Nonexistence of global solutions for a weakly coupled system of semilinear damped wave equations in the scattering case with mixed nonlinear terms

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Abstract. In this paper we consider the blow-up of solutions to a weakly coupled system of semilinear damped wave equations in the scattering case with nonlinearities of mixed type. The proof of the blow-up results is based on an iteration argument. We find as critical curve for the pair of exponents \((p, q)\) in the nonlinear terms the same one found for the weakly coupled system of semilinear wave equations with the same kind of nonlinearities. In the critical and not-damped case we combine an iteration argument with the so-called slicing method to show the blow-up dynamic of a weighted version of the functionals used in the subcritical case.

Mathematics Subject Classification. Primary 35L71, 35B44; Secondary 35G55, 35L05.

Keywords. Semilinear weakly coupled system, Mixed nonlinearities, Damped wave equation, Blow-up, Scattering producing damping, Critical curve.

1. Introduction

In this paper we consider a weakly coupled system of wave equations with time-dependent and scattering producing damping terms and with mixed kinds of power nonlinearity, namely,

\[
\begin{align*}
    u_{tt} - \Delta u + b_1(t)u_t &= |v|^q, & x \in \mathbb{R}^n, \ t > 0, \\
    v_{tt} - \Delta v + b_2(t)v_t &= |\partial_t u|^p, & x \in \mathbb{R}^n, \ t > 0, \\
    (u, u_t, v, v_t)(0, x) &= (\varepsilon u_0, \varepsilon u_1, \varepsilon v_0, \varepsilon v_1)(x) & x \in \mathbb{R}^n,
\end{align*}
\]

(1.1)

where \(b_1, b_2 \in C([0, \infty)) \cap L^1([0, \infty))\) are nonnegative functions, \(\varepsilon\) is a positive parameter describing the size of initial data and \(p, q > 1\). More precisely, we
will focus on blow-up phenomena for local solutions and we will derive the corresponding upper bound for the lifespan.

In order to motivate the study of (1.1), let us recall some semilinear models which are strongly related to this weakly coupled system.

Let us begin with the Cauchy problem for the semilinear wave equation with power nonlinearity
\[
\begin{aligned}
  &u_{tt} - \Delta u = |u|^p, \\
  &(u, u_t)(0, x) = (\varepsilon u_0, \varepsilon u_1)(x), 
\end{aligned}
\]
\[x \in \mathbb{R}^n, \\ t > 0,
\]
\[\tag{1.2}
\]

After John’s pioneering paper [16], it was conjectured by Strauss in [33] that the critical exponent for the Cauchy problem (1.2) is the positive root of the equation
\[
\frac{1 + p^{-1}}{p - 1} = \frac{n - 1}{2},
\]
which is nowadays named after him Strauss exponent and denoted in this paper by \(p_{\text{Str}}(n)\). Here, critical exponent means that for \(1 < p \leq p_{\text{Str}}(n)\) local in time solutions blow up in finite times under certain sign assumptions on the initial data and regardless of the smallness of these, while for \(p > p_{\text{Str}}(n)\) the global in time existence of small data solutions holds in suitable function spaces. We refer to the introductions of [15,34] for a detailed overview on the proof of the validity of Strauss’ conjecture and on the sharp lifespan estimates for local solutions both in the subcritical case and in the critical case.

A similar situation has been studied in the case of the Cauchy problem for the semilinear wave equation of derivative type as well, namely,
\[
\begin{aligned}
  &u_{tt} - \Delta u = |\partial_t u|^p, \\
  &(u, u_t)(0, x) = (\varepsilon u_0, \varepsilon u_1)(x), 
\end{aligned}
\]
\[x \in \mathbb{R}^n, \\ t > 0,
\]
\[\tag{1.3}
\]
For (1.3) it has been proved that the critical exponent is the so-called Glassey exponent \(p_{\text{Gla}}(n) = \frac{n+1}{n-1}\), that satisfies the equation
\[
\frac{1}{p - 1} = \frac{n - 1}{2}
\]
although the global in time existence in the supercritical case for non radial solutions is still open for spatial dimensions \(n \geq 4\), see also [1,17,25,30,31,43] for the blow-up results and [9,10,32,35] for the global existence results.

Concerning the weakly coupled systems of semilinear wave equations
\[
\begin{aligned}
  &u_{tt} - \Delta u = G_1(v, \partial_t v), \\
  &v_{tt} - \Delta v = G_2(u, \partial_t u), \\
  &(u, u_t, v, v_t)(0, x) = (\varepsilon u_0, \varepsilon u_1, \varepsilon v_0, \varepsilon v_1)(x), 
\end{aligned}
\]
\[x \in \mathbb{R}^n, \\ t > 0,
\]
\[\tag{1.4}
\]
the cases \(G_1(v, \partial_t v) = |v|^p, G_2(u, \partial_t u) = |u|^q\) and \(G_1(v, \partial_t v) = |\partial_t v|^p, G_2(u, \partial_t u) = |\partial_t u|^q\) have been studied in [2,4–6,8,19–21] and in [7,14,18,41], respectively. While in the case of power nonlinearities (that is, for \(G_1(v, \partial_t v) = |v|^p,\)
\( G_2(u, \partial_t u) = |u|^q \) the critical curve is given by
\[
\max \left\{ \frac{p + 2 + q^{-1}}{pq - 1}, \frac{q + 2 + p^{-1}}{pq - 1} \right\} = \frac{n - 1}{2},
\]
the case of semilinear terms of derivative type (that is, for \( G_1(v, \partial_t v) = |\partial_t v|^p \), \( G_2(u, \partial_t u) = |\partial_t u|^q \) the critical curve is
\[
\max \left\{ \frac{p + 1}{pq - 1}, \frac{q + 1}{pq - 1} \right\} = \frac{n - 1}{2},
\]
even though the global existence part has been studied so far only in the three dimensional and radial symmetric case. Recently, the case with mixed nonlinear terms \( G_1(v, \partial_t v) = |v|^q \), \( G_2(u, \partial_t u) = |\partial_t u|^p \) has been investigated for (1.4) in [11,14]. In this paper we shall prove that the for same range of exponents \( p, q > 1 \) as in [14] a blow-up result can be proved in the subcritical case even when we add as lower order terms in the linear part damping terms with time-dependent and scattering producing coefficients (see [38–40] for this classification of a damping term with time-dependent coefficients for wave models). Furthermore, the same upper bound for the lifespan can be derived. In the critical case, we will restrict our considerations to the not-damped case, improving in some cases the upper bound for the lifespan with respect to [14], by using a quite different method.

Recently, several results for semilinear wave equations and for weakly coupled systems of semilinear wave equations have been proved in presence of time-dependent and scattering-producing coefficients for damping terms by Lai-Takamura, Wakasa-Yordanov and Palmieri-Takamura. More precisely, the blow-up dynamic for local solutions of
\[
\begin{align*}
  \begin{cases}
    u_{tt} - \Delta u + b(t)u_t = G(u, \partial_t u), & x \in \mathbb{R}^n, \ t > 0, \\
    (u, u_t)(0, x) = (\varepsilon u_0, \varepsilon u_1)(x), & x \in \mathbb{R}^n,
  \end{cases}
\end{align*}
\]
has been considered in [22,37] for power nonlinearity \( G(u, \partial_t u) = |u|^p \), in [23] for the case of derivative type \( G(u, \partial_t u) = |\partial_t u|^p \) and in [24] for the case of combined nonlinearity \( G(u, \partial_t u) = |\partial_t u|^p + |u|^q \). Finally, really recently the weakly coupled system of semilinear damped wave equations in the scattering case
\[
\begin{align*}
  \begin{cases}
    u_{tt} - \Delta u + b_1(t)u_t = G_1(v, \partial_t v), & x \in \mathbb{R}^n, \ t > 0, \\
    v_{tt} - \Delta v + b_2(t)v_t = G_2(u, \partial_t u), & x \in \mathbb{R}^n, \ t > 0, \\
    (u, u_t, v, v_t)(0, x) = (\varepsilon u_0, \varepsilon u_1, \varepsilon v_0, \varepsilon v_1)(x), & x \in \mathbb{R}^n,
  \end{cases}
\end{align*}
\]
has been considered in [27] for \( G_1(v, \partial_t v) = |v|^p \), \( G_2(u, \partial_t u) = |u|^q \) and in [28] for \( G_1(v, \partial_t v) = |\partial_t v|^p \), \( G_2(u, \partial_t u) = |\partial_t u|^q \).

In this paper our approach is based on the following methods: in the subcritical case we employ two multipliers, that are introduced in [22], in order to apply a standard iteration argument based on lower bound estimates for the spatial integrals of the nonlinear terms and on a coupled system of ordinary integral inequalities; in the critical case, we modify the approach introduced by Wakasa-Yordanov in [36,37] and adapted to weakly coupled systems in
Let \( \varepsilon u_0(x), v(0, x) = \varepsilon v_0(x) \) in \( H^1(\mathbb{R}^n) \) and the integral relations

\[
\int_{\mathbb{R}^n} \partial_t u(t, x) \phi(t, x) \, dx - \int_{\mathbb{R}^n} \varepsilon u_1(x) \phi(0, x) \, dx - \int_0^t \int_{\mathbb{R}^n} \partial_t u(s, x) \phi_s(s, x) \, dx \, ds \\
+ \int_0^t \int_{\mathbb{R}^n} \nabla u(s, x) \cdot \nabla \phi(s, x) \, dx \, ds + \int_0^t \int_{\mathbb{R}^n} b_1(s) \partial_t u(s, x) \phi(s, x) \, dx \, ds \\
= \int_0^t \int_{\mathbb{R}^n} |v(s, x)|^q \phi(s, x) \, dx \, ds
\]
(1.5)

and

\[
\int_{\mathbb{R}^n} \partial_t v(t, x) \psi(t, x) \, dx - \int_{\mathbb{R}^n} \varepsilon v_1(x) \psi(0, x) \, dx - \int_0^t \int_{\mathbb{R}^n} \partial_t v(s, x) \psi_s(s, x) \, dx \, ds \\
+ \int_0^t \int_{\mathbb{R}^n} \nabla v(s, x) \cdot \nabla \psi(s, x) \, dx \, ds + \int_0^t \int_{\mathbb{R}^n} b_2(s) \partial_t v(s, x) \psi(s, x) \, dx \, ds \\
= \int_0^t \int_{\mathbb{R}^n} |\partial_t u(s, x)|^p \psi(s, x) \, dx \, ds
\]
(1.6)

for any test functions \( \phi, \psi \in C_0^\infty([0, T) \times \mathbb{R}^n) \) and any \( t \in [0, T) \).

Let us state the blow-up result for (1.1) in the subcritical case.

**Theorem 1.2.** Let \( b_1, b_2 \) be continuous, nonnegative and summable functions. Let us consider \( p, q > 1 \) satisfying

\[
\max \left\{ \frac{q + 1 + p^{-1}}{pq - 1}, \frac{2 + q^{-1}}{pq - 1} \right\} > \frac{n - 1}{2}.
\]
(1.7)
Assume that \( u_0, v_0 \in H^1(\mathbb{R}^n) \) and \( u_1, v_1 \in L^2(\mathbb{R}^n) \) are nonnegative and compactly supported in \( B_R \) functions such that \( u_1 \neq 0 \) and \( v_0 \neq 0 \).

Let \((u, v)\) be an energy solution of (1.1) with lifespan \( T = T(\varepsilon) \) such that
\[
supp u, supp v \subset \{(t, x) \in [0, T) \times \mathbb{R}^n : |x| \leq t + R\}. \tag{1.8}
\]

Then, there exists a positive constant \( \varepsilon_0 = \varepsilon_0(u_0, u_1, v_0, v_1, n, p, q, b_1, b_2, R) \) such that for any \( 0 < \varepsilon \leq \varepsilon_0 \) the solution \((u, v)\) blows up in finite time. Moreover, the upper bound estimate for the lifespan
\[
T(\varepsilon) \leq C\varepsilon^{-\max\{\Theta_1(n, p, q), \Theta_2(n, p, q)\}^{-1}} \tag{1.9}
\]
holds, where \( C \) is an independent of \( \varepsilon \), positive constant and
\[
\Theta_1(n, p, q) = \frac{q + 1 + p^{-1}}{pq - 1} - \frac{n - 1}{2},
\]
\[
\Theta_2(n, p, q) = \frac{2 + q^{-1}}{pq - 1} - \frac{n - 1}{2}. \tag{1.10}
\]

Remark 1.3. The upper bound estimates (1.9) for the lifespan coincide with the ones for the case \( b_1 = b_2 = 0 \), for more details see also [14, Section 9].

The main result in the critical and not-damped case is Theorem 1.5. Before stating this result, we recall the definition of weak solution to (1.1) in the not-damped case \( b_1 = b_2 = 0 \).

Definition 1.4. Let \( u_0, v_0 \in H^1(\mathbb{R}^n) \) and \( u_1, v_1 \in L^2(\mathbb{R}^n) \). We say that \((u, v)\)

is a weak solution to
\[
\begin{cases}
  u_{tt} - \Delta u = |v|^q, & x \in \mathbb{R}^n, \ t > 0, \\
  v_{tt} - \Delta v = |\partial_t u|^p, & x \in \mathbb{R}^n, \ t > 0, \\
  (u, u_t, v, v_t)(0, x) = \varepsilon(u_0, u_1, v_0, v_1)(x) & x \in \mathbb{R}^n,
\end{cases} \tag{1.11}
\]
on \([0, T)\) if
\[
u \in \mathcal{C}([0, T), H^1(\mathbb{R}^n)) \cap \mathcal{C}^1([0, T), L^2(\mathbb{R}^n)) \quad \text{and} \quad \partial_t u \in L^p_{\text{loc}}([0, T) \times \mathbb{R}^n),
\]
\[
v \in \mathcal{C}([0, T), H^1(\mathbb{R}^n)) \cap \mathcal{C}^1([0, T), L^2(\mathbb{R}^n)) \quad \text{and} \quad v \in L^q_{\text{loc}}([0, T) \times \mathbb{R}^n)
\]
satisfy the equalities
\[
\begin{align*}
\int_{\mathbb{R}^n} \partial_t u(t, x) \phi(t, x) \, dx - \int_{\mathbb{R}^n} u(t, x) \phi_s(t, x) \, dx & - \varepsilon \int_{\mathbb{R}^n} u_1(x) \phi(0, x) \, dx \\
+ \varepsilon \int_{\mathbb{R}^n} u_0(x) \phi_s(0, x) \, dx + \int_0^t \int_{\mathbb{R}^n} u(s, x) \left( \phi_{ss}(s, x) - \Delta \phi(s, x) \right) \, dx \, ds \\
= \int_0^t \int_{\mathbb{R}^n} |v(s, x)|^q \phi(s, x) \, dx \, ds \tag{1.12}
\end{align*}
\]
and
\[
\int_{\mathbb{R}^n} \partial_t v(t,x)\psi(t,x)\,dx - \int_{\mathbb{R}^n} v(t,x)\psi_s(t,x)\,dx - \varepsilon \int_{\mathbb{R}^n} v_1(x)\psi(0,x)\,dx \\
+ \varepsilon \int_{\mathbb{R}^n} v_0(x)\psi_s(0,x)\,dx + \int_0^t \int_{\mathbb{R}^n} v(s,x)\left(\psi_{ss}(s,x) - \Delta \psi(s,x)\right)\,dx\,ds \\
= \int_0^t \int_{\mathbb{R}^n} |\partial_t u(s,x)|^p \phi(s,x)\,dx\,ds
\] (1.13)
for any test functions \(\phi, \psi \in C_0^\infty([0,T) \times \mathbb{R}^n)\) and any \(t \in [0,T)\).

**Theorem 1.5.** Let \(n \geq 2\) and \(b_1 = b_2 = 0\). Let us assume that \(p, q > 1\) satisfy
\[
\max \left\{ \frac{q + 1 + p^{-1}}{pq - 1}, \frac{2 + q^{-1}}{pq - 1} \right\} = \frac{n - 1}{2},
\] (1.14)
Assume that \(u_0, v_0 \in H^1(\mathbb{R}^n)\) and \(u_1, v_1 \in L^2(\mathbb{R}^n)\) are nonnegative and compactly supported in \(B_R\) functions such that \(u_1 \neq 0\) and \(v_0 \neq 0\). Let \((u, v)\) be a weak solution of (1.11) satisfying (1.8) with lifespan \(T = T(\varepsilon)\) (cf. Definition 1.4). Then, there exists a positive constant \(\varepsilon_0 = \varepsilon_0(u_0, u_1, v_0, v_1, n, p, q, R)\) such that for any \(\varepsilon \in (0, \varepsilon_0]\) the solution \((u, v)\) blows up in finite time. Moreover, the upper bound estimates for the lifespan
\[
T(\varepsilon) \leq \begin{cases} 
\exp \left( C\varepsilon^{-p(pq-1)} \right) & \text{if } \Theta_1(n, p, q) = 0, \\
\exp \left( C\varepsilon^{-q(pq-1)} \right) & \text{if } \Theta_2(n, p, q) = 0, \\
\exp \left( C\varepsilon^{-\frac{q}{pq-1}(pq-1)} \right) & \text{if } \Theta_1(n, p, q) = \Theta_2(n, p, q) = 0,
\end{cases}
\] (1.15)
hold, where \(C\) is an independent of \(\varepsilon\), positive constant.

