Approach to exact inflation in modified Friedmann equation

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Abstract. We study inflationary universe models that are characterized by a single scalar inflaton field. The study of these models are based on two dynamical equations; one corresponding to the Klein-Gordon equation for the inflaton field, and the other, to a generalized Friedmann equation. After describing the kinematics and dynamics of the models, we determine in some detail scalar density perturbations and relic gravitational waves. We apply this approach to the Friedmann-Chern-Simons and the brane-world inflationary models.

Keywords: Cosmic Inflation, Primordial Features, Cosmic Microwave Background
1 Introduction

The early accelerated expansion of the universe (inflation) has been proposed as a good approach for solving most of the cosmological "puzzles" [1, 2]. This brief accelerated expansion serves, apart from solving most of the cosmological problems, to produce the seeds for the large scale structure formation. In fact, the present popularity of inflation is entirely due to its ability to generate a spectrum of density perturbations which lead to structure formation in the universe. In essence, the conclusion that all the observations of microwave background anisotropy performed so far support inflation, rests on the consistency of the anisotropy with an almost Harrison-Zel’dovich power spectrum predicted by most of the inflationary universe scenarios [3].

The implementation of inflationary models rests on the introduction of a scalar inflaton field, $\phi$. The evolution of this field becomes governed by its scalar potential, $V(\phi)$, via the Klein-Gordon equation. In this way, this equation of motion, together with the Friedmann equation, form the most simple set of field equations, which can be used to study inflationary solutions.

The previous implementation works if we give an explicit expression for the scalar inflaton potential, $V(\phi)$. However, in most cases result complicated to find solutions, even in the situation in which it is applied the so-called slow-roll approximation, where the kinetics terms is much smaller than the potential energy, i.e. $\ddot{\phi}^2 \ll V(\phi)$, together with the approximation $|\dddot{\phi}| \ll H |\dot{\phi}|$.

Another way to find inflationary solutions, out of the slow-roll approximation, is giving the functional form of the Hubble parameter in term of the inflaton field, i.e. $H(\phi)$ [4]. At first glance this approach looks quite unsuitable, since the function $H(\phi)$ could be chosen arbitrarily. Of course not any Hubble function, $H(\phi)$, will yield an appropriated inflationary solution. The same it is found when the slow-roll approximation is taken into account. There, specific scalar potentials are chosen in order that the model could be implemented. For instance, if the scalar potential does not present a minimum, then, the exit from inflation becomes a problem, since the usual reheating process can not be carried out.

It seems that the description of inflation in term of the Hubble parameter as a function of the scalar field, i.e. $H(\phi)$, looks "more natural"[5]. In this way, this approach known as
the Hamilton-Jacobi formalism (H-J) provides us with a straight-forward way of exploring the inflationary scenario\textsuperscript{[6, 7]}.

The H-J approach presents some advantages when compared with the slow-roll approximation: first, in the H-J formalism the form of the potential is deduced, and second, since an exact solution is obtained, then, application to the final period of inflation is possible, where the kinetic term of the inflaton field in the Friedmann equation becomes important, i.e. when studying the final stage of inflation. In this approach we can use the scalar field, $\phi$, as a ”time variable”, and for that, we demand that this field increases monotonically, i.e. its time derivative, $\dot{\phi}$, should not change of sign along the inflationary evolution.

In this article we would like to study the consequences that result when considering a modified Friedmann equation expressed by

$$\mathcal{F}(H) \equiv \left( \frac{8\pi}{3m_{Pl}^2} \right) \rho_\phi = \left( \frac{8\pi}{3m_{Pl}^2} \right) \left[ \frac{1}{2} \dot{\phi}^2 + V(\phi) \right]$$

where $\mathcal{F} \geq 0$ is an arbitrary function of the Hubble parameter $H = \dot{a}/a$, with $a$ the scale factor, the prime represents a derivative with respect to the scalar inflaton field $\phi$, i.e. $\prime \equiv \frac{d}{d\phi}$ and the dots represent derivatives with respect to the cosmological time, $t$. Here, the inflaton potential is expressed by $V(\phi)$ and $m_{Pl}^2 \equiv 1/G$ represents the Planck mass.

The motivation for using this kind of equation lies in the fact that in the literature several models have been studied which can be reduced to such modified Friedman equation. In a $L(R)$-theory of gravity in which $L(R) = R - \frac{\alpha^2}{3!}$, where $R$ is the scalar curvature and $\alpha$ is a constant with dimension of mass square, the Friedmann equation becomes modified by the expression $\mathcal{F}(H) = \frac{6H^2 - \alpha}{8} - \frac{\alpha}{4\alpha}H^2$\textsuperscript{[8]}.

Also, it is possible to consider $\mathcal{F}(H) = H^2 - \alpha H^4$, where $\alpha$ is a constant with dimension of $[mass]^{-2}$. There exist various forms in arriving to this expression for $\mathcal{F}(H)$. This has been derived by considering a quantum corrected entropy-area relation of the type $S_A = m_{Pl}^2 \frac{A}{4} - \tilde{\alpha} \ln \left( m_{Pl}^2 \frac{A}{4} \right)$\textsuperscript{[9]}, where $A$ is the area of the apparent horizon, and $\tilde{\alpha}$ is a dimensionless positive constant determined by the conformal anomaly of the fields. This conformal anomaly is interpreted as a quantum correction to the entropy of the apparent horizon\textsuperscript{[10]}. On the other hand, this modified Friedmann equation could be obtained when an AdS-Schwarzschild black-hole via holographic renormalization is considered, together with mixed boundary conditions corresponding to the Einstein field equations in four dimension\textsuperscript{[11]}. Also, this could be derived in terms of spacetime thermodynamics together with a generalized uncertainty principle of quantum gravity\textsuperscript{[12]}. A Chern-Simons type of theory yields to this modification too\textsuperscript{[13]}. The resulting Friedmann equation when considering this type of modification we will call the Friedmann-Chern-Simons equation\textsuperscript{[14]}.

On the other hand, superstring and M-Theory bring the possibility of considering our universe as a domain wall embedded in a higher dimensional space. In this scenario the standard model of particle is confined to the brane, while gravitation propagate into the bulk spacetime. The effect of extra dimensions induces a change in the Friedmann equation.

