Generalized problem of two and four Newtonian centers

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Abstract

We consider integrable spherical analogue of the Darboux potential, which appear in the problem (and its generalizations) of the planar motion of a particle in the field of two and four fixed Newtonian centers. The obtained results can be useful when constructing a theory of motion of satellites in the field of an oblate spheroid in constant curvature spaces.

Keywords and phrases: spherical two (and four) centers problem, Newtonian potential, spherico-conical coordinates, separation of variables
1 Introduction.

The two-center problem is well known in classical celestial mechanics: two fixed
centers, with masses $m_1$ and $m_2$, attract some “massless” particle, moving in their
field according to Newton’s law. The integrability of this problem was proved by
Euler, by means of the separation of variables [15].

A qualitative analysis of the plane two-center problem was offered by C. Char-
lier [7] (see also [11]); a qualitative analysis of the spatial two-center problem can be
found in the paper by V. M. Alekseev [2]. Note also that it was Lagrange who ob-
served that the two-center problem remains integrable upon addition of the potential
of an elastic spring attached to the midpoint of the rectilinear segment connecting
both centers. Lagrange also studied the limiting case of the problem, where one of
the two-centers and its mass tend to infinity. In the limit we have the problem of
a point subject to the superposition of the field of a Newtonian center (the Kepler
problem) and a homogeneous field. The corresponding separation of variables and
a qualitative study of this problem was done by M. Born in his book on atomic
mechanics [6], in connection with the splitting of a hydrogen atom’s spectral lines
observed when it is put into an electric field (the Stark effect).

A more general two-dimensional integrable system, which incorporates the two-
center problem, was found by G. Darboux (1901) [10], by means of the method of the
separation of variables. In this paper Darboux also obtained the existence conditions
for an additional quadratic integral in case of a natural system on a plane. Later, these conditions were described by Whittaker [27].

Consider a particle of unit mass, moving on $\mathbb{R}^2 = \{x, y\}$ in the potential field

$$V = \frac{A}{x^2} + \frac{A'}{y^2} + \frac{B}{r} + \frac{B'}{r'} + \frac{B_1}{r_1} + \frac{B'_1}{r'_1} + C\rho^2,$$

(1)

where $A, A', B, B', B_1, B'_1, C = \text{const}$. Here, $r$ and $r'$ are the real distances between $m$ and the two identical real centers positioned at the points $(-c, 0), (c, 0)$ on the abscissa axis, $r = \sqrt{(x - c)^2 + y^2}$, $r' = \sqrt{(x + c)^2 + y^2}$; $\rho$ is the distance between $m$ and the origin $O$; $r_1$ and $r'_1$ are the “complex distances” between $m$ and the two imaginary centers $(0, d_1)$ and $(0, -d_1)$, $r_1 = \sqrt{x^2 + (y - id)^2}$, $r'_1 = \sqrt{x^2 + (y + id)^2}$ (see Fig. 1).

For the potential (1) to be real, it is necessary that $B'_1$ and $B_1$ are complex conjugate: $\overline{B'_1} = B_1$. As is shown in [10], if $d = c$, the system (1) allows the separation of variables in elliptic coordinates

$$x = c \cosh v \cos u, \quad y = c \sinh v \sin u$$

and has an additional first integral, which is quadratic in the momenta.

Let us examine some special cases of the potential (1). The case of $B_1 = B'_1 = 0$ was studied by G. Liouville (a more special case of $A = A' = B_1 = B'_1 = 0$ was, as we already mentioned, described by Lagrange).

It was shown in the paper [1] that the problem of a particle moving in the field of two complex conjugate centers, i.e., when $A = A' = B = B' = C = 0$ in (1), is
integrable in the three-dimensional space and makes a good approximation to the problem of a satellite’s motion in the field of an oblate spheroid (e.g., the motion of an artificial satellite of the Earth).

In the paper by I. S. Kozlov [17], the problem of the plane motion of a particle in the field of four fixed centers (two real and two complex) was integrated in terms of quadratures and further studied. Besides, in [17], several interpretations of this problem were offered, with reference to actual problems of applied celestial mechanics.

