Fermi transport of spinors and free QED states in curved spacetime

Daniel Canarutto

Dipartimento di Matematica Applicata “G. Sansone”,
Via S. Marta 3, 50139 Firenze, Italia
email: daniel.canarutto@unifi.it
http://www.dma.unifi.it/~canarutto
25 November 2008

Abstract

Fermi transport of spinors can be precisely understood in terms of 2-spinor geometry. By using a partly original, previously developed treatment of 2-spinors and classical fields, we describe the family of all transports, along a given 1-dimensional timelike submanifold of spacetime, which yield the standard Fermi transport of vectors. Moreover we show that this family has a distinguished member, whose relation to the Fermi transport of vectors is similar to the relation between the spinor connection and spacetime connection. Various properties of the Fermi transport of spinors are discussed, and applied to the construction of free electron states for a detector-dependent QED formalism introduced in a previous paper.

2000 MSC: 53B05, 53B21, 81Q99.
Keywords: Fermi transport, 2-spinors, Dirac spinors, free electron states.

Contents

1 Two-spinors and Dirac spinors 1
2 Two-spinor bundle and field theories 5
3 Fermi transport 9
4 An application: free QED states 16

1 Two-spinors and Dirac spinors

This section and the next one contain a sketch of the two-spinor approach to Dirac algebra and field theories referred to in the Introduction. See [3, 4, 7] for details.

1.1 Hermitian tensors

If $V$ is a finite dimensional complex vector space, then we indicate by $V^*$ its dual space, by $\overline{V}^* \cong V^\tau$ its anti-dual space (namely the vector space of all anti-linear maps $V \to \mathbb{C}$)
and by $\mathbf{V} \cong V^{**} \cong V^{**}$ its conjugate space. One then has natural anti-isomorphisms $V \to V^* : v \mapsto \bar{v}$ and $V^* \to V^{**} : \lambda \mapsto \bar{\lambda}$. Following a rather standard usage, we use “dotted indices” for vector and tensor components in $V$ and $V^*$.

The space $V \otimes \overline{V}$ has a natural real linear (complex anti-linear) involution $w \mapsto \bar{w}^\dagger$, which on decomposable tensors reads

$$ (u \otimes \bar{v})^\dagger = v \otimes \bar{u} . $$

Hence one has the natural decomposition of $V \otimes \overline{V}$ into the direct sum of the real eigenspaces of the involution with eigenvalues $\pm 1$, respectively called the Hermitian and anti-Hermitian subspaces, namely

$$ V \otimes \overline{V} = (V \overline{V}) \oplus i (V \overline{V}) . $$

In other terms, the Hermitian subspace $V \overline{V}$ is constituted by all $w \in V \otimes \overline{V}$ such that $w^\dagger = w$, while an arbitrary $w$ is uniquely decomposed into the sum of an Hermitian and an anti-Hermitian tensor as

$$ w = \frac{1}{2} (w + w^\dagger) + \frac{1}{2} (w - w^\dagger) . $$

In terms of components in any basis, $w = w^{AB} b_A \otimes \bar{b}_B$ is Hermitian (anti-Hermitian) iff the matrix $(w^{AB})$ of its components is of the same type, namely $\bar{w}^{BA} = \pm w^{AB}$.

### 1.2 Two-spinor space

Let $S$ be a 2-dimensional complex vector space. Then $\wedge^2 S$ is a 1-dimensional complex vector space. The Hermitian subspace of $(\wedge^2 S) \otimes (\wedge^2 S)$ is a 1-dimensional real vector space with a distinguished orientation, whose positively oriented semispace

$$ \mathbb{L}^2 := [(\wedge^2 S) \overline{\wedge^2 S}]^+ := \{ w \otimes \bar{w}, w \in \wedge^2 S \} $$

has the square root semi-space $\mathbb{L}$, called the space of length units. The complex 2-dimensional space

$$ U := \mathbb{L}^{-1/2} \otimes S $$

is called the 2-spinor space. Observe that the 1-dimensional space

$$ Q := \wedge^2 U = \mathbb{L}^{-1} \otimes \wedge^2 S $$

has a distinguished Hermitian metric, defined as the unity element in

$$ \overline{Q}^* \overline{Q}^* \equiv (\wedge^2 U^*) \overline{\wedge^2 U^*} = \mathbb{L}^{-2} \otimes (\wedge^2 S^*) \overline{\wedge^2 S^*} \cong \mathbb{R} . $$

Hence there is the distinguished set of normalized “symplectic” forms on $U$, any two of them related by a phase factor.

Consider an arbitrary basis $(\xi_A)$ of $S$, and let $(x^A)$ be its dual basis of $S^*$. This determines the mutually dual bases

$$ w := \varepsilon^{AB} \xi_A \wedge \xi_B , \quad w^{-1} := \varepsilon_{AB} x^A \wedge x^B , $$

\[ A \text{ unit space is defined to be a 1-dimensional real semi-space, namely a positive semi-field associated with the semi-ring } \mathbb{R}^+ \text{ (see } \cite{10} \text{ for details). The square root } \mathbb{U}^{1/2} \text{ of a unit space } \mathbb{U}, \text{ is defined by the condition that } \mathbb{U}^{1/2} \otimes \mathbb{U}^{1/2} \text{ be isomorphic to } \mathbb{U}. \text{ More generally, any rational power of a unit space is defined up to isomorphism (negative powers correspond to dual spaces). Here we only use the unit space } \mathbb{L} \text{ of lengths and its powers; essentially, this means that we take } \hbar = c = 1. \]
1.3 From 2-spinors to Minkowski space

respectively of $\lambda^2 S$ and $\lambda^2 S^\ast$ (here $\varepsilon^{AB}$ and $\varepsilon_{AB}$ both denote the antisymmetric Ricci matrix), and the basis

$$l := \sqrt{w \otimes w} \quad \text{of} \quad L \, .$$

Then one also has the induced mutually dual, \textit{normalized} bases

$$(\zeta_A) := (l^{-1/2} \otimes \xi_A) \, , \quad (z^A) := (l^{1/2} \otimes x^A)$$

of $U$ and $U^\ast$, and also

$$\varepsilon := l \otimes w^{-1} = \varepsilon_{AB} z^A \wedge z^B \in Q^\ast \equiv \lambda^2 U^\ast \, ,$$

$$\varepsilon^{-1} \equiv l^{-1} \otimes w = \varepsilon^{AB} \zeta_A \wedge \zeta_B \in Q \equiv \lambda^2 U \, .$$

Remark.\quad In contrast to the usual 2-spinor formalism, no symplectic form is fixed. The 2-form $\varepsilon$ is unique up to a phase factor which depends on the chosen 2-spinor basis, and determines isomorphisms

$$\varepsilon^b : U \to U^\ast : u \mapsto u^b \, , \quad \langle u^b, v \rangle := \varepsilon(u, v) \Rightarrow (u^b)_B = \varepsilon_{AB} v^A \, ,$$

$$\varepsilon^\# : U^\ast \to U : \lambda \mapsto \lambda^\# \, , \quad \langle \mu, \lambda^\# \rangle := \varepsilon^{-1}(\lambda, \mu) \Rightarrow (\lambda^\#)_B = \varepsilon^{AB} \lambda_A \, .$$

1.3 From 2-spinors to Minkowski space

Though a normalized element $\varepsilon \in Q^\ast$ is unique only up to a phase factor, the tensor product $g \equiv \varepsilon \otimes \bar{\varepsilon} \in Q^\ast \otimes \bar{Q}^\ast$ is a naturally distinguished object. This can also be seen as a bilinear form on $U \otimes \bar{U}$, acting on decomposable elements as

$$g(p \otimes \bar{q}, r \otimes \bar{s}) = \varepsilon(p, r) \bar{\varepsilon}(\bar{q}, \bar{s}) \, .$$

The fact that any $\varepsilon$ is non-degenerate implies that $g$ is non-degenerate too. In a normalized 2-spinor basis $(\zeta_A)$ one writes $w = w^{A\bar{A}} \zeta_A \otimes \bar{\zeta}_{\bar{A}} \in U \otimes \bar{U}$; $g_{A\bar{A}B\bar{B}} = \varepsilon_{AB} \bar{\varepsilon}_{A\bar{A}B\bar{B}}$ and

$$g(w, w) = \varepsilon_{AB} \bar{\varepsilon}_{A\bar{A}} w^{A\bar{A}} \bar{w}^{B\bar{B}} = 2 \det w \, .$$

The Hermitian subspace

$$H := U \sqrt{\bar{U}} \subset U \otimes \bar{U}$$

is a 4-dimensional \textit{real} vector space, and the restriction of $g$ to $H$ turns out to be a Lorentz metric with signature $(+, -, -, -)$. Actually, for any given normalized basis $(\zeta_A)$ of $U$ consider the \textit{Pauli basis} $(\tau_\lambda)$ of $H$ associated with $(\zeta_A)$, namely

$$\tau_\lambda \equiv \tau_\lambda^{A\bar{A}} \zeta_A \otimes \bar{\zeta}_\bar{A} \equiv \frac{1}{\sqrt{2}} \sigma_\lambda^{A\bar{A}} \zeta_A \otimes \bar{\zeta}_\bar{A} \, , \quad \lambda = 0, 1, 2, 3 \, ,$$

where $\sigma_\lambda$ denotes the $\lambda$-th Pauli matrix; then one easily finds $g(\tau_\lambda, \tau_\mu) = 2 \delta_\lambda^0 \delta_\mu^0 - \delta_\lambda^\mu$. Conversely, any orthonormal basis of $H$ can be written as the Pauli basis associated with an appropriate two-spinor basis.