Here, the function $\mathcal{F}(H)$ results to be $\mathcal{F}(H) = \left( \frac{8\pi \lambda}{3m_{Pl}^2} \right) \left[ \sqrt{1 + \left( \frac{3m_{Pl}^2}{4\pi \lambda} \right) H^2 - 1} \right]$, where $\lambda$ represents the brane tension\textsuperscript{[15]}.

Here, in this paper, after giving a general approach to the study of inflation based on the modified Friedmann equation, we will describe in some detail the latter two cases speci-
fied previously, i.e. the Friedmann-Chern-Simons and the brane-world inflationary universe models.

2 The exact solution approach

Our study will be based on considering Eq. (1.1) together with the scalar field equation

\[
\ddot{\phi} + 3H\dot{\phi} + V'(\phi) = 0. \tag{2.1}
\]

In obtaining this latter equation we have assumed that the matter, specified by the inflaton scalar field, enters into the action Lagrangian in such a way that its variation in a Friedmann-Robertson-Walker background metric leads to the Klein-Gordon equation, expressed by Eq. (2.1). Therefore, we are considering constrained sort of models, in which the background (together with the perturbed equation, see Eq. (3.3)) are not modified. In this context theory of gravity, such that Hořava-Lifshitz[16] lies outside of the approach followed here. Also, in this study we will use the scalar field \( \phi \) as a "time variable". The requirement imposed in this approach is that the scalar field increases monotonically and its time derivative, \( \dot{\phi} \), should not change of sign along the path evolution.

From the field Eqs. (1.1) and (2.1) it is found that

\[
\dot{\phi} = - \left( \frac{m^2_{Pl}}{8\pi} \right) \mathcal{F}_{,H} \left( \frac{H'}{H} \right), \tag{2.2}
\]

where \( \mathcal{F}_{,H} \equiv d\mathcal{F}/dH \). This latter equation allows us to write down an explicit expression for the scalar potential

\[
V(\phi) = \frac{3m^2_{Pl}}{8\pi} \mathcal{F} \left[ 1 - \frac{m^2_{Pl}}{48\pi} \left( \frac{H'}{H} \right)^2 \mathcal{F}_{,H}^2 \right]. \tag{2.3}
\]

It is not hard to show that

\[
a H = \left( \frac{m^2_{Pl}}{8\pi} \right) \mathcal{F}_{,H} a' H', \tag{2.4}
\]

from which we get

\[
a(\phi) = a_i \exp \left\{ \frac{8\pi}{m^2_{Pl}} \int_{\phi_i}^{\phi} \frac{H^2}{H' \mathcal{F}_{,H}} d\phi \right\}. \tag{2.5}
\]

where \( a_i = a(\phi_i) \).

On the other hand, we see that the acceleration quation for the scale factor results to be

\[
\frac{\ddot{a}}{a} = H^2 [1 - \epsilon_H], \tag{2.6}
\]

where the function \( \epsilon_H \) corresponds to

\[
\epsilon_H \equiv - \frac{d\ln H}{d\ln a} = \left( \frac{m^2_{Pl}}{8\pi} \right) \mathcal{F}_{,H} \left( \frac{H'}{H} \right)^2. \tag{2.7}
\]

From this latter expression we can see that this definition, called the first Hubble slow-roll parameter, gives information about the acceleration of the universe. During inflation we have
that $\epsilon_H < 1$, and this period ends when $\epsilon_H$ takes the value equal to one. In the next section we will use this parameter for describing scalar and tensor perturbations.

One interesting quantity in characterizing inflationary universe models is the amount of inflation. Usually, this quantity is defined by

$$N(t) = \ln \frac{a(t_e)}{a(t)}$$

where $a(t_e)$ corresponds to the scale factor evaluated at the end of inflation. For the modified Friedmann Equation it becomes, in terms of the scalar field

$$N(\phi) = \int_{t_e}^{t_e} H dt = \left( \frac{8\pi}{m_{pl}^2} \right) \int_{\phi_e}^{\phi} \frac{H^2}{F_H F_{,HH}} d\phi = \int_{\phi_e}^{\phi} \frac{1}{\epsilon_H} \frac{H'}{H} d\phi. \quad (2.9)$$

Here, $\phi_e$ represents the value of the scalar field at the end of inflation. Its value is determined by imposing that $\epsilon_H(\phi_e) = 1$.

It seems to be more appropriated to describe the amount of inflation in terms of the comoving Hubble length, $1/(aH)$ than in terms of the scale factor only. In this case the amount of inflation becomes defined as[17]

$$\bar{N} \equiv \ln \frac{a(t_e) H(t_e)}{a(t) H(t)}, \quad (2.10)$$

which results into

$$\bar{N}(\phi) = \int_{\phi_e}^{\phi} \left( \frac{1}{\epsilon_H} - 1 \right) \frac{H'}{H} d\phi. \quad (2.11)$$

Note that, in general, $\bar{N}(\phi)$ is smaller that $N(\phi)$ and only in the slow-roll limit they coincide.

In the description of inflation it is convenient to show that their solutions are independent from their initial conditions. This ensure the true predictive power that presents any inflationary universe model, otherwise the corresponding physical quantities associated with the inflationary phase, such that the scalar or tensor spectra, would depend on these initial conditions. Thus, with the purpose of being predictive, any inflationary model needs that their solutions present an attractor behavior, in the sense that solutions with different initial conditions should tend to a unique solution[5].

In order to study the corresponding inflationary attractor solutions for our case, we follow Ref. [5]. We start by considering a linear perturbation, $\delta H(\phi)$, around a given inflationary solution, expressed by $H_0(\phi)$. In the following we will refer to this quantity as the background solution, and any quantity with the subscript zero is assumed to be evaluated taking into account the background solution. Therefore, at first order on $\delta H(\phi)$, we get from the field Equations (1.1) and (2.1) that

$$\left[ 1 + \frac{1}{3} \epsilon_H \left( 1 - H \frac{F_{,HH}}{F_{,H}} \right) \right]_0 \delta H \simeq \frac{1}{3} \left( \frac{m_{pl}^2}{8\pi} \right) \frac{F_{,H}}{H^2} \frac{H'}{H} \mid_0^\phi \delta H', \quad (2.12)$$

