GLOBAL ASYMPTOTIC STABILITY OF TRAVELING WAVES TO THE ALLEN-CAHN EQUATION WITH A FRACTIONAL LAPLACIAN

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Abstract. In this paper, we study the asymptotic stability of traveling wave fronts to the Allen-Cahn equation with a fractional Laplacian. The main tools that we used are super- and subsolutions and squeezing methods.

1. Introduction. This paper is concerned with the global asymptotic stability of traveling wave fronts for the Allen-Cahn equation with a fractional Laplacian:

\[
\begin{cases}
\partial_t u + (-\Delta)^s u = f(u), & \forall t > 0, \ x \in \mathbb{R}, \\
u(x, 0) = \varphi(x), & x \in \mathbb{R},
\end{cases}
\]

where \(0 < s < 1\), and the nonlinearity \(f\) is of bistable type satisfying

\((H)\): \(f \in C^1(\mathbb{R}, \mathbb{R}), \ f(0) = f(1) = 0, \ f'(0) < 0, \ f'(1) < 0\) and \(\int_0^1 f(u)du \neq 0\);

\(\exists a \in (0, 1), \) such that \(f(a) = 0; \ f(u) > 0, \ \forall u \in (a, 1); \ f(u) < 0, \ \forall u \in (0, a)\).

By the definition of \(f\), we can find \(\delta_0 \in (0, 1)\) such that

\(f(u) > 0, \ \forall u \in [-\delta_0, 0) \cup (a, 1); \ f(u) < 0, \ \forall u \in (0, a) \cup (1, 1 + \delta_0]\). (2)

As we all know, the fractional Laplacian \((-\Delta)^s u\) can be defined as follows:

\[(-\Delta)^s u(x) = C_{n,s} \text{P.V.} \int_{\mathbb{R}^n} \frac{u(x) - u(y)}{|x - y|^{n+2s}} dy, \ \forall x \in \mathbb{R}^n,\]

where

\[C_{n,s} = \frac{2^{2s} \Gamma(\frac{n}{2} + s)}{\Gamma(1 - s) \pi^{\frac{n}{2}}}\]

and P.V. stands for the Cauchy principal value, see for example [12]. The fractional Laplacian can also be defined by Fourier transformation, see [12].

In the past few years, many researchers paid their attention to the traveling waves for reaction-diffusion equations with a fractional Laplacian, see [5, 15, 1, 16, 17]. Mellet et al. [15] considered a fractional reaction-diffusion equation with a combustion nonlinearity, in which the sliding argument was used to establish the existence of traveling waves of the problem where the fractional power \(s \in (\frac{1}{2}, 1)\). Gui and Huan [10] proved the nonexistence of traveling wave fronts for fractional reaction-diffusion equation with a combustion nonlinearity with \(s \in (0, \frac{1}{2}]\).

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Gui and Huan [10] also proved the existence and nonexistence of traveling wave fronts for the generalized Fisher-KPP model with the fractional Laplacian. For the bistable type model, Gui and Zhao [11] got the existence and uniqueness of monotonically increasing traveling wave fronts for this problem, where they focused on the unbalanced case (namely, \( \int_0^1 f(u) du \neq 0 \) and \( s \in (0, 1) \)). The main tool they used is the continuation method. Other related works on this topic have been considered in [3, 4, 5, 6, 7] and references therein.

For the classical Allen-Cahn equation, its traveling wave fronts have been widely studied. Fife and Mcleod [9] studied the global exponential stability of one-dimensional traveling waves. Ninomiya and Taniguchi [18] studied the existence and the stability of V-shaped traveling wave fronts in two-dimensional space. Taniguchi [21, 22] proved the existence, uniqueness and asymptotic stability of pyramidal traveling fronts in three-dimensional space. Matano et al. [14] studied the stability of planar waves in high-dimensional space. There were many follow-up studies for this problem.

We also note that Chen [8] studied the existence, uniqueness and global exponential stability of traveling wave fronts of a class of nonlinear and nonlocal evolution equations. Smith and Zhao [20] established the global exponential stability of traveling wave fronts in delayed reaction diffusion equations. Wang et al. [23] established the existence, uniqueness and global exponential stability of traveling wave fronts in reaction diffusion equations with nonlocal delays. Their methods give us a lot of inspiration. To the best of our knowledge, there is no result for the global asymptotic stability of traveling wave fronts for the Allen-Cahn equation with a fractional Laplacian. The aim of this paper is to present the global exponential stability of traveling waves of the problem (1) under the assumption (H).

In this paper, we care about the one dimension case, which means \( x \in \mathbb{R} \). Traveling wave fronts of (1) are the solutions with the form \( u(x, t) = U(x - ct) \), where \( c \) is called the speed of the traveling waves. When the assumption (H) holds, it follows from [11] that (1) admits a unique traveling wave front \((U, c)\) satisfying

\[
\begin{cases}
(-\Delta)^s U(y) - cU'(y) = f(U(y)), & \forall y \in \mathbb{R}, \\
\lim_{y \to -\infty} U(y) = 0, & \lim_{y \to \infty} U(y) = 1,
\end{cases}
\]

Furthermore, by [11], the traveling wave front \( U \) of (1) has the following properties:

(H1): \( c \neq 0, \) \( U \) is monotonically increasing, \( 0 < U(\xi) < 1, \) \( \xi \in \mathbb{R} \) and \( \lim_{\xi \to \pm \infty} U'(\xi) = 0. \)

In the next section, we first build the comparison principle for (1) and introduce a very important inequality. Then we establish several pairs of the super- and subsolutions. In the last section, we will use the squeezing method to prove our main result. Our main result reveals the global exponential stability of traveling wave fronts. The methods we used are based on [8, 20].

2. Comparison principle and super- subsolutions. Define a space

\[ C_{u,b}(\mathbb{R}) := \{ u : \mathbb{R} \to \mathbb{R} : u \text{ is bounded and uniformly continuous in } \mathbb{R} \}. \]
$C_{u,b}(\mathbb{R})$ is a Banach space with the $L^\infty(\mathbb{R})$ norm. In the following section of this article, let $X = C_{u,b}(\mathbb{R})$. For $0 < s < 1$, let $(-\Delta)^s = A$, as in [2], we get a continuous function $m = m(x,t)$, $x \in \mathbb{R}$, $t > 0$, such that

(A1) $0 < m \in C(\mathbb{R} \times (0, +\infty))$, $\int_{\mathbb{R}} m(x,t)dx = 1$ for all $t > 0$.

(A2) $m(\cdot, t) * m(\cdot, r) = m(\cdot, t + r)$, $(t, r) \in (0, \infty)^2$.

(A3) For $s \in (0, 1)$, $B > 1$ and $x \in \mathbb{R}$, $t > 0$, we have

$$B^{-1} \frac{t^\frac{s}{2}}{(1 + t^{-\frac{s}{2}}|x|^{1+2s})} \leq m(x,t) \leq B \frac{t^\frac{s}{2}}{(1 + t^{-\frac{s}{2}}|x|^{1+2s})}.$$ 

For $u \in X$, as in [2], define $T(t)$ as follows:

$$T(t)u(x) := (m(\cdot, t) * u)(x) = \int_{\mathbb{R}} m(y,t)u(x-y)dy.$$ 

It follows from [2] that $T(t)$ is a strongly continuous semigroup. In particular, $-A$ is the infinitesimal generator of $T(t)$, that is

$$-Au = \lim_{t \to 0^+} \frac{T(t)u - u}{t}, \forall u \in D(A) \subset X,$$ 

(4)

where the domain $D(A)$ of $A$ is defined by

$$D(A) := \{u \in X \text{ for which the limit in } X \text{ as } t \to 0 \text{ in (4) exists}\}.$$ 

By the definition, it is observed that $C^2_{u,b}(\mathbb{R}) \subset D(A)$, where

$$C^2_{u,b}(\mathbb{R}) := \{u \in C_{u,b}(\mathbb{R})|u', u'' \in C_{u,b}(\mathbb{R})\}.$$ 

Now we have the following lemma.

**Lemma 2.1.** Assume that $u(x,t) = u(t)(\cdot) \in C^1([0, T), X) \cap C([0, T), D(A))$ satisfies

$$\partial_t u(x,t) + (-\Delta)^s u(x,t) = h(x,t), \quad t \in [0, T), \quad x \in \mathbb{R},$$ 

(5)

where $h(x,t) \in C([0, T), X)$. Then

$$u(t) = T(t)u(0) + \int_0^t T(t-r)h(r)dr \quad \forall t \in [0, T).$$ 

(6)

The lemma can be proved by applying the standard theory of the strongly continuous semigroup [19, 24], so we omit the details.

