Do products of compact complex manifolds admit LCK metrics?

Liviu Ornea\textsuperscript{1,2}  |  Misha Verbitsky\textsuperscript{3,4}  |  Victor Vuletescu\textsuperscript{1}

Abstract
An locally conformally Kähler (LCK) manifold is a Hermitian manifold which admits a Kähler cover with deck group acting by holomorphic homotheties with respect to the Kähler metric. The product of two LCK manifolds does not have a natural product LCK structure. It is conjectured that a product of two compact complex manifolds is never LCK. We classify all known examples of compact LCK manifolds onto three not exclusive classes: LCK with potential, a class of manifolds we call of Inoue type, and those containing a rational curve. In the present paper, we prove that a product of an LCK manifold and an LCK manifold belonging to one of these three classes does not admit an LCK structure.

MSC 2020
32J27, 53C55 (primary)

Contents
1. INTRODUCTION .................................... 757
2. PRELIMINARIES .................................... 758
   2.1. Definitions and basic results ...................... 758
   2.2. LCK manifolds with potential .................... 759
   2.3. Manifolds of Inoue type .......................... 760
   2.4. Manifolds with rational curves ................... 762
3. PRODUCTS WITH ONE FACTOR LCK WITH POTENTIAL ARE NOT LCK ...... 763
1 | INTRODUCTION

A locally conformally Kähler (LCK) manifold is a Hermitian manifold $(M, I, g)$ which admits a Kähler cover such that the deck group acts by holomorphic homotheties with respect to the Kähler metric. Equivalently, the fundamental form $\omega(x, y) := g(Ix, y)$, $x, y \in TM$, satisfies the integrability condition $d\omega = \theta \wedge \omega$ for some closed 1-form $\theta$, called the Lee form.

Typical examples are the Hopf manifolds, the Kodaira surfaces, the Kato manifolds. We refer to the book [17] for details about LCK geometry. In Section 2 of this paper, we present the necessary background of LCK geometry (Subsection 2.1), and we describe all the known classes of examples of LCK manifolds (Subsections 2.2–2.4).

One can easily see that if $(M, I, g)$ is LCK with Lee form $\theta$, then $(M, I, e^{f}g)$ is LCK with Lee form $\theta + df$. As such, LCK geometry is a part of conformal geometry. Since a product of conformal groups is not a conformal group ($CO(n) \times CO(m) \not\subset CO(n + m)$), the product of two LCK manifolds does not have a natural product LCK structure.

Note that assuming from the beginning that the factors of the product are LCK is not restrictive. Indeed, if the product admits an LCK structure, then both factors have induced LCK structures, because a complex submanifold of an LCK manifold is again LCK.

The question then arises whether the product of two LCK manifolds does admit any LCK structure compatible with the product complex structure. Even more generally (see [17, Question 47.6]), does the product of two complex manifolds admit any LCK structure, not necessarily compatible with the product complex structure? Only partial answers to this question were given up to now. The following products are known to not bear strict LCK metrics: the product of two compact Vaisman manifolds ([22, Corollary 3.3]), the product of a compact Kähler manifold of dimension at least 2 and a compact strict LCK manifold ([12, Corollary 2]), the product of a compact complex smooth curve with a compact manifold carrying no LCK metric with potential ([7, Proposition 7]).

We observe that all known examples of compact LCK manifolds fall in one of the following three classes: (1) LCK manifolds with potential, (2) LCK manifolds that we call of Inoue type, a class containing all LCK Inoue surfaces and the LCK Oeljeklaus–Toma manifolds and (3) LCK manifolds containing rational curves. We define these classes and provide appropriate examples in Subsection 2.2, Subsection 2.3 and Subsection 2.4, respectively. Note that these three classes are not mutually exclusive, see Remark 2.28.

