

A New Method for Band-limited Imaging with Undersampled Detectors

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ABSTRACT. Since its original use on the Hubble Deep Field, “Drizzle” has become a de facto standard for the combination of images taken by the Hubble Space Telescope. However, the Drizzle algorithm was developed with small, faint, partially resolved sources in mind and is not the best possible algorithm for unresolved objects with high signal-to-noise ratios. Here, a new method for creating band-limited images from undersampled data is presented. The method uses a drizzled image as a first-order approximation and then rapidly converges toward a band-limited image that fits the data, given the statistical weighting provided by the drizzled image. The method, named iDrizzle, for iterative Drizzle, effectively eliminates both the small high-frequency artifacts and convolution with an interpolant kernel that can be introduced by drizzling. The method works well in the presence of geometric distortion and can easily handle cosmic rays, bad pixels, or other missing data. It can combine images taken with random dithers, though the number of dithers required to obtain a good final image depends, in part, on the quality of the dither placements. iDrizzle may prove most beneficial for producing high-fidelity point-spread functions from undersampled images and could be particularly valuable for future dark energy missions such as Wide-Field Infrared Survey Telescope and Euclid, which will likely attempt to do high-precision supernova photometry and lensing experiments with undersampled detectors.

1. INTRODUCTION

Astronomical detectors are often undersampled. A wide field of view is often most economically obtained by using large pixels, and large pixels produce a minimum of read noise. Large pixels, however, result in undersampling. In order to fully sample an image with an undersampled detector, one must dither and combine multiple images. However, distortions in the field of view may make it impossible to perform shifts that equally well sample different parts of the detector. In practice, then, the combined pixels from dithering of astronomical detectors often produce irregular sampling of the image plane.

In order to combine the irregularly sampled data from the Hubble Deep Field (HDF; Williams et al. 1996), a new image algorithm, Drizzle (Fruchter and Hook 2002) was developed. Drizzle combines dithered images in a statistically optimal fashion. However, as can be seen in Figure 1, Drizzle adds small high-frequency artifacts to the image. On scales larger than an original pixel, these rapidly average out. Thus, for the prime purpose of the HDF, the study of small faint galaxies, Drizzle is an excellent algorithm. However, for the analysis of images of point sources with high signal-to-noise ratios (S/N), or other cases where preservation of the true point-spread function (PSF) is essential, one might prefer an algorithm which more exactly reproduces the highest-frequency features in the image.

In order to produce higher-fidelity images than are created by Drizzle, Lauer (1999) introduced a method that attempts to analytically predict the values of a regularly subsampled image based on the values of dithered images. This method works well when the dithers are nearly interlaced (that is, when the dither positions nearly fall on an evenly spaced grid) or when the data can be broken up into multiple sets of nearly interlaced data sets, though a loss in accuracy is seen as the samples diverge from interlacing. It cannot handle substantial image distortions, however, but must instead create local reconstructions. This reflects a general problem in the field—at present, there is no well-behaved analytical method for reconstructing a band-limited function from irregularly sampled data, even when that sampling everywhere reaches the Nyquist rate.

The spatial frequencies in an astronomical image are strongly limited by the optics of the telescope and convolution with the detector pixel. The effective angular power frequency cutoff is typically no larger than $|k| \leq D/\lambda_s$, where $D$ is the (maximum) aperture of the telescope and $\lambda_s$ is the shortest wavelength in the passband. Optical images are therefore band-limited (see Fig. 2). Given a set of undersampled individual exposures that are distorted and whose dithers therefore result in varying patterns of sampling across the field, solving for a single image that is consistent with all the exposures is equivalent to reconstructing a band-limited function from irregularly sampled data.

The reconstruction of a band-limited function from regularly sampled data through sinc interpolation is a well-known result of information theory due to Shannon (1948), though the result goes back to earlier work by Whittaker (1935) and in the Russian literature to Kotelnikov (1933). In the case of irregular sampling, obtaining an analytical representation of the function
is more complicated and involves the use of a type of separable Hilbert space called a “frame.” An introduction to the use of frame theory to solve the irregular sampling problem can be found in Benedetto (1992).

