The problem of finding the radon flux density by its concentration at different depths of the earth's surface

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Abstract. The model of radon density distribution on the basis of the fractional diffusion–wave equation (FDWE) is considered. The technique of parametric identification of the model is proposed. Comparison of the results of model calculations with experimental data allows to draw a conclusion about the adequacy of the technique. Relevance of the work is determined by the fact that the forecast of radon density distribution in the Earth's crust allows to determine the places of earthquakes in advance.

1. Introduction
In the past decades fractional calculus has attracted considerable interests in the fields of engineering and science. One can refer to the treatises [1,2] for elementary knowledge of fractional calculus. Roughly speaking fractional space derivatives usually can be used to model anomalous diffusion or dispersion while fractional time derivatives to model some processes with ‘memory’ effects. As a counterpart of traditional integer order differential equation, fractional differential equation can be obtained by replacing the integer order derivatives with fractional ones in integer order differential equation. Fractional differential equations such as fractional kinetic equations, fractional diffusion equation, fractional advection–diffusion equation and fractional Fokker–Planck equation can be used to model the transport dynamics in complex systems. Specifically when replacing the first or second order time derivative in diffusion or wave equation respectively with fractional (a > 0 order) time derivative we get time fractional diffusion equation as 0 < α < 1 and time FDWE as 1 < α < 2. Specially, time FDWE can be viewed as the interpolation of diffusion and wave equation and be used to model the process which exhibits heat diffusion and wave propagation.

To solve the problem of finding the radon flux density [3] by its concentration at different depths of the earth's surface, we present the method of approximate solution of the first boundary value problem for the fractional differential equation of advection-diffusion [4]

$$\frac{\partial u(x,t)}{\partial t} = D_{0+}^{\alpha} u(x,t),$$

As is known [3, 5], the problem of finding the radon flux density by its concentration at different depths of the earth's surface is set as follows: to find a solution to the boundary value problem

$$\frac{\partial u(x,t)}{\partial t} = D_{0+}^{\alpha} u(x,t),$$
where $\mathcal{D}_θ^α u(x, t)$ – Riemann-Liouville fractional derivative of the order $α$, with boundary conditions

$$
\begin{align*}
&u(x, 0) = k, \\
&-\mathcal{D}_θ^α \frac{\partial u(x, t)}{\partial x} \bigg|_{x=0} = q(t), \\
&u(x_i, t_j) = c_{ij}
\end{align*}
$$

where $k$ is background concentration at a given depth for a given monitoring area, $q(t)$ - radon flux density. As shown in [6-13] eigenfunctions of the problem $x^{α-1}E_{α, β}(λ_n x^α)$ is full in $L_2(0,1)$ and the following formula takes place:

$$
u(x, t) = \sum_{n=1}^{∞} δ_n \exp(λ_n t)x^{α-1}E_{α, β}(λ_n x^α).
$$

We replace infinite sum by first $N = 50$ summands in (1).

$$u(x, t) \approx \sum_{n=1}^{50} δ_n \exp(λ_n t)x^{α-1}E_{α, β}(λ_n x^α)
$$

2. Parametric identification in FDWE model

To identify the parameter $α$ (fractional derivative order), the following theorem is used. If there is only one solution

$$u(x, t) = \sum_{n=1}^{∞} e^{λ_n t} \left[ \int_0^t f_n(t) e^{-λ_n t} + φ_n \right] x^{α-1}E_{α, β}(λ_n x^{α-1})
$$

of the boundary value problem

$$
\begin{align*}
\frac{\partial u(x, t)}{\partial t} &= \frac{\partial u^α(x, t)}{\partial x^α} + f(x, t) \\
&u(0, t) = u(1, t) = 0, \\
u(x, 0) &= φ(x)
\end{align*}
$$

then the fractional order $α$ can be defined unambiguously by the function $u(x, 0) = φ(x)$. Let’s consider the proof of the above theorem. Having subordinated solution (2) of the boundary value problem (3)-(4)-(5) to the initial condition (5), we come to equality:

$$φ(x) = \sum_{n=1}^{∞} φ_nω_n(x),
$$

where as mentioned above, $ω_n(x) = x^{α-1}E_{α, β}(λ_n x^α)$ – a system of functions forming a non-orthogonal basis in the Hilbert space $L_2(0,1)$. To determine the coefficients $φ_n$ in (6) together with the system $\{ω_n\}_{n=1}^{∞} = x^{α-1}E_{α, β}(λ_n x^α)$ we will consider the system of functions $\{z_n\}_{n=1}^{∞} = (1 - x)^{α-1}E_{α, β}(λ_n(1 - x)^α)$ – biorthogonal to the system $\{ω_n\}$. $\{z_n\}$ – is the contiguous boundary problem eigenfunctions system. Multiply both parts of the (6) by the system of functions $z_n(x)$, we get:
\[ \varphi(x)z_n(x) = \sum_{n=1}^{\infty} \varphi_n \omega_n(x)z_n(x). \] (7)

