Non-orientable surface-plus-one-relation groups

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March 21, 2009

To the memory of Karl Gruenberg

Abstract

Recently Dicks-Linnell determined the $L^2$-Betti numbers of the orientable surface-plus-one-relation groups, and their arguments involved some results that were obtained topologically by Hempel and Howie. Using algebraic arguments, we now extend all these results of Hempel and Howie to a larger class of two-relator groups, and we then apply the extended results to determine the $L^2$-Betti numbers of the non-orientable surface-plus-one-relation groups.

2000 Mathematics Subject Classification. Primary: 20F05; Secondary: 16S34, 20J05.

Key words. $L^2$-Betti number, surface-plus-one-relation group

1 Notation

In this section, we collect together the conventions and basic notation we shall use.

Let $G$ be a (discrete) multiplicative group, fixed throughout the article.

For two subsets $A$, $B$ of a set $X$, the complement of $A \cap B$ in $A$ will be denoted by $A - B$ (and not by $A \setminus B$ since we let $G \setminus Y$ denote the set of $G$-orbits of a left $G$-set $Y$).

By an ordering, $<$, of a set, we shall mean a binary relation which totally orders the set.

A sequence is a set endowed with a specified listing of its elements, usually represented as a vector in which the coordinates are the elements of (the underlying set of) the sequence. For two sequences $A$, $B$, their concatenation will be denoted $A \vee B$.

We use $\mathbb{R} \cup \{-\infty, \infty\}$ with the usual conventions, such as $\frac{1}{\infty} = 0$.

We will find it useful to have a notation for intervals in $\mathbb{Z}$ that is different from the notation for intervals in $\mathbb{R}$.

Let $i$, $j \in \mathbb{Z}$.

We write

$$[i \uparrow j] := \begin{cases} (i, i + 1, \ldots, j - 1, j) \in \mathbb{Z}^{j-i+1} & \text{if } i \leq j, \\ () \in \mathbb{Z}^0 & \text{if } i > j. \end{cases}$$

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Also, \( [i \uparrow \infty] := (i, i + 1, i + 2, \ldots) \) and \( [i \uparrow \infty] := [i \uparrow \infty] \setminus \{\infty\} \). We define \( [j \downarrow i] \) to be the reverse of the sequence \([i \downarrow j] \), that is, \((j, j - 1, \ldots, i + 1, i) \).

We shall use sequence notation to define families of indexed symbols. Let \( v \) be a symbol. For each \( k \in \mathbb{Z} \), we let \( v_k \) denote the ordered pair \((v, k) \). We let

\[
v_{[i \downarrow j]} := \begin{cases} (v_i, v_{i+1}, \ldots, v_{j-1}, v_j) & \text{if } i \leq j, \\ () & \text{if } i > j. \end{cases}
\]

Also, \( v_{[i \uparrow \infty]} := (v_i, v_{i+1}, v_{i+2}, \ldots) \). We define \( v_{[j \downarrow i]} \) to be the reverse of the sequence \( v_{[i \downarrow j]} \).

Now suppose that \( v_{[i \downarrow j]} \) is a sequence in the group \( G \), that is, there is specified a map of sets \( v_{[i \downarrow j]} \to G \). We treat the elements of \( v_{[i \downarrow j]} \) as elements of \( G \), possibly with repetitions, and we define

\[
\Pi v_{[i \downarrow j]} := \begin{cases} v_i v_{i+1} \cdots v_{j-1} v_j \in G & \text{if } i \leq j, \\ 1 \in G & \text{if } i > j. \end{cases}
\]

\[
\Pi v_{[j \downarrow i]} := \begin{cases} v_j v_{j-1} \cdots v_{i+1} v_i \in G & \text{if } j \geq i, \\ 1 \in G & \text{if } j < i. \end{cases}
\]

For elements \( a, b \) of \( G \), we write \( \overline{a} := a^{-1} \), \( ab := ab \overline{a} \), and \([a, b] := ab \overline{a} b\).

For any subsets \( R \) and \( X \) of \( G \), we let \( \langle R \rangle \) denote the subgroup of \( G \) generated by \( R \), we write \( ^X R := \{ r^x \mid r \in R, x \in X \} \), and \( G/\langle R \rangle := G/\langle \langle R \rangle \rangle \). If \( R = \{ r \} \), we write simply \( \langle r \rangle, ^X r \) and \( G/\langle r \rangle \), respectively.

For each set \( X \), the vague cardinal of \( X \), denoted \( |X| \), is the element of \([0 \uparrow \infty] \) defined as follows. If \( X \) is a finite set, then \( |X| \) is defined to be the cardinal of \( X \), an element of \([0 \uparrow \infty] \). If \( X \) is an infinite set, then \( |X| \) is defined to be \( \infty \).

The rank of \( G \), denoted \( \text{rank}(G) \), is the smallest element of the subset of \([0 \uparrow \infty] \) which consists of vague cardinals of generating sets of \( G \).

Mappings of left modules will usually be written on the right of their arguments.

## 2 Background and summary of results

In outline, the article has the following structure. More detailed definitions can be found in the appropriate sections.

Let \( S \) be a surface group, that is, the fundamental group of a closed surface. Here, there exists some \( k \in [0 \uparrow \infty] \) and a presentation \( S = \langle x_{[1 \downarrow k]} \mid w \rangle \) where either \( k \) is even and \( w = \prod_{i \in [1 \downarrow k]} x_{2i-1} x_{2i} \) (the orientable case), or \( k \geq 1 \) and \( w = \prod x_{[1 \downarrow k]}^2 \) (the non-orientable case). Let \( r \in \langle x_{[1 \downarrow k]} \mid \rangle \) and let \( G := \langle x_{[1 \downarrow k]} \mid w, r \rangle \); we say that \( G \) is a surface-plus-one-relation group. A simple example is \( \langle a, b, c \mid a^2 b c^2, a b c \rangle \simeq \langle a, b \mid [a, b] \rangle \). Under the name
'one-relator surface groups', the orientable surface-plus-one-relation groups were introduced and studied by Hempel [15], and further investigated by Howie [19], with the aim of carrying some of the known theory of one-relator groups over to these two-relator groups. In the same area, Bogopolski [4] extended to orientable-surface groups the result of Magnus for free groups that two elements have the same normal closure if and only if the two elements are equivalent modulo conjugation or conjugation plus inversion.

The purpose of this article is to study $G$ algebraically, and generalize the work of Hempel and Howie to include the non-orientable case. If $k \leq 2$, then $G$ is virtually abelian of rank at most two, and we consider such groups to be well understood. Thus we assume that $k \geq 3$, and here the closed surface is said to be hyperbolic. Recall the following.

2.1 Lemma. A free-generating sequence of $\langle a, b, c \mid \rangle$ is given by $(x, y, z) := (baba, ab, cab)$, and here $abcab(x, y) z^2 = (abcab)(baba)(ab)(cab)(cab)(baba) = a^2b^2c^2$. □

Thus, on setting $d = k - 2 \in [1] \cup [\infty]$, we can change the generating sequence from $x_{[1]}$ to $(x, y) \vee z_{[1]}$ and arrange for $w$ to take the form $[x, y] u$, with $u \in \langle z_{[1]} \rangle$.

This leads us to consider the class of two-relator groups described as follows. Let $d \in [0] \cup [\infty]$, let $F := \langle (x, y) \vee z_{[1]} | \rangle$, let $u \in \langle z_{[1]} | \rangle \leq F$, let $r \in F$, let $S := \langle (x, y) \vee z_{[1]} | [x, y] u \rangle$, and let $G := \langle (x, y) \vee z_{[1]} | [x, y] u, r \rangle$. Notice that our surface group has been changed to a form which includes many new groups, while we lose three closed surfaces, namely the sphere, the projective plane, and the Klein bottle.

In Section 3, we introduce a rewriting procedure for $S$.

In Section 4, we recall the concepts of potency and ‘being residually a finite $p$-group for each prime $p$’, and we show that $S$ enjoys these properties, and we discuss connections with early work of Karl Gruenberg. The potency of $S$ is used later in Section 6 to show that $G$ is virtually torsion free.

In Section 5, we shall see that by changing $(x, y)$ to a different free-generating sequence of $\langle x, y \mid \rangle$ without changing $[x, y]$, and then by carefully changing $r$ without changing the conjugacy class of the image of $r$ in $S$, we may assume that we have a presentation in which either $r = x^m$ for some $m \in [0] \cup [\infty]$, or $d \geq 1$ and $r$ is what we shall call a ‘Hempel relator for the presentation $\langle (x, y) \vee z_{[1]} | [x, y] u \rangle$’. Notice that we use a double bar to distinguish a presentation from the group being presented.

In the case where $r = x^m$, we shall see that $G$ is virtually one-relator, and we consider such groups to be well understood.

The main part of the article then examines the case where $r$ is a Hempel relator; here, we can generalize the results that Hempel and Howie obtained for hyperbolic orientable surface-plus-one-relation groups.

In Section 6, for $r$ a Hempel relator, we perform what Howie calls ‘Hempel’s trick’ and express $G$ as an HNN-extension of a one-relator group over an isomorphism between two free subgroups. We deduce that if $r$ generates a maximal cyclic subgroup of $F$, then $G$ is locally
indicable. The proof uses a deep result of Howie which he proved by topological methods; we present a shorter proof, based on Bass-Serre Theory, in Appendix A. Let \( m \) denote the index of \( \langle r \rangle \) in a maximal cyclic subgroup of \( F \); we show that
\[
G \text{ is } (((\text{free}) \rtimes \text{(cyclic of order } m\text{)}) \text{ by } (\text{locally indicable})).
\]

In Section 7, for \( r \) a Hempel relator, we verify the exactness of a sequence suitable for building an \( \mathbb{E} G \), a classifying space for proper \( G \)-actions.

In Section 8, we recall the concept of VFL and show that \( G \) enjoys this property, and we calculate Euler characteristics.

In Section 9, we apply all the foregoing Hempel-Howie-type results to calculate the \( L^2 \)-Betti numbers of the surface-plus-one-relation groups; for the orientable case this was done in [12, Theorem 5.1], in essentially the same way, using the original Hempel-Howie results.

In Appendix A, as mentioned, we use Bass-Serre Theory to simplify proofs of some important results of Howie on local indicability.

3 Rewriting in \( S = \langle (x, y) \lor z_{[1]} \mid [x, y]u \rangle \)

We shall use the following at various points in the article.

3.1 Notation. Let \( d \in [0, \infty[ \), let \( x, y \) and \( z \) be symbols, let \( F := \langle (x, y) \lor z_{[1]} \mid \rangle \), and let \( u \) and \( r \) be elements of \( F \). Suppose that \( u \in \langle z_{[1]} \rangle \). Let \( S := \langle (x, y) \lor z_{[1]} \mid [x, y]u \rangle \), and let \( G := \langle (x, y) \lor z_{[1]} \mid [x, y]u, r \rangle \). We shall denote the natural map \( F \to S \) by \( w \mapsto w \mod [x, y]u \).

The two-relator group \( G \) is a one-relator quotient of \( S \), with relator \( r \mod [x, y]u \).

The remaining notations do not concern \( r \) or \( G \).

Let \( N(F) := \langle \mathbb{Z} \times ((x) \lor z_{[1]}) \mid \rangle \). For each \( i \in \mathbb{Z} \), we shall denote the natural map
\[
\langle (x) \lor z_{[1]} \mid \rangle \simeq \langle \{ i \} \times ((x) \lor z_{[1]}) \mid \rangle \leq \langle \mathbb{Z} \times ((x) \lor z_{[1]}) \mid \rangle
\]
by \( w \mapsto iw \). Let \( y \) denote the automorphism of \( N(F) \) determined by the shifting bijection
\[
(i^{i}x) \lor i^{z_{[1]}} \to (i^{i+1}x) \lor i^{z_{[1]}}
\]
for all \( i \in \mathbb{Z} \). Hence \( C_{\infty} := \langle y \mid \rangle \) acts on \( N(F) \). If we form the semidirect product \( N(F) \rtimes C_{\infty} \), we get the group \( \langle (x, y) \lor z_{[1]} \mid \rangle \). Thus we may identify \( N(F) \rtimes C_{\infty} \) with \( F \), and \( N(F) \) then becomes the normal subgroup of \( F \) generated by \( (x) \lor z_{[1]} \), that is, the set of elements whose \( y \)-exponent sum, with respect to \( (x, y) \lor z_{[1]} \), is zero. Notice that, for each \( (i, w) \in \mathbb{Z} \times ((x) \lor z_{[1]}) \), we have identified \( iw = v_w \). Bearing in mind that \( \mathbb{Z} \) is an abbreviation for \( y_{\mathbb{Z}} = C_{\infty} \), we shall write \( y_{\mathbb{Z}}(x) \) for \( \mathbb{Z} \times (x) \), and \( \mathbb{Z} z_{[1]} \) for \( \mathbb{Z} \times z_{[1]} \).
Let \( N(S) := \langle (\bar{z}_x) \vee z_{[1, d]} \mid (i^{t+1} \cdot i \cdot u \cdot i \cdot x \mid i \in \mathbb{Z}) \rangle \). The shifting action of \( C_\infty = \langle y \mid \rangle \) on \( N(F) \) induces a \( C_\infty \)-action on \( N(S) \), and we find that we can identify \( N(S) \times C_\infty \) with \( S \). Thus \( N(S) \) is the image of \( N(F) \) in \( S \).

For each \( j \in \mathbb{Z} \), let \( y \) act on \( \langle (j_x) \vee z_{[1, d]} \mid \rangle \) by \( j_x \mapsto j_x \cdot i u \cdot j_x \), and, for each \( i z_s \in z_{[1, d]} \), \( i z_s \mapsto i+1 z_s \). We find that we can make the identification

\[ S = \langle (j_x) \vee z_{[1, d]} \mid \rangle \rtimes C_\infty. \]

Thus \( \langle (j_x) \vee z_{[1, d]} \mid \rangle \) can be viewed as a free factor of \( N(F) \leq F \) which maps bijectively to \( N(S) \) in \( S \). By varying \( j \), we get a family of embeddings of \( N(S) \) in \( N(F) \leq F \). \( \square \)

## 4 Potency of \( S = \langle (x, y) \vee z_{[1, d]} \mid [x, y]u \rangle \)

In this section, we prove a useful fact about \( S \) and recall some related history.

### 4.1 Definition. A group \( S \) is said to be potent if, for each \( s \in S - \{1\} \) and each \( m \in [2^{1\infty}[ \), there exists a homomorphism from \( S \) to some finite group which sends \( s \) to some element of order exactly \( m \); in this event, we also say that \( S \) is a potent group.

The following fact, whose proof invokes two results of Allenby \[1\], will be applied in Section 6 to show that certain two-relator groups are virtually torsion free.

### 4.2 Lemma. For \( u \in \langle z_{[1, d]} \mid \rangle \), the group \( S = \langle (x, y) \vee z_{[1, d]} \mid [x, y]u \rangle \) is potent.

**Proof.** In the case where \( u \neq 1 \), \( S = \langle x, y \mid \rangle \rtimes \langle z_{[1, d]} \mid \rangle \), and then \( S \) is potent because the free product of two free groups amalgamating a non-trivial cyclic group is potent \[1, \text{Section 4}\]. (We can also use the latter result as a perverse reference for the fact that free groups are potent.) Now we may assume that \( u = 1 \), and, hence, \( S \) is the free product of a rank-two, free-abelian group and a free group. Observe that free-abelian groups are potent. Recall that the free product of two potent groups is potent \[1, \text{Theorem 2.4}\]. (We can also use the latter result as a less perverse reference for the fact that free groups are potent.) The result now follows. \( \square \)

We dedicate the remainder of this section to setting the foregoing results of Allenby into a historical context, with particular emphasis on connections with the early research of Karl Gruenberg. This digression will allow us to explain how some of the techniques involved can be used to prove the potency of \( S \) directly.

### 4.3 Definition. Let us say that a group \( S \) is \( \text{r3p}\forall p \) if, for each prime \( p \), \( S \) is residually a finite \( p \)-group, that is, \( S \) embeds in a direct product of finite \( p \)-groups; in this event, we also say that \( S \) is an \( \text{r3p}\forall p \)-group.
Surface-plus-one-relation groups

The history begins with free groups.
In 1935, Magnus obtained the following important result.

(H1) Every free group is residually torsion-free nilpotent.

Let us recall the method that Magnus used.

