Abstract. The notion of a $C$-filtered object, where $C$ is some (typically small) collection of objects in a Grothendieck category, has become ubiquitous since the solution of the Flat Cover Conjecture around the year 2000. We introduce the $C$-Filtration Game of length $\omega_1$ on a module, paying particular attention to the case where $C$ is the collection of all countably-generated projective modules. We prove that Martin’s Maximum implies the determinacy of many $C$-Filtration Games of length $\omega_1$, which in turn imply the determinacy of certain Ehrenfeucht-Fraisse games of length $\omega_1$; this allows a significant strengthening of a theorem of Mekler-Shelah-Vaananen [20]. Also, Martin’s Maximum implies that if $R$ is a countable hereditary ring, the class of $\sigma$-closed potentially projective modules—i.e., those modules that are projective in some $\sigma$-closed forcing extension of the universe—is closed under $<\aleph_2$-directed limits. We also give an example of a (ZFC-definable) class of abelian groups that, under the ordinary subgroup relation, constitutes an Abstract Elementary Class (AEC) with Löwenheim-Skolem number $\aleph_1$ in some models in set theory, but fails to be an AEC in other models of set theory.

1. Introduction

The classic Gale-Stewart Theorem ensures that 2-player open games (of length $\omega$) are always determined. A frequently used example of such a game is the Ehrenfeucht-Fraissé game of length $\omega$ on a pair of first-order structures, which is an open game for Spoiler (using the terminology of Spencer [26]). It is natural to wonder to what extent does the Gale-Stewart Theorem extend to open games of length $\omega_1$. In this context, “open for Player X” means that every win by Player X is known at some countable stage of play. The $\omega_1$-length analogue of Gale-Stewart Theorem fails, however, even when restricted to Ehrenfeucht-Fraissé games. For example, Mekler-Shelah-Vaananen [20] provide a ZFC example of an Ehrenfeucht-Fraissé game of length $\omega_1$ that is not determined. They did, however, show:

\[\text{More precisely: let } W_X \text{ denote the set of maximal branches through the game tree that represent wins for Player X. The game (of length } \omega_1\text{) is open for Player X if } W_X \text{ is open with respect to the topology generated by sets of the form } B_t := \text{“the maximal branches passing through } t\text{” where } t \text{ is a node in the game tree of countable height.}\]
**Theorem 1.1** (Mekler et al. [20], Theorem 10). Assume (*)

Every structure has an $L(aa)$-elementary substructure of size $<\aleph_2$, where $L(aa)$ is Shelah’s Stationary Logic. Then whenever $G$ is a group and $F$ is a free abelian group, the Ehrenfeucht-Fraïssé game of length $\omega_1$ on the pair $(G,F)$ is determined.

The assumption (*) is consistent relative to the consistency of a supercompact cardinal; Ben-David [3] shows that it holds after the countably-closed Levy collapse to turn a supercompact cardinal into $\aleph_2$. Mekler et al. use assumption (*) to show that if $G$ is abelian, uncountable, and $\aleph_2$-free—i.e., if all its subgroups of size $<\aleph_2$ are free—then *Duplicator* must have a winning strategy in the Ehrenfeucht-Fraïssé game of length $\omega_1$ on the pair $(G,F)$ whenever $F$ is an uncountable free abelian group. The other cases—e.g., where $F$ is countable, or when $G$ is not $\aleph_2$-free—are easily determined in ZFC alone.

We strengthen Theorem 1.1 in two ways:

1. We weaken the assumption (*) to the Fuchino-Usuba [13] principle $RP_{\text{internal}}$ that, unlike (*), follows from Martin’s Maximum ([5], [7]). This improvement is due to a simple observation: that a certain sentence in Stationary Logic used in [20] is equivalent, over $\aleph_1$-generated modules, to a different syntactic form (see Corollary 3.2 and subsequent discussion).

2. Given a module $M$ and a collection $C$ of countably-presented modules, we introduce the $C$-filtration game of length $\omega_1$ on $M$ (Definition 4.1). Player 2 attempts to prove that $M$ is “$C$-filtered” (see below), while Player 1 tries to prevent her from doing so; Player 2 wins if she lasts $\omega_1$ rounds. These games have many interesting properties:

   a. $RP_{\text{internal}}$ implies their determinacy (if $C$ has some nice quotient behavior, see Theorem 1.2), and their determinacy—in the case where the ring is $\mathbb{Z}$ and $C = \{\mathbb{Z}\}$—implies the conclusion of Mekler et al.’s Theorem 1.1 (see Corollary 4.17).

   b. Winning strategies of Players 1 and 2 are related, respectively, to Abstract Elementary Classes and “potential” membership in the class of $C$-filtered modules (see below).

“$M$ is $C$-filtered” means that there exists a $\subseteq$-increasing and continuous sequence $\langle M_\xi : \xi < \eta \rangle$ with union $M$, such that $M_0 = \{0\}$, and each quotient of the form $M_\xi/M_{\xi+1}$ is isomorphic to a member of $C$ (but $M_\xi$ for

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2The proof of Theorem 10 in [20] seems to actually use the stronger assumption that (*) holds for the infinitary version $L_{\omega_1\omega_1}(aa)$ of Stationary Logic. The version of (*) for $L_{\omega_1\omega_1}(aa)$ does hold after Levy collapsing to turn a supercompact cardinal into $\aleph_2$ (as in [3]), but implies the Continuum Hypothesis. The infinitary version seems to be needed to express freeness in the language of groups, as in the sentence labeled (+) before Proposition 5 of [20]. The use of infinitary logic can be circumvented by augmenting the structure with enough set theory, as in (3) and (4) on page 11 of this paper.
\( \xi > 0 \) is not required to be a member of \( C \). This is a significant weakening of the assertion that \( M \) is a direct sum of modules from \( C \), and has become ubiquitous in approximation theory since the proof of the Flat Cover Conjecture around the year 2000 ([4], [10], [27]). The key fact ([22]) is that if there exists a cardinal \( \kappa \) such that every member of \( C \) is \( < \kappa \)-presented,\(^3\) then the class of all \( C \)-filtered modules is a “precovering” class, which allows one to replicate many of the constructions from classical homological algebra “relative” to the class of \( C \)-filtered modules.

**Theorem 1.2.** Assume the stationary reflection principle \( RP_{\text{internal}} \). Assume \( R \) is a ring of size at most \( \aleph_1 \). Then for any “quotient-hereditary”\(^4\) collection \( C \) of countably-presented \( R \)-modules, the \( C \)-Filtration Game of length \( \omega_1 \) on any \( R \)-module is determined.

In particular,\(^5\) if \( R \) is a hereditary ring of size at most \( \aleph_1 \) and either \( C = \{ R \} \) or \( C = \) the collection of countably-generated projective modules,\(^6\) then \( C \)-Filtration Games of length \( \omega_1 \) are always determined.

If \( C \) is the collection of countably-generated projective modules, we often refer to the \( C \)-filtration games as **Projective Filtration Games**, since (by Kaplansky [16]) the class of projective modules is exactly the class of \( C \)-filtered modules. Similarly, if \( C = \{ R \} \), we often refer to the \( C \)-filtration games as **Free Filtration Games**.

Winning strategies for Duplicator in Ehrenfeucht-Fraïssé games are closely related to the notion of *potential isomorphisms* (i.e., isomorphisms introduced by forcing); this was first investigated in Nadel-Stavi [21]. This connection shows up in Filtration Games too; e.g., Player 2 has a winning strategy in the Projective Filtration Game of length \( \omega_1 \) on \( M \) if and only if \( M \) is \( \sigma \)-closed potentially projective, meaning that \( M \) is projective in some \( \sigma \)-closed forcing extension of the universe. This is discussed in Section 4.1.

The notion of \( \sigma \)-closed potential projectivity arises in another context too. Šaroch-Trlifaj [23] proved that if \( \kappa \) is a strongly compact cardinal and \( R \) is a ring of cardinality less than \( \kappa \), then the direct limit of any \( < \kappa \)-directed system of projective \( R \)-modules is projective. It is natural to ask what fragment of that result can possibly hold for, say, \( \kappa = \aleph_n \) \((n \geq 1)\). We cannot hope to prove that projective modules are closed under \( < \aleph_n \)-directed limits for such \( n \), even for \( R = \mathbb{Z} \), because of well-known ZFC results about almost free groups ([19]). However, we can consistently get a version of their theorem (for \( n = 2 \)) if we replace “projective” with “\( \sigma \)-closed potentially projective”:

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\(^3\)In which case the class of \( C \)-filtered modules is called *\( \kappa \)-deconstructible*.

\(^4\)See Definition 4.9 for the meaning of “\( C \) is quotient-hereditary”.

\(^5\)See Lemma 4.11 for why this is a special instance of the first part of the theorem.

\(^6\)For projective modules, countably-generated and countably-presented are equivalent (Section 2).
**Theorem 1.3.** Assume $RP_{\text{internal}}$. Suppose $R$ is a ring of size at most $\aleph_1$ and $\mathcal{C}$ is a collection of countably-presented $R$-modules that is quotient-hereditary. Then the class of $\sigma$-closed potentially $\mathcal{C}$-filtered modules is closed under $< \aleph_2$-directed limits.

In particular: if $R$ is a countable, hereditary ring, then the class of $\sigma$-closed potentially projective $R$-modules is closed under $< \aleph_2$-directed limits.

Finally, we give some applications of Filtration Games to *Abstract Elementary Classes (AEC)*, which were introduced by Shelah [24]. An AEC is a (possibly class-sized) partial order satisfying some axioms that generalize certain properties possessed by the elementary submodel ordering in many logics. Given a ring $R$, a regular cardinal $\mu$, and a collection $\mathcal{C}$ of $< \mu$-presented $R$-modules, let $\Gamma_{\mu,P1}^{\text{Filt(\mathcal{C})}}$ denote the class of $R$-modules $M$ such that Player 1 has a winning strategy in the $\mathcal{C}$-filtration game of length $\mu$ on $M$ (the main case of interest is $\mu = \omega_1$). Define $M \prec_R N$ to mean there is an injective $R$-module homomorphism from $M$ to $N$. First we prove a ZFC theorem:

**Theorem 1.4.** If $\mathcal{C}$ is quotient-hereditary, then the partial order $\left( \Gamma_{\mu,P1}^{\text{Filt(\mathcal{C})}}, \prec_R \right)$ is (always) an AEC, with Löwenheim-Skolem number at most $\max(|R|, 2^{< \mu})$.

In particular, if $R$ is a hereditary ring, then the class of $R$-modules such that Player 1 has a winning strategy in the Projective Filtration Game of length $\omega_1$ is an AEC.

Consider the special case $\mu = \omega_1$. If $RP_{\text{internal}}$ holds and $R$ and $\mathcal{C}$ satisfy the hypothesis of Theorem 1.2, then the determinacy ensures that $\Gamma_{\omega_1,P1}^{\text{Filt(\mathcal{C})}}$ is the same as the class of modules such that Player 2 does not have a winning strategy in the $\mathcal{C}$-filtration game of length $\omega_1$. But asserting that the latter class is an AEC turns out to be independent of ZFC:

**Theorem 1.5.** The class of abelian groups $G$ for which Player 2 does not have a winning strategy in the Free Filtration game of length $\omega_1$ on $G$, ordered by the subgroup (or pure subgroup, or elementary subgroup) relation, is an AEC with Löwenheim-Skolem number $\aleph_1$ in some models of set theory, but is not even an AEC in others.

Section 2 covers preliminaries. Section 3 provides the connection between stationary sets and potentially $\mathcal{C}$-filtered modules, including a useful consequence of Hill’s Lemma. Section 4 introduces Filtration Games, the Dual Basis Game, and analyzes the relationship between these games and Ehrenfeucht-Fraïssé games. Section 4 also includes the proof of Theorem 1.4. Section 5 proves an “$\aleph_2$-compactness” result (under $RP_{\text{internal}}$) that is.

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For the purposes of the following it would not matter if we placed the additional constraints that the embedding is pure, or even elementary in the language of $R$-modules.
in turn, used to prove Theorems 1.2, 1.3, and 1.5. Section 6 lists some open problems.

2. Preliminaries

Unless otherwise indicated, our terminology agrees with Jech [15] (for the set-theoretic background) and Eklof-Mekler [9] and Göbel-Trlifaj [14] (for the algebra background). By “$R$-module” we will officially mean a left $R$-module. If $X$ is a subset of an $R$-module $M$, $\langle X \rangle^M_R$ denotes the $R$-submodule of $M$ generated by $X$; the $M$ and $R$ decorations will be omitted when it is clear from the context. A module $M$ is $\mu$-generated if $M = \langle X \rangle$ for some $X \subseteq M$ with $|X| = \mu$; equivalently, there exists a short exact sequence $0 \to K \to F \to M \to 0$ where $F$ is free of rank $\mu$. $M$ is $\mu$-presented if there exists a short exact sequence $0 \to K \to F \to M \to 0$ where $F$ is free of rank $\mu$, and $K$ is $\mu$-generated. A module $P$ is projective if it is a direct summand of some free module. Equivalently, $P$ is projective if every short exact sequence of the form

$$0 \to K \to M \to P \to 0$$

splits, in which case $M \cong K \oplus P$. In particular, if $P$ is projective and $\mu$-generated as witnessed by a short exact sequence

$$0 \to K \to F \to P \to 0$$

where $F$ is free of rank $\mu$, then $F \cong K \oplus P$, and hence $K \cong F/P$. So $K$ is $\mu$-generated, yielding:

**Fact 2.1.** If $P$ is projective and $\mu$-generated, then $P$ is $\mu$-presented.

Occasionally we will refer to the notion of a pure embedding: an $R$-submodule $M$ of $N$ is called a pure submodule if for every (finite) matrix $A$ with entries from $R$, and every vector $\vec{b}$ of elements from $M$, if $A\vec{x} = \vec{b}$ has a solution in $N$, then it has a solution in $M$. An embedding $\rho : M \to N$ is called a pure embedding if its image is a pure submodule of $N$.

