EXISTENCE RESULTS OF HILFER INTEGRO-DIFFERENTIAL EQUATIONS WITH FRACTIONAL ORDER

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Abstract. The paper is relevance with Hilfer derivative with fractional order which is generalized case of R-L and Caputo's sense. We ensured the solution using noncompact measure and Mönch’s fixed point technique. Illustrative examples are included for the applicability of presented technique.

1. Introduction. In the recent years, scientific community renders more attention on fractional calculus for its wider applications in science and engineering. Especially, the applications of interdisciplinary fields can be effectively designed using fractional derivatives. This is a emerging field in this scenario even though fractional derivatives have a long mathematical history. It is developed by Kilbas et al. [12], Miller and Ross [14], Podulbny [16], Zhou [32, 33, 34], Trujillo[10], the papers [1, 2, 4, 5, 6, 7, 8, 9, 13, 17, 18, 19, 20, 21, 22, 23, 24, 25, 26, 27, 28, 29, 34] and the references with in.

A strong provocation to approach fractional calculus comes from physics. In some recent papers, the solutions of two parameter FDE are investigated by Laplace transforms with probability density function which was discussed by Gu and Trujillo [10], Wang and Zhang [30].

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The couple of parameter family of fractional derivative with \( \zeta, \eta \) permits one to interpolate between the R-L and the Caputo derivative discussed in Hilfer [8, 10, 11, 22, 31] fractional derivative.

Stimulated by the above discussion, Hilfer fractional derivative with nonlocal condition will be analyzed in this paper using Laplace transforms and probability density function. Initially, we will introduce a mild solution, and derive some results which is used to prove the mild solution by noncompactness measure. In section three, we obtain the solution using Mönch’s fixed point technique [15]. Finally, we discussed some applications based on this study.

The Hilfer integro-differential equation of fractional order is,

\[
D^{\zeta, \eta}_{0+} y(\omega) = Ay(\omega) + R(\omega, y_\omega, \int_0^\omega h_1(\omega, s, y_s) ds), \quad \omega \in D = [0, b_1], \quad (1.1)
\]

\[
I^{(1-\eta)}_{0+} y(0) = \sum_{m=1}^n c_m y(\omega_m), \quad \omega \in [-r, 0], \quad (1.2)
\]

where \( D^{\zeta, \eta}_{0+} \) Hilfer derivative of order \( 0 \leq \zeta \leq 1, 0 < \eta < 1 \) and \( y(\cdot) \) takes the value in \( X \) with \( \| \cdot \|_x \).

\( A \) be the infinitesimal generator of bounded linear operator in strongly semigroup theory.

2. Preliminaries. Some basic facts and lemmas, properties are introduced in this section which will be used effectively in upcoming sections. Let \( D = [0, b_1] \) and \( D' = (0, b_1] \), by \( B(D, Y) \) and \( B(D', Y) \) depicts all continuous functions from \( D \) to \( Y \) and \( D' \) to \( Y \), respectively. Now, consider

\[ X = \left\{ y \in B(D', Y) : \lim_{\omega \to 0} \omega^{(1-\zeta)}(1-\eta) y(\omega) \right\} \]

exists and infinite with \( \| \cdot \|_x \) identified as

\[ \| Y \|_x = \sup_{\omega \in D'} \left\{ \omega^{(1-\zeta)}(1-\eta) \right\}. \]

Also,

(i) If \( \zeta = 1 \), then \( X = B(D, Y) \) and \( \| \cdot \|_x = \| \cdot \|_y \).

(ii) Let \( y(\omega) = \omega^{(1-\eta)} x(\omega) \), for \( \omega \in D' \), \( y \in X \) iff \( x \in B(D, Y) \).

The closed bounded and convex subsets of \( B(D, Y) \) and \( X \) are given by \( E_r(D) = \{ x \in B(D, Y) / \| x \| \leq r \} \), \( E^2_r(D') = \{ y \in X / \| Y \|_x \leq r \} \).

**Definition 2.1.** [16, 32] Fractional equation of order \( q \) for \( f : [a, \infty) \to R \) is

\[ I^q_{a+} f(\omega) = \frac{1}{\Gamma q} \int_a^\omega (\omega - s)^{q-1} f(s) ds, \quad \omega > a, \quad q > 0, \]

equipped with the RHS is point-wise characterized on \( [a, \infty) \).

**Definition 2.2.** [16, 32] The R-L equation of order \( q > 0 \) is characterized as

\[ D^q_{a+} f(\omega) = \frac{1}{\Gamma(k-q)} \int_a^\omega (\omega - s)^{k-q-1} f(s) ds, \quad \omega > a, \quad k-1 < q < k. \]
Definition 2.3. [16, 32] For $f$, the Caputo’s equation of order $q > 0$ is characterized as
\[ ^cD_{a+}^q f(\omega) = D_{a+}^q \left[ f(\omega) - \sum_{k=0}^{n-1} \frac{\omega^k}{k!} f^{(k)}(0) \right], \quad \omega > a, \ n - 1 < q < n. \]

Definition 2.4. (HFD) [10, 11] If $\zeta \in [0, 1]$ and $\eta \in (0, 1)$, then the generalized R-L fractional derivative is given by
\[ D^\zeta_\eta f(\omega) = I^{(1-\eta)}_{\alpha+} \frac{d}{d\omega} I^{(1-\zeta)(1-\eta)}_{\alpha+} f(\omega), \]
where $a$ is the lower limit.