Let $R$ be an associative ring, let $X$ be a set and let $R\langle\langle X\rangle\rangle$ denote the ring of formal power series in the set $X$ of non-commuting variables. Each $f \in R\langle\langle X\rangle\rangle$ has a unique expression as a formal sum $\sum f_m[x]$ where, for each $m \in [0,\infty[$, $f_m$ is homogeneous of degree $m$. For each $n \in [0,\infty[$, let $I_n$ denote $I_n(R, X) := \{ f \in R\langle\langle X\rangle\rangle | f_m = 0 \text{ for all } m \in [0, (n-1)] \}$, an ideal in $R\langle\langle X\rangle\rangle$. The natural map $R\langle\langle X\rangle\rangle \to \lim_{\leftarrow} R\langle\langle X\rangle\rangle/I_n$ is an isomorphism and endows $R\langle\langle X\rangle\rangle$ with the inverse-limit topology, in which $I_n$ is a basis for the neighborhoods of 0. For each $n \in [1,\infty[$, let $U_n$ denote $U_n(R, X) := 1 + I_n$, a subgroup of the group of units of $R\langle\langle X\rangle\rangle$. If $m \in [1,\infty[$, then the natural map from $U_m$ to the group of units of $R/\langle X\rangle/I_{n+1}$ has kernel $U_{n+1}$ and image which can be denoted $1 + (I_m/I_{n+1})$. It follows that $U_{n+1}$ is a normal subgroup of $U_1$ and that $U_n/U_{n+1}$ is a free $R$-module which is central in $U_1/U_{n+1}$.

Consider now the case where $R = \mathbb{Z}$. Here, $U_n/U_{n+1}$ is a free-abelian group, and, hence, $U_1(\mathbb{Z}, X)$ is residually a finite $p$-group, since the $U_n/U_{n+1}$ are finite $p$-groups. In the case where $X$ is infinite, we can retract $R\langle\langle X\rangle\rangle$ onto $R\langle\langle Y\rangle\rangle$ for any subset $Y$ of $X$, and it follows that $U_1(\mathbb{Z}, X)$ is again residually a finite $p$-group. Now, as Luis Paris pointed out to us, a leading-coefficient argument again shows that $1 + X$ freely generates a free subgroup of $U_1(\mathbb{Z}, X)$; see [24, I]. This completes the outline of Magnus’ proof of (H1).

Gruenberg [14, p. 29] remarks that A. Mal’cev in 1949, M. Hall in 1950, and Takehasi in 1951, independently found the following result.

(H2) Every free group is residually torsion-free nilpotent.

Let us recall how (H2) is related to power series. Let $p$ be a prime number and let $R = \mathbb{Z}_p$, the integers modulo $p$. In the case where $X$ is finite, $U_1(\mathbb{Z}_p, X)$ is residually a finite $p$-group, since the $U_1/U_{n+1}$ are finite $p$-groups. In the case where $X$ is infinite, we can retract $R\langle\langle X\rangle\rangle$ onto $R\langle\langle Y\rangle\rangle$ for any subset $Y$ of $X$, and it follows that $U_1(\mathbb{Z}_p, X)$ is again residually a finite $p$-group. Now, as Luis Paris pointed out to us, a leading-coefficient argument again shows that $1 + X$ freely generates a free subgroup of $U_1(\mathbb{Z}_p, X)$, and this proves (H2).

Since every non-trivial finite $p$-group has a central, hence normal, subgroup of order $p$, it is not difficult to see the following result.

(H3) Every free group is residually torsion-free nilpotent.

The result (H3) was presented implicitly by Fischer, Karrass, and Solitar in 1972; see [13, Proof of Theorem 2]. It was presented explicitly by Kim and McCarron in 1993; see [20, Lemma 2.2]. The facts (H2) and (H3) together prove the following result.

(H4) Every free group is potent.

The result (H4) was presented explicitly by Stebe in 1971 with a proof, attributed to Passman, based on (H1); see [26, Lemma 1]. Independently, it was presented implicitly by
Fischer, Karrass, and Solitar in 1972, with a proof based on (H2) and (H3); see [13, Proof of Theorem 2].

4.4 Remark. Let \( d \in [1, \infty[ \), let \( u \in \langle z_{11d} \mid \rangle \), and let \( S := \langle (x, y) \vee z_{11d} \mid [x, y]u \rangle \).

We sketch an argument that shows how power series can be used to prove that \( S \) is \( \mathcal{RF}_p \).

Let \( p \) be a prime number.

As in Notation 3.1, \( S = N(S) \rtimes C_\infty \) where \( C_\infty = \langle y \mid \rangle \) and \( N(S) = \langle (x) \vee z_{11d} \mid \rangle \).

Here \( y \) acts on \( N(S) \) by \( x \mapsto 0 \cdot u \cdot x \), \( i z_* \mapsto i + 1 \cdot z_* \). Let \( b \) be a symbol and choose an identification of the countably infinite set \( (x) \vee Zz_{11d} \) with the subset \( 1 + b_{01\infty} \) of \( \mathbb{Z}_p \langle b_{01\infty} \rangle \). Hence, by the \( p \)-analogue of Magnus’ result, we get an embedding of \( N(S) \) in the group of units of \( \mathbb{Z}_p \langle b_{01\infty} \rangle \). Invoking the topology, we see that the action of \( y \) on \( N(S) \) extends uniquely to a continuous automorphism of the topological ring \( \mathbb{Z}_p \langle b_{01\infty} \rangle \). Hence, we can form the skew-group-ring, or skew Laurent-polynomial ring, \( \mathbb{Z}_p \langle b_{01\infty} \rangle \langle C_\infty \rangle \). It is then clear that \( S \) embeds in the group of units of \( \mathbb{Z}_p \langle b_{01\infty} \rangle \langle C_\infty \rangle \).

Consider any \( n \in [1, \infty[ \).

Let \( q := p^n \) and let \( C_{q^2} := \langle y \mid y^{q^2} \rangle \).

We have the closed, \( y \)-invariant ideal \( I_{n+1} = I_{n+1}(\mathbb{Z}_p, b_{01\infty}) \) of \( \mathbb{Z}_p \langle b_{01\infty} \rangle \), and we can form the skew-group-ring \( \mathbb{Z}_p \langle b_{01\infty} \rangle / I_{n+1} \langle C_\infty \rangle \). It is not difficult to check that the image of \( (N(S))^q \) is trivial, where \( (N(S))^q \) denotes the subgroup of \( N(S) \) generated by the \( q \)th powers of elements of \( N(S) \).

Let \( J_{n+1} \) denote the (closed, \( y \)-invariant) ideal of \( \mathbb{Z}_p \langle b_{01\infty} \rangle \) generated by

\[
I_{n+1} \cup \{ i z_* - q \cdot z_* \mid i z_* \in z_{11d} \}.
\]

Again, we can form the skew-group-ring \( \mathbb{Z}_p \langle b_{01\infty} \rangle / J_{n+1} \langle C_\infty \rangle \). The \( y^{q^2} \)-action fixes the image \( z \cdot z_{11d} \) of \( z_{11d} \) and carries the image of \( x \) to the image of \( (\prod (q-1)0_{1u}) \cdot x \), and, hence, the \( y^{q^2} \)-action carries the image of \( x \) to the image of \( (\prod (q-1)0_{1u})^q \cdot x \in (N(S))^q \cdot x \), which equals the image of \( x \). Thus we can form the skew-group-ring \( \mathbb{Z}_p \langle b_{01\infty} \rangle / J_{n+1} \langle C_{q^2} \rangle \) and find that the image of \( S \) is a finite \( p \)-group. By varying \( n \), we see that \( S \) is residually a finite \( p \)-group. By varying \( p \), we see that \( S \) is \( \mathcal{RF}_p \). By (H3), \( S \) is potent.

In 1957, Gruenberg proved the following two results; see [14, Theorem 2.1(i) and Part (iii) of the Corollary on p.44].

(H5) Every residually torsion-free nilpotent group is \( \mathcal{RF}_p \).

(H6) The free product of any family of \( \mathcal{RF}_p \)-groups is \( \mathcal{RF}_p \).

Each of these sheds important light on (H2).

In 1981, Allenby obtained the following result; see [1, Theorem 2.4].

(H7) The free product of any family of potent groups is potent. This sheds light on (H4).
We next consider a class of groups which contains \( S \) when \( u \neq 1 \).

Let \( A \) and \( B \) be free groups, let \( a \in A - \{1\} \), and let \( b \in B - \{1\} \). The amalgamated free product \( A \ast_{a=b} B \) is called a \textit{cyclically-pinched one-relator group}.

In 1968, G. Baumslag used power series to obtain the following result; see [3, Theorem 1].

(H8) If \( a \) is not a proper power, and \( B \) is cyclic, then \( A \ast_{a=b} B \) is residually torsion-free nilpotent.

In particular, by (H5), \( A \ast_{a=b} B \) is then \textit{RFP}\(\forall p\). In 1998, Kim and Tang used this to obtain the following result; see [21, Corollary 3.6].

(H9) \( A \ast_{a=b} B \) is \textit{RFP}\(\forall p\) if and only if \( a \) or \( b \) is not a proper power.

This gives the earliest proof we know of that \( S \) is \textit{RFP}\(\forall p\), since the case where \( u \neq 1 \) follows from (H9) with \( A = \langle x, y \mid \rangle, B = \langle z^{1+d} \mid \rangle, a = [y, x], b = u, \) since \([y, x]\) is not a proper power, while the case where \( u = 1 \) follows from (H6) and the fact that free-abelian groups are \textit{RFP}\(\forall p\).

In 1981, Allenby obtained the following result; see [1, Section 4].

(H10) \( A \ast_{a=b} B \) is potent.

We have seen that (H10) and (H7) together imply that \( S \) is potent.

Finally, let us mention fundamental groups of closed surfaces.

In 1968, Chandler [8] obtained the following result.

(H11) For every closed surface except the projective plane and the Klein bottle, the fundamental group is residually torsion-free nilpotent and, hence, by (H5), is \textit{RFP}\(\forall p\).

The following is a consequence of (H10).

(H12) For every closed surface except the projective plane, the fundamental group is potent.

5 Hempel relators

In this section, we introduce Hempel relators and show how to find them.

5.1 Definitions. Let \( F \) be a group, let \( r \) be an element of \( F \), and let \( R \) be a subset of \( F \).

We say that \( r \) \textit{has a unique root} in \( F \) if and only if either \( r = 1 \) or \( r \) lies in a unique maximal infinite cyclic subgroup of \( F \).

If \( r \neq 1 \) and \( r \) lies in a unique maximal infinite cyclic subgroup \( C \) of \( F \), we define \textit{the unique root} of \( r \) in \( F \), denoted \( \sqrt[\ast]{r} \), to be the unique generator of \( C \) of which \( r \) is a positive power, and we define \( \log r := [C; \langle r \rangle] \). Here, if \( s = \sqrt[\ast]{r} \) and \( m = \log r \), then \( s^m = r \).

We say that \( 1 \) is \textit{the unique root} of \( 1 \) in \( F \), and we define \( \sqrt[\ast]{1} := 1 \) and \( \log 1 := \infty \).

We say that \( F \) \textit{has unique roots} if each element of \( F \) has a unique root in \( F \). For example, a free group has unique roots.
We say that $R$ has unique roots in $F$ if each element of $R$ has a unique root in $F$; in this event, we let $\sqrt[n]{R}$ denote the set of unique roots of elements of $R$ in $F$.

5.2 Lemma. With Notation 3.1, there is a free-generating sequence $(x', y')$ of $\langle x, y \mid \rangle$ such that $[x', y'] = [x, y]$, and the $y'$-exponent sum of $r$ with respect to $(x', y') \lor z_{[1; d]}$ is zero, and the $x'$-exponent sum of $r$ with respect to $(x', y') \lor z_{[1; d]}$ is non-negative.

Moreover, $S$ has unique roots and any element of the free subgroup $N(S)$ has the same unique root with respect to both $S$ and $N(S)$.

Proof. Let $a$ and $b$ respectively denote the $x$- and $y$-exponent sums of $r$ with respect to $(x, y) \lor z_{[1; d]}$. We want $a \geq 0$ and $b = 0$.

Replacing $x$ with $xy \pm 1$ fixes $[x, y]$ and changes $(a, b)$ to $(a, b \pm a)$. Replacing $y$ with $yx \pm 1$ fixes $[x, y]$ and changes $(a, b)$ to $(a \pm b, b)$. If $ab \neq 0$, then one or more of these four operations reduces $|a| + |b|$. By repeating such operations, we can arrange that $ab = 0$. If $a = 0$, then we can alter $(0, b)$ to $(b, b)$ and then to $(b, 0)$; thus we may assume that $b = 0$. Finally, if $a < 0$, we can successively alter $(a, 0)$ to $(a, a)$, $(0, a)$, $(-a, a)$, and $(-a, 0)$. Thus we may also assume that $a \geq 0$.

We next show that $r \mod [x, y]u$ has a unique root in $S$. We may assume that $r \mod [x, y]u$ lies in $S\{1\}$. Let $(x', y')$ be a free-generating sequence of $\langle x, y \mid \rangle$ such that $[x', y'] = [x, y]$, and the $y'$-exponent sum of $r$ with respect to $(x', y') \lor z_{[1; d]}$ is zero. To simplify notation, we forget the original $(x, y)$, use $(x', y')$ as the new generating sequence, and name it $(x, y)$. Thus we may assume that $r \mod [x, y]u$ lies in $N(S)$, a free subgroup of $S$. Hence $r \mod [x, y]u$ has a unique root in $N(S)$. Any cyclic subgroup of $S$ containing $r \mod [x, y]u$ maps to a finite, hence trivial, subgroup in $S/N(S) = C_\infty = \langle y \mid \rangle$. Hence, any cyclic subgroup of $S$ containing $r \mod [x, y]u$ lies in $N(S)$. Thus the unique root of $r \mod [x, y]u$ in the free group $N(S)$ is the unique root in $S$.

Since $r \mod [x, y]u$ is an arbitrary element of $S$, this completes the proof.

5.3 Definition. With Notation 3.1, let $X_1 := \langle 1 \rangle \lor [0] \lor [z_{[1; d]}]$. We say that $r$ is a Hempel relator for the presentation $\langle x, y \mid \rangle \lor [x, y]u$ and that $\langle (x, y) \lor z_{[1; d]} \rangle \lor [x, y]u, r \rangle$ is a Hempel presentation if the following hold.

(R1) $r \in \langle X_1 \mid \rangle \leq N(F) \leq F$,

(R2) In $\langle X_1 \mid \rangle$, $r$ is not conjugate to any element of $\langle 0u^{-1} \mid 1 \rangle$.

(R3) With respect to $X_1$, $r$ is cyclically reduced.

(R4) With respect to $X_1$, $r$ involves some element of $0z_{[1; d]}$, that is, $r$ does not lie in the free factor $\langle (1x) \lor [1] \lor [z_{[1; d]}] \rangle$.

We now want to show that for our purposes we can assume that $r$ is a Hempel relator.
5.4 Lemma. With Notation 3.1, there exist an element \( w \) of \( F \), an element \( v \) of \( \langle F([x, y]u) \rangle \), and an automorphism \( \alpha \) of \( \langle x, y \mid \ . \ . \ . \rangle \) which fixes \([x, y] \), such that the element \( r' \) formed from \( r \) by first applying \( \alpha \) and then left multipying by \( v \) and then left conjugating by \( w \) is either a non-negative power of \( x \) or a Hempel relator for \( \langle (x, y) \lor z_{[1:d]} \mid [x, y]u \rangle \).

Here, \( G \simeq \langle (x, y) \lor z_{[1:d]} \mid [x, y]u, r' \rangle \) and \( \log_F r' = \log_S(r \text{ mod}[x, y]u) \).

Proof. We shall successively alter \( r \), and to avoid extra notation we shall use the same symbol \( r \) to denote the altered element of \( F \) at each stage.

We first alter \( r \) by automorphisms of \( \langle x, y \mid \ . \ . \ . \rangle \) which fix \([x, y] \). By Lemma 5.2, we may assume that the \( y \)-exponent sum of \( r \) with respect to \( (x, y) \lor z_{[1:d]} \) is zero, and the \( x \)-exponent sum of \( r \) with respect to \( (x, y) \lor z_{[1:d]} \) is non-negative. Hence \( r \text{ mod}[x, y]u \) lies in the free subgroup \( N(S) \).

For each \( j \in \mathbb{Z} \), we can view \( (jx) \lor z_{[1:d]} \) as a free-generating sequence of \( N(S) \) and express \( r \text{ mod}[x, y]u \) as a word therein and lift the word back to a new element of \( N(F) \); this multiplies \( r \) by an element of \( \langle F([x, y]u) \rangle \). We then reduce the expression cyclically, which corresponds to conjugating \( r \) in \( F \).

