{gf}(x) + X_{ce}(x) + \Delta(\text{terms proportional to } \partial_\mu \partial_\nu A_{\nu}(x) \text{ or } \partial_\psi(x))
+ \frac{1}{g_0} f^{abc} \partial_\mu c^a(x) c^b(x) \gamma_\mu \bar{\psi}^c(x) + \Delta \mathcal{O}(A^2_{\mu R})
\]  

in the one-loop order; here we have used the Feynman gauge \( \lambda_0 = 1 \). For the insertion of this combination in the on-shell correlation functions of gauge invariant operators, the first line of the right-hand side can be neglected because of the BRS exactness (2.18)\(^1\) and the tree-level equations of motion in the Feynman gauge. If we consider only operators that do not contain the ghost field and the anti-ghost field, the term \( \Delta f^{abc} \partial_\mu c^a(x) c^b(x) \gamma_\mu \bar{\psi}^c(x) \) can also be neglected because there is no tree-level diagram in which this term can contribute.

From these considerations, for the insertion of the combination (2.47) in the on-shell correlation functions of gauge-invariant operators that do not contain the ghost field and the anti-ghost field, we can set

\[
\partial_\mu S_{\mu R}(x) + X_{gfR}(x) + X_{ceR}(x) \to \partial_\mu S_\mu(x) + \Delta \mathcal{O}(A^2_{\mu R}) \to \partial_\mu S_\mu(x),
\]

in the one-loop order; in the last step, we have invoked gauge invariance of the whole expression.

The bottom line of the above lengthy analysis is that, under dimensional regularization, the properly normalized conserved supercurrent in (on-shell) correlation functions of gauge-invariant operators (which do not contain the ghost field and the anti-ghost field) is given

\(^1\) The presence of the counterterm \( S' \) in Eq. (2.24) does not influence this argument, because \( S' \) is invariant under the BRS transformation (2.9).
by
\[ S_{\mu R}(x) = -\frac{1}{2g_0} \sigma_{\rho\sigma} \gamma_\mu \psi^a(x) F_{\rho\sigma}^a(x) + O(g_0^3). \] (2.52)

This is the result that we have already announced in Eq. (1.2).

Having obtained the expression for the correctly normalized conserved supercurrent (2.52), in the next section we will construct a composite operator of the flowed fields that reproduces Eq. (2.52) in the small flow-time limit.

3. Representation of the supercurrent in terms of flowed fields

3.1. Flow equations

The flow equations we adopt in this paper are identical to those of Refs. [2, 3, 5]. The flow of the gauge field along the flow time \( t \geq 0 \) is defined by
\[ \partial_t B_\mu(t, x) = D_\nu G_{\nu\mu}(t, x), \quad B_\mu(t = 0, x) = A_\mu(x), \] (3.1)
where
\[ D_\mu \equiv \partial_\mu + [B_\mu, \cdot], \quad G_{\mu\nu}(t, x) \equiv \partial_\mu B_\nu(t, x) - \partial_\nu B_\mu(t, x) + [B_\mu(t, x), B_\nu(t, x)], \]
and the flow for the fermion (gaugino) fields is defined by
\[ \partial_t \chi(t, x) = D^2 \chi(t, x), \quad \chi(t = 0, x) = \psi(x), \] (3.3)
\[ \partial_t \bar{\chi}(t, x) = \bar{\chi}(t, x) D^2, \quad \bar{\chi}(t = 0, x) = \bar{\psi}(x), \] (3.4)
where the covariant derivatives in the adjoint representation are defined by
\[ D^{ab}_\mu \equiv \delta^{ab} \partial_\mu + f^{abc} B^c_\mu(t, x) \equiv \delta^{ab} \partial_\mu + B^{ab}_\mu(t, x), \]
\[ D^a_\mu \equiv \delta^a \partial_\mu - f^{abc} B^c_\mu(t, x) \equiv \delta^a \partial_\mu - B^a_\mu(t, x). \]
(3.5)
(3.6)
The fields, \( B(t, x), \chi(t, x) \), and \( \bar{\chi}(t, x) \), are referred to as flowed fields throughout this paper.

