FINITE Σ-RICKART MODULES

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Abstract. In this article, we study the notion of a finite Σ-Rickart module, as a module theoretic analogue of a right semi-hereditary ring. A module $M$ is called finite Σ-Rickart if every finite direct sum of copies of $M$ is a Rickart module. It is shown that any direct summand and any direct sum of copies of a finite Σ-Rickart module are finite Σ-Rickart modules. We also provide generalizations in a module theoretic setting of the most common results of semi-hereditary rings. Also, we have a characterization of a finite Σ-Rickart module in terms of its endomorphism ring. In addition, we introduce $M$-coherent modules and provide a characterization of finite Σ-Rickart modules in terms of $M$-coherent modules. At the end, we study when Σ-Rickart modules and finite Σ-Rickart modules coincide. Examples which delineate the concepts and results are provided.

Key Words: semi-hereditary ring, finite Σ-Rickart module, Rickart module, $M$-coherent module, $f$-injective, $M$-pure epimorphism

1. Introduction

After hereditary rings were introduced by Kaplansky in the earliest 50’s, many mathematicians studied semi-hereditary rings as a natural generalization of hereditary rings. Recall that a ring $R$ is said to be right semi-hereditary if every finitely generated right ideal of $R$ is projective. In [20] L. Small gives an example of a ring $R$ which is right semi-hereditary but $R$ is not right hereditary. Following the research on hereditary rings, many characterizations of semi-hereditary rings also have been made. For instance, in [2] is proved that a ring $R$ is right semi-hereditary if and only if every finitely generated submodule of a projective $R$-module is projective. Later, in [15 Theorem 2] is shown that a ring $R$ is right semi-hereditary if and only if factor modules of absolutely pure $R$-modules are absolutely pure (also, if and only if factor modules of injective $R$-modules are $f$-injective). Chase in [3] characterizes a right semi-hereditary ring as a right coherent ring whose right ideals are flat. Very close to right semi-hereditary rings are right Rickart rings. A ring $R$ is said to be right Rickart if the right annihilator of any element of $R$ is generated by an idempotent. Small in [21 Proposition] proves that a ring $R$ is right semi-hereditary if and only if $\text{Mat}_n(R)$ is a right Rickart ring for every positive integer $n$. In 2012 Lee, Rizvi, and Roman in [13] extend Small’s result with the theory of Rickart modules. A right $R$-module $M$ is called Rickart if $\text{Ker} \varphi$ is a direct summand of $M$ for any $\varphi \in \text{End}_R(M)$. They prove that a ring $R$ is right semi-hereditary if and only if $R^{(n)}$ is a Rickart module for every positive integer $n$ [13 Theorem 3.6]. Inspired by the last result, in this paper we define finite Σ-Rickart modules. A right $R$-module $M$ is called finite Σ-Rickart if $M^{(n)}$ is a Rickart module for every $n > 0$. We present many properties of these modules extending those for right semi-hereditary rings. For a module $M$, we will focus on finitely $M$-generated modules as a more general concept of finitely generated modules. We study a finite Σ-Rickart module $M$ using the class , add($M$), which is the analogue to that of finitely generated projective modules.
add(R), in the case of the ring R. To get the module theoretic version of Megibben’s result [15, Theorem 2] mentioned above, we introduce the class \( \mathfrak{S}_M \): in the case of the ring \( M = R \), \( \mathfrak{S}_R \) is the class of absolutely pure modules. We will compare \( \mathfrak{S}_M \) with the class \( \mathfrak{E}_M \) which is introduced in [11]. Note that \( \mathfrak{E}_R \) is the class of injective modules.

After the introduction and some preliminary background, in Section 2, finite \( \Sigma \)-Rickart modules are defined, some examples are presented, and the general properties of these modules are studied. It is proved that direct summands and finite direct sums of copies of a finite \( \Sigma \)-Rickart module inherit the properties (Lemma 2.4). It is shown that \( M \) is finite \( \Sigma \)-Rickart if and only if every finitely \( M \)-generated submodule of an element in add(\( M \)) has \( D_2 \) condition (Theorem 2.13). We introduce the class \( \mathfrak{S}_M \) for a right \( R \)-module \( M \) and characterize finite \( \Sigma \)-Rickart modules in terms of this new class (Theorem 2.26) which is a module theoretic version of [15, Theorem 2]. At the end of the section we provide a characterization of an endoregular module in terms of a finite \( \Sigma \)-Rickart module as well as a characterization of von Neumann regular rings as a corollary (Theorem 2.29 and Corollary 2.30 respectively).

Our focus in Section 3 is on the endomorphism ring of a finite \( \Sigma \)-Rickart module. We introduce the concept of \( M \)-coherent modules and we link it under intrinsically projective modules. That is, when \( M \) is intrinsically projective, two characterizations for an \( M \)-coherent module \( M \) in terms of the intersection property of finitely \( M \)-generated submodules of \( M \) (Theorem 3.10) and in terms of when \( \text{End}_R(M) \) is a right coherent ring (Theorem 3.15) are provided. These results will help us to characterize a finite \( \Sigma \)-Rickart module in terms of its endomorphism ring. A module \( M \) is finite \( \Sigma \)-Rickart if and only if \( S = \text{End}_R(M) \) is a right semi-hereditary ring and \( S \) is flat if and only if \( M \) is intrinsically projective, \( M \)-coherent and all right \( S \)-ideals are flat (Theorem 3.20).

When a module \( S \) is flat where \( S = \text{End}_R(M) \) in Section 4, we prove that \( M \in \mathfrak{E}_M \) if and only if \( S \) is a right self-injective ring, and \( M \in \mathfrak{S}_M \) if and only if \( S \) is a right \( f \)-injective ring (Corollary 4.3). Also, for \( M = \bigoplus_{i=1}^{n} H_i \), if \( H_i \) is an indecomposable endoregular module and \( H_j \) is a \( H \)-Rickart for all \( 1 \leq i, j \leq n \) then \( \text{End}_R(M) \) is a semiprimary PWD (Corollary 4.8). Therefore as an application, if \( M \) is a finite \( \Sigma \)-Rickart module and \( P \) is any simple module such that \( \text{Hom}_R(M, P) = 0 \) then \( M^{(\ell)} \oplus P^{(n)} \) is a finite \( \Sigma \)-Rickart module for any \( \ell, n > 0 \) (Proposition 4.10). At the end, we characterize those finite \( \Sigma \)-Rickart modules with semiprimary endomorphism ring (Proposition 4.15). This allows us to determine when the concepts of \( \Sigma \)-Rickart and finitie \( \Sigma \)-Rickart coincide (Corollary 4.19).

Throughout this paper, \( R \) is an associating ring with unity and \( M \) is a unitary right \( R \)-module. For a right \( R \)-module \( M \), \( S = \text{End}_R(M) \) will denote the endomorphism ring of \( M \); thus \( M \) can be viewed as a left \( S \)-right \( R \)-bimodule. For \( \varphi \in S \), \( \ker \varphi \) and \( \text{im} \varphi \) stand for the kernel and the image of \( \varphi \), respectively. The notations \( N \leq M, N \leq M \), \( N \leq_{\text{ess}} M \), and \( N \leq_{\oplus} M \) mean that \( N \) is a submodule, a fully invariant submodule, an essential submodule, and a direct summand of \( M \), respectively. We use \( M^{(n)} \) to denote the direct sum of \( n \) copies of \( M \). By \( \mathbb{Q}, \mathbb{Z}, \) and \( \mathbb{N} \) we denote the set of rational, integer, and natural numbers, respectively. For \( 1 < n \in \mathbb{N}, \mathbb{Z}_n \) denotes the \( \mathbb{Z} \)-module \( \mathbb{Z}/n\mathbb{Z} \). We also denote \( r_S(N) = \{ r \in R \mid Nr = 0 \} \) and \( I_S(N) = \{ \varphi \in S \mid \varphi N = 0 \} \) for \( N \leq M \), and \( r_S(I) = \{ \varphi \in S \mid \varphi I = 0 \} \) for \( I_S \leq S \).

In [12] was introduced the concept of Rickart modules and were presented many properties of them.

**Definition 1.1.** A right \( R \)-module \( M \) is called Rickart if \( \ker \varphi \) is a direct summand of \( M \) for every endomorphism \( \varphi \in \text{End}_R(M) \) [12]. \( M \) is called endoregular if \( \text{End}_R(M) \) is a von Neumann regular ring [14].
Recall that a module $M$ is said to have $D_2$ condition if $\forall N \leq M$ with $M/N \cong M' \leq \oplus M$, we have $N \leq \oplus M$. Note that any Rickart module and any projective module satisfies $D_2$ condition. Dually, $M$ is said to have $C_2$ condition if $\forall N \leq M$ with $N \cong M' \leq \oplus M$, we have $N \leq \oplus M$.

**Proposition 1.2.** The following statements hold true for a right $R$-module $M$:

(i) ([12] Proposition 2.11) $M$ is a Rickart module if and only if $M$ has $D_2$ condition and $\text{Im} \varphi$ is isomorphic to a direct summand of $M$ for all $\varphi \in \text{End}_R(M)$.

(ii) ([14] Theorem 1.11) $M$ is an endoregular module if and only if $M$ is a Rickart module and $M$ has $C_2$ condition.

(iii) ([14] Proposition 2.26) $M$ is a projective Rickart module if and only if $\text{Im} \varphi$ is projective for each $\varphi \in \text{End}_R(M)$.

A module $M$ is said to be $N$-Rickart (or relatively Rickart to $N$) if $\text{Ker} \rho \leq \oplus M$ for every homomorphism $\rho \in \text{Hom}_R(M, N)$ [19].

**Theorem 1.3** ([13] Theorem 2.6). Let $M$ and $N$ be modules. Then $M$ is $N$-Rickart if and only if for any direct summand $M' \leq \oplus M$ and any submodule $N' \leq N$, $M'$ is $N'$-Rickart.

In some results we will assume that $M$ has some projectivity conditions in order to get deeper results. The next lemma will be useful.

**Lemma 1.4** ([22] 18.2). The following statements hold true for a right $R$-module $M$:

(i) Consider $0 \to N' \to N \to N'' \to 0$ as a short exact sequence. If $M$ is an $N$-projective module then $M$ is $N'$- and $N''$-projective.

(ii) If $M$ is $N_i$-projective for right $R$-modules $N_1, \ldots, N_\ell$, then $M$ is $\bigoplus_{i=1}^\ell N_i$-projective.

Recall that a right $R$-module $M$ is called $\Sigma$-Rickart if every direct sum of copies of $M$ is a Rickart module [11]. Also, Add$(M)$ denotes the class of all right $R$-modules $K$ such that $K$ is isomorphic to a direct summand of $M$ if $K$ for some nonempty index set $I$ (see [11] Definition 2.9)).

**Proposition 1.5** ([11] Proposition 2.11). Let $M$ be a right $R$-module such that $R \in \text{Add}(M)$. Then $M$ is a $\Sigma$-Rickart module if and only if $M$ is a projective $R$-module and $R$ is a right hereditary ring.

**Theorem 1.6** ([11] Theorem 4.6)). The following conditions are equivalent for a finitely generated module $M$:

(a) $M$ is a $\Sigma$-Rickart module;

(b) $S = \text{End}_R(M)$ is a right hereditary ring and $SM$ is flat.

## 2. Finite $\Sigma$-Rickart Modules

In this section, after we introduce $\Sigma$-Rickart modules in 2020 [11], we present another natural generalized notion which is called finite $\Sigma$-Rickart modules and obtain some of its basic properties. Note that, since proofs of some results are similar to those in [11], we will omit or include proofs for the convenience of the reader.