If \( r = jx^m \) for some \( m \in \mathbb{Z} \), then \( m \geq 0 \). By replacing \( r \) with \( y^{-j}r \), we may assume that \( r = x^m \), and the desired conclusions hold.

Thus, we may assume that, for each \( j \in \mathbb{Z} \), the cyclically reduced expression for \( r \) in \( (jx) \lor z_{[1:d]} \) involves some element of \( z_{[1:d]} \), and, hence, there is a unique smallest non-empty interval \( [\mu_j, \nu_j] \) in \( \mathbb{Z} \) such that the cyclically reduced expression for \( r \) in \( (jx)\lor z_{[1:d]} \) is a word in \( (jx) \lor [\mu_j, \nu_j]z_{[1:d]} \).

We claim that for some \( k \in \mathbb{Z} \), \( \mu_{k+1} = k \).

To pass from \( (j+1)x \lor z_{[1:d]} \) to \( (jx) \lor z_{[1:d]} \), we replace \( j+1x \) with \( ju \cdot jx \) and reduce cyclically. In passing from \([\mu_{j+1}, \nu_{j+1}] \) to \([\mu_j, \nu_j] \), we may add \( j \), we may delete some values, and we then take the convex hull in \( \mathbb{Z} \). By repeating this change sufficiently often, we find that, for \( j \ll 0, j \leq \mu_j \). Among all \( j \in \mathbb{Z} \) such that \( j \leq \mu_j \), let us choose one that minimizes \( \nu_j - \mu_j \in [0, \infty[ \).

To pass from \( (jx) \lor z_{[1:d]} \) to \( (j+1)x \lor z_{[1:d]} \), we replace \( jx \) with \( j \cdot \mu_j \cdot j+1x \) and reduce cyclically. In passing from \([\mu_j, \nu_j] \subseteq [j, \infty[ \) to \([\mu_{j+1}, \nu_{j+1}] \), we may add \( j \), we may delete some values, and we then take the convex hull in \( \mathbb{Z} \). Hence, \([\mu_{j+1}, \nu_{j+1}] \subseteq [j, \nu_j] \); that is, \( \mu_{j+1} \geq j \) and \( \nu_{j+1} \leq \nu_j \).

If \( \mu_{j+1} = j \), we take \( k = j \).

To prove the claim, it remains to consider the case where \( \mu_{j+1} \geq j+1 \). By the minimality assumption, \( \mu_{j+1} \leq \mu_j \). Hence, \( j+1 \leq \mu_{j+1} \leq \mu_j \). Since \( j+1 \leq \mu_j \), when we replace \( jx \) with \( j \cdot \mu_j \cdot j+1x \) the occurrences of \( j \cdot \mu_j \) survive unaffected by (cyclic) reduction. Since \( j+1 \leq \mu_{j+1} \), we see that \( j \cdot \mu_j = 1 \). Hence \( u = 1 \) and, hence, \( \mu_j \) does not depend on \( j \). We take \( k = \mu_0 \), and then \( \mu_{k+1} = \mu_0 = k \).

In all cases, then, we have some \( k \in \mathbb{Z} \) such that \( \mu_{k+1} = k \). By replacing \( r \) with \( y^{-k}r \), we may arrange that \( k = 0 \). Now \( r \) has become a Hempel relator. Notice that \( r \) is not conjugate.
to a power of $0^u \cdot 1x$ in $\langle (1x) \vee [0]^\infty \zeta_1 \rangle$ because $r \mod [x, y]u$ is not conjugate to a power of $0^x$ in $\langle (1x) \vee \zeta_1 \rangle = N(S)$.

In Examples 8.3, we shall see that, in the case where $r \in \langle x \rangle$, $G$ is virtually one-relator; we consider such groups to be well understood.

6 HNN decomposition, local indicability and torsion

In this section, we shall extend three types of results of Hempel and Howie, namely the HNN decomposition, the local indicability, and the analysis of torsion.

6.1 Notation. We shall use a left-right twisting of the notation in [10, Examples I.3.5(v)], and write $G_v * t_e$ to denote an HNN extension, where it is understood that $G_v$ is a group, $G_e$ is a subgroup of $G_v$, there is specified some injective homomorphism $t_e : G_e \rightarrow G_v$, $g \mapsto \tilde{t}_e g$, and the associated HNN extension is

$$G_v * t_e := (G_v * \langle t_e \rangle) / \langle \{ \tilde{t}_e g t_e \tilde{t}_e g \mid g \in G_e \} \rangle.$$

If $G = G_v * t_e$, then the Bass-Serre $G$-tree has vertex set $G/G_v$ and edge set $G/G_e$, with $gG_e$ joining $gG_v$ to $gt_eG_v$, for each $g \in G$.

In the case where $r$ is a Hempel relator we shall now see that we get an HNN extension of a one-relator group over a free group.

6.2 Notation. With Notation 3.1, suppose that $r$ is a Hempel relator for the presentation $\langle (x, y) \vee z_1 \rangle || [x, y]u$; thus, for $X_1 := (1x) \vee [0]^\nu \zeta_1$, the following hold.

(R1) $r \in \langle X_1 \rangle \leq N(F) \leq F$.

(R2) In $\langle X_1 \rangle$, $r$ is not conjugate to any element of $\langle x \rangle$, where $x := 0^\nu \cdot 1x$. In particular, $r \neq 1$ and, hence, $d \neq 0$.

(R3) With respect to $X_1$, $r$ is cyclically reduced.

(R4) With respect to $X_1$, $r$ involves some element of $0^\nu \zeta_1$.

(R5) With respect to $X_1$, $r$ involves some element of $\nu \zeta_1$, where $\nu = \nu(r)$ denotes the least element of $[0]^\infty \zeta_1$ such that $r \in \langle (1x) \vee [0]^\nu \zeta_1 \rangle$.

Since $1x = 0u \cdot x$, we can identify $\langle (x) \vee [0]^\nu \zeta_1 \rangle = \langle (1x) \vee [0]^\nu \zeta_1 \rangle$, and thus view $r$ as an element of a free group with two specified free-generating sequences. With
respect to \((x) \lor [0^\nu]z_{[11]d}\), \(r\) involves some element of \([\nu]z_{[11]d}\), even if \(\nu = 0\), by (R5) and (R2). With respect to \((1^x) \lor [0^\nu]z_{[11]d}\), \(r\) involves some element of \([\nu]z_{[11]d}\), by (R4). We define

\[
G_{[0^\nu]} := \langle (x) \lor [0^\nu]z_{[11]d} \mid r \rangle = \langle (1^x) \lor [0^\nu]z_{[11]d} \mid r \rangle,
\]

\[
G_{[0^\nu]} := \langle (x) \lor [0^\nu]z_{[11]d} \mid r \rangle = \langle (1^x) \lor [0^\nu]z_{[11]d} \mid r \rangle.
\]

By Magnus’ Freiheitssatz, which appears as Corollary A.3.2 in Appendix A, the natural maps from \(G_{[0^\nu]}\) and \(G_{[1]}\) to \(G_{[0^\nu]}\) are injective.

We have an isomorphism \(y: G_{[0^\nu]} \rightarrow G_{[1]}\) given by the natural bijection on the specified free-generating sequences. We can then form the HNN extension \(G_{[0^\nu]} \ast G_{[1]}\). On simplifying the presentation we recover the presentation of \(G\) and thus obtain the HNN decomposition \(G = G_{[0^\nu]} \ast G_{[1]}\).

In the case where \(G\) is a hyperbolic orientable surface-plus-one-relation group, this HNN decomposition was obtained topologically by Howie [19, Proposition 2.1.2(c)] who attributed it to an argument implicit in [15, Proof of Theorem 2.2] which in turn is attributed to Howie.

### 6.3 Definition

A group is said to be **indicable** if either it is trivial or it has some infinite, cyclic quotient. A group is said to be **locally indicable** if each finitely generated subgroup is indicable.

### 6.4 Theorem

Let \(d \in [0, \infty[^\cup\), let \(F := \langle (x, y) \lor z_{[11]d} \mid \rangle\), and let \(u\) and \(r\) be elements of \(F\). Suppose that \(u \in \langle z_{[11]d} \rangle\), and that \(r\) is a Hempel relator for \(\langle (x, y) \lor z_{[11]d} \mid [x, y, u] \rangle\). Let \(G := \langle (x, y) \lor z_{[11]d} \mid [x, y, u, r] \rangle\). If \(\xi \lor r = r\), then \(G\) is locally indicable.

**Proof.** We use Notation 6.2.

In \(G\), define, for each \(i \in \mathbb{Z}\),

\[
G_{[i(i+\nu-1)]} := \langle (i^x) \lor [i(i+\nu-1)]z_{[11]d} \mid \rangle,
\]

\[
G_{[i(i+\nu)]} := \langle (i^x) \lor [i(i+\nu)]z_{[11]d} \mid i^r \rangle.
\]

We can then write

\[
(i) \qquad G_{[i(i+\nu)]} = (G_{[i(i+\nu-1)]} \ast \langle \langle (i+\nu)z_{[11]d} \rangle \rangle')\langle i^r \rangle.
\]

Since \(\xi \lor r = r\), \(i^r\) is not a proper power in \(G_{[i(i+\nu-1)]} \ast \langle \langle (i+\nu)z_{[11]d} \rangle \rangle\).

By (R2), (R3) and (R5), \(i^r \in G_{[0^\nu]} \ast \langle \langle \nu \rangle \rangle\) is not conjugate to any element of \(G_{[0^\nu]}\). Hence \(i^r \in G_{[i(i+\nu-1)]} \ast \langle \langle (i+\nu)z_{[11]d} \rangle \rangle\) is not conjugate to any element of \(G_{[i(i+\nu-1)]}\).
By using \( i + \nu + 1 \), we can form the free product with amalgamation
\[
G_{\uparrow (i+\nu+1)} := G_{\uparrow (i+\nu)} * \langle i \cdot x = i+1 \cdot x \rangle \vee \langle [(i+1)\uparrow (i+\nu)] z_{[1,d]} \rangle \quad \text{for each} \quad G_{\uparrow (i+\nu+1)} = G_{\uparrow (i+\nu)} * G_{\uparrow (i+1)\uparrow (i+\nu+1)}.
\]

By varying \( i \), we get a bi-infinite chain of free products with amalgamation
\[
N(G) := \cdots G_{\uparrow (i-1)\uparrow (i+\nu-1)} * G_{\uparrow (i+\nu)} * G_{\uparrow (i+1)\uparrow (i+\nu)} * G_{\uparrow (i+1)\uparrow (i+\nu+1)} * G_{\uparrow (i+\nu+1)} \cdots.
\]

Here \( C_\infty = \langle y \mid \rangle \) acts by shifting and we find that \( N(G) \rtimes C_\infty = G \). For any finite, non-empty interval \([j,1]\) in \( \mathbb{Z} \) let us define
\[
G_{\uparrow (i+\nu)} := G_{\uparrow (j+\nu)} * \cdots * G_{\uparrow (i+1)\uparrow (i+\nu)} * G_{\uparrow (i+\nu-1)}\uparrow (i+\nu),
\]
a subgroup of \( G \). By using (1), we see that
\[
G_{\uparrow (i+\nu)} = (G_{\uparrow (i+\nu-1)} * \langle i+\nu z_{[1,d]} \rangle) / \langle i \rangle.
\]

Now \( \langle i \rangle \in G_{\uparrow (i+\nu-1)} * \langle i+\nu z_{[1,d]} \rangle \subseteq G_{\uparrow (i+\nu-1)} * \langle i+\nu z_{[1,d]} \rangle \), and \( \langle i \rangle \) has the same cyclically reduced expression in both free products. In particular, \( \langle i \rangle \) is not a proper power and is not conjugate to any element of \( G_{\uparrow (i+\nu-1)} \). By a result of Howie given as Corollary A.3.6 in Appendix A, the subgroup \( G_{\uparrow [1,\infty]} := \bigcup_{i \in [1,\infty]} G_{\uparrow (i+\nu)} \) is then locally indicable. It follows that \( \bigcup_{j \in [0,1](-\infty]} G_{\uparrow [j,\infty]} \) is locally indicable, that is, \( N(G) \) is locally indicable. Hence, \( N(G) \rtimes C_\infty \) is locally indicable, that is, \( G \) is locally indicable. This completes the proof.

In the case where \( G \) is a hyperbolic orientable surface-plus-one-relation group, the above result was given by Hempel [15, Theorem 2.2] with its proof attributed to Howie.

We now discuss torsion.

6.5 Theorem. Let \( d \in [0,\infty] \), let \( F := \langle (x,y) \vee z_{[1,d]} \rangle \), and let \( u \) and \( r \) be elements of \( F \). Suppose that \( u \in \langle z_{[1,d]} \rangle \) and that \( r \) is a Hempel relator for \( \langle (x,y) \vee z_{[1,d]} \parallel \text{[x,y]}u \rangle \).

Let \( G := \langle (x,y) \vee z_{[1,d]} \parallel [x,y]u, r \rangle \).

Let \( m = \log_F r \) and let \( C_m := \langle \sqrt[d]{r} \rangle / \langle \rangle \). Then the following hold.

(i) \( C_m \) can be identified with the subgroup of \( G \) generated by the image of \( \sqrt[d]{r} \).

(ii) \( G/\langle C_m \rangle = \langle (x,y) \vee z_{[1,d]} \parallel [x,y]u, \sqrt[d]{r} \rangle \) which is locally indicable.

(iii) There exists a subset \( X \) of \( G \) such that \( \langle C_m \rangle = \ast (X \cdot C_m) := \ast (g \cdot C_m) \). Let \( K \) denote the kernel of the homomorphism \( \ast (X \cdot C_m) \to C_m \) which acts as conjugation by \( \overline{g} \) on \( g \cdot C_m \), for each \( g \in X \). Then \( K \) is a free group, and \( (G \cdot C_m) = K \times C_m \).
(iv). Each torsion subgroup of $G$ lies in some conjugate of $C_m$.

(v). Every torsion-free subgroup of $G$ is locally indicable.

(vi). $G$ has some torsion-free finite-index subgroup.

Proof. We again use Notation 6.2.

Notice that $d \neq 0$ and $r \neq 1$, by (R2).

Notice that $\sqrt[\nu]{r}$ too is a Hempel relator for $\langle (x, y) \lor z_{[1 \mid d]} \mid [x, y] u \rangle$, and that $\nu(\sqrt[\nu]{r})=\nu(r)$ by Lemma 5.2.

(i). By results of Magnus, we can view $C_m$ as a subgroup of the one-relator group $G_{[0 \mid \nu]}$. By results of Higman, Neumann, and Neumann, we can view $G_{[0 \mid \nu]}$ as a subgroup of $G$. Thus (i) holds.

(ii) follows easily from Theorem 6.4.

(iii). It is straightforward to see the following.

$$G/\langle G^{C_m} \rangle = \langle (x, y) \lor z_{[1 \mid d]} \mid [x, y] u, \sqrt[\nu]{r} \rangle$$

$$= \langle (x) \lor [0^{[\nu]}] z_{[1 \mid d]} \mid \sqrt[\nu]{r} \rangle \ast y$$

$$= (G_{[0 \mid \nu]}/\langle G_{[0 \mid \nu]}^{C_m} \rangle) \ast y_{G_{[1 \mid d]}}$$

If we consider $\langle G^{C_m} \rangle$ acting on the Bass-Serre tree for the HNN decomposition of $G$, we now see that $\langle G^{C_m} \rangle$ acts freely on the edge set, and that the quotient graph is the Bass-Serre tree for the above HNN decomposition of $G/\langle G^{C_m} \rangle$. Hence $\langle G^{C_m} \rangle$ is a free product of conjugates of $\langle G_{[0 \mid \nu]}^{C_m} \rangle$, with one conjugate for each vertex of the latter tree. By [13, Theorem 1], $\langle G_{[0 \mid \nu]}^{C_m} \rangle$ in turn is a free product of conjugates of $C_m$. Hence, $\langle G^{C_m} \rangle$ is a free product of conjugates of $C_m$. Hence we have the desired homomorphism $\langle G^{C_m} \rangle \to C_m$, and its kernel $K$. Clearly $\langle G^{C_m} \rangle = K \ltimes C_m$. If we consider $K$ acting on the Bass-Serre tree associated with the graph-of-groups decomposition of $\langle G^{C_m} \rangle$ as a free product of copies of $C_m$, we see that $K$ acts freely on the vertex set, and, hence, $K$ is a free group.

(iv). Suppose that $H$ is some torsion subgroup of $G$. The image of $H$ in the torsion-free quotient $G/\langle G^{C_m} \rangle$ is then trivial, that is, $H \subseteq \langle G^{C_m} \rangle = K \ltimes C_m$. Since $H \cap K$ is necessarily trivial, $H$ embeds in $C_m$ and, in particular, $H$ is finite. Also, $H$ lies in $\langle G^{C_m} \rangle = \ast (X^{C_m})$. By Bass-Serre Theory, or the Kurosh Subgroup Theorem, $H$ lies in a conjugate of $C_m$.

