NONABELIAN LOCALIZATION IN EQUIVARIANT $K$-THEORY AND RIEMANN-ROCH FOR QUOTIENTS

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Abstract. We prove a localization formula in equivariant algebraic $K$-theory for an arbitrary complex algebraic group acting with finite stabilizer on a smooth algebraic space. This extends to non-diagonalizable groups the localization formulas of H.A. Nielsen [Nie] and R. Thomason [Tho5].

As an application we give a Riemann-Roch formula for quotients of smooth algebraic spaces by proper group actions. This formula extends previous work of B. Toen [Toe] and the authors [EG3].

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1. Introduction

Equivariant $K$-theory was developed in the late 1960’s by Atiyah and Segal as a tool for the proof of the index theorem for elliptic operators invariant under the action of a compact Lie group. In the late 1980’s and early 1990’s Thomason constructed an algebraic equivariant $K$-theory modeled on Quillen’s earlier construction of higher $K$-theory for schemes.

In both the topological and algebraic contexts equivariant $K$-theory is studied using its structure as a module for the representation ring of the group $G$. The fundamental theorem of equivariant $K$-theory is the localization theorem for actions of diagonalizable groups. We describe a version of this theorem in the complex algebraic setting. Let $R(G)$ be the representation ring of $G$ tensored with $\mathbb{C}$. If $X$ is a $G$-space we let $G(G, X)$ be the equivariant $K$-groups of the category of $G$-equivariant coherent sheaves, also tensored with $\mathbb{C}$. If $G$ is diagonalizable and $h \in G$, let $\iota: X^h \to X$ be the inclusion of the fixed locus of $h$. Let $m_h \subset R(G)$ denote the maximal ideal of representations whose virtual characters vanish at $h$. The localization theorem states that the natural map

$$\iota_*: G(G, X^h)_{m_h} \to G(G, X)_{m_h},$$

is an isomorphism. Much of the power of this theorem comes from the fact that if $X$ is regular, then so is $X^h$, and the localization isomorphism has an explicit inverse, arising from the self-intersection formula for the regular embedding $X^h \hookrightarrow X$: If $\alpha \in G(G, X)_{m_h}$ is an element of localized equivariant $K$-theory then

$$\alpha = \iota_* \left( \frac{\iota^* \alpha}{\lambda_{-1}(N^*_\iota)} \right).$$

Here $N^*_\iota$ is the conormal bundle to $\iota$, and $\lambda_{-1}(N^*_\iota)$ is defined to be the element in $K$-theory corresponding to the formal sum $\sum_{l=0}^{\text{rank} N^*_\iota} (-1)^l \Lambda^l(N^*_\iota)$.

The formula of Equation (1) is extremely useful because it reduces global calculations to those on the fixed locus. It has been applied in a wide range of contexts. For example, the localization theorem on the flag variety $G/B$ can be used to give a proof of the Weyl character formula. In [EG3] we used the localization theorem to prove a
Kawasaki-Riemann-Roch formula for quotients by diagonalizable group actions (similar ideas had been introduced earlier by Atiyah [Ati]).

If we try to generalize Equation (1) to the nonabelian case we immediately run into the problem that $X_h$ is not in general $G$-invariant. However, the locus $X_{\Psi} = \overline{G \cdot h}$ is $G$-invariant; it is the closure of the union of the fixed point loci of elements in the conjugacy class $\Psi$ of $h$. Let $m_{\Psi} \subset R(G)$ denote the maximal ideal of representations whose virtual characters vanish on $\Psi$, and let $i : X_{\Psi} \to X$ denote the inclusion. Then $i^* : G(G, X_{\Psi}) \to G(G, X)_{m_{\Psi}}$ is an isomorphism (see Theorem 3.3; this is a variant of a result of Thomason [Tho5], adapting a result of Segal [Seg2] from topological $K$-theory). Unfortunately, as Thomason observed [Tho4], $X_{\Psi}$ can be singular even when $X$ is smooth, so the self-intersection formula does not apply. To obtain a nonabelian version of Equation (1) new ideas are needed.

Although $X_h$ is not $G$-invariant, it is $Z$-invariant, where $Z = Z_G(h)$ is the centralizer in $G$ of $h$. In [VV] Vezzosi and Vistoli proved that if $G$ acts on $X$ with finite stabilizers then there is an isomorphism between a localization of $G(G, S_{\Psi})$ and a localization of $G(G, X)$. (Note that their theorem holds in arbitrary characteristic.) In this paper we work with the added hypothesis that the projection $f$ from the global stabilizer $S_X = \{(g, x) | gx = x\}$ to $X$ is a finite morphism. Our main result states that there is a natural pushforward $\iota : G(G, S_{\Psi}) \to G(G, X)$, such that when $X$ is smooth, the following formula holds for $\alpha \in G(G, X)_{m_{\Psi}}$:

$$\alpha = \iota_{\text{c}} \left( \frac{\lambda^{-1}(\mathfrak{g}/\mathfrak{z})^* \cap (\iota^* \alpha)_h}{\lambda^{-1}(Nf^*_{\text{c}})} \right).$$

Here $\iota_{\text{c}}$ is the composition of the restriction map $G(G, X) \to G(Z, X)$ with the pullback $G(Z, X) \xymatrix{\ar[r]^{\iota} & \ar[l]_{\iota'} G(Z, X^h)$; $(\iota^* \alpha)_h$ is the image of $\iota^* \alpha$ in $G(Z, X^h)_{m_{\Psi}}$, and $\mathfrak{g}, \mathfrak{z}$ are the Lie algebras of $G$ and $Z$ respectively.

To prove this result, we use an equivalent formulation involving the global stabilizer. Let $S_{\Psi} \subset S_X$ be the closed subspace of pairs $(g, x)$ with $g \in \Psi$. The finite map $f : S_{\Psi} \to X$ has image $X_{\Psi}$, but unlike $X_h$, the space $S_{\Psi}$ is regular (if $X$ is). There is a natural identification of $G(G, S_{\Psi})$ with $G(Z, X^h)$, and the map $\iota_{\text{c}}$ is identified with the pushforward $f_*$ in $G$-equivariant $K$-theory. Moreover, the natural map $f : S_{\Psi} \to X$ is a local complete intersection morphism, so it has a normal bundle $N_f$. There is a distinguished "central summand" $G(G, S_{\Psi})_{c_{\Psi}}$ of $G(G, S_{\Psi})$; if $\beta \in G(G, S_{\Psi})$, we let $\beta_{c_{\Psi}}$ denote the component of $\beta$ in the central summand. Equation (2) is equivalent to the
statement that if \( \alpha \in G(G, X)_{m\Psi} \), then

\[
\alpha = f_\ast \left( \frac{(f^*\alpha)_{c\Psi}}{\lambda_{-1}(N^*_f)} \right).
\]

This formula looks similar to the formula that would hold if \( f \) were a regular embedding (the only change would be to replace \((f^*\alpha)_{c\Psi}\) by \(f^*\alpha\)). However, that formula is not correct, and indeed, the main difficulty in proving this is that \( f \) is not a regular embedding, so we cannot apply the self-intersection formula. The proof given here is less direct; we first prove the result when \( G \) is a product of general linear groups, and then use a change of groups argument to deduce the general case.

The main application of Equations (2) and (3) is to give refined formulas for the Todd classes of sheaves of invariant sections on quotients of smooth algebraic spaces. If \( X \) is a smooth, separated, algebraic space and \( G \) is an algebraic group acting properly (and thus with finite stabilizer) then the theorem of Keel and Mori \([KM]\) implies that there is a (possibly singular) geometric quotient \( Y = X/G \). For such quotients there is a map in \( K \)-theory \( \pi_G: G(G, X) \to G(Y) \) induced by the exact functor which takes a \( G \)-equivariant coherent sheaf to its subsheaf of invariant sections (Lemma 6.2). Let \( \tau_Y: G_0(G, X) \to A_*(Y) \) be the Riemann-Roch map defined by Baum, Fulton, MacPherson \([BFM] [Ful]\). If \( \alpha \in G_0(G, X) \) then we obtain explicit expressions (Theorems 6.7 and 6.8 for \( \tau_Y(\pi_G(\alpha)) \)) in terms of the restriction of \( \alpha \) to a class in \( G_0(Z, X^h) \) where \( h \in \Psi \) is any element. If we sum over all conjugacy classes \( \Psi \) we obtain formulas for \( \tau_Y(\pi_G(\alpha)) \). When the quotient is quasi-projective our formulas for \( \tau_Y(\pi_G(\alpha)) \) can be deduced from the Riemann-Roch formula for stacks due to B. Toen \([Toe]\). Our method of proof is quite different, and makes no essential use of stacks.

The proof of our Riemann-Roch theorem is essentially the same as the proof for diagonalizable \( G \) given in \([EG3]\), with the nonabelian localization theorem of this paper in place of the localization theorem for diagonalizable groups. A key element of the proof in \([EG3]\) was the fact that, if \( G \) is diagonalizable, \( X^h \) is \( G \)-invariant and we may define an \( h \)-action on \( G_0(G, X^h) \) which we called “twisting by \( h \)”. Intuitively, this twist comes from the \( h \)-action on the sections of any \( G \)-equivariant coherent sheaf on \( X^h \). In the nonabelian setting one can still twist by a central element, so there is an action of \( h \) on \( G_0(G, X^h) \). This can be viewed as a twist of \( G(G, S_\Psi) \), which intuitively comes from the tautological action of the element \( g \) on the fiber at \((g, x)\) of any \( G \)-equivariant vector bundle on \( S_\Psi \). Versions of this twist and the global
stabilizer appear in the Riemann-Roch theorems of Kawasaki and Toen, and motivated our approach to the localization theorem and Riemann-Roch theorem. Interestingly, in the Riemann-Roch formula obtained from (3), one might expect a term involving \((f^*\alpha)_c\). However, we prove that the contributions from \((f^*\alpha)_c\) and \(f^*\alpha\) are equal, so our formula does not mention the central summand.

In this paper we work over \(\mathbb{C}\) and tensor all \(K\)-groups with \(\mathbb{C}\). The reason we do this is so that we can identify the representation ring \(R(G)\) with class functions on \(G\); this idea, which goes back to Atiyah and Segal, allows us to directly relate \(G\)-equivariant \(K\)-theory to conjugacy classes in \(G\). By working over \(\mathbb{C}\) we hope that the geometric techniques used to prove our main results are not obscured by technical details.

Nevertheless, we believe that versions of the nonabelian localization and Riemann-Roch theorems should hold over an arbitrary algebraically closed field provided we assume that all stabilizer groups are reduced. In this situation, instead of localizing at maximal ideals \(m_h \in R(G) \otimes \mathbb{C}\) where \(h \in G\) has finite order, we may localize at the multiplicatively closed set \(S_H\) defined on p. 10 of [VV], where \(H\) is the cyclic group generated by \(h\). In a different direction, there should be topological versions of these results for actions of compact Lie groups. This will be pursued elsewhere.

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1.1. Conventions and notation. We work entirely over the ground field of complex numbers \(\mathbb{C}\). All algebraic spaces are assumed to be of finite type over \(\mathbb{C}\). For a reference on the theory of algebraic spaces, see the book [Kn]. All algebraic groups are assumed to be linear. A basic reference for the theory of algebraic groups is Borel’s book [Bor].

If \(G\) is an algebraic group then \(Z(G)\) denotes the center of \(G\). If \(h \in G\) is any element then \(Z_G(h)\) denotes the centralizer of \(h\) in \(G\). The conjugacy class of \(h\) in \(G\) is denoted \(C_G(h)\). The map \(G \to C_G(h), g \mapsto ghg^{-1}\) identifies \(C_G(h)\) with the homogeneous space \(G/Z_G(h)\).

If \(G\) is an algebraic group then \(R(G)\) denotes the representation ring of \(G\) tensored with \(\mathbb{C}\).

1.1.1. Group actions. Let \(G\) be an algebraic group acting on an algebraic space \(X\). We consider three related conditions on group actions.

(i) We say that \(G\) acts properly if the map 
\[ G \times X \to X \times X, (g, x) \mapsto (x, gx) \]

is proper.
Let \( G \times X \to X \times X \) be the map we just defined and let \( X \to X \times X \) be the diagonal. Then \( S_X = G \times X \times_{X \times X} X \) is called the global stabilizer. As a set, \( S_X = \{(g, x) | gx = x\} \).

(ii) We say that \( G \) acts with finite stabilizer if the projection \( S_X \to X \) is a finite morphism.

(iii) We say that \( G \) acts with finite stabilizers if the projection \( S_X \to X \) is quasi-finite; i.e. for every point \( x \in X \) the isotropy group \( G_x \) is finite.

Since \( G \) is affine, the map \( G \times X \to X \times X \) is finite if it is proper. It follows that any proper action has finite stabilizer. If \( G \) acts properly on \( X \), then the diagonal morphism of \( X \) factors as the composition of two proper maps \( X \xrightarrow{(e,1_X)} G \times X \to X \times X \). This implies that the diagonal is a closed embedding, so \( X \) is automatically separated.

2. Groups, representation rings, and conjugacy classes

This section collects a number of facts about algebraic groups, representation rings, and conjugacy classes which are used in the proof of the localization theorem. We have included proofs of some results that are essentially known but are difficult to find in the literature for groups that are not connected or semisimple.

2.1. The representation ring and class functions. Let \( G \) be an algebraic group over \( \mathbb{C} \). The group \( G \) is called reductive if the radical of its identity component \( G_0 \) is a torus. Any reductive algebraic group is the complexification of a maximal compact subgroup \([OV, \text{Theorem 8, p. 244}]\). (This result can be deduced from the corresponding result for connected groups, using facts about maximal compact subgroups of Lie groups with finitely many connected components \([Hoc, \text{Theorem XV.3.1}]\).) By the unipotent radical of \( G \) we mean the unipotent radical of the identity component of \( G \); this is a normal subgroup of \( G \), and \( G \) modulo its unipotent radical is reductive. Any complex algebraic group \( G \) has a Levi subgroup \( L \). This means \( G \) is the semidirect product of \( L \) and the unipotent radical of \( G \) \([OV, \text{Chapter 6, Theorem 4}]\); \( L \) is necessarily reductive.