The remaining part of this paper is organized as follows: in Sect. 2.1 we derive the coupled system of ODIIs (ordinary differential inequalities) that the spatial averages of the components of a local solution has to satisfy, then, using a suitable pair of multipliers \((m_1, m_1)\) (cf. (2.1) below) we derive the corresponding integral iteration frame from this system of ODIIs; in Sect. 2.2 we prove suitable lower bounds for the space integrals of the nonlinearities; hence, in Sect. 2.3 we combine the results from Sects. 2.1 to 2.2 in an iterative procedure which allows us to determine a sequence of lower bound estimates for the above cited spatial averages; finally, in Sect. 2.4 we conclude the proof of Theorem 1.2 proving the blow-up result thanks to the sequence of lower bounds obtained via the iteration argument and deriving the upper bound for the lifespan of a local solution. Finally, in Sect. 3 we prove Theorem 1.5. The intermediate steps are similar to the ones for the subcritical case, yet due to the presence of logarithmic factors in the lower bound estimates, the iteration procedure is combined with the slicing method. The crucial difference consists in the choice of the functionals, whose blow-up dynamic is considered. Indeed, differently from the subcritical case, we do not consider spatial averages of the components of a local solution rather weighted spatial averages of these components.
Notations
Throughout this paper we will use the following notations: $B_R$ denotes the ball around the origin with radius $R$; $f \lesssim g$ means that there exists a positive constant $C$ such that $f \leq Cg$ and, analogously, for $f \gtrsim g$; moreover, $f \sim g$ means $f \lesssim g$ and $f \gtrsim g$; finally, as in the introduction, $p_{\text{Str}}(n)$ and $p_{\text{Gla}}(n)$ denote the Strauss exponent and the Glassey exponent, respectively.

2. Subcritical case
2.1. Iteration frame
Let us recall the definition of some multipliers related to our model, which have been introduced in [22], and some properties of them, that we will employ throughout the remaining sections.

Definition 2.1. Let $b_1, b_2 \in \mathcal{C}([0, \infty)) \cap L^1([0, \infty))$ be the nonnegative, time-dependent coefficients in (1.1). We define the multipliers
\[ m_j(t) = \exp \left( - \int_t^\infty b_j(\tau) d\tau \right) \quad \text{for } t \geq 0 \text{ and } j = 1, 2. \] (2.1)

As $b_1, b_2$ are nonnegative functions, then, $m_1, m_2$ are increasing functions. Moreover, due to the summability of $b_1, b_2$, the multipliers are bounded and
\[ m_j(0) \leq m_j(t) \leq 1 \quad \text{for } t \geq 0 \text{ and } j = 1, 2. \] (2.2)

Finally, a remarkable property of these multipliers is the following one:
\[ m'_j(t) = b_j(t) m_j(t) \quad \text{for } j = 1, 2. \] (2.3)

The properties given in (2.2) and (2.3) are essential in order to handle and somehow to “neglect” the damping terms.

Henceforth, we assume that $u_0, u_1, v_0, v_1$ satisfy the assumptions of Theorem 1.2. Let $(u, v)$ be an energy solution of (1.1) on $[0, T)$ in the sense of Definition 1.1. Then, we introduce the following pair of functionals
\[ U(t) = \int_{\mathbb{R}^n} u(t, x) \, dx, \quad V(t) = \int_{\mathbb{R}^n} v(t, x) \, dx. \] (2.4)

Let us point out that the pair of functionals whose dynamic will investigated in Sect. 2.3 is actually $(U', V)$ due the nonlinearity in (1.1).

Therefore, using the support condition (1.8) and employing cutoff functions $\phi, \psi$ to localize the light-cone in (1.5)-(1.6), it results that $U, V$ satisfy
\[ U''(t) + b_1(t) U'(t) = \int_{\mathbb{R}^n} |v(t, x)|^q \, dx, \] (2.5)
\[ V''(t) + b_2(t) V'(t) = \int_{\mathbb{R}^n} |\partial_t u(t, x)|^p \, dx. \] (2.6)

Let us derive first integral lower bound estimates for $V$ from (2.6). Multiplying both sides of (2.6) by $m_2$ and using (2.3), we get
\[ m_2(t)V''(t) + m_2(t)b_2(t)V'(t) = \frac{d}{dt} (m_2(t)V'(t)) = m_2(t) \int_{\mathbb{R}^n} |\partial_t u(t, x)|^p \, dx. \]
Hence, integrating over $[0,t]$ the last relation and rearranging the resulting equation, we have

$$V'(t) = \frac{m_2(0)}{m_2(t)} V'(0) + \int_0^t \frac{m_2(s)}{m_2(t)} \int_{\mathbb{R}^n} |\partial_t u(s, x)|^p \, dx \, ds$$

$$\geq m_2(0)V'(0) + m_2(0) \int_0^t \int_{\mathbb{R}^n} |\partial_t u(s, x)|^p \, dx \, ds,$$

where in the last step we used (2.2). A further integration over $[0,t]$ provides

$$V(t) \geq V(0) + m_2(0)V'(0) t + m_2(0) \int_0^t \int_0^s \int_{\mathbb{R}^n} |\partial_t u(\tau, x)|^p \, dx \, d\tau \, ds \quad (2.7)$$

for any $t \geq 0$. Using again the support property for $u_t(t, \cdot)$ and Hölder’s inequality, we find that (2.7) implies

$$V(t) \geq C \int_0^t \int_0^s (1 + \tau)^{-n(p-1)} (U'(\tau))^p \, d\tau \, ds \quad \text{for any } t \geq 0. \quad (2.8)$$

for a suitable positive constant $C = C(n, p, b_2, R)$.

Proceeding in a similar way, we derive now two lower bound estimates for $U'$. A multiplication by $m_1$ in (2.5) and a successive integration over $[0,t]$ lead to

$$U'(t) = \frac{m_1(0)}{m_1(t)} U'(0) + \int_0^t \frac{m_1(s)}{m_1(t)} \int_{\mathbb{R}^n} |v(s, x)|^q \, dx \, ds.$$

Employing again (2.2), from the last estimate we derive

$$U'(t) \geq m_1(0) U'(0) + m_1(0) \int_0^t \int_{\mathbb{R}^n} |v(s, x)|^q \, dx \, ds. \quad (2.9)$$

Finally, thanks to the support condition for $v(t, \cdot)$, by Hölder’s inequality we find

$$U'(t) \geq K \int_0^t (1 + s)^{-n(q-1)} (V(s))^q \, ds \quad \text{for any } t \geq 0. \quad (2.10)$$

for a suitable positive constant $K = K(n, q, b_1, R)$.

In Sect. 2.3 we employ (2.8) and (2.10) in the iteration schemes.

### 2.2. Lower bounds for the spatial integral of the nonlinearities

The goal of this section is to determine lower bound estimates for the integrals of the semilinear terms. According to this purpose, we need to take into account the analysis of further auxiliary functionals related to the local solution $(u, v)$ of (1.1). More specifically, we are going to estimate the functionals

$$U_1(t) \doteq \int_{\mathbb{R}^n} u(t, x) \Psi(t, x) \, dx,$$

$$V_1(t) \doteq \int_{\mathbb{R}^n} v(t, x) \Psi(t, x) \, dx,$$

$$U_2(t) \doteq \int_{\mathbb{R}^n} \partial_t u(t, x) \Psi(t, x) \, dx.$$
In the definition of the functionals $U_1, V_1, U_2$, we used the function $\Psi = \Psi(t, x) = e^{-t}\Phi(x)$, where

$$
\Phi = \Phi(x) = \begin{cases} 
    e^x + e^{-x} & \text{for } n = 1, \\
    \int_{S^{n-1}} e^{\omega \cdot x} dS_\omega & \text{for } n \geq 2 
\end{cases} \tag{2.14}
$$

is an eigenfunction of the Laplace operator, as $\Delta \Phi = \Phi$. The test function $\Psi$ has been introduced for the first time in [42] in the study of the blow-up result for the semilinear classical wave equation with power nonlinearity in the critical case for high space dimension.

**Lemma 2.2.** Let $(w, \tilde{w})$ be a local energy solution of the Cauchy problem

$$
\begin{align*}
    w_{tt} - \Delta w + b_1(t)w_t &= G_1(t, x, w, w_t, \tilde{w}, \tilde{w}_t), & x \in \mathbb{R}^n, & t \in (0, T), \\
    \tilde{w}_{tt} - \Delta \tilde{w} + b_2(t)\tilde{w}_t &= G_2(t, x, w, w_t, \tilde{w}, \tilde{w}_t), & x \in \mathbb{R}^n, & t \in (0, T), \\
    (w, w_t, \tilde{w}, \tilde{w}_t)(0, x) &= (\varepsilon w_0, \varepsilon w_1, \varepsilon \tilde{w}_0, \varepsilon \tilde{w}_1)(x), & x \in \mathbb{R}^n,
\end{align*}
$$

where the time-dependent coefficients $b_1, b_2 \in C([0, \infty)) \cap L^1([0, \infty))$ and the nonlinear terms $G_1, G_2$ are nonnegative. Furthermore, we assume that $w_0, w_1, \tilde{w}_0, \tilde{w}_1$ are nonnegative, nontrivial and compactly supported and that $w, \tilde{w}$ satisfy a support condition as in (1.8). Let $W_1, \tilde{W}_1$ be defined by

$$
W_1(t) \doteq \int_{\mathbb{R}^n} w(t, x)\Psi(t, x) \, dx \quad \text{and} \quad \tilde{W}_1(t) \doteq \int_{\mathbb{R}^n} \tilde{w}(t, x)\Psi(t, x) \, dx
$$

for any $t \geq 0$. Then, for any $t \geq 0$ the following estimates hold

$$
W_1(t) \geq \varepsilon \frac{m_1(0)}{2} \int_{\mathbb{R}^n} w_0(x)\Phi(x) \, dx \quad \text{and} \quad \tilde{W}_1(t) \geq \varepsilon \frac{m_2(0)}{2} \int_{\mathbb{R}^n} \tilde{w}_0(x)\Phi(x) \, dx.
$$

**Proof.** See Lemma 2.2 in [28].

In particular, from Lemma 2.2 we get immediately the lower bound estimates

$$
U_1(t) \geq \varepsilon I_1[u_0] \quad \text{for any } t \geq 0, \tag{2.15}
$$

$$
V_1(t) \geq \varepsilon I_2[v_0] \quad \text{for any } t \geq 0, \tag{2.16}
$$

where

$$
I_j[f] \doteq \frac{m_j(0)}{2} \int_{\mathbb{R}^n} f(x)\Phi(x) \, dx \quad \text{for } j = 1, 2.
$$

In the next step we follow the main ideas of [23, Section 3] and [24, Section 4] in order to control the functional $U_2$ from below.

**Lemma 2.3.** Let $U_2$ be defined by (2.13). Under the same assumptions of Theorem 1.2, the following estimate holds

$$
U_2(t) \geq \varepsilon I_1[u_1] \quad \text{for any } t \geq 0. \tag{2.17}
$$
Proof. Let us begin pointing out that
\[
\frac{d}{dt} \left( m_1(t) \int_{\mathbb{R}^n} (\partial_t u(t, x) + u(t, x)) \Psi(t, x) \, dx \right)
= b_1(t) m_1(t) \int_{\mathbb{R}^n} (\partial_t u(t, x) + u(t, x)) \Psi(t, x) \, dx
+ m_1(t) \frac{d}{dt} \int_{\mathbb{R}^n} (\partial_t u(t, x) + u(t, x)) \Psi(t, x) \, dx.
\] (2.18)

Choosing \( \psi \equiv \Psi \) in (1.6), we have
\[
\int_{\mathbb{R}^n} \partial_t u(t, x) \Psi(t, x) \, dx - \int_{\mathbb{R}^n} \varepsilon u_1(x) \Phi(x) \, dx - \int_0^t \int_{\mathbb{R}^n} \partial_t u(s, x) \Psi_s(s, x) \, ds \, dx
+ \int_0^t \int_{\mathbb{R}^n} \nabla u(s, x) \cdot \nabla \Psi(x) \, ds \, dx
+ \int_0^t \int_{\mathbb{R}^n} b_1(s) \partial_t u(s, x) \Psi(s, x) \, ds \, dx
= \int_0^t \int_{\mathbb{R}^n} |v(s, x)|^q \Psi(s, x) \, dx \, ds.
\]
Differentiating both sides of the previous equality with respect to \( t \), we arrive at
\[
\int_{\mathbb{R}^n} |v(t, x)|^q \Psi(t, x) \, dx = \frac{d}{dt} \int_{\mathbb{R}^n} \partial_t u(t, x) \Psi(t, x) \, dx
+ \int_{\mathbb{R}^n} ( - \partial_t u(t, x) \Psi_t(t, x) + \nabla u(t, x) \cdot \nabla \Psi(t, x) + b_1(t) \partial_t u(t, x) \Psi(t, x)) \, dx.
\] (2.19)

Using \( \Delta \Psi = \Psi \) and \( \Psi_t = -\Psi \), (2.19) yields
\[
\int_{\mathbb{R}^n} |v(t, x)|^q \Psi(t, x) \, dx = \frac{d}{dt} \int_{\mathbb{R}^n} (\partial_t u(t, x) + u(t, x)) \Psi(t, x) \, dx
+ b_1(t) \int_{\mathbb{R}^n} \partial_t u(t, x) \Psi(t, x) \, dx.
\] (2.20)

If we combine (2.18) and (2.20), we obtain
\[
\frac{d}{dt} \left( m_1(t) \int_{\mathbb{R}^n} (\partial_t u(t, x) + u(t, x)) \Psi(t, x) \, dx \right)
= b_1(t) m_1(t) U_1(t) + m_1(t) \int_{\mathbb{R}^n} |v(t, x)|^q \Psi(t, x) \, dx,
\] (2.21)
where \( U_1 \) is defined by (2.11). Thanks to (2.15) we have that \( U_1 \) is nonnegative. Then, integrating (2.21) over \([0, t]\), we get the estimate
\[
m_1(t) \int_{\mathbb{R}^n} (\partial_t u(t, x) + u(t, x)) \Psi(t, x) \, dx
\geq \varepsilon m_1(t) \int_{\mathbb{R}^n} (u_0(x) + u_1(x)) \Phi(x) \, dx + \int_0^t m_1(s) \int_{\mathbb{R}^n} |v(s, x)|^q \Psi(s, x) \, dx.
\] (2.22)
Furthermore, we may rewrite (2.19) as follows
\[\int_{\mathbb{R}^n} |v(t,x)|^q \Psi(t,x) \, dx = \frac{d}{dt} \int_{\mathbb{R}^n} \partial_t u(t,x) \Psi(t,x) \, dx \]
\[+ b_1(t) \int_{\mathbb{R}^n} \partial_t u(t,x) \Psi(t,x) \, dx + \int_{\mathbb{R}^n} (\partial_t u(t,x) - u(t,x)) \Psi(t,x) \, dx. \tag{2.23}\]
If we multiply both sides of (2.23) by \(m_1(t)\), we find
\[\frac{d}{dt} \left( m_1(t) \int_{\mathbb{R}^n} \partial_t u(t,x) \Psi(t,x) \, dx \right) + m_1(t) \int_{\mathbb{R}^n} (\partial_t u(t,x) - u(t,x)) \Psi(t,x) \, dx \]
\[= m_1(t) \int_{\mathbb{R}^n} |v(t,x)|^q \Psi(t,x) \, dx. \tag{2.24}\]
Adding (2.22) and (2.24), we find
\[\frac{d}{dt} \left( m_1(t) \int_{\mathbb{R}^n} \partial_t u(t,x) \Psi(t,x) \, dx \right) + 2m_1(t) \int_{\mathbb{R}^n} \partial_t u(t,x) \Psi(t,x) \, dx \]
\[
\geq \varepsilon m_1(0) \int_{\mathbb{R}^n} (u_0(x) + u_1(x)) \Phi(x) \, dx + m_1(t) \int_{\mathbb{R}^n} |v(t,x)|^q \Psi(t,x) \, dx \]
\[+ \int_0^t m_1(s) \int_{\mathbb{R}^n} |v(s,x)|^q \Psi(s,x) \, dx. \tag{2.25}\]
Let us set the auxiliary functional
\[U_3(t) = m_1(t) \int_{\mathbb{R}^n} \partial_t u(t,x) \Psi(t,x) \, dx - \frac{m_1(0)}{2} \int_{\mathbb{R}^n} u_1(x) \Phi(x) \, dx - \frac{1}{2} \int_0^t m_1(s) \int_{\mathbb{R}^n} |v(s,x)|^q \Psi(s,x) \, dx \, ds.\]
Clearly, \(U_3(0) = \varepsilon I_1[u_1]\). Besides, (2.25) implies
\[U_3'(t) + 2U_3(t) \geq \varepsilon m_1(0) \int_{\mathbb{R}^n} u_0(x) \Phi(x) \, dx + \frac{1}{2} m_1(t) \int_{\mathbb{R}^n} |v(t,x)|^q \Psi(t,x) \, dx \]
\[\geq 0. \tag{2.26}\]
Hence, multiplying (2.26) by \(e^{2t}\) and integrating over \([0,t]\), we get the estimate
\[U_3(t) \geq e^{-2t} U_3(0) \geq 0.\]
Therefore, as \(U_3\) is nonnegative we may write
\[m_1(t) \int_{\mathbb{R}^n} \partial_t u(t,x) \Psi(t,x) \, dx \]
\[\geq \frac{\varepsilon m_1(0)}{2} \int_{\mathbb{R}^n} u_1(x) \Phi(x) \, dx + \frac{1}{2} \int_0^t m_1(s) \int_{\mathbb{R}^n} |v(s,x)|^q \Psi(s,x) \, dx \, ds \]
\[\geq \frac{\varepsilon m_1(0)}{2} \int_{\mathbb{R}^n} u_1(x) \Phi(x) \, dx \]
which implies immediately (2.17) due to (2.2).

