Gravitational Waves on Conductors

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Abstract

We consider a gravitational wave impinging on either a normal or a super-conductor. Fermi normal coordinates are derived for waves of arbitrary frequency. The transverse-traceless wave amplitude expressed in these coordinates is used to derive the gravitationally induced electromagnetic fields in the conductor. In the superconducting case, using a time-dependent Ginzburg-Landau effective Lagrangian, we derive the perturbations in supercarrier density and phase. As examples, we calculate the outward propagating EM waves produced by a low frequency gravitational wave impinging on a small sphere and by a high frequency wave normally incident on a large disk. We estimate for both targets the GW to EM conversion efficiencies and also the magnitude of the superconductor’s phase perturbation.
I. INTRODUCTION

There has been interest expressed lately in the possible use of a superconductor as a transducer between high frequency gravitational and electromagnetic waves\cite{1}. In the following we investigate the effect of a gravitational wave of arbitrary frequency on both normal and super-conductors. We attempt to make a minimum of special assumptions, however we do use time dependent Ginzburg-Landau equations for the superconducting case, which may only be strictly valid for temperatures near the superconducting transition. The superconductor is a kind of “generic” superconductor, with Ginzburg-Landau length taken as equal to the London penetration depth, about 100 nm. We also assume there are no normal electrons present in the superconductor, a highly idealised condition. Only the direct effect of the gravitational wave on the conducting particles is considered. The possibility of electrostrictive effects are ignored, but could conceivably be important\cite{2}.

In section II we find the Fermi normal coordinate system for an object in free-fall in a gravitational wave. The wavelength could be short compared to the object’s size. In section III the single particle Hamiltonian for a non-relativistic particle in the presence of gravitational and electromagnetic waves is derived and used to obtain general expressions for the gravitationally induced fields in both normal and super-conductors. As an example, in section IV the electromagnetic wave generated by a plane, long wavelength gravitational wave impinging on a sphere is derived. We show that in this case the outgoing EM wave is the same for both normal and super-conductor, and the energy conversion efficiency is extremely small. We also show that for typical estimates of the gravitational wave intensity, the phase perturbation in the case of the superconductor is very sensitive to the presence of the gravitational wave and might be measurable. In section V we consider a short wavelength plane gravitational wave normally incident on a large, thin disk. We show that in this case the energy conversion efficiency is still extremely small. For reasonable estimates of the gravitational wave amplitude the magnitude of the phase perturbation is unchanged.
II. THE GRAVITATIONAL WAVE

A. Transverse-Traceless Gauge

We start with the coordinate metric for a gravitational wave in transverse-traceless
gauge\(^3\),

\[
\tilde{g}_{\mu\nu} = \eta_{\mu\nu} + \tilde{h}_{\mu\nu},
\]

(1)

where

\[
\partial_\mu (\tilde{h}_\nu^\mu - \frac{\delta_\mu^\nu}{2} \tilde{h}_\alpha^\alpha),
\]

(2)

\[
\partial^\alpha \partial_\alpha \tilde{h}_{\mu\nu} = 0,
\]

(3)

and has the plane wave solutions:

\[
\tilde{h}_{\mu\nu} = R[\varepsilon_{\mu\nu} e^{ikx}],
\]

(4)

where \(\varepsilon_{\mu\nu}\) is a constant tensor. We will take the wave as propagating along the z-axis, and
then, in complex notation, the wave can be expressed as

\[
\tilde{h}_{ij} = [h_{+ij} + h_{\times ij}] e^{-ik(t-z)},
\]

(5)

where

\[
h_{+11} = -h_{+22} = h_+, h_{\times 12} = h_{\times 21} = h_\times,
\]

all other components being zero.

The curvature tensor is, in first order,

\[
R_{\mu\lambda\kappa\nu} = \frac{1}{2} [\tilde{h}_{\lambda\nu,\mu\kappa} - \tilde{h}_{\mu\nu,\lambda\kappa} - \tilde{h}_{\lambda\kappa,\mu\nu} + \tilde{h}_{\mu\kappa,\lambda\nu}].
\]

(7)

Which gives, in TT gauge,

\[
R_{0101} = R_{3131} = -R_{0202} = -R_{3232} = -R_{0131} = R_{0232} = -\frac{k^2}{2} h_+,
\]

(8)

\[
R_{0102} = R_{3132} = -R_{0132} = -R_{3102} = -\frac{k^2}{2} h_\times,
\]

(9)

and all the other independent components are zero.
B. Fermi Normal Coordinates

Manasse and Misner[4] constructed Fermi normal coordinates to second order in the displacement from a point in free fall. We generalize their approach here by deriving the coordinates to all orders in the displacement from a freely falling center of mass, but to first order in the gravitational wave amplitude.

Starting with a point on the CM’s world line with proper time $\tau$, we run spacelike geodesics out a distance $\lambda$. These geodesics start in the direction $\alpha^i$, $i = 1, 2, 3$, so any point $P$ in a neighborhood of the CM can be considered to have the coordinates $(x^0, x^i) = (\tau, \lambda \alpha^i)$.

(10)

Also, at the CM, the metric can be taken to be $\eta_{\mu\nu}$ with zero first derivatives.

