INvariants and semi-invariants in the cohomology of the complement of a reflection arrangement

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Abstract. Suppose $V$ is a finite dimensional, complex vector space, $\mathcal{A}$ is a finite set of codimension one subspaces of $V$, and $G$ is a finite subgroup of the general linear group $GL(V)$ that permutes the hyperplanes in $\mathcal{A}$. In this paper we study invariants and semi-invariants in the graded $\mathbb{Q}G$-module $H^\ast(M(\mathcal{A}))$, where $M(\mathcal{A})$ denotes the complement in $V$ of the hyperplanes in $\mathcal{A}$ and $H^\ast(\cdot)$ denotes rational singular cohomology, in the case when $\mathcal{A}$ is a reflection arrangement and the pair $(\mathcal{A}, G)$ arises from a reflection coset. The main result is the construction of an explicit, natural (from the point of view of Coxeter groups) basis of the space of invariants, $H^\ast(M(\mathcal{A}))^G$. In addition to leading to a proof of the description of the space of invariants conjectured by Felder and Veselov for Coxeter groups that does not rely on computer calculations, this construction provides an extension of this description of the space of invariants to arbitrary finite, complex reflection groups. The main result also leads to simplifications of some cohomology computations of Lehrer, Callegaro-Marín, and Marin.

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1. Introduction

1.1. Let $V$ be a finite dimensional, complex vector space. An arrangement-group pair with underlying vector space $V$ is a pair $(\mathcal{A}, G)$, where $\mathcal{A}$ is a central hyperplane arrangement in $V$, or more simply an “arrangement,” that is, $\mathcal{A}$ is a finite set of codimension one subspaces

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of $V$, and $G$ is a finite subgroup of the general linear group $\text{GL}(V)$ that permutes the hyperplanes in $\mathcal{A}$. The “complement of $\mathcal{A}$” is the open submanifold

$$M(\mathcal{A}) = V \setminus \bigcup_{H \in \mathcal{A}} H$$

of $V$. Clearly, $M(\mathcal{A})$ is a $G$-stable submanifold of $V$. Let

$$H^*(M(\mathcal{A})) = \bigoplus_{k \geq 0} H^k(M(\mathcal{A}))$$

denote the rational, singular cohomology of $M(\mathcal{A})$.

The rule $g \mapsto (g^{-1})^*$ endows $H^*(M(\mathcal{A}))$ with the structure of a graded $\mathbb{Q}G$-algebra and the space of invariants, $H^*(M(\mathcal{A}))^G$, can be compactly encoded in the \textit{Poincaré polynomial}

$$P(\mathcal{A}, G; t) = \sum_{k \geq 0} \dim \left( H^k(M(\mathcal{A}))^G \right) t^k.$$

In this paper we study invariants and semi-invariants in graded $\mathbb{Q}G$-modules $H^*(M(\mathcal{A}))$ that arise in the context of complex reflection groups, and more generally, reflection cosets.

The main results are (1) the construction of an explicit, natural (from the point of view of Coxeter groups) basis of $H^*(M(\mathcal{A}))^G$, where $(\mathcal{A}, G)$ arises as described below from a reflection coset, and (2) a short new proof that determinant-like characters do not occur in $H^*(M(\mathcal{A}(G)))^G$ for a complex reflection group $G$.

1.2. In more detail, recall that a non-identity linear transformation with finite order in $\text{GL}(V)$ is a \textit{reflection} if it fixes a hyperplane in $V$ pointwise, a finite subgroup of $\text{GL}(V)$ is a \textit{(complex or unitary) reflection group} if it is generated by reflections, and a \textit{reflection coset} is a finite subset, $C \subseteq \text{GL}(V)$, such that (1) $CC^{-1} = \{ gh^{-1} \mid g, h \in C \}$ is a reflection subgroup of $\text{GL}(V)$ that is normal in $\langle C \rangle$, the subgroup generated by $C$, and (2) the quotient $\langle C \rangle/CC^{-1}$ is a finite cyclic group. In particular, $\langle C \rangle$ is a finite subgroup of $\text{GL}(V)$, which may or may not be a reflection group.

Technically arrangements, reflection groups, and reflection cosets are pairs, namely $(\mathcal{A}, V)$, $(G, V)$, and $(C, V)$, respectively. In practice the ambient vector space is omitted when it is clear from context. Notice that the identity linear transformation is not a reflection but that the trivial subgroup of $\text{GL}(V)$ is a reflection (sub)group.

For a finite subset $S \subseteq \text{GL}(V)$, define $\mathcal{A}(S)$ to be the (possibly empty) arrangement of hyperplanes in $V$ that are fixed point sets of reflections in $S$.

$$\mathcal{A}(S) = \{ \text{Fix}(r) \mid r \in S \text{ is a reflection} \},$$

where $\text{Fix}(r) = \text{Fix}_V(r)$ is the set of fixed points of $r$ in $V$. If $S = G$ is a group, then $\mathcal{A}(G)$ is a \textit{reflection arrangement}. Obviously $\mathcal{A}(G)$ is determined by the subgroup generated by all reflections in $G$.

1.3. \textbf{Reflection pairs}. Let $C \subseteq \text{GL}(V)$ be a reflection coset. The groups $G_0 = CC^{-1}$ and $G_1 = \langle C \rangle$ both act on the reflection arrangements $\mathcal{A}_0 = \mathcal{A}(G_0)$ and $\mathcal{A}_1 = \mathcal{A}(G_1)$. In this way, $C$ gives rise to four arrangement-group pairs as in 1.1, namely

$$(\mathcal{A}(G_0), G_0), \quad (\mathcal{A}(G_0), G_1), \quad (\mathcal{A}(G_1), G_0), \quad \text{and} \quad (\mathcal{A}(G_1), G_1).$$

Notice that if $z \in \text{GL}(V)$ is a scalar transformation with finite order, then $Cz$ is also a reflection coset. Set $G_0' = (Cz)(Cz)^{-1}$ and $G_1' = \langle Cz \rangle$. Then clearly $G_0' = G_0$ and $G_1' = G_1\langle z \rangle$. If $\mathcal{A}$ is any arrangement in $V$, then $z$ acts trivially on $\mathcal{A}$ and it is easy to
check (see 2.3) that $z$ acts trivially on $H^* (M(\mathcal{A}))$ as well. Thus, with the obvious notational

convention,

$$
H^* (M(\mathcal{A}_0'))^{G_0} = H^* (M(\mathcal{A}_0'))^{G_0}, \quad H^* (M(\mathcal{A}_0'))^{G_1} = H^* (M(\mathcal{A}_0'))^{G_1},
H^* (M(\mathcal{A}_1'))^{G_0} = H^* (M(\mathcal{A}_1'))^{G_0}, \quad H^* (M(\mathcal{A}_1'))^{G_1} = H^* (M(\mathcal{A}_1'))^{G_1}.
$$

In particular, replacing $C$ by $Cz$ only has the affect of introducing one new arrangement, namely $\mathcal{A}((Cz)_0)$, but the groups of linear transformations of cohomology spaces do not change.

For the purposes of this paper we define a reflection pair to be an arrangement-group pair, $(\mathcal{A}, G)$, with the property that there is a reflection coset, say $C$, such that with the notation above,

$$
H^* (M(\mathcal{A}))^G = H^* (M(\mathcal{A}_i))^G,
$$

for some $i, j \in \{0, 1\}$. In this case, we say that the reflection pair $(\mathcal{A}, G)$ arises from $C$. Obviously a given reflection pair can arise from many reflection cosets.

Notice that if $(\mathcal{A}, G)$ is a reflection pair with underlying vector space $V$, then there is a reflection subgroup, $\tilde{G} \subseteq GL(V)$, such that $G$ normalizes $\tilde{G}$ and $\mathcal{A} = \mathcal{A}(\tilde{G})$. In particular, although $G$ might not be a reflection group, $\mathcal{A}$ is a reflection arrangement.

1.4. A reflection coset, $C \subseteq GL(V)$, is called irreducible if $CC^{-1}$ acts irreducibly on $V$. Otherwise $C$ is reducible. Obviously if $C$ is irreducible, then $\langle C \rangle$ also acts irreducibly on $V$. A reflection pair $(\mathcal{A}, G)$ is called irreducible if it arises from an irreducible reflection coset.

Irreducible reflection cosets have been classified, up to multiplication by a scalar transformation, by Broué, Malle, and Michel [4]. This classification is recalled in 4.3. As indicated above, the classification of irreducible reflection pairs is somewhat more involved. This classification is given in Corollary 4.12 and 4.13.

1.5. With these preliminaries in hand, we can more precisely formulate the main results in the paper. First, the Poincaré polynomials for all reflection pairs can be determined from Theorem 5.5 (see Corollary 5.6). Next, explicit bases of $H^k (M(\mathcal{A}))^G$, when $(\mathcal{A}, G)$ is an irreducible reflection pair, are constructed in Theorem 7.4. Finally, an immediate consequence of Theorem 9.1 is that certain linear characters do not occur in $H^* (M(\mathcal{A}(G)))$ for any complex reflection group $G$.

1.6. **Poincaré polynomials and prior work.** The computation of the Poincaré polynomials, $P(\mathcal{A}(G), G; t)$, for most of the irreducible reflection pairs can be extracted from work of Lehrer [11], Callegaro, and Marin [6], [14]. When $\mathcal{A}$ is the arrangement of an imprimitive complex reflection group the argument in [11] relies on reduction to positive characteristic and the Grothendieck trace formula. When $\mathcal{A}$ is the arrangement of a primitive complex reflection group one needs to use the deep result that the orbifold $M(\mathcal{A})/G$ is a $K(\pi, 1)$-space. The approach used here, which is an extension of the computations for Coxeter groups in [3], is more elementary, less computationally intensive (for primitive groups), and yields a more refined description of the invariants $H^* (M(\mathcal{A}(G)))^G$. The arguments do not rely on any machine computations, although case-by-case calculations are still necessary.

The main results in this paper (see Corollary 5.6, Corollary 5.8, and Theorem 7.4) suggest the existence of some, as yet unidentified, geometric or algebraic phenomenon that would provide a conceptual explanation of the structure of the cohomology ring $H^* (M(\mathcal{A}(G)))^G$. 
1.7. The Felder-Veselov construction. The explicit bases of the spaces of invariants constructed in §7 also lead to a simple proof of a conjecture for Coxeter groups made by Felder and Veselov [8], as well as a natural extension of the resulting theorem to all reflection pairs (see Theorem 7.4).

In more detail, consider the special case when \( G \subseteq \GL(V) \) is a reflection group that contains a Coxeter system. The (ungraded) character of \( G \) on \( H^*(M(\A(G))) \) has been computed by various authors. Felder and Veselov characterize the conjugacy classes in the support of this character as the conjugacy classes of so-called special involutions. Special involutions can be classified for each Coxeter type and it turns out that the number of such classes is equal to the dimension of the invariant subspace \( H^*(M(\A))^G \). Suppose \( g \in G \) and let \( Z = Z_G(\Fix(g)) \) be the pointwise stabilizer of the space \( \Fix(g) \). By a theorem of Steinberg, \( Z \) is a reflection subgroup of \( \GL(V) \), and hence a Coxeter group. Let \( c \) be a Coxeter element in \( Z \). Then \( c \) determines a cohomology class \( \zeta_c \in H^*(M(\A(G))) \) in a natural way (see [8] or §7). Obviously the average of \( \zeta_c \) over \( G \), say \( \Av_G(\zeta_c) \), lies in \( H^*(M(\A(G)))^G \), and it is easy to see that \( \Av_G(\zeta_c) \) does not depend on the choice of the Coxeter element \( c \). Felder and Veselov conjectured that the rule \( g \mapsto \Av_G(\zeta_c) \) defines a bijection between the set of conjugacy classes of special involutions and a basis of \( H^*(M(\A(G)))^G \).

Felder and Veselov proved their conjecture in many cases. The remaining cases were settled in [7]. Computer calculations were used to verify the conjecture for the exceptional Coxeter groups. The approach used in this paper to compute \( H^*(M(\A))^G \), when \( (\A, G) \) is a reflection pair, leads to a proof of the Felder-Veselov conjecture that does not rely on computer calculations. The key observation is that when \( G \) is a Coxeter group, the parabolic subgroups that arise in Theorem 7.4 are precisely the subgroups that arise as pointwise stabilizers of fixed points of special involutions. The proof of Theorem 7.4 is a case-by-case computation using the classification of complex reflection groups.

1.8. Broué, Malle, and Rouquier [5] define specific generating sets for non-Coxeter complex reflection groups that have some of the properties of Coxeter generating sets of a Coxeter group. In this paper, these generating sets are used to construct bases of \( H^*(M(\A(G)))^G \) that for Coxeter groups agree with the Felder-Veselov bases. For well-generated complex reflection groups that are not Coxeter groups, the construction of the basis of \( H^*(M(\A(G)))^G \) is the same as for Coxeter groups. For groups that are not well-generated, the situation is more complicated, but still manageable.

Computations for non-Coxeter complex reflection groups show that the number of conjugacy classes in the support of the character of \( G \) on \( H^*(M(\A(G))) \) can be much larger than the dimension of \( H^*(M(\A(G)))^G \). It would be interesting to find a concise formula for the character of \( G \) on \( H^*(M(\A(G))) \) in the spirit of [8], as well as to identify a suitable replacement of the notion of special involutions that would be valid for all complex reflection groups.

1.9. The rest of this paper is organized as follows. In the next section we recall the general results about hyperplane arrangements and reflection arrangements needed in this paper. In §3 we describe the reduction of the computation of \( H^*(M(\A))^G \), when \( (\A, G) \) is a reflection pair, to the case of irreducible reflection pairs. The classification of irreducible reflection pairs is given in §4. Theorem 5.5, which gives a decomposition of the spaces of invariants \( H^*(M(\A))^G \) for irreducible reflection pairs with rank greater than two, is stated in §5 and proved in §6. In §7 explicit bases of the invariants \( H^*(M(\A))^G \) are described.
for each irreducible reflection pair with rank greater than two. In §8 we show how the
bases constructed in §7 lead to a refinement of a theorem proved by Lehrer [11]. In §9
we give a short alternative proof of another result of Lehrer [10] that irreducible characters
which do not vanish on subgroups generated by generating reflections of $G$ do not occur in
$H^*(M(\mathcal{A}(G)))$ when $G$ is a complex reflection group. The appendix contains computations
for rank two reflection pairs.

1.10. Conventions and notation. Basic references are Bourbaki [2] and Lehrer-Taylor
[13] for the theory of complex reflection groups and the book of Orlik and Terao [17] for the
theory of arrangements.

For the imprimitive reflection groups we need notation for roots of unity. Define

$$\mu_r \text{ to be the group of } r^{\text{th}} \text{ roots of unity and } \omega_r = e^{2\pi \sqrt{-1}/r},$$

when $r$ is a positive integer.

If $G \subseteq \text{GL}(V)$ is a complex reflection group, we may choose a $G$-invariant unitary form
on $V$. Normally we assume that such a form has been chosen. Thus, if $X$ is a subspace of
$V$, then $X^\perp$ denotes the orthogonal complement of $X$.

Except in §9, where $\mathbb{Q}$ is replaced by $\mathbb{C}$, if $X$ is a topological space, then $H^*(X)$ denotes
the (total) singular cohomology of $X$ with coefficients in $\mathbb{Q}$. If a group $G$ acts on $X$ by
homeomorphisms, we frequently denote the induced action on cohomology, $h \mapsto (g^{-1})^*(h)$
for $g \in G$ and $h \in H^*(X)$, by $g \cdot h$, or simply $gh$. The meaning should always be clear by
context.

Suppose $\mathcal{A}$ is an arrangement in a finite dimensional complex vector space $V$. For a
subspace $X \subseteq V$ define

$$\text{cd } X = \text{codim}_V X = \dim X^\perp \quad \text{and} \quad \mathcal{A}_X = \{ H \in \mathcal{A} \mid X \subseteq H \},$$

so $\mathcal{A}_X$ is an arrangement $V$. The center of $\mathcal{A}$ is $\text{Cent}(\mathcal{A}) = \bigcap_{H \in \mathcal{A}} H$ and $\mathcal{A}$ is essential if
$\text{Cent}(\mathcal{A}) = 0$. The rank of $\mathcal{A}$, denoted by $\text{rk } \mathcal{A}$, is defined by

$$\text{rk } \mathcal{A} = \text{cd Cent}(\mathcal{A}) = \dim \text{Cent}(\mathcal{A})^\perp.$$

The lattice of $\mathcal{A}$, denoted by $L(\mathcal{A})$, is the set of subspaces of $V$ of the form $H_1 \cap \cdots \cap H_k$,
where $\{H_1, \ldots, H_k\}$ is a subset of $\mathcal{A}$. Notice that if $X \in L(\mathcal{A})$, then $\text{rk } \mathcal{A}_X = \text{cd } X$ and
$\text{Cent}(\mathcal{A}_X) = X$. For a non-negative integer $k$ set

$$L(\mathcal{A})_k = \{ X \in L(\mathcal{A}) \mid \text{cd } X = k \}.$$

For a subgroup $G \subseteq \text{GL}(V)$ and a subset $X \subseteq V$ define

$$Z_G(X) = \{ g \in G \mid X \subseteq \text{Fix}_V(g) \},$$

the pointwise stabilizer of $X$ in $G$, and

$$N_G(X) = \{ g \in G \mid gX = X \},$$

the setwise stabilizer of $X$ in $G$.

If in addition $G$ is a reflection subgroup, then the rank of $G$ is defined to be the rank of
$\mathcal{A}(G)$, so

$$\text{rk } G = \text{cd Cent}(\mathcal{A}(G)).$$

1.11. Reduction to essential arrangements. Suppose $(\mathcal{A}, G)$ is an arrangement-group
pair with underlying vector space $V$. Let $V_1$ be a subspace of $V$ contained in $\text{Cent}(\mathcal{A})$, set
$V_0 = V/V_1$, and let $p: V \to V_0$ be the projection. Then

$\bullet$ $p(\mathcal{A}) = \{ p(H) \mid H \in \mathcal{A} \}$ is an arrangement in $V_0$ with $L(p(\mathcal{A})) = p(L(\mathcal{A}))$, which
is essential if $V_1 = \text{Cent}(\mathcal{A})$,}
• $p$ induces a bijection $\mathcal{A} \to p(\mathcal{A})$ and a lattice isomorphism $L(\mathcal{A}) \to L(p(\mathcal{A}))$, 
• $M(p(\mathcal{A})) = p(M(\mathcal{A}))$, and 
• $p^*: H^*(p(M(\mathcal{A}))) \cong H^*(M(\mathcal{A}))$ is an isomorphism.

Clearly, Cent$(\mathcal{A})$ is a $G$-stable subspace. Suppose that $V_1$ is also $G$-stable. Abusing notation slightly, let $p(G)$ denote the image of $G$ in $\text{GL}(V_0)$. Then 

• $(p(\mathcal{A}), p(G))$ is an essential arrangement-group pair with underlying vector space $V_0$, 
• the isomorphism $p^*$ intertwines the action of $p(G)$ on $H^*(M(\mathcal{A}))$ with the action of $G$ on $H^*(M(\mathcal{A}))$, and 
• $p^*$ induces an isomorphism of vector spaces $H^*(M(\mathcal{A}))^{p(G)} \cong H^*(M(\mathcal{A}))^G$.

2. Preliminaries

In this section we fix notation and review some background results for future reference. Unless otherwise indicated, $(\mathcal{A}, G)$ is an arrangement-group pair with underlying vector space $V$.

2.1. The Orlik-Solomon presentation of $H^*(M(\mathcal{A}))$. The presentation of $H^*(M(\mathcal{A}))$, given by Orlik and Solomon [15] for any arrangement $\mathcal{A}$, underlies the computations below. For $H \in \mathcal{A}$ choose a linear form $\alpha_H$ in the dual space, $V^*$, of $V$ such that $H = \ker \alpha_H$, and define $h$ to be the cohomology class of the 1-form $(1/2\pi i)(d\alpha_H/\alpha_H) = \alpha^*_H((1/2\pi i)(dz/z))$ in the de Rahm cohomology group $H^1_{\text{dr}}(M(\mathcal{A}))$.

We use the convention that hyperplanes are denoted using upper case $H$, possibly with decorations, and the corresponding Orlik-Solomon generators of $H^*(M(\mathcal{A}))$ are denoted using lower case $h$, with the same decorations. For example, for any two hyperplanes $H_1$ and $H_2$ in $\mathcal{A}$, $h_1h_2$ denotes the product of the Orlik-Solomon generators $(1/2\pi i)(d\alpha_{H_1}/\alpha_{H_1})$ and $(1/2\pi i)(d\alpha_{H_2}/\alpha_{H_2})$ in $H^1_{\text{dr}}(M(\mathcal{A}))$. With this convention, the $\mathbb{Q}$-subalgebra of $H^*_{\text{dr}}(M(\mathcal{A}))$ generated by $\{1\}$ and $\{h | H \in \mathcal{A}\}$ is isomorphic as a graded algebra to $H^*(M(\mathcal{A}))$, and

\[
\sum_{i=1}^{m} (-1)^i h_1 \cdots \hat{h}_i \cdots h_m = 0 \quad (\hat{h}_i \text{ is deleted})
\]

whenever $\{\alpha_{H_1}, \ldots, \alpha_{H_m}\}$ is a linearly dependent subset of $V^*$. Condition (a), together with the anti-commutativity condition $h_1h_2 = -h_2h_1$ for $H_1, H_2 \in \mathcal{A}$, is the Orlik-Solomon presentation of $H^*(M(\mathcal{A}))$.

2.2. It follows easily from 2.1(a) that $H^k(M(\mathcal{A}))$ is spanned by the $k$-fold products $h_1 \cdots h_k$ where $\{\alpha_{H_1}, \ldots, \alpha_{H_k}\}$ is a linearly independent subset of $V^*$. In particular, if $\text{cd} X = k$, then using Brieskorn’s Lemma we identify $H^{\text{cd} X}(M(\mathcal{A}_X))$ with the subspace of $H^k(M(\mathcal{A}))$ spanned by all products $h_1 \cdots h_k$, where $H_1, \ldots, H_k \in \mathcal{A}$ and $H_1 \cap \cdots \cap H_k = X$.

2.3. If $(\mathcal{A}, G)$ is an arrangement-group pair, then $G$ acts as graded algebra automorphisms on $H^*(M(\mathcal{A}))$. Thus, if $g \in G$ and $H_1, \ldots, H_k \in \mathcal{A}$, then

\[
g \cdot h_1 \cdots h_k = (g \cdot h_1) \cdots (g \cdot h_k).
\]

If $r \in G$ is a reflection and $H_r = \text{Fix}(r) \in \mathcal{A}$, then

\[
gH_r = H_{grg^{-1}} \in \mathcal{A} \quad \text{and} \quad g \cdot h_r = h_{grg^{-1}} \in H^1(M(\mathcal{A})).
\]

It follows easily that $H^1(M(\mathcal{A}))$ affords the permutation representation arising from the action of $G$ on $\mathcal{A}$.
The equality (b) is used without comment in the computations below.