For a general algebraic group \( G \), let \( \hat{G} \) denote the set of isomorphism classes of irreducible (finite-dimensional) algebraic representations of \( G \). Let \( \mathbb{C}[G] \) denote the coordinate ring of \( G \), and \( \mathbb{C}[G]^G \) the ring of class functions, i.e., the functions on \( G \) which are invariant under conjugation. There is a map \( R(G) \to \mathbb{C}[G]^G \) which takes \([V]\) to \( \chi_V \), where \( V \) is a representation of \( G \) and \( \chi_V \) its character.
Let $V$ be a representation of $G$, and let $V^*$ denote the dual representation. There is a map

\[ V^* \otimes V \to \mathbb{C}[G] \]

(4)

taking $\lambda \otimes v$ to the function $\lambda/v$ defined by

\[ \lambda/v(x) = \lambda(xv) \]

for $x \in G$. This map is $G \times G$-equivariant, where $G \times G$ acts on $V^* \otimes V$ via the $G$-action on each factor, and $G \times G$ acts on functions by

\[ ((g_1, g_2) \cdot f)(x) = f(g_1^{-1}xg_2), \]

for $g_1, g_2, x \in G$ and $f \in \mathbb{C}[G]$. Functions of the form $\lambda/v$ are called representative functions; we denote the algebra they generate by $T(G)$. If $G$ is a linear algebraic group, then the action of $G \times G$ on $\mathbb{C}[G]$ is locally finite, and this can be used to show that $T(G) = \mathbb{C}[G]$. If $G$ is the complexification of $K$, then the map $T(G) \to T(K)$ is an isomorphism (cf. [BtD]). As a consequence, we obtain the algebraic Peter-Weyl theorem, attributed to Hochschild and Mostow:

**Proposition 2.1.** If $G$ is a complex reductive algebraic group, then the map

\[ \oplus_{V \in G} V^* \otimes V \to \mathbb{C}[G] \]

(5)

is an isomorphism as representations of $G \times G$.

This follows from the usual Peter-Weyl theorem for compact groups ([Ros, p. 201]), in view of the isomorphism $\mathbb{C}[G] \to T(K)$.

In this paper, we frequently use the following result. For connected groups, it can be proved using restriction to a maximal torus.

**Proposition 2.2.** If $G$ is reductive, the map $R(G) \to \mathbb{C}[G]^G$ taking a representation to its character is an isomorphism. In particular, for any $G$, $R(G)$ is a finitely generated algebra over $\mathbb{C}$.

**Proof.** View $G$ as embedded diagonally in $G \times G$. The isomorphism (5) induces an isomorphism of $G$-invariants in the source and target. If $V$ is any representation of $G$, then $(V^* \otimes V)^G = \text{Hom}_G(V, V)$, and if $V$ is irreducible representation, Schur’s lemma implies that this is 1-dimensional, spanned by the identity map $\text{id}_V$. But $\phi(\text{id}_V) = \chi_V$, proving the first statement of the proposition. The second statement follows because the representation rings of $G$ and any Levi factor are isomorphic.

As another application of the relationship between $G$ and $K$, we have the following proposition.
Proposition 2.3. If $H \hookrightarrow G$ is an embedding of groups, then $R(H)$ is a finite $R(G)$-module.

Proof. If $H_1$ is a Levi subgroup of $H$ then the restriction map $R(H) \to R(H_1)$ is an isomorphism, so if necessary replacing $H$ by $H_1$, we may assume $H$ is reductive. Let $G'$ be the quotient of $G$ by its unipotent radical; then the natural map $R(G') \to R(G)$ is an isomorphism. Moreover, the kernel of $H \to G'$ is a unipotent normal subgroup of $H$, hence is trivial. Hence the map $H \to G'$ is injective. Therefore, if necessary replacing $G$ by $G'$, we may assume $G$ is reductive.

Let $L$ be a maximal compact subgroup of $H$; we may assume $L$ is contained in a maximal compact subgroup $K$ of $G$. Then the restriction maps $R(G) \to R(K)$ and $R(H) \to R(L)$ are isomorphisms (cf. [BtD]). Therefore the proposition follows from the analogous result of [Seg1] for compact groups. □

2.2. Conjugacy classes and representation rings. If $H$ is a subgroup of $G$, let $C_H(g)$ denote the $H$-conjugates of $g$, i.e., the image of the map $H \to G$ taking $h$ to $hgh^{-1}$.

Proposition 2.4. Let $G$ be a reductive algebraic group. A conjugacy class $\Psi$ in $G$ is closed if and only if it is semisimple.

Proof. In any algebraic group, the conjugacy class of a semisimple element is closed [Bor, Theorem 9.2]. Conversely, suppose $\Psi$ is a closed conjugacy class. Let $g = su$ be the Jordan decomposition of some $g \in \Psi$. We want to show that $u = 1$, or in other words, that $C_G(s) = C_G(su)$. A general result of Mumford about reductive groups acting on affine schemes ([MFK, Ch. 1, Cor. 1.2]) implies that given two closed disjoint conjugation-invariant subsets of $G$, there is a class function which is 1 on one subset and 0 on the other. Therefore, to show that $C_G(s) = C_G(su)$ it suffices to show that any class function takes the same value on $C_G(s)$ as on $C_G(su)$. But this holds because it is true for characters, which span the space of class functions. □

Note that if $G$ is not reductive, there may be closed conjugacy classes which are not semisimple, for example if $G = \mathbb{G}_a$.

If $\Psi = C_G(g)$ is a semisimple conjugacy class, let $m_\Psi \subset R(G)$ be the ideal of virtual characters which vanish on $\Psi$. Then $m_\Psi$ is the kernel of the homomorphism of $\mathbb{C}$-algebras $R(G) \to \mathbb{C}$ defined by the property that $[V] \mapsto \chi_V(\Psi)$. Since the trivial character does not vanish on $\Psi$, $m_\Psi$ is a maximal ideal of $R(G)$. 


Proposition 2.5. Let $G$ be an arbitrary algebraic group. The assignment $\Psi \mapsto m_\Psi$ gives a bijection between the set of semisimple conjugacy classes in $G$ and the maximal ideals in $R(G)$.

Proof. First observe that the result holds if $G$ is reductive by Propositions 2.2 and 2.4. For general $G$, let $G = LU$ be a Levi decomposition and let $r: R(G) \rightarrow R(L)$ be the restriction map. Since $U$ is unipotent, $r$ is an isomorphism.

If $m \subset R(G)$ is a maximal ideal then, since $L$ is reductive, $r(m) = m_\Phi$ for some semisimple conjugacy class $\Phi = C_L(l)$ in $L$. If $\Psi = C_G(l)$ then $r(m_\Psi) \subset m_\Phi$. Thus $m_\Psi \subset m$. But $m_\Psi$ is maximal so it equals $m$.

Suppose $\Psi_1$ and $\Psi_2$ are two semisimple conjugacy classes in $G$ such that $m_{\Psi_1} = m_{\Psi_2}$. Since the proposition holds for the reductive subgroup $L$, $r(m_{\Psi_i}) = m_\Phi$ for some semisimple conjugacy class $\Phi \subset L$. It follows that $\Psi_1 \cap L = \Psi_2 \cap L = \Phi$. This means that $\Psi_1 \cap \Psi_2$ is nonempty, so $\Psi_1 = \Psi_2$. □

Given an embedding of groups $G \rightarrow H$, let $r: R(H) \rightarrow R(G)$ be the restriction map. Let $\Psi = C_H(h)$ be a semisimple conjugacy class in $H$. By Proposition 2.3, the ideal $m_\Psi R(G)$ is contained in a finite number of maximal ideals of Spec $R(G)$. If $\Psi'$ is a conjugacy class in $\Psi \cap G$, then the restriction of any virtual character vanishing on $\Psi$ also vanishes on $\Psi'$; that is, $m_\Psi R(G) \subset m_{\Psi'}$. Thus, $\Psi \cap G$ decomposes into a finite number of semisimple conjugacy classes $\Psi_1, \ldots, \Psi_l$.

Proposition 2.6. Let $G \hookrightarrow H$ be an embedding of groups, and let $\Psi'$ and $\Psi$ be semisimple conjugacy classes in $G$ and $H$, respectively. Then $m_{\Psi'} R(G) \subset m_{\Psi'}$ if and only if the conjugacy class $\Psi'$ is contained in $(\Psi \cap G)$.

Proof. The if direction follows from the discussion immediately preceding the statement of the proposition. Conversely, if $\Psi'$ is a conjugacy class in $G$ not contained in $\Psi \cap G$, then $\Psi'$ is disjoint from the conjugacy classes $\Psi_1, \ldots, \Psi_l$ in $\Psi \cap G$. By Proposition 2.3, $m_{\Psi'}$ is distinct from each of the $m_{\Psi_i}$. Therefore there is a virtual character $f \in m_{\Psi'}$ such $f$ is not in some $m_{\Psi_i}$. Thus, the restriction of $f$ to $\Psi \cap G$ is not zero, so $f \notin m_{\Psi} R(G)$. □

Remark 2.7. Proposition 2.6 implies that if $\Psi \subset H$ is a semisimple conjugacy class then $R(G)_{m_{\Psi}}$ is a semilocal ring with maximal ideals $m_{\Psi_1} R(G)_{m_{\Psi}}, \ldots, m_{\Psi_l} R(G)_{m_{\Psi}}$.

As noted above, if $G$ is a subgroup of $H$ and $g \in G$, the intersection of $C_H(g)$ with $G$ may consist of more than one $G$-conjugacy class in $G$. The following result shows that it is possible to find embeddings where this does not occur.
Proposition 2.8. Suppose $G$ is an algebraic group and $\Psi = C_G(g)$ is a semisimple conjugacy class in $G$. There is an embedding $G \rightarrow H$, where $H = \prod_i \text{GL}_{m_i}$, such that $C_H(g) \cap G = C_G(g)$.

Proof. Since $R(G) = R(L)$, where $L$ is a Levi factor of $G$, $R(G)$ is Noetherian. Therefore we can find a finite set $f_1, \ldots, f_l$ of elements which generate the maximal ideal $m_\Psi \subset R(G)$. Each function $f_i$ can be written as a finite sum $f_i = \sum_j a_{ij} \chi_{ij}$ where $\chi_{ij}$ is the character of a $G$-module $V_{ij}$. Let $V$ be a faithful representation of $G$ and set $H = \text{GL}(V) \times \prod_{i,j} \text{GL}(V_{ij})$. Then $G$ embeds as a subgroup of $H$ since it embeds as a subgroup of the first factor, $\text{GL}(V)$.

Let $g'$ be an element of $G$ such that $g'$ is conjugate to $g$ in $H$. Since $H$ is a direct product, the image of $g$ and $g'$ in each $\text{GL}(V_{ij})$ must have the same trace. Thus, $\chi_{ij}(g) = \chi_{ij}(g')$ for all $i, j$. Hence $f_i(g) = f_i(g') = 0$ for each generator $f_i$ of $m_\Psi$ (since $g$ is, by definition, an element of the conjugacy class $\Psi$). Therefore, by Proposition 2.3, $g'$ is in the conjugacy class of $g$ in $G$. \qed

3. Equivariant $K$-theory

This section contains some $K$-theoretic results needed for the proof of our main result, the nonabelian localization theorem.

3.1. Basic facts and notation. Let $G$ be an algebraic group acting on an algebraic space $X$. We use the notation $\text{coh}_X^G$ to denote the abelian category of $G$-equivariant coherent $\mathcal{O}_X$-modules. Let

$$G(G, X) = \bigoplus_{i=0}^\infty G_i(G, X) \otimes \mathbb{C}$$

where $G_i(G, X)$ is the $i$-th Quillen $K$-group of the category $\text{coh}_X^G$. Since all our coefficients are taken to be complex, we will simply write $G_0(G, X)$ (rather than $G_0(G, X) \otimes \mathbb{C}$) for the Grothendieck group of $G$-equivariant coherent sheaves, tensored with $\mathbb{C}$. Likewise, we write $K_0(G, X)$ for the Grothendieck ring of $G$-equivariant bundles, also tensored with $\mathbb{C}$. When $X$ is a smooth scheme, Thomason’s equivariant resolution theorem implies that $K_0(G, X) = G_0(G, X)$ (this is true even without tensoring with $\mathbb{C}$). We use analogous notation in the non-equivariant setting, writing $\text{coh}_X$ for the category of coherent $\mathcal{O}_X$-modules, and write $G(X), G_0(X),$ and $K_0(X)$ for the non-equivariant versions of $K$-theory.

If $\mathcal{E}$ is a $G$-equivariant locally free sheaf then the assignment $\mathcal{F} \mapsto \mathcal{E} \otimes F$ defines an exact functor $\text{coh}_X^G \rightarrow \text{coh}_X^G$. This implies that there is an action of $K_0(G, X)$ on $G(G, X)$. 
If $X$ and $Y$ are $G$-spaces and $p: X \to Y$ is a $G$-map, then there is a pullback $p^*: K_0(G, X) \to K_0(G, X)$. When $Y = \text{Spec } \mathbb{C}$, this pullback makes $G(G, X)$ an $R(G) = K_0(G, \text{Spec } \mathbb{C})$ module.

If $p: X \to Y$ is a $G$-equivariant proper morphism then there is a pushforward $p_*: G(G, X) \to G(G, Y)$ which is an $R(G)$-module homomorphism [Tho2 1.11-12]. The pushforward is defined on the level of Grothendieck groups by the formula $p_*[\mathcal{F}] = \sum (-1)^i [R^i p_* \mathcal{F}]$.

If $p: X \to Y$ is flat and $G$-equivariant then there is a pullback $p^*: G(G, X) \to G(G, Y)$ which is also an $R(G)$-module homomorphism. It is induced by the exact functor which takes a coherent sheaf $\mathcal{F}$ to its pullback $p^* \mathcal{F}$.

More generally, if $X$ is a regular algebraic space then Vezzosi and Vistoli proved [VV] Theorem A4 that $G(G, X)$ is isomorphic to the Waldhausen $K$-theory of the category $\mathcal{W}_{3, X}$ of complexes of flat quasi-coherent $G$-equivariant $\mathcal{O}_X$ modules with bounded coherent cohomology. It follows that if $p: X \to Y$ is a map of regular algebraic spaces then there is a pullback $p^*: G(G, X) \to G(G, Y)$ which is an $R(G)$-module homomorphism. If $p$ is a regular embedding of smooth algebraic spaces then there is a self-intersection formula for $\alpha \in G(G, X)$:

$$p^* p_* \alpha = \lambda_{-1}(N_p^*) \cap \alpha,$$

where $N_p^*$ is the normal bundle to map $p$ and $\lambda_{-1}(N_p^*) \in K_0(G, X)$ is the formal sum $\sum_{i=0}^{rk} (-1)^i [\lambda^i(N_p^*)]$. This fact is proved in the course of the proof of Theorem 3.7 of [VV].

3.2. Morita equivalence. In this paper we make extensive use of a particular instance of Morita equivalence, which we briefly describe. If $Z \subset G$ is a closed subgroup and $X$ is a $Z$-space then we may consider the $G \times Z$-space $G \times X$ where $(k, z) \cdot (g, x) = (kgz^{-1}, zx)$. We write $G \times_Z X$ for the quotient of $G \times X$ by the free action of the subgroup $1 \times Z$. The space $G \times_Z X$ will often be referred to as a mixed space. If the $Z$ action on $X$ is the restriction of a $G$ action, then the automorphism of $G \times X$ given by $(g, x) \mapsto (g, gx)$ induces an isomorphism of quotients $G \times_Z X \to G/Z \times X$.

