AMPLITUHEDRON CELLS AND STANLEY SYMMETRIC FUNCTIONS

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Abstract. The amplituhedron was recently introduced in the study of scattering amplitudes in $N = 4$ super Yang-Mills. We compute the cohomology class of a tree amplituhedron subvariety of the Grassmannian to be the truncation of an affine Stanley symmetric function.

1. Introduction

Let $\text{Gr}(k, n)$ denote the Grassmannian of $k$-planes in $\mathbb{C}^n$. It has a stratification by positroid varieties $\Pi_f \text{ [Pos]} \text{ [KLS]}$, where $f$ ranges over the finite set $\text{Bound}(k, n)$ of $(k, n)$-bounded affine permutations (defined in Section 4). Each positroid variety is the intersection of $n$ cyclically rotated Schubert varieties. In [KLS], Knutson, Lam, and Speyer identified the cohomology class of a positroid variety with the affine Stanley symmetric function $\tilde{F}_f \text{ [Lam]}$.

The totally nonnegative part $\text{Gr}(k, n)_{\geq 0}$ of the real Grassmannian is the locus where all Plücker coordinates take nonnegative values $\text{ [Lus] [Pos]}$, and was studied extensively by Postnikov. Arkani-Hamed and Trnka [AT], motivated by the study of scattering amplitudes in $N = 4$ super Yang-Mills, proposed that $\text{Gr}(k, n)_{\geq 0}$ should be considered a Grassmannian-analogue of a simplex. Arbitrary convex polytopes are images of simplices under affine or linear maps, and Arkani-Hamed and Trnka proposed to study the amplituhedron $\mathcal{A}$: the image of the totally nonnegative Grassmannian induced by a linear map $Z : \mathbb{R}^n \to \mathbb{R}^{k+m}$ (which in turn gives a rational map $Z_{\text{Gr}} : \text{Gr}(k, n) \to \text{Gr}(k, k+m)$). In addition, physical considerations suggested the study of triangulations of the amplituhedron, obtained as unions of images of the positroid cells $(\Pi_f)_{\geq 0} := \Pi_f \cap \text{Gr}(k, n)_{\geq 0}$, again under the map $Z_{\text{Gr}}$. Specifically, the scattering amplitude can be obtained by summing differential forms over cells of a triangulation of the amplituhedron.

The behavior of positroid cells under the map $Z_{\text{Gr}}$ exhibit a number of features not present in usual convex geometry, including:

(1) Even when $Z$ is generic, the image $Z_{\text{Gr}}((\Pi_f)_{\geq 0})$ may not have the expected dimension. For example, even if $\dim((\Pi_f)_{\geq 0}) = \dim(\text{Gr}(k, k+m))$ we may have $\dim(Z_{\text{Gr}}((\Pi_f)_{\geq 0})) < \dim(\text{Gr}(k, k+m))$ for generic $Z$.

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1 In this paper we shall only consider the “tree” amplituhedron, leaving the “loop” amplituhedron for later work.
(2) The map $Z_{Gr}|_{((\Pi_f)_\geq 0)} : ((\Pi_f)_\geq 0) \to Z_{Gr}((\Pi_f)_\geq 0)$ can be dimension-preserving, but have degree $d$ greater than one.

In this paper, we study the complex geometry of the behavior of the stratification $Gr(k, n) = \bigcup_f \Pi_f$ under the map $Z_{Gr}$, from a Schubert calculus perspective. Let $Y_f$ denote the closure of the image of $\Pi_f$ under $Z_{Gr}$. We call $Y_f$ an amplituhedron variety in the case that it has the same dimension as $\Pi_f$.

Recall that the cohomology ring $H^* (Gr(k, n), \mathbb{Z})$ can be identified with a quotient of the ring of symmetric functions, and that the basis of Schubert classes correspond to the Schur functions $s_\lambda$, labeled by partitions $\lambda \subseteq (n-k)^k$ that fit inside a $k \times (n-k)$ rectangle. Let $\ell = n - k - m$. For $\mu \subseteq (m)^k$ we let $\mu^+ \subseteq (n-k)^k$ be the partition obtained from $\mu$ by adding $\ell$ columns of height $k$ to the left of $\mu$. For example, with $\ell = 2$ and $k = 4$, we may have

$$\mu = \begin{array}{ccc|c} & & & \hline \end{array} \quad \mu^+ = \begin{array}{ccc|c} & & & \hline \end{array}$$

Given $f = \sum c_\lambda s_\lambda$ representing a cohomology class in $H^* (Gr(k, n))$, we define the truncation $\tau_{k+m}(f) \in H^* (Gr(k, k+m))$ by

$$\tau_{k+m}(f) = \sum_{\mu \subseteq (m)^k} c_{\mu^+} s_\mu.$$

Let $d_f$ denote the degree of the map $Z_{Gr}|_{\Pi_f} : \Pi_f \to Y_f$.

**Theorem 1.1.** The cohomology class of the amplituhedron variety $Y_f$ is equal to $\frac{1}{d_f} \tau_{k+m}(\tilde{F}_f)$.

We also prove that $\tau_{k+m}(\tilde{F}_f) = 0$ if and only if $\dim Y_f < \dim \Pi_f$. As a corollary, we deduce a criterion for $Z_{Gr}((\Pi_f)_\geq 0)$ to have the same dimension as $(\Pi_f)_\geq 0$, which corresponds to the physical notion of “kinematical support”. We also obtain some estimates on the degree $d_f$. We present a number of possible further directions in Section 5.

