Causality and dimensionality in geometric scattering

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Abstract: The scattering matrix which describes low-energy, non-relativistic scattering of
spin-1/2 fermions interacting via finite-range potentials can be obtained from a geometric
action principle in which space and time do not appear explicitly [1]. In the case of zero-
range forces, causality leads to constraints on scattering trajectories in the geometric picture.
The effect of spatial dimensionality is also investigated by considering scattering in two and
three dimensions. In the geometric formulation it is found that dimensionality is encoded in
the phase of the harmonic potential that appears in the geometric action.

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1 Introduction

The scattering matrix (S-matrix) encodes observable consequences of the quantum mechanical interaction of two particles. While the S-matrix is typically obtained by solving an effective quantum field theory (EFT) —usually in perturbation theory— the end-product S-matrix is strictly a function of on-shell kinematical variables. Therefore, the effects of space and time, which are naturally embedded in any EFT calculation of the S-matrix, are entirely encoded in the kinematical variables. Motivated by the non-local correlations between the scattered particles due to quantum entanglement, a question one might ask is: can the S-matrix be obtained in a spacetime-independent formulation without direct reference to an EFT?\footnote{For recent efforts in this direction from the perspective of perturbative gauge theories, see Refs. \cite{2, 3}.}

Recent work \cite{1} by the authors has answered the above question in the affirmative by providing a geometric construction of the non-relativistic scattering of spin-1/2 fermions\footnote{This simple system, with the fermions identified as nucleons, has the advantage of being the low-energy EFT of the Standard Model at distances greater than the Compton wavelength of the pion, and has widespread and important applications in nuclear physics. See Ref. \cite{4, 5} for a review.}. In the geometric theory, the S-matrix is a trajectory in the space of all phase shifts allowed by unitarity, and is parameterized by a kinematical energy-momentum variable that is determined up to Galilean transformations. A critical observation in Ref. \cite{1} is that the S-matrix
allows a momentum inversion symmetry which is not manifest in the EFT of contact operators. This symmetry maps low- and high-energy scattering processes into each other and so is a UV/IR symmetry. This UV/IR symmetry leaves classes of observables invariant and is conformal in the sense that various combinations of phase shifts are left unchanged. In the geometric picture this corresponds to reflection symmetries of the $S$-matrix trajectories. The UV/IR symmetry allows the construction of exact solutions in the geometric theory, including specification of the harmonic forces which deform the geodesics in the geometric space. These exact solutions provide a forum for the exploration of spacetime constraints on the $S$-matrix in the geometric formulation of scattering.

The $S$-matrix evolves a state vector from the boundary of space in the infinite past, to the boundary of space in the infinite future, and must do so in a manner consistent with causality and with awareness of the number of spatial dimensions in which it is acting. Constraints due to causality on non-relativistic scattering have implications for the analytic structure of the $S$-matrix and, for systems arising from finite-range potentials, for the range of allowed values of the scattering parameters. These bounds, known as Wigner bounds, provide powerful constraints on the exact $S$-matrix solutions implied by conformal invariance. In the geometric theory it is found that these bounds manifest themselves as constraints on the tangent vectors of scattering trajectories. In addition, as quantum mechanics depends strongly on spatial dimensionality, the differences between scattering in two and three dimensions are explored in the geometric formulation. The resulting $S$-matrix in two spatial dimensions again has a solution implied by conformal invariance. Despite the strikingly different physics that it gives rise to, the form of the two-dimensional harmonic potential differs from its three-dimensional counterpart only by a change of coupling strength and phase.

This paper is organized as follows. Section 2 sets up the $S$-matrix framework, focusing on the properties of the most general $S$-matrix consistent with finite-range forces. The $S$-matrix is shown to allow conformal symmetries that are not manifest in the EFT action, and which provide powerful geometric constraints. In Section 3, the $S$-matrix of contact forces is shown to be the solution of a dynamical system which evolves the two-particle state in a two-dimensional space defined by the two phase shifts and bounded by unitarity. The conformal symmetries allow an exact determination of the forces that determine the $S$-matrix in this space. These first two sections expand on material first presented in Ref. [1]. Section 4 is new material which explores the manner in which spacetime features of scattering manifest themselves in the geometric theory of scattering. Constraints due to causality are considered, and the dependence on spatial dimensionality is found by varying between three and two dimensions. Finally, Section 5 summarizes and concludes.

