SYMMETRIC POLYNOMIALS VANISHING ON THE DIAGONALS SHIFTED BY ROOTS OF UNITY.

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Abstract. For a pair of positive integers \((k, r)\) with \(r \geq 2\) such that \(k + 1\) and \(r - 1\) are relatively prime, we describe the space of symmetric polynomials in variables \(x_1, \ldots, x_n\) which vanish at all diagonals of codimension \(k\) of the form \(x_i = tq^a x_{i-1}, i = 2, \ldots, k + 1,\) where \(t\) and \(q\) are primitive roots of unity of orders \(k + 1\) and \(r - 1\).

1. Introduction: the wheel condition

Fix a finite set \(S\) of non-zero complex numbers called the wheel set. A symmetric polynomial \(f \in \mathbb{C}[x_1, \ldots, x_n]^{S_n}\) satisfies the wheel condition relative to \(S\) if \(f(x_1, \ldots, x_n) = 0\) on all planes which have the form

\[
x_2 = t_1 x_1, \quad x_3 = t_2 x_2, \quad \ldots, \quad x_l = t_l x_{l-1}, \quad x_1 = t_l x_l,
\]

where \(t_1, \ldots, t_l \in S\). Note that \((1)\) implies that \(\prod_{i=1}^{l} t_i = 1\). This condition is called the resonance condition. We denote the space of all symmetric polynomials satisfying the wheel condition associated to \(S\) by \(F_S\).

Note that the space of symmetric polynomials in \(n\) variables satisfying the wheel condition relative to \(S\) is an ideal in \(\mathbb{C}[x_1, \ldots, x_n]\).

Let \(g_{nm}\) be the dimension of the symmetric polynomials in \(n\) variables of degree \(m\) satisfying the wheel condition. Then the character is given by

\[
\chi(F_S^k) = \sum_{n,m} g_{nm} z^n v^m.
\]

The basic question we are interested in is the computation of the character and a construction of an explicit basis in \(F_S\).

The study of the polynomials satisfying the wheel condition was initiated in \cite{FJMM2}. We briefly describe the results of that paper.

For natural numbers \(k, r, r \geq 2\), we fix \(t, q \in \mathbb{C}\) such that the resonance condition \(q^a t^b = 1\) is valid if and only if \(a = (k + 1)s, b = (r - 1)s\) for some \(s\). Define the wheel set \(S_r(q, t) \subset \mathbb{C}\) by

\[
S_r(q, t) = \{t, tq, \ldots, tq^{r-1}\}.
\]

The number of variable \(l\) in \((1)\) in this case, can be a multiple of \(k + 1\). However, it is easy to see that it is enough to impose the zero condition only for \(l = k + 1\). This remark is also valid in the root of unity case which we will consider in this paper.

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It is proved in [FJMM2] that the space \( F_{S_r(q,t)} \) has a basis of Macdonald polynomials \( \{P_{\lambda}(x_1, \ldots, x_n; q, t)\} \) where \( \lambda \) ranges over the \((k, r, n)\)-admissible partitions, satisfying

\[
\lambda_i - \lambda_{i+k} \geq r \quad (i = 1, \ldots, n-k).
\]

In particular, the Macdonald polynomials corresponding to the admissible partitions are well defined. Admissible partitions appeared first in the work [FJLMM], a bosonic formula for the corresponding character is given in [FJMM].

The case \( k + 1 \) and \( r - 1 \) are relatively prime is of special interest. In this case we can assume

\[
t = u^{r-1}, \quad q = u^{-(k+1)},
\]

for some \( u \) which is not a root of unity.

The Jack limit \( u \to 1 \) of the space \( F_{S_r(q,t)}^k \) is spanned by the set of Jack polynomials \( \{J_{\lambda}(x_1, \ldots, x_n; \beta)\} \) where \( \beta = -(r-1)/(k+1) \) and \( \lambda \) ranges over \((k, r, n)\)-admissible partitions. If \((k+1, r-1) = 1\) then it is expected that the limiting space coincides with the space of correlation functions of an abelian current in a vertex operator algebra, associated with the minimal series \((k+1, k+r)\) of the \( W_k \) algebra, see [FJMM].

\( \beta \) is a nonincreasing sequence of nonnegative integers

\[
\lambda = (\lambda_1, \ldots, \lambda_n), \quad \lambda_1 \geq \cdots \geq \lambda_n \geq 0.
\]

The sum \( |\lambda| = \sum_{i} \lambda_i \) is called weight of \( \lambda \).

We denote the space of symmetric function satisfying the wheel condition related to the wheel set \( S_{r-1}(q,t) \) with \( k \) by \( F^{(k,r)} \) and the corresponding character by \( \chi_{k,r}(z,v) \).

