Space–time statistical solutions
for an inhomogeneous chain of harmonic oscillators

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Abstract

We consider an one-dimensional inhomogeneous harmonic chain consisting of two different semi-infinite chains of harmonic oscillators. We study the Cauchy problem with random initial data. Under some restrictions on the interaction between the oscillators of the chain and on the distribution of the initial data, we prove the convergence of space-time statistical solutions to a Gaussian measure.

*Key words: inhomogeneous chain of harmonic oscillators, Cauchy problem, random initial data, space-time statistical solutions, weak convergence of measures*

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1 Introduction

We consider an infinite one-dimensional harmonic chain of particles having nearest-neighbor interactions and unit mass. We assume that the particles located at points \( x = 1, 2, \ldots \) have the same interaction force constants \( \nu_+ > 0 \), and the same external harmonic forces with constants \( \kappa_+ \geq 0 \) act on them. The particles located at points \( x = -1, -2, \ldots \) have constants \( \nu_- > 0 \) and \( \kappa_- \geq 0 \), respectively. In addition, an external force with a constant \( \kappa_0 \geq 0 \) acts on the particle located at the origin, and \( \kappa_0 \neq \kappa_\pm \), in general. Therefore, the displacement of the particle located at a point \( x \in \mathbb{Z} \) from its equilibrium position obeys the following equations:

\[
\begin{cases}
\ddot{u}(x, t) = (\nu_+^2 \Delta_L - \kappa_+^2)u(x, t), & x \geq 1, \quad t > 0, \\
\ddot{u}(0, t) = \nu_+^2 (u(1, t) - u(0, t)) + \nu_-^2 (u(-1, t) - u(0, t)) - \kappa_0^2 u(0, t), & t > 0, \\
\ddot{u}(x, t) = (\nu_-^2 \Delta_L - \kappa_-^2)u(x, t), & x \leq -1, \quad t > 0.
\end{cases}
\]

(1.1)

Here \( u(x, t) \in \mathbb{R}, \Delta_L \) denotes the second derivative on \( \mathbb{Z} = \{0, \pm 1, \pm 2, \ldots \} \):

\[\Delta_L u(x) = u(x + 1) - 2u(x) + u(x - 1), \quad x \in \mathbb{Z}.\]

For system (1.1), we study the Cauchy problem with the initial data

\[u(x, 0) = u_0(x), \quad \dot{u}(x, 0) = v_0(x), \quad x \in \mathbb{Z}.\]

(1.2)

Formally, this system is Hamiltonian with the Hamiltonian functional of the form

\[H(u, \dot{u}) = H_+(u, \dot{u}) + H_-(u, \dot{u}) + H_0(u, \dot{u}),\]

\[H_\pm(u, \dot{u}) := \frac{1}{2} \sum_{\pm x \geq 1} \left( |\dot{u}(x, t)|^2 + \nu_\pm^2 |u(x \pm 1, t) - u(x, t)|^2 + \kappa_\pm^2 |u(x, t)|^2 \right),\]

\[H_0(u, \dot{u}) := \frac{1}{2} \left( |\dot{u}(0, t)|^2 + \sum_{\pm} \nu_\pm^2 |u(\pm 1, t) - u(0, t)|^2 + \kappa_0^2 |u(0, t)|^2 \right).\]

We consider two cases of equations (1.1). In the first case, we assume that the harmonic chain is homogeneous and \( \kappa_\pm, \kappa_0 > 0 \), i.e.,

\[\nu_\pm := \nu > 0 \quad \text{and} \quad \kappa_\pm = \kappa_0 =: \kappa > 0.\]

(1.3)

In the second one, we impose condition C on the coefficients \( \nu_\pm > 0 \) and \( \kappa_0, \kappa_\pm \geq 0 \). To state this condition, we introduce the following notation. For simplicity, we assume that \( \kappa_- \leq \kappa_+ \).

Put \( a_\pm := \sqrt{4\nu_\pm^2 + \kappa_\pm^2} \) and

\[K_\pm(\omega) := \frac{1}{2} \left( \kappa_-^2 + \kappa_+^2 \right) + \frac{1}{2} \sqrt{\omega^2 - \kappa_-^2} \sqrt{\omega^2 - \kappa_+^2} \quad \text{for} \quad \omega \in \mathbb{R} : \quad |\omega| \geq a_\pm;\]

\[K_0(\omega) := \frac{1}{2} \left( \kappa_-^2 + \kappa_+^2 \right) - \frac{1}{2} \sqrt{\kappa_-^2 - \omega^2} \sqrt{a_-^2 - \omega^2} \quad \text{for} \quad \omega \in \mathbb{R} : \quad |\omega| \leq \kappa_+ \quad \text{(if} \quad \kappa_+ > 0).\]

**Condition C.** For different values of \( \kappa_\pm \) and \( \nu_\pm \), the constant \( \kappa_0 \) satisfies the following restrictions:

\[\kappa_0^2 < K_+(a_-), \quad \text{if} \quad a_- \geq a_+; \quad \kappa_0^2 < K_-(a_+), \quad \text{if} \quad a_+ \geq a_-;\]

\[\kappa_0^2 > K_0(\kappa_-), \quad \text{if} \quad \kappa_- \neq 0;\]

\[\kappa_0^2 > K_-(\kappa_+) \quad \text{or} \quad \kappa_0^2 < K_0(\kappa_-), \quad \text{if} \quad a_- \leq \kappa_+;\]

\[\kappa_0 \neq 0, \quad \text{if} \quad \kappa_- = \kappa_+ = 0.\]
Note that if $\kappa_+ = \kappa_-$, then condition C implies that

$$
\kappa_0^2 \in \left( \kappa_-^2, \kappa_-^2 + 2 \max \{ \nu_-, \nu_+ \} \sqrt{\nu_-^2 - \nu_+^2} \right) \quad \text{and} \quad \nu_- \neq \nu_+.
$$

Thus, condition C excludes the case when two semi-infinite parts of the chain are identical, i.e., when $\kappa_+ = \kappa_-$ and $\nu_+ = \nu_-$. We assume that the initial data $Y_0$ belong to the phase space $\mathcal{H}_\alpha$, $\alpha \in \mathbb{R}$, defined below.

**Definition 1.1** (i) $\ell^2_\alpha \equiv \ell^2_\alpha(\mathbb{Z})$, $\alpha \in \mathbb{R}$, is the Hilbert space of real-valued sequences $u(x)$, $x \in \mathbb{Z}$, with the norm

$$
\|u\|_\alpha = \left( \sum_{x \in \mathbb{Z}} \langle x \rangle^{2\alpha} u^2(x) \right)^{1/2} < \infty, \quad \langle x \rangle := (1 + x^2)^{1/2}.
$$

Below we use also the notation $\ell^2 = \ell^2_0$.

(ii) $\mathcal{H}_\alpha = \ell^2_\alpha \times \ell^2_\alpha$ is the Hilbert space of pairs $Y = (u(x), v(x))$ of real-valued sequences $u(x)$ and $v(x)$ endowed with the norm

$$
\|Y\|_\alpha^2 = \|u\|_\alpha^2 + \|v\|_\alpha^2 < \infty.
$$

(iii) Write $\mathcal{C}^k_\alpha = C^k(\mathbb{R}; \ell^2_\alpha)$, $k = 0, 1$, $\alpha \in \mathbb{R}$. Introduce the seminorms in $\mathcal{C}^k_\alpha$ by the rule

$$
|u(\cdot, \cdot)|^2_{\alpha,k,T} = \max_{|t| \leq T} \sum_{r=0}^k \|\partial_t^r u(\cdot, t)\|_{\alpha}^2, \quad T > 0. \quad (1.4)
$$

(iv) Denote by $R$ the operator $R : \mathcal{H}_\alpha \to \mathcal{C}^1_\alpha$ such that

$$
(RY_0)(x,t) = u(x,t), \quad (1.5)
$$

where $u(x,t)$ is the solution to problem (1.1)–(1.2) with the initial data $Y_0 = (u_0, v_0)$.

Below we assume that $\alpha < -3/2$ if condition C holds and $\alpha < -1/2$ if condition (1.3) holds. We suppose that the initial date $Y_0$ is a random function. Denote by $\mu_0$ a Borel probability measure on $\mathcal{H}_\alpha$ giving the distribution of $Y_0$.

**Definition 1.2** Introduce a Borel probability measure $P$ on the space $\mathcal{C}^1_\alpha$ by the rule

$$
P(\omega) = \mu_0(R^{-1}\omega) \quad \text{for any Borel set} \quad \omega \in \mathcal{B}(\mathcal{C}^1_\alpha).
$$

Here and below $\mathcal{B}(X)$ denotes the $\sigma$-algebra of Borel sets of a topological space $X$. The measure $P$ is called a space-time statistical solution to problem (1.1)–(1.2) corresponding to the initial measure $\mu_0$. Denote by $\{P_\tau, \tau \in \mathbb{R}\}$ the following family of measures

$$
P_\tau(\omega) = P(S^{-1}_\tau \omega) \quad \text{for any} \quad \omega \in \mathcal{B}(\mathcal{C}^1_\alpha), \quad \tau \in \mathbb{R}.
$$

Here $S_\tau$ denotes the shift operator in time,

$$
S_\tau(u(x,t)) = u(x,t + \tau), \quad \tau \in \mathbb{R}. \quad (1.6)
$$
The main goal of the paper is to prove that the measures $P_\tau$ weakly converge as $\tau \to \infty$ to a limit on the space $\mathcal{C}_0^{\alpha}$,

$$P_\tau \to P_\infty, \quad \tau \to \infty. \tag{1.7}$$

This means the convergence of the integrals

$$\int_{\mathcal{C}_0^{\alpha}} f(u) P_\tau(du) \to \int_{\mathcal{C}_0^{\alpha}} f(u) P_\infty(du) \quad \text{as} \quad \tau \to \infty$$

for any bounded continuous functional $f$ on $\mathcal{C}_0^{\alpha}$. Furthermore, the limit measure $P_\infty$ is a Gaussian measure on the space $\mathcal{C}_1^{\alpha}$ supported by the solutions to problem (1.1). Thus, the convergence (1.7) can be considered as an analog of the central limit theorem for a class of solutions to the equations (1.1). The proof of convergence (1.7) is based on the results of [6] and used the technique of [11, 16]. Also, we check that the group $S_\tau$ is mixing w.r.t. the measure $P_\infty$, i.e., for any $f, g \in L^2(\mathcal{C}_\alpha, P_\infty)$,

$$\lim_{\tau \to \infty} \int f(S_\tau u) g(u) P_{\infty}(du) = \int f(u) P_{\infty}(du) \int g(u) P_{\infty}(du). \tag{1.8}$$

For models described by partial differential equations, the long-time behavior of space-time statistical solutions was studied by Komech and Ratanov [11] for wave equations and Ratanov [14] for parabolic equations. For Klein–Gordon equations, the result was obtained in [2]. The time evolution and ergodic properties of infinite harmonic crystals were studied by Lanford, Lebowitz [12] and by van Hemmen [8]. For the one-dimensional chains of harmonic oscillators, the behavior of statistical solutions $\mu_t := [U(t)]^* \mu_0$ as $t \to \infty$, where $U(t)$ stands for the solving operator of problem (1.1)–(1.2), was investigated in [6]. In this paper, we extend these results to the space-time statistical solutions of problem (1.1).

## 2 Main results

Introduce the notation

$$Y_0(x) = (Y_0^0(x), Y_1^0(x)) \equiv (u_0(x), v_0(x)), \quad Y(t) = (Y^0(t), Y^1(t)) \equiv (u(\cdot, t), \dot{u}(\cdot, t)).$$

**Theorem 2.1** (see [4, Theorem 2.1]) Let $\kappa_\pm, \kappa_0 \geq 0$, $\nu_\pm > 0$ and $Y_0 \in \mathcal{H}_\alpha, \alpha \in \mathbb{R}$. Then the Cauchy problem \((1.1)–(1.2)\) has a unique solution $Y(t) \in C(\mathbb{R}, \mathcal{H}_\alpha)$. The operator $U(t) : Y_0 \to Y(t)$ is continuous in $\mathcal{H}_\alpha$. Furthermore, there exist constants $C, B < \infty$ such that

$$\|U(t)Y_0\|_\alpha \leq Ce^{B|t|}\|Y_0\|_\alpha, \quad t \in \mathbb{R}. \tag{2.1}$$

**Corollary 2.2** It follows from (2.1) that for any $Y_0 \in \mathcal{H}_\alpha$,

$$|R Y_0|_{\alpha,1,T} \leq C(T)\|Y_0\|_\alpha, \quad \forall T > 0,$$

where the operator $R$ is defined in (1.3).

Below we assume that $\alpha < -1/2$ if condition (1.3) holds and $\alpha < -3/2$ if condition C holds.
2.1 Conditions on the initial measure

Definition 2.3 (i) A measure $\mu$ is called translation invariant (or space homogeneous) if $\mu(S_hB) = \mu(B)$ for any $B \in \mathcal{B}(\mathcal{H}_\alpha)$ and $h \in \mathbb{Z}$, where $S_hY(x) = Y(x + h)$, $x \in \mathbb{Z}$.

(ii) For a probability measure $\mu$ on $\mathcal{H}_\alpha$, we denote by $\hat{\mu}$ its characteristic functional (Fourier transform),

$$\hat{\mu}(\Psi) \equiv \int \exp(i(Y, \Psi)) \mu(dY), \quad \Psi \in \mathcal{S}.$$  

Here $\Psi = (\Psi^0, \Psi^1) \in \mathcal{S} := S \oplus S$, $S := S(\mathbb{Z})$, where $S(\mathbb{Z})$ denotes a space of real quickly decreasing sequences,

$$\langle Y, \Psi \rangle = \sum_{i=0,1} \sum_{x \in \mathbb{Z}} Y_i(x) \Psi_i(x), \quad Y = (Y^0, Y^1), \quad \Psi = (\Psi^0, \Psi^1).$$  

Below we use also the notation $\langle Y, \Psi \rangle_\pm := \sum_{i=0,1} \sum_{x \in \mathbb{Z}_\pm} Y_i(x) \Psi_i(x)$, $\mathbb{Z}_\pm := \{x \in \mathbb{Z} : \pm x \geq 0\}$.

