Research Article

Numerical Solution of a Class of Time-Fractional Order Diffusion Equations in a New Reproducing Kernel Space

Xiaoli Zhang,1,2 Haolu Zhang,3 Lina Jia,2 Yulan Wang,2 and Wei Zhang1

1Institute of Economics and Management, Jining Normal University, Jining 012000, Inner Mongolia, China
2Department of Mathematics, Inner Mongolia University of Technology, Hohhot 010051, China
3School of Civil Engineering, Inner Mongolia University of Technology, Hohhot 010051, China

Correspondence should be addressed to Yulan Wang; wylnei@163.com

Received 17 December 2019; Accepted 10 March 2020; Published 10 April 2020

Academic Editor: Wilfredo Urbina

Copyright © 2020 Xiaoli Zhang et al. This is an open access article distributed under the Creative Commons Attribution License, which permits unrestricted use, distribution, and reproduction in any medium, provided the original work is properly cited.

In this paper, we structure some new reproducing kernel spaces based on Jacobi polynomial and give a numerical solution of a class of time fractional order diffusion equations using piecewise reproducing kernel method (RKM). Compared with other methods, numerical results show the reliability of the present method.

1. Introduction

In this paper, we consider the following time-fractional order diffusion equation:

\[
\begin{aligned}
&D_t^c u(x, t) + \beta_1(x, t) \frac{\partial u(x, t)}{\partial x} + \beta_2(x, t) \frac{\partial^2 u(x, t)}{\partial x^2} \\
&= f(x, t), (x, t) \in \Omega = [0, 1] \times [0, 1], \\
&u(x, 0) = \phi_1(x), 0 \leq x \leq 1, \\
&u(0, t) = \varphi_1(t), u(1, t) = \varphi_2(t), 0 \leq t \leq 1,
\end{aligned}
\]  

where \( \beta_1(x, t), \beta_2(x, t), f(x, t) \) are known functions, and \( D_t^c u(x, t) \) is the variable order Caputo fractional derivative of order \( c \):

\[
D_t^c u(x, t) = \frac{1}{\Gamma(1-c)} \int_0^t (t-\tau)^{-c} \frac{\partial u(x, \tau)}{\partial \tau} \, d\tau.
\]

The time-fractional order equation [1–4] has widespread applications in viscoelastic materials, signal processing, fluid mechanics, and dynamic of viscoelastic materials. The analytic solution to this equation is almost impossible to obtain. In recent years, several numerical methods [5–11] have been proposed. In previous work, the author used Taylor’s formula or Delta function to construct reproducing kernel space [12–17]. In this paper, we structure some new reproducing kernel spaces based on Jacobi polynomials and give a numerical solution of a class of time-space fractional order diffusion equation using piecewise reproducing kernel method.

Definition 1. Let \( H \) be a real Hilbert spaces of functions \( f: \Omega \longrightarrow R \). A function \( K: \Omega \times \Omega \longrightarrow R \) is called reproducing kernel for \( H \) if

\[
\begin{align*}
(i) & \ K(x, \cdot) \in H \text{ for all } x \in \Omega, \\
(ii) & \ f(x) = \langle f, K(\cdot, x) \rangle_H \text{ for all } f \in H \text{ and all } x \in \Omega.
\end{align*}
\]

2. Structuring Reproductive Kernel Space Based on the Shifted Jacobi Polynomials

The shifted Jacobi polynomials \( P^{\alpha, \beta}_{i,j}(x) \) of degree \( i \) is given [18] by

\[
P^{\alpha, \beta}_{i,j}(x) = \sum_{k=0}^{i} (-1)^{(i-k)} \frac{\Gamma(i+\beta+1)\Gamma(i+k+1+\alpha+\beta)}{\Gamma(k+1+\beta)\Gamma(i+\alpha+\beta+1)(i-k)!} x^{i-k},
\]

where
where $P_{i,j}^{α,β}(x)$ is a weight function and

$$h_k = \begin{cases} \Gamma(k + \alpha + 1) / \Gamma(k + \alpha + \beta + 1), & n = m, \\ 0, & n \neq m. \end{cases}$$