Remark 2.1. (i) Hilfer derivative becomes the classical R-L derivative of fractional order when $\zeta = 0$, $0 < \eta < 1$ and $a = 0$,
\[ D^0_\eta f(\omega) = \frac{d}{d\omega} I^{(1-\eta)}_{\alpha+} f(\omega) = D^0_{\alpha+} f(\omega). \]
(ii) Hilfer derivative becomes the classical Caputo derivative when $\zeta = 1$, $0 < \eta < 1$ and $a = 0$,
\[ D^1_\eta f(\omega) = I^{(1-\eta)}_{\alpha+} \frac{d}{d\omega} f(\omega) = {^cD}_\eta^1 f(\omega). \]

Now we give the Hausdorff noncompact measure $\alpha(\cdot)$ defined on $\wedge$ which is bounded subset of Banach space $Y$ is $\alpha(\Lambda) = \inf \{ \varepsilon > 0, \Lambda \}$ has a finite $\varepsilon - \text{net}$.

Lemma 2.1. [3, 10] The noncompact measure $\alpha(\cdot)$ fulfills
(i) $\alpha(\mathcal{B}_1) \leq \alpha(\mathcal{B}_2)$ for all bounded subsets of $\mathcal{B}_1$ and $\mathcal{B}_2$ of $Y$ and $\mathcal{B}_1 \subseteq \mathcal{B}_2$.
(ii) For each $x \in X$ and each nonempty subset $\mathcal{B} \subseteq X$, $\alpha(\{y \in \mathcal{B}\}) = \alpha(\mathcal{B})$.
(iii) If $\mathcal{B}$ is relatively compact iff $\alpha(\mathcal{B}) = 0$.
(iv) If $\mathcal{B}_1 + \mathcal{B}_2 = \{ y + x; y \in \mathcal{B}_1, x \in \mathcal{B}_2 \}$, then $\alpha(\mathcal{B}_1 + \mathcal{B}_2) \leq \alpha(\mathcal{B}_1) + \alpha(\mathcal{B}_2)$.
(v) $\alpha(\mathcal{B}_1 \cup \mathcal{B}_2) \leq \max\{\alpha(\mathcal{B}_1), \alpha(\mathcal{B}_2)\}$.
(vi) $\alpha(\mu \mathcal{B}) \leq |\mu| \alpha(\mathcal{B})$, for any $\mu \in R$.

Let us construct
\[ \int_0^\omega w(\tau)d\tau = \left\{ \int_0^\omega u(\tau)d\tau; u \in W \right\}, \text{ for any } w \subset B(D, Y) \text{ and } \omega \in D, \]
where $w(\tau) = \{ u(\tau) \in Y; u \in w \}$.

Lemma 2.2. [10] If $w \subset B(D, Y)$ is equicontinuous, $\omega \rightarrow \alpha(w(\omega))$ is continuous on $D$, and
\[ \alpha(w) = \max_{\omega \in D} \alpha(w(\omega)), \alpha \left( \int_0^\omega W(\tau)d\tau \right) = \int_0^\omega (w(\tau))d\tau, \text{ for } \omega \in D. \]

Lemma 2.3. [15] Let $\{ U_n(\omega) \}_{n=1}^\infty$ be the Bochner functions from $D \rightarrow Y$ provided $|U_n(\omega)| \leq m(\omega)$ for almost all $\omega \in D$ and for every $n \geq 1$, $m \in L(D, R_+)$, then $\Phi(\omega) = \alpha(U_n(\omega))$ belongs to $L(D, R_+)$ and fulfills
\[ \alpha \left( \left\{ \int_0^\omega U_n(\tau)d\tau; n \geq 1 \right\} \right) \leq 2 \int_0^\omega \Phi(\tau)d\tau. \]

Lemma 2.4. [10] Suppose $\nu > 0$, a nonnegative function $a(\omega)$ is locally integrable on $0 \leq \omega \leq (T \leq +\infty)$ and $g(\omega)$ is nonnegative, increasing continuous and characterized on $0 \leq \omega < T$, $g(\omega) \leq k$ (constant), and $u(\omega)$ is locally integrable on
\[0 \leq \omega < T \text{ together}
\]
\[u(\omega) \leq a(\omega) + g(\omega) \int_{0}^{\omega} (\omega - \tau)^{\nu-1} u(\tau) d\tau.
\]
Then
\[u(\omega) \leq a(\omega) + \int_{0}^{\omega} \left( \sum_{n=1}^{\infty} \frac{(g(\omega)\Gamma(\beta))^{n}}{\Gamma(n\nu)} (\omega - \tau)^{\nu-1} a(\tau) \right) d\tau, \quad 0 \leq \omega \leq T.
\]
Particularly, when \[a(\omega) = 0\], then \[u(\omega) = 0\], for all \[0 \leq \omega < T\].