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2Henceforth, we shall always take cohomologies with $\mathbb{Z}$-coefficients.
2. Projection maps and Schubert varieties

We fix positive integers \(n, k, m\) satisfying \(n > k + m\), and we let \(\ell = n - k - m\). Let 
\(\text{Mat}(n, k + m)\) denote the space of \(n \times (k + m)\) matrices, and let 
\(\text{Mat}(n, k + m)\) denote the open subset of full rank \(n \times (k + m)\) matrices. We think of 
\(Z \in \text{Mat}(n, k + m)\) as a linear map \(Z : \mathbb{C}^n \to \mathbb{C}^{k+m}\). The map \(Z\) induces a rational map 
\(Z_{Gr} : \text{Gr}(k, n) \to \text{Gr}(k, k + m)\) given by \(X \mapsto X \cdot Z\). The exceptional locus \(E_Z\) of 
\(Z_{Gr}\) is the subset of \(\text{Gr}(k, n)\) where the map \(Z_{Gr}\) is not defined:
\[
E_Z = \{X \in \text{Gr}(k, n) \mid X \cap \ker(Z) \neq (0)\}.
\]
Here \(\ker(Z) \subset \mathbb{C}^n\) is the usual kernel of a linear map. The exceptional locus \(E_Z\) is 
in fact a Schubert variety which has codimension \(m + 1\).

**Lemma 2.1.** The morphism \(Z_{Gr} : \text{Gr}(k, n) \setminus E_Z \to \text{Gr}(k, k + m)\) is a fiber bundle 
with fiber \(\mathbb{C}^{k(n-k-m)}\).

**Proof.** We use the \(\text{GL}(n)\) actions on \(Z\) and on \(\text{Gr}(k, n)\) to reduce to the case that \(Z\) 
is the orthogonal projection of \(\text{span}(e_1, e_2, \ldots, e_n)\) onto \(\text{span}(e_1, e_2, \ldots, e_{k+m})\). Then 
the map \(Z_{Gr}\) looks like
\[
(Y|\ast) \mapsto Y
\]
where \(Y\) is a \(k \times (k+m)\) matrix representing a point in \(\text{Gr}(k, k+m)\), and \(\ast\) denotes 
the \(\mathbb{C}^{k(n-k-m)}\) fiber. \(\square\)

We use the notation \([a] := \{1, 2, \ldots, a\}\). Let \(I \in \binom{[n]}{k}\) be a \(k\)-element subset of \([n]\). Let 
\(F_\bullet = \{0 = F_0 \subset F_1 \subset \cdots F_{n-1} \subset F_n = \mathbb{C}^n\}\) be a flag in \(\mathbb{C}^n\), so that \(\dim F_i = i\). 
The Schubert variety \(X_I(F_\bullet)\) is given by
\[
(1) \quad X_I(F_\bullet) = \{X \in \text{Gr}(k, n) \mid \dim(X \cap F_j) \geq \#(I \cap [n-j+1, n]) \text{ for all } j \in [n]\}.
\]
Thus \(X_{[k]}(F_\bullet) = \text{Gr}(k, n)\) and \(\text{codim}(X_I(F_\bullet)) = i_1 + i_2 + \cdots + i_k - (1+2+\cdots+k)\), where 
\(I = \{i_1, i_2, \ldots, i_k\}\). Here and elsewhere, we always mean complex (co)dimension 
when referring to complex subvarieties.

Let \(G_\bullet\) be a flag in \(\mathbb{C}^{k+m}\). Then \(Z^{-1}(G_\bullet)\) is the partial flag 
\(Z^{-1}(G_\bullet) := \{\ker(Z) = Z^{-1}(G_0) \subset Z^{-1}(G_1) \subset \cdots \subset Z^{-1}(G_{k+m}) = Z^{-1}(\mathbb{C}^{k+m}) = \mathbb{C}^n\}\)
of subspaces with successive dimensions \(n - (k + m), n - (k + m) + 1, \ldots, n\). We 
denote by \(Y_J(G_\bullet)\) a Schubert variety in \(\text{Gr}(k, k+m)\), where \(J\) is a \(k\)-element subset 
of \([k+m]\). A full-flag extension of \(Z^{-1}(G_\bullet)\) is simply any flag \(F_\bullet\) in \(\mathbb{C}^n\) whose 
\(n - (k + m), n - (k + m) + 1, \ldots, n\)-dimensional pieces give \(Z^{-1}(G_\bullet)\).

**Lemma 2.2.** We have
\[
Z_{Gr}^{-1}(Y_J(G_\bullet)) = X_I(F_\bullet)
\]
where \(F_\bullet\) is any full-flag extension of \(Z^{-1}(G_\bullet)\), and \(I \subset [k+m]\) is considered a subset 
of \([n]\) via the natural inclusion \([k+m] = \{1, 2, \ldots, k+m\} \subset \{1, 2, \ldots, n\} = [n]\).
Proof. Suppose $X \in Z_{Gr}^{-1}(Y_I(G_*))$. Then $\dim(Z_{Gr}(X) \cap G_j) \geq \#(I \cap [k+m-j+1, k+m])$, and so $\dim(X \cap F_{j+n-k-m}) \geq \#(I \cap [k+m-j+1, n])$ for all $j \in [1, k+m]$. That is, $\dim(X \cap F_{j'}) \geq \#(I \cap [n-j'+1, n])$ for all $j' \in [n-k-m+1, n]$. Since membership in $X_I(F_{j'})$ imposes no condition on $X \cap F_{j'}$ for $j' \in [1, n-k-m]$, we conclude that $Z_{Gr}^{-1}(Y_I(G_*)) \subset X_I(F_{j'})$. But using Lemma 2.1, we see that $Z_{Gr}^{-1}(Y_I(G_*))$ and $X_I(F_{j'})$ are closed irreducible subvarieties of $Gr(k, n)$ of the same dimension, and so must be identical. □

If $J = \{m+1, m+2, \ldots, k+m\}$ then $Y_J(G_*)$ is a single point $Y = G_k \in Gr(k, k+m)$. Lemma 2.2 then says that $Z_{Gr}^{-1}(Y_I(G_*)) = Gr(k, Z^{-1}(Y))$ is a subGrassmannian of $Gr(k, n)$.

