ARVESON’S CRITERION FOR UNITARY SIMILARITY

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ABSTRACT. This paper is an exposition of W.B. Arveson’s complete invariant for the unitary similarity of complex, irreducible matrices.

INTRODUCTION

Forty years ago W.B. Arveson announced an important theorem concerning the unitary similarity problem [2]. His proof of the theorem appeared two years later as a consequence of a deep study [1, 3] that profoundly influenced the subsequent development of operator algebra theory. With the richness of the operator-algebraic results in these seminal papers, Arveson’s significant and novel contribution to linear algebra has been somewhat overshadowed. Therefore, my aims with this exposition are to draw attention again to this remarkable result and to give a self-contained proof of it.

The method of proof is different from Arveson’s (and from Davidson’s treatment [7] of Arveson’s approach), and so may be considered new. However, the arguments draw upon known results, adapted to the setting, language, and notation of linear algebra. The significant ideas are due to other mathematicians; I have merely reconfigured them in a package accessible to readers with a background in core linear algebra.

The paper is intended to be self-contained. Results that have found their way into textbooks are merely recalled for the reader’s benefit. The standard references used here are the books of Horn and Johnson [11] (for linear algebraic analysis) and Paulsen [16] (for completely positive linear transformations of matrix spaces). I provide proofs for results that may be well known (Dunford’s Ergodic Theorem [8], Kadison’s Isometry Theorem [12]), but are not in standard textbooks. In such cases, the proofs treat the problem at hand rather than the most general situation.

We shall use the following terminology and notation. The set of $n \times n$ matrices over the field $\mathbb{C}$ of complex numbers is denoted by $M_n$, and for every $X \in M_n$ the conjugate transpose of $X$ is denoted by $X^*$. A matrix $X \in M_n$ is: hermitian if $X^* = X$; positive semidefinite if $X = Y^*Y$ for some $Y \in M_n$; unitary if $X$ is invertible and $X^{-1} = X^*$. The spectral (or operator) norm of $X \in M_n$ is given by

$$
\|X\| = \sqrt{\text{spr}(X^*X)},
$$

where $\text{spr}(Y)$ denotes the spectral radius of $Y \in M_n$. The closed unit ball of $M_n$ is the set

$$
\text{Ball}(M_n) = \{X \in M_n : \|X\| \leq 1\},
$$

which is a convex set whose set of extreme points is $U_n$ [12], [11 §3.1, Problem 27]. In the metric topology of $M_n$ induced by the spectral norm, the sets $\text{Ball}(M_n)$ and $U_n$ are compact.
1. The Unitary Similarity Problem

Two matrices $A, B \in M_n$ are said to be unitarily similar if $B = U^*AU$ for some $U \in U_n$.

**Definition 1.1.** Let $\mathcal{O} \subseteq M_n$ be fixed, nonempty subset of matrices. The unitary similarity problem for $\mathcal{O}$ is to find a countable family $\mathcal{F}_\mathcal{O}$ of functions defined on $\mathcal{O}$ with the following two properties:

1. $f(U^*AU) = f(A)$, for all $A \in \mathcal{O}, \ U \in U_n, \ f \in \mathcal{F}_\mathcal{O}$;
2. $f(A) = f(B)$, for fixed $A, B \in \mathcal{O}$ and for all $f \in \mathcal{F}_\mathcal{O}$, if and only if $B = U^*AU$ for some $U \in U_n$.

Condition 1 above asserts that the functions $f \in \mathcal{F}_\mathcal{O}$ are invariant under unitary similarity and condition 2 says that these invariants are complete in the sense that if matrices $A, B \in \mathcal{O}$ are not unitarily equivalent, then $f(A) \neq f(B)$ for at least one of the invariants $f \in \mathcal{F}_\mathcal{O}$.

In the best of circumstances, the set $\mathcal{O}$ is $M_n$, but that is not always to be the case, and instead one may require that the set $\mathcal{O}$ be an algebraic variety or possess some good topological properties. The set $\mathcal{O}$ considered by Arveson is of the latter type: it has the topology of a second countable complete metric space.