The following hypotheses are introduced for further existence results.

**H1** Let \(\{R(\omega)\}_{\omega \geq 0}\) is uniformly bounded where \(R(\omega)\) is continuous in the uniform operator topology for \(\omega \geq 0\), (i.e) there is \(K \geq 1\) provided
\[
\sup_{\omega \in [0, +\infty)} |R(\omega)| < K.
\]

**H2** For every \(\omega \in D'\), \(f(\omega, \cdot) = Y \rightarrow Y\) is continuous, for all \(y \in Y\), \(f(\cdot, y) = D' \rightarrow Y\) is strongly measurable.

**H3** For \(k \in L(D', R^+))\) provided \(I_{\omega+}^{1} k \in B(D', R^+)\), \(\omega \rightarrow 0^+\), \(\omega(\zeta-1)(\eta-1) I_{\omega+}^{1} k(\omega) = 0\), and \(|f(\omega, y)| \leq k(\omega)\), for all \(y \in Y\) and almost \(\omega \in D\); Evidently, if \(H_3\) holds, for some \(p > 0\) provided
\[
k \left( \frac{\left| \sum_{m=1}^{n} c_m y(\omega_m) \right|}{\Gamma(1 - \eta) + \eta} + \sup_{\omega \in D} \left\{ \frac{\omega(\zeta-1)(\eta-1)}{\Gamma(1 - \eta) + \eta} \int_{0}^{\omega} (\omega - s)^{\eta-1} k(s) ds \right\} \right) \leq p.
\]

**H4** For any bounded set \(G \subseteq Y\) and a constant \(b\), we have
\[
\alpha(f(\omega, G)) \leq b \omega(\zeta-1)(\eta-1) \alpha(G), \text{ for all } \omega \in [0, b_1].
\]

**Lemma 2.5.**[10] The integral equation (2.1) is same as the Cauchy problem (1.1)-(1.2)
\[
y(\omega) = \sum_{m=1}^{n} c_m y(\omega_m) \frac{\omega(\zeta-1)(1-\eta)}{\Gamma(1 - \eta) + \eta} + \frac{1}{\Gamma(\eta)} \int_{0}^{\omega} (\omega - s)^{\eta-1} [Ay(s) + R(s, y_s, G(y(s)))] ds,
\]
where \(G(y(s)) = \int_{0}^{s} h_1(s, u, y_u) du\).

The wright function \(M_{\eta}(\varphi)\) is characterized as
\[
M_{\eta}(\varphi) = \sum_{n=1}^{\infty} \frac{(-\varphi)^{n-1}}{(n-1)!\Gamma(1 - n\eta)}, \quad 0 < \eta < 1, \ \varphi \in \Theta
\]
satisfies the inequality
\[
\int_{0}^{\infty} \varphi^n M_{\eta}(\varphi) d\varphi = \frac{\Gamma(1 + \kappa)}{\Gamma(1 + n\kappa)}, \text{ for } \varphi \geq 0.
\]
Lemma 2.6.\cite{10} If (2.1) holds, then
\begin{equation}
y(\omega) = I_{0+}^{(1-\eta)} T_{\eta}(\omega) \sum_{m=1}^{n} c_m S_m(\omega_m) + \int_{0}^{\omega} T_{\eta}(\omega - s) \mathcal{R}\left(s, y_s, G(s)\right) ds,
\end{equation}
\begin{equation}
y(\omega) = N_{\zeta, \eta}(\omega) \sum_{m=1}^{n} c_m S_m(\omega_m) + \int_{0}^{\omega} T_{\eta}(\omega - s) \mathcal{R}\left(s, y_s, \int_{0}^{s} h_1(s, u, y_u) du\right) ds. \quad (2.2)
\end{equation}

Where, $T_{\eta}(\omega) = \omega^{\eta-1} S_\eta(\omega)$, $S_\eta(\omega) = \int_{0}^{\infty} \eta \varphi M_\eta(\varphi) R(\omega^{\eta}\varphi) d\varphi$

and $N_{\zeta, \eta}(\omega) = I_{0+}^{(1-\eta)} T_{\eta}(\omega)$.

Remark 2.2. By (2.7) and remark (2.1),
(i) $D_{0+}^{(1-\eta)} N_{\zeta, \eta}(\omega) = T_{\eta}(\omega)$, $\omega \in D'$,
(ii) $N_{\eta, \eta}(\omega) = T_{\eta}(\omega) = \omega^{\eta-1} S_\eta(\omega)$, $\omega \in D'$,
(iii) $N_{1, \eta}(\omega) = N_\eta(\omega)$, $\omega \in D$.

