BOUNDEDNESS AND GLOBAL SOLVABILITY TO A CHEMOTAXIS-HAPTOTAXIS MODEL WITH SLOW AND FAST DIFFUSION

CHUNHUA JIN*

School of Mathematical Sciences, South China Normal University
Guangzhou, 510631, China

(Communicated by Michael Winkler)

Abstract. In this paper, we deal with the following coupled chemotaxis-haptotaxis system modeling cancer invasion with nonlinear diffusion,

\[
\begin{align*}
    u_t &= \Delta u^m - \chi \nabla \cdot (u \cdot \nabla v) - \xi \nabla \cdot (u \cdot \nabla \omega) + \mu u (1 - u - \omega), \quad \text{in } \Omega \times \mathbb{R}^+, \\
    v_t - \Delta v + v &= u, \quad \text{in } \Omega \times \mathbb{R}^+, \\
    \omega_t &= -v \omega, \quad \text{in } \Omega \times \mathbb{R}^+, \\
    \frac{\partial (\nabla u^m - \chi u \cdot \nabla v - \xi u \cdot \nabla \omega) \cdot \mathbf{n}}{\partial \Omega} &= \frac{\partial v}{\partial \mathbf{n}}|_{\partial \Omega} = 0,
\end{align*}
\]

where \( \Omega \subset \mathbb{R}^N \) is a bounded domain. Under zero-flux boundary conditions, we showed that for any \( m > 0 \), the problem admits a global bounded weak solution for any large initial datum if \( \frac{\mu}{\lambda} \) is appropriately small. The slow diffusion case \( (m > 1) \) of this problem have been studied by many authors \([14, 7, 19, 23] \), in which, the boundedness and the global in time solution are established for \( m > \frac{2N}{N+2} \), but the cases \( m \leq \frac{2N}{N+2} \) remain open.

1. Introduction. In this paper, we consider the following coupled chemotaxis-haptotaxis model

\[
\begin{align*}
    u_t &= \Delta u^m - \chi \nabla \cdot (u \cdot \nabla v) - \xi \nabla \cdot (u \cdot \nabla \omega) + \mu u (1 - u - \omega), \quad \text{in } Q, \\
    v_t - \Delta v + v &= u, \quad \text{in } Q, \\
    \omega_t &= -v \omega \\
    u(x, 0) &= u_0(x), \quad v(x, 0) = v_0(x), \quad \omega(x, 0) = \omega_0(x), \quad x \in \Omega,
\end{align*}
\]

where \( m > 0, Q = \Omega \times \mathbb{R}^+, \Omega \subset \mathbb{R}^N \) is a bounded domain with smooth boundary. \( u, v, \omega \) represent the cancer cell density, urokinase Plasminogen Activator (uPA) protease concentration, and the extracellular matrix (ECM) density respectively, \( \chi, \xi \geq 0 \) are the chemotactic and haptotactic coefficients respectively, \( \mu u (1 - u - \omega) \) with \( \mu > 0 \) is the proliferation or death of cancer cell according to a generalized logistic law including competition for space with the ECM, \( -v + u \) is the

2010 Mathematics Subject Classification. Primary: 92C17, 35B65; Secondary: 35M10.

Key words and phrases. Chemotaxis-haptotaxis, slow diffusion, fast diffusion, global weak solution, boundedness.

The author is supported by NSFC(11471127), Guangdong Natural Science Funds for Distinguished Young Scholar (2015A030306029).

* Corresponding author: Chunhua Jin.
kinetics term, \(-v\) represents decay of protease, while, \(+u\) represents the spontaneous production of protease, \(-v\omega\) represents the degradation of ECM.

Since Keller and Segel [6] introduced the classical chemotaxis model in 1970, the Keller-Segel model and some modified Keller-Segel models have been widely studied by some researchers. It is well known that the formation of a cell aggregate will result in the finite-time blow-up. For example, for the Keller-Segel model of this form

\[
\begin{cases}
    u_t = \Delta u - \nabla \cdot (u \nabla v), \\
    \tau v_t - \Delta v + v = u,
\end{cases}
\]

we refer to [3, 4, 11, 18] for the study of global solutions and blow up solutions. However, in addition to migration, the cell density should also be affected by the proliferation or death of cells, so a logistic damping term \(\mu u (1 - u)\) characterizing the death and proliferation of cells is introduced into this model. It has been detected that the logistic damping of cancer cell densities will weaken the aggregation behavior, and prevent blow up. In fact, in the two dimensional space, it has been shown that the solutions will always exist globally [17]; and in the three dimensional case, the solutions will exist globally for large \(\mu\) [17, 22].

This chemotaxis-haptotaxis model was first introduced by Chaplain and Lolas [2], it described the process of cancer cell invade the surrounding normal tissue. In this model, the random diffusion of cancer cells is characterized by linear isotropic diffusion, that is the model (1) with \(m = 1\). For this case, the global classical solution is obtained in the two dimensional space by [13], and for the three dimensional case, the global classical solution is obtained only for large \(\mu\) [1], and remains open for small \(\mu\). However, from a physical point of view, the equation modelling the migration of cancer cells should rather be regarded as nonlinear diffusion [10, 14], so some researchers are led to consider the nonlinear diffusive model, in which, the cell mobility is described by a nonlinear function of the cancer cell density, and the most common form is the porous medium diffusion. In 2011, Tao and Winkler [14] considered the nonlinear diffusion case without degenerate, that is the model (1) with \(\Delta u^m\) replaced by \(\Delta ((u + \varepsilon)^{m-1} u)\), and the global in time classical solutions are established for the following cases

\[m > m^* = \begin{cases} 
    \frac{2N^2 + 4N - 4}{N(N + 4)}, & \text{if } N \leq 8, \\
    \frac{2N^2 + 3N + 2 - \sqrt{8N(N + 1)}}{N(N + 1)}, & \text{if } N \geq 9,
\end{cases}\]

where \(N\) is the space dimension. However, they leave a question here: “whether the global solutions are bounded”. Recently, Li, Lankeit [7] and Wang [19], showed the global solvability of classical solution with \(\varepsilon > 0\) (or weak solution with \(\varepsilon = 0\)) for any \(m > \frac{2N-2}{N}\), in particular, the boundedness of the solutions also are established. Recently, Zheng [23] extended these results to the cases \(m > \frac{2N}{N+2}\). But the cases \(1 < m \leq \frac{2N}{N+2}\) remain unknown. On the other hands, for the fast diffusion cases, that is \(0 < m < 1\), as far as we know, there are no relevant researches.

In the present paper, we consider this problem in \(N\) dimensional space with \(N \geq 3\), and we extend the above results to any \(m > 0\). Firstly, we consider the regularized problem (10), and showed the boundedness and global solvability of classical solutions for any \(m > 0\) under the assumption that \(\frac{\mu}{\chi}\) is appropriately
small. Then, by letting $\varepsilon \to 0$, we further obtain the boundedness and global solvability of weak solutions.