We also denote by \( F^{(k,l)}_n \) the subspace of \( F^{(k,l)} \) consisting of the functions of \( n \) variables. Note that \( q^{r-1} = 1 \), and the new wheel set with \( r - 1 \) is thus related to the previous one \( S_r(q,t) \). However, in the present case, it is not true that the resonance condition \( q^{r-1} = 1 \) necessarily restricts to the case \( a = (k+1)s, b = (r-1)s \) for some \( s \). In other words, not all vanishing planes \( \emptyset \) at the roots of unity case are limits of vanishing planes for generic \( u \). Therefore, we have new phenomena in the root of unity case, which is independent of generic results of [FJMM2].

The plan of paper is as follows. In Section 2 we prepare some special polynomials from MacDonald’s book [M]. In Section 3 we give a basis of the space \( F^{(k,2)} \). In Section 4 we give the main results of this paper, Theorems 4.6 and 4.7, which give an explicit basis in \( F^{(k,r)} \) and the character \( \chi_{k,r}(z,v) \), respectively.

## 2. Macdonald Polynomials

In this section we recall some facts we need about the theory of Macdonald polynomials and fix our notations.

### 2.1. Partitions

We denote \( \pi_n \) the set of partitions of length at most \( n \). By this we mean that an element \( \lambda \) of \( \pi_n \) is a nonincreasing sequence of nonnegative integers

\[
\lambda = (\lambda_1, \ldots, \lambda_n), \quad \lambda_1 \geq \cdots \geq \lambda_n \geq 0.
\]

The sum \( |\lambda| = \sum_{i} \lambda_i \) is called weight of \( \lambda \).

We denote the number of parts \( \lambda_j \) which are equal to \( i \) by \( m_i = m_i(\lambda) \) and write \( \lambda = (0^{m_0}, 1^{m_1}, 2^{m_2}, \ldots) \).
A partition $\lambda \in \pi_n$ is called $(k,r,n)$-admissible if $\lambda_i - \lambda_{i+k} \geq r$ for $i = 1, \ldots, n-k$. We will use $(k,1,n)$-admissibility in this paper. This is equivalent to the condition that $m_i(\lambda) \leq k$ for all $i$. In the below we simply call $\lambda$ is (non)-admissible with the understanding that it means the (non)-admissibility for $(k,1,n)$ with a prescribed value of $k$.

There is a partial dominance order of partitions. For two partitions $\lambda, \mu$ of the same weight we write $\lambda > \mu$ iff $\lambda \neq \mu$, $\lambda_1 + \cdots + \lambda_i \geq \mu_1 + \cdots + \mu_i$ for all $i$.

We also use the total lexicographical order of partitions. For two partitions $\lambda, \mu$ of the same weight we write $\lambda \prec \mu$ iff for some $i$ we have $\lambda_i < \mu_i$ and $\lambda_j = \mu_j$, $j = 1, \ldots, i-1$.

2.2. Macdonald polynomials. The Macdonald operators $D^r_n(q,t)$, $0 \leq r \leq n$, are mutually commuting $q$-difference operators acting on the ring of symmetric polynomials $\mathbb{C}(q,t)[x_1, \ldots, x_n]$:

$$D_n^r = \sum_{|I|=r} A_I(x;t) T_I,$$

where

$$A_I(x;t) = t^{(r-1)/2} \prod_{i \in I} \prod_{j \notin I} \frac{tx_i - x_j}{x_i - x_j},$$

$$T_I = \prod_{i \in I} T_{q,x_i}, \quad (T_{q,x_i})(x_1, \ldots, x_n) = f(x_1, \ldots, qx_i, \ldots, x_n),$$

and $I \subset \{1, \ldots, n\}$ runs over subsets of cardinality $r$. Let $D_n(X;q,t) = \sum_{r=0}^n D_n^r X^r$ be their generating function.

For a partition $\lambda \in \pi_n$ the Macdonald polynomial $\{P_\lambda\}$ is defined as a unique eigenvector of $D_n(X;q,t)$ of the form

$$P_\lambda = m_\lambda + \sum_{\mu < \lambda} u_{\lambda\mu} m_\mu \quad (u_{\lambda\mu} \in \mathbb{C}(q,t)),$$

where

$$m_\lambda = \prod_{i=1}^n \frac{1}{m_i!} \sum_{w \in \mathcal{S}_n} w \left( x_1^{\lambda_1} \cdots x_n^{\lambda_n} \right)$$

is the monomial symmetric function. Here a permutation $w \in \mathcal{S}_n$ acts on a function by permuting the arguments $w f(x_1, \ldots, x_n) = f(x_{w^{-1}(1)}, \ldots, x_{w^{-1}(n)})$.

Such a polynomial is unique and the coefficients $u_{\lambda\mu}$ are rational functions of $q$ and $t$ with possible poles of the form

$$q^a t^b = 1 \quad (a \in \mathbb{Z}_{\geq 0}, b \in \mathbb{Z}_{> 0}),$$

see [Int].

The corresponding eigenvalues are given by the formula:

$$D_n(X;q,t) P_\lambda = \prod_{i=1}^n (1 + Xq^\lambda t^{n-i}) P_\lambda,$$

see [M], VI,3,4.

