Expressing the cone radius in the relational calculus with real polynomial constraints

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Abstract

We show that there is a query expressible in first-order logic over the reals that returns, on any given semi-algebraic set \( A \), for every point a radius around which \( A \) is conical. We obtain this result by combining famous results from calculus and real algebraic geometry, notably Sard’s theorem and Thom’s first isotopy lemma, with recent algorithmic results by Rannou.

1 Introduction

The framework of constraint databases, introduced by Kanellakis, Kuper and Revesz [8], provides a nice theoretical model for spatial databases [11]. A spatial dataset is modeled using real polynomial inequality constraints; such sets are also known as semi-algebraic sets [1, 3]. The relational calculus (first-order logic) with real polynomial constraints then serves as a basic query language, denoted here as FO.

The study of the expressive power of query languages for constraint databases is an active domain of research [14]. One of the problems in particular that received attention in recent years is that of determining the exact power of FO in expressing topological properties of spatial databases [2, 5, 7, 9, 13]. One such property, which is central in this research, is that locally around each point, a semi-algebraic set has the topology of a cone. A radius at which this behavior shows is called a cone radius around the point for the set.

Accordingly, a cone radius query is a query that returns, for a semi-algebraic set \( A \), a set of pairs \((\vec{p}, r)\) giving for every point \( \vec{p} \) a cone radius \( r \) in \( \vec{p} \) for \( A \). In this paper, we show that there exists an FO formula expressing a cone radius query.
far, this was only known in two dimensions [3]. Expressibility of the cone radius, apart from being a natural question in itself, has also applications: for example, it has been linked to the expressibility of piecewise linear approximations [4].

2 Preliminaries

2.1 Spatial databases and Queries

A semi-algebraic set in \( \mathbb{R}^n \) is a finite union of sets definable by conditions of the form

\[
f_1(\vec{x}) = f_2(\vec{x}) = \cdots = f_k(\vec{x}) = 0, \; g_1(\vec{x}) > 0, \; g_2(\vec{x}) > 0, \ldots, \; g_\ell(\vec{x}) > 0,
\]

with \( \vec{x} = (x_1, \ldots, x_n) \in \mathbb{R}^n \), and where \( f_1(\vec{x}), \ldots, f_k(\vec{x}), g_1(\vec{x}), \ldots, g_\ell(\vec{x}) \) are multi-variate polynomials in the variables \( x_1, \ldots, x_n \) with real coefficients. A database schema \( S \) is a finite set of relation names, each with a given arity. A database over \( S \) assigns to each \( S \in S \) a semi-algebraic set \( S_D \) in \( \mathbb{R}^n \), if \( n \) is the arity of \( S \). A \( k \)-ary query over \( S \) is a function mapping each database over \( S \) to a semi-algebraic set in \( \mathbb{R}^k \).

As query language we use first-order logic (FO) over the vocabulary \( (+, \cdot, 0, 1, <) \) expanded with the relation names in \( S \). A formula \( \varphi(x_1, \ldots, x_k) \) expresses the \( k \)-ary query defined by

\[
\varphi(D) := \{(a_1, \ldots, a_k) \in \mathbb{R}^k \mid \langle \mathbb{R}, D \rangle \models \varphi(a_1, \ldots, a_k)\},
\]

for any database \( D \). Note that \( \varphi(D) \) is always semi-algebraic because all relations in \( D \) are; indeed, by Tarski’s theorem [13], the relations that are first-order definable on the real ordered field are precisely the semi-algebraic sets.

An example of a query expressed in FO is the following: Let \( S \) be a schema containing the relation name \( S \). Consider the FO-formula

\[
\varphi_{\text{int}}(\vec{x}) := (\exists \varepsilon > 0)(\forall x'_1) \cdots (\forall x'_n)(\|\vec{x} - \vec{x}'\| < \varepsilon \rightarrow S(x'_1, \ldots, x'_n)).
\]

For any database \( D \), \( \varphi_{\text{int}}(D) \) equals the interior of \( S^D \). However, not every query is first-order expressible: the query which asks whether a set is connected is not expressible in FO. This result and other results related to constraint databases have recently been collected in a single volume [10].