3.2. The small flow-time expansion of composite operators

We want to express the composite operator (2.52) in the original gauge theory in terms of the small flow-time \( t \to 0 \) limit of flowed fields, because of the nice renormalization property of flowed fields. This is achieved if we can find the coefficients in the small flow-time expansion (Ref. [4]). To be explicit, we need to know the coefficients \( \zeta_i(t) (i = 1, 2, 3) \) in the following asymptotic expansion for \( t \to 0 \):
\[ \chi^a(t, x) G^{a}_{\mu\nu}(t, x) = \zeta_1(t) \psi^a(x) F^a_{\mu\nu}(x) \]
\[ + \zeta_2(t) \left[ \gamma_\mu \gamma_\rho \psi^a(x) F^a_{\rho\nu}(x) - \gamma_\nu \gamma_\rho \psi^a(x) F^a_{\rho\mu}(x) \right] \]
\[ + \zeta_3(t) \sigma_{\rho\sigma} \psi^a(x) F^a_{\rho\sigma}(x) + O(t), \] (3.7)
where the composite operators in the right-hand side possess the same mass dimension (\( = 7/2 \)) and the same gauge, Lorentz, and parity structures as the left-hand side. For \( t \to 0 \), perturbation theory is justified owing to the asymptotic freedom (see below) and the
coefficients have the following loop expansion,
\[ \zeta_1(t) = 1 + \zeta_1^{(1)}(t) + \cdots, \quad \zeta_2(t) = \zeta_2^{(1)}(t) + \cdots, \quad \zeta_3(t) = \zeta_3^{(1)}(t) + \cdots, \] (3.8)
where \( \zeta_i^{(1)}(t) \) are one-loop-order coefficients. Thus, to the one-loop order, we can invert the relation \( \text{(3.7)} \) with respect to \( \psi^a(x)F^a_{\mu\nu}(x) \) as
\[ \psi^a(x)F^a_{\mu\nu}(x) = \left[ 1 - \zeta_1^{(1)}(t) \right] \chi^a(t, x)G^a_{\mu\nu}(t, x) \]
\[ - \zeta_2^{(1)}(t) \left[ \gamma_\mu \gamma_\rho \chi^a(t, x)G^a_{\rho\nu}(t, x) - \gamma_\nu \gamma_\rho \chi^a(t, x)G^a_{\rho\mu}(t, x) \right] \]
\[ - \zeta_3^{(1)}(t) \sigma_{\rho\sigma} \sigma_{\mu\nu} \chi^a(t, x)G^a_{\rho\sigma}(t, x) + O(t), \] (3.9)
and then to the one-loop order the supercurrent \( \text{(2.32)} \) is expressed as
\[ S_{\mu R}(x) = -\frac{1}{2g_0} \sigma_{\rho\sigma} \gamma_\mu \psi^a(x)F^a_{\rho\sigma}(x) + O(g_0^3) \]
\[ = -\frac{1}{2g_0} \left[ 1 - \zeta_1^{(1)}(t) - 2(D - 3)\zeta_2^{(1)}(t) + (D - 9)(D - 4)\zeta_3^{(1)}(t) \right] \]
\[ \times \sigma_{\rho\sigma} \gamma_\mu \chi^a(t, x)G^a_{\rho\sigma}(t, x) \]
\[ - \frac{1}{2g_0} \left[ 4(D - 4)\zeta_2^{(1)}(t) - 4(D - 5)(D - 4)\zeta_3^{(1)}(t) \right] \]
\[ \times \gamma_\nu \chi^a(t, x)G^a_{\nu\mu}(t, x) + O(t) + O(g_0^3). \] (3.10)
This is the relation that we wanted to have. Thus, our next task is to compute the one-loop expansion coefficients \( \zeta_i^{(1)} \).

