PLURICLOSED FLOW ON GENERALIZED KÄHLER MANIFOLDS WITH
SPLIT TANGENT BUNDLE

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Abstract. We show that the pluriclosed flow preserves generalized Kähler structures with the extra condition \( [J_+, J_-] = 0 \), a condition referred to as “split tangent bundle.” Moreover, we show that in this case the flow reduces to a nonconvex fully nonlinear parabolic flow of a scalar potential function. We prove a number of a priori estimates for this equation, including a general estimate in dimension \( n = 2 \) of Evans-Krylov type requiring a new argument due to the nonconvexity of the equation. The main result is a long time existence theorem for the flow in dimension \( n = 2 \), covering most cases. We also show that the pluriclosed flow represents the parabolic analogue to an elliptic problem which is a very natural generalization of the Calabi conjecture to the setting of generalized Kähler geometry with split tangent bundle.

1. Introduction

A generalized Kähler structure on a compact manifold \( M \) is a triple \((g, J_+, J_-)\) of a Riemannian metric and two integrable complex structures \( J_\pm \) so that
\[
\begin{align*}
d^c_+ \omega_+ &= -d^c_- \omega_- \\
d^c_+ \omega_- &= -d^c_- \omega_+ = 0.
\end{align*}
\]
These equations first arose in \cite{9} through investigations of supersymmetric sigma models. Later these equations were given a purely geometric context through Hitchin’s generalized geometric structures \cite{13}, \cite{10}. One interpretation of the generalized Kähler condition is as a pair of “pluriclosed” or “strong Kähler with torsion (SKT)” structures, satisfying further compatibility conditions. In \cite{20} \cite{21} the author and Tian introduced a parabolic flow of pluriclosed structures. Later we discovered that this flow preserves generalized Kähler structure, suitably interpreted \cite{22}. A crucial observation of \cite{22} is that in order to preserve generalized Kähler structure the complex structures \( J_\pm \) must themselves flow as well. The construction of this flow will be reviewed in \cite{22}.

In this paper we analyze this flow in the special case that the generalized Kähler structure satisfies the further condition \( [J_+, J_-] = 0 \). The first main result is that with this extra initial condition, the complex structures \( J_\pm \) remain fixed, and the flow moreover reduces to a flow of a scalar potential satisfying a nonconvex fully nonlinear parabolic equation.

**Theorem 1.1.** Let \((M^{2n}, g, J_\pm)\) be a generalized Kähler manifold satisfying \([J_+, J_-] = 0\). Let \( \omega_t \) denote the solution to pluriclosed flow on \((M, J_\pm)\) with initial condition \( \omega_0 = g(J_+, \cdot) \). Then \( \omega_t \) is a generalized Kähler metric on \((M, J_\pm)\). Moreover, in this setting the pluriclosed flow reduces to a fully nonlinear parabolic flow of a scalar potential function \( f \) (see \cite{3.4}).

**Remark 1.2.** The evolution equation for \( f \) is closely related to generalized Calabi-Yau equations which have appeared in physics literature (\cite{4} \cite{13} \cite{11} \cite{12} \cite{18}). In a way, Theorem 1.1 treats the generalized Kähler-Ricci flow of \cite{22} in the “most classical case” of generalized Kähler manifolds.

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described in [9]. Already in [9] a local potential in the case \([J_+, J_-] = 0\) is described, which is essentially the one we use here. In later works, culminating in [12], a local potential for generalized Kähler structures is shown to exist in full generality. A fascinating direction for future work is to reduce the generalized Kähler-Ricci flow of [22] to a flow of a potential function.

Theorem 1.1 allows us to make a more tractable refinement of the existence conjecture for pluriclosed flow ([21] Conjecture 5.2), where it is suggested that the natural cohomological obstructions to the long time existence of the flow are the only obstructions (see Conjecture 3.11). Moreover, it also suggests a very natural analogue of Calabi’s conjecture [5] in this setting, which directly generalizes the classical conjecture in the Kähler setting, resolved by Yau [25]. While we do not address this Calabi-type conjecture directly here, it is clear that the a priori estimates we establish can be used in this setting as well. The second main theorem of this paper is to obtain the smooth long time existence of the pluriclosed flow on many generalized Kähler complex surfaces, mostly confirming Conjecture 3.11.

Theorem 1.3. Let \((M^4, g_0, J_+)\) be a generalized Kähler surface satisfying \([J_+, J_-] = 0\) and \(\text{Rank}_{J_\pm} = 1\). Suppose \((M^4, J_+)\) is biholomorphic to one of:

1. A ruled surface over a curve of genus \(g \geq 1\).
2. A bi-elliptic surface,
3. An elliptic fibration of Kodaira dimension 1,
4. A compact complex surface of general type, whose universal cover is biholomorphic to \(\mathbb{H} \times \mathbb{H}\),
5. An Inoue surface of type \(S_M\).

In case (1) the maximal existence time of the flow is determined by cohomological data associated to \(g_0\) (see §3.3). In all other cases the solution to pluriclosed flow with initial condition \(g_0\) exists on \([0, \infty)\). In the case of bi-elliptic surfaces the flow converges exponentially to a flat metric. In the case of surfaces of general type, the normalized pluriclosed flow exists for all time and converges to the unique Kähler-Einstein metric on \((M, J_+)\).

Remark 1.4. Generalized Kähler surfaces satisfying \([J_+, J_-] = 0\) with \(\text{Rank}_{J_\pm} = 1\) (see Definition 2.2) were classified by Apostolov-Gualtieri [1], building on work of Beauville [2]. The only cases we are not able to treat are ruled surfaces over \(\mathbb{C}P^1\) and the Hopf surfaces. The reason for this is the lack of a certain a priori estimate which requires certain conditions on a background metric to be satisfied (see the proof in §7). It seems likely that the flow always exists smoothly for the natural conjectural existence time (see Conjecture 3.11), and indeed many of our estimates, including the crucial Harnack estimate, apply in full generality.

Remark 1.5. In the cases \(\text{Rank}_{J_\pm} \in \{0, 2\}\), it follows that in fact \(J_+ = \pm J_-\), and generalized Kähler metrics are automatically Kähler. Thus solutions to pluriclosed flow in this setting are solutions to Kähler-Ricci flow, where for instance the sharp long time existence result is already known in [23].

In the course of the proof of Theorem 1.3 we establish a number of a priori estimates. First we establish a general \(L^\infty\) estimate for the potential function along the flow (Lemma 5.1) in any
By exploiting special structure in the case $n = 2$ we establish a uniform bound on the time derivative of the potential function as well (Proposition 5.2). One crucial step is to establish an upper bound for the metric in the presence of a lower bound. This requires the introduction of a kind of “torsion potential” which has a miraculously simply evolution equation (Lemma 4.5), and can be used to control the difficult torsion terms appearing in various evolution equations. This torsion potential has applications to understanding pluriclosed flow in full generality, and will be expounded upon in future work. The final key estimate is a general Evans-Krylov type estimate for the pluriclosed flow in this setting for dimension $n = 2$ (cf. Theorem 6.1).

Here is an outline of the rest of this paper: in §2 we recall the relationship of pluriclosed flow and generalized Kähler geometry, and also give some background on generalized Kähler structures with $[J_+, J_-] = 0$. In §3 we establish that the pluriclosed flow preserves the generalized Kähler condition in the “natural gauge,” and give the proof of Theorem 1.1. We also motivate a sharp local existence conjecture specializing (21 Conjecture 5.2) to this special case. In §4 we record a number of evolution equations along the pluriclosed flow special to this setting. Next in §5 we obtain a number of a priori estimates on the potential function, its derivatives, and the metric tensor. Section 6 has the proof of Evans-Krylov type regularity for our equation in dimension $n = 2$. We bring these estimates together to prove Theorem 1.3 in §7.

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2. Background

2.1. Pluriclosed flow. Let $(M^{2n}, J)$ be a complex manifold. Suppose $g$ is a Riemannian metric compatible with $J$, with Kähler form $\omega = g(J\cdot, \cdot)$. We say that the metric $g$ is pluriclosed if

$$\partial \bar{\partial} \omega = 0.$$  

In [20] the author and Tian defined a parabolic flow of Hermitian metrics which preserves the pluriclosed condition. To define it, let

$$P(\omega) := -\partial^c \omega - \bar{\partial}^c \omega - \sqrt{-1} \frac{\partial}{2} \partial \bar{\partial} \log \det g,$$

where the last term stands in for the representative of the first Chern class arising from the Chern connection. The pluriclosed flow is then the evolution equation

$$\frac{\partial}{\partial t} \omega = -P(\omega).$$

One can phrase this flow equation using the curvature of the Chern connection. In particular, let $T$ denote the torsion of the Chern connection, and let $S = \text{tr}_\omega \Omega$ denote the curvature endomorphism of the Chern connection. Lastly, set

$$Q_{ij} = g^{ik} g^{jm} T_{ikm} T_{jlm}.$$  

It follows ([20] Proposition 3.3) that the associated Hermitian metrics $g_t$ satisfy

$$\frac{\partial}{\partial t} g = -S + Q.$$
We also here define the normalized flow. We say that a one-parameter family of pluriclosed metrics \( g_t \) satisfies the normalized pluriclosed flow if

\[
\frac{\partial}{\partial t} \omega = -P(\omega) - \omega.
\] (2.4)

In some applications of Ricci flow the “normalized flow” means to fix the volume to be constant along the flow. The normalization of (2.4) generally does not preserve the volume. However, for various applications in Kähler Ricci flow the scaling choice as in (2.4) is useful, and is the one most useful to us here.

2.2. Pluriclosed flow and generalized Kähler geometry. We briefly recall the construction of solutions to the generalized Kahler Ricci flow. Let \((M^{2n}, g, J_{\pm})\) be a generalized Kahler structure. Let \(\omega_{\pm}\) denote the associated Kahler forms, both of which are pluriclosed. Let \(\omega_{\pm}(t)\) denote the solution to pluriclosed flow with these initial conditions. By \([21]\) we know that the associated metrics \(g_{\pm}(t)\) and three-forms \(H_{\pm}(t) = d_{\pm}^c \omega_{\pm}\) are solutions to the (gauge-modified) \(B\)-field flow:

\[
\frac{\partial}{\partial t} g_{\pm} = -2 Rc + \frac{1}{2} H + L_{\theta^t_{\pm}} g
\]  
\[
\frac{\partial}{\partial t} H_{\pm} = \Delta_{\theta} H + L_{g^t_{\pm}} H
\] (2.5)

Now let \((\phi_{\pm})_t\) denote the one-parameter families of diffeomorphisms generated by \(\theta^t_{\pm}\), respectively. It follows that \((\phi_{\pm})_t^* (g_{\pm}(t), (\phi_{\pm})_t^* H_{\pm}(t))\) are both solutions to the \(B\)-field flow. Moreover, one observes that \(((\phi_-)_t^* (g_-)_t, (\phi_-)_t^* H_-(T))\) is ALSO a solution to the \(B\)-field flow. Since \(g_-(0) = g_-\) and \(H_+(0) = -H_-(0)\), by uniqueness of solutions to (2.5) one has \((g_+(t), H_+(t)) = (g_-(t), -H_-(t))\) for all \(t\) such that the flows exist. Indeed, it furthermore follows that \(g\) is compatible with \((J_{\pm})_t := (\phi_{\pm})_t^* J_{\pm}\), and is generalized Kähler with respect to these two complex structures. These complex structures \((J_{\pm})_t\) are moving via diffeomorphism, but when the evolution equation is expressed with respect to the “moving gauge,” i.e. in terms of the pullback data, one observes that they in fact satisfy a PDE of the form

\[
\frac{\partial}{\partial t} J_{\pm} = \Delta J + \text{Rm} * J + DJ^{*2},
\]

where one can consult \([22]\) Proposition 3.1 for the precise formula.

The presence of evolving complex structures makes this flow of generalized Kähler structures perhaps a bit intimidating. One purpose of Theorem [11] is to show that if the initial generalized Kähler structure satisfies \([J_+, J_-] = 0\), then we in fact have that pluriclosed flow remains compatible with these given \(J_{\pm}\), and thus we do not need to consider the diffeomorphism modified flows, and hence we can treat \(J_{\pm}\) as fixed.

2.3. Generalized Kähler manifolds with commuting complex structures. In this subsection we recall some fundamental structural results concerning generalized Kähler manifolds with commuting complex structures. Most of our discussion here is adopted from \([1]\), which the reader can consult for further information.

Let \((M^{2n}, g, J_{\pm})\) be a generalized Kähler manifold satisfying the further condition

\[
[J_+, J_-] = 0.
\] (2.6)

Let

\[
\Pi := J_+ J_- \in \text{End}(TM).
\] (2.7)
Using (2.6), one easily shows that \( \Pi^2 = \text{Id} \). It follows that the eigenvalues of \( \Pi \) are \( \pm 1 \), and the corresponding eigenspace decomposition yields a splitting of the tangent bundle

\[
TM = T_+ M \oplus T_- M.
\]

This is the source of the terminology, “generalized Kähler manifolds with split tangent bundle.” More precisely, one can define

\[
T_\pm M = \ker(Q \mp I) = \ker(J_\pm \mp J_-).
\]

We observe here an important general convention, which is that from this point on many objects will be labeled with \( \pm \), and this will almost always refer to the splitting above, and NOT to the usage of distinct complex structures \( J_\pm \). In §2.2 we referred to the operators \( \partial \pm \), which do in fact refer to \( J_\pm \), and these will arise a few more times below. With this one exception, the notation \( \pm \) signifies the decomposition (2.8), and in any case there is never any actual overlap of notation, only a potential for confusion which it seems we are stuck with by existing literature.

In particular, associated to (2.8) one has an associated decomposition \( d = \delta_+ + \delta_- \) induced by \( T^* M = T^*_+ M \oplus T^*_+ M \). This splitting further refines the decomposition of differential forms into types with respect to \( J_+ \), and hence defines a decomposition

\[
d = \delta_+ + \delta_- = \partial_+ + \bar{\partial}_+ + \partial_- + \bar{\partial}_-.
\]

Also, we can decompose the operator \( \partial \bar{\partial} : C^\infty \to \Lambda^{1,1} \) into types, with the only two relevant pieces being

\[
\delta_\pm \delta_\pm^c = \pi_\pm \sqrt{-1} \partial \bar{\partial},
\]

where \( \pi_\pm \) denotes the projections onto \( \Lambda^{1,0}_\pm \Lambda^{0,1}_\pm \), respectively. Next we record a theorem showing that generalized Kähler metrics have a very nice structure with respect to this splitting.

**Theorem 2.1.** ([1] Theorem 4) Let \( (M, g, J_+, J_-) \) be a generalized Kähler manifold with \( [J_+, J_-] = 0 \). Let \( \Pi = J_+ J_- \). Then the \( \pm 1 \)-eigenspaces of \( \Pi \) are \( g \)-orthogonal \( J_\pm \)-holomorphic foliations on whose leaves \( g \) restricts to a Kähler metric.