However, the frame theory solution for irregular sampling often leads to functional inversions that are numerically difficult and that, when the data are not well-oversampled, can be ill-conditioned, even in the absence of noise. This produces very large computational costs (Werther 1999; Aldroubi & Gröchenig 2001). Thus, the frame theory approach is rarely used in practice. Instead, iterative reconstruction techniques have been developed to solve for a band-limited function in the case of irregularly sampled data (e.g., Feichtinger & Gröchenig 1994; Werther 1999; Gröchenig and Strohmer 2001).

Here, I present a new iterative reconstruction method. In essence, it is an improvement upon a simple but powerful iterative technique, the Voronoi or nearest-neighbor approximation. In the method introduced here, the astronomical imaging algorithm Drizzle replaces the nearest-neighbor approximation. Drizzle allows this method to handle geometric distortion and combines the data using the full statistical power of the individual images. In the absence of noise, this method converges directly to the band-limited image. In the presence of noise, the method introduces a small increase in statistical noise, but the systematic high-frequency noise introduced by Drizzle (see Fig. 1) is removed.

1 See “A First Guided Tour on the Irregular Sampling Problem,” by Tobias Werther (http://www.math.ucdavis.edu/~strohmer/research/sampling/irsampling.html).

2. THE METHOD

One might imagine that one could take the total set of irregularly sampled data that everywhere meets the Nyquist criterion, perform a direct Fourier transform, remove any frequencies above the cutoff frequency, and use the inverse Fourier transform to arrive at the true band-limited image. Unfortunately, a direct Fourier transform of irregularly sampled data throws a great deal of power out of the original passband. This simple method thus fails terribly. It is, in fact, more effective to first put the data onto a regular Nyquist grid by simply taking the value of the nearest neighbor before doing the Fourier transform. The inverted band-limited function turns out to be a much truer approximation than the direct transform case. This approximation is known as the Voronoi approximation.

The Voronoi approximation is a band-limited function and thus can be sinc-interpolated to the irregular grid of the data. One can therefore subtract the Voronoi approximation from the original function at all of the data points. Furthermore, this smaller difference function is itself a band-limited function, so one can repeat the process and get a further-refined approximation to the underlying band-limited function. This procedure is known to converge geometrically (see Theorem 8.13 in Feichtinger and Gröchenig 1994).

The nearest-neighbor approximation is, however, far from ideal for astronomical imaging. Only one data sample is used at any point on the regular grid, even if several nearby data
samples could provide information, and there is no means to weight the data according to its statistical significance. Therefore, in the method proposed here the nearest-neighbor approximation is replaced by Drizzle in each iteration. While Drizzle introduces small artifacts (as does the nearest-neighbor method), the iterative comparison with the original data serves to remove these.

The proposed procedure is described in detail next. If the reader is not already familiar with the ideas discussed in the previous paragraphs and, in particular, the iterative use of the Voronoi approximation to obtain a band-limited image, following the flow of this procedure may not be easy. To these readers I strongly recommend a truly excellent and fairly short online tutorial. After understanding this tutorial, it will be clear that the procedure presented here is but a variation on a theme.

1. Drizzle the $N$ dithered images of a field, \{I_1, I_2, I_3 \ldots I_N\}, onto an oversampled output grid to produce the image $D_1$. The 1 in the subscript of $D$ represents that this is the first iteration. The drizzling is performed using weight files, which provide the statistical weights of the individual image and zero-out bad pixels, cosmic rays, and other significant defects.

2. Fourier transform the image $D_1$ to create $\tilde{D}_1 = F(D_1)$, where $F(\cdot)$ is the Fourier operator.

3. Produce a band-limited version of the transform of the image, $\tilde{B}_1 = L * \tilde{D}_1$, where $L(k) = 1$ when $\hat{k} = 0$, $L(\hat{k}) = 0$ when $\hat{k}$ lies outside of the band-limited region, and $L(\hat{k})$ tapers to 0 as $\hat{k}$ approaches the boundary of the band-limited region.

4. Transform the Fourier plane back to the image plane to obtain the first approximation to the true image, $A_1 = F^{-1}(\tilde{B}_1)$.

5. Map the first approximation to the combined image, $A_1$, back to the frames of the original individual images using blot (Fruchter and Hook 2002), producing a series of approximations to the original images, $A_1^m = blot(A_1 \Rightarrow I_1^m)$.

6. Subtract the blotted approximations from the corresponding original images to produce a series of new images $I_1^m = I_1^m - A_1^m$.