Theorem 2. Let

$$\mathcal{H}_n[0, 1] = \left\{ P_{i,j}^{α,β}(x) \mid \int_0^1 \omega(x) P_{i,j}^{α,β}(x) dx < \infty, \right\}$$

where $i = 1, \ldots, n$, is the weighted inner product space of the shifted Jacobi polynomials on [0, 1]. The inner product and norm are defined as

$$\langle P_{i,j}^{α,β}(x), P_{i,j}^{α,β}(x) \rangle = \int_0^1 \omega(x) P_{i,j}^{α,β}(x) P_{i,j}^{α,β}(x) dx,$$

where

$$\left\| P_{i,j}^{α,β}(x) \right\|_{\mathcal{H}_n} = \sqrt{\langle P_{i,j}^{α,β}(x), P_{i,j}^{α,β}(x) \rangle_{\mathcal{H}_n}},$$

and $\mathcal{H}_n[0, 1]$ is a reproducing kernel Hilbert space. Its reproducing kernel is

$$R_n(x, y) = \sum_{i=0}^n e_i(x)e_i(y),$$

where $e_i(x) = \frac{1}{\Gamma((2k + \alpha + \beta + 1) / \Gamma(k + \alpha + 1) \Gamma(k + \beta + 1))} P_{i,j}^{α,β}(x)$. Using Ref. [5] and the reproducing kernel of $\mathcal{H}_n[0, 1]$, we can get following reproducing kernel spaces.

(1) Space $H_2 = \{ u(x) \mid u(x) \in \mathcal{H}_n, u(0) = 0 \}$, $H_n$ with the same inner product as $\mathcal{H}_n$, is a reproducing kernel space and its reproducing kernel is

$$K_2(x, y) = R_n(x, y) - \frac{R_{n+1}(x,y)}{\left\| R_{n+1}(0,0) \right\|^2}.$$

(2) Space $H_{n+1} = \{ u(x) \mid u(x) \in \mathcal{H}_{n+1}, u(0) = 0, \ u(1) = 0 \}$, $H_{n+1}$ with the same inner product as $\mathcal{H}_{n+1}$, is a reproducing kernel space and its reproducing kernel is

$$K_{n+1}(x, y) = \frac{k_{n+1}(1,x)k_{n+1}(y,1)}{\left\| k_{n+1}(1,1) \right\|^2}.$$

where $k_{n+1}(x, y) = R_{n+1}(x, y) - (R_{n+1}(0,x)R_{n+1}(y,0)/\left\| R_{n+1}(0,0) \right\|^2)$.

(3) Space $H(Ω) = H_2 \otimes H_3 = \{ u(x,t) \mid u(x,t) \in H_2 \otimes H_3 \}$, $u(0,t) = u(1,t) = u(x,0) = 0$, and its reproducing kernel is

$$K(x, t, y, s) = K_n(x, y) \times K_{n+1}(t, s), (x, y), (t, s) \in Ω,$$

where $K_2(x, y), K_{n+1}(x, y)$ from (11) and (12). Reproducing kernels with different $α, β$ are shown in Figures 1–8.

### 3. Piecewise Reproducing Kernel Method

After homogenization, equation (1) is converted to the following form:

$$\begin{cases} \mathcal{H}_{(x,t)} v(x,t) = g(x,t), (x,t) \in Ω = [0, 1] \times [0, 1], \\ v(x, 0) = 0, \ 0 \leq x \leq 1, \\ v(0, t) = v(1, t) = 0, \ 0 \leq t \leq 1, \end{cases}$$

where $\mathcal{H}$ is a operator, and

$$\mathcal{H}_{(x,t)} u(x,t) = D^2_{11} u(x,t) + \beta_1(x,t) \frac{\partial u(x,t)}{\partial x} + \beta_2(x,t) \frac{\partial^2 u(x,t)}{\partial x^2}$$

$$v(x,t) = u(x,t) - U(x,t) - u_0(x) + U_0(x), \quad U(x,t) = \phi_1(t)(1 - x) + \varphi_2(t),$$

where $\mathcal{H}_{(x,t)} = \mathcal{H}_{(x)} \mathcal{H}_{(t)}$, $\psi_i(x,t) = \mathcal{H}_{(x)}\mathcal{H}_{(t)}$, $i = 1, 2, \ldots, \infty$.

$$\mathcal{H}_{(x)} = \sum_{k=1}^{\infty} \beta_{ik} \psi_{ik}$$

where the $\beta_{ik}$ are the coefficients resulting from Gram–Schmidt orthonormalization.