(v). Suppose that $H$ is some torsion-free subgroup of $G$. Then, by Bass-Serre Theory, or the Kurosh Subgroup Theorem, $H \cap \ast (X^{C_m}) = H \cap \langle G^{C_m} \rangle$ is free, and, hence, locally indicable. Now $H/(H \cap \langle G^{C_m} \rangle)$ embeds in the locally indicable group $G/\langle G^{C_m} \rangle$, and hence is locally indicable. It follows that $H$ is locally indicable.

(vi). We imitate the proof of [13, Theorem 2]. By Lemma 4.2, or Remark 4.4, there exists some finite group $\Phi$ and some homomorphism $\alpha: \langle (x, y) \lor z_{[1 \mid d]} \mid [x, y] u \rangle \to \Phi$ such that $\alpha(\sqrt[\nu]{r})$ has order exactly $m$. This induces a homomorphism

$$\beta: \langle (x, y) \lor z_{[1 \mid d]} \mid [x, y] u, r \rangle \to \Phi$$
which is injective on $C_m$. Let $N$ denote the kernel of $\beta: G \to \Phi$. If $H$ is some torsion subgroup of $N$, then, by (iv), $H \subseteq ^gC_m \cap N$ for some $g \in G$. Since $\beta$ is injective on $^gC_m$ and vanishes on $N$, we see that $^gC_m \cap N = \{1\}$. Thus $N$ is torsion free and of finite index in $G$. □

In the case where $G$ is a hyperbolic orientable surface-plus-one-relation group, Howie [19] proved several of these results.

7 Exact sequences

In this section, given a Hempel presentation we verify the exactness of a sequence suitable for building a classifying space for proper actions.

7.1 Remark. We shall find ourselves considering diagrams of abelian groups of the forms that appear in Fig 7.1.1, where maps are written on the right of their arguments. If the left-hand diagram is commutative and its two columns and long row are exact, then the right-hand diagram is an exact sequence; this implication follows from the theory of total complexes of double complexes, and is easy to check by chasing diagrams. □

Figure 7.1.1: A double complex and its total complex
7.2 Notation. Let $W$ be a set and let $G$ be a group.

We identify $\mathbb{Z}[G \times W] = \mathbb{Z}G \otimes_{\mathbb{Z}} \mathbb{Z}W$, and, hence, for each $(p, w) \in \mathbb{Z}G \times W$, we view $p \otimes w$ as an element of $\mathbb{Z}[G \times W]$.

We also identify $\mathbb{Z}[G \times W]$ with the direct sum of a family of copies of $\mathbb{Z}G$ indexed by $W$.

For any subgroup $C$ of $G$, we consider $C$ as acting trivially on $W$ and write $\mathbb{Z}[G \times_C W] = \mathbb{Z}G \otimes_{\mathbb{Z}C} \mathbb{Z}W$.

For any map of sets $\alpha: W \to G$, $w \mapsto \alpha(w)$, the map of sets

$$W \to (\mathbb{Z}[G \times W]) \times G = \begin{pmatrix} 1 & 0 \\ \mathbb{Z}[G \times W] & G \end{pmatrix}, \quad w \mapsto (1 \otimes w, \alpha(w)) = \begin{pmatrix} 1 \\ 1 \otimes w \end{pmatrix} \begin{pmatrix} 0 \\ \alpha(w) \end{pmatrix},$$

extends uniquely to a group homomorphism $(W \mid \cdot) \to (\mathbb{Z}[G \times W]) \times G$, denoted $r \mapsto (\frac{\partial r}{\partial W}, \alpha(r))$. We call $\frac{\partial r}{\partial W}$ the total Fox derivative of $r$ (with respect to $W$, relative to $\alpha$). If the map $\alpha$ is understood, then, for each $w \in W$, we write $\frac{\partial r}{\partial w} \otimes w$ for the summand of $\frac{\partial r}{\partial W}$ corresponding to $w$.

7.3 Theorem. Let $d \in [0, \infty[$, let $F := \langle (x, y) \vee z_{[1, d]} \mid \cdot \rangle$, and let $u$ and $r$ be elements of $F$. Suppose that $u \in \langle z_{[1, d]} \rangle$ and that $r$ is a Hempel relator for $\langle (x, y) \vee z_{[1, d]} \parallel [x, y]u \rangle$. Let $m := \log_F r$ and $G := \langle (x, y) \vee z_{[1, d]} \mid [x, y]u, r \rangle$.

Let $C$ denote the subgroup of $G$ generated by the image $c$ of $\xi/r$. Let $(x, y, z_{[1, d]})$ denote $(x, y) \vee z_{[1, d]}$, and let $G \times \{[x, y]u, cr\}$ denote $(G \times \{[x, y]u\}) \cup (G \times_C \{r\})$. Then the sequence of left $\mathbb{Z}G$-modules given by

$$0 \to \mathbb{Z}[G \times \{[x, y]u, cr\}] \to \mathbb{Z}[G \times (x, y, z_{[1, d]})] \to \mathbb{Z}[G/1] \xrightarrow{(1-G)} \mathbb{Z}[G/G] \to 0$$

is exact.

Before we give a proof, let us digress to emphasize the classical topological significance of having an exact sequence of left $\mathbb{Z}G$-modules as in Theorem 7.3.

7.4 Review. (i). Let $G$ be an arbitrary discrete group.

By the Hurewicz-Whitehead Theorem, a CW-complex is contractible if and only if it is acyclic and simply connected; for our purposes we could take this as the definition of contractible.

A $G$-CW-complex is a CW-complex on which each element of $G$ acts continuously, permuting the open cells of each dimension fixing only those cells which it fixes pointwise.
Within the context of $G$-CW-complexes, an $EG$ (pronounced e.g.) is a contractible $G$-CW-complex on which $G$ acts freely. An $EG$ is also called a classifying space for free $G$-actions.

The CW-complex quotient of an $EG$ by the $G$-action is called a $BG$, and is also called a $K(G, 1)$, and is also called a classifying space for $G$, and is also called an Eilenberg-Mac Lane space for $G$. The fundamental group of a $BG$ is isomorphic to $G$, and the higher homotopy groups of a $BG$ are trivial.

In the context of $G$-CW-complexes, an $EG$ (pronounced E-underbar $G$) is a $G$-CW-complex with the property that, for each subgroup $H$ of $G$, the subcomplex fixed pointwise by $H$ is contractible if $H$ is finite, and is empty if $H$ is infinite. The latter condition means precisely that all cell-stabilizers are finite, and, in this event, we say that the $G$-action is proper. An $EG$ is also called a classifying space for proper $G$-actions.

The CW-orbifold quotient of an $EG$ by the $G$-action is sometimes called a $BG$. In the sense of K. S. Brown’s two-dimensional extension of Bass-Serre Theory, the fundamental orbifold group of a $BG$ is isomorphic to $G$; see [6].

(ii). Let $\langle X \parallel R \rangle$ be a finite presentation, and let $G = \langle X \mid R \rangle$. We use $g$, $x$, and $r$ to denote variables ranging over $G$, $X$, and $R$, respectively, and we use the same symbol $x$ to denote the image of $x$ in $\langle X \mid \; \rangle$, and, also, to denote the image of $x$ in $G$.

Recall that the Cayley complex of $\langle X \parallel R \rangle$, denoted $\mathcal{C} = \mathcal{C}(\langle X \parallel R \rangle)$, is a two-dimensional CW-complex with exactly one 0-cell, denoted $[1]$, and with set of 1-cells, denoted $[X]$, in bijective correspondence with $X$ by a map denoted $X \to [X]$, $x \mapsto [x]$, and with set of 2-cells, denoted $[R]$, in bijective correspondence with $R$ by a map denoted $R \to [R]$, $r \mapsto [r]$. The attaching maps are forced for the 1-cells. Each $r$ is a word in $X \cup X^{-1}$, and we take the closure of the 2-cell $[r]$ to be a polygon whose (counter-clockwise) boundary has the corresponding labelling in the 1-cells and their inverses, and this labelling gives the attaching map for $[r]$; the inverse of a 1-cell is the same 1-cell with the opposite orientation. The fundamental group of $\mathcal{C}$, with base-point the unique 0-cell, has a natural identification with $G = \langle X \mid R \rangle$.

We define $\mathcal{C} = \mathcal{C}(\langle X \parallel R \rangle)$ to be the following $G$-CW-complex. The $G$-sets of 0-, 1-, and 2-cells are $G \times \{[1]\}$, $G \times [X]$, and $G \times [R]$, respectively. The initial 0-cell of $(g, [x])$ is $(g, [1])$. The terminal 0-cell of $(g, [x])$ is $(gx, [1])$. We introduce the notation $(g, [x]) := (g\bar{\tau}, [x])^{-1}$.

The boundary label of the 2-cell $(g, [r])$ is the sequence

$$(g \prod x_{[1:|i-1|]}[x_i] \mid i \in [1\uparrow N]),$$

where $x_{[1\uparrow N]}$ is the sequence in $X \cup X^{-1}$ which gives the reduced expression $r = \prod x_{[1\uparrow N]}$. The boundary label of $(g, [r])$ is a closed path at $(g, [1])$ that reads the element $r$ of $\langle X \mid \; \rangle$. If we choose a path from the base-point $(1, [1])$ to $(g, [1])$ in the one-skeleton of $\mathcal{C}$, then we can form a closed path in $\mathcal{C}$ at $(1, [1])$ that reads $g\tau r$ and is made null-homotopic by $(g, [r])$.

It is not difficult to show that $\mathcal{C}$ is simply connected, as follows. Each closed path at $(1, [1])$ in $\mathcal{C}$ can be homotoped to a closed path at $(1, [1])$ that lies in the one-skeleton of $\mathcal{C}$.
Each closed path at $(1, [1])$ in the one-skeleton of $\mathcal{C}$ reads an element of $\langle R \rangle$ in $\langle X \mid \rangle$ and, hence, can be homotoped to a concatenation of closed paths at $(1, [1])$ in the one-skeleton of $\mathcal{C}$ each of which reads a conjugate of an element of $R$ or its inverse and is null-homotopic. This implies that $\mathcal{C}$ is simply connected.

It is clear that $G \setminus \mathcal{C} = \overline{\mathcal{C}}$. It can be checked that $\mathcal{C}$ is a universal cover of $\overline{\mathcal{C}}$, and that the augmented cellular chain complex of $\mathcal{C}$ is the $ZG$-complex

$$0 \to Z[G \times [R]] \xrightarrow{1 \otimes [r] - \frac{\partial r}{\partial x}} Z[G \times [X]] \xrightarrow{1 \otimes [x] - (x-1) \otimes [1]} Z[G \times \{[1]\}] \xrightarrow{1 \otimes [1] - 1} Z \to 0.$$  

(iii). We now use the notation of Theorem 7.3; in particular, $x$, $r$, and $g$ do not represent variables.

Let $X := (x, y) \vee z_{[1,d]}$, let $R := \{[x, y]u, r\}$, let $\overline{\mathcal{C}} := \overline{\mathcal{C}}\langle X \parallel R \rangle$, and let $\mathcal{C} := \mathcal{C}\langle X \parallel R \rangle$.

Let $\mathcal{C}'$ denote the CW-complex that is obtained from $\mathcal{C}$ by identifying the 2-cells $(g, [r])$ and $(gc, [r])$, for each $g \in G$. The boundary of $(gc, [r])$ is a cyclic shift of the boundary of $(g, [r])$, and together $(g, [r])$ and $(gc, [r])$ form a 2-sphere in $\mathcal{C}$ (with identifications of at most finitely many 0-cells on the equator); we are collapsing this 2-sphere by identifying the two 2-cells without altering their common boundaries.

Consider the augmented cellular chain complex of $\mathcal{C}$; identifying $gc \otimes [r]$ with $g \otimes [r]$, for each $g \in G$, yields the augmented cellular chain complex of $\mathcal{C}'$, and this is the exact sequence of Theorem 7.3; hence $\mathcal{C}'$ is acyclic. The collapsing of 2-spheres by identifying opposite faces affects neither the one-skeleton nor the fundamental group, and, hence, $\mathcal{C}'$ is simply connected. By the above-mentioned Hurewicz-Whitehead Theorem, $\mathcal{C}'$ is contractible.

If $m = 1$, then $\mathcal{C} = \mathcal{C}'$, $\mathcal{C}$ is an $EG$, and $\overline{\mathcal{C}}$ is a $BG$.

If $m \geq 2$, although $G$ does permute the open cells of $\mathcal{C}'$, $\mathcal{C}'$ is not a $G$-CW-complex since $c$ fixes the 2-cell that is the equivalence class of $(1, [r])$ but does not fix it pointwise; to remedy this, we subdivide the new 2-cells.

Let $\mathcal{C}''$ denote the CW-complex obtained from $\mathcal{C}$ by deleting the orbit of 2-cells $G \times \{[r]\}$, and adding one orbit of 0-cells $(G/C) \times \{[C]\}$, and one orbit of 1-cells $G \times \{e\}$, and one orbit of 2-cells $G \times \{\sqrt{r}C\}$. Let $g \in G$. The initial 0-cell of $(g, e)$ is $(g, [1])$. The terminal 0-cell of $(g, e)$ is $(gC, [C])$. The boundary of $(g, [\sqrt{r}C])$ is the concatenation of, first, the path from $(g, [1])$ to $(gc, [1])$ that reads $\sqrt{r}$ in the one-skeleton of $\mathcal{C}$, which lies in the one-skeleton of $\mathcal{C}''$, and, then, the path from $(gc, [1])$ to $(g, [1])$ given by $(gc, e), (g, e)^{-1}$, which can be thought of as reading the element $\overline{c}$ of $C$.

It can be seen that $\mathcal{C}''$ is a $G$-CW-complex. It can also be seen that $\mathcal{C}''$ is obtained from $\mathcal{C}'$ by subdivision, and is therefore contractible.

The one-skeleton of $\mathcal{C}''$ is a quotient of the Bass-Serre tree that arises from the graph of groups which has $\{1, C = \langle c \mid c^m \rangle\}$ as its set of vertices and its set of vertex-groups, and $X \cup \{e\}$ as its set of edges, with each element of $X$ joining 1 to 1, and $e$ joining 1 to $C$; the fundamental group of this graph of groups is $F \ast C$, and $G$ is the quotient $(F \ast C) / \langle [x, y]u, \sqrt{rC} \rangle$. 

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In \( \mathcal{C}' \), the cell-stabilizers are finite. For each subgroup of \( G \) of order a prime number, its fixed subcomplex in \( \mathcal{C}' \) has no 1-cells, and it cannot have more than one 0-cell, by a diagram-chasing argument from P. A. Smith Theory. For each non-trivial finite subgroup of \( G \), its fixed subcomplex in \( \mathcal{C}' \) then consists of a single 0-cell, since it is non-empty by Theorem 6.5(iv). Thus \( \mathcal{C}' \) is an \( \mathbb{E}G \). With respect to Theorem 6.5(iv), we now see that, if \( H \) is a non-trivial torsion subgroup of \( G \), then there exists a unique element \( gC \in G/C \) such that \( H \leq \langle gC \rangle \).

In \( G\setminus \mathcal{C}' \), which is a \( \mathcal{B}G \), the 0-cell \( G(C, [C]) \) is a cone-point of angle \( \frac{2\pi}{m} \) and the 2-cell \( G(1, [\xi r \xi]) \) forms a cone. A closed path travelling \( m \) times in a row around the 0-cell \( G(C, [C]) \) in a small neighbourhood is null-homotopic. It is particularly easy to view \( G \) as the fundamental orbifold group of \( G\setminus \mathcal{C}' \) where the base-point is the 0-cell \( G(1, [1]) \).

Let \( \mathcal{C}'' \) denote the contractible \( G \)-CW-complex obtained from \( \mathcal{C}' \) collapsing the 1-cell \( (g, [e]) \) and identifying the two 0-cells \( (g, [1]) \) and \( (gC, [C]) \), for each \( g \in G \). The paths in the one-skeleton of \( \mathcal{C}'' \) read words in \( X \vee X^{-1} \vee (C - \{1\}) \). Again, \( \mathcal{C}'' \) is an \( \mathbb{E}G \) and \( G \setminus \mathcal{C}'' \) is a \( \mathcal{B}G \).