\footnote{The inversion symmetry does manifest itself in the renormalization group (RG) evolution of the EFT couplings.}
2 S-matrix theory

2.1 S-matrix theory of contact forces

It is a simple matter to write down the $S$-matrix without reference to any underlying field theory by directly imposing general physical principles and symmetries. Consider two species of equal-mass, spin-1/2 fermions, which we label as neutrons and protons (i.e. nucleons), that interact at low energy via forces that are strictly of finite range. The spins of the two-body system can be either aligned or anti-aligned. Therefore, near threshold the $S$-matrix is dominated by the s-wave and can be written as [14]

$$
\hat{S}(p) = \frac{1}{2} \left( e^{i2\delta_1(p)} + e^{i2\delta_0(p)} \right) \hat{I} + \frac{1}{2} \left( e^{i2\delta_1(p)} - e^{i2\delta_0(p)} \right) \hat{P}_{12}
$$

where the SWAP operator is

$$
\hat{P}_{12} = \frac{1}{2} \left( \hat{1} + \hat{\sigma} \cdot \hat{\sigma} \right)
$$

and, in the direct-product space of the nucleon spins,

$$
\hat{1} \equiv \hat{I}_2 \otimes \hat{I}_2 \quad , \quad \hat{\sigma} \cdot \hat{\sigma} \equiv \sum_{\alpha=1}^{3} \hat{\sigma}^\alpha \otimes \hat{\sigma}^\alpha ,
$$

where $\hat{I}_2$ is the 2 $\times$ 2 unit matrix, and the $\hat{\sigma}^\alpha$ are the Pauli matrices. The $\delta$ s are s-wave phase shifts with $s = 0$ corresponding to the spin-singlet ($^1S_0$) channel and $s = 1$ corresponding to the spin-triplet ($^3S_1$) channel.

The SWAP operator takes an initial unentangled product state, say $|p \uparrow \rangle |n \downarrow \rangle$, and scatters it into the unentangled product state $|p \downarrow \rangle |n \uparrow \rangle$. In general, the $S$-matrix is an entangling operator as the basis which diagonalizes the interaction is different from the single-particle basis which describes the initial product state. Therefore, it is imperative to treat the $S$-matrix as the fundamental object of study, rather than the EFT action or the scattering amplitude, when addressing issues related to quantum entanglement.

The entangling character of the $S$-matrix is captured by its entanglement power (EP) [1, 14, 15]

$$
\mathcal{E}(\hat{S}) = \frac{1}{6} \sin^2(2(\delta_1 - \delta_0)) ,
$$

which is a state-independent measure of the entanglement generated by the $S$-matrix acting on an initial product state. Note that this object manifestly couples the two spin states in a manner that is quite distinct from the Lorentz-invariant, spin decoupled interactions that are encoded in the EFT action. Indeed, when it vanishes there is an enhanced $SU(4)$ spin-flavor symmetry (Wigner’s supermultiplet symmetry [16–18]) which explicitly relates the singlet and triplet scattering channels.
The two angular degrees of freedom, the phase shifts $\delta_{0,1}$, are characterized by the scattering lengths, $a_{0,1}$ and effective ranges $r_{0,1}$, near threshold via the effective range expansion

$$p \cot \delta_s(p) = -\frac{1}{a_s} + \frac{1}{2} r_s p^2 + O(p^4)$$

with $\vec{p}$ (with $p = |\vec{p}|$) chosen to be the center-of-mass (c.o.m.) momentum. The omitted $O(p^4)$ corrections are known as shape parameters. Near threshold, the $S$-matrix can be written as

$$\hat{S} = \frac{1}{2} (S_1 + S_0) \hat{1} + \frac{1}{2} (S_1 - S_0) \hat{P}_{12},$$

where the $S$-matrix elements are

$$S_s = e^{2i\delta_s(p)} = \frac{1 - ia_s(p)p}{1 + ia_s(p)p},$$

and a momentum-dependent scattering length is defined as

$$a_s(p) \equiv \frac{a_s}{1 - \frac{1}{2} a_s r_s p^2 + O(p^4)}.$$  

In terms of phase shifts

$$\phi \equiv 2\delta_0 = -2 \tan^{-1}(a_0(p)p), \quad \theta \equiv 2\delta_1 = -2 \tan^{-1}(a_1(p)p).$$

Here the phase shifts have been expressed in terms of the angular variables $\phi \in [0, 2\pi]$ and $\theta \in [0, 2\pi]$.

The $S$-matrix of Eq. (2.6) is specified by the two angular variables $\phi(p)$ and $\theta(p)$ that are determined by the Schrödinger equation once the finite-range quantum mechanical potential is specified. As these variables are periodic, the two-dimensional “phase space” that these variables define is a flat torus manifold, illustrated in Fig. (1). The range of values that $\phi(p)$ and $\theta(p)$ can take are bounded by unitarity, with boundary values determined by the four RG fixed points, which occur at $\hat{S} = \pm \hat{1}$ and $\pm \hat{P}_{12}$ when the $s$-wave scattering lengths are either vanishing or infinite (at unitarity) [1, 14]. Generally, in effective range theory, the $S$-matrix trajectory on the flat torus will originate at the trivial fixed point at scattering threshold and trace out a curve that exits the flat-torus and enters a bulk space at the first inelastic threshold [1]. In what follows, all inelasticities will be pushed to infinite momentum and $S$-matrix trajectories will begin and end at an RG fixed point.