In what follows, we give the assumptions of this paper.

$$
(H) \quad \begin{cases}
  u_0, \nabla \sqrt{\omega_0} \in L^\infty(\Omega), \nabla u_0^m \in L^2(\Omega), v_0, \omega_0 \in W^{2,\infty}(\Omega), \\
  u_0, v_0, \omega_0 \geq 0, \quad \frac{\partial u_0^m}{\partial n} \bigg|_{\partial \Omega} = \frac{\partial v_0}{\partial n} \bigg|_{\partial \Omega} = \frac{\partial \omega_0}{\partial n} \bigg|_{\partial \Omega} = 0,
\end{cases}
$$

It is easy to verify that $\frac{\partial \omega}{\partial n} \bigg|_{\partial \Omega} = 0$ due to $\frac{\partial u_0^m}{\partial n} \bigg|_{\partial \Omega} = 0$ and $\frac{\partial n}{\partial \Omega} = 0$. Therefore, the zero-flux boundary conditions are equivalent to

$$
\frac{\partial u^m}{\partial n} \bigg|_{\partial \Omega} = \frac{\partial \omega}{\partial n} \bigg|_{\partial \Omega} = \frac{\partial v}{\partial n} \bigg|_{\partial \Omega} = 0. \quad (2)
$$

**Theorem 1.1.** Assume (H) hold, and $m > 0$. Then when $L_\mu$ is appropriately small, the problem (1) admits a nonnegative weak solution $(u, v, \omega)$ with $u \in X_1, v \in X_2, \omega \in X_3$, where

$$
X_1 = \{ u \in L^\infty(\Omega \times R^+) ; \nabla u^m \in L^\infty_{loc}([0, \infty); L^2(\Omega)), \left( \frac{u^{m+1}}{m} \right)_t , \nabla u^m \in L^2_{loc}([0, \infty); L^2(\Omega)) \},
$$

$$
X_2 = \{ v \in L^\infty((0, \infty); W^{1,\infty}(\Omega)); v_1, \Delta v \in L^p_{loc}((0, \infty); L^p(\Omega)) \text{ for any } p > 1 \},
$$

$$
X_3 = \{ \omega \in L^\infty(\Omega \times R^+) ; w_l \in L^\infty(\Omega \times R^+), \omega \in L^\infty_{loc}([0, \infty); W^{2,\infty}(\Omega)) \},
$$

such that

$$
\sup_{t \in (0, \infty)} \|u(\cdot, t)\|_{L^\infty} + \|v(\cdot, t)\|_{W^{1,\infty}} + \|\omega(\cdot, t)\|_{L^\infty} + \|\omega_l(\cdot, t)\|_{L^\infty} \leq M_1, \quad (3)
$$

$$
\sup_{t \in (0, T)} \int_\Omega |\nabla u^m|^2 dx + \| \frac{u^{m+1}}{m} \|_{W^{2,1}(\Omega)} + \| \nabla \omega \|_{L^\infty(\Omega)} + \| \Delta \omega \|_{L^\infty(\Omega)} \leq M_2(T), \quad (4)
$$

$$
\sup_{t \in R^+} \| u^m \|^2_{W^{2,1}_p(Q_1(t))} + \sup_{t \in R^+} \| v \|^2_{W^{2,1}_p(Q_1(t))} \leq M_3, \quad \text{for any } p > 1. \quad (5)
$$

Here $M_i$ ($i = 1, 2, 3$) depend on $\mu, \chi, \xi, v_0, \omega_0, M_2(T)$ also depends on $T$.

**Remark 1.** For simplicity, in what follows, we always assume that $m < 2$, since that the case $m > 2N/4N$ have been completely solved.

2. Preliminaries. We first give some notations, which will be used throughout this paper.

**Notations:** $\| \cdot \|_{L^p} = \| \cdot \|_{L^p(\Omega)}, Q_\tau(t) = \Omega \times (t, t + \tau), Q_T := Q_\tau(0) = \Omega \times (0, T)$.

Next, we give the definition of weak solutions.

**Definition 2.1.** $(u, v, \omega)$ is called a weak solution of (1), if $u, v, \omega \geq 0, (u, v, \omega) \in X_1 \times X_2 \times X_3$, such that for any given $T > 0$

$$
- \int_{Q_T} u \varphi_t dx dt - \int_{\Omega} u(x, 0) \varphi(x, 0) dx + \int_{Q_T} \nabla u^m \nabla \varphi dx dt - \chi \int_{Q_T} u \nabla v \nabla \varphi dx dt - \xi \int_{Q_T} u \nabla \omega \varphi dx dt = \mu \int_{Q_T} u(1 - u - \omega) \varphi dx dt,
$$

$$
- \int_{Q_T} v \varphi_t dx dt - \int_{\Omega} v(x, 0) \varphi(x, 0) dx + \int_{Q_T} (\nabla v \nabla \varphi + (v - u) \varphi) dx dt = 0,
$$
\[-\iint_{Q_T} \omega \varphi_t \, dx \, dt - \int_{\Omega} \omega(x, 0) \varphi(x, 0) \, dx + \iint_{Q_T} v \omega \varphi \, dx \, dt = 0\]

for any \( \varphi \in C^\infty(\overline{Q}_T) \) with \( \frac{\partial \varphi}{\partial n} \bigg|_{\partial \Omega} = 0 \) and \( \varphi(x, T) = 0 \).

Before going further, we list some lemmas, which will be used throughout this paper. Firstly, we give the Gagliardo-Nirenberg interpolation inequality \([8]\) as follows

**Lemma 2.2.** For functions \( u : \Omega \to \mathbb{R} \) defined on a bounded Lipschitz domain \( \Omega \in \mathbb{R}^N \), we have

\[
\| D^j u \|_{L^p} \leq C \| D^m u \|_{L^r}^{\alpha} \| u \|_{L^q}^{1-\alpha} + C \| u \|_{L^s},
\]

where \( 1 \leq q, r \leq \infty, \frac{1}{m} \leq \alpha \leq 1, \)

\[
\frac{1}{p} = \frac{j}{N} + \left( \frac{1}{r} - \frac{m}{N} \right) \alpha + \frac{1-\alpha}{q},
\]

and \( s > 0 \) is arbitrary.

We list the following lemma \([12]\).

**Lemma 2.3.** Let \( T > 0, \tau \in (0, T), a > 0, b > 0, \) and suppose that \( y : [0, T) \to [0, \infty) \) is absolutely continuous such that

\[
y'(t) + ay(t) \leq h(t), \quad \text{for } t \in [0, T),
\]

where \( h \geq 0, h(t) \in L^1_{\text{loc}}([0, T)) \) and

\[
\int_{t-\tau}^{t} h(s) \, ds \leq b, \quad \text{for all } t \in [\tau, T).
\]

Then

\[
y(t) \leq \max\{y(0) + b, \frac{b}{a\tau} + 2b\} \text{ for all } t \in [0, T).\]

We also give a generalized lemma of Lemma 2.3.

**Lemma 2.4.** Let \( T > 0, \tau \in (0, T), \sigma \geq 0, a > 0, b \geq 0, \) and suppose that \( f : [0, T) \to [0, \infty) \) is absolutely continuous, and satisfies

\[
f'(t) + af^{1+\sigma}(t) \leq h(t), t \in \mathbb{R},\quad (6)
\]

where \( h \geq 0, h(t) \in L^1_{\text{loc}}([0, T)) \) and

\[
\int_{t-\tau}^{t} h(s) \, ds \leq b, \quad \text{for all } t \in [\tau, T).
\]

Then

\[
\sup_{t \in (0, T)} f(t) + a \sup_{t \in (\tau, T)} \int_{t-\tau}^{t} f^{1+\sigma}(s) \, ds \leq b + 2 \max\{f(0) + b + a\tau, \frac{b}{a\tau} + 1 + 2b + 2a\tau\}.\quad (7)
\]

**Proof.** Noticing that \( f \leq f^{1+\sigma} + 1 \), then

\[
f'(t) + af \leq a + h(t), t \in \mathbb{R}.
\]

By Lemma 2.3, we obtain

\[
f(t) \leq \max\{f(0) + b + a\tau, \frac{b}{a\tau} + 1 + 2b + 2a\tau\} \text{ for all } t \in [0, T).\]
By (6), we further have
\[
a \int_{t-\tau}^{t} f^{1+\sigma}(s)ds \leq \sup_{t} f(t) + b.
\]
(7) is obtained. □

By [5], we give the following Lemma.

