Matter loops corrected modified gravity in Palatini formulation

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Abstract

Recently, corrections to the standard Einstein-Hilbert action are proposed to explain the current cosmic acceleration in stead of introducing dark energy. In the Palatini formulation of those modified gravity models, there is an important observation due to Arkani-Hamed: matter loops will give rise to a correction to the modified gravity action proportional to the Ricci scalar of the metric. In the presence of such term, we show that the current forms of modified gravity models in Palatini formulation, specifically, the $1/R$ gravity and $\ln R$ gravity, will have phantoms. Then we study the possible instabilities due to the presence of phantom fields. We show that the strong instability in the metric formulation of $1/R$ gravity indicated by Dolgov and Kawasaki will not appear and the decay timescales for the phantom fields may be long enough for the theories to make sense as effective field theory. On the other hand, if we change the sign of the modification terms to eliminate the phantoms, some other inconsistencies will arise for the various versions of the modified gravity models. Finally, we comment on the universal property of the Palatini formulation of the matter loops corrected modified gravity models and its implications.

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It now seems well-established that the expansion of our universe is currently in an accelerating phase. The most direct evidence for this is from the measurements of type Ia supernova \[1\]. Other indirect evidences such as the observations of CMB by the WMAP satellite \[2\], large-scale galaxy surveys by 2dF and SDSS \[3\] also seem supporting this.

But now the mechanisms responsible for this acceleration are not very clear. Many authors introduce a mysterious cosmic fluid called dark energy to explain this (see Refs.\[4, 5, 6\] for a review). On the other hand, some authors suggest that maybe there does not exist such mysterious dark energy, but the observed cosmic acceleration is a signal of our first real lack of understanding of gravitational physics \[7\]. An example is the DGP model \[8\]. Recently, there are active discussions in this direction by modifying the action for gravity \[8-22\]. Specifically, a \(1/R\) term is suggested to be added to the action \[9, 10\]: the so called \(1/R\) gravity. It is interesting that such term may be predicted by string/M-theory \[11\]. Then the modifications by adding \(1/R + R^2\) \[12\] or \(\ln R\) terms \[13\] are also suggested. For \(1/R\) gravity, Vollick \[14\] used Palatini variational principle to derive the field equations. The modified Friedmann equations in Palatini formulation and their properties are discussed in our previous papers \[15\]. The Palatini formulation of the \(1/R + R^2\) and \(\ln R\) theory are discussed in Ref.\[16\] and Ref.\[17\] respectively. Flanagan \[18\] derived the equivalent scalar-tensor theory of the Palatini formulation. Furthermore, in Ref.\[19\], Flanagan derived the equivalent scalar-tensor theory of a more general modified gravity framework. His results provide a fundamental and powerful tool for discussing modified gravity models in Palatini formulation. In the following discussions, we will use the sign conventions for various quantities in Flanagan’s papers.

In general, when handled in Palatini formulation, one considers the action to be a functional of the metric \(g_{\mu\nu}\) and a connection \(\nabla_\mu\) which is independent of the metric. The resulting modified gravity action can be written as
\[
S[g_{\mu\nu}, \nabla_\mu] = \frac{1}{2\kappa^2} \int d^4x \sqrt{-\bar{g}} L(\hat{R}),
\]
where \(\kappa^2 = 8\pi G\), \(\hat{R} = \bar{g}^{\mu\nu} \hat{R}_{\mu\nu}\) and \(\hat{R}_{\mu\nu}\) is the Ricci tensor of the connection \(\nabla_\mu\). All the current existed forms of \(L\) can be written as \(L(\hat{R}) = \hat{R} + f(\hat{R})\). For \(1/R\) gravity, \(f(\hat{R}) = -\alpha^2/(3\hat{R})\) \[9, 14\]; for \(1/R + R^2\) gravity, \(f(\hat{R}) = -\alpha^2/(3\hat{R}) + R^2/(3\beta)\) \[12, 16\];
for ln $R$ gravity, $f(\hat{R}) = \beta \ln(\hat{R}/\alpha)$ \cite{13, 17}. As shown by Flanagan \cite{18}, these theories, when written in a canonical form, contain a scalar field with no kinetic term. Thus they can be seen as General Relativity coupled to modified matter actions \cite{19}.

