Non-properly embedded $H$-planes in $\mathbb{H}^2 \times \mathbb{R}$

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Abstract For any $H \in (0, \frac{1}{2})$, we construct complete, non-proper, stable, simply-connected surfaces embedded in $\mathbb{H}^2 \times \mathbb{R}$ with constant mean curvature $H$.

1 Introduction

In their ground breaking work [2], Colding and Minicozzi proved that complete minimal surfaces embedded in $\mathbb{R}^3$ with finite topology are proper. Based on the techniques in [2], Meeks and Rosenberg [5] then proved that complete minimal surfaces with positive injectivity embedded in $\mathbb{R}^3$ are proper. More recently, Meeks and Tinaglia [7]...
proven that complete constant mean curvature surfaces embedded in $\mathbb{R}^3$ are proper if they have finite topology or have positive injectivity radius.

In contrast to the above results, in this paper we prove the following existence theorem for non-proper, complete, simply-connected surfaces embedded in $\mathbb{H}^2 \times \mathbb{R}$ with constant mean curvature $H \in (0, 1/2)$. The convention used here is that the mean curvature function of an oriented surface $M$ in an oriented Riemannian three-manifold $N$ is the pointwise average of its principal curvatures.

Theorem 1.1 For any $H \in (0, 1/2)$ there exists a complete, stable, simply-connected surface $\Sigma_H$ embedded in $\mathbb{H}^2 \times \mathbb{R}$ with constant mean curvature $H$ satisfying the following properties:

(1) The closure of $\Sigma_H$ is a lamination with three leaves, $\Sigma_H$, $C_1$ and $C_2$, where $C_1$ and $C_2$ are stable catenoids of constant mean curvature $H$ in $\mathbb{H}^3$ with the same axis of revolution $L$. In particular, $\Sigma_H$ is not properly embedded in $\mathbb{H}^2 \times \mathbb{R}$.

(2) Let $K_L$ denote the Killing field generated by rotations around $L$. Every integral curve of $K_L$ that lies in the region between $C_1$ and $C_2$ intersects $\Sigma_H$ transversely in a single point. In particular, the closed region between $C_1$ and $C_2$ is foliated by surfaces of constant mean curvature $H$, where the leaves are $C_1$ and $C_2$ and the rotated images $\Sigma_H(\theta)$ of $\Sigma$ around $L$ by angle $\theta \in [0, 2\pi)$.

When $H = 0$, Rodriguez and Tinaglia [10] constructed non-proper, complete minimal planes embedded in $\mathbb{H}^2 \times \mathbb{R}$. However, their construction does not generalize to produce complete, non-proper planes embedded in $\mathbb{H}^2 \times \mathbb{R}$ with non-zero constant mean curvature. Instead, the construction presented in this paper is related to the techniques developed by the authors in [3] to obtain examples of non-proper, stable, complete planes embedded in $\mathbb{H}^3$ with constant mean curvature $H$, for any $H \in (0, 1)$.

There is a general conjecture related to Theorem 1.1 and the previously stated positive properness results. Given $X$ a Riemannian three-manifold, let $\text{Ch}(X) := \inf_{S \in S} \frac{\text{Area}(\partial S)}{\text{Volume}(S)}$, where $S$ is the set of all smooth compact domains in $X$. Note that when the volume of $X$ is infinite, $\text{Ch}(X)$ is the Cheeger constant.

Conjecture 1.2 Let $X$ be a simply-connected, homogeneous three-manifold. Then for any $H \geq \frac{1}{2} \text{Ch}(X)$, every complete, connected $H$-surface embedded in $X$ with positive injectivity radius or finite topology is proper. On the other hand, if $\text{Ch}(X) > 0$, then there exist non-proper complete $H$-planes in $X$ for every $H \in [0, \frac{1}{2} \text{Ch}(X)]$. 

By the work in [2], Conjecture 1.2 holds for $X = \mathbb{R}^3$ and it holds in $\mathbb{H}^3$ by work in progress in [6]. Since the Cheeger constant of $\mathbb{H}^2 \times \mathbb{R}$ is 1, Conjecture 1.2 would imply that Theorem 1.1 (together with the existence of complete non-proper minimal planes embedded in $\mathbb{H}^2 \times \mathbb{R}$ found in [10]) is a sharp result.

2 Preliminaries

In this section, we will review the basic properties of $H$-surfaces, a concept that we next define. We will call a smooth oriented surface $\Sigma_H$ in $\mathbb{H}^2 \times \mathbb{R}$ an $H$-surface if
it is embedded and its mean curvature is constant equal to $H$; we will assume that $\Sigma_H$ is appropriately oriented so that $H$ is non-negative. We will use the cylinder model of $\mathbb{H}^2 \times \mathbb{R}$ with coordinates $(\rho, \theta, t)$; here $\rho$ is the hyperbolic distance from the origin (a chosen base point) in $\mathbb{H}_t^2$, where $\mathbb{H}_t^2$ denotes $\mathbb{H}^2 \times \{t\}$. We next describe the $H$-catenoids mentioned in the Introduction.

The following $H$-catenoids family will play a particularly important role in our construction.

### 2.1 Rotationally invariant vertical $H$-catenoids $C_d^H$

We begin this section by recalling several results in [8,9]. Given $H \in (0, \frac{1}{2})$ and $d \in [-2H, \infty)$, let

$$\eta_d = \cosh^{-1}\left(\frac{2dH + \sqrt{1 - 4H^2 + d^2}}{1 - 4H^2}\right)$$

and let $\lambda_d : [\eta_d, \infty) \to [0, \infty)$ be the function defined as follows.

$$\lambda_d(\rho) = \int_{\eta_d}^{\rho} \frac{d + 2H \cosh r}{\sqrt{\sinh^2 r - (d + 2H \cosh r)^2}} dr. \quad (1)$$

Note that $\lambda_d(\rho)$ is a strictly increasing function with $\lim_{\rho \to \infty} \lambda_d(\rho) = \infty$ and derivative $\lambda_d'(\eta_d) = \infty$ when $d \in (-2H, \infty)$.

In [8] Nelli, Sa Earp, Santos and Toubiana proved that there exists a 1-parameter family of embedded $H$-catenoids $\{C_d^H \mid d \in (-2H, \infty)\}$ obtained by rotating a generating curve $\lambda_d(\rho)$ about the $t$-axis. The generating curve $\tilde{\lambda}_d$ is obtained by doubling the curve $(\rho, 0, \lambda_d(\rho))$, $\rho \in [\eta_d, \infty)$, with its reflection $(\rho, 0, -\lambda_d(\rho))$, $\rho \in [\eta_d, \infty)$. Note that $\lambda_d$ is a smooth curve and that the necksize, $\eta_d$, is a strictly increasing function in $d$ satisfying the properties that $\eta_{-2H} = 0$ and $\lim_{d \to \infty} \eta_d = \infty$.

If $d = -2H$, then by rotating the curve $(\rho, 0, \lambda_d(\rho))$ around the $t$-axis one obtains a simply-connected $H$-surface $E_H$ that is an entire graph over $\mathbb{H}_0^2$. We denote by $-E_H$ the reflection of $E_H$ across $\mathbb{H}_0^2$.

We next recall the definition of the mean curvature vector.

**Definition 2.1** Let $M$ be an oriented surface in an oriented Riemannian three-manifold and suppose that $M$ has non-zero mean curvature $H(p)$ at $p$. The **mean curvature vector** at $p$ is $\mathbf{H}(p) := H(p)\mathbf{N}(p)$, where $\mathbf{N}(p)$ is its unit normal vector at $p$. The mean curvature vector $\mathbf{H}(p)$ is independent of the orientation on $M$.

Note that the mean curvature vector $\mathbf{H}$ of $C_d^H$ points into the connected component of $\mathbb{H}^2 \times \mathbb{R} - C_d^H$ that contains the $t$-axis. The mean curvature vector of $E_H$ points upward while the mean curvature vector of $-E_H$ points downward.

In order to construct the examples described in Theorem 1.1, we first obtain certain geometric properties satisfied by $H$-catenoids. For example, in the following lemma, we show that for certain values of $d_1$ and $d_2$, the catenoids $C_{d_1}^H$ and $C_{d_2}^H$ are disjoint.
Given \( d \in (-2H, \infty) \), let \( b_d(t) := \lambda_d^{-1}(t) \) for \( t \geq 0 \); note that \( b_d(0) = \eta_d \). Abusing the notation let \( b_d(t) := b_d(-t) \) for \( t \leq 0 \).

**Lemma 2.1 (Disjoint \( H \)-catenoids)** Given \( d_1 > 2 \), there exist \( d_0 > d_1 \) and \( \delta_0 > 0 \) such that for any \( d_2 \in [d_0, \infty) \), then

\[
\inf_{t \in \mathbb{R}} (b_{d_2}(t) - b_{d_1}(t)) \geq \delta_0.
\]

In particular, the corresponding \( H \)-catenoids are disjoint, i.e. \( \mathcal{C}^H_{d_1} \cap \mathcal{C}^H_{d_2} = \emptyset \).

Moreover, \( b_{d_2}(t) - b_{d_1}(t) \) is decreasing for \( t > 0 \) and increasing for \( t < 0 \). In particular,

\[
\sup_{t \in \mathbb{R}} (b_{d_2}(t) - b_{d_1}(t)) = b_{d_2}(0) - b_{d_1}(0) = \eta_{d_2} - \eta_{d_1}.
\]

The proof of the above lemma requires a rather lengthy computation that is given in the Appendix.

We next recall the well-known mean curvature comparison principle.