### 3.3. One-loop computation of expansion coefficients

The background field method developed in Ref. [71] is a powerful method to compute the coefficients in the small flow-time expansion. See also Ref. [72]. In this method, we decompose fields into the background \( e \)-number part and the quantum fluctuating part as
\[ A_\mu(x) = \hat{A}_\mu(x) + a_\mu(x), \quad B_\mu(t, x) = \hat{B}_\mu(t, x) + b_\mu(t, x), \] (3.11)
\[ \psi(x) = \hat{\psi}(x) + p(x), \quad \chi(t, x) = \hat{\chi}(t, x) + k(t, x), \] (3.12)
\[ \tilde{\psi}(x) = \tilde{\psi}(x) + \tilde{p}(x), \quad \tilde{\chi}(t, x) = \tilde{\chi}(t, x) + \tilde{k}(t, x). \] (3.13)
Quantities with a hat (\( \hat{\cdot} \)) are background ones. Then we introduce the background gauge-covariant gauge-fixing term
\[ S_{gf} = \frac{\lambda_0}{2g_0^2} \int d^4x \hat{D}_\mu a^a_\mu(x)\hat{D}_\nu a^a_\nu(x), \] (3.14)
where \( \hat{D}_\mu \) is the covariant derivative with respect to \( \hat{A}_\mu(x) \) instead of Eq. \( \text{(2.7)} \). We also adopt the following “gauge-fixed” flow equations:
\[ \partial_t b_\mu(t, x) = D_\mu G_{\nu\mu}(t, x) + \alpha_0 D_\mu \hat{D}_\nu b_\nu(t, x), \quad B_\mu(t = 0, x) = A_\mu(x), \] (3.15)
\[ \partial_t \chi^a(t, x) = \left( (D^2)^{ab} - \alpha_0 f^{acb}[\hat{D}_\mu b_\nu(t, x)]^c \right) \chi^b(t, x), \quad \chi^a(t = 0, x) = \psi^a(x), \] (3.16)
\[ \partial_t \tilde{\chi}^a(t, x) = \tilde{\chi}^b(t, x) \left( (\bar{D}^2)^{ba} + \alpha_0 f^{bca}[\hat{D}_\mu b_\nu(t, x)]^c \right), \quad \tilde{\chi}^a(t = 0, x) = \tilde{\psi}^a(x). \] (3.17)
In what follows, we work with the “Feynman gauge”, \( \lambda_0 = \alpha_0 = 1 \). We postulate that the background fields obey their own flow equations (Ref. \[71\]):

\[
\begin{align*}
\partial_t \hat{B}_\mu(t, x) &= \hat{D}_\nu \hat{G}_{\nu \mu}(t, x), \\
\hat{B}_\mu(t = 0, x) &= \hat{A}_\mu(x), \\
\partial_t \hat{\chi}(t, x) &= \hat{D}^2 \hat{\chi}(t, x), \\
\hat{\chi}(t = 0, x) &= \hat{\psi}(x), \\
\partial_t \hat{\chi}(t, x) &= \hat{\chi}(t, x) \hat{D}^2, \\
\hat{\chi}(t = 0, x) &= \hat{\psi}(x),
\end{align*}
\]

where the quantities with a hat are given by corresponding quantities with the replacement \( B_\mu(t, x) \to \hat{B}_\mu(t, x) \). We assume \( \hat{D}_\nu \hat{F}_{\nu \mu}(x) = 0 \) (Ref. \[71\]) so that the background gauge field does not evolve, \( \hat{B}_\mu(t, x) = \hat{A}_\mu(x) \). We further assume that \( \hat{p}^{ab} \hat{\psi}^b(x) = \hat{\psi}^a(x) \hat{D}^{ab} = 0 \) to suppress the tree-level tadpoles, \( \langle p^a(x) \rangle^{(0)} = \langle \hat{p}^a(x) \rangle^{(0)} = 0 \), where the superscript \( (0) \) implies that the expectation values are computed in the tree-level approximation. These assumptions also imply \( \langle a^a(x) \rangle^{(0)} = \mathcal{O}(\hat{\psi}^2) \) and, through the flow equations of quantum fields (Ref. \[71\]), \( \langle b^a(t, x) - a^a(x) \rangle^{(0)} = \mathcal{O}(t, \hat{\psi}^2) \) and \( \langle k^a(t, x) \rangle^{(0)} = \langle \hat{k}^a(t, x) \rangle^{(0)} = \mathcal{O}(t, \hat{\psi}^3) \) (see Eq. \[3.27\] below).

Now, to find the expansion coefficients \( \zeta^{(1)}(t) \), we substitute the decompositions \(3.11\)–\(3.13\) and Eq. \(3.8\) into Eq. \(3.7\) and take the expectation value of both sides in the presence of the background fields. Since the flow-time evolution of the background fermion field is given by

\[
\hat{\chi}(t, x) = e^{t \hat{D}^2} \hat{\psi}(x) = \hat{\psi}(x) + \mathcal{O}(t),
\]

we can set \( \hat{\chi}(t, x) \to \hat{\psi}(x) \) in our calculation which neglects \( \mathcal{O}(t) \) terms. This yields