The augmented cellular chain complex of \( \mathcal{C} \) has underlying \( \mathbb{Z}G \)-structure

\[
0 \rightarrow \mathbb{Z}G^2 \rightarrow \mathbb{Z}G^{d+2} \rightarrow \mathbb{Z}G \rightarrow \mathbb{Z} \rightarrow 0,
\]

and the (exact) augmented cellular chain complexes of \( \mathcal{C}' \), \( \mathcal{C}'', \mathcal{C}'' \) have underlying \( \mathbb{Z}G \)-structures

\[
\begin{align*}
0 & \rightarrow \mathbb{Z}G \oplus \mathbb{Z}[G/C] \rightarrow \mathbb{Z}G^{d+2} \rightarrow \mathbb{Z}G \rightarrow \mathbb{Z} \rightarrow 0, \\
0 & \rightarrow \mathbb{Z}G^2 \rightarrow \mathbb{Z}G^{d+3} \rightarrow \mathbb{Z}G \oplus \mathbb{Z}[G/C] \rightarrow \mathbb{Z} \rightarrow 0, \\
0 & \rightarrow \mathbb{Z}G^2 \rightarrow \mathbb{Z}G^{d+2} \rightarrow \mathbb{Z}[G/C] \rightarrow \mathbb{Z} \rightarrow 0,
\end{align*}
\]

respectively.

Proof of Theorem 7.3. In Notation 6.2, let \((x, [0^{|\nu|}]_{z[1:1]}{d})\) denote \((x) \vee [0^{|\nu|}]_{z[1:1]} \), let \((1x, [1^{|\nu|}]_{z[1:1]}{d})\) denote \((1x) \vee [1^{|\nu|}]_{z[1:1]} \), and let

\[
\langle x, [0^{|\nu|}]_{z[1:1]} \rangle := G_{0^{|\nu|}} := \langle (x) \vee [0^{|\nu|}]_{z[1:1]} \mid r \rangle,
\]

\[
\langle 1x, [1^{|\nu|}]_{z[1:1]} \rangle := G_{1^{|\nu|}} := \langle (1x) \vee [1^{|\nu|}]_{z[1:1]} \mid \_ \rangle.
\]

Recall that \( y : \langle x, [0^{|\nu|-1}]_{z[1:1]} \rangle \rightarrow \langle 1x, [1^{|\nu|}]_{z[1:1]} \rangle \) acts by \( x \mapsto 1x, i^{|\nu|}_z \mapsto i^{|\nu|+1}_z \), and that \( G = \langle x, [0^{|\nu|}]_{z[1:1]} \rangle \rightarrow \langle 1x, [1^{|\nu|}]_{z[1:1]} \rangle \) acts by \( x \mapsto 1x, i^{|\nu|}_z \mapsto 1^{|\nu|+1}_z \). All the notation involved in Fig. 7.4.1 has now been explained.

We now make five observations about Fig. 7.4.1.

(i). The long row is the exact augmented cellular chain complex of the Bass-Serre tree corresponding to the HNN graph-of-groups decomposition of \( G \) with one vertex and one
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\[ \begin{array}{c}
0 \\
\downarrow \\
\mathbb{Z}[G/C] \\
\uparrow \\
\mathbb{Z}[G \times \langle x, [1]^{\nu}z_{[1]} \rangle] \\
\downarrow \\
\mathbb{Z}[G \times \langle x, [0]^{\nu}z_{[1]} \rangle] \\
\downarrow \\
\mathbb{Z}[G \times (y)] \\
\downarrow \\
\mathbb{Z}[G \times (y)] \\
\end{array} \]

\[ \begin{array}{c}
\mathbb{Z}[G \times \langle x, [1]^{\nu}z_{[1]} \rangle] \\
\rightarrow \\
\mathbb{Z}[G] \\
\rightarrow \\
\langle 1 \rangle \\
\rightarrow \\
\mathbb{Z}[G/\langle x, [1]^{\nu}z_{[1]} \rangle] \\
\rightarrow \\
\mathbb{Z}[G/\langle x, [0]^{\nu}z_{[1]} \rangle] \\
\rightarrow \\
\mathbb{Z}[G/G] \\
\rightarrow \\
0 \end{array} \]

Figure 7.4.1: A commuting diagram.

(ii). The left column is the exact sequence obtained by applying the exact functor \( \mathbb{Z}G \otimes \mathbb{Z}[\langle x, [1]^{\nu}z_{[1]} \rangle] \) to the exact augmented cellular chain complex of the Cayley tree of the free group \( \langle x, [1]^{\nu}z_{[1]} \rangle \), or, equivalently, the Bass-Serre tree corresponding to the graph-of-groups decomposition with one vertex and \( 1 + \nu d \) edges. See, for example, [10, Examples I.3.5(i) and Theorem I.6.6].

(iii). The right column is the exact sequence obtained by applying the exact functor \( \mathbb{Z}G \otimes \mathbb{Z}[\langle x, [0]^{\nu}z_{[1]} \rangle] \) to Lyndon’s exact sequence for the one-relator group \( \langle x, [0]^{\nu}z_{[1]} \mid r \rangle \);
see, for example, [9, (∗) on p. 167].

(iv). It is clear that the lower square commutes.

(v). To see that the upper square commutes, we note that along the upper route in the upper square,

\[1 \otimes 1 \mapsto (y - 0u) \otimes x - \sum_{w \in \mathbb{Z}_{[1]d}} \frac{\partial(0u)}{\partial w} \otimes w\]

\[\mapsto (y - 0u)(x - 1) - (0u - 1) = yx - 0ux - y + 1\]

\[= 1x \cdot y - 1x - y + 1 = (1x - 1)(y - 1),\]

and

\[1 \otimes \overline{z}_* \mapsto y \otimes \overline{z}_* - 1 \otimes \overline{z}_*\]

\[\mapsto y(\overline{z}_* - 1) - (\overline{z}_* - 1) = y \cdot \overline{z}_* - y - \overline{z}_* + 1\]

\[= \overline{z}_* \cdot y - y - \overline{z}_* + 1 = (\overline{z}_* - 1)(y - 1).\]

It is now clear that the upper square commutes.

With these five observations in mind, we can apply Remark 7.1 to Fig. 7.4.1, and what we get is the exact sequence which appears as the left column of Fig. 7.4.2. After some adjustments, it becomes the right column of Fig. 7.4.2, which can be viewed as the augmented cellular chain complex of the complex obtained from the universal cover of the Cayley complex of the presentation

\[\langle (x, y) \vee [0^1\nu z_{[1]d}] \parallel [x, y]u, (y \cdot \overline{z}_* \cdot \overline{y} \cdot \overline{z}_*, i \overline{z}_* \in [1^1\nu z_{[1]d}], r(x, [0^1\nu z_{[1]d}]\rangle\]

by collapsing 2-spheres.

If \(\nu = 0\), we have the desired exact sequence.

If \(\nu \geq 1\), then we shall delete \([1^1\nu z_{[1]d}]\) from the set of generators and delete

\[(y \cdot \overline{z}_* \cdot \overline{y} \cdot \overline{z}_*, i \overline{z}_* \in [1^1\nu z_{[1]d}]\)

from the set of relators, and understand, henceforth, that \(i \overline{z}_*\) denotes \(y^i \overline{z}_*\) in the new generators. We consider \(G\) as being unaltered, but we alter the exact sequence by dividing out by the exact subcomplex

\[0 \to \mathbb{Z}[G \times ([1^1\nu z_{[1]d}])] \left\langle \frac{1 \otimes 1 \otimes \overline{z}_* \cdots (y \cdot \overline{z}_* \cdot \overline{y} \cdot \overline{z}_*) \cdots \frac{\partial(0u, i \overline{z}_* \cdot \nu \overline{z}_*)}{\partial(x, y, [0^1\nu z_{[1]d}])} \right\rangle \mathbb{Z}[G \times (\frac{\partial(0u, i \overline{z}_* \cdot \nu \overline{z}_*)}{\partial(x, y, [0^1\nu z_{[1]d}]})] \to 0 \to 0.\]

Thus, for each \(i \in [1^1d]\), in the quotient of \(\mathbb{Z}[G \times (\frac{1x, [1^1\nu z_{[1]d}], cr})]\), we are identifying \(1 \otimes 1 \otimes \overline{z}_*\) with 0, while, in the quotient of \(\mathbb{Z}[G \times (x, y, [0^1\nu z_{[1]d}])]\), we are identifying \(1 \otimes 1 \otimes \overline{z}_*\) with
Figure 7.4.2: An exact sequence rewritten.

\[(1 - i z^*) \otimes y + y \otimes i^{-1} z^*, \text{which, by induction, is identified with } \frac{\partial(y^* z^* x)}{\partial(y, z^*)}. \text{It follows that, in the quotient of } \mathbb{Z}[G \times (x, y, [0^1]z_{[1,d]}^1)], \frac{\partial(r(x,y,z_{[1,d]}))}{\partial(\zeta)} \text{becomes identified with } \frac{\partial(r(x,y,z_{[1,d]}))}{\partial(x,y,z_{[1,d]}^1)} , \text{and the exact quotient sequence is the augmented cellular chain complex of the complex obtained from the universal cover of our original Cayley complex by collapsing 2-spheres.} \]

In the case where \( G \) is a hyperbolic orientable surface-plus-one-relation group, this result was obtained by Howie [19, Corollary 3.6].
8 VFL and Euler characteristics

In this section, we calculate the Euler characteristics of the surface-plus-one-relation groups.

8.1 Definitions. Consider any resolution of $\mathbb{Z}$ by projective, left $\mathbb{Z}G$-modules

$\cdots \rightarrow P_2 \rightarrow P_1 \rightarrow P_0 \rightarrow \mathbb{Z} \rightarrow 0$. 

The *length* of the projective $\mathbb{Z}G$-resolution (2) is the supremum, in $[0^{\uparrow} \infty]$, of the non-empty set $\{ n \in [0^{\uparrow} \infty] \mid P_n \neq 0 \}$.

Let $P$ denote the unaugmented complex $\cdots \rightarrow P_2 \rightarrow P_1 \rightarrow P_0 \rightarrow 0$, and view $\mathbb{Q}$ as a $\mathbb{Q}$-$\mathbb{Z}G$-bimodule. For each $n \in [0^{\uparrow} \infty]$, $H_n(G; \mathbb{Q}) := \text{Tor}_n^G(\mathbb{Q}, \mathbb{Z}) := H_n(\mathbb{Q} \otimes_{\mathbb{Z}G} P)$; for the purposes of this article, it will be convenient to understand that $H_n(G; -)$ applies to right $\mathbb{Z}G$-modules. The $n$th Betti number of $G$ is $b_n(G) := \dim_{\mathbb{Q}} H_n(G; \mathbb{Q}) \in [0^{\uparrow} \infty]$. The value of the Betti numbers does not depend on the choice of projective $\mathbb{Z}G$-resolution (2).

The *cohomological dimension* of $G$, denoted $\text{cd}_G$, is the smallest element of the subset of $[0^{\uparrow} \infty]$ which consists of lengths of projective $\mathbb{Z}G$-resolutions of $\mathbb{Z}$. The *virtual cohomological dimension* of $G$, denoted $\text{vcd}_G$, is the smallest element of the subset of $[0^{\uparrow} \infty]$ which consists of cohomological dimensions of finite-index subgroups of $G$. The *cohomological dimension of $G$ with respect to an associative ring $Q$*, denoted $\text{cd}_Q G$, is the smallest element of the subset of $[0^{\uparrow} \infty]$, which consists of lengths of projective $QG$-resolutions of $Q$.

If there exists a resolution (2) of finite length such that all the $P_n$ are finitely generated, free left $\mathbb{Z}G$-modules, then we say that $G$ is of type FL and we define the *Euler characteristic of $G$* to be

$$\chi(G) := \sum_{n \in [0^{\uparrow} \infty]} (-1)^n b_n(G).$$

If $G$ has some finite-index subgroup $H$ of type FL, then we say that $G$ is of type VFL and, if $G$ is not of type FL, we define the *Euler characteristic of $G$* to be

$$\chi(G) := \frac{1}{[G:H]} \chi(H);$$

this value, which is sometimes called the ‘virtual Euler characteristic’, does not depend on the choice of $H$.

8.2 Remark. If $G$ has some finitely generated, one-relator, finite-index subgroup $H$, say $H = \langle X \mid r \rangle$, then $G$ is of type VFL and

$$-\chi(G) = -\frac{1}{[G:H]} \chi(H) = \frac{1}{[G:H]} (|X| - 1 - \frac{1}{\log(|X|)}).$$

See, for example, [9, (∗) on p. 167].
We now discuss the case where \( r = x^m \).

### 8.3 Examples

Let \( d, m \in [0, \infty[ \), let \( F := \langle (x, y) \mid z_{[1]} \rangle \), and let \( u \) and \( r \) be elements of \( F \). Suppose that \( u \in \langle z_{[1]} \rangle - \{1\} \) and that \( r = x^m \). Let \( G = \langle (x, y) \mid z_{[1]} \rangle \). In \([1, \infty[\), let \( m_F := \log_F x^m \in \{m, \infty\} \), let \( m' \) denote the supremum of the orders of the finite subgroups of \( G \), and let \( m'' := \max(m_F, m') \).

It is straightforward to verify the following assertions. It then follows that \( G \) is virtually one-relator, and hence of type VFL.

(i) If \( m = 0 \), then \( G = \langle (x, y) \mid z_{[1]} \rangle \).

Here, \( m_F = \infty \), \( m' = 1 \), \( m'' = \infty \), and \( -\chi(G) = d = d - \frac{1}{m''} \geq 0 \).

(ii) If \( m = 1 \), then \( G = \langle (y) \mid z_{[1]} \rangle \).

Here, \( m_F = 1 \), \( m' = \log_F u \), \( m'' = \log_F u \), and \( -\chi(G) = d = d - \frac{1}{\log_F u} = d - \frac{1}{m''} \geq 0 \).

(iii) If \( m \geq 2 \), then \( G = \langle (y) \mid z_{m} \rangle \).

Here, \( m_F = m' = m'' = m \), and \( -\chi(G) = d = d - \frac{1}{m} = d - \frac{1}{m''} \geq 0 \).

It is now convenient to go back to writing \((x, y) \cup z_{[1]} \) in the form \( x_{[1]} \) with \( k = d + 2 \).

### 8.4 Corollary

Let \( k \in \{3, \infty\} \), let \( F := \langle x_{[1]} \rangle \), and let \( w \) and \( r \) be elements of \( F \). Suppose that \( w \in [x_1, x_2][x_{[3]}] \) and that \( r \) is a Hempel relator for \( \langle x_{[1]} \rangle \). Let \( G := \langle x_{[1]} \rangle \).

Let \( m := \log_F r \). Then \( \text{vcd} G \leq 2 \), \( G \) is of type VFL and \( -\chi(G) = k - 2 - \frac{1}{m} \geq 0 \).

Let \( C \) denote the subgroup of \( G \) generated by the image of \( \sqrt[k]{r} \), let \( Q \) be any associative ring such that \( mQ = Q \), let \( R := QG \), and let \( e := \frac{1}{m} \sum_{c \in C} c \in R \). Then the sequence of left \( R \)-modules

\[
0 \to R \oplus Re \xrightarrow{egin{pmatrix} \frac{\partial w}{\partial x_1} & \cdots & \frac{\partial w}{\partial x_k} \\ \frac{\partial r}{\partial x_1} & \cdots & \frac{\partial r}{\partial x_k} \end{pmatrix}} R^k \xrightarrow{\begin{pmatrix} x_{1}^{-1} \\ \vdots \\ x_{k}^{-1} \end{pmatrix}} R \xrightarrow{G_{\{1\}}} Q \to 0
\]

is exact, and \( \text{cd}_{Q} G \leq 2 \).

**Proof.** By Theorem 6.5(vi), there exists some torsion-free finite-index subgroup \( H \) of \( G \). Clearly, \( H \) acts freely on \( G \) on the left and the number of orbits is \( |G : H| = |H \backslash G| \). Since \( H \) meets each conjugate of \( C \) trivially, \( C \) acts freely on \( H \backslash G \) on the right. \( H \) acts virtually on \( G/C \) on the left, and the number of orbits in each case is \( |H \backslash G| = \frac{|H \backslash G|}{|C|} = \frac{|H|}{m} \).
By Theorem 7.3, with $d = k - 2$, we see that $cdH \leq 2$, $H$ is of type FL, and $\chi(H) = [G:H] - (d + 2)[G:H] + ([G:H] + \frac{[G:H]}{m})$. Hence $vcdG \leq 2$, $G$ is of type VFL, and $\chi(G) = 1 - (d + 2) + (1 + \frac{1}{m})$. Since $(QC)e \simeq Q$ as left $QC$-modules, we see that 

$$Q[G/C] = QG \otimes_{QC} Q \simeq QG \otimes_{QC} (QC)e \simeq Re.$$ 

It is now clear that the results hold.

