From quantum gravity to quantum field theory via noncommutative geometry

Johannes Aastrup\textsuperscript{1} and Jesper Møller Grimstrup\textsuperscript{2}

\textsuperscript{1} Mathematisches Institut, Universität Hannover, Welfengarten 1, D-30167 Hannover, Germany
\textsuperscript{2} The Niels Bohr Institute, University of Copenhagen, Blegdamsvej 17, DK-2100 Copenhagen, Denmark

E-mail: aastrup@uni-math.gwdg.de and grimstrup@nbi.dk

Received 18 April 2013, revised 8 November 2013
Accepted for publication 25 November 2013
Published 10 January 2014

Abstract

A link between canonical quantum gravity and fermionic quantum field theory is established in this paper. From a spectral triple construction, which encodes the kinematics of quantum gravity, we construct semi-classical states which, in a semi-classical limit, give a system of interacting fermions in an ambient gravitational field. The emergent interaction involves flux tubes of the gravitational field. In the additional limit, where all gravitational degrees of freedom are turned off, a free fermionic quantum field theory emerges.

Keywords: quantum gravity, noncommutative geometry, quantum field theory

PACS numbers: 02.40.Gh, 04.60.−m, 11.10.Ef, 12.10.−g

1. Introduction

In this paper we establish a link between the kinematical sector of canonical quantum gravity and fermionic quantum field theory. The starting point is a mathematical construction—a spectral triple—which utilizes Connes’ noncommutative geometry to recombine the kinematical sector of canonical quantum gravity into a geometrical construction over a configuration space of general relativity [1–3]. From this spectral triple an infinite system of interacting fermions in an ambient gravitational field emerge in a semi-classical approximation. We discuss two different types of semi-classical states, one of which entail a fermionic interaction which involves flux tubes of the gravitational field. Finally, a free fermionic quantum field theory emerges from the construction in the limit where all gravitational degrees of freedom are turned off.

The fermionic degrees of freedom encountered in the semi-classical approximation emerge as degrees of freedom related to a certain labeling of the semi-classical states. The spectral triple construction itself involves, a priori, only gravitational degrees of freedom.
A spectral triple \((B, H, D)\) is the central ingredient in a generalization of Riemannian geometry known as noncommutative geometry. Its constituents are a \(*\)-algebra \(B\) represented as bounded operators on a Hilbert space \(H\) together with an unbounded, self-adjoint operator \(D\) called the Dirac operator. For a commutative algebra the spectral triple is, under certain requirements \([4, 5]\), equivalent to a Riemannian geometry on the state space of the algebra.

The spectral triple which we investigate in this paper is a geometrical construction over a configuration space of connections. It is constructed over an infinite system of cubic lattices and the algebra is generated by loops running in these lattices. This algebra is inherently noncommutative. A copy of \(SU(2)\) is assigned to each edge in these lattices—much alike lattice gauge theory—and with the refinement of lattices the number of copies of \(SU(2)\) grows to infinity in a way which captures information of a configuration space of \(SU(2)\) connections. This configuration space is related to canonical quantum gravity via Ashtekar variables \([6, 7]\) and the algebra of loops represent holonomy transforms of the Ashtekar connection. The Dirac type operator is then essentially given by a sum of Dirac operators on these copies of \(SU(2)\).

In \([2, 3]\) it was proven that a semi-finite spectral triple emerge from these constituents, and in \([8]\) it was shown that the interaction between the algebra of loops and the Dirac type operator reproduces the structure of the Poisson bracket of general relativity formulated in terms of Ashtekar variables. This means that the spectral triple encodes information of the kinematics of quantum gravity.

The new insight presented in this paper is that a natural class of semi-classical states, labeled by a certain loop order, resides naturally in the Hilbert space associated to the spectral triple. In \([8, 9, 10]\) the states at first loop order were identified and were shown to entail the Dirac Hamiltonian in a semi-classical approximation. Hence these states were interpreted as one-particle states of a fermion in an ambient gravitational field dictated by the semi-classical approximation. Correspondingly, in this paper we find that the \(n\)th order states entail, in a semi-classical approximation, a system of \(n\) coupled fermions. In fact, we discuss two different types of semi-classical states. The first type, which we discuss in detail, gives rise to a fermion interaction which involves flux tubes of the Ashtekar connection, whereas this interaction is absent when working with the second type of states. Finally, in the limit where all gravitational degrees of freedom are turned off, a free fermionic quantum field theory emerges. The first type of states entail a certain \(n\)th order Hamiltonian in a semi-classical approximation, whereas the second type of states does not have this feature.

The construction of the semi-classical states follows a certain logic which takes its point of departure in the choice of algebra: the choice of a noncommutative algebra of loops necessitates the addition of a matrix factor in the Hilbert space to accommodate a representation of the algebra. Also, the algebra of loops comes with a dependency on a choice of basepoint needed in order to define a product between loops. To free the construction from this basepoint dependency one is lead to consider a type of states which spread out the basepoint. These states involve a certain matrix degree of freedom which will later merge into spinor degrees of freedom. It is the additional matrix factor in the Hilbert space which permit these matrices to appear. Finally, the combination of these basepoint-independent states with coherent states over a classical point in the phase-space of gravity entails in a semi-classical approximation an infinite system of fermions which interact both among themselves and with the ambient gravitational fields given by the classical phase-space point. Thus, there is a direct line of reason from the choice of a noncommutative algebra of holonomy loops (as opposed, for
example, to a commutative algebra of functions on the space of connections) to the emergence of fermionic degrees of freedom.

The paper is organized as follows. In section 2 we briefly introduce elements of noncommutative geometry and its relation to the standard model. In section 3 we briefly recall the construction of spectral triples over a space of connection. In section 4 we comment on the Poisson bracket of General Relativity which is encoded in the spectral triple through the interaction between the Dirac type operator and the algebra of loops. Then, with the basic construction presented, we are in section 5 ready to address the question regarding the semi-classical approximation. We start the analysis by reviewing first the issue of a certain basepoint dependency of the spectral triple. The resolution of this problem leads directly to states which, in a semi-classical limit, entail the Dirac Hamiltonian of a single particle. In section 6 we then show that also higher order states reside within the spectral triple which, in a semi-classical approximation, leads to a system of interacting fermions. We show that in the special limit where all gravitational degrees of freedom are turned off, a Fock space emerges. In section 7 we discuss a possible additional grading and, finally, in section 8 we give a conclusion. Detailed computations are given in the appendix.

2. Noncommutative geometry

Noncommutative geometry is based on the insight, due to Connes, that the metric on a compact Riemannian manifold can be recovered from the Dirac operator $D$ and its interaction with the algebra of smooth functions on the manifold [4]. This means that the metric data is completely determined by the triple
\[(C^\infty(M), L^2(M, S), D).\] (1)

This result entails a natural generalization of Riemannian geometry where one considers also noncommutative algebras. The central object in this generalization is the spectral triple $(A, H, D)$ where $A$ is a not necessarily commutative $*$-algebra, $H$ is a Hilbert space carrying a representation of $A$ and $D$ is an unbounded, self-adjoint operator called the Dirac operator. Such a triple is called spectral if it satisfies two conditions.

1. The resolvent of $D$, $(1 + D^2)^{-1}$, is a compact operator in $H$.
2. The commutator $[D, a]$ is bounded for all elements $a \in A$.

A noncommutative geometry consists of a spectral triple which is required to satisfy an additional number of rules which generalize the interactions of the constituents in (1) so that the construction coincides with Riemannian geometry whenever $A$ is commutative. These rules strongly restrict the choice of the Dirac operator $D$.

It turns out that the standard model of particle physics coupled to the gravitational field provides an example of a noncommutative geometry. In this case the algebra is an almost commutative algebra of the form
\[A = C^\infty(M) \otimes A_F, \quad A_F = \mathbb{C} \oplus \mathbb{H} \oplus M_3(\mathbb{C}),\] (2)

which interacts with a Dirac operator that consist of two parts
\[D = D_M + D_F,\]

where $D_M$ is the standard Dirac operator on the manifold $M$ and $D_F$ is a matrix valued function on $M$ which encodes the metric data of the states over $A_F$. It was Connes who realized that the entire structure of the standard model coupled to general relativity is encoded in a spectral triple that involve this algebra and Dirac operator [5, 11–13]. In fact, the abstract requirements for the Dirac operator, as mentioned above, entails that $D_F$ contains both the non-Abelian
gauge fields and the Higgs field of the standard model together with their couplings to the elementary fermions. This very remarkable fact provides the Higgs field with a geometrical interpretation as a carrier of metric information on a noncommutative space. Furthermore, the action of the standard model coupled to the Einstein–Hilbert action emerges from this spectral triple construction through the so-called spectral action principle [14–16] which involves a heat-kernel expansion of the Dirac operator $D$.

What emerges from this spectral triple construction is essentially the classical action: quantization of the fields of the standard model—barring the gravitational field—is applied after the heat kernel expansion. Thus, the work of Connes and Chamseddine raises the question whether quantum theory should not play a more prominent role in this approach to high-energy physics. Furthermore, since the construction of Connes and Chamseddine is fundamentally gravitational, one would expect that the answer to this question should, somehow, involve elements of quantum gravity. Thus, one might speculate that the spectral triple construction due to Connes should be interpreted as a low-energy limit of a theory of quantum gravity.

It was this line of reasoning that motivated the construction in [1–3] of a semi-finite spectral triple over a configuration space of connections, see also [17]. There, the idea is to use noncommutative geometry to identify a natural non-perturbative construction within the framework of canonical quantum gravity and then, subsequently, identify a semi-classical limit which coincides with known physics. Ultimately, the goal is to make contact to Connes work on the standard model.

Before we present the construction of this semi-finite spectral triple we will briefly review Ashtekar variables and holonomy loops since these play a key role in the physical interpretation of the semi-finite spectral triple.

### 3. The spectral triple

In this section we will shortly recall the construction of spectral triples over a space of connection. We do this to setup the notation. For full detail see [8], section 3–6.

#### 3.1. Connection variables of gravity

Let $M$ be a four-dimensional globally hyperbolic manifold and consider a foliation of $M$ according to $M = \mathbb{R} \times \Sigma$, where $\Sigma$ is a spatial manifold. Let $\mathcal{A}$ be the space of connections in the trivial $SU(2)$ principal bundle over $\Sigma$. Given a loop or path $l$, we denote by $h_l$ the holonomy function on $\mathcal{A}$, i.e.

$$h_l(A) = \text{Hol}(l, A),$$

where $\text{Hol}(l, A)$ denotes the holonomy of $A$ along $l$.

