HEAT EQUATION AND POISSON EQUATION IN A MODEL MATRIX GEOMETRY

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Abstract. In this paper, we study the Poisson equation and heat equation in a model matrix geometry $M_n$. Our main results are about the Poisson equation and global behavior of the heat equation on $M_n$. We can show that if $c_0$ is the initial positive definite matrix in $M_n$, then $c(t)$ exists for all time and is positive definite too. We can also show the entropy stability of the solutions to the heat equation.

Mathematics Subject Classification 2000: 01Cxx
Keywords: heat equation, poisson equation, global flow, matrix geometry

1. Introduction

In the Riemannian geometry, the spectrum of the Laplacian on manifold gives the geometric and topological information about the manifold. The heat equation proof of Atiyah-Singer index theorem is one of the most famous example [30]. In particular, through the use of the Heat equation, one can define the curvature of the compact n-dimensional Riemannian manifold $(M,g)$ as below. Let $H(x,y,t)$ be the heat kernel of Laplacian operator [29]. Let $(\lambda_j)$ be the spectrum and $\{\phi_j(x)\}$ the corresponding eigenfunctions on $M$. Then

$$H(x,y,t) = \sum_j e^{-\lambda_j t} \phi_j(x)\phi_j(y).$$

We have the expansion

$$(4\pi t)^{n/2}H(x,x,t) = 1 + \frac{t}{3}R + O(t^2)$$

near $t = 0$. Here $R$ is the scalar curvature of the metric $g$. This implies that we can define the scalar curvature by

$$R = 3\frac{d}{dt}|_{t=0}[(4\pi t)^{n/2}H(x,x,t)].$$

Hence, it is nature to use the heat equation to define scalar curvature in the non-commutative geometry [10][11][12]. The aim of this paper is to explore this interesting part in a simple case, which has been recently studied by R.Duvenhage in [14]. In [14], the author introduces the Ricci flow and his main result can be

The research is partially supported by the National Natural Science Foundation of China (No.)
briefly stated as follows. Let $M_n$ be the $C^*$ algebra generated by the two matrices

$$a = e^{2\pi i x/n}, \quad b = e^{2\pi i y/n},$$

where $x, y$ are two Hermitian matrices. Define the derivations

$$\delta_1 := [y, \cdot], \quad \delta_2 := -[x, \cdot]$$

and the Laplacian operator

$$\Delta = -(\delta_1^2 + \delta_2^2).$$

Then the Ricci flow can be defined by [14]

$$\frac{d}{dt} c(t) = -\Delta \log c(t).$$

For any positive definite matrix $c_0 \in M_n$, there is a global solution and it converges to the scalar matrix determined by $c_0$. Furthermore, along this flow, the von Neumann entropy of $c(t)$ is increasing except $c_0$ is a scalar matrix.

We shall introduce in the same $M_n$ the eigenvalues and eigenfunctions of the Laplacian operator $\Delta$ and define the heat kernel and the scalar curvature as above. Then we can introduce the Ricci flow by the standard way that

$$\frac{d}{dt} c(t) = R(c(t))c(t)$$

for the positive definite matrix $c(t)$. This gives the fourth way to define the Ricci flow in non-commutative geometry. However, since there is no explicit relation about the scalar curvature and the matrix $c(t)$, this approach may be very complicated for us to get a global Ricci flow [19][20][17][18]. The Ricci flow found many interesting applications in physics. It appears as the renormalization group equations of 2-dimensional sigma models [11][2][3][4][5]. It also be used to study the evolution of the ADM mass in asymptotically flat spaces [13]. More recently, it appears in studying the contribution of black holes in Euclidean quantum gravity [22][21]. In [6], the paper describes an appropriate analog of Hamiltons Ricci flow for the noncommutative two tori, which are the prototype example of noncommutative manifolds. It is still of interest to find more way to define the Ricci flow in noncommutative geometry.

Our main results are about the Poisson equation and global behavior of the heat equation on $M_n$. We can show that if $c_0$ is the initial positive definite matrix in $M_n$, then $c(t)$ exists for all time and is positive definite too. We can also show the entropy stability of the solutions to the heat equation.

2. ELEMENTARY NONCOMMUTATIVE DIFFERENTIAL GEOMETRY

Let $X, Y$ be two Hermitian matrices on $C^n$. Define $U = e^{2\pi i X}$, $V = e^{2\pi i Y}$. We use $M_n$ to denote the algebra of all $n \times n$ complex matrices generated by $U$ and $V$ with the bracket $\{u, v\} = uv - vu$. Then $CI$, which is the scalar multiples of the
identity matrices $I$, is the commutant of the operation \{u, v\}. Sometimes we simply use $1$ to denote the $n \times n$ identity matrix.