It is known [13, 14], that for zeros \( \{\lambda_n\} \) of the function \( E_{\alpha, \beta}(\lambda) \) \((\alpha < 2, \beta - \text{random real number})\) the kind that \( \mu \leq |\arg(\lambda)| \leq \pi, \text{r.e.} \ \mu \in \left(\frac{\pi}{2}, \min\{\pi, \alpha \pi\}\right) \) the following assessment is fair

\[ |E_{\alpha, \beta}(\lambda)| \leq \frac{C}{1 + |\lambda_n|} \] (8)

where \( C \) – random real constant.

For sufficiently large (absolute value) zeros \( \{\lambda_n\} \) of the function \( E_{\alpha, \beta}(\lambda) \) is also true the following

\[ \frac{1}{\lambda_n^2} = 2n\pi i - (1 + \alpha) \left[ \ln(2n\pi) + \frac{\pi}{2} i \right] + \ln \frac{\alpha}{\Gamma(-\alpha)} + O(1) = \]

\[ = \left[ -(1 + \alpha) \ln(2n\pi) + \ln \frac{\alpha}{\Gamma(-\alpha)} + O(1) \right] + \left[ 2n\pi - \frac{\pi}{2} (1 + \alpha) \right] i, \]

where it follows that

\[ |\lambda_n^2| = \left[ -(1 + \alpha) \ln(2n\pi) + \ln \frac{\alpha}{\Gamma(-\alpha)} + O(1) \right]^2 + \left[ 2n\pi - \frac{\pi}{2} (1 + \alpha) \right]^2 \sim \]

\[ \sim [-(1 + \alpha) \ln(2n\pi)]^2 + [2n\pi]^2 \sim O(n^2). \]

Thus, the equivalence is as follows

\[ |\lambda_n| \sim O(n^{\alpha}) \] (9)

Using (8) and (9) for the systems of functions \( \{\omega_n\} \) and \( \{z_n\} \) we get the following

\[ |\omega_n(x)| = |x^{\alpha-1}E_{\alpha, \alpha}(\lambda_n x^{\alpha})| \leq \frac{C_1 x^{\alpha-1}}{1 + |\lambda_n x^{\alpha}|} \leq \frac{C_1}{|\lambda_n|} \leq \frac{C_1}{n^\alpha x}, \quad C_1 = \text{const} \] (10)

\[ |z_n(x)| = |(1 - x)^{\alpha-1}E_{\alpha, \alpha}(\lambda_n (1 - x)^{\alpha})| \leq \]

\[ \leq \frac{C_2 (1 - x)^{\alpha-1}}{1 + |\lambda_n (1 - x)^{\alpha}|} \leq \frac{C_2}{|\lambda_n| (1 - x)} \leq \frac{C_2}{n^\alpha (1 - x)}, \quad C_2 = \text{const}. \] (11)

Hence, it follows that the functions \( \{\omega_n\} \) and \( \{z_n\} \) in (7) are limited, so (9) can be integrated along the segment \([0,1]::\)

\[ \int_0^1 \varphi(x)z_n(x)dx = \sum_{n=1}^{\infty} \varphi_n \int_0^1 \omega_n(x)z_n(x)dx. \] (12)

Equality (12) will be rewritten in the form

\[ (\varphi, z_n) = \sum_{n=1}^{\infty} \varphi_n (\omega_n, z_n), \] (13)

where

\[ (\varphi, z_n) = \int_0^1 \varphi(x)z_n(x)dx, \ (\omega_n, z_n) = \int_0^1 \omega_n(x)z_n(x)dx. \]

Due to the biorthogonality of the function systems \( \{\omega_n(x)\} \) and \( \{z_n(x)\} \) from (13) it follows that \( (\varphi, z_n) \equiv \varphi_n (\omega_n, z_n) \). It follows from here that
\[ \varphi_n = \frac{\varphi, z_n}{\omega_n, z_n} \]  

(14)