Theorem 1 (see [11, 19–23]). If $\mathcal{H}^{-1}$ is existing and $\{x_i, t_j\}_{i=1}^\infty$ is denumerable dense points in $Ω$, then

$$v(x,t) = \sum_{i=1}^{\infty} \sum_{k=1}^{\infty} \beta_{ik} g(x_k, t_k) \psi_i(x,t),$$

is an analytical solution of (14). Deriving from the form of (17), we get the approximate solution of (14) as

$$v_N(x, t) = \sum_{i=1}^{N} \sum_{k=1}^{\infty} \beta_{ik} g(x_k, t_k) \psi_i(x,t).$$

However, the direct application of (17) could not have a good numerical simulation effect possibly for (1). The focus of this paper is to fill this gap, so we use the piecewise reproducing kernel method. The main technique of the piecewise reproducing kernel method see Ref. [16, 21, 24, 25].
Figure 1: Reproducing kernel $R_n(x,y)$ of space $\mathcal{H}_3[0,1]$ with $\alpha = \beta = 1$.

Figure 2: The set of reproducing kernel $R_n(x,y)$ with $n = 1,2,\ldots,7$, $\alpha = \beta = 1$.

Figure 3: Reproducing kernel $R_3(x,y)$ of space $\mathcal{H}_3[0,1]$ with $\alpha = 2, \beta = 3$.

Figure 4: The set of reproducing kernel $R_n(x,y)$ with $n = 1,2,\ldots,7$, $\alpha = 2, \beta = 3$.

Figure 5: Reproducing kernel $R_3(x,y)$ of space $\mathcal{H}_3[0,1]$ with $\alpha = -0.5, \beta = 1$.

Figure 6: The set of reproducing kernel $R_n(x,y)$ with $n = 1,2,\ldots,7$, $\alpha = 0.5, \beta = 1$. 

Figure 7: Reproducing kernel $R_3(x,y)$ of space $\mathcal{H}_3[0,1]$ with $\alpha = 1, \beta = 1$.
More about convergence theorem and error estimation, those detailed proof can be seen in [23–26].

4. Numerical Experiments

In this section, some numerical experiments are studied to demonstrate the accuracy of the present method.

**Experiment 1.** We consider the following time fractional reaction-diffusion equation:

\[
D_t^{\alpha} u(x,t) = \frac{x^2}{\Gamma(3-\gamma)} \frac{\partial^3 u(x,t)}{\partial x^3} - \frac{x}{\Gamma(3-\gamma)} \frac{\partial u(x,t)}{\partial x} + f(x,t), \quad 0 < x < 1, 0 \leq t \leq 1,
\]

\[
\begin{align*}
  u(x,0) &= x^2 (1-x), \quad u(0,t) = 0, \quad u(1,t) = 0,
\end{align*}
\]
Absolute errors

Table 1: Comparison of absolute errors obtained for Experiment 1 at $\alpha = \beta = 0.5, \gamma = 0.8, t = 0.01, n = 4$.

| $x$  | Exact solution | Traditional RKM $N = 2$ | Present method $(h = 0.001) N = 2$ | Present method $(h = 0.000001) N = 2$ |
|------|----------------|--------------------------|-------------------------------------|----------------------------------------|
| 0.1  | 0.0090         | 2.272e−03                | 4.565e−06                           | 6.536e−11                              |
| 0.2  | 0.0320         | 4.032e−03                | 9.332e−06                           | 1.164e−09                              |
| 0.3  | 0.0630         | 5.284e−03                | 1.384e−05                           | 3.207e−09                              |
| 0.4  | 0.0960         | 6.029e−03                | 1.765e−05                           | 5.585e−09                              |
| 0.5  | 0.1251         | 6.270e−03                | 2.028e−05                           | 7.818e−09                              |
| 0.6  | 0.1441         | 6.010e−03                | 2.129e−05                           | 9.425e−09                              |
| 0.7  | 0.1471         | 5.250e−03                | 2.023e−05                           | 9.927e−09                              |
| 0.8  | 0.1281         | 3.994e−03                | 1.663e−05                           | 8.843e−09                              |
| 0.9  | 0.0810         | 2.243e−03                | 1.004e−05                           | 5.694e−09                              |

where $f(x, t) = ((8x^2(1 - x)t^{2-\gamma} + 3x^3(4t^2 + 1))/\Gamma(3 - \gamma))$, the exact solution $u_T(x, t) = x^2(1 - x)(4t^2 + 1)$. Numerical solution of Experiment 1 is shown in Figures 9–11 and Table 1. From Table 1, we can see that the absolute error is getting smaller and smaller when $h$ is smaller. Figure 10 shows the relationship between absolute error and reproducing kernel. Figure 11 shows the relationship between absolute error and $\gamma$.