2.4. Brieskorn’s Lemma, group actions, and semi-invariants. The main general result regarding the \( \mathbb{Q}G \)-module structure of the cohomology ring \( H^*(M(\mathscr{A})) \) used in this paper is an equivariant version of Brieskorn’s lemma that describes each subspace \( H^k(M(\mathscr{A})) \) as a sum of induced modules due to Lehrer and Solomon.

Let \( X(\mathscr{A}, G) \) be a fixed set of orbit representatives and set \( X(\mathscr{A}, G)_k = X(\mathscr{A}, G) \cap L(\mathscr{A})_k \).

Let \( \xi: G \to \mathbb{Q}^\times = \text{GL}(\mathbb{Q}^1) \) be a rational, linear character of \( G \) and let

\[
e_\xi = \frac{1}{|G|} \sum_{g \in G} \xi(g^{-1})g
\]

be the centrally primitive idempotent in \( \mathbb{Q}G \) corresponding to \( \xi \). For a \( \mathbb{Q}G \)-module \( M \), set

\[
M^\xi = e_\xi M = \{ m \in M \mid \forall g \in G, \ g m = \xi(g)m \}.
\]

In the special case when \( \xi = 1_G \) is the trivial representation, \( M^{1_G} = M^G \) as usual. To minimize subscripts, set \( e_\xi = e_{1_G} \). If \( H \) is a subgroup of \( G \), then \( \xi|_H \) is a linear character of \( H \) with centrally primitive idempotent \( e_{\xi|H} \) in \( \mathbb{Q}H \). Note that \( e_\xi e_{\xi|H} = e_\xi \).

In the next proposition, for \( Y \in L(\mathscr{A}) \) we identify \( H^k(M(\mathscr{A}_Y)) \) with a subspace of \( H^k(M(\mathscr{A})) \) as in 2.2.

**Proposition 2.5.** Suppose \( k \geq 0 \) and set \( X_k = X(\mathscr{A}, G)_k \).

1. The inclusions \( M(\mathscr{A}) \subseteq M(\mathscr{A}_X) \) induce isomorphisms of \( \mathbb{Q}G \)-modules,

\[
H^k(M(\mathscr{A})) \cong \bigoplus_{X \in L(\mathscr{A})_k} H^k(M(\mathscr{A}_X)) \cong \bigoplus_{X \in X_k} \text{Ind}_{NG(X)}^G(H^k(M(\mathscr{A}_X))).
\]

2. If \( X \in L(\mathscr{A}) \), then multiplication by \( e_\xi \) defines an isomorphism

\[
e_\xi_X H^k(M(\mathscr{A}_X)) \cong H^k(M(\mathscr{A}_X))^{\xi_X} \xrightarrow{e_\xi(\cdot)} \left( \bigoplus_{Y \in GX} H^k(M(\mathscr{A}_Y)) \right)^\xi,
\]

where \( GX \) denotes the \( G \)-orbit of \( X \). Summing over \( X \in X_k \), the first isomorphism in (1) restricts to the equality

\[
H^k(M(\mathscr{A}))^\xi = \sum_{X \in X_k} e_\xi \cdot H^k(M(\mathscr{A}_X))^{\xi_X},
\]

where the sum on the right-hand side is an internal direct sum.

3. Suppose that \( \mathscr{A} = \mathscr{A}(\bar{G}) \) is a reflection arrangement, where \( \bar{G} \subseteq \text{GL}(V) \) is a reflection group that is normalized by \( G \). If \( X \in L(\mathscr{A}) \), then \( \mathscr{A}_X = \mathscr{A}(Z_{\bar{G}}(X)) \), and so

\[
H^k(M(\mathscr{A}))^G \cong \bigoplus_{X \in X(\mathscr{A}, G)_k} H^k(M(\mathscr{A}(Z_{\bar{G}}(X))))^{N_G(X)}.
\]

**Proof.** The first isomorphism in (1) is due to Brieskorn [3]. The second is stated and proved by Lehrer and Solomon [12].

It is straightforward to check that the projection mapping from \( \bigoplus_{Y \in GX} H^k(M(\mathscr{A}_Y)) \) to \( H^k(M(\mathscr{A}_X)) \) restricts to a left inverse to the multiplication mapping from \( H^k(M(\mathscr{A}_X))^{\xi_X} \) to \( e_\xi \cdot \left( \bigoplus_{Y \in GX} H^k(M(\mathscr{A}_Y)) \right) \). The assertions in (2) then follow from (1) and Frobenius reciprocity.
With the hypotheses of (3), if \( X \in L(\mathcal{A}) \), it follows from a theorem of Steinberg [20] that 
\( Z_G(X) \) is generated by the reflections it contains, which are precisely the reflections that fix 
the hyperplanes in \( \mathcal{A}_X \) pointwise. Thus, \( Z_G(X) \) is a reflection group with 
\( \mathcal{A}(Z_G(X)) = \mathcal{A}_X \) and \( \text{Cent}(\mathcal{A}(Z_G(X))) = X \). In particular, \( \text{rk} Z_G(X) = \text{cd} X \). \( \square \)

Notice that the proposition holds when the base field, \( \mathbb{Q} \), is replace by \( \mathbb{C} \) throughout.

2.6. Some special cases for small and large \( k \) are important in the sequel.

- In degree 0, \( L(\mathcal{A})_0 = \{V\} = \mathcal{X}(\mathcal{A}, G)_0 \). It is easy to see that \( G \) acts trivially on
  \( H^0(V) \cong H^0(M(\mathcal{A})) \). Thus \( H^0(M(\mathcal{A}))^G \) is one-dimensional and \( H^0(M(\mathcal{A}))^\xi = 0 \)
  for \( \xi \neq 1_G \).

- In degree 1, \( L(\mathcal{A})_1 = \mathcal{A} \) and \( \mathcal{X}(\mathcal{A}, G)_1 \) is a set of orbit representatives for the action of \( G \) on \( \mathcal{X} \).
  It was observed in 2.3 that \( H^1(M(\mathcal{A})) \) is simply the permutation module arising from the action of \( G \) on \( \mathcal{A} \) and that if \( H \in \mathcal{X}(\mathcal{A}, G)_1 \), then \( H^1(M(\mathcal{A}_H))^{N_G(H)} \)
  is one-dimensional with basis the \( N_G(H) \)-average of the Orlik-Solomon generator \( h \).

- At the other extreme, if \( k > \text{rk} \mathcal{A} \), then \( H^k(M(\mathcal{A})) = 0 \), and thus \( H^k(M(\mathcal{A}))^G = 0 \).

- The “top” degree is \( \text{rk} \mathcal{A} \), in which case \( L(\mathcal{A})_{\text{rk} \mathcal{A}} = \{\text{Cent}(\mathcal{A})\} = \mathcal{X}_{\text{rk} \mathcal{A}} \).
  Clearly, \( \mathcal{A}_{\text{Cent}(\mathcal{A})} = \mathcal{A} \) and \( N_G(\text{Cent}(\mathcal{A})) = G \), so Brieskorn’s Lemma does not provide any insight into the \( \mathbb{Q}G \)-module structure of \( H_{\text{rk} \mathcal{A}}^k(M(\mathcal{A})) \), or \( \dim H_{\text{rk} \mathcal{A}}^k(M(\mathcal{A}))^G \).
  However, together with 2.8(a), the isomorphism in Proposition 2.5(3) provides a powerful inductive, or recursive, tool to compute \( H_{\text{rk} \mathcal{A}}^k(M(\mathcal{A}))^G \) when \( (\mathcal{A}, G) \) is an irreducible reflection pair.

We say that \( (\mathcal{A}, G) \) has top degree invariants if \( H_{\text{rk} \mathcal{A}}^k(M(\mathcal{A}))^G \neq 0 \). Roughly speaking, when \( (\mathcal{A}, G) \) is an irreducible reflection pair, Proposition 2.5(3) reduces the computation of \( H^k(M(\mathcal{A}))^G \) to those \( X \) in \( \mathcal{X}(\mathcal{A}, G)_k \) such that the pair \( (\mathcal{A}_X, Z_G(X)) \) has top degree invariants.

2.7. Contraction with the Euler vector field. The last general construction we need is the acyclic complex from [17, §3.1] that arises from contracting (classes of) differential forms in \( H^* (M(\mathcal{A})) \) with the Euler vector field. Precisely, there is a graded derivation of \( H^* (M(\mathcal{A})) \) with degree \(-1\) that maps each Orlik-Solomon generator \( h \in H^1(M(\mathcal{A})) \) to \( 1 \in H^1(M(\mathcal{A})) \):

\[
\partial : H^* (M(\mathcal{A})) \to H^* (M(\mathcal{A})) \quad \text{by} \quad \partial (h_1 \cdots h_k) = \sum_{i=1}^k (-1)^{i-1} h_1 \cdots \hat{h}_i \cdots h_k,
\]

for hyperplanes \( H_1, \ldots, H_k \in \mathcal{A} \). It is shown in [17, Lem. 3.13] that \( (H^* (M(\mathcal{A})), \partial) \) is an acyclic chain complex and it follows from 2.3(a) that \( \partial \) is \( G \)-equivariant. In particular,

\[
\partial : H_{\text{rk} \mathcal{A}}^{\text{rk} - 1}(M(\mathcal{A})) \to H_{\text{rk} \mathcal{A}}^{\text{rk} - 1}(M(\mathcal{A}))
\]

is injective and \( G \)-equivariant.

Define

\[
\mu : H^* (M(\mathcal{A})) \to H^* (M(\mathcal{A})) \quad \text{by} \quad \mu (x) = (|\mathcal{A}|^{-1} \sum_{H \in \mathcal{A}} h) x,
\]

so \( \mu \) is left multiplication by the class \( |\mathcal{A}|^{-1} \sum_{H \in \mathcal{A}} h \in H^1(M(\mathcal{A})) \). It is straightforward to check that \( \mu \) is a graded, \( G \)-equivariant, linear transformation with degree 1 and that \( \mu \partial + \partial \mu = \text{id} \).
For $0 \leq k \leq \text{rk}(\mathcal{A})$ set
\[ \overline{H}^k = \partial(H^{k+1}(M(\mathcal{A}))). \]
Then the restriction of $\mu$ to $\overline{H}^k$ is a $\mathbb{Q}G$-module isomorphism onto its image, with inverse given by the restriction of $\partial$ to $\mu(\overline{H}^k)$. Moreover there is a canonical direct sum decomposition of $\mathbb{Q}G$-modules,
\[ H^k(M(\mathcal{A})) \cong \mu(\overline{H}^{k-1}) \oplus \overline{H}^k, \]
where by convention we take $H^{-1}(M(\mathcal{A})) = 0$.

2.8. Two immediate consequences of 2.7(a) are
- if $Y$ is an irreducible $\mathbb{Q}G$-module, then $Y$ occurs in the graded module $H^*(M(\mathcal{A}))$ in pairs of degrees that differ by 1, and
- if $\xi$ is a linear character of $G$, then $\sum_{k \geq 0}(-1)^k \dim H^k(M(\mathcal{A}))^\xi = 0$.

It follows from the second assertion and Proposition 2.5(2) that
\[ \sum_{X \in \mathcal{A} \setminus \{\text{Cent}(\mathcal{A})\}} (-1)^{cd_X} \dim H^{cd_X}(M(\mathcal{A}_X))^\xi + (-1)^{rk_{\mathcal{A}}} \dim H^{rk_{\mathcal{A}}}(M(\mathcal{A}))^\xi = 0. \]
Thus, if $\dim H^{cd_X}(M(\mathcal{A}_X))^\xi$ has been computed for $X$ different from Cent($\mathcal{A}$), then $\dim H^{rk_{\mathcal{A}}}(M(\mathcal{A}))^\xi$ is determined by (a).

Together with Brieskorn’s Lemma and Theorem 5.5, the map $\partial$ can be used to construct bases of $H^*(M(\mathcal{A}))^G$ for irreducible reflection pairs. The case when $\text{rk} \mathcal{A} = 2$ is completed in the next proposition.

**Proposition 2.9.** Suppose that $(\mathcal{A}, G)$ is an arrangement-group pair with $\text{rk} \mathcal{A} = 2$.

1. If $G$ acts on $\mathcal{A}$ with a orbits, then the Poincaré polynomial of $(\mathcal{A}, G)$ is
\[ P(\mathcal{A}, G; t) = 1 + at + (a - 1)t^2. \]
2. If $\{H_1, \ldots, H_a\}$ is a set of orbit representatives for the action of $G$ on $\mathcal{A}$, then
\[ \{1\} \amalg \{e_G \cdot h_1, \ldots, e_G \cdot h_a\} \amalg \{e_G \cdot h_1 h_2, \ldots, e_G \cdot h_1 h_a\} \]
is a basis of $H^*(M(\mathcal{A}))^G$.

**Proof.** For $k = 0$, we’ve noted above that $\dim H^0(M(\mathcal{A}))^G = 1$ and so $\{1\}$ is a basis.

If $k = 1$, then it follows from 2.3(a) that $\dim H^1(M(\mathcal{A}))^G = a$ and $\{e_G \cdot h_1, \ldots, e_G \cdot h_a\}$ is a basis of $H^1(M(\mathcal{A}))^G$.

Suppose $k = 2$. By 2.8(a), $\dim H^2(M(\mathcal{A}))^G = a - 1$, so $P(\mathcal{A}, G; t) = 1 + at + (a - 1)t^2$. To complete the proof, it suffices to show that $\{e_G \cdot h_1 h_2, \ldots, e_G \cdot h_1 h_a\}$ is linearly independent. Because $\partial: H^2(M(\mathcal{A})) \to H^1(M(\mathcal{A}))$ is injective and $G$-equivariant, it is enough to show that the span of $\{\partial(e_G \cdot h_1 h_2), \ldots, \partial(e_G \cdot h_1 h_a)\}$ has dimension $a - 1$. But this last assertion is clear because $\partial(e_G \cdot h_1 h_i) = e_G \cdot h_i - e_G \cdot h_1$ for $2 \leq i \leq a$. \hfill \Box

3. Reduction to irreducible reflection pairs

In this section $C \subseteq \text{GL}(V)$ is a reflection coset, $\sigma \in C$ is a fixed coset representative, $G_0 = CC^{-1}$, and $G_1 = \langle C \rangle = \langle G_0 \sigma \rangle$. We sketch how the computation of $H^*(M(\mathcal{A}_i))^G_j$ for $i, j \in \{0, 1\}$ can be reduced to the case when $C$ is irreducible.
3.1. $G_1$ acts reducibly. First, suppose that $G_1$ does not act irreducibly on $V$. A standard argument (see [4, §3E]) shows that if $V'$ and $V''$ are complementary, $G_1$-stable, proper subspaces of $V$, then $G_0$ is the internal product of $G_0' = Z_{G_0}(V')$ and $G_0'' = Z_{G_0}(V')$ and $C$ is isomorphic to the product reflection coset $C' \times C''$, where

$$C' = (Z_{G_0}(V')\sigma)|_{V'} \subseteq \text{GL}(V') \quad \text{and} \quad C'' = (Z_{G_0}(V')\sigma)|_{V''} \subseteq \text{GL}(V'').$$

It follows from the $G_1$-invariance of the decomposition $V = V' + V''$ and the Künneth theorem that

$$H^*(M(\mathcal{A}_i))^G_i \cong H^*(M(\mathcal{A}_i'))^G_i \otimes H^*(M(\mathcal{A}_i''))^G_i,$$

for $i, j \in \{0, 1\}$. This reduces the computation of $H^*(M(\mathcal{A}_i))^G_i$ to the case when $G_1$ acts irreducibly on $V$.

3.2. $G_1$ acts irreducibly. Now suppose that $G_1$ acts irreducibly on $V$. Fix an irreducible $G_0$-invariant subspace $V_0 \subseteq V$. A standard argument (see the proof of [1, Prop. 6.9]) shows that there is a minimal positive integer, say $r$, such that $V = V_0 + \sigma V_0 + \cdots + \sigma^{r-1}V_0$. Since $\sigma$ permutes the subspaces $\sigma^i V_0$ cyclically and $G_0$ acts completely reducibly on $V$, it follows from the minimality of $r$ that $V$ is the internal direct sum of $V_0, \sigma V_0, \ldots, \sigma^{r-1}V_0$, and that $\sigma^r V_0 = V_0$.

Define $Z_0 = Z_{G_0}(\sigma^r V_0 + \cdots + \sigma V_0)$. Then $Z_0$ is generated by the reflections in $G_0$ that fix $\sigma^r V_0 + \cdots + \sigma V_0$ pointwise, $\sigma^r$ normalizes $Z_0$, and $G_0$ is the internal direct product of the reflection subgroups $Z_0, \sigma Z_0 \sigma^{-1}, \ldots, \sigma^{r-1}Z_0 \sigma^{-r}$. More precisely, the map

$$m : Z_0 \to G_0 \quad \text{by} \quad m(z_1, \ldots, z_r) = z_1(\sigma z_2 \sigma^{-1}) \cdots (\sigma^{r-1} z_r \sigma^{-r})$$

is a group isomorphism. Moreover, the restriction mapping from $Z_0$ to $Z_0|_{V_0}$ is an isomorphism and the action of $G_0$ on $V_0$ factors through $Z_0$, so $Z_0$ acts irreducibly on $V_0$ and $(Z_0, V_0)$ is an irreducible reflection group.

**Lemma 3.3.** With the preceding notation,

(a) $\mathcal{A}(Z_0) = \{ H \in \mathcal{A}(G_0) \mid V_0 \not\subseteq H \}$,

(b) $\mathcal{A}_0 = \mathcal{A}(Z_0) \amalg \sigma(\mathcal{A}(Z_0)) \amalg \cdots \amalg \sigma^{r-1}(\mathcal{A}(Z_0))$,

and

(c) $\mathcal{A}_1 = \mathcal{A}((Z_0 \sigma^r)) \amalg \sigma(\mathcal{A}((Z_0 \sigma^r))) \amalg \cdots \amalg \sigma^{r-1}(\mathcal{A}((Z_0 \sigma^r)))$,

unless $r = 2$ and dim $V_0 = 1$. If $r = 2$ and dim $V_0 = 1$, then up to isomorphism, $G_1$ is equal to $G(q, 1, 2)$ and $G_0 \cong \mathbb{A}^2_2$ is the group of diagonal matrices in $G(d, 1, 2)$, where $d$ is a divisor of $q$.

**Proof.** First, if $H \in \mathcal{A}(G_0)$, then either $\sum_{j=1}^{r-1} \sigma^j V_0 \subseteq H$ or $V_0 \not\subseteq H$, and not both. The equality in (a) follows immediately. The equality (b) follows from the decompositions $V = V_0 + \sigma V_0 + \cdots + \sigma^{r-1}V_0$ and $G_0 = Z_0 \cdot (\sigma Z_0 \sigma^{-1}) \cdots (\sigma^{r-1} Z_0 \sigma^{-r})$ using a similar argument. The equality (c) follows from analyzing the condition $\text{cd} \text{Fix}_V(g\sigma^m) = 1$ for $g \in G_0$ and $m > 0$. One shows that (1) if $g \sigma^{rk}$ is a reflection, then

$$\mathcal{A}(G_0 \sigma^{rk}) = \mathcal{A}(Z_0 \sigma^{rk}) \amalg \sigma(\mathcal{A}(Z_0 \sigma^{-1} \sigma^{rk})) \amalg \cdots \amalg \sigma^{r-1}(\mathcal{A}(Z_0 \sigma^{-r} \sigma^{rk}))$$

and (2) if $m$ is not divisible by $r$, then $\mathcal{A}(G_0 \sigma^m) = \emptyset$ unless $r = 2$ and dim $V_0 = 1$. Finally, when $r = 2$ and dim $V_0 = 1$, the final assertion is easily verified. □
3.4. Define
\[ \mathcal{Z}_0 = Z_0|_{V_0}, \quad \overline{\sigma} = \sigma^r|_{V_0}, \quad \text{and} \quad \overline{C} = \mathcal{Z}_0\overline{\sigma}. \]

Then \((\mathcal{Z}_0, V_0)\) is an irreducible reflection group, \(\mathcal{A}(\mathcal{Z}_0) = \{ V_0 \cap H \mid H \in \mathcal{A}(G_0), V_0 \not\subseteq H \}\), and \(\overline{C}\) is an irreducible reflection coset in \(\text{GL}(V_0)\). Obviously \(\overline{C}\overline{C}^{-1} = \mathcal{Z}_0\). Define
\[ Z_1 = \langle Z_0\sigma^r \rangle \quad \text{and} \quad \overline{Z}_1 = \langle \overline{C} \rangle = \langle \mathcal{Z}_0\overline{\sigma} \rangle. \]

To simplify the notation somewhat, set
\[ \overline{\mathcal{A}}_i = \mathcal{A}(\overline{Z}_i) \quad \text{for} \quad i = 0, 1. \]

We want to compute the space of invariants \(H^*(\mathcal{A}_i)^G_i\), for \(i, j \in \{0, 1\}\), in terms of spaces \(H^*(\mathcal{A}_i)\overline{Z}_j\).

3.5. Next, define
\[ f: V_0^r \to V \quad \text{by} \quad f(v_1, v_2, \ldots, v_r) = v_1 + \sigma v_2 + \cdots + \sigma^{r-1} v_r \]
and
\[ \overline{\sigma}: V_0^r \to V_0^r \quad \text{by} \quad \overline{\sigma}(v_1, \ldots, v_r) = (\overline{\sigma^r} v_r, v_1, \ldots, v_{r-1}), \]
and let
\[ m: \mathcal{Z}_0^r \to G_0 \quad \text{be the composition} \quad \mathcal{Z}_0^r \approx \to \mathcal{Z}_0^r \overset{m}{\xrightarrow{\simeq}} G_0, \]
where the first isomorphism is the inverse of the \(r\)-fold product of the restriction isomorphism \(Z_0 \xrightarrow{\cong} \overline{Z}_0\).

**Lemma 3.6.**
1. The maps \(f\) and \(\overline{\sigma}\) are vector space isomorphisms.
2. \(f \circ \overline{\sigma} = \sigma \circ f\).
3. Via the isomorphism \(m\),
   (a) \(f\) intertwines the product action of \(\mathcal{Z}_0^r\) on \(V_0^r\) with the action of \(G_0\) on \(V\) and
   (b) \(f\) intertwines the action of \((\mathcal{Z}_0^r\overline{\sigma})\) on \(V_0^r\) with the action of \(G_1\) on \(V\).
4. \(f\) induces isomorphisms of arrangements \((\mathcal{A}_i, V_0^r) \cong \mathcal{A}_i, V\) for \(i = 0, 1\).
5. \(f\) restricts to homeomorphisms \(M(\mathcal{A}_i)^{G_0^r} \cong M(\mathcal{A}_i)\) for \(i = 0, 1\).

