DARBOUX COORDINATES AND LIOUVILLE-ARNOLD INTEGRATION IN LOOP ALGEBRAS

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Abstract

Darboux coordinates are constructed on rational coadjoint orbits of the positive frequency part $\tilde{g}^+$ of loop algebras. These are given by the values of the spectral parameters at the divisors corresponding to eigenvector line bundles over the associated spectral curves, defined within a given matrix representation. A Liouville generating function is obtained in completely separated form and shown, through the Liouville-Arnold integration method, to lead to the Abel map linearization of all Hamiltonian flows induced by the spectral invariants. The results are formulated in terms of sheaves to allow for singularities due to a degenerate spectrum. Serre duality is used to define a natural symplectic structure on the space of line bundles of suitable degree over a permissible class of spectral curves, and this is shown to be equivalent to the Kostant-Kirillov symplectic structure on rational coadjoint orbits, reduced by the group of constant loops. A similar construction involving a framing at infinity is given for the nonreduced orbits. The general construction is given for $g = gl(r)$ or $sl(r)$, with reductions to orbits of subalgebras determined as invariant fixed point sets under involutive automorphisms. As illustrative examples, the case $g = sl(2)$, together with its real forms, is shown to reproduce the classical integration methods for finite dimensional systems defined on quadrics, with the Liouville generating function expressed in hyperellipsoidal coordinates, as well as the quasi-periodic solutions of the cubically nonlinear Schrödinger equation. For $g = sl(3)$, the method is applied to the computation of quasi-periodic solutions of the two component coupled nonlinear Schrödinger equation. This case requires a further symplectic constraining procedure in order to deal with singularities in the spectral data at $\infty$.

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Introduction.

In a series of recent papers [AHP, AHH1, AHH2] isospectral Hamiltonian flows on rational coadjoint orbits of loop algebras arising from the Adler, Kostant, Symes (AKS) theorem were studied. A systematic way of representing both finite dimensional integrable Hamiltonian systems and quasi-periodic solutions of integrable PDE’s within this framework was developed using moment map embeddings. A representation of such flows in terms of rational matricial functions of the loop parameter determines an invariant spectral curve and associated linear flows of eigenvector line bundles [vMM, AvM]. The Dubrovin-Krichever-Novikov technique [K, KN, Du] may be used to solve the equations of motion in terms of theta functions [RS, AHH1], but the Hamiltonian content of this approach is not evident.

The link between the algebro-geometric method of integration and the Hamiltonian point of view was made within the context of differential algebras by Gel’fand and Dickey [GD, D], the linearization being based upon the Liouville quadrature method. Within the loop algebra setting, however, the symplectic content of the algebro-geometric integration method has until now only been developed in specific examples, without a general theory encompassing all generic cases. Certain aspects, of course, are easy to see; the constants of motion, for example, are essentially the coefficients of the defining equations of the spectral curves, and their level sets are affine subvarieties of the associated Jacobians. The algebro-geometric approach thus gives a foliation of the phase space by tori, which may be seen in the symplectic framework as Lagrangian. However the Hamiltonian content of the actual integration procedure requires further clarification within the general framework of loop algebras. (For related developments in the case of rapidly decreasing boundary conditions, see [FT, BS1, BS2], where the link between the scattering transform and Darboux coordinates on Poisson Lie groups is developed.)

In this paper we use the Lie algebraic and algebro-geometric structures associated with isospectral flows of the AKS type on the dual $\tilde{\mathfrak{g}}^{+,*}$ of the positive frequency half of a loop algebra $\tilde{\mathfrak{g}}$ to introduce a set of Darboux coordinates on rational coadjoint orbits: the “spectral divisor coordinates” associated with line bundles over spectral curves embedded in a specific ruled surface. These essentially give a complete separation of variables in the Hamilton-Jacobi problem for all AKS flows on such orbits - or, more precisely, a Liouville generating function for the canonical transformation to linearizing coordinates in completely separated form. The transformation is expressed in terms of abelian integrals and hence this procedure derives the Abel map linearization within the Hamiltonian setting provided by loop algebras, with the Liouville-Arnold
torus identified with the Jacobi variety of the underlying spectral curve. The general approach is developed only for the loop algebras $\tilde{\mathfrak{gl}}(r)^+$, $\tilde{\mathfrak{sl}}(r)^+$, but the integrated AKS flows for more general loop algebras may be obtained by restricting to invariant symplectic submanifolds obtained as fixed point sets under involutive automorphisms.

Section 1 gives the classical Hamiltonian approach to the introduction of spectral divisor coordinates on rational coadjoint orbits and the construction of the completely separated Liouville generating function (eq. (1.76)). Certain partial reductions of the orbits that arise in most applications are also dealt with. This section is essentially classical in spirit, and may in principle be understood without going beyond the tools of nineteenth century mathematics (Riemann surfaces, abelian integrals, Lie Poisson structures and Hamilton-Jacobi theory). The only additional notion needed as a unifying factor is that of finite dimensional Poisson subspaces of the dual of a loop algebra, consisting of elements that are rational in the loop parameter. Such spaces can, however, be given a natural Poisson structure without any further familiarity with loop algebras.

More specifically, the spectral curve $\mathcal{S}$ of $\mathcal{N}(\lambda)$, an $r \times r$ matrix valued rational function of the complexified loop parameter $\lambda$, is determined by the characteristic equation

$$\det(\mathcal{N}(\lambda) - \lambda \zeta I) = 0,$$

and so sits naturally in $\mathbb{C}^2$ or, after compactification, in an ambient ruled surface $\mathcal{T}$. The line bundle corresponding to $\mathcal{N}(\lambda)$ is associated to a divisor on the curve $\mathcal{S}$, given by the finite zeroes of a component of the eigenvector, and hence determines a set of functions given by the corresponding coordinate pairs $(\lambda_\mu, \zeta_\mu)_{\mu=1,...,g}$ ($g = \text{genus of } \mathcal{S}$) on the ambient surface. (The bundle in question is actually of degree $g+r-1$, but the associated divisor can be normalized with $r-1$ points chosen over $\lambda = \infty$.) The key point in understanding the link between the symplectic and algebraic geometry is that the set of functions $(\lambda_\mu, \zeta_\mu)_{\mu=1,...,g}$ turn out in general to form a Darboux coordinate system on a reduced version of the coadjoint orbits (Theorem 1.4), and can very simply be augmented using invariants related to the spectrum over $\lambda = \infty$ to form a Darboux system on the full orbits (Theorem 1.5). From this fact, the canonical linearization of flows induced by the spectral invariants leads, through the Liouville-Arnold method (eqs. (1.82a,b)), to the Abel map (Theorem 1.6) and hence, to $\theta$-function formulae for the integrated flow (Corollary 1.7).

The development of Section 1 is essentially self-contained and sufficiently explicit to lead directly to the examples of Section 3, but a deeper understanding of the underlying approach requires the constructions of Section 2. These are aimed at an intrinsic
explanation of the underlying symplectic structure from an algebro-geometric viewpoint and involve some more modern machinery, such as Serre duality. It also is convenient to formulate the results in terms of sheaves, in order to have a setting in which smooth and singular curves may be treated on an equal footing. Such a generalization is required to allow for the types of spectra occurring in the rational matrix valued functions that arise in some of the more interesting applications, such as the coupled nonlinear Schrödinger (CNLS) equation.

The naturally defined class of spectral divisor coordinates, leads to the following intrinsic characterization, identifying the (Lie algebraic) Kostant-Kirillov symplectic form on rational coadjoint orbits in terms of purely algebro-geometric data. On the reduced orbits, the infinitesimal variations of the spectral curves correspond to sections of the normal bundle of $S \subset T$, constrained to vanish at the poles of $N(\lambda)$. The space of such sections may be identified with $V := H^0(S, K_S)$; i.e. to sections of the canonical bundle. On the other hand, variations of the line bundles, representing the tangent space to the isospectral foliation, are given by the cohomology group $W := H^1(S, O)$, and Serre duality tells us that $W = V^*$. The tangent space to the reduced orbit is thus identified with $V \oplus V^*$, which leads to a natural symplectic form corresponding to this decomposition. The remarkable fact is that this coincides with the Kostant-Kirillov form (Theorem 2.7) on the orbits reduced under the action of the subgroup of constant loops. A similar construction, involving supplementary data over $\lambda = \infty$, holds on the nonreduced orbits as well (Theorem 2.8). The AKS theorem, together with this identification between Lie algebraic and algebro-geometric symplectic forms, seems to be at the root of the ubiquitous presence of algebro-geometric constructions in the theory of integrable systems.

Although the main contents of Sections 1 and 2 concern the loop algebras $\widetilde{\mathfrak{gl}}(r)^+$ or $\widetilde{\mathfrak{sl}}(r)^+$, reductions to other algebras, obtained as fixed point sets under involutive automorphisms, are also placed in the symplectic framework (Theorems 2.9-2.11). This allows a determination of the flows induced by spectral invariants through restriction to the corresponding invariant symplectic submanifolds, but does not produce an intrinsic formulation in terms of separating Darboux coordinates on the coadjoint orbits of these subalgebras. The analogous coordinates have yet to be derived in the general case, though one obtains, by restriction from the case $\widetilde{\mathfrak{gl}}(r)^+$, extensions of the results of Section 2 applicable to these subalgebras.

In Section 3, these results are illustrated in a number of examples. As a first application, Darboux coordinates are computed on generic coadjoint orbits of the Lie algebras $\mathfrak{sl}(2, \mathbb{C})$ and $\mathfrak{sl}(3, \mathbb{C})$, viewed as rational coadjoint orbits of the corresponding
loop algebras having only one simple pole. Next, the $\tilde{\mathfrak{sl}}(2)$ case with $n$ simple poles is shown to reproduce the standard linearization results for well-known classical examples of finite dimensional systems (cf. [M]). For this case, the “spectral divisor coordinates” are essentially just the hyperellipsoidal coordinates, the reductions corresponding to fixing spectral data at $\infty$ lead to constraints defined by quadrics, and the spectral curve is always hyperelliptic. This case also includes the “finite gap” quasi-periodic solutions of familiar systems of PDE’s such as the cubically nonlinear Schrödinger (NLS) equation. The higher rank “spectral divisor coordinates” are thus really generalizations of hyperellipsoidal coordinates.

Finally, as an illustration of the $\tilde{\mathfrak{sl}}(3)$ case, involving trigonal curves, the finite gap solutions of the coupled 2-component nonlinear Schrödinger (CNLS) equation are also obtained by the Liouville-Arnold integration technique. In this case the particular structure of the spectrum at infinity leads to further singularities in the curve, and hence a decrease of the arithmetic genus relative to the generic case, and an incomplete set of Darboux coordinates on the coadjoint orbit. This example is used to indicate how such problems may be dealt with by restricting to an invariant symplectic submanifold on which the spectral curves share the same generic type of singularities. The linearization on the constrained manifold then proceeds in the same way as in the unconstrained case.

Background and Acknowledgements: For a more complete account of the moment map construction leading to isospectral flows on rational coadjoint orbits and the algebro-geometric method of integration, the reader should consult [AHP, AHH1, AHH2]. The present work is the first complete account of the spectral Darboux coordinate construction and the Liouville–Arnold integration method on loop algebras, but earlier summaries and announcements of the main results communicated at various conferences and workshops may be found in [H, AHH3, AHH4, AHH5]. The authors are pleased to acknowledge helpful discussions with L. Dickey, B. Dubrovin, H. Flaschka, P. van Moerbeke, E. Previato and A. Reyman relating to this material.
1. Darboux Coordinates and Linearization of Flow.

1a. Rational Orbits and Spectral Curves

The Hamiltonian systems to be considered here involve isospectral flows of matrices determined by equations of Lax type:

\[ \frac{d\mathcal{N}(\lambda)}{dt} = [\mathcal{B}(\lambda), \mathcal{N}(\lambda)], \]

where \( \mathcal{N}(\lambda), \mathcal{B}(\lambda) \) are \( r \times r \) matrices depending on a complex parameter \( \lambda \). The matrix \( \mathcal{N}(\lambda) \) is taken to be of the form

\[ \mathcal{N}(\lambda) = \lambda Y + \lambda \sum_{i=1}^{n} \frac{N_i}{\lambda - \alpha_i}, \]

where \( Y \in \mathfrak{gl}(r) \), and \( \{\alpha_i \in \mathbb{C}\}_{i=1,...,n} \) are constants. Thus, we are considering rational \( \mathcal{N}(\lambda) \) with fixed, simple poles at the finite points \( \{\alpha_i\} \) and possibly at \( \infty \). Rational matrices with higher order poles may be dealt with similarly, but will not be considered here for the sake of notational simplicity.

The particular form (1.2) arises naturally as the translate by \( \lambda Y \) of the image

\[ \mathcal{N}_0(\lambda) = \lambda \sum_{i=1}^{n} \frac{N_i}{\lambda - \alpha_i} \]

of a moment map from a symplectic vector space parametrizing rank–\( r \) perturbations of a fixed \( N \times N \) matrix with eigenvalues \( \{\alpha_i\}_{i=1,...,n} \) into the dual \( (\mathfrak{g}^+)^* \) of a loop algebra, represented by \( r \times r \) matrix functions of the complexified loop parameter \( \lambda \), holomorphic in a suitable domain [AHP, AHH2]. This serves to embed a large class of integrable systems as Lax pair flows in \( (\mathfrak{g}^+)^* \). The image space for such maps is a Poisson subspace of \( (\mathfrak{g}^+)^* \), with respect to the Lie Poisson structure, the symplectic leaves (coadjoint orbits) consisting of rational functions of \( \lambda \). Since a specific \( r \times r \) matrix representation is involved, we view \( \mathfrak{g} \) as a subalgebra of \( \mathfrak{gl}(r, \mathbb{C}) \) or \( \mathfrak{sl}(r, \mathbb{C}) \), obtained generally by reductions under involutive automorphisms (cf. Section 2c).

The loop algebra elements \( X \in \tilde{\mathfrak{gl}}(r) \) are viewed as smooth maps \( X : S^1 \mapsto \mathfrak{gl}(r) \) from a fixed circle \( S^1 \) in the complex \( \lambda \) - plane, containing the points \( \{\alpha_i\} \) in its interior, and the subalgebra \( \tilde{\mathfrak{gl}}(r^+) \) consists of those \( X(\lambda) \) that extend as holomorphic functions to the interior of \( S^1 \). The loop group \( \tilde{\mathcal{G}}l(r) \) similarly consists of smooth maps \( g : S^1 \mapsto \mathcal{G}l(r) \), while the subgroup \( \tilde{\mathcal{G}}l(r^+) \) consists again of those \( g(\lambda) \) that extend holomorphically inside \( S^1 \). The subspace \( \tilde{\mathfrak{g}}l(r)_- \subset \tilde{\mathfrak{g}}l(r) \) of loops extending
holomorphically outside $S^1$ to $\infty$ is identified with a dense subspace of the dual space $\tilde{\mathfrak{gl}}(r)^{++}$ through the dual pairing:

$$< \mu, X > := \frac{1}{2\pi i} \oint_{S^1} \text{tr} (\mu(\lambda)X(\lambda)) \frac{d\lambda}{\lambda},$$  \hspace{1cm} (1.4)

$$\mu \in \tilde{\mathfrak{gl}}(r), \; X \in \tilde{\mathfrak{gl}}(r)^{+}.$$  

The matrix $B(\lambda)$ has the form:

$$B(\lambda) = (d\Phi(N(\lambda)))_+,$$  \hspace{1cm} (1.5)

where $\Phi \in \mathcal{I}(\tilde{\mathfrak{gl}}(r)^*)$ is an element of the ring of $Ad^*$-invariant polynomials on $\tilde{\mathfrak{gl}}(r)^*$ and the subscript $+$ means projection to the subspace $\tilde{\mathfrak{gl}}(r)^{+}$. In general, no notational distinction will be made between $\tilde{\mathfrak{gl}}(r)^{++}$ and $\tilde{\mathfrak{gl}}(r)^{-}$. The coadjoint action of $\tilde{\mathfrak{gl}}(r)^{+}$ on rational elements $N_0$ of the form (1.3) is given by:

$$g : \tilde{\mathfrak{gl}}(r)^{-} \rightarrow \tilde{\mathfrak{gl}}(r)^{-}$$

$$g : \lambda \sum_{i=1}^{n} \frac{N_i}{\lambda - \alpha_i} \mapsto \lambda \sum_{i=1}^{n} \frac{g(\alpha_i)N_i g(\alpha_i)^{-1}}{\lambda - \alpha_i}.$$  \hspace{1cm} (1.6)

Equation (1.1) is Hamilton’s equation on the coadjoint orbit $Q_{N_0} \subset \tilde{\mathfrak{gl}}(r)^{++}$, with respect to the orbital (Kostant-Kirillov) symplectic form $\omega_{\text{orb}}$, corresponding to the Hamiltonian:

$$\phi(\mu) = \Phi(\mu + \lambda Y).$$  \hspace{1cm} (1.7)

The Poisson commutative ring of such functions on $Q_{N_0}$ will be denoted $\mathcal{F}_Y$. According to the “shifted” version [FRS] of the Adler-Kostant-Symes theorem [A, Ko, S], such systems generate commuting Lax pair flows. Moreover, they may be shown to be completely integrable on “generic” coadjoint orbits [RS, AHP, AHH1] in $\tilde{\mathfrak{gl}}(r)^{++}$. On such orbits, the (AKS) ring of commuting invariants is generated by the coefficients of the characteristic polynomial of $N(\lambda)$.

In analyzing the spectrum, it is convenient to deal with matricial polynomials in $\lambda$, so we define

$$\hat{L} := \frac{a(\lambda)}{\lambda} N(\lambda)$$

$$= Ya(\lambda) + L_0 \lambda^{n-1} + \cdots + L_{n-1},$$  \hspace{1cm} (1.8)

where

$$a(\lambda) := \prod_{i=1}^{n} (\lambda - \alpha_i).$$  \hspace{1cm} (1.9)
The matrix
\[ L_0 = \lim_{\lambda \to \infty} N_0(\lambda) = \sum_{i=1}^{n} N_i \] (1.10)
may be viewed as a moment map generating the conjugation action of \( Gl(r) \) on \( \tilde{\mathfrak{gl}}(r)^{++} \):
\[ Gl(r) \times \tilde{\mathfrak{gl}}(r)^{++} \to \tilde{\mathfrak{gl}}(r)^{++} \]
\[ (g, X(\lambda)) \mapsto gX(\lambda)g^{-1}. \] (1.11)
The matrix \( \hat{L} \) satisfies the same Lax equation (1.1) as \( N(\lambda) \), and the coefficients of its characteristic polynomial:
\[ P(\lambda, z) := \det(\hat{L}(\lambda) - zI) \] (1.12)
generate the same ring of invariants as that of \( N(\lambda) \).

Remark: It is also possible to view
\[ \lambda^{-n+1}[\hat{L} - Ya(\lambda)] := \hat{L}(\lambda) \] (1.13)
directly as an element of an orbit in \( \tilde{\mathfrak{gl}}(r)^{++} \) (polynomial in \( \lambda^{-1} \)). Since the ring of invariants is the same, the results are equivalent, with a suitable redefinition of the Hamiltonians and parametrization of the spectral curve (cf. [AHP]). We retain our present conventions, with \( N_0(\lambda) \) viewed as the point in \( \tilde{\mathfrak{gl}}(r)^{++} \) undergoing Hamiltonian flow, since these are adapted to examining the particular spectral constraints occurring at the finite values \( \{\lambda = \alpha_i\} \) that appear in specific examples (cf. Section 3).

The spectral curve \( S_0 \subset \mathbb{C}^2 \) defined by the characteristic equation
\[ P(\lambda, z) = 0 \] (1.14)
is invariant under the AKS Hamiltonian flows. Let \( m \) be the degree of \( \hat{L}(\lambda) \), \( (m = n \text{ if } Y \neq 0 \text{ or } m = n - 1 \text{ if } Y = 0) \) and let \( \{k_i\} \) denote the ranks of the matrices \( \{N_i\}_{i=1}^{n} \) in (1.2) (coadjoint invariants, and hence invariants of any Hamiltonian flow in \( Q_{N_0} \)).

Lemma 1.1. The spectral polynomial \( P(\lambda, z) \) has the form:
\[ P(\lambda, z) = (-z)^r + z^{r-1}P_1(\lambda) + \sum_{j=2}^{r} A_j(\lambda)P_j(\lambda)z^{r-j}, \] (1.15)
where
\[ A_j(\lambda) := \prod_{i=1}^{n} (\lambda - \alpha_i)^{\max(0, j-k_i)} \] (1.16)
and
\[ \text{deg } P_j(\lambda) = \sum_{i=1}^{n} \min(j, k_i) - j(n - m) =: \kappa_j \] (1.17)

**Remark:** This means that \( P(\lambda, z) \), and all its partial derivatives in \( \lambda \) or \( z \) up to order \( r - k_i - 1 \) vanish at \((\alpha_i, 0)\).

**Proof.** This follows immediately by expanding \( \text{det}(\hat{L}(\lambda) - zI) \) and using the fact that \( \hat{L}(\alpha_i) = N_i \prod_{j=1, j \neq i}^{n} (\alpha_i - \alpha_j) \) has rank \( k_i \).

The structure of \( P(\lambda, z) \) implies that on \( S_0 \), \( z \sim O(\lambda^m) \) as \( \lambda \to \infty \). This suggests assigning \( z \) a homogeneity degree \( m \), thereby giving \( P(\lambda, z) \) an overall degree \( rm \). We may then compactify \( S_0 \), regarding it as the affine part of an \( r \)-sheeted branched cover of \( P^1 \), by embedding it in the total space \( T \) of \( O(m) \), the \( m \)th power of the hyperplane section bundle over \( P^1 \), whose sections are homogeneous functions of degree \( m \) (cf. [AHH1] and Sec. 2). The pair \((\lambda, z)\) is viewed as the base and fibre coordinates over the affine neighborhood \( U_0 := \pi^{-1}(P^1 - \{\infty\}) \). Over \( U_1 := \pi^{-1}(P^1 - \{0\}) \), we have coordinates \((\tilde{\lambda}, \tilde{z})\) related to \((\lambda, z)\) on \( U_0 \cap U_1 \) by:
\[ \tilde{\lambda} = \frac{1}{\lambda}, \quad \tilde{z} = \frac{z}{\lambda^m}. \] (1.18)

Re-expressing (1.14) as a polynomial equation in \((\tilde{\lambda}, \tilde{z})\) extends \( S_0 \) to \( U_1 \), thereby defining its compactification \( S \subset T \).

