Energy and Momentum of a Stationary Beam of Light in the New General Relativity

Gamal G.L. Nashed and Mohamed M. Mourad

Mathematics Department, Faculty of Science, Ain Shams University, Cairo, Egypt
Mathematics Department, Faculty of Science, Minia University, Minia, Egypt

e-mail:nasshed@asunet.shams.edu.eg

We give an exact solution to the gravitational field in the new general relativity. The solution creates Bonnor spacetime. This spacetime describes the gravitational field of a stationary beam of light. The energy and momentum of this solution is calculated using the energy-momentum complex given by Møller in (1978) within the framework of the Weitzenböck spacetime.
1. Introduction

Since Einstein proposed the general theory of relativity, relativists have not been able to agree upon a definition of the energy-momentum complexes associated with the gravitational field [1]. Bondi [2] argued that general relativity does not permit a non-localizable form of energy, so, in principle we should expect to be able to find an acceptable definition.

The tetrad formulation of gravitation was considered by Møller in connection with attempts to define the energy of gravitational field [3, 4]. For a satisfactory description of the total energy of an isolated system it is necessary that the energy density of the gravitational field is given in terms of first- and/or second-order derivatives of the gravitational field variables. It is well-known that there exists no covariant, nontrivial expression constructed out of the metric tensor. However, covariant expressions that contain a quadratic form of first-order derivatives of the tetrad field are feasible. Thus it is legitimate to conjecture that the difficulties regarding the problem of defining the gravitational energy-momentum are related to the geometrical description of the gravitational field rather than are an intrinsic drawback of the theory [5, 6].

New general relativity (N.G.R.) is a gravitational theory which is formulated by gauging external (spacetime) translation [7] and underlain with the Weitzenböck [8] spacetime characterized by the metricity condition and by the vanishing of curvature tensor. N.G.R with the gravitational Lagrangian (3) given below describes well all the observed gravitational phenomena in the same level as the general relativity (G.R.). Schwarzschild metric, Reissner-Nordström metric, Weyl metric solutions, Kerr and Kerr Newman metric [9] have been known also in the N.G.R.

Bonner [10] gave an exact solution in the framework of general relativity theory which describes a stationary beam of light in the z-direction. Bringley [11] have shown that this solution is of Kerr-Schild class, then he calculated its associated energy using the energy momentum complexes of Einstein, Landau-Lifshitz, Papapetrou and Weinberg (ELLPW). It is the aim of the present work to derive a solution of an axially symmetric tetrad in the N.G.R. In section 2 we gave a brief review of the N.G.R. The axially symmetric tetrad is applied to the field equations of the N.G.R. in section 3. The solution of the field equations is also given in section 3. In section 4 we calculated the energy and momentum densities associated with the obtained solution using the definition of the energy-momentum complex given by Møller [3, 12]. The final section is devoted to discussion and main results.

2. The new general relativity theory of gravitation

In a spacetime with absolute parallelism the parallel vector fields $e_i^\mu$ define the nonsymmetric connection

$$\Gamma^\lambda_{\mu\nu} \overset{\text{def}}{=} e_i^\lambda e_i^{\mu\nu},$$

(1)
where \( e_{i\mu} = \partial_{\nu}e_{i\mu} \). The curvature tensor defined by \( \Gamma_{\mu\nu}^\lambda \) is identically vanishing, however. The metric tensor \( g_{\mu\nu} \) is given by

\[
g_{\mu\nu} = \eta_{ij} b^i_\mu b^j_\nu, \tag{2}\]

with the Minkowski metric \( \eta_{ij} = \text{diag}(+1, -1, -1, -1) \). We note that, associated with any tetrad field \( e_i^\mu \) there is a metric field defined uniquely by (2), while a given metric \( g^{\mu\nu} \) does not determine the tetrad field completely; for any local Lorentz transformation of the tetrads \( e_i^\mu \) leads to a new set of tetrads which also satisfy (2). The gravitational Lagrangian \( \mathcal{L}_G \) has the form [9]

