Yilmaz Cancels Newton*

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Abstract

A central tenet of the new theory of gravity proposed by H. Yilmaz is the inclusion of a gravitational stress-energy tensor $-t_{\mu \nu}$ along with the matter stress-energy tensor $T_{\mu \nu}$ on the right hand side of the Einstein field equations. This change does not effect the Newtonian limit of the field equations since these terms are quadratic in potential gradients. From the Bianchi identities, however, important changes appear in any equations of motion consistent with these field equations. For matter described as a perfect fluid, and with Yilmaz’s choice of signs when introducing these quadratic terms, we find that the Euler hydrodynamic equation in the Newtonian limit is modified to remove all gravitational forces. This allows, e.g., a solar system in which the Sun and the planets are permanently at rest, but does not explain how fluid bodies such as the Sun or Jupiter could form or be prevented from dispersing.

PACS 04.20.Cv – Fundamental problems and general formalism.
PACS 04.50.+h – . . . other theories of gravitation.
PACS 04.25.-g – Approximation methods; equations of motion.

1 Introduction

Recent publications by Yilmaz [1, 2] claim that General Relativity (GR) is inconsistent and that its Newtonian limit is unsatisfactory. Because these claims have resulted in some widely distributed notice [4, 5], and because

*UMD PP-95-115; gr-qc/9504050
they may be having an invalid influence on experimental proposals [3], it
seems appropriate to restate that they are incorrect. But the straightforward
and long established arguments (or more rigorous recent work [3]) showing
GR to have a good Newtonian limit leave little room for including Newtonian
stresses in still another way in the field equations, as Yilmaz has proposed.
We will see here that the Yilmaz theory makes changes to GR that are so
profound they remove all gravitational interactions at the Newtonian level
in any exact solution of the field equations of the modified theory.

In notation that differs in detail from that used by Yilmaz in his papers,
but adopts the MTW conventions [8], the Einstein field equations are
\[
R^{\mu\nu} - \frac{1}{2} g^{\mu\nu} R \equiv G^{\mu\nu} = (8\pi G/c^4) T^{\mu\nu},
\]
(1.1)
where all components of \( T^{\mu\nu} \) have units of energy density (i.e., pressure)
and this tensor describes the stress-energy of all non-gravitational fields. In
the Yilmaz theory this equation is modified to read
\[
G^{\mu\nu} = (8\pi G/c^4)(T^{\mu\nu} - t^{\mu\nu}),
\]
(1.2)
where in every instance \( t^{\mu\nu} \) as used here differs in sign from the quantity
Yilmaz represents by this symbol. The exact definition of the Yilmaz grav-
itational stress-energy tensor \(-t^{\mu\nu}\) is in some cases unclear, but explicit
expressions have been given in situations that include the Newtonian limit.
In suitable (harmonic) coordinate systems chosen by Yilmaz the definition
becomes \( t_{\mu\nu} = u_{\mu\nu} \) where \( u_{\mu\nu} \) is the Einstein stress-energy pseudotensor;
my meaning of \( t_{\mu\nu} \) avoids a minus sign in this definition, as well as in the
relationship (4.3) below of \( t_{\mu\nu} \) to the Newtonian gravitational stresses.

2 Relativistic Matter

To compare the GR and Yilmaz theories in the Newtonian limit one needs a
theory of matter so there is something to exhibit gravitational interactions.
I will take for a theory of matter the relativistic hydrodynamics of a perfect
fluid. [Newtonian hydrodynamics with gravitational forces is summarized
in appendix [4].] This is a suitable test case as it avoids dealing with the
singularities of point particles, yet includes some nongravitational (pressure
gradient) forces which are needed in the Newtonian theory for static solu-
tions to exist. In this case one takes the stress-energy tensor of matter to be
\[
T^{\mu\nu} = \rho c^2 u^\mu u^\nu + P(g^{\mu\nu} + u^\mu u^\nu).
\]
(2.1)
The fluid 4-velocity is normalized by the condition
\[ g_{\mu\nu} u^\mu u^\nu = -1 \] (2.2)
so that in quasi-rectangular coordinates the 4-velocity can be written
\[ u^\mu = (1, v/c) \gamma \] (2.3)
where \( v^k = dx^k/dt \) is the coordinate velocity of the fluid particles, \( x^0 = ct \) is the fourth coordinate in tensor expressions, and \( \gamma \) will be determined by the normalization (2.2). All components \( g_{\mu\nu} \) and \( u^\mu \) are dimensionless, as \( \gamma \) will then be.