Lastly, let us record a useful definition.

**Definition 2.2.** Let \( (M^{2n}, g, J_\pm) \) be a generalized Kähler manifold with \( [J_+, J_-] = 0 \). The rank of \( (J_+, J_-) \) is

\[
\text{Rank}_{J_\pm} := \dim_{\mathbb{C}} T^C_\pm M.
\]

Observe that if \( \text{Rank}_{J_\pm} \in \{0, n\} \), then \( J_+ = \pm J_- \), and the structure is Kähler.

2.4. Adapted local coordinates. In this subsection we describe local coordinates which are adapted to the local product decomposition \( TM = T_+ M \oplus T_- M \) described above. This local description of generalized Kähler structures with \( [J_+, J_-] = 0 \) was first discovered in physical investigations into supersymmetry equations [11], where it is frequently referred to as “bihermitian local product geometry.”

**Lemma 2.3.** Let \( (M^{2n}, g, J_\pm) \) be a generalized Kähler structure satisfying \( [J_+, J_-] = 0 \). Given \( p \in M \) there exist local complex coordinates adapted to the transverse foliations \( T_\pm \). That is, there is a neighborhood \( U = U_1 \times U_2 \subset \mathbb{C}^{n_1} \times \mathbb{C}^{n_2} \) such that \( T_- U = TU_1, T_+ U = TU_2 \).

**Proof.** This follows from Theorem 2.1, see ([1] §3).
Let us now describe the construction of these coordinates. As the eigenspaces of $\Pi$ are integrable, one can choose real coordinates $(x_1, \ldots, x_{2m}, x_{2m+1}, \ldots, x_{2n})$ so that $\Pi$ locally takes the form

$$\Pi = \begin{pmatrix} I_{2m} & 0 \\ 0 & -I_{2n-2m} \end{pmatrix},$$

where here $2m = \dim \mathbb{R} T_\pm M$. As the complex structures preserve the eigenspaces of $\Pi$, we can refine these coordinates so that the complex structures take the form

$$J_{\pm} = \begin{pmatrix} \pm J_m \\ 0 \end{pmatrix}, \quad J_{n-m} \begin{pmatrix} 0 & J_{n-m} \\ J_{n-m} & 0 \end{pmatrix},$$

where

$$J_1 := \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}, \quad J_k := \begin{pmatrix} J_1 & \cdots \\ & \ddots \end{pmatrix}.$$

We thus have complex coordinates

$$z_i = x_{2i-1} + e^{-i/2} x_{2i},$$

which observe are only naturally oriented for the complex structure $J_\pm$. All of our computations in local complex coordinates are on the complex manifold $(M, J_\pm)$.

**Remark 2.4.** As noted above, when performing local coordinate calculations with these coordinates, dummy indices which are decorated with a $\pm$ will refer to only to summations over vectors spanning $T^{\pm}_C M$, respectively. More generally, a quantity like $g_{\alpha \pm \beta \pm}$ means the metric evaluated at two vectors $\frac{\partial}{\partial z^\alpha \pm}, \frac{\partial}{\partial z^\beta \pm} \in T^{\pm}_C M$.

With this decomposition in hand we can write a local coordinate expression for the operators $\delta_{\pm} \delta_{\pm}^\pm$. In particular, one has

$$\delta_{\pm} \delta_{\pm}^\pm f = f_{,\alpha_\pm \beta_\pm} \sqrt{-1} dz_{\pm_\alpha} \partial_{\beta_\pm}.$$

To finish, we observe how the structure of a generalized Kähler metric is reflected in these coordinates. In particular, since the decomposition $TM = T_+ M \oplus T_- M$ is orthogonal for any generalized Kähler metric, it follows that, locally,

$$g_{\alpha_+ \beta_-} = 0.$$

Also, the partial Kähler condition of Theorem [2.11] implies that in these coordinates one has, locally,

$$g_{\alpha_+ \beta_-} - g_{\mu_+ \bar{\beta}_-, \alpha_+} = g_{\alpha_+ \bar{\beta}_+, \mu_+} - g_{\alpha_+ \bar{\mu}_+, \bar{\beta}_+} = 0$$

$$g_{\alpha_- \beta_-} - g_{\mu_- \bar{\beta}_-, \alpha_-} = g_{\alpha_- \bar{\beta}_-, \mu_-} - g_{\alpha_- \bar{\mu}_-, \bar{\beta}_-} = 0.$$

Together (2.11) and (2.12) imply that the only nonvanishing components of the torsion are

$$T_{\alpha_+ \beta_-} = g_{\beta_- \bar{\beta}_-, \alpha_+}, \quad T_{\alpha_- \beta_+} = g_{\beta_+ \bar{\beta}_+, \alpha_-}$$

Finally, the pluriclosed condition combined with (2.11) implies the useful coordinate identity

$$g_{\alpha_+ \bar{\beta}_+, \mu_-} + g_{\mu_- \bar{\beta}_-, \alpha_+} \bar{\beta}_+ = 0.$$
3. PLURICLOSED FLOW AND COMMUTING GENERALIZED KÄHLER GEOMETRY

In this section we establish that pluriclosed flow on generalized Kähler manifolds with split tangent bundle preserves the generalized Kähler condition, without the gauge fixing issue described in §2.2. The first step is to establish a curvature identity in this setting which expresses $P$ in terms of the first Chern classes of $T^+_M$. We can then use this to reduce the pluriclosed flow to a potential function flow in §3.2 and motivate our Calabi-type conjecture in §3.4.

3.1. A curvature identity. To begin, note from Theorem 2.11 that $T^+_M$ are indeed holomorphic vector bundles over $M$. Moreover, as these are subbundles of $T^c_M$, by restriction, any generalized Kähler metric $g$ induces Hermitian metrics on these two bundles. In the adapted local coordinates the local coefficients of these two metrics are $q_{\alpha \pm \beta \pm}$. These metrics of course have associated curvature tensors, and in particular given a generalized Kähler metric we will refer to the first Chern class representatives for the induced bundle metrics as $\rho_\pm$. By the well known transgression formula for the first Chern class of a complex line bundle, we have the local expression

\[(\rho_\pm)_j = - \sqrt{-1} \partial_j \overline{\partial_j} \log g_{\pm}, \]

where we have added a factor of 2 for notational convenience. Furthermore, exploiting the decomposition of $\Lambda^{1,1}_R$ into types via $T^c_M = T^+_M \oplus T^-_M$, we can consider the corresponding decomposition of $\rho_\pm$. We will only need for the pieces of the form $\Lambda^{0,1}_\pm \wedge \Lambda^{1,0}_\pm$, and we will denote the corresponding pieces of $\rho_\pm$ decomposition by a $\pm$ superscript. For concreteness sake, in the adapted local coordinates one has the expressions

\[(\rho_+^\pm)_{\alpha \pm \beta \pm} = - g^{\alpha \pm \delta} g_{\delta + \tau \pm, \alpha \pm \beta \pm} + g^{\alpha \pm \mu \pm} g_{\mu \pm \tau \pm, \alpha \pm \beta \pm} g_{\tau \pm, \alpha \pm}, \]

\[(\rho_-^\pm)_{\alpha \pm \beta \pm} = - g^{\alpha \pm \delta} g_{\delta - \tau \pm, \alpha \pm \beta \pm} + g^{\alpha \pm \mu \pm} g_{\mu \pm \tau \pm, \alpha \pm \beta \pm} g_{\tau \pm, \alpha \pm}, \]

\[(\rho_+^\pm)_{\alpha \pm \beta \pm} = - g^{\alpha \pm \delta} g_{\delta + \tau \pm, \alpha \pm \beta \pm} + g^{\alpha \pm \mu \pm} g_{\mu \pm \tau \pm, \alpha \pm \beta \pm} g_{\tau \pm, \alpha \pm}, \]

\[(\rho_-^\pm)_{\alpha \pm \beta \pm} = - g^{\alpha \pm \delta} g_{\delta - \tau \pm, \alpha \pm \beta \pm} + g^{\alpha \pm \mu \pm} g_{\mu \pm \tau \pm, \alpha \pm \beta \pm} g_{\tau \pm, \alpha \pm}. \]

With this background in place we give our calculation of $P$ in this special setting. To begin we recall a general coordinate formula for $P$.

Lemma 3.1. (20) Let $(M^{2n}, g, J)$ be a complex manifold with pluriclosed metric $g$. In local complex coordinates, one has

\[P(\omega)_j = - \sqrt{-1} \left[ g^{\alpha \beta} g_{\beta \gamma} \partial_\gamma g^{\alpha \beta} \left( g_{\alpha \beta} g_{\gamma \beta} - g_{\alpha \beta} g_{\gamma \beta} - g_{\alpha \beta} g_{\gamma \beta} \right) \right]. \]

Proposition 3.2. Let $(M^{2n}, g, J_\pm)$ be a generalized Kähler manifold such that $[J_+, J_-] = 0$. Then

\[P(\omega) = \rho_+^+ - \rho_+^- - \rho_-^+ + \rho_-^-. \]

Proof. We perform calculations in local coordinates as described above. We will frequently make use of (2.11) - (2.14), sometimes without explicitly saying so. To begin we observe a simplification of the first order terms in the local expression for $P$. In particular, since $g_{\alpha + \beta^-} = 0$, it follows
It follows similarly that

\[
\gamma\nu\gamma\rho\left( g_{\gamma\nu\gamma\rho,\delta} - g_{\gamma\nu\gamma\rho,\delta} - g_{\gamma\nu\gamma\rho} g_{\gamma\rho,\delta} \right) \\
= g_{\gamma\nu}^{\alpha} g_{\gamma\delta}^{\alpha} \left( g_{\alpha+\gamma\nu,\delta} - g_{\alpha+\gamma\nu,\delta} - g_{\alpha+\gamma\nu} g_{\alpha+\gamma\nu,\delta} \right) \\
+ g_{\gamma\nu}^{\delta} \left( g_{\alpha-\gamma\nu,\delta} - g_{\alpha-\gamma\nu,\delta} - g_{\alpha-\gamma\nu} g_{\alpha-\gamma\nu,\delta} \right) 
\]

Using this identity we can compute

\[
-P_{\mu+\nu+} = \sqrt{-1} \left[ g_{\nu+}^{\alpha,\gamma} g_{\nu,\alpha} + g_{\nu,\alpha} g_{\nu+}^{\gamma,\delta} + g_{\nu+}^{\gamma,\delta} \left( g_{\alpha+\nu,\delta} - g_{\alpha+\nu,\delta} - g_{\alpha+\nu} g_{\alpha+\nu,\delta} \right) \\
+ g_{\nu}^{\gamma,\delta} \left( g_{\alpha-\nu,\delta} - g_{\alpha-\nu,\delta} - g_{\alpha-\nu} g_{\alpha-\nu,\delta} \right) \right] 
\]

\[
= \sqrt{-1} \left[ g_{\nu+}^{\alpha,\gamma} g_{\nu,\alpha} + g_{\nu,\alpha} g_{\nu+}^{\gamma,\delta} + g_{\nu}^{\gamma,\delta} \left( g_{\alpha-\nu,\delta} - g_{\alpha-\nu,\delta} - g_{\alpha-\nu} g_{\alpha-\nu,\delta} \right) \right] 
\]

Observe that in the second to last line we applied the partial Kähler condition (2.12) for the term \( g_{\mu+\nu,\alpha} \), and the pluriclosed condition (2.14) for the term \( g_{\mu+\nu,\alpha} \). Next we compute

\[
-P_{\mu-\nu-} = \sqrt{-1} \left[ g_{\nu-}^{\alpha,\gamma} g_{\nu,\alpha} + g_{\nu,\alpha} g_{\nu-}^{\gamma,\delta} + g_{\nu}^{\gamma,\delta} \left( g_{\alpha-\nu,\delta} - g_{\alpha-\nu,\delta} - g_{\alpha-\nu} g_{\alpha-\nu,\delta} \right) \right] 
\]

\[
= \sqrt{-1} \left[ g_{\nu-}^{\alpha,\gamma} g_{\nu,\alpha} + g_{\nu,\alpha} g_{\nu-}^{\gamma,\delta} + g_{\nu}^{\gamma,\delta} \left( g_{\alpha-\nu,\delta} - g_{\alpha-\nu,\delta} - g_{\alpha-\nu} g_{\alpha-\nu,\delta} \right) \right] 
\]

\[
= 0. 
\]

It follows similarly that \( P_{\mu-\nu+} = 0 \). Lastly we compute

\[
-P_{\mu-\nu-} = \sqrt{-1} \left[ g_{\nu-}^{\alpha,\gamma} g_{\nu,\alpha} + g_{\nu,\alpha} g_{\nu-}^{\gamma,\delta} + g_{\nu}^{\gamma,\delta} \left( g_{\alpha-\nu,\delta} - g_{\alpha-\nu,\delta} - g_{\alpha-\nu} g_{\alpha-\nu,\delta} \right) \right] 
\]

\[
= \sqrt{-1} \left[ g_{\nu-}^{\alpha,\gamma} g_{\nu,\alpha} + g_{\nu,\alpha} g_{\nu-}^{\gamma,\delta} + g_{\nu}^{\gamma,\delta} \left( g_{\alpha-\nu,\delta} - g_{\alpha-\nu,\delta} - g_{\alpha-\nu} g_{\alpha-\nu,\delta} \right) \right] 
\]

\[
= \sqrt{-1} \left[ g_{\nu-}^{\alpha,\gamma} g_{\nu,\alpha} + g_{\nu,\alpha} g_{\nu-}^{\gamma,\delta} + g_{\nu}^{\gamma,\delta} \left( g_{\alpha-\nu,\delta} - g_{\alpha-\nu,\delta} - g_{\alpha-\nu} g_{\alpha-\nu,\delta} \right) \right] 
\]

\[
= [-\rho_+ + \rho_-]_{\mu-\nu-}. 
\]
Using this formula, we can easily observe that in this setting the tensor $P$ satisfies the conditions of a generalized Kähler metric, excepting positivity. One notes easily using the defining equations for $\omega_\pm$ that $\omega_+ = -\omega_+(\Pi, \cdot)$, therefore if we want to think of our flow solely in terms of $\omega_+$ evolving by $P(\omega_+)$, we get an induced equation on $\omega_-$ flowing by $-P(\omega_+)(\Pi, \cdot)$. The corollary below represents a check that this will preserve the generalized Kähler condition. The full proof is in §3.2.

