Doubly singular matrix variate beta type I and II and singular inverted matricvariate \( t \) distributions

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Abstract
In this paper, the densities of the doubly singular beta type I and II distributions are found, and the joint densities of their corresponding nonzero eigenvalues are provided. As a consequence, the density function of a singular inverted matricvariate \( t \) distribution is obtained.

1 Introduction

Let \( A \) and \( B \) be independent Wishart matrices or, alternatively, let \( B = YY' \) where \( Y \) has a matrix variate normal distribution. Then the matrix variate beta type I distribution is defined in the literature as

\[
U = \begin{cases} 
(A + B)^{-1/2}B(A + B)^{-1/2}, \\
B^{1/2}(A + B)^{-1/2}B^{1/2}, \\
Y'(A + B)^{-1}Y.
\end{cases}
\] (1)

Analogously, the matrix variate beta type II distribution is defined as

\[
F = \begin{cases} 
A^{-1/2}BA^{-1/2}, \\
B^{1/2}A^{-1}B^{1/2}, \\
Y'A^{-1}Y.
\end{cases}
\] (2)

These can be classified as central, noncentral or doubly noncentral, depending on whether \( A \) and \( B \) are central, \( B \) is noncentral or \( A \) and \( B \) are noncentral, respectively; see Díaz-García and Gutiérrez-Jáimez

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In addition, such a distribution can be classified as nonsingular, singular or doubly singular, when $A$ and $B$ are nonsingular, $B$ is singular or $A$ and $B$ are singular, respectively, see Díaz-García and Gutiérrez-Jáimez (2008a). Under definitions (1) and (2) and their classifications, mentioned above, the matrix variate beta type I and II distributions have been studied by different authors; see Olkin and Rubin (1964), Dickey (1967), Mitra (1970), Khatri (1970), Srivastava and Khatri (1979), Muirhead (1982), Uhlig (1994), Cadet (1996), Díaz-García and Gutiérrez-Jáimez (1997, 2006, 2007, 2008a,b), among many others. However, the density functions of doubly singular beta type I and II distributions have not been studied. In such cases the inverses that appear in definitions (2) and (1) must be replaced by the Moore-Penrose inverse. The doubly singular beta type I distribution was briefly considered by Mitra (1970), but this distribution has been the object of considerable recent interest. Observe that this distribution appears in a natural way when the number of observations is smaller than the dimension in multivariate analysis of variance; see Srivastava (2007).

Furthermore, observe that if $Z$ is any random matrix $m \times r$, it is generically called as matrix variate if the kernel of its density function is in terms of the trace operator. But if the kernel of the density function is in terms of the determinant alone or in terms of both, then the determinant and the trace operator, $Z$, are generically called matricvariate, see Dickey (1967) and Díaz-García and Gutiérrez-Jáimez (2008c). Alternatively, when the random matrix $A : m \times m$ is symmetric by definition (e.g. Wishart and beta matrices, etc.), they are generically referred to as matrix variate.

In Section 2 of the present study, we review singular matricvariate $t$ distributions and determine the distribution of a linear transformation and the joint density function of its nonzero singular values. In Section 3, an expression is provided for the density function of the doubly singular matrix variate beta type II distribution, and the joint density function of its nonzero eigenvalues is determined.

Similar results are provided for the doubly singular matrix variate beta type I distribution; see Section 4. Finally, in Section 5, we study the inverted matricvariate $t$ distribution, also termed the matricvariate Pearson type II distribution.

### 2 Preliminary results

Let $\mathcal{L}_{r,m}(q)$ be the linear space of all $m \times r$ real matrices of rank $q \leq \min(m, r)$ and let $\mathcal{L}_{r,m}^+(q)$ be the linear space of all $m \times r$ real matrices of rank $q \leq \min(m, r)$, with $q$ distinct singular values. The set of matrices $H_1 \in \mathcal{L}_{r,m}$ such that $H_1'H_1 = I_r$ is a manifold denoted as $\mathcal{V}_{r,m}$, termed the Stiefel manifold. In particular, $\mathcal{V}_{m,m}$ is the group of orthogonal matrices $O(m)$. The invariant measure on a Stiefel manifold is given by the differential form