Let us assume that \( S \) has no multiple components. Let \((\lambda_0, z_0)\) belong to \( S \), and suppose that the multiplicity of the eigenvalue \( z_0 \) of \( \hat{L}(\lambda_0) \) is \( k > 1 \). It follows from the constructions of [AHH1] (cf. also Section 2) that there is a partial desingularisation \( \tilde{S} \) of \( S \) such that, generically, the number of points (with multiplicity) in \( \tilde{S} \) over \((\lambda_0, z_0)\) equals the number of Jordan blocks of \( \hat{L}(\lambda_0) \) with eigenvalue \( z_0 \), and \( \tilde{S} \) is smooth over \((\lambda_0, z_0)\). If, for example, \( \hat{L}(\lambda_0) \) has only one Jordan block of size \( k \) with eigenvalue \( z_0 \), then \( \tilde{S} = S \) and, generically, \( S \) has a smooth \( k \)-fold branch point over \( P^1 \). In the opposite extreme, if \( \hat{L}(\lambda_0) \) has \( k \) independent eigenvectors with eigenvalue \( z_0 \) then generically there are \( k \) points (with multiplicity) over \((\lambda_0, z_0)\) and \( S \) has a \( k \)-fold node.

These remarks are of particular importance when \( \lambda_0 = \alpha_i \), since the Jordan form of \( \hat{L}(\alpha_i) \) is an invariant of the coadjoint orbit. Thus, if \( \hat{L}(\alpha_i) \) is diagonalisable with multiple eigenvalues, the generic spectral curve for the orbit will be singular.

**Genericity Conditions**

In what follows, we only consider the singularities that follow from the specific structure (1.2), (1.8) assumed for \( \hat{L}(\lambda_0) \). We shall make the simplifying assumption
that the $N_i$ (and hence $\hat{L}(\alpha_i)$) are diagonalizable, with the only multiple eigenvalue being $z = 0$, with multiplicity $r - k_i$. This property is, of course, “generic” for orbits with rank($N_i$) = $k_i$, but is only assumed in order to simplify the exposition. If other Jordan forms are allowed for the $N_i$’s, the only effect is to change the specific form (1.22) for the spectral polynomial $P(\lambda, z)$, (1.27) for the genus formula determining the dimension of $Q_{N_0}$ and the explicit expressions (1.83), (1.85) for the abelian differentials. All these can easily be modified to hold for other cases. The main results, contained in Theorems 1.3-1.6, Corollary 1.7 and the subsequent sections, remain valid mutatis mutandis.

We also assume that one of the following two conditions hold:

Case (i): $Y = 0$ and $L_0$ has a simple spectrum ($m = n - 1$).

Case (ii): $Y \neq 0$ and has a simple spectrum ($m = n$).

Again, these conditions are generic and invariant on coadjoint orbits, but in section 3 it will be indicated how they may be relaxed.

Finally, we make a further spectral genericity assumption regarding the singularities of the curve $S$; namely, that the only singularities occur at the points $(\alpha_i, 0)$, where there is an $r - k_i$-fold node with $r - k_i$ distinct branches intersecting transversally. This amounts to requiring that the eigenspaces of $\hat{L}(\lambda)$ all be 1-dimensional except at $\lambda = \alpha_i$, where, by the structure of $N_0(\lambda)$, the eigenvalue $z = 0$ has an eigenspace of dimension $r - k_i$. The desingularization $\tilde{S}$ is then smooth and is isomorphic to $S$ away from $(\alpha_i, 0)$. This condition is generic in the space of $N_0$’s of the form (1.3) and, if satisfied at any point of $Q_{N_0}$, it is also valid in a neighborhood of the isospectral manifold through that point. (In particular, it is invariant under the AKS flows.)

The coefficients of the polynomials $P_j(\lambda)$ generate the AKS ring on each coadjoint orbit $Q_{N_0} \subset \tilde{gl}(r)^{++}$ and should be viewed as functions on the Poisson submanifold consisting of rational elements of the form (1.3) (with rank($N_i$) = $k_i$). Note that

$$P_1(\lambda) = \text{tr}\hat{L}(\lambda), \quad (1.19)$$

and hence its coefficients are Casimir invariants (i.e. constants on all coadjoint orbits). The nonzero eigenvalues $\{z_{ik}\}_{i=1,...,n, k=1,...,k_i}$ over the points $\{\lambda = \alpha_i\}_{i=1,...,n}$ are also Casimir invariants, since they are determined as the nonzero roots of the characteristic equation:

$$\det[N_i \prod_{j=1, j\neq i}^n (\alpha_i - \alpha_j) - zI] = 0, \quad (1.20)$$
which is invariant under the coadjoint action (1.6). The \( N := \sum_{i=1}^{n} k_{i} \) trivial invariants \( \{ z_{\kappa i} \} \) determine, in particular, the coefficients of \( P_{1}(\lambda) \), since
\[
\sum_{\kappa=1}^{k_{i}} z_{\kappa i} = \text{tr} \hat{\mathcal{L}}(\alpha_{i}) = P_{1}(\alpha_{i}), \quad i = 1, \ldots n.
\] (1.21)
(For case (i), this is sufficient to determine the degree \( n - 1 \) polynomial \( P_{1}(\lambda) \); for case (ii), the degree \( n \) coefficient is just \( \text{tr} Y \).) This may all be summarized by noting that the spectral curves \( \tilde{S} \) on the orbit \( Q_{N_{0}} \) are constrained to pass through the \( N + n \) points \( \{(\alpha_{i}, z_{\kappa i}), (\alpha_{i}, 0)\} \), with \( r - k_{i} \) branches intersecting at the singular points \( \{(\alpha_{i}, 0)\} \), the values \( \{ z_{\kappa i} \} \) being fixed. It should also be noted that for \( Y \neq 0 \) the leading (deg \( \kappa_{i} \) terms in the polynomials \( P_{j}(\lambda) \) are constants, determined entirely by the symmetric invariants of \( Y \). For \( Y = 0 \), the leading terms are not constants, but they are determined as symmetric invariants of \( L_{0} \), and hence are constant on its level sets.

A way to express \( P(\lambda, z) \) in terms of independent, non-Casimir invariants is to choose a reference point \( N_{R} \in Q_{N_{0}} \) on the orbit and parametrize the difference between \( P(\lambda, z) \) and its value \( P_{R}(\lambda, z) \) at \( N_{R} \).

**Proposition 1.2.** In a neighborhood of the point \( N_{R} \in Q_{N_{0}} \), the characteristic polynomial has the form:
\[
P(\lambda, z) \equiv P_{R}(\lambda, z) + a(\lambda) \sum_{j=2}^{r} a_{j}(\lambda)p_{j}(\lambda)z^{r-j}
\] (1.22)
where
\[
a_{j}(\lambda) = \prod_{i=1}^{n}(\lambda - \alpha_{i})^{\max(0, j - k_{i} - 1)},
\] (1.23)
\[
p_{j}(\lambda) =: \sum_{a=0}^{\delta_{j}} P_{ja}\lambda^{a}
\] (1.24)
and \( \{p_{j}(\lambda)\}_{j=1,\ldots,r} \) are polynomials of degree:
\[
\delta_{j} \equiv \text{deg } p_{j}(\lambda) = \begin{cases} d_{j} - j & \text{if } Y = 0 \\ d_{j} & \text{if } Y \neq 0 \end{cases}
\] (1.25a)
\[
d_{j} \equiv \sum_{i=1}^{n} \min(j - 1, k_{i}).
\] (1.25b)

For \( Y = 0 \), the leading coefficients \( P_{j\delta_{j}} \) are constant translates of the elementary symmetric invariants of \( L_{0} \), while for \( Y \neq 0 \), the leading coefficients \( P_{j\delta_{j}} \) are
all constants; namely, the elementary symmetric invariants of $Y$ (translated by the corresponding leading terms in $P_R(\lambda, z)$). The number of spectral parameters $\{P_{ja}\}$, $(a = 0, \ldots, \delta_j + n - m - 1, \ j = 2, \ldots, r)$ defining the polynomials $p_j(\lambda)$ on generic orbits is thus:

$$d = \sum_{j=2}^{r} (d_j - (n - m)(j - 1))$$

$$= \tilde{g} + r - 1,$$

where

$$\tilde{g} = \frac{1}{2}(r - 1)(mr - 2) - \frac{1}{2} \sum_{i=1}^{n} (r - k_i)(r - k_i - 1).$$

In a neighborhood of any generic point on $Q_{N_0}$, these spectral invariants are all independent.

**Proof.** The structure of $P(\lambda)$ follows Lemma 1.1, plus the fact that $P(\lambda, z) - P_R(\lambda, z)$ vanishes at each $\lambda = \alpha_i$, while $z$ vanishes at least linearly in $\lambda - \alpha_i$ along each branch through $(\alpha_i, 0)$. From formula (1.6) and the above genericity conditions regarding the residues $N_i$, the dimension of the coadjoint orbit $Q_{N_0}$ is easily computed to be $2d$. From the proof of complete integrability of the AKS flows on such orbits given in [AHH1], it follows that the isospectral foliation is Lagrangian, and hence the $d$ spectral parameters $\{P_{ja}\}$ are independent. The expression of $P_j \delta_j$ in terms of the elementary symmetric invariants of $L_0$ or $Y$ follows directly from the fact that the leading term in $\hat{L}(\lambda)$ in eq. (1.12) is either $L_0 \lambda^{n-1}$ or $Y \lambda^n$.

It follows from the adjunction formula applied to the curve $\tilde{S}$ obtained by blowing up $\mathcal{T}$ once at each point $(\alpha_i, 0)$ (cf. [AHH1, GH]) that $\tilde{g}$ in eq. (1.27) is also equal to the (arithmetic) genus of $\tilde{S}$. If we reduce such an orbit under the $\text{Gl}(r, \mathbb{C})$ action (1.9) for case (i), or the action of the stabilizer $G_Y \subset \text{Gl}(r, \mathbb{C})$ of $Y$ for case (ii), the dimension of the reduced space is precisely $2\tilde{g}$, and the projected spectral invariants again define completely integrable Hamiltonian systems [AHH1]. These facts suggest exploiting the orbital symplectic structure further so as to explicitly integrate the isospectral flows via Hamiltonian methods. This will be the content of the following subsections.

**1b. Divisor Coordinates on Reduced Orbits**

Define

$$\mathcal{K}(\lambda, z) := \hat{L}(\lambda) - zI,$$

(1.28)
and let $\widetilde{K}(\lambda, z)$ denote its classical adjoint (matrix of cofactors). Let $V_0 \in \mathbb{C}^r$ be an eigenvector of $L_0$ in case (i), or of $Y$ in case (ii). From the results of [AHH1], it follows that the set of polynomial equations:

$$\widetilde{K}(\lambda, z)V_0 = 0 \quad (1.29)$$

have, away from $(\alpha_i, 0)$, precisely $\tilde{g}$ generically distinct finite solutions $\{(\lambda_\mu, z_\mu)\}_{\mu=1,...,\tilde{g}}$ that may be viewed as functions on the coadjoint orbit $Q_{N_0}$. (Changing to the coordinates $(\tilde{\lambda}, \tilde{z})$, there are also $r - 1$ further solutions with $\tilde{\lambda} = 0$, i.e., $\lambda = \infty$. If $V_0$ is not chosen as an eigenvector of $L_0$ or $Y$, the remaining $r - 1$ solutions will generically also be at finite values of $(\lambda, z)$.)

The significance of these functions in terms of the algebraic geometry of the spectral curves $\tilde{S}$ may be summarized as follows (cf. [AHH1] and Section 2a below for the detailed construction). To each matricial polynomial $\hat{L}(\lambda)$ is associated a degree $\tilde{g} + r - 1$ line bundle $\tilde{E} \to \tilde{S}$ over the partly desingularized spectral curve $\tilde{S}$. Away from the degenerate eigenvalues this coincides with the dual of the bundle of eigenvectors of $\hat{L}^T(\lambda)$ over $S$. At a smooth point $(\lambda, z)$ of $S$, the fibre of $\tilde{E}$ is the cokernel of the map $K(\lambda, z)$:

$$0 \longrightarrow \mathbb{C}^r \xrightarrow{K(\lambda, z)} \mathbb{C}^r \longrightarrow \tilde{E} \longrightarrow 0. \quad (1.30)$$

More generally, this exact sequence defines the direct image of $\tilde{E}$ over $\tilde{S}$ (cf. Sec. 2a). Vectors $V_0$ in $\mathbb{C}^r$ then give sections of $\tilde{E}$ by projection. These sections vanish precisely at the points where $V_0$ is in the image of $K(\lambda, z)$. Since $K(\lambda, z)K(\lambda, z) = P(\lambda, z)I$, this is equivalent to (1.29), at least over the open set of points in $S$ corresponding to nondegenerate eigenvalues, for which the corank of $K(\lambda, z)$ is one. From [AHH1], the degree of $\tilde{E}$ is $\tilde{g} + r - 1$, so sections of $\tilde{E}$ have $\tilde{g} + r - 1$ zeroes. The choice of $V_0$ as an eigenvector of the leading term in $\hat{L}(\lambda)$ implies that $r - 1$ of these are over $\lambda = \infty$, and the coordinates of the remaining $\tilde{g}$ points are the finite solutions $\{(\lambda_\mu, z_\mu)\}_{\mu=1,...,\tilde{g}}$.

In evaluating Poisson brackets, it is preferable to introduce another normalization, corresponding to the eigenvalues of $\frac{N(\lambda)}{\lambda}$ rather than $\hat{L}(\lambda)$, by defining:

$$\zeta := \frac{z}{a(\lambda)} \quad (1.31)$$

and

$$\mathcal{M}(\lambda, \zeta) := \frac{N(\lambda)}{\lambda} - \zeta I, \quad (1.32)$$

with classical adjoint $\mathcal{M}(\lambda, \zeta)$. Then

$$\widetilde{K}(\lambda, z) = [a(\lambda)]^{r-1}\mathcal{M}(\lambda, \zeta) \quad (1.33)$$
and eq. (1.29) is equivalent to:

\[ \tilde{M}(\lambda, \zeta)V_0 = 0. \]  

(1.34)

The \( \tilde{g} \) solutions \( \{ (\lambda_\mu, z_\mu) \}_{\mu = 1, \ldots, \tilde{g}} \) are thus related to the solutions \( \{ (\lambda_\mu, \zeta_\mu) \}_{\mu = 1, \ldots, \tilde{g}} \) of (1.34) by:

\[ \zeta_\mu = \frac{z_\mu}{a(\lambda_\mu)}. \]  

(1.35)

Viewing \( \{ (\lambda_\mu, \zeta_\mu) \}_{\mu = 1, \ldots, \tilde{g}} \) as functions on \( Q_{N_0} \), we may evaluate their Poisson brackets with respect to the orbital (Kostant-Kirillov) symplectic structure \( \omega_{orb} \).

**Theorem 1.3.** The Poisson brackets of the functions \( (\lambda_\mu, \zeta_\mu)_{\mu = 1, \ldots, \tilde{g}} \) are:

\[ \{ \lambda_\mu, \lambda_\nu \} = 0, \quad \{ \zeta_\mu, \zeta_\nu \} = 0, \quad \{ \lambda_\mu, \zeta_\nu \} = \delta_{\mu \nu}. \]  

(1.36)

**Proof.** Choose a basis in which the leading term in \( \hat{L}(\lambda) \) (i.e. \( L_0 \) for case (i) and \( Y \) for case (ii)) is diagonal, and let \( V_0 = (1, 0 \ldots 0)^T \). Let \( \tilde{M}_{ij}(\lambda, \zeta) \) denote the \( ij^{th} \) component of \( \tilde{M}(\lambda, \zeta) \). The points \( (\lambda_\nu, \zeta_\nu) \) are then determined by the conditions

\[ \tilde{M}_{k1}(\lambda_\nu, \zeta_\nu) = 0 \]  

(1.37)

for all \( k \). Generically, these points are cut out by only two of these equations, say

\[ \tilde{M}_{11} = \tilde{M}_{21} = 0. \]  

(1.37a)

That is, generically the matrix

\[ F_\nu := \left( \begin{array}{cc} \frac{\partial \tilde{M}_{11}}{\partial \lambda} & \frac{\partial \tilde{M}_{11}}{\partial \zeta} \\ \frac{\partial \tilde{M}_{21}}{\partial \lambda} & \frac{\partial \tilde{M}_{21}}{\partial \zeta} \end{array} \right)(\lambda_\nu, \zeta_\nu) \]  

(1.38)

is invertible. By implicit differentiation, the Poisson brackets of the functions \( (\lambda_\nu, \zeta_\nu) \) are then:

\[
\begin{pmatrix}
\{ \lambda_\nu, \lambda_\mu \} & \{ \lambda_\nu, \zeta_\mu \} \\
\{ \zeta_\nu, \lambda_\mu \} & \{ \zeta_\nu, \zeta_\mu \}
\end{pmatrix}
= 
(F_\nu)^{-1}
\begin{pmatrix}
\{ \tilde{M}_{11}(\lambda_\nu, \zeta_\nu), \tilde{M}_{11}(\lambda_\mu, \zeta_\mu) \} & \{ \tilde{M}_{11}(\lambda_\nu, \zeta_\nu), \tilde{M}_{21}(\lambda_\mu, \zeta_\mu) \} \\
\{ \tilde{M}_{21}(\lambda_\nu, \zeta_\nu), \tilde{M}_{11}(\lambda_\mu, \zeta_\mu) \} & \{ \tilde{M}_{21}(\lambda_\nu, \zeta_\nu), \tilde{M}_{21}(\lambda_\mu, \zeta_\mu) \}
\end{pmatrix}
(F_\mu)^T. 
\]  

(1.39)

To determine the brackets in the matrix on the right hand side of equation (1.39) we first recall that if \( f \) and \( g \) are functions on the orbit \( Q_{N_0} \), their Poisson bracket at a point \( \mu \in Q_{N_0} \subset \mathfrak{g}l(r) \) is given by

\[ \{ F, G \} = \langle \mu, \frac{\delta f}{\delta \mu}, \frac{\delta g}{\delta \mu} \rangle, \]  

(1.40)
where $\frac{\delta f}{\delta \mu}$ is the differential of $f$ at $\mu$, considered as an element of $\tilde{\mathfrak{gl}}(r)^+$, and the pairing $< \ , >$ is defined by eq. (1.4). The $ij$th coefficient of $\mathcal{M}(\lambda, \zeta)$ evaluated at the point $(\lambda_0, \zeta_0)$, viewed as a function of $\mu \in \tilde{\mathfrak{gl}}(r)$, may be written

$$\mathcal{M}_{ij}(\lambda_0, \zeta_0) = - < \mu, \frac{e_{ji}}{\lambda - \lambda_0} > - \zeta_0 I, \tag{1.41}$$

where $e_{ji}$ is the matrix with a 1 in the $ji$th place and zeroes elsewhere. It follows that

$$\frac{\delta \mathcal{M}_{ij}(\lambda_0, \zeta_0)}{\delta \mu} = - \frac{e_{ji}}{\lambda - \lambda_0} \tag{1.42}$$

and hence, dropping the 0 subscripts,

$$\{ \mathcal{M}_{ij}(\lambda, \zeta), \mathcal{M}_{kl}(\sigma, \eta) \} = \frac{1}{\lambda - \sigma} \left[ (\mathcal{M}_{il}(\lambda, \zeta) - \mathcal{M}_{il}(\sigma, \eta)) \delta_{jk} - (\mathcal{M}_{kj}(\lambda, \zeta) - \mathcal{M}_{kj}(\sigma, \eta)) \delta_{il} \right]. \tag{1.43}$$

Since $\tilde{\mathcal{M}}(\lambda, \zeta)$ is the classical adjoint of $\mathcal{M}(\lambda, \zeta)$ we have

$$\tilde{\mathcal{M}}(\lambda, \zeta) \mathcal{M}(\lambda, \zeta) = \det(\mathcal{M}(\lambda, \zeta)) I. \tag{1.44}$$

Differentiating with respect to a parameter $t$ yields

$$\frac{d\tilde{\mathcal{M}}(\lambda, \zeta)}{dt} = \tilde{\mathcal{M}}(\lambda, \zeta) \text{tr} \left( (\frac{d}{dt} \tilde{\mathcal{M}}(\lambda, \zeta)) \tilde{\mathcal{M}}(\lambda, \zeta) \right) - \tilde{\mathcal{M}}(\lambda, \zeta) \left( \frac{d}{dt} \mathcal{M}(\lambda, \zeta) \right) \tilde{\mathcal{M}}(\lambda, \zeta) \tag{1.45}$$

away from points $(\lambda, \zeta)$ where $\det(\mathcal{M}(\lambda, \zeta)) = 0$; i.e., points on the spectral curve. Thus, away from the spectral curve,

$$\frac{\partial \tilde{\mathcal{M}}_{ij}(\lambda, \zeta)}{\partial \mathcal{M}_{pq}(\lambda, \zeta)} = \frac{\tilde{\mathcal{M}}_{qp}(\lambda, \zeta) \tilde{\mathcal{M}}_{ij}(\lambda, \zeta) - \tilde{\mathcal{M}}_{ip}(\lambda, \zeta) \tilde{\mathcal{M}}_{qj}(\lambda, \zeta)}{\det \mathcal{M}(\lambda, \zeta)}. \tag{1.46}$$

The derivation property of the bracket

$$\{ \tilde{\mathcal{M}}_{ij}(\lambda, \zeta), \tilde{\mathcal{M}}_{kl}(\sigma, \eta) \} = \sum_{pqrs} \frac{\partial \tilde{\mathcal{M}}_{ij}(\lambda, \zeta)}{\partial \mathcal{M}_{pq}(\lambda, \zeta)} \frac{\partial \tilde{\mathcal{M}}_{kl}(\sigma, \eta)}{\partial \mathcal{M}_{rs}(\sigma, \eta)} \{ \mathcal{M}_{pq}(\lambda, \zeta), \mathcal{M}_{rs}(\sigma, \eta) \} \tag{1.47}$$

then gives

$$\{ \tilde{\mathcal{M}}_{i1}(\lambda, \zeta), \tilde{\mathcal{M}}_{k1}(\sigma, \eta) \} = \frac{1}{\lambda - \sigma} \left[ \frac{1}{\det \mathcal{M}(\sigma, \eta)} \left[ (\tilde{\mathcal{M}}(\sigma, \eta) \tilde{\mathcal{M}}(\lambda, \zeta))_{k1} \tilde{\mathcal{M}}_{i1}(\sigma, \eta) ight. ight. \\ - (\tilde{\mathcal{M}}(\sigma, \eta) \tilde{\mathcal{M}}(\lambda, \zeta))_{i1} \tilde{\mathcal{M}}_{k1}(\sigma, \eta) \\ + \left. \left. \frac{1}{\det \mathcal{M}(\lambda, \zeta)} \left[ (\tilde{\mathcal{M}}(\lambda, \zeta) \tilde{\mathcal{M}}(\sigma, \eta))_{i1} \tilde{\mathcal{M}}_{k1}(\lambda, \zeta) ight. ight. \\
- (\tilde{\mathcal{M}}(\lambda, \zeta) \tilde{\mathcal{M}}(\sigma, \eta))_{k1} \tilde{\mathcal{M}}_{i1}(\lambda, \zeta) \right] \right]. \tag{1.48}$$
By eq. (1.37), $\widetilde{M}_{k1}(\lambda_\nu, \zeta_\nu)$ vanishes for all $k, \nu$. Taking the limits $(\lambda, \zeta) \to (\lambda_\mu, \zeta_\mu)$, $(\sigma, \eta) \to (\lambda_\nu, \zeta_\nu)$ along any path transversal to the curve $S$, the right hand side of equation (1.48) has limit zero for $\nu \neq \mu$ (the simple zero in $\det M$ is cancelled by a double zero in the numerator), implying

$$\{\widetilde{M}_{11}(\lambda_\nu, \zeta_\nu), \widetilde{M}_{21}(\lambda_\mu, \zeta_\mu)\} = \{\widetilde{M}_{11}(\lambda_\nu, \zeta_\nu), \widetilde{M}_{11}(\lambda_\mu, \zeta_\mu)\} = 0 \quad (1.49)$$

when $\mu \neq \nu$. Hence $\{\lambda_\nu, \lambda_\mu\}$, $\{\zeta_\nu, \zeta_\mu\}$ and $\{\lambda_\nu, \zeta_\mu\}$ all vanish when $\nu \neq \mu$.

