ON PETRENKO’S DEVIATIONS AND SECOND ORDER
DIFFERENTIAL EQUATIONS

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ABSTRACT. New results on the oscillation of solutions of $f'' + A(z)f = 0$ and on the growth of solutions of $f'' + A(z)f' + B(z)f = 0$ are obtained, where $A$ and $B$ are entire functions. Petrenko’s magnitudes of deviation of $g$ with respect to $\infty$ play a key role in the results, where $g$ represents one of the coefficients $A$ or $B$. These quantities are defined by

$$\beta^- (\infty, g) = \liminf_{r \to \infty} \frac{\log M(r, g)}{T(r, g)}, \hspace{1cm} \beta^+ (\infty, g) = \limsup_{r \to \infty} \frac{\log M(r, g)}{T(r, g)},$$

where $M(r, g) = \max_{|z| = r} |g(z)|$ and $T(r, g)$ is the Nevanlinna characteristic of $g$.

Keywords: Asymptotic growth, growth of solutions, order of growth, oscillation of solutions, Petrenko’s deviation.

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1. Introduction

We consider the oscillation of solutions of

$$f'' + A(z)f = 0 \quad (1.1)$$

and the growth of solutions of

$$f'' + A(z)f' + B(z)f = 0 \quad (1.2)$$

where $A$ and $B$ are entire functions. It is well known that all solutions of either equation are entire. If $g$ represents either of the coefficients $A$ or $B$, our results on the equations (1.1) and (1.2) rely on the magnitudes of deviation of $g$ with respect to $\infty$ introduced by Petrenko [23]. These quantities are given by

$$\beta^- (\infty, g) = \liminf_{r \to \infty} \frac{\log M(r, g)}{T(r, g)} \hspace{1cm} \text{and} \hspace{1cm} \beta^+ (\infty, g) = \limsup_{r \to \infty} \frac{\log M(r, g)}{T(r, g)},$$

where $M(r, g) = \max_{|z| = r} |g(z)|$ and $T(r, g)$ is the Nevanlinna characteristic of $g$.

Recall that any entire function $g$ satisfies the inequalities

$$T(r, g) \leq \log M(r, g) \leq \frac{R + r}{R - r} T(R, g) \quad (1.4)$$

for all $0 < r < R < \infty$ [14, Theorem 1.6]. Choosing $R = 2r$, it is easy to obtain the following conclusion [14, Theorem 1.7]: The functions $T(r, g)$ and $\log M(r, g)$ have the same order $\rho$. Moreover, if $0 < \rho < \infty$, then $T(r, g)$ and $\log M(r, g)$ are simultaneously of minimal type, mean type or maximal type.

As for the quantities in (1.3), if $g$ is of finite lower order $\mu$, then [23, Theorem 1] shows that

$$1 \leq \beta^- (\infty, g) \leq \mathcal{B} (\mu),$$

where

$$\mathcal{B} (\mu) := \begin{cases} \frac{\pi \mu}{\sin(\pi \mu)}, & \text{if } 0 \leq \mu \leq \frac{1}{2}, \\ \pi \mu, & \text{if } \mu \geq \frac{1}{2}. \end{cases}$$

Both of the inequalities in (1.5) are sharp – see [6] regarding the first inequality, and [23, §4] regarding the second inequality. The construction in [23, §4] also shows that...
$\beta^-(\infty, g) < \beta^+(\infty, g)$ may happen. The case $1 < \beta^-(\infty, g) < B(\mu)$ is also possible – for example, the Airy integral $\text{Ai}(z)$ has lower order $3/2$ and satisfies [10, 12]

$$1 < \lim_{r \to \infty} \frac{\log M(r, \text{Ai})}{T(r, \text{Ai})} = \frac{3\pi}{4} < \frac{3\pi}{2}.$$ 

If $g$ is of infinite order, then $\beta^-(\infty, g)$ need not be finite. For example, if $g(x) = \exp(e^x)$, then $T(r, f) \sim e^r(2\pi r)^{-1/2}$ and $\log M(r, f) = e^r$ [14, pp. 19–20].

