Bosonized Formulation of Lattice QCD.

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Abstract

Problems in lattice gauge models with fermions are discussed. A new bosonic Hermitean effective action for lattice QCD with dynamical quarks is presented. In distinction of the previous versions [1], [2] it does not include constraints and is better suited for Monte-Carlo simulations.

1 Problems in lattice QCD

Lattice formulation of QCD seems at present to be the only regular method to study nonperturbative effects in this model. Wilson’s formulation of lattice Yang-Mills theory [3] provides a solid basis for computations in pure gluodynamics. However real calculations pose certain requirements on the size and spacing of a lattice. To suppress finite volume effects the size of a lattice should be bigger than typical hadron diameter, which gives the estimate $L \geq 3 \text{fermi}$. The experience shows that on such lattices many physical effects can be described with sufficient accuracy for lattice spacing $\lesssim 0.1 \text{fermi}$.

Modern computer facilities allow to consider the lattices of the size $32^4$ (record size is $64^4$), which seems to be sufficient for reliable calculations in QCD without quarks or in the quenched sector neglecting dynamical quark loops. In this way reasonably good results were obtained in calculation of hadron spectrum and form-factors, studying QCD at finite temperature and density.

However, contrary to the common lore that "all difficulties in gauge theories are related to Yang-Mills sector and inclusion of matter fields makes no problem", in

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lattice QCD the situation is different. Whereas the pure gauge sector is well understood, treating of dynamical fermions is far from satisfactory and both principal and technical problems are still present.

Well known problem of fermion spectrum degeneracy on the lattice makes difficult to consider chiral fermions as well as to study chiral effects in QCD (For detailed discussion and references see e.g. reviews [4], [5]). Recently a new formulation of chiral fermion models on the lattice was proposed [6], [7], which allows to remove unwanted fermion doublers preserving chiral invariance of the theory at least in the framework of weak coupling expansion. Another approach, pretending on nonperturbative solution of the problem was proposed in papers [8].

However to estimate a real value of all these proposals extensive nonperturbative calculations are needed. Here comes a major technical problem, which is common for all models with dynamical fermions, in particular QCD. Calculation of physical quantities on the lattice can be done by computing path integrals which are approximated by corresponding finite sums. But in fermion models one encounters path integrals over anticommuting variables which cannot be treated in this manner.

The action we are interested in is usually quadratic in Fermi fields and the corresponding integral is Gaussian. It is equal to the determinant of the quadratic form in the action, whereas integration over Bose fields produces the inverse determinant. Hence the most direct way to compute a fermion determinant is to calculate firstly the corresponding bosonic determinant and then to invert it. Alternatively one can perform integration over Fermi fields analytically to get a nonlocal bosonic effective action, thus avoiding integration over anticommuting variables. This procedure includes inversion of huge matrices and is very costly and time consuming. However considerable progress in this direction has been done, mainly using hybrid Monte-Carlo algorithm [9]. (For recent review of this and other methods see [10]).

Another possibility was formulated by M.Lüscher [11], [12], who proposed to calculate a fermion determinant replacing it by an infinite series of boson determinants. This method is based on the equation of the type

$$\det(Q)^2 = \lim_{n \to \infty} \left[ \det(P_n(Q^2)) \right]^{-1}$$

where $P_n(s)$ is a sequence of polynomials such that

$$\lim_{n \to \infty} P_n(s) = 1/s, \quad 0 < s \leq 1$$

By cutting the sequence at some finite $n = N$ one gets an approximate value of fermion determinant expressed in terms of integrals over $N$ bosonic fields. This idea was developed further in papers [13], [14]. It appears however that computational efforts needed in this approach are comparable with hybrid Monte Carlo algorithm mentioned above.

Another possibility was discussed recently in our papers [1], [2]. In our approach a $D$-dimensional lattice fermion determinant was presented as a path integral of exponent of $D + 1$-dimensional constrained bosonic effective action. This representation is exact in the limit when the size of extra dimension becomes infinite and
lattice spacing zero. For a finite lattice correction terms are present, whose value depends on the particular choice of effective bosonic action. In this talk I shall use the freedom in the choice of effective action to formulate a new version of the bosonization algorithm for lattice QCD which seems to be better suited for Monte Carlo simulations, in particular it allows to get rid off the constraint equation.

2 Bosonic effective action for lattice QCD

We consider Euclidean QCD on the four-dimensional cubic lattice with spacing $a$. The quark fields are four-dimensional spinors $\psi^i(x)$, $i$ being a flavour index, $U_\mu(x)$ is a gauge field. Let us introduce bosonic fields $\phi(x,t)$ defined on a five-dimensional lattice, which have the same spinorial and internal structure as $\psi$. The fifth component $t$ is defined on the one-dimensional chain of the length $L$ with the lattice spacing $b$

$$L = Nb, \quad 0 \leq n < N$$

The effective bosonic action may be written as follows

$$S = S_W(U) + a^4 \sum_x \left\{ b \sum_{n=0}^{N-1} \left\{ \left[ b^{-2}(\phi^*_n(x)\phi_n(x) + \phi^*_n(x)\phi_{n+1}(x) - 2\phi^*_n\phi_n) - \right. \right. \\
$$

$$
- i[\phi^*_{n+1}(x)\gamma_5 \hat{D}\phi_n(x) - h.c.] [b]^{-1} + \\
+ \phi^*_n(x)D^2\phi_n(x) - \sqrt{L}(\phi^*_n(x)(m - i\hat{D}\gamma_5)\chi(x)e^{-mbn} + h.c.) + \\
+ \frac{L}{2m}\chi^*(x)\chi(x) \right\}.$$ 

Here $S_W(U)$ is the standard Wilson action for lattice Yang-Mills field [3], $\chi(x)$ are four-dimensional lattice bose fields having the same spinorial and internal structure as $\psi$. Lattice covariant derivative is denoted by $D_\mu$

$$D_\mu \psi(x) = \frac{1}{a}[U_\mu(x)\psi(x + a_\mu) - \psi(x)]$$

$$\hat{D} = 1/2\gamma_\mu(D^*_\mu - D_\mu)$$

(For simplicity we consider naive fermions, but all the construction is extended in a straightforward way to Wilson fermions).

