Bound states of the Klein-Gordon equation for vector and scalar
genral Hulthén-type potentials in $D$-dimension

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Abstract

We solve the Klein-Gordon equation in any $D$-dimension for the scalar and vector general Hulthén-type potentials with any $l$ by using an approximation scheme for the centrifugal potential. Nikiforov-Uvarov method is used in the calculations. We obtain the bound state energy eigenvalues and the corresponding eigenfunctions of spin-zero particles in terms of Jacobi polynomials. The eigenfunctions are physical and the energy eigenvalues are in good agreement with those results obtained by other methods for $D = 1$ and 3 dimensions. Our results are valid for $q = 1$ value when $l \neq 0$ and for any $q$ value when $l = 0$ and $D = 1$ or 3. The s-wave ($l = 0$) binding energies for a particle of rest mass $m_0 = 1$ are calculated for the three lower-lying states ($n = 0, 1, 2$) using pure vector and pure scalar potentials.

Keywords: Bound states, Klein-Gordon equation, Hulthén potential, Nikiforov-Uvarov method, Approximation schemes.

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I. INTRODUCTION

In nuclear and high energy physics [1,2], it is necessary to obtain the exact bound state energy spectrum of non-relativistic or relativistic wave equations for various potentials. In relativistic quantum mechanics, the Klein-Gordon and Dirac wave equations are most frequently used to understand the dynamics of a particle in large class of potentials. Therefore, much works have been done to solve these relativistic wave equations for various radial and angular potentials. Unfortunately, some quantum mechanical equations with these potentials can be solved analytically only for s-wave equation (i.e., $l = 0$ states). Some other works have been done to study the $l \neq 0$ bound state solutions of the Klein-Gordon equation with pure scalar, pure vector and mixed vector and scalar potentials in various dimensional space using different methods. The scalar coupling constant is almost taken to be equal to the vector coupling constant (i.e., $S_0 = V_0$) [2-14]. For this case, the Klein-Gordon equation reduces to a Schrödinger-like equation and thereby the bound state solutions are easily obtained by using the well-known methods developed in nonrelativistic quantum mechanics [2]. On the other hand, it has been shown that when $S_0 \geq V_0$, there exist real bound state solutions (i.e., the radial wave function must satisfy the boundary condition that it becomes zero when $r \to \infty$ and also finite at $r = 0$). However, there are very few exact solvable cases [15-19]. The bound state solutions for the last case is obtained for the s-wave Klein-Gordon equation with the exponential [15] and some other classes of potentials [16]. Chen et al. [19] used the exponential function transformation approach along with an approximation for the centrifugal potential to find bound-state solutions of the Klein-Gordon equation with the vector and scalar Hulthén potentials for different $l$.

The $D$-dimensional Schrödinger and Klein-Gordon wave equations have been solved for various types of angular and radial dependent potentials using the Nikiforov-Uvarov (N-U) method [20-25]. The exact bound state solutions for the Schrödinger, Klein-Gordon and Salpeter wave equations, in one-dimension (1D), with the generalized Woods-Saxon potential are also obtained [26-29]. These solutions of the $D$-dimensional Klein-Gordon equation describing a spin-zero particle are obtained for the case $V_0 = S_0$ ring-shaped Kratzer-type and pseudoharmonic potentials [21,22]. Furthermore, analytic solution of radial Schrödinger equation has been given for the $l \neq 0$ general Hulthén potential within an approximation to the centrifugal barrier term [25,29]. Simsek and Egrifes [30] have investigated the reality of
exact bound states of the 1D Klein-Gordon equation with complex and/or PT-symmetric non-Hermitian exponential-type like general Hulthén potential. Berkdemir [31] has also applied the method to solve the Klein-Gordon equation of a spin-zero particle for \( V(r) = S(r) \) Kratzer-type potentials. Qiang et al. [32] have given an approximate analytic solution of the \( l \neq 0 \) Klein-Gordon equation for the general Hulthén-type potential within the approximation given in [29]. Saad [33] has presented an approximate solution of the \( l \neq 0 \) bound states of the Klein-Gordon equation in \( D \)-dimensions with the general Hulthén-type potentials following the model used in [29,32]. Chen et al. [34] have employed two semiclassical methods to determine the bound state energy spectrum of Klein-Gordon equation when \( S_0 \geq V_0 \). Dong et al [35] have also obtained the bound state solutions of the Schrödinger equation for an exponential-type potential following Refs. [29,32]. Very recently, we have obtained the bound-state solutions for the \( D \)-dimensional Schrödinger equation with the Manning-Rosen potential [36] which can be reduced to the Hulthén potential and the exponential-type hyperbolic potential [37] using a novel approximation to the centrifugal term [29,36,37].

The purpose of the present paper is to study the relativistic characteristics of the scalar and vector general Hulthén-type potentials written, respectively, as follows [32,33]:

\[
V(r) = -\frac{V_0 e^{-r/r_0}}{1 - q e^{-r/r_0}}, \quad S(r) = -\frac{S_0 e^{-r/r_0}}{1 - q e^{-r/r_0}}, \quad r_0 = \alpha^{-1}, \quad q \neq 0
\]  

where \( V_0 \) and \( S_0 \) represent the coupling constants of the vector and scalar general Hulthén-type potentials, respectively, \( \alpha \) is the screening range parameter, \( r_0 \) represents the spatial range and \( q \) is the deformation parameter [26-28]. This potential has been widely used in a number of areas of physics. In atomic physics, \( V_0 = Ze^2/r_0 \) where \( Z \) is the atomic number. Equation (1) behaves like a Coulomb potential \( V_C(r) = -Ze^2/r \) when \( r \ll r_0 \), but decreases exponentially in the asymptotic region when \( (r \gg r_0) \), so its capacity for bound state is smaller than the Coulomb potential. The Klein-Gordon equation has been solved analytically in [30] only for the s-wave. This equation cannot be solved analytically for \( l \neq 0 \) because of the centrifugal term, \( 1/r^2 \). Therefore, for \( l \neq 0 \), we must use an approximation for the centrifugal term similar to other authors [29,38-43] to find approximate analytical solutions of the \( D \)-dimensional Klein-Gordon equation with vector and scalar general Hulthén-type potentials. However, the resulting analytic solutions are only valid for \( q = 1 \) in the \( l \neq 0 \) case and for any \( q \) value in the \( l = 0 \) and \( D = 1, 3 \).

This paper is organized as follows: In Section II, we will derive \( l \neq 0 \) state solutions
within the approximation given in [29] for the $D$-dimensional Klein-Gordon equation describing spin-zero particle with the vector and scalar general Hulthén-type potentials. We further separate the wave equation into radial and angular parts. Section III is devoted to a brief description of the N-U method. In Section IV, we present general solutions to the radial and angular equations in $D$-dimensions. We also give discussions of the solution in various dimensions and the constraints for obtaining the real bound state solutions. Further, numerical values for the three lower-lying states when pure vector and pure scalar potentials are used with coupling constants $V_0 = 0.25$ and $S_0 = 0.25$ and unity mass particle. Finally, the relevant conclusions are given in Section V.