\[
\mathcal{L}_G = \sqrt{-g}L_G = \sqrt{-g}\left( -\frac{1}{3\kappa} (t^{\mu\nu\lambda}t_{\mu\nu\lambda} - \Phi^{\mu}\Phi_\mu) + \xi a^\mu a_\mu \right), \tag{3}\]

where \( t_{\mu\nu\lambda}, \Phi_\mu \) and \( a_\mu \) are irreducible representation of the torsion tensor defined by

\[
t_{\mu\nu\lambda} \overset{\text{def.}}{=} \frac{1}{2}(T_{\mu\nu\lambda} + T_{\nu\mu\lambda}) + \frac{1}{6}(g_{\lambda\mu}\Phi_\nu + g_{\lambda\nu}\Phi_\mu) - \frac{1}{3}g_{\mu\nu}\Phi_\lambda, \tag{4}\]

\[
\Phi_\mu \overset{\text{def.}}{=} T^\lambda_{\mu\lambda}, \quad a_\mu \overset{\text{def.}}{=} \frac{1}{6}\epsilon_{\mu\nu\rho\sigma}T_{\nu\rho\sigma}, \tag{4}\]

with \( \epsilon_{\mu\nu\rho\sigma} \) is a totally antisymmetric tensor normalized to

\[
\epsilon_{0123} = -\sqrt{-g}, \quad \text{with} \quad g \overset{\text{def.}}{=} \det(g_{\mu\nu}),
\]

and \( T_{\mu\nu\lambda} \) is the torsion tensor defined by

\[
T^\lambda_{\mu\nu} \overset{\text{def.}}{=} \Gamma_{\mu\nu}^\lambda - \Gamma_{\nu\mu}^\lambda. \tag{5}\]

\( \kappa \) and \( \xi \) are the Einstein gravitational constant and a free dimensionless parameter\(^1\).

The gravitational field equations for the system described by \( L_G \) are the following:

\[
G_{\mu\nu}(\{}\{}) + H_{\mu\nu} = -\kappa T_{\mu\nu}, \tag{6}\]

\[
\partial_{\mu} \left( J^{ij\mu} \right) = 0, \tag{7}\]

where the Einstein tensor \( G_{\mu\nu}(\{}\{}) \) is defined by

\[
G_{\mu\nu}(\{}\{}) \overset{\text{def.}}{=} R_{\mu\nu}(\{}\{}) - \frac{1}{2}g_{\mu\nu}R(\{}\{}), \tag{8}\]

and \( R_{\mu\nu}(\{}\{}) \) is the Ricci tensor and \( R(\{}\{} \) is the Ricci scalar. We assume that the energy-momentum tensor of matter fields is symmetric. The energy-momentum tensor of a source field with Lagrangian \( L_M \):

\[
\sqrt{-g}T^{\mu\nu} \overset{\text{def.}}{=} e^{i\mu} \frac{\delta(-\sqrt{-g}L_M)}{\delta e^i_\nu}. \tag{9}\]

\(^*\)Latin indices are raising and lowering with the aid of \( \eta_{ij} \) and \( \eta^{ij} \).

\(^1\)Throughout this paper we use the relativistic units, \( c = G = 1 \) and \( \kappa = 8\pi \).
Here $H_{\mu\nu}$ and $J_{ij\mu}$ are given by

$$H_{\mu\nu} \text{ def.} \frac{k}{\lambda} \left[ \frac{1}{2} \left\{ \epsilon^{\mu\rho\sigma\lambda}(T^\nu_{\rho\sigma} - T^\nu_{\sigma\rho}) + \epsilon^{\nu\rho\sigma\lambda}(T^\mu_{\rho\sigma} - T^\mu_{\sigma\rho}) \right\} a_{\lambda} - \frac{3}{2} a_{\mu} a^\nu - \frac{3}{4} g^{\mu\nu} a^\lambda a_{\lambda} \right], \quad (10)$$

