Continuity of States on Non-Unital Differential Algebras in Loop Quantum Cosmology

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Abstract: In a recent paper Engle et al. (Commun Math Phys 354:231–246, 2017) showed that there is a unique state on the reduced holonomy–flux ∗-algebra of homogeneous isotropic loop quantum cosmology, that is invariant under residual diffeomorphisms. This result has been claimed to be true both for the Ashtekar–Bojowald–Lewandowski framework and for that introduced by the present author. Unfortunately, the uniqueness proof relies on an incorrect argument which spoils the second case. In our short note, we are going to patch this issue, this way keeping the nice uniqueness result in both cases. Moreover, we will even extend the underlying operator algebraic statements as this might help later for studying higher-dimensional models.

1. Introduction

Representation theory has turned out indispensable for many mathematically rigorous quantum theories. Particularly strong statements come from uniqueness results like the celebrated Stone–von Neumann theorem, giving uniqueness in quantum mechanics, or the recent results in loop quantum gravity on the holonomy–flux [4] as well as on the Weyl algebra [3]. So it comes with no surprise that one is looking for their counterparts also in the realm of loop quantum cosmology. Indeed, Engle et al. have recently claimed [2] that there is a unique invariant state on the holonomy–flux ∗-algebra also for homogeneous isotropic cosmologies. Here, invariance is understood w.r.t. so-called residual diffeomorphisms, i.e., those diffeomorphisms that do not destroy the symmetry (here: homogeneity and isotropy, taking their action on the fiducial cell into account). Although we believe that their uniqueness result is correct, its proof contains a flaw concerning the continuity of some state. To explain the problem, let us consider a state ω on some ∗-subalgebra D of some abelian C∗-algebra C = Cb(X). To prove continuity of ω, Engle et al. used that it is sufficient to show (see below) that √1 + ϕ is in D for each real-valued ϕ ∈ D with ∥ϕ∥∞ < 1. Unfortunately, √1 + ϕ ∈ D implies that D is unital. Unitality, however, is not given for D = C0(ℝ) which is the second case considered...
in [2] and is needed to prove uniqueness for the embeddable loop quantum cosmology case.

Fortunately, it is not very difficult to modify this step in the proof without modifying the ultimate uniqueness claim as we will show in this short notice. We will prove results that are somewhat more general than needed for just closing the gap in [2] as these extensions might become useful for investigations of models with more degrees of freedom like Bianchi I.

2. Relation to the Engle–Hanusch–Thiemann Paper

In the whole article, let there be

\[ M \ldots \text{some open set in a Banach space} \]
\[ \mathfrak{A} \ldots \text{some } *\text{-algebra} \]
\[ \mathfrak{B} \ldots \text{some unital Banach } *\text{-algebra} \]
\[ \mathfrak{C} \ldots \text{some Banach } *\text{-subalgebra of } C_b(M, \mathfrak{B}) \]

Note that we assume the norm on \( \mathfrak{B} \) to fulfill \( \| b^* \| = \| b \| \) for all \( b \in \mathfrak{B} \). Moreover, \( C_b(M, \mathfrak{B}) \) denotes the set of bounded continuous functions from \( M \) to \( \mathfrak{B} \) and is equipped with the usual supremum norm. Finally, observe that our results comprise, in particular, the situation

\[ M \ldots \mathbb{R} \]
\[ \mathfrak{A} \ldots \text{quantum reduced holonomy–flux } *\text{-algebra} \]
\[ \mathfrak{B} \ldots \mathfrak{C} \]
\[ \mathfrak{C} \ldots \text{either } C_0(\mathbb{R}) \text{ or } C_{\text{AP}}(\mathbb{R}) \text{ or } C_0(\mathbb{R}) \oplus C_{\text{AP}}(\mathbb{R}) \]

which is exactly the situation studied by Engle et al. in [2].

3. States

**Definition 3.1.** A state on \( \mathfrak{A} \) is a \( *\)-linear functional \( \omega : \mathfrak{A} \rightarrow \mathbb{C} \) which is weakly positive, i.e. it fulfills

\[ \omega(a^*a) \geq 0 \quad \text{for all } a \in \mathfrak{A}. \]

For our purposes, we do not require a state to be normalized (i.e. to fulfill \( \omega(1) = 1 \)) as soon as \( \mathfrak{A} \) is unital.