Since the systems of functions \( \{\omega_n(x)\} \) and \( \{z_n(x)\} \) are not only biorthogonal to each other, but are also orthonormal, then

\[ \left(\omega_n, z_n\right) = \int_0^1 \omega_n(x)z_n(x)\,dx = 1 \]

and formula (14), respectively, takes the form

\[ \varphi_n = (\varphi, z_n). \]

Thus, the Fourier coefficients \( \varphi_n \) in the solution \( u(x, t) \) of the boundary value problem (3)-(4)-(5) are expressed as a scalar product of the functions \( \varphi(x) \) and \( z_n(x) \) in the form

\[ \varphi_n = (\varphi, z_n) = \int_0^1 \varphi(x)z_n(x)\,dx. \]

Lets consider the following boundary value problem now

\[ \frac{\partial u(x, t)}{\partial t} = \frac{\partial u^\beta(x, t)}{\partial x^\beta} + f(x, t), \quad \frac{\partial u(x, 0)}{\partial x} = \varphi(x), \quad u(x, 0) = \phi(x), \]

(15)

\[ u(0, t) = u(1, t) = 0, \]

(16)

\[ u(0, t) = \varphi(x), \]

(17)

where \( \frac{\partial u^\beta(x, t)}{\partial x^\beta} = \frac{1}{\Gamma(2-\beta)} \frac{\partial^2}{\partial x^2} \int_0^x u(\tau, t)\,d\tau \)

Riemann-Liouville fractional derivative of the order \( \alpha \leq 1 < \beta < 2 \).

If \( u(x, t) = \varphi(x) \) boundary value problem (3)-(4)-(5) is equal to the initial condition \( v(x, t) = \varphi(x) \) краевой задачи (15)-(16)-(17), to

\[ \sum_{n=1}^\infty \varphi_n x^{\alpha-1}E_{\alpha,\alpha}(\lambda_n x^\alpha) = \sum_{n=1}^\infty \overline{\varphi}_n x^{\beta-1}E_{\beta,\beta}(\lambda_n x^\beta) \quad \forall x \in (0,1) \]

(18)

Considering (9), the right part of (18) can be assessed as follows

\[ |\varphi_n x^{\alpha-1}E_{\alpha,\alpha}(\lambda_n x^\alpha)| \leq C x^{\alpha-1} \frac{1}{1 + |\lambda_n x^\alpha|^\gamma}. \]

(19)

In this case, both parts of equality (19) are analytic in \( Re(x) > 0 \), so

\[ \sum_{n=1}^\infty \varphi_n x^{\alpha-1}E_{\alpha,\alpha}(\lambda_n x^\alpha) = \sum_{n=1}^\infty \overline{\varphi}_n x^{\beta-1}E_{\beta,\beta}(\lambda_n x^\beta) \quad \forall x > 0 \]

(20)

It is known [15], that the following equality is true for the function \( z^{\alpha-1}E_{\alpha,\alpha}(\lambda z^\alpha) \), where \( 0 < \alpha < 2 \) and \( \frac{\pi \alpha}{2} < \mu < min\{\pi, \pi \alpha \} \)

\[ z^{\alpha-1}E_{\alpha,\alpha}(\lambda z^\alpha) = -\sum_{k=1}^{N-1} \lambda^{-k-1} \frac{1}{\Gamma(-\alpha k) z^{\alpha k+1}} + O\left(\frac{1}{z^{\alpha N+1}}\right), \]

(21)

where \( |z| \to \infty, N \in \mathbb{N}\{1\} \) and \( \mu \leq |arg(\lambda z^\alpha)| \leq \pi. \)

Taking into account (21) for the left side of the equality (20), we obtain
\[
\sum_{n=1}^{\infty} \varphi_n x^{-1} E_{\alpha,\alpha}(\lambda_n x^\alpha) = \sum_{n=1}^{\infty} \varphi_n \left[ -\frac{1}{\Gamma(-\alpha)} \cdot \frac{1}{\lambda_n^{-\alpha+1}} + O\left(\frac{1}{x^{\alpha+1}}\right) \right].
\] (22)

It follows from (22) that

\[
\sum_{n=1}^{\infty} -\varphi_n \cdot \frac{1}{\Gamma(-\alpha)} \cdot \frac{1}{\lambda_n^{1+\alpha}} + \sum_{n=1}^{\infty} \varphi_n \cdot O\left(\frac{1}{x^{\alpha+1}}\right) = \\
= \sum_{n=1}^{\infty} -\varphi_n \cdot \frac{1}{\Gamma(-\beta)} \cdot \frac{1}{\lambda_n^{1+\beta}} + \sum_{n=1}^{\infty} \varphi_n \cdot O\left(\frac{1}{x^{\beta+1}}\right)
\] (23)

when \( x \to \infty \). This means that \( \alpha = \beta \).

