THE GREEN POLYNOMIALS VIA VERTEX OPERATORS

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Abstract. An iterative formula for the Green polynomial is given using the vertex operator realization of the Hall-Littlewood function. Based on this, (1) a general combinatorial formula of the Green polynomial is given; (2) several compact formulas are given for Green’s polynomials associated with upper partitions of length \( \leq 3 \) and the diagonal lengths \( \leq 3 \); (3) a Murnaghan-Nakayama type formula for the Green polynomial is obtained; and (4) an iterative formula is derived for the bitrace of the finite general linear group \( G \) and the Iwahori-Hecke algebra of type \( A \) on the permutation module of \( G \) by its Borel subgroup.

1. Introduction

The Green polynomials \( Q^\lambda_{\mu}(q) \) were introduced by Green [3] to compute irreducible characters of the finite general linear group \( GL_n(\mathbb{F}_q) \). When \( q = \infty \), they are exactly the irreducible character value \( \chi^\lambda(C_\mu) \) of the symmetric group \( S_n \). According to Hotta and Springer [4], the \( t \)-coefficients \( \psi^{\mu,t} \) of \( Q^\lambda_{\mu}(t) \) are certain characters of \( S_n \) that afford the \( S_n \)-action on the rational cohomology \( H^*(X_\mu) \) of the variety \( X_\mu \), the subvariety fixed by the unipotent elements of type \( \mu \) of the flag variety.

The Green polynomial or its variant \( X^\lambda_{\mu}(t) = t^{\mu(\mu)}Q^\lambda_{\mu}(t^{-1}) \) is defined as the transition coefficient of the power-sum symmetric function \( p_\mu \) in terms of the Hall-Littlewood symmetric function \( P_\lambda(t) \) [14]. Let \( f^\nu_{\lambda\mu}(t) \) be the structure constants (Hall polynomial) of the Hall algebra generated by the \( P_\lambda \). By Green’s original definition, the Green polynomial can be written as a sum of products of lower degree Green polynomials with weights \( f^\nu_{\lambda\mu}(t) \):

\[
Q^\nu_{\rho\tau}(t) = \sum_{\lambda,\mu} f^\nu_{\lambda\mu}(t)Q^\lambda_{\rho}(t)Q^\mu_{\tau}(t).
\]

Based on this iteration, Green has given a table for \( n \leq 5 \). Morris [11] used an implicit iteration of the Kostka-Foulkes polynomial to provide a table for \( n = 6, 7 \). As far as we know, no explicit formula is known for \( Q^\lambda_{\mu}(t) \) in the general case.

Lascoux-Leclerc-Thibon [8] has proved a formula (LLT) for the Green polynomials at roots of unity, conjectured by Morris-Sultana [13]. Morita

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[10] has generalized the LLT formula and given a formula for $Q^\lambda_\mu(\omega)$ for $\lambda$ being hook-shaped at a root of unity $\omega$.

Recently, Bryan and one of us [2] have used vertex operators to derive a direct iterative formula for the Kostka-Foulkes polynomial. In the first part of the paper, we will use the same idea to give an iterative formula for the Green polynomials. Using this formula, we are able to recover all previously known compact formulas for the Green polynomials and offer some new ones.

First of all, we obtain a general combinatorial formula:

$$X^\lambda_\mu(t) = \sum_{\{\rho^i\},\{\tau^i\}} \frac{(-1)^{l(\rho^i)}}{z_{\rho^i}(t)}$$

summed over interlacing sequences $\{\tau^i\}$ and $\{\rho^i\}$ of partitions (see Theorem 2.7). In particular, this also gives a combinatorial formula of the irreducible character value for the symmetric group $S_n$ ($t = 0$ or $q = \infty$).

Special cases of our general formula recover previously known formulas for $X^\lambda_\mu(t)$. In fact, our compact formula $Q^{k,1n-k}_\mu(t)$ for all values of $t$ recovers and generalizes Morita’s formula. As examples, we check that our formula recovers the known ones for $Q^{1n}_\mu(t)$ and $Q^{1\mu}_\mu(t)$. Moreover, we also derive several general formulas for the Green polynomials, namely, compact formulas of $Q^\lambda_\mu(t)$ for $l(\mu) \leq 3$ as well as with Frobenius diagonal of $\mu \leq 3$ are obtained.

Our method relies upon applications of dual vertex operators on the vertex operator realization of the Hall-Littlewood functions [7] and straightening out the general Hall-Littlewood operators associated with compositions to those with partitions. One application is to derive a Murnaghan-Nakayama formula for the Green polynomial in the general case.

The second part of the paper deals with an important application of our method. Let $G = GL_n(F_q)$ and $H_n(q)$ the Iwahori-Hecke algebra in type $A$. Let $B$ be the upper Borel subgroup of $G$, the algebra $H_n(q)$ is naturally realized via the permutation module $\text{Ind}_{G}^{1}B$, where 1 is the trivial $B$-module. In fact, this model also gives an alternative derivation of the Frobenius character formula of $H_n(q)$ [15].

Note that the general linear group $G$ acts on $\text{Ind}_{B}^{G}1$ by left multiplication, which commutes with the natural action of the Iwahori-Hecke algebra $H_n(q)$. By Green’s theory the multiplicity of the irreducible $H_q(n)$-module appearing in $\text{Ind}_{B}^{G}1$ indexed by $\lambda$ is controlled by the Kostka-Foulkes polynomial $K_{\lambda\mu}(t)$, and the latter is exactly the irreducible character value $\chi^\lambda(u_\mu)$ of $GL_n(F_q)$ at the unipotent element $u_\mu$ in type $\mu$. Halverson and Ram [5] derived a combinatorial formula for the $(G, H_q(n))$-bitrace of the permutation module $\text{Ind}_{B}^{G}1$ using the Bruhat decomposition. In this paper, we will derive an iterative formula for the bitrace on $\text{Ind}_{B}^{G}1$ and then a general formula for the bitrace. Based on the iterative formula we also give a table of the bitrace for $n \leq 5$. 
The paper is naturally divided into three parts. In Sect. 2 we first recall the vertex operator realization of the Hall-Littlewood functions and express the Green polynomial \( X^{\lambda}_{\mu}(t) \) as the transition coefficients between the Hall basis and the power-sum basis. Using the technique of vertex operators, we derive a useful iterative formula for \( X^{\lambda}_{\mu}(t) \). Then we derive a general formula of \( X^{\lambda}_{\mu}(t) \) as well as several compact formulas in special cases. In Sect. 3 we derive a Murnaghan-Nakayama type formula for the Green polynomial by using a straightening formula of the Hall-Littlewood functions indexed by compositions. Finally in Sect. 4 we compute the bitrace of the finite general linear group \( G \) and the Iwahori-Hecke algebra of type \( A \) on the permutation module of \( G \), and derive an iterative formula as well as the general formula. The paper is concluded with a table of the bitrace for \( n \leq 5 \).