**Proposition 2.2 (Mean curvature comparison principle)** Let \( M_1 \) and \( M_2 \) be two complete, connected embedded surfaces in a three-dimensional Riemannian manifold. Suppose that \( p \in M_1 \cap M_2 \) satisfies that a neighborhood of \( p \) in \( M_1 \) locally lies on the side of a neighborhood of \( p \) in \( M_2 \) into which \( H_2(p) \) is pointing. Then \( |H_1|(p) \geq |H_2|(p) \). Furthermore, if \( M_1 \) and \( M_2 \) are constant mean curvature surfaces with \( |H_1| = |H_2| \), then \( M_1 = M_2 \).

### 3 The examples

For a fixed \( H \in (0, 1/2) \), the outline of construction is as follows. First, we will take two disjoint \( H \)-catenoids \( \mathcal{C}_1 \) and \( \mathcal{C}_2 \) whose existence is given in Lemma 2.1. These catenoids \( \mathcal{C}_1, \mathcal{C}_2 \) bound a region \( \Omega \) in \( \mathbb{H}^2 \times \mathbb{R} \) with fundamental group \( \mathbb{Z} \).

In the universal cover \( \widetilde{\Omega} \) of \( \Omega \), we define a piecewise smooth compact exhaustion \( \Delta_1 \subset \Delta_2 \subset \cdots \subset \Delta_n \subset \cdots \) of \( \widetilde{\Omega} \). Then, by solving the \( H \)-Plateau problem for special curves \( \Gamma_n \subset \partial \Delta_n \), we obtain minimizing \( H \)-surfaces \( \Sigma_n \) in \( \Delta_n \) with \( \partial \Sigma_n = \Gamma_n \).

In the limit set of these surfaces, we find an \( H \)-plane \( \Sigma_H \subset \mathbb{H}^2 \times \mathbb{R} \).

#### 3.1 Construction of \( \widetilde{\Omega} \)

Fix \( H \in (0, \frac{1}{2}) \) and \( d_1, d_2 \in (2, \infty) \), \( d_1 < d_2 \), such that by Lemma 2.1, the related \( H \)-catenoids \( \mathcal{C}^H_{d_1} \) and \( \mathcal{C}^H_{d_2} \) are disjoint; note that in this case, \( \mathcal{C}^H_{d_1} \) lies in the interior of the simply-connected component of \( \mathbb{H}^2 \times \mathbb{R} - \mathcal{C}^H_{d_2} \). We will use the notation \( \mathcal{C}_i := \mathcal{C}^H_{d_i} \). Recall that both catenoids have the same rotational axis, namely the \( t \)-axis, and recall that the mean curvature vector \( H_i \) of \( \mathcal{C}_i \) points into the connected component of
$\mathbb{H}^2 \times \mathbb{R} - \mathcal{C}_i$ that contains the $t$-axis. We emphasize here that $H$ is fixed and so we will omit describing it in future notations.

Let $\Omega$ be the closed region in $\mathbb{H}^2 \times \mathbb{R}$ between $\mathcal{C}_1$ and $\mathcal{C}_2$, i.e., $\partial \Omega = \mathcal{C}_1 \cup \mathcal{C}_2$ (Fig. 1-left). Notice that the set of boundary points at infinity $\partial_{\infty} \Omega$ is equal to $S_{\infty}^1 \times \{-\infty\} \cup S_{\infty}^1 \times \{\infty\}$, i.e., the corner circles in $\partial_{\infty} \mathbb{H}^2 \times \mathbb{R}$ in the product compactification, where we view $\mathbb{H}^2$ to be the open unit disk $\{(x, y) \in \mathbb{R}^2 \mid x^2 + y^2 < 1\}$ with base point the origin $\vec{0}$.

By construction, $\Omega$ is topologically a solid torus. Let $\tilde{\Omega}$ be the universal cover of $\Omega$. Then, $\partial \tilde{\Omega} = \tilde{\mathcal{C}}_1 \cup \tilde{\mathcal{C}}_2$ (Fig. 1-right), where $\tilde{\mathcal{C}}_1, \tilde{\mathcal{C}}_2$ are the respective lifts to $\tilde{\Omega}$ of $\mathcal{C}_1, \mathcal{C}_2$. Notice that $\tilde{\mathcal{C}}_1$ and $\tilde{\mathcal{C}}_2$ are both $H$-planes, and the mean curvature vector $H$ points outside of $\tilde{\Omega}$ along $\tilde{\mathcal{C}}_1$ while $H$ points inside of $\tilde{\Omega}$ along $\tilde{\mathcal{C}}_2$. We will use the induced coordinates $(\rho, \tilde{\theta}, t)$ on $\tilde{\Omega}$ where $\tilde{\theta} \in (-\infty, \infty)$. In particular, if

$$\Pi: \tilde{\Omega} \to \Omega$$

is the covering map, then $\Pi(\rho_0, \tilde{\theta}_0, t_0) = (\rho_0, \theta_0, t_0)$ where $\theta_0 \equiv \tilde{\theta}_0 \mod 2\pi$.

Recalling the definition of $b_i(t)$, $i = 1, 2$, note that a point $(\rho, \theta, t)$ belongs to $\Omega$ if and only if $\rho \in [b_1(t), b_2(t)]$ and we can write

$$\tilde{\Omega} = \{(\rho, \tilde{\theta}, t) \mid \rho \in [b_1(t), b_2(t)], \tilde{\theta} \in \mathbb{R}, \ t \in \mathbb{R}\}.$$

### 3.2 Infinite bumps in $\tilde{\Omega}$

Let $\gamma$ be the geodesic through the origin in $\mathbb{H}_0^2$ obtained by intersecting $\mathbb{H}_0^2$ with the vertical plane $\{\theta = 0\} \cup \{\theta = \pi\}$. For $s \in [0, \infty)$, let $\varphi_s$ be the orientation preserving hyperbolic isometry of $\mathbb{H}_0^2$ that is the hyperbolic translation along the geodesic $\gamma$ with $\varphi_s(0, 0) = (s, 0)$. Let

$$\hat{\varphi}_s: \mathbb{H}^2 \times \mathbb{R} \to \mathbb{H}^2 \times \mathbb{R}, \ \hat{\varphi}_s(\rho, \theta, t) = (\varphi_s(\rho, \theta), t)$$

be the related extended isometry of $\mathbb{H}^2 \times \mathbb{R}$.
Let $C_d$ be an embedded $H$-catenoid as defined in Sect. 2.1. Notice that the rotation axis of the $H$-catenoid $\hat{\Omega}_0(C_d)$ is the vertical line $\{(s_0, 0, t) \mid t \in \mathbb{R}\}$.

Let $\delta := \inf_{t \in \mathbb{R}} (b_2(t) - b_1(t))$, which gives an upper bound estimate for the asymptotic distance between the catenoids; recall that by our choices of $C_1, C_2$ given in Lemma 2.1, we have $\delta > 0$. Let $\delta_1 = \frac{1}{2} \min{\delta, \eta_1}$ and let $\delta_2 = \delta - \frac{\delta_1}{2}$. Let $\hat{C}_1 := \hat{\varphi}_{\delta_1}(C_1)$ and $\hat{C}_2 := \hat{\varphi}_{-\delta_2}(C_2)$. Note that $\delta_1 + \delta_2 > \delta$.

Claim 3.1 The intersection $\Omega \cap \hat{C}_i, i = 1, 2$, is an infinite strip.

Proof Given $t \in \mathbb{R}$, let $\mathbb{H}^2$ denote $\mathbb{H}^2 \times \{t\}$. Let $\tau_i^1 := C_i \cap \mathbb{H}^2$ and $\tau_i^2 := \hat{\mathcal{C}}_i \cap \mathbb{H}^2$. Note that for $i = 1, 2$, $\tau_i^1$ is a circle in $\mathbb{H}^2$ of radius $b_1(t)$ centered at $(0,0,t)$ while $\tau_i^2$ is a circle in $\mathbb{H}^2$ of radius $b_2(t)$ centered at $p_1, t := (\delta, 0, t)$ and $\tau_i^2$ is a circle in $\mathbb{H}^2$ of radius $b(t)$ centered at $p_2, t := (-\delta, 0, t)$. We claim that for any $t \in \mathbb{R}$, the intersection $\tau_i^1 \cap \Omega$ is an arc with an end point in $\tau_i^1$, $i = 1, 2$. This result would give that $\Omega \cap \hat{C}_i$ is an infinite strip. We next prove this claim.

Consider the case $i = 1$ first. Since $\delta_1 < \eta_1 \leq b_1(t)$, the center $p_1$, $t$ is inside the disk in $\mathbb{H}^2$ bounded by $\tau_i^1$. Since the radii of $\tau_i^1$ and $\tau_i^2$ are both equal to $b_1(t)$, then the intersection $\tau_i^1 \cap \tau_i^2$ is nonempty. It remains to show that $\tau_i^2 \cap (\delta_1 + \eta_1, 2)$. Let $\delta_1 < \delta_1 = \inf_{t \in \mathbb{R}} (b_2(t) - b_1(t))$.

This argument shows that $\Omega \cap \hat{C}_1$ is an infinite strip.

Consider now the case $i = 2$. Since $\delta_2 < \delta < b_2(t)$, the center $p_2, t$ is inside the disk in $\mathbb{H}^2$ bounded by $\tau_i^2$. Since the radii of $\tau_i^2$ and $\tau_i^2$ are both equal to $b_2(t)$, then the intersection $\tau_i^2 \cap \tau_i^2$ is nonempty. It remains to show that $\tau_i^2 \cap \tau_i^2 = \emptyset$, namely that $b_2(t) - \delta_2 > b_1(t)$. This follows because

$$b_2(t) - b_1(t) \geq \inf_{t \in \mathbb{R}} (b_2(t) - b_1(t)) = \delta > \delta_2$$

This completes the proof that $\Omega \cap \hat{C}_2$ is an infinite strip and finishes the proof of the claim.

