Frank-Wolfe variants for minimization of a sum of functions

Suhail M. Shah

September 7, 2020

Abstract

We propose several variants of the Frank-Wolfe algorithm to minimize a sum of functions. The main proposed algorithm is inspired from the dual averaging scheme of Nesterov adapted for Frank Wolfe in a stochastic setting. A distributed version of this scheme is also suggested. Additionally, we also propose a Frank-Wolfe variant based on incremental gradient techniques. The convergence rates for all the proposed algorithms are established. The performance is studied on least squares regression and multinomial classification.

1 Introduction

In this paper, we consider the optimization problem

$$\min_{x \in \Omega} \{ f(x) := E_\xi[f(x,\xi)] = \int f(x, \omega) dP(\omega) \} ,$$

(1)

where $P$ is the probability law of the random variable $\xi$, $x \in \mathbb{R}^d$, $f : \mathbb{R}^d \to \mathbb{R}$ and $\Omega \subset \mathbb{R}^d$ is compact convex set. We also consider a closely related problem to the above, where the objective function can be expressed as a sum of functions:

$$\min_{x \in \Omega} \{ f(x) = \frac{1}{m} \sum_{i=1}^{m} f^i(x) \} ,$$

(2)

where $f^i : \mathbb{R}^d \to \mathbb{R}$ for all $i$. This situation arises quite frequently in machine learning in the context of stochastic optimization. While it may be desirable to minimize (1), such a goal is sometimes untenable when one does not have access to the law $P$ or when one cannot draw from an infinite population sample set. In that case, a practical approach is to instead seek the solution of a problem that involves an estimate of the the expectation in (1) giving rise to (2). Then, $f^i(x) = f(x, \xi_i)$ in (2) for some realization $\xi_i$ of $\xi$.

Optimization problems of the form (2) usually have a large $m$, which makes finding a possible solution by first order methods a computationally intensive and time consuming task. A possible remedy is to use a stochastic approach. Also, because of high dimensions, they are generally beyond the capability of second order methods due to extremely high iteration complexity (see [1]). Note that (2) is a constrained optimization problem, so the conventional approaches (e.g., projected gradient descent) to solve it require a projection on the constraint set at every step of the algorithm. This can be quite an expensive operation and may make the problem intractable.

Recently there has been a lot of interest in the Frank-Wolfe method (FW) [2], also known as conditional gradient, for solving constrained optimization problems. In many problems where
the domain $\Omega$ admits fast linear optimization while having slow projections, FW has the potential to be much more efficient than projection based algorithms. Such problems include linear regression with constraints, multiclass classification, matrix completion, etc.

**Contributions:** Our contributions are as follows. We suggest three algorithms related to Frank-Wolfe method to deal with (1) and (2):

- The first algorithm is based on the dual averaging technique of [3], [4] and deals with (1). The essential idea of [3] is to use a weighted average of all the gradients till the current step instead of just the gradient at the current iterate. This is done to overcome a significant disadvantage of the latter approach. Since a diminishing time step is used in most gradient related algorithms, this implies new gradient computations enter the algorithm with decreasing weights (see (1.6), [3]). We leverage this idea to propose a stochastic version of FW, where $\mathcal{O}(\frac{1}{\epsilon^3})$ gradient evaluations are required while having a convergence rate of $\mathcal{O}(\frac{1}{\epsilon})$.

- We also propose an incremental gradient approach to solve (2). This requires only $\mathcal{O}(\frac{1}{\epsilon})$ gradient evaluations while maintaining the same convergence rate. We remark here that under our assumptions (convex and smooth), this is the best known convergence rate with the minimal number of gradient evaluations to deal with a sum of functions form. The incremental method involves storing the last $m$ gradient evaluations and using a running average of these $m$ evaluations instead of just the gradient at the current iterate. The essential idea why the incremental version works is that because of a small time step which is diminishing, the last $m$ iterates generated by the algorithm are not very different from each other. Together with the smoothness of the function, this implies a good enough approximation of the running average to the gradient at the current iterate. We also remark here the proof of the main convergence result (Theorem 3) is remarkably easy and may have application elsewhere.

- Finally, we also propose a distributed version of FW with dual averaging. The proposed algorithm is inspired from the ideas of [5], where distributed dual averaging is developed for the proximal gradient method. This algorithm also achieves optimal convergence rates which could be achieved for a distributed implementation of FW.

**Related Work:** The classical Frank-Wolfe method using a line search was originally proposed in [2]. The convergence rate for a smooth convex function with a polyhedral domain was shown to be $\mathcal{O}(\frac{1}{\epsilon})$. This rate was later improved in [6] for such domains by suggesting a variant known as the ”away step FW”. A more recent treatment of the method was given in [7], where it was extended to atomic domains. For the classical Frank-Wolfe algorithm, [8] showed a linear rate for the special case of quadratic objectives when the optimum is in the strict interior of the domain, a result also proved earlier in [9]. Building upon the work of [10], it was shown in [11] that for strongly convex sets, FW achieves a rate of $\mathcal{O}(\frac{1}{\sqrt{\epsilon}})$. We remark here that any improvement in the convergence rate of FW depends upon the geometric properties (being a polytope, strong convexity etc.) of the constraint set. A primal-dual averaging algorithm for deterministic optimization was studied in [12]. We also mention [13] where a deterministic algorithm similar to dual averaging was presented in a game theoretic setting of online learning. We also mention the related work [14] in the context of stochastic FW, where the authors consider a stochastic substitute version of FW in the context of structured convex optimization.

---

1 By this we mean $\mathcal{O}(\frac{1}{\epsilon})$ steps are needed to reach any $\epsilon$-accurate solution, so that $f(x) - \min_{y \in \Omega} f(y) \leq \epsilon$ for any output $x$ of the algorithm.
Compared to the deterministic version, the stochastic version is relatively less studied. One of the main works to study FW in an online learning setting is [15]. The vanilla version of stochastic FW requires $O(\frac{1}{\epsilon^3})$ stochastic gradient evaluations to achieve an $\epsilon$-accurate optimum. The convergence rate is $O(\frac{1}{\epsilon^2})$. The alternative to the vanilla version suggested in [15] is based on combing gradient sliding techniques [16] with variance reduction algorithms [17], [18]. Although the convergence rate remains the same, the number of gradient evaluations is brought down significantly. In [19], a similar approach (variance reduction) is taken to handle non-convex functions. The convergence rate and gradient evaluations required are both of the order $O(\frac{1}{\epsilon^2})$.

The literature on distributed algorithms is vast, mostly building upon the seminal work of [20]. The most relevant works for our purposes are [21] and [22]. In [21], a distributed FW is proposed in order to solve the associated optimization problem in the setting where the elements to be combined are not centrally located but spread over a network. A decentralized version (where at each iteration an approximate average is obtained) is studied in [22]. Furthermore, a much more communication efficient version is also proposed which utilizes the gradient tracking concept of [23]. The distributed version we study here is inspired by [5], which studies distributed dual averaging for proximal gradient methods. Broadly speaking, this work also leverages two time scale algorithm theory whose broad range of applications can be found in [24–30] among others.

2 Background

In this section we discuss the basics of Frank-Wolfe, dual averaging and distributed algorithms [31–33].

2.1 Frank-Wolfe

We first review the standard Frank-Wolfe algorithm, the pseudo code for which is given in Algorithm 1.

