The Convexity and Concavity of Envelopes of the Minimum-Relative-Entropy Region for the DSBS

Lei Yu

Abstract

In this paper, we prove that for the doubly symmetric binary distribution, the lower increasing envelope and the upper envelope of the minimum-relative-entropy region are respectively convex and concave. We also prove that another function induced the minimum-relative-entropy region is concave. These two envelopes and this function were previously used to characterize the optimal exponents in strong small-set expansion problems and strong Brascamp–Lieb inequalities. The results in this paper, combined with the strong small-set expansion theorem derived by Yu, Anantharam, and Chen (2021), and the strong Brascamp–Lieb inequality derived by Yu (2021), confirm positively Ordentlich–Polyanskiy–Shayevitz’s conjecture on the strong small-set expansion (2019) and Polyanskiy’s conjecture on the strong Brascamp–Lieb inequality (2016). The proofs in this paper are based on the equivalence between the convexity of a function and the convexity of the set of minimizers of its Lagrangian dual.

Index Terms

Convexity, minimum-relative-entropy region, DSBS, strong small-set expansion conjecture, strong Brascamp–Lieb inequality conjecture

I. INTRODUCTION

Consider a doubly symmetric binary distribution $P_{XY}$ with correlation $\rho \in (0, 1)$, i.e.,

$$P_{XY} = \begin{pmatrix} X \setminus Y & 0 & 1 \\ 0 & \frac{1+\rho}{4} & \frac{1-\rho}{4} \\ 1 & \frac{1-\rho}{4} & \frac{1+\rho}{4} \end{pmatrix}. \quad (1)$$

Denote $k = \left(\frac{1+\rho}{1-\rho}\right)^2$. Define

$$D_2 (a) := D ( (a, 1 - a) \| P_X ) = 1 - H_2 (a),$$

$$D_2^{(a,b)} (p) := D \left( \left[ \begin{array}{c} 1 + p - a - b \\ a - p \\ b - p \end{array} \right] \| P_{XY} \right),$$

$$\mathbb{D}_2 (a, b) := \min_{0, a+b-1 \leq p \leq a, b} D_2^{(a,b)} (p)$$

$$= D_2^{(a,b)} (p^*_{a,b}),$$

where $D (Q \| P)$ denotes the relative entropy from $Q$ to $P$, $H_2 : t \in [0, 1] \mapsto -t \log t - (1 - t) \log (1 - t)$ is the binary entropy function, and

$$p^*_{a,b} = \frac{(k - 1) (a + b) + 1 - \sqrt{((k - 1) (a + b) + 1)^2 - 4 k (k - 1) ab}}{2 (k - 1)}.$$

Define the minimum-relative-entropy region of $P_{XY}$ as

$$D (P_{XY}) := \bigcup_{a,b \in [0,1]} \{(D_2 (a), D_2 (b), \mathbb{D}_2 (a, b))\}.$$

L. Yu is with the School of Statistics and Data Science, LPMC & KLMDASR, Nankai University, Tianjin 300071, China (e-mail: leiyu@nankai.edu.cn).

1Throughout this paper, the bases of all logarithms are set to 2.
Denote $H_2^{-1}$ as the inverse of the restriction of the binary entropy function $H_2$ to the set $[0, \frac{1}{2}]$. Denote $D_2^{-1}(s) := H_2^{-1}(1 - s)$ which is the inverse of $D_2$. Then, the lower and upper envelopes of $\mathcal{D}(P_{XY})$ respectively are

$$\varphi(s, t) = D_2\left(D_2^{-1}(s), D_2^{-1}(t)\right)$$

$$\psi(s, t) = D_2\left(D_2^{-1}(s), 1 - D_2^{-1}(t)\right)$$

$$\varphi_q(s) = \min_{0 \leq t \leq 1} \varphi(s, t) - \frac{t}{q}$$

$$\psi_q(s) = \max_{0 \leq t \leq 1} \psi(s, t) - \frac{t}{q}.$$

Define the lower and upper increasing envelopes of $\mathcal{D}(P_{XY})$ respectively as

$$\tilde{\varphi}(\alpha, \beta) = \min_{s \geq \alpha, t \geq \beta} \varphi(s, t)$$

$$\tilde{\psi}(\alpha, \beta) = \max_{s \leq \alpha, t \leq \beta} \psi(s, t)$$

$$\tilde{\varphi}_q(\alpha) = \min_{s \geq \alpha} \varphi_q(s), \ q \geq 1$$

$$\tilde{\psi}_q(\alpha) = \begin{cases} \max_{s \leq \alpha} \varphi_q(s) & q < 0 \\ \max_{s \leq \alpha} \psi_q(s) & 0 < q < 1 \end{cases}.$$

Denote $\Theta, \Theta_q$, respectively as the lower convex envelopes of $\tilde{\varphi}, \tilde{\varphi}_q$, and $\overline{\Theta}, \overline{\Theta}_q$ respectively as the upper concave envelopes of $\tilde{\psi}, \tilde{\psi}_q$. In fact, $\Theta$ and $\overline{\Theta}$ are the optimal exponents in the (forward and reverse) small-set expansion problems, and they, together with $\Theta'_q, \overline{\Theta}_q$, are the optimal exponents in the (forward and reverse) strong Brascamp–Lieb inequalities [1], [2]. Note that $q$ in this paper in fact corresponds to its Hölder conjugate $q'$ in [2].

We have the following properties of $\Theta, \Theta_q$, and $\overline{\Theta}_q$, which implies that the directional gradients of $\Theta$ along the $x$-axis and $y$-axis are both not greater than 1, and those of $\overline{\Theta}$ and that of $\overline{\Theta}_q$ are all not smaller than 1.

**Lemma 1.** [2] For all $\alpha, \beta \in [0, 1]$ and $0 \leq s \leq 1 - \alpha, 0 \leq t \leq 1 - \beta$, we have

$$\Theta(\alpha + s, \beta + t) - \Theta(\alpha, \beta) \leq s + t$$

$$\overline{\Theta}(\alpha + s, \beta + t) - \overline{\Theta}(\alpha, \beta) \geq s + t$$

$$\overline{\Theta}_q(\alpha + s) - \overline{\Theta}_q(\alpha) \geq s \text{ for } q < 0.$$ 

Moreover, it has been shown that $\tilde{\psi} = \psi$ which means that $\psi$ is nondecreasing.

**Lemma 2.** [1] We have $\tilde{\psi} = \psi$.

Similarly, we also have the following lemma, i.e., $\tilde{\varphi}_q$ is nondecreasing.

**Lemma 3.** [2] For $q < 0$, we have $\overline{\Theta}_q = \tilde{\varphi}_q$.

We next introduce the main results in this paper.