It’s not difficult to prove that an element $w \in U \otimes \bar{U} = \mathbb{C} \otimes H$ is null, that is $g(w, w) = 0$, \textit{iff} it is a decomposable tensor: $w = u \otimes \bar{s}, u, s \in U$. A null element in $U \otimes \bar{U}$ is also in $H$ \textit{iff} it is of the form $\pm u \otimes \bar{u}$. Hence the \textit{null cone} $N \subset H$ is constituted exactly by such elements. Note how this fact yields a way of distinguishing between time orientations: by convention, one chooses the \textit{future} and \textit{past} null-cones in $H$ to be, respectively,

$$N^+ := \{ u \otimes \bar{u}, \ u \in U \} \, , \quad N^- := \{ -u \otimes \bar{u}, \ u \in U \} \, .$$
1.4 From 2-spinors to Dirac spinors

Next observe that an element of $U \otimes \overline{U}$ can be seen as a linear map $\overline{U}^* \rightarrow U$, while an element of $\overline{U}^* \otimes U^*$ can be seen as a linear map $U \rightarrow \overline{U}^*$. Then one defines the linear map

$$\gamma : U \otimes \overline{U} \rightarrow \text{End}(U \oplus \overline{U}^*) : y \mapsto \gamma(y) := \sqrt{2} \left< y, y^b \right>^*,$$

i.e. $\gamma(y)(u, \chi) = \sqrt{2}(\chi, y^b u)$, where $y^b := y^b(y) \in U^* \otimes \overline{U}^*$ and $y^b \in \overline{U}^* \otimes U^*$ is the transposed tensor. In particular for a decomposable $y = p \otimes \bar{q}$ one has

$$\gamma(p \otimes \bar{q})(u, \chi) = \sqrt{2}(\bar{q}, p, (p^b, u) q^b).$$

It’s not difficult to see that, for all $y, y' \in U \otimes \overline{U}$, one has

$$\gamma(y) \circ \gamma(y') + \gamma(y') \circ \gamma(y) = 2 g(y, y') \mathbb{1},$$

namely $\gamma$ is a Clifford map relatively to $g$; thus one is led to regard

$$W := U \oplus \overline{U}^*$$

as the space of Dirac spinors, decomposed into its Weyl subspaces. The restriction of $\gamma$ to the Minkowski space $H$ is called the Dirac map.

The 4-dimensional complex vector space $W$ is naturally endowed with a further structure: the obvious anti-isomorphism

$$W \rightarrow W^* : (u, \chi) \mapsto (\bar{\chi}, \bar{u}).$$

Namely, if $\psi = (u, \chi) \in W$ then $\bar{\psi} = (\bar{u}, \bar{\chi}) \in \overline{W}$ can be identified with $(\bar{\chi}, \bar{u}) \in W^*$; this is the so-called ‘Dirac adjoint’ of $\psi$. This operation can be seen as the “index lowering anti-isomorphism” related to the Hermitian product

$$k : W \times W \rightarrow \mathbb{C} : ((u, \chi), (u', \chi')) \mapsto \langle \bar{\chi}, u' \rangle + \langle \chi', \bar{u} \rangle,$$

which is obviously non-degenerate; its signature turns out to be $(+ + -)$, as it can be seen in a “Dirac basis” (below).

Let $(\zeta_{\alpha})$ be a normalized basis of $U$; the Weyl basis of $W$ is defined to be the basis $(\zeta_{\alpha})$, $\alpha = 1, 2, 3, 4$, given by

$$(\zeta_1, \zeta_2, \zeta_3, \zeta_4) := (\zeta_1, \zeta_2, -\bar{z}_1, -\bar{z}_2).$$

The Dirac basis $(\zeta'_{\alpha})$, $\alpha = 1, 2, 3, 4$, is given by

$$(\zeta'_1 = \frac{1}{\sqrt{2}}(\zeta_1, \bar{z}_1) \equiv \frac{1}{\sqrt{2}}(\zeta_1 - \zeta_3), \quad \zeta'_2 = \frac{1}{\sqrt{2}}(\zeta_2, \bar{z}_2) \equiv (\zeta_2 - \zeta_4),$$

$$\zeta'_3 = \frac{1}{\sqrt{2}}(\zeta_1, -\bar{z}_1) \equiv (\zeta_1 + \zeta_3), \quad \zeta'_4 = \frac{1}{\sqrt{2}}(\zeta_2, -\bar{z}_2) \equiv (\zeta_2 + \zeta_4).$$

Setting

$$\gamma_\lambda := \gamma(\tau_\lambda) \in \text{End}(W)$$

one recovers the usual Weyl and Dirac representations as the matrices $(\gamma_\lambda)$, $\lambda = 0, 1, 2, 3$, in the Weyl and Dirac bases respectively.
It should be noted that no distinguished Hermitian metric exists either on $U$ or $W$; assigning such structure is equivalent to fixing an observer (this point remains somewhat obscured in most traditional treatments of spinors). In fact, a Hermitian 2-form $h$ on $U$ is an element in $\bar{U}^* \vee U^* \cong H^*$. One says that $h$ is normalized if it is non-degenerate, positive and $g^\#(h) = h^{-1}$; the latter condition is equivalent to $g(h, h) = 2$. If $h$ is normalized then it is necessarily a future-pointing timelike element in $H^*$. For example, if $(\tau_\lambda)$ is a Pauli basis and $(t^\lambda)$ is the dual basis, then $\sqrt{2}t^0 = i^1 \otimes z^1 + i^2 \otimes z^2$ is normalized; conversely, every positive-definite normalized Hermitian metric $h$ can be expressed in the above form for some suitable normalized 2-spinor bases. In 4-spinor terms: if $h$ is assigned, then it extends naturally to a Hermitian metric $h$ on $W$, which can be characterized by

$$h(\psi, \phi) = k(\gamma_0 \bar{\psi}, \phi) .$$

**Remark.** Some other operations on 4-spinor space, commonly used in the literature, actually depend on particular choices or conventions. Charge conjugation, in particular, is the antilinear involution

$$\mathcal{C} : W \to W : (u, \chi) \mapsto e^{-it}(\bar{\epsilon}(\bar{\chi}), -\bar{\epsilon}(\bar{u}))$$

determined by the choice of a normalized element $\omega = e^{i\tau} \epsilon \in \Lambda^2 U^*$. Parity is the endomorphism $\gamma_0 \equiv \gamma(\tau_0)$, so it depends on the choice of an observer (here written as the element $\tau_0$ of a suitable Pauli frame). Time-reversal is the composition $\gamma_\eta \gamma_0 \mathcal{C}$, where $\gamma_\eta$ (in a Pauli basis: $\gamma_\eta = \gamma_0 \gamma_1 \gamma_2 \gamma_3$) is the endomorphism corresponding, via $\gamma$, to the volume form $\eta$ determined by $g$ on $H$.

## 2 Two-spinor bundle and field theories

### 2.1 Two-spinor connections

Consider any real manifold $M$ and a vector bundle $S \to M$ with complex 2-dimensional fibres. Denote base manifold coordinates as $(x^a)$; choose a local frame $(\xi_\lambda)$ of $S$, determining linear fibre coordinates $(x^\lambda)$. According to the constructions of the previous sections, one now has the bundles $Q$, $L$, $U$, $H$, $W$ over $M$, with smooth natural structures; the frame $(\xi_\lambda)$ yields the frames $\epsilon$, $l$, $(\xi_\lambda)$ and $(\tau_\lambda)$, respectively. Moreover for any rational number $r \in \mathbb{Q}$ one has the semi-vector bundle $L^r$.