Since the actions of $G$ and $Z$ on $G \times X$ commute, the action of $G \times 1$ on $G \times X$ descends to an action on the quotient $G \times_Z X$. The Morita equivalence we use is the equivalence of categories between the category of $Z$-equivariant coherent sheaves on $X$ and the category $G$-equivariant coherent sheaves on $G \times_Z X$. The equivalence is given by pulling a $Z$-module on $X$ back to $G \times X$ to obtain a $G \times Z$-module on $G \times X$ and then taking the subsheaf of $1 \times Z$-invariant sections to obtain a $G$-module on $G \times_Z X$.
Remark 3.1. When $X$ is a point, Morita equivalence between the categories of $G$-equivariant coherent sheaves on $G/Z$ and $Z$-modules is obtained by taking the fiber of a sheaf at the identity coset $Z$. Under this equivalence the tangent bundle of $G/Z$ corresponds to the $Z$-module $g/\mathfrak{z}$, where $g$ and $\mathfrak{z}$ denote the Lie algebras of $G$ and $Z$ respectively [Bor, Proposition 6.7]. If $X$ is a $G$-space then the identification $G \times Z X = G/Z \times X$ identifies the relative tangent bundle of the projection $p: G \times Z X \to X$ with the bundle $G \times Z (X \times g/\mathfrak{z})$.

Remark 3.2. The Morita equivalence of categories above induces an $R(G)$-module isomorphism in $K$-theory $G(Z, X) \to G(G, G \times Z X)$. Here $R(G)$ acts on $G(Z, X)$ via the restriction map $R(G) \to R(Z)$. This observation will be used repeatedly in the sequel.

3.3. Localization in equivariant $K$-theory. In this section we extend the localization theorem of [Tho5, Theorem 2.2] to arbitrary algebraic groups. The proof is similar to Thomason’s. However, because we work with ideals in $R(G) \otimes \mathbb{C}$ which may have zero intersection with the integral representation ring, we cannot directly quote his results.

Theorem 3.3. Let $G$ be an algebraic group acting on an algebraic space $X$. Let $\Psi = C_G(h)$ be a semisimple conjugacy class and let $X_\Psi$ be the closure of $GX^h$ in $X$.

(a) The proper pushforward

$$i_*: G(G, X_\Psi) \to G(G, X)$$

is an isomorphism of $R(G)$-modules after localizing at $m_\Psi$.

(b) If $X$ is smooth and $h \in Z(G)$ (so $\Psi = h$ and $X^h$ is $G$-invariant) then the map of $R(G)$-modules

$$\cap \lambda_{-1}(N_i^*)^{-1}: G(G, X^h) \to G(G, X^h)$$

is invertible after localizing at $m_h$, and if $\alpha \in G(G, X)_m h$, then

$$\alpha = i_* \left( \lambda_{-1}(N_i^*)^{-1} \cap i^* \alpha \right).$$

Here the notation $\lambda_{-1}(N_i^*)^{-1} \cap i^* \alpha$ means the image of $i^* \alpha$ under the inverse of the isomorphism $\cap \lambda_{-1}(N_i^*)$.

Remark 3.4. If $X$ is a smooth scheme then $X^h$ is as well. By Thomason’s equivariant resolution theorem we may identify $K_0(G, X^h)$ with $G_0(G, X^h)$ [Tho3, Theorem 5.7] and view $(\lambda_{-1}(N_i^*))^{-1}$ as an element in $G(T, X^h)_m h$.

Proof. Step 1: $G = T$ is a torus. If $T$ is a torus then $C_G(h) = h$. Using Noetherian induction and the localization long exact sequence [Tho3, Theorem 2.7] it suffices to prove that $G(T, X)_m h = 0$ whenever
$X^h$ is empty. Since $T$ is diagonalizable, we can, by Proposition 4.10, identify $R(T)$ with the coordinate ring of $T$, and $m_h$ with the maximal ideal of $h \in T$. If $T' \subset T$ is a closed subgroup then $R(T'_m) = 0$ unless $h \in T'$, By Thomason’s generic slice theorem, there is a $T$-invariant open set $U$, a closed subgroup $T'$ acting trivially on $U$ and a $T$-equivariant isomorphism $U \simeq T/T' \times U/T$. Since $T'$ acts trivially on $U$, and $X^h$ is empty, $h \notin T'$, By Morita equivalence, $G(T, U) = G(T', U/T)$. But $T'$ acts trivially on $U/T$, so $G(T', U/T) = R(T') \otimes G(U/T)$. Thus $G(T', U/T)_m = 0$. Applying Noetherian induction and using the localization long exact sequence we conclude that $G(T, X)_m = 0$. This proves part (a).

Next we must show that when $X$ is smooth, the multiplication map

$$\cap \lambda_{-1}(N^*_i) : G(T, X^h)_m \to G(T, X^h)_m$$

is an isomorphism. This may be done using Noetherian induction on $X^h$. Again we apply Thomason’s generic slice theorem. Thus we may assume that $X^h = T/T' \times X^h/T$. As above $G(T, X^h)_m = R(T')_m \otimes G(X^h/T)$.

Let $N_x$ be the fiber of $N^*_i$ over a point $x \in X^h$. Since $T'$ acts trivially on $X^h$, $N_x$ is a $T'$-module. As in the proof of [Tho5, Lemma 3.2], the identification $G(T, X^h)_m = R(T')_m \otimes G(X^h/T)$ implies that the action of $\lambda_{-1}(N^*_i)$ on $G(T, X^h)_m$ is invertible if and only if $\lambda_{-1}(N_x)$ is invertible in $R(T')_m$ for some $x$ in each connected component of $X^h$; i.e. $\lambda_{-1}(N_x) \notin m_h$. Now $N_x$ decomposes into 1-dimensional eigenspaces for the action of $T'$ so we can write $\lambda_{-1}(N_x) = \prod_{i=1}^s (1 - \chi_i)$, where $\chi_1, \ldots, \chi_s$ are (not necessarily distinct) characters of $T'$.

Let $H$ be the closure of the cyclic subgroup of $T$ generated by $h$. Then $X^H = X^h$. Since $X^h = T/T' \times X^h/T$, $T'$ is the biggest subgroup of $T$ acting trivially on $X^h = X^H$. Thus, we see that $H \subset T'$ and the characters in the $T'$-module decomposition restrict to characters of $H$. None of these characters can be trivial on $H$ since the normal space to $X^H$ at $x$ is the quotient of the tangent space $T_{x,X}$ by the invariant subspace $T_{x,X}^H$. Since the cyclic subgroup generated by $h$ is dense in $H$, we see that, for each $\chi_i$, we have $\chi_i(h^n) = \chi_i(h)^n \neq 1$ for some exponent $n$. Thus $(1 - \chi_i) \notin m_h$. Therefore, the action of $\lambda_{-1}(N^*_i)$ on $G(T, X^h)_m$ is invertible.

Finally, the formula in Equation (6) can be deduced as follows. Since $i_*$ is surjective, $\alpha = i_* \beta$ for some $\beta \in G(T, X^h)_m$. Thus,

$$i^* \alpha = i^* i_* \beta = \lambda_{-1}(N^*_i) \cap \beta.$$
where the second equality follows from the self-intersection formula in equivariant $K$-theory. Since the action of $\lambda_{-1}(N_i^*)$ is invertible after localizing at $m_h$, the formula follows.

Step 2. $G$ is connected and reductive. We use a standard reduction to a maximal torus argument. Let $T$ be any maximal torus in $G$ and let $B$ be a Borel subgroup of $G$ containing $T$. Since $B/T$ is isomorphic to affine space, the restriction $G(B, X) \rightarrow G(T, X)$ is an isomorphism [Tho4, Proof of Theorem 1.13]. The same proof implies that the projection $p: G \times_B X \rightarrow X$ induces a pullback $p^!: G(G, X) \rightarrow G(G, G \times_B X) = G(T, X)$ and a pushforward $p_!: G(T, X) \rightarrow G(G, X)$, with the properties that $p^!1 = 1$, and if $\beta \in K_0(G, X)$, then $p!(p^!\beta \cap \alpha) = \beta \cap p_!\alpha$. In particular, $p^!$ is a split monomorphism. Moreover, standard functorial properties of equivariant $K$-theory imply that $p^!$ and $p_!$ are functorial for $G$-equivariant morphisms.

To prove (a), as in the torus case it suffices to prove that if $X^h$ is empty then $G(G, X)_{m_{pq}} = 0$. Since $G(G, X)$ embeds in $G(T, X)$ it actually suffices to prove that $G(T, X)_{m_{pq}} = 0$. Let $h_1, h_2, \ldots, h_n$ be the conjugates of $h$ contained in the maximal torus $T$. By Remark 2.7 $R(T)_{m_{pq}}$ is a semilocal ring whose maximal ideals are the $m_h, R(T)_{m_{pq}}$. Hence, if $M$ is any $R(T)$-module, to show that $M_{m_{pq}} = 0$ it suffices to show that $M_{m_{h_i}} = 0$ for all $i$.

Since each $h_i$ is $G$-conjugate to $h$, each $X^{h_i}$ is empty. By the torus case, $G(T, X)_{m_{h_i}} = 0$ for each $h_i$. This implies that $G(G, X)_{m_{pq}} = 0$, proving (a).

Now suppose that $h \in Z(G)$. To avoid confusion, we write $m_h^G$ for the maximal ideal in $R(G)$ corresponding to the one-element conjugacy class $h \in G$ and $m_h^T$ the maximal ideal in $R(T)$ corresponding to the one-element conjugacy class $h \in T$. By Remark 2.7 $R(T)_{m_h^G} = R(T)_{m_h^T}$. If $X$ is smooth then, by Step 1, the action of $\lambda_{-1}(N_i^*) \in K_0(G, X^h)$ on $G(T, X^h)_{m_h^G} = G(T, X^h)_{m_h^T}$ is invertible. Thus the action of $\lambda_{-1}(N_i^*)$ on $G(G, X^h)_{m_h^G} = p(G(T, X^h)_{m_h^G})$ is as well.

Once we know that the action of $\lambda_{-1}(N_i^*)$ is invertible after localizing at $m_h^G$, the formula of equation (6) follows from the self-intersection formula.

Step 3: $G$ is arbitrary. As in the previous steps, to prove part (a) it suffices to show that if $\Psi = C_G(h)$ and $X^h$ is empty, then $G(G, X)_{m_{pq}} = 0$. By Proposition 2.8 there is an embedding of $G$ into a product of general linear groups $Q$ such that if $\Psi_1 = C_Q(h)$, then $\Psi_1 \cap Q = \Psi$. This implies that if $X^h$ is empty then so is $(Q \times_G X)^h$. Thus, by Remark 3.2
and Step 2, we conclude that $G(G, X)_{m_{2i}} = 0$. Since $m_{2i} R(G) \subset m_{2}$ it follows that $G(G, X)_{m_{2}} = 0$ as well.

Now suppose that $h$ is central in $G$; then $h$ is central in $Q$ as well. Let $i: X^h \to X$ be the inclusion of the fixed locus and let $i_h: Q \times_G X^h \to Q \times_G X^h$ be the corresponding inclusion of mixed spaces. By Morita equivalence and Remark 2.7 it suffices to show that the action of $\lambda^{-1}(N^*_f)$ on $G(Q, Q \times_G X^h)$ is invertible after localizing at $m^Q_h \subset R(Q)$. This follows from Step 2 and the following lemma.

**Lemma 3.5.** Let $G$ be a closed subgroup of an algebraic group $Q$ and let $h \in G \cap Z(Q)$. If $X$ is a smooth $G$-space then $(Q \times_G X)^h = (Q \times_G X)^h$ as closed subspaces of $Q \times_G X$.

**Proof of Lemma 3.5.** It is clear that $Q \times_G X^h \subset (Q \times_G X)^h$, so we need only show the reverse inclusion. Since $Q \times_G X^h$ and $(Q \times_G X)^h$ are closed smooth subspaces of the algebraic space $Q \times_G X$ it suffices to show that they have the same closed points (since we work over the algebraically closed field $\mathbb{C}$).

A point corresponding to the $G$-orbit of $(q, x) \in Q \times X$ is fixed by $h$ if and only if $h(q, x) = (hq, x)$ is in the same $G$ orbit as $(q, x)$. This means that there is an element $g \in G$ such that $(hq, x) = (qg^{-1}, gx)$. Thus $g$ fixes $x$ and $g^{-1} = q^{-1}hq$. Since $h \in Z(Q)$ this means $g = h^{-1}$ since $h$ and $h^{-1}$ have the same fixed locus we conclude that $x \in X^h$; i.e. $(q, x) \in Q \times X^h$. $\square$

This concludes the proof of Theorem 3.3. $\square$

3.4. **Decomposition of equivariant $K$-theory.** Let $G$ be an algebraic group acting on an algebraic space $X$. Assume that $G$ acts with finite stabilizers. In this case, there is a decomposition of $G(G, X)$ into a direct sum of pieces, which we now describe.

Since $X$ is assumed to be Noetherian there is a finite set of conjugacy classes $\Phi_1, \ldots, \Phi_m$ of elements of finite order such that $X^g$ is nonempty if and only if $g \in \Phi_i$ for some $i$ [VV, Theorem 5.4].

**Proposition 3.6.** With the assumptions above, the localization maps $G(G, X) \to G(G, X)_{m_{2i}}$ induce a direct sum decomposition

$$G(G, X) = \bigoplus_i G(G, X)_{m_{2i}}.$$  

**Proof.** By [EG2, Remark 5.1], there is a ideal $J \subset R(G)$ that annihilates $G(G, X)$ and such that $R(G)/J$ is supported at a finite number of points of $\text{Spec } R(G)$. This implies that $G(G, X) = \bigoplus_i G(G, X)_{m_{2i}}$.  

\[\text{If } G(G, X) \text{ is a finitely generated } R(G)\text{-module this also follows from Theorem and [VV] Theorem 5.4].}\]
for some set of semi-simple conjugacy classes \( \{\Phi_1, \Phi_2, \ldots, \Phi_m\} \). If \( X^h \) is empty for \( h \in \Phi \), then by Theorem 3.3, \( G(G, X)_{m_{\Psi}} = 0 \).

Remark 3.7. Suppose that \( G \) acts on \( X \) with finite stabilizers. If \( \alpha \in G(G, X) \), then we denote the component of \( \alpha \) in the summand \( G(G, X)_{m_{\Psi}} \) by \( \alpha_{\Psi} \). Note that if \( \beta \in K_0(G, X) \) then

\[
(\beta \cap \alpha)_{\Psi} = \beta \cap (\alpha_{\Psi}).
\]

Also, suppose that \( f \) is a \( G \)-equivariant morphism of algebraic spaces such that \( G \) acts with finite stabilizers on the source and target. If \( f \) is proper morphism, then \( f_*(\alpha_{\Psi}) = (f_\ast \alpha)_{\Psi} \). Likewise, if \( f \) is flat or a map of regular algebraic spaces then \( f^*\alpha_{\Psi} = (f^*\alpha)_{\Psi} \). These basic facts follow immediately from the fact that \( f_* \) and \( f^* \) are \( R(G) \)-module homomorphisms. They will be used repeatedly in the proof of Theorem 5.1.