**Lemma 2.5.** Assume that \( u_0 \in W^{2,p}(\Omega) \), and \( f \in L^p_{loc}((0, +\infty); L^q(\Omega)) \) with

\[
\sup_{t \in (\tau, +\infty)} \int_{t-\tau}^{t} \|f\|_{L^p}^p ds \leq A,
\]

where \( \tau > 0 \) is a fixed constant. Then the following problem

\[
\begin{align*}
\begin{cases}
u_t - \Delta \nu + \nu = f(x, t), \\
\frac{\partial \nu}{\partial \mathbf{n}} |_{\partial \Omega} = 0, \\
u(x, 0) = u_0(x)
\end{cases}
\end{align*}
\]

admits a unique solution \( \nu \) with \( \nu \in L^p_{loc}((0, +\infty); W^{2,p}(\Omega)) \) and \( \nu_t \in L^p_{loc}((0, +\infty); L^p(\Omega)) \) with

\[
\sup_{t \in (\tau, +\infty)} \int_{t-\tau}^{t} (\|\nu\|_{W^{2,p}}^p + \|\nu_t\|_{L^p}^p) ds \leq A M e^{\frac{p}{2\tau} - 1} + M e^{\frac{p}{2\tau}} \|u_0\|_{W^{2,p}},
\]

where \( M \) is a constant independent of \( \tau \).

3. **Boundedness and global existence of weak solution.** We first consider the approximate problems given by

\[
\begin{align*}
\begin{cases}
u_{et} = \Delta (\varepsilon + u_e) + \chi \nabla \cdot (u_e \cdot \nabla v_e) - \xi \nabla \cdot (u_e \cdot \nabla \omega_e)
+ \mu u_e (1 - u_e - \omega_e), & \text{in } Q, \\
v_{et} = \Delta v_e - v_e + u_e, & \text{in } Q, \\
\omega_{et} = -v_e \omega_e, \\
\left\langle (\nabla (\varepsilon + u_e))^{m-1} u_e - \chi u_e \cdot \nabla v_e - \xi u_e \cdot \nabla \omega_e \right\rangle \cdot \mathbf{n} \bigg|_{\partial \Omega} = 0,
\end{cases}
\end{align*}
\]

where

\[
\begin{align*}
\begin{cases}
u_{e0}, \omega_{e0} \in C^{2+\alpha} (\bar{\Omega}), \\
\frac{\partial \omega_{e0}}{\partial \mathbf{n}} |_{\partial \Omega} = 0, \\
u_{e0} \rightarrow u_0, v_{e0} \rightarrow v_0, \omega_{e0} \rightarrow \omega_0 \quad \text{uniformly,}
\end{cases}
\end{align*}
\]

so that

\[
\begin{align*}
\|u_{e0}\|_{L^\infty} + \|\nabla u_{e0}\|_{L^2} + \|v_{e0}\|_{W^{2,\infty}} + \|\omega_{e0}\|_{W^{2,\infty}} + \|\nabla \omega_{e0}\|_{L^\infty}
\leq 2 \|u_0\|_{L^\infty} + \|\nabla u_0\|_{L^2} + \|v_0\|_{W^{2,\infty}} + \|\omega_0\|_{W^{2,\infty}} + \|\nabla \omega_0\|_{L^\infty}.
\end{align*}
\]

It is easy to see that

\[
\omega_e = \omega_{e0} e^{-\int_0^t \psi_s (x, s) ds},
\]

which implies that \( \frac{\partial \omega_{e0}}{\partial \mathbf{n}} |_{\partial \Omega} = 0 \) since \( \frac{\partial \omega_{e0}}{\partial \mathbf{n}} |_{\partial \Omega} = \frac{\partial \omega_{e0}}{\partial \mathbf{n}} |_{\partial \Omega} = 0 \). Note that \( \nabla ((\varepsilon + u_e)^{m-1} u_e) \cdot \mathbf{n} = (\varepsilon + u_e)^{m-2} (\varepsilon + m u_e) \frac{\partial u_e}{\partial \mathbf{n}} \), then the boundary conditions of (10)
are equivalent to
\[ \frac{\partial u_\varepsilon}{\partial n} \bigg|_{\partial \Omega} = \frac{\partial v_\varepsilon}{\partial n} \bigg|_{\partial \Omega} = \frac{\partial \omega_\varepsilon}{\partial n} \bigg|_{\partial \Omega} = 0 \]
since \( \frac{\partial \omega_\varepsilon}{\partial n} |_{\partial \Omega} = 0 \) and \( \frac{\partial v_\varepsilon}{\partial n} |_{\partial \Omega} = 0 \). Furthermore, by a direct calculation, it is not difficult to get
\[ -\Delta \omega_\varepsilon = -\Delta \omega_0 e^{-\int_0^t \nu_\varepsilon(x,s)ds} + 2e^{-\int_0^t \nu_\varepsilon(x,s)ds} \nabla \omega_0 \cdot \int_0^t \nabla v_\varepsilon(x,s)ds \\
- \omega_0 e^{-\int_0^t \nu_\varepsilon(x,s)ds} \int_0^t \nabla v_\varepsilon(x,s)ds \int_0^t \Delta v_\varepsilon(x,s)ds. \tag{11} \]

By [16], we have the following estimate
\[ -\Delta \omega_\varepsilon(x,t) \leq \|\omega_0\|_{L^\infty} v_\varepsilon(x,t) + \|\Delta \omega_0\|_{L^\infty} + 4\|\nabla \sqrt{\omega_0}\|_{L^\infty}^2 + \frac{1}{\varepsilon}\|\omega_0\|_{L^\infty}. \tag{12} \]

Before going further, we first state the local existence result of classical solution to (10) as follows. In fact, by a fixed point argument similar to [9, 21, 15], it can be proved.

**Lemma 3.1.** Assume (B) hold. Then there exists \( T_{\max} \in (0, +\infty) \) such that the problem (1) admits a unique classical solution \( (u, v, \omega) \in C^{2+\alpha,1+\alpha/2}(\Omega \times (0, T_{\max})) \) with
\[ u_\varepsilon \geq 0, \quad v_\varepsilon \geq 0, \quad \omega_\varepsilon \geq 0 \quad \text{for all } (x,t) \in \Omega \times (0, T_{\max}), \]
such that either \( T_{\max} = \infty \), or
\[ \limsup_{t \to T_{\max}} (\|u_\varepsilon(\cdot, t)\|_{L^\infty} + \|v_\varepsilon\|_{W^{1,\infty}}) = \infty. \]

In what follows, we show the global existence of classical solution. We give the following Proposition.

**Proposition 1.** Assume (B) hold and \( \frac{\lambda}{\mu} \) is appropriately small. Then for any \( \varepsilon > 0 \), the problem (10) admits a unique classical global solution \( (u_\varepsilon, v_\varepsilon, \omega_\varepsilon) \), and
\[ \sup_{t \in (0, \infty)} (\|u_\varepsilon(\cdot, t)\|_{L^\infty} + \|v_\varepsilon(\cdot, t)\|_{W^{1,\infty}} + \|\omega_\varepsilon(\cdot, t)\|_{L^\infty} + \|\omega_\varepsilon(\cdot, t)\|_{L^\infty}) \leq M_1, \tag{13} \]
\[ \sup_{t \in (0, T)} \int_{\Omega} (\varepsilon + u_\varepsilon)^{(m-1)} |\nabla u_\varepsilon|^2 dx + (\varepsilon + u_\varepsilon)^{\frac{m+1}{2}} \|W^{1,1}_{\varepsilon}(Q_T)\|^2 \leq M_2(T), \tag{14} \]
\[ \sup_{t \in \mathbb{R}^+} \|u_\varepsilon\|^2_{W^{1,\alpha}_{\varepsilon}(Q_1(t))} + \sup_{t \in \mathbb{R}^+} \|v_\varepsilon\|^2_{W^{p,1}_{\varepsilon}(Q_1(t))} \leq M_3, \quad \text{for any } p > 1, \tag{15} \]
where \( Q_1(t) = \Omega \times (t, t+1), \quad Q_T = \Omega \times (0, T), \quad M_i(i = 1, 2, 3) \) are independent of \( \varepsilon, \quad M_2(T) \) depends on \( T \).