**Experiment 2.** We consider the following time fractional reaction-diffusion equation [7]

$$D_t^\alpha u(x, t) + x \frac{\partial u(x, t)}{\partial x} - \frac{\partial^2 u(x, t)}{\partial x^2} = f(x, t),$$

$$0 < x < 1, 0 \leq t \leq 1,$$

$$u(x, 0) = x^2 - x^3, u(0, t) = u(1, t) = 0,$$

where $f(x, t) = ((2t^{2-\gamma})/\Gamma(3 - \gamma))(x^2 - x^3) + (t^2 + 1)(2x^2 - 3x^3 + 6x - 2)$, the exact solution $u_T(x, t) = (t^2 + 1)(x^2 - x^3)$. Numerical solution of Experiment 2 is shown in Table 2. From Table 2, we can see that the absolute
error obtained by the present method is smaller than the absolute error obtained by Ref. [7].

**Experiment 3.** We consider the following time-space fractional advection-reaction-diffusion equation:

\[
D_0^\gamma u(x,t) = \alpha x^2 t^\gamma - \beta u(x,t) \frac{\partial u(x,t)}{\partial x^\gamma} + \frac{\Gamma(2 - \gamma) }{\Gamma(3 - \gamma) } \frac{\partial^\gamma u(x,t)}{\partial x^\gamma} + f(x,t),
\]

\[
0 < x < 1, 0 \leq t \leq 1,
\]

\[
u(x,0) = x^2, u(0,0) = 0, u(1,0) = 0.05 x^2 + 1,
\]

where \( f(x,t) = -32 x^2 t^\gamma - \beta / \Gamma(3 - \gamma) \), the exact solution \( \nu(x,t) = x^2 (4t^2 + 1) \). Numerical solution of Experiment 3 is shown in Figure 12 and Table 3.

**Experiment 4.** We consider the following time-space fractional advection-reaction-diffusion equation:

\[
D_0^\gamma u(x,t) = \frac{\partial^\gamma u(x,t)}{\partial x^\gamma} + f(x,t), \quad 0 < x < 1, 0 \leq t \leq 1,
\]

\[
u(x,0) = 0, u(0,0) = u(1,0) = 0,
\]

where \( f(x,t) = (3 \sqrt{\pi} / 4l^4 (2.5 - \alpha) )t^{1.5 - \alpha} x^4 (x - 1) \), the exact solution \( \nu(x,t) = x^2 (1 - x) t^{1.5} \). By mathematical 8.0, the numerical comparison of absolute errors by present method are given in Figures 13–20 at \( n = 2, N = 5 \). The reproducing kernel of Experiment 4 with different \( \alpha, \beta \) is shown in Table 4 by present method at \( n = 2, N = 5 \). Figures 13 and 14 show the relationship between absolute error and \( \alpha, \beta \). From Figures 15 and 16, we can see that the absolute error is small. Figures 17–20 show the relationship between absolute error and \( \gamma \).
Table 4: The reproducing kernel of Experiments 1 and 4 with different $\alpha, \beta$.

| $\alpha$ | $\beta$ | $K_1(x, y)$ | $K_3(x, y)$ |
|---------|---------|-------------|-------------|
| $\alpha = 0$ | $\beta = 0$ | $4xy(12 - 15y + 5x(-3 + 4y))$ | $60xy(-1 + x)(-1 + y)(4 - 7y + 7x(-1 + 2y))$ |
| $\alpha = 1/2$ | $\beta = 1/2$ | $(128xy/7\pi)(21 - 28y + 4x(-7 + 10y))$ | $(1024y/5\pi)(-1 + x)y(-1 + y)(9 - 16y + 16x(-1 + 2y))$ |
| $\alpha = 2$ | $\beta = 1$ | $30xy(10 - 14y + 7x(-2 + 3y))$ | $280(-1 + x)y(-1 + y)y(5 - 9y + 9x(-1 + 2y))$ |
| $\alpha = 1/2$ | $\beta = 1$ | $(1155/512)xy(80 - 104y + 13x(-8 + 11y))$ | $(6435/512)(-1 + x)y(-1 + y)y(80 - 136y + 17x(-8 + 15y))$ |
| $\alpha = 2$ | $\beta = 3$ | $504xy(7 - 10y + 5x(-2 + 3y))$ | $2772(-1 + x)y(-1 + y)y(7 - 12y + 2x(-6 + 11y))$ |