3. Cohomology class of a projection

We shall need the following version of Kleiman transversality.

**Theorem 3.1** ([Kle, Corollary 4]). Assume the base field is $\mathbb{C}$. Let $X$ be an integral algebraic scheme with a transitive action of an algebraic group $G$. Let $Y, Z \subset X$ be integral subschemes. Then

(1) There exists a dense subset $U \subset G$ such that for $g \in U$, the intersection $gY \cap Z$ is proper, that is, each component has dimension $\dim(Y) + \dim(Z) - \dim(X)$.

(2) If in addition $Y$ and $Z$ are smooth, then $U$ can be chosen so that for all $g \in U$, the subschemes $gY$ and $Z$ intersect transversally, that is, the intersection $gY \cap Z$ is smooth and each component has dimension $\dim(Y) + \dim(Z) - \dim(X)$.

We remark that if $gY$ and $Z$ intersect transversely then the intersection $gY \cap Z$, being smooth, must be contained in the smooth locus of both $gY$ and $Z$. We shall also need the following technical result which appears in the proof of Theorem 3.1.

**Lemma 3.2** ([Kle, Lemma 1]). Assume the base field is $\mathbb{C}$. Consider a diagram with integral algebraic schemes:

\[
\begin{array}{ccc}
W & \rightarrow & Z \\
p & \swarrow & \searrow r \\
S & \rightarrow & X
\end{array}
\]

(1) Assume $q$ is flat. Then, there exists a dense open subset $U$ of $S$ such that for each point $s \in U$, either the fibered product, $p^{-1}(s) \times_X Z$, is empty or it is equidimensional and its dimension is given by the formula,

$$\dim(p^{-1}(s) \times_X Z) = \dim(p^{-1}(s)) + \dim(Z) - \dim(X).$$

(2) Assume $q$ is flat with smooth fibers. Assume $Z$ is smooth. Then $p^{-1}(s) \times_X Z$ is smooth for each point $s$ in an open dense subset of $S$. 

**Diagram:**
Let $W \in \text{Gr}(k, n)$ be an irreducible subvariety. For $Z \in \text{Mat}(n, k + m)$, we define

$$\tau_Z(W) := \overline{Z_{\text{Gr}}(W \setminus E_Z)}.$$

For a generic $Z$, the subscheme $W \setminus E_Z$ is irreducible and dense in $W$. Thus $\tau_Z(W)$ is itself an irreducible subvariety. There is a $GL(n)$-action on $\text{Gr}(k, n)$ and a $GL(n)$-action on $\text{Mat}(n, k + m)$. We choose compatible conventions so that $gE_{Z_0} = E_{gZ_0}$.

Let $Y_I = Y_I(G\bullet) \subset \text{Gr}(k, k + m)$ be a Schubert subvariety. For $Z \in \text{Mat}(n, k + m)$ a full-rank matrix, let $X_I^Z = Z^{-1}_{\text{Gr}}(Y_I) \subset \text{Gr}(k, n)$ be as in Lemma 2.2.

**Lemma 3.3.** Fix $W \subset \text{Gr}(k, n)$ an irreducible subvariety, $Y_I \subset \text{Gr}(k, k + m)$ a Schubert variety, satisfying $\dim(W) + \dim(Y_I) = km$. Then there exists a Zariski-open subset $U \subset \text{Mat}(n, k + m)$ such that:

1. for all $Z \in U$, we have $W \setminus E_Z$ is open and dense in $W$;
2. (a) either for all $Z \in U$ we have $\dim(\tau_Z(W)) < \dim(W)$ and $\tau_Z(W) \cap Y_I = \emptyset$,
   (b) or for all $Z \in U$ we have $\dim(\tau_Z(W)) = \dim(W)$, the intersection $\tau_Z(W) \cap Y_I$ is transversal, and all intersection points lie in the locus inside $Z_{\text{Gr}}(W \setminus E_Z)$ where the map $Z_{\text{Gr}}|_{W \setminus E_Z} : W \setminus E_Z \to Z_{\text{Gr}}(W \setminus E_Z)$ has fibers of cardinality exactly $d_Z$, where $d_Z$ is the degree of the map $Z_{\text{Gr}}|_{W \setminus E_Z}$. Furthermore, $d_Z$ is constant for all $Z \in U$.
3. for all $Z \in U$, we have $W$ intersects $X_I^Z$ transversally, and all intersection points lie in $W \setminus E_Z$.

**Proof.** In the following, we shall use the fact that a morphism between irreducible varieties is generically flat, and a morphism between smooth irreducible varieties is generically smooth. (Here all varieties are over $\mathbb{C}$.) Similar results are used throughout [Kle], and we refer the reader there for precise references.

For a fixed full-rank $Z_0$, by Theorem 3.1, there exists an open subset $V \subset GL(n)$ such that $W$ and $gE_{Z_0}$ intersect properly for any $g \in V$. By dimension considerations we will have $W \setminus gE_{Z_0}$ is open and dense in $W$ for any $g \in V$. But $gE_{Z_0} = E_{gZ_0}$. The map $g \mapsto gZ_0$ gives a surjective map from $GL(n)$ to $\text{Mat}(n, k + m)$. It follows that the image of $V$ contains a Zariski open subset $U_1 \subset \text{Mat}(n, k + m)$.

