\textbf{Λ-SUBMODULES OF FINITE INDEX OF ANTICYCLOTOMIC PLUS AND MINUS SELMER GROUPS OF ELLIPTIC CURVES}

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\textbf{Abstract.} Let $p$ be an odd prime and $K$ an imaginary quadratic field where $p$ splits. Under appropriate hypotheses, Bertolini showed that the Selmer group of a $p$-ordinary elliptic curve over the anticyclotomic $\mathbb{Z}_p$-extension of $K$ does not admit any proper Λ-submodule of finite index, where Λ is a suitable Iwasawa algebra. We generalize this result to the plus and minus Selmer groups (in the sense of Kobayashi) of $p$-supersingular elliptic curves. In particular, in our setting the plus/minus Selmer groups have Λ-corank one, so they are not Λ-cotorsion. As an application of our main theorem, we prove results in the vein of Greenberg–Vatsal on Iwasawa invariants of $p$-congruent elliptic curves, extending to the supersingular case results for $p$-ordinary elliptic curves due to Hatley–Lei.

\section{Introduction}

When studying Selmer groups in the context of Iwasawa theory, it is often desirable to show that these Selmer groups have no proper Λ-submodules of finite index or, equivalently, that their Pontryagin duals have no nontrivial finite Λ-submodules, where Λ is an appropriate $p$-adic Iwasawa algebra for a prime number $p$ (to fix ideas, in this introduction we can take Λ to be the $\mathbb{Z}_p$-algebra $\mathbb{Z}_p[[X]]$ of formal power series over the $p$-adic integers $\mathbb{Z}_p$). For instance, a finitely generated Λ-module $M$ (which, in our context, will always be the Pontryagin dual of a suitable Selmer group) admits a map

$$M \rightarrow \Lambda^{\oplus r} \oplus \bigoplus_{i=1}^{s} \Lambda/(p^{a_i}) \oplus \bigoplus_{j=1}^{t} \Lambda/(F_{j}^{n_j})$$

with finite kernel and cokernel, for integers $r, s, t \geq 0$, $a_i, n_j \geq 1$ and irreducible Weierstrass polynomials $F_j \in \mathbb{Z}_p[X]$. If $M$ has no nontrivial finite Λ-submodules, then this map is in fact injective.

Such non-existence theorems have played a crucial role in certain strategies for proving cases of the Iwasawa Main Conjecture, as in \cite{9}, \cite{13}, \cite{18}, and by now they have been studied in great generality (see, e.g., \cite{12}), although many specific cases of interest remain open to inquiry.

In \cite{2} and \cite{3}, Bertolini approached this question for the Selmer groups attached to rational elliptic curves with good ordinary reduction at a prime $p$ along the anticyclotomic $\mathbb{Z}_p$-extension of a suitable imaginary quadratic field $K$, which is the unique $\mathbb{Z}_p$-extension of $K$ that is Galois and non-abelian over $\mathbb{Q}$.

The splitting behavior in $K$ of $p$ and of the prime factors of the conductor $N$ of $E$ exerts a tremendous influence on the structure of the Selmer groups that one can attach to $E$ and $K_\infty$.

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In particular, when $N$ is divisible by an odd number of primes that are inert in $K$ the situation is remarkably similar to the cyclotomic (over $\mathbb{Q}$ or over $K$) setting. When the number of such primes is even, however, the sign of the functional equation for the relevant $L$-function is $-1$, which forces the Selmer groups to grow and thus become more complicated (see, e.g., [2] and [23]).

Another situation in which the Iwasawa theory for elliptic curves is more delicate is when $E$ has good supersingular reduction at $p$. In this case, even in the cyclotomic setting over $\mathbb{Q}$ the usual Selmer groups fail to satisfy a control theorem as in [26], so the Selmer group over the full cyclotomic extension is unable to provide bounds on Selmer coranks at finite layers of the cyclotomic tower. Thus, for supersingular primes, it becomes necessary to consider special restricted plus/minus Selmer groups that turn out to be more amenable to classical arguments. This general program was first proposed and carried out over $\mathbb{Q}$ by Kobayashi in [22] and has admitted since then many generalizations such as [19] and [16]. It is worth remarking that the $p$-adic analytic counterpart of Kobayashi’s theory was provided by Pollack in [30].

In this paper we take up the task of generalizing Bertolini’s results ([2]) to the case of elliptic curves with good supersingular reduction at $p$. Thus, we consider the complications that arise both from having supersingular reduction and from working over the anticyclotomic $\mathbb{Z}_p$-extension of an imaginary quadratic field satisfying the Heegner hypothesis with respect to the conductor of our elliptic curve.

1.1. Setup and notation. Throughout this article, $p$ will denote an odd prime number. Let $E/\mathbb{Q}$ be an elliptic curve of conductor $N$ with good supersingular reduction at $p$ and $a_p(E) = 0$, where $a_p(E)$ is the $p$-th Fourier coefficient of the weight 2 newform attached to $E$ by modularity.

Let $K$ be an imaginary quadratic field such that all the primes dividing $pN$ split in $K$ (in particular, $K$ satisfies the Heegner hypothesis relative to $N$), write $O_K$ for the ring of integers of $K$ and let

$$pO_K = pp^c$$

be the factorization of $p$ as a product of (distinct) maximal ideals of $O_K$ (here $c$ denotes the nontrivial element of $\text{Gal}(K/\mathbb{Q})$). Fix algebraic closures $\overline{\mathbb{Q}}$ and $\overline{\mathbb{Q}}_p$ of $\mathbb{Q}$ and $\mathbb{Q}_p$, respectively, write $\mathbb{C}_p$ for the completion of $\overline{\mathbb{Q}}_p$, choose an embedding $\iota_p : \overline{\mathbb{Q}} \hookrightarrow \mathbb{C}_p$ and suppose that $p$ is the prime above $p$ that lands inside the valuation ideal of $\mathbb{C}_p$. Let $T$ denote the $p$-adic Tate module of $E$, set $V := T \otimes_{\mathbb{Z}_p} \mathbb{Q}_p$ and $A := V/T = E[p^\infty]$. For every integer $m \geq 1$, define

$$A_m := T/p^mT = E[p^m]$$

Let $K_\infty$ be the anticyclotomic $\mathbb{Z}_p$-extension of $K$, whose $n$-th finite layer will be denoted by $K_n$, and let $\Lambda$ be the associated Iwasawa algebra. We assume that the two primes of $K$ above $p$ are totally ramified in $K_\infty$; this is a natural condition to require when working in the supersingular setting (cf., e.g., [8 Assumptions 1.7, (2)], [16 Hypothesis (S)], [31 Theorem 1.2, (2)]) and holds if $p$ does not divide the class number of $K$. We shall denote the unique primes (of the finite layers) of $K_\infty$ above $p$ and $p^c$ by the same symbols. For more details, see Section 2.

Finally, for any compact or discrete $\Lambda$-module $M$ we write $M^\vee := \text{Hom}^\text{cont}_{\mathbb{Z}_p}(M, \mathbb{Q}_p/\mathbb{Z}_p)$ for its Pontryagin dual, which may be equipped with the compact-open topology (here $\text{Hom}^\text{cont}_{\mathbb{Z}_p}(\bullet, \bullet)$ denotes continuous homomorphisms of $\mathbb{Z}_p$-modules and $\mathbb{Q}_p/\mathbb{Z}_p$ is equipped with the quotient, i.e., discrete, topology).

1.2. Main results. For every $n \in \mathbb{N}$ we define plus and minus Mordell-Weil groups $E^\pm(K_n)$, which provide the local conditions in terms of which we introduce our plus and minus Selmer groups $\text{Sel}^\pm_{p^n}(E/K_\infty)$ à la Kobayashi. It is worth remarking that our construction is inspired
by Kim ([18]); in particular, we define plus/minus Selmer groups over the finite layers of $K_\infty/K$ somewhat differently from most of the literature (see, e.g., [7], [8], [16], [24]). This construction has the advantage of yielding a nicer control theorem and also some important duality properties that may be of independent interest.

Our main result, which corresponds to Theorem 5.9 can be stated as follows.

**Theorem 1.1.** Under an appropriate set of hypotheses, the $\Lambda$-module $\text{Sel}^\pm_{p^\infty}(E/K_\infty)$ admits no proper $\Lambda$-submodule of finite index.

As a sample application of this result, we are able to prove a theorem regarding the variation of Iwasawa invariants among $p$-congruent rational elliptic curves, that is, rational elliptic curves with Galois-isomorphic $p$-torsion subgroups. From here on, set $G_\mathbb{Q} := \text{Gal}(\bar{\mathbb{Q}}/\mathbb{Q})$.

**Theorem 1.2.** Let $E_1/\mathbb{Q}$ and $E_2/\mathbb{Q}$ be elliptic curves with good supersingular reduction at $p$. Suppose that $E_1[p] \cong E_2[p]$ as $G_\mathbb{Q}$-modules. Then under an appropriate set of hypotheses

$$\mu(\text{Sel}^\pm_{p^\infty}(E_1/K_\infty)\hat{\otimes}) = 0 \iff \mu(\text{Sel}^\pm_{p^\infty}(E_2/K_\infty)\hat{\otimes}) = 0.$$ 

Furthermore, when these $\mu$-invariants vanish the $\lambda$-invariants of the Pontryagin duals of these Selmer groups are related by an explicit formula.

The “explicit formula” alluded to in the theorem above is determined by the sizes of certain local Galois cohomology groups as described in (6.6). Theorem 1.2 is in the vein of work by Greenberg–Vatsal ([13]) and extends to the supersingular setting results of Hatley–Lei for $p$-ordinary elliptic curves ([15]). The reader is referred to the introduction to Section 6 for the definitions of $\mu$- and $\lambda$-invariants, and to Theorems 6.8 and 6.16 for the full statement.