**Definition 2.1.** A right $R$-module $M$ is called finite $\Sigma$-Rickart if every finite direct sum of copies of $M$ is a Rickart module.

**Example 2.2.** (i) $R_R$ is a finite $\Sigma$-Rickart module iff $R$ is a right semi-hereditary ring.

(ii) Any $K$-nonsingular continuous module is finite $\Sigma$-Rickart.

(iii) Every $\Sigma$-Rickart module and every endoregular module are finite $\Sigma$-Rickart.
Any submodule of $Q_\mathbb{Z}$ is a finite $\Sigma$-Rickart module. For, let $N$ be any submodule of $Q_\mathbb{Z}$ and $\varphi : N^{(n)} \rightarrow N^{(n)}$ be any endomorphism for any $0 < n \in \mathbb{N}$. Then $\text{Im} \, \varphi$ is a torsion-free group. Hence $\text{Ker} \, \varphi$ is a pure subgroup of $N^{(n)}$ by [5, Ch.V, 26(d)]. Therefore $\text{Ker} \, \varphi \leq \oplus N^{(n)}$ by [6, Lemma 86.8]. Thus $N$ is a finite $\Sigma$-Rickart module.

(v) Let $R$ be a Dedekind domain and $M$ be a direct sum of finitely generated torsion-free $R$-modules of rank one. Then every submodule of $M$ is a finite $\Sigma$-Rickart module ([9, Theorems 3 and 4]).

(vi) Every finitely generated free (projective) module over a right semi-hereditary ring is a finite $\Sigma$-Rickart module.

(vii) When $M = \bigoplus_{i \in \mathcal{I}} M_i$ with $M_i \leq M$ for all $i \in \mathcal{I}$, $\bigoplus_{i \in \mathcal{I}} M_i$ is a finite $\Sigma$-Rickart module if and only if $M_i$ is a finite $\Sigma$-Rickart module for all $i \in \mathcal{I}$.

We have the implications for right $R$-modules:

\[
\begin{align*}
\Sigma\text{-Rickart} & \quad \Downarrow \\
\text{Endoregular} & \quad \Rightarrow \quad \text{finite } \Sigma\text{-Rickart} \quad \Rightarrow \quad \text{Rickart}
\end{align*}
\]

The next examples show that each converse of the above implications is not true, in general.

**Example 2.3.** (i) $\mathbb{Z}[x]_{[x]}$ is Rickart but it is not finite $\Sigma$-Rickart.

(ii) The localization of integers at a prime $p$, $\mathbb{Z}(p) = \{a/b \mid a, b \in \mathbb{Z}, p \nmid b\}$, is a finite $\Sigma$-Rickart $\mathbb{Z}$-module which is not endoregular.

(iii) If a module has $C_2$ condition then the three concepts in the low part of (2.1) coincide by Proposition [12(ii)]

(iv) Consider the $\mathbb{Z}$-module $M = Q \oplus \mathbb{Z}$. Since $\mathbb{Z}^{(n)}$ is a nonsingular extending module for any $n \in \mathbb{N}$ and $E(\mathbb{Z}^{(n)}) = \mathbb{Q}^{(n)}$, from [13, Theorem 2.16] $M^{(n)}$ is a Rickart module for any $n \in \mathbb{N}$. Thus, $M$ is a finite $\Sigma$-Rickart module. However, $M$ is not a $\Sigma$-Rickart module. For, assume that $M$ is a $\Sigma$-Rickart module. Since $\mathbb{Z} \leq \oplus M$, by Proposition [13] $M$ is a projective module, a contradiction.

**Lemma 2.4.** (i) Every direct summand of a finite $\Sigma$-Rickart module is finite $\Sigma$-Rickart.

(ii) Every finite direct sum of copies of a finite $\Sigma$-Rickart module is finite $\Sigma$-Rickart.

**Proof.** (i) Let $M$ be a finite $\Sigma$-Rickart module and $N$ be a direct summand of $M$. Then $N^{(n)}$ is a direct summand of $M^{(n)}$ for all $0 < n \in \mathbb{N}$. Since $M^{(n)}$ is a Rickart module, so is $N^{(n)}$. Thus, $N$ is a finite $\Sigma$-Rickart module.

(ii) Let $M$ be a finite $\Sigma$-Rickart module. Consider $M^{(n)}$ a direct sum of copies of $M$ for any $n \in \mathbb{N}$. Then $(M^{(n)})^{(m)} = M^{(nm)}$ is a Rickart module for all $n, m \in \mathbb{N}$. Therefore $M^{(n)}$ is a finite $\Sigma$-Rickart module.

**Definition 2.5.** Let $M$ be a right $R$-module. Denote by $\text{add}(M)$ the class of all right $R$-modules $K$ such that $K$ is isomorphic to a direct summand of $M^{(n)}$ for some $0 < n \in \mathbb{N}$. Note that $\text{add}(R)$ consists of all finitely generated projective right modules over a ring $R$.

**Remark 2.6.** If $M$ is a right $R$-module such that $R \in \text{add}(M)$, then $M$ is a projective left $S$-module where $S = \text{End}_R(M)$. For, since $R$ is in $\text{add}(M)$, $M^{(n)} \cong R \oplus N$ for some right $R$-module $N$ and some $n \in \mathbb{N}$. Applying the functor $\text{Hom}_R(-, M)$ we get $S^{(n)} \cong \text{Hom}_R(R, M) \oplus \text{Hom}_R(N, M) \cong M \oplus \text{Hom}_R(N, M)$ as left $S$-modules. Thus, $S\,M$ is projective. In addition, for the case of $\text{Add}(M)$, if $M_R$ is finitely generated such that $R \in \text{Add}(M)$, then $S\,M$ is projective.

The next proposition generalizes Lemma [2.4(ii)].
Proposition 2.7. A module $M$ is finite $\Sigma$-Rickart if and only if every element in $\text{add}(M)$ is a finite $\Sigma$-Rickart module.

Recall that a module $N$ is said to be finitely $M$-generated if there exists an epimorphism $M^{(n)} \to N$ for some $0 < n \in \mathbb{N}$.

Lemma 2.8. For a finite $\Sigma$-Rickart module $M$, the following statements hold true:

(i) $M^{(n)}$ is $M^{(n)}$-Rickart for every $0 < m, n \in \mathbb{N}$.

(ii) For given $K \in \text{add}(M)$, the intersection of two finitely $M$-generated submodules of $K$ is finitely $M$-generated.

(iii) The intersection of two finitely $M$-generated submodules of $M$ is finitely $M$-generated.

Proof. (i) It directly follows from Theorem 1.3. (ii) The proof is similar to that of [11, Lemma 2.13]. (iii) It is the special case of (ii) (see also Theorems 3.10 and 3.20). □

Corollary 2.9 (e.g., [10, Corollary 4.60]). For a right semi-hereditary ring $R$, the intersection of two finitely generated ideals of $R$ is finitely generated.

Proof. It directly follows from Lemma 2.8 iii) (see also Corollary 3.12 and Lemma 3.19). □

Theorem 2.10. The following conditions are equivalent for a module $M$:

(a) $M$ is a finite $\Sigma$-Rickart module;

(b) every $K \in \text{add}(M)$ satisfies the following two statements:

(1) any finitely $M$-generated submodule of $K$ is in $\text{add}(M)$; and

(2) any epimorphism $N \to K$ with $N$ finitely $M$-generated splits.

Proof. The proof is similar to that of [11, Theorem 2.12]. □

The following examples show that Conditions (b1) and (b2) of Theorem 2.10 are independent.

Example 2.11. (i) Let $M_Z = \mathbb{Z}_{p\infty}$ for a prime $p \in \mathbb{Z}$ and let $K \in \text{add}(M)$ be arbitrary. Consider $P$ as a finitely $M$-generated submodule of $K$. Then there exists an epimorphism $\rho: M^{(n)} \to P$ for some $n > 0$. Since $M$ is divisible, so is $M^{(n)}$. It is a fact that epimorphic images of divisible groups are divisible, hence $P$ is divisible. This implies that $P \leq^\oplus K$. Thus $P \in \text{add}(M)$. Therefore $M = \mathbb{Z}_{p\infty}$ satisfies Theorem 2.10 b1).

Now, consider the epimorphism $\varphi: M \to M$ given by $\varphi(a) = ap$. Since $\varphi$ is not a monomorphism and $M$ is uniform, $\varphi$ does not split. Thus $M$ does not satisfy Theorem 2.10 b2). Note that $M$ is not finite $\Sigma$-Rickart because $M$ is not a Rickart $\mathbb{Z}$-module.

(ii) Consider the ring

$$R = \left\{ \begin{pmatrix} a & (x,y) \\ 0 & a \end{pmatrix} \mid a \in \mathbb{Z}_2, (x,y) \in \mathbb{Z}_2 \oplus \mathbb{Z}_2 \right\}$$

with the usual addition and multiplication of matrices. Then $R$ is a commutative local artinian ring with maximal ideal $I = \left\{ \begin{pmatrix} 0 & (x,y) \\ 0 & 0 \end{pmatrix} \mid (x,y) \in \mathbb{Z}_2 \oplus \mathbb{Z}_2 \right\}$. Let $M$ be a finitely generated free $R$-module. Then $M$ satisfies Theorem 2.10 b2) because every element in $\text{add}(M)$ is projective. However, let $N$ be a simple submodule of $M$. Since $M$ is a free module, $N$ is finitely $M$-generated. Since $R$ is local, $r_R(N) \leq^\text{ess} R$. Thus $N$ is a singular simple right $R$-module. Hence $N$ is not projective, that is, $N$ is not in $\text{add}(M)$. Therefore $M$ does not satisfy Theorem 2.10 b1). Note that $R$ is not a Rickart $R$-module because $\text{Ker} \begin{pmatrix} 0 & (1,1) \\ 0 & 0 \end{pmatrix} \leq^\text{ess} R_R$, hence $M$ is not finite $\Sigma$-Rickart.

Theorem 2.12. If $M$ is a finite $\Sigma$-Rickart module then every finitely $M$-generated submodule $P$ of any element in $\text{add}(M)$ is isomorphic to a direct sum of finitely $M$-generated submodules of $M$. 

Proof. The proof is similar to that of [11, Theorem 2.14]. □

**Theorem 2.13.** The following conditions are equivalent for a module $M$:

(a) $M$ is a finite $\Sigma$-Rickart module;
(b) every finitely $M$-generated submodule of any element in $\text{add}(M)$ has $D_2$ condition.

Proof. The proof is similar to that of [11, Theorem 2.17]. □

**Corollary 2.14.** The following conditions are equivalent for a module $M$:

(a) $M$ is a quasi-projective finite $\Sigma$-Rickart module;
(b) every finitely $M$-generated submodule of any element in $\text{add}(M)$ is $M$-projective;
(c) every finitely $M$-generated submodule of any element in $\text{add}(M)$ is quasi-projective.

Proof. (a)$\Rightarrow$(b) Let $K$ be a finitely $M$-generated submodule of an element in $\text{add}(M)$. By Theorem 2.10 $K \in \text{add}(M)$, that is, $K$ is isomorphic to a direct summand of $M^{(n)}$ for some $n > 0$. Since $M$ is quasi-projective, $M^{(n)}$ is $M$-projective. Hence $K$ is $M$-projective.

(b)$\Rightarrow$(c) Let $K$ be a finitely $M$-generated submodule of an element in $\text{add}(M)$. Then there exists an epimorphism $M^{(n)} \rightarrow K$ for some $n > 0$. Since $K$ is $M$-projective, $K$ is $K$-projective by Lemma 1.4. Therefore $K$ is quasi-projective. (c)$\Rightarrow$(a) It follows from Theorem 2.13. □

**Corollary 2.15.** Let $R$ be a Dedekind domain which is a complete discrete valuation ring. Then every torsion-free module of finite rank is a quasi-projective finite $\Sigma$-Rickart module.