**Corollary 3.3.** Let $(M^{2n}, g, J_\pm)$ be a generalized Kähler manifold satisfying $[J_+, J_-] = 0$. Then

\begin{equation}
\frac{d}{dt} P(\omega_+) = d_- [P(\omega_+)(\Pi, \cdot)]
\end{equation}

**Proof.** First we compute, using that $d\rho_\pm = 0$,

$$dP = (\delta_+ + \delta_-) (\rho_+^+ - \rho_- - \rho_+^- + \rho_-^-) = \delta_- \rho_+^+ - \delta_+ \rho_-^+ - \delta_+ \rho_+^- + \delta_+ \rho_-^-$$

Next we observe using the definition of $T_\pm$ that

$$P(\omega_+)(\Pi, \cdot) = \rho_+^+ - \rho_- + \rho_+^- - \rho_-^-,$$

and hence

$$d [P(\Pi, \cdot)] = \delta_- \rho_+^+ - \delta_+ \rho_-^+ + \delta_+ \rho_+^- - \delta_+ \rho_-^-.$$

Using (2.9), it follows that the actions of $J_\pm$ on $\delta_+ \rho_-^+ - \delta_- \rho_+^+$ are equal, whereas they differ by a sign on $\delta_+ \rho_- - \delta_+ \rho_-^-$. The corollary follows. □

### 3.2. Reduction to a scalar potential.

Our next goal is to develop a formal picture of the existence time for solutions to the pluriclosed flow on generalized Kähler manifolds with $[J_+, J_-] = 0$. We first show that after the selection of background data the flow in this setting can be reduced to a flow of a scalar potential function. We then exhibit a natural cohomology space with an associated “positive cone” in analogy with the Kähler cone which suggests a conjectural maximal existence time for the pluriclosed flow in this setting, specializing (21 Conjecture 5.2)

**Lemma 3.4.** Let $(M^{2n}, g_0, J_\pm)$ be a generalized Kähler manifold. Let $h_\pm$ denote metrics on $T_\pm M$, respectively, and let

$$P = P(h_\pm) = \rho_+^+(h_+) - \rho_-^+(h_-) + \rho_+^-(h_+) + \rho_-^-(h_-).$$

Furthermore, set $\omega_t = \omega_0 - tP$, and for a smooth function $f$ let

$$\omega_t^f := \omega_t + (\delta_+ \delta_+^c - \delta_- \delta_-^c) f.$$

Lastly, suppose $f_t$ solves

\begin{equation}
\frac{\partial}{\partial t} f = \log \frac{\det g_+^f \det h_-}{\det h_+ \det g_-^f},
\end{equation}

$$f(0) = 0.$$

Then $\omega_t^f$ is the unique solution to pluriclosed flow with initial condition $\omega_0$. 
Proof. Using the given definitions, (3.1) and Proposition 3.2 we compute
\[
\frac{\partial}{\partial t} \omega_t^f = -\rho^+(h_+) + \rho^-(h_-) + \rho^-(h_-) - \rho^-(h_-) + \sqrt{-1} (\delta_+ \delta_- - \delta_- \delta_+) \log \left( \frac{\det g_+^f \det h_-}{\det h_+ \det g_-^f} \right)
\]
\[
= \sqrt{-1} \left[ \delta_+ \delta_- \left( \log \det h_+ - \log \det h_- + \log \left( \frac{\det g_+^f \det h_-}{\det h_+ \det g_-^f} \right) \right) \right] 
+ \delta_- \delta_- \left( \log \det h_- - \log \det h_+ - \log \left( \frac{\det g_+^f \det h_-}{\det h_+ \det g_-^f} \right) \right) \right]
\]
\[
= - \rho^+_+ + \rho^+_+ + \rho^-_- - \rho^-_- 
= - P(\omega_t^f).
\]
The lemma follows. \qed

Remark 3.5. Equation (3.3) is a fully nonlinear, parabolic equation for \(f\). Crucially however, it is nonconvex, and therefore the well-understood theories of Evans-Krylov [8, 15] on \(C^{2,\alpha}\) regularity do not apply to this equation, nor does the direct method of “Calabi’s third order estimate.” Obtaining this estimate for this equation in general is a serious challenge, which we overcome in the case \(n = 2\) using delicate maximum principle arguments in [6].

Remark 3.6. The corresponding fixed-point equation for (3.3) has appeared in physics literature as a “generalized Calabi-Yau” condition [1, 11, 12, 18].

Proof of Theorem 1.1. Since \(\omega_0 > 0\) we can fix some background metrics \(h\) as in Lemma 3.4 and some \(\epsilon > 0\) so that \(\omega_0 - \epsilon P(h) > 0\). From a calculation identical to that in Lemma 4.2 one can show that equation (3.3) is strictly parabolic, and thus, for a smaller \(\epsilon > 0\) there is a solution to (3.3) on \([0, \epsilon]\). By Lemma 3.4 the family of metrics \(\omega_t^f = \omega_0 - \epsilon P(h) + \sqrt{-1} (\delta_+ \delta_- - \delta_- \delta_+) f\) is the unique solution to pluriclosed flow with initial condition \(\omega_0\). Moreover, by Corollary 3.3 these metrics are generalized Kähler with respect to \(J\), finishing the proof. \qed

Remark 3.7. In [21] the author and Tian showed that in general the pluriclosed flow can be reduced to a degenerate parabolic equation for a \((0,1)\)-form. In this case, due to the extra complex structure, we have reduced all the way to a scalar equation. By using the formulas above, one can easily recover that the \((0,1)\)-form potential, in this case, is
\[
\alpha = \sqrt{-1} (\bar{\partial}_+ f - \bar{\partial}_- f)
\]

3.3. The positive cone. In this subsection we define a notion of a “positive cone” for generalized Kähler metrics in analogy with the Kähler cone in Kähler geometry.

Definition 3.8. Let \((M^{2n}, J_+, J_-)\) be a bihermitian manifold with \([J_+, J_-] = 0\). Given \(\phi_+ \in \Lambda^1_{J_+, \mathbb{R}}\), let \(\phi_- = -\phi(\Pi, \cdot) \in \Lambda^1_{J_-, \mathbb{R}}\). We say that \(\phi_+\) is formally generalized Kähler if
\[
d_J^c \phi_+ = -d_J^c \phi_- \\
\dd_J^c \phi_+ = 0.
\]

This definition is meant to codify the formal properties that a generalized Kähler metric satisfies, other than positivity. The content of Corollary 3.3 is that \(P\) is formally generalized Kähler, and in particular the path \(\omega_0 - tP(h)\) consists of formally generalized Kähler forms. We
next study how long this path can possibly represent a class which contains genuine generalized Kähler metrics.

**Definition 3.9.** Let \((M^{2n}, g, J_{\pm})\) denote a generalized Kähler manifold such that \([J_+, J_-] = 0\). Let

\[ \mathcal{H} := \left\{ \phi \in \Lambda^{1,1}_{J_+}, \mathbb{R} \mid \phi \text{ is formally generalized Kähler} \right\} / \left\{ \delta_+ \delta_+^c f - \delta_- \delta_-^c f \right\}. \]

Furthermore, set

\[ \mathcal{P} := \{ [\phi] \in \mathcal{H} \mid \exists \omega \in [\phi], \omega > 0 \}. \]

We will refer to \(\mathcal{P}\) as the positive cone, and should be interpreted in direct analogy with the Kähler cone on a Kähler manifold.

**Definition 3.10.** Let \((M^{2n}, g_0, J_{\pm})\) denote a generalized Kähler manifold such that \([J_+, J_-] = 0\). Let \(h_{\pm}\) denote background metrics on \(T^\mathbb{C}_\pm M\), and let \(P = P(h_{\pm})\) as above. We define

\[ \tau^* := \sup\{ t > 0 \mid [\omega_0 - tP] \in \mathcal{P} \}. \] (3.4)

Our calculations above show that \(\tau^*\) does not depend on the choice of \(h_{\pm}\). Certainly \(\tau^*\) represents the maximum possible existence time for a solution to pluriclosed flow in this setting. In analogy with the Theorem of Tian-Zhang [23] for Kähler-Ricci flow, and as a specialization of the corresponding general conjecture for pluriclosed flow [21], we make the conjecture that \(\tau^*\) does indeed represent the smooth maximal existence time.

**Conjecture 3.11.** Let \((M^{2n}, g_0, J_{\pm})\) denote a generalized Kähler manifold with \([J_+, J_-] = 0\). Let \(\tau^*\) be defined as in (3.4). Then the solution to pluriclosed flow with initial condition \(g_0\) exists on \([0, \tau^*)\).

Our Theorem 1.3 verifies this conjecture in \(n = 2\), for most cases.

3.4. **A Calabi-type conjecture.** Our discussion above shows that \(P\) is a fully nonlinear elliptic operator in the potential function \(f\). Moreover, the image of \(P\) always represents the same cohomology class in \(\mathcal{H}\), generalizing the usual first Chern class associated to a complex manifold. While pluriclosed flow is the natural parabolic flow associated to this operator, it seems natural to also consider the elliptic PDE problem suggested by this picture:

**Conjecture 3.12.** Let \((M^{2n}, g, J_{\pm})\) denote a generalized Kähler structure satisfying \([J_+, J_-] = 0\). Given \(\phi \in [P] \in \mathcal{H}\), there exists a unique \(\omega_f \in [\omega]\) such that \(P(\omega_f) = \phi\).

If a generalized Kähler structure satisfies \(J_+ = \pm J_-\), then in fact \((M, J_{\pm})\) are both Kähler manifolds, and generalized Kähler metrics on \((M, J_{\pm})\) are the same as Kähler metrics on the individual complex manifolds. In particular, the whole discussion above reduces to the well-known setting of Kähler-Ricci flow, and Conjecture 3.12 reduces to the famous Calabi conjecture [6, 7], resolved by Yau [25].

4. **Evolution Equations**

In this section we record a number of evolution equations for some relevant geometric quantities along a solution to pluriclosed flow in this context. Moreover we introduce the “torsion potential” as remarked upon in the introduction, and exhibit its remarkably simple evolution equation. To begin let us record some useful definitions and notational conventions. First, we will let \(g\) denote a solution to pluriclosed flow as in Theorem 1.1. We will denote the induced Hermitian metrics on \(T^\mathbb{C}_\pm\) as \(g_{\pm}\), respectively. We use \(h\) to denote an arbitrary fixed background
metric, with corresponding decomposition \( h = h_+ + h_- \). For Lemmas involving the potential function \( f \) we assume that \( \tilde{g} \) is a smooth one parameter family of generalized Kähler metrics, and

\[
g = \tilde{g} + (\delta_+ \delta_+^c f - \delta_- \delta_-^c f),
\]

where \( \tilde{g} = g_0 - tP(h) \) and \( f \) evolves according to (3.3). The notation \( \Delta \) will refer to the Chern Laplacian, acting via

\[
\Delta A = g^\nabla A \nabla_A - \nabla_A \nabla A,
\]

where \( A \) is a section of some tensor bundle and \( \nabla \) are the induced Chern connections on these bundles. Also, \( \Gamma^{\pm} \) will refer to the local coefficients of the Chern connection on \( T^{\pm} \) associated to the metrics \( g^{\pm} \), with \( \Gamma^h \) defined analogously. In this context we let

\[
\Upsilon(g^\pm, h^\pm) := \nabla g^\pm - \nabla h^\pm
\]

denote the tensor representing the difference of these two connections. This tensor arises in the evolution equations below, as do different ways to measure its size. In particular, as a section of \( \Lambda^{1,0} \otimes \Lambda^{1,0} \otimes T^{1,0} \), one in principle could use different metrics to contract each pair of indices.

We will denote these three choices as subscripts. For instance, \( \log \det g_+ \) and \( \log \det h_+ \) will refer to the local coefficients of the Chern connection on \( T^+_\pm \) associated to the metrics \( g_\pm \), with \( \Gamma^h \) defined analogously. In this context we let

\[
\Upsilon(g^\pm, h^\pm) := \nabla g^\pm - \nabla h^\pm
\]

denote the tensor representing the difference of these two connections. This tensor arises in the evolution equations below, as do different ways to measure its size. In particular, as a section of \( \Lambda^{1,0} \otimes \Lambda^{1,0} \otimes T^{1,0} \), one in principle could use different metrics to contract each pair of indices.

We will denote these three choices as subscripts. For instance,

\[
|\Upsilon(g^+, h^+)|_{g^+, g^-, h} = g_+^\nabla \nabla_+ - \nabla_+ \nabla_+ = \frac{1}{2} |T|^2 - \text{tr}_g \rho(h^+)\]

Lemma 4.1. Given the setup above,

\[
\partial_t \log \frac{\det g_+}{\det h_+} = \Delta \log \frac{\det g_+}{\det h_+} + \frac{1}{2} |T|^2 - \text{tr}_g \rho(h^+).
\]

Proof. We give the proof for the case of \( g_+ \), the other case being analogous. To begin we compute

\[
\partial_t \log \frac{\det g_+}{\det h_+} = g_+^\nabla_+ \nabla_+ - \nabla_+ \nabla_+ + \frac{1}{2} |T|^2 - \text{tr}_g \rho(h^+).
\]

Next we observe

\[
\Delta \log \frac{\det g_+}{\det h_+} = g_+^\nabla_+ \nabla_+ - \nabla_+ \nabla_+ + \frac{1}{2} |T|^2 - \text{tr}_g \rho(h^+).
\]

It follows that

\[
\partial_t \log \frac{\det g_+}{\det h_+} = \Delta \log \frac{\det g_+}{\det h_+} + \sum_{i=1}^5 A_i,
\]
where the terms $A_i$ are defined by the equality. Next we observe that

$$A_1 + A_4 = g^{g+} g^{\tilde{g}+} g^g + g^g g^{\tilde{g}+} g^g = g^{g+} g^{\tilde{g}+} g^g = 0$$

Also we have, using (2.11), (2.12),

$$A_2 + A_3 = -g^{\tilde{g}+} g^g g^{\tilde{g}+} g^g = -g^{\tilde{g}+} g^g g^{\tilde{g}+} g^g = 0$$

Combining the above calculations yields the result. □

**Lemma 4.2.** Given the setup above, one has

$$\frac{\partial}{\partial t} \hat{f} = \Delta \hat{f} + \text{tr}_{g_+} P(h_+) - \text{tr}_{g_-} P(h_-).$$

**Proof.** We directly compute using Lemma 1.1

$$\frac{\partial}{\partial t} \frac{\partial}{\partial t} = \frac{\partial}{\partial t} \frac{\partial}{\partial t} \log \frac{\det g_+ \det h_-}{\det h_+ \det g_-}$$

$$= \Delta \log \frac{\det g_+ \det h_-}{\det h_+ \det g_-} + \frac{1}{2} |T|^2 - \text{tr}_{g_+} \rho(h_+) - \Delta \log \frac{\det g_- \det h_+}{\det h_- \det g_+} - \frac{1}{2} |T|^2 + \text{tr}_{g_-} \rho(h_-)$$

$$= \Delta \hat{f} + \text{tr}_{g_+} (\rho_+ - \rho_-) + \text{tr}_{g_-} (\rho_+ - \rho_-)$$

$$= \Delta \hat{f} + \text{tr}_{g_+} P(h_+) - \text{tr}_{g_-} P(h_-).$$

□

**Lemma 4.3.** Given the setup above,

$$\frac{\partial}{\partial t} \text{tr}_{g_+} h_+ = \Delta \text{tr}_{g_+} h_+ - |Y(g_+, h_+)|^2 g_+^{-1} g_-^{-1} h_+ - \text{tr} g_+^{-1} h_+ g_+^{-1} Q + g^g g^{\tilde{g}+} \left( \Omega h_+ \right)_{\rho_+ \rho_+}.$$