Let $E$ be a inverse densitized triad field. We define

$$dF_a = \epsilon_{mnp}E_m^a \, dx^n \wedge dx^p$$

and given an oriented surface $S$ in $\Sigma$ we define

$$F^a_S = \int_S dF_a. \quad (3)$$

The Poisson bracket of gravity reads

$$\{h_l, F^a_S\} = \iota(S, l) \iota h_l \tau_a h_l; \quad (4)$$

where $\tau$ denote the generators of the Lie algebra of $SU(2)$. Here, $\iota$ is given by $\iota(S, l) = \pm 1, 0$ depending on the intersection between $S$ and $l$, see [9].
3.2. The triple on a cubic lattice

Let $\Gamma$ be a three-dimensional, finite, cubic lattice in $\Sigma$. Let $\{x_i\}$ and $\{l_i\}$ denote vertices and edges in $\Gamma$, respectively. We associate with each edge a copy of $SU(2)$. We denote $G = SU(2)$. We put $\mathcal{A}_\Gamma = G^{n(\Gamma)}$, where $n(\Gamma)$ denotes the number of edges in $\Gamma$. We define

$$\mathcal{H}_\Gamma = L^2(G^{n(\Gamma)}, Cl(T^*G^{n(\Gamma)}) \otimes M_2(\mathbb{C})),$$

where $G^{n(\Gamma)}$ is equipped with the Haar measure and the Clifford bundle $Cl(T^*G^{n(\Gamma)})$ is with respect to a left and right invariant metric on $G$.

Given a loop $l$ in $\Gamma$, we have a holonomy operator $h_l$ acting on $\mathcal{H}_\Gamma$ in the following way: Decompose $l$ into edges $l = l_1 \cdots l_n$. Then

$$(h_l(\xi))(g_1, \ldots, g_{n(\Gamma)}) = g_{l_1} \cdots g_{l_n} \xi(g_1, \ldots, g_{n(\Gamma)}), \quad \xi \in \mathcal{H}_\Gamma,$$

where $g_{l_1} \cdots g_{l_n}$ acts on the $M_2(\mathbb{C})$ factor. The algebra in the spectral triple is the $\ast$-algebra generated by the $h_l$'s, where $l$ runs over all loops with a given base point. We denote this by $B_\Gamma$.

The Dirac type operator on $\mathcal{H}_\Gamma$ is of the form

$$D_\Gamma = \sum_j a_{\Gamma} e^i_j \cdot L_{e^i_j},$$

where the product is Clifford multiplication, $a_{\Gamma} \in \mathbb{R} \setminus \{0\}$. In equation (5) $\{e^i_j\}$ denotes a left-translated orthonormal basis of $T^*G$, where $G$ is the $i$th copy in $G^{n(\Gamma)}$. $L_{e^i_j}$ denotes the corresponding left-invariant vectorfield.

The triple $(B_\Gamma, \mathcal{H}_\Gamma, D_\Gamma)$ is a spectral triple.

3.3. The limiting triple

Let $\{\Gamma_i\}, i \in \mathbb{N}$, be an infinite sequence of nested three-dimensional, finite, cubic lattices.

Corresponding to this sequence of cubic lattices there is a projective system $\{\mathcal{A}_{\Gamma_i}\}$ of spaces obtained from the graphs $\{\Gamma_i\}$, together with natural projections between these spaces

$$P_{i+1} : \mathcal{A}_{\Gamma_{i+1}} \to \mathcal{A}_{\Gamma_i}.$$

Consider now a system of triples

$$(B_\Gamma, \mathcal{H}_\Gamma, D_\Gamma).$$

Under certain restrictions on the $(a_{\Gamma_i})$ a direct limit of the triples can be constructed, see [8] for details and subtleties. The resulting triple

$$(\mathcal{B}, \mathcal{H}, D)$$

is not a spectral triple, but when the sequence $(a_{\Gamma_i})$ converges to infinity it is a semi-finite spectral triple.
4. Link to canonical quantum gravity

This spectral triple encodes information about the kinematics of quantum gravity. Consider first the spaces $A_{\Gamma_i}$ and their projective limit. Denote by

$$\overline{A} := \lim_{\Gamma_i} A_{\Gamma_i}. $$

Further, given a trivial principal $G$-bundle denote by $A$ the space of all smooth connections herein. In [3] we prove that $A$ is densely embedded in $\overline{A}$:

$$A \hookrightarrow \overline{A}. $$

This fact justifies the terminology generalized connections for the completion $\overline{A}$ and shows that the semi-finite spectral triple $(\mathcal{B}, \mathcal{H}, \mathcal{D})$ is indeed a geometrical construction over the space $A$ of smooth connections.

Next, in [8] we have shown that the interaction between the Dirac type operator $D$ and the algebra $\mathcal{B}$ in the semi-finite spectral triple $(\mathcal{H}, \mathcal{H}, D)$ reproduces the structure of the Poisson bracket (4).

What we find is the following: consider the $n$th level of subdivision of an edge $l_i$ and consider a vector field $L_{e_j}$ which corresponds to a copy of $G$ assigned to a segment of $l_i$ that emerges at this level of subdivision. Then $L_{e_j}$ corresponds to a quantization of a flux variable sitting at the endpoint of this segment

$$F^a_{\Delta S} \xrightarrow{\text{quantization}} i_P^a L_{\theta_j} + i_P^a \Omega^a_{\theta_k}, $$

(7)

where $\Omega^a_{\theta_k}$ is a correction term that consist of a certain combination of twisted, right-invariant vector fields acting on the copies of $G$ assigned to segments of $l_i$ that are situated ‘higher’ in the inductive system of lattices. Put differently, $\Omega^a_{\theta_k}$ probes information which is more coarse grained relative to the line segment to which the $(i+s)$th copy of $G$ is assigned. In (7) $\Delta S$ refers to a surface located at the endpoint of the specific line segment, with the size of $2^{-2n}$. For details we refer to [8].

In the following we shall ignore the correction terms $\Omega^a_{\theta_k}$ when we apply relation (7) to translate quantized quantities involving the Dirac type operator $D$ to their classical counterparts. The reason for this will become clear in the next section where we construct semi-classical states. These states have the property that any dependency on finite parts of the inductive system of lattices vanishes in the semi-classical limit.

In the limit of repeated subdivision of lattices we find that the semi-finite spectral triple $(\mathcal{B}, \mathcal{H}, D)$ encodes information tantamount to a representation of the Poisson bracket of general relativity. Thus, the triple captures information about the kinematical sector of quantum gravity.

5. A semi-classical approximation and a continuum limit

In this section we review and further develop the analysis of semi-classical states found in $\mathcal{H}$. The relevant references are [8, 10]. Again we set $G = SU(2)$ and choose the two-by-two matrix representation hereof.

The semi-classical analysis comes together with a certain continuum limit in which we discard information related to finite graphs. Thus, in this limit we ‘zoom in’ on infitesimal edges only.

---

3 This size refers to the coordinate system which emerges from the projective system of cubic lattices. Thus, an edge which appears in the graph $\Gamma_0$ is of length ‘1’.
5.1. Dependency on the choice of basepoint

The semi-classical analysis start with the realization that the spectral triple construction comes with a dependency on the choice of basepoint $x_0$. This was first pointed out in [9]. Had we instead chosen to work with traced holonomy loops this dependency would not show up since the basepoint dependency vanishes due to the cyclicity of the trace. In [10] this observation was taken as the point of departure for a construction which ultimately entail the emergence of the Dirac Hamiltonian for a single fermion and—as we shall see in the following—for a system of interacting fermions, in a semi-classical approximation.

Consider first the graph $\Gamma_n$ and an edge $l_i$ in $\Gamma_n$. Associated to $l_i$ the operator

$$U_i := \frac{i}{2} \left( e_l^e g_i e_l^e d + e_l^e e_l^d g_i e_l^e \right)$$

and check that $U_i^* U_j = U_j^* U_i = I_2$ (see the appendix). Here, $g_i = \nabla (l_i)$ is an element in the copy of $G$ assigned to the edge $l_i$. Given an element $a \in \mathcal{B}_{\Gamma_n}$, we compute

$$\text{Tr}_{\text{Cl}}(U_i a U_i^*) = a_0 I_2,$$

where we write $a = a_0 I_2 + a' \sigma^i$ with $\sigma^i$ being the Pauli matrices, and where $\text{Tr}_{\text{Cl}}$ denotes the trace over the Clifford algebra. Thus, conjugating with $U_i$ singles out the matrix trace of $a$. Next, let $p = \{l_{i_1}, l_{i_2}, \ldots, l_{i_k}\}$ be a path in $\Gamma_n$ and define the associated operators by

$$U_p := U_{i_1} U_{i_2} \cdots U_{i_k}, \quad U_p := \nabla (l_{i_1}) \cdot \nabla (l_{i_2}) \cdots \nabla (l_{i_k}),$$

where $U_p$ is the ordinary parallel transport along $p$. The operators $U_p$ form a family of mutually orthogonal operators labeled by paths in $\Gamma_n$

$$\text{Tr}_{\text{Cl}}(U_p^* U_{p'}) = \delta_{p, p'}.$$

This relation is due to the presence of the Clifford algebra elements in $U_p$. Here $\delta_{p, p'}$ equals one when the paths $p$ and $p'$ are identical and zero otherwise.

Consider now states in $\mathcal{H}_{\Gamma_n}$ of the form

$$\Psi_n(\psi) = 2^{-3n} \sum_i U_{p_i} \psi(x_i) U_{p_i}^{-1},$$

where the sum runs over vertices $x_i$ in $\Gamma_n$ and where the path $p_i$ connects the basepoint $x_0$ with vertices $x_i$. Here $\psi(x)$ denotes an element in $M_2(\mathbb{C})$ associated to the vertex $x$. Later $\psi(x)$ will be seen to form a spinor degree of freedom at the point $x_i$.

For the purpose of this section the sum in (10) may run over all vertices. Later, however, it will be necessary to restrict this sum to a subclass of vertices. First, consider edges in $\Gamma_n/\Gamma_{n-1}$, that is, edges which lie in $\Gamma_n$ but not in $\Gamma_{n-1}$. To each edge is associated two vertices, and therefore two different paths, where one is a single ‘step’ longer than the other. Let us denote them $p_i$ and $p_{i+1}$. When we consider the operators $U_{p_i}$ and $U_{p_{i+1}}$, connecting the basepoint with such pairs of vertices, then we shall only consider pairs where the shortest path $p_i$ corresponds to an even operator $U_{p_i}$ (‘even’ is with respect to the Clifford algebra). Thus, we let the sum in (10) run over these vertices. This will be important when we develop the semi-classical analysis.