**Theorem 1.** $\tilde{\varphi}$ is convex on $[0, 1]^2$.

**Theorem 2.** $\psi$ is concave on $[0, 1]^2$.

**Theorem 3.** For $q < 0$, $\varphi_q$ is concave on $[0, 1]$.

The proofs of Theorems 1-3 are respectively given in Sections II-IV. These proofs are based on the equivalence between the convexity of a function and the convexity of the set of minimizers of its Lagrangian dual; see Lemma 5. Note that a more common way to prove convexity of a function is based on the following equivalence: A twice differentiable function of several variables is convex on a convex set if and only if its Hessian matrix of second partial derivatives is positive semidefinite on the interior of the convex set. Compared with this common equivalence, the equivalence used in this paper sometimes is easier to verify, especially, for a function whose second derivative is complicated. To verify the convexity of the set (or usually the uniqueness) of minimizers of its Lagrangian dual, it usually suffices to check the stationary points of the Lagrangian dual, which only involves the first derivative of
the function. Furthermore, when we verify the uniqueness of the minimizer, sometimes by changing variables, one can convert the minimization of the Lagrangian dual to a (strictly) convex optimization problem, and hence, the uniqueness follows directly.

By Theorems 1 and 2, as well as Lemma 2, we know that \( \Theta = \tilde{\varphi} \) and \( \Theta = \psi \). This, combined with the strong small-set expansion theorem [1], [2], resolves Ordentlich–Polyanskiy–Shayevitz’s conjecture on the strong small-set expansion [3]. It is easy to check that for \( q \geq 1 \), \( \varphi_q(s) = \min_{0 \leq t \leq 1} \tilde{\varphi}(s, t) - \frac{t}{q} \). Hence, \( \varphi_q \) is nondecreasing, which implies \( \tilde{\varphi}_q = \varphi_q \). By Theorems 1 and 2, we have the following corollary.

**Corollary 1.** For \( q \geq 1 \), \( \tilde{\varphi}_q = \varphi_q \) is convex; and for \( q \in (-\infty, 0) \cup (0, 1) \), \( \tilde{\psi}_q \) is concave.

This, combined with the strong Brascamp–Lieb inequality [2, Corollary 7], independently resolves Polyanskiy’s conjecture on strong Brascamp–Lieb inequality stated in [4]. Note that as mentioned in [4], Polyanskiy’s original conjecture was already solved by himself in an unpublished paper [5].

Summarizing all the above, we have that \( \tilde{\varphi} \) and \( \varphi_q \) with \( q \geq 1 \) are nondecreasing and convex; \( \psi, \varphi_q \) with \( q < 0 \), and \( \psi_q \) with \( 0 < q < 1 \) are nondecreasing and concave. This further implies that

\[
\Theta = \tilde{\varphi} \leq \varphi, \\
\Theta = \tilde{\psi} = \psi, \\
\Theta_q = \tilde{\varphi}_q = \varphi_q, q \geq 1, \\
\Theta_q = \tilde{\psi}_q = \begin{cases} \varphi_q & q < 0 \\ \psi_q & 0 < q < 1 \end{cases}.
\]

The functions \( \varphi, \tilde{\varphi}, \Theta, \psi, \tilde{\psi}, \varphi_q, \tilde{\varphi}_q, \Theta_q \) and \( \psi_q, \tilde{\psi}_q, \Theta_q \) for \( \rho = 0.9 \) are plotted in Fig. 1.

**II. PROOF OF THEOREM 1**

For brevity, we denote

\[
f(\alpha, \beta) := \tilde{\varphi}(\alpha, \beta) = \inf_{s \geq \alpha, t \geq \beta} D_2 \left( D_2^{-1}(s), D_2^{-1}(t) \right) = \inf_{a \leq D_2^{-1}(\alpha), b \leq D_2^{-1}(\beta)} D_2(a, b)
\]

Observe that given \( a \leq 1/2 \), \( D_2(a, b) \) is convex in \( b \), and the minimum \( D_2(a) \) is attained at \( b = a \ast \frac{1 - \rho}{2} \geq a \). Here \( \ast \) denote the binary convocation, i.e., \( x \ast y = x(1 - y) + y(1 - x) \). Hence,

\[
\inf_{b \leq D_2^{-1}(\beta)} D_2(a, b) = \begin{cases} D_2(a) & a \ast \frac{1 - \rho}{2} \leq D_2^{-1}(\beta) \\ D_2 \left( a, D_2^{-1}(\beta) \right) & a \ast \frac{1 - \rho}{2} > D_2^{-1}(\beta) \end{cases}
\]

WLOG, we assume \( \alpha \geq \beta \) (i.e., \( D_2^{-1}(\alpha) \leq D_2^{-1}(\beta) \)). For this case,

\[
f(\alpha, \beta) = \inf_{a \leq D_2^{-1}(\alpha)} \begin{cases} D_2(a) & a \ast \frac{1 - \rho}{2} \leq D_2^{-1}(\beta) \\ D_2 \left( a, D_2^{-1}(\beta) \right) & a \ast \frac{1 - \rho}{2} > D_2^{-1}(\beta) \end{cases}
\]

\[
= \min \left\{ \inf_{a \leq D_2^{-1}(\alpha), a \ast \frac{1 - \rho}{2} \leq D_2^{-1}(\beta)} D_2(a), \inf_{a \leq D_2^{-1}(\alpha), a \ast \frac{1 - \rho}{2} > D_2^{-1}(\beta)} D_2 \left( a, D_2^{-1}(\beta) \right) \right\}
\]

We denote \( \ast^{-1} \) as the deconvolution operation, i.e., \( z \ast^{-1} y = \frac{z - y}{1 - 2y} \) is the solution to \( x \ast y = z \) with \( x \) unknown.

If \( D_2^{-1}(\beta) \ast^{-1} \frac{1 - \rho}{2} \geq D_2^{-1}(\alpha) \),

\[
f(\alpha, \beta) = \inf_{a \leq D_2^{-1}(\alpha), a \ast \frac{1 - \rho}{2} \leq D_2^{-1}(\beta)} D_2(a)
\]

\[
= \alpha
\]

If \( D_2^{-1}(\beta) \ast^{-1} \frac{1 - \rho}{2} < D_2^{-1}(\alpha) \), then the first term in (9) satisfies that