II. THE KLEIN-GORDON EQUATION WITH SCALAR AND VECTOR POTENTIALS

The time-independent Klein-Gordon equation (in any arbitrary $D$-dimension) with scalar $S(r)$ and vector $V(r)$ potentials, $r = |r|$, describing spin-zero particle of rest mass, $m_0$, can be written as (in the relativistic natural units $\hbar = c = 1$) [2,21,22]

$$\left\{ \nabla_D^2 + [E_{nl} - V(r)]^2 - [m_0 + S(r)]^2 \right\} \psi_{l_1 \cdots l_{D-2}}^{(l_{D-1} = l)}(x) = 0,$$

$$\nabla_D^2 = \sum_{j=1}^{D} \frac{\partial^2}{\partial x_j^2},$$

where $E_{nl}$ denotes the relativistic energy and $\nabla_D^2$ denotes the $D$-dimensional Laplacian. Usually, it is required that $S_0 \geq V_0$ and $m_0 > E_{nl}$ for the existence of real bound states [15-17]. Furthermore, the $x$ is a $D$-dimensional position vector, with the unit vector along $x$ is usually denoted by $\hat{x} = x/r$, is the hyperspherical Cartesian components $x_1, x_2, \cdots, x_D$ given as follows (cf. Refs. [20-25] and the references therein):

$$x_1 = r \cos \theta_1 \sin \theta_2 \cdots \sin \theta_{D-1},$$

$$x_2 = r \sin \theta_1 \sin \theta_2 \cdots \sin \theta_{D-1},$$

$$x_3 = r \cos \theta_2 \sin \theta_3 \cdots \sin \theta_{D-1},$$

$$\vdots$$

$$x_j = r \cos \theta_{j-1} \sin \theta_j \cdots \sin \theta_{D-1}, \ 3 \leq j \leq D - 1,$$

$$\vdots$$
We obtain the following Schrödinger-like equation:

\[ x_{D-1} = r \cos \theta_{D-2} \sin \theta_{D-1}, \]

\[ x_D = r \cos \theta_{D-1}, \quad \sum_{j=1}^{D} x_j^2 = r^2, D \geq 2. \]  

(3)

We have \( x_1 = r \cos \theta_1, \) \( x_2 = r \sin \theta_1, \) \( \theta_1 = \varphi, \) for \( D = 2 \) and \( x_1 = r \cos \theta_1, \) \( x_2 = r \sin \theta_1 \sin \theta_2, \) \( x_3 = r \cos \theta_2, \) \( \theta_2 = \theta, \) for \( D = 3. \) The volume element of the configuration space is given by

\[ \prod_{j=1}^{D} dx_j = r^{D-1} d^3r d\Omega, \quad d\Omega = \prod_{j=1}^{D-1} (\sin \theta_j)^{j-1} d\theta_j, \]  

(4)

where \( r \in [0, \infty), \) \( \theta_1 \in [0, 2\pi] \) and \( \theta_j \in [0, \pi], j \in [2, D-1]. \)

The wave function \( \psi_{l_1 \cdots l_{D-2}}^{(l_{D-1}=l)}(x) \) with a given angular momentum \( l \) can be decomposed as a product of a radial wave function \( R_l(r) \) and the normalized hyper-spherical harmonics \( Y_{l_1 \cdots l_{D-2}}^{(l)}(\theta_1, \theta_2, \cdots, \theta_{D-1}) \) as

\[ \psi_{l_1 \cdots l_{D-2}}^{(l_{D-1}=l)}(x) = R_l(r)Y_{l_1 \cdots l_{D-2}}^{(l)}(\theta_1, \theta_2, \cdots, \theta_{D-1}), \]  

(5)

where

\[ Y_{l_1 \cdots l_{D-2}}^{(l)}(\theta_1, \theta_2, \cdots, \theta_{D-1}) = H(\theta_1)H(\theta_2, \cdots, \theta_{D-2})H(\theta_{D-1}), \]  

(6)

which is the simultaneous eigenfunction of \( L_j^2 : \)

\[ L_j^2 Y_{l_1 \cdots l_{D-2}}^{(l)}(\theta_1, \theta_2, \cdots, \theta_{D-1}) = l_j(l_j + j - 1)Y_{l_1 \cdots l_{D-2}}^{(l)}(\theta_1, \theta_2, \cdots, \theta_{D-1}), \]

\[ l = 0, 1, \cdots, l_k = 0, 1, \cdots, l_{k+1}, j \in [1, D-1], k \in [2, D-2], \]

\[ l_1 = -l_2, -l_2 + 1, \cdots, l_2 - 1, l_2, \]

\[ L_{D-1}^2 Y_{l_1 \cdots l_{D-2}}^{(l)}(\theta_1, \theta_2, \cdots, \theta_{D-1}) = l(l + D - 2)Y_{l_1 \cdots l_{D-2}}^{(l)}(\theta_1, \theta_2, \cdots, \theta_{D-1}). \]  

(7)

The angular momentum operators \( L_j^2 \) are defined as

\[ L_1^2 = -\frac{\partial^2}{\partial \theta_1^2}, \]

\[ L_k^2 = \sum_{a<b=2} L_{a}^2 = -\frac{1}{\sin^{k-1} \theta_k} \frac{\partial}{\partial \theta_k} \left( \sin^{k-1} \theta_k \frac{\partial}{\partial \theta_k} \right) + \frac{L_{k-1}^2}{\sin^2 \theta_k}, \quad 2 \leq k \leq D - 1, \]

\[ L_{ab} = -i \left[ x_a \frac{\partial}{\partial x_b} - x_b \frac{\partial}{\partial x_a} \right]. \]  

(8)

Employing the method of separation of variables and substituting Eqs. (5)-(7) into Eq. (2), we obtain the following Schrödinger-like equation:

\[ R_l''(r) + \frac{(D-1)}{r} R_l'(r) + \left\{ \frac{l(l + D - 2)}{r^2} + [E_n - V(r)]^2 - [m_0 + S(r)]^2 \right\} R_l(r) = 0, \]

(9)
where \( l(l + D - 2)/r^2 \) is known as the centrifugal term. Furthermore, using \( R_t(r) = r^{-(D-1)/2}g(r) \), it is straightforward to find out the radial wave equation:

\[
g''(r) + \left\{ \left[ E_{nt} - V(r) \right]^2 - \left[ m_0 + S(r) \right]^2 - \frac{(D + 2l - 1)(D + 2l - 3)}{4r^2} \right\} g(r) = 0. \tag{10} \]

Obviously, Eq. (10) can be analytically solved \([17, 27, 30]\) only for \( l = 0 \) (s-wave) case. For \( l \neq 0 \) case, we have to use an approximation for the centrifugal term similar to the non-relativistic cases which is valid for \( q = 1 \) value \([29, 32, 33, 38-43]\):

\[
\frac{1}{r^2} \approx \frac{\alpha^2 e^{-\alpha r}}{(1 - q e^{-\alpha r})^2}. \tag{11} \]

We follow the model used by Qiang et al. \([32]\) and rewrite Eq. (10) for the scalar and vector general Hulthén-type potentials (1) as,

\[
g''(r) + \left[ E_{nt}^2 - m_0^2 + \frac{2(m_0 S_0 + E_{nt} V_0) e^{-\alpha r}}{1 - q e^{-\alpha r}} - \frac{(S_0^2 - V_0^2) e^{-2\alpha r}}{(1 - q e^{-\alpha r})^2} - \frac{(D + 2l - 1)(D + 2l - 3)}{4r^2} \right] \times g(r) = 0. \tag{12} \]

On the other hand, the other angular equations, obtained from the separation procedures, are \([21, 22]\):

\[
\left[ \frac{1}{\sin^{j-1} \theta_j} \frac{d}{d \theta_j} \left( \sin^{j-1} \theta_j \frac{d}{d \theta_j} \right) + l_j(l_j + j - 1) - \frac{l_{j-1}(l_{j-1} + j - 2)}{\sin^2 \theta_j} \right] H(\theta_2, \cdots, \theta_{D-2}) = 0,
\]

\[ j \in [2, D - 2], \tag{13} \]

\[
\left[ \frac{1}{\sin^{D-2} \theta_{D-1}} \frac{d}{d \theta_{D-1}} \left( \sin^{D-2} \theta_{D-1} \frac{d}{d \theta_{D-1}} \right) + l(l + D - 2) - \frac{L_{D-2}^2}{\sin^2 \theta_{D-1}} \right] H(\theta_{D-1}) = 0, \tag{14} \]

\[
\frac{d^2 H((\theta_1))}{d \theta_1^2} + l_1^2 H((\theta_1)) = 0. \tag{15} \]

It is well-known that the solution of Eq. (15) is

\[
H_{l_1}(\theta_1) = \frac{1}{\sqrt{2\pi}} \exp(\pm il_1 \theta_1), \quad l_1 = 0, 1, 2, \cdots. \tag{16} \]

Hence, the above Eqs. (12)-(14), have to be solved by using N-U method \([20-31, 36]\) which is reviewed briefly in the following Section.
III. NIKIFOROV-UVAROV METHOD