**Proof.** To begin we compute

$$\frac{\partial}{\partial t} \text{tr}_{g_+} h_+ = \frac{\partial}{\partial t} \left( g^{\tilde{g}+} h_+ \right)$$

$$= -g^{\tilde{g}+} g^{\tilde{g}+} g^g = -g^{\tilde{g}+} g^g$$

$$= -g^{\tilde{g}+} \left[ g^g g^{\tilde{g}+} + g^g g^{\tilde{g}+} \left( g^g g^{\tilde{g}+} + g^g g^{\tilde{g}+} \right) \right] g^{\tilde{g}+} h_+.$$
Next we observe
\[
\Delta \text{tr}_{g_+} h_+ = g^{\rho} \left( g^{\beta+} h_{\alpha+} \right) \frac{\partial}{\partial t} \text{tr}_{g_+} h_+ = g^{\rho} \left( -g^{\beta+} + g^{\beta+} \right) \frac{\partial}{\partial t} \text{tr}_{g_+} h_+.
\]
Combining these yields
\[
\frac{\partial}{\partial t} \text{tr}_{g_+} h_+ = \Delta \text{tr}_{g_+} h_+ - g^{\beta+} \left( g^{\beta+} - g^{\beta+} \right) \text{tr}_{g_+} h_+ + \sum_{i=1}^{7} A_i.
\]
We simplify some terms. First, using (2.11) and (2.12) we have:

\[A_1 = -g^{\beta+} g^{\beta+} g^{\beta+} h_{\alpha+} \frac{\partial}{\partial t} \text{tr}_{g_+} h_+ + g^{\beta+} g^{\beta+} g^{\beta+} \text{tr}_{g_+} h_+ + g^{\beta+} g^{\beta+} g^{\beta+} \text{tr}_{g_+} h_+.
\]

Next
\[A_2 + A_3 = g^{\beta+} g^{\beta+} g^{\beta+} h_{\alpha+} \frac{\partial}{\partial t} \text{tr}_{g_+} h_+ + g^{\beta+} g^{\beta+} g^{\beta+} \text{tr}_{g_+} h_+ + g^{\beta+} g^{\beta+} g^{\beta+} \text{tr}_{g_+} h_+.
\]
It follows that
\[\sum_{i=1}^{4} A_i = g^{\beta+} g^{\beta+} g^{\beta+} h_{\alpha+} \frac{\partial}{\partial t} \text{tr}_{g_+} h_+ + g^{\beta+} g^{\beta+} g^{\beta+} \text{tr}_{g_+} h_+ + g^{\beta+} g^{\beta+} g^{\beta+} \text{tr}_{g_+} h_+.
\]
\[= -g^{\beta+} g^{\beta+} g^{\beta+} h_{\alpha+} \frac{\partial}{\partial t} \text{tr}_{g_+} h_+ + g^{\beta+} g^{\beta+} g^{\beta+} \text{tr}_{g_+} h_+ + g^{\beta+} g^{\beta+} g^{\beta+} \text{tr}_{g_+} h_+.
\]

\[= -g^{\beta+} g^{\beta+} g^{\beta+} h_{\alpha+} \frac{\partial}{\partial t} \text{tr}_{g_+} h_+ + g^{\beta+} g^{\beta+} g^{\beta+} \text{tr}_{g_+} h_+ + g^{\beta+} g^{\beta+} g^{\beta+} \text{tr}_{g_+} h_+.
\]
Next we compute

\[ A_7 = -g^\tau \eta^\tau_{\alpha +} h_{\alpha +, \beta +} \]

\[ = g^\tau \eta^\tau_{\alpha +} \left( \Omega^{h_+} \right)_{\mu +, \beta +} - g^\tau \eta^\tau_{\alpha +} h_{\alpha +, \beta +, \gamma}, \]

It follows that

\[ \sum_{i=5}^7 A_i = -g^\tau \eta^\tau_{\alpha +} \left( \Gamma^h \right)_\mu p_{\alpha +} \]

\[ - g^\tau \eta^\tau_{\alpha +} \left( \Gamma^h \right)_\mu p_{\alpha +} \left( \Gamma^h \right)_p g^\tau_{\alpha +} h_{\mu +, \beta +} \]

\[ + g^\tau \eta^\tau_{\alpha +} \left( \Omega^{h_+} \right)_{\mu +, \beta +} \]

\[ = - |\mathcal{Y}(g_+, h_+)|^2_{g^{-1}, g^{-1}} + g^\tau \eta^\tau_{\alpha +} \left( \Omega^{h_+} \right)_{\mu +, \beta +}. \]

Collecting the above calculations yields the result. \( \square \)

**Lemma 4.4.** Given the setup above,

\[ \frac{\partial}{\partial t} \text{tr}_{h_+} g_{\pm} = \Delta \text{tr}_{h_+} g_{\pm} - |\mathcal{Y}(g_+, h_+)|^2_{g^{-1}, g} + \text{tr}_{h_+} Q - g^\tau(h_{\pm}^{-1} g h_{\pm}^{-1}) \left( \Omega^{h_+} \right)_{\mu +, \beta +}. \]

**Proof.** Again we only give the proof for the case of \( \text{tr}_{h_+} g_+ \), the other case being analogous. To begin we compute

\[ \frac{\partial}{\partial t} \text{tr}_{h_+} g_+ = h^\tau \eta_{\alpha +} \left( P(h)_{\alpha +, \beta +} \right) \]

\[ = h^\tau \eta_{\alpha +} \left( \log \frac{\det g_+ \det h_+}{\det h_+ \det g_-} \right)_{\alpha +, \beta +} + \text{tr}_{h_+} P(h_+). \]

Next we observe using (2.11), (2.14),

\[ \left( \log \frac{\det g_+ \det h_+}{\det h_+ \det g_-} \right)_{\alpha +, \beta +} = g^\tau_{\rho +} g_{\rho +, \tau +, \alpha +, \beta +} - g^\tau_{\rho +} g^\tau_{\tau +, \alpha +, \beta +} g_{\delta +, \tau +, \alpha +, \beta +} + g^\tau_{\rho +} g^\tau_{\tau +, \alpha +, \beta +} g_{\delta +, \tau +, \alpha +, \beta +} \]

\[ + h^\tau_{\rho +} g_{\rho +, \tau +, \alpha +, \beta +} - h^\tau_{\rho +} g_{\rho +, \tau +, \alpha +, \beta +} g_{\delta +, \tau +, \alpha +, \beta +} \]

\[ - h^\tau_{\rho +} h_{\rho +, \tau +, \alpha +, \beta +} + h^\tau_{\rho +} h^\tau_{\tau +, \alpha +, \beta +} g_{\delta +, \tau +, \alpha +, \beta +} + h^\tau_{\rho +} h^\tau_{\tau +, \alpha +, \beta +} g_{\delta +, \tau +, \alpha +, \beta +} \]

\[ = g^\tau_{\rho +} g_{\rho +, \tau +, \alpha +, \beta +} - g^\tau_{\rho +} g^\tau_{\tau +, \alpha +, \beta +} g_{\delta +, \tau +, \alpha +, \beta +} + g^\tau_{\rho +} g^\tau_{\tau +, \alpha +, \beta +} g_{\delta +, \tau +, \alpha +, \beta +} \]

\[ + h^\tau_{\rho +} h_{\rho +, \tau +, \alpha +, \beta +} - h^\tau_{\rho +} h^\tau_{\tau +, \alpha +, \beta +} g_{\delta +, \tau +, \alpha +, \beta +} + h^\tau_{\rho +} h^\tau_{\tau +, \alpha +, \beta +} g_{\delta +, \tau +, \alpha +, \beta +} \]

\[ - h^\tau_{\rho +} h_{\rho +, \tau +, \alpha +, \beta +} + h^\tau_{\rho +} h^\tau_{\tau +, \alpha +, \beta +} g_{\delta +, \tau +, \alpha +, \beta +} + h^\tau_{\rho +} h^\tau_{\tau +, \alpha +, \beta +} g_{\delta +, \tau +, \alpha +, \beta +}. \]
Next we observe
\[
\Delta \text{tr}_{h^+} g_+ = g^T_i \left( h^{\alpha+}_{+\gamma} g_{+\beta_+} \right)_{i\bar{j}} - (h_-)_{i\bar{j}} = g^T_i \left( -h^\alpha_+ \delta_+ h_{+\gamma_+,i} h^\gamma+_{+\alpha} g_{+\beta_+} + h^\alpha_+ g_{+\beta_+} \right)_{i\bar{j}}
\]
\[
= g^T_i \left[ h^{\alpha+}_+ h_{\rho_+,\sigma_+} i h^\sigma_+ \delta_+ h_{+\gamma_+,i} h^\gamma+_{+\alpha} g_{+\beta_+} - h^\alpha_+ \delta_+ h_{+\gamma_+,i} h^\gamma+_{+\alpha} g_{+\beta_+} \right.
\]
\[
+ h^\beta_+ \delta_+ h_{+\gamma_+,i} h^\gamma+_{+\alpha} g_{+\beta_+} - h^\alpha_+ \delta_+ h_{+\gamma_+,i} h^\gamma+_{+\alpha} g_{+\beta_+} \left. \right]_{i\bar{j}}
\]
\[
- h^\beta_+ \delta_+ h_{+\gamma_+,i} h^\gamma+_{+\alpha} g_{+\beta_+} + h^\beta_+ \delta_+ h_{+\gamma_+,i} h^\gamma+_{+\alpha} g_{+\beta_+} \]
\[
- h^\beta_+ \delta_+ h_{+\gamma_+,i} h^\gamma+_{+\alpha} g_{+\beta_+} + h^\beta_+ \delta_+ h_{+\gamma_+,i} h^\gamma+_{+\alpha} g_{+\beta_+} \left]_{i\bar{j}} \right.
\]
\[
= \Delta \text{tr}_{h^+} g_+ + \sum_{i=1}^{11} A_i.
\]

Combining the two above calculations yields
\[
h^{\alpha+}_{+\gamma} \left( \log \frac{\det g_+ \det h_-}{\det h_+ \det g_-} \right)_{,\alpha+\beta_+}
\]
\[
= \Delta \text{tr}_{h^+} g_+ + \sum_{i=1}^{11} A_i.
\]

Now we identify some terms. First of all
\[
A_3 + A_4 = h^{\alpha+}_{+\gamma} h^{\alpha-\mu} \left[ h_{\rho_-,\sigma_-,\gamma} - h^\alpha_- h_{\rho_-,\sigma_-} h_{\rho_-,\sigma_-} \right] = - \text{tr}_{h^+} \rho^+(h_-).
\]

Next we observe
\[
A_5 + A_6 = h^{\alpha+}_{+\gamma} h^{\alpha+\mu} \left[ -h_{\rho_+,\sigma_+,\gamma} + h^\alpha_+ h_{\rho_+,\sigma_+,\gamma} \right] = \text{tr}_{h^+} \rho^+(h_+).
\]

Next
\[
A_8 + A_9 = g^T_i \left( h^{\alpha+}_+ h_{\rho_+,\sigma_+,\gamma} h^\gamma+_{+\alpha} g_{+\beta_+} - h^\alpha_+ h_{+\gamma_+,i} h^\gamma+_{+\alpha} g_{+\beta_+} \right)_{i\bar{j}}
\]
\[
= g^T_i \left( h^{\alpha+}_+ g_{+\gamma_+,i} h^\gamma+_{+\alpha} g_{+\beta_+} \right)_{i\bar{j}}
\]
\[
= - g^T_i \left( h^{\alpha+}_+ g_{+\gamma_+,i} h^\gamma+_{+\alpha} g_{+\beta_+} \right)_{i\bar{j}}
\]
\[
= - g^T_i \left( h^{\alpha+}_+ g_{+\gamma_+,i} h^\gamma+_{+\alpha} g_{+\beta_+} \right)_{i\bar{j}}
\]
\[
= \Delta \text{tr}_{h^+} g_+ + \sum_{i=1}^{11} A_i.
\]

Also, using (2.11) we have that
\[
A_2 = h^{\alpha+}_{+\gamma} g^{\gamma-\mu} h^\gamma-_- h_{\rho_-,\sigma_-} g_{+\beta_+} \]
\[
= h^{\alpha+}_{+\gamma} g^{\gamma-\mu} T_{\alpha+\beta_-} T_{\gamma+\alpha} T^\gamma+_{\gamma_- \mu_-}.
\]
We now consider the piece of $A_7 + A_{10} + A_{11}$ coming from the contraction with the metric $g_-$. In particular, we have

$$(A_7 + A_{10} + A_{11})_+ = g^\rho_- \mu_- \left[ -h^\beta_+^\rho \rho_+ \varpi_+ \varpi_- h^\beta_+ \delta_+ h^\beta_+ \varpi_+ \varpi_- h^\beta_+ \varpi_+ \varpi_- \right] \delta_+ h^\beta_+ \varpi_+ \varpi_- h^\beta_+ \varpi_+ \varpi_- + h^\beta_+ \delta_+ h^\beta_+ \delta_+ h^\beta_+ \varpi_+ \varpi_- + h^\beta_+ \delta_+ h^\beta_+ \delta_+ h^\beta_+ \varpi_+ \varpi_-$$

Collecting up the above calculations yields the result.

As we remarked in [31], the tensor playing the role of the 1-form version of pluriclosed flow (see [21]) is

$$\alpha = \frac{\sqrt{-1}}{2} \left( \bar{\varpi} - \varpi \right) f.$$
It follows that the torsion potential takes the form
\[ \partial \sigma = (\partial^+ + \partial^-) \frac{\sqrt{-1}}{2} (\partial^+ - \partial^-) f = \sqrt{-1} \partial^- \partial^+ f. \]

This tensor obeys a remarkable evolution equation. While the term \( \Psi \) in the Lemma below looks formidable, observe that it is at worst linear in the flowing connection, and moreover completely vanishes against a Kähler background.