Next, we give some estimates of \( (u_\varepsilon, v_\varepsilon, \omega_\varepsilon) \), we take \( \tau = \min \{ 1, \frac{T_{\max}}{2} \} \). In what follows, these constants \( C, \tilde{C}, M, C_1, M_i \) denote some different constants, which are independent of \( \varepsilon, \tau \) and \( T_{\max} \). If no special explanation, they depend at most on \( m, N, \mu, \xi, \chi, u_0, v_0, \omega_0 \) and \( \Omega \). In fact, it is easy to see that \( \tau = 1 \) if \( T_{\max} \geq 2 \). It is worth mentioning that if \( \tau < 1 \), meaning \( T_{\max} < 2 \), and all these estimates of this form \( \int_{t-\tau}^t \cdots ds \) in this paper can be replaced by \( \int_0^{T_{\max}} \cdots ds \). So all these constants are independent of \( \tau \).
Lemma 3.2. Assume (B) hold. Let \((u_\varepsilon, v_\varepsilon, \omega_\varepsilon)\) be the classical solution of (10) in \((0, T_{max})\). Then

\[
\sup_{t \in (0, T_{max})} \|\omega_\varepsilon(\cdot, t)\|_{L^\infty} \leq \|\omega_0\|_{L^\infty} \leq 2\|\omega_0\|_{L^\infty},
\]

(16)

\[
\sup_{t \in (0, T_{max})} \|u_\varepsilon(\cdot, t)\|_{L^1} + \sup_{t \in (\tau, T_{max})} \int_{t-\tau}^{t} \int_{\Omega} u_\varepsilon^2 dx \leq C_1,
\]

(17)

\[
\sup_{t \in (\tau, T_{max})} \int_{t-\tau}^{t} \|v_\varepsilon\|_{W^{2,2}}^2 ds \leq C_2,
\]

(18)

where \(C_1, C_2\) are independent of \(T_{max}, \tau\) and \(\varepsilon\).

Proof. By a direct calculation, it is easy to see that (16) holds. Integrating the first equation of (10) directly, we obtain

\[
\frac{d}{dt} \int_{\Omega} u_\varepsilon dx + \mu \int_{\Omega} (u_\varepsilon^2 + u_\varepsilon \omega_\varepsilon) dx = \mu \int_{\Omega} u_\varepsilon dx \leq \frac{\mu}{2} \int_{\Omega} u_\varepsilon^2 dx + C.
\]

Then by lemma 2.4 we obtain (17). Recalling the second equation of (10), using (17) and combining with Lemma 2.5, we obtain (18). The proof is complete.

Lemma 3.3. Assume (B) hold. Let \((u_\varepsilon, v_\varepsilon, \omega_\varepsilon)\) be the classical solution of (10) in \((0, T_{max})\). Then

\[
\sup_{t \in (\tau, T_{max})} \int_{t-\tau}^{t} \int_{\Omega} \left[(u_\varepsilon + \varepsilon)^{m-2} |\nabla u_\varepsilon|^2 + u_\varepsilon^2 \ln(1 + u_\varepsilon)\right] dx ds
\]

\[
+ \sup_{t \in (0, T_{max})} \int_{\Omega} u_\varepsilon \ln u_\varepsilon dx \leq C,
\]

(19)

where \(C\) depends on \(\mu\).

Proof. Multiplying the first equation of (10) by \(1 + \ln u_\varepsilon\), then integrating it over \(\Omega\), and using (12), we obtain

\[
\frac{d}{dt} \int_{\Omega} u_\varepsilon \ln u_\varepsilon dx + m \int_{\Omega} (u_\varepsilon + \varepsilon)^{m-2} |\nabla u_\varepsilon|^2 dx + \frac{\mu}{2} \int_{\Omega} u_\varepsilon^2 \ln(1 + u_\varepsilon) dx
\]

\[
\leq -\chi \int_{\Omega} u_\varepsilon \Delta v_\varepsilon dx - \xi \int_{\Omega} u_\varepsilon \Delta \omega_\varepsilon dx + C
\]

\[
\leq \chi \int_{\Omega} (u_\varepsilon^2 + |\Delta v_\varepsilon|^2) dx + \xi \int_{\Omega} u_\varepsilon (C_1 v_\varepsilon + C_2) dx + C
\]

\[
\leq C_3 \int_{\Omega} (u_\varepsilon^2 + |\Delta v_\varepsilon|^2 + v_\varepsilon^2) dx + C.
\]

Recalling (17), (18) and using Lemma 2.3, we obtain (19).

Lemma 3.4. Assume (B) hold. Let \((u_\varepsilon, v_\varepsilon, \omega_\varepsilon)\) be the classical solution of (10) in \((0, T_{max})\). Then for any \(r > 0\),

\[
\sup_{t \in (0, T_{max})} \int_{\Omega} u_\varepsilon^{r+1} dx + \frac{\mu}{4} \sup_{t \in (\tau, T_{max})} \int_{t-\tau}^{t} \int_{\Omega} u_\varepsilon^{r+1} dx ds
\]

\[
+ \frac{\mu}{4} \int_{t-\tau}^{t} \int_{\Omega} u_\varepsilon^{r+2} dx ds
\]

\[
\leq M_1 \varepsilon^{r+2} \mu^{-\frac{r+1}{2}} \sup_{t \in (\tau, T_{max})} \int_{t-\tau}^{t} \int_{\Omega} |u_\varepsilon|^{r+2} dx ds + M_2.
\]

(20)
where $M_1$ is independent of $\mu$, $M_2$ depends on $\mu$, and both of them depend on $r$.

**Proof.** Multiplying the second equation of (10) by $v_\varepsilon$, $\Delta v_\varepsilon$ respectively, and integrating them over $\Omega$, we obtain

$$
\frac{1}{2} \frac{d}{dt} \int_\Omega (v_\varepsilon^2 + |\nabla v_\varepsilon|^2) dx + \frac{1}{2} \int_\Omega (v_\varepsilon^2 + |\Delta v_\varepsilon|^2) dx + 2 \int_\Omega |\nabla v_\varepsilon|^2 dx \leq \frac{1}{2} \int_\Omega u_\varepsilon^2 dx.
$$

By (17) and using Lemma 2.3, we obtain

$$
\sup_{t \in (0, T_{max})} \|v_\varepsilon(\cdot,t)\|_{H^1}^2 + \sup_{t \in (\tau, T_{max})} \int_{t-\tau}^t \|v_\varepsilon(\cdot,t)\|_{H^2}^2 \leq C.
$$

(21)

By (21), when $r \leq \frac{4}{N-2}$ ($r + 2 \leq \frac{2N}{N-2}$), we see that

$$
\sup_{t \in (0, T_{max})} \|v_\varepsilon(\cdot,t)\|_{L^{r+2}} \leq C \sup_{t \in (0, T_{max})} \|v_\varepsilon(\cdot,t)\|_{H^1} \leq C.
$$

When $r > \frac{4}{N-2}$, we obtain

$$
\|v_\varepsilon\|_{L^{r+2}} \leq C \|v_\varepsilon\|_{L^{\frac{4(r+2)}{N-2}}} \|\Delta v_\varepsilon\|_{L^{\frac{(N-2)(r+2)}{N-2}}} + C \|v_\varepsilon\|_{L^{\frac{2N}{N-2}}} \leq \tilde{C} (1 + \|\Delta v_\varepsilon\|_{L^{\frac{(N-2)(r+2)}{N-2}}}^{\frac{(N-2)(r+2)}{2}}).
$$

Summing up, we have

$$
\|v_\varepsilon(\cdot,t)\|_{L^{r+2}} \leq C (1 + \|\Delta v_\varepsilon\|_{L^{\frac{(N-2)(r+2)}{2}}}).
$$