### 2.2 UV/IR symmetries of the $S$-matrix

#### Out-state density matrix

In this section $S$-matrices with a momentum inversion symmetry that interchanges the IR and the UV will be studied. The $S$-matrix is defined as the operator which evolves the incoming (“in”) state before scattering into the outgoing (“out”) state after scattering, i.e. $\hat{S}|\text{in}\rangle = \dots$
Figure 1. The flat-torus manifold on which the phase shifts propagate. The blue dots correspond to the four fixed points of the RG. The dotted lines forming the diagonals of the square are axes of symmetry for $S$-matrix trajectories. The purple dashed curve is an example of an $S$-matrix trajectory in the scattering length approximation corresponding to the fourth row of Table 1 (with $a_1, a_0 > 0$ and $a_1/a_0 = 5$). The red curves are examples of $S$-matrix trajectories with range parameter given in the fifth row of Table 2. These trajectories leave $\phi + \theta$ invariant (with $a_1, a_0 < 0$ (bottom left quadrant) and with $a_1, a_0 > 0$ (top right quadrant), $a_0/a_1 = 15$ and $\lambda = 0.01$).

$\rho = |\text{out}\rangle \langle \text{out}| = \hat{S}|\text{in}\rangle \langle \text{in}|\hat{S}^\dagger$.

The first kind of symmetry transformation that will be considered is $\rho \rightarrow \rho$ which leaves all spin-observables invariant. In the following sections it will be shown that there are non-trivial instances of this symmetry where the $S$-matrix itself is not invariant. The second kind of symmetry transformation that will be considered is $\rho \rightarrow \tilde{\rho}$ where

$\tilde{\rho} \equiv \hat{S}^*|\text{in}\rangle \langle \text{in}|\hat{S}^T$.

This is the density matrix that would be produced if all phase shifts change sign, or, equivalently, if attractive interactions are replaced with repulsive interactions of equal magnitude, and vice-versa.

The scattering-length approximation

The $S$-matrix takes constant values at the fixed points of the RG, which implies that at both the trivial and unitary fixed points, the underlying EFT has non-relativistic conformal invariance (Schrödinger symmetry). The scattering trajectories are bridges which connect various conformal field theories. In addition to this conformal invariance of the EFT action
at the RG fixed points, there is a UV/IR symmetry which acts directly on the $S$-matrix and which is present at finite values of the scattering lengths.

Consider the momentum inversion transformation

$$ p \mapsto \frac{1}{|a_1a_0|p} . \quad (2.12) $$

As this maps $p = 0$ to $p = \infty$ it is a transformation which interchanges the IR and UV. It corresponds to the transformations on the phase shifts and density matrices shown in Table 1. This momentum inversion is a conformal invariance that leaves the combination of angular variables $\phi + \theta$ ($a_1a_0 < 0$) or $\phi - \theta$ ($a_1a_0 > 0$) invariant and implies a reflection symmetry of the $S$-matrix trajectory, as illustrated for a specific case in Fig. (1) (purple curve). When $a_1a_0 > 0$, the two phase shifts conspire to leave the density matrix unchanged despite neither phase shift separately being invariant. This demonstrates that, in multi-channel scattering, there exist symmetries which are not manifest at the level of the scattering amplitude and appear as reflections of $S$-matrix trajectories. It is notable that the EP is invariant with respect to the transformation of Eq. (2.12) [1].

| $\phi$ | $\theta$ | $\rho$ | $a_0$ | $a_1$ |
|-------|-------|-------|-------|-------|
| $-\pi + \theta$ | $\pi + \phi$ | $\bar{\rho}$ | $+$ | $-$ |
| $\pi + \theta$ | $-\pi + \phi$ | $\bar{\rho}$ | $-$ | $+$ |
| $\pi - \theta$ | $\pi - \phi$ | $\rho$ | $-$ | $-$ |
| $-\pi - \theta$ | $-\pi - \phi$ | $\rho$ | $+$ | $+$ |

Table 1. Action of the momentum inversion transformation in the scattering length approximation.

**Including range corrections**

The scattering length approximation is a part of a larger class of UV/IR symmetric $S$-matrix models which include range effects that are strictly correlated with the scattering lengths. These UV/IR symmetries have a distinct character as the range effects necessarily arise from derivative operators in the EFT. Consider the general momentum inversion

$$ p \mapsto \frac{1}{\lambda|a_1a_0|p} , \quad (2.13) $$

where the real parameter $\lambda > 0$. One can ask: what is the most general $S$-matrix for which this inversion symmetry gives rise to interesting symmetries? This transformation rules out all shape-parameter effects and correlates the effective ranges with the scattering lengths in a specific way. Table 2 gives the effective-range parameters for all $S$-matrix models with a symmetry under the momentum inversion of Eq. (2.13). Note that the first four rows correlate the singlet effective range with the triplet scattering length and vice-versa.

An interesting and well-known feature of the effective-range expansion in nucleon-nucleon (NN) scattering is the smallness of the shape parameter corrections (see, for instance, Ref. [19,
Table 2. Action of the momentum-inversion transformation on models that have a non-zero effective range. Here $\eta \equiv \lambda |a_0 a_1|$ and $\rho_\pm$ is the density matrix projected onto the total spin $\frac{1}{2} (1 \pm 1)$ subspace.