2.2 Cones

Let \( A \subseteq \mathbb{R}^n \) be a semi-algebraic set and \( \vec{p} \in \mathbb{R}^n \) a point not in \( A \). We define the cone with base \( A \) and top \( \vec{p} \) as the union of all closed line segments between \( \vec{p} \) and points in \( A \). We denote this set with \( \text{Cone}(A, \vec{p}) := \{t\vec{b} + (1 - t)\vec{p} \mid \vec{b} \in A, \; 0 \leq t \leq 1\} \). For a point \( \vec{p} \in \mathbb{R}^n \) and \( \varepsilon > 0 \), denote the closed ball centered at
\( \vec{p} \) with radius \( \varepsilon \) by \( B^n(\vec{p}, \varepsilon) \), and denote the sphere centered at \( \vec{p} \) with radius \( \varepsilon \), by \( S^{n-1}(\vec{p}, \varepsilon) \). The following well-known theorem says that, locally around each point, a semi-algebraic set has the topology of a cone.

**Theorem ([1, 3]).** Let \( A \subseteq \mathbb{R}^n \) be a semi-algebraic set and \( \vec{p} \) a point of \( A \). Then there is a real number \( \varepsilon > 0 \) such that the intersection \( A \cap B^n(\vec{p}, \varepsilon) \) is homeomorphic to the set \( \text{Cone}(A \cap S^{n-1}(\vec{p}, \varepsilon), \vec{p}) \).

Any real number \( \varepsilon > 0 \) as in the lemma is called a \textit{cone radius} of \( A \) in \( \vec{p} \).

Let \( S \) be a schema containing a relation name \( S \) of arity \( n \). A \textit{cone radius query} \( Q_{\text{radius}} \) is a query which maps any database \( D \) over \( S \) to a set of pairs \((\vec{p}, r) \in \mathbb{R}^n \times \mathbb{R}\) such that for every point \( \vec{p} \in S_D \) there exists at least one pair \((\vec{p}, r) \in Q_{\text{radius}}(D) \), and for every \((\vec{p}, r) \in Q_{\text{radius}}(D) \), \( r \) is a cone radius in \( \vec{p} \) for \( S_D \).

We will use the following notation: Let \( A \subseteq B \subseteq \mathbb{R}^n \), the closure of \( A \) in \( B \) is denoted by \( \text{cl}_B(A) \), and \( \text{int}_B(A) \) indicates the interior of \( A \) in \( B \). When the ambient space \( B \) is \( \mathbb{R}^n \), we omit the subscript \( B \). We denote \( \text{cl}(A) - A \) (the frontier of \( A \)) with \( \partial A \).

### 2.3 Whitney stratification

Every semi-algebraic set \( A \subseteq \mathbb{R}^n \) can be “nicely” decomposed in a finite sequence \( Z \) of \( n+1 \) semi-algebraic sets \( Z_0, \ldots, Z_n \), called \textit{strata}, with the following properties. For each \( i = 0, \ldots, n \):

1. \( Z_i \) is either a \( C^1 \) semi-algebraic set in \( \mathbb{R}^n \) of dimension \( i \), or an empty set; and

2. each triple \((Z_i, \vec{p}, Z_j) \) for \( i < j \) and \( \vec{p} \in Z_i \) has the \textit{Whitney property}.

Such a decomposition is called a \( C^1 \)-Whitney stratification of \( A \). We will not need the precise definitions of when a set is \( C^1 \) and of the Whitney property. We refer to the paper of Rannou [12] for more details.

We remark that in this paper we do not require the \textit{frontier condition}, which says that the frontier of a stratum is the union of lower dimensional strata, and also do not suppose strata to be connected. Both properties are not necessary for Thom’s first isotopy lemma [3, 14], which we will use in our proof in Section 4.

### 3 Constructing a Whitney stratification

Let \( A \) be a semi-algebraic set in \( \mathbb{R}^n \). We shall construct a \( C^1 \)-Whitney stratification \( Z \) of the closure \( \text{cl}(A) \) such that \( A \) is the union of connected components of strata of \( Z \). We then say that \( Z \) is \textit{compatible with} \( A \). This construction will be expressible in FO.
Our construction is an adaptation of the construction given by Shiota [14, Lemma 1.2.2].

