On controlling simple dynamics by a disagreement function.

Katarzyna Sznajd-Weron
Institute of Theoretical Physics, University of Wrocław
January 13, 2022

We introduce a formula for the disagreement function which is used to control a recently proposed dynamics of the Ising spin system. This leads to four different phases of the Ising spin chain in a zero temperature. One of these phases is doubly degenerated (anti- and ferromagnetic states are equally probable). On the borders between the phases two types of transitions are observed: infinite degeneration and instability lines. The relaxation of the system depends strongly on the phase.

1 Introduction

The Ising spin system is one of the most frequently used models of statistical mechanics. Its simplicity (binary variables) makes it appealing to researchers from other branches of science including biology [1], sociology [2] and economy [3, 4]. In sociophysics models of opinion formation based on the social impact theory (reviewed in [5]), the individual opinion is described by the Ising spin. This corresponds not only to typical "yes"-"no" questions, but also to important issues where the distribution of opinion seems to be bimodal, peaked on extreme values. In general, in these models the influence flows inward from the border to the center, like in the majority rules, where the site in the middle takes the state of the majority of neighbouring sites.
In contrast, in our (USDF) model [6] (reviewed in [7]) an outward flow of influence is imposed. In the USDF model, an isolated person does not convince others; however, a group of people sharing the same opinion influences their neighbors. In spite of simple rules the model exhibited complicated dynamics in one [6] and more (references in [7]) dimensions. In less than a year, this model has found several applications: e.g. it was used to explain the distribution of votes among candidates in Brazilian local election [8] and to model the price dynamics [9].

In this paper we introduce the "disagreement function" [3] which is used to control the dynamics of the model. We show that for a one dimensional Ising spin chain in zero temperature this leads to four different phases: ferromagnetic, antiferromagnetic, (2,2) antiphase and a doubly degenerated phase in which both ferromagnet and antiferromagnet are equally probable stable steady states of the system. Apart of structural differences between phases the difference in relaxation will be shown. The system in general will relax in two different ways depending on the phase. Moreover, a sharp change of the relaxation time on borders of the phases will be observed.

2 The model

Recently a simple model for opinion evolution in closed community was proposed [6]. In this model the community is represented by a horizontal chain of Ising spins, which are either up or down. A pair of parallel neighbors forces its two neighbors to have the same orientation (in random sequential updating), while for an antiparallel pair, the left neighbor takes the orientation of the right part of the pair, and left neighbor follows the right part of the pair. Thus the model can be described by two simple dynamic rules:

- \( D_1: S_{i-1}(t+1) = S_i(t) \) and \( S_{i+2}(t+1) = S_i(t) \) if \( S_i(t) \neq S_{i+1} \)
- \( D_2: S_{i-1}(t+1) = S_{i+1}(t) \) and \( S_{i+2}(t+1) = S_i(t) \) if \( S_i(t) \neq S_{i+1} \)

In contrast to usual majority rules [10], in this model the influence does not flow inward from the surrounding neighbors to the center site, but spreads outward from the center to the neighbors. The model thus describes the spread of opinions. The dynamic rules leads to two different stable steady states (ferromagnetic and antiferromagnetic) with equal probability. The
second dynamic rule \((D_2)\) of the model has been already changed in two different ways. In case of antiparallel spins the neighboring spins can either flip with probability \(1/2\) \([9]\) \((D_{2A})\) or remain unchanged \([7]\) \((D_{2B})\). In both cases \((D_{2A} \text{ and } D_{2B})\) the only final state is ferromagnet. It is worth to mention that the ferromagnetic state for both rules, \(D_{2A}\) and \(D_{2B}\), is always reached (even in two dimensions) in contrast to the Ising spin system under Glauber dynamics \([11, 12]\). In the case of \(D_{2B}\) besides of ferromagnetic stable steady states, the antiferromagnetic unstable steady state exists.

Since, we have up till now three different rules for the case of antiparallel spins, we propose a generalization of the previous models. The generalized model consists of two components, hence the name TC model:

- The dynamics: choose a pair of spins \(S_{i+1}\) and \(S_{i+2}\) and change its next nearest neighbors \(S_i\) and \(S_{i+3}\).
- The rules: control the dynamics of the \(i\)-th and \((i + 3)\)-th spins by the disagreement function.

In the next sections we introduce the disagreement function and show that TC model includes as a special cases all earlier proposed models \([6, 9, 7]\). Moreover, TC model consists of more then those three subcases which we present on its phase diagram. Using Monte Carlo simulations we show how the system described by TC model relax.

3 How to control dynamics?

Let us assume for a while that we have the formula for a function that can control TC dynamics and denote it by \(E\). We choose at random a pair of spins \(S_{i+1}\) and \(S_{i+2}\) and we calculate \(E^+ = E(S_i, S_{i+1}, S_{i+2})\). Next we calculate \(E^- = E(-S_i, S_{i+1}, S_{i+2})\) in the case of flipped \(i\)-th spin. If \(E^- < E^+\) then we will flip the \(i\)-th spin, if not the spin will remain unchanged. We do the same for the second neighbor of the chosen pair i.e. for the spin \(S_{i+3}\).

