Continuants with equal values, a combinatorial approach

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Abstract

A regular continuant is the denominator $K$ of a terminating regular continued fraction, interpreted as a function of the partial quotients. We regard $K$ as a function defined on the set of all finite words on the alphabet $1 < 2 < 3 < \ldots$ with values in the positive integers. Given a word $w = w_1 \cdots w_n$ with $w_i \in \mathbb{N}$ we define its multiplicity $\mu(w)$ as the number of times the value $K(w)$ is assumed in the Abelian class $\mathcal{X}(w)$ of all permutations of the word $w$. We prove that there is an infinity of different lacunary alphabets of the form \{ $b_1 < \cdots < b_l < l + 1 < l + 2 < \cdots < s$ \} with $b_j, l, s \in \mathbb{N}$ and $s$ sufficiently large such that $\mu$ takes arbitrarily large values for words on these alphabets. The method of proof relies in part on a combinatorial characterisation of the word $w_{\text{max}}$ in the class $\mathcal{X}(w)$ where $K$ assumes its maximum.

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Introduction. Given a sequence $w = (w_1, \ldots, w_n)$, of positive $w_i$, let $K(w)$ be the continuant of $w$, i.e., the denominator of the finite regular continued fraction $\frac{1}{w_1 + \frac{1}{w_2 + \cdots + \frac{1}{w_n}}}$. We shall regard $w$ as a word of length $n$ over the alphabet $\{1 < 2 < 3 < \ldots\}$ and write $w = w_1 \cdots w_n$. Since $K(w) = K(\bar{w})$, where $\bar{w} = w_n \cdots w_1$ denotes the reversal of $w$, we shall henceforth identify each word $w$ with its reverse $\bar{w}$. Let $\mathcal{X}(w)$ denote the Abelian class of $w$ consisting of all permutations of $w$. The following problem has attracted much attention and led to a number of applications (see e.g. \cite{1 4 5 7 8}): Let $A = \{a_1 < \cdots < a_s\}$ be a finite ordered alphabet with $a_j \in \mathbb{N}$. Given a word $w = w_1 w_2 \cdots w_n$ with $w_i \in A$, find the arrangements $w_{\text{max}}, w_{\text{min}} \in \mathcal{X}(w)$ maximizing resp. minimizing the function $K(\cdot)$ on $\mathcal{X}(w)$. The first author \cite{3} gave an explicit description of both extremal arrangements $w_{\text{max}}$ and $w_{\text{min}}$ and showed that in each case the arrangement is unique (up to reversal) and independent of the actual values of the positive integers $a_i$. He also investigated the analogous problem for the semi-regular continuant $K'$ defined as the denominator of the semi-regular continued fraction $\frac{1}{K'} = \frac{1}{w_1 - \frac{1}{w_2 - \cdots - \frac{1}{w_n - \frac{1}{w_{n-1} - \frac{1}{w_0}}}}}$ with entries $w_i \in \{2, 3, \ldots\}$. He gave a fully combinatorial description of the minimizing arrangement $w'_{\text{min}}$ for $K'(\cdot)$ on $\mathcal{X}(w)$ and showed that the arrangement is unique (up to reversal) and independent of the actual values of the positive integers $a_i$. However, the determination of the maximizing arrangement $w'_{\text{max}}$ for the semi-regular continuant turned out to be more difficult. He showed that in the special case of a 2-digit alphabet $\{(2 \leq a_1 < a_2)\}$, the maximizing arrangement $w'_{\text{max}}$ is a Sturmian word and is independent of the values of the $a_i$. Recently the second author together with M. Edson and A. De Luca \cite{8} developed an algorithm for constructing $w'_{\text{max}}$ over any ternary alphabet $\{(2 \leq a_1 < a_2 < a_3)\}$, and showed that the maximizing arrangement is independent of the choice of the digits. In contrast, they exhibited examples of words $w = w_1 \cdots w_n$ over a 4-digit alphabet $A = \{(2 \leq a_1 < a_2 < a_3 < a_4)\}$

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for which the maximizing arrangement for \( K'(\cdot) \) is not unique and depends on the actual values of the positive integers \( a_1 \) through \( a_s \). In the course of these investigations the following problem came up: given an alphabet \( A \) of positive integers, we say that a word \( w \) on \( A \) has multiplicity \( \mu = \mu(w) \) if the value \( K(w) \) occurs precisely \( \mu \) times in the multi-set \( \{ K(x) : x \in \mathcal{X}(w) \} \). The multiplicity \( \mu'(w) \) is defined analogously for the semi-regular continuant \( K'(w) \). Thus each Abelian class \( \mathcal{X}(w) \) is split into subclasses of equally valued words. Question: is it true that \( \mu \) can take arbitrarily large values for infinitely many alphabets and is there a combinatorial proof of this? Our aim here is to give a positive answer to this question in the case of regular continued fractions.