The starting point of the current work is an observation due to Nima Arkani-Hamed (see Ref.\cite{19} for a more detailed discussion of Arkani-Hamed’s idea): the theory \cite{11} has fine-tuning problems as an effective quantum field theory. Specifically, matter loops will give rise to a correction to the action \cite{11} proportional to the Ricci scalar $\bar{R}$ of the metric $\bar{g}_{\mu\nu}$. Thus in this paper, we will study the following matter loops corrected modified gravity action

$$S_{\text{loop}}[\bar{g}_{\mu\nu}, \hat{\nabla}_\mu] = \frac{1}{2\kappa^2}\int d^4x\sqrt{-\bar{g}}[\bar{R} + f(\hat{R})] \equiv S_{\text{EH}} + S_{\text{Palatini}},$$

(2)

where $S_{\text{EH}}$ is the familiar Einstein-Hilbert action. When written in the canonical form, the presence of a $\hat{R}$ term will induce a kinetic term for the scalar field. Note that for simplicity of discussions, we ignored the linear $\hat{R}$ term. Since the main motivation of modified gravity models is to explain the current cosmic acceleration, the $f(\hat{R})$ term necessarily dominates over the $\hat{R}$ term in realistic applications of the models. Thus we believe it is sufficient and more illuminating to consider only the combined effects of the induced Ricci scalar $\bar{R}$ and $f(\hat{R})$. However, it can also be checked explicitly that the inclusion of a linear $\hat{R}$ term will not change the main conclusions drawn from action \cite{2}.

Following Refs.\cite{11, 20}, we can write the action $S_{\text{Palatini}}$ in an equivalent form. Introduce a scalar field $\varphi$ and define the action

$$S_\varphi[\bar{g}_{\mu\nu}, \hat{\nabla}_\mu, \varphi] = \frac{1}{2\kappa^2}\int d^4x\sqrt{-\bar{g}}[f'(\varphi)(\hat{R} - \varphi) + f(\varphi)].$$

(3)

Then if $f''(\hat{R}) \neq 0$, the equation of motion of $\varphi$ implies $\varphi = \hat{R}$. Thus action \cite{3} is classically equivalent to the action $S_{\text{Palatini}}$ (Unless indicated explicitly, when we speak of two actions equivalent, we always mean equivalence in classical level).

Define

$$\varepsilon = \text{sign} f'(\varphi),$$

(4)

and define the scalar field $\sigma$ by

$$f'(\varphi) = \varepsilon e^{-2\sigma}.$$

(5)
Then a slight generalization of Flanagan’s derivation [18] gives that the action (3) is equivalent to
\[ \tilde{S}[\tilde{g}_{\mu\nu}, \sigma] = \frac{1}{2\kappa^2} \int d^4x \sqrt{-\tilde{g}}[\varepsilon\tilde{R} - V(\sigma)], \]
where
\[ \tilde{g}_{\mu\nu} = e^{-2\sigma}\tilde{g}_{\mu\nu}, V(\sigma) = \varepsilon e^{2\sigma}\varphi - e^{4\sigma} f(\varphi), \]
and \( \varphi \) is given in terms of \( \sigma \) through Eq.(5). Thus, when considering the theory (1), with \( f \) replaced by \( L \) in the above expressions, in order to have a positive coefficient of the Ricci scalar, we must have \( \text{sign}L' = 1 \). This is just the case for the Palatini formulation of \( 1/R \) \[14, 15\] and \( \ln R \) theory \[17\], where we have \( \varepsilon = 1 \), so that \( \text{sign}L' = \text{sign}(1 + f') = 1 \) is obviously satisfied.

Also transform the Einstein-Hilbert action \( S_{EH} \) to the \( \tilde{g}_{\mu\nu} \) frame, and add it with action (6), the action (2) is equivalent to
\[ \tilde{S}_{\text{loop}}[\tilde{g}_{\mu\nu}, \sigma] = \frac{1}{2\kappa^2} \int d^4x \sqrt{-\tilde{g}}[(e^{2\sigma} + \varepsilon)\tilde{R} + 6e^{2\sigma}(\tilde{\nabla}\sigma)^2 - V(\sigma)]. \]

Define
\[ \eta = \text{sign}(e^{2\sigma} + \varepsilon), \]
and define the scalar field \( \chi \) as
\[ e^{2\sigma} + \varepsilon = \eta e^{-2\chi}. \]

Let us conformally transform the metric as
\[ \tilde{g}_{\mu\nu} = e^{-2\chi}\tilde{g}_{\mu\nu}. \]
Then the action (9) is transformed to
\[ \tilde{S}_{\text{loop}}[\tilde{g}_{\mu\nu}, \chi] = \frac{1}{2\kappa^2} \int d^4x \sqrt{-\tilde{g}}[\eta\tilde{R} - 6\eta(\tilde{\nabla}\chi)^2 + \frac{6e^{-2\chi}}{\eta e^{-2\chi} - \varepsilon}(\tilde{\nabla}\chi)^2 - \tilde{V}(\chi)], \]
where \( \tilde{V}(\chi) = e^{4\chi}V(\sigma) \) and \( \sigma \) are given in terms of \( \chi \) through Eq.(10).