The goal of this paper is to show that if $X$ is a compact LCK manifold in any of these classes above and $Y$ is an arbitrary compact complex manifold, then the product $M := X \times Y$ does not carry any LCK metric (see Theorems 3.3, 4.1 and 5.1).

Note that if we admit the Global Spherical Shell conjecture (GSS conjecture) (Subsection 2.4), then the three classes above cover all LCK surfaces. It follows that if the GSS conjecture is true, then a product of a compact LCK surface and any compact complex manifold is not of LCK type.
2  |  PRELIMINARIES

In this section, we gather the very basic definitions and results in locally conformally Kähler geometry. We refer to the recent monograph [17].

2.1  |  Definitions and basic results

Let \((M, I, g, \omega)\) be a Hermitian manifold, \(\dim \mathbb{C} M \geq 2\). Here, \(\omega(\cdot, \cdot) = g(I \cdot, \cdot)\).

**Definition 2.1.** The Hermitian manifold \((M, I, g, \omega)\) is LCK if there exists a closed 1-form \(\theta\) such that \(d\omega = \theta \wedge \omega\). The 1-form \(\theta\) is called the Lee form and the \(g\)-dual vector field \(\theta^\sharp\) is called the Lee field.

If the Lee form is exact, \((M, I, g, \omega)\) is called globally conformally Kähler (GCK). An LCK structure which is not GCK will be called strict.

**Definition 2.2.** More generally, if \((M, I)\) is a complex manifold, a weak lck structure (WLCK) on \(M\) is a \((1,1)\)-form \(\omega\) which obeys a relation \(d\omega = \theta \wedge \omega\) for some closed 1-form \(\theta\) and such that \(h(\cdot, \cdot) := \omega(I \cdot, \cdot)\) is positive definite outside a proper analytic subspace \(\mathcal{B}(\omega)\) (called also ‘the bad locus’ of \(\omega\)). A globally WLCK (GWLCK) structure is one such that \(\theta\) is exact.

**Theorem 2.3** [24, Remark, p. 236; 17, Section 3.4.2]. The Hermitian manifold \((M, I, g, \omega)\) is LCK if and only if it admits a Kähler cover \(\Gamma \longrightarrow (\tilde{M}, \tilde{\omega}) \longrightarrow M\) such that the deck group \(\Gamma\) acts by holomorphic homotheties.

It follows that for any LCK manifold \(M\), one can define a homothety character \(\chi : \Gamma \longrightarrow \mathbb{R}^{>0}\) associating the scale factor \(\frac{\pi_1(M)}{\omega}\) to each deck transform \(\gamma\). The homothety character is a representation of \(\pi_1(M)\) and corresponds to the class of the Lee form \([\theta]\) under the canonical isomorphisms \(H^1(M, \mathbb{R}) \cong \text{Hom} \mathbb{Z}(H_1(M), \mathbb{R}) \cong \text{Hom} \mathbb{Z}(\pi_1(M)[\pi_1(M)], \mathbb{R})\).

**Example 2.4.** The classical Hopf manifold is a quotient of \(\mathbb{C}^n \setminus 0\) by the \(\mathbb{Z}\)-action generated by a linear map \(z \mapsto \lambda z\), where \(\lambda \in \mathbb{C}, |\lambda| > 1\). This \(\mathbb{Z}\)-action multiplies the standard flat Kähler form \(-\sqrt{-1} \sum dz_i \wedge d\bar{z}_i\) by \(|\lambda|^2\), and hence defines an LCK form \(-\sqrt{-1} \sum \frac{dz_i \wedge d\bar{z}_i}{|z|^2}\) on the quotient \(\mathbb{C}^n \setminus 0\).

**Remark 2.5.** The LCK condition is conformally invariant: if \((M, I, g, \omega)\) is LCK with Lee form \(\theta\) and \(f : M \rightarrow \mathbb{R}\) is smooth, then \((M, I, e^f g, e^f \omega)\) is LCK with Lee form \(\theta + df\).