2.1. Dual Basis characterization of projectivity. Recall that projective modules are, by definition, direct summands of free modules. Several classical characterizations of projectivity appear in Lam [18]. For example, a module is projective if and only if it has a dual basis. A dual basis for an $R$-module $M$ is a pair

$$D = \left( B, (f_b)_{b \in B} \right)$$

such that $B \subseteq M$, each $f_b : M \to R$ is $R$-linear, and for all $x \in M$,

$$\text{sprt}_D(x) := \{ b \in B : f_b(x) \neq 0 \}$$

is finite and

$$x = \sum_{b \in \text{sprt}(x)} f_b(x)b.$$
Existence of a dual basis for $M$ is equivalent to projectivity of $M$ (see e.g. Lam [18]). The dual basis characterization of projectivity yields the following nice consequence, that projectivity is absolute for countably-generated modules:

**Lemma 2.2.** Let $\mathcal{H} \subset \mathcal{H}'$ be transitive ZFC$^-$ models. Then for any ring $R \in \mathcal{H}$ and any $R$-module $M \in \mathcal{H}$ such that $\mathcal{H} \models "M \text{ is countably generated}"$, 

\[
\mathcal{H} \models M \text{ is projective} \iff \mathcal{H}' \models M \text{ is projective}.
\]

**Proof.** The forward direction holds regardless of the size of $M$ in $\mathcal{H}$, because projectivity of $M$ is witnessed by a dual basis for $M$, which is easily upward absolute from $\mathcal{H}$ to $\mathcal{H}'$ (alternatively, $\mathcal{H}$ sees that $M$ is a direct summand of a free module, and this is clearly upward absolute to $\mathcal{H}'$).

For the $\iff$ direction, in $\mathcal{H}$ fix a countable set $Z$ generating $M$, and in $\mathcal{H}'$ let $T$ be the tree of finite attempts to build a dual basis for $M = \langle Z \rangle_M$. In other words, in $\mathcal{H}$ fix an enumeration \{\(z_n : n < \omega\)\} of $Z$, and let $T$ be the tree whose nodes are triples 

\[
t = \left(n^t, B^t, (f^t_b)_{b \in B^t}\right)
\]

such that

1. $n^t \in \omega$;
2. $B^t$ is a finite subset of $M$;
3. Each $f^t_b$ is a function from \{\(z_n : n < n^t\)\} to $R$ that lifts to an $R$-module homomorphism from $\langle \{z_n : n < n^t\} \rangle \to R$;
4. For each $n < n^t$, $z_n = \sum_{b \in B^t} f^t_b(z_n)b$.

with tree ordering as follows: $t > s$ if $n^t > n^s$, $B^t \supseteq B^s$, the $f^t_b$'s extend the $f^s_b$'s for all $b \in B^s$; and, importantly, for all $n < n^s$, \{\(b \in B^t : f^t_b(z_n) \neq 0\)\} $\subseteq B^s$.

The last requirement ensures that branches through the tree won't make the support of any member of $Z$ infinite.

Now since $M$ is projective in $\mathcal{H}'$, there is a dual basis for $M$ in $\mathcal{H}'$, and this dual basis can be used to recursively build an infinite branch through $T$ in $\mathcal{H}'$. Since $T \in \mathcal{H}$, absoluteness of wellfoundedness between transitive ZFC$^-$ models ensures that there is an infinite branch through $T$ in $\mathcal{H}$ too. And any infinite branch through $T$ yields a dual basis for $M$.

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8 It is clear that a cofinal branch through $T$ yields a set $B \subseteq M$ and homomorphisms $f_b : \langle Z \rangle_M \to R$ such that for every $z \in Z$, $\text{sprt}(z) := \{b \in B : f_b(z) \neq 0\}$ is finite, and $z = \sum_{b \in \text{sprt}(z)} f_b(z)b$. It is then routine to check that the latter also holds for all $x \in \langle Z \rangle$; i.e., if $x \in \langle z_1, \ldots, z_k \rangle$, then $\text{sprt}(x) \subseteq \bigcup_{i \leq k} \text{sprt}(z_i)$, and $x = \sum_{b \in \bigcup_{i \leq k} \text{sprt}(z_i)} f_b(x)b$. 

An easier argument, just using closure of \( \mathcal{H} \) in \( \mathcal{H}' \) to absorb the dual basis, yields the following. Note that for \( \mu = \omega_1 \) the following lemma is superfluous, since Lemma 4.3 got the same conclusion without assuming \( < \omega_1 \)-closure of \( \mathcal{H} \) in \( \mathcal{H}' \).

**Lemma 2.3.** Let \( \mathcal{H} \subset \mathcal{H}' \) be transitive ZFC\(^-\) models, and assume \( \mu \) is an infinite regular cardinal from the point of view of \( \mathcal{H} \) such that \( \mathcal{H} \) is closed under \( < \mu \)-length sequences from \( \mathcal{H}' \). Then for any ring \( R \in \mathcal{H} \) and any \( R \)-module \( M \in \mathcal{H} \) such that \( \mathcal{H} \models "M \text{ is } < \mu \text{-generated}" \),

\[
\mathcal{H} \models M \text{ is projective} \iff \mathcal{H}' \models M \text{ is projective}.
\]

2.2. **Filtrations.** If \( X \) is a set, a **filtration of** \( X \) is a sequence \( \langle X_i : i < \eta \rangle \), for some ordinal \( \eta \), such that

- \( X_i \subseteq X_j \) whenever \( i < j \) (not necessarily proper containment);
- \( X_i = \bigcup_{k<i} X_k \) whenever \( i \) is a limit ordinal; and
- \( X = \bigcup_{i<\eta} X_i \).

**Note:** we do not require that \( \eta \) is a cardinal, nor that each \( X_i \) is of smaller cardinality than \( X \). Our definition above adheres to the common notion of filtration from the module theory literature (e.g., Göbel-Trlifaj [14]), though some set theory literature (including previous papers of the author) have included those additional assumptions in the definition of filtration. We will often want filtrations to have these additional properties, however, as discussed in Section 3.

Let \( \mathcal{C} \) be a class of \( R \)-modules. An \( R \)-module \( M \) is said to be **\( \mathcal{C} \)-filtered** if there exists a filtration \( \bar{M} = \langle M_i : i < \eta \rangle \) of \( M \) such that \( M_0 = \{0\} \) and for all \( i < \eta \), if \( i+1 < \eta \) then

\[
\frac{M_{i+1}}{M_i}
\]

is isomorphic to an element of \( \mathcal{C} \). We say that \( \mathcal{C} \) is closed under transfinite extensions if every \( \mathcal{C} \)-filtered module is an element of \( \mathcal{C} \). Common examples of classes closed under transfinite extensions are the class of projective modules, the class of free modules, and the roots of \( \text{Ext}(\mathcal{C}, N) \) for any fixed module \( N \) (“Eklof’s Lemma”; see [8]).

**Lemma 2.4.** If \( \langle M_i : i < \eta \rangle \) is a \( \mathcal{C} \)-filtration of \( M \), then for all \( i < j < \eta \), \( M_j/M_i \) is \( \mathcal{C} \)-filtered.

**Proof.** Fix \( i < j \) where \( j < \eta \). Then

\[
\langle M_k/M_i : k \in [i,j) \rangle
\]

is a filtration of \( M_j/M_i \) if \( j \) is a limit ordinal, and

\[
\langle M_k/M_i : k \in [i,j] \rangle
\]

is a filtration of \( M_j/M_i \) if \( j \) is a successor ordinal. And each has consecutive factors that are (isomorphic to an element) in \( \mathcal{C} \), since \( \frac{M_{k+1}}{M_k} \simeq \frac{M_{k+1}}{M_k} \). \( \square \)
2.3. **Stationary sets.** If $X$ is a set and $\mu$ is a regular uncountable cardinal, $\wp_\mu(X)$ denotes the set
\[ \{ Z \subset X : |Z| < \mu \}. \]
We sometimes write $[X]^{<\omega}$ instead of $\wp_\omega(X)$. We also use $[X]^{<\omega}$ to denote all finite subsets of $X$. A set $C$ is called **closed and unbounded (club)** in $\wp_\mu(X)$ if and only if $C \subseteq \wp_\mu(X)$ and:

- $C$ is $\subseteq$-cofinal in $\wp_\mu(X)$; and
- If $\vec{Z}$ is a $\subseteq$-increasing chain of members of $C$ of length strictly less than $\mu$, then the union of the chain is in $C$.

A set $S$ is called **stationary in** $\wp_\mu(X)$ if it meets every closed unbounded subset of $\wp_\mu(X)$. We need a lemma due to Kueker (see [15]):

**Lemma 2.5.** Assume $X$ is a set and $X \supseteq \mu$. Then $D$ contains a club in $\wp_\mu(X)$ if and only if there exists an $F : [X]^{<\omega} \to X$ such that
\[ C_{F,\mu,X} := \{ z \in \wp_\mu(X) : z \cap \mu \text{ is an ordinal and } z \text{ is closed under } F \} \]
is contained in $D$. For $\mu = \omega_1$, the “$z \cap \mu$ is an ordinal” is unnecessary in the definition of $C_{F,\mu,X}$.

For a cardinal $\theta$, $H_\theta$ denotes the collection of sets of hereditary cardinality $< \theta$. If $\theta$ is regular and uncountable, then $H_\theta$ is transitive and $(H_\theta, \in)$ satisfies all axioms of ZFC except possibly the powerset axiom.

**Corollary 2.6.** Suppose $\mu$ is a regular uncountable cardinal, $X$ is a set of cardinality at least $\mu$, $C$ is a closed unbounded subset of $\wp_\mu(X)$, $\theta$ is a regular cardinal with $(X,C,\mu) \in H_\theta$, and $W$ is an elementary substructure of $(H_\theta, \in)$ such that $(X,C,\mu) \in W$ and $\mu \subseteq W$. Then $C \cap \wp_\mu(W \cap X)$ contains a club in $\wp_\mu(W \cap X)$.

**Proof.** We can without loss of generality assume that $X \supseteq \mu$, since any elementary substructure of $(H_\theta, \in)$ that has $X$ as an element will see a bijection between $X$ and an ordinal. Then by Lemma 2.5, there is an $F : [X]^{<\omega} \to X$ such that $C_{F,\mu,X} \subseteq C$. By elementarity of $W$ in $H_\theta$ and the fact that $C \subseteq W$, we can assume $F \in W$. Then by elementarity of $W$,
\[ W \cap X \text{ is closed under } F. \]
In other words, $F|W := F \upharpoonright [W \cap X]^{<\omega}$ maps into $W \cap X$, and hence the set $C_{F|W,\mu,W \cap X}$ contains a closed unbounded subset of $\wp_\mu(W \cap X)$. And, clearly, this set is contained in $C_{F,\mu,X}$ (and therefore in $C$).

We also need another standard corollary of Kueker’s characterization of club sets (see [11]):

\[9\text{For } \mu \geq \omega_2 \text{ this technical requirement is essential, due to the possibility of Chang’s Conjecture holding (see [11]).}\]
Lemma 2.7. Suppose $\mu$ is a regular uncountable cardinal, $X$ is any set, and $D$ contains a closed unbounded subset of $\wp_\mu(X)$. Then for all regular $\theta$ such that $X, D \in H_\theta$, and all $W \prec (H_\theta, \in, X, D)$: if $|W| < \mu$ and $W \cap \mu$ is an ordinal, then $W \cap X \in D$.

3. Stationary sets and potentially filtered modules

In this section we investigate the notion of being $\mathcal{C}$-filtered in certain forcing extensions. To motivate the main technical lemmas of this section, suppose $\mathcal{C}$ is a collection of countably presented modules, and suppose $M$ is a module that is $\aleph_1$-generated and $\mathcal{C}$-filtered. Recall that "$M$ is $\mathcal{C}$-filtered" means that there exists a filtration $\vec{M} = \langle M_i : i < \eta \rangle$ of $M$ such that $M_0 = \{0\}$ and $M_{i+1}/M_i$ is (isomorphic to a module) in $\mathcal{C}$ for all $i$. But the definition does not require that $\eta$ is a cardinal, much less that $\eta = \omega_1$ or that the $M_i$'s themselves are countably generated. On the other hand, since $M$ has a generating set of size $\aleph_1$, there exists a $\subseteq$-increasing and continuous sequence $\vec{Z} = \langle Z_i : i < \omega_1 \rangle$ of countable subsets of $M$, whose union generates $M$. For various reasons, it would be nice to know that we can replace $\vec{M}$ with another $\mathcal{C}$-filtration that looks more like $\vec{Z}$; in particular, which has length exactly $\omega_1$ and whose entries are countably generated. The following lemma, which follows from the highly versatile Hill Lemma, will allow us to do exactly that (see also Corollary 3.2):

Lemma 3.1. Suppose $\mu$ is a regular uncountable cardinal and $\mathcal{C}$ is a class of $< \mu$-presented modules. Suppose $M$ is $\mathcal{C}$-filtered. Then there is a closed unbounded subset $D$ of $\wp_\mu(M)$ such that whenever $Z \subseteq Z'$ are both in $D$, $\langle Z \rangle$ and $\langle Z' \rangle$ are $\mathcal{C}$-filtered (via filtrations of length $< \mu$).