The N-U method is briefly outlined here and the details can be found in [20-31,36,37,44]. N-U method is proposed to solve the second-order differential equation of the hypergeometric type:

\[
\psi''_n(s) + \frac{\tau(s)}{\sigma(s)} \psi'_n(s) + \frac{\bar{\sigma}(s)}{\sigma'(s)} \psi_n(s) = 0,
\]

(17)

where \(\sigma(s)\) and \(\bar{\sigma}(s)\) are polynomials, at most of second-degree, and \(\tau(s)\) is a first-degree polynomial. Using a wave function, \(\psi_n(s)\), of the simple form

\[
\psi_n(s) = \phi_n(s)y_n(s),
\]

(18)

reduces Eq. (17) into an equation of a hypergeometric type

\[
\sigma(s)y''_n(s) + \tau(s)y'_n(s) + \lambda y_n(s) = 0,
\]

(19)

where

\[
\sigma(s) = \pi(s) \frac{\phi(s)}{\phi'(s)},
\]

(20)

\[
\tau(s) = \bar{\tau}(s) + 2\pi(s), \quad \tau'(s) < 0,
\]

(21)

and \(\lambda\) is a parameter defined as

\[
\lambda = \lambda_n = -n\tau'(s) - \frac{n(n-1)}{2} \sigma''(s), \quad n = 0, 1, 2, \cdots.
\]

(22)

The polynomial \(\tau(s)\) with the parameter \(s\) and prime factors show the differentials at first degree be negative. The other part \(y_n(s)\) of the wave function Eq. (18) is the hypergeometric-type function whose polynomial solutions are given by Rodrigues relation

\[
y_n(s) = \frac{B_n}{\rho(s)} \frac{d^n}{ds^n} \left[ \sigma^n(s) \rho(s) \right],
\]

(23)

where \(B_n\) is the normalization constant and the weight function \(\rho(s)\) can be found by [44]

\[
\omega'(s) - \frac{\tau(s)}{\sigma(s)} \omega(s) = 0, \quad \omega(s) = \sigma(s)\rho(s).
\]

(24)

The function \(\pi(s)\) and the parameter \(\lambda\) are defined as

\[
\pi(s) = \frac{\sigma'(s) - \bar{\tau}(s)}{2} \pm \sqrt{\left(\frac{\sigma'(s) - \bar{\tau}(s)}{2}\right)^2 - \bar{\sigma}(s) + k\sigma(s)},
\]

(25)
\[ \lambda = k + \pi'(s), \quad (26) \]

where \( \pi(s) \) has to be a polynomial of degree at most one. To obtain \( k \), the expression under the square root sign in Eq. (25) can be arranged to be the square of a polynomial of first degree [44]. This is possible only if its discriminant is zero. Finally, the energy eigenvalues are obtained by comparing Eqs. (22) and (26).

IV. SOLUTIONS OF THE RADIAL AND ANGLE-DEPENDENT EQUATIONS

A. The \( D \)-dimensional angular equations

At the beginning, we rewrite Eqs. (13) and (14) representing the angular wave equations in the following simple forms [21, 22]:

\[
\frac{d^2 H(\theta_j)}{d\theta_j^2} + (j-1)\cot\theta_j \frac{dH(\theta_j)}{d\theta_j} + \left( \Lambda_j - \frac{\Lambda_j-1}{\sin^2 \theta_j} \right) H(\theta_j) = 0, \quad j \in [2, D-2], \quad D > 3, \quad (27)
\]

\[
\frac{d^2 H(\theta_{D-1})}{d\theta_{D-1}^2} + (D-2)\cot\theta_{D-1} \frac{dH(\theta_{D-1})}{d\theta_{D-1}} + \left[ l(l+D-2) - \frac{\Lambda_{D-2}}{\sin^2 \theta_{D-1}} \right] H(\theta_{D-1}) = 0, \quad (28)
\]

where \( \Lambda_p = l_p(l_p+p-1), \quad p = j-1, \quad j \) with \( l_p = 0, 1, 2, \cdots \) are the angular quantum numbers in \( D \)-dimensions. Employing \( s = \cos \theta_j \), we transform Eq. (27) to the associated-Legendre equation

\[
\frac{d^2 H(s)}{ds^2} - \frac{js}{1-s^2} \frac{dH(s)}{ds} + \Lambda_j - \Lambda_j-1 - \Lambda_j s^2 \left( 1-s^2 \right)^{-2} H(s) = 0, \quad j \in [2, D-2], \quad D > 3. \quad (29)
\]

By comparing Eqs. (29) and (17), the corresponding polynomials are obtained

\[
\tilde{\tau}(s) = -js, \quad \sigma(s) = 1 - s^2, \quad \sigma'(s) = -\Lambda_j s^2 + \Lambda_j - \Lambda_j-1. \quad (30)
\]

Inserting the above expressions into Eq. (25) and taking \( \sigma'(s) = -2s \), one obtains the following function:

\[
\pi(s) = \frac{(j-2)}{2} s \pm \sqrt{\left( \frac{(j-2)}{2} \right)^2 + \Lambda_j - k} s^2 + k - \Lambda_j + \Lambda_j-1. \quad (31)
\]

Following the method, the polynomial \( \pi(s) \) is found to have the following four possible values:

\[
\pi(s) = \begin{cases} 
\left( \frac{j-2}{2} + \tilde{\Lambda}_{j-1} \right) s & \text{for } k_1 = \Lambda_j - \Lambda_{j-1}, \\
\left( \frac{j-2}{2} - \tilde{\Lambda}_{j-1} \right) s & \text{for } k_1 = \Lambda_j - \Lambda_{j-1}, \\
\frac{(j-2)}{2} s + \tilde{\Lambda}_{j-1} & \text{for } k_2 = \Lambda_j + \left( \frac{j-2}{2} \right)^2, \\
\frac{(j-2)}{2} s - \tilde{\Lambda}_{j-1} & \text{for } k_2 = \Lambda_j + \left( \frac{j-2}{2} \right)^2, 
\end{cases} \quad (32)
\]
where $\tilde{\Lambda}_p = l_p + (p-1)/2$, with $p = j - 1$, $j$ and $j \in [2, D - 2]$, $D > 3$. Imposing the condition $\tau'(s) < 0$, where $\tau' = \tau'(l_{j-1})$, we may select the following physical solutions:

$$k_1 = \Lambda_j - \Lambda_{j-1} \quad \text{and} \quad \pi(s) = -l_{j-1}s, \ j \in [2, D - 2], D > 3,$$

which yields

$$\tau(s) = -2(1 + \tilde{\Lambda}_{j-1})s. \quad (34)$$

Using Eqs. (22) and (26), give

$$\lambda_{n_j} = 2n_j(1 + \tilde{\Lambda}_{j-1}) + n_j(n_j - 1), \quad (35)$$

$$\lambda = \Lambda_j - \Lambda_{j-1} - l_{j-1}, \quad (36)$$

and setting $\lambda = \lambda_{n_j}$, we obtain

$$n_j = l_j - l_{j-1}. \quad (37)$$

Substituting Eqs. (30), (33) and (34) into Eqs. (20)-(21) and (23)-(24), give

$$\phi(s) = (1 - s^2)^{l_{j-1}/2}, \quad \rho(s) = (1 - s^2)^{l_{j-1}+(j-2)/2}. \quad (38)$$

Further, substituting the weight function $\rho(s)$ given in Eq. (38) into the Rodrigues relation (23) gives

$$y_{n_j}(s) = A_{n_j} \left(1 - s^2\right)^{-\tilde{\Lambda}_{j-1}} \frac{d^{n_j}}{ds^{n_j}} \left(1 - s^2\right)^{n_j+\tilde{\Lambda}_{j-1}}, \quad (39)$$

where $A_{n_j}$ is the normalization factor. Finally, the angular wave functions are

$$H_{n_j}(\theta_j) = N_{n_j} \left(\sin \theta_j\right)^{l_{j-1}-n_j} P_n^{(l_j-n_j+(j-2)/2, l_j-n_j+(j-2)/2)}(\cos \theta_j), \quad (40)$$

where the quantum numbers $n_j$ are defined in Eq. (37) and the normalization factor is

$$N_{n_j} = \sqrt{\frac{(2l_j + j - 1)!}{2\Gamma(l_j + l_{j-1} + j - 2)}}, \ j \in [2, D - 2], D > 3. \quad (41)$$