**Remark 1.3.** Our arguments for obtaining these results have some significant differences from the strategy developed by Greenberg in [10], which has been generalized to many different settings (see, e.g., [14] [15] [17] [19] [21] [33] [32]). In [11], Greenberg showed that his strategy works for very general Selmer groups that can be defined by a surjective global-to-local map in cohomology. In the setting studied in this paper, the plus and minus Selmer groups cannot be defined in this way, since they fail Greenberg’s so-called CRK hypothesis. On the other hand, both strategies utilize non-primitive Selmer groups, and the influence of Greenberg’s approach is amply evident in our own.

1.3. **Future directions.** The original motivation for [3] was to strengthen some of the results from [2]. It would be interesting to study generalizations of results in [2] in the supersingular setting. For example, on adapting Bertolini’s strategy in [3] to our plus/minus case, we have imposed a technical hypothesis on the vanishing of certain relative Shafarevich–Tate groups of $E$ along finite layers of $K_\infty/K$, denoted by $\text{III}_{p^\infty}(E, K_{n+1}/K_{n})$ (see Lemma 5.7 for more details). It would be worthwhile to investigate this assumption further, e.g., to find sufficient conditions and give explicit examples.

The results of this paper should also have implications for the annihilators of anticyclotomic plus/minus Selmer groups, in the spirit of [1]; we plan to tackle this question in a future project.

Finally, it would be interesting to generalize the results in [1], [2], [3], as well as those in the present article, to the case of higher weight modular forms.

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2. Anticyclotomic Iwasawa algebras

2.1. The anticyclotomic $\mathbb{Z}_p$-extension of $K$. For every $m \in \mathbb{N}$ let $H_{p^m}$ denote the ring class field of $K$ of conductor $p^m$, then set $H_p^\infty := \cup_{m \in \mathbb{N}} H_{p^m}$. There is an isomorphism

$$\text{Gal}(H_p^\infty/K) \simeq \mathbb{Z}_p \times \Delta,$$

where $\Delta$ is a finite group. The anticyclotomic $\mathbb{Z}_p$-extension $K^\infty/K$ is the unique $\mathbb{Z}_p$-extension of $K$ contained in $H_p^\infty$. It can be characterized as the unique $\mathbb{Z}_p$-extension of $K$ that is Galois and non-abelian (in fact, generalized dihedral) over $\mathbb{Q}$. We can write $K^\infty := \cup_{m \geq 0} K_n$, where $K_n$ is the unique subfield of $K^\infty$ such that $G_n := \text{Gal}(K_n/K) \simeq \mathbb{Z}/p^n\mathbb{Z}$.

In particular, $K_0 = K$. We also define

$$G_\infty := \lim_{\leftarrow m} G_m = \text{Gal}(K^\infty/K) \simeq \mathbb{Z}_p,$$

where the inverse limit is taken with respect to the natural restriction maps. Finally, for all $n, n' \in \mathbb{N} \cup \{\infty\}$ with $n \leq n'$ we set

$$G_{n'/n} := \text{Gal}(K_{n'/n}/K_n).$$

In particular, $G_{n/0} = G_n$ for all $n \in \mathbb{N}$.

2.2. The Iwasawa algebra $\Lambda$. Throughout our article, we fix a topological generator $\gamma_\infty$ of $G_\infty$. Furthermore, we consider the Iwasawa algebra

$$\Lambda := \lim_{\rightarrow n} \mathbb{Z}_p[G_n] = \mathbb{Z}_p[G_\infty]$$

attached to $K^\infty/K$, the inverse limit being taken with respect to the maps that are induced by restriction. As is well known (see, e.g., [28, Proposition 5.3.5]), $\gamma_\infty$ determines an isomorphism of topological $\mathbb{Z}_p$-algebras $\Lambda \xrightarrow{\sim} \mathbb{Z}_p[X]$ such that $\gamma_\infty \mapsto 1 + X$. In the rest of the paper, we shall tacitly identify $\Lambda$ with $\mathbb{Z}_p[X]$ in this way.

3. Selmer groups and control theorems

3.1. Plus and minus Selmer groups over $K^\infty$. We introduce plus and minus norm groups à la Kobayashi (see [22]). First of all, define the two sets of indices

$$S_+^n := \{0, 1, \ldots, n\} \cap 2\mathbb{Z};$$
$$S_0^n := \{0, 1, \ldots, n\} \cap (2\mathbb{Z} + 1).$$

Let $n \geq 1$ be an integer. For $v \in \{p, p^c\}$, set

$$\hat{E}(K_{n,v}) := \left\{ P \in \hat{E}(\mathfrak{M}_{K_{n,v}}) \mid \text{Tr}_{n/m+1}(P) \in \hat{E}(\mathfrak{M}_{K_{m,v}}) \text{ for all } m \in S_0^n \right\},$$

where we write $\hat{E}(\mathfrak{M}_{K_{n,v}})$ for the formal group of $E$ whose points are defined over the maximal ideal $\mathfrak{M}_{K_{n,v}}$ of the valuation ring of $K_{n,v}$ and

$$\text{Tr}_{n/m+1} : \hat{E}(\mathfrak{M}_{K_{n,v}}) \rightarrow \hat{E}(\mathfrak{M}_{K_{m+1,v}})$$

is the trace map on formal groups.

Let $v \in \{p, p^c\}$. Following [28, §3.3], we define

$$\mathbb{H}_v^\pm := \bigcup_{n \geq 0} \hat{E}(K_{n,v}) \otimes \mathbb{Q}_p/\mathbb{Z}_p,$$

(3.1)
For $n \in \mathbb{N}$, we also set
\[ \mathbb{H}^\pm_{n,v} := (\mathbb{H}^\pm_v)^{\text{Gal}(K_{\infty,v}/K_{n,v})}. \]

**Remark 3.1.** Via the Kummer map, we may identify $\mathbb{H}^\pm_v$ as a $\Lambda$-submodule of $H^1(K_{\infty,v}, A) := \varprojlim_n H^1(K_{n,v}, A)$. In turn, we may identify $\mathbb{H}^\pm_{n,v}$ and $\mathbb{H}^\pm_{n,v}[p^n]$ as submodules of $H^1(K_{n,v}, A)$ and $H^1(K_{n,v}, A_m)$, respectively.

In line with our general notational conventions, let $(\mathbb{H}^\pm_v)^\vee$ be the Pontryagin dual of $\mathbb{H}^\pm_v$.

**Proposition 3.2.**
(a) The $\Lambda$-module $(\mathbb{H}^\pm_v)^\vee$ is free of rank one.
(b) Let $m, n \in \mathbb{N}$. Under the local Tate pairing
\[ H^1(K_{n,v}, A_m) \times H^1(K_{n,v}, A_m) \rightarrow \mathbb{Z}/p^n\mathbb{Z}, \]
the exact annihilator of $\mathbb{H}^\pm_{n,v}[p^n]$ is $\mathbb{H}^\pm_{n,v}[p^n]$.

**Proof.** Part (a) is \[18\] Proposition 3.13] when the sign is $\cdot$. This has been subsequently generalized to the $+$ case in \[20\] Proposition 2.11], as our ground field is $\mathbb{Q}_p$. Part (b) then follows from the proof of \[18\] Proposition 3.15].

Now we define Selmer groups for $E$ over $K_\infty$ and over the finite layers of $K_\infty/K$. First of all, for every $n \in \mathbb{N}$ let $S_n$ be the set of places of $K_n$ dividing $Np\infty$, let $K_nS_n$ be the maximal extension of $K_n$ unramified outside $S_n$ and write $H^1_{S_n}(K_n, *)$ as a shorthand for $H^1(K_nS_n/K_n, *)$. In the following, it is convenient to set also $A_\infty := A = E[p^\infty]$.

**Definition 3.3.** Let $m \in \mathbb{N}\cup\{\infty\}$ and $n \in \mathbb{N}$. The $p^m$-Selmer group of $E$ over $K_n$ is
\[ \text{Sel}_{p^m}(E/K_n) := \ker \left( H^1(K_{n,v}, A_m) \rightarrow \prod_{v\in S_n} \frac{H^1(K_{n,v}, A_m)}{E(K_{n,v})/p^n} \right). \]
The $p^m$-Selmer group of $E$ over $K_\infty$ is
\[ \text{Sel}_{p^m}(E/K_\infty) := \varprojlim_n \text{Sel}_{p^m}(E/K_n), \]
the direct limit being taken with respect to the restriction maps.

We also introduce plus/minus Selmer groups à la Kobayashi, along the lines of \[18\].

**Definition 3.4.** Let $m \in \mathbb{N}\cup\{\infty\}$ and $n \in \mathbb{N}$. The plus/minus Selmer groups of $E$ over $K_n$ are
\[ \text{Sel}_{p^m}^\pm(E/K_n) := \ker \left( \text{Sel}_{p^m}(E/K_n) \rightarrow \prod_{v\mid p} \frac{H^1(K_{n,v}, A_m)}{\mathbb{H}^\pm_{n,v}[p^n]} \right). \]
The plus/minus Selmer groups of $E$ over $K_\infty$ are
\[ \text{Sel}_{p^m}^\pm(E/K_\infty) := \varprojlim_n \text{Sel}_{p^m}^\pm(E/K_n), \]
the direct limit being taken with respect to the restriction maps.

**Remark 3.5.** Our definitions are different from those given in \[22\] and \[16\], unless the base field is $K_0 = K$ or $K_\infty$. We have followed \[18\] §4.4 because we would like our Selmer conditions at $p$ to satisfy part (b) of Proposition 3.2 for our applications later.