\[
\begin{align*}
\langle \hat{\psi}^a(x) \left[ \hat{D}^{ab} b_b^a(t, x) - \hat{D}^{ab} b_b^a(t, x) \right] \rangle^{(1)} &- \langle \hat{\psi}^a(x) \left[ \hat{D}^{ab} a_b^a(t, x) - \hat{D}^{ab} a_b^a(t, x) \right] \rangle^{(1)} \\
&+ \langle \hat{\psi}^a(x) f^{abc} b_b^c(t, x) b^a_b(t, x) - \hat{\psi}^a(x) f^{abc} a_b^a(t, x) a^a_b(t, x) \rangle^{(1)} \\
&+ \langle k^a(t, x) \hat{F}_{\mu \nu}(x) \rangle^{(1)} - \langle p^a(x) \hat{F}_{\mu \nu}(x) \rangle^{(1)} \\
&+ \langle k^a(t, x) \left[ \hat{D}^{ab} b_b^a(t, x) - \hat{D}^{ab} b_b^a(t, x) \right] - p^a(x) \left[ \hat{D}^{ab} a_b^a(t, x) - \hat{D}^{ab} a_b^a(t, x) \right] \rangle^{(1)} \\
&= \zeta_1^{(1)}(t) \hat{\psi}^a(x) \hat{F}_{\mu \nu}^a(x) \\
&+ \zeta_2^{(1)}(t) \left[ \gamma_{\mu \rho} \hat{\psi}^a(x) \hat{F}_{\mu \nu}^a(x) - \gamma_{\nu \rho} \hat{\psi}^a(x) \hat{F}_{\rho \mu}^a(x) \right] \\
&+ \zeta_3^{(1)}(t) \sigma_{\rho \sigma} \gamma_{\mu \nu} \hat{\psi}^a(x) \hat{F}_{\rho \sigma}^a(x) + \mathcal{O}(t),
\end{align*}
\]

where the superscript \( (1) \) implies that the expectation values are computed in the one-loop order.

Now, in Eq. \(3.22\), \( \langle b^a_b(t, x) \rangle^{(1)} \) in the first term is a dimension 1 combination of background fields that behaves as the adjoint representation under the background gauge transformation; it possesses one vector index. The lowest-dimensional operator of such properties is \( t \hat{D}^{ba} \hat{F}_{ba}^a(x) \), which is already \( \mathcal{O}(t) \); thus we can neglect this term because we are neglecting \( \mathcal{O}(t) \) terms in the present calculation.

On the other hand, by carrying out a calculation of the type explained below, one finds that \( \langle a^a_b(x) \rangle^{(1)} \) in the second term of Eq. \(3.22\) identically vanishes under dimensional regularization, because there is no mass scale (such as \( t \)) that makes the loop integral nonzero.
for any $D$. For the same reason, $\langle p^a(x)\rangle^{(1)}$ in the third line also identically vanishes under dimensional regularization\(^\text{12}\).

The second line of Eq. (3.22) is evaluated as follows: The tree-level $bb$ propagator \textit{in the presence of the background fields} is given by (Ref. [71])

\[\langle b^a_\mu(t, x)b^b_\nu(s, y)\rangle_0 = g_0^2 \int_{t+s}^\infty d\xi \left( e^{\xi \Delta_x} \right)^{ab}_{\mu\nu} \delta(x-y) + \mathcal{O}(\hat{\psi}^2), \tag{3.23}\]

where

\[\Delta^{ab}_{\mu\nu} = \delta_{\mu\nu}(\hat{D}^2)^{ab} + 2\hat{F}^{ab}_{\mu\nu}, \quad \hat{F}^{ab}_{\mu\nu} = f^{acb} \hat{F}^c_{\mu\nu}(x). \tag{3.24}\]

In Eq. (3.23), we have discarded the contribution coming from $\xi = \infty$ because $\Delta_x < 0$ at least in perturbation theory. The tree-level $aa$ propagator is given simply by setting $t = s = 0$ in this expression. Then we have

\[\langle \hat{\psi}^a(x) f^{abc} b^b_\mu(t, x)b^c_\nu(t, x) - \hat{\psi}^a(x) f^{abc} a^b_\mu(t, x)a^c_\nu(x) \rangle^{(1)} = g_0^2 \frac{1}{(4\pi)^2} C_2(G) \frac{-4}{D-4} (8\pi t)^{2-D/2} \hat{\psi}^a(x) \hat{F}^a_{\mu\nu}(x). \tag{3.25}\]

To obtain this, we first use the above $bb$ and $aa$ propagators for the contractions in Eq. (3.25). Then we express the delta function in Eq. (3.23) as $\delta(x-y) = \int \frac{d^Dp}{(2\pi)^D} e^{ip(x-y)}$ and shift the plain wave $e^{ipx}$ in the left of the differential operators (Ref. [71]). Finally the loop momentum integration yields Eq. (3.25).