Consider an arbitrary $\mathbb{C}$-linear connection $\Gamma$ of $S \to M$, called a 2-spinor connection. In the fibred coordinates $(x^a, x^\lambda)$ $\Gamma$ is expressed by the coefficients $\Gamma_a^\lambda : M \to \mathbb{C}$, namely the covariant derivative of a section $s : M \to S$ is expressed as

$$\nabla s = (\partial_a s^A - \Gamma_a^\lambda s^B) \, dx^a \otimes \xi_\lambda .$$

The rule $\nabla s = \nabla s$ yields a connection $\bar{\Gamma}$ on $\bar{S} \to M$, whose coefficients are given by

$$\bar{\Gamma}_a^\lambda \bar{\theta}^\lambda = \bar{\Gamma}_a^\lambda \bar{\theta}^\lambda .$$

Actually, $\Gamma$ determines linear connections on each of the above said induced vector bundles over $M$. Denote by $2G$ and $2Y$ the connections induced on $L$ and $Q$ (this notation makes sense because the fibres are 1-dimensional), namely

$$\nabla l = -2G_a \, dx^a \otimes l , \quad \nabla \epsilon = 2iY_a \, dx^a \otimes \epsilon ,$$

$$\nabla w^{-1} \equiv \nabla (l^{-1} \otimes \epsilon) = 2(G_a + iY_a) \, dx^a \otimes l^{-1} \otimes \epsilon .$$

---

2 In the traditional notation, $\gamma_\lambda^A$ indicates the $h$-adjoint of $\gamma_\lambda$, and then depends on the chosen observer.
and the like. By direct calculation we find

\[ G_a = \frac{1}{4} (\Gamma_a^A + \bar{\Gamma}_a^A) , \quad Y_a = \frac{1}{4i} (\Gamma_a^A - \bar{\Gamma}_a^A) . \]

Since \( Y_a \) is real, the induced linear connection on \( Q \) is Hermitian (preserves its natural Hermitian structure).

The coefficients of the induced connections \( \tilde{\Gamma} \) on \( U \), and \( \tilde{\Gamma} \) on \( H \), turn out to be

\[ \tilde{\Gamma}_a^{AB} = \Gamma_a^{AB} - G_a \delta_B^A , \]
\[ \tilde{\Gamma}_a^{A\dot{A}} = \Gamma_a^{A\dot{A}} \delta^{A\dot{A}} + \delta^A_B \bar{\Gamma}_a^{A\dot{A}} - 2 G_a \delta^{A\dot{A}} \delta^{B\dot{B}} . \]

Since its coefficients are real, \( \tilde{\Gamma} \) turn out to be reducible to a real connection on \( H \). Moreover this connection \( \tilde{\Gamma} \) turns out to be metric, namely \( \nabla[\tilde{\Gamma}] g = 0 \). Hence, its coefficients are antisymmetric and traceless, namely

\[ \tilde{\Gamma}^{\lambda\mu}_a + \tilde{\Gamma}^{\mu\lambda}_a = 0 , \quad \tilde{\Gamma}^{\lambda}_a = 0 . \]

The above relations between \( \Gamma \) and the induced connections can be inverted as

\[ \Gamma_a^{AB} = (G_a + i Y_a) \delta^A_B + \frac{1}{2} \bar{\Gamma}_a^{AB} , \]

and a similar relation holds among the curvature tensors, namely

\[ R^A_{ab} = -2 (dG + i dY)_{ab} \delta^A_B + \frac{1}{2} \bar{R}^{A\dot{A}}_{ab} \delta^{A\dot{A}} . \]

### 2.2 Two-spinor tetrad

Henceforth I’ll assume that \( M \) is a real 4-dimensional manifold. Consider a linear morphism

\[ \Theta : T M \rightarrow S \otimes \overline{S} = \mathbb{C} \otimes L \otimes H , \]

namely a section

\[ \Theta : M \rightarrow \mathbb{C} \otimes L \otimes H \otimes T^* M \]

(all tensor products are over \( M \)). Its coordinate expression is

\[ \Theta = \Theta^a_{\lambda} \tau^\lambda \otimes dx^a = \Theta^{A\dot{A}}_a \zeta^A \otimes \bar{\xi}^{A\dot{A}} \otimes dx^a , \quad \Theta^a_{\lambda}, \Theta^{A\dot{A}}_a : M \rightarrow \mathbb{C} \otimes L . \]

We’ll assume that \( \Theta \) is non-degenerate and valued in the Hermitian subspace \( \mathbb{L} \otimes H \subset S \otimes \overline{S} \); then \( \Theta \) can be viewed as a ‘scaled’ tetrad (or soldering form, or vierbein); the coefficients \( \Theta^a_{\lambda} \) are real (i.e. valued in \( \mathbb{R} \otimes L \)) while the coefficients \( \Theta^{A\dot{A}}_a \) are Hermitian, i.e. \( \bar{\Theta}^{A\dot{A}}_a = \Theta^{A\dot{A}}_a \). Through a tetrad, the geometric structure of the fibres of \( H \) is carried to a similar, scaled structure on the fibres of \( TM \). It will then be convenient, from now on, to distinguish by a tilda the objects defined on \( H \), so I’ll denote by \( \tilde{g} \), \( \tilde{\eta} \) and \( \tilde{\gamma} \) the Lorentz metric, the \( \tilde{g} \)-normalized volume form and the Dirac map of \( H \), and set

\[ g := \Theta^* \tilde{g} : M \rightarrow \mathbb{C} \otimes L^2 \otimes T^* M \otimes T^* M , \]
\[ \eta := \Theta^* \tilde{\eta} : M \rightarrow \mathbb{C} \otimes L^4 \otimes \wedge^4 T^* M , \]
\[ \gamma := \tilde{\gamma} \circ \Theta : TM \rightarrow \mathbb{L} \otimes \text{End}(W) , \]
which have the coordinate expressions
\[ g = \eta_{\lambda\mu} \Theta^\lambda_a \Theta^\mu_b \, dx^a \otimes dx^b = \varepsilon_{AB} \varepsilon_{\lambda\mu} \Theta^{AB}_a \Theta^{\lambda\mu}_b \, dx^a \otimes dx^b , \]
\[ \eta = \det(\Theta) \, dx^0 \wedge dx^1 \wedge dx^2 \wedge dx^3 , \]
\[ \gamma = \sqrt{2} \Theta^{AB}_a (\zeta^A \otimes \zeta^B + \varepsilon_{AB} \varepsilon_{\lambda\mu} \bar{z}^B \otimes z^\mu) \otimes dx^a . \]
The above objects turn out to be a Lorentz metric, the corresponding volume form and a Clifford map. Moreover
\[ \Theta_b^\mu := \Theta^\lambda_a \eta_{\lambda\mu} g^{ab} = (\Theta^{-1})^b_\mu : M \to \mathbb{C} \otimes \mathbb{L}^{-1} , \quad g^{ab} : M \to \mathbb{C} \otimes \mathbb{L}^{-2} . \]

A non-degenerate tetrad, together with a two-spinor frame, yields mutually dual orthonormal frames \( \Theta_\lambda \) of \( \mathbb{L}^{-1} \otimes TM \) and \( \Theta^\lambda \) of \( \mathbb{L} \otimes T^*M \), given by
\[ \Theta_\lambda := \Theta^{-1}(\tau_\lambda) = \Theta^\rho_a \partial x_a , \quad \tilde{\Theta}_\lambda := \Theta^*(t^\lambda) = \Theta^\rho_a dx^a . \]

We also write
\[ \gamma = \gamma_\lambda \otimes \tilde{\Theta}^\lambda = \gamma_\lambda \otimes dx^a , \quad \gamma_\lambda := \gamma(\Theta_\lambda) : M \to \text{End}(W) , \]
\[ \gamma_a := \gamma(\partial x_a) = \Theta^\rho_a \gamma_\lambda : M \to \mathbb{L} \otimes \text{End}(W) . \]