This latter expression can be solved for getting

$$\delta H(\phi) = \delta H(\phi_i) \exp \int_{\phi_i}^{\phi} \left( \frac{3}{\epsilon_H} \right) \left[ 1 + \frac{1}{3} \epsilon_H \left( 1 - H \frac{F_{,HH}}{F_{,H}} \right) \right] \frac{H'}{H} \mid_0^\phi d\phi, \quad (2.13)$$

where $\phi_i$ corresponds to some arbitrary initial value of $\phi$. By considering theories in which $F_{,H} > H F_{,HH}$ we find that the integrand within the exponential term will be negative, since $d\phi$ and $H'$ have opposite signs (assuming that $\dot{\phi}$ does not change sign due to the perturbation $\delta H$). Thus, all the linear perturbations tend to vanish quickly[17].
3 Scalar and tensor perturbations

Inflation generates perturbations through the amplification of quantum fluctuations, which are stretched to astrophysical scales by the brief, but rapid inflationary expansion. The simplest models of inflation generate two types of perturbations, density perturbations which come from quantum fluctuations of the scalar field\[^{18-21}\], and relic gravitational waves which are tensor metric fluctuations\[^{22-26}\]. The former experience gravitational instability and lead to structure formation\[^{27}\], while the latter predicts a stochastic background of relic gravitational waves which could influence the cosmic microwave background anisotropy via the presence of polarization in this\[^{28}\]. The upcoming experiments such as Planck satellite will characterize polarization anisotropy to a higher accuracy\[^{29}\]. It is very timely to develop the tools which can optimally utilize the polarization information to constrain models of the early universe. Specifically, the magnetic modes (B-modes) are signal from cosmic inflation and suggest the presence of gravitational waves\[^{30}\].

In order to describe these perturbations let us consider the Hamilton-Jacobi slow-roll parameters. The first slow-roll parameter \( \epsilon_H \) was already defined in the previous section, Eq. (2.7).

The second slow-roll parameter, \( \eta_H \), is defined as

\[
\eta_H \equiv - \frac{d \ln H'}{d \ln a} = \left( \frac{m_{Pl}^2}{8 \pi} \right) \frac{F_H H'}{H'}. \tag{3.1}
\]

We should note here that both \( \epsilon_H \) and \( \eta_H \) are exact quantities, although we call them "slow-roll parameters". In the slow-roll limit these parameters becomes\[^{17, 31}\]

\[
\epsilon_H \rightarrow \epsilon, \quad \eta_H \rightarrow \eta - \epsilon, \tag{3.2}
\]

where the quantities \( \epsilon \) and \( \eta \) are the common slow-roll parameters which satisfies \( \epsilon \ll 1 \) and \( \eta \ll 1 \), in agreement with the slow-roll approximation.

The evolution equation for the Fourier modes of the scalar perturbations (quantum mode functions) at some comovil wave number scale \( k \) is governed by\[^{32, 33}\]

\[
d^2 u_k d\tau^2 + \left( k^2 - \frac{1}{z} \frac{d^2 z}{d\tau^2} \right) u_k = 0, \tag{3.3}
\]

where \( \tau \) represents the conformal time defined by \( \tau = \int \frac{1}{a} dt \) and \( u_k \) corresponds to the Fourier transformed of the Mukhanov variable, which is defined by \( u = z R \), with \( z = a \dot{\varphi} \) and \( R \) defining the gauge-invariant comovil curvature perturbation. This latter amount remains constant outside the horizon, i.e. metric perturbations with wavelengths larger than the Hubble radius will be frozen\[^{34}\].

During inflation it is expected that \( k^2 \gg \frac{1}{z \frac{d^2 z}{d\tau^2}} \), i.e. the physical modes are assumed to have a wavelength much smaller than the curvature scale, and thus Eq. (3.3) can be solved to achieve

\[
u_k(\tau) \sim e^{-ik\tau} \left( 1 + \frac{\mathcal{A}_k}{\tau} + \ldots \right), \tag{3.4}
\]

On the other hand, when \( k^2 \ll \frac{1}{z \frac{d^2 z}{d\tau^2}} \), we have that the physical modes correspond to wavelengths much bigger than the curvature scale.
The mass term \( \frac{1}{z} \frac{d^2 z}{d \tau^2} \) becomes in our case

\[
\frac{1}{z} \frac{d^2 z}{d \tau^2} = 2a^2 H^2 \left\{ 1 - \frac{3}{2} \eta_H + \frac{1}{2} \epsilon_H \left( 5 - 3 \frac{F_{HH}}{F_H} \right) \right. \\
+ \left. \epsilon_H^2 \left( 1 - H \frac{F_{HH}}{F_H} + \frac{1}{2} H^2 \frac{F_{HHH}}{F_H} \right) - \frac{1}{2} \eta_H \left( 3 - 2 H \frac{F_{HH}}{F_H} \right) + \frac{1}{2} \eta_H^2 \right. \\
+ \left. \frac{1}{2 H} \epsilon_H^2 \left( 3 - 2 H \frac{F_{HHH}}{F_H} \right) - \frac{1}{2 H} \eta_D \right\}. 
\]

(3.5)

For \( F(H) = H^2 \) we obtain that[35]

\[
\frac{1}{z} \frac{d^2 z}{d \tau^2} = 2a^2 H^2 \left\{ 1 + \epsilon_H - \frac{3}{2} \eta_H - \frac{1}{2} \epsilon_H \eta_H + \frac{1}{2} \eta_H^2 + \frac{1}{2} \eta_D^2 - \frac{1}{2 H} \eta_H \right\}. 
\]

(3.5)

It has long been known that Eq. (3.3) could be solved exactly in the case in which the mass term \( \frac{1}{z} \frac{d^2 z}{d \tau^2} \) is proportional to \( \tau^{-2} \), in which case this equation reduces to a Bessel equation, where the standard solution becomes \( u_k \sim \sqrt{-k} e^{-i \eta} \) as \( -k \eta \to \infty \), and \( A_k \) as \( -k \eta \to 0 \).

(3.8)

In order to obtain \( u_k \) by solving equation (3.3), we need to impose some boundary conditions. These asymptotic conditions are usually taken to be the so-called Bunch-Davies vacuum states[38]

\[
u \to \begin{cases} \\
\frac{1}{\sqrt{2 \pi}} e^{-ik \eta} & \text{as } -k \eta \to \infty, \\
\mathcal{A}_k \frac{1}{2} & \text{as } -k \eta \to 0. \\
\end{cases} 
\]

(3.8)

This ensures that perturbations that are generated well inside the horizon, i.e. in the region where \( k \ll aH \), the modes approach plane waves and those that are generated well outside the horizon, i.e. in the region where \( k \gg aH \), remain unchanged.

