On convergence properties of infinitesimal generators of scaled multi-type CBI processes

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Abstract

It is a common method for proving weak convergence of a sequence of time-homogeneous Markov processes towards a time-homogeneous Markov process first to show convergence of the corresponding infinitesimal generators and then to check some additional conditions. The aim of the present paper is to investigate convergence properties of infinitesimal generators of appropriately scaled random step functions formed from a multi-type continuous state and continuous time branching process with immigration.

1 Introduction

Studying weak convergence of Markov processes has a long tradition and history. It is a common method for proving weak convergence of a sequence of time-homogeneous Markov processes towards a time-homogeneous Markov process first to show convergence of the corresponding infinitesimal generators and then to check some additional conditions, see, e.g., Ethier and Kurtz [7, Chapter 4, Section 8]. In a recent paper, we proved that, under some fourth order moment assumptions, a sequence of scaled random step functions \( (n^{-1}X_{⌊nt⌋})_{t≥0}, n≥1 \), formed from a critical, irreducible multi-type continuous state and continuous time branching process with immigration (CBI process) \( X \) converges weakly towards a squared Bessel process supported by a ray determined by the Perron vector of a matrix related to the branching mechanism of \( X \), see Barczy and Pap [4, Theorem 4.1], and Section 2 as well. This convergence result has been shown not by infinitesimal generators, that is why we consider in Section 3 the sequences of infinitesimal generators of \( (n^{-1}X_{⌊nt⌋})_{t≥0}, n≥1 \), and \( (n^{-1}X_{nt})_{t≥0}, n≥1 \), formed from a (not necessarily critical or irreducible) multi-type CBI process \( X \). Adding some correction terms
to these sequences of infinitesimal generators, under some second order moment assumptions, we show their convergence, see Propositions 3.4 and 3.7. As a consequence, the sequences of infinitesimal generators (without correction terms) do not converge in general. We also apply Proposition 3.4 to irreducible and critical multi-type CBI processes, see Corollary 3.5 and Remark 3.6. In Remark 3.8 we specialize Proposition 3.7 to single-type irreducible and critical CBI processes.

In Section 2, for completeness and better readability, from Barczy et al. [2] and [4], we recall some notions and statements for multi-type CBI processes such as the form of their infinitesimal generator, a formula for their first moment, the definition of irreducible CBI processes and a classification of them, namely we recall the notion of subcritical, critical and supercritical irreducible CBI processes, see Definitions 2.7 and 2.8, respectively.

2 Multi-type CBI processes

Let $\mathbb{Z}_+^d$, $\mathbb{N}$, $\mathbb{R}$, $\mathbb{R}_+$ and $\mathbb{R}_{++}$ denote the set of non-negative integers, positive integers, real numbers, non-negative real numbers and positive real numbers, respectively. For $x, y \in \mathbb{R}$, we will use the notations $x \wedge y := \min\{x, y\}$ and $x^+ := \max\{0, x\}$. By $\|x\|$ and $\|A\|$, we denote the Euclidean norm of a vector $x \in \mathbb{R}^d$ and the induced matrix norm of a matrix $A \in \mathbb{R}^{d \times d}$, respectively. The natural basis in $\mathbb{R}^d$ will be denoted by $e_1, \ldots, e_d$. By $C^2_c(\mathbb{R}_+^d, \mathbb{R})$ we denote the set of twice continuously differentiable real-valued functions on $\mathbb{R}_+^d$ with compact support.

2.1 Definition. A matrix $A = (a_{i,j})_{i,j \in \{1, \ldots, d\}} \in \mathbb{R}^{d \times d}$ is called essentially non-negative if $a_{i,j} \in \mathbb{R}_+^d$ whenever $i, j \in \{1, \ldots, d\}$ with $i \neq j$, i.e., if $A$ has non-negative off-diagonal entries. The set of essentially non-negative $d \times d$ matrices will be denoted by $\mathbb{R}_{+}^{d \times d}$.

2.2 Definition. A tuple $(d, c, \beta, B, \nu, \mu)$ is called a set of admissible parameters if

(i) $d \in \mathbb{N},$

(ii) $c = (c_i)_{i \in \{1, \ldots, d\}} \in \mathbb{R}_+^d,$

(iii) $\beta = (\beta_i)_{i \in \{1, \ldots, d\}} \in \mathbb{R}_+^d,$

(iv) $B = (b_{i,j})_{i,j \in \{1, \ldots, d\}} \in \mathbb{R}_{+}^{d \times d},$

(v) $\nu$ is a Borel measure on $U_d := \mathbb{R}_+^d \setminus \{0\}$ satisfying $\int_{U_d} (1 \wedge \|z\|) \nu(dz) < \infty,$

(vi) $\mu = (\mu_1, \ldots, \mu_d)$, where, for each $i \in \{1, \ldots, d\}$, $\mu_i$ is a Borel measure on $U_d$ satisfying

$$\int_{U_d} \left[ (1 \wedge z_i)^2 + \sum_{j \in \{1, \ldots, d\} \setminus \{i\}} (1 \wedge z_j) \right] \mu_i(dz) < \infty.$$
2.3 Remark. Our Definition 2.2 of the set of admissible parameters is a special case of Definition 2.6 in Duffie et al. [5], which is suitable for all affine processes, see Barczy et al. [2, Remark 2.3].

2.4 Theorem. Let \((d, c, \beta, B, \nu, \mu)\) be a set of admissible parameters. Then there exists a unique transition semigroup \((P_t)_{t \in \mathbb{R}_+}\) acting on the Banach space (endowed with the supremum norm) of real-valued bounded Borel-measurable functions on the state space \(\mathbb{R}_+^d\) such that its infinitesimal generator is

\[
(Af)(x) = \sum_{i=1}^d c_i x_i f''(x) + \langle \beta + Bx, f'(x) \rangle + \int_{U_d} (f(x + z) - f(x)) \nu(dz) \tag{2.1}
\]

\[
+ \sum_{i=1}^d x_i \int_{U_d} (f(x + z) - f(x) - f'_i(x)(1 \land z_i)) \mu_i(dz)
\]

for \(f \in C^2_b(\mathbb{R}_+^d, \mathbb{R})\) and \(x \in \mathbb{R}_+^d\), where \(f'_i\) and \(f''_i\), \(i \in \{1, \ldots, d\}\), denote the first and second order partial derivatives of \(f\) with respect to its \(i\)-th variable, respectively, and \(f'(x) := (f'_1(x), \ldots, f'_d(x))^T\). Moreover, the Laplace transform of the transition semigroup \((P_t)_{t \in \mathbb{R}_+}\) has a representation

\[
\int_{\mathbb{R}_+^d} e^{-\langle \lambda, y \rangle} P_t(x, dy) = e^{-\langle x, \nu(t, \lambda) \rangle} f'_i \psi(\nu(s, \lambda)) ds, \quad x \in \mathbb{R}_+^d, \quad \lambda \in \mathbb{R}_+^d, \quad t \in \mathbb{R}_+,
\]

where, for any \(\lambda \in \mathbb{R}_+^d\), the continuously differentiable function \(\mathbb{R}_+ \ni t \mapsto \nu(t, \lambda) = (\nu_1(t, \lambda), \ldots, \nu_d(t, \lambda))^T \in \mathbb{R}_+^d\) is the unique locally bounded solution to the system of differential equations

\[
\partial_t \nu_i(t, \lambda) = -\varphi_i(\nu(t, \lambda)), \quad \nu_i(0, \lambda) = \lambda_i, \quad i \in \{1, \ldots, d\},
\]

with

\[
\varphi_i(\lambda) := c_i \lambda_i^2 - \langle Be_i, \lambda \rangle + \int_{U_d} (e^{-\langle \lambda, z \rangle} - 1 + \lambda_i (1 \land z_i)) \mu_i(dz)
\]

for \(\lambda \in \mathbb{R}_+^d\) and \(i \in \{1, \ldots, d\}\), and

\[
\psi(\lambda) := \langle \beta, \lambda \rangle - \int_{U_d} (e^{-\langle \lambda, z \rangle} - 1) \nu(dz), \quad \lambda \in \mathbb{R}_+^d.
\]

2.5 Remark. This theorem is a special case of Theorem 2.7 of Duffie et al. [5] with \(m = d, n = 0\) and zero killing rate. □

2.6 Definition. A Markov process with state space \(\mathbb{R}_+^d\) and with transition semigroup \((P_t)_{t \in \mathbb{R}_+}\) given in Theorem 2.4 is called a multi-type CBI process with parameters \((d, c, \beta, B, \nu, \mu)\).