\[
\inf_{a \leq D_2^{-1}(\alpha), a \ast \frac{1 - \rho}{2} \leq D_2^{-1}(\beta)} D_2(a) = D_2 \left( D_2^{-1}(\beta) \ast^{-1} \frac{1 - \rho}{2} \right)
\]
and the second term satisfies
\[
\inf_{a \leq D_2^{-1}(\alpha), a + \frac{\rho}{2} > D_2^{-1}(\beta)} \mathbb{D}_2 \left( a, D_2^{-1}(\beta) \right) \leq \mathbb{D}_2 \left( D_2^{-1}(\beta) \ast \frac{1 - \rho}{2}, D_2^{-1}(\beta) \right) = D_2 \left( D_2^{-1}(\beta) \ast \frac{1 - \rho}{2} \right)
\]
By the convexity of $\mathbb{D}_2$, the second term in (9) also satisfies
\[
\inf_{a \leq D_2^{-1}(\alpha), a + \frac{\rho}{2} > D_2^{-1}(\beta)} \mathbb{D}_2 \left( a, D_2^{-1}(\beta) \right) = \mathbb{D}_2 \left( D_2^{-1}(\alpha), D_2^{-1}(\beta) \right)
\]
Therefore,
\[
f(\alpha, \beta) = \mathbb{D}_2 \left( D_2^{-1}(\alpha), D_2^{-1}(\beta) \right)
\]
Summarizing the above, we have
\[
f(\alpha, \beta) = \begin{cases} 
\alpha & (\alpha, \beta) \in S_0 \\
\beta & (\alpha, \beta) \in S_0^\top \\
\mathbb{D}_2 \left( D_2^{-1}(\alpha), D_2^{-1}(\beta) \right) & \text{otherwise}
\end{cases}
\] (10)
where $S_0 := \{(\alpha, \beta) : D_2^{-1}(\beta) \geq D_2^{-1}(\alpha) \ast \frac{1 - \rho}{2}\}$ and $S_0^\top := \{(\alpha, \beta) : (\beta, \alpha) \in S_0\}$. The planes $x \mapsto x$ and $(x, y) \mapsto y$ are tangent planes of $f$ at points in $S_0$ and in $S_0^\top$, respectively. Denote
\( \tilde{f}(a, b) = \begin{cases} 
D_2(a) & b \geq a + \frac{1-\epsilon}{2} \\
D_2(b) & a \geq b + \frac{1-\epsilon}{2} \\
D_2(a, b) & \text{otherwise} \end{cases} \)

Note that \( f \) and \( \tilde{f} \) are differentiable.

Define

\[ g(s, t) := f(s, t) - \lambda s - \mu t \]

and

\[ \Gamma := \min_{s, t \in [0, 1]} g(s, t) = \min_{a, b \in [0, 1/2]} \tilde{f}(a, b) - \lambda D_2(a) - \mu D_2(b). \]

We have the following two lemmas.

**Lemma 4.** Let \( S := [0, 1]^2 \). Let \( f \) be the function given in (10), and \( \tilde{f} \) be its lower convex envelope. Then, for any subgradient \( (\lambda, \mu) \) of \( \tilde{f} \) at a point in \( S^n \), the set of minimizers of the function \( g : (x, y) \mapsto f(x, y) - \lambda x - \mu y \) is a convex subset of \( S \).

**Lemma 5.** Let \( S \) be a compact convex subset of \( \mathbb{R}^n \), and \( f : S \to \mathbb{R} \) be a continuous function. For each point \( y \in S \), let \( \lambda_y \) be a subgradient of \( \tilde{f} \) at \( y \). For any \( A \subseteq S \), let \( G(A) := \{ \lambda_y : y \in A \} \). Then, the following hold.

1) For any \( \lambda \in G(S^o) \), the set of minimizers of the function \( g : x \in S \mapsto f(x) - \langle \lambda, x \rangle \) is a convex subset of \( S \), if and only if \( f \) is convex on \( S \).

2) For any \( \lambda \in G(S^o) \), the minimizer of the function \( g : x \in S \mapsto f(x) - \langle \lambda, x \rangle \) is unique, if and only if \( f \) is strictly convex on \( S^o \).

By these two lemmas, we have that \( f \) is convex. We next prove these two lemmas.

**A. Proof of Lemma 4**

Since \( \tilde{f} \) is increasing, any subgradient \((\lambda, \mu)\) of \( \tilde{f} \) must satisfy that \( \lambda, \mu \geq 0 \). By Lemma 1, any subgradient \((\lambda, \mu)\) of \( \tilde{f} \) must satisfy that \( \lambda, \mu \leq 1 \). Hence, we only need to consider \( 0 \leq \lambda, \mu \leq 1 \). In the following, we denote \( \lambda = 1/p, \mu = 1/q \) with \( p, q \in [1, \infty] \).

If \( q = 1, p = \infty \), then

\[ g(s, t) = f(s, t) - t. \]

For this case,

\[ \Gamma = \min_{a, b \in [0, 1/2]} \tilde{f}(a, b) - D_2(b) = 0. \]

This implies that the minimizers are the points in \( S_0 \). The set \( S_0 \) is convex, since it corresponds to the set of \( 0 \leq \beta \leq D_2(D_2^{-1}(\alpha) + \frac{1-\epsilon}{2}) \), and \( \alpha \mapsto D_2(D_2^{-1}(\alpha) + \frac{1-\epsilon}{2}) = 1 - D_2(H_2^{-1}(1 - \alpha) + \frac{1-\epsilon}{2}) \) is concave [6, Problem 2.5]. Similarly, for \( p = 1, q = \infty \), the set of minimizers is also convex.

If \( 1 < q \leq \infty, p = \infty \), then

\[ g(s, t) = f(s, t) - \frac{t}{q}. \]

\(^3\)If \( f : S \to \mathbb{R} \) is a real-valued convex function defined on a convex set \( S \) in the Euclidean space \( \mathbb{R}^n \), a vector \( v \) in that space is called a subgradient of \( f \) at a point \( x_0 \) in \( S^n \) if for any \( x \) in \( S \) one has \( f(x) - f(x_0) \geq \langle v, x - x_0 \rangle \) where the \( \langle \cdot, \cdot \rangle \) denotes the inner product. Equivalently, \( x \in S \mapsto f(x) + \langle v, x - x_0 \rangle \) forms a supporting hyperplane of \( f \) at \( x_0 \). A vector \( v \) is called a supergradient of a concave function \( g \) at a point \( x_0 \) if \( -v \) is a subgradient of \( -g \) at \( x_0 \).

\(^4\)We use \( S^o \) to denote the interior of \( S \).
For this case,

\[
\Gamma = \min_{a,b \in [0,1/2]} \tilde{f}(a, b) - \frac{1}{q} D_2(b)
\]

\[
= \min_{b \in [0,1/2]} \left( 1 - \frac{1}{q} \right) D_2(b)
\]

\[
= 0.
\]

This implies that the minimizer of \( g \) is \((0, 0)\), and is unique. Similarly, for \( 1 < p \leq \infty, q = \infty \), the minimizer is also unique.

We now consider the case of \( 1 \leq p < q < \infty \). Denote \( Q_{XY} := \begin{bmatrix} q_{00} & q_{01} \\ q_{10} & q_{11} \end{bmatrix} \) as a distribution (i.e., \( q_{x,y} \geq 0 \) for all \( x,y \in \{0,1\} \) and \( \sum_{x,y \in \{0,1\}} q_{x,y} = 1 \)). Consider the following minimization problem.

\[
\Gamma = \min_{Q_{xy} : q_{11} + q_{01} \leq 1/2, q_{11} + q_{01} \leq 1/2} d(Q_{xy}),
\]

where

\[
d(Q_{xy}) := D(Q_{xy} || P_{xy}) - \frac{1}{p} D_2(q_{11} + q_{10}) - \frac{1}{q} D_2(q_{11} + q_{01}).
\]

We have the following two facts.

Fact 1. If the minimizer of (11) is unique, then the minimizer of \( g \) on \([0,1]^2 \setminus (S_0 \cup S_0^T)\) is also unique.