Let \( X \) be a \( Z \)-space where \( Z \) acts with finite stabilizers and let \( Z \subset G \) be an embedding of \( Z \) into another algebraic group \( G \). Morita equivalence (Section 3.2) identifies \( G(G, G \times_Z X) \) with \( G(Z, X) \) giving an \( R(Z) \)-module structure on \( G(G, G \times_Z X) \). As a result we may obtain a more refined decomposition of \( G(G, G \times_Z X) \).

Proposition 3.8. Let \( Z \) act on \( X \) with finite stabilizers. If \( \Psi \) is a semisimple conjugacy class in \( G \) and \( (\Psi \cap Z) \) decomposes into the union of conjugacy classes \( \Psi_1, \ldots, \Psi_t \), then

\[
G(G, G \times_Z X)_{m_{\Psi}} = \bigoplus_{i=1}^t G(G, G \times_Z X)_{m_{\Psi_i}}.
\]

Proof. By Remark 2.7, \( R(Z)_{m_{\Psi_i}} \) is a semi-local ring with maximal ideals \( m_{\Psi_1}, \ldots, m_{\Psi_t} \) where \( \Psi_1, \ldots, \Psi_t \) are the conjugacy classes in \( \Psi \cap Z \). The proposition follows.

3.5. Projection formulas for flag bundles. In this section we prove some projection formulas for maps of flag bundles which will be needed in the proof of the nonabelian localization theorem. We begin with a lemma which is certainly known in greater generality, but for which we lack a suitable reference.

Lemma 3.9. Let \( p: P_1 \to P_2 \) be a proper, flat, \( G \)-equivariant map of quasi-projective schemes such that action of \( G \) is linearized with respect to an ample line bundle on each \( P_i \). Let \( X \) be an algebraic space with a \( G \)-action. Let \( p_1: P_1 \times X \to P_1 \) and \( p_2: P_2 \times X \to P_2 \) be the projections. Set \( \phi = (p \times 1): P_1 \times X \to P_2 \times X \). If \( A \in K_0(G, P_1) \) and \( \alpha \in G(G, P_2 \times X) \) then

\[
(7) \quad \phi_*(p_1^*A \cap \phi^*\alpha) = p_2^*p_*A \cap \alpha
\]
Proof. We use the ideas in the proof of the projection formula given in Qui Proposition 7.2.

By assumption the action of $G$ on $P_1$ is linearized with respect to an ample line bundle. Hence $K_0(G, P_1)$ is generated by classes of equivariant vector bundles $E$ with $R^ip_*E = 0$ for $i > 0$ [Qui, Section 7.2]. Thus we may assume $A = [E]$ and $R^ip_*E = 0$ for $i > 0$. Since $p$ is flat it follows that $p_*E$ is a locally free $G$-equivariant sheaf on $P_2$. Thus, if $F$ is any coherent sheaf on $P_2 \times X$ then $R^i\phi_*(p_*E \otimes \phi^*F) = 0$. (This can be checked locally in the étale topology so we may assume $X$ is an affine scheme. The proof in that case is given on p. 59 of Srinivas’s book [Sri].) Thus the functor

$$\text{coh}^G_{(P_2 \times X)} \to \text{coh}^G_{(P_2 \times X)}, \quad \mathcal{F} \mapsto \phi_*(p_*E \otimes \phi^*\mathcal{F})$$

is exact. This functor induces the endomorphism of $G(G, X)$ given by $\alpha \mapsto \phi_*(p_*E \cap \phi^*\alpha)$.

On the other hand, the endomorphism of $G(G, P_2 \times X)$ given by $\alpha \mapsto p_2^*p_*A \cap \alpha$ is induced by the exact functor

$$\text{coh}^G_{(P_2 \times X)} \to \text{coh}^G_{(P_2 \times X)}, \quad \mathcal{F} \mapsto p_2^*p_*E \otimes \mathcal{F}.$$ 

Srinivas also proves that there is a natural (and hence $G$-equivariant) isomorphism $\phi_*(p_*E \otimes \phi^*\mathcal{F}) \simeq \phi_*p^*_1E \otimes \mathcal{F}$. Since $\phi_*p^*_1E$ is naturally (and thus $G$-equivariantly) isomorphic to $p_2^*p_*E$ the two exact functors we have defined are isomorphic. Therefore, the formula of Equation (7) holds. 

Let $G$ be a connected group and let $P \subset G$ be a parabolic subgroup containing a Borel subgroup $B$ and having Levi factor $Z$. Choose a maximal torus $T \subset Z$. Since $P$ is parabolic, $T$ is a maximal torus of $G$ as well. Let $W(G, T) = N_G(T)/Z_G(T)$ and $W(Z, T) = N_Z(T)/Z_Z(T)$ be the Weyl groups of $G$ and $Z$ respectively. If $X$ is a $G$-space then we have projections

$$G \times_B X \xrightarrow{p} G \times P X \xrightarrow{q} X.$$ 

Set $\pi = q \circ p$. The flag bundle projection formulas are given by the next result.

**Proposition 3.10.** If $\alpha \in G(G, X)$, $\beta \in G(G, G \times P X)$ then the following identities hold:

(i) $\pi_*(\lambda_1(T^*_\pi) \cap \pi^*\alpha) = |W(G, T)|\alpha.$

(ii) $q_*(\lambda_1(T^*_q) \cap q^*\alpha) = \frac{|W(G, T)|}{|W(Z, T)|}\alpha.$
(iii) $p_*(\lambda_1(T^*_q) \cap p^*\beta) = |W(Z, T)|(\lambda_1(T^*_q) \cap \beta)$.

Proof. Since $G$ acts on $X$, the mixed spaces $G \times_B X$ and $G \times_P X$ are isomorphic to $G/B \times X$ and $G/P \times X$ respectively. The maps $p, q, \pi$ are all smooth and projective and the bundles $T_\pi, T_q$ and $T_p$ are all obtained by pullback from the smooth projective schemes $G/B$ or $G/P$. Therefore, by Lemma 3.9 we may assume that $X = \text{Spec } \mathbb{C}$. By Theorem 3.3, the equation holds after localizing at any prime ideal in $R(T)$. By Theorem 3.3, the equation holds after localizing at $m_a \in R(T)$ where $a \in T$ is any element with $(G/P)^a = (G/P)^T$. This proves (3). Pushing forward to $G(T, \text{Spec } k) = R(T)$ completes the proof of (ii).

If $G$ is connected then $G$-equivariant $K$-theory is a summand in $T$-equivariant $K$-theory [Tho1, Theorem 1.13]. Thus, to prove (iii) we may again work in $T$-equivariant $K$-theory. The $T$ action on $G/B$ has $|W(T)|$ fixed points while the $T$ action on $G/P$ has $|W(G, T)/W(Z, T)|$. For each fixed point $P \in G/P$ the fiber $p^{-1}(P)$ contains exactly $|W(Z, T)|$ of the $T$-fixed points in $G/B$. Applying (3) to $G/B$ and $G/P$, we see that $p_*(\lambda_1(T^*_G) \cap p^*\beta) = |W(Z, T)|(\lambda_1(T^*_G) \cap \beta)$. \qed

4. The global stabilizer and its equivariant $K$-theory

4.1. General facts about the global stabilizer. If $X$ is a $G$-space then $G$ acts on $G \times X$ by conjugation on the first factor and by the original action on the second factor. The global stabilizer $S_X \subset G \times X$ is a $G$-invariant subspace and the projection $f: S_X \to X$ is $G$-equivariant.
**Definition 4.1.** Let $\Psi$ be a semisimple conjugacy class in $G$. Define $S_\Psi \subset S_X$ to be the inverse image of $\Psi$ under the projection $S_X \rightarrow G$. Set theoretically, $S_\Psi = \{(g, x) | g \in \Psi \text{ and } gx = x\}$.

**Remark 4.2.** Since semisimple conjugacy classes are closed, $S_\Psi$ is closed in $S_X$. Thus, if $G$ acts with finite stabilizer then the projection $f : S_\Psi \rightarrow X$ is also finite.

If $\Psi$ and $\Psi'$ are disjoint conjugacy classes then $S_\Psi \cap S_{\Psi'} = \emptyset$. When $G$ acts with finite stabilizers then $S_X$ is the disjoint sum of the closed subspaces $S_{\Psi_1} \bigsqcup S_{\Psi_2} \ldots \bigsqcup S_{\Psi_l}$ where $\{\Psi_1, \ldots, \Psi_l\}$ is the set of conjugacy classes whose elements have non-trivial stabilizer (these conjugacy classes are necessarily semisimple because in characteristic 0 any element of finite order is semisimple). Hence $S_\Psi$ is also open in $S_X$ in this case.

Let $h$ be an element of $\Psi$ and let $Z = Z_G(h)$ be the centralizer of $h$. The map $G \times X^h \rightarrow S_\Psi$ given by $(g, x) \mapsto (ghg^{-1}, gx)$ is invariant under the free action of $Z$ on $G \times X^h$ given by $z(g, x) = (gz^{-1}, zx)$.

**Lemma 4.3.** $G \times X^h \rightarrow S_\Psi$ is a $Z$-torsor. Hence $S_\Psi$ is identified with the quotient $G \times Z X^h$.

**Proof.** The map $G \rightarrow \Psi$, $g \mapsto ghg^{-1}$ identifies $\Psi$ with $G/Z$. Thus, by base change, the map $\Phi_h : G \times X \rightarrow \Psi \times X$ given by $(g, x) \mapsto (ghg^{-1}, gx)$ is also a torsor. Since $\Phi_h^{-1}(S_\Psi) = G \times X^h$ the lemma follows. \qed

**Remark 4.4.** If $X$ is smooth then the identification of $S_\Psi$ with $G \times Z X^h$ implies that $S_\Psi$ is smooth and the projection $f : S_\Psi \rightarrow X$ factors as the composition of the regular embedding $S_\Psi \rightarrow G \times Z \rightarrow G \times X$. If $X$ is smooth, the map $S_\Psi \rightarrow G \times Z X$ is a regular embedding but the projection $G \times Z X \rightarrow Z$ is not proper. This makes it difficult to use $G \times Z X$ in the proof of our main result. However, suppose that $G$ is connected and $Z$ is a Levi factor of a parabolic subgroup $P$. Let $\rho : G \times Z X \rightarrow G \times P X$ be the projection, and let $j = \rho \circ i : S_\Psi \rightarrow G \times P X$. The following result holds.

**Lemma 4.5.** Let $G$ be connected and let $\Psi = C_G(h)$, where $h$ is semisimple. Assume that $Z = Z_G(h)$ is a Levi factor of a parabolic subgroup $P$. If $G$ acts on $X$ with finite stabilizer then $j : S_\Psi \rightarrow G \times P X$ is a regular embedding.
Proof. Since the morphisms of algebraic spaces $i$ and $\rho$ are representable\footnote{We say that a morphism of algebraic space $f: X \to Y$ is representable if for any scheme $Y'$ and map $Y' \to Y$, the fiber product $X \times_Y Y'$ is a scheme.} the composite $j = \rho \circ i$ is as well. By \cite[Proposition 19.1.1]{EGA4} any closed immersion of regular schemes is a regular embedding. The property of being a regular embedding is local for the étale topology of the target so we may apply this proposition to representable morphisms of algebraic spaces. Since $S_\Psi$ and $G \times_P X$ are smooth we are reduced to showing that $j$ is a closed immersion.

The finite map $f: S_\Psi \to X$ factors as $p \circ j$ where $p: G \times_P X \to X$ is the projection, so $j$ is finite and thus closed. To prove that it is an immersion we will show that is unramified and injective on geometric points. These two conditions suffice because by \cite[Cor. 18.4.7]{EGA4} an unramified morphism (Zariski) locally factors as the composition of an étale morphism with a closed immersion. If, in addition, the map is injective on geometric points then the étale morphism is an open immersion by \cite[Theorem 17.9.1]{EGA4}. Hence these conditions will imply that our morphism is locally the composition of an open immersion with a closed immersion.

To show that $j$ is unramified it suffices to show that the map $f = j \circ q: S_\Psi \to X$ is unramified. Since $S_\Psi$ is open in $S_X$ we need only show that the projection $S_X \to X$ is unramified. This last fact follows from the fact the fiber of $S_X \to X$ over a geometric point $x \to X$ is a group scheme over $x$. Since we work in characteristic 0 this must be reduced.

Next we show that $j$ is injective on geometric points. Consider the morphism $G \times X^h \to G \times_Z X^h \to G \times_P X$. Suppose that $(g_1, x_1)$ and $(g_2, x_2)$ have the same image in $G \times_P X$. By definition this means that there is an element $p \in P$ such that $x_2 = px_1$ and $g_2 = g_1p^{-1}$. Since $P$ is a semidirect product of $Z$ and $U$ with $U$ unipotent and normal, we can write $p = uz$ with $u \in U$ and $z \in Z$. Thus $x_2 = uzx_1$; we want to show that $u = 1$. Because $zx_1$ and $x_2$ are both in $X^h$, it suffices to show that if $u$ is an element of $U$ with $x$ and $ux$ both in $X^h$, then $u = 1$. This last assertion follows because the assumptions imply that $hux = huh^{-1}x = ux$, or $u^{-1}(huh^{-1})$ fixes $x$. Now, $u^{-1}(huh^{-1})$ is unipotent since it is in $U$. However, this element has nontrivial fixed locus, so by Remark \ref{remark:unipotent} it is semisimple. Hence $u^{-1}(huh^{-1}) = 1$, so $u \in Z$. Since $Z \cap U = \{1\}$ we obtain $u = 1$, as desired.

4.2. Localization theorems related to the global stabilizer. Let $\Psi = C_G(h)$ be a semisimple conjugacy class, and let $Z = Z_G(h)$. Recall that by Lemma \ref{lemma:localization} $\Phi_h: G \times X^h \to S_\Psi$ is a $Z$-torsor.

By Morita equivalence, \( G(G, S_\Psi) \) is isomorphic to \( G(Z, X^h) \), so \( G(G, S_\Psi) \) is an \( R(Z) \)-module. The restriction homomorphism \( R(G) \to R(Z) \) is compatible with the \( R(G) \) and \( R(Z) \)-module structures on \( G(G, S_\Psi) \) (cf. Section 32). Since \( h \) is central in \( Z \) it is a one-element conjugacy class with corresponding maximal ideal \( m_h \subset R(Z) \).

The next lemma shows that the action of \( R(Z) \) on \( G(G, S_\Psi) \) is independent of the choice of \( h \in \Psi \). This fact is crucial for the proof of the nonabelian localization theorem.