**Remark 2.6.** Let \(\iota : N \rightarrow M\) be a submanifold of an LCK manifold \((M, I, g, \omega, \theta)\). Then, \(N\) is LCK with induced complex structure and metric, and with Lee form \(\iota^* \theta\). If \(\iota^* \theta\) is exact, the submanifold is called induced globally conformally Kähler (IGCK). We stress that not all Kähler submanifolds of an LCK manifold are IGCK.

**Example 2.7.** Let \(M\) be a classical Hopf manifold, Example 2.4, and \(E = \frac{\mathbb{C} \setminus 0}{\mathbb{Z}} \subset M\) an elliptic curve obtained from a complex line in \(\mathbb{C}^n \setminus 0\). Clearly, \(E\) is Kähler, but the Lee form \(\theta = -d \log |z|\) is clearly not exact on \(E\); hence, \(E\) is not IGCK. On the other hand, a blow-up of a point in an
LCK manifold is again LCK ([20, Proposition 2.4], [25, Theorem 1]), and the exceptional divisor is IGCK, as well as all its submanifolds.

**Remark 2.8.** By contrast to the Kähler case, the class of manifolds of LCK type is not closed under blow-ups ([18, Theorem 1.3, Claim 1.5]). Still, one obviously has that the blow-up of a manifold carrying a WLCK structure also carries a WLCK structure.

The next result, proven by Vaisman, states the dichotomy between Kähler and LCK manifolds.

**Theorem 2.9** [23, Theorem 2.1], [17, Section 4.3 for a different proof]. Let \((M, I, \omega, \theta)\) be a compact LCK manifold, \(\dim C M \geq 2\). Assume that \((M, I)\) admits also a Kähler structure: then \([\theta] = 0\), that is, \((M, I, g, \omega)\) is GCK.

**Remark 2.10** [1, Lemma 2.5]. If \((M, I, \omega, \theta)\) is assumed to be only WLCK, but still compact and carrying a Kähler metric, it follows similarly that \(M\) must be GWLCK.

In our proofs, we shall use the following theorem.

**Theorem 2.11** [18, Lemma 3.1], [12, Lemma]. Let \(M\) be an LCK manifold, \(B\) a connected differentiable manifold, \(\dim B < \dim C M\) and \(\pi : M \to B\) a continuous, proper map. Assume that either:

(i) \(B\) is an irreducible complex variety, and \(\pi\) is holomorphic, or

(ii) \(\pi\) is a locally trivial fibration with fibres complex subvarieties of \(M\).

Suppose that any Lee class on \(M\) is in the image of \(\pi^*\) and the fibres of \(\pi\) are positive dimensional. Then, any LCK structure on \(M\) is GCK.

**Remark 2.12** [1, Lemma 2.4]. If \((M, I, \omega, \theta)\) is assumed to be only WLCK and such that the ‘bad locus’ \(B(\omega)\) contains no fibre of \(\pi\), it follows similarly that in fact \(M\) is GWLCK.

In the next three subsections, we provide more examples of LCK manifolds. All known LCK manifolds fall within one of the following three classes that we now describe.

### 2.2 LCK manifolds with potential

**Definition 2.13** [13, 17, Chapter 12]. An LCK manifold has LCK potential if it admits a Kähler covering on which the Kähler form \(\bar{\omega}\) has a global and positive potential function \(\psi, \bar{\omega} = dd^c \psi\), such that the deck group multiplies \(\psi\) by a constant. In this case, \(M\) is called an LCK manifold with potential.

**Example 2.14.** Let \((M, I, \omega, g, \theta)\) be an LCK manifold with the Lee form parallel with respect to the Levi–Civita connection of \(g\). Then, \((M, I, \omega, g, \theta)\) is called a Vaisman manifold. Let \(\pi : (\tilde{M}, \tilde{\omega}) \to (M, \omega, \theta)\) be a Kähler cover of a Vaisman manifold. Then, one can see that the squared norm of \(\pi^* \theta\) with respect to a Kähler metric \(\tilde{\omega}\) is a global Kähler potential satisfying the conditions in Definition 2.13. Hence, Vaisman manifolds are particular examples of LCK manifolds with potential. Among the examples of Vaisman manifolds, we mention the following.
(i) All elliptic surfaces ([3, Theorem 1]).