Now, $\Psi_n$ is a state in $\mathcal{H}_{\Gamma_n}$ which does not show the dependency on the choice of basepoint $x_0$ mentioned above. This means that the expectation value of an element $a$ in $\mathcal{B}_{\Gamma_n}$ on this state will depend only on the trace of $a$

$$\langle \Psi_n | a | \Psi_n \rangle = \langle \Psi_n | \text{Tr}(a) | \Psi_n \rangle = \langle \text{Tr}(a) \rangle \sum_i \psi^*(x_i) \psi(x_i)$$

and since the trace of an element of $\mathcal{B}_{\Gamma_n}$ is basepoint independent, these states circumvent this problem.
5.2. Coherent states in \( \mathcal{H} \)

The strategy is to combine states of the form (10) with a semi-classical approximation. To do this we need to introduce coherent states in \( \mathcal{H} \). We first recall results for coherent states on various copies of \( SU(2) \). This construction uses results of Hall [19, 20] and is inspired by the articles [21–23].

First pick a point \((A^n_m, E^m_n)\) in the phase space of Ashtekar variables\(^4\) on a 3-manifold \(\Sigma\). The states which we construct will be coherent states peaked over this point. Consider first a single edge \(l_i\) and thus one copy of \(SU(2)\). Let \(\{e^i_l\}\) be a basis for \(su(2)\). There exist families \(\phi^i_{l} \in L^2(SU(2))\) such that

\[
\lim_{t \to 0} \langle \phi^i_{l}, tL^{i}_{\psi} \phi^i_{l} \rangle = 2^{-2n}iE^m_i(x_i),
\]

where \(v \in \mathbb{C}^2\), and \(,\) denotes the inner product hereon; \(x_i\) denotes the ‘right’ endpoint of \(l_i\) (we assume that \(l_i\) is oriented to the ‘right’), and the index ‘\(m\)’ in the \(E^m_i\) refers to the direction of \(l_i\). The factor \(2^{-2n}\) is due to the fact that \(L^i_{\psi}\) corresponds to a flux operator with a surface determined by the lattice [8]. Corresponding statements hold for operators of the type

\[
f(\nabla(l_i))P(tL^{i}_{\psi}, tL^{i'}_{\psi}, tL^{j}_{\psi}),
\]

where \(P\) is a polynomial in three variables, and \(f\) is a smooth function on \(SU(2)\), i.e.

\[
\lim_{t \to 0} \langle \phi^i_{l}, f(\nabla(l_i))P(tL^{i}_{\psi}, tL^{i'}_{\psi}, tL^{j}_{\psi}) \phi^i_{l} \rangle = f(h_i(A))P(iE^m_1, iE^m_2, iE^m_3).
\]

The states \(\phi^i_{l}\) have further important physical properties which we are however not going to use at the present stage of the analysis. Also, the precise construction of these states, in particular the choice of complexifier [22], is irrelevant for the results presented in this paper.

Let us now consider the graph \(\Gamma_n\). We split the edges into \(\{l_i\}\) and \(\{l'_i\}\), where \(\{l_i\}\) denotes the edges appearing in the \(n\)th subdivision but not in the \(n - 1\)th subdivision, and \(\{l'_i\}\) the rest. Let \(\phi^i_{l'}\) be the coherent state on \(SU(2)\) defined above and define the states \(\phi^i_{l}\) by

\[
\lim_{t \to 0} \langle \phi^i_{l'}, \nabla(l_i) \phi^i_{l'} \otimes v \rangle = (v, h_i(A)v),
\]

and

\[
\lim_{t \to 0} \langle \phi^i_{l'}, tL^{i}_{\psi} \phi^i_{l'} \rangle = 0.
\]

Finally define \(\phi^i_{l} \) to be the product of all these states as a state in \(L^2(A_{\Gamma_n})\). These states are essentially identical to the states constructed in [22] except that they are based on cubic lattices and a particular mode of subdivision.

In the limit \(n \to \infty\) these states produce the right expectation value on all loop operators in the infinite lattice. The reason for the split of edges in \(l_i\) and \(l'_i\) in the definition of the coherent state is to pick up only those degrees of freedom which ‘live’ in the continuum limit \(n \to \infty\). In this way we shall, once the continuum limit is taken, partially have eliminated dependences on finite parts of the lattices. In a classical setup, this amounts to information which has measure zero in a Riemann integral.

\(^4\) Recall that we work here with a real \(SU(2)\) connection.
5.3. The Dirac operator in three dimensions

We are now ready to combine the content of the previous two subsections. Consider first the states
\[ \Psi_n^i(\psi) := \Psi_n(\psi)\phi_n^i \] (14)
composed by the states (10) and the coherent states introduced in the previous section. Furthermore, we will restrict the series \( \{a_j\}_{j \in \mathbb{N}} \) of parameters in \( D \); we now require all \( a_j \)'s associated to edges appearing in the \( k \)-th subdivision but not in the \( k-1 \)-th subdivision to be equal. With this restriction the parameters \( a_j \) represent a scaling degree of freedom.

We find
\[
\langle \Psi_n^i | D | \Psi_n^j \rangle = \frac{\alpha_n}{2^{n+1}} \sum_i \left( \psi^*(x_i) E_{a} g^a \sigma^a \psi(x_{i+1}) g_i^{-1} + (g^a \sigma^a \psi(x_{i+1}) \right) g_i^{-1} g_{n-1}^m E_{a} \psi(x_i)), \tag{15}
\]
where \( x_i \) and \( x_{i+1} \) denote start and endpoint of an edge \( l_i \) and where the sum runs over edges appearing in the \( n \)-th but not in the \( n-1 \)-th subdivision. The evaluation over coherent states has been performed, see equation (12) and (13), and therefore \( E_{a}^m \) appears in (15) where the index \( 'm' \) refers to the direction of the edge \( l_i \). Expand \( g_i \) according to
\[
g_i = L_2 + \epsilon A_m(x_i) + \mathcal{O}(\epsilon^2),
\]
where \( 'm' \) is again the direction of the edge \( l_i \). This expansion is permitted whenever we apply the coherent states and take the continuum limit \( n \to \infty \) together with the semi-classical limit \( t \to 0 \). All together the term \( E_{a}^m g^a \sigma^a \psi(x_{i+1}) g_i^{-1} \), in the combined semi-classical and continuum limit, give a covariant derivative
\[
\lim_{n \to \infty} \lim_{t \to 0} \left( E_{a}^m g^a \sigma^a \psi(x_{i+1}) g_i^{-1} \right) = E_{a}^m \sigma^a (1 + \epsilon A_m)(\psi(x) + \epsilon \partial_\alpha \psi(x)) + \mathcal{O}(\epsilon^2)
\]
where there is, at this point, no summation over \( 'm' \) since it denotes the direction of the edge \( l_i \). Finally, equation (15) gives
\[
\lim_{n \to \infty} \lim_{t \to 0} \left( \langle \Psi_n^i | D | \Psi_n^j \rangle \right) = \frac{1}{2} \int_\Sigma d^3x \psi^*(x) \left( E_{a}^m \nabla_m \sigma^a + \nabla_mE_{a}^m \sigma^a \right) \psi(x), \tag{16}
\]
provided we set \( \epsilon = 2^{-n} \) and write \( \nabla_m = \partial_m + [A_m, \cdot] \) which is the covariant derivative. Crucially, to obtain (16) we are required to fix the parameters \( \{\alpha_n\} \) to \( \alpha_n = 2^{3n} \). Thus, the expectation value of the Dirac type operator \( D \) on the states (14) renders, in a combined semi-classical and continuum limit, the expectation value of an ordinary spatial Dirac operator on a manifold \( \Sigma \). The spinor degrees of freedom emerged from the matrix factor which was introduced in the Hilbert space \( \mathcal{H} \) in order to accommodate a representation of the algebra of loops.

The expression (16) is formulated with respect to a certain coordinate system which emerges from the graphs \( \Gamma_n \). Thus, the cubic lattices can, in this specific limit, be interpreted as an emergent coordinate system. In particular, this coordinate system is exactly the coordinate system in which the classical Ashtekar variables are formulated\(^5\).

Notice that in order to obtain expression (16) we had to fix the parameter \( a_n \). Thus, the freedom which we encountered when we constructed the Dirac type operator \( D \) is eliminated in order to obtain a sensible semi-classical limit.

Notice also that \( E_{a}^m \) is the densitized Dreibein \( E_{a}^m = \sqrt{g} e_{a}^m \) where \( e_{a}^m \) is the dreibein and \( g \) is the determinant of the 3-metric \( g \) given by the dreibein. This means that the inverse measure \( d^3x \sqrt{g} \) appears naturally in (16). However, a problem regarding the normalization of spinors does arise in this framework, see [8].

\(^5\) Of course, the Ashtekar variables do not depend on a particular choice of a coordinate system. However, the way they enter the analysis in this paper, they are written down with respect to a coordinate system.
5.4. The Dirac Hamiltonian

In order to obtain the Dirac Hamiltonian instead of the Dirac operator in (16) we must introduce additional degrees of freedom which, in the appropriate limit, can correspond to the lapse and shift fields. This will then encode what amounts to a choice of foliation of space-time. These degrees of freedom are, seen from the spectral triple construction, related to the relationship between the Clifford algebra and the matrix factor in $H$. There are alternative ways to introduce these additional degrees of freedom, see [9, 18], and at present it is not clear which approach is more natural. For instance, following [18] we can introduce a modification of the Dirac operator with a matrix factor associated to each edge, written

$$D_M = \sum_{i,a} a_i M_i e_i$$  \hspace{1cm} (17)

where $M_i$ is an arbitrary, self-adjoint two-by-two matrix associated to the edge $l_i$. In the sum in (17) $k$ runs over the different copies of the group according to the change of variables introduced in section (4.4) and $a$ is an $SU(2)$ index. One finds that the expectation value of $D_M$ on the states (14) gives

$$\lim_{n \to \infty} \lim_{t \to 0} \langle \Psi_1^n | t D_M | \Psi_1^n \rangle = \frac{1}{2} \int_{\Sigma} d^3x \psi^* (x) \left( (N + N^a \sigma^a) (E^a_\mu \nabla_\mu \sigma^a + \nabla_\mu E^a_\mu \sigma^a) \right) \psi (x) + \text{zero order terms},$$  \hspace{1cm} (18)

where we wrote $M_i$ as $N(x)I_2 + i N^a(x) \sigma^a$ with ‘$x$’ referring to the point which $l_i$ singles out in this limit. Here, $N(x)$ and $N^a(x)$ are seen to give the lapse and shift fields. In (18) we have omitted certain zero-order terms and equation (18) is therefore seen to equal the principal part of the Dirac Hamiltonian in $3 + 1$ dimensions. Thus, the states (14) can be interpreted as one-particle states on which the Dirac type operator $D_M$ gives the Hamiltonian in the semi-classical approximation. We refer to [8] for a more detailed discussion of this interpretation.

