INTEGRABLE GEOMETRY and SOLITON EQUATIONS IN 2+1 DIMENSIONS

R. Myrzakulov

Centre for Nonlinear Problems, PO Box 30, 480035, Alma-Ata-35, Kazakhstan
E-mail: cnlpmyra@satsun.sci.kz

Abstract

Using the differential geometry of curves and surfaces, the L-equivalent soliton equations of the some (2+1) - dimensional integrable spin systems are found. These equations include the modified Novikov-Veselov, Kadomtsev- Petviashvili, Nizhnik-Novikov-Veselov and other equations. Some aspects of the connection between geometry and multidimensional soliton equations are discussed.
1 Introduction

The relation between geometry and soliton equations has been the subject of continued interest in recent years\[1-9\]. In general the connection between geometry and nonlinear partial differential equations (NPDE) has the long history (for a historical review, see, e.g., [4]). Spin systems (which are the important from physical and mathematical point of views subclass of NPDE) are a good laboratory to demonstrate and to understand the connection of geometry and NPDE. The first representative of integrable spin systems is the isotropic Landau-Lifshitz equation (LLE)\[2,10\]

\[\vec{S}_t = \vec{S} \wedge \vec{S}_{xx}\]  

(1)

where \( \vec{S}^2 = E = \pm 1 \). Here and hereafter subscripts denote partial derivatives. Pioneering work by Lakshmanan [2] showed that the LLE (1) is equivalent to the known nonlinear Schrodinger equation (NLSE)

\[iq_t + q_{xx} + 2E |q|^2 q = 0\]  

(2)

for the case \( E = +1 \) (for the case \( E = -1 \) the such equivalence was established in [17]). This equivalence in [17] we called the Lakshmanan equivalence or shortly \( \textbf{L-equivalence} \). As well known between equations (1) and (2) take place the gauge equivalence [15].

Let us, first, we briefly recall some of the well known facts on the geometrical formalism that used in [2], with the minor modifications (including the case \( E = -1 \)) of ref. [17]. Consider the motion of curves which are given by

\[
\begin{pmatrix}
\vec{e}_1 \\
\vec{e}_2 \\
\vec{e}_3
\end{pmatrix}_x = C \begin{pmatrix}
\vec{e}_1 \\
\vec{e}_2 \\
\vec{e}_3
\end{pmatrix},
\]

(3a)

\[
\begin{pmatrix}
\vec{e}_1 \\
\vec{e}_2 \\
\vec{e}_3
\end{pmatrix}_t = G \begin{pmatrix}
\vec{e}_1 \\
\vec{e}_2 \\
\vec{e}_3
\end{pmatrix},
\]

(3b)

with

\[
C = \begin{pmatrix}
0 & k & 0 \\
-Ek & 0 & \tau \\
0 & -\tau & 0
\end{pmatrix}, \quad G = \begin{pmatrix}
0 & \omega_3 & -\omega_2 \\
-E\omega_3 & 0 & \omega_1 \\
E\omega_2 & -\omega_1 & 0
\end{pmatrix}.
\]

Hence we have

\[C_t - G_x + [C,G] = 0\]  

(4)

Here \( \vec{e}_1, \vec{e}_2, \vec{e}_3 \) denote respectively the unit tangent, normal and binormal vectors, defined in the usual way; \( k \) and \( \tau \) are respectively the curvature and torsion of the curve. Note that (3a) is the usual Serret-Frenet equation (SFE). If \( \vec{S} = \vec{e}_1 \), then the function \( q = \frac{1}{2} e^{-\alpha^2 \tau} \) satisfies the NLSE (2), that is, equations (1) and (2) are L-equivalent each to other.
As known recently many efforts have been made to study the (2+1)-dimensional integrable NPDE\[11-14\]. Here we have the interesting fact: the (1+1)-dimensional integrable NPDE admit some number (not one) integrable (and nonintegrable) (2+1)-dimensional generalizations. So, for example, the LLE(1) has the following (2+1)-dimensional integrable and nonintegrable extensions:

1°. The Myrzakulov I (M-I) equation\[17\]

\[
\vec{S}_t = (\vec{S} \wedge \vec{S}_y + u\vec{S})_x \tag{5a}
\]

\[
u_x = -\vec{S}(\vec{S}_x \wedge \vec{S}_y) \tag{5b}
\]

2°. The Myrzakulov VIII (M-VIII) equation\[17\]

\[
iS_t = \frac{1}{2}[S_{xx}, S] + iwS_x \tag{6a}
\]

\[
w_y = \frac{1}{4i}tr(S[S_x, S_y]) \tag{6b}
\]

3°. The Ishimori equation\[17\]

\[
iS_t + \frac{1}{2}[S, (\frac{1}{4}S_{xx} + \alpha^2S_{yy})] + iu_yS_x + iu_xS_y = 0 \tag{7a}
\]

\[
\alpha^2 u_{yy} - \frac{1}{4}u_{xx} = \frac{\alpha^2}{4i}tr(S[S_y, S_x]) \tag{7b}
\]