(ii) The diagonal Hopf manifolds \( \mathbb{C}^n \setminus 0 \langle A \rangle \), where \( A \in \text{GL}(n, \mathbb{C}) \) is diagonalisable and its eigenvalues \( a_i \in \mathbb{C} \) satisfy \( 0 < |a_i| < 1 \) ([5, Theorem 1], [14, Section 2.5], [17, Chapter 15]).

**Remark 2.15.** Vaisman manifolds are endowed with a canonical foliation locally generated by the Lee and anti-Lee fields \( \theta^\# \) and \( I\theta^\# \) (e.g. [24, Theorem 3.1]). On a compact Vaisman manifold, any complex subvariety is tangent to the canonical foliation ([21, Theorem 3.2], [17, Theorem 7.34]).

The following theorem provides a useful criterion for a compact LCK manifold to be of Vaisman type.

**Theorem 2.16** [7, Proposition 3]. Let \((M, I, \omega, \theta)\) be a compact LCK manifold, not GCK. Consider a compact torus \( T \) acting on \((M, I)\) by biholomorphic diffeomorphisms. Let \( t \subset TM \) be the Lie algebra of vector fields tangent to this action. Assume that \( I(t) \cap t \neq 0 \). Then \((M, I)\) is of Vaisman type. Moreover, the Lie algebra \( I(t) \cap t \) coincides with the Lie algebra generated by the Lee and the anti-Lee fields: \( \theta^\#, I\theta^\# \).

**Remark 2.17.** LCK manifolds with potential are stable to small deformations ([13, Theorem 2.6]). It follows that all linear Hopf manifolds \( \mathbb{C}^n \setminus 0 \langle A \rangle \), where \( A \in \text{GL}(n, \mathbb{C}) \), with eigenvalues \( a_i \in \mathbb{C}, 0 < |a_i| < 1 \), are LCK with potential. On the other hand, linear but non-diagonal Hopf manifolds are not Vaisman ([14, Theorem 2.16, Example 2.18]).

**Theorem 2.18** [13, Theorem 3.4], [17, Chapter 13]. A compact LCK manifold with potential admits a holomorphic embedding into a linear Hopf manifold. Moreover, all compact LCK manifolds with potential contain an smooth elliptic curve.

**Remark 2.19.** Clearly, any complex submanifold of an LCK manifold with potential is LCK with potential. It was recently proven ([16, Corollary 4.3]) that the non-linear Hopf manifolds \( \mathbb{C}^n \setminus 0 \langle \gamma \rangle \), where \( \gamma \) is an invertible, holomorphic contraction with origin in \( 0 \in \mathbb{C}^n \), can be holomorphically embedded into linear Hopf manifolds; in particular, also the non-linear Hopf manifolds are LCK with potential.

**Remark 2.20.** One can easily prove that if some cohomology class \([\theta] \in H^1(M, \mathbb{R})\) is the class of the Lee form of some LCK structure with potential \( \psi \), then \( u[\theta] \) is also the Lee class of an LCK metric \( \frac{dd^c \psi^u}{\psi^u} \) for any \( u \in \mathbb{R}^{\geq 1} \). In fact, the set of Lee classes on a compact LCK manifold with potential is an open half-space in \( H^1(M, \mathbb{R}) \) ([15, Theorem 8.4]).

## 2.3 Manifolds of Inoue type

Recall that a real \((p, p)\)-form \( A \) on a complex \( n \)-manifold \( M \) is called weakly positive if \( A \wedge \alpha^{n-p} \) is a non-negative top form for any Hermitian form \( \alpha \) on \( M \).