Now, let $Y^+ := \Omega \cap \hat{C}_2$ and let $Y^- := \Omega \cap \hat{C}_1$. In light of Claim 3.1 and its proof, we know that $Y^+ \cap C_1 = \emptyset$ and $Y^- \cap C_2 = \emptyset$.

**Fig. 2** The position of the bumps $B^\pm$ in $\hat{\Omega}$ is shown in the picture. The small arrows show the mean curvature vector direction. The $H$-surfaces $\Sigma_n$ are disjoint from the infinite strips $B^\pm$ by construction.
Remark 3.2 Note that by construction, any rotational surface contained in Ω must intersect \( C_1 \cup C_2 \). In particular, \( Y^+ \cup Y^- \) intersects all \( H \)-catenoids \( C_d \) for \( d \in (d_1, d_2) \) as the circles \( C_d \cap \mathbb{H}^2 \) intersect either the circle \( \hat{\tau}_t \) or the circle \( \hat{\tau}^1_t \) for some \( t > 0 \) since \( \delta_1 + \delta_2 > \delta \).

In \( \tilde{\Omega} \), let \( B^+ \) be the lift of \( Y^+ \) in \( \tilde{\Omega} \) which intersects the slice \( \{ \tilde{\theta} = -10\pi \} \). Similarly, let \( B^- \) be the lift of \( Y^- \) in \( \tilde{\Omega} \) which intersects the slice \( \{ \tilde{\theta} = 10\pi \} \). Note that each lift of \( Y^+ \) or \( Y^- \) is contained in a region where the \( \tilde{\theta} \) values of their points lie in ranges of the form \( (\theta_0 - \pi, \theta_0 + \pi) \) and so \( B^+ \cap B^- = \emptyset \). See Fig. 2.

The \( H \)-surfaces \( B^\pm \) near the top and bottom of \( \tilde{\Omega} \) will act as barriers (infinite bumps) in the next section, ensuring that the limit \( H \)-plane of a certain sequence of compact \( H \)-surfaces does not collapse to an \( H \)-lamination of \( \tilde{\Omega} \) all of whose leaves are invariant under translations in the \( \tilde{\theta} \)-direction.

Next we modify \( \tilde{\Omega} \) as follows. Consider the component of \( \tilde{\Omega} - (B^+ \cup B^-) \) containing the slice \( \{ \tilde{\theta} = 0 \} \). From now on we will call the closure of this region \( \tilde{\Omega}^* \).

### 3.3 The compact exhaustion of \( \tilde{\Omega}^* \)

Consider the rotationally invariant \( H \)-planes \( E_H, -E_H \) described in Sect. 2. Recall that \( E_H \) is a graph over the horizontal slice \( \mathbb{H}^2_0 \) and it is also tangent to \( \mathbb{H}^2_0 \) at the origin. Given \( t \in \mathbb{R} \), let \( E'_H = -E_H + (0, 0, t) \) and \( -E'_H = E_H - (0, 0, t) \). Both families \( \{E'_H\}_{t \in \mathbb{R}} \) and \( \{-E'_H\}_{t \in \mathbb{R}} \) foliate \( \mathbb{H}^2 \times \mathbb{R} \). Moreover, there exists \( n_0 \in \mathbb{N} \) such that for any \( n > n_0 \), \( n \in \mathbb{N} \), the following holds. The highest (lowest) component of the intersection \( S^+_n := E^n_H \cap \Omega \) (\( S^-_n := -E^n_H \cap \Omega \)) is a rotationally invariant annulus with boundary components contained in \( C_1 \) and \( C_2 \). The annulus \( S^+_n \) lies “above” \( S^-_n \) and their intersection is empty. The region \( U_n \) in \( \Omega \) between \( S^+_n \) and \( S^-_n \) is a solid torus, see Fig. 3-left, and the mean curvature vectors of \( S^+_n \) and \( S^-_n \) point into \( U_n \).

Let \( \tilde{U}_n \subset \tilde{\Omega} \) be the universal cover of \( U_n \), see Fig. 3-right. Then, \( \partial \tilde{U}_n - \partial \tilde{\Omega} = \tilde{S}^+_n \cup \tilde{S}^-_n \) where can view \( \tilde{S}^+_n \) as a lift to \( \tilde{U}_n \) of the universal cover of the annulus \( S^+_n \). Hence,
$\tilde{S}_n^\pm$ is an infinite $H$-strip in $\tilde{\Omega}$, and the mean curvature vectors of the surfaces $\tilde{S}_n^+$, $\tilde{S}_n^-$ point into $\tilde{U}_n$ along $\tilde{S}_n^\mp$. Note that each $\tilde{U}_n$ has bounded $t$-coordinate. Furthermore, we can view $\tilde{U}_n$ as $(U_n \cap \mathcal{P}_0) \times \mathbb{R}$, where $\mathcal{P}_0$ is the half-plane $\{\theta = 0\}$ and the second coordinate is $\tilde{\theta}$. Abusing the notation, we redefine $\tilde{U}_n$ to be $\tilde{U}_n \cap \tilde{\Omega}^*$, that is we have removed the infinite bumps $B^\pm$ from $\tilde{U}_n$.

Now, we will perform a sequence of modifications of $\tilde{U}_n$ so that for each of these modifications, the $\theta$-coordinate in $\tilde{U}_n$ is bounded and so that we obtain a compact exhaustion of $\tilde{\Omega}^*$. In order to do this, we will use arguments that are similar to those in Claim 3.1. Recall that the necksize of $C_2$ is $\eta_2 = b_2(0)$. Let $\tilde{C}_3 = \tilde{\varphi}_{\eta_2}(C_2)$, see equation (3) for the definition of $\tilde{\varphi}_{\eta_2}$. Then, $\tilde{C}_3$ is a rotationally invariant catenoid whose rotational axis is the line $(\eta_2, 0) \times \mathbb{R}$ (Fig. 4-left).

**Lemma 3.3** The intersection $\tilde{C}_3 \cap \Omega$ is a pair of infinite strips.

**Proof** It suffices to show that $\tilde{C}_3 \cap \tilde{C}_1$ and $\tilde{C}_3 \cap \tilde{C}_2$ each consists of a pair of infinite lines. Now, consider the horizontal circles $\varphi_i^1$, $\varphi_i^2$, and $\varphi_i^3$ in the intersection of $\tilde{H}_i^2$ and $C_1$, $C_2$, and $\tilde{C}_3$ respectively, where $\tilde{H}_i^2 = \tilde{H}_i^2 \times \{t\}$. For any $t \in \mathbb{R}$, $\varphi_i^1$ is a circle of radius $b_1(t)$ in $\tilde{H}_i^2$ with center $(0, 0, t)$. Similarly, $\varphi_i^3$ is a circle of radius $b_2(t)$ in $\tilde{H}_i^2$ with center $(\eta_2, 0, t)$, see Fig. 4-right. Hence, it suffices to show that for any $t \in \mathbb{R}$ each of the intersection $\varphi_i^1 \cap \varphi_i^3$ and $\varphi_i^2 \cap \varphi_i^3$ consists of two points.

By construction, it is easy to see $\varphi_i^2 \cap \varphi_i^3$ consists of two points. This is because $\varphi_i^2$ and $\varphi_i^3$ have the same radius, $b_2(t)$ and $\eta_2 + b_2(t) > b_2(t)$ and $\eta_2 - b_2(t) > -b_2(t)$. Therefore, it remains to show that $\varphi_i^1 \cap \varphi_i^3$ consists of two points. By construction, this would be the case if $\eta_2 - b_2(t) < b_1(t)$ and $\eta_2 - b_2(t) > -b_1(t)$. The first inequality follows because $\eta_2 = \inf_{t \in \mathbb{R}} b_2(t)$. The second inequality follows from Lemma 2.1 because

$$\eta_2 > \eta_2 - \eta_1 = \sup_{t \in \mathbb{R}} (b_2(t) - b_1(t)).$$

$\square$

![Diagram](image-url)
Now, let $\hat{C}_3 \cap \Omega = T^+ \cup T^-$, where $T^+$ is the infinite strip with $\theta \in (0, \pi)$, and $T^-$ is the infinite strip with $\theta \in (-\pi, 0)$. Note that $T^\pm$ is a $\theta$-graph over the infinite strip $\hat{P}_0 = \Omega \cap P_0$ where $P_0$ is the half plane $\{ \theta = 0 \}$. Let $V$ be the component of $\Omega - \hat{C}_3$ containing $\hat{P}_0$. Notice that the mean curvature vector $H$ of $\partial V$ points into $V$ on both $T^+$ and $T^-$.

Consider the lifts of $T^+$ and $T^-$ in $\tilde{\Omega}$. For $n \in \mathbb{Z}$, let $\tilde{T}_n^+$ be the lift of $T^+$ which belongs to the region $\tilde{\theta} \in (2n\pi, (2n + 1)\pi)$. Similarly, let $\tilde{T}_n^-$ be the lift of $T^-$ which belongs to the region $\tilde{\theta} \in ((2n - 1)\pi, 2n\pi)$. Let $V_n$ be the closed region in $\tilde{\Omega}$ between the infinite strips $\tilde{T}_n^-$ and $\tilde{T}_n^+$. Notice that for $n$ sufficiently large, $B^\pm \subset V_n$.

Next we define the compact exhaustion $\Delta_n$ of $\tilde{\Omega}^e$ as follows: $\Delta_n := \tilde{\mathcal{U}}_n \cap V_n$. Furthermore, the absolute value of the mean curvature of $\partial\Delta_n$ is equal to $H$ and the mean curvature vector $H$ of $\partial\Delta_n$ points into $\Delta_n$ on $\partial\Delta_n = [(\partial\Delta_n \cap \tilde{C}_1) \cup B^-]$.

