ON THE RIGIDITY OF ARNOUX-RAUZY WORDS

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Abstract. An infinite word generated by a substitution is rigid if all the substitutions which fix
this word are powers of the same substitution. Sturmian words as well as characteristic Arnoux-
Rauzy words generated by substitutions are known to be rigid. In the present paper, we prove
that all Arnoux-Rauzy words generated by substitutions are rigid. The proof relies on two main
ingredients: first, the fact that the primitive substitutions that fix an Arnoux-Rauzy word share
a common power, and secondly, the notion of normal form of an episturmian substitution (i.e.,
a substitution that fixes an Arnoux-Rauzy word). The main difficulty is then of a combinatorial
nature and relies on the normalization process when taking powers of episturmian substitutions:
the normal form of a square is not necessarily equal to the square of the normal forms.

1. Introduction

Rigidity property is an algebraic property of infinite words that are fixed by substitution: an
infinite word generated by a substitution is rigid if all the substitutions which fix this word
are powers of a same substitution. This property extends naturally to sets of substitutions fixing
an infinite word. This is a natural property that occurs for several prominent families of infinite
words. Rigidity has been first considered for the Thue-Morse word in [Pan81] (see also [S02] for
its generalization as Prouhet words), for generalized Fibonacci words in [Pan83], then for the class
Sturmian words in [S08] together with [RS12] and [RW10] (see also [BFS12]), and lastly for strict
epistandard words in [Kri08], which also called characteristic Arnoux-Rauzy words. This brief
overview of the literature shows that if there exist numerous results on the two-letter case, the
situation is more contrasted as soon as the size of the alphabet increases. For instance, over a
ternary alphabet, the monoid of morphisms generating a given infinite word by iteration can be
infinitely generated, even when the word is generated by iterating an invertible primitive morphism
(see [DK09, Kri08]).

The aim of this paper is to prove rigidity for Arnoux-Rauzy words (see Theorem 3.1). This class
of infinite words, introduced in [AR91], provides a generalization of Sturmian words, the latter
corresponding to the case of a two-letter alphabet. They are defined in combinatorial terms (see
Section 2.1) and belong to the family of infinite words having linear factor complexity, and more
precisely \((d - 1)n + 1\) factors of length \(n\) for all \(n\), when defined over a \(d\)-letter alphabet. They have
been further generalized as episturmian words; see the survey [GJ09]. Despite the fact that they
share many properties of Sturmian words, Arnoux-Rauzy words display a more complex behavior.
For example, while Sturmian words are 1-balanced (i.e., the numbers of occurrences of a letter
in any two factors of the same length differ by at most 1), Arnoux-Rauzy words do not have to
be balanced [BCS13]. Arnoux-Rauzy words also show weaker geometric properties than Sturmian
words [AI01], and their abelian subshifts have much more complicated structure [KPW18]. The
rigidity of the subclass of characteristic Arnoux-Rauzy words has been established in [Kri08]. We
extend this result to any Arnoux-Rauzy word.

Our proof follows the general line of the proofs in [S08] and [RS12] for Sturmian words. However,
the structure of the monoid of epiSturmian substitutions (i.e., the substitutions that fix

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1By substitution, it is meant here a non-erasing morphism.

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Arnoux-Rauzy words) is more complicated over a larger alphabet; see e.g. [Ric03]. Episturmian substitutions are described in terms of a normalized directive word \([BDD+18]\), which itself relies on the notion of block-equivalence. Our proof first uses the fact that if an Arnoux-Rauzy is fixed by two primitive substitutions, then these substitutions coincide up to powers (see Theorem 3.2 proved in [BDD+18]). We then go from the existence of a common power to the following property (stated in Lemma 3.6): let \(\sigma\) and \(\tau\) be two episturmian substitutions such that \(\sigma^n = \tau^m\), for \(n \geq m \geq 1\); then, there exists an episturmian substitution \(\rho\) such that \(\tau = \sigma \circ \rho\). The main issue we will have to face comes from the fact that normalization does not behave well with respect to taking squares and powers: the normalized form of a square of a word in normal form is a priori not the square of its normal form. This thus requires a careful study of normalization of powers of normal forms, which is the main step of the proof. This is handled in Sections 4 and 5.

Let us sketch the contents of this paper. Basic notions are introduced in Section 2. In particular, we recall the notions of block-equivalence, block-normalization and normal form of an episturmian substitution. The main statement and the general strategy are discussed in Section 3. The normalization of powers of normal forms is discussed in details in Section 4 by focusing on the block-normalization of powers and by introducing several types of errors (i.e., factors which are forbidden in the normal form) that are introduced when taking powers. The proof of the main step (Lemma 3.5) is handled in Section 5 through several decomposition lemmas. We lastly introduce and discuss in Section 6 the notion of weak rigidity.

2. Basic notions

2.1. Words, substitutions and rigidity. Let \(A\) be a finite alphabet. We let \(\varepsilon\) denote the empty word of the free monoid \(A^\ast\), \(A^+\) the free semigroup and \(A^\infty\) the set of infinite words over \(A\). For any word \(w\) in the free monoid \(A^\ast\) (endowed with the concatenation as operation), \(|w|\) denotes the length of \(w\), and \([w]_j\) stands for the number of occurrences of the letter \(j\) in the word \(w\). A factor of a (finite or infinite) word \(w\) is defined as the concatenation of consecutive letters occurring in \(w\). In other words, the word \(u\) is a factor of the finite word \(w\) if there exist words \(p\) and \(s\) such that \(w = pus\). If \(p = \varepsilon\) (resp., \(s = \varepsilon\)) we say that \(u\) is a prefix (resp., suffix) of \(w\). For \(w = w_1 \cdots w_n \in A^\ast\), the notation \(\text{pref}_k(w)\) stands for the prefix of length \(k\) of \(w\), i.e., \(\text{pref}_k(w) = w_1 \cdots w_k\), and the \(k\)-th letter of \(w\) is denoted by \(w[k]\). For \(i, j\) integers with \(i < j\), the set of integers \(\{i, i + 1, \ldots, j\}\) is denoted as \([i, j]\). We use the notation \(w[i, j]\) for the factor \(u = w_i w_{i+1} \cdots w_j\) of \(w\). The set of integers \([i, j]\) is then called the support of this occurrence in \(w\) of the factor \(w_i w_{i+1} \cdots w_j\), and \(w[i, j] = w_i w_{i+1} \cdots w_j\). We say that \(i\) is an index of an occurrence of the factor \(u\) in \(w\). The notation \(\text{pref}_\ell u\) stands for the prefix of \(u\) of length \(\ell\), i.e. \(\text{pref}_\ell u = u_1 \cdots u_\ell\). For a letter \(a\), the notation \([u]_a\) stands for the number of occurrences of \(a\)'s in \(u\). The reversal (also called mirror image) of a word \(w = w_1 \cdots w_n \in A^n\) is the word \(w_n \cdots w_1\).

Let \(x\) be an infinite word in \(A^\infty\). A factor \(w\) of \(x\) is said to be left special if there exist at least two distinct letters \(a, b\) of the alphabet \(A\) such that \(aw\) and \(bw\) are factors of \(x\). The set of factors of the infinite word \(x\) is denoted by \(L_x\).

An infinite word \(x \in A^\infty\) is an Arnoux-Rauzy word if the set of its factors \(L_x\) is closed under reversal and for all \(n\) it has exactly one left special factor of length \(n\). Arnoux-Rauzy words are also called strict episturmian words. An infinite word \(x\) is an episturmian word if the set of its factors \(L_x\) is closed under reversal and for all \(n\) it has at most one left special factor of length \(n\). An Arnoux-Rauzy word is thus an episturmian word, but an episturmian word is not necessarily an Arnoux-Rauzy word. An episturmian word is called characteristic if all of its left special factors are prefixes of it. A characteristic Arnoux-Rauzy word is also called a standard episturmian word, or else epistandard). For more on episturmian words, see the survey [GJ09]. An infinite word is said to be uniformly recurrent if every factor appears infinitely often and with bounded gaps. Arnoux-Rauzy words are known to be uniformly recurrent.

A substitution \(\sigma : A^\ast \to A^\ast\) is a monoid morphism that is assumed to be non-erasing, that is, the image of every non-empty element is non-empty. If there exists a letter \(a \in A\) such that the word \(\sigma(a)\) begins with \(a\) and if \(|\sigma^n(a)|\) tends to infinity, then there exists a unique infinite word, denoted by \(\sigma^\omega(a)\), which has all words \(\sigma^n(a)\) as prefixes. Such an infinite word is called a fixed
point of the substitution $\sigma$ or a word generated by the substitution $\sigma$. All the morphisms that are considered in the present paper are non-erasing and so they are substitutions.

Let $\sigma$ and $\tau$ be two substitutions on $\mathcal{A}$. The substitution $\tau$ is a conjugate of $\sigma$ if there exists $v \in \mathcal{A}^*$ such that either $\sigma(vw) = \sigma(w)v$ for all $w \in \mathcal{A}^*$, or $\tau(w)v = \sigma(w)v$ for all $w \in \mathcal{A}^*$.

A substitution $\sigma : \mathcal{A}^* \to \mathcal{A}^*$ is called primitive if there is a positive integer $k$ such that for all $a, b \in \mathcal{A}$, the letter $b$ appears in $\sigma^k(a)$. If $\sigma$ is a primitive substitution, then there exists a power $\sigma^n$ that admits a fixed point, and the set of factors of any fixed point of $\sigma^n$ is uniformly recurrent (see for example Proposition 1.2.3 in [Fog02]). Furthermore, all these fixed points have the same language that we call the language of the substitution.

The stabilizer of an infinite word $x \in \mathcal{A}^\mathbb{N}$, denoted by $\text{Stab}(x)$, is the monoid of substitutions $\sigma$ defined on the alphabet $\mathcal{A}$ that satisfy $\sigma(x) = x$. Words that have a cyclic stabilizer are called rigid. A word $x$ is thus rigid if there exists a substitution $\sigma$ such that for any substitution $\tau$ such that $\tau(x) = x$, then there exists a non-negative integer $n$ such that $\tau = \sigma^n$. In the present paper, we concentrate on the iterative stabilizer according to the terminology of [Kri08]: we focus here on substitutions (i.e., non-erasing morphisms) and on infinite words generated by iterating a substitution. For general results on the possible growth of elements of the stabilizer, we refer to [DK09] and [DR09]. It is shown in particular that polynomial and exponential growth cannot co-exist in the stabilizer for aperiodic words. We discuss weaker notions of rigidity in Section 6.

### 2.2. Episturmian substitutions and their normal form.

Episturmian substitutions have been introduced in [JP02] as generalizations to larger alphabets of Sturmian substitutions, which correspond to the case of the two-letter alphabet. We consider the following substitutions

$$
\psi_a(b) = \begin{cases} 
ab & \text{if } b \neq a, \\
a & \text{if } b = a.
\end{cases}
$$

and the permutation

$$
\theta_{ab} : \begin{cases} 
a & \to b \\
b & \to a \\
c & \to c \text{ if } c \neq a, b.
\end{cases}
$$

The monoid of episturmian substitutions over $\mathcal{A}$ is the monoid generated by the permutations $\theta_{ab}$, for $a, b \in \mathcal{A}$, together with the set of substitutions $\psi_a, \overline{\psi}_a$, for $a \in \mathcal{A}$. The pure episturmian substitutions are the substitutions obtained by compositions of elements of the form $\psi_a$ and $\overline{\psi}_a$, for $a \in \mathcal{A}$ (no permutation is allowed besides the identity). The epistandard substitutions are the substitutions obtained by compositions of the permutations together with the set of substitutions $\psi_a$ (that is, no $\overline{\psi}_b$ is allowed). We use $\mathcal{S}_\mathcal{A}$ as a notation for the set of permutations over the alphabet $\mathcal{A}$.

The monoid of episturmian substitutions has been thoroughly investigated, see e.g. [Ric03]. We will use in particular the following properties. Note that in both statements below, $x$ is assumed to contain all the letters of the alphabet on which the substitution $\sigma$ is defined.

**Theorem 2.1.** [JP02] Theorem 3.13, [DJP01] Theorem 11

*Let $\sigma$ be a substitution. If $x$ is an Arnoux-Rauzy word and $\sigma(x) = x$, then $\sigma$ is an episturmian substitution.*

Episturmian substitutions over the alphabet $\mathcal{A}$ can be viewed as automorphisms of the free group generated by the alphabet $\mathcal{A}$. This implies the following property of cancellativity.

**Proposition 2.2.** [JP02, Ric03] Lemma 7.2

*The monoid of episturmian substitutions is left-cancellative and right-cancellative, i.e., for any episturmian substitutions $\sigma, \tau, \rho$, if $\sigma \circ \tau = \sigma \circ \rho$, then $\tau = \rho$, and if $\rho \circ \sigma = \tau \circ \sigma$, then $\rho = \tau$.*

Episturmian words can be infinitely decomposed over the set of episturmian substitutions with the decomposition being described in terms of spinned directive words (see [JP02] Theorem 3.10). However, an episturmian word can have several decompositions. There is a way to normalize the directive words of an episturmian word so that any episturmian word can be defined uniquely by

\footnote{Note that this notion of rigidity has no relation with the ergodic notion of rigidity.}
its so-called normalized directive word \([\text{JP02}]\). This normalization relies on the notion of block-equivalence. This leads to the notion of directive words of an episturmian substitution as discussed in details in \([\text{GLR08}]\). We use here this normalization in order to produce a unique decomposition to any episturmian substitution.

We follow the notation of \([\text{JP02}]\) \([\text{GLR08}]\) \([\text{GLR09}]\). We provide letters with a notion of spin and introduce for each letter \(a\) its spinned version \(\overline{a}\). The letter \(\overline{a}\) is considered as having spin 1 while \(a\) is considered as having spin 0. We then consider the new alphabet \(\overline{A} = \{\overline{a} | a \in A\}\). A (finite or infinite) word over the alphabet \(A \cup \overline{A}\) is called a spinned word. For \(w \in (A \cup \overline{A})^*\), the opposite of \(w\) is the word \(\overline{w}\) obtained from \(w\) by exchanging all spins in \(w\). Let \(w\) be a spinned word. The word \(w\) is called a spinned version of the the word \(w'\) over the alphabet \(A\) obtained by replacing in \(w\) each occurrence of a letter with spin 1 (which thus belongs to \(\overline{A}\)) by its counterpart with spin 0 in \(A\). For instance, if \(A = \{1, 2, 3\}\), then \(\overline{22}\) is a spinned word and it is a spinned version of the word 322. A word in \(\overline{A}\) is said barred. For a spinned word \(x\) we use the notation \(x = \overline{x_1}\overline{x}_2 \cdots\) where \(\overline{x} = x_i\) or \(\overline{x}\) when the spins are not explicitly given.

The notion of spin extends to the episturmian substitutions \(\psi_{w}\) and \(\overline{\psi}_{w}\), by using the convention that \(\psi_{\overline{w}} = \overline{\psi}_{w}\) for all \(a \in A\).

If \(w = w_1 \cdots w_k \in (A \cup \overline{A})^*\), we then define the (pure) substitution \(\psi_{w}\) with directive word \(w\) as

\[
\psi_{w} = \psi_{w_1} \circ \cdots \circ \psi_{w_k}.
\]

One checks that \(\psi_{\overline{w}} = \overline{\psi}_{w}\).

A block-transformation is the replacement in a spinned word of an occurrence of a factor of the form \(xv\), where \(x \in A\), \(v \in (A \setminus \{x\})^*\), by \(\overline{xv}\) or vice-versa. We write it for short

\[
(1) \quad x\overline{v} \rightarrow \overline{xv} \quad \text{or} \quad \overline{xv} \rightarrow x\overline{v} \quad (x \in A, v \in (A \setminus \{x\})^*).
\]

Two finite spinned words \(w, w'\) are said to be block-equivalent, and we write \(w \equiv w'\), if we can pass from one to the other by a (possibly empty) chain of block-transformations. The block-equivalence is an equivalence relation over spinned words, and if \(w \equiv w'\), then \(w\) and \(w'\) are spinned versions of a common word over \(A\).

A finite spinned word \(w\) is said to be in normal form if \(w\) has no factor in \(\cup_{a \in A} \overline{a} \in A\). By block-normalization of a spinned word (we also say normalization for short), we mean a succession of block-transformations that produces its normal form. The interest of this notion comes from the following theorem.

**Theorem 2.3.** \([\text{GLR09}]\) Theorem 3.1 Let \(w\) and \(w'\) be two spinned words over \(A \cup \overline{A}\). One has \(\psi_{w} = \psi_{w'}\) if and only if \(w \equiv w'\).

In particular, if

\[
\psi_{a_1} \circ \psi_{a_2} \circ \cdots \psi_{a_k} = \psi_{b_1} \circ \psi_{b_2} \circ \cdots \psi_{b_k},
\]

where, for all \(i, j\) with \(i \in \{1, \cdots, k\}, j \in \{1, \cdots, \ell\}\), one has \(\overline{\psi}_{a_i} \in \{\psi_{a_i}, \overline{\psi}_{a_i}\}\), \(\overline{\psi}_{b_j} \in \{\psi_{b_j}, \overline{\psi}_{b_j}\}\), and \(a_i = b_j\), for all \(i\).

Any pure episturmian substitution is known to have a unique directive word which is in normal form by \([\text{GLR09}]\) Lemma 5.3. Now consider the general case of episturmian substitutions (not only pure ones, that is, episturmian substitutions that have no permutation in their decomposition). Let \(\theta\) be a permutation on the alphabet \(A\). It is readily verified that for all \(a \in A\), one has

\[
(2) \quad \theta \circ \psi_{a} = \psi_{\theta(a)} \circ \theta, \quad \theta \circ \overline{\psi}_{a} = \overline{\psi}_{\theta(a)} \circ \theta.
\]

According to this property, if \(\sigma\) is an episturmian substitution, then \(\sigma\) admits a unique decomposition as \(\sigma = \mu_{w} \circ \theta_{\sigma}\), where \(\mu_{w}\) is a pure episturmian substitution and \(\theta_{\sigma}\) is a permutation.

**Definition 2.4.** The normal decomposition of the episturmian substitution \(\sigma\) is defined as the (unique) decomposition

\[
\sigma = \overline{\psi}_{a_1} \circ \cdots \circ \overline{\psi}_{a_n} \circ \theta_{\sigma},
\]
where the spinned word $w_σ$ defined as $w_σ = a_1 \cdots a_n$ is in normal form. The spinned word $w_σ$ is called the normalized directive word of $σ$ and the permutation $θ_σ$ is called the normal permutation of $σ$.

We use the notation $$[σ] = \hat{a}_1 \cdots \hat{a}_nθ_σ$$ as a word over the alphabet $A \cup \overline{A} \cup S_A$. We call this word the normal form of $σ$. If $σ$ is pure, then $θ_σ$ is equal to the identity, denoted as $Id$. By abuse of notation, when $θ_σ = Id$, we write $[σ] = \hat{a}_1 \cdots \hat{a}_n$ for short. If $σ$ is epistandard, then all the $\hat{a}_i$’s are equal to $a_i$. By block-normalization for an episturmian substitution $σ = ψ_{a_1} \circ \cdots \circ ψ_{a_n} \circ θ_σ$, it is meant a succession of block-transformations that produces the normal form of its directive word $a_1 \cdots a_n$.