Although now twenty years old, the survey paper by Shapiro [17] remains a good reference for an overview of the unitary similarity problem. Perhaps the most celebrated of all contributions to the problem are two classical results: Specht’s trace invariants [18] and Littlewood’s algorithm [15].

2. Statement of Arveson’s Theorem

**Definition 2.1.** Assume $X, P \in M_n$.

1. $P$ is a projection if $P^* = P$ and $P^2 = P$.
2. $X \in M_n$ is irreducible if $XP = PX$, for a projection $P$, holds only if $P \in \{0, I\}$, where $I \in M_n$ denotes the identity matrix.
3. $\mathcal{O}_{irr}$ denotes the set of all irreducible matrices in $M_n$.

Equivalently, $X \in M_n$ is irreducible if and only if the algebra generated by the set $\{I, X, X^*\}$ is $M_n$. The set $\mathcal{O}_{irr}$ is a dense $G_δ$-set [9]. Therefore, $\mathcal{O}_{irr}$ is a Polish space, which is to say that (in the relative topology) $\mathcal{O}_{irr}$ is a second countable complete metric space.

The set $\mathcal{S}$ of pairs $(H, K)$ of $n \times n$ matrices with entries in $\mathbb{Q} + i\mathbb{Q}$ is countable and dense in $M_n \times M_n$. Let $\mathcal{F}_{\mathcal{O}_{irr}}$ be the family of functions $f_{(H,K)}, \ (H, K) \in \mathcal{S}$, defined on $M_n$ by

$$f_{(H,K)}(A) = \|A \otimes H + I \otimes K\|, \ A \in M_n.$$ 

Because $U \otimes I \in U_{n^2}$ (the unitary group of $M_n \otimes M_n$) for all $U \in U_n$, it is clear that $f_{(H,K)}(U^*AU) = f(A)$ for all $U \in U_n$ and $A \in M_n$. Hence, $\mathcal{F}_{\mathcal{O}_{irr}}$ is a countable family of unitary similarity invariants for $M_n$. The following theorem shows, using the fact that $\mathcal{S}$ is dense in $M_n \times M_n$, that $\mathcal{F}_{\mathcal{O}_{irr}}$ is a complete invariant for unitary similarity for the class $\mathcal{O}_{irr}$.

**Theorem 2.1.** (Arveson) The following statements are equivalent for $A, B \in M_n$ such that $A \in \mathcal{O}_{irr}$:

1. $\|A \otimes H + I \otimes K\| = \|B \otimes H + I \otimes K\|$, for all $H, K \in M_n$;
2. $B = U^*AU$ for some $U \in U_n$.
Note that if neither $A$ nor $B$ is assumed to be irreducible, then (i) does not imply (ii). In particular, if $X$ is any irreducible matrix and if $A = X \oplus X$ and $B = X \oplus 0$, then $A$ and $B$ satisfy (i) but not (ii).

The key steps in the proof of Theorem 2.1 are:

1. to show that there are unital completely positive linear transformations $\phi, \psi : M_n \to M_n$ such that $\phi(A) = B$ and $\psi(B) = A$;
2. to show that, for the transformation $\omega = \psi \circ \phi$ on $M_n$, the condition $\omega(A) = A$ implies that $\omega(X) = X$ for every $X \in M_n$ (this is the heart of the argument and is called the Boundary Theorem);
3. to show that if a unital completely positive linear transformation of $M_n$ is an isometry, then it must be a unitary similarity transformation (this result is known as Kadison’s Isometry Theorem);
4. to use $X = \psi(\phi(X))$ for all $X \in M_n$ to show that $\phi$ is an isometry and, hence, a unitary similarity transformation.

3. Completely Positive Linear Transformations of Matrix Spaces

For a fixed $n \in \mathbb{N}$, our interest is with linear transformations $\phi : M_n \to M_n$ that leave certain matrix cones invariant, not just at the level of $M_n$ itself, but at the level of all matrix rings over $M_n$.