(iii) A measure $\mu$ is called Gaussian (of zero mean) if its characteristic functional has the form $\hat{\mu}(\Psi) = \exp\{-Q(\Psi, \Psi)/2\}$, where $Q$ is a real-valued nonnegative quadratic form in $S$.

We assume that the initial data $Y_0(x)$ in (1.2) is a measurable random function with values in $(\mathcal{H}_\alpha, \mathcal{B}(\mathcal{H}_\alpha))$. Recall that $\mu_0$ is a Borel probability measure on $\mathcal{H}_\alpha$ which is the distribution of $Y_0$. Let $\mathbb{E}$ stand for the mathematical expectation w.r.t. this measure. Denote by $Q_0(x, y) = (Q^0_{ij}(x, y))_{i,j=0,1}$ the correlation matrix of the measure $\mu_0$, where

$$Q^0_{ij}(x, y) := \mathbb{E}(Y_i^0(x)Y_j^0(y)) \equiv \int Y_i^0(x)Y_j^0(y) \mu_0(dY_0), \quad x, y \in \mathbb{Z}, \quad i, j = 0, 1,$$

and by $Q_0(\Psi, \Psi)$ a real-valued quadratic form on $\mathcal{S}$ with the matrix kernel $Q_0(x, y)$.

We impose conditions S1–S4 on the initial measure $\mu_0$.

S1 $\mu_0$ has zero mean value, i.e., $\mathbb{E}(Y_0(x)) = 0$, $x \in \mathbb{Z}$.

S2 The correlation functions $Q^0_{ij}(x, y)$ satisfy the bound

$$|Q^0_{ij}(x, y)| \leq h(|x - y|), \quad (2.2)$$

where $h$ is a nonnegative bounded function and $h(r) \in L^1(0, +\infty)$.

S3 The correlation matrix $Q_0(x, y)$ satisfies the following condition

$$Q_0(x + y, y) \to \begin{cases} q_-(x) & \text{as } y \to -\infty \\ q_+(x) & \text{as } y \to +\infty \end{cases} \quad x \in \mathbb{Z}. \quad (2.3)$$

Here $q_\pm(x) = (q^0_{ij}(x))_{i,j=0,1}$ stand for correlation matrices of some translation invariant measures $\mu_\pm$ with zero mean in $\mathcal{H}_\alpha$.

Definition 2.4 Let $A$ be an interval in $\mathbb{Z}$. Denote by $\sigma(A)$ a $\sigma$-algebra in $\mathcal{H}_\alpha$ generated by the initial data $Y_0(x)$ with $x \in A$. Introduce the Ibragimov mixing coefficient of the measure $\mu_0$ by the rule

$$\varphi(r) \equiv \sup \frac{|\mu_0(A \cap B) - \mu_0(A)\mu_0(B)|}{\mu_0(B)}.$$  

4
Here the supremum is taken over all sets $A \in \sigma(A), B \in \sigma(B)$ with $\mu_0(B) > 0$, and all intervals $A, B \subset \mathbb{Z}$ with distance $d(A, B) \geq r$. The measure $\mu_0$ satisfies Ibragimov’s strong uniform mixing condition if $\varphi(r) \to 0$ as $r \to \infty$ (cf. [4, Definition 17.2.2]).

**S4** $\mu_0$ has a finite “mean energy density”, i.e., $\sup_{x \in \mathbb{Z}} E|Y_0(x)|^2 \leq \epsilon_0 < \infty$. Moreover, $\mu_0$ satisfies Ibragimov’s strong uniform mixing condition, and $\varphi^{3/2}(r) \in L^1(0, +\infty)$.

**Lemma 2.5** (i) Condition **S2** implies that for any $\Phi, \Psi \in \mathcal{H}_0$,

$$|Q_0(\Phi, \Psi)| \equiv |\langle Q_0(x, y), \Phi(x) \otimes \Psi(y) \rangle| \leq C \|\Phi\|_0 \|\Psi\|_0. \tag{2.4}$$

This follows from the bound (2.2) applying either the Shur test (see, e.g., [13, p.223]) or Young’s inequality (see, e.g., [13, Theorem 0.3.1]).

(ii) It follows from conditions **S1**–**S3** that $q^{ij}_{\pm} \in \ell^1$, $i, j = 0, 1$. Hence, $q^{\pm}_{ij} \in C(\mathbb{T})$.

Assertions (i) and (ii) are proved in [4, Lemma 5.1].

(iii) Conditions **S1** and **S4** imply the bound (2.2) with the function $h(r) = C\epsilon_0 \varphi^{3/2}(r)$. This follows from [4, Lemma 17.2.3].

(iv) The correlation functions $Q_0^{ij}$ have the property: $Q_0^{ij}(x, y) = Q_0^{j i}(y, x)$, $i, j = 0, 1$. Then, the correlation functions $q^{\pm}_{ij}$ from condition **S3** satisfy the relation

$$q^{\pm}_{ij}(-x) = q^{ij}_{\pm}(x), \quad q^{10}_{\pm}(x) = q^{01}_{\pm}(-x), \quad x \in \mathbb{Z}. \tag{2.5}$$

### 2.2 The convergence of space–time statistical solutions

Denote by $\mathcal{P}$ a space of real-valued functions $v(x, t)$ which are infinite differentiable in $t$ and quickly decrease in $t$ and $x$,

$$\sup_{t \in \mathbb{R}} \sup_{x \in \mathbb{Z}} |v(x, t)|^M \leq C < \infty \quad \text{for any } M, N \text{ and } r \geq 0.$$

Let $\langle \cdot, \cdot \rangle$ stand for the inner product in $L^2(\mathbb{R}; \ell^2)$ (or in its extensions),

$$\langle u_1, u_2 \rangle = \sum_{x \in \mathbb{Z}} \int_{-\infty}^{+\infty} u_1(x, t) u_2(x, t) \, dt.$$

**Definition 2.6** Denote by $Q^P_\tau(x_1, x_2, t_1, t_2), x_1, x_2 \in \mathbb{Z}$, $t_1, t_2 \in \mathbb{R}$, the correlation functions of the measures $P_\tau$, $\tau \in \mathbb{R}$, introduced in Definition [13] i.e., for any $v_1, v_2 \in \mathcal{P}$,

$$Q^P_\tau(v_1, v_2) := \langle Q^P_\tau, v_1 \otimes v_2 \rangle = \int [u, v_1][u, v_2] \, P_\tau(du)$$

$$= \sum_{x_1, x_2 \in \mathbb{Z}} \int_{-\infty}^{+\infty} \int_{-\infty}^{+\infty} Q^P_\tau(x_1, x_2, t_1, t_2) v_1(x_1, t_1) v_2(x_2, t_2) \, dt_2, \quad \tau \in \mathbb{R}.$$

The main result of the paper is the following theorem.
Theorem 2.7 Let \( \alpha < -3/2 \) and condition C hold. Then the following assertions hold.

(i) Let conditions S1–S3 be fulfilled. Then the correlation functions of \( P_\tau \) converge to a limit as \( \tau \to \infty \). Moreover, for any \( v_1, v_2 \in \mathcal{P} \),

\[
Q^P_\tau(v_1, v_2) \to Q^P_\infty(v_1, v_2) \quad \text{as} \quad \tau \to \infty, \tag{2.6}
\]

where

\[
Q^P_\infty(v_1, v_2) = Q^{P,\nu}_\infty(T\Omega'\vec{v}_1, T\Omega'\vec{v}_2), \tag{2.7}
\]

\( \vec{v}_i := (v_i, 0) \), the quadratic form \( Q^{P,\nu}_\infty \) is defined in (3.18) below, the operators \( \Omega' \) and \( T \) are defined in (3.32) and (3.39), respectively.

(ii) Let conditions S1, S3, and S4 be fulfilled. Then the convergence (1.7) holds. The limit measure \( P_\infty \) is a Gaussian measure on the space \( \mathcal{C}_\alpha \) supported by the solutions to problem (1.1).

(iii) The measure \( P_\infty \) is invariant w.r.t. the shifts in time, and the convergence (1.8) holds.

Remark. If the initial measure \( \mu_0 \) is Gaussian, then convergence (1.7) follows from convergence (2.6). Furthermore, the weak convergence of the measures \( P_\tau \) doesn’t imply, in general, the convergence of their correlation matrices. Therefore, the last fact we prove separately.

Theorem 2.7 is proved in Section 3. In Appendix, we consider the homogeneous case (1.3) and prove the similar results.

Theorem 2.8 Let \( \alpha < -1/2 \) and condition (1.3) hold. Then all assertions of Theorem 2.7 remain true with the limiting correlation function \( Q^P_\infty(x_1, x_2, t_1, t_2) \) of the following form

\[
Q^P_\infty(x_1, x_2, t_1, t_2) = q^P_\infty(x_1 - x_2, t_1 - t_2). \tag{2.8}
\]

The Fourier transform of \( q^P_\infty(x,t) \) w.r.t. variable \( x \) \( (x \to \theta) \) is of the form

\[
\hat{q}^P_\infty(\theta, t) = \cos(\phi(\theta)t) q^{00}_\infty(\theta) - \sin(\phi(\theta)t) \phi^{-1}(\theta) q^{01}_\infty(\theta), \tag{2.8}
\]

where \( \phi(\theta) := \sqrt{\nu^2(2 - 2\cos \theta) + \kappa^2} \), and

\[
\hat{q}^{ij}_\infty(\theta) = \hat{q}^{ij}_\infty(\theta) + \hat{q}^{ij}_\infty(\theta), \quad i, j = 0, 1, \tag{2.9}
\]

with \( \hat{q}^{ij}_\infty(\theta) \) defined similarly to (3.11) but with \( \phi(\theta) \) instead of \( \phi(\pm(\theta)). \)

3 Proof: Inhomogeneous case of the chain

We divide the proof of Theorem 2.7 into two steps:

**Step 1**: Instead of problem (1.1)–(1.2) we first study a simpler “unperturbed” problem (3.1) with zero condition at origin and prove the results similar to Theorem 2.7, see Section 3.1.

**Step 2**: In Section 3.2 we introduce a “wave” operator \( \Omega \), which allows us to reduce the “perturbed” problem (1.1)–(1.2) to the problem (3.1).
3.1 Unperturbed problem

Consider the following problem

\[
\begin{cases}
\ddot{z}(x, t) = (\nu^2 - \Delta_L - \kappa_\pm^2)z(x, t), & \pm x \geq 1, \quad t > 0, \\
z(0, t) = 0, & t \geq 0, \\
z(x, 0) = u_0(x), \quad \dot{z}(x, 0) = v_0(x), & x \neq 0.
\end{cases}
\]  

(3.1)

Lemma 3.1 (see [6, Lemma 2.1]) Let \( \alpha \in \mathbb{R} \). Then for any \( Y_0 \equiv (u_0, v_0) \in \mathcal{H}_\alpha \) there exists a unique solution \( Z(t) \equiv (z(\cdot, t), \dot{z}(\cdot, t)) \in C(\mathbb{R}, \mathcal{H}_\alpha) \) to problem (3.1). Furthermore, the operator \( U_0(t) : Y_0 \mapsto Z(t) \) is continuous in \( \mathcal{H}_\alpha \), and \( \|U_0(t)Y_0\|_\alpha \leq Ce^{B|t|}\|Y_0\|_\alpha, \ t \in \mathbb{R} \).

The solution to problem (3.1) consists of two solutions to the initial–boundary value problems in \( \mathbb{Z}_+ \) and \( \mathbb{Z}_- \) with zero boundary condition at \( x = 0 \). Therefore, the solution to (3.1) has a form

\[
(U_0(t)Y)_0(x) = \begin{cases}
\sum_{j=0}^{\infty} \sum_{y \in \mathbb{Z}_+} G_{t,+}^{ij}(x, y) Y_0^{ij}(y) & \text{for } x \in \mathbb{Z}_+, \\
\sum_{j=0}^{\infty} \sum_{y \in \mathbb{Z}_-} G_{t,-}^{ij}(x, y) Y_0^{ij}(y) & \text{for } x \in \mathbb{Z}_-.
\end{cases}
\]  

(3.2)

where \( Y_0^0(x) \equiv u_0(x), \ Y_0^1(x) \equiv v_0(x) \), and the Green function \( G_{t,\pm}(x, y) = (G_{t,\pm}^{ij}(x, y))_{i,j=0} \) is a matrix-valued function with the entries of the form

\[
G_{t,\pm}^{ij}(x, y) := G_{t,\pm}^{ij}(x - y) - G_{t,\pm}^{ij}(x + y), \ x, y \in \mathbb{Z}_\pm, \quad G_{t,\pm}^{ij}(x) \equiv \frac{1}{2\pi} \int_0^\pi e^{-ix\theta} \hat{G}_{t,\pm}^{ij}(\theta) \, d\theta,
\]  

(3.3)

\[
\left(\hat{G}_{t,\pm}^{ij}(\theta)\right)_{i,j=0} = \begin{pmatrix}
\cos \left(\phi_\pm(\theta) t\right) & \text{sin} \left(\phi_\pm(\theta) t\right) \\
-\phi_\pm(\theta) \text{sin} \left(\phi_\pm(\theta) t\right) & \cos \left(\phi_\pm(\theta) t\right)
\end{pmatrix},
\]  

(3.4)

\[
\phi_\pm(\theta) = \sqrt{\nu_\pm^2 (2 - 2 \cos \theta) + \kappa_\pm^2}.
\]  

(3.5)

In particular, \( \phi_\pm(\theta) = 2\nu_\pm \big| \text{sin} \theta/2 \big| \) if \( \kappa_\pm = 0 \). Note that \( G_{t,\pm}^{ij}(0, y) \equiv 0 \), since \( G_{t,\pm}^{ij}(-x) = G_{t,\pm}^{ij}(x) \).