Using (2.15) and (2.17), we may finally derive the lower bounds for the integrals with respect to the spatial variables of the semilinear terms.
Proposition 2.4. Let \((u,v)\) be an energy solution of (1.1) on \([0,T]\) with non-negative, continuous and summable coefficients of the damping terms \(b_1, b_2\). Furthermore, we require the same assumptions on \(u_0, u_1, v_0, v_1\) as in Theorem 1.2. Then, the following estimates hold

\[ \int_{\mathbb{R}^n} |v(t,x)|^q \, dx \geq \tilde{C} \varepsilon^q (1 + t)^{n-1 - \frac{n-1}{2} q}, \]  
\[ \int_{\mathbb{R}^n} |\partial_t u(t,x)|^p \, dx \geq \tilde{K} \varepsilon^p (1 + t)^{n-1 - \frac{n-1}{2} p} \]

for any \(t \geq 0\), where the constants \(\tilde{C}, \tilde{K} > 0\) depend on \(n, p, q, b_1, b_2, R, u_1, v_0\).

Remark 2.5. Let us underline explicitly that the conditions \(u_1 \neq 0\) and \(v_0 \neq 0\) guarantee that the multiplicative constants in (2.16) and (2.17) are positive. This fact will play a fundamental role in the proof of Proposition 2.4.

Proof. Let us prove (2.27). By Hölder’s inequality and \(\supp v(t,\cdot) \subset B_{R+t}\), since \(V_1\) is a nonnegative function, it follows

\[ \int_{\mathbb{R}^n} |v(t,x)|^q \, dx \geq (V_1(t))^q \left( \int_{B_{R+t}} (\Psi(t,x))^q \, dx \right)^{-(q-1)} \geq \left( \varepsilon I_2[v_0] \right)^q (1 + t)^{n-1 - \frac{n-1}{2} q}, \]

where in the second inequality we used (2.16) and the following estimate (cf. [42, estimate (2.5)]):

\[ \int_{B_{R+t}} (\Psi(t,x))^q \, dx \lesssim (1 + t)^{n-1 - \frac{n-1}{2} q'}. \]

Using (2.17), we can prove (2.28) in a completely analogous way. \(\square\)

2.3. Iteration argument

In this section we combine the results from Sects. 2.1 to 2.2 by using an iteration procedure to get a sequence of lower bound estimates for the functionals \(V\) and \(U'\) (for the definition of \(U\) and \(V\) see (2.4) in Sect. 2.1).

More precisely, we want to prove that

\[ V(t) \geq C_j (1 + t)^{-b_j} t^{a_j} \quad \text{for any } t \geq 0, \]
\[ U'(t) \geq K_j (1 + t)^{-\beta_j} t^{\alpha_j} \quad \text{for any } t \geq 0, \]

where \(\{C_j\}_{j \in \mathbb{N}}, \{a_j\}_{j \in \mathbb{N}}, \{b_j\}_{j \in \mathbb{N}}, \{K_j\}_{j \in \mathbb{N}}, \{\alpha_j\}_{j \in \mathbb{N}}\) and \(\{\beta_j\}_{j \in \mathbb{N}}\) are suitable sequences of nonnegative numbers that we will determine afterwards.

Our strategy is to prove (2.29) and (2.30) by induction.

Let us begin with the base case \(j = 0\). Plugging (2.27) in (2.9), it results

\[ U'(t) \geq m_1(0) \tilde{C} \varepsilon^q \int_0^t (1 + s)^{n-1 - \frac{n-1}{2} q} \, ds \geq \frac{m_1(0) \tilde{C}}{n} \varepsilon^q (1 + t)^{-\frac{n-1}{2} q} t^n \]

which is (2.30) for \(j = 0\) provided that \(K_0 \equiv \frac{m_1(0) \tilde{C}}{n} \varepsilon^q, \alpha_0 = n, \beta_0 = \frac{n-1}{2} q\). We pointed out that we used that \(u_1\) is a nonnegative function to guarantee
\[ U'(0) \geq 0. \] Analogously, combining (2.28) and (2.7) and using \( V(0) \geq 0 \) (thanks to the nonnegativity of \( v_0 \)), we find
\[
V(t) \geq m_2(0)\tilde{K} \varepsilon^p \int_0^t \int_0^s (1 + \tau)^{n-1-\frac{n-1}{q}p} d\tau \, ds \\
\geq \frac{m_2(0)K}{n(n+1)} \varepsilon^p (1 + t)^{-\frac{n-1}{q}p} t^{n+1}.
\]

So, we proved also (2.29) for \( j = 0 \) provided that \( C_0 = \frac{m_2(0)K}{n(n+1)} \varepsilon^p \), \( a_0 = n + 1 \), \( b_0 = \frac{n-1}{q}p \).

Let us proceed now with the inductive step. If we plug (2.29) in (2.10), then, for any \( t \geq 0 \) we have
\[
U'(t) \geq KC_j^q \int_0^t (1 + s)^{-n(q-1) - bjqs_{ajq}} ds \\
\geq KC_j^q (1 + t)^{-n(q-1) - bjq} \int_0^t s_{ajq} ds \\
= KC_j^q (a_jq + 1)^{-1} (1 + t)^{-n(q-1) - bjqa_jq+1}.
\]

Thus, using the last lower bound in (2.8), we obtain for \( t \geq 0 \)
\[
V(t) \geq CK^pC_j^{pq}(a_jq + 1)^{-p} \int_0^t \int_0^s (1 + \tau)^{-n(pq-1) - bj pq \tau a_j p + p} d\tau \, ds \\
\geq CK^pC_j^{pq}(a_jq + 1)^{-p} (1 + t)^{-n(pq-1) - bj pq} \int_0^t \int_0^s a_j p + p \tau a_j p + p d\tau ds \\
= CK^pC_j^{pq}(a_jq + 1)p(a_jpq + p + 1)(a_jpq + p + 2)(1 + t)^{-n(pq-1) - bj pq a_jpq + p + 2}.
\]

Also, we proved (2.29) for \( j + 1 \) provided that
\[
C_{j+1} = CK^pC_j^{pq}(a_jq + 1)^{-p} (a_jpq + p + 1)^{-1}(a_jpq + p + 2)^{-1}, \\
a_{j+1} = pqa_j + p + 2, \quad b_{j+1} = b_j + n(pq - 1).
\]

Similarly, if we plug (2.30) in (2.8), then, for any \( t \geq 0 \) we get
\[
V(t) \geq CK_j^p \int_0^t \int_0^s (1 + \tau)^{-n(p-1) - \beta p \tau} d\tau ds \\
\geq CK_j^p (1 + t)^{-n(p-1) - \beta p} \int_0^t \int_0^s a_j p \tau d\tau ds \\
= CK_j^p (a_j p + 1)^{-1}(a_j p + 2)^{-1} (1 + t)^{-n(p-1) - \beta p a_j p + 2}.
\]

Consequently, a combination of the last lower bound with (2.10) yields
\[
U'(t) \geq KC_j^qK_j^{pq}(a_j p + 1)^{-q}(a_j p + 2)^{-q} \int_0^t (1 + s)^{-n(pq-1) - \beta pq s_{ajpq + 2}} ds \\
\geq \frac{KC_j^qK_j^{pq}}{(a_jp + 1)^{q}(a_jp + 2)^{q}(a_jpq + 2q + 1)}(1 + t)^{-n(pq-1) - \beta pq a_jpq + 2q + 1}.
\]
for any \( t \geq 0 \). Hence, we proved (2.30) for \( j + 1 \) provided that
\[
\alpha_{j+1} = pq\alpha_j + 2q + 1, \quad \beta_{j+1} = \beta_j + n(pq - 1)
\]
\[
K_{j+1} = KC^qK_j^{pq}(\alpha_j p + 1)^{-q}(\alpha_j p + 2)^{-q}(\alpha_j pq + 2q + 1)^{-1}.
\]

It is clear, from the recursive relations and from the nonnegative values of the initial constants \( C_0, K_0, a_0, b_0, \alpha_0, \beta_0 \), that \( C_j, K_j, a_j, b_j, \alpha_j, \beta_j \) are nonnegative real numbers for all \( j \in \mathbb{N} \).

Next we determine the explicit expressions for \( a_j, b_j, \alpha_j, \beta_j \) and lower bound estimates for \( C_j, K_j \). As \( a_j = pq\alpha_{j-1} + p + 2 \), employing iteratively this condition and the value \( a_0 = n + 1 \), we find
\[
a_j = pq\alpha_{j-1} + p + 2 = \cdots = a_0(pq)^j + (p + 2)\sum_{k=0}^{j-1}(pq)^k
\]
\[
= \left(n + 1 + \frac{p+2}{pq-1}\right)(pq)^j - \frac{p+2}{pq-1}.
\]

Analogously,
\[
\alpha_j = \alpha_0(pq)^j + (2q + 1)\sum_{k=0}^{j-1}(pq)^k = \left(n + \frac{2q+1}{pq-1}\right)(pq)^j - \frac{2q+1}{pq-1},
\]
\[
b_j = b_0(pq)^j + n(pq - 1)\sum_{k=0}^{j-1}(pq)^k = \left(n\frac{1}{2} - p + n\right)(pq)^j - n,
\]
\[
\beta_j = \beta_0(pq)^j + n(pq - 1)\sum_{k=0}^{j-1}(pq)^k = \left(n\frac{1}{2} - q + n\right)(pq)^j - n.
\]

In particular, using the representation formulas for \( a_j \) and \( \alpha_j \), we may derive lower bounds for \( C_j \) and \( K_j \). Indeed, due to
\[
\alpha_{j-1}pq + p + 2 \leq a_j \leq \left(n + 1 + \frac{p+2}{pq-1}\right)(pq)^j,
\]
\[
\alpha_{j-1}pq + 2q + 1 \leq \alpha_j \leq \left(n + \frac{2q+1}{pq-1}\right)(pq)^j,
\]
we have
\[
C_j = CK^pC_{j-1}^{pq}(\alpha_{j-1}p + 1)^{-p}(\alpha_{j-1}pq + p + 1)^{-1}(\alpha_{j-1}pq + 2q + 1)^{-1}
\]
\[
\geq CK^pC_{j-1}^{pq}(\alpha_{j-1}pq + p + 2)^{-p+2} \geq M(pq)^{-p+2j}C_{j-1}^{pq},
\]
(2.31)

and
\[
K_j = KC^qK_{j-1}^{pq}(\alpha_j p + 1)^{-q}(\alpha_j p + 2)^{-q}(\alpha_j pq + 2q + 1)^{-1}
\]
\[
\geq KC^qK_{j-1}^{pq}(\alpha_j pq + 2q + 1)^{-2q+1} \geq \tilde{M}(pq)^{-2q+1}K_{j-1}^{pq},
\]
(2.32)

where \( M \doteq CK^p\left(n + 1 + \frac{p+2}{pq-1}\right)^{-(p+2)} \) and \( \tilde{M} \doteq KC^q\left(n + \frac{2q+1}{pq-1}\right)^{-(2q+1)} \).
Applying the logarithmic function to both sides of (2.31) and using in an iterative way the resulting estimate, we arrive at

\[
\log C_j \geq pq \log C_{j-1} - j \log((pq)^{p+2}) + \log M \\
\geq (pq)^2 \log C_{j-2} - (j + (j - 1)pq) \log((pq)^{p+2}) + (1 + pq) \log M \\
\geq \cdots \geq (pq)^j \log C_0 - \sum_{k=0}^{j-1} (j - k)(pq)^k \log((pq)^{p+2}) + \sum_{k=0}^{j-1} (pq)^k \log M \\
= (pq)^j \left( \log C_0 - \frac{pq}{(pq-1)^2} \log((pq)^{p+2}) + \frac{\log M}{pq-1} \right) \\
+ (j + 1) \frac{\log((pq)^{p+2})}{pq-1} + \frac{\log((pq)^{p+2})}{(pq-1)^2} - \frac{\log M}{pq-1},
\]

(2.33)

where we used the formulas

\[
\sum_{k=0}^{j-1} (pq)^k = \frac{(pq)^j - 1}{pq - 1}, \quad \sum_{k=0}^{j-1} (j - k)(pq)^k = \frac{1}{pq - 1} \left( \frac{(pq)^{j+1} - 1}{pq - 1} - (j + 1) \right),
\]

(2.34)

that can be proved via an inductive argument.

Therefore, for \( j \geq j_1 \doteq \left[ \frac{\log M}{\log((pq)^{p+2})} - 1 - \frac{1}{pq-1} \right] \) by (2.33) we get

\[
\log C_j \geq (pq)^j \left( \log C_0 - \frac{pq}{(pq-1)^2} \log((pq)^{p+2}) + \frac{\log M}{pq-1} \right) \\
= (pq)^j \log(N \varepsilon^p),
\]

(2.35)

where \( N \doteq \frac{m_2(0)\tilde{K}}{n(n+1)} ((pq)^{p+2}) - \frac{pq}{(pq-1)^2} M \frac{1}{pq-1} \). Analogously, from (2.32) we derive the estimate

\[
\log K_j \geq (pq)^j \left( \log K_0 - \frac{pq}{(pq-1)^2} \log((pq)^{2q+1}) + \frac{\log M}{pq-1} \right) \\
= (pq)^j \log(\tilde{N} \varepsilon^q)
\]

(2.36)

for \( j \geq j_2 \doteq \left[ \frac{\log M}{\log((pq)^{2q+1})} - 1 - \frac{1}{pq-1} \right] \), where the multiplicative constant inside the logarithmic function is given by \( \tilde{N} \doteq \frac{m_1(0)\tilde{C}}{n} ((pq)^{2q+1}) - \frac{pq}{(pq-1)^2} \tilde{M} \frac{1}{pq-1} \).

In the next section we will combine (2.29), (2.35) and (2.36) to complete the proof of Theorem 1.2 in the case \( \Theta_1(n,p,q) > 0 \) and in the case \( \Theta_2(n,p,q) > 0 \), respectively.