2. VERTEX OPERATOR REALIZATION OF HALL-LITTLEWOOD POLYNOMIALS

A composition \( \lambda = (\lambda_1, \lambda_2, \ldots, \lambda_l) \), denoted by \( \lambda \vdash n \), is a sequence of nonnegative integers \( \lambda_i \) (the parts) that sum up to \( n \). If the sequence is weakly decreasing, then \( \lambda \) is called a partition and denoted as \( \lambda \vdash n \). The total sum \( \sum \lambda_i = n \) is the weight of \( \lambda \) and the number of (nonzero) parts is denoted by \( l(\lambda) \). A partition \( \lambda \) of weight \( n \) is usually denoted by \( \lambda \vdash n \), and the set of partitions will be denoted by \( \mathcal{P} \). Sometimes \( \lambda \) is arranged in the ascending order: \( \lambda = (1^{m_1} \ 2^{m_2} \ \cdots) \) with \( m_i \) being the multiplicity of \( i \) in \( \lambda \). For partition \( \lambda \), let \( z_\lambda = \prod_{i \geq 1} i^{m_i(\lambda)} m_i(\lambda)! \) and denote

\[
(2.1) \quad z_\lambda(t) = \frac{z_\lambda}{\prod_{i \geq 1} (1 - t^{\lambda_i})}
\]

The Young or Ferrers diagram of partition \( \lambda \) is the diagram of \( l(\lambda) \) rows of boxes aligned to the left where the \( i \)th row consists of \( \lambda_i \) boxes. The partition \( \lambda' = (\lambda'_1, \ldots, \lambda'_l) \) corresponding to the reflection of the Young diagram of \( \lambda \) along the diagonal is called the dual partition of \( \lambda \).

The juxtaposition \( \lambda \cup \mu \) of partitions \( \lambda \) and \( \mu \) is defined as the union of all parts of \( \lambda \) and \( \mu \) and then arranged in the descending order.

Let \( \Lambda \) be the ring of symmetric functions over the ring of integers. Let \( F = \mathbb{Q}(t) \) be the field of rational functions in \( t \), and we will be mainly working with the ring \( V = \Lambda_F \). The space \( \Lambda \) has several well-known bases indexed by partitions: elementary symmetric functions, monomial symmetric functions, homogeneous symmetric functions, and Schur functions. The set of power sum symmetric functions is a linear basis of \( \Lambda_Q \). Here the \( n \)th degree power-sum symmetric function \( p_n = \sum x^n_i \), and the power sum function \( p_\lambda = p_{\lambda_1} p_{\lambda_2} \cdots \). Using the degree gradation, \( V \) becomes a graded ring

\[
(2.2) \quad V = \bigoplus_{n=0}^{\infty} V_n.
\]

A linear operator \( A \) is of degree \( n \) if \( A(V_m) \subset V_{m+n} \).
The space $V$ is equipped with the Hall-Littlewood bilinear form $\langle \ , \ \rangle$ defined by
\begin{equation}
\langle p_{\lambda}, p_{\mu} \rangle = \delta_{\lambda \mu} z_{\lambda}(t).
\end{equation}
As $\{z_{\lambda}(t)^{-1} p_{\lambda}\}$ is the dual basis of the power sum basis, the dual operator of the multiplication operator $p_{n}$ is the differential operator $p_{n}^{*} = \frac{n}{(1-t)^{n}} \frac{\partial}{\partial p_{n}}$ of degree $-n$. Note that $^{*}$ is $\mathbb{Q}(t)$-linear and anti-involutive satisfying
\begin{equation}
\langle H_{n} u, v \rangle = \langle u, H_{n}^{*} v \rangle
\end{equation}
for $u, v \in V$.

We now recall the vertex operator realization of the Hall-Littlewood symmetric functions from [7].

The vertex operators $H(z)$ and its dual $H^{*}(z)$ are $t$-parameterized linear maps: $V \longrightarrow V[[z, z^{-1}]]$ defined by
\begin{equation}
H(z) = \exp \left( \sum_{n \geq 1} \frac{1-t}{n} p_{n} z^{n} \right) \exp \left( - \sum_{n \geq 1} \frac{\partial}{\partial p_{n}} z^{-n} \right)
\end{equation}
\begin{equation}
= \sum_{n \in \mathbb{Z}} H_{n} z^{n},
\end{equation}
\begin{equation}
H^{*}(z) = \exp \left( - \sum_{n \geq 1} \frac{1-t}{n} p_{n} z^{n} \right) \exp \left( \sum_{n \geq 1} \frac{\partial}{\partial p_{n}} z^{-n} \right)
\end{equation}
\begin{equation}
= \sum_{n \in \mathbb{Z}} H_{n}^{*} z^{-n}.
\end{equation}
Here $V[[z, z^{-1}]] = F[[z, z^{-1}]] \otimes V$ is the vector space over formal Laurent series in $z$. The components $H_{n}$ and $H_{n}^{*}$ are endomorphisms of $V$ with degree $n$, thus $H_{-n}$ and $H_{n}^{*}$ are annihilation operators for $n > 0$. We collect their relations as follows.

**Proposition 2.1.** [7] The operators $H_{n}$ and $H_{n}^{*}$ satisfy the following relations
\begin{equation}
H_{m} H_{n} - t H_{n} H_{m} = t H_{m+1} H_{n-1} - H_{n-1} H_{m+1},
\end{equation}
\begin{equation}
H_{m}^{*} H_{n}^{*} - t H_{n}^{*} H_{m}^{*} = t H_{m-1}^{*} H_{n+1}^{*} - H_{n+1}^{*} H_{m-1}^{*},
\end{equation}
\begin{equation}
H_{m} H_{n}^{*} - t H_{n}^{*} H_{m} = t H_{m-1} H_{n}^{*} - H_{n}^{*} H_{m-1} + (1-t)^{2} \delta_{m,n},
\end{equation}
\begin{equation}
H_{-n}^{*} 1 = \delta_{n,0}, \quad H_{n} 1 = \delta_{n,0},
\end{equation}
where $\delta_{m,n}$ is the Kronecker delta function.

We remark that the indexing of $H_{n}$ and $H_{n}^{*}$ is different from that of [7], where $H_{n}$ was denoted as $H_{-n}$ for instance.

As the vacuum vector $1$ is annihilated by $p_{n}^{*}$, we have that
\begin{equation}
H(z).1 = \exp(\sum_{n=1}^{\infty} \frac{1-t}{n} p_{n} z^{n}) = \sum_{n=0}^{\infty} q_{n} z^{n}
\end{equation}
where \( q_n \) is a symmetric function of degree \( n \) in \( V \), called the Hall-Littlewood polynomial associated with one-row partition \( (n) \):

\[
q_n = H_n.1 = \sum_{\lambda \vdash n} \frac{1}{z_\lambda(\lambda)} p_\lambda.
\]

The proposition implies that

\[
H_n H_{n+1} = t H_{n+1} H_n,
\]

\[
H_n^* H_{n-1} = t H_{n-1}^* H_n^*,
\]

\[
\langle H_n.1, H_n.1 \rangle = \sum_{\lambda \vdash n} \frac{1}{z_\lambda(t)} = 1 - t, \quad n > 0
\]

\[
\langle H_n.1, H_{n-1}^* \rangle = \sum_{\lambda \vdash n} \frac{(-1)^{l(\lambda)}}{z_\lambda(t)} = t^n - t^{n-1}, \quad n > 0
\]

where the last two identities follow from (2.9) and (2.7) by induction.

Note that \( H_n.1 = q_n(t) \) can be generalized to all situations as the vertex operator realization of the Hall-Littlewood functions [14]. For each partition \( \lambda \), denote \( q_\lambda = q_{\lambda_1} q_{\lambda_2} \cdots \), then \( \{q_\lambda\} \) also forms a basis of \( V \).