$$ (H_1'dH_1) \equiv \bigwedge_{i=1}^{r} \bigwedge_{j=i+1}^{m} h_i' d h_i $$

written in terms of the exterior product ($\bigwedge$), where we choose an $m \times (m - r)$ matrix $H_2$ such that $H$ is an $m \times m$ orthogonal matrix, with $H = (H_1'H_2)$ and where $dh$ is an $m \times 1$ vector of differentials; see Muirhead (1982, Section 2.1.4). Moreover

$$ \int_{H_1 \in \mathcal{V}_{r,m}} (H_1'dH_1) = \frac{2^r \pi^{m r/2}}{\Gamma_r[m/2]} \quad \text{and} \quad \int_{H \in O(m)} (H'dH) = \frac{2^m \pi^{m^2/2}}{\Gamma_m[m/2]} \quad (3) $$

Denote by $S_m$, the homogeneous space of $m \times m$ positive definite symmetric matrices; and by $S_m(q)$, the $(mq - q(q - 1)/2)$-dimensional manifold of rank $q$ positive semidefinite
Let \( \mathbf{T} \in \mathcal{L}_{r,m}(r_{\mathbb{X}}) \) be the random matrix

\[
\mathbf{T} = (A^{1/2})^+ \mathbf{Y} + \mu
\]

where \( A^{1/2}A^{1/2} = A \in S_m^+(q) \sim \mathcal{W}_{n,q}^q(n, \Theta), \Theta \in \mathcal{S}_m(r_{\Theta}) \) with \( r_{\Theta} \leq m \) and \( q = \min(m, n) \), independent of \( \mathbf{Y} \in \mathcal{L}_{r,m}(r_{\mathbb{X}}) \sim \mathcal{N}_{m \times r}(0, \mathbf{I}_m \otimes \Xi) \), \( \Xi \in \mathcal{S}_r(r_{\mathbb{X}}) \) with \( r_{\mathbb{X}} \leq r \) and \( m \geq q \geq r_{\mathbb{X}} > 0 \). Then the singular matrixvariate \( \mathbf{T} \) has the density

\[
\frac{c(m, n, q, q_1, r_{\Theta}, r_{\mathbb{X}}, r_{\alpha})}{\prod_{i=1}^{r_{\mathbb{X}}} \chi_i(\Xi)^{n/2} \prod_{j=1}^{r_{\Theta}} \chi_j(\Theta)^{n/2}} \prod_{l=1}^{r_{\alpha}} \chi_l [\Theta^{-} + (\mathbf{T} - \mu)\Xi^{-}(\mathbf{T} - \mu)^{\prime}]^{-(n+r_{\mathbb{X}})/2} (d\mathbf{T}),
\]

with

\[
c(m, n, q, q_1, r_{\Theta}, r_{\mathbb{X}}, r_{\alpha}) = \frac{\pi^{nq-r_{\Theta}}(n+r_{\mathbb{X}})^{q_1-1}r_{\alpha}^2}{2^{(n+r_{\mathbb{X}})+r_{\Theta}}r_{\alpha}^2} \prod_{l=1}^{r_{\alpha}} \Gamma_{|l|/2}[(n+r_{\mathbb{X}})/2],
\]

where \( q_1 = \min(m, n+r_{\mathbb{X}}) \), \( r_{\alpha} = \text{rank} [\Theta^{-} + (\mathbf{T} - \mu)\Xi^{-}(\mathbf{T} - \mu)^{\prime}] \leq m \), and \((d\mathbf{T})\) denotes the Hausdorff measure.

In particular observe that if \( \Xi = \mathbf{I}_r \), i.e. \( r_{\mathbb{X}} = r \), \( r_{\Theta} = r_{\alpha} = m \), \( \Theta = \mathbf{I}_m \), \( q = n \) and \( q_1 = n + r_{\mathbb{X}} \), we have

\[
dF_{\mathbf{T}}(\mathbf{T}) = \pi^{-r(n+r)/2} \frac{\Gamma_{n+r}[(n+r)/2]}{\Gamma_{n/2}} |\mathbf{I}_m + (\mathbf{T} - \mu)(\mathbf{T} - \mu)^{\prime}]^{-(n+r)/2} (d\mathbf{T})
\]

where \( \mathbf{T} \in \mathcal{L}_{r,m}(r_{\mathbb{X}}) \) and now \((d\mathbf{T})\) denotes the Lebesgue measure.