(22)

Multiplying the first equation of (10) by $u_\varepsilon$ for any $r > 0$, integrating it over $\Omega$, and combining with (12), (22), we obtain

$$
\frac{1}{r + 1} \frac{d}{dt} \int_\Omega u_\varepsilon^{r+1} dx + \tilde{m} r \int_\Omega u_\varepsilon^{r-1} (u_\varepsilon + \varepsilon)^{m-1} |\nabla u_\varepsilon|^2 dx + \mu \int_\Omega u_\varepsilon^{r+2} dx
\leq \int_\Omega \chi u_\varepsilon^r \nabla u_\varepsilon \cdot \nabla v_\varepsilon dx - \frac{\varepsilon r}{r + 1} \int_\Omega u_\varepsilon^{r+1} \Delta \omega_\varepsilon dx + \mu \int_\Omega u_\varepsilon^{r+1} dx
\leq - \frac{\varepsilon r}{r + 1} \int_\Omega u_\varepsilon^{r+1} \Delta v_\varepsilon dx + 2 \xi \|\omega_\varepsilon\|_{L^\infty} \int_\Omega u_\varepsilon^{r+1} v_\varepsilon(x,t) dx + \frac{\mu}{4} \int_\Omega u_\varepsilon^{r+2} dx + C
\leq C \chi^{r+2} \mu^{-r+1} \int_\Omega |\Delta v_\varepsilon|^{r+2} dx + C \mu^{-(r+1)} \int_\Omega v_\varepsilon^{r+2} dx + \frac{\mu}{2} \int_\Omega u_\varepsilon^{r+2} dx + C
\leq C \chi^{r+2} \mu^{-(r+1)} \int_\Omega |\Delta v_\varepsilon|^{r+2} dx + C \mu^{-(r+1)} \int_\Omega |\Delta v_\varepsilon|^{\frac{(N-2)(r+2)}{N-2}} dx
+ \frac{\mu}{2} \int_\Omega u_\varepsilon^{r+2} dx + C,
$$

where $\tilde{m} = \min \{1, m\}$. Using Lemma 2.4, Lemma 2.5, we obtain

$$
\sup_{t \in (0, T_{max})} \int_\Omega u_\varepsilon^{r+1} dx + \tilde{m} r \int_\Omega u_\varepsilon^{r-1} (u_\varepsilon + \varepsilon)^{m-1} |\nabla u_\varepsilon|^2 dx ds
+ \frac{\mu}{2} \sup_{t \in (\tau, T_{max})} \int_{t-\tau}^t \int_\Omega u_\varepsilon^{r+2} dx ds
\leq C \chi^{r+2} \mu^{-(r+1)} \sup_{t \in (\tau, T_{max})} \int_{t-\tau}^t \int_\Omega |\Delta v_\varepsilon|^{r+2} dx ds
+ C \mu^{-(r+1)} \sup_{t \in (\tau, T_{max})} \int_{t-\tau}^t \int_\Omega |\Delta v_\varepsilon|^{\frac{(N-2)(r+2)}{N-2}} dx ds + C.
$$
which implies (20), and the proof is complete. □

Assume Lemma 3.5. Note that

\[ C \frac{(N-2)(r+2)}{n+2} dxds + C_3. \]

which implies (20), and the proof is complete.

Using this lemma, we can further prove that

**Lemma 3.5.** Assume (B). Then if \( \frac{\lambda}{\mu} \) is sufficiently small, we have

\[
\sup_{t \in (0, T_{\text{max}})} \left( \| \|v\|_{W^{1,1}} + \|\omega_{\varepsilon}\|_{L^\infty} \right) \leq C, \tag{23}
\]

\[
\sup_{t \in (0, T_{\text{max}})} \|u_{\varepsilon}\|_{L^\infty} \leq C, \tag{24}
\]

where \( C \) is independent of \( T_{\text{max}} \) and \( \varepsilon \).

**Proof.** By Lemma 3.4, it is easy to see that if \( \frac{\lambda}{\mu} \) is sufficiently small, then

\[
\sup_{t \in (0, T_{\text{max}})} \int_\Omega u_{\varepsilon}^{r+1} dx + \sup_{t \in (\tau, T_{\text{max}})} \int_{t-\tau}^t \int_\Omega u_{\varepsilon}^{r-1}(u_{\varepsilon} + \varepsilon)^m - 1 |\nabla u_{\varepsilon}|^2 dxds dxds
\]

\[
+ \sup_{t \in (\tau, T_{\text{max}})} \int_{t-\tau}^t \int_\Omega u_{\varepsilon}^{r+2} dxds \leq C(r) \tag{25}
\]

for any \( r > 0 \). By Duhamel’s principle, we see that the solution \( v_{\varepsilon} \) can be expressed as follows

\[
v_{\varepsilon} = e^{-t} e^{t \Delta} v_{\varepsilon 0} + \int_0^t e^{-(t-s)} e^{(t-s) \Delta} u_{\varepsilon}(s) ds,
\]

where \( \{e^{t \Delta}\}_{t \geq 0} \) is the Neumann heat semigroup in \( \Omega \), for more details of Neumann heat semigroup, please refer to [20]. By (25) with \( r = N \), we obtain for any \( t \in (0, T_{\text{max}}) \)

\[
\|v_{\varepsilon}(\cdot, t)\|_{L^\infty} \leq e^{-t} \|v_{\varepsilon 0}\|_{L^\infty} + \int_0^t e^{-(t-s)} (t-s)^{-\frac{N}{2} - \frac{1}{m+1}} \|u_{\varepsilon}(s)\|_{L^{N+1}} ds
\]

\[
\leq e^{-t} \|v_{\varepsilon 0}\|_{L^\infty} + \sup_{s \in (0, T_{\text{max}})} \|u_{\varepsilon}(\cdot, s)\|_{L^{N+1}} \int_0^\infty e^{-s} s^{-\frac{N+1}{m+1}} ds \leq C,
\]

and

\[
\|\nabla v_{\varepsilon}(\cdot, t)\|_{L^\infty} \leq e^{-t} \|\nabla v_{\varepsilon 0}\|_{L^\infty} + \int_0^t e^{-(t-s)} (t-s)^{-\frac{N}{2} - \frac{1}{m+1} - \frac{1}{2}} \|u_{\varepsilon}(s)\|_{L^{N+1}} ds
\]

\[
\leq e^{-t} \|\nabla v_{\varepsilon 0}\|_{L^\infty} + \sup_{s \in (0, T_{\text{max}})} \|u_{\varepsilon}(\cdot, s)\|_{L^{N+1}} \int_0^\infty e^{-s} s^{-\frac{N+1}{m+1}} ds \leq C.
\]

Note that

\[
\|\omega_{\varepsilon}\|_{L^\infty} \leq \|\omega_{\varepsilon} v_{\varepsilon}\|_{L^\infty}.
\]
Multiplying the first equation of (10) by \( u_{e}^{-1} \) with \( r > \max\{2N, 4m\} \), then integrating it over \( \Omega \), and combining with (23), we obtain

\[
\frac{d}{dt} \int_{\Omega} u_{e}^{r} dx + m r (r - 1) \int_{\Omega} u_{e}^{r-2}(u_{e} + \varepsilon)^{m-1} |\nabla u_{e}|^{2} dx + r \mu \int_{\Omega} u_{e}^{r+1} dx + \int_{\Omega} u_{e}^{r} dx \\
\leq \chi r (r - 1) \int_{\Omega} u_{e}^{r-1} \nabla u_{e} \nabla v_{e} dx - \xi (r - 1) \int_{\Omega} u_{e}^{r} \Delta v_{e} dx + (m r + 1) \int_{\Omega} u_{e}^{r} \\
\leq \frac{m r (r - 1)}{2} \int_{\Omega} u_{e}^{r-2}(u_{e} + \varepsilon)^{m-1} |\nabla u_{e}|^{2} dx + \frac{\chi^{2} r (r - 1)}{2 m} \int_{\Omega} u_{e}^{r}(u_{e} + \varepsilon)^{1-m} |\nabla u_{e}|^{2} dx \\
+ (m r + 1) \int_{\Omega} u_{e}^{r} dx + (r - 1) \int_{\Omega} u_{e}^{r} (2 \|\omega_{0}\|_{L^{\infty}} v_{e} + M) dx \\
\leq \frac{m r (r - 1)}{2} \int_{\Omega} u_{e}^{r-2}(u_{e} + \varepsilon)^{m-1} |\nabla u_{e}|^{2} dx + C r \int_{\Omega} u_{e}^{r} dx + C r^{2} \int_{\Omega} u_{e}^{r}(u_{e} + \varepsilon)^{1-m} dx.
\]