2.4. Conclusion of the Proof of Theorem 1.2

Let us start with the case \( \Theta_1(n,p,q) > 0 \). Combining (2.29) and (2.35), we have for \( t \geq 0 \) and \( j \geq j_1 \)

\[
V(t) \geq \exp \left( (pq)^j \log(N \varepsilon^p) \right) (1 + t)^{-b_j} t^{a_j} \\
= \exp \left( (pq)^j \log \left( N \varepsilon^p (1 + t)^{-\left( \frac{n-1}{2}p+n \right) t^{n+1+\frac{p+2}{pq-1}} \right) \right) (1 + t)^{n t^{\frac{p+2}{pq-1}}}.
\]
As for \( t \geq 1 \) it holds \((1 + t) \leq 2t\), the previous estimate yields
\[
V(t) \geq \exp \left( (pq)^j \log \left( 2^{-\left(\frac{n-1}{2}p+n\right) N\varepsilon^p t^{\frac{pq+p+1}{pq-1} - \frac{n-1}{2}p} \right) \right) (1 + t)^n t^{-\frac{p+2}{pq-1}}
\]
\[
= \exp \left( (pq)^j \log \left( 2^{-\left(\frac{n-1}{2}p+n\right) N\varepsilon^p t^{\frac{n+p}{pq-1}} \right) \right) (1 + t)^n t^{-\frac{p+2}{pq-1}}
\]
\[
= \exp \left( (pq)^j \log \left( \varepsilon^p J(t) \right) \right) (1 + t)^n t^{-\frac{p+2}{pq-1}} \tag{2.37}
\]
for \( t \geq 1 \), where \( J(t) = 2^{-\left(\frac{n-1}{2}p+n\right) N t^p \Theta_1(n,p,q) \). Consequently, we may choose \( \varepsilon_0 \) sufficiently small such that
\[
2^{\left(\frac{n-1}{2} + \frac{n}{pq} \right) \Theta_1(n,p,q) - 1} N^{-\left(\frac{p \Theta_1(n,p,q)}{pq} - 1 \right) \varepsilon_0} - \Theta_1(n,p,q) \geq 1.
\]
So, for \( \varepsilon \in (0, \varepsilon_0] \) and for \( t \geq 2^{\left(\frac{n-1}{2} + \frac{n}{pq} \right) \Theta_1(n,p,q) - 1} N^{-\left(\frac{p \Theta_1(n,p,q)}{pq} - 1 \right) \varepsilon_0} - \Theta_1(n,p,q) \) it holds \( \varepsilon^p J(t) > 1 \). Consequently, letting \( j \to \infty \) in \( \tag{2.37} \), the lower bound of \( V(t) \) blows up and, then, \( V(t) \) cannot be finite. Also, we proved that \( V \) may be definite only for \( t \leq \varepsilon^{-\Theta_1(n,p,q)} \).

Now, we prove the result in the case \( \Theta_2(n,p,q) > 0 \). Combining \( \tag{2.30} \) and \( \tag{2.36} \), we have for \( t \geq 0 \) and \( j \geq j_2 \)
\[
U'(t) \geq \exp \left( (pq)^j \log(\tilde{N} \varepsilon^q) \right)(1 + t)^{-\beta_j t^{\alpha_j}}
\]
\[
= \exp \left( (pq)^j \log \left( \tilde{N} \varepsilon^q (1 + t)^{-\left(\frac{n-1}{2}q+n\right) t^{n+\frac{2q+1}{pq}}} \right) \right)(1 + t)^n t^{-\frac{2q+1}{pq-1}}.
\]
Then, for \( t \geq 1 \) it holds
\[
U'(t) \geq \exp \left( (pq)^j \log \left( 2^{-\left(\frac{n-1}{2}q+n\right) \tilde{N} \varepsilon^q t^{q \Theta_2(n,p,q)} \right) \right)(1 + t)^n t^{-\frac{2q+1}{pq-1}}
\]
\[
= \exp \left( (pq)^j \log \left( \varepsilon^q \tilde{J}(t) \right) \right)(1 + t)^n t^{-\frac{p+2}{pq-1}} \tag{2.38}
\]
for \( t \geq 1 \), where \( \tilde{J}(t) = 2^{-\left(\frac{n-1}{2}q+n\right) \tilde{N} t^q \Theta_2(n,p,q) \}. \) Hence, we can take \( \varepsilon_0 \) so small that
\[
2^{\left(\frac{n-1}{2} + \frac{n}{pq} \right) \Theta_2(n,p,q) - 1} \tilde{N}^{-\left(\frac{q \Theta_2(n,p,q)}{pq} - 1 \right) \varepsilon_0} - \Theta_2(n,p,q) \geq 1.
\]
Thus, for \( \varepsilon \in (0, \varepsilon_0] \) and for \( t \geq 2^{\left(\frac{n-1}{2} + \frac{n}{pq} \right) \Theta_2(n,p,q) - 1} \tilde{N}^{-\left(\frac{q \Theta_2(n,p,q)}{pq} - 1 \right) \varepsilon_0} - \Theta_2(n,p,q) \) it holds \( \varepsilon^q \tilde{J}(t) > 1 \). Also, as \( j \to \infty \) in \( \tag{2.38} \), the lower bound of \( U'(t) \) diverges and \( U'(t) \) is not finite. In this second case, we proved that \( U' \) can be finite only for \( t \leq \varepsilon^{-\Theta_2(n,p,q)} \). Combining the two possible cases, we proved the result and the upper bound estimate for the lifespan \( \tag{1.9} \).

3. Critical case

In the critical case, we restrict our considerations to the not-damped case. Therefore, we shall consider the weakly coupled system of semilinear wave equations \( \tag{1.11} \) in the critical case \( \max\{\Theta_1(n,p,q), \Theta_2(n,p,q)\} = 0 \). We will generalize the approach from \cite{36, 37} for a single semilinear equation and from \cite{27} for a weakly coupled system with power nonlinearities, in order to deal with the mixed type of nonlinear terms.

This section is organized as follows: first, in Sect. \ref{3.1} we recall some auxiliary functions from \cite{36} and we use them to introduce the functionals for
the critical case; in Sect. 3.2 we derive the iteration frame for these functionals, that is, a coupled system of nonlinear ordinary integral inequalities; in Sect. 3.3 lower bound estimates for the functionals, that allow to start with the iteration procedure, are derived; then, in Sect. 3.4 we combine the iteration frame from Sect. 3.2 and the lower bounds from Sect. 3.3 with a slicing method; hence, in Sect. 3.5 we use the sequences of lower bounds for the functionals from Sect. 3.4 to prove the blow-up result and to establish the upper bound for the lifespan; finally, in Sect. 3.6 we compare our results with those proved in Section 9 of [14] and we provide the analytic expression of the coordinates of the cusp point in the $p$–$q$ plane for the critical curve.

3.1. Introduction of the functionals for the critical case

Throughout the treatment of the critical case we will employ the auxiliary functions

$$
\eta_r(t, s, x) = \int_0^{\lambda_0} e^{-\lambda(R+t)} \frac{\sinh(\lambda(t-s))}{\lambda(t-s)} \Phi(\lambda x) \lambda^r \, d\lambda,
$$

$$
\xi_r(t, s, x) = \int_0^{\lambda_0} e^{-\lambda(R+t)} \cosh(\lambda(t-s)) \Phi(\lambda x) \lambda^r \, d\lambda,
$$

for $t > s \geq 0$ and $x \in \mathbb{R}^n$, where $r > -1$, $\lambda_0$ is a fixed positive constant and $\Phi$ is defined by (2.14). These auxiliary functions have been introduced in [36] as generalizations of the test function considered by Zhou (see [44, equation (3.2)]) in the treatment of the critical case for the semilinear wave equation with power nonlinearity in the higher dimensional case. Let us underline that the assumption on $r$ is done in order to guarantee the integrability of the function $\lambda^r$ in a neighborhood of $0$.

As functionals to study the blow-up dynamic we will consider

$$
\mathcal{U}(t) = \int_{\mathbb{R}^n} \partial_t u(t, x) \eta_{r_1}(t, t, x) \, dx, \quad (3.1)
$$

$$
\mathcal{V}(t) = \int_{\mathbb{R}^n} v(t, x) \eta_{r_2}(t, t, x) \, dx, \quad (3.2)
$$

where $\eta_r(t, t, x)$ denotes the continuous extension of the function $\eta_r(t, s, x)$ as $s \to t$.

We point out that the choice of the conditions for the pair $(r_1, r_2)$ will depend on the critical case we deal with. More specifically, we have to distinguish among the three possible subcases $\Theta_1(n, p, q) = 0 > \Theta_2(n, p, q)$, $\Theta_1(n, p, q) < 0 = \Theta_2(n, p, q)$ and $\Theta_1(n, p, q) = \Theta_2(n, p, q) = 0$.

First, we derive two fundamental identities for $\mathcal{U}$ and $\mathcal{V}$, which involve the initial data and the nonlinear terms.

**Proposition 3.1.** Let $(u, v)$ be a weak solution of (1.11) on $[0, T)$ and let $\mathcal{U}, \mathcal{V}$ denote the functionals defined by (3.1), (3.2). Then, the following identities
are satisfied for any $t \geq 0$:

$$
\mathcal{U}(t) = \varepsilon t \int_{\mathbb{R}^n} u_0(x) \eta_{r_1+2}(t, 0, x) \, dx + \varepsilon \int_{\mathbb{R}^n} u_1(x) \xi_{r_1}(t, 0, x) \, dx \\
+ \int_0^t \int_{\mathbb{R}^n} |v(s, x)|^q \xi_{r_1}(t, s, x) \, dx \, ds,
$$

(3.3)

$$
\mathcal{V}(t) = \varepsilon \int_{\mathbb{R}^n} v_0(x) \xi_{r_2}(t, 0, x) \, dx + \varepsilon t \int_{\mathbb{R}^n} v_1(x) \eta_{r_2}(t, 0, x) \, dx \\
+ \int_0^t (t-s) \int_{\mathbb{R}^n} |\partial_t u(s, x)|^p \eta_{r_2}(t, s, x) \, dx \, ds.
$$

(3.4)

**Proof.** In order to show the validity of (3.3) and (3.4) we will employ the definition of weak solution for (1.11) with a suitable choice of the test functions $(\phi, \psi)$ in (1.12) and (1.13). If we assume that $(u, v)$ satisfies (1.8), then, supp $u(t, \cdot)$, supp $v(t, \cdot) \subset B_{R+t}$ for any $t \geq 0$. Therefore, we may remove the assumption of compactness for the supports of the test functions in Definition 1.4. Hence, it is possible to consider

$$
\phi = \phi(t; t, s, x) = \cosh(\lambda(t-s)) \Phi(\lambda x),
$$

$$
\psi = \psi(t; t, s, x) = \frac{\sinh(\lambda(t-s))}{\lambda} \Phi(\lambda x).
$$

Since $\Delta \Phi(\lambda x) = \lambda^2 \Phi(\lambda x)$, then, $\phi, \psi$ are solutions of the homogeneous free wave equation. Moreover,

$$
\phi(t; t, x) = \Phi(\lambda x), \quad \psi(t; t, x) = 0, \quad \phi_s(t; t, x) = 0, \quad \psi_s(t; t, x) = -\Phi(\lambda x),
$$

and

$$
\phi(t; 0, x) = \cosh(\lambda t) \Phi(\lambda x), \quad \phi_s(t; 0, x) = -\lambda \sinh(\lambda t) \Phi(\lambda x),
$$

$$
\psi(t; 0, x) = \lambda^{-1} \sinh(\lambda t) \Phi(\lambda x), \quad \psi_s(t; 0, x) = -\cosh(\lambda t) \Phi(\lambda x).
$$

Consequently, from (1.12) and (1.13) we obtain

$$
\int_{\mathbb{R}^n} \partial_t u(t, x) \Phi(\lambda x) \, dx = \varepsilon \lambda \sinh(\lambda t) \int_{\mathbb{R}^n} u_0(x) \Phi(\lambda x) \, dx \\
+ \varepsilon \cosh(\lambda t) \int_{\mathbb{R}^n} u_1(x) \Phi(\lambda x) \, dx \\
+ \int_0^t \int_{\mathbb{R}^n} |v(s, x)|^q \cosh(\lambda(t-s)) \Phi(\lambda x) \, dx \, ds
$$

(3.5)

$$
\int_{\mathbb{R}^n} v(t, x) \Phi(\lambda x) \, dx = \varepsilon \cosh(\lambda t) \int_{\mathbb{R}^n} v_0(x) \Phi(\lambda x) \, dx \\
+ \varepsilon \lambda^{-1} \sinh(\lambda t) \int_{\mathbb{R}^n} v_1(x) \Phi(\lambda x) \, dx \\
+ \int_0^t \int_{\mathbb{R}^n} |\partial_t u(s, x)|^p \lambda^{-1} \sinh(\lambda(t-s)) \Phi(\lambda x) \, dx \, ds.
$$

(3.6)
Multiplying both sides of (3.5) by $e^{-\lambda(R+t)}\lambda^{r_1}$, integrating the resulting relation with respect to $\lambda$ over $[0,\lambda_0]$ and, finally, applying Fubini’s theorem, we get (3.3). Similarly, from (3.6) we find (3.4). This concludes the proof.  

The next step is to derive from (3.3) and (3.4) the iteration frame. In order to do so, we need to estimate sharply the auxiliary functions $\eta_r$ and $\xi_r$.

Lemma 3.2. Let $n \geq 2$. There exist $\lambda_0 > 0$ such that the following properties hold:

- (i) if $r > -1$, $|x| \leq R$ and $t \geq 0$, then,
  $$\xi_r(t,0,x) \geq A_0,$$
  $$\eta_r(t,0,x) \geq B_0(t)^{-1};$$

- (ii) if $r > -1$, $|x| \leq s + R$ and $t > s \geq 0$, then,
  $$\xi_r(t,s,x) \geq A_1\langle s\rangle^{-r-1},$$
  $$\eta_r(t,s,x) \geq B_1(t)^{-1}\langle s\rangle^{-r};$$

- (iii) if $r > \frac{n-3}{2}$, $|x| \leq t + R$ and $t > 0$, then,
  $$\eta_r(t,t,x) \leq B_2\langle t\rangle^{-\frac{n-1}{2}}(t - |x|)^{\frac{n-3}{2}-r}.$$ 

Here $A_0, A_1$ and $B_k$, $k = 0, 1, 2$, are positive constants depending only on $\lambda_0$, $r$ and $R$ and we denote $\langle y \rangle = 3 + |y|$.

Remark 3.3. Let us stress that differently from [36, Lemma 3.1] we require in the statement of (i) and (ii) the condition of $r > -1$ instead of $r > 0$. Nonetheless, the proofs from [36] of (i) and of the lower bound for $\eta_r(t,s,x)$ in (ii) are still valid even for $r > -1$.

Proof. We can restrict our considerations to the lower bound estimate for $\xi(t,s,x)$ in (ii), as the other properties are already proved in [36, Lemma 3.1]. Since $\langle s \rangle \geq 2$, we may shrink the domain of integration in the definition of $\xi_r(t,s,x)$ as follows

$$\xi_r(t,s,x) \geq \int_{\lambda_0/\langle s \rangle}^{2\lambda_0/\langle s \rangle} e^{-\lambda(R+t)} \cosh(\lambda(t-s)) \Phi(\lambda x) \lambda^r d\lambda.$$ 

We remark that the condition

$$\Phi(x) \asymp \langle x \rangle^{-\frac{n-1}{2}} e^{|x|} \text{ for any } x \in \mathbb{R}^n$$ 

implies that the infimum

$$\inf_{\lambda \in [\lambda_0, 2\lambda_0/\langle s \rangle]} \inf_{|x| \leq s + R} e^{-\lambda(s+R)} \Phi(\lambda x)$$
can be estimated from below by a constant $A = A(\lambda_0, R) > 0$ that does not depend on $\lambda, s$ and $x$. Therefore, we may estimate

$$
\xi_r(t, s, x) \geq \int_{\lambda_0 / (s)}^{2\lambda_0 / (s)} e^{-\lambda(t-s)} \cosh(\lambda(t - s)) e^{-\lambda(R+s)} \Phi(\lambda x) \lambda^r \, d\lambda \\
= \int_{\lambda_0 / (s)}^{2\lambda_0 / (s)} \frac{1}{2} \left(1 + e^{-2\lambda(t-s)}\right) e^{-\lambda(R+s)} \Phi(\lambda x) \lambda^r \, d\lambda \\
\geq A \int_{\lambda_0 / (s)}^{2\lambda_0 / (s)} \frac{1}{2} \left(1 + e^{-2\lambda(t-s)}\right) \lambda^r \, d\lambda \\
\geq \frac{A}{2} \int_{\lambda_0 / (s)}^{2\lambda_0 / (s)} \lambda^r \, d\lambda = \frac{A \lambda_0^{r+1}}{2 (r+1)} (2^{r+1} - 1) (s)^{-r-1},
$$

which is the desired lower bound estimate for $\xi_r(t, s, x)$.

\[ \square \]

3.2. Derivation of the iteration frame in the critical case

In order to derive the iteration scheme, we have to consider separately the three critical cases. In each case we will fix suitable conditions on the pair $(r_1, r_2)$, that will influence, on the one hand, the structure of the scheme itself with the possible presence of a logarithmic factor in the integral inequalities and, on the other hand, the functional $\mathcal{U}$ and/or $\mathcal{V}$ for which we can derive a lower bound containing a logarithmic factor.