Fact 2. Any strictly interior stationary points of (11) satisfy the following Lagrangian conditions:

\[
Q_{XY}(x, y) = \frac{\Pi(x, y)}{\sum_{x,y \in \{0,1\}} \Pi(x, y)}, \forall x, y \in \{0,1\}
\]

where

\[
\Pi(x, y) = P_{XY}(x, y) \left( \frac{Q_X(x)}{P_X(x)} \right)^{1/p} \left( \frac{Q_Y(y)}{P_Y(y)} \right)^{1/q}.
\]

From Fact 2,

\[
\left( \frac{q_{00} + q_{01}}{q_{10} + q_{11}} \right)^{1/p} = \frac{q_{00}}{q_{10}} \frac{1}{\theta} = \frac{q_{01}}{q_{11}} \theta =: z
\]

(12)

with \( \frac{q_{00} + q_{01}}{q_{10} + q_{11}} \geq 1 \), and

\[
\left( \frac{q_{00} + q_{10}}{q_{01} + q_{11}} \right)^{1/q} = \frac{q_{00}}{q_{01}} \theta = \frac{q_{10}}{q_{11}} \frac{1}{\theta}
\]

(13)

with \( \frac{q_{00} + q_{10}}{q_{01} + q_{11}} \geq 1 \).

If \( p = 1 \leq q < \infty \), then solving the above equations, we have \( (q_{00}, q_{01}, q_{10}, q_{11}) = \left( \frac{1+\rho}{4}, \frac{1-\rho}{4}, \frac{1-\rho}{4}, \frac{1+\rho}{4} \right) \). Obviously, this stationary point is not a minimizer. Hence, the minimizers are on the boundary \( s = 0 \). Observe that \( f(0,t) \) is convex in \( t \) for \( t \geq D_2(\frac{1-\rho}{2}) \). Hence, the minimizer is unique. Similarly, for \( q = 1 \leq p < \infty \), the minimizer is also unique.

It remains to consider the case \( 1 < p, q < \infty \). Denote \( r = (p-1)(q-1) \). If \( r > \rho^2 \) and \( 1 < p, q < \infty \), then by the information-theoretic characterization of the hypercontractivity region [7], \((0, 0)\) is the unique minimizer of \( g \). Hence, Lemma 4 is satisfied for this case.

We next consider the case \( 0 < r \leq \rho^2, 1 < p, q < \infty \). Denote \( \theta = \frac{1-\rho}{1+\rho} \), \( u := \frac{1}{p-1}, v := \frac{1}{q-1} \). For this case, we have \( u > 1 \) or \( v > 1 \). By symmetry, WLOG, we assume \( u > 1 \) here. Solving (12) and (13), we have

\[
z^{uv} = \left( \frac{1+\theta z}{\theta + z} \right)^u \theta + (\theta + z)^u.
\]

(14)

By using a proof idea similar to [8], we show the uniqueness of the root of the equation above.

Lemma 6. Let \( |v| > 1 \) and \( \theta \in (0, 1) \). If \( 0 < r \leq \rho^2 \), then the equation (14) has a unique root (w.r.t. \( z \)) for \( z \in (1, \infty) \).
Remark 1. Lemma 6 for the case $v > 1$ is sufficient to prove Lemma 4. However, Lemma 6 for $v < -1$ will be used to prove Theorems 2 and 3 in the next two sections.

Proof: Here we only focus on the case $v > 1$. The case $v < -1$ follows similarly.

Let $z = e^h$ and $g (h) = \log \left( (1 + \theta e^h)^v \theta + (\theta + e^h)^v \right)$. Then, the equation (14) is equivalent to

$$rvh = g (h) - g (-h) - hv.$$

We define

$$\varphi (h) = g (h) - g (-h) - hv.$$

Its derivative is

$$\varphi' (h) = g' (h) + g' (-h) - v - rv.$$

where

$$g' (h) = v \left( 1 - \theta \left( \frac{(1 + \theta e^h)^{v-1} + (\theta + e^h)^{v-1}}{(1 + \theta e^h)^v \theta + (\theta + e^h)^v} \right) \right).$$

We next show that $\varphi' (h) = 0$ has a unique root on $h > 0$.

Observe that $\varphi' (h) > 0$ is equivalent to

$$r \left( \eta^v + \eta^{-v} \right) + \eta + \eta^{-1} < (1 - r) \left( \theta + \theta^{-1} \right)$$

(15)

where $\eta = \frac{1 + \theta e^h}{\theta + e^h}$. (The equivalence still holds if both the inequalities above change to the other direction.)

For $v > 1$, $\eta \mapsto r \left( \eta^v + \eta^{-v} \right) + \eta + \eta^{-1}$ is convex. Hence, for $0 < r \leq \rho^2$,

$$r \left( \eta^v + \eta^{-v} \right) + \eta + \eta^{-1} = (1 - r) \left( \theta + \theta^{-1} \right)$$

has a unique root on $\eta < 1$, denoted as $\eta_0$. Moreover, for $\eta = 1$, (15) holds. Denote $h_0$ such that $\eta_0 = \frac{1 + \theta e^{h_0}}{\theta + e^{h_0}}$. Hence, $\varphi' (h) \geq 0$ is equivalent to that $\eta_0 \leq \eta \leq 1$, which implies $\varphi$ is increasing on $[0, h_0]$ and decreasing on $(h_0, \infty)$. Observe that $\varphi (0) = 0$. Hence, $\varphi (h) = 0$ has a unique root for $h > 0$.

We now verify that this stationary point is the unique minimizer. Consider a boundary point $(1, t)$. For this point, denoting $b = D_2^{-1} (t)$, we have

$$g (1, t) = D_2 (0, b) - \frac{1}{p} - \frac{1}{q} D_2 (b)$$

$$= D_2^{(0, b)} (p_0^*, \alpha) - \frac{1}{p} - \frac{1}{q} D_2 (b)$$

Denoting $a = D_2^{-1} (s)$, by the Taylor expansion of $g (s, t)$ at $s = 1$ (for fixed $t$), we have

$$D_2^{(a, b)} (p_a^*, \alpha) = D_2^{(0, b)} (p_0^*, \alpha) + a \log a + o_{a \to 0} (a \log a)$$

and

$$D_2 (a) = 1 + a \log a + o_{a \to 0} (a \log a) .$$

Hence,

$$g (s, t) = g (1, t) + \left( 1 - \frac{1}{p} \right) a \log a + o_{a \to 0} (a \log a) ,$$

which implies that $(1, t)$ is not a minimizer of $g$.

Consider a boundary point $(0, t)$. For this point,

$$g (0, t) = t - \frac{1}{p} t \geq 0$$

Moreover, by the information-theoretic characterization of the hypercontractivity region [7], we know that for $r < \rho^2$, $\frac{r}{p} + \frac{q}{q}$ is not a supporting plane of $f$, which means that there is a point $(x, y)$ such that $f (x, y) < \frac{r}{p} + \frac{q}{q}$. Hence, any boundary point $(0, t)$ cannot be a minimizer.