If $\phi(r)$ is any increasing function and convex in $\log r$ such that $\phi(r)/\log r \to \infty$, then there is an entire function $g$ satisfying $T(r, g) \sim \phi(r) \sim \log M(r, g)$ [6]. This result allows us to construct many examples of functions $g$ for which $\beta^-(\infty, g) = \beta^+(\infty, g)$.

An entire function $g(z) = \sum_{n=0}^{\infty} a_n z^{\lambda_n}$ is said to have Fejér gaps if $\sum \lambda_n^{-1} < \infty$ and Fabry gaps if $\lim \lambda_n/n = \infty$. A function $g$ with Fejér gaps has no finite deficient values, and satisfies

$$T(r, g) \sim \log M(r, g)$$

as $r \to \infty$ outside a set of finite logarithmic measure. A function $g$ with Fabry gaps satisfies

$$\log L(r, g) \sim \log M(r, g), \quad L(r, g) = \min_{|z|=r} |g(z)|,$$

as $r \to \infty$ outside a set of zero logarithmic density [7]. Consequently, $g$ satisfies (1.6) as $r \to \infty$ outside a set of zero logarithmic density. Value distribution of entire functions $g$ satisfying (1.6) as $r \to \infty$ on a set of positive density is studied in [15].

Recall that the density and the lower density of a set $F \subset [1, \infty)$ are respectively

$$\overline{\text{dens}}(F) = \limsup_{r \to \infty} \frac{\int_{F \cap [1, r]} dt}{r - 1} \quad \text{and} \quad \underline{\text{dens}}(F) = \liminf_{r \to \infty} \frac{\int_{F \cap [1, r]} dt}{r - 1},$$

while the corresponding logarithmic densities are

$$\overline{\text{logdens}}(F) = \limsup_{r \to \infty} \frac{\int_{F \cap [1, r]} \frac{dt}{\log r}}{\log r} \quad \text{and} \quad \underline{\text{logdens}}(F) = \limsup_{r \to \infty} \frac{\int_{F \cap [1, r]} \frac{dt}{\log r}}{\log r}.$$ 

The following inequalities are known:

$$0 \leq \underline{\text{dens}}(F) \leq \underline{\text{logdens}}(F) \leq \overline{\text{logdens}}(F) \leq \overline{\text{dens}}(F) \leq 1.$$

If $\beta^-(\infty, g) = \beta^+(\infty, g) < \infty$, then there exists an $\alpha \in (0, 1]$ such that

$$T(r, g) \sim \alpha \log M(r, g)$$

as $r \to \infty$ without an exceptional set. Allowing exceptional sets, this generalizes (1.6). For example, $g(z) = e^z$ satisfies (1.8) for $\alpha = 1/\pi$, while exponential polynomials in general satisfy a condition of the form (1.8) as $r \to \infty$ outside of a set of zero density [22].

The main results on the equations (1.1) and (1.2) in terms of Petrenko’s magnitudes of deviation are stated and discussed in Sections 2 and 3, while the proofs of the main results are given in Sections 4 and 5.

2. Oscillation theory

Bank and Laine [2, 3] have proved that if $A$ is a transcendental entire function of order $\rho(A) < 1/2$, then $\lambda(E) = \infty$, where $E$ is a product of two linearly independent solutions of the equation (1.1) and $\lambda(g)$ denotes the exponent of convergence of zeros of $g$. Moreover, if $\rho(A) \notin \mathbb{N}$, then

$$\lambda(E) \geq \rho(A).$$

(2.1)

For the case $1/2 \leq \rho(A) < 1$, Rossi [24] improved the inequality (2.1) to

$$\lambda(E) \geq \frac{\rho(A)}{2\rho(A) - 1}.$$ 

(2.2)
where \( \lambda(E) = \infty \) if \( \rho(A) = 1/2 \). The case \( \rho(A) = 1/2 \) was proved independently by Shen [25]. Recently, Bergweiler and Eremenko have showed that (2.2) is the best possible in the case \( 1/2 < \rho(A) < 1 \) [4], and that (2.1) is the best possible in the case \( \rho(A) \geq 1 \) [5]. However, under additional assumptions on the coefficient \( A \), the inequality (2.1) can be improved to

\[
\lambda(E) \geq \frac{N \mu(A)}{2 \mu(A) - N},
\]

where \( N \) is the number of the unbounded components of the set \( \{ z \in \mathbb{C} : |A(z)| > K|z|^p \} \), where \( K > 0 \) and \( p > 0 \), and the lower order \( \mu(A) \) of \( A \) satisfies \( N/2 \leq \mu(A) < N \) [4]. In the same paper [4], Bergweiler and Eremenko showed that (2.3) is the best possible.