4°. The Myrzakulov IX (M-IX) equation\[17\]

\[
iS_t + \frac{1}{2}[S, M_1 S] + A_2 S_x + A_1 S_y = 0 \tag{8a}
\]

\[
M_2u = \frac{\alpha^2}{4i}tr(S[S_y, S_x]) \tag{8b}
\]

5°. The Myrzakulov XVIII (M-XVIII) equation\[17\]

\[
iS_t + \frac{1}{2}[S, (\frac{1}{4}S_{xx} - \alpha(2b + 1)S_{xy} + \alpha^2S_{yy})] + A_{20} S_x + A_{10} S_y = 0 \tag{9a}
\]

\[
\alpha^2 u_{yy} - \frac{1}{4}u_{xx} = \frac{\alpha^2}{4i}tr(S[S_y, S_x]) \tag{9b}
\]

6°. The (2+1)-dimensional LLE

\[
\vec{S}_t = \vec{S} \wedge (\vec{S}_{xx} + \vec{S}_{yy}) \tag{10}
\]

All of these equations in 1+1 dimension reduce to the LLE(1). Note that here the Ishimori(7), M-I(5), M-VIII(6), M-IX(8) and M-XVIII(9) equations are integrable, at the same time equation (10) is not integrable.

In the study of the (2+1)-dimensional NPDE naturally arise the following questions:

1°. What is the analog of the SFE(3a) in 2+1 dimensions?
Answer: The analog of the SFE(3a) in 2+1 dimensions is the following set of equations:

\[
\begin{pmatrix}
\vec{e}_1 \\
\vec{e}_2 \\
\vec{e}_3
\end{pmatrix}_x = C
\begin{pmatrix}
\vec{e}_1 \\
\vec{e}_2 \\
\vec{e}_3
\end{pmatrix},
\begin{pmatrix}
\vec{e}_1 \\
\vec{e}_2 \\
\vec{e}_3
\end{pmatrix}_y = D
\begin{pmatrix}
\vec{e}_1 \\
\vec{e}_2 \\
\vec{e}_3
\end{pmatrix}
\]

with

\[
C = \begin{pmatrix}
0 & k & 0 \\
-Ek & 0 & \tau \\
0 & -\tau & 0
\end{pmatrix},
D = \begin{pmatrix}
0 & m_3 & m_2 \\
Em_3 & 0 & m_1 \\
Em_2 & -m_1 & 0
\end{pmatrix}.
\]

(11)

In [17], this set of equations was called the Serret-Frenet-Myrzakulov equation (SFME). [Compare the SFME(11) with the usual SFE(3a)]. Note that from the SFME(11) follows the following conditions

\[C_y - D_x + [C, D] = 0,
\]

(13a)

or in terms of elements

\[m_1 = \partial_x^{-1}(\tau y + Ekm_2),
\]

(13b)

\[m_2 = \partial_x^{-1}(km_1 - \tau m_3)
\]

(13c)

\[m_3 = \partial_x^{-1}(k_y - \tau m_2).
\]

(13d)

These conditions play a key role in the geometrical interpretation of the (2+1) - dimensional NPDE from the integrability point of view. In particular, as it seems to us, these conditions suggest to us, how find the integrable reductions of the (2+1) - dimensional NPDE. Let us return to questions.

2°. What is the (2+1)-dimensional version of the L-equivalence?

3°. How find the L-equivalent counterparts of the known and new integrable and nonintegrable NPDE?

and so on. May be the some answers of the some these questions were given in [17]. It should be noted that the SFME(11) plays a key role in our construction. The other new moment of our formalism, that is, it contain both cases: the focusing version when \( E = +1 \) (the Euclidean case) and the defocusing case when \( E = -1 \) (the Minkowski case).

Third, the central element of our construction is the Myrzakulov - 0(M-0) equation. Let us consider a \( n \) - dimensional space \( \mathbb{R}^n \) with the unit basic vectors \( e_j, j = 1, 2, ..., n \). Let \( \vec{S} \equiv \vec{e}_1, \vec{S}^2 \equiv \vec{e}_1^2 = E = \pm 1 \). Then, for example, the (3+1) - dimensional M-0 equation reads as [9]

\[
\vec{S}_t = \sum_{j=2}^{n} a_j \vec{e}_j
\]

(14a)

\[
\vec{S}_x = \sum_{j=2}^{n} b_j \vec{e}_j
\]

(14b)

\[
\vec{S}_y = \sum_{j=2}^{n} c_j \vec{e}_j
\]

(14c)
In this paper we consider the some integrable reductions of the (2+1) -
dimensional M-0 equation. Using the geometrical formalism the L-equivalent
counterparts of the some (2+1) - dimensional integrable spin systems were
found.

