ABSTRACT

For realistic values of the Higgs boson mass the high temperature electroweak phase transition cannot be described perturbatively. The symmetric phase is governed by a strongly interacting $SU(2)$ gauge theory. Typical masses of excitations and scales of condensates are set by the “high temperature confinement scale” $\approx 0.2 T$. For a Higgs boson mass around 100 GeV or above all aspects of the phase transition are highly nonperturbative. Near the critical temperature strong electroweak interactions are a dominant feature also in the phase with spontaneous symmetry breaking. Depending on the value of the Higgs boson mass the transition may be a first order phase transition or an analytical crossover.

1. RUNNING COUPLINGS NEAR PHASE TRANSITIONS

Phase transitions in the very early universe may be tumultuous periods out of thermodynamic equilibrium. Some traces of such a transition may still be observable today, giving a hint on what happened shortly after the big bang. Remnants of the electroweak phase transition may help to understand the universe at the age of about $10^{-12}$ s. One of the most prominent features of the standard model of electroweak interactions is spontaneous symmetry breaking. The masses of the gauge bosons and fermions are proportional to the vacuum expectation value of the Higgs doublet. According to the original argument of Kirzhnits and Linde\textsuperscript{1} this expectation value vanishes at sufficiently high temperature and the electroweak symmetry is restored. Such high temperatures were presumably realized in the very early universe immediately after the big bang. As the universe cooled there must have been a phase transition from the symmetric to the spontaneously broken phase of the standard model. This phase transition may have important consequences for our present universe, one example being the possible creation of the excess of matter compared to antimatter (baryon asymmetry)\textsuperscript{2}. The physical implications of the high temperature electroweak phase transition depend
strongly on its nature (whether it is second or first order) and its details. More specifically, the possibility of generating the baryon asymmetry imposes the requirement of out of equilibrium conditions\(^3\), which is satisfied only if the phase transition is of the first order; the exact amount of produced baryon number is very sensitive to the details of the fluctuations which drive the transition (profile of bubbles, velocity of the wall etc.)\(^4\); and avoiding the washing out of any generated baryon asymmetry requires a sufficiently strong first order phase transition\(^5\).

Most theoretical studies of the electroweak phase transition use high temperature perturbation theory\(^6\) for a computation of the temperature-dependent effective potential for the Higgs field. This is often supplemented by an appropriate resummation of graphs or a solution of a corresponding gap equation for the mass terms\(^7\). Near a phase transition, however, the use of perturbation theory becomes questionable. Roughly speaking, one can trust perturbation theory only to the extent that mean field theory gives a qualitatively correct description of the corresponding statistical system. This is known to be not always the case for critical phenomena. A good example for the breakdown of high temperature perturbation theory in the vicinity of the phase transition are scalar field theories\(^8\). The second order character of the transition and the corresponding critical exponents are not reproduced by perturbation theory.

The deeper reason for the breakdown of perturbation theory lies in the effective three-dimensional character of the high temperature field theory\(^9\). Field theory at nonvanishing temperature \(T\) can be formulated in terms of an Euclidean functional integral where the “time dimension” is compactified on a torus with radius \(T^{-1}\)\(^10\). For phenomena at distances larger than \(T^{-1}\) the Euclidean time dimension cannot be resolved. Integrating over modes with momenta \(p^2 > (2\pi T)^2\) or, alternatively, over the higher Fourier modes on the torus (the \(n \neq 0\) Matsubara frequencies) leads to “dimensional reduction” to an effective three-dimensional theory. This is very similar to dimensional reduction in Kaluza-Klein theories\(^11\) for gravity. The change of the effective dimensionality for distances larger than \(T^{-1}\) is manifest in the computation\(^8\) of the temperature-dependent effective potential in scalar theories. The scale dependence of the renormalized couplings is governed by the usual perturbative \(\beta\)-functions only for \(p^2 > (2\pi T)^2\). In contrast, for smaller momenta \(p^2 < (2\pi T)^2\) the running of the couplings was found to be determined by three-dimensional \(\beta\)-functions instead of the perturbative four-dimensional ones - as proposed in a different setting in ref. 12. The effect of the three-dimensional running is clearly manifest in the temperature-dependence of the couplings shown\(^8\) in fig. 1, especially for the renormalized quartic scalar coupling \(\lambda_R\), which vanishes for \(T\) approaching the critical temperature \(T_c\). As an alternative to integrating out all modes with \(p^2 > (2\pi T)^2\) an effective three-dimensional theory for the long distance electroweak physics has been obtained in ref. 13 by integrating out the higher Matsubara frequencies*.

If the three-dimensional running of the couplings becomes important, the physics of the phase transition is dominated by classical statistics even in case of a quantum field theory. A second order phase transition is characterized by an infinite correlation length. The critical exponents which describe the behaviour near the critical temperature are always those of the corresponding classical statistical system. Since the fixpoints of the

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* For an earlier treatment of dimensional reduction in high temperature QCD see ref. 14.
three-dimensional $\beta$-functions are very different from the four-dimensional (perturbative) fixpoints, we conclude that high temperature perturbation theory is completely misleading in the vicinity of a second-order phase transition. This argument extends to sufficiently weak first order transitions. A second related example for the breakdown of perturbation theory is the symmetric phase of the electroweak gauge theory. The gauge bosons are massless in perturbation theory and the three-dimensional running always dominates at large distances\textsuperscript{15}.

