A generalisation of Pisier homogeneous Banach algebra

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Abstract

In 1979 Pisier proved remarkably that a sequence of independent and identically distributed standard Gaussian random variables determines, via random Fourier series, a homogeneous Banach algebra \( \mathcal{P} \) strictly contained in \( C(\mathbb{T}) \), the class of continuous functions on the unit circle \( \mathbb{T} \) and strictly containing the classical Wiener algebra \( \mathcal{A}(\mathbb{T}) \) that is, \( \mathcal{A}(\mathbb{T}) \subseteq \mathcal{P} \not\subseteq C(\mathbb{T}) \). This improved some previous results obtained by Zafran in solving a long-standing problem raised by Katznelson. In this paper we extend Pisier’s result by showing that any probability measure on the unit circle defines a homogeneous Banach algebra contained in \( C(\mathbb{T}) \). Thus Pisier algebra is not an isolated object but rather an element in a large class of Pisier-type algebras. We consider the case of spectral measures of stationary sequences of Gaussian random variables and obtain a sufficient condition for the boundedness of the random Fourier series \( \sum_{n \in \mathbb{Z}} \hat{f}(n) \xi_n \exp(2\pi i n t) \) in the general setting of dependent random variables \( (\xi_n) \).

Key words: homogeneous Banach algebra, random Fourier series, Gaussian processes.

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1 Introduction

Pisier [8] constructed a homogeneous Banach algebra \( \mathcal{P} \) of continuous functions on the unit circle \( \mathbb{T} = \mathbb{R}/\mathbb{Z} \) such that \( \mathcal{A}(\mathbb{T}) \not\subseteq \mathcal{P} \not\subseteq C(\mathbb{T}) \) where \( C(\mathbb{T}) \) is the Banach algebra of continuous functions on \( \mathbb{T} \) and \( \mathcal{A}(\mathbb{T}) \) is the classical Wiener algebra of continuous functions on \( \mathbb{T} \) with absolutely convergent Fourier series. He further obtained that all 1-Lipschitzian functions operate on \( \mathcal{P} \). To fully describe \( \mathcal{P} \), we shall consider a fixed probability space \( (\Omega, \mathcal{F}, \mathbb{P}) \) and a Gaussian Hilbert space \( \mathcal{H} \), that is, a closed subspace of \( L^2(\mathbb{P}) \) whose elements are all Gaussian random variables. Pisier algebra \( \mathcal{P} \) is determined by a sequence \( Z = (Z_n)_{n \in \mathbb{Z}} \) of independent standard Gaussian random variables. We recall that a Gaussian process on \( \mathbb{T} \) is simply a family \( \{X(t) : t \in \mathbb{T}\} \) of random variables in \( \mathcal{H} \). For \( f \in C(\mathbb{T}) \) and \( t \in \mathbb{T} \), the random Fourier series \( \sum_{n \in \mathbb{Z}} \hat{f}(n) Z_n e^{2\pi i n t} \) converges obviously in \( L^2(\mathbb{P}) \) because \( \sum_{n \in \mathbb{Z}} |\hat{f}(n)|^2 < \infty \). The class \( \mathcal{P} \) is defined as the class of functions \( f \in C(\mathbb{T}) \) such that the Gaussian process \( S(f, Z) = \{S(f, Z, t) : t \in \mathbb{T}\} \) consisting of the \( L^2 \)-sums \( S(f, Z, t) = \sum_{n \in \mathbb{Z}} \hat{f}(n) Z_n e^{2\pi i n t} \) has bounded sample paths almost surely. Equivalently \( \mathcal{P} \) is the class of functions \( f \in C(\mathbb{T}) \) such that the Fourier random series \( \sum_{n \in \mathbb{Z}} \hat{f}(n) Z_n e^{2\pi i n t} \) converges in \( C(\mathbb{T}) \). Other equivalent characterisations of \( \mathcal{P} \) can be obtained from the classical Billard theorem (see Cohen and Cuny [2] and Kahane [3, p 58]).

Given a Borel probability measure \( \mu \) on \( \mathbb{T} \) and a continuous function \( f \in C(\mathbb{T}) \), we shall associate a Gaussian process \( X_f = \{X_f(t) : t \in \mathbb{T}\} \) such that for all \( t, s \in \mathbb{T} \),

\[
E(|X_f(t) - X_f(s)|^2) = \int_{\mathbb{T}} |f(t + u) - f(s + u)|^2 d\mu(u)
\]

and consider the class \( \mathcal{P}(\mu) \) of those continuous functions \( f \) on \( \mathbb{T} \) for which the process \( X_f \) has bounded paths (almost surely) and set

\[
\|f\|_{\mathcal{P}(\mu)} = E \left( \sup_{s, t \in \mathbb{T}} |X_f(t) - X_f(s)| \right) + \|f\|_\infty.
\]
Pisier algebra $\mathcal{P}$ is $\mathcal{P}(\mu)$ where $\mu$ is the Lebesgue measure on $\mathbb{T}$. We shall obtain that $\mathcal{P}(\mu)$ is a homogeneous Banach algebra on which all 1-Lipschitz functions operate and further we shall obtain a sufficient condition for $\mathcal{P}(\mu) \subset \mathcal{P}$.