3.2.1. Case $\Theta_1(n, p, q) = 0$. In this case we consider $r_1 = \frac{n-1}{2} - \frac{1}{p}$ and $r_2 \in \left(\frac{n-3}{2}, \frac{n-1}{2} - \frac{1}{q}\right)$. The purpose of this section is to derive the frame for the iteration argument, which is a coupled system of integral inequalities for the functionals $\mathcal{U}, \mathcal{V}$. In order to get this system we will combine the fundamental identities (3.3), (3.4) and the estimates for the auxiliary functions in Lemma 3.2. Combining (1.8), (3.1) and Hölder’s inequality, we find

$$
\mathcal{U}(s) \leq \left( \int_{\mathbb{R}^n} |\partial_t u(s, x)|^p \eta_{r_2}(t, s, x) \, dx \right)^{\frac{1}{p}} \left( \int_{B_{R+\delta}} \frac{\eta_{r_1}(s, s, x)^{p'}}{\eta_{r_2}(t, s, x)^{\frac{p'}{p}}} \, dx \right)^{\frac{1}{p'}}. \tag{3.7}
$$

Using Lemma 3.2 (ii)–(iii) and the condition $r_1 = \frac{n-1}{2} - \frac{1}{p}$, we may estimate

$$
\int_{B_{R+\delta}} \frac{\eta_{r_1}(s, s, x)^{p'}}{\eta_{r_2}(t, s, x)^{\frac{p'}{p}}} \, dx \leq \langle t \rangle^{\frac{p'}{p}} \langle s \rangle^{\frac{p'}{p} - \frac{n-1}{2}} R^{\frac{p'}{p} - \frac{n-1}{2}p'} \int_{B_{R+\delta}} \langle s - |x| \rangle^{\frac{n-3}{2} - r_1} \, dx \\
\leq \langle t \rangle^{\frac{p'}{p}} \langle s \rangle^{\frac{p'}{p} - \frac{n-1}{2}p' + n-1} \log(s).
$$

Therefore, we get

$$
\int_{\mathbb{R}^n} |\partial_t u(s, x)|^p \eta_{r_2}(t, s, x) \, dx \gtrsim \langle \mathcal{U}(s) \rangle^p \left( \int_{B_{R+\delta}} \frac{\eta_{r_1}(s, s, x)^{p'}}{\eta_{r_2}(t, s, x)^{\frac{p'}{p}}} \, dx \right)^{-\frac{1}{p'}} \\
\gtrsim \langle t \rangle^{-1} \langle s \rangle^{-r_2 + \frac{n-1}{2}p - (n-1)(p-1)(\log(s))^{-(p-1)}} \langle \mathcal{U}(s) \rangle^p.
$$
Consequently, from (3.4) we obtain
\[ \mathcal{V}(t) \gtrsim \langle t \rangle^{-1} \int_0^t (t-s) \langle s \rangle^{-r_2 + \frac{n-1}{2}p - (n-1)(p-1)} \langle \log \langle s \rangle \rangle^{-(p-1)} \langle \mathcal{U}(s) \rangle^p \, ds. \]  
(3.8)

Now we will derive an analogous integral lower bound for \( \mathcal{U} \). By (3.2) and Hölder’s inequality we have
\[ \mathcal{V}(s) \leq \left( \int_{\mathbb{R}^n} |v(s,x)|^q \xi_1(t,s,x) \, dx \right)^{1/q} \left( \int_{B_{R+s}} \frac{\eta_{r_2}(s,s,x)}{\xi_1(t,s,x)} q' \, dx \right)^{-\frac{1}{q'}}. \]  
(3.9)

Employing again Lemma 3.2 and the condition \( r_2 < \frac{n-1}{2} - \frac{1}{q} \), we arrive at
\[ \int_{B_{R+s}} \frac{\eta_{r_2}(s,s,x)}{\xi_1(t,s,x)} q' \, dx \lesssim \langle s \rangle^{-r_2 + \frac{n-1}{2}q'} \int_{B_{R+s}} \langle s - |x| \rangle^{-\frac{n-3}{2} - r_2} q' \, dx \]
\[ \lesssim \langle s \rangle^{-r_2 + \frac{n-1}{2}q'} + n \langle \frac{n-3}{2} - r_2 \rangle q', \]
which implies in turn
\[ \int_{\mathbb{R}^n} |v(s,x)|^q \xi_1(t,s,x) \, dx \, ds \gtrsim \langle \mathcal{V}(s) \rangle^q \left( \int_{B_{R+s}} \frac{\eta_{r_2}(s,s,x)}{\xi_1(t,s,x)} q' \, dx \right)^{-\frac{q'}{q}} \]
\[ \gtrsim \langle s \rangle^{-\frac{1}{2} + \frac{n}{2} - (n-1)q} q + r_2 q \langle \mathcal{V}(s) \rangle^q ds. \]  
(3.10)

### 3.2.2. Case \( \Theta_2(n,p,q) = 0 \)

For this critical case we assume \( r_2 = \frac{n-1}{2} - \frac{1}{q} \)
and \( r_1 \in (\frac{n-3}{2}, \frac{n-1}{2} - \frac{1}{p}) \). Due to the fact that we switch in some sense the role of \( r_1 \) and \( r_2 \) with respect to the previous critical case \( \Theta_1(n,p,q) = 0 \), somehow also the structure of the iteration frame is reversed with respect to the previous section.

By Lemma 3.2 (ii)-(iii) and the condition \( r_1 < \frac{n-1}{2} - \frac{1}{q} \) it follows
\[ \int_{B_{R+s}} \frac{\eta_{r_1}(s,s,x)}{\eta_{r_2}(t,s,x)} p' \, dx \lesssim \langle t \rangle^{\nu' p'} \langle s \rangle^{-r_2 + \frac{n-1}{2}p'} \int_{B_{R+s}} \langle s - |x| \rangle^{-\frac{n-3}{2} - r_1} p' \, dx \]
\[ \lesssim \langle t \rangle^{\nu' p'} \langle s \rangle^{-\frac{n-1}{2}p' + n + (\frac{n-3}{2} - r_1)p'}. \]

Then, from (3.7) we get
\[ \int_{\mathbb{R}^n} |\partial_t u(s,x)|^p \eta_{r_2}(t,s,x) \, dx \gtrsim \langle t \rangle^{-1} \langle s \rangle^{-r_2 + \frac{n-1}{2}p - (n-1)(p-1)} \langle \mathcal{U}(s) \rangle^p \]
\[ \gtrsim \langle t \rangle^{-1} \langle s \rangle^{-r_2 + \frac{n}{p} + n + r_1 p} \langle \mathcal{U}(s) \rangle^p. \]
Also, (3.4) yields

\[
\mathcal{Y}(t) \gtrsim \langle t \rangle^{-1} \int_0^t (t - s) \langle s \rangle^{-r_2 - (n-1)p + n + r_1 p} (\mathcal{U}(s))^p \, ds.
\]  

(3.11)

We determine now the integral lower bound for \( \mathcal{U} \). By using Lemma 3.2 and the condition \( r_2 = \frac{n-1}{2} - \frac{1}{q} \), we arrive at

\[
\int_{B_{R+s}} \frac{n r_2(s,s,x)^{q'}}{\zeta_1(t,s,x) r_1^{q'}} \, dx \lesssim \langle s \rangle^{(r_1 + 1)q' - \frac{n-1}{2}q'} \int_{B_{R+s}} \langle s - |x| \rangle^{(\frac{n-3}{2} - r_2)q'} \, dx \
\lesssim \langle s \rangle^{(r_1 + 1)q' - \frac{n-1}{2}q' + n - 1 \log(s)}.
\]

The last estimate together with (3.9) provides

\[
\int_{\mathbb{R}^n} |v(s,x)|^q \xi_1(t,s,x) \, dx \gtrsim (\mathcal{Y}(s))^q \left( \int_{B_{R+s}} \frac{n r_2(s,s,x)^{q'}}{\zeta_1(t,s,x) r_1^{q'}} \, dx \right)^{-\frac{q}{q'}} \
\gtrsim \langle s \rangle^{-(r_1 + 1) + \frac{n-1}{2}q - (n-1)(q - 1)(\log(s))^{-(q - 1)} (\mathcal{Y}(s))^q}.
\]

Thus, (3.3) and the last estimate imply

\[
\mathcal{U}(t) \gtrsim \int_0^t \langle s \rangle^{-(r_1 + 1) + \frac{n-1}{2}q - (n-1)(q - 1)(\log(s))^{-(q - 1)} (\mathcal{Y}(s))^q \, ds.
\]  

(3.12)

3.2.3. Case \( \Theta_1(n,p,q) = \Theta_2(n,p,q) = 0 \). In this case we choose \( r_1 = \frac{n-1}{2} - \frac{1}{p} \) and \( r_2 = \frac{n-1}{2} - \frac{1}{q} \). In particular, one can prove the identities

\[
n - \frac{n-1}{2} - \frac{1}{p} = n - 1 - \frac{n-1}{2} q, \quad (3.13)
\]

\[
n - \frac{n-1}{2} - \frac{1}{q} = n - \frac{n-1}{2} p, \quad (3.14)
\]

due to the fact that the pair \( (p,q) \) satisfies both the critical conditions \( \Theta_1(n,p,q) = \Theta_2(n,p,q) = 0 \). Indeed, if we denote \( \kappa_1 = n - 1 - \frac{n-1}{2} q - \frac{n-1}{2} + \frac{1}{p} \) and \( \kappa_2 = n - \frac{n-1}{2} p - \frac{n-1}{2} + \frac{1}{q} \), then

\[
\kappa_1 + q \kappa_2 = (pq - 1) \Theta_1(n,p,q) = 0,
\]

\[
p \kappa_1 + \kappa_2 = (pq - 1) \Theta_2(n,p,q) = 0.
\]

As \( pq \neq 1 \), then, trivially \( \kappa_1 = \kappa_2 = 0 \), but this means exactly the validity of (3.13)-(3.14).

Since \( r_1 = \frac{n-1}{2} - \frac{1}{p} \) as in Sect. 3.2.1, we can prove (3.8). However, thanks to (3.14) we see that the power of \( \langle s \rangle \) in the right hand side of (3.8) is exactly \(-1 \), that is,

\[
\mathcal{Y}(t) \gtrsim \langle t \rangle^{-1} \int_0^t (t - s) \langle s \rangle^{-1} (\log(s))^{-p - 1} (\mathcal{U}(s))^p \, ds.
\]  

(3.15)

Similarly, since \( r_2 = \frac{n-1}{2} - \frac{1}{q} \) as in Sect. 3.2.2 it holds (3.12). Yet, due to (3.13) we find again that the power of \( \langle s \rangle \) in the right hand side of (3.12) is exactly \(-1 \), that is,

\[
\mathcal{U}(t) \gtrsim \int_0^t \langle s \rangle^{-1} (\log(s))^{-q - 1} (\mathcal{Y}(s))^q \, ds.
\]  

(3.16)
3.3. Lower bound estimates for the functionals containing a logarithmic factor

Purpose of this section is to derive lower bounds for $\mathcal{U}$ and/or $\mathcal{V}$ of logarithmic type. As in the previous section, we shall consider separately the three critical cases. We point out that the assumptions on the pair $(r_1, r_2)$ are the same as in Sect. 3.2 and they depend on the critical case that we consider.

3.3.1. Case $\Theta_1(n, p, q) = 0$. In this case we will derive a lower bound for the functional $\mathcal{U}$ in two step. From (3.4), Lemma 3.2 (ii) and Proposition 2.4, we get for $t \geq 0$

$$\mathcal{U}(t) \geq \int_0^t (t - s) \int_{\mathbb{R}^n} |\partial_t u(s, x)|^p \eta_{r_2}(t, s, x) \, dx \, ds$$

$$\geq (t)^{-1} \int_0^t (t - s)(s)^{-r_2} \int_{\mathbb{R}^n} |\partial_t u(s, x)|^p \, dx \, ds$$

$$\geq \varepsilon^p (t)^{-1} \int_0^t (t - s)(s)^{-r_2 + n - \frac{n-1}{2}p} \, ds. \quad (3.17)$$

Consequently, for $t \geq 1$

$$\mathcal{U}(t) \geq \varepsilon^p (t)^{-1 - r_2 - \frac{n-1}{2}p} \int_0^t (t - s)(s)^{n-1} \, ds$$

$$\geq \varepsilon^p (t)^{-1 - r_2 - \frac{n-1}{2}p} \int_0^t \frac{1}{2} (t - s)(s)^{n-1} \, ds$$

$$\geq \varepsilon^p (t)^{-1 - r_2 - \frac{n-1}{2}p} \int_0^t \frac{1}{2} (t - s)(s)^{n-1} \, ds \geq \varepsilon^p (t)^{-r_2 - \frac{n-1}{2}p+n}. \quad (3.18)$$

Plugging the last lower bound for $\mathcal{U}$ in (3.10), we have for $t \geq 1$

$$\mathcal{U}(t) \geq \varepsilon^{pq} \int_0^t \frac{1}{2} (s)^{-r_1 + n - 1 + q + r_2 q + (-r_2 - \frac{n-1}{2}p+n)q} \, ds$$

$$\geq \varepsilon^{pq} \int_0^t \frac{1}{2} (s)^{-r_1 + n - 1 + q - \frac{n-1}{2}pq} \, ds \geq \varepsilon^{pq} \int_0^t \frac{1}{2} (s)^{q + p - \frac{n-1}{2}(p-q)} \, ds$$

$$\geq \varepsilon^{pq} \int_0^t \frac{1}{2} (s)^{q-1} \, ds \geq \varepsilon^{pq} \int_0^t \frac{1}{2} (s)^{-1} \, ds \geq \varepsilon^{pq} \log t, \quad (3.19)$$

where we used in the third inequality the actual value of $r_1$ and in the fourth one the critical condition $\Theta_1(n, p, q) = 0$.

3.3.2. Case $\Theta_2(n, p, q) = 0$. Let us determine a lower bound for $\mathcal{V}$ in two step. From (3.3), Lemma 3.2 (ii) and Proposition 2.4 we obtain for $t \geq 0$

$$\mathcal{V}(t) \geq \int_0^t \int_{\mathbb{R}^n} |v(s, x)|^q \xi_{r_1}(t, s, x) \, dx \, ds \geq \int_0^t \int_{\mathbb{R}^n} |v(s, x)|^q \, dx \, ds$$

$$\geq \varepsilon^q \int_0^t \frac{1}{2} (s)^{-(r_1+1) + n - 1 - \frac{n-1}{2}q} \, ds. \quad (3.20)$$
Also, for $t \geq 0$
\[
\mathcal{W}(t) \gtrsim \varepsilon^q \int_0^t \langle s \rangle^{-(r_1+1)+n-\frac{n-1}{q}} ds \gtrsim \varepsilon^q \langle t \rangle^{-(r_1+1)-\frac{n-1}{q}} \int_0^t \langle s \rangle^{n-1} ds \gtrsim \varepsilon^q \langle t \rangle^{-(r_1+1)-\frac{n-1}{q}}. 
\]

Plugging the last lower bound for $\mathcal{W}$ in (3.11), we have for $t \geq \frac{3}{2}$
\[
\mathcal{V}(t) \gtrsim \varepsilon^{pq} \langle t \rangle^{-1} \int_0^t (t-s)\langle s \rangle^{-r_2-(n-1)p+n+r_1 p+(-r_1+1)-\frac{n-1}{q}} ds 
= \varepsilon^{pq} \langle t \rangle^{-1} \int_0^t (t-s)\langle s \rangle^{1+q-\frac{n-1}{q}(pq-1)} ds 
= \varepsilon^{pq} \langle t \rangle^{-1} \int_0^t (t-s)\langle s \rangle^{-1} ds \gtrsim \varepsilon^{pq} \langle t \rangle^{-1} \int_1^1 \frac{t-s}{s} ds 
\gtrsim \varepsilon^{pq} \langle t \rangle^{-1} \int_1^1 \log s ds \gtrsim \varepsilon^{pq} \langle t \rangle^{-1} \int_1^1 \log s ds \gtrsim \varepsilon^{pq} \log \left(\frac{2t}{3}\right). 
\] (3.21)

where we employed in the third step the actual value of $r_2$ and in the fourth one the critical condition $\Theta_2(n, p, q) = 0$.

3.3.3. Case $\Theta_1(n, p, q) = \Theta_2(n, p, q) = 0$. In this last case we can improve both (3.19) and (3.21) thanks to (3.13), (3.14). Indeed, combining (3.20) and (3.13), for any $t \geq 0$ we obtain
\[
\mathcal{W}(t) \gtrsim \varepsilon^q \int_0^t \langle s \rangle^{-1} ds \gtrsim \varepsilon^q \log t. 
\] (3.22)

Analogously, using (3.18) and (3.14), for any $t \geq \frac{3}{2}$ we have
\[
\mathcal{V}(t) \gtrsim \varepsilon^{p} \langle t \rangle^{-1} \int_0^t (t-s)\langle s \rangle^{-1} ds \gtrsim \varepsilon^{p} \log \left(\frac{2t}{3}\right). 
\] (3.23)

3.4. Iterated lower bound estimates: slicing method
In this section we derive iteratively a sequence of lower bound estimates for $\mathcal{W}$ or $\mathcal{V}$. Then, in Sect. 3.5 we will employ these iterated lower bounds to prove the blow-up and to derive the upper bound for the lifespan of the local solution $(u, v)$.