To compute the bracket for $\nu = \mu$ we first note that the brackets on the diagonal of the matrix on the right hand side of equation (1.39) are zero in this case. Thus, to show that $\{\lambda_\nu, \zeta_\nu\} = 1$ it suffices to show that

$$\{\widetilde{M}_{11}(\lambda_\nu, \zeta_\nu), \widetilde{M}_{21}(\lambda_\nu, \zeta_\nu)\} = \det(F_\nu). \quad (1.50)$$

To compute the left hand side of (1.50) we first take the limit $(\lambda, \zeta) \to (\sigma, \eta)$ in (1.43) using the derivation property (1.47) of the bracket to show

$$\{\widetilde{M}_{11}(\lambda, \zeta), \widetilde{M}_{21}(\lambda, \zeta)\} = \sum_{prs} \left( \frac{\partial \widetilde{M}_{11}}{\partial M_{pr}} \frac{\partial \widetilde{M}_{21}}{\partial M_{rs}} - \frac{\partial \widetilde{M}_{11}}{\partial M_{qs}} \frac{\partial \widetilde{M}_{21}}{\partial M_{pr}} \right) dM_{ps}. \quad (1.51)$$

On the other hand

$$\det \left( \frac{\partial \widetilde{M}_{11}}{\partial \lambda} \frac{\partial \widetilde{M}_{11}}{\partial \zeta} \right) (\lambda, \zeta) = \sum_{prq} \left( \frac{\partial \widetilde{M}_{11}}{\partial M_{pr}} \frac{\partial \widetilde{M}_{21}}{\partial M_{pq}} - \frac{\partial \widetilde{M}_{11}}{\partial M_{pr}} \frac{\partial \widetilde{M}_{21}}{\partial M_{pq}} \right) dM_{pq}. \quad (1.52)$$

Equation (1.50) now follows by substituting eq. (1.46) into eqs. (1.51) and (1.52) and using the fact that $\widetilde{M}(\lambda_\nu, \zeta_\nu)$ has rank 1. \qed

The implication of Theorem 1.3 is that the functions $\{(\lambda_\mu, \zeta_\mu)\}_{\mu=1, \ldots, \bar{g}}$ nearly provide a Darboux coordinate system on the coadjoint orbit $Q_{\mathcal{N}_0}$. However, the dimensions are not quite right. For case (i), we have

$$\dim Q_{\mathcal{N}_0} = 2\bar{g} + (r + 2)(r - 1) \quad (1.53a)$$

for generic orbits, while for case (ii)

$$\dim Q_{\mathcal{N}_0} = 2(\bar{g} + r - 1). \quad (1.53b)$$

(Note that in these formulae, it is the value of $\bar{g}$ that is different, according to eq. (1.27), not the dimension of $Q_{\mathcal{N}_0}$ which, of course, is the same.)
On the other hand, for case (i), the Marsden-Weinstein reduced coadjoint orbit \(Q_{\text{red}}\), obtained by fixing the value of the \(Gl(r)\) moment map \(L_0\) and quotienting by its stabilizer \(G_{L_0} \subset Gl(r)\), is of dimension \(2\tilde{g}\). Similarly, for case (ii) we may reduce by the stabilizer \(G_Y \subset Gl(r)\) of \(Y\), since the shifted AKS Hamiltonians of the form (1.7) are invariant under this subgroup and the restriction of \(L_0\) to the corresponding subalgebra \(g_Y\) is conserved under the flows. The reduced orbit under this action, also denoted \(Q_{\text{red}}\), is again of dimension \(2\tilde{g}\). (Note again that the value of \(\tilde{g}\) for the latter case is, by eq. (1.27), \(\frac{1}{2}r(r-1)\) greater than for the former.) Thus, if the coordinates \((\lambda_\mu, \zeta_\mu)\) could be shown to be projectable to the reduced spaces, and if the reduced Poisson brackets remain the same as in eq. (1.36), we would have Darboux coordinates on \(Q_{\text{red}}\).

For case (ii) this may be seen immediately. Since \(V_0\) was assumed to be an eigenvector of \(Y\), with no degeneracy allowed, the defining equation (1.34) is invariant under the stabilizer \(G_Y \subset Gl(r)\) (an \(r-1\) dimensional abelian group under our hypotheses). Thus \((\lambda_\mu, \zeta_\mu)\) are all invariant under the Hamiltonian \(G_Y\)-action, and the Poisson brackets of their projection to \(Q_{\text{red}}\) are the same as on \(Q_{N_0}\).

For case (i), we cannot quite apply Hamiltonian symmetry reduction under \(Gl(r)\), since the functions \((\lambda_\mu, \zeta_\mu)\) are only invariant under the stabilizer subgroup \(G_{L_0}\). However, we may still compute the Poisson brackets on the reduced space by the procedure used for constrained Hamiltonian systems. Let us first choose the reduction condition given by the level set:

\[
L_0 = \text{diag}\{l_i\}, \tag{1.54}
\]

where the eigenvalues \(\{l_i\}\) are, by our genericity assumption, distinct. The diagonal terms in eq. (1.54) are the first class constraints, which generate the Hamiltonian \(G_{L_0}\)-action, and the terms with \(i > 1\) may be chosen as the independent generators. Applying the standard procedure of modifying the Hamiltonian by adding a linear combination of the remaining, second class constraints, we see that the following modified functions generate flows that are tangential to the constrained submanifold:

\[
\hat{\lambda}_\mu = \lambda_\mu - \sum_{i,j=1,i\neq j}^r \frac{\{\lambda_\mu, (L_0)_{ij}\}}{l_i - l_j} (L_0)_{ji}, \tag{1.55a}
\]

\[
\hat{\zeta}_\mu = \zeta_\mu - \sum_{i,j=1,i\neq j}^r \frac{\{\zeta_\mu, (L_0)_{ij}\}}{l_i - l_j} (L_0)_{ji}. \tag{1.55b}
\]

Evaluating their Poisson brackets, we find, again:

\[
\{\hat{\lambda}_\mu, \hat{\lambda}_\nu\} = 0, \quad \{\hat{\zeta}_\mu, \hat{\zeta}_\nu\} = 0, \quad \{\hat{\lambda}_\mu, \hat{\zeta}_\nu\} = \delta_{\mu\nu} \tag{1.56}
\]
since, by implicit differentiation of the defining equations (1.37), the second factor in (1.55a,b) involves terms of the form \( \{ \tilde{M}_k, (L_0)_{ij} \} \) which, applying the chain rule and eq. (1.46), vanish unless \( i = 1 \). The cross terms in the Poisson brackets (1.56) therefore all contain terms proportional to \( \{ \tilde{M}_k, (L_0)_{i1} \} \), \( i \neq 1 \), which vanish at \( (\lambda, \zeta) = (\lambda_\mu, \zeta_\mu) \). Since the functions \( (\hat{\lambda}_\mu, \hat{\zeta}_\mu) \) coincide with \( (\lambda_\mu, \zeta_\mu) \) on the constrained manifold and generate tangential flow, it follows that the projections of \( (\lambda_\mu, \zeta_\mu) \) to \( Q_{\text{red}} \) (the quotient of the constrained manifold by \( G_{L_0} \)) satisfy the same Poisson bracket relations as (1.56). Finally, for other values of \( L_0 \) than (1.54), we just repeat the same argument with respect to a diagonalizing basis of eigenvectors.

Combining these results we obtain, for both cases (i) and (ii):

**Theorem 1.4.** The projections of \( (\lambda_\mu, \zeta_\mu)_{\mu=1,...,\tilde{g}} \) to the reduced orbit \( Q_{\text{red}} \), in both case (i) \((Y = 0)\) and case (ii) \((Y \neq 0, \text{ with distinct eigenvalues})\), are Darboux coordinates; that is, the reduced symplectic form is:

\[
\omega_{\text{red}} = \sum_{\mu=1}^{\tilde{g}} d\lambda_\mu \wedge d\zeta_\mu. \quad (1.57)
\]

**Remark**: The proof of Theorem 1.3 did not depend on the fact that there are \( \tilde{g} \) finite points in the spectral divisor. If the vector \( V_0 \) is not chosen as an eigenvector of \( Y \), the number of such finite points, and corresponding coordinate pairs \( (\lambda_\mu, \zeta_\mu) \), may be between \( \tilde{g} \) and \( \tilde{g} + r - 1 \). The number of points over \( \lambda = \infty \) equals the number of eigenvalues \( \tilde{z} \) of the asymptotic form of \( \hat{L}(\lambda) \) (i.e., \( Y \) for case (ii) and \( L_0 \) for case (i)), for which \( V_0 \) is in the image of \( Y - \tilde{z}I \) for case (ii) (resp. \( L_0 - \tilde{z}I \) for case (i)). This is zero for generically chosen (non-diagonal) \( Y \) (or \( L_0 \)) or, equivalently, if \( Y \) is taken as a diagonal matrix, and \( V_0 \) chosen as a vector with no vanishing components (e.g. \( V_0 = (1,1,...,1)^T \)). In this case, the number of finite spectral divisor coordinate pairs \( (\lambda_\mu, \zeta_\mu) \) will actually be \( \tilde{g} + r - 1 \), sufficient to provide a Darboux coordinate system for the full orbit in case (ii) and an \( r(r-1) \) co-dimensional symplectic submanifold in case (i) (cf. Section 1c). However, for the examples involving integrable systems that will be of interest to us (cf. Section 3), it is not this type of spectral Darboux system that is needed for directly determining solutions, but those derived in the following subsection. The problem lies with the invertibility of the Abel map (cf. Section 1d), which requires a degree \( \tilde{g} \) divisor. The remaining \( r - 1 \) points of the spectral divisor are related to the singular differentials having pole singularities over \( \lambda = \infty \).

---

1 Thanks are due to B. Dubrovin for raising the point discussed in this remark.
There remains then the question of the nonreduced orbits \( Q_{N_0} \). Can the functions \((\lambda, \zeta)_{\mu=1,..,\tilde{g}}\) somehow be completed to provide a Darboux coordinate system on \( Q_{N_0} \)? The answer is: yes, for case (ii), and partially for case (i). The construction is given in the following subsection.

1c. Darboux Coordinates on Unreduced Orbits

In case (i) we shall obtain Darboux coordinates, not on the complete coadjoint orbit \( Q_{N_0} \), but on a constrained submanifold \( Q_{N_0}^0 \subset Q_{N_0} \) consisting of elements for which the off-diagonal elements of \( L_0 \) vanish:

\[
(L_0)_{ij} = 0 \quad \text{if} \quad i \neq j. \tag{1.58}
\]

By our earlier genericity assumptions, the diagonal elements \((L_0)_{ii}\) are hence distinct, and it is easily verified that \( Q_{N_0}^0 \subset Q_{N_0} \) is a symplectic submanifold of dimension

\[
\dim Q_{N_0}^0 = 2(\tilde{g} + r - 1). \tag{1.59}
\]

(Note that \( m = n - 1 \) in the genus formula (1.27) and we are dealing with case (1.53a), not (1.53b).) For case (ii), we choose a basis in which \( Y \) is diagonal:

\[
Y = \text{diag}\{Y_i\}. \tag{1.60}
\]

Thus in both cases, the leading term of \( \hat{L}(\lambda) \) is diagonal. As in the proof of Theorem 1.3, we also choose the eigenvector \( V_0 \) in (1.29) to be \( V_0 = (1, 0, 0, ..., 0)^T \). In both cases, let

\[
P_i := (L_0)_{ii}, \quad i = 1, \ldots r. \tag{1.61}
\]

These generate the action of the group \( D \) of diagonal matrices, which equals \( G_{L_0} \) and \( G_Y \), respectively, for cases (i) and (ii). The generator \( P_1 \) is not independent of the others, since the sum:

\[
\sum_{i=1}^{r} P_i = \text{tr}L_0 \tag{1.62}
\]

is a Casimir. These generators Poisson commute amongst themselves and also with the \( D \)-invariant functions \((\lambda, \zeta)_{\mu=1,..,\tilde{g}}\), since equation (1.37), which determines them, is \( D \)-invariant. In case (i), let

\[
q_i := \ln(L_1)_{i1} + \frac{1}{2} \sum_{j \neq i, j > 1}^{r} \ln(P_i - P_j), \tag{1.63}
\]
while for case (ii), let
\[ q_i := \ln(L_0)_{i1}. \]  
(1.64)

With these definitions, we have:

**Theorem 1.5.** The coordinate functions \((\lambda_\mu, \zeta_\mu, q_i, P_i)_{\mu=1,...,\tilde{g}; i=2,...,r}\) form a Darboux system on \(Q_{N_0}^0\) in case (i), and \(Q_{N_0}\) in case (ii); that is, the only nonvanishing Poisson brackets between them are given by:

\[
\{\lambda_\mu, \zeta_\nu\} = \delta_{\mu\nu}, \quad \{q_i, P_j\} = \delta_{ij}.
\]  
(1.65)

Equivalently,
\[
\omega_{orb} = \sum_{\mu=1}^{\tilde{g}} d\lambda_\mu \wedge d\zeta_\mu + \sum_{i=2}^{r} dq_i \wedge dP_i,
\]  
(1.66)

where the equality refers to the full orbit \(Q_{N_0}\) in case (ii), and the restriction of \(\omega_{orb}\) to \(Q_{N_0}^0\) in case (i).

**Proof.** The proof proceeds in two steps. First, as in Theorem 1.3, the Poisson brackets are computed on the full coadjoint orbits. In case (i), we then reduce this to the constrained submanifold, which is symplectic. From the Poisson brackets (1.43) used in the proof of Theorem 1.3 follows:

\[
\{(L_0)_{ij}, (L_s)_{kl}\} = (L_s)_{kj} \delta_{il} - (L_s)_{il} \delta_{jk},
\]  
(1.67a)

\[
\{M_{ij}(\lambda, \zeta), (L_0)_{kl}\} = (Y - M(\lambda, \zeta))_{il} \delta_{jk} - (Y - M(\lambda, \zeta))_{kj} \delta_{il}
\]  
(1.67b)

\[
\{M_{ij}(\lambda, \zeta), (L_1)_{kl}\} = \lambda - \sum_{m=1}^{g} \alpha_m \[(Y - M(\lambda, \zeta))_{il} \delta_{jk} - (Y - M(\lambda, \zeta))_{kj} \delta_{il}] + [(L_0)_{il} \delta_{kj} - (L_0)_{kj} \delta_{il}].
\]  
(1.67c)

This implies, in addition to the relations (1.36), the brackets:

\[
\{\lambda_\mu, q_i\} = \{\zeta_\mu, q_i\} = 0
\]  
(1.68a)

\[
\{\lambda_\mu, P_i\} = \{\zeta_\mu, P_i\} = 0
\]  
(1.68b)

\[
\{P_i, P_j\} = 0
\]  
(1.68c)

\[
\{q_i, P_j\} = \delta_{ij}
\]  
(1.68d)

\[
\{q_i, q_j\} = \begin{cases} (P_i - P_j)^{-1} & \text{for case (i)} \\ 0 & \text{for case (ii)} \end{cases}
\]  
(1.68e)

where (1.68a) holds only on the constrained manifold \(Q_{N_0}^0\) for case (i). As in the proof of Theorem 1.3, we must use the fact that \(\widetilde{M}_{i1}\) is zero at \((\lambda_\mu, \zeta_\mu)\). In case (ii)
this completes the proof. For case (i), the constraints must be taken into account.
As in the proof of case (i) of Theorem 1.4, we shift the functions \((\lambda, \zeta, q_i, P_j)\) by terms proportional to the second class constraints \((L_0)_{ij} = 0, i \neq j\) to get functions \((\hat{\lambda}, \hat{\zeta}, \hat{q}_i, \hat{P}_j)\) which agree with \((\lambda, \zeta, q_i, P_j)\) on \(Q^0_{N_0}\) and which generate flows in \(Q_{N_0}\) that are tangential to \(Q^0_{N_0}\).

Since \(\{L_0, L_0\}_{ij}, \{L_0, L_0\}_{kl}\) = \((L_0)_{kj}\delta_{il} - (L_0)_{il}\delta_{jk}\), it suffices, for a general function \(f\) on \(Q_{N_0}\), to take

\[
\hat{f} = f - \sum_{i,j=1, i \neq j}^r \frac{\{f, (L_0)_{ij}\}}{P_i - P_j} (L_0)_{ji}.
\]  

As in the proof of Theorem 1.4, the Poisson brackets (1.36) remain unchanged on the constrained manifold. From eq. (1.69), it follows (as in Theorem 1.4), that the \(P_j\)'s already generate tangential flows and hence the Poisson brackets (1.68b-d) remain unchanged. Eq. (1.68a) also is unchanged since, by the same arguments as in the proof of Theorem 1.4, the additional cross terms obtained after constraining are all proportional to terms of the form \(\{q_i, (L_0)_{j1}\}\), which vanish on the constrained manifold. Using eq. (1.67a), we see that the remaining Poisson bracket (1.68e) gets shifted to zero.

**Remarks:**

i) The submanifold \(Q^0_{N_0} \subset Q_{N_0}\) is, in fact, the relevant phase space for many interesting examples of integrable systems, such as the finite gap solutions of the cubically nonlinear Schrödinger equation (cf. [AHP, P] and Sections 3c, 3d).

ii) If, in formulae (1.2), (1.3), we choose \(n = 1, \alpha_1 = 0\) and Y \(\neq 0\), then \(Q_{N_0}\) is really a coadjoint orbit in \(gl(r)^*\) or \(sl(r)^*\) and Theorem 1.5, together with the \(Ad^*\) invariants (Casimirs), provides Darboux coordinate systems for these finite dimensional Lie algebras (cf. Sec. 3a).

1d. Liouville - Arnold Integration and the Abel Map

We now turn to the integration of the Hamiltonian systems (1.1) generated either by elements of the Poisson commutative ring \(F_Y\) of functions of the form (1.7), with \(\Phi\) in the ring \(I(\widetilde{gl(r)}^*)\) of \(Ad^*\) - invariants on \(\widetilde{gl(r)}^*\), or its extension \(F_Y(P)\) by the generators \(\{P_i\}_{i=2,\ldots,r}\). Thus, our Hamiltonians are all expressible as functions of the invariants \(\{P_{ia}, P_i\}\). The notational conventions of the preceding sections allow us to treat cases
(i) and (ii) simultaneously, although it should be remembered that the spectral curves and ring of invariants $\mathcal{F}_Y(P)$ depend on the choice of $Y$, and the relevant symplectic manifold is $\mathcal{Q}_{N_0}^0$ for case (i) and the entire orbit $\mathcal{Q}_{N_0}$ for case (ii). The reduced spaces, though both denoted $\mathcal{Q}_{\text{red}}$, are also different, their dimensions $2\tilde{g}$ being given by the genus formula (1.27) with $m = n - 1$ for case (i) and $m = n$ for case (ii). For case (i), $\mathcal{Q}_{\text{red}}$ signifies the generic $Gl(r)$ - reduction of $\mathcal{Q}_{N_0}^0$ or, equivalently, the reduction of $\mathcal{Q}_{N_0}$ by the abelian $r - 1$ - dimensional group action generated by $\{P_i\}_{i=2,...,r}$. For case (ii), $\mathcal{Q}_{\text{red}}$ is the reduction of the full orbit $\mathcal{Q}_{N_0}$ by the latter action.