As already indicated, Pisier algebra arose from the study of random Fourier series of i.i.d Gaussian random variables. We shall consider throughout a discrete-time complex Gaussian process $\xi = (\xi_n)_{n \in \mathbb{Z}}$ in the Gaussian Hilbert space $\mathcal{H}$ such that $E(\xi_n) = 0$ and $E(|\xi_n|^2) = 1$ for all $n \in \mathbb{Z}$. We shall further assume that for all $n, m \in \mathbb{Z}$, $E(\xi_n \xi_m)$ depends only on $n - m$ and $E(\xi_n \xi_m) = 0$. This implies that the process $\xi = (\xi_n)_{n \in \mathbb{Z}}$ is stationary. Clearly for any $\mathbb{Z} \in \text{span}(\xi)$, $Z$ is a complex Gaussian variable with distribution $f(z) = \exp(-|z|^2/\sigma^2)/(\pi \sigma^2)$ where $\sigma^2 = E(|Z|^2)$. The covariance function of $\xi$ is the function $\gamma : \mathbb{Z} \to \mathbb{C}$ defined by $\gamma(n) = E(\xi_n \overline{\xi_m})$. Clearly $\gamma(-n) = \gamma(n)$ and $\gamma(0) = 1$. Such sequence $(\xi_n)$ can be obtained by taking a sequence $(\zeta_n)$ of real standard Gaussian random variables such that $E(\zeta_n \zeta_m)$ depends only on $(n - m)$ and thereafter take $\xi_n = (\zeta_n + i \zeta_n)/\sqrt{2}$ where $(\zeta_n)$ is an independent copy of $(\zeta_n)$. Since the covariance function $\gamma$ is positive semidefinite, the classical Bochner theorem implies that there exists a Borel probability measure $\mu$ on the unit circle $\mathbb{T}$ such that

$$
E(\xi_n \overline{\xi_m}) = \gamma(n-m) = \int_{\mathbb{T}} e^{2\pi i (n-m)t} \, d\mu(t), \quad n, m \in \mathbb{Z}.
$$

(1)

The measure $\mu$ is the spectral measure of the process $\xi$. If $\mu$ is absolutely continuous with respect to the Lebesgue measure on $\mathbb{T}$, its density $\varphi$ is called the spectral density function of the process $\xi$. In the corresponding Banach algebra $\mathcal{P}(\mu)$ (where $\mu$ is the spectral measure of $\xi$), it is easy to see that the formal random Fourier series $\sum_{n \in \mathbb{Z}} \hat{f}(n) \xi_n e^{2\pi i nt}$ is almost surely the formal Fourier series of the function $t \to X_f(t)$. This is true because the Gaussian sequence $(\hat{f}(n)\xi_n)_{n \in \mathbb{Z}}$ has the same distribution with the sequence of the Fourier coefficients $(\hat{X}_f(n))_{n \in \mathbb{Z}}$ of $X_f$ given by

$$
\hat{X}_f(n) = \int_{\mathbb{T}} X_f(t) e^{-2\pi i nt} \, dt.
$$

If $\mu$ has a density $\varphi$ such that $\varphi \in L^p(du)$ for some $1 < p < \infty$, then for any $f \in \mathcal{P}(\mu)$ and $t \in \mathbb{T}$, the formal Fourier random series $\sum_{n \in \mathbb{Z}} \hat{f}(n) \xi_n e^{2\pi i nt}$ converges in $L^2(\mathbb{T})$. We shall obtain a sufficient condition on the covariance matrix of the sequence $(\xi_n)$ for the almost surely boundedness of the random Fourier series $\sum_{n \in \mathbb{Z}} \hat{f}(n) \xi_n e^{2\pi i nt}$ that generalises a classical condition only known for i.i.d variables. This is in particular applied to the classical fractional Gaussian noise.

The study of random Fourier series $\sum_{n \in \mathbb{Z}} \hat{f}(n) \xi_n \exp(2\pi i nt)$ where the random variables $(\xi_n)$ are interdependent is important as it builds a bridge between the classical random series $\sum_{n \in \mathbb{Z}} \hat{f}(n)Z_n \exp(2\pi i nt)$ of independent variables $(Z_n)$ and the deterministic Fourier series $\sum_{n \in \mathbb{Z}} \hat{f}(n) \exp(2\pi i nt)$ which is the limit case where all the random variables $(\xi_n)$ are the same. For this limit case the problem of $L^2$-convergence becomes the problem of characterisation of the classical class $\mathcal{C}$ of functions $f \in C(\mathbb{T})$ with everywhere convergent Fourier series. (This is very complex since $\mathcal{C}$ is beyond the Borel hierarchy in $C(\mathbb{T})$ (see Ajtai and Kechris [1]).) This paper can be seen as a step toward the study of random Fourier series in the Banach space $C(\mathbb{T})$ in the presence of interdependent random variables which constitutes an important open problem.