If \( \Gamma \) is a complex-linear connection on \( S \), and \( \tilde{\Gamma} \) and \( \tilde{\Gamma} \) are the induced connections on \( \mathbb{L} \) and \( H \), then a non-degenerate tetrad \( \Theta : TM \to \mathbb{L} \otimes H \) yields a unique connection \( \Gamma \) on \( TM \), characterized by the condition
\[ \nabla[\Gamma \otimes \tilde{\Gamma} \Theta] = 0 . \]
Moreover \( \Gamma \) is metric, i.e. \( \nabla[\Gamma]g = 0 \). Denoting by \( \Gamma^\lambda_{a\mu} \) the coefficients of \( \Gamma \) in the frame \( \Theta_\lambda \equiv \Theta^{-1}(l \otimes \tau_\lambda) \) one obtains
\[ \Gamma^\lambda_{a\mu} = \tilde{\Gamma}^\lambda_{a\mu} + 2G_a^\delta \delta_{\lambda} . \]
The curvature tensors of \( \Gamma \) and \( \tilde{\Gamma} \) are related by
\[ R_{ab\mu} = \tilde{R}_{ab\mu} , \quad R_{ab\mu} = \tilde{R}_{ab\mu} \Theta^\rho_a \Theta^\nu_d . \]
Hence the Ricci tensor and the scalar curvature are given by
\[ R_{ad} = R_{ab\mu} = \tilde{R}_{ab\mu} \Theta^\rho_a \Theta^\nu_d , \quad R^a = \tilde{R}^\lambda_{ab\mu} \Theta^\rho_a \Theta^\nu_d . \]
In general, the connection \( \Gamma \) will have non-vanishing torsion, which can be expressed as
\[ \Theta^\rho_a T_{ab}^c = \tilde{\partial}^c_{a \Theta^\rho_b} + \Theta^\rho_a \tilde{\Theta}_b^\mu + 2 \Theta^\rho_a G_b . \]

2.3 Einstein-Cartan-Maxwell-Dirac field theory

In this section I'll give an essential sketch of a “minimal geometric data” field theory which has been presented in previous papers [3, 4]. The quoted words refer to the fact that the unique “geometric datum” is a vector bundle \( S \rightarrow M \) with complex 2-dimensional fibres and real 4-dimensional base manifold. All other bundles and fixed geometric objects are determined just by this datum through functorial constructions, as we saw in the previous
sections; no further background structure is assumed. Any considered bundle section which is not functorially fixed by our geometric datum is a field. A natural Lagrangian can then be written, yielding a field theory which turns out to be essentially equivalent to a classical theory of Einstein-Cartan-Maxwell-Dirac fields.

The fields are taken to be the tetrad \( \Theta \), the 2-spinor connection \( \Gamma \), the electromagnetic field \( F \) and the electron field \( \psi \). The gravitational field is represented by \( \Theta \) (which can be viewed as a ‘square root’ of the metric) and the traceless part of \( \Gamma \), namely \( \tilde{\Gamma} \), seen as the gravitational part of the connection. If \( \Theta \) is non-degenerate one obtains, as in the standard metric-affine approach \([9, 8]\), essentially the Einstein equation and the equation for torsion; the metricity of the spacetime connection is a further consequence. But note that the theory is non-singular also in the degenerate case. The connection \( G \) induced on \( \mathbb{L} \) will be assumed to have vanishing curvature, \( dG = 0 \), so that one can always find local charts such that \( G_a = 0 \); this amounts to gauging away the conformal (‘dilaton’) symmetry. Coupling constants will arise as covariantly constant sections of \( \mathbb{L} \), which now becomes just a vector space.

The Dirac field is a section \( \psi \equiv (u, \chi) : M \to \mathbb{L}^{-3/2} \otimes W \) assumed to represent a semi-classical particle with one-half spin, mass \( m \in \mathbb{L}^{-1} \) and charge \( q \in \mathbb{R} \).

The electromagnetic potential can be thought of as the Hermitian connection on \( \Lambda^2 U \) determined by \( \Phi \), whose coefficients are indicated as \( iY_a \); locally one writes \( Y_a \equiv qA_a \), where \( A : M \to T^* M \) is a local 1-form.

The electromagnetic field is represented by a spinor field \( \tilde{F} : M \to \mathbb{L}^{-2} \otimes \Lambda^2 H^* \) which, via \( \Theta \), determines the 2-form \( F := \Theta^* \tilde{F} : M \to \Lambda^2 T^* M \). The relation between \( Y \) and \( F \) will follow as one of the field equations.

The total Lagrangian density is the sum of a gravitational, an electromagnetic and a Dirac term: \( \mathcal{L} = \mathcal{L}_g + \mathcal{L}_{em} + \mathcal{L}_D = (\ell_g + \ell_{em} + \ell_D) d^4x : M \to \Lambda^4 T^* M \), where

\[
\begin{align*}
\ell_g &= \frac{1}{8k} \tilde{\eta} \left| (\tilde{R}^\# \wedge \Theta \wedge \Theta) \right|, \\
\ell_{em} &= -\frac{i}{2} \tilde{\eta} \left| [\Theta \wedge \Theta \wedge (dA \otimes \tilde{F})] + \frac{1}{4} (\tilde{F} \cdot \tilde{F}) \eta \right|, \\
\ell_D &= \Im \left[ \frac{1}{48} \langle \tilde{\psi}, \tilde{\eta} \rangle (\tilde{\eta} \# \nabla \psi \wedge \Theta \wedge \Theta \wedge \Theta) \right] - m \langle \tilde{\psi}, \psi \rangle \eta.
\end{align*}
\]

In the above expressions, \( k \) is Newton’s gravitational constant; a superscript \( \# \) denotes “index raising” relatively to the Lorentz metric \( \tilde{g} \), and the “exterior” product among vector valued forms is naturally defined\(^3\). One has the coordinate expressions

\[
\begin{align*}
\ell_g &= \frac{1}{8k} \varepsilon_{\lambda \mu \rho \sigma} \varepsilon^{abcd} \tilde{R}_{ab}^{\lambda \mu} \Theta_c^{\nu} \Theta_d^{\rho}, \\
\ell_{em} &= -\frac{1}{4} \varepsilon^{abcd} \varepsilon_{\lambda \mu \rho \sigma} \partial_a A_b \tilde{F}^{\lambda \mu} \Theta_c^{\nu} \Theta_d^{\rho} + \frac{1}{4} \tilde{F}^{\alpha \beta} \tilde{F}_{\alpha \beta} \det \Theta, \\
\ell_D &= \frac{1}{\sqrt{2}} \tilde{\Theta}_{A A'} \left( \nabla_a u^a \tilde{u}^{A'} - u^a \nabla_a \tilde{u}^{A'} + \varepsilon^{AB} \varepsilon^{\tilde{A} \tilde{B}} (\tilde{x}_B \nabla_a \chi^{B} - \nabla_a \tilde{x}_B \chi^B) \right) \\
&\quad - m \left( \tilde{x}_A u^a + \chi_A \tilde{u}^a \right) \det \Theta,
\end{align*}
\]

where

\[
\tilde{\Theta}_{A A'} \equiv \frac{1}{\sqrt{2}} \sigma^\lambda_{A A'} \tilde{\Theta}_{\lambda} \equiv \frac{1}{\sqrt{2}} \sigma^\lambda_{A A'} \left( \frac{1}{48} \varepsilon_{\lambda \mu \rho \sigma} \Theta_b^{\mu} \Theta_d^{\rho} \right).
\]

Writing down the Euler-Lagrange equation\(^4\) for the Lagrangian density \( \mathcal{L} \) is a straightforward (though not short) task. Summarizing the basic results:

\[^3\] For example, \((x \otimes \alpha) \wedge (y \otimes \beta) = (x \wedge y) \otimes (\alpha \wedge \beta)\), \(\alpha, \beta \in T^* M, x, y \in H\), and the like.

\[^4\] One has to calculate the variational derivatives relatively to all the fields \( F, \Theta, \tilde{F}, \psi \equiv (u, \chi) \).
• The $\Theta$-component corresponds (in the non-degenerate case) to the Einstein equation.
• The $\tilde{\Gamma}$-component gives the equation for torsion. From this one sees that the spinor field is a source for torsion, and that in this context one cannot formulate a torsion-free theory.
• The $\tilde{F}$-component reads $F = 2 \, dA$ in the non-degenerate case, and of course this yields the first Maxwell equation $dF = 0$.
• The $A$-component reduces, in the non-degenerate case, to the second Maxwell equation $\frac{1}{2} * dF = j$, where $j : M \to T^*M$ is the Dirac current.
• The $\bar{u}$- and $\bar{\chi}$-components give a generalized form of the standard Dirac equation, which can be written in compact form as
  \[(i \nabla - m + \frac{1}{2} \gamma^a (\tilde{T})) \psi = 0 .\]

Here, $\tilde{T}$ denotes the 1-form obtained from the torsion by contraction, with coordinate expression $\tilde{T}_a = T^b_{ab}$.

3 Fermi transport

A 1-dimensional timelike submanifold $L \subset M$ can be seen as a ‘pointlike observer’, or as the world-line of a ‘detector’. Note that there is a natural inclusion $TL \subset TM$. The restriction of the spacetime time metric is a Riemann metric on $L$, which yields the detector’s ‘proper time’.