2 The Myrzakulov XVII equation

The Myrzakulov XVII(M-XVII) equation

\[ S_t = \frac{1}{4} S_{xxx} - \frac{3}{4} S_{xyy} + C_1 S_x + C_2 S_y + C_3 S, \]  

\[ V_x + i V_y = \frac{1}{8} [(S_x^2 + S_y^2)_x - i(S_x^2 + S_y^2)_y] \]  

was introduced in [17] and arises from the compatibility conditions of the linear
problem

\[ \Phi_y = S \Phi_x, \quad \Phi_t = \Phi_{xxx} + B_2 \Phi_{xx} + B_1 \Phi_x \]  

Here

\[ B_2 = \frac{3}{2} S S_x, \quad B_1 = \frac{3}{4} S S_{xx} + \frac{3}{16} (S_x^2 + S_y^2 + 8V + 8V^2) + \frac{3}{2} (V - V) S + \frac{3i}{8} S \{ S_x, S_y \} + \frac{3i}{4} S_{xy} \]  

\[ C_1 = \frac{3}{16} (S_x^2 + S_y^2 + 8V + 8V^2), \quad C_2 = \frac{3}{4} (2iV - 2iV - S_x S_y), \]  

\[ C_3 = -\frac{3}{2} \{ S + V - \frac{3}{8} S_x^2 \}_x + \frac{1}{2} (S_x S_y)_y \} S = \frac{1}{4} S (3 S_{xyy} - S_{xxx}), \]  

\[ \vec{S} = (S_1, S_2) \]  

is the spin vector, \( S^2 = E = \pm 1 \), \( V \) is scalar function, \( S = \sum_{k=1}^3 S_k \sigma_k (S_3 = 0) \), \( \sigma_k \) are Pauli matrix,

\[ \sigma_1 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \quad \sigma_2 = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}, \quad \sigma_3 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}. \]

It is convenient sometimes the following form of the M-XVII equation

\[ \vec{S}_t = \vec{S}_{zzz} + \vec{S}_{zz} + C^- \vec{S}_z + C^+ \vec{S}_\bar{z} + C_3 \vec{S}, \]  

\[ V_z = \frac{1}{2} (\vec{S}_z \vec{S}_\bar{z} )_z \]  

where \( z = x + iy \).

3 The Serret-Frenet-Myrzakulov equation and the
mNVE as the L-equivalent counterpart of the M-
XVII equation

In this section we find the L-equivalent counterpart of the M-XVII equation(15).
To this purpose, following[17] we consider the motion of 2-dimensional curves.
The Serret-Frenet-Myrzaluov equation (SFME) in this case has the form [17]

\[
\left( \begin{array}{c}
\tilde{e}_1 \\
\tilde{e}_2
\end{array} \right)_x = C \left( \begin{array}{c}
\tilde{e}_1 \\
\tilde{e}_2
\end{array} \right),
\left( \begin{array}{c}
\tilde{e}_1 \\
\tilde{e}_2
\end{array} \right)_y = D \left( \begin{array}{c}
\tilde{e}_1 \\
\tilde{e}_2
\end{array} \right),
\] (18)

with

\[
C = \begin{pmatrix}
0 & k \\
-Ek & 0
\end{pmatrix},
D = \begin{pmatrix}
0 & m \\
-Em & 0
\end{pmatrix},
\] (19)

where \( m = \partial_x^{-1}k_y \). [Compare the SFME (18) with the usual SFE

\[
\left( \begin{array}{c}
\tilde{e}_1 \\
\tilde{e}_2
\end{array} \right)_x = C \left( \begin{array}{c}
\tilde{e}_1 \\
\tilde{e}_2
\end{array} \right),
C = \begin{pmatrix}
0 & k \\
-Ek & 0
\end{pmatrix}
\] ]

At the same time, the time evolution of curve is specified by

\[
\left( \begin{array}{c}
\tilde{e}_1 \\
\tilde{e}_2
\end{array} \right)_t = G \left( \begin{array}{c}
\tilde{e}_1 \\
\tilde{e}_2
\end{array} \right).
\] (20)

Here \( \tilde{e}_k \) are a unit basic vectors,

\[
G = \begin{pmatrix}
0 & \omega \\
-E\omega & 0
\end{pmatrix},
\] (21)

\( k \) is the curvature of curve. From these equations we have

\[
C_y - D_x + [C, D] = 0,
C_t - G_x + [C, G] = 0,
D_t - G_y + [D, G] = 0.
\] (22)

So we get

\[
k_t = \omega_x,
m_t = \omega_y.
\] (23)

Let \( \tilde{S} \equiv \tilde{e}_1 \). Then we obtain

\[
\omega = \frac{1}{4}(k_{xx} - 3k_{yy}) - \frac{1}{4}(k^3 - km^2) + c_1k + c_2m.
\] (24)

with

\[
c_1 = \frac{3}{16}(k^2 + m^2 + 8\bar{V} + 8V),
c_2 = \frac{3}{4}(2i\bar{V} - 2iV - km).
\]