However, before starting with this iterative procedure, we summarize the estimates that we proved in Sects. 3.2 and 3.3.
In Sect. 3.2 we proved for any \( t \geq 0 \) the coupled system of integral inequalities
\[
\mathcal{U}(t) \geq C \int_0^t (s)^{-r_1+n-1-(n-1)q+r_2q} \mathcal{V}(s)^q \, ds \tag{3.24}
\]
\[
\mathcal{V}(t) \geq K(t)^{-1} \int_0^t (t-s)(s)^{-r_2+n+(n-1)(p-1)(\log(s))-(p-1)} \mathcal{U}(s)^p \, ds \tag{3.25}
\]
if \( \Theta_1(n,p,q) = 0 \),
\[
\mathcal{U}(t) \geq C \int_0^t (s)^{-(r_1+1)+\frac{n-1}{2}q-(n-1)(q-1)}(\log(s))^{-q-1} \mathcal{V}(s)^q \, ds \tag{3.26}
\]
\[
\mathcal{V}(t) \geq K(t)^{-1} \int_0^t (t-s)(s)^{-r_2-(n-1)p+n+r_1p} \mathcal{U}(s)^p \, ds \tag{3.27}
\]
if \( \Theta_2(n,p,q) = 0 \) and
\[
\mathcal{U}(t) \geq C \int_0^t (s)^{-1}(\log(s))^{-q-1} \mathcal{V}(s)^q \, ds \tag{3.28}
\]
\[
\mathcal{V}(t) \geq K(t)^{-1} \int_0^t (t-s)(s)^{-1}(\log(s))^{-(p-1)} \mathcal{U}(s)^p \, ds \tag{3.29}
\]
if \( \Theta_1(n,p,q) = \Theta_2(n,p,q) = 0 \), where \( C, K \) are positive constants depending on \( n, p, q, R \). Let us underline that the range for the pair \((r_1, r_1)\) is implicitly fixed by the corresponding critical case according to Sect. 3.2.

On the other hand, the lower bound estimates (3.19), (3.21), (3.22) and (3.23) from Sect. 3.3 can be summarized as follows
\[
\begin{cases}
\mathcal{U}(t) \geq \tilde{C} e^{pq} \log t & \text{if } \Theta_1(n,p,q) = 0, \\
\mathcal{U}(t) \geq \tilde{C} e^q \log t & \text{if } \Theta_1(n,p,q) = \Theta_2(n,p,q) = 0,
\end{cases}
\]
for any \( t \geq 1 \) and
\[
\begin{cases}
\mathcal{V}(t) \geq \tilde{K} e^{pq} \log \left( \frac{2t}{3} \right) & \text{if } \Theta_2(n,p,q) = 0, \\
\mathcal{V}(t) \geq \tilde{K} e^p \log \left( \frac{2t}{3} \right) & \text{if } \Theta_1(n,p,q) = \Theta_2(n,p,q) = 0,
\end{cases}
\]
for any \( t \geq \frac{3}{2} \), where \( \tilde{C}, \tilde{K} \) are positive constants depending on \( n, p, q, R, u_0, u_1, v_0, v_1 \).

Now we can start with the iteration argument. As in the previous sections, we consider separately the three critical cases.

**3.4.1. Case \( \Theta_1(n,p,q) = 0 \).** Let us introduce the sequence of positive real numbers \( \{\ell_j\}_{j \in \mathbb{N}} \), where \( \ell_j = 2 - 2^{-j} \), that will be used to split the time interval in the slicing method. In this case the goal is to prove that 
\[
\mathcal{U}(t) \geq C_j \left( \log(t) \right)^{-b_j} \left( \log \left( \frac{t}{\ell_j} \right) \right)^{a_j} \quad \text{for } t \geq \ell_j, \tag{3.32}
\]
for any \( j \in \mathbb{N} \), where \( \{C_j\}_{j \in \mathbb{N}}, \{a_j\}_{j \in \mathbb{N}} \) and \( \{b_j\}_{j \in \mathbb{N}} \) are sequences of nonnegative real numbers that we shall determine throughout the iteration argument. Thanks to (3.30) we see that (3.32) is satisfied for \( j = 0 \), provided that the
initial values of the sequences are given by $a_0 = 1, b_0 = 0$ and $C_0 = \tilde{C}_{\varepsilon}^{pq}$. Hence, we employ an inductive argument to prove the validity of (3.32) for any $j \in \mathbb{N}$. We proceed now with the inductive step. Let us plug (3.32) in (3.25), after shrinking the domain of integration, we obtain for $s \geq \ell_{j+1}
abla$(s) ≥ K\langle s \rangle^{-(r_2+1)-\frac{n-1}{2}} p (\log(s))^{-(p-1)} \int_{\ell_j}^{s} (s-\tau)\langle \tau \rangle^{n} (\mathcal{H}(\tau))^{p} d\tau \
abla KC_j^p \langle s \rangle^{-(r_2-2)-\frac{n-1}{2}} p (\log(s))^{-(p-1)-b_j} p \int_{\ell_j}^{s} (s-\tau)\langle \tau \rangle^{n} (\log(\frac{\tau}{\ell_j}))^{a_j p} d\tau.
(3.33)

Let us estimate the last integral. For $s \geq \ell_{j+1}$ if we shrink further the domain of integration we get
\[
\int_{\ell_j}^{s} (s-\tau)\langle \tau \rangle^{n} (\log(\frac{\tau}{\ell_j}))^{a_j p} d\tau \geq \int_{\ell_j s}^{s} (s-\tau)\langle \tau \rangle^{n} (\log(\frac{\tau}{\ell_j}))^{a_j p} d\tau \\
\geq \left(\frac{\ell_j s}{\ell_{j+1}}\right)^n (\log(\frac{s}{\ell_{j+1}}))^{a_j p} \int_{\ell_j s}^{s} (s-\tau) d\tau \\
\geq \frac{1}{2} \left(\frac{\ell_j s}{\ell_{j+1}}\right)^n (1 - \frac{\ell_j}{\ell_{j+1}})^2 (\log(\frac{s}{\ell_{j+1}}))^{a_j p} s^{n+2}.
\]

Using the inequalities $2\ell_j \geq \ell_{j+1}$, $1 - \frac{\ell_j}{\ell_{j+1}} \geq 2^{-(j+2)}$ and $4s \geq \langle s \rangle$ for any $s \geq 1$, from the last estimate we arrive at
\[
\int_{\ell_j}^{s} (s-\tau)\langle \tau \rangle^{n} (\log(\frac{\tau}{\ell_j}))^{a_j p} d\tau \geq 2^{-2j-3(n+3)} (\log(\frac{s}{\ell_{j+1}}))^{a_j p} \langle s \rangle^{n+2}.
\]
(3.34)

Plugging (3.34) in (3.33), we find the following estimate for $\nabla(s)$ for $s \geq \ell_{j+1}$
\[
\nabla(s) \geq 2^{-2j-3(n+3)} KC_j^p \langle s \rangle^{-(r_2-2)-\frac{n-1}{2}} p n (\log(s))^{-(p-1)-b_j} p (\log(\frac{s}{\ell_{j+1}}))^{a_j p}
\]

Using the above lower bound for $\nabla(s)$ in (3.24), for $t \geq \ell_{j+1}$ we get
\[
\mathcal{H}(t) \geq C \int_{\ell_{j+1}}^{t} \langle s \rangle^{-(r_1+n-1-(n-1)q+r_2 q)} (\nabla(s))^q d s \\
\geq 2^{-2qj-3q(n+3)} CK^q C_j^{pq} (\log(t))^{-q(p-1)-b_j} p q \\
\times \int_{\ell_{j+1}}^{t} \langle s \rangle^{-(r_1+n-1+q-\frac{n-1}{2}} p q (\log(\frac{s}{\ell_{j+1}}))^{a_j pq} d s \\
\geq 2^{-2qj-3q(n+3)-2} KC_j^q C_j^{pq} (\log(t))^{-q(p-1)-b_j} p q \int_{\ell_{j+1}}^{t} \frac{1}{s} (\log(\frac{s}{\ell_{j+1}}))^{a_j pq} d s \\
\geq 2^{-2qj-3q(n+3)-2} \frac{KC_j^q C_j^{pq}}{(a_j pq + 1)} (\log(t))^{-q(p-1)-b_j} p q (\log(\frac{t}{\ell_{j+1}}))^{a_j pq+1},
\]
where in the second last step we used that the critical relation \( \Theta_1(n, p, q) = 0 \)
is equivalent to
\[
-r_1 + n - 1 + q - \frac{n-1}{2}pq = -\frac{n-1}{2} + \frac{1}{p} + n - 1 + q - \frac{n-1}{2}pq \\
= q + \frac{1}{p} - \frac{n-1}{2}(pq - 1) = -1.
\]
So, we proved (3.32) for \( j + 1 \), if we define
\[
C_{j+1} \doteq 2^{-2aqj-3q(n+3)-2}CK^qC_j^{pq}(a_jpq + 1)^{-1},
\]
\[
a_{j+1} \doteq a_jpq + 1, \quad b_{j+1} \doteq q(p-1) + b_jpq.
\]
In order to derive the upper bound estimate for the life span of the solution, it is convenient to derive an estimate from below of \( C_j \), where the dependence on \( j \) in the lower bound is more explicit than the one in the definition of \( C_j \) itself.

But first, let us derive the explicit expression for \( a_j \) and \( b_j \). Using iteratively the recursive relations between two successive elements that we just proved, we find
\[
a_j = a_{j-1}pq + 1 = \cdots = a_0(pq)^j + \sum_{k=0}^{j-1}(pq)^k = (pq)^j + \frac{(pq)^{j+1}-1}{pq-1},
\]
\[
b_j = b_{j-1}pq + q(p-1) = \cdots = b_0(pq)^j + q(p-1)\sum_{k=0}^{j-1}(pq)^k
\]
\[= \frac{q(p-1)}{pq-1}((pq)^j - 1).\] (3.35)
Therefore,
\[
a_{j-1}pq + 1 \leq \frac{pq}{pq-1}(pq)^j.
\]
In particular, the previous inequality implies
\[
C_j \geq MN^{-j}C_{j-1}^{pq}, \quad (3.36)
\]
where \( M \doteq 2^{-q(3n+7)-2}CK^q(pq-1)^{-pq} \) and \( N \doteq 2^{2q}pq \). Applying the logarithmic function to both sides of (3.36) and using iteratively the resulting inequality, we get
\[
\log C_j \geq (pq)\log C_{j-1} - j \log N + \log M
\]
\[
\geq (pq)^2\log C_{j-2} - \left(j + (j - 1)(pq)\right)\log N + \left(1 + pq\right)\log M
\]
\[
\geq \cdots \geq (pq)^j\log C_0 - \sum_{k=0}^{j-1}(j - k)(pq)^k\log N + \sum_{k=0}^{j-1}(pq)^k\log M
\]
\[
= (pq)^j\log C_0 - (pq)^j\sum_{k=1}^{j}k(pq)^{-k}\log N + \frac{(pq)^{j+1}}{pq-1}\log M
\]
\[
= (pq)^j\left(\log C_0 - S_j\log N + \frac{\log M}{pq-1}\right) - \frac{\log M}{pq-1},
\]
where $S_j = \sum_{k=1}^{j} k(pq)^{-k}$. As $\{S_j\}_{j \geq 1}$ is a sequence of the partial sums of a convergent series, if we denote by $S$ the limit of this sequence, because of $S_j \uparrow S$ as $j \to \infty$, then, we may estimate

$$C_j \geq M^{-(pq-1)} \exp \left( (pq)^j \log \left( C_0 N^{-S} M^{pq-1} \right) \right). \quad (3.37)$$

### 3.4.2. Case $\Theta_2(n, p, q) = 0$.

In this second critical case we shall prove that

$$\mathcal{V}(t) \geq K_j \left( \log(t) \right)^{-\beta_j} \left( \log \left( \frac{t}{\ell_{2j+1}} \right) \right)^{\alpha_j} \quad \text{for} \quad t \geq \ell_{2j+1} \quad (3.38)$$

for any $j \in \mathbb{N}$, where $\{K_j\}_{j \in \mathbb{N}}$, $\{\alpha_j\}_{j \in \mathbb{N}}$ and $\{\beta_j\}_{j \in \mathbb{N}}$ are sequences of nonnegative real numbers that we will be fixed during the iterative procedure. Due to (3.31) we see that (3.38) is satisfied for $j = 0$, supposed that the initial values of the sequences are given by $\alpha_0 = 1$, $\beta_0 = 0$ and $K_0 = \tilde{K}_{pq}$. Also in this case it remains to prove the inductive step in order to show the validity of (3.38) for any $j \in \mathbb{N}$. For this purpose we plug in (3.38) in (3.26), so that, after a restriction of the domain of integration, for $s \geq \ell_{2j+1}$ we have

$$\mathcal{V}(s) \geq C \langle s \rangle^{-(r_1+1)-\frac{n-1}{2}q} (\log(s))^{-(q-1)} \int_{\ell_{2j+1}}^{s} \langle \tau \rangle^{n-1} \left( \frac{\tau}{\ell_{2j+1}} \right)^{\alpha_j q} d\tau$$

$$\geq C K_j^q \langle s \rangle^{-(r_1+1)-\frac{n-1}{2}q} (\log(s))^{-(q-1)-\beta_j q} \int_{\ell_{2j+1}}^{s} \langle \tau \rangle^{n-1} \left( \frac{\tau}{\ell_{2j+1}} \right)^{\alpha_j q} d\tau. \quad (3.39)$$

For $s \geq \ell_{2j+1}$ we may estimate the last integral as follows:

$$\int_{\ell_{2j+1}}^{s} \langle \tau \rangle^{n-1} \left( \frac{\tau}{\ell_{2j+1}} \right)^{\alpha_j q} d\tau \geq \int_{\ell_{2j+1}}^{s} \tau^{n-1} \left( \frac{\tau}{\ell_{2j+1}} \right)^{\alpha_j q} d\tau$$

$$\geq \left( \frac{\ell_{2j+1}}{\ell_{2j+2}} \right)^{n-1} \left( 1 - \frac{\ell_{2j+1}}{\ell_{2j+2}} \right) s^n \left( \log \left( \frac{s}{\ell_{2j+2}} \right) \right)^{\alpha_j q}$$

$$\geq 2^{-2j-(3n+2)} \langle s \rangle^n \left( \log \left( \frac{s}{\ell_{2j+2}} \right) \right)^{\alpha_j q}. \quad (3.40)$$

Combining (3.39) and (3.40), we obtain

$$\mathcal{V}(s) \geq 2^{-2j-(3n+2)} C K_j^q \langle s \rangle^{-(r_1+1)-\frac{n+1}{2}q+n} (\log(s))^{-(q-1)-\beta_j q} \left( \log \left( \frac{s}{\ell_{2j+2}} \right) \right)^{\alpha_j q}$$
for $s \geq \ell_{2j+2}$. Combining this lower bound for $\mathcal{U}(s)$ and (3.27), for $t \geq \ell_{2j+3}$ we arrive at

$$\mathcal{E}(t) \geq K(t)^{-1} \int_{\ell_{2j+2}}^{t} (t - s)(s)^{-r_2 - (n-1)p + n + r_1p} (\mathcal{U}(s))^p \, ds$$

$$\geq 2^{-2pj-p(3n+2)} K_{j}^p K_{j}^{pq} (\log(s))^{-(q-1)-\beta_j q} \langle t \rangle^{-1}$$

$$\times \int_{\ell_{2j+2}}^{t} (t - s)(s)^{-r_2 + n - \frac{n-1}{2}pq} \left( \log \left( \frac{s}{\ell_{2j+2}} \right) \right)^{\alpha_j pq} \, ds$$

$$\geq 2^{-2pj-p(3n+2)-2} K_{j}^p K_{j}^{pq} (\log(s))^{-(q-1)-\beta_j q} \langle t \rangle^{-1}$$

$$\times \int_{\ell_{2j+2}}^{t} \frac{t-s}{s} \left( \log \left( \frac{s}{\ell_{2j+2}} \right) \right)^{\alpha_j pq} \, ds. \quad (3.41)$$

Let us estimate from below the integral on the right-hand side of (3.41). Performing a step of integration by parts and then shrinking the domain of integration, we have

$$\int_{\ell_{2j+2}}^{t} \frac{t-s}{s} \left( \log \left( \frac{s}{\ell_{2j+2}} \right) \right)^{\alpha_j pq} \, ds = (\alpha_j pq + 1)^{-1} \int_{\ell_{2j+2}}^{t} \left( \log \left( \frac{s}{\ell_{2j+2}} \right) \right)^{\alpha_j pq + 1} \, ds$$

$$\geq (\alpha_j pq + 1)^{-1} \int_{\ell_{2j+2}}^{t} \left( \log \left( \frac{s}{\ell_{2j+2}} \right) \right)^{\alpha_j pq + 1} \, ds$$

$$\geq (\alpha_j pq + 1)^{-1} \left( 1 - \frac{\ell_{2j+2}}{\ell_{2j+3}} \right) t \left( \log \left( \frac{t}{\ell_{2j+3}} \right) \right)^{\alpha_j pq + 1}$$

$$\geq 2^{-2j-6} (\alpha_j pq + 1)^{-1} \langle t \rangle \left( \log \left( \frac{t}{\ell_{2j+3}} \right) \right)^{\alpha_j pq + 1}. \quad (3.42)$$