In order to understand the high temperature behaviour of a theory we should understand the qualitative features of the $\beta$-functions in three dimensions. These $\beta$-functions have nothing to do with the ultraviolet regularization of the field theory - in this respect there is no difference between vanishing and nonvanishing temperature. They are rather related to the infrared behaviour of the theory or the dependence of Green functions on some sort of infrared cutoff. According to Wilson’s concept of the renormalization group these $\beta$-functions describe the scale dependence of the couplings if one looks at the system on larger and larger distances. For an understanding of systems with approximate scaling in a certain range it is useful to define dimensionless couplings. One divides out an appropriate power of the infrared cutoff $k$ which plays the role of the renormalization scale. For example, the dimensionless quartic scalar coupling $\lambda$ in the effective three-dimensional theory is related to the four-dimensional coupling $\lambda_4$ and the temperature by

$$\lambda = \frac{\lambda_3}{k} = \lambda_4 \frac{T}{k}$$

For a pure $N$-component scalar theory the qualitative behaviour of the function $\beta_\lambda = \frac{\partial \lambda}{\partial t}$, $t = \ln k$, is shown in fig. 2. Arrows denote the flow of $\lambda$ with decreasing $k$ and we have assumed a massless theory. One clearly sees an infrared stable fixpoint corresponding to the second-order phase transition.

Next we turn to scalar QED which should describe the phase transition of superconductors or of the abelian Higgs model at high temperatures. The supposed form of the functions $\beta_{\epsilon^2} = \frac{\partial \epsilon^2}{\partial t}$ and $\beta_{\lambda/\epsilon^2} = \frac{\partial \left( \frac{\lambda}{\epsilon^2} \right)}{\partial t}$ is shown* in fig. 3. One observes a fixpoint for $\epsilon^2$ but no fixpoint for $\lambda/\epsilon^2$ or $\lambda$. Therefore $\lambda$ decreases until it vanishes and the corresponding phase transition is presumably of the first order. For comparison we also show the $\beta$-function for a number $M$ of complex scalar fields exceeding a critical value $M_{cr}$. Here an ultraviolet and an infrared stable fixpoint appear. For small initial $\lambda$ ($\lambda/\epsilon^2$ smaller than the UV fixpoint) the phase transition remains of the first order. The UV fixpoint corresponds to a triple point and for initial $\lambda/\epsilon^2$ larger than the UV fixpoint one has a second-order transition with critical behaviour governed by the IR fixpoint. The critical number of charged scalars $M_{cr}$ (above which a second order transition is possible) is not very well known.

A nonabelian gauge theory like the electroweak theory is confining also in three dimensions. We have depicted the running of the gauge coupling $g^2$ and the ratio $\lambda/g^2$ in fig. 4. For sufficiently small initial $\lambda$ (small physical Higgs boson mass) $\lambda(k)$ reaches zero for $k$ much larger than the three-dimensional confinement scale. One then expects

* The true phase diagrams are multidimensional and the diagrams of fig. 3 may be interpreted as projections on appropriate trajectories. The fixpoint structure is independent of particular trajectories.
a first-order transition which is analogous to the four-dimensional Coleman-Weinberg scenario\textsuperscript{16}. Typical mass scales are of the order $k_{cw}$ where $\lambda(k_{cw}) = 0$. In this case it is possible that high temperature perturbation theory gives reliable results. On the other hand, if the three-dimensional confinement scale $\Lambda_{con}^{(3)}$ (the value of $k$ for which the gauge coupling diverges or becomes very large) is reached with $\lambda(\Lambda_{con}^{(3)}) > 0$ the behaviour near the phase transition is described by a strongly interacting electroweak theory. Then strong effective coupling constants appear not only in the symmetric phase, but also in the phase with spontaneous symmetry breaking. The phase transition may either be of the first order or an analytical crossover may replace the phase transition\textsuperscript{15}. A second-order transition seems unlikely.\textsuperscript{*} In any case, the “strongly interacting phase transition” will be very different from perturbative expectations.

### 2. PERTURBATIVE INFRARED DIVERGENCES AND “CUBIC TERMS”

In high temperature perturbation theory the electroweak phase transition is predicted to be of the first order. This follows from a “cubic term” in the effective potential generated by thermal fluctuations. In order to get an idea for which values of the scalar field and the temperature perturbation theory may be applied, we have to estimate the reliability of the cubic term. For a complex two-component Higgs scalar $\varphi$ the cubic term is proportional $(\varphi^\dagger \varphi)^{3/2}$ and we should understand the origin of such a nonanalytic behaviour of the effective potential. We will see that it is closely linked to the issue of infrared divergences for the quartic scalar coupling\textsuperscript{8}. Indeed, a term $\sim (\varphi^\dagger \varphi)^{3/2}$ implies that the quartic scalar coupling diverges for $\varphi \to 0$. This infrared divergence\textsuperscript{17} is immediately apparent by inspection of the diagram in fig. 5 for which the $n = 0$ Matsubara frequency gives a contribution

$$\Delta \lambda_4 \sim T \int d^3q \frac{g_4^4}{(q^2 + \frac{1}{2} g_4^2 \varphi^\dagger \varphi)^2} \sim \frac{g_4^3 T}{(\varphi^\dagger \varphi)^{1/2}}$$