Proof. Let $M$ be a torsion-free $R$-module of finite rank. Let $N$ be a finitely $M$-generated submodule of an element in $\text{add}(M)$. Then $N$ is torsion-free and has finite rank. Then $N$ is quasi-projective by [18, Theorem 5.8]. From Corollary 2.14, $M$ is a quasi-projective finite $\Sigma$-Rickart module.

It is well known that a ring $R$ is right semi-hereditary if and only if every finitely generated submodule of a right projective module is projective ([2, Proposition 6.2]). In the next result, we give more characterizations for right semi-hereditary rings.

**Corollary 2.16.** The following conditions are equivalent for a ring $R$:

(a) $R$ is a right semi-hereditary ring;
(b) every finitely generated submodule of any projective right $R$-module is projective;
(c) every finitely generated submodule of any projective right $R$-module is $R$-projective;
(d) every finitely generated submodule of any projective right $R$-module is quasi-projective;
(e) every finitely generated submodule of any projective right $R$-module has $D_2$ condition.

In [11] $\Sigma$-Rickart modules were characterized using a class of modules called $\mathcal{E}_M$. For a right $R$-module $M$, it is denoted by $\mathcal{E}_M$ the class of all right $R$-modules $A$ such that for any monomorphism $\alpha : N \rightarrow M$ with $N$ an $M$-generated module and for any homomorphism $\beta : N \rightarrow A$, there exists $\gamma : M \rightarrow A$ such that $\beta = \gamma \alpha$. For the analogue of the above class related to finite $\Sigma$-Rickart modules, we introduce the following.

**Definition 2.17.** Let $M$ be a right $R$-module. Denote by $\mathfrak{F}_M$ the class of all right $R$-modules $A$ such that for any monomorphism $\alpha : N \rightarrow M$ with $N$ a finitely $M$-generated module and for any homomorphism $\beta : N \rightarrow A$, there exists $\gamma : M \rightarrow A$ such that $\beta = \gamma \alpha$.

For a right $R$-module $M$, a right $R$-module $A$ is said to be $f_M$-injective if for any finitely $M$-generated submodule $N$ of $M$ and for any homomorphism $\beta : N \rightarrow A$, there exists a homomorphism $\gamma : M \rightarrow A$ such that $\gamma|_N = \beta$. Note that in the case of $M = R_R$, $A$ is said to be $f$-injective if $A$ is $f_R$-injective (see [8]). We can easily see that the every element in $\mathfrak{F}_M$ is exactly $f_M$-injective as the following.
Proposition 2.18. For a right $R$-module $M$, a module $A$ is in $\mathfrak{F}_M$ iff $A$ is $f_M$-injective.

Proposition 2.19. For a right $R$-module $M$, a module $A$ is in $\mathfrak{F}_M$ if and only if for any monomorphism $\alpha : N \to K$ with $N$ a finitely $M$-generated module and $K \in \operatorname{add}(M)$, and for any homomorphism $\beta : N \to A$, there exists $\gamma : K \to A$ such that $\beta = \gamma \alpha$.

Proof. The proof is similar to that of [11, Proposition 3.2]. □

Remark 2.20. (i) We have the following contentions,

$$\mathfrak{E}_R \subseteq \{\text{all } M\text{-injective modules}\} \subseteq \mathfrak{E}_M \subseteq \mathfrak{F}_M = \{\text{all } f_M\text{-injective modules}\}$$

where $\mathfrak{E}_R = \{\text{all injective modules}\}$. Note that $\mathfrak{F}_R = \{\text{all } f\text{-injective modules}\}$.

(ii) If every submodule of $M$ is finitely $M$-generated then every module in $\mathfrak{F}_M$ is $M$-injective.

Proposition 2.21. The following statements hold true for a right $R$-module $M$:

(i) $\mathfrak{F}_M$ is closed under direct products.

(ii) $\mathfrak{F}_M$ is closed under direct summands.

(iii) If $M$ is in $\mathfrak{F}_M$ then $M$ has $C_2$ condition.

(iv) If every finitely $M$-generated submodule of $A$ is in $\mathfrak{F}_M$, then $A$ is in $\mathfrak{F}_M$.

Proof. All proofs are similar to those of [11, Proposition 3.6]. However, we give the proof of (iv) for the convenience of the reader. (iv) Let $\alpha : N \to M$ be a monomorphism with $N$ finitely $M$-generated and let $\beta : N \to A$ be any homomorphism. Since $N$ is finitely $M$-generated, $\operatorname{Im} \beta$ is finitely $M$-generated. Because $\operatorname{Im} \beta \subseteq A$, by hypothesis there exists $\gamma : M \to \operatorname{Im} \beta$ such that $\beta(N) = \gamma \alpha(N)$. Therefore $A \in \mathfrak{F}_M$. □

Corollary 2.22. If every finitely generated submodule of $M$ is $f$-injective then $M$ is also $f$-injective.

Proposition 2.23. The following conditions are equivalent for a module $M$:

(a) $M$ is an endoregular module;

(b) $M$ has $D_2$ condition and $\mathfrak{F}_M = \operatorname{Mod-R}$.

Proof. (a)⇒(b) It is clear that $M$ has $D_2$ condition. Let $L$ be any right $R$-module and let $N$ a finitely $M$-generated submodule of $M$. Then there exists an epimorphism $\rho : M^{(n)} \to N$ for some $n > 0$. Also, let $\alpha : N \to M$ be any monomorphism and $\beta : N \to L$ be any homomorphism. By [11, Corollary 3.15], $M^{(n)}$ is an endoregular module, and hence $\alpha \rho(M^{(n)}) = \alpha(N)$ is a direct summand of $M$. Take $\gamma = \beta \alpha^{-1} \oplus 0$. Then $\gamma : M \to L$ is a homomorphism such that $\gamma \alpha = \beta$. Therefore $L$ is in $\mathfrak{F}_M$.

(b)⇒(a) Let $\varphi : M \to M$ be any endomorphism of $M$. Then $\operatorname{Im} \varphi$ is finitely $M$-generated. Since $\operatorname{Im} \varphi$ is in $\mathfrak{F}_M$, the canonical inclusion $j : \operatorname{Im} \varphi \to M$ splits, that is, $\operatorname{Im} \varphi$ is a direct summand of $M$. By the $D_2$ condition, we can infer that $\ker \varphi$ is a direct summand of $M$. Thus, $M$ is an endoregular module. □

An epimorphism $\mu : A \to B$ is called an $M$-pure epimorphism if for any homomorphism $\beta : M \to B$, there exists $\gamma : M \to A$ such that $\mu \gamma = \beta$ [22] (see also [11, Proposition 3.10]).

Remark 2.24. It is not difficult to see that:

(i) An epimorphism $\mu : A \to B$ is an $M$-pure epimorphism if and only if $\mu$ is a $K$-pure epimorphism for any $K$ in $\operatorname{add}(M)$.

(ii) For a projective module $M$, every epimorphism is an $M$-pure epimorphism.

(iii) If $\mu : A \to B$ and $\nu : C \to D$ are $M$-pure epimorphisms, then $\mu \oplus \nu : A \oplus C \to B \oplus D$ is also an $M$-pure epimorphism.
Lemma 2.25 ([11] Lemma 3.11). Let $M$ be $M(I)$-projective for any (resp., finite) index set $I$. If $A$ is an (resp., finitely) $M$-generated module then every epimorphism $\mu : A \to B$ is an $M$-pure epimorphism.

This results is a module theoretic version of [15] Theorem 2.

Theorem 2.26. Consider the following conditions for a module $M$:

(i) $M$ is a finite $\Sigma$-Rickart module.
(ii) $\mathfrak{F}_M$ is closed under $M$-pure epimorphisms.
(iii) If $\mu \in \text{Hom}_R(A, A')$ is an $M$-pure epimorphism with $A$ $M$-injective, then $A' \in \mathfrak{F}_M$.

Then the implications (i)$\Rightarrow$(ii)$\Rightarrow$(iii) hold true. In addition, if $M$ is $M(I)$-projective for any index set $I$, then the three conditions are equivalent.

Proof. The proofs are similar to those of [11] Theorem 3.12. \qed

Remark from [11] Example 3.13 that the converse of (i)$\Rightarrow$(ii) is not true, in general. Recall that a submodule $N$ of a right $R$-module $M$ is said to be pure if for every left $R$-module $K$, the canonical homomorphism $\iota \otimes 1 : N \otimes_R K \to M \otimes_R K$ is a monomorphism, where $\iota : N \to M$ is the canonical inclusion. $M$ is said to be absolutely pure if $M$ is a pure submodule of any module which contains $M$ as a submodule (see [15]).

Lemma 2.27 ([15] Corollary 2 and Remark 2.20(i)). A right $R$-module $M$ is absolutely pure if and only if $M$ is $f$-injective if and only if $M$ is in $\mathfrak{F}_R$.

As a corollary, we have a characterization for right semi-hereditary rings including [15] Theorem 2.

Corollary 2.28. The following conditions are equivalent for a ring $R$:

(a) $R$ is a right semi-hereditary ring;
(b) Every factor module of any $f$-injective $R$-module is $f$-injective;
(c) Every factor module of any absolutely pure $R$-module is absolutely pure;
(d) Every factor module of any injective $R$-module is absolutely pure.

Now, we are going to give a module theoretic version of [8] Theorem 3.4.

Theorem 2.29. The following conditions are equivalent for a right $R$-module $M$:

(a) $M$ is an endoregular module;
(b) $M$ is a finite $\Sigma$-Rickart module and $M$ is in $\mathfrak{F}_M$;
(c) $M$ is strongly $D_2$ condition (i.e., $M^{(n)}$ has $D_2$ condition for all $n > 0$) and any finitely $M$-generated submodule of $M^{(n)}$ is a direct summand for all $n > 0$.

Proof. (a)$\Leftrightarrow$(b) Let $M$ be an endoregular module. Then by [14] Corollary 3.15, $M^{(n)}$ is an endoregular module, which is Rickart. Thus, $M$ is finite $\Sigma$-Rickart. Also, from Proposition 2.28 (iii) $M$ is in $\mathfrak{F}_M$. Conversely, let $M \in \mathfrak{F}_M$. Then $M$ has $C_2$ condition from Proposition 2.21(iii). Hence $M$ is an endoregular module by [12] Theorem 3.17.

(a)$\Rightarrow$(c) Since each endoregular module is finite $\Sigma$-Rickart, from Theorem 2.13 it is easy to see that $M$ is strongly $D_2$ condition. Now, let $N$ be a finitely $M$-generated submodule of $M^{(n)}$ for some $n > 0$. Then there exists an epimorphism $\rho : M^{(\ell)} \to N$ for some $\ell > 0$. On the other hand, let $j : N \to M^{(n)}$ be the canonical inclusion. Then there is a homomorphism $j\rho : M^{(\ell)} \to M^{(n)}$. Therefore $\text{Im}(j\rho) = N$ is a direct summand of $M^{(n)}$.

(c)$\Rightarrow$(b) Let $\varphi : M^{(n)} \to M^{(n)}$ be any endomorphism. By hypothesis $\text{Im} \varphi \leq \oplus M^{(n)}$. Since $M^{(n)}$ has $D_2$ condition, $\text{Ker} \varphi \leq \oplus M^{(n)}$. Thus $M$ is a finite $\Sigma$-Rickart module. In addition, let $N$ be a finitely $M$-generated submodule of $M$ and let $\beta : N \to M$ be any homomorphism. By hypothesis, $N$ is a direct summand of $M$. This implies that $\beta$ can be extended to a homomorphism $\gamma : M \to M$. Thus, $M$ is in $\mathfrak{F}_M$. \qed
Corollary 2.30. The following conditions are equivalent for a ring $R$:
(a) $R$ is a von Neumann regular ring;
(b) $R$ is a right semi-hereditary ring and $R_R$ is an $f$-injective module;
(c) $R$ is a right semi-hereditary ring and $R_R$ is an absolutely pure module;
(d) every finitely generated submodule of $R^{(n)}$ is a direct summand for all $n > 0$.