Let \((X_t)_{t \in \mathbb{R}_+}\) be a multi-type CBI process with parameters \((d, c, \beta, B, \nu, \mu)\) such that the moment conditions

\[
\int_{U_d} \|z\| 1_{\{\|z\| \geq 1\}} \nu(dz) < \infty, \quad \int_{U_d} \|z\| 1_{\{\|z\| \geq 1\}} \mu_i(dz) < \infty, \quad i \in \{1, \ldots, d\}
\]
hold. Then, by (3.4) in Barczy et al. [2],

\[ \mathbb{E}(X_t \mid X_0 = x) = e^{t\tilde{B}}x + \int_0^t e^{u\tilde{B}}\tilde{\beta}du, \quad x \in \mathbb{R}^d, \quad t \in \mathbb{R}_+, \]

where

\[ \tilde{B} := (b_{i,j})_{i,j \in \{1, \ldots, d\}}, \quad \tilde{b}_{i,j} := b_{i,j} + \int_{U_d} (z_i - \delta_{i,j})^+ \mu_j(dz), \]

\[ \tilde{\beta} := \beta + \int_{U_d} z \nu(dz), \]

with \( \delta_{i,j} := 1 \) if \( i = j \), and \( \delta_{i,j} := 0 \) if \( i \neq j \). Note that \( \tilde{B} \in \mathbb{R}^{d \times d}_+ \) and \( \tilde{\beta} \in \mathbb{R}^d_+ \), since

\[ \int_{U_d} \|z\|\nu(dz) < \infty, \quad \int_{U_d} (z_i - \delta_{i,j})^+ \mu_j(dz) < \infty, \quad i, j \in \{1, \ldots, d\}, \]

see Barczy et al. [2, Section 2].

Next we recall a classification of multi-type CBI processes. For a matrix \( A \in \mathbb{R}^{d \times d} \), \( \sigma(A) \) will denote the spectrum of \( A \), i.e., the set of the eigenvalues of \( A \). Then \( r(A) := \max_{\lambda \in \sigma(A)} |\lambda| \) is the spectral radius of \( A \). Moreover, we will use the notation

\[ s(A) := \max_{\lambda \in \sigma(A)} \text{Re}(\lambda). \]

A matrix \( A \in \mathbb{R}^{d \times d} \) is called reducible if there exist a permutation matrix \( P \in \mathbb{R}^{d \times d} \) and an integer \( r \) with \( 1 \leq r \leq d - 1 \) such that

\[ P^TA = \begin{bmatrix} A_1 & A_2 \\ 0 & A_3 \end{bmatrix}, \]

where \( A_1 \in \mathbb{R}^{r \times r} \), \( A_3 \in \mathbb{R}^{(d-r) \times (d-r)} \), \( A_2 \in \mathbb{R}^{r \times (d-r)} \), and \( 0 \in \mathbb{R}^{(d-r) \times r} \) is a null matrix. A matrix \( A \in \mathbb{R}^{d \times d} \) is called irreducible if it is not reducible, see, e.g., Horn and Johnson [9, Definitions 6.2.21 and 6.2.22]. We do emphasize that no 1-by-1 matrix is reducible.

**2.7 Definition.** Let \( (X_t)_{t \in \mathbb{R}_+} \) be a multi-type CBI process with parameters \( (d, c, \beta, B, \nu, \mu) \) such that the moment conditions (2.3) hold. Then \( (X_t)_{t \in \mathbb{R}_+} \) is called irreducible if \( B \) is irreducible.

**2.8 Definition.** Let \( (X_t)_{t \in \mathbb{R}_+} \) be a multi-type CBI process with parameters \( (d, c, \beta, B, \nu, \mu) \) such that \( \mathbb{E}(\|X_0\|) < \infty \) and the moment conditions (2.3) hold. Suppose that \( (X_t)_{t \in \mathbb{R}_+} \) is irreducible. Then \( (X_t)_{t \in \mathbb{R}_+} \) is called

\[
\begin{cases}
\text{subcritical} & \text{if } s(\tilde{B}) < 0, \\
\text{critical} & \text{if } s(\tilde{B}) = 0, \\
\text{supercritical} & \text{if } s(\tilde{B}) > 0.
\end{cases}
\]
For motivations of Definitions 2.7 and 2.8 see Barczy et al. [4, Section 3].

Next we will recall a convergence result for irreducible and critical multi-type CBI processes.

A function $f : \mathbb{R}_+ \to \mathbb{R}^d$ is called càdlàg if it is right continuous with left limits. Let $\mathcal{D}(\mathbb{R}_+, \mathbb{R}^d)$ and $\mathcal{C}(\mathbb{R}_+, \mathbb{R}^d)$ denote the space of all càdlàg and continuous functions on $\mathbb{R}_+$, respectively. Let $\mathcal{D}_\infty(\mathbb{R}_+, \mathbb{R}^d)$ denote the Borel $\sigma$-field in $\mathcal{D}(\mathbb{R}_+, \mathbb{R}^d)$ for the metric characterized by Jacod and Shiryaev [11, VI.1.15] (with this metric $\mathcal{D}(\mathbb{R}_+, \mathbb{R}^d)$ is a complete and separable metric space). For $\mathbb{R}^d$-valued stochastic processes $(\mathcal{Y}_t)_{t \in \mathbb{R}_+}$ and $(\mathcal{Y}_t^n)_{t \in \mathbb{R}_+}$, $n \in \mathbb{N}$, with càdlàg paths we write $\mathcal{Y}_t^n \xrightarrow{D} \mathcal{Y}$ as $n \to \infty$ if the distribution of $\mathcal{Y}_t^n$ on the space $(\mathcal{D}(\mathbb{R}_+, \mathbb{R}^d), \mathcal{D}_\infty(\mathbb{R}_+, \mathbb{R}^d))$ converges weakly to the distribution of $\mathcal{Y}$ on the space $(\mathcal{D}(\mathbb{R}_+, \mathbb{R}^d), \mathcal{D}_\infty(\mathbb{R}_+, \mathbb{R}^d))$ as $n \to \infty$.

The proof of the following convergence theorem can be found in Barczy and Pap [4, Theorem 4.1].