We note that, independently on the conditions imposed for the coefficient \( A \), the lower bound for \( \lambda(E) \) in the results stated above always depend on either \( \rho(A) \) or \( \mu(A) \). We proceed to search for conditions on \( A \) such that the lower bounds for \( \lambda(E) \) are independent on \( \rho(A) \) or \( \mu(A) \). The following result by Laine and Wu is in this direction.

**Theorem A** ([20]). Let \( A \) be a transcendental entire function of finite order satisfying

\[
T(r, A) \sim \log M(r, A)
\]

as \( r \to \infty \) outside an exceptional set \( G \) of finite logarithmic measure. If \( E \) is a product of two linearly independent solutions of (1.1), then \( \lambda(E) = \infty \).

We prove the following generalization of Theorem A, which also improves the inequality (2.1) when \( 1 \leq \rho(A) < \frac{1 - \beta}{2(1 - \alpha)} \).

**Theorem 2.1.** Let \( \alpha \in (0, 1) \), and let \( A \) be a transcendental entire function satisfying

\[
T(r, A) \sim \alpha \log M(r, A)
\]

as \( r \to \infty \) outside a set \( G \) with \( \log \text{denc}(G) = \beta < 1 \). Suppose further that one of the following holds:

1. \( \rho(A) \notin \mathbb{N} \),
2. \( \mu(A) < \rho(A) \),
3. \( \rho(A) < \frac{1 - \beta}{2(1 - \alpha)} \).

If \( E \) is a product of two linearly independent solutions of (1.1), then

\[
\lambda(E) \geq \frac{1 - \beta}{2(1 - \alpha)}.
\]

In particular, if \( \alpha = 1 \), then \( \lambda(E) = \infty \).

If \( A \) is Mittag-Leffler’s function of order \( \rho \in (1/2, (2 + \pi)/(2\pi)) \), then \( A \) satisfies (2.4) with \( \alpha = \frac{1 - \rho}{\pi \rho} \), see [14, p. 19]. Such functions \( A \) are examples of entire functions with the property \( \rho(A) < \frac{1}{\pi(1 - \rho)} \). Examples of entire functions \( A \) satisfying (1) or (2) in Theorem 2.1 are standard. Each of the conditions (1)–(3) is necessary since the equation

\[
f'' + (e^z - 1/16)f = 0
\]

has two linearly independent solutions \( f_1 \) and \( f_2 \) such that \( \lambda(f_1 f_2) = 0 \), see [18, p. 107]. Here the coefficient \( A(z) = e^z - 1/16 \) has order \( \rho(A) = 1 \) and satisfies (2.4) for \( \alpha = 1/\pi \).

The lower bound of \( \lambda(E) \) in Theorem 2.1 does not depend on \( \rho(A) \), and we can see that \( \lambda(E) \) can be arbitrarily large provided only that \( \alpha \) is close enough to 1 independently on the magnitude of \( \rho(A) \).

**Corollary 2.2.** Let \( A \) be a transcendental entire function, and let \( \alpha = 1/\beta^+(\infty, A) \). Suppose further that one of (1)–(3) with \( \beta = 0 \) in Theorem 2.1 holds. If \( E \) is a product of two linearly independent solutions of (1.1), then

\[
\lambda(E) \geq \frac{1}{2(1 - \alpha)}.
\]

In particular, if \( \alpha = 1 \), then \( \lambda(E) = \infty \).
3. Growth of solutions

It is well known that the coefficients $A$ and $B$ of (1.2) are polynomials if and only if the solutions of (1.2) are of finite order. The possible orders in terms of the degrees of $A$ and $B$ can be found in [13]. Hence, if $A$ is transcendental and if $f_1$ and $f_2$ are linearly independent solutions of (1.2), then at least one of them is of infinite order. Finite order solutions are also possible – for example, $f(z) = e^{-z}$ solves (1.2) with $A(z) = e^z$ and $B(z) = e^z - 1$. This background led to asking the following research question in [9]: What conditions on $A$ and $B$ will guarantee that every solution $f \neq 0$ of (1.2) has infinite order? Examples of such conditions are

(i) $\rho(A) < \rho(B)$,
(ii) $A$ is a polynomial and $B$ is transcendental,
(iii) $\rho(B) < \rho(A) \leq 1/2$,
(iv) $A$ is transcendental with $\rho(A) = 0$ and $B$ is a polynomial,

see Theorems 2 and 6 in [9] and the main result in [16].