4. Engle–Hanusch–Thiemann Strategy

Let us recall the main argument in question (somewhat adapted to our notation). In Subsection “Continuity” of Section 4 of [2], Engle et al. considered \( \mathfrak{d} \) to contain those functions in \( C_0(\mathbb{R}) \) or \( C_{\text{AP}}(\mathbb{R}) \) that are smooth and any of their derivatives are in \( C_0(\mathbb{R}) \) or \( C_{\text{AP}}(\mathbb{R}) \), respectively, again. Moreover, there is a \( *\)-homomorphism \( I \) from \( \mathfrak{d} \) to the reduced holonomy–flux \( *\)-algebra \( \mathfrak{A} \) and a state \( \omega \) on \( \mathfrak{A} \). Now, the authors from [2] claim that this already implies that \( \hat{\omega} := \omega \circ I \) is continuous. The idea for the proof was to use that \( \sqrt{1 \pm t} \) is analytic for \( |t| < 1 \), whence \( 1 \pm \varphi \) for real-valued \( \varphi \) on \( \mathbb{R} \) can be written (either by functional calculus or even more direct by Taylor expansion) as \( \psi \psi \) with \( \psi := \sqrt{1 \pm \varphi} \in \mathfrak{d} \), provided \( \| \varphi \|_{\infty} < 1 \). Now, the state property implies
\( \omega(1 \pm \varphi) = \omega(\overline{\varphi}) = \omega(I(\psi)I(\psi)) \geq 0 \). A straightforward argument shows now that \( \omega(\varphi) \) is at most 1 for normalized \( \omega \).

At a first glance, the proof above is nice and complete. However, the existence of \( \psi \) is only given for \( d \) being \( C_{\text{AP}}(\mathbb{R}) \), but not for \( C_0(\mathbb{R}) \). In fact, in the latter case, \( \psi \) goes to 1 at infinity, but not to 0 as required. More abstractly, the argument only goes through if \( d \) is unital itself. That unitality is indeed needed in general, will be shown in Sect. 5 where we construct a non-continuous state in a non-unital example.

Nevertheless, fixing that issue in [2] is not very difficult. Indeed, in the situation of [2], the overall framework is unital. More concretely, the relevant commutative algebra (denoted by \( D \) there) is in both cases unital, namely \( C_{\text{AP}}(\mathbb{R}) \) or \( C_0(\mathbb{R}) \oplus C_{\text{AP}}(\mathbb{R}) \), respectively, after completion. The solution is now just to “steal” the unit from the upper level in order to get the desired continuity. This will be done in the proof of Proposition 8.1.

5. Example of Non-Continuous State

Before going to do this, let us construct a non-continuous state on some normed \(*\)-algebra.

For this, let us consider the product space \( X \) of countably many unit intervals \([0, 1]\). Then the canonical projections \( x_n : X \rightarrow [0, 1] \) are, of course, bounded continuous functions, i.e., \( x_n \in C_b(X) \). Denote by \( D \) the \(*\)-subalgebra of \( C_b(X) \) generated by all these \( x_n \). It is clear that the monomials, i.e. the finite products of \( x_n \) with repetitions admitted, form a vector space basis for \( D \). Note that the unit function 1 is not contained in \( D \). Now, we define \( \omega : D \rightarrow \mathbb{C} \) to be the linear functional that maps \( x_n \) to \( n \) and any monomial of degree 2 or more to 0. Obviously, \( \omega \) is linear. It is even a state. In fact, if \( f \in D \), then \( f^*f \) is a sum of monomials of degree at least 2, giving \( \omega(f^*f) \equiv 0 \). On the other hand, we see that \( \|x_n\|_{\infty} = 1 \) for all \( n \), but \( \omega(x_n) = n \), giving non-continuity of \( \omega \).

6. Differential Algebra

Let us now come back to the general situation. We introduce

**Definition 6.1.** The **differential algebra** \( D(\mathcal{C}) \) of \( \mathcal{C} \) is given by

\[
D(\mathcal{C}) := \{ \varphi \in \mathcal{C} \mid \text{all partial derivatives (of any order) of } \varphi \text{ are in } \mathcal{C} \}.
\]

Note that we consider partial derivatives w.r.t. subspaces of \( M \) with real dimension 1. Moreover, \( \varphi \in D(\mathcal{C}) \) tacitly includes the assumption that all the partial derivatives of \( \varphi \) exist. Obviously, we have

**Lemma 6.1.** \( D(\mathcal{C}) \) is a \(*\)-subalgebra of \( \mathcal{C} \).

**Remark.** 1. For homogeneous isotropic cosmology, \( D(C_{\text{AP}}(\mathbb{R})) \) and \( D(C_0(\mathbb{R})) \) correspond to \( d_{\text{AP}} \) and \( d_0 \), respectively, in [2].