**Definition 3.1.** (Two Identifications of Matrix Spaces) Fix $n \in \mathbb{N}$. For every $p \in \mathbb{N}$ the ring $M_{pn}$ of $pn \times pn$ matrices is considered in the following two equivalent ways:

1. as block matrices—namely $M_{pn} = M_p(M_n)$, the ring of $p \times p$ matrices over the ring $M_n$;
2. as tensor (Kronecker) products—that is, $M_{pn} = M_n \otimes M_p$.

The identity matrix of $M_{pn}(M_n)$ is denoted by $I_n \otimes I_p$. Likewise, if $\mathcal{T} \subseteq M_n$ is any subspace, then $M_p(\mathcal{T})$ denotes the vector space of all $p \times p$ matrices with entries from $\mathcal{T}$ and is identified with $\mathcal{T} \otimes M_p$.

**Definition 3.2.** (Matricial Cones and Orderings) If $\mathcal{R} \subseteq M_n$ is a subspace of matrices with the properties

1. $I \in \mathcal{R}$ and
2. $X^* \in \mathcal{R}$ for every $X \in \mathcal{R}$,

then the canonical matricial cones of $\mathcal{R}$ are the sets

$$M_p(\mathcal{R})_+ = \{ H \in M_p(\mathcal{R}) : H \text{ is a positive semidefinite matrix} \}.$$ 

If $M_p(\mathcal{R})_{sa}$ denotes the real vector space of hermitian matrices of $M_p(\mathcal{R})$ and if $X, Y \in M_p(\mathcal{R})_{sa}$, then $X \leq Y$ denotes $Y - X \in M_p(\mathcal{R})_+$; this is called the canonical matricial ordering of $\mathcal{R}$.

The matricial cones of $\mathcal{R}$ have extremely good cone-theoretic properties. First, the set $M_p(\mathcal{R})_+$ is a cone in the usual sense of being closed under multiplication by positive scalars and finite sums. Moreover: this cone is pointed, which is to say that $M_p(\mathcal{R})_+ \cap (-M_p(\mathcal{R})_+) = \{0\}$; it is reproducing in that $M_p(\mathcal{R})_{sa}$ is obtained by taking all differences $H - K$, for $H, K \in M_p(\mathcal{R})_+$; and it is closed in the topology of $M_p(M_n)$. Such a cone is said to be proper. Since $M_p(\mathcal{R}) = M_p(\mathcal{R})_{sa} + iM_p(\mathcal{R})_{sa}$, the cone $M_p(\mathcal{R})_+$ spans $M_p(\mathcal{R})$.

The identity matrix of $M_p(M_n)$ is an Archimedean order unit for $M_p(\mathcal{R})_{sa}$: for every $H \in M_p(\mathcal{R})_{sa}$ there is a $t > 0$ such that $-t(I_n \otimes I_p) \leq H \leq t(I_n \otimes I_p)$ and $t(I_n \otimes I_p) + H \in M_p(\mathcal{R})_+$ for all $t > 0$ if and only if $H \in M_p(\mathcal{R})_+$. 

Lastly, there is an intimate relationship between the norm and the ordering: for every $Z \in M_p(\mathbb{R})$,

$$
\|Z\| = \inf \left\{ t > 0 : \begin{bmatrix} t(I_n \otimes I_p) & Z \\ Z^* & t(I_n \otimes I_p) \end{bmatrix} \in M_{2p}(\mathbb{R})^+ \right\}.
$$

**Definition 3.3.** Assume that $\mathcal{R} \subseteq M_n$ is a subspace that is closed under the conjugate transpose $X \mapsto X^*$ and contains the identity matrix, and let $\phi : \mathcal{R} \to M_n$ be any linear transformation.

1. The *norm* of $\phi$ is defined by $\|\phi\| = \max\{\|\phi(X)\| : X \in \mathcal{R}, \|X\| = 1\}$.
2. If $\mathcal{R} = M_n$, then $\phi^k$ denotes $\phi \circ \cdots \circ \phi$, the composition of $\phi$ with itself $k$ times.
3. For any $p \in \mathbb{N}$, $\phi^{(p)}$ denotes the linear transformation $\phi^{(p)} : M_p(\mathcal{R}) \to M_p(M_n)$, $\phi^{(p)}([X_{ij}]) = [\phi(X_{ij})]$.
4. If $\phi^{(p)}$ maps $M_p(\mathcal{R})^+$ into $M_p(M_n)^+$, for every $p \in \mathbb{N}$, then $\phi$ is called a completely positive linear transformation.
5. If $\phi$ is completely positive and if $\phi(I) = I$, then $\phi$ is called a ucp map (unital completely positive).
6. If $\mathcal{R} = M_n$ and if $\phi$ is a ucp map, then $\phi$ is called a conditional expectation if $\phi^2 = \phi$.