We define $Z_n$ as the subset of $A$ where $A$ is locally $C^1$ and of dimension $n$. Now suppose that the strata $Z_n, \ldots, Z_{k+1}$ have already been constructed. Then the stratum $Z_k$ is constructed as follows. Define $A_0 = A$ and $A_1 = \partial A$. For $i = 0, 1$ construct

\[ R^i_k := \{ \vec{p} \in A_i - \bigcup_{j=k+1} Z_j \mid A_i \text{ is } C^1 \text{ and of dimension } k \} \quad \text{in a neighborhood of } \vec{p} \]  

\[ W^i_k := \bigcap_{j=k+1} \text{int} R^i_k (\{ \vec{p} \in R^i_k \mid (R^i_k, \vec{p}, Z_j) \text{ has the Whitney property} \}) \]  

\[ Z^i_k := W^i_k - \text{cl}(R^{1-i}_k). \]

Then we define $Z_k := Z^0_k \cup Z^1_k$.

The stratum $Z_k$ has indeed the desired properties: By (1) it is $C^1$ and of dimension $k$, (2) guarantees that for all points in $Z_k$, and for any $j > k$, the triples $(Z_k, \vec{p}, Z_j)$ have the Whitney property, and (3) ensures that the connected components of $Z_k$ lie either completely in $A$ or completely in $\partial A$.

It is well known [17, 14] that the dimension of $A_i - \bigcup_{j=k+1}^n Z_j$ is strictly less than the dimension of $A_i - \bigcup_{j=k+1}^n Z_j$ for $i = 0, 1$. Hence, the stratification $Z$ will consists of exactly $n + 1$ strata $Z_k$, some of which may be empty.

We now show that the above construction is in FO.

**Theorem 1.** Let $S$ be a database schema containing a relation name $S$ of arity $n$. The $n$-ary query $Q_{k\text{-stratum}}$, which takes as input a database $D$ over $S$, and returns the $k$th stratum $Z_k$ of the stratification $Z$ constructed above for $A = S^D$, is expressible in FO.

In order to prove FO-expressibility of these queries, it is sufficient to show that the sets $R^i_k$, $W^i_k$, and $Z^i_k$ occurring in the construction are FO-expressible. But this follows immediately from the work of Rannou [12]. Indeed, from that work we can deduce the following lemma, which immediately implies Theorem 1:

**Lemma 1.** (i) Let $S$ be a database schema containing a relation name $S$ of arity $n$. The $n + 1$ queries of arity $n$, defined as

\[ Q_{k\text{-reg}}(D) := \{ x \in \mathbb{R}^n \mid x \in S^D \land (S^D \text{ is } C^1 \text{ and of dimension } k \text{ in an open neighborhood of } x) \}, \]

for any database $D$ over $S$, are all expressible in FO.

(ii) Let $S$ be a database schema containing two relation names $S_1$ and $S_2$ of arity
n. The $n$-ary query, defined as

$$Q_{Whitney}(D) := \{ \bar{x} \in \mathbb{R}^n \mid S^D_1, S^D_2 \text{ are } C^1 \text{ and } (S^D_1, \bar{x}, S^D_2) \text{ has the Whitney property} \}.$$  

for any database $D$ over $\mathcal{S}$, is expressible in FO.

4 Expressing the cone radius in FO

We are now ready to prove the main result of this paper.

**Theorem 2.** There exists an FO-expressible cone radius query.

**Proof.** Consider a semi-algebraic set $A$ in $\mathbb{R}^n$, and let $\mathcal{Z}$ be the $C^1$-Whitney stratification of $\text{cl}(A)$ compatible with $A$. Let $\bar{p} \in \text{cl}(A)$ and define the $C^1$-map

$$f_{\bar{p}} : \text{cl}(A) \rightarrow \mathbb{R} : \bar{x} \mapsto \|\bar{x} - \bar{p}\|^2.$$

We will need Thom’s First Isotopy Lemma [14]. Applied to the $C^1$-map $f_{\bar{p}}$ and the $C^1$-Whitney stratification $\mathcal{Z}$, this lemma can be stated as follows: For any $a < b$,