(26)

Here \( C \) is independent of \( r \). In what follows, we estimate the last two terms of (26). When \( m \geq 1 \), by (26), we further have

\[
\frac{d}{dt} \int_{\Omega} u_{e}^{r} dx + \frac{r(r - 1)}{2} \int_{\Omega} u_{e}^{r+3} |\nabla u_{e}|^{2} dx + \int_{\Omega} u_{e}^{r} dx \\
\leq C r \int_{\Omega} u_{e}^{r} dx + C r^{2} \int_{\Omega} u_{e}^{r+1-m} dx.
\]

(27)

By Gagliardo-Nirenberg interpolation inequality, it follows

\[
C r \|u_{e}\|_{L^{r}} = C r \|u_{e}^{\frac{r-m-1}{2}} \|_{L^{r(N-m)2}}^{\frac{r-m-1}{2}} + C r \|u_{e}\|_{L^{r}}^{\frac{r-m-1}{2}} \\
\leq C_{1} r \|\nabla u_{e} \|_{L^{r}}^{\frac{r-m-1}{2}} + C_{2} \|u_{e}\|_{L^{(2r(1+m)-2m)/(r-m)}}^{\frac{r-m-1}{2}} + C_{2} r \|u_{e}\|_{L^{r}}^{\frac{r-m-1}{2}} \\
\leq \eta \|\nabla u_{e} \|_{L^{r}}^{\frac{r-m-1}{2}} + C_{2} \|u_{e}\|_{L^{(2r(1+m)-2m)/(r-m)}}^{\frac{r-m-1}{2}} + C_{2} r \|u_{e}\|_{L^{r}}^{\frac{r-m-1}{2}} \\
\leq \eta \|\nabla u_{e} \|_{L^{r}}^{\frac{r-m-1}{2}} + C_{2} \|u_{e}\|_{L^{(2r(1+m)-2m)/(r-m)}}^{\frac{r-m-1}{2}} + C_{2} r \|u_{e}\|_{L^{r}}^{\frac{r-m-1}{2}} + C_{2} r \|u_{e}\|_{L^{r}}^{\frac{r-m-1}{2}}.
\]

and

\[
C r^{2} \|u_{e}\|_{L^{r+1-m}}^{\frac{r+1-m}{2}} = C r^{2} \|u_{e}^{\frac{r+1-m}{2}} \|_{L^{(2r(1+m)-2m)/(2r(1+m)-m)}}^{\frac{r+1-m}{2}} \leq C_{2} r^{2} \|u_{e}\|_{L^{r}}^{\frac{r+1-m}{2}} \leq C_{2} r^{2} \|u_{e}\|_{L^{r}}^{\frac{r+1-m}{2}} \\
+ C_{2} r^{2} \|\nabla u_{e} \|_{L^{r}}^{\frac{r+1-m}{2}} + C_{2} r^{2} \|u_{e}\|_{L^{(2r(1+m)-2m)/(2r(1+m)-m)}}^{\frac{r+1-m}{2}} \\
\leq \eta \|\nabla u_{e} \|_{L^{r}}^{\frac{r+1-m}{2}} + C_{2} \|u_{e}\|_{L^{(2r(1+m)-2m)/(2r(1+m)-m)}}^{\frac{r+1-m}{2}} + C_{2} r^{2} \|u_{e}\|_{L^{r}}^{\frac{r+1-m}{2}} \\
\leq \eta \|\nabla u_{e} \|_{L^{r}}^{\frac{r+1-m}{2}} + C_{2} \|u_{e}\|_{L^{(2r(1+m)-2m)/(2r(1+m)-m)}}^{\frac{r+1-m}{2}} + C_{2} r^{2} \|u_{e}\|_{L^{r}}^{\frac{r+1-m}{2}}.
\]

Substituting the above two inequalities into (27), we obtain

\[
\frac{d}{dt} \int_{\Omega} u_{e}^{r} dx + \int_{\Omega} u_{e}^{r} dx \\
\leq C_{3} r^{N+2} \|u_{e}\|_{L^{(2r(1+m)-2m)/(2r(1+m)-m)}}^{\frac{r+1-m}{2}} + C_{4} r \|u_{e}\|_{L^{r}}^{\frac{r+1-m}{2}} \\
+ C_{5} r^{N+2} \|u_{e}\|_{L^{r}}^{\frac{r+1-m}{2}} + C_{6} r^{2} \|u_{e}\|_{L^{r}}^{\frac{r+1-m}{2}},
\]

where \( C \) is independent of \( r \).
Taking $r_j = 2r_0$, $r_0 = \max\{2N, 4m\}$, $M_j = \max\left\{1, \sup_{t \in (0, T_{\max})} \|u_\varepsilon\|_{L^r} \right\}$, then when $m \geq 1$, we have

\[
M_j \leq (C_3 + C_4 + C_5 + C_6) \frac{1}{r_j} \int_\Omega \left( \int_0^t \left( \int_\Omega \left( u_{\varepsilon} + \varepsilon \right)^{m-1} |\nabla u_{\varepsilon}|^2 dx + r_0 \int_\Omega u_{\varepsilon}^{r-1} dx + \int_\Omega u_{\varepsilon}^2 dx \right) dt \right) .
\]

\[
\leq (C_3 + C_4 + C_5 + C_6) \frac{1}{r_j} \int_\Omega \left( \int_0^t \left( \int_\Omega \left( u_{\varepsilon} + \varepsilon \right)^{m-1} |\nabla u_{\varepsilon}|^2 dx + r_0 \int_\Omega u_{\varepsilon}^{r-1} dx + \int_\Omega u_{\varepsilon}^2 dx \right) dt \right) .
\]

where $\hat{M}$ is independent of $j$. Letting $j \to \infty$, (24) is derived.

When $0 < m < 1$, noticing that $u_{\varepsilon}(u_{\varepsilon} + \varepsilon)^{1-m} \leq u_{\varepsilon}^{r+1-m} + \varepsilon^{1-m} u_{\varepsilon}$, then by (26), we have

\[
\frac{d}{dt} \int_\Omega u_{\varepsilon}^2 dx + \frac{mr(r-1)}{2} \int_\Omega u_{\varepsilon}^{r-2}(u_{\varepsilon} + \varepsilon)^{m-1} |\nabla u_{\varepsilon}|^2 dx + r_0 \int_\Omega u_{\varepsilon}^{r-1} dx + \int_\Omega u_{\varepsilon}^2 dx
\]

\[
\leq \hat{C}_r^2 \left( \int_\Omega u_{\varepsilon}^{N(r-1)} dx \right)^{\frac{N-1}{N}} + \hat{C}_r^2 \left( \int_{u_0}^{N(r-1)} u_{\varepsilon}^{N(r-1)} dx \right)^{\frac{N-1}{N}}.
\]