3. $M$-coherent modules and the endomorphism ring of a finite $\Sigma$-Rickart module

The next result can be seen as a generalization of Schanuel’s Lemma \cite[5.1]{schanuel}.

Lemma 3.1. Let $M$ be a right $R$-module and $K \in \add(M)$. Let
\[
0 \longrightarrow D \overset{\sigma}{\longrightarrow} K \overset{\rho}{\longrightarrow} C \longrightarrow 0
\]
\[
0 \longrightarrow A \overset{\alpha}{\longrightarrow} B \overset{\beta}{\longrightarrow} C \longrightarrow 0
\]
be short exact sequences with $\beta$ an $M$-pure epimorphism. Then there exists a short exact sequence
\[
0 \longrightarrow D \overset{\delta}{\longrightarrow} K \oplus A \overset{\eta}{\longrightarrow} B \longrightarrow 0.
\]
Moreover, if $\rho$ is also an $M$-pure epimorphism, then so is $\eta$.

Proof. Consider the following diagram:
\[
\begin{array}{ccc}
0 & \longrightarrow & D \overset{\sigma}{\longrightarrow} K \overset{\rho}{\longrightarrow} C \longrightarrow 0 \\
& \downarrow & \gamma \\
0 & \longrightarrow & A \overset{\alpha}{\longrightarrow} B \overset{\beta}{\longrightarrow} C \longrightarrow 0.
\end{array}
\]
Since $K \in \add(M)$ and $\beta$ is an $M$-pure epimorphism, there exists $\gamma : K \to B$ such that $\rho = \beta \gamma$. We claim that the following sequence is exact
\[
(3.1) \quad 0 \longrightarrow D \overset{\delta}{\longrightarrow} K \oplus \Im \alpha \overset{\eta}{\longrightarrow} B \longrightarrow 0
\]
where $\delta(d) = (\sigma(d), \gamma(\sigma(d)))$ and $\eta(k, x) = \gamma(k) - x$ for $d \in D$, $k \in K$ and $x \in \Im \alpha$: It is clear that $\gamma(\sigma(d)) \in \Im \alpha$, $\delta$ is a monomorphism, and $\eta \delta = 0$. Let $(k, x) \in K \oplus \Im \alpha$ such that $\eta(k, x) = 0$. That is, $\gamma(k) = x$. Then $0 = \beta(x) = \beta(\gamma(k)) = \rho(k)$. Since $\Im \sigma = \Ker \rho$ there exists $d \in D$ such that $\sigma(d) = k$. Thus $\delta(d) = (\sigma(d), \gamma(\sigma(d))) = (k, \gamma(k)) = (k, x)$. Hence $\Im \delta = \Ker \eta$. Now, it remains to show that $\eta$ is an epimorphism. Let $b \in B$. Since $\rho$ is an epimorphism, there exists $\ell \in K$ such that $\rho(\ell) = \beta(b)$. Hence $\beta(\gamma(\ell) - b) = 0$, that is, there exists $a \in A$ such that $\alpha(a) = \gamma(\ell) - b$. Thus $\eta(\ell, \alpha(a)) = b$, proving the claim. Since $A \cong \Im \alpha$, we have an exact sequence
\[
0 \longrightarrow D \longrightarrow K \oplus A \longrightarrow B \longrightarrow 0.
\]
Moreover, suppose $\rho$ is also an $M$-pure epimorphism. Let $\zeta : M \to B$ be any homomorphism. Then, $\beta \zeta : M \to C$. Since $\rho$ is an $M$-pure epimorphism, there exist $\varphi : M \to K$ such that $\rho \varphi = \beta \zeta$. This implies that $\beta \gamma \varphi = \beta \zeta$ and so $\gamma \varphi(m) - \zeta(m) \in \Ker \beta = \Im \alpha$ for all $m \in M$. Define $\overline{\varphi} : M \to K \oplus \Im \alpha$ as $\overline{\varphi}(m) = (\varphi(m), \gamma \varphi(m) - \zeta(m))$. It is clear that $\eta \overline{\varphi} = \zeta$.

Lemma 3.2. Let $M$ be a right $R$-module and
\[
0 \longrightarrow A \overset{\alpha}{\longrightarrow} B \overset{\beta}{\longrightarrow} C \longrightarrow 0
\]
be an exact sequence with $A$ and $C$ finitely $M$-generated modules. If $\beta$ is an $M$-pure epimorphism then $B$ is finitely $M$-generated.
Proof. Since \( C \) is finitely \( M \)-generated, there exists an epimorphism \( \rho : M^{(n)} \to C \) for some \( n \in \mathbb{N} \). Consider the pull-back \( P \) of \((\beta, \rho)\), that is, \( P = \{(b, m) \in B \oplus M^{(n)} \mid \beta(b) = \rho(m)\} \):

\[
\begin{array}{ccc}
P & \xrightarrow{\beta} & M^{(n)} \\
\rho' \downarrow & & \downarrow \rho \\
0 & \xrightarrow{\alpha} & A & \xrightarrow{\beta} & B & \xrightarrow{\rho} & C & \xrightarrow{\gamma} & 0.
\end{array}
\]

Note that both \( \rho' \) and \( \beta' \) are epimorphisms. On the other hand, \( \ker \beta' = \ker \beta \oplus 0 \cong A \). Since \( \beta \) is an \( M \)-pure epimorphism, there exists \( \kappa : M^{(n)} \to B \). Therefore, \( \beta' \) splits. Hence \( P \cong A \oplus M^{(n)} \). This implies that \( P \) is finitely \( M \)-generated because \( A \) is finitely \( M \)-generated. Since \( \rho' \) is an epimorphism, \( B \) is finitely \( M \)-generated. \( \square \)

**Proposition 3.3.** Let \( C \) be a module such that there exists an exact sequence \( 0 \to D \to M^{(n)} \xrightarrow{\pi} C \to 0 \) with \( \pi \) an \( M \)-pure epimorphism and \( D \) finitely \( M \)-generated. If \( \rho : B \to C \) is an \( M \)-pure epimorphism with \( B \) finitely \( M \)-generated, then \( \ker \rho \) is finitely \( M \)-generated.

Proof. Consider the exact sequence

\[
0 \to D \to M^{(n)} \xrightarrow{\pi} C \to 0
\]

with \( D \) finitely \( M \)-generated. Since \( \rho \) and \( \pi \) are \( M \)-pure epimorphisms, by Lemma 3.1 we get an exact sequence

\[
0 \to D \to M^{(n)} \oplus \ker \rho \xrightarrow{\eta} B \to 0
\]

with \( \eta \) an \( M \)-pure epimorphism. From Lemma 3.2 \( \ker \rho \) is finitely \( M \)-generated. \( \square \)

Recall that a right \( R \)-module \( M \) is said to be **intrinsically projective** if for every diagram

\[
\begin{array}{ccc}
M & \xrightarrow{\gamma} & N \\
\downarrow{\alpha} & & \downarrow{\beta} \\
M^{(n)} & \xrightarrow{\rho} & 0
\end{array}
\]

with \( n > 0 \) and \( N \leq M \), there exists \( \gamma : M \to M^{(n)} \) such that \( \alpha \gamma = \beta \) (see [23]). Note that every finite \( \Sigma \)-Rickart module and every quasi-projective module is intrinsically projective. In addition, a right \( R \)-module \( M \) is intrinsically projective if and only if \( I = \text{Hom}_R(M, IM) \) for all finitely generated right ideals \( I \leq \text{End}_R(M) \) ([23], 5.7).

**Lemma 3.4.** Let \( M \) be an intrinsically projective module. Then the following statements hold true:

(i) Any epimorphism \( \rho : L \to C \), with \( C \leq M \) and \( L \) finitely \( M \)-generated, is an \( M \)-pure epimorphism.

(ii) For any finitely \( M \)-generated submodule \( N \) of \( M^{(n)} \) with any \( n \in \mathbb{N} \), \( \text{Hom}_R(M, N) \) is a finitely generated right \( S \)-module where \( S = \text{End}_R(M) \).

Proof. (i) Let \( \rho : L \to C \) be any epimorphism with \( C \leq M \) and \( L \) finitely \( M \)-generated. Let \( \alpha : M \to C \) be any homomorphism. Since \( L \) is finitely \( M \)-generated there exists an epimorphism \( \beta : M^{(n)} \to L \) for some \( n > 0 \). Since \( M \) is intrinsically projective, there exists \( \gamma : M \to M^{(n)} \) such that \( \alpha = (\rho \beta) \gamma = \rho(\beta \gamma) \).

(ii) Let \( N \leq M^{(n)} \) be finitely \( M \)-generated. Hence there exist an integer \( k > 0 \) and an epimorphism \( \rho : M^{(k)} \to N \). Let \( \ell = \max\{k, n\} \), then we can see \( \rho : M^{(\ell)} \to N \) and \( N \leq M^{(\ell)} \). Let \( \pi_i : M^{(\ell)} \to M \) denote the canonical projection for each \( 1 \leq i \leq \ell \). Let \( \varphi : M \to N \) be any homomorphism and consider the epimorphisms \( \pi_i \rho : M^{(\ell)} \to \pi_i(N) \) for \( 1 \leq i \leq \ell \). Since \( M \) is intrinsically projective there exists \( \gamma_i : M \to M^{(\ell)} \) such that
Thus, \( \text{Hom}_R(M, N) \) is generated by \( \rho_1, \rho_2, \ldots, \rho_\ell \).

For a right \( R \)-module \( M \), a right \( R \)-module \( N \) is said to be \textit{finitely \( M \)-presented} if there exists an exact sequence \( M^{(n)} \to M^{(\ell)} \to N \to 0 \) for some \( n, \ell > 0 \) ([22]).

\[ \text{Lemma 3.5.} \text{ Let } M \text{ be an intrinsically projective module and } C \text{ be a \textit{finitely \( M \)-presented} submodule of } M. \text{ If } \rho : B \to C \text{ is an epimorphism with } B \text{ \textit{finitely} \( M \)-generated}, \text{ then } \text{Ker } \rho \text{ is \textit{finitely} \( M \)-generated.} \]

\[ \text{Proof.} \text{ Since } C \text{ is \textit{finitely} \( M \)-presented there exists an exact sequence } 0 \to D \to M^{(n)} \xrightarrow{\pi} C \to 0 \]

with \( D \) \textit{finitely} \( M \)-generated. Note that \( \rho \) and \( \pi \) are \( M \)-pure epimorphisms from Lemma 3.4(i). Therefore, the result follows from Proposition 3.3. \( \square \)

\[ \text{Proposition 3.6. Let } M \text{ be an intrinsically projective module and } A, B \leq M \text{ be \textit{finitely} \( M \)-presented submodules. Consider the following exact sequence } 0 \to A \cap B \to A \oplus B \xrightarrow{\pi} A + B \to 0. \]

Then \( A + B \) is \textit{finitely} \( M \)-presented if and only if \( A \cap B \) is \textit{finitely} \( M \)-generated.

\[ \text{Proof.} \text{ Let } A \text{ and } B \text{ be \textit{finitely} \( M \)-presented submodules of } M. \text{ Then there exist epimorphisms } \rho_1 : M^{(n_1)} \to A \text{ and } \rho_2 : M^{(n_2)} \to B \text{ for some } n_1, n_2 \in \mathbb{N}. \text{ So, } \rho = \rho_1 \oplus \rho_2 : M^{(n_1)} \oplus M^{(n_2)} \to A \oplus B \text{ is an epimorphism. That is, } A \oplus B \text{ is \textit{finitely} \( M \)-generated}. \]

Suppose \( A + B \) is \textit{finitely} \( M \)-presented. Since \( M \) is intrinsically projective and \( A \oplus B \) is \textit{finitely} \( M \)-generated, \( A \cap B \) is \textit{finitely} \( M \)-generated by Lemma 3.5.