The next result deals with the $n = 0$ case.
Lemma 3.6. For all \( m \in \mathbb{N} \cup \{ \infty \} \), there is an equality
\[
\Sel_{p^m}(E/K) = \Sel_{p^m}(E/K).
\]

Proof. In light of Remark 3.1, it suffices to show that \( \mathbb{H}_{0,v}^+[p^m] \) coincides with the image of \( E(K_v)/p^mE(K_v) \) in \( H^1(K_v, A_m) \) under the Kummer map. Indeed, \( \mathbb{H}_{0,v}^+[p^m], E(K_v)/p^mE(K_v) \) and \( H^1(K_v, A_m)/(E(K_v)/p^mE(K_v)) \) are all free of rank one over \( \mathbb{Z}/p^m\mathbb{Z} \). On the other hand, \( E(K_v)/p^mE(K_v) \) is contained in \( \mathbb{H}_{0,v}^+[p^m] \), as explained in the proof of [18 Proposition 3.15], and the result follows. \( \square \)

3.2. Control theorems. Our goal is to prove a result (Proposition 3.8) that will play the role in our context of control theorems for classical Selmer groups (see, e.g., [2, §2.3]).

Lemma 3.7. Let \( v \in \{ p, p^c \} \) and \( m, n, n' \in \mathbb{N} \cup \{ \infty \} \) with \( n \leq n' \). The natural maps
\[
(3.2) \quad \frac{H^1(K_{n,v}, A_m)}{\mathbb{H}_{n,v}^+[p^m]} \to \frac{H^1(K_{n',v}, A_{m})}{\mathbb{H}_{n',v}^+[p^m]}, \quad \frac{H^1(K_{n,v}, A_n)}{\mathbb{H}_{n,v}^+[p^m]} \to \frac{H^1(K_{n,v}, A)}{\mathbb{H}_{n,v}^+[p^m]}
\]
are injective.

Proof. Since \( E \) has supersingular reduction at \( p \), \( H^0(K_{n',v}, A_m) = 0 \) by the proof of [22 Proposition 7] (see also [16, Lemma 4.6]). The inflation-restriction exact sequence gives an isomorphism
\[
(3.3) \quad \text{res} : H^1(K_{n,v}, A_m) \xrightarrow{\sim} H^1(K_{n',v}, A_m)G_{n'/n}.
\]
In particular, it gives an injection \( H^1(K_{n,v}, A_m) \hookrightarrow H^1(K_{n',v}, A_m) \). To show that the first map in (3.2) is injective, it is enough to show that \( \mathbb{H}_{n,v}^+[p^m] = \mathbb{H}_{n',v}^+[p^m]G_{n'/n} \).

But Proposition 3.2(a) tells us that there is an isomorphism of \( \Lambda \)-modules
\[
\mathbb{H}_{n,v}^+[p^m] \cong (\mathbb{Z}/p^m\mathbb{Z})[G_n]^\vee
\]
and similarly for \( n' \). Thus (3.1) holds.

Now we study the second map in (3.2). Consider the short exact sequence
\[
0 \to A_m \to A \xrightarrow{p^m} A \to 0.
\]
The fact that \( H^0(K_{n,v}, A) = 0 \) implies that
\[
(3.5) \quad H^1(K_{n,v}, A_m) = H^1(K_{n,v}, A)[p^m],
\]
which gives the second injection. \( \square \)

We are now in a position to prove

Proposition 3.8. Let \( m, m', n, n' \in \mathbb{N} \cup \{ \infty \} \) with \( m \leq m' \) and \( n \leq n' \). The restriction map induces an isomorphism of \( \Lambda \)-modules
\[
\Sel_{p^m}(E/K_n) \cong \Sel_{p^m}(E/K_{n'})[p^m]G_{n'/n}.
\]

Proof. By the identity given in equation (3.5), it suffices to prove the result when \( m = \infty \). Consider the defining exact sequence
\[
0 \to \Sel_{p\infty}(E/K_{n'}) \to \Sel_{p\infty}(E/K_{n'}) \to \prod_{v | p} \frac{H^1(K_{n',v}, A)}{\mathbb{H}_{n',v}^+[p^m]}.
\]
Taking $G_{n'/n}$-invariants yields an exact sequence

$$0 \to \text{Sel}_{p^\infty}^+(E/K_n)_{G_{n'/n}} \to \text{Sel}_{p^\infty}(E/K_n)_{G_{n'/n}} \to \left(\prod_{v|p} H^1(K_{n',v}, A)\right)^{G_{n'/n}}.$$  

By [2, Lemma 1], the middle term is isomorphic to $\text{Sel}_{p^\infty}(E/K_n)$ and, in light of the proof of Lemma 3.7 (in particular, the identifications given in (3.3) and (3.4)), we obtain an exact sequence

$$0 \to \text{Sel}_{p^\infty}^+(E/K_n)_{G_{n'/n}} \to \text{Sel}_{p^\infty}(E/K_n) \to \left(\prod_{v|p} H^1(K_{n,v}, A)\right)^{G_{n'/n}}.$$  

But this is the exact sequence defining $\text{Sel}_{p^\infty}^+(E/K_n)$, hence there is an isomorphism of $\Lambda$-modules

$$\text{Sel}_{p^\infty}^+(E/K_n) \simeq \text{Sel}_{p^\infty}^+(E/K_{n'})_{G_{n'/n}},$$

as was to be shown. \hfill \square

4. Plus and minus Heegner points

4.1. Plus/minus Mordell–Weil groups. Let $n \geq 1$ be an integer. We define plus and minus norm subgroups of $E(K_n)$ that are global counterparts of the plus/minus norm groups introduced in §3.1. For every $m \in S_n^\pm$ let

$$\text{Tr}_{n/m+1} : E(K_n) \to E(K_{m+1})$$

be the Galois trace map with respect to the group law on $E$.

**Definition 4.1.** The plus/minus Mordell–Weil groups of $E$ over $K_n$ are

$$E^\pm(K_n) := \{P \in E(K_n) \mid \text{Tr}_{n/m+1}(P) \in E(K_m) \text{ for all } m \in S_{n}^\pm\}.$$  

Similar to (3.1), we also define

$$\mathbb{H}_{\infty}^\pm := \bigcup_{n \geq 0} E^\pm(K_n) \otimes \mathbb{Q}_p/\mathbb{Z}_p, \quad \mathbb{H}_n^\pm := \left(\mathbb{H}_{\infty}^\pm\right)^{\text{Gal}(K_\infty/K_n)}.$$  

**Remark 4.2.** The $\Lambda$-module $\mathbb{H}_{\infty}^\pm$ (respectively, $\mathbb{H}_n^\pm$) may be identified with a submodule of $\text{Sel}_{p^\infty}^+(E/K_\infty)$ (respectively, $\text{Sel}_{p^\infty}^+(E/K_n)$) via the usual Kummer map in Galois cohomology. Analogously, given an integer $m \geq 1$, we may view $\mathbb{H}_n^\pm[p^m]$ as a $\Lambda$-submodule of $\text{Sel}_{p^m}^+(E/K_n)$.

4.2. Plus/minus Heegner points. Let $\{z_n \in E(K_n)\}_{n \geq 1}$ be a compatible family of Heegner points as in [24, §4.2]. Following [24, §4.3], we give

**Definition 4.3.** The plus/minus Heegner points are

$$z_n^\pm := \begin{cases} z_n & \text{if } n \text{ is even}, \\ z_{n-1}^\pm & \text{if } n \text{ is odd}. \end{cases}, \quad z_n := \begin{cases} z_{n-1}^- & \text{if } n \text{ is even}, \\ z_n & \text{if } n \text{ is odd}. \end{cases}$$  

Since we have assumed that $a_p(E) = 0$, it is a consequence of the formulas in [24, §3.1, Proposition 1] that the points $z_n^\pm$ satisfy the following relations:

(a) $\text{tr}_{K_m/K_{m-1}}(z_n^\pm) = -z_n^\pm_{m-1}$ for every even $m \geq 2$;
(b) $\text{tr}_{K_m/K_{m-1}}(z_n^\pm) = p z_n^\pm_{m-1}$ for every odd $m \geq 1$;
(c) $\text{tr}_{K_m/K_{m-1}}(z_n^-) = p z_n^-_{m-1}$ for every even $m \geq 2$;
(d) $\text{tr}_{K_m/K_{m-1}}(z_n^-) = -z_n^-_{m-1}$ for every odd $m \geq 3$.
(e) \( \text{tr}_{K_1/K_0}(z_1^-) = \frac{p-1}{2}z_0^- = \frac{p-1}{2}z_0. \)

In particular, we see that \( z_m^\pm \in E^\pm(K_n). \)

For all \( m, n \in \mathbb{N}, \) set \( R_{m,n} := (\mathbb{Z}/p^m)[G_n]. \) As in [24 §4.4], we define \( E^\pm_{m,n} \) to be the \( R_{m,n} \)-submodule of \( \text{Sel}_{p^m}(E/K_n) \) generated by \( z_m^\pm. \) This in turn defines \( \Lambda \)-submodules

\[
E^\pm := \lim_{\rightarrow m} E^\pm_{m,m} \subset \text{Sel}_{p^m}(E/K_n)
\]
as well as the Pontryagin duals

\[
H^\pm := (E^\pm_{\infty})^\vee = \lim_{\rightarrow m}(E^\pm_{m,m})^\vee.
\]

Finally, we introduce the \( R_{m,n} \)-module

\[
E^\pm_{m,n} := (E^\pm_{\infty})_{G_n}/n^\vee.
\]

When \( m = 1, \) we shall omit the index \( m \) from the notation and simply write \( R_n, E^\pm_n, E^\pm. \)

We recall the following result on \( H^\pm_{\infty}. \)

**Proposition 4.4.** The \( \Lambda \)-module \( H^\pm_{\infty} \) is finitely generated, torsion-free and of rank 1.

**Proof.** This is [24 Proposition 4.7].

We strengthen this slightly in

**Proposition 4.5.** The \( \Lambda \)-module \( H^\pm_{\infty} \) is free of rank one.

**Proof.** By definition, \( E^\pm_{m,m} \) is a cyclic \( R_{m,m} \)-module. Thus, \( \lim_{\rightarrow m}(E^\pm_{m,m})^\vee \) is a cyclic \( \lim_{\rightarrow m} R_{m,m} \)-module. Thus, \( H^\pm_{\infty} \) is cyclic over \( \Lambda. \) Since it is also torsion-free, and since \( \Lambda \) is an integral domain, this implies \( H^\pm_{\infty} \) is free of rank 1.