Vanishing shape corrections is a key signature of an $S$-matrix with momentum-inversion symmetry. Indeed, as the NN s-wave effective ranges are positive while the scattering lengths have opposite sign, the model given in the second row of Table 2, with $\lambda$ fitted to the data, provides a description of the low-energy s-wave phase shifts that improves upon the scattering-length approximation. As will be seen below, models that arise from zero-range forces with exact momentum-inversion symmetry and a positive effective range strictly violate causality. However, relaxing the zero-range condition can lead to interesting results for nuclear physics, as is considered in Ref. [21].

3 Geometric scattering theory

3.1 Metric on the flat-torus

As the space on which the two phase shifts, $\theta$ and $\phi$, propagate is a two-dimensional flat space, the line element should take the form $ds^2 \propto d\phi^2 + d\theta^2$. This metric can be obtained formally by parameterizing the $S$-matrix of Eq. (2.6) as

$$\hat{S} = \left[ x(p) + i y(p) \right] \mathbf{1} + \left[ z(p) + i w(p) \right] \mathbf{P}_{12},$$

with

$$x = \frac{1}{2} \left[ \cos(\phi) + \cos(\theta) \right], \quad y = \frac{1}{2} \left[ \sin(\phi) + \sin(\theta) \right],$$

$$z = \frac{1}{2} \left[ -\cos(\phi) + \cos(\theta) \right], \quad w = \frac{1}{2} \left[ -\sin(\phi) + \sin(\theta) \right].$$

Then, as an embedding in $\mathbb{R}^4$, with line element

$$ds^2 = dx^2 + dy^2 + dz^2 + dw^2,$$

one finds the flat two-dimensional Euclidean line element

$$ds^2 = \frac{1}{2} \left( d\phi^2 + d\theta^2 \right).$$

With $\phi$ and $\theta$ periodic, the corresponding metric describes the flat torus $\mathbb{T}^2 \sim S^1 \times S^1 \hookrightarrow \mathbb{R}^4$, where $S^1$ is the circle. From this line element, one can read off the flat-torus metric tensor $g_{ab}$, with $a, b = 1, 2$. 

| $\phi \mapsto$ | $\theta \mapsto$ | $\rho \mapsto$ | $r_0$ | $r_1$ |
|----------------|----------------|-------------|------|------|
| $\phi$         | $\theta$       | $\rho$      | $-2\eta/a_0$ | $-2\eta/a_1$ |
| $\phi$         | $-\theta$      | $\rho_+ + \rho_-$ | $-2\eta/a_0$ | $+2\eta/a_1$ |
| $-\phi$        | $\theta$       | $\rho_+ + \rho_-$ | $+2\eta/a_0$ | $-2\eta/a_1$ |
| $-\phi$        | $-\theta$      | $\rho$      | $-2\eta/a_1$ | $-2\eta/a_0$ |
| $\theta$       | $\phi$         | $\rho$      | $+2\eta/a_1$ | $+2\eta/a_0$ |
| $-\theta$      | $-\phi$        | $\rho$      | $+2\eta/a_1$ | $+2\eta/a_0$ |
3.2 Geometric action

The action for a general parameterization of a curve on a space with coordinates $X^1 = \phi$ and $X^2 = \theta$ and metric tensor $g_{ab}$ can be expressed as [22]

$$
\int L \left( X, \dot{X} \right) d\sigma = \int \left( N^{-2} g_{ab} \dot{X}^a \dot{X}^b - \nabla(X) \right) N d\sigma ,
$$

(3.5)

where $\sigma$ parameterizes the curve (affine or inaffine), $L$ is the Lagrangian, $\dot{X} \equiv dX/d\sigma$, and $\nabla(X)$ is an external geometric potential which is assumed to be a function of $X$ only. The corresponding Euler-Lagrange equations give the trajectory equations

$$
\ddot{X}^a + g^{abc} \dot{X}^b \dot{X}^c = \kappa(\sigma) \dot{X}^a - \frac{1}{2} N^2 g^{ab} \partial_b \nabla(X) ,
$$

(3.6)

where $g^{abc}$ are the Christoffel symbols for the metric $g_{ab}$, and

$$
\kappa(\sigma) \equiv \frac{\dot{N}}{N} = \frac{d}{d\sigma} \ln \frac{d\tau}{d\sigma} .
$$

(3.7)

Here $\kappa$ is the inaffinity [23], which vanishes when $\sigma = \tau$ with $\tau$ an affine parameter.

An interesting feature of the geometric construction of scattering is that the relative momentum that describes the motion of the center-of-mass is a non-affine parameter. For a constant geometric potential, the trajectory equations reduce to the geodesic equations which describe straight-line trajectories on the flat-torus. Any curvature indicates the presence of a non-constant geometric potential. Now, a priori, if a solution for $\phi$ and $\theta$ is specified, there are two equations of motion for three unknowns, the inaffinity and the two force components in the $\phi$ and $\theta$ directions. However, the presence of UV/IR symmetries can reduce the number of unknowns to two and thus allows an explicit construction of the geometric potential [1].