**Theorem 2.2.** [7] Let \( \lambda = (\lambda_1, \ldots, \lambda_l) \) be a partition. The vertex operator products \( H_{\lambda_1} \cdots H_{\lambda_l}.1 \) is the Hall-Littlewood function \( Q_\lambda(t) \):

\[
H_{\lambda_1} \cdots H_{\lambda_l}.1 = Q_\lambda(t) = \prod_{i<j} \frac{1 - R_{ij}}{1 - t R_{ij}} q_{\lambda_1} \cdots q_{\lambda_l}
\]

where the raising operator \( R_{ij} q_\lambda = q(\lambda_1, \ldots, \lambda_{i-1}, \lambda_i + 1, \ldots, \lambda_{j-1}, \lambda_j, \ldots, \lambda_l) \). Moreover, \( H_{\lambda}.1 = H_{\lambda_1} \cdots H_{\lambda_l}.1 \) are orthogonal in \( V \):

\[
\langle H_{\lambda}.1, H_\mu.1 \rangle = \delta_{\lambda\mu} b_\lambda(t),
\]

where \( b_\lambda(t) = (1 - t)^{l(\lambda)} \prod_{i \geq 1} [m_i(\lambda)]! \) and \( [n] = \frac{1 - t^n}{1 - t} \).

As a result, the transition matrix between the bases \( \{p_\lambda\} \) and \( \{H_\lambda\} \) gives rise to Green’s polynomials. More precisely, for \( \lambda, \mu \vdash n \), let \( X_\mu^\lambda(t) \) be the coefficient of \( P_\lambda = b_\lambda(t)^{-1} Q_\lambda(t) \) in \( p_\mu \):

\[
p_\mu = \sum_\lambda X_\mu^\lambda(t) P_\lambda(t).
\]

It is known that \( X_\mu^\lambda(t) \) is a polynomial in \( t \) of degree \( n(\lambda) \), and the *Green polynomials* are defined as \( Q_\mu^\lambda(t) = t^{n(\lambda)} X_\mu^\lambda(t)^{-1} \) for all partitions \( \lambda, \mu \) of the same weight [3] [14]. In the following we simply regard \( X_\mu^\lambda(t) \) as Green’s polynomials.

Using theorem 2.2 we can write Green’s polynomials as:

\[
X_\mu^\lambda(t) = \langle H_\lambda.1, p_\mu \rangle.
\]

Thus \( X_\mu^\lambda(t) = 0 \) unless \( |\lambda| = |\mu| \), and \( X_\mu^n(t) = \delta_{n,|\lambda|} \) by (2.12).
Now let’s discuss how to compute $X^\mu_{\lambda}(t)$. Consider the following linear maps $V \rightarrow V[[z, z^{-1}]]$:

\[(2.21) \quad P(z) = \sum_{n \geq 1} p_n z^n,\]
\[(2.22) \quad P^*(z) = \sum_{n \geq 1} p^*_n z^{-n},\]

where the operator $p_n$ and the dual $p^*_n$ are of degree $n$ and $-n$ respectively.

The normal ordering of vertex operators are defined as usual, so

\[\begin{align*}
: H^*(z) P(w) : &= P(w) H^*(z), \\
: H(z) P^*(w) : &= P^*(w) H(z).
\end{align*}\]

By the usual techniques of vertex operators, we have the following operator product expansions:

\[(2.23) \quad H^*(z) P(w) = P(w) H^*(z) + H^*(z) \frac{w}{z-w},\]
\[(2.24) \quad P^*(z) H(w) = H(w) P^*(z) + H(w) \frac{w}{z-w}.\]

Taking coefficients of the above expressions, we immediately get the following commutation relations.

**Proposition 2.3.** The commutation relations between the Hall-Littlewood vertex operators and power sum operators are:

\[(2.25) \quad H^*_m p_n = p_n H^*_m + H^*_m H^*_{m-n},\]
\[(2.26) \quad p^*_m H_n = H_n p^*_m + H_n H_{n-m}.\]

To proceed we need some notations. For each partition $\lambda = (\lambda_1, \ldots, \lambda_l)$, we define that

\[(2.27) \quad \lambda[i] = (\lambda_{i+1}, \ldots, \lambda_l), \quad i = 0, 1, \ldots, l\]

So $\lambda[0] = \lambda$ and $\lambda[l] = \emptyset$. We define a subpartition $\tau$ of $\lambda$, denoted $\tau \triangleleft \lambda$, if the parts of $\tau$ are also parts of $\lambda$, i.e. $\tau = (\lambda_{i_1}, \ldots, \lambda_{i_s})$ for some $1 \leq i_1 < \cdots < i_s \leq l$. Note that $\tau$ could be $\emptyset$ or $\lambda$.

Let $D^{(i)}(\lambda)$ be the number of subpartitions of $\lambda$ with weight $i$, then the generating function of subpartitions of $\lambda$ is given by

\[(2.28) \quad D_\ell(\lambda) = \sum_{i \geq 0} D^{(i)}(\lambda) t^i = \sum_{\tau \triangleleft \lambda} t^{\left|\tau\right|} = (1 + t)^{m_1(\lambda)} (1 + t^2)^{m_2(\lambda)} \cdots = \prod_{i \geq 1} [2]_{t^i}^{m_i(\lambda)}.
\]

In particular, the total number of subpartitions of $\lambda$ is $2^l(\lambda)$. 
Theorem 2.4. For partition $\mu \vdash n$ with $l(\mu) = l$ and integer $k$,

\[
H_k^* p_{\mu} = \sum_{\tau \ll \mu} p_{\tau} H_{k+|\tau|-n}^* = \sum_{i=0}^{n-k} \sum_{\tau \ll \mu, \tau^+ i} p_{\tau} H_{k+i-n}^*.
\]  

(2.29)

\[
p_k^* H_{\mu} = \sum_{i=1}^{l} H_{\mu_1} \ldots H_{\mu_{i-1}} \ldots H_{\mu_{i}}.
\]  

(2.30)

Proof. The second relation (2.30) follows from (2.29) by taking $\ast$. We argue by induction on $l(\lambda)$ for the first relation. The initial step is clear. Now assume that (2.29) holds for any partition with length $< l(\lambda)$, so it follows from Proposition 2.3 and induction hypothesis that

\[
H_k^* p_{\lambda} = p_{\lambda_1} H_k^* p_{\lambda_2} \ldots p_{\lambda_l} + H_k^* p_{\lambda_{l+1}} \ldots p_{\lambda_l}.
\]

(2.29)

\[
= p_{\lambda_1} \sum_{\rho < \lambda^{[1]}} p_{\rho} H_{k+|\lambda_1|+|\tau|-n}^* + \sum_{\rho < \lambda^{[1]}} p_{\rho} H_{k+|\rho|-n}^*
\]

(2.30)

\[
= \sum_{\tau \ll \lambda} p_{\tau} H_{k+|\tau|-n}^*.
\]

Note that when $n > |\lambda|$ (cf. (2.29))

\[
H_n^* p_{\lambda_1} p_{\lambda_2} \ldots p_{\lambda_l} = 0.
\]

To effectively use our result, let us also compute the following symmetric function in $V$. For $n \geq 0$, we have $H_n^*.1 = \delta_{n,0}$ and

\[
H_{n-1}^*.1 = \sum_{\lambda \vdash n} \frac{(-1)^{l(\lambda)}}{z_{\lambda}(t)} p_{\lambda},
\]

(2.31)

which implies that $\langle p_{\lambda}, H_{n-1}^*.1 \rangle = (-1)^{l(\lambda)} \delta_{n,|\lambda|}$.