We define two derivations $\delta_1$ and $\delta_2$ on the algebra $M_n$ by the commutators
\[
\delta_1 := [y, \cdot], \quad \delta_2 := -[x, \cdot].
\]

Define the Laplacian operator on $M_n$ by
\[
\Delta = \delta_1^*\delta_1 + \delta_2^*\delta_2 = -\delta_1^2 - \delta_2^2 = -\delta_\mu^2\delta_\mu,
\]
where we have used the Einstein sum convention. We use the Hilbert-Schmidt norm defined by the inner product
\[
< a, b > = \tau(a^*b)
\]
on the algebra $M_n$. Here $a^*$ is the complex conjugate of the matrix $a$, $\tau$ denotes the usual trace function on $M_n$. We now state basic properties of $\delta_1$, $\delta_2$ and $\Delta$ (see also [14]).

**Proposition 2.1.** For $a \in M_n$, We have the following properties
(a) If $\delta_1 a = \delta_2 a = 0$, then $a \in CI$. Conversely, $\delta_1 a = \delta_2 a = 0$.

(b) $\tau(a\delta_\mu b) = -\tau(b\delta_\mu a)$, that is $< a^*, \delta_\mu b >= -< b^*, \delta_\mu a >$.

Furthermore, if $a, b$ are Hermitian matrices, then
\[
< \delta_\mu^*a, b >= -< \delta_\mu a, b >, \quad (\delta_\mu a)^* = -\delta_\mu a.
\]

(c) $\exists c > 0$, such that
\[
c|a - \bar{a}|^2 \leq < \delta_\mu(a - \bar{a}, \delta_\mu(a - \bar{a}) > \leq c^{-1}|a - \bar{a}|^2,
\]
where $\bar{a} = \frac{\tau(a)}{n}I$.

(d) The operators $-\delta_1^2$, $-\delta_2^2$ and $\Delta$ on the Hilbert space $M_n$ are positive, that is
\[
< a, \delta_\mu^2 a > \geq 0 \quad \text{and} \quad < a, \Delta a > \geq 0.
\]

(e) If $< a, \delta_\mu^2 a >= 0$, then $\delta_\mu a = 0$.

(f) $\ker \Delta = CI$.

(g) $\tau(\Delta a) = 0$.

For completeness, we give the detail proof.

**Proof.** (a) If $\delta_1 a = \delta_2 a = 0$, then $[y, a] = -[a, x] = 0$, that is $a$ commutes with $x, y$. so $a$ can commute with the algebra generators $u, v$. Hence $a \in u, v = CI$.

The converse $\delta_1 a = \delta_2 a = 0$ is trivial.

(b) We only prove the conclusion for $\mu = 1$.

Compute,
\[
\tau(a\delta_1 b) = \tau(a[y, b]) = \tau(ayb - aby) = \tau(bay - bya)
\]
\[
= -\tau(bya - bay) = -\tau(b[y, a]) = -\tau(b\delta_1 a).
\]
The similar computation gives result for $\mu = 2$.

(c) Define
\[
|a - \bar{a}|_1 = < \delta_\mu(a - \bar{a}), \delta_\mu(a - \bar{a}) >^\frac{1}{2},
\]
we verify that $| \cdot |_1$ is a norm on $M_n/CI$.

We need to verify the following three.

1. $\delta(a) = 0$, if $|a|_1 = 0 \iff \delta_\mu(a) = 0$, $\mu = 1, 2$.

2. $\forall \lambda > 0, |\lambda a|_1 = |\lambda||a|_1$ is clear true.

3. $\forall a, b \in M_n/CI$, it is also true that $|a + b|_1 \leq |a|_1 + |b|_1$.

Since $M_n/CI$ is a finite dimension vector space, the Hilbert Schmidt norm $| \cdot |$ is equivalent to $| \cdot |_1$ on $M_n/CI$.

(d) Compute directly that $<a, \delta_\mu a > = <a, -\delta^*_\mu a > = -<\delta_\mu a, \delta_\mu a > \leq 0$

for $\mu = 1, 2$. The result implies that the positivity of the Laplacian operator.

(e) By the definition of $\delta_\mu$, we have $<a, \delta_\mu a > = <a, -\delta^*_\mu a > = -<\delta_\mu a, \delta_\mu a > = 0$.

So we obtain $\delta_\mu a = 0$.

(f) On one hand, for $a \in CI$, by the fact $\Delta a = -\delta_1^2 a - \delta_2^2 a = 0$, it is easy to know that $CI \subset \ker \Delta$. On the other hand, if $a \in \ker \Delta$, that is, $\Delta a = 0$, we derive $0 = <a, \Delta a > = -<a, \delta_\mu^2 >$ by (d), so $\delta_\mu a = 0$ by (e). It follows that $a \in CI$ by (a), so $\ker \Delta \subset CI$.