Proof. Let $\vec{M} = \langle M_i : i < \eta \rangle$ witness that $M$ is $\mathcal{C}$-filtered. By the Hill Lemma (Theorem 7.10 of Göbel-Trlifaj [14]), there is a family $\mathcal{F}$ of submodules of $M$ such that (i) $\mathcal{F}$ contains $\{M_i : i < \eta\}$; (ii) $\mathcal{F}$ is closed under arbitrary sums and intersections; (iii) whenever $N \subseteq N'$ are both in $\mathcal{F}$ then $N'/N$ is $\mathcal{C}$-filtered; and (iv) whenever $N \in \mathcal{F}$ and $X$ is a $< \mu$-sized subset of $M$, there is an $N' \in \mathcal{F}$ containing $\langle N \cup X \rangle$ such that $N'/N$ is $< \mu$-presented.

Set $D := \{Z \subseteq M : |Z| < \mu$ and $\langle Z \rangle \in \mathcal{F}\}$, i.e., $D$ is the collection of all subsets of $M$ of size $< \mu$ that generate an element of $\mathcal{F}$. We claim that $D$ is as desired. Note first that $D$ is nonempty, because $M_\lambda = M_\lambda/\{0\}$ is (isomorphic to) an element of $\mathcal{C}$ and hence $< \mu$-presented (and an element of $\mathcal{F}$, by (i)). To see closure, suppose $\langle Z_k : k < \zeta \rangle$ is a $\subseteq$-increasing sequence from $D$, where $\zeta < \mu$. Part (ii) ensures that $\bigcup_{k<\zeta} \langle Z_k \rangle = \langle \bigcup_{k<\zeta} Z_k \rangle$ is an element of $\mathcal{F}$; and since $\mu$ is regular and $\zeta < \mu$,
$\bigcup_{k<\zeta} Z_k$ has size $<\mu$. Hence, $\bigcup_{k<\zeta} Z_k \in D$. To see that $D$ is $\subseteq$-cofinal in $\wp_\mu(M)$, fix a $<\mu$-sized subset $X$ of $M$. By \textit{[iv]} (taking $N = M_0 = \{0\}$) there is some $N' \in \mathcal{F}$ such that $X \subseteq N'$ and $N'/\{0\} \simeq N'$ is $<\mu$-presented; say $Z$ is a $<\mu$-sized set of generators for $N'$. Then $N' = \langle Z \rangle = \langle X \cup Z \rangle$, and hence $X \cup Z \in D$. Hence, $D$ is closed and unbounded in $\wp_\mu(M)$.

Now suppose $Z \subset Z'$ are both in $D$; then $\langle Z \rangle \subseteq \langle Z' \rangle$ are both in $\mathcal{F}$, and so by \textit{[iii]} $\langle Z' \rangle/\langle Z \rangle$ is $\mathcal{C}$-filtered; say by

$$\langle U_\xi/(Z) : \xi < \zeta \rangle.$$  

We can without loss of generality assume there are no superfluous indices; i.e. that $U_\xi$ is a proper subset of $U_{\xi+1}$ for all $\xi < \zeta$. Since the union of the $U_\xi$’s equals $\langle Z' \rangle$, $|Z'| < \mu$, and $\mu$ is regular, it follows that $\zeta < \mu$. So it is a $\mathcal{C}$-filtration of $\langle Z' \rangle/\langle Z \rangle$ of length $<\mu$.

Now \textit{[iii]}, together with the fact that $M_0 = \{0\} \in \mathcal{F}$, ensure that every element of $\mathcal{F}$ is $\mathcal{C}$-filtered; so, in particular, $\langle Z \rangle$ is $\mathcal{C}$-filtered whenever $Z \in D$.

\begin{proof}
Let $X$ be an $\mu$-sized generating set for $M$, and fix an enumeration $\{x_i : i < \mu\}$ of $X$.

For the \textit{[1]} implies \textit{[2]} direction, suppose $M$ is $\mathcal{C}$-filtered. By Lemma \textit{3.1} there is a closed unbounded $D \subseteq \wp_\mu(M)$ such that whenever $Z \subseteq Z'$ are both in $D$, $\langle Z' \rangle/\langle Z \rangle$ is $\mathcal{C}$-filtered, via a filtration of length strictly less than $\mu$.

\begin{enumerate}
\item \text{There is a closed unbounded set $D \subseteq \wp_\mu(M)$ such that for every $Z \in D$, $S_Z := \{Z' \in D : Z \subseteq Z' \text{ and } \langle Z' \rangle/\langle Z \rangle \text{ is } \mathcal{C}\text{-filtered via a } <\mu\text{-length filtration}\}$ is stationary in $\wp_\mu(M)$.
\end{enumerate}

\begin{enumerate}
\item $M$ is $\mathcal{C}$-filtered.
\item $M$ is $\mathcal{C}$-filtered by a filtration of (ordinal) length at most $\mu$, all of whose entries are $<\mu$-presented.
\item There is a closed unbounded $D \subseteq \wp_\mu(M)$ such that whenever $Z \subseteq Z'$ are both in $D$, $\langle Z' \rangle/\langle Z \rangle$ is $\mathcal{C}$-filtered, via a filtration of length strictly less than $\mu$.
\item there is a closed unbounded set $D \subseteq \wp_\mu(M)$ such that for every $Z \in D$, $S_Z := \{Z' \in D : Z \subseteq Z' \text{ and } \langle Z' \rangle/\langle Z \rangle \text{ is } \mathcal{C}\text{-filtered via a } <\mu\text{-length filtration}\}$ is stationary in $\wp_\mu(M)$.
\end{enumerate}

\textbf{Proof.} Let $X$ be an $\mu$-sized generating set for $M$, and fix an enumeration $\{x_i : i < \mu\}$ of $X$.

For the \textit{[1]} implies \textit{[2]} direction, suppose $M$ is $\mathcal{C}$-filtered. By Lemma \textit{3.1} there is a closed unbounded $D \subseteq \wp_\mu(M)$ such that whenever $Z \subseteq Z'$ are both in $D$, $\langle Z' \rangle/\langle Z \rangle$ is $\mathcal{C}$-filtered. Using that $D$ is closed unbounded in $\wp_\mu(M)$, recursively build a $\subseteq$-increasing and continuous chain $\{Z_i : i < \mu\}$ of $<\mu$-sized subsets of $M$ such that for all $i < \mu$:

\begin{itemize}
\item $x_i \in Z_{i+1}$;
\item $Z_i \in D$
\end{itemize}
Now \( \langle Z_i : i < \mu \rangle \) is a filtration of \( M \), but probably there are many \( i \)'s such that \( \langle Z_{i+1} \rangle / \langle Z_i \rangle \) is not in \( \mathcal{C} \), but merely \( \mathcal{C} \)-filtered\(^{10}\). However, since \( Z_i \subset Z_{i+1} \) are both in \( D \), there is a \( \mathcal{C} \)-filtration
\[
(2) \quad \langle U^i_{\xi} / \langle Z_i \rangle : \xi < \zeta_i \rangle
\]
of the quotient \( \langle Z_{i+1} \rangle / \langle Z_i \rangle \) such that \( \zeta_i < \mu \). Then concatenating all sequences of the form
\[
\langle U^i_{\xi} : \xi < \zeta_i \rangle
\]
across all \( i < \mu \) yields a \( \mathcal{C} \)-filtration of \( M \) of length at most \( \mu \); it is a \( \mathcal{C} \)-filtration because for each \( i \) and \( \xi \) such that \( \xi + 1 < \zeta_i \),
\[
\frac{U^i_{\xi+1} / U^i_{\xi}}{U^i_{\xi+1} / \langle Z_i \rangle} \simeq \frac{U^i_{\xi+1} / \langle Z_i \rangle}{U^i_{\xi} / \langle Z_i \rangle}
\]
which is in \( \mathcal{C} \) because (2) is a \( \mathcal{C} \)-filtration.

The (2) implies (3) direction follows from Lemma 3.1. The (3) implies (4) is trivial. Finally, assume (4); we want to build a \( \mathcal{C} \)-filtration for \( M \).

Fix a club \( D \subseteq \mathcal{P}_\mu(M) \) and a stationary \( S_Z \subseteq \mathcal{P}_\mu(M) \) for each \( Z \in D \), as in the assumptions of (4). Recursively construct a \( \subseteq \)-increasing sequence \( \langle Z_i : i < \mu \rangle \) such that:

- \( x_i \in Z_{i+1} \cap S_Z \cap D \) for all \( i < \mu \), provided that \( Z_i \in D \) (otherwise halt the construction; note that if \( S_Z \) is defined, then \( S_Z \cap D \) is stationary in \( \mathcal{P}_\mu(M) \));
- \( Z_i = \bigcup_{k<i} Z_k \) for all limit ordinals \( i \).

Note that if \( i \) is a limit ordinal below \( \mu \) and \( Z_k \in D \) for all \( k < i \), then \( Z_i \in D \) by closure of \( D \) in \( \mathcal{P}_\mu(M) \). So the construction never breaks down at any ordinal before \( \mu \). We have constructed a filtration \( \tilde{Z} = \langle Z_i : i < \mu \rangle \) of \( M \), such that for all \( i < \mu \), \( \langle Z_{i+1} \rangle / \langle Z_i \rangle \) is \( \mathcal{C} \)-filtered. And this filtration of \( \langle Z' \rangle / \langle Z \rangle \) is of length less than \( \mu \), assuming no redundant entries. \( \square \)

We remark that parts (3) and (4) in the statement of Corollary 3.2 can be rephrased in terms of Stationary Logic (see [1]), respectively, as
\[
(H_\theta, \in, R, M, \mathcal{C} \cap H_\theta) \models \text{aa}Z \text{ aaZ' } \langle Z \cap M \rangle \quad \text{and}
\]
\[
(3) \quad \langle Z' \cap M \rangle / \langle Z \cap M \rangle \text{ are } \mathcal{C} \text{-filtered}
\]
and
\[
(H_\theta, \in, R, M, \mathcal{C} \cap H_\theta) \models \text{aa}Z \text{ statZ' } \langle Z \cap M \rangle \quad \text{and}
\]
\[
(4) \quad \langle Z' \cap M \rangle / \langle Z \cap M \rangle \text{ are } \mathcal{C} \text{-filtered}
\]

\(^{10}\)Of course if \( \mathcal{C} \) is closed under transfinite extensions, or even just countable transfinite extensions, it does not matter.
for any $\theta$ such that $(R,M) \in H_\theta$; here the interpretation of $aa$ and $stat$ depend on $\mu$, though the most interesting case for us is $\mu = \omega_1$. The ability to replace the second $aa$ quantifier with a $stat$ quantifier is a key observation, since it allows us to prove Theorem 1.2 from a "non-diagonal" version of stationary set reflection that, unlike assumption (*) of Theorem 1.1, follows from Martin’s Maximum.

Note that if $C$ happens to be closed under transfinite extensions of length $< \mu$, then in the equivalences above, one can also require that $\langle Z \rangle \in C$ for club-many $Z \in \wp(\mu)$.

**Definition 3.3.** Let $\mu$ be a regular (not necessarily uncountable) cardinal. Given a class $C$ modules, a module $M$ will be called $< \mu$-closed potentially $C$-filtered if $M$ is $C$-filtered in some $< \mu$-closed forcing extension of the universe. We will say that $M$ is $< \mu$-closed potentially projective if $M$ is projective in some $< \mu$-closed forcing extension of the universe. For the case $\mu = \omega$, we will sometimes just say “potentially $C$-filtered” instead of “$< \omega$ potentially $C$-filtered”, since every forcing poset is, trivially, $< \omega$ closed.

**Remark 3.4.** The parameter $C$ in the definition above refers to the actual class in the ground model, not to any interpretation in the forcing extension. That is, “$M$ is $< \mu$-closed potentially $C$-filtered” officially means that there is a $< \mu$-closed poset $\mathbb{P}$ forcing “$\mathbb{M}$ is $\mathbb{C}$-filtered” (where $C$ is a class in the ground model). However, in some cases the interpretations will coincide, e.g. when $C$ is the collection of countably-generated or countably presented projective modules; see Corollary 3.7 below.

**Lemma 3.5.** Let $\mu$ be regular uncountable and $C$ be a collection of $< \mu$-presented modules. If $M$ is $< \mu$-closed potentially $C$-filtered, then there is a $< \mu$-closed forcing extension of $V$ in which $|M| \leq \mu$ and there is a $C$-filtration of $M$ of length at most $\mu$.

**Proof.** Let $W$ be a $< \mu$-closed extension of the ground model $V$ where $M$ is $C$-filtered. Let $W'$ be a further $< \mu$-closed forcing extension of $W$ where $|M| \leq \mu$. Clearly the $C$-filtration of $M$ is upward absolute to $W'$, and every member of $C$ is $< \mu$-presented in $W'$. Note that $W'$ is a $< \mu$-closed forcing extension of $V$. And by Corollary 3.2 $W'$ has a $C$-filtration of $M$ of length at most $\mu$. $\Box$

**Lemma 3.6.** Suppose $V \subset W$ are transitive ZFC models, $\mu$ is a cardinal in $V$, $V$ is $\mu$-closed in $W$ (i.e., $W \cap \mu^+ V \subset V$), and $R$ is a ring in $V$. Then:

1. For every fixed $R$-module $M$ in $V$, every $\mu$-generated submodule of $M$ in $W$ is also in $V$.

---

11 Statements (3) and (4) involve some slight abbreviations; e.g. (3) should really say the following, where lowercase variables are first order:

$(H_\theta, \in, R, M, C \cap H_\theta) \models aaZ \exists p' \exists p = (Z \cap M)$, $p' = (Z \cap M)$, and $p'/p \in C$. 