Now introduce the following new real function

\[
q = [k^2/4 + (\partial_x^{-1}k_y)^2]^{\frac{1}{4}}
\] (25)

It is not difficult check that this function satisfies the following modified Novikov-Veselov equation (mNVE)

\[
q_t = (q_{zzz} + 3Vq_z + \frac{3}{2}V_zq) + (q_{zzz} + 3Vq_z + \frac{3}{2}V_zq)
\] (26a)

\[
V_z = (q^2)_z.
\] (26b)
where \( z = x + iy \). As well known this equation was introduced in [26] and is associated with the Lax representation

\[
L^{mNV} \Psi = \left( \frac{\partial}{\partial q} - q \frac{\partial}{\partial \bar{q}} \right) \Psi = 0 \quad (27a)
\]

\[
\Psi_t = (A^+ + A^-) \Psi \quad (27b)
\]

where

\[
A^+ = \partial^3 + 3 \begin{pmatrix} 0 & -q_z \\ 0 & V \end{pmatrix} \partial + \frac{3}{2} \begin{pmatrix} 0 & 2qV \\ 0 & V_z \end{pmatrix}, \quad A^- = \bar{\partial}^3 + 3 \begin{pmatrix} \bar{V} \bar{\partial} & 0 \\ q_{\bar{z}} & 0 \end{pmatrix} \bar{\partial} + \frac{3}{2} \begin{pmatrix} \bar{V}_z & 0 \\ -2q\bar{V} & 0 \end{pmatrix}.
\]

So we have proved that the mNVE(26) and the M-XVII equation(15) are L-equivalent each to other that coincide with the fact that these equations too are gauge equivalent each to other[16].

### 4 The related spin systems and their L-equivalents

#### 4.1 The Myrzakulov XI equation

The mNVE equation(26) is the some modification of the Novikov - Veselov equation(NVE)

\[
q_t = q_{zzz} + q_{zzz} + (Vq)_z + (\bar{V}q)_{\bar{z}}, \quad (28a)
\]

\[
V_z = 3(q^2)_z. \quad (28b)
\]

The NV equation(28) was introduced in [24-25]. In contrast with the mNVE, the NVE(28) is associated with the \((L, A, B)\)- triple

\[
\frac{\partial}{\partial t} L^{NV} + [L^{NV}, A] - BL^{NV} = 0, \quad (29)
\]

\[
L^{NV} = \bar{\partial} \bar{q} + q, \quad A = (\bar{\partial}^3 + V \partial) + (\bar{\partial}^3 + \bar{V} \bar{\partial}), \quad B = \partial V + \bar{\partial} \bar{V}. \quad (30)
\]

Let us consider the other form of the NVE

\[
q_t = \alpha q_{xxx} + \beta q_{yyy} - 3\alpha(vq)_x - 3\beta(wq)_y \quad (31a)
\]

\[
w_x = q_y \quad (31b)
\]

\[
v_y = q_x. \quad (31c)
\]

This equation is the compatibility condition of the linear problem

\[
\phi_{xy} = q\phi \quad (32a)
\]

\[
\phi_t = \alpha \phi_{xxx} + \beta \phi_{yyy} - 3\alpha v\phi_x - 3\beta w\phi_y \quad (32b)
\]

and introduced in [24-25]. Here note the NVE(31) is the Lakshmanan equivalent to the following (2+1)-dimensional integrable spin model - the so-called M-XI equation[17]

\[
\bar{S}_t = \frac{\omega}{k} \bar{S}_x \quad (33)
\]
with
\[ \omega = \alpha q_{xxx} + \beta q_{yyy} - 3\alpha(vq)_x - 3\beta(wq)_y \quad (34a) \]
\[ v_y = k \quad (34b) \]
\[ w_x = m \quad (34c) \]
where \( q = \partial_x^{-1} k \).

4.2 The Myrzakulov X equation
Similarly we can show that the M-X equation
\[ \vec{S}_t = (3k^2 + k_{xx} + 3\alpha^2 w)k^{-1}\vec{S}_x \quad (35a) \]
\[ w_{xx} = k_{yy} \quad (35b) \]
is equivalent to the famous Kadomtsev - Petviashvili equation (KPE)
\[ (q_t - 6qq_x - q_{xxx})_x = 3\alpha^2 q_{yy} \quad (36) \]
with \( q = k \).

5 Conclusion
We have discussed the differential geometrical approach to study the some properties of soliton equations in 2+1 dimensions. Using the geometrical formalism which was presented in [17], the L-equivalent counterparts of the some integrable (2+1) - dimensional spin systems are obtained. So we have shown that the Lakshmanan equivalent soliton equations of the M-XVII, M-XI and M-X equations are the well known modified Novikov - Veselov, Novikov - Veselov and Kadomtsev - Petviashvili equations respectively. Finally note that the some other aspects of the some spin systems were considered in [19-23].