In order to define formally the surjective map between the words on the alphabet $A \cup \overline{A} \cup S_A$ and corresponding episturmian substitutions, we introduce the morphism $μ$ which maps letters to substitutions with $μ : \hat{a} \rightarrow ψ_σ$. Moreover, we use the same notation for a permutation considered as a permutation and as a letter. One has $σ = μ([σ])$. Note also that one has $σ \circ θ_σ^{-1} = μ(w_σ)$, where $w_σ$ stands for the normalized directive word of $σ$. The length of a spinned word $w$ corresponds to the usual notion of length over the alphabet $A \cup \overline{A}$. The length of an episturmian substitution is defined as the length of any of its directive words, denoted as $|σ|$. It is well defined by Theorem [CLR09 Theorem 3.1] above. We stress the fact that the length of $σ = ψ_{a_1} \circ \cdots \circ ψ_{a_n} \circ θ_σ$ is $k$.

The main issue we will have to face comes from the fact that normalization does not behave well with respect to taking squares and powers: the normalized form of a square of a spinned word in normal form is a priori not the square of its normal form. Indeed, consider the spinned word $aσaσ$. Its square $aσaσ$ is not in normal form because of the occurrence of $σ$. This issue will be handled in details in Sections 4 and 5.

The following example shows that an Arnoux-Rauzy substitution may involve only one letter in its normalized directive word. For more on the one-letter case, see Section 4.3.

Example 2.5. Consider the Fibonacci substitution $σ$ defined over the two-letter alphabet $\{a, b\}$ as $σ : a \mapsto ab$, $b \mapsto a$. This is the most classical example of a Sturmian substitution, i.e., of a two-letter Arnoux-Rauzy substitution. Over $\{a, b\}$, one has $ψ_σ : a \mapsto a$, $b \mapsto ab$. Let $θ_{ab}$ be the two-letter permutation that exchanges letters, i.e., $θ_{ab} : a \mapsto b$, $b \mapsto a$. One has $σ = ψ_a \circ θ_{ab}$, and its normalized directive word $w_σ$ is $a$.

3. Main result and strategy

Let us state and describe the general proof for the main result of this paper. We recall that Theorem 3.1 extends the result from [Kri08 Theorem 15] which proves rigidity for strict epistandard words, i.e., characteristic Arnoux-Rauzy words.

Theorem 3.1. Arnoux-Rauzy words are rigid.

The proof of Theorem 3.1 is based on the following results. We first know from [BDD+18] the following theorem, whose proof relies on the notion of return words. Note that this statement implies that the Perron-Frobenius eigenvalues of $σ$ and $τ$ are multiplicatively dependent, which is also a consequence of Cobham’s Theorem [Dur11].

Theorem 3.2. [BDD+18 Theorem 9] Let $x$ be an Arnoux-Rauzy word that is a fixed point of both $σ$ and $τ$ primitive substitutions. Then there exist $i,j \geq 1$ such that $τ^i = σ^j$.

We then prove primitivity for the substitutions that fix Arnoux-Rauzy words.

Lemma 3.3. Let $x$ be an Arnoux-Rauzy word over the alphabet $A$ and let $σ$ be a substitution such that $σ(x) = x$, with $σ$ not equal to the identity. Then $σ$ is primitive.

Proof. The infinite word $x$ is uniformly recurrent due to being an Arnoux-Rauzy word. We now prove by contradiction that $(|σ^n(a)|)_n$ tends to infinity for each letter $a ∈ A$. Suppose indeed that for some letter $a$, the sequence of lengths $(|σ^n(a)|)_n$ does not tend to infinity. With the notation of Definition 2.4 consider $n$ such that $θ^n_σ = Id$, and take $σ' = σ^n$. We have $θ^n_σ = Id$. We also have
that \( ((\sigma')^k(a))_i \) does not tend to infinity. From the normal form of Arnoux-Rauzy substitutions, one deduces that no power of \( \sigma' \) contains some \( \psi_b \) in its normal form, with \( b \neq a \): otherwise, applying \( \sigma' \) to a word adds at least one letter for each occurrence of \( a \) in the word. This is only possible when \( \sigma' = \psi_b \circ \overline{\psi_a} \), but this substitution does not have an infinite fixed point, which gives us the desired contradiction. We thus deduce from the fact that \( \langle (\sigma^n(a)) \rangle \) tends to infinity for each letter \( a \in \mathcal{A} \) together with \( x \) being uniformly recurrent that \( \sigma \) is primitive from [Que10 Proposition 5.5].

\[ \square \]

Next lemma provides a commutation property for substitutions fixing a common Arnoux-Rauzy word.

**Lemma 3.4.** Let \( \sigma \) and \( \tau \) be two substitutions. If \( x \) is an Arnoux-Rauzy word such that \( \sigma(x) = \tau(x) = x \), then \( \tau \circ \sigma = \sigma \circ \tau \).

**Proof.** Let \( x \) be an Arnoux-Rauzy word over the alphabet \( \mathcal{A} \) such that \( \sigma(x) = \tau(x) \), with \( \sigma \) and \( \tau \) being substitutions. According to [DJP01], there exist an Arnoux-Rauzy word \( y \) that has the same language as \( x \), and \( \tau' \) and \( \sigma' \) epistandard substitutions that are conjugate respectively to \( \tau \) and \( \sigma \) such that \( \sigma'(y) = \tau'(y) \). By [Kri08 Theorem 15], there exists a substitution \( \theta \) such that \( \sigma' = \theta \) and \( \tau' = \theta \), by using rigidity for fixed point of epistandard substitutions. Then, one has \( \sigma' \circ \tau' = \tau' \circ \sigma' = \theta^k \). This implies by conjugation that \( |\sigma \circ \tau(a)| = |\tau \circ \sigma(a)| \) for each \( a \in \mathcal{A} \). Now since \( \tau \) and \( \tau' \) are conjugate, we have that \( |\tau(a)| \) and \( |\tau'(a)| \) are of the same length and moreover are abelian equivalent, i.e., contain the same numbers of occurrences of each letter. So, we have that \( |\sigma \circ \tau(a)| = |\tau \circ \sigma(a)| \) for each \( a \in \mathcal{A} \). Now since \( \sigma \) and \( \sigma' \) are conjugate, the images of abelian equivalent words under \( \sigma \) and \( \sigma' \) are of the same length (in fact, they are also abelian equivalent), hence \( |\sigma \circ \tau(a)| = |\tau \circ \sigma(a)| \) for each \( a \in \mathcal{A} \). We deduce from \( \sigma \circ \tau(x) = \tau \circ \sigma(x) \) that \( \sigma \circ \tau = \tau \circ \sigma \) by considering the respective images of each letter in \( x \) and by recalling that all the letters of \( \mathcal{A} \) occur in \( x \). \[ \square \]

We now want to go from the existence of a common power (Theorem 3.2) to a “prefix property”. This is the object of next lemma which is the main step in the proof of Theorem 3.1. This is the analogue of [RS12 Proposition 4.1] which is proved for Sturmian words.

**Lemma 3.5.** Let \( \sigma \) and \( \tau \) be two episturmian substitutions such that \( \sigma^n = \tau^m \), for \( n \geq m \geq 1 \). Then there exists an episturmian substitution \( \theta \) such that \( \tau = \sigma \circ \theta \).

The proof of this lemma is quite involved; we provide it in Section 5. Using this lemma, we now can prove Theorem 3.1.

**Proof.** Let \( x \) be an Arnoux-Rauzy word. Let \( \sigma \), \( \tau \) be two substitutions distinct from the identity such that \( \sigma(x) = \tau(x) = x \). By Theorem 2.1 they are both episturmian. By Lemma 3.3 \( \sigma \) and \( \tau \) are primitive. By Lemma 3.4 they commute. By Theorem 3.2 they have a common power, i.e., there exist \( n, m \geq 1 \) such that \( \sigma^n = \tau^m \).

We now prove by induction on \( \max(|\sigma|, |\tau|) \) that if there exist \( n, m \geq 1 \) such that \( \sigma^n = \tau^m \), then there exist integers \( k, \ell \) and an episturmian substitution \( \theta \) such that \( \sigma = \theta^k \), \( \tau = \theta^\ell \). Note that this step is the analogue of [RS12 Corollary 4.3] which holds for Sturmian words (i.e., for the two-letter case).

We have \( |\sigma| \geq 1 \) and \( |\tau| \geq 1 \) since they are primitive.

If \( |\sigma| = |\tau| = 1 \), then \( \sigma = \tau \). Indeed, since the lengths are equal to 1, due to Lemma 3.3 and to the uniqueness of the normal form, there exist a letter \( a \) and permutations \( \theta_\sigma \) and \( \theta_\tau \) such that \( \sigma \) and \( \tau \) are both either of the form \( \sigma = \psi_a \circ \theta_\sigma \), \( \tau = \psi_a \circ \theta_\tau \), or they both involve \( \psi_a \), with both cases being symmetric. We assume that they both involve \( \psi_a \). Now since \( x = \sigma(x) = \tau(x) \), we have \( \psi_a(\theta_\sigma(x)) = \psi_a(\theta_\tau(x)) \). Notice that the form of the substitution \( \psi_a \) implies that if \( \psi_a(w) = \psi_a(w') \) for some infinite words \( w \) and \( w' \), then \( w = w' \) (since \( \psi_a \) only inserts a letter \( a \) before each letter which is not \( a \)). So, \( \theta_\sigma(x) = \theta_\tau(x) \). Since \( \theta_\sigma \) and \( \theta_\tau \) are permutations, it follows that \( \theta_\sigma = \theta_\tau \).
Suppose w.l.o.g. that \( n \geq m \) and that the induction property holds for substitutions with lengths smaller than \( \max(\{|\sigma|, |\tau|\}) \) which is equal to \( |\tau| \) (since \( n|\sigma| = m|\tau| \) by the definition of the length of substitution). By Lemma 3.3, there exists an episturmian substitution \( \varphi \) such that \( \tau = \sigma \circ \varphi \). From \( \tau = \sigma \circ \varphi \), we deduce that \( \sigma \circ \varphi \circ \sigma = \sigma \circ \sigma \circ \varphi \), and thus \( \varphi \circ \sigma = \sigma \circ \varphi \) by left-cancellativity (see Proposition 2.2). If \( \varphi = \text{Id} \), then \( \sigma = \tau \), which provides the desired induction property with \( \sigma = \tau \). The case where \( \varphi \) is a permutation is similar to the case where both substitutions have length 1 above. We now assume \( |\varphi| \geq 1 \), and thus \( |\tau| > \max(|\sigma|, |\varphi|) \), which yields in particular \( m < n \). One has \( \tau^m = \sigma^m \circ \varphi^m = \sigma^n \), and consequently \( \sigma^{n-m} = \varphi^m \) (again by left-cancellativity). We now can apply the induction hypothesis to \( \sigma \) and \( \varphi \). Hence there exist \( \varphi' \) episturmian substitution, \( k, \ell \) integers such that \( \sigma \equiv \varphi'^k \), \( \varphi \equiv (\varphi')^\ell \), which also yields \( \tau = (\varphi')^{k+\ell} \).

This ends the induction proof.

\[ \square \]

4. Block-normalization of powers

We recall that the normalized form of a square (power) of a spinned word in normal form may not be equal to the square (power) of its normal form. We study in this section how normalization behaves with respect to powers \( \sigma^n \) for an episturmian substitution \( \sigma \).

4.1. First notation. We recall that \( \sigma = a_1 \cdots a_k \theta_{\sigma} \) is the word over the alphabet \( A \cup \overline{A} \cup S_A \) representing \( \sigma \) in its normal form as defined in Section 2.2. From now on, we will keep the notation \( k \) for the length of \( \sigma \). The normalized directive word of \( \sigma \) is \( \nu_{\sigma} = a_1 \cdots a_k \) and \( \theta_\sigma \) is the normal permutation of \( \sigma \). We also recall that the letters \( a_i \) belong to \( A \cup \overline{A} \).

Let \( n \) be a positive integer. Let us now consider \( \sigma^n \). We want to compare the words \( [\sigma]^n \) and \( [\sigma^n] \). As mentioned above (see also Example 4.2 and 4.12), the word \( [\sigma]^n \equiv [\sigma] \) does not have to be the normal form of \( \sigma^n \). In order to get the normal form \( [\sigma^n] \) of \( \sigma^n \), we first handle the occurrences of permutations, which we shift to the end, and secondly, we perform block-transformations. The latter is the main step that we will have to work out.

To get the normalized form of \( [\sigma^n] \), we start by shifting the \( n \) occurrences of the permutation \( \theta_{\sigma} \) to the end of the word using the relations (2). In particular, the normal permutation of \( \nu_{\sigma}^n \) is equal to the \( n \)-th power \( \theta_{\sigma}^n \) of the normal permutation of \( \sigma \). We recall that we use the notation \( \theta_{\sigma}^n \) for the letter in bijection with the permutation \( \theta_{\sigma}^n \). This gives a word \( Z_{\sigma^n} \) over the alphabet \( A \cup \overline{A} \cup S_A \), which is a priori not the normal form \([\sigma]^n\), defined by

\[ Z_{\sigma^n} = \overline{a_1} \cdots \overline{a_k} \theta_{\sigma}(a_1) \cdots \theta_{\sigma}(a_k) \cdots \theta_{\sigma}^{n-1}(a_1) \cdots \theta_{\sigma}^{n-1}(a_k) \theta_{\sigma}^n, \]

with

\[ \sigma^n = \overline{\psi}_1 \cdots \overline{\psi}_k \overline{\psi}_{\sigma}(a_1) \cdots \overline{\psi}_{\sigma}(a_k) \cdots \overline{\psi}_{\sigma}^{n-1}(a_1) \cdots \overline{\psi}_{\sigma}^{n-1}(a_k) \theta_{\sigma}^n = \mu(Z_{\sigma^n}). \]

One has

\[ \sigma^n = \mu([\sigma^n]) = \mu(Z_{\sigma^n}) \text{ and } [\sigma]^n \equiv Z_{\sigma^n}. \]

If \( n = 1 \), then \( Z_{\sigma} = [\sigma] \).

Observe that the values of spins (i.e., the occurrence or not of a bar for \( \overline{a}_i \)) are \( k \)-periodic in \( Z_{\sigma^n} \) (as above, \( k \) is the length of \( \sigma \)). A factor of length \( k \) at an index congruent to 1 modulo \( k \) in \( Z_{\sigma^n} \) of the form \( \theta_{\sigma}^i(a_1) \cdots \theta_{\sigma}^i(a_k) \) \( (i = 0, \ldots, n - 1) \) in \( Z_{\sigma^n} \) is called a \( k \)-period of \( Z_{\sigma^n} \); it starts at an index congruent to 1 modulo \( k \) in \( Z_{\sigma^n} \). We use vertical bars to mark periods, which gives

\[ Z_{\sigma^n} = \overline{a_1} \cdots \overline{a_k} | \theta_{\sigma}(a_1) \cdots \theta_{\sigma}(a_k) | \cdots | \theta_{\sigma}^{n-1}(a_1) \cdots \theta_{\sigma}^{n-1}(a_k) | \theta_{\sigma}^n. \]

Bars are thus located between letters with indices \( kj \) and \( kj + 1 \). Moreover periods are labeled with indices \( j \) running from 1 to \( n \).

By abuse of notation, a word of the form \( \overline{a_0} \cdots \overline{a_\ell} \) with \( \ell = 0 \) stands for the word \( \overline{a} \).

We now consider normalization via block-transformations. If the word \( Z_{\sigma^n} \) is not the normal form of \( \sigma^n \), then this means that there are errors, i.e., occurrences of non-normal spinned words of the form \( \overline{a_0} \cdots \overline{a_\ell} \) in \( Z_{\sigma^n} \), with the letter \( a \) and the letters \( b_i \)'s in \( A \). We call the errors in \( Z_{\sigma^n} \) of this form simple \( a \)-errors. More precisely, a \textit{simple \( a \)-error} is an occurrence in \( Z_{\sigma^n} \) of a factor of the form

\[ \overline{a_0} \cdots \overline{a_\ell} a, \]
where the block $b_1 \cdots b_l$ can be empty. The letter $a$ is referred to as the letter of the error.

In order to get the normal form of $Z_{\sigma^n}$ (which is also the normal form of $\sigma^n$) we will perform series of block-transformations on $Z_{\sigma^n}$. This will be described in detail in Section 4.2. Section 4.3 handles the particular case where the normal form of $\sigma$ admits only one letter. The details of the process of normalization and more precise statements are provided in Sections 4.4 and 4.5.

4.2. On the propagation of simple errors. In this section, we describe two successive levels of block-transformations that will be performed in $Z_{\sigma^n}$ in order to get its normal form $[\sigma^n]$, and we distinguish the types of errors that occur in this process. We assume in all this section that there exists a simple $a$-error in $Z_{\sigma^n}$, i.e., an error of the form $\pi(\Lambda) a$, where $\Lambda = \mathcal{A} \setminus \{a\}$. The leftmost occurrence of a simple error plays a significant role as stressed by the following lemma, which basically says that all simple errors in $Z_{\sigma^n}$ are defined by the leftmost error:

**Lemma 4.1.** Let $\sigma$ be an episturmian substitution. We assume that there exists a simple error in $Z_{\sigma^n}$. The leftmost simple error occurs at the boundary between the first and second periods. In particular, it contains in its support the integer $k = |\sigma|$. Let $a$ be the letter of this error. At the boundaries of the $i$-th and the $i+1$-th periods, one also gets a simple $\theta^{-1}_n(a)$-error in $Z_{\sigma^n}$. Moreover, the supports simple errors in $Z_{\sigma^n}$ are $k$-periodic, and there is only one possible simple error at the boundary of two periods.

**Proof.** Simple errors cannot occur inside the $k$-periods (blocks of length $k = |\sigma|$), since the word $Z_{\sigma^n}$ is a concatenation of words of length $k$ in normal form. Due to $k$-periodicity of bars and letters (modulo the permutation $\theta_k$), simple errors occur with period $k$. So, the leftmost simple error occurs at the boundary between the first and second periods.

We will use from now on the following convention for the vertical bars which mark periods in $ab_1 \cdots b_l a$: we write $\overline{ab_1} \cdots \overline{b_l} a$ to indicate that the vertical bar which marks periods in $Z_{\sigma^n}$ can be located in this occurrence either after any of the $b_i$'s, or after $\overline{a}$.

After correction of the simple errors using block-transformations $\Pi$, new errors (i.e. factors in $\bigcup_{a \in \mathcal{A}} \pi(\Lambda) a$) might occur. We first give an example that illustrates the first level block-transformations that has to be performed.

**Example 4.2.** Let $\sigma$ with normal form

$$[\sigma] = \overline{ab}\overline{ac}\overline{da}e$$

with $a, b, c, d, e$ being distinct letters and $\theta_k$ being the identity.