The effective action (4) allows to calculate in a standard way any gauge invariant correlation function. In particular we shall show that the square of the quark determinant can be presented as the following path integral

$$\int \exp\left\{ a^4 \sum_{i=1}^2 \sum_x \bar{\psi}_i(x)(\hat{D} + m)\psi_i(x) \right\} d\bar{\psi}d\psi = \lim_{L \to \infty,b \to 0} \int \exp\{ -S \} d\phi^*_n d\phi_n d\chi^* d\chi$$

where free boundary conditions in $t$ are assumed

$$\phi_n = 0, \quad n < 0, \quad n \geq N$$
For finite $b$ and $L$ this equality has to be corrected by the terms of order $O(b/a)^2$, $O(\exp\{-mL\})$. We consider the case of two flavours to provide positivity of the effective bosonic action. An analogous representation can be written for the modulus of quark determinant.

To prove the equation (7) we consider the following integral

\[
I = \int \exp\{a^4 \sum_x b \sum_{n=0}^{N-1} [b^{-2}(\phi_{n+1}^\star(x) \exp\{-i\gamma_5 \hat{D}b\})\phi_n(x) + h.c. - 2\phi_n^\star \phi_n)] - L^{1/2}(\phi_n^\star(x)(m - i\hat{D}\gamma_5)\chi(x) + h.c.) \exp\{-mbn\} \}
\]

\[
+ \frac{L}{2m} \chi^\star(x)\chi(x) \}
\]

To calculate the integral (10) we make the following change of variables:

\[
\phi_n^\alpha \rightarrow \exp\{-iD^\alpha nb\}\phi_n^\alpha, \quad \phi_n^\star \rightarrow \exp\{iD^\alpha nb\}\phi_n^\star
\]

Then the integral (10) acquires the form

\[
I = \int \exp\{\sum_\alpha b \sum_{n=0}^{N-1} [(b^{-2})(\phi_{n+1}^\alpha \exp\{-iD^\alpha b\})\phi_n^\alpha + h.c. - 2\phi_n^\star\phi_n^\alpha)] - L^{1/2}(\phi_n^\star\exp\{-(m - iD^\alpha)bn\}(m - iD^\alpha)\chi^\alpha + h.c.)
\]

\[
+ \frac{L}{2m} \chi^\star\chi^\alpha \}
\]

Now the quadratic form in the exponent does not depend on $D^\alpha$ and to calculate the integral it is sufficient to find a stationary point of the exponent. For small $b$ the sum over $n$ can be replaced by the integral and the stationary equations by the following differential ones:

\[
\ddot{\phi}^\star = -L^{1/2}\chi^\star(m + iD^\alpha) \exp\{-iD^\alpha + m\} = 0
\]
\[ \ddot{\phi}^\alpha - L^{1/2} \chi^\alpha (m - i D^\alpha) \exp \{ i D^\alpha - m \} = 0 \]
\[ \phi^\alpha (L) = \phi^\alpha (0) = 0, \quad \phi^{\alpha*} (L) = \phi^{\alpha*} (0) = 0 \]

The solution of these eq.s is
\[ \phi^\alpha_{st} = \frac{L^{1/2} \chi^\alpha \exp \{ -(m + i D^\alpha)t \} }{(m + i D^\alpha)} + \]
\[ + L^{-1/2} \frac{\chi^\alpha (t - L)}{m + i D^\alpha} - L^{-1/2} \frac{t \exp \{ -(m + i D^\alpha)L \} }{(m + i D^\alpha)} \]
\[ \phi^{\alpha*}_{st} = \frac{L^{1/2} \chi^{\alpha*} \exp \{ -(m - i D^\alpha)t \} }{(m - i D^\alpha)} + \]
\[ + L^{-1/2} \frac{\chi^{\alpha*} (t - L)}{m - i D^\alpha} - L^{-1/2} \frac{\exp \{ -(m - i D^\alpha)L \} \chi^{\alpha} }{(m - i D^\alpha)} \]

Substituting this solution into eq(12) and integrating over \( t \) we get
\[ I = \int \exp \{ - \sum_\alpha \frac{\chi^{\alpha*} \chi^{\alpha} }{m^2 + (D^\alpha)^2} \} d\chi^{\alpha*} d\chi^{\alpha} + O(e^{-mL}) \]

Integrating over \( \chi \) and omitting exponentially small corrections one gets
\[ I = \prod_\alpha (m^2 + (D^\alpha)^2) = \det \{ -(\hat{D})^2 + m^2 \} = \det (\hat{D} + m)^2 \]

As was discussed above the integral in the r.h.s. of eq.(7) is a linearized version of the integral \( I \), the differences being \( O(b^2/a^2) \). At the same time it is easy to show that replacing the sum in the eq.(12) by the integral over \( t \) also produces corrections of order \( O(b^2/a^2) \). Therefore we proved that the square of quark determinant indeed can be presented as the path integral of the exponent of bosonic effective action (11). For a finite lattice the corrections are of the order
\[ O(b^2/a^2) + O(e^{-mL}) \]

In distinction of the previous version [2] the effective action described here does not include constraints, which makes easier it’s Monte-Carlo simulations. At present such simulations for simplified lower dimensional model are in progress.

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