The seminal paper [9] has prompted a considerable amount of interest in studying the growth of solutions of complex linear differential equations having well over one hundred citations in the MathSciNet database in 2020. The following result by Laine and Wu is of particular interest from the point of view of this paper.

**Theorem B** ([19]). Suppose that $A$ and $B \neq 0$ are entire functions such that $\rho(B) < \rho(A) \leq \infty$ and

$$T(r, A) \sim \log M(r, A)$$

as $r \to \infty$ outside a set $G$ of finite logarithmic measure. Then every non-trivial solution of (1.2) is of infinite order.

Kwon and Kim [17] have shown that the conclusion of Theorem B still holds if the set $G$ satisfies $\log \text{dens}(G) < (\rho(A) - \rho(B))/\rho(A)$. If the condition $\rho(B) < \rho(A) \leq \infty$ is replaced with $\mu(B) < \mu(A) \leq \infty$, where $A$ satisfies (2.4) as $r \to \infty$ outside a set $G$ satisfying $\log \text{dens}(G) = 0$, then [22, Theorem 1.5] shows that

$$\rho(f) \geq \frac{\mu(A) - \mu(B)}{21(\mu(A) + \mu(B)) \sqrt{2\pi} (1 - \alpha)} - 1$$

for every solution $f \neq 0$ of (1.2). In particular, if $\alpha = 1$, then $\rho(f) = \infty$.

For an entire function $g$, we define

$$\xi(g) := \frac{1}{2\pi} \cdot \text{m} \left( \left\{ \theta \in [0, 2\pi) : \limsup_{r \to \infty} \frac{\log^+ \left| g(re^{i\theta}) \right|}{\log r} < \infty \right\} \right).$$

Clearly $0 \leq \xi(g) \leq 1$. We have $\xi(g) = 1$ if $g$ is a polynomial, and $\xi(g) = 0$ if $g(z) = e^z + e^{-z}$. A transcendental entire function $g$ with $\xi(g) = 1$ exists, see [14, Lemma 4.1]. If $g$ is a Mittag-Leffler’s function of order $\rho > 1/2$, then $\xi(g) = 1 - \frac{1}{2\pi}$, see [14, p. 19].

The following result gives a new condition on the coefficients of (1.2) in terms of Petrenko’s deviation forcing the solutions to be of infinite order.

**Theorem 3.1.** Let $A$ be an entire function such that $\xi(A) > 0$, and let $B$ be a transcendental entire function satisfying $\beta^{-}(\infty, B) < \frac{1}{1 - \xi(A)}$. Then every non-trivial solution of (1.2) is of infinite order.

It follows from (1.7) that an entire function $g$ with Fabry gaps satisfies $\beta^{-}(\infty, g) = 1$. This gives raise to the following immediate consequence of Theorem 3.1.

**Corollary 3.2.** Let $A$ and $B$ be entire functions. Suppose there exists a sector where $\log^+ \left| A(z) \right| \lesssim \log \left| z \right|$, and suppose that $B$ is transcendental with Fabry gaps. Then every non-trivial solution of (1.2) is of infinite order.
Corollary 3.2 improves [21, Theorem 1.3] in the case when $A$ is a shortage solution of $w'' + P(z)w = 0$ for a non-constant polynomial $P$ [10]. Indeed, for such $A$ there is a sector in which $A$ tends to zero exponentially. This is particularly true if $A$ is the Airy integral $\text{Ai}(z)$ that solves the equation $w'' - zw = 0$. More generally, if $A \neq 0$ is a contour integral solution of
\[ w^{(n)} + (-1)^{n+1}bw^{(k)} + (-1)^{n+1}zw = 0, \quad n \geq 2, \quad n > k > 0, \quad b \in \mathbb{C}, \]
then [11, Theorem 3] reveals that $\xi(A) \geq \frac{1}{2\pi} \cdot \frac{\pi}{n+1} \geq \frac{1}{3} > 0.$

Corollary 3.2 also improves [21, Theorem 1.7]. Indeed, if $A$ is extremal for Yang’s inequality, that is, if $p = q/2$, where $p$ denotes the number of finite deficient values and $q$ denotes the number of Borel’s directions of order of $\geq \mu(A)$ of $A$, then [27, Theorem 4] asserts that there exists a sector where $A$ decays to a certain value $a \in \mathbb{C}$.