**Proof.** Statements (1), (2), and (3) are direct computations, (4) follows from Lemma 3.3, and (5) follows from (4). \(\square\)

3.7. Using Lemma 3.6 we may identify the groups \(G_0 \equiv \mathcal{Z}_0^r\) and \(G_1 \equiv \overline{C} = \mathcal{Z}_1\), the reflection cosets \(C \subseteq \text{GL}(V)\) and \(\overline{C}^r \subseteq \text{GL}(V_0^r)\), and the reflection pairs
\[ (\mathcal{A}_0^r, G_0) \equiv (\mathcal{Z}_0^r, \overline{C}^r) \quad \text{and} \quad (\mathcal{A}_1^r, G_0) \equiv (\mathcal{Z}_1^r, \overline{C}^r) , \]
\[ (\mathcal{A}_0^r, G_1) \equiv (\mathcal{A}_1^r, \overline{C}^r) , \quad \text{and} \quad (\mathcal{A}_1^r, G_1) \equiv (\mathcal{A}_1^r, \overline{C}^r). \]

Since \(\text{Cent}(\mathcal{A}(Z_0)) = \sum_{j=1}^{r-1} \sigma^j V_0\), there is an isomorphism \(H^*(\mathcal{A}(Z_0)) \cong H^*(\mathcal{A}_i)\overline{Z}_j\), for \(i = 0, 1\), that on elements we denote by \(h \leftrightarrow \overline{h}\). The isomorphism in Lemma 3.6(5), together with the Künneth Theorem, shows that there is a commutative diagram of isomorphisms of graded vector spaces
\[ \begin{array}{ccc}
H^*(\mathcal{A}_i) & \overset{f_i^*}{\longrightarrow} & H^*(\mathcal{A}_i)^{\overline{\sigma}^r} \\
\sigma^* \downarrow & & \overline{\sigma}^* \downarrow \\
H^*(\mathcal{A}_i) & \overset{f_i^*}{\longrightarrow} & H^*(\mathcal{A}_i)^{\overline{\sigma}^r} 
\end{array} \]
such that for homogeneous elements \( h_1, \ldots, h_r \in H^*(M(\mathcal{A}(Z_i))) \) we have

\[
\begin{align*}
\pm \sigma^r(h_r) \sigma(h_1) \cdots \sigma^{r-1}(h_1) & \xrightarrow{f^\rho_r} \pm \sigma^r(h_r) \otimes h_1 \otimes \cdots \otimes h_{r-1},
\end{align*}
\]

where the sign is determined by the degrees of \( h_1, \ldots, h_r \).

**Proposition 3.8.** For \( i \in \{0, 1\} \), the isomorphisms \( f^\rho_r \) intertwine the action of \( G_0 \) with the action of \( Z_0^r \), and the action of \( G_1 \) with the action of \( Z_1 = \langle Z_0^r \sigma \rangle \), and induce isomorphisms

\[
\begin{align*}
H^*(M(\mathcal{A}_0))^{G_0} & \cong \left( H^*(M(\mathcal{A}_0)) Z_0 \right)^{\otimes r}, & H^*(M(\mathcal{A}_0))^{G_1} & \cong \left( \left( H^*(M(\mathcal{A}_0)) Z_0 \right)^{\otimes r} \sigma^* \right), \\
H^*(M(\mathcal{A}_1))^{G_0} & \cong \left( H^*(M(\mathcal{A}_1)) Z_0 \right)^{\otimes r}, & H^*(M(\mathcal{A}_1))^{G_1} & \cong \left( \left( H^*(M(\mathcal{A}_1)) Z_0 \right)^{\otimes r} \sigma^* \right). 
\end{align*}
\]

**Proof.** The first assertion follows by direct computation and the second follows from the first by taking invariants and using that \( Z_0^r \) acts componentwise on \( r \)-fold tensor products. \( \square \)

**3.9.** Proposition 3.8 shows how to compute \( H^*(M(\mathcal{A}_i))^{G_j} \) for \( i, j \in \{0, 1\} \) starting from the reflection coset \( \overline{C} = Z_0^r \sigma \). For \( H^*(M(\mathcal{A}_0))^{G_0} \) and \( H^*(M(\mathcal{A}_1))^{G_0} \) this is clear from the Künneth Theorem and the compatible factorizations of \( \mathcal{A}_0, \mathcal{A}_1, \) and \( G_0 \). For example \( H^*(M(\mathcal{A}_0))^{G_0} \cong H^*(M(\mathcal{A}_0)) Z_0 \). For \( H^*(M(\mathcal{A}_0))^{G_1} \) and \( H^*(M(\mathcal{A}_1))^{G_1} \), choose a homogeneous basis of \( \left( H^*(M(\mathcal{A}_0)) Z_0 \right)^{\sigma^*} \) consisting of eigenvectors for \( (\sigma^* \sigma) \). Then an explicit basis of \( \left( H^*(M(\mathcal{A}_1)) Z_0 \right)^{\sigma^*} \) is given by certain orbit sums (indexed by a set of Lyndon words).

Further details of the construction are omitted as this basis does not play a role in the sequel.

### 4. Irreducible reflection pairs

The next task is to compile a list of irreducible reflection pairs, \( (\mathcal{A}, G) \), with the property that if \( C \) is an irreducible reflection coset with \( G_0 = CC^{-1} \) and \( G_1 = \langle C \rangle \), then each \( H^*(M(\mathcal{A}_i))^{G_j} = H^*(M(\mathcal{A}))^G \) for some pair \( (\mathcal{A}, G) \) on the list. The first step is to establish notation for the irreducible reflection groups, arrangements, and cosets.

**4.1. Irreducible complex reflection groups and reflection types.** Because reflection groups are pairs, and not abstract groups, the classification of reflection groups takes into account the underlying vector space \( V \). Following [16, §3], we say that two reflection groups \( (G, V) \) and \( (G', V') \) have the same reflection type if there are isomorphisms \( G \to G' \) and \( V \to V' \) such that \( (gv)' = g'v' \) for \( g \in G \) and \( v \in V \). The reflection type of a non-trivial complex reflection group is the concatenation of the reflection types of its non-trivial, irreducible factors. The reflection types of the irreducible reflection groups are given by the Shephard-Todd classification.

The notation we use for the irreducible complex reflection groups and their reflection types is mostly consistent with [17, App. C]. In particular, we use the Coxeter label for symmetric groups and exceptional groups. If \( (G, V) \) is an irreducible complex reflection group, then with the Shephard-Todd grouping, \( (G, V) \) has the same reflection type as one of the following reflection groups:
(1) A Coxeter group of type $A_n$ for $n \geq 0$: $A_0$ is the trivial group acting on $V = \mathbb{C}$. For $n > 0$, let $W_{n+1} = G(1,1,n+1)$ be the group of permutation matrices in $GL_{n+1}(\mathbb{C})$ and set $V = v_1^\perp$, where $v_1$ is the vector in $\mathbb{C}^n$ with all entries equal to 1. Then $W_{n+1}$ stabilizes $V$ and $A_n$ is $(W_{n+1}|v,V)$.

(2) $G$ is in the infinite family of groups $G(r,p,n)$, acting on $V = \mathbb{C}^n$, where $r$, $p$, and $n$ are positive integers, $p$ divides $r$, $r \geq 2$, $n \geq 2$, and $(r,p,n) \neq (2,2,2)$. To enhance readability, in the tables below we use subscripts and write $G_{r,p,n}$ instead of $G(r,p,n)$.

(3) $G$ is a group of roots of unity, $\mu_r$ for some $r > 2$, acting on $V = \mathbb{C}$.

(4) $G$ is one of thirty-four exceptional groups labeled $G_4$, $G_5$, $\ldots$, $G_{37} = E_8$, each acting on an explicitly defined complex vector space (see [13, Ch. 8]).

4.2. Irreducible reflection arrangements. It is not that uncommon for non-isomorphic reflection groups to have the same reflection arrangement. Parallel to the classification of irreducible reflection groups, we have the following classification of irreducible reflection arrangements.

(1) We may realize $\mathcal{A}(A_n)$ as the arrangement of $W_{n+1}$ acting on $\mathbb{C}^{n+1}$, which is the braid arrangement $B_{n+1}$.

Notice that $B_1 = \mathcal{A}(A_0)$ is the empty arrangement and that $B_2 = \mathcal{A}(A_1)$ is the unique irreducible reflection arrangement with rank 1.

(2) Using the notation in [17, Ch. 6], for $n, r \geq 2$,

\[
\mathcal{A}(G(r,r,n)) = \mathcal{A}_n^0(r), \quad \text{and} \quad \mathcal{A}(G(r,p,n)) = \mathcal{A}_n(r) \quad \text{for} \quad p < r.
\]

(3) For $r > 1$, $\mathcal{A}(\mu_r) = B_2$.

(4) The arrangements of the exceptional complex reflection groups are denoted using script instead of roman characters, for example $\mathcal{G}_4$ and $\mathcal{E}_8$. There are equalities of arrangements,

\[
\mathcal{G}_7 = \mathcal{G}_{10}, \quad \mathcal{A}_2(4) = \mathcal{G}_8, \quad \mathcal{G}_9 = \mathcal{G}_{13}, \quad \text{and} \quad \mathcal{G}_{11} = \mathcal{G}_{15}.
\]

4.3. Irreducible reflection cosets. Broué, Malle, and Michel [4] have classified irreducible reflection cosets. If $C = G\sigma \subseteq GL(V)$ is an irreducible reflection coset with $\dim V > 1$, then up to scalar multiples of the identity, the pair $(G, \sigma)$ appears in the following list.

(1) $G$ is irreducible and $\sigma$ is the identity linear transformation,

(2) $G = G(r,p,n) \subseteq GL(\mathbb{C}^n)$ with $r > 1$ and $\sigma = \varphi^q$, where $\varphi$ is the diagonal matrix with entries $(\omega_r, 1, \ldots, 1)$ and $0 < q < r$,

(3) $G = F_4$ and $\sigma = \gamma$ induces the graph automorphism of $F_4$,

(4) $G = D_4 = G(2,2,4)$ and $\sigma = \tau$ induces the triality automorphism of $D_4$,

(5) $G = G(3,3,3)$, embedded as a normal reflection subgroup in $G_{26}$ and $\sigma \in G_{26}$ is any element with order 4 that normalizes $G$,

(6) $G = G_5$, or $G = G_7$, embedded as normal reflection subgroups of $G_{15} = \langle G_7, G_{14} \rangle$, and $\sigma = \rho_2$ is any reflection in $G_{14}$ with order 2,

(7) $G = G_5$, or $G = G_7$, with $G_5 \subseteq G_7$ and $G_7$ embedded as a normal reflection subgroup of $G_{10}$, and $\sigma = \rho_4$ is any reflection in $G_{10}$ with order 4, or

(8) $G = G(4,2,2)$, embedded as a normal reflection subgroup in $G_6$, and $\sigma = \rho_3$ is any reflection in $G_6$ with order 3.
4.4. **Irreducible reflection pairs.** As noted above, each reflection coset $C = G\sigma$ leads to a family of irreducible reflection cosets $Cz = G\sigma z$, where $z$ runs over the group of scalar transformations of the underlying vector space with finite order. For each such $G\sigma z$ we want to compute

\[ H^*(M(\mathcal{A}(G)))^G, \quad H^*(M(\mathcal{A}(G)))^{(G\sigma z)} = H^*(M(\mathcal{A}(G)))^{(G\sigma)}, \]

\[ H^*(M(\mathcal{A}((G\sigma z))))^G, \quad H^*(M(\mathcal{A}((G\sigma z))))^{(G\sigma z)} = H^*(M(\mathcal{A}((G\sigma z))))^{(G\sigma)}. \]

Thus, for each irreducible reflection coset $G\sigma$ as in 4.3(1)-4.3(8), and scalar transformation $z$, we need to compute $H^*(M(\mathcal{A}))^G$, where $(\mathcal{A}, G)$ is a reflection pair with

(a) \[ \mathcal{A} \in \{ \mathcal{A}(G), \mathcal{A}((G\sigma z)) \} \quad \text{and} \quad G \in \{ G, (G\sigma) \}. \]

To compile the list of pairs $(\mathcal{A}, G)$ in (a) we first compute the arrangements $\mathcal{A}((G\sigma z))$ and the groups $(G\sigma)$ for each of the reflection cosets $G\sigma$ in 4.3(1)-4.3(8), and then organize the results by the isomorphism type of $\mathcal{A}$.

**Theorem 4.5.** Suppose $G\sigma$ is one of the reflection cosets in 4.3(1)-4.3(8) and $z$ is a scalar transformation with finite order.

1. If $\text{rk} \, G = 2$, the arrangements $\mathcal{A}((G\sigma z))$ with $\mathcal{A}((G\sigma z)) \neq \mathcal{A}(G)$, and the groups $(G\sigma)$, are given in Table 5 in the appendix.
2. If $\text{rk} \, G > 2$, then $\mathcal{A}((G\sigma z)) = \mathcal{A}(G)$, unless $G = G(r, r, n)$ or $G = 25$.
3. If $G = G(r, r, n)$ with $n > 2$ and $\mathcal{A}((G\varphi z)) \neq \mathcal{A}(G)$, then $\mathcal{A}((G\varphi z)) = \mathcal{A}(G)$.
4. If $G = 25$ and $\mathcal{A}((Gz)) \neq \mathcal{A}(G)$, then $\mathcal{A}((Gz)) = \mathcal{A}(G)$.

4.6. The proof of the theorem is a routine calculation, using Springer’s theory of regular elements, for each $\sigma$ in 4.3(1)-4.3(8). The computations in rank two are left to the reader. Here we sketch the main idea and work out some examples to give the flavor of the calculations.

Suppose $G$ is any reflection group. Define a reflection coset, $C$, to be regular if $\mathcal{A}(C) \not\subseteq \mathcal{A}(CC^{-1})$. Equivalently, $C$ is regular if and only if $C$ contains a reflection, $r$, such that $\text{Fix}(r) \notin \mathcal{A}(CC^{-1})$. Clearly, if $C = G\sigma z$ and the index of $G$ in $\langle G\sigma z \rangle$ is $m$, then $\mathcal{A}((G\sigma z)) = \bigcup_{k=1}^{m} \mathcal{A}(G\sigma z^k)$. Thus, $\mathcal{A}((G\sigma z)) \neq \mathcal{A}(G)$ if and only if $G\sigma z^k$ is a regular reflection coset for some $k$ with $0 < k < m$.

Suppose $G\sigma z$ is regular. Let $\zeta$ denote the eigenvalue of $z$ and choose $g \in G$ so that $g\sigma z$ is a reflection with $\text{Fix}(g\sigma z) \notin \mathcal{A}(G)$. Then $\zeta$ is a root of unity and $\text{Fix}(g\sigma z)$ is equal to the $\zeta^{-1}$-eigenspace of $g\sigma$. Because $\text{Fix}(g\sigma z) \notin \mathcal{A}(G)$, the $\zeta^{-1}$-eigenspace of $g\sigma$, contains a regular vector in $V$. The proof of the next theorem is a straightforward application of [19, Thm. 6.4].

**Theorem 4.7.** Suppose that $G\sigma z$ is a regular $G$-reflection coset, the degrees of $G$ are $d_1, \ldots, d_n$, and $f_1, \ldots, f_n$ are basic polynomial invariants of $G$ that are also eigenfunctions for $\sigma$. Say $\text{deg} \, f_i = d_i$ and $\sigma \cdot f_i = \epsilon_i f_i$ for $1 \leq i \leq n$. Choose $g \in G$ so that $g\sigma z$ is a reflection. Then

1. the eigenvalues of the reflection $g\sigma z$ are $\epsilon_1^{-1}\zeta^{d_1}, \ldots, \epsilon_n^{-1}\zeta^{d_n}$ (so exactly one of $\epsilon_1^{-1}\zeta^{d_1}, \ldots, \epsilon_n^{-1}\zeta^{d_n}$ is not equal to one) and
2. if $h \in G$ and $h\sigma z$ is a reflection, then $\text{Fix}(g\sigma z)$ and $\text{Fix}(h\sigma z)$ are in the same $G$-orbit. In particular, $\mathcal{A}(G\sigma z)$ is a single $G$-orbit of hyperplanes in $V$, which is disjoint from $\mathcal{A}(G)$.

The eigenvalues $\epsilon_i$ are computed in [13, App. D.5].
4.8. Using [13, Thm. 12.23] the theorem can be rephrased to include a characterization of regular reflection cosets. This stronger result is not needed in full generality, but it is convenient to use the special case when \( \sigma \) is the identity.

For an integer \( d \), let \( a(d) \) denote the number of degrees of \( G \) that are divisible by \( d \). Similarly, define \( a^*(d) \) to be the number of codegrees of \( G \) that are divisible by \( d \). We say that \( d \) is a regular reflection number of \( G \) if \( a(d) = a^*(d) = n - 1 \). The degrees and codegrees of the exceptional reflection groups are given in [13, App. D.2], from which the regular reflection numbers can easily be computed.

Suppose \( z \) has order \( d \). It follows from [13, Thm. 12.23] that \( Gz \) is a regular reflection coset if and only if \( d \) is a regular reflection number for \( G \). If so, then \( \mathcal{A}(Gz) \) is a single \( G \)-orbit of hyperplanes, which is obviously disjoint from \( \mathcal{A}(G) \).

Suppose \( G \) is irreducible. If \( Gz \) and \( Gz' \) are two reflection cosets, then \( Gz = Gz' \) if and only if \( z^{-1}z' \) is in the center of \( G \), and if so, then \( \mathcal{A}(Gz) = \mathcal{A}(Gz') \). Thus scalar representatives of reflection cosets, and in particular regular reflection cosets, are determined modulo the center of \( G \).

It turns out that if the rank of \( G \) is greater than two, \( G \) has regular reflection numbers if and only if \( G = G(r, r, n) \) or \( G = G_{25} \). These two cases are discussed in more detail below.

In rank two, the only irreducible reflection groups with no regular reflection numbers are the groups \( G(p, 2) \) with \( p \) even, \( G_7, G_{11}, G_{15}, \) and \( G_{19} \).

4.9. Examples. In these examples, as well as in the appendix, for a positive integer, \( r \), let \( z_r \in \text{GL}(\mathbb{C}^2) \) denote the scalar matrix with eigenvalue \( \omega_r \).

A first example is the group \( G_{25} \). The degrees of \( G_{25} \) are \( 6, 9, \) and \( 12 \) and the codegrees are \( 0, 3, 6 \). Thus, the regular reflection numbers of \( G_{25} \) are \( 2 \) and \( 6 \). Moreover, \( \gcd(6, 9, 12) = 3 \), so the center of \( G_{25} \) has order three and hence \( G_{25}z_2 = G_{25}z_6 \). It follows that there is a unique regular \( G_{25} \)-reflection coset, namely \( G_{25}z_2 \). It is well-known that \( \langle G_{25}z_2 \rangle = G_{26} \) (see [13, Thm. 8.42]).

4.10. Next suppose that \( G = G(r, p, n) \) with \( n > 2 \).

The degrees of \( G \) are \( r, 2r, \ldots, (n-1)r, nr/p \) and the codegrees are \( 0, r, \ldots, (n-1)r \). Thus \( G(r, p, n) \) has no regular reflection numbers if \( p \neq r \) and the reflection numbers of \( G(r, r, n) \) are the divisors of \( r \) that do not divide \( n \). If \( d \) divides \( r \) and not \( n \), then \( \langle G(r, r, n)z_d \rangle \subseteq G(r, 1, n) \) and \( \mathcal{A}(\langle G(r, r, n)z_d \rangle) \neq \mathcal{A}(G(r, r, n)) \), so \( \mathcal{A}(\langle G(r, r, n)z_d \rangle) = \mathcal{A}_n(r) \).

Now consider \( \sigma = \varphi^q \) with \( 0 < q < r \). If \( p \neq r \), then
\[
(\sigma)_n(r) = \mathcal{A}(G) \subseteq \mathcal{A}(\langle G\varphi^qz \rangle) \subseteq \mathcal{A}(\langle G(r, 1, n)\varphi^qz \rangle) = \mathcal{A}(\langle G(r, 1, n)z \rangle) = \mathcal{A}_n(r),
\]
where the last equality holds because \( G(r, 1, n) \) has no regular reflection numbers. Hence \( \mathcal{A}(\langle G\varphi^qz \rangle) = \mathcal{A}(G) \) for every scalar transformation \( z \). If \( p = r \) and \( \mathcal{A}(\langle G\varphi^qz \rangle) \neq \mathcal{A}(G) \), then the computation in (a) becomes
\[
\mathcal{A}_n^0(r) = \mathcal{A}(G) \subseteq \mathcal{A}(\langle G\varphi^qz \rangle) \subseteq \mathcal{A}(\langle G(r, 1, n)\varphi^qz \rangle) = \mathcal{A}_n(r),
\]
from which it again follows that \( \mathcal{A}(\langle G\varphi^qz \rangle) = \mathcal{A}_n(r) \).

4.11. As a final example, suppose \( G = D_4 = G(2, 2, 4) \) and that conjugation by \( \sigma = \tau \) induces the triality automorphism of \( G \) as in 4.3(4). Consider a reflection coset \( G\tau\tau z \). The degrees of \( D_4 \) are \( 2, 4, 4, 6 \), and a basic set of polynomial invariants of \( G \) may be chosen so that the eigenvalues of \( \tau \) on these invariants are \( 1, \omega_3, \omega_3^{-1}, \) and \( 1 \), respectively. Just suppose that \( G\tau \tau z \) is regular and choose a reflection \( g\tau z \in G\tau z \). By Theorem 4.7 the eigenvalues of
$g \tau z$ are $\zeta^2, \omega_3^{-1} \zeta^4, \omega_3 \zeta^4,$ and $\zeta^6,$ where $\zeta$ is the eigenvalue of $z.$ It is never true that exactly three of $\zeta^2, \omega_3^{-1} \zeta^4, \omega_3 \zeta^4,$ and $\zeta^6$ are equal to one. This is a contradiction, so $G \tau z$ is not a regular reflection coset for any $z.$

Sorting the computations in Theorem 4.5 by reflection arrangement we obtain the classification of irreducible reflection pairs in the next corollary.

**Corollary 4.12.** Suppose $C$ is an irreducible reflection coset with $G_0 = CC^{-1}$ and $G_1 = \langle C \rangle.$ Then for $i, j \in \{0, 1\},$ one of the following statements holds.

1. $H^*(M(G_i))^G_j = H^*(M(A^j(G)))^G_i,$ where $G$ is irreducible.
2. The rank of $G_0$ is equal to two and $H^*(M(G_i))^G_j = H^*(M(A^j(G)))^G,$ where $(\mathcal{A}, G)$ is one of the pairs in Table 6 in the appendix. The table also records a reflection coset, $C,$ that gives rise to the reflection pair $(\mathcal{A}, G).$
3. The rank of $G_0$ is greater than two and $H^*(M(G_i))^G_j = H^*(M(A^j(G)))^G,$ where $(\mathcal{A}, G)$ is one of the following pairs:
   - $(\mathcal{A}, G) = (\mathcal{A}_n(r), G(r, r, n)), (\mathcal{A}, G) = (\mathcal{A}_n^0(r), G(r, p, n)),$ where $p < r.$ Both pairs arise from the coset $G(r, r, n)\varphi^p.$
   - $(\mathcal{A}, G) = (\mathcal{F}_4, \langle F_4 \gamma \rangle).$ This pair arises from the coset $F_4 \gamma.$
   - $(\mathcal{A}, G) = (\mathcal{A}_3^0(2), \langle D_4 \tau \rangle).$ This pair arises from the coset $D_4 \tau.$
   - $(\mathcal{A}, G) = (\mathcal{A}_3(3), \langle G(3, 3, 3) \sigma \rangle).$ This pair arises from the coset $G(3, 3, 3) \sigma.$

4.13. **Families of irreducible reflection pairs.** For the purpose of computing the spaces $H^*(M(\mathcal{A}^j(G))^G),$ the collection of irreducible reflection pairs naturally divides into five families.