**2.9 Theorem.** Let $(X_t)_{t \in \mathbb{R}_+}$ be a multi-type CBI process with parameters $(d, c, \beta, B, \nu, \mu)$ such that $\mathbb{E}(\|X_0\|^4) < \infty$ and

$$
(2.8) \quad \int_{U_d} \|z\|^4 1_{\{\|z\| \geq 1\}} \nu(dz) < \infty, \quad \int_{U_d} \|z\|^4 1_{\{\|z\| > 1\}} \mu_i(dz) < \infty, \quad i \in \{1, \ldots, d\}.
$$

Suppose that $(X_t)_{t \in \mathbb{R}_+}$ is irreducible and critical. Then

$$(\mathcal{X}_t^n)_{t \in \mathbb{R}_+} := (n^{-1}X_{nt})_{t \in \mathbb{R}_+} \xrightarrow{D} (\mathcal{X}_t)_{t \in \mathbb{R}_+} := (\mathcal{X}_t u_{\text{right}})_{t \in \mathbb{R}_+} \quad \text{as} \quad n \to \infty
$$

in $\mathcal{D}(\mathbb{R}_+, \mathbb{R}^d)$, where $u_{\text{right}} \in \mathbb{R}^d_{++}$ is the right Perron vector of $e^B$ corresponding to the eigenvalue 1 (see Barczy and Pap [4, Lemma A.3]), $(\mathcal{X}_t)_{t \in \mathbb{R}_+}$ is the unique strong solution of the stochastic differential equation (SDE)

$$
d\mathcal{X}_t = (u_{\text{left}}, \beta) dt + \sqrt{\langle C u_{\text{left}}, u_{\text{left}} \rangle} \mathcal{X}_t^+ d\mathcal{W}_t, \quad t \in \mathbb{R}_+, \quad \mathcal{X}_0 = 0,
$$

where $u_{\text{left}} \in \mathbb{R}^d_{++}$ is the left Perron vector of $e^B$ corresponding to the eigenvalue 1 (see Barczy and Pap [4, Lemma A.3]), $(\mathcal{W}_t)_{t \in \mathbb{R}_+}$ is a standard Brownian motion and

$$
\overline{C} := \sum_{k=1}^d \langle e_k, u_{\text{right}} \rangle C_k \in \mathbb{R}^{d \times d}
$$

with

$$
(2.9) \quad C_k := 2c_k e_k e_k^T + \int_{U_d} zz^T \mu_k(dz) \in \mathbb{R}^{d \times d}, \quad k \in \{1, \ldots, d\}.
$$

**3 Non-convergence of infinitesimal generators**

We will need some differentiability properties of the functions $\psi$ and $\nu$ introduced in Theorem 2.4.
3.1 Lemma. Let \((X_t)_{t \in \mathbb{R}^+}\) be a multi-type CBI process with parameters \((d, c, \beta, B, \nu, \mu)\) such that the moment conditions \([2,3]\) hold. Then

\[
\partial_\lambda \psi(\lambda) = \langle \tilde{\beta}, e_i \rangle - \int_{U_d} (-z_i)(e^{-(\lambda, z)} - 1) \nu(dz), \quad \lambda \in \mathbb{R}^d_+,
\]

(3.2) \[
\lim_{\lambda \downarrow 0} \partial_\lambda \psi(\lambda) = \langle \tilde{\beta}, e_i \rangle
\]

for all \(i \in \{1, \ldots, d\}\), where the function \(\psi : \mathbb{R}^d_+ \rightarrow \mathbb{R}_+\) is defined in Theorem 2.4.

Proof. Under the moment conditions \([2,3]\) together with part (v) of Definition 2.2 we can write the function \(\psi\) in the form

\[
\psi(\lambda) = \langle \tilde{\beta}, \lambda \rangle - \int_{U_d} (e^{-(\lambda, z)} - 1 + \langle \lambda, z \rangle) \nu(dz), \quad \lambda \in \mathbb{R}^d_+.
\]

Indeed, by \([2,7]\),

\[
\langle \tilde{\beta}, \lambda \rangle - \int_{U_d} (e^{-(\lambda, z)} - 1 + \langle \lambda, z \rangle) \nu(dz) - \psi(\lambda)
\]

\[
= \langle \tilde{\beta} - \beta, \lambda \rangle - \int_{U_d} \langle \lambda, z \rangle \nu(dz) = \left( \int_{U_d} z \nu(dz), \lambda \right) - \int_{U_d} \langle \lambda, z \rangle \nu(dz) = 0.
\]

By the dominated convergence theorem one can derive

\[
\partial_\lambda \psi(\lambda) = \lim_{h \downarrow 0} h^{-1}(\psi(\lambda + he_i) - \psi(\lambda)) = \langle \tilde{\beta}, e_i \rangle - \lim_{h \downarrow 0} \int_{U_d} \left( e^{-(\lambda, z)} \frac{e^{-hz_i} - 1}{h} + z_i \right) \nu(dz)
\]

\[
= \langle \tilde{\beta}, e_i \rangle - \int_{U_d} (-z_i)(e^{-(\lambda, z)} - 1) \nu(dz)
\]

for all \(\lambda \in \mathbb{R}^d_+\) and \(i \in \{1, \ldots, d\}\), since \(|e^{-(\lambda, z)} - \frac{e^{-hz_i} - 1}{h}| \leq z_i \leq \|z\|\), \(\int_{U_d} \|z\| \nu(dz) < \infty\) and \(\lim_{h \downarrow 0} \frac{e^{-hz_i} - 1}{h} = -z_i\). Again by the dominated convergence theorem, we have

\[
\lim_{\lambda \downarrow 0} \int_{U_d} (-z_i)(e^{-(\lambda, z)} - 1) \nu(dz) = 0.
\]

In order to derive differentiability properties of the function \(v\), we need the following simple observation; for the 1-dimensional case, see, e.g., Feller [8 page 435].

3.2 Lemma. Let \(\xi = (\xi_1, \ldots, \xi_d)^T\) be a random vector such that \(P(\xi \in \mathbb{R}^d_+) = 1\). Consider its Laplace transform \(g : \mathbb{R}^d_+ \rightarrow \mathbb{R}_+\) defined by \(g(\lambda) := E(e^{-(\lambda, \xi)})\) for \(\lambda = (\lambda_1, \ldots, \lambda_d)^T \in \mathbb{R}^d_+\). Then \(g\) is infinitely differentiable on \(\mathbb{R}^d_+\), and for all \((k_1, \ldots, k_d)^T \in \mathbb{Z}^d_+\), we have

\[
\partial_{\lambda_1}^{k_1} \cdots \partial_{\lambda_d}^{k_d} g(\lambda) = (-1)^{k_1 + \cdots + k_d} E(\xi_1^{k_1} \cdots \xi_d^{k_d} e^{-(\lambda, \xi)}), \quad \lambda \in \mathbb{R}^d_+,
\]

(3.4) \[
E(\xi_1^{k_1} \cdots \xi_d^{k_d}) = (-1)^{k_1 + \cdots + k_d} \lim_{\nu \downarrow 0} \partial_{\lambda_1}^{k_1} \cdots \partial_{\lambda_d}^{k_d} g(\lambda) \in \mathbb{R}_+ \cup \{\infty\}.
\]