3.3 Solvable models

In the scattering length approximation, the UV/IR symmetry of Eq. (2.12) determines the geometric potential exactly. It is given by [1]

$$
\nabla(\phi, \theta) = \frac{|a_0 a_1|}{(|a_0| + |a_1|)^2} c_1^2 \tan^2 \left( \frac{1}{2} (\phi + \epsilon \theta) \right) ,
$$

(3.8)

where $\epsilon = -1$ for $a_1 a_0 > 0$ and $\epsilon = +1$ for $a_1 a_0 < 0$, and $c_1$ is an integration constant. The inaffinity associated with a trajectory parameterized by the c.o.m momentum is constructed from

$$
N = \frac{c_1}{p} (\sin \phi - \epsilon \sin \theta) .
$$

(3.9)

The $S$-matrix trajectory is independent of the parameterization and can be simply described—with vanishing inaffinity and choice $N = c_1 = 1$—by an affine parameter $\tau$ via the simple Lagrangian

$$
L = \frac{1}{2} \left( \dot{\phi} + \dot{\theta} \right) - \nabla(\phi, \theta) ,
$$

(3.10)
where the dots denote differentiation with respect to \( \tau \). Of course \( \tau \) has no interpretation as a momentum or energy in a scattering process; such a parameter is not affine.

The conformal \( S \)-matrix models with the UV/IR symmetry of Eq. (2.13), and \( \lambda \)-dependent effective ranges, also lead to solvable geometric potentials. The general solution is cumbersome, however in the special case where the effective ranges are correlated to the scattering lengths as in the last two rows of Table 2 —and \( \lambda = 1/4 \)— the geometric potential is identical to Eq. (3.8) except for an overall factor of \( 1/2 \) and a rescaled argument

\[
\frac{1}{2}(\phi + \epsilon \theta) \rightarrow \frac{1}{4}(\phi + \epsilon \theta).
\] (3.11)

It will be seen in section 4.2 why this case is special.

4 Spacetime constraints

4.1 Galilean invariance

The parameter \( p \), which labels the c.o.m. momentum of the scattering process, is related to the total energy, \( E \), in the system by \( p = \sqrt{ME} \). If the incoming particle momenta are \( \vec{p}_1 \) and \( \vec{p}_2 \) then in the c.o.m. frame, \( \vec{p}_1 = -\vec{p}_2 \equiv \vec{p} \) and \( p = |\vec{p}| \). Other Galilean frames can be reached from the c.o.m. frame via a combined rotation \( \mathcal{R} \) and boost by a velocity \( \vec{v} \):

\[
\vec{p} \rightarrow \mathcal{R} \vec{p} + M \vec{v} \tag{4.1}
\]

which implies the transformation

\[
p \rightarrow |\vec{p}| \sqrt{1 + \frac{\vec{v}^2}{\vec{p}^2}}, \tag{4.2}
\]

with \( x \equiv (M \vec{v})^2/\vec{p}^2 \). Varying \( x \) between 0 and 1 corresponds to transforming between the c.o.m. and laboratory frames. Hence, Galilean invariance allows arbitrary reparameterizations of the \( S \)-matrix of the form \( p \rightarrow \Omega p \) with \( 1 \leq \Omega \leq \infty \) and \( \Omega \) interpolating between the c.o.m at rest and boosted to infinite momentum. As this is just a rescaling of \( p \), changing to another inertial frame does not affect the inaffinity, Eq. (3.7).

4.2 Causality bounds on zero-range scattering

Wigner bounds

In non-relativistic scattering with finite-range forces, causality places bounds on physical scattering parameters by way of Wigner bounds [10–13]. Consider a two-body s-wave wave function both free and in the presence of an interaction potential of range \( R \). The difference in phase between the scattered and free spherical wave is defined to be twice the phase shift. The most negative phase shift is obtained when the scattered wave does not penetrate the potential and reflects off the boundary at \( r = R \). This provides a lower bound on the phase shift \( \delta(p) \geq -Rp \), see Fig. (2). Now consider a plane wave at an infinitesimally
Figure 2. An illustration of the minimum phase shift for scattering off a potential of range $R$. The red (blue) curve represents the phase of the incoming (outgoing) $s$-wave spherical wave.