**Theorem 2.1.** Under the condition of Lemma 2.1 let \( \mu = 0 \) and \( \mathbf{X} = \mathbf{T}\mathbf{C}^{+} \in \mathcal{L}_{r_{\mathbb{X}},r}^+(r_{\mathbb{X}}) \), where \( \Xi = \mathbf{C} \mathbf{C}^{\prime} \) such that \( \mathbf{C} \in \mathcal{L}_{r_{\mathbb{X}},r}^+(r_{\mathbb{X}}) \). Then the density function of \( \mathbf{X} \) is

\[
\frac{c(m, n, q, q_1, r_{\Theta}, r_{\mathbb{X}}, r_{\alpha})}{\prod_{j=1}^{r_{\mathbb{X}}} \chi_j(\Theta)^{n/2}} \prod_{l=1}^{r_{\alpha}} \chi_l [\Theta^{-} + \mathbf{X}\mathbf{C}^{\prime}]^{-(n+r_{\mathbb{X}})/2} (d\mathbf{X}),
\]

where \( q_1 = \min(m, n+r_{\mathbb{X}}) \), \( r_{\alpha} = \text{rank} [\Theta^{-} + \mathbf{X}\mathbf{C}^{\prime}] \leq m \), and now \((d\mathbf{X})\) denotes the Lebesgue measure.

**Proof.** Make the change of variables \( \mathbf{T} = \mathbf{X}\mathbf{C}^{\prime} \) as described by Díaz-García (2007)

\[
(d\mathbf{T}) = \prod_{i=1}^{r_{\mathbb{X}}} \chi_i(\mathbf{C}\mathbf{C}^{\prime})^{n/2} (d\mathbf{X}) = \prod_{i=1}^{r_{\mathbb{X}}} \chi_i(\Xi)^{n/2} (d\mathbf{X}).
\]
Also, observe that
\[
C^t \Xi C = (C^t C)^{1/2} C^t C (C^t C)^{1/2} = C^t C (C^t C)^{1/2} C = I_{r_n},
\]
(9)
since \( C \) and \( C^t C \) have the same rank \( r_n \) and \( C^t C \) is of the order \( r_n \times r_n \). Finally, note that by (9)
\[
\Xi_n = \frac{1}{\pi} \int_0^\infty \frac{e^{-(r_n + r)}}{\pi r_n} \frac{dr_n}{r_n} = \frac{1}{\pi} \int_0^\infty \frac{e^{-(r_n + r)}}{\pi r_n} \frac{dr_n}{r_n}.
\]
(10)
By substituting (9) and (10) in (8) we obtain the desired result.
\[
\Xi \Xi^t = X C^t \Xi C X^t = X (C^t C)^{-1} X^t = X \Xi^2 = XX^t.
\]

Now let \( \kappa_i, i = 1, \ldots, r \) be the nonzero singular value of \( T \). Then taking \( \mu = 0 \) in (9) we have

**Theorem 2.2.** The joint density function of \( \kappa_1, \ldots, \kappa_r \), the singular values of \( T \) is
\[
\pi^{-(2n-2m)/2} \frac{2^r \Gamma_{n+r}[(n+r)/2]}{\Gamma_n[r/2] \Gamma_r[m/2]} \prod_{i=1}^r \frac{\kappa_i^{m-r}}{(1 + \kappa_i^2)^{(n+r)/2}} \prod_{i<j} (\kappa_i^2 - \kappa_j^2) \left( \int_{\kappa_i}^{\kappa_j} d\kappa_i \right)
\]
(11)
where \( \kappa_1 > \cdots > \kappa_r > 0 \).