The fourth term of Eq. (3.22) is evaluated as follows: We see that the tree-level $kb$ propagator is given by

\[\langle k^a(t, x)b^b_\mu(s, y) \rangle_0 = -g_0^2 \left( e^{t\hat{D}} x \frac{1}{\hat{D}_x} \right)^{ac} f^{cde} \hat{\psi}^e(x) \int_s^\infty du \left( e^{u\Delta_x} \right)^{db}_{\mu\nu} \delta(x-y) + \mathcal{O}(\hat{D}\hat{\psi}, \hat{\psi}^3). \tag{3.26}\]

Then using Eqs. (3.23) and (3.26) for the contractions in the solution of the flow equation (Ref. [71]),

\[k^a(t, x) = \left( e^{t\hat{D}_x} \right)^{ab} p^b(x)
\]

\[+ \int_0^t ds \left[ e^{(t-s)\hat{D}} x \right]^{ab} \left[ 2 f^{bcd} b^c_\mu(s, x) \hat{D}^{de}_{\mu\nu} + f^{bde} f^{efg} b^e_\nu(s, x) b^{fg}_\mu(s, x) \right]
\]

\[\times \left\{ e^{s\hat{D}} \hat{\psi}^g(x) + k^e(s, x) \right\}, \tag{3.27}\]

we have

\[\langle k^a(t, x) \hat{F}^a_{\mu\nu}(x) \rangle^{(1)} = g_0^2 \frac{1}{(4\pi)^2} C_2(G) \frac{2}{D-4} (8\pi t)^{2-D/2} \hat{\psi}^a(x) \hat{F}^a_{\mu\nu}(x). \tag{3.28}\]

\(^{12}\)More specifically, one ends up with one-loop integrals of the form $\int \frac{d^Dp}{(2\pi)^D} \frac{1}{(p^2)^n}$ that identically vanish under dimensional regularization.
Finally, after some careful calculation using the above relations, we have
\[ \left\langle k^a(t, x) \left[ \tilde{D}_\mu^a \tilde{b}_\nu^b(t, x) - \tilde{D}_\nu^b \tilde{b}_\mu^a(t, x) \right] - \frac{g_0^2}{(4\pi)^2} C_2(G) \left( \frac{2}{(D-4)(D-2)} \right)(8\pi t)^{2-D/2} \right\rangle^{(1)} \]
\[ = \frac{g_0^2}{(4\pi)^2} C_2(G) \left( \frac{2}{(D-4)(D-2)} \right)(8\pi t)^{2-D/2} \times \left[ D\gamma_\mu \gamma_\nu \psi^a(x)\tilde{F}_{\mu\nu}^a(x) - D\gamma_\nu \gamma_\rho \psi^a(x)\tilde{F}_{\rho\mu}^a(x) + 2\sigma_{\rho\sigma} \sigma_{\mu\nu} \psi^a(x)\tilde{F}_{\rho\sigma}^a(x) \right]. \] (3.29)

Thus, from Eqs. (3.22), (3.25), (3.28), and (3.29), we have the one-loop coefficients,
\[ \zeta_1^{(1)}(t) = \frac{g_0^2}{(4\pi)^2} C_2(G) \frac{2}{D-4} (8\pi t)^{2-D/2}, \]
\[ \zeta_2^{(1)}(t) = \frac{g_0^2}{(4\pi)^2} C_2(G) \frac{2}{D-4} (8\pi t)^{2-D/2}, \]
\[ \zeta_3^{(1)}(t) = \frac{g_0^2}{(4\pi)^2} C_2(G) \frac{4}{D-4} (8\pi t)^{2-D/2}. \] (3.30)

Note that these coefficients themselves have a pole at \( D = 4 \).

### 3.4. Final steps

Substituting Eqs. (3.30)–(3.32) into Eq. (3.10), we have an expression for the supercurrent:
\[ S_{\mu R}(t, x) = -\frac{1}{2g_0} \left[ 1 + \frac{g_0^2}{(4\pi)^2} C_2(G) \frac{2}{D-2} (8\pi t)^{2-D/2} \right] \sigma_{\rho\sigma} \gamma_\mu \chi^a(t, x) G_{\rho\sigma}^a(t, x) \]
\[ + \frac{1}{2g_0} \frac{g_0^2}{(4\pi)^2} C_2(G) \frac{8}{D-2} (8\pi t)^{2-D/2} \gamma_\nu \chi^a(t, x) G_{\nu\sigma}^a(t, x) + O(t) + O(g_0^3). \] (3.33)