(2)

The $\varphi$-dependent gauge boson mass plays here the role of the effective infrared cutoff and we may associate

$$k = m_W(\varphi) = \frac{1}{\sqrt{2}} g_4 (\varphi^\dagger \varphi)^{1/2}$$

(3)

Inserting the correction to the quartic term $\sim \Delta \lambda (\varphi^\dagger \varphi)^2$ in the effective potential gives exactly the cubic term mentioned before. The question arises now to which extent the perturbative treatment of the effectively three-dimensional behaviour can be trusted. A similar cubic term appears also in the perturbative treatment of the pure scalar theory and has been shown\textsuperscript{8} to be completely misleading in the vicinity of the critical temperature. (For $T = T_c$ the effective scalar potential has not a divergent, but a

\textsuperscript{*} A second-order transition requires a massless scalar degree of freedom at the critical temperature. The Higgs scalar presumably acquires a mass through strong electroweak interactions both in the symmetric and SSB phase. Chiral condensation phenomena of quarks and leptons similar as for quarks in QCD seem not very plausible for the high temperature electroweak theory.
vanishing quartic coupling and is dominated by the six-point function, \( U \sim (\varphi^\dagger \varphi)^3 \).
What about gauge theories?

At the scale \( k_T = 2\pi T \) the three-dimensional gauge coupling and quartic scalar coupling are given by

\[
\bar{\lambda}_3(k_T) = \lambda_4 T, \quad \bar{g}_3^2(k_T) = g_4^2 T
\]

(4)

For \( k < k_T \) the running of the couplings becomes effectively three-dimensional. In the limit of small \( \bar{\lambda}_3 \ll \bar{g}_3^2 \) we neglect the contributions from scalar loops and the running of \( \bar{\lambda}_3 \) follows* from gauge boson loops (fig. 5)

\[
\frac{\partial \bar{\lambda}_3}{\partial t} = \frac{9 \bar{g}_3^4}{64\pi k} \quad (5)
\]

Let us for a moment neglect the running of the gauge coupling. With the infrared cutoff identified with \( m_W(\varphi) \) (3) and defining \( \bar{\lambda}_3 \) by an appropriate derivative of the effective potential \( U \)

\[
\bar{\lambda}_3(k) = T \frac{\partial^2 U}{\partial \rho^2}
\]

(6)

\[ \rho = \varphi^\dagger \varphi = \varphi^2, \quad k^2 = \frac{1}{2} g_4^2 \rho \]

the flow equation (5) can be turned into a differential equation for \( U(\rho) \). Solving

\[
\varphi \frac{\partial}{\partial \varphi} \bar{\lambda}_3 = \frac{9\sqrt{2}}{64\pi} g_4^3 T^2 \quad (7)
\]

for \( \varphi < \varphi_T, \frac{1}{2} g_4^2 \varphi_T^2 = k_T^2 = (2\pi T)^2 \) one obtains

\[
\bar{\lambda}_3(\varphi) = \left( \lambda_4 + \frac{9g_4^4}{128\pi^2} \right) T - \frac{9\sqrt{2}g_4^3}{64\pi} T^2 \quad (8)
\]

From the definition (6) of \( \bar{\lambda}_3 \) follows the second order differential equation

\[
\frac{\partial^2 U}{\partial \rho^2} = \lambda_4 + \frac{9g_4^4}{128\pi^2} - \frac{9\sqrt{2}g_4^3}{64\pi} T \rho^{-\frac{1}{2}} \quad (9)
\]

The general solution has two integration constants, one of them being irrelevant here

\[
U(\rho) = -\mu^2(T)\rho + \frac{1}{2} \left( \lambda_4 + \frac{9g_4^4}{128\pi^2} \right) \rho^2 - \frac{3\sqrt{2}}{16\pi} g_4^3 T \rho^{3/2} \quad (10)
\]

The temperature-dependent mass term \( \mu^2(T) \) appears here as an integration constant and can be determined from high temperature perturbation theory (which should be valid for \( \varphi = \varphi_T \))

\[
-\mu^2(T) = -\mu_0^2 + \frac{3g_4^2 T^2}{16} \quad (11)
\]

* If the contribution of the Debye-screened \( A_0 \) mode is omitted, the r.h.s. of eq. (5) should be reduced by a factor 2/3, see sect. 5.
The potential (10) is nothing else than the result of high temperature perturbation theory. We conclude that the latter is reliable if the scalar loops can be omitted ($\bar{\lambda}_3 \ll \bar{g}_2^3$) and if the running of the gauge coupling can be neglected. Additional terms from the scalar loops could be included in the evolution equation (5) without changing the qualitative picture. The crucial question concerns the neglect of the running of the gauge coupling. This will determine the range of validity of perturbation theory and we will turn back to this question below.