Proposition 2.1. From $(H_1)$ and for $\omega > 0$, $S_\eta(\omega)$ is continuous.

Proof. Using $(H_1)$, $R(\omega)_{\omega \geq 0}$ is uniformly bounded. $\sup_{\omega \in D'} |R(\omega)| < K$;

Now, $S_\eta(\omega) = \int_{0}^{\infty} \eta g M_\eta(g) R(\omega^{\eta} g) dg$. For $\omega, h > 0$ and $y \in Y$,
$$|S_\eta(\omega + h) y - S_\eta(\omega) y| \leq 2K \int_{0}^{\infty} \eta g M_\eta(g) |y| dg = \frac{2K}{\Gamma \eta} |y|,$$
due to Lebesgue dominated convergence theorem, $|S_\eta(\omega + h) y - S_\eta(\omega) y| \to 0$ as $h \to \infty$. Hence $S_\eta(\omega)$ is continuous in the uniform operator topology.

Proposition 2.2. Some $\omega > 0$, $y \in Y$ and by $(H_1)$, the operators $\{T_{\eta}(\omega)\}$ and $\{N_{\zeta, \eta}(\omega)\}$ are linear.

$$|T_{\eta}(\omega) y| \leq \frac{K \omega^{\eta-1}}{\Gamma \eta} |y| \quad \text{and} \quad |N_{\zeta, \eta}(\omega) y| \leq \frac{K \omega^{(\zeta-1)(\eta-1)}}{\Gamma_\zeta(1-\eta) + \eta} |y|.$$

Proof. Since $\{S_\eta(\omega)\}_{\omega > 0}$ is continuous, the first result is obvious. For $\omega \in D'$ and $y \in Y$
$$|N_{\zeta, \eta}(\omega) y| = \left| \frac{1}{\Gamma_\zeta(1-\eta)} \int_{0}^{\omega} (\omega - s)^{1-\zeta(\eta-1)} T_{\eta}(s) y ds \right|$$
$$\leq \frac{\omega^{(\zeta-1)(1-\eta)}}{\Gamma_\zeta(1-\eta) \Gamma \eta} \int_{0}^{\omega} (\omega - s)^{1-\zeta(\eta-1)} s^\eta |y| ds$$
$$= \frac{\omega^{(\zeta-1)(1-\eta)} K}{\Gamma_\zeta(1-\eta) \Gamma \eta} |y|.$$

$|S_\eta(\omega + h) y - S_\eta(\omega) y| \to 0.$ \quad (2.3)
Proposition 2.3. By hypotheses $(H_1)$, $\{T_\eta(\omega)\}_{\omega > 0}$ and $\{N_\zeta, \eta(\omega)\}_{\omega > 0}$ are strongly continuous, for any $y \in Y$ and $0 < \omega < \bar{\omega} < b_1$, $|T_\eta(\omega)y - T_\eta(\bar{\omega})y| \to 0$ and $|N_\zeta, \eta(\bar{\omega})y - N_\zeta, \eta(\omega)y| \to 0$ as $\omega \to \bar{\omega}$.

Proof. Since $\{S_\eta(\omega)\}_{\omega > 0}$ is strongly continuous by proposition (2.1), $|T_\eta(\omega)y| \leq \frac{2K\omega^{\eta-1}}{\Gamma \eta} |y|$.

By Lebesgue dominated convergence theorem, $\{T_\eta(\omega)\}_{\omega > 0}$ is strongly continuous. For any $y \in Y$ and $0 < \omega < \bar{\omega} < b_1$, 

$$\left|N_\zeta, \eta(\bar{\omega})y - N_\zeta, \eta(\omega)y\right|$$

$$= \frac{1}{\Gamma \zeta(1 - \eta)} \left|\int_0^\omega (\bar{\omega} - s)^{1-\zeta(\eta-1)}S_\eta(s)ysds - \int_0^\omega (\bar{\omega} - s)^{1-\zeta(\eta-1)}S_\eta(s)ysds\right|$$

$$\leq \frac{1}{\Gamma \zeta(1 - \eta)} \left|\int_0^\omega (\bar{\omega} - s)^{1-\zeta(\eta-1)}s^{\eta-1}S_\eta(s)ysds\right|$$

$$+ \frac{1}{\Gamma \zeta(1 - \eta)} \left|\int_0^\omega [(\bar{\omega} - s)^{1-\zeta(\eta-1)} - (\bar{\omega} - s)^{1-\zeta(\eta-1)}] s^{\eta-1}S_\eta(s)ysds\right|$$

$$\leq \frac{K(\bar{\omega})^{\eta-1}}{\Gamma \zeta(1 - \eta)\Gamma \eta} \frac{1}{\zeta(1 - \eta)(\omega - \bar{\omega})} |y|$$

$$+ \frac{K}{\Gamma \zeta(1 - \eta)\Gamma \eta} \left|\int_0^\omega [(\bar{\omega} - s)^{1-\zeta(\eta-1)} - (\bar{\omega} - s)^{1-\zeta(\eta-1)}] s^{\eta-1}ds\right| |y|.$$ (2.4)

Since

$$\left|\int_0^\omega [(\bar{\omega} - s)^{1-\zeta(\eta-1)} - (\bar{\omega} - s)^{1-\zeta(\eta-1)}] s^{\eta-1}ds\right| \leq 2 \int_0^\omega (\omega - s)^{1-\zeta(\eta-1)} s^{\eta-1}ds$$

exists.