Combining all the cases above, we have Lemma 4.
B. Proof of Lemma 5

Here we only prove the first statement. The other statement follows similarly.

The “only if” part is obvious. Here we only focus on the “if” part. For a function $g$, denote its epigraph as $\text{epi}(g)$. Denote $\tilde{f}$ as the lower convex envelope of $f$. Let $M$ be a finite value such that $M > \max_{x \in S} f(x)$. Hence, $U := (S \times (-\infty, M]) \cap \text{epi}(\tilde{f})$ is convex and compact in $\mathbb{R}^{n+1}$. By Krein–Milman theorem, $U$ has extreme points, and if the set of extreme points is denoted by $E \subseteq \{(x, f(x)) : x \in S \} \cup (S \times \{M\})$, then $U$ is the closed convex hull of $E$.

Suppose that there exists $x_0 \in S^0$ such that $\tilde{f}(x_0) < f(x_0)$. Then, $(x_0, \tilde{f}(x_0))$ is in the interior of $U$, and hence is not in $E$. Hence, there exist $n + 2$ points $x_i \in E, 1 \leq i \leq n + 2$ such that $(x_0, \tilde{f}(x_0)) = \sum_{i=1}^{n+2} q_i(x_i, \tilde{f}(x_i))$ where $\sum_{i=1}^{n+2} q_i = 1, q_i \geq 0, 1 \leq i \leq n + 2$, and at least two of $q_i \geq 0, 1 \leq i \leq n + 2$ are strictly positive. Let $x \mapsto (\lambda, x) + c$ be a supporting hyperplane of $\tilde{f}$ at $x_0$. Then all the points $(x_i, \tilde{f}(x_i))$ with $q_i > 0$ are on this hyperplane, and all of them are minimizers of $g : x \in S \mapsto f(x) - (\lambda, x)$. By the assumption that the set of minimizers is convex, $x_0$ is also a minimizer of $g$, which implies that $(x_0, f(x_0))$ is on that hyperplane. Hence, $\tilde{f}(x_0) = f(x_0)$, contradicting with the assumption $\tilde{f}(x_0) < f(x_0)$. Therefore, $\tilde{f}(x_0) = f(x_0)$ for all $x_0 \in S^0$. That is, $f$ is convex on $S^0$. By the continuity of $f$, $f$ is convex on $S$.

III. PROOF OF THEOREM 2

For brevity, we denote

$$f(\alpha, \beta) := \bar{\psi}(\alpha, \beta) = \sup_{s \leq \alpha, t \leq \beta} \mathbb{D}_2 \left(D_2^{-1}(s), 1 - D_2^{-1}(t)\right)$$

$$= \mathbb{D}_2 \left(D_2^{-1}(\alpha), 1 - D_2^{-1}(\beta)\right)$$

(16)

Denote

$$\bar{f}(a, b) = \mathbb{D}_2(a, b)$$

Note that $f$ and $\bar{f}$ are differentiable. Define

$$g(s, t) := f(s, t) - \lambda s - \mu t$$

$$= \mathbb{D}_2 \left(D_2^{-1}(s), 1 - D_2^{-1}(t)\right) - \lambda s - \mu t$$

and

$$\Gamma := \max_{s,t \in [0,1]} g(s, t)$$

$$= \max_{a \in [0,1/2], b \in [1/2,1]} \mathbb{D}_2(a, b) - \lambda D_2(a) - \mu D_2(b).$$

(17)

We have the following lemma.

Lemma 7. Let $S := [0, 1]^2$. Let $f$ be the function given in (16), and $\tilde{f}$ be its upper concave envelope. Then, for any supergradient $(\lambda, \mu)$ of $\tilde{f}$ at a point in $S^0$, the set of maximizers of the function $g : (x, y) \mapsto f(x, y) - \lambda x - \mu y$ is a convex subset of $S$.

Combining Lemmas 7 and 5, we have that $f$ is strictly concave. We next prove Lemma 7.

A. Proof of Lemma 7

By Lemma 1, any supergradient $(\lambda, \mu)$ of $\tilde{f}$ at a point in $S^0$ must satisfy that $\lambda, \mu \in [1, \infty)$. Hence, we only need to consider $\lambda, \mu \in [1, \infty)$. In the following, we denote $\lambda = 1/p, \mu = 1/q$ with $p, q \in (0, 1]$. In the following, we denote $Q_X = \{1 - a, a\}, Q_Y = \{1 - b, b\}$, and hence, $\max_{Q_X}$ and $\min_{Q_Y}$ denote optimizations over the probability simplex $\{(a_0, a_1) \in \mathbb{R}^2_{\geq 0} : a_0 + a_1 = 1\}$. Since the objective function in (17) is continuous, and the domain of feasible solutions is compact, we know that the maximum in (17) is attained.

In fact, (17) can be rewritten as

$$\Gamma = \max_{Q_X, Q_Y} \mathbb{D}(Q_X, Q_Y || P_{X \mid Y}) - \lambda D(Q_X || P_X) - \mu D(Q_Y || P_Y)$$

(18)
where
\[ \mathbb{D}(Q_X, Q_Y \| P_{XY}) = \min_{R_{XY} \in C(Q_X, Q_Y)} D(R_{XY} \| P_{XY}). \] (19)

The Lagrangian of the minimization in (19) is
\[ L_{Q_X, Q_Y}(\eta_X, \eta_Y, R_{XY}) = D(R_{XY} \| P_{XY}) + \sum_x \eta_X(x) (R_X(x) - Q_X(x)) + \sum_y \eta_Y(y) (R_Y(y) - Q_Y(y)). \]

Since the minimization in (19) is a convex optimization problem with linear constraints, Slater’s condition is satisfied, which in turn implies that the strong duality holds and the optimal solution of the dual problem exists [9]. Hence,
\[ \mathbb{D}(Q_X, Q_Y \| P_{XY}) = \max_{\eta_X, \eta_Y} \min_{R_{XY}} L_{Q_X, Q_Y}(\eta_X, \eta_Y, R_{XY}) \]
and the maximum is attained. Substituting this into (18) yields
\[ \Gamma = \max_{Q_X, Q_Y} \min_{\eta_X, \eta_Y, R_{XY}} K(Q_X, Q_Y, \eta_X, \eta_Y, R_{XY}) \] (20)
where
\[ K(Q_X, Q_Y, \eta_X, \eta_Y, R_{XY}) := L_{Q_X, Q_Y}(\eta_X, \eta_Y, R_{XY}) - \lambda D(Q_X \| P_X) - \mu D(Q_Y \| P_Y). \]

Let \((Q_X^*, Q_Y^*, \eta_X^*, \eta_Y^*)\) be a maximizer in (20), and given \((Q_X^*, Q_Y^*, \eta_X^*, \eta_Y^*)\), \(R_{XY}^*\) is a minimizer for the inner minimization. Observe that
\[ \Gamma = \max_{Q_X, Q_Y} \min_{\eta_X, \eta_Y, R_{XY}} K(Q_X, Q_Y, \eta_X, \eta_Y, R_{XY}) = \min_{R_{XY}} \left\{ D(R_{XY} \| P_{XY}) + \sum_x \eta_X^*(x) R_X(x) + \sum_y \eta_Y^*(y) R_Y(y) \right\} \]
- \[ \min_{Q_X, Q_Y} \left\{ \lambda D(Q_X \| P_X) + \mu D(Q_Y \| P_Y) + \sum_x \eta_X^*(x) Q_X(x) + \sum_y \eta_Y^*(y) Q_Y(y) \right\} \].