2 The Kepler problem. The two-center problem on a sphere and a pseudosphere. Historical notes

Systematic generalization of various problems from classical and celestial mechanics to constant curvature spaces (a three-dimensional sphere $S^3$, as well as a pseudosphere $L^3$, or Lobachevsky space) was done by W. Killing in his extensive, but, unfortunately, almost forgotten paper [16].

Note also that beside Killing, in the 19th century, non-Euclidean mechanics in constant curvature spaces was studied by R. Lipschitz, F. Schering and H. Liebmann. It is interesting that though a whole chapter from Liebmann’s textbook on non-
Euclidean geometry [20] concerned the generalization of Newton’s law of attraction, study of the Kepler problem, and reformulation of Kepler’s laws for the cases of a sphere and a pseudosphere, similar results were independently and almost simultaneously rediscovered in the 20th century by several authors [19, 8, 13, 18, 14, 23, 9]. The classical paper by E. Schrödinger [22] should also be mentioned, where he studied a quantum analog of the Kepler problem in a curved space, implicitly assuming that the corresponding classical problem was integrable. An analogue of Newton’s law of attraction for $L^3$ was known to J. Bolyai, and N. I. Lobachevsky, and for $S^3$ — to P. Serret.

W. Killing in [16] studied, among other things, the problems of $n$-dimensional dynamics in constant curvature spaces, including the dynamics of an $n$-dimensional rigid body. An up-to-date analysis can be found in [12] (see also [5]).

The generalization of the two-center problem to constant curvature spaces is also due to W. Killing, who integrated this problem, using the method of the separation of variables. It was independently solved in [19], where a more general problem was studied, similarly to what Lagrange did by introducing an elastic interaction potential in the plane two-center problem. In the papers [26, 25], a bifurcational analysis of the two-center problem on a sphere and on the Lobachevsky plane was offered. In [5] we examined the spatial two-center problem from the standpoint of its reduction and integrability; we also studied other integrable and non-integrable problems of celestial mechanics in curved spaces (including the restricted two-
three-body problems, behaviour of libration points, dynamics of rigid bodies).

In this paper, we will offer an explicit algebraic expression for the first integral
of the generalized two-center problem from [16, 19], and show a new analogue of
the problem of four Newtonian centers and \( n \) Hookian centers. In this paper, we
study only the case of a two-dimensional sphere \( S^2 \), though all the reasoning can
easily be extended to a pseudosphere \( L^2 \). Certain (not all) results are generalized
to a three-dimensional sphere \( S^3 \) (or a pseudosphere \( L^3 \)).

3 Generalization of the two-center problem to \( S^2 \).

Additional quadratic integral

Assume that a unit sphere \( S^2 \) is given in the three-dimensional space \( \mathbb{R}^3 = \{ q_1, q_2, q_3 \} \)
by \( |q|^2 = q_1^2 + q_2^2 + q_3^2 = 1 \) and denote by \( q = (q_1, q_2, q_3), \ p = (p_1, p_2, p_3) \) the
redundant coordinates and momenta, respectively. Now if we introduce the an-
gular momentum vector \( M = p \times q \) and put \( \gamma = q \), it is easy to show [3, 4, 4]
that the equations of motion in an arbitrary potential \( V = V(q) = V(\gamma) \) can be
presented as a Hamiltonian system with the Poisson bracket defined by the alge-
bra \( e(3) = so(3) \oplus \mathbb{R}^3 \):

\[
\{ M_i, M_j \} = \varepsilon_{ijk} M_k, \quad \{ M_i, M_j \} = \varepsilon_{ijk} M_k, \quad \{ \gamma_i, \gamma_j \} = 0 \quad (2)
\]
and the Hamiltonian

\[ H = \frac{1}{2} (\mathbf{M}, \mathbf{M}) + V(\gamma). \]  

(3)

From (2), (3) we obtain the equations

\[ \dot{\mathbf{M}} = \gamma \times \frac{\partial V}{\partial \gamma}, \quad \dot{\gamma} = \gamma \times \mathbf{M}, \]

which coincide with the equations of motion of a spherical top in the potential \( V(\gamma) \) [3].