Figure 13: Absolute errors of Experiment 4 with different reproducing kernels for $t = 0.1, h = 0.01, \gamma = 0.5$.

Figure 14: Absolute errors of Experiment 4 with different reproducing kernels for $t = 0.1, h = 0.0001, \gamma = 0.5$.

Figure 15: Absolute errors of Experiment 4 by the present method for $t = 0.1, h = 0.0001, \alpha = \beta = 1$.

Figure 16: Absolute errors of Experiment 1 by the present method for $t = 0.1, \alpha = \beta = 1, \gamma = 0.5$.

Figure 17: Absolute errors of Experiment 4 with different fractional derivatives at $h = 0.001, t = 0.1, \alpha = (1/2), \beta = 1$.

Figure 18: Absolute errors of Experiment 4 with different fractional derivatives at $h = 0.001, t = 0.1, \alpha = \beta = 0$. 
Absolute errors of Experiment 4 with different fractional derivatives at $h = 0.01, t = 0.1, \alpha = \beta = 1.$

5. Conclusions

In this paper, some new reproductive kernels are given. The numerical results of some models show that the present method has high precision compared with traditional RKLM, and has a better convergence for this kind of model. Besides, the method can also be used to study other time variable fractional order advection-dispersion model.

Data Availability

The data used to support the findings of this study are available from the corresponding author upon request.

Conflicts of Interest

The authors declare that there are no conflicts of interest regarding the publication of this article.

Acknowledgments

The authors would like to express our deep thanks to the anonymous reviewers whose valuable comments and suggestions helped us improve this article greatly. This paper was supported by the Natural Science Foundation of Inner Mongolia (2017MS0103).

References

[1] A. Saadatmandi, M. Dehghan, and M.-R. Azizi, “The Sinc-Legendre collocation method for a class of fractional convection-diffusion equations with variable coefficients,” *Communications in Nonlinear Science and Numerical Simulation*, vol. 17, no. 11, pp. 4125–4136, 2012.

[2] H. Dehestani, Y. Ordokhani, and M. Razzaghi, “Application of the modified operational matrices in multiterm variable-order time-fractional partial differential equations,” *Mathematical Methods in the Applied Sciences*, vol. 42, no. 18, pp. 7296–7313, 2019.

[3] Y. Chen, Y. Wu, Y. Cui, Z. Wang, and D. Jin, “Wavelet method for a class of fractional convection-diffusion equation with variable coefficients,” *Journal of Computational Science*, vol. 1, no. 3, pp. 146–149, 2010.

[4] H. Dehestani, Y. Ordokhani, and M. Razzaghi, "Fractional-order Legendre-Laguerre functions and their applications in fractional partial differential equations," *Applied Mathematics and Computation*, vol. 336, pp. 433–453, 2018.

[5] Y. Wang, M. Du, F. Tan, Z. Li, and T. Nie, "Using reproducing kernel for solving a class of fractional partial differential equation with non-classical conditions," *Applied Mathematics and Computation*, vol. 219, no. 11, pp. 5918–5925, 2013.

[6] H. Dehestani, Y. Ordokhani, and M. Razzaghi, "A numerical technique for solving various kinds of fractional partial differential equations via Genocchi hybrid functions," *Revista de la Real Academia de Ciencias Exactas, Fisicas y Naturales. Serie A. Matematicas*, vol. 113, no. 4, pp. 3297–3321, 2019.

[7] F. Zhou and X. Xu, "The third kind Chebyshev wavelets collocation method for solving the time-fractional convection-diffusion equations with variable coefficients," *Applied Mathematics and Computation*, vol. 280, pp. 11–29, 2016.

[8] W. Jiang and Y. Lin, "Representation of exact solution for the time-fractional telegraph equation in the reproducing kernel space," *Communications in Nonlinear Science and Numerical Simulation*, vol. 16, no. 9, pp. 3639–3645, 2011.