**Lemma 4.5.** Given the above setup,
\[ \frac{\partial}{\partial t} \partial_- \partial_+ f = \Delta \partial_- \partial_+ f + \Psi, \]

where
\begin{align*}
\Psi_{\mu_+ \mu_-} &:= (\text{tr} h^- \nabla h T^h)_{\mu_+ \mu_-} - h_+^{\alpha \alpha_+} h_+^{\gamma_+ \gamma_+} T^h_{\mu_+ \mu_-} T^h_{\gamma_+ \alpha_+ \beta_+} \\
&\quad - (\text{tr} h^+ \nabla h T^h)_{\mu_+ \mu_-} + h_+^{\beta_+ \alpha_+} h_+^{\gamma_- \gamma_-} T^h_{\mu_+ \mu_-} T^h_{\gamma_- \alpha_+ \beta_+} \\
&\quad - (\text{tr} g^- \nabla g T^\gamma)_{\mu_+ \mu_-} + g_+^{\alpha_+ \alpha_-} g_+^{\gamma_+ \gamma_+} T_{\mu_+ \mu_-} T_{\gamma_+ \alpha_+ \beta_+} \\
&\quad + (\text{tr} g^+ \nabla g T^\gamma)_{\mu_+ \mu_-} + g_+^{\beta_+ \alpha_+} g_+^{\gamma_- \gamma_-} T_{\mu_+ \mu_-} T_{\gamma_- \alpha_+ \beta_+} \\
&\quad + g_+^{\alpha_+ \alpha_-} g_+^{\delta_+ \delta_+} T_{\alpha_+ \gamma_+} T_{\delta_+ \mu_-} T_{\beta_+} - g_+^{\alpha_+ \alpha_-} \nabla \partial_+ \partial_- T_{\alpha_+ \gamma_+} T_{\delta_+ \mu_-} T_{\beta_+}.
\end{align*}

**Proof.** To begin we compute
\[ \frac{\partial}{\partial t} f_{\mu_+ \mu_-} = \left( \frac{\log \det g_+ \det h_-}{\det h_+ \det g_-} \right)_{\mu_+ \mu_-} = \sum_{i=1}^8 A_i. \]

Let us simplify the terms coming from \( h \) above. First of all, we note that
\[ (\text{tr} h^- \nabla h T^h)_{\mu_+ \mu_-} = h_+^{\gamma_+ \gamma_+} T^h_{\mu_+ \mu_-} T^h_{\gamma_+ \alpha_+ \beta_+}. \]

A similar calculation yields
\[ - (\text{tr} h^+ \nabla h T^h)_{\mu_+ \mu_-} = A_7 + A_8 - h_+^{\alpha_+ \alpha_-} h_+^{\gamma_- \gamma_-} T^h_{\mu_+ \mu_-} T^h_{\gamma_- \alpha_+ \beta_+}. \]
Next we observe that
\[
[\Delta \partial_+ \partial_+ f]_{\mu_+ \mu_-} = g^\alpha_+ \left[ \nabla \nabla \partial_+ \partial_+ f \right]_{\alpha_- \mu_+ \mu_-} \\
= g^\alpha_+ \left[ (\partial_+ \partial_+ f)_{\mu_+ \mu_-} - \Gamma^\delta_{\mu_+} (\partial_+ \partial_+ f)_{\mu_\delta \mu_-} - \Gamma^\delta_{\mu_-} (\partial_+ \partial_+ f)_{\delta_+ \mu_-} \right] \\
= g^\alpha_+ \left[ f_{\mu_+ \mu_-} - g^{\gamma_+} \delta_+ g_{\mu_+ \gamma_+ \alpha} f_{\delta_+ \mu_-} - g^{\gamma_-} \delta_+ g_{\mu_- \gamma_- \alpha} f_{\delta_- \mu_+} \right].
\]

First we express
\[
g^\alpha_+ f_{\mu_+ \mu_-} - g^{\gamma_+} \delta_+ g_{\mu_+ \gamma_+ \alpha} + g^{\gamma_-} \delta_+ g_{\mu_- \gamma_- \alpha} = A_1 + A_3 + g^{\gamma_-} \delta_+ g_{\alpha_- \beta_+ \mu_-} - g^{\gamma_+} \delta_+ g_{\alpha_+ \beta_- \mu_+}.
\]

Next we simplify
\[
-g^\beta_- \left[ g^{\gamma_+} \delta_+ g_{\mu_+ \gamma_+ \alpha} f_{\delta_+ \mu_-} + g^{\gamma_-} \delta_+ g_{\mu_- \gamma_- \alpha} f_{\delta_- \mu_+} \right] \\
= -g^\beta_- \left[ g^{\gamma_+} \delta_+ g_{\mu_+ \gamma_+ \alpha} \left( g_{\delta_+ \gamma_+ \mu_-} - \tilde{g}_{\delta_+ \gamma_+ \mu_-} \right) \right] \\
= g^\beta_- \left[ g^{\gamma_+} \delta_+ g_{\mu_+ \gamma_+ \alpha} \left( g_{\delta_+ \gamma_+ \mu_-} - \tilde{g}_{\delta_+ \gamma_+ \mu_-} \right) \right] \\
= A_2 + A_4 + g^{\gamma_-} \delta_+ g_{\mu_+ \gamma_+ \alpha} \tilde{g}_{\delta_+ \gamma_+ \mu_-} - g^{\gamma_-} \delta_+ g_{\mu_+ \gamma_+ \alpha} \tilde{g}_{\delta_+ \gamma_+ \mu_-} \\
+ g^\beta_- \left[ g^{\gamma_-} \delta_+ g_{\mu_- \gamma_- \alpha} \left( g_{\delta_- \gamma_- \mu_+} - \tilde{g}_{\delta_- \gamma_- \mu_+} \right) \right].
\]

where in the penultimate line we observed a cancellation and applied (2.12) to two terms.

Combining our calculations above yields
\[
\frac{\partial}{\partial t} (\partial_+ \partial_+ f)_{\mu_+ \mu_-} = \left[ \Delta \partial_+ \partial_+ f + \nabla^h T^h + T^h \ast_h T^h \right]_{\mu_+ \mu_-} + \sum_{i=1}^{6} B_i.
\]
Lastly we simplify the $B_i$ terms. First

$$- \left( \nabla^g \tilde{T} \right)_{\mu_+ \mu_-} = - g^{g,-\alpha} \nabla^g \tilde{T}_{\mu_+ \alpha_+ \beta_-} - g^{\gamma_+, \alpha_-} \left( \tilde{T}_{\mu_+ \alpha_+ \beta_-} \right)_{\mu_-} + (\Gamma^g)_{\mu_+ \mu_- \gamma_+} \tilde{T}_{\gamma_+ \alpha_+ \beta_-} + (\Gamma^g)_{\mu_+ \alpha_- \gamma_+} \tilde{T}_{\mu_+ \gamma_+ \beta_-}$$

$$= g^{\gamma_+, \alpha_-} \left[ - (\tilde{T}_{\mu_+ \alpha_+ \beta_-})_{\mu_-} + (\Gamma^g)_{\mu_+ \mu_- \gamma_+} \tilde{T}_{\gamma_+ \alpha_+ \beta_-} + (\Gamma^g)_{\mu_+ \alpha_- \gamma_+} \tilde{T}_{\mu_+ \gamma_+ \beta_-} \right]$$

$$= g^{\gamma_+, \alpha_-} \left[ - g_{\alpha_+ \beta_- \mu_+ \mu_-} + g^{\gamma_+} g_{\mu_+ \gamma_+ \alpha_+ \beta_-} + g^{\gamma_+} g_{\gamma_+ \alpha_+ \sigma_+ \mu_+} - g_{\gamma_+ \beta_- \mu_+ \mu_-} \right]$$

$$= B_1 + B_6 + g^{\gamma_+, \alpha_-} g^{\gamma_+} T_{\mu_+ \alpha_+ \sigma_+} \tilde{T}_{\gamma_+ \alpha_+ \beta_-}.$$

Next

$$\left( \nabla^g \tilde{T} \right)_{\mu_+ \mu_-} = g^{\gamma_+, \alpha_+} \nabla^g \tilde{T}_{\mu_+ \alpha_+ \beta_-} + g^{\gamma_+} \left[ (\tilde{T}_{\mu_+ \alpha_+ \beta_-})_{\mu_-} - (\Gamma^g)_{\mu_+ \mu_- \gamma_+} \tilde{T}_{\gamma_+ \alpha_+ \beta_-} - (\Gamma^g)_{\mu_+ \alpha_- \gamma_+} \tilde{T}_{\mu_+ \gamma_+ \beta_-} \right]$$

$$= g^{\gamma_+, \alpha_+} \left[ \tilde{T}_{\mu_+ \alpha_+ \beta_-} - (\Gamma^g)_{\mu_+ \mu_- \gamma_+} \tilde{T}_{\gamma_+ \alpha_+ \beta_-} - (\Gamma^g)_{\mu_+ \alpha_- \gamma_+} \tilde{T}_{\mu_+ \gamma_+ \beta_-} \right]$$

$$= g^{\gamma_+, \alpha_+} \left[ \tilde{T}_{\mu_+ \alpha_+ \beta_-} - g^{\gamma_+} g_{\mu_+ \gamma_+ \alpha_+ \beta_-} - g^{\gamma_+} g_{\gamma_+ \alpha_+ \sigma_+ \mu_+} + g_{\gamma_+ \beta_- \mu_+ \mu_-} \right]$$

$$= B_2 + B_3 - g^{\gamma_+, \alpha_+} g^{\gamma_+} T_{\mu_+ \alpha_+ \sigma_+} \tilde{T}_{\gamma_+ \alpha_+ \beta_-}.$$

Lastly

$$B_4 + B_5 = g^{\gamma_+, \alpha_+} g^{\gamma_+} T_{\delta_+ \mu_+ \beta_-} \tilde{T}_{\gamma_+ \alpha_+ \beta_-} - g^{\gamma_+, \alpha_+} g^{\gamma_+} T_{\alpha_+ \mu_+ \gamma_+ \beta_-} - g^{\gamma_+, \alpha_+} g^{\gamma_+} T_{\alpha_+ \mu_+ \gamma_+ \beta_-} - g^{\gamma_+, \alpha_+} g^{\gamma_+} T_{\delta_+ \mu_+ \beta_-}.$$

The result follows.

**Remark 4.6.** Observe that this evolution equation has the remarkable property that the quadratic first order nonlinearity present throughout most evolution equations associated to this flow has in this case completely disappeared. This will play a crucial role in obtaining estimates in this setting. Indeed, this property holds more generally for solutions to the pluriclosed flow, and will be presented in future work.

**Lemma 4.7.** Let $(M^{2n}, \omega_t, J)$ be a solution to pluriclosed flow, and suppose $\beta_t \in \Lambda^{p,0}$ is a one-parameter family satisfying

$$\frac{\partial}{\partial t} \beta = \Delta g_t \beta + \Psi,$$

where $\Psi_t \in \Lambda^{p,0}$. Then

$$\frac{\partial}{\partial t} |\beta|^2 = \Delta |\beta|^2 - |\nabla \beta|^2 - |\nabla \beta|^2 - p \langle Q, \nabla \beta \otimes \beta \rangle + 2 \Re \langle \beta, \Psi \rangle.$$

**Proof.** By direct computation we have

$$\frac{\partial}{\partial t} |\beta|^2 = \frac{\partial}{\partial t} g^{2i_1 \ldots i_p} g^{2j_1 \ldots j_p} \beta_{i_1 \ldots i_p} \beta_{j_1 \ldots j_p} - p \langle S - Q, \nabla \beta \otimes \beta \rangle + \langle \Delta \beta, \beta \rangle + \langle \beta, \Delta \beta \rangle + 2 \Re \langle \beta, \Psi \rangle.$$

$$= p \langle S - Q, \nabla \beta \otimes \beta \rangle + \langle \Delta \beta, \beta \rangle + \langle \beta, \Delta \beta \rangle + 2 \Re \langle \beta, \Psi \rangle.$$
Next we observe the commutation formula
\[ \overline{\Delta \beta}_{j_1 \ldots j_p} = g R^{i} \nabla_{R}^{i} \overline{\beta}_{j_1 \ldots j_p} \]
\[ = g R^{i} \nabla_{R}^{i} \overline{\beta}_{j_1 \ldots j_p} - \sum_{r=1}^{p} g R^{i} \Omega_{j_1 \ldots j_{r-1}, j_r \ldots j_p} \overline{\beta}_{j_1 \ldots j_{r-1} j_{r+1} \ldots j_p} \]
\[ = \Delta \overline{\beta}_{j_1 \ldots j_p} - \sum_{r=1}^{p} S_{j_1 \ldots j_{r-1}, j_r \ldots j_p} \overline{\beta}_{j_1 \ldots j_{r-1} j_{r+1} \ldots j_p} . \]

It follows that
\[ \langle \beta, \overline{\Delta \beta} \rangle = g R^{i_1} \ldots g R^{i_p} \beta_{i_1 \ldots i_p} \overline{\Delta \beta}_{i_1 \ldots i_p} \]
\[ = g R^{i_1} \ldots g R^{i_p} \beta_{i_1 \ldots i_p} \left[ \Delta \overline{\beta}_{j_1 \ldots j_p} - \sum_{r=1}^{p} S_{j_1 \ldots j_{r-1}, j_r \ldots j_p} \overline{\beta}_{j_1 \ldots j_{r-1} j_{r+1} \ldots j_p} \right] \]
\[ = \langle \beta, \Delta \overline{\beta} \rangle - p \langle S, \beta \otimes \overline{\beta} \rangle . \]

Lastly observe the identity
\[ \Delta |\beta|^2 = \langle \Delta \beta, \overline{\beta} \rangle + \langle \beta, \Delta \overline{\beta} \rangle + |\nabla \beta|^2 + |\nabla \overline{\beta}|^2 . \]

Combining the above calculations yields the lemma. \qed

**Remark 4.8.** This lemma yields a particularly clean estimate for \((p, 0)\) forms evolving by the heat equation against a wide class of Hermitian curvature flows, as considered in [19]. In the case of pluriclosed flow we have natural, geometrically meaningful \((p, 0)\) forms to which this observation can be applied. The crucial cancellation of the \("S\) term arising from the variation of the norm itself is similar to the very useful cancellation of the Ricci curvature terms which occurs in the evolution of the gradient of a function evolving by the heat flow against a Ricci-flow background.

**Lemma 4.9.** Given the setup above, one has
\[ \frac{\partial}{\partial t} |\partial_+ \partial_- f|^2 = \Delta |\partial_+ \partial_- f|^2 - |\nabla \partial_+ \partial_- f|^2 - |\nabla \partial_+ \partial_- f|^2 - 2 \langle Q, \partial_+ \partial_- f \otimes g \overline{\partial_+ \partial_- f} \rangle + 2 \Re \langle \partial_+ \partial_- f, \Psi \rangle , \]
where \(\Psi\) is as in (4.2).