Then, by Lebesgue dominated convergence theorem,

$$\left|\int_0^\omega [(\bar{\omega} - s)^{1-\zeta(\eta-1)} - (\bar{\omega} - s)^{1-\zeta(\eta-1)}] s^{\eta-1}ds\right| \to 0 \text{ as } \omega \to \bar{\omega}.$$ 

Consequently $\left|N_\zeta, \eta(\bar{\omega})y - N_\zeta, \eta(\omega)y\right| \to 0$ as $\omega \to \bar{\omega}$, i.e. $\{N_\zeta, \eta(\omega)\}_{\omega > 0}$ is strongly continuous.

The operator $Q$ is structured as, for $y \in X$, $(Qy)\omega = (Q_1y)(\omega) + (Q_2y)(\omega)$, where

$$(Q_1y)(\omega) = N_\zeta, \eta(\omega) \sum_{m=1}^\infty c_m S_\eta(t_m), \quad (Q_2y)(\omega) = \int_{0}^{\omega} T_\eta(\omega - s) \Re \left(s, y_s, G(s)\right) ds,$$

for every $\omega \in D'$.

By (2.3) and $(H_3)$, we have

$$\lim_{\omega \to 0^+} \omega^{\zeta(\eta-1)} N_\zeta, \eta(\omega) \sum_{m=1}^\infty c_m S_\eta(t_m)$$

$$= \frac{1}{\Gamma \zeta(1 - \eta)\Gamma \eta} \int_{0}^{\omega} (\omega - s)^{1-\zeta(\eta-1)} s^{\eta-1} \sum_{m=1}^n c_m S_\eta(t_m) ds.$$
The conditions (Theorem 3.1. On the side of outcomes, we need the following theorems.

Existence results. For any \( \omega \in (0, b_1] \), let \( y(\omega) = \omega^{(\zeta-1)(1-\eta)} x(\omega) \) such that

\[
(\Upsilon x)(\omega) = (\Upsilon_1 x)(\omega) + (\Upsilon_2 x)(\omega).
\]

Clearly, \( y \) is a mild solution of (1.1)-(1.2) in \( X \) iff \( x = \Upsilon x \) has a solution \( x \in B(D, Y) \).

3. Existence results. Here, we formulate and prove the existence results of our system (1.1)–(1.2) by measure of noncompactness and Mönch fixed point technique. On the side of outcomes, we need the following theorems.

**Theorem 3.1.** The conditions \((H_1)-(H_3)\) holds, then \( \{ \Upsilon x : x \in E_r(D) \} \) is equicontinuous.

**Proof.**

**Step 1.** \( \{ \Upsilon x : x \in E_r(D) \} \). Let \( \omega x = \omega^{(\zeta-1)(1-\eta)} x(\omega) \), for any \( x \in E_r(D) \) and \( \omega \in (0, b_1] \). For \( 0 \leq \bar{\omega}_1 \leq \bar{\omega}_2 \leq b_1 \),

\[
| (\Upsilon_1 x)(\bar{\omega}_2) - (\Upsilon_1 x)(\bar{\omega}_1) | 
\leq \left| \omega^{(\zeta-1)(1-\eta)} N_{\zeta, \eta}(\bar{\omega}_2) S_\eta(\bar{\omega}_2) - \omega^{(\zeta-1)(1-\eta)} N_{\zeta, \eta}(\bar{\omega}_1) S_\eta(\bar{\omega}_1) \right| \sum_{m=1}^{n} c_m.
\]

By (2.5) and proposition(2.3), \( \omega^{(1-\zeta)(1-\eta)} \) is uniformly continuous on \( D \).

Consequently \( \{ \Upsilon_1 x : x \in E_r(D) \} \) is equicontinuous.

**Step 2.** For any \( x \in E_r(D) \)

\[
| (\Upsilon_2 x)(\bar{\omega}_2) - (\Upsilon_2 x)(\bar{\omega}_1) | 
\leq \left| \omega^{(\zeta-1)(1-\eta)} \int_{\bar{\omega}_1}^{\bar{\omega}_2} T_\eta(\bar{\omega}_2 - s) \Re \left( s, y_s, G(s) \right) ds \right| 
\leq \frac{K \omega^{(\zeta-1)(1-\eta)}}{\Gamma \eta} \int_{0}^{\bar{\omega}_2} (\bar{\omega}_2 - s) k(s) ds \to 0 \text{ as } \bar{\omega}_2 \to 0.
\]