We can now describe the Euler characteristics of the surface-plus-one-relation groups.

8.5 Remarks. Let $k \in [0, \infty[$, let $F := \langle x_{\lceil k\rceil} | \rangle$, and let $w$ and $r$ be elements of $F$. Suppose either that $k$ is even and $w = \prod_{i \in \lceil k \rceil^2} x_{2i-1}, x_{2i}$, or that $k \geq 1$ and $w = \prod x_{\lceil k \rceil}^2$. Let $S := \langle x_{\lceil k \rceil} | w \rangle$ and let $G := \langle x_{\lceil k \rceil} | w, r \rangle$.

In $[1, \infty[$, let $m := \log_S r \mod w$, let $m'$ denote the supremum of the orders of the finite subgroups of $G$, and let $m'' := \max(m, m')$.

Case 1. $k \leq 2$.

Here, $G$ is virtually abelian of rank at most two. In particular, $G$ is virtually one-relator, and it follows from Remark 8.2 that $G$ is of type VFL and $\chi(G) = \frac{1}{\lvert G \rvert} \geq 0$.

Case 2. $k \geq 3$, and $w = \prod x_{\lceil k \rceil}^2$.

Case 2.1. For some $m \in [0, \infty[$, $r \mod w$ is conjugate in $S$ to the $m$th power of some term of some free-generating sequence of the subgroup $\langle x_1, x_2, x_3 \rangle$ of $S$.

Here, by using Lemma 2.1 and Examples 8.3, we find that $G$ is virtually one-relator, that $G$ is of type VFL, and that $-\chi(G) = k - 2 - \frac{1}{m'} \geq 0$.

Let us list the cases that arise in Examples 8.3.

Case 2.1.1. $(k, m) = (3, 1)$.

Here, $m = 1$, and $m' = m'' = 2$, and $G \simeq C_\infty \ast C_2$, and $-\chi(G) = \frac{1}{2} \geq 0$.

Case 2.1.2. $(k, m) \neq (3, 1)$ and $r \mod w = 1$.

Here, $m' = 1$, and $m = m'' = \infty$.

Case 2.1.3. $(k, m) \neq (3, 1)$ and $r \mod w \neq 1$.

Here, $m' = m = m'' < \infty$.

Case 2.2. $r \mod w$ is not conjugate in $S$ to any power of any term of any free-generating sequence of the subgroup $\langle x_1, x_2, x_3 \rangle$ of $S$.

Here, by using Lemmas 2.1 and 5.4, we find that there exists some presentation for $G$ as in Corollary 8.4.

Hence, $G$ is of type VFL, $-\chi(G) = k - 2 - \frac{1}{m} \geq 0$, and $m' = m = m'' < \infty$. 
Case 3. $k \geq 4$, and $k$ is even, and $w = \prod_{i \in [1:]^2} [x_{2i-1}, x_{2i}]$.

Case 3.1. $r \mod w = 1$.

Here, $G$ is of type VFL, $-\chi(G) = k - 2 - \frac{1}{m} \geq 0$, $m' = 1$, and $m = m'' = \infty$.

Case 3.2. $r \mod w \neq 1$.

Here, there exists some $i \in [1:]^2$ such that the conjugacy class of $r \mod w$ in $S$ is disjoint from the subgroup $\langle x_{(2i-1)(2i)} \rangle$ of $S$.

By using Lemmas 2.1 and 5.4, we find that there exists some presentation for $G$ as in Corollary 8.4.

Hence, $G$ is of type VFL, $-\chi(G) = k - 2 - \frac{1}{m} \geq 0$, and $m' = m = m'' < \infty$. \hfill \Box

We summarize the preceding, in which one can use $m'' = m$ except for Case 2.1.1, and one can use $m'' = m'$ if $m \neq \infty$, that is, if $r \mod w \neq 1$.

8.6 Corollary. Let $k \in [0:]^\infty \cup [1:]^\infty$, let $F := \langle x_{[1:k]} \rangle$, and let $w$ and $r$ be elements of $F$. Suppose either that $k$ is even and $w = \prod_{i \in [1:]^2} [x_{2i-1}, x_{2i}]$, or that $k \geq 1$ and $w = \prod x_{[1:k]}^2$.

Let $S := \langle x_{[1:k]} \rangle$ and let $G := \langle x_{[1:k]} \rangle w, r \rangle$. In $[1:]^\infty$, let $m := \log_S r \mod w$, let $m'$ denote the supremum of the orders of the finite subgroups of $G$, and let $m'' := \max(m, m')$.

Then

$$\chi(G) = \begin{cases} \frac{1}{|G|} \geq 0 & \text{if } k \leq 2, \\ -k + 2 + \frac{1}{mm'} \leq 0 & \text{if } k \geq 3. \end{cases} \hfill \Box$$

9 $L^2$-Betti numbers of surface-plus-one-relation groups

Let us begin with a brief algebraic review of Atiyah’s theory of $L^2$-Betti numbers of groups.

9.1 Review. Let $\mathbb{C}[[G]]$ denote the set of all functions from $G$ to $\mathbb{C}$ expressed as formal sums, that is, a function $x : G \rightarrow \mathbb{C}$, $g \mapsto x_g$, will be written as $\sum_{g \in G} x_g g$. Then $\mathbb{C}[[G]]$ has a natural $\mathbb{C}G$-bimodule structure, and contains a copy of $\mathbb{C}G$ as $\mathbb{C}G$-sub-bimodule. For each $x \in \mathbb{C}[[G]]$, let $||x|| := \sqrt{\sum_{g \in G} |x_g|^2} \in [0, \infty]$, and let $\text{tr}(x) := x_{1_G} \in \mathbb{C}$.

Let $\ell^2(G) := \{ x \in \mathbb{C}[[G]] : ||x|| < \infty \}$. For $x, y \in \ell^2(G)$, $g \in G$, and $S$ a finite subset of $G$, it follows from the Cauchy-Schwarz inequality that $\sum_{h \in S} |x_h y_{\bar{g}}| \leq ||x|| \cdot ||y||$; hence, there exists a well-defined limit $\sum_{h \in G} (x_h y_{\bar{g}}) \in \mathbb{C}$; hence, there exists a well-defined element $x \cdot y := \sum_{g \in G} ((\sum_{h \in G} x_h y_{\bar{g}}) g) \in \mathbb{C}[[G]]$, called the external product of $x$ and $y$. 
The group von Neumann algebra of $G$ is defined as the additive abelian group

$$N(G) := \{ p \in \ell^2(G) \mid p \cdot \ell^2(G) \subseteq \ell^2(G) \}$$

endowed with the ring structure induced by the external product; it can be shown that this agrees with the definition of the group von Neumann algebra given in [23, Section 1.1], and that $N(G) = \{ p \in \ell^2(G) \mid \ell^2(G) \cdot p \subseteq \ell^2(G) \}$. We then have a chain of $\mathbb{C}G$-bimodules $\mathbb{C}G \subseteq N(G) \subseteq \ell^2(G) \subseteq \mathbb{C}[[G]]$, and $\mathbb{C}G$ is a subring of $N(G)$, and $\ell^2(G)$ is an $N(G)$-bimodule containing $N(G)$ as $N(G)$-sub-bimodule.

It can be shown that the elements of $N(G)$ which act faithfully on the left $N(G)$-module $\ell^2(G)$ are precisely the two-sided non-zerodivisors in $N(G)$, and that these form a left and right Ore subset of $N(G)$; see [23, Theorem 8.22(1)]. The ring of unbounded operators affiliated to $N(G)$, denoted $\mathcal{U}(G)$, is defined as the left, and the right, Ore localization of $N(G)$ at the set of its two-sided non-zerodivisors; see [23, Section 8.1]. For example, it is then clear that

(3) if $g$ is an element of $G$ of infinite order, then $g-1$ is invertible in $\mathcal{U}(G)$.

It can be shown that $\mathcal{U}(G)$ is a von Neumann regular ring in which one-sided inverses are two-sided inverses, and, hence, one-sided zerodivisors are two-sided zerodivisors; see [23, Section 8.2].

It can be shown that there exists a continuous, additive von Neumann dimension that assigns to every left $\mathcal{U}(G)$-module $M$ a value $\dim_{\mathcal{U}(G)} M \in [0, \infty]$; see Definition 8.28 and Theorem 8.29 of [23]. For example, if $e$ is an idempotent element of $N(G)$, then $\dim_{\mathcal{U}(G)} \mathcal{U}(G)e = \text{tr}(e)$; see Theorem 8.29 and Sections 6.1-2 of [23].

For each $n \in [0 \uparrow \infty[$, the $n$th $L^2$-Betti number of $G$ is defined as

$$b_n^{(2)}(G) := \dim_{\mathcal{U}(G)} H_n(G; \mathcal{U}(G)) \in [0, \infty],$$

where $\mathcal{U}(G)$ is to be viewed as a $\mathcal{U}(G)$-$\mathbb{Z}G$-bimodule; see Definition 6.50, Lemma 6.51 and Theorem 8.29 of [23].

It is easy to show that if $G$ is finite, then, for each $n \in [0 \uparrow \infty[$,

$$b_n^{(2)}(G) = \begin{cases} \chi(G) = \frac{1}{|G|} & \text{if } n = 0, \\ 0 & \text{if } n \in [1 \uparrow \infty[. \end{cases}$$

By [23, Theorem 6.54(8b)],

(4) $b_0^{(2)}(G) = \frac{1}{|G|}$.

By [23, Theorem 1.9(8)], if $H$ is a finite-index subgroup of $G$, then

(5) $b_n^{(2)}(H) = [G : H] b_n^{(2)}(G)$. 

In general, there is little relation between the \( n \)th \( L^2 \)-Betti number and the \( n \)th (ordinary) Betti number, \( b_n(G) = \dim_{\mathbb{Q}} H_n(G; \mathbb{Q}) \in [0, \infty] \). However, by [23, Remark 6.81], if \( G \) is of type VFL, we can define and calculate the \( L^2 \)-Euler characteristic

\[
(6) \quad \chi^{(2)}(G) := \sum_{n \in [0, \infty]} (-1)^n b_n^{(2)}(G) = \sum_{n \in [0, \infty]} (-1)^n b_n(G) =: \chi(G).
\]

Recall that, for any finitely generated, virtually one-relator group \( G \), a formula for \( \chi(G) \) was given in Remark 8.2.

9.2 Lemma. If \( G \) is a finitely generated, virtually one-relator group, then \( G \) is of type VFL and, for each \( n \in [0, \infty] \),

\[
b_n^{(2)}(G) = \begin{cases} 
\max\{\chi(G), 0\} = \frac{1}{|G|} & \text{if } n = 0, \\
\max\{-\chi(G), 0\} & \text{if } n = 1, \\
0 & \text{if } n \in [2, \infty].
\end{cases}
\]

Proof. If \( G \) is itself a finitely generated, one-relator group, the conclusions hold, by [12, Theorem 4.2]. By equation (5) in Review 9.1, this then extends to overgroups of finite index on dividing by the index. \( \square \)

Recall that a right \( \mathbb{Z}G \)-module is \( G \)-projective if it is a \( \mathbb{Z}G \)-summand of \( A \otimes_{\mathbb{Z}} \mathbb{Z}G \) for some \( \mathbb{Z} \)-module \( A \). This includes all right \( QG \)-projective \( QG \)-modules for all associative rings \( Q \).

We shall use the following, which is the left-right dual of [11, Corollary 5.6].

9.3 Theorem ([11]). If \( G \) is any group, and \( M \) is any right \( \mathbb{Z}G \)-module, and \( P \) is any \( G \)-projective \( \mathbb{Z}G \)-submodule of \( M \), and \( v \) is any element of \( M \) such that the subset \( v + P \) of \( M \) is a \( G \)-subset of \( M \), then there exists a right \( G \)-tree whose edge stabilizers are finite and whose vertex set is the right \( G \)-set \( v + P \).

For Hempel presentations we have the following information.

9.4 Lemma. Let \( k \in [3, \infty] \), let \( F := \langle x_{[1,k]} \ | \ \rangle \), and let \( w \) and \( r \) be elements of \( F \). Suppose that \( w \in [x_1, x_2] \langle x_{[3,k]} \rangle \) and that \( r \) is a Hempel relator for \( \langle x_{[1,k]} \ | \ w \rangle \). Let \( G := \langle x_{[1,k]} \ | \ w, r \rangle \). Let \( m := \log_F r \), let \( C_m \) denote the subgroup of \( G \) generated by the image of \( \sqrt{F} \), and let \( e := \frac{1}{m} \sum_{c \in C_m} c \in CG \).

(i). For all \( q \in \mathcal{U}(G) \) and all \( a \in CG \), if \( qea = 0 \) then either \( qe = 0 \) or \( ea = 0 \).
(ii). The homology of

\[
0 \to \mathcal{U}(G) \oplus \mathcal{U}(G) e \xrightarrow{\begin{pmatrix} \frac{\partial w}{\partial x_1} & \cdots & \frac{\partial w}{\partial x_k} \\ \frac{\partial r}{\partial x_1} & \cdots & \frac{\partial r}{\partial x_k} \end{pmatrix}} \mathcal{U}(G)^k \xrightarrow{\begin{pmatrix} x_1-1 \\ \vdots \\ x_k-1 \end{pmatrix}} \mathcal{U}(G) \to 0,
\]

is \(H_*(G; \mathcal{U}(G))\), and, for each \(n \in (0) \vee [3|\infty[\), \(b^{(2)}_n(G) = 0\).

Proof. (i). Recall that \([12, \text{Theorem 3.1(iii)}]\), which depends on results in \([7]\) and \([22]\), asserts that if \(G\) is ((free \(\rtimes\) \(C_m\)) by (locally indicable)), then (i) holds. By Theorem 6.5(ii),(iii), we now see that (i) holds.

(ii). By using Corollary 8.4 and equation (4) of Review 9.1, it is not difficult to see that (ii) holds. \(\square\)

Let us now consider the special case of hyperbolic surface-plus-one-relation groups.

9.5 Lemma. Let \(k \in [3|\infty[\), let \(F := \langle x_{1[k]} \rangle\), and let \(w\) and \(r\) be elements of \(F\). Suppose either that \(w = [x_1, x_2, \prod x_j^{2}]_{[3|k]}\) or that \(k\) is even and \(w = \prod_{j \in [1, k]} [x_{2j-1}, x_{2j}]\). Suppose that \(r\) is a Hempel relator for \(\langle x_{1[k]} \rangle \rtimes w\). Let \(G := \langle x_{1[k]} \rangle \rtimes w, r\). Then \(b^{(2)}_2(G) = 0\).

Proof. Let \(m := \log_F r\), let \(C\) denote the subgroup of \(G\) generated by the image of \(\sqrt[n]{r}\), and let \(e := \frac{1}{m} \sum c \in CG\). If the map

\[
\mathcal{U}(G) \oplus \mathcal{U}(G) e \xrightarrow{\begin{pmatrix} \frac{\partial w}{\partial x_1} & \cdots & \frac{\partial w}{\partial x_k} \\ \frac{\partial r}{\partial x_1} & \cdots & \frac{\partial r}{\partial x_k} \end{pmatrix}} \mathcal{U}(G)^k
\]

is injective, then \(b^{(2)}(G) = 0\), by Lemma 9.4(ii). It remains to consider the case where there exists some \((p, qe) \neq (0, 0)\) in the kernel of this map, that is, \(p, q \in \mathcal{U}(G)\) and \((7)\)

\[p \frac{\partial w}{\partial x_j} + qe \frac{\partial r}{\partial x_j} = 0 \text{ in } \mathcal{U}(G).\]

Let \(G\) act by right multiplication on the set of all subsets of \(\mathcal{U}(G)\), let \(V := p + qe \mathcal{C}G\), a subset of \(\mathcal{U}(G)\), and let \(G_V := \{g \in G \mid V g = V\} = \{g \in G \mid p(g - 1) \in qe \mathcal{C}G\}\), a subgroup of \(G\). The subset \(V\) of \(\mathcal{U}(G)\) is then closed under the right \(G_V\)-action on \(\mathcal{U}(G)\). By Lemma 9.4(i), the surjective map \(e \mathcal{C}G \to qe \mathcal{C}G, ea \mapsto qea\), is either injective or zero. In either event, \(qe \mathcal{C}G\) is a projective right \(\mathcal{C}G\)-module, and hence a projective right \(\mathcal{C}G\)-module. By Theorem 9.3, there exists a right \(G_V\)-tree \(T\) with finite edge stabilizers and vertex set the right \(G_V\)-set \(p + qe \mathcal{C}G\).