The following theorem captures a few of the most important features of completely positive linear transformations of matrix spaces.

**Theorem 3.1.** Assume that $\mathcal{R} \subseteq M_n$ is a subspace that is closed under the conjugate transpose and contains the identity matrix, and let $\phi : \mathcal{R} \to M_n$ be a completely positive linear transformation.

1. (Arveson Extension Theorem) There is a completely positive linear transformation $\Phi : M_n \to M_n$ such that $\Phi|_{\mathcal{R}} = \phi$.
2. (Stinespring–Kraus–Choi Representation) There are linearly independent matrices $V_1, \ldots, V_r \in M_n$ such that

$$
\phi(X) = \sum_{j=1}^{r} V_j^* XV_j, \quad \forall X \in \mathcal{R}.
$$

3. If $\mathcal{R} = M_n$ and if $\phi$ is a conditional expectation with range $\mathcal{S}$, then

$$
\phi(YZ) = \phi(Y\phi(Z)), \quad \forall Y \in \mathcal{S}, Z \in M_n.
$$

Proofs for the assertions in Theorem 3.1 are given, respectively, in Theorem 7.5, Theorem 4.1, and Theorem 15.2 of [16].

4. **An Ergodic Theorem**

The following result is special case of a theorem of Dunford [8].

**Theorem 4.1.** (Ergodic Theorem) If $\omega : M_n \to M_n$ is a linear transformation of norm 1 and has 1 as an eigenvalue, then

$$
\lim_{m \to \infty} \frac{1}{m} \sum_{k=0}^{m-1} \omega^k
$$

exists and the limit $\Omega$ in (2) is an idempotent linear transformation with range $\ker(\omega - \text{id}_{M_n})$ and kernel $\text{ran}(\omega - \text{id}_{M_n})$. 
Proof. If \( X \in \ker(\omega - \text{id}_{M_n}) \), then \( \omega^k(X) = X \) for every \( k \in \mathbb{N} \) and so \( \frac{1}{m} \sum_{k=0}^{m-1} \omega^k(X) = X \) for every \( m \in \mathbb{N} \). Thus, on the subspace \( \ker(\omega - \text{id}_{M_n}) \), the limit in (2) exists and coincides with the identity on \( \ker(\omega - \text{id}_{M_n}) \).

Suppose that \( Y = (\omega - \text{id}_{M_n})(X) \), for some \( X \in M_n \). Thus,

\[
\left\| \frac{1}{m} \sum_{k=0}^{m-1} \omega^k(Y) \right\| = \left\| \frac{1}{m} \sum_{k=0}^{m-1} \omega^k (\omega - \text{id}_{M_n}) (X) \right\|
\]

\[
= \left\| \frac{1}{m} (\omega^m - \text{id}_{M_n}) (X) \right\|
\]

\[
\leq \frac{1}{m} \|\omega^m - \text{id}_{M_n}\| \|X\|
\]

\[
\leq \frac{2}{m} \|X\|.
\]

Hence, on the subspace \( \text{ran}(\omega - \text{id}_{M_n}) \), the limit in (2) exists and coincides with the zero transformation on \( \text{ran}(\omega - \text{id}_{M_n}) \).