**Theorem.** Fix positive integers \( 1 \leq t \leq l < s, b_1 < ... < b_l \leq l \) and let \( A \) be an ordered alphabet of the form \( \{ b_1 < ... < b_l < l + 1 < ... < s \} \). Then for all \( s \) sufficiently large, there exists an infinite sequence of words \( w_k \) over \( A \) with multiplicities \( \mu(w_k) \to \infty \) as \( k \to \infty \).

It should be noted that for fixed \( s \) one obtains the largest possible alphabet \( A' = \{ 1 < 2 < \cdots < s \} \) by choosing \( b_1 = t = l = 1 \) \( (< s) \). Our proof makes use of the combinatorial structure of \( w_{\text{max}} \) found by the first author in [3].

**Preliminaries.** We introduce some notation. Let \( w = w_1 \cdots w_n \) be a word of length \( n \geq 2 \) with \( w_j \in \mathbb{N} \) \( (j = 1, \ldots, n) \). The regular continuant of \( w \) has a matrix representation

\[
K(w_1) = w_1 \quad \text{and} \quad K(w) = \det \begin{pmatrix} w_1 & -1 & 0 & \cdots & 0 \\ 1 & w_2 & -1 & \ddots & \vdots \\ \vdots & 1 & \ddots & \ddots & 0 \\ \vdots & \ddots & \ddots & \ddots & 0 \\ 0 & \cdots & 0 & 1 & w_n \end{pmatrix}, \quad n \geq 2
\]

It can also be defined recursively by \( K(\{ \}) = 1 \) \( (\{ \} \) empty word), \( K(w_1) = w_1 \) and \( K(w_1 \cdots w_j) = w_j K(w_1 \cdots w_{j-1}) + K(w_1 \cdots w_{j-2}) \) for \( j \geq 2 \). For each \( 1 \leq k \leq m \leq n \) we set \( w_{k,m} := w_k \cdots w_m \) and \( W := W_{1,n}, W_{k,m} := K(w_{k,m}) \). The following fundamental formula goes back to the late 19th century and can be found in Perron [2], p.11, (4): \( (W =) \ W_{1,n} = W_{1,j} W_{j+1,n} + W_{1,j-1} W_{j+2,n} \) \((j \in \{ 1, \ldots, n-1 \})\). From this we infer the simple but useful inequality

\[
W_{1,n} < 2W_{1,j} W_{j+1,n}.
\]

(1)

Let \( A = \{ a_1 < \cdots < a_s \} \subset \mathbb{N} \). We consider a word \( w = w_1 \cdots w_n := a_1^{p_1} \cdots a_s^{p_s} \) of length \( n \) with Parikh vector \( p = (p_1, \ldots, p_s) \) with \( p_1 + \cdots + p_s = n \) where

\[
a^r := \underbrace{a \cdots a}_r
\]

denotes a sequence of \( r \) equal elements \( a \). Let \( \mathcal{X} = \mathcal{X}(A,p) \) denote the set of all permutations of \( w \) where we identify each word \( v \) with its reverse \( \bar{v} \). Let \( N(A,p) \) denote the cardinality of \( \mathcal{X} \). Then, \( N(A,p) \geq \frac{n!}{a_1^{p_1} \cdots a_s^{p_s}} \). We put \( W_{\text{max}} = W_{\text{max}}(A,p) := \max \{ K(v) : v \in \mathcal{X} \} \). It was shown in [3] (see (3), p. 190) that \( W_{\text{max}} \) is uniquely attained (up to reversal) by the arrangement

\[
a_s L_{s-1} a_{s-2} L_{s-3} \cdots a_1^{p_1} \cdots a_{s-3} L_{s-2} a_{s-1} L_s
\]

(2)
where \( L_i = a_i^{n_i - 1} \). Let \( P = P(A, p) = \#\{K(v) : v \in X\} \).

**Proof of the Theorem.** Our first goal is to describe how to specify the last digit \( s (\geq 2) \) in an alphabet \( A : \{b_1 < \cdots < b_t < l + 1 < \cdots < s\} \). We consider ‘equipartitioned’ words

\[
w = w_1 \cdots w_n := b_1^m \cdots b_t^m (l + 1)^m \cdots s^m.
\]

Corresponding to the Parikh vector \( p = (m, m, \ldots, m) \) in which each digit of \( A \) occurs precisely \( m \)-times in \( w \). We will give a lower bound for \( s \) (see (7) below). To this end, we introduce the quantities \( Q_{r,m} - 1 := K(r^m) \) \((r \in 1, 2, \ldots)\). They are the elements of the \( r \)-th generalised Fibonacci sequence which is determined by the recursion \( Q_{r,0} := 1, Q_{r,1} := K(r) = r, Q_{r,j+1} := rQ_{r,j} + Q_{r,j-1} \) \((j = 1, 2, \ldots)\).