Note that the degree of $P_\lambda$ is equal to the weight of $\lambda$. 

2.3. **Hall-Littlewood polynomials.** Let \( \lambda \) be a partition of length \( n \) and let \( t \in \mathbb{C} \). The **Hall-Littlewood polynomial** \( P_\lambda(x_1, \ldots, x_n; t) \) is defined by

\[
P_\lambda(x_1, \ldots, x_n; t) = \prod_{i=1}^{n} \prod_{j=1}^{m_i} \frac{1-t^{i-j}}{1-t} \sum_{w \in S_n} w \left( x_1^{\lambda_1} \cdots x_n^{\lambda_n} \prod_{1 \leq i < j \leq n} \frac{x_i - tx_j}{x_i - x_j} \right).
\]

(4)

We will construct a basis of \( F_n^{(k,2)} \) by using these polynomials in Proposition 3.3.

It is well known that for generic \( t \) the function \( P_\lambda \) is a symmetric polynomial of the form

\[
P_\lambda = m_\lambda + \sum_{\mu < \lambda} a_{\lambda \mu} m_\mu, \quad a_{\lambda \mu} \in \mathbb{C},
\]

(5)

see [M], III 1.2.

It is also well known that the Hall-Littlewood polynomials are obtained from the Macdonald polynomials by setting \( q = 0 \): \( P_\lambda(x_1, \ldots, x_n; t) = P_\lambda(x_1, \ldots, x_n; 0, t) \), see [M], p.324. Although for \( r = 2 \) we are dealing with the case \( q = 1 \) in the wheel condition, the solution is given in terms of the special case of Macdonald polynomials with \( q = 0 \). This is not surprising because as we have already remarked the root of unity case is not a specialization of the generic case. (See Proposition 3.10.)

### 3. The case \( r = 2 \)

3.1. **Preliminaries.** Let \( r = 2 \). Then we have \( q = 1 \), \( t \) a primitive root of unity of order \( k + 1 \), the wheel set is \( \{ t \} \), and \( F_n^{(k,2)} \) is the space of symmetric functions satisfying a single condition:

\[
f(x, tx, t^2x, \ldots, t^kx, x_{k+2}, \ldots, x_n) = 0.
\]

(6)

The condition (6) appears as the initial condition for the recurrence relation of “deformed cycles” in the construction of form factors in \( SU(k+1) \) invariant Thirring model with values in tensor products of vector representations, see Proposition 7.2 in [T].

Consider the example \( k = 1 \) and \( r = 2 \). We have \( t = -1 \), and the space \( F_n^{(1,2)} \) consists of polynomials of the form

\[
\prod_{1 \leq i < j \leq n} (x_i + x_j)g(x_1, \ldots, x_n),
\]

where \( g \) is any symmetric function. In the generic case for \( k = 1 \) and \( r = 2 \) the wheel set is \( \{ t, t^{-1} \} \) with \( t \neq -1 \). The functions satisfying the wheel condition have the form

\[
\prod_{1 \leq i < j \leq n} (x_i - tx_j)(x_j - tx_i)g(x_1, \ldots, x_n).
\]

The space \( F_n^{(1,2)} \) is greater than the space of functions for the generic case in the limit \( t = -1 \).
3.2. The space $E^{(k,2)}$. We describe the dual space $E^{(k,2)}_n = (F^{(k,2)}_n)^*$ to $F^{(k,2)}_n$.

Let $F_n = \mathbb{C}[x_i]_{i=1}^{\pi_n}$ be a commutative algebra of symmetric polynomials in variables $x_1, \ldots, x_n$. We have $F_n^{(k,2)} \subset F_n$. The space $F_n$ has a basis of elementary monomial functions $\{m_\lambda\}_{\lambda \in \pi_n}$. We use the usual degree, i.e., $\deg x_i = 1$ and the degree of $m_\lambda$ is equal to the weight of partition $\lambda$.

Let $E = \mathbb{C}[e_i]_{i \in \mathbb{Z}_{\geq 0}}$ be a commutative algebra of polynomials in variables $e_i$. For a partition $\lambda = (\lambda_1, \ldots, \lambda_n) \in \pi_n$, $\lambda_1 \geq \cdots \geq \lambda_n$, set $e_\lambda = \prod_{i=1}^n e_{\lambda_i}$. We define the degree of $e_i$ to be $i$. Then, the degree of $e_\lambda$ is equal to the weight of $\lambda$. For $\mu \in \pi_m$ and $\nu \in \pi_n$ we define $\lambda = \mu \cup \nu \in \pi_{m+n}$ by joining the parts of $\mu$ and $\nu$. Then, we have $e_\lambda = e_\mu e_\nu$.

Monomials $e_\lambda \in \pi_n$ are linearly independent. Let $E_n \subset E$ be the subspace with the basis $\{e_\lambda\}_{\lambda \in \pi_n}$.