Example 2.5. Using Theorem 2.4 and (2.16) we can easily compute some Green’s polynomials.

\[
X_{(42)}^{(12)}(t) = \langle p_2 p_2 p_1 p_1, H_4 H_2.1 \rangle
\]

(2.16)

\[
= \langle H_4^* p_2 p_2 p_1 p_1, H_2.1 \rangle
\]

(2.29)

\[
= 2 \langle p_2, H_2.1 \rangle + \langle H_{-2}.1, H_2.1 \rangle + 2 \langle p_1 H_{-1}.1, H_2.1 \rangle + \langle p_1 p_1, H_2.1 \rangle
\]

(2.17)

\[
= 2 + t^2 - t + 2(t - 1) + 1
\]

(2.18)

\[
= t^2 + t + 1.
\]

Theorem 2.6. For partition $\lambda, \mu \vdash n$,

\[
X_{\mu}^\lambda(t) = \sum_{\tau \ll \mu} \sum_{|\tau| \leq n - \lambda_{1}} \frac{(-1)^{l(\rho)}}{z_{\rho}(t)} X_{\tau \cup \rho}^{\lambda^{[1]}}(t)
\]

(2.17)
\[(2.32) \quad = \sum_{i=0}^{n-\lambda_1} \sum_{\tau \prec \mu} \sum_{\rho \in (n-\lambda_1-i)} \frac{(-1)^l(\rho)}{z_{\rho}(t)} X_{\tau \cup \rho}^{(1)}(t).\]

**Proof.** This recurrence formula follows from \[(2.29)\] and \[(2.31)\]. \(\square\)

When \(\lambda = (n)\), the summation is empty, so \(X_{\mu}^{(n)}(t) = 1\) for any \(\mu \vdash n\). When \(\lambda = (m, n)\)

\[X_{\mu}^{(m, n)}(t) = \sum_{\tau \prec \mu} \sum_{|\tau| \leq n} \frac{(-1)^l(\rho)}{z_{\rho}(t)} \]

\[= \sum_{\tau \prec \mu} (t^{n-|\tau|} - t^{n-|\tau|-1}) + \sum_{|\tau| = n} 1\]

\[= (t-1)[D_{t-1}(\mu)^{n-1}]_+ + D^{(n)}(\mu)\]

\[= (t-1)[D_{t}(\mu)^{m-1}]_+ + D^{(n)}(\mu),\]

where \([f(t)]_+\) is the regular part of the function \(f(t)\) in \(t\).

One can use the compact formula as follows.

\[X_{(21^{12})}^{(42)}(t) = (t-1)[(1 + t)^2 + t^2 t^{-5}]_+ + D^{(2)}(2^2 1^2)\]

\[= (t-1)(t+2) + 3 = t^2 + t + 1.\]

Using the iteration and \(X_{\mu}^{(n)}(t) = 1\) it follows that

\[X_{\mu}^{(\lambda_1, \lambda_2, \lambda_3)}(t)\]

\[= \sum_{\tau^1 \prec \mu, |\tau^1| \leq |\lambda^{[1]}|} \frac{(-1)^l(\rho^1)}{z_{\rho^1}(t)} \langle p_{\rho^1 \cup \tau^1}, H_{\lambda^{[1]}} \rangle \]

\[= \sum_{\tau^1 \prec \mu, |\tau^1| \leq |\lambda^{[1]}|} \frac{(-1)^l(\rho^1)}{z_{\rho^1}(t)} \left( \sum_{\rho^2 \vdash \lambda^2} \frac{(-1)^l(\rho^2)}{z_{\rho^2}(t)} \right)\]

\[= \sum_{\tau^1 \prec \mu, |\tau^1| \leq |\lambda^{[1]}|} \frac{(-1)^l(\rho^1)}{z_{\rho^1}(t)} ([D_{t-1}(\rho^1 \cup \tau^1)t^{\lambda_3-1}]_+(t-1) + D^{(\lambda_3)}(\rho^1 \cup \tau^1))\]

\[= \sum_{\rho^1 \vdash |\lambda^{[1]}|-|S(\mu)|} \frac{(-1)^l(\rho^1)}{z_{\rho^1}(t)} ([D_{t-1}(\rho^1 \cup S(\mu))t^{\lambda_3-1}]_+(t-1) + D^{(\lambda_3)}(\rho^1 \cup S(\mu)))\]

Let \(\lambda\) and \(\mu\) be two partitions of \(n\) and \(l = l(\lambda)\). Let \(\rho^i, \tau^i\) be two sequences of \(l-1\) partitions such that \(|\tau^i| \leq |\lambda^{[i]}|\) and

\[\tau^1 \prec \mu, \rho^1 \vdash |\lambda^{[1]}| - |\tau^1|; \quad \tau^2 \prec \rho^1 \cup \tau^1, \rho^2 \vdash |\lambda^{[2]}| - |\tau^2|; \quad \ldots \ldots ;\]
\[\tau^{l-1} \triangleleft \rho^{l-2} \cup \tau^{l-2}, \rho^{l-1} \vdash |\lambda^{[l-1]}| - |\tau^{l-1}|.\]

One starts with a subpartition \(\tau^1\) of \(\mu\) with weight \(\leq |\lambda[1]|\), then picks any partition \(\rho^1\) of weight of the difference \(|\lambda[1]| - |\tau^1|\). Then one selects the next subpartition \(\tau^2\) of \(\tau^1 \cup \rho^1\), and picks any partition \(\rho^2\) of weight \(|\lambda[2]| - |\tau^2|\), and continue to form \(\{\tau^3, \rho^3\}, \cdots\), etc. So the weights of \(\tau^i \cup \rho^i\) are decreasing as \(|\lambda[i]|\).

By the same method, we have the general formula:

**Theorem 2.7.** Let \(\lambda, \mu\) be two partitions. Then the Green polynomial

\[
X^{(\lambda)}_{\mu}(t) = \sum_{\{\rho^i, \tau^i\}} \frac{l(\lambda)-1}{\prod_{j=1}^{l(\lambda)-1}} \frac{(-1)^{l(\rho^i)}}{z_{\rho^i}(t)}
\]

where the sum runs through all sequences of \(l(\lambda) - 1\) pairs of partitions \(\{\rho^i, \tau^i\}\) such that \(|\tau^i| \leq |\lambda[i]|\), \(\tau^i \triangleleft \tau^{i-1} \cup \rho^{i-1}\) and \(\rho^i \vdash |\lambda[i]| - |\tau^i|\), where \(i = 1, \cdots, l(\lambda) - 1\) and \(\tau^0 \cup \rho^0 = \mu\).

**Proof.** This follows from repeatedly using (2.32), and notice that for any \(\mu \vdash m\), \(X^{(m)}_{\mu}(t) = 1\) by above. \(\square\)

**Lemma 2.8.** For partition \(\lambda\) of \(n\), we have that

\[
\sum_{\tau < \lambda, \tau \neq \emptyset} \prod_{j \geq 1} (t^\tau_j - 1) = t^n - 1,
\]

\[
\sum_{\tau < \lambda} \prod_{j \geq 1} \frac{1}{t^\tau_j - 1} = \frac{t^n}{\prod_{i \geq 1} (t^{\lambda_i} - 1)}
\]

For partition \(\lambda\), define \([\lambda] = \prod_{i \geq 1} (t^{\lambda_i} - 1) \text{ and } [\emptyset] = 1\). Then the function \(\sum_{\tau < \lambda} [\tau]\) is strictly multiplicative for \(\lambda\) with respect to its parts. Note that

\[
\sum_{\tau < \lambda} [\tau] = \sum_{j=0}^m \binom{m}{j} (t^i - 1) = t^m.
\]

Therefore

\[
\sum_{\tau < \lambda} [\tau] = \prod_{i \geq 1} \left( \sum_{\tau < \lambda} [\tau] \right) = \prod_{i \geq 1} t^{im} = [\lambda[1]].
\]

The other identity can be proved similarly.

We can compute more Green’s polynomials, for example some well-known formulas in [14, §III.7].