In order to normalize $Z_{\sigma^3}$ with

$$Z_{\sigma^3} = \overline{ab}\overline{ac}\overline{da}e | \overline{ab}\overline{ac}dae | \overline{ab}\overline{ac}dae,$$

one first has to handle the simple $a$-errors that occur at the boundaries of the periods, i.e., $\overline{ac}a$. The letters of their supports are marked with blue. Their block-normalization gives

$$Z_{\sigma^3} = \overline{ab}\overline{ac}d | \overline{ab}acda | \overline{ab}acdae \equiv \overline{ab}acdae | \overline{ab}acdae | \overline{ab}acdae.$$

We see that, after having normalized the simple errors, new errors (i.e., factors $\overline{da} a$ and $\overline{da} a$) occur, which induces a propagation in the block-transformations to be performed; the involved letters are marked with red:

$$Z_{\sigma^3} \equiv \overline{ab}acdae | \overline{ab}acdae | \overline{ab}acdae.$$

After performing the corresponding block-normalization, and by noticing that the letter $c$ with spin 0 stops the propagation, we get the normal form of $\sigma^3$:

$$Z'_{\sigma^3} = [\sigma^3] = \overline{ab}acdae | \overline{ab}acdae | \overline{ab}acdae.$$

This example leads us to introduce the following definition.
Definition 4.3 (Propagated error of the first level). Let $\overline{A} = A \setminus \{a\}$. A maximal factor of the form
\[
\overline{a}(\overline{A}) \overline{\sigma}(\overline{A})^3 \cdots \overline{a}(\overline{A})^n a(\overline{A})^m a \cdots a(\overline{A})^n a = (\overline{a}(\overline{A})^*)^+ (a(\overline{A})^*)^+ a
\]
is called a propagated error of the first level.

By maximal, it is meant here that its occurrence cannot be extended to the left by $\overline{a}(\overline{A})^*$ or to the right by $(\overline{A})^* a$. The terminology “first level” in “propagated error of the first level” refers to the fact that there will be possibly two levels of block-transformations, even though we will not use a specific terminology for errors involved in the second level.

The following lemma states that the supports of such factors do not intersect from a period to the next one. The outcome of the block-normalization of all the propagated errors of the first level is called the first level of block-normalization and it is denoted by $Z^*_{\sigma^n}$. Next lemma shows that the block-normalization is performed at the boundary of each pair of successive k-periods.

Lemma 4.4. Let $\sigma$ be an episturian substitution of positive length $k$ such that $Z_{\sigma^n}$ contains a simple error. We consider an occurrence of a propagated error of the first level which contains $k$ in its support. Let $i$ stand for the index of its first letter in the word $Z_{\sigma^n}$ and let $p$ be such that $p+k$ is the index of its last letter. One has $p < i$.

Proof. In $Z_{\sigma^n}$ letters $\tilde{a}$ occur with 0 spin from index $k+1$ to $k+p$ (they are not barred), whereas all letters $\bar{a}$ occur with spin 1 from index $i+k$ to $2k$, by $k$-periodicity. Hence if the letter of the next error is also $a$, the next propagated error of the first level starts (at $i+k$) after the previous one ends (at $p+k$). Consider now the case where the letter of the next error is $b$ with $b \neq a$. The letter $a$ has spin 0 at position $k+p$, and a $b$-error contains only barred letters for letters distinct from $b$. So, the next propagated error of the first level also starts at position $k+i > k+p$. □

Consider the leftmost propagated error of the first level. The support of the leftmost propagated error of the first level in $Z_{\sigma^n}$ is the interval $[i, p+k]$ of integers, with $1 \leq i \leq k, 1 \leq p \leq k$. There is also a $\theta_{\sigma^n}(a)$-propagated error of the first level that starts at index $i+k$. So, in fact we have $i > 1$ and $p < k$. Similarly, there is also a $\theta_{\sigma^n}(a)$-propagated error of the first level that starts at index $i+jk$, for $0 \leq j < n−1$. The error is said to contain the position $\ell$ if the integer $\ell$ belongs to is support. The initial positions of errors of the first level are $i+jk, 0 \leq j < n−1$, and their support is $[i+jk, p+k(j+1)]$, for $0 \leq j < n−1$. We recall that $Z^*_{\sigma^n}$ denote the word obtained after the first level of block-normalizations, i.e., block-transformations performed inside the supports $[i+jk, p+k(j+1)]$ of the errors of the first level. The exact form of this word is provided in Lemma 4.4 in Section 4.3; this section also contains more details on the first level of block-transformations.

We assume now that $\sigma$ admits at least two distinct letters in its normal form (the one-letter case is considered in Section 4.3.3). It might occur that $Z^*_{\sigma^n}$ is not in normal form. The second level of block-normalizations consists in this last round of block-transformations to be performed. Consider indeed the following example as an illustration.

Example 4.5. Let $\sigma$ with $[\sigma] = \overline{abc}a$, with $a, b, c$ distinct letters and the normal permutation $\theta_{\sigma}$ being equal to the identity. Letters in red below indicate simple errors that occur at the boundary between two periods (and they coincide with the leftmost propagated errors of the first level), whereas letters in blue refer to new errors created after performing this first level of block-transformations on letters in red. One has
\[
\sigma^3 = Z_{\sigma^n} = \overline{abc}a | \overline{abc}a | \overline{abc}a,
\]
\[
Z'^*_{\sigma^3} = \overline{abc}a | \overline{abc}a | \overline{abc}a.
\]
We notice that $Z'^*_{\sigma^3}$ is not in normal form because of the factor $\overline{abc}a$. Lastly we get for the normal form $[\sigma^3]$ of $\sigma^3$
\[
[\sigma^3] = \overline{abc}a | abc|a | \overline{abc}a.
\]

The following lemma provides a characterization of the case where a second level of block-transformations has to be performed. We recall that the supports of the propagated errors of the first level in $Z_{\sigma^n}$ are $[i+jk, p+kj]$ for $j = 1, \ldots, n−1$. 


Lemma 4.6. Let \( \sigma \) be an episturmian substitution of positive length which admits at least two distinct letters in its normal form. Let \( n \geq 2 \). Let \( a \) be the letter of the leftmost error in \( Z_\sigma^n \). The word \( Z_\sigma^n \) obtained after the first level of block-normalizations is not in normal form if and only if

1. \( \theta'_n(a) = a \) and \( n \geq 3 \);
2. between two consecutive errors propagated errors of the first level in \( Z_\sigma^n \), one has \( Z_\sigma^n[p + kj + 1, i + kj - 1] = b_1 \cdots b_t \), for \( j = 1, \ldots, n - 2 \) (with \( t \) possibly equal to 0), and the letters \( b_j \)'s are distinct from \( a \) for all \( j \).

We provide a proof of this lemma in Section 4.5.

We deduce in particular that all propagated errors of the first level have the same type, in the sense that, after the first level of block-transformations, if a new simple error is created in the \( j \)-th \( k \)-period, with \( j = 1, \ldots, n - 2 \), then new simple errors are created in each \( k \)-period of index \( j \), with \( j = 1, \ldots, n - 2 \). This justifies the following definition.

Definition 4.7 (Errors of type I and II). Let \( Z_\sigma^n \) be the word obtained after performing the first level of block-transformations. Propagated errors of the first level are said of type I if \( Z_\sigma^n \) is in normal form, i.e., \( Z_\sigma^n = [\sigma^n] \). They are said of type II otherwise. In other words, all propagated errors of the first level of the word \( Z_\sigma^n \) are of type II if \( Z_\sigma^n \) is not in normal form. For sake of simplicity in the terminology, an error of type I (resp. II) refers to a propagated error of the first level of type I (resp. II). We also say that \( Z_\sigma^n \) has errors of type I (resp. II).

In particular, Lemma 4.6 implies the following:

Corollary 4.8. If the errors are of type I and \( \theta'_n(a) = a \), then \( p < i - 1 \), and there exists an occurrence of a letter \( b \neq a \) in \([k + p + 1, k + i - 1]\), and thus also in \([p + 1, i - 1]\).

We will later prove that \( Z_\sigma^n[i + kj] = a \) and \( Z_\sigma^n[p + kj + 1, i + kj - 1] = \pi j \) for \( j = 0, \ldots, k - 2 \). So, Lemma 4.6 in fact gives the description of the simple errors that occur in \( Z_\sigma^n \) when errors are of type II: they are factors of the form \( a b_1 \cdots b_t a \) occurring at positions \([p + kj, i + kj]\) for \( j = 1, \ldots, k - 2 \). These simple errors can also propagate to blocks of \( \bar{a} \) of length greater than 2 and incuding positions \( i + kj \) and \( p + kj \). The second level of block-normalisation consists in normalizing these errors, after which we obtain the normal form of \( \sigma^n \). The process of the second level of block-normalization is described in Lemma 4.10 in Assertion 3.

We now summarize the algorithm described above to get the normal form of \([\sigma^n]\) when the normal form of \( \sigma \) admits more than one letter and \( \sigma \) is assumed to have at least one simple error. We first move the permutation to the end. This gives the word \( Z_\sigma^n \). Simple errors can propagate when performing block-transformations and we consider propagated errors of the first level; these are maximal factors of the form \( a(\bar{A}^r) a \cdots a(\bar{A}^r) a \cdot \cdot \cdot a(\bar{A}^r) a \). By lemma 4.4, their supports do not overlap. The first level corresponds to the block-normalization of propagated errors of the first level; it involves each pair of consecutive \( k \)-periods of \( Z_\sigma^n \). The word \( Z_\sigma^n \) is then transformed into a word denoted by \( Z_\sigma^n \). Propagated errors of the first level are of type I if, after this first level of block-transformations, the word \( Z_\sigma^n \) is in normal form, i.e., \( Z_\sigma^n = [\sigma^n] \).

If not, then propagated errors of the first level are of type II, and we apply the second level of block-transformations in order to get the normal form \([\sigma^n]\) by replacing in \( Z_\sigma^n \) the maximal factors of the form \( a \bar{b} \bar{b} \cdots \bar{b} a^q \) by \( a \bar{b} a^q^{-1} b_1 \cdots b_t a^q^{-1} \).

We now end this section with a few comments, the notation used in the following sections and more examples of errors.

Remark 4.9. (1) Consider the leftmost propagated error of the first level in \( Z_\sigma^n \) and let \( a \) be its letter. Then, all propagated errors of the first level are of the same type and there exists a propagated error of the first level in each pair of consecutive periods of letter \( \theta^{-1}_n(a) \), for \( i = 1, \ldots, n - 1 \).

(2) We assume that \( Z_\sigma^n \) admits a simple error. Let \( i \) stand for the index of the leftmost propagated error of the first level in \( Z_\sigma^n \). The prefix of length \( i - 1 \) of the normal form \([\sigma] \) is a prefix of \( Z_\sigma^n \).

(3) In the full process of block-normalization we only make block-transformations of the form \( ab_1 \cdots b_t a \rightarrow ab_1 \cdots b_t \bar{a} \). Hence, once the spin of a letter \( b \neq a \) is equal to 0 in the full
normalization process at some index in the support of this error, it will never change back
to spin 1 at this index.

(4) Let \( n, m \) be positive integers, and let \( \sigma, \tau \) be episturmian substitutions such that \( \sigma^n = \tau^m \).

At each index of \( Z_{\sigma^n} \) and \( Z_{\tau^m} \), we have \( b \) for the same \( b \in \mathcal{A} \) but possibly with different
spins.

(5) One has \( Z_{\sigma^n} \equiv [\sigma^n] \).

**Notation 4.10.** The letter of the leftmost error (if any) is denoted by \( a \). We use the notation \( k \)
for the length of \( \sigma \), \( i \) for the index of occurrence of the leftmost error in \( Z_{\sigma^n} \) and \( k + p \) for the last
index of the leftmost propagated error of the first level, i.e., \( [i, k + p] \) is the support of the leftmost
maximal factor of the form \( \overline{\mathcal{A}}^i \mathcal{A}^j \mathcal{A}^k \cdots \mathcal{A}^{p} \mathcal{A}^i a(\mathcal{A}) \cdots a(\mathcal{A}) \ a \).

When working with two episturmian substitutions \( \sigma \) and \( \tau \), as in Section 5 we use letters with
primes for \( \tau \). For instance, in the case where there are errors in both \( Z_{\sigma^n} \) and \( Z_{\tau^m} \), \( a \) stands for
the letter of the leftmost error in \( \sigma^n \) and \( a' \) stands for the letter of the leftmost error in \( \tau^m \).

**Example 4.11** (Error of type I with a permutation). Let \( \sigma \) with normal form
\[
[\sigma] = \overline{\text{cabc}} \text{cabc} \theta_\sigma
\]
with \( a, b, c \) being distinct letters, and \( \theta_\sigma \) being the permutation \( \theta_{ac} \) exchanging \( a \) and \( c \).

We have
\[
Z_{\sigma^3} = \overline{\text{cabc}} \text{cabc} \mid \overline{\text{acbc}} \text{acbc} \mid \overline{\text{cabc}} \text{cabc} \mid \theta_3^3.
\]

This word contains two propagated errors of the first level; the letters of their supports are
marked with red: the first one is of letter \( a \), and the second one is of letter \( c \). Normalizing these
errors, we get the normal form \( [\sigma^3] \) of \( Z_{\sigma^3} \):
\[
Z_{\sigma^3}' = [\sigma^3] = \overline{\text{cabc}} \text{cabc} \mid \overline{\text{acbc}} \text{acbc} \mid \overline{\text{cabc}} \text{cabc} \mid \theta_3^3.
\]

In this example \( k = |\sigma| = 8 \), and the support of the leftmost \( a \)-error of type I is \( [7, 9] \), with
\( i = 7 \), \( p = 1 \).

**Example 4.12** (Error of type II). Let \( \tau \) with normal form
\[
[\tau] = \overline{\text{abacadae}}
\]
with pairwise distinct letters \( a, b, c, d, e \) and \( \theta_\tau \) being the identity. The difference with \( \sigma \) with
normal form \( [\sigma] = \overline{\text{abacadae}} \) (handled previously in Example 1.2) is that the letter \( c \) occurs with a
spin equal to \( 1 \) (i.e., as \( \overline{c} \)). Let us normalize \( \tau^3 \). The first level is the same as for \( \sigma \) (up to bars on
c's). We thus first normalize the blocks \( \overline{\text{ac}} \). The letters in the supports of the propagated errors of
the first level are then marked by red; the letter that causes the second level of block-normalization
is marked by blue (before and after normalization):
\[
Z_{\tau^3} = \overline{\text{abacadae}} \overline{\text{abacadae}} \overline{\text{abacadae}}.
\]
The first level of block-normalizations produces propagated errors of the first level, which gives as
before:
\[
Z_{\tau^3}' = \overline{\text{abacadae}} \overline{\text{abacadae}} \overline{\text{abacadae}}.
\]

This creates the \( a \)-error \( \overline{\text{aca}} \), and the second level of block-normalizations produces the normal
form \( [\tau^3] \) of \( \tau^3 \):
\[
[\tau^3] = \overline{\text{abacadae}} \overline{\text{abacadae}} \overline{\text{abacadae}}.
\]

One has \( k = 8 \), \( i = 5 \), \( p = 3 \). We see that the full propagation of the errors during the
normalization process involves letters whose supports form an interval that overlaps all the periods,
namely \([i, p + (n - 1)k] = [5, 19] \).
4.3. The case of a one-letter normal word. We assume that the normalized directive word of the episturmian substitution \( \sigma \), also assumed to be of positive length, contains only one letter \( \tilde{a} \), i.e., all its letters are equal to \( \tilde{a} \). We thus have for the normal form of \( \sigma \):

\[
[\sigma] = a^{\tilde{a}} \theta_{\sigma}^s
\]

for some integers \( s, t \geq 0 \), with \( s + t \geq 1 \).

Consider first the case where \( \theta_{\sigma}(a) = a \). After applying several times the block-transformation \( \overline{\nu}a \to a\overline{\nu} \), we get for the normal form of \( \sigma^n \), for \( n \geq 1 \):

\[
[\sigma^n] = a^{\tilde{a}} \overline{\nu}^n \theta_{\sigma}^n.
\]

Now consider the case where \( \theta_{\sigma}(a) \neq a \). To get the normal form of \( [\sigma^n] \) from \( [\sigma]^n \), we only have to move \( k \) to the permutation to the right, using the rule (2). So,

\[
[\sigma^n] = a^{\tilde{a}} | \theta_{\sigma}^s(a) \theta_{\sigma}(a) | (\theta_{\sigma}^s(a))^{\overline{\nu}^n} \theta_{\sigma}(a) \mid \cdots | (\theta_{\sigma}^{n-1}(a))^{\overline{\nu}^n} \theta_{\sigma}^n.
\]

4.4. More on the first level of block-transformations. Let \( \sigma \) be an episturmian substitution of positive length \( k \). Lemma 4.14 below describes the changes of spins performed during the first level of block-transformations. We recall that an important property is that, in the first level, propagated errors do not affect each other from one \( k \)-period to another, since their supports do not intersect (see Lemma 4.3). Consequently, if there are no more errors after this first level of block-normalizations, then the changes in the process of normalization are local within the intervals \( [i + k, p + k(j + 1)] \) \( (j = 0, \ldots, n - 2) \). In particular, for the corresponding substitutions, one gets

\[
\mu(\text{pref}_{p+k}([\sigma^n])) = \mu(\text{pref}_{p+k}([\sigma]^n)).
\]

Before stating Lemma 4.14, we first discuss and illustrate a special case of the propagated error of the first level, namely when it contains a simple error \( \overline{\nu}a \) (clearly, this is only possible at the index \( k \), since \( Z_{\sigma^n}[1, k] \) and \( Z_{\sigma^n}[k + 1, 2k] \) do not have errors inside them). The central part of a propagated \( a \)-error of the first level is then defined as the longest factor of the form \( \overline{\nu}^i | a^+ \) that contains \( k \) in its support (in particular \( Z_{\sigma^n}(k) = \overline{\nu} \) and \( Z_{\sigma^n}(k + 1) = a \)).

Example 4.13. (Error of type 1 with a central part)

This example illustrates Assertion 8 in Lemma 4.14.

Let \( \sigma \) with normal form \( [\sigma] = a^3ba^{3}ca^{3}da^{3} \). By considering \( Z_{\sigma^3} \), we get errors \( \overline{\nu}a \) of the first level. The supports of the propagated errors of the first level are marked in red below:

\[
Z_{\sigma^3} = a^3ba^{3}ca^{3}da^{3} | a^3ba^{3}ca^{3}da^{3} | a^3ba^{3}ca^{3}da^{3}.
\]

During the normalization, first we normalize the central part \( \overline{\nu}^i|a^3 \), which gives:

\[
\overline{\nu}^3 | a^3 \to \overline{\nu}^2a | \overline{\nu}a^2 \to \overline{\nu}aa | a\overline{\nu}a \to a\overline{\nu}a | a\overline{\nu}a \to a^2 | a^2a \to a^3 | \overline{\nu}^3.
\]

Then we treat its propagation to the left (with the error involving \( d \)):

\[
a^3a^2ca^2da^{3} | a^3a^2ca^2da^{3} | a^3a^2ca^2da^{3} | a^3a^2ca^2da^{3} \to \overline{\nu}a^2 | ca^2da^{3} | a^3a^2ca^2da^{3} \to \overline{\nu}a^2 | ca^2da^{3} | a^3a^2ca^2da^{3}.
\]

Now we correct twice the error \( \overline{\nu}a \) in the part \( \overline{\nu}^2a^3 \):

\[
a^3ba^{3}ca^{2}da^{2} | a^3ba^{3}ca^{2}da^{2} | a^3ba^{3}ca^{2}da^{2} | a^3ba^{3}ca^{2}da^{2} | a^3ba^{3}ca^{2}da^{2} \to \overline{\nu}a^2 | ca^2da^{3} | a^3a^2ca^2da^{3} \to \overline{\nu}a^2 | ca^2da^{3} | a^3a^2ca^2da^{3}.
\]

Now we correct the error involving \( c \):

\[
a^3ba^{3}ca^{2}da^{2} | a^3ba^{3}ca^{2}da^{2} | a^3ba^{3}ca^{2}da^{2} | a^3ba^{3}ca^{2}da^{2} | a^3ba^{3}ca^{2}da^{2} \to \overline{\nu}a^2 | ca^2da^{3} | a^3a^2ca^2da^{3} | a^3a^2ca^2da^{3} \to \overline{\nu}a^2 | ca^2da^{3} | a^3a^2ca^2da^{3} \to \overline{\nu}a^2 | ca^2da^{3} | a^3a^2ca^2da^{3} \to \overline{\nu}a^2 | ca^2da^{3} | a^3a^2ca^2da^{3}.
\]

and finally the part \( \overline{\nu}^2a \) between \( b \) and \( c \), as well as the part \( \overline{\nu}a^2 \) after \( d \), which gives:

\[
[\sigma^3] = a^3ba^{3}ca^{3}da^{3} | a^3ba^{3}ca^{3}da^{3} | a^3ba^{3}ca^{3}da^{3} | a^3ba^{3}ca^{3}da^{3} | a^3ba^{3}ca^{3}da^{3}.
\]
Let us now state Lemma 4.14 which characterizes the word $Z'_{\sigma}$ obtained after the first level of block-transformations. We assume that $Z_{\sigma}$ contains a propagated error of the first level and that the letter of its leftmost one is $a$, and we use Notation 4.10. See also Figure 1 below for an illustration of the case where there is no central part.

