Coulomb interaction and magnetic catalysis in the quantum Hall effect in graphene

To cite this article: E V Gorbar et al 2012 Phys. Scr. 2012 014018

View the article online for updates and enhancements.
Coulomb interaction and magnetic catalysis in the quantum Hall effect in graphene

E V Gorbar\textsuperscript{1}, V P Gusynin\textsuperscript{1}, V A Miransky\textsuperscript{2} and I A Shovkovy\textsuperscript{3}

\textsuperscript{1} Bogolyubov Institute for Theoretical Physics, 03680 Kiev, Ukraine
\textsuperscript{2} Department of Applied Mathematics, University of Western Ontario, London, ON, N6A 5B7, Canada
\textsuperscript{3} Department of Applied Sciences and Mathematics, Arizona State University, Mesa, AZ 85212, USA

E-mail: gorbar@bitp.kiev.ua, vgusynin@bitp.kiev.ua, vmiransk@uwo.ca and igor.shovkovy@asu.edu

Received 14 April 2011
Accepted for publication 8 June 2011
Published 31 January 2012
Online at stacks.iop.org/PhysScr/T146/014018

Abstract

The dynamics of symmetry breaking responsible for lifting the degeneracy of the Landau levels (LLs) in the integer quantum Hall (QH) effect in graphene is studied in a low-energy model with the Coulomb interaction. The gap equation for Dirac quasiparticles is analyzed for both the lowest and higher LLs, taking into account the LL mixing. It is shown that the characteristic feature of the long-range Coulomb interaction is the dependence of the gap parameters on the LL index \(n\) (‘running’ gaps). The renormalization (running) of the Fermi velocity as a function of \(n\) is also studied. The solutions of the gap equation reproduce correctly the experimentally observed integer QH plateaus in graphene in strong magnetic fields.

PACS numbers: 73.22.Pr, 71.70.Di, 71.70.–d

(Some figures may appear in colour only in the online journal)

1. Introduction

As is well known, the low-energy dynamics of electrons in graphene \cite{Novoselov2004} is described by the Dirac equation in \((2+1)\) dimensions \cite{Novoselov2005}. Perhaps the most direct confirmation of the pseudorelativistic character of electron motion in graphene is given by the experimental observation \cite{Dean2008, Geim2007} of the anomalous quantum Hall (QH) effect theoretically predicted in \cite{Klitzing1980, Klaui1980, deShalit1980}. The anomalous QH plateaus are observed at the filling factors \(\nu = \pm 4(n + 1/2)\), where \(n = 0, 1, 2, \ldots\) is the Landau level (LL) index. The factor 4 in the filling factor is due to a fourfold (spin and sublattice-valley) degeneracy of each QH state in graphene. The presence of the anomalous (from the viewpoint of more standard condensed matter systems) term 1/2 in the filling factor unmistakably reveals the relativistic-like character of electron motion in graphene \cite{Novoselov2005, Novoselov2008, Novoselov2006}.

Later experiments \cite{Koenig2007, Liu2009} in strong magnetic fields \((B \gtrsim 20\, T)\) observed new QH plateaus, with integer filling factors \(\nu = 0, \pm 1\) and \(\pm 4\). More recent experiments \cite{Rycerz2009, Pustilnik2009} discovered additional plateaus, with \(\nu = \pm 3\) and \(\nu = \pm 1/3\). While the latter corresponds to the fractional QH effect, the plateaus with \(\nu = 0, \pm 1, \pm 4\) and \(\pm 3\) are intimately connected with a breakdown of the \(U(4)\) symmetry of the low-energy effective quasiparticle Hamiltonian in graphene (connected with the spin and sublattice-valley degeneracy mentioned above) \cite{Gorbar2011}. Strictly speaking, because of the Zeeman effect, \(U(4)\) is reduced to \(U_1(2) \times U_1(2)\), with \(U_1(2)\) being the sublattice-valley symmetry at a fixed spin \((s = \uparrow\) or \(s = \downarrow\)). However, taking into account that the Zeeman interaction is rather weak for realistic magnetic fields, \(U(4)\) is a good approximate symmetry guaranteeing that at weak magnetic fields, only QH plateaus with the filling factors \(\nu = \pm 4(n + 1/2)\) appear. The observed new QH plateaus \(\nu = 0, \pm 1, \pm 4\) and \(\pm 3\) occur clearly due to the electron–electron interaction, leading to (quasi-)spontaneous \(U(4)\) symmetry breaking that removes the degeneracy of the \(n = 0\) and \(n = 1\) LLs.

To describe the new QH plateaus, the following two theoretical scenarios were suggested. One of them is QH ferromagnetism (QHF) \cite{Gorbar2011, Gorbar2012}, whose order parameters are the spin and valley charge densities (the dynamics of a Zeeman spin splitting enhancement considered in \cite{Gorbar2011} is intimately connected with the QHF). This scenario is
related to the theory of exchange-driven spin splitting of LLs [21]. The second one is the magnetic catalysis (MC) scenario, whose order parameters are excitonic condensates, responsible for the generation of the Dirac masses of charge carriers [22–25]. The essence of the MC, which is connected with the effective dimensional reduction in the dynamics of charged fermions in an external magnetic field, was revealed in [26]. It was first applied for a single layer of graphite in [9, 10].

One may think that the QHF and MC order parameters should compete with each other. However, an analysis performed in [27] in an effective model with the local four-fermion interaction showed that these two sets of the order parameters necessarily coexist (this feature has recently been discussed also in [28]). This fact clearly indicates that these two sets of order parameters have a common dynamical origin, i.e. they are two sides of the same coin. Their simultaneous consideration qualitatively reproduces all the QH plateaus observed experimentally in strong magnetic fields.

Certainly, it would be important to extend the analysis of [27] to the case of realistic long-range Coulomb interactions. In the present paper, the dynamics of $U(4)$ symmetry breaking responsible for the appearance of QH plateaus with $v = 0, \pm 1, \pm 3, \pm 4$ and $\pm 5$ in strong magnetic fields is studied in a low-energy model with Coulomb interactions. While the symmetric structure of the solutions describing these plateaus is similar to that of the solutions in the model with the local four-fermion interaction [27], there are essential qualitative differences between them. In particular, the long-range Coulomb interaction leads to the dependence of the gap parameters on the LL index $n$ (‘running’ gaps). Recently, the dynamics with the Coulomb interaction in the $v = 0$ quantum Hall state was studied in [28, 29] by utilizing different approaches to the present one. In these papers, the important role of the LL mixing effects was also revealed. Note that these mixing effects are important both in models with long-range interactions and those with short-range ones, as the model considered in [7].

The rest of this paper is organized as follows. In section 2, the general features of the model, in particular the structure of the order parameters, are described. The Dirac quasiparticle propagator and energy dispersion relations are considered in section 3. The Schwinger–Dyson (gap) equation for the quasiparticle propagator is derived in section 4. In section 5, we present our numerical results, which include (i) the renormalization of the LL-dependent Fermi velocity parameter, (ii) the solutions of the gap equation at the lowest Landau level (LLL) and (iii) the solutions of the gap equation at the $n = 1$ LL, respectively. A general discussion of the main results and the summary are given in section 6. The appendices at the end of the paper contain some of our derivations and technical details used in the main text.

Note that in most of the paper, we use units with $\hbar = 1$ and $e_B = 1$.

2. The model

The low-energy quasiparticle excitations in graphene are conveniently described in terms of a four-component Dirac spinor $\Psi_s = (\Psi_{K_A}, \Psi_{K_B}, \Psi_{K'A}, \Psi_{K'B})$, which combines the Bloch states with spin indices $s = \uparrow$ or $s = \downarrow$ on the two different sublattices ($A$, $B$) of the hexagonal graphene lattice and with momenta near the two inequivalent points ($K$, $K'$) at the opposite corners of the two-dimensional Brillouin zone. The free quasiparticle Hamiltonian has a pseudorelativistic form with the Fermi velocity $v_F \approx 10^6$ m/s1 playing the role of the speed of light

$$H_0 = v_F \int d^2 r \, \bar{\Psi}_s (\gamma^1 \pi_x + \gamma^2 \pi_y) \Psi_s,$$  \hspace{1cm} (1)

where $\mathbf{r} = (x, y)$ and $\bar{\Psi}_s = \Psi^\dagger_s \gamma^0$ is the Dirac-conjugated spinor, and summation over spin is understood. In equation (1), $\gamma^\mu$ with $\mu = 0, 1, 2$ are $4 \times 4$ gamma matrices belonging to a reducible representation $\gamma^\mu = \tilde{\tau}_3 \otimes (\tau_3, i\tau_2, -i\tau_1)$ where the Pauli matrices $\tilde{\tau}, \tau$ act in the subspaces of the valley $(K, K')$ and sublattice $(A, B)$ indices, respectively. (For the Dirac $\gamma$-matrices, we use the same representation as in [22], appendix C.) The $\gamma$-matrices satisfy the usual anticommutation relations $[\gamma^\mu, \gamma^n] = 2g^{\mu n}$, $g^{\mu n} = \text{diag} (1, -1, -1)$, $\gamma^\mu, \nu = 0, 1, 2$. The canonical momentum $\pi = (\pi_x, \pi_y) = -i\hbar \nabla + eA/c$ includes the vector potential $A$ corresponding to a magnetic field $\mathbf{B}$, which is the component of the external magnetic field $\mathbf{B}$ orthogonal to the graphene $xy$-plane.

The Coulomb interaction term has the form

$$H_C = \frac{1}{2} \int d^2 r \, d^2 r' \, \Psi_i^\dagger (\mathbf{r}) \Psi_j (\mathbf{r}) U_C (\mathbf{r} - \mathbf{r}') \Psi_i^\dagger (\mathbf{r}') \Psi_j (\mathbf{r}),$$  \hspace{1cm} (2)

where $U_C (\mathbf{r})$ is the Coulomb potential in a magnetic field. The corresponding potential with the polarization effects taken into account was represented, for example, in equation (46) of [10]. The Hamiltonian $H = H_0 + H_C$ possesses the $U(4)$ symmetry discussed above. The electron chemical potential $\mu$ is introduced through adding the term $-\mu \Psi^\dagger \Psi$ in $H$ (this term preserves the $U(4)$ symmetry). The Zeeman term $\mu_B \mathbf{B} \Psi \sigma_3 \Psi$, where $\mathbf{B} \equiv \mathbf{B}$ and $\mu_B = e\hbar / (2mc)$ is the Bohr magneton, and $\sigma_3$ is the third Pauli matrix, breaks the $U(4)$ symmetry down to the $U(2) \times U(2)$ symmetry. The generators of the $U(2) \times U(2)$ subgroup are given by

$$P_s \otimes I_A, \quad P_s \otimes i\gamma^3, \quad P_s \otimes \gamma^5, \quad P_s \otimes \frac{1}{2} [\gamma^3, \gamma^5],$$  \hspace{1cm} (3)

where $\gamma^5 \equiv i\gamma^0 \gamma^1 \gamma^2 \gamma^3$ and $P_s = (1 + \sigma_3)/2$ are the projectors on spin-$\uparrow$ ($s = +1$) and spin-$\downarrow$ ($s = -1$) states.