Consider the initial value problem

$$\begin{cases}
\partial_t u(x,t) + (-\Delta)^s u(x,t) = f(u(x,t)), \quad \forall t > 0, \quad x \in \mathbb{R}, \\
u(x,0) = \varphi(x), \quad x \in \mathbb{R},
\end{cases}$$

(7)

where $\varphi(x) \in X$. We say that a function $u(x,t, \varphi) = u(t, \varphi)(x) \in C([0, T), X)$ is a mild solution of (7) with initial value $\varphi \in X$ on $t \in [0, T)$, if $u(x,t, \varphi)$ satisfies

$$u(t, \varphi) = T(t)\varphi + \int_0^t T(t-r)f(u(r))dr, \forall t \in [0, T).$$

Accordingly, if $u \in C([0, T), X)$ satisfies (6) for all $t \in [0, T)$, then we say that $u$ is a mild solution of (5) with initial value $u(0) \in X$ on $t \in [0, T)$.

**Definition 2.2.** A function $u(x,t) = u(t)(x) \in C([0, T), X)$ is called a mild solution to the sub-solution of (7) on $t \in [0, T)$, if $u(x,t)$ satisfies

$$u(t) \geq (\leq)T(t-t_\tau)u(\tau) + \int_\tau^t T(t-r)f(u(r))dr, \text{for all } 0 \leq \tau < t < T.$$
Clearly, if a function \( u(x,t) = u(t)(x) \in C([0,T), X) \) is both a mild supersolution and a mild subsolution, then it is a mild solution.

**Definition 2.3.** A function \( u(x,t) = u(t)(\cdot) \in C^1([0,T), X) \cap C([0,T), D(A)) \) is called a classical supersolution (subsolution) of \( (7) \) on \( t \in [0,T) \), if \( u(x,t) \) satisfies

\[
\partial_t u(x,t) + (-\Delta)^s u(x,t) \geq (\leq) f(u(x,t)), \quad \forall t > 0, \ x \in \mathbb{R}.
\]

Due to Lemma 2.1, it is easy to see that if a function \( u(x,t) \) is a classical supersolution (subsolution) of \( (7) \) on \( t \in [0,T) \), then it must be a mild supersolution (subsolution) of \( (7) \) on \( t \in [0,T) \).

The following theorem gives the existence of solutions, the comparison principle and a key inequality.

**Theorem 2.4.** Assume that \( (H) \) holds. Let \( \delta_0 > 0 \) be defined by (2). If \( \varphi \in X \) and \(-\delta_0 \leq \varphi(x) \leq 1 + \delta_0 \) for all \( x \in \mathbb{R} \), then \( (1) \) admits a unique mild solution \( u(x,t,\varphi) \) on \( [0,\infty) \). Furthermore, if \( \varphi \in D(A) \), then the mild solution \( u(x,t,\varphi) \) is a classical solution. In addition, assume that \( u^+(x,t) \) and \( u^-(x,t) \) are a mild supersolution and a mild subsolution of \( (1) \) on \( [0,\infty) \) respectively, and satisfy \( u^+(x,0) \geq u^-(x,0) \) for \( x \in \mathbb{R} \) and \(-\delta_0 \leq u^-(x,t), u^+(x,t) \leq 1 + \delta_0 \) for \( x \in \mathbb{R} \) and \( t \geq 0 \), then one has \( u^+(x,t) \geq u^-(x,t) \) for \( x \in \mathbb{R} \) and \( t > 0 \), and

\[
\psi(J,t) = \frac{B^{-1}}{t^{\frac{s}{2}}(1 + |t^{-\frac{s}{2}}(J + 1)|^{1+2s})},
\]

for any \( J \geq 0 \), \( t > 0 \), \( x \) and \( z \) with \( |x - z| \leq J \), where

\[
\psi(J,t) = \max_{u \in [-\delta_0,1+\delta_0]} [f'(u)] \quad \text{and} \quad B > 1 \text{ is defined by (A3) above.}
\]

**Proof.** Clearly, \( u(x,t) \equiv 1 + \delta_0 \) and \( u(x,t) \equiv -\delta_0 \) are the mild supersolution and the mild subsolution of \( (1) \), respectively. Let \( \varphi \in X \) satisfy \( -\delta_0 \leq \varphi(x) \leq 1 + \delta_0 \) for all \( x \in \mathbb{R} \). Note that the nonlinearity \( f \) satisfies the assumption \( (H) \). Then it follows from [13, Corollary 5] that \( (1) \) admits a unique mild solution \( u(x,t,\varphi) \) on \( t \in [0,\infty) \) satisfying \(-\delta_0 \leq u(x,t,\varphi) \leq 1 + \delta_0 \) for \( x \in \mathbb{R} \) and \( t \geq 0 \), see also \[20, 23\]. Due to the standard theory of the strongly continuous semigroup \[19, 24\], the mild solution \( u(x,t,\varphi) \) is a classical solution provided \( \varphi \in D(A) \), see also \[2, Remark 2.6\].

Let \( u^+(x,t) \) and \( u^-(x,t) \) be a mild supersolution and a mild subsolution of \( (1) \) on \( [0,\infty) \) respectively, and satisfy \( u^+(x,0) \geq u^-(x,0) \) for \( x \in \mathbb{R} \) and \(-\delta_0 \leq u^-(x,t), u^+(x,t) \leq 1 + \delta_0 \) for \( x \in \mathbb{R} \) and \( t \geq 0 \). Then again by [13, Corollary 5], we have

\[
-\delta_0 \leq u^-(x,t) \leq u(x,t,u^-(0)) \leq u(x,t,u^+(0)) \leq u^+(x,t) \leq 1 + \delta_0
\]

for all \( x \in \mathbb{R} \) and \( t \geq 0 \), where \( u(x,t,u^-(0)) \) and \( u(x,t,u^+(0)) \) denote the mild solutions of \( (1) \) with initial values \( u^-(x,0) \) and \( u^+(x,0) \) respectively. Thus, the remainder is to show the inequality. Since \( u^+(x,t) \) and \( u^-(x,t) \) are a mild supersolution and a mild subsolution of \( (1) \) on \( [0,\infty) \), we have

\[
u^+(t) \geq T(t - \tau)u^+(\tau) + \int_{\tau}^{t} \int_{\tau}^{r} T(t - r) f(u^+(r)) dr, \quad \forall t > \tau \geq 0
\]
Consequently, we get
\[ u^-(t) \leq T(t - \tau)u^-(\tau) + \int_\tau^t T(t - r)f(u^-(r))dr, \quad \forall t > \tau \geq 0. \]

Let \( v(x,t) = u^+(x,t) - u^-(x,t) \) for \( x \in \mathbb{R} \) and \( t \in [0, \infty) \). Then we have that
\[ v(t) \geq T(t - \tau)v(\tau) - E \int_\tau^t T(t - r)v(r)dr, \quad \forall t > \tau \geq 0. \]

Therefore, \( v(x,t) \) is a mild supersolution of the following equation
\[
\begin{cases}
\partial_t u + (-\Delta)^s u = -Eu, & \forall t > 0, \ x \in \mathbb{R}, \\
u(x,0) = u^+(x,0) - u^-(x,0), & x \in \mathbb{R}.
\end{cases}
\]  

Let \( z(t) \) be the mild solution of (8), that is,
\[ z(t) = T(t) (u^+(x,0) - u^-(x,0)) - E \int_0^t T(t - \tau)u(\tau)dr. \]

Obviously, we have \( z(t) = e^{-Et}T(t)(u^+(0) - u^-(0)) \). Using the comparison principle (see [13, Corollary 5] and [2, Subsection 2.3]), we obtain \( v(t) \geq z(t) \). Thus,
\[ u^+(x,t) - u^-(x,t) \geq e^{-Et}T(t)(u^+(0) - u^-(0))(x), \quad \forall x \in \mathbb{R}, \ t > 0. \]