Let $s^\alpha$ denote the tangent to such a space like geodesic, and let $n^\alpha$ denote a deviation vector. Then the equation of geodesic deviation, Manasse and Misner’s Eq. (63), becomes

$$\frac{d^2 n^\mu}{d s^2} + 2\frac{d n^\sigma}{d s} \Gamma^\mu_{\sigma\alpha} s^\alpha + n^\sigma s^\alpha s^\beta [R^\mu_{\alpha\sigma\beta} + \Gamma^\mu_{\sigma\alpha,\beta} + \Gamma^\tau_{\alpha\tau} \Gamma^\mu_{\tau\beta} - \Gamma^\mu_{\sigma\tau} \Gamma^\tau_{\alpha\beta}] = 0 .$$

(11)

We take the gravitational waves to be small of first order and set

$$R^\mu_{\alpha\sigma\beta} \rightarrow R^\mu_{\alpha\sigma\beta} e^{ik \cdot x} ,$$

(12)

taking the real part at the end of the calculation, and, since the Riemann tensor is small of first order, we may take in the exponent the metric tensor in $k \cdot x$ to be of zeroth order,

$$k \cdot x = k^\gamma x_\gamma = k^\gamma \eta_{\gamma\delta} x^\delta .$$

(13)

The $\Gamma^2$ terms are of second order and may be neglected.

We will denote the metric and its perturbation in Fermi normal coordinates as

$$g_{\mu\nu} = \eta_{\mu\nu} + h_{\mu\nu} ,$$

(14)

With $n$ a vector between neighboring space-like geodesics,

$$n = \frac{\partial}{\partial \alpha} , \quad n^\sigma = \lambda \delta^\sigma_i , \quad \frac{d n^\sigma}{d \lambda} = \delta^\sigma_i , \quad \frac{d^2 n^\sigma}{d \lambda^2} = 0 ,$$

and $s$ the tangent to the geodesic,

$$s^\sigma = \partial_\lambda x^\sigma = \partial_\lambda (\lambda \alpha^i \delta^\sigma_i) = \alpha^i \delta^\sigma_i ,$$

(15)
Eq. (11) becomes
\[ \lambda \Gamma_{ij,k}^\mu \alpha_j^i \alpha_k^i + 2 \Gamma_{ij}^\mu \alpha_j^i + \lambda \alpha_j^i \alpha_k^i R_{jk}^\mu e^{ikx} = 0 , \] (16)
which is just an ordinary differential equation,
\[ \partial_\lambda (\lambda^2 \Gamma_{ij}^\mu \alpha_j^i) = -\lambda^2 \alpha_j^i \alpha_k^i R_{jk}^\mu e^{i\lambda \hat{k} \cdot \hat{x}} = 0 , \] (17)
and can be integrated to give the \( \Gamma_{ij}^\mu \) as functions of \( \lambda \). Then, using the first order affine condition,
\[ \partial g_{ij} / \partial \lambda = \alpha_k^i g_{ij,k} = \alpha_k^i [\Gamma_{ik}^\mu \eta_{\mu j} + \Gamma_{jk}^\mu \eta_{\mu i}] , \] (18)
we get the space-space portion of the metric. A similar process can be applied to the other metric components, and the results for TT gauge are:
\[ h_{00} = -k^2 / 2 [h_+(x^2 - y^2) + 2h_x xy] q(kz) e^{-ik\tau} , \] (19)
and, defining the vector \( \vec{h} = h_{0i} \), we get
\[ \vec{h} = k^2 / 3 \left[ h_+ \begin{pmatrix} -zx \\ zy \\ x^2 - y^2 \end{pmatrix} + h_x \begin{pmatrix} -zy \\ -zx \\ 2xy \end{pmatrix} \right] p(kz) e^{-ik\tau} , \] (20)
also
\[ (h_{ij}) = k^2 / 2 \begin{pmatrix} -h_+ z^2 & -h_x z^2 & (h_+ x + h_x y) z \\ -h_x z^2 & h_+ z^2 & (h_+ x - h_x y) z \\ (h_+ x + h_x y) z & (h_+ x - h_x y) z & -(h_+ (x^2 - y^2) + 2h_x xy) \end{pmatrix} g(kz) e^{-ik\tau} , \] (21)
where
\[ g(w) = -6 / w^2 \left[ -2( e^{iw} - 1 ) /iw + e^{iw} + 1 \right] , \] (22)
\[ p(w) = 3 / 2 \left[ ( e^{iw} - 1 ) / (iw)^2 - 1 /iw + g(w) / 6 \right] , \] (23)
\[ q(w) = 2 \left[ ( e^{iw} - 1 ) / (iw)^2 - 1 /iw \right] . \] (24)
These weight functions all \( \rightarrow 1 \) in the long wavelength limit, and then Eqs. (19) - (21) become identical to Manasse and Misner’s results. In this limit
\[ \vec{\nabla} \cdot \vec{h} = \nabla^2 h_{00} = 0 . \] (25)
III. FIELDS IN CONDUCTORS

A. Charged Particles in Curved Space

The Lagrangian for a particle of charge $q$, mass $m$, interacting with the electromagnetic field in a curved space is:

$$L = -mc \left[ -g_{\alpha\beta} \dot{x}^\alpha \dot{x}^\beta \right]^{\frac{1}{2}} + qg_{\alpha\beta} \dot{x}^\alpha A^\beta . \quad (26)$$

Linearising, this is, for non-relativistic motion,

$$L = -mc^2 \left[ 1 - \frac{h_{00}}{2} - \frac{\vec{h} \cdot \vec{v}}{c} - \frac{v^2}{2c^2} \right] - q\Phi + q\vec{v} \cdot \vec{A} . \quad (27)$$

With both $h_{\mu\nu}$ and $\vec{A}$ small of first order, the canonical conjugate momentum is

$$\vec{p} = m\vec{v} + mc\vec{h} + q\vec{A} , \quad (28)$$

and the Hamiltonian is

$$H = \frac{1}{2m} \left[ \vec{p} - q\vec{A} - mc\vec{h} \right]^2 + q\Phi - \frac{mc^2}{2} h_{00} . \quad (29)$$