**Lemma 4.14.** Let $\sigma$ be an episturmian substitution of length $k \geq 1$.

1. Any letter $b \neq a$ occurs as $\overline{b}$ (i.e., with spin 1) in the support $[i, k + p]$ of the leftmost propagated error of the first level in $Z_{\sigma}$, and as $b$ in the support of the leftmost error in $Z'_{\sigma}$.
2. The first letter of the leftmost propagated error of the first level occurs (at index $i$) as $\overline{a}$ in $Z_{\sigma}$, and as $a$ in $Z'_{\sigma}$.
3. The last letter of the leftmost error propagated error of the first level occurs (at index $k + p$) as $\overline{a}$ in $Z_{\sigma}$, and as $a$ in $Z'_{\sigma}$.
4. The spins of the letter $a$ in the leftmost propagated error of the first level are equal to 1 before index $k$ included and to 0 after index $k$ in $Z'_{\sigma}$.
5. If the leftmost propagated error of the first level has no central part, then the spins of all the occurrences of $a$ in $[i, k + p]$, except the first and last ones, coincide in $Z_{\sigma}$ and $Z'_{\sigma}$.
6. Otherwise, the leftmost propagated error of the first level contains a central part of the form $\overline{a}^+ | a^+$ in $Z_{\sigma}$ with $Z_{\sigma}(k) = a$ and $Z_{\sigma}(k + 1) = a$. Let $r, q, r', q'$ be such that the central part is of the form $\overline{a}^+ | a^+$ and its normalization is of the form $a^+ \overline{a}^+$. Then
   (a) if the leftmost propagated error of the first level contains only letters $a$, then $r' = r$ and $q' = q$;
   (b) if the leftmost propagated error of the first level contains an occurrence of a letter $\overline{b}$ with $b \neq a$ at an index smaller than $k$ and if it contains only occurrences of the letter $a$ after index $k + 1$, then $r' = r + 1$ and $q' = q - 1$;
   (c) if the leftmost propagated error of the first level contains an occurrence of a letter $\overline{c}$ with $c \neq a$ at an index bigger than $k$ and if it contains only occurrences of the letter $a$ before index $k$, then $r' = r - 1$ and $q' = q + 1$;
   (d) if the leftmost propagated error of the first level contains an occurrence of a letter $\overline{b}$ with $b \neq a$ at an index smaller than $k$ and an occurrence of a letter $\overline{c}$ with $c \neq a$ at an index bigger than $k$, then $r' = r$ and $q' = q$. Moreover, except for the central part, the first and the last occurrences of $a$, the spins of the occurrences of the letter $a$ inside the leftmost propagated error of the first level remain unchanged.
7. The substitution $\mu(\text{pref}_{k+p} Z_{\sigma})$ coincides with the substitution $\mu(\text{pref}_{k+p} Z'_{\sigma})$. In particular, there exists an episturmian substitution $\phi$ such that the substitution $\mu(\text{pref}_{k+p} Z'_{\sigma})$ is equal to $\phi \circ \sigma$.
8. Assume that the errors are of type I, i.e. by definition $[\sigma^n] = Z'_{\sigma}$. If a letter $a$ changes its spin in $[i + 1, k]$, then there exists a central part, and its index of occurrence belongs to the central part. In addition, the changes in the process of normalization are local within the interval $[i + kj, p + k(j + 1)]$ ($j = 0, \ldots, n - 2$). In particular, $\mu(\text{pref}_{p+k}([\sigma^n])) = \mu(\text{pref}_{p+k}([\sigma^n]))$.

**Proof.** The first part of Assertion (1) and Assertion (3) follow from the definition of the propagated error of the first level.

The other assertions are derived from the block-normalization process which works as follows. First, we consider the case where there is no central part. The block-normalization of

$$ac_{11} \cdots c_{1m_1} \cdots \overline{ac_{11}} \cdots c_{1m_1} \overline{ad_1} \cdots \overline{b} \overline{ad_1} \cdots d_{1s} \overline{a} \cdots \overline{d_{us}} \overline{a}$$

yields

$$ac_{11} \cdots c_{1m_1} \cdots \overline{ac_{11}} \cdots c_{1m_1} \overline{ad_1} \cdots b \overline{ad_1} \cdots d_{1s} \overline{a} \cdots d_{us} \overline{a}.$$

Indeed, we first treat the simple error $\overline{ac_{11}} \cdots c_{1m_1} \overline{ad_1}$ between two consecutive periods by applying block-normalization. Then, we consecutively treat in the same way the newly occurred errors $\overline{ac_{11}} \cdots c_{1m_1} \overline{a}$ to the left, one by one, for $j = t, \ldots, j = 1$, and also the errors $\overline{ad_1} \cdots d_{1s} \overline{a}$ to the right, for $j' = 1, \ldots, j' = u$, as in Example 122. The first and last occurrences of $\overline{a}$ change their
spin once, hence Assertions (2) and (3). Except for the first and last occurrences of \(\tilde{a}\), letters \(\tilde{a}\) inside the propagated error change their spin twice, and hence remain unchanged. So, we proved the second part of Assertion (1) and Assertion (5).

Now, consider the case where there is a central part (Assertion (6)), i.e., we need to normalize

\[
\begin{align*}
ac_{11} \cdots ac_{1m_1} \cdots ac_{1t} \cdots ac_{tm} \cdots ac_{1} \cdots ac_{t} \cdots ac_{k+1}.
\end{align*}
\]

Cases (c) and (d) are proved exactly in the same way. Along the lines, we also proved Assertions (1), (2) and (3) in the case when we have a central part.

For the proof of Assertion (7), we use the fact that we can move the permutations via (2).

Assertion (8) is straightforward.

\[\square\]

The following technical proposition will be used in the proof of Lemma 5.7. We state it for convenience, although it is a direct consequence of Lemma 4.14, Assertion 6.

**Proposition 4.15.** Let \(\sigma\) be an episturmian substitution. We assume that \(Z_{\sigma^n}\) contains an \(a\)-error having a central part. Let \([i, k + p]\) stand for the support of its leftmost propagated error of the first level. Let \(t\) be the starting position of the central part \((i \leq t \leq k)\) and let \(q\) be such that its ending position is of the form \(k + \ell\) (with \(k + q \leq k + p\)), i.e.,

\[
Z_{\sigma^n}[t, k] = \overline{a}^{k-t+1}, \quad Z_{\sigma^n}[k + 1, k + q] = a^q, \quad Z_{\sigma^n}[t - 1] \neq \overline{a}, \quad Z_{\sigma^n}[k + q + 1] \neq a.
\]

Let \(\ell\) be a position in the central part, i.e., \(t \leq \ell \leq k + q\). If \([\sigma^n][\ell] = a\), then the following holds:

1. If \(t > i\) (the leftmost propagated error of the first level does not start with the central part), then \(q \geq \ell - t + 1\).
2. If \(t = i\) (the leftmost propagated error of the first level starts with the central part), then \(q \geq \ell - t\).

**Proof.** The first part follows from Lemma 4.14, Assertion 6 Parts 2 and 4. Indeed, one has in the normal form at least \(k - t + 1\) \(\overline{a}\)’s before index \(k + q\). Since \([\sigma^n][\ell] = a\), this gives \(k + q - \ell \geq k - t + 1\), i.e., \(q \geq \ell - t + 1\).

Similarly, the second part follows from Lemma 4.14, Assertion 6 Parts 1 and 3. \[\square\]
4.5. More on the second level of block-normalizations. We first give a proof of Lemma 4.6 stated in Section 4.2. We then provide an extended version of this lemma, namely Lemma 4.16.

Proof of Lemma 4.6. Suppose that there is a simple error in $Z'_{\sigma}$ once we have finished the first level of block-transformations. Let $[\ell, \ell']$ be the support of the leftmost occurrence of a simple error in $Z'_{\sigma}$ and let $b$ stand for its letter.

We first assume $b \neq a$. By $k$-periodicity, $[\ell, \ell']$ intersects the support $[i, k+p]$ of the $a$-propagated error we just normalized during the first level of block-transformations. Moreover, $[\ell, \ell']$ is not included in $[i, k+p]$, since there are no errors in $Z'_{\sigma}[i, k+p]$. Hence $k+1 \leq \ell \leq k+p$. But there are no occurrences of $\overline{a}$ in $Z'_{\sigma}[i, k+p]$ by Lemma 4.14 Assertion (1) since we assumed $b \neq a$. We thus get a contradiction with $b \neq a$, and the letter of this simple error is equal to $a$.

So, the new simple error is of the form $\overline{ab_1 \ldots b_{t}a}$ with $b_i \in A \setminus a$ with $t \geq 0$. Since $k+1 \leq \ell \leq k+p$, the error contains only two occurrences of $\overline{a}$ and is not contained completely in the support $[i, k+p]$ of the leftmost error of the first level, hence this simple error starts at index $k+p$ (the last occurrence of $\overline{a}$ in the propagated $a$-error), i.e., $\ell = p+k$. Moreover, the occurrence of $a$ in it must be at index $k+i$, i.e., $\ell = k+i$ (so that $Z_{\sigma}[k+i] = a$ and $Z'_{\sigma}[k+i] = a$). Otherwise the factor $Z_{\sigma}[i, k+p]$ could be continued to the right keeping the form (3) and hence would not be maximal, and the propagated error of the first level would not actually end at the index $k+p$.

This implies that $\theta_2(a) = a$ and $n \geq 3$. We then use the $k$-periodicity for the proof of Assertion 2.

Lemma 4.16 below describes a word $Z_{\sigma}$ containing errors of type II (see Definition 4.7) and the action of the second level of block-transformations on $Z'_{\sigma}$. According to Lemma 4.6, all errors to be normalized have the same letter $a$ (i.e., they are occurrences of non-normal spinned words of the form $\overline{ab_1 \ldots b_{t}a}$, with the $b_i$’s being distinct from $a$ and $t$ possibly equal to 0), and between two consecutive errors of the first level in $Z'_{\sigma}$ (at indices $[p+1+jk, i+1-jk]$), with $j = 1, \ldots, n-2$, one has a factor of the form $\overline{b_1 \ldots b_{t}a}$ (possibly empty). Whereas the supports of the errors of type I are disjoint, next lemma shows that an error of type II involves errors whose supports form an interval that overlaps all the periods. Indeed, we see below that in type II the leftmost propagated error of the first level that occurs in $Z_{\sigma}$ error propagates on on all the $k$-periods of $Z_{\sigma}$, i.e., on $[i, p+k(n-1)]$ (with Notation 4.10).

According to Lemma 4.6 one has $\theta_2(a) = a$, $n \geq 3$, and between two consecutive errors propagated errors of the first level in $Z'_{\sigma}$, one has $Z'_{\sigma}[p+jk, i+jk] = \overline{ab_1 \ldots b_{t}a}$, for $j = 1, \ldots, n-2$, with $t$ possibly equal to 0, with the letters $b_i$’s being distinct from $a$ for all $j$ (here we also use Lemma 4.14 Assertions (2) and (3)).

Lemma 4.16. Let $\sigma$ be an episturmian substitution of positive length which admits at least two distinct letters in its normal form and let $n \geq 3$. Assume that the errors in $Z_{\sigma}$ are of type II. Let $a$ be the letter of the leftmost error in $Z_{\sigma}$. Let $Z'_{\sigma}$ be the word obtained after the first level of block-normalizations (as described in Lemma 4.14). Then the following holds:

(1) any occurrence of a letter $b$, with $b \neq a$, has a spin equal to 1 in $Z_{\sigma}$;
(2) inside the first period of $Z_{\sigma}$, each occurrence of $\overline{a}$ has spin 0 at indices smaller than or equal to $p$ and spin 1 after that;
(3) The second level of block-transformations is performed in at most two steps:

- First, we correct the simple errors $\overline{ab_1 \ldots b_{t}a}$ in $Z'_{\sigma}$ at indices $p+jk, i+jk$, for $j = 1, \ldots, n-2$. The factors $\overline{b_1 \ldots b_{t}a}$ in $Z'_{\sigma}$ at indices $p+jk+1, i+jk-1, \ldots, jk+1, i+1-jk$, for $j = 1, \ldots, n-2$ become $b_1 \ldots b_{t}a$.

- Secondly, consider the maximal factors of the form $\overline{a^s}$ with support $[p-s+1+jk, p+jk]$ for $j = 1, \ldots, n-2$ (maximal in the sense that $Z'[p-s] \neq \overline{a}$ and $\overline{a^s}$ with support $[i+jk, i+q-1+jk]$ (so that $Z'[i+q] \neq \overline{a}$). They are normalized using block-transformations $\overline{a^s} \rightarrow \overline{a^s}$.

More precisely, the following happens:

- if $s = 1$, then in the normal form $[\sigma^n]$ we have $[\sigma^n][p+jk] = a$, for $j = 1, \ldots, n-2$. Symmetrically, if $q = 1$, then $[\sigma^n][i+jk] = \overline{a}$ for $j = 1, \ldots, n-2$.

If $s = 1$ and $q = 1$, then the second level of block-normalization is over.
If $s > 1$ or $q > 1$, then we continue the block-normalization as follows:

- If there is a letter $c$ with $c \neq \hat{a}$ in $Z_{\sigma}^n[i, k + p]$, then the intervals $[p - s + 1 + kj + k, p + kj + k]$ and $[i + kj, i + q - 1 + kj]$ do not intersect, for $j = 1, \ldots, n - 2$. If $s > 1$ and $Z_{\sigma}^n[i - s + 1 + kj, p + kj] = a^{i - s}$ for $j = 1, \ldots, n - 2$, then $[\sigma^n][p - s + 1 + kj, p + kj] = a^{i - s [\sigma^n] - t}$ (since $s > \ell$ due to Assertion (3) from Lemma 4.14). Symmetrically, if $q > 1$ and $Z_{\sigma}^n[i + kj, i + q - 1 + kj] = a^{i - q [\sigma^n] - t}$ for $j = 1, \ldots, n - 2$, then $[\sigma^n][i + kj, i + q - 1 + kj] = a^{i - q [\sigma^n] - t}$ (due to Assertion (1) due to Lemma 4.14); 

- If $Z_{\sigma}^n[i, k + p]$ contains only occurrences of $\hat{a}$ (there is thus a central part), then $Z_{\sigma}^n[i + kj, p + kj + k] = a^{i + kj + p - i q - \ell + 1}$ ($j = 1, \ldots, n - 2$); moreover, these positions are not changed for $j = 1, \ldots, n - 3$, i.e., $[\sigma^n][i + kj, p + kj + k] = a^{i + kj + p - i q - \ell + 2}$ ($j = 0$) and $[\sigma^n][i + k(n - 3), p + k(n - 2)] = a^{i + k(n - 3) + p - i q - \ell + 2}$ (if $n \geq 4$).

(4) in $[\sigma^n]$, any occurrence of a letter $b$, with $b \neq a$, occurs with spin 0 in $[i, p + k(n - 1)]$, and with spin 1 in both the prefix of length $i - 1$ and the suffix of length $k - p$;

(5) one has $[\sigma^n][i] = a$, $[\sigma^n][k(n - 1) + p] = \sigma$;

(6) the spin of a letter $\hat{a}$ at any index which neither belongs to the central part (if there is a central part), nor equals $i$, nor equals $p + (n - 1)k$, is the same in $Z_{\sigma}^n$ and in $[\sigma^n]$.

Proof. Assertion (1) follows from Lemma 4.14 Assertion (2) (for indices between $p + kj$ and $i + kj$), and Lemma 4.14 Assertion (1) (for indices between $i + kj$ and $p + kj$).

Assertion (2) follows from Lemma 4.14 Assertion (4) and the fact that there are no occurrences of $\hat{a}$ between indices $p$ and $i$ when we have errors of type II (Lemma 4.14 Assertion (1)).

The normalization process described in Assertion (3) is a direct result of block-transformations applied first to the simple errors $\overline{a}_1 \cdots \overline{a}_l a$, and then to the neighbouring factors of the form $\hat{a}^+$ (if they are of length greater than 1). So, for Assertion (3) we only need to prove that after these block-transformations the obtained word is in normal form, i.e., does not contain any errors.

Now we proceed as follows. We first prove that the claims of Assertions (4)–(6) hold for the word obtained after the second level of block-transformations described in Assertion (3). Then we prove Assertion (3) (that there are no errors after these transformations). As a corollary, we'll have that after the second level of block-transformations we get the normal form, so the Assertions (3)–(6) hold true for normal forms as stated.

Assertion (1) follows from Assertion (5) for indices between $p + kj$ and $i + kj$, $j = 1, \ldots, n - 2$, and from Lemma 4.14 Assertion (1) for indices in the support of the error of the first level (i.e., between $i + kj$ and $p + kj + k$, $j = 0, \ldots, n - 2$). The letters at indices before $i$ and after $k(n - 1) + p$ are not touched by the normalization of the first levels by definition and also by the normalization described in Assertion (3), so they stay unchanged (and hence have spin 1 by Assertion (1)).

Assertion (2) follows from Lemma 4.14 Assertions (3) and (5) and $k$-periodicity, since these indices are not touched by the block-transformations described in Assertion (3).

Assertion (3) follows from Assertion (8) and from Lemma 4.14 Assertions (4) and (6).

Now we prove that there are no more errors in the obtained word (Assertion (7)). Suppose there are errors after the two levels of block-normalizations.

First note that the errors must be of the letter $a$. Indeed, errors of another letter $b \neq a$ must be of the form $\overline{b}a \cdots a$ (say at index $j$). Due to Assertion (1), we must have either $j < i$ or $j > p + (n - 1)k$, since only at these indices we can have $b$ for $b \neq a$. The second case is impossible due to absence of letters $b$ of spin 0 after $p + (n - 1)k$, and the first case is impossible since the first letter of spin 0 in the conjectured simple error must be $b$ while it is $a$ at index $i$. So, we indeed cannot have error of a letter $b \neq a$.