Throughout this § we’ll assume a tetrad $\Theta$ and a spinor connection $\Gamma$ to be fixed, namely we’ll work in a given gravitational field background. Moreover, for simplicity, we’ll assume $G_a = 0$ as it is in the standard field theories (§2.3).

Since $\Theta$ is fixed, it will be convenient to make the identification $TM \cong L \otimes H$ (and the like) in order to simplify our notations. Note (§2.2) that the scalar product of elements in $TM$ is valued into $R \otimes L^2$, while the tensor product of elements in $H$ is real valued; we express this fact by saying that the spacetime metric $g$ is $L^2$-scaled, while the metric on $H$ is unscaled (or ‘conformally invariant’).

3.1 Rivisitation of the standard Fermi transport

Denote as $T_L M \to L$ and $H_L \to L$ the restrictions of the bundles $TM \to M$ and $H \to M$ to the base $L$ (the fibres over elements in $L$ are the same). Then $H_L \cong L^{-1} \otimes T_L M$ has one distinguished section, namely the unit future-pointing scaled vector field

\[\tau : L \to L^{-1} \otimes TL \subset L^{-1} \otimes T_L M \cong H_L .\]

We now consider the linear morphism over $L$

\[\Phi : TL \to \wedge^2 H_L : v \mapsto \Phi_v \equiv 2 (\nabla_v \tau) \wedge \tau .\]

Choose base coordinates $(x^a) \equiv (x^1, x^2, x^3, x^4)$ adapted to $L$, namely such that $\partial x_4 \equiv \partial / \partial x^4$ is tangent to $L$ at the points of $L$; let moreover $(\tau_\lambda)$ be any orthonormal frame of $H_L$ such that $\tau_0 \equiv \tau$; then one gets the coordinate expression

\[\Phi = 2 \tilde{\Gamma}^{04}_j \, dx^4 \otimes \tau_j \wedge \tau_0 .\]
By ‘lowering the second index’ of $\Phi$ through the metric one gets a linear morphism

$$\Phi^\flat : L \rightarrow H_L \otimes H_L^* \equiv \text{End}(H_L),$$

namely

$$\Phi^\flat_v = \nabla_v \tau_0 \otimes t^0 - \tau_0 \otimes \nabla_v t^0,$$

with coordinate expression

$$\Phi^\flat_v = -dx^4 \otimes (\tilde{\Gamma}_{i40}^j \tau_j \otimes t^0 + \tilde{\Gamma}_{4j}^0 \tau_0 \otimes t^j),$$

where the dual frame of $(\tau_\lambda)$ was denoted as $(t^\lambda)$.

Note that $(\Phi^\flat_v)^{040} = (\Phi^\flat_v)^{j4j} = (\Phi^\flat_v)^{\lambda4\lambda} = 0$.

The bundle $H_L \hookrightarrow L$ has of course the connection naturally induced by $\tilde{\Gamma}$: the covariant derivative of any section $X : L \rightarrow H_L$ is defined to be the map $v \mapsto \nabla_v X : L \rightarrow H_L$, namely $\nabla_v X \equiv \nabla_v [\tilde{\Gamma}] X$ is the restriction of $\nabla_v [\tilde{\Gamma}] X'$ for any local extensions $v'$ and $X'$ of $v$ and $X$. But now we observe that $\Phi^\flat_v$ can be viewed as a section $L \rightarrow T^*L \otimes \text{End}(H_L)$, according to $\Phi^\flat_v = v | \Phi^\flat$. Hence we are able to introduce a new connection of $H_L \hookrightarrow L$, namely the Fermi connection $\tilde{\Gamma}^F := \tilde{\Gamma} + \Phi^\flat$.

The covariant derivative associated with $\tilde{\Gamma}^F$ turns out to have the expression

$$D_v X \equiv \nabla_v [\tilde{\Gamma}^F] X = \nabla_v X - \Phi^\flat_v(X) =$$

$$= \nabla_v X + g(\nabla_v \tau, X) \tau - g(\tau, X) \nabla_v \tau : L \rightarrow H_L,$$

for any sections $v : L \rightarrow TL$ and $X : L \rightarrow H_L$.

The usual Fermi derivative is defined as a derivation with respect to the detector’s proper time, that is

$$DX \equiv D_v X : L \rightarrow L^{-1} \otimes H_L, \quad X : L \rightarrow H_L.$$

**Proposition 3.1** For any $v : L \rightarrow TL$ and $X, Y : L \rightarrow H_L$ one has

$$v.(X \cdot Y) = (D_v X) \cdot Y + X \cdot D_v Y.$$

**Proof:** It follows from the fact that $\tilde{\Gamma}$ is metric and $\Phi$ is anti-symmetric, so that $\Phi^\flat$ is valued into the Lorentz group (see §3.4 for more details about that). We can directly verify our statement by observing that the Lie derivative along $v$ of the scalar field $X \cdot Y$ is well-defined on $L$ independently of extensions. We then have

$$v.(X \cdot Y) = (\nabla_v X) \cdot Y + X \cdot \nabla_v Y = [D_v X + \Phi^\flat_v(X)] \cdot Y + X \cdot [D_v Y + \Phi^\flat_v(Y)] =$$

$$= (D_v X) \cdot Y + X \cdot D_v Y + \Phi^\flat_v(X^\flat, Y^\flat) + \Phi^\flat_v(Y^\flat, X^\flat) = (D_v X) \cdot Y + X \cdot D_v Y,$$

since $\Phi^\flat_v$ is antisymmetric. \hfill \Box

---

5 We recall that the difference between any two connections on a vector bundle $E \rightarrow B$ is a tensor field $B \rightarrow T^*B \otimes \text{End}(E)$. 
3.1 Rivisitation of the standard Fermi transport

A section $X : L \to H_L$ which is covariantly constant relatively to $\Gamma_F$ (namely $DX = 0$, or $D_v X = 0$ for all $v : L \to TL$) is said to be Fermi-transported along $L$; a Fermi-transported section is uniquely determined by the value it takes at any point of $L$.

A few points are worth stressing:

- The scalar product of Fermi-transported vectors is constant along $L$ (this follows at once from the above proposition).
- $\tau$ itself is Fermi-transported; if $f : L \to \mathbb{R}$ then $D(f \tau) = (\tau.f) \tau$.
- If $X : L \to H^\perp_L$ (the subbundle of $H_L$ orthogonal to $\tau$) then also $D_v X : L \to H^\perp_L$, coinciding with the orthogonal projection onto $H^\perp_L$ of the ordinary covariant derivative $\nabla_v X$.

Thus $\Gamma_F$ preserves the splitting of $H_L$ into the direct sum of its subbundles parallel and orthogonal to $\tau$. Moreover one also has Fermi-transported orthonormal frames $(\tau_\lambda)$ of $H_L$ such that $\tau_0 \equiv \tau$ (one only has to fix the frame at some point of $L$ and ‘Fermi-transport’ it).

In any orthonormal frame (not necessarily Fermi-transported) one has the coordinate expressions

$$D_v X = v^4 (\partial_4 X^\lambda \tau_\lambda - X^k \tilde{\Gamma}^j_{4k} \tau_j) \equiv v.X^0 \tau_0 + (v.X^j - v^4 X^k \tilde{\Gamma}^j_{4k}) \tau_j,$$

$$DX = \Theta^0_4 (\partial_4 X^\lambda \tau_\lambda - X^k \tilde{\Gamma}^j_{4k} \tau_j),$$

which are independent of any extensions of $v$ and $X$.

**Remark.** The definition of the Fermi derivative could be extended to the case when $L$ is spacelike, but cannot be immediately extended to a derivative along a null 1-dimensional submanifold, since in the latter case there exists no normalized tangent vector ($\tau$). Moreover, the Fermi transport along an arbitrary timelike curve cannot be seen as parallel transport relatively to some connection on $H \rightarrow M$.

However, a different kind of extension can be devised. For this purpose, we first note that the section $\Phi^b : L \to T^*L \otimes \text{End}(H_L)$ can be extended via the spacetime metric to a section $\Phi^b : L \to T^*_L M \otimes \text{End}(H_L)$.

Namely we set

$$v \cdot \Phi^b := g(v, \tau) \Phi^b, \quad v \in T_L M.$$ 

Suppose that $M$ is filled with congruence of timelike 1-dimensional submanifolds, with normalized tangent vector field $\tau : M \to H$. Then considering the above said extension for all said submanifolds we obtain a section

$$\Phi^b : M \to T^*M \otimes \text{End}(H) \cong T^*M \otimes \text{End}(TM).$$

Consider now the new spacetime connection $\tilde{\Gamma} + \Phi^b$. This has the property that the parallel transport along lines of the chosen congruence coincides with Fermi transport there; the same is *not* true, however, for lines which do not belong to the chosen congruence. Also, note that the transport along spacelike lines orthogonal to the lines of the congruence coincides with ordinary parallel transport.