(3.5) \[
Consequently, \ E(\xi_1^{k_1} \cdots \xi_d^{k_d}) < \infty \ \text{if and only if} \ (-1)^{k_1 + \cdots + k_d} \lim_{\nu \downarrow 0} \partial_{\lambda_1}^{k_1} \cdots \partial_{\lambda_d}^{k_d} g(\lambda) < \infty.
\]
Proof. First we prove (3.4) by induction. If \( k_1 = \ldots = k_d = 0 \), then (3.4) holds trivially. Suppose that (3.4) holds for \( (k_1, \ldots, k_d)^\top \in \mathbb{Z}^d_+ \). Then for all \( \lambda = (\lambda_1, \ldots, \lambda_d)^\top \in \mathbb{R}^d_+ \), \( i \in \{1, \ldots, d\} \) and \( h \in \mathbb{R} \) with \( h \neq 0 \) and \( h \geq -\lambda_i/2 \) we have

\[
\frac{\partial^{k_1}_i \ldots \partial^{k_d}_i g(\lambda + he_i) - \partial^{k_1}_i \ldots \partial^{k_d}_i g(\lambda)}{h} = (-1)^{k_1 + \ldots + k_d} \mathbb{E} \left( \xi_1^{k_1} \ldots \xi_d^{k_d} \left( \frac{e^{-\langle \lambda, \xi \rangle} - e^{-\langle \lambda, \xi_i \rangle}}{h} \right) \right),
\]

where the mean value theorem and \( \min\{\lambda_i + h, \lambda_i\} \geq \lambda_i/2 \) yields

\[
\mathbb{E} \left( \xi_1^{k_1} \ldots \xi_i^{k_i} \left| \frac{e^{-\langle \lambda, \xi \rangle} - e^{-\langle \lambda, \xi_i \rangle}}{h} \right| \right) \leq \mathbb{E} \left( \xi_1^{k_1} \ldots \xi_i^{k_i - 1} \xi_i^{k_i+1} \xi_{i+1}^{k_{i+1}} \ldots \xi_d^{k_d} e^{-\langle \lambda, \xi \rangle + \lambda \xi_i/2} \right) < \infty,
\]

since the random variable \( \xi_1^{k_1} \ldots \xi_i^{k_i - 1} \xi_i^{k_i+1} \xi_{i+1}^{k_{i+1}} \ldots \xi_d^{k_d} e^{-\langle \lambda, \xi \rangle + \lambda \xi_i/2} \) is bounded. By the dominated convergence theorem, we obtain (3.4) for \( \lambda \in \mathbb{R}^d_+ \) and \( (k_1, \ldots, k_i - 1, k_i + 1, k_{i+1}, \ldots, k_d)^\top \).

The monotone convergence theorem yields (3.5). \( \square \)

3.3 Lemma. Let \( (X_t)_{t \in \mathbb{R}_+} \) be a multi-type CBI process with parameters \((d, c, \beta, B, \nu, \mu)\). Then

\[
\nu(t, \lambda) = 0 \quad \text{as} \quad \lambda \downarrow 0
\]

for all \( t \in \mathbb{R}_+ \), where the function \( \nu : \mathbb{R}_+ \times \mathbb{R}_d^+ \rightarrow \mathbb{R}_d^+ \) is defined in Theorem 2.4.

If the moment conditions \( 2.3 \) hold, then for all \( t \in \mathbb{R}_+ \), the function \( \mathbb{R}_d^+ \ni \lambda \mapsto \nu(t, \lambda) \) is infinitely differentiable, and

\[
\lim_{\lambda \downarrow 0} \partial_{\lambda_i} v_k(t, \lambda) = e_i^\top e^t B e_k
\]

for all \( t \in \mathbb{R}_+ \) and \( i, k \in \{1, \ldots, d\} \). Moreover, if

\[
\int_{U_d} \|z\|^2 1_{\{|z| \geq 1\}} \nu(dz) < \infty, \quad \int_{U_d} \|z\|^2 1_{\{|z| \geq 1\}} \mu_i(dz) < \infty, \quad i \in \{1, \ldots, d\},
\]

then

\[
\lim_{\lambda \downarrow 0} \partial_{\lambda_i} \partial_{\lambda_j} v_k(t, \lambda) = -e_i^\top e^t B^\top \int_0^t e^{-uB^\top} \sum_{\ell=1}^d e_{i\ell} e_j^\top e^{uB} C e^{uB^\top} e_j du
\]

for all \( t \in \mathbb{R}_+ \), \( i, j, k \in \{1, \ldots, d\} \) and \( \lambda \in \mathbb{R}_d^+ \).

Proof. Let \( (Z_t)_{t \in \mathbb{R}_+} \) be a multi-type CBI process with parameters \((d, c, 0, B, 0, \mu)\) (which is, in fact, a continuous state and continuous time branching process without immigration). Then, by (2.2), its Laplace transform takes the form

\[
g_{t,z}^i(\lambda) := \mathbb{E}(e^{-\langle \lambda, Z_t \rangle} | Z_0 = z) = e^{-\langle z, v(t, \lambda) \rangle}, \quad \lambda, z \in \mathbb{R}_d^+, \quad t \in \mathbb{R}_+.
\]

By Lemma 3.2, \( g_{t,z}^i \) is infinitely differentiable on \( \mathbb{R}_d^+ \) for each \( t \in \mathbb{R}_+ \) and \( z \in \mathbb{R}_d^+ \), and the limit \( \lim_{\lambda \downarrow 0} (-1)^{k_1 + \ldots + k_d} \partial_{\lambda_1}^{k_1} \ldots \partial_{\lambda_d}^{k_d} g_{t,z}(\lambda) \in \mathbb{R}_+ \cup \{\infty\} \) exists for all \( (k_1, \ldots, k_d)^\top \in \mathbb{Z}_+^d \),
4.4 in Barczy et al. [3] implies for all \(i, j, k\) the moment conditions (3.8) hold. Note that (3.8) for any bounded and Borel measurable function \(E\) implies Section 3, formula (3.1). For \(\lambda\) in a similar way, and the proof is complete. ✷

By monotone convergence theorem, \(g_{t,z}(\lambda) \uparrow g_{t,z}(0) = 1\) as \(\lambda \downarrow 0\) for all \(z \in \mathbb{R}^d_+\) and \(t \in \mathbb{R}_+\), hence \(v(t, \lambda) \downarrow v(t, 0) = 0\) as \(\lambda \downarrow 0\) for all \(t \in \mathbb{R}_+\). Clearly,

\[
\partial_{\lambda} v_k(t, \lambda) = \frac{-\partial_{\lambda} g_{t, e_k}(\lambda)}{g_{t, e_k}(\lambda)^2} \quad t \in \mathbb{R}_+, \quad \lambda \in \mathbb{R}^d_+, \quad i, k \in \{1, \ldots, d\}.
\]

With the notation \(Z_t = (Z_{t,1}, \ldots, Z_{t,d})^\top\), under the moment conditions (2.3), formula (2.4) implies \(E(Z_t | Z_0 = z) = e^{tB}z\), hence by Lemma 3.2,

\[
\lim_{\lambda \downarrow 0} \partial_{\lambda} v_k(t, \lambda) = \lim_{\lambda \downarrow 0} \partial_{\lambda} g_{t, e_k}(\lambda) = E(Z_{t,i} | Z_0 = e_k) = e^{tB}e_k.
\]

In a similar way,

\[
\partial_{\lambda} \partial_{\lambda} v_k(t, \lambda) = \frac{-g_{t, e_k}(\lambda)\partial_{\lambda} \partial_{\lambda} g_{t, e_k}(\lambda) + \partial_{\lambda} g_{t, e_k}(\lambda)\partial_{\lambda} g_{t, e_k}(\lambda)}{g_{t, e_k}(\lambda)^2}, \quad t \in \mathbb{R}_+, \quad \lambda \in \mathbb{R}^d_+
\]

for all \(i, j, k \in \{1, \ldots, d\}\). Under the moment conditions (3.8), Theorem 4.2 and Proposition 4.4 in Barczy et al. [3] implies \(\mathbb{E}(\|Z_t\|^2 | Z_0 = z) < \infty\) and

\[
\text{Var}(Z_t | Z_0 = z) = \sum_{\ell=1}^d \int_0^t (e^{(t-u)B}e_k)e^{uB}C_\ell e^{uB^\top}du,
\]

hence, by Lemma 3.2

\[
\lim_{\lambda \downarrow 0} \partial_{\lambda} \partial_{\lambda} v_k(t, \lambda) = -\lim_{\lambda \downarrow 0} \partial_{\lambda} \partial_{\lambda} g_{t, e_k}(\lambda) \partial_{\lambda} g_{t, e_k}(\lambda) = -\text{Cov}(Z_{t,i}, Z_{t,j} | Z_0 = e_k)
\]

\[
= -\sum_{\ell=1}^d \int_0^t (e^{(t-u)B}e_k)e^{uB}C_\ell e^{uB^\top}e_j du,
\]

and the proof is complete.