Now we are going to prove that we cannot have errors of the letter $a$ either. First note that there are no errors $\overline{a}_1 \cdots \overline{a}_l a$ for $l > 0$: indeed, by the previous assertions, series of occurrences of letters $\overline{a}$ for $c \neq \overline{a}$ can only occur in the first period before index $i$ and in the last period after index $k(n - 1) + p$.

If such a series occurs in the first period, this means that the simple error $\overline{a}_1 \cdots \overline{a}_l a$ either ends at the position $i$ or is entirely contained in the prefix of length $(i - 1)$ of the word. The first case
is impossible since due to Lemma 4.6 the occurrence of \( \hat{a} \) preceding the position \( p \) and it has spin 0 by Assertion (1) and due to the fact that this position is not touched by the first and second level of transformations, whereas it must be 1 to have an error. The second case is impossible since we did not touch the prefix of length \( (i - 1) \) on the first and second level of block-transformations, so it coincides with the prefix of \( [\sigma] \) which is in normal form and hence cannot contain errors. For the last period the proof is symmetric.

It remains to prove that we cannot have errors \( \overline{\sigma a} \). Note that we can only have such errors at positions \( p - s + kj \) and \( i + q - 1 + kj \), since at other positions we are inside normalized parts: either after first level of normalization (between \( p - s + kj \) and \( i + q - 1 + k(j + 1) \)) or after second level of normalization described in Assertion (3) (between \( p + q - 1 + kj \) and \( i - s + kj \)) or inside prefixes or suffixes of \( [\sigma] \) (before \( i \) and after \( p + k(n - 1) \)). And we cannot have errors \( \overline{\sigma a} \) at positions \( p - s + kj \) and \( i + q - 1 + kj \), since we defined \( s \) and \( q \) such that \( Z_{\sigma^n}[p + k - s] \neq \hat{a} \), and \( Z_{\sigma^n}[i + kj + q] \neq \hat{a} \). We remark that we use the fact that we have at least two letters in the normal form (also when we define the maximal intervals of the form \( \hat{a} \): when there are at least two letters, then their lengths are smaller than \( k \), otherwise if there is only one letter, they appear to be one interval covering the whole word), this is why this case is considered separately in Section 4.3.

So, we proved that after the second level of block-transformations we have normalized form, hence Assertions (4)–(6) hold for normal forms.

Figure 2 corresponds to the case where there is no central part.

The interval \([i, p + k(n - 1)]\) is called the support of the full propagated error. Next examples illustrate Lemma 4.16.

**Example 4.17.** (Error of type II: first case of normalisation from Lemma 4.16, Assertion (3), i.e. there is a letter \( d \) with \( d \neq \hat{a} \) in \( Z_{\sigma^n}[i, k + p] \))

Let \( \sigma \) be a substitution with normal form

\[
[\sigma] = a^{3}bcad
\]

with \( a, b, c, d \) distinct letters and \( \theta_{\sigma} \) being the identity. The letters of the supports of the propagated errors of the first level errors are marked by red; this coincides with the central part; letters in blue indicate letters that are involved in the second level of block-normalizations. One has

\[
Z_{\sigma} = a^3bcad|a^3bcad|a^3bcad|a^3bcad|a^3bcad.
\]

The first level of block-normalizations (which deals with the red part) yields:

\[
Z'_{\sigma} = a^3bcad|a^2abcab|a^2abcab|a^2abcab|a^2abcab.
\]

And we create another error \( \overline{abcd} \) (involved letters are marked by red below), and treating it we get:

\[
Z'_{\sigma} = a^3bcad|a^2abcab|a^2abcab|a^2abcab|a^2abcab.
\]
In this example there are no errors in the series of \( \hat{a} \)'s, so we got the normal form on the previous step:

\[
[\sigma^4] = a^2bca|a^3bc|a^2bca|a^2bc|a|bca|a|bca|a|bca\).
\]

**Example 4.18. (Error of type II: second case of normalisation from Lemma 4.16)**

**Assertion (3), i.e.** \( Z_{\sigma^n}[i,k+p] \) **contains only occurrences of \( \hat{a} \)** \( \) Consider now \( n = 4 \) in the previous example. We thus consider \( \sigma \) with normal form

\[
[\sigma] = a^3bca^3
\]

with \( a, b, c \) distinct letters and \( \theta_a \) being the identity. The letters of the supports of the propagated errors of the first level errors are first marked by red; this coincides with the central part; letters in blue indicate letters that are involved in the second level of block-normalizations. One has

\[
Z_{\sigma^4} = a^3bca^3|a^3bca^3|a^3bca^3|a^3bca^3|a^3bca^3|a^3bca^3.
\]

The first level of block-normalizations (which deals with the red part) yields:

\[
Z_{\sigma^4}' = a^3bca^3|a^3bca^3|a^3bca^3|a^3bca^3|a^3bca^3.
\]

And we create as before errors \( \overline{bca} \); treating them as in the previous example we get:

\[
a^3bca^3|\overline{bca}a^3|\overline{bca}a^2|\overline{bca}a^2|\overline{bca}a^3.
\]

So here it remains to correct errors \( \overline{a} \) in the series of \( \hat{a} \) (their supports are depicted in red below):

\[
[\sigma^4] = a^3bca^3|\overline{a}a^2bca|\overline{a}a^2bca^2|\overline{a}a^2bca^3|\overline{a}a^3bca^3 = a^3bca^3|a^2bca^3|a^2bca^3|a^3bca^3.
\]

### 4.6. More about normal forms

In the proof of Lemma 5.3 we will use the following technical general proposition about normal forms. It is used in particular in the proof of Lemma 5.11.

**Proposition 4.19.** Let \( \sigma \) and \( \tau \) be episturmian substitutions. We assume that \( \sigma^n = \tau^m \) for \( n, m \) positive integers. We assume that \( Z_{\sigma^n} \) and \( Z_{\tau^m} \) both contain errors and that both leftmost errors are of the same letter \( a \). We also assume that all the errors are of letter \( a \) in \( Z_{\sigma^n} \) and \( Z_{\tau^m} \), i.e., \( \theta_a(a) = a \) and \( \theta_\tau(a) = a \).

1. If one has \( |\text{pref}_{\ell}(Z_{\sigma^n})|_a = |\text{pref}_{\ell}(Z_{\tau^m})|_a \) for some positive integer \( \ell \), then the substitutions \( \mu(\text{pref}_{\ell}(Z_{\sigma^n})) \) and \( \mu(\text{pref}_{\ell}(Z_{\tau^m})) \) are equal.

2. If, for some index \( \ell \), \( Z_{\sigma^n}[\ell] = Z_{\tau^m}[\ell] = \overline{b} \) for some letter \( b \neq a \), then \( \mu(\text{pref}_{\ell}(Z_{\sigma^n})) = \mu(\text{pref}_{\ell}(Z_{\tau^m})) \).

3. If, for some index \( \ell \), \( Z_{\sigma^n}[\ell] = b \) and \( Z_{\tau^m}[\ell] = \overline{b} \) for some letter \( b \neq a \), then \( |\text{pref}_{\ell}(Z_{\sigma^n})|_a = |\text{pref}_{\ell}(Z_{\tau^m})|_a + 1 \) and \( |\text{pref}_{\ell}(Z_{\sigma^n})|_\tau = |\text{pref}_{\ell}(Z_{\tau^m})|_\tau - 1 \).

**Proof.** First observe that \( [\sigma^n] = [\tau^m] \). The proof of **Assertion (1)** is by induction on \( \ell \). Suppose that for each \( \ell' < \ell \) the statement holds. The induction property clearly holds for \( \ell = 1 \).

Note first that \( |\text{pref}_{\ell}(Z_{\sigma^n})|_a = |\text{pref}_{\ell}(Z_{\tau^m})|_a \) implies that \( |\text{pref}_{\ell}(Z_{\sigma^n})|_\tau = |\text{pref}_{\ell}(Z_{\tau^m})|_\tau \), by **Assertion (2)** in **Remark 4.9**.

We first assume that at index \( \ell \) in \( Z_{\sigma^n} \), the letter is some \( \hat{b} \) with \( b \neq a \). At index \( \ell \) in \( Z_{\tau^m} \), one has the same letter \( \hat{b} \) (possibly with a different spin), by **Assertion (1)** in **Remark 4.9**. If the occurrences at index \( \ell \) of the letter \( b \) in \( Z_{\sigma^n}(\ell) \) and \( Z_{\tau^m}(\ell) \) have the same spin, the proof follows from the fact that the episturmian monoid is left-cancellative (see **Proposition 2.2**), by considering the prefix of length \( \ell - 1 \) together with the induction hypothesis for \( \ell - 1 \). And they cannot have different spins, since otherwise during the normalization of the one with \( \overline{b} \) we must apply a rule involving this position, which changes the number of \( a \)'s in the prefix, and we can do it only once by **Assertion (3)** of **Remark 4.9**. Since \( [\sigma^n] = [\tau^m] \), this is not possible.

We now assume that \( Z_{\sigma^n}[\ell] = \overline{a} \) (and hence \( Z_{\tau^m}[\ell] \) is equal to \( a \) or \( \overline{a} \) by the above). As above, if \( Z_{\sigma^n}[\ell] \) and \( Z_{\tau^m}[\ell] \) have the same spin, the proof follows by left-cancellativity and from the induction hypothesis for \( \ell' = \ell - 1 \). So, we have to consider the case where in one of the representations we have \( a \), and in the other one \( \overline{a} \). Without loss of generality, assume that \( Z_{\sigma^n}[\ell] = a \) and \( Z_{\tau^m}[\ell] = \overline{\tau} \).
• If, for some $\ell' < \ell$, $|\text{pref}_{\ell'}(Z_{\sigma^n})|_a = |\text{pref}_{\ell'}(Z_{\tau^m})|_a$ and $|\text{pref}_{\ell'}(Z_{\sigma^n})|_{\overline{a}} = |\text{pref}_{\ell'}(Z_{\tau^m})|_{\overline{a}}$, then we are done by induction and left-cancellativity.

• So, it remains to consider the case where, for each length $\ell' < \ell$, we have $|\text{pref}_{\ell'}(Z_{\sigma^n})|_a \neq |\text{pref}_{\ell'}(Z_{\tau^m})|_a$. Since $Z_{\sigma^n}[\ell] = a$ and $Z_{\tau^m}[\ell] = \overline{a}$, then $|\text{pref}_{\ell'}(Z_{\sigma^n})|_a < |\text{pref}_{\ell'}(Z_{\tau^m})|_a$ and $|\text{pref}_{\ell'}(Z_{\sigma^n})|_{\overline{a}} > |\text{pref}_{\ell'}(Z_{\tau^m})|_{\overline{a}}$, for all $1 \leq \ell' < \ell$.

In particular, for $\ell' = 1$ this means that $Z_{\sigma^n}(1) = a$ and $Z_{\tau^m}(1) = \overline{a}$. Since $[\sigma^n] = [\tau^m]$ and the only modifications during normalization are of the form $aw_1 \cdots w_n a \rightarrow ab_1 \cdots b_n \overline{a}$ (see Assertion 3 of Remark 4.19), we have $[\sigma^n][1] = [\tau^m][1] = a$. Now, to convert the first symbol $a$ in $Z_{\sigma^n}$ to $a$ during the full process of block-normalization from $Z_{\sigma^n}$ to $[\sigma^n]$, we need to have a prefix of the form $\overline{a}[c \in A]^a$ of length at most $\ell' = 1$ ($c \neq a$). Indeed, consider the first occurrence of a $a$ in $Z_{\sigma^n}$. Before it, we cannot have an occurrence of a letter $b$ with $b \neq a$ with spin 0, since it would block the prefix for the modifications that have to be done (by Assertion 3 of Remark 4.19, contradicting the existence of a $a$ as a first letter for $[\sigma^n]$). Hence there is a propagated $a$-error of the first level starting at index 1 in $Z_{\sigma^n}$ and with support included in $[1, \ell]$. By normalizing this error, we obtain that the first letter of the word $Z_{\sigma^n}'$ is $a$, and again by left-cancellativity applied to $\mu(Z_{\sigma^n}') = \mu(Z_{\tau^m})$ for the prefix of length 1 (which is $a$) and by induction hypothesis for $\ell' = \ell - 1$. This ends the proof of Assertion 1.

We now prove Assertion 2. Assume first that both $Z_{\sigma^n}$ and $Z_{\tau^m}$ have $b$ at the position $\ell$. The normalization is conducted independently before and after index $\ell$. Assume now that both $Z_{\sigma^n}$ and $Z_{\tau^m}$ have $\overline{b}$ at the position $\ell$. Then, during the normalization, it can only change to $b$ once and in both $Z_{\sigma^n}$ and $Z_{\tau^m}$. As a result of application of the rule which involves $b$, the number of $a$’s in the prefix increases by 1, and no other rules change the number of $a$’s in this prefix. So, by Assertion 1, the prefixes correspond to equal substitutions.

For Assertion 3, it is enough to notice that the only rule changing the number of $a$’s and $\overline{a}$’s in the prefix of length $\ell$ is the one changing the spin of the occurrence of $\overline{b}$ at the position $\ell$ to $b$, which changes the numbers of $a$’s and $\overline{a}$’s in the prefix by 1.

The following proposition is a generalization of Proposition 4.19 when the errors inside $Z_{\sigma^n}$ or $Z_{\tau^m}$ do not have possibly the same letter.

Proposition 4.20. We assume that $Z_{\sigma^n}$ and $Z_{\tau^m}$ both contain errors and that both leftmost errors have the same letter $a$. Let $[i, k + p]$ and $[i', k' + p']$ stand for the respective supports of the leftmost errors ($|\sigma| = k$ and $|\tau| = k'$). All the assertions of Proposition 4.19 also hold if

1. $\theta_r(a) \neq a$ and $\theta_r(a) = a$ and $\ell < k + i$,  
2. or (symmetrically) if $\theta_r(a) \neq a$ and $\theta_r(a) = a$ and $\ell < k' + i'$,  
3. or (both) if $\theta_r(a) \neq a$ and $\theta_r(a) \neq a$ and $\ell < \min(k + i, k' + i')$.

Proof. The proof basically repeats the proof of Proposition 4.19 (and the assertions of Remark 4.9 also hold for the index $\ell$ in the indicated prefix).

5. Decomposition lemmas and proof of Lemma 5.3

The aim of this section is to prove Lemma 5.3. We use Notation 4.10. We assume that $\sigma^n = \tau^m$, with $n \geq m \geq 1$. Observe that $[\sigma^n] = [\tau^m], Z_{\sigma^n} \equiv [\sigma^n] = [\tau^m] \equiv Z_{\tau^m}$.

We want to prove that there exists an episturmian substitution $\varphi$ such that $\tau = \sigma \circ \varphi$. The letter $k$ stands for the length $|\sigma|$ of $\sigma$ and $k'$ stands for the length $|\tau|$ of $\tau$. One has $kn = k'm$, $k' \geq k$.

The proof of Lemma 5.3 is given in Section 5.1 and relies on a succession of lemmas stated and proved in Section 5.2. The following remark will be used at several places in the proof.

Remark 5.1. We recall from Section 2.2 that $[\sigma] = w_n \theta_r$, where $w_n$ is the normalized directive word of $\sigma$. If the normalized directive word $w_n$ of $\sigma$ is a prefix of the normalized directive word
are both prefixes of $Z$ provide the list of lemmas treating all the cases needed for the proof of Lemma 3.5. The precise

namely an error. Indeed, otherwise, one has $Z$.

We then remark that it is enough to consider the case when at least one of $Z_{\sigma^n}$ or $Z_{\tau^m}$ has an error. Indeed, otherwise, one has $Z_{\sigma^n} = [\sigma^n]$ and $Z_{\tau^m} = [\tau^m]$. Since $[\sigma^n] = [\tau^m]$, this gives $Z_{\sigma^n} = Z_{\tau^m}$. This implies that $[\sigma]$ and $[\tau]$ coincide up to the permutations $\theta_\sigma$ and $\theta_\tau$, since they are both prefixes of $Z_{\sigma^n} = Z_{\tau^m}$. We then can move $\theta_\sigma$ inside $\mu(Z_{\tau^m})$ in such a way that $\tau = \sigma \circ \varrho$.

So, we now assume that $n \geq m \geq 2$ and at least one of $Z_{\sigma^n}$ or $Z_{\tau^m}$ has an error. Here we provide the list of lemmas treating all the cases needed for the proof of Lemma 3.5. The precise statements and proofs are provided in the next section.

- **Lemma 5.2** states that if $Z_{\sigma^n}$ or $Z_{\tau^m}$ has an error, then so does the other one.
- **Lemma 5.3** handles the case where the left propagated errors start at the same index $(i = i')$.
- **Lemma 5.4** handles the case where errors occur in both $Z_{\sigma^n}$ and $Z_{\tau^m}$ with different letters ($\alpha \neq \alpha'$).
- **Lemma 5.5** handles the case where $\sigma$ and $\tau$ have the same length.
- **Lemma 5.6** handles the case where $Z_{\sigma^n}$ and $Z_{\tau^m}$ both have an error with the same letter, with both being of type I.
- **Lemma 5.7** handles the case where $Z_{\sigma^n}$ and $Z_{\tau^m}$ both have an error with the same letter, but with different types.
- Lastly, **Lemma 5.12** handles the case where one of the substitutions $\sigma$ and $\tau$ admits only one letter in its normal form.

### 5.2. Decomposition lemmas

We now state and prove the lemmas that are used in the proof of Lemma 5.5. All lemmas stated in this section, except Lemma 5.12, involve substitutions that admit at least two distinct letters in their normalized directive sequence. The statements and proof rely on Notation 4.10.

**Lemma 5.2.** Let $\sigma$ and $\tau$ be episturmian substitutions such that $\sigma^n = \tau^m$, for $n, m \geq 2$, with $\sigma$ and $\tau$ both admitting at least two distinct letters in their normal form. If one of the two words $Z_{\sigma^n}$ or $Z_{\tau^m}$ has an error, then the other one also has an error.

**Proof.** We assume w.l.o.g. that $Z_{\tau^m}$ has an error. We work by contradiction and we assume that $Z_{\sigma^n}$ has no error. Let us show that $[\tau^m]^2 \neq [\tau^m]$ by proving that at index $k'(m-1) + i'$ (which belongs to the last period of $\tau^m$) the letter has spin 1 in $[\tau^m]$ and spin 0 in $[\tau^m]$. Indeed, one has $i' > p'$ by Lemma 4.13 and thus $k'(m-1) + i' > k'(m-1) + p'$. Hence, if the error is of type I, this comes from Assertion (2) of Lemma 4.14 together with $k$-periodicity; otherwise, if the error is of type II, since $i' > p'$, then $k'(m-1)+i'>k'(m-1)+p'$; the suffix of support $[k'(m-1)+p'-1,k'm]$ is not changed by normalization from $Z_{\sigma^n}$ to $[\sigma^n]$, hence $Z_{\sigma^n} = [\sigma^n]$. It is thus easy to see that $Z_{\sigma^n}$ contains no error, the words made of the spins are the same for $[\sigma^{2n}]$ and for $[\sigma^m]^2$, which yields a contradiction with the words made of the spins for $[\tau^m]^2$ and $[\tau^m]$ which are not the same.