Define $L \subset W \times U_1$ by

$$L := \{(X, Z) \mid X \notin E_Z\}.$$

Obviously $L$ is an irreducible and open subset of $W \times U_1$, and the fiber of $L$ over $Z \in U_1$ is $W \setminus E_Z$. Define $\mu : L \to \text{Gr}(k, k + m)$ by $\mu(X, Z) = X \cdot Z \in \text{Gr}(k, k + m)$. Define

$$S := (\mu \times \text{id})(L) = \{\mu(X, Z), Z \mid (X, Z) \in L\} \subset \text{Gr}(k, k + m) \times U_1.$$

Obviously, $S$ is an irreducible subset of $\text{Gr}(k, k + m) \times U_1$. We now assume that $\dim(S) = \dim(L)$ so that for $Z$ in a dense open subset of $U_1$, we have $\dim(\tau_Z(W)) = \dim(W)$, and we are in case (b) of part (2) of the Lemma. (The case where $\dim(S) < \dim(L)$ is easier since one expects $\tau_Z(W)$ and $Y_I$ not to intersect.) Let $d$ be the degree
of the map $\mu \times \id : L \to S$. For a dense open subset $S' \subset S$, the fiber $(\mu \times \id)^{-1}(s)$ for $s \in S'$ will have exactly $d$ points.

Let $S$ denote the closure of $S$ in $\Gr(k, k + m) \times U_1$. Over an open subset $U_2 \subset U_1$, the map $p : \tilde{S} \to U_2$ will be flat, and the fiber over $Z \in U_2$ will be reduced and equal to $\tau_Z(W)$, and all these fibers will have the same dimension. By shrinking $U_2$ if necessary, we may assume that for $Z \in U_2$, the fiber $\tau_Z'(W) := p^{-1}(Z) \cap S'$ is an open dense subset of $\tau_Z(W)$ that is contained in $Z_{\Gr}(W \setminus E)$, with the additional property that for $Y \in \tau_Z(W)$, the fiber $Z_{\Gr}^{-1}(Y)$ has cardinality equal to exactly $d$.

Let $T = p^{-1}(U_2) \cap S' \subset S$. By replacing $T$ by its smooth locus, and shrinking $U_2$ if necessary, we may in addition assume that $T$ is smooth.

For $h \in G = GL(k+m)$, we have $E_Z = E_{Z \cdot h}$ and $X \cdot (Z \cdot h) = h(X \cdot Z)$. Thus if the above properties hold for $U_2 \subset \Mat(n, k + m)$ and $T \subset \Gr(k, k + m) \times \Mat(n, k + m)$, they also hold for $U_2 \cdot h$ and $T \cdot h$ (where $h \in G$ sends $(Y, Z)$ to $(hY, Z \cdot h)$). So we may assume that $U_2$ and $T$ are closed under the $G$-action.

Consider the natural map $q : T \to \Gr(k, k + m)$. Since $T$ is smooth and irreducible, $q$ is flat with smooth fibers over a dense subset $V$ of $\Gr(k, k + m)$. The map $q$ commutes with the actions of $G$ on $T'$ and on $\Gr(k, k + m)$. Since $G$ acts transitively on $\Gr(k, k + m)$, the translations $gV$ for $g \in G$ obviously cover $\Gr(k, k + m)$, and it follows that $q$ is flat over $\Gr(k, k + m)$.

Now we apply Lemma 3.2(1,2) to the family

![Diagram](where $\iota : Y_I^{\sm} \to \Gr(k, k + m)$ is the inclusion of the smooth locus $Y_I^{\sm}$. We deduce that there is a dense open subset $U_3 \subset U_2$ such that for each $Z \in U_3$, we have that $\tau_Z(W)$ and $Y_I^{\sm}$ intersect transversally.

To finish obtaining the statement of (2)(b), it remains to show that we can find $U_4 \subset U_3$ so that for $Z \in U_4$, we have $\tau_Z(W) \cap Y_I = \tau_Z'(W) \cap Y_I^{\sm}$. That is, there are no intersection points in $\tau_Z(W) \setminus \tau_Z'(W)$ or $Y_I \setminus Y_I^{\sm}$. To do so, we repeat the argument (using Lemma 3.2(1))) for the family $(\tilde{S} \setminus T) \to U_2$ and the inclusion $\iota : Y_I \to \Gr(k, k + m)$. The typical fiber of $(\tilde{S} \setminus T) \to U_2$ has lower dimension than $\tau_Z(W)$ since $\tau_Z'(W)$ is open dense in $\tau_Z(W)$. Thus we expect $\tau_Z(W) \setminus \tau_Z'(W)$ not to intersect $Y_I$. We deduce that there exists a dense open subset of $U_3$ where all the intersection points of $\tau_Z(W)$ and $Y_I$ lie in $\tau_Z'(W) \cap Y_I$. Repeating the argument, we can also find a dense open subset of $U_3$ such that the intersection points of $\tau_Z(W)$ and $Y_I$ lie in $\tau_Z'(W) \cap Y_I^{\sm}$. Thus we can find $U_4 \subset \Mat(n, k + m)$ satisfying conditions (1) and (2) of the Lemma.
Finally, the condition (3) holds in an open subset $U' \subset \text{Mat}(n, k + m)$: the argument here only requires applying Theorem 3.1(1,2). We then set $U := U' \cap U_4$. \hfill □

An irreducible subvariety $W \subset \text{Gr}(k, n)$ of complex codimension $d$ has a cohomology class $[W] \in H^{2d}(\text{Gr}(k, n))$, which must be non-zero. Transverse intersections allow one to compute products in cohomology.

**Theorem 3.4** ([Ful, Appendix B]). Let $X$ be a nonsingular variety. Let $Y, Z \subset X$ be closed irreducible subvarieties. Suppose $Y$ and $Z$ intersect transversally. Then we have

$$[Y] \cdot [Z] = [Y \cap Z]$$

in the cohomology ring $H^*(X)$.

When $Y \cap Z$ is a finite set of $r$ (reduced) points, we have $[Y \cap Z] = r[\text{pt}] \in H^*(X)$.