We deduce

**Corollary 4.6.** The \( R_n \)-module \( E^\pm_n \) is cyclic.

**Proof.** Immediate from Proposition 4.5.

**Lemma 4.7.** There is an injection of \( R_n \)-modules

\[
E^\pm_{m,n} \hookrightarrow \text{Sel}_{p^m}(E/K_n).
\]

**Proof.** We may identify \( E^\pm_{m,n} \) with a submodule of \( H^1(K_n, A_m) \) via the Kummer map. On the other hand, the inclusion \( E^\pm_{\infty} \subset \mathbb{H}^\pm_{\infty} \) induces an inclusion \( E^\pm_{m,n} \subset \mathbb{H}^\pm_{n}[p^m], \) and the lemma is proved.

From now on we shall view \( E^\pm_{m,n} \) as a submodule of \( \text{Sel}_{p^m}(E/K_n) \) via Lemma 4.7 without further notice.

**Lemma 4.8.** Let \( n \geq 1 \) be an integer. There is a natural injection \( E^\pm_{m,n} \hookrightarrow E^\pm_{m,n+1} \) whose image is \( (E^\pm_{m,n+1})^G_{n+1}/n^\vee. \)

**Proof.** As in the proof of Lemma 4.7, we may identify \( E^\pm_{m,n} \) and \( E^\pm_{m,n+1} \) as submodules of \( H^1(K_n, A_m) \) and \( H^1(K_{n+1}, A_m), \) respectively. Furthermore, as in the proof of Lemma 4.7 the inflation-restriction exact sequence gives an isomorphism

\[
H^1(K_n, A_m) \cong H^1(K_{n+1}, A_m)^G_{n+1}/n,
\]
and the result follows.
5. On proper $\Lambda$-submodules of finite index

5.1. Cohomology, universal norms and perfect pairings. For all $m, n \in \mathbb{N}$, write $I_{m,n}$ for the augmentation ideal of $R_{m,n}$. The following result generalizes [3 Proposition 6.3] to our plus/minus setting.

**Proposition 5.1.** Let $m, n \in \mathbb{N}$. There is a perfect pairing of Tate cohomology groups

$$\langle \cdot, \cdot \rangle_{m,n} : \hat{H}^0(G_n, \text{Sel}^\pm_{p^m}(E/K_n)) \times \hat{H}^{-1}(G_n, \text{Sel}^\pm_{p^m}(E/K_n)) \to I_{m,n}/I_{m,n}^2.$$

**Proof.** Write $\text{Sel}^\pm_{p^m}(E/K_n)^0$ for the kernel of the corestriction map $\text{Sel}^\pm_{p^m}(E/K_n) \to \text{Sel}^\pm_{p^m}(E/K)$. Fix a generator $\gamma_n$ of $G_n$. Since, by [4, Theorem 3.2], the $R_{m,n}$-module $H^1_S(K_n, A_m)$ is free, the kernel of the corestriction $H^1_S(K_n, A_m) \to H^1_S(K, A_m)$ is $(\gamma_n - 1)H^1_S(K_n, A_m)$. Therefore if $y \in \text{Sel}^\pm_{p^m}(E/K_n)^0$, then there exists $z \in H^1_S(K_n, A_m)$ satisfying

$$y = (\gamma_n - 1)z.$$

We define a pairing

$$[\cdot, \cdot]_{m,n} : \text{Sel}^\pm_{p^m}(E/K) \times \text{Sel}^\pm_{p^m}(E/K)^0 \to I_{m,n}/I_{m,n}^2$$

by sending $(x, y)$ to

$$\sum_v [x_v, z_v](\gamma_n - 1) \mod I_{m,n}^2,$$

where the sum runs over all primes of $K$ underlying those in $S$, $x_v$ is the natural image of $x$ in $E(K_v)/p^mE(K_v)$, $z_v$ is the natural image in $H^1(K_v, E)[p^m]$ of the $z \in H^1_S(K_n, A_m)$ from [5.1] and $[\cdot, \cdot]_v$ is the local Tate pairing

$$E(K_v)/p^mE(K_v) \times H^1(K_v, E)[p^m] \to \mathbb{Z}/p^m\mathbb{Z}.$$

The argument in [3 Proposition 6.3] to show that this pairing is independent of the choice of $z$ or $\gamma_n$ carries over to our setting, as it relies on algebraic properties of Galois cohomology only.

To check that $[\cdot, \cdot]_{m,n}$ induces a perfect pairing $\langle \cdot, \cdot \rangle_{m,n}$ on the Tate cohomology groups it is enough to show that we have a right non-degenerate pairing, as the two groups have the same order. We extend the proof of [27 Lemma 6.15] to our setting. Consider the commutative diagram

$$
\begin{array}{ccccccc}
\bigoplus_{v \in S} H^1(K_{n,v}, A_m) \\
\downarrow & & & & & & \\
\bigoplus_{v \in S} H^1(K_{n,v}, A_m) & \leftarrow & H^1_S(K_n, A_m) & \leftarrow & \text{Sel}^\pm_{p^m}(E/K_n) & \leftarrow & 0,
\end{array}
$$

where we put $\mathbb{H}^\pm_{n,v}[p^m] := E(K_{n,v})/p^mE(K_{n,v})$ for $v \nmid p$. If we take $\mathbb{Z}/p^m\mathbb{Z}$-linear duals, then Proposition [3.2] gives us a commutative diagram

$$
\begin{array}{ccccccc}
\bigoplus_{v \in S} H^1(K_{n,v}, A_m) \\
\downarrow & & & & & & \\
\bigoplus_{v \in S} \mathbb{H}^\pm_{n,v}[p^m] & \rightarrow & H^1_S(K_n, A_m)^* & \rightarrow & \text{Sel}^\pm_{p^m}(E/K_n)^* & \rightarrow & 0.
\end{array}
$$

Let $y$ be an element of the right kernel of $[\cdot, \cdot]_{m,n}$ and let $z \in H^1_S(K_n, A_m)$ be as in [5.1]. As can be checked by a diagram chase, the image of $z$ in $\bigoplus_{v \in S} H^1(K_{n,v}, A_m)$ splits as $a_1 + a_2$, where $a_1$ is in the image of $\bigoplus_{v \in S} \mathbb{H}^\pm_{n,v}[p^m]$ and $a_2$ is in the image of $H^1_S(K, A_m)$. Therefore $z$ itself
decomposes as $z_1 + z_2$, where $z_1 \in \text{Sel}^p(E/K_n)$ and $z_2 \in \text{im}(H^1_S(K, A_m) \hookrightarrow H^1_S(K_n, A_m))$. This shows that $y \in (\gamma_n - 1)\text{Sel}^p(E/K_n)$. In particular, the image of $y$ in $H^{-1}(G_n, \text{Sel}^p(E/K_n))$ is zero. This shows that the pairing $\langle \cdot, \cdot \rangle_{m,n}$ is right non-degenerate, as required.

For every $n \in \mathbb{N}$, define

$$S^+_p(E/K_n) := \lim_{\leftarrow m} \text{Sel}^+_p(E/K_n),$$

the inverse limit being taken with respect to the multiplication-by-$p$ maps $E[p^{m+1}] \xrightarrow{\cdot p} E[p^m]$. For all $n, n' \in \mathbb{N}$ with $n \leq n'$ there is a natural corestriction map

$$\text{cores}_{K_{n'}/K_n}: S^+_p(E/K_{n'}) \rightarrow S^+_p(E/K_n).$$

We can then define the $\Lambda$-module

$$\hat{S}^+_p(E/K_\infty) := \lim_{\leftarrow n} S^+_p(E/K_n),$$

the inverse limit being taken with respect to the corestriction maps in (5.2).

The next proposition will be used in the proof of our main result (Theorem 5.9).

**Proposition 5.2.** There is a canonical isomorphism of $\Lambda$-modules

$$\hat{S}^+_p(E/K_\infty) \simeq \text{Hom}_\Lambda(\text{Sel}^+_P(E/K_\infty)'^\vee, \Lambda).$$

**Proof.** One can proceed as in the proof of [29, Lemme 5], replacing the control theorem used in [29] with Proposition 3.8. □

The universal norm submodule of $S^+_p(E/K)$ is

$$US^+_p(E/K) := \bigcap_{n \geq 1} \text{cores}_{K_n/K} \left(S^+_p(E/K_n) \right) \subset S^+_p(E/K).$$

The following theorem is the counterpart for plus/minus Selmer groups of [3, Theorem 6.1].