6. Many-particle states

In the states (10) the sum runs over parallel transports which start at the basepoint, travels up to a spinor degree of freedom (the matrix factor $\psi(x_i)$) at a vertex, and then back to the basepoint. The parallel transport going up to the spinor utilizes the operators $U_p$ whereas the parallel transport going back uses the operators $U_d$, see (9). The two path up and down are in (10) taken to be identical, but one could also permit different paths. Thus, the spinor degrees of freedom arise together with a loop composed of $U_p$ and $U_d$ operators and is located at the point where these are joined.

In this section we analyze objects in $\mathcal{H}$ which involves products of loops composed of $U_p$’s and $U_d$’s. Each of these loops carry spinor degrees of freedom, and thus the number of spinor degrees of freedom will grow together with the number of loop factors. These loop objects will be seen to form a system of many-particle states in the semi-classical continuum limit.

We start the analysis by constructing states which are built from $\Psi_n(\psi)$ and the coherent states $\phi^i$. Thus, we write down the anti-symmetrized state

$$\Psi_{m,n}(\psi_1, \ldots, \psi_m, \phi^i) := \sum_{\sigma \in S_m} (-1)^{\sigma} \Psi_n(\psi_{\sigma(1)} \ldots \psi_{\sigma(m)}) \phi^i_{\text{grad} \leq 1}$$  \hspace{1cm} (19)

where $S_m$ is the group of $m$ permutations and where ‘grad $\leq 1$’ means that we only allow terms in this sum which involve at most one path $U_p$, which is odd with respect to the Clifford algebra. This restriction appears to be rather unnatural but will play a central role in the following analysis. We shall comment on this below.
Before we continue let us comment briefly on the structure of (19). When we construct states in \( \mathcal{H} \) which we wish to assign the prefix ‘physical’ then gauge invariance must be taken into consideration. \textit{A priori}, it is not obvious how one should define an action of the gauge group to the matrix factor in \( \mathcal{H} \)—and thereby to the spinor degrees of freedom which turn up in (19)—but in the continuum an semi-classical limit one must demand that only gauge covariant and invariant quantities emerge (in this limit one can readily define a gauge transformation for the spinors). Therefore, the ‘physical’ states, to the extend that they are not trivial with respect to the Clifford algebra and the matrix factor in \( \mathcal{H} \), can only involve loops. This is why loops, build of \( U_p \)'s and \( U_p \)'s, must appear in (19).

We now proceed to compute both the inner product and the expectation value of \( \mathcal{D} \) on such states in the semi-classical approximation. In the following we shall not be concerned with the lapse and shift fields and therefore only work with the Dirac type operator \( \mathcal{D} \) (and not \( \mathcal{D}_M \)).

### 6.1. The two-particle sector

Consider the case \( m = 2 \)

\[
\Psi'_{2,n}(\psi_1, \psi_2, \phi_p) = 2^{-6n} \sum_{i,j} (U_{p_i} \psi_1(x_i) U_{p_j}^{-1} U_{p_i} \psi_2(x_j) U_{p_j}^{-1}) \phi_p = (\psi_1 \leftrightarrow \psi_2).
\]

A long and tedious computation, which is carried out in the appendix, gives first

\[
\langle \Psi'_{2,n}(\psi_1, \psi_2, \phi_p) | \Psi'_{2,n}(\psi_1, \psi_2, \phi_p) \rangle = \langle \Psi(\psi_2) | \Psi(\psi_2) \rangle \langle \Psi(\psi_1) | \Psi(\psi_1) \rangle + \langle \Psi(\psi_2) | \Psi(\psi_2) \rangle \langle \Psi(\psi_1) | \Psi(\psi_1) \rangle - \frac{2}{4} \sum_{i,j} \text{Tr}_{M_2} \left( U_{p_i} \psi_2^*(x_j) \psi_1(x_j) U_{p_j}^{-1} U_{p_i} \psi_1^*(x_i) \psi_2(x_i) U_{p_i}^{-1} \right)
\]

+ terms which vanish in the semi-classical continuum limit

+ terms from anti-symmetrization.

(20)

The first two terms in (20) are identical to expressions one would obtain from the one-particle sectors of \( \psi_1 \) and \( \psi_2 \) respectively. The last term, however, is a ‘cross-term’ which involves parallel transports \( U_p \)'s. If we take the combined semi-classical and continuum limit of the last term we find

\[
\lim_{n \to \infty} \frac{1}{8} \int \Sigma dx \int \Sigma dy \text{Tr}_{M_2} \left( U(x,y) \psi_2^*(x) \psi_1(x) U(x,y) \psi_1^*(y) \psi_2(y) \right)
\]

+ terms from anti-symmetrization,

(21)

where \( U(x,y) = U^{-1}(y,x) \) is the parallel transport between points \( x \) and \( y \) in the continuum lattice, given by the composition of parallel transports \( U_{p_i} \)'s. Thus, in the continuum limit we find terms where two fermions are connected by a gravitational flux tube. In the definition (10) of \( \Psi_n(\psi) \) we could have included a normalized sum over paths connecting the basepoint with each matrix \( \psi(x_i) \). In that case \( U(x,y) \) in (25) would involve a sum over paths connecting \( x \) and \( y \) and thus the interaction terms would involve several flux tubes.

The computation of (20) depends crucially on the presence of the Clifford algebra elements in \( U_p \) and the fact that conjugation with \( U_p \) corresponds to a trace over the matrices, see (8). Also, the computation relies on the peculiar restriction in (19) to products which involve at most one odd \( U_p \). If we permitted also products of two odd \( U_p \)'s then we would encounter terms which would diverge in the continuum and semi-classical approximation. Thus, this restriction is found to be vital. The factor \( \frac{1}{4} \) in (20) comes from a trace over \( M_2 \) that emerge from a conjugation with \( U_p \), as shown in the appendix.
Finally, in the particular limit where we let the coherent states $\phi'_n$ be peaked over the classical phase space point which corresponds to a flat space-time with $U(x, y) \equiv 1$, this expression gives
\[
\text{flat-space} \xrightarrow{\text{asy}} \frac{1}{4} \int d\Sigma \int d\Sigma' \text{Tr}_F \left( \psi^2_2(x) \psi_1(x) \psi^*_1(y) \psi_2(y) \right) + \text{terms from anti-symmetrization.}
\]
(22)

Ignoring the factor $\frac{1}{2}$ this term resemble a cross term coming from an anti-symmetrized Fock space. However, such a term would be of the form
\[
\int d\Sigma \text{Tr} \left( \psi^2_2(x) \psi_1(x) \right) \int d\Sigma' \text{Tr} \left( \psi^*_1(y) \psi_2(y) \right) + \cdots
\]
(23)
and we see that the difference is the occurrence of an additional trace. This means that the cross-term term (21) can, in the flat-space limit, almost be interpreted in terms of anti-symmetrization of a fermionic Fock space, except that an odd mixing of spinor degrees of freedom happen due to the missing trace in (26). We shall further analyze this issue in section 6.2.

Let us now work out the expectation value of the Dirac type operator $D$ on the states (19). Again, the computations are given in the appendix. First, we find
\[
\langle \Psi_{2,n}(\psi_1, \psi_2, \phi'_n) | D | \Psi_{2,n}(\psi_1, \psi_2, \phi'_n) \rangle = \langle \Psi(\psi_2) | D | \Psi(\psi_1) \rangle + \langle \Psi(\psi_2) | \Psi(\psi_1) \rangle + \frac{2 - \alpha_n}{4} \sum_{ij} \Omega(p_i, p_j)
\]
\[
\times \left( \text{Tr}_M \left( U_{p_i} E_{p_j}^m g_{j+1}^* \sigma^a \psi^*_2(x_{j+1}) g^*_j \psi_1(x_j) U^{-1}_{p_j} U_{p_i} \psi^*_1(x_i) \psi_2(x_i) U_{p_i} \right) + \text{terms which vanish in the semi-classical continuum limit} + \text{terms from anti-symmetrization,}
\]
(24)
where expressions like $E_{p_j}^m \sigma^a g_{j+1}^* \psi^*_2(x_{j+1}) \psi_1(x_j)$ (sum over $a$) refer again to an edge which starts where $U_{p_i}$ ends and which has the matrix $\psi^*_2$ associated to its endpoint, with the direction of the edge denoted by ‘$m$’. In (24) $\Omega(p_1, p_2)$ is a factor which depends both on the lengths of $p_1$ and $p_2$ and on how large their intersection is. This factor can be obtained from the analysis in the appendix.

The first two terms in (24) are terms which give the one-particle Hamiltonians in the semi-classical limit for the spinors $\psi_1$ and $\psi_2$. The computation of the one-particle sector was worked out in the previous section. The last four terms look like an interaction between particles $\psi_1$ and $\psi_2$. If we take the continuum and semi-classical limit of these last terms, we find (ignoring an overall factor due to $\Omega(p_1, p_2)$)
\[
\text{cl+cont.} \xrightarrow{\text{asy}} \frac{1}{4} \int d\Sigma \int d\Sigma' \text{Tr} \left( U(y, x) \psi^*_2(x) \nabla \psi_1(x) U(x, y) \psi^*_1(y) \psi_2(y) \right) + \text{terms from anti-symmetrization.}
\]
(25)
Thus, again we find terms where two fermions are connected by a gravitational flux tube.
Finally, we again consider the particular limit where the coherent states $\phi'_n$ are peaked over the classical phase space point which corresponds to a flat space-time with $U(x, y) \equiv 1$ and $\nabla \equiv \not \! 0$, then this expression gives

$$\text{flat-space} \sim \frac{1}{2} \int \Sigma \, dx \int \Sigma \, dy \, \text{Tr} \left( \frac{\partial}{\partial x^a} \left( \psi_2^* (x) \sigma^a \psi_1 (x) \right) \psi_1^*(y) \psi_2 (y) \right)$$

$$+ \frac{1}{2} \int \Sigma \, dx \int \Sigma \, dy \, \text{Tr} \left( \psi_2^* (x) \psi_1 (x) \frac{\partial}{\partial y^a} \left( \psi_1^*(y) \sigma^a \psi_2 (y) \right) \right)$$

$$+ \text{terms from anti-symmetrization} \equiv 0. \quad (26)$$

Thus, the interaction term vanishes\(^6\) when gravity is ‘turned off’.