By the definition of \( m(x,t) \), for any \( z \in \mathbb{R} \), we know that
\[
T(t)(u^+(0) - u^-(0))(x) = \int_{\mathbb{R}} m(t,x-y)(u^+(y,0) - u^-(y,0))dy \\
\geq \int_{\mathbb{R}} \frac{B^{-1}}{t^{\frac{1}{2}}(1 + |t^{-\frac{1}{2}}(x-y)|^{1+2s})} (u^+(y,0) - u^-(y,0))dy \\
\geq \int_{z}^{z+1} \frac{B^{-1}}{t^{\frac{1}{2}}(1 + |t^{-\frac{1}{2}}|^{1+2s})} (u^+(y,0) - u^-(y,0))dy.
\]

Now for any \( J \geq 0 \), for any \( |x-z| \leq J \), if \( z < y < z + 1 \), then \(-J - 1 \leq x - y \leq J \), hence
\[
T(t)(u^+(x,0) - u^-(x,0)) \geq \int_{z}^{z+1} \frac{B^{-1}}{t^{\frac{1}{2}}(1 + |t^{-\frac{1}{2}}(J+1)|^{1+2s})} (u^+(y,0) - u^-(y,0))dy \\
= \frac{B^{-1}}{t^{\frac{1}{2}}(1 + |t^{-\frac{1}{2}}(J+1)|^{1+2s})} \int_{z}^{z+1} (u^+(y,0) - u^-(y,0))dy \\
= \Psi(J,t) \int_{z}^{z+1} (u^+(y,0) - u^-(y,0))dy,
\]
where
\[ \Psi(J,t) = \frac{B^{-1}}{t^{\frac{1}{2}}(1 + |t^{-\frac{1}{2}}(J+1)|^{1+2s})}. \]

Consequently, we get
\[ u^+(x,t) - u^-(x,t) \geq \Psi(J,t)e^{-Et} \int_{z}^{z+1} (u^+(y,0) - u^-(y,0))dy. \]

The proof is complete. \( \Box \)

In the following we establish two pairs of super- and subsolutions of (1) on \([0, \infty)\), which is very important to prove our main result in next section.
Lemma 2.5. Assume that (H) holds. Let \((U, c)\) be the traveling wave front of (1) satisfying (3) and (H1). Then there exist positive numbers \(\beta_0, \sigma_0\) and \(\delta\) such that for every \(\delta \in (0, \delta]\) and \(\xi_0 \in \mathbb{R}\), the functions \(u^+\) and \(u^-\) defined by

\[
u^\pm(x, t) := U(x - ct + \xi_0 \pm \sigma_0 \delta(1 - e^{-\beta_0 t})) \pm \delta e^{-\beta_0 t}
\]

are a classical supersolution and a classical subsolution of (1) on \([0, \infty)\), respectively.

Proof. We only prove \(u^+(x, t)\) is a supersolution of (1). The other case can be treated in a similar way. Since

\[
\lim_{(u, \beta) \to (0, 0)} (f'(u) + \beta) = f'(0) < 0
\]

and

\[
\lim_{(u, \beta) \to (1, 0)} (f'(u) + \beta) = f'(1) < 0,
\]

we can give \(\beta_0 > 0\) and \(\delta^* \in (0, \delta_0)\) such that

\[
f'(u) < -\beta_0, \quad \forall u \in [-\delta^*, \delta^*] \cup [1 - \delta^*, 1 + \delta^*].
\]

By (H1), we can find \(\Phi = \Phi(\delta^*) > 0\) such that

\[
U(\xi) \geq 1 - \frac{\delta^*}{2} \quad \text{for all } \xi \geq \Phi \quad \text{and} \quad U(\xi) \leq \frac{\delta^*}{2} \quad \text{for all } \xi \leq -\Phi.
\]

Let

\[
c_1 = c_1(\delta^*) = \max \{|f'(u)|; u \in [-\delta^*, 1 + \delta^*]\}
\]

and

\[
\varphi_0 = \varphi_0(\beta_0, \delta^*) = \min \{U'(\xi) ||\xi| \leq \Phi\} > 0.
\]

Define

\[
L[u] := \partial_t u(x, t) + (\Delta)^* u(x, t) - f(u(x, t))
\]

and take

\[
\sigma_0 = \sigma_0(\beta_0, \delta^*) = \frac{\beta_0 + c_1}{\varphi_0(\beta_0, \delta^*)} > 0, \quad \bar{\delta} = \bar{\delta}(\delta^*) = \frac{\delta^*}{2} > 0.
\]

We may assume that \(\xi_0 = 0\). For any \(\delta \in (0, \bar{\delta}]\), let \(\xi(x, t) = x - ct + \sigma_0 \delta(1 - e^{-\beta_0 t})\).

Since \(U'(\xi) > 0\), we have that

\[
L[u^+] = \partial_t u^+(x, t) + (\Delta)^* u^+(x, t) - f(u^+(x, t))
\]

\[
= U'(\xi(x, t)) (-c + \sigma_0 \delta \beta_0 e^{-\beta_0 t}) - \delta \beta_0 e^{-\beta_0 t}
\]

\[
+ (\Delta)^* U(\xi(x, t)) - f(U(\xi(x, t))) + \delta e^{-\beta_0 t}
\]

\[
= U'(\xi(x, t)) (-c + \sigma_0 \delta \beta_0 e^{-\beta_0 t}) - \delta \beta_0 e^{-\beta_0 t} + c U'(\xi(x, t))
\]

\[
+ f(U(\xi(x, t))) - f(U(\xi(x, t))) + \delta e^{-\beta_0 t}
\]

\[
= \delta e^{-\beta_0 t} \left[ U'(\xi(x, t)) \sigma_0 \beta_0 - \beta_0 - \int_0^1 f(U(\xi(x, t))) + \tau \delta e^{-\beta_0 t} \right] d\tau
\]

for any \(t \geq 0\) and \(x \in \mathbb{R}\). We consider it in three cases.

Case 1: \(|\xi(x, t)| \leq \Phi\). By the choices of \(\Phi\) and \(c_1\), it is easy to see that

\[
\left| \int_0^1 f(U(\xi(t))) + \tau \delta e^{-\beta_0 t} \right| d\tau \leq c_1.
\]

By the definition of \(\sigma_0\), we obtain

\[
L[u^+] \geq \delta e^{-\beta_0 t} (\varphi_0 \sigma_0 \beta_0 - \beta_0 - c_1) = 0.
\]
Case II: $\xi(x, t) \geq \Phi$. By (10), we can see that $1 - \frac{\delta^*}{2} \leq U(\xi(x, t)) < 1$. Then
$$1 - \frac{\delta^*}{2} \leq U(\xi(x, t)) + \delta e^{-\beta_{0}t} < 1 + \frac{\delta^*}{2}.$$ 
Thus by (9), we have
$$L[u^+] \geq \delta e^{-\beta_{0}t}[U'(\xi(x, t))\sigma_0\beta_0 - \beta_0 - (-\beta_0)] \geq 0.$$

Case III: $\xi(x, t) \leq -\Phi$. In fact, by (10), $0 < U(\xi(x, t)) \leq \frac{\delta^*}{2}$. Therefore, one has
$$0 \leq U(\xi(x, t)) + \delta e^{-\beta_{0}t} \leq \frac{\delta^*}{2} + \delta e^{-\beta_{0}t} \leq \frac{\delta^*}{2} + \delta = \frac{\delta^*}{2} + \delta \leq \delta^*.$$
By (9), we obtain $\partial_t u^+(x, t) + (-\Delta)^s u^+(x, t) - f(u^+(x, t)) \geq 0$.
Combining Cases I-III, we have
$$L[u^+] \geq 0 \text{ for } x \in \mathbb{R}, t \geq 0.$$
The proof is complete. \hfill \Box

Define a function $\rho(\cdot) \in C^\infty(\mathbb{R})$ such that $\rho$ satisfies the following conditions:
$$\rho(x) = 0 \text{ for } x \leq 0; \ \rho(x) = 1 \text{ for } x \geq 4;$$
$$|\rho''(x)| \leq 1; \ 0 < \rho'(x) < 1 \text{ for } x \in (0, 4).$$
Then we can get the following lemma.