**Theorem 5.3.** There is a perfect pairing

$$\langle \cdot, \cdot \rangle: S^+_p(E/K)/US^+_p(E/K) \times \text{Sel}^+_P(E/K_\infty)G_\infty \rightarrow G_\infty \otimes \mathbb{Z}_p \mathbb{Q}_p/\mathbb{Z}_p.$$

**Proof.** Using Proposition 5.1, this follows as in the proof of [3, Theorem 6.1], where we replace the control theorem used to prove [3, Lemma 6.4] with the plus/minus analogue provided by Proposition 3.8. □

This in turn gives the following

**Corollary 5.4.** The $\Lambda$-module $\text{Sel}^+_P(E/K_\infty)$ admits no proper $\Lambda$-submodule of finite index if and only if $S^+_p(E/K)/US^+_p(E/K)$ has no $\mathbb{Z}_p$-torsion.

**Proof.** Using Theorem 5.3 in place of [3, Theorem 6.1], one can proceed exactly as in the proof of [3, Corollary 6.2], which is completely algebraic in nature and does not depend on the definition of the Selmer groups involved. □
5.2. Computation of universal norms. Under appropriate hypotheses, we now compute the universal norm submodule $US_p^\pm(E/K)$, thus extending [3, Theorem 7.1] to our setting. In doing so, we follow [3, §7] closely. We begin with a generalization of [1, Lemma 9]. Recall that $R_n = R_{1,n} = (\mathbb{Z}/p\mathbb{Z})[G_n]$ and $E_n^\pm = E_{1,n}^\pm$.

**Lemma 5.5.** If $E_n^\pm \neq 0$, then $Sel_p^\pm(E/K_n)$ admits a free $R_n$-submodule $U_n^\pm$ such that $E_n^\pm \subset U_n^\pm$.

**Proof.** For $m \in \{n, n+1\}$, fix a generator $\gamma_{m}$ of $G_m$. Recall from Corollary 4.6 that $E_n^\pm$ is a cyclic $R_m$-module. Consequently, [1, Lemma 3] says that

(5.3) \[ E_n^\pm \simeq R_m/(\gamma_{m} - 1)p^n - t_n^\pm \simeq (\gamma_{m} - 1)t_n^\pm R_m, \]

where $t_n^\pm = p^n - \dim_{\mathbb{F}_p}(E_n^\pm)$. Note that $G_{n+1/n} = \langle \gamma_{n+1} \rangle$, which is cyclic of order $p$. Then

\[ ((\gamma_{n+1} - 1)^s R_{n+1})^G = ((1 + \gamma_{n+1} + \ldots + \gamma_{n+1}^{(p-1)}) (\gamma_{n+1} - 1)^s R_{n+1} = (\gamma_{n+1} - 1)^{p^n+s} R_{n+1} \]

for all $0 \leq s \leq p^n+1$. In particular, isomorphism (5.3) tells us that

\[ \dim_{\mathbb{F}_p} ((\gamma_{n+1} - 1)^s R_{n+1})^G = p^n - s. \]

Recall from Lemma 4.5 that

\[ (E_{n+1}^\pm)^{G_{n+1/n}} = E_n^\pm. \]

Since $\dim_{\mathbb{F}_p} E_{n+1}^\pm = p^n+1 - t_{n+1}^\pm$, we have

\[ \dim_{\mathbb{F}_p} (E_{n+1}^\pm)^{G} = p^n - t_{n+1}^\pm = p^n - t_n^\pm, \]

so $t_{n+1}^\pm = t_n^\pm$. Let us write $t_n^\pm$ for this common value and define the cyclic $R_{n+1}$-module

\[ U_n^\pm := (\gamma_{n+1} - 1)^{p^n+s - t_n^\pm} E_{n+1}^\pm \simeq (\gamma_{n+1} - 1)^{p^n+s} R_{n+1}. \]

Note that $U_n^\pm$ is contained in $E_{n+1}^\pm$ by definition and is isomorphic to $R_{n+1}^{G_{n+1/n}}$. Therefore $U_n^\pm$ is invariant under $G_{n+1/n}$, which makes it an $R_n$-module. On the one hand, $U_n^\pm$ contains $(E_{n+1}^\pm)^{G_{n+1/n}} = E_n^\pm$; on the other hand, $U_n^\pm$ is contained in $Sel_p^\pm(E/K_{n+1})^{G_{n+1/n}} \simeq Sel_p^\pm(E/K_n)$. Finally, we deduce from (5.3) that $\dim_{\mathbb{F}_p} U_n^\pm = p^n$, hence $U_n^\pm$ is free of rank one over $R_n$. \qed

Now we introduce plus and minus analogues of Shafarevich–Tate groups over finite layers of $K_{\infty}/K$ and, following [3, §7], relative versions of them as well.

**Definition 5.6.** The plus and minus $p^n$-Shafarevich–Tate groups of $E$ over $K_n$ are

\[ \Pi_{p^n}(E/K_n) := Sel_p^\pm(E/K_n)/\mathbb{H}_n^\pm[p^n]. \]

The relative plus and minus $p^n$-Shafarevich–Tate groups are

\[ \Pi_{p^n}(E, K_{n+1}/K_n) := \ker \left( \Pi_{p^n}(E/K_n) \rightarrow \Pi_{p^n}(E/K_{n+1}) \right), \]

where the map on the right is induced by restriction.

Let $U_n^\pm$ be the free $R_n$-submodule of $Sel_p^\pm(E/K_n)$ from Lemma 5.5.

**Lemma 5.7.** Suppose that $\Pi_{p^n}(E, K_{n+1}/K_n) = 0$ and $E_n^\pm \neq 0$. Then $U_n^\pm \subset \mathbb{H}_n^\pm[p]$. 
Proof. Consider the commutative diagram

\[
\begin{array}{cccc}
0 & \rightarrow & \mathbb{H}^+_{n}[p] & \rightarrow & \text{Sel}_{p}^{+}(E/K_{n}) & \rightarrow & \text{III}^{+}_{p}(E/K_{n}) & \rightarrow & 0 \\
& & \downarrow & & \downarrow & & \downarrow & & \downarrow \\
0 & \rightarrow & (\mathbb{H}^+_{n+1}[p])^{G_{n+1}/n} & \rightarrow & \text{Sel}_{p}^{+}(E/K_{n+1})^{G_{n+1}/n} & \rightarrow & \text{III}^{+}_{p}(E/K_{n+1})^{G_{n+1}/n} & \rightarrow & 0
\end{array}
\]

in which the middle vertical isomorphism comes from Proposition 3.8. By applying the snake lemma, one checks that

(5.4) \[ \mathbb{H}^+_{n}[p] = (\mathbb{H}^+_{n+1}[p])^{G_{n+1}/n}. \]

By construction, \( U^{+}_{n} \) is contained in the module on the right-hand side of (5.4), and the result follows.

Lemma 5.8. There is an equality

\[ S_{p}^{+}(E/K) = \lim_{\rightarrow \ n} \mathbb{H}^+_{0}[p^{m}]. \]

Proof. By Lemma 3.6 \( \mathbb{H}^+_{0}[p^{m}] = E(K)/p^{m}E(K) \) and \( \text{Sel}_{p}^{m}(E/K) = \text{Sel}_{p}^{m}(E/K) \), therefore there is a short exact sequence

(5.5) \[ 0 \rightarrow \mathbb{H}^+_{0}[p^{m}] \rightarrow \text{Sel}_{p}^{m}(E/K) \rightarrow \text{III}^{m}(E/K) \rightarrow 0. \]

But \( \text{III}^{m}(E/K) \) is finite and bounded independently of \( m \) (23, Theorem A), hence the result follows upon taking inverse limits in (5.5).

The following theorem is the main result of this article.

Theorem 5.9. Suppose that \( \text{III}^{+}_{p}(E,K_{n+1}/K_{n}) = 0 \) and \( \mathfrak{C}^{+}_{n} \neq 0 \) for all \( n \in \mathbb{N} \). Then \( US_{p}^{+}(E/K) \) is free of rank one over \( \mathbb{Z}_{p} \) and the \( \Lambda \)-module \( \text{Sel}_{p}^{+}(E/K_{\infty}) \) admits no proper \( \Lambda \)-submodule of finite index.

Proof. We first observe that \( E(K) \) is \( p \)-torsion free; this is a consequence of our hypotheses that \( E \) has good supersingular reduction at primes above \( p \) and that \( p \) is unramified in \( K \). Thus it follows from the proof of Lemma 5.8 that \( S_{p}^{+}(E/K) \simeq E(K) \otimes_{\mathbb{Z}} \mathbb{Z}_{p} \) is a free \( \mathbb{Z}_{p} \)-module of finite rank. Furthermore, by [24, Theorem 1.4] (see also [6, Theorem A]), the \( \Lambda \)-module \( \text{Sel}_{p}^{+}(E/K_{\infty}) \) has rank one. Combining this with Proposition 5.2, and arguing as in [2, §3.2], one sees that

\[ \text{rank}_{\mathbb{Z}_{p}} US_{p}^{+}(E/K) = \text{rank}_{\Lambda} S_{p}^{+}(E/K_{\infty}) = 1. \]

Then it suffices to show that \( US_{p}^{+}(E/K) \) contains a non-trivial element of \( S_{p}^{+}(E/K) \) not divisible by \( p \). This would imply that \( US_{p}^{+}(E/K) \simeq \mathbb{Z}_{p} \) and that the \( \mathbb{Z}_{p} \)-module \( S_{p}^{+}(E/K)/US_{p}^{+}(E/K) \) is torsion-free. The last statement of the theorem would then follow from Corollary 5.4.