(a) If $f_{\bar{p}}$ is proper, i.e., $f_{\bar{p}}^{-1}([a, b])$ is compact, and

(b) if for each stratum $Z \in \mathcal{Z}$, the restriction

$$f_{\bar{p}}| (Z \cap \text{int}(B^n(\bar{p}, b) - B^n(\bar{p}, a))) \rightarrow (a, b) \subseteq \mathbb{R}$$

has no critical points (this will be explained later),

then for any $c \in (a, b)$, there exists a homeomorphism

$$h : \text{cl}(A) \cap \text{int}(B^n(\bar{p}, b) - B^n(\bar{p}, a)) \rightarrow (\text{cl}(A) \cap S^{n-1}(\bar{p}, c)) \times (a, b).$$

Moreover, this homeomorphism satisfies the following two properties:

(i) For each $d \in (a, b)$, $h(\text{cl}(A) \cap S^{n-1}(\bar{p}, d)) = (\text{cl}(A) \cap S^{n-1}(\bar{p}, c)) \times \{d\}$, and

(ii) $h(Z \cap \text{int}(B^n(\bar{p}, b) - B^n(\bar{p}, a))) = (Z \cap S^{n-1}(\bar{p}, c)) \times (a, b)$ is a homeomorphism for every connected component $Z$ of a stratum of $\mathcal{Z}$.

This statement of Thom’s First Isotopy Lemma is a specialized form of Theorem II.6.2 in Shiota [14].

Remark that condition (a) is automatically satisfied. Indeed, the inverse image by $f_{\bar{p}}$ of any interval $[a, b] \subseteq \mathbb{R}$ is equal to $\text{cl}(A) \cap (B^n(\bar{p}, b) - \text{int}(B^n(\bar{p}, a)))$ which is closed and bounded in $\mathbb{R}^n$. 

5
**Claim 1.** If condition (b) is satisfied for $0 < b$ (so $a = 0$), then every $c \in (0, b)$ is a cone radius of $A$ in $\vec{p}$.

**Proof of Claim.** Take an arbitrary real number $c \in (0, b)$. The lemma gives a homeomorphism

$$h_0 : \text{cl}(A) \cap \text{int}(B^n(\vec{p}, b) - \{\vec{p}\}) \rightarrow (\text{cl}(A) \cap S^{n-1}(\vec{p}, c)) \times (0, b).$$

By property (i), we obtain a homeomorphism

$$h_1 : \text{cl}(A) \cap (B^n(\vec{p}, c) - \{\vec{p}\}) \rightarrow (\text{cl}(A) \cap S^{n-1}(\vec{p}, c)) \times (0, c],$$

which equals the restriction $h_0|\text{cl}(A) \cap (B^n(\vec{p}, c) - \{\vec{p}\})$. Since the cylinder $(\text{cl}(A) \cap S^{n-1}(\vec{p}, c)) \times (0, c]$ is clearly homeomorphic to the cone $\text{Cone}(\text{cl}(A) \cap S^{n-1}(\vec{p}, c), \vec{p}) - \{\vec{p}\}$, e.g., by the homeomorphism

$$h_2(\vec{x}, t) := (1 - \frac{t}{c})\vec{p} + (\frac{t}{c})\vec{x} \quad \text{for } t \in (0, c],$$

we obtain a homeomorphism

$$h_3 := h_2 \circ h_1 : \text{cl}(A) \cap (B^n(\vec{p}, c) - \{\vec{p}\}) \rightarrow \text{Cone}(\text{cl}(A) \cap S^{n-1}(\vec{p}, c), \vec{p}) - \{\vec{p}\}.$$

It is easily verified that $h_2 : (Z \cap S^{n-1}(\vec{p}, c)) \times (0, c] \rightarrow \text{Cone}(Z \cap S^{n-1}(\vec{p}, c), \vec{p}) - \{\vec{p}\}$ is a homeomorphism for each connected component $Z$ of a stratum of $Z$. Since $h_1$ also satisfies property (ii), we have that $h_3 : (Z \cap (B^n(\vec{p}, c) - \{\vec{p}\})) \rightarrow \text{Cone}(Z \cap S^{n-1}(\vec{p}, c), \vec{p}) - \{\vec{p}\}$ is a homeomorphism for each connected component $Z$ of a stratum of $Z$.