12 The parameter $\theta$ is the same as in (1), and $H_\theta$ is the transitive collapse of the transitive closure of $(\theta, R, M, C \cap H_\theta)$. 

---
(2) Every $R$-module in $W$ whose relations are $\mu$-generated is (isomorphic to) an element of $V$.

(3) Every $\mu$-presented $R$-module in $W$ is (isomorphic to) an $R$-module in $V$ that is $\mu$-presented in $V$.

Proof. For part (1), if $Z \in W$ is a $\mu$-sized subset of $M$, then $Z \in V$ by $\mu$-closure of $V$ in $W$, and hence $\langle \langle Z \rangle \rangle_M$ is an element of $V$, since $Z$, $M$, and $R$ are in $V$.

For part (2): in $W$, suppose $N \cong F/K$ where $F$ is free and $K$ is a $\mu$-generated submodule of $F$; say $K = \langle Z \rangle_F$ where $Z$ is (in $W$) a $\mu$-sized subset of $F$. Note that $F \in V$, since the ring $R$ is in $V$ and $F$ is a direct sum of copies of $R$; then $Z \in V$ too, by the $\mu$-closure of $V$ in $W$. Hence $K = \langle Z \rangle_F \in V$, and therefore $F/K \in V$.

For part (3): if the $F$ from the previous paragraph was also $\mu$-generated in $W$, then by the $\mu$-closure of $V$ in $W$, $F$ is also $\mu$-generated in $V$. Hence the quotient $F/K$ is a $\mu$-presentation from the point of view of $V$ too. \[\square\]

Our focus will often be on the case $\mu = \omega_1$, in which case we will say “$\sigma$-closed potentially $C$-filtered” instead of “$\omega_1$-closed potentially $C$-filtered” (and similarly $\sigma$-closed instead of $\omega_1$ closed forcing).

Corollary 3.7. For any ring $R$, and any $R$-module $M$, the following are equivalent:

(1) $M$ is $\sigma$-closed potentially projective;

(2) $M$ is $\sigma$-closed potentially $C$-filtered, where $C$ is the collection of all countably generated, projective modules.

Proof. For (1) $\Rightarrow$ (2): suppose $M \in V$ is an $R$-module and $W$ is a $\sigma$-closed forcing extension of $V$ where $M$ becomes projective. Then by Kaplansky’s Theorem (in $W$), $M$ is $C^W$-filtered, where $C^W$ is the collection of countably-generated projective modules in $W$. But every element of $C^W$ is an element of $V$ by Lemma 3.6 and is projective in $V$ by Lemma 2.2. Hence, from the point of view of $W$, $C^W \subseteq C^V$ (in fact they are equal), and so $W \models \text{"}M$ is $C^V$-filtered".

For the converse, suppose $M$ is an $R$-module in $V$, and $W$ is a $\sigma$-closed forcing extension in which $M$ becomes $C^V$-filtered. Since projectivity is upward absolute, each module in $C^V$ is projective in $W$, and hence $W$ sees that $M$ is a transfinite extension of projective modules, and is hence projective. \[\square\]

An important fact about $\sigma$-closed forcing is:

Fact 3.8 (Jech [15]). If $\mathbb{P}$ is $\sigma$-closed, then for all uncountable $X$ in the ground model:

\[\text{12}\] Which by Fact 2.1 is the same as the collection of all countably presented, projective modules.
• Closed unbounded subsets of $\wp_{\omega_1}(X)$ in the ground model remain closed unbounded in $V^P$.
• Stationary subsets of $\wp_{\omega_1}(X)$ in the ground model remain stationary in $V^P$.

The second bullet is specific to $\mu = \omega_1$; in general, $<\omega_2$-closed forcing can kill the stationarity of some subset of $\wp_{\omega_2}(X)$. The next lemma is similar to the $\mu = \omega_1$ instance of Corollary 3.2. The difference is that Corollary 3.2 assumed $M$ was $\aleph_1$-generated, while in Corollary 3.9 there is no cardinality restriction on $M$.

**Corollary 3.9.** Let $C$ be a collection of countably presented modules, and $M$ be a module. The following are equivalent:

1. $M$ is $\sigma$-closed potentially $C$-filtered;
2. In some $\sigma$-closed forcing extension, there is a $C$-filtration of $M$ of (ordinal) length at most $\omega_1$;
3. There is a closed unbounded $D \subseteq \wp_{\omega_1}(M)$ such that whenever $Z \subseteq Z'$ are both in $D$, $(Z')/(Z)$ is $C$-filtered, via a filtration of length less than $\omega_1$.
4. There is a closed unbounded set $D \subseteq \wp_{\omega_1}(M)$ such that for every $Z \in D$,

   $$S_Z := \{Z' \in D : Z \subset Z' \text{ and } (Z')/(Z) \text{ is } C\text{-filtered} \text{ via a filtration of length } <\omega_1\}$$

   is stationary in $\wp_{\omega_1}(M)$.

**Proof.** This follows directly from Corollary 3.2 and Fact 3.8.

4. **Filtration Games**

In Section 4.1 we introduce the notion of a $C$-Filtration Game, played on a single module, and prove some basic facts about these games, including their connection to potentially $C$-filtered modules. Section 4.2 shows that, in many cases, the class of modules for which Player 1 has a winning strategy in a Filtration Game constitutes an AEC. Section 4.3 introduces the Dual Basis Game on a module, which gives an alternate (and somewhat more concrete) description of the $C$-Filtration Game in the case where $C$ is the collection of countably generated, projective modules. Section 4.4 shows that determinacy of Filtration Games implies determinacy of certain Ehrenfeucht-Fraïssé games.

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13 This is true for larger $\mu$ too, simply because $< \mu$-closed forcings add no new sets of size less than $\mu$. 
4.1. Filtration games and potential filtrations.

Definition 4.1. Let $\mu$ be a regular (not necessarily uncountable) cardinal, $C$ be a collection of $R$-modules, and $M$ an $R$-module. The $C$-filtration game of length $\mu$ on $M$, denoted $G^\text{Filt}(C)$, is a game between two players with at most $\mu$ innings, as follows:

| Player 1 | $X_0$ | $X_1$ | ... | $X_i$ | ... |
|----------|-------|-------|-----|------|-----|
| Player 2 | $Z_0$ | $Z_1$ | ... | $Z_i$ | ... |

At the beginning of round $i$ (where $i < \mu$), Player 1 plays an $X_i \subseteq M$ of size $< \mu$, and Player 2 responds with a $Z_i \subseteq M$ of size $< \mu$ such that

$$X_i \cup \bigcup_{k<i} Z_k \subseteq \langle Z_i \rangle$$

and

$$(+) \quad \frac{\langle Z_i \rangle}{\langle \bigcup_{k<i} Z_k \rangle} \text{ is } C\text{-filtered.}$$

Player 2 wins if she lasts $\mu$ rounds; otherwise Player 1 wins.

Note that for successor ordinals $i$, $(\H)$ just means that $\langle Z_i \rangle/\langle Z_{i_0} \rangle$ is $C$-filtered, where $i = i_0 + 1$. If $C$ is closed under transfinite extensions of length $< \mu$, then the game from Definition 4.1 is the same as if we replaced requirement $(\H)$ with the apparently stronger requirement that

$$(++) \quad \frac{\langle Z_i \rangle}{\langle \bigcup_{k<i} Z_k \rangle} \text{ is (isomorphic to an element) in } C.$$  

Remark 4.2. When $C$ is the collection of all $< \mu$-presented, projective modules, we may sometimes refer to the game $G^\text{Filt}(_C)(M)$ as the Projective Filtration game of length $\mu$ on $M$. When $C = \{R\}$, we may sometimes refer to the game $G^\text{Filt}(C)$ as the Free Filtration game of length $\mu$ on $M$.

The following lemma dispenses with a technicality. Recall that the definition of “$C$-filtered” requires successive factors in the filtration to be “merely isomorphic to an element of $C$, not necessarily in $C$; this is the convention, since it is convenient to sometimes work with $C$’s that are not closed under isomorphism. However, even if one did require $C$ to be closed under isomorphism, one would still run into trouble again when going to a forcing extension of the universe, which may add new modules and cause $C$ to no longer be closed under isomorphisms. The following lemma says this isn’t an issue in the situations we will encounter:

Lemma 4.3. Suppose $C$ is a collection of $< \mu$-generated modules in the ground model $V$, $N \subseteq V$ is a module, and $W$ is a $< \mu$-closed forcing extension of $V$. Suppose $\langle U_{\xi} : \xi < \zeta \rangle$ is, in $W$, a $C$-filtration of $N$, and that $\zeta < \mu$. Then $\langle U_{\xi} : \xi < \zeta \rangle$ is an element of $V$, and $V \models "U_{\xi} \text{ is a } C\text{-filtration of } N\".$
Proof. Let \( \langle \hat{U}_\xi : \xi < \zeta \rangle \) be a name for the sequence. Since \( \mathcal{C} \) consists of \(< \mu\)-generated modules, and \( \zeta < \mu \), it is forced that for all \( \xi < \zeta \), \( \hat{U}_\xi \) is a \(< \mu\)-generated submodule of \( N \), and hence (by \(< \mu\)-closure of \( V \) in \( W \) and because \( N \in V \)) an element of \( V \). Again by \(< \mu\) closure of \( V \) in \( W \), the entire sequence \( \hat{U} \) is forced to be in \( V \). Moreover, if \( \xi + 1 < \zeta \), then \( W \models "U_{\xi+1}/U_\xi \) is isomorphic to an element of \( \mathcal{C}" \); let \( N_\xi \in \mathcal{C} \) witness this fact, and note that \( N_\xi \in V \) because \( \mathcal{C} \in V \). Finally, since \( N_\xi \) and \( U_{\xi+1}/U_\xi \) are both in \( V \), and are both \(< \mu\)-generated, any isomorphism between them in \( W \) is already in \( V \) (since it is determined by what it does to the generating set of the domain of size \( < \mu \)). So \( V \models "N_\xi \) is isomorphic to \( U_{\xi+1}/U_\xi \). \( \square \)

Lemma 4.4. Let \( \mu \) be a regular (not necessarily uncountable) cardinal, \( \mathcal{C} \) a collection of \(< \mu\)-presented modules, and \( M \) be a module. The following are equivalent:

1. \( M \) is \(< \mu\)-closed potentially \( \mathcal{C} \)-filtered.
2. Player 2 has a winning strategy in the game \( \mathcal{G}_\mu^{\text{Filt}(\mathcal{C})}(M) \).

Proof. First suppose \( M \) is \(< \mu\)-closed potentially \( \mathcal{C} \)-filtered. By Lemma 3.5, there is a \(< \mu\)-closed forcing \( \mathbb{P} \) that forces the existence of a \( \mathcal{C} \)-filtration of \( M \) of length at most \( \mu \). Let \( \hat{Z} = \langle \hat{Z}_i : i < \mu \rangle \) be a name for this filtration, where each \( \hat{Z}_i \) is forced to be a set of size \(< \mu \). By \(< \mu\)-closure of \( \mathbb{P} \), every proper initial segment of \( \hat{Z} \) is forced to be in the ground model, and each \( \langle \hat{Z}_{i+1} \rangle / \langle \hat{Z}_i \rangle \) is forced to be \( \mathcal{C} \)-filtered via a filtration of length \(< \mu \). By Lemma 4.3 each such filtration is forced to be in the ground model, and moreover to be a \( \mathcal{C} \)-filtration there. Then Player 2 can use the name \( \hat{Z} \) to obtain a winning strategy in \( \mathcal{G}_\mu^{\text{Filt}(\mathcal{C})}(M) \), as follows: suppose \( \xi < \mu \) and the game so far looks like the following:

| Player 1 | \( X_0 \) | \( X_1 \) | \ldots | \( X_\xi \) | \ldots | \( X_\mu \) |
|----------|-----------|-----------|--------|-----------|--------|-----------|
| Player 2 | \( U_0 \) | \( U_1 \) | \ldots | \( U_\xi \) | \ldots | \( U_\mu \) |

Suppose also that Player 2 has created a descending sequence \( \langle p_\zeta : \zeta < \xi \rangle \) of conditions in \( \mathbb{P} \), and a strictly increasing sequence \( \langle i_\zeta : \zeta < \xi \rangle \) of ordinals below \( \mu \), such that for all \( \zeta < \xi \),

\[ p_\zeta \Vdash \hat{U}_\zeta = \hat{Z}_{i_\zeta} \text{ and } \langle \hat{Z}_{i_\zeta} \rangle \supseteq \hat{X}_\zeta \]

By \(< \mu\)-closure of \( \mathbb{P} \) there is a lower bound \( p^*_\zeta \) of the sequence \( \langle p_\zeta : \zeta < \xi \rangle \). Now \( p^*_\zeta \) forces “there is some \( i < \omega_1 \) such that \( \langle \hat{Z}_i \rangle \supseteq \hat{X}_\zeta \). \( \hat{Z}_i \) is in the ground model, and \( i > i_\zeta \) for all \( \zeta < \xi \)”. Pick some \( p_\xi \leq p^*_\zeta \) that decides the value of such an \( i \) and also decides a ground model set \( Z \) for \( \hat{Z}_i \). Player 2 then responds to \( X_\xi \) with the set \( Z \). It is routine to check that this procedure yields a valid response at any round before \( \mu \), and hence Player 2 wins.