---

6 This follows from a well-known result about general connections, since $\Gamma_F$ is a true connection on the restricted bundle $H_L \rightarrow L$ for fixed $L$.

7 Samuel and Nityananda [12] have introduced a somewhat different transport law for polarization vectors along non-geodesic null curves.
3.2 Fermi transport of 2-spinors

Let \( U_L \to L \) be the restriction of the bundle \( U \to M \) to the base manifold \( L \).

Introducing an appropriate Fermi transport for spinors amounts essentially to defining a modification \( \Gamma' \) of the connection \( \Gamma \) on \( U_L \to L \), in such a way that the induced connection \( \Gamma' \otimes \Gamma' \) on \( H_L \to L \) coincides with \( \Gamma' \). The solution to the problem of determining \( \Gamma' \) is not unique, as we’ll see, so one has got to describe the family of all solutions (in the next section, the results obtained here will be extended to 4-spinors).

There is a natural procedure we can follow: writing down an analogous of the relation between \( \tilde{\Gamma} \) and \( \Gamma \) (§3.1). We start from the two-spinor form of \( \Phi^b \), namely

\[
\Phi^b = \Phi_4^{A\lambda} \, d\xi^4 \otimes \tilde{\zeta}_A \otimes z^b \otimes \tilde{z}^B : M \to T^*L \otimes \text{End}(H),
\]

with \( \Phi_4^{A\lambda} = \Phi_4^\lambda \tau_\lambda^{A\lambda} \, t^\mu_{BB} = \frac{1}{2} \Phi_4^\lambda \sigma_\lambda^{A\lambda} \, \sigma_{BB} \).

By taking half the trace of \( \Phi^b \) relatively to its conjugate 2-spinor indices we get the section

\[
\phi : L \to T^*L \otimes \text{End}(U_L)
\]

which has the coordinate expression \( \phi = \phi_4^{A\lambda} \, d\xi^4 \otimes \tilde{\zeta}_A \otimes z^b \) with

\[
\phi_4^{A\lambda} = \frac{1}{2} \Phi_4^{A\lambda}. \]

A simple calculation, using the properties of the Pauli matrices, gives then

\[
\phi_4^{A\lambda} = \frac{1}{2} \frac{1}{4} \delta_{jB} \sigma^A_{jB}.
\]

Note that

\[
\text{Tr}(\phi) = \phi_4^{A\lambda} \, d\xi^4 = 0
\]

(in agreement with \( \Phi_4^\lambda = \Phi_4^{A\lambda} = 0 \)). Conversely, it’s not difficult to show—by standard 2-spinor algebra—that

\[
\Phi_4^{A\lambda} = \Phi_4^{A\lambda} \delta_{AB} + \delta_{AB} \Phi_4^{A\lambda}. \]

Next, we introduce the spinor Fermi connection on \( U_L \to L \),

\[
\Gamma' := \Gamma + \phi,
\]

which has the coordinate expression

\[
(\Gamma')_4^{AB} = \Gamma_4^{AB} + \phi_4^{AB} = (G_4 + iy_4) \delta^A_B + \frac{1}{2} \Gamma_4^{A\lambda} \, B_A + \frac{1}{2} \Phi_4^{A\lambda}.
\]

If \( v : L \to TL \) and \( u : L \to U_L \) are sections, then

\[
\nabla_u(\Gamma')_4^A = \nabla_u(\Gamma)_4^A - v^A \phi_4^{AB} u_B = v^A (\partial_4 u^A - \Gamma_4^{AB} u_B - \frac{1}{2} \Gamma_4^{AB} \sigma_{jB} u_B).
\]

**Proposition 3.2** The connection \( \Gamma' \otimes \Gamma' \) induced by \( \Gamma' \) on \( H_L \to L \) coincides with the the Fermi connection \( \Gamma' \). Moreover, any other linear connection \( \Gamma'_f \) of \( U_L \to L \) yielding \( \Gamma' \) differs from \( \Gamma' \) by a term of the type \( i \alpha \otimes \mathbb{1} \) with \( \alpha : L \to T^*L \), namely

\[
(\Gamma'_f)_4^{AB} = (\Gamma')_4^{AB} + i \alpha_4 \delta^A_B.
\]

---

\(^8\) For simplicity, we denote the restriction of \( \Phi \) by the same symbol.
3.3 Fermi transport of 4-spinors

PROOF: The coefficients of $\Gamma_\phi$ and $\bar{\Gamma}_\phi$ are

$$\left(\Gamma_\phi \otimes \bar{\Gamma}_\phi\right)_{4B} = \Gamma_{4B}^A \delta^A_{B'} + \delta_A^B \bar{\Gamma}_{4B}^{A'} = (\Gamma_{4B}^A + \phi_{4B}^A) \delta^A_{B'} + \delta_A^B \bar{\Gamma}_{4B}^{A'} =$$

$$= (\Gamma_{4B}^A \delta^A_{B'} + \delta_A^B \bar{\Gamma}_{4B}^{A'}) + (\phi_{4B}^A \delta^A_{B'} + \delta_A^B \bar{\phi}_{4B}^{A'}) =$$

$$= \hat{\Gamma}_{4B}^{A'} \delta_A^B + \Phi_{4B}^{A'} \delta_A^B .$$

Now we observe that any other connection $\bar{\Gamma}_\phi'$ of $U_L \rightarrow L$ can be written as $\bar{\Gamma}_\phi + \Xi$, with $\Xi : L \rightarrow T^*L \otimes \text{End}(U_L)$. The condition that $\bar{\Gamma}_\phi'$ yields $\bar{\Gamma}_\phi$ can be written as

$$\Gamma_{4B}^A \delta^A_{B'} + \delta_A^B \bar{\Gamma}_{4B}^{A'} = (\Gamma_{4B}^A + \Xi_{4B}^A) \delta^A_{B'} + \delta_A^B \bar{\Gamma}_{4B}^{A'} \Xi_{4B}^{A'} ,$$

$$\Rightarrow \Xi_{4B}^A \delta^A_{B'} + \delta_A^B \Xi_{4B}^{A'} = 0 .$$

A short discussion then shows that $\Xi_{4B}^A = \xi \delta_A^B$ with $\xi : L \rightarrow i\mathbb{R}$. □

Conclusion: we obtained a family of connections of the restricted bundle $U_L \rightarrow L$. Each element of the family yields the standard Fermi transport, and is characterized by the arbitrary choice of an imaginary function on $L$. $\Gamma_\phi$ is a distinguished element of the family, so we see it as the natural generalization of Fermi transport to 2-spinors.

3.3 Fermi transport of 4-spinors

The coefficients of the connection induced by $\Gamma$ on $U^*_L \rightarrow M$ (namely the dual of $\bar{\Gamma}$, see §2.1) are

$$\hat{\Gamma}_\phi^{aA} = -\bar{\Gamma}_\phi^{aA} .$$

The couple $(\Gamma, \hat{\Gamma})$ then constitutes the induced (4-spinor) connection on the bundle $U \oplus U^* \equiv W \rightarrow M$. Its coefficients can be expressed in the form

$$\Gamma^\alpha_{aB} = iy_a \delta^\alpha_{B} + \frac{1}{4} \hat{\Gamma}^a_{\alpha} (\gamma^\alpha \gamma^\beta)_{aB} , \quad \alpha, \beta = 1, 2, 3, 4 .$$

Its restriction to $W_L \rightarrow L$ can then be modified in order to obtain a 4-spinor Fermi connection, that is the connection

$$(\Gamma_\phi, \hat{\Gamma}_\phi) = (\Gamma + \phi, \hat{\Gamma} - \bar{\phi})$$

obtained from $\bar{\Gamma}_\phi$ by a similar procedure. Namely, this new connection differs from $(\Gamma, \hat{\Gamma})$ by the section

$$\left(\phi, -\bar{\phi}^*\right) : L \rightarrow T^*L \otimes \text{End}(W_L) ,$$

where the transpose conjugate

$$\bar{\phi}^* : L \rightarrow T^*L \otimes U^*_L \otimes \bar{U}_L \equiv T^*L \otimes \text{End}(U^*_L) ,$$

has the coordinate expression

$$\left(\bar{\phi}^*\right)_{4B}^{A'} = \frac{1}{2} \bar{\phi}_{4B}^{A'} = \frac{1}{2} \hat{\Gamma}_{4B}^{0j} \delta_j^{A'} .$$