We may rewrite this in terms of renormalized quantities. The renormalized gauge coupling in the MS scheme is given by
\[ g_0^2 = \mu^2 g^2 \left[ 1 + \frac{g^2}{(4\pi)^2} C_2(G) \frac{1}{\epsilon}(-3) + O(g^4) \right]. \] (3.34)

For the gaugino field, we use a “ringed variable” (Ref. [14]):
\[ \tilde{\chi}(t, x) = \sqrt{\frac{-\dim(G)}{(4\pi)^2 t^2 \left\langle \tilde{\chi}(t, x) \bar{\mathcal{D}} \chi(t, x) \right\rangle}} \chi(t, x) \]
\[ = \frac{1}{(8\pi t)^{D/2}} \left[ 1 + \frac{g^2}{(4\pi)^2} C_2(G) \left\{ \frac{3}{2} \ln(8\pi \mu^2 t) - \frac{1}{2} \ln(432) \right\} + O(g^4) \right] \chi(t, x), \] (3.35)

where
\[ \bar{\mathcal{D}}_\mu = \mathcal{D}_\mu - \bar{D}_\mu, \] (3.36)

which is free from the wave-function renormalization of the flowed fermion field (see Ref. [5]).

Then, we have
\[ S_{\mu R}(t, x) = -\frac{1}{2g} \left[ 1 + \frac{g^2}{(4\pi)^2} C_2(G) \left\{ -\frac{7}{2} - \frac{3}{2} \ln(8\pi \mu^2 t) + \frac{1}{2} \ln(432) \right\} \right] \sigma_{\rho\sigma} \gamma_\mu \tilde{\chi}^a(t, x) G_{\rho\sigma}^a(t, x) \]
\[ - \frac{g}{(4\pi)^2} C_2(G) 3 \gamma_\nu \tilde{\chi}^a(t, x) G_{\nu\sigma}^a(t, x) + O(t) + O(g^3). \] (3.37)
This expression is manifestly UV finite because local products of the flowed gauge field and the ringed fermion field are free from UV divergences. This must be so because the supercurrent is a physical Noether current that must be free from UV divergences.

Finally, since $S_{\mu R}(x)$ is totally composed from bare quantities as, e.g., Eq. (3.33) shows, it is independent of the renormalization scale $\mu$ when expressed by the running gauge coupling $\bar{g}(\mu)$, defined by

$$\mu \frac{d\bar{g}(\mu)}{d\mu} = \beta(\bar{g}(\mu)), \quad \beta(g) \equiv \lim_{\epsilon \to 0} \mu \frac{\partial}{\partial \mu} g \bigg|_{g_0 \text{ fixed}},$$

(3.38)

explicitly,

$$\beta(g) = -b_0 g^3 - b_1 g^5 + \mathcal{O}(g^7), \quad b_0 = \frac{1}{(4\pi)^2} 3C_2(G), \quad b_1 = \frac{1}{(4\pi)^4} 6C_2(G)^2. \quad (3.39)$$

Thus we set $\mu = 1/\sqrt{8t}$. Then,

$$S_{\mu R}(x) = \frac{1}{2\bar{g}(1/\sqrt{8t})} \left\{ 1 + \bar{g}(1/\sqrt{8t})^2 C_2(G) \left[ -\frac{7}{2} - \frac{3}{2} \ln \pi + \frac{1}{2} \ln(432) \right] \right\} \times \sigma_{\rho\sigma} \gamma_{\mu} \chi^a(t,x) G^a_{\rho\sigma}(t,x)$$

$$- \frac{\bar{g}(1/\sqrt{8t})}{(4\pi)^2} C_2(G) 3\gamma_{\nu} \chi^a(t,x) G^a_{\mu\nu}(t,x) + \mathcal{O}(t) + \mathcal{O}(\bar{g}(1/\sqrt{8t})^3).$$

(3.40)

This shows that the perturbative determination of the expansion coefficients is justified for $t \to 0$. Taking the $t \to 0$ limit to get rid of higher-order terms, we arrive at the announced expression,

$$S_{\mu R}(x) = \lim_{t \to 0} \left( -\frac{1}{2\bar{g}(1/\sqrt{8t})} \left\{ 1 + \bar{g}(1/\sqrt{8t})^2 C_2(G) \left[ -\frac{7}{2} - \frac{3}{2} \ln \pi + \frac{1}{2} \ln(432) \right] \right\} \times \sigma_{\rho\sigma} \gamma_{\mu} \chi^a(t,x) G^a_{\rho\sigma}(t,x) \right.$$  

$$- \frac{\bar{g}(1/\sqrt{8t})}{(4\pi)^2} C_2(G) 3\gamma_{\nu} \chi^a(t,x) G^a_{\mu\nu}(t,x) \right).$$