3. AVERAGE ACTION

A useful tool for describing the running of couplings in arbitrary dimension is the average action\(^{18}\). Consider a simple model with a real scalar field $\chi$. The average scalar field is easily defined by

$$\phi_k(x) = \int d^d y f_k(x - y) \chi(y)$$

with $f_k$ decreasing rapidly for $(x - y)^2 > k^{-2}$ and properly normalized. The average is taken over a volume of size $\sim k^{-d}$. The average action $\Gamma_k[\varphi]$ obtains then by functional integration of the “microscopic variables” $\chi$ with a constraint forcing $\phi_k(x)$ to equal $\varphi(x)$ up to small fluctuations. It is the effective action for averages of fields and therefore the analogue in continuous space of the block spin action\(^{19}\) on the lattice. All modes with momenta $q^2 > k^2$ are effectively integrated out. Lowering $k$ permits to explore the theory at longer and longer distances. The average action has the same symmetries as the original action. As usual it may be expanded in derivatives, with average potential $U_k(\rho), \rho = \frac{1}{2} \varphi^2$, kinetic term, etc.

$$\Gamma_k = \int d^d x \left\{ U_k(\rho) + \frac{1}{2} Z_k(\rho) \partial_\mu \varphi \partial^\mu \varphi + ... \right\}$$

In a suitable formulation\(^{20}\) the effective average action becomes the generating functional for 1PI Green functions with an infrared cutoff set by the scale $k$. It interpolates between the classical action for $k \to \infty$ and the effective action for $k \to 0$. In this version an exact nonperturbative evolution equation describes the dependence of $\Gamma_k$ on the infrared cutoff $k$ ($t = \ln k$)

$$\frac{\partial}{\partial t} \Gamma_k = \frac{1}{2} \int \frac{d^d q}{(2\pi)^d} \left( \Gamma_k^{(2)} + R_k \right)^{-1} \frac{\partial}{\partial t} R_k$$

Here $R_k(q)$ is a suitable infrared cutoff which may depend on $q^2$, as, for example, $R_k = q^2 \exp \left( -\frac{q^2}{k^2} \right) \left( 1 - \exp \left( -\frac{q^2}{k^2} \right) \right)^{-1}$ or $R_k = k^2$. The two-point function $\Gamma_k^{(2)}$ obtains by second functional variation of $\Gamma_k$

$$\Gamma_k^{(2)}(q', q) = \frac{\delta^2 \Gamma_k}{\delta \varphi(-q') \delta \varphi(q)}$$

Therefore $(\Gamma_k^{(2)} + R_k)^{-1}$ is the exact propagator in presence of the infrared cutoff $R_k$ and the flow equation (14) takes the form of the scale variation of a renormalization
group-improved one-loop expression. We emphasize that the evolution equation is fully nonperturbative and no approximations are made. A simple proof can be found in ref. 20. The exact flow equation (14) can be shown to be equivalent with earlier versions of “exact renormalization group equations”21 and it encodes the same information as the Schwinger-Dyson equations 22.

An exact nonperturbative evolution equation is not yet sufficient for an investigation of nonperturbative problems like high temperature field theories. It is far too complicated to be solved exactly. For practical use it is crucial to have a formulation that allows to find reliable nonperturbative approximative solutions. Otherwise speaking, one needs a description of $\Gamma_k$ in terms of only a few $k$-dependent couplings. The flow equations for these couplings can then be solved numerically or by analytical techniques. It is on the level of such truncations of the effective average action that suitable approximations have to be found. In this respect the formulation of the effective average action20 offers important advantages: The average action has a simple physical interpretation and eq. (14) is close to perturbation theory if the couplings are small. The formulation is in continuous space and all symmetries - including chiral symmetries or gauge symmetries15 - can be respected. Since $\Gamma_k$ has a representation as a functional integral alternative methods (different from solutions of the flow equations) can be used for an estimate of its form. Furthermore, the flow equation (14) is directly sensitive to the relevant infrared physics since the contribution of particles with mass larger than $k$ is suppressed by the propagator on the r.h.s. of eq. (14). The closed form of this equation does not restrict one a priori to given expansions, like in 1PI $n$-point functions. In addition the momentum integrals in eq. (14) are both infrared and ultraviolet convergent if a suitable cutoff $R_k$ is chosen. Only modes in the vicinity of $q^2 = k^2$ contribute substantially. This feature is crucial for gauge theories where the formulation of a gauge-invariant ultraviolet cutoff is difficult without dimensional regularization.