### 6.2. The flat-space limit: Weyl spinors

Let us examine the terms in the normalization (23) which emerge in the limit where we approximate the construction around a flat space-time geometry. These terms have the form

$$\int \Sigma \, dx \int \Sigma \, dy \, \text{Tr} (\psi^* (x) \phi (x) \phi^* (y) \psi (y)),$$

where $\psi (x)$ and $\phi (x)$ are fields which takes value in two-by-two matrices and which should be interpreted as spinor fields. If we rewrite this expression in terms of Weyl spinors

$$\psi = (\psi_1, \psi_2), \quad \phi = (\phi_1, \phi_2), \quad (27)$$

where $\psi_i, \phi_i$ are the columns in $\psi$ and $\phi$ which represent Weyl components, then (27) becomes

$$\int \Sigma \, (\psi_1^* \cdot \phi_1) \int \Sigma \, (\phi_1^* \cdot \psi_1) + \int \Sigma \, (\psi_2^* \cdot \phi_2) \int \Sigma \, (\phi_2^* \cdot \psi_2)$$

$$+ \int \Sigma \, (\psi_1^* \cdot \phi_2) \int \Sigma \, (\phi_2^* \cdot \psi_1) + \int \Sigma \, (\psi_2^* \cdot \phi_1) \int \Sigma \, (\phi_1^* \cdot \psi_2). \quad (28)$$

On the other hand, consider the case where $\phi$ and $\psi$ appear in a two-particle state in a fermionic Fock space and consider in particular the terms coming from the anti-symmetrization of this state:

$$\int \Sigma \, (\psi_1^* \cdot \phi_1) \int \Sigma \, (\phi_1^* \cdot \psi_1) + \int \Sigma \, (\psi_2^* \cdot \phi_2) \int \Sigma \, (\phi_2^* \cdot \psi_2) + 2 \int \Sigma \, (\psi_2^* \cdot \phi_2) \int \Sigma \, (\phi_1^* \cdot \psi_1). \quad (29)$$

The two expressions (28) and (29) are very similar, their difference being an odd mixing of Weyl components in the last two terms of (28) and the absence in (28) of the last term in (29). The mixing of Weyl components in (28) appears to be unphysical, but one notices that if we restrict our construction to describe Weyl spinors only (thus, letting one of each columns in (27) be zero), then (28) and (29) coincide. The result is then, due to anti-symmetrization, that the entire normalization of the two-particle sector obtains an overall factor. This feature generalizes to many-particle states. Therefore we conclude

1. The construction coincides with a free fermionic quantum field theory in the flat-space limit if one restricts it to involve only Weyl spinors.

2. When we consider four-spinors a mixing of Weyl components appear in the normalization.

The restriction to Weyl spinors does not appear very natural. Rather, we suspect that additional structure should be introduced to the construction which causes (28) and (29) to be equal. Further, one might speculate whether this is related to the fact that we are working with $SU(2)$ connections and not with complexified $SU(2)$ connections, the latter being the original

---

\(^6\) We ignore here possible issues with boundary terms.
Ashtekar connection. In [8] we have already proposed the idea that complexified SU(2)
connections may be introduced in our framework via a doubling of the Hilbert space together
with the introduction of a real structure. It may be that such an extension—implemented in a
natural way—will cause (28) and (29) to be equal.

6.3. Many particle states

The computations of sectors which involve more than two particles are very similar to the
computations from the two-particle sector. Some details are given in the appendix. In general,
what we find is that the continuum semi-classical limit of the expectation value of an n-particle
state will, in addition to the n-particle sector also give all the n−k-sectors (k ∈ {1, 2, . . . n−1})
and will involve pairs of fermions connected by one flux tubes. Again, in the flat space-time
limit the interaction terms vanish, with the same mixing of Weyl components appearing in the
normalization of states. We find that by restricting the construction to Weyl spinors we obtain
what amounts to a n-particle sector coming from an anti-symmetric Fock-space.

6.4. A second type of semi-classical states

From the computation of the expectation value of D on the states (19) one is lead to consider
whether an alternative construction of many-particle states in H is possible. In fact, with the
infinite dimensional Clifford bundle in H there is some room to play. In the following we shall
discuss one possibility.

Consider the two loops in Γn where one is build out of U’s except for one edge, and the
other is build entirely from U’s. Both loops have a matrix degree of freedom ψ inserted:

\[ \Xi_1(\psi) = U_{p_1} \psi(x_i) U_{p_2}, \quad \Xi_2(\psi) = U_{p_1} U_i \psi(x_{i+1}) U_{p_1}, \]

so that the composition (p_1, l_i, p_2) form a based loop in BΓ_n, and where x_i and x_{i+1} are the
vertices where l_i starts and ends respectively. Thus, \( \Xi_1 \) involves a loop build out of U’s and
with a single ‘hole’, and \( \Xi_2 \) involves the same loop, but without the hole. If we take the
expectation of D on the sum \( \Xi_1(\psi) + \Xi_2(\psi) \) we get again the basic building block of a spatial
Dirac operator, corresponding to one of the term in (15). Thus, with a sum over different loops
we can again recover expressions like (16) and (18).

The difference between this line of construction and what we did in the previous sections
appears when we proceed to construct also many-particle states. Either we do so by permitting
products of loops, or we consider loops with more than one hole. In any case, the presence
of U’s ‘on both sides of’ \( \psi(x_i) \) in (30) has the effect that the parallel transports in (21) and
(25) is absent and that instead a double trace appears. Thus, the whole issue with mixing of
different Weyl spinor components disappear with this alternative construction. Therefore, with
this approach it is possible to work with 4-spinors.

Thus, we find that there are two types of semi-classical states: the ones discussed in the
previous sections, see (19), which entail a fermion interaction with gravitational flux tubes,
and the ones described in this section, see (30), which come without this fermion interaction.
Also, if we permit in (19) a normalized sum over several paths connecting the matrix \( \psi(x_i) \)
with the basepoint, then the result will be a fermion interaction which involves more than one
gravitational flux tube. Whenever flux tubes are present an odd mixing of Weyl components
appear, a feature which is not present if we restrict the construction to Weyl spinors. Note also
that the states (19) and products of states (30) will span the same Hilbert space. The difference
between the two types of states is solely the way the matrices \( \psi(x_i) \) are inserted.

Further analysis is needed to understand better the form of these semi-classical states. It
seems clear, however, that the Hilbert space H should only be regarded as an intermediate
structure which we for now use to construct the spectral triple. A more appropriate Hilbert space should most likely be found in the continuum limit as the Hilbert space spanned by states of the form (19) (probably with both symmetric and anti-symmetric sectors). One might speculate whether we are in fact recovering a GNS construction around the semi-classical states \( \lim_{n \to \infty} \phi_n \).

Clearly, a change of basis in \( \mathcal{H} \), which links the two types of states, must exist. This puts into question how the fermion interaction arises with this change of basis. However, since \( \mathcal{H} \) can only be regarded as an intermediate construction, it does not seem crucial what change of basis in \( \mathcal{H} \) links these different states, but rather which structures in \( \mathcal{H} \) will remain once a continuum limit Hilbert space has been found. Clearly, more analysis is needed to clarify these questions.

7. An additional grading

In the course of the analysis presented in the last sections several issues turned up which, we believe, indicate that additional structure should be introduced to the spectral triple construction. In particular, we think that a real structure might be what we are looking for. For now, however, let us first list these issues and second introduce an additional grading which will solve some of them.

1. The computations in the appendix are highly sign-sensitive. the grading of the \( U_i \)'s depend on their length and entail complicated factors. We do not believe that this dependency on the number of edges in a path being even or odd should play any role in the construction.
2. In the same line of reasoning, the restriction in (19) to include only products of \( U_i \)'s with certain gradings seems odd.
3. We are working with \( SU(2) \) connections whereas the original Ashtekar connection takes value in complexified \( SU(2) \). We have already in [8] suggested that a doubling of the Hilbert space would be a viable method to introduce this complexification. Such a doubling would come with a grading.
4. The fact that we are forced to restrict the construction to Weyl spinors is not satisfactory and calls for a solution. Again, the shift from 2- to 4-spinors involve a grading.

In a first step, one can introduce a \( Cl(1) \) factor in the Hilbert space and add a \( \gamma_0 \) from this Clifford algebra to both the Dirac operator \( D \) and to the \( U_i \)'s

\[
D \rightarrow \gamma_0 D, \quad U_i \rightarrow \gamma_0 U_i.
\]

This additional grading would solve the problems 1 and 2 since \( \gamma_0 \epsilon_i \) and \( U_i \) now commute (unless \( i = j \)). This additional grading amounts to a doubling of the construction where \( D \) acts like

\[
\begin{pmatrix}
0 & D \\
-D & 0
\end{pmatrix}
\text{, with } \gamma_0 = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}.
\]

Whether such an additional structure might resolve problems 3 and 4 requires further analysis.

8. Discussion

The key new insight presented in this paper is that central elements of fermionic quantum field theory can be derived from the kinematical sector of canonical quantum gravity in a semi-classical approximation by utilizing tools from noncommutative geometry.

Specifically, we show that an infinite system of interacting fermions emerges in a semi-classical approximation from a previously constructed semi-finite spectral triple over
holonomy loops. Further, when all gravitational degrees of freedom are turned off a free fermionic quantum field theory emerge. This result is a strong indication that this model should be interpreted as describing quantized gravitational fields coupled to quantized matter fields. It is important to realize that the spectral triple construction does, a priori, only involve gravitational degrees of freedom. Therefore, matter degrees of freedom are, to some extend, emergent and canonical.

A further indication that this model describes quantized gravitational fields is that it encodes the kinematics of quantum gravity [8]. This should be understood in the sense that the interaction between the algebra of holonomy loops and the Dirac type operator reproduces the structure of the Poisson bracket of general relativity formulated in terms of Ashtekar variables [6, 7]. Thus, the spectral triple is essentially a rearrangement of canonical quantum gravity cast in terms of Ashtekar variables.

This paper is a continuation of a series of papers [8, 9, 10, 18] devoted to the study of the semi-classical analysis of the spectral triple construction. Here, semi-classical refer to the gravitational fields and is computed in orders of the Planck length. In [8], semi-classical states were constructed which, in a semi-classical approximation, entails the Dirac Hamiltonian in 3+1 dimensions. Thus, these states were interpreted as one-fermion states in a given foliation and a given background gravitational field. In this paper we recognize that these one-fermion states are the simplest examples of a much larger class of states which are labeled by a certain loop order. The one-particle states found in [10] lie in the first order sector. At nth order the states entail, in a semi-classical approximation, a system of n interacting fermions. Depending on the exact way these states are constructed they may come with an interaction which involves flux-tubes of the Ashtekar connection. These flux tubes connect fermion degrees of freedom located at different points in space. Furthermore, we find that in the additional limit where all gravitational degrees of freedom are turned off a free fermionic quantum field theory emerge. Again, depending on the exact way the semi-classical states are constructed this free theory may involve a certain mixing of different spinor degrees of freedom. This mixing can be circumvented by restricting to Weyl spinors.

