Active suspensions have non-monotonic flow curves and multiple mechanical equilibria

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We point out unconventional mechanical properties of confined active fluids, such as bacterial suspensions, under shear. Using a minimal model of an active liquid crystal with no free parameters, we predict the existence of a window of bacteria concentration for which a suspension of *E. Coli* effectively behaves, at steady-state, as a negative viscosity fluid and reach quantitative agreement with experimental measurements. Our theoretical analysis further shows that a negative apparent viscosity is due to a non-monotonic local velocity profile, and is associated with a non-monotonic stress vs. strain rate flow curve. This implies that fixed stress and fixed strain rate ensembles are not equivalent for active fluids.

Active suspensions, such as swarms of bacteria and the cytoskeleton of living cells, consist of interacting self-driven particles that individually consume energy and collectively generate motion and mechanical stresses in the bulk [1–5]. Due to the orientable nature of their constituents, active suspensions can exhibit liquid crystalline order and have been modeled as active liquid crystals (LCs) [2, 3, 5]. An astonishing property of confined active LCs is their ability to spontaneously flow in the absence of any mechanical forcing [6–12]. It is therefore expected that active LCs exhibit, under external driving, unusual mechanical properties.

The most commonly sought mechanical property of a complex fluid is the apparent (shear) viscosity, which expresses the macroscopic fluid’s resistance to flow and can be defined through the idealized configuration depicted in Fig. 1. A uniform shear is applied to the fluid by pressing the macroscopic fluid’s resistance to flow and can be defined through the idealized configuration depicted in Fig. 1. A uniform shear is applied to the fluid by

\[ \eta = \frac{\sigma}{\dot{\gamma}} \]

where \( \sigma \) is the macroscopic shear stress and \( \dot{\gamma} \) is the macroscopic shear strain rate. We emphasize that \( \sigma \) and \( \dot{\gamma} \) are spatially averaged over the gap, they are not equivalent to their local counterparts which may not be uniform.

There is now ample evidence that the apparent viscosity of suspensions can be reduced by activity [4, 13–22]. But only recently has it been taken seriously that this phenomenon can continue until even a negative apparent viscosity is achieved. Using a highly sensitive rheometer, Lopez et al. [22] were able to measure zero and possibly negative values of the apparent viscosity in a suspension of *E. Coli* at steady-state, thereby demonstrating that microscopic bacterial activity can be converted into macroscopic useful mechanical power. Several theories have been proposed since to rationalize \( \eta_{app} \leq 0 \), either based on kinetic models [23, 24] or on a generalized Navier-Stokes equation [19], yet comparison with [22] is, at best, qualitative.

In this Letter, we show that a minimal model of an active LC can predict quantitatively the transition to \( \eta_{app} \leq 0 \) reported by [22]. Our model, which has no free parameters, explains both the observed non-monotonic evolution of \( \eta_{app} \) and the decreasing response time of the system with increasing bacterial concentration (Fig. 2a).

Importantly \( \eta_{app} < 0 \) is due to a non-monotonic velocity profile (Fig. 1), the local viscosity being always positive (essentially that of the solvent). We further show that the steady state with \( \eta_{app} < 0 \) reached by [22] is a structurally stable mechanical equilibrium, and is associated with a non-monotonic flow curve \( \sigma(\dot{\gamma}) \) (Fig. 2b).

As a consequence, if the suspension were to be sheared by tuning the stress rather than the strain rate, experiments would yield different results: \( \sigma(\dot{\gamma}) \neq \dot{\gamma}(\sigma) \). This also implies history dependent mechanics.

The hydrodynamic theory of active matter provides a now well-accepted continuum description of active LCs in terms of a reduced number of slowly-varying fields [3]. The local coarse-grained orientation of the particles is represented by the polarization vector. The magnitude of the polarization vector is a fast variable, therefore we take it to be constant on long timescales and assume without loss of generality that the polarization is a unit vector \( p \) (the effect of its magnitude being absorbed in the phenomenological coefficients). The other relevant slow variables are the fluid velocity \( u \) and the particle number density. For simplicity we shall assume nematic symmetry and homogeneous density, this hypothesis is relaxed and shown to be unimportant in Supplemental Material.