Likewise, in solving Eq. (28), we introduce a new variable $s = \cos \theta_{D-1}$. Thus, we can also rearrange it as the universal associated-Legendre differential equation

$$\frac{d^2 H(s)}{ds^2} - \frac{(D - 1)s}{1 - s^2} \frac{d H(s)}{ds} + \frac{\nu(1 - s^2) - \Lambda_{D-2}}{(1 - s^2)^2} H(s) = 0, \quad (42)$$

where

$$\nu = l(l + D - 2). \quad (43)$$
Equation (42) has been recently solved in 2D and 3D by the N-U method in [45,46]. By comparing Eqs. (42) and (17), the corresponding polynomials are obtained

\[
\tilde{\tau}(s) = -(D - 1)s, \quad \sigma(s) = 1 - s^2, \quad \tilde{\sigma}(s) = -\nu s^2 + \nu - \Lambda_{D-2}.
\] (44)

Inserting the above expressions into Eq. (25) and taking \(\sigma'(s) = -2s\), one obtains the following function:

\[
\pi(s) = \frac{(D - 3)}{2} s \pm \sqrt{\left(\frac{D - 3}{2}\right)^2 + \nu} s^2 + k - \nu + \Lambda_{D-2},
\] (45)

which gives the possible solutions:

\[
\pi(s) = \begin{cases} 
\left(\frac{D - 3}{2} + m'\right)s & \text{for } k_1 = \nu - \Lambda_{D-2}, \\
\left(\frac{D - 3}{2} - m'\right)s & \text{for } k_1 = \nu - \Lambda_{D-2}, \\
\left(\frac{D - 3}{2}\right)s + m' & \text{for } k_2 = \nu + \left(\frac{D - 3}{2}\right)^2, \\
\left(\frac{D - 3}{2}\right)s - m' & \text{for } k_2 = \nu + \left(\frac{D - 3}{2}\right)^2,
\end{cases}
\] (46)

where \(m' = l_{D-2} + \frac{D - 3}{2}\). Imposing the condition \(\tau'(s) < 0\), where \(\tau' = \tau'(l_{D-2})\), one may select:

\[
k_1 = \nu - \Lambda_{D-2} \quad \text{and} \quad \pi(s) = -l_{D-2}s,
\] (47)

which yields

\[
\tau(s) = -2(1 + m')s.
\] (48)

The following expressions for \(\lambda\) are obtained, respectively,

\[
\lambda = \lambda_{n_{D-1}} = 2n_{D-1}(1 + m') + n_{D-1}(n_{D-1} - 1),
\] (49)

\[
\lambda = l(l + D - 2) - l_{D-2}(l_{D-2} + D - 2),
\] (50)

and from \(\lambda = \lambda_{n_{D-1}}\), we obtain the angular momentum quantum number:

\[
l = l_{D-2} + n_{D-1}.
\] (51)

where \(n_{D-1} = n\) and \(l_1 = m\) in 3D space [46]. Using Eqs. (20)-(21) and (23)-(24), we obtain

\[
\phi(s) = \left(1 - s^2\right)^{l_{D-2}/2}, \quad \rho(s) = \left(1 - s^2\right)^{m'}.\]
(52)

The Rodrigues relation (23) gives the following wave functions:

\[
y_{n_{D-1}}(s) = B_{n_{D-1}} \left(1 - s^2\right)^{-m'} \frac{d^{n_{D-1}}}{ds^{n_{D-1}}} \left(1 - s^2\right)^{n_{D-1} + m'},
\] (53)
where $B_{n_{D-1}}$ is the normalization factor. Finally, the angular wave functions are

$$H_{n_{D-1}}(\theta_{D-1}) = N_{n_{D-1}} (\sin \theta_{D-1})^{l_{D-2}} P_{n_{D-1}}^{(m',m')}(\cos \theta_{D-1}), \quad (54)$$

where the normalization factor is

$$N_{n_{D-1}} = \sqrt{\frac{(2n_{D-1} + 2m' + 1) n_{D-1}!}{2\Gamma(n_{D-1} + 2m')}}. \quad (55)$$

with $m'$ is defined after Eq. (46).

B. Bound States of the $D$-dimensional Radial Equation

The $D$-dimensional Klein-Gordon radial energy eigenvalue equation with the vector and scalar general Hulthén-type potentials can be rewritten as

$$g''(r) + \frac{1}{r^2} \left[ E_{nl}^2 - m_0^2 + \frac{2 (m_0 S_0 + E_{nl} V_0) e^{-ar}}{1 - q e^{-ar}} \right] - \frac{\frac{1}{4} (D + 2l - 1)(D + 2l - 3) \alpha^2 e^{-ar} + (S_0^2 - V_0^2) e^{-2ar}}{(1 - q e^{-ar})^2} g(r), \quad g(0) = 0, \quad (56)$$

On account of the wave function $g(r)$ satisfying the standard bound state condition, which is when $r \to \infty$, the wave function $g(r) \to 0$. Equation (56) can be further transformed by using a new variable $s = q e^{-ar}$ ($r \in [0, \infty), \quad s \in [q, 0])$,

$$g''(s) + \frac{(1 - s)}{s(1 - s)} g'(s) + \frac{1}{[s(1 - s)]^2} \left[ -\varepsilon_{nl}^2 + (\beta_1 - \gamma + 2\varepsilon_{nl}) s - (\beta_1 + \beta_2 + \varepsilon_{nl}^2) s^2 \right] g(s) = 0, \quad (57)$$

where

$$\varepsilon_{nl} = \sqrt{m_0^2 - E_{nl}^2} / \alpha, \quad \beta_1 = \frac{2 (m_0 S_0 + E_{nl} V_0)}{\alpha^2 q}, \quad \beta_2 = \frac{S_0^2 - V_0^2}{\alpha^2 q^2}, \quad \gamma = \frac{(D + 2l - 1)(D + 2l - 3)}{4q}. \quad (58)$$

For bound states, $E_{nl} \leq m_0, \quad \varepsilon_{nl} \geq 0 \quad [17,47,48]$. Comparing Eqs. (57) and (17), the corresponding polynomials are obtained:

$$\bar{\tau}(s) = 1 - s, \quad \sigma(s) = s(1 - s), \quad \bar{\sigma}(r) = -\varepsilon_{nl}^2 + (\beta_1 - \gamma + 2\varepsilon_{nl}) s - (\beta_1 + \beta_2 + \varepsilon_{nl}^2) s^2. \quad (59)$$

The substitution of Eq. (59) into Eq. (25) and taking $\sigma'(s) = 1 - 2s$, give the polynomial:

$$\pi(s) = -\frac{s}{2} \pm \frac{1}{2} \sqrt{[1 + 4(\beta_1 + \beta_2 + \varepsilon_{nl}^2 - k)] s^2 + 4(k - \beta_1 + \gamma - 2\varepsilon_{nl}^2) s + 4\varepsilon_{nl}^2}. \quad (60)$$
If the expression under the square root in Eq. (60) is set equal to zero and solved for \( k \), we obtain:

\[
k = \beta_1 - \gamma \pm \varepsilon_{nl} \sqrt{1 + 4(\beta_2 + \gamma)}.
\]

(61)

In view of that, we arrive at the following four possible functions of \( \pi(s) \):

\[
\pi(s) = \begin{cases} 
-\frac{s}{2} + \varepsilon_{nl} - \left[ \varepsilon_{nl} - \frac{1}{2} \sqrt{1 + 4(\beta_2 + \gamma)} \right] s & \text{for } k_1 = \beta_1 - \gamma + \varepsilon_{nl} \sqrt{1 + 4(\beta_2 + \gamma)}, \\
-\frac{s}{2} - \varepsilon_{nl} + \left[ \varepsilon_{nl} + \frac{1}{2} \sqrt{1 + 4(\beta_2 + \gamma)} \right] s & \text{for } k_1 = \beta_1 - \gamma + \varepsilon_{nl} \sqrt{1 + 4(\beta_2 + \gamma)}, \\
-\frac{s}{2} + \varepsilon_{nl} - \left[ \varepsilon_{nl} + \frac{1}{2} \sqrt{1 + 4(\beta_2 + \gamma)} \right] s & \text{for } k_2 = \beta_1 - \gamma - \varepsilon_{nl} \sqrt{1 + 4(\beta_2 + \gamma)}, \\
-\frac{s}{2} - \varepsilon_{nl} + \left[ \varepsilon_{nl} - \frac{1}{2} \sqrt{1 + 4(\beta_2 + \gamma)} \right] s & \text{for } k_2 = \beta_1 - \gamma - \varepsilon_{nl} \sqrt{1 + 4(\beta_2 + \gamma)}. 
\end{cases}