Let

$$e(z) = \sum_{i \geq 0} e_i z^i,$$

be a formal power series in $z$. Then $E_n$ is spanned by the coefficients of power series $\prod_{i=1}^n e(z_i)$. Moreover, we have

$$e(z_1) \cdots e(z_n) = \sum_{\lambda \in \pi_n} e_\lambda m_\lambda(z_1, \ldots, z_n).$$

Define a bilinear pairing $\langle \cdot, \cdot \rangle : E_n \otimes F_n \to \mathbb{C}$ by setting

$$\langle e(z_1) \cdots e(z_n), f(x_1, \ldots, x_n) \rangle = f(z_1, \ldots, z_n),$$

for any symmetric polynomial $f(x_1, \ldots, x_n) \in F_n$.

The following lemma is clear.

**Lemma 3.1.** The pairing $\langle \cdot, \cdot \rangle$ is a well defined bilinear nondegenerate pairing of graded spaces. Moreover, the bases $e_\lambda$ and $m_\lambda$ are dual, $\langle e_\lambda, m_\mu \rangle = \delta_{\lambda\mu}$. □

Let $t$ be a primitive root of unity of order $k + 1$. We have a subspace $F_n^{(k,2)} \subset F_n$. Let $J$ be the space spanned by the coefficients of formal power series $\prod_{i=0}^k t^i z$.

**Lemma 3.2.** The orthogonal complement $(F_n^{(k,2)})^\perp \subset E_n$ with respect to the pairing $\langle \cdot, \cdot \rangle$ coincides with the subspace $J \cdot E_{n-k-1}$.

**Proof.** For $f \in F_n$, we have

$$\langle \prod_{i=0}^k (t^i z) \prod_{i=k+2}^n e(z_i), f(x_1, \ldots, x_n) \rangle = f(z, tz, \ldots, t^k z, z_{k+2}, \ldots, z_n) = 0,$$

if and only if $f \in F_n^{(k,2)}$. Therefore $(J \cdot E_{n-k-1})^\perp = F_n^{(k,2)}$. Since the graded components are finite-dimensional and the pairing respects the grading, we obtain $(J \cdot E_{n-k-1})^{\perp \perp} = J \cdot E_{n-k-1}$ and the lemma is proved. □

Denote by $E_n^{(k,2)}$ the quotient space of $E_n$ by the space $J \cdot E_{n-k-1}$,

$$E_n^{(k,2)} = E_n / (J \cdot E_{n-k-1}).$$
Since the space of relations is graded, the subspace $E_n^{(k,2)}$ inherits grading from $F_n$.

The following is clear.

Lemma 3.3. The pairing $\langle \cdot, \cdot \rangle$ induces a well defined nondegenerate bilinear pairing of graded spaces

$$\langle \cdot, \cdot \rangle : E_n^{(k,2)} \otimes F_n^{(k,2)} \rightarrow \mathbb{C}.$$ 

3.3. The spanning set of $E_n^{(k,2)}$. Denote the series $\prod_{i=0}^{k} e(t^i z)$ by $E_k(z)$. We have $E_k(tz) = E_k(z)$ and therefore $E_k(z)$ has the form

$$E_k(z) = \sum_{i \geq 0} \epsilon_i z^{i(k+1)} \quad (\epsilon_i \in E_{k+1}).$$

We have

$$\epsilon_i = (-1)^{ik} \epsilon_i^{k+1} + \sum_{\lambda \succ (i^{k+1})} c_{\lambda,i} e_\lambda \quad (c_{\lambda,i} \in \mathbb{C}). \quad (7)$$

Note that $\epsilon_i^{k+1} = e_{(i^{k+1})}$.

We denote $\bar{e}$ the image of an element $e \in E_n$ in the quotient space $E_n^{(k,2)}$.

Lemma 3.4. The elements $\{\bar{e}_\lambda\}$ with admissible partitions $\lambda \in \pi_n$, i.e., those satisfying the condition $m_i(\lambda) \leq k$ for all $i$, span $E_n^{(k,2)}$.

Proof. We fix nonnegative integers $d, n$ and work with a finite-dimensional space generated by $e_\lambda$ with partitions $\lambda \in \pi_n$ of weight $d$.

We claim that if $\lambda$ is nonadmissible, i.e., $m_i > k$ for some $i \geq 0$, then $\bar{e}_\lambda$ is a linear combination of $e_\nu$ such that $\nu \succ \lambda$.

Indeed, if $i = 0$ then $\bar{e}_\lambda = 0$ and there is nothing to prove. Otherwise let $\tilde{\lambda} \in \pi_{n-k-1}$ be a partition obtained from $\lambda$ by deleting $k + 1$ parts equal to $i$. Then we have $e_\lambda = e_{(i^{k+1})} e_{\tilde{\lambda}}$ where $e_{\tilde{\lambda}} \in E_{n-k-1}$.