4. STRONG ELECTROWEAK INTERACTIONS

We are now ready to discuss the running of the three-dimensional gauge coupling. We start from the effective average action for a pure $SU(N_c)$ Yang-Mills theory. It is a gauge-invariant functional of the gauge field $A$ and obeys the exact evolution equation15 (with $\text{Tr}$ including a momentum integration)

$$\frac{\partial}{\partial t} \Gamma_k[A] = \frac{1}{2} \text{Tr} \left\{ \frac{\partial R_k[A]}{\partial t} \left( \Gamma_k^{(2)}[A] + GF[A] + R_k[A] \right)^{-1} \right\} - \epsilon_k[A]$$

(16)

Here $GF[A]$ is the contribution from a generalized gauge-fixing term in a covariant background gauge

$$GF[A] = \Gamma_k^{\text{gauge}(2)}[A, \bar{A}]|_{\bar{A}=A}$$

(17)

and $\epsilon_k[A]$ is the ghost contribution15. The infrared cutoff $R_k$ is in general formulated in terms of covariant derivatives. We make the simple truncation

$$\Gamma_k[A] = \frac{1}{4} \int d^d x Z_{F,k} F_{\mu\nu} F^{\mu\nu}$$

$$\Gamma_k^{\text{gauge}}[A, \bar{A}] = \frac{1}{2} \int d^d x Z_{F,k} (D_\mu[\bar{A}] (A^\mu - \bar{A}^\mu))^2$$

(18)
and observe that in our formulation the gauge coupling $\hat{g}$ appearing in $F_{\mu\nu}$ and $D_\mu$ is a constant independent of $k$. The only $k$-dependent coupling can be associated with the dimensionless renormalized gauge coupling

$$g^2(k) = k^{d-4} \hat{g}^2(k) = k^{d-4} Z_{F,k}^{-1} \hat{g}^2.$$  \hspace{1cm} (19)

The running of $g^2$ is related to the anomalous dimension $\eta_F$

$$\eta_F = - \frac{\partial}{\partial t} \ln Z_{F,k}$$

$$\frac{\partial g^2}{\partial t} = \beta g^2 = (d-4)g^2 + \eta_F g^2$$  \hspace{1cm} (20)

Using configurations with constant magnetic field it was found\textsuperscript{15} to obey

$$\frac{\partial g^2}{\partial t} = (d-4)g^2 - \frac{44}{3} N_c v_d a_d g^4 - \frac{20}{3} N_c v_d b_d g^2$$  \hspace{1cm} (21)

with*\textsuperscript{23}

$$v_d^{-1} = 2^{d+1/2} \pi^{d/2} \Gamma \left( \frac{d}{2} \right)$$

$$a_d = \frac{(26 - d)(d-2)}{44} n_1^{d-4}$$

$$b_d = \frac{(24 - d)(d-2)}{40} l_1^{d-2}$$  \hspace{1cm} (22)

Only the momentum integrals ($x \equiv q^2$)

$$n_1^d = -\frac{1}{2} k^{-d} \int_0^\infty \frac{dx}{x^{d-4}} \frac{\partial}{\partial t} \left( \frac{\partial P}{\partial x} P^{-1} \right)$$

$$l_1^d = -\frac{1}{2} k^{2-d} \int_0^\infty \frac{dx}{x^{d/2-1}} \frac{\partial}{\partial t} P^{-1}$$  \hspace{1cm} (23)

depend on the precise form of the infrared cutoff $R_k$ appearing in

$$P(x) = x + Z_{k^{-1}} R_k(x)$$  \hspace{1cm} (24)

In four dimensions one has $a_4 = 1$, $v_4 = 1/32\pi^2$ and eq. (21) reproduces the one-loop result for $\beta g^2$ in lowest order $g^4$. For an appropriate choice of $R_k$ with $b_4 = 1$ an expansion of eq. (21) also gives 93% of the perturbative $g^6$ coefficient. For $d \neq 4$ the solution of eq. (21) is

$$\frac{g^2(k)}{(1 + \delta g^2(k))^\gamma} = \frac{g^2(k_0)}{(1 + \delta g^2(k_0))^\gamma} \left( \frac{k}{k_0} \right)^{d-4}$$

$$\delta = \frac{N_c v_d}{3} \left( \frac{44}{4 - d} a_d - 20 b_d \right)$$

$$\frac{1}{\gamma} = 1 - \frac{5}{11} (4 - d) \frac{b_d}{a_d}$$  \hspace{1cm} (25)

* The vanishing of the $\beta$-function for $\hat{g}^2$ for $d = 2$ and $d = 26$ is no accident. In lowest order in the $\epsilon$-expansion\textsuperscript{23} the denominator in the last term is absent and $v_3 a_3$ is replaced by $v_4 a_4$. 

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Concerning the high temperature field theory we should use the three-dimensional $\beta$-function and associate $k_0$ with the scale $k_T = 2\pi T$, where the three-dimensional running sets in. The “initial value” of the gauge coupling reads $g^2(k_T) = 2\alpha(k_T)$ with $\alpha$ the four-dimensional fine structure constant. For $k < k_T$ the three-dimensional gauge coupling increases with a power behaviour instead of the four-dimensional logarithmic behaviour. The three-dimensional confinement scale $\Lambda_{\text{conf}}^{(3)}$ - where $g^2$ diverges - is proportional to the temperature. For the electroweak theory ($\alpha_w = 1/30$) and the choice $P(x) = x/(1 - \exp(-x/k^2))$ one finds\(^{15}\)

$$\Lambda_{\text{conf}}^{(3)} = 0.23T \quad (0.1T)$$

(26)

Here the number in brackets corresponds to a “lowest order approximation” where $b_3$ is put to zero in the $\beta$-function (21).