Thus let us set \( T^{+}_{n} := \lim_{\rightarrow \ n} \mathbb{H}^+_{0}[p^{m}] \) for the \( p \)-adic Tate module of \( \mathbb{H}^+_{n} \); then

\[ T^{+}_{n}/pT^{+}_{n} = \mathbb{H}^+_{n}[p]. \]

Lemma 5.7 says that the free \( R_{n} \)-module \( U^{+}_{n} \) from Lemma 5.5 lies inside \( \mathbb{H}^+_{n}[p] \). Let \( \bar{U}^{+}_{n} \) be a free \( \mathbb{Z}_{p}[G_{n}] \)-submodule of \( T^{+}_{n} \) of rank one lifting \( U^{+}_{n} \) modulo \( p \), generated by an element \( v_{n} \). Then \( \text{cores}_{K_{n}/K}(v_{n}) \) is not divisible by \( p \), thanks to the freeness of \( \bar{U}^{+}_{n} \). Lemma 5.8 tells us that

\[ \text{cores}_{K_{n}/K}(v_{n}) \in T^{+}_{0} = US_{p}^{+}(E/K). \]
By compactness, we may find a subsequence \((\text{cores}_{K_n/K(v_{n_i})})_{i \geq 0}\) converging to an element of \(S^+_p(E/K)\) that lies in \(US^+_p(E/K)\) and is not divisible by \(p\), as required. \(\Box\)

6. An application: variation of Iwasawa invariants

6.1. \(p\)-congruent elliptic curves. In this section, we illustrate one application of Theorem 5.9. Recall that two elliptic curves \(E/\mathbb{Q}\) and \(E'/\mathbb{Q}\) are \(p\)-congruent (or congruent modulo \(p\)) if \(E[p] \cong E'[p]\) as \(G_{\mathbb{Q}}\)-modules. In [13], Greenberg and Vatsal showed that the Iwasawa invariants (defined over the cyclotomic \(\mathbb{Z}_p\)-extension of \(\mathbb{Q}\)) of \(p\)-congruent, \(p\)-ordinary elliptic curves are related by an explicit formula, and they used this to prove many cases of the Iwasawa Main Conjecture. These results were later extended to Hida families by Emerton, Pollack and Weston ([9]), while results in the non-ordinary case were established for elliptic curves by B. D. Kim ([19]) and for more general modular forms by the first two named authors ([14]).

An analogue for the anticyclotomic \(\mathbb{Z}_p\)-extension \(K_{\infty}\) of an imaginary quadratic field \(K\) of the results in [9] was obtained by Castella, C.-H. Kim and Longo ([5]) under the assumption that the tame level \(N\) of the Hida family is divisible by an \textit{odd} number of primes that are inert in \(K\); in this setting, results on the vanishing of the \(\mu\)-invariant were obtained by Pollack and Weston ([31]). When \(N\) satisfies this divisibility hypothesis, the usual Selmer groups over \(K_{\infty}\) are expected to be \(\Lambda\)-cotorsion, and the arguments regarding the variation of Iwasawa invariants closely mirror the cyclotomic case.

As we have seen in previous sections, when all the primes dividing \(N\) split in \(K\) the classical Selmer groups are not \(\Lambda\)-cotorsion, and extra care must be taken (this happens, more generally, when \(N\) is divisible by an \textit{even} number of inert primes). Recently, an analogue in this setting of the Greenberg–Vatsal result was proved for \(p\)-ordinary elliptic curves by the first two named authors ([15]). In this final section of the paper, we will extend the results by Hatley and Lei to the supersingular case.

6.2. Iwasawa invariants. Let us briefly recall the definition of the Iwasawa invariants. As explained in [2.2] we identify \(\Lambda\) with \(\mathbb{Z}_p[[X]]\) via our fixed topological generator \(\gamma_{\infty}\) of \(G_{\infty}\). Let \(M\) be a finitely generated \(\Lambda\)-module; there is a pseudo-isomorphism, i.e., a map with finite kernel and cokernel

\[
M \sim \Lambda^\oplus_r \oplus \bigoplus_{i=1}^s \Lambda/(p^{a_i}) \oplus \bigoplus_{j=1}^t \Lambda/(F_j^{n_j})
\]

for suitable integers \(r, s, t \geq 0, a_i, n_j \geq 1\) and irreducible Weierstrass polynomials \(F_j \in \mathbb{Z}_p[X]\) (see, e.g., [28, Theorem 5.3.8]). The \(\mu\)-invariant and the \(\lambda\)-invariant of \(M\) are

\[
\mu(M) := \sum_{i=1}^s a_i, \quad \lambda(M) := \sum_{j=1}^t n_j \deg(F_j).
\]

When \(M\) is a cofinitely generated \(\Lambda\)-module, which means that the Pontryagin dual \(M^\vee\) of \(M\) is finitely generated over \(\Lambda\), we abuse notation and write \(\mu(M)\) and \(\lambda(M)\) for \(\mu(M^\vee)\) and \(\lambda(M^\vee)\), respectively. Furthermore, we still call \(\mu(M)\) and \(\lambda(M)\) the invariants “of \(M\)”. In our context, such an \(M\) will always be a suitable Selmer group.

6.3. Auxiliary Selmer groups. We describe some auxiliary Selmer groups that are useful for studying the Iwasawa invariants of the \(\pm\)-Selmer groups.
6.3.1. \( p \)-depleted Selmer groups. Recall that we defined the \( \pm \)-Selmer groups from the full Selmer groups by imposing stricter local conditions at the places above \( p \). By relaxing these conditions, or by making them as strict as possible, we obtain the \( p \)-depleted Selmer groups.

Recall from §1.1 that \( A = E[p^\infty] \). For a place \( v \mid p \) and \( L_v \in \{ \emptyset, \pm, 0 \} \), we set

\[
H^1_{L_v}(K_\infty, v, A) := \begin{cases} 
H^1(K_\infty, v, A) & \text{if } L_v = \emptyset, \\
H^\pm_\infty & \text{if } L_v = \pm, \\
\{0\} & \text{if } L_v = 0.
\end{cases}
\]

For a local condition \( L_v \) at a place \( v \mid p \), we set

\[
H^1_{/L_v}(K_\infty, v, A) := \frac{H^1(K_\infty, v, A)}{H^1_{L_v}(K_\infty, v, A)}.
\]

For all other places, we define \( H^1_{/f}(K_\infty, v, A) \) to be the quotient of \( H^1(K_\infty, v, A) \) by the usual (unramified) local condition. Then for \( L = \{L_v\} \mid p \) and \( S \) a finite set of places including those dividing \( Np_\infty \), we define

\[
\text{Sel}_L(E/K_\infty) := \ker\left(H^1_\Lambda(K_\infty, A) \xrightarrow{\gamma_L} \prod_{v \mid p} H^1_{/f}(K_\infty, v, A) \times \prod_{v \mid p} H^1_{/L_v}(K_\infty, v, A)\right)
\]

Finally, we write \( \mathcal{X}_L(E) \) for the Pontryagin dual of \( \text{Sel}_L(E/K_\infty) \).

**Remark 6.1.** If \( L = \{\pm, \emptyset\} \) with the same choice of sign at each place, then we recover our plus/minus Selmer groups, so we will just write \( \text{Sel}^\pm(E/K_\infty) \). In other words, one has

\[
\text{Sel}^\pm(E/K_\infty) := \text{Sel}_{\pm, \pm}(E/K_\infty) = \text{Sel}^{\pm}_{p_\infty}(E/K_\infty).
\]

Furthermore, we write \( \mathcal{X}_\pm(E) \) for the corresponding Pontryagin duals. Note that, to simplify notation, in this section we are dropping the \( p_\infty \) subscript.

From here on we assume that \( p \geq 5 \). Recall from §1.1 that \( T \) denotes the \( p \)-adic Tate module of \( E \). The next result concerns the \( \Lambda \)-module structure of some of our Selmer groups that we introduced above.

**Theorem 6.2.** Suppose that the conductor \( N \) of the elliptic curve \( E/\mathbb{Q} \) is square-free (i.e., \( E \) is semistable) and that the Galois representation \( \rho_E : \text{Gal}(\bar{\mathbb{Q}}/K) \to \text{Aut}_{\mathbb{Z}_p}(T) \) is surjective. Then

\[
\begin{align*}
(i) \quad & \text{rank}_\Lambda \mathcal{X}_\pm(E) = 1 \\
(ii) \quad & \text{rank}_\Lambda \mathcal{X}_{\emptyset, 0}(E) = 0 \\
(iii) \quad & \text{rank}_\Lambda \mathcal{X}_{\pm, 0}(E) = 0 \\
(iv) \quad & \text{rank}_\Lambda \mathcal{X}_{\pm, \emptyset}(E) = 1
\end{align*}
\]

**Proof.** Parts (i) and (ii) are contained in [6, Theorem 6.1] (see also [24, Theorem 5.1]), while part (iii) is contained in [6, Theorem 6.2]. Finally, part (iv) is then a consequence of [6, Lemma 3.8]. \( \square \)

The previous theorem has important consequences for the \( \Lambda \)-module structure of some of our Selmer groups, as the next proposition shows.

**Proposition 6.3.** Let \( E/\mathbb{Q} \) be an elliptic curve satisfying the hypotheses of Theorem 6.2. For the local conditions \( L = \{\pm, \emptyset\} \) and \( L = \{\emptyset, 0\} \), the following statements are true:

1. the global-to-local map \( \gamma_L \) defining \( \text{Sel}_L(E/K_\infty) \) is surjective;
(2) Sel\(_{L}(E/K_{\infty})\) has no proper finite-index \(\Lambda\)-submodules.

Proof. This can be proved in the same way as [15, Lemma 3.10 and Proposition 3.12]. In the proofs of the results in [15], it suffices to verify a list of conditions that are due to Greenberg ([11]). The first four of those, called RFX\((\mathcal{D})\), LOC\(_{v}^{(1)}(\mathcal{D})\), LOC\(_{v}^{(2)}(\mathcal{D})\), and LEO\((\mathcal{D})\), go through without change, while the final one, labeled CRK\((\mathcal{D}, \mathcal{L})\), follows immediately from Theorem 6.2 □

6.3.2. Residual Selmer groups. Since the ultimate goal of this section is to compare \(\pm\)-Selmer groups for \(p\)-supersingular elliptic curves \(E_1\) and \(E_2\) such that \(E_1[p] \simeq E_2[p]\) as Galois modules, it is convenient to define local cohomology groups that are simply the \(E[p]\)-valued counterparts of (6.2).