**Definition 2.21.** Let \( A \) be a weakly positive, non-zero \((p, p)\)-form on a complex manifold \( M \) admitting an LCK structure, \( \dim_{\mathbb{C}} M > p > 0 \). We say that \( A \) consumes the LCK structures if for any LCK structure \((\omega, \theta)\) on \( M, \theta \) is cohomologous to a closed 1-form \( \theta_1 \) such that \( A \wedge \theta_1 = 0 \). We
DO PRODUCTS OF COMPACT COMPLEX MANIFOLDS ADMIT LCK METRICS?

will say that a compact complex manifold is of Inoue type if it admits such an $A$ which is also closed, $dA = 0$.

**Remark 2.22.** As $A \neq 0$, it follows that the inequality $\int_M A \wedge \alpha^{n-p} \geq 0$ is strict,

$$\int_M A \wedge \alpha^{n-p} > 0$$

(2.1)

for any Hermitian form $\alpha$.

**Example 2.23.** Firstly, we recall the LCK OT manifolds (see [11] for details). Let $n \in \mathbb{N}_{>1}$ and fix a number field $K$ having $n-1$ real embeddings $\sigma_1, \ldots, \sigma_{n-1}$ and a single (up to conjugation) complex one, say $\sigma_n$. Let $O_K$ be the ring of integers of $K$ and $U \subset O_K^*$ be a subgroup of finite index formed by *positive units*, that is, for any $u \in U$, one has $\sigma_1(u), \ldots, \sigma_{n-1}(u) > 0$. Consider the semidirect product $\Gamma := O_K \rtimes U$ acting on

$$\mathbb{H}^{n-1} \times \mathbb{C} = \{(w_1, \ldots, w_{n-1}, z) \mid w_i \in \mathbb{H}, z \in \mathbb{C}\}$$

as

$$a \cdot (w_1, \ldots, w_{n-1}, z) := (w_1 + \sigma_1(a), \ldots, w_{n-1} + \sigma_{n-1}(a), z + \sigma_n(a))$$

for any $a \in O_K$ and, respectively,

$$u \cdot (w_1, \ldots, w_{n-1}, z) := (\sigma_1(u)w_1, \ldots, \sigma_{n-1}(u)w_{n-1}, \sigma_n(u)z)$$

for any $u \in U$. The resulting manifold $M := (\mathbb{H}^{n-1} \times \mathbb{C})/\Gamma$ is a compact manifold that admits an LCK metric whose Lee form is

$$\theta_0 := d \log \left( \prod_{i=1}^{n-1} \text{Im}(w_i) \right).$$

(2.2)

Moreover, one can prove ([19, Theorem 3.11], [8, Proposition 6.5]) that for any LCK metric on $M$, the associated Lee form is cohomologous to the form $\theta_0$ defined above.

**Remark 2.24.** Notice that if in the previous example one takes $n = 2$, one retrieves the LCK structure of the Inoue surfaces of type $S^0$ (for the description of these surfaces, see [6], also [20] and [17, Chapters 22, 44]).

**Example 2.25.** Next, we briefly recall the definition of Inoue surfaces of types $S^+$ and $S^−$ ([6], see also [17, Chapter 44]). Namely, an Inoue surface of type $S^+$ is a quotient $M := \mathbb{H} \times \mathbb{C}/\Gamma$ where $\Gamma$ is a group of affine transformations generated by $g_0, g_1, g_2, g_3$ which are of the form

$$g_0(w, z) := (\alpha w, z + t)$$
and
\[ g_i(w, z) := (w + a_i, z + b_i w + c_i), \quad i = 1, 2, 3, \]
where \( \alpha > 1 \) is a algebraic quadratic integer, \( t \in \mathbb{C} \) is some complex number and \( a_i, b_i, c_i \) are appropriately chosen complex numbers. It is known that \( M \) has an LCK metric if and only if the parameter \( t \) is real ([20, Proposition 3.4]). Similarly to the previous case, the associated Lee form of any LCK metric on \( M \) is cohomologous to the Lee form given by (2.2) (the result is implicit in [3, Proof of Proposition 18]; see also [2, Proposition 5.2]).