and

$$J_{ij\mu} \text{ def.} \left[ -\frac{1}{2} \epsilon^i_{\;\rho} \epsilon^j_{\;\sigma} \epsilon^{\rho\sigma\mu\nu} a_{\nu} \right], \quad (11)$$

respectively, where

$$\lambda \text{ def.} \frac{4}{9} \xi + \frac{1}{3\kappa}. \quad (12)$$

### 3. An exact solution

In this section we will seek a solution satisfying the following conditions: The parallel vector fields having the form

$$b^k_{\mu} = \delta^k_{\mu} + M(x,y)l^k l_{\mu}. \quad (13)$$

Here $M(x,y)$ is a function of $(x,y)$ and $l_{\mu}$ is a quantity satisfying the conditions

$$\eta^{\mu\nu} l_{\mu} l_{\nu} = 0, \text{ and } l^k \text{ is defined by } l^k \text{ def.} \delta^k_{\mu} \eta^{\mu\nu} l_{\nu}. \quad (14)$$

Applying (13) to the field equations (6) and (7) one can obtains the values of $l_{\mu}$ and $l^k$ in the form

$$l_0 = 1/\sqrt{2}, \quad l_1 = l_2 = 0, \quad l_3 = -1/\sqrt{2}. \quad (15)$$

Writing explicitly the tetrad (13) using (15) one obtains

$$b^{(0)}_0 = 1 + \frac{M(x,y)}{2}, \quad b^{(0)}_3 = -\frac{M(x,y)}{2},$$

$$b^{(1)}_1 = b^{(2)}_2 = 1,$$

$$b^{(3)}_0 = -b^{(0)}_3, \quad b^{(3)}_3 = 1 - \frac{M(x,y)}{2}. \quad (16)$$

The metric associated with solution (16) has the form

$$ds^2 = -dx^2 - dy^2 - [1 - M(x,y)]dz^2 - 2M(x,y)dzdt + [1 + M(x,y)] dt^2, \quad (17)$$

which is the Bonner spacetime that describe a stationary beam of flow in the $z$-direction. For solution (16) to satisfy (6) and (7) then the left hand side of (6) must has the form

$$\nabla^2 M(x,y) = 16\pi \rho, \quad \text{with} \quad \rho = -T^3_3 = -T^0_3 = T^3_0 = T^0_0. \quad (18)$$
4. Energy and momentum

The superpotential is given by [3, 12]

\[ U_{\mu}^{\nu\lambda} = \frac{(-g)^{1/2}}{2\kappa} P_{\chi\rho\sigma}^{\tau\nu\lambda} \left[ \Phi^\rho g^\sigma \chi_{\mu\tau} - \lambda g_{\tau\mu} \gamma_{\lambda\rho\sigma} - (1 - 2\lambda) g_{\tau\mu} \gamma_{\rho\chi} \right], \tag{19} \]

where \( P_{\chi\rho\sigma}^{\tau\nu\lambda} \) is

\[ P_{\chi\rho\sigma}^{\tau\nu\lambda} \overset{\text{def.}}{=} \delta_{\chi}^\tau g_{\rho\sigma}^{\nu\lambda} + \delta_{\tau}^\rho g_{\sigma\chi}^{\nu\lambda} - \delta_{\sigma}^\tau g_{\chi\rho}^{\nu\lambda} \tag{20} \]

with \( g_{\rho\sigma}^{\nu\lambda} \) being a tensor defined by

\[ g_{\rho\sigma}^{\nu\lambda} \overset{\text{def.}}{=} \delta_{\rho}^\nu \delta_{\sigma}^\lambda - \delta_{\sigma}^\nu \delta_{\rho}^\lambda. \tag{21} \]

The energy-momentum density is defined by [3]

\[ \tau_{\mu}^{\nu} = U_{\mu}^{\nu\lambda}, \lambda, \tag{22} \]

where comma denotes ordinary differentiation. The energy \( E \) contained in a sphere with radius \( R \) is expressed by the volume integral [13]

\[ P_\mu(R) = \int_{r=R} \int \int U_{\mu}^{0\alpha} d^3x = \int_{r=R} \int \int \tau_{\mu}^0 d^3x, \tag{23} \]

with \( P_0(R) = E(R) \) which is the energy and \( P_\alpha(R) \) is the spatial momentum. Calculating the necessary components of (22) one can obtains

\[ \tau^{00} = \tau^{03} = \frac{\nabla^2 M(x, y)}{16\pi} = \rho, \tag{24} \]

where (18) is used in (24).