Our dynamics looks now similar to the Glauber dynamics in zero temperature, where \(E\) plays the role of energy. However, there are three main differences between these two dynamics:

- In Glauber dynamics we flip the \(i\)-th spin according to the interactions with \((i - 1)\)-th and \((i + 1)\)-th spins, here we look at \((i + 1)\)-th and \((i + 2)\)-th spins.
In Glauber dynamics the flip is done even if the old energy is equal to the new one. It doesn’t seem natural in zero temperature but it is needed to get the ground state (in two dimensions even this is not enough [12]).

In our case $E$ is called the disagreement function, since it is not the energy.

Now we will look for the formula for $E$. We shall deal with the lattice model where each lattice site $i$ is occupied by an ising spin $S_i = \pm 1$. Usually, the spins are assumed to interact through pairwise coupling of the form $-J_{ij}S_iS_j$, where $J_{ij}$ are exchange integrals. Of course, the ordering of the spins is determined by the interactions. One of the best studied examples is the nearest neighbour (nn) Ising model with ferromagnetic coupling, i.e. $J_{ij} = J > 0$ for neighbour spins $S_i$ and $S_j$, while $J_{ij} = 0$ for more distant spins. Certainly, in a such model, the spins form the ferromagnetic state (all spins up or all spins down) in low temperature. For $J < 0$ the antiferromagnetic state is formed in low temperature.

In TC model the $i$-th spin interacts with its two neighbors, and the 1D hamiltonian can be written in the following form:

$$H = -J_1 \sum_i S_iS_{i+1} - J_2 \sum_i S_iS_{i+2}.$$  \hspace{1cm} (1)

For $J_1 > 0$ and $J_2 < 0$ this is the well known ANNNI (axial next-nearest neighbour Ising) model introduced in [13] and reviewed in [14]. It describes the Ising spin chain with ferromagnetic interaction $J_1 > 0$ between nearest neighbours (nn) and antiferromagnetic interactions between next nearest neighbours (nnn). Of course, in the one-dimensional case truly ordered states are stable only in zero temperature $T = 0$. If we introduce the competition ratio $r = -J_2/J_1$ we get in $T = 0$ ferromagnetic state for $r < 1/2$ and (2,2) structure for $r > 1/2$.

Now, we will use the nnn Ising hamiltonian [11] to construct the disagreement function $E$. Chowdbury and Stauffer introduced similiarly a disagreement function based on the simple nn Ising hamiltonian to the model of financial market [3]. We write $E$ in the following form:

$$E = -J_1 S_iS_{i+1} - J_2 S_iS_{i+2}. \hspace{1cm} (2)$$
Each individual would like to minimize the corresponding disagreement function. In the TC dynamics we choose a pair $S_{i+1}$ and $S_{i+2}$ and we change its neighbour $S_i$ (we also change $S_{i+3}$ spin calculating $E = -J_1 S_{i+3} S_{i+2} - J_2 S_{i+3} S_{i+1}$, but for simplicity we further write only about $i$-th spin). For these three spins ($S_i, S_{i+1}, S_{i+2}$) we have four values of $E$:

1. $+++$ or $---$ gives $E_1 = -(J_1 + J_2)$
2. $+++$ or $---$ gives $E_2 = J_1 + J_2$
3. $+-+$ or $--+$ gives $E_3 = J_1 - J_2$
4. $--+$ or $++-$ gives $E_4 = J_2 - J_1$

It is worth to notice that the possible transitions are only between states 1 and 2 or between 3 and 4. Now we can derive from the TC model all previous models:

• USDF model [9]:
  if $S_{i+1}(t) * S_{i+2}(t) = 1$ then $S_i(t + 1) = S_{i+1}(t)$ i.e. $E_1 < E_2$
  if $S_{i+1}(t) * S_{i+2}(t) = -1$ then $S_i(t + 1) = S_{i+2}(t)$ i.e. $E_3 < E_4$.
  Thus USDF model correspond to the TC model with $-J_2 < J_1 < J_2$.

• The model of financial market [9]:
  if $S_{i+1}(t) * S_{i+2}(t) = 1$ then $S_i(t + 1) = S_{i+1}(t)$
  if $S_{i+1}(t) * S_{i+2}(t) = -1$ then $S_i(t + 1) = -S_i(t)$ with probability 1/2.
  This corresponds to the TC model with $E_1 < E_2$ and $E_4 < E_3 \Rightarrow -J_2 < J_1$ and $J_1 > J_2$.

• Other models reviewed in [7]:
  if $S_{i+1}(t) * S_{i+2}(t) = 1$ then $S_i(t + 1) = S_{i+1}(t)$ i.e. $E_1 < E_2$
  if $S_{i+1}(t) * S_{i+2}(t) = -1$ then $S_i(t + 1) = S_i(t)$ i.e. $E_3 = E_4$
  These models correspond to the TC model with $J_1 = J_2$.