### 3.4 The sequence of $H$-surfaces

We next define a sequence of compact $H$-surfaces $\{\Sigma_n\}_{n \in \mathbb{N}}$ where $\Sigma_n \subset \Delta_n$. For each $n$ sufficiently large, we define a simple closed curve $\Gamma_n$ in $\partial\Delta_n$, and then we solve the $H$-Plateau problem for $\Gamma_n$ in $\Delta_n$. This will provide an embedded $H$-surface $\Sigma_n$ in $\Delta_n$ with $\partial\Sigma_n = \Gamma_n$ for each $n$.

The Construction of $\Gamma_n$ in $\partial\Delta_n$:

First, consider the annulus $A_n = \partial\Delta_n - (\tilde{C}_1 \cup \tilde{C}_2 \cup B^+ \cup B^-)$ in $\partial\Delta_n$. Let $\tilde{l}_n^+$ be the lift of $\tilde{T}_n^+$ and $\tilde{l}_n^-$ be the lift of $\tilde{T}_n^-$. Let $l_n^\pm = \tilde{l}_n^\pm \cap A_n$. Let $\mu_n^+$ be an arc in $S_n^+ \cap A_n$, whose $\tilde{\theta}$ and $\rho$ coordinates are strictly increasing as a function of the parameter and whose endpoints are $l_n^+ \cap S_n^+$ and $l_n^- \cap S_n^-$ (Fig. 5-left). Similarly, define $\mu_n^-$ to be a monotone arc in $S_n^- \cap A_n$ whose endpoints are $l_n^+ \cap S_n^-$ and $l_n^- \cap S_n^-$. Note that these arcs $\mu_n^+$ and $\mu_n^-$ are by construction disjoint from the infinite bumps $B^\pm$. Then, $\Gamma_n = \mu_n^+ \cup l_n^+ \cup \mu_n^- \cup l_n^-$ is a simple closed curve in $A_n \subset \partial\Delta_n$ (Fig. 5-right).

Next, consider the following variational problem ($H$-Plateau problem): Given the simple closed curve $\Gamma_n$ in $A_n$, let $M$ be a smooth compact embedded surface in $\Delta_n$ with $\partial M = \Gamma_n$. Since $\Delta_n$ is simply-connected, $M$ separates $\Delta_n$ into two regions. Let $Q$ be the region in $\Delta_n - \Sigma$ with $Q \cap \tilde{C}_2 \neq \emptyset$, the “upper” region. Then define the functional $\mathcal{I}_H = \text{Area}(M) + 2H \text{ Volume}(Q)$.

![Fig. 5](image-url) In the left, $\mu_n^+$ is pictured in $S_n^+$. On the right, the curve $\Gamma_n$ is described in $\partial\Delta_n$. Springser
By working with integral currents, it is known that there exists a smooth (except at the 4 corners of $\Gamma_n$), compact, embedded $H$-surface $\Sigma_n \subset \Delta_n$ with $\text{Int}(\Sigma_n) \subset \text{Int}(\Delta_n)$ and $\partial \Sigma_n = \Gamma_n$. Note that in our setting, $\Delta_n$ is not $H$-mean convex along $\Delta_n \cap \tilde{C}_1$. However, the mean curvature vector along $\Sigma_n$ points outside $Q$ because of the construction of the variational problem. Therefore $\Delta_n \cap \tilde{C}_1$ is still a good barrier for solving the $H$-Plateau problem. In fact, $\Sigma_n$ can be chosen to be, and we will assume it is, a minimizer for this variational problem, i.e., $I(\Sigma_n) \leq I(M)$ for any $M \subset \Delta_n$ with $\partial M = \Gamma_n$; see for instance [12, Theorem 2.1] and [1, Theorem 1]. In particular, the fact that $\text{Int}(\Sigma_n) \subset \text{Int}(\Delta_n)$ is proven in Lemma 3 of [4]. Moreover, $\Sigma_n$ separates $\Delta_n$ into two regions.

Similarly to Lemma 4.1 in [3], in the following lemma we show that for any such $\Gamma_n$, the minimizer surface $\Sigma_n$ is a $\tilde{\theta}$-graph.

**Lemma 3.4** Let $E_n := A_n \cap \tilde{T}_n^+$. The minimizer surface $\Sigma_n$ is a $\tilde{\theta}$-graph over the compact disk $E_n$. In particular, the related Jacobi function $J_n$ on $\Sigma_n$ induced by the inner product of the unit normal field to $\Sigma_n$ with the Killing field $\partial_{\tilde{\theta}}$ is positive in the interior of $\Sigma_n$.

**Proof** The proof is almost identical to the proof of Lemma 4.1 in [3], and for the sake of completeness, we give it here. Let $T_\alpha$ be the isometry of $\tilde{\Omega}$ which is a translation by $\alpha$ in the $\tilde{\theta}$ direction, i.e.,

$$T_\alpha(\rho, \tilde{\theta}, t) = (\rho, \tilde{\theta} + \alpha, t).$$  \hfill (4)

Let $T_\alpha(\Sigma_n) = \Sigma_n^\alpha$ and $T_\alpha(\Gamma_n) = \Gamma_n^\alpha$. We claim that $\Sigma_n^\alpha \cap \Sigma_n = \emptyset$ for any $\alpha \in \mathbb{R} \setminus \{0\}$ which implies that $\Sigma_n$ is a $\tilde{\theta}$-graph; we will use that $\Gamma_n^\alpha$ is disjoint from $\Sigma_n$ for any $\alpha \in \mathbb{R} \setminus \{0\}$.

Arguing by contradiction, suppose that $\Sigma_n^\alpha \cap \Sigma_n \neq \emptyset$ for a certain $\alpha \neq 0$. By compactness of $\Sigma_n$, there exists a largest positive number $\alpha'$ such that $\Sigma_n^{\alpha'} \cap \Sigma_n \neq \emptyset$.

Let $p \in \Sigma_n^{\alpha'} \cap \Sigma_n$. Since $\partial \Sigma_n^{\alpha'} \cap \partial \Sigma_n = \emptyset$ and the interior of $\Sigma_n$, respectively $\Sigma_n^{\alpha'}$, lie in the interior of $\Delta_n$, respectively $T_{\alpha'}(\Delta_n)$, then $p \in \text{Int}(\Sigma_n^{\alpha'}) \cap \text{Int}(\Sigma_n)$. Since the surfaces $\text{Int}(\Sigma_n^{\alpha'})$, $\text{Int}(\Sigma_n)$ lie on one side of each other and intersect tangentially at the point $p$ with the same mean curvature vector, then we obtain a contradiction to the mean curvature comparison principle for constant mean curvature surfaces, see Proposition 2.2. This proves that $\Sigma_n$ is graphical over its $\tilde{\theta}$-projection to $E_n$.

Since by construction every integral curve, $(\bar{\rho}, s, \bar{t})$ with $\bar{\rho}, \bar{t}$ fixed and $(\bar{\rho}, s_0, \bar{t}) \in E_n$ for a certain $s_0$, of the Killing field $\partial_{\tilde{\theta}}$ has non-zero intersection number with any compact surface bounded by $\Gamma_n$, we conclude that every such integral curve intersects both the disk $E_n$ and $\Sigma_n$ in single points. This means that $\Sigma_n$ is a $\tilde{\theta}$-graph over $E_n$ and thus the related Jacobi function $J_n$ on $\Sigma_n$ induced by the inner product of the unit normal field to $\Sigma_n$ with the Killing field $\partial_{\tilde{\theta}}$ is non-negative in the interior of $\Sigma_n$. Since $J_n$ is a non-negative Jacobi function, then either $J_n \equiv 0$ or $J_n > 0$. Since by construction $J_n$ is positive somewhere in the interior, then $J_n$ is positive everywhere in the interior. This finishes the proof of the lemma. \hfill $\square$
4 The proof of Theorem 1.1

With $\Gamma_n$ as previously described, we have so far constructed a sequence of compact stable $H$-disks $\Sigma_n$ with $\partial \Sigma_n = \Gamma_n \subset \partial \Delta_n$. Let $J_n$ be the related non-negative Jacobi function described in Lemma 3.4.

By the curvature estimates for stable $H$-surfaces given in [11], the norms of the second fundamental forms of the $\Sigma_n$ are uniformly bounded from above at points which are at intrinsic distance at least one from their boundaries. Since the boundaries of the $\Sigma_n$ leave every compact subset of $\tilde{\Omega}^*$, for each compact set of $\tilde{\Omega}^*$, the norms of the second fundamental forms of the $\Sigma_n$ are uniformly bounded for values $n$ sufficiently large and such a bound does not depend on the chosen compact set. Standard compactness arguments give that, after passing to a subsequence, $\Sigma_n$ converges to a (weak) $H$-lamination $\tilde{\mathcal{L}}$ of $\tilde{\Omega}^*$ and the leaves of $\tilde{\mathcal{L}}$ are complete and have uniformly bounded norm of their second fundamental forms, see for instance [5].

Let $\beta$ be a compact embedded arc contained in $\tilde{\Omega}^*$ such that its end points $p_+$ and $p_-$ are contained respectively in $B^+$ and $B^-$, and such that these are the only points in the intersection $[B^+ \cup B^-] \cap \beta$. Then, for $n$-sufficiently large, the linking number between $\Gamma_n$ and $\beta$ is one, which gives that, for $n$ sufficiently large, $\Sigma_n$ intersects $\beta$ in an odd number of points. In particular $\Sigma_n \cap \beta \neq \emptyset$ which implies that the lamination $\tilde{\mathcal{L}}$ is not empty.