**Algorithm 1 Classical Frank-Wolfe (FW)**

**Input :** Objective Function $f$; parameter $\alpha_k$

**Initialize :** $x_0 \in \Omega$, $w_0 = \arg\min_{x \in \Omega} \nabla f(x_0)^T x$.

**For** $k = 1, 2, \ldots$ **do**:

a) Compute the gradient $\nabla f(x_k)$. Call the linear optimization oracle to compute:

$$w_k \in \arg\min_{x \in \Omega} \langle \nabla f(x_k), x \rangle.$$  

b) Set

$$x_{k+1} = (1 - \alpha_k)x_k + \alpha_kw_k.$$  

**end for**

At each instant $k$, the algorithm makes a call to a linear optimization (LO) oracle, followed by a convex averaging step of the current iterate with the output of the LO oracle (Step b). Note
that as long as \( x_0 \) is initialized in the interior of \( \Omega \), the sequence \( \{x_k\}_{k \geq 1} \) is guaranteed to stay inside \( \Omega \). Since the algorithm substitutes the projection problem (which can be thought of as a constrained quadratic minimization problem) with a linear optimization one, FW is sometimes referred to as a “projection-free” algorithm. This can make it very desirable for structured constraints (see [34]). The deterministic Frank-Wolfe has a convergence rate of \( O(\frac{1}{k}) \) (Theorem 1, [7]). Besides the advantages of a low iteration cost and ease of implementation, the iterates generated by FW enjoy many structural properties. In particular, since the iterates \( \{x_k\} \) can be written as a convex combination of a smaller number of extreme points of \( \Omega \), sparsity and low rank (for matrix constraints) are preserved at every step (Section 3, [7]).

2.2 Dual Averaging

We review the idea of dual averaging in the context of mirror descent. Mirror descent is based on a proximal function \( \psi : \Omega \to \mathbb{R} \) assumed to be 1-strongly convex with respect to the Euclidean norm \( \| \cdot \| \):

\[
\psi(y) \geq \psi(x) + \langle \nabla \psi(x), y-x \rangle + \frac{1}{2} \|x-y\|^2.
\]

An example of such a proximal function is the quadratic function \( \psi(x) = \frac{1}{2} \|x\|^2 \). Define the mapping:

\[
\Pi^\psi_{\Omega}(g, \alpha) = \arg \min_{x \in \Omega} \left\{ \langle g, x \rangle + \frac{1}{\alpha} \psi(x) \right\}.
\]

We state the mirror descent algorithm with dual averaging in Algorithm 2.

**Algorithm 2** Mirror descent with Dual Averaging (FW)

**Input**: Objective Function \( f \); parameter \( \alpha_k \)

**Initialize**: Set \( x_0 = \arg \min_{x \in \Omega} \psi(x) \) and \( g_0 = 0 \).

**For** \( k = 1, 2, \ldots \) **do**: At each node \( i \),

\( a) \) Compute the gradient \( \nabla f(x_k) \).

\( b) \) Update the average gradient:

\[
g_k = \frac{k-1}{k} g_{k-1} + \frac{1}{k} \nabla f(x_k).
\]

\( c) \) Set

\[
x_{k+1} = \Pi^\psi_{\Omega}(g_k, \alpha_k).
\]

**end for**

We note that,

\[
g_k = \frac{1}{k} \sum_{i=1}^{k} \nabla f(x_i).
\]

This implies that we use the average gradient upto time \( k \) in Step (b) of Algorithm 2. For the
stochastic setting, instead of using the entire sum present in \( f(\cdot) \) one can randomly sample \( f^i \) and use \( \nabla f^i(\cdot) \) instead of \( \nabla f(\cdot) \) in Step (c). We refer the reader to [4] for more details on dual averaging in an online setting. We remark that the above algorithm is equivalent to the standard projected gradient descent with the average gradient when \( \psi(x) = \frac{1}{2}\|x\|^2 \) (Proposition 2.1, [35]).

### 2.3 Distributed Algorithms

Suppose we have a network of \( m \) agents indexed by \( 1, \ldots, m \). We associate with each agent \( i \), the function \( f^i : \mathbb{R}^d \to \mathbb{R} \) and a global convex constraint set \( \Omega \). The (global) function which we aim to minimize is \( f : \mathbb{R}^d \to \mathbb{R} \).

Let the communication network be modelled by a static undirected graph \( G = (V, E) \) where \( V = \{1, \ldots, m\} \) is the node set and \( E \subset V \times V \) is the set of links \((i, j)\) indicating that agent \( j \) can send information to agent \( i \). All of the arguments presented here can be extended to a time-varying graph under suitable assumptions. Here we deal only with a static network for ease of notation.

We associate with the network a non-negative weight matrix \( Q = [q_{ij}]_{i,j \in V} \) such that \( q_{ij} > 0 \iff (i, j) \in E \).

**Assumption 1:**

i) [Double Stochasticity] \( 1^T Q = 1^T \) and \( Q 1 = 1 \).

ii) [Irreducibility and aperiodicity] We assume that the underlying graph is irreducible, i.e., there is a directed path from any node to any other node, and aperiodic, i.e., the g.c.d. of lengths of all paths from a node to itself is one. It is known that the choice of node in this definition is immaterial. This property can be guaranteed, e.g., by making \( q_{ii} > 0 \) for some \( i \).

The objective of distributed optimization is to minimize \( f(\cdot) \) subject to staying in the constraint \( \Omega \) while simultaneously achieving consensus, i.e.

\[
\begin{align*}
\text{minimize} & \quad \sum_{i=1}^{m} f^i(x^i) \\
\text{subject to} & \quad x^i \in \Omega \\
& \quad \sum_{j \in N(i)} q_{ij} \|x^i - x^j\|^2 = 0 \quad \forall i.
\end{align*}
\]

The most popular way to solve this is studied in for the unconstrained case in [20] and subsequently in [36] for the constrained case. It involves the following two steps:

1. **[Consensus Step]** This step involves local averaging at each node and is aimed at achieving consensus,

\[
u^i_k = \sum_{j \in N(i)} q_{ij} x^j_k.
\]

\[\text{(S1) [Consensus Step]}\]}

Recall that \( f(\cdot) = \sum_{i=1}^{m} f^i(\cdot).\)

3In many applications, one may also have \( f^i(\cdot) = f(\cdot) \forall i.\)
Gradient Descent Step] This step is the gradient descent part aimed at minimizing \( f^i \) at each node:

\[ x_{k+1}^i = P_{\Omega}(v_k^i - a_k \nabla f^i(v_k^i)), \]

where \( P_{\Omega}(\cdot) \) is the projection on the set \( \Omega \) and \( a_k \) is a positive scalar decaying at a suitable rate to zero.

The convergence properties of the above algorithm have been extremely well studied for convex as well as the non-convex case.