larger momentum, $\bar{p}$ with $\delta(\bar{p}) \geq -R\bar{p}$. The difference between $\delta(\bar{p})$ and $\delta(p)$ provides a semi-classical bound on the derivative of the phase shift with respect to momentum

$$ \frac{d\delta}{dp} \geq -R. \quad (4.3) $$

By time evolving the plane waves, the above becomes a bound on the time delay between the incident and scattered wave, $\Delta t \geq -MR/p$. It is in this sense that causality constrains non-relativistic scattering. A more careful derivation, which includes quantum mechanical effects, induces a second term on the right hand side of Eq. (4.3) and leads to the bound [10]:

$$ \frac{d\delta}{dp} \geq -R + \frac{\sin (2\delta + 2pR)}{2p}. \quad (4.4) $$

Evaluated at threshold this becomes a constraint on the effective range parameter:

$$ r \leq 2 \left[ R - \frac{R^2}{a} + \frac{R^3}{3a^2} \right]. \quad (4.5) $$

In the Wilsonian EFT paradigm, an $S$-matrix element derived from EFT is dependent on a momentum cutoff, $\Lambda \sim 1/R$, which is kept finite and varied to ensure cutoff-independence to a given order in the perturbative EFT expansion. What occurs above this scale is irrelevant to the infrared physics that is encoded by the $S$-matrix and compared to experiment. An explicit calculation of the Wigner bound in the EFT of contact operators with cutoff regularization can be found in Ref. [11]. As the bound depends explicitly on the EFT cutoff, its relevance in physical scenarios is somewhat ambiguous as the EFT can violate causality bounds as long as the violations occur above the cutoff, and the bound itself weakens as higher-order corrections in the EFT expansion are included [24].
The $S$-matrix models with momentum-inversion symmetry can originate from zero-range or finite-range forces. Here it will be assumed that the underlying theory has strictly zero-range forces. This then implies strong causality bounds whose geometric interpretation can be studied. Explicitly, with zero-range forces, causality requires $r_s \leq 0$ and the tangent vectors to $S$-matrix trajectories satisfy

$$
\dot{\phi}(p) \geq \frac{\sin \phi(p)}{p}, \quad \dot{\theta}(p) \geq \frac{\sin \theta(p)}{p},
$$

where dot represents differentiation with respect to momenta. The allowed tangent vectors clearly depend on the quadrant of the flat torus in which they lie. In addition, by enforcing continuity of the tangent vectors at the boundary of each quadrant, it is found that an $S$-matrix trajectory can only exit a quadrant through the upper or right edge. These various geometric constraints are illustrated in Fig. (3) using the examples of Fig. (1). It is notable that the large momentum behavior of any $S$-matrix curve which ends at the trivial fixed point must be in the top-right quadrant, which is also the only place where a trajectory can have loops. Since the Wigner bound segregates by quadrant and indicates a direction of preferred $S$-matrix evolution, causality introduces an asymmetry which breaks the homogeneity and discrete isotropy of a generic flat torus. Applying the Wigner bound to the symmetric $S$-matrix models in Table 2 restricts the allowed signs of the scattering lengths as shown in Table 3.

### Causal singularities of the $S$-matrix

In addition to Wigner bounds, causality in non-relativistic scattering is manifest in various constraints on the analytic structure of the $S$-matrix in the complex-momentum plane [7–9]. The simplicity of the $S$-matrix models with momentum-inversion symmetry reveal these constraints and their relation with the Wigner bound in straightforward fashion. The $s$-wave $S$-matrix elements with momentum inversion symmetry are ratios of polynomials of second degree and can thus be expressed as

$$
S_s \equiv \frac{(p + p_s^{(1)})(p + p_s^{(2)})}{(p - p_s^{(1)})(p - p_s^{(2)})},
$$

### Table 3. Action of the momentum-inversion transformation on causal models that have a non-zero effective range.

| $\phi \mapsto$ | $\theta \mapsto$ | $\rho \mapsto$ | $r_0$ | $r_1$ |
|----------------|----------------|----------------|-------|-------|
| $\phi$        | $\theta$      | $\rho$        | $-2a_1 \lambda$ $(a_1 > 0)$ | $-2a_0 \lambda$ $(a_0 > 0)$ |
| $\phi$        | $-\theta$     | $\rho_- + \bar{\rho}_+$ | $+2a_1 \lambda$ $(a_1 < 0)$ | $-2a_0 \lambda$ $(a_0 > 0)$ |
| $-\phi$       | $\theta$      | $\rho_+ + \bar{\rho}_-$ | $-2a_1 \lambda$ $(a_1 > 0)$ | $+2a_0 \lambda$ $(a_0 < 0)$ |
| $-\phi$       | $-\theta$     | $\bar{\rho}$  | $+2a_1 \lambda$ $(a_1 < 0)$ | $+2a_0 \lambda$ $(a_0 < 0)$ |
| $\theta$      | $\phi$        | $\bar{\rho}$  | $-2a_0 \lambda$ $(a_0 > 0)$ | $-2a_1 \lambda$ $(a_1 > 0)$ |
| $-\theta$     | $-\phi$       | $\rho$        | $+2a_0 \lambda$ $(a_0 < 0)$ | $+2a_1 \lambda$ $(a_1 < 0)$ |
Figure 3. Left panel: the range of tangent vectors on the flat torus allowed by the Wigner bound is superposed on Fig. (1). The purple trajectory in the bottom left quadrant has zero effective range and is seen to be consistent with the tangent-vector conditions. The red trajectory in the bottom left quadrant has positive effective ranges and is seen to violate the tangent-vector conditions, while the trajectory in the top right quadrant has negative effective ranges and is consistent with the conditions. Right panel: by matching the allowed tangent vectors at the boundaries of each quadrant it is found that $S$-matrix trajectories can only exit a quadrant via the upper or right edge.