Let \((X_t)_{t \in \mathbb{R}_+}\) be a multi-type CBI process with parameters \((d, c, \beta, B, \nu, \mu)\) such that the moment conditions (3.8) hold. Note that \((n^{-1}X_k)_{k \in \mathbb{Z}_+}\) is a Markov chain with state space \(\mathbb{R}^d_+\) for all \(n \in \mathbb{N}\). The (discrete) infinitesimal generator of the process \((X_t^{(n)})_{t \in \mathbb{R}_+} = (n^{-1}X_{[nt]})_{t \in \mathbb{R}_+}\) is defined by

\[
(\mathcal{A}X^{(n)})(f)(x) := n[E(f(n^{-1}X_1) | n^{-1}X_0 = x) - f(x)], \quad x \in \mathbb{R}^d_+,
\]

for any bounded and Borel measurable function \(f : \mathbb{R}^d_+ \to \mathbb{R}\), see, e.g., Kato [12, Chapter IX, Section 3, formula (3.1)]. For \(\lambda \in \mathbb{R}^d_+\), let us introduce the function

\[
e_\lambda(x) := e^{-\langle \lambda, x \rangle}, \quad x \in \mathbb{R}^d_+.
\]
3.4 Proposition. Let \((X_t)_{t \in \mathbb{R}_+}\) be a multi-type CBI process with parameters \((d, c, \beta, B, \nu, \mu)\) such that the moment conditions (3.8) hold. Then
\[
\lim_{n \to \infty} \left[ (A_{X^{(n)}} e_\lambda)(x) + n(e^{-\langle \lambda, x \rangle} - e^{-\langle \lambda, e^B x \rangle}) \right] = e_\lambda(e^B x) \left[ \frac{1}{2} \sum_{\ell=1}^d \int_0^1 (e_\ell e^{(1-s)B} x) \lambda^\top e^{sB} C_x e^{sB} \lambda \, ds - \lambda^\top \int_0^1 e^{sB} \beta \, ds \right]
\]
for all \(x \in \mathbb{R}_+^d\) and \(\lambda \in \mathbb{R}_+^d\), where \(X^{(n)}_t = n^{-1} X_{[nt]}, \; t \in \mathbb{R}_+, \; n \in \mathbb{N}\). Consequently, given \(x \in \mathbb{R}_+^d\) and \(\lambda \in \mathbb{R}_+^d\), the sequence \((A_{X^{(n)}} e_\lambda)(x)\) converges as \(n \to \infty\) if and only if \(\langle \lambda, x \rangle = \langle \lambda, e^B x \rangle\).

Proof. By (2.2), for each \(\lambda \in \mathbb{R}_+^d\) and \(x \in \mathbb{R}_+^d\), we obtain
\[
(A_{X^{(n)}} e_\lambda)(x) = n \left[ \mathbb{E}(e_\lambda(n^{-1} X_1) \mid X_0 = n x) - e_\lambda(x) \right] = n \left[ \mathbb{E}(e^{-\langle \lambda, n^{-1} X_1 \rangle} \mid X_0 = n x) - e^{-\langle \lambda, x \rangle} \right]
\]
Applying (3.7) and L’Hôpital’s rule, we obtain
\[
\lim_{h \downarrow 0} h^{-1} \langle x, \nu(1, h \lambda) \rangle = \sum_{k=1}^d x_k \lim_{h \downarrow 0} h^{-1} \nu_k(1, h \lambda) = \sum_{k=1}^d x_k \lim_{h \downarrow 0} \partial_h \nu_k(1, h \lambda)
\]
\[
= \sum_{k=1}^d x_k \sum_{i=1}^d \lambda_i \lim_{h \downarrow 0} \partial_{\lambda_i} \nu_k(1, h \lambda) = \sum_{k=1}^d x_k \sum_{i=1}^d \lambda_i e_i^\top e^B e_k = \lambda^\top e^B x = \langle \lambda, e^B x \rangle.
\]
Applying (3.3), we have
\[
\int_0^1 \psi(\nu(s, h \lambda)) \, ds = \int_0^1 \left( \langle \beta, \nu(s, h \lambda) \rangle - \int_{U_d} (e^{-\langle \nu(s, h \lambda), z \rangle} - 1 + \langle \nu(s, h \lambda), z \rangle) \nu(dz) \right) \, ds \to 0
\]
as \(h \downarrow 0\), since, by continuity of \([0, 1] \ni s \mapsto \nu(s, h \lambda) \in \mathbb{R}_+^d\), \(h \in \mathbb{R}_+\), by (3.6) and by monotone convergence theorem, we have \(\int_0^1 \nu(s, h \lambda) \, ds \downarrow 0\) as \(h \downarrow 0\), and
\[
0 \leq \int_0^1 \left( \int_{U_d} (e^{-\langle \nu(s, h \lambda), z \rangle} - 1 + \langle \nu(s, h \lambda), z \rangle) \nu(dz) \right) \, ds \leq \frac{1}{2} \int_{U_d} \|z\|^2 \nu(dz) \int_0^1 \|\nu(s, h \lambda)\|^2 \, ds \downarrow 0
\]
as \(h \downarrow 0\). Consequently,
\[
(3.11) \quad \lim_{h \downarrow 0} \exp \left\{ -h^{-1} \langle x, \nu(1, h \lambda) \rangle - \int_0^1 \psi(\nu(s, h \lambda)) \, ds \right\} = \exp \left\{ -\langle \lambda, e^B x \rangle \right\} = e_\lambda(e^B x).
\]

Hence, applying again L’Hôpital’s rule, we obtain
\[
\lim_{n \to \infty} \left[ (A^{(n)} \nu_{\lambda})(x) + n(e^{-\lambda x} - e^{-\lambda e^{B} x}) \right] = \lim_{n \to \infty} n \left[ \exp \left\{ - (nx, v(1, n^{-1} \lambda)) - \int_{0}^{1} \psi(s, n^{-1} \lambda) \, ds \right\} - \exp \{ - (\lambda, e^{B} x) \} \right]
\]
(3.12)
\[
= \lim_{\Delta \to 0} \left[ \Delta \psi(s, h\lambda) \right] = \lim_{h \to 0} \partial_{h} \exp \left\{ - h^{-1} (x, v(1, h\lambda)) - \int_{0}^{1} \psi(s, h\lambda) \, ds \right\}.
\]

For each \( h \in \mathbb{R}^{++} \) and \( \lambda \in \mathbb{R}^{d} \), by dominated convergence theorem, we have
\[
\partial_{h} \int_{0}^{1} \psi(s, h\lambda) \, ds = \lim_{\Delta \to 0} \int_{0}^{1} \frac{\psi(s, (h + \Delta)\lambda) - \psi(s, h\lambda)}{\Delta} \, ds
\]
(3.13)
\[
= \int_{0}^{1} \partial_{h} \psi(s, h\lambda) \, ds.
\]