The order parameters that describe the breakdown of the $U(4)$ symmetry are the same as those in [27]. The charge densities (and corresponding chemical potentials), which span the QHF order parameters, are given by

$$\mu_3 : \Psi^\dagger \sigma_3 \Psi = \frac{1}{2} \sum_{K, K' = A, B} \sum_{a = A, B} (\Psi^\dagger_{ka\uparrow} \Psi_{ka\uparrow} - \Psi^\dagger_{ka\downarrow} \Psi_{ka\downarrow}),$$  \hspace{1cm} (4)

$$\tilde{\mu}_3 : \Psi^\dagger \gamma^3 \gamma^5 P_s \Psi \equiv \Psi^\dagger_{K'A} \Psi_{K'A} + \Psi^\dagger_{K'B} \Psi_{K'B} - \Psi^\dagger_{K'A} \Psi_{K'B} - \Psi^\dagger_{K'B} \Psi_{K'A},$$  \hspace{1cm} (5)

where

$$\gamma^3 \gamma^5 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$$  \hspace{1cm} (6)
is the diagonal valley matrix related to the $SU(2)$ symmetry. The chemical potentials $\mu_3 = (\mu_+ - \mu_-)/2$ and $\mu_\perp$ are related to the spin and valley densities, respectively. The latter describes a charge density imbalance between the two valleys in the Brillouin zone (or the anomalous magnetic moment in the language of relativistic field theory).

Their MC cousins are connected with the charge density wave (CDW) and the valley polarized CDW. The corresponding masses are the Dirac mass and the Haldane mass [8],

$$\Delta_s : \tilde{\Psi} P_s \Psi = \psi^K_{K,A} \psi^K_{K,A} - \psi^K_{K,B} \psi^K_{K,B} + \psi^K_{K'} \psi^K_{K'}$$

$$\Delta_s : \tilde{\Psi} \gamma^5 P_s \Psi = \psi^K_{K,A} \psi^K_{K,A} - \psi^K_{K,B} \psi^K_{K,B}$$

(7)

respectively.

While the generation of the CDW in equation (7) breaks spontaneously the $U_s(2)$ (more precisely, its subgroup $SU_s(2)$), the generation of valley polarized CDW in equation (8) does not break this symmetry. On the other hand, while the Dirac mass term (7) is even under time reversal $\mathcal{T}$, the Haldane mass term (8) is $\mathcal{T}$-odd (for a recent review of the transformation properties of different mass terms in graphene, see [30]). It is noteworthy that the mass $\Delta$ was first discussed long ago in connection with inducing the Chern–Simons term in the effective action of $(2+1)$-dimensional relativistic gauge field theories [31].

3. Quasiparticle propagator

The inverse bare quasiparticle propagator in the mixed $(\omega, \mathbf{r})$-representation is given by

$$iS^{-1}(\omega; \mathbf{r}, \mathbf{r}') = [(\omega + \mu) \gamma^0 - \mu B \gamma^1 \gamma^5 \sigma^3 + \text{tr}(\mathbf{F} \cdot \mathbf{r} - \mathbf{r}')] \delta(\mathbf{r} - \mathbf{r}')$$

(9)

(To simplify the representation of all formulae, we omit the spin index in this section.) Similarly, the general structure of the full fermion propagator for quasiparticles of a fixed spin has the following form:

$$i G^{-1}(\omega; \mathbf{r}, \mathbf{r}') = [(\omega + \mu) \gamma^0 - \mu B \gamma^1 \gamma^5 \sigma^3 + \text{tr}(\mathbf{F} \cdot \mathbf{r} - \mathbf{r}')] \delta(\mathbf{r} - \mathbf{r}')$$

(10)

where $\tilde{\mathbf{F}}$ and $\tilde{\Sigma}$ are viewed as generalized wavefunction renormalization and self-energy operators, respectively. In the special case of the bare propagator in equation (9), the corresponding functions are $\tilde{F}_{\text{bare}} = 1$ and $\tilde{\Sigma}_{\text{bare}} = (\mu - \mu_B \sigma^3) \gamma^0$.

By definition, $\tilde{\mathbf{F}}$ and $\tilde{\Sigma}$ are functions of the three mutually commuting dimensionless operators: $(\pi \cdot \gamma)^2 \ell^2$, $\gamma^0$ and $i \sigma \gamma^1 \gamma^2$, where $\sigma = \text{sgn}(\mathbf{B})$ and $\ell = \sqrt{BC}/|B|$ is the magnetic length. Taking into account that $(\gamma^0)^2 = 1$ and $(i \sigma \gamma^1 \gamma^2)^2 = 1$, the operators $\tilde{\mathbf{F}}$ and $\tilde{\Sigma}$ can be written in the following form:

$$\tilde{\mathbf{F}} = f + \gamma^0 g + i s \gamma^1 \gamma^2 \tilde{g} + i s \gamma^0 \gamma^1 \gamma^2 \tilde{f}$$

(11)

$$\tilde{\Sigma} = \Delta + \gamma^0 \mu + i s \gamma^1 \gamma^2 \tilde{\mu} + i s \gamma^0 \gamma^1 \gamma^2 \Delta$$

(12)

where $f$, $\tilde{f}$, $g$, $\tilde{g}$, $\Delta$, $\Delta$, $\mu$ and $\tilde{\mu}$ are functions of only one operator, $(\pi \cdot \gamma)^2 \ell^2$. Note that for the functions $\mu$, $\tilde{\mu}$ and $\Delta$, we keep the same notation as for the parameters $\mu$, $\tilde{\mu}$ and $\Delta$ in equations (4), (5) and (7), respectively.

It is obvious from the representations in equations (11) and (12) that $\tilde{\mathbf{F}}$ and $\tilde{\Sigma}$ do not necessarily commute with $(\pi \cdot \gamma)$. It is convenient, therefore, to introduce two other functions $\tilde{\mathbf{F}}^-$ and $\tilde{\Sigma}^-$, which satisfy the relations

$$\tilde{\mathbf{F}}^- (\pi \cdot \gamma) = (\pi \cdot \gamma) \tilde{\mathbf{F}}^-,$$

$$\tilde{\Sigma}^- (\pi \cdot \gamma) = (\pi \cdot \gamma) \tilde{\Sigma}^-.$$

(13)

(14)

As follows from its definition, the explicit representations of these functions read

$$\tilde{\mathbf{F}}^- = f - \gamma^0 g - i s \gamma^1 \gamma^2 \tilde{f} + i s \gamma^0 \gamma^1 \gamma^2 \tilde{g},$$

(15)

$$\tilde{\Sigma}^- = \Delta - \gamma^0 \mu - i s \gamma^1 \gamma^2 \tilde{\mu} + i s \gamma^0 \gamma^1 \gamma^2 \Delta.$$}

(16)

These are obtained from $\tilde{\mathbf{F}}^+$ and $\tilde{\Sigma}^+$ by reversing the signs in front of the two terms that anticommute with $(\pi \cdot \gamma)$.

The physical meaning of the functions $\Delta$, $\Delta$, $\mu$ and $\tilde{\mu}$ that appear in the definition of $\tilde{\Sigma}$ is straightforward: $\Delta$ is the Dirac mass function, $\Delta$ is the Haldane (time-reversal odd) mass function, $\mu$ is the charge density chemical potential and $\tilde{\mu}$ is the chemical potential for charge density imbalance between the two valleys in the Brillouin zone. As for the functions $f$, $\tilde{f}$, $g$ and $\tilde{g}$ that appear in the definition of $\tilde{F}^+$, they are various structures in the wavefunction renormalization operator.

It may appear that in the most general case, the full propagator (10) can also include another wavefunction renormalization, multiplying the frequency term $\gamma^0 \omega$. This is not the case, however, because this Dirac structure is already included in the self-energy $\tilde{\Sigma}$, which may depend on $\omega$ in general. We note at the same time that the solution for $\tilde{\Sigma}$ will turn out to be independent of $\omega$ in the instantaneous approximation utilized in this study. This fact is also one of the reasons why it is particularly convenient to separate the term $\gamma^0 \omega$ from the generalized self-energy operator $\tilde{\Sigma}$ in (10).

As mentioned earlier, the functions $f$, $\tilde{f}$, $g$, $\tilde{g}$, $\Delta$, $\Delta$, $\mu$ and $\tilde{\mu}$ are functions of $(\pi \cdot \gamma)^2 \ell^2$, whose eigenvalues are non-positive even integers: $-2n \equiv -(2N + 1 + s_1 s_2)$, where $N = 0, 1, 2, \ldots$ is the orbital quantum number and $s_2 = \pm 1$ is the sign of the pseudospin projection. Therefore, in what follows, it will be convenient to use the following eigenvalues of the operators $\tilde{\mathbf{F}}^+$ and $\tilde{\Sigma}^+$ (for more details see appendix A):

$$F^0_{n_1 s_1} \equiv f_n + s_0 g_n + s_1 \tilde{g}_n + s_0 s_1 \tilde{f}_n,$$

(17)

$$\Sigma^0_{n_1 s_1} \equiv \Delta_n + 2 s_0 \mu_n + s_1 \tilde{\mu}_n + 2 s_0 s_1 \Delta_n,$$

(18)

where $f_n$, $\tilde{f}_n$, $g_n$, $\tilde{g}_n$, $\Delta_n$, $\Delta_n$, $\mu_n$ and $\tilde{\mu}_n$ are the eigenvalues of the corresponding coefficient operators in the $n$th LL state. Further, $s_0 \equiv \pm 1$ and $s_2 \equiv \pm 1$ are the eigenvalues of $\gamma^0$ and $i s \gamma^0 \gamma^1 \gamma^2$, respectively.

In terms of eigenvalues, the inverse propagator is derived in appendix A. Its final form reads

$$i G^{-1}(\omega; \mathbf{r}, \mathbf{r}') = e^{i \mathbf{g}(\mathbf{r} - \mathbf{r}')} \cdot i G^{-1}(\omega; \mathbf{r} - \mathbf{r}'),$$

(19)
where $L_n^\alpha$ are the Laguerre polynomials ($L_n^\alpha \equiv L_n$). We also introduced the following shorthand notation:

\[ \xi = \frac{(r - r)^2}{2l^2}, \quad \Phi(r, r') = -\frac{1}{2l^2}(x + x')(y - y') \]  
(Schwarzschild phase)  

and

\[ \mu_{n, \sigma} = \mu_n + \gamma \mu_n, \quad \Delta_{n, \sigma} = \Delta_n + \gamma \Delta_n, \]

\[ f_{n, \sigma} = f_n + \gamma f_n, \quad g_{n, \sigma} = g_n + \gamma g_n. \]

Note that by definition, the Laguerre polynomials $L_n^\alpha$ with negative $n$ are identically zero. Finally, $\mathcal{P}_{n, n'}$ are the projectors in the Dirac space,

\[ \mathcal{P}_{n, n'} = \xi (1 + s_0 0) (1 + s_1 2 i s_2 1 y^2) \]  
with $s_0, s_1 = \pm 1$.  

Similarly, the expression for the propagator itself reads

\[ G(o; r, r') = e^{i\Phi(r, r')} G(o; r - r'), \]  

\[ \tilde{G}(o; r, r') = \frac{e^{-\xi/2}}{2\pi l^2} \sum_{n=0}^{\infty} \sum_{\sigma = \pm 1} \sum_{n' = \pm 1} \left[ s_{o, n} L_n(\xi) + [s_0 \mu_{n, \sigma} + \Delta_{n, \sigma}] \delta_{n, n'} L_n(\xi) + \frac{i\nu}{\xi^2} (\gamma \cdot r)(f_{n, \sigma} - s_0 g_{n, \sigma}) L_{n-1}(\xi) \right] \mathcal{P}_{n, n'}, \]  

where the energies in the lowest and higher LLs are

\[ E_{0, \sigma} = \sigma \Delta_{0, \sigma} = \Delta_0 + \sigma \Delta_0, \]

\[ E_{n, \sigma} = \sqrt{2n(v_F^2 l^2)} |f_{n, \sigma}^2 - g_{n, \sigma}^2| + \Delta_{n, \sigma} \]  
for $n \geq 1$.  