**Lemma 5.3.** Let $\sigma$ and $\tau$ be episturmian substitutions such that $\sigma^n = \tau^m$, with $n \geq m \geq 2$, with $\sigma$ and $\tau$ both admitting at least two distinct letters in their normal form and with $Z_{\sigma^n}$ or $Z_{\tau^m}$ both having an error. If both leftmost errors start at the same index $(i = i')$, then there exists an episturmian substitution $\varrho$ such that $\tau = \sigma \circ \varrho$. 


Proof. The normal form $|\sigma|$ deprived from its last permutation letter $\theta_\sigma$ (i.e., the normalized directive word $w_\sigma$), is a prefix of $Z_\sigma$ by definition (by Remark 4.9). It is also a prefix of $Z_\tau$. Indeed, we have the same spins in $Z_{\sigma^n}$ and $Z_{\tau^m}$ before the beginning $i$ of the leftmost error. By Lemma 4.14 between $i$ and $k$, all the spins are equal to 1 in $Z_{\sigma^n}$, and similarly between $i$ and $k'$ (we have $i' = i$), all the spins are equal to 1 in $Z_{\tau^m}$. This implies that $w_\sigma$ is a prefix of $Z_\tau$. The desired conclusion comes from Remark 5.1. □

Lemma 5.4. Let $\sigma$ and $\tau$ be episturmian substitutions such that $\sigma^n = \tau^m$, for $n \geq m \geq 2$, with $\sigma$ and $\tau$ both admitting at least two distinct letters in their normalized directive sequences. We assume that $Z_{\sigma^n}$ and $Z_{\tau^m}$ both contain errors and that the leftmost errors have different letters. Then there exists an episturmian substitution $\varphi$ such that $\tau = \sigma \circ \varphi$.

Proof. Let $a$ be the letter of the leftmost error in $Z_{\sigma^n}$ for $\sigma$ and let $a'$ be the letter of the leftmost error for $\tau$. We assume $a \neq a'$. We recall that since $n \geq m$, then $k' \geq k$. Consider three cases according to the types of errors.

- Assume first that the errors are of type I for both substitutions. One thus has $Z_{\sigma^n} = [\sigma^n] = [\tau^m] = Z_{\tau^m}$, by Assertion 3 from Lemma 4.14. The support $[i', k' + p']$ of the leftmost propagated error in $Z_{\tau^m}$ cannot contain $k + p$. Indeed, at position $k + p$, one has $a$ in $Z_{\sigma^n} = [\sigma^n]$ by Assertion 1 ($a' \neq a$), and $a$ in $[\sigma^n]$ by Assertion 3. Similarly, the support $[i, k + p]$ of the leftmost propagated error in $Z_{\sigma^n}$ cannot contain $k' + p'$. Since $k' \geq k \geq i$, this implies $k' + p' > i$, and thus $k' + p' > k + p$, and $i' > k + p$ from what precedes. Hence the leftmost propagated error of the first level in $Z_{\tau^m}$ starts strictly after $k + p$. In particular $k + p \leq k'$ and $\text{pref}_{k+p}Z_{\tau^m} = \text{pref}_{k+p}[\tau]$. Moreover, the prefix of length $k + p$ in $Z_{\sigma^n}$ is not modified by block-normalization since $k + p < i'$. This yields $\text{pref}_{k+p}Z_{\tau^m} = \text{pref}_{k+p}[\tau^m] = \text{pref}_{k+p}[\sigma^n] = \text{pref}_{k+p}Z_{\sigma^n}$. By Assertion 4, there exists an episturmian substitution $\varphi_1$ such that $\mu(\text{pref}_{k+p}Z_{\sigma^n}) = \sigma \circ \varphi_1$. Since $k + p \leq k'$, there exists an episturmian substitution $\varphi_2$ such that $\tau = \mu(\text{pref}_{k+p}Z_{\tau^m}) \circ \varphi_2$, which gives the existence of an episturmian substitution $\varphi$ such that $\tau = \sigma \circ \varphi$.

- The case where the errors are of type II for both substitutions is impossible. Let us prove it by contradiction and consider the indices in $Z_{\sigma^n}$ and $Z_{\tau^m}$ where the errors end, namely $k(n - 1) + p$ and $k'(m - 1) + p'$, respectively, by Lemma 4.14. Note that $a \neq a'$ implies that $k(n - 1) + p \neq k'(m - 1) + p'$. Suppose that $k(n - 1) + p > k'(m - 1) + p'$. By Assertion 3 from Lemma 4.14, the letter at index $k'(m - 1) + p$ in $[\tau^m]$ is $a'$. Since $a' \neq a$ and since $k'(m - 1) + p$ is inside the error for $\sigma$, it occurs with spin 1 in $[\sigma^n]$ by Assertion 1, and we get the desired contradiction from $[\sigma^n] = [\tau^m]$. The same reasoning applies if the error in $\tau$ ends later than in $\sigma$.

- Now, suppose that errors are of different types. Assume first that $Z_{\sigma^n}$ admits errors of type I. We have $Z_{\sigma}(i) = \varpi$ and $[Z_{\sigma}(i)] = a$, as it is the first letter of the error, and $Z_{\sigma}(k + p) = a$ and $[Z_{\sigma}(k + p)] = \varpi$. By Assertion 4 of Lemma 4.14 and since $a \neq a'$, $k + p$ cannot belong to the support $[i', p' + k'(m - 1)]$ of the full propagated error in $Z_{\tau}$, so either $k + p < i'$, or $k + p > k'(m - 1) + i'$. In the first case we have a contradiction with the same assertion at position $i < k + p < i'$. In the second case we can take squares $\sigma^{2n}$ and $\tau^{2m}$ (or bigger powers), so that the inequality does not hold with $2m$ instead of $m$. The case when $Z_{\tau^m}$ admits errors of type I is symmetric. □

Lemma 5.5. Let $\sigma$ and $\tau$ be episturmian substitutions such that $\sigma^n = \tau^m$, for $n, m \geq 2$, with $\sigma$ and $\tau$ both admitting at least two distinct letters in their normal form. If both substitutions have the same length, then there exists an episturmian substitution $\varphi$ such that $\tau = \sigma \circ \varphi$ (and $\varphi$ is a permutation).

Proof. First note that since $|\sigma| = |\tau|$, then $n = m$. We also note that in the case of equal lengths, we must have, for each letter $b$ occurring in $Z_\sigma$, that it also occurs in $Z_\tau$ at the same positions, and moreover $\theta_\sigma(b) = \theta_\tau(b)$. Otherwise, in the normal form we have different letters in the second period at a same position in $[\sigma^n]$ and in $[\tau^n]$. 

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By Lemma 5.2, \(Z_{\sigma^n}\) and \(Z_{\tau^n}\) both have an error. We distinguish two cases according to the fact that they start at the same position, or not.

- Let us assume that the leftmost errors in \(Z_{\sigma^n}\) and \(Z_{\tau^n}\) start at the same position, i.e., \(i = i'\). Lemma 5.3 together with \(|\sigma| = |\tau|\) imply that there exists a permutation \(\varrho\) such that \(\tau = \sigma \circ \varrho\).

- Assume now that \(Z_{\sigma^n}\) and \(Z_{\tau^n}\) are such that their leftmost errors start at different positions, i.e., \(i \neq i'\). We are going to prove that this case is impossible.

We assume \(i' < i\) w.l.o.g. This implies that \(Z_{\sigma^n}\) and \(Z_{\tau^n}\) coincide before \(i'\) (i.e., in \([1, i' - 1]\)), because nothing changes here during the normalization. Let \(c\) be the first non-barred letter. The letter of the error in \(Z_{\tau}\) is the first non-barred letter in the second period, which is \(\theta_{\tau}(c)\). The same argument applies for \(\sigma\), so the letters of the leftmost errors coincide in both substitutions.

Now consider two further cases.

1. We assume \(\theta_{\sigma}(a) = a\) (hence by the above \(\theta_{\tau}(a) = a\)).
   Since all the errors are \(a\)-errors, the respective total numbers of occurrences of \(a\)'s and \(\varrho\)'s do not change during the normalization. Hence they are equal in both \(Z_{\sigma^n}\) and \(Z_{\tau^n}\), and since \(|\sigma| = |\tau|\), they are also equal in the first period, i.e., \(|\text{pref}_k Z_{\sigma^n}| = |\text{pref}_k Z_{\tau^n}|\).
   We recall that \(Z_{\sigma^n}\) and \(Z_{\tau^n}\) coincide before \(i'\) (i.e., in \([1, i' - 1]\)), and in \([i, k]\) (this is due to the structure of errors: all the letters there have spin 1). In \(Z_{\tau^n}\), all the letters have spin 1 also between \(i'\) and \(i\). After normalization we have \([\tau^n][i'] = a\), so we must have \(Z_{\tau^n}[i'] = a\) (since \(i' < i\), i.e., in \(Z_{\tau^n}\), the position \(i'\) is before the occurrences of the errors, so it does not change during normalization). So, we have \(|\text{pref}_k Z_{\sigma^n}| > |\text{pref}_k Z_{\tau^n}|\), a contradiction.

2. We now assume \(\theta_{\sigma}(a) \neq a\) (and hence \(\theta_{\tau}(a) \neq a\)). In particular, we have no type II error by Lemma 4.16 in \(Z_{\sigma^n}\) and in \(Z_{\tau^n}\), and the letter of the error changes in every period. Consider the position \(i'\). As in the previous case, we have \(Z_{\sigma^n}[i'] = a\) (since \(i' < i\)), and thus, in the last period, this gives \(Z_{\sigma^n}[k(n - 1) + i'] = \theta_{\sigma}^{n - 1}(a)\). We have \(Z_{\tau^n}[k(n - 1) + i'] = [Z_{\tau^n}[k(n - 1) + i'] = \theta_{\tau}^{n - 1}(a)\) (since \(k(n - 1) + i' > k(n - 1) + p'\)), we are after the end of the last period in \(Z_{\tau^n}\) and we use Assertion 6 from Lemma 4.16. So, we must have \([Z_{\tau^n}[k(n - 1) + i'] = \theta_{\tau}^{n - 1}(a)\), while \(Z_{\sigma^n}(k(n - 1) + i') = \theta_{\sigma}^{n - 1}(a)\).
   This implies that during the normalization for \(\sigma^n\), the letter at index \(k(n - 1) + i'\) had to change its spin. This can happen only if this position is inside the error between the \((n - 1)\)-th and the \(n\)-th periods for \(\sigma^n\), and only if this letter is the letter of the error. But the letter of the error is \(\theta_{\tau}^{n - 2}(a)\), and we are in the case \(\theta_{\sigma}(a) \neq a\), which yields the desired contradiction.

\[\square\]

In the proof of the next lemma (namely Lemma 5.7) we will use several times the following argument.

**Claim 5.6.** Let \(\sigma\) and \(\tau\) be episturmian substitutions such that \(\sigma^n = \tau^m\), for \(n > m \geq 2\), with \(\sigma\) and \(\tau\) both admitting at least two distinct letters in their normal form. We assume that both substitutions contain an error with the same letter error \(a\) (with possibly different types of errors).

Let the first propagated error of the first level in \(Z_{\sigma^n}\) start before the end of the first period in \(Z_{\tau^m}\), i.e., \(i' \leq k\). Then, either there is no central part in \(Z_{\tau^m}\), or, for any \(i' \leq t \leq k\), the position \(t\) is not in the central part of the propagated error of the first level in \(Z_{\tau^m}\). In particular, there exists a letter \(b \neq a\) (that occurs as \(b\)) in \([i', k']\) in \(Z_{\tau^m}\).

**Proof.** We assume that there exists a central part in \(Z_{\tau^m}\). Assume by contradiction that the position \(t\), with \(i' \leq t \leq k\), is in the central part of \(Z_{\tau^m}\). Clearly, we have \(\varrho\)'s at positions \(t, \ldots, k'\), since we are in the central part of \(Z_{\tau^m}\). Consider the largest position \(j < t\) with an occurrence of \(\overline{b}\) for \(b \neq a\) (such an occurrence exists since \(\sigma\) contains letters distinct from \(a\) by assumption).

We now look at the last periods in \(Z_{\sigma^n}\) and \(Z_{\tau^m}\), by exploiting \(k\)-periodicity and \(k'\)-periodicity, respectively. For \(Z_{\sigma^n}\) we get that \(Z_{\sigma^n}[k(n - 1) + j, kn] = \overline{b}a' \cdots a'\), where \(b' \neq a'\); in fact,
\[ a' = \theta_{\sigma}^{-1}(a) \text{ and } b' = \theta_{\sigma}^{-1}(b). \] On the other hand, \( Z_{\tau_m}[k'(m-1)+j,k'm] = b' a'' \cdots a'' \), where \( b'' \neq a'' \); in fact, \( a'' = \theta_{\sigma}^{-1}(a) \) and \( b'' = \theta_{\sigma}^{-1}(b) \). Moreover, one has \( a'' = a' \) (by looking at the last letter).

Now consider the position \( k(n-1)+j \) in \( Z_{\tau_m} \). We have \( k(n-1)+j > k'(m-1)+j \), since \( k < k' \) and \( kn = k'm \). Therefore, we have \( Z_{\tau_m}[k(n-1)+j] = a'' = a' \), but since the letters in \( Z_{\tau_m} \) and \( Z_{\sigma} \) at the same positions coincide (up to spins), we have \( Z_{\tau_m}[k(n-1)+j] = b' = \theta_{\sigma}^{-1}(b) \neq a' = \theta_{\sigma}^{-1}(a) \), a contradiction.

This implies that there cannot be only a’s in \([i',k]\), otherwise \( i' \) would be in the central part. \( \square \)

**Lemma 5.7.** Let \( \sigma \) and \( \tau \) be episturmian substitutions such that \( \sigma^n = \tau^m \), for \( n > m \geq 2 \), with \( \sigma \) and \( \tau \) both admitting at least two distinct letters in their normal form. We suppose that \( \sigma \) and \( \tau \) have the same letter \( a \) for their leftmost error and that the \( a \)-error is of type I for both. Then, there exists an episturmian substitution \( \bar{q} \) such that \( \tau = \sigma \circ \bar{q} \).

**Proof.** According to Notation 4.10 the intervals of propagated errors are \([i+k,j,p+k(j+1)]\), with \( j \leq n-1 \).

If \( i = i' \), the desired conclusion comes from Lemma 5.3. If \( k + p < i' \), we denormalize \( \tau \) to find the normal form. Indeed, the process of normalization of the prefix of length \( k + p \) for \( \sigma \) is independent of the rest of the word. One has \( k + p \neq i' \) (hence \( k + p > i' \)) since otherwise, in the normalized form, we would have \( a \) at \( i' \) in \([\tau^m]\). On the other hand, it is a final position for an error, hence we would have \( \bar{a} \) at \( k + p \) in \([\sigma^n]\).

We thus assume \( k + p > i' \) and \( i \neq i' \) and we distinguish the three following cases, namely \( i' < i \), \( i < i' \leq k \), and \( k < i' < k + p \).

**Case 1.** We first assume that \( i' < i \). One has \( Z_{\tau_m}[i] = \bar{a} \), since \( i' < i \leq k' \) (indeed \( i < k < k' \)) and due to Assertion 4.14 of Lemma 4.14. Moreover, \([\tau^m][i] = [\sigma^n][i] = a \) (by Assertion 4.14 of Lemma 4.14 since \( i \) is the index of the first letter of an error in \( Z_{\sigma} \)). Since \( \sigma \) the letter \( a \) has different spins in \( Z_{\tau_m} \) and \( [\tau^m] \) and \( i \in [i' + 1, k'] \), then there is an \( a \)-central part \( \sigma^a | \sigma^a \) in \( Z_{\tau_m} \), and the index \( i \) belongs to it by Assertion 4.14 of Lemma 4.14. But, due to Claim 5.6, applied to \( t = i \), this is impossible. We thus reach a contradiction.

**Case 2.** We now assume that \( i < i' \leq k \). At index \( i' \), we are in the central part of \( Z_{\tau_m} \). Indeed, we use an argument symmetric to the one used in the previous case: \( Z_{\sigma}([i']) = \bar{\pi} \) and \([\sigma^n][i'] = a \).

There thus exists a central part in \( Z_{\sigma} \). Let \( s \geq 0 \) be such that the central part in \( Z_{\sigma} \) starts at \( i' - s \); one has \( i < i' - s \leq i' \).

The strategy is to use Proposition 4.19 Assertion 4.14 to compare the numbers of \( a \)-’s in prefixes of the same length of \( Z_{\sigma} \) and \( Z_{\tau_m} \), we first split the prefixes of length \( k \) into four parts, namely \([1, i - 1], [i, i' - s - 1], [i', s, i' - 1] \), and \([i', k] \). The numbers of \( a \)’s in the first part \([1, i - 1] \) (before the beginning of errors) and in the fourth parts \([i', k] \) (made of \( \bar{\pi} \)’s since \( i' \) is in the central part of \( Z_{\tau_m} \)) coincide. In the third part, we have \( Z_{\tau_m}[i' - s, i' - 1] = \bar{\pi} \), \( Z_{\tau}[i' - s, i' - 1] = a^s \) (one has \([\sigma^n][i' - s, i' - 1] = a^s \) by Lemma 4.14 and this part is unchanged in the normalization of \( Z_{\tau_m} \)). Hence \([Z_{\tau_m}[i' - s, i' - 1]]_a = 0 \) and \([Z_{\tau_m}[i' - s, i']][a] = s \).

For the second part, we distinguish the two cases \( i < i' - s \) and \( i = i' - s \).

- **We first handle the case \( i < i' - s \).** We will apply Proposition 4.19 Assertion 4.14 to prefixes of length \( k + s + 1 \).

  In the second part \([i, i' - s - 1] \) (which is outside of the central part), the only index of occurrence of the letter \( a \) where \( Z_{\sigma} \) and \( Z_{\tau_m} \) differ is \( i \) by Assertion 4.14 of Lemma 4.14 \( Z_{\sigma}[i] = \bar{\pi} \) and \( Z_{\tau_m}[i] = a \). By summing up, this gives from above \([|\text{pref}_k(Z_{\sigma})|_a - |\text{pref}_k(Z_{\tau_m})|_a = -s - 1 \).

  We now apply Assertion 4.14 of Proposition 4.14 to \( \ell = i' \) and \( t = i' - s \). Let \( r \) stand for the number of \( \bar{\pi} \)’s after \( k + 1 \). One has \( r \geq i - (i' - s) + 1 \), i.e., \( r \geq s + 1 \). This gives \( s + 1 \) occurrences of \( \pi \)’s in \( Z_{\tau_m} \) starting from index \( k + 1 \).

  Note that \( k' > k + s + 1 \). Otherwise, \( k' \leq k + s + 1 \) implies that \( k \) is in the central part of \( Z_{\tau_m} \), yielding a contradiction with Claim 5.6.
It remains to compare the parts \([k+1, k+s+1]\). We have \(Z_{\sigma^n}[k+1, k+s+1] = a^{s+1}\), \(Z_{\tau^m}[k+1, k+s+1] = \overline{a}^{s+1}\).

Summing up, we have equal numbers of \(a\)'s in the prefixes of length \(k+s+1\), and hence they are equal as substitutions by Assertion 1 of Proposition 4.20.