For the symmetric phase of the electroweak theory we can neglect the scalar fluctuations in a good approximation and the estimate (26) directly applies. One has to deal with a strongly interacting gauge theory with typical nonperturbative mass scales only somewhat below the temperature scale! Similar to QCD one expects that condensates like $\langle F_{ij}F^{ij} \rangle$ play an important role\(^{15,24}\). More generally, the physics of the symmetric phase corresponds to a strongly coupled $SU(2)$ Yang-Mills theory in three dimensions: The relevant excitations are “$W$-balls” (similar to glue balls) and scalar bound states. All “particles” are massive (except the “photon”) and the relevant mass scale is set by $\Lambda_{\text{conf}}^{(3)} \sim T$. Also the values of all condensates are given by appropriate powers of the temperature. Since the temperature is the only scale available the energy density must have the same $T$-dependence as for an ideal gas

$$\rho = cT^4$$

(27)

Only the coefficient $c$ should be different from the value obtained by counting the perturbative degrees of freedom*. We expect that quarks and leptons form $SU(2)$ singlet bound states similar to the mesons in QCD*. A chiral condensate seems, however, unlikely in the high temperature regime and we do not think that fermions play any important role for the dynamics of the electroweak phase transition. The “photon” (or rather the gauge boson associated to weak hypercharge) decouples from the $W$-balls. Its effective high temperature coupling to fermion and scalar bound states is renormalized to a very small value. As for the phase with spontaneous symmetry breaking, the fermions and “photon” can be neglected for the symmetric phase. We conclude that the high temperature phase transition of the electroweak theory can be described by an effective three-dimensional Yang-Mills-Higgs system. It is strongly interacting in the symmetric phase. Depending on the value of the mass of the Higgs boson it may also be strongly interacting in the phase with spontaneous symmetry breaking if the temperature is near the critical temperature. A more detailed investigation of this issue will be given in the next section.

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* A similar remark also applies to high temperature QCD. We expect quantitative modifications of early cosmology due to the difference between $c$ and the ideal gas value.

* We use here a language appropriate for the excitations of the three-dimensional Euclidean theory. Interpretation in terms of relativistic particles has to be used with care!
5. RENORMALIZATION GROUP-IMPROVED EFFECTIVE POTENTIAL FOR THE ELECTROWEAK PHASE TRANSITION

Before performing a complete study of the average action for the standard model at high temperature, the most important issues can be addressed in a simplified treatment. For large enough \( \rho, \rho \geq \varphi_2^2 T^2 / g_4^2 \) (see sect. 2) we believe high temperature perturbation theory and take the one-loop expression \( U_3(\rho_3) = -\mu^2(T) \rho_3 + \frac{1}{2} \left( \lambda_3 + \frac{9g_3^4}{128\pi^2 T} \right) \rho_3^2 - \frac{1}{12\pi} \left( 6m_B^3 + 3m_E^3 + m_1^3 + 3m_2^3 \right) \) \( (28) \)

with

\[
\begin{align*}
m_B^2 &= \frac{1}{2} \bar{g}_3^2 \rho_3 \\
m_E^2 &= m_D^2(T) + \frac{1}{2} \bar{g}_3^2 \rho_3 \\
m_1^2 &= 3\lambda_3 \rho_3 - \mu^2(T) \\
m_2^2 &= \lambda_3 \rho_3 - \mu^2(T)
\end{align*}
\]

Here we use already a three-dimensional language with \( U_3 = U/T, \rho_3 = \rho/T, \bar{\lambda}_3 = \lambda_4 T, \bar{g}_3^2 = g_4^2 T \). The couplings \( \lambda_4 \) and \( g_4^2 \) are evaluated at the scale \( k_T = 2\pi T \) and \( \mu^2(T), m_B^2(T) \) take the perturbative values. For \( \rho < \varphi_2^2 T \) we can compute a renormalization group-improved potential by using a flow equation similar to (5) with \( k \equiv m_B \).

In the language of the average action this amounts to an infrared cutoff \( P(x) = x + k^2 \).

This yields in the \( \alpha = 0 \) gauge the approximative evolution equation \( \partial \bar{\lambda}_3(k) = \frac{3}{64\pi} k^{-1} \left\{ 2\bar{g}_3^4(k) + \bar{g}_3^2(k) \frac{m_B}{m_E} + 12\bar{\lambda}_3^2(k) \frac{m_B}{m_1} + 4\bar{\lambda}_3^2(k) \frac{m_B}{m_2} \right\} \) \( (30) \)

We recover eq. (5) if we neglect \( \bar{\lambda}_3 \) on the r.h.s. and approximate \( m_B = m_E \). For an alternative derivation we could obtain eq. (30) by taking appropriate derivatives of the potential \( U_3(\rho_3) \) \( (28) \) using the definitions \( (6) \) and treating the rations \( m_B/m_E \) etc. as independent of \( k \). For the correct interpretation of the flow equation we should also adapt the relation between \( k \) and \( \rho \) to the situation with running gauge coupling

\[
k^2 = m_B^2 = \frac{1}{2} \bar{g}_3^2(k) \rho_3 \]

\[
\bar{\lambda}_3 = \frac{\partial^2 U_3}{\partial \rho_3^2}
\]

The solution of the flow equation (30) with initial values at \( k_T \) specified by (28) and the subsequent solution of the second order differential equation (32) for \( U_3(\rho_3) \) should give a good approximation for the temperature dependent effective potential in a range of \( \rho \) to be discussed below.