**Claim:** \( Q_{r,j-1} < (r + 1)^j \) for each fixed \( r \geq 1 \) and all \( j \geq 1 \).

To prove the claim, we proceed by induction on \( j \): This is obviously true for \( j = 1 \) and \( j = 2 \). Then by the induction hypothesis

\[
Q_{r,j-1} = rQ_{r,j-2} + Q_{r,j-3} < r(r + 1)^{j-1} + (r + 1)^{j-2}
\]

\[
= (r + 1)^{j-2}(r(r + 1) + 1) < (r + 1)^{j-2}(r + 1)^2
\]

\[
= (r + 1)^j.
\]

In order to obtain an upper bound for the number \( P(A, p) \), it suffices to consider words over the largest allowed \( s \)-digit alphabet \( A' : \{1 < \cdots < s\} \), \( b_1 = t = l = 1 (< s) \), with Parikh vector \( p' = (m, m, \ldots, m) \). Clearly

\[
P(A, p) \leq W_{\text{max}}(A, p) \leq W_{\text{max}}(A', p')
\]

and by (2)

\[
w_{\text{max}}(A', p') = s \cdot (s - 1)^{m-1} \cdot (s - 2) \cdots 1 \cdot 1^{m-1} \cdots (s - 2)^{m-1} \cdot (s - 1) \cdot s^{m-1}.
\]

By iteration of (1) applied to the decomposition in (3) we obtain the inequalities

\[
W_{\text{max}}(A', p') = K(w_{\text{max}}(A', p'))
\]

\[
< 2^{2s} s \cdot (s - 1) \cdots 3 \cdot 2 \prod_{j=1}^s K(j^{-1})
\]

\[
= 2^{2s} s! \prod_{j=1}^s Q_{j,m-1}
\]

\[
< 2^{2s} s! \prod_{j=1}^s (j + 1)^m
\]

\[
= 2^{2s} s!( (s + 1)! )^m
\]
and hence
\[ P(A, p) < 2^{2s} \cdot s! \cdot (s + 1)!^m. \] (4)

For each \( s \geq 2 \) we define \( m_0 = m_0(s) \) to be the smallest positive integer such that
\[ 2^{2s} \cdot s! \leq \left( \frac{100}{99} \right)^{m_0}. \]

Then
\[ P(A, p) < \left( \frac{100}{99} \right)^{(s + 1)!} \] for all \( m \geq m_0(s). \) (5)

On the other hand, we have the following lower bound for the number of different words in \( \mathcal{X}(w) \):
\[ N(A, p) \geq \frac{(s - l + t)\cdot m!}{2(m!)^{s-l+t}}. \] (6)

Based on the condition (7) below, we will later make a choice of \( s = s'(t, l) \) depending on the parameters \( t, l \). We apply the estimates provided by Sterling’s formula to the factorial terms occurring in relations (5) and (6) to obtain
\[
(P(A, p))^{1/m} < \frac{100}{99} \cdot (s + 1)! < \frac{100\cdot12}{99} \cdot e^{-(s+1)} \cdot (s + 1)^{s+1} \cdot \sqrt{2\pi(s+1)}.
\]
\[
(((s-l+t)\cdot m)!)^{1/m} > e^{-(s-l+t)} \cdot (s - l + t) \cdot m^{s-l+t} \cdot \sqrt{2\pi(s-l+t)\cdot m^{1/m}}.
\]
\[
(2\cdot m!)^{s-l+t} \cdot (s-l+t) \cdot m^{s-l+t} \cdot (2\cdot \frac{12}{11} \cdot (\sqrt{2\pi(m)})^{s-l+t})^{1/m}.
\]

When we put the right hand sides of the last two inequalities together, the terms \( e^{s-l+t} \) and \( m^{s-l+t} \) cancel out, and if we keep the parameters \( t, l \) fixed for the moment, the terms of the form \( \sqrt{\cdot}^{1/m} \) tend to 1 as \( m \to \infty \). Letting \( m \to \infty \) we get
\[
\lim_{m \to \infty} \left( \frac{N(A, p)}{P(A, p)} \right)^{1/m} \geq \frac{99 \cdot 12}{100} \cdot \frac{e^{s+1} (s - l + t) \cdot (s + 1)^{s+1}}{\sqrt{2\pi(s+1)(s + 1)^{s+1}}} = \frac{363 \cdot e^{s+1} (s + 1 - l + t - 1)^{(s+1-l+t-1)}}{400 \cdot \sqrt{2\pi(s + 1)(s + 1)^{s+1}}} = \frac{363}{400} \cdot \frac{e^{s+1}}{\sqrt{2\pi(s+1)(s - l + t)^{l-t+1}}} \left( 1 - \frac{l - t + 1}{s + 1} \right)^{(s+1)}.
\]