There are of course more subcases of the TC model depending on interaction coefficients $J_1$ and $J_2$. On the Figure 1 all possible phases, depending on interaction coefficients, are presented. The North (doubly degenerated) phase corresponds to the original rule $D_2$. The East (ferromagnetic) phase corresponds to rule $D_{2A}$ (the flip in case of antiparallel spins is made at random).
The line between these two phases corresponds to rule $D_{2B}$ (the flip is possible only in case of parallel spins). On this line the antiferromagnetic steady state still exists but it becomes unstable and we never reach it outside of this state. It is also interesting to see what happens on other border lines. The border between the ferromagnetic state and the (2,2) antiphase (see Fig.2) is infinitely degenerated. Let us define (after [14]) a $k$-band formed by $k$ adjacent, identically oriented spins, terminated at the both ends by opposite oriented spins. With such a definition, the ferromagnetic structure is zero-band, antiferromagnetic is one-band and (2,2) antiphase is a two-band structure. On the line between ferromagnet and (2,2) antiphase any sequence of $k$-band ($k \geq 2$) is equally probable (see Fig.2). The line between (2,2) antiphase and antiferromagnet is also degenerated, and any sequence of $k$-band (with $k=1,2$) is the steady state (see Fig.2).

There is also another interesting feature which differs phases from each other - the time and the style in which the system relax. We will describe it in the next section.

4 How does the system relax?

What happens when we suddenly cool our system from a high temperature to zero temperature? As we mentioned previously the system will relax to one of the possible final states described by the phase diagram (Fig.1). But how does it relax? We studied this using Monte Carlo simulations. We found out that the relaxation process strongly depends on phase. The system can reach antiferromagnetic state in the West (antiferromagnetic) phase as well as in the North (degenerated) phase. However, it will relax to this state differently in each case. In the antiferromagnetic phase the system will be almost totally ordered after several Monte Carlo Steps (MCS). Then the system will oscillate around the final state. These oscillations will decrease in time and finally the system will reach the steady state. In the degenerated phase the system will order very slowly.

In Figure 3 the examples of relaxations in all four phases are presented. To show this relaxation we choose the opinion changes, since the model was proposed to investigate the opinion dynamics. We defined the opinion [6] as
a magnetization of the system:

\[ m = \sum_{i=1}^{N} S_i, \]

For such a choice the system will relax to \(|m| = 1\) (ferromagnet) or \(|m| = 0\) (antiferromagnet or \((2,2)\) antiphase). Of course, one could also choose the two point correlation function \(g = \langle S_i S_{i+1} \rangle\) to see how the system relaxes. We have done it to recognize the final state \((g = 1, -1\) or \(0\) for ferromagnet, antiferromagnet and \((2,2)\) antiphase respectively). For \(J_1 > -J_2\) (North and East part of the diagram in Fig.1) the ordering of the system is very slow. Sometimes the opinion can change dramatically in a short time (see Fig.3). The long time trends are observed, which reminds very much of the real sociological processes. For \(J_1 < -J_2\) the system is almost ordered after several Monte Carlo steps, however, then it takes a long time to reach the real final steady state. The opinion is fluctuating around zero and these fluctuations are decreasing in time (see Fig.3). Although the way in which the system relaxes in the North and East phases is the same, the relaxation time in each of these phases is different. About a two times shorter (in average) time is needed to reach the final state in the degenerated phase. The relaxation time changes very sharply on the border between these two phases (Fig.4). A similar effect is observed also on the border between the antiferromagnetic and degenerated phases.

5 Summary

We proposed the new generalized model of opinion formation. The disagreement function was introduced to control the simple dynamics of an Ising spin chain in zero temperature. This allowed to generalize the previous model of opinion dynamics. It was shown that the phase diagram for that system described by such a model consists of four different phases. The most interesting is the existence of the doubly degenerated phase in which the system can reach the antiferromagnetic steady state or the ferromagnetic steady state with the same probability. Moreover, it was shown that the system can relax in two different ways depending on the interaction coefficients. Surprisingly the system can reach the antiferromagnetic state in two different ways. In the antiferromagnetic phase the system will be almost ordered after several
Monte Carlo steps and then decreasing oscillations around the final state will lead the system into this state. In the degenerated phase, the system will behave "blindly" making a long "random" walk to the final state. It would be probably worth to look at the system described by such a model in higher dimensions and higher temperature. We also hope that the generalized TC model will find so many applications as its older brothers [6].

I would like to thank the Foundation for Polish Science (FNP) for the financial support.

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Figure 1: The phase diagram of the TC model
Figure 2: Examples of 3 different steady states of the TC model are presented. Bright lines denote spins up and dark lines denote spins down.
Figure 3: Examples of the relaxation for 1000 spins system are presented. Two kinds of relaxations were observed depending on interaction coefficients. For $J_1 > -J_2$ the system makes long "random" walk to the final state, while for $J_1 < -J_2$ the system makes decreasing oscillations around the final state.
Figure 4: Relaxation time for $J_2 = 1$. On this figure we present results for the system of 1000 spins averaged over 10000 samples.