(62)

The correct value of \( \pi(s) \) is chosen such that the function \( \tau(s) \) in Eq. (21) must have negative derivative [44]. So we can select the physical values to be

\[
k = \beta_1 - \gamma - \varepsilon_{nl} \sqrt{1 + 4(\beta_2 + \gamma)} \quad \text{and} \quad \pi(s) = -\frac{s}{2} + \varepsilon_{nl} - \left[ \varepsilon_{nl} + \frac{1}{2} \sqrt{1 + 4(\beta_2 + \gamma)} \right] s,
\]

(63)

which yield

\[
\tau(s) = 1 + 2\varepsilon_{nl} - 2 \left[ 1 + \varepsilon_{nl} + \frac{1}{2} \sqrt{1 + 4(\beta_2 + \gamma)} \right] s,
\]

\[
\tau'(s) = -2 \left[ 1 + \varepsilon_{nl} + \frac{1}{2} \sqrt{1 + 4(\beta_2 + \gamma)} \right] < 0.
\]

(64)

Using Eqs. (22) and (26), the following expressions for \( \lambda \) are obtained, respectively,

\[
\lambda = \lambda_n = n^2 + \left[ 1 + 2\varepsilon_{nl} + \sqrt{1 + 4(\beta_2 + \gamma)} \right] n, \quad (n = 0, 1, 2, \ldots), \quad \text{(65)}
\]

\[
\lambda = \beta_1 - \gamma - \frac{1}{2}(1 + 2\varepsilon_{nl}) \left[ 1 + \sqrt{1 + 4(\beta_2 + \gamma)} \right], \quad \text{(66)}
\]

where \( n \) is the radial quantum number. Solving the last two equations and using \( \beta_2 = \delta_+^2 - \delta_-^2 - \gamma \), give

\[
\varepsilon_{nl}^{(D)} = \frac{\beta_1 - \gamma - n^2 - (2n + 1)\delta_\pm}{2(n + \delta_\pm)} = \frac{4q[\beta_1 - n^2 - (2n + 1)\delta_\pm - (D + 2l - 1)(D + 2l - 3)]}{8q(n + \delta_\pm)}, \quad (n = 0, 1, 2, \ldots),
\]

\[
= \frac{2q(m_{0}S_{0} + E_{nl}V_{0}) + S_{0}^2 - V_{0}^2}{2q^2a^2(n + \delta_\pm)} - \frac{n + \delta_\pm}{2}, \quad (n = 0, 1, 2, \ldots),
\]

\[
\delta_\pm = \frac{1}{2} \left( 1 \pm \frac{a}{q} \right), \quad a = \sqrt{q^2 + \frac{4(S_{0}^2 - V_{0}^2)}{a^2} + q(D + 2l - 1)(D + 2l - 3)}, \quad (l = 0, 1, 2, \ldots),
\]

(67)
with $\delta = \delta_+$ for $q > 0$ and $\delta = \delta_-$ for $q < 0$. Furthermore, from setting $\varepsilon^{(D)}_{nl} = \varepsilon_{nl}$ and using Eqs. (58) and (67), we obtain the energy equation

$$
\sqrt{m_0^2 - E_{nl}^2} = \frac{2q(m_0 S_0 + E_{nl} V_0) + S_0^2 - V_0^2}{2q^2 \alpha(n + \delta)} - \frac{\alpha(n + \delta)}{2},
$$

(68)

or equivalently the explicit expression of the energy eigenvalues

$$
E^{(D)}_{nl} = \frac{\eta_{nl} V_0}{2q} \pm \kappa_{nl} \sqrt{m_0^2 + \left(\frac{\eta_{nl}}{4q}\right)^2 - \left(\frac{\eta_{nl}}{4q}\right)^2},
$$

$$
\eta_{nl} = \frac{4(V_0^2 - S_0^2) + \kappa_{nl}^2 - 8q m_0 S_0}{4V_0^2 + \kappa_{nl}^2},
$$

$$
\kappa_{nl} = q \alpha (2n + 1) \pm \sqrt{q^2 \alpha^2 + 4(S_0^2 - V_0^2) + q \alpha^2 (D + 2l - 1)(D + 2l - 3)},
$$

(69)

and

$$
q^2 \alpha^2 + q \alpha^2 (D + 2l - 2)^2 + 4S_0^2 \geq q \alpha^2 + 4V_0^2,
$$

(70)

is a constraint over the potential parameters. In the case of pure vector potential ($S_0 = 0, V_0 \neq 0$), the energy equation (69) reduces to

$$
E^{(D)}_{nl} = \frac{V_0}{2q} \pm \kappa_{nl} \sqrt{m_0^2 + \left(\frac{1}{4q}\right)^2 - \left(\frac{1}{4q}\right)^2},
$$

$$
\kappa_{nl} = q \alpha (2n + 1) \pm \sqrt{q^2 \alpha^2 + q \alpha^2 (D + 2l - 1)(D + 2l - 3) - 4V_0^2},
$$

(71)

with the constraint

$$
q^2 \alpha^2 + q \alpha^2 (D + 2l - 2)^2 \geq q \alpha^2 + 4V_0^2,
$$

(72)

for real bound states. From the above result, it is not difficult to conclude that the two energy solutions are valid for the particle and the second one corresponds to the anti-particle energy.

The restriction that gives the critical coupling value leads to the result

$$
n \leq \frac{1}{q \alpha} \left( \sqrt{4q^2 m_0^2 - V_0^2} - \sqrt{\frac{q^2 \alpha^2}{4} [q + (D + 2l - 1)(D + 2l - 3)] - V_0^2} \right) - \frac{1}{2},
$$

(73)

i.e., there are only finitely many eigenvalues. In order that at least one level might exist, it is necessary that the inequality:

$$
q \alpha + \sqrt{q \alpha^2 [q + (D + 2l - 1)(D + 2l - 3)] - 4V_0^2} \leq 2 \sqrt{4q^2 m_0^2 - V_0^2},
$$

(74)

13
is fulfilled. As can be seen from Eq. (73), there are at most only two lower-lying states \((n = 0, 1)\) for the Klein-Gordon particle of mass unity when the parameter \(\alpha = 1\) and \(q = \pm 1\) for any arbitrary value of \(V_0\). Therefore, we have the inequality:

\[
 n \leq \pm \left( \sqrt{4 - V_0^2} - \sqrt{\left( \frac{D + 2l - 2}{2} \right)^2 - V_0^2} \right) - \frac{1}{2} \tag{75}
\]

For instance, if one selects \(V_0 = (D + 2l - 2)/2\), then the real bound states must be restricted by \(n \leq \left( \sqrt{16 - (D + 2l - 2)^2} - 1 \right) / 2\).

Having solved the \(D\)-dimensional Klein-Gordon equation for scalar and vector general Hulthén-type potentials, we should make some remarks.