**Example 2.9.** For each partition \(\lambda\) of \(n\), we have that

\[
X^{(n)}_{\lambda}(t) = \frac{\prod_{i \geq 1} (t^i - 1)}{t \prod_{j \geq 1} (t^{\lambda_j} - 1)} = \frac{[n]!}{|\lambda|!}.
\]
This can be checked by induction using Theorem 2.4 and Lemma 2.8. The initial step of $n = 1$ is clear. Now for $\lambda \vdash n$

\[
X_\lambda^{(1^n)}(t) = \sum_{\tau \prec \lambda, |\tau| \leq n-1} \sum_{\rho \vdash (n-1-|\tau|)} \frac{(-1)^{l(\rho)}}{z_\rho(t)} X^{(1^{n-1})}_{\tau \cup \rho}(t)
\]

\[
= \sum_{\tau \prec \lambda, |\tau| \leq n-1} \sum_{\rho \vdash (n-1-|\tau|)} \frac{(-1)^l(\rho)}{z_\rho(t)} \frac{n-1}{\prod_{j=1}^{n-1} (t_j - 1)}
\]

\[
= \sum_{\tau \prec \lambda, |\tau| \leq n-1} \sum_{\rho \vdash (n-1-|\tau|)} \frac{\prod_{j=1}^{n-1} (t_j - 1)}{\prod_{j \geq 1} (t_j^{\rho_j} - 1)}
\]

where the last identity has used (2.35).

Summarizing the above, we have that

**Theorem 2.10.** For partition $\lambda \vdash n$, one have

\[
X^{(n-k, k)}_\lambda(t) = \sum_{\tau \prec \lambda, |\tau| \leq k} \frac{(-1)^l(\rho)}{z_\rho(t)}
\]

\[
X^{(k, 1^{n-k})}_\mu(t) = \prod_{i=1}^{n-k} \frac{(t_i - 1)}{\prod_{j \geq 1} (t_j^{\mu_j} - 1)} \sum_{\tau \prec \mu, |\tau| \geq k} \prod_{j \geq 1} (t_j^{\tau_j} - 1)
\]

\[
X^{(k_1, k_2, k_3)}_\lambda(t) = \sum_{\tau \prec \lambda, |\tau| \leq k_3} \sum_{\rho \vdash (k_3 - |\tau|)} \frac{(-1)^l(\xi) + l(\rho)}{z_\xi(t)z_\rho(t)}
\]

\[
X^{(h_1, h_2, 1^{n-h_1-h_2})}_\lambda(t) = \prod_{i=1}^{n-h_1-h_2} (t_i - 1) \prod_{j \geq 1} (t_j^{\mu_l} - 1)
\]

\[
= \sum_{\rho \vdash (n-h_1-|\tau|)} \sum_{\tau \prec \lambda, |\tau| \leq n-h_1} \frac{\prod_{i=1}^{n-h_1-h_2} (t_i - 1)}{\prod_{j \geq 1} (t_j^{\mu_l} - 1)z_\rho(t)}
\]
Proof. The first identity follows from (2.32). The second identity follows from (2.32) (2.33) and (2.36). The third identity follows from (2.32) and (2.37). The last identity follows from (2.32) and (2.38). □

Remark 2.11. Morita [10] has given a different formula for the hook case at the root of unity.

Example 2.12. Given \( \lambda = (2^2, 1^2) \) and \( \mu = (3, 1^3) \), our formula says that

\[
X^{(3, 1^3)}_{(2^2, 1^2)}(t)
\]

\[
= \frac{(t - 1)(t^2 - 1)(t^3 - 1)}{(t^2 - 1)(t^3 - 1)} [4(t^2 - 1)(t - 1) + (t - 1)(t^2 - 1) + 2(t^2 - 1)(t - 1) + 2(t^2 - 1)(t - 1)]
\]

\[
= (t^3 + t^2 + 2t + 2)(t^3 - 1).
\]

3. A Murnaghan-Nakayama rule

Let \( k \) be a nature number and \( \mu \) be a partition, we consider \( p_k^{\ast}H_{\mu} \). To do this, we need to express \( H_{\lambda}, \lambda \vdash n \) in terms of the basis elements \( H_{\mu}, \mu \in \mathcal{P} \).

For any \( m, n \), repeatedly using (2.7) gives that

\[
[H_m, H_n]_t = (t^2 - 1)H_{n-1}H_{m+1} - t[H_{n-2}, H_{m+2}]_t
\]

\[
= \sum_{i=1}^{s-1} t^{i-1}(t^2 - 1)H_{n-i}H_{m+i} - t^{s-1}[H_{n-s}, H_{m+s}]_t.
\]

For \( m < n \), let \( \epsilon = 0, 1 \) be the parity of \( n - m \), i.e. \( \epsilon \equiv n - m \pmod{2} \), then \( \frac{n-m}{2} = \frac{n-m-\epsilon}{2} \) and

\[
H_m H_n = t H_n H_m + \sum_{i=1}^{\frac{n-m}{2}-1} (t^{i+1} - t^{i-1})H_{n-i}H_{m+i} + t^{\frac{n-m}{2}-1}(t^{1+\epsilon} - 1)H_{\frac{n+m+2}{2}} H_{\frac{n-m-2}{2}}
\]

i.e. for \( n - i > m + i \), the coefficient of \( H_{n-i}H_{m+i} \) is \( (t^{i+1} - t^{i-1}) \); if \( n - i = m + i \) the coefficient of \( H_{n-i}H_{m+i} \) is \( (t^{i} - t^{-i}) \). Note that at any stage if \( \lambda_i > \lambda_{i+1} + \cdots, H_{(\cdots, -\lambda_i, \lambda_{i+1}, \cdots)} = 0. \)

For a composition \( \lambda \) such that \( \lambda_i < \lambda_{i+1} \), let \( S_{i,a} \) be the transformation \( (\lambda_1, \cdots, \lambda_i, \lambda_{i+1}, \cdots) \mapsto (\lambda_1, \cdots, \lambda_{i+1} - a, \lambda_i + a, \cdots) \), where \( 0 < a \leq \frac{\lambda_{i+1} - \lambda_i}{2} \). Define

\[
C(S_{i,a}) = \begin{cases} 
1 & a = 0 \\
\left(\frac{\lambda_{i+1} - \lambda_i}{2} - a\right) & a > 0 \leq \frac{\lambda_{i+1} - \lambda_i}{2} \\
\left(a - \frac{\lambda_{i+1} - \lambda_i}{2}\right) & a < 0 < \frac{\lambda_{i+1} - \lambda_i}{2}
\end{cases}
\]

where \( \epsilon \) is the parity of \( \lambda_i - \lambda_{i+1} \). For \( i = (i_1, \ldots, i_r) \) and \( a = (a_1, \ldots, a_r) \) define

\[
C(S_{i,a}) = C(S_{i_1,a_1})C(S_{i_2,a_2}) \cdots C(S_{i_r,a_r})
\]
where the product order follows the action order of $S_{i_1,a_1}S_{i_2,a_2} \cdots S_{i_r,a_r}$ from right to left. The following are two special cases: (i) if $t = 0$, then $C(\iota,a) = 0$ unless all $a_i = 1$. When all $a_i = 1$ for $1 \leq i \leq r$ (which only happens when $\lambda_{i+1} - \lambda_i \geq 2$), then $C(\iota,a) = (-1)^r$; (ii) if $t = -1$, then $C(\iota,a) = 0$ unless all $a_i = 0$ in which $C(\iota,a) = (-1)^r$.

**Proposition 3.1.** Suppose $\lambda$ is a composition, then

$$H_\lambda = \sum_{\iota,a} C(S_{\iota,a}) H_{S_{i_1,a_1}S_{i_2,a_2} \cdots S_{i_r,a_r}\lambda}$$

summed over $\iota = (i_1, \ldots, i_r), a = (a_1, \ldots, a_r) \in \mathbb{Z}_+^r$ such that $S_{i_1,a_1}S_{i_2,a_2} \cdots S_{i_r,a_r}\lambda \in \mathcal{P}$.