The Inoue surfaces of type \( S^- \) are quotients of order 2 of surfaces of type \( S^+ \) above with real parameter \( t \); in particular, all of them admit LCK metrics and the de Rham class of the Lee form of any LCK metric on them is still unique, as in the previous cases.

**Proposition 2.26.** All LCK OT manifolds and all LCK Inoue surfaces are of Inoue type, in the sense of Definition 2.21.

**Proof.** Let \( M \) be any of the manifold as in the statement, and \( \mathbb{H}^{n-1} \times \mathbb{C} = \{(w_1, \ldots, w_{n-1}, z) \mid w_i \in \mathbb{H}, z \in \mathbb{C}\} \) its universal cover. Define
\[ A := \left( \prod_{i=1}^{n-1} \frac{1}{y_i^2} \right) dx_1 \wedge dy_1 \wedge \cdots \wedge dx_{n-1} \wedge dy_{n-1}, \]
where \( w_i = x_i + \sqrt{-1} y_i \). Clearly, \( A \) is weakly positive, and satisfies \( A \wedge \vartheta = 0 \), where \( \vartheta = d \sum \log y_i \) is the standard LCK form.

On the other hand, keeping in mind the uniqueness of the Lee class of \( \vartheta_0 \) in (2.2), we immediately see that \( A \) consumes the LCK structures. \( \square \)

## 2.4 Manifolds with rational curves

The ‘GSS conjecture’ claims that any minimal class VII surface \( M \) with \( b_2 > 0 \) contains an open complex subvariety \( U \subset M \) biholomorphic to a neighbourhood of the standard sphere \( S^3 \subset \mathbb{C}^2 \), and \( M \setminus U \) is connected. Surfaces which satisfy the GSS conjecture are called *Kato surfaces*. This conjecture is widely believed to be true. Once the GSS conjecture is proven, this finishes the classification of the compact complex surfaces. If it is true, all non-Kähler surfaces are LCK, except for a particular class of \( S^+ \) Inoue surfaces. See also [26] and [17, Chapters 24, 25] for an up to date treatment of LCK geometry on compact complex surfaces.

The interesting feature about the remaining types of LCK manifolds known so far is that they all carry rational curves.

An easy example of LCK manifold which is not with potential is the blow-up of an LCK manifold with potential (obviously, the existence of a rational curve on the universal cover of the blow-up prevents the existence of a global plurisubharmonic function).

Recall that the blow-up \( \hat{M} \) of an LCK manifold \( M \) at a point (or, more generally, along an induced IGCK submanifold) is still LCK ([20, Proposition 2.4], [25, Theorem 1], [18, Theorem 1.3]). Moreover, it can be seen that for any Lee class \([\vartheta] \in H^1(M)\), its pullback to \( \hat{M} \) is the class of a Lee form.
Example 2.27. A notable class of examples are the Kato manifolds (higher dimensional analogues of the Kato surfaces, see [4, 9, 10]). Since they also contain rational curves, it follows that they cannot be LCK with potential. Indeed, let $M$ be an LCK manifold with potential and $C \subset M$ a rational curve. By Theorem 2.18, $M$ can be holomorphically embedded in a linear Hopf Manifold $H$, hence $C \subset H$, a contradiction because all curves on Hopf manifolds are elliptic.

Remark 2.28. One may see that the blow-up at a point of an Inoue-type LCK manifold is also of Inoue type and obviously contains a rational curve.

3 | PRODUCTS WITH ONE FACTOR LCK WITH POTENTIAL ARE NOT LCK

Lemma 3.1. Let $M$ be a compact Vaisman manifold. Then $M$ is not biholomorphic to a product of complex manifolds.