For the \( \Leftarrow \) direction, suppose \( \tau \) is a winning strategy for Player 2 in the game. Define a poset \( \mathbb{Q} \) where conditions are \(< \mu\)-length partial runs of the
game where Player 2 has used $\tau$ along the way, ordered by end-extension. Clearly $Q$ is $< \mu$-closed. Furthermore, an easy density argument ensures that for every $x \in M$ and every $q \in Q$, there is a $q'$ extending $q$ such that $x$ is in one of the sets played by Player 1 at some time in the partial run $q'$. It follows that if $G$ is generic for $Q$ and $\vec{Z}_G$ is the union of Player 2’s moves, $\vec{Z}_G$ is a $C$-filtration of $M$ in $V[G]$.

\[\square\]

Lemma 4.5. Suppose $\mu$ is a regular cardinal, $C$ is a collection of $< \mu$-presented modules, and $M$ is a $\mu$-generated module. The following are equivalent:

1. $M$ is $C$-filtered.
2. $M$ is $< \mu$-closed potentially $C$-filtered.
3. Player 2 has a winning strategy in $G_{\text{Filt}}(C)_\mu(M)$.

Proof. Equivalence of (2) with (3) is true for all modules, by Lemma 4.4. The implication (1) $\Rightarrow$ (2) is trivial. Finally, to see that (3) implies (1), fix a $\mu$-sized generating set $X$ for $M$, let $\tau$ be a winning strategy for Player 2 in the $C$-filtration game of length $\mu$, and play a run of the game where Player 1 enumerates $X$ and Player 2 uses $\tau$. Since $\tau$ is a winning strategy, this will yield a $C$-filtration of $M$.

\[\square\]

4.2. Winning Strategies for Player 1 and AECs. Recall the definition of an Abstract Elementary Class (AEC), due to Shelah:

Definition 4.6. Given a language $\mathcal{L}$ and a class $K$ of $\mathcal{L}$-structures, a partial order $K = (K, \prec_K)$ is called an Abstract Elementary Class (AEC) iff:

1. If $M \in K$ and $M \cong N$, then $N \in K$;
2. If $M \prec_K N$, $M \cong M'$, and $N \cong N'$, then $M' \prec_K N'$;
3. If $M \prec_K N$, then $M$ is a substructure of $N$;
4. If $M_0, M_1$, and $M_2$ are elements of $K$, $M_0$ is a substructure of $M_1$, $M_0 \prec_K M_2$, and $M_1 \prec_K M_2$, then $M_0 \prec_K M_1$.
5. (Tarski-Vaught axioms) If $\alpha$ is an ordinal and $\langle M_i : i < \alpha \rangle$ is a $\subseteq$-continuous and increasing $\prec_K$-chain of elements of $K$, then:
   a. $M := \bigcup_{i<\alpha} M_i$, with the obvious interpretations of $\mathcal{L}$-symbols, is an element of $K$;
   b. $M_i \prec_K M$ for each $i < \alpha$; and
   c. if $N \in K$ is such that $M_i \prec_K N$ for all $i < \alpha$, then $M \prec_K N$.
6. (Löwenheim-Skolem number) There exists a cardinal $\text{LS}(K)$ such that whenever $A \subseteq B$ and $B \in K$, there exists an $A' \in K$ such that $A \subseteq A' \prec_K B$ and $|A'| \leq |A| + \text{LS}(K)$.

Given a ring $R$, a regular infinite cardinal $\mu$, and a collection $C$ of $< \mu$-presented $R$-modules, recall from the introduction that $\Gamma_{\mu,P1}^{\text{Filt}(C)}$ denotes the
class of \( R \)-modules \( M \) such that Player 1 has a winning strategy in the game \( \mathcal{G}^{\text{Filt}(C)}_{\mu}(M) \). And \( \prec_R \) denotes the embeddability relation for \( R \)-modules.

**Remark 4.7.** Everything we discuss in this section would also work if we had instead defined \( M \prec_R N \) to mean that there is a **pure** embedding from \( M \) to \( N \), or even an elementary embedding in with respect to the language of \( R \)-modules.

We show that under certain assumptions on \( C \), the class

\[
\left( \Gamma_{\mu, P_1}^{\text{Filt}(C)}, \prec_R \right)
\]

is an AEC. The only requirements from Definition 4.6 that are not trivial to verify are the requirements (v) and (vi).

**Lemma 4.8.** For any infinite regular \( \mu \), any ring \( R \), and any class \( C \) of \( R \)-modules, the partial order \( \left( \Gamma_{\mu, P_1}^{\text{Filt}(C)}, \prec_R \right) \) satisfies the Löwenheim-Skolem axiom of Definition 4.6, as witnessed by the cardinal

\[
\kappa := \max \left( |R|, 2^{<\mu} \right).
\]

**Proof.** Suppose \( M \) is an \( R \)-module and Player 1 has a winning strategy in the game \( \mathcal{G}^{\text{Filt}(C)}_{\mu}(M) \); let \( \tau \) be such a strategy. Let \( A \) be a subset of \( M \) of cardinality at most \( \kappa \). We need to find a submodule of \( M \) of size at most \( \kappa \), containing \( A \), for which Player 1 still has a winning strategy. Fix a regular cardinal \( \theta \) such that \( M, R, \tau, A \in H_\theta \). Since \( \kappa^{<\mu} = \kappa \), there exists an

\[
X < (H_\theta, \in, M, R, \tau, A, C \cap H_\theta)
\]

such that \( |X| = \kappa \subseteq X \) and \( X \) is closed under \( <\mu \)-length sequences. Since \( R \in X \) and \( |R| \leq \kappa \subseteq X \), it follows that \( R \subseteq X \), and hence that \( X \cap M \) is closed under multiplication from \( R \). Hence \( X \cap M \) is an \( R \)-submodule of \( M \); in fact, it is an elementary submodule in the language of \( R \)-modules. Similarly, since \( A \in X \) and \( |A| \leq \kappa \subseteq X \), it follows that \( A \subseteq X \). So \( X \cap M \supseteq A \).

We claim that Player 1 has a winning strategy in \( \mathcal{G}^{\text{Filt}(C)}_{\mu}(X \cap M) \). This follows from the fact that \( \tau \in X \), \( X \) is closed under \( <\mu \)-sequences, and the definition of the game requires both players to play \( <\mu \)-sized subsets of \( M \) at each stage. More precisely, the winning strategy is just the restriction of \( \tau \) to partial, \( <\mu \)-length runs of the game \( \mathcal{G}^{\text{Filt}(C)}_{\mu}(M) \) such that Player 2’s moves are always contained in \( X \cap M \). The model \( X \) is closed under such plays, since \( \tau \in X \) and \( X \) is \( <\mu \)-closed. \( \square \)

**Definition 4.9.** Let us say that a class \( C \) of \( R \)-modules is **quotient-hereditary** if the following holds: whenever \( Z \subseteq Z' \) are \( R \)-modules such that

\[
Z \text{ and } Z'/Z \text{ are } C-\text{filtered},
\]

then for all submodules \( U' \) of \( Z' \),

\[
U' \cap Z \text{ and } (U' \cap Z')/(U' \cap Z) \text{ are } C-\text{filtered}.
\]
Lemma 4.10. Suppose $C$ is a quotient-hereditary class of $R$-modules. Then for any infinite regular $\mu$, $\Gamma_{C}^{\text{Filt}(C)}_{\mu,P_1}$ is upward closed under the submodule relation.

Proof. Assume $M$ is a submodule of $N$ and Player 1 has a winning strategy $\tau$ in the game on $M$. Define a strategy $\tau^N$ for Player 1 in the game on $N$ as follows:

• $\tau^N(\emptyset) := \tau(\emptyset)$; note this is a $<\mu$-sized subset of $M$.
• Suppose $j < \mu$ and $p = \langle (X_i, Z_i) : i < j \rangle$ is a partial run of the game on $N$ of length $j$, where the $X_i$’s are Player 1’s moves and the $Z_i$’s are Player 2’s responses.
  - If at least one $X_i$ fails to be contained in $M$, then $\tau^N(p)$ is undefined.
  - Otherwise, let $p|_M := \langle (X_i, Z_i \cap M) : i < j \rangle$.
    Since $p$ is a partial run of the game, for each $i < j$ the modules $\bigcup_{k<i} \langle Z_k \rangle$ and $\langle Z_i \rangle / \bigcup_{k<i} \langle Z_k \rangle$ are $C$-filtered; and since $C$ is quotient-hereditary, it follows that $M \cap \bigcup_{k<i} \langle Z_k \rangle$ and $M \cap \langle Z_i \rangle / \bigcup_{k<i} \langle Z_k \rangle$ are $C$-filtered. Furthermore, since the $X_i$’s are assumed to be contained in $M$ by our case, and $Z_i \supset X_i$, it follows that each $M \cap \langle Z_i \rangle \cap M \cap \bigcup_{k<i} \langle Z_k \rangle$ contains $X_i$. It follows that $p|_M$ is a run of the game on $M$. If $p|_M$ is a game according to $\tau$, then set $\tau^N(p) := \tau(p|_M)$.

(Otherwise leave $\tau^N(p)$ undefined).

It follows that $\tau^N$ is a winning strategy for Player 1 in the game on $N$. If not, there would be a game

$\langle X_i, Z_i : i < \mu \rangle$

on $N$ lasting $\mu$ rounds, where Player 1 used $\tau^N$ along the way (in particular, $\tau^N$ was defined at every stage). But then

$\langle X_i, Z_i \cap M : i < \mu \rangle$

is a game of length $\mu$ on $M$, where Player 1 used $\tau$. This contradicts that $\tau$ was a winning strategy for Player 1 in the game on $M$. \qed

We can now complete the proof of Theorem 1.3 (except the “in particular” part, which is taken care of by Lemma 4.11 below). Lemma 4.10 immediately implies clause ((v)a) of the Tarski-Vaught axiom of Definition 4.6. The fact that our ordering $<_R$ is just the submodule (or pure submodule, or elementary submodule) relation ensures that parts ((v)b) and
of the Tarski-Vaught axiom are straightforward. Lemma 4.8 ensures that the Löwenheim-Skolem axiom (vi) of Definition 4.6 holds. The other requirements in Definition 4.6 are routine.

Recall that a ring $R$ is hereditary if submodules of projective $R$-modules are always projective. We next justify the various “in particular” clauses that appear in the statements of the theorems in the introduction.

**Lemma 4.11.** If $R$ is a hereditary ring, then the class of countably generated, projective $R$-modules is quotient-hereditary. The same is true for the class of countably presented, projective $R$-modules.

**Proof.** By Kaplansky’s Theorem, being filtered by countably generated (or countably presented) projective modules is equivalent to being projective.\(^{14}\)

So we just need to show that, if $Z \subset Z'$ are $R$-modules such that $Z$ and $Z'/Z$ are projective, then for every submodule $U'$ of $Z'$, both $U' \cap Z$ and $U'/(U' \cap Z)$ are projective.

Now since $Z$ and $Z'/Z$ are projective, so is $Z'$, and hence (by hereditary property of $R$) $U'$ and $U' \cap Z$ are projective. But projectivity of their quotient seems to require a little argument, which is a slight modification of an argument of Kaplansky (see page 42 of Lam \([13]\)).

Since $Z$ and $Z'/Z$ are projective, then $Z' \cong Z \oplus Z'/Z$; by amalgamating dual bases for $Z$ and $Z'/Z$ in the obvious way, there exists a dual basis $D' = (B', (f'_b)_{b \in B'})$ for $Z'$ such that

\[
\forall x \in Z \text{ sprt}_{D'}(x) := \{ b \in B' : f'_b(x) \neq 0 \} \text{ is contained in } B' \cap Z.
\]

Fix an enumeration $\langle b_\xi : \xi < |B'| \rangle$ of $B'$ such that members of $B' \cap Z$ are enumerated first; say $\zeta_Z \leq |B'|$ is such that

\[
B' \cap Z = \{ b_\xi : \xi < \zeta_Z \}
\]

Let $f'_\xi$ denote $f'_{b_\xi}$ for each $\xi < |B'|$. For each $\mu \leq |B'|$ let $\text{span}(D' \mid \mu)$ denote the set of $x \in Z'$ such that

\[
\text{sprt}_{D'}(x) \subseteq \{ b_\xi : \xi < \mu \}.
\]

For each $\xi < |B'|$ let $U'_\xi := U' \cap \text{span}(D' \mid \xi)$. Note that the sequence

\[
\langle U'_\xi : \xi < |B'| \rangle
\]

is a filtration of $U'$, and the union of the first $\zeta_Z$ many entries is $U' \cap Z$ by (5).