6 Appendix A: On the some integrable (2+1)-dimensional spin systems
6.1 The Myrzakulov IX equation
We note that the M-XVII equation (15) is related with the Myrzakulov - IX (M-IX) hierarchy
\[ is_t + \frac{1}{2}[S,M_1 S] + A_2 S_x + A_1 S_y = 0 \quad (37a) \]
\[ M_2 u = \frac{\alpha^2}{4i} tr(S[S_y , S_x ]) \quad (37b) \]
where \( \alpha, b, a = \text{consts} \) and
\[ S = \begin{pmatrix} S_3 & rS^- \\ rS^+ & -S_3 \end{pmatrix}, \quad S^\pm = S_1 \pm iS_2, \quad S^2 = I, \quad r^2 = \pm 1 \]
\[ M_1 = \alpha^2 \frac{\partial^2}{\partial y^2} - 2\alpha(b - a) \frac{\partial^2}{\partial x \partial y} + (a^2 - 2ab - b) \frac{\partial^2}{\partial x^2}; \]
\[ M_2 = \alpha^2 \frac{\partial^2}{\partial y^2} - \alpha(2a + 1) \frac{\partial^2}{\partial x \partial y} + a(a + 1) \frac{\partial^2}{\partial x^2}; \]
\[ A_1 = 2i\{(2ab + a + b)u_x - (2b + 1)\alpha u_y\} \]
\[ A_2 = 2i\{(2ab + a + b)u_y - \alpha^{-1}(2a^2b + a^2 + 2ab + b)u_x\}. \]

These set of equations is integrable and the Lax representation of the M-IX equation (37) is given by:
\[ \alpha \Phi_y = \frac{1}{2} [S + (2a + 1)I] \Phi_x \quad (38a) \]
\[ \Phi_t = \frac{i}{2} [S + (2b + 1)I] \Phi_{xx} + \frac{i}{2} W \Phi_x \quad (38b) \]

with
\[ W_1 = W - W_2 = (2b + 1)E + (2b - a + \frac{1}{2})SS_x + (2b + 1)FS \]
\[ W_2 = W - W_1 = FI + \frac{1}{2}S_x + ES + \alpha SS_y \]
\[ E = -\frac{i}{2\alpha} u_x, \quad F = \frac{i}{2} \left( \frac{(2a + 1)u_x}{\alpha} - 2u_y \right) \]

It is well known that the M-IX equation (37) is equivalent to the following Zakharov equation [13]
\[ iq_t + M_1 q + v q = 0, \quad (39a) \]
\[ ip_t - M_1 p - v p = 0, \quad (39b) \]
\[ M_2 v = -2M_1(pq), \quad (39c) \]
where \( v = i(c_{11} - c_{22}), \quad p = \bar{E}q \). It is interest note that the M-IX equation (37) admits the some integrable reductions. Let us now present these particular integrable cases.

### 6.2 The Myrzakulov VIII equation

Let \( b = 0 \). Then equations (37) take the form
\[ iS_t = \frac{1}{2} [S_{\xi\xi}, S] + iwS_{\xi} \quad (40a) \]
\[ w_{\eta} = \frac{1}{4i} tr(S[S_{\xi}, S_{\eta}]) \quad (40b) \]

where
\[ \xi = x + \frac{a + 1}{\alpha} y, \quad \eta = -x - \frac{a}{\alpha} y, \quad w = u_{\xi}, \]
which is the M-VIII equation [17]. The equivalent counterpart of the M-VIII equation (40) we obtain from (39) as \( b = 0 \)
\[ iq_t + q_{\xi\xi} + v q = 0, \quad (41a) \]
\[ v_{\eta} = -2r^2(\bar{q}q)_{\xi}, \quad (41b) \]

which is the other Zakharov equation [13].
6.3 The Ishimori equation

Now consider the case: \( a = b = -\frac{1}{2} \). In this case equations (37) reduces to the well known Ishimori equation

\[
iS_t + \frac{1}{2}[S, (\frac{1}{4}S_{xx} + \alpha^2 S_{yy})] + iu_y S_x + iu_x S_y = 0 \quad (42a)
\]

\[\alpha^2 u_{yy} - \frac{1}{4}u_{xx} = \frac{\alpha^2}{4i} tr(S[S_y, S_x]) \quad (42b)\]

The equivalent counterpart of the equation (42) is the Davey-Stewartson equation

\[
iq_t + \frac{1}{4}q_{xx} + \alpha^2 q_{yy} + vq = 0 \quad (43a)
\]

\[\alpha^2 v_{yy} - \frac{1}{4}v_{xx} = -2\{\alpha^2 (pq)_{yy} + \frac{1}{4}(pq)_{xx}\} \quad (43b)\]

that follows from the ZE (39). Note that equations (42) and (43) are gauge equivalent each to other[11].