If we combine (3.41) and (3.42), then, we conclude

$$\mathcal{E}(t) \geq 2^{-2(p+1)j-p(3n+2)-8} \frac{K_{j}^p K_{j}^{pq} (\log(s))^{-(q-1)-\beta_j q} \left( \log \left( \frac{t}{\ell_{2j+3}} \right) \right)^{\alpha_j pq + 1}}{(\alpha_j pq + 1)^{-1}}$$

that is (3.38) for $j + 1$, provided that

$$K_{j+1} = 2^{-2(p+1)j-(3n+2)p-8} K_{j}^p K_{j}^{pq} (\alpha_j pq + 1)^{-1},$$

$$\alpha_{j+1} = \alpha_j pq + 1, \quad \beta_{j+1} = \beta_j q + p(q-1).$$

Let us point out that in the previous chain of inequalities we used the relation

$$-r_2 + n - \frac{n-1}{2}pq = -\frac{n-1}{2} + \frac{1}{q} + n - \frac{n-1}{2}pq$$

$$= 1 + \frac{1}{q} - \frac{n-1}{2}(pq - 1) = -1$$

which follows from the critical relation $\Theta_2(n, p, q) = 0$. Analogously to what we did in the first critical case, we derive now a lower bound for $K_j$. Let us find first the expression of $\alpha_j$ and $\beta_j$. Applying iteratively the definitions of
\( \alpha_j \) and \( \beta_j \), we end up with the representation formulas
\[
\alpha_j = \alpha_{j-1}pq + 1 = \cdots = \alpha_0(pq)^j + \frac{(pq)^{j-1}}{pq-1} = \frac{(pq)^{j+1}-1}{pq-1},
\]
\[
\beta_j = \beta_{j-1}pq + p(q-1) = \cdots = \beta_0(pq)^j + \frac{p(q-1)}{pq-1}((pq)^j - 1)
\]
(3.43)

In particular, it holds the inequality \( \alpha_{j-1}pq + 1 \leq \frac{pq}{pq-1}(pq)^j \), that implies in turn
\[
K_j \geq M_1 N_1^{-j} K_j^{pq},
\]
(3.44)

where \( M_1 = 2^{-3np-6}KCP(pq-1)^2pq \) and \( N_1 = 2^{2(p+1)pq} \). Analogously to the derivation of (3.37) via (3.36), from (3.44) we obtain
\[
K_j \geq M_1^{-(pq-1)} \exp\left((pq)^j \log\left(K_0 N_1^{-S M_1^{pq-1}}\right)\right).
\]
(3.45)

### 3.4.3. Case \( \Theta_1(n, p, q) = \Theta_2(n, p, q) = 0 \)

In this third critical case the goal is to prove that
\[
\mathcal{U}(t) \geq D_j \left( \log(t) \right)^{-h_j} \left( \log \left( \frac{t}{\ell_j} \right) \right)^{g_j} \quad \text{for} \quad t \geq \ell_j
\]
(3.46)

for any \( j \in \mathbb{N} \), where \( \{D_j\}_{j \in \mathbb{N}}, \{g_j\}_{j \in \mathbb{N}} \) and \( \{h_j\}_{j \in \mathbb{N}} \) are sequences of nonnegative real numbers that we shall determine throughout the iteration argument.

Due to the iteration scheme for \( \Theta_1(n, p, q) = \Theta_2(n, p, q) = 0 \) in (3.28) and (3.29), the inductive step will have somehow a more symmetric scheme than the ones in the previous cases. We point out that (3.30) implies the validity of (3.46) in the base case \( j = 0 \) if we consider
\[
D_0 = \tilde{C} \varepsilon, \quad g_0 = 1, \quad h_0 = 0.
\]

Let us proceed with the inductive step. If we plug (3.46) in (3.29), then for \( s \geq \ell_{j+1} \) we get
\[
\mathcal{U}(s) \geq KD_j^p(s)^{-1} \int_{\ell_j}^s (s - \tau) (\tau)^{-1} \left( \log(\tau) \right)^{-(p-1)-h_jp} \left( \log \left( \frac{\tau}{\ell_j} \right) \right)^{g_jp} d\tau
\]
\[
\geq 2^{-2} KD_j^p \left( \log(s) \right)^{-(p-1)-h_jp} \left( \log \left( \frac{s}{\ell_j} \right) \right)^{g_jp} d\tau
\]
\[
= 2^{-2} KD_j^p (g_jp + 1)^{-1} \left( \log(s) \right)^{-(p-1)-h_jp} \left( s \right)^{-1} \int_{\ell_j}^s \left( \log \left( \frac{s}{\ell_j} \right) \right)^{g_jp+1} d\tau
\]
\[
\geq 2^{-2} KD_j^p (g_jp + 1)^{-1} \left( \log(s) \right)^{-(p-1)-h_jp} \left( \ell_j \right)^{-1} \int_{\ell_j}^{s \ell_j} \left( \log \left( \frac{s}{\ell_j} \right) \right)^{g_jp+1} d\tau
\]
\[
\geq 2^{-4} KD_j^p (g_jp + 1)^{-1} \left( 1 - \frac{\ell_j}{\ell_{j+1}} \right) \left( \log(s) \right)^{-(p-1)-h_jp} \left( \log \left( \frac{s}{\ell_{j+1}} \right) \right)^{g_jp+1} d\tau
\]
\[
\geq 2^{-(j+6)} KD_j^p (g_jp + 1)^{-1} \left( \log(s) \right)^{-(p-1)-h_jp} \left( \log \left( \frac{s}{\ell_{j+1}} \right) \right)^{g_jp+1}.
\]
A combination of the previous lower bound for $\mathcal{V}(s)$ and (3.28) yields for $t \geq \ell_{j+1}$

$$\mathcal{V}(t) \geq 2^{-(j+6)q} \frac{C}{(g_jp + 1)q} \int_{\ell_{j+1}}^{t} (s)^{-1}(\log(s))^{-(pq-1) - h_jpq} \left( \log \left( \frac{s}{\ell_{j+1}} \right) \right)^{g_jpq+q} ds \geq 2^{-(j+6)q-2} \frac{C}{(g_jp + 1)q} (\log(t))^{-(pq-1) - h_jpq} \int_{\ell_{j+1}}^{t} \frac{1}{s} \left( \log \left( \frac{s}{\ell_{j+1}} \right) \right)^{g_jpq+q} ds = 2^{-(j+6)q-2} \frac{C}{(g_jp + 1)q} (\log(t))^{-(pq-1) - h_jpq} \left( \log \left( \frac{t}{\ell_{j+1}} \right) \right)^{g_jpq+q+1}.$$ 

The last inequality is (3.46) for $j + 1$ provided that

$$D_{j+1} = 2^{-(j+6)q-2} C K^q D_j^{pq} (g_jp + 1)^{-q} (g_jpq + q + 1)^{-1},$$

$$g_{j+1} = g_jpq + q + 1, \quad h_{j+1} = h_jpq + pq - 1.$$ 

Finally, we find a lower bound for the coefficient $D_j$. First, we have

$$g_j = g_{j-1}pq + q + 1 = \cdots = g_0(pq)^j + (q + 1) \sum_{k=0}^{j-1} (pq)^k$$

$$= \left( 1 + \frac{q+1}{pq-1} \right) (pq)^j - \frac{q+1}{pq-1}, \quad (3.47)$$

$$h_j = h_{j-1}pq + pq - 1 = \cdots = h_0(pq)^j + (pq - 1) \sum_{k=0}^{j-1} (pq)^k = (pq)^j - 1.$$ 

Hence,

$$g_{j-1}p + 1 \leq g_{j-1}pq + q + 1 \leq \frac{q(p+1)}{pq-1} (pq)^j$$

implies

$$D_j \geq M_2 N_2^{-j} D_{j-1}^{pq}, \quad (3.48)$$

where $M_2 \doteq 2^{-5q-2} C K^q \left( \frac{pq-1}{q(p+1)} \right)^{q+1} \frac{1}{(pq)^q}$ and $N_2 \doteq 2^q (pq)^{q+1}$. In an analogous way as in the derivation of (3.37) through (3.36), by (3.48) we have

$$D_j \geq M_2^{-(pq-1)} \exp \left( (pq)^j \log \left( D_0 N_2^{-S} M_2^{pq-1} \right) \right). \quad (3.49)$$

### 3.5. Upper bound for the lifespan of local solutions

In this section we finally prove that a local solution $(u, v)$ of (1.11) blows up in finite time under the assumption of Theorem 1.5. As in the previous sections we will consider separately the three critical cases.
3.5.1. Case $\Theta_1(n, p, q) = 0$. If we combine (3.32), (3.35) and (3.37), then, for $t \geq 2 \geq \ell_j$ we have

\[
\mathcal{U}(t) \geq M^{-(pq-1)} \exp \left( (pq)^j \log \left( C_0 N^{-S} M^{pq-1} \right) \left( \log(t) \right)^{4j} \left( \log \left( \frac{1}{2j} \right) \right)^{a_j} \right) \\
\geq M^{-(pq-1)} \exp \left( (pq)^j \log \left( C_0 N^{-S} M^{pq-1} \right) \left( \log(t) \right)^{4j} \left( \log \left( \frac{1}{2j} \right) \right)^{a_j} \right) \\
\geq M^{-(pq-1)} \left( \frac{\log(t)^{q(p-1)}}{\log \left( \frac{t}{2j} \right)^{1/pq-1}} \right) \\
\times \exp \left( (pq)^j \log \left( (C_0 N^{-S} M^{pq-1}) - \frac{q(p-1)}{pq-1} \log(t) + \frac{pq}{pq-1} \log \left( \frac{1}{2j} \right) \right) \right).
\]

Since for $t \geq 4$ the inequalities $\log(t) \leq \log(2t) \leq 2 \log t$ and $\log \left( \frac{t}{2j} \right) \geq \frac{1}{2} \log t$ hold, then for $t \geq 4$ the last estimate from below for $\mathcal{U}(t)$ implies

\[
\mathcal{U}(t) \geq M^{-(pq-1)} \left( \frac{\log(t)^{q(p-1)}}{\log \left( \frac{t}{2j} \right)^{1/pq-1}} \right) \exp \left( (pq)^j \log \left( E \varepsilon^{pq} \left( \log(t) \frac{q}{pq-1} \right) \right) \right),
\]

(3.50)

where $E = 2^{-\frac{q(p-1)}{pq-1}} C N^{-S} M^{pq-1}$.

Let us point out that $H(t, \varepsilon) = E \varepsilon^{pq} \left( \log(t) \frac{q}{pq-1} \right) > 1$ if and only if $t > \exp \left( E \frac{pq-1}{q} \varepsilon^{-p(pq-1)} \right)$. Consequently, we can fix a sufficiently small $\varepsilon_0$ such that

\[
\exp \left( E \frac{pq-1}{q} \varepsilon_0^{-p(pq-1)} \right) \geq 4.
\]

Then, for any $\varepsilon \in (0, \varepsilon_0]$ and $t \geq \exp \left( E \frac{pq-1}{q} \varepsilon^{-p(pq-1)} \right) \geq 4$ it holds $H(t, \varepsilon) > 1$; so, letting $j \to \infty$ in (3.50), we find that the lower bound for $\mathcal{U}(t)$ blows up. Thus, we proved that $\mathcal{U}(t)$ can be finite only for

\[
t \leq \exp \left( E \frac{pq-1}{q} \varepsilon^{-p(pq-1)} \right),
\]

(3.51)

which is the upper bound estimate for the lifespan in (1.15) in the critical case $\Theta_1(n, p, q) = 0$.

3.5.2. Case $\Theta_2(n, p, q) = 0$. Combining (3.38), (3.43) and (3.45), then, for $t \geq 2 \geq \ell_{2j+1}$ we have

\[
\mathcal{V}(t) \geq M_1^{-(pq-1)} \exp \left( (pq)^j \log \left( K_0 N_1^{-S} M_1^{pq-1} \right) \right) \left( \log(t) \right)^{4j} \left( \log \left( \frac{t}{2j+1} \right) \right)^{a_j} \\
\geq M_1^{-(pq-1)} \exp \left( (pq)^j \log \left( K_0 N_1^{-S} M_1^{pq-1} \right) \right) \left( \log(t) \right)^{4j} \left( \log \left( \frac{1}{2} \right) \right)^{a_j} \\
\geq M_1^{-(pq-1)} \left( \frac{\log(t)^{q(p-1)}}{\log \left( \frac{t}{2j+1} \right)^{1/pq-1}} \right) \\
\times \exp \left( (pq)^j \log \left( (K_0 N_1^{-S} M_1^{pq-1}) - \frac{q(p-1)}{pq-1} \log(t) + \frac{pq}{pq-1} \log \left( \frac{1}{2} \right) \right) \right).
\]
Analagously as in the last section, for \( t \geq 4 \) this estimate from below for \( \mathcal{V}(t) \) provides
\[
\mathcal{V}(t) \geq M_{1}^{-\rho(q-1)} \left( \frac{(\log(t))^{\rho(q-1)}}{\log \left( \frac{t}{\tau_{0}} \right)} \right)^{1/p_{q-1}} \exp \left( (pq)^{j} \log \left( E_{1}^{\rho(q) \log(t) \frac{p_{q-1}}{p_{q}}} \right) \right),
\]
(3.52)
where \( E_{1} = 2^{-\frac{\mu(2q-1)}{p_{q}}} \tilde{N}_{1}^{2-S} M_{1}^{pq-1} \). If we denote \( H_{1}(t, \varepsilon) = E_{1}^{\rho(q) \log(t) \frac{p_{q-1}}{p_{q}}} \), then, \( H_{1}(t, \varepsilon) > 1 \) if and only if \( t > \exp \left( E_{1}^{-\frac{\mu(q-1)}{p_{q}}} \varepsilon^{-q(pq-1)} \right) \). Therefore, we can choose a sufficiently small \( \varepsilon_{0} \) such that \( \exp \left( E_{1}^{-\frac{\mu(q-1)}{p_{q}}} \varepsilon_{0}^{-q(pq-1)} \right) \geq 4 \). Also, for any \( \varepsilon \in (0, \varepsilon_{0}] \) and \( t > \exp \left( E_{1}^{-\frac{\mu(q-1)}{p_{q}}} \varepsilon^{-q(pq-1)} \right) \) we have \( H_{1}(t, \varepsilon) > 1 \); thus, taking the limit as \( j \to \infty \) in (3.52), we find that the lower bound for \( \mathcal{V}(t) \) blows up. Hence, we showed that \( \mathcal{V}(t) \) may be finite just for
\[
t \leq \exp \left( E_{1}^{-\frac{\mu(q-1)}{p_{q}}} \varepsilon^{-q(pq-1)} \right),
\]
(3.53)
which is exactly the upper bound estimate for the lifespan in (1.15) for \( \Theta_{2}(n, p, q) = 0 \).