For each $\xi < |B'|$, define $J_\xi$ to be the image of the $R$-module homomorphism

\[ f'_\xi \mid U'_{\xi+1} \]

\(^{14}\)Kaplansky’s Theorem usually refers to the “projective implies filtered by countably generated projectives” direction; the other direction, i.e. that projectives are closed under transfinite extensions, is a special case of Eklof’s Lemma \([8]\).
and observe that the kernel of this map is exactly $U'_\xi$. Hence,

\begin{equation}
\frac{U'_{\xi+1}}{U'_\xi} \cong J_{\xi}.
\end{equation}

Now $J_{\xi}$ is an $R$-submodule of the (free) module $R$ (i.e., $J_{\xi}$ is an ideal in $R$), and hence by the assumption that $R$ is hereditary, (6) is projective for each $\xi < |B'|$. So

\[ \langle U'_\xi \mid \xi < \zeta \rangle \text{ is a projective filtration of } U' \cap Z, \]

and

\[ \langle U'_\xi \mid \xi < |B'| \rangle \text{ is a projective filtration of } U'. \]

It follows from splitting properties of projective modules that

\[ U' \cap Z \cong \bigoplus_{\xi < \zeta} U'_{\xi+1}/U'_\xi \cong \bigoplus_{\xi < \zeta} J_{\xi}, \]

\[ U' \cong \bigoplus_{\xi < |B'|} U'_{\xi+1}/U'_\xi \cong \bigoplus_{\xi < |B'|} J_{\xi}, \]

and

\[ U'/(U' \cap Z) \cong \bigoplus_{\xi \in [\zeta, |B'|]} J_{\xi}. \]

Since each $J_{\xi}$ is projective, and direct sums of projective modules are projective, this completes the proof. \qed

**Corollary 4.12.** If $R$ is a hereditary ring, then the class of $R$-modules for which Player 1 has a winning strategy in the Projective Filtration game of length $\omega_1$ (see Remark 4.2), under the embeddability (or pure embeddability) ordering, is an AEC.

The same statement holds if “Projective Filtration Game” is replaced by “Free Filtration Game”.

In Section 4.3 we give an alternate, more concrete description of the Projective Filtration Game.

### 4.3. The Dual Basis Game

This section is not essential to the rest of the paper, but we describe a slightly more constructive/intuitive game that is equivalent to the Projective Filtration Game; in particular, each move in this equivalent game involves just finitely many decisions. We focus on $\mu = \omega_1$ for concreteness; i.e., we give an alternate description of the game $\mathcal{G}^{\text{Filt}(C)}_{\omega_1}(\cdot)$, where $\mathcal{C}$ is the collection of countably presented, projective modules.

Recall the definition of dual basis from Section 2, and the fact that a module is projective if and only if it has a dual basis. We introduce weaker variant of a dual basis. Given an $R$-module $M$ and a subset $X$ of $M$, say that a pair $D = \left( \mathcal{B}, (f_b)_{b \in \mathcal{B}} \right)$ is an $\textbf{M-Dual Basis for } \langle X \rangle$ if it satisfies the usual requirements of being a dual basis for $\langle X \rangle$, except that we do not
require $B \subseteq \langle X \rangle$; merely that $B \subseteq M$ (hence the $M$-prefix in the notation). More precisely, $\mathcal{D}$ is an $M$-Dual Basis for $\langle X \rangle$ iff:

1. $B \subseteq M$;
2. Each $f_b$ is an $R$-linear map from $\langle X \rangle \to R$;
3. For each $x \in X$, $\text{sprt}_D(x) := \{ b \in B : f_b(x) \neq 0 \}$ is finite, and $x = \sum_{b \in \text{sprt}_D(x)} f_b(x)b$.

If $X \subseteq Y \subseteq M$, $\mathcal{D}^X = (B^X, (f^X_b)_{b \in B_X})$ is an $M$-Dual Basis for $\langle X \rangle$, and $\mathcal{D}^Y = (B^Y, (f^Y_b)_{b \in B_Y})$ is an $M$-Dual Basis for $\langle Y \rangle$, we say that $\mathcal{D}^Y$ is a conservative extension of $\mathcal{D}^X$ if $B^Y \supseteq B^X$, $f^Y_b$ extends $f^X_b$ for all $b \in B^X$, and for every $x \in X$, $\text{sprt}_{\mathcal{D}^X}(x) = \text{sprt}_{\mathcal{D}^Y}(x)$.

**Definition 4.13.** Let $R$ be a ring and $M$ an $R$-module. The Dual Basis Game on $M$ of length $\omega_1$, denoted $DB_{\omega_1}(M)$, is the two-player game where, at each countable round $i$:

- Player 1 plays an element $x_i$ of $M$.
- Player 2 responds, if possible, with an $M$-Dual Basis $\mathcal{D}^i = (B^i, (f^i_b)_{b \in B^i})$ for $\langle \{x_k : k \leq i\} \cup \bigcup_{k < i} B^k \rangle$ such that:
  1. $\mathcal{D}^i$ conservatively extends $\mathcal{D}^k$ for all $k < i$;
  2. $B^i \setminus \bigcup_{k < i} B^k$ is finite.

Player 2 wins if she lasts $\omega_1$ rounds; otherwise Player 1 wins.

The conservativity requirement for the extensions ensures that supports don’t grow into an infinite set as the game progresses.

The game $DB_{\omega_1}(M)$ is equivalent to the Projective Filtration Game of length $\omega_1$ on $M$ introduced in Remark 4.2. Since we will not use this fact, but only mention it as a curiosity, we will omit the proof. The basic reason the games are equivalent is:

**Fact 4.14.** For any modules $Z \subset Z'$, the following are equivalent:

1. $Z$ and $Z'/Z$ are projective;
2. There is a dual basis $\mathcal{D}$ for $Z$ that can be conservatively extended to a dual basis $\mathcal{D}'$ for $Z'$; this means that $\mathcal{D}'$ extends $\mathcal{D}$ in the obvious way and, moreover, for all $z \in Z$, $\text{sprt}_{\mathcal{D}'}(z) = \text{sprt}_{\mathcal{D}}(z)$. (Note: here we are referring to ordinary dual bases, not $M$-dual bases).
3. $Z$ is projective and every dual basis for $Z$ can be conservatively extended to a dual basis for $Z'$.

Although they are equivalent, the games $DB_{\omega_1}(M)$ and the Projective Filtration Game of length $\omega_1$ on $M$ proceed at different paces; the former

\[^{15}\text{It is an easy exercise to verify that if this requirement holds of every element of } X, \text{ it also holds for every element of } \langle X \rangle.\]
involves finitely many new objects at each round, while the latter involves countably many.

4.4. Relation to Ehrenfeucht-Fraïssé games. In this section we show that determinacy of Filtration Games implies determinacy of certain Ehrenfeucht-Fraïssé games, including the kind appearing in Mekler et al. [20].

If $\mathfrak{A} = (A, \ldots)$, $\mathfrak{B} = (B, \ldots)$ is a pair of structures in the same (relational) signature, and $\mu$ is an ordinal, the **Ehrenfeucht-Fraïssé game of length $\mu$ on the pair $\mathfrak{A}, \mathfrak{B}$**, denoted $\text{EF}_\mu(\mathfrak{A}, \mathfrak{B})$, is the game lasting (at most) $\mu$ innings, where at the top of inning $i$, Spoiler plays an element of $A \cup B$, and Duplicator then responds with an element of the other structure, in such a way that the pairs of elements chosen so far constitutes a partial isomorphism between $\mathfrak{A}$ and $\mathfrak{B}$. Duplicator wins if she lasts $\mu$ innings; otherwise Spoiler wins.

By the Gale-Stewart Theorem, $\text{EF}_\omega(-, -)$ is always determined. Moreover, this is related to “potential isomorphisms”: Player 2 has a winning strategy in $\text{EF}_\omega(\mathfrak{A}, \mathfrak{B})$ if and only if $\mathfrak{A}$ is isomorphic to $\mathfrak{B}$ in some forcing extension (see Nadel-Stavi [21]).

Determinacy of $\text{EF}_\omega(-, -)$ does not generalize to higher cardinals: Mekler et al. [20] give a ZFC example of a non-determined $\text{EF}_{\omega_1}(-, -)$ game. On the other hand, Mekler et al. [20] did prove that it is relatively consistent with ZFC plus large cardinals that $\text{EF}_{\omega_1}(G, F)$ is determined whenever $G$ is a group and $F$ is a free abelian group. Our Theorem 1.2 can be viewed as a strengthening of their theorem, in light of the results below.

If $M$ and $N$ are $R$-modules, for the purposes of the game $\text{EF}_\mu(M, N)$ we will view them as structures in the $(|R| + \aleph_0)$-sized language that includes predicates describing the scalar multiplication by $R$ on the modules. Then if $X \subset M$, $Y \subset N$, and $\sigma : X \to Y$ is a function, $\sigma$ is a partial isomorphism from $M \to N$ if and only if $\sigma$ lifts to an $R$-module isomorphism

$$\hat{\sigma} : \langle X \rangle^M_R \to \langle Y \rangle^N_R.$$

**Theorem 4.15.** Suppose $\mu$ is a regular uncountable cardinal, $R$ is a ring, $M$ is an $R$-module, and $C$ is a class of $< \mu$-presented $R$-modules. If Player 1 has a winning strategy in the game $G^\text{Filt}(C)_\mu(M)$, then for every $C$-filtered module $P$, Spoiler has a winning strategy in $\text{EF}^R_\mu(M, P)$.

**Proof.** Let $\tau$ be a winning strategy for Player 1 in $G^\text{Filt}(C)_\mu(M)$, and suppose $P$ is a $C$-filtered module. We use $\tau$ to define a winning strategy for Spoiler in $\text{EF}_\mu(M, P)$. This relies heavily on Lemma 3.4 which guarantees that there is a club $D \subseteq \wp_\mu(P)$ such that

$$\forall Z \subset Z' \text{ both in } D, \langle Z \rangle \text{ and } \frac{\langle Z' \rangle}{\langle Z \rangle} \text{ are } C\text{-filtered}.$$

We now describe a strategy for Spoiler in the game $\text{EF}_\mu(M, P)$, which we will denote by $\psi$. 
Let \( X_0 := \tau(\emptyset) \) be Player 1’s opening move according to \( \tau \) in the \( C \)-filtration game on \( M \). \textit{Spoiler} begins the \( \text{EF}_\mu(M, P) \) game by simply enumerating \( X_0 \) in his first \(|X_0|\)-many moves. Let \( P_0 \) be the set of \textit{Duplicator}’s responses. Fix a \( Z_0 \in D \) such that \( P_0 \subseteq Z_0 \). \textit{Spoiler} then uses his next \(|Z_0|\)-many moves to enumerate \( Z_0 \). Assuming \textit{Duplicator} has survived so far (i.e., through these first \(|X_0| + |Z_0|\) stages of the EF game), she has constructed a bijection

\[
\sigma_0 : Y_0 \rightarrow Z_0
\]

where \( Y_0 \) is \(< \mu\)-sized subset of \( M \) containing \( X_0 \), \( \langle Z_0 \rangle_R^P \) is \( C \)-filtered (because \( Z_0 \in D \)), and \( \sigma_0 \) lifts to an isomorphism

\[
\tilde{\sigma}_0 : \langle Y_0 \rangle_R^M \rightarrow \langle Z_0 \rangle_R^P.
\]

Now go back to the Filtration Game, where it is the bottom of the 0-th inning, and make Player 2 play the set \( Y_0 \); this is a valid move because \( \langle Y_0 \rangle \) is isomorphic to \( \langle Z_0 \rangle \) and hence \( C \)-filtered, and \( X_0 \subseteq Y_0 \).

In general, suppose we are at a position \( t \) in the game tree of \( \text{EF}_{\omega_1}(M, P) \), and that it is \textit{Spoiler}’s turn to play.

Suppose we have constructed:

1. a \( \subseteq \)-increasing (but not necessarily continuous) sequence

\[
\langle Z_i : i < \alpha_t \rangle
\]

of elements of \( D \), for some \( \alpha_t < \mu \), such that for all \( i < \alpha_t \),

\[
\bigcup_{k < i} \langle Z_k \rangle
\]

is \( C \)-filtered;

2. a partial play

\[
p(t) = \langle (X_i, Y_i) : i < \alpha_t \rangle
\]

of the game \( G_{\mu}^{\text{Filt}(C)}(M) \) according to \( \tau \);

3. a coherent sequence of bijections \( \langle \sigma_i : Y_i \rightarrow Z_i : i < \alpha_t \rangle \) arising from the plays so far of the EF game, such that each \( \sigma_i \) lifts to an isomorphism

\[
\tilde{\sigma}_i : \langle Y_i \rangle_R^M \rightarrow \langle Z_i \rangle_R^P.
\]

Then let \( X_{\alpha_t} := \tau(p(t)) \). In the next \(|X_{\alpha_t}|\)-many rounds of the EF game, \textit{Spoiler} enumerates \( X_{\alpha_t} \). Letting \( P_{\alpha_t} \) be \textit{Duplicator}’s responses, let \( Z_{\alpha_t} \) be an element of \( D \) such that

\[
P_{\alpha_t} \cup \bigcup_{i < \alpha_t} Z_i \subseteq Z_{\alpha_t}.
\]

Observe that by closure of \( D \), \( \bigcup_{i < \alpha_t} Z_i \in D \), and hence

\[
\langle Z_{\alpha_t} \rangle \bigcup_{i < \alpha_t} \langle Z_i \rangle
\]

is \( C \)-filtered.
(So (1) has been maintained). Spoiler then enumerates $Z_{\alpha_t}$ at his turn in the next $|Z_{\alpha_t}|$ rounds of the EF game. Assuming Duplicator has survived, the play of the EF game so far which is $|X_{\alpha_t}| + |Z_{\alpha_t}|$ (in ordinal arithmetic) many rounds beyond node $t$, yields some $< \mu$-sized $Y_{\alpha_t} \subseteq M$ containing $X_{\alpha_t}$, and a bijection 

$$\sigma_{\alpha_t} : Y_{\alpha_t} \to Z_{\alpha_t}$$

extending $\sigma_i$ for all $i < \alpha_t$ that lifts to an isomorphism $\tilde{\sigma}_{\alpha_t} : \langle Y_{\alpha_t} \rangle^M \to \langle Z_{\alpha_t} \rangle^N$, and such that 

$$\bigcup_{k < \alpha_t} \langle Z_k \rangle$$

is $\mathcal{C}$-filtered.