For every \( m \in \mathbb{N} \), \( \frac{1}{m} \omega^m \| \leq \frac{1}{m} \|\omega\| \|m = \frac{1}{m} \), and so \( \frac{1}{m} \omega^m \to 0 \). If \( J \) is the Jordan canonical form of \( \omega \), then \( \frac{1}{m} J^m \to 0 \) as well. This is true for every Jordan block of \( J \) and in particular for every \( \ell \times \ell \) Jordan block \( J_{\ell}(1) \) for the eigenvalue 1 of \( \omega \). But if \( \ell > 1 \), then \( \frac{1}{m} J_{\ell}(1)^m \) fails to converge to the zero matrix, and so it must be that \( \ell = 1 \). This proves that

\[
(3) \quad \ker((\omega - \text{id}_{M_n})^2) = \ker(\omega - \text{id}_{M_n}).
\]

The Rank-Plus-Nullity Theorem asserts that the dimensions of \( \ker(\omega - \text{id}_{M_n}) \) and \( \text{ran}(\omega - \text{id}_{M_n}) \) sum to \( n^2 = \text{dim} M_n \). Equation (3) shows that \( \ker(\omega - \text{id}_{M_n}) \) and \( \text{ran}(\omega - \text{id}_{M_n}) \) have zero intersection. Hence, \( M_n \) is an algebraic direct sum of \( \ker(\omega - \text{id}_{M_n}) \) and \( \text{ran}(\omega - \text{id}_{M_n}) \), which proves that the limit (2) exists and that the limit \( \Omega \) is an idempotent. \( \square \)

**Corollary 4.2.** If \( \omega : M_n \to M_n \) is a linear transformation such that \( \|\omega\| = 1 \), and if \( \lambda \) is an eigenvalue of \( \omega \) such that \( |\lambda| = 1 \), then \( \lambda \) is a semisimple eigenvalue in the sense that

\[
\ker((\omega - \lambda \text{id}_{M_n})^2) = \ker(\omega - \lambda \text{id}_{M_n}).
\]

**Proof.** Let \( \omega' = \frac{1}{\lambda} \omega \) and apply Theorem 1.1. \( \square \)

Our main application of the Ergodic Theorem is:

**Corollary 4.3.** If \( \omega : M_n \to M_n \) is a unital completely positive linear transformation, then \( \Omega = \lim_{m \to \infty} \frac{1}{m} \sum_{k=0}^{m-1} \omega^k \) is a conditional expectation with range \( \{X \in M_n : \omega(X) = X\} \), the set of fixed points of \( \omega \).

A second application of the Ergodic Theorem is drawn from quantum information theory \[14\].

**Corollary 4.4.** If \( \omega : M_n \to M_n \) is a ucp map, then there is a sequence \( \{k_j\}_{j \in \mathbb{N}} \) and a conditional expectation \( \Phi \) on \( M_n \) such that

\[
\Phi = \lim_{j \to \infty} \omega^{k_j}.
\]
Moreover, $\Phi$ is the unique conditional expectation in the set of cluster points of the set $\{\omega^k\}_{k \in \mathbb{N}}$.

**Proof.** Suppose that $\omega$ is in Jordan canonical form $J$. By Corollary 4.2, every eigenvalue $\lambda$ of $\omega$ of modulus 1 is semisimple, which is to say that the size of every Jordan block of $\lambda$ in $J$ is $1 \times 1$. Hence, we may choose any sequence $\{k_j\}_{j \in \mathbb{N}}$ so that the eigenvalues of $J^{k_j}$ accumulate around 1 and 0 as $j \to \infty$, thereby yielding a limiting matrix that is idempotent. Clearly this is the only such idempotent cluster point of $\{J^k\}_{k \in \mathbb{N}}$. Going back from the Jordan form $J$ to $\omega$, one concludes that $\Omega$ is a idempotent, unital, and completely positive. \qed

5. Completely Positive Isometries of $M_n$

A special case of a theorem of Kadison [12, Theorem 10] is:

**Theorem 5.1.** (Kadison’s Isometry Theorem) If $\phi : M_n \to M_n$ is a unital completely positive linear transformation such that $\|\phi(X)\| = \|X\|$ for all $X \in M_n$, then there exists $U \in U_n$ such that $\phi(X) = U^*XU$ for all $X \in M_n$.