(3.41)

If one prefers the $\overline{\text{MS}}$ scheme instead of the MS scheme assumed in this expression, it suffices to make the replacement

$$\ln \pi \to \gamma - 2 \ln 2,$$

(3.42)

where $\gamma$ is Euler’s constant.\textsuperscript{13}

4. Conclusion

In this paper, we have obtained a representation of the properly normalized conserved supercurrent in the 4D $\mathcal{N} = 1$ SYM, in terms of the small flow-time expansion of the gradient flow,

\textsuperscript{13}The factor $\ln \pi$ in (3.41) comes from the product of the factor $(4\pi)^{\epsilon}$ arising from the one-loop momentum integral and the pole $1/\epsilon$ as $1/\epsilon + \ln \pi + \ln 4$. Since the gauge couplings in the MS scheme and in the $\overline{\text{MS}}$ scheme are related as

$$g_{\text{MS}}^2 = \pi^{-\epsilon} e^{\gamma - 2\ln 2} g_{\overline{\text{MS}}}^2,$$

(3.43)

the factor $(4\pi)^{\epsilon}$ is replaced by $(4e^{\gamma - 2\ln 2})^{\epsilon}$ in the $\overline{\text{MS}}$ scheme, thus resulting in the rule (3.42).
Eq. (3.41). Because of remarkable renormalization properties of the gradient flow (Refs. [4–6]), this representation possesses a meaning independent of the adopted regularization. This in particular implies that the representation can be used in lattice numerical simulations, as a similar representation of the energy–momentum tensor can be (Refs. [19–21]). An important application would be to determine the supersymmetric point in the parameter space, for which the conservation of the supercurrent provides a definite criterion.

For more general supersymmetric theories, one has to take into account the flow of the scalar field. We may adopt a simple flow equation,

\[ \partial_t \varphi(t, x) = D_\mu D_\mu \varphi(t, x), \quad \varphi(t = 0, x) = \phi(x), \]  

(4.1)

because the inclusion of further terms corresponding to mass, self-interaction, and Yukawa terms in the right-hand side would break the renormalizability (Ref. [73]). By using this setup, it must be possible to obtain a representation of the properly normalized conserved supercurrent in general supersymmetric theories; for general theories the parameter tuning in lattice numerical simulations will be really demanding. We hope to study this problem in the near future.

Acknowledgements

We would like to thank David B. Kaplan and Tetsuya Onogi for discussions that motivated the present work. This work was supported by JSPS KAKENHI Grant Numbers 16J02259 (A. K.) and 16H03982 (H. S.).

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A. Charge conjugation matrix in dimensional regularization

In this appendix, we consider the charge conjugation matrix in dimensional regularization. In the $D = 4$ Euclidean space, the charge conjugation matrix $C$ is defined such that

\[ C^{-1} \gamma_\mu C = -\gamma_\mu^T, \quad C^T = -C, \]  

(A1)

where $T$ denotes the transpose and thus

\[ C^{-1} \sigma_{\mu \nu} C = -\sigma_{\mu \nu}^T, \quad C^{-1} \gamma_5 C = \gamma_5^T. \]  

(A2)

For a general complex spacetime dimension $D$, we postulate

\[ C^{-1} \gamma_\mu C = -s_1(D) \gamma_\mu^T, \quad C^T = -s_2(D) C, \]  

(A3)

where coefficients $s_1(D)$ and $s_2(D)$ are meromorphic functions of $D$. Requiring the usual properties of the transpose, we find

\[ s_1(D)^2 = s_2(D)^2 = 1. \]  

(A4)

However, since $s_1(D = 4) = s_2(D = 4) = +1$, we have $s_1(D) = s_2(D) = +1$ for general $D$; the charge conjugation matrix in dimensional regularization also satisfies Eqs. (A1) and (A2). These relations are fully employed in our computation in the main text.
It should be noted that the above definition obtained by the analytic continuation from $D = 4$ does not necessarily coincide with the conventional charge conjugation matrix at a fixed integer dimension. For example, for $D = 5$, we have $C^{-1} \gamma_\mu C = +\gamma_\mu^T$ as implied by the latter relation of Eq. (A2). Nevertheless, the above definition is perfectly legitimate from the perspective of dimensional regularization.