We use the relation $\bar{e}_i \bar{e}_{\tilde{\lambda}} = 0$ and get (see (7))

$$\bar{e}_\lambda = \bar{e}_{(i^{k+1})} \bar{e}_{\tilde{\lambda}} = -(-1)^{ik} \sum_{\rho \succ (i^{k+1})} c_{\rho,i} \bar{e}_{\rho \cup \tilde{\lambda}};$$

and our claim is proved.

Now the lemma follows. Indeed, we rewrite $\bar{e}_\lambda$ in terms of $\bar{e}_\nu$ with larger $\nu$ (with respect to $\succ$ ordering). Then we rewrite nonadmissible $\bar{e}_\rho$ appearing in this sum in terms of $\bar{e}_\rho$ with even larger partitions $\rho$ and so on. Since our space is finite-dimensional, after finitely many repetitions we obtain a sum with only admissible partitions. \qed

In fact we will show (see Corollary 3.7) that the elements $\{\bar{e}_\lambda\}$ with admissible partitions $\lambda \in \pi_n$ are linearly independent and therefore form a basis in $E_n^{(k,2)}$.
3.4. A basis in $F^{(k,2)}$. Let $t$ be a primitive root of unity of order $k + 1$.

**Proposition 3.5.** The set of Hall-Littlewood polynomials $\{P_\lambda(x_1, \ldots, x_n; t)\}$, where $\lambda$ ranges over all admissible partitions is a basis of $F^{(k,2)}$.

**Proof.** The admissibility of $\lambda$ is nothing but $m_\lambda(\lambda) \leq k$. It implies that polynomials $\{P_\lambda(x_1, \ldots, x_n; t)\}$ have no pole when the variable $t$ in (4) specializes to the root of unity, $t^{k+1} = 1$. Because of (5) they are linearly independent. It is also clear from the definition that they satisfy the condition (6), since every term in (4) satisfies it.

On the other hand, by Lemma 3.3, the dimension of the space of polynomials in $F^{(k,2)}$ of degree $d$ is equal to the dimension of the subspace of degree $d$ in $E^{(k,2)}$. By Lemma 3.4 it is bounded from above by the number of admissible partitions in $\pi_n$ of weight $d$.

Since we have as many linearly independent function $P_\lambda$ of degree $d$ as the number of admissible partitions in $\pi_n$ of weight $d$, they form a basis and the bounds are actually equalities.  

3.5. The character $\chi_{k,2}$. From Proposition 3.5 we have the following immediate corollary.

**Proposition 3.6.** The character of the space $F^{(k,2)}$ is given by

$$\chi_{k,2}(z, v) = \prod_{s=0}^\infty (1 + v^s z + v^{2s} z^2 + \cdots + v^{ks} z^k) = \prod_{s=0}^\infty \frac{1 - (v^s z)^k + 1 + v^s z}{1 - v^s z}. \quad (8)$$

Note that according to [FJMM1] for generic $t$ a basis in the space of functions satisfying the wheel condition related to the wheel set $\{t, t^{-k-1}\}$ is given by Macdonald polynomials parametrized by $(k,2,n)$-admissible partitions. The corresponding character cannot be written in such a simple factored form.

Now from the comparison of dimensions we also obtain:

**Corollary 3.7.** The elements $\{\bar{e}_\lambda\}$ with admissible partitions $\lambda \in \pi_n$ form a basis of $E^{(k,2)}$.  

Define formal power series in $v$, $b_n^{(k)}(v)$, by

$$\chi_{k,2}(z, v) = \sum_{n=0}^\infty b_n^{(k)}(v) z^n.$$

For $k = 1$, we have the following result.

**Lemma 3.8.** We have

$$b_n^{(1)}(v) = \frac{v^{n(n-1)/2}}{\prod_{i=1}^n (1 - v^i)}.$$

**Proof.** The lemma follows from the identities

$$\prod_{s=0}^\infty (1 + v^s z) = \sum_{n=0}^\infty \left( \sum_{\lambda} v^{\lambda} \right) z^n = \sum_{n=0}^\infty \frac{v^{n(n-1)/2}}{\prod_{i=1}^n (1 - v^i)} z^n,$$
where in the second expression the sum is over all partitions $\lambda \in \pi_n$ such that $\lambda_i > \lambda_{i+1}$, $i = 1, \ldots, n - 1$.

Now we give a formula for the coefficients $b_n^{(k)}$.

**Lemma 3.9.** We have

$$b_n^{(k)}(v) = \sum_{a,b; (k+1)a+b=n} (-1)^a v^{(k+1)a(a-1)/2} \prod_{i=1}^a (1 - v^{(k+1)i}) \prod_{j=1}^b (1 - v^j).$$

**Proof.** We have the identity

$$\prod_{s=0}^{\infty} (1 - v^s z) = \sum_{b=0}^{\infty} \left( \sum_{\lambda \in \pi_b} v^{\lambda|} \right) z^b = \sum_{b=0}^{\infty} \prod_{j=1}^b (1 - v^j).$$

We obtain the lemma multiplying this identity by the identity

$$\prod_{s=0}^{\infty} (1 - (v^s z)^{k+1}) = \chi_{1,2}(-z^{k+1}, v^{k+1}) = \sum_{a=0}^{\infty} \frac{(-1)^a v^{(k+1)a(a-1)/2} z^{(k+1)a}}{\prod_{i=1}^a (1 - v^{(k+1)i})},$$

which follows from Lemma 3.8. \hfill \Box

3.6. **Other bases in $F^{(k,2)}$.** The following more general proposition is proved by the methods of [FJMM2], Theorem 2.4. We skip the details of the proof and do not use this result in any other part of the paper.