Conversely, since \( A \) and \( B \) are \textit{finitely} \( M \)-presented, there is an exact sequence

\[ 0 \to \text{Ker } \rho \to M^{(n_1)} \oplus M^{(n_2)} \xrightarrow{\rho} A \oplus B \to 0 \]

with \( \text{Ker } \rho \) \textit{finitely} \( M \)-generated and \( \rho = \rho_1 \oplus \rho_2 \) an \( M \)-pure epimorphism by Lemma 3.4(i) and Remark 2.24(iii). Consider the exact sequence

\[ 0 \to \text{Ker } \pi \rho \to M^{(n_1+n_2)} \xrightarrow{\pi \rho} A + B \to 0. \]

Note that \( \pi \) is an \( M \)-pure epimorphism by Lemma 3.4(i) because \( A \oplus B \) is \textit{finitely} \( M \)-generated. Hence from Lemma 3.1, we have an exact sequence

\[ 0 \to \text{Ker } \pi \rho \to (A \cap B) \oplus M^{(n_1+n_2)} \xrightarrow{\eta} A \oplus B \to 0 \]

with \( \eta \) an \( M \)-pure epimorphism. Since \( A \cap B \) is \textit{finitely} \( M \)-generated, \( (A \cap B) \oplus M^{(n_1+n_2)} \) is \textit{finitely} \( M \)-generated. Also since \( \text{Ker } \rho \) is \textit{finitely} \( M \)-generated, \( \text{Ker } \eta \cong \text{Ker } \pi \rho \) is \textit{finitely} \( M \)-generated by Proposition 3.3. This implies that \( A + B \) is \textit{finitely} \( M \)-presented. \( \square \)

\[ \text{Definition 3.7. Let } M \text{ be a right } R \text{-module and } N \text{ be a \textit{finitely} \( M \)-generated module. The module } N \text{ is called \textit{\( M \)-coherent} if for any } n > 0 \text{ and every homomorphism } \rho : M^{(n)} \to N, \text{Ker } \rho \text{ is \textit{finitely} \( M \)-generated.} \]

Remark that if a right \( R \)-module \( M \) is \( M \)-coherent then \( sM \) is flat where \( S = \text{End}_R(M) \). Also, a ring \( R \) is said to be \textit{right coherent} if \( R_R \) is \( R \)-coherent. In addition, \( M \) is a coherent right \( R \)-module if and only if \( M \) is an \( R \)-coherent right \( R \)-module ([10] 4G]).
Lemma 3.8. Let $M$ be an $M$-coherent module. If $\rho : B \to C$ is an epimorphism with $B$ finitely $M$-generated and $C \leq M$, then $\ker \rho$ is finitely $M$-generated.

Proof. Let $\rho : B \to C$ be an epimorphism with $B$ finitely $M$-generated and $C \leq M$. Hence there exists an epimorphism $\pi : M^{(n)} \to B$ for some $n > 0$. Since $M$ is $M$-coherent, $\ker \rho \pi = \pi^{-1}(\ker \rho)$ is finitely $M$-generated. Since $\ker \rho$ is a factor module of $\ker \rho \pi$, $\ker \rho$ is finitely $M$-generated. \hfill \Box

Proposition 3.9. $M$ is an $M$-coherent module if and only if every finitely generated submodule of $M$ is finitely $M$-presented.

Proof. It directly follows from Lemma 3.8 and the definition of an $M$-coherent module. \hfill \Box

Theorem 3.10. Consider the following conditions for a module $M$:

(i) $M$ is an $M$-coherent module.

(ii) The intersection of two finitely $M$-generated submodules of $M$ is finitely $M$-generated and $\ker \varphi$ is finitely $M$-generated for all $\varphi \in \operatorname{End}_R(M)$.

Then (i)$\Rightarrow$(ii) holds. In addition, if $M$ is intrinsically projective then the two conditions are equivalent.

Proof. (i)$\Rightarrow$(ii) By the definition of an $M$-coherent module, $\ker \varphi$ is finitely $M$-generated for every $\varphi \in \operatorname{End}_R(M)$. Now, let $A$ and $B$ be finitely $M$-generated submodules of $M$. Hence $A \oplus B$ is finitely $M$-generated. Consider the natural exact sequence

$$0 \to A \cap B \to A \oplus B \to A + B \to 0.$$ 

It follows from Lemma 3.8 that $A \cap B$ is finitely $M$-generated.

In addition, suppose $M$ is intrinsically projective. For (ii)$\Rightarrow$(i), we are going to prove by induction on $n$ that the kernel of any homomorphism $\varphi : M^{(n)} \to M$ is finitely $M$-generated. If $n = 1$, the kernel of any $\varphi \in \operatorname{End}_R(M)$ is finitely $M$-generated by hypothesis. Suppose $n > 1$ and for all homomorphisms $\rho : M^{(\ell)} \to M$ with $\ell < n$, $\ker \rho$ is finitely $M$-generated. Let $\psi : M^{(n)} \to M$ be any homomorphism. We have that $\operatorname{Im} \psi = \psi(0 \oplus M^{(n-1)}) + \psi(M \oplus 0)$. Since $\ker(\psi|_{\operatorname{End}_R(M^{(n-1)})})$ is finitely $M$-generated by the induction hypothesis and $\ker(\psi|_{\operatorname{End}_M(0)})$ is finitely $M$-presented, $\psi(0 \oplus M^{(n-1)})$ and $\psi(M \oplus 0)$ are finitely $M$-presented. By hypothesis, $\psi(0 \oplus M^{(n-1)}) \cap \psi(M \oplus 0)$ is finitely $M$-generated. It follows from Proposition 3.6 that $\operatorname{Im} \psi = \psi(0 \oplus M^{(n-1)}) + \psi(M \oplus 0)$ is finitely $M$-presented. Hence $\ker \psi$ is finitely $M$-generated by Lemma 3.5. Thus, $M$ is an $M$-coherent module. \hfill \Box

Corollary 3.11 (\cite[Corollary 4.60]{10}). A ring $R$ is a right coherent ring if and only if the intersection of two finitely generated ideals of $R$ is finitely generated and $r_R(a)$ is finitely generated for all $a \in R$.

Corollary 3.12. Let $M$ be an intrinsically projective Rickart module. Then $M$ is $M$-coherent if and only if the intersection of two finitely $M$-generated submodules of $M$ is finitely $M$-generated.

Chase (\cite[4.60]{10}) shows that a domain $R$ is right coherent if and only if the intersection of two finitely generated right ideals of $R$ is finitely generated. In the next result, we extend to a right Rickart ring.

Corollary 3.13. A right Rickart ring $R$ is right coherent if and only if the intersection of two finitely generated right ideals of $R$ is finitely generated.

Lemma 3.14 (\cite[15.9]{22}). The following conditions are equivalent for a module $M$:

(a) $SM$ is flat where $S = \operatorname{End}_R(M)$;

(b) for any homomorphism $\rho : M^{(n)} \to M^{(k)}$ with $n, k > 0$, $\ker \rho$ is $M$-generated;
(c) for any homomorphism $\rho : M^{(n)} \to M$ with $n > 0$, $\text{Ker} \rho$ is $M$-generated.

Note that if a module $N$ is $M$-generated then $N = \text{Hom}_R(M, N)M$. For an intrinsically projective module $M$, there is another characterization when $M$ is $M$-coherent as well as Theorem 3.10.

**Theorem 3.15.** Consider the following conditions for a module $M$:

(i) $S$ is a right coherent ring and $S^M$ is flat.

(ii) $M$ is an $M$-coherent module.

Then (i) implies (ii). In addition, if $M$ is intrinsically projective then the two conditions are equivalent.

**Proof.** (i)$\Rightarrow$(ii) Let $N \leq M$ be finitely $M$-generated. Consider the exact sequence $0 \to K \to M^{(n)} \xrightarrow{\rho} N \to 0$. Applying $\text{Hom}_R(M, \_)$, we get

$$0 \to \text{Hom}_R(M, K) \to \text{Hom}_R(M, M^{(n)}) \xrightarrow{\rho^*} \text{Hom}_R(M, N).$$

Since $N \leq M$, $\text{Hom}_R(M, N)$ embeds in $S$. Note that $\text{Im} \rho_*$ is finitely generated as a right $S$-module. This implies that $\text{Hom}_R(M, K)$ is finitely generated as a right $S$-module because $S$ is a right coherent ring. Hence there exists an epimorphism $S^{(\ell)} \to \text{Hom}_R(M, K)$ for some $\ell > 0$. Note that $K$ is $M$-generated because $S^M$ is flat from Lemma 3.14. Applying $\_ \otimes_S M$,

$$S^{(\ell)} \otimes_S M \xrightarrow{\cong} \text{Hom}_R(M, K) \otimes_S M \xrightarrow{\cong} M^{(\ell)} \xrightarrow{\cong} K \xrightarrow{\cong} 0.$$

Thus $K$ is finitely $M$-generated. This implies that $M$ is $M$-coherent.

In addition, suppose $M$ is intrinsically projective. For (ii)$\Rightarrow$(i), it is easy to see that $S^M$ is flat from Lemma 3.14. Let $I$ be a finitely generated right ideal of $S$. Then there is an exact sequence

$$0 \to \text{Ker} \eta \to S^{(n)} \xrightarrow{\eta} I \to 0$$

for some $n > 0$. It is enough to show that $J = \text{Ker} \eta$ is finitely generated as a right $S$-module. Applying the functor $\_ \otimes_S M$, since $S^M$ is flat we get

(3.2) $$0 \longrightarrow J \otimes_S M \longrightarrow S^{(n)} \otimes_S M \xrightarrow{\eta \otimes 1} I \otimes_S M \longrightarrow 0$$

where $\alpha, \beta$ are the canonical homomorphisms and $J' = \text{Ker} \beta(\eta \otimes 1)\alpha^{-1}$. Since $M$ is intrinsically projective and $IM \leq M$, the functor $\text{Hom}_R(M, \_)$ is exact in (3.2). Therefore, the following diagram has exact rows:

$$0 \longrightarrow J \longrightarrow S^{(n)} \longrightarrow I \longrightarrow 0$$

$0 \longrightarrow \text{Hom}_R(M, J \otimes_S M) \longrightarrow \text{Hom}_R(M, S^{(n)} \otimes_S M) \longrightarrow \text{Hom}_R(M, I \otimes_S M) \longrightarrow 0$

$$0 \longrightarrow \text{Hom}_R(M, J') \longrightarrow \text{Hom}_R(M, M^{(n)}) \longrightarrow \text{Hom}_R(M, IM) \longrightarrow 0.$$
Hence $J \cong \text{Hom}_R(M, J')$. Since $M$ is $M$-coherent, $J'$ is finitely $M$-generated. Therefore $J$ is a finitely generated right $S$-module by Lemma 3.14(ii).

**Corollary 3.16.** The following are equivalent for an intrinsically projective module $M$:

(a) $M$ is an $M$-coherent module;
(b) $S = \text{End}_R(M)$ is a right coherent ring and $SM$ is flat;
(c) The intersection of two finitely $M$-generated submodules of $M$ is finitely $M$-generated and $\ker \varphi$ is finitely $M$-generated for all $\varphi \in \text{End}_R(M)$.