The following is a Murnaghan-Nakayama rule for the Green polynomial. The result generalizes a formula of Morris [12] which corresponds to our result in the case of $l(\lambda) = 2$.

**Theorem 3.2.** Let $\lambda, \mu \in \mathcal{P}_n$, then

$$X_\mu^\lambda(t) = \sum_{j=1}^{l(\mu)} \sum_{\iota,a} C(S_{\iota,a}) X_{\mu^{[j]}}^{S_{i_1,a_1}S_{i_2,a_2} \cdots S_{i_r,a_r}(\lambda-\mu_1 \varepsilon_j)}(t)$$

summed over $\iota = (i_1, \ldots, i_r), a = (a_1, \ldots, a_r) \in \mathbb{Z}_+^r$ such that $S_{i_1,a_1}S_{i_2,a_2} \cdots S_{i_r,a_r}(\lambda-\mu_1 \varepsilon_j) \in \mathcal{P}_{n-\mu}$. Here $\varepsilon_j$ is the composition with 1 at the $j$-th position 0 elsewhere.

**Proof.** By Prop. 3.1 it follows that

$$p_{\mu,\lambda}^* = \sum_{i=1}^{l(\mu)} H_{\lambda-\mu_1 \varepsilon_i} \cdot 1$$

$$= \sum_{i=1}^{l(\mu)} \sum_{\iota,a} C(S_{\iota,a}) H_{S_{i_1,a_1}S_{i_2,a_2} \cdots S_{i_r,a_r}(\lambda-\mu_1 \varepsilon_i)} \cdot 1,$$

where the sum runs through all $\iota = (i_1, \ldots, i_r), a = (a_1, \ldots, a_r) \in \mathbb{Z}_+^r$ such that $S_{i_1,a_1}S_{i_2,a_2} \cdots S_{i_r,a_r}(\lambda-\mu_1 \varepsilon_j) \in \mathcal{P}_{n-\mu}$, which immediately implies the theorem.

**Example 3.3.** Given $\lambda = (9,5,2)$ and $\mu = (8,4,2,2)$, then

$$p_{\mu,\lambda}^* H_{\lambda} \cdot 1 = H_{(1,5,2)} \cdot 1 + H_{(9,-3,2)} \cdot 1 + H_{(9,5,-6)} = H_{(1,5,2)} \cdot 1$$

$$= C(S_2S_1) H_{S_2S_1(1,5,2)} \cdot 1 + C(S_1) H_{S_1(1,5,2)} \cdot 1 + C(S_2) H_{S_2(1,5,2)} \cdot 1$$

$$= t^2 H_{(5,2,1)} \cdot 1 + (t^2 - 1) H_{(4,2,2)} \cdot 1 + (t^2 - t) H_{(3,3,2)} \cdot 1.$$ 

Therefore

$$X_{(9,5,2)}^{(8,4,2,2)}(t) = t^2 X_{(4,2,2)}^{(5,2,1)}(t) + (t^2 - 1) X_{(4,2,2)}^{(4,2,2)}(t) + (t^2 - t) X_{(4,2,2)}^{(3,3,2)}(t).$$
Example 3.4. Now let’s consider the special case $X_{(1^n)}(t)$. It is easy to see that for $\lambda = (1^{m_1}2^{m_2}\cdots)$,

$$p_i^*H_\lambda = \sum_{i=1}^{\lambda_i} H_{\lambda-i\varepsilon_i}, 1 = \sum_{i \geq 1} [m_i] H_{(1^{m_1}\cdots(i-1)^{m_i-1+1/i^m_i-1})}, 1$$

Repeating the process, we have that

$$p_i^*H_\lambda = \sum_{i,j} H_{\lambda-i\varepsilon_i-j\varepsilon_j}, 1$$

$$= \sum_{|i-j| \geq 2} [m_i][m_j] H_{(\cdots(i-1)^{m_i-1+1/i^m_i-1}\cdots(j-1)^{m_j-1+1/j^m_j-1})}, 1$$

$$+ \sum_{|i-j| = 1} [m_i][m_j] H_{(\cdots(i-1)^{m_i-1+1/i^m_i-1}\cdots(j-1)^{m_j-1+1/j^m_j-1})}, 1$$

$$+ \sum_{i} [m_i][m_i-1] H_{(\cdots(i-1)^{m_i-1+1/i^m_i-1})}, 1$$

Continuing in this way, $p_i^*H_\lambda$ will eventually turn each summand into a function in $t$ and we get that

$$p_i^*H_\lambda, 1 = \sum_T \phi_T(t)$$

where $T$ runs through all standard tableaux of shape $\lambda$. The function $\phi_T(t)$ is defined as follows. For a skew horizontal strip $\theta = \lambda - \mu$, we define

$$\phi_\theta(t) = \prod_{i \in \theta} [m_i(\lambda)]$$

where $\theta$ is a union of skew horizontal strips $\theta^{(i)} = \lambda^{(i)} - \mu^{(i)}$, then define $\phi_T(t) = \prod_{i=1}^l \phi_{\theta^{(i)}}(t)$. As a result $X_{(1^n)}(t) = \sum_T \phi_T(t)$, where $T$ runs through all standard tableaux of shape $\lambda$.

As we mentioned before when $t = 0$, $H_\lambda, 1 = S_\lambda, 1$ is the Schur function associated with partition $\lambda$ and the Schur function. In this case the straightening rule (3.3) reduces to $S_mS_n = -S_{n-1}S_{m+1}$. This can be reformulated as follows. Let $\delta = (l-1,\cdots,1,0)$, we say two $l$–tuples $\mu$ and $\lambda$ are related if $\mu + \delta = \sigma(\lambda + \delta)$ for some permutation $\sigma$. We denote by $\pi(\mu)$ the associated non-increasing integral tuple $\lambda$ of $\mu$. If there exists an odd permutation $\sigma$ such that $\mu + \delta = \sigma(\mu + \delta)$, then we say that $\mu$ is degenerate, then

$$S_\mu, 1 = \begin{cases} 
\text{sgn}(\sigma)S_{\pi(\mu)}, 1 & \text{if } \pi(\mu) \in \mathcal{P} \\
0 & \text{if } \mu \text{ is degenerate or } \pi(\mu) \notin \mathcal{P}
\end{cases}$$

We can easily recover the usual Murnaghan-Nakayama rule (cf. [16]) using vertex operators.
**Example 3.5.** Let $\mu = (\mu_1, \ldots, \mu_l)$ be a partition and $k$ a positive integer. Then one has that

$$p_k^* S_{\lambda, 1} = \sum_{\mu} (-1)^{ht(\lambda - \mu)} S_{\mu, 1}$$

summed over all partitions $\mu \subset \lambda$ such that $\lambda - \mu$ is a border strip of length $k$.

**Proof.** By (2.30),

$$p_k^* S_{\lambda, 1} = \sum_{i=1}^{l} S_{\lambda_1} \cdots S_{\lambda_i-k} \cdots S_{\lambda_l, 1}.$$ 

Note that $S_{\lambda_1} \cdots S_{\lambda_i-k} \cdots S_{\lambda_l, 1} = 0$ unless $(\lambda_1 + l - 1, \ldots, \lambda_i - k + l - i, \ldots, \lambda_l) \in \mathbb{Z}_q^+$ has no identical terms. We rearrange the sequence $\lambda + \delta$ in descending order, and we may assume that for some $j > i$

$$\lambda_j + l - j < \lambda_i - k + l - i < \lambda_{j-1} + l - (j - 1),$$

in which case the related partition $\mu$ is

$$(\lambda_1, \ldots, \lambda_{i-1}, \lambda_{i+1} - 1, \ldots, \lambda_{j-2} - 1, \lambda_i - k - i + j - 1, \lambda_j, \ldots, \lambda_l)$$

therefore $\theta = \lambda - \mu$ is a border strip of length $k$ and $ht(\theta) = j - i - 1$, $\text{sgn}(\sigma) = (-1)^{l(\sigma)} = (-1)^{j-i-1}$.