where

$$p_s^{(1,2)} = \frac{1}{r_s} \left( i \pm \sqrt{\frac{2r_s}{a_s} - 1} \right).$$

Consider the evolution of the singularities in the complex-$p$ plane as $\lambda$ is varied [25] for the causal model given in the last row of Table 3. This model, with both scattering lengths negative, leaves $\phi - \theta$ invariant, and has poles at

$$p_s^{(1,2)} = -\frac{1}{2|a_s|\lambda} \left( i \pm \sqrt{4\lambda - 1} \right).$$

There are three distinct cases, illustrated in Fig. (4).

$\lambda > 1/4$: there are two resonance poles in the lower-half complex plane on opposite sides of the imaginary axis. Dropping the partial-wave subscript on the scattering length,

$$p^{(1)} = -p_R - ip_I, \quad p^{(2)} = p_R - ip_I,$$

with

$$p_R = \frac{1}{2|a|\lambda} \sqrt{4\lambda - 1}, \quad p_I = \frac{1}{2|a|\lambda};$$

$\lambda = 1/4$: there is a double pole corresponding to a virtual state on the negative imaginary axis at

$$p^{(1)} = p^{(2)} = -\frac{i}{2|a|\lambda};$$

$\lambda < 1/4$: there are no poles in the lower-half complex plane, and the corresponding $S$-matrix trajectories are bounded above and below by the Wigner bound and the top boundary of the Wigner cone respectively.
\[ \lambda < 1/4: \] there are two poles corresponding to virtual states on the negative imaginary axis at
\[ p^{(1)} = -ip_-, \quad p^{(2)} = -ip_+, \quad \text{ (4.13)} \]
with
\[ p_\pm = \frac{1}{2|a_s|\lambda} \left( 1 \pm \sqrt{1 - 4\lambda} \right). \quad \text{ (4.14)} \]

It is clear that the Wigner bound implies that the poles of the S-matrix elements lie in the lower-half of the complex-momentum plane as one would expect of states that decay with time. Note that the special case of the double pole (and vanishing square root) is in correspondence with the special case geometric potential which takes the simple form, Eq. (3.11).

### 4.3 Spatial dependence of scattering

The S-matrix is clearly aware of the number of spatial dimensions that it is acting in and one therefore expects that this is reflected in the geometric theory of scattering via a modified geometric potential. It is straightforward to carry out the analysis that was done above in two spatial dimensions. In two dimensions, low energy scattering arising from short-range forces is enhanced in the IR due to an apparent scaling symmetry of the Schrödinger equation. One consequence of this is that there is only a single fixed point S-matrix, the identity, which is reached at both zero and infinite scattering length. Spin and particle statistics are also distinct in two dimensions, however, for our purposes, all that will be required is a scattering process with two independent low energy channels. One way this could be achieved is by placing the three-dimensional scattering system in a strongly anisotropic harmonic potential which effectively confines one of the spatial dimensions [26–30]. This allows the two-fermion system to be continuously deformed from three to two dimensions and provides a means of studying

**Figure 4.** Singularities of S-matrix elements in the complex-p plane as \( \lambda \) is varied. The arrows indicate direction of decreasing \( \lambda \).
the dependence of the geometric theory, constructed above, on spatial dimensionality. A qualitatively equivalent and simpler way of achieving this reduction of dimensionality is by periodically identifying and compactifying one of the spatial dimensions [31, 32], say in the z-direction.

Regardless of how the two dimensional system is obtained, the ERE is [33–35]

$$\cot \delta_s(p) = \frac{1}{\pi} \log(a_s^2 p^2) + \sigma_{2,s} p^2 + \mathcal{O}(p^4)$$  \hspace{1cm} (4.15)

where the $a_s$ and $\sigma_{2,s}$ are the two-dimensional scattering lengths and areas, respectively. The full S-matrix can be constructed from the phase shifts as in Eq. (2.1). Retaining just the first term in the ERE gives rise to the scattering length approximation which, in terms of periodic variables on the flat torus, is

$$\phi = 2 \cot^{-1} \left( \frac{1}{\pi} \log(a_0^2 p^2) \right) \quad , \quad \theta = 2 \cot^{-1} \left( \frac{1}{\pi} \log(a_1^2 p^2) \right) \hspace{1cm} (4.16)$$

where the higher order effective area and shape parameters have been set to zero$^4$. Note that there is an IR enhancement in two dimensions, as made evident by the logarithmic dependence on the c.o.m. momenta, and that there exists a bound state for either sign of coupling strength [6, 34, 35].