For a place \(v \mid p\) and \(\mathcal{L}_v \in \{0, \pm, \varnothing\}\), we set

\[
H^1_{\mathcal{L}_v}(K_{\infty, v}, E[p]) := \begin{cases} 
H^1(K_{\infty, v}, E[p]) & \text{if } \mathcal{L}_v = \varnothing, \\
\mathbb{H}_{\infty}^+[p] & \text{if } \mathcal{L}_v = \pm, \\
\{0\} & \text{if } \mathcal{L}_v = 0.
\end{cases}
\]

For a local condition \(\mathcal{L}_v\) at a place \(v \mid p\), we set

\[
H^1_{/\mathcal{L}_v}(K_{\infty, v}, E[p]) := \frac{H^1(K_{\infty, v}, E[p])}{H^1_{\mathcal{L}_v}(K_{\infty, v}, E[p])}.
\]

For all other places, we define \(H^1_{/f}(K_{\infty, v}, E[p])\) to be the quotient of \(H^1(K_{\infty, v}, E[p])\) by the usual (unramified) local condition. Then for \(\mathcal{L} = \{\mathcal{L}_v\}_{v \mid p}\) and \(S\) a finite set of places including those dividing \(Np\infty\), we define

\[
\text{Sel}_{\mathcal{L}}(E[p]/K_{\infty}) := \ker \left( H^1_S(K_{\infty}, E[p]) \xrightarrow{\gamma_L} \prod_{v \in S} H^1_{/f}(K_{\infty, v}, E[p]) \times \prod_{v \mid p} H^1_{/\mathcal{L}_v}(K_{\infty, v}, E[p]) \right).
\]

6.3.3. Non-primitive Selmer groups. In general, it is not the case that the Selmer groups defined in 6.3.2 are the \(p\)-torsion subgroups of the corresponding Selmer groups from 6.3.1. For this reason, we define the following non-primitive Selmer groups.

Let \(S\) be a finite set of places of \(K_{\infty}\) containing those dividing \(Np\infty\); observe that, since all the primes dividing \(Np\) split in \(K\), there are only finitely many places of \(K_{\infty}\) that divide \(Np\). Moreover, let \(\Sigma\) be a subset of \(S\) not containing the archimedean primes or the primes above \(p\). Fix a choice \(\mathcal{L} = \{\mathcal{L}_v\}_{v \mid p}\) of local conditions.

**Definition 6.4.** The non-primitive Selmer groups are

\[
\text{Sel}^S_E(E/K_{\infty}) := \ker \left( H^1_S(K_{\infty}, A) \xrightarrow{\gamma_L} \prod_{v \in S \setminus \Sigma} H^1_{/f}(K_{\infty, v}, A) \times \prod_{v \mid p} H^1_{/\mathcal{L}_v}(K_{\infty, v}, A) \right)
\]

and

\[
\text{Sel}^S_K(E[p]/K_{\infty}) := \ker \left( H^1_S(K_{\infty}, E[p]) \xrightarrow{\gamma_L} \prod_{v \in S \setminus \Sigma} H^1_{/f}(K_{\infty, v}, E[p]) \times \prod_{v \mid p} H^1_{/\mathcal{L}_v}(K_{\infty, v}, E[p]) \right).
\]
We set $\mathcal{X}_E^\Sigma(E) := \text{Sel}_E^\Sigma(E/K_\infty)^\vee$ for the Pontryagin dual of $\text{Sel}_E^\Sigma(E/K_\infty)$. To ease notation, when $\mathcal{L} = \{\pm\}$ we simply write $\text{Sel}^{\pm,\Sigma}(E/K_\infty)$ and $\mathcal{X}_E^\Sigma(E)$.

Let us now explain the significance of these non-primitive Selmer groups. There is a short exact sequence induced by (6.3) yields a short exact sequence
\begin{equation}
0 \longrightarrow E[p] \longrightarrow A \overset{p}{\longrightarrow} A \longrightarrow 0.
\end{equation}

Since, as in the proof of Lemma 3.7, $H^0(K_\infty, A_m) = 0$ for all $m$, this exact sequence induces an isomorphism
\begin{equation}
H^1_\Sigma(K_\infty, E[p]) \simeq H^1_\Sigma(K_\infty, A)[p].
\end{equation}

Now we show that the local conditions at $v \in S \setminus \Sigma$ are also compatible with taking $p$-torsion.

**Theorem 6.5.** *If $\Sigma$ contains all the primes at which $A$ is ramified, then there is an isomorphism of $\Lambda$-modules*
\[
\text{Sel}_E^\Sigma(E[p]/K_\infty) \simeq \text{Sel}_E^\Sigma(E/K_\infty)[p].
\]

**Proof.** In light of isomorphism (6.4), it suffices to check the compatibility of the local conditions defining each Selmer group. For $v \mid p$ this is clear from the definitions, so we consider $v \in S \setminus \Sigma$ with $v \nmid p$.

Since $A$ is unramified at $v$, the corresponding inertia group $I_v$ acts trivially on $A$, hence the long exact sequence induced by (6.3) yields a short exact sequence
\[
0 \longrightarrow A/pA \longrightarrow H^1(I_v, E[p]) \longrightarrow H^1(I_v, A)[p] \longrightarrow 0.
\]

Since $A$ is divisible, the first term in this sequence is zero, giving an isomorphism
\[
H^1(I_v, E[p]) \simeq H^1(I_v, A)[p].
\]

It follows that the “unramified” local conditions are compatible, as desired. \qed

We record the following result, which is an immediate consequence of the definitions of the Selmer groups involved.

**Corollary 6.6.** *Let $E_1/\mathbb{Q}$ and $E_2/\mathbb{Q}$ be elliptic curves with good supersingular reduction at $p$ such that $E_1[p] \simeq E_2[p]$ as $G_{\mathbb{Q}}$-modules. If $\Sigma$ is a set of places of $K_\infty$ containing all those above which either $E_1[p]$ or $E_2[p]$ is ramified, then for any choice of local conditions $\mathcal{L}$ we have*
\[
\text{Sel}_E^\Sigma(E_1[p]/K_\infty) \simeq \text{Sel}_E^\Sigma(E_2[p]/K_\infty).
\]

**6.4. Results on Iwasawa invariants.** For the rest of this paper, we fix $p$-supersingular elliptic curves $E_1$ and $E_2$ with square-free conductors $N_1$ and $N_2$, respectively, and $E_1[p] \simeq E_2[p]$ as $G_{\mathbb{Q}}$-modules. Moreover, we assume that the imaginary quadratic field $K$ satisfies the Heegner hypothesis with respect to both $N_1$ and $N_2$. We take $S$ to be the finite set of places of $K_\infty$ dividing $N_1N_2p\infty$ and we fix a subset $\Sigma \subset S$ that does not contain the archimedean primes or the primes above $p$ and that does include all the primes above which either $E_1[p]$ or $E_2[p]$ is ramified.

In this section of the paper we relate the Iwasawa invariants of $\text{Sel}^{\pm}(E_i/K_\infty)$ for $i = 1, 2$. In light of Corollary 6.6, what we need now is a way to determine the Iwasawa invariants of the aforementioned Selmer groups from their non-primitive counterparts. Keeping in mind our notational convention (see §6.4) for Iwasawa invariants of cofinitely generated $\Lambda$-modules, set
\[
\mu_\mathcal{L}(E) := \mu(\text{Sel}_\mathcal{L}(E/K_\infty)), \quad \lambda_\mathcal{L}(E) := \lambda(\text{Sel}_\mathcal{L}(E/K_\infty))
\]
and define \( \mu^\Sigma_L(E) \) and \( \lambda^\Sigma_L(E) \) similarly. As before, when \( L = \{ \pm, \pm \} \) we simply write \( \mu_\pm(E) \) and \( \mu^\Sigma_\pm(E) \).

**Lemma 6.7.** For any choice of local conditions \( L \), there are equalities

1. \( \mu_L(E) = \mu^\Sigma_L(E) \)
2. \( \text{rank}_\Lambda X_L(E) = \text{rank}_\Lambda X^\Sigma_L(E) \).

**Proof.** Directly from the definitions, there is an exact sequence of \( \Lambda \)-modules

\[
0 \to \text{Sel}_L(E/K_\infty) \to \text{Sel}^\Sigma_L(E/K_\infty) \to \prod_{v \in \Sigma} H^1_f(K_\infty, v, A).
\]

By [17, Proposition 4.2], the rightmost term is \( \Lambda \)-cotorsion with \( \mu \)-invariant 0. Upon taking Pontryagin duals in (6.5), the lemma follows from [15, Proposition 2.1]. \( \square \)

Recall that, when \( E \) is supersingular, \( \mu_\pm(E) \) is always expected to vanish, and it is known to vanish in some cases (see, e.g., [25, Theorem B]). Under our own running hypotheses, we have the following result.

**Theorem 6.8.** For any choice of local conditions \( L \), the following are equivalent:

1. \( \mu_L(E_1) = 0 \);
2. \( \mu_L(E_2) = 0 \);
3. \( \mu^\Sigma_L(E_1) = 0 \);
4. \( \mu^\Sigma_L(E_2) = 0 \).