The $\tilde{g} + r - 1$ independent spectral invariants for case (i) may be chosen to be $(P_{\alpha_i}, P_j)_{i,j=2,...,r;a_i=0,...,\delta_i - 1}$, since the coefficients $P_{j\delta_j}$ occurring in Proposition 1.2 may be expressed as translates of the elementary symmetric invariants of $L_0 = \text{diag}\{P_i\}$

$$P_{j\delta_j} = (-1)^{r-j} \sum_{1 \leq i_1 < ... < i_j} P_{i_1}...P_{i_j} + m_j,$$  \hspace{1cm} (1.71)

where the constants $\{m_j\}$ depend on the reference polynomial $\mathcal{P}_R(\lambda, z)$. For case (ii), the leading coefficients $\{P_{j\delta_j}\}$ are constants (translates of the elementary symmetric invariants of $Y$) and the next to leading coefficients are translates of linear combinations of the $P_i$’s:

$$P_{j,\delta_j-1} = (-1)^{r-j} \sum_{i=1}^r P_i \sum_{1 \leq i_1 < ... < i_j-1 \neq i} Y_{i_1}...Y_{i_{j-1}} + n_j,$$  \hspace{1cm} (1.72)

where again, the constants $n_j$ depend on $\mathcal{P}_R(\lambda, z)$ and the constants $Y_i$. Thus, the $\tilde{g} + r - 1$ independent invariants may be chosen to be $\{P_{\alpha_i}, P_j\}_{i,j=2,...,r,a_i=1,...,\delta_i-2}$. The Hamiltonians may be viewed in the two cases as functions of the independent invariants:

$$h = h(P_{\alpha_i}, P_j) \quad i, j = 2, ... , r, \quad a_i = 1, ... , \delta_i - 1 - \epsilon.$$  \hspace{1cm} (1.73)

with $\epsilon = 0$ for case (i) and $\epsilon = 1$ for case (ii). We can use the Darboux coordinates of Theorem 1.5 to express the symplectic forms $\omega_{\text{orb}}$ or $\omega_{\text{orb}|\mathcal{Q}_{N_0}^0}$ as (minus) the exterior derivative of a 1-form:

$$\theta := \sum_{\mu=1}^{\tilde{g}} \zeta_\mu d\lambda_\mu + \sum_{i=2}^r P_i dq_i.$$  \hspace{1cm} (1.74)

Restricting to the invariant Lagrangian manifolds $\mathcal{L}$ obtained by fixing the level sets of $\{P_{\alpha_i}, P_j\}$, there exists (within a suitable neighbourhood of such $\mathcal{L}$’s) a Liouville generating function $S(\lambda_\mu, q_i, P_{\alpha_i}, P_i)$ such that:

$$\theta|_{\mathcal{L}} = dS.$$  \hspace{1cm} (1.75)
Integrating from an arbitrary initial point thus gives

\[ S(\lambda, q_i, P_{ia}, P_i) = \sum_{\mu=1}^{\tilde{g}} \int_{\lambda_0}^{\lambda_\mu} \frac{z(\lambda, P_{ia}, P_j)}{a(\lambda)} d\lambda + \sum_{i=2}^{r} q_i P_i, \]  

where the \( \lambda \) integrals are evaluated within a chosen polygonization of the spectral curve \( \tilde{S} \) and the function

\[ z = z(\lambda, P_{ia}, P_j) \]  

is determined implicitly along \( L \) by the spectral equation:

\[ \mathcal{P}(\lambda, z(\lambda, P_{ia}, P_j)) = 0. \]  

Applying the standard canonical transformation procedure, the coordinates \((Q_{ia}, Q_i)\) canonically conjugate to the invariants \((P_{ia}, P_i)\) are then

\[ Q_{ia} = \frac{\partial S}{\partial P_{ia}} = \sum_{\mu=1}^{\tilde{g}} \int_{\lambda_0}^{\lambda_\mu} \frac{1}{a(\lambda) \partial P_{ia}} d\lambda \]  

\[ Q_i = \frac{\partial S}{\partial P_i} = \sum_{\mu=1}^{\tilde{g}} \int_{\lambda_0}^{\lambda_\mu} \frac{1}{a(\lambda) \partial P_i} d\lambda + q_i. \]

Evaluating the integrands by implicit differentiation of eq. (1.78) with respect to the invariants \(\{P_{ia}, P_i\}\), and using eqs (1.71), (1.72), we have

\[ \frac{\partial z}{\partial P_{ia}} = -a(\lambda) \frac{a_i(\lambda) z^{r-i} \lambda^a}{\mathcal{P}_z(\lambda, z)}, \quad i = 2, \ldots r, \quad a = 1, \ldots \delta_i - 1 - \epsilon \]  

\[ \frac{\partial z}{\partial P_i} = -a(\lambda) \sum_{j=2}^{r} \frac{R_{ij} a_j(\lambda)(-z)^{r-j} \lambda^{\delta_j - \epsilon}}{\mathcal{P}_z(\lambda, z)}, \quad i = 2, \ldots r \]

where

\[ R_{ij} := \begin{cases} (P_1 - P_i) \sum_{2 \leq i_1 < i_2 \ldots < i_{j-2} \neq i} P_{i_1} \ldots P_{i_{j-2}} \\
\quad \text{and} \quad \epsilon = 0 \quad \text{for case (i)} \\
(Y_1 - Y_i) \sum_{2 \leq i_1 < i_2 \ldots < i_{j-2} \neq i} Y_{i_1} \ldots Y_{i_{j-2}} \\
\quad \text{and} \quad \epsilon = 1 \quad \text{for case (ii)} \end{cases} \]

The flow is then given in implicit form by the linear equations:

\[ \sum_{\mu=1}^{\tilde{g}} \int_{\lambda_0}^{\lambda_\mu} \frac{a_i(\lambda) z^{r-i} \lambda^a}{\mathcal{P}_z(\lambda, z)} d\lambda = C_{ia} - \frac{\partial h}{\partial P_{ia}} t \]  

\[ \sum_{\mu=1}^{\tilde{g}} \int_{\lambda_0}^{\lambda_\mu} \sum_{j=2}^{r} \frac{R_{ij} a_j(\lambda)(-z)^{r-j} \lambda^{\delta_j - \epsilon}}{\mathcal{P}_z(\lambda, z)} d\lambda = q_i + c_i - \frac{\partial h}{\partial P_i} t. \]
where \( \{ C_{ia}, c_i \}_{i=2,\ldots,r; \; a=1,\ldots,\delta_i-1-\epsilon} \) are integration constants and a fixed base point \( \lambda_0 \) has been used in the integration.

**Remark:** On any given level set of the \( P_i \)'s the Hamiltonians \( h(P_{ia}, P_j) \) project to the reduced space \( Q_{\text{red}} \) and eq. (1.82a) alone gives the corresponding linearization of the reduced flow.

We note that the linearizing map defined by eqs. (1.82a,b) involves \( \tilde{g} + r - 1 \) abelian integrals on \( \tilde{S} \). If we knew that the \( \tilde{g} \) differentials

\[
\omega_{ia} := \frac{a_i(\lambda)z^{r-i}\lambda^\alpha}{P_z(\lambda, z)}d\lambda
\]  

appearing as integrands in (1.82a) were all holomorphic (i.e. abelian differentials of the first kind) and independent then, up to a normalizing change of basis, (1.82a) would just be the statement that the Abel map

\[
A : S^\tilde{g}\tilde{S} \longrightarrow J(\tilde{S})
\]

taking the unordered set of \( \tilde{g} \) points \( \{ p_\mu \} \in \tilde{S} \) with coordinates \( (\lambda_\mu, z_\mu) \) to its image in the Jacobi variety \( J(\tilde{S}) \) linearizes the flow - the familiar type of result usually obtained from algebro-geometric methods of integration [AvM, KN, Du, AHH1]. In fact, this is exactly the case. Moreover the remaining \( r - 1 \) integrands

\[
\omega_i := \sum_{j=2}^{r} R_{ij} a_j(\lambda)(-z)^{r-j}\lambda^{\delta_j-\epsilon} \frac{d\lambda}{P_z(\lambda, z)}
\]  

appearing in (1.82b) are abelian differentials of the third kind with simple poles at the \( r \) points \( \{ \infty_i \} \) over \( \lambda = \infty \) with local coordinates

\[
\infty_i \Leftrightarrow \begin{cases} 
(\tilde{\lambda} = 0, \tilde{z} = P_i) & \text{for case (i)} \\
(\tilde{\lambda} = 0, \tilde{z} = Y_i) & \text{for case (ii)}.
\end{cases}
\]  

**Theorem 1.6.** The \( \tilde{g} \) differentials \( \{ \omega_{ia} \}_{i=1,\ldots,\tilde{g}} \) in eq. (1.83) form a basis for the space \( H^0(\tilde{S}, K_{\tilde{S}}) \) of abelian differentials of the first kind (where \( K_{\tilde{S}} \) denotes the canonical bundle). The linear flow equation (1.82a) may therefore be expressed as:

\[
A(D) = B + Ut,
\]

where \( B, U \in \mathbb{C}^{\tilde{g}} \) are obtained by applying the inverse of the \( \tilde{g} \times \tilde{g} \) normalizing matrix \( M \), with elements

\[
M_{\mu,(ia)} := \oint_{a_\mu} \omega_{ia},
\]
to the vectors $C, H \in \mathbb{C}^g$ with components $C_{ia}$ and $-\frac{\partial h}{\partial P_{ia}}$, respectively (the pair $(ia)$ viewed as a single coordinate label in $\mathbb{C}^g$).

The $r-1$ differentials \{\omega_i\}_{i=2,...,r}$ in eq. (1.85) are abelian differentials of the third kind with simple poles at $\infty_i$ and $\infty_1$, and residues $+1$ and $-1$, respectively. After a suitable translation by elements of $H^0(\tilde{S}, K_{\tilde{S}})$ to obtain the standard normalization with respect to a canonical homology basis \{\(a_\mu, b_\mu \in H_1(\tilde{S}, \mathbb{Z})\)\}_{\mu=1,...,\tilde{g}}$, these provide a basis for the $r-1$ dimensional space of normalized differentials with simple poles over $\lambda = \infty$.

Remark: Combining these results with the remark following eq. (1.82a,b), we see that for Hamiltonian flows on the reduced orbit (or equivalently for Hamiltonians that are independent of the $P_i$'s), the linearization map only involves abelian differentials of the first kind. For flows on the unreduced orbit it is necessary to introduce the differentials of the third kind in order to determine the time dependence of the additional coordinates \{\(q_i\)\} (viz. Corollary 1.7).

Proof. Every holomorphic 1–form on $\tilde{S}$ can be obtained by evaluating the Poincaré residue of a meromorphic 2–form on $T$ with pole divisor at $\tilde{S}$. Over the affine coordinate neighborhood $U_0$ such a residue has the form

$$\omega = \frac{f(\lambda, z)d\lambda}{P_z(\lambda, z)},$$

where, for holomorphicity at $\lambda = \infty$, the total weighted degree of the polynomial $f(\lambda, z)$ must not exceed $m(r-1) - 2$, and for holomorphicity at the points $\{(\alpha_i, 0)\}$, the function $f(\lambda, z)$ must vanish to sufficiently high order so as to cancel the zeroes of $P_z(\lambda, z)$. Since at these points $P_z(\lambda, z)$ vanishes like $(\lambda - \alpha_i)^{r-k_i}$, while $z$ vanishes along each intersecting branch like $\lambda - \alpha_i$, $f(\lambda, z)$ must be a sum of terms of the form

$$z^{r-k_i} \prod_{j=1}^{n}(\lambda - \alpha_j)^{\max(0, i-k_j-1)} \lambda^a, \quad j = 2, \ldots r,$$

where, in order to have total degree at most $m(r-1) - 2$,

$$0 \leq a \leq \delta_j - 1 - \epsilon.$$  \hspace{1cm} (1.90)

But these are precisely the 1–forms $\omega_{ia}$ of eq. (1.83), which therefore span the entire $\tilde{g}$–dimensional space of holomorphic 1–forms $H^0(\tilde{S}, K_{\tilde{S}})$. Since, by Proposition 1.2, there are exactly $\tilde{g}$ such $\omega_i$'s, they are necessarily linearly independent.

Turning to the remaining $r-1$ differentials \{\omega_i\} of eq.(1.85), these have the same structure near the points $(\alpha_i, 0)$ as the $\omega_{ia}$'s, and hence are holomorphic there, but
since the numerator polynomial is of degree \(m(r - 1) - 1\), they have simple poles over \(\lambda = \infty\). To obtain the exact location of these poles and their residues, recall that the numerator of (1.85) was obtained by evaluating
\[
\frac{1}{a(\lambda)} \frac{\partial z}{\partial P_i} = -\frac{\partial P}{\partial P_i} a(\lambda) P_z.
\]
(1.91)

Near \(\lambda = \infty\), \(P(\lambda, z)\) is of the form:
\[
\begin{cases}
\frac{1}{\lambda(u-1)} \prod_{i=1}^r (P_i - \tilde{z}) + O(\bar{\lambda}) & \text{for case (i)} \\
\frac{1}{\lambda^r} \prod_{i=1}^r (Y_i - \tilde{z}) + \bar{\lambda} \sum_{i=1}^r (P_i - \sum_{j=1}^n \alpha_j Y_i) \prod_{k \neq i} (Y_k - \tilde{z}) + O(\bar{\lambda}^2) & \text{for case (ii)},
\end{cases}
\]
(1.92)

where the change of coordinates (1.18) has been used. Hence, for case (i), near \(\lambda = \infty\),
\[
\frac{1}{a(\lambda)} \frac{\partial z}{\partial P_i} d\lambda \sim \frac{\prod_{k \neq i} (P_k - \tilde{z}) - \prod_{k \neq i} (P_k - \tilde{\lambda})}{\sum_{l=1}^r \prod_{k \neq l} (P_k - \tilde{z})} \frac{d\tilde{\lambda}}{\lambda},
\]
(1.93a)

and this has simple poles at \(\infty_i (\tilde{\lambda} = 0, \tilde{z} = P_i), i > 1\) and \(\infty_1 (\tilde{\lambda} = 0, \tilde{z} = P_1)\) with residues +1 and -1, respectively. Similarly, for case (ii),
\[
\frac{1}{a(\lambda)} \frac{\partial z}{\partial P_i} d\lambda \sim \frac{\prod_{k \neq i} (Y_k - \tilde{z}) - \prod_{k \neq i} (Y_k - \tilde{\lambda})}{\sum_{l=1}^r \prod_{k \neq l} (Y_k - \tilde{z})} \frac{d\tilde{\lambda}}{\lambda},
\]
(1.93b)

giving again simple poles at \(\infty_i (\tilde{\lambda} = 0, \tilde{z} = Y_i), i > 1\) and \(\infty_1 (\tilde{\lambda} = 0, \tilde{z} = Y_1)\) with residues +1 and -1. \(\Box\)

It follows, since (1.82a) is essentially the Abel map, that any function on the 
Lagrangian manifold \(\mathcal{L}\) that is symmetric in the coordinates \((\lambda_\mu)\) may be expressed along the flow lines in terms of quotients of theta functions on the curve \(\tilde{S}\). In particular, for the coordinates \(\{q_i(t)\}\) themselves, we have

**Corollary 1.7.** For a suitable choice of constants \(\{e_i, f_i\}_{i=2,...,r}\), the coordinate functions \(\{q_i(t)\}\) satisfying eq.(1.82b) are given by:
\[
q_i(t) = \ln \frac{\theta(B + tU - A(\infty_i) - K)}{\theta(B + tU - A(\infty_1) - K)} + e_i t + f_i,
\]
(1.94)

where \(K \in \mathbb{C}^g\) is the Riemann constant.

**Proof.** We use the standard method underlying the reciprocity theorems relating different types of abelian differentials (cf. [GH]). Namely, on the polygonization of \(\tilde{S}\)
obtained by cutting along a canonical basis \( \{ a_\mu, b_\mu \} \) of cycles, we define the meromorphic differential

\[
d\psi(p) := d \left( \ln \theta(A(D) - A(p) - K) \right),
\]

where \( D \) is the divisor \( \sum_{\mu=1}^{\tilde{g}} p_\mu \) formed from the \( \tilde{g} \) points \( (p_\mu)_{\mu=1,...,\tilde{g}} \) with coordinates \( (\lambda_\mu, z_\mu) \) and \( p \) denotes the point of evaluation on \( \tilde{S} \). Since \( d\psi \) has simple poles with residues 1 at the \( p_\mu \)'s, we may express the abelian sum appearing in eq. (1.82b) as an integral

\[
\sum_{\mu=1}^{\tilde{g}} \int_{\lambda_0}^{\lambda_\mu} \omega_i = \oint_{C} \left[ \int_{\lambda_0}^{p} \omega_i \right] d\psi
\]

around a contour \( C \) enclosing only these singularities of the integrand, and not the ones at \( p = \{ \infty \} \), which are logarithmic branch points. Integrating by parts and deforming the contour to the boundary \( B \) of the polygonization of \( \tilde{S} \) gives

\[
\sum_{\mu=1}^{\tilde{g}} \int_{\lambda_0}^{\lambda_\mu} \omega_i = \sum_{j=2}^{r} \oint_{C_j} \ln \theta(A(D) - A(p) - K) \omega_i - \oint_{B} \ln \theta(A(D) - A(p) - K) \omega_i.
\]

where the \( C_j \) are small loops enclosing the poles at \( \infty_j \) and no other singularities. If the differentials \( \omega_i \) were normalized, the contributions to the boundary integral from the pairs \( \pm a_\mu \) in \( B \) would just be constants (the discontinuity given by the theta multiplier over the \( b_\mu \) cycle), and the contributions from the \( b_\mu \) terms would vanish. However, our differentials \( \{ \omega_i \} \) differ from the normalized ones by linear combinations of the holomorphic differentials \( \{ \omega_{ia} \} \) in eqs. (1.83). It follows from eq. (1.87) that these differences contribute linear terms in \( t \) with constant coefficients. The remaining terms in eq. (1.97) may be evaluated by taking residues at \( \{ \infty_i \}_{i=1,...,r} \), using the results of Theorem 1.6 and eq. (1.87) to yield the logarithmic theta function term in eq. (1.94). The constants and linear terms in (1.94) are then obtained by summing those from the normalizing shift with those already present in eqs. (1.82b).

Remark: The LHS of (1.82a,b) may be interpreted as an extended Abel map from \( S^{\tilde{g}} \tilde{S} \) to a generalized Jacobi variety \( \mathcal{J}(\hat{S}) \) associated to the singularized curve \( \hat{S} \) obtained by identifying the points \( \{ \infty \} \), where \( \mathcal{J}(\hat{S}) \) is a \( (\mathbb{C}^*)^{r-1} \) extension of \( \mathcal{J}(\tilde{S}) \). The extended theta function for \( \hat{S} \) is obtained by multiplying the ordinary theta function for the nonsingular curve by exponential factors in the extended directions \([C]\). This may be viewed as the source of the additional linear terms in eq. (1.94).
2. The Algebraic Geometry of the Symplectic Form.

2a. The Geometric Structure of Coadjoint Orbits

In this section, we give a geometric description of the coadjoint orbits, based on the results of [AHH1] (cf. also [B]). This will be done both in the reduced case, treated above in 1b, and the unreduced case, treated in 1c. There are two parametrizations, corresponding to the choices \( Y = 0 \) (case (i)) and \( Y = \text{diag} (Y_i), Y_i \text{ distinct} \) (case (ii)). In the reduced case, we consider the reduced orbits \( Q_{\text{red}} \) of Theorem 1.4, while in the unreduced case, we consider, for case (i), the “restricted” orbit \( Q_{N_0}^0 \) defined by eq. (1.58), and for case (ii), the full orbit \( Q_{N_0} \). Note that with these choices the highest order term of the matricial polynomial \( \hat{L}(\lambda) \) is diagonal in both cases.

To each element of the unreduced orbit \( Q_{N_0} \), we can associate a certain set of geometrical data. First, there is the spectral curve \( S \subset T \), defined in (1.14), which has the following properties (see [AHH1] for a more detailed discussion):

**Lemma 2.1.** Let \( \{ z_{ia} \}_{i=1,...,n,a=1,...,r-k_i} \) be the non zero spectrum of the residue matrices \( N_i \prod_{j=1,j\neq i}^n (\alpha_i - \alpha_j) \) at \( \lambda = \alpha_i \). (The \( \{ N_i \} \) are assumed to satisfy the genericity conditions given in Section 1a.)

1. \( S \) passes through the points \( (\alpha_i, z_{ia}) \), is compact and lies in the linear system \( | \mathcal{O}(rm) | \).

2. Generically \( S \) has an \( (r-k_i) \)-fold ordinary singular point at \( (\alpha_i, 0) \); desingularising \( S \) at these \( \{(\alpha_i, 0)\} \) yields a smooth curve \( \rho : \tilde{S} \rightarrow S \).

3. In case (ii), \( S \) passes through \( (\tilde{\lambda}, \tilde{z}) = (0, Y_i) \) (see (1.60)). In case (i), \( S \) generically has \( r \) distinct points over \( \tilde{\lambda} = 0 \).

A second element has already been alluded to. Let \( \pi : T \rightarrow \mathbf{P}_1 \) be the natural projection, where \( T \) is, as above, the total space of \( \mathcal{O}(m) \). We can lift the bundles \( \mathcal{O}(j) \) on \( \mathbf{P}_1 \) to \( T \), and so to \( S \) and \( \tilde{S} \). Let all these bundles be denoted \( \mathcal{O}(j) \) and, if \( V \) is a sheaf, let \( V \otimes \mathcal{O}(j) \) be denoted \( V(j) \). In a natural trivialisation over \( U_0 \subset T \), \( \mathcal{O}(m) \) has a basis of sections \( \{ z, 1, \lambda, \cdots \lambda^m \} \). Given \( \hat{L}(\lambda) \), define a sheaf \( E \), supported over \( \tilde{S} \), by the following exact sequence over \( T \) (cf. (1.30))

\[
0 \rightarrow \mathcal{O}(-m)^{\oplus r} \xrightarrow{\mathcal{K}(z, \lambda)} \mathcal{O}^{\oplus r} \xrightarrow{\mu} E \rightarrow 0, \quad (2.1)
\]

where \( \mathcal{K}(z, \lambda) = (\hat{L}(\lambda) - z1) \) (i.e. \( \mu \) is projection to the cokernel of \( (\hat{L}(\lambda) - z1) \)). If \( S \) is reduced one has
Lemma 2.2 [AHH1]. (1) $E$ is a torsion free sheaf over $S$ and is generically the direct image of a line bundle $\tilde{E}$ over $\tilde{S}$, with

$$\deg(\tilde{E}) = \tilde{g} + r - 1 \tag{2.2a}$$

(2) One has:

$$H^0(S, E \otimes O(-1)) = 0. \tag{2.2b}$$

In consequence, if $F_\lambda$ is a fibre of $\pi : T \to \mathbb{P}_1$, the restriction map

$$H^0(S, E) \to H^0(S \cap F_\lambda, E) \tag{2.2c}$$

is an isomorphism, and so $H^0(S, E)$ is $r$-dimensional.

Remark: For the spectral curve $S$ of any element of $Q_{N_0}$, we can construct the partly desingularised curve $\tilde{S}$ mapping to $S$ and the line bundle $\tilde{E}$ over $\tilde{S}$ whose direct image is $E$ as follows. There is an exact sequence of bundles over $\mathbb{P}_r - 1$:

$$0 \to T^* \mathbb{P}_{r-1}(1) \xrightarrow{\rho} \mathcal{O}^{\oplus r} \xrightarrow{\phi} \mathcal{O}(1) \to 0, \tag{2.3}$$

where $\phi$ is just the evaluation map of sections. We consider the map of sheaves over $T \times \mathbb{P}_{r-1}$:

$$T^* \mathbb{P}_{r-1}(1) \oplus \mathcal{O}(-m) \oplus \mathcal{O}^{\oplus r}. \tag{2.4}$$

The cokernel of this map is supported over the union of a curve $\tilde{S}$ mapping to $S$ and some projective spaces over the singular points of $S$ where the corank of $\mathcal{K}(z, \lambda)$ is greater than one. We take $\tilde{E}$ to be the restriction of this cokernel to $\tilde{S}$.

A third datum that can be associated to an element of $Q_{N_0}$ is a trivialisation of $E$ over $\lambda = \infty$. Let $\{e_i\}$ denote the standard basis of $H^0(T, \mathcal{O}^{\oplus r})$. If $\infty_j$ is the point of the curve over $\lambda = \infty$ corresponding to the $j$-th eigenvector, then from (2.1), since the leading order term of $\hat{L}(\lambda)$ is diagonal, $\mu(e_i)$ is nonzero over $\infty_j$ only when $i = j$. Then $\mu(e_i)(\infty_i)$ defines a trivialisation $\tau$ of the fibre of $E$ at $\infty_i$, i.e. over $\lambda = \infty$.