In the Yilmaz theory the hydrodynamic equations have gravitational influences that differ from those in GR. In this case of a matter theory with only four independent fields (\( \rho \) and the three \( v^k \) with \( P(\rho) \) given by an equation of state) the Bianchi identities which enforce local energy and momentum conservation constrain the matter equations of motion sufficiently to determine them completely. The contracted Bianchi identities \( G^{\mu\nu;\nu} = 0 \) from equation (1.2) imply \( (T^{\mu\nu} - t^{\mu\nu})_{;\nu} = 0 \) in the Yilmaz theory.

When split into components parallel and perpendicular to the fluid 4-velocity \( u^\mu \) the local conservation equations \( (T^{\mu\nu} - t^{\mu\nu})_{;\nu} = 0 \) are an energy equation
\[ (\rho c^2 u^\mu + u_\nu t^{\nu\mu})_{;\mu} = -Pu^{;\nu} + t^{\mu\nu}u_{;\nu} \] (2.4)
from \(-u_\mu(T^{\mu\nu} - t^{\mu\nu})_{;\nu} = 0 \) and the generalized Euler equation
\[ (\rho c^2 + P)u^{\nu;\nu}u^\nu = -(g^{\mu\nu} + u^\mu u^\nu)(P_{;\nu} - t^{\alpha ;\nu}) \] (2.5)
from \((\delta_{\nu}^{\mu} + u^\mu u_\nu)(T^{\nu\alpha} - t^{\nu\alpha})_{;\alpha} = 0 \). The terms involving \( t^{\mu\nu} \) are absent for the Einstein theory of gravity. In equation (2.5) this allows geodesic motion \( u^{\mu;\nu}u^\nu = 0 \) in GR when the pressure gradient forces on the right are also absent. Although the \( t^{\mu\nu} \) terms, like the pressure terms, should be negligible in the Newtonian limit in the gravitational field equations (1.2) and also negligible in the equation of continuity (2.4) they may, like the pressure gradient terms, be important in the Newtonian limit of the Euler equation (2.5). We will find that they cancel the terms from \( \rho c^2 u^{\mu;\nu}u^\nu \) that in Einstein’s theory give the Newtonian gravitational force term in the Euler equation.

The generalized Euler equation (2.5) is not optional or modifiable in the Yilmaz theory. If a \( t^{\mu\nu} \) term is inserted in the Einstein equations, the Bianchi identities show that it must appear in the hydrodynamic equations of motion.
satisfied in any exact solution. Thus to leading order it must be satisfied by any motion that approximates an exact solution of equations (1.2) with a hydrodynamic $T^{\mu\nu}$ stress-energy tensor. This argument makes no use of any additional equations that may be part of the Yilmaz theory, as we have to this point made use only of the field equations (1.2) and the assumption that the theory is applicable to selfgravitating systems of fluid objects with the stress-energy tensor (2.1).