**Lemma 2.6.** Assume that (H) holds and let $\delta_0 = \min\{\frac{1-s}{2}, \frac{\delta}{2}, \delta_0\}$. Then for any $\delta \in (0, \delta_0]$, there exist two positive numbers $\varepsilon = \varepsilon(\delta)$ and $C = C(\delta)$ such that, for every $\xi \in \mathbb{R}$, the functions $z^+$ and $z^-$ defined by
$$z^+(x, t) = 1 + \delta - [1 - (a - 2\delta)e^{-\varepsilon t}]\rho(-\varepsilon x + \varepsilon \xi - \varepsilon Ct),$$
$$z^-(x, t) = -\delta + [1 - (1 - a - 2\delta)e^{-\varepsilon t}]\rho(\varepsilon x - \varepsilon \xi - \varepsilon Ct)$$
are a classical supersolution and a classical subsolution of (1) on $[0, \infty)$, respectively.

**Proof.** Consider $z^-(x, t)$. By a translation, we assume $\xi = 0$. Fix $\delta \in (0, \delta_0]$. Define
$$m_1 = m_1(\delta) = \min \left\{ \rho'(x); \frac{\delta}{2} \leq \rho(x) \leq 1 - \frac{\delta}{2} \right\} > 0.$$
Then we have
$$m_1 R_\delta^{-2s} \leq \frac{C_{1,s}}{s} R_\delta^{-2s} < \frac{1}{2} \min \left\{ f(u); u \in \left[ -\delta, -\frac{\delta}{2} \right] \right\} \leq \frac{1}{2} \min \left\{ f(u); u \in \left[ a + \frac{\delta}{2}, 1 - \frac{\delta}{2} \right] \right\}. \quad (11)$$
and
$$2C_{1,s} R_\delta^{-2s} < \frac{1}{2} \min \left\{ f(u); u \in \left[ -\delta, -\frac{\delta}{2} \right] \right\} \leq \frac{1}{2} \min \left\{ f(u); u \in \left[ a + \frac{\delta}{2}, 1 - \frac{\delta}{2} \right] \right\}. \quad (12)$$
Let $\varepsilon = \varepsilon(\delta) > 0$ such that
$$\frac{\varepsilon^2 C_{1,s}}{2 - 2s} R_\delta^{-2s} - \frac{1}{2} \min \left\{ f(u); u \in \left[ -\delta, -\frac{\delta}{2} \right] \right\} < 0 \quad (13)$$
and
$$\frac{\varepsilon^2 C_{1,s}}{2 - 2s} R_\delta^{-2s} - \frac{1}{2} \min \left\{ f(u); u \in \left[ a + \frac{\delta}{2}, 1 - \frac{\delta}{2} \right] \right\} < 0. \quad (14)$$
Take $C = C(\delta) > 0$ such that
$$-\varepsilon am_1 C + \varepsilon \frac{2C_{1,s}}{s} R_\delta^{-2s} + \frac{\varepsilon^2 C_{1,s}}{2 - 2s} R_\delta^{-2s} + \max \left\{ f(u); u \in \left[ -\frac{\delta}{2}, 1 - \frac{3\delta}{2} \right] \right\} < 0. \quad (15)$$
Now we estimate $\partial_t z^-(x, t)$ and $(-\Delta)^s z^-(x, t)$. For any $t \geq 0$ and $x \in \mathbb{R}$, one has that

$$\partial_t z^-(x, t) = -\varepsilon C[1 - (1 - a - 2\delta)e^{-\varepsilon t}]\rho'(\varepsilon x - \varepsilon Ct) + \varepsilon(1 - a - 2\delta)e^{-\varepsilon t}\rho(\varepsilon x - \varepsilon Ct) \leq \varepsilon(1 - a - 2\delta)e^{-\varepsilon t}\rho(\varepsilon x - \varepsilon Ct) \leq \varepsilon.$$

By [12], we have

$$(-\Delta)^s z^-(x, t) = [1 - (1 - a - 2\delta)e^{-\varepsilon t}]C_{1,s}\int_{|x-y| > R\delta} \frac{\rho(\varepsilon x - \varepsilon Ct) - \rho(\varepsilon y - \varepsilon Ct)}{|x-y|^{1+2s}} dy + [1 - (1 - a - 2\delta)e^{-\varepsilon t}]C_{1,s}P.V.\int_{|x-y| \leq R\delta} \frac{\rho(\varepsilon x - \varepsilon Ct) - \rho(\varepsilon y - \varepsilon Ct)}{|x-y|^{1+2s}} dy.$$

Let

$$B_1 := \int_{|x-y| > R\delta} \frac{\rho(\varepsilon x - \varepsilon Ct) - \rho(\varepsilon y - \varepsilon Ct)}{|x-y|^{1+2s}} dy,$$

$$B_2 := P.V.\int_{|x-y| \leq R\delta} \frac{\rho(\varepsilon x - \varepsilon Ct) - \rho(\varepsilon y - \varepsilon Ct)}{|x-y|^{1+2s}} dy.$$

Since $0 \leq \rho(x) \leq 1$ for all $x \in \mathbb{R}$, we have

$$|B_1| \leq \int_{|x-y| > R\delta} \frac{2}{|x-y|^{1+2s}} dy = \frac{2}{s} R\delta^{-2s}.$$

For $B_2$, we have

$$B_2 = P.V.\int_{|x-y| \leq R\delta} \frac{\rho(\varepsilon x - \varepsilon Ct) - \rho(\varepsilon y - \varepsilon Ct)}{|x-y|^{1+2s}} dy$$

$$= - \int_{|x| < R\delta} \rho(\varepsilon(x+w) - \varepsilon Ct) - \rho(\varepsilon x - \varepsilon Ct) - \varepsilon w\rho'(\varepsilon x - \varepsilon Ct)) \frac{1}{|w|^{1+2s}} dw$$

$$= \int_{|x| < R\delta} \int_0^1 \frac{\varepsilon^2 w^2 \rho''(\varepsilon x + \varepsilon wp - \varepsilon Ct)}{|w|^{1+2s}} dr dp dw.$$

Because of $|\rho''| \leq 1$, we get

$$B_2 \leq \frac{\varepsilon^2}{2 - 2s} R\delta^{2-2s}.$$

By the above estimates, we have that

$$L[z^-] := \partial_t z^-(x, t) + (-\Delta)^s z^-(x, t) - f(z^-(x, t))$$

$$\leq -\varepsilon C[1 - (1 - a - 2\delta)e^{-\varepsilon t}]\rho'(\varepsilon x - \varepsilon Ct) + \varepsilon(1 - a - 2\delta)e^{-\varepsilon t}\rho(\varepsilon x - \varepsilon Ct) + \frac{2C_{1,s}^2 R\delta^{-2s}}{2 - 2s} + \frac{\varepsilon^2 C_{1,s} R\delta^{2-2s}}{2 - 2s} - f(z^-(x, t))$$

$$\leq -\varepsilon C[1 - (1 - a - 2\delta)e^{-\varepsilon t}]\rho'(\varepsilon x - \varepsilon Ct) + \varepsilon + \frac{2C_{1,s}^2 R\delta^{-2s}}{2 - 2s} + \frac{\varepsilon^2 C_{1,s} R\delta^{2-2s}}{2 - 2s} - f(z^-(x, t)).$$

We consider it in three cases.
Lemma 3.1. Assume that $x \in \mathbb{R}$ for all $t$. We first give the following two lemmas.

\[
L[z^-] \leq \varepsilon + \frac{2C_{1,s}}{s} R_{2s}^{-2s} + \frac{C_{1,s}}{2} R_{2s}^{-2s} - \min \left\{ f(u); u \in \left[ -\delta, -\frac{\delta}{2} \right] \right\} < 0.
\]

Case II: If $\rho(x - Ct) > 1 - \frac{\delta}{2}$, then $a + \frac{\delta}{2} \leq z^-(x, t) \leq 1 - \delta$. By (12) and (14), we have

\[
L[z^-] \leq \varepsilon + \frac{2C_{1,s}}{s} R_{2s}^{-2s} + \frac{C_{1,s}}{2} R_{2s}^{-2s} - \min \left\{ f(u); u \in \left( a + \frac{\delta}{2}, 1 - \delta \right) \right\} < 0.
\]

Case III: If $\frac{\delta}{2} \leq \rho(x - Ct) \leq 1 - \frac{\delta}{2}$, then $-\frac{\delta}{2} \leq z^-(x, t) \leq 1 - \frac{3\delta}{2}$. By (15), we have

\[
L[z^-] \leq -\varepsilon C \rho'(\varepsilon x - \varepsilon Ct) + \varepsilon + \frac{2C_{1,s}}{s} R_{2s}^{-2s} + \frac{C_{1,s}}{2} R_{2s}^{-2s} \quad + \max \left\{ |f(u)|; u \in \left[ -\frac{\delta}{2}, 1 - \frac{3\delta}{2} \right] \right\} < 0.
\]

In summary, we get

\[
\partial_t z^-(x, t) + (-\Delta)^s z^-(x, t) - f(z^-(x, t)) \leq 0, \quad \forall x \in \mathbb{R}, \quad t \geq 0.
\]

Then $z^-(x, t)$ is a subsolution of (1) on $[0, \infty)$. We can prove $z^+(x, t)$ is a supersolution of (1) on $[0, \infty)$ in a similar way. This completes the proof.