These new results raises many questions. First, one should clarify what happens to the free quantum field theory when one perturbs the background around the flat-space limit. The interesting question is what kind of interaction will emerge. One might speculate whether it will fall within the setup of an interacting quantum field theory. On a similar note, one should also analyze how the quantum corrections—which goes beyond the semi-classical approximation—will affect the free theory. Again, what interactions will arise? Will they be local and will they fall within the setup of an interacting quantum field theory?

Also, one need to determine whether the fermionic interaction which emerges from the first (out of two) set of semi-classical states is plausible as a physical interaction, just as one should aim to understand better the particular mixing of different Weyl components which comes from this interaction. The restriction to Weyl spinors which we identify as a possible way to circumvent this mixing does not appear natural. Rather, it seems that additional structure should be introduced to remedy this problem. Note also that the second set of semi-classical states which we construct avoid the mixing problem.

The computations leading to the many-particle sectors are highly sign-sensitive and depend on a complicated system of gradings related to lengths of paths. This sensitivity does not appear very natural and one might speculate whether some additional grading—for instance in the form of a real structure—could alter this. Such an additional grading should also render superfluous the odd restriction made in the definition of many-particle states which permits only terms with certain types of grading. Furthermore, the issue concerning the normalization of spinors discussed in [8] is still unresolved and requires attention.
The semi-classical analysis presented in this paper operates with a double limit: first a semi-classical limit is taken at a finite level of discretization whereupon a continuum limit is taken. At the present level of analysis it is not known whether the continuum limit can be taken without the semi-classical approximation and what constraints such a limit might entail. Further, it is important to note that the semi-classical approximation discussed in this paper is with respect to the Planck length $l_P$ and is a semi-classical limit of quantized gravitational fields. When elements of quantum field theory emerge in this limit the question arises how $\hbar$ might emerge from the construction.

The semi-classical states have a very particular form. They involve a sum over holonomy loops composed of two different kinds of parallel transports: one is the ordinary one, the other involves also elements of the infinite dimensional Clifford algebra associated to the spectral triple construction. The latter resemble an $n$-form (at a finite level) and one might wonder what geometrical significance these states have, in relation to the spectral triple construction. Also, in the construction of the semi-classical states we imposed an anti-symmetrization. Thus, a priori there will also be a symmetric sector in the Hilbert space. Whether this sector could somehow be related to bosonic degrees of freedom is an interesting question.

The spectral triple construction works for a compact Lie group, which we set equal to $SU(2)$. The Ashtekar connection, however, is in its original form a complexified $SU(2)$ connection, corresponding to $SL(2, \mathbb{C})$, which is non-compact. A choice of an $SU(2)$ connection corresponds either to a Euclidean setting or to a formulation where the Hamilton acquires an additional term. Clearly, it is desirable to extend the spectral triple construction to the complexified group and because we operate with a particular continuum limit which favors infinitesimal edges, we believe this might be done by doubling the Hilbert space and introducing a real structure.

Another interesting issue is the appearance of the lapse and shift fields in the construction. So far we have found several ways to introduce these fields via matrices associated to edges in the infinite lattice. It is, however, essential to understand what geometrical role—in terms of the spectral triple construction—these degrees of freedom play. If this is understood one might be lead to a natural formulation of a Wheeler–DeWitt equation.

Indeed, despite the advances made in this paper it should be stressed that the Hilbert space in the spectral triple construction can only be regarded as a kinematical Hilbert space. So far no Wheeler–DeWitt constraint has been constructed nor implemented. Thus, the analysis carried out in this paper should be interpreted as dealing with a fermionic sector in a larger yet to be understood framework which involves also a purely gravitational sector.

Acknowledgments

We are thankful to Mario Paschke and Ryszard Nest for numerous enlightening discussions. Also, we would like to thank Patrizia Vitale and Fedele Lizzi for hospitality and fruitful discussions during a visit.

Appendix. Computations

We adopt the convention:

$$(e^a_i)^* = -e^a_i \quad \text{and} \quad (\sigma^a)^* = -\sigma^a.$$ 

This convention is realized by

$$\sigma^1 = \begin{pmatrix} 0 & -i \\ -i & 0 \end{pmatrix}, \quad \sigma^2 = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}, \quad \sigma^3 = \begin{pmatrix} -i & 0 \\ 0 & i \end{pmatrix}.$$
With this realization we have
\[ \sigma^a \sigma^b = -\delta^{ab} + \epsilon^{abc} \sigma^c. \]

We begin by defining:
\[ U_i = \frac{i}{2}(e^i \sigma^a + e^*_i \sigma^a). \]

Notice that:
\[ U_i U_i^* = \frac{i}{2}(e^i \sigma^a + e^*_i \sigma^a)(e^*_i \sigma^a + e^i \sigma^a) = \frac{i}{2}(\sigma^a e^i - e^i \sigma^a e^*_i). \]

Using (A.1) we get
\[ e^i e^j e^k = \frac{1}{4} (4 + \sum_{a \neq b} e^i \sigma^a e^j e^k - e^j \sigma^a e^i e^k + e^i e^j e^k a^* e^a). \]

Next we compute for \( n \)
\[ \text{Tr}_{\mathcal{A}_1}(U_n^* A U_n) \]
\[ = \frac{i}{2} (\sigma^a e^i \sigma^a + A) = \text{Tr}_{\mathcal{M}_2}(A) 12, \]
\[ \text{Tr}_{\mathcal{A}_1}(U_1^* A U_1) = \frac{1}{2} (\sigma^a e^i \sigma^a + A) = \text{Tr}_{\mathcal{M}_2}(A) 12, \]
\[ \text{since writing} \]
\[ A = a_1 12 + a_n \sigma^a \]
\[ \text{and seeing that} \]
\[ \sigma^a e^i \sigma^b e^k \sigma^a + \sigma^b = 0. \]

We will now make the convention \( e^1 := e^1 e^2 \) and \( \sigma_0 := 12. \) The form of \( U_1 \) then reads
\[ \frac{i}{2} e^1 \sigma^a, \]

and we furthermore have
\[ \{e^i, e^j\} = -2 \delta^{ij}. \]

### A.1 The basic trick

We will demonstrate the basic trick for the computations we are going to do, by first computing the example
\[ \text{Tr}(A_n^* U_n \cdots A_1^* U_1 B^* U^* U_1 \cdots U_n C_n U D). \]

We note, that \( U_1 C_1 \cdots U_n C_n \) is sandwiched between \( U^* \) and \( U. \) We write
\[ U_1 C_1 \cdots U_n C_n = \frac{i^n}{2^n} e^1 \cdots e^n \sigma^{a_1} C_1 \cdots \sigma^{a_n} C_n. \]

Using (A.1) we get
\[ \text{Tr}(A_n^* U_n \cdots A_1^* U_1 B^* U^* U_1 \cdots U_n C_n U D) \]
\[ = \text{Tr}(A_n^* U_n \cdots A_1^* U_1 B^* U^* \left( \frac{i^n}{2^n} e^1 \cdots e^n \sigma^{a_1} C_1 \cdots \sigma^{a_n} C_n \right) U D) \]
\[ = (-1)^n \text{Tr}(A_n^* U_n \cdots A_1^* U_1 \text{Tr}_{\mathcal{M}_2}(\frac{i^n}{2^n} e^1 \cdots e^n \sigma^{a_1} C_1 \cdots \sigma^{a_n} C_n) B^* U^* U D) \]
\[ = (-1)^n \text{Tr}(A_n^* U_n \cdots A_1^* U_1 \text{Tr}_{\mathcal{M}_2}(\frac{i^n}{2^n} e^1 \cdots e^n \sigma^{a_1} C_1 \cdots \sigma^{a_n} C_n) B^* D). \]
We next write
\[
\text{Tr}_{M_2}\left( \frac{i^n}{2^{n-1}} e_1^{a_1} \cdots e_n^{a_n} C_1 \cdots C_n \right) = \frac{i}{2} e_1^{a_1} \left( \frac{i^{n-1}}{2^{n-1}} e_2^{a_2} \cdots e_n^{a_n} C_1 \cdots C_n \right)^a,
\]
where we are summing over \( a \) and \( (X)^a \) means the \( \sigma^a \)-component of \( X \). Since
\[
\frac{i^{n-1}}{2^{n-1}} e_2^{a_2} \cdots e_n^{a_n} \sigma^{a_2} C_2 \cdots \sigma^{a_n} C_n = C_1 U_2 C_2 \cdots U_n C_n,
\]
we will also write
\[
\left( \frac{i^{n-1}}{2^{n-1}} e_2^{a_2} \cdots e_n^{a_n} \sigma^{a_2} C_2 \cdots \sigma^{a_n} C_n \right)^a = (C_1 U_2 C_2 \cdots U_n C_n)^a.
\]
Continuing the computation we get
\[
\text{Tr} \left( A_n^* U_n^* \cdots A_1^* U_1^* \text{Tr}_{M_2} \left( \frac{i^n}{2^n} e_1^{e_1} \cdots e_n^{e_n} \sigma^{e_1} C_1 \cdots \sigma^{e_n} C_n \right) B^* D \right) = \text{Tr} \left( A_n^* U_n^* \cdots A_1^* U_1^* \left( U_1^* \frac{i}{2} e_A^{A} \right) \left( C_1 U_2 C_2 \cdots U_n C_n \right)^a B^* D \right) = \text{Tr} \left( A_n^* U_n^* \cdots A_1^* \frac{1}{4} \left( C_1 U_2 C_2 \cdots U_n C_n \right) B^* D \right) = \frac{1}{4} \text{Tr}(A_n^* C_n B^* D) \text{Tr}(A_1^* C_1) \cdots \text{Tr}(A_{n-1}^* C_{n-1}).
\]
All together we get
\[
\text{Tr}(A_n^* U_n^* \cdots A_1^* U_1^* B^* U^* U_1 C_1 \cdots U_n C_n U D) = \frac{(-1)^{n}}{4} \text{Tr}(A_n^* C_n B^* D) \text{Tr}(A_1^* C_1) \cdots \text{Tr}(A_{n-1}^* C_{n-1}).
\]
What is fundamental in this computation is the following (which we will use subsequently): we want to compute a term of the form
\[
\text{Tr}(A U_1^* B U^* U_1 C U D),
\]
where \( \tilde{A}, \tilde{B}, \tilde{C}, \tilde{D} \) are almost general elements (see below) in \( \text{Cl}(T_{2n} \mathbb{R}) \otimes M_2 \). We can compute this as
\[
\frac{(-1)^{\#}}{4} (\tilde{A} \tilde{C} \tilde{B} U^* U D) = \frac{(-1)^{\#}}{4} (\tilde{A} \tilde{C} \tilde{B} D),
\]
where \( (-1)^{\#} \) is the sign we pick up by commuting the Clifford part of \( U_1 \tilde{A} \) past the Clifford part of \( B U^* \), i.e. \( \# \) is the Clifford degree of \( U_1 \tilde{C} \) multiplied with the Clifford degree of \( B U^* \).