Definition 3.2 Introduce a measure \( \nu_0 = \mu_0 \{ Y_0 \in \mathcal{H}_\alpha : Y_0(0) = 0 \} \). Denote by \( \nu_t \), \( t \in \mathbb{R} \), a Borel probability measure on \( \mathcal{H}_\alpha \) giving the distribution of the solution \( U_0(t)Y_0 \) to problem (3.1), i.e., \( \nu_t(B) = \nu_0(U_0(-t)B) \) for any \( B \in \mathcal{B}(\mathcal{H}_\alpha) \). The correlation matrix of \( \nu_t \) is denoted as

\[
Q_t^\nu(x, y) = \left( Q_t^{ij}(x, y) \right)_{i,j=0,1}, \quad Q_t^{ij}(x, y) := \int Y_i(x)Y_j(y) \nu_t(dY), \quad x, y \in \mathbb{Z}, \ t \in \mathbb{R}.
\]

The correlation matrix \( Q_t^\nu \) has the following property.

Lemma 3.3 Let condition S2 hold. Then

\[
\sup_{t \in \mathbb{R}} |Q_t^\nu(x, y)| \leq \sqrt{C_1 + C_2 |x|C_1 + C_2 |y|}, \quad x, y \in \mathbb{Z},
\]  

(3.6)

where the constants \( C_1 \) and \( C_2 \) do not depend on \( x, y \), and \( C_2 = 0 \) if \( \kappa_- \kappa_+ \neq 0 \).
Proof We check (3.6) only for $x, y \in \mathbb{Z}_+$. For another values of $x, y$ the proof is similar. Using Definition 3.2 and representation (3.2), we obtain that for $x, y \in \mathbb{Z}_+$, $t \in \mathbb{R}$, $i, j = 0, 1$,

$$Q^\nu_{i,j}(x, y) = \int (U_0(t)Y_0)^i(x)(U_0(t)Y_0)^j(y)\nu_0(dy)$$

$$= \sum_{k,l=0,1} \sum_{x',y' \in \mathbb{Z}_+} G^\nu_{i,k}(x,x')Q_0^{\nu,k}(x',y')G^{\nu,l}_{i,j}(y,y') = \langle Q_0^\nu(\cdot, \cdot), \Phi^i(\cdot, t) \otimes \Phi^j(\cdot, t) \rangle_+,$$

where $\Phi^i(x', t) := \left(G^0_{i,k}(x, x'), G^1_{i,k}(x, x')\right)$. Hence, applying (2.4), one obtains

$$\left| Q^\nu_{i,j}(x, y) \right| \leq C \| \Phi^i(\cdot, t) \|_0 \| \Phi^j(\cdot, t) \|_0,$$

where the constant $C$ does not depend on $x, y, t$. On the other hand, the Parseval identity and (3.3) imply

$$\| \Phi^i(\cdot, t) \|_0^2 = \frac{1}{\pi} \int \sin^2(x\theta) \left( |\hat{G}^0_{i,k}(\theta)|^2 + |\hat{G}^1_{i,k}(\theta)|^2 \right) d\theta, \quad x \in \mathbb{Z}_+, \quad i = 0, 1.$$

Hence, by (3.4), we have $\| \Phi^0(\cdot, t) \|_0^2 \leq C < \infty$ and

$$\| \Phi^0(\cdot, t) \|_0^2 \leq \int \sin^2(x\theta) \left( C_1 + C_2 \frac{1}{\phi^2_\pm(\theta)} \right) d\theta \leq C_3 + C_4 x, \quad x \in \mathbb{N}, \quad (3.7)$$

where the constants $C_3$ and $C_4$ do not depend on $t \in \mathbb{R}$ and $x \in \mathbb{N}$. Moreover, $C_4 = 0$ if $\kappa_+ \neq 0$, by (3.5). If $\kappa_+ = 0$, then $\phi^2_\pm(\theta) = 4\nu^2 \sin^2(\theta/2)$ and the bound in the r.h.s. of (3.7) follows from Fejér’s theorem (see, e.g., [10]).

Corollary 3.4 Let $\alpha < -1/2$ if $\kappa_- \kappa_+ \neq 0$, and $\alpha < -1$ otherwise. Then

$$\sup_{t \in \mathbb{R}} \int \| Y \|_0^2 \nu_t(dy) \leq C < \infty. \quad (3.8)$$

Indeed, applying the bound (3.6) gives

$$\int \| Y \|_0^2 \nu_t(dy) = \sum_{x \in \mathbb{Z}} \langle x \rangle^{2\alpha} (Q^\nu_{0,0}(x, x) + Q_{0,1}^\nu(x, x)) \leq \sum_{x \in \mathbb{Z}} \langle x \rangle^{2\alpha} (C_1 + C_2|x|) \leq C(\alpha) < \infty,$$

by the choice of the $\alpha$.

Introduce the limiting matrix $Q^\nu_\infty(x, y)$ by the rule

$$Q^\nu_\infty(x, y) = \begin{cases} Q_{\infty,+}(x, y) & \text{if } x, y > 0, \\ Q_{\infty,-}(x, y) & \text{if } x, y < 0, \\ 0 & \text{otherwise}, \end{cases} \quad (3.9)$$

where

$$Q_{\infty,\pm}(x, y) := q_{\infty,\pm}(x - y) - q_{\infty,\pm}(x + y) - q_{\infty,\pm}(-x - y) + q_{\infty,\pm}(-x + y), \quad x, y \in \mathbb{Z}_+. \quad (3.10)$$
The Fourier transforms of the entries of \( q_{\infty, \pm}(x) \), \( x \in \mathbb{Z} \), have the form

\[
\begin{align*}
\hat{q}_{\infty, \pm}^{00} & = \frac{1}{T} \left( \hat{q}_{\pm}^{00}(\theta) + \hat{q}_{\pm}^{11}(\theta) \phi_{\pm}^2(\theta) \right) \pm \frac{i}{2} \text{sign}(\theta) \phi_{\pm}^{-1}(\theta) \left( \hat{q}_{\pm}^{10}(\theta) - \hat{q}_{\pm}^{01}(\theta) \right), \\
\hat{q}_{\infty, \pm}^{11} & = \phi_{\pm}^2(\theta) \hat{q}_{\infty, \pm}^{00}(\theta), \\
\hat{q}_{\infty, \pm}^{01} & = -\hat{q}_{\infty, \pm}^{10} = \pm i \text{sign}(\theta) \phi_{\pm}(\theta) \hat{q}_{\infty, \pm}^{00}(\theta),
\end{align*}
\]  

(3.11)

where \( \theta \in \mathbb{T} \) if \( \kappa_{\pm} \neq 0 \) and \( \theta \in \mathbb{T} \setminus \{0\} \) otherwise, the functions \( \hat{q}_{\pm}^{ij} \), \( i, j = 0, 1 \), are the entries of the matrices \( q_{\pm} \) from condition (2.3), \( \phi_{\pm}(\theta) \) are defined in (3.3).

**Remark.** By (2.5), \( \hat{q}_{\pm}^{ii}(-\theta) = \hat{q}_{\pm}^{ii}(\theta) \) and \( \hat{q}_{\pm}^{ij}(-\theta) = \hat{q}_{\pm}^{ji}(\theta) \) if \( i \neq j \). Therefore, by (2.9) and (3.10), \( Q_{\infty, \pm}(x, y) = 0 \) and \( Q_{\infty}^{\nu, \pm}(x, y) = 0 \) if \( i \neq j \).

\[
Q_{\infty, \pm}(x, y) = \frac{2}{\pi} \int_{\mathbb{T}} \hat{q}_{\infty, \pm}(\theta) \sin(x\theta) \sin(y\theta) d\theta = Q_{\infty, \pm}^{\nu}(x, y), \quad x, y \in \mathbb{Z}_{\pm}.
\]

Denote by \( Q_{\nu}^{\nu}(\Psi, \Psi) \), \( t \in \mathbb{R} \), a real-valued quadratic form on \( \mathcal{S} = [S(\mathbb{Z})]^2 \) with the matrix kernel \( Q_{\nu}^{\nu}(x, y) \). Using (3.9), we have

\[
Q_{\nu}^{\nu}(\Psi, \Psi) = \langle Q_{\infty, \pm}^{\nu}(x, y), \Psi(x) \otimes \Psi(y) \rangle = \sum_{\pm} \langle Q_{\infty, \pm}(x, y), \Psi(x) \otimes \Psi(y) \rangle. \quad (3.12)
\]

The following theorem was proved in [6].

**Theorem 3.5** Let \( \alpha < -1/2 \) if \( \kappa_{\pm} \neq 0 \), and \( \alpha < -1 \) otherwise. Then the following assertions hold. (i) Let conditions S1–S3 be fulfilled. Then the correlation functions of the measures \( \nu_{\pm} \) converge to a limit:

\[
Q_{\nu}^{\nu}(x, y) := \int \left( Y(x) \otimes Y(y) \right) \nu_{\pm}(dY) \to Q_{\infty, \pm}^{\nu}(x, y), \quad t \to \infty, \quad x, y \in \mathbb{Z},
\]

where the limiting correlation matrix \( Q_{\infty, \pm}^{\nu}(x, y) \) is of the form (3.9).

(ii) Let conditions S1, S3 and S4 be fulfilled. Then the measures \( \nu_{\pm} \) converge weakly to a limit measure as \( t \to \infty \) on the space \( \mathcal{H}_{\alpha} \). The limit measure \( \nu_{\infty} \) is Gaussian with zero mean value and its characteristic functional is

\[
\hat{\nu}_{\infty}(\Psi) := \int e^{i(Y, \Psi)} \nu_{\infty}(dY) = \exp \left\{ -\frac{1}{2} Q_{\infty}^{\nu}(\Psi, \Psi) \right\}, \quad \Psi \in \mathcal{S},
\]

where the quadratic form \( Q_{\infty}^{\nu} \) is defined in (3.12).

Below we will use an additional property of the quadratic form \( Q_{\nu}^{\nu} \), \( t \in \mathbb{R} \). To state it we first introduce auxiliary spaces.

**Definition 3.6** For any sequence \( \psi \), we introduce odd sequences \( \psi_{-} \) and \( \psi_{+} \) by the rule

\[
\psi_{\pm}(x) = \begin{cases} 
\psi(x) & \text{for } \pm x > 0, \\
0 & \text{for } x = 0, \\
-\psi(-x) & \text{for } \pm x < 0.
\end{cases}
\]  

(3.13)
Define the Hilbert space \( \ell^2(\kappa) := \{ \psi \in \ell^2 : \hat{\psi}_\perp \hat{\phi}^{-1}_\perp(\theta), \hat{\psi}_\parallel \hat{\phi}^{-1}_\parallel(\theta) \in L^2(\mathbb{T}) \} \) with the norm
\[
\| \psi \|_{\ell^2(\kappa)} := \| \psi \|_{\ell^2} + \sum_{\pm} \left\| F_{\theta \rightarrow x}^{-1}[\hat{\phi}^{-1}_\pm(\theta)] * \hat{\psi}_\pm \right\|_{\ell^2}.
\]
Introduce the space \( \mathcal{H}(\kappa) := \ell^2(\kappa) \times \ell^2 \) with the norm
\[
\| \Psi \|_{\mathcal{H}(\kappa)} := \| \Psi_0 \|_{\ell^2(\kappa)} + \| \Psi_1 \|_{\ell^2}, \quad \Psi = (\Psi_0, \Psi_1).
\]
In particular, if \( \kappa_\perp \kappa_\parallel \neq 0 \), then \( \mathcal{H}(\kappa) = \mathcal{H}_0 = \ell^2 \times \ell^2 \).

**Remark.** By \((3.13)\), in the Fourier transform, \( |\hat{\psi}_\perp(\theta)| \leq C \sin \| \theta \| \sum_{\pm x > 0} |x| |\psi(x)|, \ \theta \in \mathbb{T}. \)

Hence, if \( \sum_{x \in \mathbb{Z}} |x||\psi(x)| < \infty \), then \( \hat{\psi}_\perp \hat{\phi}^{-1}_\perp \in C(\mathbb{T}). \) In particular, \( \ell^2_\alpha \subset \ell^2(\kappa) \) for \( \alpha < -3/2 \), since \( \sum_{x \in \mathbb{Z}} |x| |\psi(x)| \leq C \| \psi \|_{-\alpha} \) by the Cauchy–Schwartz inequality.