This implies that an electron with $q = -e$ in a resistive medium with collision frequency $\tau$ would have the equation of motion

$$m\ddot{\vec{v}} + \frac{mc\vec{v}}{\tau} = -e\vec{E} - mc\dot{\vec{h}} + \frac{mc^2}{2} \vec{\nabla} h_{00} . \quad (30)$$

The normal electron current is then

$$\vec{J}_n = \sigma_n \left[ \vec{E} + \frac{mc}{e} \dot{\vec{h}} - \frac{mc^2}{2e} \vec{\nabla} h_{00} \right] , \quad (31)$$

where for frequency $\omega << \frac{1}{\tau}$ the normal electron conductivity is

$$\sigma_n = \frac{e^2 \tau n_n}{m} , \quad (32)$$

where $n_n$ is the normal electron density.

B. Field Solutions

For both the normal and super-conductors, the field equations inside the conductors are of the form

$$\nabla^2 \phi - \kappa^2 \phi = f(\vec{x}) , \quad (33)$$
where \( \phi \) is a field amplitude, \( f(\vec{x}) \) is the source proportional to the gravitational field, and \( \frac{1}{\kappa} = \lambda \) is either the skin depth, for normal conductors, or the London penetration depth, for superconductors. This depth can vary from mm in normal conductors to some 100 nm in superconductors, and can be expected to be much smaller than both the scale \( \lambda_W \) over which the gravitational wave varies significantly and also much smaller than the conductor’s size \( R \). The general solution for \( \phi \) is

\[
\phi(\vec{x}) = -\int d^3x' \frac{e^{-\kappa|x-x'|}}{4\pi|x-x'|} f(x') + \phi_H(x),
\]

(34)

where \( \phi_H(x) \) is a homogeneous solution. Under our scale assumptions this reduces to

\[
\phi(\vec{x}) = -\frac{f(\vec{x})}{\kappa^2} + \phi_H e^{\kappa \zeta} + O\left(\frac{\lambda}{\lambda_W}\right),
\]

(35)

where \( \zeta \) denotes the distance from the conductor surface, (taken as positive in the outwards direction) and \( \phi_H \) is independent of \( \zeta \) but may depend on the coordinates parallel to the surface.

The amplitude for the homogeneous solution has to be determined by fitting the interior solution to an outgoing wave at the boundary.

C. Fields in a Normal Conductor

We assume that the gravitational field oscillates with frequency \( \omega \), then

\[
\vec{J}_n = \sigma_n \left[ \vec{E} - i\omega \frac{mc}{e} \vec{h} - \frac{mc^2}{2e} \vec{\nabla}h_{00} \right].
\]

(36)

With a typical value of \( \sigma_n \sim 10^8 / \Omega m \) and with \( \omega < 10^8 \text{Hz} \), we can neglect the displacement current, and Ampere’s law becomes

\[
\vec{\nabla} \times \vec{B} = \mu_0 \sigma_n \left[ \vec{E} - i\omega \frac{mc}{e} \vec{h} - \frac{mc^2}{2e} \vec{\nabla}h_{00} \right].
\]

(37)

This implies

\[
\vec{\nabla} \cdot \vec{E} = i\omega \frac{mc}{e} \vec{\nabla} \cdot \vec{h} + \frac{mc^2}{2e} \vec{\nabla}^2 h_{00}.
\]

(38)

Eq. (37) implies

\[
\nabla^2 \vec{B} + i\omega \mu_0 \sigma_n \vec{B} = i\omega \mu_0 \sigma_n \frac{mc}{e} \vec{\nabla} \times \vec{h}.
\]

(39)

For \( 10^3 \text{Hz} < \omega < 10^8 \text{Hz} \) the skin depth

\[
\lambda_S = \sqrt{\frac{2}{\omega \mu_0 \sigma_n}},
\]
runs between 5 mm and 10 \( \mu m \) and is much smaller than the gravitational wavelength. With the conductor’s size on the order of a meter, the considerations of the previous section apply, and
\[
\vec{B} = \frac{mc}{e} \vec{\nabla} \times \vec{h} + \vec{B}_H e^{\kappa \zeta} ,
\]
where \( \kappa = \frac{1}{\kappa_S} \). The time derivative of Eq. (37), Faraday’s law and Eq. (38) lead to
\[
\vec{E} = i\omega \frac{mc}{e} \vec{h} + \frac{mc^2}{2e} \vec{\nabla} h_00 + \frac{1}{i\omega \mu_0 \sigma_n} \vec{\nabla} \cdot \vec{E} + \vec{E}_H e^{\kappa \zeta} .
\]

When the gravitational wavelength is much larger than the conductor size, Eq. (25) applies, leading to \( \vec{\nabla} \cdot \vec{E} = 0 \) and
\[
\vec{E} = i\omega \frac{mc}{e} \vec{h} + \frac{mc^2}{2e} \vec{\nabla} h_00 + \vec{E}_H e^{\kappa \zeta} .
\]