1. A first family is the collection of all rank two irreducible reflection pairs. These reflection pairs are listed in Table 6, along with reflection cosets that give rise to them.

   When $(\mathcal{A}, G)$ has rank two, the Poincaré polynomial $P(\mathcal{A}, G; t),$ and a basis of $H^*(M(\mathcal{A}^j(G))^G,$ can be computed using Proposition 2.9, once the orbits of $G$ on $\mathcal{A}$ are known. These orbits are straightforward to compute for each rank two, irreducible reflection pair. In the rest of this paper we take $P(\mathcal{A}, G; t)$ and $H^*(M(\mathcal{A}^j(G))^G$ as known for rank two reflection pairs.

2. A second family is the set of pairs $(\mathcal{A}(n), A_n)$ for $n > 2.$ It was observed above that if $n > 1,$ then $H^*(M(A_{n-1}))^{A_{n-1}} \cong H^*(M(\mathcal{B}_n))^W_n.$ Brieskorn [3] has shown that $P(\mathcal{B}_n, W_n; t) = 1 + t.$ Because $H^1(M(\mathcal{B}_n))^W_n$ affords the permutation representation of $W_n$ on $\mathcal{B}_n,$ a basis of $H^*(M(\mathcal{B}_n))^W_n$ is given by $1 \in H^0(M(\mathcal{B}_n))^W_n$ and the orbit sum of the Orlik-Solomon generator, $h,$ in $H^1(M(\mathcal{B}_n))^W_n,$ where $H$ is any hyperplane in $\mathcal{B}_n.$

3. Third is the collection of pairs $(\mathcal{A}(G), G),$ where $G$ is an exceptional, irreducible, reflection group with rank greater than two. As a matter of terminology, in the rest of this paper we call these pairs *primitive.*

   The computation of $P(\mathcal{A}, G; t)$ for primitive reflection pairs proceeds by recursion. This argument is an extension of that given by Brieskorn for the exceptional Coxeter groups.

4. Next are the three irreducible reflection pairs in Corollary 4.12(3b), (3c), (3d), which we call *very exceptional.*

   The computations for the very exceptional reflection pairs is most easily dealt with case-by-case, using results from the other families.
(5) Finally, the irreducible reflection pairs, \((\mathcal{A}, G)\), where \(\mathcal{A} \in \{\mathcal{A}_n^0(r), \mathcal{A}_n(r)\}\) and \(G \in \{G(r, r, n), G(r, p, n)\}\) with \(r > 1\), \(p\) a proper divisor of \(r\), and \(n > 2\). We call these pairs **imprimitive**.

The computation of \(P(\mathcal{A}, G; t)\) for imprimitive reflection pairs proceeds by induction on \(n\). Again, this argument is an extension of that given by Brieskorn for \((\mathcal{P}_n, W_n)\).

5. **Invariants and Poincaré Polynomials for Irreducible Reflection Pairs**

5.1. Suppose \((\mathcal{A}, G)\) is an irreducible reflection pair. Then \(\mathcal{A} = \mathcal{A}(\tilde{G})\), where either (1) \(\tilde{G} = G\) or (2) \(\tilde{G} \neq G\) and there is an irreducible reflection coset, \(C\), in Table 6 or Corollary 4.12(3), such that \(\tilde{G} \in \{CC^{-1}, \langle C \rangle\}\).

Recall from Proposition 2.5(2) that

\[
H^k(M(\mathcal{A}))^G \cong \bigoplus_{X \in \mathcal{X}(\mathcal{A}, G)_k} H^k(M(\mathcal{A}_X))^{N_G(X)}.
\]

Clearly, to compute \(H^k(M(\mathcal{A}))^G\) it is sufficient to determine the set of \(X \in \mathcal{X}(\mathcal{A}, G)_k\) such that \(H^k(M(\mathcal{A}_X))^{N_G(X)} \neq 0\) and compute \(H^k(M(\mathcal{A}_X))^{N_G(X)}\) for each such \(X\).

Define

\[
\mathcal{X}(\mathcal{A}, G)^{tdi}_k = \{ X \in \mathcal{X}(\mathcal{A}, G)_k \mid \dim H^k(M(\mathcal{A}_X))^{N_G(X)} \neq 0 \}
\]

(tdi for “top degree invariants”). In practice it turns out that \(\mathcal{X}(\mathcal{A}, G)^{tdi}_k\) has at most two elements and that if \(X \in \mathcal{X}(\mathcal{A}, G)^{tdi}_k\), then \(H^k(M(\mathcal{A}_X))^{N_G(X)}\) is at most two, except when \(\mathcal{A} = \mathcal{A}_2(2r), G = G(r, p, 2),\) and \(p\) is even (in which case the dimension is equal to 3).

5.2. **Orbit representatives for primitive and imprimitive reflection pairs.** Suppose \((\mathcal{A}, G)\) is an irreducible reflection pair and \(X \in L(\mathcal{A})\). Then \(Z_G(X)\) acts faithfully on \(X^\perp\). We may identify \(Z_G(X)\) with \(Z_G(X)|_{X^\perp}\) and thus consider \(Z_G(X)\) as a subgroup of GL\((X^\perp)\) generated by reflections. Then \((Z_G(X), X^\perp)\) is an effective reflection group, that is, no irreducible factor of the reflection type of \((Z_G(X), X^\perp)\) is equal to \(A_0\).

Let \(\mathcal{T}\) denote the set of reflection types of effective complex reflection groups and define

\[
rt: L(\mathcal{A}) \to \mathcal{T} \text{ by } rt(X) = \text{“the reflection type of } (Z_G(X), X^\perp)\text{”}.
\]

It is clear that if \(g \in G\), then \((Z_G(X), X^\perp)\) and \((Z_G(gX), (gX)^\perp)\) have the same reflection type, so \(rt\) is constant on \(G\)-orbits. Define the **reflection type** of a \(G\)-orbit to be \(rt(X)\) for any \(X\) in the orbit.

5.3. Suppose that \((\mathcal{A}, G) = (\mathcal{A}(G), G)\) is a primitive reflection pair. Orlik and Solomon have shown that the map \(rt\) is very close to being an injection when restricted to \(\mathcal{X}(\mathcal{A}, G)\) and have given a classification of the orbits of \(G\) in \(L(\mathcal{A}(G))\) based on the reflection types of pointwise stabilizers of orbit representatives (see [17, App. C]). This classification is used in Theorem 5.5 and its proof as follows: Let \(\mathcal{T}(\mathcal{A}, G)\) denote the set of orbit types in the Orlik-Solomon classification. For each \(T \in \mathcal{T}(\mathcal{A}, G)\) choose \(X_T \in L(\mathcal{A})\) with \(rt(X_T) = T\). We may take \(\mathcal{X}(\mathcal{A}, G) = \{ X_T \mid T \in \mathcal{T}(\mathcal{A}, G) \}\). Then \(\mathcal{T}(\mathcal{A}, G)\) indexes both \(\mathcal{X} = \mathcal{X}(\mathcal{A}, G)\) and \(L(\mathcal{A})/G\). Define

\[
\mathcal{T}(\mathcal{A}, G)_k = \{ T \in \mathcal{T}(\mathcal{A}, G) \mid X_T \in \mathcal{X}_k \} \text{ and } \mathcal{T}(\mathcal{A}, G)^{tdi}_k = \{ T \in \mathcal{T}(\mathcal{A}, G) \mid X_T \in \mathcal{X}^{tdi}_k \}.
\]

The rule that maps an orbit in \(L(\mathcal{A}(G))\) to the isomorphism type of the pointwise stabilizer of an orbit representative is not always one-to-one. For primitive reflection pairs it turns out
that if $T$ occurs as the reflection type of the pointwise stabilizer for more than one orbit, then $T \notin \mathcal{T}(\mathcal{A}, G)^{tdi}$, so this ambiguity does not affect the computations in this paper. For example, the group $G_{27}$ has seven orbits in $L(\mathcal{G}_{27})$,

$$\mathcal{T}(\mathcal{G}_{27}, G_{27}) = \{A_0, A_1, A'_2, A''_2, B_2, I_2(5), G_{27}\},$$

so $\mathcal{T}(\mathcal{G}_{27}, G_{27})_0 = \{A_0\}$, $\mathcal{T}(\mathcal{G}_{27}, G_{27})_1 = \{A_1\}$, $\mathcal{T}(\mathcal{G}_{27}, G_{27})_2 = \{A'_2, A''_2, B_2, I_2(5)\}$, and $\mathcal{T}(\mathcal{G}_{27}, G_{27})_3 = \{G_{27}\}$. It turns out that $\mathcal{T}(\mathcal{G}_{27}, G_{27})^{tdi} = \{A_0, A_1, B_2, G_{27}\}$.

It follows from Theorem 5.5 that $|\mathcal{T}(\mathcal{A}, G)^{tdi}|$ tends to be relatively small when compared with the cardinality of $|\mathcal{T}(\mathcal{A}, G)|$. For example, $|\mathcal{T}(\mathcal{G}_{34}, G_{34})| = 24$ and $|\mathcal{T}(\mathcal{S}_8, E_8)| = 41$, but it turns out that $|\mathcal{T}(\mathcal{G}_{34}, G_{34})^{tdi}| = |\mathcal{T}(\mathcal{S}_8, E_8)^{tdi}| = 4$.

5.4. Next suppose that $(\mathcal{A}, G)$ is imprimitive. It is not unusual for several orbits to have the same reflection type. As explained in 6.11, it is more natural for computations to label the orbits of $G$ in $L(\mathcal{A})$ by partitions, specifically, partitions of $m$ with $0 \leq m \leq n$, together with a second parameter in case $m = n$.

For a partition, $\lambda$, $X_\lambda$ denotes a $G$-orbit representative with type $\lambda$. It turns out that the restriction of $rt$ to $X^{tdi}$ is injective, hence a bijection. Consequently, we may identify $\mathcal{X}(\mathcal{A}, G)^{tdi}$ and $\mathcal{T}(\mathcal{A}, G)^{tdi}$, and label these orbits by partitions, $\lambda$, or reflection types, $T$, without ambiguity. With the notation in 6.11, the partitions indexing orbit representatives in $\mathcal{X}(\mathcal{A}, G)^{tdi}$ and the corresponding reflection types in $\mathcal{T}(\mathcal{A}, G)^{tdi}$ for imprimitive reflection pairs are given in the top rows in Table 2.

**Theorem 5.5.** Suppose $(\mathcal{A}, G)$ is an irreducible reflection pair with rank greater than two.

1. If $(\mathcal{A}, G)$ is primitive or very exceptional, then $\dim H^{rkT}(M(\mathcal{A}_{X_T}))^{NG(X_T)} = 1$ for $T \in \mathcal{T}(\mathcal{A}, G)^{tdi}$. The sets $\mathcal{T}(\mathcal{A}, G)^{tdi}$ and Poincaré polynomials $P(\mathcal{A}(G), G; t)$ are given in Table 1.

2. If $(\mathcal{A}, G)$ is imprimitive, then the partitions $\lambda \in \mathcal{X}(\mathcal{A}, G)^{tdi}$, reflection types $T \in \mathcal{T}(\mathcal{A}, G)^{tdi}$, and $\dim H^{rkT}(M(\mathcal{A}_{X_T}))^{NG(X_T)}$ for $T \in \mathcal{T}_G^{tdi}$, are given in Table 2.

A first consequence of the theorem is that the polynomials $P(\mathcal{A}(G), G; t)$ have only four possible patterns.

**Corollary 5.6.** Suppose that $G$ is a non-trivial, irreducible, complex reflection group with rank at least two.

1. If $G$ is one of either $G(r, r, n)$, where either $n$ or $r$ is odd; $G_4; G_8; G_{12}; G_{16}; G_{20}; G_{22}; G_{25}; G_{32};$ or $E_6$, then

$$P(\mathcal{A}(G), G; t) = 1 + t.$$

2. If $G$ is one of either $G(r, r, n)$, where both $n$ and $r$ are even; $G_{23}; G_{24}; G_{27}; G_{29}; G_{30}; G_{31}; G_{33}; G_{34}; G_{36};$ or $G_{37}$, then

$$P(\mathcal{A}(G), G; t) = 1 + t + t^{n-1} + t^n.$$

3. If $G$ is one of either $G(r, p, n)$, where $p < r$ and either $n$ or $p$ is odd; $G_5; G_6; G_9; G_{10}; G_{13}; G_{14}; G_{17}; G_{18}; G_{21}; G_{26};$ or $G_{28}$, then

$$P(\mathcal{A}(G), G; t) = 1 + 2t + \cdots + 2t^{n-1} + t^n.$$

4. If $G$ is one of either $G(r, p, n)$, where $p < r$ and both $n$ and $p$ are even; $G_7; G_{11}; G_{15};$ or $G_{19}$, then

$$P(\mathcal{A}(G), G; t) = 1 + 2t + \cdots + 2t^{n-2} + 3t^{n-1} + 2t^n.$$
### Table 1. \((\mathcal{A}, G)\) primitive or very exceptional

| rk | \((\mathcal{A}, G)\)                              | \(\mathcal{T}(\mathcal{A}, G)^{\text{tdi}}\) | \(P(\mathcal{A}, G; t)\)  |
|----|--------------------------------------------------|---------------------------------------------|--------------------------|
| 3  | \((G_{25}, G_{25})\)                            | \(A_0, C_3\)                               | \(1 + t\)                |
| 3  | \((\mathcal{A}_3^0(3), \langle G_{3,3,3}\rangle)\) | \(A_0, A_1\)                               |                          |
| 4  | \((G_{32}, G_{32})\)                            | \(A_0, C_3\)                               |                          |
| 6  | \((E_6, E_6)\)                                  | \(A_0, A_1\)                               |                          |
| 3  | \((H_3, H_3)\)                                  | \(A_0, A_1, A_1^2, H_3\)                   | \(1 + t + t^{n-1} + t^n\) |
| 3  | \((G_{24}, G_{24})\)                            | \(A_0, A_1, B_2, G_{24}\)                  |                          |
| 3  | \((G_{27}, G_{27})\)                            | \(A_0, A_1, B_2, G_{27}\)                  |                          |
| 4  | \((G_{29}, G_{29})\)                            | \(A_0, A_1, B_3, G_{29}\)                  |                          |
| 4  | \((H_4, H_4)\)                                  | \(A_0, A_1, H_3, H_4\)                     |                          |
| 4  | \((G_{31}, G_{31})\)                            | \(A_0, A_1, G_{4,2,3}, G_{31}\)           |                          |
| 4  | \((\mathcal{F}_4, \langle F_4\rangle)\)        | \(A_0, A_1, B_3, F_4\)                     |                          |
| 5  | \((G_{33}, G_{33})\)                            | \(A_0, A_1, D_4, G_{33}\)                  |                          |
| 5  | \((\mathcal{A}_4^0(2), \langle D_4\rangle)\)   | \(A_0, A_1, A_1^3, D_4\)                   |                          |
| 6  | \((G_{34}, G_{34})\)                            | \(A_0, A_1, G_{33}, G_{34}\)               |                          |
| 7  | \((E_7, E_7)\)                                  | \(A_0, A_1, D_6, E_7\)                     |                          |
| 8  | \((E_8, E_8)\)                                  | \(A_0, A_1, E_7, E_8\)                     |                          |
| 3  | \((G_{26}, G_{26})\)                            | \(A_0, A_1, C_3, A_1 C_3, G_{3,1,2}, G_{26}\) | \(1 + 2t + 2t^2 + t^3\) |
| 4  | \((F_4, F_4)\)                                  | \(A_0, A_1, \tilde{A}_1, A_1 \tilde{A}_1, B_2, C_3, B_3, F_4\) | \(1 + 2t + 2t^2 + 2t^3 + t^4\) |

### Table 2. \((\mathcal{A}, G)\) imprimitive, \(G = G(r, p, n)\) with \(r \geq 2, n \geq 3,\) and \(p \leq r\)

| deg \(k\) \(\lambda\) \(\lambda^{\text{tdi}}\) \(T^{\text{tdi}}\) | \(k = 0\) | \(k = 1\) | \(2 \leq k \leq n - 2\) | \(k = n - 1\) | \(k = n\) |
|--------------------------|--------------------------|--------------------------|--------------------------|--------------------------|--------------------------|
| \(\mathcal{A}_n(r)\)    | \(p, n\) even           | 1                         | 1                         | 1                         | 2                         | 1                         | 2                         |
|                          | else                     | 1                         | 1                         | 1                         | 1                         | 1                         | 1                         |
| \(\mathcal{A}_n^0(r)\)  | \(p, n\) even           | 1                         | 1                         | –                         | –                         | 1                         | –                         | 1                         |
|                          | else                     | 1                         | 1                         | –                         | –                         | –                         | –                         | –                         |

5.7. If \(n = 2\), then \(1 + t + t^{n-1} + t^n = 1 + 2t + t^2\). Hence the formula in the second statement in the corollary applies to the groups \(G(r, p, 2)\) with \(p\) odd and \(p < r\), \(G_5, G_6, G_9, G_{10}, G_{13}, G_{14}, G_{17}, G_{18},\) and \(G_{21}\) as well.
The next corollary is the analog of Corollary 5.6 for the reflection pairs \((A, G(r, p, n))\), where \(A \in \{A_n^0(r), A_n(r)\}\).

**Corollary 5.8.** Suppose \(n, r, \) and \(p\) are positive integers such that \(p\) divides \(r\) and \(n \geq 2\). Then

\[
P(A_n^0(r), G(r, p, n); t) = \begin{cases} 
1 + t & \text{if } p \text{ or } n \text{ is odd}, \\
1 + t + t^{n-1} + t^n & \text{if } p \text{ and } n \text{ are even},
\end{cases}
\]

and

\[
P(A_n(r), G(r, p, n); t) = \begin{cases} 
1 + 2t + \cdots + 2t^{n-2} + 2t^{n-1} + t^n & \text{if } p \text{ or } n \text{ is odd}, \\
1 + 2t + \cdots + 2t^{n-2} + 3t^{n-1} + 2t^n & \text{if } p \text{ and } n \text{ are even}.
\end{cases}
\]

Another consequence of the theorem is the curious dichotomy implicit in the next corollary.

**Corollary 5.9.** Suppose \((A, G)\) is an irreducible reflection pair with rank at least two. Then the following are equivalent:

1. \(H^{rk G}(M(A)) = 0\), that is \((A, G)\) does not have top degree invariants.
2. \(P(A, G; t) = 1 + t\).

6. **Proof of Theorem 5.5**

In this section we prove Theorem 5.5, first for primitive reflection pairs, then for imprimitive pairs. The proof for very exceptional reflection pairs uses the results of Theorem 7.4 (for primitive and imprimitive reflection pairs) and is given in 7.9.

6.1. **Preparing for the proof.** Before giving the proof of Theorem 5.5 we state some preliminary results and work out an example. In the proof of Theorem 5.5(1) extensive use is made of the presentations given by Broué, Malle, and Rouquier. As a matter of terminology, the diagrams in [5, Tab. 1-4] that encode these presentations are called BMR diagrams.

The proofs of the next two lemmas are immediate.

**Lemma 6.2.** Suppose that \((A, G)\) and \((A', G')\) are reflection pairs. If \((A, G)\) does not have top degree invariants, then neither does \((A \times A', G \times G')\).

**Lemma 6.3.** If \((A, G)\) is an arrangement-group pair, \(X \in L(A)\), and there is a \(g \in N_G(X)\) that acts on \(H^{cd X}(M(A_X))^{Z_G(X)}\) as \(-1\), then \(H^{cd X}(M(A_X))^{N_G(X)} = 0\).

In practice, the element \(g\) frequently arises as either the long word, or an analog of the long word (in the non-Coxeter case), in a subgroup that contains \(Z_G(X)\) that is suggested by the BMR diagram of \(G\).

6.4. Suppose \((A, G)\) is an arrangement-group pair with underlying vector space \(V\) such that \(Z(G)\), the center of \(G\), acts on \(V\) as scalars. Then \(Z(G)\) acts trivially on both \(A\) and \(H^*(M(A))\), and \(Z(G) \subseteq N_G(X)\) for \(X \in L(A)\).

**Lemma 6.5.** Suppose \((A, G)\) is an arrangement-group pair such that \(Z(G)\) acts on \(V\) as scalars and \(X \in L(A)\). If \(N_G(X) = Z_G(X) \cdot Z(G)\), then \(N_G(X)\) acts trivially on \(H^{cd X}(M(A_X))^{Z_G(X)}\). In particular, if \(X \neq 0\) and \(|G| = |Z_G(X)| \cdot |Z(G)| \cdot |GX|\), then \(N_G(X)\) acts trivially on \(H^{cd X}(M(A_X))^{Z_G(X)}\).
6.6. Example: the group $G_{32}$. To illustrate the ideas in the proof of Theorem 5.5, in this subsection we prove the theorem for the reflection pair $(\mathcal{G}_{32}, G_{32})$, assuming that the theorem has been proved for groups with rank less than four. Set

$$G = G_{32}, \quad \mathcal{G} = \mathcal{A}(G) = \mathcal{G}_{32}, \quad \text{and} \quad \mathcal{T} = \mathcal{T}(\mathcal{G}, G).$$

The orbits in $L(\mathcal{G})/G$ are indexed by reflection types, and as in 5.3,

$$\mathcal{T} = \{ A_0, \mu_3, G_4, \mu_3^2, \mu_3 G_4, G_{25}, G_{32} \}.$$

By Proposition 2.5(3) it is sufficient to compute $H^{rkT}(M(\mathcal{G}_{X_T}))^{N(T)}$, for $T \in \mathcal{T}$, where to minimize parentheses and subscripts we have set $N(T) = N_G(X_T)$. This computation is done recursively by rank, building up from $\mathcal{T}_0$ and $\mathcal{T}_1$.

It was observed in 2.6 that $\mathcal{T}_0 \cup \mathcal{T}_1 \subseteq \mathcal{T}(\mathcal{G}, G)^{rdi}$. Because $\mathcal{T}_1 = \{ \mu_3 \}$, $G$ acts transitively on $\mathcal{G}$, and so $dim H^{rkT}(M(\mathcal{G}_{X_T}))^{N(T)} = 1$ for $T \in \mathcal{T}_0 \cup \mathcal{T}_1$.

We are assuming that we have already shown that the reflection pair $(\mathcal{G}_4, G_4)$ does not have high degree invariants, so $G_4 \notin \mathcal{T}^{rdi}$. It follows from Lemma 6.2 that $\mu_3 G_4 \notin \mathcal{T}^{rdi}$.