The evolution of the director field, in a globally ordered or isotropic phase (bearing in mind that local alignment is always present), is governed by

\[
(\partial_t + u_j \partial_j) p_i + \Omega_{ij} p_j = \lambda E_{ij} p_j + \Gamma h_i
\]

(1)

where \( E_{ij} = (\partial_i u_j + \partial_j u_i)/2 \), \( \Omega_{ij} = (\partial_i u_j - \partial_j u_i)/2 \), \( \lambda \) is the flow alignment parameter which determines the response of \( p \) to simple unbounded shear (\( |\lambda| > 1 \) for alignment, \( |\lambda| < 1 \) for tumble), \( 1/\Gamma \) is the rotational viscosity, and \( h_i = K \nabla^2 p_i + h_{i\parallel} p_i \) is the molecular field where \( K \) is the Frank elasticity (one-constant approximation) and \( h_{i\parallel} \) is a Lagrange multiplier enforcing \( |p| = 1 \). The fluid is assumed incompressible (\( \nabla \cdot u = 0 \)) and the flow field obeys, upon neglecting fluid inertia, the Stokes flow equa-
tion $\nabla \cdot \mathbf{σ} = 0$ with

$$
\sigma_{ij} = 2\eta E_{ij} - \Pi \delta_{ij} - \frac{\lambda + 1}{2} p_i p_j - \frac{\lambda - 1}{2} p_j h_i - \alpha p_i p_j \quad (2)
$$

where $\eta$ is the bulk fluid viscosity ($\eta > 0$), $\Pi$ is the bulk pressure, and $\alpha$ is the activity coefficient. This coefficient is related to the active stresses in a suspension of particles modeled as force dipoles: the magnitude of $\alpha$ is proportional to the strength of the force pair and the sign of $\alpha$ depends on whether the induced flow is extensile ($\alpha > 0$) or contractile ($\alpha < 0$). Our geometry is a two-dimensional slab of thickness $L$ (Fig. 1) with translational invariance in the direction parallel to the walls. We use no-slip and parallel anchoring as boundary conditions. The fluid is subject to a macroscopic shear rate $\dot{\gamma} = 2V/L$ and the shear stress (simply denoted $\sigma$) is uniform across the film.

To determine the apparent viscosity of a fluid we have the choice of two ensembles: prescribed $\dot{\gamma}$ or prescribed $\sigma$. In passive fluids, both ensembles are equivalent: the steady shear response is characterized uniquely by $\sigma(\dot{\gamma})$ or by $\dot{\gamma}(\sigma)$. We emphasize here that this is not generally true in active fluids. This peculiar property may be seen in the limiting cases $\dot{\gamma} = 0$ or $\sigma = 0$. In the absence of external mechanical forcing, a confined quasi-one-dimensional active LC exhibits a spontaneous transition from a homogeneous immobile state to an inhomogeneous flowing state at a critical value of the activity coefficient $\alpha$ [6]. Crucially, the spontaneous flow transition depends on the mechanical constraint: it occurs at $\alpha_{c,\sigma}$ for $\sigma = 0$ but at a larger value $\alpha_{c,\dot{\gamma}} = 4\alpha_{c,\sigma}$ for $\dot{\gamma} = 0$, where $\alpha_{c,\sigma} = 2\pi^2 \eta \Gamma K L^{-2} (1 + \xi^2)/(\lambda - 1)$ with $\xi = (\lambda - 1)^2/(4\eta \Gamma)$ [6]. The immediate consequence is that $\sigma(\dot{\gamma} = 0) = 0$ whereas $\dot{\gamma}(\sigma = 0) \neq 0$ for $\alpha_{c,\sigma} < \alpha < \alpha_{c,\dot{\gamma}}$ [25]. Note that the spontaneous flow transition is only seen in flow-aligning extensile systems ($\lambda > 1, \alpha > 0$) or flow-tumbling contractile systems ($0 < \lambda < 1, \alpha < 0$) [9, 11] [26].

By studying numerically the same active LC in the presence of external shear (see Supplemental Material for methodology details), we found that this system can also undergo a bifurcation under weak applied shear and exhibit two asymmetric stable branches beyond $\alpha_{c,\sigma}$ for imposed $\sigma$ (Fig. 3b) or $\alpha_{c,\dot{\gamma}}$ for imposed $\dot{\gamma}$ (not shown): as such a high level of activity is irrelevant to experiments, results are deferred to Supplemental Material). This results in the non-equivalence of these two ensembles and in the unconventional flow curves $\sigma(\dot{\gamma})$ and $\dot{\gamma}(\sigma)$ displayed in Fig. 3a.