So, we can denormalize the prefix of length \(k+s+1\) in \(Z_{\tau^m}\) and find \(Z_{\sigma}\) as a prefix of \(\tau\). Indeed, the substitution which corresponds to \(\text{pref}_{k+s+1} Z_{\tau^m}\) equals to the substitution which corresponds to \(\text{pref}_{k+s+1} Z_{\sigma^n}\). There exists \(h\) such that \(\mu(\text{pref}_{k+s+1} Z_{\sigma^n}) = \sigma \circ h = \sigma \circ \mu(\text{pref}_{k+s+1} Z_{\sigma^n})\). And \(\mu(\text{pref}_{k+s+1} Z_{\tau^m})\) is a prefix of \(\tau' (k' > k+s+1)\).

- We now assume \(i = i' - s\). One has from above \(|\text{pref}_k(Z_{\sigma^n})|_a - |\text{pref}_k(Z_{\tau^m})|_a = -s\).

We now apply Assertion (2) of Proposition 4.15 to \(l = i'\) and \(t = i' - s\). Let \(r\) stand for the number of \(\overline{a}\)'s after \(k+1\). One has \(r \geq i' \ast (i' - s)\), i.e., \(r \geq s\). Hence there are at least \(s\) occurrences of the letter \(a\) in \(Z_{\sigma^n}\) starting from index \(k+1\). This gives \(s\) occurrences of \(\overline{a}\)'s in \(Z_{\tau^m}\) starting from index \(k+1\) (\(k' > k+s+1\) from above). Hence, we have \(Z_{\sigma^n}[k+1, k+s] = a^s\), \(Z_{\tau^m}[k+1, k+s] = \overline{a}^s\). Summing up, we have equal numbers of \(a\)'s in the prefixes of length \(k+s\), and hence they are equal as substitutions by Assertion (1) of Proposition 4.20.

Case 3. We now assume that \(k < i' < k+p\). We distinguish two cases, whether \(i'\) belongs to the central part of \(Z_{\sigma^n}\) or not.

Case 3.1. Let us assume that \(i'\) is not in the central part of \(\sigma\). We will show that we can denormalize the prefix of length \(i'\) of \(Z_{\tau^m}\) to find \(\sigma\). To do this, we count the numbers of occurrences of \(a\) and \(\overline{a}\) in the prefix of length \(i'\), show that they are the same and apply Proposition 4.20.

Consider the largest index \(t\) of occurrence of a letter distinct from \(a\) before the position \(i'\); let \(b\) stand for this letter. Since \(k < i'\) by the conditions of Case 3 and since \(i'\) is not in the central part of \(\sigma\), we have \(k < t < i'\).

Since \(t\) is inside the error in \(Z_{\sigma^n}\), we have \(Z_{\sigma^n}(t) = \overline{a}\) and \([\sigma^n][t] = b\) by Lemma 4.14, Assertion 1. Moreover, since \(t < i'\), we have \([\tau^m][t] = Z_{\tau^m}(t)\), and thus \(Z_{\tau^m}(t) = b\) by \([\sigma^n] = [\tau^m]\). So, applying Part 3 of Proposition 4.20, we get \(|\text{pref}_t(Z_{\sigma^n})|_a = |\text{pref}_t(Z_{\tau^m})|_a - 1\).

Now in the part \([t+1, i' - 1]\) (which might as well be empty) we have \(a's\) in \(Z_{\sigma^n}\) (since we are in the second part of the error with \(k < t\)), as well as in the normal form (as occurrences of \(a\) which are neither central, nor initial, nor final, according to Lemma 4.14), so the same is true for \(Z_{\tau^m}\) (we are before the beginning \(i\) of the leftmost error, so everything there coincides with the normal form).

Finally, since \(k < i' < k+p\) is in the second part of the error, by Assertion 1 of Lemma 4.14 we get \(Z_{\sigma^n}[i'] = a\). We also have \(Z_{\tau^m}[i'] = \overline{a}\) (as the first letter of an error by Assertion 2).

Summing up, we get equal numbers of occurrences of \(a\)'s in the prefixes of length \(i'\) of \(Z_{\tau^m}\) and \(Z_{\sigma^n}\). So, by Proposition 4.20 Part 1, they are equal as permutations, and we can denormalise the prefix of length \(i'\) of \(Z_{\tau}\) to find \(Z_{\sigma}\) as a prefix of \(\tau\).

Case 3.2. We now assume that \(i'\) is in the central part of \(Z_{\sigma^n}\). Let \(s\) (with \(s \geq 1\) and \(i \leq k-s\)) be the length of the longest suffix of \(Z_{\sigma}\) filled with \(\overline{a}\)'s. We have \(k-s+1 \leq k < i' < k+p\).

One has \(Z_{\sigma^n}[k+1, i'] = a^{i' - k}\), \(Z_{\tau^m}(i') = \overline{a}\), \([\sigma^n][i'] = [\sigma^n][i'] = a\), and \(Z_{\sigma^n}[i'] = a\). At index \(k\) in \(Z_{\sigma^n}\), one has \(\tau\) (by Assertion 2 of Lemma 4.14). In \(Z_{\tau^m}\), from index \(k-s+1\) to \(i' - 1\), one has only \(a's\); otherwise it would create an error and we also use that \(i'\) was not the beginning of the error.

Similarly as in the previous case, to compare the numbers of \(a's\) in prefixes of the same length, we split the prefixes of length \(i'\) into four parts, namely \([1, i-1]\), \([i, k-s]\), \([k-s+1, k]\), \([k+1, i']\). For the first part \([1, i-1]\), one has the same number of \(a's\) in \(Z_{\sigma^n}\) and \(Z_{\tau^m}\) (by being before the errors). For \([k-s+1, k]\), one has \(|Z_{\sigma^n}(k-s+1, k)| - |Z_{\tau^m}(k-s+1, k)|_a = -s\). For \([k+1, i']\), we have only \(a's\) for both; moreover, \(Z_{\tau^m}(i') = \overline{a}\) and \(Z_{\sigma^n}[i'] = a\). This yields \(|Z_{\sigma^n}(k+1, i')|_a - |Z_{\tau^m}(k+1, i')|_a = +1\).

For the second part, we distinguish the two cases \(i < k-s\) and \(i = k-s+1\). Note that the case \(i = k-s\) is impossible. Indeed, if the error starts before the index \(k-s+1\) \((i < k-s+1)\), we have \(Z_{\sigma}[k-s] = \overline{b}\) for some \(b \neq a\).
We assume $i < k - s$.

In the second part $[i, k - s]$, the only index of occurrence of the letter $a$ where $Z_{\sigma^n}$ and $Z_{\tau^n}$ differ is $i$ with $Z_{\sigma^n}(i) = \overline{a}$. By summing up, this gives from above $|\text{pref}_{i'}(Z_{\sigma^n})|_a - |\text{pref}_{i'}(Z_{\tau^n})|_a = -s$.

We apply Proposition 4.15 to $\ell = i'$ and $\ell = k - s + 1$. We recall that $r$ stands for the number of $a$'s after $k + 1$ in $Z_{\sigma^n}$. This gives $r \geq i' - (k - s + 1) + 1$. Hence one has at least $s$ a's after $i' + 1$ (included). This gives $|\text{pref}_{i'+s}(Z_{\sigma^n})|_a = |\text{pref}_{i'+s}(Z_{\tau^n})|_a$.

We now assume $s = k - i + 1$. Applying Proposition 4.15 one has at least $s - 1$ a's after $i' + 1$ (included). This gives $|\text{pref}_{i'+s-1}(Z_{\sigma^n})|_a = |\text{pref}_{i'+s-1}(Z_{\tau^n})|_a$.

We will make use of the following observation:

Claim 5.8. Let $\sigma$ and $\tau$ be episturmian substitutions such that $\sigma^n = \tau^m$, with $n \geq m \geq 2$, with $\sigma$ and $\tau$ both admitting at least two distinct letters in their normal form. We assume that $Z_{\sigma^n}$ and $Z_{\tau^m}$ have an error with the same letter $a$ for their leftmost error. If errors are of type II in $Z_{\tau^m}$, then $\theta_\sigma(a) = a$. Similarly, if errors are of type II in $Z_{\sigma^n}$, then $\theta_\tau(a) = a$.

Proof. We assume that errors are of type II in $Z_{\tau^m}$. In particular, this yields $m \geq 3$ by Assertion 1 of Lemma 4.1. So, one has $n \geq 3$. Assume that $\theta_\sigma(a) \neq a$. Then errors in $Z_{\sigma^n}$ are of type I (by Assertion 1 of Lemma 4.1). Now, consider the index $2k + p$. It is the end of the error in the third period in $Z_{\sigma^n}$, so one has the letter $\theta_\tau(a)$. It has spin 1 after normalization in $[\sigma^n]$ by Assertion 3 of Lemma 4.13. Due to Assertion 4 of Lemma 4.16, this can only happen if $2k + p \notin [i', m(k' - 1) + p']$. If $2k + p \geq m(k' - 1) + p'$, we can then take squares $Z_{\sigma^n}$ and $Z_{\tau^m}$ to yield a contradiction by having $m$ large enough. If $2k + p < i'$, consider the position $k + i$ (and thus $k + i < i'$). We have $Z_\sigma[k + i] = \overline{\theta_\sigma(a)}$ (as it is the first letter of a propagated error of the first level). After normalization, we have $[\sigma^n][k + i] = \overline{\theta_\tau(a)}$. And in the normalized form $[\tau^m]$, for letters which are not equal to $a$, occurrences with spin 0 can only be inside the error (due to Assertion 2 of Lemma 4.14). We reach a contradiction in both cases.

We assume now that the error is of type II in $Z_{\sigma^n}$. By considering squares, i.e., $\sigma^2n$ and $\tau^2m$, we can assume $m \geq 3$ and the proof works as above.

The following lemmas treat the case analogous to Lemma 5.7 for the case when both errors are of type II (Lemma 5.9) or not (Lemma 5.11).

Lemma 5.9. Let $\sigma$ and $\tau$ be episturmian substitutions such that $\sigma^n = \tau^m$, with $n \geq m \geq 2$, with $\sigma$ and $\tau$ both admitting at least two distinct letters in their normal form. We suppose that $\sigma$ and $\tau$ have the same letter $a$ for their leftmost error, and that errors are of type II for both substitutions. Then, there exists an episturmian substitution $\rho$ such that $\tau = \sigma \circ \rho$.

Proof. Since the errors are of type II, it means that for each letter $b \neq a$ each occurrence of $\overline{b}$ in both $Z_{\tau^n}$ and in $Z_{\sigma^n}$ has spin 1 by Assertion 1 of Lemma 4.16. After normalization, each such occurrence inside the support of the full propagated error goes from spin 1 to spin 0, and outside the full propagated error, it keeps spin 1 by Lemma 4.16. This means that each occurrence of $\overline{b}$ should be either inside both supports of full propagated errors, or outside both. So, both errors start with the same block of $\overline{a}$'s. Now notice that since we have only block-transformations of letter $a$, the proportion of $a$ and $\overline{a}$ must be the same in both $[\sigma]$ and $[\tau]$, hence $i < i'$ (by Assertion 2 of Lemma 4.11). Now since $[\sigma^n(i')] = [\tau^m(i')] = a$, the spin of $i'$ must change during the normalization of $Z_\sigma$, which means that it is in the central part of $Z_\sigma$, i.e., $Z_\sigma[i', k] = \overline{\sigma} \cdots \overline{a}$. Now the indices in $[i', k]$ cannot belong to the central part of $Z_\tau$ (by Claim 5.6), so there exists a letter $b \neq a$ and an index $\ell$ of occurrence of $\overline{b}$, with $k < \ell \leq k'$, which is in fact $\overline{b}$ in both $Z_\sigma$ and $Z_\tau$. So, by Assertion 2 of Proposition 4.19, the substitutions corresponding to the prefixes of $Z_\sigma$ and $Z_\tau$ of length $\ell$ are equal. It follows that we can denormalize the prefix of $Z_\tau$ of length $\ell$ (which is also a prefix of $[\tau]$), introducing $\tau$ as a composition of $\sigma$ and an episturmian substitution.

Claim 5.10. Let $\sigma$ and $\tau$ be episturmian substitutions such that $\sigma^n = \tau^m$, for $n > m \geq 2$, with $\sigma$ and $\tau$ both admitting at least two distinct letters in their normal form. We assume that all the
errors in $\sigma^n$ and $\tau^m$ are of letter $a$, and let $i$ and $j$ be integers with $ik \neq jk'$, $i < n$ and $j < m$. Let $M = \min(ik, jk')$ and $N = \max(ik, jk')$. Then there exists a letter $b \neq a$ that occurs in $[M, N]$, and a letter $c \neq a$ (which can be the same as $b$) that occurs in $[M + 1, N + 1]$.

Proof. The proof is similar to the one of Claim 5.6.

Lemma 5.11. Let $\sigma$ and $\tau$ be episturmian substitutions such that $\sigma^n = \tau^m$, for $n > m \geq 2$, with $\sigma$ and $\tau$ both admitting at least two distinct letters in their normal form. We suppose that $\sigma$ and $\tau$ have the same letter $a$ for their leftmost error, with different types of errors. Then, there exists an episturmian substitution $\theta$ such that $\tau = \sigma \circ \theta$.

Proof. By Lemma 5.4, the letters of the leftmost errors in $Z_\sigma$ and $Z_\tau$ are the same; let $a$ stand for this letter. Note that Claim 5.8 implies that $\theta_\sigma(a) = \theta_\tau(a) = a$.

We now prove that the case where $Z_{\sigma^n}$ has errors of type I and $Z_{\tau^m}$ has errors of type II cannot hold. Since in $\tau$ we have errors of type II, we have $m \geq 3$ (by Assertion (11) of Lemma 4.10) and hence $n \geq 4$ (since $n > m$). Now consider parts outside the support of the errors of type I in $Z_{\sigma^n}$ (i.e., at positions $k\ell + p + 1, \ldots, k\ell + k + i − 1$ for $\ell = 1, \ldots, n − 2$). In particular, this means that $p \leq i + 2$ (we have $p < i$ by definition, and $p = i + 1$ means an error of type II). Since errors are of type I, each part corresponding to a value of $\ell$ should contain $b$ for some $b \neq a$ (as described in different parts due to the permutation $\theta_\sigma$). Let $k + t$ stand for the index of such an occurrence in the second period (hence $p < t < i$), i.e., $Z_\sigma^n(k + t) = b$. Since the spin is the same in all the periods of $\sigma$, and at least one of them is covered by the support of the full propagated error in $Z_\tau^m$ ($m \geq 3$ and $n \geq 4$), $[\sigma^n][k + t] = b$ (by Lemma 4.10, Assertion (1)). This implies $Z_\sigma^n[k + t] = b$ (since the spin is not changed after normalization outside the supports of the errors of type I). By $k$-periodicity, this implies that in the first period in $Z_{\sigma^n}$, one has $Z_{\sigma^n}[t] = b$. We also have $[\sigma^n][t] = b$. Due to the structure of the error of type II, all the occurrences of the $b$'s have spin 1 in $Z_{\tau^m}$ (by Assertion (11) of Lemma 4.10) and in $[\tau^m]$ all the occurrences of the $b$'s are inside the error (by Assertion (4) of Lemma 4.10); outside nothing changes, so they keep their spin equal to 1. One thus has $i' < t$ and $k(n − 1) + t < k'(m − 1) + p'$, which implies $t − k < p' − k'$ since $kn = k'm$. However, $t > i'$ together with $i' > p'$ implies $t > p'$. The inequality $k − t > k' − p'$ implies $k > k'$, which contradicts $k' > k$.

We thus assume that $Z_{\sigma^n}$ has errors of type II, and $Z_{\tau^m}$ has errors of type I. Symmetrically to the above arguments, we get that each occurrence of $b$ at a position $t$ in $Z_{\tau^m}$, with $k' + p' < t < k' + i'$, $l = 0, \ldots, m − 1$, must be an occurrence of $b$ and hence must be inside the type II error in $Z_{\sigma^n}$. These considerations work for $m > 2$, and to make them work in the case $m = 2$, it is enough to take squares of $Z_{\sigma^n}$ and $Z_{\tau^m}$. For each such $t$ with $p' < t < i'$, one has $t > i$ (by considering the first period) and $k' − t > k − p$ (by considering the last period). But, contrary to the previous case, this does not complete the proof. In particular, $t > i'$ implies $i' > i$, as well as $k' > i + k − p$.

We now distinguish two cases, namely $i < i' \leq k$ and $i' > k$.

Case 1. We first assume that $i < i' \leq k$.

By Claim 5.6 there exists an occurrence $t$ in $[i' + 1, k']$ of a letter different from $a$, say $b$. One has $Z_{\tau^m}(t) = \tilde{b} = Z_{\sigma^n}(t)$. We then use Proposition 4.19 for $t − 1$.

Let us prove that $t > k$. The position $i'$ belongs to the central part of $Z_{\sigma^n}$. Indeed, $Z_{\tau^m}(i') = \overline{\sigma}$ and $[Z_{\tau^m}][i'] = a$ by Assertion (2) of Lemma 4.11. But $Z_{\sigma^n}(i') = \overline{\sigma}$ since $i' \leq k$ together with Assertion (4) of Lemma 4.14. The position $i'$ belongs to the support of the leftmost error of $Z_{\sigma^n}$, so by Lemma 4.11 there can be a change in the spin only if $i'$ is in the central part of the error in $Z_{\sigma^n}$, so we have $Z_{\sigma^n}[i', k] = \overline{\sigma} \cdots \overline{\sigma}$ and $Z_{\sigma^n}(k + 1) = a$. This corresponds to $Z_{\tau^m}[i', k + 1] = \overline{\sigma} \cdots \overline{\sigma}$ since $k + 1 \leq k'$. By Assertion (4) of Lemma 4.14, Hence $t > k$. Now we use Proposition 4.19 for $t − 1$. This gives pref$_{t−1} Z_{\tau^m} \equiv$ pref$_{t−1} Z_{\tau^m}$ and so we can denormalize pref$_{t−1} Z_{\tau^m}$ to find $[\sigma]$.

Case 2. We assume now $i' > k$. We distinguish two cases according to the occurrence or not of a letter $b \neq a$ in $[i' + 1, k']$.

- Assume that there exists an index $s$, with $i' < s \leq k' + 1$, such that $Z_{\tau^m}[s] = \tilde{b}$ for some $b \neq a$. Clearly, we then have $Z_{\tau^m}[s] = \tilde{b}$ by Assertion (11) of Lemma 4.14 and $Z_{\sigma^n}[s] = \tilde{b}$.
Lemma 5.12. Assume that either \( \sigma \) one letter (the case where both words contain only one letter is treated inside Case 1).

We now consider the one-letter case described in Section 4.3.

Case 1. Assume first that \( \sigma \) contains only one letter in its normal form. We have \( [\sigma] = a^s a \theta_\sigma \), with \( s, t \geq 0, s + t \geq 1 \) (see Section 4.3).

We now distinguish two further cases according to the fact that \( \theta_\sigma(a) = a \) or not.
Assume first that $\theta_\tau(a) = a$. One has $|\sigma^n| = a^{s_m} a^{-m} \theta^n_\tau$. We have only letters $\tilde{a}$’s in $|\sigma^n|$, so in $|\tau^m|$ as well. It follows that $\theta_\tau(a) = a$, $|\tau| = a^{s_m} a^{-m} \theta^n_r$ for some integers $s', t' \geq 0$, $s' + t' \geq 1$, and $|\tau^m| = a^{s_m} a^{-m} \theta^n_{\tau^{-1}}$. Hence $s'm = sn$, $t'm = tn$, and since $n \geq m$, we have $s \leq s'$, and $t' \leq t$. Applying several times the block-transformation $a\sigma \rightarrow \sigma a$, we can denormalize $\tau$, which gives $|\tau| = a^{s_m} a^{-m} a^{s'} \theta^n_{\tau^{-1}} \theta_\tau$ and we conclude as in Remark 5.1 by inserting $\tau^{-1}$.