**Proposition 3.10.** Let $t$ be a primitive root of unity of order $k + 1$. The set of Macdonald polynomials $\{P(x_1, \ldots, x_n; t, q)\}$ where $\lambda$ ranges over all admissible partitions is a basis of $F_n^{(k,2)}$ if $q$ is not a root of unity. \hfill \Box

Proposition 3.5 is just $q = 0$ case of Proposition 3.10.

4. **The case $r > 2$**

4.1. **Frobenius homomorphism.** Fix $k \in \mathbb{Z}_{>0}$, $r \in \mathbb{Z}_{>1}$ such that $k+1$ and $r-1$ are relatively prime. Fix primitive roots of unity $t$ and $q$ of order $k+1$ and $r-1$, respectively. We consider the wheel set $S_{r-1}(q, t) = \{t, tq, \ldots, tq^{r-2}\}$. It is invariant under the multiplication by $q$.

The space $F^{(k,r)}$ is the space of symmetric polynomials $f(x_1, \ldots, x_n)$ which vanish at

$$x_i = r^{i-1}q^{s_i}x_1 \quad (1 < i \leq k + 1),$$

for all $s_2, \ldots, s_{k+1} \in \{0, 1, \ldots, r - 2\}$.

Define the *Frobenius homomorphism*

$$\mathcal{F} : \mathbb{C}[x_1, \ldots, x_n] \to \mathbb{C}[x_1, \ldots, x_n],$$

$$f(x_1, \ldots, x_n) \mapsto f(x_1^{-1}, \ldots, x_n^{-1}).$$

We have an obvious lemma.
Lemma 4.1. The Frobenius homomorphism induces an imbedding of the vector spaces
\[ \mathcal{F}: \ F^{(k,2)} \to F^{(k,r)}. \]

We denote the image \( \mathcal{F}(F^{(k,2)}) \) by \( R \).

Let \( F_n^{(k,r)} \subset F^{(k,r)} \) be the subspace of polynomials with \( n \) variables. Let \( R_n = R \cap F_n^{(k,r)} \).

A symmetric polynomial \( f \in F_n^{(k,r)} \) is in the Frobenius image \( R_n \subset F_n^{(k,r)} \) if and only if there exists a symmetric polynomial \( \tilde{f}(x_1^r, x_2^r, \ldots, x_n^r) \) such that \( f(x_1, x_2, \ldots, x_n) = \tilde{f}(x_1^r, \ldots, x_n^r) \). In such a case \( \tilde{f} \in F_n^{(k,2)} \).

The space \( F_n^{(k,r)} \) is an ideal in the algebra of symmetric polynomials. We view the Frobenius image \( R_n \) as a subring in \( \mathbb{C}[x_1, \ldots, x_n] \).

Our goal is to prove that \( F_n^{(k,r)} \) is a free \( R \)-module of rank \( N = (r-1)^n \) and to describe the generating set (see Proposition 4.5).

4.2. Slim partitions. We describe the division with remainders of partition by positive integers.

By definition, the result of addition of two partitions \( \lambda, \mu \in \pi_n \) is a partition \( (\lambda+\mu) \in \pi_n \) with components \( \lambda_i + \mu_i, \ i = 1, \ldots, n \). In particular, the result of multiplication of a partition \( \lambda \) by a positive integer \( r-1 \) is the partition \( (r-1)\lambda \) with components \( (r-1)\lambda_i \). A partition \( \lambda \) is called divisible by a positive integer \( r-1 \) if all parts \( \lambda_i \) are divisible by \( r-1 \).

Let a symmetric polynomial \( f(x_1, x_2, \ldots, x_n) \) be expressed in terms of monomial functions \( f = \sum c_{\lambda} m_{\lambda}, \ c_{\lambda} \in \mathbb{C} \). Then \( f(x_1, x_2, \ldots, x_n) = \tilde{f}(x_1^{r-1}, \ldots, x_n^{r-1}) \) for some symmetric polynomial \( \tilde{f} \) if and only if \( c_{\lambda} = 0 \) for all \( \lambda \) which are not divisible by \( r-1 \).

A partition \( \lambda \in \pi_n \) is called \((r-1)\)-slim if \( \lambda_i - \lambda_{i+1} < r-1 \) for \( i = 1, \ldots, n-1 \) and \( \lambda_n < r-1 \). We denote the subset of \( \pi_n \) consisting of all \((r-1)\)-slim partitions by \( \pi_n^s \).