**Proof.** This now follows directly from Lemmas 4.5 and 4.7. \qed

5. **A priori estimates**

In this section and the next we obtain a number of a priori estimates which are the main content of Theorem 1.3. First in [5.1] we set up the scalar reduction inside the positive cone. Then we obtain an estimate for the potential function in general dimensions in Lemma 5.1. Then we establish an upper bound for the metric in the presence of a lower bound. This requires controlling potentially troublesome torsion terms arising in the evolution of \(\text{tr}_h g\), which is where the very helpful evolution equation of Lemma 4.9 plays a crucial role.
5.1. Setup. Fix a time $\tau < \tau^*$, and fix arbitrary metrics $\tilde{h}_\pm$ on $T_\pm M$ respectively, and observe that by construction $\omega_0 - \tau P(\tilde{h}_\pm) \in \mathcal{P}_+$, and so there exists $a \in C^\infty(M)$ such that
$$\omega_0 - \tau P(\tilde{h}_\pm) + [\delta_+ \delta^+_\pm - \delta_- \delta^-_\pm] a > 0.$$ Now set $h_\pm = e^{\mp \frac{a}{\tau}} \tilde{h}_\pm$. It follows that
$$\omega_0 - \tau P(h_\pm) > 0.$$ Moreover, by convexity one has a smooth one-parameter family of background metrics
$$\tilde{\omega}_t := \omega_0 - t P(h_\pm) > 0.$$ With this choice of background data we let $f_t$ be the solution to (3.3) as in Lemma 3.2. In the remainder of this section we assume this setup, and all constants $C$ will depend on all of the choices $n, \tau, h, \tilde{g}.$

5.2. Estimate for the potential. 

Lemma 5.1. Given the setup above, there exists a constant $C$ such that
$$|f| \leq C(1 + t).$$

Proof. Fix some constant $A > 0$ and let
$$\phi(x, t) := f(x, t) + tA.$$ By direct computations we have
$$\frac{\partial}{\partial t} \phi = \log \det g_+ \det h_- \det h_+ \det g_- + A$$
$$= \log \det g_+ \det \tilde{g}_- + \log \det \tilde{g}_- \det g_- + \log \det \tilde{g}_+ \det h_- \det g_- + A$$
$$\geq \log \det g_+ \det \tilde{g}_- \det g_-$$
where the last inequality follows by choosing $A$ sufficiently large with respect to the background data $g, h_\pm$. At a minimum point for $\phi$, it follows that $f$ is at a minimum, and so $\delta_+ \delta^+_\pm f > 0$. It now follows from the maximum principle that the minimum of $\phi$ is nondecreasing, and so the lower bound for $\phi$, and hence $f$, follows. A similar argument yields the upper bound. □

5.3. Estimate for the time derivative of the potential. 

Proposition 5.2. Given the setup above in the case $n = 2, \text{Rank} = 1$, there exists a constant $C > 0$ so that
$$\left| \frac{\partial f}{\partial t} \right| \leq C.$$ 

Proof. To begin we observe the identity
$$\Delta f = g^{\alpha_+ \alpha_-} f_{, \alpha_+ \alpha_-} + g^{\alpha_- \alpha_+} f_{, \alpha_- \alpha_+}$$
$$= g^{\alpha_+ \alpha_-} (g_{\alpha_+ \alpha_-} - \tilde{g}_{\alpha_+ \alpha_-}) - g^{\alpha_- \alpha_+} (\tilde{g}_{\alpha_+ \alpha_-} - g_{\alpha_+ \alpha_-})$$
$$= -g^{\alpha_+ \alpha_-} \tilde{g}_{\alpha_+ \alpha_-} + g^{\alpha_- \alpha_+} \tilde{g}_{\alpha_+ \alpha_-}.$$ Now let
$$\Phi(x, t) = (T - t) \dot{f} + f.$$
Combining the above observation with Lemma 4.2 yields the evolution equation

\[ \frac{\partial}{\partial t} \Phi = (T - t) \hat{f} - \hat{f} + \hat{f} \]

\[ = (T - t) \left[ \Delta \hat{f} + g^{\alpha_+ \alpha_-} P(h_\pm)_{\alpha_+ \alpha_-} - g^{\alpha_- \alpha_+} P(h_\pm)_{\alpha_- \alpha_+} \right] \]

\[ = \Delta \Phi - \Delta f + (T - t) \left[ g^{\alpha_+ \alpha_-} P(h_\pm)_{\alpha_+ \alpha_-} - g^{\alpha_- \alpha_+} P(h_\pm)_{\alpha_- \alpha_+} \right] \]

\[ = \Delta \Phi + g^{\alpha_+ \alpha_-} ((T - t) P(h_\pm) + \hat{g})_{\alpha_+ \alpha_-} - g^{\alpha_- \alpha_+} ((T - t) P(h_\pm) + \hat{g})_{\alpha_- \alpha_+} \]

\[ = \Delta \Phi + g^{\alpha_+ \alpha_-} (\hat{g}_T)_{\alpha_+ \alpha_-} - g^{\alpha_- \alpha_+} (\hat{g}_T)_{\alpha_- \alpha_+} \]

We will now apply the maximum principle to this identity. Fix some constant $2A > 0$. Since $|f|$ has a uniform bound by Lemma 5.1 if a maximum of $\Phi$ is sufficiently large we can conclude that $\frac{\partial \Phi}{\partial t} > A$. Using the evolution equation for $f$ this means that there is a uniform constant $C$, if we fix coordinates such that $\hat{g} = \text{Id}$ at the point in consideration, one has

\[ g_{\alpha_+ \alpha_-} > e^A g_{\alpha_- \alpha_+}, \]

which in turn implies that

\[ -g^{\alpha_- \alpha_+} \leq -e^A g^{\alpha_+ \alpha_-} \]

In particular, if $A$ is sufficiently large with respect to the background data $\omega_T$ this implies that

\[ g^{\alpha_+ \alpha_-} (\hat{g}_T)_{\alpha_+ \alpha_-} - g^{\alpha_- \alpha_+} (\hat{g}_T)_{\alpha_- \alpha_+} \leq g^{\alpha_+ \alpha_-} \left( (\hat{g}_T)_{\alpha_+ \alpha_-} - e^A (\hat{g}_T)_{\alpha_- \alpha_+} \right) \]

\[ \leq 0. \]

By the maximum principle we conclude that a uniform upper bound for $\Phi$ holds on any compact subinterval of $[0, T)$. A similar line of reasoning yields the lower bound for $\Phi$, finishing the proposition.

5.4. Metric upper bound.

**Proposition 5.3.** Given the setup above, suppose there exists a constant $K > 0$ so that

\[ g_t \geq \frac{1}{K} h. \]

Then there exists a constant $C$ depending on $K$ and the background data so that

\[ \text{tr}_h g + |\partial_+ \partial_- f|^2 \leq C. \]

**Proof.** Fix a constant $A > 0$ yet to be determined, and let

\[ \Phi := \log \text{tr}_h g + \text{tr}_h h + A \left| \partial^+ \partial^- f \right|^2. \]

By combining Lemmas 4.3, 4.4 and 4.9 we obtain

\[ \left( \frac{\partial}{\partial t} - \Delta \right) \Phi \leq \frac{1}{\text{tr}_h g} \left[ \text{tr}_h Q - g^{\gamma \beta} (h^{-1} g h^{-1})^{\gamma \beta} \left( \Omega^h \right)_{\gamma \beta} \right] \]

\[ - |\Upsilon(g, h)| g^{\gamma \beta} g_{\gamma \beta - 1, h} - \text{tr} g^{-1} h g^{-1} Q + g^{\gamma \beta} g^{\alpha \gamma} \left( \Omega^h \right)_{\alpha \gamma \beta} \]

\[ + A \left[ - |\nabla \partial_+ \partial_- f|^2 - |\nabla \partial_+ \partial_- f|^2 - 2 \langle Q, \partial_+ \partial_- f \otimes g, \nabla \partial_+ \partial_- f \rangle + 2 \Re \langle \partial_+ \partial_- f, \Psi \rangle \right]. \]
It remains to establish an a priori upper bound for the right hand side. To facilitate estimates we choose complex coordinates at a point where \( h^{-1}_\gamma = \delta^\gamma_i \). First we estimate

\[
(5.2) \quad \frac{1}{\text{tr}_hg} \text{tr}_h Q \leq \frac{C}{\text{tr}_hg} \text{tr}_h \left( |T_{g}^{-2}g| \right) \leq C |T_g|^2.
\]

Next we have

\[
(5.3) \quad \frac{1}{\text{tr}_hg} g^{\gamma\delta}(h^{-1}gh^{-1})^{\gamma\delta} \left( \Omega^h \right)_{\gamma\delta\tau} \leq \frac{CK}{\text{tr}_hg} h^{\gamma\delta} \left( h^{-1}gh^{-1} \right)^{\gamma\delta} \left( \Omega^h \right)_{\gamma\delta\tau} \leq \frac{CK}{\text{tr}_hg} \text{tr}_h g \left| \Omega^h \right|_h \leq CK.
\]

Next, since \( Q \geq 0 \) we have

\[
(5.4) \quad - \text{tr} g^{-1}h g^{-1}Q \leq 0.
\]

Also we estimate

\[
(5.5) \quad g^{\gamma\mu} g^{\delta\alpha} \left( \Omega^h \right)_{\gamma\mu\delta\alpha} \leq CK^2 \left| \Omega^h \right|_h \leq CK^2.
\]

Next, using that \( Q \) and \( \partial_+ \partial_- f \otimes_g \overrightarrow{\partial}_+ \partial_- f \) are positive definite we obtain

\[
(5.6) \quad - |\nabla \partial_+ \partial_- f|^2 - 2 \langle Q, \partial_+ \partial_- f \otimes_g \overrightarrow{\partial}_+ \partial_- f \rangle \leq 0.
\]

Plugging (5.2) - (5.5) into (5.1) yields the preliminary inequality

\[
(5.7) \quad \left( \frac{\partial}{\partial t} - \Delta \right) \Phi \leq C |T|^2 - |\Upsilon(g,h)|^2_{g,h} - A \left| \overrightarrow{\nabla} \partial_+ \partial_- f \right|^2 + 2A \Re \langle \partial_+ \partial_- f, \mu \rangle + C(K).
\]

With this inequality one can more clearly see why the quantity \( \Phi \) is chosen as it is. The main troublesome term is the \( C |T|^2 \) arising from the evolution of \( \log \text{tr}_h \omega \), the main quantity we want control over. The favorable \( \left| \overrightarrow{\nabla} \partial_+ \partial_- f \right|^2 \) term will be used to control this, which is why the term \( |\partial_+ \partial_- f|^2 \) is introduced into \( \Phi \). On the other hand this also introduces some “junk” terms arising from the quantity \( \Psi \), which involve the full Chern connection associated to \( g \) (as opposed to just the torsion). We balance these out with the favorable \( |\Upsilon|^2 \) term, which is the reason for the introduction of the term \( \text{tr}_g h \) into \( \Phi \). We now make this precise. We first observe the equality

\[
T_{\alpha_+\beta_+\gamma_+} = g_{\beta_+\gamma_+\alpha_-} = g_{\beta_+\gamma_+\alpha_-} + f_{\beta_+\alpha_-} = T_{\alpha_+\beta_+\gamma_+}.
\]

We conclude by the Cauchy-Schwarz inequality that

\[
(5.8) \quad C |T|^2 \leq C \left( |\overrightarrow{T}|_{g}^2 + |\overrightarrow{\nabla} \partial_+ \partial_- f|^2 \right) \leq C(K) + C \left| \overrightarrow{\nabla} \partial_+ \partial_- f \right|^2.
\]

It remains to estimate the term \( 2A \Re \langle \partial_+ \partial_- f, \Psi \rangle \). These estimates are all similar. For instance, we have

\[
g^{\gamma_+\mu_+} g^{\gamma_-\mu_-} \nabla_{\mu_-} \partial_+ \partial_- f_{\gamma_+\mu_+\gamma_-} = g^{\gamma_+\mu_+} g^{\gamma_-\mu_-} \left[ \nabla_{\mu_-} \partial_+ \partial_- f_{\gamma_+\mu_+\gamma_-} - \Upsilon_{\mu_-}^{\rho_+} \partial_+ \partial_- f_{\gamma_+\mu_+\gamma_-} + \Upsilon_{\mu_+}^{\rho_-} \partial_+ \partial_- f_{\gamma_+\mu_+\gamma_-} \right] f_{\gamma_+\mu_+}.
\]
Considering coordinates where $h_{ij} = \delta_{ij}$ and $g_{ij} = \lambda_i \delta_{ij}$ we have
\[
g^{\alpha + \mu} g^{\beta - \nu} g^{- \alpha - \beta} \mu^{\alpha + \mu} \rho^{\beta - \nu} \sigma^{\alpha + \mu} \rho^{\beta - \nu} f_{\alpha + \mu} f_{\beta - \nu} \]
\[
\leq g^{\alpha + \mu} g^{\beta - \nu} \left[ \delta \mu^{\alpha + \mu} \rho^{\beta - \nu} \sigma^{\alpha + \mu} \rho^{\beta - \nu} f_{\alpha + \mu} f_{\beta - \nu} \right] \]
\[
\leq \delta |\rho(g, h)|^2 g^{-1, g^{-1}, h} + C(K, \delta^{-1}, \tilde{T}) |\partial_+ \partial_- f|^2.
\]

By applying analogous estimates of this kind one arrives at the estimate, for any choice of $\delta > 0$,
\[
(5.9) \quad 2A (\partial_+ \partial_- f, \Psi) \leq 2A \delta |\rho(g, h)|^2 g^{-1, g^{-1}, h} + C(K, \delta^{-1}, \tilde{T}) |\partial_+ \partial_- f|^2.
\]

Plugging (5.8) and (5.9) into (5.5) yields
\[
\left( \frac{\partial}{\partial t} - \Delta \right) \Phi \leq (\delta A - 1) |\rho(g, h)|^2 g^{-1, g^{-1}, h} + (C - A) |\nabla \partial_+ \partial_- f|^2 + C(K, \delta^{-1}, \tilde{T}) \left( 1 + |\partial_+ \partial_- f|^2 \right).
\]

We choose $A$ large with respect to controlled constants and then $\delta = \frac{1}{A}$, to arrive finally at the differential inequality
\[
\left( \frac{\partial}{\partial t} - \Delta \right) \Phi \leq C(K, \delta^{-1}, \tilde{T}) \left( 1 + |\partial_+ \partial_- f|^2 \right) \leq C(K, \delta^{-1}, \tilde{T}) (1 + \Phi).
\]

It follows from the maximum principle that there is a constant $C$ such that
\[
\sup_{M \times [0, \tau]} \Phi \leq C.
\]

Since $\text{tr}_g h + A |\partial_+ \partial_- f|^2 > 0$, this immediately implies the upper bound for $\text{tr}_h g$. Also, since $g_t$ is bounded below, we have that $\log \text{tr}_t g$ is bounded below, and so this implies the upper bound for $|\partial^+ \partial^- f|^2$.

\[\square\]

**Remark 5.4.** This estimate in fact holds generally for solutions to pluriclosed flow, where a more general “torision potential” quantity plays the role of $\partial_+ \partial_- f$. This will be expounded upon in a future work.