The bracket (2) is degenerate and has two Casimir functions: \( F_1 = (\mathbf{M}, \gamma) \), \( F_2 = (\gamma, \gamma) = 1 \). For the problem of a point moving on a sphere, it is necessary that \( F_1 = (\mathbf{M}, \gamma) = (\mathbf{p} \times \gamma, \gamma) = 0 \).

It is well known that the analogues of the Newtonian and Hookian potentials on \( S^2 \) are, respectively, \( U_1 = \mu \cot \theta \) and \( U_2 = c \tan^2 \theta \), \( \mu, c = \text{const} \), where \( \theta \) is measured from a certain fixed pole on the sphere [16, 19].

Consider the potential

\[ V = -\mu_1 \cot \theta_1 - \mu_2 \cot \theta_2, \]  

(4)

where \( \mu_1, \mu_2 \) are the intensities of the Newtonian centers, while \( \theta_i \) is the angle between the radius-vector of a particle and the radius-vector of the \( i \)-th center. Place the Newtonian centers at the points \( \mathbf{r}_1 = (0, \alpha, \beta), \mathbf{r}_2 = (0, -\alpha, \beta), \alpha^2 + \beta^2 = 1 \), and add to (4), for the sake of generality, the potentials of three Hookian centers located at the mutually perpendicular axes \( \frac{1}{2} \sum c_i / \gamma_i^2 \) (\( c_i = \text{const} \)). We should also introduce
the quadratic potential \(C(\alpha^2\gamma_2^2 - \beta^2\gamma_3^2)\) \(C \neq 0\), which is a special case of Neumann’s potential. On the level \((M, \gamma) = 0\) we find two commuting functions \(\{H, F\} = 0\), quadratic in \(M\) [4, 21]:

\[
H = \frac{1}{2}M^2 - \mu_1 \frac{\beta \gamma_3 + \alpha \gamma_2}{\sqrt{\gamma_1^2 + \beta_1 \gamma_2^2 + \alpha^2 \gamma_3^2 - 2\alpha\beta \gamma_2 \gamma_3}} - \mu_2 \frac{\beta \gamma_3 - \alpha \gamma_2}{\sqrt{\gamma_1^2 + \beta_1 \gamma_2^2 + \alpha^2 \gamma_3^2 + 2\alpha\beta \gamma_2 \gamma_3}} + \frac{1}{2} \frac{\gamma_2^2 + \gamma_3^2}{\gamma_1^2} + \\
+ \frac{1}{2} c_1 \frac{\gamma_1^2 + \gamma_2^2}{\gamma_2^2} + \frac{1}{2} c_2 \frac{\gamma_1^2 + \gamma_3^2}{\gamma_3^2} + \frac{1}{2} c_3 \frac{\gamma_1^2 + \gamma_3^2}{\gamma_3^2} + C(\alpha^2\gamma_2^2 - \beta^2\gamma_3^2),
\]

(5)

\[
F = \alpha^2 M_2^2 - \beta^2 M_3^2 + 2\alpha\beta(V_1 - V_2) - \\
- \frac{c_1}{\gamma_1^2}(\beta^2\gamma_2^2 - \alpha^2\gamma_3^2) - \frac{c_2}{\gamma_2^2} \beta^2 \gamma_1^2 + \frac{c_3}{\gamma_3^2} \alpha^2 \gamma_1^2 + 2C \alpha^2 \beta^2 \gamma_1^2,
\]

(6)

where \(\mu_1, \mu_2, \alpha, \beta, c_1, c_2, c_3, C = \text{const}\), and the functions \(V_1, V_2\) are:

\[
V_1 = \frac{\mu_1 (\beta \gamma_2 + \alpha \gamma_3)}{\sqrt{\gamma_1^2 + \beta^2 \gamma_2^2 + \alpha^2 \gamma_3^2 - 2\alpha\beta \gamma_2 \gamma_3}},
\]

(6)

\[
V_2 = \frac{\mu_2 (\beta \gamma_2 - \alpha \gamma_3)}{\sqrt{\gamma_1^2 + \beta^2 \gamma_2^2 + \alpha^2 \gamma_3^2 + 2\alpha\beta \gamma_2 \gamma_3}}.
\]