**Proposition 3.17.** Let $M$ be a finite $\Sigma$-Rickart module. Then the following statements hold true:

(i) $\text{End}_R(M)$ is a right semi-hereditary ring.
(ii) $\text{End}_R(M)$ is a right coherent ring.
(iii) Every finitely $M$-generated submodule of $M$ is $M$-coherent.

**Proof.** (i) Since $M^{(n)}$ is Rickart, $\text{Mat}_n(S)$ is a right Rickart ring for all $0 < n \in \mathbb{N}$ with $S = \text{End}_R(M)$. Then $S$ is a right semi-hereditary ring by [21 Proposition]. (ii) It is trivial (see Lemma 3.19).

(iii) Let $N$ be a finitely $M$-generated submodule of $M$ and let $\varphi : M^{(n)} \to N$ be any homomorphism. Since $N \leq M$ and $M$ is finite $\Sigma$-Rickart, $\ker \varphi \leq \oplus M^{(n)}$. Hence $\ker \varphi$ is finitely $M$-generated.

**Lemma 3.18 ([22 39.10(2)]).** If $S = \text{End}_R(M)$ is a right Rickart ring then $r_S(\varphi)M \leq \oplus M$ for all $\varphi \in S$.

**Lemma 3.19 (Chase [3 Theorem 4.1]).** A ring $R$ is right semi-hereditary if and only if $R$ is a right coherent ring and all right ideals of $R$ are flat.

As a finitely generated $\Sigma$-Rickart module is characterized in terms of its endomorphism ring (Theorem 3.20), we obtain the characterization of a finite $\Sigma$-Rickart module using its endomorphism ring.

**Theorem 3.20.** The following conditions are equivalent for a module $M$ and $S = \text{End}_R(M)$:

(a) $M$ is a finite $\Sigma$-Rickart module;
(b) $S$ is a right semi-hereditary ring and $SM$ is flat;
(c) $M$ is an intrinsically projective $M$-coherent module and all right $S$-ideals are flat.

**Proof.** (a)$\Rightarrow$(b) It follows from Proposition 3.17(i) and Lemma 3.14

(b)$\Rightarrow$(c) Since $S$ is a right semi-hereditary ring, $S$ is a right coherent ring and every right $S$-ideal is flat by Lemma 3.19. It follows from [24 Examples 5.6(2)] that $M$ is intrinsically projective. From Theorem 3.15 $M$ is $M$-coherent because $SM$ is flat.

(c)$\Rightarrow$(a) Since $M$ is intrinsically projective and $M$-coherent, by Theorem 3.15 $S$ is a right coherent ring and $SM$ is flat. From Lemma 3.19 $S$ is a right semi-hereditary ring because all right $S$-ideals are flat. Let $\varphi : M^{(n)} \to M^{(n)}$ be any endomorphism. Since $SM$ is flat as above, $\ker \varphi$ is $M$-generated by Lemma 3.14. Hence $\ker \varphi = \text{Hom}_R(M^{(n)}, \ker \varphi)M^{(n)} = r_{\text{Mat}_n(S)}(\varphi)M^{(n)} \leq \oplus M^{(n)}$ from Lemma 3.18. Therefore $M$ is a finite $\Sigma$-Rickart module. □

The "$SM$ is flat" condition in (b)$\Rightarrow$(a) is not superfluous as shown next.

**Example 3.21.** (i) Consider $Z_p\infty$ as a $Z$-module. Then $S = \text{End}_Z(Z_p\infty)$ is a right semi-hereditary ring. But $Z_p\infty$ is neither a finite $\Sigma$-Rickart $Z$-module nor a flat left $S$-module.

(ii) The $Z$-module $Z_4$ is $Z_4$-coherent, however $Z_4$ is not finite $\Sigma$-Rickart.

An explicit application of Theorem 3.20 is exhibited in the next example.
Example 3.22. Let $R$ be the ring of $n \times n$ upper triangular matrices over a right semi-hereditary ring $A$. Let $e \in R$ be a unit matrix with 1 in the $(1,1)$-position and 0 elsewhere. Then $\text{End}_R(eR) \cong A$ and $eR \cong A^n$ as projective left $A$-modules. Therefore $eR$ is a finite $\Sigma$-Rickart module by Theorem 3.20. For example, while $R = \left( \begin{array}{cc} 0 & Z \\ Z & 0 \end{array} \right)$ is not a right hereditary ring, $eR = \left( \begin{array}{cc} Z & 0 \\ 0 & Z \end{array} \right)$ is a finite $\Sigma$-Rickart $R$-module for $e = (1 \, 0)$. 

Since every finitely generated projective module over a right semi-hereditary ring is a finite $\Sigma$-Rickart module, its endomorphism ring is a right semi-hereditary ring as a consequence of Theorem 3.20.

Corollary 3.23. The following statements hold true:

(i) (\cite{[4]} Theorem 2.10) If $R$ is a right semi-hereditary ring and $P$ is a finitely generated projective $R$-module, then $\text{End}_R(P)$ is a right semi-hereditary ring.

(ii) If $R$ is a right semi-hereditary ring, so is $eRe$ for any idempotent $e \in R$.

4. Applications

Proposition 4.1. Let $M$ be a right $R$-module with $S = \text{End}_R(M)$ such that $SM$ is flat. Then the following equivalences hold true:

(i) A right $R$-module $A$ is in $\mathfrak{E}_M$ iff $\text{Hom}_R(M, A)$ is an injective right $S$-module.

(ii) A right $R$-module $A$ is in $\mathfrak{F}_M$ iff $\text{Hom}_R(M, A)$ is an $f$-injective right $S$-module.

Proof. (i) Suppose $A \in \mathfrak{E}_M$. Let $I_S$ be a right ideal of $S$ and let $\alpha : I \to \text{Hom}_R(M, A)$ be any $S$-homomorphism. Hence we have the following diagram of right $R$-modules

$$
\begin{array}{cccc}
0 & \longrightarrow & I \otimes_S M & \longrightarrow & S \otimes_S M \cong M \\
& & \alpha \otimes 1 & \downarrow & \ \\
& & \text{Hom}_R(M, A) \otimes_S M \downarrow j & \ \\
& & A \ \\
\end{array}
$$

where $\iota : I_S \to S$ is the canonical inclusion and $j : \text{Hom}_R(M, A) \otimes_S M \to A$ is given by $j(f \otimes m) = f(m)$. Note that $I \otimes_S M$ is $M$-generated and since $SM$ is flat, $\iota \otimes 1$ is a monomorphism. Let $\theta$ denote the canonical isomorphism $S \otimes_S M \to M$. By the definition of $\mathfrak{E}_M$, there exists an $R$-homomorphism $g : M \to A$ such that $g\theta(\iota \otimes 1) = j(\alpha \otimes 1)$. Define $\tilde{\alpha} : S \to \text{Hom}_R(M, A)$ as $(\tilde{\alpha}(f))(m) = gf(m)$ for $f \in S$. Let $h \in I$ and $m \in M$. Hence

$$(\tilde{\alpha}(h))(m) = gh(m) = g(\theta(h \otimes m)) = (g\theta(\iota \otimes 1))(h \otimes m)$$

$$= j(\alpha \otimes 1)(h \otimes m) = j(\alpha(h) \otimes m) = (\alpha(h))(m).$$

This implies that $\tilde{\alpha}(h) = \alpha(h)$ for all $h \in I$. Thus, $\text{Hom}_R(M, A)$ is an injective right $S$-module.

Conversely, let $A_R$ be an $R$-module such that $\text{Hom}_R(M, A)$ is an injective right $S$-module. Let $N$ be an $M$-generated submodule of $M$ and $f : N \to A$ be an $R$-homomorphism. Hence we have the following diagram of right $S$-modules

$$
\begin{array}{cccc}
0 & \longrightarrow & \text{Hom}_R(M, N) & \longrightarrow & S \\
& & f_* & \downarrow & \ \\
& & \text{Hom}_R(M, A) \downarrow \alpha & \ \\
\end{array}
$$
where \( i : N \to M \) is the canonical inclusion. By hypothesis there exists an \( S \)-homomorphism \( \alpha : S \to \text{Hom}_R(M, A) \) such that \( \alpha_i = f_i \). Define \( \bar{\alpha} : M \to A \) as \( \bar{\alpha}(m) = (\alpha(\text{Id}_M))(m) \). Let \( n \in N \) be arbitrary. Since \( N \) is \( M \)-generated, \( n = \sum_{i=1}^k g_i(m_i) \) with \( g_i \in \text{Hom}_R(M, N) \) and \( m_i \in M \). Then

\[
\bar{\alpha}(n) = (\alpha(\text{Id}_M))(i(n)) = (\alpha(\text{Id}_M))i \left( \sum_{i=1}^k g_i(m_i) \right) = \sum_{i=1}^k (\alpha(\text{Id}_M))i g_i(m_i) = \sum_{i=1}^k (\alpha(i g_i))(m_i)
\]

because \( g_i \in S \). This implies that \( f = \bar{\alpha}i \). Thus, \( A_R \) is in \( \mathfrak{E}_M \).

(ii) The proof is similar to that of (i). Note that if \( I_S \) is a finitely generated ideal of \( S \), then \( I \otimes_S M \) is a finitely \( M \)-generated right \( R \)-module. \( \square \)

**Remark 4.2.** (i) For any endoregular module \( M \), \( \text{Hom}_R(M, A) \) is an \( f \)-injective right \( S \)-module for any module \( A \).

(ii) ([15, Theorem 5 and Corollary 2]) Every module over a von Neumann regular ring is \( f \)-injective.

**Corollary 4.3.** Let \( M \) be a right \( R \)-module with \( S = \text{End}_R(M) \) such that \( S \) is flat. Then the following equivalences hold true:

(i) \( M \in \mathfrak{E}_M \) if and only if \( S \) is a right self-injective ring.

(ii) \( M \in \mathfrak{F}_M \) if and only if \( S \) is a right \( f \)-injective ring.

Here we have an alternative proof of Theorem 2.29.

**Corollary 4.4.** The following conditions are equivalent for a right \( R \)-module \( M \):

(a) \( M \) is a finite \( \Sigma \)-Rickart module and \( M \in \mathfrak{F}_M \);

(b) \( \text{End}_R(M) \) is a von Neumann regular ring.

**Proof.** The proof follows from Theorems 2.29 and 3.20 and Corollaries 2.30 and 4.3(ii). \( \square \)

**Corollary 4.5.** The following conditions are equivalent for a finitely generated module \( M \):

(a) \( M \) is a \( \Sigma \)-Rickart module and \( M \in \mathfrak{E}_M \);

(b) \( \text{End}_R(M) \) is semisimple artinian.

**Proof.** (a)\(\Rightarrow\)(b) By [11] Theorem 4.6] \( S = \text{End}_R(M) \) is a right hereditary ring and \( S \) is flat. It follows from Corollary 4.3(i) that \( S \) is right self-injective. Thus, \( S_S \) is semisimple artinian by [7] Corollary]. (b)\(\Rightarrow\)(a) Since \( S \) is von Neumann regular, from Theorems 2.29 and 3.20 \( S \) is flat. So, the proof follows from [11] Theorem 4.6] that \( M \) is \( \Sigma \)-Rickart and from Corollary 4.3(i) that \( M \in \mathfrak{E}_M \). \( \square \)

Continuing the study of the endomorphism ring of a finite \( \Sigma \)-Rickart module, we study the case when \( \text{End}_D(M) \) is a semiprimary ring (Theorem 4.10). Recall that a ring \( R \) is said to be semiprimary if its Jacobson radical, \( \text{Rad} R \), is nilpotent and \( R/\text{Rad} R \) is a semisimple artinian ring.