**Remark 4.1** By Remark 3.2, a leaf of $\tilde{\mathcal{L}}$ that is invariant with respect to $\tilde{\theta}$-translations cannot be contained in $\tilde{\Omega}^*$. Therefore none of the leaves of $\tilde{\mathcal{L}}$ are invariant with respect to $\tilde{\theta}$-translations.

Let $\tilde{\mathcal{L}}$ be a leaf of $\tilde{\mathcal{L}}$ and let $J_{\tilde{\mathcal{L}}}$ be the Jacobi function induced by taking the inner product of $\partial_{\tilde{\mathcal{L}}}$ with the unit normal of $\tilde{\mathcal{L}}$. Then, by the nature of the convergence, $J_{\tilde{\mathcal{L}}} \geq 0$ and therefore since it is a Jacobi field, it is either positive or identically zero. In the latter case, $\tilde{\mathcal{L}}$ would be invariant with respect to $\theta$-translations, contradicting Remark 4.1. Thus, by Remark 4.1, we have that $J_{\tilde{\mathcal{L}}}$ is positive and therefore $\tilde{\mathcal{L}}$ is a Killing graph with respect to $\partial_{\tilde{\mathcal{L}}}$.

**Claim 4.2** Each leaf $\tilde{\mathcal{L}}$ of $\tilde{\mathcal{L}}$ is properly embedded in $\tilde{\Omega}^*$.

**Proof** Arguing by contradiction, suppose there exists a leaf $\tilde{\mathcal{L}}$ of $\tilde{\mathcal{L}}$ that is NOT proper in $\tilde{\Omega}^*$. Then, since the leaf $\tilde{\mathcal{L}}$ has uniformly bounded norm of its second fundamental form, the closure of $\tilde{\mathcal{L}}$ in $\tilde{\Omega}^*$ is a lamination of $\tilde{\Omega}^*$ with a limit leaf $\Lambda$, namely $\Lambda \subset \tilde{\mathcal{L}} - \tilde{\mathcal{L}}$. Let $J_\Lambda$ be the Jacobi function induced by taking the inner product of $\partial_{\tilde{\mathcal{L}}}$ with the unit normal of $\Lambda$.

Just like in the previous discussion, by the nature of the convergence, $J_\Lambda \geq 0$ and therefore, since it is a Jacobi field, it is either positive or identically zero. In the latter case, $\Lambda$ would be invariant with respect to $\tilde{\theta}$-translations and thus, by Remark 4.1, $\Lambda$ cannot be contained in $\tilde{\Omega}^*$. However, since $\Lambda$ is contained in the closure of $\tilde{\mathcal{L}}$, this would imply that $\tilde{\mathcal{L}}$ is not contained in $\tilde{\Omega}^*$, giving a contradiction. Thus, $J_\Lambda$ must be positive and therefore, $\Lambda$ is a Killing graph with respect to $\partial_{\tilde{\mathcal{L}}}$. However, this implies that $\tilde{\mathcal{L}}$ cannot be a Killing graph with respect to $\partial_{\tilde{\mathcal{L}}}$. This follows because if we fix a point $p$ in $\Lambda$ and let $U_p \subset \Lambda$ be neighborhood of such point, then by the nature of
the convergence, $U_p$ is the limit of a sequence of disjoint domains $U_{p_n}$ in $\widetilde{L}$ where $p_n \in \widetilde{L}$ is a sequence of points converging to $p$ and $U_{p_n} \subset \widetilde{L}$ is a neighborhood of $p_n$. While each domain $U_{p_n}$ is a Killing graph with respect to $\partial \widetilde{g}$, the convergence to $U_p$ implies that their union is not. This gives a contradiction and proves that $\Lambda$ cannot be a Killing graph with respect to $\partial \tilde{g}$. Since we have already shown that $\Lambda$ must be a Killing graph with respect to $\partial \tilde{g}$, this gives a contradiction. Thus $\Lambda$ cannot exist and each leaf $\widetilde{L}$ of $\widetilde{L}$ is properly embedded in $\Omega^*$. 

Arguing similarly to the proof of the previous claim, it follows that a small perturbation of $\beta$, which we still denote by $\beta$ intersects $\Sigma_n$ and $\widetilde{L}$ transversally in a finite number of points. Note that $\widetilde{L}$ is obtained as the limit of $\Sigma_n$. Indeed, since $\Sigma_n$ separates $B^+$ and $B^-$ in $\Omega^*$, the algebraic intersection number of $\beta$ and $\Sigma_n$ must be one, which implies that $\beta$ intersects $\Sigma_n$ in an odd number of points. Then $\beta$ intersects $\widetilde{L}$ in an odd number of points and the claim below follows.

**Claim 4.3** The curve $\beta$ intersects $\widetilde{L}$ in an odd number of points.

In particular $\beta$ intersects only a finite collection of leaves in $\widetilde{L}$ and we let $\mathcal{F}$ denote the non-empty finite collection of leaves that intersect $\beta$.

**Definition 4.1** Let $(\rho_1, \tilde{\theta}_0, t_0)$ be a fixed point in $\widetilde{L}_1$ and let $\rho_2(\tilde{\theta}_0, t_0) > \rho_1$ such that $(\rho_2(\tilde{\theta}_0, t_0), \tilde{\theta}_0, t_0)$ is in $\widetilde{L}_2$. Then we call the arc in $\Omega$ given by

$$
(\rho_1 + s(\rho_2 - \rho_1), \tilde{\theta}_0, t_0), \ s \in [0, 1].
$$

the vertical line segment based at $(\rho_1, \tilde{\theta}_0, t_0)$.

**Claim 4.4** There exists at least one leaf $\widetilde{L}_\beta$ in $\mathcal{F}$ that intersects $\beta$ in an odd number of points and the leaf $\widetilde{L}_\beta$ must intersect each vertical line segment at least once.

**Proof** The existence of $\widetilde{L}_\beta$ follows because otherwise, if all the leaves in $\mathcal{F}$ intersected $\beta$ in an even number of points, then the number of points in the intersection $\beta \cap \mathcal{F}$ would be even. Given $\widetilde{L}_\beta$ a leaf in $\mathcal{F}$ that intersects $\beta$ in an odd number of points, suppose there exists a vertical line segment which does not intersect $\widetilde{L}_\beta$. Then since by Claim 4.2 $\widetilde{L}_\beta$ is properly embedded, using elementary separation arguments would give that the number of points of intersection in $\beta \cap \widetilde{L}_\beta$ must be zero mod 2, that is even, contradicting the previous statement. 

Let $\Pi$ be the covering map defined in equation (2) and let $\mathcal{P}_H := \Pi(\widetilde{L}_\beta)$. The previous discussion and the fact that $\Pi$ is a local diffeomorphism, implies that $\mathcal{P}_H$ is a stable complete $H$-surface embedded in $\Omega$. Indeed, $\mathcal{P}_H$ is a graph over its $\theta$-projection to $\text{Int}(\Omega) \cap \{(\rho, 0, t) \mid \rho > 0, t \in \mathbb{R}\}$, which we denote by $\theta(\mathcal{P}_H)$. Abusing the notation, let $J_{\mathcal{P}_H}$ be the Jacobi function induced by taking the inner product of $\partial_\theta$ with the unit normal of $\mathcal{P}_H$, then $J_{\mathcal{P}_H}$ is positive. Finally, since the norm of the second fundamental form of $\mathcal{P}_H$ is uniformly bounded, standard compactness arguments imply that its closure $\overline{\mathcal{P}}_H$ is an $H$-lamination $\mathcal{L}$ of $\Omega$, see for instance [5].

**Claim 4.5** The closure of $\mathcal{P}_H$ is an $H$-lamination of $\Omega$ consisting of itself and two $H$-catenoids $L_1, L_2 \subset \Omega$ that form the limit set of $\mathcal{P}_H$. 

\[ \text{Springer} \]
Remark 4.6  Note that these two $H$-catenoids are not necessarily the ones which determine $\partial \Omega$.

Proof  Given $(\rho_1, \tilde{\theta}_0, t_0) \in \tilde{C}_1$, let $\tilde{\gamma}$ be the fixed vertical line segment in $\tilde{\Omega}$ based at $(\rho_1, \tilde{\theta}_0, t_0)$, let $\tilde{p}_0$ be a point in the intersection $\tilde{L}_\beta \cap \tilde{\gamma}$ (recall that by Claim 4.4 such intersection is not empty) and let $p_0 = \Pi(\tilde{p}_0) = \Pi(\tilde{\gamma}) \cap \mathcal{P}_H$. Then, by Claim 4.4, for any $i \in \mathbb{N}$, the vertical line segment $T_{2\pi i}(\tilde{\gamma})$ intersects $\tilde{L}_\beta$ in at least a point $\tilde{p}_i$, and $\tilde{p}_{i+1}$ is above $\tilde{p}_i$, where $T$ is the translation defined in equation (4). Namely, $\tilde{p}_0 = (r_0, \tilde{\theta}_0, t_0)$, $\tilde{p}_i = (r_i, \tilde{\theta}_0 + 2\pi i, t_0)$ and $r_i < r_{i+1} < \rho_2(\tilde{\theta}_0, t_0)$. The point $\tilde{p}_i \in \tilde{L}_\beta$ corresponds to the point $p_i = \Pi(\tilde{p}_i) = (r_i, \tilde{\theta}_0 \mod 2\pi, t_0) \in \mathcal{P}_H$. Let $r(2) := \lim_{i \to \infty} r_i$ then $r(2) \leq \rho_2(\tilde{\theta}_0, t_0)$ and note that since $\lim_{i \to \infty} (r_{i+1} - r_i) = 0$, then the value of the Jacobi function $J_{\mathcal{P}_H}$ at $p_i$ must be going to zero as $i$ goes to infinity. Clearly, the point $Q := (r(2), \theta_0 \mod 2\pi, t_0) \in \Omega$ is in the closure of $\mathcal{P}_H$, that is $L$. Let $L_2$ be the leaf of $L$ containing $Q$. By the previous discussion $J_{L_2}(Q) = 0$. Since by the nature of the convergence, either $J_{L_2}$ is positive or $L_2$ is rotational, then $L_2$ is rotational, namely an $H$-catenoid.