**Proof.** Let \( T = Q_1 D_T P' \) be the nonsingular part of the singular value decomposition (SVD), where \( Q_1 \in V_{r,m} \), \( P \in O(r) \) and \( D_T = \text{diag}(\kappa_1, \ldots, \kappa_r) \), \( \kappa_1 > \cdots > \kappa_r > 0 \). Then, from
\[
(dT) = 2^{-r} |D_T|^{m-r} \prod_{i<j} (\kappa_i^2 - \kappa_j^2) (dD_T)(Q_1 dQ_1)(P' dP).
\]
The result is then obtained immediately from (9), noting that
\[
|I_m + Q_1 D_T P' P D_T Q_1'| = |I_m + D_T^2|
\]
and integrating over \( Q_1 \in V_{r,m} \) and \( P \in O(r) \) using (9).
\[
\square
\]

### 3 Doubly singular beta type II distribution

**Theorem 3.1.** Let \( A^{1/2} A^{1/2} = A \in S^+_m(q_A) \sim W^{q_A}_m(n, \Theta) \), \( \Theta \in S_m(r_\Theta) \) with \( r_\Theta \leq m \) and \( q_A = \min(m, n) \), independent of \( Y \in L^+_{r,m}(r_\Xi) \sim N^+_m(r_\Xi, r_\Xi, \Xi) \), \( \Xi \in S_r(r_\Xi) \) with \( r_\Xi \leq r \) and \( m \geq q \geq r_\Xi > 0 \). The doubly singular matrix variate beta type II distribution is defined as
\[
F = (A^{1/2})^T (Y \Xi Y) (A^{1/2})^T = (A^{1/2})^T B A^{1/2}
\]
where \( B = (Y \Xi Y) \in S^+_m(r_\Xi) \sim P W^{q_A}_m(r_\Xi, I_m) \). The density function of \( F \) is
\[
\pi^{-r_\Xi^2/2} \Gamma_{r_\Xi} [r_\Xi/2] \prod_{j=1}^{r_\Xi} \chi_j(\Theta^- + F)^{-(n+r_\Xi)/2} (dF),
\]
(12)
where \( F = Q_1 D_F Q_1' \) is the nonsingular part of the spectral decomposition, with \( Q_1 \in V_{r_\Xi,m} \) and \( D_F = \text{diag}(\delta_1, \ldots, \delta_{r_\Xi}) \), \( \delta_1 > \cdots > \delta_{r_\Xi} > 0 \) and where \((dF)\) denotes the Hausdorff measure.
Proof. By (10), observe that
\[ F = (A^{1/2})^+ (Y \Xi^2 Y) (A^{1/2})^+ = T \Xi^2 T = XX' \]

Let \( X = Q_1 D_X P' \) be the nonsingular part of the singular value decomposition (SVD), where \( Q_1 \in \mathcal{V}_{r,m} \), \( P \in \mathcal{O}(r) \) and \( D_X = \text{diag}(\kappa_1, \ldots, \kappa_m) \), \( \kappa_1 > \cdots > \kappa_m > 0 \). Then \( F = XX' = Q_1 D_X P' D_X Q_1' = Q_1 D_F Q_1', \) with \( D_F = \text{diag}(\delta_1, \ldots, \delta_r) = D_X^2 \). Then, from [Díaz-García et al. (1997)],
\[ (dX) = 2^{-r} |D_F|^{(r_m - m - 1)/2} (dF) (P'dP). \]  

Substituting (13) in (7) and integrating over \( P \in \mathcal{O}(r) \) using (4) gives the stated density function for \( F \).

In particular, if \( \Theta = I_m \), \( q_1 = n + r_m \) and \( q = n \), then \( r_a = m \) and
\[ \pi^{-nr_m} \frac{\Gamma(n + r_m)/2}{\Gamma(n/2)\Gamma(r_m/2)} |D_F|^{(r_m - m - 1)/2} |I_m + F|^{-(n + r_m)/2} (dF). \]  

An alternative way to obtain (14) when \( \Xi = I_r \) and therefore \( r_m = r \) is by using the third definition of \( F \) given in (2). This denotes the matrix variate as \( \tilde{F} \).