3.5.3. Case \( \Theta_{1}(n, p, q) = \Theta_{2}(n, p, q) = 0 \). For \( t \geq 2 \geq \ell_{2j+1} \) the combination of (3.46), (3.47) and (3.49) leads to
\[
\mathcal{U}(t) \geq M_{2}^{-\rho(q-1)} \exp \left( (pq)^{j} \log \left( D_{0} N_{2}^{-S} M_{2}^{pq-1} \right) \right) \left( \log(t) \right)^{\frac{1}{p_{q}}} \left( \log \left( \frac{t}{\tau_{0}} \right) \right)^{g_{j}}
\geq M_{2}^{-\rho(q-1)} \exp \left( (pq)^{j} \log \left( D_{0} N_{2}^{-S} M_{2}^{pq-1} \right) \right) \left( \log(t) \right)^{\frac{1}{p_{q}}} \left( \log \left( \frac{t}{\tau_{0}} \right) \right)^{g_{j}}
\geq M_{2}^{-\rho(q-1)} \log(t) \left( \log \left( \frac{t}{\tau_{0}} \right) \right)^{-\frac{p_{q-1}}{p_{q}}} \times \exp \left( (pq)^{j} \log \left( D_{0} N_{2}^{-S} M_{2}^{pq-1} \right) \right) \log(t) + \left( 1 + \frac{q+1}{p_{q}} \right) \log \left( \frac{t}{\tau_{0}} \right) \right) \right)
\]
Similarly to the last sections, for \( t \geq 4 \) the above estimate from below for \( \mathcal{U}(t) \) yields
\[
\mathcal{U}(t) \geq M_{2}^{-\rho(q-1)} \log(t) \left( \log \left( \frac{t}{\tau_{0}} \right) \right)^{-\frac{p_{q-1}}{p_{q}}} \exp \left( (pq)^{j} \log \left( E_{2}^{\rho(q) \log(t) \frac{p_{q-1}}{p_{q}}} \right) \right),
\]
(3.54)
where \( E_{2} = 2^{-2\frac{\mu}{p_{q}}} \tilde{C} N_{2}^{-S} M_{2}^{pq-1} \). Let us denote \( H_{2}(t, \varepsilon) = E_{2}^{\rho(q) \log(t) \frac{p_{q-1}}{p_{q}}} \). Then, \( H_{2}(t, \varepsilon) > 1 \) if and only if \( t > \exp \left( E_{2}^{-\frac{\mu(q-1)}{p_{q}}} \varepsilon^{-q(pq-1)} \right) \). Therefore, as before we can choose a sufficiently small \( \varepsilon_{0} \) such that
\[
\exp \left( E_{2}^{-\frac{\mu(q-1)}{p_{q}}} \varepsilon_{0}^{-q(pq-1)} \right) \geq 4.
\]
Also, for any \( \varepsilon \in (0, \varepsilon_{0}] \) and \( t > \exp \left( E_{2}^{-\frac{\mu(q-1)}{p_{q}}} \varepsilon^{-q(pq-1)} \right) \) it holds \( H_{2}(t, \varepsilon) > 1 \); thus, taking the limit as \( j \to \infty \) in (3.54), we see that the lower bound for \( \mathcal{U}(t) \) diverges. So, we proved that if \( \mathcal{U}(t) \) is finite, then,
\[
t \leq \exp \left( E_{2}^{-\frac{\mu(q-1)}{p_{q}}} \varepsilon^{-q(pq-1)} \right),
\]
(3.55)
that is, we proved (1.15) in the critical case \( \Theta_{1}(n, p, q) = \Theta_{2}(n, p, q) = 0 \).
3.6. Final remarks on the critical case

3.6.1. Comparison with other results. As we have already mentioned in the introduction, Ikeda-Sobajima-Wakasa very recently proved a blow-up result for the semilinear weakly coupled system (1.11) both in the subcritical case and in the critical case, by using a revised test function method. While in the subcritical case we obtained exactly the same result (but including damping terms in the scattering case), in the critical case we got quite different estimates for the lifespan in all three subcases. Let us compare our results with theirs.

In the first critical case $\Theta_1(n, p, q) = 0$ we proved the estimate (3.51), while in [14] the upper bound estimate

$$T(\varepsilon) \leq \exp \left( C\varepsilon^{-q(pq-1)} \right)$$

is proved. Let us point out that in the critical case $\Theta_1(n, p, q) = 0 > \Theta_2(n, p, q)$ it is not possible to determine, in general, which exponent among $p$ and $q$ is the biggest one. So far, the best estimate for the lifespan that we can get is the one obtained combining (3.51) and (3.56), that is,

$$T(\varepsilon) \leq \exp \left( C\varepsilon^{-\min\{p(pq-1), q(pq-1)\}} \right) \quad \text{if} \quad \Theta_1(n, p, q) = 0.$$

On the contrary, in the case $\Theta_2(n, p, q) = 0$ we obtained (3.53), which is an improvement of the estimate $T(\varepsilon) \leq \exp \left( C\varepsilon^{-p(pq-1)} \right)$ proved in [14] in the same critical case. Indeed, in this case we have

$$\frac{q+1+p^{-1}}{pq-1} - \frac{n-1}{2} < 0 = \frac{2+q^{-1}}{pq-1} - \frac{n-1}{2}$$

which provides $q - q^{-1} < 1 - p^{-1} < p - p^{-1}$, that implies in turn $q < p$.

We consider now the case $\Theta_1(n, p, q) = \Theta_2(n, p, q) = 0$. We point out explicitly that in this critical case we could employ an iteration argument for the functional $V$ as well in the last section. Nevertheless, we would find as upper bound for the lifespan

$$T(\varepsilon) \leq \exp \left( C\varepsilon^{-\frac{p}{pq+1}(pq-1)} \right)$$

which is weaker than the one that we derived by working with $W$, namely, (3.55). Indeed, the comparison of the two critical conditions $\Theta_1(n, p, q) = 0$ and $\Theta_2(n, p, q) = 0$ leads to $q - q^{-1} = 1 - p^{-1} < p - p^{-1}$, which implies as above $q < p$. Moreover, we emphasize that we have improved the estimate for this case in comparison to the one in [14] for the corresponding case, namely, $T(\varepsilon) \leq \exp \left( C\varepsilon^{-(pq-1)} \right)$.

Finally, we mention that in [3] the authors propose an improvement for the critical line for (1.11) in the $p-q$ plane in the three dimensional and radially symmetric case on the base of a global existence result for small data solutions that they proved (cf. [3, Theorem 1.3]). Nonetheless, in the results of the above quoted paper for the 3d case a gap is present between the global existence result and the blow-up result.
3.6.2. The intersection point of the critical curves. Finally, we remark that in the critical case \( \Theta_1(n, p, q) = \Theta_2(n, p, q) = 0 \), we can determine the expression of \( p \) and \( q \), that is, we determine the coordinates of the intersection point of the critical curves in the \( p - q \) plane. By straightforward calculations, we get that \( \Theta_1(n, p, q) = \Theta_2(n, p, q) \) implies
\[
p = (1 + q^{-1} - q)^{-1}.
\] (3.57)
We underline that we should require \( 1 < q < \frac{1 + \sqrt{5}}{2} \), in order to get an admissible \( p \). If we plug in (3.57) in \( \Theta_2(n, p, q) = 0 \), we find that \( q \) satisfies the cubic equation
\[
0 = (n + 1)q^3 - \frac{n + 1}{2}q^2 - \frac{n + 5}{2}q - 1 = (2q + 1)\left(\frac{n + 1}{2}q^2 - \frac{n + 1}{2}q - 1\right).
\] (3.58)
Therefore, the only admissible solution of (3.58) is
\[
q_{\text{mix}}(n) = \frac{1}{2} \left(1 + \sqrt{\frac{n + 9}{n + 1}}\right).
\]
It is easy to check that \( q_{\text{mix}}(n) < \frac{1 + \sqrt{5}}{2} \) for any \( n \geq 2 \). Plugging this expression for \( q_{\text{mix}}(n) \) in (3.57), we get
\[
p_{\text{mix}}(n) = \frac{q_{\text{mix}}(n)}{1 + q_{\text{mix}}(n) - (q_{\text{mix}}(n))^2} = \frac{n + 1 + \sqrt{(n + 9)(n + 1)}}{2(n - 1)}.
\]
It is interesting to compare these exponents, \( p_{\text{mix}}(n) \) and \( q_{\text{mix}}(n) \), with the critical exponent for the semilinear wave equation with power nonlinearity, i.e., the Strauss exponent
\[
p_{\text{Str}}(n) = \frac{n + 1 + \sqrt{n^2 + 10n - 7}}{2(n - 1)}
\]
and with the exponent for the semilinear wave equation of derivative type, i.e., the Glassey exponent
\[
p_{\text{Gla}}(n) = \frac{n + 1}{n - 1}.
\]
Elementary computations show that
\[
q_{\text{mix}}(n) < p_{\text{Gla}}(n) < p_{\text{Str}}(n) < p_{\text{mix}}(n)
\]
for any \( n \geq 2 \). Therefore, we may conclude that for the cusp point of the critical curve for (1.11) the power of the nonlinear term \( |\partial_t u|^p \) is bigger than the critical power for the semilinear wave equation of derivative type, while the power of the nonlinear term \( |v|^q \) is smaller than the critical power for the semilinear wave equation with power nonlinearity. In this sense, we have a balance between \( p \) and \( q \) for the cusp point of the critical curve for the weakly coupled system of semilinear wave equations with mixed nonlinear terms, in comparison to the cases with power nonlinearities and of derivative type.
Acknowledgements

The first author is member of the Gruppo Nazionale per L’Analisi Matematica, la Probabilità e le loro Applicazioni (GNAMPA) of the Instituto Nazionale di Alta Matematica (INdAM). This paper was written partially during the stay of the first author at Tohoku University within the period October to December 2018. He thanks the Mathematical Institute of Tohoku University for the worm hospitality and the excellent working conditions during this period. The first author is supported by the University of Pisa, Project PRA 2018 49. The second author is partially supported by the Grant-in-Aid for Scientific Research (B)(No.18H01132).

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References

[1] Agemi, R.: Blow-up of solutions to nonlinear wave equations in two space dimensions. Manuscr. Math. 73, 153–162 (1991)

[2] Agemi, R., Kurokawa, Y., Takamura, H.: Critical curve for $p$–$q$ systems of nonlinear wave equations in three space dimensions. J. Differ. Equ. 167(1), 87–133 (2000)

[3] Dai, W., Fang, D., Wang, C.: Global existence and lifespan for semilinear wave equations with mixed nonlinear terms. J. Differ. Equ. 267(5), 3328–3354 (2019)

[4] Del Santo, D.: Global existence and blow-up for a hyperbolic system in three space dimensions. Rend. Istit. Mat. Univ. Trieste 29(1/2), 115–140 (1997)

[5] Del Santo, D., Mitidieri, E.: Blow-up of solutions of a hyperbolic system: the critical case. Differ. Equ. 34(9), 1157–1163 (1998)

[6] Del Santo, D., Georgiev, V., Mitidieri, E.: Global existence of the solutions and formation of singularities for a class of hyperbolic systems, In: Colombini F., Lerner N. (eds) Geometrical Optics and Related Topics. Progress in Nonlinear Differential Equations and Their Applications, vol 32. Birkhäuser, Boston, MA (1997). https://doi.org/10.1007/978-1-4612-2014-5_7

[7] Deng, K.: Blow-up of solutions of some nonlinear hyperbolic systems. Rocky Mt. J. Math. 29, 807–820 (1999)

[8] Georgiev, V., Takamura, H., Zhou, Y.: The lifespan of solutions to nonlinear systems of a high-dimensional wave equation. Nonlinear Anal. 64(10), 2215–2250 (2006)

[9] Hidano, K., Tsutaya, K.: Global existence and asymptotic behavior of solutions for nonlinear wave equations. Indiana Univ. Math. J. 44, 1273–1305 (1995)

[10] Hidano, K., Wang, C., Yokoyama, K.: The Glassey conjecture with radially symmetric data. J. Math. Pures Appl. 98(9), 518–541 (2012)
[11] Hidano, K., Yokoyama, K.: Life span of small solutions to a system of wave equations. Nonlinear Anal. 139, 106–130 (2016)

[12] Ikeda, M., Sobajima, M.: Life-span of solutions to semilinear wave equation with time-dependent critical damping for specially localized initial data. Math. Ann. 372, 1017–1040 (2018)

[13] Ikeda, M., Sobajima, M.: Sharp upper bound for lifespan of solutions to some critical semilinear parabolic, dispersive and hyperbolic equations via a test function method. Nonlinear Anal. 182, 57–74 (2019)

[14] Ikeda, M., Sobajima, M., Wakasa, K.: Blow-up phenomena of semilinear wave equations and their weakly coupled systems. J. Differ. Equ. 267(9), 5165–5201 (2019)

[15] Imai, T., Kato, M., Takamura, H., Wakasa, K.: The sharp lower bound of the lifespan of solutions to semilinear wave equations with low powers in two space dimensions. In: Kato, K., Ogawa, T., Ozawa, T. (eds.), Asymptotic Analysis for Nonlinear Dispersive and Wave Equations, Advanced Studies in Pure Mathematics, vol. 81, pp, 31–53 (2019)

[16] John, F.: Blow-up of solutions of nonlinear wave equations in three space dimensions. Manuscripta Math. 28(1–3), 235–268 (1979)

[17] John, F.: Blow-up of solutions for quasi-linear wave equations in three space dimensions. Commun. Pure Appl. Math. 34, 29–51 (1981)

[18] Kubo, H., Kubota, K., Sunagawa, H.: Large time behavior of solutions to semilinear systems of wave equations. Math. Ann. 335, 435–478 (2006)

[19] Kurokawa, Y.: The lifespan of radially symmetric solutions to nonlinear systems of odd dimensional wave equations. Tsukuba J. Math. 60(7), 1239–1275 (2005)

[20] Kurokawa, Y., Takamura, H.: A weighted pointwise estimate for two dimensional wave equations and its applications to nonlinear systems. Tsukuba J. Math. 27(2), 417–448 (2003)

[21] Kurokawa, Y., Takamura, H., Wakasa, K.: The blow-up and lifespan of solutions to systems of semilinear wave equation with critical exponents in high dimensions. Differ. Integral Equ. 25(3/4), 363–382 (2012)

[22] Lai, N.A., Takamura, H.: Blow-up for semilinear damped wave equations with subcritical exponent in the scattering case. Nonlinear Anal. 168, 222–237 (2018)

[23] Lai, N.A., Takamura, H.: Nonexistence of global solutions of nonlinear wave equations with weak time-dependent damping related to Glassey’s conjecture. Differ. Integral Equ. 32(1–2), 37–48 (2019)

[24] Lai, N.A., Takamura, H.: Nonexistence of global solutions of wave equations with weak time-dependent damping and combined nonlinearity. Nonlinear Anal. Real World Appl. 45, 83–96 (2019)

[25] Masuda, K.: Blow-up solutions for quasi-linear wave equations in two space dimensions. Lect. Notes Numer. Appl. Anal. 6, 87–91 (1983)
[26] Palmieri, A.: A note on a conjecture for the critical curve of a weakly coupled system of semilinear wave equations with scale-invariant lower order terms. Math. Methods Appl. Sci. 43(11), 6702–6731 (2020)

[27] Palmieri, A., Takamura, H.: Blow-up for a weakly coupled system of semilinear damped wave equations in the scattering case with power nonlinearities. Nonlinear Anal. 187, 467–492 (2019)

[28] Palmieri, A., Takamura, H.: Nonexistence of global solutions for a weakly coupled system of semilinear damped wave equations of derivative type in the scattering case. Mediterr. J. Math. 17, 13 (2020). https://doi.org/10.1007/s00009-019-1445-4

[29] Palmieri, A., Tu, Z.: Lifespan of semilinear wave equation with scale invariant dissipation and mass and sub-Strauss power nonlinearity. J. Math. Anal. Appl. 470(1), 447–469 (2019)

[30] Rammaha, M.A.: Finite-time blow-up for nonlinear wave equations in high dimensions. Commun. Partial Differ. Equ. 12, 677–700 (1987)

[31] Schaeffer, J.: Finite-time blow-up for $u_{tt} - \Delta u = H(u_r, u_t)$ in two space dimensions. Commun. Partial Differ. Equ. 11, 513–543 (1986)

[32] Sideris, T.C.: Global behavior of solutions to nonlinear wave equations in three space dimensions. Commun. Partial Differ. Equ. 8, 1219–1323 (1983)

[33] Strauss, W.A.: Nonlinear scattering theory at low energy. J. Funct. Anal. 41(1), 110–133 (1981)

[34] Takamura, H.: Improved Kato’s lemma on ordinary differential inequality and its application to semilinear wave equations. Nonlinear Anal. 125, 227–240 (2015)

[35] Tzvetkov, N.: Existence of global solutions to nonlinear massless Dirac system and wave equations with small data. Tsukuba Math. J. 22, 198–211 (1998)

[36] Wakasa, K., Yordanov, B.: Blow-up of solutions to critical semilinear wave equations with variable coefficients. J. Differ. Equ. 266(9), 5360–5376 (2018)

[37] Wakasa, K., Yordanov, B.: On the blow-up for critical semilinear wave equations with damping in the scattering case. Nonlinear Anal. 180, 67–74 (2019)

[38] Wirth, J.: Solution representations for a wave equation with weak dissipation. Math. Methods Appl. Sci. 27(1), 101–124 (2004)

[39] Wirth, J.: Wave equations with time-dependent dissipation I. Non-effective dissipation. J. Differ. Equ. 222, 487–514 (2006)

[40] Wirth, J.: Wave equations with time-dependent dissipation II. Effective dissipation. J. Differ. Equ. 232, 74–103 (2007)

[41] Xu, W.: Blowup for systems of semilinear wave equations with small initial data. J. Partial Differ. Equ. 17(3), 198–206 (2004)

[42] Yordanov, B.T., Zhang, Q.S.: Finite time blow up for critical wave equations in high dimensions. J. Funct. Anal. 231(2), 361–374 (2006)
[43] Zhou, Y.: Blow up of solutions to the Cauchy problem for nonlinear wave equations. Chin. Ann. Math. Ser. B 22(3), 275–280 (2001)

[44] Zhou, Y.: Blow up of solutions to semilinear wave equations with critical exponent in high dimensions. Chin. Ann. Math. Ser. B 28(2), 205–212 (2007)

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Received: 2 January 2020.
Accepted: 15 October 2020.