7. Return to the first step and now drizzle the new set of images \{I_1^1, I_1^2, I_1^3 \ldots I_1^N\} to produce the image $D_2$.

8. Continue as before, with one modification. Now at step 4, in the $N$th iteration, $A_N = F^{-1}(\tilde{B}_N) + \sum_{j=1}^{N-1} A_j$, since for $j > 1$, $D_j$ does not estimate the true image, but rather the difference between the true image and previous approximation.

9. After the iterations are complete, the final approximation is an oversampled image. This can be sinc-interpolated back down to critical sampling or, indeed, to an even-coarser scale if preferred.

With each iteration, the algorithm creates a band-limited approximation to the true underlying function and subtracts the approximation from the original data, leaving a difference function that must be approximated in the next iteration. However, the difference function, like the original true function, is band-limited. If the data points were on a regularly sampled grid, these iterations would not be necessary. One could compute the function on a set of desired output points simply by using the sinc interpolation. However, it is the fact the samples are irregular that makes this procedure necessary. Although the method creates a band-limited function at each step, the need to approximate the values on a regular grid causes an error. However, this error drops as the amplitude of the remaining function falls with each successive iteration.

In Figure 3 I show this procedure applied to simulated noiseless Advanced Camera for Surveys (ACS) images. A comparison with Figure 1 shows the large reduction of the drizzling artifacts, even in the first full iteration, and the rapid convergence after that. The reason for replacing the Voronoi approximation with Drizzle, however, was to handle cases with noise, and therefore in Figure 4 I show the same result with simulated noisy Wide-Field and Planetary Camera 2 (WFPC2) images. Again after a few iterations, all evidence of the stars is gone, and one sees only the noise of the sky.

Even though the WFPC2 is heavily undersampled, the moderate $S/N$ for the original stars used in Figure 4 does not push the method to the limit of possible observations. This is done more closely in the left-hand side of Figure 5, where I show the same subtraction performed on a combination of 12 simulated ACS images of the stellar field, with each star near saturation in a 1200 s exposure, with appropriate Poisson and read noise added. This tests the method in a situation of extremely high $S/N$: in essence, the highest $S/N$ it is likely to face in practice.

The residuals are close to that expected from noise statistics, and the peak errors are reduced from the drizzled subtraction by a factor of $\sim 20$. However, the introduction of noise greatly slows convergence—24 iterations were used to produce the output shown here.

On the right-hand side of Figure 5 is a central region of the Hubble Ultra Deep Field (UDF; Beckwith 2006). The bright star in this image is near saturation in each of the 12 1200 s individual exposures combined with the new method to form the final image. When this image is mapped back onto the individual input exposures and subtracted, residuals of approximately 2% peak are found under the stellar image. These small but measurable errors are most likely caused by temporal variations in the PSF produced by the change in the insolation of the telescope as it orbits the Earth. Here, the residuals on the bright star were not improved by going beyond a few iterations.

### 3. IMAGE PROPERTIES

#### 3.1. Image Fidelity

As noted in the discussion of the method, iDrizzle will converge exactly in the absence of noise. In the presence of noise, however, successive iterations eventually lose their ability to improve the image as the power in the statistical noise overwhelms any systematic errors caused by the failure to converge to the correct underlying band-limited function. For example, the
noise in the combined image tested in left-hand side of Figure 5 is dominated by the Poisson noise of the bright stars in the original exposures. This statistical noise limits the convergence of the method, and prevents the complete removal of the faint “ringing” seen surrounding the centers of the (subtracted) point sources. This is because two pixels from two different images may lie very close to one another on the sky, but have very different noise values. Thus, the noise breaks the assumption that the data is truly band-limited, and the effect of this is most apparent in the Poisson noise of bright point sources. However, the ringing in the image is only $\sim 1 \times 10^{-4}$ of peak and is limited to a radius of a few times the full width at half-maximum (FWHM) of the PSF. Without the subtraction of the original point source, the ringing is not visible.