This implies that the restriction $h = h_3|A \cap (B^n(\vec{p}, c) - \{\vec{p}\})$ is a homeomorphism from $h(A \cap (B^n(\vec{p}, c) - \{\vec{p}\})$ to $\text{Cone}(A \cap S^{n-1}(\vec{p}, c), \vec{p}) - \{\vec{p}\}$, because $A$ is the union of connected components of strata of $Z$.

The homeomorphism $h$ can easily be extended to the point $\vec{p}$, hence $c$ is indeed a cone radius as desired. \qed

Let $S$ be a schema containing a relation name $S$ of arity $n$, and let $D$ be a database over $S$. By Claim 1, we can define the following cone radius query

$$Q_{\text{radius}}(D) := \{ (\vec{p}, r) \in \mathbb{R}^n \times \mathbb{R} \mid \vec{p} \in S^D \text{ and } r \in (0, b) \},$$

where the interval $(0, b)$ satisfies condition (b) for the map $f_{\vec{p}}$ and semi-algebraic set $A = S^D$. Let us express this query in FO.

We define the critical point query as

$$Q_{\text{crit}}(D) := \{ (\vec{p}, \vec{x}) \in \mathbb{R}^n \times \mathbb{R}^n \mid \vec{p} \in S^D \text{ and } \vec{x} \in Q_{k-\text{stratum}}(D) \text{ for a certain } k \}
\text{ and } \vec{x} \text{ is a critical point of } f_{\vec{p}} \text{ restricted to } Q_{k-\text{stratum}}(D) \}.$$

The **critical points** of $f_{\vec{p}}$ restricted to a stratum $Z$, are the points $\vec{x} \in Z$ where the differential map $d_{\vec{x}}(f_{\vec{p}}|Z)$ is not surjective.
Claim 2. A point $\vec{x} \in \mathbb{R}^n$ is a critical point of $f_{\vec{p}}|Z$ if and only if the tangent space of $Z$ in $\vec{x}$ is orthogonal to the vector $\vec{x} - \vec{p}$.

Proof of Claim. We compute the differential $d_{\vec{x}}(f_{\vec{p}}|Z)$ as follows: Locally around $\vec{x}$, we may assume that the projection on the first $k$ coordinates $\Pi : Z \to U \subset \mathbb{R}^k$, is a homeomorphism, where $k$ is the dimension of $Z$ in $\vec{x}$. By definition of the differential, $d_{\vec{x}}(f_{\vec{p}}|Z) = (d_{(x_1, \ldots, x_k)}g)(d_{(x_1, \ldots, x_k)}\Pi^{-1})^{-1}$, where $g = (f|Z) \circ \Pi^{-1}$. By the $C^1$ Inverse Function Theorem, we may assume that $\Pi^{-1} : U \to Z : (x_1, \ldots, x_k) \mapsto (x_1, \ldots, x_k, \varphi_{k+1}, \ldots, \varphi_n)$, where $\varphi_i(x_1, \ldots, x_k)$ are $C^1$-maps, and hence $g : U \to \mathbb{R} : (x_1, \ldots, x_k) = \sum_{i=1}^k (x_i - p_i)^2 + \sum_{j=k+1}^n (\varphi_j(x_1, \ldots, x_k) - p_j)^2$. An elementary calculation shows that the differential of $f_{\vec{p}}|Z$ in $\vec{x}$ is the vector

$$d_{\vec{x}}(f_{\vec{p}}|Z) = 2 \left( (x_i - p_i) + \sum_{j=k+1}^n (x_j - p_j) \frac{\partial \varphi_j}{\partial x_i}(x_1, \ldots, x_k) \right)_{i=1, \ldots, k \text{ n-k times}},$$