In fact, assuming that \( x \to \infty \) at \( \alpha > \beta \), multiplication of both parts of the (23) by \( x^{\beta+1} \) leads us to

\[
\sum_{n=1}^{\infty} -\varphi_n \cdot \frac{1}{\Gamma(-\alpha)} \cdot \frac{1}{\lambda_n^{x^\alpha-\beta}} + \sum_{n=1}^{\infty} \varphi_n \cdot O\left(\frac{1}{x^{\alpha-\beta}}\right) = \\
= \sum_{n=1}^{\infty} -\varphi_n \cdot \frac{1}{\Gamma(-\beta)} \cdot \frac{1}{\lambda_n^{x^\beta+\beta}} + \sum_{n=1}^{\infty} \varphi_n \cdot O\left(\frac{1}{x^{\beta+\beta}}\right)
\] (24)

Let us now consider the majorant rows obtained in the left and right parts of the (24). We have

\[
\sum_{n=1}^{\infty} \left| \frac{1}{\lambda_n^{x^\alpha}} \right| = \sum_{n=1}^{\infty} \left| \frac{1}{\lambda_n^{x^\beta}} \right|
\] (25)

Taking into account the equivalence (10), for equality (25) we get

\[
\sum_{n=1}^{\infty} \frac{1}{\lambda_n^{x^\alpha}} = \sum_{n=1}^{\infty} \frac{1}{\lambda_n^{x^\beta}} = \sum_{n=1}^{\infty} \frac{1}{O(n^2)}
\]

which suggests the convergence of the majorant \( \sum_{n=1}^{\infty} \left| \frac{1}{\lambda_n^{x^\alpha}} \right| \) and \( \sum_{n=1}^{\infty} \left| \frac{1}{\lambda_n^{x^\beta}} \right| \), and therefore rows in both parts of the equality (25).

Provided that \( x \to \infty \) for the left side of equality (25) we get a value equal to zero, and for the right side - a value other than zero. That is, at \( \alpha > \beta \) or, in general, while \( \alpha \neq \beta \) we get a contradiction to equation.

From all the above it follows that only when the fractional parameters \( \alpha \) and \( \beta \) are equal, \( \alpha = \beta \), boundary value problems (3)-(4)-(5) and (15)-(16)-(17) may have the same solutions under equal initial conditions. This means that the boundary value problem (3)-(4)-(5) has the only solution in a given order \( \alpha \)-fractional derivative.

3. Numerical calculation, results and conclusions.
To build an approximate solution to the problem (3)-(4)-(5), an algorithm has been developed in Matlab R2017b. Next, we will show its results.

The following characteristics of the solution of this equation are obtained. The first 10 zeros of the function at replacement of its kind \( E_{\alpha,\alpha}(\lambda) \) while \( \alpha = 3/2 \):

\[
E_{\alpha,\alpha}(\lambda) = \frac{1}{\Gamma(\alpha)} + \frac{\lambda}{\Gamma(\alpha + \alpha)} + \frac{\lambda^2}{\Gamma(\alpha + 2\alpha)} + \cdots + \frac{\lambda^{10}}{\Gamma(\alpha + 10\alpha)}
\]

\( z1=-5.075432607289122 + 0.000000000000000i \)
\( z2=-14.693125447872484 + 0.000000000000000i \)
\( z_3 = -15.001776654315286 - 6.175190776985366i \)
\( z_4 = -15.001776654315286 + 6.175190776985366i \)
\( z_5 = -12.702880346631002 - 14.665791021568882i \)
\( z_6 = -12.702880346631002 + 14.665791021568882i \)
\( z_7 = -5.118110131503308 - 24.274070062273850i \)
\( z_8 = -5.118110131503308 + 24.274070062273850i \)
\( z_9 = 12.940470206313410 - 33.473411767087732i \)
\( z_{10} = 12.940470206313410 + 33.473411767087732i \)

Arrange the zero modules in ascending order: \( z_{10}, z_9, z_8, \ldots, z_2, z_1 \).