We can reobtain $\hat{L}(\lambda)$ from the triple $(S, E, \tau)$ as follows. The trivialisation $\tau$, along with condition (2) of Lemma (2.2) allows us to fix a basis $\{f_i\}$ of $H^0(S, E)$ by the condition:

$$f_i(\infty_j) = \delta_{ij}. \tag{2.5}$$

Then $\hat{L}(\lambda)$ is defined as the endomorphism of $H^0(S, E)$ (expressed in the basis $\{f_i\}$) defined for each $\lambda$ by the diagram:

$$\begin{align*}
H^0(S, E) & \longrightarrow H^0(S \cap F_\lambda, E) \\
\downarrow \hat{L}(\lambda) & \downarrow \times z \\
H^0(S, E) & \longrightarrow H^0(S \cap F_\lambda, E)
\end{align*} \tag{2.6}$$
where $\times z$ denotes multiplication by the fibre coordinate $z$. This is equivalent to building a resolution:

$$
0 \to \left[ \mathcal{O}(-m)^{\oplus r} \simeq H^0(S, E) \otimes \mathcal{O}(-m) \right] \xrightarrow{K(z, \lambda)} \left[ \mathcal{O}^{\oplus r} \simeq H^0(S, E) \otimes \mathcal{O} \right] \to E \to 0.
$$

(2.7)

Let $\mathcal{U}$ be the variety of equivalence classes of triples $(S, E, \tau)$ such that

1. $S$ satisfies the conditions (1)-(3) of Lemma 2.1, is generic in the sense of conditions (2), (3) and is such that the curve $\tilde{S}$ obtained by desingularising at the $(\alpha_i, 0)$ is smooth.

2. $E$ satisfies conditions (1) and (2) of Lemma 2.2, and is generic in the sense of condition (1).

3. $\tau$ is a trivialisation of $E$ over the $r$ points of $S$ at infinity.

**Theorem 2.3 [AHH1].** There is a biholomorphic equivalence of a non-empty Zariski open subset of $Q_{N_0}$ in case (i) and of $Q_0$ in case (ii) with $\mathcal{U}$.

The geometric picture then consists of a $2(\tilde{g} + r - 1)$ dimensional space $\mathcal{U}$ of triples $(S, E, \tau)$ that projects to a $(\tilde{g} + r - 1)$ dimensional space $\mathcal{W}$ of curves $S$:

$$
\sigma : \mathcal{U} \to \mathcal{W}.
$$

(2.8)

From the discussion of the previous section, it follows that the fibres of this map form a Lagrangian foliation of $\mathcal{U}$, since the coefficients of the spectral curve provide a $(\tilde{g} + r - 1)$-parameter family of commuting Hamiltonians defining this foliation. The infinitesimal aspects of this picture are described by the following theorem.

**Theorem 2.4.** (1) The tangent space at $S$ to the space $\mathcal{W}$ of curves is $H^0(\tilde{S}, K_{\tilde{S}}(1))$, where $K_{\tilde{S}}$ is the canonical bundle of $\tilde{S}$.

(2) Fixing $S$, the tangent space to the fibre $\sigma^{-1}(S)$ is $H^1(\tilde{S}, \mathcal{O}(-1))$.

**Proof.** (1) First order variations of a curve $\tilde{S}$ immersed in $\mathcal{T}$ are in one to one correspondence with sections of the normal bundle $N$ of $\tilde{S}$ in $\mathcal{T}$. These variations are not, however, entirely free: they are constrained to vanish at $\{\lambda = \alpha_i\}_{i=1, \ldots, n}$, and in case (ii) at $\lambda = \infty$. In other words, they must vanish at the zeroes of a section of $\mathcal{O}(m+1)$, and so the permissible variations of $\tilde{S}$ correspond to $H^0(\tilde{S}, N \otimes \mathcal{O}(-m-1))$. However, $\mathcal{O}(-m-2)$ is isomorphic to the canonical bundle $K_{\mathcal{T}}$ of the surface $\mathcal{T}$. Using the notation of eq. (1.18), this can be seen from the fact that

$$
d\tilde{z} \wedge d\tilde{\lambda} = -\lambda^{-m-2}dz \wedge d\lambda.
$$

(2.9)
From the adjunction formula, we have $K_\tilde{S} = N \otimes K_\tau$, and hence the tangent space to $W$ at $S$ is $T_S(W) = H^0(\tilde{S}, K_{\tilde{S}}(1))$.

(2) Line bundles $\tilde{E}$ on $\tilde{S}$ are classified by $H^1(\tilde{S}, \mathcal{O}^*)$. Using the exponential exact sequence,

$$0 \to \mathbb{Z} \to \mathcal{O} \to \mathcal{O}^* \to 0,$$

we can express infinitesimal variations of $\tilde{E}$ as elements of $H^1(\tilde{S}, \mathcal{O})$. If we assume now a fixed trivialisation of $\tilde{E}$ at $\lambda = \infty$, then in considering an infinitesimal variation of an equivalence class represented by a cocycle $\beta$ over $U_0 \cap U_1$, we can only modify $\beta$ by functions on $U_0$, and functions on $U_1$ which vanish at $\lambda = \infty$. This means in effect that such variations are represented by the cohomology group $H^1(S, \mathcal{O}(-1))$. □

We can write down explicit formulae as follows. The image $S$ of $\tilde{S}$ lies in the linear system $| \mathcal{O}(rm) |$ in $TP_1$. If $S$ were smooth, sections of the normal bundle of $S$ would be sections of $\mathcal{O}(rm)$ and hence represented by polynomials $f(\lambda, z)$ of degree $(rm)$, with the convention $\deg(\lambda) = 1, \deg(z) = m$. The normal vector field corresponding to $f$ would be:

$$f(\lambda, z) \frac{\partial}{\partial z}.$$

The fact that $S$ may be singular at $(\alpha_i, 0)$ forces us to require that $f$ vanish to an appropriate degree at $(\alpha_i, 0)$, in order that (2.11) be finite along each branch of $S$ at $(\alpha_i, 0)$, and thus represent a section of $N_{\tilde{S}}$. (See the discussion in Section 1d.) Furthermore, since we want $S$ to remain fixed at $\lambda = \alpha_i$ and, in case (ii), at $\lambda = \infty$, we must require that (2.11) vanish at $\lambda = \alpha_i$ and, in case (ii), at $\lambda = \infty$. Thus (2.11) is divisible by $a(\lambda)$. The explicit identification $\mathcal{O}(-m-2) \simeq K_\tau$ used here then tells us that the 1–form with a pole at $\lambda = \infty$ corresponding to (2.11) is:

$$f(\lambda, z) \frac{\partial}{\partial z} \frac{d\lambda}{a(\lambda) \partial P/\partial z(\lambda, z)}.$$

At a point $(S, E, \tau)$ of $U$, let $\beta(\lambda, z)$ be a cocycle on $U_0 \cap U_1$ representing a class in $H^1(S, \mathcal{O}(-1))$ and let $T(\lambda, z)$ be a transition function for $E$ over $U_0 \cap U_1$ with respect to trivialisations over $U_0, U_1$ compatible with the trivialisation $\tau$. We can write down a one-parameter family $(E_s, \tau_s)$ deforming $(E_0, \tau_0) := (E, \tau)$, with derivative at $s = 0$ equal to the class $[\beta]$, by choosing for $E_s$ the transition function (with respect to trivialisations compatible with $\tau_s$)

$$T(\lambda, z, s) = T(\lambda, z)e^{s\beta(\lambda, z)}.$$
A geometric parametrization of the reduced orbits can be constructed in a similar fashion. In reducing, the following extra constraints must be imposed on the spectrum of $\hat{L}(\lambda)$, and so on the spectral curve $S$:

(3') In case (i), $S$ intersects infinity ($\bar{\lambda} = 0$) at the points $\bar{z} = P_i$, where $P_i$ are distinct constants. In case (ii), the branches of $S$ at infinity have expansions $\bar{z} = Y_i + \bar{\lambda}P_i + O(\bar{\lambda}^2)$, $P_i$ constants.

We then quotient by the action of the diagonal group. This action can be thought of in terms of changing the trivialisation $\tau$, so we define $U_{\text{red}}$ to be the variety of equivalence classes of pairs $(S, E)$, where:

1. The curve $S$ in $T$ satisfies conditions (1) and (2) of Lemma 2.1 and condition (3'), is generic in the sense of conditions (2) and is such that the curve $\bar{S}$ obtained by desingularising at the $(\alpha_i, 0)$ is smooth.

2. The torsion free sheaf $E$ satisfies conditions (1) and (2) of Lemma 2.2 and is generic in the sense of condition (1)

**Theorem 2.5.** There is a biholomorphic equivalence of a non-empty Zariski open subset of $Q_{\text{red}}$ with the set $U_{\text{red}}$.

Again, we have a submersion $\sigma_{\text{red}} : U_{\text{red}} \to W_{\text{red}}$ onto a space $W_{\text{red}}$ of curves, with $U_{\text{red}}, W_{\text{red}}$ of dimensions $2\tilde{g}, \tilde{g}$ respectively. The fibre of this map at $S$ is a Lagrangian submanifold, and is in essence the complement of the theta-divisor in the Jacobian of $S$.

For the infinitesimal picture, we can repeat the reasoning of Theorem 2.4. Since we are imposing one extra order of constraint on the curve at $\lambda = \infty$, the first order variations of the spectral curve now correspond to sections of $N \otimes \mathcal{O}(-m - 2) = K_S$. On the other hand, we are just considering infinitesimal variations of the bundles $E$ since there is no trivialization $\tau$ fixed at infinity, and so we obtain

**Theorem 2.6.** The tangent space at $S$ to $W_{\text{red}}$ is $H^0(\bar{S}, K_{\bar{S}})$. Fixing $S$, the tangent space of $\sigma_{\text{red}}^{-1}(S)$ is given by $H^1(\bar{S}, \mathcal{O})$.

2b. Symplectic and Algebraic Geometry

The maps $\mathcal{U} \to \mathcal{W}$, $U_{\text{red}} \to W_{\text{red}}$, the theorems in the preceding sections, and Serre duality give us exact sequences, at a point $p$ of $\mathcal{U}$ or $U_{\text{red}}$:

\begin{align*}
0 \to H^0(\bar{S}, K_{\bar{S}}(1))^* &\to T_p \mathcal{U} \to H^0(\bar{S}, K_{\bar{S}}(1)) \to 0 \quad (2.14a) \\
0 \to H^0(\bar{S}, K_{\bar{S}})^* &\to T_p U_{\text{red}} \to H^0(\bar{S}, K_{\bar{S}}) \to 0. \quad (2.14b)
\end{align*}
Given splittings of these sequences the tangent spaces can be written as sums of vector spaces of the form \( A^* \oplus A \), where \( A = H^0(\tilde{S}, K_{\tilde{S}}(1)) \) in the unreduced case and \( A = H^0(\tilde{S}_1, K_{\tilde{S}}) \) in the reduced case. A natural skew form \( \omega \) can then be defined on such a sum:

\[
\omega((a, v)(b, u)) = a(u) - b(v). \tag{2.15}
\]

A splitting of (2.14a,b) amounts to finding some way, infinitesimally, of fixing the line bundle (and trivialisation) while varying the curve. There is a natural geometric way of doing this. At a point \((S, E, \tau)\) of \( U \) (resp. \((S, E)\) of \( U_{\text{red}} \)), we extend the bundle \( E \) to a neighbourhood of \( S \) in \( T \). Similarly, we extend the trivialisation to a neighbourhood of \( S \cap F_{\infty} \) in \( F_{\infty} \). This gives us a “background” \((E, \tau)\) (resp. \( E \)) to restrict to variations of the curve. These extensions, and the splittings of (2.14a,b) that they define, are not unique. They do, however, define the same 2-forms via (2.15). (This will appear as a direct consequence of the proofs of Theorems 2.7 and 2.8 below). For the time being, let us suppose that some arbitrary choice has been made. Since these forms involve Serre duality, we denote them by \( \omega_S \) and \( \omega_{S,\text{red}} \). Let \( \omega_{\text{orb}}, \omega_{\text{red}} \) denote the Kostant-Kirillov forms on the orbits as in Section 1.

**Theorem 2.7.** Under the identification between \( U_{\text{red}} \) and a Zariski open set in the reduced orbit \( Q_{\text{red}} \) given in Theorem 2.5, we have

\[
\omega_{\text{red}} = \omega_S_{\text{red}}. \tag{2.16}
\]

**Proof.** Let \((S(x, t), E(x, t))\) be a two parameter family in our space, with \( S(x, t) \) defined by the equation \( P(\lambda, z, x, t) = 0 \), and the family of sections of \( E(x, t) \) given over \( \{U_i\}_{i=0,1} \) by the functions \( \{s_i(\lambda, z, x, t)\}_{i=0,1} \). We shall now evaluate the symplectic 2-form on the tangent vectors \( \frac{\partial}{\partial t}, \frac{\partial}{\partial x} \) to \( U \) at \((x, t) = (0, 0)\). To do this, we use the description given by Theorem 1.4. If \( \sum_{\nu}(\lambda_{\nu}, z_{\nu}) \) is the divisor of \( s \) away from \( \lambda = \infty \), so that the points \((\lambda_{\nu}, z_{\nu})\) are given by the simultaneous vanishing of \( f \) and \( s_0 \), then the reduced Kostant-Kirillov form evaluated on \( \frac{\partial}{\partial t}, \frac{\partial}{\partial x} \) is

\[
\omega_{\text{red}} \left( \frac{\partial}{\partial t}, \frac{\partial}{\partial x} \right) = \sum_{\nu} \frac{(\lambda_{\nu})_t(z_{\nu})_x - (\lambda_{\nu})_x(z_{\nu})_t}{a(\lambda_{\nu})}. \tag{2.17}
\]

where the subscripts \( x, t \) denote differentiation. Now define new variables \( \hat{\lambda}_{\nu}, \hat{z}_{\nu} \) by

\[
\begin{align*}
\hat{\lambda}_{\nu}(0, 0) &= \lambda_{\nu}(0, 0) \\
P(\hat{\lambda}_{\nu}(x, t), z(\hat{\lambda}_{\nu}(x, t)), 0, 0) &= 0 \\
s_0(\hat{\lambda}_{\nu}(x, t), z(\hat{\lambda}_{\nu}(x, t)), x, t) &= 0 \\
\hat{z}_{\nu}(0, 0) &= z_{\nu}(0, 0) \\
P(\lambda_{\nu}(0, 0), \hat{z}_{\nu}(x, t), x, t) &= 0.
\end{align*} \tag{2.18}
\]
Here $\hat{\lambda}_\nu(x,t)$ is the $\lambda$-coordinate of the point cut out on $\mathcal{S}(0,0)$ by the equation $s_0(\lambda, z, x, t) = 0$ and $\hat{z}_\nu$ represents the variation of the $z$-coordinate of $\mathcal{S}(x, t)$ over $\lambda = \lambda_\nu(0,0)$. Implicit differentiation of the equations (2.18) and of the corresponding equations for $\lambda_\nu, z_\nu$, allows one to show that at $(x,t) = (0,0)$,

\begin{equation}
(\lambda_\nu)_t(z_\nu)_x - (\lambda_\nu)_x(z_\nu)_t = (\hat{\lambda}_\nu)_t(\hat{z}_\nu)_x - (\hat{\lambda}_\nu)_x(\hat{z}_\nu)_t. \tag{2.19}
\end{equation}

Let $d$ denote exterior differentiation along the curve $\mathcal{S}(x, t)$, with $(x,t)$ fixed. We have, for $(x,t)$ small,

\begin{equation}
\hat{\lambda}_\nu = \frac{1}{2\pi i} \oint_{C_\nu} \lambda \, d \ln s_0 \tag{2.20}
\end{equation}

for some suitable contour $C_\nu$, and so

\begin{equation}
(\hat{\lambda}_\nu)_t \text{ or } x = \frac{1}{2\pi i} \oint_{C_\nu} \lambda (d \ln s_0)_t \text{ or } x = -\frac{1}{2\pi i} \oint_{C_\nu} (\ln s_0)_t \text{ or } x d\lambda. \tag{2.21}
\end{equation}

At $(x,t) = (0,0)$

\begin{equation}
(\hat{z}_\nu)_t \text{ or } x = -\frac{\mathcal{P}_t \text{ or } x(\lambda_\nu, z_\nu, 0,0)}{\mathcal{P}_z(\lambda_\nu, z_\nu, 0,0)}. \tag{2.22}
\end{equation}

Let

\begin{equation}
F = \left[ \left( \frac{\mathcal{P}_x}{a(\lambda)\mathcal{P}_z} \right) d\lambda \right] \left( \ln s_0 \right)_t - [x \leftrightarrow t]. \tag{2.23}
\end{equation}

Then at $(x,t) = (0,0)$, we have

\begin{equation}
\frac{(\lambda_\nu)_t(z_\nu)_x - (\lambda_\nu)_x(z_\nu)_t}{a(\lambda_\nu)} = \frac{1}{2\pi i} \oint_{C_\nu} F. \tag{2.24}
\end{equation}

Choose a base point $\lambda_0$ and cut open the Riemann surface $\tilde{\mathcal{S}}(0,0)$ into a $4\tilde{g}$-gon ($\tilde{g} = \text{genus}(\tilde{\mathcal{S}})$) in the standard fashion. Then,

\begin{equation}
\omega_{\text{red}} \left( \frac{\partial}{\partial t}, \frac{\partial}{\partial x} \right) = \frac{1}{2\pi i} \sum_\nu \oint_{C_\nu} F
= \frac{1}{2\pi i} \left( \oint_{\text{edge of } 4\tilde{g}-\text{gon}} F - \sum_j \oint_{D_j} F \right), \tag{2.25}
\end{equation}

where the $D_j$ are contours around the points $\infty_j$ over $\lambda = \infty$. Since the expression $F$ in (2.23) is defined on the curve itself along the cut locus, the contributions of the two sides of the cut to (2.25) cancel, so the integral along the edge is zero. Also, the 1-form
\( \frac{P_x}{a(\lambda)P_z} \) \( d\lambda \) is holomorphic at \( \lambda = \infty \). As above, we then write the transition function for the line bundle \( E(x,t) \) to first order in \((x,t)\) as \( T(\lambda, z) e^{t\beta_t + x\beta_x} \), so that

\[
\begin{align*}
  s_0(\lambda, z, x, t) &= s_1(\lambda, z, x, t) T(\lambda, z) e^{t\beta_t + x\beta_x} (1 + r(\lambda, z, x, t)P(\lambda, z, x, t)) \\
  \text{(2.26)}
\end{align*}
\]

for some function \( r \) on \( U_0 \cap U_1 \), where \( s_0, s_1 \) represent sections along the curve. Substituting into (2.25) gives

\[
\begin{align*}
  \omega_{\text{red}} \left( \frac{\partial}{\partial t}, \frac{\partial}{\partial x} \right) &= \frac{1}{2\pi i} \sum_j \oint_{D_j} \left( \frac{P_x \, d\lambda}{a(\lambda)P_z} \right) \left( \beta_t + (\ln s_1)_t + \ln(1 + rP)_t \right) - (x \leftrightarrow t) \\
  \text{(2.27)}
\end{align*}
\]

Of the three terms in the integrand, the second gives zero since \( (\ln s_1)_t \) is holomorphic at \( \lambda = \infty \), and the third vanishes after antisymmetrization \((x \leftrightarrow t)\). Thus:

\[
\begin{align*}
  \omega_{\text{red}} \left( \frac{\partial}{\partial t}, \frac{\partial}{\partial x} \right) &= \frac{1}{2\pi i} \sum_j \oint_{D_j} \frac{P_x}{a(\lambda)P_z} d\lambda \cdot \beta_t - (x \leftrightarrow t) \\
  \text{(2.28)}
\end{align*}
\]

To complete the proof, we note that:

1) \( \frac{P_x \, d\lambda}{a(\lambda)P_z}, \frac{P_x \, d\lambda}{a(\lambda)P_z} \) are simply the elements of \( H^0(\tilde{S}, K_{\tilde{S}}) \) corresponding to the variations of the curves in the \( x, t \) directions, and \( \beta_t, \beta_x \) are representative cocycles for the elements of \( H^1(\tilde{S}, \mathcal{O}) \) representing the variations of the line bundles \( E(x,t) \), as in Theorem 2.4.

2) The sum of the contour integrals around the \( D_j \) is the explicit representation of the Serre duality pairing, when the cocycles are chosen with respect to the \( U_0, U_1 \) covering. \( \square \)

For the unreduced case, the corresponding result is:

**Theorem 2.8.** Under the identification between \( \mathcal{U} \) and a Zariski open set in the orbit \( Q_{N_0} \) for case (ii), and the symplectic submanifold \( Q_{N_0}^0 \) for case (i) given in Theorem 2.3, we have

\[
\begin{align*}
  \text{Case (i)} \quad \omega_{\text{orb}}|_{Q_{N_0}^0} &= \omega_S + \frac{1}{2} \sum_{i \neq j} \frac{dP_i \wedge dP_j}{P_i - P_j} \quad \text{over} \quad Q_{N_0}^0 \\
  \text{(2.29a)} \\
  \text{Case (ii)} \quad \omega_{\text{orb}} &= \omega_S. \\
  \text{(2.29b)}
\end{align*}
\]

**Proof.** Repeating verbatim the proof of Theorem 2.7, we have, instead of (2.27)

\[
\sum_{\mu} \frac{dz_\mu}{a(\lambda_\mu)} \left( \frac{\partial}{\partial t}, \frac{\partial}{\partial x} \right) = \frac{1}{2\pi i} \sum_j \oint_{D_j} \frac{P_x \, d\lambda}{a(\lambda)P_z} \left( \beta_t + (\ln s_1)_t + (\ln(1 + rP))_t \right) - (x \leftrightarrow t) \\
\text{(2.30)}
\]
Again, the first term corresponds to \( \omega_S \), and the third term disappears after antisymmetrization. The second term, however, does not, since now the 1-form \( \frac{\rho_a d\lambda}{a(\lambda) P_x} \) can have a pole at \( \lambda = \infty \). The residue of this pole at the point \( \infty_i \) is precisely \( -(P_1)_x \).