After some calculations we also find

$$(\phi, -\bar{\phi}^*) = \frac{1}{2} \hat{\gamma}(\Phi) = \frac{1}{2} \hat{\Gamma}_{4B}^{0j} d\chi^4 \otimes (\gamma_0 \gamma_j - \gamma_j \gamma_0) ,$$
where \( \hat{\gamma} : \wedge H \to \text{End} W \) is the natural extension of the Dirac map. For simplicity, let us indicate a connection on \( U_L \to L \) and the induced connection on \( W_L \to L \) by the same symbol; then the induced 4-spinor Fermi connection of \( W_L \to L \) can be written as

\[
\Gamma_F = \Gamma + (\phi, -\bar{\phi}^\ast) = \Gamma + \frac{1}{4} \hat{\gamma}(\Phi) .
\]

Any other member \( \Gamma'_F \) of the family of 4-spinor Fermi connections are obtained from the above expression via the replacement \( \phi \to \phi + i \alpha \otimes \mathbb{1} \), namely

\[
\Gamma'_F = \Gamma + i \alpha \otimes \mathbb{1} .
\]

### 3.4 Group considerations

A detailed study of the relations between 2-spinor groups, 4-spinor groups and the Lorentz group was exposed in [7]. In this section I’ll just recall a few results which are relevant in the present discussion.

In the algebraic setting of [112] consider the group

\[
\text{Sl}(U) := \{ K \in \text{Aut}(U) : \det K = 1 \} ,
\]

which preserves the two-spinor structure. Its relations with the special orthochronous Lorentz group and the orthochronous Spin group are described by the commutative diagram

\[
\begin{array}{ccc}
\text{Sl}(U) & \longrightarrow & \text{Lor}^1(H) \\
\downarrow & & \downarrow \\
\text{Spin}^1(W) & \longrightarrow & K \otimes \bar{K} \\
& \downarrow & \downarrow \downarrow (K, (\bar{K}^\ast)^{-1}) \\
& \text{Spin}^1(W) & \otimes \bar{\Phi}
\end{array}
\]

One has the isomorphic Lie algebras \( \mathfrak{L}_{\text{Lor}} \equiv \mathfrak{L}_{\text{Lor}}(H) \), \( \mathfrak{L}_{\text{Spin}} \equiv \mathfrak{L}_{\text{Spin}}(W) \) and

\[
\mathfrak{L}_{\text{Sl}} \equiv \mathfrak{L}_{\text{Sl}}(U) \cong \{ \phi \in \text{End}(U) : \text{Tr} \phi = 0 \} .
\]

Furthermore \( \mathfrak{L}_{\text{Lor}}(H) \) is isomorphic, as a vector space, to \( \wedge^2 H \). Thus one has the diagram of isomorphisms

\[
\begin{array}{ccc}
\mathfrak{L}_{\text{Sl}}(U) & \leftarrow & \mathfrak{L}_{\text{Lor}}(H) \\
\downarrow & & \downarrow \\
\mathfrak{L}_{\text{Spin}}(W) & \leftarrow & \wedge^2 H \\
& \downarrow & \downarrow \downarrow (\phi, -\bar{\phi}^\ast) \\
& \mathfrak{L}_{\text{Spin}}(W) & \leftarrow \Phi
\end{array}
\]

where the relations among the above objects are as follows.

**a)** \( \Phi^\phi \in \mathfrak{L}_{\text{Lor}}(H) \subset \text{End}(H) = H \otimes H^\ast \) is obtained from \( \Phi \in \wedge^2 H \subset H \otimes H \) through the isomorphism \( g^\phi : H \to H^\ast \) determined by the Lorentz metric.

**b)** \( \phi \in \mathfrak{L}_{\text{Sl}}(U) \subset U \otimes U^\ast \) is one-half the trace of \( \Phi^\phi \) relatively to the conjugate factors. Conversely, \( \Phi^\phi = \phi \otimes \mathbb{1}_U + \mathbb{1}_U \otimes \bar{\phi} \). Hence \( \text{Tr} \phi = 0 \).
3.4 Group considerations

c) \( \frac{1}{2} \hat{\gamma}(\Phi) = (\phi, -\phi^*) \in \text{End } U \oplus \text{End } U^* \subset \text{End } W \), where \( \hat{\gamma} : \wedge H \to \text{End } W \) is the natural extension of the Dirac map to the exterior algebra of \( H \).

Furthermore, it should be observed that the biggest group which preserves the two-spinor structure is not \( \text{Sl}(U) \) but rather the ‘complexified’ group

\[
\text{Sl}^c(U) := \{ K \in \text{Aut}(U) : |\det K| = 1 \} = (U(1) \times \text{Sl}(U)) / \mathbb{Z}_2 ,
\]

which leaves any symplectic form of \( U \) invariant up to a phase factor. Its Lie algebra is

\[
\mathfrak{L}\text{Sl}^c(U) \cong \{ A \in \text{End}(U) : \Re \text{Tr } A = 0 \} = i\mathbb{R} \oplus \mathfrak{L}\text{Sl}(U) .
\]

Accordingly, any \( \theta \in \mathfrak{L}\text{Sl}^c(U) \) can be uniquely decomposed as

\[
\theta = \frac{1}{2} (\text{Tr } \theta) \mathbb{1} + (\theta - \frac{1}{2} (\text{Tr } \theta) \mathbb{1}) \equiv i\alpha \mathbb{1} + \phi , \quad \alpha \in \mathbb{R} , \quad \phi \in \mathfrak{L}\text{Sl}(U) ,
\]

with \( \mathbb{1} \equiv \mathbb{1}_U \), and one has

\[
\theta \otimes \mathbb{1} + \mathbb{1} \otimes \bar{\theta} = \phi \otimes \mathbb{1} + \mathbb{1} \otimes \bar{\phi} .
\]

In other words, \( \theta \in \mathfrak{L}\text{Sl}^c(U) \) determines an element \( \Phi^b \in \mathfrak{L}\text{Lor}(H) \) via its traceless part \( \phi \).

In the bundle setting of §2 the above spaces and groups become vector bundles and group bundles over \( M \). Consider sections

\[
\phi : M \to T^* M \otimes \mathfrak{L}\text{Sl}(U) ,
\]

\[
\theta \equiv i\alpha \otimes \mathbb{1}_U + \phi : M \to T^* M \otimes \mathfrak{L}\text{Sl}^c(U) ,
\]

\[
\Phi^b : M \to T^* M \otimes \mathfrak{L}\text{Lor}(H) ,
\]

\[
\frac{1}{2} \hat{\gamma}(\Phi) = (\phi, -\phi^*) : M \to T^* M \otimes \mathfrak{L}\text{Spin}(W) ,
\]

fulfilling the same mutual relations \( a, b, c \) and \( d \) as the previously considered algebraic objects with the same names (\( \alpha : M \to T^* M \) is now a real 1-form).

Such Lie-algebra-bundle valued 1-forms can be seen as differences between linear connections preserving the respective vector bundle structures (while the curvature tensors are 2-forms valued in the same Lie-algebra-bundles). More precisely, it’s not difficult to prove:

**Proposition 3.3** Let \( \Gamma \) and \( \Gamma' \) be 2-spinor connections, and \( \tilde{\Gamma}, \tilde{\Gamma}' \) the respectively induced connections of \( H \to M \). Then

\[
\theta \equiv i\alpha \otimes \mathbb{1} + \phi := \Gamma - \Gamma' : M \to T^* M \otimes \mathfrak{L}\text{Sl}^c(U)
\]

and

\[
\Phi^b := \tilde{\Gamma} - \tilde{\Gamma}' : M \to T^* M \otimes \mathfrak{L}\text{Lor}(H)
\]

fulfil the above relations \( a, b, c, d \). In particular, \( \Phi^b \) only depends on the traceless part \( \phi \) of \( \theta \).
4 An application: free QED states

Though a kind of ‘covariance’ can be achieved in flat spacetime, current quantum theory remains essentially observer-dependent. This feature is most evident when one tries to formulate QFT in curved spacetime, where one is unable to define a distinguished, observer-independent set of free states (e.g. see Birrel and Davies [1]).

In a previous paper [6] I studied a quantum formalism, in momentum space, carried by a pointlike observer: connections on underlying ‘classical’ bundles determine ‘quantum connections’ on ‘distributional bundles’, namely bundles over spacetime whose fibres are distributional spaces, and restriction to a given observer worldline does the job. I also hinted at the possibility that free electron states, for the observer’s formalism, be described in terms of a Fermi transport of spinors, rather than by ordinary covariant transport. This seems natural in view of the standard interpretation of the usual Fermi transport of vectors in relation to small gyroscopes carried by the observer.