3 An Exact Solution

Yilmaz has given an exact solution of his field equations [2, 6] which I derive here to verify that, in spite of changes in notation and sign conventions, I am stating equations from his theory correctly. When the consequences (2.5) of the field equations are considered in this example one sees a failure to reproduce Newtonian gravitational effects. The metric Yilmaz proposes is

$$ds^2 = -e^{2\Phi}d(ct)^2 + e^{-2\Phi}(dx^2 + dy^2 + dz^2). \quad (3.1)$$

For this choice of metric and coordinates Yilmaz evaluates his gravitational stress-energy tensor $-t^{\mu\nu}$ to give

$$t^{\mu\nu} = \lambda(c^4/4\pi G)[\Phi,\mu\Phi,\nu - \frac{1}{2}\delta^{\nu}_{\mu}\Phi,\alpha\Phi,\alpha] \quad (3.2)$$

with $\lambda = 1$. We have inserted the tag $\lambda$ here so that by setting $\lambda = 0$ in later equations we can recover the corresponding equation in Einstein’s theory.

I now use the computer algebra aid GRTensor II [9, 10] to construct the Yilmaz field equations in this case. The result is

$$Y^{00} : \quad 2\Phi,_{kk} + (\lambda - 1)\Phi,_{k}\Phi,_{k} + (\lambda + 3)e^{-4\Phi}(\Phi,_{0})^2$$
$$= (8\pi G/c^2)[\rho\gamma^2 + (P/c^2)(e^{-2\Phi} - \gamma^2)] \quad (3.3)$$

$$Y^{0j} : \quad -2\Phi,_{j0} - 2(\lambda - 1)\Phi,_{j}\Phi,_{0}$$
$$= (8\pi G/c^3)\gamma^2 v^j(\rho + P/c^2) \quad , (3.4)$$

$$Y^{ij} : \quad (1 - \lambda)e^{4\Phi}[\delta^{ij}\Phi,_{k}\Phi,_{k} - 2\Phi,_{i}\Phi,_{j}]$$
$$+ \delta^{ij}[2\Phi,_{00} + (\lambda - 5)(\Phi,_{0})^2]$$
$$= (8\pi G/c^4)[\gamma^2 v^i v^j(\rho + P/c^2) + \delta^{ij}P e^{2\Phi}] \quad . (3.5)$$

The left hand sides of these equations are the gravitational quantities $Y^{\mu\nu} \equiv G^{\mu\nu} + (8\pi G/c^4)t^{\mu\nu}$ while all the matter terms are collected on the right.
To find a static solution we set $\lambda = 1$ (Yilmaz theory), drop all time derivatives, set $v^j = 0$, and from the normalization condition (2.2) get $\gamma^2 = e^{-2\Phi}$. The resulting equations are

$$
Y^{00} : \quad 2\Phi_{,kk} = (8\pi G/c^2)\rho e^{-2\Phi}, \quad (3.6)
$$

$$
Y^{0j} : \quad 0 = 0, \quad (3.7)
$$

$$
Y^{ij} : \quad 0 = (8\pi G/c^4)\delta^{ij}P e^{2\Phi}. \quad (3.8)
$$

To construct solutions to these equations one chooses any desired density distribution $\rho_{\text{eff}}(x, y, z)$ representing one or more fluid bodies and then solves the Poisson equation $U_{,kk} = 4\pi G \rho_{\text{eff}}$. Then the choice $\Phi = U/c^2$ reduces equation (3.6) to the form $\rho_{\text{eff}} = \rho e^{-2U/c^2}$ which serves to display the invariant density $\rho = u_\mu u_\nu T^{\mu\nu}/c^2$ in terms of the given effective density. To solve equation (3.8) one must set $P = 0$.

The existence of this exact solution is disastrous for the Yilmaz theory. Any equation of motion for fluid matter adjoined to the Yilmaz field equations (1.2) cannot differ from the generalized Euler equation (2.5) since that equation is just a combination of the field equations and their derivatives. Thus this static solution with $P = 0$ satisfies also the fluid equations of motion (2.5) in any consistent theory incorporating the Yilmaz field equations (1.2). One such solution could represent two fluid planets (Jupiter and Saturn) placed near a fluid Sun with none of these bodies in motion. No gravitational forces act to accelerate any of these bodies, no mechanical (pressure) forces are needed to balance gravity, and the only gravitational effect is the renormalization between effective density $\rho_{\text{eff}}$ and local invariant density $\rho$.