The corresponding energies of quasiparticles are determined by the location of the poles of propagator (26), i.e.

\[ \omega_{0, \sigma} = -\mu_{0, \sigma} + E_{0, \sigma}, \]

\[ \omega_{n, \sigma} = -\mu_{n, \sigma} + E_{n, \sigma}, \]  
for $n \geq 1$.  

Let us note that $\sigma = \pm 1$ is the eigenvalue of the matrix $s_x y^0 y^1 y^2 \equiv s_x y^0 y^1$, which up to the overall sign $s_z$ is the quantum number associated with the valley. This follows from the explicit representation in equation (6) and from our convention for the four-component Dirac spinor, whose first two components are associated with valley $K$ and the last two components with valley $K'$.  

**Figure 1.** Diagrammatic representation of the Schwinger–Dyson equations for the electron and photon propagators in the mean-field approximation.

### 4. The Schwinger–Dyson equation

The Schwinger–Dyson (gap) equation for the fermion propagator in the random-phase approximation (RPA) is shown diagrammatically in figure 1. Note that in contrast to the naive mean-field approximation, the RPA Coulomb interaction includes the polarization (screening) effects, which are not negligible in the dynamics responsible for symmetry breaking in graphene.

It is important to emphasize that the gap equation for the fermion propagator in figure 1 contains two tadpole diagrams. One of them is connected with the Hartree contribution due to dynamical charge carriers, while the other takes into account the background charge from the ions in graphene and in the substrate. The presence of both tadpoles is essential to ensure the overall neutrality of the sample. Indeed, the equation for the gauge field implies that the two tadpole contributions should exactly cancel yielding (the Gauss law)  

\[ j^0_{\text{ext}} - e \text{Tr} [\gamma^0 G] = 0. \]

As is clear from the above arguments, this is directly related to the gauge symmetry in the model. (Since the Gauss law does not take place in models with contact interactions [27], there is an analogue of only one tadpole diagram describing the Hartree interaction, which contributes to the gap equation.) Thus, the resulting Schwinger–Dyson equation for the fermion propagator takes the form

\[ G^{-1}(t - t'; r, r') = S^{-1}(t - t'; r, r') + e^2 \gamma^0 G(t - t'; r, r') \gamma^0 \times D(t' - t; r' - r), \]

where $G(t; r, r')$ is the full fermion propagator and $D(t; r)$ is the propagator mediating the Coulomb interaction.

At this point, it is instructive to compare the cases of a local four-fermion interaction and a non-local Coulomb interaction. In the case of a local four-fermion interaction, the right-hand side of the Schwinger–Dyson equation contains $\delta(r - r')$ and the fermion propagator only at the point of coincidence $G(0; r, r)$. This means that the right-hand side of the Schwinger–Dyson equation is a constant in the momentum space and, hence, it does not renormalize the kinetic $\pi \cdot \gamma$ part of the fermion propagator (i.e. $F' = 1$). Also, in the case of a local four-fermion interaction, $\Sigma^*$ does not depend on the LL index $n$. Clearly, this simplifies to a great extent the analysis of the gap equation. The situation changes in the case of the non-local Coulomb interaction.

Following [10], we consider the instantaneous approximation for the Coulomb interaction by neglecting the...
dependence of the photon polarization function \( \Pi(\omega, k) \) on \( \omega \). Then, in momentum space, the photon propagator takes the following form:

\[
D(\omega, k) \approx D(0, k) = \frac{i}{\epsilon_0[k + \Pi(0, k)]},
\]

where \( \Pi(0, k) \) is the static polarization function and \( \epsilon_0 \) is a dielectric constant. In essence, the instantaneous approximation neglects the retardation of the interaction. This is a dielectric constant. In essence, the instantaneous approximation has a tendency to underestimate the strength of the Coulomb interaction \([3, 26] \).

Unfortunately, it is difficult to find exact solutions of equation (32). Therefore, one has to use some approximations. Here, we will study the dynamical symmetry breaking in the model under consideration retaining contributions only of the lowest and several first LLs. Obviously, this approximation is consistent only if the dynamically generated gaps are suppressed compared to the Landau scale \( \varepsilon_L \equiv \sqrt{\hbar v_F^2 |eB|/c} \) (it characterizes the energy spectrum of the free theory and is the gap between the lowest and first LLs).

In the instantaneous approximation, the photon propagator reads

\[
D(t, r) = \int \frac{d^3k}{(2\pi)^3} \int \frac{d\omega}{2\pi} D(\omega, k) e^{-i\omega t + i\mathbf{k} \cdot \mathbf{r}} = \frac{i}{\epsilon_0} \int_0^\infty \! dk \frac{k J_0(kr)}{2\pi k + \Pi(0, k)} \delta(t).
\]

By noting that the Schwinger phase on both sides of the gap equation (32) is the same, we arrive at the following gap equation for the translation invariant part of the propagator:

\[
iG^{-1}(\omega; \mathbf{r}) = i\tilde{S}^{-1}(\omega; \mathbf{r}) = \frac{\epsilon^2}{\epsilon_0} \int_0^\infty \! d\Omega \int_0^\infty \! dk \frac{k J_0(kr)}{2\pi k + \Pi(0, k)} \times \gamma^0 \tilde{G}(\Omega; \mathbf{r}) \gamma^0.
\]

As shown in appendix B, this is equivalent to an infinite set of algebraic equations; see equations (B.23), (B.24), (B.26) and (B.27).

5. Numerical results

In this section, we present our numerical solutions to the truncated set of gap equations. In contrast to our previous analysis in [27], where a model with a contact interaction was studied in detail, here we investigate the effect of the long-range interaction on the dynamics of symmetry breaking in the QH effect in graphene. For this purpose, we will neglect the screening effects, captured by the polarization function \( \Pi(0, k) \). Note that this approximation is justified in the case of a strong magnetic field, which is of main interest to us. Indeed, since \( \Pi(0, k) \sim \kappa(k\ell) \) in infrared [10], it is suppressed with respect to the bare Coulomb interaction. This point greatly simplifies the numerical analysis because now the kernel of the gap equation takes an analytical form. A more rigorous study of the dynamics with screening (e.g. in the RPA) will be presented elsewhere [34]. Note that the strength of the Coulomb interaction is characterized by graphene’s ‘fine structure constant’ \( \alpha \equiv e^2/\epsilon_0|V_r| \), which is approximately equal to \( 2.2/\epsilon_0 \). In our numerical calculations below, we use \( \epsilon_0 = 1 \).

In order to use numerical calculations efficiently, it is important to understand all energy scales in the problem at hand. Ignoring the large energy cutoff due to a finite width of the conductance band, there are essentially only two characteristic energy scales in the action of graphene: (i) the LL scale \( \varepsilon_L \equiv \sqrt{\hbar v_F^2 |eB|/c} \) and (ii) the much smaller Zeeman energy \( Z \equiv \mu_B B \). In our numerical calculations, we measure all physical quantities with energy in units of \( \varepsilon_L \). Numerically, these are

\[
\varepsilon_L = \sqrt{\hbar v_F^2 |eB|/c} = 26\sqrt{B[T]} \text{ meV},
\]

\[
Z = \mu_B B = 5.8 \times 10^{-2} B[T] \text{ meV},
\]

where \( B[T] \) is the value of the magnetic field measured in teslas. The corresponding temperature scales are \( \varepsilon_L/k_B = 300\sqrt{B[T]} \) K and \( Z/k_B = 0.67 B[T] \) K.

Note that the magnetic length \( \ell = \sqrt{\hbar c}/|eB| \) and the Landau energy scale \( \varepsilon_L \) are related through the Fermi velocity as follows: \( \ell = \hbar v_F/\varepsilon_L \). Thus, the LL scale \( \varepsilon_L \) is of the order of the Landau scale \( \ell \). The corresponding temperature scales are \( \varepsilon_L/k_B = 300\sqrt{B[T]} \) K and \( Z/k_B = 0.67 B[T] \) K.

5.1. Renormalization of the Fermi velocity (weak field)

Let us start from the simplest analysis, when the role of the dynamical mass parameters is negligible and there is no significant splitting of the LLs. This is presumably the case when the magnetic field is not so strong. Even in this case, however, there is a very interesting dynamics responsible for the renormalization of the Fermi velocity. This is also interesting from the experimental point of view because the renormalized value of the Fermi velocity parameter, which is also a function of the Landau index \( n \), affects the energies of optical transitions [35–37].

To this end, let us consider only the subset of the gap equations, which involve the wavefunction renormalization \( f_n \); see equation (B.19). Even this subset contains an infinite number of gap equations for each choice of spin, i.e.

\[
f_n = 1 + \frac{\alpha}{2} \sum_{n' = 1}^{12} \kappa_{n', n}^{(1)} \frac{[1 - n_F(E_{n'} - \mu)] - n_F(E_n + \mu)}{\sqrt{2\pi}}
\]

for \( n \geq 1 \),

where we took into account the definition of the coefficients \( \kappa_{n', n}^{(1)} \) in equation (B.11) and used the approximate expression for the LL energies: \( E_{n, \sigma} \approx E_n = \sqrt{2n\varepsilon_L f_{n, \sigma}} \), which are independent of the valley quantum number \( \sigma \equiv s_{0, \pm 2} \). Note that the spin index is also omitted, which is justified especially at weak fields. Here we use the notation \( n_F(x) \equiv 1/(e^x + 1) \) for the Fermi–Dirac distribution function.

Before analyzing the set of equation for the wavefunction renormalization \( f_n \) numerically, it is instructive to note that the expression on the right-hand side of equation (37) is formally divergent. Indeed, by taking into account the
Table 1. Values of the wavefunction renormalization \( f_n \) for several values of the chemical potentials.

| \(|\mu| < \sqrt{2}E_L\)| | \(|\sqrt{2}E_L < \mu < \sqrt{3}E_L\)| | \(|\sqrt{3}E_L < \mu < \sqrt{5}E_L\)| | \(|\sqrt{5}E_L < \mu < \sqrt{8}E_L\)| | \(|\sqrt{8}E_L < \mu < \sqrt{10}E_L\)| | \(|\sqrt{10}E_L < \mu < \sqrt{12}E_L\)| |
|---|---|---|---|---|---|
| 1.270 | 1.243 | 1.227 | 1.214 | 1.205 | 1.197 | 1.190 |
| 1.194 | 1.224 | 1.217 | 1.208 | 1.201 | 1.194 | 1.188 |
| 1.193 | 1.224 | 1.217 | 1.208 | 1.201 | 1.194 | 1.188 |
| 1.166 | 1.177 | 1.200 | 1.199 | 1.194 | 1.189 | 1.184 |
| 1.150 | 1.156 | 1.165 | 1.184 | 1.185 | 1.182 | 1.179 |
| 1.149 | 1.155 | 1.164 | 1.182 | 1.184 | 1.182 | 1.179 |
| 1.138 | 1.142 | 1.148 | 1.156 | 1.172 | 1.174 | 1.173 |

There we show many sets of the results which correspond to different values of the chemical potentials. The points are the actual data, while the lines connecting the points are shown as a guide to the eye for the data for fixed values of the chemical potentials. The data on the top line correspond to small values of the chemical potential, \(|\mu| < \sqrt{2}E_L\). The other lines correspond to the chemical potentials in the energy gaps between the \(n\)th and the \((n + 1)\)th LL (with \(n = 0, 1, 2, \ldots\) from top to bottom). Part of the same data are also given in table 1.

In general, the renormalized Fermi velocity \(\tilde{v}_{F,n}\) is about 10–20\% larger than its non-renormalized value \(v_F\). These results seem to be somewhat smaller than the predictions in [40]. One should keep in mind, however, that there are considerable uncertainties in the theoretical predictions for the renormalized values of the Fermi velocity. In part, these are associated with a relatively large value of the coupling constant in graphene and with the logarithmic running of the wavefunction renormalization itself.