The cardinality of \( \pi_n^s \) is \( N = (r-1)^n \).

The following lemma is straightforward.

Lemma 4.2. Let \( \lambda \in \pi_n \). There is a unique way to represent \( \lambda \) in the form
\[ \lambda = (r-1)\mu + \nu, \]
where \( \mu, \nu \in \pi_n \) are partitions and \( \nu \) is \((r-1)\)-slim. □

The partitions \( \mu, \nu \) in the lemma are called the quotient and the remainder of partition \( \lambda \) divided by integer \( r-1 \).

4.3. Wheel condition is broken if the highest partition is slim. A symmetric polynomial \( f(x_1, \ldots, x_n) \) is called of highest partition \( \lambda \) if it has the form
\[ f = c_{\lambda} m_{\lambda} + \sum_{\mu < \lambda} c_{\mu} m_{\mu}, \quad (c_{\mu} \in \mathbb{C}), \]
where \( c_{\lambda} \) is a nonzero complex number.
Let $\mu \in \pi_n$ be an $(r - 1)$-slim partition and let $g_\mu$ be of highest partition $\mu$. We claim that $g$ does not satisfy the wheel condition, that is $g \not\in F^{(k,r)}$. We prove a slightly stronger statement.

**Proposition 4.3.** Let $\mu \in \pi_n^a$ be an $(r - 1)$-slim partition and let $g_\mu$ be of highest partition $\mu$. If $n > k$ then for any non-zero complex number $c$ there exist $s_1, \ldots, s_k \in \{0,1,\ldots,r-2\}$ such that $g(cq^{s_1}, ctq^{s_2}, \ldots, ct^kq^{s_k+1}, x_{k+2}, \ldots, x_n) \neq 0$.

**Proof.** We fix a non-zero complex number $c$. Let $\mathbb{C}^N (N = (r-1)^n)$ be the vector space with basis $\{u_s\}$ where the index $s = (s_1, \ldots, s_n)$ ranges over all sequences such that $s_i \in \{0, \ldots, r-2\}$. Let $m = n - k - 1 \geq 0$. For $y = (y_1, \ldots, y_m) \in \mathbb{C}^m$, define the evaluation map

$$\kappa_y : \mathbb{C}[x_1, \ldots, x_n]^{\mathcal{S}_n} \to \mathbb{C}^N$$

$$f \mapsto \sum_s f(cq^{s_1}, ctq^{s_2}, \ldots, ct^kq^{s_k+1}, y_1q^{s_k+2}, \ldots, y_mq^n)u_s.$$

We need to show that there exists $y$ such that $\kappa_y(g_\mu) \neq 0$.

If $f \in F_n^{(k,r)}$ then $\kappa_y(f) = 0$ for all $y$. If $f$ is in the image of the Frobenius homomorphism then $\kappa_y(f) = a(y) \sum u_s$ for some $a(y) \in \mathbb{C}$.

For each $\lambda \in \pi_n$ choose a symmetric polynomial $f_\lambda$ with highest partition $\lambda$. Then $\{f_\lambda\}_{\lambda \in \pi_n}$ is a basis in $\mathbb{C}[x_1, \ldots, x_n]^{\mathcal{S}_n}$. For each slim partition $\mu \in \pi_n^a$ choose a symmetric polynomial $g_\mu$ with highest partition $\mu$. Then $\{g_\mu f_\lambda\}_{\lambda \in \pi_n, \mu \in \pi_n^a}$ is also a basis in $\mathbb{C}[x_1, \ldots, x_n]^{\mathcal{S}_n}$.

In particular, the algebra $\mathbb{C}[x_1, \ldots, x_n]^{\mathcal{S}_n}$ is a free module of rank $N$ with generators $g_\mu$ over the image of the Frobenius homomorphism. Therefore, if $\kappa_y(g_\mu) = 0$ for some slim partition $\mu$ and all $y$, then for all $y$ the map $\kappa_y$ is not surjective. We claim it is impossible.

Indeed, for generic choice of $(y_1, \ldots, y_m) \in \mathbb{C}^m$, the images of the $N$ evaluation points $(cq^{s_1}, ctq^{s_2}, \ldots, ct^kq^{s_k+1}, y_1q^{s_k+2}, \ldots, y_mq^n)$ in the quotient space $A = \mathbb{C}^n/\mathcal{S}_n$ are all distinct. Therefore the algebra of polynomials $\mathbb{C}[A] = \mathbb{C}[x_1, \ldots, x_n]^{\mathcal{S}_n}$ separates these points and the evaluation map is surjective. \hfill $\square$

### 4.4. The main results.

We still have $t$ and $q$ as in (3). Recall Macdonald’s operators $D_r^n$.

**Lemma 4.4.** If $f \in F_n^{(k,r)}$ satisfies the wheel condition then $D_r^n(q, \tilde{t})f$ satisfies the wheel condition for all $r \in \mathbb{Z}_{>0}$ and $\tilde{t} \in \mathbb{C}$.