Consider the orbit indexed by the reflection type $T = \mu_3^2$. Referring to the BMR diagram in [5, Tab. 1] for $G_{32}$, define

$$X_T = \text{Fix}(\langle s, u \rangle) \quad \text{and} \quad g = (stu)^2.$$

Then $Z_G(X_T) = \langle s, u \rangle$ and by Proposition 2.9, $H^2(M(\mathcal{G}_{X_T}))^{Z_G(X_T)}$ is one-dimensional with basis $\{ h_s h_u \}$. Using the fact that $g^2$ is a generator of the center of $G_{25}$ (see the row indexed by $G_{25}$ in [5, Tab. 1]), it is straightforward to check that conjugation by $g$ interchanges $s$ and $u$, so $g$ acts as $-1$ on $H^2(M(\mathcal{G}_{X_T}))^{Z_G(X_T)}$, and so $H^2(M(\mathcal{G}_{X_T}))^{N(T)} = 0$ by Lemma 6.3. Therefore $\mu_3^2 \notin \mathcal{T}^{rdi}$.

We are assuming that we have already shown that the reflection pair $(\mathcal{G}_{25}, G_{25})$ does not have high degree invariants, so $G_{25} \notin \mathcal{T}^{rdi}$.

Finally, set $Z_0 = Z_G(X_{A_0})$, $N_0 = N_G(X_{A_0})$, $Z_1 = Z_G(X_{\mu_3})$, and $N_1 = N_G(X_{\mu_3})$. Then by 2.8(a),

$$0 = \dim H^0(M(\mathcal{G}(Z_0)))^{N_0} - \dim H^1(M(\mathcal{G}(Z_1)))^{N_1} + (-1)^4 \dim H^4(M(\mathcal{G}))^G$$

$$= 1 - 1 + \dim H^4(M(\mathcal{G}))^G.$$

Therefore, $H^4(M(\mathcal{G}))^G = 0$. It follows that $\mathcal{T}^{rdi} = \{ A_0, \mu_3 \}$ and $\dim H^{rkT}(M(\mathcal{G}_{X_T}))^{N(T)} = 1$ for $T \in \mathcal{T}^{rdi}$.

6.7. Primitive reflection pairs. In this subsection $(\mathcal{G}, G)$ is a primitive reflection pair. Thus $G$ is an exceptional complex reflection group with rank at least three, $\mathcal{G} = \mathcal{A}(G)$, and $\mathcal{T}(\mathcal{G}, G)$ indexes $L(\mathcal{G})/G$, as described in 5.3. To simplify the notation and help keep the subscripts under control, set

$$\mathcal{T} = \mathcal{T}(\mathcal{G}, G), \quad \mathcal{X} = \mathcal{X}(\mathcal{G}, G) = \{ X_T \mid T \in \mathcal{T} \}, \quad \text{and} \quad \mathcal{G}_T = \mathcal{G}_{X_T} = \mathcal{A}(Z_G(X_T)).$$

We compute $\dim H^{rkT}(M(\mathcal{G}_T))^{N_T}$ for $T \in \mathcal{T}$ recursively, following the same line of reasoning as for $G_{32}$ in 6.6.

To begin, as noted in 2.6, $\mathcal{T}_0 \cup \mathcal{T}_1 \subseteq \mathcal{T}^{rdi}$ and $\dim H^{rkT}(M(\mathcal{G}_T))^{N_T} = 1$ for $T \in \mathcal{T}_0 \cup \mathcal{T}_1$. 

Proof. The first assertion is clear. With the hypotheses in the second assertion, $Z_G(X)$ intersects $Z(G)$ trivially, so it follows from the equality $|G| = |Z_G(X)| \cdot |Z(G)| \cdot |GX|$ that $N_G(X) = Z_G(X) \cdot Z(G)$. □
Next consider orbits indexed by reflection types $T$ with $1 < \text{rk} T < \text{rk} G$. Again, it turns out that most of the time it follows from Lemma 6.2 that $\dim H^{\text{rk} T}(M(\mathcal{G}_T))^{N_T} = 0$. Table 3 contains the pairs $(G, T)$ where $G$ is one of the groups under consideration and $T$ is the reflection type indexing an orbit of $G$ in $L(\mathcal{G})$ to which Lemma 6.2 does not apply. In all but four of these cases, the dimension of $H^{\text{rk} T}(M(\mathcal{G}_T))^{N_T}$ can be determined using Lemma 6.3 or Lemma 6.5. The entries in the table should be interpreted as follows:

- Dots are included to improve readability.
- The reflection type $T = A_1^k$ denotes $k$ copies of $A_1$ for some $k \geq 2$. As indicated by “−1”, Lemma 6.3 applies in all cases.
- An entry of “−1” indicates that Lemma 6.3 applies, so $\dim H^{\text{rk} T}(M(\mathcal{G}_T))^{N_T} = 0$.
- An entry of “1” indicates that Lemma 6.5 applies and $N_T$ acts on $H^{\text{rk} T}(M(\mathcal{G}_T))^Z_T$ trivially. In all cases it turns out, by recursion, that $\dim H^{\text{rk} T}(M(\mathcal{G}_T))^{N_T} = 1$.
- An entry of “ib” indicates that the corresponding parabolic subgroup is “bulky.” It follows from [18] that for these two pairs, $N_T$ centralizes $Z_T$, and so by recursion, $\dim H^{\text{rk} T}(M(\mathcal{G}_T))^{N_T} = \dim H^{\text{rk} T}(M(\mathcal{G}_T))^Z_T = \dim H^{\text{rk} T}(M(\mathcal{G}_T))^Z_T = 1$.
- The entry “6.8” for the pair $(G_{31}, G(4, 2, 2))$ is a reference to that subsection, where it is shown that $N_T$ acts on $H^2(M(\mathcal{G}_T))^{N_T}$ as the symmetric group $W_3$ acting on its reflection representation, and so $\dim H^2(M(\mathcal{G}_T))^{N_T} = 0$.
- The entry “6.9” for the pair $(G_{33}, D_4)$, is a reference to that subsection, where it is shown that $N_T$ acts trivially on $H^4(M(\mathcal{G}_T))^{N_T}$, and so by recursion we have $\dim H^4(M(\mathcal{G}_T))^{N_T} = 1$.

Finally, $\dim H^{\text{rk} G}(M(\mathcal{G}))^G$ can be computed in each case from the rows in Table 3 using the values of $|\mathcal{G}/G|$ given in the tables in [17, App. C] and 2.8(a). For example, for the group $G = G_{34}$ we have $|\mathcal{G}/G| = 1$ and $1 − 1 + 1 − \dim H^{\text{rk} G}(M(\mathcal{G}))^G = 0$, so $\dim H^{\text{rk} G}(M(\mathcal{G}))^G = 1$.

### Table 3. Orbit reflection types not covered by Lemma 6.2

| $G$ | $A_1^k$ | $A_1 \tilde{A}_1$ | $A_1 \mu_3$ | $\mu_3^2$ | $B_2$ | $G_{3,1,2}$ | $G_{4,2,2}$ | $B_3$ | $\mu_3$ | $G_{4,2,3}$ | $H_3$ | $D_4$ | $G_{33}$ | $A_1 D_4$ | $D_6$ | $E_7$ |
|-----|---------|-----------------|-------------|----------|------|-------------|-------------|------|-------|-------------|------|------|--------|----------|------|------|
| $H_3$ | $-1$    | .               | .           | .        | .    | .           | .           | .    | .     | .           | .    | .    | .      | .         | .    | .    |
| $G_{24}$ | $-$    | .               | .           | .        | .    | .           | .           | .    | .     | .           | .    | .    | .      | .         | .    | .    |
| $G_{25}$ | $-$    | .               | .           | $-1$     | .    | .           | .           | .    | .     | .           | .    | .    | .      | .         | .    | .    |
| $G_{26}$ | $-$    | .               | .           | $1$      | .    | .           | .           | .    | .     | .           | .    | .    | .      | .         | .    | .    |
| $G_{27}$ | $-$    | .               | .           | $1$      | .    | .           | .           | .    | .     | .           | .    | .    | .      | .         | .    | .    |
| $F_4$  | $-1$    | $ib$            | .           | $ib$     | .    | .           | .           | .    | .     | .           | .    | .    | .      | .         | .    | .    |
| $G_{29}$ | $-1$    | $-$             | .           | $-1$     | .    | .           | .           | .    | .     | .           | .    | .    | .      | .         | .    | .    |
| $H_4$  | $-1$    | .               | .           | .        | .    | .           | .           | .    | .     | .           | .    | .    | .      | .         | .    | .    |
| $G_{31}$ | $-1$    | .               | .           | 6.8      | .    | .           | .           | .    | .     | .           | .    | .    | .      | .         | .    | .    |
| $G_{32}$ | $-$    | .               | .           | $-1$     | .    | .           | .           | .    | .     | .           | .    | .    | .      | .         | .    | .    |
| $G_{33}$ | $-1$    | .               | .           | .        | .    | .           | .           | .    | .     | .           | .    | .    | .      | .         | .    | .    |
| $G_{34}$ | $-1$    | .               | .           | .        | .    | .           | .           | .    | .     | .           | .    | .    | .      | .         | .    | .    |
| $E_6$  | $-1$    | .               | .           | .        | .    | .           | .           | .    | .     | .           | .    | .    | .      | .         | .    | .    |
| $E_7$  | $-1$    | .               | .           | .        | .    | .           | .           | .    | .     | .           | .    | .    | .      | .         | .    | .    |
| $E_8$  | $-1$    | .               | .           | .        | .    | .           | .           | .    | .     | .           | .    | .    | .      | .         | .    | .    |
6.8. The orbit with reflection type \( G(4, 2, 2) \) in \( G_{31} \). In this subsection \( G = G_{31} \), \( \mathcal{G} = \mathcal{G}_{31} \), \( T = G(4, 2, 2) \), and we sketch the calculation that \( H^2(M(\mathcal{G}_T))^{N_T} = 0 \).

Referring to the BMR diagram in [5, Tab. 3] for \( G_{31} \), take

\[ Z_T = (s, t, u) \]

so \( Z_T \) is a reflection group with reflection type \( G(4, 2, 2) \).

We need a description of \( H^2(M(\mathcal{G}_T)) \). To that end, set

\[ s' = tst = usu, \quad t' = st = utu, \quad \text{and} \quad u' = tut = sus. \]

Then

\[ \mathcal{G}_T = \{ H_s, H_t, H_u, H_{s'}, H_{t'}, H_{u'} \}, \]

where \( H_x = \text{Fix}(x) \), and \( \{ h_s h_s', h_s h_t, h_s h_{t'}, h_s h_u, h_s h_{u'} \} \) is a basis of \( H^2(M(\mathcal{G}_T)) \).

Define

\[ z = stu, \quad w_1 = svtwsv \quad \text{and} \quad w_2 = wtwuuw, \]

where \( w \) is as in the BMR diagram in [5, Tab. 3] for \( G_{31} \). Proofs of the following assertions are straightforward and are omitted.

- The center of \( Z_T \) is a cyclic group of order 4 generated by \( z \), and \( \{ 1, s, t, u \} \) is a cross section of \( \langle z \rangle \) in \( Z_T \).
- The set \( \{ e_{Z_T} \cdot h_s h_s, e_{Z_T} \cdot h_t h_u \} \) is a basis of \( H^2(M(\mathcal{G}_T))^{Z_T} \).
- Conjugation by \( w_1 \) maps \( s \) to \( t z^2 \), \( t \) to \( s \), and fixes \( u \). Hence \( w_1 \cdot h_s = h_t, w_1 \cdot h_t = h_s, \) and \( w_1 \cdot h_u = h_u \). Similarly, \( w_2 \cdot h_s = h_t, w_2 \cdot h_t = h_u, \) and \( w_2 \cdot h_u = h_s \).
- The subgroup of \( N_T / Z_T \) generated by \( w_1 Z_T \) and \( w_2 Z_T \) is isomorphic to the symmetric group \( S_3 \) and acts on \( H^2(M(\mathcal{G}_T))^{Z_T} \) as the reflection representation of \( S_3 \).

It follows from the last assertion that \( H^2(M(\mathcal{G}_T))^{N_T} = 0 \), as claimed.

6.9. The orbit with reflection type \( D_4 \) in \( G_{33} \). In this subsection \( G = G_{33} \), \( \mathcal{G} = \mathcal{G}_{33} \), \( T = D_4 \), and we sketch an argument that shows that \( N_T \) acts trivially on \( H^4(M(\mathcal{G}_T))^{Z_T} \).

Referring to the first BMR diagram in [5, Tab. 4] for \( G_{33} \), define

\[ z_1 = stwtst, \quad z_2 = wtwvuw, \quad \text{and} \quad g = sttwz_g \]

Then conjugation by \( z_1 \) interchanges \( s \) and \( w \) and fixes \( t \), and conjugation by \( z_2 \) interchanges \( w \) and \( v \) and fixes \( u \). Therefore, conjugation by \( g \) maps \( s \) to \( v \), \( v \) to \( w \), and \( w \) to \( s \).

Now set \( r = tut \). It is straightforward to check that the subgroup generated by \( s, w, v, \) and \( r \) is a Coxeter group of type \( D_4 \) with \( \{ s, w, v, r \} \) as a Coxeter generating set. Take

\[ Z_T = \langle s, w, v, r \rangle. \]

A short calculation using the relation \( utwutw = wtuwtu \) shows that \( grg^{-1} = r \). Hence, \( g \in N_T \) and conjugation by \( g \) acts on \( Z_T \) as the triality automorphism of \( D_4 \) with respect to the Coxeter generating set \( \{ s, w, v, r \} \). It is straightforward to check that \( |\langle g \rangle| = 6 = |N_T : Z_T| \) and that \( \langle g \rangle \cap Z_T \) is the trivial subgroup, so \( \langle g \rangle \) is a complement to \( Z_T \) in \( N_T \).

It follows from [8] and the isomorphism

\[ H^4(M(\mathcal{G}_T)^{Z_T})^{D_4} \]

that \( H^4(M(\mathcal{G}_T))^{Z_T} \) is one-dimensional with basis vector \( e_{Z_T} h_s h_w h_v h_r \). Then

\[ g e_{Z_T} h_s h_w h_v h_r = e_{Z_T} g h_s h_w h_v h_r = e_{Z_T} h_v h_s h_w h_r = e_{Z_T} h_s h_w h_v h_r, \]

and it follows that \( N_T \) acts trivially on \( H^4(M(\mathcal{G}_T))^{Z_T} \), as asserted.

This completes the proof of Theorem 5.5(1) for primitive reflection pairs.
6.10. **Imprimitive reflection pairs.** We now turn to the proof of Theorem 5.5(2). To continue we need to establish some more notation related to the groups $G(r, p, n)$ and the arrangements $\mathcal{A}_n^0(r)$ and $\mathcal{A}_n(r)$. When considering one of these arrangements, the underlying vector space is always $V = \mathbb{C}^n$ and unless otherwise noted, $r, n \geq 2$.

For $1 \leq i \leq n$ let $x_i \in V^*$ denote projection on the $i^{th}$ coordinate. Suppose $1 \leq i \neq j \leq n$ and $\zeta \in \mu_r$. Define hyperplanes

$$H_i = \ker x_i = \{ v \in V \mid v_i = 0 \} \quad \text{and} \quad H_{ij}(\zeta) = \ker(x_i - \zeta x_j) = \{ v \in V \mid v_i = \zeta v_j \},$$

where $v_k = x_k(v)$. It is convenient to set $H_{ij} = H_{ij}(1)$. Then

$$\mathcal{A}_n^0(r) = \{ H_{ij}(\zeta) \mid 1 \leq i < j \leq n, \ \zeta \in \mu_r \} = \mathcal{A}(G(r, r, n))$$

and

$$\mathcal{A}_n(r) = \{ H_{ij}(\zeta) \mid 1 \leq i < j \leq n, \ \zeta \in \mu_r \} \amalg \{ H_i \mid 1 \leq i \leq n \} = \mathcal{A}(G(r, p, n))$$

when $p < r$. By convention, $\mathcal{A}_1^0(r) = \emptyset$ is the empty arrangement. A special case is $r = 1$: $\mathcal{A}_n^0(1) = B_n = \mathcal{A}(W_n)$ is the $n^{th}$ braid arrangement.

Let $D_{r,n}$ denote the subgroup of $\text{GL}_n(\mathbb{C})$ consisting of diagonal matrices with entries in $\mu_r$. The determinant map restricts to a surjective group homomorphism from $D_{r,n}$ to $\mu_r$. If $p$ is a divisor of $r$, define $D(r, p, n)$ to be the preimage of the subgroup $\mu_{r/p}$. Then $G(r, p, n) = W_n D(r, p, n)$. With this notation we have $W_n = G(1,1,1) \subseteq G(r, p, n)$.

6.11. **Orbit representatives, centralizers, and subarrangements.** Suppose $n \geq 2$ and let $\lambda = (\ell_1, \ldots, \ell_a)$ be a partition of $m$ for an integer $m$ with $0 \leq m \leq n$. Set $\ell_0 = n-m$ and for $i > 0$ let $\bar{\ell}_i = n-m+\ell_1+\cdots+\ell_i$ denote the sum of $n-m$ and the $i^{th}$ partial sum of $\lambda$. Let $\{e_1, \ldots, e_n\}$ be the standard basis of $\mathbb{C}^n$ and for $1 \leq i \leq a$ define $b_i = e_{\bar{\ell}_{i-1}+1} + \cdots + e_{\bar{\ell}_i}$. Finally, define

$$X_\lambda = \text{span}\{ b_1, \ldots, b_a \}.$$ 

Then $X_\lambda$ is an $a$-dimensional subspace of $\mathbb{C}^n$. Notice that $X_\emptyset = 0$ and $X_{(1^n)} = V$, and that $\text{cd}X_\lambda = n-a$.

For a positive integer $p$, define

$$\delta_{\lambda;p} = \gcd(p, \ell_1, \ldots, \ell_a)$$

and define $d_0 \in \text{GL}(\mathbb{C}^n)$ to be the diagonal matrix with entries $\omega_r, 1, \ldots, 1$.

It follows from the computations in [17, §6.4] that

$$\mathcal{X}(\mathcal{A}_n^0(r), G(r, p, n)) = \{ X_\lambda \mid \lambda \vdash m, \ 0 \leq m \leq n-2 \} \amalg \{ d_0^n X_\lambda \mid \lambda \vdash n, \ 0 \leq u < \delta_{p,\lambda} \}$$

and

$$\mathcal{X}(\mathcal{A}_n(r), G(r, p, n)) = \{ X_\lambda \mid \lambda \vdash m, \ 0 \leq m \leq n-1 \} \amalg \{ d_0^n X_\lambda \mid \lambda \vdash n, \ 0 \leq u < \delta_{p,\lambda} \}$$

are sets of orbit representatives for the action of $G(r, p, n)$ on $\mathcal{A}_n^0(r)$ and $\mathcal{A}_n(r)$, respectively.

6.12. With $\lambda = (\ell_1, \ldots, \ell_a)$ as above, it is straightforward to check that $Z_{G(r, p, n)}(X_\lambda)$ is the “diagonal” subgroup of $G(r, p, n)$ isomorphic to $G(r, p, n-m) \times W_{\ell_1} \times \cdots \times W_{\ell_a}$, consisting of block diagonal matrices with blocks $g, w_1, \ldots, w_a$, where $g \in G(r, p, n-m)$ and $w_i \in W_{\ell_i}$. Then

(a) \[ Z_{G(r, p, n)}(X_\lambda) = G(r, p, n-m) \boxtimes W_{\ell_1} \boxtimes \cdots \boxtimes W_{\ell_a} \subseteq G(r, p, n), \]

where $\boxtimes$ denotes block diagonal direct sum. In particular, $Z_{G(r, p, n)}(X_\lambda)$ is a reflection subgroup of $\text{GL}(\mathbb{C}^n)$.
It is also straightforward to check that
\[ \mathcal{A}_n^0(r)_{X_{\lambda}} \cong \mathcal{A}_{n-m}(r) \times B_{\ell_1} \times \cdots \times B_{\ell_a} \]
and
\[ \mathcal{A}_n(r)_{X_{\lambda}} \cong \mathcal{A}_{n-m}(r) \times B_{\ell_1} \times \cdots \times B_{\ell_a}. \]

(b) Recall that \( W_1 \) is the trivial group acting on \( \mathbb{C} \) and that \( B_1 = \emptyset \). Thus, if \( \lambda \) has \( b \) parts of size greater than 1, then the reflection type of \( Z_{G(r,p,n)}(X_{\lambda}) \) is \( G(r,p,n-m)B_{\ell_{t_1}} \cdots B_{\ell_{t_b}} \).

6.13. The reflection pairs \((\mathcal{A}_n(r), G(r,p,n))\). In the rest of this section we take
\[ G = G(r,p,n), \quad A = \mathcal{A}_n(r), \quad \text{and} \quad X = \mathcal{X}(A,G), \]
where \( p \leq r \) and \( r,n \geq 2 \). To prove Theorem 5.5(2) for the reflection pair \((A,G)\) we need to compute \( \dim H^{cd,X}(M(\mathcal{A}_X))^N_G(X) \) for \( X \in \mathcal{X} \). The proof is by induction on \( n \). The induction begins with the base cases \( n = 2 \) and \( n = 3 \). The case \( n = 2 \) follows from Proposition 2.9. Details of the argument when \( n = 3 \) are left to the reader. For the inductive step, we assume that \( n \geq 4 \), and so \( G(r,p,n-2) \) has rank at least two.

The strategy is to first show that \( \dim H^{cd,X}(M(\mathcal{A}_X))^N_G(X) = 0 \) unless \( X = X_{\lambda} \), where \( \lambda \) is either the empty partition, corresponding to the reflection type \( G \), or a partition with largest part at most 2, and at most one part equal to 2. For these latter partitions, \( \dim H^{cd,X}(M(\mathcal{A}_X))^N_G(X) \) is computed using induction and some elbow grease. Finally, once \( \dim H^{cd,X}(M(\mathcal{A}_X))^N_G(X) \) has been computed for \( X \neq X_{\emptyset} \) (or \( T \neq G \)), then \( \dim H^{cd,X}(M(\mathcal{A}_X))^N_G(X) = \dim H^n(M(\mathcal{A}))^G \) can be determined using 2.8 (a).

The proof of Theorem 5.5(2) when \( \mathcal{A} = \mathcal{A}_n^0(r) \) is similar, but uses no new ideas and is technically less complicated, so is omitted.

6.14. First notice that if \( \lambda \) is a partition of \( n \) and \( d \) is a diagonal matrix in \( G(r,1,n) \), then \( \dim H^{cd,X}(M(\mathcal{A}_{dX_{\lambda}}))^N_G(X_{\lambda}) = \dim H^{cd,X}(M(\mathcal{A}_{X_{\lambda}}))^N_G(X_{\lambda}) \), so it is sufficient to compute \( \dim H^{cd,X}(M(\mathcal{A}_{X_{\lambda}}))^N_G(X_{\lambda}) \) for \( X_{\lambda} \in \mathcal{X} \).