Using the cos $\pi p$ -theorem, one can easily see that if $\xi(A) > 0$ and $\mu(B) < 1/2$, then every non-trivial solution of (1.2) is of infinite order. The same conclusion holds if
\[ 1/2 \leq \mu(B) < \frac{1}{\pi(1 - \xi(A))}. \tag{3.1} \]
This follows by (1.5) and Theorem 3.1. Next we show that the condition (3.1) can be weakened to $\mu(B) < \frac{1}{2(1 - \xi(A))}.$

**Theorem 3.3.** Let $A$ be an entire function such that $\xi(A) > 0$, and let $B$ be a transcendental entire function satisfying $\mu(B) < \frac{1}{2(1 - \xi(A))}$. Then every non-trivial solution of (1.2) is of infinite order.

To illustrate this theorem, let $A$ be Mittag-Leffler’s function of order $\rho(A) > 1/2$, and let $B$ be a transcendental entire function with $\mu(B) \neq \rho(A)$. Then $\xi(A) = 1 - \frac{1}{2\rho(A)}$ [14, p. 19], so that either $\mu(B) < \frac{1}{2(1 - \xi(A))}$ or $\rho(B) \geq \mu(B) > \rho(A)$. It follows from Theorem 3.3 and [9, Corollary 1] that every non-trivial solution of (1.2) is of infinite order.

4. Proofs of Theorem 2.1 and Corollary 2.2

Let $g$ be an entire function, and let $D = \{z \in \mathbb{C} : |g(z)| > 1\}$. For any $r > 0$, let $A_k(r)$ for $k = 1, 2, \ldots, n(r)$ be the arcs of $|z| = r$ contained in $D$, and let $\rho\theta_k(r)$ be their lengths. Define $\theta(r) = \infty$ if the entire circle $|z| = r$ lies in $D$. Otherwise, define $\theta(r) = \max_k \theta_k(r)$.

**Lemma 4.1** ([1]). For any entire function $g$ and for any $0 < \eta < 1$, we have
\[ \log \log M(r, g) > \pi \int_{r_0}^{\eta r} \frac{dt}{t \theta(t)} - c(\mu, r_0), \tag{4.1} \]
where $0 < r_0 < \eta r$ and $c(\mu, r_0)$ is a constant independent of $r$.

**Lemma 4.2** ([8]). Let $g$ be a meromorphic function of finite order $\rho$, and let $\varepsilon > 0$ be a given constant. Then there exists a set $\mathcal{F} \subset [1, \infty)$ of finite logarithmic measure, such that for all $z$ satisfying $|z| \notin \mathcal{F} \cup [0, 1]$ and for all integers $k > j > 0$, we have
\[ \left| \frac{g^{(k)}(z)}{g^{(j)}(z)} \right| < |z|^{(k-j)(\rho-1+\varepsilon)}. \]

**Lemma 4.3** ([26]). Let $A$ and $E$ be entire functions satisfying (4.2) below. Suppose that $\lambda(E) < \rho(E)$. Then
\[ \mu(E) = \rho(E) = \mu(A) = \rho(A), \]
and these numbers are equal to an integer or $\infty$. 