The test shown on the left-hand side of Figure 5 was conducted using 12 simulated ACS images from Hubble Space Telescope (HST) with random subpixel dithers. Good subtractions were seen with the ACS PSF and sampling with as few as eight random positions. When the same test was repeated with simulated WFPC2 images, which are undersampled to a far greater degree, a minimum of 12 random positions was required before one saw good convergence. At the same time, eight optimally placed dithers gave excellent results using WFPC2 sampling. The ability of iDrizzle to accurately reproduce the PSF will depend on the ratio of the FWHM of the PSF to the size of the detector pixels, the number of samples that are available, and the pattern of those samples. Additionally, because the removal of the systematic or pattern errors can be limited by the statistical noise power, deeper integrations will not only improve the statistical errors, they will also generally improve the systematic fidelity of the images.

3.2. Statistical Noise Amplification

Drizzle places the center of a pixel output exactly where it was observed. However, the average weight of an output pixel will not necessarily fall at the center of the pixel, and there is thus a jitter between the represented and effective position of a pixel. Furthermore, the peak of a drizzled PSF will never be greater than the greatest value in the appropriate region of the input images. By contrast, iDrizzle attempts to predict the true value of the image at the center of the output pixel, and thus the peak of a PSF will sometimes be brighter than any value of the input images in the appropriate region. iDrizzle essentially uses estimates of the derivatives of the data to extrapolate to a position (the center of the output pixel) that is not necessarily exactly sampled in any of the input images. This will produce some noise amplification, which will vary with the quality of the dithering. In typical tests performed on this method, the noise amplification has been in the region of 10%. This noise amplification, however, does not prevent the method from obtaining extraordinary results in high-S/N images. For instance, when the PSF fitting software DAOPHOT (Stetson 1987) is applied to the PSF based on the UDF bright star, it finds statistically identical photometry when measuring the true representation of ACS PSFs (Fig. 1) with purely statistical noise introduced and when measuring the iDrizzle reconstruction. Both are at millimagnitude accuracy and challenge the precision of the photometry software. As the S/N level is lowered by using fainter simulated stars against a constant noise background, both the introduced noise and the noise surplus of about 10% caused by the method become evident. Nonetheless, as can be seen in Figure 3, the image reconstruction remains very accurate. The noise is dominated by statistical, rather than systematic, errors.

3.3. Correlated Noise and the Choice of the Inversion Filter

Images produced by Drizzle show correlated noise. Part of this correlation is caused by the drizzling process itself—a non-zero value of $\text{pixfrac}$ causes Drizzle to place a given input pixel value down on a region of linear size $p \times l$, where $p$ is the value of $\text{pixfrac}$ and $l$ is the linear size of an input pixel. As the iteration proceeds, iDrizzle effectively forces $p$ to zero. However, as noted earlier, noise correlation is introduced to the image by the application of the tapering function $L(\tilde{k})$. White noise is turned into reddish (correlated) noise by this procedure.
Thus, there is a competition between suppressing ringing near bright unresolved sources while maintaining optimal noise properties across the field. This ringing is not caused by any power in the band-limited image passed by the telescope, but rather by the Poisson noise of the bright sources. The limiting angular frequency of this power is not determined by the Nyquist limit of the original images, but rather by the spacing of the dithers of the combined images. A gentle taper in Fourier space will best suppress this ringing. However, to pass all of the information allowed by the optics, while limiting noise correlation, one prefers a sharp cutoff near the Nyquist frequency of |k| ≈ λ/D. For the examples shown here, a circular mask (in Fourier space) that falls from 90% to 10% transmission over a width of 0.1λ/D is used. The sampling of the drizzled image was nearly twice the Nyquist sampling of the band-limited image. This was found to give good suppression of ringing in the simulated WFPC2 and ACS images, while limiting correlated noise and minimizing the loss of true information in the image. The center of the taper function was slightly to the blue of the Nyquist frequency. This essentially eliminates correlated noise (when a final image is created with Nyquist sampling), at the expense of allowing a slightly greater amount of noise through the filter. No claim is made that this particular choice of cutoff, apodization, or sampling is optimal, but it was found to be effective.

3.4. Determining Convergence

The algorithm described here is part of a class of techniques that vary in their use of specific weightings or interpolation schemes to iteratively arrive at a solution of the irregular sampling problem. These all tend to converge geometrically to the correct solution, with the speed of convergence being dependent on the value of 2lν0, where l is the maximal distance between samples and ν0 is the Nyquist frequency of the data. Where 2lν0 ≪ 1, convergence is rapid; however, where 2lν0 ∼ 1, convergence is slow or fails (see Werther 1999, eq. [50], for a specific case, and the online tutorial\footnote{http://drizzle.stsci.edu} and Feichtinger & Gröchenig 1994 for broader discussions).