We claim that \(0 \not\in p + qe \mathcal{C}G\). Since \((p, qe) \neq (0, 0)\), we may assume that \(qe \neq 0\) for this argument. Consider any \(a \in \mathcal{C}G\). By Corollary 8.4, with \(Q = \mathbb{C}\), the map
is injective, and, in particular, \((-ea, e)\) does not lie in the kernel, since \(e \neq 0\). Hence there exists some \(j \in [1 \mid k]\), such that \(0 \neq -ea \frac{\partial w}{\partial x_j} + e \frac{\partial r}{\partial x_j} = e(-a \frac{\partial w}{\partial x_j} + \frac{\partial r}{\partial x_j})\). By Lemma 9.4(i),

\[
0 \neq qe(-a \frac{\partial w}{\partial x_j} + \frac{\partial r}{\partial x_j}) = (-qea) \frac{\partial w}{\partial x_j} + qe \frac{\partial r}{\partial x_j}.
\]

It is now clear from (7) that \(p \neq -qea\). This proves that \(0 \notin p + qeCG\), as claimed.

By using equation (3) of Review 9.1, we now find that each vertex stabilizer for \(T\) is torsion. Hence, by Theorem 6.5(iv), each vertex stabilizer for \(T\) has order at most \(m\). It follows that \(G_V\) is virtually free; see, for example, [10, Theorem IV.1.6.(b) \(\Rightarrow\) (c)].

Let

\[
G_{\pm V} := \{g \in G \mid \text{either } p(g - 1) \in qeCG \text{ or } p(g + 1) \in qeCG\} = \{g \in G \mid Vg = \pm V\},
\]

a subgroup of \(G\). We shall see that \(G_{\pm V} = G\).

By (7), we see that, for each \(j \in [1 \mid k]\), \(p \frac{\partial w}{\partial x_j} \in qeCG\), and, hence,

\[
\begin{align*}
(8) & \quad p(1 - x_1 x_2 \bar{x}_1) = p \frac{\partial w}{\partial x_1} \in qeCG, \\
(9) & \quad p(x_1 - [x_1, x_2]) = p \frac{\partial w}{\partial x_2} \in qeCG, \\
(10) & \quad \text{for all } j \in [3 \mid k], \quad p \frac{\partial w}{\partial x_j} \in qeCG.
\end{align*}
\]

By (8), \(x_1 x_2 \in G_V\). Right multiplying (9) by \(\bar{x}_1\), we see that \(p(1 - x_1 x_2 \bar{x}_1) \in qeCG\). Thus \(x_1 x_2\) and \(x_1 x_2 \bar{x}_1\) lie in \(G_V\). Hence, their product \(x_1 x_2\) lies in \(G_V\), and then \(x_1\) and \(x_2\) lie in \(G_V\). In particular, \(V[x_1, x_2] = V\).

We claim that \(x_{[3 \mid k]} \subseteq G_{\pm V}\). Arguing inductively, we suppose that \(j \in [3 \mid k]\) and \(x_{[1 \mid (j - 1)]} \subseteq G_{\pm V}\). For this step, we shall consider only the non-orientable case, \(w = [x_1, x_2] \prod x_{[3 \mid k]}^2\); the orientable case is similar, and was done in [12, Lemma 5.15]. Let

\[
u = [x_1, x_2] \prod x_{[3 \mid (j - 1)]}^2.\]

Then \(u \in G_V\), that is, \(p - pu \in qeCG\). Now \(\frac{\partial w}{\partial x_j} = u(1 + x_j)\), and, by (10), \(pu(1 + x_j) \in qeCG\). Summing these two elements of \(qeCG\), we see that \(p + pu x_j \in qeCG\). Hence \(ux_j \in G_{\pm V}\), and, hence, \(x_j \in G_{\pm V}\).

By induction, the claim is proved.

Hence \(G_{\pm V} = G\), and, hence \([G; G_V] \leq 2\), and, hence, \(G\) is virtually free. Thus \(\text{vcd} G \leq 1\), and, hence, \(b_2^{(2)}(G) = 0\).

Recall that, for any surface-plus-one-relation group \(G\), a formula for \(\chi(G)\) was given in Corollary 8.6.
9.6 Theorem. Let $G$ be a surface-plus-one-relation group. Then $G$ is of type VFL and, for each $n \in \mathbb{N}$,

$$b_n^{(2)}(G) = \begin{cases} 
\max\{\chi(G), 0\} = \frac{1}{|G|} & \text{if } n = 0, \\
\max\{-\chi(G), 0\} & \text{if } n = 1, \\
0 & \text{if } n \in \mathbb{N}.
\end{cases}$$

Proof. If $G$ is virtually one-relator, then, by Lemma 9.2, the desired conclusions hold. Thus, we may assume that $G$ is not virtually one-relator. It then follows from Remarks 8.5 that we may assume that $G$ has a presentation as in Lemma 9.5 and that $G$ is of type VFL. Then, by Lemmas 9.5 and 9.4(ii), $b_2^{(n)}(G) = 0$ for all $n \in \mathbb{N}$. By equation (6) of Review 9.1, $\chi(G) = -b_1^{(2)}(G)$, as desired. 

APPENDIX A

Howie towers via Bass-Serre Theory

In this appendix, we use Bass-Serre Theory to prove some of Howie’s results on local indicability.

We shall use [10] as our reference for Bass-Serre Theory.
Throughout, let $F$ be a group.

A.1 Actions on trees

A.1.1 Notation. Let $E$ be a subset of an $F$-set $X$.

Two elements of $E$ that lie in the same $F$-orbit in $X$ are said to be glued together by $F$, and we write $\text{glue}(F, E) := \{f \in F \mid fE \cap E \neq \emptyset\}$. 

A.1.2 Definitions. Let $T = (T, V, ET, \iota, \tau)$ be an $F$-tree.

(i). Let $r$ be an element of $F$ that fixes no vertex of $T$.

The smallest $\langle r \rangle$-subtree of $T$, denoted $\text{axis}(r)$, has the form of the real line, and $r$ acts on it by shifting it; see, for example, [10, Proposition I.4.11]. We write $E\text{axis}(r) := E(\text{axis}(r))$.

The $(F, ET)$-support of $r$ is defined as

$$\text{supp}(r) := \{Fe \in F \setminus ET \mid e \in E\text{axis}(r)\}.$$ 

For all $f \in F$, $\text{axis}(fr) = f\text{axis}(r)$ and $\text{supp}(fr) = \text{supp}(r)$.

For all $n \in \mathbb{N}$, $\text{axis}(r^n) = \text{axis}(r)$.

If $F$ acts freely on $ET$, then $r$ has a unique root in $F$. 

If \( \text{glue}(F, E\text{axis}(r)) = \langle r \rangle \), then \( r \) has a unique root in \( F \) and \( \sqrt[r]{r} = r \).

(ii). Let \( < \) be a (total) ordering of \( F \setminus ET \).

A subset \( R \) of \( F \) is said to be \((F, T, <)-\text{staggered}\) if each element of \( R \) fixes no vertex of \( T \), and, for each \((r_1, r_2) \in R \times R\), exactly one of the following three conditions holds:

\[
\text{min}(\text{supp}(r_1), <) < \text{min}(\text{supp}(r_2), <) \quad \text{and} \quad \text{max}(\text{supp}(r_1), <) < \text{max}(\text{supp}(r_2), <); \\
\text{min}(\text{supp}(r_2), <) < \text{min}(\text{supp}(r_1), <) \quad \text{and} \quad \text{max}(\text{supp}(r_2), <) < \text{max}(\text{supp}(r_1), <).
\]

When either of the latter two conditions holds, we say that \( r_1 \) and \( r_2 \) have \text{staggered supports} (with respect to \( < \)).

(iii). A subset \( R \) of \( F \) is said to be \((F, T)-\text{staggerable}\) if there exists an ordering \( < \) of \( F \setminus ET \) such that \( R \) is \((F, T, <)-\text{staggered}\).

A.1.3 Example. For a free product \( F = A \ast B \), the \text{Bass-Serre tree} is the \( F \)-graph \( T \) with vertex set the disjoint union of \( F/A \) and \( F/B \), and edge set \( F \), in which each \( f \in ET = F \) has initial vertex \( fA \) and terminal vertex \( fB \). It can be shown that \( T \) is a tree; see, for example, [10, Theorem I.7.6]. Notice that \( F \) acts \text{freely} on the edge set of \( T \).

Let \( r \in F \).

Clearly, \( r \) fixes some vertex of \( T \) if and only if \( r \) lies in some conjugate of \( A \) or some conjugate of \( B \).

If \( r \) fixes no vertex of \( T \), then there exists some \( n \in [1 \uparrow \infty[ \), and some sequence \( a_{\uparrow n} \) in \( A - \{1\} \), and some sequence \( b_{\uparrow n} \) in \( B - \{1\} \) such that some conjugate of \( r \) can be expressed in the form \( f = \prod_{i \in [1 \uparrow n]} (a_i b_i) \). The entire conjugacy class \( f \) can be represented by writing \( \prod_{i \in [1 \uparrow n]} (a_i b_i) \) cyclically.

Here, \( \text{supp}(r) = \{F\} = F \setminus ET \). If \( < \) denotes the unique ordering of \( F \setminus ET \), then \( \{r\} \) is \((F, T, <)-\text{staggered}\).

If \( r = \prod_{i \in [1 \uparrow n]} (a_i b_i) \), then \( \langle r \rangle \setminus E\text{axis}(r) = \{\langle r \rangle w \mid w \text{ is an initial subword of } r \} \) and there is a direct description of \( \sqrt[r]{r} \) and \( \log_F r \), as follows. We can write \( \log_F r = \frac{n}{m} \), where \( m \) is the smallest divisor of \( n \) with the property that, for each \( i \in [1 \uparrow (n - m)] \), \( a_i = a_{i+m} \) and \( b_i = b_{i+m} \). Here, \( \sqrt[r]{r} = \prod_{i \in [1 \uparrow m]} (a_i b_i) \).

\[
\text{A.2 Staggerability}
\]

In this section, we will show that if \( F \) is a locally indicable group, and \( T \) is an \( F \)-tree with trivial edge stabilizers, and \( R \) is an \((F, T)-\text{staggerable}\) subset of \( F \), (and, hence, \( R \) has unique roots in \( F \)), and \( \sqrt[r]{R} = R \), then \( F/\langle fR \rangle \) is locally indicable, and \( \langle fR \rangle \) acts freely on \( T \), or, equivalently, the natural map \( F \rightarrow F/\langle fR \rangle \) is injective on the vertex stabilizers. The following result deals with an extreme case.
A.2.1 Lemma. Let $F$ be a locally indicable group, let $T$ be an $F$-tree with trivial edge stabilizers, and let $R$ be an $(F, T)$-staggerable subset of $F$ such that $\sqrt{R} = R$.

Suppose that $F$ is finitely generated and that $F \setminus T$ is finite.

Then $F/\langle FR \rangle$ is indicable; if $F/\langle FR \rangle$ is trivial then $F$ acts freely on $T$ and, for each $r \in R$, glue($F, E\text{axis}(r)$) = $\langle r \rangle$.

Proof. We argue by induction on $|F \setminus ET|$. Since $F$ is a finitely generated, locally indicable group, $F$ is indicable, which means that the implications hold when $R$ is empty. Thus we may assume that $R$ is non-empty; it is then clear from Definitions A.1.2 that $|F \setminus ET| \geq 1$.

By induction, we may suppose that the implications hold for all smaller values of $|F \setminus ET|$. We may assume that $R = FR$, and we may assume that we are given an ordering $<$ of the finite set $F \setminus ET$ such that $R$ is $(F, T, <)$-staggered. In particular, there exists some $e_{\max} \in \bigcup_{r \in R} E\text{axis}(r)$ such that

$$F e_{\max} = \max(\{Fe \mid e \in \bigcup_{r \in R} E\text{axis}(r)\}, \langle r \rangle).$$

There then exists some $r_{\max} \in R$ such that $e_{\max} \in E\text{axis}(r_{\max})$, and, by the definition of $(F, T, <)$-staggered, $F e_{\max}$ does not meet the axis of any element of $R - FR(r_{\max})$. Thus there exists some pair $(r, e)$, for example, $(r_{\max}, e_{\max})$, such that the following hold.

11. $r \in R, e \in E\text{axis}(r)$, and $Fe$ does not meet the axis of any element of $R - Fr$.

12. glue($F, E\text{axis}(r)$) = $\langle r \rangle$ and/or $(r, e) = (r_{\max}, e_{\max})$.

In the forest $T - Fe$, let $T_i$ denote the component containing $te$, and let $T_r$ denote the component containing $re$. Let $F_i$ denote the $F$-stabilizer of $\{T_i\}$, and let $F_r$ denote the $F$-stabilizer of $\{T_r\}$. Let $R_i := R \cap F_i$ and $R_r := R \cap F_r$. By (11), for each $r' \in R - \{Fr\}$, axis($r'$) lies in $T - Fe$ and hence lies in a component of $T - Fe$. It follows that $FR_i \cup FR_r \cup Fr$ is all of $R$. Notice that if $F\{T_i\} = F\{T_r\}$ then $FR_i = FR_r$.

By applying the Bass-Serre Structure Theorem to the $F$-tree whose vertices are the components of $T - Fe$, and whose edge set is $Fe$, with $f$ joining $FT_i$ to $FT_r$, we see that

$$F = \begin{cases} F_iF_r & \text{if } F\{T_i\} \neq F\{T_r\}, \\ F_i\langle f \mid \rangle & \text{if } f \in F \text{ and } fT_i = T_r. \end{cases}$$

Hence,

$$F/\langle R - \{r\} \rangle = \begin{cases} (F_i/\langle R_i \rangle)(F_r/\langle R_r \rangle) & \text{if } F\{T_i\} \neq F\{T_r\}, \\ (F_i/\langle R_i \rangle)(f \mid) & \text{if } f \in F \text{ and } fT_i = T_r. \end{cases}$$

Consider the case where both $F_i/\langle R_i \rangle$ and $F_r/\langle R_r \rangle$ have infinite, cyclic quotients. By (14), $F/\langle R - \{r\} \rangle$ has a rank-two, free-abelian quotient. On incorporating $r$, we see that $F/\langle R \rangle$ has an infinite, cyclic quotient, and the desired conclusion holds.
Thus, it remains to consider the case where one of $F_i/\langle R_i \rangle$, $F_r/\langle R_r \rangle$ does not have an infinite, cyclic quotient; by replacing $e$ with $\tau$, if necessary, we may assume that $F_i/\langle R_i \rangle$ does not have an infinite, cyclic quotient.

By the induction hypothesis applied to $(F_i, T_i, R_i)$, we see that $F_i/\langle R_i \rangle$ is trivial, that $F_i$ acts freely on $T_i$, and, for each $r_i \in R_i$, glue($F_i, Eaxis(r_i)$) = $\langle r_i \rangle$, and, hence, glue($F, Eaxis(r_i)$) = $\langle r_i \rangle$.

By replacing $r$ with $\tau$ if necessary, we may assume the following.

(15) There exists a segment of axis($r$) of the form $e, p, re$.

If $F\{T_i\} \neq F\{T_r\}$, then, by (15), we have a path $\tau p$ in axis($r$) from $\tau \tau e \in \tau T_r \neq T_i$ to $ie \in T_i$. Now $\tau p$ necessarily enters $T_i$ through an edge of the form $g\bar{e}$ where $g \in F_i$ and $\bar{e}$ is the inverse of the edge $e$. Notice that $g \in$ glue($F_i, Eaxis(r)$) − $\langle r \rangle$. This proves the following.

(16) If glue($F_i, Eaxis(r)$) ⊆ $\langle r \rangle$ then $F\{T_i\} = F\{T_r\}$.

Consider the case where $R_i$ is empty. Here, $F_i = F_i/\langle R_i \rangle = \{1\}$. By (16), $F\{T_i\} = F\{T_r\}$, and, then, by (13), there exists some $t \in F$ such that $F = \langle t \mid \tau \rangle$. Now $\{t, t\} = \sqrt{F} - \{1\} \supseteq R = \{r\}$. Hence, $F = \langle r \rangle$, and, hence, glue($F, Eaxis(r)$) = $\langle r \rangle$. This proves the following.

(17) If glue($F, Eaxis(r)$) ≠ $\langle r \rangle$ then $R_i$ is non-empty.