We proceed to prove Theorem 2.1 by modifying the reasoning in [20]. Let \( f_1 \) and \( f_2 \) be two linearly independent solutions of (1.1), and set \( E = f_1 f_2 \). From the results of Bank-Laine, Rossi and Shen, if \( \rho(A) := \rho \leq 1/2 \), then \( \lambda(E) = \infty \). Therefore, we may assume that \( \rho > 1/2 \). We make use of the famous Bank-Laine formula [3]

\[
-4A(z) = \frac{c^2}{E^2} + 2 \frac{E''}{E} - \left( \frac{E'}{E} \right)^2, \tag{4.2}
\]

where \( c \) is non-zero constant. Hence,

\[
T(r, E) = N(r, 1/E) + 2^{-1}T(r, A) + S(r, E), \quad r \to \infty. \tag{4.3}
\]

It follows from (4.3) that \( \rho(E) \) and \( \lambda(E) \) are both finite or both infinite. If \( \rho(E) = \infty \), then there is nothing to prove, and for that reason we suppose that \( \rho(E) = \varrho < \infty \).

From Lemma 4.2, choosing \( \varepsilon = \frac{1}{2} \), we have

\[
2 \left| \frac{E''(z)}{E(z)} \right| + \left| \frac{E'(z)}{E(z)} \right|^2 < 3|z|^{2\varepsilon - 1}, \tag{4.4}
\]

for all \( z \) satisfying \( |z| \notin \mathcal{F} \cup [0, 1] \), where \( \mathcal{F} \) is a set of finite logarithmic measure. Set

\[
D_1 := \{ z \in \mathbb{C} : |E(z)| > 1 \} \quad \text{and} \quad \mathcal{F}^* := \{ z \in \mathbb{C} : |z| \in \mathcal{F} \},
\]

and let \( r \theta_1(r) \) be the length of the longest arc of \( |z| = r \) in \( D_1 \). Hence, from (4.2) and (4.4), there is a constant \( r_0 > 1 \), such that for all \( z \in D_1 \setminus \mathcal{F}^* \), \( |z| > r_0 \), we have

\[
|A(z)| < |z|^{2\varrho}. \tag{4.5}
\]

Define the sets

\[
D_2 := \{ z \in \mathbb{C} : |A(z)| > |z|^{2\varrho} \} \quad \text{and} \quad H(r) := \{ \theta \in [0, 2\pi) : re^{i\theta} \in D_2 \}.
\]

From (4.5), it’s clear that

\[
(D_1 \setminus (\mathcal{F}^* \cup \{ |z| \leq r_0 \})) \bigcap (D_2 \setminus (\mathcal{F}^* \cup \{ |z| \leq r_0 \})) = \emptyset. \tag{4.6}
\]

Then

\[
2\pi T(r, A) = \int_{H(r)} \log^+ |A(re^{i\theta})|d\theta + \int_{L H(r)} \log^+ |A(re^{i\theta})|d\theta \leq m(H(r)) \log M(r, A) + 2\varrho(2\pi - m(H(r))) \log r,
\]

which gives

\[
2\pi \leq m(H(r)) \frac{\log M(r, A)}{T(r, A)} + 2\varrho(2\pi - m(H(r))) \frac{\log r}{T(r, A)}.
\]

Since \( A \) is transcendental and satisfies (2.4) outside \( G \), we obtain from the latter inequality that

\[
\liminf_{r \to \infty} m(H(r)) \geq 2\pi \alpha. \tag{4.7}
\]

Given \( \varepsilon > 0 \), from (4.6) and (4.7), there exists \( r_1 > r_0 \), such that for all \( r \notin G \cup \mathcal{F} \cup [0, r_1] \), we have

\[
\theta_1(r) \leq m \left( \left\{ \theta \in [0, 2\pi) : re^{i\theta} \in D_1 \right\} \right) \leq 2(1 - \alpha) \pi + \varepsilon. \tag{4.8}
\]

Set \( J_r := [r_1, r/2] \setminus (G \cup \mathcal{F}) \). Then it follows from Lemma 4.1 and (4.8), that

\[
\log \log M(r, E) \geq \pi \int_{r_1}^{r/2} \frac{dt}{t \theta_1(t)} - c(1/2, r_1) \geq \frac{\pi}{2(1 - \alpha) \pi + \varepsilon} \int_{J_r} \frac{dt}{t} - c(1/2, r_1).
\]
Hence,
\[
\rho(E) \geq \frac{\pi}{(1-\alpha)2\pi+\varepsilon} \left(1 - \log \text{dens}(G \cup \mathcal{F})\right) \\
\geq \frac{\pi}{(1-\alpha)2\pi+\varepsilon} \left(1 - \log \text{dens}(G) - \log \text{dens}(\mathcal{F})\right) = \frac{(1-\beta)\pi}{(1-\alpha)2\pi+\varepsilon}.
\]

Letting \( \varepsilon \to 0^+ \), we get \( \rho(E) = \infty \) if \( \alpha = 1 \) and \( \rho(E) \geq \frac{1-\beta}{2(1-\alpha)} \) if \( \alpha < 1 \). Since \( A \) satisfies one of (1)–(3), it follows from Lemma 4.3 that \( \rho(E) = \lambda(E) \), which in turn implies the conclusion of Theorem 2.1.