2 Banach algebras defined by probability measures

Given a Borel probability measure $\mu$ on $\mathbb{T}$ and a $f \in C(\mathbb{T})$, we can associate to $f$ the pseudo-distance $d_f$ on $\mathbb{T}$ defined by

$$
d_f(t, s) = \left( \int_{\mathbb{T}} |f(t+u) - f(s+u)|^2 d\mu(u) \right)^{1/2}, \quad t, s \in \mathbb{T}.
$$

Since the space $L^2(\mu, \mathbb{C})$ embeds in a Gaussian Hilbert space, there exists a Gaussian process $X_f = \{X_f(t) : t \in \mathbb{T}\}$ such that for all $t, s \in \mathbb{T}$,

$$
E \left( X_f(t) X_f(s) \right) = \int_{\mathbb{T}} f(t+u) f(s+u) d\mu(u)
$$

2
and hence
\[ E(|X_f(t) - X_f(s)|^2) = d_f^2(t,s) \quad \text{and} \quad E(|X_f(t)|^2) = \int \|f(t + u)|^2d\mu(u). \]

More precisely, one may consider an orthonormal basis \((e_n)_{n \in A}\) of \(L^2(\mu, \mathbb{C})\) (where \(A\) is a subset of \(\mathbb{Z}\)), a sequence of independent and identically distributed (i.i.d) standard complex Gaussian random variables \((Z_n)\) and set for all \(t \in \mathbb{T}\),
\[ X_f(t) = \sum_{n \in A} \langle f_t, e_n \rangle Z_n \]
where \(f_t\) is the function defined on \(\mathbb{T}\) by \(f_t(u) = f(t + u)\) and
\[ \langle g, e_n \rangle = \int_{\mathbb{T}} g(u)\overline{e_n(u)}d\mu(u), \; g \in L^2(\mu, \mathbb{C}). \]

(The convergence of the series \(\sum_{n \in A} \langle f_t, e_n \rangle Z_n\) in \(L^2(\mathbb{P})\) is guaranteed by the fact that \((Z_n)\) are independent \(N(0,1)\) variables.) Clearly the process \(X_f\) depends linearly on \(f\). Let \(\mathcal{P}(\mu)\) be the class of continuous functions \(f\) on \(\mathbb{T}\) such that the process \(X_f\) has almost surely bounded paths and set
\[ \|f\|_{\mathcal{P}(\mu)} = E \left( \sup_{s,t \in \mathbb{T}} |X_f(t) - X_f(s)| \right) + \|f\|_\infty. \]

The classical Pisier algebra \(\mathcal{P}\) is \(\mathcal{P}(\mu)\) where \(\mu\) is the Lebesgue measure on \(\mathbb{T}\). We have the following result.

**Theorem 1** For every Borel probability measure \(\mu\), the class \(\mathcal{P}(\mu)\) is a homogeneous Banach algebra on which all 1-Lipschitz functions operate.

**Proof** The proof is a replication of Pisier’s original argument in [8] and the following classical comparison principle: Let \(X = \{X(t) : t \in \mathbb{T}\}\) and \(Y = \{Y(t) : t \in \mathbb{T}\}\) be two real Gaussian processes. If
\[ E|Y(t) - Y(s)|^2 \leq E|X(t) - X(s)|^2 \quad \text{for all} \; t, s \in \mathbb{T}, \]
then
\[ E \left( \sup_{s,t \in \mathbb{T}} |Y(t) - Y(s)| \right) \leq E \left( \sup_{s,t \in \mathbb{T}} |X(t) - X(s)| \right). \]

In particular if \(X\) has bounded paths almost surely then \(Y\) has also bounded paths almost surely. Here
\[ E \left( \sup\{|X(t)| : t \in \mathbb{T}\} \right) = \sup \{E\sup\{|X(t)| : t \in F\} : F \text{ finite subset of } \mathbb{T}\}. \]

The comparison principle extends easily to complex Gaussian processes. Indeed, write \(X_1 = \Re(X), X_2 = \Im(X)\) (both \(X_1\) and \(X_2\) are real Gaussian processes) and set \(X' = X_1 + X_2'\) where \(X_2'\) is a real Gaussian process which is an independent copy of \(X_2\). Then clearly
\[ E \left( \sup_{t,s} |X(t) - X(s)| \right) \leq 2E \left( \sup_{t,s} |X'(t) - X'(s)| \right), \]
\[ E \left( \sup_{t,s} |X'(t) - X'(s)| \right) \leq 2E \left( \sup_{t,s} |X(t) - X(s)| \right) \]
and
\[ E|X'(t) - X'(s)|^2 = E|X_1(t) - X_1(s)|^2 + E|X_2'(t) - X_2'(s)|^2 = E|X(t) - X(s)|^2. \]
Similarly for \( Y \), define \( Y_1, Y_2, Y'_2 \) and write \( Y' = Y_1 + Y'_2 \) so that \( \mathbb{E}|Y'(t) - Y'(s)|^2 = \mathbb{E}|Y(t) - Y(s)|^2 \). Hence \( \mathbb{E}|Y'(t) - Y'(s)|^2 \leq \mathbb{E}|X'(t) - X'(s)|^2 \) which implies

\[
\mathbb{E} \left( \sup_{t,s} |Y'(t) - Y'(s)| \right) \leq \mathbb{E} \left( \sup_{t,s} |X'(t) - X'(s)| \right).
\]