6.15. Let \( \lambda = (\ell_1, \ldots, \ell_a) \) be a partition of \( m \), with \( 0 \leq m \leq n \), that has \( b \) parts of size greater than 1. Set
\[ V_0 = \text{span}\{ e_1, \ldots, e_{n-a+b} \} \quad \text{and} \quad V_1 = \text{span}\{ e_{n-a+b+1}, \ldots, e_n \} \]
and let \( p: \mathbb{C}^n \to V_0 \) be the projection. Then
\[ V_0 = \mathbb{C}^{n-m} \times \mathbb{C}^{\ell_1} \times \cdots \times \mathbb{C}^{\ell_b}, \quad p(\mathcal{A}_{X_{\lambda}}) = \mathcal{A}_{n-m}(r) \times B_{\ell_1} \times \cdots \times B_{\ell_b}, \]
and
\[ p(Z_G(X_{\lambda})) = G(r,p,n-m) \times W_{\ell_1} \times \cdots \times W_{\ell_b}, \]
where as in 1.11 \( p(Z_G(X_{\lambda})) \) denotes the image of \( Z_G(X_{\lambda}) \) in \( \text{GL}(V_0) \). Also as in 1.11, the Künneth Theorem induces an isomorphism
\[ (a) \quad H^{n-a}(M(\mathcal{A}_{X_{\lambda}}))^Z_G(X_{\lambda}) \cong H^{n-m}(M(\mathcal{A}_{n-m}(r)))^{G(r,p,n-m)} \otimes H^{\ell_1-1}(M(\mathcal{B}_{\ell_1}))^{W_{\ell_1}} \otimes \cdots \otimes H^{\ell_b-1}(M(\mathcal{B}_{\ell_b}))^{W_{\ell_b}}. \]

Lemma 6.16. If \( H^{cd,X}(M(\mathcal{A}_{X_{\lambda}}))^N_G(X_{\lambda}) \neq 0 \), then either \( \lambda = \emptyset \) or \( \lambda = (2^b, 1^{a-b}) \), where \( b \leq 1 \) and \( a + b = m \).
Proof. Brieskorn [3] has shown that $P(\mathcal{B}_t, W_i; t) = 1 + t$ for $\ell \geq 2$. Thus, if $\ell_1 > 2$, then $(\mathcal{B}_t, W_i)$ does not have top degree invariants, so it follows from 6.15(a) and Lemma 6.2 that $H^{n-a}(M(\mathcal{A}_X),)^{Z_G(X)} = 0$. Therefore, if $\dim H^{cd\mathcal{X}}(M(\mathcal{A}_X),)^{N_G(X)} \neq 0$, then $\lambda = (2^b, 1^{a-b})$, where $a + b = m$.

Suppose $\lambda = (2^b, 1^{a-b})$ with $a + b = m$ and $b > 1$. For the moment, let $s_i \in W^n$ denote the reflection that interchanges the basis vectors $e_i$ and $e_{i+1}$ and define $\tilde{H}_1 = \text{Fix}(s_{n-m+1})$, $\tilde{H}_3 = \text{Fix}(s_{n-m+1})$, and $g$ to be the longest element in the Coxeter group of type $A_3$ generated by $\{s_{n-m+1}, s_{n-m+2}, s_{n-m+3}\}$. Then $\tilde{H}_1, \tilde{H}_3 \subset \mathcal{A}_X$, $g \in N_G(X)$, and $g$ acts on $\mathcal{A}_X$ by interchanging $\tilde{H}_1$ and $\tilde{H}_3$, and fixing $\mathcal{A}_X \setminus \{\tilde{H}_1, \tilde{H}_3\}$ elementwise. By (a), $H^{cd\mathcal{X}}(M(\mathcal{A}_X),)^{N_G(X)}$ is spanned by products of the form $x\tilde{h}_1\tilde{h}_3y$, where $x \in H^{n-m}(M(\mathcal{A}_n-m(r)))^{G(r,p,n-m)}$, $\tilde{h}_1$ and $\tilde{h}_3$ are the Orlik-Solomon generators corresponding to $\tilde{H}_1$ and $\tilde{H}_3$, respectively, and $y \in H^1(M(\mathcal{B}))_{(a-b)}$, where we have used the identification

$$Z_G(X) = G(r, p, n - m) \otimes W_2 \otimes \cdots \otimes W_b \otimes W_1 \otimes \cdots \otimes W_a \subseteq G$$

in 6.12(a). Then $g \cdot x\tilde{h}_1\tilde{h}_3y = x\tilde{h}_3\tilde{h}_1y = -x\tilde{h}_1\tilde{h}_3y$, and so $H^{cd\mathcal{X}}(M(\mathcal{A}_X),)^{N_G(X)} = 0$, by Lemma 6.3.

6.17. In a further effort to keep levels of subscripts under control, in the rest of this section, for a partition, $\lambda$, set

$$cd \, \lambda = cd \, X_\lambda \quad Z(\lambda) = Z_G(X_\lambda), \quad \text{and} \quad N(\lambda) = N_G(X_\lambda).$$

There are $2n - 1$ non-empty partitions of the form $(2^b 1^{a-b})$ with $0 \leq a + b \leq n$, namely

$$\eta_k = (1^{n-k}), \text{ for } 0 \leq k \leq n - 1, \quad \text{and} \quad \tau_k = (1^{n-k-1}), \text{ for } 1 \leq k \leq n - 1,$$

where the notation is chosen so that $cd \eta_k = cd \tau_k = k$.

Clearly, the only partitions of $n$ in this collection are $\eta_0$ and $\tau_1$. Since $\delta_{p, \eta_0} = \delta_{p, \tau_1} = 1$, each of $\eta_0$ and $\tau_1$ indexes a unique orbit in $L(\mathcal{A})$.

Next, corresponding to codimension 0 and 1, there are three special cases: $\eta_0 = (1^n)$, $\eta_1 = (1^{n-1})$, and $\tau_1 = (1^{n-2})$.

- $X_{\eta_0} = X_{(1^n)} = V$, $Z(\eta_0)$ is the trivial subgroup, with reflection type $A_0$, and we have seen that $\dim H^0(M(\mathcal{A}_r)) = 1$.
- $X_{\eta_1} = X_{(1^{n-1})}$ is the hyperplane $H_n \subset \mathbb{C}^n$, $Z(\eta_1) \cong G(r, p, 1)$ and has reflection type $\mu_p$ if $p < r$ and $A_0$ if $p = r$, and clearly $\dim H^1(M(\mathcal{A}_H))^{Z(\eta_1)} = 1$.
  When $p = r$, the reflection type $A_0$ indexes two orbits, namely those indexed by $\eta_0 = (1^n)$ and $\eta_1 = (1^{n-1})$. In order to distinguish these two orbits we denote the reflection type of $Z_G(r, r, n)(X_{(n-1)})$ by $G$, $r, r, 1$.
- $X_{\tau_1} = X_{(2^{n-2})}$ is the hyperplane $H_{1,2} \subset \mathbb{C}^n$, $Z(\tau_1)$ has reflection type $A_1$, and it follows from 1.11 that $\dim H^1(M(\mathcal{A}_{H_{1,2}}))^{Z(\tau_1)} = 1$.

For $k > 1$, the reflection type of the pointwise stabilizer $Z(\eta_k)$ is $G(r, p, k)$ and the reflection type of the pointwise stabilizer $Z(\tau_k)$ is $G(r, p, k-1)A_1$.

Lemma 6.18. Suppose that $2 \leq k \leq n - 1$.

1. $H^k(M(\mathcal{A}_{\eta_k}))^{N(\eta_k)} \cong H^k(M(\mathcal{A}_{\tau_k}(r)))^{G(r, 1,k)}$.
2. If $k < n - 1$, then $H^k(M(\mathcal{A}_{\tau_k}))^{N(\tau_k)} \cong H^{k-1}(M(\mathcal{A}_{\tau_{k-1}}))^{G(r, 1,k-1)} \otimes H^1(M(\mathcal{B}_2))$.
(3) If $k = n - 1$, then
\[ H^{n-1}(M(\mathcal{A}_{r-1}))^{N_{\tau(n-1)}} = H^{n-2}(M(\mathcal{A}_{r-1}))^{G(r,p,n-2)} \oplus H^1(M(\mathcal{B}_2)). \]

Proof. The proof of (1) is similar to the proof of (2) and is omitted. We prove (2) and (3).

Set $\tau = \tau_k$. In coordinates, $X_\tau$ is the set of all vectors in $\mathbb{C}^n$ of the form $[0 \ a \ a \ v]^t$, where $0 \in \mathbb{C}^{k-1}$, $a \in \mathbb{C}$, $v \in \mathbb{C}^{n-k-1}$, and the superscript $t$ denotes transpose.

It follows from 6.12(a) that
\begin{itemize}
  \item $Z(\tau) = G(r, p, k - 1) \boxtimes W_2$ is the group of block diagonal matrices $\begin{bmatrix} g_{k-1} & w_2 & I_{n-k-1} \end{bmatrix}$, where $g_{k-1} \in G(r, p, k - 1)$, $w_2 \in W_2$, and $I_{n-k-1}$ is the identity matrix, and
  \item $N(\tau)$ is the group of block diagonal matrices $\begin{bmatrix} w_{k-1} & w_2 & e & w_{n-k-1} \end{bmatrix}$, where $w_j \in W_j$, $d \in D_{r,k-1}$, $e$ is a scalar matrix in $D_{r,2}$, $f \in D_{r,n-k-1}$, and $d \cdot e \cdot det f \in \mu_{r/p}$.
\end{itemize}

For any divisor $q$ of $r$, let
\[ Z^q = Z_{G(r,q,n)}(X_\tau) \quad \text{and} \quad N^q = N_{G(r,q,n)}(X_\tau), \]
be the pointwise and setwise stabilizers of $X_\tau$ in $G(r, q, n)$, respectively.

By 6.12(b), $\mathcal{A}_r \cong \mathcal{A}_{r-1}(r \times \mathcal{B}_2)$, and taking $p = 1$, it follows from the Künneth Theorem that there is an isomorphism
\begin{itemize}
  \item[(a)] $H^{k-1}(M(\mathcal{A}_{k-1}(r))) \otimes H^1(M(\mathcal{B}_2)) \cong H^k(M(\mathcal{A}_r))$
\end{itemize}
that intertwines the $G(r, 1, k - 1) \times W_2$-action on the left with the $N^1$-action on the right. Hence, there is a subspace, $U$, of $H^k(M(\mathcal{A}_r))$ such that $U \cong H^{k-1}(M(\mathcal{A}_{k-1}(r)))$ and
\[ H^k(M(\mathcal{A}_r)) = U \cdot (\mathcal{C}h_{k,k+1}), \]
where the right hand side denotes the set of sums of products. If $g_{k-1} \in G(r, 1, k - 1)$, $g_{n-k-1} \in G(r, 1, n - k - 1)$, and $u \in U$, then
\begin{itemize}
  \item[(b)] $\begin{bmatrix} g_{k-1} & I_2 & g_{n-k-1} \end{bmatrix} \cdot u \cdot h_{k,k+1} = (g_{k-1} \cdot u) \cdot h_{k,k+1}$. \end{itemize}

Suppose $k < n - 1$. Then, given $g_{k-1} \in G(r, 1, k - 1)$, it is possible to find a diagonal matrix $d_{n-k-1} \in D_{r,n-k-1}$ so that $\begin{bmatrix} g_{k-1} & I_2 & d_{n-k-1} \end{bmatrix} \in N^p = N(\tau)$. Hence, it follows from (a) and (b) that
\[ H^k(M(\mathcal{A}_r))^{N(\tau)} \cong H^{k-1}(M(\mathcal{A}_{k-1}(r)))^{G(r,1,k-1)} \otimes H^1(M(\mathcal{B}_2)). \]

This proves (2).

Now suppose $k = n - 1$. Then $\tau = (2)$ and in coordinates, $X_\tau$ is the set of all vectors in $\mathbb{C}^n$ of the form $[0 \ a \ a \ v]^t$, where $0 \in \mathbb{C}^{n-2}$ and $a \in \mathbb{C}$, and
\[ H^{n-1}(M(\mathcal{A}_2))^{Z(\tau)} \cong H^{n-2}(M(\mathcal{A}_{n-2}(r)))^{G(r,p,n-2)} \otimes H^1(M(\mathcal{B}_2)). \]
It is easy to check that $N(\tau)$ acts trivially on $\mathcal{A}_r$, when $n = 2$ and $n = 3$, and so
\[ H^{n-1}(M(\mathcal{A}_r))^{N(\tau)} = H^{n-1}(M(\mathcal{A}_r))^{Z(\tau)} \]
in these cases.
In the rest of the proof we assume that \( n \geq 4 \). Define \( \epsilon = 1 \) if \( p \) or \( n \) is odd and \( \epsilon = 2 \) otherwise. With this notation there are containments

\[
\begin{array}{cccc}
Z^c_p & \longrightarrow & Z^c_n & \longrightarrow & H^{n-1}(M(\mathcal{A}_r))^N & \longrightarrow & H^{n-1}(M(\mathcal{A}_r))^{Z^c} \\
N^c_p & \longleftarrow & N^c_n & \longleftarrow & H^{n-1}(M(\mathcal{A}_r))^{N^c} & \longleftarrow & H^{n-1}(M(\mathcal{A}_r))^{Z^c}.
\end{array}
\]

(c)

Notice that \( p \) and \( n \) are both even if and only if \( \epsilon \) and \( n - 2 \) are both even, so by induction \( H^{n-1}(M(\mathcal{A}_r))^{Z^c} \) and \( H^{n-1}(M(\mathcal{A}_r))^{Z^c} \) have the same dimension and hence must be equal. Moreover, \( N^c = Z^c \cdot Z(G(r, 1, n)) \), and so it follows from Lemma 6.5 that \( H^{n-1}(M(\mathcal{A}_r))^{N^c} = H^{n-1}(M(\mathcal{A}_r))^{Z^c} \). This shows that all four spaces in the second diagram in (c) are equal. Therefore,

\[
H^{n-1}(M(\mathcal{A}_r))^{N^p} = H^{n-1}(M(\mathcal{A}_r))^{Z^p} \cong H^{n-2}(M(\mathcal{A}_{n-2}(r)))^{G(r,p,n-2)} \otimes H^1(M(\mathcal{B}_2)).
\]

This proves (3). \( \square \)

6.19. Now consider the entries in Table 2 in the rows indexed by \( \mathcal{A} = \mathcal{A}_n(r) \).

Suppose \( k = 0 \) and \( \eta = \eta_0 \). Then we have seen that

\[
\dim H^0(M(\mathcal{A}_\eta))^N = \dim H^0(C^n) = 1.
\]

This justifies the entries in the column indexed by \( k = 0 \) in Table 2.

Suppose \( 1 \leq k \leq n - 2 \), \( \eta = \eta_k \), and \( \tau = \tau_k \). Then it follows from Lemma 6.18 and induction that

\[
\dim H^k(M(\mathcal{A}_\eta))^N = \dim H^k(M(\mathcal{A}_\eta(r)))^{G(r,1,k)} = 1
\]

and

\[
\dim H^k(M(\mathcal{A}_\tau))^N = \dim H^{k-1}(M(\mathcal{A}_{k-1}(r)))^{G(r,1,k-1)} = 1.
\]

This justifies the entries in the column indexed by \( 1 \leq k \leq n - 2 \) in Table 2.

Suppose \( k = n - 1 \), \( \eta = \eta_{n-1} \) and \( \tau = \tau_{n-1} \). Then it follows from Lemma 6.18 and induction that

\[
\dim H^{n-1}(M(\mathcal{A}_\eta))^N = \dim H^{n-1}(M(\mathcal{A}_{n-1}(r)))^{G(r,1,n-1)} = 1
\]

and

\[
\dim H^{n-1}(M(\mathcal{A}_\tau))^N = \dim H^{n-2}(M(\mathcal{A}_{n-2}(r)))^{G(r,p,n-2)} = \begin{cases} 2 & \text{if } p \text{ and } n \text{ are even}, \\ 1 & \text{if } p \text{ or } n \text{ is odd}, \end{cases}
\]

because

\[
\dim H^1(M(\mathcal{A}_2(r)))^N = \begin{cases} 2 & \text{if } p \text{ is even}, \\ 1 & \text{if } p \text{ is odd}. \end{cases}
\]

This justifies the entries in the columns indexed by \( k = n - 1 \) in Table 2.

Finally, suppose \( k = n \). Then it follows from 2.8(a) and what has already been proved that

\[
\dim H^n(M(\mathcal{A}))^G = \begin{cases} 2 & \text{if } p \text{ and } n \text{ are even}, \\ 1 & \text{if } p \text{ or } n \text{ is odd}. \end{cases}
\]
This justifies the entries in the columns indexed by \( k = n \) in Table 2 and completes the proof of Theorem 5.5(2) when \( \mathcal{A} = \mathcal{A}_n(r) \).

7. A BASIS OF \( H^*(M(\mathcal{A}))^G \)

In this section we turn to the construction of a basis of \( H^*(M(\mathcal{A}))^G \) for each reflection pair \((\mathcal{A}, G)\). We may assume that \((\mathcal{A}, G)\) is irreducible. The strategy is to find bases of \( H^{kT}(M(\mathcal{A}_{X_T}))^{N_G(X_T)} \), for \( T \in T(\mathcal{A}, G)^{tdi} \), and then use the equality

\[
H^k(M(\mathcal{A}))^G = \sum_{T \in T(\mathcal{A}, G)^{tdi}} e_G \cdot H^k(M(\mathcal{A}_{X_T}))^{N_G(X_T)}
\]

in Proposition 2.5(2). Most of the time \( H^{kT}(M(\mathcal{A}_{X_T}))^{N_T} \) is one-dimensional, so it is enough to identify a non-zero element in \( H^{kT}(M(\mathcal{A}_{X_T}))^{N_T} \). The key technical result we use is the acyclic complex in 2.7.

7.1. Suppose \((\mathcal{A}, G)\) is a reflection pair with underlying vector space \( V \). As observed in 2.6 that if \( T = A_0 \), then \( \{T\} = T(\mathcal{A}, G)^{0}_{A_0} \), \( M(\mathcal{A}_{X_T}) = V \), \( N_G(X_T) = G \), and \( \{1 = e_G \cdot 1\} \) is a basis of \( H^0(M(\mathcal{A}_{X_T}))^{N_G(X_T)} = H^0(M(\mathcal{A}))^G \). Define

\[
\text{cx}_{A_0} = 1 \in H^0(M(\mathcal{A}_{A_0})) \quad \text{and} \quad B^{A_0}_{G} = \{\text{cx}^{A_0}_{G}\}
\]

(cx for Coxeter). Similarly, if \( T = A_1 \) and \( H = X_T \), then \( T \in T(\mathcal{A}, G)^{1}_{A_1} \), \( e_{N_G(H)} \cdot \{h\} \) is a basis of \( H^1(M(\mathcal{A}_{X_T}))^{N_G(X_T)} \), and \( e_G e_{N_G(X_T)} \cdot h = e_G \cdot h \) is a non-zero element in \( H^1(M(\mathcal{A}))^G \). Define

\[
\text{cx}_{A_1} = h \in H^1(M(\mathcal{A}_{X_T})) \quad \text{and} \quad B^{A_1}_{G} = \{\text{cx}^{A_1}_{G}\}.
\]

For any \( k \) it follows from Proposition 2.5 that

\[
H^k(M(\mathcal{A})) \cong \sum_{T \in T(\mathcal{A}, G)^{tdi}} \left( \sum_{Y \in G_{X_T}} H^k(M(\mathcal{A}_Y)) \right),
\]

that \( \sum_{Y \in G_{X_T}} H^k(M(\mathcal{A}_Y)) \) is a \( \mathbb{Q}G \)-submodule of \( H^k(M(\mathcal{A})) \), isomorphic to the induced module \( \text{Ind}_{N_G(X_T)}^G H^k(M(\mathcal{A}_{X_T})) \), and that

\[
\left( \sum_{Y \in G_{X_T}} H^k(M(\mathcal{A}_Y)) \right)^G = e_G \cdot H^k(M(\mathcal{A}_{X_T})) = e_G \cdot H^k(M(\mathcal{A}_{X_T}))^{N_G(X_T)}.
\]

Suppose for the moment that \( \dim V = 2 \) and that \( \{H_1, \ldots, H_a\} \) is a set of \( G \)-orbit representatives in \( \mathcal{A} \). If \( a > 1 \), then \( \{G\} = T(\mathcal{A}, G)^{2}_{A_1} \). Define

\[
\text{cx}_{G} = h_1 h_2 \quad \text{and} \quad B^{G} = \{h_1 h_2, \ldots, h_1 h_n\}.
\]

The next corollary follows from Proposition 2.9.

**Corollary 7.2.** Suppose \((\mathcal{A}, G)\) is an irreducible reflection pair with \( \text{rk} \mathcal{A} = 2 \) and set \( T = T(\mathcal{A}, G) \).

1. For \( T \in T^{tdi} \), the set \( e_{N_G(X_T)} \cdot B^{G}_{T} \) is a basis of \( H^{rkT}(M(\mathcal{A}_{X_T}))^{N_G(X_T)} \).
2. For \( T \in T^{tdi} \), the set \( e_G B^{G}_{T} \) is a basis of \( \left( \sum_{Y \in G_{X_T}} H^{rkX_T}(M(\mathcal{A}_Y)) \right)^G \).
3. For \( k \geq 0 \), the disjoint union \( \bigcup_{T \in T^{tdi}} e_G B^{G}_{T} \) is a basis of \( H^k(M(\mathcal{A}))^G \).
The main result in this section is the analog of the statement in Corollary 7.2, but without the assumption that \( \text{rk } \mathcal{A} = 2 \). In order for this to make sense, we need to define the sets \( B_{T}^{\mathcal{A},G} \) when \( (\mathcal{A}, G) \) has rank greater than two and \( T \in T(\mathcal{A}, G)^{\text{di}} \). It turns out that when \( \dim H^{kT}(M(\mathcal{A}_{X_T}))^{N_G(X_T)} = 1 \) we can take \( B_{T}^{\mathcal{A},G} = \{ \text{cx}_T^{\mathcal{A},G} \} \), where \( \text{cx}_T^{\mathcal{A},G} \in H^{kT}(M(\mathcal{A}_{X_T})) \) is a “Coxeter-like” cohomology class.

7.3. Until 7.9, \((\mathcal{A}, G)\) is an irreducible reflection pair with rank at least three that is either primitive or imprimitive.

(1) First suppose \((\mathcal{A}, G)\) is primitive, so \( \mathcal{A} = \mathcal{A}(G) \) and \( G \) is an exceptional reflection group.

Let \( T \in T(\mathcal{A}, G)^{\text{di}} \setminus \{ A_0, A_1 \} \). Then \( \dim H^{kT}(M(\mathcal{A}_{X_T}))^{N_G(X_T)} = 1 \), by Theorem 5.5(1), and so

\[ H^{k}(M(\mathcal{A}_{X_T}))^{N_{G}(X_T)} \cong H^{k}(M(\mathcal{A}(Z_{G}(X_T))))^{Z_{G}(X_T)}, \]

where \( k = \text{rk } T \). There is a BMR diagram for \( G \), say \( \mathcal{D} \), such that a BMR diagram for \( T \) appears as an admissible subdiagram of \( \mathcal{D} \) (see [5, §1.B]). Let \( s_1, \ldots, s_k \) be generators of \( G \) indexed by the nodes in the admissible subdiagram for \( T \). For \( 1 \leq i \leq k \) define hyperplanes \( H_i = \text{Fix}(s_i) \in \mathcal{A} \) with Orlik-Solomon generators \( h_i \in H^{1}(M(\mathcal{A})). \) Then \( Z_{G}(X_T) = \langle s_1, \ldots, s_k \rangle \) by [5, 17(1)].