**Theorem 3.2.** Let \( A^{1/2} A^{1/2} = A \in \mathcal{S}_n^+(n) \sim \mathcal{P}\mathcal{W}_n^m(n, I_m) \), independent of \( Y \in \mathcal{L}_{r,m}^+(r) \sim \mathcal{N}_{m,r}^+(0, I_m \otimes I_r) \), and \( m \geq n \geq r > 0 \). The doubly singular matrix variate beta type II distribution is defined as
\[ \tilde{F} = Y'A^+ Y, \]
which has the density function
\[ \pi^{-r(r + 2n - m)/2} \frac{\Gamma(n + r)/2}{\Gamma(n/2)\Gamma(r/2)} |\tilde{F}|^{(m - r - 1)/2} |I_r + \tilde{F}|^{-(n + r)/2} (d\tilde{F}), \]  

where \( (d\tilde{F}) \) denotes the Lebesgue measure.

Proof. Observe that
\[ \tilde{F} = Y'A^+ Y = T'T, \]
where the density function of \( T \) is given by (10). Now as in the proof of Theorem 2.2, let \( T = Q_1 D_T P' \); hence \( \tilde{F} = T'T = P'D_T Q_1 D_T P' = P'D_F P' \), with \( D_F = D_T^2 \). Then by [Díaz-García et al. (1997)]
\[ (dT) = 2^{-r} |\tilde{F}|^{(r_m - m - 1)/2} (d\tilde{F}) (Q_1' dQ_1). \]

The proof is completed by substituting (10) in (6) and integrating over \( Q_1 \in \mathcal{V}_{r,m} \) using (3).

Note that Theorems 3.1 and 3.2 are generalizations of Theorems 10.4.1 and 10.4.4 in Muirhead (1982) for the central doubly singular case.

**Theorem 3.3.** Let \( \delta_1, \ldots, \delta_r \) be the nonzero eigenvalues of \( F \) (with density function (14)) and let \( r_m = r \) or equivalent be the nonzero eigenvalues of \( \tilde{F} \) (with density function (15)). Then the joint density function of \( \delta_1, \ldots, \delta_r \) is
\[ \pi^{-r(2n - m)/2} \frac{\Gamma(n + r)/2}{\Gamma(n/2)\Gamma(r/2)\Gamma(m/2)} \prod_{i=1}^r \frac{\delta_i^{(m - r - 1)/2}}{(1 + \delta_i)^{(n + r)/2}} \prod_{i<j} (\delta_i - \delta_j)^{-r} \]  

where \( \delta_1 > \cdots > \delta_r > 0 \).
Proof. This result can be obtained by any of the following methods:

i) By making the transformation $F = Q_1 D_F Q'_1$ in (14) with $r_\Xi = r$ and integrating over $Q_1 \in V_{r,m}$ using (3).

ii) Alternatively, by making the transformation $\tilde{F} = P D_F P'$ in (15) and integrating over $P \in O(r)$ using (3), the proof is completed.

iii) A third proof is derived immediately from Theorem 2.2 observing that $\kappa_i = \sigma_i^{1/2}$, $i = 1, \ldots, r$ and so

$$\left( \prod_{i=1}^r d\kappa_i \right) = 2^{-r} \prod_{i=1}^r \sigma_i^{-1/2} \left( \prod_{i=1}^r d\sigma_i \right)$$

Remark 3.1. Finally, from Mitra (1970, Lemma 3.1) observe that:

a) The density function (14) is invariant under any arbitrary matrix $\Theta \in S_m$ and $Y \in L_{r,m}(r_\Xi) \sim \mathcal{N}_{m \times r}(0, \Theta \otimes \Xi)$.

b) Similarly, Theorem 2.2 is invariant if $A \in S_m^+(n) \sim \mathcal{P}_{W_m}(n, \Theta)$, independent of $Y \in L_{r,m}(r) \sim \mathcal{N}_{m \times r}(0, \Theta \otimes I_r)$, where $\Theta \in S_m$.

c) Therefore, Theorem 3.3 too, is invariant under conditions established in items a) and b).