The description of the primordial curvature perturbation presents a standard result given by[20, 21, 39]

\[
\mathcal{P}_R(k) = \left( \frac{H}{|\phi|} \right)^2 \left( \frac{H}{2 \pi} \right)^2 \bigg|_{aH=k} 
\]

(3.7)

This perturbation is in general a function of the wave number \( k \), which is evaluated for \( aH = k \), i.e. when a given mode crosses outside the horizon during inflation. Since the
modes do not evolve outside the horizon, the amplitude of the modes when they cross back inside the horizon, coincides with the value that they had when they left the horizon.

By using the primordial scalar perturbations we can introduce the scalar spectral index $n_s$ defined by

$$n_s - 1 \equiv \frac{d \ln P_R}{d \ln k}. \quad (3.10)$$

This quantity becomes

$$n_s - 1 = 2 \eta_H - 2 \left(3 - H \frac{\mathcal{F}_{,HH}}{\mathcal{F}_{,H}}\right) \epsilon_H. \quad (3.11)$$

In the same way we define the *running scalar spectral index*, $\alpha_s \equiv \frac{dn_s}{d \ln k}$ which results to be given by

$$\alpha_s = 2 \left(8 - 3H \frac{\mathcal{F}_{,HH}}{\mathcal{F}_{,H}}\right) \epsilon_H \eta_H - 2 \left(9 - 5H \frac{\mathcal{F}_{,HH}}{\mathcal{F}_{,H}} + H^2 \frac{\mathcal{F}_{,HHH}}{\mathcal{F}_{,H}}\right) \epsilon_H^2 - 2 \xi_H, \quad (3.12)$$

where $\xi_H$ corresponds to the *third slow-roll parameter* and becomes given by

$$\xi_H \equiv \left(\frac{m_{pl}^2}{4 \pi}\right)^2 \left(\frac{\mathcal{F}_{,H}}{H}\right)^2 \frac{H'' H'}{H^2}. \quad (3.13)$$

In addition to the scalar curvature perturbations, transverse-traceless tensor perturbations can also be generated from quantum fluctuations during inflation [23, 27]. The tensor perturbations do not couple to matter and consequently they are only determined by the dynamics of the background metric, so the standard results for the evolution of tensor perturbations of the metric remains valid. The two independent polarizations evolve like minimally coupled massless fields with spectrum\(^1\)

$$\mathcal{P}_T = \left. \frac{16 \pi m^2_{pl}}{m^2_{pl}} \right| \frac{H}{2 \pi} \left|_{aH=k}. \quad (3.14)$$

Similarly to the case of scalar perturbations, we evaluate the expression on the right hand side of Eq. (3.14) when the comoving scale $k$ leaves the horizon during inflation. Furthermore, we can introduce the *gravitational wave spectral index* $n_T$ defined by $n_T \equiv \frac{d \ln \mathcal{P}_T}{d \ln k}$, which results to be

$$n_T = -2 \epsilon_H. \quad (3.15)$$

Here, we can also introduce the *running tensor spectral index* $\alpha_T$ defined by

$$\alpha_T \equiv \frac{dn_T}{d \ln k} = 4 \eta_H \eta_H - 2 \left(3 - H \frac{\mathcal{F}_{,HH}}{\mathcal{F}_{,H}}\right) \epsilon_H^2 \quad (3.16)$$

At this point we can define the *tensor-to-scalar amplitude ratio* $r \equiv \frac{\mathcal{P}_T}{\mathcal{P}_R}$ which becomes

$$r = 2 \frac{\mathcal{F}_{,H}}{H} \epsilon_H, \quad (3.17)$$

\(^1\)We mention here that this expression should be implemented with a factor such that $F_x^2(H/\mu)$, where $F_x^2(x) = \sqrt{1 + x^2} - \left(\frac{1 - \sqrt{1 + x^2}}{x^2}\right) x^2 \sinh^{-1} \frac{x}{2}$, when a braneworld with a Gauss-Bonnet term is considered [40].
and combining equations (3.15) and (3.17) we find that

\[ n_T = - \left( \frac{H}{\mathcal{F}_H} \right) r. \] (3.18)

This latter expression corresponds to the inflationary consistency condition\[35, 41\]. Note that for standard cosmology, in which \( \mathcal{F}(H) = H^2 \), Eq. (3.18) reduces to \( r = -\frac{1}{2} n_T \). However, this relation could be violated in some cases\[42, 43\]. Note that this relation depends on the kind of theory that we are dealing with.

4 The hierarchy of the slow-roll parameters and the flow equations

There exist a different way of studying inflationary universe models, which is subtended by a sort of hierarchy imposed on the slow-roll parameters\[44, 45\]. In fact, the set of equations in this approach is based on derivatives with respect to the e-folding number over the slow-roll parameters.

We have previously introduced the slow-roll parameters, such as \( \epsilon_H, \eta_H \) and \( \xi_H \), to which we have given a sort of hierarchy calling them first, second and third slow-roll parameters, respectively. Each of these parameters is characterized by their dependence on the order of the scalar field derivative of the Hubble ratio, \( H(\phi) \), such as \( \epsilon_H \sim (H')^2, \eta_H \sim H'' \) and \( \xi_H \sim H''' \), as we can see from equations (2.7), (3.1) and (3.13), respectively. It is possible extend this definition to higher derivatives of the Hubble parameter so that we can introduce the following parameter

\[ l \lambda_H = \left( \frac{m_{pl}^2}{4\pi} \right) l^l \left( \frac{\mathcal{F}_H}{H} \right)^l \left( \frac{H'}{H} \right)^{l-1} \frac{d^{l+1}H}{d\phi^{l+1}}, \quad (l \geq 1), \] (4.1)

where for \( l = 1 \) we have that \( l^1 \lambda_H \equiv \eta_H \) and \( l = 2 \) corresponds to \( l^2 \lambda_H \equiv \xi_H \).

It not hard to show that the following set of equations is satisfied

\[ \frac{d\epsilon_H}{dN} = \left( \frac{H}{\mathcal{F}_H} \right)^2 - 3 \epsilon_H + 2\eta_H \right] \epsilon_H, \] \[ \frac{d\lambda_H}{dN} = \left( \frac{H}{\mathcal{F}_H} \right)^2 - 2 \epsilon_H + (l-1)\eta_H \right] l \lambda_H + \frac{l+1}{2} \lambda_H. \quad (l \geq 2) \]

where \( \lambda \equiv 2\eta_H - 4\epsilon_H \)[45].