**Proof.** Assume that $\phi$ has a Stinespring–Kraus–Choi representation that is given by

$$\phi(X) = \sum_{i=1}^r V_i^* XV_j, \quad X \in M_n,$$

for some linearly independent $V_1, \ldots, V_r \in M_n$. Let $\{e_1, \ldots, e_n\}$ be the standard orthonormal basis for $\mathbb{C}^r$ and consider the function $V : \mathbb{C}^n \to \mathbb{C}^n \otimes \mathbb{C}^r$ for which

$$V \xi = \sum_{i=1}^r V_i \xi \otimes e_i, \quad \xi \in \mathbb{C}^n.$$

Define an injective unital homomorphism $\pi : M_n \to M_n \otimes M_r$ by $\pi(X) = X \otimes I_r$. Thus,

$$\phi(X) = V^* \pi(X) V = \sum_{i=1}^r V_i^* XV_j, \quad X \in M_n. \quad (4)$$

Furthermore, because $V_1, \ldots, V_r \in M_n$ are linearly independent,

$$\text{Span} \{\pi(X)V \xi \mid X \in M_n, \xi \in \mathbb{C}^n\} = \mathbb{C}^n \otimes \mathbb{C}^r. \quad (5)$$

The linear map $\phi$ is an isometry of a finite-dimensional space; thus, $\phi$ has an isometric inverse. Therefore, if $W \in U_n$, then $\phi(W)$ is the midpoint between $X, Y \in \text{Ball}(M_n)$ only if $W$ is the midpoint between $\phi^{-1}(X), \phi^{-1}(Y) \in \text{Ball}(M_n)$, which is possible only if $\phi^{-1}(X) = \phi^{-1}(Y) = W$ because unitary matrices are extreme points of $\text{Ball}(M_n)$. Thus, $\phi(W)$ is an extreme point of $\text{Ball}(M_n)$, which is to say that $\phi(W) \in U_n$ for all $W \in U_n$.

Decompose $\mathbb{C}^n \otimes \mathbb{C}^r$ as $\text{ran} V \oplus (\text{ran} V)^\perp$ and choose $W \in U_n$. With respect to this decomposition of $\mathbb{C}^n \otimes \mathbb{C}^r$, the unitary matrix $\pi(W)$ has the form

$$\pi(W) = \begin{bmatrix} \phi(W) & Z_{12} \\ Z_{21} & Z_{22} \end{bmatrix}.$$

Since

$$\begin{bmatrix} I_n & 0 \\ 0 & I_{(r-1)n} \end{bmatrix} = \pi(W)^* \pi(W) = \begin{bmatrix} \phi(W)^* \phi(W) + Z_{21}^* Z_{21} & * \\ * & * \end{bmatrix},$$

...
we have $Z_{21}^\ast Z_{21} = I_n - \phi(W)^* \phi(W) = 0$ (as $\phi(W)$ is unitary). Thus, $Z_{21} = 0$. Likewise, from $\pi(W)\pi(W)^* = I_n$, we deduce that $Z_{12} = 0$. Therefore, the off-diagonal blocks of $\pi(W)$ must be zero. This is true for every $W \in U_n$, and because $U_n$ spans $M_n$, it is also true that

$$\pi(X) = \begin{bmatrix} \phi(X) & 0 \\ 0 & * \end{bmatrix},$$

for every $X \in M_n$. That is, the subspace $\operatorname{ran} V = \pi(X)$-invariant, for every $X \in M_n$. But in light of (5), this implies that the range of $V$ is $C^n \otimes C^r$, which is possible only if $r = 1$. Thus, $V_1$ is unitary and taking $U = V_1$ completes the proof of the theorem.

\[\square\]

6. Fixed Points

The deepest aspect of Arveson’s criterion for unitary similarity is the following theorem concerning the set $\{X \in M_n : \omega(X) = X\}$ of fixed points of a unital completely positive linear transformation $\omega$ of $M_n$.

**Theorem 6.1.** (Boundary Theorem) If $A \in M_n$ is irreducible and if $\omega : M_n \to M_n$ is a unital completely positive linear transformation such that $\omega(A) = A$, then $\omega(X) = X$ for every $X \in M_n$.