For fixed \( t, l \) \((l - t \geq 1)\) the function \( f(t, l, s) = \left( 1 - \frac{l + t + 1}{s + 1} \right)^{(s+1)} \) in the variable \( s \) is strictly increasing on the interval \([l - t + 1, \infty)\) with \( f(t, l, s) \nearrow e^{-(l-t)-1} \) as \( s \to \infty \). We define \( s_0 \) to be the lowest integer such that \( f(t, l, s_0) \geq \frac{1}{2} \cdot e^{-(l-t)-1} \). Then
\[
\lim_{m \to \infty} \left( \frac{N(A, p)}{P(A, p)} \right)^{1/m} \geq \frac{363}{400} \cdot \frac{e^{s+1}}{\sqrt{2\pi(s+1)(s - l + t)^{l-t+1}}} \cdot \frac{1}{2} \cdot e^{-(l-t)-1} =: H(t, l, s)
\]
for all \( s \geq s_0 \). Obviously there exists some sufficiently large \( s' = s'(t, l) \geq s_0 \) such that

\[
H(t, l, s') > 1.
\]

Therefore the right hand side of

\[
\left( \frac{N(A, p)}{P(A, p)} \right) > (H(t, l, s'))^m
\]

can be made arbitrarily large by letting \( m \rightarrow \infty \). We call an \((s' - l + t)\)-digit alphabet \( A = \{1 \leq b_1 < \cdots < b_t < \cdots < s'\} \) admissible if \( s' = s'(t, l) \) fulfills condition (7). We consider the word \( u(A, p_1) = (b_1)^{m_1} \cdots (b_t)^{m_1} (l + 1)^{m_1} \cdots (s')^{m_1} \) of length \( n = (s' - l + t)m_1 \) with Parikh vector \( p_1 = (m_1)_{s'}^{s'} \) where we choose \( m_1 \geq m_0 \) such that \( \left( \frac{N(A, p_1)}{P(A, p_1)} \right) > (H(t, l, s'))^{m_1} \). The multi-set \( \mathcal{X}_1 = \mathcal{X}(A, p_1) \) is made up of the \( N(A, p_1) = \#\mathcal{X}_1 \) permuted arrangements of \( u \). There exists at least one word \( w_1 \in \mathcal{X}_1 \) with multiplicity \( \mu \geq 2 \) because otherwise we would have \( N(A, p_1) = P(A, p_1) \) which contradicts (8) with \( m = m_1 \). Let \( \mu_1 (\geq 2) \) be the maximal multiplicity attained by words \( w \in \mathcal{X}_1 \). Next choose \( m_2 > m_1(s') \) such that \( H(t, l, s')^{m_2} > \mu_1 \). We claim that at least one word \( w_2 \) from \( \mathcal{X}_2 = \mathcal{X}(A, p_2) \), \( p_2 = ((m_2)^{s' - l + t}) \) has multiplicity \( \mu > \mu_1 \). Otherwise we would have \( N(A, p_2) \leq \mu_1 P(A, p_2) \) which contradicts (8) with \( m = m_2 \). Next let \( \mu_2 (\geq \mu_1) \) be the maximal multiplicity attained by words \( w \in \mathcal{X}_2 \). Proceeding with this construction step by step we end up with a sequence of words \( w_k \) on \( A \) with multiplicities \( \mu_k \rightarrow \infty \) as \( k \rightarrow \infty \). The construction can be carried out for infinitely many different admissible alphabets. This completes the proof of the Theorem.

The question remains largely unsolved in the case of semi-regular continuants though it seems certain that the behavior is quite similar to the regular case.

There is some evidence supporting the following

**Conjecture.** Given any ordered alphabet \( A = \{a_1 < \cdots < a_s\} \) \( (a_j \in \mathbb{N}, \ s \geq 2) \), let \( \mu \geq 2 \) be a positive integer. Then there exist infinitely many words on \( A \) whose multiplicity is precisely \( \mu \). The problem appears to require a difficult investigation into the values of continuants. Most likely our theorem and the conjecture also hold for continuants of semi-regular continued fractions. Unfortunately no higher-dimensional analogue of the theorem is available at present for \( s \geq 4 \) due to the fact that very little is known about the maximizing arrangements \( w_{\max}^t \) for \( s \geq 4 \) (see [8]).
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