2. For Bianchi I, the differential algebra \( D(C_{\text{AP}}(\mathbb{R}^3)) \) corresponds to \( D \) in Section 5 of [2] for the standard LQC configuration space. As for the isotropic case, this differential algebra is dense in \( \mathcal{C} \) [2].

The standard LQC description of Bianchi I, however, comes with a drawback: there the LQC configuration space is not embeddable into that of full LQG as long as
all (piecewise) analytic paths are considered in the full theory. In order to fix this issue, a modification [3] of the LQC configuration space has been introduced that is canonical and guarantees embeddability. More precisely, \( \mathcal{C} \) is then generated by all parallel transport matrix functions of homogeneous connections along analytic paths (not just axis-parallel straight lines). Unfortunately, in this “embeddable” LQC version it is an open question whether the differential algebra is dense in \( \mathcal{C} \). At a first glance, this appears to be a surprise as \( \mathcal{C} \) is generated by smooth functions; indeed, the underlying differential equation depends analytically on the parameters \( c_1, c_2 \) and \( c_3 \). But, it remains unknown whether the partial derivatives are in \( \mathcal{C} \) again. Even worse: to the best of our knowledge, the explicit form of \( \mathcal{C} \) remains unknown (as already mentioned in [2]).

7. Closedness of \( \mathcal{D}(\mathcal{C}) \) under Analytic Functions

Extending the arguments of [2], we get

**Lemma 7.1.** Let \( f \) be analytic at 0 with convergence radius \( r \) and \( f(0) = 0 \). Then \( f \circ \varphi \in \mathcal{D}(\mathcal{C}) \) for all \( \varphi \in \mathcal{D}(\mathcal{C}) \) with \( \|\varphi\|_\infty < r \).

**Remark.** 1. By means of holomorphic functional calculus,\(^1\) \( f \) above is to be understood also as a mapping from (some subspace of) the Banach algebra \( \mathcal{B} \) to itself. In our particular situation, we simply have \( f(b) = \sum_k c_k b^k \) for \( f(z) = \sum_k c_k z^k \) with \( c_k \in \mathbb{C} \) and \( \|b\| < r \); unless otherwise noted, the index \( k \) is running over \( \mathbb{N} \). Moreover, the series converges uniformly on any \( B_\rho \) with \( \rho < r \); this is true for both \( f \) on \( \mathbb{C} \) and \( f \) on \( \mathcal{B} \).

2. Note that the lemma above is no longer true if we drop the condition \( f(0) = 0 \) in the non-unital case. In fact, the simplest case \( f \equiv 1 \), which is perfectly analytic, gives \( f \circ \varphi \equiv 1 \) for all \( \varphi \in \mathcal{D}(\mathcal{C}) \). But, 1 is not in \( \mathcal{C} \supseteq \mathcal{D}(\mathcal{C}) \) in the non-unital case.

3. In the unital case, one easily sees that the assumption \( f(0) = 0 \) can indeed be dropped. Setting \( g(z) := f(z) - f(0) \), we see that \( g(0) = 0 \), hence \( g \circ \varphi \in \mathcal{D}(\mathcal{C}) \) for all \( \varphi \in \mathcal{D}(\mathcal{C}) \). Now, \( f \circ \varphi \equiv g \circ \varphi + f(0) \cdot 1 \) is in \( \mathcal{D}(\mathcal{C}) \) as well.

**Proof.** Choose \( c_k \in \mathbb{C} \) such that \( f(z) = \sum_k c_k z^k \) for \( |z| < r \). From \( f(0) = 0 \), we get \( c_0 = 0 \). As \( \mathcal{D}(\mathcal{C}) \) is an algebra, it contains with \( \varphi \) also \( \varphi^k \) for all \( k \geq 1 \). Now, \( f \circ \varphi \in \overline{\mathcal{D}(\mathcal{C})} \) as for \( \|\varphi\|_\infty < r \)

\[
\left\| f \circ \varphi - \sum_{k=1}^{n} c_k \varphi^k \right\|_\infty \equiv \left\| \sum_{k=n+1}^{\infty} c_k \varphi^k \right\|_\infty \leq \sum_{k=n+1}^{\infty} |c_k| \|\varphi\|_\infty^k \to 0.
\]