3 The algorithms

We assume here that the function \( f \) in (1) (and each \( f^i \) in (2)) is convex and \( L \)-smooth in \( \mathbb{R}^d \). By smoothness, we mean that the gradient is \( L \)-Lipschitz continuous, i.e.

\[ \| \nabla f(x) - \nabla f(y) \| \leq L \| x - y \|. \]

For a general \( f : \mathcal{X} \to \mathbb{R} \), the norm on the LHS will be the dual norm. Together with the convexity, the \( L \)-smoothness of \( f \) implies:

\[ f(y) \leq f(x) + \langle \nabla f(x), y - x \rangle + \frac{L}{2} \| x - y \|^2. \]

We use \( L_f(x, \cdot) \) to denote the linear approximation of \( f(\cdot) \) at the point \( x \):

\[ L_f(x, y) := f(x) + \langle \nabla f(x), y - x \rangle. \]

Let \( C_{\Omega} \) denote the following bound:

\[ C_{\Omega} := \sup_{x, y \in \Omega} \| x - y \|. \]

3.1 Frank-Wolfe with stochastic dual averaging (FW-SDA)

Our first algorithm combines the Frank-Wolfe algorithm with stochastic dual averaging to solve (1). We assume here that we have access to a stochastic first order oracle (SFO). The oracle takes a point \( x \) and returns an unbiased sample \( \nabla f(x, \xi') \), where \( \xi' \) is a sample drawn i.i.d from \( P \). By an unbiased sample \( \nabla f(x, \xi') \), we mean that

\[ \mathbb{E}[\nabla f(x, \xi')] = \nabla f(x). \]

The essential idea here is to generate enough of these random samples at each iteration and use it to update the average gradient employed in a standard dual averaging scheme. The pseudo code is provided in Algorithm 3. In addition to the sequence \( \{x_k\} \) and \( \{w_k\} \) which are present in the Frank-Wolfe algorithm, Algorithm 3 has two additional sequences \( \{z_k\} \) and \( \{g_k\} \). The auxiliary sequence \( \{z_k\} \) is again standard for Nesterov’s algorithm. The sequence \( \{g_k\} \) keeps track of the average gradient upto time \( k \) and constitutes the dual averaging part of the algorithm. We note that \( g_k \) can be written as:

\[ g_k = \frac{1}{\sum_{i=1}^{k} \beta_i} \sum_{i=1}^{k} \beta_i \nabla_k(z_{i-1}), \]

where \( \nabla_k \) is as in (8).
Algorithm 3 Stochastic Frank-Wolfe with Dual averaging (FW-SDA)

Input: Initial point $x_0$; parameters $\alpha_k$, $p_k$ and $\beta_k$.

Initialize: $x_0 \in \Omega$, $w_0 = \arg\min_{x \in \Omega} \langle \nabla f(x_0), x \rangle$ and $g_0 = 0$.

For $k = 1, 2, \ldots$ do:

a) Set
$$z_{k-1} = (1 - \alpha_k)x_{k-1} + \alpha_kw_{k-1}. \quad (7)$$

b) Draw $p_k$ i.i.d samples $\xi_1, \ldots, \xi_{p_k}$ according to the distribution $P$ independent of the past history $F_{k-1}$ and set
$$\nabla_k(z_{k-1}) := \frac{1}{p_k} \sum_{j=1}^{p_k} \nabla f(z_{k-1}, \xi_j). \quad (8)$$

c) Update the average weighted gradient:
$$g_k = \frac{1}{\sum_{i=1}^{k} \beta_i} \left\{ \left( \sum_{i=1}^{k-1} \beta_i \right) g_{k-1} + \beta_k \nabla_k(z_{k-1}) \right\}. \quad (9)$$

d) Call the Linear Optimization (LO) oracle to compute:
$$w_k \in \arg\min_{x \in \Omega} \langle g_k, x \rangle.$$

e) Set
$$x_k = (1 - \alpha_k)x_{k-1} + \alpha_kw_k. \quad (10)$$

end for

Let $L_{f_k}$ denote the following (with $\nabla_k(x)$ as in (8)),
$$L_{f_k}(x, y) := f(x) + \langle \nabla_k(x), y - x \rangle. \quad (11)$$

for any $x, y \in \Omega$. We have,
$$\mathbb{E}[L_f(x, y) - L_{f_k}(x, y)] = \mathbb{E}[\langle \nabla f(x) - \nabla_k(x), y - x \rangle].$$

From Cauchy Schwarz inequality,
$$\mathbb{E}[L_f(x, y) - L_{f_k}(x, y)] = C_\Omega \mathbb{E}[\|\nabla f(x) - \nabla_k(x)\|]. \quad (12)$$

where $C_\Omega = \sup_{x,y \in \Omega} \|x - y\|$. Step (d) of Algorithm 3 implies
$$w_k \in \arg\min_w \Phi_k(w), \quad (13)$$

where $\Phi_k(w)$ is the accumulated linear model till time $k$, i.e., $\Phi_k(w) = 0$ if $k = 0$ and
$$\Phi_k(w) := \frac{1}{\sum_{i=1}^{k} \beta_i} \sum_{i=1}^{k} \beta_i L_f(z_{i-1}; w), k \geq 1. \quad (14)$$
with $L_{f^k}(\cdot, \cdot)$ defined in (11). The next theorem gives the rate of convergence of Algorithm 3
and builds upon the proof of its deterministic counterpart [12].

**Theorem 1.** Let $\alpha_k = \frac{2}{k+1}$. If the parameter $\beta_k$ is chosen such that

$$\alpha_k = \frac{\beta_k}{\sum_{i=1}^k \beta_i},$$

and the number of samples in $p_k$ in (8) is such that the following bound holds,

$$\mathbb{E}[\|\nabla_k(x) - \nabla f(x)\|] \leq \frac{1}{k},$$

for any $x \in \mathbb{R}^d$, then Algorithm 3 ensures that

$$\mathbb{E}[f(x_k) - f(x^*)] = \mathcal{O}\left(\frac{1}{k}\right)$$

for any $k > 0$.

**Proof.** (i) We have from (3),

$$f(x_k) \leq L_f(z_{k-1}; x_k) + \frac{L}{2}\|x_k - z_{k-1}\|^2,$$

$$= (1 - \alpha_k)L_f(z_{k-1}; x_{k-1}) + \alpha_k L_f(z_{k-1}; w_k) + \frac{L}{2}\|x_k - z_{k-1}\|^2,$$

where we have used (10) and the linearity of $L_f(\cdot; \cdot)$ in the second argument. Subtracting (7) from (10) we have,

$$x_k - z_{k-1} = \alpha_k(w_k - w_{k-1}).$$

Since $C_\Omega = \sup_{x,y \in \Omega} \|x - y\|$, we get

$$f(x_k) \leq (1 - \alpha_k)L_f(z_{k-1}; x_{k-1}) + \alpha_k L_f(z_{k-1}; w_k) + \frac{L}{2}\alpha_k^2 C_\Omega^2.$$

Using the convexity of $f(\cdot)$ in the first term in the RHS in the above we get,

$$f(x_k) \leq (1 - \alpha_k)f(x_{k-1}) + \alpha_k L_f(z_{k-1}; w_k) + \frac{L}{2}\alpha_k^2 C_\Omega^2.$$

Taking expectation we have,

$$\mathbb{E}[f(x_k)] \leq (1 - \alpha_k)\mathbb{E}[f(x_{k-1})] + \alpha_k \mathbb{E}[L_f(z_{k-1}; w_k)] + \alpha_k \mathbb{E}\{L_f(z_{k-1}; w_k) - L_{f^k}(z_{k-1}; w_k)\} + \frac{L}{2}\alpha_k^2 C_\Omega^2.$$