Let $E_\bullet$ be the standard flag in $\mathbb{C}^n$. The cohomology ring $H^*(\text{Gr}(k, n))$ vanishes in odd degrees, and the set $\{[X_I(E_\bullet)] \mid \text{codim}(X_I) = d\}$ of Schubert classes forms a $\mathbb{Z}$-basis of $H^{2d}(\text{Gr}(k, n))$.

Recall that $H^*(\text{Gr}(k, n))$ is isomorphic to the quotient of the ring $\Lambda$ of symmetric functions by an ideal $I_{k,n}$ (see [Ful]). Under this identification, we have

$$[X_I] = s_{\lambda(I)}$$

where $\lambda(I) = (i_k - k, i_{k-1} - (k - 1), \ldots, i_1 - 1)$, and $s_\lambda$ denotes a Schur function. Thus $[\text{Gr}(k, n)] = s_{(0)}$ and $[\text{pt}] = s_{(n-k)^k}$. Let $\lambda^c$ denote the 180 degree rotation of the complement of $\lambda$ inside the $(n-k)^k$ rectangle. Then $\lambda^c(J) = \lambda(I)$ where $I = J^c := \{(n+1) - j \mid j \in J\}$. Inside $H^*(\text{Gr}(k, n))$, we have the equality

$$s_\lambda s_\mu = \begin{cases} 1 & \mu = \lambda^c \\ 0 & \text{otherwise} \end{cases}$$

for $|\lambda| + |\mu| = k(n-k)$. To summarize, a class $\sigma \in H^{2r}(\text{Gr}(k, n))$ is determined by calculating $\sigma s_\mu$ for all $\mu$ satisfying $|\mu| = k(n - k) - r$.

Let $W \subset \text{Gr}(k, n)$ be an irreducible subvariety. Recall that in Section 2, we defined the truncation $\tau_{k+m}([W]) \in H^*(\text{Gr}(k, k+m))$.

**Proposition 3.5.** Let $U_I \in \text{Mat}(n, k+m)$ denote the Zariski-open subset of Lemma 2.3 for $Y_I$, and let $U = \bigcap_I U_I$ where the intersection is over all $I$ such that $\text{dim}(W) + \text{dim}(Y_I) = km$.

1. If $\tau_{k+m}([W]) = 0$ then $\text{dim}(\tau_Z(W)) < \text{dim}(W)$ for all $Z \in U$.
2. If $\tau_{k+m}([W]) \neq 0$ then for all $Z \in U$, we have $\text{dim}(\tau_Z(W)) = \text{dim}(W)$ and

$$[\tau_Z(W)] = \frac{1}{d} \tau_{k+m}([W])$$

where $d$ is the degree of $Z_{\text{Gr}}|_{W \setminus E_Z}$. 


that the coefficient of $s$ find $I$ a non-zero number of points. The image of these points under $Z$ the conditions of Lemma 3.3, we deduce that $s$ 

\[ \text{Proof.} \text{ Suppose } r \in U. \text{ If } \tau_{k+m}([W]) \neq 0 \text{ then by condition (3) of Lemma 3.3 we can find } I \in \binom{[k+m]}{k} \text{ satisfying } \dim(W) + \dim(Y_I) = km \text{ so that } X_I^Z \text{ intersects } W \setminus E \text{ in a non-zero number of points. The image of these points under } Z_{Gr} \text{ lie in } \tau(Z(W)) \cap Y_I, \text{ and since this intersection is transverse, we must have } \dim(\tau(Z(W))) = \dim(W). \text{ For each } I, \text{ we have that } \tau(Z(W)) \text{ intersects } Y_I \text{ transversally in a finite number of points } r_I. \text{ Also } W \text{ intersects } X_I^Z \text{ transversally in a finite number of points } s_I, \text{ and from the conditions of Lemma 3.3 we deduce that } s_I = dr_I \text{ from Lemma 2.2. Let } \lambda(I)^c \text{ be the complement of } \lambda(I) \text{ in the } k \times m \text{ rectangle. It follows from Theorem 3.4 that the coefficient of } s_{\lambda(I)^c} \in [\tau(Z(W))] \text{ is equal to } r_I \text{ which is equal to } 1/d \text{ times the coefficient of } s_{(\lambda(I)^c)^+e} \text{ in } [W]. \text{ Claim (2) follows.} \]

Now suppose $\tau_{k+m}([W]) = 0$. Then by a similar argument, we deduce that $\tau(Z(W))$ does not intersect any $Y_I$. This is impossible if $\dim(\tau(Z(W))) = \dim(W)$ since $[\tau(Z(W))]$ has a non-zero cohomology class and the intersections $\tau(Z(W)) \cap Y_I$ are transversal. It follows that $\tau_{k+m}([W]) = 0$ implies that $\dim(\tau(Z(W))) < \dim(W)$. Claim (1) follows. \hfill \Box

4. Amplituhedron varieties and affine Stanley symmetric functions

4.1. Affine Stanley symmetric functions. Let $W_n$ denote the affine Coxeter group of type $A$, with generators $s_0, s_1, \ldots, s_{n-1}$, and relations

\[
\begin{align*}
    s_i^2 &= 1 \\
    s_is_j &= s_js_i & \text{if } |i-j| > 1 \\
    s_is_{i+1}s_i &= s_{i+1}s_is_{i+1} 
\end{align*}
\]

where all indices are taken modulo $n$. The length $\ell(w)$ of $w \in W_n$ is the length of the shortest expression of $w$ as a product of the $s_i$.