For $\alpha < \alpha_{c,\sigma}$, the flow curves are identical and increase monotonically, as in passive fluids. When $\alpha = \alpha_{c,\sigma}$, $\dot{\gamma}(\sigma)$ has a vertical tangent at zero, which corresponds to the bifurcation from a single steady state to bistability for imposed $\sigma$. For $\alpha_{c,\sigma} < \alpha < \alpha_{c,\dot{\gamma}}$, $\sigma(\dot{\gamma})$ exhibits local minima $\pm \sigma_m$ connected by a negative slope $d\sigma/d\dot{\gamma} < 0$. The associated solutions are unique and stable when $\dot{\gamma}$ is imposed (blue square), but are unstable (not shown) when $\sigma$ is imposed. Instead $\dot{\gamma}(\sigma)$ has two stable non-equivalent solutions (leftward and rightward orange triangles) in the range $[-\sigma_m, +\sigma_m]$. While these features were identified separately by [27] for the former and by [12] for the latter, here we provide the conceptual framework that shows they are simply different sides of the
FIG. 3. Steady-state rheology of a weakly sheared active suspension of flow-aligning particles ($\lambda = 1.9$). (a) Flow curves in stress-controlled (circles) and strain-rate-controlled (lines) conditions for increasing $\alpha$ (from left to right); (b-c) bifurcation diagrams in stress-controlled (b) and strain-rate-controlled (c) conditions. The symbols show the correspondence between the flow curves and the bifurcation diagrams, associated velocity profiles are provided in (d) (for comparison, dotted lines show the behavior of a Newtonian passive fluid). Only stable solutions are shown. In the absence of applied shear, a pitchfork bifurcation occurs at $\alpha = \alpha_{c,\sigma}$ for $\sigma = 0$ and at $\alpha = \alpha_{c,\dot{\gamma}}$ for $\dot{\gamma} = 0$. These bifurcations correspond to the singularities $d\dot{\gamma}/d\sigma|_{\sigma=0} = \infty$ and $d\sigma/d\dot{\gamma}|_{\dot{\gamma}=0} = -\infty$, respectively) in the flow curves. The stars denote dimensionless quantities: $\alpha^* = \alpha L^2/K$, $\sigma^* = \sigma L^2/K$, $\dot{\gamma}^* = \dot{\gamma} L^2/(\Gamma K)$, and $u^* = uL/(\Gamma K)$. These results are qualitatively independent of $\lambda$.

same coin. When $\alpha = \alpha_{c,\dot{\gamma}}$, $\sigma(\dot{\gamma})$ admits a vertical tangent at zero, which corresponds to the bifurcation for imposed $\dot{\gamma}$. For $\alpha > \alpha_{c,\dot{\gamma}}$, $\sigma(\dot{\gamma})$ exhibits discontinuous branches which shape depends on $\lambda$ as explained in Supplemental Material.

A related, and striking, property of weakly sheared active films is the existence, for $\alpha_{c,\sigma} < \alpha < \alpha_{c,\dot{\gamma}}$, of a structurally stable mechanical equilibrium with $\eta_{app} < 0$ if (and only if) $\dot{\gamma}$ is imposed (blue square in Fig. 3). It is accommodated through a sinusoidal modulation of the linear velocity profile (Fig. 3d, blue squares) rather than through well-defined shear bands [28, 29], and corresponds to the regime reached in the experiments of [22]. If one would tune the applied stress instead, $\dot{\gamma}(\sigma)$ would exhibit hysteresis and at $\sigma = 0$, the plates would move.

When an active film is strongly sheared, its rheology can be either Newtonian or strongly nonlinear depending on the magnitude of $\lambda$ (Fig. 4). For $|\lambda| > 1$, the particles respond to shear by aligning at a well-defined angle with respect to the flow direction. Suspensions in this flow-aligning regime behave as Newtonian fluids under strong shear. For $|\lambda| < 1$, the particles rotate and form rolls within the gap as applied shear increases. The flow curves of suspensions in this flow-tumbling regime are characterized by multiple regions of locally reduced apparent viscosity which coincide with the completion of a half turn by the director at the center of the film. While these nonlinearities are also present in passive systems, non-monotonic flow curves with multiple local extrema or discontinuities as in Fig. 4 are peculiar to active systems.

The sign of $\lambda$, typically (but not necessarily) related to the particle shape, plays no role here (see Supplemental Material for $\lambda < 0$).

These theoretical predictions have been obtained from generic equations which apply to a broad class of active LCs [3]. The price to pay for this generality is the introduction of several system-dependent coefficients. In the following we will show how these phenomenological
coefficients can be extracted from shear experiments to allow testable quantitative predictions.

For that purpose we consider the transient response of an active film to applied shear and restrict our analysis to small $\sigma$ or $\dot{\gamma}$. By solving the linearized governing equations (see Supplemental Material) we find that the strain rate response to a step of stress from 0 to $\sigma$, valid for $\alpha/\alpha_{c,\sigma} < 1$, is

$$\frac{\sigma(t)}{\dot{\gamma}} = \eta \left[ \frac{\eta}{\eta_{\text{app}}} - \left( \frac{\eta}{\eta_{\text{app}}} - 1 \right) \exp \left( - \frac{t}{\tau_{\sigma}} \right) \right]^{-1}$$  (3a)