A convenient quantitative measure of the many-particle effects in the transition energies is given by the prefactors \(C_{n,n'}\), which are introduced as deviations from the non-interacting carriers in graphene,

\[
\Delta E_{n,n'} \equiv E_{n'} \pm E_n = (\sqrt{2n'} \pm \sqrt{2n}) \varepsilon_L + \alpha \varepsilon_L C_{n,n'},
\]

where, once again, a small Zeeman splitting of the energy levels is ignored. Note that both terms in (38) scale as \(\sim \sqrt{B}\) and experimental data on infrared spectroscopy of LLs of graphene clearly confirm this behavior [35–37]. On the other hand, the dependence of the coefficients \(C_{n,m}\) on the LL pair allows one to obtain information on many-body effects. By making use of our notation for the wavefunction renormalization, we obtain

\[
C_{n,n'} = \frac{\sqrt{2n'}}{\alpha} (f_{n'} - 1) \pm \frac{\sqrt{2n}}{\alpha} (f_n - 1),
\]

where \(n' \neq 0\) and \(n' \neq 0\). (For transitions from the \(n = 0\) level and for transitions to the \(n' = 0\) level, the LLL never gives any contribution to the corresponding prefactors.) By making use of our results for \(f_n\), we obtain the values of prefactors \(C_{n,n'}\). For transitions between several low-lying LLs, the values of the prefactors are listed in table 2.

It is clear that the Coulomb interaction contribution to the LL transitions slightly increases the transition energies above their non-interacting values in agreement with experimental data (e.g. see figure 3(a) in [36]).

![Figure 2](image-url)

**Figure 2.** The numerical values of the wavefunction renormalization coefficients \(f_n\) versus the LL index \(n\) for several fixed values of the chemical potential.
where we took into account the exact value of $\kappa_{0,0}^{(0)} = 1/(2\sqrt{2\pi})$; see table B.1. One of the solutions to this set of equations is of the ‘magnetic catalysis’ type, i.e.

$$\mu_0^{\text{eff}} = \mu - \frac{\alpha \varepsilon_f}{4\sqrt{2\pi}},$$

$$|\Delta_0^{\text{eff}}| = \frac{\alpha \varepsilon_f}{4\sqrt{2\pi}}.$$  \hspace{1cm} (45)

(Note that, according to this simplified estimate in the LLL approximation, the value of the gap $|\Delta_0^{\text{eff}}|$ ranges from 30 to 190 K when the magnetic field changes from 1 to 50 T, assuming that the coupling constant $\alpha$ is of the order of 1.)

The other two solutions are

$$-\infty < \mu < \frac{\alpha \varepsilon_f}{4\sqrt{2\pi}}, \quad \mu_0^{\text{eff}} = \mu - \frac{\alpha \varepsilon_f}{4\sqrt{2\pi}}, \quad \Delta_0^{\text{eff}} = 0,$$ \hspace{1cm} (47)

$$\frac{\alpha \varepsilon_f}{4\sqrt{2\pi}} < \mu < \infty, \quad \mu_0^{\text{eff}} = \mu + \frac{\alpha \varepsilon_f}{4\sqrt{2\pi}}, \quad \Delta_0^{\text{eff}} = 0.$$ \hspace{1cm} (48)

All of these solutions can easily be found numerically. In addition, the numerical study of the finite-temperature equations shows two extra (probably unstable) solutions satisfying the approximate condition $\mu_0^{\text{eff}} \approx |\Delta_0^{\text{eff}}|$. These solutions seem to survive even when one approaches the limit $T \to 0$. They are lost in the analytical study of the above zero-temperature equations because they correspond to the vanishing value of the argument in the step-functions. The ambiguity of the step-function in this case prevents us from finding the same solutions. Note that all five solutions can be continuously continued into each other through a sort of hysteresis loop without any discontinuities in the values of $\mu_0^{\text{eff}}$ and $|\Delta_0^{\text{eff}}|$. This fact alone strongly indicates that some ‘intermediate’ solutions are metastable or even unstable.

### 5.3. Numerical solutions for $\mu$ near the lowest Landau level

Let us now perform a more realistic numerical analysis by including several LLs and accounting for the spin degree of freedom in the analysis. The explicit form of the gap equations is presented in appendix B; see equations (B.23)–(B.27). Because of the large number of dynamical parameters, in this section, we will use a rather small number for the cutoff $n_{\text{max}} = 5$ in the summation over the LLs.

The gap equations for both spins are similar except for the value of the chemical potential: they are $\mu_\uparrow = \mu - Z$ and $\mu_\downarrow = \mu + Z$. By repeating the same analysis as above, we find that there exist many more solutions around the vanishing value of $\mu$. Keeping only a subset of several qualitatively different solutions with lowest energies, we find among them a pure Dirac mass solution, four types of the Haldane mass solutions and four types of hybrid solutions (see below). The pure Dirac mass solution has nonzero Dirac masses for both spins and no Haldane masses (i.e. the order parameters are triplets with respect to both $SU(2)$ and $SU_1(2)$). Four Haldane mass solutions are determined by four possible sign combinations of the two time-reversal breaking masses: (i) $\Delta_0,\uparrow > 0$ and $\Delta_0,\downarrow > 0$;
(ii) $\Delta_{0,\uparrow} > 0$ and $\Delta_{0,\downarrow} < 0$; (iii) $\Delta_{0,\uparrow} < 0$ and $\Delta_{0,\downarrow} > 0$; (iv) $\Delta_{0,\uparrow} < 0$ and $\Delta_{0,\downarrow} < 0$. All of these are characterized by singlet order parameters with respect to both $SU_1(2)$ and $SU_\downarrow(2)$ symmetry groups. Similarly, four hybrid solutions are determined by the following conditions: (i) $\tilde{\Delta}_{0,\uparrow} \neq 0$ and $\Delta_{0,\downarrow} > 0$; (ii) $\Delta_{0,\uparrow} \neq 0$ and $\Delta_{0,\downarrow} < 0$; (iii) $\Delta_{0,\uparrow} > 0$ and $\tilde{\Delta}_{0,\downarrow} \neq 0$; (iv) $\Delta_{0,\uparrow} < 0$ and $\tilde{\Delta}_{0,\downarrow} \neq 0$. A common feature of the hybrid solutions is that one of their order parameters is a triplet with respect to the $SU_1(2)$ or $SU_\downarrow(2)$ group, while the other order parameter is a singlet with respect to the other group.

Including many ‘intermediate’ branches of solutions, in our numerical analysis we were able to identify several dozens of non-equivalent solutions in the vicinity of the LLL. The free energies for all of these solutions can be easily calculated using equation (C.5) in appendix C.

The free energies of several lowest energy solutions are plotted in figure 3. Four singlet-type solutions and one triplet solution are shown by solid lines in the figure. Four lowest energy hybrid solutions are shown by dashed lines. There are three qualitatively different regions, in which the lowest energy states are different, i.e.

$$\mu < -Z : \Delta_{0,\uparrow} > 0 \& \Delta_{0,\downarrow} < 0,$$

$$-Z < \mu < Z : \Delta_{0,\uparrow} > 0 \& \Delta_{0,\downarrow} < 0,$$

$$\mu > Z : \Delta_{0,\uparrow} < 0 \& \Delta_{0,\downarrow} < 0.$$  

At the points $\mu = \pm Z$, there also exist hybrid solutions with the same lowest values of energy as the two Haldane mass solutions.

- $\nu = -2$ (the LLL is empty). This is a singlet solution, whose free energy as a function of $\mu$ is shown by the green solid line in figure 3. This solution corresponds to the unbroken $U_1(2) \times U_\downarrow(2)$ symmetry and has the lowest energy for $\mu < -Z$

$$\tilde{\Delta}_{0,\uparrow}^{\text{eff}} = 0, \quad \mu_{0,\uparrow}^{\text{eff}} = \mu - Z - \Delta_{0,\uparrow}, \quad \Delta_{0,\uparrow} \approx 0.2252\ell_\epsilon,$$

$$\tilde{\Delta}_{0,\downarrow}^{\text{eff}} = 0, \quad \mu_{0,\downarrow}^{\text{eff}} = \mu + Z - \Delta_{0,\downarrow}, \quad \Delta_{0,\downarrow} \approx 0.2252\ell_\epsilon.$$  

In this case, the LLL quasiparticle energies are

$$\omega_{0,\uparrow} = -\mu + Z + |\Delta_{0,\uparrow}| > 0,$$  

$$\omega_{0,\downarrow} = -\mu - Z + |\Delta_{0,\downarrow}| > 0.$$  

Since none of the LLL sublevels are occupied, this solution corresponds to a $v = -2$ state.

- $\nu = -1$ (the LLL is one-quarter filled). This is a hybrid solution, whose free energy as a function of $\mu$ is shown by the red dashed line in figure 3. The symmetry in the corresponding state is spontaneously broken down to $U_1(2) \times U_{\downarrow}(1) \times U_{\uparrow}(1)$, where the two latter factors describe $U(1)$ transformations at a fixed spin and a fixed valley. At $\mu = -Z$, this solution is degenerate in energy with the solutions for the $v = -2$ and $v = 0$ cases and is given by

$$\tilde{\Delta}_{0,\uparrow}^{\text{eff}} = 0, \quad \mu_{0,\uparrow}^{\text{eff}} = \mu - Z - \Delta_{0,\uparrow}, \quad \Delta_{0,\uparrow} \approx 0.2252\ell_\epsilon,$$

$$\tilde{\Delta}_{0,\downarrow}^{\text{eff}} \approx 0.2252\ell_\epsilon, \quad \mu_{0,\downarrow}^{\text{eff}} = \mu + Z, \quad \Delta_{0,\downarrow} \approx 0.$$  

In this case, the LLL quasiparticle energies are

$$\omega_{0,\uparrow} = -\mu + Z + |\Delta_{0,\uparrow}| > 0,$$  

$$\omega_{0,\downarrow} = -\mu - Z + |\Delta_{0,\downarrow}| > 0.$$  

Since two of the LLL sublevels are occupied, this solution corresponds to a $v = 0$ state.

- $\nu = 0$ (the half-filled LLL; neutral point). This is a singlet solution. Its free energy as a function of $\mu$ is shown by the blue solid line in figure 3. The symmetry in the corresponding state is $U_1(2) \times U_\downarrow(2)$, but the Zeeman splitting is dynamically enhanced. This solution has the lowest energy for $-Z < \mu < Z$

$$\tilde{\Delta}_{0,\uparrow}^{\text{eff}} = 0, \quad \mu_{0,\uparrow}^{\text{eff}} = \mu - Z - \Delta_{0,\uparrow}, \quad \Delta_{0,\uparrow} \approx 0.2252\ell_\epsilon,$$

$$\tilde{\Delta}_{0,\downarrow}^{\text{eff}} \approx 0.2252\ell_\epsilon, \quad \mu_{0,\downarrow}^{\text{eff}} = \mu + Z, \quad \Delta_{0,\downarrow} \approx 0.$$

In this case, the LLL quasiparticle energies are

$$\omega_{0,\uparrow} = -\mu + Z + |\Delta_{0,\uparrow}| > 0,$$  

$$\omega_{0,\downarrow} = -\mu - Z - |\Delta_{0,\downarrow}| < 0.$$  

Since only one LLL sublevel is occupied, this solution corresponds to a $v = -1$ state.
two neighboring levels, the LLL and the \( n = 2 \) LL (recall that, according to equations (17) and (18), \( \Delta_n, \Delta_{nl}, \mu_n, f_n \) are functions of the LL index). Physically, the information concerning the gaps in other LLs is relevant for experiments connected with transitions between LLs [36].