Now let us recall the explicit rebuilding of \( \hat{L}(\lambda) \) from the sections of \( E \) (see [AHH1]). If \( f_j \) is the basis of sections of \( E \), normalised as in (2.5), then for each \( \lambda \), we can evaluate \( f_j \) in the \( U_0 \) trivialisation at the \( r \) points \( (\lambda, z_i(\lambda)) \) of \( S \) above \( \lambda \), and set:

\[
(\psi(\lambda))_{ij} = f_j(\lambda, z_i(\lambda)).
\] (2.31)

Then

\[
\hat{L}(\lambda) = \psi^{-1} \cdot \text{diag}(z_i(\lambda)) \cdot \psi.
\] (2.32)

Near \( \lambda = \infty \), \( \psi \) can be expanded:

\[
(\psi)_{ij} = \delta_{ij} + \bar{\lambda} \gamma_{ij} + O(\bar{\lambda}^2).
\] (2.33)

Dividing (2.32) by \( \lambda^m \), we have, in case (i)

\[
\hat{L}(\lambda) \cdot \lambda^{-n+1} = L_0 + \bar{\lambda} L_1 + \cdots
\]

\[
= \text{diag}(P_i) + \bar{\lambda} ([\text{diag}(P_i), \gamma] + \text{diag}((L_1)_{ii})) + O(\bar{\lambda}^2),
\] (2.34a)

and, in case (ii)

\[
\hat{L}(\lambda) \lambda^{-n} = Y + \bar{\lambda}(L_0 - \sum_{j=1}^{n} \alpha_j Y) + \cdots
\]

\[
= \text{diag}(Y_i) + \bar{\lambda} ([\text{diag}(Y_i), \gamma] + \text{diag} \left( P_i - \sum_{j=1}^{n} \alpha_j Y_i \right)) + O(\bar{\lambda}^2).
\] (2.34b)

The section \( s = (s_0, s_1) \) used in our calculations is just \( f_1 \), so that

\[
s_1(\lambda, \bar{z}_i(\lambda)) = \delta_{i1} + \bar{\lambda} \gamma_{i1} + O(\bar{\lambda}^2).
\] (2.35)

Thus, at \( \bar{\lambda} = 0 \) (\( \lambda = \infty \)),

\[
\left( \ln s_1(\lambda, \bar{z}_1(\lambda)) \right)_t = 0
\] (2.36)

and for \( i \neq 1 \), at \( \bar{\lambda} = 0 \), in case (i), using (1.63),

\[
\left( \ln s_1(\lambda, \bar{z}_i(\lambda)) \right)_t = \left( \ln (L_1)_{i1} \right)_t - \left( \ln (P_i - P_1) \right)_t
\]

\[
= (q_i - \frac{1}{2} \sum_{j \neq i, j > 1} \ln (P_i - P_j))_t - \left( \ln (P_i - P_1) \right)_t;
\] (2.37a)
while in case (ii), using (1.64),

\[
\ln s_1(\tilde{\lambda}, \tilde{z}_i(\tilde{\lambda}))_t = \ln((L_0)_{ii})_t
= (q_i)_t,
\]

(2.37b)

and similarly for \((\ln s_1)_x\). Referring to Theorem 1.5, in case (i), the expression (2.30) becomes:

\[
\omega_{orb} \left( \frac{\partial}{\partial t}, \frac{\partial}{\partial x} \right) - m \sum_{i=2}^{m} \left( (q_i)_t(P_i)_x - (q_i)_x(P_i)_t \right) \\
= \omega_S \left( \frac{\partial}{\partial t}, \frac{\partial}{\partial x} \right) - m \sum_{i=2}^{m} ((q_i)_t(P_i)_x - (q_i)_x(P_i)_t) \\
+ \frac{1}{2} \sum_{i \neq j} (P_i)_t(P_j)_x - (P_i)_x(P_j)_t. \quad (2.38a)
\]

and in case (ii):

\[
\omega_{orb} \left( \frac{\partial}{\partial t}, \frac{\partial}{\partial x} \right) - m \sum_{i=2}^{m} \left( (q_i)_t(P_i)_x - (q_i)_x(P_i)_t \right) \\
= \omega_S \left( \frac{\partial}{\partial t}, \frac{\partial}{\partial x} \right) - m \sum_{i=2}^{m} ((q_i)_t(P_i)_x - (q_i)_x(P_i)_t), \quad (2.38b)
\]

thus proving the theorem.

\[\square\]

\textbf{2c. Reductions to subalgebras}

Reductions to subalgebras of \(\tilde{\mathfrak{gl}}(r)^+\) can be obtained by considering the fixed point sets of one or several involutions \(\tilde{\sigma}\) on \(\tilde{\mathfrak{gl}}(r)^+\) (cf. [AHP, AHH1, HHM]). This procedure can be used to obtain all the “classical” loop algebras. Coadjoint orbits in the reduced algebras correspond to unions of components of the fixed point sets on the orbits of the unreduced algebra.

\textbf{2c.1. Involution on Loop Algebras Induced by Involution on \(\mathfrak{gl}(r, \mathbb{C})\).}

Let \(\sigma : \mathfrak{gl}(r, \mathbb{C}) \to \mathfrak{gl}(r, \mathbb{C})\) be an involutive automorphism. We can define corresponding linear (resp. antilinear) involutions on \(\tilde{\mathfrak{gl}}(r)^+\) by

\[
\tilde{\sigma}(\tilde{\mathcal{L}})(\lambda) = \sigma(\tilde{\mathcal{L}}(\lambda)) \quad \text{(resp.} \sigma(\tilde{\mathcal{L}}(\lambda)),
\]

(2.39)
depending on whether \( \sigma \) is linear or antilinear. The involutions \( \sigma \) that occur in reductions to the classical algebras fall into three types.

*Type (i) (antilinear involution)*

\[
\sigma(x) = \gamma x \gamma^{-1}, \quad \gamma^2 = \pm 1
\]  
(2.40)

(This gives, e.g., the reduction to \( \widetilde{\mathfrak{gl}}(r, \mathbb{R})^+ \).) The corresponding \( \tilde{\sigma} \) induces an antiholomorphic involution \( i \) on the “spectral surface” \( T \):

\[
i(z, \lambda) = (\bar{z}, \bar{\lambda}).
\]  
(2.41)

A fixed point of \( \tilde{\sigma} \) has its spectrum fixed by \( i \). From the exact sequence (2.12) defining \( E \), we have:

\[
0 \longrightarrow \mathcal{O}(-m)^{\oplus r} \xrightarrow{(z1 - \bar{L}(\lambda))} \mathcal{O}^{\oplus r} \xrightarrow{i^*E} 0,
\]  
(2.42a)

whereas, the sheaf \( \tilde{\sigma}(E) \) corresponding to \( \tilde{\sigma}(\hat{L}) \) is given by:

\[
0 \longrightarrow \mathcal{O}(-m)^{\oplus r} \xrightarrow{\gamma(z1 - \bar{L}(\lambda))\gamma^{-1}} \mathcal{O}^{\oplus r} \xrightarrow{\tilde{\sigma}(E)} 0.
\]  
(2.42b)

The maps \( \gamma : \mathcal{O}^{\oplus r} \to \mathcal{O}^{\oplus r}, \gamma : \mathcal{O}(-m)^{\oplus r} \to \mathcal{O}(-m)^{\oplus r} \) then give us, by (2.42a,b), an isomorphism between \( \tilde{\sigma}(E) \) and \( \tilde{i}^*E \). If we assume that \( \gamma \) preserves the diagonal form at infinity, we obtain (see also [HHM]):

**Theorem 2.9.** For involutions (2.40) of type (i),

1. The fixed point set of \( \tilde{\sigma} \) on the orbit \( Q_{N_0} \), or on \( Q_{\text{red}} \), is a real symplectic manifold, with (real) symplectic form given by the restriction of the symplectic form on the ambient space.

2. Under the identification of \( Q_{N_0} \) with the set \( \mathcal{U} \) of triplets \((S, E, \tau)\) given in Theorem 2.3, the action of \( \tilde{\sigma} \) is

\[
\tilde{\sigma}(S, E, \tau) = (i(S), \tilde{i}^*E, \tilde{i}^*\tau).
\]  
(2.43)

For this case, real Darboux coordinates may also be obtained. If we choose an eigenvector \( V_0 \) in (1.29) which is invariant under the corresponding involution on \( \mathbb{C}^r \), we obtain an \( i \)-invariant divisor \( \sum(\lambda_\mu, z_\mu) \). The points in the divisor can be ordered so that

\[
(\lambda_{2\mu}, \bar{z}_{2\mu}) = (\lambda_{2\mu+1}, \bar{z}_{2\mu+1}), \quad \mu = 1, \ldots, s
\]

\[
(\lambda_\nu, \bar{z}_\nu) = (\lambda_\nu, \bar{z}_\nu), \quad \nu = 2s + 2, \ldots, \tilde{g}
\]  
(2.44)
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for some $s$. It follows that the real and imaginary parts

\[
\left( \frac{1}{\sqrt{2}} \text{Re}(\lambda_{2\mu}), \frac{1}{\sqrt{2}} \text{Re}(z_{2\mu}), \left( \frac{1}{\sqrt{2}} \text{Im}(\lambda_{2\mu}), \frac{1}{\sqrt{2}} \text{Im}(z_{2\mu}) \right)_{\mu=1,\ldots,s}, \ (\lambda_{\nu}, z_{\nu})_{\nu=2s+2,\ldots,g} \right)
\]

(2.45)

are Darboux coordinates on the real submanifold of fixed points in $Q_{\text{red}}$. For the unreduced case $Q_{\bar{N}_0}$, the remaining Darboux coordinates are similarly obtained from $(q_i, P_i)_{i=2,\ldots,r}$.

Type (ii) (linear involution)

\[
\sigma(x) = -\gamma T \gamma^T, \quad \gamma = \pm \gamma^{-1} = \pm \gamma^T
\]

(2.46)

(This gives, e.g., reductions to $o((r, C))$ and $sp((\frac{r}{2}, C))$. Here, $\tilde{\sigma}$ induces on $T$ the involution:

\[
i(\lambda, z) = (\lambda, -z),
\]

(2.47)

and so determines a corresponding map on spectral curves. For the line bundles the map is slightly more complicated. From the defining sequence (2.1) for $E$ we have, for $i^*E$, $\sigma(E)$:

\[
0 \to \mathcal{O}(-m) \oplus \mathcal{O}(\mathcal{O} \oplus \mathcal{O} \to i^*E \to 0
\]

(2.48a)

\[
0 \to \mathcal{O}(-m) \oplus \mathcal{O}(-1) \oplus \mathcal{O} \to \sigma(E) \to 0.
\]

(2.48b)

Locally, we can use (2.48a,b) to represent sections of $i^*E$, $\sigma(E)$ by $a, b \in \mathcal{O} \oplus \mathcal{O}$, respectively. If $<, >$ denotes the standard bilinear pairing $\mathcal{O} \times \mathcal{O} \to \mathcal{O}$, consider:

\[
< \gamma(z \mathbf{1} - \hat{L}(\lambda))a, b >
\]

(2.49)

where $\sim$ denotes, as above, the classical adjoint. It is easy to check, from (2.48a,b), that this projects to give a pairing of $i^*E$ with $\sigma(E)$ over $i(S)$. (Remember that $i(S)$ has equation $\det(\mathbf{1} - \hat{L}(\lambda)) = 0$.) Since the entries of $(-z \mathbf{1} - \hat{L}(\lambda))$ lie in $H^0(S, \mathcal{O}((r-1)m))$, this gives a globally defined map:

\[
i^*E \otimes \sigma(E) \rightarrow \mathcal{O}((r-1)m).
\]

(2.50)

When $i(S)$ is smooth, the adjunction formula tells us that $\mathcal{O}((r-1)m)$ is $K_{i(S)}(2)$. Since in this case, the adjoint matrix $(-z \mathbf{1} - \hat{L}(\lambda))$ is everywhere of rank one, we can show directly from the sequences (2.48) that (2.49) is surjective, and so

\[
\sigma(E) \simeq K_{i(S)}(2) \otimes (i^*E)^*.
\]

(2.51)
When $i(S)$ is not smooth then, in the generic case considered, we can identify the canonical bundle $K_{i(\tilde{S})}$ of the desingularisation $i(\tilde{S})$ of $i(S)$ with $\mathcal{O}((r-1)m)[-D]$, where $D$ is a positive divisor supported by the singularities of $i(S)$ (see [AHH1]). On the other hand, $(-z1 - \hat{L}(\lambda))$ also vanishes at the singular points of $i(S)$ in such a way that the image of (2.50) in $\mathcal{O}((r-1)m)$ is also $\mathcal{O}((r-1)m)[-D]$. Therefore again, with a slight abuse of notation:

$$\sigma(E) \simeq K_{i(\tilde{S})}(2) \otimes (i^*(E))^\ast.$$ (2.52)

Now recall that over $\lambda = \infty$, $\hat{L}$ is diagonal. We assume that $\gamma$ preserves this form. The trivialisation of $\sigma(E)$ is then the same as that of $i^*E$. Summing up:

**Theorem 2.10.** For involutions (2.46) of type (ii),

1. The fixed point set of $\tilde{\sigma}$ on the orbit $Q$ (or $Q_{\text{red}}$) is a complex symplectic manifold.

2. Under the identifications of Theorem 2.3, the action of $\tilde{\sigma}$ on the triplet $(S, E, \tau)$ is

$$\tilde{\sigma}(S, E, t) = (i(S), (i^*(E))^\ast \otimes K_{i(\tilde{S})}(2), i^*\tau).$$ (2.53)

**Remarks:**

1. If $i(S) = S$, the fixed point set of the action of $\tilde{\sigma}$ on the Jacobian of $S$ is a translate of the Prym variety associated to $i$.

2. It is not clear geometrically what the Darboux coordinates should be in this case. When $S$ is hyperelliptic (so that there is an extra involution $j$) such coordinates can be found [AvM].

**Type (iii) (antilinear involution)**

$$\sigma(x) = -\gamma x^T \gamma^{-1}, \quad \gamma = \pm \gamma^{-1} = \pm \gamma^T$$ (2.54)

(This gives, e.g., the reduction to $u(p,q)$.) For this case, $\tilde{\sigma}$ induces on $T$ the involution

$$i(\lambda, z) = (\overline{\lambda}, -\overline{z}),$$ (2.55)

and so determines a map on spectral curves. Proceeding as above, we have:

**Theorem 2.11.** For involution (2.54) of type (iii),

1. The fixed point set of $\tilde{\sigma}$ in the orbits $Q$ (or $Q_{\text{red}}$) is a real symplectic manifold, with (real) symplectic form given by restriction of the symplectic form on the ambient space.
2) The action of $\tilde{\sigma}$ on the triplet $(S, E, \tau)$ is

$$\tilde{\sigma}(S, E, \tau) = \left( i(S), K_{i(S)}(2) \otimes (i^*E)^*, i^*\tau \right).$$ (2.56)

2c.2 Twisted Involutions

Given an involution $\sigma : \mathfrak{gl}(r) \to \mathfrak{gl}(r)$ we can define a “twisted” involution $\hat{\sigma}$ on $\tilde{\mathfrak{gl}}(r)$ by

$$\hat{\sigma}(\hat{\mathcal{L}})(\lambda) = \sigma(\hat{\mathcal{L}}(-\lambda)).$$ (2.57)

The case by case study of types (i), (ii) and (iii) above can be repeated. For each of these, the map $i$ induced by $\hat{\sigma}$ on $\mathcal{T}$ is that induced by $\tilde{\sigma}$, composed with $(\lambda, z) \mapsto (-\lambda, z)$. With this modification of $i$, Theorems 2.9, 2.10 and 2.11 again hold verbatim.

3. Examples.

In the following, we examine four applications of the above analysis: computation of Darboux coordinates on generic coadjoint orbits of $\mathfrak{sl}(2)$ and $\mathfrak{sl}(3)$; finite dimensional integrable systems involving isospectral flows in $\tilde{\mathfrak{sl}}(2)^*$; the cubically nonlinear Schrödinger equation (NLS) and the coupled 2-component nonlinear Schrödinger system (CNLS). Details on how these systems arise through moment map embeddings from a space of rank 2 or 3 perturbations of $N \times N$ matrices may be found in [AHP].

3a. Darboux Coordinates for $\mathfrak{sl}(2)^*$ and $\mathfrak{sl}(3)^*$

As a first application of the results of Section 1, we compute Darboux coordinates on generic coadjoint orbits of the algebras $\mathfrak{sl}(r)$, $r = 2, 3$ by viewing these as Poisson subspaces of the corresponding loop algebra $\tilde{\mathfrak{sl}}(r)^{**}$. Thus, we choose $n = 1$ in eqs. (1.2), (1.3) and, without loss of generality, $\alpha_1 = 0$. The residue matrix $N_1 = L_0$ is identified as an element of $\mathfrak{sl}(r)^*$ and we consider the spectral curve

$$\mathcal{P}(\lambda, z) = \det(\hat{\mathcal{L}}(\lambda) - zI) = 0$$ (3.1)

for matrices of the form

$$\hat{\mathcal{L}}(\lambda) = \lambda Y + L_0 \in \tilde{\mathfrak{sl}}(r)^*, \quad r = 2, 3$$ (3.2)

where $Y \in \mathfrak{sl}(r)^*$ is a fixed matrix with simple spectrum. (Only case (ii), with $Y \neq 0$ will be considered, since we want coordinates on the full coadjoint orbits.)
The Darboux coordinates \( \{\lambda_{\mu}, \zeta_{\mu}\}_{\mu=1,...,g} \) are determined by eq. (1.29) with \( V_0 = (1, 0, \ldots)^T \in \mathbb{C}^r \), and the “missing” coordinates \( \{q_i, P_i\}_{i=2,...,r} \) corresponding to the spectral points over \( \lambda = \infty \) are given by eqs. (1.61), (1.64). According to the remark following Theorem 1.4, for each eigenvalue \( \tilde{z} \) of \( Y \) for which \( V_0 \) is in the image of \( Y - \tilde{z}I \), one of the \( g + r - 1 \) points in the spectral divisor will appear over \( \lambda = \infty \), requiring the addition of a pair \( (q_i, P_i) \) of “missing” coordinates to complete the Darboux system. In particular, if \( V_0 \) is chosen as an eigenvector of \( Y \), as in Theorem 1.5, there will be \( r - 1 \) such pairs associated to points at \( \infty \) and \( g \) pairs of “finite” spectral Darboux coordinates.

For \( r = 2 \), the ring of Casimir invariants is generated by \( \text{tr}(L_0^2) \) and the generic orbits are 2-dimensional. Let
\[
Y = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}, \quad L_0 = \begin{pmatrix} -a & r \\ u & a \end{pmatrix}.
\]
(3.3)
The spectral curve in this case has genus \( g = 0 \) and \( V_0 \) is an eigenvector of \( Y \); hence, there are no finite spectral divisor coordinates, only the pair of “missing” Darboux coordinates
\[
q_2 = \ln u, \quad P_2 = a,
\]
(4.4)
(valid for \( u \neq 0 \)) corresponding to the eigenvalue \( \tilde{z} = -1 \). Alternatively, choosing \( Y \) to be
\[
Y = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix},
\]
(3.5)the curve still has \( g = 0 \), but \( V_0 \) is not an eigenvector of \( Y \), so the spectral divisor consists of one point at finite \( \lambda \). Eq. (1.29) reduces to the linear system
\[
\lambda + u = 0 \\
z - a = 0,
\]
(3.6)providing the Darboux coordinates
\[
\lambda_1 = -u, \quad \zeta_1 = \frac{z_1}{\lambda_1} = -\frac{a}{u}.
\]
(3.7)

For \( r = 3 \), the ring of Casimir invariants is generated by \( \text{tr}L_0^2, \text{tr}L_0^3 \) and the generic orbits are 6-dimensional. Let
\[
Y = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & -1 \end{pmatrix}, \quad L_0 = \begin{pmatrix} -a - b & r & s \\ u & a & e \\ v & f & b \end{pmatrix}.
\]
(3.8)
The spectral curve in this case is generically elliptic \((g = 1)\) and \(V_0\) is an eigenvector of \(Y\). The first pair \((\lambda_1, \zeta_1 = \frac{z_1}{\lambda_1})\) of Darboux coordinates is obtained by solving the system:

\[
(\lambda - z + a)(\lambda + z - b) + ef = 0 \quad (3.9a)
\]

\[
u(\lambda + z - b) + ev = 0 \quad (3.9b)
\]

\[
v(\lambda - z + a) - uf = 0. \quad (3.9c)
\]

The second and third of these equations imply the first, and determine the coordinates:

\[
\lambda_1 = \frac{1}{2} \left( b - a - \frac{ev}{u} + \frac{uf}{v} \right) \quad (3.10a)
\]

\[
\zeta_1 = \frac{z_1}{\lambda_1} = \frac{uva + uvb - ev^2 - fu^2}{-uva + uvb - ev^2 + fu^2}, \quad (3.10b)
\]

when \(u, v\) and \(\lambda_1\) are nonzero. The remaining two points of the spectral divisor, corresponding to the eigenvalues \(\tilde{z} = 1, -1\) of \(Y\), lie over \(\lambda = \infty\). To complete the system, we must therefore add the two pairs of “missing” coordinates:

\[
q_2 = \ln u \quad P_2 = a \quad (3.11a)
\]

\[
q_3 = \ln v \quad P_3 = b. \quad (3.11b)
\]

Alternatively, we may pick \(Y\) so that \(V_0\) does not lie in the image of \(Y - \tilde{z}I\) for any eigenvalue \(\tilde{z}\) of \(Y\); e.g.

\[
Y = \begin{pmatrix} 0 & 1 & 0 \\ 1 & 0 & 1 \\ 0 & 1 & 0 \end{pmatrix}. \quad (3.12)
\]

The genus is still \(g = 1\), but the number of finite pairs of spectral divisor coordinates is now 3. These may be obtained by solving the pair of equations

\[
z^2 + z(v - a - b) + \lambda(u - e) + ab - ef + fu - av = 0 \quad (3.13a)
\]

\[
\lambda z + uz + (v - b)\lambda + ev - bu = 0, \quad (3.13b)
\]

which reduces to a cubic for \(z\), with generically distinct roots \((z_1, z_2, z_3)\). Setting

\[
\lambda_i = \frac{z_i^2 + (v - a - b)z_i + ab - av + fu - fe}{u - e}, \quad \zeta_i = \frac{z_i}{\lambda_i}, \quad i = 1, 2, 3 \quad (3.14)
\]

gives the Darboux system.
Continuing similarly for higher $r$, the ring of Casimirs for $\mathfrak{sl}(r)$ is generated by $\{\text{tr}L_i^t\}_{i=2,\ldots,r}$ and the generic orbits are $r(r-1)$-dimensional. The genus of the generic spectral curve is $g = \frac{1}{2}(r-2)(r-1)$ and there are, in principle, $r$ classes of spectral Darboux coordinates possible, in which the number of finite coordinate pairs is between $g$ and $g + r - 1$. In each case their determination involves the solution of a polynomial equation of corresponding degree.