**Remark 1.** It is worth remarking that $z^+(x, t)$ and $z^-(x, t)$ in Lemma 2.6 have the following properties:

(i) $z^+(x, 0) = 1 + \delta$ for all $x \geq \xi$; $z^+(x, 0) \geq a - \delta$ for all $x \in \mathbb{R}$. For all $x \leq \xi - Ct - \frac{\delta}{2}$, we have $z^+(x, t) \leq \delta + (a - 2\delta)e^{-\alpha t}$;

(ii) $z^-(x, 0) = -\delta$ for all $x \leq \xi$; $z^-(x, 0) \leq a + \delta$ for all $x \in \mathbb{R}$. For all $x \geq \xi + Ct + \frac{\delta}{2}$, we have $z^-(x, t) \geq 1 - \delta - (1 - a - 2\delta)e^{-\alpha t}$.

3. Stability of traveling wave front. In this section, we establish the global asymptotic stability of traveling wave front for (1). In order to prove the main result, we first give the following two lemmas.

Throughout this section, let $U(x - ct)$ be the traveling wave front of (1) satisfying (3) and (H1). By Lemma 2.5, we define the following two functions:

\[
w^\pm(x, t, \xi, \delta) = U \left( x - ct + \eta \pm \sigma_0 \delta \left( 1 - e^{-\beta_0 t} \right) \right) \pm \delta e^{-\beta_0 t},
\]

for $x \in \mathbb{R}$, $t \in [0, \infty)$, $\xi \in \mathbb{R}$ and $\delta \in (0, \tilde{\delta})$, where $\tilde{\delta}$, $\sigma_0$, $\beta_0$ are as in Lemma 2.5.

**Lemma 3.1.** Assume that (H) holds. Then we can find a positive number $\varepsilon^*$, such that, if $u(x, t)$ is a mild solution of (1) with initial value $u(x, 0) \in X$ satisfying $0 \leq u(x, 0) \leq 1$ for all $x \in \mathbb{R}$, and for some $\xi \in \mathbb{R}$, $h > 0$, $T > 0$ and $0 < \delta < \delta_0$, where $\delta_0 = \min \left\{ \frac{1 - \alpha}{T}, \frac{3}{2}, \delta, \frac{1}{\sigma_0} \right\}$, the following inequality

\[
w^- (x, 0, cT + \xi, \delta) \leq u(x, t) \leq w^+ (x, 0, cT + \xi + h, \delta), \quad x \in \mathbb{R}
\]

holds, then for every $t \geq T + 1$, there exist $\tilde{\xi}(t), \tilde{\delta}(t), \tilde{h}(t)$ such that

\[
w^- \left( x, 0, -ct + \tilde{\xi}(t), \tilde{\delta}(t) \right) \leq u(x, t) \leq w^+ \left( x, 0, -ct + \tilde{\xi}(t) + \tilde{h}(t), \tilde{\delta}(t) \right), \quad x \in \mathbb{R},
\]

where $\tilde{\xi}(t), \tilde{\delta}(t), \tilde{h}(t)$ are as follows:

\[
\tilde{\xi}(t) \in \left[ \xi + \varepsilon^* \min(1, h) - \sigma_0(2\delta + \varepsilon^* \min(1, h)), \xi + \varepsilon^* \min(1, h) - \sigma_0(1 - e^{-\beta_0 h}) \right],
\]

\[
\tilde{\delta}(t) \in \left[ 1 + \varepsilon^* \min(1, h) - \sigma_0(2\delta + \varepsilon^* \min(1, h)), 1 + \varepsilon^* \min(1, h) - \sigma_0(1 - e^{-\beta_0 h}) \right],
\]

\[
\tilde{h}(t) \in \left[ 1 - \varepsilon^* \min(1, h) + \sigma_0(2\delta + \varepsilon^* \min(1, h)), 1 - \varepsilon^* \min(1, h) + \sigma_0(1 - e^{-\beta_0 h}) \right].
\]
Proof. As described in Lemma 2.5, \( w^+(x, t, -cT + \xi + h, \delta) \) and \( w^-(x, t, -cT + \xi, \delta) \) are super- and subsolutions of (1), respectively. By Theorem 2.4, we have

\[
\begin{align*}
&\left(\delta e^{-\beta\sigma} + \varepsilon^* \min(1, h)\right) e^{-\beta_0(t-(T+1))}, \\
&\tilde{h}(t) \in [0, -\varepsilon^* \min(1, h) + h + \sigma_0 \delta e^{-\beta\sigma} + \varepsilon^* \min(1, h) + 2\sigma_0 \delta].
\end{align*}
\]

In other words, \( \tilde{h}(t) \in [0, -\varepsilon^* \min(1, h) + h + \sigma_0 \delta e^{-\beta\sigma} + \varepsilon^* \min(1, h) + 2\sigma_0 \delta] \).

Thus at least one of the following is true:

\[
\text{Let } z = cT - \xi. \text{ Using Theorem 2.4 once again, we have that, for any } J \geq 0, x \in \mathbb{R} \text{ with } |x - z| \leq J \text{ and } t \geq 0, \text{ there is }
\]

\[
U(x - c(T + t) + \xi - \sigma_0 \delta(1 - e^{-\beta_0 t})) - \delta e^{-\beta_0 t} \leq u(x, T + t)
\]

\[
\leq U(x - c(T + t) + \xi + h + \sigma_0 \delta(1 - e^{-\beta_0 t})) + \delta e^{-\beta_0 t}, \quad x \in \mathbb{R}, \quad t \geq 0.
\]

(16)

Let \( z = cT - \xi \). Using Theorem 2.4 once again, we have that, for any \( J \geq 0, x \in \mathbb{R} \) with \( |x - z| \leq J \) and \( t \geq 0 \), there is

\[
u(x, T + t) - w^-(x, t, -cT + \xi, \delta) \geq \Psi(J, t) e^{-Et} \int_z^{z+1} (u(y, T) - w^-(y, 0, -cT + \xi, \delta)) dy
\]

\[
= \Psi(J, t) e^{-Et} \int_z^{z+1} (u(y, T) - (U(y - cT + \xi) - \delta)) dy
\]

\[
= \Psi(J, t) e^{-Et} \int_z^{z+1} (u(y, T) - U(y - cT + \xi)) dy + \delta.
\]

By (H1), we obtain \( \lim_{|\eta| \to \infty} U'(\eta) = 0 \). Then we can choose a positive number \( N_0 \) such that \( U'(\eta) \leq 1 \) for all \( |\eta| \geq N_0 \). Let \( J = N_0 + |c| + 1, \varepsilon_1 = \frac{1}{2} \min\{U'(x); |x| \leq 2\} > 0, \)

\[
h = \min(1, h), \text{ then }
\]

\[
\int_z^{z+1} (U(y - cT + \xi + h) - U(y - cT + \xi)) dy = \int_0^1 (U(y + h) - U(y)) dy \geq 2\varepsilon_1 h.
\]

Thus at least one of the following is true:

(i) \( \int_z^{z+1} (u(y, T) - U(y - cT + \xi)) dy \geq \varepsilon_1 h, \)

(ii) \( \int_z^{z+1} (U(y - cT + \xi + h) - u(y, T)) \geq \varepsilon_1 h. \)

In fact, we only need to consider case (i). For all \( |x - z| \leq J \), let \( t = 1, \Psi_0(J) = \Psi(J, 1), \) then

\[
u(x, T + 1) \geq U(x - c(T + 1) + \xi - \sigma_0 \delta(1 - e^{-\beta_0}))
\]

\[
- \delta e^{-\beta_0} + \Psi(J, 1) e^{-E} \int_z^{z+1} (u(y, T) - w^-(y, 0, -cT + \xi, \delta)) dy
\]

\[
\geq U(x - z - c - \sigma_0 \delta(1 - e^{-\beta_0})) - \delta e^{-\beta_0} + \Psi_0(J) e^{-E} \varepsilon_1 h.
\]