### 6. Harnack estimate

The purpose of this section is to establish $C^\infty$-regularity of a solution to the pluriclosed flow in this setting assuming uniform equivalence of the metric. As stated in the introduction, the corresponding issue for Kähler-Ricci flow can be resolved in at least two ways. First, one can try to directly apply the maximum principle to the norm of the Chern connection potential, which is known as “Calabi’s third-order estimate” [5]. Alternatively, one can obtain an estimate on the full real Hessian of the Kähler potential and then apply the Evans-Krylov [8] [15] theory. Both of these methods rely on the “convexity” of the Monge-Ampere operator. In particular, the Evans-Krylov estimates apply only in the setting of a convex operator, whilst in Calabi’s approach the convexity appears in the form of a favorable quadratic nonlinearity which arises in the relevant maximum principle arguments. Conversely, equation (3.3) is nonconvex, and so neither of these approaches can succeed directly.

In what follows we prove the relevant estimate in the case $n = 2$. The restriction to dimension $n = 2$ comes from Lemma [6.4], where the special structure of the torsion in this dimension is exploited to obtain a favorable inequality for the evolution of the gradient of solutions to the heat equation against a pluriclosed flow background.
Theorem 6.1. Let \((M^4, g_0, J_{\pm})\) be a generalized Kähler surface. Let \(g_t\) denote the solution to pluriclosed flow with initial condition \(g_0\). Suppose the solution exists on \([0, T)\), and there is a constant \(A > 0\) so that
\[
A^{-1} g_0 \leq g_t \leq A g_0.
\]
Then given \(k \in \mathbb{N}\) there exists \(C(A, g_0, k)\) such that
\[
\sup_{M \times [0, T)} |g|_{C^k} \leq C .
\]
(6.1)
In particular, \(g_T := \lim_{t \to T} g_t\) exists and is smooth, and the flow extends smoothly past time \(T\).

Remark 6.2. The proof of Theorem 6.1 proceeds in three steps. First we show that if the statement were false one can construct an ancient solution to pluriclosed flow on \(\mathbb{C}^2\) via a blowup argument, which has the same uniform metric bounds, and is nonflat. Next we show that, for such a solution, any bounded ancient solution to the time-dependent Chern-Laplacian heat equation is constant. We then apply this rigidity to some special geometric quantities which imply flatness of the underlying pluriclosed flow, which is a contradiction which finishes the proof.

6.1. Construction of blowup limits.

Proposition 6.3. Let \((M^{2n}, g_t, J)\) be a solution to pluriclosed flow on a compact manifold \(M\). Suppose the solution exists on \([0, \tau)\), and there is a constant \(A > 0\) so that
\[
A^{-1} g_0 \leq g_t \leq A g_0.
\]
Suppose furthermore that, setting \(\Upsilon := \nabla^C g_t - \nabla^C g_0\), one has
\[
\limsup_{t \to \tau} |\Upsilon|_{g_t}^2 = \infty.
\]
(6.2)
There exists a blowup sequence of solutions which converges to a nonflat solution of pluriclosed flow on \((-\infty, 0] \times \mathbb{C}^n\) such that
\[
A^{-1} g_E \leq g_t \leq A g_E, \quad |g_t|_{C^k} \leq C(k, n, A).
\]
(6.3)
where \(g_E\) denotes the standard Euclidean metric on \(\mathbb{C}^n\).

Proof. Given a solution to pluriclosed flow as in the statement, we set
\[
\Phi(x, t) = |\Upsilon|^2 + |\Omega| + |\nabla T|.
\]
Assuming (6.2) holds, we choose a sequence of points \(\{(x_i, t_i)\}\) such that \(t_i \to \tau\) and
\[
\Phi(x_i, t_i) = \sup_{M \times [0, t_i)} \Phi.
\]
Certainly \(\lim_{i \to \infty} \Phi(x_i, t_i) = \infty\). Since \(M\) is compact there exists \(x_\infty \in M\) such that \(\lim_{i \to \infty} x_i = x_\infty\). By fixing local complex coordinates around \(x_\infty\) (mapping \(x_\infty\) to 0) such that at \(x_\infty\) one has \((g_0)_{ij} = \delta_{ij}\). Now set \(\lambda_i = \Phi(x_i, t_i)^{-\frac{1}{2}}\), and in the fixed coordinates we construct rescaled solutions \(g_i(x, t) = g_i(x_i + \lambda_i x, t_i + \lambda_i^2 t_i)\). By construction one obtains that the metrics \(g_i\) have uniform \(C^2\) bounds on \(B_{\lambda_i^{-1}(0)} \times [-\lambda_i^{-2} t_i, 0]\). Moreover, by the smoothing estimates of [19] Theorem 1.1 we obtain uniform estimates on all derivatives of the Chern curvature and its torsion, which implies uniform \(C^k\) bounds on the metric. Also by construction, the rescaled metrics satisfy \(\Phi(0, 0) = 1\). One obtains a limit using the Arzela-Ascoli theorem which satisfies the condition (6.3) by construction. \(\square\)
6.2. Rigidity of ancient heat equation solutions. The crucial vanishing result we need is a general result for ancient solutions to the heat equation against a pluriclosed flow background which is uniformly equivalent to the standard metric on \( \mathbb{C}^2 \).

**Lemma 6.4.** Let \((M^{2n}, \omega_t, J)\) be a solution to pluriclosed flow, and suppose \(f_t \in \Lambda^{0,1} \) is a one-parameter family satisfying

\[
\frac{\partial}{\partial t} f = \Delta_{g_t} f.
\]

Then

\[
\frac{\partial}{\partial t} \left| \overline{\partial} f \right|^2 = \Delta \left| \overline{\partial} f \right|^2 - \left| \nabla \overline{\partial} f \right|^2 - \left| \nabla f \right|^2 - \langle Q, \overline{\partial} f \otimes \partial f \rangle + 2 \Re \langle \overline{\partial} f, T \circ \partial \overline{\partial} f \rangle
\]  

(6.4)

**Proof.** To begin we observe an identity

\[
(\overline{\partial}\Delta f)_{\overline{\partial}} = g^i_{\overline{\partial}} \nabla_{\overline{\partial}} \nabla_i \nabla_{\overline{\partial}} f
\]

\[
= g^i_{\overline{\partial}} \nabla_{\overline{\partial}} \nabla_{\overline{\partial}} \nabla_i f
\]

\[
= g^i_{\overline{\partial}} \left[ \nabla_{\overline{\partial}} \nabla_i \nabla_{\overline{\partial}} f + T_{\overline{\partial} \overline{\partial}} \nabla_{\overline{\partial}} \nabla_i f \right]
\]

\[
= g^i_{\overline{\partial}} \left[ \nabla_{\overline{\partial}} \nabla_j \nabla_{\overline{\partial}} f + T_{\overline{\partial} \overline{\partial}} \nabla_{\overline{\partial}} \nabla_j f \right]
\]

\[
= \overline{\Delta} \nabla_{\overline{\partial}} f + g^i_{\overline{\partial}} T_{\overline{\partial} \overline{\partial}} \nabla_{\overline{\partial}} \nabla_i f
\]

\[
= (\overline{\Delta} \nabla_{\overline{\partial}} f + T \circ \partial \overline{\partial} f)_{\overline{\partial}}
\]

Thus we may compute

\[
\frac{\partial}{\partial t} \overline{\partial} f = \overline{\partial} \Delta f = \overline{\nabla} g_t \overline{\partial} f + T \circ \partial \overline{\partial} f.
\]

The result now follows from Lemma 4.7.

**□**

**Lemma 6.5.** Let \((M^4, \omega_t, J)\) be a solution to pluriclosed flow, and suppose \(f_t \in C^\infty(M)\) is a one-parameter family satisfying

\[
\frac{\partial}{\partial t} f = \Delta_{g_t} f.
\]

Then

\[
\frac{\partial}{\partial t} \left| \overline{\partial} f \right|^2 \leq \Delta \left| \overline{\partial} f \right|^2 - \left| \nabla \overline{\partial} f \right|^2
\]

**Proof.** We derive a special inequality for a \((0,1)\)-form \(\beta\) in this context. Since \(n = 2\), (20) Lemma 4.4) implies that \(Q = \frac{1}{k} |T|^2 g\), thus \(-\langle Q, \beta \otimes \overline{\beta} \rangle = -\frac{1}{k} |T|^2 |\beta|^2\). Also note that, in a unitary frame for \(g\), using the skew-symmetry of \(T\) one has

\[
|T|^2 = \sum_{i,j,k=1}^2 T_{ijk} T_{\overline{i} \overline{j} \overline{k}} = 2(|T_{121}|^2 + |T_{122}|^2)
\]
Let us now analyze the quantity $\Re \langle \beta, T \circ \partial \beta \rangle$. Working again in a unitary frame for $g$ we see

$$\langle \beta, T \circ \partial \beta \rangle = \sum_{i,j,l=1}^{2} T_{ji} \nabla_i \beta_l \beta_j$$

$$= \beta_1 \left( \sum_{l=1}^{2} T_{2l1} \nabla_l \beta_2 + \sum_{l=1}^{2} T_{2l1} \nabla_l \beta_1 \right)$$

$$= \beta_1 T_{211} \nabla_2 \beta_1 + \beta_1 T_{212} \nabla_2 \beta_2 + \beta_2 T_{211} \nabla_1 \beta_1 + \beta_2 T_{212} \nabla_1 \beta_2$$

$$\leq \frac{1}{2} \left[ |\beta_1|^2 |T_{211}|^2 + |\nabla_2 \beta_1|^2 + |\beta_1|^2 |T_{212}|^2 + |\nabla_2 \beta_2|^2 \right.$$  

$$+ |\beta_2|^2 |T_{211}|^2 + |\nabla_1 \beta_1|^2 + |\beta_2|^2 |T_{212}|^2 + |\nabla_1 \beta_2|^2 \right]$$

$$= \frac{1}{2} \left[ (|\beta_1|^2 + |\beta_2|^2) \left( |T_{211}|^2 + |T_{212}|^2 \right) + |\nabla \beta|^2 \right]$$

$$= \frac{1}{2} |\nabla \beta|^2 + \frac{1}{4} |T|^2 |\beta|^2.$$

Collecting the above discussion it follows that

$$- |\nabla \beta| + 2\Re \langle \beta, T \circ \partial \beta \rangle - \langle Q, \beta \otimes \overline{\beta} \rangle \leq 0.$$  

Using this inequality in (6.4) with $\beta = \overline{\partial f}$ yields the result. □

**Proposition 6.6.** Let $g_t$ be a solution to pluriclosed flow on $\mathbb{C}^2 \times (-\infty, 0]$ satisfying (6.3). Suppose $\phi(x, t)$ satisfies

$$\sup_{(-\infty, 0] \times \mathbb{C}^2} |\phi| \leq C,$$

$$\frac{\partial}{\partial t} \phi = \Delta \phi.$$

Then $\phi$ is constant.

**Proof.** Fix a constant $R > 1$, and let $\eta$ denote a cutoff function for the ball of radius $R > 0$. Using the uniform bounds on the metric and connection from (6.3) it follows that

$$|\nabla \eta| + |\nabla^2 \eta| \leq \frac{C}{R}.$$

Now let

$$\Phi(x, t) = \eta \left[ (t + R) |\overline{\partial f}|^2 + f^2 \right].$$

We directly compute

$$\frac{\partial}{\partial t} \Phi \leq \eta \left[ |\overline{\partial f}|^2 + (t + R) |\Delta |\overline{\partial f}|^2 + \Delta (f^2) - |\overline{\partial f}|^2 \right]$$

$$= \Delta \Phi - \Delta \eta \left[ (t + R) |\overline{\partial f}|^2 + f^2 \right] - \langle \nabla \eta, \nabla \left[ (t + R) |\overline{\partial f}|^2 + f^2 \right] \rangle$$

$$\leq \Delta \Phi + \frac{C |t + \sqrt{R}|}{R}.$$

Applying the maximum principle to this inequality on $[-\sqrt{R}, 0]$ yields the estimate

$$\sup_{\mathbb{C}^2} \Phi(x, 0) \leq \sup_{\mathbb{C}^2} \Phi(x, -\sqrt{R}) + C \leq \sup_{\mathbb{C}^2} f^2 + C \leq A + C.$$
It follows that, for all $R$,

$$R |\bar{\partial}f|^2 (0, 0) \leq \Phi(0, 0) \leq C$$

It follows that $|\bar{\partial}f|^2 (0, 0) = 0$. But this estimate can be repeated using cutoff functions centered at an arbitrary point in $\mathbb{C}^2 \times (-\infty, 0]$, and so $\bar{\partial}f \equiv 0$ identically. Thus at each time slice $f$ is constant, and then from the evolution equation for $f$ it follows that $f^{\partial\bar{\partial}}f \equiv 0$, and so $f$ is constant on $\mathbb{C}^2 \times (-\infty, 0]$.

6.3. Rigidity of blowup limits. In this subsection we establish a rigidity theorem for ancient solutions in the case $n = 2$. One key input is a further identity special to the this dimension.

**Lemma 6.7.** When $n = 2$ and $h_+$ is a flat metric, one has

$$\frac{\partial}{\partial t} \text{tr}_{h_+} g_+ = \Delta \text{tr}_{h_+} g_+ - \left\langle \partial^+ \log \frac{\det g_+}{\det h_+} \cdot h_+ \cdot \bar{\partial} \log \frac{\det g_+}{\det h_+}, h_+ \right\rangle_h.$$  

**Proof.** Beginning with the result of Lemma 4.4 we first observe that since $h_+$ is flat the last term involving $\Omega^h$ will vanish. Using that $n = 2$ we proceed to simplify the first order terms. First of all,

$$-|\Upsilon(g_+, h_+)|^2 g_{h, h} = -h^{\alpha_+\alpha_-} g^{\alpha_-\delta h} g_{\mu_+} \bar{\Gamma}^{\mu_+}_{\alpha_+\alpha_-} \bar{\Gamma}^{\mu_+}_{\mu_+} - h^{\mu_+\alpha_+} g^{\alpha_-\delta} g_{\mu_+} \bar{\Gamma}^{\mu_+}_{\delta \alpha_+} \bar{\Gamma}^{\mu_+}_{\alpha_+}$$

$$= -h^{\alpha_+\alpha_-} g^{\alpha_-\alpha_+} g_{\alpha_+\alpha_-} \bar{g}_{\alpha_+\alpha_-} - h^{\alpha_+\alpha_-} g^{\alpha_-\alpha_+} g_{\alpha_+\alpha_-} \bar{g}_{\alpha_+\alpha_-} + h^{\alpha_+\alpha_-} g^{\alpha_-\alpha_+} g_{\alpha_+\alpha_-} \bar{g}_{\alpha_+\alpha_-} + h^{\alpha_+\alpha_-} g^{\alpha_-\alpha_+} g_{\alpha_+\alpha_-} \bar{g}_{\alpha_+\alpha_-}.$$

Also

$$\text{tr}_{h_+} Q = h^{\alpha_+\alpha_-} Q_{\alpha_+\alpha_-}$$

$$= h^{\alpha_+\alpha_-} \left( g^{\alpha_-\alpha_+} g_{\alpha_+\alpha_-} T_{\alpha_+\alpha_-} T_{\alpha_+\alpha_-} + g^{\alpha_-\alpha_+} g_{\alpha_+\alpha_-} T_{\alpha_+\alpha_-} T_{\alpha_+\alpha_-} \right) + h^{\alpha_+\alpha_-} g^{\alpha_-\alpha_+} g_{\alpha_+\alpha_-} \bar{g}_{\alpha_+\alpha_-}$$

Combining the above two expressions yields the inner product claimed above. □

**Theorem 6.8.** Let $g_t$ be a solution to pluriclosed flow on $\mathbb{C}^2 \times (-\infty, 0]$ satisfying (6.3). Then $g_t \equiv g_0$ is a flat metric.