By Lagrangian conditions,
\[ R_{XY}^*(x, y) = \frac{P_{XY}(x, y) e^{-\eta_X^*(x) - \eta_Y^*(y)}}{\sum_{x,y} P_{XY}(x, y) e^{-\eta_X^*(x) - \eta_Y^*(y)}} \] (21)
\[ Q_X^*(x) = \frac{P_X(x) e^{-\eta_X^*(x)/\lambda}}{\sum_x P_X(x) e^{-\eta_X^*(x)/\lambda}} \] (22)
\[ Q_Y^*(y) = \frac{P_Y(y) e^{-\eta_Y^*(y)/\mu}}{\sum_y P_Y(y) e^{-\eta_Y^*(y)/\mu}}. \] (23)

Observe that \(K(Q_X, Q_Y, \eta_X, \eta_Y, R_{XY})\) is convex in \(R_{XY}\), and concave in \((\eta_X, \eta_Y)\), by the strong duality,
\[ \Gamma = \min_{R_{XY}} \max_{\eta_X, \eta_Y} K(Q_X^*, Q_Y^*, \eta_X, \eta_Y, R_{XY}). \] (24)

Let \(R_{XY}^{**}\) be a minimizer for the minimization in (24). Given \((Q_X^*, Q_Y^*)\), by the strong duality, \((\eta_X^*, \eta_Y^*, R_{XY}^{**})\) is a saddle point of \((\eta_X, \eta_Y, R_{XY}) \mapsto K(Q_X^*, Q_Y^*, \eta_X, \eta_Y, R_{XY})\), which hence satisfies
\[ R_{XY}^{**}(x, y) = \frac{P_{XY}(x, y) e^{-\eta_X^*(x) - \eta_Y^*(y)}}{\sum_{x,y} P_{XY}(x, y) e^{-\eta_X^*(x) - \eta_Y^*(y)}} \] (25)
\[ R_X^* = Q_X^* \] (26)
\[ R_Y^* = Q_Y^*. \] (27)
Comparing (21) with (25) we have $R^*_X = R^*_Y$. Therefore, from (21)-(23) and (25)-(27), we have

$$
R^*_{XY} (x,y) = \frac{P_{XY} (x,y) e^{-\eta^*_{XY} (x,y)}}{\sum_{x,y} P_{XY} (x,y) e^{-\eta^*_{XY} (x,y)}}
$$

$$
R^*_X = Q^*_X
$$

$$
Q^*_X (x) = \frac{P_X (x) e^{-\eta^*_X (x)}}{\sum_x P_X (x) e^{-\eta^*_X (x)}}
$$

$$
Q^*_Y (y) = \frac{P_Y (y) e^{-\eta^*_Y (y)}}{\sum_y P_Y (y) e^{-\eta^*_Y (y)}}
$$

To prove the uniqueness of optimal $(Q^*_X, Q^*_Y)$, it suffices to show the uniqueness of the solution of the following equations with $R_{XY}$ unknown.

$$
R_{XY} (x,y) = \frac{\hat{\Pi} (x,y)}{\sum_{x,y \in \{0,1\}} \hat{\Pi} (x,y)}, \forall x, y \in \{0,1\}
$$

where

$$
\hat{\Pi} (x,y) = P_{XY} (x,y) \left( \frac{R_X (x)}{P_X (x)} \right)^{\lambda} \left( \frac{R_Y (y)}{P_Y (y)} \right)^{\mu}
$$

Denote $R_{XY} := \begin{bmatrix} q_{00} & q_{01} \\ q_{10} & q_{11} \end{bmatrix}$. From (28), we have (12) and (13) but with $\frac{q_{00} + q_{01}}{q_{10} + q_{11}} \geq 1$ and $\frac{q_{00} + q_{01}}{q_{10} + q_{11}} \leq 1$.

If $p = 1, q \in [0,1]$, then solving (12) and (13), we have $(q_{00}, q_{01}, q_{10}, q_{11}) = \left( \frac{1+\rho}{4}, \frac{1-\rho}{4}, \frac{1-\rho}{4}, \frac{1+\rho}{4} \right)$. Obviously, this stationary point is not a maximizer. Hence, all the maximizers are on the boundary $s = 1$. For this case,

$$
g(1,t) = f(1,t) - 1 - \frac{t}{q} = \left( 1 - \frac{1}{q} \right) D_2(b) - b \log \frac{1 - \rho}{2} - (1 - b) \log \frac{1 + \rho}{2},
$$

where $b = D_2^{-1}(t)$. If $q \in (0,1)$, then (29) is strictly concave in $b$. Hence, the maximizer of $g$ is unique for this case. If $q = 1$, then (29) is maximized uniquely at $b = 1$, and hence, the maximizer of $g$ is also unique (i.e., $s = t = 1$) for this case. Hence, the maximizer of $g$ is unique for $p = 1, q \in [0,1]$. By symmetry, the maximizer of $g$ is also unique for $q = 1, p \in [0,1]$.

It remains to consider the case $p, q \in (0,1)$. Denote $r = (p - 1)(q - 1)$. If $r > \rho^2$ and $p, q \in (0,1)$, then by the information-theoretic characterization of the reverse hypercontractivity region, $(0,0)$ is the unique maximizer of $g$. Hence, Lemma 4 is satisfied for this case.

We next consider the case $0 < r \leq \rho^2, p, q \in (0,1)$. Denote $\theta = \frac{1-\rho}{1+\rho}$, $u := \frac{1}{p-1}$, $v := \frac{1}{q-1}$. For this case, we have $u, v < -1$. Solving (12) and (13), we have (14). By Lemma 6, there is only one stationary point of $g$.

We now verify that this stationary point is the unique maximizer. By the information-theoretic characterization of the reverse hypercontractivity region, $\Gamma = \max_{s,t \in [0,1]} g(s,t)$ is positive. Hence, $(0,0)$ is not a maximizer of $g$. We next consider a boundary point $(1,t)$. Denoting $a = D_2^{-1}(s), b = D_2^{-1}(t), by the Taylor expansion of f(s,t) at s = 1 (for fixed t), we have

$$
D_2^{a,b} \left( p^*_{a,b} \right) = D_2^{0,b} \left( p^*_{0,b} \right) + a \log a + o_{a \to 0} (a \log a)
$$

and

$$
D_2 (a) = 1 + a \log a + o_{a \to 0} (a \log a).
$$

That is, $\frac{\partial}{\partial a} f(s,t) |_{s=1} = 1$ for any $t \in [0,1]$. Hence,

$$
g(s,t) = g(1,t) + \left( 1 - \frac{1}{p} \right) a \log a + o_{a \to 0} (a \log a),
$$

which implies that $(1,t)$ is not a maximizer of $g$. Similarly, any point $(s,1)$ is also not a maximizer of $g$. 