4 Doubly singular beta type I distribution

Theorem 4.1. Let $A^{1/2} A^{1/2} = A \in S_m^+(n) \sim \mathcal{P}_{W_m}(n, I_m)$ be independent of $Y \in L_{r,m}(r_\Xi) \sim \mathcal{N}_{m \times r}(0, I_m \otimes \Xi)$, $\Xi \in S_r(r_\Xi)$ with $r_\Xi \leq r$ and $m \geq q \geq r_\Xi > 0$. The doubly singular matrix variate beta type I distribution is defined as

$$U = \left( (A + Y \Xi - Y)^{1/2} (Y \Xi - Y) \left( A + Y \Xi - Y \right)^{1/2} \right)^+$$

$$= \left( A + B \right)^{1/2} B \left( A + B \right)^{1/2}$$

where $B = (Y \Xi - Y) \in S_m^+(r_\Xi) \sim \mathcal{P}_{W_m}(r_\Xi, I_m)$. The density function of $U$ is

$$\pi^{-nm} \Gamma_n [n (r_\Xi + m)/2] \Gamma_n [n/2] \Gamma_{r_\Xi} [(r_\Xi + m)/2] D_U \left( r_\Xi - m - 1 \right)/2 |I_m - U|^{-\left( n - m - 1 \right)/2} (dU),$$

(18)

where $U = Q_1 D_U Q'_1$ is the nonsingular part of the spectral decomposition, with $Q_1 \in V_{r_\Xi, m}$ and $D_U = \text{diag}(\lambda_1, \ldots, \lambda_{r_\Xi})$, $1 > \lambda_1 > \cdots > \lambda_{r_\Xi} > 0$ and where $(dU)$ denotes the Hausdorff measure.

Proof. From Díaz-García and Gutiérrez-Jáimez (2006) it is known that if $F \in S_m^+(r_\Xi)$ is a singular matrix variate beta type II distribution, then the matrix variate $U = I_m - (I_m + F)^{-1}$ has a singular matrix type I distribution; moreover

$$\left( dF \right) = |I_m - D_U|^{-\left( m + 1 - r_\Xi \right)/2} |I_m - U|^{-\left( m + 1 + r_\Xi \right)/2} (dU)$$

where $U = Q_1 D_U Q'_1$ with $Q_1 \in V_{r_\Xi, m}$ and $D_U = \text{diag}(\lambda_1, \ldots, \lambda_{r_\Xi})$, $1 > \lambda_1 > \cdots > \lambda_{r_\Xi} > 0$. The proof follows from making the transformation $F = (I_m - U)^{-1} - I_m$ in (14), noting that $F = (I_m - U)^{-1} U$ and $D_F = (I_m - D_U)^{-1} D_U$; see Díaz-García and Gutiérrez-Jáimez (2006). \(\square\)
Theorem 4.2. Let $A^{1/2}A^{1/2} = A \in S_m^+(n) \sim \mathcal{W}_m^r (n, I_m)$, independent of $Y \in \mathcal{L}_{m,r}^+(r) \sim \mathcal{N}_{m \times r}(0, I_m \otimes I_r)$, and $m \geq n \geq r > 0$. The doubly singular matrix variate beta type II distribution is defined as

$$\tilde{U} = Y'(A + YY')^+ Y,$$

and its density function is given by

$$\pi^{-r+2n-m/2} \frac{\Gamma_{n+r}((n+r)/2)}{\Gamma_{n/2}\Gamma_{r/2}\Gamma_{m/2}} \prod_{i=1}^r \lambda_i^{(m-r-1)/2} (1 - \lambda_i)^{(n-m-1)/2} \prod_{i < j} (\lambda_i - \lambda_j) \left( \prod_{i=1}^r d\lambda_i \right),$$

(19)

where $(d\tilde{U})$ denotes the Lebesgue measure.

Proof. The proof is obtained immediately by making the transformation $\tilde{F} = (I_m - \tilde{U})^{-1} \tilde{U}$ with $(d\tilde{F}) = |I_m - \tilde{U}|^{-(r+1)}$ in (18).