For $m \in \mathbb{L}^{-1}$ let $P_m \subset T^* M$ be the subbundle over $M$ of all future-pointing $p \in T^* M$ such that $g^R(p, p) = m^2$. Then $P_m$ is the classical momentum bundle for a particle of mass $m$ (the limit case $m = 0$ can also be considered). Consider the 2-fibred bundle

$$[\left(\wedge^3 T^* M \right)^+]^{1/2} \equiv V^{-1/2} P_m \rightarrow P_m \rightarrow M$$

whose upper fibres are the spaces of half-densities of the momentum spaces. If $V \rightarrow P_m$ is a complex vector bundle (whose fibres represent the internal degrees of freedom of the particle) then for each $x \in M$ one has the vector spaces $V^1_x$ of all generalized sections (in a distributional sense)

$$(P_m)_x \rightarrow (V^{-1/2} P_m \otimes V)_x ,$$

which can be assembled into a smooth bundle $V^1 \rightarrow M$ (smoothness being defined in a certain, appropriate way). A Fock bundle can be constructed as $V := \bigoplus_{i=0}^\infty V^i$, where $V^i$ is defined to be either $\wedge^i V^1$ or $\vee^i V^1$ (respectively, antisymmetrized and symmetrized tensor products for fermions and bosons). Thus one particle states are represented as $V$-valued generalized half densities.

The spacetime connection determines a connection $\Gamma_m$ on $P_m \rightarrow M$; moreover, in the usual physical situations one has a connection $V \rightarrow P_m \rightarrow M$ which is linear projectable over $\Gamma_m$. These determine a connection on $V^1 \rightarrow M$ and hence on $V \rightarrow M$.

For each $p \in (P_m)_x$ , $x \in M$, let $\delta_p$ be the Dirac density with support $\{ p \}$ in $(P_m)_x$, $\omega_m$ the Leray density of $(P_m)_x \subset T_x^* M$ and $(b_\alpha(p))$ a basis of $V_p$. Let moreover $l \in \mathbb{L}$ be a ‘length unit’. Then the set $(B_{p\alpha})$, with

$$B_{p\alpha} := \frac{1}{\sqrt{2l^3 \sqrt[p_0]{\omega_m^{-1/2}} \otimes b_\alpha} ,$$

constitutes a generalized frame of $V^1$. The above said connection on $V \rightarrow M$ yields parallel transport of such frames along curves in $M$; in particular, 4-momentum $p$ is parallely transported.

Consider now the bundle $W \rightarrow M$ of Dirac spinors. For each $p \in P_m$ one has a splitting

$$W = W^+_p \oplus W^-_p , \quad W^{\pm}_p := \ker(\gamma[p] \mp m) .$$

\footnote{At least locally the chosen worldline determines, via exponentiation, a splitting space+time enabling a position space representation.}

\footnote{The restrictions of the Hermitian metric $k$ (1.1) to these two subspaces turn out to have the signatures $(+, +)$ and $(-, -)$, respectively.}
Thus for each $m \in \mathbb{L}^{-1}$ one has the 2-fibred bundles $W^+_m \rightarrow P_m \rightarrow M$ defined by

$$W^+_m := \bigcup_{p \in P_m} W^+_p \subset P_m \times M.$$ 

$W^+_m$ and $W^-_m$ are then the electron bundle and the positron bundle, respectively.

All the above constructions can be restricted to timelike one-dimensional base manifold $L \subset M$. In order to introduce appropriate generalized frames for free electron and positron states along $L$ one needs, for each $p \in (P_m)_L$ a frame $$(u_A (p), v_A (p)), \quad A = 1, 2$$
of $W_p$ which is adapted to the splitting $W_p = W^+_p \oplus W^-_p$. A consistent choice can be made by the following procedure.

Fix any point $x_0 \in L$. Let $\tau_{x_0} \in \mathbb{L}^{-1} \otimes T_{x_0} L$ be the unit future-pointing scaled vector field. Choose any 2-spinor basis $(\zeta_A)$ such that the timelike element $\tau_0$ of the induced Pauli basis coincides with $\tau_{x_0}$. Then the corresponding Dirac basis (§1.4), constituted by the elements

$$u_1 := \frac{1}{\sqrt{2}} (\zeta_1, \bar{z}^1), \quad u_2 := \frac{1}{\sqrt{2}} (\zeta_2, \bar{z}^2), \quad v_1 := \frac{1}{\sqrt{2}} (\zeta_1, -\bar{z}^1), \quad v_2 := \frac{1}{\sqrt{2}} (\zeta_2, -\bar{z}^2),$$
is adapted to the splitting determined by $p \equiv m g^b (\tau_{x_0})$. Next we Fermi-transport this basis along $L$. $\tau$ itself is Fermi-transported, and so is the 1-form $\tau^b := g^b (\tau)$ corresponding to $\tau$ via the spacetime metric. Thus we get a Dirac frame $(u_A (m \tau^b), v_A (m \tau^b))$ which is adapted to splitting determined by $p \equiv m \tau^b$.

Now we have to extend this to a Dirac frame of $W_L$ for all $p \in (P_m)_L$. This can be done, at each $x \in L$, essentially by the usual procedure of the flat inertial case. Namely, if $p \in (P_m)_x$ is now an arbitrary 4-momentum at $x$ then take the unique boost $\Lambda$ such that $\Lambda (\tau_x) = g^b (p)/m$; up to sign there is a unique transformation $K \in \text{Spin}(W)$ which projects over $\Lambda$, and an overall sign can be fixed by continuity. This $K$ transforms the Dirac frame $(u_A (m \tau^b), v_A (m \tau^b))$ into the new Dirac frame $(u_A (p), v_A (p))$.

The introduction of free photon states has subtleties of a different nature, while their transport along the observer’s world line is performed via ordinary Fermi transport. Then one modifies the induced free-particle connection on the Fock bundle of QED by an interaction (not deduced from any underlying classical structure) which yields the full picture of electrodynamics (see [6] for details).

References

[1] Birrel, N.D. and Davies, P.C.W.: Quantum fields in curved space, Cambridge University Press, Cambridge (1982).

[2] Canarutto, D. and Jadczyk, A.: ‘Two-spinors and Einstein-Cartan-Maxwell-Dirac fields’, Il Nuovo Cimento 113 B (1997), 49–67.

[3] Canarutto, D.: ‘Possibly degenerate tetrad gravity and Maxwell-Dirac fields’, J. Math. Phys. 39, N.9 (1998), 4814–4823.

11 This is related to the structure of boosts. See [7] for a detailed account of the relations among spinor groups and the Lorentz group in terms of 2-spinors.
REFERENCES

[4] Canarutto, D.: ‘Two-spinors, field theories and geometric optics in curved space-time’, Acta Appl. Math. 62 N.2 (2000), 187–224.

[5] Canarutto, D.: ‘Connections on distributional bundles’, Rend. Semin. Mat. Univ. Padova 111 (2004), 71–97.

[6] Canarutto, D.: ‘Quantum bundles and quantum interactions’, Int. J. Geom. Met. Mod. Phys., 2 N.5, (2005), 895–917; arXiv:math-ph/0506058v2 (2006).

[7] Canarutto, D.: ‘“Minimal geometric data” approach to Dirac algebra, spinor groups and field theories’, Int. J. Geom. Met. Mod. Phys., 4 N.6, (2007), 1005–1040; http://arxiv.org/abs/math-ph/0703003v2 (2008).

[8] Ferraris, M. and Kijowski J.: ‘Unified Geometric Theory of Electromagnetic and Gravitational Interactions’, Gen. Rel. Grav. 14, 1 (1982), 37–47.

[9] Hehl, F.W., McCrea, J.D., Mielke, E.W. and Ne’eman, Y.: ‘Metric-affine gauge theory of gravity: field equations, Noether identities, world spinors, and breaking of dilaton invariance’, Phys. Rep. 258 (1995), 1–171.

[10] Janyška, J., Modugno, M. and Vitolo, R.: ‘Semi–vector spaces and units of measurement’, Preprint (2008), arXiv:0710.1313v1.

[11] Penrose, R. and Rindler, W.: Spinors and space-time. I: Two-spinor calculus and relativistic fields, Cambridge Univ. Press, Cambridge (1984).

[12] Samuel, J. and Nityananda, R.: ‘Transport along null curves’, J. Phys. A, 33 (2000), 2895-2905.