Consider a boundary point \((0, t)\). For this point, \(\frac{\partial}{\partial s} f(s, t)|_{s=0} = \infty\) for all \(t \in [0, 1]\). Hence, any point \((0, t)\) is not a maximizer of \(g\). Similarly, any point \((s, 0)\) is also not a maximizer of \(g\).

Combining all the cases above, we have Lemma 4.

IV. Proof of Theorem 3

For brevity, we denote
\[
f(s) := \varphi_q(s) = \min_{0 \leq t \leq 1} \varphi(s, t) - \frac{t}{q} = \min_{0 \leq t \leq 1} D_2 \left( D_2^{-1}(s), D_2^{-1}(t) \right) - \frac{t}{q}.
\] (30)

Denote
\[
\tilde{f}(a, b) = D_2(a, b)
\]

Note that \(f\) and \(\tilde{f}\) are differentiable. Define
\[
g(s) := f(s) - \lambda s
\]
and
\[
\Gamma := \max_{s \in [0, 1]} g(s)
\]
\[
= \max_{a \in [0, 1/2]} \min_{b \in [0, 1/2]} D_2(a, b) - \frac{D_2(b)}{q} - \lambda D_2(a).
\] (31)

We have the following lemma.

Lemma 8. Let \(S := [0, 1]^2\). Let \(f\) be the function given in (30), and \(\hat{f}\) be its upper concave envelope. Then, for any supergradient \((\lambda, \mu)\) of \(\hat{f}\) at a point in \(S^o\), the set of maximizers of the function \(g : (x, y) \mapsto f(x, y) - \lambda x - \mu y\) is a convex subset of \(S\).

Combining Lemmas 8 and 5, we have that \(f\) is strictly concave. We next prove Lemma 8.

A. Proof of Lemma 8

By Lemma 3, any supergradient \(\lambda\) of \(\hat{f}\) at a point in \(S^o\) must satisfy that \(\lambda \in [1, \infty)\). In the following, we denote \(\lambda = 1/p, \mu = 1/q\) with \(p \in (0, 1], q < 0\).

In fact, (31) can be rewritten as
\[
\Gamma = \max_{Q_X, Q_Y} \min \mathbb{D} (Q_X, Q_Y \| P_{XY}) - \mu D (Q_Y \| P_Y) - \lambda D (Q_X \| P_X)
\]
\[
= \max_{Q_X} \left\{ \min_{Q_Y, R_{XY} : R_X = Q_X, R_Y = Q_Y} \left\{ D (R_{XY} \| P_{XY}) - \mu D (Q_Y \| P_Y) \right\} - \lambda D (Q_X \| P_X) \right\}
\] (32)
(33)

where \(\mathbb{D} (Q_X, Q_Y \| P_{XY})\) is defined in (19). The Lagrangian of the inner minimization in (33) is
\[
L_{Q_X} (Q_Y, R_{XY}, \eta_X, \eta_Y) = D (R_{XY} \| P_{XY}) - \mu D (Q_Y \| P_Y) + \sum_x \eta_X(x) (R_X(x) - Q_X(x))
\]
\[+ \sum_y \eta_Y(y) (R_Y(y) - Q_Y(y)).
\]

Since the inner minimization in (33) is a convex optimization problem with linear constraints, the strong duality holds. Hence, the inner minimization in (33) is equal to
\[
\max_{\eta_X, \eta_Y} \min_{Q_X, R_{XY}} L_{Q_X} (Q_Y, R_{XY}, \eta_X, \eta_Y).
\]
Substituting this into (33) yields
\[
\Gamma = \max_{Q_x, Q_Y, \eta_x, \eta_y} \min_{Q_{XY}, R_{XY}} K(Q_X, Q_Y, \eta_x, \eta_y, R_{XY}) \tag{34}
\]
where
\[
K(Q_X, Q_Y, \eta_x, \eta_y, R_{XY}) := L_{Q_X}(Q_Y, R_{XY}, \eta_x, \eta_y) - \lambda D(Q_X \| P_X).
\]
Let \((Q_X^*, \eta_x^*, \eta_y^*)\) be a maximizer in (34), and given \((Q_X^*, \eta_x^*, \eta_y^*)\), \((Q_Y^*, R_{XY}^*)\) is a minimizer for the inner minimization. Observe that
\[
\Gamma = \max_{Q_x, Q_Y} \min_{R_{XY}} K(Q_X, Q_Y, \eta_x^*, \eta_y^*, R_{XY})
\]
\[
= \min_{R_{XY}} \left\{ D(R_{XY} \| P_{XY}) + \sum_x \eta_x^*(x)R_X(x) + \sum_y \eta_y^*(y)R_Y(y) \right\}
\]
\[
- \min_{Q_X} \left\{ \lambda D(Q_X \| P_X) + \sum_x \eta_x^*(x)Q_X(x) \right\}
\]
\[
- \max_{Q_Y} \left\{ \mu D(Q_Y \| P_Y) + \sum_y \eta_y^*(y)Q_Y(y) \right\}.
\]
By Lagrangian conditions,
\[
R_{XY}^* (x, y) = \frac{P_{XY} (x, y) e^{-\eta_x^*(x) - \eta_y^*(y)}}{\sum_{x,y} P_{XY} (x, y) e^{-\eta_x^*(x) - \eta_y^*(y)}} \tag{35}
\]
\[
Q_X^*(x) = \frac{P_X (x) e^{-\eta_x^*(x)/\lambda}}{\sum_x P_X (x) e^{-\eta_x^*(x)/\lambda}} \tag{36}
\]
\[
Q_Y^*(y) = \frac{P_Y (y) e^{-\eta_y^*(y)/\mu}}{\sum_y P_Y (y) e^{-\eta_y^*(y)/\mu}} \tag{37}
\]
Observe that \(K(Q_X, Q_Y, \eta_x, \eta_y, R_{XY})\) is convex in \((Q_Y, R_{XY})\), and concave in \((\eta_x, \eta_y)\), by the strong duality,
\[
\Gamma = \min_{Q_Y, R_{XY}} \max_{\eta_x, \eta_y} K(Q_X^*, Q_Y, \eta_x^*, \eta_y^*, R_{XY}^*). \tag{38}
\]
Let \((Q_Y^{**}, R_{XY}^{**})\) be a minimizer for the minimization in (38). Given \(Q_X^*\), by the strong duality, \((\eta_x^*, \eta_y^*, R_{XY}^{**})\) is a saddle point of \((\eta_x, \eta_y, Q_Y, R_{XY}) \mapsto K(Q_X^*, Q_Y, \eta_x, \eta_y, R_{XY})\), which hence satisfies
\[
R_{XY}^{**} (x, y) = \frac{P_{XY} (x, y) e^{-\eta_x^*(x) - \eta_y^*(y)}}{\sum_{x,y} P_{XY} (x, y) e^{-\eta_x^*(x) - \eta_y^*(y)}} \tag{39}
\]
\[
R_X^{**} = Q_X^* \tag{40}
\]
\[
R_Y^{**} = Q_Y^{**} \tag{41}
\]
\[
Q_Y^{**} (y) = \frac{P_Y (y) e^{-\eta_y^*(y)/\mu}}{\sum_y P_Y (y) e^{-\eta_y^*(y)/\mu}}. \tag{42}
\]
Comparing (35) with (39) we have \(R_{XY}^* = R_{XY}^{**}, \ Q_Y^* = Q_Y^{**}\). Therefore, from (35)-(37) and (39)-(41), we have
\[
R_{XY}^* (x, y) = \frac{P_{XY} (x, y) e^{-\eta_x^*(x) - \eta_y^*(y)}}{\sum_{x,y} P_{XY} (x, y) e^{-\eta_x^*(x) - \eta_y^*(y)}}
\]
\[
R_X^* = Q_X^*
\]
\[
R_Y^* = Q_Y^*
\]
\[
Q_X^*(x) = \frac{P_X (x) e^{-\eta_x^*(x)/\lambda}}{\sum_x P_X (x) e^{-\eta_x^*(x)/\lambda}}
\]
\[
Q_Y^*(y) = \frac{P_Y (y) e^{-\eta_y^*(y)/\mu}}{\sum_y P_Y (y) e^{-\eta_y^*(y)/\mu}}.
To prove the uniqueness of $Q^*_X$, it suffices to show the uniqueness of the solution of the following equations with $R_{XY}$ unknown.