Then

\[
\mathbb{E} \left( \sup_{t,s} |Y(t) - Y(s)| \right) \leq 2 \mathbb{E} \left( \sup_{t,s} |Y'(t) - Y'(s)| \right) \leq 2 \mathbb{E} \left( \sup_{t,s} |X'(t) - X'(s)| \right) \leq 4 \mathbb{E} \left( \sup_{t,s} |X(t) - X(s)| \right).
\]

Using the comparison principle, it follows that for \( f, g \in C(T) \), if

\[
\mathbb{E}((X_f(t) - X_f(s))^2) \leq \mathbb{E}((X_g(t) - X_g(s))^2),
\]

then

\[
\mathbb{E} \left( \sup_{s,t} |X_f(t) - X_f(s)| \right) \leq 4 \mathbb{E} \left( \sup_{s,t} |X_g(t) - X_g(s)| \right).
\]

This implies that if \( g \in \mathcal{P}(\xi) \) then \( f \in \mathcal{P}(\xi) \). It is now easy to use the same argument as in the paper by Pisier \cite{Pisier} to obtain the complete proof of Theorem \[1\]

In the case where the probability measure \( \mu \) is absolutely continuous with density \( \varphi \), that is, \( d\mu(u) = \varphi(u)du \), it is clear that if \( \varphi \) is bounded, then \( \mathcal{P} \subset \mathcal{P}(\mu) \) where \( \mathcal{P} \) is the original Pisier algebra (associated to the Lebesgue measure on \( T \)).

**Theorem 2** Let \( \mu \) be a Borel probability measure on \( T \). If there is an interval \( I \subset T \) and a constant \( C > 0 \) such that for any continuous function \( g \geq 0 \) on \( T \)

\[
\int_I g(u)du \leq C \int_I g(u)d\mu(u),
\]

then \( \mathcal{P}(\mu) \subset \mathcal{P} \). In particular if \( \mu \) admits a density \( \varphi \) that is continuous and strictly positive on some open interval \( I \) of \( T \), then \( \mathcal{P}(\mu) \subset \mathcal{P} \).

**Proof** Clearly there is a finite set of translations of \( I \) that covers \( T \), that is, there exist \( t_1, t_2, \ldots, t_N \) in \( T \) such that \( T \subset (I + t_1) \cup \ldots \cup (I + t_N) \). Set \( \mu' = (\sum_{k=1}^{N} \mu_j)/N \) with \( \mu_j = \delta_{t_j} \ast \mu \).

(Since each \( \mu_j \) is a probability measure then \( \mu' \) is also a probability measure.) Then for any continuous function \( g \geq 0 \) on \( T \),

\[
\int_T g(u)du = \sum_{j=1}^{N} \int_{I+t_j} g(u)du = \sum_{j=1}^{N} \int_I g(u + t_j)du \leq C \sum_{j=1}^{N} \int_I g(u + t_j)d\mu(u)
\]

\[
= C \sum_{j=1}^{N} \int_I g(u)d\mu_j(u) = CN \int_I g(u)d\mu'(u)
\]

\[
\leq CN \int_T g(u)d\mu'(u).
\]

In particular for any \( t, s \in T, f \in C(T) \),

\[
\int_T |f(t + u) - f(s + u)|^2du \leq CN \int_T |f(t + u) - f(s + u)|^2d\mu'(u).
\]

This implies by the comparison principle that \( \mathcal{P}(\mu') \subset \mathcal{P} \). Moreover \( \mathcal{P}(\mu') = \mathcal{P}(\mu) \). Indeed, for any \( t_0 \in T \), it is clear that \( \mathcal{P}(\mu) = \mathcal{P}(\delta_{t_0} \ast \mu) \) since the corresponding processes \( X_{t_0} \) are obtained from the other by a time translation by \( t_0 \). This implies that \( \mathcal{P}(\mu') \subset \mathcal{P}(\mu) \). To see
that $\mathcal{P}(\mu) \subset \mathcal{P}(\mu')$ also holds, consider independent processes $X_1^f, X_2^f, \ldots, X_N^f$ such that $X_j^f$ is the process associated to the measure $\delta_{\theta_j} \ast \mu$ for all $1 \leq j \leq N$. Set $X_f = (X_1^f + \ldots + X_N^f)/N$. Clearly $X_f$ can be viewed as the process associated to the measure $\mu'$ and

$$
\mathbb{E}(\sup_{s,t} |X_f(t) - X_f(s)|) \leq \frac{1}{N} \sum_{j=1}^{N} \mathbb{E}(\sup_{s,t} |X_j^f(t) - X_j^f(s)|) = \mathbb{E}(\sup_{s,t} |X_f(t) - X_f(s)|)
$$

since each $X_j^f$ has the same distribution as $X_f$ (for all $j$). The comparison principle yields $\mathcal{P}(\mu) \subset \mathcal{P}(\mu')$.