(i) For \(s\)-wave \((l = 0)\), the exact energy eigenvalues of the 1D Klein-Gordon equation becomes

\[
 E_n = \frac{\eta_n V_0}{2q} \pm \frac{\kappa_n}{4q(4V_0^2 + \kappa_n^2)} \sqrt{\left[ \kappa_n^2 + 4 \left( V_0^2 - S_0^2 \right) \right] \left[ (2S_0 + 4qm_0)^2 - \kappa_n^2 - 4V_0^2 \right]}, \tag{76}
\]

where

\[
\kappa_n = q\alpha(2n + 1) + \sqrt{q^2\alpha^2 + 4(S_0^2 - V_0^2)}, \quad \eta_n = \frac{4(V_0^2 - S_0^2) + \kappa_n^2 - 8qm_0S_0}{4V_0^2 + \kappa_n^2}. \tag{77}
\]

In order that at least one level might exist, it is necessary that the inequality

\[
16q^2m_0^2 \geq \eta_n^2 \left( 4V_0^2 + \kappa_n^2 \right), \quad q^2\alpha^2 + 4S_0^2 \geq 4V_0^2, \tag{78}
\]

is fulfilled. In the case of pure vector potential \((S_0 = 0, V_0 \neq 0)\), the energy spectrum

\[
 E_n = \frac{V_0}{2q} \pm \left[ q\alpha(2n + 1) + \sqrt{q^2\alpha^2 - 4V_0^2} \right] \sqrt{\frac{m_0^2}{4V_0^2 + \left[ q\alpha(2n + 1) + \sqrt{q^2\alpha^2 - 4V_0^2} \right]^2} - \frac{1}{16q^2}, \tag{79}
\]

with

\[
16q^2m_0^2 \geq 4V_0^2 + \left[ q\alpha(2n + 1) + \sqrt{q^2\alpha^2 - 4V_0^2} \right]^2, \quad q\alpha \geq 2V_0. \tag{80}
\]

We notice that the result given in Eq. (79) is identical to Eq. (31) of Ref. [30]. There are only two lower-lying states \((n = 0, 1)\) for the Klein-Gordon particle of a rest mass \(m_0 = 1.0\) and \(\alpha = 1.0\). For example, one may calculate the ground state energy for the coupling strength \(V_0 = q\alpha/2\) as

\[
 E_0 = \frac{V_0}{2q} \left[ 1 \pm \sqrt{\frac{2m_0q^2}{V_0^2} - 1} \right]. \tag{81}
\]
Further, in the case of pure scalar potential \((V_0 = 0, S_0 \neq 0)\), the energy spectrum

\[
E_n = \pm \frac{1}{4q} \frac{1}{q\alpha(2n + 1) + \sqrt{q^2 \alpha^2 + 4S_0^2}} \sqrt{[q\alpha(2n + 1) + \sqrt{q^2 \alpha^2 + 4S_0^2}]^2 - 4S_0^2} \times \sqrt{(2S_0 + 4m_0^2)^2 - [q\alpha(2n + 1) + \sqrt{q^2 \alpha^2 + 4S_0^2}]^2}. \tag{82}
\]

In the above equation, all bound states appear in pairs, with energies \(\pm E_n\). Since the Klein-Gordon equation is independent of the sign of \(E_n\) for scalar potentials, the wavefunctions become the same for both energy values. We notice that Eq. (82) is identical to Eq. (24) of Ref. [47] obtained by N-U method and to Eq. (20) of Ref. [48] obtained by supersymmetric method. If the range parameter \(\alpha\) is chosen to be \(\alpha = 1/\lambda_c\), where \(\lambda_c = \hbar/m_0c = 1/m_0\) denotes the Compton wavelength of the Klein-Gordon particle. It can be seen easily that while \(S_0 \rightarrow 0\) in ground state \((n = 0)\), all energy eigenvalues tend to the value \(E_0 \approx 0.866 m_0\).

(ii) For \(D = 3\), the mixed scalar and vector Hulthén potentials \((q = 1, l \neq 0)\), the energy eigenvalues are

\[
E_{nl} = \frac{\eta_{nl} V_0}{2} \pm \left[ \alpha(2n + 1) + \sqrt{\alpha^2 + 4B} \right] \sqrt{\frac{m_0^2}{4V_0^2 + [\alpha(2n + 1) + \sqrt{\alpha^2 + 4B}]^2} - \left( \frac{\eta_{nl}}{4} \right)^2}, \tag{83}
\]

where

\[
\eta_{nl} = \frac{4(V_0^2 - S_0^2) + \left[ \alpha(2n + 1) + \sqrt{\alpha^2 + 4B} \right]^2 - 8m_0^2S_0}{4V_0^2 + \left[ \alpha(2n + 1) + \sqrt{\alpha^2 + 4B} \right]^2}, \quad B = S_0^2 - V_0^2 + \alpha^2 l(l + 1). \tag{84}
\]

In order that at least one level might exist, it is necessary that the inequality

\[
16m_0^2 \geq \eta_{nl}^2 \left\{ 4V_0^2 + \left[ \alpha(2n + 1) + \sqrt{\alpha^2 + 4B} \right]^2 \right\}, \quad \alpha^2 + 4B \geq 0, \tag{85}
\]

is fulfilled. In the case of pure vector \((S_0 = 0, V_0 \neq 0)\), the energy eigenvalues are

\[
E_{nl} = \frac{V_0}{2} \pm \left[ \alpha(2n + 1) + \sqrt{\alpha^2(2l + 1)^2 - 4V_0^2} \right] \times \sqrt{\frac{m_0^2}{4V_0^2 + [\alpha(2n + 1) + \sqrt{\alpha^2(2l + 1)^2 - 4V_0^2}]^2} - \frac{1}{16}}, \tag{86}
\]

with the following constraint over the potential parameters:

\[
16m_0^2 \geq 4V_0^2 + \left[ \alpha(2n + 1) + \sqrt{\alpha^2(2l + 1)^2 - 4V_0^2} \right]^2,
\]

15
\[(2l + 1)\alpha \geq 2V_0.\] (87)

A preliminary analysis about the possibility of obtaining real bound states for the Klein-Gordon particle of mass unity is done for various dimensions as follows: when \(D = 3\), we have only two lower-lying real bound states \((n = 0, 1)\) for the angular quantum number \(l = 0\), one lower-lying state \((n = 0)\) for \(l = 1\) and no states for \(l \geq 2\). When \(D = 4\), we have two lower-lying states for \(l = 0\) and no states for \(l \geq 1\). Further, when \(D = 5\), we have one lower-lying state for \(l = 0\) and no states for \(l \geq 1\). Finally, when \(D \geq 6\), there are no real bound states for \(l \geq 0\), however, only scattering states will be possible.

(iii) When \(D = 3\) and \(l = 0\), the centrifugal term \(\frac{(D+2l-1)(D+2l-3)}{4r^2}\) = 0, and the approximation term \(\frac{(D+2l-1)(D+2l-3)\alpha e^{-\alpha r}}{4(1-\eta e^{-\alpha r})^2}\) = 0, too. Thus, letting \(l = 0\) and \(D = 3\) in Eq. (10), it reduces to the exact spectrum formula and normalized radial eigenfunctions of the Klein-Gordon equation for vector and scalar general Hulthén-type potentials:

\[
\sqrt{m_0^2 - E_n^2} = \frac{2qr_0(m_0S_0 + E_nV_0) + r_0(S_0^2 - V_0^2)}{2q^2(n + \delta) - \frac{n + \delta}{2r_0}}.
\]

\[
\delta = \frac{1}{2} \left[ 1 + \frac{1}{q} \sqrt{q^2 + 4r_0^2(S_0^2 - V_0^2)} \right], \quad (n = 0, 1, 2, 3, \ldots)
\] (88)

which gives

\[
E_n = \frac{\eta V_0}{2q} + \left( q\alpha(2n + 1) + \sqrt{q^2\alpha^2 + 4(S_0^2 - V_0^2)} \right)
\times \sqrt{\frac{m_0^2}{4V_0^2 + \left( q\alpha(2n + 1) + \sqrt{q^2\alpha^2 + 4(S_0^2 - V_0^2)} \right)^2} - \left( \frac{\eta}{4q} \right)^2}, \quad (n = 0, 1, 2, 3, \ldots)
\]

\[16q^2m_0^2 \geq \eta_n^2 \left[ 4V_0^2 + \left( q\alpha(2n + 1) + \sqrt{q^2\alpha^2 + 4(S_0^2 - V_0^2)} \right)^2 \right], \quad q^2\alpha^2 + 4(S_0^2 - V_0^2) \geq 0,
\]

\[
\eta_n = \frac{4(V_0^2 - S_0^2) + \left( q\alpha(2n + 1) + \sqrt{q^2\alpha^2 + 4(S_0^2 - V_0^2)} \right)^2}{4V_0^2 + \left( q\alpha(2n + 1) + \sqrt{q^2\alpha^2 + 4(S_0^2 - V_0^2)} \right)^2} - 8qm_0S_0.
\] (89)