**Lemma 3.7** The quadratic forms \( Q_t^\prime(\Psi, \Psi) \) and the characteristic functionals \( \hat{\nu}_t(\Psi), t \in \mathbb{R}, \) are equicontinuous in \( \mathcal{H}(\kappa). \)

**Proof** For \( t \in \mathbb{R}, \) introduce a formal adjoint operator \( U_0^t(t) \) to the solving operator \( U_0(t), \)
\[
\langle U_0(t)Y, \Psi \rangle = \langle Y, U_0^t(t)\Psi \rangle, \quad Y \in \mathcal{H}_\alpha, \quad \Psi \in \mathcal{S}.
\]
Then, the action of the group \( U_0^t(t) \) coincides with the action of \( U_0(t) \) up to the order of the components. Namely, \( U_0^t(t)\Psi = \left( \hat{\psi}(\cdot, t), \psi(\cdot, t) \right) \), where \( \psi(x, t) \) is a solution to problem \((3.1)\) with the initial data \( (u_0, v_0) = (\Psi_1, \Psi_0) \). Using \((2.4)\), we have
\[
Q_t^\prime(\Psi, \Psi) = Q_0^\prime(U_0^t(t)\Psi, U_0^t(t)\Psi) \leq C \| U_0^t(t)\Psi \|_0^2.
\]
On the other hand, by \((3.2)\) and \((3.3),\)
\[
(U_0^t(t)\Psi)^j(y) = \sum_{i=0}^1 \sum_{x \in \mathbb{Z}_\pm} G_{i|^j}(x, y) \Psi^i(x) = \sum_{i=0}^1 \sum_{x \in \mathbb{Z}_\pm} G_{i|^j}(x - y) \Psi^i(x) \quad \text{for} \quad y \in \mathbb{Z}_\pm.
\]
Here we use notation \((3.13)\). Therefore, applying the Parseval identity and \((3.4),\) we obtain
\[
\| U_0^t(t)\Psi \|_0^2 \leq C \sum_{\pm} \int_T \left( (1 + \phi^{-1}_\perp(\theta)) |\hat{\psi}_\perp^0(\theta)|^2 + |\hat{\psi}_\perp^1(\theta)|^2 \right) d\theta \leq C \| \Psi \|_{\mathcal{H}(\kappa)}^2.
\]
Hence,
\[
Q_t^\prime(\Psi, \Psi) \leq C \| \Psi \|_{\mathcal{H}(\kappa)}^2 \quad \text{uniformly in} \quad t \in \mathbb{R}. \tag{3.14}
\]
This implies the equicontinuity of the characteristic functionals \( \hat{\nu}_t(\Psi), t \in \mathbb{R}. \) Indeed, by the Cauchy–Schwartz inequality and \((3.14),\) one obtains
\[
|\hat{\nu}_1(\Psi_1) - \hat{\nu}_2(\Psi_2)| = \left| \int (e^{iY\Psi_1} - e^{iY\Psi_2}) \nu_t(dY) \right| \leq \int |e^{i(Y\Psi_1 - \Psi_2)} - 1| \nu_t(dY)
\leq \int |\langle Y, \Psi_1 - \Psi_2 \rangle| \nu_t(dY) \leq \sqrt{\int |\langle Y, \Psi_1 - \Psi_2 \rangle|^2 \nu_t(dY)}
= \sqrt{Q_t^\prime(\Psi_1 - \Psi_2, \Psi_1 - \Psi_2)} \leq C \| \Psi_1 - \Psi_2 \|_{\mathcal{H}(\kappa)}. \quad \square
\]
Definition 3.8 (i) Denote by \( R_0 \) the operator \( R_0 : \mathcal{H}_\alpha \to \mathfrak{c}_\alpha^1 \) such that

\[
(R_0 Y_0)(x, t) = z(x, t),
\]

where \( z(x, t) \) is the solution to problem (3.1) with the initial data \( Y_0 = (Y_0^0, Y_0^1) \equiv (u_0, v_0) \). Then, by (3.2),

\[
(R_0 Y_0)(x, t) = \begin{cases} 
\sum_{j=0}^{1} \sum_{y \in \mathbb{Z}_+} G^{0j}_{1n}(x, y) Y_0^j(y) & \text{for } x \in \mathbb{Z}_+, \\
\sum_{j=0}^{1} \sum_{y \in \mathbb{Z}_-} G^{0j}_{1n}(x, y) Y_0^j(y) & \text{for } x \in \mathbb{Z}_-.
\end{cases}
\]

In particular, \( (R_0 Y_0)(0, t) = 0 \) for all \( t \).

(ii) Introduce a Borel probability measure \( P^\nu \) on the space \( \mathfrak{c}_\alpha^1 \) as

\[
P^\nu(\omega) = \nu_0(R_0^{-1}\omega), \quad \forall \omega \in \mathcal{B}(\mathfrak{c}_\alpha^1).
\]

This measure is called a space-time statistical solution to problem (3.1).

(iii) Denote by \( \{P^\nu_\tau, \tau \in \mathbb{R}\} \) the family of measures defined by the rule

\[
P^\nu_\tau(\omega) = P^\nu(S_\tau^{-1}\omega), \quad \forall \omega \in \mathcal{B}(\mathfrak{c}_\alpha^1), \quad \tau \in \mathbb{R}.
\]

In this section, we prove the following theorem.

Theorem 3.9 Let \( \alpha < -1/2 \) if \( \kappa_- \kappa_+ \neq 0 \), and \( \alpha < -1 \) otherwise. Then the following assertions hold. (i) Let conditions S1 and S2 be fulfilled. Then the bounds are true:

\[
\sup_{\tau \geq 0} \int |z|^2_{\alpha,1,T} P^\nu_\tau(dz) \leq C(\alpha) < \infty, \quad \forall T > 0,
\]

where the constant \( C(\alpha) \) does not depend on \( T > 0 \).

(ii) Let conditions S1–S3 be fulfilled. Then for any \( v_1, v_2 \in \mathcal{P} \),

\[
Q^\nu_\tau(v_1, v_2) := \int [z, v_1][z, v_2] P^\nu_\tau(dz) \to Q^\nu_\infty(v_1, v_2), \quad \tau \to \infty.
\]

Here

\[
Q^\nu_\infty(v_1, v_2) := [Q^\nu_\infty, v_1 \otimes v_2],
\]

where the limiting correlation matrix \( Q^\nu_\infty \) is of a form

\[
Q^\nu_\infty(x_1, x_2, t_1, t_2) = \begin{cases} 
Q^\nu_\infty^+(x_1, x_2, t_1, t_2) & \text{if } x_1, x_2 > 0, \\
Q^\nu_\infty^-(x_1, x_2, t_1, t_2) & \text{if } x_1, x_2 < 0, \\
0 & \text{otherwise},
\end{cases}
\]

Here

\[
Q^\nu_\infty^\pm(x_1, x_2, t_1, t_2) := \frac{2}{\pi} \int T \cos(\phi \pm(\theta)(t_1 - t_2)) \hat{q}^{00}_{\infty, \pm}(\theta) \sin(x_1\theta) \sin(x_2\theta) d\theta,
\]

where \( \hat{q}^{00}_{\infty, \pm} \) is defined in (3.11).

(iii) Let conditions S1, S3 and S4 be fulfilled. Then the measures \( P^\nu_\tau \) converge weakly to a limiting measure \( P^\nu_\infty \) on the space \( \mathfrak{c}_\alpha^0 \) as \( \tau \to \infty \). The characteristic functional of \( P^\nu_\infty \) is

\[
\hat{P}^\nu_\infty(v) \equiv \exp \left\{ -\frac{1}{2} Q^\nu_\infty(v, v) \right\}, \quad v \in \mathcal{P},
\]

where the quadratic form \( Q^\nu_\infty \) is defined in (3.11–3.20).
Proof (i) At first, note that
\[ P_\tau^\nu(\omega) = \nu_\tau(R_0^{-1}\omega) \quad \text{for any } \omega \in \mathcal{B}(\mathcal{C}_\alpha^1) \text{ and } \tau > 0, \] (3.22)
where \( \nu_\tau \) is defined in Definition 3.2. Hence, the bound (3.19) follows from (3.8), because
\[
\int |z|_{\alpha,1,T}^2 P_\tau^\nu(dz) = \int |R_0Y|_{\alpha,1,T}^2 \nu_\tau(dY) = \sup_{|s| \leq T} \int \|U_0(s)Y\|_\alpha^2 \nu_\tau(dY)
\]
\[
= \sup_{|s| \leq T} \int \|Y\|_\alpha^2 \nu_{s+\tau}(dY) \leq C(\alpha) < \infty.
\]

(ii) Let \( z \equiv z(\cdot, t) \) be a solution to problem (3.1). Then, for any \( v \in \mathcal{P} \),
\[
[z, v] = [R_0Y_0, v] = \langle Y_0, R_0'v \rangle,
\] (3.23)
where \( R_0' \) is an adjoint operator to the operator \( R_0 \), \( R_0'v = ((R_0'v)^0, (R_0'v)^1) \), and
\[
(R_0'v)^j(y) = \begin{cases} 
\sum_{x \in \mathbb{Z}_+} \int_{-\infty}^{+\infty} G_{t,x}^{0j}(x, y)v(x, t) \, dt & \text{if } y \in \mathbb{Z}_+, \\
\sum_{x \in \mathbb{Z}_-} \int_{-\infty}^{+\infty} G_{t,x}^{0j}(x, y)v(x, t) \, dt & \text{if } y \in \mathbb{Z}_-,
\end{cases}
\] (3.24)
by (3.15). In particular, using (3.3), we have \( (R_0'v)^j(0) = 0 \). Below we use the notation
\[
\|v\|_{L^1(\mathbb{R}; X)} := \int_{-\infty}^{+\infty} \|v(\cdot, t)\|_X \, dt \quad \text{for } v(\cdot, t) \in L^1(\mathbb{R}; X) \text{ with } X = \ell^2 \text{ or } X = \ell^2(\kappa).
\]
We state the additional properties of the operator \( R_0' \) in the following lemma.

Lemma 3.10 (i) If \( \kappa-\kappa_+ \neq 0 \), then \( R_0'v \in \mathcal{S} \) for any \( v \in \mathcal{P} \). (ii) For any \( v \in L^1(\mathbb{R}; \ell^2(\kappa)) \),
\[
\|R_0'v\|_{\mathcal{H}(\kappa)} \leq C\|v\|_{L^1(\mathbb{R}; \ell^2(\kappa))}. \] (3.25)

Proof The first assertion follows from (3.24) and formulas (3.3)–(3.5). To prove assertion (ii), we apply (3.24), notation (3.13) for \( v(x, t) \), and equations (3.3) and obtain
\[
(R_0'v)^j(y) = \sum_{x \in \mathbb{Z}} \int_{-\infty}^{+\infty} G_{t,x}^{0j}(x - y)v_{\pm}(x, t) \, dt \quad \text{for } y \in \mathbb{Z}_\pm.
\]
Hence, by the Parseval identity and (3.3), we have
\[
\|(R_0'v)^j\|_{L^2} \leq \sum_{\pm} \int_{-\infty}^{+\infty} \|G_{t,\pm}^{0j}(\theta)\|_{L^2(\mathbb{T})} \, d\theta \leq \sum_{\pm} \int_{-\infty}^{+\infty} \|\phi_{\pm}^{-j}(\theta)\|_{L^2(\mathbb{T})} \, d\theta.
\]
Therefore, \(\| (R_0'v)^0 \|_{\ell^2} \leq C \| v \|_{L^1(\mathbb{R}, \ell^2)}\). If \(\kappa_\pm \neq 0\), then the same bound is valid for \((R_0'v)^1\).

If \(\kappa_\pm = 0\), then \(\| (R_0'v)^1 \|_{\ell^2} \leq C \| v \|_{L^1(\mathbb{R}, \ell^2(\kappa))}\). Furthermore,

\[
\| (R_0'v)^0 \|_{\ell^2(\kappa)} = \| (R_0'v)^0 \|_{\ell^2} + \sum_{\pm} \left\| F^{-1}[\hat{\phi}_{\pm}^*] * ((R_0'v)^0)^{\pm} \right\|_{\ell^2}
\]

\[
\leq \int_{-\infty}^{+\infty} \| v(\cdot, t) \|_{\ell^2} dt + \sum_{\pm} \int_{-\infty}^{+\infty} \| \hat{\phi}_{\pm}^{-1}(\theta) \hat{v}_{\pm}(\theta, t) \|_{L^2(T)} dt.
\]

This implies the bound (3.25). In particular, \(R_0'v \in \mathcal{H}(\kappa)\) for any \(v \in \mathcal{P}\). Lemma 3.10 is proved. □

We return to the proof of assertion (ii) of Theorem 3.9. For any \(v_1, v_2 \in \mathcal{P}\),

\[
Q^\nu_\tau(v_1, v_2) = \int \langle R_0 Y, v_1 \rangle [R_0 Y, v_2] \nu_\tau(dY) = \int \langle Y, R_0'v_1 \rangle \langle Y, R_0'v_2 \rangle \nu_\tau(dY)
\]

\[
= \langle Q^\nu_\tau(x, y), (R_0'v_1)(x) \otimes (R_0'v_2)(y) \rangle = Q^\nu_\tau(R_0'v_1, R_0'v_2).
\]

If \(\kappa_\pm \neq 0\), then \(R_0'v_i \in \mathcal{S}\) and Theorem 3.5 (i) implies

\[
Q^\nu_\tau(v_1, v_2) = Q^\nu_\tau(R_0'v_1, R_0'v_2) \rightarrow Q^\nu_\infty(R_0'v_1, R_0'v_2), \quad \tau \rightarrow \infty. \tag{3.26}
\]

If \(\kappa_\pm = 0\), then convergence (3.26) follows from Theorem 3.5 (i) and Lemmas 3.7 and 3.10 because the space \(\mathcal{S}\) is dense in \(\mathcal{H}(\kappa)\).