**Proof.** Let $\mathcal{R} = \text{Span}\{I, A, A^\ast\}$ so that $M_n$ is the algebra generated by $\mathcal{R}$ and $\omega_{|\mathcal{R}} = \text{id}_{\mathcal{R}}$. Let $\mathcal{S} = \{X \in M_n : \omega(X) = X\}$, which is a unital subspace of $M_n$ that contains the identity matrix and is closed under the involution $Z \mapsto Z^\ast$. Because $\mathcal{S} \supseteq \mathcal{R}$, the algebra generated by $\mathcal{S}$ is $M_n$.

The Ergodic Theorem asserts that $\Omega = \lim_{m \to \infty} \frac{1}{m} \sum_{k=0}^{m-1} \omega^k$ is a conditional expectation that maps $M_n$ onto the fixed point space $\mathcal{S}$. Thus, by the Choi–Effros Theorem [6, 15, Theorem 15.2], the linear space $\mathcal{S}$ is an algebra under the product $\odot$ defined by

$$X \odot Y = \Omega(XY), \quad X, Y \in \mathcal{S}.$$  
(6)

If $Y \in \mathcal{S}$ and $Z \in M_n$, then by Theorem 6.1.3,

$$\Omega(YZ) = \Omega(Y\Omega(Z)) = Y \odot \Omega(Z) = \Omega(Y) \odot \Omega(Z).$$

Likewise, if $Y_1, Y_2 \in \mathcal{S}$ and $Z \in M_n$, then

$$\Omega((Y_1Y_2)Z) = \Omega(Y_1\Omega(Y_2Z)) = Y_1 \odot \Omega(Y_2Z) = (\Omega(Y_1) \odot \Omega(Y_2)) \odot \Omega(Z).$$

By induction, if $a$ is any word in $2q$ noncommuting variables, and if $Y_1, \ldots, Y_q \in \mathcal{S}$ and $Z \in M_n$, then

$$\Omega(a(Y_1, \ldots, Y_q, Y_1^*, \ldots, Y_q^*)Z) = (a \odot (\Omega(Y_1), \ldots, \Omega(Y_q), \Omega(Y_1)^*, \ldots, \Omega(Y_q)^*)) \odot \Omega(Z),$$

where $a \odot (\Omega(Y_1), \ldots, \Omega(Y_q), \phi(Y_1)^*, \ldots, \phi(Y_q)^*)$ denotes the $\odot$-product of the letters of the word $a$. Because the algebra generated by $\mathcal{S}$, namely $M_n$, is given by linear combinations of elements of the form $a(Y_1, \ldots, Y_q, Y_1^*, \ldots, Y_q^*)$ for various positive integers $q$, words $a$, and elements $Y_j \in \mathcal{S}$, the linear transformation $\Omega$ satisfies

$$\Omega(WZ) = \Omega(W) \odot \Omega(Z), \quad \forall W, Z \in M_n.$$

That is, $\Omega$ is a homomorphism of the associative algebra $M_n$ onto the associative algebra $\mathcal{S}$ with product $\odot$. Because $M_n$ has no nontrivial ideals and $\mathcal{S} \neq \{0\}$, $\Omega$ must in fact be an isomorphism. Thus, $\ker \Omega = \{0\}$, which implies that the
idempotent $\Omega$ is the identity transformation. Therefore, the range of $\Omega$, namely the fixed point set $\mathcal{S}$, is all of $M_n$.

**Corollary 6.2.** If $\omega$ is a unital completely positive linear transformation of $M_n$ for which $\ker(\omega - \text{id}_{M_n}) \cap \mathcal{O}_{\text{irr}} \neq \emptyset$, then $\omega$ is the identity transformation.

**Corollary 6.3.** (Noncommutative Choquet Theorem) If $A \in M_n$ is irreducible and $\mathcal{R} = \text{Span}\{I, A, A^*\}$, then the unital completely positive linear transformation $\iota : \mathcal{R} \rightarrow M_n$ defined by $\iota(X) = X$, for $X \in \mathcal{R}$, has a unique completely positive extension to $M_n$.

7. **Proof of Theorem 6.1**

If $A, B \in M_n$ are unitarily similar, then a straightforward calculation verifies that $\|A \otimes H + I \otimes K\| = \|B \otimes H + I \otimes K\|$ for all $H, K \in M_n$.