We note that the third term in the RHS of the above inequality can be bounded using (12) and (16) as

$$\mathbb{E}\{L_f(z_{k-1}; w_k) - L_{f^k}(z_{k-1}; w_k)\} \leq \frac{C_\Omega}{k} \leq C_\Omega \alpha_k,$$

so that

$$\mathbb{E}[f(x_k)] \leq (1 - \alpha_k)\mathbb{E}[f(x_{k-1})] + \alpha_k \mathbb{E}[L_{f^k}(z_{k-1}; w_k)] + C_\Omega \alpha_k^2 + \frac{L}{2}\alpha_k^2 C_\Omega^2.$$
Let $\tilde{\beta}_k = \sum_{i=1}^{k} \beta_i$. We next bound the second term in the RHS of (17). We have from (14),

$$\tilde{\beta}_k \Phi_k(w_k) = \beta_k L_{f_k}(z_{k-1}; w_k) + \tilde{\beta}_{k-1} \Phi_{k-1}(w_k).$$

From (13), $w_{k-1} \in \text{argmin}_w \Phi_{k-1}(w)$, so that $\Phi_{k-1}(w_{k-1}) \leq \Phi_{k-1}(w_k)$. This gives

$$\tilde{\beta}_k \Phi_k(w_k) \geq \beta_k L_{f_k}(z_{k-1}; w_k) + \tilde{\beta}_{k-1} \Phi_{k-1}(w_{k-1}).$$

We also have from (15) that $\alpha_k \tilde{\beta}_k^{-1} = \tilde{\beta}_k^{-1}$. So,

$$\alpha_k \tilde{\beta}_k^{-1} [\tilde{\beta}_k \Phi_k(w_k) - \tilde{\beta}_{k-1} \Phi_{k-1}(w_k)] = \Phi_k(w_k) - \alpha_k \tilde{\beta}_k^{-1} \tilde{\beta}_{k-1} \Phi_{k-1}(w_{k-1}),$$

$$= \Phi_k(w_k) - \tilde{\beta}_{k-1} \tilde{\beta}_k^{-1} \Phi_{k-1}(w_{k-1}).$$

Noting that $\tilde{\beta}_{k-1} = \tilde{\beta}_k - \beta_k$, we have $\tilde{\beta}_{k-1} \tilde{\beta}_k^{-1} = (1 - \tilde{\beta}_k^{-1} \beta_k) = (1 - \alpha_k)$ (using (15)). So,

$$\alpha_k \tilde{\beta}_k^{-1} [\tilde{\beta}_k \Phi_k(w_k) - \tilde{\beta}_{k-1} \Phi_{k-1}(w_k)] = \Phi_k(w_k) - (1 - \alpha_k) \Phi_{k-1}(w_{k-1}).$$

Using the above equality in (19), we have

$$\alpha_k \mathbb{E}[L_{f_k}(z_{k-1}, w_k)] \leq \mathbb{E}[\Phi_k(w_k) - (1 - \alpha_k) \Phi_{k-1}(w_{k-1})].$$

Then using the above in (18),

$$\mathbb{E}[f(x_k)] \leq (1 - \alpha_k) \mathbb{E}[f(x_{k-1})] + \mathbb{E}[\Phi_k(w_k) - (1 - \alpha_k) \Phi_{k-1}(w_{k-1})] + C_2 \alpha_k^2 + \frac{L}{2} \alpha_k^2 C_1^2.$$

Rearranging, we get

$$\mathbb{E}[f(x_k) - \Phi_k(w_k)] \leq (1 - \alpha_k) \mathbb{E}[(f(x_{k-1}) - \Phi_k(w_{k-1})] + C_2 \alpha_k^2 + \frac{L}{2} \alpha_k^2 C_1^2. \quad (20)$$

Set $\nu_k = \mathbb{E}[f(x_k) - \Phi_k(w_k)]$ and define

$$\Gamma_k := \begin{cases} (1 - \alpha_k) \Gamma_{k-1}, & \text{if } k \geq 2 \\ 1, & \text{if } k = 1. \end{cases}$$

With the above notation, we can write (20) as :

$$\nu_k \leq (1 - \alpha_k) \nu_{k-1} + \frac{L}{2} \alpha_k^2 C_1^2 + C_2 \alpha_k^2,$$

$$\Rightarrow \frac{\nu_k}{\Gamma_k} \leq \frac{\nu_{k-1}}{\Gamma_{k-1}} + \frac{L}{2} \frac{\alpha_k^2}{\Gamma_k} C_1^2 + C_2 \frac{\alpha_k^2}{\Gamma_k}.$$

A simple calculation gives :

$$\Gamma_k = \frac{2}{k(k+1)} \text{ and } \frac{\alpha_k^2}{\Gamma_k} \leq 2.$$

To conclude the proof we note that,

$$\nu_k \leq \Gamma_k \left\{ \frac{\nu_1}{\Gamma_1} + \sum_{i=1}^{k} \left( \frac{L}{2} C_1^2 + C_2 \right) \left( \frac{\alpha_i^2}{\Gamma_i} \right) \right\},$$

$$\nu_k \leq \frac{2}{k(k+1)} \left\{ \frac{\nu_1}{\Gamma_1} + k(2C_1^2 + 2C_2) \right\},$$

$$= O\left( \frac{1}{k} \right).$$
From (13) we have,
$$\Phi_k(w_k) \leq \Phi_k(w) \forall w \in \Omega.$$ 
Then, by the definition of $$\Phi_k(w_k)$$,
$$\mathbb{E}[\Phi_k(w_k)] \leq \frac{1}{\sum_{i=1}^{k} \beta_i} \mathbb{E}\left[\sum_{i=1}^{k} \beta_i f(z_{i-1}) + \beta_i \langle \nabla f(z_{i-1}), w - z_{i-1} \rangle\right].$$
From the unbiasedness of $$\nabla f(x, \xi)$$ and the independence of $$z_{i-1}$$ and $$\xi$$ (see Step (b)), we have
$$\mathbb{E}[\Phi_k(w_k)] \leq \frac{1}{\sum_{i=1}^{k} \beta_i} \sum_{i=1}^{k} \{\beta_i f(z_{i-1}) + \beta_i \langle \nabla f(z_{i-1}), w - z_{i-1} \rangle\},$$
which gives from the convexity of $$f$$,
$$\mathbb{E}[\Phi_k(w_k)] \leq f(w).$$
We have for $$w = x^*$$ in the above,
$$\mathbb{E}[\Phi_k(w_k)] \leq f(x^*),$$
which gives
$$\mathbb{E}[f(x_k) - f(x^*)] \leq \mathbb{E}[f(x_k) - \Phi_k(w_k)] = \nu_k = O\left(\frac{1}{k}\right).$$

Remark 2. The condition (16) can be satisfied by setting $$p_k = k^2$$ and the condition (15) can be satisfied by using $$\beta_k = k, k \geq 1.$$

3.2 Incremental Method:
In this section we consider an incremental approach to Frank-Wolfe for solving (2). For a survey of incremental methods in the context of gradient descent and proximal algorithms, we refer the reader to [35]. The driving idea behind incremental methods for problems of form (2) is to keep track of the running average of the last $$m$$-gradient evaluations. Let $$[k]_m := (k \text{ modulo } m) + 1$$. These $$m$$ gradient evaluations can be constructed in a number of ways. The most obvious way is to go through each $$f^i$$ in a cyclic manner, so that at each instant $$k$$, we have the following average:
$$d_k = \sum_{i=1}^{m} \nabla f^{[k]_m}(x_{k-i+1}).$$
Another possible technique, popular in neural network training practice, is to reshuffle randomly the order of the component functions after a cycle of $$m$$ steps. Algorithm 4 gives the pseudo-code for incremental Frank-Wolfe. We take the cyclic approach here. The convergence analysis for the random shuffling algorithm differs only in trivial ways from the cyclic one.
Algorithm 4 Incremental Frank-Wolfe

Input : Objective Function \( f = \frac{1}{m} \sum_{i=1}^{m} f^i \); parameters \( \alpha_k \)

Initialize : \( x_0 \in \Omega, w_0 = \arg\min_{x \in \Omega} \langle \nabla f(x_0), x \rangle \) and \( d_0 = 0 \).