Now we can see from the action (12) that, since the sign of the coefficient of the Ricci scalar should be positive, we must have \( \eta = 1 \). Then, in order that the \( \chi \) field have a right sign for the kinetic term, we must have \( \varepsilon = -1 \). Thus when adding the matter loops induced Ricci scalar into the action, the \( 1/R \) gravity and \( \ln R \) gravity in Palatini
formulation studied in Refs. [14, 15, 17], where \( \varepsilon = 1 \), will have phantoms or negative energy excitations, which may make the theory unstable classically or (and) quantum mechanically. However, it is now well-known that the presence of phantom fields does not necessarily mean that such models is excluded as viable cosmological models. As shown explicitly in Ref. [25] by field theoretic computations, the decay timescale for the phantom field with a gaussian potential can be long enough for the theory to make sense as an effective field theory (see also Ref. [26, 27] for more works on phantom cosmology). Below we will argue that this is also hold for the matter loops corrected \( 1/R \) gravity in Palatini formulation. Before that, let us first consider another sort of instability present in the metric formulation of the \( 1/R \) gravity as indicated by Dolgov and Kawasaki [23], which is much more serious than the phantom decay instability. We will show that, for matter loops corrected \( 1/R \) theory in Palatini formulation, this sort of instability will not appear.

For the matter loops corrected \( 1/R \) theory, \( f(\hat{R}) = -\alpha^2/(3\hat{R}) \), the contracted field equations are (one can derive those equations with a slight generalization of the framework in Ref. [14])

\[
-\hat{R} + \frac{\alpha^2}{\hat{R}} = \kappa T, \tag{13}
\]

and

\[
\hat{R} = \bar{\hat{R}} + 3(f')^{-1}\bar{\nabla}_0 \nabla_0 f' - \frac{3}{2}(f')^{-2}(\bar{\nabla}_0 f')^2. \tag{14}
\]

Adding them together we can get

\[
6\bar{\nabla}_0 \nabla_0 \hat{R} - \frac{12}{\hat{R}}(\bar{\nabla}_0 \hat{R})^2 + \hat{R}^2 + \kappa T \hat{R} = \alpha^2. \tag{15}
\]

Let us consider the internal solution with time dependent matter density as in Ref. [23]. The first order correction to the curvature \( \hat{R} = \hat{R}_0 + \hat{R}_1 \) satisfies the equation

\[
6\bar{\nabla}_0 \nabla_0 \hat{R}_1 - \frac{24\bar{\nabla}_0 \hat{R}_0}{\hat{R}_0} \bar{\nabla}_0 \hat{R}_1 + (12(\bar{\nabla}_0 \hat{R}_0)^2 + 2\hat{R}_0 + \kappa T)\hat{R}_1 =
\]

\[
-6\bar{\nabla}_0 \nabla_0 \hat{R}_0 + \frac{12(\bar{\nabla}_0 \hat{R}_0)^2}{\hat{R}_0} - \hat{R}_0^2 - \kappa T \hat{R}_0 + \alpha^2. \tag{16}
\]

Since we expect \( \hat{R}_0 \sim -\kappa T \) and \( T < 0 \), the coefficient of \( \hat{R}_1 \) is positive in Eq.(16). Moreover, since \( \hat{R} \) and \( \bar{\hat{R}} \) are related to each other through the algebraic equation [13], thus the matter loops corrected \( 1/R \) theory will not suffer from the instability of the
sort indicated by Dolgov and Kawasaki [23], which is due to a large negative coefficient of the $\hat{R}_1$ term. The absence of this sort of instability also happens in the Palatini formulation of the $1/R$ theory without matter loops induced Ricci scalar [15].