D. The Superconductor

The Ginzburg-Landau free energy for a superconductor with supercarriers of mass \( m^* = 2m \), charge \( e^* = -2e \), is[5],
\[
F = \int_V d^3x \left[ \frac{1}{2m^*} \left| (-i\hbar \vec{\nabla} - e^* \vec{A}) \Psi \right|^2 + e^* \Phi |\Psi|^2 - \alpha |\Psi|^4 + \frac{\beta}{2} |\Psi|^4 \right] .
\]
Comparision with the Hamiltonian of Eq. (29) suggests that it would be reasonable to extend this to an effective Lagrangian,
\[
L = \int_V d^3x \left[ i\hbar \Psi^* \frac{\partial \Psi}{\partial t} - \mathcal{H}_S \right] ,
\]
where
\[
\mathcal{H}_S = \frac{1}{2m^*} \left| (-i\hbar \vec{\nabla} - e^* \vec{A} - m^* \hbar c \vec{h}) \Psi \right|^2 + \left( e^* \Phi - \frac{m^* c^2 h_{00}}{2} \right) |\Psi|^2 - \alpha |\Psi|^4 + \frac{\beta}{2} |\Psi|^4 .
\]
Varying the corresponding action leads to the equation for the order parameter \( \Psi \),
\[
\left( 1 - i\hbar \vec{\nabla} - e^* \vec{A} - m^* \hbar c \vec{h} \right)^2 + e^* \Phi - \frac{m^* c^2}{2} h_{00} - \alpha + \beta |\Psi|^2 \right] \Psi ,
\]
and a boundary condition at the superconductor surface,
\[
\hat{n} \cdot \left( -i\hbar \vec{\nabla} - e^* \vec{A} - m^* \hbar c \vec{h} \right) \Psi = 0 .
\]
The supercurrent is

\[
\vec{J}_S = -\frac{ie^*}{2m^*} \left( \Psi^* \vec{\nabla} \Psi - \vec{\nabla} \Psi^* \Psi \right) - \frac{e^2}{m^*} \left( \vec{A} + \frac{m^* c}{e^*} \vec{h} \right) |\Psi|^2
\]  

(48)

and the boundary condition Eq. (47) implies that at the surface

\[
\hat{n} \cdot \vec{J}_S = 0 .
\]  

(49)

Linearising,

\[
\Psi = |\Psi_0| (1 + s) e^{i\theta} ,
\]  

(50)

and introducing the gauge invariant phase \( \tilde{\theta} \),

\[
\theta = \tilde{\theta} - \frac{e^*}{\hbar} \int \Phi dt ,
\]  

(51)

Eq. (46) yields the linearised equations

\[
i\hbar \omega \tilde{\theta} = -\frac{\hbar^2}{2m^*} \nabla^2 s - \frac{m^* c^2}{2} h_{00} + 2\alpha s ,
\]  

(52)

\[
i\hbar \omega s = \frac{1}{2m^*} \left[ \hbar^2 \nabla^2 \tilde{\theta} + \hbar \nabla \cdot \left( \frac{e^*}{-i\omega} \vec{E} - m^* c \vec{h} \right) \right] ,
\]  

(53)

and the boundary conditions at the surface,

\[
\hat{n} \cdot \left( \hbar \nabla \tilde{\theta} + \frac{e^*}{-i\omega} \vec{E} - m^* c \vec{h} \right) = 0 ,
\]  

(54)

\[
\hat{n} \cdot \nabla s = 0 .
\]  

(55)

The London penetration depth is,

\[
\lambda_L = \sqrt{\frac{m^*}{\mu_0 (e^*)^2 |\Psi_0|^2}} ,
\]  

(56)

and in our generic superconductor, \( \lambda_L \sim 10^{-7} m \). We also use the London frequency,

\[
\omega_L = \frac{1}{\lambda_L \sqrt{\mu_0 \epsilon}} ,
\]  

(57)

which is about \( 10^{15} Hz \). Linearized, the current of Eq. (48) is now

\[
\vec{J}_S = \frac{1}{\mu_0 e^* \lambda_L^2} \left( \hbar \nabla \tilde{\theta} + \frac{e^* E}{-i\omega} - m^* c \vec{h} \right) .
\]  

(58)

With no normal electrons present, Gauss' law becomes

\[
\vec{\nabla} \cdot \vec{E} = \frac{2e^* |\Psi_0|^2 s}{\epsilon} = \frac{2m^* \omega_L^2 e^*}{\epsilon} s .
\]  

(59)
and with this Eq. (53) becomes

\[ i\hbar \omega s = \frac{\hbar^2}{2m^*} \nabla^2 \tilde{\theta} + i\hbar \frac{\omega_L^2}{\omega} s - \frac{\hbar c}{2} \vec{\nabla} \cdot \vec{h}. \]  

In the long wavelength limit, the divergence of \( \vec{h} \) can be neglected, moreover, \( \frac{\omega_L^2}{\omega} > 10^{14} \), for \( \omega > 10^3 Hz \), so Eq. (60) reduces to

\[ s = i\xi_L^2 \nabla^2 \tilde{\theta}, \]

where

\[ \xi_L = \sqrt{\frac{\hbar^2}{2m^* \omega_L^2}}. \]  

This varies from \( 10^{-16} \) to \( 10^{-13} m \) as \( \omega \) goes from \( 10^3 \) to \( 10^8 Hz \). Using this in Eq. (52) leads to

\[ i\hbar \frac{\omega}{\alpha} \tilde{\theta} = -i\xi^2 \xi_L^2 \nabla^4 \tilde{\theta} - \frac{m^* c^2}{2\alpha} h_{00} + 2i\xi_L^2 \nabla^2 \tilde{\theta}, \]

where

\[ \xi = \sqrt{\frac{\hbar^2}{2m^* \alpha}}, \]

is the Ginzburg-Landau coherence length, \( \sim 10^{-7} m \) in our generic superconductor.