**Remark 1:** If $\mu$ is the spectral measure of a stationary complex Gaussian process $(\xi_n)$ and $\mu$ has a density $\varphi$ such that $\varphi \in L^p(du)$ for some $1 < p < \infty$, then for any $f \in \mathcal{P}(\mu)$ and $t \in \mathbb{T}$, the formal Fourier random series $\sum_{n \in \mathbb{Z}} \hat{f}(n) \xi_n e^{int}$ converges in $L^2(\mathbb{P})$. Indeed, for $t \in \mathbb{T}$ let $U : \text{span}(e^{2\pi nt}) \to \text{span}(\xi_n)$ be the linear map such that $U(e^{2\pi nt}) = \xi_n$ for all $n \in \mathbb{Z}$. By relation (1) $U$ defines an isometry from $L^2(\mu)$ to $L^2(\mathbb{P})$. Then by H"older’s inequality $L^r(du) \subset L^2(\mu)$ for some $1 < r < \infty$ and hence $U$ is bounded from $L^r(du)$ to $L^2(\mathbb{P})$. Since the Fourier transform of any $f \in L^r(du)$ converges in $L^r(du)$ it follows that for any $f$ in $L^r(du)$ the series $\sum_{n \in \mathbb{Z}} \hat{f}(n) \xi_n$ converges in $L^2(\mathbb{P})$. Applying this to the translates $f_t(u) = f(t + u)$ of $f$ yields that the random Fourier series $\sum_{n \in \mathbb{Z}} \hat{f}(n) \xi_n e^{2\pi nt}$ converges in $L^2(\mathbb{P})$ for every $t \in \mathbb{T}$.

# Banach algebras spanned by the fractional Gaussian noise

## 3.1 Spectral function density of fractional Gaussian noise

Now we provide explicit examples of Pisier-type Banach algebras $\mathcal{P}(\mu)$. The classical discrete complex fractional Gaussian noise (fGn) of parameter $0 \leq H < 1$ is the complex Gaussian process $\Delta = (\Delta_n)_{n \in \mathbb{Z}}$ with mean $0$ and covariance function:

$$
\mathbb{E}(\Delta_n \overline{\Delta_m}) = \gamma(n-m) = \frac{1}{2} |n-m|^{2H} + \frac{1}{2} |n-m-1|^{2H} - |n-m|^H, \quad m, n \in \mathbb{Z}.
$$

The process $\Delta$ can be realised as $\Delta_n = (\delta_n^1 + i \delta_n^2)/\sqrt{2}$ where $\delta^1 = (\delta_n^1) = (\delta_n^1)$ is the classical real fractional Gaussian noise and $\delta^2$ is an independent copy of $\delta^1$. (Here $\mathbb{E}(\delta_n^1 \delta_n^2) = \gamma(n-m).$) The process $\Delta$ reduces to a sequence of i.i.d complex Gaussian random variables for $H = 1/2$ but generally for $H \neq 1/2$ the random variables $(\Delta_n)$ are *interdependent*. (Fractional Gaussian noise is an important model with wide range applications in physics and signal processing.) It is well-known that the sequence $\Delta$ has a spectral density function and we shall denote it by $\varphi_H$. The covariance function (2) implies that for any $\ell \in \mathbb{Z}$, the sequence $(\Delta_{\ell+k})_{k \in \mathbb{Z}}$ has the same distribution as the initial sequence $\Delta = (\Delta_k)_{k \in \mathbb{Z}}$, that is the distribution of $\Delta$ is stationary. Moreover for any $\ell \in \mathbb{Z}$, the sequence $\Delta^\ell = (\Delta_k^\ell)_{k \in \mathbb{Z}}$ defined by

$$
\Delta_k^\ell = \frac{1}{\ell^H} \left( \sum_{k\ell \leq j < (k+1)\ell} \Delta_j \right), \quad \text{for all } k \in \mathbb{Z}
$$

has the same distribution as the initial sequence $\Delta = (\Delta_k)_{k \in \mathbb{Z}}$. This means that the distribution of $\Delta$ is self-similar. (See Sinai [10] for some details on self-similar distributions.) This can be seen by noting that the fGn $(\Delta_k)$ is a sequence of the increments $(X(k+1) - X(k))$ of the fractional Brownian motion of index $H$ and it is well-known that it is a self-similar process in the sense that, for any $\alpha > 0$, the processes $\{X(\alpha t) : t \in \mathbb{R}\}$ has the same distribution as the process $\{\alpha H X(t) : t \in \mathbb{R}\}$. Indeed,

$$
\Delta_k^\ell = \frac{1}{\ell^H} (X((k+1)\ell - X(k\ell)) \sim X(k+1) - X(k) = \Delta_k
$$
by the self-similarity of fractional Brownian motion. Since the distribution of $\Delta$ is stationary and self-similar, then by a result of Sinai \cite[Theorem 2.1]{11}, it admits a spectral density function $\varphi_H$ on the unit circle $\mathbb{T}$ given by

$$\varphi_H(t) = C(H) |e^{2\pi i t} - 1|^2 \left( \sum_{n=-\infty}^{\infty} \frac{1}{|t+n|^{2H+1}} \right), \quad t \in \mathbb{T}, t \neq 0 \tag{3}$$