For \( k = 0,1,\ldots \) do :

a) Compute the gradient \( \nabla f^{[k]}(x_k) \).

b) Update the aggregated gradient:

\[
d_k = \frac{1}{m} \left\{ m d_{k-1} - \nabla f^{[k]}(x_{k-m}) + \nabla f^{[k]}(x_k) \right\}
\]

. 

c) Call the LO oracle to compute:

\[
w_k \in \arg\min_{x \in \Omega} \langle d_k, x \rangle
\]

. 

d) Set

\[
x_{k+1} = (1 - \alpha_k) x_k + \alpha_k w_k. \tag{21}
\]

end for

Theorem 3. With \( \alpha_k = \frac{2}{k+1} \), Algorithm 4 ensures that

\[
f(x_k) - f(x^*) = O\left(\frac{1}{k}\right)
\]

for any \( k > 0 \).

Proof. We have

\[
f(x_{k+1}) \leq f(x_k) + \langle \nabla f(x_k), x_{k+1} - x_k \rangle + \frac{L}{2} \| x_{k+1} - x_k \|^2 \quad \text{(from (3))}
\]

\[
= f(x_k) + \alpha_k \langle \nabla f(x_k), w_k - x_k \rangle + \frac{L}{2} \alpha_k^2 \| w_k - x_k \|^2 \quad \text{(from (21))}
\]

\[
= f(x_k) + \alpha_k \langle \nabla f(x_k) - d_k, w_k - x_k \rangle + \alpha_k \langle d_k, w_k - x_k \rangle + \frac{L}{2} \alpha_k^2 \| w_k - x_k \|^2.
\]

From the optimality of \( w_k \), we have

\[
f(x_{k+1}) \leq f(x_k) + \alpha_k \langle \nabla f(x_k) - d_k, w_k - x_k \rangle + \alpha_k \langle d_k, x^* - x_k \rangle + \frac{L}{2} \alpha_k^2 C_\Omega^2. \tag{22}
\]

By the definition of \( d_k \),

\[
\|d_k - \nabla f(x_k)\| = \frac{1}{m} \sum_{i=k-m+1}^{k} \| \nabla f^{[i]}(x_i) - \sum_{i=1}^{m} \nabla f^i(x_k) \|.
\]
Since \( \sum_{i=1}^{m} \nabla f^i(x_k) = \sum_{i=k-m+1}^{k} \nabla f^i(x_k) \), we have
\[
\|d_k - \nabla f(x_k)\| = \frac{1}{m} \| \sum_{i=k-m+1}^{k} (\nabla f^i(x_i) - \nabla f^i(x_k)) \|.
\]
From the smoothness of \( f^i(\cdot) \) for all \( i \), we have
\[
\|d_k - \nabla f(x_k)\| \leq \frac{L}{m} \sum_{i=k-m+1}^{k} \|x_i - x_k\| = \frac{L}{m} \sum_{i=1}^{m} \|x_{k-i+1} - x_k\|. \tag{23}
\]
We note from (21), that for \( 1 \leq i \leq m-1 \),
\[
x_k - x_{k-i} = \sum_{j=1}^{i} \alpha_{k-j}(w_{k-j} - x_{k-j}) \tag{24}
\]
so that
\[
\|x_k - x_{k-i}\| \leq \sum_{j=1}^{i} \alpha_{k-j}\|w_{k-j} - x_{k-j}\| \leq C_\Omega \sum_{j=1}^{i} \alpha_{k-j}. \tag{25}
\]
For \( k > 2m \), we have for \( 1 \leq j \leq m \)
\[
\frac{\alpha_{k-j}}{\alpha_k} \leq \frac{\alpha_{k-m}}{\alpha_k} \leq 2,
\]
so that
\[
\sum_{j=1}^{i} \alpha_{k-j} \leq \sum_{j=1}^{i} \left( \frac{\alpha_{k-j}}{\alpha_k} \right) \alpha_k \leq 2 \sum_{j=1}^{i} \alpha_k \leq 2i\alpha_k.
\]
We use the above bound in (23) to get
\[
\|x_k - x_{k-i}\| \leq 2C_\Omega \times i\alpha_k.
\]
Using this in (22),
\[
\|d_k - \nabla f(x_{k-1})\| \leq \frac{2LC_\Omega\alpha_k}{m} \sum_{j=1}^{m} j = LC_\Omega\alpha_k(m+1). \tag{26}
\]
We use the above bound in (22) :
\[
f(x_{k+1}) \leq f(x_k) + \alpha_k \langle d_k, x^* - x_k \rangle + \alpha_k \langle \nabla f(x_k) - d_k, w_k - x_k \rangle + \frac{L}{2} \alpha_k^2 C_\Omega^2,
\]
\[
\leq f(x_k) + \alpha_k \langle \nabla f(x_k), x^* - x_k \rangle + \alpha_k \langle \nabla f(x_k) - d_k, w_k - x^* \rangle + \frac{L}{2} \alpha_k^2 C_\Omega^2,
\]
\[
\leq f(x_k) + \alpha_k \langle \nabla f(x_k), x^* - x_k \rangle + \alpha_k C_\Omega \| \nabla f(x_k) - d_k \| + \frac{L}{2} \alpha_k^2 C_\Omega^2,
\]
\[
\leq f(x_k) + \alpha_k \langle \nabla f(x_k), x^* - x_k \rangle + \frac{LC_\Omega^2 \alpha_k^2 (m+1)}{2} + \frac{L}{2} \alpha_k^2 C_\Omega^2.
\]
This gives from the convexity of \( f(\cdot) \),
\[
f(x_{k+1}) \leq f(x_k) + \alpha_k \left( f(x^*) - f(x_k) \right) + \frac{LC_\Omega^2 \alpha_k^2 (m+2)}{2},
\]
12
so that
\[ f(x_{k+1}) - f(x^*) \leq (1 - \alpha_k)(f(x_k) - f(x^*)) + \frac{LC_\Omega^2 \alpha_k^2}{2}(m + 2). \]
Setting \( \nu_k = f(x_k) - f(x^*) \) we have,
\[ \nu_{k+1} \leq (1 - \alpha_k)\nu_k + \mathcal{O}(\alpha_k^2). \]
Proceeding the same way as in Theorem 3 we get
\[ f(x_k) - f(x^*) = \mathcal{O}\left(\frac{1}{k}\right). \]

3.3 Dual Averaging in a distributed setting

The distributed version of FW-SDA is given in Algorithm 5. We assume the same setup as in Section 2.3, namely Assumption 1 holds true for the network.

**Algorithm 5** Distributed dual averaging for Frank-Wolfe

**Input**: Objective Function \( f = \frac{1}{m} \sum_{i=1}^{m} f^i \); parameters \( \alpha_k \).

**Initialize**: \( x_0^i \in \Omega, w_0^i = \arg\min_{x \in \Omega} \langle \nabla f^i(x_0^i), x \rangle \) and \( g_0^i = 0 \) for all \( i \).