**Proof.** Since we are working on $\mathbb{C}^2$ against a flat background metric, by Lemma 4.2 we see that $\partial f$ satisfies the time-dependent heat equation. Moreover, using the uniform equivalence of $g$ with $g_E$ certainly $|\partial f| \leq C$. It follows from Proposition 6.6 that $\partial f \equiv c$. Using this, one observes from Lemma 6.7 that $\text{tr}_{h_+} g_\pm$ satisfy the heat equation, and are moreover bounded. Thus by Proposition 6.6 we obtain that $\text{tr}_{h_+} g_\pm$ are constant functions, and hence $g$ is flat. □

6.4. Proof of Theorem 6.1.

**Proof of Theorem 6.7** Let $\Upsilon = \nabla g_C - \nabla g_0$. First we show an estimate for $\Upsilon$. Assuming that $|\Upsilon|^2$ blows up at time $\tau$, by Proposition 6.3 we obtain a nonflat blowup solution to pluriclosed flow on $(-\infty, 0) \times \mathbb{C}^2$ satisfying (6.3). However, by Theorem 6.8 this solution is indeed flat, which is a contradiction. A similar blowup argument can be used to establish all of the higher $C^k$ estimates as well. □

7. Proof of Theorem 1.3

In this section we establish Theorem 1.3. Our proof consists of putting together the a priori estimates of 5 to prove the main analytic result, Theorem 7.1. Then we verify case by case that the manifolds stated in Theorem 1.3 satisfy the hypotheses of Theorem 7.1 finishing the proof of long time existence. We then prove the convergence statements in 7.2.
7.1. Long time existence statements.

Theorem 7.1. Let \((M, g, J_+)\) be a compact generalized Kähler surface satisfying \([J_+, J_-] = 0\). Suppose further that \(c_1(T^C_+M) \leq 0\) or \(c_1(T^C_-M) \leq 0\). Then Conjecture 7.1 holds for \((M, g, J_+)\). In particular, given \(g_0\) a generalized Kähler metric on \(M\), and defining \(\tau^*\) via (3.4), the solution to pluriholomorphic flow with initial condition \(g_0\) exists smoothly on \([0, \tau^*)\).

Proof. Suppose without loss of generality that \(c_1(T^C_+M) \leq 0\). Using the transgression formula for the first Chern class, we can choose a metric \(h_+\) on \(T^C_+M\) such that \(\rho(h_+) \leq 0\). Fix a time \(\tau < \tau^*\) and consider the setup as in [5]. Since \(\rho(h_+) \leq 0\), we can directly apply the maximum principle to the evolution equation of Lemma 4.1 to conclude an a priori lower bound for \(\det g_+\), which is just a lower bound for \(g_+\) since \(\text{Rank} T^C_+M = 1\). Next, from Proposition 5.2 we obtain a uniform estimate on the ratio \(\frac{\det g_+}{\det h_+}\), which since \(g_+\) is bounded below implies a uniform lower bound for \(\det g_-\), which again implies that \(g_-\) is bounded below since \(\text{Rank} T^C_-M = 1\). From Proposition 5.3 we thus conclude uniform upper and lower bounds for \(g_\tau\). Applying Theorem 6.1 we conclude uniform \(C^\infty\) estimates for \(g_\tau\) on \([0, \tau]\). The theorem follows. \(\square\)

Proof of Theorem 7.1. Let us treat the cases of long time existence in turn. As remarked upon in the introduction, generalized Kähler surfaces with commuting complex structure with rank 1 were classified in [1], building on work of Beauville [2]. We consider the different parts of this classification case by case.

**Ruled surfaces:** As shown in ([1], [2]), in this case \((M, J_+)\) is the projectivization of a projectively flat holomorphic plane bundle over a compact Riemann surface, which we have suppose to be of genus \(g \geq 1\). As computed in ([2] (4.2)), for our (indeed ANY) splitting of \(T^C(M)\), one has \(c_1(T^C_+M) \cdot c_1(T^C_-M) = c_2(TM) = 4(1 - g) \leq 0\). It follows that either \(T^C_+M \leq 0\), and so Theorem 7.1 yields the existence on \([0, \tau^*)\).

**Bi-elliptic surfaces:** In this case the surface is finitely covered by a torus. In particular, \(c_1(M, J_+) = 0\), and Theorem 7.1 guarantees the long time existence.

**Elliptic fibrations:** As explained in the proof of ([1], Theorem 1), the universal cover of \(M\) is \(\mathbb{C} \times \mathbb{H}\), with the fundamental group acting diagonally by isometries of the canonical product metric. Moreover, the splitting into \(T^C_+M\) corresponds to the obvious splitting of the tangent bundle of \(\mathbb{C} \times \mathbb{H}\), and so depending on the orientation, either \(c_1(T^C_+M) < 0\), and so Theorem 7.1 guarantees the long time existence.

**General type:** As shown in ([1], Theorem 1), the universal cover of these surfaces is \(\mathbb{H} \times \mathbb{H}\), with the fundamental group acting diagonally by biholomorphisms of \(\mathbb{H}\) on each factor. Moreover, the splitting of \(T^C(M)\) corresponds to the natural product structure of \(\mathbb{H} \times \mathbb{H}\). The canonical metric on \(\mathbb{H}\) is invariant under the full biholomorphism group, and so in particular the canonical product metric descends to \(M\), showing that \(c_1(T^C_+M) < 0\). Thus Theorem 7.1 guarantees the long time existence.

**Inoue surfaces:** To begin we briefly recall the construction of Inoue surfaces [14]. Fix a matrix \(M \in \text{SL}(3, \mathbb{Z})\) with one real eigenvalue \(\alpha > 0\) and two eigenvalues \(\beta \neq \overline{\beta}\). We fix \((a_1, a_2, a_3)\) a real eigenvector for eigenvalue \(\alpha\) and \((b_1, b_2, b_3)\) an eigenvector for the eigenvalue \(\beta\). Let \(\mathbb{H} = \{z | \Im z > 0\}\) denote the upper half space, and let \(\Gamma\) be the group of automorphisms of \(\mathbb{H} \times \mathbb{C}\) generated by

\[
f_0(z, w) = (\alpha z, \beta w), \quad f_j(z, w) = (z + a_j, w + b_j), \quad 1 \leq j \leq 3.
\]

The manifold \(S_M = (\mathbb{H} \times \mathbb{C}) / \Gamma\) was discovered in [14], and is called an Inoue surface. This is the simplest of three classes defined by Inoue, and the only one admitting generalized Kähler structure with commuting complex structures. In this case the generalized complex structures
come from the corresponding splitting $\mathbb{H} \times \mathbb{C}$. The two complex structures correspond to choosing $J_\pm = J_H \oplus (\pm J_C)$. On these Inoue surfaces there is a simple pluriclosed metric constructed by Tricerri,

$$\omega = \frac{\sqrt{-1}}{y^2}dz \wedge d\bar{z} + \sqrt{-1}ydw \wedge d\bar{w}.$$  

(7.1)

It is easily checked that in fact this metric is generalized Kähler. Moreover, we have

$$\rho(g_+) = -\sqrt{-1}\partial\bar{\partial} \log y = \frac{1}{4y^2}dz \wedge d\bar{z}$$

$$\rho(g_-) = -\sqrt{-1}\partial\bar{\partial} \log y^{-2} = -\frac{\sqrt{-1}}{2y^2}dz \wedge d\bar{z}$$

In particular, $c_1(T^C M) \leq 0$. Thus we have verified the hypotheses of Theorem 7.1 and the long time existence claim in this case follows.

7.2. Convergence statements. In this subsection we establish the convergence statements of Theorem 1.3.

**Proposition 7.2.** Let $(M^4, g_0, J_\pm)$ be a generalized Kähler surface such that $(M^4, J_+)$ is biholomorphic to a bi-elliptic surface. The solutions $g_t$ to pluriclosed flow with initial condition $g_0$ exists for all time and converges exponentially to a flat metric.

**Proof.** The long time existence follows from Theorem 1.3 and so we establish the convergence statements. Bi-elliptic surfaces admit tori as finite covers, and so by lifting to such a cover it suffices to establish the statement for the case $(M^4, J_+)$ is biholomorphic to a torus. Moreover, the complex structure $J_+$ respects a splitting $T^4 = T^2 \times T^2$, and the complex structures take the form $J_\pm = (\pm J_1) \oplus J_2$. Moreover, there is a flat background generalized Kähler metric $h = h_+ \oplus h_-$. □

**Proposition 7.3.** Let $(M^4, g_0, J_\pm)$ be a generalized Kähler surface such that $(M^4, J_+)$ is biholomorphic to surface of general type whose universal cover is $\mathbb{H} \times \mathbb{H}$, with fundamental group acting diagonally. The solutions $g_t$ to the normalized pluriclosed flow with initial condition $g_0$ exists for all time and converges exponentially to the unique Kähler-Einstein metric on $M$.

**Proof.** The long time existence for the unnormalized flow follows from Theorem 1.3. It follows easily that the normalized flow also exists for all time, and so we establish the convergence statement. Let $h$ denote the quotient of the standard product metric on $\mathbb{H} \times \mathbb{H}$, as described above. We first establish a more precise upper bound along the unnormalized flow along the lines of Proposition 5.3 but exploiting the favorable background metric. First observe that the term $\Psi$ of (4.2) vanishes in this case since we have a Kähler background. Thus, by applying the maximum principle to the result of Lemma 4.1 directly we obtain that

$$\sup_{M \times [0, \infty)} |\partial^+ \partial^- f|^2 \leq \sup_{M \times \{0\}} |\partial^+ \partial^- f|^2.\tag{7.2}$$

Let

$$\Phi = \log \frac{\det g_+}{\det h_+} + |\partial^+ \partial^- f|^2 - \log t - A,$$

where we choose $A$ sufficiently large below. We aim to show that $\Phi \leq 0$. Observe that the term $\Psi$ of (4.2) vanishes in this case since we have a Kähler background. Combining Lemmas 4.1 and
(4.9) yields

\[
\frac{\partial}{\partial t} \Phi \leq \Delta \Phi + \frac{1}{2} |T|^2 - |\nabla \partial^+ \partial^- f|^2 - \text{tr}_g(\rho(h_+)) - \frac{1}{t}.
\]

First we observe that since the metrics \(\tilde{g}\) in the setup of \(\S 5\) are all Kähler, and the Chern connection \(\nabla\) acts as \(\bar{\partial}\) on \(\Lambda^{2,0}\), we have that

\[
\nabla \partial^+ \partial^- f = \bar{\partial} \partial^+ \partial^- f = \partial \omega = T,
\]

and thus

\[
|\nabla \partial^+ \partial^- f|^2 \geq |T|^2.
\]

Also, by construction of \(h\) one certainly has \(\rho(h_+) \geq -Ch_+\), and so

\[
-\text{tr}_g \rho(h_+) \leq C \text{tr}_{g_+} h_+ = Ce^{-\log \frac{\det g_+}{\det h_+}}
\]

Now suppose \((x, t)\) is a spacetime maximum of \(\Phi\) such that \(\Phi(x, t) = 0\). Using (7.2) we conclude that if \(A\) is chosen sufficiently large with respect to \(g_0\) we have

\[
\log \frac{\det g_+}{\det h_+} = \log t + A - \frac{1}{2} \left| \partial^+ \partial^- f \right|^2 \geq \log t + \frac{A}{2}.
\]

Combining (7.3)-(7.5) yields that, at \((x, t)\),

\[
\frac{\partial}{\partial t} \Phi \leq \Delta \Phi + C \exp \left[ -\log t - \frac{A}{2} \right] - \frac{1}{t}.
\]

\[
= \Delta \Phi + \frac{C}{e^{\frac{A}{2} t}} - \frac{1}{t}
\]

\[
\leq \Delta \Phi,
\]

where the last line follows by choosing \(A\) larger still. It follows from the maximum principle that

\[
\sup_{[0, \infty)} \Phi \leq 0.
\]

Rearranging this inequality implies the metric upper bound

\[
\text{tr}_{h_+} g_+ = \frac{\det g_+}{\det h_+} \leq Ct.
\]

A similar, but easier, argument using the quantity \(\Psi = \log \frac{\det g_+}{\det h_+} - \log t + A\) shows that a similar linear lower bound holds for the metric, i.e. there is a constant \(c > 0\) so that

\[
c t \leq \frac{\det g_+}{\det h_+}.
\]

Similar arguments yield upper and lower bounds for \(g_-\). It follows that if \(g_t\) now denotes the solution to the normalized flow, then

\[
C^{-1} h \leq g_t \leq Ch.
\]

We now briefly sketch how to adapt Theorem 6.1 to the normalized flow. The starting point is Proposition 6.3, which constructs nontrivial blowup solutions to pluriclosed flow assuming the higher regularity fails. The extra term in the normalized flow causes no problem for this argument, and in fact is going to zero along the blowup sequence. In particular, one can still construct the blowup limit of Proposition 6.3 and the limiting solution is a solution to unnormalized pluriclosed flow. Then Theorem 6.8 finishes the proof as before.
Thus the metrics \( g_t \) along the normalized flow have uniform \( C^k \) bounds for all time. It follows that any sequence of times \( \{t_j\} \to \infty \) converges subsequentially in \( C^\infty \) to a limiting metric \( g_\infty \). Now one observes that \( \{g_t\} \) consists of a blowdown sequence of the original flow. Using the expanding entropy functional for pluriclosed flow and the attendant rigidity statements (Theorem 6.5, Corollary 6.8, Corollary 6.11), it follows that \( g_\infty \) is a Kähler-Einstein metric, in fact the unique Kähler-Einstein metric which \( M \) admits. In particular, every such sequence of metrics \( \{g_t\} \) converges to the same limit, and hence the whole flow converges to the unique Kähler-Einstein metric, as claimed.

\[\square\]

**Remark 7.4.** In [3] Boling studied homogeneous solutions to the pluriclosed flow on Inoue surfaces, and found that these always exist for all time, and after appropriate rescaling converge in the Gromov-Hausdorff topology to circles. It is reasonable to expect that this always happens, and one approach to answering this would be to establish that the solutions are “Type III” with bounded diameter (after rescaling), and then exploit properties of the expanding entropy functional ([21]) to exhibit the required convergence, in analogy with the classification of bounded diameter type III solutions of Ricci flow on three-manifolds given by Lott [17].

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