where $C(H)$ is a normalising constant. Clearly,

$$\varphi_H(t) = 4C(H) \left( \sin^2 \pi t \right) \left( \sum_{n=0}^{\infty} \frac{1}{(n+t)^{2H+1}} + \frac{1}{(n+1-t)^{2H+1}} \right)$$

$$= 4C(H) \left( \sin^2 \pi t \right) \left( \zeta(2H + 1, t) + \zeta(2H + 1, 1-t) \right) \tag{4}$$

where $\zeta(\ldots)$ is the classical Hurwitz zeta function. For $H = 1/2$, obviously $\varphi_H = 1$ (constant function on $\mathbb{T}$). For $H \neq 1/2$, $\varphi_H$ is clearly continuous on $(0, 1)$. Moreover it is clear that $\zeta(2H + 1, t) = O(t^{-(2H+1)})$ for $t$ near 0 and $\zeta(2H + 1, 1-t) = O(1-t^{-(2H+1)})$ for $t$ near 1. Since also $\sin^2 \pi t = O(t^2 (1-t)^2)$ for $t$ near 0 or 1, it follows from (4) that

$$\varphi_H(t) = O(t^{1-2H}(1-t)^{-2H}), \quad t \in (0, 1). \tag{5}$$

This implies that for $H \leq 1/2$, the function $\varphi_H$ is continuous on the whole unit circle $\mathbb{T}$ and for $H > 1/2$, $t = 0$ and $t = 1$ are singular points (actually $0 = 1$ in $\mathbb{T}$). However even in the case where $H > 1/2$, $\varphi_H$ is integrable and in fact \cite{10} implies that

$$\varphi_H \in L^p(\mathbb{T}) \text{ for all } p < 1/(2H - 1), \ H > 1/2 \tag{6}$$

3.2 The Banach algebra $\mathcal{P}(H)$

The class $\mathcal{P}(\mu_\Delta)$ where $\mu_\Delta$ is the spectral measure of the discrete fractional Gaussian noise $\Delta = (\Delta_n)$ of parameter $H$ will simply be denoted $\mathcal{P}(H)$. Its study was initiated in \cite{11} where it was proven that for $0 \leq H \leq 1/2$, $\mathcal{P}(1/2) \subset \mathcal{P}(H)$ and it was conjectured that $\mathcal{P}(H) \subset \mathcal{P}(1/2)$ for $1/2 \leq H < 1$. The question whether $\mathcal{P}(H)$ is in general a Banach algebra with the usual multiplication of functions was also raised. With the discussion above, it is now clear that for all $0 \leq H < 1$, $\mathcal{P}(H)$ is a homogeneous Banach algebra on which all 1-Lipschitzian functions operate. Moreover for $0 \leq H < 1/2$, the continuity and boundedness of $\varphi_H$ and the fact that $\varphi_H(t) > 0$ everywhere on $\mathbb{T}$ (except for $t = 0 = 1$) implies by Theorem \cite{2} that $\mathcal{P}(H) = \mathcal{P}(1/2) = \mathcal{P}$. Finally for $H$ varying in the interval $[0, 1)$, $\mathcal{P}(H)$ is a decreasing family of homogeneous Banach algebras. That is,

**Theorem 3** For any $0 \leq H_1 < H_2 < 1$, $\mathcal{P}(H_2) \subset \mathcal{P}(H_1)$.

**Proof** This follows from the structure of the spectral density function $\varphi_H$ of the fractional Gaussian noise: For for $H_1 \leq H_2$, it is the case that the quotient $\varphi_{H_1}/\varphi_{H_2}$ is bounded. Indeed, write

$$\frac{\varphi_{H_1}}{\varphi_{H_2}} = K \frac{\zeta(1 + 2H_1, t) + \zeta(1 + 2H_1, 1-t)}{(1 + 2H_2, t) + \zeta(1 + 2H_2, 1-t)), \quad t \in (0, 1)$$

with $K = C(H_1)/C(H_2)$. Clearly $\varphi_{H_1}/\varphi_{H_2}$ is continuous in the open interval $(0, 1)$. For $t$ in a neighborhood of 0 in $\mathbb{T}$,

$$\frac{\varphi_{H_1}}{\varphi_{H_2}} = O(t^{-(1+2H_1)}/t^{-(1+2H_2)}) = O(t^{2(H_2-H_1)}),$$

which implies that $\lim_{t \to 0} \varphi_{H_1}(t)/\varphi_{H_2}(t) = 0$. Similarly for $t$ near 1, $\lim_{t \to 1} \varphi_{H_1}(t)/\varphi_{H_2}(t) = 0$. This implies that $\varphi_{H_1}/\varphi_{H_2}$ is continuous in the closed interval $[0, 1]$ and hence $\varphi_{H_1}/\varphi_{H_2}$ is continuous and bounded on the unit circle $\mathbb{T}$. Then for a constant $M$ depending only on $H_1$ and $H_2$ we have that

$$\varphi_{H_1}(t) \leq M \varphi_{H_2}(t) \text{ for all } t \in \mathbb{T}. \tag{7}$$
Then for each \( f \in C(\mathbb{T}) \),
\[
\int_{\mathbb{T}} |f(t + u) - f(s + u)|^2 \varphi_H_1(u) du \leq M \int_{\mathbb{T}} |f(t + u) - f(s + u)|^2 \varphi_H_2(u) du
\]
which yields by the comparison principle that \( \mathcal{P}(H_2) \subset \mathcal{P}(H_1) \).