6.4 The Myrzakulov XVIII equation

Consider the reduction: \( a = -\frac{1}{2} \). Then (37) reduces to the M-XVIII equation[17]

\[
iS_t + \frac{1}{2}[S, (\frac{1}{4}S_{xx} - \alpha(2b + 1)S_{xy} + \alpha^2 S_{yy})] + A_{0} S_x + A_{10} S_y = 0 \quad (44a)
\]

\[\alpha^2 u_{yy} - \frac{1}{4}u_{xx} = \frac{\alpha^2}{4i} tr(S[S_y, S_x]) \quad (44b)\]

where \( A_{j0} = A_j \) as \( a = -\frac{1}{2} \). The corresponding gauge equivalent equation obtain from (39) and looks like

\[
iq_t + \frac{1}{4}q_{xx} - \alpha(2b + 1)q_{xy} + \alpha^2 q_{yy} + vq = 0 \quad (45a)
\]

\[\alpha^2 v_{yy} - \frac{1}{4}v_{xx} = -2\{\alpha^2 (pq)_{yy} - \alpha(2b + 1)(pq)_{xy} + \frac{1}{4}(pq)_{xx}\} \quad (45b)\]

Note that the Lax representations of equations (40), (42) and (44) we can get from (38) as \( b = 0, a = b = -\frac{1}{2} \) and \( a = -\frac{1}{2} \) respectively.

7 Appendix B: Differential geometry of surfaces and the M-XXII equation

Consider the Myrzakulov (M-XXII) equation[17]

\[-iS_t = \frac{1}{2}([S, S_y] + 2iuS)_x + \frac{i}{2} V_1 S_x - 2ib^2 S_y \quad (46a)\]

\[u_x = -\vec{S}(\vec{S}_x \wedge \vec{S}_y), \quad V_{1x} = \frac{1}{4b^2} (\vec{S}_x^2)_y, \quad (46b)\]
where \( \vec{S} = (S_1, S_2, S_3) \) is a spin vector, \( S^2 = E = \pm 1 \). These equations are integrable. The corresponding Lax representation is given by

\[
\Phi_x = \{-i(\lambda^2 - b^2)S + \frac{\lambda - b}{2b}SS_x\}\Phi \tag{47a}
\]

\[
\Phi_t = 2\lambda^2\Phi_y + \{(\lambda^2 - b^2)(2A + B) + (\lambda - b)C\}\Phi \tag{47b}
\]

with

\[
A = \frac{1}{4}([S, S_y] + 2iuS) + \frac{i}{4}V_1S, \quad B = \frac{i}{2}V_1S, \quad C = -\frac{V_1}{4b^2}SS_x + \frac{i}{2b}\{S_{xy} - [S_x, A]\}.
\]

Here spin matrix has the form

\[
S = \begin{pmatrix} S_3 & rS^- & -S_3 \\ rS^+ & S_3 & \end{pmatrix}, \quad S^2 = I, \quad r^2 = \pm 1, \quad S^\pm = S_1 \pm iS_2.
\]

Now find the Lakshmanan equivalent counterpart of the M-XXII equation (46) for the case \( E = +1 \) (for the case \( E = -1 \), see, e.g., [17]). To this end we can use the two geometrical approaches (D- and C-approaches). Let us use the C-approach, i.e., the surface approach. Consider the motion of surface in the 3-dimensional space which generated by a position vector \( \vec{r}(x, y, t) = \vec{r}(x_1, x_2, t) \).

According to the C-approach, \( x \) and \( y \) are local coordinates on the surface. The first and second fundamental forms in the usual notation are given by

\[
I = d\vec{r}d\vec{r} = Edx^2 + 2Fdx dy + Gdy^2, \quad II = -d\vec{r}d\vec{n} = Ldx^2 + 2Mdx dy + Ndy^2 \tag{48}
\]

where

\[
E = \vec{r}_x^2 = g_{11}, \quad F = \vec{r}_x\vec{r}_y = g_{12}, \quad G = \vec{r}_y^2 = g_{22},
\]

\[
L = \vec{n}\vec{r}_{xx} = b_{11}, \quad M = \vec{n}\vec{r}_{xy} = b_{12}, \quad N = \vec{n}\vec{r}_{yy} = b_{22}, \quad \vec{n} = (\vec{r}_x \wedge \vec{r}_y) / |\vec{r}_x \wedge \vec{r}_y|.
\]