**Remark 3.6.** When $t = -1$, $H_{\mu, 1}$ is the Schur Q-function. In this case, one can also obtain a result similar to Example 3.5. See [14, §III.8, Ex.11] for details.

### 4. Bitraces for $GL_n(\mathbb{F}_q)$ and the Hecke Algebra of Type $A_{n-1}$

Let $H_n(q)$ be the Iwahori-Hecke algebra of the symmetric group $S_n$ and $G = GL_n(\mathbb{F}_q)$ the general linear group over the finite field $\mathbb{F}_q$, where $q = p^m$. Let $W = Ind_B^G 1$ be the permutation module of $G$ induced from the Borel subgroup $B$ consisting of upper triangular matrices. Then $H_n(q)$ naturally acts on $W$ which commutes with that of $G$, so $W$ becomes a $G$-$H_n(q)$ bimodule. Following [5], we define the bitrace of $(g, h) \in G \times H_n(q)$ on $W$ as follows:

**Definition 4.1.** Let $u \in G$ and $h \in H_n(q)$. The trace of the action of $uh$ on $Ind_B^G 1$ is

$$\text{btr}(u, h) = \sum_{gB \in G/B} u(gB)|_{gB} h|_{gB}$$

where $u(gB)|_{gB}$ denotes the coefficient of $gB$ in $u(gB)h$.

A combinatorial formula for $\text{btr}(u_\lambda, T_\mu)$ is known by using the explicit description of the action of $H_n(q)$ in terms of the Bruhat decomposition of $G$ [5]. We now give an algebraic iterative formula for $\text{btr}(u_\lambda, T_\mu)$. 

As a bimodule and in view of double centralizer property, $W$ decomposes itself into:

$$W = \bigoplus_{\lambda} G^\lambda \otimes H^\lambda$$

where $G^\lambda$ (resp. $H^\lambda$) is an irreducible $G$ (resp. $H_n(q)$)-module. Taking trace gives rise to

$$btr(g, h) = \sum_{\lambda \vdash n} \chi^\lambda(g) \zeta^\lambda(h)$$

where $\chi^\lambda$ (resp. $\zeta^\lambda$) is the irreducible character of $G$ (resp. $H_n(q)$).

By [3] the irreducible character $\chi^\lambda$ of $G$ is given by

$$\chi^\lambda(u_\nu) = q^{-n(\nu)} K_{\lambda, \nu}(q^{-1})$$

where $u_\nu$ is a unipotent element of $GL_n(\mathbb{F}_q)$ with Jordan normal form of blocks size $\nu_i$ and $K_{\lambda, \nu}(t)$ is the Kostka-Foulkes polynomial defined by expanding the Schur function $s_\lambda$ in terms of the dual Hall-Littlewood functions $P_\mu(t) = b^{-1}_\mu(t) Q_\lambda(t)$ ($t = q^{-1}$):

$$s_\lambda = \sum_{\nu \vdash n} K_{\lambda, \nu}(t) P_\nu(t).$$

(4.3)

Also the Frobenius formula for the Hecke algebra [15] says that

$$\frac{q^{\mu}}{(q-1)^{l(\mu)} q^\mu(q^{-1})} = \sum_{\lambda \vdash n} \zeta^\lambda(T_{\gamma^\mu}) s_\lambda.$$  

(4.4)

Combining (4.3) and (4.4), we see that the bitrace is expressed as the matrix coefficient:

$$btr(u_\nu, T_{\gamma^\mu}) = \left\langle P_{\nu}(q^{-1}), q^\mu(q^{-1}) \right\rangle.$$  

(4.5)

Let $B_{\mu}^\nu(t) = \left\langle Q_{\nu}(t), q^\mu(t) \right\rangle$, by Theorem 2.22 it follows that

$$B_{\mu}^\nu(t) = \langle H_{\nu, 1}, q_\mu \rangle.$$  

(4.6)

Thus we can use vertex operator technique to compute the bitrace as follows.

First of all, it is easy to see the operator product expansions as in (2.23)-(2.24):

$$H^*(z) q^*(w) = q^*(w) H^*(z) \frac{z - tw}{z - w},$$  

(4.7)

$$q^*(z) H(w) = H(w) q^*(z) \frac{z - tw}{z - w}.$$  

(4.8)

Taking coefficients of $z^{-a} w^{-m}$ in (4.7) and (4.8), we get the following commutation relations.
Proposition 4.2. For any \( m, n \in \mathbb{Z} \),

\[
H_n^* q_m = q_m H_n^* + (1 - t) \sum_{k=1}^{m} q_{m-k} H_{n-k}^*,
\]

(4.9)

\[
q_n^* H_m = H_m q_n^* + (1 - t) \sum_{k=1}^{n} H_{m-k} q_n^*.
\]

(4.10)

Using the same method of Theorem 2.4, we immediately have the following result.

Theorem 4.3. For partitions \( \lambda, \mu \vdash n \) and integer number \( k \),

\[
q_k^* H_\nu = \sum_{|\tau| = k} (1 - t)^{|(\tau)|} H_{\nu - \tau},
\]

(4.11)

\[
H_k^* q_\mu = \sum_{\tau \in \mathbb{Z}_+} (1 - t)^{|(\tau)|} q_{\mu - \tau} H_k^*|_\tau|
\]

(4.12)

where \( \mathbb{Z}_+ \) is the set of non-negative integer.

Let \( \mu \) be a composition and \( \lambda \) be a partition. Recall (3.3) and set

\[
B(\lambda, \mu) = \sum_{\lambda \vdash n} C(S_{\lambda, \mu})
\]

summed over \( \lambda = (i_1, i_2, \ldots, i_r), a = (a_1, a_2, \ldots, a_r) \) such that \( S_{\lambda, a} \mu = \lambda \).

We remark that \( B(\lambda, \mu) = 0 \), unless \( |\lambda| = |\mu| \). If \( \mu_i + \mu_{i+1} + \cdots < 0 \) at any stage, then \( B(\lambda, \mu) = 0 \). And if \( \mu \) is also a partition, then \( B(\lambda, \mu) = \delta_{\lambda, \mu} \). Let \( \lambda, \nu \) be partitions and \( \tau \) be a non-negative composition, then \( B(\lambda, \nu - \tau) = 0 \) unless \( \lambda \subseteq \nu \).