5. Main results and discussion

The tetrad formulation of gravitation was considered by Møller in connection with attempts to define the energy of gravitational field [3, 14]. For a satisfactory description of the total energy of an isolated system it is necessary that the energy-density of the gravitational field is given in terms of first- and/or second-order derivatives of the gravitational field variables. It is well-known that there exists no covariant, nontrivial expression constructed out of the metric tensor. However, covariant expressions that contain a quadratic form of
first-order derivatives of the tetrad field are feasible. Thus it is legitimate to conjecture that the difficulties regarding the problem of defining the gravitational energy-momentum are related to the geometrical description of the gravitational field rather than are an intrinsic drawback of the theory [5, 6].

An exact solution (16) which gives the Bonner metric spacetime has been given in the new general relativity [9]. This solution is axially symmetric and describes a stationary beam of light flowing in the z-direction. The spacetime of this solution does not have a singularities at all.

It was shown by Møller [15] that the tetrad description of the gravitational field allows a more satisfactory treatment of the energy-momentum complex than does general relativity. Unlike the usual energy momentum complexes in the classical field theory [16], the energy-momentum complex considered above is unique, in the sense that it does not allow a redefinition, i.e., the addition of extra quantities, because these quantities would violate the field equations. Therefore, we have used the superpotential (19) to calculate the energy and spatial momentum densities (22). The energy and momentum densities calculated from the complex (22) coincide and are equal to the energy and momentum density components of $T^{\mu}_{\nu}$. This result is what would be expected from purely physical arguments.
References

[1] C. W. Misner, K. S. Thorne and J. A. Wheeler, *Gravitation* (Freeman, San Francisco, 1973), p. 603; F. I. Cooperstock and R. S. Sarracino, *J. Phys.* A11, 877 (1978); S. Chandrasekhar and V. Ferrari, *Proceeding of Royal Socity of London* A435, 645 (1991).

[2] H. Bondi, *Proc. Roy. Soc. London* A427, (1990) 249.

[3] C. Møller, *Mat. Fys. Medd. Dan. Vid. Selsk.* 39 (1978), 13.

[4] C. Møller, *Nucl. Phys.* 57 (1964), 330.

[5] J. W. Maluf, *J. Math. Phys.* 35 (1994), 335.

[6] J. W. Maluf, J. F. DaRocha-neto, T. M. L. Toribio and K. H. Castello-Branco, *Phys. Rev.* D65 (2002), 124001.

[7] K. Hayashi and T. Nakano, *Prog. Theor. Phys.* 38 (1967), 491.

[8] K. Hayashi and T. Shirafuji, *Phys. Rev.* D19 (1979), 3524.

[9] T. Kawai and N. Toma, *Prog. Theor. Phys.* 87 (1992), 583.

[10] W. B. Bonnor, *Gen. Relativ. Grav.* 32, (2000) 1627.

[11] T. Bringley gr-qc/0204006.

[12] F.I. Mikhail, M.I. Wanas, A. Hindawi and E.I. Lashin, *Int. J. Theor. Phys.* 32 (1993), 1627.

[13] C. Møller, *Ann. of Phys.* 4 (1958), 347; 12 (1961), 118.

[14] C. Møller, “Tetrad fields and conservation laws in general relativity” in Proc. International School of Physics “Enrico Fermi” ed. C. Møller, (Academic Press, London, 1962).

[15] C. Møller, *Mat. Fys. Medd. Dan. Vid. Selsk.* 35 (1966), no.3.

[16] L.D. Landau and E.M. Lifshitz, *The classical Theory of Fields* (Pergamon Press, Oxford, 1980).