Since $d_{(x_1, \ldots, x_k)}\Pi^{-1}$ is an isomorphism between the tangent space $T_{(x_1, \ldots, x_k)}U$ of $U$ in the projection $\Pi(\vec{x})$, and the tangent space $T_{\vec{x}}Z$ of $Z$ in $\vec{x}$, any tangent vector $(v_1, \ldots, v_n) \in T_{\vec{x}}Z$ is of the form $(d_{(x_1, \ldots, x_k)}\Pi^{-1})(v_1, \ldots, v_k)$. More specifically, any tangent vector $\vec{v} \in T_{\vec{x}}Z$ can be written as

$$(v_1, \ldots, v_n) = (v_1, \ldots, v_k, \sum_{i=1}^k \frac{\partial \varphi_{k+1}}{\partial x_i}(x_1, \ldots, x_k)v_i, \ldots, \sum_{i=1}^k \frac{\partial \varphi_n}{\partial x_i}(x_1, \ldots, x_k)v_i).$$

Hence, the product

$$d_{\vec{x}}(f_{\vec{p}}|Z) \cdot \vec{v} = 2 \sum_{i=1}^k (x_i - p_i)v_i + 2 \sum_{j=k+1}^n (x_j - p_j) \left( \sum_{i=1}^k \frac{\partial \varphi_j}{\partial x_i}(x_1, \ldots, x_k)v_i \right),$$

is equal to $2 \sum_{i=1}^n (x_i - p_i)v_i$. This implies that the differential map $d_{\vec{x}}(f_{\vec{p}}|Z)$ is not surjective if and only if $2 \sum_{i=1}^n (x_i - p_i)v_i = 0$ for all tangent vectors $\vec{v} \in T_{\vec{x}}Z$. This proves the Claim. 

The proof of the theorem now continues as follows. The tangent space query

$$Q_{\text{tangent}}(D) := \{(\vec{x}, \vec{v}) \in \mathbb{R}^n \times \mathbb{R}^n \mid S^D \text{ is } C^1, \ \vec{x} \in S^D \text{ and } \vec{v} \in T_{\vec{x}}S^D \},$$

is expressible in FO [12, Lemma 2]. Because the orthogonality of two vectors can be easily expressed in FO, the formula

$$\varphi_{\text{crit}}(\vec{p}, \vec{x}) := S(\vec{p}) \land \bigvee_{j=0}^n \forall \vec{v} \left( \varphi_{\text{tangent}}(\varphi_{\text{j-stratum}}(S))(\vec{x}, \vec{v}) \rightarrow (\vec{x} - \vec{p}) \cdot \vec{v} = 0 \right)$$

is valid.
expresses $Q_{\text{crit}}$ correctly by Claim 2. Here, $\varphi_{j\text{-stratum}}$ denotes an FO-formula expressing $Q_{j\text{-stratum}}$ for $j = 0, \ldots, n$, and $\varphi_{\text{tangent}}$ is an FO formula expressing $Q_{\text{tangent}}$.

A critical value of $f_{\vec{p}}$ is the image by $f_{\vec{p}}$ of a critical point. The query which returns the set of critical values is expressible in FO by the formula

$$\varphi_{\text{val}}(\vec{p}, r) := \exists \vec{x} (\varphi_{\text{crit}}(\vec{p}, \vec{x}) \land r = f_{\vec{p}}(\vec{x})).$$

We now observe that $\{r \in \mathbb{R} \mid \varphi_{\text{val}}(\vec{p}, r)\}$ is finite for each $\vec{p}$. Indeed, the set of critical points $\{\vec{x} \in \mathbb{R}^n \mid \varphi_{\text{crit}}(\vec{p}, \vec{x})\}$ is semi-algebraic and hence admits a $C^1$-cell decomposition $C = \{C_1, \ldots, C_m\}$ such that $f|C_i$ is $C^1$ \cite{16}. Sard’s Theorem for $C^1$-maps \cite{15} implies that each $f_{\vec{p}}|C_i$ attains only a finite number of values. Hence the image by $f_{\vec{p}}$ of the set of critical points is finite.

This implies that either there are no critical values, or there exists a minimal critical value. By Claim 1, any value smaller than this minimal value is a cone radius. We therefore conclude that the query expressed in FO as

$$\varphi_{\text{radius}}(\vec{p}, r) := (\forall r')(\varphi_{\text{val}}(\vec{p}, r') \rightarrow r < r'),$$

is a cone radius query, as desired. \hfill \Box

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