Consequently, as \( \mathcal{C} \) is Banach, we have \( f \circ \varphi \in \mathcal{C} \). It remains to prove that any of its partial derivatives is in \( \mathcal{C} \) again. For this, consider \( h(z) := f(z) - f(0) \), which is analytic for \( |z| < r \) and fulfills \( h(0) = 0 \). Hence, \( h \circ \varphi \) is in \( \mathcal{C} \) as shown above. As \( \partial_{\alpha} \varphi \) is in \( \mathcal{C} \) by assumption, we have

\[
\partial_{\alpha} [f \circ \varphi] = (f \circ \varphi) \cdot \partial_{\alpha} \varphi = (h \circ \varphi) \cdot \partial_{\alpha} \varphi + f(0) \partial_{\alpha} \varphi \in \mathcal{C}.
\]

Inductively, we see that any partial derivative of \( f \circ \varphi \) is in \( \mathcal{C} \). \( \square \)

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\(^1\) As suggested by one of the referees, we have added a very short survey on holomorphic functional calculus and its ideas in Appendix A, in order to make the paper more self-contained.
8. Continuity Criterion

Proposition 8.1. Let

- ω be a state on the unital *-algebra \( \mathcal{A} \);
- \( I : \mathcal{D}(\mathcal{C}) + \mathbb{C} \mathbf{1} \longrightarrow \mathcal{A} \) be a unital *-homomorphism.

Then \( \omega \circ I \) is continuous with norm \( \omega(I(\mathbf{1})) \).

The following proof is inspired by the corresponding proof for unital \( \mathcal{C} \) in [2].

Proof.  

- Obviously, \( \hat{\omega} := \omega \circ I : \mathcal{D}(\mathcal{C}) + \mathbb{C} \mathbf{1} \longrightarrow \mathbb{C} \) is *-linear.
- Let \( \varphi \in \mathcal{D}(\mathcal{C}) \) and \( r > \| \varphi \|_\infty \). Define \( g(z) := \sqrt{r \pm z} \) and \( f(z) := g(z) - g(0) \).
- As \( f \) is analytic at 0 with convergence radius \( r \) and \( f(0) = 0 \), we see from Lemma 7.1 that \( f \circ \varphi \) is in \( \mathcal{D}(\mathcal{C}) \) again, hence \( g \circ \varphi \equiv f \circ \varphi + \sqrt{r} \mathbf{1} \in \mathcal{D}(\mathcal{C}) + \mathbb{C} \mathbf{1} \).
- Let now additionally \( \varphi^* = \varphi \).
  - As \( g \) is real on \( (-r, r) \), we have \( (g \circ \varphi)^* = g \circ \varphi \).
  - From \( (g \circ \varphi)^* \cdot (g \circ \varphi) = (g \circ \varphi)^2 = g^2 \circ \varphi = r \mathbf{1} \pm \varphi \), we get

\[
\rho \hat{\omega}(\mathbf{1}) \pm \hat{\omega}(\varphi) = \hat{\omega}(r \mathbf{1} \pm \varphi) = \hat{\omega}(g \circ \varphi)^* \cdot (g \circ \varphi) = \omega(I(g \circ \varphi)^* \cdot I(g \circ \varphi)) \geq 0.
\]

- From \( \hat{\omega}(\mathbf{1}) = \hat{\omega}(\mathbf{1}^* \mathbf{1}) = \omega(I(\mathbf{1})^* I(\mathbf{1})) \geq 0 \), we get hence

\[
|\hat{\omega}(\varphi)| \leq \hat{\omega}(\mathbf{1}) \| \varphi \|_\infty,
\]

as \( r > \| \varphi \|_\infty \) has been arbitrary.

- Let us now drop the reality assumption on \( \varphi \) and choose \( \lambda \in U(1) \) with \( \lambda \hat{\omega}(\varphi) \in \mathbb{R} \).
- Then \( \hat{\omega}(\varphi) = \hat{\omega}(\lambda \varphi) = \hat{\omega}(\text{Re}[\lambda \varphi]) + i\hat{\omega}(\text{Im}[\lambda \varphi]) \)

implies \( \hat{\omega}(\text{Im}[\lambda \varphi]) = 0 \), whence by \( \| \varphi^* \|_\infty = \| \varphi \|_\infty \)

\[
|\hat{\omega}(\varphi)| = |\hat{\omega}(\text{Re}[\lambda \varphi])|
\]

\[
\leq \hat{\omega}(\mathbf{1}) \| \text{Re}[\lambda \varphi] \|_\infty \leq \hat{\omega}(\mathbf{1}) \| \lambda \varphi \|_\infty = \hat{\omega}(\mathbf{1}) \| \varphi \|_\infty.
\]

The statement on the norm of \( \omega \circ I \) is now obvious. \( \square \)

Remark.  