Indeed, for all \( s, h \in \mathbb{R}^{++} \) and \( \Delta \in (-h, h) \) with \( \Delta \neq 0 \), by mean value theorem,
\[
\left| \frac{\psi(s, (h + \Delta)\lambda) - \psi(s, h\lambda)}{\Delta} \right| \leq \| \lambda \| \sup_{\delta \in [-|\Delta|, h + |\Delta|]} |\partial_{\delta} \psi(s, \delta\lambda)|,
\]
where
\[
\partial_{\delta} \psi(s, \delta\lambda) = \sum_{k=1}^{d} \partial_{\lambda_{k}} \psi(s, \delta\lambda) \partial_{\delta} v_{k}(s, \delta\lambda) = \sum_{k=1}^{d} \partial_{\lambda_{k}} \psi(s, \delta\lambda) \sum_{i=1}^{d} \lambda_{i} \partial_{\lambda_{i}} v_{k}(s, \delta\lambda)
\]
for all \( \lambda \in \mathbb{R}^{d} \) and \( \delta \in \mathbb{R}^{++} \). By (3.1),
\[
|\partial_{\lambda_{k}} \psi(s, \delta\lambda)| \leq \| \vec{\beta} \| + \int_{U_{d}} \| z \| \nu(dz), \quad \lambda \in \mathbb{R}^{d}, \quad k \in \{1, \ldots, d\}.
\]

By (3.10) and Lemma 3.2,
\[
0 \leq \partial_{\lambda_{k}} v_{k}(s, \delta\lambda) = - \frac{\partial_{\lambda_{k}} g_{s,e_{k}}(\delta\lambda)}{g_{s,e_{k}}(\delta\lambda)} = \frac{\mathbb{E}(Z_{s,i} e^{-\delta(\lambda, Z_{s})} | Z_{0} = e_{k})}{\mathbb{E}(e^{-\delta(\lambda, Z_{s})} | Z_{0} = e_{k})} \leq \frac{\mathbb{E}(Z_{s,i} | Z_{0} = e_{k})}{\mathbb{E}(e^{-|\Delta| \lambda^{2}(\lambda, Z_{s})} | Z_{0} = e_{k})} \leq \frac{\mathbb{E}(Z_{s,i} | Z_{0} = e_{k})}{\mathbb{E}(e^{-\Delta(\lambda, Z_{s})^{2}} | Z_{0} = e_{k})} = \frac{e_{i}^{\top} e^{B} e_{k}}{g_{s,e_{k}}(2h\lambda)}
\]
for all \( \delta \in (h - |\Delta|, h + |\Delta|) \subset \mathbb{R}^{++} \), \( \lambda \in \mathbb{R}^{++} \), and \( i, k \in \{1, \ldots, d\} \), where \( (Z_{t})_{t \in \mathbb{R}^{+}} \) is a multi-type CBI process with parameters \((d, c, 0, B, 0, \mu)\). Consequently,
\[
\left| \frac{\psi(s, (h + \Delta)\lambda) - \psi(s, h\lambda)}{\Delta} \right| \leq \| \lambda \| \left( \| \vec{\beta} \| + \int_{U_{d}} \| z \| \nu(dz) \right) \sum_{k=1}^{d} \sum_{i=1}^{d} \lambda_{i} \partial_{\lambda_{i}} v_{k}(s, \delta\lambda)
\]
where the functions \( \mathbb{R}^{+} \ni s \mapsto e_{i}^{\top} e^{B} e_{k} \in \mathbb{R}^{+} \) and \( \mathbb{R}^{+} \ni s \mapsto g_{s,e_{k}}(2h\lambda) = e^{-v_{k}(s,2h\lambda)} \in \mathbb{R}^{++} \) are continuous, hence we conclude (3.13).
Applying (3.12), (3.13) and (3.11), we have
\[
\lim_{n \to \infty} \left[ (A_{\epsilon^{(n)}} e_{\lambda})(x) + n(e^{-(\lambda, x)} - e^{-(\lambda, e_{\tilde{B}} x)}) \right]
\]
\[
= e_{\lambda}(e_{\tilde{B}} x) \lim_{h \downarrow 0} \left[ h^{-2} \sum_{k=1}^{d} x_k \left( v_k(1, h \lambda) - h \sum_{i=1}^{d} \lambda_i \partial_{\lambda_i} v_k(1, h \lambda) \right) \right.
\]
\[
- \sum_{k=1}^{d} \sum_{i=1}^{d} \lambda_i \int_{0}^{1} \partial_{\lambda_k} \psi(v(s, h \lambda)) \partial_{\lambda_i} v_k(s, h \lambda) ds \right], \quad \lambda \in \mathbb{R}^d_{++}.
\]

By L'Hôpital's rule and by (3.9),
\[
\lim_{h \downarrow 0} h^{-2} \sum_{k=1}^{d} x_k \left( v_k(1, h \lambda) - h \sum_{i=1}^{d} \lambda_i \partial_{\lambda_i} v_k(1, h \lambda) \right)
\]
\[
= \sum_{k=1}^{d} x_k \lim_{h \downarrow 0} \frac{\sum_{i=1}^{d} \lambda_i \partial_{\lambda_i} v_k(1, h \lambda) - \sum_{i=1}^{d} \lambda_i \partial_{\lambda_i} v_k(1, h \lambda) - h \sum_{i=1}^{d} \sum_{j=1}^{d} \lambda_i \lambda_j \partial_{\lambda_i} \partial_{\lambda_j} v_k(1, h \lambda)}{2h}
\]
\[
= - \sum_{k=1}^{d} x_k \lim_{h \downarrow 0} \frac{1}{2} \sum_{i=1}^{d} \sum_{j=1}^{d} \lambda_i \lambda_j \partial_{\lambda_i} \partial_{\lambda_j} v_k(1, h \lambda)
\]
\[
= \frac{1}{2} \sum_{k=1}^{d} \sum_{i=1}^{d} \sum_{j=1}^{d} \lambda_i \lambda_j e_k e_i e_j e_j \int_{0}^{1} e^{u \tilde{B}} \sum_{\ell=1}^{d} e_{\epsilon} e_{\epsilon}^T C e^{u \tilde{B}} e_{\epsilon} du
\]
\[
= \frac{1}{2} \sum_{k=1}^{d} \int_{0}^{1} x^T e^{u \tilde{B}} e^{u \tilde{B}} e_{\epsilon}^T C e^{u \tilde{B}} e_{\epsilon} \lambda du, \quad \lambda \in \mathbb{R}^d_{++}.
\]

For each $i, k \in \{1, \ldots, d\}$ and $\lambda \in \mathbb{R}^d_{++}$, by dominated convergence theorem, we have
\[
(3.15) \quad \lim_{h \downarrow 0} \int_{0}^{1} \partial_{\lambda_k} \psi(v(s, h \lambda)) \partial_{\lambda_i} v_k(s, h \lambda) ds = \int_{0}^{1} \lim_{h \downarrow 0} \partial_{\lambda_k} \psi(v(s, h \lambda)) \partial_{\lambda_i} v_k(s, h \lambda) ds.
\]