Arguing similarly but considering the intersection of $\tilde{L}_\beta$ with the vertical line segments $T_{-2\pi i}(\tilde{\gamma})$, $i \in \mathbb{N}$, one obtains another $H$-catenoid $L_1$, different from $L_2$, in the lamination $\tilde{\mathcal{L}}$. This shows that the closure of $\mathcal{P}_H$ contains the two $H$-catenoids $L_1$ and $L_2$.

Let $\Omega_g$ be the rotationally invariant, connected region of $\Omega - [L_1 \cup L_2]$ whose boundary contains $L_1 \cup L_2$. Note that since $\mathcal{P}_H$ is connected and $L_1 \cup L_2$ is contained in its closure, then $\mathcal{P}_H \subset \Omega_g$. It remains to show that $\mathcal{L} = \mathcal{P}_H \cup L_1 \cup L_2$, i.e. $\overline{\mathcal{P}_H} = \mathcal{P}_H \cup L_1 \cup L_2$. If $\overline{\mathcal{P}_H} \neq L_1 \cup L_2$ then there would be another leaf $L_3 \in \mathcal{L} \cap \Omega_g$ and by previous argument, $L_3$ would be an $H$-catenoid. Thus $L_3$ would separate $\Omega_g$ into two regions, contradicting that fact that $\mathcal{P}_H$ is connected and $L_1 \cup L_2$ are contained in its closure. This finishes the proof of the claim. □

Note that by the previous claim, $\mathcal{P}_H$ is properly embedded in $\Omega_g$.

Claim 4.7  The $H$-surface $\mathcal{P}_H$ is simply-connected and every integral curve of $\partial \theta$ that lies in $\Omega_g$ intersects $\mathcal{P}_H$ in exactly one point.

Proof  Let $D_g := \text{Int}(\Omega_g) \cap \{(\rho, 0, t) \mid \rho > 0, t \in \mathbb{R}\}$, then $\mathcal{P}_H$ is a graph over its $\theta$-projection to $D_g$, that is $\theta(\mathcal{P}_H)$. Since $\theta : \Omega_g \to D_g$ is a proper submersion and $\mathcal{P}_H$ is properly embedded in $\Omega_g$, then $\theta(\mathcal{P}_H) = D_g$, which implies that every integral curve of $\partial \theta$ that lies in $\Omega_g$ intersects $\mathcal{P}_H$ in exactly one point. Moreover, since $D_g$ is simply-connected, this gives that $\mathcal{P}_H$ is also simply-connected. This finishes the proof of the claim. □

From this claim, it clearly follows that $\Omega_g$ is foliated by $H$-surfaces, where the leaves of this foliation are $L_1$, $L_2$ and the rotated images $\mathcal{P}_H(\theta)$ of $\mathcal{P}_H$ around the $t$-axis by angles $\theta \in [0, 2\pi)$. The existence of the examples $\Sigma_H$ in the statement of Theorem 1.1 can easily be proven by using $\mathcal{P}_H$. We set $\Sigma_H = \mathcal{P}_H$, and $C_i = L_i$ for $i = 1, 2$. This finishes the proof of Theorem 1.1.

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Appendix: Disjoint $H$-catenoids

In this section, we will show the existence of disjoint $H$-catenoids in $\mathbb{H}^2 \times \mathbb{R}$. In particular, we will prove Lemma 2.1. Given $H \in (0, \frac{1}{2})$ and $d \in [-2H, \infty)$, recall that $\eta_d = \cosh^{-1}(\frac{2dH+\sqrt{1-4H^2+d^2}}{1-4H^2})$ and that $\lambda_d : [\eta_d, \infty) \to [0, \infty)$ is the function defined as follows.

$$\lambda_d(\rho) = \int_{\eta_d}^{\rho} \frac{d + 2H \cosh r}{\sqrt{\sinh^2 r - (d + 2H \cosh r)^2}} dr. \quad (6)$$

Recall that $\lambda_d(\rho)$ is a monotone increasing function with $\lim_{\rho \to \infty} \lambda_d(\rho) = \infty$ and that $\lambda'_d(\eta_d) = \infty$ when $d \in (-2H, \infty)$. The $H$-catenoid $C^H_d$, $d \in (-2H, \infty)$, is obtained by rotating a generating curve $\Lambda^H_d$ about the $t$-axis. The generating curve $\Lambda^H_d$ is obtained by doubling the curve $(\rho, 0, \lambda_d(\rho))$, $\rho \in [\eta_d, \infty)$, with its reflection $(\rho, 0, -\lambda_d(\rho))$, $\rho \in [\eta_d, \infty)$.

Finally, recall that $b_d(t) := \lambda^{-1}_d(t)$ for $t \geq 0$, hence $b_d(0) = \eta_d$, and that abusing the notation $b_d(t) := b_d(-t)$ for $t \leq 0$.

Lemma 2.1 (Disjoint $H$-catenoids) Given $d_1 > 2$ there exist $d_0 > d_1$ and $\delta_0 > 0$ such that for any $d_2 \in [d_0, \infty)$ and $t > 0$ then

$$\inf_{t \in \mathbb{R}} (b_{d_2}(t) - b_{d_1}(t)) \geq \delta_0.$$  

In particular, the corresponding $H$-catenoids are disjoint, i.e., $C^H_{d_1} \cap C^H_{d_2} = \emptyset$.

Moreover, $b_{d_2}(t) - b_{d_1}(t)$ is decreasing for $t > 0$ and increasing for $t < 0$. In particular,

$$\sup_{t \in \mathbb{R}} (b_{d_2}(t) - b_{d_1}(t)) = b_{d_2}(0) - b_{d_1}(0) = \eta_{d_2} - \eta_{d_1}.$$  

Proof We begin by introducing the following notations that will be used for the computations in the proof of this lemma,

$$c := \cosh r = \frac{e^r + e^{-r}}{2}, \quad s := \sinh r = \frac{e^r - e^{-r}}{2}.$$  

Recall that $c^2 - s^2 = 1$ and $c - s = e^{-r}$. Using these notations,

$$\lambda_d(\rho) = \int_{\eta_d}^{\rho} \frac{d + 2H \cosh r}{\sqrt{\sinh^2 r - (d + 2H \cosh r)^2}} dr \quad (7)$$

can be rewritten as

$$\lambda_d(\rho) = \int_{\eta_d}^{\rho} \frac{d + 2H(s + e^{-r})}{\sqrt{s^2 - (d + 2Hc)^2}} dr = f_d(\rho) + J_d(\rho), \quad (8)$$
where

\[ f_d(\rho) = \int_{a_d}^{\rho} \frac{2Hs}{\sqrt{s^2 - (d + 2Hc)^2}} \, dr \quad \text{and} \quad J_d(\rho) = \int_{a_d}^{\rho} \frac{d + 2He^{-r}}{\sqrt{s^2 - (d + 2Hc)^2}} \, dr \]

First, by using a series of substitutions, we will get an explicit description of \( f_d(\rho) \). Then, we will show that for \( d > 2 \), \( J_d(\rho) \) is bounded independently of \( \rho \) and \( d \).

**Claim 4.8**

\[ f_d(\rho) = \frac{2H}{\sqrt{1 - 4H^2}} \cosh^{-1} \left( \frac{(1 - 4H^2) \cosh \rho - 2dH}{\sqrt{d^2 + 1 - 4H^2}} \right). \]  

(9)

**Remark 4.9** After finding \( f_d(\rho) \), we used Wolfram Alpha to compute the derivative of \( f_d(\rho) \) and verify our claim. For the sake of completeness, we give a proof.

**Proof of Claim 4.8** The proof is a computation with requires several integrations by substitution. Consider

\[ \int \frac{2Hs}{\sqrt{s^2 - (d + 2Hc)^2}} \, dr \]

By using the fact that \( s^2 = c^2 - 1 \) and applying the substitution \( \{ u = c, du = \frac{dc}{dr} \, dr = sdr \} \) we obtain that

\[ \int \frac{2Hs}{\sqrt{s^2 - (d + 2Hc)^2}} \, dr = \int \frac{2H}{\sqrt{u^2 - 1 - (d + 2Hu)^2}} \, du. \]

Note that

\[
\begin{align*}
&u^2 - 1 - (d + 2Hu)^2 = u^2 - 1 - (d^2 + 4dHu + 4H^2u^2) \\
&= (1 - 4H^2)u^2 - 4dHu - d^2 - 1 \\
&= (1 - 4H^2) \left( u^2 - \frac{4dH}{1 - 4H^2}u + \frac{4d^2H^2}{(1 - 4H^2)^2} \right) - \frac{4d^2H^2}{1 - 4H^2} - d^2 - 1 \\
&= (1 - 4H^2) \left[ \left( u - \frac{2dH}{(1 - 4H^2)} \right)^2 - \frac{d^2 + 1}{1 - 4H^2} \right] \\
&= (1 - 4H^2) \left[ \left( u - \frac{2dH}{(1 - 4H^2)} \right)^2 - \frac{4d^2H^2 + (1 - 4H^2)(d^2 + 1)}{(1 - 4H^2)^2} \right] \\
&= (1 - 4H^2) \left[ \left( u - \frac{2dH}{(1 - 4H^2)} \right)^2 - \frac{d^2 + 1 - 4H^2}{(1 - 4H^2)^2} \right].
\end{align*}
\]
Therefore, by applying a second substitution, \( w = u - \frac{2dH}{1 - 4H^2} \), \( dw = du \), and letting \( a^2 = \left( \frac{d^2 + 1 - 4H^2}{1 - 4H^2} \right) \) we get that

\[
\int \frac{2H}{\sqrt{u^2 - 1 - (d + 2Hu)^2}} \, du = \int \frac{2H}{\sqrt{1 - 4H^2} \sqrt{w^2 - a^2}} \, dw
\]