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In order to solve the infinite set of equations (4.2) the series is truncated by imposing a vanishing value to a given high enough slow-roll parameter. This corresponds to take that $M+1 \lambda_H = 0$, for an appropriated large number $M$ (for instance, in the literature has been used $M = 5^{[45]}$). With this truncation the set of equations has been solved both numerically$^{[44–47]}$ and analytically$^{[48–50]}$.

With respect to possible solutions of these equations and their relations with the inflationary paradigm, it was emphasized in Ref. $^{[48]}$ that there is no a clear connection between them. Actually, it is not clear that a particular solution of the flow equations corresponds directly to some type of inflationary solution. To achieve this task we must add additional ingredients to the corresponding solutions. The main ingredient that has been left out of this scheme has been the Friedmann equation itself. Thus, in solving the flow equations we could get $\epsilon_H$ as a function of the scalar field $\phi$ (imposing the condition that this parameter will satisfy the range $0 \leq \epsilon_H \leq 1$), and then we could get $H(\phi)$ through the following relation

$$\int_{H_i}^{H(\phi)} \sqrt{\frac{F_H}{H^3}} dH = \sqrt{\frac{8\pi}{m_{Pl}^2}} \int_{\phi_i}^{\phi} \sqrt{\epsilon_H(\phi)} d\phi. \quad (4.4)$$

Thus, in principle, we could give the function $F(H)$ so that we could get $\epsilon_H$ as a function of $\phi$ by solving the set of Eqs. (4.2) we could obtain the Hubble parameter, $H(\phi)$, as a function of the scalar field through Eq. (4.4)$^2$. With this in hand, we can obtain an explicit expression for the scalar potential, $V(\phi)$, given by

$$V(\phi) = \left(\frac{3m_{Pl}^2}{8\pi}\right) F \left[1 - \frac{1}{6} H \left(\frac{F_H}{F}\right) \epsilon_H\right]. \quad (4.5)$$

In short, for a given function $F(H)$, we could say that the Hubble flow formalism allows us to determinate the scalar field potential, $V(\phi)$, associated to some inflationary universe model. In order to realize this task we first solve the flow equations, Eqs. (4.2), from which we can obtain the first slow-roll parameter, $\epsilon_H$, under the condition that this parameter must satisfy the bound $0 \leq \epsilon_H \leq 1$. Then, by using Eq. (4.4), we get the corresponding Hubble parameter as a function of the scalar field, $H(\phi)$, from which we could obtain all the other quantities associated to the inflationary scenario.

5 Two interesting cases

5.1 The Friedmann-Chern-Simons model

As stated in the introduction we would like to consider here a model in which the Friedmann equation modifies to$^3$

$$F(H) \equiv H^2 - \alpha H^4 = \left(\frac{8\pi}{3 m_{Pl}^2}\right) \rho_\phi, \quad (5.1)$$

$^2$In the standard case in which $F = H^2$ it is obtained that

$$H(\phi) = H_i \exp \left[ \frac{8\pi}{m_{Pl}^2} \int_{\phi_i}^{\phi} \sqrt{\epsilon_H(\phi)} d\phi \right].$$

$^3$For more details on this case see Ref. [14].
where $\alpha$ is an arbitrary constant with dimension of $m_{Pl}^{-2}$. Here, we assume that during the inflationary evolution the Hubble parameter, $H$, satisfies the bound $H < 1/\sqrt{\alpha}$, so that the energy density associated to the scalar field $\phi$ is positive.

It is possible to choose for the generating function a polynomial like $H(\phi) = H_0(1 + \beta \phi + \beta_2 \phi^2 + \ldots + \beta_N \phi^N)$, where $H_0$ and the different $\beta$ are constants. This sort of solution was used to generate suitable functions of slow-roll parameters[48]. Here, just for simplicity, and in order to show how the this approach work, we shall take the previous polynomial, but, up to first order in the scalar field $\phi$, i.e. $H(\phi) = H_0(1 + \beta \phi)$, with $\beta$ an arbitrary constant with dimension of $m_{Pl}^{-1}$. In this case, the scalar potential becomes

$$V(\phi) = \left(\frac{3m_{Pl}^2}{8\pi}\right) H_0^2 \phi^2 \left[1 - \alpha H_0^2 \phi^2\right]$$

$$\times \left[1 - \frac{m_{Pl}^2}{12\pi} \phi^2 \left(\frac{1 - 2\alpha H_0 \phi}{\sqrt{1 - \alpha H_0^2 \phi^2}}\right)^2\right], \quad (5.2)$$

where $\overline{\phi} \equiv 1 + \beta \phi$.

In the slow-roll approximation, i.e. where $\phi^2 \ll V(\phi)$ together with $|\ddot{\phi}| |dV(\phi)/d\phi|$, it is found that the scalar potential becomes

$$V(\phi) \approx \left(\frac{3m_{Pl}^2}{8\pi}\right) H_0^2 \phi^2 \left[1 - \alpha H_0^2 \phi^2\right]. \quad (5.3)$$

Figure 1 depicts the shape of the potential for the exact case (thick line), expressed by Eq. (5.2), together with the approximated slow-roll case, Eq. (5.3). In the same figure the dotted line represents the exact case in which the $\alpha$-parameter is vanished.

The scalar field results to be given by

$$\phi(t) = \frac{1}{\sqrt{2\alpha \beta H_0}} \cosh \left[2 \tanh^{-1} \left(\tanh \left[\frac{1}{2} \cosh^{-1}(\sqrt{2\alpha H_0})\right] e^{\mathcal{H}(t-t_0)}\right)\right] - \frac{1}{\beta}, \quad (5.4)$$

where $\mathcal{H} \equiv \sqrt{2\alpha} (\beta H_0)^2 m_{Pl}^2 4\pi$ and $\phi(t_0) = 0$. This latter expression allows us to write down the Hubble parameter as a function of time. From this result we get the scale factor, $a(t)$, which results to be

$$a(t) = a_0 \left(\frac{\sinh \left[2 \tanh^{-1} \left(\tanh \left(\frac{1}{2} \cosh^{-1}(\sqrt{2\alpha H_0})\right) e^{\mathcal{H}(t-t_0)}\right)\right]}{\sinh \left[\cosh^{-1}(\sqrt{2\alpha H_0})\right]}\right)^\frac{4\sqrt{2\alpha}}{\mathcal{H}}. \quad (5.5)$$

In order to see if this latter expression describes an accelerated phase, for given values of the parameters, we plot in Figure 2 the deceleration parameters $q$, which is defined as $q = -\frac{\ddot{a}}{a}$. For this plotting, we have taken the value $\sqrt{\alpha} H_0 = 10^{17}$ eV. The different curves correspond to different values of the exponent that appears in the scale factor $a$, i.e. $4 \sqrt{2\alpha} \mathcal{H}$. These curves show that the universe is accelerating, since the parameter $q(t)$ turns out to be negative as time passes. Therefore, our model presents a period of inflation, at least for the values of the parameters that we have considered here.