3b. Finite Dimensional Systems and Isospectral Flows in $\widetilde{\mathfrak{sl}}(2)^{++}$

The moment map embedding of finite dimensional integrable systems as isospectral flows in loop algebras developed in [AHP] leads, in the case $\widetilde{\mathfrak{sl}}(2)$, to the following parametrization. In eqs. (1.1-1.3), let

\[
Y = \begin{pmatrix} a & b \\ c & -a \end{pmatrix} \in \mathfrak{sl}(2, \mathbb{C})
\]

and rank($N_i$) = $k_i$ = 1. Then $N_0(\lambda) \in \widetilde{\mathfrak{gl}}(2)^{++}$ may be taken of the form:

\[
N_0(\lambda) = \lambda \sum_{i=1}^n \frac{G_i^T F_i}{\alpha_i - \lambda} = \lambda G^T (A - \lambda I)^{-1} F,
\]

where $(F_i, G_i)_{i=1,\ldots,n}$ are the rows of a pair $F, G \in M^{n \times 2}$ of $n \times 2$ complex matrices and $A = \text{diag}(\alpha_i) \in M^{n \times n}$ is a diagonal matrix with distinct eigenvalues $(\alpha_i)_{i=1,\ldots,n}$. Imposing the trace-free conditions $\text{tr}(G_i^T F_i) = 0$ and using the freedom of replacing

\[
F_i \mapsto d_i F_i, \quad G_i \mapsto d_i^{-1} G_i, \quad d_i \in \mathbb{C} - 0,
\]

we may take $(F, G)$, without loss of generality, to be of the form:

\[
F = \frac{1}{\sqrt{2}}(x, y), \quad G = \frac{1}{\sqrt{2}}(y, -x),
\]

where $x, y \in \mathbb{C}^n$ are viewed as column vectors. (This amounts to a symplectic reduction with respect to the center of $\widetilde{\mathfrak{gl}}(2, \mathbb{C})^{++}$, giving flows in $\mathfrak{sl}(2, \mathbb{C})^{++}$.) The reduced orbital symplectic form is then just

\[
\omega = dx^T \wedge dy
\]

and $\mathcal{N}(\lambda)$ has the form

\[
\mathcal{N}(\lambda) = \lambda \begin{pmatrix} a & b \\ c & -a \end{pmatrix} + \frac{\lambda}{2} \left( -\sum_{i=1}^n \frac{x_i y_i}{\lambda - \alpha_i} - \sum_{i=1}^n \frac{y_i^2}{\lambda - \alpha_i} \right),
\]
where \((x_i, y_i)_{i=1,...,n}\) are the components of \((x, y)\). We now also impose the reality conditions
\[
x = \overline{x}, \quad y = \overline{y}, \quad Y = \overline{Y},
\] (3.21)
to obtain flows in \(\widetilde{\mathfrak{sl}}(2, \mathbb{R})^{++}\). The Hamiltonian systems obtained by pulling back the AKS ring \(\widetilde{\mathcal{I}}(\mathfrak{sl}(2)^*)\) through the map
\[
\widetilde{J}_Y : \mathbb{R}^n \times \mathbb{R}^n \longrightarrow \widetilde{\mathfrak{sl}}(2)^*,
\]
\[
\widetilde{J}_Y : (x, y) \longmapsto \lambda Y + \lambda G^T (A - \lambda I)^{-1} F = \mathcal{N}(\lambda)
\] (3.22)
are then Poisson commutative and, with the possible addition of certain quadratic constraints, coincide with those studied by Moser in [M] (cf. also [AHP, AHH1]). (The fibres of this map are generated by the finite group of reflections \((x_i, y_i) \mapsto (-x_i, -y_i)\) of the coordinate axes. Since the points with \((x_i = 0, y_i = 0)\) are excluded from the inverse image of the orbit \(Q_N\) by the condition \(\text{rank}(N_i) = 1\), the resulting ambiguity is resolved along the flows by continuity.)

As an illustrative example, consider the C. Neumann system [N]. This has been amply studied by a variety of methods in the literature [AvM, F, Kn, M, Sch, Ra1]. We include it here to show how our general approach reduces to the familiar results for this case, giving a complete separation of variables in hyperellipsoidal coordinates and linearization via a hyperelliptic Abel map. To obtain this system, we choose the matrix \(Y\) to be
\[
Y = \begin{pmatrix} 0 & -\frac{1}{2} \\ 0 & 0 \end{pmatrix},
\] (3.23)
and the Hamiltonian \(\phi\) to be:
\[
\phi(x, y) = -\text{tr}(\mathcal{N}(\lambda)^2)_0 = \frac{1}{2}[(x^T x)(y^T y) + x^T A x - (x^T y)^2],
\] (3.24)
where the subscript \((\ )_0\) signifies the \(\lambda^0\) term in the Laurent expansion around \(\lambda = 0\) for large \(\lambda\). To obtain the appropriate phase space, we must also add the symplectic constraints:
\[
x^T x = 1, \quad y^T x = 0,
\] (3.25)
defining the cotangent bundle \(T^* S^{n-1} \sim TS^{n-1} \subset \mathbb{R}^{2n}\). The Neumann oscillator Hamiltonian is
\[
H(x, y) = \frac{1}{2}[y^T y + x^T A x],
\] (3.26)
which coincides with \(\phi(x, y)\) on the constrained manifold. The constraints (3.25) may be viewed as a Marsden-Weinstein reduction under the stabilizer \(\text{Stab}(Y) \subset \mathfrak{sl}(2, \mathbb{R})\).
(cf. Section 1b), in which $x^T x = 1$ defines a level set of the moment map generating the flow

$$ (x, y) \mapsto (x, y + tx) \quad (3.27) $$

induced by the one-parameter subgroup $\text{Stab}(Y)$, while $y^T x = 0$ defines a section over the quotient of the level set by this flow (i.e., of the null foliation it generates). It follows that the $H$-flow of the constrained system is obtained from the $\phi$-flow of the free system simply by orthogonal projection of the momentum $y$ relative to $x$:

$$ (x(t), y(t))_{\text{free}} \mapsto (\ddot{x}(t), \ddot{y}(t))_{\text{constr.}} := \left( (x(t), y(t)) - \left( \frac{x^T(t) y(t)}{x^T(t)x(t)} \right) x(t) \right) \quad (3.28) $$

from the invariant manifold defined by $x^T x = 1$. The equations of motion for the unconstrained system are

$$ \frac{d}{dt} x = (x^T x) y - (x^T y) x \quad (3.29a) $$

$$ \frac{d}{dt} y = -(y^T y) x - Ax + (x^T y) y. \quad (3.29b) $$

These are equivalent (within a quotient by the finite group of reflections in the coordinate axes) to the Lax equation

$$ \frac{dN}{dt} = [B, N], \quad (3.30a) $$

where

$$ B = d\phi(N)_+ = \begin{pmatrix} x^T y & \lambda + y^T y \\ -x^T x & -x^T y \end{pmatrix}. \quad (3.30b) $$

The invariant spectral curve is thus given by the characteristic equation

$$ \det \left( \frac{N(\lambda)}{\lambda} - \zeta I \right) = 0, \quad (3.31) $$

which, defining as in Section 1

$$ z := a(\lambda) \zeta, \quad a(\lambda) := \prod_{i=1}^{n} (\lambda - \alpha_i), \quad (3.32) $$

determines a genus $g = n - 1$ hyperelliptic curve defined by (cf. eqs. (1.15), (1.22):

$$ z^2 - a(\lambda) P(\lambda) = 0, \quad (3.33) $$
where

\[ P(\lambda) := -P_2(\lambda) = -\frac{a(\lambda)}{4} \sum_{i=1}^{n} \frac{I_i}{\lambda - \alpha_i} \]

\[ = P_{n-1} \lambda^{n-1} + P_{n-2} \lambda^{n-2} + \cdots + P_0 \]  

(3.34a)

\[ P_{n-1} = -\frac{1}{4} \sum_{i=1}^{n} I_i = -\frac{1}{4} x^T x \]  

(3.34b)

\[ P_{n-2} = \frac{1}{4} \sum_{i=1}^{n} \alpha_i I_i = \frac{1}{2} \phi \]  

(3.34c)

and

\[ I_i := \sum_{j=1, j \neq i}^{n} \frac{(x_i y_j - x_j y_i)^2}{\alpha_i - \alpha_j} + \frac{x_i^2}{2} \]  

(3.35)

are the Devaney-Uhlenbeck invariants (cf. [M]).

Applying the prescription of Section 1b, with \( V_0 = (1, 0)^T \), the reduction with respect to the stabilizer of \( Y \) gives rise to the constraints (3.11) as discussed above, and the solutions to eq. (1.34) give us the Darboux coordinates \((\lambda_\mu, \zeta_\mu)_{\mu=1,...,n-1}\) defined by

\[ \sum_{i=1}^{n} \frac{x_i^2}{\lambda - \alpha_i} = \frac{\prod_{\mu=1}^{n-1}(\lambda - \lambda_\mu)}{a(\lambda)} \]  

(3.36a)

\[ \zeta_\mu = \frac{1}{2} \sum_{i=1}^{n} \frac{x_i y_i}{\lambda_\mu - \alpha_i} = \sqrt{\frac{P(\lambda_\mu)}{a(\lambda_\mu)}}. \]  

(3.36b)

Thus, the spectral divisor coordinates here are just the usual hyperellipsoidal coordinates \((\lambda_\mu)\), together with their conjugate momenta \((\zeta_\mu)\). The Liouville generating function on the isospectral foliation thus becomes

\[ S = \sum_{\mu=1}^{n-1} \zeta_\mu d\lambda_\mu |_{P_i=\text{cst.}} = \sum_{\mu=1}^{n-1} \int_{0}^{\lambda_\mu} \sqrt{\frac{P(\lambda)}{a(\lambda)}} d\lambda, \]  

(3.37)

and the canonically conjugate coordinates undergoing linear flow are

\[ Q_j := \frac{\partial S}{\partial P_j} = \frac{1}{2} \sum_{\mu=1}^{n-1} \int_{0}^{\lambda_\mu} \frac{\lambda^j d\lambda}{\sqrt{a(\lambda)P(\lambda)}} = b_j t, \quad j = 0, \ldots, n-2, \]  

(3.38)

where, for our Hamiltonian \( \phi = 2P_{n-2} \),

\[ b_{n-2} = 2, \quad b_j = 0, \quad j < n-2. \]  

(3.39)
This reproduces the familiar linearization via the hyperelliptic Abel map obtained through the classical methods of Jacobi (cf. [M]).

The other classical systems treated in [M] as isospectral flows of rank 2 perturbations of a fixed matrix $A$, such as geodesic flow on hyperellipsoids or the Rosochatius system, follow identically (cf. also [GHHW, AHP, Ra2]). In all these cases, the spectral divisor Darboux coordinates $(\lambda, \zeta)$ will coincide with the usual hyperellipsoidal coordinates, or some complexification thereof, the curves will be hyperelliptic and the spectral invariants will be an analogue of the Devaneys-Uhlenbeck invariants (3.21) encountered in this case.

3c. The NLS System

The NLS equation has two distinct forms
\[ u_{xx} + \sqrt{-1}u_t = 2|u|^2u \]  
\[ u_{xx} + \sqrt{-1}u_t = -2|u|^2u. \]  
Here we shall discuss (3.40a), but (3.40b) can be dealt with in a similar fashion. The finite genus quasi-periodic solutions of (3.40a) are determined by a pair of commuting flows in Poisson submanifolds of $\tilde{su}(1,1)^{++}$ consisting of $su(1,1)$-valued rational or polynomial functions of $\lambda$. Let $\hat{L}(\lambda)$ be a matricial polynomial of the form
\[ \hat{L}(\lambda) = \frac{a(\lambda)}{\lambda} \mathcal{N}(\lambda) = L_0 \lambda^{n-1} + L_1 \lambda^{n-2} + \cdots + L_{n-1}, \]  
where $\mathcal{N}(\lambda) \in \tilde{gl}(2,\mathbb{C})^{++}$ is an element of a rational coadjoint orbit of the form (1.2), with $Y = 0$, simple poles at $(\lambda = \alpha_i)_{i=1,...,n}$ and residues $N_i$ of rank $k_i = 1$. The reality condition implying that $\mathcal{N} \in \tilde{su}(1,1)^{++}$ is equivalent to the conditions $N_i, L_i \in su(1,1)$.

The spectral curve for this case (cf. Lemma 1.1 and Proposition 1.2) is given by
\[ \det(\hat{L}(\lambda) - zI) = z^2 + a(\lambda)\mathcal{P}_2(\lambda) = 0, \]  
where $\mathcal{P}_2(\lambda)$ is a polynomial of degree $n - 2$:
\[ \mathcal{P}_2(\lambda) = P_{20} + P_{21}\lambda + \cdots + P_{2,n-2}\lambda^{n-2}. \]  
Choosing Hamiltonians of the AKS type (with $Y = 0)$:
\[ H_x = \frac{1}{2} \left[ \frac{a(\lambda)}{\lambda^n} \lambda \text{tr}(\mathcal{N}(\lambda)^2) \right]_0 = -P_{2,n-3} \]  
\[ H_t = \frac{1}{2} \left[ \frac{a(\lambda)}{\lambda^n} \lambda^2 \text{tr}(\mathcal{N}(\lambda)^2) \right]_0 = -P_{2,n-4} \]
gives the Lax equations

\[
\frac{d}{dx} \hat{\mathcal{L}}(\lambda) = [\lambda L_0 + L_1, \hat{\mathcal{L}}(\lambda)] \tag{3.45a}
\]
\[
\frac{d}{dt} \hat{\mathcal{L}}(\lambda) = [\lambda^2 L_0 + \lambda L_1 + L_2, \hat{\mathcal{L}}(\lambda)]. \tag{3.45b}
\]

The leading term \( L_0 \) in \( \hat{\mathcal{L}}(\lambda) \) is the \( \mathfrak{su}(1,1) \) moment map and hence is invariant under all AKS flows. We choose it to be

\[
L_0 = \frac{i}{2} \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}. \tag{3.46a}
\]

Parametrizing \( L_1, L_2 \) as

\[
L_1 = \begin{pmatrix} s & \overline{u} \\ u & -s \end{pmatrix} \tag{3.46b}
\]
\[
L_2 = i \begin{pmatrix} S & -\overline{U} \\ U & -S \end{pmatrix}, \tag{3.46c}
\]

it is easily verified that \( s \) is a constant of motion, which may be set equal to zero. On this level set we have

\[
U = u_x \tag{3.47}
\]

and \( S - |u|^2 \) is constant, so we may choose

\[
s = 0 \tag{3.48a}
\]
\[
S = |u|^2. \tag{3.48b}
\]

With these values for the matrices \( L_0, L_1 \) and \( L_2 \), the commutativity conditions for the flows determined by eqs. (3.45a,b) are equivalent to the condition that \( u(x,t) \) satisfy eq. (3.40a), and eq. (3.45a) determines \( L_3, \ldots, L_{n-1} \) in terms of \( u \) and its \( x \)-derivatives, up to choices of integration constants.

In [AHP] these flows were related to reduced canonical Hamiltonian flows via rank 2 isospectral perturbations of matrices as follows (cf. also [AHH1]). Consider the subspace \( W \subset M^{n \times 2} \times M^{n \times 2} \) of the space of pairs of complex \( n \times 2 \) matrices consisting of pairs \( (F,G) \) with columns of the form

\[
F = \frac{1}{\sqrt{2}}(z, i\overline{z}), \quad G = \frac{1}{\sqrt{2}}(-i\overline{z}, z) \tag{3.49}
\]
where $z \in \mathbb{C}^n$ is a column vector with components $(z_i)_{i=1,...,n}$. The space $W$ is a real symplectic subspace of $M^{n \times 2} \times M^{n \times 2}$ that may be identified with $\mathbb{C}^n$, with symplectic form

$$\omega = -i \, dz^T \wedge d\bar{z} = -i \sum_{j=1}^n dz_j \wedge d\bar{z}_j.$$  \hfill (3.50)

Fixing $n$ real constants $\{\alpha_i\}_{i=1,...,n}$, and defining as above, $A := \text{diag}(\alpha_i) \in M^{n \times n}$, one constructs a moment map

$$\tilde{J} : \mathbb{C}^n \rightarrow \tilde{\mathfrak{su}}(1,1)^{**}$$

$$\tilde{J} : z \rightarrow \lambda G^T (\lambda I - A)^{-1} F$$

$$= \frac{\lambda}{2} \left( i \sum_{j=1}^n \frac{|z_j|^2}{\lambda - \alpha_j} - \sum_{j=1}^n \frac{z_j^2}{\lambda - \alpha_j} \right) \left( - \sum_{j=1}^n \frac{z_j^2}{\lambda - \alpha_j} - i \sum_{j=1}^n \frac{|z_j|^2}{\lambda - \alpha_j} \right) =: \mathcal{N}(\lambda).$$  \hfill (3.51b)

This is analogous to the map defined by eqs. (3.20), (3.22) above, with different reality conditions. The fibres are again generated by the finite group of reflections in the coordinate hyperplanes $\{z_i = 0\}$. Thus, via $\tilde{J}$, $\mathbb{C}^n$ (minus the coordinate hyperplanes) provides a canonical model for an orbit $Q_N$ in $\tilde{\mathfrak{su}}(1,1)^{**}$. This consists of elements with simple poles at $(\alpha_i)_{i=1,...,n}$ with rank $k_i = 1$ matrix residues, whose kernels and images are conjugate 2-vectors, null with respect to the hermitian form $H(w,z) = |w|^2 - |z|^2$. The symplectic form $\omega$ in eq. (3.50) is just the Kostant-Kirillov form $\omega_{orb}$ on this orbit.

The pull back under $\tilde{J}$ of the AKS Hamiltonians (3.44a,b) generate commuting flows on the symplectic space $W$. The condition (3.46a) gives invariant constraints

$$\mathbf{z}^T \mathbf{z} = 1$$

$$\mathbf{z}^T \mathbf{z} = 0,$$  \hfill (3.52a)

while eqs. (3.48a,b) are equivalent to

$$\mathbf{z}^T A \mathbf{z} = \sum_{j=1}^n \alpha_j$$

$$|\mathbf{z}^T A \mathbf{z}|^2 - \mathbf{z}^T A^2 \mathbf{z} = \sum_{j=1}^n \alpha_j^2 - 2\left( \sum_{j=1}^n \alpha_j \right)^2.$$  \hfill (3.52c)

The function $u$ entering in eq. (3.40a) is, by eq. (3.46b),

$$u = -\frac{1}{2} \mathbf{z}^T A \mathbf{z}.$$  \hfill (3.53)
The coefficients $P_{20}, \ldots, P_{2, n-2}$ in the spectral polynomial (3.43) give $n - 1$ real functions in involution on $\mathbb{C}^n \sim W$. Adding the further invariant $P_2 = (L_0)_{22} = -\frac{1}{2} \sum_{j=1}^{n} |z_j|^2$ gives a completely integrable system on $W$.

The genus of the spectral curve (3.42) is $g = n - 2$. Since the translation term $Y$ is zero, the flows on the Jacobi variety of this curve linearize the above completely integrable systems on $\mathbb{C}^n$ reduced by the $\mathfrak{su}(1, 1)$ action. The reduction is given in this case by fixing the value (3.46a) of the $\mathfrak{su}(1, 1)$ moment map $L_0$, and then dividing by the isotropy group of $L_0$. The isotropy group is $S^1$, whose action

$$e^{i\psi} : z \rightarrow e^{i\psi}z$$

is generated by the Hamiltonian $P_2$. With $u$ given by (3.53), we see that this action maps $u$ to $e^{2i\psi}u$ and hence integration of the reduced system only determines $u$ up to a phase factor.

By considering instead the constrained submanifold, $Q^0_N \subset Q_N$, described for case (i) of Section 1d, we can obtain the linearization of the flows implicitly in terms of complex hyperelliptic coordinates and determine $u$ explicitly without any arbitrariness of phase. Since $Q^0_N$ is given by setting the off-diagonal terms in $L_0$ equal to zero, the two real constraints given by eq. (3.52b) give a symplectic submanifold $X \subset \mathbb{C}^n$ (minus the coordinate hyperplanes) which models $Q^0_N$. The functions $P_{20}, \ldots, P_{2, n-2}$ give a complete set of integrals on the $2n - 2$ dimensional space $X$. (On this space, we have $P_{2, n-2} = -P_2^2$.) The constraint (3.52a) is equivalent to choosing

$$P_{2, n-2} = \frac{1}{4},$$

while (3.52c,d) are then equivalent to choosing

$$P_{0, n-3} = \frac{1}{4}(\sum_{j=1}^{n} \alpha_j)$$

$$P_{0, n-4} = \frac{1}{8} \left[ \sum_{j=1}^{n} \alpha_j^2 + (\sum_{j=1}^{n} \alpha_j)^2 \right].$$

Following the prescription of Section 1b, again choosing $V_0 = (1, 0)^T$ and requiring it to be in the kernel of the matrix of cofactors of $(\hat{L}(\lambda) - zI)$, the divisor coordinates $\{\lambda_\mu, \zeta_\mu\}_{\mu=1, \ldots, n-2}$ are given by

$$\sum_{i=1}^{n} \frac{z_i^2}{\lambda - \alpha_i} = -\frac{2u \prod_{\mu=1}^{n-2}(\lambda - \lambda_\mu)}{a(\lambda)}$$

(3.56)
and

\[ \zeta_\mu = -\frac{i}{2} \sum_{i=1}^{n} \frac{|z_i|^2}{\lambda_\mu - \alpha_i} = \sqrt{-P_2(\lambda_\mu)/a(\lambda_\mu)}, \]  

(3.57)

which define complex hyperelliptic coordinates \((\lambda_\mu)_{\mu=1,\ldots,n-2}\) and their canonically conjugate momenta \((\zeta_\mu)_{\mu=1,\ldots,n-2}\).

The function \(u\) in equation (3.57) is defined in eq. (3.53) and gives the off-diagonal term in \(L_1\). By the results of Theorem 1.5, \(q_2 = \ln(u)\) must be included, along with its canonical conjugate \(P_2\), to complete the Darboux coordinate system on \(X\). Thus, the restriction of the orbital symplectic form (3.50) to \(X\) is just

\[ \omega = \sum_{\mu=1}^{n-2} d\lambda_\mu \wedge d\zeta_\mu + dq_2 \wedge dP_2. \]  

(3.58)

Notice that \((\lambda_\nu, \zeta_\nu)_{\nu=1,\ldots,n-2}, q_2, P_2\) appear to give \(2n-2\) complex functions on \(X\), which has real dimension \(2n - 2\). However, since we have reduced the loop algebra to \(\tilde{\mathfrak{su}}(1,1)\) there are reality conditions satisfied by these functions (see Section 2c, case (iii)) which will be preserved by the flows. Similarly, the conditions (3.52a-d) (or equivalently, (3.52b), (3.55a-c)) may be imposed on the initial data, and will be preserved under the flows.