Therefore $\ker \Delta = CI$.

(g) $\tau(\Delta a) = <1, \Delta a > = <1, -\delta_\mu^2 a > = -<\delta_\mu a, \delta_\mu a > = 0$ for $\mu_1 = 0$.

□

**Proposition 2.2.** For any positive definite matrix $a \in M_n$, $\forall m \in Z$ we have $\tau(a^m \Delta a) \geq 0$

with equality if and only if $a \in CI$, i.e. if and only if $a$ is a scalar multiple of the identity matrix $I$.

**Proof.** By the definitions of $\Delta$, $\delta_1$, $\delta_2$ and (b) of the above proposition (2.1), we obtain $\tau(a^m \Delta a) = -\tau(a^m \delta_\mu^2 a) = -\tau(\delta_\mu a^m \delta_\mu a)$. Consider that $\tau(\delta_\mu a^m \delta_\mu a)$ is a sum of terms of the form $\tau((\delta_\mu a) a^p (\delta_\mu a) a^q) = \tau(a^{\frac{p}{2}} (\delta_\mu a) a^{\frac{q}{2}} (\delta_\mu a) a^{\frac{m}{2}}) = -\tau(a^{\frac{q}{2}} (\delta_\mu a) a^{\frac{p}{2}} (\delta_\mu a) a^{\frac{m}{2}}) = -\tau(\delta_\mu a a^{\frac{p}{2}} (\delta_\mu a) a^{\frac{q}{2}}) \leq 0$.

by $\tau(ab) = \tau(ba)$ and the proposition (2.1) (b) for $a$ is positive definite, where $p, q \in 0, 1, 2, \cdots$ with $p + q + 1 = m$. Therefore, $\tau(a^m \Delta a) \geq 0$. 


Now suppose $\tau(a^m \Delta a) = 0$.
Since $\tau(a^m \delta_\mu^2 a) \leq 0$, it follows that $\tau(a^m \delta_\mu^2 a) = 0$ for $\mu = 1, 2$. In particular, $m = 1$, we derive

$$0 = \tau(\delta_\mu a \delta_\mu a) = \langle (\delta_\mu a)^*, \delta_\mu a \rangle = -\langle \delta_\mu a, \delta_\mu a \rangle = -\langle \delta_\mu a, \delta_\mu a \rangle.$$ 

So $\delta_\mu a = 0$, hence $a \in CI$ by proposition 2.1(a). The converse is trivial.

The proof is complete.

3. Poisson equation

We study for a given $b \in M_n$, the solvability of the Poisson equation

$$\Delta a = b. \tag{3.1}$$

Since

$$\langle 1, \Delta a \rangle = \tau(\Delta a) = \tau(b) = 0,$$

we know that the necessary condition to solve the Poisson equation is $\bar{b} = 0$.

We can show that it is also the sufficient condition in the class $M_n/CI$.

**Theorem 3.1.** The Poisson equation (3.1) is solvable in $M_n/CI$ if and only if $\bar{b} = 0$.

**Proof.** Assume $\bar{b} = 0$, by the result from linear algebra, we only need to show the homogeneous equation

$$\Delta a = 0$$

has only zero solution in $M_n/CI$. In fact, if $\Delta a = 0$, then

$$\langle a, \Delta a \rangle = 0.$$

Note $\langle a, \Delta a \rangle = -\langle a, \delta_1^2 a \rangle = -\langle a, \delta_2^2 a \rangle = 0$, then $\langle a, \delta_\mu^2 a \rangle = 0$.

Hence, $\delta_\mu a = 0$. Then $a \in CI$.

We also have the following result for the eigenvalue of the Laplacian operator.

**Lemma 3.2.**

$$\lambda_1 = \inf\left\{ \frac{\langle \Delta a, a \rangle}{\langle a, a \rangle}, a \neq 0, \right\} \tag{3.2}$$

is the least eigenvalue of $\Delta$ on $M_n/CI$.

**Proof.** Let $a_n \in M_n/CI$, $|a_n| = 1$, such that

$$\langle \Delta a_n, a_n \rangle \rightarrow \lambda_1.$$ 

By Weierstrass compactness theorem for bound sequences in finite vector space, we may assume

$$|a_n - a_\infty| \rightarrow 0.$$ 

Hence, $|a_\infty| = 1$ and $\langle \Delta a_n, a_n \rangle \rightarrow \langle \Delta a_\infty, a_\infty \rangle = \lambda_1$. 