Indeed, again by (3.10) and Lemma 3.2,
\[
0 \leq \partial_{\lambda_i} v_k(s, h \lambda) = - \partial_{\lambda_i} g_{s, e_k}(h \lambda) = \frac{\mathbb{E}(Z_{s,i} e_{-h} | Z_0 = e_k)}{\mathbb{E}(e_{-(h, Z_s)} | Z_0 = e_k)} \leq \frac{\mathbb{E}(Z_{s,i} | Z_0 = e_k)}{\mathbb{E}(e_{-(h, Z_s)} | Z_0 = e_k)} = \frac{e_{i} e_{\tilde{B}} e_{k}}{g_{s, e_k}(\lambda)}
\]

for all $h \in (0, 1)$, $\lambda \in \mathbb{R}^d_{++}$, $s \in \mathbb{R}^d$ and $i, k \in \{1, \ldots, d\}$, hence, applying (3.14),
\[
|\partial_{\lambda_k} \psi(v(s, h \lambda)) \partial_{\lambda_i} v_k(s, h \lambda)| \leq \left( \|\tilde{\beta}\| + \int_{U_d} \|z\| \nu(dz) \right) \frac{e_{i} e_{\tilde{B}} e_{k}}{g_{s, e_k}(\lambda)},
\]

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hence we conclude (3.15). Applying (3.2), (3.6) and (3.7), we have

\[
\sum_{k=1}^{d} \sum_{i=1}^{d} \lambda_i \int_0^1 \lim_{\lambda \downarrow 0} \partial_{\lambda_k} \psi(s,h\lambda) \partial_{\lambda_i} v_k(s,h\lambda) \, ds
\]

\[
= \sum_{k=1}^{d} \sum_{i=1}^{d} \lambda_i \int_0^1 \tilde{\beta}_k(e^\top \tilde{B} e_k) \, ds = \lambda^\top \int_0^1 e^{\tilde{B} u} \tilde{\beta} \, du,
\]

hence we obtain the statement. \(\square\)

3.5 Corollary. Let \((X_t)_{t \in \mathbb{R}_+}\) be an irreducible and critical multi-type CBI process with parameters \((d,c,\beta,B,\nu,\mu)\) such that the moment conditions \((2.8)\) hold and \(\tilde{B}\) given in \((2.5)\) is not \(0\) (implying \(d \geq 2\)). Then \(\{(X_t^{(n)})_{t \in \mathbb{R}_+}\} \xrightarrow{D} (X_t, u_{\text{right}})_{t \in \mathbb{R}_+}\) as \(n \to \infty\), and, given \(x \in \mathbb{R}^d_+\) and \(\lambda \in \mathbb{R}^d_+\), the sequence \((A_{X^{(n)}}e_{\lambda})(x)\) converges as \(n \to \infty\) if and only if \(\langle \lambda, x \rangle = \langle \lambda, e^\top \tilde{B} x \rangle\). In particular,

(i) there exist \(x \in \mathbb{R}^d_+\) and \(\lambda \in \mathbb{R}^d_+\) such that the sequence \((A_{X^{(n)}}e_{\lambda})(x)\) does not converge as \(n \to \infty\),

(ii) the sequence \((A_{X^{(n)}}e_{\lambda})(x)\) converges as \(n \to \infty\) for all \(\lambda \in \mathbb{R}^d_+\) if and only if \(x = \delta u_{\text{right}}\) with some \(\delta \in \mathbb{R}\).

Proof. First, we note that there exists a multi-type CBI process which satisfies the conditions of the corollary. Namely, every 2-type CBI process with parameters \((2,c,\beta,B,\nu,\mu)\) satisfying the moment conditions \((2.8)\) with

\[
\tilde{B} = \begin{bmatrix} -1 & 1 \\ 1 & -1 \end{bmatrix}
\]

serves us as an example. The convergence \(\{(X_t^{(n)})_{t \in \mathbb{R}_+}\} \xrightarrow{D} (X_t, u_{\text{right}})_{t \in \mathbb{R}_+}\) as \(n \to \infty\) follows by Theorem 2.9. Proposition 3.4 yields that the sequence \((A_{X^{(n)}}e_{\lambda})(x)\) converges as \(n \to \infty\) if and only if \(\langle \lambda, x \rangle = \langle \lambda, e^\top \tilde{B} x \rangle\). Next we prove that there exist \(x \in \mathbb{R}^d_+\) and \(\lambda \in \mathbb{R}^d_+\) such that the sequence \((A_{X^{(n)}}e_{\lambda})(x)\) does not converge as \(n \to \infty\). By Proposition 3.4 if \(\langle \lambda, x \rangle \neq \langle \lambda, e^\top \tilde{B} x \rangle\) with some \(x \in \mathbb{R}^d_+\) and \(\lambda \in \mathbb{R}^d_+\), then the sequence \((A_{X^{(n)}}e_{\lambda})(x)\) does not converge as \(n \to \infty\). Using Dunford and Schwartz [6, Theorem VII.1.8], one can easily check that the following statements are equivalent:

- \(\langle \lambda, e^\top \tilde{B} x \rangle = \langle \lambda, x \rangle\) for all \(x \in \mathbb{R}^d_+\) and \(\lambda \in \mathbb{R}^d_+\);
- \(e^\top \tilde{B} x = x\) for all \(x \in \mathbb{R}^d_+\);
- \(\sigma(e^\top \tilde{B}) = \{1\}\);
- \(\sigma(\tilde{B}) = \{0\}\);
- \(\tilde{B} = 0\).
Since \( \tilde{B} \neq 0 \), there exist some \( x \in \mathbb{R}^d_+ \) and \( \lambda \in \mathbb{R}^d_+ \) such that \( \langle \lambda, e^\tilde{B} x \rangle \neq \langle \lambda, x \rangle \), implying (i). Given \( x \in \mathbb{R}^d_+ \), we have \( \langle \lambda, e^\tilde{B} x \rangle = \langle \lambda, x \rangle \) for all \( \lambda \in \mathbb{R}^d_+ \) if and only if \( e^\tilde{B} x = x \), which holds if and only if \( x = \delta u_{\text{right}} \) with some \( \delta \in \mathbb{R} \), yielding (ii). \( \square \)

### 3.6 Remark.
Rosenkrantz [13], [14] provided an example for a sequence of one-dimensional diffusion processes given by SDEs which converges weakly to a Markov limit process, however the drift coefficients of the corresponding SDEs do not converge, and consequently, the corresponding sequence of infinitesimal generators does not converge either. He also provided an example for one-dimensional diffusion processes given by SDEs which converge weakly to a Markov limit process, and the drift and diffusion coefficients of the corresponding SDEs converge, but their limits are not the ones that are expected to appear in the infinitesimal generator of the limit Markov process. On the one hand, Corollary 3.5 can be considered as a non-trivial multi-dimensional example, which resembles the phenomena described by Rosenkrantz. On the other hand, part (ii) of Corollary 3.5 is in accordance with Theorem 2.9 since there the degenerate limit process is concentrated on the ray determined by \( u_{\text{right}} \). It is an open question whether Theorem 2.9 might be proved by the help of infinitesimal generators. \( \square \)

It is also interesting to investigate the sequence \( \mathcal{Y}_n^{(n)} := n^{-1} X_{nt}, \ t \in \mathbb{R}_+, \ n \in \mathbb{N}, \) of scaled CBI processes.