In the case being discussed here, however, there is no unique solution because of the presence of noise. Further, because of geometric distortion, the value of l (and thus 2lν0) is likely to vary across the image (and thus so will the rapidity of local convergence).

In practice, it is fairly simple for the user to test whether a particular observing plan with a given instrument is likely to produce acceptable images. To do this, the user should create a well-sampled output image. In the test cases shown here, PSFs placed at regular intervals with a small random dither were used. The user can vary the amplitude of the PSF as well, to see how the technique works in different noise regimes. This simulated image is then mapped to the input image scale using the Drizzle package routine, blot. Blot will, with the correct distortion files, map the image back as it would have appeared on the distorted field of the detector. The user can create as many input images as he or she desires, with a dither pattern emulating the one expected to be obtained from the observations. Noise can then be added to each image to match the level expected in the observations. Finally, one can combine the images using iDrizzle. This will allow the user to check how many iterations it will take to obtain a reasonable result. As the amplification of the noise tends to grow with the number of iterations, greatly exceeding the number of required iterations is not advised. Usually, a few to a dozen iterations will do. Indeed, in the case of the real UDF data, convergence stopped after only a few iterations. As mentioned earlier, this is because the technique was limited by the variability of the HST PSF.

4. DISCUSSION AND FUTURE DEVELOPMENTS

In this article, a new method for the combination of dithered astronomical images has been introduced. The method, iDrizzle, has the ability to handle shifts, distortions, and missing data and converges rapidly to an accurate representation of the underlying image. It provides a dramatic improvement in fidelity over Drizzle, particularly on resolved objects, at the cost of a small increase in statistical noise. It requires, however, that the combined images come close to Nyquist sampling across the combined image.

The algorithm upon which this work is in part based, Drizzle, by itself creates excellent images of resolved objects (such as galaxies in a deep HST image) and will produce images of slightly higher-S/N images than iDrizzle in cases when statistical noise is larger than the small artifacts of using Drizzle. Similarly, observers attempting precise stellar photometry may use Drizzle and/or iDrizzle to locate their stars and remove image defects, and they can do photometry on each individual image and combine results to avoid the small increase in statistical noise produced by iDrizzle. However, this approach would require a substantial amount of effort to avoid the inherent...
photometric biases of working with undersampled data—something that iDrizzle does directly. Where iDrizzle truly stands out is in creating accurate images of objects with unresolved or nearly unresolved components. As the test with the UDF data showed, iDrizzle allows far more accurate image subtraction and thus may be of great use to observers trying to remove powerful point sources underlying extended objects, such as active galactic nuclei, from their host galaxies. In the long run though, an important use of iDrizzle may be the most direct application: the creation of point-source functions.

Future space-based wide-field imagers are likely to be undersampled. Nonetheless, they will be used (and some will be expressly intended) for applications such as lensing, which require excellent knowledge of the PSF. As shown here, with a moderate number of dithers, even with random sampling, iDrizzle can produce high-fidelity PSFs. The desire to cover a large area of sky, however, may mean that some projects will prefer to keep the number of dithers in a given filter to a bare minimum—four exposures may be a common preference. While four dithers is too few for iDrizzle to estimate the image in the absence of near regular sampling, iDrizzle could be used to recover the PSF if the telescope has either temporal or spatial stability. With temporal stability, images of different stars with similar colors taken near the same location on the detector could be combined to form a single PSF. Similarly, if the PSF is stable over a sufficiently wide field, different stars of similar color from a single image may be combined as a substitute for an increased number of dithers. While in both cases one will have to fit for the magnitudes and relative positions of the stars, as magnitudes and positions are identical (up to a known shift) for all dither sets of the same star, the small number of parameters fits should typically add little noise, particularly if the fitting is done iteratively while solving for the shape of the PSF.

The new method presented here, iDrizzle, may find use in a number of applications where high image fidelity is required. In order that the community may test this algorithm and expand upon its use, the Pyraf script used to create the UDF image, along with the UDF data used in this test, is posted online for download by interested readers.2

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2 See http://www.stsci.edu/~fruchter/iDrizzle/UDF-iDrizzle.