Deep inside the superconductor, a depth much greater than the characteristic lengths \( \xi, \xi_L \), we will have

\[ \tilde{\theta} = i\frac{m^* c^2}{2\hbar \omega} h_{00}, \]

and \( s \to 0 \), since \( \nabla^2 h_{00} = 0 \) in the long wavelength limit. Near the surface \( \tilde{\theta} \) will be of the form

\[ \tilde{\theta} = i\frac{m^* c^2}{2\hbar \omega} h_{00} + \tilde{\theta}_0 e^{iq \zeta}, \]

where \( q \) satisfies

\[ i\hbar \frac{\omega}{\alpha} = -i\xi^2 \xi_L^2 q^4 + 2i\xi_L^2 q^2. \]

The roots corresponding to attenuation as one goes into the conductor are

\[ q_{\pm} = \frac{1}{\xi} \left[ 1 \pm \sqrt{\frac{\xi^2 \hbar \omega}{\xi_L^2 \alpha} - 1} \right]^\frac{1}{4}, \]

for our generic superconductor,

\[ q_{\pm} = \sqrt{\pm i} q_0, \]

where

\[ q_0 = \left[ \frac{2m^* c^2}{\alpha} \right]^\frac{1}{4} \frac{1}{\sqrt{\xi \lambda_L}}. \]
Now
\[ \tilde{\theta} = i \frac{m^* c^2}{2 \hbar \omega} h_{00} + \tilde{\theta}_+ e^{q+\zeta} + \tilde{\theta}_- e^{q-\zeta}, \tag{71} \]
and, by Eq. (61),
\[ s = -\xi^2 \eta^2 \left[ \tilde{\theta}_+ e^{q+\zeta} - \tilde{\theta}_- e^{q-\zeta} \right]. \tag{72} \]

The boundary condition, Eq. (55) implies that
\[ \tilde{\theta}_- = i \tilde{\theta}_+ , \tag{73} \]
so we can write
\[ \tilde{\theta} = i \frac{m^* c^2}{2 \hbar \omega} h_{00} + \tilde{\theta}_H \left( e^{q+\zeta} + ie^{q-\zeta} \right), \tag{74} \]
and
\[ s = -\xi^2 \eta^2 \tilde{\theta}_H \left[ e^{q+\zeta} - ie^{q-\zeta} \right]. \tag{75} \]

E. Fields in the Superconductor

Neglecting the displacement current, Ampere’s law is now
\[ \nabla \times \vec{B} = \frac{1}{\lambda_L^2 e^*} \left( \hbar \nabla \tilde{\theta} + \frac{e^* E}{-i\omega} - m^* \vec{c} \hbar \right), \tag{76} \]
taking the curl, this becomes
\[ \nabla^2 \vec{B} - \frac{1}{\lambda_L^2} \vec{B} = \frac{m^* c}{e^* \lambda_L^2} \nabla \times \hbar, \tag{77} \]
which has the solution,
\[ \vec{B} = -\frac{m^* c}{e^*} \nabla \times \hbar + \vec{B}_He^{\kappa \zeta}, \tag{78} \]
where \( \kappa = \frac{1}{\lambda_L} \). Now Eqs. (76) and (78) give for the E field,
\[ \vec{E} = i \frac{\omega}{e^*} \left( \hbar \nabla \tilde{\theta} - m^* \vec{c} \hbar \right) - i \omega \lambda_L^2 \nabla \times \left( \vec{B}_He^{\kappa \zeta} \right) + \frac{i \omega \lambda_L^2 m^* c}{e^*} \nabla \times \left( \nabla \times \hbar \right). \tag{79} \]

In the long wavelength limit the last term in this equation will be zero. In practice, it is convenient to first write E in the Eq. (79) form, obtain B from Faraday’s equation, and then obtain the surface terms by fitting to an outward propagating radiation field.

It should be noted that with the use of Eq. (74), in the long wavelength limit the interior part of Eq. (79) becomes identical to that of Eq. (42) with the substitutions \( e \rightarrow -e^* \) and \( m \rightarrow m^* \).
IV. EXAMPLE: THE SPHERE

A. Vector Spherical Harmonics

Let \( \vec{L} = -i\vec{x} \times \vec{\nabla} \), then the vector \( \vec{h} \) can be written as

\[
\vec{h} = \frac{ik^2 r^2}{6} \tilde{L}C(\theta, \phi) ,
\]

where the function

\[
C(\theta, \phi) = \sin^2(\theta) [h_+ \sin(2\phi) - h_- \cos(2\phi)] ,
\]

is a superposition of \( \ell = 2 \) spherical harmonics. Defining a similar function

\[
D(\theta, \phi) = \sin^2(\theta) [h_+ \cos(2\phi) + h_- \sin(2\phi)] ,
\]

we can write

\[
\vec{\nabla} h_{00} = \vec{\nabla} \times \frac{ik^2 r^2}{6} \tilde{L}D .
\]

A useful vector identity is, for an arbitrary function \( f(r) \), and an \( \ell = 2 \) spherical harmonic \( Y_{2,m}(\theta, \phi) \),

\[
\vec{\nabla} \times \vec{L} f(r) Y_{2,m}(\theta, \phi) = \left( f' + \frac{f}{r} \right) \hat{n} \times \vec{L} + \frac{6i\hat{n}}{r} f \right] Y_{2,m}(\theta, \phi) .
\]

B. The Normal Conductor

We now write the E field of Eq. (42) as

\[
\vec{E} = -\frac{\omega mc k^2}{6e} r^2 \tilde{L}C + \frac{imc k^2}{12e} \vec{\nabla} \times \left( r^2 \tilde{L}D \right) + E_{0C} e^{\kappa(r-R)} \tilde{L}C - \frac{i\omega}{\kappa} B_{0D} \vec{\nabla} \times \left( e^{\kappa(r-R)} \tilde{L}D \right) ,
\]

where we have taken the surface terms as having the same angular dependence as the volume gravitational source terms.