It remains to check formula (3.20). Using (3.26), (3.12), (3.9) and (3.24), we have

\[
Q^\nu_\infty(v_1, v_2) = Q^\nu_\infty(R_0'v_1, R_0'v_2) = \langle Q^\nu_\infty(y_1, y_2), R_0'v_1(y_1) \otimes R_0'v_2(y_2) \rangle
\]

\[
= \sum_{\pm} \sum_{x_1, x_2 \in \mathbb{Z}_{\pm}} \int_{-\infty}^{+\infty} dt_1 \int_{-\infty}^{+\infty} dt_2 Q^\nu_\infty,\pm(x_1, x_2, t_1, t_2) v_1(x_1, t_1) v_2(x_2, t_2) dt_1 dt_2,
\]

where, by definition,

\[
Q^\nu_\infty,\pm(x_1, x_2, t_1, t_2) := \sum_{i, j = 0, 1} \sum_{y_1, y_2 \in \mathbb{Z}_{\pm}} Q^{ij}_{\infty, \pm}(y_1, y_2) G_{t_1, \pm}(x_1, y_1) G_{t_2, \pm}(x_2, y_2)
\]

for \(x_1, x_2 \in \mathbb{Z}_{\pm}, t_1, t_2 \in \mathbb{R}\). Hence, (3.19) holds. Using formulas (3.3) and (3.10) and the Parseval identity, we obtain

\[
Q^\nu_\infty,\pm(x_1, x_2, t_1, t_2) = \sum_{i, j = 0, 1} \sum_{y_1, y_2 \in \mathbb{Z}} q^{ij}_{\infty, \pm}(y_1 - y_2) G_{t_1, \pm}(x_1, y_1) G_{t_2, \pm}(x_2, y_2)
\]

\[
= \frac{4}{2\pi} \sum_{i, j = 0, 1} \frac{1}{\pi} \int_{-\pi/2}^{\pi/2} \hat{q}_{\infty, \pm}(\theta) \hat{G}_{t_1, \pm}(\theta) \hat{G}_{t_2, \pm}(\theta) \sin(x_1 \theta) \sin(x_2 \theta) d\theta \tag{3.27}
\]

for \(\pm x_1, \pm x_2 > 0, t_1, t_2 \in \mathbb{R}\). Applying (3.11) and (3.4), we obtain

\[
Q^\nu_\infty,\pm(x_1, x_2, t_1, t_2) = \frac{2}{\pi} \int_{T} \left\{ \cos(\phi_\pm(\theta)(t_1 - t_2)) \frac{\hat{q}_{\infty, \pm}(\theta)}{\hat{\phi}_\pm(\theta)} - \frac{\sin(\phi_\pm(\theta)(t_1 - t_2))}{\phi_\pm(\theta)} \right\} 
\]

\[
\times \sin(x_1 \theta) \sin(x_2 \theta) d\theta.
\]

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This implies relation (3.20) since the functions \( q^{01}_{\infty,\pm}(\theta) \) are odd and \( \phi_{\pm}(\theta) \) are even.

(iii) According to the methods of [16], to establish the weak convergence of the measures \( P_{\tau}^\nu \) on the space \( \mathcal{C}_0^\nu \) it is enough to prove the following two assertions:

(A1) **The family of measures** \( \{P_{\tau}^\nu, \tau \in \mathbb{R}\} \) **is weakly compact in** \( \mathcal{C}_0^\nu \);

(A2) **The characteristic functionals of** \( P_{\tau}^\nu \) **converge to a limit as** \( \tau \to \infty \).

The first (second) assertion provides the existence (resp., uniqueness) of the limit measures \( P_{\infty}^\nu \).

**Proof of assertion (A1):** To prove the weak compactness of the family \( \{P_{\tau}^\nu, \tau \in \mathbb{R}\} \), we verify that this family satisfies the following conditions (a) and (b) of the Prokhorov Theorem (see, e.g., [1]):

(a) \( \sup \{P_{\tau}^\nu, \tau \in \mathbb{R}\} < \infty \),

(b) for any \( \varepsilon > 0 \) there is a compact \( K_{\varepsilon} \) in \( \mathcal{C}_0^\nu \) such that \( \sup_{\tau} P_{\tau}^\nu(\mathcal{C}_0^\nu \setminus K_{\varepsilon}) < \varepsilon \).

Condition (a) holds since \( P_{\tau}^\nu \) are probability measures. To check condition (b), we apply the technique of [16, Theorem XII.5.2]. For \( k = 0, 1 \) and \( T > 0 \), denote by \( \mathcal{C}_0^\nu_{\alpha,T} \) the space of the functions \( t \to u(\cdot, t) \in \ell_2^\alpha \), \( t \in [0,T] \), for which the norm (1.4) is finite. For any \( T > 0 \) and \( M > 0 \), introduce sets

\[
K(T, M) := \{ u \in \mathcal{C}_0^1_{\alpha,T} : |u|_{\alpha,1,T} \leq M \}.
\]

Below we choose \( M \equiv M(T) \) by a special way. The sets \( K(T, M) \) are uniformly bounded and uniformly equicontinuous. Since the embedding of the spaces \( \ell_2^\alpha \) in \( \ell_2^\beta \) is compact if \( \alpha > \beta \), then the sets \( K(T, M) \) are precompact in \( \mathcal{C}_0^\nu_{\beta,T} \) by the Dubinskii embedding theorems (see, e.g., [16, Theorem IV.4.1]) using the Arzela–Ascoli theorem (see, e.g., [17, Ch.3, §3]). For \( T > 0 \), introduce the operator \( J_T : \mathcal{C}_0^\nu_\beta \to \mathcal{C}_0^\nu_{\beta,T} \) of the restriction of the functions \( u(x,t) \in \mathcal{C}_0^\nu_\beta \) from \( \mathbb{Z} \times \mathbb{R} \) into \( \mathbb{Z} \times [-T, T] \). Applying the Chebyshev inequality and the bound (3.16), we obtain

\[
P_{\tau}^\nu \{ \mathcal{C}_0^\nu_\beta \setminus J_T^{-1}K(T, M) \} \leq \int |u|^2_{\alpha,1,T} P_{\tau}(du)/M^2 \leq C(\alpha)/M^2,
\]

where by \( K \) we denote the closure of \( K \) in the topology of the metrizable space \( \mathcal{C}_0^\nu_\beta \). For any \( \varepsilon > 0 \), we choose the positive constants \( M = M(\varepsilon(T)) \) such that

\[
C(\alpha) \sum_{T=1}^\infty \frac{1}{M^2_\varepsilon(T)} < \varepsilon.
\]

Set \( K_{\varepsilon} := \bigcap_{T=1}^\infty J_T^{-1}K(T, M(T)) \). Then the bound (3.28) implies the condition (b).

**Proof of assertion (A2):** Applying (3.22), (3.23), Theorem 3.5 (ii) and Lemmas 3.7 and 3.10 we obtain that for every \( v \in \mathcal{P} \),

\[
\hat{P}_\tau^\nu(v) = \int e^{i|z|v} P_\tau^\nu(dz) = \int e^{i(Y_0,R_0^v)}\nu_\tau(dY_0) \to \exp \left\{ -\frac{1}{2} Q_{\infty}^\nu (R_0^v, R_0^v) \right\} \quad \tau \to \infty.
\]

The assertion (iii) of Theorem 3.9 is proved. □
3.2 Perturbed problem

The key role in the proof of convergence (1.7) for problem (1.1) plays the following lemma.

**Lemma 3.11** (see [6, Lemma 4.3]) Let \( Y_0 \in \mathcal{H}_\alpha \), \( \alpha < -3/2 \), and conditions C, S1, and S2 hold. Then there exists a linear bounded operator \( \Omega : \mathcal{H}_0 \to \mathcal{H}_\alpha \) such that the following representation holds

\[
U(t)Y_0(x) = \Omega(U_0(t)Y_0)(x) + \delta(x, t), \quad \text{where} \quad \mathbb{E}\|\delta(\cdot, t)\|_{\alpha}^2 \leq C(t)^{-1}. \tag{3.29}
\]

Here \( U(t)Y_0 \equiv (u(\cdot, t), \dot{u}(\cdot, t)) \) is a solution to problem (1.1)–(1.2), the operator \( \Omega \) is of the form

\[
\Omega Y = Y + \Gamma Y, \quad (\Gamma Y)(x) := \langle Y, \bar{\Gamma}^0(x, \cdot) \rangle, \quad x \in \mathbb{Z}, \tag{3.30}
\]

where \( \bar{\Gamma}^j(x, y), \ j = 0, 1, \) is a vector-valued function of the form

\[
\bar{\Gamma}^j(x, y) = \begin{cases} \int_0^{+\infty} \Gamma_x^j(s) \left( U_s^0(-s) \bar{G}_j^j(y) \right) ds & \text{if } x > 0, \\ \bar{G}_j^j(y) & \text{if } x = 0,
\end{cases} \quad y \in \mathbb{Z}.
\]

Here \( \bar{G}_j^j(y) := v_2^j \bar{G}_j^j(y) \) for \( \pm y \geq 0 \), \( \bar{G}_j^j(y) := \int_0^{+\infty} N_0^j(s) \bar{g}_j^0(y, -s) ds, \ y \in \mathbb{Z}, \quad \bar{g}_j^0(y, t) := \left( g_{t, j}^{00}(\pm 1, y), g_{t, j}^{01}(\pm 1, y) \right) \),

\[ N_0^j(s) \equiv N(s), \ N_0^1(s) \equiv \dot{N}(s), \ \text{the functions } N(s) \ \text{and } \Gamma_x^j(s) \ \text{are constructed in [6]. They satisfy the following bounds:}
\]

\[
|N(s)| \leq C(s)^{-3/2}, \quad \sum_{x \in \mathbb{Z}\setminus\{0\}} \langle x \rangle^{2\alpha} |\Gamma_x^j(s)|^2 \leq C(s)^{-3}, \quad s \in \mathbb{R}, \ \alpha < -3/2.
\]

**Corollary 3.12** Let \( \alpha < -3/2 \). Then there is a bounded linear operator \( \Omega' : \mathcal{H}_{-\alpha} \to \mathcal{H}_0 \) such that for any \( \Psi \in \mathcal{S} \) we have

\[
\langle U(t)Y_0, \Psi \rangle = \langle U_0(t)Y_0, \Omega' \Psi \rangle + \delta(t), \quad \text{where} \quad \mathbb{E}\|\delta(t)\|_\alpha^2 \leq C(t)^{-1}\|\Psi\|_{-\alpha}^2. \tag{3.31}
\]

The operator \( \Omega' \) is of a form

\[
\Omega' \Psi = \Psi + \Gamma' \Psi, \quad (\Gamma' \Psi)(y) := \sum_{j=0}^{1} \langle \Gamma^j(\cdot, y), \Psi^j(\cdot) \rangle, \quad \Psi = (\Psi^0, \Psi^1). \tag{3.32}
\]

**Remark.** As shown in [6], \( \|\Gamma^j(x, \cdot)\|_0 \in \mathcal{H}_\alpha \ \forall \alpha < -3/2 \), where \( \| \cdot \|_0 \equiv \| \cdot \|_{\mathcal{H}_0} \). Furthermore, using the similar reasonings as in [6] one can check that

\[
\|\Gamma^j(x, \cdot)\|_{\mathcal{H}_\alpha} \in \mathcal{H}_\alpha \ \text{for any } \alpha < -3/2, \ j = 0, 1.
\]

Therefore,

\[
\|\Gamma' \Psi\|_{\mathcal{H}_\alpha} \leq C\|\Psi\|_{-\alpha} \quad \text{and} \quad \|\Omega' \Psi\|_{\mathcal{H}_\alpha} \leq C\|\Psi\|_{-\alpha} \quad \forall \Psi \in \mathcal{H}_{-\alpha}. \tag{3.33}
\]

Before to prove Theorem 2.7 we state the results concerning the statistical solutions \( \mu_t \) to problem (1.1).
Definition 3.13 \( \mu_t \) is a Borel probability measure in \( \mathcal{H}_\alpha \) which gives the distribution of \( Y(t) \), \( \mu_t(B) = \mu_0(U(-t)B) \) for any \( B \in \mathcal{B}(\mathcal{H}_\alpha) \), \( t \in \mathbb{R} \). The correlation functions of the measure \( \mu_t \) are defined as

\[
Q_t^{ij}(x,y) = \mathbb{E}(Y^i(x,t)Y^j(y,t)), \quad i,j = 0,1, \quad x,y \in \mathbb{Z}, \quad t \in \mathbb{R}.
\]

Here \( Y^i(x,t) \) are the components of the solution \( Y(t) = (Y^0(\cdot,t), Y^1(\cdot,t)) = (u(\cdot,t), \dot{u}(\cdot,t)) \). Denote by \( Q_t \) the quadratic form with the matrix kernel \( (Q_t^{ij}(x,y))_{i,j=0,1} \),

\[
Q_t(\Psi,\Psi) = \int ||Y||^2 \mu_t(dY) = \sum_{i,j=0,1} \langle Q_t^{ij}(x,y), \Psi^i(x)\Psi^j(y) \rangle, \quad t \in \mathbb{R}, \quad \Psi = (\Psi^0, \Psi^1) \in \mathcal{S}.
\]

Lemma 3.14 Let \( \alpha < -3/2 \) and conditions \( C, S1, \) and \( S2 \) be fulfilled. Then the following bound holds

\[
\sup_{t \in \mathbb{R}} \int ||Y||^2 \mu_t(dY) = \sup_{t \in \mathbb{R}} \mathbb{E}||U(t)Y||^2 < C < \infty. \tag{3.34}
\]

Proof. As shown in [4], for any \( \alpha < -3/2 \), \( \|U_0(t)\dot{\Gamma}^j(x,\cdot)\|_0 \in \mathcal{H}_\alpha \) uniformly in \( t \in \mathbb{R} \), i.e.,

\[
\sup_{t \in \mathbb{R}} \left\| \left( \|U_0'(t)\dot{\Gamma}^j(x,\cdot)\|_0 \right) \right\|^2 \equiv \sup_{t \in \mathbb{R}} \sum_{x \in \mathbb{Z}} \langle x \rangle^{2\alpha} \|U_0'(t)\dot{\Gamma}^j(x,\cdot)\|^2_0 < \infty. \tag{3.35}
\]

We check that

\[
\sup_{t \in \mathbb{R}} \mathbb{E}\|\Omega U_0(t)Y_0\|^2 \leq C < \infty. \tag{3.36}
\]

Indeed, applying (3.30), (3.8), (2.4) and (3.35) gives

\[
\mathbb{E}\|\Omega U_0(t)Y_0\|^2 \leq \mathbb{E}\|U_0(t)Y\|^2 + \sum_{j=0}^{1} \mathbb{E}||\langle Y(\cdot), U_0'(t)\dot{\Gamma}^j(x,\cdot) \rangle||^2
\leq C_1 + \sum_{j=0}^{1} \sum_{x \in \mathbb{Z}} \langle x \rangle^{2\alpha} Q_0 \langle U_0'(t)\dot{\Gamma}^j(x,\cdot), U_0'(t)\dot{\Gamma}^j(x,\cdot) \rangle
\leq C_1 + C_2 \sum_{j=0}^{1} \sum_{x \in \mathbb{Z}} \langle x \rangle^{2\alpha} \|U_0'(t)\dot{\Gamma}^j(x,\cdot)\|^2_0 \leq C < \infty.
\]

Therefore, (3.29) and (3.36) imply the bound (3.34). \( \square \)

Theorem 3.15 (see [6, Theorems 2.3, 2.4]) Let \( \alpha < -3/2 \) and condition \( C \) hold. Then the following assertions are fulfilled.