Next we'll consider \( \phi_n \), with the given
\[
\phi(x) = \begin{cases} 
2x, & 0 \leq x \leq 0.5 \\
4 - 2x, & 0.5 \leq x \leq 1 
\end{cases}
\]

\( \phi_1 = 0.204044249996967 + 0.000000000000000i \)
\( \phi_2 = 0.059191298033412 + 0.000000000000000i \)
\( \phi_3 = 0.041445425608307 - 0.029133472405196i \)
\( \phi_4 = 0.041445425608307 + 0.029133472405196i \)
\( \phi_5 = -0.007734306888190 - 0.028282816498061i \)
\( \phi_6 = -0.007734306888190 + 0.028282816498061i \)
\( \phi_7 = -0.006571026762936 + 0.185853948184785i \)
\( \phi_8 = -0.006571026762936 - 0.185853948184785i \)
\( \phi_9 = 4.837789236364265 + 5.171710200942336i \)
\( \phi_{10} = 4.837789236364265 - 5.171710200942336i \)

To estimate the order of accuracy of the solution, let's take in the infinite sum \( N=20 \). The calculation will be made according to the same scheme as before. Let's get

\( z_1 = -5.07543002954341 \)
\( z_2 = -17.4720153397254 \)
\( z_3 = -32.5092964896613 \)
\( z_4 = -35.4584486727518 \)
\( z_5 = -36.5912859512712 - 10.693578910293i \)
\( z_6 = -36.5912859512712 + 10.693578910293i \)
\( z_7 = -35.5040145232784 - 20.9453869870746i \)
\( z_8 = -35.5040145232784 + 20.9453869870746i \)
\( z_9 = -31.9699760345811 - 32.0144183584324i \)
\( z_{10} = -31.9699760345811 + 32.0144183584324i \)
\( z_{11} = -25.2320876582448 - 43.7562793536097i \)
\( z_{12} = -25.2320876582448 + 43.7562793536097i \)
\( z_{13} = -14.2265649968089 - 55.8034941647279i \)
\( z_{14} = -14.2265649968089 + 55.8034941647279i \)
\( z_{15} = 2.77221443730327 - 67.4936280421291i \)
\( z_{16} = 2.77221443730327 + 67.4936280421291i \)
\( z_{17} = 29.1062261104992 - 77.4531074654252i \)
\( z_{18} = 29.1062261104992 + 77.4531074654252i \)
\( z_{19} = 73.7226858034134 - 81.9572626204321i \)
\( z_{20} = 73.7226858034134 + 81.9572626204321i \)

Next we calculate \( \phi_n \)
\[\phi_1 = 0.137401421100116\]
\[\phi_2 = -0.0308223487266898\]
\[\phi_3 = -0.00272043831357143\]
\[\phi_4 = -0.000862402968477541\]
\[\phi_5 = 0.00327285462119446 + 0.00253480897689706i\]
\[\phi_6 = 0.00327285462119446 - 0.00253480897689706i\]
\[\phi_7 = 0.00860177592247567 - 0.00259899756689706i\]
\[\phi_8 = 0.00860177592247567 + 0.00259899756689706i\]
\[\phi_9 = 0.00104788292634568 + 0.010800335702171i\]
\[\phi_10 = 0.00104788292634568 - 0.010800335702171i\]
\[\phi_11 = 0.00128137962410719 + 0.112630001132604i\]
\[\phi_12 = 0.00128137962410719 - 0.112630001132604i\]
\[\phi_13 = 1.52094820160761 + 1.98293743502147i\]
\[\phi_14 = 1.52094820160761 - 1.98293743502147i\]
\[\phi_15 = 58.7183734083227 + 34.8297281262009i\]
\[\phi_16 = 58.7183734083227 - 34.8297281262009i\]
\[\phi_17 = 3176.87182573119 + 2643.67183509621i\]
\[\phi_18 = 3176.87182573119 - 2643.67183509621i\]
\[\phi_19 = -491228.693249359 + 1245129.74242596i\]
\[\phi_20 = -491228.693249359 - 1245129.74242596i\]

Let's evaluate the difference

\[\left| \sum_{n=1}^{20} \delta_n \exp(\lambda_n t)x^{\alpha-1}E_{\alpha,\alpha}(\lambda_n x^{\alpha}) - \sum_{n=1}^{10} \delta_n \exp(\lambda_n t)x^{\alpha-1}E_{\alpha,\alpha}(\lambda_n x^{\alpha}) \right| \leq 0.17\]

The latter inequality shows that the radon flux density can be considered by the formula

\[u(x, t) \approx \sum_{n=1}^{N} \delta_n \exp(\lambda_n t)x^{\alpha-1}E_{\alpha,\alpha}(\lambda_n x^{\alpha})\]

The results obtained by numerical methods are comparable with the experimental data [16], which proves the adequacy of the model (3)-(4)-(5) and the methods of parametric identification.

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