An element $v \in W_n$ is called cyclically decreasing if it has a reduced word $v = s_{i_1}s_{i_2} \cdots s_{i_k}$ such that $i_1, i_2, \ldots, i_k$ are distinct, and if both $i$ and $i+1$ occur then $i+1$ occurs before $i$. For example, $s_3s_1s_1s_0s_0$ is cyclically decreasing if $n = 7$. A cyclically decreasing factorization of $v$ is a factorization $v = v_1v_2 \cdots v_r$ where $\ell(v) = \ell(v_1) + \ell(v_2) + \cdots + \ell(v_r)$ and each $v_i$ is cyclically decreasing. For $v \in W_n$, we define the affine Stanley symmetric function

\[
\tilde{F}_v(x_1, x_2, \ldots) = \sum_{v=v_1v_2\cdots v_r} x_1^{\ell(v_1)}x_2^{\ell(v_2)} \cdots x_r^{\ell(v_r)}.
\]

In [Lam] it is shown that $\tilde{F}_v$ is a symmetric function.

An affine permutation is a bijection $f : \mathbb{Z} \rightarrow \mathbb{Z}$ satisfying

\[
\begin{align*}
    (1) \quad f(i+n) &= f(i) + n \\
    (2) \quad \sum_{i=1}^{n} (f(i) - i) &= kn
\end{align*}
\]

A $(k, n)$-bounded affine permutation is an affine permutation satisfying

\[
i \leq f(i) \leq i + n.
\]
We denote the (finite) set of \((k, n)\)-bounded affine permutations by \(\text{Bound}(k, n)\). The group \(W_n\) acts on the set of affine permutations on the right, with \(s_i\) acting by swapping \(f(i + rn)\) and \(f(i + rn + 1)\) for all \(r \in \mathbb{Z}\).

Let \(f_0 : \mathbb{Z} \to \mathbb{Z}\) denote the bounded affine permutation given by \(f_0(i) = i + k\). Each bounded affine permutation \(f\) has an expression as \(f_0s_{i_1}s_{i_2}\cdots s_{i_k} = f_0v\) for \(v \in W_n\). The length \(\ell(f)\) of \(f\) is declared to be equal to the length of \(v\). We define \(\tilde{F}_f := \tilde{F}_v\).

### 4.2. The cohomology class of a positroid variety

Let \(X \in \text{Gr}(k, n)\). Pick a \(k \times n\) matrix representing \(X\), with columns \(v_1, v_2, \ldots, v_n\), and using \(v_i = v_{i+n}\) we define \(v_i\) for all \(i \in \mathbb{Z}\). Define a function \(f_X : \mathbb{Z} \to \mathbb{Z}\) by

\[
 f_X(i) = \min_{j \geq i} (v_i \in \text{span}(v_{i+1}, v_{i+2}, \ldots, v_j)).
\]

Note that if \(v_i = 0\) then \(f_X(i) = i\). It is not too hard to show [KLS] [Pos] that \(f_X \in \text{Bound}(k, n)\).

Let \(f \in \text{Bound}(k, n)\). We define the open positroid variety

\[
 \Pi_f := \{X \in \text{Gr}(k, n) \mid f_X = f\}
\]

and the positroid variety \(\Pi_f := \overline{\Pi_f}\). We have a decomposition \(\text{Gr}(k, n) = \bigsqcup_{f \in \text{Bound}(k, n)} \Pi_f\).

Let \(\tilde{F}_f \in H^*(\text{Gr}(k, n))\) be the image of the affine Stanley symmetric function in the quotient \(\Lambda/I_{k,n} \simeq H^*(\text{Gr}(k, n))\).

**Theorem 4.1** ([KLS]). We have \([\Pi_f] = \tilde{F}_f \in H^*(\text{Gr}(k, n))\).

We define the **truncated affine Stanley symmetric function** to be \(\tau_{k+m}(\tilde{F}_f)\), where \(\tilde{F}_f\) is thought of as an element of \(H^*(\text{Gr}(k, n))\).

### 4.3. The main theorem

Let \(f \in \text{Bound}(k, n)\) be a \((k, n)\)-bounded affine permutation, and \(\Pi_f\) be the positroid variety labeled by \(f\) [Pos] [KLS]. For a general \(Z\), we define

\[
 Y_f := \overline{Z_{\text{Gr}(\Pi_f \setminus E_Z)}}
\]

to be the closure of the image of \(\Pi_f \setminus E_Z\) under \(Z_{\text{Gr}}\). Obviously \(Y_f\) depends on \(Z\), but we will suppress this from the notation. Define \(Z_f\) to be \(Z_f = Z_{\text{Gr}|_{\Pi_f \setminus E_Z}} : (\Pi_f \setminus E_Z) \to Y_f\).

Our main result follows from Theorem 4.1 and Proposition 3.5 applied to \(W = \Pi_f\).

**Theorem 4.2.** There exists a Zariski-open set \(U \subset \text{Mat}(n, k + m)\) such that

1. if \(\tau_{k+m}([\Pi_f]) = 0\) then \(\dim(Y_f) < \dim(\Pi_f)\) for all \(Z \in U\), and
2. if \(\tau_{k+m}([\Pi_f]) \neq 0\) then for all \(Z \in U\), we have \(\dim(Y_f) = \dim(\Pi_f)\) and

\[
 [Y_f] = \frac{1}{d} \tau_{k+m}([\Pi_f]) \in H^*(\text{Gr}(k, k + m))
\]

where \(d = \deg(Z_f)\) is the degree of \(Z_f\), which is constant for all \(Z \in U\).
Definition 4.3. If \( \dim Y_f = \dim \Pi_f \) for a general \( Z \), then we declare \( Y_f \) to be an amplituhedron variety, and say that the affine permutation \( f \) has kinematical support.

From now on, \( Y_f, Z_f \) and \( \deg(Z_f) \) will always refer to an amplituhedron variety, the corresponding map and its degree, for \( Z \in U \).