(iv) In the case \(S_0 = V_0\) Hulthén potential, Eqs. (67) and (68) can be reduced to the relativistic energy equation:

\[
\sqrt{m_0^2 - E_R^2} = \frac{r_0V_0(m_0 + E_R)}{(n + \delta)} - \frac{n + \delta}{2r_0}, \quad \delta = \frac{(D + 2l - 1)}{2}, \quad (n, l = 0, 1, 2, \ldots).
\] (90)

which is Eq. (22) of Ref. [19].
(v) We discuss non-relativistic limit of the energy equation (90). When \( V_0 = S_0 \), Eq. (10) reduces to a Schrödinger-like equation for the potential \( 2V(r) \). In other words, the non-relativistic limit is the Schrödinger equation for the potential \(-2V_0e^{-r/r_0}/[1-e^{-r/r_0}]\). Hence, using the transformation \( m_0 + E_R \to 2m_0 \) and \( m_0 - E_R \to -E_{NR} \) [22], we obtain the non-relativistic energy equation:

\[
E_{NR} = -\frac{1}{8m_0\alpha^2} \left[ \frac{4m_0V_0 - \alpha^2(n + \delta)^2}{(n + \delta)} \right]^2, \quad \alpha = r_0^{-1}
\]  

(91)

which is Eq. (23) of Ref. [19] with \( \delta \) is given in Eq. (90). It is worthwhile to remark that Eq. (91) is identical to Eq. (59) of Ref. [30] when the potential is \( 2V(r) \) and \( \alpha \) becomes pure imaginary, i.e., \( \alpha \to i\alpha \).

Thus, in the weak coupling condition, \([ (n + \delta)/m_0r_0]^2 \ll 1, [V_0r_0/(n + \delta)]^2 \ll 1\), expanding the energy equation (90), retaining only the term containing the power of \((1/m_0r_0)^2\) and \((r_0V_0)^4\), we have the relativistic energy

\[
E_R \approx E_{NR} + m_0 + 4m_0 \left( \frac{V_0r_0}{q(n + \delta)} \right)^4,
\]

(92)

which is Eq. (24) of Ref. [19] with \( \delta \) is given in Eq. (90). The first term is the non-relativistic energy and third term is the relativistic approximation to energy.

(vi) For a more specific case where \( q = -1 \), the usual Hulthén potential is reduced to the shifted usual Woods-Saxon (WS) potential

\[
V(r) = -V_0 + \frac{V_0}{1 + e^{-r/r_0}}, \quad S(r) = -S_0 + \frac{S_0}{1 + e^{-r/r_0}}, \quad r_0 = \alpha^{-1},
\]

(93)

and hence the energy eigenvalues, Eq. (69), for the general WS-type potentials, i.e., \( q \to -q \) and then \( q = 1 \), become

\[
E_{nl}^{(D)} = -\xi_{nl}V_0 - \frac{\xi_{nl}V_0}{\sqrt{2\alpha^2 + 4(S_0^2 - V_0^2) - \alpha^2(D + 2l - 2)^2 - \alpha(2n + 1)}} \pm \sqrt{\frac{m_0^2}{4V_0^2 + \left[ 2\alpha^2 + 4(S_0^2 - V_0^2) - \alpha^2(D + 2l - 2)^2 - \alpha(2n + 1) \right]^2}} + \left( \frac{\xi_{nl}}{4} \right)^2,
\]

(94)

\[
16m_0^2 \geq \xi_{nl}^2 \left\{ 4V_0^2 + \sqrt{\frac{m_0^2}{4V_0^2 + \left[ 2\alpha^2 + 4(S_0^2 - V_0^2) - \alpha^2(D + 2l - 2)^2 - \alpha(2n + 1) \right]^2}} \right\}.
\]

(95)

where

\[
\xi_{nl} = \frac{4(V_0^2 - S_0^2) + \sqrt{2\alpha^2 + 4(S_0^2 - V_0^2) - \alpha^2(D + 2l - 2)^2 - \alpha(2n + 1)}}{4V_0^2 + \sqrt{2\alpha^2 + 4(S_0^2 - V_0^2) - \alpha^2(D + 2l - 2)^2 - \alpha(2n + 1)}} + 8m_0S_0.
\]
and the inequality
\[ 2\alpha^2 + 4S_0^2 \geq 4V_0^2 + \alpha^2(D + 2l - 2)^2, \] (96)
must be fulfilled. In the pure vector potential \((S_0 = 0, V_0 \neq 0)\), the energy eigenvalues are
\[ E_{nl}^{(D)} = -\frac{V_0}{2} \pm \left[ \sqrt{2\alpha^2 - \alpha^2(D + 2l - 2)^2 - 4V_0^2} - \alpha(2n + 1) \right] \]
\times \frac{m_0^2}{\sqrt{4V_0^2 + \left[ \sqrt{2\alpha^2 - \alpha^2(D + 2l - 2)^2 - 4V_0^2} - \alpha(2n + 1) \right]^2} - \frac{1}{16},}
\[ 16m_0^2 \geq 4V_0^2 + \left[ \sqrt{2\alpha^2 - \alpha^2(D + 2l - 2)^2 - 4V_0^2} - \alpha(2n + 1) \right]^2, \] (97)
where
\[ 2\alpha^2 \geq 4V_0^2 + \alpha^2(D + 2l - 2)^2. \] (98)

From the above equation, for any given \(\alpha\), the Klein-Gordon equation with the usual shifted WS potential has negative eigenvalues, i.e, \(E_{nl} < 0\). The given restriction in (99) imposes the critical coupling value and thus leads to the result
\[ n \leq \frac{1}{\alpha} \left( \frac{2\alpha^2}{4} \left[ 2 - (D + 2l - 2)^2 \right] - V_0^2 - \sqrt{4m_0^2 - V_0^2} - \frac{1}{2}, \right) \] (99)
i.e., there are only finitely many eigenvalues. Further, it is necessary that the inequality:
\[ \sqrt{\alpha^2 \left[ 2 - (D + 2l - 2)^2 \right] - 4V_0^2} \geq \alpha + 2\sqrt{4m_0^2 - V_0^2}, \] (100)
must be fulfilled.

Now, let us find the wave function \(y_{nl}(s)\), which is the polynomial solution of hypergeometric-type equation, we multiply Eq. (19) by the weight function \(\rho(s)\) so that it can be rewritten in self-adjoint form [36]
\[ [\omega(s)y_{nl}'(s)]' + \lambda \rho(s)y_{nl}(s) = 0. \] (101)
The weight function \(\rho(s)\) which satisfies Eqs. (24) and (102) has the form
\[ \rho(s) = s^{2\varepsilon_{nl}}(1 - s)^{\sqrt{1+4(\beta_2+\gamma)}}, \] (102)
and consequently from the Rodrigues relation (23), we obtain
\[ y_{nl}(s) = B_{nl}s^{-2\varepsilon_{nl}}(1 - s)^{-\sqrt{1+4(\beta_2+\gamma)}} \frac{d^n}{ds^n} \left[ s^{n+2\varepsilon_{nl}}(1 - s)^{n} \right]. \]
\( = B_{nl} P_n^{(2\varepsilon_{nl}; \sqrt{1+4(\beta_2+\gamma)})} (1 - 2s). \) 

On the other hand, inserting the values of \( \sigma(s), \pi(s) \) and \( \tau(s) \) given in Eqs. (59), (63) and (64) into Eq. (20), one can find the other part of the wave function as

\[
\phi(s) = s^{\varepsilon_{nl}} (1 - s)^{1/2} \frac{1}{1 + \sqrt{1+4(\beta_2+\gamma)}}.
\]

Hence, the wave function in Eq. (18) becomes

\[
g(s) = C_{nl} s^{\varepsilon_{nl}} (1 - s)^{1/2} \frac{1}{1 + \sqrt{1+4(\beta_2+\gamma)}} P_n^{(2\varepsilon_{nl}; \sqrt{1+4(\beta_2+\gamma)})} (1 - 2s)
\]

\[
= C_{nl} s^{\varepsilon_{nl}^{(D)}} (1 - s)^{1/2} \frac{1}{1 + \sqrt{1+4(\beta_2+\gamma)}} P_n^{(2\varepsilon_{nl}^{(D)},2\beta-1)} (1 - 2s), \ s \in [q, 0).
\]