In eq. (30) the factor \( m_B/m_E \) accounts properly for the Debye screening of the \( A_0 \) mode. We take here \( m_B/m_E = 0 \) for simplification of the discussion. We also observe that our equation becomes invalid for vanishing or negative \( \bar{\lambda}_3 \rho_3 - \mu^2(T) \). This problem is related to the approach to convexity of the effective scalar potential and absent in a
more complete treatment\textsuperscript{25}. We simplify by taking \( \mu^2(T) = 0 \) in the mass ratios such that \( m_B^2/m_1^2 = \frac{1}{6} \tilde{g}_3^2/\lambda_3, m_B^2/m_2^2 = \frac{1}{2} \tilde{g}_3^2/\lambda_3 \) and

\[
\frac{\partial}{\partial t} \lambda_3 = \frac{3}{32\pi k} \left\{ \tilde{g}_3^4 + (\sqrt{6} + \sqrt{2}) \tilde{\lambda}_3^4 \tilde{g}_3^2 \right\} + 2\eta \tilde{\lambda}_3 \tag{33}\]

Here we have included the effects of wave function renormalization and switched to a renormalized coupling \( \tilde{\lambda}_3 \). The dominant contribution to the anomalous dimension \( \eta \) comes from gauge boson fluctuations and reads\textsuperscript{15}

\[
\eta = -\frac{1}{4\pi} \tilde{g}_3^2 k^{-1} \tag{34}\]

For the three-dimensional running of the gauge coupling we use eq. (21) with \( b_3 = 0 \) and a correction factor \( \tau \) accounting for the difference between the \( \alpha = 0 \) gauge needed here and the \( \alpha = 1 \) gauge for which (21) was computed and for the contribution from scalar loops

\[
\frac{\partial}{\partial t} \tilde{g}_3^2 = -\frac{23}{24\pi} \tau \tilde{g}_3^4 k^{-1} \tag{35}\]

We expect a value of \( \tau \) near one. We can only use the system of equations (33) and (35) for \( k \) larger than the three-dimensional confinement scale. This gives a lower bound\textsuperscript{15} on \( \rho, \rho > \rho_{np} \): Let us assume that renormalization group-improved perturbation theory breaks down at a scale

\[
k_{np} = 1.1 \Lambda_{\text{conf}}^{(3)} = 1.1 T \frac{23 \tau \alpha_w}{6 + \frac{23 \tau \alpha_w}{2\pi}} \approx 0.14 \tau T \tag{36}\]

At this scale \( \tilde{g}_3^2(k_{np}) \) has increased by a factor of about ten as compared to \( \tilde{g}_3^2(k_T) \)

\[
\frac{\tilde{g}_3^2(k_{np})}{\tilde{g}_3^2(k_T)} = \frac{11}{1 + \frac{23 \tau \alpha_w}{12\pi}} \approx 10.8 \tag{37}\]

and the dimensionless gauge coupling \( g^2(k) = \tilde{g}_3^2(k)/k \) has reached the nonperturbative region

\[
\frac{g^2(k_{np})}{4\pi^2} = \frac{60}{23 \tau \pi} = 0.83 \tau^{-1} \tag{38}\]

From (31)(36) and (37) we infer

\[
\rho_{np} = 0.26 \frac{\alpha_w \tau^2}{1 + \frac{23 \tau \alpha_w}{12\pi}} T^2 \approx (0.09 \tau T)^2 \tag{39}\]

For values of \( \rho \) below \( \rho_{np} \) we expect a complete breakdown even for renormalization group-improved perturbation theory due to condensates and similar strong interaction phenomena. We also observe that the nonperturbative improvement of the evolution equation (21) for \( \tilde{g}^2 \) enhances \( \rho_{np} \) by a factor of about 4.8. Our best estimate for the onset of strong nonperturbative phenomena is therefore\textsuperscript{*}

\[
\rho_{np} = \left( \frac{T}{5} \right)^2 \tag{40}\]

\* This does not mean that high temperature perturbation theory can be trusted for \( \rho > \rho_{np} \). Only the renormalization group-improved potential discussed here has a chance to remain valid at such low values of \( \rho \)!
For $\rho > \rho_{np}$ it is instructive to consider the evolution of the ratio $\tilde{\lambda}_3/\tilde{g}_3^2$ and use $\tilde{g}_3^2(k)$ instead of $k$ as a variable

$$\frac{\partial}{\partial \ln \tilde{g}_3^2} \left( \frac{\tilde{\lambda}_3}{\tilde{g}_3^2} \right) = -\frac{9}{92\tau} \left\{ 1 + \frac{92\tau - 48 \tilde{\lambda}_3}{9} \tilde{g}_3^2 + (\sqrt{6} + \sqrt{2}) \left( \frac{\tilde{\lambda}_3}{\tilde{g}_3^2} \right)^{3/2} \right\}$$

(41)