It remains to prove Corollary 2.2. We argue similarly as above up to (4.7), which now reads without the exceptional set \( G \). If \( \alpha > 0 \), we get (4.8), which yields the lower bound \( \rho(E) \geq \frac{\pi}{(1-\alpha)2\pi+\varepsilon} \), where we may let \( \varepsilon \to 0^+ \). If \( \alpha = 0 \), that is, if \( \beta^+(\infty, A) = \infty \), we use \( \theta_1(r) \leq 2\pi \), and obtain \( \lambda(E) \geq \frac{1}{2} \). Finally we apply Lemma 4.3.

5. Proof of Theorems 3.1 and 3.3

We recall the following lemmas.

**Lemma 5.1** ([8]). Let \( g \) be a meromorphic function of finite order \( \mu \), and let \( \varepsilon > 0 \) be a given constant. Then there exists a set \( E \subset [0,2\pi) \) that has linear measure zero, such that if \( \psi_0 \in [0,2\pi) \setminus E \), then there exists a constant \( R_0 = R_0(\psi_0) > 1 \) such that for all \( z \) satisfying \( \arg z = \psi_0 \) and \( |z| \geq R_0 \), and for all integers \( k > j \geq 0 \), we have
\[
\left| \frac{g^{(k)}(z)}{g^{(j)}(z)} \right| < |z|^{(k-j)(\mu-1+\varepsilon)}.
\]

**Lemma 5.2** ([28]). Let \( g \) be an entire function of lower order \( \mu(g) \in [1/2, \infty) \). Then there exists a sector \( S(\alpha, \beta) = \{ z : \alpha < \arg z < \beta \} \) with \( \beta - \alpha > \frac{\pi}{\mu(g)} \) and \( 0 \leq \alpha < \beta \leq 2\pi \), such that
\[
\limsup_{r \to \infty} \frac{\log \log |g(re^{i\theta})|}{\log r} \geq \mu(g)
\]
holds for all rays \( \arg z = \theta \in (\alpha, \beta) \).

We proceed to prove Theorem 3.1. Suppose on the contrary to the assertion that there exists a non-trivial solution \( f \) of (1.2) with \( \rho(f) = \rho < \infty \). Then, from Lemma 5.1, we have for any \( \theta \in [0,2\pi) \setminus E \), where \( E \in [0,2\pi) \) has linear measure zero, that there is a constant \( R(\theta) > 1 \) such that for any \( r > R(\theta) \),
\[
\left| \frac{f^{(j)}(re^{i\theta})}{f(re^{i\theta})} \right| \leq \varepsilon^{2\theta}, \quad j = 1, 2.
\]

Recall that \( 1 \leq \beta^-(\infty, B) < 1/(1 - \xi(A)) \), where \( \xi(A) > 0 \). Given constants
\[
0 < \varepsilon < \frac{1}{\beta^-(\infty, B)} - (1 - \xi(A)) \quad \text{and} \quad \frac{2}{2 + \varepsilon} < d < 1,
\]
define
\[
I_d(r) := \{ \theta \in [0,2\pi) : \log |B(re^{i\theta})| \geq (1 - d) \log M(r, b) \}.
\]

Then,
\[
2\pi T(r, B) = \int_{I_d(r)} \log^+ |B(re^{i\theta})|d\theta + \int_{\mathcal{I}_d(r)} \log^+ |B(re^{i\theta})|d\theta \\
\leq m(I_d(r)) \log M(r, B) + \left(2\pi - m(I_d(r))\right)(1 - d) \log M(r, B).
\]