Corollary 4.4. Let \( f \) be a \((k, n)\)-bounded affine permutation. Then \( f \) has kinematical support if and only if for some partition \( \lambda \) satisfying \( \ell^k \subseteq \lambda \subseteq (n-k)^k \), the coefficient of \( s_\lambda \) in the affine Stanley symmetric function \( \tilde{F}_f \) is non-zero.

When \( \dim(\Pi_f) = km \), this says that \( f \) has kinematical support if and only if \( s_{\ell^k} \) appears in \( \tilde{F}_f \) with non-zero coefficient. In this case, the coefficient of \( s_{\ell^k} \) in \( \tilde{F}_f \) can be computed using the affine Pieri rule for the flag variety (see Remark 4.10).

Example 4.5. Let \( k = 2, m = 4, \) and \( n = 8 \). Suppose \( f = [4, 3, 6, 5, 8, 7, 10, 9] \in \text{Bound}(2, 8) \), which can be written as \( f = s_0s_1s_3s_5s_7 \). Then from the definitions we have \( \tilde{F}_f = (\sum_{i=1}^{\infty} x_i)^4 \). The coefficient of \( s_{(2,2)} \) in \( \tilde{F}_f \) is equal to 2. So \( f \) has kinematical support and the map \( Z_f : (\Pi_f \setminus E_2) \rightarrow Y_f \) has degree 2.

In a similar manner we can easily produce maps \( Z_f \) of arbitrarily high finite degree.

Remark 4.6. Each positroid variety \( \Pi_f \) comes from a canonical meromorphic top-form \( \omega_{\Pi_f} \). If \( f \) has kinematical support then \( Z_f \) is generically finite and we can define a canonical form \( \omega_{Y_f} \) by pushing forward the canonical form \( \omega_{\Pi_f} \) of the positroid variety. See [Lam+].

Remark 4.7. If \( m = 4 \) and \( \dim Y_f = \dim \Pi_f = 4k = \dim \text{Gr}(k, k+4) \), then our notion of kinematical support essentially agrees with the notion from the theory of scattering amplitudes [ABGPT], though we caution that the work [ABGPT] is mostly set in “momentum space”, while the present work is set in “momentum-twistor space”. Physically, it is clear that when considering amplituhedron cells of dimension \( 4k \), one should restrict to those cells with kinematical support.

4.4. Degree of \( Z_f \). For the cells \( Y_f \) of dimension \( km \) that are used to triangulate the amplituhedron, we have an easy criterion for the degree. For \( m = 4 \), this is presumably the same combinatorial criterion discussed in [ABGPT], after translating from “momentum space” to “momentum-twistor space”.

Proposition 4.8. Suppose \( \dim(\Pi_f) = km \). Then the degree of \( Z_f \) is the coefficient of \( s_{\ell^k} \) in \( \tilde{F}_f \), if this coefficient is positive. If this coefficient is 0, then \( f \) does not have kinematical support.

Problem 4.9. Let \( k \) and \( m \) be fixed, and allow \( n \) to vary. Is there a uniform bound on \( d_f \) for all \( f \) with kinematical support?

Remark 4.10. Suppose \( f \) has kinematical support and \( \dim(\Pi_f) = km \). The Pieri rule for the affine flag manifold conjectured in [LLMS], and proved in [Lee] can be used to give a manifestly positive formula for the degree \( d_f = \deg(Z_f) \).
Specifically, the (dual) affine Pieri rule gives an identity

\[ e_k \tilde{F}_f = \sum_g c_{k,f}^g \tilde{F}_g \]

where the nonnegative numbers \( c_{k,f}^g \) count objects called strong strips [LLMS]. Now, we have

\[ [s_{ek}] \tilde{F}_f = [s_{(n-k)k}] (e_k)^m \tilde{F}_f \]

where \([s_{ek}] \tilde{F}_f\) denotes the coefficient of \( s_{ek} \) when \( \tilde{F}_f \) is expanded in terms of Schur functions. By [KLS] Theorem 7.8,

\[ [s_{(n-k)k}] \tilde{F}_g = \begin{cases} 1 & g \in \text{Bound}(k,n) \text{ and } \ell(g) = k(n-k) \\ 0 & \text{otherwise} \end{cases} \]

Thus \( d_f = [s_{ek}] \tilde{F}_f \) can be obtained by counting iterated strong strips.

The following result also gives a criterion for the degree of \( Z_f \) to be 1.

**Proposition 4.11.** Suppose \( f \) has kinematical support and \( \tilde{F}_f = \sum_\lambda c_\lambda s_\lambda \in H^*(\text{Gr}(k,n)) \). Let \( c = \gcd_{\mu \subset \mu_f}(c_{\mu^*}) \). Then the degree of \( Z_f \) divides \( c \). In particular, if \( c = 1 \), then \( \deg(Z_f) = 1 \).

For the next result, we will use the following version of Zariski’s main theorem.

**Theorem 4.12.** If \( Y \) is a quasi-compact separated scheme and \( f : X \to Y \) is a separated, quasi-finite, finitely presented morphism then there is a factorization into \( X \to Z \to Y \), where the first map is an open immersion and the second one is finite.

The following result roughly says that taking boundaries reduces the degree. So the intuition is that lower-dimensional cells tend to have smaller degree. Denote by \( \partial \Pi_f \) the boundary of a positroid variety \( \Pi_f \). This is the union of all positroid varieties \( \Pi_f \subset \Pi_f \) of strictly lower dimension. For more details on the closure partial order of positroid varieties see [KLS] Pos.

**Proposition 4.13.** Suppose \( \Pi_f \subset \partial \Pi_f \) and both \( f' \) and \( f \) have kinematical support. Then \( \deg(Z_{f'}) \leq \deg(Z_f) \).