Recall that a ring \( R \) is called a PWD (piecewise domain) if it possesses a complete set \( \{e_1, \ldots, e_n\} \) of orthogonal idempotents such that \( xy = 0 \) implies \( x = 0 \) or \( y = 0 \) whenever \( x \in e_i Re_k \) and \( y \in e_k Re_j \) (see [7]).

**Theorem 4.6.** The following conditions are equivalent for a module \( M \):

(a) \( M \) has a decomposition \( \bigoplus_{i=1}^n H_i^{(e_i)} \) with \( H_i \) an indecomposable endoregular module, \( H_i \) is \( H_i \)-Rickart for all \( 1 \leq i, j \leq n \), and \( H_i \not\cong H_j \) for \( i \neq j \);
(b) $S = \text{End}_R(M)$ is isomorphic to a upper triangular matrix ring

$$
\begin{pmatrix}
\text{Mat}_{\ell_1}(D_1) & V_{12} & V_{13} & \cdots & V_{1n} \\
0 & \text{Mat}_{\ell_2}(D_2) & V_{23} & \cdots & V_{2n} \\
0 & 0 & \text{Mat}_{\ell_3}(D_3) & \cdots & V_{3n} \\
\vdots & \vdots & \vdots & \ddots & \vdots \\
0 & 0 & 0 & \cdots & \text{Mat}_{\ell_n}(D_n)
\end{pmatrix}
$$

where $D_i$ is a division ring for $1 \leq i \leq n$, and $V_{ij}$ is a $\text{Mat}_{\ell_i}(D_i)\cdot \text{Mat}_{\ell_j}(D_j)$-bimodule for all $1 \leq i < j \leq n$ satisfying $I_S(x) \cap V_{ij} = 0$ for any $0 \neq x \in H_j$. In particular, $S$ is a semiprimary PWD.

**Proof.** (a)$\Rightarrow$(b) Since each $H_i$ is indecomposable endoregular and $H_i$ is $H_j$-Rickart module, every nonzero homomorphism $\rho : H_i \rightarrow H_j$ is a monomorphism. This implies that $S$ is a PWD \cite{7}. On the other hand, since $H_i \not\cong H_j$ for all $1 \leq i \neq j \leq n$, $\text{Hom}_R(H_i, H_j) = 0$ or $\text{Hom}_R(H_j, H_i) = 0$ from \cite{24} Proposition 18. Without loss of generality we can assume that the decomposition $M = \bigoplus_{i=1}^{n} H_i$ is such that $\text{Hom}_R(H_i, H_j) = 0$ for all $i < j$. Consider the complete set of orthogonal primitive idempotents $\{e_1, \ldots, e_n\}$ of $S$ such that $H_i = e_i M$. It follows from \cite{24} Main Theorem that

$$
S = \begin{pmatrix}
P_1 & V_{12} & \cdots & V_{1n} \\
0 & P_2 & \cdots & V_{2n} \\
\vdots & \vdots & \ddots & \vdots \\
0 & 0 & \cdots & P_n
\end{pmatrix}
$$

where $V_{ij}$ is a $P_i \cdot P_j$-bimodule and

$$
P_i = \begin{pmatrix}
D_i & W_{12} & \cdots & W_{1\ell_i} \\
W_{21} & D_i & \cdots & W_{2\ell_i} \\
\vdots & \vdots & \ddots & \vdots \\
W_{\ell_i1} & W_{\ell_i2} & \cdots & D_i
\end{pmatrix}
$$

with $D_i$ a division ring and each $W_{jk} = D_i$ as $D_i \cdot D_i$-bimodule. That is, $P_i \cong \text{Mat}_{\ell_i}(D_i)$. Suppose that $\text{Hom}_R(H_j, H_i) \neq 0$ with $1 \leq i < j \leq n$. It follows from \cite{13} Corollary 2.10 that $H_j$ is $H_i$-Rickart. Therefore, every nonzero homomorphism in $\text{Hom}_R(H_j, H_i)$ is a monomorphism. Let $0 \neq x \in H_j$ and $\varphi \in S$ such that $\varphi \in I_S(x) \cap V_{ij}$. Then $\varphi(x) = 0$. If $V_{ij} = 0$, there is nothing to prove. Suppose that $V_{ij} = \text{Hom}_R(H_j^{(\ell_i)}, H_i^{(\ell_i)}) \neq 0$. Assume that $\varphi \neq 0$. Since $\varphi|_{H_j}$ is a monomorphism by the above comment, $x = 0$, a contradiction.

Hence $\varphi = 0$. Therefore $I_S(x) \cap V_{ij} = 0$. Note that $\text{Rad}(S) = \begin{pmatrix}
0 & V_{12} & \cdots & V_{1n} \\
0 & 0 & \cdots & V_{2n} \\
\vdots & \vdots & \ddots & \vdots \\
0 & 0 & \cdots & 0
\end{pmatrix}$ is nilpotent and $S/\text{Rad}(S) \cong \text{Mat}_{\ell_1}(D_1) \times \cdots \times \text{Mat}_{\ell_n}(D_n)$ which is a semisimple artinian ring. Hence $S$ is a semiprimary ring.

(b)$\Rightarrow$(a) Let $\{e_{ij} \mid 1 \leq i, j \leq m\}$ denote the matrix units where $m = \ell_1 + \cdots + \ell_n$. Hence $M$ has a decomposition $M = e_{11} M \oplus \cdots \oplus e_{mm} M$. Denote $H_i = e_{ii} M$. Since $\text{End}_R(H_i) \cong e_{ii} S e_{ii} \cong D_i$ is a division ring, $H_i$ is an indecomposable endoregular $R$-module for all $1 \leq i \leq m$. By hypothesis, $H_j \cong H_k$ for $m_i - 1 < j, k \leq m_i$ where $m_i = \sum_{k=0}^{i} \ell_k$ and $\ell_0 = 0$ and for each $1 \leq i \leq n$. Hence $M = M_1 \oplus \cdots \oplus M_n$ where $M_i = \bigoplus_{k=m_{i-1}+1}^{m_i} H_k \cong H_i^{(\ell_i)}$. Without loss of generality we can take a summand of each $M_i$ and assume that $M = H_i^{(\ell_i)} \oplus \cdots \oplus H_n^{(\ell_n)}$. Let $\rho : H_j \rightarrow H_i$ be any nonzero homomorphism for $1 \leq i < j \leq n$. Assume that $0 \neq x \in H_j$ such that $\rho(x) = 0$. Consider $\rho \oplus 0 : H_j^{(\ell_j)} \rightarrow H_i^{(\ell_i)}$. Then
\((\rho \oplus 0)(x) = 0\). This implies that \((\rho \oplus 0) \in \mathcal{S}(x) \cap V_{ij} = 0\), a contradiction. Therefore \(x = 0\). This implies that \(\rho\) is a monomorphism. Hence it is easy to see that \(H_i\) is \(H_j\)-Rickart for all \(1 \leq i, j \leq n\). \(\square\)

**Remark 4.7.** If \(M\) is a finite direct sum of indecomposable endoregular modules, then \(\text{End}_R(M)\) is a semi-perfect ring.

**Corollary 4.8.** Suppose that \(M = \bigoplus_{i=1}^{n} H_i\) with \(H_i\) an indecomposable endoregular module, \(H_i\) is \(H_j\)-Rickart for all \(1 \leq i, j \leq n\) and \(H_i \neq H_j\) for \(i \neq j\). If there exists an ordering \(\mathcal{I}_n = \{1, 2, \ldots, n\}\) for the class \(\{H_i\}_{i \in \mathcal{I}_n}\), such that \(H_i\) is \(H_j\)-injective for all \(i < j\), then \(M\) is a Rickart module and \(\text{End}_R(M)\) is isomorphic to a upper triangular matrix ring

\[
\begin{pmatrix}
D_1 & V_{12} & V_{13} & \cdots & V_{1n} \\
0 & D_2 & V_{23} & \cdots & V_{2n} \\
0 & 0 & D_3 & \cdots & V_{3n} \\
\vdots & \vdots & \vdots & \ddots & \vdots \\
0 & 0 & 0 & \cdots & D_n
\end{pmatrix}
\]

where \(D_i\) is a division ring for \(1 \leq i \leq n\), and \(V_{ij}\) is a \(D_i-D_j\)-bimodule for all \(1 \leq i < j \leq n\). In particular, \(\text{End}_R(M)\) is a semiprimary PWD.

**Proof.** It follows directly from [13, Corollary 2.13] and Theorem 4.6. \(\square\)

The next example illustrates Theorem 4.6 and Corollary 4.8.

**Example 4.9.** Let \(F\) be a field and \(R = \left( \begin{array}{cc} F & F \\ 0 & F \end{array} \right)\). Consider the right \(R\)-module

\[
M = \left( \begin{array}{cc} F & F \\ 0 & 0 \end{array} \right) \oplus \left( \begin{array}{cc} 0 & 0 \\ F & 0 \end{array} \right) \oplus \left( \begin{array}{cc} 0 & 0 \\ 0 & F \end{array} \right) = \left( \begin{array}{cc} F & F \\ 0 & 0 \end{array} \right) \oplus \left( \begin{array}{cc} 0 & 0 \\ 0 & F \end{array} \right)^{(2)}.
\]

Denote \(H_1 = \left( \begin{array}{cc} F & F \\ 0 & 0 \end{array} \right)\) and \(H_2 = \left( \begin{array}{cc} 0 & 0 \\ F & 0 \end{array} \right)\). Then \(\text{End}_R(H_1) = F = \text{End}_R(H_2), \text{Hom}_R(H_2, H_1) = F\) and \(\text{Hom}_R(H_1, H_2) = 0\). Thus, \(M\) satisfies the condition (a) of Theorem 4.6. Hence the endomorphism ring of \(M\) is

\[
\text{End}_R(M) \cong \left( \begin{array}{cc} F & F \\ 0 & 0 \end{array} \right) \oplus \left( \begin{array}{cc} 0 & 0 \\ 0 & F \end{array} \right).
\]

Moreover, \(H_1\) is \(H_2\)-injective because \(H_2\) is simple, therefore \(M = H_1 \oplus H_2\) is a Rickart module. In particular, \(R\) is a right Rickart ring.

Inspired by the last example, we have the following proposition.

**Proposition 4.10.** Let \(M\) be a finite \(\Sigma\)-Rickart module and \(P\) be any simple module such that \(\text{Hom}_R(M, P) = 0\). Then \(M^{(\ell)} \oplus P^{(n)}\) is a finite \(\Sigma\)-Rickart module for any \(\ell, n > 0\).

**Proof.** Let \(k > 0\). Hence \((M^{(\ell)} \oplus P^{(n)})(k) = M^{(k\ell)} + P^{(kn)}\). It is clear that \(P^{(kn)}\) is \(M^{(k\ell)}\)-Rickart and by hypothesis, \(M^{(k\ell)}\) is \(P^{(kn)}\)-Rickart. Since \(P^{(kn)}\) is semisimple, \(M^{(k\ell)}\) is \(P^{(kn)}\)-injective. It follows from [13, Corollary 2.13] that \(M^{(k\ell)} \oplus P^{(kn)}\) is a Rickart module. Thus, \(M^{(\ell)} \oplus P^{(n)}\) is a finite \(\Sigma\)-Rickart module for any \(\ell, n > 0\). \(\square\)

**Remark 4.11.** It follows from Proposition 4.10 that the module \(M\) in Example 4.9 is not just Rickart, but finite \(\Sigma\)-Rickart. In particular, the ring \(R = \left( \begin{array}{cc} F & F \\ 0 & F \end{array} \right)\) is right hereditary.