- **\( \nu = 3 \)** (the \( n = 1 \) LL is one-quarter filled). Hybrid solution, which is valid for \( \mu \simeq \sqrt{2} \epsilon - Z \). The symmetry in this state is spontaneously broken down to \( U_{12}^{(K)}(1) \times U_{12}^{(K)}(1) \).

\[
\tilde{\Delta}^{\text{eff}}_{0, \uparrow} = 0, \quad \mu^{\text{eff}}_{0, \uparrow} = \mu_{\uparrow} - \Delta_{0, \uparrow}, \quad \Delta_{0, \uparrow} = -0.225 \epsilon_{\ell},
\]

\[
\tilde{\Delta}^{\text{eff}}_{0, \downarrow} = 0, \quad \mu^{\text{eff}}_{0, \downarrow} = \mu_{\downarrow} - \Delta_{0, \downarrow}, \quad \Delta_{0, \downarrow} = -0.277 \epsilon_{\ell},\quad (74)
\]

- **\( \nu = 4 \)** (the \( n = 1 \) LL is half-filled). Singlet solution, which is valid for \( \sqrt{2} \epsilon - Z \lesssim \mu \lesssim \sqrt{2} \epsilon + Z \). While formally the symmetry of this state is the same as in the action, \( U_{12}(2) \times U_{12}(2) \), it is characterized by a dynamically enhanced Zeeman splitting.

\[
\tilde{\Delta}^{\text{eff}}_{0, \uparrow} = 0, \quad \mu^{\text{eff}}_{0, \uparrow} = \mu_{\uparrow} - \Delta_{0, \uparrow}, \quad \Delta_{0, \uparrow} = -0.225 \epsilon_{\ell},
\]

\[
\tilde{\Delta}^{\text{eff}}_{0, \downarrow} = 0, \quad \mu^{\text{eff}}_{0, \downarrow} = \mu_{\downarrow} - \Delta_{0, \downarrow}, \quad \Delta_{0, \downarrow} = -0.328 \epsilon_{\ell},\quad (80)
\]

- **\( \nu = 5 \)** (the \( n = 1 \) LL is three-quarter filled). Hybrid solution, which is valid for \( \mu \simeq \sqrt{2} \epsilon + Z \). The symmetry

\[
\tilde{\Delta}^{\text{eff}}_{0, \uparrow} = 0, \quad \mu^{\text{eff}}_{0, \uparrow} = \mu_{\uparrow} - \Delta_{0, \uparrow}, \quad \Delta_{0, \uparrow} = -0.393 \epsilon_{\ell},
\]

\[
\tilde{\Delta}^{\text{eff}}_{0, \downarrow} = 0, \quad \mu^{\text{eff}}_{0, \downarrow} = \mu_{\downarrow} - \Delta_{0, \downarrow}, \quad \Delta_{0, \downarrow} = -1.092 \epsilon_{\ell},\quad (85)
\]

\[
\tilde{\Delta}^{\text{eff}}_{0, \uparrow} = 0, \quad \mu^{\text{eff}}_{0, \uparrow} = \mu_{\uparrow} + 0.048 \epsilon_{\ell},
\]

\[
\tilde{\Delta}^{\text{eff}}_{0, \downarrow} = 0, \quad \mu^{\text{eff}}_{0, \downarrow} = \mu_{\downarrow} + 0.048 \epsilon_{\ell},\quad (87)
\]

- **\( \nu = 6 \)** (the \( n = 1 \) LL is fully filled). Hybrid solution, which is valid for \( \mu \simeq 2 \epsilon + Z \). The symmetry

\[
\tilde{\Delta}^{\text{eff}}_{0, \uparrow} = 0, \quad \mu^{\text{eff}}_{0, \uparrow} = \mu_{\uparrow} + 0.048 \epsilon_{\ell},
\]

\[
\tilde{\Delta}^{\text{eff}}_{0, \downarrow} = 0, \quad \mu^{\text{eff}}_{0, \downarrow} = \mu_{\downarrow} + 0.048 \epsilon_{\ell},\quad (89)
\]

- **\( \nu = 7 \)** (the \( n = 1 \) LL is seven-quarter filled). Hybrid solution, which is valid for \( \mu \simeq 2 \epsilon + Z \). The symmetry

\[
\tilde{\Delta}^{\text{eff}}_{0, \uparrow} = 0, \quad \mu^{\text{eff}}_{0, \uparrow} = \mu_{\uparrow} + 0.048 \epsilon_{\ell},
\]

\[
\tilde{\Delta}^{\text{eff}}_{0, \downarrow} = 0, \quad \mu^{\text{eff}}_{0, \downarrow} = \mu_{\downarrow} + 0.048 \epsilon_{\ell},\quad (91)
\]

- **\( \nu = 8 \)** (the \( n = 1 \) LL is eight-quarter filled). Hybrid solution, which is valid for \( \mu \simeq 2 \epsilon + Z \). The symmetry

\[
\tilde{\Delta}^{\text{eff}}_{0, \uparrow} = 0, \quad \mu^{\text{eff}}_{0, \uparrow} = \mu_{\uparrow} + 0.048 \epsilon_{\ell},
\]

\[
\tilde{\Delta}^{\text{eff}}_{0, \downarrow} = 0, \quad \mu^{\text{eff}}_{0, \downarrow} = \mu_{\downarrow} + 0.048 \epsilon_{\ell},\quad (93)
\]
in this state is spontaneously broken down to \( U^{(g)}_\uparrow \times U^{(g)}_\downarrow \times U^{(g)}_3 \times U^{(g)}_4 \).

\[
\tilde{\Delta}^{\text{eff}}_{0,\uparrow} = 0.052\varepsilon_\ell, \quad \mu^{\text{eff}}_{0,\uparrow} = \mu_\uparrow - \Delta_{0,\uparrow}, \quad \Delta_{0,\uparrow} = -0.277\varepsilon_\ell, \quad (86)
\]

\[
\tilde{\Delta}^{\text{eff}}_{0,\downarrow} = 0, \quad \mu^{\text{eff}}_{0,\downarrow} = \mu_\downarrow - \Delta_{0,\downarrow}, \quad \Delta_{0,\downarrow} = -0.328\varepsilon_\ell, \quad (87)
\]

\[
\tilde{\Delta}_{1,\uparrow} = -0.018\varepsilon_\ell, \quad \mu_{1,\uparrow} = \mu_\uparrow + 0.148\varepsilon_\ell, \quad \Delta_{1,\uparrow} = -0.049\varepsilon_\ell, \quad f_{1,\uparrow} = 1.103, \quad (88)
\]

\[
\tilde{\Delta}_{1,\downarrow} = 0, \quad \mu_{1,\downarrow} = \mu_\downarrow + 0.244\varepsilon_\ell, \quad \Delta_{1,\downarrow} = -0.031\varepsilon_\ell, \quad f_{1,\downarrow} = 1.065, \quad (89)
\]

\[
\tilde{\Delta}_{2,\uparrow} = -0.006\varepsilon_\ell, \quad \mu_{2,\uparrow} = \mu_\uparrow + 0.091\varepsilon_\ell, \quad \Delta_{2,\uparrow} = -0.045\varepsilon_\ell, \quad f_{2,\uparrow} = 1.102, \quad (90)
\]

\[
\tilde{\Delta}_{2,\downarrow} = 0, \quad \mu_{2,\downarrow} = \mu_\downarrow + 0.142\varepsilon_\ell, \quad \Delta_{2,\downarrow} = -0.039\varepsilon_\ell, \quad f_{2,\downarrow} = 1.092, \quad (91)
\]

- \( \nu = 6 \) (the \( n = 1 \) LL is filled). Singlet solution with the unbroken \( U^{(g)}_\uparrow \times U^{(g)}_\downarrow \times U^{(g)}_3 \times U^{(g)}_4 \) symmetry. It is valid for \( \mu \gtrsim \sqrt{2\varepsilon_\ell + 2} \) that was mentioned in the previous subsection.

\[
\tilde{\Delta}^{\text{eff}}_{0,\uparrow} = 0, \quad \mu^{\text{eff}}_{0,\uparrow} = \mu_\uparrow - \Delta_{0,\uparrow}, \quad \Delta_{0,\uparrow} = -0.328\varepsilon_\ell, \quad (92)
\]

\[
\tilde{\Delta}^{\text{eff}}_{0,\downarrow} = 0, \quad \mu^{\text{eff}}_{0,\downarrow} = \mu_\downarrow - \Delta_{0,\downarrow}, \quad \Delta_{0,\downarrow} = -0.328\varepsilon_\ell, \quad (93)
\]

\[
\tilde{\Delta}_{1,\uparrow} = 0, \quad \mu_{1,\uparrow} = \mu_\uparrow + 0.244\varepsilon_\ell, \quad \Delta_{1,\uparrow} = -0.031\varepsilon_\ell, \quad f_{1,\uparrow} = 1.065, \quad (94)
\]

\[
\tilde{\Delta}_{1,\downarrow} = 0, \quad \mu_{1,\downarrow} = \mu_\downarrow + 0.244\varepsilon_\ell, \quad \Delta_{1,\downarrow} = -0.031\varepsilon_\ell, \quad f_{1,\downarrow} = 1.065, \quad (95)
\]

\[
\tilde{\Delta}_{2,\uparrow} = 0, \quad \mu_{2,\uparrow} = \mu_\uparrow + 0.142\varepsilon_\ell, \quad \Delta_{2,\uparrow} = -0.039\varepsilon_\ell, \quad f_{2,\uparrow} = 1.092, \quad (96)
\]

\[
\tilde{\Delta}_{2,\downarrow} = 0, \quad \mu_{2,\downarrow} = \mu_\downarrow + 0.142\varepsilon_\ell, \quad \Delta_{2,\downarrow} = -0.039\varepsilon_\ell, \quad f_{2,\downarrow} = 1.092, \quad (97)
\]

Before concluding this section, it is appropriate to mention that, while all the results for the QH states associated with filling the \( n = 1 \) LLs are qualitatively similar to those obtained in [27], the Coulomb long-range interaction makes all gaps and other dynamical parameters functions of the LL index \( n \).

6. Discussion

The present analysis of integer QH plateaus connected with lifting the fourfold degeneracy of LLs yields to the following results. The \( \nu = 0 \) QH state is a QH ferromagnet with a significantly dynamically enhanced Zeeman splitting. The \( \nu = 1 \) QH plateau is described by a CDW for the electrons with a spin polarized along the magnetic field. Such a CDW is absent for electrons with the opposite polarization. The solution corresponding to the \( \nu = 2 \) QH plateau is a singlet and it necessarily includes the Haldane mass. At the \( n = 1 \) LL, the dynamically generated gaps are fewer less than those at the LLL. Therefore, the corresponding QH states could be observed only in rather strong magnetic fields and very clean samples.

It is noteworthy that the present results obtained in the mean-field approximation with the long-range Coulomb interaction are qualitatively similar to those obtained in [27], where a short-range four-fermion interaction was used. The main reasons for this fact are (a) the universality of the MC phenomenon and (b) in both these models, there is essentially one dimensionful parameter, the Landau scale \( \varepsilon_\ell = \sqrt{h m^2_e B^2 / c^4} \), through which the energy spectra of the QH dynamics are expressed. In this context, we recall that the dimensionful coupling constant \( G_{\text{int}} \) in the model in [27] is taken as \( G_{\text{int}} \sim \lambda / \varepsilon_\ell \), and the free dimensionless coupling \( \lambda \) plays the role of the Coulomb coupling there. On the other hand, it is noticeable that while the gaps are constant for the short-range interaction, they depend on the LL index \( n \) in the case of the long-range Coulomb one. Note that without magnetic field energy, gaps are functions of a frequency \( \omega \) and a momentum \( k \) for long-range interactions [10]. The discrete index \( n \) replaces the momentum when magnetic field is switched on.