Let

\[
J_1 = J + |c| + 3, \varepsilon^* = \min \left\{ \min_{|x| \leq J_1} \varepsilon_1 \Psi_0(J) e^{-E} \frac{U'(x)}{U'(x)}, h \right\}.
\]

By the mean value theorem, we have that for all \( x \) with \( |x - z| \leq J, \)

\[
u(x - z - c + \varepsilon^* h - \sigma_0 \delta(1 - e^{-\beta_0})) - U(x - z - c - \sigma_0 \delta(1 - e^{-\beta_0}))
\]

\[
= U'(\eta_1) e^{\varepsilon_1 h} \leq \varepsilon_1 h \Psi_0(J) e^{-E}.
\]
Thus,
\[ u(x, T + 1) \geq U \left( x - z - e^{*}\tilde{h} - \sigma_0 \delta(1 - e^{-\beta_0}) \right) - \delta e^{-\beta_0} \]
\[ = U \left( x - c(T + 1) + \xi + e^{*}\tilde{h} - \sigma_0 \delta(1 - e^{-\beta_0}) \right) - \delta e^{-\beta_0}. \quad (17) \]

By the mean value theorem and the choice of \( N_0 \) and \( J \), we know that, for all \( |x - z| \geq J \),
\[ U(x - c(T+1)+\xi-\sigma_0 \delta(1-e^{-\beta_0}))-U(x-c(T+1)+\xi+e^{*}\tilde{h}-\sigma_0 \delta(1-e^{-\beta_0})) \geq -e^{*}\tilde{h}. \]

Then, for all \( |x - z| \geq J \),
\[ U(x - c(T+1)+\xi-\sigma_0 \delta(1-e^{-\beta_0})) \geq U(x - c(T+1)+\xi+e^{*}\tilde{h}-\sigma_0 \delta(1-e^{-\beta_0})) - e^{*}\tilde{h}. \]

Therefore, let \( t = 1 \) in (16), we have
\[ u(x, T + 1) \geq U \left( x - c(T + 1) + \xi + e^{*}\tilde{h} - \sigma_0 \delta(1 - e^{-\beta_0}) \right) - (e^{*}\tilde{h} + \delta e^{-\beta_0}), \quad (18) \]

for all \( |x - z| \geq J \). By (17) and (18), we obtain that, for all \( x \in \mathbb{R} \),
\[ u(x, T + 1) \geq U \left( x - c(T + 1) + \xi + e^{*}\tilde{h} - \sigma_0 \delta(1 - e^{-\beta_0}) \right) - e^{*}\tilde{h} - \delta e^{-\beta_0}, \]

That is,
\[ u(x, T + 1) \geq U \left( x - c(T + 1) + \xi + e^{*}\tilde{h} - \sigma_0 \delta(1 - e^{-\beta_0}) \right) - (e^{*}\tilde{h} + \delta e^{-\beta_0}), \]

Then
\[ u(x, T + 1) \geq w^-(x, 0, \eta, \delta e^{-\beta_0} + e^{*}\tilde{h}), \]

where
\[ \eta = -c(T + 1) + \xi + e^{*}\tilde{h} - \sigma_0 \delta(1 - e^{-\beta_0}). \]

Let \( \tilde{\xi} = -\sigma_0 \delta(1 - e^{-\beta_0}) \), we have
\[ \eta = -c(T + 1) + \xi + e^{*}\tilde{h} + \tilde{\xi}. \]

By Theorem 2.4, for all \( t' \geq 0 \), we obtain
\[ u(x, T + 1 + t') \geq w^-(x, t', \eta, \delta e^{-\beta_0} + e^{*}\tilde{h}). \quad (19) \]

For any \( t \geq T + 1 \), let \( t' = t - (T + 1) \) in (19), then
\[ u(x, t) \geq w^-(x, t - (T + 1), \eta, \delta e^{-\beta_0} + e^{*}\tilde{h}) \]
\[ = U \left( x - c(t - (T + 1)) + \eta - \sigma_0 (\delta e^{-\beta_0} + e^{*}\tilde{h})(1 - e^{-\beta_0(t-(T+1))}) \right) \]
\[ - (\delta e^{-\beta_0} + e^{*}\tilde{h}) e^{-\beta_0(t-(T+1))} \]
\[ = U \left( x - c(t - (T + 1)) + \eta - \sigma_0 (\delta e^{-\beta_0} + e^{*}\tilde{h})(1 - e^{-\beta_0(t-(T+1))}) \right) \]
\[ - \tilde{\delta}(t) e^{-\beta_0}, \]

where \( \tilde{\delta}(t) = (\delta e^{-\beta_0} + e^{*}\tilde{h}) e^{-\beta_0(t-(T+1))} \). By the choice of \( \eta \) and the monotonicity of \( U(\cdot) \), we have
\[ u(x, t) \geq w^- \left( x, 0, -ct + \tilde{\xi}(t), \tilde{\delta}(t) \right), \]

where
\[ \tilde{\xi}(t) = \xi + e^{*}\tilde{h} - \sigma_0 \delta(1 - e^{-\beta_0}) - \sigma_0 (\delta e^{-\beta_0} + e^{*}\tilde{h})(1 - e^{-\beta_0(t-(T+1))}). \]

It is easy to see that
\[ \tilde{\xi}(t) \geq \xi + e^{*}\tilde{h} - \sigma_0 (2\tilde{\delta} + e^{*}\tilde{h}) \]
and
\[ \tilde{\xi}(t) \leq \xi + e^{*}\tilde{h} - \sigma_0 (1 - e^{-\beta_0}). \]
This completes the proof.

\[ \tilde{\zeta}(t) \in \left[ \xi + \varepsilon^* \tilde{h} - \sigma_0(2\delta + \varepsilon^* \tilde{h}), \xi + \varepsilon^* \tilde{h} - \sigma_0(1 - e^{-\beta_0}) \right]. \]

For any \( t \geq T \), by (3.2), it follows that

\[ u(x, t) \leq U \left( x - ct + \xi + h + \sigma_0\delta(1 - e^{-\beta_0(t-T)}) \right) + \delta e^{-\beta_0(t-T)}. \]

Then for all \( t \geq T + 1 \),

\[ u(x, t) \leq U \left( x - ct + \xi + h + \sigma_0\delta(1 - e^{-\beta_0(t-T)}) \right) + \tilde{\delta}(t) = U \left( x - ct + \xi + h + \sigma_0\delta(1 - e^{-\beta_0(t-T)}) \right) + \tilde{\delta}(t), \quad e^{-\beta_0}. \]

Therefore for all \( t \geq T + 1 \),

\[ u(x, t) \leq w^+ \left( x, 0, -ct + \tilde{\zeta}(t) + \tilde{h}(t), \tilde{\delta}(t) \right), \]

where

\[ \tilde{h}(t) = -\varepsilon^* \tilde{h} + h + \sigma_0(\delta e^{-\beta_0} + \varepsilon^* \tilde{h})(1 - e^{-\beta_0(t-(T+1))}) + \sigma_0\delta(1 - e^{-\beta_0}) \]

\[ = -\varepsilon^* \tilde{h} + h + \sigma_0(\delta e^{-\beta_0} + \varepsilon^* \tilde{h})(1 - e^{-\beta_0(t-(T+1))}) + 2\sigma_0\delta - \sigma_0\delta(e^{-\beta_0} + e^{-\beta_0(t-T)}). \]

Notice that \( h \geq \varepsilon^* \), we know that \( \tilde{h}(t) \geq 0 \), and

\[ \tilde{h}(t) \leq -\varepsilon^* \tilde{h} + h + \sigma_0(\delta e^{-\beta_0} + \varepsilon^* \tilde{h}) + 2\sigma_0\delta. \]

This completes the proof. \( \square \)

**Lemma 3.2.** Assume that (H) holds. Let \( \varphi \in X \) satisfy \( 0 \leq \varphi \leq 1 \) and

\[ \liminf_{x \to \infty} \varphi(x) > a, \quad \limsup_{x \to \infty} \varphi(x) < a. \]