Then we turn to the analysis of whether the decay timescales for the phantom fields can be long enough for the theories to make sense as effective field theory [25]. We still consider the matter loops corrected $1/R$ theory as an illuminating example. The potential $\tilde{V}$ is given by

$$\tilde{V}(\chi) = e^{4\chi}V(\sigma) = \frac{2\alpha}{\sqrt{3}}e^{4\chi}(e^{-2\chi} - 1)^{\frac{3}{2}}. \quad (17)$$

To apply the analysis of Ref. [25], we need first to transform the kinetic term in action (12) to the standard form of phantom field. Define $\Phi = M_P l\arcsin(e^{\chi})$, then action (12) with $\eta = \epsilon = 1$ can be rewritten as:

$$\tilde{S}_{\text{loop}}[\tilde{g}_{\mu
u}, \Phi] = \frac{1}{2\kappa^2} \int d^4x \sqrt{-\tilde{g}} \tilde{R} - \int d^4x \sqrt{-\tilde{g}}[-3(\nabla\Phi)^2 + \tilde{V}(\Phi)], \quad (18)$$

where

$$\tilde{V}(\Phi) = \frac{M_P^2\alpha}{\sqrt{3}} \sin\left(\frac{\Phi}{M_P l}\right) \cos^3\left(\frac{\Phi}{M_P l}\right). \quad (19)$$

Note since $\exp(-2\chi) = \exp(2\sigma) + 1$, we have $\chi < 0$, then $0 < \Phi < \pi/2$. See Fig.1.

Then we will follow closely the analysis of Ref. [25]. But in the current case, there is a difference with the case considered in Ref. [25]: here the phantom field is conformally coupled to matter and thus phantoms may decay to matter particles. We will ignore those decay channels in the present analysis and restrict our attentions to only gravitons and phantoms. Then, the phantom decay channel involving the smallest number of particles is

$$\Phi \rightarrow h + \Phi_1 + \Phi_2 \quad (20)$$

where $h$ is a graviton and $\Phi, \Phi_1, \Phi_2$ is phantom particles.

Since scalar fields with negative kinetic terms evolve to the maxima of their classical potential, where from Eq. (19) and Fig.1 we can see $\Phi_{\max} \sim M_P l$ and $V(\Phi_{\max}) \sim M_P^2\alpha$. Thus we will consider the potential (19) expanded as a power series around the background value $\Phi_{\max}$. To study the decay (20), the interaction part of the Lagrangian is required to be first order in $h$ and third order in $\Phi$, which is

$$L_I = \frac{1}{M_P l} (M_P l h) \frac{1}{3!} V'''(\Phi_{\max}) \Phi^3 = \lambda_{\text{eff}}(M_P l h) \Phi^3. \quad (21)$$
The scalar potential given by Eq. (19). \( \Phi_{\text{norm}} \equiv \Phi/M_{Pl} \) and \( V_{\text{norm}} \equiv \sqrt{3}V/(M_{Pl}^2 \alpha) \).

The effective coupling constant can be estimated as

\[
\lambda_{\text{eff}} \sim \frac{V(\Phi_{\text{max}})}{M_{Pl}^4} \sim \frac{\alpha}{M_{Pl}^2} \sim 10^{-120},
\]

where we have used in the last the step the condition: \( \alpha \sim \left(10^{-33}eV\right)^2 \), see Refs. [9, 14].

This is just the same order of magnitude as in Ref. [25]. Then the remaining part of the analysis is the same as in Ref. [25]: such a small coupling constant will lead to a decay timescale that can be larger than the Hubble time; moreover, the total decay timescale remains of order of (22) as long as the momentum cut-off is not larger than \( M_{Pl} \), which is a reasonable assumption. Thus we can conclude that the decay timescale for the phantom field with potential (19) can also be long enough for the theory to make sense as an effective field theory.

Next, we will explore whether one can modify the current modified gravity models to eliminate the phantom field, i.e. \( \varepsilon = -1 \), without other inconsistencies.

For the \( 1/R \) theory, if one changes the sign of the action to be \( f(\dot{R}) = \alpha^2/(3\dot{R}) \) so that \( \varepsilon = -1 \), another more serious problem arises. The vacuum field equation for such
an action is
\[-\hat{R} - \frac{\alpha^2}{\hat{R}} = 0,\] (23)
and the equation by varying the connection will give \(\hat{R} = \hat{\hat{R}}\). Thus there is no real solution of the vacuum equation (23). This obviously can not be a physical theory.

For the \(\ln R\) theory, if one changes the sign of the action to be \(f(\hat{R}) = -\beta \ln(\hat{R}/\alpha)\) so that \(\varepsilon = -1\). The situation is not as serious as the \(1/R\) theory. In the \(\varepsilon = 1\) case, there is an unique correspondence between \(\hat{R}\) and any constant value of \(T\); while in the \(\varepsilon = -1\) case, the correspondence might be two-to-one, or no solution for a given constant \(T\), depending on the value \(\alpha/\beta\), but never one-to-one (see the discussion in Ref. [13] for more details on this case in metric formulation). This is also not a good sign for the theory to become a physically promising candidate theory.