- Of course, \( \omega \circ I \) restricted to \( \mathcal{D}(\mathcal{C}) \) is continuous as well. Note, however, that for non-unital \( \mathcal{C} \) the norm of this restriction need no longer be \( \omega(I(\mathbf{1})) \); in general, it is smaller. In fact, consider \( \mathcal{C} := C_0(\mathbb{R}) \) and \( \mathcal{A} := \mathcal{C} + \mathbb{C} \mathbf{1} \subseteq C_b(\mathbb{R}) \) together with \( \omega(c + \mathbf{1}) := \lambda \). Obviously, \( \omega \) is a state on \( \mathcal{A} \) that vanishes on \( \mathcal{C} \), hence on \( \mathcal{D}(\mathcal{C}) \) as well. Thus, if \( I \) is the usual embedding, we get \( \omega \circ I|_{\mathcal{D}(\mathcal{C})} \equiv 0 \), in contrast to \( \omega(I(\mathbf{1})) = 1 \).
- Our proof gives a sharper and more general bound for the norm than that given in the subsection on continuity in [2]. Indeed, on the one hand, we do no longer require that \( \omega \) is normalized nor that \( \mathcal{C} \) is unital; on the other hand, we were able to drop the factor 2 in [2]. The argument above allows also to remove the factor 2 in the estimate in Lemma 3.2 in [4]. There the authors conjectured that this might be possible, but as it had been irrelevant for their overall result (as it had in [2]), they refrained from proving it.

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2 Note that we have chosen the square-root to be holomorphic on \( \mathbb{C} \setminus (-\infty, 0] \) and positive on \( (0, \infty) \).

3 As usual, we let \( \text{Re} b := \frac{1}{2}(b + b^*) \) and \( \text{Im} b := \frac{1}{2i}(b - b^*) \).
9. Conclusions

The results derived above are surely not the maximal extension of those claimed in [2]. It is quite obvious, that $M$ can be replaced by a manifold or some locally convex space. But even more: a closer look to the proofs shows that actually the only things we have really needed have been the derivation properties of the partial derivatives $\partial_\alpha$ and some norm estimates. Also, holomorphic functional calculus is much more than just replacing $z$ in the Taylor series by some algebra element. So we expect that the findings above are just moderate extensions of [2].

One might now ask why we have chosen this intermediate level. Of course, we could have restricted ourselves just to the cases in [2]. Or we could have searched for the maximal extension. Well, the latter is hard to find (if it exists at all), so we decided to let us be guided by further applications to be expected in loop quantum cosmology. In particular, we are looking for information about models of more degrees of freedom like Bianchi I. There, to the best of our knowledge, the explicit form of the restriction algebra underlying the quantum configuration space is still completely unknown. It is just known that $\mathcal{C}$ is no longer a subset of $C_b(\mathbb{R})$, but of $C_b(\mathbb{R}^3)$. Nevertheless, continuity of states on $\mathcal{C}$ is still given as shown above—as soon as one can prove that $\mathcal{D}(\mathcal{C})$ is dense in $\mathcal{C}$.

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Appendix

A. Holomorphic Functional Calculus

Let $A$ be a unital Banach algebra (it need not have an involution).

The introduction of polynomials of Banach algebra elements is trivial: just take $f(a) := a^3 + 42a$ if $f(z) = z^3 + 42z$ is a complex polynomial in $z$. Surely, there is no confusion arising from denoting both functions by the same letter. If $f$ is holomorphic at 0, hence given locally around 0 by a power series $\sum c_n z^n$ with some coefficients $c_n \in \mathbb{C}$ and some convergence radius $r$, we can define $f(a)$ by $\sum c_n a^n$ for $\|a\| < r$. This works best if $f$ is entire (e.g. $f = \exp$), as then $f$ will be defined on full $A$. If, however, we are given a holomorphic function with a more sophisticated open domain $U \subset \mathbb{C}$, the construction is more subtle.