The Einstein equations under these same static assumptions have no exact fluid solution in agreement with the Newtonian impossibility of a motionless solar system. The usual Oppenheimer-Volkoff static fluid solutions (spherical symmetry only) are obtained by allowing $-g_{00}$ and $g^{xx}$ to differ and do require pressure gradients to support the matter against its own gravitational fields. See, for example, [8, §23.7]. To see that allowing additional independent components in $g_{\mu\nu}$ does not help in the Yilmaz theory we treat the Newtonian limit below without assuming static configurations nor restrictions beyond harmonic coordinate conditions on the metric components.
4 Newtonian limit

In the Newtonian limit one assumes that velocities are small, \( |v/c| \ll 1 \), that gravitational potentials \( g_{\mu\nu} \) are near their Minkowski values, and that pressures or other mechanical stresses are negligible compared to energy densities \( |P| \ll \rho c^2 \). By arguments known since Einstein and Hilbert, one finds that only one combination of the \( g_{\mu\nu} \) is generated by the dominant term in \( T^{\mu\nu} \) which is \( \rho c^2 u^0 u^0 \). (See, e.g., [8, §18.4].) Neglecting terms in \( g_{\mu\nu} \) smaller than this leading term by factors of \( v/c \) or \( \Phi \) one finds that the metric is the same as equation (3.1) in which one may set \( e^{\pm 2\Phi} = 1 \pm 2\Phi \). This metric will be time dependent with \( \Phi_\theta = \partial \Phi/\partial ct \) smaller than \( \Phi,_{k} \) by a factor of \( v/c \) where \( v \) is the velocity of the matter sources. [Consider, for example, the Newtonian potential \( U = -GM/|r - vt| \) of a moving point mass.] The equality of \( -g_{00} \) and \( g^{xx} \) is deduced rather than assumed, and holds only as a first approximation. These arguments hold equally well for the Yilmaz theory or for the Einstein theory since \( t^{\mu\nu} \) is of order \( (\partial \Phi)^2 \) and thus smaller than the leading \( \partial^2 \Phi \) terms since \( |\Phi| \ll 1 \). Thus both the Yilmaz theory and the Einstein theory lead to

\[
2\Phi,_{kk} = (8\pi G/c^2)\rho \tag{4.1}
\]

from the \( G^{00} \) equation, while all terms in the other field equations are negligible in comparison. The solution of this equation is

\[
\Phi = U/c^2 \tag{4.2}
\]

where \( U \) is the Newtonian potential satisfying equation (A.1) for the same matter distribution \( \rho(x, y, z, t) \).

Since both the Einstein and Yilmaz field equations give the same metric in the Newtonian limit, it had long been assumed that the Yilmaz theory had a satisfactory Newtonian limit. But Yilmaz [3] has recently suggested that the theories differ in this limit, prompting this study which localizes the difference to the equations of motion required by the Bianchi identities. Thus we proceed to study approximations to equations (2.4) and (2.5).

The reduction of the relativistic continuity equation (2.4) is relatively straightforward using the metric (3.1). The terms on the right by which restmass is changed by mechanical or gravitational work are unimportant for Newtonian strength forces, and in the mass current vector \( \rho c^2 u^\mu + u_\nu t^\nu{}^\mu \) the gravitational binding energy contribution to the rest mass from \( t^\nu{}^\mu \) is also small. This leaves \( (\rho c u^\nu),_\mu = 0 \) in which the approximation \( u^\mu = (1, v^j/c) \)
will suffice. All terms involving $\partial \Phi$ are smaller by a factor $\Phi$ than those in the Newtonian equation and thus negligible, so equation (A.2) results.