If \( T \) is well-generated, then \( X_T = \text{Fix}(Z_{G}(X_T)) = H_1 \cap \cdots \cap H_k \). Define

\[ \text{cx}_{T}^{\mathcal{A},G} = h_1 \cdots h_k \in H^{k}(M(\mathcal{A}_{X_T})). \]

If \( T \) is not well-generated, then \( G = G_{31} \) and \( T \in \{ G(4, 2, 3), G_{31} \} \). In this case, with the notation in [5, Tab. 3], define

\[ \text{cx}_{T}^{\mathcal{A},G} = \begin{cases} h_u h_v h_w \in H^{3}(M(\mathcal{A}_{X_T})) & \text{if } T = G(4, 2, 3), \\ h_u h_v h_w \in H^{4}(M(\mathcal{A})) & \text{if } T = G_{31}. \end{cases} \]

(2) Now suppose \((\mathcal{A}, G)\) is imprimitive and \( T \in T(\mathcal{A}, G)^{\text{di}} \setminus \{ A_0, A_1 \} \). Then \( G = G(r, p, n) \) and there are two cases.

\( \mathcal{A} = \mathcal{A}_n(r) \): By Theorem 5.5(2),

\[ T(\mathcal{A}, G)^{\text{di}} = \{ A_0, A_1 \} \cup \{ G(r, p, k) \mid 1 \leq k \leq n \} \]

\[ \cup \{ G(r, p, k - 1)A_1 \mid 2 \leq k \leq n - 1 \}. \]

With the notation in [5, 3.A] and [5, Tab. 1], the group \( G(r, 1, n) \) (denoted by \( G(d, 1, r) \) in [5]) has generators \( s, t_2, \ldots, t_n \). These define hyperplanes \( \text{Fix}(s), \text{Fix}(t_2), \ldots, \text{Fix}(t_n) \) in \( \mathcal{A} \) with corresponding Orlik-Solomon generators \( h_\varphi, h_2, \ldots, h_n \) in \( H^{1}(M(\mathcal{A})). \) Using the notation in 4.3 and 6.10, we may take \( s = \varphi \) and \( \text{Fix}(t_j) = H_{j-1,j}(1) \) for \( 2 \leq j \leq n \).

If \( \dim H^{kT}(M(\mathcal{A}_{X_T}))^{N_{G}(X_T)} = 1 \), define

\[ \text{cx}_{T}^{\mathcal{A},G} = \begin{cases} h_\varphi h_2 h_3 \cdots h_k, & \text{if } T = G(r, p, k) \text{ and } 1 \leq k \leq n - 1, \\ h_\varphi h_2 h_3 \cdots h_{k-1} h_{k+1}, & \text{if } T = G(r, p, k - 1)A_1 \text{ and } 2 \leq k \leq n - 1, \\ h_\varphi h_2 h_3 \cdots h_n, & \text{if } T = G \text{ and either } p \text{ or } n \text{ is odd}. \end{cases} \]

If \( \dim H^{kT}(M(\mathcal{A}_{X_T}))^{N_{G}(X_T)} > 1 \), then \( T \in \{ G(r, p, n - 2)A_1, G \} \) and \( p \) and \( n \) are even. To simplify the notation, set \( \tilde{T} = G(r, p, n - 2)A_1 \). Because \( p \) is even, \( G(r, p, n) \) is not well-generated unless \( p = r \), which is not under consideration because
\[ \mathcal{A} = \mathcal{A}_n(r) = \mathcal{A}(G). \]

Define a hyperplane \( H'_2 = \text{Fix}(st_2s^{-1}) \in \mathcal{A} \), corresponding to the BMR generator \( t'_2 \) in [5, Tab. 2], and then define

\[
\begin{align*}
\text{cx}_{T}^{G} &= h_\varphi h_2 h_3 \cdots h_{n-2} h_1, \\
\text{cx}_{T,1}^{G} &= h_\varphi h_2 h_3 \cdots h_{n-2} h_n,
\end{align*}
\]

and

\[
B_T^{G} = \{ \text{cx}_T^{G}, \text{cx}_{T,1}^{G} \} \subseteq H^{rkT}(\mathcal{M}(\mathcal{A}^G_X))
\]

for \( T \in \{\mathcal{T}, G\} \).

\[ \mathcal{A} = \mathcal{A}^0_n(r) : \] By Theorem 5.5(2), because \( T \notin \{A_0, A_1\} \) we have that \( p \) and \( n \) are even, \( T \in \{G(r, p, n - 2)A_1, G\} \), and \( \dim H^{rkT}(\mathcal{M}(\mathcal{A}^G_X))^{N_G(X_T)} = 1 \). With the notation just introduced, notice that the hyperplanes \( H_2, H'_2, H_3, \ldots, H_n \) lie in \( \mathcal{A}^0_n(r) \). Define

\[
\text{cx}_T^{G} = \begin{cases} 
    h_2 h_3 \cdots h_{n-2} h_n, & \text{if } T = G(r, p, n - 2)A_1, \\
    h_2 h_3 \cdots h_n, & \text{if } T = G.
\end{cases}
\]

Finally, for \( (\mathcal{A}, G) \) primitive or imprimitive, define

\[ B_T^{G} = \{ \text{cx}_T^{G} \} \quad \text{for all } T \text{ with } \dim H^{rkT}(\mathcal{M}(\mathcal{A}^G_X))^{N_G(X_T)} = 1. \]

We can now state the main result in this section.

**Theorem 7.4.** Suppose \( (\mathcal{A}, G) \) is an irreducible reflection pair and set \( T = T(\mathcal{A}, G) \).

1. For \( T \in T^{\text{dir}} \), the set \( e_{N_G(X_T)} \cdot B_T^{G} \) is a basis of \( H^{rkT}(\mathcal{M}(\mathcal{A}^G_X))^{N_G(X_T)} \).
2. For \( T \in T^{\text{dir}} \), the set \( e_G \cdot B_T^{G} \) is a basis of \( \left( \sum_{Y \in G_X T} H^{rkX_T}(\mathcal{M}(\mathcal{A}^G_Y)) \right)^G \).
3. For \( k \geq 0 \), the disjoint union \( \coprod_{T \in T^{\text{dir}}} e_G \cdot B_T^{G} \) is a basis of \( H^k(\mathcal{M}(\mathcal{A}))^G \).

The next corollary contains as a special case a conjecture of Felder and Veselov [8] for Coxeter groups.

**Corollary 7.5.** If \( G \) is a well-generated complex reflection group, then

\[
\{ e_G \cdot \text{cx}_T^{G} | T \in T(\mathcal{A}^G) \}^{\text{dir}}
\]

is a basis of \( H^*(\mathcal{M}(\mathcal{A}))^G \).

### 7.6. Proof of Theorem 7.4.

The assertions in Theorem 7.4(2) and Theorem 7.4(3) follow from Theorem 7.4(1) and Proposition 2.5.

As with Theorem 5.5, the proof of Theorem 7.4(1) is by recursion for primitive reflection pairs, by induction on \( n \) for imprimitive reflection pairs, and case-by-case for very exceptional reflection pairs. To illustrate the method, we provide details for the reflection pairs \( (\mathcal{A}_{34}, G_{34}) \) and \( (\mathcal{A}_4(r), G(r, p, 4)) \) when \( p \) is even. The extension to other cases is straightforward and is omitted.
7.7. Consider the reflection pair \((\mathcal{A}, G) = (\mathcal{A}_{34}, G_{34})\). By Theorem 5.5(1), \(\mathcal{T}(\mathcal{A}, G)^{tdi} = \{A_0, A_1, G_{33}, G\}\). We have seen that \(e_{NG}(X_T) \cdot cx^{\mathcal{A},G}_T \neq 0\) for \(T \in \{A_0, A_1\}\). In the rest of this subsection, set \(T = G_{33}\).

Using the second BMR diagram for \(G_{34}\) ([5, Tab. 4]) we have
\[
cx^{\mathcal{A},G}_G = h_\varphi h_t h_u h_v h_w h_x \quad \text{and} \quad cx^{\mathcal{A},G}_{G_{33}} = h_\varphi h_t h_u h_v h_w.
\]
Assuming that we have proved the theorem for all \(T \in \mathcal{T}(\mathcal{A}, G) \setminus \{G\}\), we know that \(e_{G_{33}} \cdot cx^{\mathcal{A},G}_{G_{33}} \neq 0\) in \(H^5(M(\mathcal{A}_{33}))^{G_{33}}\). Using Theorem 5.5(1) again, there is an isomorphism
\[
H^5(M(\mathcal{A}_{33}))^{G_{33}} \cong H^5(M(\mathcal{A}_{33}))^{G_{33}} = H^5(M(\mathcal{A}_{33}))^{NG(X_T)},
\]
that maps \(e_{G_{33}} \cdot cx^{\mathcal{A},G}_{G_{33}} \neq 0\) to \(e_{NG(X_T)} \cdot cx^{\mathcal{A},G}_T\). Hence \(e_{NG(X_T)} \cdot cx^{\mathcal{A},G}_T \neq 0\).

It remains to show that \(e_G \cdot cx^{\mathcal{A},G}_G \in H^6(M(\mathcal{A}))^G\) is not equal to zero. Applying the map \(\partial\) in 2.7 to \(e_G \cdot cx^{\mathcal{A},G}_G\) we get
(a) \(\partial(e_G \cdot h_\varphi h_t h_u h_v h_w h_x) = e_G \cdot h_t h_u h_v h_w h_x - e_G \cdot h_\varphi h_t h_u h_v h_w h_x + e_G \cdot h_\varphi h_t h_u h_v h_w h_x - e_G \cdot h_\varphi h_t h_u h_v h_w h_x - e_G \cdot h_\varphi h_t h_u h_v h_w h_x\).

Consider the first summand, \(e_G \cdot h_t h_u h_v h_w h_x\). Set \(X = H_t \cap H_u \cap H_v \cap H_w \cap H_x\). It follows from Steinberg’s Theorem that the pointwise stabilizer of \(X\) is \(Z_G(X) = \langle t, u, v, w', x \rangle\), which has reflection type \(D_5\). Thus, for the reflection type \(\tilde{T} = D_5\) in \(\mathcal{T}(\mathcal{A}, G)\), we may take \(X = X\). Then \(e_{NG(X_T)} \cdot cx^{\mathcal{A},G}_{T} = e_{Z_G(X)} \cdot h_\varphi h_t h_u h_v h_w h_x \in H^5(M(\mathcal{A}_5))^{Z_G(X)}\). But \(H^5(M(\mathcal{A}_5))^{Z_G(X)} \cong H^5(M(\mathcal{A}(D_5)))^{D_5} = 0\), where the last equality follows from Theorem 5.5(2). Thus \(e_{Z_G(X)} \cdot h_t h_u h_v h_w h_x = 0\), and hence \(e_G \cdot h_t h_u h_v h_w h_x = 0\).

Similar arguments show that all the summands in (a) are equal to zero except the last. Arguing as in the preceding paragraph, with \(X = H_t \cap H_u \cap H_v \cap H_w \cap H_{w'}\), we have \(Z_G(X) = \langle s, t, u, v, w' \rangle\), which has reflection type \(G_{33}\). Therefore, \(e_G \cdot h_\varphi h_t h_u h_v h_w = e_G \cdot cx^{\mathcal{A},G}_{G_{33}}\).

Putting the pieces together, \(\partial(e_G \cdot cx^{\mathcal{A},G}_G) = -e_G \cdot cx^{\mathcal{A},G}_{G_{33}} \neq 0\). Since \(\partial\) is injective, we conclude that \(e_G \cdot cx^{\mathcal{A},G}_G \neq 0\), as desired.

7.8. Consider the reflection pair \((\mathcal{A}, G) = (\mathcal{A}_k(r), G(r, p, 4))\), where \(p\) is even. By induction, we may assume that the conclusions of the theorem hold for \(n = 2\) and \(n = 3\).

It follows from Theorem 5.5(2) that
\[
\mathcal{T}(\mathcal{A}, G)^{tdi} = \{A_0, A_1, G(r, p, 1), G(r, p, 1)A_1, G(r, p, 2), G(r, p, 2)A_1, G(r, p, 3), G\},
\]
and that \(\dim H^{rkT}(M(\mathcal{A}_{X_T}))^{NG(X_T)} = 1\) unless \(T \in \{G(r, p, 2)A_1, G\}\). For the rest of this subsection, set \(\tilde{T} = G(r, p, 2)A_1\). The elements \(cx^{\mathcal{A},G}_T\) and the sets \(B^{\mathcal{A},G}_{\tilde{T}}\) and \(B^{\mathcal{A},G}_G\) are given in Table 4.

We need to show that
\[
\text{• if } T \in \mathcal{T}(\mathcal{A}, G)^{tdi} \text{ and } \dim H^{rkT}(M(\mathcal{A}_{X_T}))^{NG(X_T)} = 1, \text{ then } e_{NG(X_T)} \cdot cx^{\mathcal{A},G}_T \neq 0, \text{ and}
\]
\[
\text{• if } T \in \{\tilde{T}, G\}, \text{ then } e_{NG(X_T)} \cdot B^{\mathcal{A},G}_{\tilde{T}} \text{ is a basis of } H^{rkT}(M(\mathcal{A}_{X_T}))^{NG(X_T)}.
\]
As in the preceding subsection, the results are known for \(T \in \{A_0, A_1, G(r, p, 1)\}\) (recall from 6.17 that \(G\) has two orbits on \(\mathcal{A}\), indexed by \(A_1\) and \(G(r, p, 1)\)).

Suppose that \(T = G(r, p, k)\) and \(k = 2, 3\). It was shown in Lemma 6.18(1) that
(a) \(H^k(M(\mathcal{A}_{X_T}))^{NG(X_T)} \cong H^k(M(\mathcal{A}_k(r)))^{G(r, 1, k)}\)
by an isomorphism that is induced by the projection \( \mathbb{C}^4 \to \mathbb{C}^4/X_T \) and that intertwines the \( N_G(X_T) \)-action on the left with the \( G(r, 1, k) \)-action on the right. By induction, the space \( H^k(M(\mathcal{A}_k(r)))^{G(r, 1, k)} \) is one-dimensional with basis \( \{ e_{G(r, 1, k)} \cdot h_\varphi \cdots h_k \} \). It is straightforward to check that the isomorphism in (a) maps \( h_\varphi \cdots h_k \) in \( H^k(M(\mathcal{A}_X)) \) to \( h_\varphi \cdots h_k \) in \( H^k(M(\mathcal{A}_k(r))) \). Therefore, \( e_{N_G(X_T)} \cdot h_\varphi \cdots h_k \) maps to \( e_{G(r, 1, k)} \cdot h_\varphi \cdots h_k \), and hence \( e_{N_G(X_T)} \cdot h_\varphi \cdots h_k \neq 0 \). This shows that \( e_G \cdot cx_T^{G} \neq 0 \) for \( T \in \{ G(r, p, 2), G(r, p, 3) \} \).

In particular

(b) \[ e_{N_G(X_G(r, p, 3))) \cdot \{ h_\varphi h_2 h_3 \} \] is a basis of \( H^3(M(\mathcal{A}_G(r, p, 3)))^{N_G(X_G(r, p, 3)))} \).

Similarly, if \( T = G(r, p, 1)A_1 \), then it follows from the isomorphism

\[ H^2(M(\mathcal{A}_X))^{N_G(X_T)} \cong H^1(M(\mathcal{A}_T))^{G(r, 1, 1)} \otimes H^1(M(\mathcal{B}_2)) \]

in Lemma 6.18(2) that \( e_{N_G(X_T)} \cdot cx_T^{G} \neq 0 \).

If \( T = G(r, p, 2)A_1 \), then by Lemma 6.18(3),

\[ H^3(M(\mathcal{A}_X))^{N_G(X_T)} \cong H^2(M(\mathcal{A}_2(r)))^{G(r, p, 2)} \otimes H^1(M(\mathcal{B}_2)) \]

By induction, \( e_{G(r, p, 2)} \cdot \{ h_\varphi h_2, h_\varphi h'_2 \} \) is a basis of \( H^2(M(\mathcal{A}_2(r)))^{G(r, p, 2)} \), and then it follows that

(c) \[ e_{N_G(X_T)} \cdot B_T^{G} = e_{N_G(X_T)} \cdot \{ h_\varphi h_2 h_4, h_\varphi h'_2 h_4 \} \] is a basis of \( H^3(M(\mathcal{A}_T))^{N_G(X_T)} \).

Notice that by (b), (c), and Theorem 5.5(2),

(d) \[ \{ e_G \cdot h_\varphi h_2 h_3, e_G \cdot h_\varphi h_2 h_4, e_G \cdot h_\varphi h'_2 h_4 \} \] is a basis of \( H^3(M(\mathcal{A})) \).

Finally, consider \( e_G \cdot B_T^{G} = e_G \cdot \{ h_\varphi h_2 h_3, h_\varphi h'_2 h_3 h_4 \} \). One easily checks that \( e_G \cdot h_\varphi h'_2 h_3 \) is a scalar multiple of \( e_G \cdot h_\varphi h_2 h_3 \), say \( e_G \cdot h_\varphi h'_2 h_3 = \xi e_G \cdot h_\varphi h_2 h_3 \). Applying the map \( \partial \) in 2.7 and arguing as in 7.7 we have

\[ \partial(e_G \cdot h_\varphi h_2 h_3 h_4) = e_G \cdot h_\varphi h_2 h_4 - e_G \cdot h_\varphi h_2 h_3 \]

and

\[ \partial(e_G \cdot h_\varphi h'_2 h_3 h_4) = e_G \cdot h_\varphi h'_2 h_4 - e_G \cdot h_\varphi h'_2 h_3 = e_G \cdot h_\varphi h'_2 h_4 - \xi e_G \cdot h_\varphi h_2 h_3. \]
Therefore, it follows from (d) that $\partial(e_G \cdot B_G(G))$ is linearly independent, and so $e_G \cdot B_G(G)$ is linearly independent and hence a basis of $H^{rk G}(M(\mathcal{A}))^G$. This completes the proof of Theorem 7.4.

7.9. **Very exceptional reflection pairs.** In this subsection we prove Theorem 5.5(1) for the very exceptional reflection pairs by using Theorem 7.4 to construct an explicit basis of $H^*(M(\mathcal{A}))^G$ for each such pair.

First consider the pair $(\mathcal{A}_2^0(3), (G(3, 3, 3)\sigma))$. Set $\mathcal{A} = \mathcal{A}_2^0(3)$ and $G = G(3, 3, 3)$. Fix a hyperplane, $H$, in $\mathcal{A}_2^0(3)$. By Theorem 5.5(2), $\{1, e_G \cdot h\}$ is a basis of $H^*(M(\mathcal{A}))^G$. Because $G$ acts transitively on $\mathcal{A}$, so does $\langle G\sigma \rangle$. It follows that $H^*(M(\mathcal{A}))^G = H^*(M(\mathcal{A}))^{\langle G\sigma \rangle}$. This justifies the entries in the row for $(\mathcal{A}_2^0(3), (G(3, 3, 3)\sigma))$ in Table 1.

Next consider the pair $(\mathcal{F}_4, \langle F_4\gamma \rangle)$. With the notation in the BMR diagram for $F_4 = G_{28}$, it follows from Theorem 7.4 that

$$e_{F_4} \cdot \{1, h_s, h_v, h_s h_v, h_t h_u, h_s h_t h_u, h_t h_u h_v, h_s h_t h_u h_v\}$$

is a basis of $H^*(M(\mathcal{F}_4))^{F_4}$. Also, the linear transformation $\gamma$ acts on $\{h_s, h_t, h_u, h_v\}$ by interchanging $h_u$ and $h_v$ and interchanging $h_t$ and $h_u$. Thus $\langle F_4\gamma \rangle$ acts transitively on $\mathcal{F}_4$.

$$\gamma \cdot h_t h_u = -h_t h_u, \quad \gamma \cdot h_t = -h_t$$

and it follows easily that

$$e_{\langle F_4\gamma \rangle} \cdot \{1, h_s, h_t, h_u, h_t h_u, h_s h_t h_u, h_s h_t h_u h_v\}$$

is a basis of $H^*(M(\mathcal{F}_4))^{\langle F_4\gamma \rangle}$. This justifies the entries in the row for $(\mathcal{F}_4, \langle F_4\gamma \rangle)$ in Table 1.

Finally, for the pair $(\mathcal{A}_4^0(2), \langle D_4\tau \rangle)$, it is easy to see that $\tau$ acts trivially on the basis of $H^*(M(\mathcal{A}_4^0(2)))^{D_4}$ in Theorem 7.4 and so $H^*(M(\mathcal{A}_4^0(2)))^{\langle D_4\tau \rangle} = H^*(M(\mathcal{A}_4^0(2)))^{D_4}$. This justifies the entries in the row for $(\mathcal{A}_4^0(2), \langle D_4\tau \rangle)$ in Table 1.

8. **Application: Computing Lehrer’s relative equivariant Poincaré polynomials**

Lehrer [11] considers the relative situation in which $G \subseteq \tilde{G} \subseteq \text{GL}(V)$ are complex reflection groups such that $G$ is irreducible and normal in $\tilde{G}$, and $\mathcal{A} = \mathcal{A}(G)$ or $\mathcal{A}(\tilde{G})$. A fundamental result in [11] is the computation of the action of the character of the representation of $\tilde{G}/G$ on $H^*(M(\mathcal{A}))^G$. The bases of $H^*(M(\mathcal{A}))^G$ described in Theorem 7.4 can easily be used to give an alternate derivation of these characters. Here we focus on the cases when

- $\mathcal{A} = \mathcal{F}_4$, $G = F_4$, and $\tilde{G} = \langle F_4\gamma \rangle$, and
- $\mathcal{A} = \mathcal{A}_n(r)$, $G = G(r, p, n)$, $\tilde{G} = G(r, 1, n)$, and $n \geq 3$.

The other cases are left to the reader. To simplify the notation a bit, for $X \in L(\mathcal{A})$ set

$$\text{Ind}_{\mathcal{A}}^{\tilde{G}} = \sum_{Y \in G X} H^{rk X}(M(\mathcal{A}_Y)).$$

8.1. The computation for $\mathcal{A} = \mathcal{F}_4$, $G = F_4$, and $\tilde{G} = \langle F_4\gamma \rangle$ follows from the computation in 7.9. We want to compute the graded character of $\langle \gamma \rangle$ on $H^*(M(\mathcal{F}_4))^{F_4}$.