**For** \( k = 1, 2, \ldots \) **do**: At each node \( i \),

a) Set \( z_{k-1}^i = (1 - \alpha_k)x_{k-1}^i + \alpha_kw_{k-1}^i \).

b) Compute the gradient \( \nabla f^i(z_{k-1}^i) \).

c) Update the weighted average gradient using the neighbour estimates:
\[
g_k^i = \frac{1}{\sum_{p=1}^{k} \beta_p} \sum_{j=1}^{m} q_{ij} \left\{ (\sum_{p=1}^{k-1} \beta_p)g_{k-1}^j + \beta_k \nabla f^j(z_{k-1}^j) \right\}. \]

d) Call the LO oracle to compute:
\( w_k^i \in \arg\min_{w \in \Omega} \langle g_k^i, w \rangle \).

e) Set
\[ x_k^i = (1 - \alpha_k) \sum_{j=1}^{m} q_{ij}x_{k-1}^j + \alpha_kw_k^i. \]

**end for**

We use the stacked vector notation in this section. Specifically for any bold faced variable \( \mathbf{x} \), we have \( \mathbf{x} := [x^1, \ldots, x^m] \). Also, we let \( \mathbf{f}(\mathbf{x}) := [f^1(x^1), \ldots, f^m(x^m)] \). We let \( \otimes \) denote the
Proof.

We prove this by induction. Suppose that

\[ g^k = (1 - \alpha_k) x_{k-1} + \alpha_k w_{k-1}. \] (27)

Then, the main steps of Algorithm 5 can be written in a stacked vector notation as follows:

\[ g^k = \frac{1}{\beta_k} \{ (\bar{g}^k) x^k + \beta_k \nabla f(z_{k-1}) \}. \] (28)

\[ x_k = \frac{1}{\beta_k} \{ (Q \otimes I_d)(1 - \alpha_k) x_{k-1} + \alpha_k w_k \}. \] (29)

Let \( L_f(z_{k-1}; w_k) := [L_f(z^1_{k-1}; w^1_k), \ldots, L_f(z^m_{k-1}; w^m_k)] \), where \( L_f(\cdot, \cdot) \) is as in [4]. We define the distributed average linear model as,

\[ \Phi_k(w) := \frac{1}{\beta_k} \sum_{i=1}^k Q^{k-t+1} \beta_k \{ L_f(z_{k-1}; w) \} = \frac{1}{\beta_k} \sum_{i=1}^k Q^{k-t+1} \beta_k \{ f(z_{i,t-1}) + \langle \nabla f(z_{i,t-1}), w - z_{i,t-1} \rangle \} \] (30)

where \( \langle \nabla f(z_i), w - z_i \rangle := \{ \langle \nabla f^1(z^i_1), w^1 - z^1_1 \rangle, \ldots, \langle \nabla f^m(z^i_m), w^m - z^i_1 \rangle \} \). Also, we can write

\[ g^k = \frac{1}{\beta_k} \{ \bar{g}^k (Q^2 \otimes I_d) g^k - Q^2 \otimes I_d \alpha_k + \beta_k (Q^2 \otimes I_d) \nabla f(z_{k-2}) + \beta_k (Q \otimes I_d) \nabla f(z_{k-1}) \}. \]

Iterating the above equation we get,

\[ g^k = \frac{1}{\beta_k} \{ \bar{g}^k (Q^k+1 \otimes I_d) g^k + \sum_{i=1}^k \beta_i (Q^{k-t+1} \otimes I_d) \nabla f(z_{i,t-1}) \}. \]

We can assume without loss of generality that \( g^0 = 0 \) for all \( i \). Comparing the above equation with (31), we have from the definition of \( w_k \) (Step (d)) \(^4\)

\[ w_k \in \text{argmin}_w \Phi_k(w). \] (31)

**Theorem 4.** With \( \alpha_k = \frac{2}{k+1} \) and any \( \beta_k \) such that \( \alpha_k = \frac{\beta_k}{\sum_{i=1}^k \beta_i} \), we have

\[ f(\bar{x}_k) - f(x^*) = O\left(\frac{1}{k}\right), \]

where

\[ \bar{x}_k = (Q^* \otimes I_d) x_k \]

and \( Q^* = \frac{11^T}{m} \).

**Proof.** Before proceeding with the main proof, we prove the following claim:

**Claim:** \( \|x_k - (Q^* \otimes I_d)x_k\| = O(\alpha_k) \) for all \( k \geq 1 \).

**Proof:** We prove this by induction. Suppose that

\[ \|x_k - (Q^* \otimes I_d)x_k\| \leq \alpha_k. \]

\(^4\)The equality is interpreted component-wise, i.e. \( w^*_k \in \text{argmin}_w \Phi_k(w) \).
We have from (29),

\[
x_{k+1} = (Q \otimes I_d)(1 - \alpha_k)x_k + \alpha_k w_{k+1}
\]

\[
\implies (Q^* \otimes I_d)x_{k+1} = (Q^* \otimes I_d)(1 - \alpha_k)x_k + \alpha_k(Q^* \otimes I_d)w_{k+1},
\]

where we have used the fact that \((Q^* \otimes I_d) \times (Q \otimes I_d) = (Q^* \otimes I_d)\). Then we have,

\[
\|x_{k+1} - (Q^* \otimes I_d)x_{k+1}\| = \|(Q \otimes I_d)(1 - \alpha_k)x_k + \alpha_kw_{k+1}
- (Q^* \otimes I_d)(1 - \alpha_k)x_k - \alpha_k(Q^* \otimes I_d)w_{k+1}\|,
\]

so,

\[
\|x_{k+1} - (Q^* \otimes I_d)x_{k+1}\| \leq (1 - \alpha_k)\|\{(Q - Q^*) \otimes I_d\}x_k\|
+ \alpha_k\|w_{k+1} - (Q^* \otimes I_d)w_{k+1}\|.
\]

Noting that \(\{(Q - Q^*) \otimes I_d\}(Q^* \otimes I_d)x_k = 0\), we get

\[
\|x_{k+1} - (Q^* \otimes I_d)x_{k+1}\| \leq (1 - \alpha_k)\|(Q \otimes I_d) - (Q^* \otimes I_d)\|
\times \|x_k - (Q^* \otimes I_d)x_k\| + \alpha_k\|w_{k+1} - (Q^* \otimes I_d)w_{k+1}\|.
\]

Since \(1 - \alpha_k \leq 1\) and \(\|(Q \otimes I_d) - (Q^* \otimes I_d)\| \leq \theta\beta\) (from Lemma 1), we have

\[
\|x_{k+1} - (Q^* \otimes I_d)x_{k+1}\| \leq (1 - \alpha_k)\theta\beta\|x_k - (Q^* \otimes I_d)x_k\| + \alpha_k\|w_{k+1} - (Q^* \otimes I_d)w_{k+1}\|.
\]

The first term is \(O(\alpha_k)\) by the induction step and so is the second one, since \(w_{k+1}\) is bounded. The claim follows.