**Lemma 4.6.** Let \( \Psi \) be a semisimple conjugacy class and let \( h_1 \) and \( h_2 = kh_1k^{-1} \) be in \( \Psi \). Set \( Z_1 = Z_G(h_1) \) and \( Z_2 = Z_G(h_2) \). Let \( C_k : Z_1 \to Z_2 \) denote conjugation by \( k \), and let \( C_k^* : R(Z_2) \to R(Z_1) \) denote the pullback. Then \( C_k^*(m_{h_2}) = m_{h_1} \) and the pullback \( C_k^* \) is compatible with the actions of \( R(Z_1) \) and \( R(Z_2) \) on \( G(G, S_\Psi) \) as defined above.

**Proof.** Since \( C_k(h_1) = h_2 \), we have \( C_k^*(m_{h_2}) = m_{h_1} \). To verify the compatibility of the \( R(Z_i) \)-actions, we must unwind the definitions. The maps \( a_i : G/Z_i \to \Psi \) defined by \( a_i(gZ_i) = gh_ig^{-1} \) give \( G \)-equivariant identifications of \( G/Z_i \) with \( \Psi \). These in turn give change of group identifications of \( R(Z_i) \) with \( K_0(G, \Psi) \), and the action of \( R(Z_i) \) on \( G(G, S_\Psi) \) is obtained composing these identifications with the pullback \( K_0(G, \Psi) \to K_0(G, S_\Psi) \). Therefore, it suffices to prove that \( C_k^* \) is compatible with the identifications of \( R(Z_i) \) with \( K_0(G, \Psi) \).

The identifications of \( G/Z_i \) with \( \Psi \) are compatible with the isomorphism \( \kappa : G/Z_1 \to G/Z_2 \) taking \( gZ_1 \) to \( gkh^{-1}Z_2 \). If \( V_2 \) is a \( Z_2 \)-module, write \( C_k^*V_2 \) for the same underlying vector space, but with \( Z_1 \)-module structure obtained by pullback via the map \( C_k \). The identification of \( R(Z_i) \) with \( K_0(G, \Psi) \) takes the class of a \( Z_i \)-module \( V_i \) to the class of the vector bundle \( G \times Z_i V_i \) on \( G/Z_i \). The pullback via \( \kappa \) of the vector bundle \( G \times Z_2 V_2 \) is \( G \)-equivariantly isomorphic to the vector bundle \( G \times Z_1 C_k^*V_2 \), which proves the result. \( \square \)

**Proposition 4.7.** Let \( G \) be an algebraic group acting on a smooth algebraic space \( X \). Let \( \Psi = C_G(h) \) be a semi-simple conjugacy class, let \( Z = Z_G(h) \), and let \( f : S_\Psi \to X \) be the projection. Let \( N_f^* = [T_{S_\Psi}^*] - [f^*(T_G^*)] \in K_0(G, S_\Psi) \) be the class of the relative cotangent bundle. The map \( \cap \lambda_{-1}(N_f^*) : G(G, S_\Psi) \to G(G, S_\Psi) \) restricts to an isomorphism after localizing at the maximal ideal \( m_h \in R(Z) \).

**Remark 4.8.** Recall that \( S_\Psi \) is smooth when \( X \) is smooth (Remark 4.4) so that \( f : S_\Psi \to X \) is a morphism of smooth algebraic spaces.

**Proof.** Let \( i : X^h \to X \) be the inclusion of the fixed locus of \( h \). By Theorem 32 the map

\[ \cap \lambda_{-1}(N_f^*) : G(Z, X^h)_{m_h} \to G(Z, X^h)_{m_h} \]
is an isomorphism. Let $i: S_\Psi \to G \times Z X$ be the corresponding inclusion of mixed spaces. By Morita equivalence the map
\[
\cap \lambda_-(N_i^*): G(G, S_\Psi)_{m_h} \to G(G, S_\Psi)_{m_h}
\]
is an isomorphism.

Let $\eta: G \times Z X \to X$ be the projection. Then $f = \eta \circ i$, so $[T_f] = [N_i] - [i^*T_\eta] \in K_0(G, S_\Psi)$ Thus $\lambda_-(N_i^*) = \lambda_-(N_j^*) \lambda_-(i^*T_\eta^*)$. Since $\lambda_-(N_j^*)$ acts by automorphisms on $G(G, S_\Psi)_{m_h}$ it follows that $\lambda_-(N_j^*)$ must as well.

Now assume that $G$ is connected and that $Z = \mathcal{Z}_G(h)$ is a Levi factor of a parabolic subgroup $P \subset G$. Since $P/Z$ is isomorphic to affine space we may identify identify $G(G, G \times P X)$ with $G(Z, X)$.

**Proposition 4.9.** Let $G$ be a connected algebraic group acting on a smooth algebraic space $X$. Let $\Psi = C_G(h)$; assume that $Z = \mathcal{Z}_G(h)$ is a Levi factor of a parabolic subgroup $P$. Let $i$ and $j$ be the maps of $S_\Psi$ into $G \times Z X$ and $G \times P X$ defined above.

(a) The map
\[
\cap \lambda_-(N_j^*): G(G, S_\Psi) \to G(G, S_\Psi)
\]
is an isomorphism after localizing at the maximal ideal $m_h \subset R(Z)$.

(b) If in addition $G$ acts on $X$ with finite stabilizer and if $\beta \in G(G, G \times P X)_{m_h}$, then
\[
\beta = j_*(\lambda_-(N_j^*)^{-1} \cap j^*\beta).
\]
where $\lambda_-(N_j^*)^{-1} \cap j^*\beta$ denotes the image of $j^*\beta$ under the inverse of the map $\cap \lambda_-(N_j^*)$.

**Proof.** (a) The map $j$ factors as $\rho \circ \iota$ where $\rho: G \times P X \to G \times Z X$ is the projection. Thus $[N_j] = [N] - [i^*T_\rho]$. Hence $\lambda_-(N_j^*) = \lambda_-(N_i^*) \lambda_-(T_\rho^*)$.

By Morita equivalence and Theorem 3.3 $\lambda_-(N_i^*)$ acts by automorphisms on $G(G, S_\Psi)_{m_h}$. Therefore, $\lambda_-(N_j^*)$ must as well.

(b) By Morita equivalence and Theorem 3.3 (for the group $Z$) the pushforward $i_*: G(G, S_\Psi)_{c_\Psi} \to G(G, G \times Z X)_{m_h}$ is an isomorphism. Since the action of $\lambda_-(N_i^*)$ on $G(G, S_\Psi)_{c_\Psi}$ is invertible, it follows from the self intersection formula that $i^*: G(G, G \times Z X)_h \to G(G, S_\Psi)_{c_\Psi}$ is also an isomorphism. Hence
\[
j^* = (\rho \circ \iota)^*: G(G, G \times P X)_{m_h} \to G(G, S_\Psi)_{c_\Psi}
\]
is as well. Thus, it suffices to prove that Equation (9) holds after applying $j^*$ to both sides. By the self-intersection formula for the regular embedding $j$, we have
\[
j^*j_*(\lambda_-(N_j^*)^{-1} \cap j^*\beta) = \lambda_-(N_j^*) \cap (\lambda_-(N_j^*)^{-1} \cap j^*\beta) = j^*\beta
\]
4.3. The central summand. Suppose that $G$ acts with finite stabilizers on $X$. Let $\Psi = C_G(h)$, and keep the notation above. Since $h$ is a one-element conjugacy class contained in $\Psi \cap Z$, Proposition 3.6 implies that $G(G, S_\Psi)_m$ is a summand in $G(G, S_\Psi)_m$. By Lemma 4.6, this summand is independent of choice of $h \in \Psi$.

Definition 4.10. Let $G$ act with finite stabilizers on $X$, and let $\Psi = C_G(h)$. With notation as above, the summand $G(G, S_\Psi)_m \subset G(G, S_\Psi)_m$ (which is independent of $h$ in $\Psi$) will be called the central summand and denoted $G(G, S_\Psi)_m$. The component of $\beta \in G(G, S_\Psi)$ in this summand will be denoted $\beta_{c_\Psi}$.

5. THE NONABELIAN LOCALIZATION THEOREM

The main theorem of our paper is the following nonabelian localization theorem.

Theorem 5.1 (Explicit nonabelian localization). Let $G$ be an algebraic group acting with finite stabilizer on a smooth algebraic space $X$. Let $\Psi = C_G(h)$ be a semisimple conjugacy class and let $f : S_\Psi \rightarrow X$ be the projection. If $\alpha \in G(G, X)$ let $\alpha_\Psi$ be the component of $\alpha$ supported at the maximal ideal $m_\Psi \subset R(G)$. Then

$$\alpha_\Psi = f_* \left( \frac{1}{\lambda_1(N_\alpha^*)^{-1} \cap (f^* \alpha)_{c_\Psi}} \right)$$

where $\lambda_1(N_\alpha^*)^{-1} \cap (f^* \alpha)_{c_\Psi}$ is the image of $(f^* \alpha)_{c_\Psi}$ under the inverse of the automorphism $\cap \lambda_1(N_\alpha^*)$ of $G(G, S_\Psi)_{c_\Psi}$.

The theorem can be restated in a way that is sometimes more useful for calculations. As usual, let $Z = Z_G(h)$. Let $\imath^! : G(G, X) \rightarrow G(Z, X^h)$ be the composition of the restriction functor $G(G, X) \rightarrow G(Z, X)$ with the pullback $G(Z, X) \rightarrow G(Z, X^h)$. Let $\beta_{h, Z}$ denote the component of $\beta \in G(Z, X^h)$ in the summand $G(Z, X^h)_{m_\Psi}$. Let $g$ (resp. $\mathfrak{g}$) be the adjoint representation of $G$ (resp. $Z$). The restriction of the adjoint representation to the subgroup $Z$ makes $g$ a $Z$-module, so $g/\mathfrak{z}$ is a $Z$-module. Let $\eta : G \times_Z X \rightarrow X$ be the projection. By Remark 3.1, $T_\eta = G \times_Z (X \times g/\mathfrak{z})$. We therefore obtain the following corollary.

Corollary 5.2. With assumptions as in Theorem 5.1, let $h$ be an element of $\Psi$. Let $Z = Z_G(h)$, and let $\imath^! : G(Z, X^h) \rightarrow G(G, X)$ be the map obtained by composing $f_\ast$ with the Morita equivalence isomorphism $G(Z, X^h) \rightarrow G(G, S_\Psi)$. If $\alpha \in G(G, X)_{m_\Psi}$, then

$$\alpha = \imath^! \left( \frac{1}{\lambda_1(N_\alpha^*)^{-1} \cap \lambda_1((g/\mathfrak{z})^*) \cap (\imath^! \alpha)_h} \right)$$.
5.1. **Proof of Theorem 5.1** if \( G \) is connected and \( Z \) is a Levi factor of a parabolic subgroup. In this section we prove Theorem 5.1 under the assumptions that \( G \) is connected and that \( Z = Z_G(h) \) is a Levi factor in a parabolic subgroup \( P \subset G \). (In fact, it would suffice for our purposes to take \( G \) equal to a product of general linear groups, and then \( Z \) is automatically such a Levi factor.) Choose a maximal torus \( T \) and and Borel subgroup \( B \) such that \( h \in T \subset B \subset P \). We have maps of mixed spaces

\[
G \times B X \xrightarrow{p} G \times P X \xrightarrow{q} X;
\]

and we write \( \pi = q \circ p \). Let \( j : S_\Psi \to G \times P X \) denote the regular embedding of Lemma 4.5.

Since \( G \times T X \to G \times B X \) is a bundle with fibers isomorphic to \( B/T \), pullback gives an isomorphism of \( G(G, G \times B X) \) with \( G(G, G \times T X) \). Now, \( \Psi \cap T \) consists of a finite number of elements \( h = h_1, \ldots, h_w \), where \( w = |W(G, T)|/|W(Z, T)| \). By Proposition 3.8, if \( \alpha \in G(G, X)_{m_\Psi} \), then there is a decomposition

\[
\pi^* \alpha = \sum_{l=1}^w (\pi^* \alpha)_{h_l},
\]

where \( (\pi^* \alpha)_{h_l} \) refers to the component supported at \( m_{h_l} \subset R(T) \). Because \( h \) is central in \( Z \), there is a also component \( (q^* \alpha)_h \) of \( q^* \alpha \) supported at the maximal ideal \( m_h \subset R(P) = R(Z) \).

The key step is given by the following proposition.

**Proposition 5.3.** Keep the assumptions of Theorem 5.1 and in addition assume that \( G \) is connected and \( Z \) is a Levi factor of a parabolic subgroup \( P \). Then the following identity holds in \( G(G, X)_{m_\Psi} \):

\[
\lambda_{-1}(N_f^*)^{-1} \cap (f^* \alpha)_{c_\Psi} = \frac{1}{|W(Z, T)|} \pi^* \lambda_{-1}(T_q^*) \cap (\pi^* \alpha)_h.
\]

**Proof.** We keep the notation introduced before the statement of the proposition. Since \( f = q \circ j \) and \( j^* \) is an \( R(Z) \)-module homomorphism \( j^*((q^* \alpha)_h) = (f^* \alpha)_h \). But \( (f^* \alpha)_h \) is independent of \( h \) and equals \( (f^* \alpha)_{c_\Psi} \) by definition of the central summand. Now, \( [N_f^*] = [N_f^*] - [j^* T_q^*] \), so

\[
\lambda_{-1}(N_f^*)^{-1} \cap (f^* \alpha)_{c_\Psi} = \lambda_{-1}(N_f^*)^{-1} \cap (\lambda_{-1}(j^* T_q^*) \cap (f^* \alpha)_{c_\Psi}) = \lambda_{-1}(N_f^*)^{-1} \cap j^* (\lambda_{-1}(T_q^*) \cap (q^* \alpha)_h).
\]
Applying $f^* = q^* \circ j^*$ to both sides of Equation (13), we obtain
\begin{equation}
(14)
f^* \left( \lambda_{-1}(N_f^*)^{-1} \cap (f^* \alpha)_{c_b} \right) = q^*j^* \left( \lambda_1(N_j^*)^{-1} \cap j^* (\lambda_{-1}(T_q^*) \cap (q^* \alpha)_h) \right)
\end{equation}
where the second equality follows from Proposition 4.9. By compatibility of pullback with support, $(\pi^* \alpha)_h = p^* ((q^* \alpha)_h)$. By Proposition 3.10,
\begin{equation}
\lambda_{-1}(T_q^*) \cap (q^* \alpha)_h = \frac{1}{|W(Z, T)|} \pi_* (\lambda_{-1}(T_{\pi}^*) \cap (\pi^* \alpha)_h)
\end{equation}
so
\begin{equation}
q_* \left( \lambda_{-1}(T_q^*) \cap (q^* \alpha)_h \right) = \frac{1}{|W(Z, T)|} \pi_* (\lambda_{-1}(T_q^*) \cap (q^* \alpha)_h).
\end{equation}

The proof of the theorem for $G$ connected and $Z$ a Levi factor of a parabolic subgroup is an easy consequence of the preceding proposition. By Proposition 3.10,
\begin{equation}
\alpha = \frac{1}{|W(G, T)|} \pi_* (\lambda_{-1}(T_{\pi}^*) \cap \pi^* \alpha).
\end{equation}
Therefore,
\begin{align*}
\alpha &= \frac{1}{|W(G, T)|} \sum_{l=1}^w \pi_* (\lambda_{-1}(T_{\pi}^*) \cap (\pi^* \alpha)_{h_i}) \\
&= \frac{|W(Z, T)|}{|W(G, T)|} \sum_{l=1}^w f_* (\lambda_{-1}(N_f^*)^{-1} \cap (f^* \alpha)_{c_b}) \\
&= f_* \left( \lambda_{-1}(N_f^*) \cap (f^* \alpha)_{c_b} \right),
\end{align*}
as desired.