Let $e_{\langle \gamma \rangle}$ denote the non-trivial character of $\langle \gamma \rangle$. The following statements are immediate consequences of the computations in 7.9:

- $H^0(M(\mathcal{F}_4))^{F_4} = (\text{Ind}_{\mathcal{A}}^{\tilde{G}})^{F_4}$, a basis is $e_{F_4} \cdot \{1\}$, and $\langle \gamma \rangle$ acts trivially.
\[H^1(M(\mathcal{F}_4))^{F_4} = (\text{Ind}_{H_3}^{F_4})^{F_4} + (\text{Ind}_{H_3}^{F_4})^{F_4},\] a basis is \(e_{F_4} \cdot \{h_s + h_v, h_a - h_v\}\), and \(\langle \gamma \rangle\) acts as \(1_{\langle \gamma \rangle} + \epsilon_{\langle \gamma \rangle}\).

\[H^2(M(\mathcal{F}_4))^{F_4} = (\text{Ind}_{H_3 \cap H_4}^{F_4})^{F_4} + (\text{Ind}_{H_3 \cap H_4}^{F_4})^{F_4},\] a basis is \(e_{F_4} \cdot \{h_s h_v, h_l h_u\}\), and \(\langle \gamma \rangle\) acts as \(\epsilon_{\langle \gamma \rangle} + \epsilon_{\langle \gamma \rangle}\).

\[H^3(M(\mathcal{F}_4))^{F_4} = (\text{Ind}_{H_3 \cap H_4 \cap H_5}^{F_4})^{F_4} + (\text{Ind}_{H_3 \cap H_4 \cap H_5}^{F_4})^{F_4},\] a basis is
\[e_{F_4} \cdot \{h_s h_l h_a + h_l h_u h_v, h_s h_l h_u - h_l h_u h_v\},\]
and \(\langle \gamma \rangle\) acts as \(1_{\langle \gamma \rangle} + \epsilon_{\langle \gamma \rangle}\).

\[H^4(M(\mathcal{F}_4))^{F_4} = (\text{Ind}_0^{F_4})^{F_4}\] is one-dimensional, a basis is \(e_{F_4} \cdot \{h_s h_l h_u h_v\}\), and \(\langle \gamma \rangle\) acts as \(1_{\langle \gamma \rangle}\).

With the notation in [11], the equivariant Poincaré polynomial is
\[P^{\langle \gamma \rangle}(X_{F_4}, t) = (1 + t + t^3 + t^4)1_{\langle \gamma \rangle} + (t + 2t^2 + t^3)\epsilon_{\langle \gamma \rangle}.\]

8.2. In the rest of this section consider the case when \(\mathcal{A} = \mathcal{A}_n(r), G = G(r, p, n), \tilde{G} = G(r, 1, n),\) and \(n \geq 3\).

In general \(T(\mathcal{A}, G) \neq T(\mathcal{A}, \tilde{G})\), but it follows from Theorem 5.5 (see Table 2) that we may canonically identify \(T(\mathcal{A}, G)^{tdi}\) and \(T(\mathcal{A}, \tilde{G})^{tdi}\) by replacing \(p\) by 1. Moreover, we can choose a set of common orbit representatives. For example, if \(\lambda = (1^{n-k})\), then \(T = G(r, p, k) \in T(\mathcal{A}, G)^{tdi}\) corresponds to \(\tilde{T} = G(r, 1, k) \in T(\mathcal{A}, \tilde{G})^{tdi}\), and we can take \(X_T = X_{\tilde{T}} = X_\lambda\). Set
\[\mathcal{T}^{tdi} = T(\mathcal{A}, G)^{tdi} = T(\mathcal{A}, \tilde{G})^{tdi}.

It is straightforward to check that if \(T \in \mathcal{T}^{tdi}\), then \(GX_T = \tilde{G}X_T\), whence
\[\text{Ind}_{X_T}^{\tilde{G}} = \sum_{Y \in GX_T} H^{rkT}(M(\mathcal{A}_T)) = \sum_{Y \in G X_T} H^{rkT}(M(\mathcal{A}_Y)) = \text{Ind}_{X_T}^{\tilde{G}}.
\]

In particular, \(\text{Ind}_{X_T}^{\tilde{G}}\) is a \(\tilde{G}\)-module and
\[(a) \hspace{1cm} e_{\tilde{G}} \cdot H^{rkT}(M(\mathcal{A}_X T)) = (\text{Ind}_{X_T}^{\tilde{G}})_{\tilde{G}} \subseteq (\text{Ind}_{X_T}^{\tilde{G}})_G = e_{G} \cdot H^{rkT}(M(\mathcal{A}_X T)).\]

If \(r\) is even, let \(\epsilon_{\tilde{G}}\) denote the linear character of \(\tilde{G}\) with order two that contains \(W_n\) in its kernel. The next theorem is a slight refinement of a result due to Lehrer [11, Thm 6.1].

**Theorem 8.3.** Suppose \(G = G(r, p, n), \tilde{G} = G(r, 1, n), \mathcal{A} = \mathcal{A}_n(r),\) and \(T \in \mathcal{T}^{tdi}\). Then the character of the \(\mathcal{Q}\tilde{G}\)-module \(\text{Ind}_{X_T}^{\tilde{G}}\) is equal to
\[
\begin{cases}
1_{\tilde{G}} + \epsilon_{\tilde{G}} & \text{if } p \text{ and } n \text{ are even and } T \in \{G(r, p, n - 2)A_1, G\}, \\
1_{\tilde{G}} & \text{otherwise}. \\
\end{cases}
\]

**Proof.** Clearly \(H^{rkT}(M(\mathcal{A}_X T))^{N_{\mathcal{G}}(X_T)} \subseteq H^{rkT}(M(\mathcal{A}_X T))^{N_{\tilde{G}}(X_T)}\). There are two cases, depending on whether or not this containment is an equality.

Suppose first that \(H^{rkT}(M(\mathcal{A}_X T))^{N_{\tilde{G}}(X_T)} = H^{rkT}(M(\mathcal{A}_X T))^{N_{\mathcal{G}}(X_T)}\). Then it follows from Theorem 5.5(2) that \(e_{\tilde{G}} \cdot H^{rkT}(M(\mathcal{A}_X T))\) and \(e_{G} \cdot H^{rkT}(M(\mathcal{A}_X T))\) are both one dimensional, so equality holds in \(a\). Therefore \(\tilde{G}\) acts trivially on \(\text{Ind}_{X_T}^{\tilde{G}}\) and so the character of the \(\mathcal{Q}\tilde{G}\)-module \(\text{Ind}_{X_T}^{\tilde{G}}\) is equal to \(1_{\tilde{G}}\), as claimed.
Now suppose that \( \dim H^{rk}(\text{M}(\mathcal{A}_{X_T})^{N_{G}(X_T)}) \neq H^{rk}(\text{M}(\mathcal{A}_{X_T})^{N_{G}(X_T)}) \). Then by Theorem 5.5(2), \( p \) and \( n \) are both even, \( T \in \{G(r,p,n-2)A_1, G\} \), \( \dim H^{rk}(\text{M}(\mathcal{A}_{X_T})^{N_{G}(X_T)}) = 1 \), and \( H^{rk}(\text{M}(\mathcal{A}_{X_T})^{N_{G}(X_T)}) = 2 \).

Consider first the case when \( T = G(r,p,n-2)A_1 \). By Theorem 7.4(2),

\[
e_{G} \cdot B^{G}_{T} = \{e_{G} \cdot \text{cx}_{T}^{G}, \text{cx}_{T,1}^{G}\} = \{e_{G} \cdot h_{\varphi}h_{2} \cdot \cdots \cdot h_{n-2}h_{n}, e_{G} \cdot h_{\varphi}h'_{2} \cdot \cdots \cdot h_{n-2}h_{n}\}
\]

is a basis of \( \text{Ind}_{T}^{G} \). Set \( G_{2} = G(r,2,n) \). Then \( G_{2}X_{T} = GX_{T} \), so \( \text{Ind}_{X_{T}}^{G_{2}} = \text{Ind}_{X_{T}}^{G} \). Moreover, \( G \subseteq G_{2} \) because \( p \) is even, so \( G_{2} \) acts on \( \text{Ind}_{X_{T}}^{G} \) and \( \langle \text{Ind}_{X_{T}}^{G} \rangle^{G_{2}} \subseteq \langle \text{Ind}_{X_{T}}^{G} \rangle^{G} \). Since both spaces are two-dimensional, they must be equal. Thus every element in \( \text{Ind}_{X_{T}}^{G} \) is \( G_{2} \)-invariant. It follows from 2.3 that

\[
\varphi \cdot h_{\varphi} = h_{\varphi}, \quad \varphi \cdot h_{2} = h'_{2}, \quad \text{and} \quad \varphi \cdot h_{j} = h_{j} \text{ for } 3 \leq j \leq n.
\]

Thus,

\[
\varphi e_{G} \cdot \text{cx}_{T}^{G, \text{cx}_{T,1}^{G}} = e_{G} \cdot \text{cx}_{T,1}^{G, \text{cx}_{T,1}^{G}} \quad \text{and} \quad \varphi e_{G} \cdot \text{cx}_{T,1}^{G} = \varphi^{2} e_{G} \cdot \text{cx}_{T,1}^{G} = e_{G} \cdot \text{cx}_{T,1}^{G},
\]

where the last equality holds because \( \varphi^{2} \in G_{2} \). Because the subgroup of \( \tilde{G} \) generated by \( \varphi \) maps surjectively onto \( \tilde{G}/G \), we see that \( e_{G} \cdot \{\text{cx}_{T}^{G, \text{cx}_{T,1}^{G}}, \text{cx}_{T,1}^{G} - \text{cx}_{T,1}^{G}\} \) is a basis of \( \text{Ind}_{T}^{G} \) such that

\[
(a) \quad e_{G} \cdot \text{cx}_{T}^{G} + e_{G} \cdot \text{cx}_{T,1}^{G} \in \langle \text{Ind}_{T}^{G} \rangle^{\tilde{G}} \quad \text{and} \quad e_{G} \cdot \text{cx}_{T}^{G} - e_{G} \cdot \text{cx}_{T,1}^{G} \in \langle \text{Ind}_{T}^{G} \rangle^{\tilde{G}}.
\]

A similar argument applied to \( T = G \) shows that \( e_{G} \cdot \{\text{cx}_{G}^{G}, \text{cx}_{G}^{G}, \text{cx}_{G}^{G} - \text{cx}_{G}^{G}\} \) is a basis of \( H^{n}(\text{M}(\mathcal{A}))^{G} \) such that

\[
(b) \quad e_{G} \cdot \text{cx}_{G}^{G} + e_{G} \cdot \text{cx}_{G,1}^{G} \in \langle \text{Ind}_{G}^{\tilde{G}} \rangle^{\tilde{G}} \quad \text{and} \quad e_{G} \cdot \text{cx}_{G}^{G} - e_{G} \cdot \text{cx}_{G,1}^{G} \in \langle \text{Ind}_{G}^{\tilde{G}} \rangle^{\tilde{G}}.
\]

It follows from (a) and (b) that the character of the \( \mathbb{Q}\tilde{G} \)-module \( \langle \text{Ind}_{X_{T}}^{G} \rangle^{G} \) is equal to \( 1 + e_{G} \) when \( p \) and \( n \) are even and \( T \in \{G(r,p,n-2)A_1, G\} \).

\[
9. \text{Application: Semi-invariants and determinant-like characters}
\]

For our last application, we consider the situation when \( G \) is a complex reflection group, \( \mathcal{A} = \mathcal{A}(G) \), and we extend the underlying scalar field for cohomology, \( G \)-modules, and representations from \( \mathbb{Q} \) to \( \mathbb{C} \). Using the same ideas as in the arguments above, we give a short proof of the following vanishing theorem due to Lehrer [10, Thm. 1.3].

**Theorem 9.1.** Suppose \( G \subseteq \text{GL}(V) \) is a complex reflection group, \( \mathcal{A} = \mathcal{A}(G) \), and \( L \) is a complex vector space that affords a representation of \( G \) such that \( L^{Z_{G}(H)} = 0 \) for every hyperplane \( H \in \mathcal{A} \). Then \( \text{Hom}_{G}(H^{*}(\text{M}(\mathcal{A})), L) = 0 \).

**Proof.** Set \( \mathcal{X} = \mathcal{X}(\mathcal{A}, G) \). The proof is by induction on \( \text{rk}G \). We have seen in 2.6 that if \( \text{rk}G = 1 \), then \( G \) acts trivially on \( H^{*}(\text{M}(\mathcal{A})) \), and so the result holds. Suppose that \( \text{rk}G > 1 \) and that the result holds for complex reflection groups \( G' \) with \( \text{rk}G' < \text{rk}G \).
By Proposition 2.5 and Frobenius reciprocity we have

\[ \text{Hom}_G(H^*(M(\mathcal{A}(G))), L) \cong \bigoplus_{X \in \mathcal{X}} \text{Hom}_G(\text{Ind}_X^G, L) \]
\[ \cong \bigoplus_{X \in \mathcal{X}} \text{Hom}_{N_G(X)}(H^c d^X(M(\mathcal{A}_X)), L|_{N_G(X)}) \]
\[ \cong \bigoplus_{X \in \mathcal{X}} \text{Hom}_{N_G(X)}(H^{rk Z_G(X)}(M(\mathcal{A}(Z_G(X)))), L|_{N_G(X)}). \]

The summand in (a) indexed by \( X = V \) is \( \text{Hom}_G(H^0(M(\mathcal{A})), L) \), which is equal to zero by assumption. Suppose \( X \in \mathcal{X} \) and \( 0 < \text{cd} X < \text{rk} G \). Clearly \( \text{Ind}_X^G(H) = 0 \) for every hyperplane in \( \mathcal{A}(Z_G(X)) \), and so by induction, \( \text{Hom}_{Z_G(X)}(H^*(M(\mathcal{A}(Z_G(X)))), L|_{Z_G(X)}) = 0 \). Every \( N_G(X) \)-equivariant homomorphism from \( H^{rk Z_G(X)}(M(\mathcal{A}(Z_G(X)))) \) to \( L \) is \( Z_G(X) \)-equivariant, so

\[ \text{Hom}_{N_G(X)}(H^{rk Z_G(X)}(M(\mathcal{A}(Z_G(X)))), L|_{N_G(X)}) = 0. \]

Finally, the argument in 2.8(a) shows that the summand in (a) indexed by \( X = 0 \), namely \( \text{Hom}_G(H^{rk G}(M(\mathcal{A})), L) \), is also equal to zero. This completes the proof. \( \Box \)

A linear character \( \xi \) of \( G \) is called a determinant-like character if, for every reflection \( s \in G \), \( \xi(s) \) is a root of unity with order equal to the order of \( s \). Such a character satisfies the hypothesis of the theorem. The determinant character, \( \det : G \to \mathbb{C}^* \), is of course a determinant-like character. If \( G \) is a Coxeter group, then \( \det \) is the sign character, and is the only determinant-like character of \( G \). For non-Coxeter reflection groups, \( \det^{-1} \) is a determinant-like character that in general is not equal to \( \det \).

It follows from a result of Lehrer [9] that the sign character of a Weyl group, \( W \), does not occur in \( H^*(M(\mathcal{A}(W))) \). The following generalization to complex reflection groups is an immediate consequence of Theorem 9.1.

**Corollary 9.2.** Suppose \( G \subseteq \text{GL}(V) \) is a non-trivial, complex reflection group and \( \xi \) is a determinant-like character of \( G \). Then \( \xi \) does not occur in \( H^*(M(\mathcal{A}(G))) \).

**Appendix A. Calculations in rank two**

In this appendix we collect the results for irreducible, rank two reflection pairs. In the tables we use the notation for scalar matrices introduced in 4.9. Boldface indicates a coset representative, say \( z \), such that \( \mathcal{A}(G z) = \emptyset \). For example \( z_{12} \) is the scalar matrix with eigenvalue \( \xi_{12} = e^{2\pi i/12} \). In the row for \( G = G_4 \) in Table 5, \( \langle G_4 z_{12} \rangle = G_7 \) and hence \( \mathcal{A}(G_4 z_{12}) = G_7 \), but \( \mathcal{A}(G_4 z_{12}) = \emptyset \).

**A.1.** Suppose \( G \sigma \) is a rank two reflection coset in 4.3. The arrangements \( \mathcal{A}(\langle G \sigma z \rangle) \) and groups \( \langle G \sigma \rangle \), where \( z \) is a scalar transformation and \( \mathcal{A}(\langle G \sigma z \rangle) \neq \mathcal{A}(G) \), are given in Table 5.

**A.2.** Suppose \( C \) is an irreducible reflection coset with rank equal to two, \( G_0 = CC^{-1} \), and \( G_1 = \langle C \rangle \). Then for \( i, j \in \{0, 1\} \), either \( H^*(M(G_i))^G_j = H^*(M(\mathcal{A}(G))^G_j \), where \( G \) is irreducible, or \( H^*(M(G_i))^G_j = H^*(M(\mathcal{A}))^G_j \), where \( (\mathcal{A}, G) \) is one of the pairs in the Table 6. The table also records a reflection coset, \( C \), that gives rise to the reflection pair \( (\mathcal{A}, G) \).
Table 5. Arrangements $\mathcal{A}(\langle G\sigma z \rangle) \neq \mathcal{A}(G)$ and groups $\langle G\sigma \rangle$ in rank two

| $G$  | $\sigma z$ | $\mathcal{A}(\langle G\sigma z \rangle)$ | $\langle G\sigma \rangle$ | Notes       |
|------|-----------|----------------------------------------|---------------------------|-------------|
| $G_{r,p,2}$ | $\varphi^p z_{2r}^{-1}$ | $\mathcal{A}_2(2r)$ | $G_{r,p,2}$ | $p$ odd    |
| $G_{r,p,2}$ | $\varphi^p z_{2r}^{-1}$ | $\mathcal{A}_2(2r)$ | $G_{r,p,2}$ | $r/p$ odd  |
| $G_{4,2,2}$ | $\rho_3$ | $G_6$ | $G_6$ |             |
| $G_4$ | $z_3, z_4, z_{12}$ | $G_5, G_6, G_7$ | $G_4$ |             |
| $G_5$ | $z_4, \rho_4, \rho_2$ | $G_7, G_{10}, G_{14}$ | $G_5, G_{10}, G_{14}$ |             |
| $G_6$ | $z_3$ | $G_7$ | $G_6$ |             |
| $G_7$ | $\rho_4, \rho_2$ | $G_{10}, G_{15}$ | $G_{10}, G_{15}$ |             |
| $G_8$ | $z_8, z_3, z_{24}$ | $G_9, G_{10}, G_{11}$ | $G_8$ |             |
| $G_9$ | $z_3$ | $G_{11}$ | $G_{11}$ |             |
| $G_{10}$ | $z_8$ | $G_{11}$ | $G_{11}$ |             |
| $G_{12}$ | $z_8, z_{24}, z_3, z_{12}$ | $G_9, G_{11}, G_{13}$ | $G_{12}$ | $G_{11} = G_{15}$ |
| $G_{13}$ | $z_8, z_{24}, z_3$ | $G_9, G_{11}, G_{15}$ | $G_{13}$ | $G_{13} = G_9, G_{11} = G_{15}$ |
| $G_{14}$ | $z_8, z_4$ | $G_{11}, G_{15}$ | $G_{14}$ | $G_{11} = G_{15}$ |
| $G_{15}$ | $z_8$ | $G_{11}$ | $G_{15}$ | $G_{11} = G_{15}$ |
| $G_{16}$ | $z_4, z_3, z_{12}$ | $G_{17}, G_{18}, G_{19}$ | $G_{16}$ |             |
| $G_{17}$ | $z_3$ | $G_{19}$ | $G_{17}$ |             |
| $G_{18}$ | $z_4$ | $G_{19}$ | $G_{18}$ |             |
| $G_{20}$ | $z_5, z_{20}, z_4$ | $G_{18}, G_{19}, G_{21}$ | $G_{20}$ |             |
| $G_{21}$ | $z_5$ | $G_{19}$ | $G_{21}$ |             |
| $G_{22}$ | $z_5, z_{15}, z_3$ | $G_{17}, G_{19}, G_{21}$ | $G_{22}$ |             |

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Table 6. Irreducible reflection pairs, rank two, not (\(\mathcal{A} G\), G)

| \(\mathcal{A}\) | \(C\) | \(G\) | Notes |
|---|---|---|---|
| \(\mathcal{A}_2^0(r)\) | \(G_{r,r,2}\varphi^p\) | \(G_{r,p,2}\) |  |
| \(\mathcal{A}_2(r)\) | | \(G_{r,r,2}\) |  |
| \(\mathcal{A}_2^0(2r)\) | \(G_{r,r,2}\varphi^p\varphi^{-1}\) | \(G_{r,p,2}\) | \(p\) odd |
| \(\mathcal{A}_2(2r)\) | \(G_{r,r,2}\varphi^p\varphi^{-1}\) | \(G_{r,r,2}\) | \(r/p\) odd |
| \(\mathcal{A}_2(4)\) | \(G_{4,2,2}\rho_3\) | \(G_6\) |  |
| \(\mathcal{B}_5\) | \(G_{4,3}, G_{5,2}, G_{5,4}\) | \(G_4, G_{14}, G_{10}\) |  |
| \(\mathcal{B}_6\) | \(G_{4,2,2}\rho_3, G_{4,2}\) | \(G_{4,2,2}, G_{4}\) |  |
| \(\mathcal{B}_7\) | \(G_{4,12,12}, G_{5,2}, G_{6,3}, G_{7,2}, G_{7,4}\) | \(G_4, G_5, G_6, G_{15}, G_{10}\) |  |
| \(\mathcal{B}_8\) | \(G_{8,28,12,28,12,28}\) | \(G_8, G_{12}, G_{13}\) |  |
| \(\mathcal{B}_9\) | \(G_{5,2}, G_{7,2}, G_{8,2}\) | \(G_5, G_7, G_8\) |  |
| \(\mathcal{B}_{10}\) | \(G_{8,24,12,24,12,24}\) | \(G_8, G_9, G_{10}, G_{12}\) |  |
| \(\mathcal{B}_{11}\) | \(G_{13,24,14,24,15,24}\) | \(G_{13}, G_{14}, G_{15}\) |  |
| \(\mathcal{B}_{12}\) | \(G_{12,4}\) | \(G_{12}\) |  |
| \(\mathcal{B}_{13}\) | \(G_{5,2}, G_{15,2}\) | \(G_5, G_{12}\) |  |
| \(\mathcal{B}_{14}\) | \(G_{7,2}, G_{12,12,12,12}, G_{13,2}, G_{14,2}\) | \(G_7, G_{12}, G_{13}, G_{14}\) |  |
| \(\mathcal{B}_{15}\) | \(G_{16,24,16,24,15,25}\) | \(G_{16}, G_{22}\) |  |
| \(\mathcal{B}_{16}\) | \(G_{16,23,16,23,20,25}\) | \(G_{16}, G_{20}\) |  |
| \(\mathcal{B}_{17}\) | \(G_{16,212,16,212,16,212}\) | \(G_{16}, G_{17,18}\) |  |
| \(\mathcal{B}_{18}\) | \(G_{20,20,20,20,20,20}\) | \(G_{20}, G_{21,22}\) |  |
| \(\mathcal{B}_{19}\) | \(G_{20,24,22,23,19,25}\) | \(G_{20}, G_{22,21}\) |  |
| \(\mathcal{B}_{20}\) | \(G_{20,24,22,23,19,25}\) | \(G_{20}, G_{22,19}\) |  |

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