Now, for \( 0 \leq \bar{\omega}_1 \leq \bar{\omega}_2 \leq b_1 \),

\[
| (\Upsilon_2 x)(\bar{\omega}_2) - (\Upsilon_2 x)(\bar{\omega}_1) | 
\leq \left| \omega^{(\zeta-1)(1-\eta)} (\bar{\omega}_2 - s)^{\eta-1} S_\eta(\bar{\omega}_2 - s) \Re \left( s, y_s, \int_{\bar{\omega}_1}^{\bar{\omega}_2} h_1(s, u, y_u) du \right) ds \right|
\]

\[
= \sum_{m=1}^{n} c_m S_\eta(t_m)
= \frac{\sum_{m=1}^{n} c_m S_\eta(t_m)}{\Gamma(1-\eta) \Gamma \eta}.
\]
where

\[ I^* = \frac{K}{\Gamma} \left| \int_{\bar{\omega}_2}^{\bar{\omega}_1} \omega_2^{(\zeta-1)(\eta-1)}(\bar{\omega}_2 - s)^{\eta-1} k(s)ds - \int_{\bar{\omega}_1}^{\bar{\omega}_1} \omega_1^{(\zeta-1)(\eta-1)}(\bar{\omega}_1 - s)^{\eta-1} k(s)ds \right| \]

\[ I^{**} = \frac{2K}{\Gamma} \left| \int_{\bar{\omega}_1}^{\bar{\omega}_1} \left[ \omega_1^{(\zeta-1)(\eta-1)}(\bar{\omega}_1 - s)^{\eta-1} - \omega_1^{(\zeta-1)(\eta-1)}(\bar{\omega}_2 - s)^{\eta-1} \right] k(s)ds \right| \]

\[ I^{***} = \left| \int_{\bar{\omega}_2}^{\bar{\omega}_1} \omega_1^{(\zeta-1)(\eta-1)}(\bar{\omega}_1 - s)^{\eta-1} \left[ S_\eta(\bar{\omega}_2 - s) - S_\eta(\bar{\omega}_1 - s) \right] \text{Re} \left( s, y_s, \int_{0}^{\bar{\omega}_1} h_1(s, u, y_u)du \right) ds \right| . \]

By \((H_3)\), \( I^* \) can reduce that \( \lim_{\bar{\omega}_2 \to \bar{\omega}_1} I^* = 0 \), since

\[ \left[ \omega_1^{(\zeta-1)(\eta-1)}(\bar{\omega}_1 - s)^{\eta-1} - \omega_1^{(\zeta-1)(\eta-1)}(\bar{\omega}_2 - s)^{\eta-1} \right] k(s) \]

\[ \leq 2\omega_1^{(\zeta-1)(\eta-1)}(\bar{\omega}_1 - s)^{\eta-1} k(s), \]

and \( \int_{\bar{\omega}_1}^{\bar{\omega}_1} \omega_1^{(\zeta-1)(\eta-1)}(\bar{\omega}_1 - s)^{\eta-1} k(s)ds \) exists, by Lebesgue dominated convergence theorem \( \int_{\bar{\omega}_2}^{\bar{\omega}_1} \left[ \omega_1^{(\zeta-1)(\eta-1)}(\bar{\omega}_1 - s)^{\eta-1} - \omega_1^{(\zeta-1)(\eta-1)}(\bar{\omega}_2 - s)^{\eta-1} \right] k(s)ds \to 0 \) as \( \bar{\omega}_2 \to \bar{\omega}_1 \). Therefore, \( \lim_{\bar{\omega}_2 \to \bar{\omega}_1} I^{**} = 0 \). For \( \delta > 0 \)

\[ I^{***} \leq \int_{0}^{\bar{\omega}_1^{\delta}} \omega_1^{(\zeta-1)(\eta-1)}(\bar{\omega}_1 - s)^{\eta-1} \left[ S_\eta(\bar{\omega}_2 - s) - S_\eta(\bar{\omega}_1 - s) \right] \]

\[ \times \text{Re} \left( s, y_s, \int_{0}^{\bar{\omega}_1^{\delta}} h_1(s, u, y_u)du \right) ds \]
Step 1. Proof.

If (Theorem 3.2) \( \omega \rightarrow \),

\[
\| \omega \| \leq 2K \Gamma \eta \frac{\Gamma}{\Gamma(1-\eta)} N_{\zeta,\eta}(\omega) \sum_{m=1}^{n} c_m
\]

\[
+ \omega^{1-\zeta(1-\eta)} T_{\eta}(\omega) \mathcal{R} \left( s, y_s, \int_0^\omega h_1(s, u, y_u)du \right) ds
\]

\[
\leq K \left( \frac{\sum_{m=1}^{n} c_m}{\Gamma(1-\eta) + \eta} + \sup_{\omega \in D} \frac{\omega^{\zeta(1-\eta)}}{\Gamma \eta} \int_0^\omega (\omega - s) k(s) ds \right) \leq p.
\]

Hence \( \| \varUpsilon x \| \leq p \), for any \( x \in E_r(D) \).