Then for any \( \delta > 0 \), there exist \( T = T(\varphi, \delta) > 0 \), \( \xi = \xi(\varphi, \delta) \in \mathbb{R} \), \( h = h(\varphi, \delta) > 0 \) such that

\[ w^-(x, 0, -cT + \xi, \delta) \leq u(x, T, \varphi) \leq w^+(x, 0, -cT + \xi + h, \delta), \quad x \in \mathbb{R}. \]

**Proof.** By Theorem 2.4, \( u(x, t, \varphi) \) exists globally on \( [0, \infty) \), and for \( x \in \mathbb{R} \), \( t \geq 0 \), we have \( 0 \leq u(x, t, \varphi) \leq 1 \). For any \( \delta > 0 \) and \( \delta_0 \) mentioned in Lemma 2.6, take \( 0 < \delta_1 = \delta_1(\delta, \varphi) < \min(\delta, \delta_0) \) such that

\[ \liminf_{x \to \infty} \varphi(x) > a + \delta_1 \] \quad and \quad \[ \limsup_{x \to \infty} \varphi(x) < a - \delta_1. \]

Then there exists \( M_0 = M_0(\varphi, \delta_1) > 0 \) such that

\[ \varphi(x) \geq a + \delta_1 \] for \( x \geq M_0 \) \quad and \quad \[ \varphi(x) \leq a - \delta_1 \] for \( x \leq -M_0. \] \quad \( (20) \)

Let \( \varepsilon = \varepsilon(\delta_1) \) and \( C = C(\delta_1) \) be defined in Lemma 2.6 with \( \delta \) replaced by \( \delta_1 \). Let \( \xi^+ = -M_0, \xi^- = M_0 \). Let \( z^+(x, t) \) be defined in Lemma 2.6 with \( \xi = \xi^+ \), respectively. By (20) and Remark 1, we have

\[ \varphi(x) \leq a - \delta_1 \leq z^+(x, 0), \quad \forall x \leq -M_0, \]

\[ \varphi(x) \leq 1 - \delta_1 = z^+(x, 0), \quad \forall x \geq -M_0, \]

and

\[ \varphi(x) \geq a + \delta_1 \geq z^-(x, 0), \quad \forall x \geq M_0, \]

\[ \varphi(x) \geq 0 \geq -\delta_1 = z^-(x, 0), \quad \forall x \leq M_0. \]
Then
\[ z^-(x, 0) \leq \varphi(x) \leq z^+(x, 0), \forall x \in \mathbb{R}. \]

By Lemma 2.6 and Theorem 2.4, we have
\[ z^-(x, t) \leq u(x, t, \varphi) \leq z^+(x, t), \ x \in \mathbb{R}. \]

By the definition of \( z^+(x, t) \), we can choose a sufficiently large number \( T > 0 \), such that when \( t \geq T \), we have
\[ \delta_1 + ( a - 2\delta_1 ) e^{-\varepsilon t} < \delta, \quad \text{and} \quad 1 - \delta_1 - ( 1 - a - 2\delta_1 ) e^{-\varepsilon t} > 1 - \delta. \]

By Remark 1,
\[ u(x, t, \varphi) \leq z^+(x, t) \leq \delta_1 + ( a - 2\delta_1 ) e^{-\varepsilon t} < \delta, \forall x \leq \xi^+ - Ct - 4\varepsilon^{-1}, \]
\[ u(x, t, \varphi) \geq z^-(x, t) \geq 1 - \delta_1 - ( 1 - a - 2\delta_1 ) e^{-\varepsilon t} > 1 - \delta, \forall x \geq \xi^- + Ct + 4\varepsilon^{-1}. \]

Let \( x^- = \xi^+ - Ct - 4\varepsilon^{-1} \), \( x^+ = \xi^- + Ct + 4\varepsilon^{-1} \), from the previous conclusion, we can see that when \( t = T \),
\[ u(x, t, \varphi) < \delta \quad \text{for} \quad x \leq x^- \quad \text{and} \quad u(x, t, \varphi) > 1 - \delta \quad \text{for} \quad x \geq x^+. \]

By (3), we can find a sufficiently large positive number \( H \), such that \( x^+ < H \frac{1}{2}, \ x^- > -H \frac{1}{2} \) and
\[ U(x) + \delta > 1 \quad \text{for} \quad x \geq H \frac{1}{2} \quad \text{and} \quad U(x) - \delta < 0 \quad \text{for} \quad x \leq -H \frac{1}{2}. \]

Since
\[ 0 < U(x) < 1, \ 0 \leq u(x, t, \varphi) \leq 1, \ \forall x \in \mathbb{R}, \ t \in [0, \infty), \]
we then have
\[ U(-H + x) - \delta \leq u(x, t, \varphi) \leq U(H + x) + \delta, \ \forall x \in \mathbb{R}, \ t = T. \]

Let \( \xi_0 = -H + cT, \ h_0 = 2H > 0 \), we have
\[ U(x - cT + \xi_0) - \delta \leq u(x, T, \varphi) \leq U(x - cT + \xi_0 + h_0) + \delta, \ \forall x \in \mathbb{R}. \]

Let \( \xi = \xi_0 \) and \( h = h_0 \), we obtain
\[ w^-(x, 0, -cT + \xi, \delta) = U(x - cT + \xi) - \delta = U(x - cT + \xi_0) - \delta \leq u(x, T, \varphi), \]
and
\[ w^+(x, 0, -cT + \xi + h, \delta) = U(x - cT + \xi + h) + \delta = U(x - cT + \xi_0 + h_0) + \delta \geq u(x, T, \varphi). \]

Thus,
\[ w^-(x, 0, -cT + \xi, \delta) \leq u(x, T, \varphi) \leq w^+(x, 0, -cT + \xi + h, \delta). \]

The proof is complete.

\[ \square \]

**Theorem 3.3.** Assume that (H) holds. Let \( U(x - ct) \) be the traveling wave front of (1) satisfying (3) and (H1), then \( U(x - ct) \) is globally asymptotically stable. That is, for every \( \varphi \in X \) with \( 0 \leq \varphi \leq 1 \) and
\[ \liminf_{x \to \infty} \varphi(x) > a, \limsup_{x \to -\infty} \varphi(x) < a, \]
there exists \( k > 0 \) such that, the solution \( u(x, t, \varphi) \) of (1) satisfies
\[ |u(x, t, \varphi) - U(x - ct + \xi)| \leq Ke^{-kt}, \ \forall x \in \mathbb{R}, \ t \geq 0, \]
where \( K = K(\varphi) > 0 \) and \( \xi = \xi(\varphi) \in \mathbb{R} \) are some constants.
Proof. We assume that $\beta_0$, $\sigma_0$, $\tilde{\delta}$ are as in Lemma 2.5, where $0<\sigma_0<1$. Choose a $\delta^* > 0$ and

$$\delta^* < \min \left\{ \min_{|t| \leq t_1} \frac{\varepsilon_1 \Psi_0(J) e^{-E(T+1)} (1-\sigma_0)}{\mathcal{U}'(x) \sigma_0}, \frac{(1-\sigma_0)\tilde{h}}{3\sigma_0} \right\},$$

such that $1 > k^* = e^\gamma - \sigma_0 (\delta^* e^{-\beta_0} + \varepsilon^*) - 2\sigma_0 \delta^* > 0$, where $\varepsilon^*$ is determined in Lemma 3.1. Then fix a $t^* \geq 1$ such that $e^{-\beta_0(t^*-1)}(1+\frac{\gamma}{t^*}) < 1 - k^*.$

We prove this theorem in following two steps.