Finally, the radial wave functions of the Klein-Gordon equation are obtained as

\[
R_{nl}(r) = N_{nl} r^{-(D-1)/2} e^{-\sqrt{m_0^2-E_0^2}r} (1 - q e^{-r/r_0})^{(q+a)/(2q)} P_n^{(2r_0 \sqrt{m_0^2-E_0^2}, a/q)} (1 - 2q e^{-r/r_0}),
\]

where \( N_{nl} \) is the radial normalization factor. Thus, the radial wave functions for the \( s \)-wave Klein-Gordon equation with pure vector Hulthén potential in 1D reduces to

\[
R_{nl}(r) = C_{nl} e^{-\sqrt{m_0^2-E_0^2}r} (1 - q e^{-r/r_0})^{(q+a)/(2q)} P_n^{(2r_0 \sqrt{m_0^2-E_0^2}, a/q)} (1 - 2q e^{-r/r_0}),
\]

where \( a = \sqrt{q^2 - 4V_0^2/\alpha^2} \) which is identical to Eq. (35) of Ref. [30]. Finally, from Eqs. (5) and (6), the total wave functions for the usual Hulthén potential is

\[
\psi_{\text{H}_{(D-1)/2}}^{(l_{D-1}-l)}(x) = N_{nl} r^{-(D-1)/2} e^{-\sqrt{m_0^2-E_0^2}r} (1 - e^{-r/r_0})^{\delta} P_n^{(2r_0 \sqrt{m_0^2-E_0^2}, 2\delta-1)} (1 - 2e^{-r/r_0})
\]

\[
\frac{1}{\sqrt{2\pi}} \exp(\pm i l_1 \theta_1) \prod_{j=2}^{D-2} \sqrt{\frac{(2l_j + j - 1) n_j!}{2\Gamma(l_j + l_{j-1} + j - 2)}} (\sin \theta_j)^{l_j-n_j} P_{n_j}^{(l_j-n_j+2(l_j-n_j)+2(2)/2)}(\cos \theta_j)
\]

\[
\sqrt{\frac{(2n_{D-1} + 2m' + 1) n_{D-1}!}{2\Gamma(n_{D-1} + 2m')}} (\sin \theta_{D-1})^{l_{D-1}} P_{n_{D-1}}^{(m',m')} (\cos \theta_{D-1}),
\]

where

\[
\delta = \frac{1}{2} \left( 1 + \sqrt{(D + 2l - 2)^2 + 4r_0^2(S_0^2 - V_0^2)} \right), \ (l = 0, 1, 2, \cdots).
\]

V. CONCLUSIONS

We have analytically found an approximate bound state energy eigenvalues and their corresponding wave functions of the \( D \)-dimensional Klein-Gordon equation for the spin-zero particle with the scalar and vector general Hulthén-type potentials using the N-U
method. The analytic energy equation and the wave functions expressed in terms of Jacobi polynomials and can be reduced to their well-known 1D and 3D solutions. The relativistic energies $E_{nl}$ in $D$-dimensions defined explicitly in Eq. (68) is for the particle and anti-particle energies for a given dimension $D$, quantum numbers $n$ and $l$ and also coupling constants satisfying the given particular constraints. For pure attractive scalar potential, all bound states appear in pairs, with energies $\pm E_{nl}$. Since the Klein-Gordon equation is independent of the sign of $E_{nl}$ for scalar potential, the wave functions become the same for both energy values. When $l = 0$, the results in this work reduce to exact solution of bound states of $s$-wave Klein-Gordon equation with vector and scalar general Hulthén-type potentials. A preliminary analysis about the possibility of obtaining finite number states for the Klein-Gordon particle of mass unity shows that the bound state energies are positive for the general Hulthén-type potentials and negative for general Woods-Saxon-type potentials.

In Table 1, we have obtained numerical results for the binding energies of ground state for mass unity 1D Klein Gordon equation with pure vector and pure scalar cases. In pure vector potential, for fixed coupling constants $V_0 = 0.25 m_0$ and various values of $\alpha = 0.5, 1.0$ and 2.0, most of the binding energies $E_0$ are decreasing with increasing deformation constant $q$. Also the values obtained with pure scalar potential, $S_0 = 0.25 m_0$, are also given in Table 1 for comparison. Furthermore, in Table 2, we give the binding energies for the $n = 1$, $\alpha = 0.5, 1.0$ and $n = 2, \alpha = 0.5$. Obviously, we have no bound states for $n = 1$ when $\alpha = 2.0$ and for $n = 2$ when $\alpha = 1.0$ and 2.0.

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TABLE I: Relativistic ground state binding energies, $E_0$, for a particle of mass, $m_0 = 1$ given by Eqs. (79) and (82) for pure vector and pure scalar potentials, respectively, as a function of $q$ for various values of $\alpha$.

| $q$ | $\alpha = 0.5$ | $\alpha = 1.0$ | $\alpha = 2.0$ | $\alpha = 0.5$ | $\alpha = 1.0$ | $\alpha = 2.0$ |
|-----|----------------|----------------|----------------|----------------|----------------|----------------|
| 0.1 | $-$            | $-$            | $-$            | 0.755639       | 0.947484       | 0.946410       |
| 0.5 | $-$            | 0.911438       | 0.500000       | 0.929812       | 0.996314       | 0.645297       |
| 1.0 | 0.820971       | 0.970804       | 0.250000       | 0.986425       | 0.964541       | 0.480683       |
| 1.5 | 0.991673       | 0.940945       | 0.166683       | 0.998606       | 0.941246       | 0.398341       |
| 2.0 | 0.999840       | 0.923893       | 0.125000       | 0.999903       | 0.926256       | 0.347332       |
| 2.5 | 0.999140       | 0.913089       | 0.100000       | 0.998350       | 0.916087       | 0.311871       |
| 5.0 | 0.988667       | 0.890301       | 0.050000       | 0.988537       | 0.892966       | 0.222136       |
| 7.5 | 0.982893       | 0.882371       | 0.033334       | 0.982996       | 0.884414       | 0.181787       |
| 10  | 0.979613       | 0.878345       | 0.025000       | 0.979771       | 0.879777       | 0.157607       |

$^a$For real bound states $q^2\alpha^2 \geq 0.25.$
TABLE II: Relativistic binding energies of the excited states, $E_1$ and $E_2$, for a particle of mass, $m_0 = 1$ given by Eqs. (79) and (82) for pure vector and pure scalar potentials, respectively, as a function of $q$ for various values of $\alpha$.

| $n = 1^a$ | $n = 2$ | $n = 1^b$ | $n = 2$ |
|-----------|---------|-----------|---------|
| $q$       | $\alpha = 0.5$ | $\alpha = 1.0$ | $\alpha = 0.5$ | $\alpha = 1.0$ | $\alpha = 0.5$ |
| 0.1       | –       | –         | -       | 0.995674 | 0.771938 | 0.922413 |
| 0.5       | –       | 0.830948  | -       | 0.984202 | 0.571823 | 0.829156 |
| 1.0       | 0.996421| 0.347292  | 0.880588| 0.954903 | 0.450227 | 0.779514 |
| 1.5       | 0.949420| 0.229259  | 0.769589| 0.935491 | 0.381304 | 0.752337 |
| 2.0       | 0.928534| 0.171407  | 0.737131| 0.922604 | 0.336188 | 0.735135 |
| 2.5       | 0.916027| 0.136934  | 0.719697| 0.913600 | 0.303882 | 0.723317 |
| 5.0       | 0.891025| 0.068342  | 0.688449| 0.892282 | 0.219316 | 0.695608 |
| 7.5       | 0.882692| 0.045546  | 0.678994| 0.884103 | 0.180255 | 0.684996 |
| 10        | 0.878525| 0.034156  | 0.674438| 0.879801 | 0.156613 | 0.679406 |

$^a$Pure vector case with coupling constant $V_0 = 0.25$.

$^b$Pure scalar case with coupling constant $S_0 = 0.25$. 