**Corollary 4.12.** The following conditions are equivalent for a ring \(R\):

(a) \(R = \bigoplus_{i=1}^{n} I_i^{(r_i)}\) with \(I_i\) an endoregular right ideal and \(I_i\) is \(I_j\)-Rickart for all \(1 \leq i, j \leq n\);
Proposition 4.15. Consider the following conditions for a Rickart module \( R \):

(b) \( R \) is isomorphic to a formal matrix ring

\[
\begin{pmatrix}
\operatorname{Mat}_{\ell_1}(D_1) & V_{12} & \cdots & V_{1n} \\
0 & \operatorname{Mat}_{\ell_2}(D_2) & \cdots & V_{2n} \\
\vdots & \vdots & \ddots & \vdots \\
0 & 0 & \cdots & \operatorname{Mat}_{\ell_n}(D_n)
\end{pmatrix}
\]

where \( D_i \) is a division ring for \( 1 \leq i \leq n \), and for all \( 1 \leq i < j \leq n \), \( V_{ij} \) is a \( \operatorname{Mat}_{\ell_i}(D_i)\)-\( \operatorname{Mat}_{\ell_j}(D_j) \)-bimodule satisfying \( I_R(x) \cap V_{ij} = 0 \) for any \( 0 \neq x \in I_j \). In particular, \( R \) is a semiprimary PWD.

To illustrate the last results, we have the following examples:

Example 4.13. (i) Let \( K \) be a field. It follows from [11, Ch.I Lemma 1.12 and Ch. VII Theorem 1.7] that every path algebra \( KQ \) of a finite, connected and acyclic quiver \( Q \) satisfies the conditions of Corollary 4.12.

(ii) Let \( K \) and \( F \) be division rings and \( U \) be any left \( K \)-right \( F \)-bimodule. Then the formal matrix ring \( R = \left( \begin{array}{cc} K & U \\ 0 & F \end{array} \right) \) trivially satisfies the condition (b) of Corollary 4.12. Moreover, the decomposition \( R = \left( \begin{array}{cc} K & 0 \\ 0 & F \end{array} \right) \oplus \left( \begin{array}{cc} 0 & 0 \\ 0 & 0 \end{array} \right) \) makes \( R \) to be a hereditary semiprimary PWD by Corollary 4.12 and Proposition 4.10.

(iii) Let \( K \) be a field and consider the ring \( R = \left( \begin{array}{ccc} K & 0 & 0 \\ K & K & 0 \\ 0 & 0 & K \end{array} \right) \). Then \( R \) has a decomposition in hollow endoregular right ideals

\[
R = \left( \begin{array}{ccc}
0 & 0 & 0 \\
K & K & 0 \\
0 & 0 & K
\end{array} \right) \oplus \left( \begin{array}{ccc}
K & 0 & 0 \\
0 & 0 & 0 \\
0 & 0 & 0
\end{array} \right) \oplus \left( \begin{array}{ccc}
0 & 0 & 0 \\
0 & 0 & 0 \\
0 & 0 & K
\end{array} \right).
\]

Denote those summands by \( I_1, I_2 \) and \( I_3 \) from the left to the right respectively. Hence \( I_2 \) and \( I_3 \) are simple \( R \)-modules. By Corollary 4.12 and applying Proposition 4.10 twice, we have that \( R \) is a (semi-)hereditary semiprimary PWD. Moreover by Corollary 4.12(b), \( R \cong \left( \begin{array}{ccc} K & 0 & 0 \\ K & K & 0 \\ 0 & 0 & K \end{array} \right) \) where the isomorphism is given by \( \left( \begin{array}{ccc}
a & 0 & 0 \\
b & c & d \\
e & 0 & 0
\end{array} \right) \mapsto \left( \begin{array}{ccc}
a & 0 & 0 \\
c & 0 & d \\
e & 0 & 0
\end{array} \right) \).

(iv) Let \( K \) be a field and \( K[x] \) be the polynomial ring with coefficients in \( K \). Consider the ring \( R = \left( \begin{array}{ccc} K & 0 \\ K & K[x]/(x) \\ 0 & 0 \end{array} \right) \) where \( (x) \) is the ideal generated by \( x \). Hence \( R \) has a decomposition in indecomposable endoregular ideals

\[
R = \left( \begin{array}{ccc} K & 0 \\ 0 & 0 \\ 0 & 0 \end{array} \right) \oplus \left( \begin{array}{ccc} 0 & 0 & 0 \\ 0 & K & 0 \\ 0 & 0 & 0 \end{array} \right) \oplus \left( \begin{array}{ccc} 0 & 0 & 0 \\ 0 & 0 & K \\ 0 & 0 & 0 \end{array} \right).
\]

Let \( I_1, I_2, I_3 \) denote those summand from the left to the right respectively. Let \( 0 \neq f(x) \in K[x]/(x) \) then \( \left( \begin{array}{ccc} 0 & x & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{array} \right) \in I_R \left( \begin{array}{ccc} 0 & 0 & 0 \\ 0 & 0 & f(x) \\ 0 & 0 & 0 \end{array} \right) \cap I_1 \). Thus, \( R \) does not satisfies the condition (b) in Corollary 4.12. Note that \( R \) is a semiprimary ring.

Definition 4.14 ([16, Definition 2.23]). A family of modules \( \{ M_\alpha \mid \alpha \in \Lambda \} \) is said to be locally-semi-Transfinitely-nilpotent \( (lsTn) \) if for any subfamily \( M_{\alpha_i} \) \( (i \in \mathbb{N}) \) with distinct \( \alpha_i \) and any family of non-isomorphisms \( \varphi_i : M_{\alpha_i} \to M_{\alpha_{i+1}} \), and for every \( x \in M_{\alpha_1} \), there exists \( n \in \mathbb{N} \) (depending on \( x \)) such that \( \varphi_n \cdots \varphi_2 \varphi_1(x) = 0 \).

Proposition 4.15. Consider the following conditions for a Rickart module \( M \):

(i) \( M = \bigoplus_{i=1}^n H_i \) with \( H_i \) a hollow endoregular module;

(ii) \( \operatorname{End}_R(M) \) is a semiprimary ring.

Then (i) \( \Rightarrow \) (ii). In addition, if \( M \) is finitely generated then the two conditions are equivalent.

Proof. (i) \( \Rightarrow \) (ii) Since \( M \) is a Rickart module, \( H_i \) is \( H_j \)-Rickart. It follows from Theorem 4.10 that \( S \) is semiprimary.
In addition, suppose $M$ is finitely generated. (ii)$\Rightarrow$(i) Suppose $S = \text{End}_R(M)$ is a semiprimary ring. Then $S$ is right perfect. From Lemma \[22\ 43.8\] $M$ has $D_1$ condition. Also $M$ is quasi-discrete because $M$ is Rickart. Therefore $M$ has an irredundant decomposition $M = \bigoplus_{i=1}^n H_i$ with $H_i$ a hollow module such that complements summands and is unique up to isomorphism by \[16\ Theorem 4.15\]. Since

$$M^{(N)} = \bigoplus_{i=1}^n H_i^{(N)}$$

and by \[22\ 43.8\], $\text{Rad}(M^{(N)}) \ll M^{(N)}$, the module $M^{(N)}$ is quasi-discrete and the family

$$\{H_i = H_{1j} \mid j \in \mathbb{N}\} \cup \cdots \cup \{H_n = H_{nj} \mid j \in \mathbb{N}\}$$

satisfies lsTn by \[16\ Theorem 4.53\] and Corollary 4.49. Since $H_i$ is indecomposable Rickart for all $1 \leq i \leq n$, every nonzero endomorphism $\varphi : H_i \to H_i$ is a monomorphism. By lsTn, $\varphi$ must be an isomorphism. Thus $H_i$ is an endoregular module for all $1 \leq i \leq n$. \[\Box\]

With the following examples we will show that the hypothesis on $M$ to be Rickart and on $H_i$ to be endoregular in Proposition \[4.15\] are not superfluous.

**Example 4.16.** (i) Set $M = \mathbb{Z}_4$ as $\mathbb{Z}$-module. It is clear that $\text{End}_\mathbb{Z}(M) = \mathbb{Z}_4$ is a semiprimary ring and $M$ is a hollow module. But $M$ is neither Rickart nor endoregular.

(ii) Let $R = \mathbb{Z}(p)$ be the localization of integers at a prime $p$ and set $M = R_R$. Then $M$ is a finitely generated Rickart module. Note that $M$ is a hollow $R$-module but is not endoregular. Moreover $R = \text{End}_R(M)$ is not a semiprimary ring because $\text{Rad}(\mathbb{Z}(p)) = p\mathbb{Z}(p)$.

It follows from an Auslander’s result \[10\ 5.72\] that a semiprimary right semi-hereditary ring is right and left hereditary. The corollaries below give an extension of Auslander’s result for the case of $\Sigma$-Rickart and fintie $\Sigma$-Rickart modules.

**Corollary 4.17.** Let $M$ be a finite $\Sigma$-Rickart module. If $M = \bigoplus_{i=1}^n H_i$ with $H_i$ a hollow endoregular module then $\text{End}_R(M^{(\ell)}) \cong \text{Mat}_\ell(S)$ is a semiprimary (right) hereditary PWD for all $\ell > 0$ where $S = \text{End}_R(M)$.

**Proof.** Let $\ell > 0$. Then $M^{(\ell)} = \bigoplus_{i=1}^n H_i^{(\ell)}$ with each $H_i$ hollow endoregular. Since $M$ is finite $\Sigma$-Rickart and finitely generated, $M^{(\ell)}$ is a Rickart module. From Theorem \[1.6\] $\text{End}_R(M^{(\ell)})$ is a semiprimary PWD. On the other hand, $\text{End}_R(M^{(\ell)}) = \text{Mat}_\ell(S)$ is a semi-hereditary ring by Theorem \[3.20\]. From \[10\ 5.72\] $\text{End}_R(M^{(\ell)})$ is a semiprimary hereditary PWD. \[\Box\]

**Corollary 4.18.** Let $M$ be a finitely generated module such that $M = \bigoplus_{i=1}^n H_i$ with $H_i$ a hollow endoregular module. The following conditions are equivalent:

(a) $M$ is a $\Sigma$-Rickart module;
(b) $M$ is a finite $\Sigma$-Rickart module.

**Proof.** (a)$\Rightarrow$(b) is clear. For, (b)$\Rightarrow$(a) $\text{End}_R(M)$ is a semiprimary hereditary ring, by Corollary \[4.17\]. It follows from \[11\ Theorem 4.6\] that $M$ is a $\Sigma$-Rickart module. \[\Box\]

**Corollary 4.19.** Consider the following conditions for a module $M$:

(i) $M$ is (finite) $\Sigma$-Rickart and $M = \bigoplus_{i=1}^n H_i$ with $H_i$ a hollow endoregular module;
(ii) $S = \text{End}_R(K)$ is a semiprimary (right) hereditary ring for every $K$ in $\text{add}(M)$ and $S_K$ is flat.

Then (i)$\Rightarrow$(ii). In addition, if $M$ is finitely generated, then the two conditions are equivalent.
Proof. (i)$\Rightarrow$(ii) Let $K \in \text{add}(M)$. Then $M^{(n)} = K \oplus L$ for some $n > 0$. Hence $K$ is a finite $\Sigma$-Rickart module. On the other hand, $M^{(n)}$ has the cancellation property, by [16, Corollary 4.20]. Therefore $K$ satisfies the hypothesis of Corollary 4.17. Thus $\text{End}_R(K)$ is a semiprimary (right) hereditary ring. (ii)$\Rightarrow$(i) follows from [11, Theorem 4.6] and Proposition 4.15. □

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