Remark 5.4. As a corollary of the proof in this case we obtain the following induction formula for $\alpha \in G(G, X)_{m^*}$.
\begin{equation}
(15)\quad \alpha = \frac{|W(G, T)|}{|W(Z, T)|} \pi_* (\lambda_{-1}(T_{\pi}^*) \cap (\pi^* \alpha)_h)
\end{equation}

5.2. Proof of Theorem 5.1 for arbitrary $G$. By Proposition 2.8, we can embed $G$ as a closed subgroup of a product of general linear groups $Q$ such that, writing $\Psi = C_G(h)$ and $\Psi_Q = C_Q(h)$, we have $\Psi_Q \cap G = \Psi$. Write $Z = Z_G(h)$ and $Z_Q = Z_Q(h)$. Let $Y = Q \times_G X$; then $Q$ acts with finite stabilizer on $Y$. The group $Q$ is connected, and direct calculation shows that $Z_Q$ is a a product of general linear
groups which is a Levi factor of a parabolic. Therefore, the nonabelian localization theorem applies to the $Q$-action on $Y$.

By Proposition $3.8$ and the fact that $\Psi_Q \cap G = \Psi$, we have $G(Q, Y)_{m_{\Psi_Q}} = G(Q, Y)_{m_{\Psi}}$. As usual, $S_{\Psi_Q}$ denotes the part of the global stabilizer $S_Y$ corresponding to $\Psi_Q$. For simplicity of notation we will write $S_Q$ for $S_{\Psi_Q}$ and $S$ for $S_{\Psi}$. We denote by $f_Q$ the morphism $S_Q \to Y$. If $\beta \in G(Q, S_Q)$ then by definition, $\beta \subset \Psi_Q$ is the component of $\beta$ supported at the maximal ideal $m_{\Psi_Q} \subset R(Z_Q)$. Likewise, if $\gamma \in G(Q, Q \times G S) = G(G, S)$ then $\gamma \subset \Psi_Q$ is the component supported at $m_h \subset R(Z)$. Since $h$ is central in $Z$ and $Z_Q$, Remark $2.7$ implies that if $M$ is any $R(Z)$-module then $M_{m_h} = M_{m_{\Psi_Q}}$, where the $R(Z)$ action on $M$ is given by the restriction homomorphism $R(Z_Q) \to R(Z)$. Thus $\gamma \subset \Psi_Q$ may also be identified with the component of $\gamma$ supported at $m_{\Psi_Q}$.

Note that $R(Z)$ acts on $G(G, S_Q)$ and $R(Z)$ acts on $G(Q, Q \times G S) = G(G, S)$. By Morita equivalence, the localization theorem for $G$ acting on $X$ is a consequence of the following lemma.

**Lemma 5.5.** Keep the assumptions and notation of this subsection. Then there is a $Q$-equivariant isomorphism $\Phi : Q \times G S \to S_Q$ such that:

1) $\Phi^* : G(Q, S_Q) \to G(Q, Q \times G S)$ is an $R(Z)$-module homomorphism where the action of $R(Z)$ on $G(Q, Q \times G S)$ is given by the restriction homomorphism $R(Z_Q) \to R(Z)$.

2) $f_Q \circ \Phi = (1 \times_G f)$.

**Proof.** Consider the map

\[ T : Q \times S \to \Psi_Q \times X, \quad (q, g, x) \mapsto (q g q^{-1}, q, x). \]

This map induces a map of quotient spaces $\tilde{\Phi} : Q \times G S \to \Psi_Q \times Y$ such that the following diagram commutes:

\[
\begin{array}{ccc}
Q \times G S & \to & Q \times G S \\
\downarrow & & \downarrow \\
Q \times G \Psi & \xrightarrow{\phi} & \Psi_Q
\end{array}
\]

Here the vertical arrows are the obvious projections and the bottom horizontal arrow is given by $\phi((q, k)) = q k q^{-1}$. Note that $S_Q$ is a closed subspace of $\Psi_Q \times Y$. We have a commutative diagram

\[
\begin{array}{ccc}
R(Z_Q) & \xrightarrow{\Phi^*} & K_0(Q, \Psi_Q) \\
\downarrow & & \downarrow \\
R(Z) & \xrightarrow{\tilde{\Phi}^*} & K_0(Q, Q \times G \Psi)
\end{array}
\]

\[
\begin{aligned}
R(Z_Q) & \xrightarrow{(1 \times_G f)} K_0(Q, Q \times G S) \\
\downarrow & & \downarrow \\
R(Z) & \xrightarrow{(1 \times_G f)} K_0(Q, Q \times G S).
\end{aligned}
\]
Here the second arrow in each row is a pullback map. The first arrow in
the top row takes \([V] \in R(Z_Q)\) to the class of the vector bundle \(Q \times_{Z_Q} V\)
on \(Q/Z_Q = \Psi_Q\). The first arrow in the bottom row takes \([W] \in R(Z)\) to the class of the vector bundle \(Q \times Z W\) on \(Q/Z = Q \times_G \Psi\). The action of \(R(Z_Q)\) on \(G(Q, S_Q)\) (resp. of \(R(Z)\) on \(G(Q, Q \times_G S)\)) is defined using the composition along the top (resp. bottom) row. The commutativity of this diagram implies that \(\bar{\Phi}^*\) is a \(R(Z_Q)\)-module homomorphism.

Next we show that \(\bar{\Phi}\) induces an isomorphism of \(Q \times_G S\) onto the
subspace \(S_Q \subset \Psi_Q \times Y\). Let \(W\) be the inverse image of \(S_Q\) in \(\Psi_Q \times Q \times X\). Then \(W\) is the closed subspace consisting of triples \((k, q, x)\) such that \(g = q^{-1}kq\) is in \(G\) and \(gx = x\). If \((g, x) \in S\) and \(q \in Q\) then clearly \((gqq^{-1}, q, x) \in W\). Thus \(\bar{\Phi}\) factors through a morphism \(\Phi: Q \times_G S \to S_Q\). Since \(\bar{\Phi}^*\) is an \(R(Z_Q)\)-module homomorphism, so is \(\Phi^*\). Next we show that \(\Phi\) is an isomorphism. Since we work over \(\mathbb{C}\) and \(\Phi\) is a
representable morphism of smooth (hence normal) algebraic spaces, it suffices, by Zariski’s main theorem (proved for algebraic spaces in [Knu, Theorem V4.2]), to prove that \(\Phi\) is bijective on geometric points.

First we show that \(\Phi\) is injective. Let \((q_1, g_1, x_1)\) and \((q_2, g_2, x_2)\)be two points of \(Q \times S\) such that \(T(q_1, g_1, x_1) = (q_1g_1q_1^{-1}, q_1, x_1)\) and \(T(q_2, g_2, x_2) = (q_2g_2q_2^{-1}, q_2, x_2)\) have the same image in \(\Psi_Q \times Y\). Then \(q_2g_2q_2^{-1} = q_1g_1q_1^{-1}\) and there is an element \(g \in G\) such that \(x_2 = gx_1\) and \(q_2 = q_1g^{-1}\). Thus \(g_2 = gq_1g^{-1}\) and hence \((q_1, g_1, x_1)\) and \((q_2, g_2, x_2)\) are in the same \(G\)-orbit in \(Q \times S\). Therefore, \(\Phi\) is injective on geometric points.

Conversely, suppose that \((k, q, x) \in W\). Then \(g = q^{-1}kq \in \Psi_Q \cap G = \Psi\). Thus \((k, q, x) = T(q, g, x)\) so \(\Phi\) is surjective on geometric points.

Finally, the fact that \(f_Q \circ \Phi = (1 \times_G f)\) is clear from the definition of \(\Phi\). This completes the proof of the lemma and with it Theorem 5.1.

6. RIEMANN-ROCH FOR QUOTIENTS

As an application of the nonabelian localization theorem, we can give an explicit formula for the Riemann-Roch map for quotients of smooth algebraic spaces by proper actions of algebraic groups. Recall, (Section [1.1.1]) that any algebraic space with a proper group action is automatically separated.

Before stating this, we need some preliminary results about invariants.

6.1. INVARIANTs AND EQUIVARIANT \(K\)-THEORY. Let \(G\) be an algebraic group acting properly on an algebraic space \(X\). The theorem of Keel
and Mori [KM] states that the quotient stack \([X/G]\) has a moduli space \(Y = X/G\) in the category of algebraic spaces. Translated in terms of group actions, this means that the map of algebraic spaces \(X \to Y\) is a categorical geometric quotient in the category of algebraic spaces.

**Definition 6.1.** Let \(X\) be an algebraic space with a proper \(G\) action and let \(X \to Y\) be the geometric quotient. If \(\mathcal{F}\) is a \(G\)-equivariant quasi-coherent \(\mathcal{O}_X\)-module, define \(\mathcal{F}^G = (\pi_*\mathcal{F})^G\). We call this the functor of taking \(G\)-invariants.

To obtain a map from equivariant \(K\)-theory to the \(K\)-theory of the quotient, we need the following lemma.

**Lemma 6.2.** Let \(X\) be an algebraic space with a proper \(G\)-action (see Section [L.1]) and let \(X \to Y\) be the geometric quotient. The assignment \(F \mapsto F^G\) defines an exact functor \(\text{coh}_X^G \to \text{coh}_Y\).

**Remark 6.3.** If \(X\) and \(Y\) are both schemes, and \(G\) is reductive, then this lemma is a consequence of the facts that the quotient map \(X \to Y\) is affine [MFK, Proposition 0.7], and that taking invariants by a locally finite action of a reductive group is an exact functor (in characteristic 0).

The theorem of Keel and Mori is proved using an étale local description of the moduli space \(Y = X/G\). In particular they prove that \([X/G]\) has a representable étale cover by quotient stacks \(\{[U_i/H_i]\}\) with \(U_i\) affine, \(H_i\) finite and such that the following diagram of stacks and moduli spaces is Cartesian:

\[
\begin{array}{ccc}
[U_i/H_i] & \to & [X/G] \\
\downarrow & & \downarrow \\
U_i/H_i & \to & X/G
\end{array}
\]

Let \(V_i = U_i \times_{[X/G]} X\). Then \(V_i\) is affine and has commuting and free actions of \(G\) and \(H_i\). Let \(X_i = Y_i/H_i\). Since \(H_i\) acts freely, \(X_i \to X\) is étale. If we let \(Y_i = X_i/G_i\), then the map of quotients \(Y_i \to Y\) is also étale, and the following diagram of spaces and quotients is Cartesian:

\[
\begin{array}{ccc}
X_i & \to & X \\
\downarrow & & \downarrow \\
Y_i & \to & Y
\end{array}
\]

(17)

The actions of \(G\) and \(H_i\) on \(Y_i\) commute so \(X_i/G = (Y_i/G_i)/H_i\). The local structure of geometric quotients given by Diagram (17) implies that the quotient map \(X \to Y\) is an affine morphism in the category of algebraic spaces.
Proof of Lemma 6.2. The question is local in the étale topology on the quotient $Y$. Thus we may assume that there is an affine scheme $V$ and a finite group $H$ such that $V$ has commuting free actions by $H$ and $G$ and $X = V/G$. It follows that the $G \times H$-quotient map $V \to Y$ factors as $p = q \circ \pi_V = \pi \circ q_V$ where $\pi_V : V \to V/G$ is a $G$-torsor, $q_V : V \to X$ is an $H$-torsor and $q : V/G \to Y$ is a quotient by $H$.

Let $\mathcal{G}$ be a $(G \times H)$-equivariant coherent sheaf on $V$. Then

$$
\mathcal{G}^{G \times H} = (\mathcal{G}^G)^H = (\mathcal{G}^H)^G
$$

The group $H$ is finite and $Y = (V/G)/H$ so the assignment $\mathcal{H} \mapsto \mathcal{H}^H$ is an exact functor $\text{coh}_{V/G}^H \to \text{coh}_Y$. Since $G$ and $H$ act freely on $V$ the assignments $\mathcal{G} \mapsto \mathcal{G}^G$ and $\mathcal{G} \mapsto \mathcal{G}^H$ define equivalences $\text{coh}_{V/G}^{G \times H} \to \text{coh}_{V/G}^H$ and $\text{coh}_{V/G}^{G \times H} \to \text{coh}_{X}^G$. Thus, the assignment $\mathcal{G} \mapsto \mathcal{G}^{G \times H}$ is an exact functor $\text{coh}_{V/G}^{G \times H} \to \text{coh}_{X}^G$. Since the assignment $\mathcal{G} \mapsto \mathcal{G}^H$ is an equivalence $\text{coh}_{V/G}^{G \times H} \to \text{coh}_{X}^G$ it follows that the assignment $\mathcal{F} \mapsto \mathcal{F}^G$ is an exact functor $\text{coh}_{V/G}^{G} \to \text{coh}_{Y}^G$.

If $X$ is an algebraic space with a proper $G$-action and geometric quotient $X \rightarrow Y$ let $\pi_G : G(G, X) \rightarrow G(Y)$ be the map on $K$-theory induced by the exact functor $\text{coh}_X^G \to \text{coh}_{Y}$ given by $\mathcal{F} \mapsto \mathcal{F}^G$. We will usually denote $\pi_G(\alpha)$ by $\alpha^G$.

If $G$ acts properly on $X$ and $q : X' \rightarrow X$ is a finite $G$-equivariant map then $G$ acts properly on $X'$ [EG3 Prop. 2.1]. Let $\pi : X \rightarrow Y$, $\pi' : X' \rightarrow Y'$ be the geometric quotients. Since the composite map $\pi \circ q : X' \rightarrow Y$ is $G$-invariant there is a map of quotient $q' : Y' \rightarrow Y$ such that the diagram commutes.

$$
\begin{array}{ccc}
X' & \rightarrow & X \\
\downarrow \pi' & & \downarrow \pi \\
Y' & \rightarrow & Y
\end{array}
$$

Lemma 6.4. The map $q'$ is finite and $\pi_G \circ q_* = q'_* \circ \pi'_* \circ \pi_G$ as maps $G(G, X') \rightarrow G(Y)$.