This gives, using Faraday’s law and neglecting terms of order \( (\delta/R) \),

\[
\vec{B} = \frac{imc k^2}{6e} \vec{\nabla} \times \left( r^2 \tilde{L}C \right) + \frac{E_{0C}}{i\omega} \vec{\nabla} \times \left( e^{\kappa(r-R)} \tilde{L}C \right) + B_{0DE} e^{\kappa(r-R)} \tilde{L}D .
\]

At the surface, the D wave produces a transverse B field, and the C wave produces a transverse E field. We therefore take for the exterior fields an outward radiating D wave TM field, and a C wave TE field[6],

\[
\vec{E} = E_1 h_1^2 (kr) + \frac{B_1 c^2}{-i\omega} \vec{\nabla} \times \left( h_1^2 (kr) \tilde{L}D \right) ,
\]

\[
\vec{B} = \frac{E_1}{i\omega} \vec{\nabla} \times \left( h_1^2 (kr) \tilde{L}C \right) + B_1 h_1^2 (kr) \tilde{L}D .
\]
C. At the Surface

When \( r = R \), the inner E field becomes

\[
\vec{E} = -\frac{\omega mc^2 k^2}{6e} R^2 \vec{L}C + \frac{imc^2 k^2}{4e} R \left( \hat{n} \times \vec{L} + 2i\hat{n} \right) D + E_{0C} \vec{L}C - \frac{i\omega}{\kappa^2} B_{0D} \left( \kappa \hat{n} \times \vec{L} + \frac{6i\hat{n}}{R} \right) D ,
\]

(89)

and the outer E field is

\[
\vec{E} = E_1 h_2^1(kR) + \frac{B_1 c^2}{-i\omega} \left[ F \hat{n} \times \vec{L} + \frac{6i\hat{n}}{R} h_2^1(kR) \right] D ,
\]

(90)

where

\[
F = k h_2^V(kR) + \frac{h_2^1(kR)}{R} .
\]

(91)

In the following we will abbreviate \( h_2^1(kR) = g \). Continuity of tangential E implies that

\[
-\frac{\omega mc^2 k^2}{6e} R^2 + E_{0C} = E_1 g ,
\]

(92)

\[
\frac{imc^2 k^2}{4e} R - i\omega \frac{B_{0D}}{\kappa} = \frac{B_1 c^2}{-i\omega} F .
\]

(93)

The inner B field at the surface becomes

\[
\vec{B} = \frac{imc^2 k^2}{2e} R \left( \hat{n} \times \vec{L} + 2i\hat{n} \right) C + \frac{E_{0C}}{i\omega} \left( \kappa \hat{n} \times \vec{L} + \frac{6i\hat{n}}{R} \right) C + B_{0D} \vec{L}D ,
\]

(94)

and the outer field is

\[
\vec{B} = \frac{E_1}{i\omega} \left[ F \hat{n} \times \vec{L} + \frac{6i\hat{n}}{R} g \right] C + B_1 g \vec{L}D .
\]

(95)

Continuity of tangential B leads to

\[
\frac{imc^2 k^2}{2e} R + \frac{E_{0C}}{i\omega} = \frac{E_1}{i\omega} F ,
\]

(96)

\[
B_{0D} = B_1 g .
\]

(97)

Continuity of normal B leads to an equation identical to Eq. (92). The amplitude coefficients are therefore:

\[
E_{0C} = -\frac{\omega mc^2 k^2}{6e} \left[ F - \frac{3g}{R} \right] \left[ g - \frac{F}{\kappa} \right]^{-1} ,
\]

(98)

\[
B_{0D} = \frac{\kappa mc^2 k^2 R}{4e\omega} \left[ 1 + \frac{\kappa F}{k^2 g} \right]^{-1} ,
\]

(99)

\[
E_1 = -\frac{\omega mc^2 k^2 R}{6e} \left[ R - \frac{3}{\kappa} \right] \left[ g - \frac{F}{\kappa} \right]^{-1} ,
\]

(100)
We consider here the case when the gravitational wavelength is much larger than both the sphere’s radius R and the London penetration depth $\lambda_L$. Then $kR \ll 1$ and $\kappa/k >> 1$ and with

$$g \to -\frac{3}{(kR)^3},$$

(102)

$$F \to \frac{6}{R(kR)^3},$$

(103)

then

$$E_{0C} = \frac{5\omega mc k^2 R}{6\kappa},$$

(104)

$$B_{0D} = -\frac{mc^2 k^4 R^2}{8e\omega},$$

(105)

$$E_1 = \frac{\omega mc k^5 R^5}{18e},$$

(106)

$$B_1 = \frac{mck^6 R^5}{24e}.$$  

(107)

### D. The Radiation Field

At large r,

$$h^1_2(kr) \to i\frac{e^{ikr}}{kr},$$

(108)

and Eq. (87) becomes

$$\vec{E} \to iE_1 \frac{e^{ikr}}{kr} \vec{L}C - iB_1 \frac{e^{ikr}}{kr} \hat{n} \times \vec{L}D.$$  

(109)

The radiation power distribution is

$$\frac{dP}{d\omega} = \frac{|r\vec{E}|^2}{2c\mu_0}.$$  

(110)

With an unpolarised beam,

$$< h^2_+ >=< h^2_\times >= \frac{h^2}{2},$$

(111)

$$< h_+ h_\times >= 0,$$  

(112)

this becomes

$$\frac{dP}{d\omega} = \frac{25c}{2592\mu_0} \left( mc \right)^2 (kR)^{10} h^2 \sin^2(\theta) \left[ \cos^2(\theta) - \frac{48}{25} \cos(\theta) + 1 \right],$$  

(113)
and the total radiated power is

\[
P = \frac{5\pi c}{162\mu_0} \left(\frac{mc}{e}\right)^2 (kR)^{10}h^2.
\]

(114)
The incident power flux in the gravitational wave is\[3\],

\[
J_0 = \frac{k^2c^5h^2}{16\pi G}.
\]

(115)
so the efficiency of gravitational to electromagnetic wave conversion is

\[
\eta = \frac{P}{J_0\pi R^2} = \frac{40\pi G}{81\mu_0} \left(\frac{m}{ce}\right)^2 (kR)^8 \sim 3 \times 10^{-44} (kR)^8,
\]

(116)
which for spheres of meter radius is a rather small number.