Now consider the case where $\theta_\tau(a) \neq a$. One has

$$|\sigma^n| = a^{s_m} a^{-m} a^{|\theta_\tau(a)|} \prod_{i=1}^{n} (\theta^n_\tau(a)) \cdots (\theta^{n-1}_\tau(a)) \theta^{n-t}_\tau(a).$$

If $|\tau| = |\sigma|$, and hence $m = n$, inspection of the first period of $|\sigma^n| = |\tau^m|$ reveals that $|\tau|$ also contains only $\tilde{a}$’s. Similarly, inspection of the second period of $|\tau^m|$ yields $\theta_\tau(a) = \theta_\tau(a) \neq a$, etc. So, $|\tau^m|$ is of the same form as $|\sigma^n|$ up to the permutations, $|\tau| = a^{s_m} a^{-m} a^{|\theta_\tau(a)|}$, and we have $\tau = \sigma \circ \rho$ with $\rho = \theta_\tau^{-1} \theta_\tau$.

Assume now $|\tau| \neq |\sigma|$. If $\tau$ also contains only one letter in its normal form, we conclude as above in the case where $\theta_\tau(a) = a$. We thus assume now that $|\tau|$ contains at least two distinct letters. If $Z_{\tau^m}$ contains no error, or if its leftmost error starts at after the index $k = |\sigma|$ (i.e., $i' > k$), then $w_\tau$ is a prefix of $w_\tau$, and we conclude by Remark 5.1. So, it remains to consider the case where there is an error in $Z_{\tau^m}$ which starts at $i' \leq k$; the letter of this error is thus $a$. Let us prove that we reach a contradiction. The letter at the ending position $k' + p'$ of the first propagated error (in $Z_{\tau^m}$) must also be $a$. Hence, the leftmost error in $Z_{\tau^m}$ (with support $[i', k' + p']$) must contain all the $b$’s (with $b = a$) from the second period of $\sigma^n$ (i.e., with support $[k + 1, 2k]$). After normalization, the letters $b$’s have spin $0$ by Lemma 4.14 and 4.16, since $b \neq a$. This implies that $t = 0$. All letters thus have spin $0$ in $\tau^m$; however, the last letter of a leftmost error must have spin $1$ in its normalized form (at index $k' + p'$ by Lemma 4.14 for an error of type I and at index $k'(m - 1) + p'$ for an error of type II by Lemma 4.16), which yields the desired contradiction.

**Case 2.** Now assume that $\tau$ contains only one letter in its normal form. Since $|\sigma| \leq |\tau|$, $\sigma$ contains only one letter in its normal form. Indeed, the condition $|\sigma^n| = |\tau^m|$ guarantees we have the same letters probably with different spins at the same positions inside both $|\sigma|$ and $|\tau|$. The case where $|\sigma|$ contains only letter has already been treated in Case 1, which ends the proof. □

### 6. More on rigidity

Rigidity is based here on the study of the set of of substitutions having a same fixed point $u$. In this section, we first discuss further relevant variations of the notion of rigidity, and then provide an example illustrating the notion of a weak rigidity.

We recall first that a shift is a closed shift-invariant set of infinite words of some $\mathcal{A}^\mathbb{N}$, where $\mathcal{A}$ is a finite alphabet. Here the set $\mathcal{A}^\mathbb{N}$ is endowed with the product topology of the discrete topology on each copy of $\mathcal{A}$ and the *shift map* $S$ is defined by $S((x_n)_{n \in \mathbb{N}}) = (x_{n+1})_{n \in \mathbb{N}}$. Given a primitive substitution $\sigma$, the subshift $X_\sigma$ generated by $\sigma$ is the set of infinite words having the same language as any infinite word fixed by some positive power of $\sigma$ (they all have the same language by primitivity). We can consider the rigidity of an infinite word $u$, of a subshift $X_\sigma$, or else of a primitive substitution $\sigma$ by considering the following sets:

$$\text{Stab}(\sigma) = \{ \tau \mid \tau \text{ is a primitive substitution with } X_\tau = X_\sigma \},$$

$$\text{Stab}(X) = \{ \tau \mid \tau \text{ is a primitive substitution with } X_\tau = X \},$$

$$\text{Stab}(u) = \{ \tau \mid \tau \text{ is a substitution with } \tau(u) = u \},$$

and by asking whether their stabilizer is cyclic, i.e., they are generated by a single element. We have focused so far on $\text{Stab}(u)$.

More precisely, consider the stabilizer of a shift $X$. We will restrict to the minimal case for ease of simplicity, that is, all infinite words in $X$ have the same set of factors. The *primitive stabilizer* of a shift $X$ is the set of all primitive substitutions $\sigma$ defined on the alphabet $\mathcal{A}$ that satisfy $X_\sigma = X$. 

In other words, these are the primitive substitutions whose language coincides with the language of $X$. The subshift $X$ is said \textit{rigid} if its stabilizer is cyclic.

Now consider weak reagidity. A shift, an infinite word, or a substitution is said \textit{weakly rigid} if for any two substitutions $\sigma$, $\tau$ in its stabilizer, $\sigma^k \sim \rho^m$ for some positive integers $k, m$, for some given equivalence relation.

Several equivalence relations can be considered for weak rigidity. We can consider weak rigidity where the equivalence relation relies on the equality, on the operation $w \mapsto w^{-1}$ (the operation $w \mapsto w^{-1}$ means that we replace 0 by 1 and 1 by 0), or else on conjugacy. (We recall that two substitutions $\sigma$ and $\tau$ over the same alphabet are conjugate ($\sigma \sim \tau$) if there exists a word $w$ such that $w\sigma(x) = \tau(x)w$ for every letter $x$.) We can also ask whether two substitutions $\sigma, \tau$ have a common power (i.e., there exist $m, n$ such that $\sigma^m = \tau^n$) or whether there are each power of a common substitution (i.e., there exist $k, \ell$ and $g$ such that $\sigma = g^k$, $\tau = g^\ell$).

Consider as an illustration the Sturmian case for two-letter alphabets; see e.g. [BFS12]. Let $u$ be the Fibonacci word generated by the Fibonacci substitution $\sigma: 0 \mapsto 01, 1 \mapsto 0$. One has $\text{Stab}(u) \neq \text{Stab}(\sigma)$. Consider indeed the conjugate $\tau$ of the Fibonacci substitution, i.e., $\tau: 0 \mapsto 10, 1 \mapsto 0$. One has $\tau \not\in \text{Stab}(u)$, but $\tau \in \text{Stab}(\sigma)$, and $\text{Stab}(u) = \{\sigma^n\}$ and $\text{Stab}(\sigma) = \{\mu \mid \exists n \mu \sim \sigma^n\}$.

We now detail an example of an infinite word $u$, obtained as a fixed point of a two-letter substitution $\sigma$, which displays some rigidity: the equivalence relation $w \mapsto w^{-1}$, in the sense that $\text{Stab}(u)$ contains only powers and products of the two substitutions $\sigma$ and $\tau$, with $\tau(0) = \sigma(1)$ and $\sigma(1) = \tau(0)$.

\textbf{Theorem 6.1.} Consider the following substitutions over the alphabet $\{0, 1\}$

\begin{equation*}
\sigma: 0 \mapsto 01, \ 1 \mapsto 100110, \quad \tau: 0 \mapsto 011001, \ 1 \mapsto 10.
\end{equation*}

Let $u$ be the fixed point of $\sigma$ starting with 0, i.e., $u = \sigma^\infty(0)$. Let $\text{Stab}(u)$ be the stabilizer of the word $u$. We have

\begin{equation*}
\text{Stab}(u) = \{\sigma^i, \tau^j, \sigma \tau^k, \tau \sigma^\ell \mid i, j, k, \ell \in \mathbb{N}\}.
\end{equation*}

To prove Theorem 6.1, we make use of several auxiliary lemmas.

\textbf{Lemma 6.2.} In $u$, the factors 00 and 11 occur only at even indices.

\textbf{Lemma 6.3.} Each factor $v$ of $u$ of length at least 5 contains a factor $aa$, where $a$ is a letter.

\textbf{Lemma 6.4.} For each $\varphi \in S_u$, the lengths $|\varphi(0)|, |\varphi(1)|$ are even.

\textbf{Proof.} Suppose first that $|\varphi(0)|$ is odd, i.e., $|\varphi(0)| = 2k + 1$ for some integer $k$. Then

\begin{equation*}
u = \varphi(0)\varphi(1)\varphi(1)\varphi(0)\cdots.
\end{equation*}

Let $|\varphi(1)| = m$. If $|\varphi(0)| \geq 3$, then $\varphi(0)$ starts with 011 and we find an occurrence of 11 at position $2k + 1 + 2m + 2$, which is odd; a contradiction with Lemma 6.2. If $|\varphi(0)| < 3$, then $\varphi(0) = 0$. Hence $\varphi(1)$ must begin with 11. We have

\begin{equation*}
\varphi(0110) = 0\varphi(1)\varphi(1)0,
\end{equation*}

which means that $\varphi(1)$ ends with 1; otherwise we have an occurrence of 00 at index $2m + 1$. But in this case we have an occurrence of 111 at index $m + 1$, which is not possible since 111 cannot be a factor of $u$. We thus have proved that $|\varphi(0)|$ is even; let $p$ be such that $|\varphi(0)| = 2p$.

Now suppose $|\varphi(1)|$ is odd, i.e., $|\varphi(1)| = 2\ell + 1$ for some integer $\ell$.

- If $|\varphi(1)| \geq 3$, then consider the prefix $\varphi(0)\varphi(1)\varphi(1)$ of $u$. Since $\varphi(1)$ contains a factor $aa$ and it is of odd length, it follows that the prefix contains $aa$ at an odd position; a contradiction.

- If $|\varphi(1)| = 3$, i.e., $\varphi(1) = abc$ for some $a, b, c \in \{0, 1\}$, then consider the prefix $\varphi(0)abcabc$ of $u$. Considering factors at odd positions $2p + 1, 2p + 3$ and $2p + 5$, we get that $a \neq b, c \neq a, b \neq c$, which is not possible over a two-letter alphabet.

- If $|\varphi(1)| = 1$, i.e., $\varphi(1) = a$ for some $a \in \{0, 1\}$, then consider the prefix $\varphi(0)aa$ of $u$. The factor $aa$ of $u$ is at odd position $2p + 1$, which is not possible by Lemma 6.2.

\hfill \blacksquare
Lemma 6.7. \(\) the desubstitution is unique in both cases. 

Proof. \(\) Suppose first that \(u\) is a concatenation of blocks \(C\) and \(D\). Indeed, by Lemma 6.2 \(u\) is a concatenation of blocks 01 and 10, and since \(\sigma(u) = u\), we have that \(\sigma(u)\) is a concatenation of blocks \(\sigma(0)\sigma(1) = CC = 01100110\) and \(\sigma(1)\sigma(0) = DD = 10011001\). This also implies that \(u\) consists of blocks \(ABAB\) and \(BABA\). We now define the following recurrence relations:

\[
\begin{align*}
T_0 &= C \\
T_n &= T_{n-1}T_{n-1}^\ast T_{n-1}^\ast T_{n-1},
\end{align*}
\]

where \(C = D, D = C\). We recall that the operation \(v \mapsto \overline{v}\) means that we replace 0 by 1 and 1 by 0. We then define the word \(u''\) over \(\{C,D\}\) as \(u'' = \lim_{n \to \infty} T_n\).

**Lemma 6.5.** One has \(u = \varrho(u'')\) where \(\varrho\) is the morphism from \(\{C,D\}^\ast\) to \(\{0,1\}^\ast\) defined by \(\varrho : C \mapsto 0110, D \mapsto 1001\).

**Proof.** The proof works by induction. One has \(\sigma(01) = \omega(T_0T_0), \sigma(10) = \omega(T_0T_0),\)

\[
\sigma^{n+2}(01) = \sigma^{n+1}(01100110) = \sigma^{n+1}(01)\sigma^{n+1}(10)\sigma^{n+1}(01)\sigma^{n+1}(10) = \\
\sigma(T_nT_n)\sigma(T_nT_n)\sigma(T_nT_n)\sigma(T_nT_n) = \sigma(T_{n+1}T_{n+1}).
\]

The same holds similarly for \(\sigma(10)\). So, we proved that arbitrarily long prefixes of \(u\) and \(\sigma(u'')\) coincide, which completes the proof. \(\square\)

**Claim 6.6.** One has \(u'' = \mu^\infty(C)\), where \(\mu\) is the substitution over \(\{C,D\}\) defined as \(\mu : C \mapsto CCDD, D \mapsto DDCC\).

**Proof.** We prove by induction on \(n\) that \(T_n = \mu^n(C), T_n = \mu^n(D)\). Indeed, \(T_1 = CCDD = \mu(C), T_1 = DDCC = \mu(D)\). Now

\[
T_{n+1} = T_nT_nT_nT_n = \mu^n(C)\mu^n(C)\mu^n(D)\mu^n(D) = \mu^n(CCDD) = \mu^{n+1}(C).
\]

The case of \(T_{n+1}\) is handled similarly. \(\square\)

We use the following notation when “desubstituting”. For \(w\) word over \(\{C,D\}\), \(\varrho^{-1}(w)\) stands for the word over \(\{0,1\}\) that satisfies \(w = \varrho(\varrho^{-1}(w))\). The same holds for \(\psi^{-1}\), by noticing that the desubstitution is unique in both cases.

**Lemma 6.7.** If \(u\) has a prefix of the form \(vv\), then there exists \(i\) such that \(|vv| = 2 \cdot 4^i\).

**Proof.** Suppose first that \(v\) is divisible by 4. Then the prefix \(vv\) corresponds to a square \(v'v' = \varrho^{-1}(v)\varrho^{-1}(v)\) in \(u''\). If \(|v'|\) is divisible by 4, we continue to desubstitute and find a smaller square \(\varrho^{-1}(v')\varrho^{-1}(v')\); we continue desubstituting until we find a square \(v_0v_0\) of length not divisible by 4. If \(|v_0| = 1\), the lemma is proved; otherwise consider the following cases on the length of \(v_0\) mod 4, for which we reach a contradiction. This will prove that \(|v_0| = 1\).

- Assume that \(|v_0| = 4k + 1\) \((k > 0)\) or \(|v_0| = 4k + 3\). If \(v_0\) ends with \(C\), then it ends with either \(DDCC\) or with \(DC\) (from the structure of the word given by Claim 6.5). The second copy of \(v_0\) starts with \(CCD\), therefore we have an occurrence of \(DC^5D\) or \(DC^5D\) in \(u''\), which is not possible. If \(v_0\) ends with \(D\), the proof is symmetric.

- Assume that \(|v_0| = 4k + 2\). For \(k = 0\) and \(k = 1\) we clearly do not have squares. For \(k > 1\), the first occurrence of \(v_0\) contains \(D^4\), and in \(u''\) the factor \(D^4\) can occur only at positions of the form \(8m - 1\). Then in the second copy of \(v_0\) the copy of the factor \(D^4\) occurs at positions of the form \(8\ell + 1\) or \(8\ell + 5\), which is not possible in \(u''\).
Now consider the case where \( v \) is not divisible by 4.

- If \( |v| = 4k + 1 \) (\( k > 0 \)) or \( |v| = 4k + 3 \), then \( v \) contains an occurrence of 11 or 00; hence its second copy occurs at an odd position, which is not possible by Lemma 6.2.
- If \( |v| = 4k + 2 \), then the contradiction comes from considering an occurrence of \( BB \) in \( u' \): it can only occur at an even position, and the second copy of \( v \) provides an occurrence at an odd position.

\[ \Box \]

**Corollary 6.8.** Let \( \varphi \in \text{Stab}(u) \). We have \( \varphi(01) = \varphi(T_nT_n) \) for some \( n \), and \( \varphi(10) = \varphi(T_nT_n). \)

**Proof.** The prefix 01100110 is a square prefix of \( u \), so \( \varphi(01100110) \) is also a square prefix of \( u \), and its length is thus of the form \( 2 \cdot 4^i \) by Lemma 6. This means that \( \varphi(01) \) has length \( 2 \cdot 4^{i-1} \), which in turn means exactly \( \varphi(01) = \varphi(T_nT_n) \) for \( n = i - 2 \).

\[ \Box \]

**Lemma 6.9.** For each \( \varphi \in \text{Stab}(u) \), there exist \( \varphi' \) and \( \varphi'' \) such that one has either \( \varphi = \sigma \circ \varphi' \), or \( \varphi = \tau \circ \varphi'' \).

**Proof.** One has \( \varphi(u) = u \). Suppose that \( |\varphi(0)| < |\varphi(1)| \) (the other case will be handled analogously if \( |\varphi(0)| \geq |\varphi(1)| \)). The word \( u' \) (with \( u = \psi(u(\varphi)) \)) consists of blocks \( ABAB \) and \( BABA \). Since \( |\varphi(0)| \) and \( |\varphi(1)| \) are even (by Lemma 6.3), we have that \( \psi^{-1}(\varphi(0)) \) consists of \( A \)'s and \( B \)'s, as well as \( \psi^{-1}(\varphi(1)) \). Since each block \( ABAB \) and \( BABA \) is a concatenation of \( A \) and \( BAB \), we get a natural factorization in these blocks. Now if this factorization is a refining of the one with \( \psi^{-1}(\varphi(0)) \) and \( \psi^{-1}(\varphi(1)) \), then the lemma is proved.

If not, then we have some position of factorization of the form \( 4k + 2, 3 \) or 4, and in addition \( \psi^{-1}(\varphi(1)) \) contains \( BB \), since the length of \( \varphi(1) \) is bigger than the length of \( \varphi(0) \) (all the short substitutions can be checked with an exhaustive search).

Now consider the prefix \( \psi^{-1}(\varphi(0),\varphi(1)) \) of \( u' \), whose length is of the form \( 2 \cdot 4^n \) by Corollary 6.8. If the length of \( \psi^{-1}(\varphi(0)) \) is divisible by 4, then all the positions of \( \psi^{-1}(\varphi(0)) \) and \( \psi^{-1}(\varphi(1)) \) are equivalent to 1 modulo 4. So, the length of \( \psi^{-1}(\varphi(0)) \) is of the form \( 4\ell + 1, 2 \) or 3 for some \( \ell \), and so is \( \psi^{-1}(\varphi(1)) \). Consider the factor \( \psi^{-1}(\varphi(1)) \) of \( u' \). Consider an occurrence of \( BB \) in the first copy of \( \psi^{-1}(\varphi(1)) \), which can only be at a position of the form \( 4m \) in \( u' \). Then the occurrence of \( BB \) in the second copy cannot be of the form \( 4m' \).

The proof of Theorem 6.1 follows from Lemma 6.9 with the observation that \( \sigma(01) = \tau(01) \), \( \sigma(10) = \tau(10) \), and hence \( \sigma \circ \sigma = \sigma \circ \tau, \tau \circ \sigma = \tau \circ \tau \) (by checking it directly on letters).

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