That \( \tilde{A}, \tilde{B}, \tilde{C}, \tilde{D} \) are almost general means, that they do not have any Clifford elements in common with \( U_1 \) and \( U \), and that \( \tilde{B}, \tilde{C} \) don’t have any Clifford elements in common.

We can also remark, that if we replace \( U_1 \tilde{C} \) with \( \tilde{C}_1 U_1 \tilde{C}_2 \), then, since we are taking the \( \text{Tr}_{M_2} \) in the computation, we can compute
\[
\text{Tr}(\tilde{A} U_1^* \tilde{B} U^* \tilde{C}_1 U_1 \tilde{C}_2 \tilde{C} U D) = \frac{(-1)^{\#}}{4} \text{Tr}(\tilde{A} \tilde{C}_1 \tilde{C}_2 \tilde{B} D),
\]
where we are picking up an extra sign, from commuting the Clifford part of \( \tilde{C}_1 \) with the Clifford part of \( U_1 \tilde{C}_2 \).
A.2. Terms appearing in the inner product

When computing the norm of a two particle state, there appear terms of the following form

\[ \text{Tr}(A^*U^*_{p_1} B^*U^*_{p_2} U_{p_1} U_{p_2} CU_{p_1} U_{p_3} D) \]  
\[ \text{Tr}(A^*U^*_{p_1} B^*U^*_{p_2} U_{p_1} U_{p_2} CU_{p_1} U_{p_3} D) \]  
\[ \text{Tr}(A^*U^*_{p_2} U_{p_1} U_{p_2} CU_{p_1} U_{p_3} D) \]

where \( p_1, p_2, p_3 \) denote paths.

The terms (A.2) are easily computed

\[ \text{Tr}(A^*U^*_{p_1} B^*U^*_{p_2} U_{p_1} U_{p_2} CU_{p_1} U_{p_3} D) = \text{Tr}(A^*D)\text{Tr}(B^*C). \]  

The terms (A.3) can be computed as follows

\[ \text{Tr}(A^*U^*_{p_1} B^*U^*_{p_2} U_{p_1} U_{p_2} CU_{p_1} U_{p_3} D) \]  
\[ = \text{Tr}(A^*U^*_{p_2} U_{p_1} B^*U^*_{p_1} U_{p_1} U_{p_2} CU_{p_1} U_{p_3} D) \]  
\[ = \text{Tr}(A^*U^*_{p_2} B^*U^*_{p_1} U_{p_2} CU_{p_1} U_{p_3} D) \]  
\[ = (-1)^{n_1+n_2+(n_1+n_2)n_3} \frac{1}{4} \text{Tr}(A^*U^*_{p_2} CB^*U^*_{p_1} U_{p_3} D) \]  
\[ = (-1)^{n_1+n_2+(n_1+n_2)n_3} \frac{1}{4} \text{Tr}(A^*CB^*D), \]

where \( n_1, n_2, n_3 \) are the length of \( p_1, p_2, p_3 \).

To compute the terms (A.4), we first write

\[ \text{Tr}(A^*U^*_{p_2} U_{p_1} B^*U^*_{p_1} U_{p_1} U_{p_2} U_{p_3} D) = \text{Tr}(A^*U^*_{p_2} B^*U^*_{p_1} U_{p_3} CU_{p_1} U_{p_3} D). \]

To proceed we note that \( U^*_i = -U_i \). We write

\[ U_{p_3} = U_{p_3} U_{p_3} \]

where \( p_{31} \) is \( p_3 \) minus the last edge. With this notation we get

\[ \text{Tr}(A^*U^*_{p_2} U_{p_1} B^*U^*_{p_1} U_{p_3} CU_{p_1} U_{p_3} D) \]  
\[ = -\text{Tr}(A^*U^*_{p_2} U^*_{p_1} B^*U^*_{p_2} U_{p_3} CU_{p_1} U_{p_3} D) \]  
\[ = (-1)^{(n_1+n_2-1)(n_1-1+n_2)} + 1 \frac{1}{4} \text{Tr}(A^*U^*_{p_2} U^*_{p_1} U_{p_3} CB^*U^*_{p_2} U_{p_3} U_{p_3} D) \]  
\[ = (-1)^{(n_1+n_2-1)(n_1-1+n_2)} + 1 \frac{1}{4} \text{Tr}(A^*U^*_{p_2} U_{p_3} U_{p_3} CB^*U^*_{p_2} U_{p_3} D). \]

We are therefore left with computing terms of the form

\[ \text{Tr}(A U_{1} \cdots U_{n} B U_{1} \cdots U_{n} C). \]
These can be computed in the following way:
\[
\text{Tr}(A U_1 \cdots U_n B U_1 \cdots U_n C) = \text{Tr}(A U_1 \cdots U_n B U_2 \cdots U_{n-1} U_n C) = \frac{(-1)^{n-1}}{4} \text{Tr}(A U_1 U_2 \cdots U_{n-1} B U_2 \cdots U_{n-1} U_n C)
\]
\[
\vdots
\]
\[
= \begin{cases} 
\frac{(-1)^{(n-1)+(n-3)+\cdots+1}}{4^{\pm 2}} \text{Tr}(ABC), & n \text{ even} \\
\frac{1}{4^{\pm 2}} \text{Tr}(AC \text{Tr}(B)), & n \text{ odd}.
\end{cases}
\]

Thus, plugging in, we finally get
\[
\langle A^* U^*_{p_2} U_{p_1} B^* U^*_{p_2} U_{p_1} | U_{p_2} U_{p_2} C U_{p_2} U_{p_2} D \rangle
\]
(12)
\[
= \begin{cases} 
\frac{(-1)^{(n+1)(n+3)+\cdots+(n-1)+(n-3)+\cdots+1}}{4^{\pm 2}} \text{Tr}(A^* C B^* D), & n_2 + n_3 \text{ odd} \\
\frac{1}{4^{\pm 2}} \text{Tr}(A^* D) \text{Tr}(C B^*), & n_2 + n_3 \text{ even}.
\end{cases}
\]
(13)

A.3. Acting with the Dirac type operator

The corresponding terms for the Dirac operator \(D\), which we need to compute, are roughly of the form
\[
\text{Tr}(A^* U^*_{p_2} U^*_{p_2} B^* U^*_{p_2} U^*_{p_1} D(U_{p_1} U_{p_2} C U_{p_2} U_{p_2} D))
\]
(14)
\[
\text{Tr}(A^* U^*_{p_2} U_{p_2} B^* U^*_{p_2} U^*_{p_1} D(U_{p_1} U_{p_2} C U_{p_2} U_{p_2} D))
\]
(15)
\[
\text{Tr}(A^* U^*_{p_2} U^*_{p_2} B^* U^*_{p_2} U^*_{p_1} D(U_{p_1} U_{p_2} C U_{p_2} U_{p_2} D)).
\]
(16)

Of course either \(p_2\) or \(p_3\) need be one edge shorter on one of the sides in order give something different from zero, and we also need to multiply with Hall coherent states, but we will omit this in the notation here.

For the computation of (14) let us first compute the term
\[
\text{Tr}(A^* U^*_{p_2} U^*_{p_2} B^* U^*_{p_2} U^*_{p_1} D(U_{p_1} U_{p_2} C U_{p_2} U_{p_2} D)).
\]

This only gives something, for the term of the form in \(D\) which include \(\sigma^i\). Multiplying \(\sigma^i\) on \(U_i\) and remembering that we are taking the trace at the end, gives \(\frac{1}{2} \sigma^i\). We therefore get
\[
\text{Tr}(A^* U^*_{p_2} U^*_{p_2} B^* U^*_{p_2} U^*_{p_1} D(U_{p_1} U_{p_2} C U_{p_2} U_{p_2} D))
\]
\[
= (-1)^{n_1 + n_2} \text{Tr} \left(A^* U^*_{p_2} U^*_{p_2} B^* U^*_{p_2} U^*_{p_1} U_{p_2} E_j \frac{-i}{2} \sigma^a C U_{p_1} U_{p_2} D \right)
\]
\[
= (-1)^{n_1 + n_2} \text{Tr}(A^* D) \text{Tr} \left(B^* E_j \frac{-i}{2} \sigma^a C \right).
\]

Here we have been sloppy with the notation, and omitting the prefactor \((-1)^n\). \(E_j\) denotes the expectation value of \(L_{\sigma^i}\) on semi-classical states, see equation (12).
Similarly, if the $U_i$ are on the other side we get
\[ (-1)^{n_1+n_2} \text{Tr}(A^*D) \text{Tr} \left( B^* E_a^i \frac{i}{2} \sigma^a C \right). \]
If we had prolonged $U_p$, instead we get
\[ (-1)^{n_1+n_2} \text{Tr} \left( A^* E_a^i \frac{i}{2} \sigma^a D \right) \text{Tr}(B^* C) \]
and
\[ (-1)^{n_1+n_2} \text{Tr} \left( A^* E_a^i \frac{i}{2} \sigma^a D \right) \text{Tr}(B^* C). \]

Similarly for (A.15) we get the terms (right-hand side indicating, where the $U_j$ has been inserted.)
\[ (-1)^{n_1+n_2+n_1+n_2+n_1+n_2+n_1+n_2} \text{Tr} \left( A^* E_a^i \frac{i}{2} \sigma^a C B^* D \right) U_{p_1} U_{p_2} C \]
\[ (-1)^{n_1+n_2+n_1+n_2+n_1+n_2+n_1+n_2} \text{Tr} \left( A^* E_a^i \frac{i}{2} \sigma^a C B^* D \right) A^* U^*_j U^*_p \]
\[ (-1)^{n_1+n_2+n_1+n_2+n_1+n_2+n_1+n_2} \text{Tr} \left( A^* C B^* E_a^i \frac{i}{2} \sigma^a D \right) U_{p_1} U_{p_2} D \]
\[ (-1)^{n_1+n_2+n_1+n_2+n_1+n_2+n_1+n_2} \text{Tr} \left( A^* C B^* E_a^i \frac{i}{2} \sigma^a D \right) B^* U^*_j U^*_p. \]

Similarly we could write the terms of the type (A.16). However in the continuum limit these terms will vanish, due to the powers of $\frac{1}{n}$ appearing in these terms.