From (3.57), we see that the Liouville generating function (1.76) on the isospectral foliation is

\[ S = \sum_{\mu=1}^{n-1} \zeta_\mu d\lambda_\mu|_{P_i=\text{cst.}} + q_2 P_2 = \sum_{\mu=1}^{n-1} \int_0^{\lambda_\mu} \frac{\sqrt{-P_2(\lambda)/a(\lambda)}}{d\lambda} + P_2 \ln u. \]  

(3.59)

and hence (cf. eqs. (1.79a,b)), the coordinates canonically conjugate to the conserved quantities \(P_{2,0}, \ldots, P_{2,n-2}\) are given by

\[ Q_{2,i} = \frac{1}{2} \sum_{\mu=1}^{n-2} \int_0^{\lambda_\mu} \frac{\lambda^i d\lambda}{\sqrt{-a(\lambda)P_2(\lambda)}}, \quad i = 0, \ldots, n-3 \]  

(3.60a)

\[ Q_{2,n-2} = \frac{1}{2} \sum_{\mu=1}^{n-2} \int_0^{\lambda_\mu} \frac{\lambda^{n-2} d\lambda}{\sqrt{-a(\lambda)P_2(\lambda)}} - \frac{\ln u}{2P_2}, \]  

(3.60b)

where the final term is derived using the relation \(P_2^2 = -P_{2,n-2}\) on \(X\). Thus, up to normalization of the \(n-2\) holomorphic differentials appearing in eq. (3.60a), this represents the hyperelliptic Abel map (cf. Theorem 1.6) It follows from Hamilton’s
equations for \( h = H_x \) and \( H_t \) given in (3.44a,b) that the \( x \) and \( t \) dependence of the \( Q_i \)'s is given by

\[
Q_{2,i} = c_i, \quad i < n - 4 \tag{3.61a}
\]
\[
Q_{2,n-4} = c_{n-4} - t \tag{3.61b}
\]
\[
Q_{2,n-3} = c_{n-3} - x \tag{3.61c}
\]
\[
Q_{2,n-2} = c_{n-2} \tag{3.61d}
\]

where \( c_0, \ldots c_{n-2} \) are constants. Expressing the singular (3rd kind) abelian differential appearing in (3.60b) in terms of the Abel map and the appropriate hyperelliptic \( \theta \)-function, we obtain, through the procedure described in Corollary 1.7, the following explicit formula for \( u(x,t) \)

\[
u(x,t) = \exp(q_2) = \tilde{K} \exp(bx + ct) \frac{\theta(A(\infty_2, p) + tU + xV - K)}{\theta(A(\infty_1, p) + tU + xV - K)}, \tag{3.62}
\]

where \( U, V \in \mathbb{C}^q \) are determined as in Theorem 1.6 from the Hamiltonians \( h = H_x, H_t \), respectively, \( b, c \) are determined as in Corollary 1.7, and the integration constants \( \tilde{K} \in \mathbb{C}, p \in S \) are determined by initial data. This agrees with the standard \( \theta \)-function solution of NLS (cf. [P]).

3d. The CNLS System and Flows in \( \mathfrak{su}(1,2)^{++} \)

The CNLS system is a 2-component generalization of eqs. (3.40a,b). The real form we consider involves two complex functions \( u(x,t), v(x,t) \) satisfying the coupled system of equations:

\[
iu_t + u_{xx} = 2u(|u|^2 + |v|^2) \tag{3.63a}
\]
\[
iu_t + v_{xx} = 2v(|u|^2 + |v|^2). \tag{3.63b}
\]

These can be obtained as the compatibility conditions for a pair of Lax equations of the type (1.1) with \( \mathcal{N}(\lambda) \in \mathfrak{su}(1,2)^{++} \) of the form given in eq. (1.2) and \( Y = 0 \), corresponding, as above, to the Hamiltonians

\[
H_x(\mathcal{N}) = \frac{1}{2} \left[ \frac{a(\lambda)}{\lambda^{n-1}} \text{tr}(\mathcal{N}(\lambda)^2) \right]_0 \tag{3.64a}
\]
\[
H_t(\mathcal{N}) = \frac{1}{2} \left[ \frac{a(\lambda)}{\lambda^{n-2}} \text{tr}(\mathcal{N}(\lambda)^2) \right]_0. \tag{3.64b}
\]

If, as above, we set:

\[
\hat{\mathcal{L}}(\lambda) = \frac{a(\lambda)}{\lambda} \mathcal{N}(\lambda) = L_0 \lambda^{n-1} + L_1 \lambda^{n-2} + L_2 \lambda^{n-3} + \cdots + L_{n-1}, \tag{3.65}
\]
then Hamilton’s equations again take the Lax form

\[
\frac{d}{dx} \hat{L}(\lambda) = [\lambda L_0 + L_1, \hat{L}(\lambda)]
\]

(3.66a)

\[
\frac{d}{dx} \hat{L}(\lambda) = [\lambda^2 L_0 + \lambda L_1 + L_2, \hat{L}(\lambda)],
\]

(3.66b)

and the flows commute. If the following invariant constraints are imposed:

\[
L_0 = \frac{i}{3} \begin{pmatrix} 2 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & -1 \end{pmatrix}
\]

(3.67a)

\[
L_1 = \begin{pmatrix} 0 & \overline{\nu} & \overline{\nu} \\ u & 0 & 0 \\ v & 0 & 0 \end{pmatrix}
\]

(3.67b)

\[
L_2 = i \begin{pmatrix} |u|^2 + |v|^2 & -\overline{u_x} & -\overline{v_x} \\ u_x & -|u|^2 & -\overline{vu} \\ v_x & -\overline{uv} & -|v|^2 \end{pmatrix}
\]

(3.67c)

the CNLS equations are obtained as the compatibility conditions for the equations (3.66a,b).

It is possible, similarly to the NLS equation discussed above, to obtain an intrinsic characterization of the orbit corresponding to residue matrices \(N_i\) of rank \(k_i = 1\) as an open, dense subset of \(C^{2n}\), viewed as a real symplectic space, (cf. [AHP, AHH1]). However, the approach developed in Section 1 allows us to treat all orbits of the type (1.3) on the same footing, regardless of the rank \(k_i = 1, 2, 3\). Only the explicit formulae (1.27) for the genus \(\tilde{g}\) of the spectral curve and (1.25a) for the degrees of the invariant polynomials will change.

By Lemma 1.1 and Proposition 1.2, the invariant spectral curve is given by a polynomial equation of the general form

\[
\det(\hat{L}(\lambda) - zI) = \mathcal{P}(\lambda, z) = \mathcal{P}_R(\lambda, z) + \mathcal{P}(\lambda, z) = 0,
\]

(3.68a)

where

\[
\mathcal{P}_R(\lambda, z) := -z^3 + z \mathcal{A}_2(\lambda) \mathcal{P}_{R2}(\lambda) + \mathcal{A}_3(\lambda) \mathcal{P}_{R3}(\lambda)
\]

(3.68b)

defines a reference curve \(S_R\), determined by the initial data for a particular solution of the CNLS system, and

\[
p(\lambda, z) := a(\lambda) \left( z a_2(\lambda) \sum_{a=0}^\rho P_{2a} \lambda^a + a_3(\lambda) \sum_{a=0}^\sigma P_{3a} \lambda^a \right)
\]

(3.68c)
is of the form given by Proposition 1.2 for neighbouring curves, with \( \rho = \delta_2, \ \sigma = \delta_3, \) given by formulae (1.25a) and \((P_{2a}, P_{3b})_{a=0,\ldots,\rho, b=0,\ldots,\sigma}\) are the Poisson commuting spectral invariants. It will be convenient to reparametrize the space of polynomials \( p(\lambda, z) \) as follows. Set

\[
a_2(\lambda) \sum_{a=0}^{\rho} P_{2a} \lambda^a = \hat{P}_{2,\rho} \lambda^{n-2} + \hat{P}_{2,\rho-1} \lambda^{n-3} + \cdots + \hat{P}_{2,0} \lambda^{n-2-\rho} + \text{(lower order)} \quad (3.68d)
\]

\[
a_3(\lambda) \sum_{a=0}^{\sigma} P_{3a} \lambda^a = \hat{P}_{3,\sigma} \lambda^{2n-3} + \hat{P}_{3,\sigma-1} \lambda^{2n-4} + \cdots + \hat{P}_{3,0} \lambda^{2n-3-\sigma} + \text{(lower order)}. \quad (3.68e)
\]

Note that the lower order terms are completely determined by the terms \( \hat{P}_{2a}, \hat{P}_{2a} \) explicitly appearing.

There can be singularities in general over the points with \( \lambda = \alpha_i \), which are assumed to be resolved as indicated in Section 1. Due to the normalizations (3.67a-c) the spectral curve of \( \hat{L}(\lambda) \) also has singularities at \( \lambda = \infty \). In terms of the coordinates \( \tilde{\lambda} = 1/\lambda \), the eigenvalues of \( \lambda^{-m} \hat{L}(\lambda) \) have the expansion around \( \tilde{\lambda} = 0 \)

\[
\tilde{z}_1(\tilde{\lambda}) = \frac{2i}{3} - (m_2 + m_3) \tilde{\lambda}^3 + \cdots
\]
\[
\tilde{z}_2(\tilde{\lambda}) = -\frac{i}{3} + m_2 \tilde{\lambda}^3 + \cdots
\]
\[
\tilde{z}_3(\tilde{\lambda}) = -\frac{i}{3} + m_3 \tilde{\lambda}^3 + \cdots
\]

(3.69)

where for generic \( \hat{L}, m_2 \neq m_3 \). This gives the curve \( S \) a triple singularity (tacnode of order 3) at \( \lambda = \infty \), and the desingularisation \( \tilde{S} \) is 3-sheeted over \( \lambda = \infty \). Furthermore, \( \hat{L}(\lambda) \) is diagonalisable in a neighbourhood of \( \lambda = \infty \). In terms of the algebro-geometric constructions of Section 2, this means that the sheaf \( E \) of (2.1) is a direct image of a line bundle \( \tilde{E} \) on \( \tilde{S} \).

The genus of the curve \( \tilde{S} \) is \( g' = \rho + \sigma - 3 \), three less than the genus \( \tilde{g} = \rho + \sigma \) given in formulae (1.26), (1.27) for the generic spectral curve in the coadjoint orbit. Similarly, the degree of \( \tilde{E} \) is \( \rho + \sigma - 1 \), three less than the generic case. Accordingly, we have six fewer divisor coordinates than in the generic case, and hence an insufficient number to provide a Darboux system on the orbit \( Q_N \).

The solution to this problem is to impose six extra constraints on elements of the orbit, so that the dimensions of the constrained submanifold coincides with the number of coordinates. This must be done in an invariant way. Note that, under the
Lax equations (3.66a,b), the matrix \( \hat{L}(\lambda) \), \( \lambda \) fixed, evolves by conjugation. In \( GL(n, \mathbb{C}) \), the spectrum is not the only invariant under conjugation; when there are multiple eigenvalues, we also have the different Jordan canonical forms. For example, among matrices with one double eigenvalue, the generic coadjoint orbits have non-diagonal canonical form, and there are orbits that are two dimensions smaller, consisting of diagonalisable matrices. With this model in mind, we impose the following further invariant constraints, defining a \( 2(\rho + \sigma - 1) \)-dimensional symplectic submanifold \( Q_s \subset Q_{N_0}^0 \):

(i) The spectral curve has genus three less than the generic curve in the orbit, and so has three extra singularities, counted with multiplicity. \( (3.70a) \)

(ii) At these extra singular points, \( \hat{L}(\lambda) \) is diagonalisable. \( (3.70b) \)

If \( \mathcal{U} \) denotes the space of triples (curves, sheaves, trivialisations over infinity) obtained from the orbit \( Q_{N_0}^0 \), and \( \mathcal{W} \) is the corresponding space of curves, with the projection map \( \delta : \mathcal{U} \to \mathcal{W} \), condition \( (3.70a) \) restricts us to the inverse image under \( \delta \) of a codimension three subvariety of curves in \( \mathcal{W} \). For curves \( \mathcal{S} \) in this variety, \( \delta^{-1}(\mathcal{S}) \) is a stratified space, with a generic stratum consisting of line bundles (+ trivialisations at \( \lambda = \infty \)) on the singular curve \( \mathcal{S} \), and other strata corresponding to direct images of line bundles on various desingularisations of the curve at subsets of the three singular points. Constraint \( (3.70b) \) then restricts us to the codimension three stratum in \( \delta^{-1}(\mathcal{S}) \) of direct images of line bundles on the curve \( \tilde{\mathcal{S}} \) obtained from \( \mathcal{S} \) by desingularising all three points.

As remarked in section 2, it is irrelevant, for the purpose of integrating the AKS flows, whether one uses the Kostant-Kirillov form \( \omega_{\text{orb}} \) or the Serre duality form \( \omega_{S} \). Let \( (\lambda_\mu, \zeta_\mu)_{\mu=1,...,\tilde{g}} \) be the divisor coordinates and let \( (q_i, P_i)_{i=2,3} \) be the “extra” Darboux coordinates as in Section 1c.

**Theorem 3.1.** The restriction to \( Q_s \) of the form \( \omega_S \) is given, over a suitable dense set, by

\[
\omega_S = \sum_{\mu=1}^{\tilde{g}} d\lambda_\mu \wedge d\zeta_\mu + \sum_{i=2}^3 dq_i \wedge dP_i + \frac{1}{2} \sum_{i \neq j}^3 \frac{dP_i \wedge dP_j}{P_i - P_j} \tag{3.71}
\]

**Proof.** One begins by noting that \( Q_s \) can be described, as in section 2, as a variety of generically smooth curves \( \tilde{\mathcal{S}} \), along with line bundles \( \tilde{E} \) (with trivialisations at \( \lambda = \infty \)) defined over \( \tilde{\mathcal{S}} \). Proceeding as in Section 2, we can define a symplectic form \( \omega_{S,s} \) on this space, using Serre duality. As in the proof of Theorem 2.8, it follows that the explicit
form of $ω_{S,s}$ is given by (3.71). There remains only to show that $ω_{S,s}$ is the restriction to $Q_s$ of $ω_S$.

This is fairly easy to see. Any line bundle $\tilde{E}$ over $\tilde{S}$ is the pull-back of a line bundle $E$ on $S$. Extending this bundle to a neighbourhood of $S$ gives a splitting of the map (curves, bundles, trivializations over $\lambda = \infty$) → (curves), both on the constrained submanifold $Q_s$ and on the ambient space $Q^0_{N_0}$. Using this splitting, we now write a pair of vector fields $\{V_i\}_{i=1,2}$ on $Q^0_{N_0}$ as $(v_i, e_i)$, $v_i \in H^0(S, K(1))$, $e_i \in H^1(S, \mathcal{O}(-1))$, where $\{V_i\}$ are tangent to $Q_s$ along $Q_s$. The $v_i$ are represented in Dolbeault cohomology at $S$ by holomorphic forms, with a simple pole at infinity. Along $Q_s$, these stay finite at the singularities; this is the condition of tangency. The $e_i$, in turn are represented by $(0, 1)$ forms, which vanish at infinity. On $V_1, V_2$, $ω_S$, $ω_{S,s}$ are both given by

$$\int_S v_1 \wedge e_2 - v_2 \wedge e_1,$$

which is obviously well behaved in a neighbourhood of $Q_s$. □

The constraints (3.70a,b) and the relations between $P_2, P_3$ and the coefficients of the polynomial $p(λ, z)$ imply that the coefficients $P_{2, r}, P_{3, σ}, P_{3, σ-1}, P_{3, σ-2}, P_{3, σ-3}$ in eq. (3.68c) can be expressed in terms of the lower coefficients $(P_{2a}, P_{3b})_{a=0,...,σ-1}$, $b=0,...,σ-4$ and $P_2, P_3$. To apply the Liouville method, we only need to know the constraints to order 1 at the CNLS curve $S_R$ corresponding to the spectral polynomial $P_R(λ, z)$. The constraint (3.70a), requiring the neighboring curve to have the same degree of singularity as $S_R$, is equivalent to first order to requiring the induced section of the normal bundle along $S_R$:

$$p(λ, z) \frac{∂}{∂P_R/∂z} \frac{∂}{∂z}. \quad (3.72)$$

to remain finite at the singular points of $S_R$.

At the CNLS curve this means that, passing to the coordinates $\tilde{z}, \tilde{λ}$, the three first terms in the Taylor expansion at $\tilde{λ} = 0$ of the expression

$$\tilde{λ}^{2n-3} \left( \tilde{z}\tilde{λ}^{-n+1}a_2(\tilde{λ}^{-1})p_2(\tilde{λ}^{-1}) + a_3(\tilde{λ}^{-1})p_3(\tilde{λ}^{-1}) \right) =: \tilde{z}f_2(\tilde{λ}) + f_3(\tilde{λ}) \quad (3.73)$$

must vanish when one substitutes $\tilde{z} = -i/3$. This yields the linearized constraints:

$$\hat{P}_{3, σ} = \frac{i}{3} \hat{P}_{2, r} \quad (3.74a)$$

$$\hat{P}_{3, σ-1} = \frac{i}{3} \hat{P}_{2, r-1} \quad (3.74b)$$

$$\hat{P}_{3, σ-2} = \frac{i}{3} \hat{P}_{2, r-2}. \quad (3.74c)$$
Setting
\[ P_j =: -\frac{i}{3} + \hat{P}_j, \quad j = 2, 3 \] (3.75)

the linear variation of the \((\hat{P}_j)\) at the CNLS curve can be computed by evaluating the limits of the normal vector field (3.72) along the two branches of the curve as one approaches the singular point. To first order:

\[ \hat{P}_j = \lim_{\tilde{\lambda} \to 0} \tilde{z}_j(\tilde{\lambda}) f_2(\tilde{\lambda}) + f_3(\tilde{\lambda}) \] (3.76)

where \((j, k) = (2, 3)\) or \((3, 2)\), and \(\tilde{z}_j(\tilde{\lambda})\) is as in (3.69). This gives

\[ \hat{P}_2 + \hat{P}_3 = i f_2(0) = iP_{2,\rho} \] (3.77)

and, if \(m_2, m_3\) are as in (3.69),

\[ m_3\hat{P}_2 + m_2\hat{P}_3 = \lim_{\tilde{\lambda} \to 0} \frac{-i}{\lambda^3} \left( -\frac{i}{3} f_2 + f_3 \right) \]
\[ = -i \left[ -\frac{i}{3} (\hat{P}_{2,\rho-3} + \hat{P}_{3,\sigma-3}) \right], \] (3.78)

where \(c_i, d_i\) are constants. Isolating \(\hat{P}_{3,\sigma-3}\) above, and using (3.77-78) to rewrite (3.74a-c) we obtain

\[ \hat{P}_{2,\rho} = -i \left( \hat{P}_2 + \hat{P}_3 \right) \] (3.79a)
\[ \hat{P}_{3,\sigma} = \frac{1}{3} (\hat{P}_2 + \hat{P}_3) \] (3.79b)
\[ \hat{P}_{3,\sigma-1} = \frac{i}{3} \hat{P}_{2,\rho-1} \] (3.79c)
\[ \hat{P}_{3,\sigma-2} = \frac{i}{3} \hat{P}_{2,\rho-2} \] (3.79d)
\[ \hat{P}_{3,\sigma-3} = i \hat{P}_{2,\rho-3} + i(m_3\hat{P}_2 + m_2\hat{P}_3). \] (3.79e)

This expresses the terms on the left in terms of the independent complete set of integrals of motion \(\hat{P}_{2,0}, \ldots, \hat{P}_{2,\rho-1}, \hat{P}_{3,0}, \ldots, \hat{P}_{3,\sigma-4}, \hat{P}_2, \hat{P}_3\) (at least to first order around the reference curve, which is all we need to apply the Liouville method). Since \((\lambda_{\nu}, z_{\nu})_{\nu=1}^{\delta}, (q_i, P_i)_{i=2,3}\) form a Darboux coordinate system “up to constants of motion”, if we define our generating function \(S(\lambda_{\nu}, q_2, q_3, P_{2a}, P_{3b}, P_2, P_3)\) as in (1.76), then the canonically conjugate coordinates

\[ \left\{ \hat{Q}_{2a} = \frac{\partial S}{\partial \hat{P}_{2a}} \right\}_{a=0,\ldots,\rho-1}, \left\{ \hat{Q}_{3b} = \frac{\partial S}{\partial \hat{P}_{3b}} \right\}_{b=0,\ldots,\sigma-4}, \left\{ \hat{Q}_i = \frac{\partial S}{\partial \hat{P}_i} \right\}_{i=2,3}. \] (3.80)
undergo linear flow:

\[
\begin{align*}
\dot{Q}_{2a} &= c_{2a} - \delta_{a,\rho-1} x - \delta_{a,\rho-2} t \\
\dot{Q}_{3b} &= c_{3b} \\
\dot{Q}_i &= c_i,
\end{align*}
\]

(3.81)

where \(c_{2a}, c_{3b}, c_i\) are constants. (Up to additive constants, \(H_x = -\dot{P}_{2,\rho-1}, H_t = -\dot{P}_{2,\rho-2}\).) Evaluating the derivatives (3.82), taking (3.79a-e) into account, we obtain:

\[
\begin{align*}
\hat{Q}_{2a} &= \sum_{\nu=1}^{\tilde{g}} \int_0^{\lambda}\frac{\nu \lambda^{n-2} + \frac{1}{3} \delta_{a,\rho-3} \lambda^{2n-6} + \delta_{a,\rho-2} \lambda^{2n-5} + \delta_{a,\rho-1} \lambda^{2n-4}}{3z^2 + A_2(\lambda)P_{R2}(\lambda)} d\lambda \tag{3.82a} \\
\hat{Q}_{3b} &= \sum_{\nu=1}^{\tilde{g}} \int_0^{\lambda}\frac{\lambda^{2n-3} + \delta_{a,\rho-3} \lambda^{2n-6}}{3z^2 + A_2(\lambda)P_{R2}(\lambda)} d\lambda \tag{3.82b} \\
\hat{Q}_2 &= \ln u + I + im_3 J \tag{3.82c} \\
\hat{Q}_3 &= \ln v + I + im_2 J \tag{3.82d}
\end{align*}
\]

where \(I, J\) are defined by

\[
\begin{align*}
I &= \sum_{\nu=1}^{\tilde{g}} \int_0^{\lambda}\frac{-iz\lambda^{n-2} + \frac{1}{3} \lambda^{2n-3}}{3z^2 + A_2(\lambda)P_{R2}(\lambda)} d\lambda \tag{3.83a} \\
J &= \sum_{\nu=1}^{\tilde{g}} \int_0^{\lambda}\frac{\lambda^{2n-6}}{3z^2 + A_2(\lambda)P_{R2}(\lambda)} d\lambda. \tag{3.83b}
\end{align*}
\]

We can check explicitly that the integrands of (3.82a,b) form a basis for the holomorphic differentials on the curve \(\tilde{S}\) which has been desingularized over \(\lambda = \alpha_i\) and over \(\lambda = \infty\). If \(\infty_1, \infty_2, \infty_3\) are the three points of \(\tilde{S}\) over \(\lambda = \infty\) corresponding to \(\tilde{z} = \frac{2i}{3}, \frac{1}{3}, \frac{-i}{3}\) respectively, the integrands of \(I + im_3 J, I + im_2 J\) have only simple poles over \(\lambda = \infty\) with residues \((-1,1,0)\) and \((-1,0,1)\) respectively at \((\infty_1, \infty_2, \infty_3)\). Proceeding as in Corollary 1.7, we obtain the \(\theta\)-function formulae

\[
\begin{align*}
u(x,t) &= \exp(q_2) = \tilde{K}_2 \exp(e_2 x + d_2 t) \frac{\theta(A(\infty_2, p) + tU + xV - K)}{\theta(A(\infty_1, p) + tU + xV - K)} \tag{3.84a} \\
v(x,t) &= \exp(q_2) = \tilde{K}_3 \exp(e_3 x + d_3 t) \frac{\theta(A(\infty_3, p) + tU + xV - K)}{\theta(A(\infty_1, p) + tU + xV - K)}, \tag{3.84b}
\end{align*}
\]

with \(U, V \in \mathbb{C}^g\) determined from the Hamiltonians \(h = H_x, H_t\) as in Theorem 1.6, \((e_i, d_i)_{i=2,3}\) as in Corollary 1.7, and the remaining integration constants determined to satisfy the appropriate initial conditions.
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