4 Boundedness of random Fourier series

**Theorem 4** Let \( \xi = (\xi_n)_{n \in \mathbb{Z}} \) be a stationary complex Gaussian process with mean 0, unit variance, covariance matrix \( \gamma(n-m) \) for \( n, m \in \mathbb{Z} \) and such that \( \mathbb{E}(\xi_n \xi_m) = 0 \) for all \( n, m \in \mathbb{Z} \). Assume that there exists a constant \( b \geq 0 \) such that the operator defined in \( \ell^2(\mathbb{Z}) \) by the bi-infinite matrix \( (a_{nm})_{n,m \in \mathbb{Z}} \) given by
\[
a_{nm} = |\gamma(n-m)| |nm|^{-b} \quad \text{for } n \neq 0, m \neq 0 \quad \text{and } a_{nn} = \begin{cases} 0 & \text{for } n = m = 0 \\ 1 & \text{for } n = m \\ |\gamma(n-m)| & \text{for } n \neq m \end{cases}
\]
is bounded. Let \( f \in L^2(\mathbb{T}) \). If
\[
\sum_{|n| \geq 2} \left( \frac{\sum_{|k| \geq |n|} |\hat{f}(k)|^2 |k|^{2b}}{|n| (\log |n|)^{1/2}} \right)^{1/2} < \infty,
\]
then the random series \( t \mapsto \sum_{n \in \mathbb{Z}} \hat{f}(n) \xi_n e^{2\pi int} \) is bounded almost surely.

In particular if \( f \in C(\mathbb{T}) \) and satisfies (8) then \( f \in \mathcal{P}(\mu) \) where \( \mu \) is the spectral measure of \( \xi \).

**Proof** Condition (8) is a generalisation of the following condition for independent variables: For a sequence \( (g_n) \) of independent standard Gaussian random variables and for \( f \in C(\mathbb{T}) \) and a sequence \( (a_n)_{n \geq 1} \) of real numbers, if
\[
\sum_{n=2}^{\infty} \frac{\sum_{k=n}^{\infty} a_k^2}{n (\log n)^{1/2}} < \infty,
\]
then the random series \( t \mapsto \sum_{n=1}^{\infty} a_n g_n e^{2\pi int} \), \( t \in \mathbb{T} \), converges uniformly. (See Marcus and Pisier [1] p. 122.) Consider the series \( S(f, \xi, t) = \sum_{n \in \mathbb{Z}} \hat{f}(n) \xi_n \exp(2\pi int), t \in \mathbb{T} \) and write for \( N \in \mathbb{N} \)
\[
S_N(t) = \sum_{|n| \leq N} \hat{f}(n) \xi_n e_n(t), \quad \text{with } e_n(t) = \exp(2\pi int), \quad \text{for all } n
\]
and \( \gamma(n-m) = \mathbb{E}(\xi_n \overline{\xi_m}) \). Then for \( N \leq M \) in \( \mathbb{N} \),
\[
\mathbb{E}(|S_M(t) - S_N(t)|^2) \leq \sum_{N < |n| \leq M} |\hat{f}(n)|^2 |e_n(t)|^2 + \sum_{N < |n| \neq |m| \leq M} |\gamma(n-m)||\hat{f}(n)||\hat{f}(m)||e_n(t)||e_m(t)|
\]
\[
\leq \sum_{N < |n| \leq M} |\hat{f}(n)|^2 |e_n(t)|^2 + \sum_{N < |n| \neq |m| \leq M} a_{nm} |n|^b |m|^b |\hat{f}(n)||\hat{f}(m)||e_n(t)||e_m(t)|.
\]
Now since the matrix \( A = (a_{nm}) \) defines a bounded operator in \( \ell^2(\mathbb{Z}) \), then
\[
\sum_{N < |n| \neq |m| \leq M} a_{nm} |n|^b |\hat{f}(n)||\hat{f}(m)||e_n(t)||e_m(t)| \leq K \sum_{N < n \leq M} |\hat{f}(n)|^2 |n|^{2b} |e_n(t)|^2
\]
for some constant \( K > 0 \). (In fact we may write this sum as \( w^T A \overline{w} \) where \( w = (w_n)_{n \in \mathbb{Z}} \) and \( w_n = |\hat{f}(n)||n|^b |e_n(t)| \)) for \( N < n \leq M \) and \( w_n = 0 \) outside the interval \( (N, M) \) in \( \mathbb{Z} \) and since \( A \) is positive semidefinite and bounded, then \( w^T A \overline{w} \leq K \|w\|^2 \). It follows that
\[
\mathbb{E}(|S_M(t) - S_N(t)|^2) \leq K' \sum_{N < n \leq M} |\hat{f}(n)|^2 |n|^{2b} |e_n(t)|^2
\]
Corollary 1