**Theorem 3.2.** If \((H_1)-(H_3)\) holds, then \( \varUpsilon : E_r(D) \rightarrow E_r(D) \) and is continuous on \( E_r(D) \).

**Proof.** Step 1. By our assumption \( \varUpsilon \) maps \( E_r(D) \) into \( E_r(D) \). For any \( \delta, x \in E_r(D) \), let \( y(\omega) = \omega^{\eta-1} x(\omega) \). Then \( y \in E_r^+(D') \). Proposition(2.2) yields

\[
|\langle \varUpsilon x \rangle(\omega)| \leq \omega^{\zeta(1-\eta)} N_{\zeta,\eta}(\omega) \sum_{m=1}^{n} c_m
\]

\[
+ \omega^{1-\zeta(1-\eta)} T_{\eta}(\omega) \mathcal{R} \left( s, y_s, \int_0^\omega h_1(s, u, y_u)du \right) ds
\]

\[
\leq K \left( \frac{\sum_{m=1}^{n} c_m}{\Gamma(1-\eta) + \eta} + \sup_{\omega \in D} \frac{\omega^{\zeta(1-\eta)}}{\Gamma \eta} \int_0^\omega (\omega - s) k(s) ds \right) \leq p.
\]
Step 2. To prove $\Upsilon$ is continuous. For any $x_m$, for $m = 1, 2, 3, \ldots$, $x \in E_r(D)$,

$$\lim_{m \to \infty} x_m = x, \lim_{m \to \infty} x_m(\omega) = x(\omega) \quad \text{and} \quad \lim_{m \to \infty} \omega^{(\zeta-1)(\eta-1)} x_m(\omega) = \omega^{(\zeta-1)(\eta-1)} x(\omega),$$

$\omega \in D'$. Then by hypotheses ($H_2$),

$$f(\omega, y_m(\omega)) = f(\omega, \omega^{(\zeta-1)(\eta-1)} x_m(\omega)) \to f(\omega, \omega^{(\zeta-1)(\eta-1)} x(\omega)) = f(\omega, y(\omega)) \quad \text{as} \quad m \to \infty$$

where $y_m(\omega) = \omega^{(\zeta-1)(\eta-1)} x_m(\omega)$ and $y(\omega) = \omega^{(\zeta-1)(\eta-1)} x(\omega)$.

Also, using hypotheses ($H_3$), for each $\omega \in D'$

$$(\omega - s)^{\eta-1} \left| \mathfrak{R} \left( s, (y)_m, \int_0^\omega h_1(s, u, (y)_m) du \right) \right| \leq (\omega - s)^{\eta-1} 2k(s) \quad \text{a.e in} \quad [0, \omega).$$

Consequently, $s \to (\omega - s)^{\eta-1} 2k(s)$ is integrable for $s \in [0, \omega)$ and $\omega \in D'$.

With reference to Lebesgue theorem,

$$\int_0^\omega (\omega - s)^{\eta-1} \left| \mathfrak{R} \left( s, (y)_m, \int_0^\omega h_1(s, u, (y)_m) du \right) \right| ds \to 0 \quad \text{as} \quad m \to \infty.$$ 

For $\omega \in D$,

$$|\Upsilon x_m(\omega) - \Upsilon x(\omega)| \leq \frac{K\omega^{(\zeta-1)(\eta-1)}}{\Gamma\eta} \times \int_0^\omega (\omega - s)^{\eta-1} \left| \mathfrak{R} \left( s, (y)_m, \int_0^\omega h_1(s, u, (y)_m) du \right) \right| ds \to 0 \quad \text{as} \quad m \to \infty.$$ 

Hence, $\Upsilon x_m \to \Upsilon x$ point wise on $D$ as $m \to \infty$ and by Theorem (3.1), $\Upsilon x_m \to \Upsilon x$ uniformly on $D$ as $m \to \infty$. So $\Upsilon$ is continuous.

**Theorem 3.3.** Let the hypothesis ($H_1$)-($H_4$) true then (1.1)-(1.2) has at least one solution in $E_r(D')$.

**Proof.** Consider a set $\mathbb{R} = \{x_m : m = 0, 1, 2, 3, \ldots\}$. For all $\omega \in D$, let $x_0(\omega) = \omega^{(\zeta-1)(\eta-1)} \times N_{\zeta, \eta}(\omega) \sum_{m=1}^n c_m$ and $x_{m+1} = \Upsilon x_m$. To examine that $\mathbb{R}$ is relatively compact.