The function \(H\) is the Hamiltonian, and \(F\) is an additional quadratic integral. As it is noted in [3], the integrability of the corresponding three-dimensional \((S^3)\) problem closely depends on whether a Hookian center (with potential \(c/\gamma_3^2\)) can be added (at some point of the arc joining the Newtonian centers) to the two-center problem [4] without violating the problem’s integrability. Indeed, the term \(c/\gamma_3^2, c = \text{const}\) appears in the three-dimensional case as a result of the Routh reduction procedure, which uses the cyclic integral. This integral is due to the equations’ invariance under rotations (group \(SO(2)\)), in the plane perpendicular to the plane of the two centers.
The system (5) is of the Liouville type and can be integrated in spherico-conical coordinates $u_1, u_2$, ($0 < u_1 < \alpha$, $0 < u_2 < \beta$) given by
\[
\gamma_1 = \sqrt{\frac{u_1 u_2}{\alpha \beta}}, \\
\gamma_2 = \sqrt{\frac{(\alpha^2 - u_1)(\alpha^2 + u_2)}{\alpha}}, \\
\gamma_3 = \sqrt{\frac{(\beta^2 + u_1)(\beta^2 - u_2)}{\beta}}.
\]
Note, however, that obtaining the integrals (5) in the algebraic form is quite a non-trivial problem, as its solution implies dealing with an inverse spherico-conical transformation.

As A. Albouy informed us, the two-center problem on $S^2$ (or $L^2$) can be transformed to the traditional Euler problem of two centers by means of the central (gnomonic) projection and a suitable transformation of time. However, we cannot yet prove this statement.

4 The problem of four Newtonian centers on $S^2$

Consider the potential on a sphere:
\[
V_{im} = \xi_1 \cot \theta_1 + \xi_2 \cot \theta_2 = \\
= \xi_1 \frac{\mu \gamma_1 + i \nu \gamma_3}{\sqrt{(\mu^2 - \nu^2)^2 - (\mu \gamma_1 + i \nu \gamma_3)^2}} + \xi_2 \frac{\mu \gamma_1 - i \nu \gamma_3}{\sqrt{(\mu^2 - \nu^2)^2 - (\mu \gamma_1 - i \nu \gamma_3)^2}},
\]
where $\mu^2 - \nu^2 = 1$, $\xi_1, \xi_2 = \text{const.}$

This potential corresponds to the two-center problem on sphere. The intensities of the centers are “complex” and the centers themselves are equidistant (complex
conjugate) from the pole (Fig. 2). For the potential to be real, it is necessary
that \( \xi_1 = \xi_2 \). As in the Euclidean case, the potential (8) can be regarded as a
certain approximation to the problem of a particle moving in the field of an oblate
spheroid in a curved space.

The system with the potential (8) is also separable in the spherico-nical coordi-
\[ \mu = \frac{\beta}{1 - \alpha^2}, \quad \nu = \frac{\alpha\beta}{1 - \alpha^2}. \]  
\[ (9) \]

In the coordinates (7), the potential \( V + V_{\text{Im}} \) is also separable. This potential
(for \( c_i = 0 \)) corresponds to the problem of four fixed centers, two imaginary and
two real, which belong to two mutually perpendicular planes through the pole (see
Fig. 2). Here, as in the planar case, when the distance between the real centers is
fixed, the distance between the complex centers is also not arbitrary: it is uniquely
defined by (9).

It is easy to show that the potentials
\[ V_G = \frac{1}{2} \left( \sum c_i/\gamma_i^2 \right), \quad V_N = C(\alpha^2\gamma_2^2 - \beta^2\gamma_3^2) \quad c_i, C = \text{const}. \]  
\[ (10) \]
can be added (without violating integrability) to the four centers problem and it
results in a more general system, separable in the coordinates (7). The poten-
tials $V, V_{\text{Im}}, V_{G}, V_N$ written in terms of the variables (7) look like:

\[
V = \frac{(\mu_1 + \mu_2) \sqrt{(\alpha^2 - u_1)(\beta^2 - u_2)} + (\mu_1 - \mu_2) \sqrt{(\alpha^2 + u_2)(\beta^2 - u_2)}}{u_1 + u_2},
\]