**Proof.** By applying Lemma 3.3 to both \( W = \Pi_f \) and \( W = \Pi_{f'} \), we see that we may assume that we are considering \( Z \in \text{Mat}(n, k+m) \) such that \( \Pi_f \setminus E_Z \) is dense in \( \Pi_f \) and \( \Pi_{f'} \setminus E_Z \) is dense in \( \Pi_{f'} \). We may suppose that \( Z_f \) has degree \( d_f \) and \( Z_{f'} \) has degree \( d_{f'} \) where both maps are dimension-preserving.

Let \( V \subset (\Pi_f \setminus E_Z) \) consist of points \( X \in \Pi_f \setminus E_Z \) where \( Z_f^{-1}(Z_f(X)) \) is finite. Since fiber dimension is upper semicontinuous on the source, the set \( V \) is open in \( \Pi_f \setminus E_Z \). But \( Z_{f'} = Z_f \mid_{\Pi_{f'} \setminus E_Z} \) so \( V \cap (\Pi_{f'} \setminus E_Z) \) is open in \( \Pi_{f'} \setminus E_Z \) as well.

By assumption \( Z_{f'}|_V \) is quasi-finite, so by Theorem 4.12, we have a factorization of \( Z_{f'}|_V \) as \( V \to S \to Y_f \), where \( V \to S \) is an open immersion and \( S \to Y_f \) is finite. Clearly, \( S \to Y_f \) has degree \( d_f \). It follows that the typical fiber of \( Z_{f'}|_V \) has exactly \( d_f \) points, and every fiber of \( Z_{f'}|_V \) has \( \leq d_f \) points. In particular, the typical fiber of \( Z_{f'} \) has \( \leq d_f \) points. Thus \( d_{f'} \leq d_f \). \( \square \)
A similar argument gives

**Proposition 4.14.** Suppose $\Pi_f' \subset \partial \Pi_f$ and $f$ does not have kinematical support. Then $f'$ does not have kinematical support.

### 4.5. Application to the amplituhedron

The totally nonnegative part $\text{Gr}(k, n)_{\geq 0}$ of the real Grassmannian is the locus of points $X \in \text{Gr}(k, n)(\mathbb{R})$ representable with nonnegative (real) Plücker coordinates $\Delta_f(X)$. The totally nonnegative part of $\Pi_f$ is defined to be $(\Pi_f)_{\geq 0} := \Pi_f \cap \text{Gr}(k, n)_{\geq 0}$.

We say that $Z$ is *positive* if all $(k + m) \times (k + m)$ minors are positive, and all entries are real. If $Z$ is positive and general (that is, $Z$ belongs to the Zariski-dense set $U$ of Theorem 4.2), we define the TNN part of $Y_f$ to be

$$(Y_f)_{\geq 0} := Z_{\text{Gr}}((\Pi_f)_{\geq 0}).$$

As shown in [AT], it is not difficult to see that in this case $(\Pi_f)_{\geq 0}$ does not intersect $E_Z$.

**Proposition 4.15.** Suppose $f$ has kinematical support. Then $\dim_{\mathbb{R}}((Y_f)_{\geq 0}) = \dim(Y_f)$.

**Proof.** It is known [Pos] that $(\Pi_f)_{\geq 0}$ has real dimension equal to the complex dimension of $\Pi_f$. Since $\Pi_f$ is irreducible (see [KLS]), it follows that $(\Pi_f)_{\geq 0}$ is Zariski-dense in $\Pi_f$. It follows that $(Y_f)_{\geq 0}$ is Zariski-dense in $Y_f$ and thus $\dim_{\mathbb{R}}((Y_f)_{\geq 0}) = \dim(Y_f)$. \qed

Suppose $f$ has kinematical support and $Z_f$ has degree $d_f$, and assume that $Z$ is positive. While the map $Z_f$ has degree $d_f$, it is not the case that the map $Z_f$ restricted to $(\Pi_f)_{\geq 0}$ is generically $d_f$ to 1. It is an interesting problem to understand the geometry of the map $Z_f$ when restricted to the real points $\Pi_f(\mathbb{R})$ or totally nonnegative points $(\Pi_f)_{\geq 0}$.

### 5. Some further directions

#### 5.1. Monomial description of truncated affine Stanley symmetric functions

Since the truncated affine Stanley symmetric function $\tau_{k+m}(\tilde{F}_f)$ is Schur-positive, it is also monomial-positive. However, it is not clear which monomials in the definition of $\tilde{F}_f$ actually contribute to the truncation.

**Problem 5.1.** Find a direct combinatorial description of the monomial expansion of $\tau_{k+m}(\tilde{F}_f)$.

Presumably this involves selecting some of the cyclically decreasing factorizations of $f$ to contribute to $\tau_{k+m}(\tilde{F}_f)$.

#### 5.2. Non-generic maps $Z$

Our results only apply to generic $Z \in \text{Mat}(n, k + m)$. However, from the point of view of convex geometry, it is interesting to consider non-generic maps. Specifically, when $k = 1$, the totally nonnegative Grassmannian $\text{Gr}(1, n)_{\geq 0}$ is a simplex embedded in projective space, and thus any polytope $P$ can
be expressed as the image $Z_{Gr}(\text{Gr}(1,n)_{>0})$ for some choice of $Z$. When $Z$ is positive, $P$ will be a cyclic polytope (see [Stu] for a related result).

**Problem 5.2.** Compute the cohomology class $[Z_{Gr}(\Pi_f \setminus E_Z)] \in H^*(\text{Gr}(k,k+m))$ for all $Z \in \text{Mat}(n,k+m)$.

A related, possibly easier, problem is the following.

**Problem 5.3.** What is the cohomology class of $X_I(F_\bullet) \cap X_J(G_\bullet)$ when the two Schubert varieties are not in generic position?

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