For the \(R^2\) theory, whose Palatini formulation is studied in Ref. [16], one should take \(f(\hat{R}) = -\hat{R}^2/\beta\), where \(\beta > 0\). Now the situation is better, the correspondence between \(\hat{R}\) and any constant \(T\) is still one-to-one. On the other hand, there is a constraint on \(\hat{\hat{R}}\): from Eq. (10), \(\sigma > 0\); and from Eq. (5), \(2\hat{R}/\beta = e^{-2\sigma}\). Thus this theory can be consistent only when \(\hat{R} < \beta/2\). The parameter \(\beta\) can be constrained by the BBN data, see Refs. [16, 28].

For a general \(f(\hat{R})\), from the vacuum equation
\[-\hat{R}_0 + f'(\hat{R}_0)\hat{R}_0 - 2f(\hat{R}_0) = 0,\] (24)
where we have substituted the relation \(\bar{R} = \hat{\hat{R}}\) followed from the equation of motion by varying the connection, one can see that if one wants the theory to be free of phantom, i.e. \(f'(\hat{R}_0) < 0\), and the vacuum equation has real positive solution, one must require \(f(\hat{R}_0) < 0\). Thus all the realistic value of \(\hat{R}\) in the presence of matter should satisfy \(f(\hat{R}) < 0\) since \(f'(\hat{R}) < 0\). But in order to drive a recent acceleration, \(f \to +\infty\) while \(\hat{R} \to 0\). All the current forms of \(f(\hat{R})\) do not satisfy those conditions all. Actually, we strongly suspect that a reasonable \(f(\hat{R})\) satisfying all those conditions exists. Thus, one have to accept the presence of phantom in the Palatini formulation of matter loops corrected modified gravity models, which is not so catastrophic as we have shown in this paper.

An important feature of the Palatini formulation of modified gravity is that it will
give the usual vacuum Einstein equations for generic Lagrangian of the form $L(R)$: the so-called 'universal property' of Palatini formulation of modified gravity model [29]. It can be easily shown that this feature is still hold for the action (2). Thus in classical level, theories of the type (1) and (2) are actually not modifications of gravity at all. As can be checked explicitly, the weak field expansions around a background spacetime (Minkowski, de Sitter and Anti-de Sitter) will give almost the same equations as for General Relativity and the only difference is in the source term. This means the behavior of graviton is unchanged. Furthermore, as shown in [30], when the trace of the energy-momentum tensor, $T$, is a constant value, the Palatini formulation of theory (1) still reduces to General Relativity. It is easy to show that this is also hold for the theory (2). The key point is that for vacuum or constant $T$, from the equation of motion by varying the connection, we have $\hat{R}_{\mu\nu} = \tilde{R}_{\mu\nu}$. This means that the non-relativistic gravitational potential generated by a static source is the standard Newtonian one. Also can be checked explicitly, the potential for a slowly moving object behaves still as $\sim 1/r$, while the nominator contains not only the mass, but also the first and second order derivatives of the trace $T$. Thus, roughly speaking, in the Palatini formulation, spacetime curvature depends not only on the value of energy-momentum tensor, but also its first and second order derivatives. The precise form of the gravitational potential and a detailed discussion of the above assertions will be presented elsewhere [31]. We also comment that the modified gravity models will give several predictions that are distinguishable with the dark energy models [7], including the power spectrum of large scale structure, the ISW effects, etc. The observations of those quantities can be used to discriminate between modified gravity models and dark energy models.

In concluding, we have shown that in the presence of a matter loops induced Ricci scalar term, the current forms of modified gravity models in Palatini formulation, specifically, the $1/R$ gravity and $\ln R$ gravity, will have phantoms. However, the phantom will not lead to the strong instability in the metric formulation of modified gravity and the decay timescale for the phantom field may be long enough for the theory to make sense as an effective field theory. Thus, based on the current analysis, the $1/R$ and $\ln R$ gravity with an induced Ricci scalar term are still viable cosmological models despite
the presence of phantom fields. On the other hand, the versions without phantoms are unacceptable because of serious inconsistencies. Note that the above conclusions are based on the classical equivalence of various conformally transformed actions. Of course, on quantum level it is well-known (see explicit examples for quantum dilatonic and higher derivative gravities [32, 33, 34]) that even classically equivalent theories are not always equivalent on quantum level.

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