\[
V_{\text{Im}} = \frac{(\xi_1 + \xi_2) \sqrt{u_2(\beta^2 - u_2)} + i(\xi_1 - \xi_2) \sqrt{u_1(\beta^2 + u_1)}}{u_1 + u_2},
\]

\[
\frac{1}{\gamma_1^2} = \frac{\beta^2(\beta^2 - u_2)^{-1} - (\beta^2 + u_1)^{-1}}{u_1 + u_2},
\]

\[
\frac{1}{\gamma_2^2} = \frac{\alpha^2(\alpha^2 - u_1)^{-1} - (\alpha^2 + u_2)^{-1}}{u_1 + u_2},
\]

\[
\frac{1}{\gamma_3^2} = \frac{\alpha\beta u_1^{-1} + u_2^{-1}}{u_1 + u_2}, V_N = C \frac{u_1^2 - u_2^2}{u_1 + u_2}.
\]

One can easily show that in the limit $R \to \infty$ (the case of Euclidean plane) the total potential $V + V_{\text{Im}} + V_{G} + V_N$ becomes the Darboux potential \( \Omega \). Note that this potential, or even $V + V_{\text{Im}}$, can no longer be generalized to the corresponding integrable potential of the three-dimensional problem ($S^2$), because there is no cyclic integral, though, taken individually, the potentials $V$ and $V_{\text{Im}}$ allow such a generalization.

5 The problem of $n$ Hookian centers on a sphere

Let us present one more integrable modification of the problem of a mass point moving in the field of the Hookian potentials $c_i/(\gamma, r_i)^2$, $c_i = \text{const}$, where the Hookian centers of attraction $r_i, i = 1, 2, \ldots, n$ do not belong to mutually orthogonal axes, but are placed arbitrarily on an equator \( \Omega \) (Fig. 3).
When \((M, \gamma) = 0\), the Hamiltonian and the additional integral are

\[
H = \frac{1}{2} M^2 + \frac{1}{2} \sum_{i=1}^{n} \frac{c_i}{(r_i, \gamma)^2} + U(\gamma_3)
\]

\[
F = M^2_3 + (1 - \gamma^2_3) \sum_{i=1}^{n} \frac{c_i}{(r_i, \gamma)^2}.
\]

There is an arbitrary function \(U(\gamma_3)\) in (11). This function means addition of an arbitrary “central” field with the center on a perpendicular to the plane of the Hookian potentials (Fig. 3). For example, one more Hookian center can be placed at the pole. This implies (see [4]) that the spatial problem of a point moving on a three dimensional sphere \(S^3\) under the action of \(n\) Hookian centers on its equator is also integrable.

Note that a Euclidean analogue of the problem in question is trivial, as even in Cartesian coordinates it yields \(n\) linear oscillators. In this case, the Hookian centers can be arbitrarily scattered in \(\mathbb{R}^2\). In the case of curved space, even on a two-dimensional sphere, the problem of motion in the field of three arbitrarily placed Hookian centers is not integrable, as simulations reveal chaos in this system. The quadratic integral \(F\) in (11) is due to the fact that the problem is separable in the spherical coordinates \((\theta, \varphi)\). Indeed, the Hamiltonian \(H\) can be written as

\[
H = \frac{1}{2} \left( p^2_\theta + \frac{p^2_\varphi}{\sin^2 \theta} \right) + \frac{1}{2} \sum_{i=1}^{n} \frac{c_i}{\sin^2 \theta \cos^2(\varphi - \varphi_i)} + U(\theta) =
\]

\[
= \frac{1}{2} p^2_\theta + \frac{1}{\sin^2 \theta} \left[ p^2_\varphi + \sum_{i=1}^{n} \frac{c_i}{\cos^2(\varphi - \varphi_i)} \right] + U(\theta),
\]

where \(\theta, \varphi\) are the coordinates of the moving mass point, while \(\varphi_i\) defines the position
of the $i$-th Hookian center on the equator (Fig. 3). The expression in square brackets is an additional integral of motion (11).

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![Figure 1: Location of real and “imaginary” centers on a plane](image)
Figure 2: Location of real and “imaginary” centers on a sphere

Figure 3: Mutual location of a particle and Hookian centers on a sphere