Its main idea comes from complex analysis, namely Cauchy’s integral formula

$$f(z) = \frac{1}{2\pi i} \int \frac{f(\zeta)}{\zeta - z} \, d\zeta.$$  

This beautiful identity allows to connect the value of $f$ at some $z \in \mathbb{C}$ with its values around $z$ or, better, its values around the set

$$\sigma(z) := \{ \zeta \in \mathbb{C} \mid \zeta - z \text{ not invertible in } \mathbb{C} \}.$$
The key observation is that even if \( z \in \mathbb{C} \) in the integral above is replaced by some \( a \in A \), this integral will still make sense. We only have to guarantee that (i) \( \zeta - a \) is invertible for any \( \zeta \) along \( \gamma \), (ii) \( \gamma \) circles around all points where \( \zeta - a \) is not invertible and (iii) \( f \) is analytic.

To make things precise, define the spectrum of \( a \in A \) by
\[
\sigma(a) := \{ \zeta \in \mathbb{C} | \zeta - a \text{ not invertible in } A \}
\]
and denote its complement by \( \varrho(a) \). One easily shows that
\[
\frac{1}{\zeta - a} = \sum_{n=0}^{\infty} (\zeta_0 - \zeta)^n \left( \frac{1}{\zeta_0 - a} \right)^{n+1}
\]
for any \( \zeta_0 \in \varrho(a) \) and any \( \zeta \) with \( |\zeta - \zeta_0| < \| (\zeta_0 - a)^{-1} \|^{-1} \). This immediately proves that \( \varrho(a) \) is open, hence \( \sigma(a) \) is compact by \( \sigma(a) \subseteq B_{\|a\|} \); moreover, \( \zeta \mapsto (\zeta - a)^{-1} \) is analytic on \( \varrho(a) \). Denoting now by \( C^\omega(a) \) the set of all complex functions \( f \) that are holomorphic on a neighbourhood of \( \sigma(a) \), we get

**Theorem A.1. Holomorphic Functional Calculus**

Let \( f \in C^\omega(a) \). For any closed path \( \gamma \) in \( \varrho(a) \) with \( \text{ind}_\gamma = 1 \) on \( \sigma(a) \), define
\[
f(a) := \frac{1}{2\pi i} \int_{\gamma} \frac{f(\zeta)}{\zeta - a} \, d\zeta.
\]
Then we have:

1. \( f(a) \) is well defined (i.e. it is in \( A \) and independent of the choice of \( \gamma \)).
2. \( f \mapsto f(a) \) is an algebra homomorphism between \( C^\omega(a) \) and \( A \).
3. \( f(z) = z^k \) implies \( f(a) = a^k \).
4. If \( f, f_n \) are holomorphic on some joint open \( U \supseteq \sigma(a) \), then
\[
f_n \to f \text{ compactly} \implies f_n(a) \to f(a).
\]

The first part is proven just using two facts: firstly, \( \frac{f(\zeta)}{\zeta - a} \) is analytic in \( \zeta \) which, in turn, is guaranteed by \( \gamma \subseteq \varrho(a) \); secondly, the path independence of the integral follows exactly as in complex analysis. For \( (f)g(a) = [fg](a) \) in the second part choose some paths \( \gamma \) for \( f \) and \( \delta \) for \( g \), such that \( \gamma \) lies completely in the interior of \( \delta \), and use some algebra, some complex analysis and the fact that double integrals of continuous functions on compact domains can be reshuffled arbitrarily. The third part is proven using some \( \gamma \) running over some \( \partial B_r \) with \( r > \|a\| \). The final part follows from
\[
2\pi \| f_n(a) - f(a) \| \leq \max_{\zeta \in \gamma} \| (\zeta - a)^{-1} \| \cdot \text{length}(\gamma) \cdot \max_{\zeta \in \gamma} |f_n(\zeta) - f(\zeta)|.
\]

In particular, it follows directly from the theorem that (with \( c_n \in \mathbb{C} \))
\[
f(z) = \sum_{n \in \mathbb{N}} c_n z^n \implies f(a) = \sum_{n \in \mathbb{N}} c_n a^n \quad \text{for } \|a\| < r
\]
where \( r \) denotes the convergence radius at 0 of the LHS complex function \( f \).

Further details, proofs, and applications can be found, e.g., in [1, starting Ch. VII.4].
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