(i) Let conditions \( S1 - S3 \) hold. Then for all \( \Psi \in \mathcal{S} \),

\[
\lim_{t \to \infty} \mathbb{E}||Y(t, \cdot)\|^2 = Q_{\infty}(\Psi, \Psi) = Q_{\infty}(\Omega^\Psi, \Omega^\Psi), \tag{3.37}
\]

where the quadratic form \( Q_{\infty} \) is introduced in (3.12).

(ii) Let conditions \( S1, S3 \) and \( S4 \) hold. Then the measures \( \mu_t \) weakly converge to a Gaussian measure \( \mu_\infty \) as \( t \to \infty \) on \( \mathcal{H}_\alpha \). The characteristic functional of \( \mu_\infty \) is of a form

\[
\hat{\mu}_\infty(\Psi) = \exp\{-Q_{\infty}(\Psi, \Psi)/2\}, \quad \Psi \in \mathcal{S}.
\]
Remark. It follows from the bounds (3.14) and (3.33) that

\[ \sup_{t \in \mathbb{R}} Q_\nu^\tau(\Omega^\tau \Psi, \Omega^\tau \Psi) \leq C \| \Omega^\tau \Psi \|_{H(\kappa)}^2 \leq C \| \Psi \|_{-\alpha}^2 \quad \forall \Psi \in \mathcal{H}_{-\alpha}. \]

In particular, the r.h.s. of (3.37) is defined for any \( \Psi \in \mathcal{S} \).

**Proof of Theorem 2.7** At first, using Lemma 3.11 we estimate \([S_\tau u, v] \), where \( v \in \mathcal{P} \), \( u \equiv u(x, t) \) is a solution to problem (1.1), \( S_\tau \) is defined in (1.6). Set \( \vec{v} := (v, 0) \). For any \( v \in \mathcal{P} \), we have

\[ [S_\tau u, v] = [S_\tau z, T \Omega^\tau \vec{v}] + \delta_\tau, \quad \text{where } \mathbb{E}(\delta_\tau^2) = o(1), \ \tau \to \infty, \quad (3.38) \]

\( z \equiv z(x, t) \) is a solution to problem (3.1), and the operator \( T \) is defined by the rule

\[ T \Phi := \Phi^0 - \Phi^1 \quad \text{for } \Phi \equiv \Phi(t) = (\Phi^0(t), \Phi^1(t)). \quad (3.39) \]

To prove (3.38), we first write \([S_\tau u, v] \) in a form

\[ [S_\tau u, v] = \int_{-\infty}^{+\infty} \langle U(t + \tau)Y_0, \vec{v}(\cdot, t) \rangle dt = \int_{-\infty}^{+\infty} \langle U_0(t + \tau)Y_0, \Omega^\tau \vec{v}(\cdot, t) \rangle dt + \delta_\tau, \quad (3.40) \]

where \( \mathbb{E}(\delta_\tau^2) = o(1), \ \tau \to \infty \). The bound (3.40) follows from Corollary 3.12 because

\[
\mathbb{E}(\delta_\tau^2) \leq \left( \int_{-\infty}^{+\infty} \sqrt{\mathbb{E} \left| \langle U(t + \tau)Y_0, \vec{v}(\cdot, t) \rangle - \langle U_0(t + \tau)Y_0, \Omega^\tau \vec{v}(\cdot, t) \rangle \right|^2 dt} \right)^2
\leq C \left( \int_{-\infty}^{+\infty} (t + \tau)^{-1/2} \|v(\cdot, t)\|_{-\alpha} dt \right)^2 = o(1) \quad \text{as } \tau \to \infty. \quad (3.41)\]

Secondly, we rewrite the integral in the r.h.s. of (3.40) using notation (3.39):

\[
\int_{-\infty}^{+\infty} \langle U_0(t + \tau)Y_0, \Omega^\tau \vec{v}(\cdot, t) \rangle dt = \int_{-\infty}^{+\infty} \langle z(\cdot, t + \tau), T \Omega^\tau \vec{v}(\cdot, t) \rangle dt = [S_\tau z, T \Omega^\tau \vec{v}].
\]

This implies representation (3.38). Further, using (3.38), we obtain that for \( v_1, v_2 \in \mathcal{P} \),

\[ Q_\nu^\tau(v_1, v_2) = \int [u, v_1][u, v_2] P_\tau(du) = Q_\nu^\tau(T \Omega^\tau \vec{v}_1, T \Omega^\tau \vec{v}_2) + \delta_\tau', \quad (3.42) \]

where the quadratic form \( Q_\nu^\tau \) is introduced in (3.17), \( \delta_\tau' = o(1) \) as \( \tau \to \infty \). Note that \( T \Omega^\tau \vec{v}_i \notin \mathcal{P} \), in general, and we can not apply convergence (3.17) immediately.

At first, using the equality \( Q_\nu^\tau(w_1, w_2) = Q_\nu^\tau(R_0^\nu w_1, R_0^\nu w_2) \), we obtain

\[ Q_\nu^\tau(v_1, v_2) = Q_\nu^\tau(R_0^\nu T \Omega^\tau \vec{v}_1, R_0^\nu T \Omega^\tau \vec{v}_2) + o(1), \quad \tau \to \infty.
\]

Then, the convergence of \( Q_\nu^\tau(v_1, v_2) \) to a limit as \( \tau \to \infty \) follows from the following facts:
(i) the quadratic form $Q^v_\nu(\Psi, \Psi)$ converges to a limit for any $\Psi \in S$ (Theorem 3.5 (i));
(ii) $S$ is dense in $\mathcal{H}(\nu);$ 
(iii) the quadratic forms $Q^\nu_\nu(\Psi, \Psi), \tau \in \mathbb{R},$ are equicontinuous in $\mathcal{H}(\nu)$ (Lemma 3.7);
(iv) $R_0' T \Omega' \bar{v} \in \mathcal{H}(\nu)$ for any $v \in \mathcal{P}$.

Hence, it remains to check the last fact. By (3.32) and (3.39),
\[ R_0' T \Omega' \bar{v} = R_0' \left( v + (\Gamma' \bar{v})^0 - \partial_t (\Gamma' \bar{v})^1 \right). \]

Due to (3.33), we have
\[ \|\Gamma' \bar{v}(\cdot, t)\|_{\mathcal{H}(\nu)} = \| (\Gamma' \bar{v})^0(\cdot, t)\|_{\mathcal{E}(\nu)} + \| (\Gamma' \bar{v})^1(\cdot, t)\|_{\mathcal{E}} \leq C\|v(\cdot, t)\|_{-\alpha}. \quad (3.43) \]

Since $v + (\Gamma' \bar{v})^0 \in L^1(\mathbb{R}; \ell^2(\nu)),$ then the bound (3.25) gives
\[ \|R_0' \left( v + (\Gamma' \bar{v})^0 \right)\|_{\mathcal{H}(\nu)} \leq C\|v + (\Gamma' \bar{v})^0\|_{L^1(\mathbb{R}; \ell^2(\nu))} \leq C_1\|v\|_{L^1(\mathbb{R}; \ell^2_{-\alpha}).} \]

However, we can not apply bound (3.25) with $(\Gamma' \bar{v})^1$ instead of $v,$ because $(\Gamma' \bar{v})^1(\cdot, t) \notin \ell^2$ for any $t$ by (3.43), but $(\Gamma' \bar{v})^1(\cdot, t) \notin \ell^2(\nu),$ in general. Now we study $R_0' \dot{w}$ with $w := (\Gamma' \bar{v})^1.$

Note first that $R_0' v(y) = R_{1, \pm} v(y)$ for $y \in \mathbb{Z}$ with
\[ (R_{1, \pm} v)(y) := \sum_{x \in \mathbb{Z}} \int_{-\infty}^{+\infty} G^{0j}_{t, \pm}(x, y) v(x, t) \, dt = \sum_{x \in \mathbb{Z}} \int_{-\infty}^{+\infty} G^{0j}_{t, \pm}(x - y, t) v(x, t) \, dt, \]

where we use notation (3.13). Then, in the Fourier transform,
\[ (\hat{R}_{1, \pm} \hat{w})(\theta) = \int_{-\infty}^{+\infty} \hat{G}^{0j}_{t, \pm}(\theta) \hat{\partial_t \hat{w}}(\theta, t) \, dt = - \int_{-\infty}^{+\infty} \hat{G}^{ij}_{t, \pm}(\theta) \hat{\hat{w}}(\theta, t) \, dt, \quad \theta \in \mathbb{T}, \]

by (3.4). Hence, using the Parseval equality, we have
\[
\|R_0' \dot{w}\|_{\mathcal{H}(\nu)} \leq C_1 \sum_{j=0,1} \| (\hat{R}_{1, \pm} \hat{w})^j \|_{L^2(\mathbb{T})} + C_2 \| (\hat{R}_{1, \pm} \hat{w})^0 \|_{L^2(\mathbb{T})}
\]
\[
\leq C \int_{-\infty}^{+\infty} \| \hat{\hat{w}}(\cdot, t) \|_{L^2(\mathbb{T})} \, dt \leq C_1 \|w\|_{L^1(\mathbb{R}; \ell^2).} \]

Therefore,
\[ \|R_0' \partial_t (\Gamma' \bar{v})^1\|_{\mathcal{H}(\nu)} \leq \sum_{\pm} \|R_{1, \pm} \partial_t (\Gamma' \bar{v})^1\|_{\mathcal{H}(\nu)} \leq C \| (\Gamma' \bar{v})^1\|_{L^1(\mathbb{R}; \ell^2)} \leq C_1 \|v\|_{L^1(\mathbb{R}; \ell^2_{-\alpha})} \]

by (3.43). Hence, $R_0' T \Omega' \bar{v} \in \mathcal{H}(\nu).$ This completes the proof of assertion (i) of Theorem 2.7.

Assertion (ii) of Theorem 2.7 follows from the following lemma.
Lemma 3.16 (1) Let conditions $C$, $S_1$, and $S_2$ hold. Then the family of the measures $\{P_\tau, \tau \in \mathbb{R}\}$ is weakly compact in the space $\mathcal{C}_\beta^0$, with any $\beta < \alpha < -3/2$, and the bound holds:

$$\sup_{\tau \geq 0} \int |u|_{\alpha,1,T}^2 P_\tau(du) \leq C(\alpha) < \infty, \quad (3.44)$$

where the constant $C(\alpha)$ does not depend on $T > 0$.

(2) Let conditions $C$, $S_1$, $S_3$, and $S_4$ hold. Then for every $v \in \mathcal{P}$, the characteristic functionals of $P_\tau$ converge to a limit as $\tau \to \infty$,

$$\hat{P}_\tau(v) \equiv \int e^{i[u,v]} P_\tau(du) \to \hat{P}_\infty(v), \quad \tau \to \infty. \quad (3.45)$$

Here $\hat{P}_\infty(v) = \hat{P}^\nu_\infty(T\Omega'\vec{v})$, where $\hat{P}^\nu_\infty$ is defined in (3.21).

Proof Similarly to (3.22), we have

$$P_\tau(\omega) = \mu_\tau(R^{-1}\omega) \quad \text{for any } \omega \in \mathcal{B}(\mathcal{C}_\alpha^1) \text{ and } \tau > 0, \quad (3.46)$$

where $\mu_\tau$ is defined in Definition 3.13. To prove the bound (3.44), we apply (3.46) and obtain

$$\int |u|_{\alpha,1,T}^2 P_\tau(du) = \int |RY|_{\alpha,1,T}^2 \mu_\tau(dY) = \sup_{|s| \leq T} \int \|U(s)Y\|_\alpha^2 \mu_\tau(dY)$$

$$= \sup_{|s| \leq T} \int \|Y\|_\alpha^2 \mu_{s+\tau}(dY) \leq C(\alpha) < \infty$$

by the bound (3.34). The bound (3.44) and the Prokhorov theorem imply the weak compactness of the measures family $\{P_\tau, \tau \in \mathbb{R}\}$ in the space $\mathcal{C}_\beta^0$, $\beta < \alpha$. This can be proved by a similar method as in the proof of Theorem 3.9 (iii).