Since $\tilde{\sigma}_{\alpha_t}$ is an isomorphism extending the previous $\tilde{\sigma}_i$’s, it follows that 

$$\bigcup_{i < \alpha_t} \langle Y_i \rangle$$

is $\mathcal{C}$-filtered,

and hence that $Y_{\alpha_t}$ is an acceptable response to $p(t)^\ast X_{\alpha_t}$ by Player 2 in the Filtration game.

We claim that $\psi$ is a winning strategy for Spoiler in the EF game. Suppose not; then by regularity of $\mu$ there exists a strictly increasing sequence 

$$\langle t_\xi : \xi < \mu \rangle$$

of nodes in the EF game tree where Spoiler has used the strategy $\psi$. Then 

$$\langle p(t_\xi) : \xi < \mu \rangle$$

is a strictly increasing sequence of nodes in the Filtration Game where Player 1 has used the strategy $\tau$; this is a contradiction, because $\tau$ is a winning strategy for Player 1 in the Filtration Game.

$$\square$$

Lemma 4.16. Let $M$ be an $R$-module. The following are equivalent:

1. Player 2 has a winning strategy in the Free Filtration Game of length $\omega_1$ on $M$ (i.e., the game $G_{\omega_1}^\text{Filt}(\mathcal{C})$ where $\mathcal{C} = \{R\}$);
2. Duplicator has a winning strategy in $\text{EF}_{\omega_1}(M,F)$, where $F$ the free $R$-module on $\omega_1$ generators;
3. $M$ is $\sigma$-closed potentially free.

If $R$ is a ring and “projective = free” for $R$-modules in all $\sigma$-closed forcing extensions—e.g., if $R = \mathbb{Z}$—then these are also equivalent to:

4. Player 2 has a winning strategy in the Projective Filtration Game of length $\omega_1$ on $M$;
5. $M$ is $\sigma$-closed potentially projective.

Proof. The equivalence of (1) with (3) is just Lemma 4.4. The equivalence of (2) with (3) is well-known, see e.g. Nadel-Stavi [21]. If freeness is equivalent to projectivity for $R$-modules in all $\sigma$-closed forcing extensions, then the
equivalence of (3) with (1) is trivial. Parts (5) and (1) are equivalent by Lemma 4.4.

Theorem 4.15 and Lemma 4.16 yield:

Corollary 4.17. Let $G$ be a $\mathbb{Z}$-module; i.e., an abelian group. Determinacy of the Free Filtration Game of length $\omega_1$ on $G$, or determinacy of the Projective Filtration Game of length $\omega_1$ on $G$, implies determinacy of $\text{EF}_{\omega_1}(G,F)$ for all free abelian groups $F$.

5. Determinacy of filtration games and other consequences of Martin’s Maximum

The key consequence of Martin’s Maximum that we will use is the following principle introduced by Fuchino-Usuba (but under a different name):

Definition 5.1. $\text{RP}_{\text{internal}}$ asserts that for all uncountable $X$, all stationary $S \subseteq [X]^{\omega_1}$, all regular $\theta$ such that $X \in H_\theta$, and all first order structures $\mathfrak{A}$ in a countable language extending $(H_\theta, \in, X, S)$, there exists a $W$ such that:

1. $|W| = \omega_1 \subset W$;
2. $W < \mathfrak{A}$; and
3. $S \cap W \cap [W \cap X]^{\omega_1}$ is stationary in $[W \cap X]^{\omega_1}$. (The fact that we require $S \cap W \cap [W \cap X]^{\omega_1}$ to be stationary, rather than just $S \cap [W \cap X]^{\omega_1}$, is why we call this kind of reflection “internal”).

The principle $\text{RP}_{\text{internal}}$ can also be characterized by a strong form of Chang’s Conjecture (Fuchino-Usuba [13]). It is also equivalent to a certain Löwenheim-Skolem-Tarski property of Stationary Logic; roughly speaking, $\text{RP}_{\text{internal}}$ is equivalent to the downward reflection of statements of the form

$\text{stat} Z \text{ aa } U_1 \ldots \text{ aa } U_k \phi(Z,U_1,\ldots,U_k)$

to a substructure of size $< \aleph_2$; here $\text{aa}$ is the “almost all” quantifier and $\text{stat}$ is the “stationarily many” quantifier of Stationary Logic, introduced by Shelah [25]. We will not prove this equivalence, but point out that it is very similar to the proof of the main theorem of Cox [6] (which was, in turn, modeled after Fuchino et al. [12]).

5.1. $\aleph_2$-compactness of potentially filtered modules. Given a class $\Gamma$ of $R$-modules, and an $R$-module $M$, let

$\mathcal{G}_{\omega_2}(M) := \{W \in \mathcal{G}_{\omega_2}(M) : (W)^M_R \in \Gamma\}$

We will say that:

- $\Gamma$ is $\omega_2$-club-compact if for every $R$-module $M$: if $\mathcal{G}_{\omega_2}(M)$ contains a closed unbounded subset of $\mathcal{G}_{\omega_2}(M)$, then $M \in \Gamma$.
- $\Gamma$ is $\omega_2$-compact if for every module $M$: if $\mathcal{G}_{\omega_2}(M) = \mathcal{G}_{\omega_2}(M)$, then $M \in \Gamma$. 
Club compactness is formally stronger than compactness, but in many natural situations they are equivalent\[16\]

The following is the key use of $\text{RP}_{\text{internal}}$:

**Theorem 5.2.** Assume $\text{RP}_{\text{internal}}$. Let $R$ be a ring of size at most $\aleph_1$ and $C$ be a collection of countably-presented $R$-modules. Then the class of $\sigma$-closed potentially $C$-filtered modules is $\aleph_2$-club-compact.

**Proof.** Suppose $M$ is an $R$-module and club-many elements of $\wp_{\omega_2}(M)$ generate $\sigma$-closed potentially $C$-filtered submodules of $M$; let $D$ denote this club. Suppose toward a contradiction that $M$ is not $\sigma$-closed potentially $C$-filtered. Then by the equivalence of parts (1) and (4) of Corollary 3.9, the set

$$\left\{ Z \in [M]^\omega : \text{For stationarily many } Z' \in [M]^\omega, \langle Z' \rangle / \langle Z \rangle \text{ is } C\text{-filtered} \right\}$$

does **not** contain a club in $[M]^\omega$; let $S$ denote its complement. Then $S$ is stationary, and for every $Z \in S$,

$$C_Z := \{ Z' \in [M]^\omega : Z' \supseteq Z \text{ and } \langle Z' \rangle / \langle Z \rangle \text{ is not } C\text{-filtered} \}$$

contains a club in $[M]^\omega$.

Fix a large regular $\theta$. By $\text{RP}_{\text{internal}}$ there exists a

$$W \prec (H_\theta, \in, D, S, \langle C_Z : Z \in S \rangle)$$

such that $|W| = \omega_1 \subset W$ and

$$S \cap W \cap [W \cap M]^\omega \text{ is stationary in } [W \cap M]^\omega. \tag{7}$$

Note that since $R \in W$ and $|R| \leq \omega_1 \subset W$, $W \cap M$ is already closed under scalar multiplication from $R$, and hence $W \cap M = \langle W \cap M \rangle^M$; i.e. $W \cap M$ is an $R$-submodule of $M$. Since $D$ contains a club in $\wp_{\omega_2}(M)$ and $D \in W$, then by Lemma 2.7, $W \cap M \in D$. Hence,

$$W \cap M \text{ is } \sigma\text{-closed potentially } C\text{-filtered}. \tag{8}$$

Since $W \cap M$ is an $R$-module of size $\omega_1$, Corollary 3.2 and (8) ensure that there is a $C$-filtration

$$\langle \langle Z_i \rangle : i < \omega_1 \rangle$$

of $W \cap M$, where each $Z_i$ is countable. By Lemma 2.6, it follows that

$$\forall k < \omega_1 \forall \ell < \omega_1 \text{ } k < \ell \implies \langle Z_i \rangle / \langle Z_k \rangle \text{ is } C\text{-filtered}. \tag{9}$$

Now $\{ Z_i : i < \omega_1 \}$ is closed unbounded in $[W \cap M]^\omega$, so by (7) there is some $i < \omega_1$ such that $Z_i \in S \cap W$. It follows that $C_{Z_i}$ is an element of $W$, and hence, by Lemma 2.6, that

$$C_{Z_i} \cap [W \cap M]^\omega \text{ contains a club in } [W \cap M]^\omega.$$

---

\[16\] E.g., if $\Gamma$ is closed under submodules, then $\wp^\Gamma_{\omega_2}(M)$ contains a club in $\wp_{\omega_2}(M)$ if and only if $\wp^\Gamma_{\omega_2}(M) = \wp_{\omega_2}(M)$.  

Then there is some \( j > i \) such that \( Z_j \in \mathcal{C}Z_i \). Then by definition of \( \mathcal{C}Z_i \), \( \langle Z_j \rangle / \langle Z_i \rangle \) is not \( \mathcal{C} \)-filtered; this contradicts (9).

□

Theorem 5.2 and Corollary 3.7 yield:

**Corollary 5.3.** Assume \( RP_{\text{internal}} \). Then for every ring \( R \) of size at most \( \aleph_1 \), the class of \( \sigma \)-closed potentially projective modules is \( \aleph_2 \)-club-compact.

5.2. **Proof of Theorem 1.2.** Theorem 1.2 from the introduction follows from Theorem 5.2 and the following theorem:

**Theorem 5.4.** Assume \( R \) is a ring and \( \mathcal{C} \) is a quotient-hereditary collection of countably presented \( R \)-modules. Suppose the class of \( \sigma \)-closed potentially \( \mathcal{C} \)-filtered modules is \( \aleph_2 \)-club-compact. Then for every \( R \)-module \( M \), the game \( \mathcal{G}_{\omega_1}^{\text{Filt}(\mathcal{C})}(M) \) is determined.

*Proof.* Suppose Player 2 does not have a winning strategy in \( \mathcal{G}_{\omega_1}^{\text{Filt}(\mathcal{C})}(M) \). By Lemma 4.4, \( M \) is not \( \sigma \)-closed potentially \( \mathcal{C} \)-filtered. By the assumed \( \aleph_2 \)-club compactness, there is a \( W \in \mathcal{W}_{\omega_2}(M) \) such that \( \langle W \rangle^M_R \) is not \( \sigma \)-closed potentially \( \mathcal{C} \)-filtered.

Note that in the game \( \mathcal{G}_{\omega_1}^{\text{Filt}(\mathcal{C})}(\langle W \rangle) \), Player 1 has an easy winning strategy: he simply enumerates \( W \). Such a game cannot last \( \omega_1 \) rounds, because if it did, it would yield a \( \mathcal{C} \)-filtration of \( \langle W \rangle \). Then by Lemma 4.10 and the assumption that \( \mathcal{C} \) is quotient-hereditary, Player 1 has a winning strategy in the game \( \mathcal{G}_{\omega_1}^{\text{Filt}(\mathcal{C})}(M) \). □

**Corollary 5.5.** \( RP_{\text{internal}} \) implies that for every hereditary ring \( R \) of size at most \( \aleph_1 \), Projective Filtration games of length \( \omega_1 \) for \( R \)-modules are determined.

The same conclusion holds for Free Filtration Games of length \( \omega_1 \).

5.3. **Proof of Theorem 1.3.** A partial order \((I, \leq)\) is called \( < \mu \)-directed (for a regular cardinal \( \mu \)) if every \( < \mu \)-sized subset of \( I \) has an upper bound. A direct system is \( < \mu \)-directed if its underlying index set is \( < \mu \)-directed.

**Theorem 5.6.** Suppose \( R \) is a ring of size at most \( \aleph_1 \) and \( \Gamma \) is a collection of \( R \)-modules such that:

1. \( \Gamma \) is \( \aleph_2 \)-club-compact;
2. \( \Gamma \) is downward closed under pure submodules.

Then \( \Gamma \) is closed under \( < \aleph_2 \)-directed limits of \( R \)-module homomorphisms.

*Proof.* Suppose \( \mathcal{D} \) is a directed system of structures from \( \Gamma \), indexed by the \( < \aleph_2 \)-directed partial order \((I, \leq)\). For \( i \leq j \) in \( I \) let \( \pi_{i,j} : M_i \to M_j \) be the associated homomorphism. Note that we do not assume the \( \pi_{i,j} \)'s are injective.

Let \( M_\mathcal{D} \) denote the direct limit of \( \mathcal{D} \), and suppose toward a contradiction that \( M_\mathcal{D} \notin \Gamma \). By \( \aleph_2 \)-club-compactness of \( \Gamma \), there is a stationary \( T \subset \)
where:

\[ \varphi_{\omega_1}(M_D) \text{ consisting of submodules of } M_D \text{ that fail to be in } \Gamma. \]

By Lemma 2.7, there is a

\[ W < (H_T, e, T, D) \]

such that \(|W| = \omega_1 \subset W\) and \(W \cap M_D \in T\); so

\[ W \cap M_D \notin \Gamma. \]

(10)

Since \(|W \cap I| \leq |W| = \aleph_1\), the \(< \aleph_2\)-directedness of \(I\) ensures there is an \(i_W \in I\) above all indices in \(W \cap I\). By assumption on \(D\), \(M_{i_W} \in \Gamma\). Since \(\Gamma\) is downward closed under pure embeddings, to obtain a contradiction it will suffice to find a pure embedding from \(W \cap M_D\) into \(M_{i_W}\).