### 3.7 Proposition.
Let \( (X_t)_{t \in \mathbb{R}_+} \) be a multi-type CBI process with parameters \( (d, c, \beta, B, \nu, \mu) \) such that the moment conditions (3.8) hold. Then

\[
\lim_{n \to \infty} (\langle A_{\mathcal{Y}^{(n)}} f(x) \rangle - n\langle \tilde{B} x, f'(x) \rangle) = \frac{1}{2} \sum_{i=1}^{d} x_i \sum_{k=1}^{d} \sum_{\ell=1}^{d} e_k^\top C_i e_\ell f''(x) + \langle \beta, f'(x) \rangle
\]

for all \( f \in C^2_c(\mathbb{R}_+, \mathbb{R}) \) and \( x \in \mathbb{R}^d_+ \) (where \( A_{\mathcal{Y}^{(n)}} \) denotes the usual infinitesimal generator of \( \mathcal{Y}^{(n)} \)). Consequently, given \( f \in C^2_c(\mathbb{R}_+, \mathbb{R}) \) and \( x \in \mathbb{R}^d_+ \), the sequence \( (A_{\mathcal{Y}^{(n)}} f)(x) \) converges as \( n \to \infty \) if and only if \( \langle \tilde{B} x, f'(x) \rangle = 0 \).

**Proof.** First note that, under the moment conditions (3.8), the infinitesimal generator (2.1) of the process \( (X_t)_{t \in \mathbb{R}_+} \) can also be written in the form

\[
(A f)(x) = \frac{1}{2} \sum_{i=1}^{d} x_i \sum_{k=1}^{d} \sum_{\ell=1}^{d} f''_{k,\ell}(x) \langle C_i e_\ell, e_k \rangle + \langle \beta + \tilde{B} x, f'(x) \rangle + \int_{U_d} (f(x + z) - f(x)) \nu(dz)
\]

\[
+ \sum_{i=1}^{d} x_i \int_{U_d} \left( f(x + z) - f(x) - \langle z, f'(x) \rangle - \frac{1}{2} \langle z, f''(x) z \rangle \right) \mu_i(dz)
\]
for \( f \in C^2_c(\mathbb{R}^d, \mathbb{R}) \) and \( x \in \mathbb{R}^d \). Indeed, by Remark 4.3 in Barczy et al. \[4\],
\[
\int_{U_d} \|z\|^2 \mu_i(\mathrm{d}z) < \infty, \ i \in \{1, \ldots, d\}, \ 
\]
and using (2.9),
\[
(Af)(x) = \frac{1}{2} \sum_{i=1}^d x_i \sum_{k=1}^d \sum_{\ell=1}^d f''_{k,\ell}(x) \langle C_i e_\ell, e_k \rangle - \langle \beta + \tilde{B}x, f'(x) \rangle - \int_{U_d} (f(x + z) - f(x)) \nu(\mathrm{d}z)
\]
\[
- \sum_{i=1}^d x_i \int_{U_d} \left( f(x + z) - f(x) - \langle z, f'(x) \rangle - \frac{1}{2} \langle z, f''(x)z \rangle \right) \mu_i(\mathrm{d}z) = D_1 + D_2,
\]
where
\[
D_1 := \frac{1}{2} \sum_{i=1}^d c_i x_i f''_{i,1}(x) + \frac{1}{2} \sum_{i=1}^d x_i \sum_{k=1}^d \sum_{\ell=1}^d f''_{k,\ell}(x) \int_{U_d} z_k z_\ell \mu_i(\mathrm{d}z)
\]
\[
- \frac{1}{2} \sum_{i=1}^d x_i \sum_{k=1}^d \sum_{\ell=1}^d f''_{k,\ell}(x)e_k^\top C_i e_\ell = 0
\]
and
\[
D_2 := \sum_{i=1}^d x_i \int_{U_d} \left( \langle z, f'(x) \rangle - f'_i(x)(1 \wedge z_i) \right) \mu_i(\mathrm{d}z) - \langle (\tilde{B} - B)x, f'(x) \rangle
\]
\[
= \sum_{i=1}^d x_i \int_{U_d} \left( f'_i(x)(z_i - (1 \wedge z_i)) + \sum_{j \in \{1, \ldots, d\} \setminus \{i\}} z_j f'_j(x) \right) \mu_i(\mathrm{d}z)
\]
\[
- \sum_{i=1}^d \sum_{j=1}^d x_j f'_i(x) \int_{U_d} (z_i - \delta_{i,j})^+ \mu_j(\mathrm{d}z) = 0.
\]
For each \( n \in \mathbb{N} \), the infinitesimal generator of the process \((Y_t^{(n)})_{t \in \mathbb{R}^+_d}\) is
\[
(A_{Y}^{(n)}f)(x) = n(Af_n)(nx), \quad x \in \mathbb{R}^d
\]
where \( f_n(x) := (n^{-1}x), \ x \in \mathbb{R}^d \), for all \( f \in C^2_c(\mathbb{R}^d, \mathbb{R}) \), see, e.g., Barczy et al. \[1\] Lemma 2.1. Consequently, by (2.6),
\[
(A_{Y}^{(n)}f)(x) = \frac{1}{2} \sum_{i=1}^d x_i \sum_{k=1}^d \sum_{\ell=1}^d f''_{k,\ell}(x) e_k^\top C_i e_\ell + \langle \beta + n\tilde{B}x, f'(x) \rangle
\]
\[
+ n \int_{U_d} \left( f(x + n^{-1}z) - f(x) - \langle n^{-1}z, f'(x) \rangle \right) \nu(\mathrm{d}z)
\]
\[
+ n^2 \sum_{i=1}^d x_i \int_{U_d} \left( f(x + n^{-1}z) - f(x) - \langle n^{-1}z, f'(x) \rangle - \frac{1}{2} \langle n^{-1}z, f''(x)n^{-1}z \rangle \right) \mu_i(\mathrm{d}z).
\]
One can show

\[
\lim_{n \to \infty} \sup_{x \in R^d_+} \left| n \int_{U_d} (f(x + n^{-1} z) - f(x) - \langle n^{-1} z, f'(x) \rangle) \nu(dz) \right| = 0,
\]

\[
\lim_{n \to \infty} \sup_{x \in R^d_+} \left| n^2 x_i \int_{U_d} \left( f(x + n^{-1} z) - f(x) - \langle n^{-1} z, f'(x) \rangle - \frac{1}{2} \langle n^{-1} z, f''(x) n^{-1} z \rangle \right) \mu_i(dz) \right| = 0
\]

for all \( i \in \{1, \ldots, d\} \), see the method of the proof of formulas (2.6) and (2.7) in Barczy et al. \[1\]. Consequently, for each \( f \in C^2_c(R^d_+, \mathbb{R}) \), we obtain (3.16).

3.8 Remark. If we consider a single-type (hence irreducible) and critical (hence \( \tilde{B} = 0 \)) CBI process with parameters \((1, c, \beta, B, \nu, \mu)\) such that the moment conditions (3.8) hold, then, by Proposition 3.7,

\[
\lim_{n \to \infty} (A_{Y(n)} f)(x) = \frac{1}{2} \tilde{C}_1 f''(x) + \tilde{\beta} f'(x), \quad f \in C^2_c(R^d_+, \mathbb{R}), \quad x \in R^d_+.
\]

Here the limit is nothing else but the infinitesimal generator of a squared Bessel process, which is in accordance with the result of Huang et al. \[10\] Theorem 2.3. In fact, Huang et al. \[10\] proved that for a critical single-type CBI process \((X_t)_{t \in R^d_+}\) satisfying the moment conditions (3.8), the sequence of scaled processes \((n^{-1} X_{nt})_{t \in R^d_+}\), \(n \in \mathbb{N}\), converges weakly to a squared Bessel process. Finally, we note that, to the best knowledge of the authors, it is not known, whether the sequence of scaled processes \((n^{-1} X_{nt})_{t \in R^d_+}\), \(n \in \mathbb{N}\), is convergent for an irreducible and critical \(d\)-type CBI process with \(d \geq 2\).

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