\section*{B. Two-loop order improvement through the superconformal anomaly}

In Ref. [13] on the energy–momentum tensor in pure Yang–Mills theory, it was possible to improve the small flow-time representation by using the information of the trace (or conformal) anomaly to the two-loop order. We can imitate this strategy for the supercurrent in the present $4D \mathcal{N} = 1$ SYM as follows.

We thus require that a formula such as Eq. (3.37) reproduces the superconformal anomaly (Refs. [62–70]),

$$\gamma_\mu S_{\mu R}(x) = -\frac{\beta(g)}{g^2} \{\sigma_\mu \psi^a F^{a}_{\mu}\} R(x),$$  \hspace{1cm} (B1)

\noindent to the two-loop order, where the beta function $\beta(g)$ is given by Eq. (3.39). First, we need to know the expression for the renormalized composite operator in the right-hand side, $\{\sigma_\mu \psi^a F^{a}_{\mu}\} R(x)$, e.g., in the MS scheme. For symmetry reasons, this operator is multiplicatively renormalized. To find the renormalization constant, we note the relation

\begin{equation}
\left(1 - \frac{3}{2}\Delta\right) \sigma_\mu \psi^a(x) F^{a}_{\mu}(x) = \left\{1 + \frac{g^2}{(4\pi)^2} C_2(G) \left[2 + \frac{3}{2} \ln(8\pi^2t) + \frac{1}{2} \ln(432)\right] + O(t)\right\} \sigma_\rho \gamma_\mu \dot{x}^a(t, x) G^{a}_{\mu\rho}(t, x) + O(t), \hspace{1cm} (B2)
\end{equation}

\noindent where $\Delta$ is defined in Eq. (2.27), which follows from Eqs. (3.4), (8.30)–(8.32), and (8.35) to the one-loop order. Since the right-hand side of this relation is manifestly finite, this is the renormalized composite operator $\{\sigma_\mu \psi^a F^{a}_{\mu}\} R(x)$ in the MS scheme. Having obtained this information, we see that the expression

\begin{equation}
S_{\mu R}(x) = \lim_{t \to 0} \left[-\frac{1}{2g} \left\{1 + \frac{g^2}{(4\pi)^2} C_2(G) \left[-\frac{7}{2} - \frac{3}{2} \ln(8\pi^2t) + \frac{1}{2} \ln(432)\right]\right\} \sigma_\rho \gamma_\mu \dot{x}^a(t, x) G^{a}_{\mu\rho}(t, x)
\right.

\noindent

\begin{equation}
\left. - b_0 g \left\{1 + \frac{b_1}{b_0} g^2 + \frac{g^2}{(4\pi)^2} C_2(G) \left[2 + \frac{3}{2} \ln(8\pi^2t) + \frac{1}{2} \ln(432)\right] + O(g^4)\right\}\right) \times \gamma_\nu \dot{x}^a(t, x) G^{a}_{\nu\mu}(t, x) + O(t), \hspace{1cm} (B3)
\end{equation}

\noindent reproduces Eq. (B1) to the two-loop order. Finally, repeating the renormalization group argument that lead to Eq. (3.41), we have

\begin{equation}
S_{\mu R}(x) = \lim_{t \to 0} \left[-\frac{1}{2g(1/\sqrt{8}t)} \left\{1 + \frac{\bar{g}(1/\sqrt{8}t)^2}{(4\pi)^2} C_2(G) \left[-\frac{7}{2} - \frac{3}{2} \ln \pi + \frac{1}{2} \ln(432)\right]\right\}\right]
\end{equation}

\noindent

\begin{equation}
\times \sigma_\rho \gamma_\mu \dot{x}^a(t, x) G^{a}_{\mu\rho}(t, x)
\end{equation}

\noindent

\begin{equation}
- \frac{\bar{g}(1/\sqrt{8}t)}{2g(1/\sqrt{8}t)} C_2(G)^2 \left\{1 + \frac{\bar{g}(1/\sqrt{8}t)^2}{(4\pi)^2} C_2(G) \left[4 + \frac{3}{2} \ln \pi + \frac{1}{2} \ln(432)\right]\right\}
\end{equation}

\noindent

\begin{equation}
\times \gamma_\nu \dot{x}^a(t, x) G^{a}_{\nu\mu}(t, x). \hspace{1cm} (B4)
\end{equation}

\section*{References