Denote $J_\varepsilon(t) = \{x \in \Omega; u_\varepsilon(x, t) \geq \varepsilon\}$, $\| \cdot \|_{L^p} = \| \cdot \|_{L^p(\Omega)}$, by Gagliardo-Nirenberg interpolation inequality, we have

\[
\frac{d}{dt} \| u_{\varepsilon} \|_{L^{\frac{r}{N-1}}(J_\varepsilon(t))} \approx \hat{C}_r^2 \| u_{\varepsilon} \|_{L^{\frac{r+1-m}{2N-1}}(J_\varepsilon(t))} \leq \hat{C}_1 \| \nabla u_{\varepsilon} \|_{L^{\frac{r+1-m}{2N-1}}(J_\varepsilon(t))}^2 + C \| u_{\varepsilon} \|_{L^{\frac{r+1-m}{2N-1}}(J_\varepsilon(t))}^{r+1-m} + \hat{C}_r^2 \| u_{\varepsilon} \|_{L^2}^{r-1}.
\]

We first need to readjust the detail of the calculation. Noticing that

\[
\| u_{\varepsilon} \|_{L^{r+1-m}} \leq \| u_{\varepsilon} \|_{L^{N(2-m)}} \| u_{\varepsilon} \|_{L^{\frac{r}{N-1}}} \leq C \| u_{\varepsilon} \|_{L^{\frac{r}{N-1}}},
\]

and

\[
\| u_{\varepsilon} \|_{L^{r+1-m}} \leq \| u_{\varepsilon} \|_{L^{N(2-m)}} \| u_{\varepsilon} \|_{L^{\frac{r}{N-1}}} \leq C \| u_{\varepsilon} \|_{L^{\frac{r}{N-1}}},
\]

substituting the above two inequalities into (28), for any $\varepsilon < 1$, we obtain

\[
\frac{d}{dt} \| u_{\varepsilon} \|_{L^{\frac{r}{N-1}}(J_\varepsilon(t))} \approx \hat{C}_r^2 \| u_{\varepsilon} \|_{L^{\frac{r+1-m}{2N-1}}(J_\varepsilon(t))}^2 + C \| u_{\varepsilon} \|_{L^{\frac{r+1-m}{2N-1}}(J_\varepsilon(t))}^{r+1-m} + \hat{C}_r^2 \| u_{\varepsilon} \|_{L^2}^{r-1}.
\]
substituting the above two inequalities into (29) gives
\[ \frac{d}{dt} \| u_\varepsilon \|_{L^r} + \| u_\varepsilon \|_{L^r}^{(N-1)r} \leq \hat{C}_1 r^2 \| u_\varepsilon \|_{L^2}^{(N-1)r} + \hat{C}_2 r^2 \| u_\varepsilon \|_{L^2}^{(N-1)r} + \hat{C}_3 r^2 (N+2) \| u_\varepsilon \|_{L^2}^{(r+Nm-m-1)}. \]

Noticing that \( \frac{(N-1)r}{2N} r^{(r+Nm-m-1)} < r \), then completely similar to the proof for the case \( m \geq 1 \), we get (24). The proof is complete. \( \square \)

**Proof of Proposition 1.** By (23), (24), and Lemma 3.1, we conclude that the problem (10) admits a unique global classical solution, and (13) holds. So, \( \tau = 1 \) in the above lemmas since the solution exists globally. By Lemma 2.5, we further have
\[ \sup_{t \in \mathbb{R}^+} \| u_\varepsilon \|_{W^{2,1}_\varepsilon (Q_1(0))} \leq C, \quad \text{for any } p > 1. \]

By (19) and (24), we obtain
\[ \sup_{t \in [T, T_{max}]} \int_{t-1}^{t} \int_{\Omega} (\varepsilon + u_\varepsilon)^r |\nabla u_\varepsilon|^2 \, dx \leq C, \quad \text{for any } r \geq m - 2. \] (30)

It means (15) holds.

By the expression of \( \omega_\varepsilon \) and (11), and we note that
\[ \int_{0}^{T} \Delta v_\varepsilon (x, s) \, ds = \int_{0}^{T} (v_{\varepsilon t} + v_\varepsilon - u_\varepsilon) \, ds = v_\varepsilon (x, t) - v_{\varepsilon 0} + \int_{0}^{T} (v_\varepsilon - u_\varepsilon) \, ds, \]
then for any \( T > 0 \),
\[ \| \nabla \omega \|_{L^\infty(Q_T)} + \| \Delta \omega \|_{L^\infty(Q_T)} \leq C_T, \] (31)
where \( C_T \) depends on \( T \) and is independent of \( \varepsilon \).

Multiplying the first equation of (10) by \( \frac{\partial(\varepsilon + u_\varepsilon)^{m-1} u_\varepsilon}{\partial t} \), and integrating it over \( \Omega \) gives
\[
\frac{1}{2} \frac{d}{dt} \int_{\Omega} |\nabla ((\varepsilon + u_\varepsilon)^{m-1} u_\varepsilon)|^2 \, dx + \tilde{m} \int_{\Omega} (\varepsilon + u_\varepsilon)^{m-1} \left| \frac{\partial u_\varepsilon}{\partial t} \right|^2 \, dx
\leq -\chi \int_{\Omega} \nabla \cdot (u_\varepsilon \cdot \nabla \omega_\varepsilon) \frac{\partial(\varepsilon + u_\varepsilon)^{m-1} u_\varepsilon}{\partial t} \, dx - \xi \int_{\Omega} \nabla \cdot (u_\varepsilon \cdot \nabla \omega_\varepsilon) \frac{\partial(\varepsilon + u_\varepsilon)^{m-1} u_\varepsilon}{\partial t} \, dx
+ \mu \int_{\Omega} u_\varepsilon (1 - u_\varepsilon - \omega_\varepsilon) \frac{\partial(\varepsilon + u_\varepsilon)^{m-1} u_\varepsilon}{\partial t} \, dx
\leq C \left( \int_{\Omega} |\nabla \cdot (u_\varepsilon \cdot \nabla \omega_\varepsilon)|^2 \, dx + C \int_{\Omega} |\nabla \cdot (u_\varepsilon \cdot \nabla \omega_\varepsilon)|^2 \, dx \right)
+ \frac{\tilde{m}}{2} \int_{\Omega} (\varepsilon + u_\varepsilon)^{m-1} \left| \frac{\partial u_\varepsilon}{\partial t} \right|^2 \, dx + C \mu \int_{\Omega} u_\varepsilon^2 (1 - u_\varepsilon - \omega_\varepsilon)^2 \, dx
\leq C_1(T) \left( 1 + \int_{\Omega} |\Delta v_\varepsilon|^2 + (\varepsilon + u_\varepsilon)^{m-1} |\nabla u_\varepsilon|^2 \right) \, dx + \frac{\tilde{m}}{2} \int_{\Omega} (\varepsilon + u_\varepsilon)^{m-1} \left| \frac{\partial u_\varepsilon}{\partial t} \right|^2 \, dx.
\]