Hence $\mathbb{R}$ is bounded uniformly and equicontinuous from theorem (3.1) and (3.2) on $D$. By ($H_4$) and lemmas (2.1), (2.3), for any $\omega \in D$ we can write $\alpha (\mathbb{R}) = \alpha (\{x_m(\omega)\}_{m=0}^\infty) = \alpha (\{x_0(\omega)\} \cup \{x_m(\omega)\}_{m=1}^\infty) = \alpha (\{x_m(\omega)\}_{m=1}^\infty)$ and

$$\alpha (\{x_m(\omega)\}_{m=0}^\infty) = \alpha (\{\Upsilon x_m(\omega)\}_{m=0}^\infty) \leq \frac{2K}{\Gamma\eta} \omega^{(\zeta-1)(\eta-1)} \int_0^\omega (\omega - s)^{1-n\eta} \alpha (\{N_{\zeta, \eta}(x_m(s))\}_{m=0}^\infty) ds$$

$$\leq \frac{2K}{\Gamma\eta} \omega^{(\zeta-1)(\eta-1)} \int_0^\omega (\omega - s)^{1-n\eta} \alpha (x_m(s)) ds,$$
then \( \alpha(\Re(\omega)) \leq \frac{2Kb}{\Gamma\eta}\omega^{(\zeta-1)(\eta-1)} \int_{0}^{\omega} (\omega - s)^{1-\eta}\alpha(\Re(s))ds \).

Therefore, by lemma (2.2) and (2.4), we get \( \alpha(\Re(\omega)) = 0 \). Then, by Arzela-Ascoli theorem \( \alpha(\Re(\omega)) \), there exists a convergent sub sequence of \( \{x_m(\omega)\}_{m=0}^{\infty} \).

Let, \( \lim_{m \to \infty} x_m = x^* \) on \( E(\Gamma(D)) \). Then by continuous operator \( T \), we have

\[
x^* = \lim_{m \to \infty} x_m = \lim_{m \to \infty} \Upsilon x_{m-1} = \Upsilon \left( \lim_{m \to \infty} x_{m-1} \right) = \Upsilon x^*
\]

Hence (1.1)-(1.2) has atleast a solution.

4. **Application.** The Hilfer fractional derivative with the initial conditions \( D_{\omega,-}^{\zeta,\eta} \)

\[
z(\omega, \theta) = \frac{\partial}{\partial y} z(\omega, \theta) + F(\omega, y(\omega, \theta), \int H(\omega, w(x, \theta - r)))ds,
\]

\[
z(\omega, 0) = z(\omega, \pi) = 0, \ (\omega, \theta) \in [0, b_1] \times (0, \pi),
\]

\[
I_0^{(\zeta-1)(\eta-1)} z(0, \theta) = \sum_{m=1}^{\infty} \arctan \left( \frac{1}{\sqrt{2m^2}} y(\pi, x), \ x \in (0, 1),
\right.
\]

where \( w > 0, \ 0 \leq \xi \leq 1 \) and \( 0 < \eta < 1 \). Here, \( Y = L^p(0, \pi) \) as the state space and \( z(\omega, \cdot) = \{z(\omega, \theta)/0 < \theta < \pi\} \) as the state. Let

\[
z(y) = \sqrt{\frac{\pi}{2}} \sin n y, \ n \in N
\]

and \( A \) is a infinitesimal generator and for all \( x \in Y, \ \{R(\omega)\}_{\omega \geq 0} \) in \( X \) is

\[
R(\omega) = \sum_{n=1}^{\infty} e^{\alpha^2 \omega} (x, x_n) x_n.
\]

Hence \( \{R(\omega)\}_{\omega \geq 0} \) is bounded. Therefore, \( (H_1) \) holds. Now \( A \subseteq D'(A) : Y \to Y \) as

\[
D'(A) : \left\{ x \in Y : \frac{\partial z}{\partial x}, \frac{\partial^2 z}{\partial x^2} \in Y \right\}
\]

are absolutely continuous also strongly measurable. Therefore, \( (H_2) \) holds. Also \( F \) satisfies \( F_1(\cdot, v) : D' \to Y \) is measurable for all \( v \in Y \) and \( F(\omega, \cdot) : Y \to Y \) is continuous almost everywhere \( \omega \in D' \) there is a function \( k \in L(D', \Re^+) \) provided

\[
||F(\omega, v)|| \leq k(\omega)||v|| \text{ for all } \omega \in D.
\]

\[
k \left( \frac{\sum_{m=1}^{n} |\arctan |}{\Gamma(1-\eta) + \eta} + \sup_{\omega \in D} \left\{ \frac{\omega^{(\zeta-1)(\eta-1)}}{\Gamma(\eta)} \int_{0}^{\omega} (\omega - s)^{\eta-1}k(s)ds \right\} \right) \leq p,
\]

and also \( \sum_{m=1}^{n} c_m = \frac{\pi}{4} < 1 \).

For any bounded set \( G \subseteq Y \), and \( b > 0, \ |G(y)| < b, \) for all \( y \in (D', \Re^+) \) (i.e) relatively compact. Hence, there exists a solution for (4.1)-(4.3).

5. **Conclusion.** In this paper, the mild solution of Hilfer fractional derivative is discussed using noncompact measure and Mönch’s fixed point technique. Also an example is provided to show the applicability of presented technique. Further, one can extend the study with the controllability factors.
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