We continue with the proof of the theorem. From (3), we have for all \(i\) :

\[
f^i(x_k^i) \leq L_{f^i}(z_{k-1}^i; x_k^i) + \frac{L}{2}\|x_k^i - z_{k-1}^i\|^2,
\]

\[
= (1 - \alpha_k)L_{f^i}(z_{k-1}^i; x_k^i) + \alpha_k L_{f^i}(z_{k-1}^i; w_k^i) + \frac{L}{2}\|x_k^i - z_{k-1}^i\|^2.
\]

The convexity of \(f^i(\cdot)\) gives

\[
f^i(x_k^i) \leq (1 - \alpha_k)f(x_k^i) + \alpha_k L_{f^i}(z_{k-1}^i; w_k^i) + \frac{L}{2}\|x_k^i - z_{k-1}^i\|^2.
\]

Summing over \(j\), we get

\[
\sum_{j=1}^{m} q_{ij} f^j(x_k^i) \leq (1 - \alpha_k)\sum_{j=1}^{m} q_{ij}\{f^j(x_k^i) + \alpha_k L_{f^j}(z_{k-1}^i; w_k^i) + \frac{L}{2}\|x_k^i - z_{k-1}^i\|^2\},
\]

so that in vector notation,

\[
Qf(x_k) \leq (1 - \alpha_k)Qf(x_{k-1}) + \alpha_k QL_f(z_{k-1}; w_k) + \frac{L}{2}\|x_k - z_{k-1}\|^2 1, \quad (32)
\]

where the inequality is interpreted component-wise. Subtracting (27) from (29), we get

\[
\|x_k - z_{k-1}\| = (1 - \alpha_k)\|x_{k-1} - (Q \otimes I_d)x_{k-1}\| + \alpha_k\|w_k - w_{k-1}\|.
\]
Using Claim (i), we have \( \|x_k - z_{k-1}\| = O(\alpha_k) \), so that \( \|x_k - z_{k-1}\| \leq \sqrt{C_{11}} \alpha_k \) for some constant \( C_{11} \). Using this in (32),

\[
Qf(x_k) \leq (1 - \alpha_k)Qf(x_{k-1}) + \alpha_k QL_f(z_{k-1}; w_k) + \frac{L C_{11} \alpha_k^2}{2}.
\]

(33)

We also have,

\[
\tilde{\beta}_k \Phi_k(w_k) = \beta_k QL_f(z_{k-1}; w_k) + \tilde{\beta}_{k-1} \Phi_{k-1}(w_k).
\]

From (31), \( w_{k-1} \in \text{argmin}_w \Phi_{k-1}(w) \), so that \( \Phi_{k-1}(w_{k-1}) \leq \Phi_{k-1}(w_k) \). This gives

\[
\tilde{\beta}_k \Phi_k(w_k) \geq \beta_k QL_f(z_{k-1}; w_k) + \tilde{\beta}_{k-1} \Phi_{k-1}(w_{k-1}),
\]

\[\implies \beta_k QL_f(z_{k-1}; w_k) \leq \tilde{\beta}_k \Phi_k(w_k) - \tilde{\beta}_{k-1} \Phi_{k-1}(w_{k-1}).\]

Proceeding in the same way as Theorem 1,

\[
\alpha_k \beta_k^{-1} |\tilde{\beta}_k \Phi_k(w_k) - \tilde{\beta}_{k-1} \Phi_{k-1}(w_k)| = \Phi_k(w_k) - \alpha_k \beta_k^{-1} \tilde{\beta}_{k-1} \Phi_{k-1}(w_{k-1}),
\]

\[= \Phi_k(w_k) - \alpha_k \beta_k^{-1} (\tilde{\beta}_k - \tilde{\beta}_{k-1}) \Phi_{k-1}(w_{k-1}),
\]

\[= \Phi_k(w_k) - (1 - \alpha_k) \Phi_{k-1}(w_{k-1}).\]

Croming the above facts we have,

\[
\alpha_k QL_f(z_{k-1}; w_k) \leq \Phi_k(w_k) - (1 - \alpha_k) \Phi_{k-1}(w_{k-1}).
\]

Using this in (33) we get,

\[
Qf(x_k) \leq (1 - \alpha_k)Qf(x_{k-1}) + (\Phi_k(w_k) - (1 - \alpha_k) \Phi_{k-1}(w_{k-1})) + \frac{L C_{11} \alpha_k^2}{2}.
\]

Rearranging, we get

\[
Qf(x_k) - \Phi_k(w_k) \leq (1 - \alpha_k) \{Qf(x_{k-1}) - \Phi_{k-1}(w_{k-1})\} + \frac{L}{2} \alpha_k^2 C_{11} \cdot 1.
\]

(34)

Set \( \nu_k = Qf(x_k) - \Phi_k(w_k) \) and define

\[
\Gamma_k := \begin{cases} (1 - \alpha_k)\Gamma_{k-1}, & \text{if } k \geq 2 \\ 1, & \text{if } k = 1. \end{cases}
\]

We can write (31) with the above notation adi:

\[
\nu_k \leq (1 - \alpha_k)\nu_{k-1} + \frac{L}{2} \alpha_k^2 C_{11},
\]

\[\implies \frac{\nu_k}{\Gamma_k} \leq \frac{\nu_{k-1}}{\Gamma_{k-1}} + \frac{L}{2} \frac{\alpha_k^2 C_{11}}{\Gamma_k} \quad \text{(since } \Gamma_k).\]

A simple calculation gives:

\[
\Gamma_k = \frac{2}{k(k + 1)} \quad \text{and} \quad \frac{\alpha_k^2}{\Gamma_k} \leq 2.
\]

5 interpreted component-wise
To conclude the proof we note that,
\[ \nu_k \leq \Gamma_k \left\{ \frac{\nu_1}{\Gamma_1} + \sum_{i=1}^{k} \left( \frac{L_2 C_2^2 + C_{\Omega}}{\Gamma_1} \right) \right\}, \]
\[ \nu_k \leq \frac{2}{k(k + 1)} \left\{ \frac{\nu_1}{\Gamma_1} + k(L C_2^2 + 2C_{\Omega}) \right\}, \]
\[ = O\left( \frac{1}{k} \right). \]

Thus we have,
\[ Qf(x_k) - F_k(w_k) \leq O\left( \frac{1}{k} \right). \]

Since \( Q^* \times Q = Q^* \), we have
\[ Q^*f(x_k) - Q^*F_k(w_k) \leq O\left( \frac{1}{k} \right). \] (35)

Let \( x^* := [x^*, ..., x^*] \). We have,
\[ F_k(w_k) \leq F_k(x^*) \leq \frac{1}{\beta_k} \sum_{i=1}^{k} \beta_i Q^{k-i+1}f(x^*) \]
so that,
\[ Q^*F_k(w_k) \leq Q^*f(x^*). \]

Using the above inequality in (35), we get,
\[ Q^*f(x_k) - Q^*f(x^*) \leq Q^*f(x_k) - Q^*F_k(w_k) \leq O\left( \frac{1}{k} \right). \]

Using any row of the above inequality we get
\[ \frac{1}{m} \sum_{i=1}^{m} f^i(x_k^i) - \frac{1}{m} \sum_{i=1}^{m} f^i(x^*) \leq O\left( \frac{1}{k} \right). \]

Then using the \( L \)-smoothness of \( f(\cdot) \),
\[ |f(x_k) - f(x^*)| = |f(x_k) - \frac{1}{m} \sum_{i=1}^{m} f^i(x_k^i)| + |\frac{1}{m} \sum_{i=1}^{m} f^i(x_k^i) - f(x^*)| \leq O(\alpha_k) + O\left( \frac{1}{k} \right) = O\left( \frac{1}{k} \right), \]
which concludes the proof. \( \Box \)

References

[1] L. Bottou, F. E. Curtis, and J. Nocedal, “Optimization methods for large-scale machine learning,” SIAM Review, vol. 60, no. 2, pp. 223–311, 2018.