**A.4. Computation applied the two-particle states**

We will now apply the computation to the two-particle states
\[ \sum_{i,j} U_{p_i} \psi_1(x_i) U_{j}^{-1} U_{p_j} \psi_2(x_j) U_{i}^{-1}. \]
We have here omitted the prefactor and Hall's coherent state in the notation. Also, we have not antisymmetrized the expression. Instead we compute the inner product with a terms of the form
\[ \sum_{i,j} U_{p_i} \psi_3(x_i) U_{j}^{-1} U_{p_j} \psi_4(x_j) U_{i}^{-1}. \]
We see, that there three types of terms are arising in the inner product:
\[ \text{Tr}(U_{p_i} \psi_2(x_i)^* U_{p_i}^* U_{j}^{-1} U_{p_j} \psi_3(x_j) U_{i}^{-1}) \] (A.17)
\[ \text{Tr}(U_{p_i} \psi_2(x_i)^* U_{p_i}^* U_{j}^{-1} U_{p_j} \psi_4(x_j) U_{i}^{-1}) \] (A.18)
\[ \text{Tr}(U_{p_i} \psi_2(x_i)^* U_{p_i}^* U_{j}^{-1} U_{p_j} \psi_3(x_j) U_{i}^{-1} U_{p_j} \psi_4(x_j) U_{i}^{-1}) \] (A.19)

According to (A.5), the terms (A.17) gives
\[ \text{Tr}(\psi_1(x_i)^* \psi_3(x_j) \psi_2(x_i)^* \psi_4(x_i)). \]

According to (A.11), the terms (A.18) gives
\[ (-1)^{m_1+m_2+n_1+n_2} \text{Tr}(U_{p_i} \psi_2(x_i)^* \psi_3(x_i) U_{j}^{-1} U_{p_j} \psi_1(x_j) \psi_4(x_i) U_{i}^{-1}) \] (7)
where $n$ denotes the length of the path $p_1$ and $p_2$ have in common, $n_1$ the length of $p_1$ minus $n$ and $n_2$ the length of $p_2$ minus $n$.

The terms (A.19) vanishes in the continuum limit due to the powers of $\frac{1}{n}$ appearing in (A.13).
A.5. Expectation value of the Dirac operator

We will now compute the terms (A.14)–(A.16) for the two particle states. Again each of these terms basically consists of four terms, in the case (A.14) these terms are

\[ \text{Tr}(U_i \psi_2(x_i)^* U_j^* U_j^{\dagger} \psi_1(x_j)^* U_i^* D(U_{i+1} \psi_3(x_{i+1}) U_{j+1} \psi_1(x_j)^* U_i^* \psi_4(x_i) U_j^{\dagger}) \]

\[ \text{Tr}(U_i \psi_2(x_i)^* U_j^* U_j^{\dagger} \psi_1(x_j)^* U_i^* D(U_{i+1} \psi_3(x_{i+1}) U_{j+1} \psi_1(x_j)^* U_i^* \psi_4(x_i) U_j^{\dagger}) \]

\[ \text{Tr}(U_i \psi_2(x_i)^* U_j^* U_j^{\dagger} \psi_1(x_j)^* U_i^* D(U_{i+1} \psi_3(x_{i+1}) U_{j+1} \psi_1(x_j)^* U_i^* \psi_4(x_i) U_j^{\dagger}) \]

where the \( i+1 \) denotes going one edge further than \( p_i \), and \( x_{i+1} \) denotes the endpoint of this edge. The terms gives

\[ -\frac{1}{2} \text{Tr}(\psi_1(x_i)^* E_a^{i+1}(x_{i+1}) \sigma^a \psi_3(x_{i+1}) \text{Tr}(\psi_2(x_i)^* \psi_4(x_i)) \]

\[ \frac{1}{2} \text{Tr}(\psi_1(x_i)^* E_a^{i+1}(x_{i+1}) \sigma^a \psi_3(x_{i+1}) \text{Tr}(\psi_2(x_i)^* \psi_4(x_i)) \]

\[ -(1)^{n_1+n_2+n_1} \frac{1}{2} \text{Tr}(\psi_1(x_i)^* \psi_3(x_i)) \text{Tr}(\psi_2(x_i)^* E_a^{i+1}(x_{i+1}) \sigma^a \psi_4(x_{i+1})) \]

\[ -(1)^{n_1+n_2+n_1} \frac{1}{2} \text{Tr}(\psi_1(x_i)^* \psi_3(x_i)) \text{Tr}(\psi_2(x_i)^* E_a^{i+1}(x_{i+1}) \sigma^a \psi_4(x_{i+1})) \]

where we have the same conventions for \( n, n_1, n_2 \) like in the computation of the inner product of the two particle state.

Similarly the terms of (A.15) give

\[ -(1)^{n_1+n_2+n_1} \frac{1}{4} \text{Tr}(U_i \psi_2(x_i)^* E_a^{i+1}(x_{i+1}) \sigma^a \psi_3(x_{i+1}) U_{i+1} \psi_1(x_j)^* \psi_4(x_j) U_j^{\dagger}) \]

\[ -(1)^{n_1+n_2+n_1} \frac{1}{4} \text{Tr}(U_i \psi_2(x_i)^* E_a^{i+1}(x_{i+1}) \sigma^a \psi_3(x_{i+1}) U_{i+1} \psi_1(x_j)^* \psi_4(x_j) U_j^{\dagger}) \]

\[ -(1)^{n_1+n_2+n_1} \frac{1}{4} \text{Tr}(U_i \psi_2(x_i)^* \psi_3(x_i) U_{i+1} \psi_1(x_j)^* E_a^{i+1}(x_{i+1}) \sigma^a \psi_4(x_{i+1}) U_{i+1}^{\dagger}) \]

\[ -(1)^{n_1+n_2+n_1} \frac{1}{4} \text{Tr}(U_i \psi_2(x_i)^* \psi_3(x_i) U_{i+1} \psi_1(x_j)^* E_a^{i+1}(x_{i+1}) \sigma^a \psi_4(x_{i+1}) U_{i+1}^{\dagger}) \]

where we again have the same conventions for \( n, n_1, n_2 \) like in the computation of the inner product of the two particle state.

Again terms of the form (A.16) will vanish in the continuum limit.

A.6. Many particle states

As we can see for the two-particle states, the computations are somewhat involved. The computations for many particle states can be done similarly. Of course the combinatorics becomes a bit more complicated. We will here outline how these computations can be done, and what the form of the outcome is.

The general form of the of the inner product, which is not vanishing in the continuum limit, is the following:

\[ \text{Tr}(A_1 U_1^* A_2 U_2^* A_3 U_3^* A_4 U_4^* A_5) \]

where \( U_1^* \) is the first \( U \) appearing adjoint when going from right to left in the expression, and where \( A_1, \ldots, A_5 \) are general elements in \( C(T_{d=4}^*) \otimes M_2 \). Omitting the signs and prefactors, this can be computed as

\[ \text{Tr}(A_1 A_3 A_5 A_2 A_4). \]
References

[1] Aastrup J, Grimstrup J M and Nest R 2009 On spectral triples in quantum gravity: I Class. Quantum Grav. 26 065011 (arXiv:0802.1783 [hep-th])

[2] Aastrup J, Grimstrup J M and Nest R 2009 On spectral triples in quantum gravity: II J. Noncommut. Geom. 3 47 (arXiv:0802.1784 [hep-th])

[3] Aastrup J, Grimstrup J M and Nest R 2009 A new spectral triple over a space of connections Commun. Math. Phys. 290 389 (arXiv:0807.3664 [hep-th])

[4] Connes A 1994 Noncommutative Geometry (San Diego, CA: Academic Press)

[5] Connes A 1996 Gravity coupled with matter and the foundation of non-commutative geometry Commun. Math. Phys. 182 155 (arXiv:hep-th/9603053)

[6] Ashtekar A 1986 New variables for classical and quantum gravity Phys. Rev. Lett. 57 2244

[7] Ashtekar A 1987 New Hamiltonian formulation of general relativity Phys. Rev. D 36 1587

[8] Aastrup J, Grimstrup J M, Paschke M and Nest R 2009 On semi-classical states of quantum gravity and noncommutative geometry arXiv:0907.5510 [hep-th]

[9] Aastrup J, Grimstrup J M and Paschke M 2010 On a derivation of the Dirac Hamiltonian from a construction of quantum gravity arXiv:1003.3802 [hep-th]

[10] Aastrup J, Grimstrup J M and Paschke M 2011 Quantum gravity coupled to matter via noncommutative geometry Class. Quantum Grav. 28 075014 (arXiv:1012.0713 [hep-th])

[11] Chamseddine A H, Connes A and Marcolli M 2007 Gravity and the standard model with neutrino mixing arXiv:hep-th/0610241

[12] Chamseddine A H and Connes A 2007 Why the standard model arXiv:0706.3688 [hep-th]

[13] Chamseddine A H and Connes A 2007 A dress for SM the beggar arXiv:0706.3690 [hep-th]

[14] Chamseddine A H and Connes A 1996 Universal formula for noncommutative geometry actions: unification of gravity and the standard model Phys. Rev. Lett. 77 4868

[15] Chamseddine A H and Connes A 1996 A universal action formula arXiv:hep-th/9606056

[16] Chamseddine A H and Connes A 1997 The spectral action principle Commun. Math. Phys. 186 731 (arXiv:hep-th/9606001)

[17] Aastrup J and Grimstrup J M 2006 Spectral triples of holonomy loops Commun. Math. Phys. 264 657 (arXiv:hep-th/0503246)

[18] Aastrup J, Grimstrup J M and Paschke M 2009 Emergent Dirac Hamiltonians in quantum gravity arXiv:0911.2404 [hep-th]

[19] Hall B C 1994 The Segal-bargmann ‘coherent state’ transform for compact Lie groups J. Funct. Anal. 122 103–51

[20] Hall B C 1997 Phase space bounds for quantum mechanics on a compact Lie group Commun. Math. Phys. 184 233–50

[21] Bahr B and Thiemann T 2009 Gauge-invariant coherent states for loop quantum gravity: I. Abelian gauge groups Class. Quantum Grav. 26 045011 (arXiv:0709.4619 [gr-qc])

[22] Thiemann T and Winkler O 2001 Gauge field theory coherent states (GCS): IV. Infinite tensor product and thermodynamical limit Class. Quantum Grav. 18 4997 (arXiv:hep-th/0005235)

[23] Bahr B and Thiemann T 2009 Gauge-invariant coherent states for loop quantum gravity: II. Non-Abelian gauge groups Class. Quantum Grav. 26 045012 (arXiv:0709.4636 [gr-qc])