To prove (3.45), we use the inequality $|e^{i\xi} - 1| \leq |\xi|$ for $\xi \in \mathbb{R}$ and bounds (3.38), (3.41) and obtain that

$$\left| \hat{P}_\tau(v) - \hat{P}^\nu_\tau(T\Omega'\vec{v}) \right| \leq E|\delta_\tau| \leq \sqrt{E(\delta^2_\tau)} \leq C \int_{-\infty}^{+\infty} (t + \tau)^{-1/2} \|v(\cdot,t)\|_{-\alpha} dt \to 0, \quad \tau \to \infty.$$

It remains to apply Theorem 3.9 (iii) and obtain that

$$\hat{P}^\nu_\tau(T\Omega'\vec{v}) \to \hat{P}^\nu_\infty(T\Omega'\vec{v}), \quad \tau \to \infty. \quad (3.47)$$

However, $T\Omega'\vec{v} \not\in \mathcal{P}$, in general. More precisely, convergence (3.47) follows from the following facts:

(i) the equality $\hat{P}^\nu_\tau(T\Omega'\vec{v}) = \hat{\nu}_\tau(R_0^\tau T\Omega'\vec{v})$ holds by (3.22) and (3.23);

(ii) $\nu_\tau(\Psi)$ converges to a limit as $\tau \to \infty$ for any $\Psi \in \mathcal{S}$ (Theorem 3.5 (ii));

(iii) $\mathcal{S}$ is dense in $\mathcal{H}(\kappa)$ (evidently);

(iv) the characteristic functionals $\hat{\nu}_\tau(\Psi)$, $\tau \in \mathbb{R}$, are equicontinuous in $\mathcal{H}(\kappa)$ (Lemma 3.7);

(v) $R_0^\tau T\Omega'\vec{v} \in \mathcal{H}(\kappa)$ for any $v \in \mathcal{P}$ (this was proved above).

Lemma 3.16 is proved. $\square$

This completes the proof of assertion (ii) of Theorem 2.7. Assertion (iii) of Theorem 2.7 is proved in next section.
3.3 Mixing property of the limit measure $P_\infty$

We first prove the convergence (1.3) for the measure $P^\nu_\infty$. The invariance of $P_\infty^\nu$ w.r.t. the group $S_\tau$, $\tau \in \mathbb{R}$, follows from Theorem 3.9 (iii).

**Lemma 3.17** The group $S_\tau$ is mixing w.r.t. the measure $P_\infty^\nu$, i.e., for any $f, g \in L^2(\mathcal{E}_1, P_\infty^\nu)$,

$$
\lim_{\tau \to \infty} \int f(S_\tau z)g(z) P^\nu_\infty(dz) = \int f(z) P^\nu_\infty(dz) \int g(z) P^\nu_\infty(dz). \quad (3.48)
$$

In particular, the group $S_\tau$ is ergodic w.r.t. the measure $P_\infty^\nu$, i.e.,

$$
\lim_{T \to \infty} \frac{1}{T} \int f(S_\tau z) \, d\tau = \int f(z) P^\nu_\infty(dz) \pmod{P^\nu_\infty}. \quad (3.49)
$$

**Proof** Since $P^\nu_\infty$ is a Gaussian measure with zero mean value, it is enough to prove (see [7]) that for any $v_1, v_2 \in \mathcal{P}$,

$$
I_\tau := \int [S_\tau z, v_1][z, v_2] P^\nu_\infty(dz) \to 0 \quad \text{as} \quad \tau \to \infty. \quad (3.49)
$$

Using (3.23), (3.24), and (3.9), we obtain

$$
I_\tau = \int \langle Y, R_0^{-1} S_\tau^{-1} v_1 \rangle \langle Y, R_0 v_2 \rangle \nu_\infty(dY) = \langle Q^\nu_\infty(y_1, y_2), (R_0^{-1} S_\tau^{-1} v_1)(y_1) \otimes (R_0 v_2)(y_2) \rangle
$$

$$
= \sum_{x_1, x_2} \sum_{\pm} \int_{-\infty}^{+\infty} dt_1 \int_{-\infty}^{+\infty} A_{\tau, \pm}(x_1, x_2, t_1, t_2) v_1(x_1, t_1) v_2(x_2, t_2) \, dt_2, \quad (3.50)
$$

where, by definition,

$$
A_{\tau, \pm}(x_1, x_2, t_1, t_2) := \sum_{i,j=0,1} \sum_{y_1, y_2} Q^{ij}_{\pm, \pm}(y_1, y_2) G^{0i}_{t_1+\tau, \pm}(x_1, y_1) G^{0j}_{t_2, \pm}(x_2, y_2). \quad (3.51)
$$

Similarly to (3.27), we have

$$
A_{\tau, \pm}(x_1, x_2, t_1, t_2) := \sum_{i,j=0,1} \sum_{y_1, y_2} q^{ij}_{\pm, \pm}(y_1 - y_2) G^{0i}_{t_1+\tau, \pm}(x_1, y_1) G^{0j}_{t_2, \pm}(x_2, y_2)
$$

$$
= \frac{2}{\pi} \sum_{i,j=0,1} \int_{-1}^{1} q^{ij}_{\pm, \pm}(\theta) \Phi^{0i}_{t_1+\tau, \pm}(\theta) \Phi^{0j}_{t_2, \pm}(\theta) \sin(x_1 \theta) \sin(x_2 \theta) \, d\theta
$$

$$
= \frac{2}{\pi} \int_{-1}^{1} \cos(\phi_{\pm}(\theta)(t_1 + \tau - t_2)) q^{00}_{\pm, \pm}(\theta) \sin(x_1 \theta) \sin(x_2 \theta) \, d\theta. \quad (3.51)
$$

Hence, applying Lemma 2.3 (ii), formulas (3.11), and Fejér’s theorem (if $\kappa_{\pm} = 0$), we obtain

$$
|A_{\tau, \pm}(x_1, x_2, t_1, t_2)| \leq C \int_{-1}^{1} |\Phi^{00}_{\pm, \pm}(\theta) \sin(x_1 \theta) \sin(x_2 \theta)| \, d\theta \leq C_1 + C_2(|x_1| + |x_2|), \quad (3.52)
$$

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where the constants $C_1$ and $C_2$ do not depend on $x_1, x_2 \in \mathbb{Z}_\pm$ and $C_2 = 0$ if $\kappa_\pm \neq 0$. Since $v_1, v_2 \in \mathcal{P}$, it follows from (3.50) and (3.52) that to prove (3.49) it suffices to check the convergence
\[ A_{\tau, \pm}(x_1, x_2, t_1, t_2) \rightarrow 0 \quad \text{as} \quad \tau \rightarrow \infty, \quad (3.53) \]
for fixed values of $x_1, x_2 \in \mathbb{Z}_\pm \setminus \{0\}$ and $t_1, t_2 \in \mathbb{R}$. We denote by $R_{\tau, \pm}$ the integrand in the r.h.s. of (3.51) and rewrite it in the form
\[ R_{\tau, \pm}(\theta) \equiv R_{\tau, \pm}(\theta; x_1, x_2, t_1, t_2) = \cos(\phi_\pm(\theta)(t_1 + \tau - t_2)) \hat{q}_{\infty, \pm}(\theta) \sin(x_1 \theta) \sin(x_2 \theta) = \cos(\phi_\pm(\theta) \tau) a_\pm(\theta) + \sin(\phi_\pm(\theta) \tau) b_\pm(\theta), \]
where
\[ a_\pm(\theta) \equiv a_\pm(\theta; x_1, x_2, t_1, t_2) = \cos(\phi_\pm(\theta)(t_1 - t_2)) \hat{q}_{\infty, \pm}(\theta) \sin(x_1 \theta) \sin(x_2 \theta), \]
\[ b_\pm(\theta) \equiv b_\pm(\theta; x_1, x_2, t_1, t_2) = -\sin(\phi_\pm(\theta)(t_1 - t_2)) \hat{q}_{\infty, \pm}(\theta) \sin(x_1 \theta) \sin(x_2 \theta), \]
and $a_\pm, b_\pm \in L^1(\mathbb{T})$. Choose a $\delta > 0$ and introduce a partition of unity, $f(\theta) + g(\theta) = 1$, where $f$ and $g$ are nonnegative functions in $C^\infty(\mathbb{T})$, $\text{supp} f \subset O_\delta(0)$, $\text{supp} g \cap O_{\delta/2}(0) = \emptyset$. We split $A_{\tau, \pm}$ into the sum of two integrals,
\[ A_{\tau, \pm} = \frac{2}{\pi} \int_\mathbb{T} f(\theta) R_{\tau, \pm}(\theta) \, d\theta + \frac{2}{\pi} \int_\mathbb{T} g(\theta) R_{\tau, \pm}(\theta) \, d\theta \equiv A_{\tau, \pm}^f + A_{\tau, \pm}^g. \]
On the one hand, $\forall \varepsilon > 0 \ \exists \delta > 0$ such that $|A_{\tau, \pm}^f| \leq C\varepsilon$ uniformly in $\tau$. On the other hand, the phase functions $\phi_\pm(\theta)$ are smooth and $\phi_\pm'(\theta) \neq 0$ on the support of the $g$. Hence, the oscillatory integrals in $A_{\tau, \pm}^g$ vanish by the Lebesgue–Riemann Theorem. Therefore, the convergence (3.53) holds and Lemma 3.17 is proved.

Now we check assertion (1.8) for the limit measure $P_\infty$. We prove that for any $v_1, v_2 \in \mathcal{P}$
\[ I_\tau' := \int [S_\tau u, v_1][u, v_2] P_\infty(du) \rightarrow 0 \quad \text{as} \quad \tau \rightarrow \infty. \quad (3.54) \]
Applying (2.7) gives
\[ I_\tau' = \int [z, T\Omega' \vec{v}_1(\cdot, t - \tau)] [z, T\Omega' \vec{v}_2] P_\infty'(dz) = \int [S_\tau z, T\Omega' \vec{v}_1][z, T\Omega' \vec{v}_2] P_\infty'(dz). \]
Put $f_i(z) := [z, T\Omega' \vec{v}_i], \ i = 1, 2$. Then, $f_i \in L^2(\mathfrak{C}_\alpha^1, P_\infty')$, since
\[ \int |f_i(z)|^2 P_\infty'(dz) = \int [u, v_i]^2 P_\infty(du) < \infty \]
by (2.7). Therefore, we apply (3.48) and obtain
\[ I_\tau' = \int f_1(S_\tau z) f_2(z) P_\infty'(dz) \rightarrow \int f_1(z) P_\infty'(dz) \int f_2(z) P_\infty'(dz) \quad \text{as} \quad \tau \rightarrow \infty. \]
Finally,
\[ \int f_i(z) P_\infty'(dz) = \int [u, v_i] P_\infty(du) = 0, \]
because $P_\infty$ has zero mean value. Hence, (3.54) holds. This completes the proof of Theorem 2.7. □
Appendix: Homogeneous harmonic chain

Let condition (1.3) hold. Then the problem (1.1)–(1.2) becomes

\[
\begin{align*}
\ddot{u}(x,t) &= (\nu^2 \Delta_L - \kappa^2)u(x,t), \quad x \in \mathbb{Z}, \quad t > 0, \\
u(x,0) &= u_0(x), \quad \dot{u}(x,0) = v_0(x), \quad x \in \mathbb{Z}.
\end{align*}
\]

At first, we state results concerning the statistical solutions \(\mu_t, \quad t \in \mathbb{R}\).

**Lemma A.1** (see 3.1 Theorem A]) Let \(\alpha < -1/2\) and condition (1.3) hold. Then all assertions of Theorem 3.13 hold, where the quadratic form \(Q_\infty\) has the matrix kernel

\[
Q_\infty(x,y) = q_\infty(x-y), \tag{A.1}
\]

\[
q_\infty(x) = F_{\theta \to x}^{-1}[\hat{q}_\infty(\theta)] \quad \text{and} \quad \hat{q}_\infty(\theta) = (\hat{q}^{ij}_\infty(\theta)) \quad \text{is defined in (2.3)}.
\]

**Proof of Theorem 2.8** Introduce the adjoint operator \(R'\) to the operator \(R\) defined in (1.5),

\[
[RY,v] = \langle Y, R'v \rangle \quad \text{for} \quad Y \in \mathcal{H}_\alpha \quad \text{and} \quad v \in \mathcal{P}. \tag{A.2}
\]

Then for \(v \in \mathcal{P}\),

\[
(R'v)(x) = \left( \sum_{y \in \mathbb{Z}}^{+\infty} \int_{\mathbb{R}}^{} G^0_t(y-x)v(y,t)dt \right) \sum_{y \in \mathbb{Z}}^{+\infty} \int_{\mathbb{R}}^{} G^1_t(y-x)v(y,t)dt,
\]

where \(G^{ij}_t\) is defined as \(G^{ij}_{t,t+}\) in (3.3) and (3.4) but with \(\phi(\theta)\) instead of \(\phi_+(\theta)\). It follows from (A.2) and (3.46) that for \(v_1, v_2 \in \mathcal{P}\),

\[
Q^R_\tau(v_1,v_2) := \int [u,v_1][u,v_2] P_\tau(du) = \int [RY,v_1][RY,v_2] \mu_\tau(dY) = \int \langle Y, R'v_1(Y, R'v_2) \mu_\tau(dY) = \langle Q_\tau(x,y), R'v_1(x) \otimes R'v_2(y) \rangle.
\]

Since \(\kappa \neq 0\), then \(R'v \in \mathcal{S}\) for any \(v \in \mathcal{P}\). Hence, we can apply Lemma A.1 and obtain

\[
Q^R_\tau(v_1,v_2) \to \langle Q_\infty(x,y), R'v_1(x) \otimes R'v_2(y) \rangle \quad \text{as} \quad \tau \to \infty.
\]

Hence, \(Q^R_\infty(v_1,v_2) = \langle Q_\infty(x,y), R'v_1(x) \otimes R'v_2(y) \rangle\). Now we check formula (2.8). Using (A.1) and (A.3), we have

\[
Q^R_\infty(x_1, x_2, t_1, t_2) = Q^R_\ast(x_1 - x_2, t_1, t_2),
\]

where in the Fourier transform \(x \to \theta\)

\[
\hat{Q}^R_\ast(\theta,t_1,t_2) = \sum_{x \in \mathbb{Z}} e^{itx} Q^R_\ast(x,t_1,t_2) = \sum_{i,j=0}^{1} \hat{G}^{0i}_t(\theta) \hat{q}^{ij}_\infty(\theta) \hat{G}^{0j}_t(\theta), \quad \theta \in \mathbb{T}, \quad t_1, t_2 \in \mathbb{R}.
\]

Using formulas \(\hat{q}^{11}_\infty(\theta) = \phi^2(\theta) \hat{q}^{00}_\infty(\theta)\), \(\hat{q}^{10}_\infty(\theta) = -\hat{q}^{01}_\infty(\theta)\), and (3.4) with \(\phi_+ \equiv \phi\), we have

\[
\hat{Q}^R_\ast(\theta,t_1,t_2) = \cos(\phi t_1) \hat{q}^{00}_\infty(\theta) \cos(\phi t_2) - \sin(\phi t_1) \phi^{-1} \hat{q}^{01}_\infty(\theta) \cos(\phi t_2) + \cos(\phi t_1) \hat{q}^{01}_\infty(\theta) \phi^{-1} \sin(\phi t_2) + \sin(\phi t_1) \hat{q}^{00}_\infty(\theta) \phi^{-1} \hat{q}^{01}_\infty(\theta).
\]
with $\phi \equiv \phi(\theta)$. This implies (2.8).