By (30), we obtain
\[ \sup_{t \in (0, T)} \int_{\Omega} |\nabla ((\varepsilon + u_\varepsilon)^{m-1} u_\varepsilon)|^2 \, dx + \int_{0}^{T} \int_{\Omega} (\varepsilon + u_\varepsilon)^{m-1} \left| \frac{\partial u_\varepsilon}{\partial t} \right|^2 \, dx dt \leq C_2(T), \]
and (14) is obtained. \( \square \)
Proof of Theorem 1.1. Since \((u_\varepsilon, v_\varepsilon, \omega_\varepsilon)\) is the classical solution of (10), then we have

\[
- \int_Q u_\varepsilon \varphi_\varepsilon dx dt - \int_\Omega u_\varepsilon(x, 0) \varphi(x, 0) dx - \int_Q (\varepsilon + u_\varepsilon)^m - u_\varepsilon \Delta \varphi dx dt,
\]

\[
- \int_Q (u_\varepsilon \nabla v_\varepsilon + u_\varepsilon \nabla \omega_\varepsilon) \nabla \varphi dx dt = \mu \int_Q u_\varepsilon(1 - u_\varepsilon - \omega_\varepsilon) \varphi dx dt,
\]

\[
- \int_Q v_\varepsilon \varphi dx dt - \int_\Omega v_\varepsilon(x, 0) \varphi(x, 0) dx + \int_Q (\nabla v_\varepsilon \nabla \varphi + (v_\varepsilon - u_\varepsilon) \varphi) dx dt = 0,
\]

\[
- \int_Q \omega_\varepsilon \varphi dx dt - \int_\Omega \omega_\varepsilon(x, 0) \varphi(x, 0) dx + \int_Q v_\varepsilon \omega_\varepsilon \varphi dx dt = 0,
\]

for any \(\varphi \in C^\infty(\bar{Q}_T)\) with \(\frac{\partial \varphi}{\partial n}\big|_{\partial Q} = 0\) and \(\varphi(x, T) = 0\). By Aubin-Lions theorem, and using Proposition 1, for any \(T > 0\), we have

\[
u_\varepsilon + \varepsilon, u_\varepsilon \to u, \text{ in } C([0, T]; L^p(\Omega)), \text{ for any } p \in (1, +\infty),
\]

\[
u_\varepsilon \rightharpoonup u, \text{ in } L^\infty(Q_T),
\]

\[
(u_\varepsilon + \varepsilon)^{-\frac{m-1}{p-1}} \to u^{-\frac{m-1}{p-1}}, \text{ in } W^{1,1}_-(Q_T),
\]

\[
v_\varepsilon \to v, \text{ uniformly},
\]

\[
v_\varepsilon \to v, \text{ in } W^{2,1}_p(Q_T), \text{ for any } p \in (1, +\infty),
\]

\[
\omega_\varepsilon \to \omega, \text{ uniformly},
\]

\[
\nabla \omega_\varepsilon \rightharpoonup \nabla \omega, \text{ uniformly},
\]

\[
\Delta \omega_\varepsilon \rightharpoonup \Delta \omega, \text{ in } L^\infty(Q_T),
\]

and for \(u, v, \omega\), we have (3)-(5) hold. Noticing that \(u_\varepsilon\) and \(u\) are bounded uniformly, then

\[
|(\varepsilon + u_\varepsilon)^{m-1}u_\varepsilon - u^m| \leq |(\varepsilon + u_\varepsilon)^{m} - u^m| + |\varepsilon (\varepsilon + u_\varepsilon)^{m-1}|
\]

\[
\leq \begin{cases} 
|\varepsilon + u_\varepsilon - u|^m + \varepsilon^m, & \text{when } m < 1, \\
C|\varepsilon + u_\varepsilon - u| + C\varepsilon, & \text{when } m \geq 1,
\end{cases}
\]

thus, we have

\[
(\varepsilon + u_\varepsilon)^{m-1}u_\varepsilon \to u^m, \text{ in } C([0, T]; L^p(\Omega)), \text{ for any } p \in (1, +\infty).
\]

Letting \(\varepsilon \to 0^+\), we also obtain

\[
- \int_Q u^m \Delta \varphi dx dt = \int_Q u^m \nabla \varphi dx dt.
\]
It means that $(u, v, \omega)$ is the global weak solution of (1).

REFERENCES

[1] X. Cao, Boundedness in a three-dimensional chemotaxis-haptotaxis model, *Z. Angew. Math. Phys.*, 67 (2016), Art. 11, 13 pp.

[2] M. A. J. Chaplain and G. Lolas, Mathematical modelling of cancer invasion of tissue: Dynamic heterogeneity, *Networks and Heterogeneous Media*, 1 (2006), 399–439.

[3] T. Cieslak and C. Stinner, Finite-time blowup and global-in-time unbounded solutions to a parabolic-parabolic quasilinear Keller-Segel system in higher dimensions, *J. Differential Equations*, 252 (2012), 5832–5851.

[4] D. Horstmann and G. Wang, Blow-up in a chemotaxis model without symmetry assumptions, *European J. Appl. Math.*, 12 (2001), 159–177.

[5] C. Jin, Boundedness and global solvability to a chemotaxis model with nonlinear diffusion, *J. Differential Equations*, 263 (2017), 5759–5772.

[6] E. Keller and A. Segel, Initiation of slime mold aggregation viewed as an instability, *J. Theoret. Biol.*, 26 (1970), 399–415. Available from: https://www.researchgate.net/publication/17711401_Initiation_of_Slime_Mold_Aggregation_Viewed_as_an_Instability.

[7] Y. Li and J. Lankeit, Boundedness in a chemotaxis-haptotaxis model with nonlinear diffusion, *Nonlinearity*, 29 (2016), 1566–1595.

[8] L. Nirenberg, On elliptic partial differential equations, *Ann. Scuola Norm. Sup. Pisa*, 13 (1959), 115–162.

[9] Y. Sugiyama, Time global existence and asymptotic behavior of solutions to degenerate quasilinear parabolic systems of chemotaxis, *Differential Integral Equations*, 20 (2007), 133–180.

[10] Z. Szymanska, C. Morales-Rodrigo, M. Lachowicz and M. Chaplain, Mathematical modelling of cancer invasion of tissue: The role and effect of nonlocal interactions, *Math. Models Methods Appl. Sci.*, 19 (2009), 257–281.

[11] T. Senba and T. Suzuki, Parabolic system of chemotaxis: Blowup in a finite and the infinite time, *Methods Appl. Anal.*, 8 (2001), 349–367.

[12] C. Stinner, C. Surulescu and M. Winkler, Global weak solutions in a PDE-ODE system modeling multiscale cancer cell invasion, *SIAM J. Math. Anal.*, 46 (2014), 1969–2007.

[13] Y. Tao, Boundedness in a two-dimensional chemotaxis-haptotaxis system, arXiv:1407.7382v1.

[14] Y. Tao and M. Winkler, A chemotaxis-haptotaxis model: The roles of nonlinear diffusion and logistic source, *SIAM J. Math. Anal.*, 43 (2011), 685–704.

[15] Y. Tao and M. Winkler, Global existence and boundedness in a Keller-Segel-Stokes model with arbitrary porous medium diffusion, *Discrete Contin. Dyn. Syst.*, 32 (2012), 1901–1914.

[16] Y. Tao and M. Winkler, Large time behavior in a multidimensional chemotaxis-haptotaxis model with slow signal diffusion, *SIAM J. Math. Anal.*, 47 (2015), 4229–4250.

[17] J. Tello and M. Winkler, A chemotaxis system with logistic source, *Comm. Partial Differential Equations*, 32 (2007), 849–877.

[18] M. Winkler, Finite-time blow-up in the higher-dimensional parabolic-parabolic Keller-Segel system, *J. Math. Pures Appl.*, 100 (2013), 748–767.

[19] Y. Wang, Boundedness in the higher-dimensional chemotaxis-haptotaxis model with nonlinear diffusion, *J. Differential Equations*, 260 (2016), 1975–1989.

[20] M. Winkler, Aggregation vs. global diffusive behavior in the higher-dimensional Keller-Segel model, *J. Differential Equations*, 248 (2010), 2889–2905.

[21] M. Winkler, Chemotaxis with logistic source: Very weak global solutions and their boundedness properties, *J. Math. Anal. Appl.*, 48 (2008), 708–729.

[22] M. Winkler, Boundedness in the higher-dimensional parabolic-parabolic chemotaxis system with logistic source, *Comm. Partial Differential Equations*, 35 (2010), 1516–1537.

[23] J. Zheng, Boundedness of solutions to a quasilinear higher-dimensional chemotaxis-haptotaxis model with nonlinear diffusion, *Discrete Contin. Dyn. Syst.*, 37 (2017), 627–643.

Received June 2017; revised August 2017.

E-mail address: jinchhua@126.com