The amount of inflation becomes in this case

$$\bar{N}(y) = \sqrt{\gamma} \left[\frac{1}{\sqrt{y}} + \frac{1}{\sqrt{2}} \left(\frac{1 - 4\gamma}{\gamma}\right) \tanh^{-1} \left(\sqrt{2} y\right)\right] - \bar{N}_e, \quad (5.6)$$
Figure 1. Plots of the scalar potentials, $V(\Phi)$, as a function of the dimensionless "scalar field", $\Phi \equiv \sqrt{\alpha} H_0 \phi$. The thick line represents the exact potential, expressed by Eq. (5.2). Dashed line represents the same potential, but in the slow-roll approximation, Eq. (5.3). The dotted line corresponds to the exact case, but when $\alpha = 0$. Here we have taken $\alpha (\beta H_0)^2 \equiv \frac{24\pi}{9m_p^2}$, and $V(\Phi)$ is expressed as a multiple of the constant $V_0 \equiv \frac{3m_p^2}{8\pi^2}$.

Figure 2. Time evolution of the deceleration parameter, $q$, as a function of $x = H(t - t_0)$. Here we have taken the value $\sqrt{\alpha} H_0 = \frac{19}{5}\sqrt{2}$. The thick, dotted, thin, dashed, dot-dashed lines correspond to the values for $4\sqrt{2} \alpha H = 1/2; 1; 3/2; 2; 5/2$, respectively. Note that, in all the cases the acceleration parameters result to be negative.

where $y$ is a dimensionless function of the scalar field defined by $y = \alpha H_0^2 (1 + \beta \phi)^2$, $\gamma$.
is a dimensionless constant given by $\gamma \equiv \left( \frac{m_{pl}^2}{4\pi} \right) \alpha (H_0 \beta)^2$ and $N_n$ corresponds to $N_n = \frac{1}{2\sqrt{1+2\sqrt{\gamma}}} + \frac{1}{\sqrt{2}} \left( \frac{1-4\sqrt{\gamma}}{\sqrt{\gamma}} \right) \tanh^{-1} \left( \sqrt{\frac{2\sqrt{\gamma}}{1+2\sqrt{\gamma}}} \right)$. Let us now consider the attractor behavior of this model. By taking into account Eq. (2.13), we get
\[
\delta H(\phi) = \delta H(\phi_i) \exp \left\{ \frac{12\pi}{m_{pl}^2} \int_{\phi_i}^{\phi} g(H_0) \frac{H_0}{H_0'} d\phi \right\},
\]
(5.7)
where $\phi_i$ represents the initial value of the scalar field $\phi$. The function $g(H_0)$ is given by
\[
g(H_0) = \left[ 1 - 2\alpha H_0^2 \left( 1 - 2\frac{2}{3} \epsilon_{H_0} \alpha H_0^2 \right) \right] / \left( 1 - 2\alpha H_0^2 \right)^2
\]
and it is positive for $2\alpha H_0^2 < 1$ (this makes sure that the energy density will be positive, as we can see from Eq. (5.1)). Thus, the integrand within the exponential term will be negative, due to that $d\phi$ and $H_0'$ have contrary signs (assuming that the perturbation $\delta H$ does not change the sign of $\dot{\phi}$)[17]. In this way, all the linear perturbations tend to vanish rapidly.

In what concern to the scalar perturbations, the scalar spectral index parameter becomes
\[
n_s - 1 = 2\eta_H - 4 \left( 1 - \frac{2\alpha H^2}{1 - 2\alpha H^2} \right) \epsilon_H,
\]
(5.8)
and the running scalar spectral index results to be
\[
\alpha_s = \left( \frac{10}{1 - 2\alpha H^2} \right) \epsilon_H \eta_H + \left( \frac{8}{1 - 2\alpha H^2} \right) \epsilon_H^2 - 2\epsilon_H^2,
\]
(5.9)
where $\epsilon_H$ is defined as
\[
\epsilon_H^2 \equiv \left( \frac{m_{pl}^2}{4\pi} \right)^2 \frac{1}{1 - 2\alpha H^2} \frac{H'^m H'}{H^2}.
\]
(5.10)
Analogously, from Eq. (3.15) we find for the gravitational wave spectral index, $n_T$, an expression given by
\[
n_T = -2\epsilon_H,
\]
(5.11)
and we find the corresponding tensor-to-scalar amplitude ratio
\[
r = 4 \left( 1 - 2\alpha H^2 \right) \epsilon_H.
\]
(5.12)
Bearing in mind the expression that we have required for $H(\phi)$, we obtain a relationship between $r$ and $n_s$ given by
\[
r(n_s) = \frac{(8\gamma + 1 - n_s)^2}{16\gamma + 1 - n_s},
\]
(5.13)
where the dimensionless constant $\gamma$ was defined previously. Note that we need to satisfy $n_s < 1 + 16\gamma$ in order to have $r > 0$. Thus, from this inequality we get a constraint on the parameter $\gamma$ given by $\gamma > \frac{1}{16} |(n_s - 1)|$.

Figure 3 shows how changes $r$ as a function of $n_s$ for two different values of the parameter $\gamma$. These values are $\gamma = 8.0 \times 10^{-3}$ and $\gamma = 16.0 \times 10^{-3}$. From this figure we see that our model can accommodate quite well the observational data. Note that this model allows the possibility of having a Harrison-Zel’dovich spectrum, i.e. $n_s = 1$, with $r \neq 0$ as could be seen from this plot.
Figure 3. This plot shows the parameter $r$ as a function of the scalar spectral index $n_s$ for two values of the constant $\gamma = \left( \frac{m^2}{4\pi} \right) (H_0 \beta)^2$, as described by Eq. (5.13). Here, we have taken the values $\gamma = 8.0 \times 10^{-3}$ and $\gamma = 16.0 \times 10^{-3}$. Note that we could have the possibility of having a Harrison-Zel’dovich spectrum ($n_S = 1$) with $r \neq 0$.