}\]
Dividing both sides of (5.2) by $\log M(r, B)$ and using the definition (1.3), we deduce

$$\limsup_{r \to \infty} m(I_d(r)) \geq \frac{2\pi}{d\beta^-(\infty, B)} - \frac{2\pi(1-d)}{d}. \tag{5.3}$$

For the choice of $\varepsilon$ and $d$, we deduce from (5.3), that there exist an infinite sequence $\{r_\nu\}$ and $R^* > 0$, such that for every non-trivial solution $\nu \in \mathbb{N}$ for which $r_\nu > R^*$, we have

$$m(I_d(r_\nu)) \geq \frac{2\pi}{d\beta^-(\infty, B)} - \frac{2\pi(1-d)}{d} - \pi \varepsilon > 2\pi(1 - \xi(A)). \tag{5.4}$$

Thus there exists an interval $(\theta_1, \theta_2)$ such that

$$(\theta_1, \theta_2) \subset I_d(r_\nu) \cap \left\{ \theta \in [0, 2\pi) : \limsup_{r \to \infty} \frac{\log |A(re^{i\theta})|}{\log r} < \infty \right\}.$$ 

Therefore, for any $\theta \in (\theta_1, \theta_2) \setminus E$, we obtain by using (1.2) and (5.1),

$$\log^+ M(r_\nu, B) \lesssim \log^+ \left| B(r_\nu e^{i\theta}) \right| \lesssim \log^+ \frac{f''(r_\nu e^{i\theta})}{f(r_\nu e^{i\theta})} + \log^+ \frac{f'(r_\nu e^{i\theta})}{f(r_\nu e^{i\theta})} + \log^+ \left| A(r_\nu e^{i\theta}) \right| + 1 \lesssim \log r_\nu, \quad \nu \to \infty.$$

This implies that $B$ is a polynomial, which contradicts the assumption that $B$ is transcendental. Thus, every non-trivial solution of (1.2) is of infinite order.

We proceed to prove Theorem 3.3. If $\mu(B) < 1/2$, then by using the cos $\pi \rho$ -theorem, we get the conclusion of the theorem. Hence we assume that $\frac{1}{2} \leq \mu(B) < \frac{1}{2(1 - \xi(A))}$. Suppose on the contrary to the assertion that there exists a non-trivial solution $f$ of (1.2) with $\rho(f) = \varrho < \infty$. Then, from Lemma 5.1, we have for any $\theta \in [0, 2\pi) \setminus E$, where $E \subseteq [0, 2\pi)$ has linear measure zero, that there is a constant $R(\theta) > 1$ such that for any $r > R(\theta)$, (5.1) holds. From Lemma 5.2, there is a sector $S(\alpha, \beta) = \{z : \alpha < \arg z < \beta\}$ with

$$\beta - \alpha \geq \frac{\pi}{\mu(B)} > 2\pi(1 - \xi(A))$$

such that

$$\limsup_{r \to \infty} \frac{\log \log |B(re^{i\theta})|}{\log r} \geq \mu(B)$$

holds for all rays $\arg z = \theta \in (\alpha, \beta)$. Thus there exists an interval $(\theta_1, \theta_2)$ such that

$$(\theta_1, \theta_2) \subset (\alpha, \beta) \cap \left\{ \theta \in [0, 2\pi) : \limsup_{r \to \infty} \frac{\log |A(re^{i\theta})|}{\log r} < \infty \right\}.$$ 

Therefore, there exists a sequence $z_n = r_n e^{i\theta}$ with $r_n \to \infty$ as $n \to \infty$ and $\theta \in (\theta_1, \theta_2) \setminus E$ such that

$$\exp \left( r_n^{\mu(B) - \varepsilon} \right) \leq |B(r_n e^{i\theta})| \leq \left| \frac{f''(r_n e^{i\theta})}{f(r_n e^{i\theta})} \right| + \left| \frac{A(r_n e^{i\theta})}{f(r_n e^{i\theta})} \right| \left| \frac{f'(r_n e^{i\theta})}{f(r_n e^{i\theta})} \right| \leq r_n^{2\mu(1 + o(1))},$$

where $n$ is large enough and $\varepsilon > 0$ is small. But this is a contradiction, and so $\rho(f) = \infty$ for every non-trivial solution $f$ of (1.2).
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