By using the fact that \( \sec^2 x - 1 = \tan^2 x \) and applying a third substitution, \( w = a \sec t \), \( dw = a \sec t \tan t \, dt \), we obtain that

\[
\int \frac{2Ha \sec t \tan t}{\sqrt{1 - 4H^2} \sqrt{a^2 \sec^2 t - a^2}} \, dt = \int \frac{2H \sec t}{\sqrt{1 - 4H^2}} \, dt \\
= \frac{2H}{\sqrt{1 - 4H^2}} \ln | \sec t + \tan t |
\]

Therefore

\[
\int \frac{2H}{\sqrt{1 - 4H^2} \sqrt{w^2 - a^2}} \, dw = \frac{2H}{\sqrt{1 - 4H^2}} \ln \left| \frac{w}{a} + \sqrt{\frac{w^2}{a^2} - 1} \right| \\
= \frac{2H}{\sqrt{1 - 4H^2}} \cosh^{-1} \left( \frac{w}{a} \right)
\]

Since \( w = u - \frac{2dH}{1 - 4H^2} \) then

\[
\int \frac{2H}{\sqrt{u^2 - 1 - (d + 2Hu)^2}} \, du = \frac{2H}{\sqrt{1 - 4H^2}} \cosh^{-1} \left( \frac{u - \frac{2dH}{1 - 4H^2}}{a} \right) \\
= \frac{2H}{\sqrt{1 - 4H^2}} \cosh^{-1} \left( \frac{u - \frac{2dH}{\sqrt{d^2 + 1 - 4H^2}}}{\sqrt{d^2 + 1 - 4H^2}} \right) \\
= \frac{2H}{\sqrt{1 - 4H^2}} \cosh^{-1} \left( \frac{(1 - 4H^2)u - 2dH}{\sqrt{d^2 + 1 - 4H^2}} \right)
\]

Finally, since \( u = \cosh r \)

\[
\int_{n_d}^{\rho} \frac{2Hs}{\sqrt{s^2 - (d + 2Hc)^2}} \, ds = \frac{2H}{\sqrt{1 - 4H^2}} \cosh^{-1} \left( \frac{(1 - 4H^2) \cosh r - 2dH}{\sqrt{d^2 + 1 - 4H^2}} \right) \bigg|_{n_d}^{\rho} \\
= \frac{2H}{\sqrt{1 - 4H^2}} \left( \cosh^{-1} \left( \frac{(1 - 4H^2) \cosh \rho - 2dH}{\sqrt{d^2 + 1 - 4H^2}} \right) \right) - \cosh^{-1} \left( \frac{(1 - 4H^2) \cosh n_d - 2dH}{\sqrt{d^2 + 1 - 4H^2}} \right)
\]
Recall that \( \eta_d = \cosh^{-1}(\frac{2dH + \sqrt{1 - 4H^2 + d^2}}{1 - 4H^2}) \) and thus

\[
\frac{(1 - 4H^2) \cosh \eta_d - 2dH}{\sqrt{d^2 + 1 - 4H^2}} = \frac{(1 - 4H^2)(\frac{2dH + \sqrt{1 - 4H^2 + d^2}}{1 - 4H^2}) - 2dH}{\sqrt{d^2 + 1 - 4H^2}} = 1.
\]

This implies that

\[
f_d(\rho) = \frac{2H}{\sqrt{1 - 4H^2}} \cosh^{-1} \left( \frac{(1 - 4H^2) \cosh \rho - 2dH}{\sqrt{d^2 + 1 - 4H^2}} \right).
\]

By Claim 4.8 we have that

\[
f_d(\rho) = \frac{2H}{\sqrt{1 - 4H^2}} \left( \cosh^{-1} \left( \frac{(1 - 4H^2) \cosh \rho - 2dH}{\sqrt{d^2 + 1 - 4H^2}} \right) \right)
\]

\[
= \frac{2H}{\sqrt{1 - 4H^2}} \left( \rho + \ln \frac{1 - 4H^2}{\sqrt{d^2 + 1 - 4H^2}} \right) + g_d(\rho),
\]

where \( \lim_{\rho \to \infty} g_d(\rho) = 0 \).

Recall that \( \lambda_d(\rho) = f_d(\rho) + J_d(\rho) \) where

\[
J_d(\rho) = \int_{\eta_d}^{\rho} \frac{d + 2He^{-r}}{\sqrt{s^2 - (d + 2Hc)^2}} dr = \int_{\eta_d}^{\rho} \frac{d + 2He^{-r}}{\sqrt{c^2 - 1 - (d + 2Hc)^2}} dr.
\]

Claim 4.10

\[
\sup_{d \in (2, \infty), \rho \in (\eta_d, \infty)} J_d(\rho) \leq \pi \sqrt{1 - 2H}.
\]

Proof of Claim 4.10 Let

\[
\alpha = \frac{2dH + \sqrt{1 - 4H^2 + d^2}}{1 - 4H^2} \quad \text{and} \quad \beta = \frac{2dH - \sqrt{1 - 4H^2 + d^2}}{1 - 4H^2}
\]

be the roots of \( c^2 - 1 - (d + 2Hc)^2 \), i.e.

\[
c^2 - 1 - (d + 2Hc)^2 = (1 - 4H^2) \left( c^2 - \frac{4dH}{1 - 4H^2}c - \frac{1 + d^2}{1 - 4H^2} \right)
\]

\[
= (1 - 4H^2)(c - \alpha)(c - \beta).
\]
Note that \( \alpha = \cosh \eta_d \) and that as \( H \in (0, \frac{1}{2}) \), \( \beta < 0 < \alpha \). Furthermore, \( 2He^{-r} < 2H < 1 < d \). Thus we have,

\[
J_d(\rho) = \int_{\eta_d}^{\rho} \frac{d + 2He^{-r}}{\sqrt{1 - 4H^2}(c - \alpha)(c - \beta)} dr < \frac{2d}{\sqrt{1 - 4H^2}} \int_{\eta_d}^{\infty} \frac{dr}{\sqrt{(c - \alpha)(c - \beta)}} < \frac{2d}{\sqrt{1 - 4H^2} \sqrt{\alpha - \beta}} \int_{\eta_d}^{\infty} \frac{dr}{\sqrt{c - \alpha}},
\]

where the last inequality holds because for \( r > \eta_d \), \( \cosh r > \alpha \) and thus \( \sqrt{\alpha - \beta} < \sqrt{c - \alpha} \). Notice that \( \alpha - \beta = 2\sqrt{\frac{1 - 4H^2 + d^2}{1 - 4H^2}} > \frac{2d}{1 - 4H^2} \). Therefore

\[
\frac{2d}{\sqrt{1 - 4H^2} \sqrt{\alpha - \beta}} < \frac{2d}{\sqrt{1 - 4H^2}} \frac{\sqrt{1 - 4H^2}}{\sqrt{2d}} = \sqrt{2d}
\]

and

\[
J_d(\rho) < \sqrt{2d} \int_{\eta_d}^{\infty} \frac{dr}{\sqrt{c - \alpha}}.
\]

Applying the substitution \( \{u = c - \alpha, du = sdr = \sqrt{(u + \alpha)^2 - 1} dr\} \), we obtain that

\[
\int_{\eta_d}^{\infty} \frac{dr}{\sqrt{c - \alpha}} = \int_{0}^{\infty} \frac{du}{\sqrt{u\sqrt{(u + \alpha)^2 - 1}}} = \int_{0}^{\infty} \frac{du}{\sqrt{u\sqrt{(u + \omega)^2 - 1}}},
\]

Let \( \omega = \alpha - 1 \). Note that since \( d \geq 1 \) then \( \alpha > 1 \) and we have that \( (u + \alpha)^2 - 1 > (u + \omega)^2 \) as \( u > 0 \). This gives that

\[
\int_{0}^{\infty} \frac{du}{\sqrt{u\sqrt{(u + \alpha)^2 - 1}}} < \int_{0}^{\infty} \frac{du}{\sqrt{u(u + \omega)}}
\]

Applying the substitution \( \{v = \sqrt{u}, dv = \frac{du}{2\sqrt{u}}\} \) we get

\[
\int_{0}^{\infty} \frac{dv}{\sqrt{v(u + \omega)}} = \int_{0}^{\infty} \frac{2dv}{\sqrt{v^2 + \omega}} = \frac{2}{\sqrt{\omega}} \arctan \frac{w}{\sqrt{\omega}} \bigg|_{0}^{\infty} < \frac{\pi}{\sqrt{\omega}}
\]

and thus

\[
J_d(\rho) < \sqrt{\frac{2d}{\omega}} \pi.
\]
Note that
\[
\omega = \alpha - 1 = \frac{2dH + \sqrt{1 - 4H^2} + d^2}{1 - 4H^2} - 1
\]
\[
> \frac{(1 + 2H)d}{1 - 4H^2} - 1 = \frac{d}{1 - 2H} - 1.
\]
Since \(d > 2\), we have \(2\omega > \frac{d}{1 - 2H}\) and \(\frac{d}{\omega} < 2(1 - 2H)\). Then \(\sqrt{\frac{2d}{\omega}} < 2\sqrt{1 - 2H}\).

Finally, this gives that
\[
J_d(\rho) < 2\pi \sqrt{1 - 2H}
\]
independently on \(d > 2\) and \(\rho > \eta_d\). This finishes the proof of the claim. \(\square\)

Using Claims 4.8 and 4.10, we can now prove the next claim.