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By variational principal,
\[ \Delta a_\infty = \lambda_1 a_\infty \]
That is to say \( a_\infty \) is the eigenvalue corresponding to the least eigenvalue \( \lambda_1 \). □

4. Heat equation

In this section we study the heat equation
\[ u_t = -\Delta u, \quad u \in M_n, \quad (4.1) \]
with boundary condition
\[ u|_{t=0} = u_0. \]
Since (4.1) is an ODE, it has a local solution \( u = u(t) \).
Note
\[ \frac{d}{dt} |u - \bar{u}|^2 = 2 < u - \bar{u}, u_t > \]
\[ = -2 < u - \bar{u}, \Delta u > \]
\[ = -2 < \delta_\mu u, \delta_\mu u > \]
\[ \leq -2C|u - \bar{u}|^2 \]
Since \( |u - \bar{u}|^2 \leq Ae^{-2Ct} \to 0 \) as \( t \to \infty \), then (4.1) has a global solution and \( \bar{u} = \lim_{t \to \infty} u(t) = \bar{u}_0. \)

Assume \( u_0 > 0 \), we claim \( u(t) > 0 \).
Since
\[ \frac{d}{dt} \det u = \tau(u^{-1}u_t) \]
\[ = -\tau(u^{-1}\Delta u) \]
\[ = - < u^{-1}, \Delta u > \]
\[ = - < \delta_\mu u^{-1}, \delta_\mu u > . \]
Note
\[ \delta_1 u^{-1} = [y, u^{-1}] = yu^{-1} - u^{-1}y = u^{-1}(wy - yu)u^{-1} = -u^{-1}\delta_1 uu^{-1}. \]
Then
\[ \frac{d}{dt} \det u = < u^{-1}\delta_1 uu^{-1}, \delta_\mu u > \]
\[ = \tau(u^{-1}\delta_1 uu^{-1}\delta_\mu u) \]
\[ = < u^{-1}\delta_\mu u, u^{-1}\delta_\mu u > \]
\[ > 0. \]
That is to say, \( \det u \) is an increasing function, hence \( \det u > 0 \). So \( u(t) > 0 \), for \( t > 0 \).
In conclusion, we have proven
Theorem 4.1. For any $u_0 \in M_n$, $[4.1]$ has a global solution $u(t)$ with its limit $\bar{u}_0$. Furthermore, if $u_0 > 0$, then $u(t) > 0, \forall t > 0$.

In below, we assume $u_0 > 0$ and define the von Neumann entropy by

$$S(u) = -\tau(u \log u)$$

for the positive solution $u = u(t)$ with $u(0) = u_0$.

We have the following result

Proposition 4.2. The entropy $S(u)$ is increasing along the heat equation $[4.1]$.

Proof.

$$\frac{d}{dt} S(u) = -\tau(u_t \log u) - \tau(uu^{-1}u_t) = \tau(\Delta u \log u) = \tau(u \Delta \log u).$$

Set $l = \log u$, then $u = e^l$.

So

$$\frac{d}{dt} S(u) = \tau(e^l \Delta l) \geq 0.$$

5. Entropy stability of the heat equation

Given two initial matrix $u_0$, $v_0$. Let $u$, $v$ be the corresponding solutions.

Proof.

$$\frac{d}{dt} |u - v|^2 = 2 < u - v, \Delta u - \Delta v >$$

$$= 2 < u - v, \Delta(u - v) >$$

$$\leq -2c|u - v|^2.$$

$$|u - v|^2 \leq Ae^{-2Ct} \to 0,$$

where $A = |u - v|^2(0)$. Remark: Similarly, we have the Trace norm stability of solutions, where the Trace norm is denoted by $T(u, v)$. This implies the Hilbert Schmidt norm stability of equation $[4.1]$.

Recall the Fannes inequality for $\forall a, b \in M_n$ and $a > 0 \ b > 0$, we have

$$|S(a) - S(b)| \leq \hat{\Delta} \log d + \eta(\hat{\Delta}),$$

where $\eta(s) = -s \log s$, $d = \text{dim} M_n$ and $\hat{\Delta} = \sum |r_i - s_i| \leq T(a, b) \leq \frac{1}{e}$.

Then we can use the Fannes inequality to get the entropy stability of the solution of $[4.1]$.
Theorem 5.1. If $T(u_0, v_0) \leq \frac{1}{e}$, $u_0 > 0$ $v_0 > 0$ in $M_n$, then the solution $u(t), v(t)$ satisfies

$$|S(u_t) - S(v_t)| \leq T(u, v)(0) \log d + \eta(T(u, v))(0).$$

Proof. By the result above we have

$$T(u, v)(t) \leq T(u, v)(0),$$

by the Fannes inequality, we have

$$|S(u_t) - S(v_t)| \leq T(u(t), v(t)) \log d + \eta(T(u(t), v(t)))$$

$$\leq T(u(0), v(0)) \log d + \eta(T(u(0), v(0))),$$

where we have used the monotonicity of the function $\eta$ in $[0, \frac{1}{e}]$. 

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