with the relaxation time

$$\frac{1}{\tau_{\sigma}} = \frac{\pi^2 G K}{L^2} \left[ 1 + \left( \frac{\eta}{\eta_{\text{app}}} \right) \xi - \frac{\eta}{\eta_{\text{app}}} \left( 1 + \xi \right) \right]$$  (3b)

whereas the stress response to a step of strain rate from 0 to $\dot{\gamma}$, valid for $\alpha/\alpha_{c,\sigma} < 1$, reads

$$\frac{\sigma(t)}{\dot{\gamma}} = \eta \left[ \frac{\eta}{\eta_{\text{app}}} - \left( \frac{\eta}{\eta_{\text{app}}} - 1 \right) \exp \left( - \frac{t}{\tau_{\sigma}} \right) \right]^{-1}$$  (4a)

with the (different) relaxation time

$$\frac{1}{\tau_{\sigma}} = \frac{\pi^2 G K}{L^2} \left[ 1 + \left( \xi - \frac{\alpha}{\alpha_{c,\sigma}} (1 + \xi) \right) \left( 1 - \frac{8}{\pi^2} \right) \right].$$  (4b)

The signal of $\sigma/\dot{\gamma}$ initially jumps to $\eta$ and then relaxes to $\eta_{\text{app}}$ given by

$$\frac{\eta_{\text{app}}}{\eta} = 1 - \frac{1}{\alpha_{c,\sigma}} \frac{\eta}{\eta_{\text{app}}} \left( 1 - \frac{8}{\pi^2} \xi + 1 \right) \left( 1 - \frac{8}{\pi^2} \right) \frac{\alpha}{\alpha_{c,\sigma}}.$$  (5)

Equation (5) shows that an active suspension behaves, at low applied strain rate, as a Newtonian fluid with $\eta_{\text{app}} \leq 0$ when $\alpha/\alpha_{c,\sigma} \geq 1$ (as also visible in Fig. 3c).

The four independent (groups of) parameters $\eta$, $\alpha(\lambda-1)$, $\Gamma K$ and $\xi$ control the suspension behavior. They can all be identified from rheological time traces using Eqs. (3–5). We applied this to the E. Coli suspensions studied by [22]; using their experimental time signals of $\sigma(t)/\dot{\gamma}$, we inferred the values of the phenomenological parameters describing their suspensions for bacteria volume fractions $\phi$ ranging from 0.11% to 2.4% (see Supplemental Material for more details about our procedure and the inferred parameters).

We obtained $\xi \approx 0$, $\eta \approx \eta_s$ with $\eta_s$ the solvent viscosity, and $\alpha(\lambda-1)$ and $\Gamma K$ were found to be positive increasing functions of $\phi$. Since E. Coli are extensile swimmers or “pushers” ($\alpha > 0$) [30], it follows that $\lambda > 1$, i.e., E. Coli behave effectively as flow-aligning rod-like particles. The dependences on $\phi$ are consistent with the expectations that (i) $\alpha = \alpha_1 \phi$ for a dilute suspension of self-propelled swimmers and (ii) $K = K_0 + K_1(\phi - \phi_c)^2$ as in conventional nematic LCs, where $\phi_c$ is the critical volume fraction (smaller than 0.1 %) associated to the isotropic-nematic transition, $K_0 > 0$ accounts for steric interactions and $K_1 > 0$ for thermodynamic interactions [31] [32]. Plugging semi-analytical expressions of $\alpha(\phi)$ and $K(\phi)$ into the governing equations yields, without any adjustable parameter, the theoretical curves presented against the experimental data in Fig. 2a.

We conclude that a minimal model of an active LC is sufficient to account quantitatively for all experimental observations, notably the emergence of a “superfluid-like” regime ($\eta_{\text{app}} \sim 0$). The non-monotonic evolution of $\eta_{\text{app}}$ with $\phi$ results from the competition between activity and elasticity, while the decrease of the response time $\tau$ with increasing $\phi$ arises from the growth of $K$. Incidentally, the predicted increase of $\eta_{\text{app}}$ at higher volume fractions, although not clearly seen in the experiments of [22], resemble that reported by [20] for a suspension of Bacillus subtilis, an analogous type of rod-like pushers.

**FIG. 4.** Steady-state flow curves for various types of active suspensions (insets: steady-state profiles of $p$). Bacteria suspensions of [22] follow the scenario displayed in the framed panel. The stars denote dimensionless quantities, see Fig. 3.
Geometrical confinement provides a powerful way to control active suspensions [33–36]. The existence of non-monotonic flow curves suggest new control mechanisms, that could find direct application in bacterial energy harvesting [37, 38]. Our results also provide strong support for models of biologically active suspensions as “living liquid crystals” [2, 3, 5] and open the way to a truly quantitative characterization of these systems.

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