In this case, the set of equations of the C-approach [17], becomes

\[
\vec{r}_t = W_1\vec{r}_x + W_2\vec{r}_y + W_3\vec{n} \tag{49a}
\]

\[
\vec{r}_{xx} = \Gamma^1_{11}\vec{r}_x + \Gamma^2_{11}\vec{r}_y + L\vec{n} \tag{49b}
\]

\[
\vec{r}_{xy} = \Gamma^1_{12}\vec{r}_x + \Gamma^2_{12}\vec{r}_y + M\vec{n} \tag{49c}
\]

\[
\vec{r}_{yy} = \Gamma^1_{22}\vec{r}_x + \Gamma^2_{22}\vec{r}_y + N\vec{n} \tag{49d}
\]

\[
\vec{n}_x = p_1\vec{r}_x + p_2\vec{r}_y \tag{49e}
\]

\[
\vec{n}_y = q_1\vec{r}_x + q_2\vec{r}_y \tag{49f}
\]

where \( W_j \) are some functions, \( \Gamma^k_{ij} \) are the Christoffel symbols of the second kind defined by the metric \( g_{ij} \) and \( g^{ij} = (g_{ij})^{-1} \) as

\[
\Gamma^k_{ij} = \frac{1}{2}g^{kl}(\frac{\partial g_{lj}}{\partial x^i} + \frac{\partial g_{li}}{\partial x^j} - \frac{\partial g_{ij}}{\partial x^l}) \tag{50}
\]

The coefficients \( p_i, q_i \) are given by

\[
p_i = -b_{1j}g^{ji}, \quad q_i = -b_{2j}g^{ji}. \tag{51}
\]
The compatibility conditions $\vec{r}_{xxy} = \vec{r}_{xyx}$ and $\vec{r}_{yyx} = \vec{r}_{xyy}$ yield the following Mainardi-Peterson-Codazzi equations (MPCE)

$$R^l_{ijk} = b_{ij}^l b_k^l - b_{ik} b_{ij}^l, \quad \frac{\partial b_{ij}}{\partial x^k} - \frac{\partial b_{ik}}{\partial x^j} = \Gamma^s_{ik} b_{js} - \Gamma^s_{ij} b_{ks} \quad (52)$$

where $b_{ij} = g^{jl} b_{il}$ and the curvature tensor has the form

$$R^l_{ijk} = \frac{\partial \Gamma^l_{ij}}{\partial x^k} - \frac{\partial \Gamma^l_{ik}}{\partial x^j} + \Gamma^s_{ij} \Gamma^l_{ks} - \Gamma^s_{ik} \Gamma^l_{js} \quad (53)$$

Let $Z = (r_x, r_y, n)^t$. Then

$$Z_x = AZ, \quad Z_y = BZ \quad (54)$$

where

$$A = \begin{pmatrix} \Gamma^1_{11} & \Gamma^2_{11} & L \\ \Gamma^1_{12} & \Gamma^2_{12} & M \\ p_1 & p_2 & 0 \end{pmatrix}, \quad B = \begin{pmatrix} \Gamma^1_{12} & \Gamma^2_{12} & M \\ \Gamma^1_{22} & \Gamma^2_{22} & N \\ q_1 & q_2 & 0 \end{pmatrix}. \quad (55)$$

Hence we get the new form of the MPCE\textup{(52)}

$$A_y - B_x + [A, B] = 0 \quad (56)$$

Let us introduce the orthogonal trihedral\textup{[17]}

$$\vec{e}_1 = \frac{\vec{r}_x}{\sqrt{E}}, \quad \vec{e}_2 = \vec{n}, \quad \vec{e}_3 = \vec{e}_1 \wedge \vec{e}_2 \quad (57)$$

Let $\vec{r}_x^2 = E = \pm 1$ and $F = 0$. Then the vectors $\vec{e}_j$ satisfy the following Serret-Frenet-Myrzakulov equation (SFME)

$$\begin{pmatrix} \vec{e}_1 \\ \vec{e}_2 \\ \vec{e}_3 \end{pmatrix}_x = C \begin{pmatrix} \vec{e}_1 \\ \vec{e}_2 \\ \vec{e}_3 \end{pmatrix}, \quad \begin{pmatrix} \vec{e}_1 \\ \vec{e}_2 \\ \vec{e}_3 \end{pmatrix}_y = D \begin{pmatrix} \vec{e}_1 \\ \vec{e}_2 \\ \vec{e}_3 \end{pmatrix} \quad (58)$$

and the time equation

$$\begin{pmatrix} \vec{e}_1 \\ \vec{e}_2 \\ \vec{e}_3 \end{pmatrix}_t = G \begin{pmatrix} \vec{e}_1 \\ \vec{e}_2 \\ \vec{e}_3 \end{pmatrix} \quad (59)$$

with

$$C = \begin{pmatrix} 0 & k & 0 \\ -Ek & 0 & \tau \\ 0 & -\tau & 0 \end{pmatrix}, \quad D = \begin{pmatrix} 0 & m_3 & -m_2 \\ -Em_3 & 0 & m_1 \\ Em_2 & -m_1 & 0 \end{pmatrix}, \quad G = \begin{pmatrix} 0 & \omega_3 & -\omega_2 \\ -E\omega_3 & 0 & \omega_1 \\ E\omega_2 & -\omega_1 & 0 \end{pmatrix},$$

where

$$k = \frac{L}{2}, \quad \tau = MG^{-1/2}. \quad (60)$$

Hence we have

$$C_y - D_x + [C, D] = 0, \quad C_t - G_x + [C, G] = 0, \quad D_t - G_y + [D, G] = 0. \quad (61)$$
Now let $\vec{S} = e_1$. Let us now introduce the new function $q$ by

$$q = \frac{k}{2b} \exp i \frac{1}{8} \partial_x^{-1} (k^2 b^{-2} - 4 \tau) - 2b^2 x$$