E. The Superconducting Sphere

The driving forces for the fields are proportional to the \(\ell = 2\) spherical harmonics C and D. We therefore set

\[
\tilde{\theta} = -i\frac{m^*c^2k^2r^2D}{4\hbar\omega} + \left(\tilde{\theta}_CC + \tilde{\theta}_DD\right) \left(e^{q+\zeta} + ie^{q-\zeta}\right).
\]

(117)
Then

\[
\nabla \tilde{\theta} = -i\frac{m^*c^2k^2\nabla \times r^2\tilde{L}D}{12\hbar\omega} + \sqrt{iq_0\hat{n}} \left(\tilde{\theta}_CC + \tilde{\theta}_DD\right) \left(e^{q+\zeta} + e^{q-\zeta}\right) + O\left(\frac{1}{qR}\right).
\]

(118)
The source of the electric field has both an \(\tilde{L}C\) and a \(\nabla \times \tilde{L}D\) term. We therefore include such terms in the harmonic solution, and take for the total electric field

\[
\tilde{E} = -i\frac{m^*c^2k^2\nabla \times r^2\tilde{L}D}{12e^*} + i\sqrt{\frac{\hbar}{e^*q_0\hat{n}}} \left(\tilde{\theta}_CC + \tilde{\theta}_DD\right) \left(e^{q+\zeta} + e^{q-\zeta}\right) + \frac{\omega m^*c}{6e^*}k^2r^2\tilde{L}C + E_{0C}e^{\zeta} \tilde{L}C - i\omega \lambda_0^2 B_{0D} \nabla \times \left(e^{\zeta} \tilde{L}D\right),
\]

(119)
from this we get, using Faraday’s law,

\[
\tilde{B} = -i\frac{m^*c}{6e^*}r^2 \nabla \times \left(r^2\tilde{L}C\right) + \frac{E_{0C}}{i\omega} \nabla \times \left(e^{\zeta} \tilde{L}C\right) + B_{0D}e^{\zeta} \tilde{L}D.
\]

(120)
We have now an \(\tilde{L}C\) TE mode plus a TM \(\tilde{L}D\) mode. We therefore take the same combination for the exterior fields, as given in Eqs. (87) and (88).
At $r = R$, the inner E field becomes
\[
\vec{E} = -i \frac{m^* c^2 k^2 R}{4 e^*} \left( \hat{n} \times \vec{L} + 2i\hat{n} \right) D + 2i\sqrt{\tilde{\nu}} e^* q_0 \tilde{\nu} \left( \hat{\theta}_C + \hat{\theta}_D D \right) + \frac{\omega m^* c}{6 e^*} k^2 R^2 \tilde{L} C + E_{0C} \tilde{L} C - i\omega \lambda_L B_{0D} \hat{n} \times \vec{L} D + O \left( \frac{\lambda_L}{R} \right),
\]
and the inner B field becomes
\[
\vec{B} = -i \frac{m^* c}{2 e^*} k^2 R \left( \hat{n} \times \vec{L} + 2i\hat{n} \right) C + \frac{E_{0C}}{i\omega} \left( \kappa \hat{n} \times \vec{L} + \frac{6i\hat{n}}{R} \right) C + B_{0D} \vec{L} D.
\]
At the surface, $\vec{B}$ and the parallel component of $\vec{E}$ must be continuous. Moreover, since the normal component of the supercurrent vanishes at the surface, there can be no surface charge, and as a result Gauss’ law requires that
\[
\hat{n} \cdot \vec{E}_{\text{in}} = \frac{\epsilon_0}{\epsilon} \hat{n} \cdot \vec{E}_{\text{out}}.
\]
Parallel E conductivity gives for the coefficients of the independent surface functions:
\[
\frac{\omega m^* c}{6 e^*} k^2 R^2 + E_{0C} = E_1 g,
\]
\[
- i \frac{m^* c^2 k^2 R}{4 e^*} - i\omega \lambda_L B_{0D} = -\frac{e^2}{i\omega} B_1 F.
\]
Eq. (123) for the normal component of E gives
\[
\hat{\theta}_C = 0,
\]
\[
\frac{m^* c^2 k^2 R}{2 e^*} + 2i\sqrt{\tilde{\nu}} e^* q_0 \tilde{\nu} \hat{\theta}_D = -\frac{6\epsilon_0 e^2}{\omega e} B_1 \frac{g}{R}.
\]
Parallel B continuity implies:
\[
B_{0D} = B_1 g,
\]
\[
\frac{m^* \kappa k^2 R}{2 e^*} + \frac{\kappa E_{0C}}{\omega} = \frac{E_1}{\omega} F.
\]
Continuity of the normal component of B leads to an equation identical to Eq. (124).