Lemma 4.4. Let \( \lambda = (\lambda_1, \lambda_2, \ldots, \lambda_l), \nu = (\nu_1, \nu_2, \ldots, \nu_m) \) be partitions. If \( l < m, \lambda_i = \nu_i, i = 1, 2, \ldots, l \) and \( \nu_l > \nu_{l+1} \), then we have

\[
\sum_{|\tau| = |\nu| - |\lambda|} (1 - t)^{|(\tau)|} B(\lambda, \nu - \tau)
\]

\[
= \sum_{|\tau| = |\nu| - |\lambda|} (1 - t)^{|(\tau)|} B(\emptyset, (\nu_{l+1}, \ldots, \nu_m) - \tau).
\]

Proof. This follows directly from (4.13) and the remark below it. \( \square \)

Recall that \( \mu \vdash n \) means \( \mu \) is a composition with weight \( n \). In this case, we can rewrite (3.3) as \( (\mu \vdash n) \):

\[
H_\mu = \sum_{\lambda \vdash n} B(\lambda, \mu) H_\lambda.
\]

(4.14)

For \( \nu \vdash n \) and using (4.11) it follows that

\[
q_k^* H_\nu = \sum_{\lambda \in \mathbb{P}_n^\nu} \sum_{|\tau| = k} (1 - t)^{|(\tau)|} B(\lambda, \nu - \tau) H_\lambda, 1
\]

(4.15)
where $\mathcal{P}_n^\nu$ is the set $\{\lambda \vdash n \mid \lambda \subset \nu\}$. Note that $\lambda$ appears in (4.15) only when $\nu/\lambda$ is a horizontal $k$-strip (cf. [14 III (5.7)]), so we have proved the following result.

**Theorem 4.5.** Let $\mu, \nu \vdash n$, then the following iterative formula holds.

\[
B^n_\mu(t) = \sum_{\lambda \in \mathcal{P}_n^\nu} \sum_{\tau = \mu_1} (1 - t)^{l(\tau)} B(\lambda, \nu - \tau) B^\lambda_{\mu_1}(t).
\]

**Proof.** This follows from (4.15) and (4.6). □

We list some of the special cases of Theorem 4.5.

**Example 4.6.** Let $\mu, \nu \vdash n$, we have

\[
\begin{align*}
B^{(n)}_\mu(t) &= (1 - t)^{l(\mu)}, \\
B^\nu_\mu(t) &= (1 - t)^{\delta_{\nu, (n)}}, \\
B^{(1^n)}_\mu(t) &= \delta_{\mu, (1^n)} \prod_{i=1}^n (1 - t^i), \\
B^\nu_\mu(t) &= (1 - t)^n X^\nu_\mu(t), \\
B^\nu_{(\mu_1, \ldots, \mu_2)}(t) &= \begin{cases} (1 - t)^3 - \delta_{\nu_1, \nu_2} - \delta_{\nu_2, 0} + t(1 - t)^2 \delta_{\nu_2, \nu_2} & \text{if } \nu \geq (\mu_1, \mu_2) \\
0 & \text{if others.} \end{cases}
\end{align*}
\]

**Proof.** (4.17), (4.19), and (4.21) follows from (4.16) by easy induction. (4.18) holds by (4.6) and (2.18). (4.20) holds by (4.6), (2.12) and (2.20). □

**Corollary 4.7.** Let $\nu, \mu \vdash n$, $\mu > \nu$, we have

\[
B^\nu_\mu(t) = 0.
\]

**Proof.** We argue by induction on $l(\mu)$. The case $l(\mu) = 1$ is (4.18). Suppose it holds for all $l(\mu) < l$ and consider $\mu = (\mu_1, \mu_2, \ldots, \mu_l), \nu < \mu$. Since $q_m q_n = q_n q_m$, by (4.16), we have

\[
B^\nu_\mu(t) = \sum_{\lambda \in \mathcal{P}_n^\nu} \sum_{\tau = \mu_1} (1 - t)^{l(\tau)} B(\lambda, \nu - \tau) B^\lambda_{(\mu_1, \ldots, \mu_l-1)}(t)
\]

By induction hypothesis and the remark above Lemma 4.3, we have

\[
B^\nu_\mu(t) = \prod_{i=1}^{l-1} \delta_{\nu_i, \mu_i} \sum_{\tau = \mu_1} (1 - t)^{l(\tau)} B((\mu_1, \ldots, \mu_l-1), \nu - \tau) B^\lambda_{(\mu_1, \ldots, \mu_l-1)}(t)
\]

It suffices to consider the case $\nu_i = \mu_i, i = 1, 2, \ldots, l - 1$. Then we have $\nu_l < \mu_l, \nu_{l+1} > 0$, so for $\bar{\mu} = (\mu_1, \ldots, \mu_{l-1})$

\[
B^\nu_\mu(t) = \sum_{\tau = \mu_1} (1 - t)^{l(\tau)} B(\bar{\mu}, (\mu_l, \nu_l, \cdots) - \tau) B^\lambda_{\mu_l}(t)
\]

\[
= \sum_{\tau = \mu_1} (1 - t)^{l(\tau)} B(\emptyset, (\nu_l, \nu_{l+1}, \cdots) - \tau) B^\lambda_{\mu_l}(t) \quad \text{(by Lemma 4.4)}
\]
$$= B_{(\nu_1, \nu_2, \cdots)} B_{\mu}(t)$$
$$= 0.$$ 

Tables for $btr(u_{\nu}, T_{\mu}), n \leq 5$. Here $[n] = 1 + \cdots + q^{n-1}$

**Table 1.** $n=2$

| $\mu \setminus \nu$ | (2) | (1$^2$) |
|-------------------|-----|--------|
| (2)               | $q$ | 0      |
| (1$^2$)           | 1   | [2]    |

**Table 2.** $n=3$

| $\mu \setminus \nu$ | (3) | (2, 1) | (1$^3$) |
|-------------------|-----|--------|--------|
| (3)               | $q^2$ | 0      | 0      |
| (2, 1)            | $q$   | $q^2$ | 0      |
| (1$^3$)           | 1    | $2q + 1$ | $\prod_{i=1}^{3} [i]$ |

**Table 3.** $n=4$

| $\mu \setminus \nu$ | (4) | (3, 1) | (2$^2$) | (2, 1$^2$) | (1$^4$) |
|-------------------|-----|--------|--------|-----------|--------|
| (4)               | $q^4$ | 0      | 0      | 0         | 0      |
| (3, 1)            | $q^3$ | $q^4$ | 0      | 0         | 0      |
| (2$^2$)           | $q^2$ | $q^4 - q^2$ | $q^4 + q^2$ | 0 | 0      |
| (2, 1$^2$)        | $q$   | $2q^2$ | $q^4 + q^2$ | $q^4 + q^4$ | 0      |
| (1$^4$)           | 1    | $3q + 1$ | $(2q + 1)[2]$ | $(3q^2 + 2q + 1)[2]$ | $\prod_{i=1}^{4} [i]$ |

**Table 4.** $n=5$

| $\mu \setminus \nu$ | (5) | (4, 1) | (3, 2) | (3, 1$^2$) | (2$^2$, 1) | (2, 1$^3$) | (1$^5$) |
|-------------------|-----|--------|--------|------------|------------|------------|--------|
| (5)               | $q^5$ | 0      | 0      | 0          | 0          | 0          | 0      |
| (4, 1)            | $q^4$ | $q^5$ | 0      | 0          | 0          | 0          | 0      |
| (3, 2)            | $q^3$ | $q^4 - q^3$ | $q^6$ | 0          | 0          | 0          | 0      |
| (3, 1$^2$)        | $q^2$ | $2q^4$ | $q^6 + q^4$ | 0         | 0          | 0          | 0      |
| (2$^2$, 1)        | $q$   | $2q^4 - q^2$ | $2q^4$ | $q^6 - q^2$ | $q^6[2]$ | 0          | 0      |
| (2, 1$^3$)        | $q$   | $3q^4 + q^2$ | $3q^4[2]$ | $q^6(2q + 1)[2]$ | $q^4[3][2]$ | 0        |
| (1$^5$)           | 1    | $4q + 1$ | $5q^2 + 4q + 1$ | $(6q^2 + 3q + 1)[2]$ | $(5q^2 + 6q^2 + 3q + 1)[2]$ | $(4q^4 + 3q^4 + 2q + 1)[3][2]$ | $\prod_{i=1}^{5} [i]$ |
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