In this case the system of flow equations (4.2) is reduced to the following set of equations

\[
\begin{align*}
\frac{d\epsilon_H}{dN} &= \left[ 2 \left( \frac{1 - 4\alpha H^2}{1 - 2\alpha H^2} \right) \epsilon_H + \sigma \right] \epsilon_H, \\
\frac{d\sigma}{dN} &= -\left[ 6 \left( \frac{2 - \alpha H^2}{1 - 2\alpha H^2} \right) \epsilon_H + \left( \frac{5 - 6\alpha H^2}{1 - 2\alpha H^2} \right) \sigma \right] \epsilon_H + 2\xi_H, \\
\frac{d^{l+1}\lambda_H}{dN} &= \left[ l \left( \frac{1 - 6\alpha H^2}{1 - 2\alpha H^2} \right) \epsilon_H + \frac{1}{2} (l - 1) \sigma \right] l \lambda_H + l^{l+1} \lambda_H & (l \geq 2).
\end{align*}
\]

In order to solve this set of equations we need to have $H = H(N)$. To get this, we start by considering Eq. (2.8), which results $N = N(\phi)$. Then, we need to invert this latter expression (if possible) to obtain $\phi = \phi(N)$. Finally, with this expression we obtain $H$ as a function of $N$, and, by introducing this function into the flow equation we proceed to solve them.

There exist another way for getting a relationship between $H$ and $N$. Let us assume that we really know the Hubble rate as a function of the scale factor, i.e., we know explicitly $H(a)$. Then, since by definition $dN = -d \ln a$, we get that $a(N) = a_e e^{(N_e - N)}$, where $a_e$ and $N_e$ are the values of the scale factor and the number of e-folding at the end of inflation. Then, by a direct substitution of $a(N)$ on the Hubble rate $H$ it is obtained $H(N)$.

As an example of the latter approach, let us consider the model in which there is a smooth exit from inflation, under the so-called decaying vacuum cosmology\cite{51}. There, it was found that the Hubble parameter as a function of the scale factor becomes

\[
H(a) = 2H_e \left( \frac{a^2_e a^2}{a^2 + a^2_e} \right),
\]

(5.15)
where \( H_e = H(a_e) \). In this case it is obtained that
\[
H(N) = H_e \left[ 1 - \tanh \left( N_e - N \right) \right].
\]  
(5.16)

Let us solve numerically the set of equation (5.14) for the first two slow-roll parameters, \( \epsilon_H \) and \( \eta_H \), when \( \xi_H \) is constant equal to 0.2. Fig. 4 shows the numerical solutions for these two slow-roll parameters. From this figure we see that the \( \epsilon_H \) remains almost constant (closed to zero) for a wide range of values of \( \tilde{N} \equiv N_e - N \). But, for \( \tilde{N} < 1 \) it increases to the value of one. Actually, for \( N = N_e \), i.e. at the end of inflation, \( \epsilon_H = 1 \). In the same range, i.e. \( \tilde{N} < 1 \), the other slow-roll parameter, \( \eta_H \), decreases from a maximum value (closed to the point \( \tilde{N} \sim 1 \)) to its final value \( \eta_H \approx 1.2 \) at the end of inflation. For this parameter, in the case \( \tilde{N} > 1 \), is it observed from the figure that it decreases lineally.

![Figure 4](image)

**Figure 4.** Numerical solutions for \( \epsilon_H \) and \( \eta_H \) from the set of equations (5.14) in the case in which \( \xi = \text{const.} = 0.2 \) and \( H(N) = H_e \left[ 1 - \tanh \left( N_e - N \right) \right] \).

### 5.2 The brane-world model

As was mentioned in the introduction we consider a five-dimensional brane scenario in which the Friedmann equation is modified to\(^{15}\)
\[
H^2 = \left( \frac{8\pi}{3m_{Pl}^2} \right) \rho_\phi \left( 1 + \frac{\rho_\phi}{2\lambda} \right),
\]  
(5.17)

where \( \lambda \) represents the brane tension.

Expression (5.17) can be written as
\[
\mathcal{F}(H) \equiv b \left[ \sqrt{1 + \left( \frac{2}{b} \right) H^2 - 1} \right] = \left( \frac{8\pi}{3m_{Pl}^2} \right) \rho_\phi,
\]  
(5.18)
where $b$ is defined by $b \equiv \frac{8\pi \lambda}{3m_{pl}^2}$.

From this latter equation and Eq. (2.3) we obtain for the scalar potential

$$V(\phi) = \lambda \left[ \sqrt{1 + \left( \frac{2}{b} \right) H^2} - 1 \right] - \frac{2}{9} \left( \frac{\lambda}{b^2} \right) \frac{(H')^2}{\left[ 1 + \left( \frac{2}{b} \right) H^2 \right]} \tag{5.19}$$

In order to obtain an explicit expression for the scalar potential we need to introduce an explicit expression for the Hubble parameter as a function of the scalar field. In this respect, we borrow the expression put forward by Hawkins and Lidsey for the Hubble parameter\[52]. Thus, we take

$$H(\phi) = \sqrt{b^2 \left[ \coth(\beta \phi) \sinh(\beta \phi) \right]} \tag{5.20}$$

respectively\[52]. Two comments are in order, first we demand that $C$ to be less that $\sqrt{6}$ in order that the potential is positive definite, and second, in the expression for the scale factor it is chosen that $t_0 = - \left( \frac{3m_{pl}^2}{14\pi AC^2} \right)^{1/2}$ in order to have $a(0) = 0$. For early time it is found that $a \sim t^{1/C^2}$, therefore for inflation to be realizable we need $C^2 < 1$.

The amount of comoving inflation becomes in this case

$$N(x) = \ln \left[ \left( \frac{\sinh(x_e)}{\sinh(x)} \right) \frac{\sqrt{2\pi C}}{m_{pl}} \frac{\cosh(x_{end})}{\cosh(x)} \right] \tag{5.22}$$

Here, $x \equiv \sqrt{2\pi C}/m_{pl}$ and $x_e = \sqrt{2\pi C}/m_{pl}$ where $\phi_e$ is the value of the scalar field at end of inflation, which corresponds to $\phi_e = \frac{m_{pl}}{\sqrt{2\pi C}} \text{sech}^{-1} \left[ \frac{1}{4} \sqrt{2 - C^2} \right]$.