The coefficients for the E and B fields are the same as in Eqs. (99) to (98) with the substitutions $m \to m^*$ and $-e \to e^*$. The new equation here is
\[
\hat{\theta}_D = -\frac{\sqrt{i} m^* c^2 k^2 R}{4h \omega q_0} \left[ \frac{3\kappa}{\epsilon_r k^2 R} \left( 1 + \frac{\kappa F}{k^2 g} \right)^{-1} - 1 \right],
\]
where $\epsilon_r$ is the superconductor’s dielectric constant.
The E and B field amplitudes are the same as for the normal conductor, with the appropriate mass, charge and penetration depth, but the phase amplitude becomes in the long wave limit

$$\tilde{\theta}_D \rightarrow \frac{\sqrt{im^*c^2k^2R}}{4\hbar\omega q_0} \left( \frac{3}{2\epsilon_r} + 1 \right).$$

(131)

The external fields are essentially the same as for the normal conductor. The efficiency of GW to EMW conversion is just as astronomically small.

The phase $\tilde{\theta}$ is of some interest. The bulk value is

$$|\tilde{\theta}| \sim \frac{m^*c^2}{2\hbar\omega} (kR)^2 \hbar.$$  

(132)

Then with $\hbar \sim 10^{-24}$, $R \sim 1m$ and waves in the megacycle range, $|\tilde{\theta}| \sim 10^{-14}$. At the surface it changes by a negligible amount,

$$|\tilde{\theta}_D| \sim \frac{m^*c(kR)^3}{\hbar q_0} \hbar \sim 10^{-19}.$$  

(133)

The phase is therefore essentially equal everywhere to its bulk value of about $10^{-14}$, a small but not astronomically small quantity, which could perhaps be measurable.

V. HIGH FREQUENCY GRAVITATIONAL WAVES

There is a possibility that high frequency gravitational waves may exist as relics of the Big Bang[7]. We show below that for waves in the gigacycle range, the conversion efficiency is still extremely low, but the phase amplitude can be somewhat larger.

The amplitude functions of Eqs. (22) - (24) fall off with large $kz$. Thus to maximise output at high frequency we require a target of small extent in the $z$-direction. We consider therefore a disk perpendicular to the x-axis, extending from $z = -a$ to $+a$, and of radius $R$. We take

$$k\lambda_L << ka << 1 << kR.$$  

(134)

Although non-zero, it turns out that $\vec{V} \cdot \vec{h}$ and $\nabla^2 h_{00}$ are still negligible compared to the other terms. Let

$$\vec{u} = x\dot{x} - y\dot{y},$$  

(135)

$$\vec{v} = y\dot{x} + x\dot{y}.$$  

(136)
We have, neglecting the surface variation in \( \tilde{\theta} \), and considering only the transverse E and B components, inside the superconductor near the \( z = +a \) side,

\[
\vec{E}_T = \frac{m^* c^2 k^2}{2 e^*} [h_+ \vec{u} + h_x \vec{v}] + [E_{0u} \vec{u} + E_{0v} \vec{v}] e^{\kappa(a-z)} ,
\]

(137)

\[
\vec{B}_T = -\frac{m^* c^2 k^2}{e^*} [-h_+ \vec{v} + h_x \vec{u}] - \frac{\kappa}{i \omega} [E_{0u} \vec{v} \vec{u} - E_{0v} \vec{u} \vec{v}] e^{\kappa(a-z)} .
\]

(138)

(Here we have also neglected the part of E coming from \( \vec{h} \) as it is negligible here in comparison to the contribution from \( \vec{\nabla} \tilde{\theta} \).) We approximate the external fields on the positive \( z \) side as

\[
\vec{E} = [E_{1u} \vec{u} + E_{1v} \vec{v}] e^{ik(z-a)} ,
\]

(139)

\[
\vec{B} = \frac{1}{c} [E_{1u} \vec{v} - E_{1v} \vec{u}] e^{ik(z-a)} .
\]

(140)

The boundary conditions lead to

\[
E_{1u} = \frac{m^* c^2 k^2}{2 e^*} h_+ + O(k\lambda_L) ,
\]

(141)

\[
E_{1v} = \frac{m^* c^2 k^2}{2 e^*} h_x + O(k\lambda_L) ,
\]

(142)

and \( E_{0u} \) and \( E_{0v} \) are similiar but smaller by a factor of \( k\lambda_L \). For an unpolarised wave, the power radiated from the positive side is

\[
P = \frac{\pi c}{24 \mu_0} \left( \frac{m^* c}{e^*} \right)^2 (kR)^4 h^2 ,
\]

(143)

and the efficiency becomes

\[
\eta = \frac{\pi}{2} \left( \frac{m^* c}{e^*} \right)^2 G(kR)^2 .
\]

(144)

Now \( \eta \approx 3 \times 10^{-44}(kR)^2 \) and is still extremely small, even for \( R \sim 10m \) with frequencies in the 100 Gigacycle range, where \( kR \sim 10^4 \). However, the phase \( \tilde{\theta} \) is now

\[
\tilde{\theta} = -i \frac{m^* c^2}{4 \hbar \omega} k^2 \left( h_+ (x^2 - y^2) + 2 h_x xy \right) ,
\]

(145)

and

\[
|\tilde{\theta}| \sim 10^{12} kR^2 h .
\]

(146)

This would be \( \sim 10^{-6} \) for \( h \sim 10^{-24} \). However, Eq. (115) gives for a wave with such an amplitude a flux of \( \sim 10^9 W/m^2 \) ! A more reasonable estimate of the cosmic background at this frequency would be \( h \sim 10^{-32} [7] \), and then \( |\tilde{\theta}| \sim 10^{-14} \), the same as for the low frequency wave.
VI. CONCLUSION

We find that the conversion of gravitational wave energy to electromagnetic energy by both normal and superconductors is a remarkably inefficient process. However in the superconducting case, the gravitational wave produces a perturbation in the order parameter phase that may perhaps be detectable.

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