**Case 1.** glue($F, Eaxis(r)$) = $\langle r \rangle$.

Here, by (16), $F\{T_i\} = F\{T_r\}$. Hence, $F(VT_i) = VT$ and $R = F_{R_i} \cup F_r$. Let $v \in VT$. We wish to show that $F_v = 1$, and, we may assume that $v \in VT_i$. Here $F_v \leq F_i$, and, since $F_i$ acts freely on $T_i$, $F_v = 1$, as desired. Thus $F$ acts freely on $T$. In (15), the path $p$ from $\tau e$ to $rue$ in axis($r$) does not meet $Fe$ since glue($F, Eaxis(r)$) = $\langle r \rangle$, and, hence, $p$ stays within $T_r$, and, hence $rue \in T_r$. Thus, $rT_i = T_r$, and, by (13), $F = F_i*\langle r \mid \tau \rangle$. Since $F_i/\langle R_i \rangle$ is trivial, we see that $F/\langle R \rangle$ is trivial, and, hence, $F/\langle R \rangle$ is indicable. Here all the required conclusions hold.

**Case 2.** glue($F, Eaxis(r)$) ≠ $\langle r \rangle$.

By (17), $R_i$ is non-empty, and, hence, there exists some $e_i \in \bigcup_{r_i \in R_i} Eaxis(r_i)$ such that $Fe_i = \min\{Fe \mid e \in \bigcup_{r_i \in R_i} Eaxis(r_i)\}$. $<$.

There then exists some $r_i \in R_i$ such that $e_i \in Eaxis(r_i)$, and we then know that glue($F, Eaxis(r_i)$) = $\langle r_i \rangle$. By (12), $(r, e) = (r_{max}, e_{max})$. Using the definition of $(F, T, <)$-staggered, one can show that $Fe_i$ does not meet the axis of any element of $F_r$, and, similarly,
$Fe_i$ does not meet the axis of any element of $F_r$ if $F_r \neq F_{r_i}$. It is clear that if $F_r \neq F_{r_i}$, then $Fe_i$ does not meet the axis of any element of $F_r$. Hence, $Fe_i$ does not meet the axis of any element of $R - F_{r_i}$. We then replace $(r,e)$ with $(r_i,e_i)$, and, by Case 1, all the required conclusions hold.

This completes the proof. \hfill \Box

We next deal with the case where $R$ is finite.

**A.2.2 Theorem.** Let $F$ be a locally indicable group, let $T$ be an $F$-tree with trivial edge stabilizers, and let $R$ be an $(F,T)$-staggerable subset of $F$ such that $\sqrt{R} = R$.

Suppose that $R$ is finite and that $H$ is a finitely generated subgroup of $F$ such that $H$ contains $R$, and $H/\langle H \rangle$ has no infinite, cyclic quotient.

Then there exists some finitely generated subgroup $F'$ of $F$ such that $F'$ contains $H$, $R$ is $(F',T)$-staggerable, $F'/\langle F' \rangle$ is trivial, $F'$ acts freely on $T$ and, for each $r \in R$, \( \text{glue}(F', E \text{axis}(r)) = \langle r \rangle \). Here, $H \leq F' = \langle F' \rangle \leq \langle F' \rangle$, and, hence, $H$ acts freely on $T$.

**Proof.** Let $v$ be an arbitrary vertex of $T$. Choose a finite generating set $S$ of $H$ such that $S$ contains $R \cup \{1\}$, and let $Y$ be the smallest subtree of $T$ containing $Sv$. Then, for each $s \in S$, $sV \cap VY$ is non-empty since it contains $sv$; thus

\[ S \subseteq \text{glue}(F, VY). \] 

If $F'$ is any subgroup of $F$ and $T'$ is any $F'$-subtree of $T$, for the purposes of this proof let us say that $(F',T')$ is an admissible pair if $F' \supseteq S$, and $T' \supseteq Y$, and $R$ is $(F',T')$-staggerable. By hypothesis, $(F,T)$ is admissible. Let $<$ be an ordering of $F \setminus ET$ such that $R$ is $(F,T,<)$-staggered.

**The Type 1 transformation.** Suppose that $F \neq \langle S \cup \text{glue}(F, EY) \rangle$ or $T \neq FY$. Define $F' := \langle S \cup \text{glue}(F, EY) \rangle$ and $T' := FY$. We shall prove that $(F',T')$ is an admissible pair and $\text{glue}(F, EY) = \text{glue}(F', EY)$.

Clearly

\[ \text{glue}(F, EY) = \text{glue}(F', EY). \]

It follows from (18) that

\[ S \cup \text{glue}(F', EY) \subseteq \text{glue}(F', Y). \]

If we consider the set of components of the $F'$-forest $F'Y$ in $T$ as an $F'$-set, we see that the component containing $Y$ is fixed by a generating set of $F'$, and hence is fixed by $F'$. This component must then be all of $F'Y$. Hence $T'$ is connected, and, hence, $T'$ is an $F'$-tree.
It is straightforward to show that
\[ (20) \quad \text{glue}(F, ET') = F'. \]

By (20), the natural map from $F\setminus ET'$ to $F\setminus ET$ is an embedding; we again denote by $<$ the ordering of $F\setminus ET'$ induced from $F\setminus ET$. We claim that, with the conjugation action by $F$ on $F$, glue($F, R$) $\subseteq F'$. Suppose that $f \in F$, that $r \in R$, and that $f^r \in R$. Then $\text{axis}(r)$ and $\text{axis}(< f r)$ lie in $T'$ and are glued together by $f$ since $\text{axis}(f r) = f \text{axis}(r)$. By (20), $f \in F'$, and the claim is proved. It follows that $R$ is $(F', T', <)-$staggered. Hence $(F', T')$ is admissible. This completes the verification of the Type 1 transformation.

**The Type 2 transformation.** Suppose that $F = \langle S \cup \text{glue}(F, EY) \rangle$, that $T = FY$, and that $F/\langle FR \rangle$ has some infinite, cyclic quotient $F/N$; here, $N$ is a normal subgroup of $F$ such that $N \geq \langle FR \rangle$. We shall prove that $(N, T)$ is admissible and glue($N, EY$) $\subset$ glue($F, EY$), where the notation denotes strict inclusion.

Since $H/\langle HR \rangle$ has no infinite, cyclic quotient, it follows that $H \subseteq N$.

Since $F/N$ is cyclic, we can choose $x \in F$ such that $xN$ generates $F/N$. Then $F = \langle x \rangle N$. Since $F/N$ is infinite, $\langle x \rangle \cap N = \{1\}$.

We now give $N\setminus ET$ an ordering. Give $(F\setminus ET) \times Z$ the lexicographic ordering and choose a left $F$-transversal $E_0$ in $ET$. Then $\langle x \rangle E_0$ is a left $N$-transversal in $ET$, and there is a bijective map

\[
(F\setminus ET) \times Z \rightarrow N\setminus ET, \quad (Fe, n) \mapsto N x^n e, (e, n) \in E_0 \times Z.
\]

Let $<_N$ denote the ordering induced on $N\setminus ET$ by this bijection.

We claim that $R$ is $(N, T, <_N)$-staggered. To see this, consider any $r_1$, $r_2 \in R$ such that $^N r_1 \neq ^N r_2$. If $^F r_1 = ^F r_2$, then $r_1$ and $r_2$ will have equal $(F, ET)$-supports, while their $(N, ET)$-supports, viewed in $(F\setminus ET) \times Z$, will differ by a non-zero shift in the second coordinate, and hence be staggered. If $^F r_1 \neq ^F r_2$, then $r_1$ and $r_2$ will have staggered $(F, ET)$-supports, and, hence, their $(N, ET)$-supports, viewed in $(F\setminus ET) \times Z$, will also be staggered.

Hence $(N, T)$ is admissible.

Since $N \not\supseteq F = \langle S \cup \text{glue}(F, EY) \rangle$, we see that $N \not\supseteq S \cup \text{glue}(F, EY)$. Since $N \supseteq S$, we see that $N \not\supseteq \text{glue}(F, EY)$, and, hence, glue($N, EY$) $\subset$ glue($F, EY$). This completes the verification of the Type 2 transformation.

Since $EY$ is finite and edge stabilizers are trivial, glue($F, EY$) is finite. Notice that transformations of Type 2 reduce glue($F, EY$), while transformations of Type 1 do not change glue($F, EY$). Notice that we cannot apply two transformations of Type 1 consecutively. Thus, after applying a finite number of transformations of Types 1 and 2, we arrive at a pair $(F', T')$ such that $F' = \langle S \cup \text{glue}(F', EY) \rangle$, $T' = F'Y$, $R$ is $(F', T')$-staggerable, and $F'/\langle FR \rangle$ has no infinite, cyclic quotient. By Lemma A.2.1, $F'/\langle FR \rangle$ is trivial, $F'$ acts
freely on $T'$, and, for each $r \in R$, glue($F', E\text{axis}(r)) = \langle r \rangle$. Since $F'$ acts freely on $T'$, it follows that $F'$ acts freely on all of $T$. Since $R$ is $(F', T')$-staggerable, it follows that $R$ is $(F', T)$-staggerable, because any ordering on $F'\setminus ET'$ can be extended to some ordering of $F'\setminus ET$, by the axiom of choice.

The finite descending chain of subgroups implicit in the above argument is the chain of subgroups considered by Howie in his tower arguments.

We now have a general result.

**A.2.3 Corollary.** Let $F$ be a locally indicable group, let $T$ be an $F$-tree with trivial edge stabilizers, and let $R$ be an $(F, T)$-staggerable subset of $F$.

Then $\langle F R \rangle$ acts freely on $T$, that is, each vertex stabilizer embeds in $F/\langle FR \rangle$ under the natural map.

If, moreover, $\sqrt[\mathbb{Q}]{R} = R$, then $F/\langle FR \rangle$ is locally indicable.

**Proof.** Recall from Definitions A.1.2 that $R$ has unique roots in $F$.

Since $\langle FR \rangle \leq \langle \sqrt[\mathbb{Q}]{FR} \rangle$, and $\sqrt[\mathbb{Q}]{R}$ is again $(F, T)$-staggerable, we may assume that $\sqrt[\mathbb{Q}]{R} = R$.

We first show that $\langle FR \rangle$ acts freely on $T$. Let $R'$ be an arbitrary finite subset of $FR$, and let $H = \langle R' \rangle$. On applying Theorem A.2.2, we see that $\langle R' \rangle$ acts freely on $T$. It then follows that all of $\langle FR \rangle$ acts freely on $T$.

We now show that $F/\langle FR \rangle$ is locally indicable. Consider an arbitrary finitely generated subgroup of $F/\langle FR \rangle$, and express it in the form $(\langle S \rangle \langle FR \rangle)/\langle FR \rangle$ where $S$ is a finite subset of $F$. It remains to show that $(\langle S \rangle \langle FR \rangle)/\langle FR \rangle$ is indicable. We may assume that $(\langle S \rangle \langle FR \rangle)/\langle FR \rangle$ has no infinite, cyclic quotient, and it remains to show that $\langle S \rangle \leq \langle FR \rangle$.

For the purposes of this proof, let $(S)'$ denote the derived subgroup of $\langle S \rangle$, and let $\langle S \rangle^{ab}$ denote the abelianization $\langle S \rangle/\langle S \rangle'$. Now

$$\frac{(\langle S \rangle \langle FR \rangle)/\langle S \rangle'(FR)}{(\langle S \rangle'(FR)}/\langle FR \rangle) = (\langle S \rangle \langle FR \rangle)/\langle FR \rangle)^{ab}$$

which is a finite abelian group by supposition. Let $d$ denotes its exponent. Then $S^d \subseteq \langle S \rangle'(FR)$ and, hence, there exists some finite subset $R_0$ of $FR$ such that $S^d$ lies in the set $\langle S \rangle'(R_0)$. Then, $(\langle S \cup R_0 \rangle/\langle R_0 \rangle)^{ab}$ is an abelian group of exponent at most $d$, and, hence, $\langle S \cup R_0 \rangle/\langle R_0 \rangle$ has no infinite, cyclic quotient, and, hence, by Theorem A.2.2, $\langle S \cup R_0 \rangle \leq \langle FR \rangle \leq \langle FR \rangle$, as desired.
A.3 Consequences

The main application is the following.

**A.3.1 Theorem (Howie).** Let $A$ and $B$ be locally indicable groups and let $r$ be an element of $A \ast B$ such that $r$ is not conjugate to any element of $(A \cup B) - \{1\}$. Then the natural maps from $A$ and $B$ to $(A \ast B)/\langle r \rangle$ are injective. If, moreover, $A^{\sqrt[\infty]{r}} = r$, then $(A \ast B)/\langle r \rangle$ is locally indicable.

*Proof.* Let $T$ be the Bass-Serre $(A \ast B)$-tree as in Example A.1.3; then $T$ is an $(A \ast B)$-tree with trivial edge stabilizers.

If $H$ is some finitely generated subgroup of $A \ast B$, then the Bass-Serre Structure Theorem for the $H$-action on $T$, or the Kurosh Subgroup Theorem, shows that $H$ is a free product of a family of finitely generated groups each of which is free, or isomorphic to a subgroup of $A$, or isomorphic to a subgroup of $B$. Hence all these free factors are indicable, and hence $H$ is indicable. Thus $A \ast B$ is locally indicable. Thus we may assume that $r \neq 1$; then $r$ is not conjugate to any element of $A \cup B$ and $\{r\}$ is $(A \ast B,T)$-staggerable.

By Corollary A.2.3, the natural maps from $A$ and $B$ to $(A \ast B)/\langle r \rangle$ are injective, and, if $A^{\sqrt[\infty]{r}} = r$, then $(A \ast B)/\langle r \rangle$ is locally indicable.

**A.3.2 Corollary (Magnus’ Freiheitssatz).** Let $F_1$ and $F_2$ be free groups. If $r$ is an element of $F_1 \ast F_2$ such that $r$ is not conjugate to any element of $F_1 - \{1\}$, then the natural map from $F_1$ to $(F_1 \ast F_2)/\langle r \rangle$ is injective.

**A.3.3 Corollary (Brodski˘ı).** Let $F$ be a free group. If $r \in F$ and $\xi r = r$, then $F/\langle r \rangle$ is locally indicable.

**A.3.4 Remarks.** The injectivity result in Theorem A.3.1 is called the *local indicability Freiheitssatz* since it generalizes Magnus’ Freiheitssatz, Corollary A.3.2.

The local indicability Freiheitssatz was proved independently by Brodski˘ı [5, Theorem 1], Howie [16, Theorem 4.3], and Short [25]. The proof by Brodski˘ı was algebraic, while the proofs by Howie and Short were topological, with Howie using topological towers and Short using diagrams. B. Baumslag [2] rediscovered Brodski˘ı’s algebraic proof.

The local indicability conclusion in Theorem A.3.1 was proved by Howie [17, Theorem 4.2(iii)⇒(i)], by topological-tower methods. The one-relator case, Corollary A.3.3, had been proved earlier by Brodski˘ı [5, Theorems 1 and 2]. Howie [18] later gave a direct proof of Brodski˘ı’s result using elementary groupoid methods, and his groupoid proof of the special case led us to the Bass-Serre proof of the general case.

**A.3.5 Corollary.** Let $G$ be a locally indicable group, let $F$ be a free group, and let $r$ be an element of $G \ast F$ that is not a proper power and that is not conjugate to any element of $G - \{1\}$. Then $(G \ast F)/\langle r \rangle$ is locally indicable.
Proof. Consider first the case where \( r \) is conjugate to an element of \( F \). By conjugating \( r \), we may assume that \( r \in F \). Here, \( F / \langle r \rangle \) is locally indicable by Corollary A.3.3. Now \( (G * F) / \langle r \rangle = G * (F / \langle r \rangle) \) is locally indicable by the degenerate case of Theorem A.3.1. Thus we may assume that \( r \) is not conjugate to any element of \( (G \cup F) - \{1\} \). Here, \( (G * F) / \langle r \rangle \) is locally indicable by Theorem A.3.1.

It is now straightforward to deduce the following special case of a result of Howie [17, Corollary 4.5] on ‘reducible presentations’.

A.3.6 Corollary (Howie). Let \( G_{[0,\infty)} \) be a family of groups such that \( G_0 = 1 \) and, for all \( n \in [0,\infty) \), \( G_{n+1} = (G_n * \langle X_{n+1} \rangle) / \langle r_{n+1} \rangle \) where \( X_{n+1} \) is a set, and \( r_{n+1} \) is an element of \( G_n * \langle X_{n+1} \rangle \) that is not a proper power and that is not conjugate to any element of \( G_n - \{1\} \). Then \( \bigcup G_{[0,\infty)} \) is locally indicable.

Acknowledgments

The research of the first- and second-named authors was jointly funded by the MEC (Spain) and the EFRD (EU) through Project MTM2006-13544.

We are grateful to Jim Howie and the conscientious referee for many excellent suggestions and corrections.

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