It should be noted that a recent study of the cyclotron mass in suspended graphene [41] suggests a very strong renormalization of the Fermi velocity (at the electron density \( n = 10^9 \text{ cm}^{-2} \), it is almost three times larger than the value commonly used). We note, however, that the effect of the Fermi velocity renormalization was studied in several other experiments, for example by photoemission spectroscopy [42], scanning single-electron transistor measurements of the local electronic compressibility [43] and optical infrared measurements in a magnetic field [35–37, 44]. While the optical measurements observed a departure from the dispersion for noninteracting particles, its value was small. The present study also suggests that the corresponding renormalization effects are much smaller than those found in [41] and, in fact, they are comparable with those observed in [36, 49].

It would be instructive to compare the present results with the realization of MC in (3 + 1)-dimensional relativistic theories, for example QED [45] and QCD [46]. It is interesting that the dynamical generation of only one type of Dirac mass is usually considered in relativistic theories in (3 + 1) dimensions. As we know, the set of order parameters in graphene is much richer. It is only recently that new order parameters analogous to the QHF order parameters and the Haldane mass have been studied in QED and QCD [47, 48].

In large part, studies of the QH effect in graphene have been responsible for those new studies on relativistic theories. (Note, however, that in (3 + 1) dimensions, the analogue of the Haldane mass term describes the axial-vector current density rather than mass [48].)

Acknowledgments

VAM is grateful to the organizers of the Nobel Symposium on Graphene and Quantum Matter for their warm hospitality. The
work of EVG and VPG was partially supported by SCOPES under grant no. IZ7320-128026 of the Swiss NSF, by grant no. SIMTECH 246937 of the European FP7 program, by the SFFR-RFBR grant ‘Application of string theory and field theory methods to nonlinear phenomena in low-dimensional systems’ and by the Program of Fundamental Research of the Physics and Astronomy Division of NAS in the Ukraine. The work of VAM was supported by the Natural Sciences and Engineering Research Council of Canada. The work of IAS was partially supported by a start-up fund from the Arizona State University and by the US National Science Foundation under grant no. PHY-0969844.

Appendix A. Fermion Green’s function

A.1. General structure

The general structure of the inverse full fermion propagator (for a single spin species) in a magnetic field is written in equation (10). Making use of the two sets of operators, $\hat{F}^\pm$ and $\hat{\Sigma}^\pm$, defined in equations (11), (12), (15) and (16), we derive the following formal representation for the full Green’s function:

$$G = i[y^00 + v_F \hat{F}^+(\pi \cdot \gamma) + \hat{\Sigma}^+)]^{-1}$$

$$= i[y^00 + v_F \hat{F}^+(\pi \cdot \gamma) - \hat{\Sigma}^-][y^00 + v_F \hat{F}^+(\pi \cdot \gamma) + \hat{\Sigma}^-]^{-1}$$

$$= i[y^00 + v_F \hat{F}^+(\pi \cdot \gamma) - \hat{\Sigma}^-][y^00 + y^00(\hat{\Sigma}^- - \hat{\Sigma}^-)]^{-1}$$

$$+ i v_F \hat{F}^+(\pi \cdot \gamma)^2 - \hat{\Sigma}^- \hat{\Sigma}^-]^{-1},$$

(A.1)

where

$$y^00(\hat{\Sigma}^- - \hat{\Sigma}^-) = 2 \omega(\mu + i s \gamma^00 \gamma^10 \gamma^20 \mu0),$$

(A.2)

$$\hat{F}^+ \hat{F}^- = (f + i s \gamma^00 \gamma^10 \gamma^20 f0)^2 - (g + i s \gamma^00 \gamma^10 \gamma^20 \mu0)^2$$

$$= f^2 + \bar{f}^2 - g^2 - \bar{g}^2 + 2 i s \gamma^00 \gamma^10 \gamma^20 (f \bar{f} - g \bar{g}),$$

(A.3)

$$\hat{\Sigma}^+ \hat{\Sigma}^- = (\Delta + i s \gamma^00 \gamma^10 \gamma^20 \Delta0) - (\mu + i s \gamma^00 \gamma^10 \gamma^20 \mu0)^2$$

$$= \Delta^2 - \mu^2 + i \mu (2 i s \gamma^00 \gamma^10 \gamma^20 (\hat{\Delta} \hat{\mu} - \hat{\mu} \hat{\Delta}),$$

(A.4)

$$\hat{\Sigma}^+ \hat{\Sigma}^- \hat{\Sigma}^- = (-e/c0 B i y^00 \gamma^10 \gamma^20 - (e/c0 B i y^00 \gamma^10 \gamma^20)^2)$$

$$= -\pi^2 \ell^2 - (e/c0 B i y^00 \gamma^10 \gamma^20)^2$$

(A.5)

and we recall the definition of the magnetic length $\ell = \sqrt{\ell_0/|v_F|}$ and $s_1 = \text{sgn}(e B)$. We also used the fact that the operator $\pi^2 \ell^2$ has the eigenvalues $(2N + 1)$ with $N = 0, 1, 2, \ldots$. In the Landau gauge, $A = (0, B x)$, utilized here, the corresponding normalized wavefunctions read

$$\psi_{Np}(\mathbf{r}) = \frac{1}{\sqrt{2\pi \ell}} \frac{1}{\sqrt{2N + 1}} \sqrt{\pi} \frac{1}{\sqrt{2N + 1}} H_N \left(\frac{x}{\ell} + p \ell\right) e^{-\frac{i}{\pi} (x + p \ell)^2} e^{i x p \ell}$$

(A.6)

where $H_N(x)$ are the Hermite polynomials. These wavefunctions satisfy the conditions of normalizability

$$\int d^2 r \psi^*_{Np}(\mathbf{r}) \psi_{Np'}(\mathbf{r}) = \delta_N \delta(p - p')$$

(A.7)

and completeness

$$\sum_{N=0}^{\infty} \int_{-\infty}^{\infty} d p \psi_{Np}(\mathbf{r}) \psi^*_{Np}(\mathbf{r}) = \delta(\mathbf{r} - \mathbf{r}')$$

(A.8)

Note that operators $\hat{F}^\pm$ and $\hat{\Sigma}^\pm$, when acting on eigenstates $|N, p, s_0, s_{12}\rangle$, reduce to their eigenvalues:

$$\hat{F}^\pm[(\pi \cdot \gamma)^2 \ell^2, \gamma_0, i s \gamma^10 \gamma^20] |N, p, s_0, s_{12}\rangle$$

$$= \hat{F}^\pm \left[-(2N + 1 + s_{12}) |N, p, s_0, s_{12}\rangle \right]$$

$$= f_{N+1/2,0}^{s_00,s_{12}} |N, p, s_0, s_{12}\rangle$$

(A.9)

$$\hat{\Sigma}^\pm \left[(\pi \cdot \gamma)^2 \ell^2, \gamma_0, i s \gamma^10 \gamma^20 \right] |N, p, s_0, s_{12}\rangle$$

$$= \hat{\Sigma}^\pm \left[-(2N + 1 + s_{12}) |N, p, s_0, s_{12}\rangle \right]$$

$$= \Sigma_{N+1/2,0}^{s_00,s_{12}} |N, p, s_0, s_{12}\rangle$$

(A.10)

Taking into account that $s_{12} = \pm 1$, we conclude that the LL index $n = N + (s_{12} + 1)/2$ is a non-negative integer. Therefore, as follows from the definitions in equations (11), (12), (15) and (16), the eigenvalues are determined by the following expressions:

$$f_{n_0,s_{12}}^{s_00} \equiv f_{n_0 + s_0, s_{12}0} + s_{12} \bar{g}_{n_0} + s_0 s_{12} \bar{f}_{n_0}$$

(A.11)

$$\sum_{n_0,s_{12}}^{s_00} \equiv \Delta_n + s_0 \mu_0 + s_{12} \bar{\mu}_0 + s_0 s_{12} \Delta_n$$

(A.12)

where $f_{n_0}, \bar{f}_{n_0}, g_{n_0}, \bar{g}_{n_0}, \Delta_n, \Delta_n, \mu_0$ and $\bar{\mu}_0$ are the eigenvalues of the corresponding coefficient operators in the nth LL state.

A.2. Inverse propagator

Let us introduce the following projectors in the Dirac space:

$$\mathcal{P}_{s_0,s_{12}} = \frac{1}{2} \left(1 + \gamma_0 \gamma_0 \right)(1 + s_{12} \gamma_0 \gamma_0 \gamma^10 \gamma^20)$$

(A.13)

Then, we can write the inverse fermion propagator as

$$i G^{-1}(\omega; \mathbf{r}, \mathbf{r}') = \sum_{s_0,s_{12}=\pm 1} i G^{-1}_{s_0,s_{12}}(\omega; \mathbf{r}, \mathbf{r}') \mathcal{P}_{s_0,s_{12}}$$

(A.14)

(note that the projector is multiplied on the right and that the ordering is important), where

$$i G^{-1}_{s_0,s_{12}}(\omega; \mathbf{r}, \mathbf{r}')$$

$$= \sum_{n_0=0}^{\infty} \int_{-\infty}^{\infty} d p \left[\gamma_00 + v_F(\pi \cdot \gamma) F_{n_0+1/2}^{s_00,s_{12}} + \Sigma^{s_00,s_{12}}_{n+1/2} \right] L_n(\xi)$$

$$+ -\frac{1}{4\pi \ell^2} \sum_{n_0=0}^{\infty} (1 + s_{12}) \psi_{n_0} F_{n+1}^{s_00} \gamma_000 \psi_{n_0'}(\mathbf{r}') L_n(\xi)$$

(A.15)
\[ G^{-1}(\omega; \mathbf{r}, \mathbf{r}') = e^{i\Phi(\mathbf{r}, \mathbf{r}')} \tilde{G}^{-1}(\omega; \mathbf{r} - \mathbf{r}') , \]  
\[ i \tilde{S}^{-1}(\omega; \mathbf{r}, \mathbf{r}') = e^{i\Phi(\mathbf{r}, \mathbf{r}')} \tilde{S}^{-1}(\omega; \mathbf{r} - \mathbf{r}') , \]
The final representation of the propagator can be written in terms of the location of the poles in equation (A.28),

\[ E_{0,\sigma} = \sigma \tilde{\Delta}_{0,\sigma} = \Delta_0 + \sigma \tilde{\Delta}_0, \]

(A.30)

where \( \tilde{\Delta}_{0,\sigma} = (\omega + \mu_{n,\sigma})^2 - E_{n,\sigma}^2 \), with \( \sigma = \pm 1 \). (A.29)

The general form of the gap equation is given in equation (32).