Theorem 4.3. Let $U \in S_m^+(r)$ be a matrix variate with density function (18) and $r = r$ (or let $\tilde{U} \in \mathcal{S}_r$, matrix variate with density function (19)). Then the joint density function of $\lambda_1, \ldots, \lambda_r$, the nonzero eigenvalues of $U$ (or $\tilde{U}$), is

$$\pi^{-r(2n-m)/2} \frac{\Gamma_{n+r}((n+r)/2)}{\Gamma_{n/2}\Gamma_{r/2}\Gamma_{m/2}} \prod_{i=1}^r \lambda_i^{(m-r-1)/2} (1 - \lambda_i)^{(n-m-1)/2} \prod_{i < j} (\lambda_i - \lambda_j) \left( \prod_{i=1}^r d\lambda_i \right),$$

(20)

where $1 > \lambda_1 > \cdots > \lambda_r > 0$.

Proof. The proof can be obtained in any of the following ways.

i) By making the transformation $U = H_1DUH_1'$ in (18) with $r = r$ and integrating over $H_1 \in V_{r,m}$ using (3).

ii) Alternatively, by making the transformation $\tilde{U} = GDG'$ in (19) and integrating over $G \in O(r)$ using (3).

iii) A third proof is derived immediately from Theorem 3.3 observing that $\lambda_i = (1 - \delta_i)^{-1} \delta_i$, $i = 1, \ldots, r$, and so

$$\left( \prod_{i=1}^r d\delta_i \right) = \prod_{i=1}^r (1 - \lambda_i)^{-2} \left( \prod_{i=1}^r d\lambda_i \right) \quad \square$$

Finally, observe that conclusions analogous to those set out in note 1 can be obtained for the case of the matrix variate beta type I distribution.

5 Matricvariate inverted $t$ distribution

Let us now find the matricvariate inverted $t$ distribution, also termed the matricvariate Pearson type II distribution. In this case, we determine its density function from the doubly singular beta type I distribution (18).
Theorem 5.1. Let $R \in L^+_{r,m}(r)$ be the random matrix,

$$R = \left[(A + YY')^+\right]^{1/2} Y,$$

where $A^{1/2}A^{1/2} = A \in S^+(n) \sim \mathcal{PW}_m(n, I_m)$, independent of $Y \in L^+_{r,m}(r) \sim N_m(0, I_m \otimes I_r)$ and $0 < r \leq n \leq m$. Then the matricvariate $R$ has the density function

$$dF_R(R) = \pi^{-r(r+2n)/2} \frac{\Gamma_{n+r}(n+r/2)}{\Gamma_n[n/2]} |I_m - RR'|^{-(n+r)/2} (dR),$$

where $I_m - RR' > 0$ and $(dR)$ denotes the Lebesgue measure.

Proof. Observe that $U = R R' = \left[(A + YY')^+\right]^{1/2} YY' \left[(A + YY')^+\right]^{1/2}$. Then from Díaz-García et al. (1997),

$$(dR) = 2^{-r} |D_U|^{(r-m-1)/2} (dU)(G'dG),$$

where $R = H_1 D_R G'$ and $U = RR' = H_1 D_R G' D_R H_1' = H_1 D_U H_1'$ with $D_U = D_R$, $H_1 \in V_{r,m}$, $G \in O(r)$ and $D_R = \text{diag}(\tau_1, \ldots, \tau_r)$, $1 > \tau_1 > \cdots > \tau_r > 0$. Thus

$$(dU) = 2^{r}|N|^{-(r-m-1)/2}(dR)(G'dG)^{-1}.$$

On substituting in (18) and integrating over $G \in O(r)$ using (3), the proof is complete. □

Alternatively, Theorem 5.1 can be proved from the distribution of $\tilde{U}$ in an analogous way.

Conclusions

Díaz-García and Gutiérrez-Jáimez (2008c) studied the density of the singular matricivariate $t$ distribution and its application to sensitivity analysis. In this work we study some properties of this distribution, namely the distribution of a linear transformation and the joint density function of its singular values. Uhlig (1994), Díaz-García and Gutiérrez-Jáimez (1997, 2008a) studied the singular matrix variate beta type I and II distributions in central and noncentral cases. Now we obtain the doubly singular matrix variate beta type I and II distributions in the central case and find the joint density function of their eigenvalues. Finally, we obtain the central density function of the Pearson type II matricvariate.

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