The momentum inversion transformation takes a similar form as in three dimensions, $p \rightarrow (a_1 a_0)^{-1} p$, and the phase shifts transform as

$$\phi(p) \mapsto -\theta(p) \quad , \quad \theta(p) \mapsto -\phi(p) \hspace{1cm} (4.17)$$

which leaves the density matrix invariant. In two spatial dimensions, the momentum-inversion symmetry implies that all effective area and shape parameters must vanish$^5$. The geometric potential on the flat torus which reproduces the phase shifts of Eq. (4.16) is found to be

$$\mathcal{V}(\phi, \theta) = -\frac{\pi^2}{4 (\log(a_0/a_1))^2} \tan^2 \left( \frac{1}{2} (\phi + \theta) + \frac{\pi}{2} \right) \hspace{1cm} (4.19)$$

Notice that the harmonic dependence is the same as in three spatial dimensions, Eq. (3.8), except for an additional phase of $\pi/2$ which causes the geometric potential to diverge when both phase shifts sum to zero. This can only occur at threshold, and can be attributed to the infinite force needed to reproduce the singular behavior of the phase shift derivatives at $p = 0$.

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$^4$This can be obtained from the compactification of a spatial dimension if the $d = 3$ effective range parameters are functions of the compactification radius [32].

$^5$In $d = 2$, causality bounds the effective area parameter [12, 13]

$$\sigma_{2,s} \leq \frac{R^2}{\pi} \left\{ \left[ \log \left( \frac{R}{2 a_s} \right) + \frac{\gamma - \frac{1}{2}}{2} \right]^2 + \frac{1}{4} \right\} \hspace{1cm} (4.18)$$

where $\gamma$ is the Euler-Mascheroni constant. Therefore, momentum-inversion implies that the Wigner bound is saturated with $\sigma_{2,s} = 0$. 

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Another property of the geometric potential is the divergence of the prefactor when \( a_0 = a_1 \). At the end of section (3.2) it was pointed out that, for UV/IR symmetric trajectories, there are two trajectory equations for two unknowns, the inaffinity and the geometric potential. However, when the scattering lengths are equal, and \( \phi = \theta \), the two equations are no longer linearly independent. In this case the trajectory is a geodesic —a straight line— on the flat torus, and no geometric potential is needed.

5 Conclusion

The \( S \)-matrix is a unitary operator that evolves a state vector from the boundary of spacetime, into the spacetime bulk to experience interaction, and then back to the spacetime boundary. In this view of scattering, all spacetime features like causality and spatial dimensionality are bulk properties, and, as the \( S \)-matrix is purely a function of kinematical variables like momentum and energy, the bulk properties must be imprinted in some way on these variables. In a general scattering process, the \( S \)-matrix evolves an initial unentangled product state into an entangled state which in general experiences non-local correlations. In order to avoid the assumption of locality, which is intrinsic to the EFT paradigm, Ref. [1] formulated a geometric theory of scattering for two species of spin-1/2 fermions interacting at low-energies via finite-range interactions. In this geometric theory the \( S \)-matrix emerges, without direct reference to spacetime, as a trajectory in an abstract space that is defined by unitarity. These \( S \)-matrix trajectories are generated by an entangling harmonic force whose form is —in certain special cases— determined exactly by a UV/IR symmetry. This paper has considered the manner in which causality and spatial dimensionality manifest themselves in the geometric picture.

The main results of this paper are:

• In \( S \)-matrix models with momentum-inversion symmetry which arise from zero-range forces, Wigner bounds, which enforce causality, severely restrict the model space by requiring that effective ranges be strictly negative. These causality bounds place restrictions on the tangent vectors of \( S \)-matrix trajectories which propagate on the flat-torus in the geometric theory of scattering. This preferred direction in the geometric space is the manifestation of spacetime causality.

• The geometric potential which determines the \( S \)-matrix in solvable models with UV/IR symmetry takes the same functional form in two and three spatial dimensions. The dependence on spatial dimensionality is manifest both in the coupling strength and phase of the geometric potential. This suggests that requiring unitarity and an ERE ensures that there are aspects of low-energy scattering, captured by the geometric formulation, that are universal and independent of spatial geometry. This could be anticipated given that the geometric potential leads to an entangling force, and the quantum correlations arising from spin entanglement are a property of an internal Hilbert space.
bullet The density matrix of the “out” state has been found to be the relevant object for identifying symmetries of a multi-channel scattering processes. All UV/IR symmetries in low-energy non-relativistic scattering with finite-range forces generated by momentum inversion that leave the density matrix invariant have been categorized and can be found in Tables 1 and 2.

This work suggests many related investigations; among them are: implications of the UV/IR symmetries of the S-matrix for the construction of EFTs of s-wave NN scattering (this is considered in Ref. [21]), as well as for EFT descriptions of few- and many-body systems of nucleons. Finally, the EPs of the \( \pi\pi \) and \( \pi N \) S-matrices were recently considered in Ref. [36] and it would be of interest to explore whether these scattering systems possess analogous geometric constructions.

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