Claim 1. There exist $T^* = T^*(\varphi) > 0$, $\xi^* = \xi^*(\varphi) \in \mathbb{R}$ such that

$$w^-(x,0,-cT^* + \xi^*,\delta^*) \leq u(x,T^*,\varphi) \leq w^+(x,0,-cT^* + \xi^* + 1,\delta^*). \quad (21)$$

By Lemma 3.2, we can find

$$w^-(x,0,-cT^* + \xi^*,\delta^*) \leq u(x,T^*,\varphi) \leq w^+(x,0,-cT^* + \xi^* + 1,\delta^*),$$

if $h \leq 1$, by the monotonicity of $U(\cdot)$, (21) satisfies. Thus, we only need to prove the case for $h > 1$. Define

$$N = \max\{m; m \in \mathbb{N}_+, \, mk^* < h\}.$$

By $0 < k^* < 1$, we obtain $N \geq 1$, $Nk^* < h \leq (N+1)k^*$, thus, $0 < h - Nk^* \leq (N+1)k^* - Nk^* = k^* < 1$. Using (22), Lemma 3.1 and the choices of $t^*$ and $k^*$, we have

$$w^-(x,0,-c(T + t^*) + \tilde{\xi}(T + t^*),\tilde{\delta}(T + t^*)) \leq u(x,T + t^*,\varphi) \leq w^+(x,0,-c(T + t^*) + \tilde{\xi}(T + t^*) + \tilde{h}(T + t^*),\tilde{\delta}(T + t^*)), \quad \forall x \in \mathbb{R}, \quad (23)$$

where

$$\tilde{\xi}(T + t^*) \in \left[ \xi + \varepsilon^* - \sigma_0 (2\delta^* + \varepsilon^*), \xi + \varepsilon^* - \sigma_0 \delta^* (1 - e^{-\beta_0}) \right],$$

$$\tilde{\delta}(T + t^*) = (\delta^* e^{-\beta_0} + \varepsilon^*) e^{-\beta_0(t^*-1)} \leq (1-k^*) \delta^* < \delta^*,$$

$$0 \leq \tilde{h}(T + t^*) \leq -\varepsilon^* + h + \sigma_0 (\delta^* e^{-\beta_0} + \varepsilon^*) + 2\sigma_0 \delta^* = h - k^*.$$  

We can repeat this process for $N$ times, then for some $\tilde{\xi} \in \mathbb{R}$, $0 < \tilde{\delta} \leq \delta^*, \hspace{2mm} 0 \leq \tilde{h} = h - Nk^* < 1$, and $T + t^*$ replaced by $T + Nt^*$, (23) holds. Let $T^* = T + Nt^*$, $\xi^* = \tilde{\xi}.$

By using the monotonicity of traveling wave front $U(\cdot)$, (21) holds.

Claim 2. Let $p = \sigma_0 (2\delta^* + \varepsilon^*), T_m = T^* + mt^*$, $\delta^*_m = (1-k^*)m\delta^*$, $h_m = (1-k^*)m$, $m \geq 0$. Then we can find a sequence $\{\xi_m\}_{m=0}^\infty$ such that $|\xi_{m+1} - \xi_m| \leq ph_m$, $m \geq 0$, and

$$w^-(x,0,-cT_m + \xi_m,\delta^*_m) \leq u(x,T_m,\varphi) \leq w^+(x,0,-cT_m + \xi_m + h_m,\delta^*_m), \quad x \in \mathbb{R}, \quad m \geq 0. \quad (24)$$

Let $m = 0$, by Claim 1, we can see that (24) holds. Thus we suppose that (24) satisfies for $m = l \geq 0$. Using Lemma 3.1, let $m = l$, $T = T_l$, $\xi = \xi_l$, $h = h_l$, $\delta_l = \delta^*_l$, $t = T_l + t^* = T_{l+1}$, $t^* \geq 1$, then

$$w^-(x,0,-cT_{l+1} + \tilde{\xi},\tilde{\delta}) \leq u(x,T_{l+1},\varphi) \leq w^+(x,0,-cT_{l+1} + \tilde{\xi} + \tilde{h},\tilde{\delta}), \quad x \in \mathbb{R},$$

where

$$\tilde{\xi}(t) \in \left[ \xi_l + \varepsilon^* h_l - \sigma_0 (2\delta^*_l + \varepsilon^* h_l), \xi_l + \varepsilon^* h_l - \sigma_0 \delta^*_l (1 - e^{-\beta_0}) \right],$$

$$\tilde{\delta} = (\delta_l e^{-\beta_0} + \varepsilon^* h_l) e^{-\beta_0(t_{l+1}-t_l-1)} \leq (1-k^*) (\delta^* + \varepsilon^*) e^{-\beta_0(t^*-1)} \leq (1-k^*) (1-k^*) \delta^*, \quad \text{that is,} \hspace{2mm} \tilde{\delta} \leq \delta^*_l.$$ 

$$\tilde{h} \leq -\varepsilon^* h_l + h_l + \sigma_0 (\delta_l e^{-\beta_0} + \varepsilon^* h_l) + 2\sigma_0 \delta_l.$$


\[
= -\varepsilon^*(1 - k^*)^l + (1 - k^*)^l + (1 - k^*)^l(\sigma_0\delta^* e^{-\beta_0} + \sigma_0\varepsilon^* + 2\sigma_0\delta^*)
\]

\[
= (1 - k^*)^l(-\varepsilon^* + 1 + \sigma_0\delta^* e^{-\beta_0} + \sigma_0\varepsilon^* + 2\sigma_0\delta^*)
\]

\[
= (1 - k^*)^{l+1}
\]

\[h_{l+1}.
\]

Let \(\xi_{l+1} = \xi_l\), it is easy to see that

\[
|\xi_{l+1} - \xi_l| \leq |\xi_l + \varepsilon^*h_l - \sigma_0\delta_l^*(1 - e^{-\beta_0}) - (\xi_l + \varepsilon^*h_l - \sigma_0(2\delta^*_l + \varepsilon^*h_l))|
\]

\[
\leq (1 - k^*)^l|\sigma_0(2\delta^* + \varepsilon^*)
\]

\[
= ph_l.
\]

Thus \(|\xi_{m+1} - \xi_m| \leq ph_m, m \geq 0\) holds for \(m = l\), (24) holds for \(m = l + 1\). Using the induction, for all \(m \geq 0\), (24) holds and \(|\xi_{m+1} - \xi_m| \leq ph_m, m \geq 0\).

\[
U\left(x - ct + \xi_m - \sigma_0\delta_m^*(1 - e^{-\beta_0(t-T_m)})\right) - \delta_m^* e^{-\beta_0(t-T_m)} \leq u(x, t, \varphi)
\]

\[
\leq U\left(x - ct + \xi_m + h_m + \sigma_0\delta_m^*(1 - e^{-\beta_0(t-T_m)})\right) + \delta_m^* e^{-\beta_0(t-T_m)}.
\]

For all \(t \geq T^*\), let \(m = \left\lfloor \frac{t-T^*}{T_*} \right\rfloor\) be the largest integer not greater than \(\frac{t-T^*}{T_*}\). Let \(\delta(t) = \delta_m^*, \xi(t) = \xi_m - \sigma_0\delta_m^*, h(t) = h_m\). Then

\[
T_m = T^* + mt^* \leq t \leq T^* + (m+1)t^* = T_{m+1}.
\]

Notice that for all \(x \in \mathbb{R}, t \geq T^*\),

\[
U(x - ct + \xi(t)) - \delta(t) \leq u(x, t, \varphi) \leq U(x - ct + \xi(t) + h(t)) + \delta(t).
\]

(25)

Furthermore, we have

\[
\delta(t) = (1 - k^*)^m \delta^* \leq \delta^* \exp\left(\left(\frac{t - T^*}{t^*} - 1\right) \ln(1 - k^*)\right),
\]

(26)

\[
h(t) = h_m + 2\sigma_0\delta_m^* = (1 - k^*)^m + 2\sigma_0(1 - k^*)^m \delta^* = (1 + 2\sigma_0\delta^*)(1 - k^*)^m.
\]

Thus,

\[
h(t) \leq (1 + 2\sigma_0\delta^*) \exp\left(\left(\frac{t - T^*}{t^*} - 1\right) \ln(1 - k^*)\right).
\]

(27)

Moreover, as proved in [20], for each \(r \geq t \geq T^*\), it holds that

\[
|\xi(r) - \xi(t)| \leq q\delta(t), \text{ where } q = \frac{p}{k^*\delta^*} + 2\sigma_0.
\]

(28)

By (28), we can see that \(\lim_{t \to \infty} \xi(t)\) exists, and define \(\xi(\infty) = \lim_{t \to \infty} \xi(t)\). We have that for every \(t \geq T^*\),

\[
|\xi(\infty) - \xi(t)| \leq q\delta(t).
\]

Thus

\[
|\xi(\infty) - \xi(t)| \leq q\delta^* \exp\left(\left(\frac{t - T^*}{t^*} - 1\right) \ln(1 - k^*)\right).
\]

(29)

By (25), (26), (27), (29), and defining \(k = -\frac{1}{T_*} \ln(1 - k^*)\), we get the desired result of the theorem.

\[
\square
\]

**Remark 2.** By the global asymptotic stability of traveling waves of (1), we can easily find that this traveling wave front is unique.

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