$$R_{XY}(x, y) = \frac{\hat{\Pi}(x, y)}{\sum_{x,y \in \{0,1\}} \hat{\Pi}(x, y)}, \forall x, y \in \{0,1\}$$

(43)

where

$$\hat{\Pi}(x, y) = P_{XY}(x, y) \left( \frac{R_X(x)}{P_X(x)} \right)^\lambda \left( \frac{R_Y(y)}{P_Y(y)} \right)^\mu.$$ 

Denote $R_{XY} := \begin{bmatrix} q_{00} & q_{01} \\ q_{10} & q_{11} \end{bmatrix}$. From (43), we have (12) and (13) with $\frac{q_{00}+q_{01}}{q_{01}+q_{11}} \geq 1$ and $\frac{q_{00}+q_{11}}{q_{01}+q_{11}} \geq 1$. However, here we denote $z$ as the expressions in (13) (rather than the ones in (12)).

If $p = 1, q < 0$, then solving (12) and (13), we have $(q_{00}, q_{01}, q_{10}, q_{11}) = \left( \frac{1+p}{4}, \frac{1-p}{4}, \frac{1+\rho}{4}, \frac{1+\rho}{4} \right)$. This is the boundary point $s = 0$ and $g(0) = 0$. For the boundary $s = 1$, $g(1) = \min_{0 \leq t \leq 1} \varphi(1, t) - \frac{t}{q} - 1$ subject to $0 \leq b \leq 1/2$.

We next consider the case $0 < r < r^2, p, q \in (0,1)$, then by the information-theoretic characterization of the reverse hypercontractivity region [10], $(0,0)$ is the unique maximizer of $g$. Hence, Lemma 4 is satisfied for this case.

We now verify that this stationary point, denoted as $s^*$, or the boundary point $s = 1$ is the unique maximizer. By the information-theoretic characterization of the reverse hypercontractivity region [10], $\Gamma = \max_{s \in [0,1]} g(s)$ is positive. This implies that the boundary point $s = 0$ is not optimal since $g(0) = 0$. Moreover, if both the stationary point $s^*$ and the boundary point $s = 1$ are maximizers, then there must exist a local minimum $s^*$ of $g$ which is strictly between $s^*$ and $1$. From the definition of $g$, we know that there exists some $\hat{s}^*$ such that $(\hat{s}^*, \hat{t}^*)$ is a local minimum of $\hat{g} : (s, t) \mapsto \varphi(s, t) - \frac{t}{q} - \frac{s}{p}$. Moreover, given $q < 0$ and $0 < a < 1/2$, $b \mapsto \mathbb{D}_2(a, b) - \frac{D_2(b)}{q}$ is convex, and the minimum of $b \mapsto \mathbb{D}_2(a, b) - \frac{D_2(b)}{q}$ is attained at a point strictly between $0 < b < 1/2$. Hence, $0 < \hat{t}^* < 1$, which implies that $(\hat{s}^*, \hat{t}^*)$ is a stationary point of $\hat{g}$. From the proof of Lemma 4, the distribution $(q_{00}, q_{01}, q_{10}, q_{11})$ induced $(\hat{s}^*, \hat{t}^*)$ must satisfy Lagrangian conditions in (12) and (13). This further implies that the $z$ defined as the expressions in (13) induced by $(\hat{s}^*, \hat{t}^*)$ satisfy (14), and it is different from the $z$ induced by $s^*$ since $\hat{s}^* \neq s^*$. This contradicts with Lemma 6. Therefore, the stationary point $s^*$ or the boundary point $s = 1$ is the unique maximizer of $g$. This completes the proof of Lemma 4.

REFERENCES

[1] L. Yu, V. Anantharam, and J. Chen. Graphs of joint types, noninteractive simulation, and stronger hypercontractivity. arXiv preprint arXiv:2102.00668, Feb. 2021. [Online]. Available: https://arxiv.org/abs/2102.00668.

[2] L. Yu. Strong brascamp-lieb inequalities. arXiv preprint arXiv:2102.06935, 2021.

[3] O. Ordentlich, Y. Polyanskiy, and O. Shayevitz. A note on the probability of rectangles for correlated binary strings. IEEE Trans. Inf. Theory, 2020.

[4] N. Kirshner and A. Samorodnitsky. A moment ratio bound for polynomials and some extremal properties of Krawchouk polynomials and Hamming spheres. arXiv preprint arXiv:1909.11929, 2019.

[5] Y. Polyanskiy. Hypercontractivity for sparse functions on the discrete hypercube. manuscript, 2019.

[6] A. El Gamal and Y.-H. Kim. Network Information Theory. Cambridge University Press, 2011.

[7] C. Nair. Equivalent formulations of hypercontractivity using information measures. In International Zurich Seminar, 2014.

[8] C. Nair and Y. N. Wang. Evaluating hypercontractivity parameters using information measures. In 2016 IEEE International Symposium on Information Theory (ISIT), pages 570–574. IEEE, 2016.
[9] S. Boyd and L. Vandenberghe. *Convex optimization*. Cambridge University Press, 2004.

[10] S. Beigi and C. Nair. Equivalent characterization of reverse Brascamp-Lieb-type inequalities using information measures. In *2016 IEEE International Symposium on Information Theory (ISIT)*, pages 1038–1042. IEEE, 2016.