For \(i \in I\) and \(x \in M_i\), \([i, x]_D\) denotes the equivalence class of the thread \(\{\pi_{i, j}(x) : j \geq i\}\) in the direct limit \(M_D\). Note that, by elementarity of \(W\), every element of \(W \cap M_D\) is of the form \([i, x]_D\) for some \(i \in W \cap I\) and some \(x \in W \cap M_i\).

Define

\[ \rho : W \cap M_D \to M_{i_W} \]

by

\[ [i, x]_D \mapsto \pi_{i, i_W}(x), \]

where \(i\) is any element of \(W \cap I\) and \(x\) is any element of \(W \cap M_i\); note that under those assumptions, \(\pi_{i, i_W}\) exists (because \(i_W\) is above every index in \(W\)). The map \(\rho\) is well-defined, because if \([i, x] = [j, y]\) where \(i, j, x, y \in W\), then by elementarity of \(W\) there is some \(k \in W \cap I\) with \(k \geq i, j\) such that \(\pi_{i, k}(x) = \pi_{j, k}(y) =: z\). Since \(i_W \geq k\), it follows from commutativity of the system \(D\) that \(\pi_{i, i_W}(x) = \pi_{k, i_W}(z) = \pi_{j, i_W}(y)\). That \(\rho\) is injective is trivial, based on the definition of the \(D\)-equivalence relation.

To see that \(\rho\) is a \(R\)-module homomorphism, first observe that \(|R| \leq \omega_1 \subset W\) and \(R \in W\), and it follows (by elementarity of \(W\)) that

(11)

\[ R \subset W. \]

Now suppose \(r, s \in R\), \(i, j \in W \cap I\), \(x \in W \cap M_i\), and \(y \in W \cap M_j\). By (11), both \(r\) and \(s\) are in \(W\); since \(x, y, i, j, k\) are also in \(W\), we have

(12)

\[ r\pi_{i, k}(x) + s\pi_{j, k}(y) \text{ is an element of } W \cap M_k. \]

Then

\[ \rho\left(r[i, x] + s[j, y]\right) = \rho\left([k, r\pi_{i, k}(x) + s\pi_{j, k}(y)]\right) = \pi_{k, i_W}(r\pi_{i, k}(x) + s\pi_{j, k}(y)) = r\pi_{i, i_W}(x) + s\pi_{j, i_W}(y) = r\rho([i, x]) + s\rho([j, y]), \]

where the second equality is by (12) and the definition of \(\rho\).

To see that \(\rho\) is a pure embedding, suppose

\[ Av = b \]

where:

- \(A\) is a (finite) matrix with entries from \(R\);
- \(b = \left(\pi_{i_1, i_W}(b_1), \ldots, \pi_{i_r, i_W}(b_r)\right)^t\) is a vector of elements of the range of \(\rho\), where each \(i_k\) and \(b_k\) come from \(W\).


• \( v \) is a vector from \( M_{iW} \).

Then the index \( i_W \) and the vector \( v \) witness that

\[
(H_{\theta, \epsilon} \models \exists i \geq i_1, \ldots, i_r \text{ and } \quad Ax = (\pi_{i_1, i}(b_1), \ldots, \pi_{i_r, i}(b_r))^t \text{ is solvable in } M_i.
\]

(13)

Now (11) (and the fact that \( A \) is a finite matrix) ensures that \( A \in W \). The other parameters from (13) are in \( W \) too, and hence the \( i \) from (13) can be taken to be in \( W \cap I \). Furthermore, the solution in \( M_i \) can be taken to come from \( W \cap M_i \), for the same reason. Say \( (u_1, \ldots, u_\ell) \) is a vector in \( W \cap M_i \) such that

\[
A(u_1, \ldots, u_\ell)^t = (\pi_{i_1, i}(b_1), \ldots, \pi_{i_r, i}(b_r))^t.
\]

Then \( (\pi_{i, iW}(u_1), \ldots, \pi_{i, iW}(u_\ell)) \) is a vector from the range of \( \rho \), and

\[
A(\pi_{i, iW}(u_1), \ldots, \pi_{i, iW}(u_\ell))^t = (\pi_{i_1, iW}(b_1), \ldots, \pi_{i_r, iW}(b_r))^t = b.
\]

\( \square \)

We can now finish the proof of Theorem 1.3. Assume \( \text{RP_{internal}} \), \( R \) is a ring of size at most \( \aleph_1 \), and \( C \) is a quotient-hereditary collection of countably-presented \( R \)-modules. Let \( \Gamma \) denote the class of \( \sigma \)-closed potentially \( C \)-filtered modules, which by Corollary 4.4 is the same as the class of modules \( M \) such that Player 2 has a winning strategy in the \( C \)-filtration game of length \( \omega_1 \) on \( M \). By Theorem 1.2 such games are determined; this determinacy, together with Lemma 4.10 implies that \( \Gamma \) is downward closed under submodules (so, in particular, under pure submodules). And Theorem 5.2 guarantees that \( \Gamma \) is \( \aleph_2 \)-club-compact. So by Theorem 5.6, \( \Gamma \) is closed under \( < \aleph_2 \)-directed limits, which completes the proof of Theorem 1.3.

**Remark 5.7.** Although we will not need the following, we note that Theorem 5.6 generalizes easily to other model-theoretic settings. For example, suppose \( L \) is a first order language, \( \Gamma \) a collection of \( L \)-structures that is \( \aleph_2 \)-club compact, and there exists a \( < \aleph_2 \)-sized collection \( \Sigma \) of \( L \)-formulas such that \( \Gamma \) is downward closed under injective \( \Sigma \)-preserving maps. Then whenever \( D \) is a \( < \aleph_2 \)-directed system of \( \Sigma \)-preserving (not necessarily injective) maps between elements of \( \Gamma \), the direct limit of \( D \) is in \( \Gamma \).

**Corollary 5.8.** Assume \( \text{RP_{internal}} \) and that \( R \) is a ring of size at most \( \aleph_1 \) that is hereditary in all \( \sigma \)-closed forcing extensions. Then the class of \( \sigma \)-closed potentially projective \( R \)-modules is closed under \( < \aleph_2 \)-directed limits.

**Proof.** If \( R \) is hereditary in all \( \sigma \)-closed forcing extensions, then the class of \( \sigma \)-closed potentially projective modules is closed under submodules (in particular, under pure submodules). The result then follows from Corollary 5.7, Theorem 5.2 and Theorem 5.6. \( \square \)
Lemma 5.9. If $R$ is a countable, hereditary ring, then $R$ remains hereditary in all $\sigma$-closed forcing extensions.

Proof. Suppose $R$ is countable and hereditary, and let $V[G]$ be a $\sigma$-distributive forcing extension of $V$. By Chapter 2E of [18], a ring $R$ is hereditary if and only if all ideals in $R$ are projective (as $R$-modules); so it suffices to show that in $V[G]$, all ideals in $R$ are projective. Suppose, in $V[G]$, that $I$ is an ideal of $R$. Since $R$ was countable in $V$ and $V[G]$ added no new countable sets, $I$ was already an element of $V$. Since $V \models "R$ is hereditary", then $I$ is projective in $V$, and this is easily upward absolute to $V[G]$. □

Corollary 5.10. Assume $RP_{\text{internal}}$ and that $R$ is a countable, hereditary ring. Then the class of $\sigma$-closed potentially projective $R$-modules is closed under $< \aleph_2$-directed limits.

Proof. This follows immediately from Lemma 5.9 and Corollary 5.8. □

5.4. Proof of Theorem 1.5. Consider the statement:

$$\Phi \equiv \text{"The class of } \mathbb{Z} \text{-modules (i.e., abelian groups) } G \text{ that are not } \sigma \text{-closed potentially free, under the pure embeddability ordering, is an AEC."}$$

We will show that $\Phi$ is independent of ZFC. One direction has basically already been done:

Theorem 5.11. $RP_{\text{internal}}$ implies that for all rings $R$ of size at most $\aleph_1$ and all quotient-hereditary classes $C$ of countably presented $R$-modules, the class of $R$-modules that are not $\sigma$-closed potentially $C$-filtered is (under the pure embeddability order) an AEC. Moreover, its Löwenheim-Skolem number is at most $\aleph_1$.

In particular, $\Phi$ holds and the Löwenheim-Skolem number is $\aleph_1$ (this is the special case where $R = \mathbb{Z}$ and $C = \{R\}$).

Proof. By Lemma 4.4, the class in the statement of the theorem is the same as the class of modules for which Player 2 has no winning strategy in the $C$-filtration game of length $\omega_1$. By Theorems 5.2 and 5.4, this game is determined, and hence the class from the previous sentence is the same as the class of modules for which Player 1 has a winning strategy in that game. And since $C$ is quotient-hereditary, Theorem 1.4 ensures this class, under the pure embeddability order, is always an AEC. Ordinarily, $2^{\aleph_0}$ is the best-known upper bound for the Löwenheim-Skolem number, but Theorem 5.2 ensures that every non-$\sigma$-closed potentially $C$-filtered module has another such submodule of size $\aleph_1$. So the Löwenheim-Skolem number of the AEC is $\aleph_1$. □

To show that $\neg \Phi$ is relatively consistent with ZFC, we will use:

Theorem 5.12 (Mekler et al. [20]). Suppose $\kappa$ is an infinite cardinal and Jensen’s $\square_\kappa$ principle holds. Then there exists an abelian group $G$ of cardinality $\kappa^+$ that is $\kappa^+$-free (i.e. every $< \kappa^+$-sized subgroup of $G$ is free), but
the game

\[ EF_{\omega_1}(F_{\omega_1}, G) \]

is not determined, where \( F_{\omega_1} \) denotes the free abelian group of size \( \omega_1 \).

**Remark 5.13.** The proof of Theorem 5.12 from [20] only dealt with the case \( \kappa = \omega_1 \), but the proof goes through for any \( \kappa \).

**Theorem 5.14.** Suppose that Jensen’s \( \square_\kappa \) principle holds for class-many \( \kappa \) (failure of this requires consistency of large cardinals). Then \( \Phi \) fails.

**Proof.** For brevity, let \( \Gamma_{-p_2} \) be the class of abelian groups such that Player 2 has no winning strategy in the Free Filtration Game of length \( \omega_1 \) on \( G \). We show that the Löwenheim-Skolem requirement (vi) from Definition 4.6 fails for this class (under the pure subgroup order).

Suppose toward a contradiction that \( \Gamma_{-p_2} \) did have an Löwenheim-Skolem number, say, \( \mu \). By assumption, there is a \( \kappa \) larger than \( \mu \) such that Jensen’s \( \square_\kappa \) holds. By Theorem 5.12 there is an almost free abelian group \( G \) of cardinality \( \kappa^+ \) such that \( EF_{\omega_1}(G, F_{\omega_1}) \) is not determined. Then by Corollary 4.17, the Free Basis game of length \( \omega_1 \) on \( G \) is not determined; so, in particular, Player 2 does not have a winning strategy in the Free Basis game of length \( \omega_1 \) on \( G \). Since \( G \) is almost free and \( |G| = \kappa^+ \), however, every \( \leq \kappa \)-sized subgroup of \( G \) is free. In particular, every \( \mu \)-sized subgroup of \( G \) is free, and hence (trivially) \( \sigma \)-closed potentially free. In other words, \( G \in \Gamma_{-p_2} \) but no \( \mu \)-sized subgroup of \( G \) is in \( \Gamma_{-p_2} \). This contradicts our assumption that \( \mu \) is the Löwenheim-Skolem number for the class \( \Gamma \). \( \square \)

### 6. Concluding remarks

The Weak Reflection Principle (WRP) is defined just like \( \text{RP}_{\text{internal}} \) (Definition 5.1), except that one only requires \( S \cap [W \cap X]^\omega \) to be stationary, rather than requiring \( S \cap [W \cap X]^\omega \) to be stationary. Clearly \( \text{RP}_{\text{internal}} \) implies WRP, but it is not known if the converse holds. Note that the “internality” part of \( \text{RP}_{\text{internal}} \) was used at crucial point near the end of the proof of Theorem 5.2 we used that \( Z_i \in W \) to ensure that \( C_{Z_i} \in W \), which was needed to conclude that \( C_{Z_i} \cap [W \cap M]^\omega \) contained a club. Furthermore, \( \text{RP}_{\text{internal}} \) can be shown to be equivalent to a kind of downward Löwenheim-Skolem-Tarki property for certain formulas in stationary logic, and the internality is used at crucial places in the equivalence (via an argument similar to the main theorem of Cox [6]).

These observations raise, yet again, an open problem that has appeared in similar forms in other places (e.g. Beaudoin [2], Krueger [17], and Cox [5]). The principles \( \text{RP}_{\text{IS}} \) and \( \text{RP}_{\text{IU}} \) are further variants whose definition we will not provide (see Krueger [17]); the principle \( \text{RP}_{\text{IU}} \) is equivalent to Fleissner’s Axiom R, by Fuchino-Usuba [13]. The following implications are known:

\[(14) \quad \text{RP}_{\text{internal}} \implies \text{RP}_{\text{IS}} \implies \text{RP}_{\text{IU}} \implies \text{WRP}.\]
Question 6.1. Which pairs of principles, if any, from the chain (14) are equivalent?

It is currently not known if any pair from (14) is equivalent. An obviously related question is:

Question 6.2. Do any of the consequences of $RP_{internal}$ from this paper follow from WRP?

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*Email address: scox9@vcu.edu*

Department of Mathematics and Applied Mathematics, Virginia Commonwealth University, 1015 Floyd Avenue, Richmond, Virginia 23284, USA