**Claim 4.11** Given \(d_2 > d_1 > 2\) there exists \(T \in \mathbb{R}\) such for any \(t > T\), we have that
\[
\frac{2H}{\sqrt{1 - 4H^2}}(\lambda_{d_2}^{-1}(t) - \lambda_{d_1}^{-1}(t))
\]
\[
> \frac{1}{2} \ln \sqrt{\frac{d_2^2 + 1 - 4H^2}{d_1^2 + 1 - 4H^2}} - 2\pi \sqrt{1 - 2H}.
\]

**Proof of Claim 4.11** Recall that \(\lambda_d(\rho) = f_d(\rho) + J_d(\rho)\) and that by Claims 4.8 and 4.10 we have that
\[
f_d(\rho) = \frac{2H}{\sqrt{1 - 4H^2}} \left( \rho + \ln \frac{1 - 4H^2}{\sqrt{d^2 + 1 - 4H^2}} \right) + g_d(\rho),
\]
where \(\lim_{\rho \to \infty} g_d(\rho) = 0\), and that
\[
\sup_{d \in (2,\infty), \rho \in (\eta_d, \infty)} J_d(\rho) \leq 2\pi \sqrt{1 - 2H}.
\]

Let \(\rho_i(t) := \lambda_{d_i}^{-1}(t)\), \(i = 1, 2\). Using this notation, since \(t = \lambda_1(\rho_1(t)) = \lambda_2(\rho_2(t))\) we obtain that
\[
0 = \lambda_2(\rho_2(t)) - \lambda_1(\rho_1(t))
\]
\[
= f_{d_2}(\rho_2(t)) + J_{d_2}(\rho_2(t)) - f_{d_1}(\rho_1(t)) - J_{d_1}(\rho_1(t))
\]
\[
= \frac{2H}{\sqrt{1 - 4H^2}} \left( \rho_2(t) + \ln \frac{1 - 4H^2}{\sqrt{d_2^2 + 1 - 4H^2}} \right) + g_{d_2}(\rho_2(t)) + J_{d_2}(\rho_2(t))
\]
\[
- \frac{2H}{\sqrt{1 - 4H^2}} \left( \rho_1(t) - \ln \frac{1 - 4H^2}{\sqrt{d_1^2 + 1 - 4H^2}} \right) - g_{d_1}(\rho_1(t)) - J_{d_1}(\rho_1(t)).
\]
Recall that \( \lim_{t \to \infty} \rho_i(t) = \infty, i = 1, 2 \), therefore given \( \varepsilon > 0 \) there exists \( T_\varepsilon \in \mathbb{R} \) such that for any \( t > T_\varepsilon \), \(|g_{d_i}(\rho_i(t))| \leq \varepsilon \). Taking

\[
4\varepsilon < \ln \frac{d_2^2 + 1 - 4H^2}{d_1^2 + 1 - 4H^2}
\]

for \( t > T_\varepsilon \) we get that

\[
\frac{2H}{\sqrt{1 - 4H^2}}(\rho_2(t) - \rho_1(t)) > \ln \frac{\sqrt{d_2^2 + 1 - 4H^2}}{\sqrt{d_1^2 + 1 - 4H^2}} + J_{d_1}(\rho_1(t)) - J_{d_2}(\rho_2(t)) - 2\varepsilon
\]

\[
> \frac{1}{2} \ln \frac{d_2^2 + 1 - 4H^2}{d_1^2 + 1 - 4H^2} + J_{d_1}(\rho_1(t)) - J_{d_2}(\rho_2(t)).
\]

Notice that \( J_{d_1}(\rho_1(t)) > 0 \) and that Claim 4.10 gives that

\[
\sup_{\rho \in (\eta_{d_2}, \infty)} J_{d_2}(\rho) \leq 2\pi \sqrt{1 - 2H}.
\]

Therefore

\[
\frac{2H}{\sqrt{1 - 4H^2}}(\rho_2(t) - \rho_1(t)) > \frac{1}{2} \ln \frac{d_2^2 + 1 - 4H^2}{d_1^2 + 1 - 4H^2} - 2\pi \sqrt{1 - 2H}.
\]

This finishes the proof of the claim. \( \Box \)

We can now use Claim 4.11 to finish the proof of the lemma. Given \( d_1 > 2 \) fix \( d_0 > d_1 \) such that

\[
\frac{\sqrt{1 - 4H^2}}{4H} \left( \ln \frac{\sqrt{d_0^2 + 1 - 4H^2}}{\sqrt{d_1^2 + 1 - 4H^2}} - 4\pi \sqrt{1 - 2H} \right) = 1.
\]

Then, by Claim 4.11, given \( d_2 \geq d_0 \) there exists \( T > 0 \) such that \( \lambda_{d_2}^{-1}(t) - \lambda_{d_1}^{-1}(t) > 1 \) for any \( t > T \). Notice that since for any \( \rho \in (\eta_2, \infty) \), \( \lambda_{d_2}'(\rho) > \lambda_{d_1}'(\rho) \), then there exists at most one \( t_0 > 0 \) such that \( \lambda_{d_2}^{-1}(t_0) - \lambda_{d_1}^{-1}(t_0) = 0 \). Therefore, since there exists \( T > 0 \) such that \( \lambda_{d_2}^{-1}(t) - \lambda_{d_1}^{-1}(t) > 1 \) for any \( t > T \) and \( \lambda_{d_2}^{-1}(0) - \lambda_{d_1}^{-1}(0) = \eta_{d_2} - \eta_{d_1} > 0 \), this implies that there exists a constant \( \delta(d_2) > 0 \) such that for any \( t > 0 \),

\[
\lambda_{d_2}^{-1}(t) - \lambda_{d_1}^{-1}(t) > \delta(d_2).
\]
A priori it could happen that \( \lim_{d_2 \to \infty} \delta(d_2) = 0 \). The fact that \( \lim_{d_2 \to \infty} \delta(d_2) > 0 \) follows easily by noticing that by applying Claim 4.11 and using the same arguments as in the previous paragraph there exists \( d_3 > d_0 \) such that for any \( d \geq d_3 \) and \( t > 0 \),

\[
\lambda_d^{-1}(t) - \lambda_{d_0}^{-1}(t) > 0.
\]

Therefore, for any \( d \geq d_3 \) and \( t > 0 \),

\[
\lambda_d^{-1}(t) - \lambda_{d_1}^{-1}(t) > \lambda_{d_0}^{-1}(t) - \lambda_{d_1}^{-1}(t) > \delta(d_0)
\]

which implies that

\[
\lim_{d_2 \to \infty} \delta(d_2) \geq \delta(d_0) > 0.
\]

Setting \( \delta_0 = \inf_{d \in [d_0, \infty)} \delta(d_2) > 0 \) gives that

\[
\inf_{t \in \mathbb{R} \geq 0} (\lambda_{d_2}^{-1}(t) - \lambda_{d_1}^{-1}(t)) \geq \delta_0.
\]

By definition of \( b_d(t) \) then

\[
\inf_{t \in \mathbb{R}} (b_{d_2}(t) - b_{d_1}(t)) = \inf_{t \in \mathbb{R} \geq 0} (\lambda_{d_2}^{-1}(t) - \lambda_{d_1}^{-1}(t)) \geq \delta_0.
\]

It remains to prove that \( b_2(t) - b_1(t) \) is decreasing for \( t > 0 \) and increasing for \( t < 0 \). By definition of \( b_d(t) \), it suffices to show that \( b_2(t) - b_1(t) \) is decreasing for \( t > 0 \). We are going to show \( \frac{d}{dt}(b_2(t) - b_1(t)) < 0 \) when \( t > 0 \).

By definition of \( b_1 \), for \( t > 0 \) we have that \( \lambda_1(b_1(t)) = t \) and thus \( b'_1(t) = \frac{1}{\lambda'_1(b_1(t))} \).

By definition of \( \lambda_d(t) \) for \( t > 0 \) the following holds,

\[
b'_1(t) = \frac{1}{\lambda'_1(b_1(t))} > \frac{1}{\lambda'_1(b_2(t))} > \frac{1}{\lambda'_2(b_2(t))} = b'_2(t).
\]

The first inequality is due to the convexity of the function \( \lambda_1(t) \) and the second inequality is due to the fact that \( \lambda'_1(\rho) < \lambda'_2(\rho) \) for any \( \rho \) near \( \eta_2 \). This proves that \( \frac{d}{dt}(b_2(t) - b_1(t)) = b'_2(t) - b'_1(t) < 0 \) for \( t > 0 \) and finishes the proof of the claim.

Note that if \( d \) is sufficiently close to \(-2H\) then \( C^H_d \) must be unstable. This follows because as \( d \) approaches \(-2H\), the norm of the second fundamental form of \( C^H_d \) becomes arbitrarily large at points that approach the “origin” of \( \mathbb{H}^2 \times \mathbb{R} \) and a simple rescaling argument gives that a sequence of subdomains of \( C^H_d \) converge to a catenoid, which is an unstable minimal surface. This observation, together with our previous lemma suggests the following conjecture.

**Conjecture:** Given \( H \in (0, \frac{1}{2}) \) there exists \( d_H > -2H \) such that the following holds. For any \( d > d' > d_H \), \( C^H_d \cap C^H_{d'} = \emptyset \), and the family \( \{C^H_d \mid d \in [d_H, \infty)\} \) gives a
The foliation of the closure of the non-simply-connected component of $H^2 \times \mathbb{R} - C^H_{d_H}$. The $H$-catenoid $C^H_{d_H}$ is unstable if $d \in (-2H, d_H)$ and stable if $d \in (d_H, \infty)$. The $H$-catenoid $C^H_{d_H}$ is a stable-unstable catenoid.

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