Proof. Working locally in the étale topology we may assume that $Y$ is affine. It follows (since $q$ is finite) that all of the other spaces in Diagram (18) are affine. Let $X = \text{Spec} \ A$, $X' = \text{Spec} \ B$. Then $Y' = \text{Spec} \ B^G$ and $Y = \text{Spec} \ A^G$. If $M$ is a $G$-equivariant $B$-module then $(A \cdot M)^G = (A \cdot M)^G$ (c.f. [EG3 Proposition 2.3]). Translated to sheaves this means that $\mathcal{F}$ is a $G$-equivariant quasi-coherent sheaf on $X'$ then $(q_* \mathcal{F})^G = q'_* (\mathcal{F}^G)$ as quasi-coherent sheaves on $Y$. 
To complete the proof of the Lemma we need to show that \( q' \) is finite. Since \( q' \) is affine we need to show that \( q'_* \mathcal{O}_{Y'} \) is coherent. Now \( \pi' \) is a geometric quotient so \( \mathcal{O}_{Y'} = \mathcal{O}_{X'}^G \). Thus \( q_* \mathcal{O}_{Y'} = (q_* \mathcal{O}_X)^G \) is coherent because \( q \) is finite.

\[ \square \]

6.2. **Twisting equivariant \( K \)-theory by a central subgroup.** Let \( Z \) be an algebraic group and let \( H \) be a subgroup (not necessarily closed) of the center of \( Z \) consisting of semisimple elements. If \( V \) is any representation of \( Z \) then, since any commuting family of semisimple endomorphisms is simultaneously diagonalizable, \( V \) can be written as a direct sum of \( H \)-eigenspaces \( V = \bigoplus \mathcal{V}_\chi \) where the sum is over all characters \( \chi : H \to \mathbb{C}^* \) and \( H \) acts on \( \mathcal{V}_\chi \) by \( h \cdot v = \chi(h)v \). Since elements of \( H \) are central in \( Z \), each eigenspace \( \mathcal{V}_\chi \) is \( Z \)-stable, and therefore we have \( V = \bigoplus \mathcal{V}_\chi \) in \( R(Z) \). We define an action of \( H \) on the representation ring \( R(Z) \), by

\[ h \cdot [V] = \sum \chi(h)^{-1}[\mathcal{V}_\chi], \]

for \( h \in H \) and \( [V] = \sum [\mathcal{V}_\chi] \in R(Z) \). Because \( H \) is a central subgroup, it acts on \( \mathbb{C}[Z]^Z \) by the rule \( (h \cdot f)(z) = f(h^{-1}z) \), for \( h \in H \), \( z \in Z \), \( f \in \mathbb{C}[Z]^Z \). With these actions, the character map from \( R(Z) \) to \( \mathbb{C}[Z]^Z \) is \( H \)-equivariant. Recall that maximal ideals of \( R(Z) \) are of the form \( m_\Psi \), where \( \Psi \) is a semisimple conjugacy class in \( Z \). The \( H \)-equivariance of the character map implies that if \( h \in H \), then

\[ h \cdot m_\Psi = m_{h\Psi}. \]

In particular,

\[ h^{-1}m_h = m_1, \]

the augmentation ideal of \( R(Z) \).

More generally, let \( X \) be an algebraic space with a \( Z \)-action, such that \( H \) acts trivially on \( X \). In this case, if \( \mathcal{E} \) is any \( Z \)-equivariant coherent sheaf on \( X \), and \( \chi \) a character of \( H \), let \( \mathcal{E}_\chi \) be the subsheaf of \( \mathcal{E} \) whose sections (on any étale open set \( U \)) are given by

\[ \mathcal{E}_\chi(U) = \{ s \in \mathcal{E}(U) \mid h \cdot s = \chi(h)s \}. \]

Then \( \mathcal{E} = \bigoplus \mathcal{E}_\chi \) is a decomposition of \( \mathcal{E} \) into a direct sum of \( H \)-eigensheaves for \( \mathcal{E} \). We define an action of \( H \) on \( G_0(Z, X) \) by

\[ h \cdot [\mathcal{E}] = \sum \chi(h)^{-1}[\mathcal{E}_\chi]. \]

If \( X \) is a point, this reduces to the previous definition of the action of \( H \) on \( R(Z) \). Thus, the actions of \( H \) on \( R(Z) \) and on \( G_0(Z, X) \) are compatible: if \( r \in R(Z) \), \( \alpha \in G_0(Z, X) \), then \( h \cdot (r \alpha) = (h \cdot r)(h \cdot \alpha) \). If \( Z \) acts on \( X \) with finite stabilizers, then we can identify \( G_0(Z, X) = \)
$\oplus G_0(Z, X)_{m_\Psi}$, where the sum is over finitely many conjugacy classes $\Psi$. The compatibility of the actions implies that

$$h^{-1}G_0(Z, X)_{m_h} = G_0(Z, X)_{m_1}.$$  

We refer to the action of $H$ on $G_0(Z, X)$ as twisting. If $\alpha \in G_0(Z, X)$, we will often write $\alpha(h)$ for $h^{-1} \cdot \alpha$ and refer to this map as twisting by $h$.

### 6.3. Twisting and the global stabilizer.

We now define a central twist. Let $\Psi = C^G_G(h)$ be a semisimple conjugacy class and let $Z = Z_G(h)$ and let $H$ be the cyclic subgroup of $Z$ generated by $h$. Since $h$ is central in $Z$, there is an action of $H$ on $G_0(Z, X^h)$. The map $\Phi_h : G \times X^h \rightarrow S_\Psi$, $(g, x) \mapsto (ghg^{-1}, gx)$ identifies $S_\Psi$ with $G \times_Z X$. Thus by Morita equivalence there is an action of $H$ on $G_0(G, S_\Psi)$.

We define a twisting map

$$G_0(G, S_\Psi) \rightarrow G_0(G, S_\Psi)$$

$$\alpha \mapsto \alpha(c_\Psi)$$

to be the map induced by Morita equivalence corresponding to the endomorphism of $G_0(Z, X^h)$ given by twisting by $h$. We will call this map the central twist.

**Lemma 6.5.** With notation as above, the map $\alpha \mapsto \alpha(c_\Psi)$ is independent of the choice of $h \in \Psi$. Moreover, if $\alpha \in G_0(G, S_\Psi)_{c_\Psi}$, then $\alpha(c_\Psi) \in G_0(G, S_\Psi)_{m_1}$. For arbitrary $\beta \in G_0(G, S_\Psi)$, the component of $\beta(c_\Psi)$ in $G_0(G, S_\Psi)_{m_1}$ equals $\beta_{c_\Psi}(c_\Psi)$.

**Proof.** Suppose that $h_1$ and $h_2 = kh_1k^{-1}$ are two elements of $\Psi$; let $Z_i = Z_G(h_i)$. Then $kX^{h_1} = X^{h_2}$ and $kZ_i k^{-1} = Z_2$. We have inclusions $f_i : X^{h_i} \rightarrow S_\Psi$ given by $f_i(x) = (h_i, x)$ for $x \in X^{h_i}$. The equivalence of categories between $\text{coh}_{S_\Psi}$ and $\text{coh}_{X^{h_i}}$ takes a $G$-equivariant coherent sheaf $\mathcal{F}$ on $S_\Psi$ to the sheaf $f_i^* \mathcal{F}$ on $X^{h_i}$ (since $f_i$ is $Z_i$-equivariant, the pullback $f_i^* \mathcal{F}$ has a natural structure of $Z_i$-equivariant sheaf). Now, $k^*$ induces an equivalence of categories between $\text{coh}_{X^{h_2}}$ and $\text{coh}_{X^{h_1}}$, and moreover $k^*f_2^* = f_1^*$. Therefore, to show the independence of the twisting map, it suffices to show that a $Z_2$-equivariant sheaf $\mathcal{E}$ on $X^{h_2}$ is an $h_2$-eigensheaf with eigenvalue $\chi$ if and only if the pullback sheaf $k^*\mathcal{E}$ on $X^{h_1}$ is an $h_1$-eigensheaf with eigenvalue $\chi$. This holds because if $U_2 \rightarrow X_2$ is any étale open set, and $U_1 = k^*U_1$, then $(k^*\mathcal{E})(U_1) = \mathcal{E}(U_2)$, and the automorphism of $(k^*\mathcal{E})(U_1)$ coming from $h_1$ coincides with the automorphism of $\mathcal{E}(U_2)$ induced by $h_2$.

Next, if $\alpha \in G_0(G, S_\Psi)_{c_\Psi}$, then under the Morita equivalence isomorphism, $\alpha$ corresponds to an element in $G_0(Z, X^h)_{m_h}$. By definition of the central twist and (19), $\alpha(c_\Psi)$ corresponds to an element
of $G_0(Z, X^h)_{m_1}$. By Proposition 3.8, the Morita equivalence isomorphism identifies $G_0(Z, X^h)_{m_1}$ with $G(G_0(G, S_\varphi)_{m_1})$. Hence $\alpha(c_\varphi) \in G_0(G, S_\varphi)_{m_1}$.

Finally, if $\beta$ is an arbitrary element of $G_0(G, S_\varphi)$, then under the Morita equivalence isomorphism the elements $\beta(c_\varphi)$ and $\beta_{c_\varphi}(c_\varphi)$ correspond to elements of $G_0(Z, X^h)$ whose components at $m_1$ are equal, so the result follows.

Now assume that $G$ acts properly on $X$. Then it also acts properly on $S_{\varphi}$, since the map $f : S_\varphi \to X$ is finite. By Lemma 6.2, there is a map in $K$-theory (the map of taking $G$-invariants) $G_0(G, S_\varphi) \to G_0(S_{\varphi}/G)$, $\alpha \mapsto \alpha^G$.

**Lemma 6.6.** If $\alpha \in G_0(G, S_\varphi)$, then $\alpha^G = (\alpha(c_\varphi))^G$.

**Proof.** Let $h \in \Psi$ and $Z = Z_G(h)$. Since the map $\Phi_h : G \times X^h \to S_\varphi$ is a $Z$-torsor, the quotients $X^h/Z$ and $S_\varphi/G$ are both identified with the quotient $M = (G \times X^h)/(G \times Z)$. We have a map $G(Z, X^h) \to G(M)$ (the map of taking $Z$-invariants) and a map $G(G, S_\varphi) \to G(M)$ (the map of taking $G$-invariants). By Morita equivalence, both $G(Z, X^h)$ and $G(G, S_\varphi)$ are isomorphic to $G \times Z, G \times X^h$ and under these isomorphisms, both maps coincide with the map $G(G \times Z, G \times X^h) \to G(M)$ (the map of taking $G \times Z$-invariants).

In view of the definition of the central twist, it suffices to prove that if $\alpha \in G_0(Z, X^h)$ then $\alpha^Z = (\alpha(h))^Z$. For this we may assume that $\alpha = [\mathcal{E}]$, where $\mathcal{E}$ is a $Z$-equivariant coherent sheaf on $X^h$. As above, write $\mathcal{E} = \oplus \mathcal{E}_\chi$. Note that $[\mathcal{E}_\chi]^Z = 0$ unless $\chi$ is the trivial character; so, denoting the trivial character by $1$, we have $\alpha^Z = \sum \mathcal{E}_\chi^Z = [\mathcal{E}_1]^Z$ and $(\alpha(h))^Z = \sum \chi(h)^{-1}[\mathcal{E}_\chi]^Z = [\mathcal{E}_1]^Z$, completing the proof. □

6.4. **Riemann-Roch for quotients: statement and proof.** If $X$ is a $G$-space let $CH^*_G(X) = \oplus_i CH^i_G(X) \otimes \mathbb{C}$ where $CH^i_G(X)$ denotes the “codimension” $i$ equivariant Chow groups of $X$ as in [EG2, p. 569]. Recall [EG1, Theorem 3] that if $G$ acts properly on $X$, with quotient $X/G$, then there is an isomorphism $\phi^G_X : CH^*_G(X) \to CH^*(X/G)$, where $CH^*(X/G) = \oplus_i CH^i(X/G) \otimes \mathbb{C}$. The map is defined as follows: Because $G$ acts with finite stabilizers $CH^*_G(X)$ is generated by fundamental classes of $G$-invariant cycles. If $V$ is a closed $G$-invariant subspace let $[V/G]$ be the image of $V$ under the quotient map. Then $\phi^G_X([V]) = e_V[V/G]$ where $e_V$ is the order of the stabilizer of a general point of $V$.

If $G$ and $X$ are understood, we may write simply $\phi$ or $\phi_X$ for $\phi^G_X$.

If $Y$ is an algebraic space let $\tau_Y : G_0(Y) \to CH^*(Y)$ be the Todd isomorphism of [Ful, Theorem 18.3] (as extended to algebraic spaces).
Likewise if $X$ is a $G$-space let \( \tau^G_Y : G_0(G, X) \to \bigoplus_{i=0}^\infty CH^i_G(X) \) be the equivariant Todd map of \cite{EG2}. When $G$ acts with finite stabilizers, \( CH^i_G(X) = 0 \) for \( i > \dim X \) so the target of the equivariant Todd map is \( CH^*_G(X) \) (cf. \cite{EG2} Cor 5.1). Note that \( \tau^G_Y \) is an isomorphism only when $G$ acts freely. However, it factors through an isomorphism \( G_0(G, X) \to \bigoplus_{i=0}^\infty CH^i_G(X) \) where $G_0(G, X)$ is the completion of $G_0(G, X)$ at the augmentation ideal of $R(G)$ \cite{EG2} Theorem 4.1.

**Theorem 6.7.** Let $G$ be an algebraic group acting properly on a smooth algebraic space $X$, and let $Y = X/G$ be the quotient. Fix a conjugacy class $\Psi$ in $G$ and let $S_\Psi$ be the corresponding part of the global stabilizer, so we have a $G$-equivariant map $\bar{f} : S_\Psi \to X$ and an induced map $g : S_\Psi / G \to Y$ on the quotients. Let $\alpha \in G_0(G, X)$ and let $\alpha_\Psi$ denote the part of $\alpha$ in $G_0(G, X)_{\Psi \Psi}$. Then

\[
\tau^*_Y((\alpha_\Psi)^G) = \phi^G_X \circ \tau^*_X \circ f_\ast \left( (\lambda^{-1}(N^*_f)^{-1} \cap f^*\alpha)(c_\Psi) \right)
\]

(20)

\[
= g_\ast \circ \phi^G_{S_\Psi} \circ \tau^G_{S_\Psi} \left( (\lambda^{-1}(N^*_f)^{-1} \cap f^*\alpha)(c_\Psi) \right).
\]

This theorem can be stated in the following equivalent form, which can be more convenient in applications.