**Proof.** If a function (not necessary symmetric) $f$ satisfies the wheel conditions then $T_{q,x_i}f$ also satisfies the wheel conditions for all $i = 1, \ldots, n$. The statement follows from this observation. \hfill $\square$

Consider a symmetric polynomial $h(x_1, \ldots, x_n)$ of the form

$$h(x_1, \ldots, x_n) = \sum_{i=1}^s f_i g_i,$$  \hspace{1cm} (11)
where the highest partitions of $g_i$ are all slim and distinct and $f_i = \mathcal{F}(\tilde{f_i})$ are all in the image of the Frobenius homomorphism.

**Proposition 4.5.** The polynomial $h$ is in $F_n^{(k,r)}$ if and only if $f_i \in R_n$, i.e., if and only if $\tilde{f}_i$ are in $F_n^{(k,2)}$.

**Proof.** The “if” part of the Proposition is obvious.

To prove the “only if” part, we first assume that $g_i = P_{\lambda^{(i)}}(q, \tilde{t})$ are Macdonald polynomials with some $\tilde{t} \in \mathbb{C}$ which is not a root of unity and all $\lambda^{(i)}$ are distinct and slim. Notice that the Macdonald polynomials are well defined.

We have

$$D_n(X; q, \tilde{t}) h = \sum_{i=1}^{s} D_n(X; q, \tilde{t}) (g_i; f_i) =$$

$$\sum_{i=1}^{s} f_i D_n(X; q, \tilde{t}) (g_i) = \sum_{i=1}^{s} \prod_{j=1}^{n} (1 + X q^{\lambda^{(i)}_j} \tilde{t}^{a_j}) f_i g_i,$$

also satisfies the wheel condition by Lemma 4.4. Since eigenvalues $\prod_{i=1}^{s} (1 + X q^{\lambda^{(i)}_j} \tilde{t}^{a_j})$, $i = 1, \ldots, s$, are all distinct, we conclude that $g_i; f_i$ satisfies the wheel condition for all $i$. Now, since for $g_i$ the wheel condition is broken by Proposition 4.3, we get $f_i$ is zero on one of the vanishing planes and therefore $\tilde{f}_i \in F_n^{(k,2)}$.

Now we prove the general case by induction on $s$. Let $\lambda^{(i)}$ be the highest partitions of $g_i$. Without loss of generality we assume that $\lambda^{(s)}$ is maximal in the set $\{\lambda^{(i)}\}$. Then $h$ can be written in the form

$$h = a P_{\lambda^{(s)}}(q, \tilde{t}) f_s + \sum_{\mu} P_{\mu}(q, \tilde{t}) f_{\mu} \quad (a \in \mathbb{C}, \ a \neq 0),$$

where $f_{\mu} = \mathcal{F}(\tilde{f}_{\mu})$ are in the image of Frobenius homomorphism and the sum is over slim partitions $\mu$ such that $\mu \neq \lambda^{(s)}$. Therefore, using the previous argument we conclude that $f_s \in R_n$. In particular $\sum_{i=1}^{s-1} f_i g_i = h - f_s g_s \in F_n^{(k,r)}$ and by the induction hypothesis we obtain $f_i \in R_n$ for all $i$.

Let $g_{\mu}, \mu \in \pi_n$, be any polynomials with highest partitions $\mu$. Let $\tilde{f}_{\lambda}$, where $\lambda$ ranges over admissible partitions, be a basis in $F_n^{(k,2)}$. We described some of such bases in Sections 3.2, 3.6. Let $f_{\lambda} = \mathcal{F}(\tilde{f}_{\lambda})$.

We have proved the following theorem.

**Theorem 4.6.** The polynomials $\{f_{\lambda} g_{\mu}\}$, where $\mu$ ranges over all $(r-1)$-slim partitions and $\lambda$ ranges over admissible partitions, form a basis in $F_n^{(k,r)}$.

As a corollary we obtain a description of the character $\chi_{k,r}(z, v)$ of the space $F^{(k,r)}$.

**Theorem 4.7.** The character of the space $F^{(k,r)}$ is given by

$$\chi_{k,r}(z, v) = \sum_{n=0}^{\infty} \left( b_n^{(k)} (v^{r-1}) \prod_{s=1}^{n} \frac{1 - v^{s(r-1)}}{1 - v^s} \right) z^n,$$

where the coefficients $b_n(v)$ are given by (3).
Proof. The $z^n$ term of the character $\chi_{k,r}(z,v)$ is the product of the character of $(r-1)$-slim partitions of length at most $n$ and the $z^n$ term of the character $\chi_{k,2}(z,v^{r-1})$. The character $\chi_{k,2}(z,v)$ is computed in Proposition 3.6, it’s $z_n$ term is $b_n^{(k)}(v)$ and the character of the $(r-1)$-slim partitions in $\pi_n$ is equal to the product $\prod_{s=1}^n(1-v^s)/(1-v^s)$. □

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