Conversely, assume that $A, B \in M_n$, $A \in \mathcal{O}_{\text{irr}}$, and $\|A \otimes H + I \otimes K\| = \|B \otimes H + I \otimes K\|$ for all $H, K \in M_n$. Define a linear map $\phi_0 : \text{Span}\{I, A\} \rightarrow M_n$ by $\phi_0(\alpha I + \alpha_1 A) = \alpha_0 I + \alpha_1 B$, $\forall \alpha, \alpha_1 \in \mathbb{C}$. Because $\|A \otimes H + I \otimes K\| = \|B \otimes H + I \otimes K\|$ for all $H, K \in M_n$, the linear map $\phi_0^{(n)} : \text{Span}\{I, A\} \otimes M_n \rightarrow M_n \otimes M_n$, in which $\phi_0^{(n)}([X_{st}]_{1 \leq s, t \leq n}) = [\phi_0(X_{st})]_{1 \leq s, t \leq n}$, is an isometry.

Let $\mathcal{R} = \text{span}\{I, A, A^*\}$. By [16] Proposition 3.5, the unital linear transformation $\phi : \mathcal{R} \rightarrow M_n$ defined by $\phi(\alpha I + \beta A + \gamma A^*) = \alpha I + \beta \phi_0(A) + \gamma \phi_0(A)^*$ is completely positive and satisfies $\phi(A) = B$. Therefore, by Theorem 5.1[14], there is a completely positive extension of $\phi$ from $\mathcal{R}$ to $M_n$; without loss of generality, let $\phi$ denote the extended completely positive transformation of $M_n$. By similar reasoning, there is a unital completely positive linear transformation $\psi : M_n \rightarrow M_n$ such that $\psi(B) = A$. Hence, $\omega = \psi \circ \phi$ is a unital completely positive linear transformation of $M_n$ with $\omega(A) = A$. By the Boundary Theorem (Theorem 6.1), $\omega = \psi \circ \phi$ is the identity transformation, and so

$$||X|| = ||\psi(\phi(X))|| \leq ||\phi(X)|| \leq ||X||$$

for every $X \in M_n$. That is, $\phi : M_n \rightarrow M_n$ is a unital completely positive isometry. Therefore, by Kadison’s Isometry Theorem (Theorem 6.1), there is a $U \in U_n$ such that $\phi(X) = U^* X U$ for every $X \in M_n$. Hence, $B = U^* A U$.

8. **Discussion**

Any proof of Arveson’s criterion for unitary similarity likely requires the Boundary Theorem (Theorem 6.1). If one compares the proof of Specht’s Theorem, as given by Kaplansky in [13], Theorem 63], with the proof of the Boundary Theorem herein, it is clear that properties of matrix rings have a crucial role in arriving at these results, even if the statements of the results are concerned only with single matrices and the proofs, for the most part, involve only linear spaces of matrices.
Our proof of the Boundary Theorem is different from Arveson’s (and from Davidson’s [7]) in that it is based on methods that are used in the study of the noncommutative Silov boundary, which was introduced by Arveson in [1] and developed further by Hamana [10] and Blecher [5]. In contrast, Arveson and Davidson approach the theorem from the perspective of the noncommutative Choquet boundary [4]. These noncommutative Silov and Choquet boundaries are used by Arveson [4] to classify, up to complete order isomorphism, all subspaces of matrices that contain the identity matrix and are closed under the conjugate transpose. Such a classification is indeed a broader, more sophisticated form of the main theorem (on unitary similarity) of the present paper, yet is still within the scope and interest of core linear algebra.

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1The Choquet boundary of a linear space $E$ of continuous complex-valued functions on a compact Hausdorff space $X$—where $E$ separates the points of $X$, contains the constant functions, and is closed under complex conjugation—is the set of all $x_0 \in X$ for which the positive linear functional $f \mapsto f(x_0)$, $f \in E$, has a unique extension to a positive linear functional on the space of all continuous functions $g : X \to \mathbb{C}$. Corollary [4] is exactly this idea, but in a noncommutative environment in which $R$ plays the role of $E$. 

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