Let $\gamma$, $\alpha$, and $\beta$ be constants such that $0 < \alpha < 1$, $0 < \beta < 1$, and $0 < \gamma < 1$. It is easy to verify that for all $t \in \mathbb{T}$, the series $\sum_{n \in \mathbb{Z}} |\hat{f}(n)|^2 |n|^{2\beta} |e_n(t) - e_n(s)|^2$ converges from which it follows that $\lim_{N,M \to \infty} \mathbb{E}(|S(t) - S_N(t)|^2) = 0$. This yields the $L^2$-convergence of the series $S(f, \xi, t)$. The same argument yields that for all $t, s \in \mathbb{T}$,

$$\mathbb{E}|S(f, \xi, t) - S(f, \xi, s)|^2 \leq K \sum_{n \in \mathbb{Z}} |\hat{f}(n)|^2 |n|^{2\beta} |e_n(t) - e_n(s)|^2$$

for some constant $K > 0$. That is

$$\mathbb{E}|S(f, \xi, t) - S(f, \xi, s)|^2 \leq K \mathbb{E} \left| \sum_{n \in \mathbb{Z}} \hat{f}(n)|n|^\beta g_n(e_n(t) - e_n(s)) \right|^2$$

where $(g_n)$ is the standard complex Gaussian sequence. Inequality (8) implies that the random series $t \mapsto \sum_{n \in \mathbb{Z}} \hat{f}(n)|n|^\beta g_n e_n(t)$ converges uniformly on $\mathbb{T}$. This yields by the comparison principle that the paths $t \mapsto S(f, \xi, t)$ are bounded almost surely. If in addition $f$ is continuous then $f \in \mathcal{P}(\xi)$. ■

An example of a covariance function $\gamma$ for which the matrix $(\gamma(n-m)nm^{-b})$ defined a bounded operator in $l^2(\mathbb{Z})$ is obtained by assuming $|\gamma(n-m)| \leq K|n-m|^{-a}$ for some constants $K > 0$ and $0 < a < 1$. This can be obtained by using the classical Schur test together with the following well-known fact: for real numbers $0 < a < 1$ and $d > 0$ such that $a + d > 1$, there exists a constant $K > 0$ (depending on $a$ and $d$) such that

$$\sum_{m \neq n} |n - m|^{-a} |m|^{-d} \leq K |n|^{1-(a+d)}.$$

Set $a_{nm} = |n - m|^{-a} |nm|^{-b}$ (with $a_{nn} = a_{n0} = a_{0n} = 0$) and consider a real number $c > 0$ such that $a + b + c > 1$ and $x = (x_n)_{n \in \mathbb{Z}}$ with $x_n = |n|^{-c}$ for $n \neq 0$ and $x_0 = 1$. It is the case that

$$\forall n \sum_{m \in \mathbb{Z}} a_{nm} x_m \leq K x_n \quad \text{and} \quad \forall m \sum_{n \in \mathbb{Z}} x_n a_{nm} \leq K x_m.$$

Hence Schur’s test yields that the operator defined by $(a_{nm})$ is bounded. In particular the covariance function $\gamma_H$ of the fractional Gaussian noise of index $H$ is such that the corresponding matrix $(a_{nm})$ given by $a_{nm} = \gamma_H(n-m)nm^{-b}$ for any $b > 0$ defines a bounded operator. Indeed it is easy to verify that $0 < \gamma_H(n-m) \leq |n-m|^{-2(1-H)}$ for $m \neq n$.

**Corollary 1** Let $f \in L^2(\mathbb{T})$ and $\frac{1}{2} \leq H < 1$. If there exist $\alpha > H$ and $K > 0$ such that

$$|\hat{f}(n)| \leq K |n|^{-\alpha}, \quad \text{for all } n \neq 0,$$

then the function $t \mapsto \sum_{n \in \mathbb{Z}} \hat{f}(n) \Delta_n e^{2 \pi int}$ is bounded almost surely. In particular if in addition $f \in C(\mathbb{T})$ but the series $\sum_{n \in \mathbb{Z}} |\hat{f}(n)|$ diverges, then $f \in \mathcal{P}(H)$ but $f \notin \mathcal{A}(\mathbb{T})$.

This follows from the fact that by taking $b = H - 1/2 > 0$ it is clear that inequality (8) holds.

**Remark 2:** It would be interesting to investigate whether inequality (8) implies the uniform convergence (almost surely) of the random Fourier series $t \mapsto \sum_{n \in \mathbb{Z}} \hat{f}(n) \xi_n e^{2 \pi int}$ as it is the case for i.i.d random variables. Pisier [3] obtained that the Banach space of $f \in L^2(\mathbb{T})$ such that $\sum_{n \in \mathbb{Z}} Z_n \hat{f}(n) e^{2 \pi int}$ (with $(Z_n)$ i.i.d Gaussian variables) is bounded a.s. is of cotype 2. It would be interesting to consider this question for the general stationary Gaussian process $(\xi_n)$. Pedersen [4] proved that spectral synthesis holds in $\mathcal{P}$. This question is also of some interest in the general setting discussed in this paper.
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