The relativistic Euler equation (2.5) similarly simplifies. (The $\mu = 0$ equation is negligible.) The left hand side with $u^0 = 1$ and $u^i = v^i/c$ reads $\rho c^2 u^\mu u_\mu u^\nu = \rho [ (\partial v^i/\partial t) + v^k v^i_{;k} + \Gamma^{i}_{00}c^2 ]$ and the right hand side is $-P_i + t^{ik}_{;k}$. But when one makes the identification $\Phi = U/c^2$ from the field equations the Yilmaz term from equation (3.2) is just given by the Newtonian stresses (A.5)

$$t^{ik} = \lambda \tau^{ik}_N$$

so from equation (A.6) one has $t^{ik}_{;k} = \lambda \rho U_{;i}$. Also one can evaluate $\Gamma^{i}_{00}c^2 = U_{,i}$ from the metric and $\Phi = U/c^2$. The result then is

$$\rho [ (\partial v^i/\partial t) + v^k v^i_{;k} ] = -P_i - \rho (1 - \lambda) U_{,i}.$$  \hfill (4.4)

This is the desired Newtonian result (A.3) in the Einstein case $\lambda = 0$ but lacks any Newtonian gravitational force in the Yilmaz case $\lambda = 1$.

\section*{Appendix}

\section{Newtonian Benchmark}

The Newtonian theory we expect to recover as a limiting case of any relativistic theory will be taken to be the hydrodynamics of a perfect fluid. This theory has as its gravitational field equation the Poisson equation

$$U_{,kk} = 4\pi G\rho$$ \hfill (A.1)

for the generation of gravitational potentials, the continuity equation

$$\partial \rho / \partial t + (\rho v^k)_k = 0$$ \hfill (A.2)

which shows how fluid motions $v^k$ affect the density, and the Euler equation

$$dv^i/\partial t = (\partial v^i/\partial t) + v_{,k} v^k = -U_{,i} - (1/\rho) P_i$$ \hfill (A.3)

which shows how the force per unit mass changes the momentum per unit mass $v^i$. (Note that for a point mass one has $U = -GM/r$ which is the opposite sign from that used for the symbol $U$ in [8, §39.7].) To have a complete deterministic theory these equations must be supplemented by an equation of state $\rho = \rho(P)$. Thermal properties including viscosity and
heat transfer are neglected here, and the equation of state is assumed not to involve the temperature.

The Euler equation can be replaced in the Newtonian theory by an equivalent form in which all forces, not just pressure forces, are described by a stress tensor or momentum flux tensor

\[ T_{ij}^N = \rho v^i v^j + P^{ij} + t_{ij}^N \]  

(A.4)

where \( \rho v^i v^j \) give the convective transport of momentum and \( P^{ij} \) are the mechanical stresses. (When two regions are in contact the forces they exert on each other represent, by action equals reaction, a momentum gain for one region and a corresponding loss for the other. Thus a contact force per unit area is a momentum transfer rate per unit contact area.) The gravitational stresses

\[ t_{ij}^N = \frac{(1/4\pi G)}{U_i U_j - \frac{1}{2} \delta_{ij} U_k U_k} \]  

(A.5)

are chosen so that, using the Poisson equation, they give the required gravitational force per unit volume:

\[ -t_{ik,k}^N = -(1/4\pi G)U_i U_{kk} = -\rho U_i \]  

(A.6)

For application to fluids one sets \( P^{ij} = P \delta^{ij} \) and finds that \( -P_{ik,k} = -P_i \) is the pressure gradient force per unit volume. Conservation of momentum is then expressed by

\[ \partial(\rho v^i)/\partial t + T_{ik}^N,k = 0 \]  

(A.7)

which treats momentum on a “per unit volume” basis in contrast to the previous Euler equation (A.3) which is on a “per unit mass” basis. It is straightforward, using the equation of continuity, to reduce equation (A.7) to equation (A.3).

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