**Theorem 6.8.** Keep the notation and hypotheses of the previous theorem, and in addition, fix $h \in \Psi$, let $Z = Z_G(h)$, and let $i : X^h \to X$ denote the closed embedding. Identify $X^h / Z$ with $S_\Psi / G$. Let $\alpha|_Z$ denote the image of $\alpha$ under the natural map of $G$-equivariant $K$-theory to $Z$-equivariant $K$-theory. Then

\[
\tau^*_Y((\alpha_\Psi)^G) = g_\ast \circ \phi^Z_{X^h} \circ \tau^Z_{X^h} \left( \lambda^{-1}(N^*_f)^{-1} \cap \lambda^{-1}(g^* / \beta^* \cap \iota^*(\alpha|_Z))(h) \right).
\]

(21)

**Remark 6.9.** Note that we do not need to compute $\alpha_\Psi$ to apply the formulas of Theorems 6.7 and 6.8. However, the answer would be the same if on the right side of these formulas we replaced $\alpha$ with $\alpha_\Psi$. The reason is that the part of the formula corresponding to $\alpha_\Psi'$ for any $\Psi' \neq \Psi$ vanishes. Also, since the invariant map is linear, $\tau^*_Y(\alpha^G) = \sum_\Psi \tau^*_Y((\alpha_\Psi)^G)$ can be computed using the formulas of Theorems 6.7 and 6.8.

**Remark 6.10.** If $X$ (and hence also $X^h$) is a smooth scheme, then $K_0(G, X) = G_0(G, X)$ and the equivariant $\tau$ maps can be calculated using the equivariant Chern character map, and the equivariant Todd class of the tangent bundle. If if $\beta \in K_0(Z, X^h)$ then using the formulas of Theorem 3.3 \cite{EG2}

\[
\tau^Z_{X^h}(\beta) = \text{ch}^Z(\beta) \tau^Z([\mathcal{O}_{X^h}])
\]

(22)
If in addition, \( Z \) is connected or \( X^h \) has a \( Z \)-linearized ample line bundle then

\[
\tau^Z(O_{X^h}) = \frac{\text{Td}^Z(T_{X^h})}{\text{Td}^Z(\mathfrak{j})}
\]

where \( \text{Td}^Z \) is the equivariant Todd class of \([\text{EG2}, \text{Definition 3.1}]\). This follows from \([\text{EG2}, \text{Theorem 3.1(d)}]\) and the following observation about the definition of the equivariant Riemann-Roch map of \([\text{EG2}]\): If a group \( G \) acts freely on a smooth space \( X \) with quotient \( X \rightarrow X/G \) then, identifying \( \text{CH}^*(G) \) with \( \text{CH}^*(X/G) \), we have \( \tau^G(O_X) = \tau(O_{X/G}) \). When \( X/G \) (and thus \( X \)) is a smooth scheme then \( \tau(O_X) = \text{Td}(T_{X/G}) \). In this case \( \tau^G(O_X) = \text{Td}^G(\pi^*T_{X/G}) \). By Lemma \( \text{[L]} \), \( T_\pi = X \times \mathfrak{g} \). Therefore, \( \text{Td}^G(\pi^*T_{X/G}) = \text{Td}^G(T_X)/\text{Td}^G(\mathfrak{g}) \).

**Proof.** It suffices to prove Theorem 6.7 since this implies Theorem 6.8 using the Morita equivalence isomorphism \( G_0(G, S_\Psi) \simeq G_0(Z, X^h) \). We also need only prove the second formula in Equation (20); this implies the first formula because the maps \( \tau \) and \( \phi \) are covariant for finite morphisms.

The proof of Theorem 6.7 is almost the same as the proof in the case where \( G \) is diagonalizable, given in \([\text{EG3}, \text{Theorem 3.1}]\). The nonabelian localization theorem replaces the localization theorem for actions of diagonalizable groups used in \([\text{EG3}]\). We give the general proof here, referring to \([\text{EG3}]\) for some omitted details. Throughout the proof we write \( \phi_M \) for \( \phi^G_M \), where \( G \) acts properly on \( M \). We write \( S \) for \( S_\Psi \).

The proof proceeds in three steps. First, we observe that the theorem is true if the action of \( G \) on \( X \) is free (and that in this case, it holds without the assumption that \( X \) is smooth). In this case, if \( \Psi \neq \{1\} \), then \( S \) is empty and \( \alpha_\Psi = 0 \), so both sides of (20) vanish. If \( \Psi = \{1\} \), then \( f : S \rightarrow X \) is an isomorphism, and the theorem amounts to the assertion that

\[
\tau_Y(\alpha^G) = \phi_X \circ \tau^G_X(\alpha),
\]

which follows from \([\text{EG2}, \text{Theorem 3.1(e)}]\).

Second, we prove the theorem for \( \Psi = \{1\} \). In this case \( f : S \rightarrow X \) is an isomorphism, and the theorem amounts to the assertion that

\[
\tau_Y((\alpha_1)^G) = \phi_X \circ \tau^G_X(\alpha_1).
\]

Since \( \tau^G \) maps components of \( K \)-theory supported at maximal ideals other than \( m_1 \) to zero (this is proved as in \([\text{EG3}, \text{Proposition 2.6}]\)), on the right side of this equation we can replace \( \alpha \) by \( \alpha_1 \). By \([\text{EG1}, \text{Proposition 10}]\) there exists a finite surjective morphism \( p : X' \rightarrow X \) of
G-spaces such that \( G \) acts freely on \( X' \) and then induced map of quotients \( q: X' \to X \) is also finite and surjective. (Note that \( X' \) need not be smooth.) By [EG3, Lemma 3.5] (which holds without the diagonalizability assumption on \( G \)), the map \( p_*: G_0(G, X')_{m_1} \to G_0(G, X)_{m_1} \) is surjective. Therefore there exists \( \beta_1 \in G_0(G, X')_{m_1} \) such that \( p_*(\beta_1) = \alpha_1 \). There is a geometric quotient \( X' \to Y' \) such that the induced map \( q: Y' \to Y \) is also finite. By Lemma 6.4 taking invariants commutes with pushforward by a finite morphism, so \( q_*((\beta_1)^G) = (\alpha_1)^G \). Therefore,

\[
\tau_Y((\alpha_1)^G) = \tau_Y(q_*((\beta_1)^G)) = q_*\tau_{Y'}((\beta_1)^G).
\]

By the first step of the proof, this equals \( q_* \circ \phi_{X'} \circ \tau_{X'}(\beta_1) \). Since \( q_* \circ \phi_{X'} = \phi_X \circ p_* \), we have

\[
q_* \circ \phi_{X'} \circ \tau_{X'}^G(\beta_1) = \phi_X \circ p_* \circ \tau_{X'}^G(\beta_1) = \phi_X \circ \tau_{X}^G \circ p_*(\beta_1) = \phi_X \circ \tau_{X}^G(\alpha_1),
\]

as desired.

Third, we prove the theorem for general \( \Psi \). Let \( \beta = \lambda - 1(N^*_f)^{-1} \cap f^* \alpha \). We need to prove that

\[
\tau_Y((\alpha_1)^G) = g_* \circ \phi_S \circ \tau^G_S((\beta(c_\Psi)).
\]

By Lemma 6.5 the components of \( \beta(c_\Psi) \) and \( \beta_{c_\Psi}(c_\Psi) \) supported at \( m_1 \in R(G) \) are equal, so arguing as in Step 2, we see that \( \tau^G_S((\beta(c_\Psi)) = \tau^G_S((\beta_{c_\Psi}(c_\Psi))). \) But \( \beta_{c_\Psi}(c_\Psi) \) is in \( G_0(S, G)_{m_1} \). Therefore we can apply the second step of the proof to \( \beta_{c_\Psi}(c_\Psi) \). Thus,

\[
\tau_Y((\alpha_1)^G) = \tau_Y \circ ((f_*(\beta_{c_\Psi}))^G) \quad \text{localization}
\]

\[
= g_* \circ \tau^G_{S/G}(\beta_{c_\Psi}) \quad \text{finite-pushforward commutes with invariants}
\]

\[
= g_* \circ \tau^G_{S/G}((\beta_{c_\Psi}(c_\Psi))^G) \quad \text{Lemma 6.6}
\]

\[
= g_* \circ \phi_S \circ \tau^G_{S}(\beta_{c_\Psi}(c_\Psi)) \quad \text{Step 2}
\]

\[
= g_* \circ \phi_S \circ \tau^G_{S}(\beta(c_\Psi)).
\]

This completes the proof. \( \square \)

7. Appendix

This appendix contains a result about the tangent bundle to a torsor which is difficult to find rigorously proved in the literature.

**Lemma 7.1.** Let \( X \xrightarrow{f} Y \) be a (left) \( G \)-torsor. Then \( T_f \) is canonically isomorphic to the \( G \)-bundle \( X \times g \) where the \( G \)-action on the Lie algebra \( g \) is the adjoint action.
Proof. By definition $T_f$ is the normal bundle to the diagonal morphism $X \xrightarrow{\Delta} X \times_Y X$. Since $f$ is a $G$-torsor the diagram
\[
\begin{array}{c}
G \times X \xrightarrow{\sigma} X \\
\downarrow \pi \quad \downarrow f \\
X \xrightarrow{f} Y
\end{array}
\]
(23)
is a cartesian, where $\sigma : G \times X \to X$ is the action map and $\pi : G \times X \to X$ is projection. If $G$ acts on $G \times X$ by conjugation on the first factor and the usual action on the second factor then all morphisms in (23) are $G$-invariant. Thus there is a canonical identification of $G$-spaces $X \times_Y X \to G \times X$. Under this identification the diagonal corresponds to the section $X \xrightarrow{(e_G, 1_X)} G \times X$. This map is obtained by base change from the $G$-equivariant inclusion $e_G \to G$ (where $G$ acts on itself by conjugation) whose normal bundle is $g$. Therefore, $T_f = X \times g$. □

Remark 7.2. Suppose $N \subset G$ is a closed normal subgroup with Lie algebra $n \subset g$. Normality of $N$ implies that if $X \xrightarrow{f} Y$ is an $N$-torsor then there is a natural left $G$-action on $Y$ such that $f$ is $G$-equivariant. Essentially the same argument as above implies that $T_f$ is naturally isomorphic to $n \otimes O_X$ where the $G$-action on $n$ is the restriction of the adjoint action to the $ad$-invariant subalgebra $n \subset g$.

References

[Ati] Michael Francis Atiyah, *Elliptic operators and compact groups*, Springer-Verlag, Berlin, 1974, Lecture Notes in Mathematics, Vol. 401.

[BFM] Paul Baum, William Fulton, and Robert MacPherson, *Riemann-Roch for singular varieties*, Inst. Hautes Études Sci. Publ. Math. (1975), no. 45, 101–145.

[Bor] Armand Borel, *Linear algebraic groups*, second ed., Springer-Verlag, New York, 1991.

[BtD] Theodor Bröcker and Tammo tom Dieck, *Representations of compact Lie groups*, Graduate Texts in Mathematics, vol. 98, Springer-Verlag, New York, 1995, Translated from the German manuscript, Corrected reprint of the 1985 translation.

[CG] Neil Chriss and Victor Ginzburg, *Representation theory and complex geometry*, Birkhäuser Boston Inc., Boston, MA, 1997.

[EG1] Dan Edidin and William Graham, *Equivariant intersection theory*, Invent. Math. 131 (1998), no. 3, 595–634.

[EG2] ———, *Riemann-Roch for equivariant Chow groups*, Duke Math. J. 102 (2000), no. 3, 567–594.

[EG3] ———, *Riemann-Roch for quotients and Todd classes of simplicial toric varieties*, Comm. in Alg. 31 (2003), 3735–3752.

[Fog] John Fogarty, *Invariant theory*, W. A. Benjamin, Inc., New York-Amsterdam, 1969.
[Ful] William Fulton, Intersection theory, Springer-Verlag, Berlin, 1984.

[EGA4] A. Grothendieck and J. Dieudonné, Éléments de Géométrie Algébrique IV. Étude locale des schemas et des morphismes de schémas, Inst. Hautes Études Sci. Publ. Math. No. 20, 24, 28, 32 (1964, 1965, 1966, 1967).

[Hoc] G. Hochschild, The structure of Lie groups, Holden-Day Inc., San Francisco, 1965.

[Hum] James E. Humphreys, Conjugacy classes in semisimple algebraic groups, Mathematical Surveys and Monographs, vol. 43, American Mathematical Society, Providence, RI, 1995.

[KM] Seán Keel and Shigefumi Mori, Quotients by groupoids, Ann. of Math. (2) 145 (1997), no. 1, 193–213.

[Knu] Donald Knutson, Algebraic spaces, Springer-Verlag, Berlin, 1971, Lecture Notes in Mathematics, Vol. 203.

[MFK] D. Mumford, J. Fogarty, and F. Kirwan, Geometric invariant theory, third ed., Springer-Verlag, Berlin, 1994.

[Nie] H. Andreas Nielsen, Diagonalizable linearized coherent sheaves, Bull. Soc. Math. France 102 (1974), 85–97.

[OV] A. L. Onishchik and È. B. Vinberg, Lie groups and algebraic groups, Springer Series in Soviet Mathematics, Springer-Verlag, Berlin, 1990, Translated from the Russian and with a preface by D. A. Leites.

[Qui] Daniel Quillen, Higher algebraic K-theory. I, Algebraic K-theory, I: Higher K-theories (Proc. Conf., Battelle Memorial Inst., Seattle, Wash., 1972), Springer, Berlin, 1973, pp. 85–147. Lecture Notes in Math., Vol. 341.

[Ros] Wulf Rossmann, Lie groups, Oxford Graduate Texts in Mathematics, vol. 5, Oxford University Press, Oxford, 2002, An introduction through linear groups.

[Seg1] Graeme Segal, The representation ring of a compact Lie group, Inst. Hautes Études Sci. Publ. Math. (1968), no. 34, 113–128.

[Seg2] Graeme Segal, Equivariant K-theory, Inst. Hautes Études Sci. Publ. Math. (1968), no. 34, 129–151.

[Sri] V. Srinivas, Algebraic K-theory, second ed., Progress in Mathematics, vol. 90, Birkhäuser Boston Inc., Boston, MA, 1996.

[Tho1] R. W. Thomason, Comparison of equivariant algebraic and topological K-theory, Duke Math. J. 53 (1986), no. 3, 795–825.

[Tho2] R. W. Thomason, Lefschetz-Riemann-Roch theorem and coherent trace formula, Invent. Math. 85 (1986), no. 3, 515–543.

[Tho3] R. W. Thomason, Algebraic K-theory of group scheme actions, Algebraic topology and algebraic K-theory (Princeton, N.J., 1983), Ann. of Math. Stud., vol. 113, Princeton Univ. Press, Princeton, NJ, 1987, pp. 539–563.

[Tho4] R. W. Thomason, Equivariant algebraic vs. topological K-homology Atiyah-Segal-style, Duke Math. J. 56 (1988), no. 3, 589–636.

[Tho5] R. W. Thomason, Une formule de Lefschetz en K-théorie équivariante algébrique, Duke Math. J. 68 (1992), no. 3, 447–462.

[Toe] B. Toen, Théorèmes de Riemann-Roch pour les champs de Deligne-Mumford, K-Theory 18 (1999), no. 1, 33–76.

[VV] Gabriele Vezzosi and Angelo Vistoli, Higher algebraic K-theory of group actions with finite stabilizers, Duke Math. J. 113 (2002), no. 1, 1–55.
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