Now we prove the convergence (1.7) by a similar way as in Theorem 2.7. Assertion (1.7) follows from the bound (3.44) and convergence (3.45). The bound (3.44) can be proved in the same way as in Theorem 2.7. Lemma A.1 implies that for any $\Psi \in \mathcal{S}$,

$$\hat{\mu}_t(\Psi) \to \hat{\mu}_\infty(\Psi) \quad t \to \infty.$$  

Using (A.2) and taking $\Psi := R'v$, we obtain

$$\hat{P}_\tau(v) = \int e^{i[u,v]} P_\tau(du) \equiv \int e^{i(Y,R'v)} \mu_\tau(dY) \to \exp \left\{-\frac{1}{2} Q_\infty(R'v,R'v)\right\}, \quad \tau \to \infty,$$

where the quadratic form $Q_\infty$ is introduced in Lemma A.1. Theorem 2.8 is proved. □

Now we verify the mixing property (1.8) for the limit measure $P_\infty$. The invariance of the measure $P_\infty$ w.r.t. the shifts in time and in space follows from convergence (1.7) and (A.1). Since the measure $P_\infty$ is Gaussian with zero mean value, it is enough to prove that for any $v_1, v_2 \in \mathcal{P}$,

$$I_\tau := \mathbb{E}_\infty (\langle S_\tau u, v_1 \rangle \langle u, v_2 \rangle) \to 0 \quad \tau \to \infty,$$  

(A.4)

where $\mathbb{E}_\infty$ denotes the integral w.r.t. the limit measure $P_\infty$.

Indeed, using (A.2) and (A.3), we obtain

$$I_\tau = \int \langle Y, R'S_\tau^{-1}v_1 \rangle \langle Y, R'v_2 \rangle \mu_\infty(dY) = \frac{1}{2\pi} \int \left( \hat{q}_\infty(\theta), (R'S_\tau^{-1}v_1)(\theta) \otimes (R'v_2)(\theta) \right) d\theta$$

$$= \frac{1}{2\pi} \int d\theta \int_{-\infty}^{+\infty} dt_1 \int_{-\infty}^{+\infty} dt_2 B_\tau(t_1, t_2, \theta) \hat{v}_1(\theta, t_1) \hat{v}_2(\theta, t_2) dt_1 dt_2,$$

where the function $B_\tau(t_1, t_2, \theta)$ is of the form

$$B_\tau(t_1, t_2, \theta) := \sum_{i,j=0}^{1} \hat{\theta}_{t_1+t_2}(\theta) \hat{q}_\infty^{ij}(\theta) \hat{G}_\tau^{0j}(\theta)$$

$$= \cos (\phi(\theta)(t_1 + \tau - t_2)) \hat{q}_\infty^{00}(\theta) - \sin (\phi(\theta)(t_1 + \tau - t_2)) \phi^{-1}(\theta) \hat{q}_\infty^{01}(\theta).$$

We represent $I_\tau$ as follows:

$$I_\tau = \sum_{\pm} \frac{1}{2\pi} \int \frac{1}{\tau} e^{i\phi(\theta) \tau} c_\pm(\theta) d\theta,$$  

(A.5)

where

$$c_\pm(\theta) := \frac{1}{2} \int_{-\infty}^{+\infty} dt_1 \int_{-\infty}^{+\infty} e^{i\phi(\theta)(t_1-t_2)} (\hat{q}_\infty^{00}(\theta) \pm i\phi^{-1}(\theta) \hat{q}_\infty^{01}(\theta)) \hat{v}_1(\theta, t_1) \hat{v}_2(\theta, t_2) dt_1 dt_2.$$

Note that $c_\pm(\theta) \in L^1(\mathbb{T})$ by Lemma 2.5 (ii) and formulas (3.11) with $\phi_\pm \equiv \phi$. Hence, the oscillatory integrals in (A.5) vanish by the Lebesgue–Riemann Theorem. Therefore, the assertions (A.4) and (1.8) hold.

Similarly to (1.8), we can check the following assertion.
Lemma A.2 Let $S_h$, $h \in \mathbb{Z}$, denote the shifts in space, $S_h u(x, t) = u(x + h, t)$. Then, for any $f, g \in L^2(\mathcal{C}_\alpha^1, P_\infty),$

$$\lim_{h \to \infty} \mathbb{E}_\infty f(S_h u) g(u) = \mathbb{E}_\infty f \mathbb{E}_\infty g.$$

**Proof** Indeed, it suffices to check that $I_h := \mathbb{E}_\infty ([S_h u, v_1][u, v_2]) \to 0$ as $h \to \infty$. Using (A.2) and (A.3), we obtain

$$I_h = \int \langle Y, R'S_h^{-1}v_1 \rangle \langle Y, R'v_2 \rangle \mu_\infty(dY) = \frac{1}{2\pi} \int_T e^{ith} D(\theta) d\theta,$$

where

$$D(\theta) := \int_{-\infty}^{+\infty} dt_1 \int_{-\infty}^{+\infty} \left( \cos(\phi(t_1-t_2)) \hat{g}_\infty^{00}(\theta) - \sin(\phi(t_1-t_2)) \hat{g}_\infty^{-01}(\theta) \right) \hat{v}_1(\theta) \hat{v}_2(\theta, t_2) dt_2.$$

Therefore, $I_h$ vanishes as $h \to \infty$ by the Lebesgue–Riemann Theorem, because $D(\theta) \in L^1(\mathbb{T})$.

The following lemma generalizes the convergence (1.8).

**Lemma A.3** The group $S_\tau$ is mixing of order $r \geq 1$ w.r.t. the measure $P_\infty$, i.e., for any $f_0, \ldots, f_r \in L^{r+1}(\mathcal{C}_\alpha^1, P_\infty),$

$$\lim_{\tau_1, \ldots, \tau_r \to \infty} \int f_0(u) f_1(S_{\tau_1} u) \cdots f_r(S_{\tau_1+\ldots+\tau_r} u) P_\infty(du) = \prod_{i=0}^r \int f_i(u) P_\infty(du).$$

**Proof** Since the measure $P_\infty$ is Gaussian with zero mean value, it is enough to prove that for any $v_0, \ldots, v_r \in \mathcal{P},$

$$I_{\tau_1, \ldots, \tau_r} := \mathbb{E}_\infty ([u, v_0][S_{\tau_1} u, v_1] \cdots [S_{\tau_1+\ldots+\tau_r} u, v_r]) \to 0 \quad \text{as } \tau \to \infty. \quad (A.6)$$

At first, note that (see [7] Ch.III, § 1)

$$\mathbb{E}_\infty ([u, v_1] \cdots [u, v_n]) = \left\{ \begin{array}{ll} 0, & \text{if } n \text{ is odd}, \\ \sum \prod \mathbb{E}_\infty ([u, v_i][u, v_j]), & \text{if } n \text{ is even}. \end{array} \right.$$  

Here the sum is taken over all partitions of $\{v_1, \ldots, v_n\}$ into pairs, the product is taken over all pairs of the partition (the pairs that differ by the permutation of elements are considered as one). For example, if $n = 4$, there are three partitions of $\{v_1, v_2, v_3, v_4\}$ into pairs and

$$\mathbb{E}_\infty ([u, v_1] \cdots [u, v_4]) = b_{12}b_{34} + b_{13}b_{24} + b_{14}b_{23}, \quad \text{where } b_{ij} := \mathbb{E}_\infty ([u, v_i][u, v_j]).$$

Hence, $I_{\tau_1, \ldots, \tau_r} = 0$ if $r$ is even. If $r$ is odd, then

$$I_{\tau_1, \ldots, \tau_r} = \mathbb{E}_\infty ([u, v_0][u, S_{\tau_1}^{-1} v_1] \cdots [u, S_{\tau_1+\ldots+\tau_r}^{-1} v_r])$$

$$= \sum_{k=1}^r \mathbb{E}_\infty ([u, v_0][u, S_{\tau_1+\ldots+\tau_k}^{-1} v_k]) \cdot \left( \sum \prod B_{ij} \right), \quad (A.7)$$
where the inner sum is taken over all partitions of \( \{v_1, \ldots, v_{k-1}, v_{k+1}, \ldots, v_r\} \) into pairs, the product is taken over all pairs of the partition, and

\[
B_{ij} := \mathbb{E}_\infty \left( \left[ u, S_{\tau_1 + \ldots + \tau_i}^{-1} v_i \right] \left[ u, S_{\tau_1 + \ldots + \tau_j}^{-1} v_j \right] \right), \quad i, j = 1, \ldots, k - 1, k + 1, \ldots, r.
\]

Since the measure \( P_\infty \) is invariant w.r.t. \( S_\tau \),

\[
B_{ij} \leq \sqrt{\mathbb{E}_\infty ([u,v_i]^2)} \sqrt{\mathbb{E}_\infty ([u,v_j]^2)} \leq C < \infty.
\]

Furthermore, \((A.4)\) implies that for any \( k \geq 1 \)

\[
\mathbb{E}_\infty \left( \left[ u, v_0 \right] \left[ u, S_{\tau_1 + \ldots + \tau_k}^{-1} v_k \right] \right) \rightarrow 0 \quad \text{as} \quad \tau_1, \ldots, \tau_k \rightarrow +\infty. \quad (A.8)
\]

Formulas \((A.7)\)–\((A.8)\) imply the convergence \((A.6)\). □

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References

[1] Yu. A. Dubinskii, “The weak convergence in the nonlinear elliptic and parabolic equations,” Math. Sb. (N.S.) 67(109), no. 4, 609–642 (1965) [in Russian].

[2] T.V. Dudnikova, “Stabilization of space-time statistical solutions of the Klein–Gordon equation,” Russian J. Math. Phys. 5(2), 176–188 (1997).

[3] T.V. Dudnikova, A.I. Komech, and N.J. Mauser, “On two-temperature problem for harmonic crystals,” J. Stat. Phys. 114(3-4), 1035–1083 (2004).

[4] T.V. Dudnikova, “Behavior for large time of a two-component chain of harmonic oscillators,” Russian J. Math. Phys. 25(4), 470–491 (2018).

[5] T.V. Dudnikova, “Convergence to stationary states and energy current for infinite harmonic crystals,” Russian J. Math. Phys. 26(4), 429–453 (2019).

[6] T.V. Dudnikova, “Stabilization of statistical solutions for an infinite inhomogeneous chain of harmonic oscillators,” Proceedings of the Steklov Institute of Mathematics 308 (2020), 168–183.

[7] I.I. Gikhman and A.V. Skorokhod, The Theory of Stochastic Processes, I, Springer, Berlin (1974).

[8] J.L. van Hemmen, “Dynamics and ergodicity of the infinite harmonic crystal,” Phys. Reports (Review Section of Phys. Letters) 65(2), 43–149 (1980).

[9] I.A. Ibragimov and Yu.V. Linnik, Independent and Stationary Sequences of Random Variables, Ed. by J. F. C. Kingman, Wolters-Noordhoff, Groningen (1971).

[10] Y. Katznelson, An Introduction in Harmonic Analysis, 3rd edition, Cambridge University Press (2004).
[11] A.I. Komech and N.E. Ratanov, “Stabilization of space-time stochastic solutions of wave equation,” p.171–187 in Statistics and Control of Stochastic Processes v.2, Steklov Seminar 1985–1986, Optimization Software Inc., Publication Division, New York–Los Angeles (1989).

[12] O.E. Lanford III and J.L. Lebowitz, “Time evolution and ergodic properties of harmonic systems,” in: “Dynamical Systems, Theory and Applications,” Lecture Notes in Physics 38 (Springer–Verlag, Berlin, 1975), pp. 144–177.

[13] N.K. Nikol’skii, Lectures on the Shift Operator, Moscow, Nauka (1980) [in Russian] (English transl.: Treatise on the Shift Operator, Springer–Verlag (1986)).

[14] N.E. Ratanov, “Stabilization of space-time statistical solutions of the parabolic equation,” Vestnik Chelyabinsk. Gos. Univ., Issue 1, 64–70 (1991) [in Russian].

[15] C.D. Sogge, Fourier Integral Operators in Classical Analysis, Cambridge University Press (1993).

[16] M.I. Vishik and A.V. Fursikov, Mathematical Problems of Statistical Hydromechanics, Kluwer Academic, New York (1988).

[17] K. Yosida, Functional Analysis, Springer–Verlag, Berlin (1965).