[2] M. Frank and P. Wolfe, “An algorithm for quadratic programming,” Naval Research Logistics (NRL), vol. 3, no. 1-2, pp. 95–110, 1956.

[3] Y. Nesterov, “Primal-dual subgradient methods for convex problems,” Mathematical Programming, vol. 120, no. 1, pp. 221–259, 2009.
[4] L. Xiao, “Dual averaging methods for regularized stochastic learning and online optimization,” *Journal of Machine Learning Research*, vol. 11, no. Oct, pp. 2543–2596, 2010.

[5] J. C. Duchi, A. Agarwal, and M. J. Wainwright, “Dual averaging for distributed optimization: Convergence analysis and network scaling,” *IEEE Transactions on Automatic Control*, vol. 57, no. 3, pp. 592–606, 2012.

[6] P. Wolfe, “Convergence theory in nonlinear programming,” *Integer and nonlinear programming*, pp. 1–36, 1970.

[7] M. Jaggi, “Revisiting Frank-Wolfe: Projection-free sparse convex optimization.” in *International Conference on Machine Learning*, 2013, pp. 427–435.

[8] A. Beck and M. Teboulle, “A conditional gradient method with linear rate of convergence for solving convex linear systems,” *Mathematical Methods of Operations Research*, vol. 39, no. 2, pp. 235–247, 2004.

[9] J. Guélat and P. Marcotte, “Some comments on Wolfe’s away step,” *Mathematical Programming*, vol. 35, no. 1, pp. 110–119, 1986.

[10] E. S. Levitin and B. T. Polyak, “Constrained minimization methods,” *USSR Computational Mathematics and Mathematical Physics*, vol. 6, no. 5, pp. 1–50, 1966.

[11] D. Garber and E. Hazan, “Faster rates for the Frank-Wolfe method over strongly-convex sets.” in *International Conference on Machine Learning*, 2015, pp. 541–549.

[12] G. Lan, “The complexity of large-scale convex programming under a linear optimization oracle,” *arXiv preprint arXiv:1309.5550*, 2013.

[13] J. D. Abernethy and J.-K. Wang, “On Frank-Wolfe and equilibrium computation,” in *Advances in Neural Information Processing Systems*, 2017, pp. 6587–6596.

[14] H. Lu and R. M. Freund, “Generalized stochastic Frank-Wolfe algorithm with stochastic” substitute” gradient for structured convex optimization,” *arXiv preprint arXiv:1807.07680*, 2018.

[15] E. Hazan and H. Luo, “Variance-reduced and projection-free stochastic optimization,” in *International Conference on Machine Learning*, 2016, pp. 1263–1271.

[16] G. Lan and Y. Zhou, “Conditional gradient sliding for convex optimization,” *SIAM Journal on Optimization*, vol. 26, no. 2, pp. 1379–1409, 2016.

[17] R. Johnson and T. Zhang, “Accelerating stochastic gradient descent using predictive variance reduction,” in *Advances in Neural Information Processing Systems*, 2013, pp. 315–323.

[18] M. Mahdavi, L. Zhang, and R. Jin, “Mixed optimization for smooth functions,” in *Advances in Neural Information Processing Systems*, 2013, pp. 674–682.

[19] S. J. Reddi, S. Sra, B. Póczos, and A. Smola, “Stochastic Frank-Wolfe methods for non-convex optimization,” in *Communication, Control, and Computing (Allerton), 2016 54th Annual Allerton Conference on*. IEEE, 2016, pp. 1244–1251.

[20] J. Tsitsiklis, D. Bertsekas, and M. Athans, “Distributed asynchronous deterministic and stochastic gradient optimization algorithms,” *IEEE Transactions on Automatic Control*, vol. 31, no. 9, pp. 803–812, 1986.
[21] A. Bellet, Y. Liang, A. B. Garakani, M.-F. Balcan, and F. Sha, “A distributed Frank–Wolfe algorithm for communication-efficient sparse learning,” in Proceedings of the 2015 SIAM International Conference on Data Mining. SIAM, 2015, pp. 478–486.

[22] H.-T. Wai, J. Lafond, A. Scaglione, and E. Moulines, “Decentralized Frank–Wolfe algorithm for convex and nonconvex problems,” IEEE Transactions on Automatic Control, vol. 62, no. 11, pp. 5522–5537, 2017.

[23] A. Nediic, A. Olshevsky, and W. Shi, “Achieving geometric convergence for distributed optimization over time-varying graphs,” SIAM Journal on Optimization, vol. 27, no. 4, pp. 2597–2633, 2017.

[24] S. M. Shah and V. S. Borkar, “Q-learning for markov decision processes with a satisfiability criterion,” Systems & Control Letters, vol. 113, pp. 45–51, 2018.

[25] S. M. Shah, “Stochastic approximation on riemannian manifolds,” Applied Mathematics & Optimization, vol. 83, no. 2, pp. 1123–1151, 2021.

[26] S. M. Shah and V. K. Lau, “Model compression for communication efficient federated learning,” IEEE Transactions on Neural Networks and Learning Systems, 2021.

[27] S. M. Shah and K. E. Avrachenkov, “Linearly convergent asynchronous distributed admm via markov sampling,” arXiv preprint arXiv:1810.05067, 2018.

[28] K. E. Avrachenkov, V. S. Borkar, S. Moharir, and S. M. Shah, “Dynamic social learning under graph constraints,” IEEE Transactions on Control of Network Systems, vol. 9, no. 3, pp. 1435–1446, 2021.

[29] S. M. Shah, “Making simulated annealing sample efficient for discrete stochastic optimization,” arXiv preprint arXiv:10.48550/ARXIV.2009.06188, 2020.

[30] S. M. Shah and V. S. Borkar, “Mean field limits through local interactions,” Advances in Dynamic and Mean Field Games: Theory, Applications, and Numerical Methods, 2017.

[31] ———, “Distributed stochastic approximation with local projections,” SIAM Journal on Optimization, vol. 28, no. 4, pp. 3375–3401, 2018.

[32] S. M. Shah, L. Su, and V. K. N. Lau, “Robust federated learning over noisy fading channels,” IEEE Internet of Things Journal, pp. 1–1, 2022.

[33] V. S. Borkar and S. M. Shah, “Distributed algorithms: Tsitsiklis and beyond,” in 2018 Information Theory and Applications Workshop (ITA). IEEE, 2018, pp. 1–9.

[34] S. Lacoste-Julien, M. Jaggi, M. Schmidt, and P. Pletscher, “Block-coordinate Frank-Wolfe optimization for structural svms,” arXiv preprint arXiv:1207.4747, 2012.

[35] D. P. Bertsekas, “Incremental gradient, subgradient, and proximal methods for convex optimization: A survey,” Optimization for Machine Learning, vol. 2010, no. 1-38, p. 3, 2011.

[36] A. Nedic, A. Ozdaglar, and P. A. Parrilo, “Constrained consensus and optimization in multi-agent networks,” IEEE Transactions on Automatic Control, vol. 55, no. 4, pp. 922–938, 2010.