**Proof.** Fix a choice of local conditions \( L \). By part (1) of Lemma 6.7, there are equalities

\[
\mu_L(E_1) = \mu^\Sigma_L(E_1), \quad \mu_L(E_2) = \mu^\Sigma_L(E_2).
\]

By Theorem 6.2 and part (2) of Lemma 6.7, \( \text{Sel}^\Sigma_L(E_1/K_\infty) \) and \( \text{Sel}^\Sigma_L(E_2/K_\infty) \) have the same \( \Lambda \)-corank, and by Theorem 6.5 and Corollary 6.6 there is an isomorphism

\[
\text{Sel}^\Sigma_L(E_1/K_\infty)[p] \simeq \text{Sel}^\Sigma_L(E_2/K_\infty)[p].
\]

Then by [15, Corollary 2.4] we have

\[
\mu^\Sigma_L(E_1) = 0 \iff \mu^\Sigma_L(E_2) = 0,
\]

completing the proof. \( \square \)

Now we turn our attention to the corresponding \( \lambda \)-invariants. Recall from [24] that \( G_\infty/n := \text{Gal}(K_\infty/K_n) \) for all \( n \in \mathbb{N} \).

**Proposition 6.9.** Let \( L = \{ \pm, \emptyset \} \) or \( L = \{ \emptyset, 0 \} \). If \( \mu_L(E_1) = \mu_L(E_2) = 0 \), then

\[
\lambda^\Sigma_L(E_1) = \lambda^\Sigma_L(E_2).
\]

**Proof.** First of all, note that, by Theorem 6.8, either both \( \mu \)-invariants vanish or neither does. Assume that both vanish. By Theorem 6.2, \( \text{Sel}_L(E_1/K_\infty) \) and \( \text{Sel}_L(E_2/K_\infty) \) have the same \( \Lambda \)-coranks, and Lemma 6.7 ensures that the same is true for their non-primitive counterparts. Thanks to Proposition 6.3 and [12, Corollary 4.1.2], we also know that neither \( \text{Sel}^\Sigma_L(E_1/K_\infty) \)
nor \( \text{Sel}_E^\Sigma(L²/K_\infty) \) has any finite-index \( \Lambda \)-submodules. Thus, by [15] Proposition 2.10 their \( \lambda \)-invariants are determined by the \( G_\infty/n \)-coinvariants of (Tate twists of) the corresponding residual non-primitive Selmer groups. Finally, by Corollary 6.6 there is an isomorphism of \( \Lambda \)-modules

\[
\text{Sel}_E^\Sigma(E[1]/K_\infty) \simeq \text{Sel}_E^\Sigma(E[2]/K_\infty),
\]

which yields the desired equality. \( \square \)

When \( \mathcal{L} = \{ \pm, \pm \} \), the \( \lambda \)-invariants of \( E_1 \) and \( E_2 \) will usually not be equal, but they will be related by an explicit formula, which we will now describe. For notational simplicity, for an elliptic curve \( E \) and a set of places \( \Sigma \) let us set

\[
\delta(\Sigma, E) := \sum_{v \in \Sigma} \lambda\left(H^1_{/f}(K_\infty, T), A\right).
\]

Note that this definition is independent of \( \mathcal{L} \).

Remark 6.10. This quantity can be computed explicitly; see [17] Proposition 4.2 for details.

Proposition 6.11. Let \( \mathcal{L} \in \{ \{ \pm, 0 \}, \{ 0, 0 \} \} \) and suppose that \( \mu_\mathcal{L}(E_1) = \mu_\mathcal{L}(E_2) = 0 \). Then

\[
\lambda_\mathcal{L}(E_1) - \lambda_\mathcal{L}(E_2) = \delta(\Sigma, E_1) - \delta(\Sigma, E_2).
\]

Proof. Letting \( E \) denote either \( E_1 \) or \( E_2 \), there is a short exact sequence

\[
0 \longrightarrow \text{Sel}_E(E/K_\infty) \longrightarrow \text{Sel}_E^\Sigma(E/K_\infty) \xrightarrow{\gamma} \prod_{v \in \Sigma} H^1_{/f}(K_\infty, T) \longrightarrow 0,
\]

where the surjectivity of \( \gamma \) follows from Proposition 6.9. By [17] Proposition 4.2] the last nonzero term in (6.7) is \( \Lambda \)-cotorsion. Dualizing and applying [15] Proposition 2.1], we have

\[
\lambda(E_i) + \delta(\Sigma, E_i) = \lambda^\Sigma_E(E_i)
\]

for \( i = 1, 2 \). But Proposition 6.9 tells us that \( \lambda^\Sigma_E(E_1) = \lambda^\Sigma_E(E_2) \), giving us the result. \( \square \)

Before proving the main result of this section, we need to introduce just a few more items of notation. First, recall that \( T \) denotes the \( p \)-adic Tate module of \( E \) and that \( p\mathcal{O}_K = pp^c \). As mentioned in Remark 6.1 for each \( v \mid p \) the Kummer map allows us to view \( \mathbb{H}_v^+ \) as a submodule of \( H^1(K_\infty, T) \). Define \( H^1_\pm(K_\infty, T) \) to be the orthogonal complement of \( \mathbb{H}_v^+ \) under the local Tate pairing

\[
H^1(K_\infty, T) \times H^1(K_\infty, T) \rightarrow \mathbb{Q}_p/\mathbb{Z}_p.
\]

For every other place \( v \), write \( H^1_{/f}(K_\infty, T) \) for the regular unramified local condition. Then we may define

\[
\text{Sel}_E^\Sigma(T/K_\infty) := \ker \left( H^0_S(K_\infty, T) \longrightarrow \prod_{v \mid p} H^1_{/f}(K_\infty, T) \times \prod_{v \mid p} \frac{H^1(K_\infty, T)}{H^1_\pm(K_\infty, T)} \right).
\]

It follows that there is a natural restriction map

\[
\text{loc}_p : \text{Sel}_E^\Sigma(T/K_\infty) \longrightarrow H^1_{/f}(K_\infty, p, T).
\]

Let

\[
\text{ck}_\pm(T) := \frac{H^1_\pm(K_\infty, p, T)}{\text{loc}_p(\text{Sel}_E^\Sigma(T/K_\infty))}
\]

be the cokernel of \( \text{loc}_p \).
Remark 6.12. In fact, we could also define a non-primitive version Sel$^±,\Sigma(T/K_\infty)$ in the obvious way, but by duality the compact local condition for $v \nmid p$ is empty, so that there is an equality Sel$^\pm(T/K_\infty) = $ Sel$^{±,\Sigma}(T/K_\infty)$.

Remark 6.13. It is shown in the proof of [6, Lemma 5.9] that $ck^\pm(T)$ is a torsion $\Lambda$-module.

To simplify the notation in the remaining results, we write $H_v := H_1^1(K_\infty,v,A)^\vee$ for the Pontryagin dual of $H_1^1(K_\infty,v,A)$.

Proposition 6.14. There is a short exact sequence

\[ 0 \to \prod_{v \in \Sigma} H_v \to X^\Sigma_\pm(E) \to X_\pm(E) \to 0. \]

Proof. There is a commutative diagram

\[
\begin{array}{cccccccc}
0 & \to & \text{ck}^\pm(T) & \to & X^\Sigma_\pm,0(E) & \to & X^\Sigma_\pm(E) & \to & 0 \\
& & \downarrow{\gamma_1} & & \downarrow{\gamma_2} & & \\
0 & \to & \text{ck}^\pm(T) & \to & X_\pm,0(E) & \to & X_\pm(E) & \to & 0 \\
\end{array}
\]

in which the rows are exact by global duality, whereas the surjectivity of $\gamma_1$ and $\gamma_2$ follows from the relevant definitions and Remark 6.12. Taking the dual of the exact sequence (6.7), we get

\[ \ker \gamma_1 = \prod_{v \in \Sigma} H_v, \]

and we conclude the proof by applying the snake lemma. \[ \square \]

We may now extend Theorem 5.9 to the non-primitive $\pm$-Selmer groups.

Theorem 6.15. The Selmer group Sel$^{±,\Sigma}(E/K_\infty)$ has no proper finite-index $\Lambda$-submodules.

Proof. By duality, it is equivalent to show that $X^\Sigma_\pm(E)$ has no nontrivial finite $\Lambda$-submodules. Suppose $M \subset X^\Sigma_\pm(E)$ is a finite $\Lambda$-submodule and consider diagram (6.8). If $\gamma_2(M)$ is not zero, then it is a nontrivial finite $\Lambda$-submodule of $X_\pm(E)$, contradicting Theorem 5.9. Thus $M \subset \ker(\gamma_2)$. By the snake lemma, $\ker(\gamma_1) \simeq \ker(\gamma_2)$, therefore we may view $M$ as a submodule of $X^\Sigma_\pm,0(E)$. But, by Proposition 6.3 and [12, Corollary 4.1.2], this also has no nontrivial finite $\Lambda$-submodules, so we must have $M = 0$ as desired. \[ \square \]

Finally, we are ready to give a result on the variation of Iwasawa invariants in the spirit of Greenberg–Vatsal.

Theorem 6.16. Assume that $\mu^\pm(E_i) = 0$ for $i = 1, 2$. Then

\[ \lambda^\pm(E_1) - \lambda^\pm(E_2) = \delta(\Sigma, E_1) - \delta(\Sigma, E_2). \]

Proof. This follows precisely as in the proof of Proposition 6.11 using instead the exact sequence from Proposition 6.14. \[ \square \]

Remark 6.17. Theorem 6.16 is a supersingular analogue of [15, Theorem 5.8], which is a result for $p$-ordinary elliptic curves. In that result, however, there is an extra term that shows up in the variation formula and corresponds to the cokernel of the local restriction map. If one imposes the extra hypotheses from [3], then it is possible to carry out the arguments in the proof of Proposition 6.14 which allow for the removal of the cokernel terms.
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