The gap equations can be equivalently rewritten as

\[ \frac{\mu_{n,\sigma} - \mu - \sigma \tilde{\Delta}_{n,\sigma} \delta_{n,\sigma}^0}{M_{n,\sigma}^2} + \frac{\mu_{n+1,\sigma} - \mu + \sigma \tilde{\Delta}_{n+1,\sigma} \delta_{n,\sigma}^0}{M_{n+1,\sigma}^2} = 0, \]

(B.2)

where \( f_{n,\sigma} \neq s_0 g_{n,\sigma} - 1 = \frac{\delta^2}{\pi} \int_0^\infty \frac{d\Omega}{2\pi} \int_0^\infty \frac{dk}{2\pi} \frac{kL_n^{(0)}(kl)}{k + \Pi(0, k)} \times \Omega + M_{n,\sigma} + \sigma \tilde{\Delta}_{n,\sigma} \delta_{n,\sigma}^0 + M_{n+1,\sigma} - \sigma \tilde{\Delta}_{n+1,\sigma} \delta_{n,\sigma}^0. \]

(B.3)

Note that the free propagator reads

\[ G(\omega; \mathbf{r}, \mathbf{r}') = e^{i\Phi(\mathbf{r}, \mathbf{r}')} \tilde{G}(\omega; \mathbf{r} - \mathbf{r}', \mathbf{r}'), \]

(A.34)

\[ \tilde{G}(\omega; \mathbf{r}, \mathbf{r}') = \frac{1}{2\pi^2} \sum_{n=0}^\infty \sum_{\sigma=\pm 1} \left\{ \frac{s_0(\omega + \mu_{n,\sigma}) - \tilde{\Delta}_{n,\sigma}}{(\omega + \mu_{n,\sigma})^2 - E_{n,\sigma}^2} \times [\delta_{n,\sigma}^0 L_{n,\sigma}(\xi) + \delta_{n,\sigma}^0 L_{n-1}(\xi)] \right\}. \]

\[ \mathcal{P}_{s_0,\sigma} \]

Note that the free propagator reads

\[ S(\omega; \mathbf{r}, \mathbf{r}') = e^{i\Phi(\mathbf{r}, \mathbf{r}') \tilde{S}(\omega; \mathbf{r} - \mathbf{r}', \mathbf{r})}, \]

(A.36)

\[ \tilde{S}(\omega; \mathbf{r}) = i \frac{e^{-i/2}}{2\pi^2} \sum_{n=0}^\infty \sum_{\sigma=\pm 1} \left\{ \frac{s_0(\omega + \mu_{n,\sigma}) + \delta_{n,\sigma}^1 L_{n,\sigma}(\xi) + \delta_{n,\sigma}^1 L_{n-1}(\xi)}{(\omega + \mu_{n,\sigma})^2 - 2n\nu_0^2/\ell^2} \times \delta_{n,\sigma}^0 \right\}. \]

(B.4)

\[ L_{m,n}^{(0)} = \frac{1}{2\ell} \int_0^\infty dr r e^{-i\xi^2/2} L_m \left( \frac{r^2}{2\ell^2} \right) L_n \left( \frac{r^2}{2\ell^2} \right) J_0(kr) \]

(A.58)

\[ = (-1)^{m+n} e^{-i\xi^2/2} L_m^{(n-m)} \left( \frac{k^2 \xi^2}{2} \right) L_n^{(m)} \left( \frac{k^2 \xi^2}{2} \right). \]

(B.5)

To obtain the results on the right-hand sides, we used the table integral 7.4222 of [50].

The gap equations can be equivalently rewritten as

\[ \mu_{n,\sigma} = \mu - \sigma \tilde{\Delta}_{n,\sigma}, \]

(B.6)
\[ \mu_{n,\sigma} = \mu + \sigma \bar{\Delta}_{n,\sigma} \]
\[ = -ia\epsilon L_n \sum_{n'=1}^{\infty} \kappa_{n',n-1}^{(0)} \int_{-\infty}^{\infty} \frac{d\omega'}{2\pi} M_{n',\sigma}, \quad \text{for } n \geq 1, \tag{B.7} \]
\[ f_{n,\sigma} = 1 + ia\epsilon L_n \sum_{n'=1}^{\infty} \frac{\kappa_{n',n-1}^{(1)}}{n} \int_{-\infty}^{\infty} \frac{d\omega'}{2\pi} M_{n',\sigma}, \quad \text{for } n \geq 1, \tag{B.8} \]

where \( \alpha = e^2/(\epsilon_0 v_F) \), \( \epsilon_L = v_F/\ell \), and

\[ \kappa_{m,n}^{(\rho)} = \int_0^{\infty} \frac{dk}{2\pi} \frac{k \ell E_{m,n}(k \ell)}{2 \pi k + \Pi(0,k)}, \quad \rho = 0, 1. \tag{B.9} \]

When the screening effects are neglected, i.e. \( \Pi(0,k) = 0 \), we can use the explicit form for \( L_{m,n}^{(\rho)} \) (with \( \rho = 0, 1 \)) in equations (B.4) and (B.5) and obtain the following analytical expressions for \( \kappa_{m,n}^{(\rho)} \) (with \( \rho = 0, 1 \)):

\[
\kappa_{m,n}^{(\rho)} \bigg|_{\Pi \to 0} = \int_0^{\infty} \frac{dx}{2\pi} L_{m,n}^{(\rho)}(x) \\
= \int_0^{\infty} \frac{dx}{2\pi} \int_0^{\infty} dt \ e^{-i t E_{m,n}(t)} L_{m,n}^{(\rho)}(t) J_0 \left( x \sqrt{2t} \right) \\
= \frac{1}{2\sqrt{\pi}} \int_0^{\infty} dt \ t^{\rho} e^{-t E_{m,n}(t)} L_{m,n}^{(\rho)}(t) \\
= \Gamma(\rho + 1/2) \Gamma(1/2 + n) \Gamma(1/2 + m) \nonumber \\
\times \sqrt{\pi} (\rho + 1/2 + n)! \\
\times F_2(-m,-n,-\rho+1/2,1/2-n,1/2-m,1/2-n;1) \\
= (-1)^{m+n} \sum_{k=0}^{\min(m,n)} \frac{\Gamma(\rho+1/2+k)}{\Gamma(1/2+k)!} \frac{1}{\Gamma(1/2-n+k)!} \\
\times \left[ (m-k)!(n-k)! \Gamma(1/2-m-k) \right]^2. \tag{B.11} \]

where we have used formula 2.19.14.15 of [51]. The values of \( \kappa_{m,n}^{(\rho)} \) (with \( \rho = 0, 1 \)) at small values of \( m \) and \( n \) are given in tables B.1 and B.2. The leading asymptotes for \( n \to \infty \) (at finite \( m \)) are

\[
\kappa_{m,n}^{(0)} \bigg|_{\Pi \to 0} \simeq \frac{1}{2\pi \sqrt{2n}} + \frac{2m-1}{8\pi (2n)^{3/2}} + \mathcal{O} \left( \frac{1}{n^{3/2}} \right) \quad \text{for } n \to \infty, \tag{B.12} \]

\[
\kappa_{m,n}^{(1)} \bigg|_{\Pi \to 0} \simeq \frac{(m+1)}{4\pi \sqrt{2n}} + \frac{(m+1)(3m-1)}{16\pi (2n)^{1/2}} + \mathcal{O} \left( \frac{1}{n^{3/2}} \right) \quad \text{for } n \to \infty. \tag{B.13} \]

The zero-temperature gap equations (B.6)–(B.9) are straightforwardly generalized to the case of non-zero temperature by making the replacement \( \Omega \to i\Omega \equiv i \tau T(2m+1) \) and using the Matsubara sums instead of the frequency integrations,

\[
\int \frac{d\Omega}{2\pi} \to \int_0^\infty \left( \cdots \right) \to i T \sum_{m=-\infty}^{\infty} \left( \cdots \right). \tag{B.14} \]

Then, we use the following table sums:

\[
T \sum_{m=-\infty}^{\infty} \frac{1}{(\Omega_{m} - i\mu)^2 + a^2} = \frac{1 - n_F(\alpha + \mu) - n_F(\alpha - \mu)}{2a}, \tag{B.15} \]

\[
T \sum_{m=-\infty}^{\infty} \frac{i\Omega_{m} + \mu}{(\Omega_{m} - i\mu)^2 + a^2} = \frac{n_F(\alpha + \mu) - n_F(\alpha - \mu)}{2}, \tag{B.16} \]

and derive the finite-temperature gap equations

\[
\mu_{n,\sigma} = \mu - \sigma \bar{\Delta}_{n,\sigma} \\
\kappa_{m,n}^{(0)} = \frac{\alpha \epsilon L_n}{2} \sum_{n'=1}^{\infty} \kappa_{n',n-1}^{(0)} \left[ n_F(E_{n',\sigma} - \mu_{n',\sigma}) - n_F(E_{n',\sigma} + \mu_{n',\sigma}) \right] \\
\kappa_{m,n}^{(1)} = \frac{\alpha \epsilon L_n}{2} \sum_{n'=1}^{\infty} \kappa_{n',n-1}^{(1)} \left[ n_F(E_{n',\sigma} - \mu_{n',\sigma}) - n_F(E_{n',\sigma} + \mu_{n',\sigma}) \right] \tag{B.17} \]

\[
\mu_{n,\sigma} = \mu - \sigma \bar{\Delta}_{n,\sigma} \\
\kappa_{m,n}^{(0)} = \frac{\alpha \epsilon L_n}{2} \sum_{n'=1}^{\infty} \kappa_{n',n-1}^{(0)} \left[ n_F(E_{n',\sigma} - \mu_{n',\sigma}) - n_F(E_{n',\sigma} + \mu_{n',\sigma}) \right] \\
\kappa_{m,n}^{(1)} = \frac{\alpha \epsilon L_n}{2} \sum_{n'=1}^{\infty} \kappa_{n',n-1}^{(1)} \left[ n_F(E_{n',\sigma} - \mu_{n',\sigma}) - n_F(E_{n',\sigma} + \mu_{n',\sigma}) \right] \tag{B.18} \]

Table B.1. Values of \( \kappa_{m,n}^{(0)} \) when the effects of the polarization tensor are neglected.

| m   | n   | \( \kappa_{m,n}^{(0)} \) |
|-----|-----|--------------------------|
| 0   | 1   | 1/2                      |
| 1   | 1   | 1/2                      |
| 2   | 1   | 1/2                      |
| 3   | 1   | 1/2                      |
| 4   | 1   | 1/2                      |
| 5   | 1   | 1/2                      |

Table B.2. Values of \( \kappa_{m,n}^{(1)} \) when the effects of the polarization tensor are neglected.

| m   | n   | \( \kappa_{m,n}^{(1)} \) |
|-----|-----|--------------------------|
| 0   | 1   | 1/2                      |
| 1   | 1   | 1/2                      |
| 2   | 1   | 1/2                      |
| 3   | 1   | 1/2                      |
| 4   | 1   | 1/2                      |
| 5   | 1   | 1/2                      |
\[ f_{n, \sigma} = 1 + \frac{\alpha \xi}{2} \sum_{n' = 1}^{\infty} \frac{k_{n', 1-n, n}}{n} E_{n', \sigma}^{\ell} [1 - n_F(E_{n', \sigma} - \mu_{n', \sigma})] - n_F(E_{n', \sigma} + \mu_{n', \sigma}) \], for \( n \geq 1 \). \hspace{1cm} (B.19)

\[ g_{n, \sigma} = \frac{\alpha \xi}{2} \sum_{n' = 1}^{\infty} \frac{k_{n', 1-n, n}}{n} E_{n', \sigma}^{\ell} [1 - n_F(E_{n', \sigma} - \mu_{n', \sigma})] - n_F(E_{n', \sigma} + \mu_{n', \sigma}) \], for \( n \geq 1 \). \hspace{1cm} (B.20)

Note that the \( n' = 0 \) term in the first equation can be rewritten in a simpler form:

\[ \frac{\alpha \xi}{2} \sum_{n' = 1}^{\infty} \frac{k_{n', 1-n, n}}{n} E_{n', \sigma}^{\ell} [1 - n_F(E_{n', \sigma} - \mu_{n', \sigma})] - 1 = \frac{\alpha \xi}{2} \kappa_{0,0}^{(0)} \tan \left( \frac{\mu_0 - \sigma \tilde{\Delta}_{0, \sigma}}{2T} \right) \]. \hspace{1cm} (B.21)