[9] H. Dehestani, Y. Ordokhani, and M. Razzaghi, "Numerical technique for solving fractional generalized pantograph-delay differential equations by using fractional-order hybrid bessel functions," *International Journal of Applied and Computational Mathematics*, vol. 6, no. 1, 2020.

[10] Y.-L. Wang, L.-N. Jia, and H.-L. Zhang, "Numerical solution for a class of space-time fractional equation by the piecewise reproducing kernel method," *Applied Mathematics and Computation*, vol. 96, no. 10, pp. 2100–2111, 2019.

[11] W. Jiang and Z. Chen, "Solving a system of linear Volterra integral equations using the new reproducing kernel method," *Applied Mathematics and Computation*, vol. 219, no. 20, pp. 10225–10230, 2013.

[12] B. Y. Wu and Y. Z. Lin, *Application of the Reproducing Kernel Space*, Science Press, Beijing, China, 2012.

[13] Y. L. Wang, C. L. Temuer, and J. Pang, "New algorithm for second-order boundary value problems of integro-differential equation," *Journal of Computational and Applied Mathematics*, vol. 229, no. 1, pp. 1–6, 2009.

[14] X. Li and B. Wu, "A new reproducing kernel collocation method for nonlocal fractional boundary value problems with non-smooth solutions," *Applied Mathematics Letters*, vol. 86, pp. 194–199, 2018.
[15] F. Geng and M. Cui, “A reproducing kernel method for solving nonlocal fractional boundary value problems,” *Applied Mathematics Letters*, vol. 25, no. 5, pp. 818–823, 2012.

[16] Y. Wang, L. Su, X. Cao, and X. Li, “Using reproducing kernel for solving a class of singularly perturbed problems,” *Computers & Mathematics with Applications*, vol. 61, no. 2, pp. 421–430, 2011.

[17] Y. Wang, T. Chaolu, and Z. Chen, “Using reproducing kernel for solving a class of singular weakly nonlinear boundary value problems,” *International Journal of Computer Mathematics*, vol. 87, no. 2, pp. 367–380, 2010.

[18] E. H. Doha, A. H. Bhrawy, and S. S. Ezz-Eldien, “A new Jacobi operational matrix: an application for solving fractional differential equations,” *Applied Mathematical Modelling*, vol. 36, no. 10, pp. 4931–4943, 2012.

[19] Y. L. Wang, Z. Y. Li, Y. Cao, and X. H. Wan, “A new method for solving a class of mixed boundary value problems with singular coefficient,” *Applied Mathematics and Computation*, vol. 217, no. 6, pp. 2768–2772, 2010.

[20] W. Jiang and T. Tian, “Numerical solution of nonlinear Volterra integro-differential equations of fractional order by the reproducing kernel method,” *Applied Mathematical Modelling*, vol. 39, no. 16, pp. 4871–4876, 2015.

[21] Y.-L. Wang, Y. Liu, Z.-Y. Li, and H.-L. Zhang, “Numerical solution of integro-differential equations of high-order Fredholm by the simplified reproducing kernel method,” *International Journal of Computer Mathematics*, vol. 96, no. 3, pp. 585–593, 2019.

[22] X. Li and B. Wu, “A new reproducing kernel method for variable order fractional boundary value problems for functional differential equations,” *Journal of Computational and Applied Mathematics*, vol. 311, pp. 387–393, 2017.

[23] W. Jiang and N. Liu, “A numerical method for solving the time variable fractional order mobile-immobile advection-dispersion model,” *Applied Numerical Mathematics*, vol. 119, pp. 18–32, 2017.

[24] F. Z. Geng and S. P. Qian, “An optimal reproducing kernel method for linear nonlocal boundary value problems,” *Applied Mathematics Letters*, vol. 77, pp. 49–56, 2018.

[25] X. Li, H. Li, and B. Wu, “Piecewise reproducing kernel method for linear impulsive delay differential equations with piecewise constant arguments,” *Applied Mathematics and Computation*, vol. 349, pp. 304–313, 2019.

[26] Y. Wang, X. Cao, and X. Li, “A new method for solving singular fourth-order boundary value problems with mixed boundary conditions,” *Applied Mathematics and Computation*, vol. 217, no. 18, pp. 7385–7390, 2011.