This corresponds to the $\beta$-function for $\lambda/g^2$ shown in fig. 4. The r.h.s. is negative and $\tilde{\lambda}_3$ is always driven from positive towards negative values as $\tilde{g}_3^2$ increases. The initial ratio $\tilde{\lambda}_3/\tilde{g}_3^2$ at $k_T$ corresponds to the zero temperature ratio between Higgs scalar mass and $W$-boson mass up to small (four-dimensional) logarithmic corrections

$$\tilde{\lambda}_3(k_T) \approx \left( \frac{m_H}{2m_W} \right)^2$$

(42)

For not too large values of $m_H/m_W$ we may, for the purpose of a simplified analytical discussion, neglect the last term in eq. (41) and use the approximative solution

$$\tilde{\lambda}_3(k) = \left( \frac{\tilde{g}_3^2(k)}{\tilde{g}_3^2(k_T)} \right)^{\frac{12}{23\tau}} \tilde{\lambda}_3(k_T) - \frac{9}{92\tau - 48 \tilde{g}_3^2(k)} \left( 1 - \left( \frac{\tilde{g}_3^2(k_T)}{\tilde{g}_3^2(k)} \right)^{1-\frac{12}{23\tau}} \right)$$

(43)

The scale $k_{cw}$ (where $\tilde{\lambda}_3(k_{cw}) = 0$) can be taken as a characteristic scale for the phase transition. It obeys

$$\tilde{g}_3^2(k_{cw}) = \left( 1 + \frac{92\tau - 48 \tilde{\lambda}_3(k_T)}{9} \tilde{g}_3^2(k_T) \right)^{-\frac{23}{23\tau-12}}$$

(44)

Equating $k_{cw}$ with $k_{np}$ (36) we find a critical value for the ratio $\tilde{\lambda}_3(k_T)/\tilde{g}_3^2(k_T)$ (for $\tau = \frac{24}{23}$)

$$\left( \frac{\tilde{\lambda}_3(k_T)}{\tilde{g}_3^2(k_T)} \right)_{cr} = 0.43$$

(45)

We conclude that for a Higgs boson mass exceeding the critical value

$$(m_H)_{cr} \approx 1.3m_W \approx 105 \text{ GeV}$$

(46)

the electroweak phase transition is described by a strongly interacting $SU(2)$ gauge theory. Not only the symmetric phase but all phenomena related to the phase transition are dominated by nonperturbative effects! We emphasize that the critical value (46) should not be interpreted as an accurate bound. Even for Higgs masses smaller than 100 GeV the strong nonperturbative effects are very important and may dominate, for example, the whole region of the effective potential between the origin and the local

** This statement is not valid for very small scalar masses when Coleman-Weinberg symmetry breaking operates.
maximum*. Our main conclusion is that for a Higgs boson mass of the order $m_W$ or even somewhat below it is impossible to give a quantitative description of the phase transition without taking the strong nonperturbative effects such as condensates into account.

One last remark concerns the “nonperturbative region” in the effective potential for $\rho < \rho_{np}$ (39). (For $m_H$ taking the critical value this concerns the region inside the turning point $\partial^2 U / \partial \rho^2 = 0$.) Even though the relatively simple renormalization group-improved treatment proposed in this section is not valid here, we can estimate the difference $\Delta U_3 = U_3(\rho_{np}) - U_3(0)$ by a simple scale argument: It has to be proportional to the third power of $k_{np}$

$$\Delta U_3 = K k_{np}^3$$  \hspace{1cm} (47)

Since there is no small dimensionless quantity in the problem and $k_{np}$ is determined relatively accurately, the constant $K$ should be near one. Restoring the four-dimensional language we therefore estimate the nonperturbative contribution to be roughly

$$\Delta U \approx 3 \cdot 10^{-3} T^4$$  \hspace{1cm} (48)

We can also offer a speculative picture how the transition could be described as an analytical crossover for very large $m_H$: As the temperature raises a condensate $\langle F_{ij} F^{ij} \rangle$ (or some other condensate) may start forming at some temperature $\tilde{T}$ for which the absolute minimum of the effective scalar potential still occurs at $\rho_0(\tilde{T}) \neq 0$. For a further increase of the temperature beyond $\tilde{T}$ the magnetic condensate $\langle F_{ij} F^{ij} \rangle$ will increase whereas $\rho_0(T)$ decreases. In the two-dimensional plane spanned by the condensate and $\rho$ the arrow $(\langle F_{ij} F^{ij} \rangle, \rho_0)$ may turn continuously from the $\rho$-direction for $T = \tilde{T}$ to the condensate direction for very large temperatures. No jump in the particle masses or other quantities would be expected for such a crossover. This picture gives a hint that a “strongly interacting electroweak transition” may need more degrees of freedom than the Higgs scalar for a meaningful description of the vacuum structure!

We conclude that for realistic values of the Higgs boson mass nonperturbative techniques are necessary for a reliable description of the electroweak phase transition. The “strongly interacting electroweak phase transition” is in several aspects close to the high temperature phase transition in QCD. Similar methods for a description of both phenomena have to be developed, in particular for an understanding of the temperature dependence of various condensates. This constitutes an interesting theoretical laboratory, with possible applications ranging from early cosmology to high energy heavy ion collisions.

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* If we use the improved evolution equation (21) instead of (35), the critical value is lowered further.
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