(62)

Then the function $q$ satisfies the following equations[17]

$$i q_t + q_{yx} + \frac{i}{2} [(V_1 q)_x - V_2 q - qpq_y] = 0$$

(63a)

$$i p_t - p_{yx} + \frac{i}{2} [(V_1 p)_x + V_2 p - qpp_y] = 0$$

(63b)

$$V_{1x} = (pq)_y, \quad V_{2x} = pqx - qpq_y$$

(63c)

where $p = \bar{E}q$. This set of equations is the L-equivalent counterpart of the M-XXII equation(46). As it seems to us, equations (63) are new integrable equation. We will call (63) the M-XXII$_q$ equation. Now let us consider the following transformation

$$q' = q \exp (-\frac{i}{2} \partial_x^{-1} |q|^2)$$

(64)

Then the new variable $q'$ satisfies the Strachan equation[18]

$$i q'_t + q'_{2xy} + i (V q')_x = 0, \quad V_x = E(|q'|^2)_y$$

(65)

We see that the M-XXII$_q$ equation(7) and the Strachan equation(22) is gauge equivalent to each other.

References

[1] A. Sym // Springer Lecture Notes in Physics, edited by R. Martini (Springer, Berlin 1985) v.239, p 154

[2] M. Lakshmanan // Phys. Lett., v.A61, 53 (1977); J. Math. Phys., v.20, 1667(1979)

[3] K. Nakayama, H. Segur and M. Wadati//Phys.Rev.Lett. v.69, N18,2603-2606(1992)

[4] A. Doliwa and P.M. Santini // Phys. Lett., v.A185, 373 (1994)

[5] A. Bobenko // Surfaces in terms of 2 by 2 matrices. Old and new integrable cases. In: Harmonic maps and integrable systems(Vieweg, Aspects Math. E23, 83-127(1994)

[6] J. Cieslinski, A. Sym and W. Wesselius //J.Phys.A: Math. Gen., v.26, 1353(1993)

[7] R. Balakrishnan, A.R. Bishop and R. Dandoloff// Phys.Rev., B47,3108(1993)

[8] W. Zakrzewski // Low dimensional sigma models (Hilger, 1989)
[9] B.G.Konopelchenko // Multidimensional integrable systems and dynamics of surfaces in space ( Preprint, Acad.Sinica, Taipei, 1993)

[10] L.A. Takhtajan // Phys. Lett., v.A64, 235(1977)

[11] B.G.Konopelchenko//Solitons in multidimensions.World Scientific. Singapore.1993.

[12] M.J. Ablowitz and P.A. Clarkson // Solitons, Nonlinear evolution equations and Inverse Scattering. (Cambridge University Press, Cambridge, 1991)

[13] V. E. Zakharov // Solitons. eds. by R.K. Bullough and P.J. Caudrey (Springer, Berlin, 1980)

[14] M.Boiti, L.Martina and F.Pempinelli // Multidimensional localized solitons(25.08.1993).

[15] V. E. Zakharov, L. A. Takhtajan //Theor. Math. Phys. 1979. v.38. p.17.

[16] G. N. Nugmanova// The Myrzakulov equations: the gauge equivalent counterparts and soliton solutions. Alma-Ata (1992)

[17] R. Myrzakulov // On some integrable and nonintegrable soliton equations of magnets (HEPI, Alma-Ata, 1987)

[18] I. A. B. Strachan // J. Math. Phys., v.34, 243 (1993)

[19] G.N.Nugmanova// On magnetoelastic solitons in ferromagnet. Preprint CNLP-1994-01. Alma-Ata. 1994.

[20] R. Myrzakulov // Soliton equations in 2+1 dimensions and Differential geometry of curves/surfaces. Preprint CNLP-1994-02. Alma-Ata. 1994.

[21] R. Myrzakulov // A (2+1) - dimensional integrable spin model ( the M-XXII equation ) and Differential geometry of curves/surfaces. Preprint CNLP-1994-03. Alma-Ata. 1994.

[22] R. Myrzakulov // The gauge equivalence of the Zakharov equations and (2+1) - dimensional continuous Heisenberg ferromagnetic models. Preprint CNLP-1994-04. Alma-Ata. 1994.

[23] R. Myrzakulov // Gauge equivalence between two - dimensional Heisenberg ferromagnets with single - site anisotropy and Zakharov equations. Preprint CNLP-1994-05. Alma-Ata. 1994.

[24] L.P. Nizhnik //DAN SSSR, v.254, 332(1980)

[25] A.P. Veselov and S.P.Novikov //DAN SSSR, v.279, 20(1984)

[26] L.V.Bogdanov // Theor. and Math. Phys., v.70, 309-314(1987)