(As defined in equation (A.30), \( E_{n, \sigma} = \sigma \tilde{\Delta}_{0, \sigma} \)) Separating the LLL from the higher LLLs, we derive

\[ \mu_0^{\text{eff}} - \mu - \sigma \tilde{\Delta}_{0, \sigma}^{\text{eff}} = \frac{\alpha \xi}{2} \kappa_{0,0}^{(0)} \tan \left( \frac{\mu_0 - \sigma \tilde{\Delta}_{0, \sigma}^{\text{eff}}}{2T} \right) + \frac{\alpha \xi}{2} \sum_{n' = 1}^{\infty} \frac{k_{n', 1-n, n}}{n} E_{n', \sigma}^{\ell} [1 - n_F(E_{n', \sigma} - \mu_{n', \sigma})] - n_F(E_{n', \sigma} + \mu_{n', \sigma}) \]

\[ - \sigma \tilde{\Delta}_{n, \sigma}^{\text{eff}} \frac{1}{E_{n, \sigma}} [1 - n_F(E_{n', \sigma} - \mu_{n', \sigma})] - n_F(E_{n', \sigma} + \mu_{n', \sigma})] \], \hspace{1cm} (B.22)

which is equivalent to the following set of gap equations for the LLL parameters:

\[ \mu_0^{\text{eff}} - \mu = \frac{\alpha \xi}{2} \kappa_{0,0}^{(0)} [n_F(\tilde{\Delta}_{0, \sigma} - \mu_{0, \sigma}) - n_F(\tilde{\Delta}_{0, \sigma} + \mu_{0, \sigma})] + \frac{\alpha \xi}{2} \sum_{n' = 1}^{\infty} \frac{k_{n', 1-n, n}}{n} E_{n', \sigma}^{\ell} [1 - n_F(E_{n', \sigma} - \mu_{n', \sigma})] - n_F(E_{n', \sigma} + \mu_{n', \sigma}) \]

\[ + \frac{\alpha \xi}{2} \sum_{n' = 1}^{\infty} \frac{k_{n', 1-n, n}}{n} E_{n', \sigma}^{\ell} [1 - n_F(E_{n', \sigma} - \mu_{n', \sigma})] - n_F(E_{n', \sigma} + \mu_{n', \sigma})] + \frac{\alpha \xi}{2} \sum_{n' = 1}^{\infty} \frac{k_{n', 1-n, n}}{n} E_{n', \sigma}^{\ell} [1 - n_F(E_{n', \sigma} - \mu_{n', \sigma})] - n_F(E_{n', \sigma} + \mu_{n', \sigma})] \]

\[ \Delta_{n, \sigma} = \frac{\alpha \xi}{2} \kappa_{0,0}^{(0)} [2n_F(\sigma \tilde{\Delta}_{0, \sigma}^{\text{eff}} - \mu_0^{\text{eff}}) - 1] \]

\[ - \sigma \tilde{\Delta}_{n, \sigma}^{\text{eff}} \frac{1}{E_{n, \sigma}} [1 - n_F(E_{n', \sigma} - \mu_{n', \sigma})] - n_F(E_{n', \sigma} + \mu_{n', \sigma})] \], \hspace{1cm} (B.26)

\[ \tilde{\Delta}_{0, \sigma}^{\text{eff}} = \frac{\alpha \xi}{2} \kappa_{0,0}^{(0)} [1 - n_F(\tilde{\Delta}_{0, \sigma}^{\text{eff}} - \mu_0^{\text{eff}}) - n_F(\tilde{\Delta}_{0, \sigma}^{\text{eff}} + \mu_0^{\text{eff}})] \]

\[ + \frac{\alpha \xi}{2} \sum_{n' = 1}^{\infty} \frac{k_{n', 1-n, n}}{n} E_{n', \sigma}^{\ell} [1 - n_F(E_{n', \sigma} - \mu_{n', \sigma})] - n_F(E_{n', \sigma} + \mu_{n', \sigma})] + \frac{\alpha \xi}{2} \sum_{n' = 1}^{\infty} \frac{k_{n', 1-n, n}}{n} E_{n', \sigma}^{\ell} [1 - n_F(E_{n', \sigma} - \mu_{n', \sigma})] - n_F(E_{n', \sigma} + \mu_{n', \sigma})] \]

\[ \text{Appendix C. Free energy density} \]

Here we calculate the free energy density \( \Omega \) in monolayer graphene with dynamically generated self-energy corrections and wavefunction renormalization by following the same approach, based on the Baym–Kadanoff formalism [52], as that used in [27] for the case of a model with a contact four-fermion interaction. In the model with Coulomb interaction studied in this paper, the corresponding effective
The diagrammatic form for the effective action at two loops.

The effective action $\Gamma$ takes the form

$$\Gamma(G, D) = -i \text{Tr} [\ln G^{-1} + S^{-1} G - 1]$$

$$+ \frac{i}{2} [\ln D_{-1}^{-1} + D_{0}^{-1} D - 1]$$

$$- \frac{i e^2}{2} \int d^3 u \int d^3 u' \text{tr}[\gamma^0 \gamma^i G(u, u') \gamma^0 G(u', u)] D(u - u'),$$

where $u = (t, \mathbf{r})$. The trace, the logarithm and the product $S^{-1} G$ are taken in the functional sense, and $G = \text{diag}(G_\uparrow, G_\downarrow)$. The diagrammatic form of this effective action is shown in figure C.1. It is instructive to compare this action with the analogous action in the model with a contact four-fermion interaction [27]. In contrast to that model, here we have only one type of two-loop diagram. It is responsible for the exchange interaction. The other 'dumbbell' diagram (i.e. two fermion loops connected by the free photon propagator), which would be responsible for the direct (Hartree) interaction, does not appear in the present model with the gauge interaction. The absence of such a diagram is a formal consequence of the Gauss neutrality condition

$$j^{0}_{\text{ext}}(u) = e \text{tr}[\gamma^0 G(u, u)] = 0,$$

where $j^{0}_{\text{ext}}(u)$ is the charge density of the substrate and carbon ions that compensate for the overall nonzero charge density of free carriers in the graphene monolayer.

The free energy density $\Omega$ is expressed through $\Gamma$ as

$$\Omega = -\Gamma / TV,$$

where $TV$ is a space–time volume. When the fermion propagator satisfies the gap equation (32), the expression for the effective action $\Gamma$ simplifies, i.e.

$$\Gamma = -i \text{Tr} [\ln G^{-1} + \frac{i}{2}(S^{-1} G - 1)] + \frac{i}{2} [\ln D_{-1}^{-1} + D_{0}^{-1} D - 1].$$

Up to the one-loop photon contribution, which we will neglect in the following, this is formally the same result as that in [27]. Therefore, we can use a similar derivation and change only the form of the Green’s function $G_i(u, u')$ in order to take into account that a wavefunction renormalization is nontrivial here and that all dynamical parameters are functions of the Landau index. We finally obtain

$$\Omega = -\frac{1}{4 \pi e^2} \sum_{\sigma = \pm 1} \left\{ \text{sgn}(\mu_{0, \sigma}) (\mu_{0, \sigma} + \mu - \sigma \tilde{\Delta}_{0, \sigma}) \right\}$$

$$\times \theta(|\mu_{0, \sigma}| - |\tilde{\Delta}_{0, \sigma}|) + \text{sgn}(\tilde{\Delta}_{0, \sigma}) (\tilde{\Delta}_{0, \sigma} - \sigma (\mu + \mu_{0, \sigma}))$$

$$\times \theta(|\tilde{\Delta}_{0, \sigma}| - |\mu_{0, \sigma}|)$$

$$+ 2 \sum_{n=1}^{\infty} \left[ \text{sgn}(\mu_{n, \sigma}) (\mu_{n, \sigma} + \mu) \theta(|\mu_{n, \sigma}| - |E_{n, \sigma}|) \right.$$}

$$+ \frac{E_{n, \sigma}^2 + 2 n f_{n, \sigma} \tilde{e}_{n}^2}{E_{n, \sigma}} \theta(|E_{n, \sigma}| - |\mu_{n, \sigma}|) - 2\sqrt{2n} \tilde{e}_{n} \right],$$

where we introduced the function $\theta(n - 1)$, which is defined so that $\theta(n - 1) = 0$ for $n < 0$ and $\theta(n - 1) = 1$ for $n \geq 1$. At a non-zero temperature, we make the replacement $\omega \sim \imath \omega_n \equiv \imath T (2m + 1)$ and use Matsubara sums instead of the frequency integrations, and obtain

$$\Omega = -\frac{1}{4 \pi e^2} \sum_{\sigma = \pm 1} \left\{ \text{sgn}(\mu_{0, \sigma}) (\mu_{0, \sigma} + \mu - \sigma \tilde{\Delta}_{0, \sigma}) \right\}$$

$$\times \theta(|\mu_{0, \sigma}| - |\tilde{\Delta}_{0, \sigma}|) + \frac{E_{n, \sigma}^2 + 2 n f_{n, \sigma} \tilde{e}_{n}^2}{E_{n, \sigma}} \theta(|E_{n, \sigma}| - |\mu_{n, \sigma}|) - 2\sqrt{2n} \tilde{e}_{n} \right],$$

C.4

$$\text{sgn}(\mu_{n, \sigma}) (\mu_{n, \sigma} + \mu) \theta(|\mu_{n, \sigma}| - |E_{n, \sigma}|)$$

$$+ \frac{E_{n, \sigma}^2 + 2 n f_{n, \sigma} \tilde{e}_{n}^2}{E_{n, \sigma}} \theta(|E_{n, \sigma}| - |\mu_{n, \sigma}|) - 2\sqrt{2n} \tilde{e}_{n} \right],$$

C.5

References

[1] Novoselov K S, Geim A K, Morozov S V, Jiang D, Zhang Y, Dubonos S V, Grigorieva I V and Firsov A A 2004 Science 306 666
[2] Semenoff G W 1984 Phys. Rev. Lett. 53 2449
[3] Novoselov K S, Geim A K, Morozov S V, Jiang D, Katsnelson M I, Grigorieva I V, Dubonos S V and Firsov A A 2005 Nature 438 197
[4] Zhang Y, Tan Y-W, Stöhrer H L and Kim P 2005 Nature 438 201
[5] Zheng Y and Ando T 2002 Phys. Rev. B 65 245420
[6] Gusynin V P and Sharapov S G 2005 Phys. Rev. Lett. 95 146480
[7] Gusynin V P and Sharapov S G 2006 Phys. Rev. B 73 245411
[8] Peres N M R, Guinea F and Castro Neto A H 2006 Phys. Rev. B 73 125411
[9] Haldane F D M 1988 Phys. Rev. Lett. 61 2015
[10] Khveshchenko D V 2001 Phys. Rev. Lett. 87 206401
[11] Khveshchenko D V 2001 Phys. Rev. Lett. 87 246802
[12] Gorbar E V, Gusynin V P, Miransky V A and Shovkovy I A 2002 Phys. Rev. B 66 045108
[13] Sharapov S G, Gusynin V P and Beck H 2004 Phys. Rev. B 69 075104
[14] Zhang Y, Jiang Z, Small J P, Purewal M S, Tan Y-W, Fazlollahi M, Chudow J D, Jaszczak J A, Stormer H L and Kim P 2006 Phys. Rev. Lett. 96 136806
[15] Jiang Z, Zhang Y, Stormer H L and Kim P 2007 Phys. Rev. Lett. 99 106802
[16] Yu X, Skachko I, Durrr F, Luican A and Andrei E Y 2009 Nature 462 192
[17] Bolotin K I, Ghalhari F, Shulman M D, Stormer H L and Kim P 2009 Nature 462 196
[18] Nomura K and MacDonald A H 2006 Phys. Rev. Lett. 96 256602
[19] Yang K, Das S and Sarma MacDonald A H 2006 Phys. Rev. B 74 075243

16