In what concern to the attractor solutions, from Eq. (2.13) we get that

$$\delta H(\phi) = \delta H(\phi_i) \exp \int_{\phi_i}^{\phi} \left[ \frac{3}{\epsilon_H} + \frac{2}{b} + \frac{H^2}{H^2} \right] \frac{(H')^2}{H^2} \right|_0 d\phi. \tag{5.23}$$

The quantity in the square bracket is positive definite, thus the difference in sign between $H'$ and $d\phi$ makes the exponential negative, and therefore, the exponential rapidly tends to zero, showing the attractor feature.

Returning to the previously introduced expression for the Hubble parameter, we obtain for the various slow-roll parameters, which result to be given by

$$\epsilon_H(\phi) = \frac{C^2}{2} \left[ 1 + \text{sech}^2(\beta \phi) \right], \tag{5.24}$$

$$\eta_H(\phi) = \frac{C^2}{2} \left[ 1 + \frac{8}{3 + \cosh(2\beta \phi)} \right] \tag{5.25}$$
and
\[ \xi_H = \frac{C^4}{4} \left[ 1 + 5 \text{sech}^2(\beta \phi) + \frac{24}{3 + \cosh(2\beta \phi)} \right]. \] (5.26)

By using these expressions we could obtain \( n_s \) and \( r \) which become
\[ n_s - 1 = -\frac{C^2}{2} \text{sech}^2(\beta \phi) [5 + \cosh(2\beta \phi)] \]

and
\[ r = 2C^2 \tanh(\beta \phi), \]

respectively. It is not hard to show that the following relation holds
\[ r = 3C^2 + (n_s - 1). \] (5.27)

Now, due to observational constraint on \( r \), which presents an upper limit, \( r < 0.20 \) (95% CL) from WMAP+BAO+SN[53], where SN is the Constitution samples compiled in Ref. [54], and since \( n_s = 0.963 \pm 0.012 \) (excluding the Harrison-Zel’dovich spectrum in a value greater than 3\( \sigma \))[55], we get that the parameter \( C \) should satisfy the upper bound \( C < 0.079 \pm 0.004 \) in order to be in agreement with the observational data.

On the other hand, the consistency condition in this case becomes
\[ r = -\frac{2}{\sqrt{1 + \frac{b}{2} H^2}} n_T, \]

where \( n_T \) results to be \( n_T = -C^2 [1 + \text{sech}(\beta \phi)] \). Note that in the limit in which \( b \to \infty \) we obtain the standard results \( r = -2n_T \).

In what concern to the hierarchy slow-roll parameters equations we find
\[
\frac{d\epsilon_H}{dN} = \left[ 2 \left( \frac{1 + \frac{b}{2} H^2}{1 + \frac{b}{5} H^2} \right) \epsilon_H + \sigma \right] \epsilon_H, \\
\frac{d\sigma}{dN} = -\left[ \frac{5 + \frac{12}{5} b H^2}{1 + \frac{2}{5} b H^2} \right] \sigma + 12 \epsilon_H \left[ \epsilon_H + 2\xi_H \right], \\
\frac{d^{l+1} \lambda_H}{dN} = \left[ \frac{l - 2 - \frac{4}{5} b H^2}{1 + \frac{2}{5} b H^2} \right] \epsilon_H + \frac{1}{2}(l - 1)\sigma \right] l\lambda_H + (l+1)\lambda_H \quad (l \geq 2). \] (5.28)

Following an approach analogous to the previous subsection we solve this set numerically in the case in which the \( \xi_H \) parameter remains constant equal to 0.2, and we use expression (5.16) for the dependence of the Hubble parameter as a function of the number of e-folding. The result is shown in Fig 5. Note that \( \eta_H \) increases enormously close to the end of inflation. With this parameter much greater that one and \( \epsilon_H \) reaches the value equal to one at the end of inflation, the slow-roll approximation becomes unsustainable at the end of inflation.

6 Conclusion

We have studied inflationary universe models in terms of a single scalar field. We have applied the exact solution approach to the modified Friedmann equations. After describing the main characteristics of the inflationary model in general terms, we described in some details two specific models. First, we have studied a model characterized by a modified Friedmann equation of the type \( H^2 - \alpha H^4 = \left( \frac{3m_{\text{pl}}^2}{8\pi} \right) \rho \). Here, it was described the kinematical evolution...
Figure 5. Numerical solutions for $\epsilon_H$ and $\eta_H$ from the set of equations (5.28) in the case in which $\xi = \text{const.} = 0.2$. Here, was used that $H(N) = H_e [1 - \tanh (N_e - N)]$.

In the case in which the Hubble parameter evolves as $H(\phi) = H_0 (1 + \beta \phi)$. With this at hand, we could obtain the scalar potential, the corresponding number of e-folding and the attractor feature of the model. For some values of the parameters that enter into the scenario, we were able to characterize inflationary universe models.

In what concern to the scalar and tensor perturbations we calculated the scalar and tensor power spectrum generated by the quantum fluctuations of the scalar and the gravitational fields. We determined the scalar and tensor spectrum indices in term of the so-called slow-roll parameters, $\epsilon_H$, $\eta_H$ and $\xi_H$. From these quantities we were able to write down explicit expressions for the different parameters. Moreover, the shape of the contours in the $r - n_s$ plane results to be in agreement with those given by the WMAP 7. In fact, we have found that the tensor-to-scalar ratio can adequately accommodate the currently available observational data for some values of the parameters.

In the case of the brane-world model the functional form for the Hubble parameter was taken to be $H(\phi) = \sqrt{\beta} \left( \frac{\coth(\beta \phi)}{\sinh(\beta \phi)} \right)$, where $\beta = \frac{2\pi C}{m_{Pl}}$. With this expression we could determine all the kinematics and dynamics of the model. On the other hand, the current astrophysical data put an upper bound on the constant $C$, which becomes $C^2 < 0.079 \pm 0.004$.

One of the important point that we did not considered here was the reheating period. Since, in general terms, inflation is a period of supercooled expansion that, when inflation ends, the temperature of the universe needs to go up to a value such that it coincides with that corresponding to the temperature of the radiation epoch, and thus matching the Big Bang model. This issue, as far as we know, has not been studied under the exact approach. Perhaps, this study may give some insight on a deeper understanding of the period of reheating. We hope to address this point in the near future.
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