Proof. Let $M = X \times Y$. Since all positive-dimensional subvarieties of a Vaisman manifold are tangent to the canonical foliation (Remark 2.15), no positive-dimensional subvarieties of a Vaisman manifold can intersect transversally. Applying this to $X \times \{y\}$ ($y \in Y$) and $\{x\} \times Y$ ($x \in X$), we arrive at contradiction.

This immediately brings the following.

Proposition 3.2. Let $M = X \times E$ be the product of a compact complex manifold and an elliptic curve. Then $M$ does not admit a strict LCK structure.

Proof. Consider the group of automorphisms of $E$ acting on $M$. Its tangent space is a Lie algebra $\mathfrak{e}$ which satisfies $\mathfrak{e} = I(\mathfrak{e})$. According to Theorem 2.16, $M$ is of Vaisman type. Now, by Lemma 3.1, we obtain that this is impossible.

The above result can be used to prove that any product of compact complex manifolds, one of which is LCK with potential, do not carry LCK metrics.

Theorem 3.3. Let $X$ be a compact LCK manifold with potential and $Y$ any compact complex manifold, $\dim Y > 0$. Then the product $M := X \times Y$ does not admit a strict LCK structure.

Proof. By Theorem 2.18, $X$ contains an elliptic curve $E$. A submanifold in an LCK manifold is also LCK. Then, by Proposition 3.2, the manifold $E \times Y$ is globally conformally Kähler. Hence, $\{x\} \times Y$ has an IGCK structure for all $x \in X$. We obtained that $M$ is fibred over $X$ with IGCK fibres. This is impossible by Theorem 2.11.

Corollary 3.4. The product of a compact complex manifold $M$ with a compact complex curve admits no strict LCK structure.

Proof. Recall that [7, Theorem 7.8] states that if a product of a compact LCK manifold $M$ with a compact complex curve has an LCK metric, then $M$ should be LCK with potential. By the above Theorem 3.3, such a situation cannot occur.
4 | PRODUCTS WITH ONE FACTOR OF INOUE TYPE ARE NOT LCK

Theorem 4.1. Let \( M = X \times Y \) be a product of two manifolds admitting a strict LCK structure, and \( A \) a closed \((p, p)\)-form on \( X \) consuming the LCK structures. Then \( M \) does not admit an LCK structure.

Proof. **Step 1**: By absurd, assume that \( M \) admits an LCK structure \((\omega, \vartheta)\). Replacing \( \vartheta \) by a cohomologous 1-form \( \vartheta' \), we can always change \( \omega \) in its conformal class resulting in an LCK structure \((\omega', \vartheta')\).

**Step 2**: Let \( \pi_1 : M \to X, \pi_2 : M \to Y \) be the projections. Using the Künneth decomposition \( H^1(M) = H^1(X) \oplus H^1(Y) \), we may assume that \( \vartheta \) is cohomologous to \( \pi_1^*\vartheta_1 + \pi_2^*\vartheta_2 \), where \( \vartheta_1, \vartheta_2 \) are closed 1-forms on \( X \), respectively \( Y \). Since the restriction of \((\omega, \vartheta)\) to \( X \approx X \times \{ y \} = \pi_2^{-1}(y) \) is an LCK structure (for any \( y \in Y \)), the cohomology class \([\vartheta_1]\) is a Lee class of a certain LCK structure. Then \([\vartheta_1]\) contains a 1-form \( \vartheta_1 \) which satisfies \( \vartheta_1 \wedge A = 0 \). Using Step 1, we can assume that \( \vartheta = \pi_1^*\vartheta_1 + \pi_2^*\vartheta_2 \), where \( \vartheta_1 \wedge A = 0 \).

**Step 3**: Let \( n = \dim \mathbb{C}X \). Consider the form \( B := \pi_1^*A \wedge \omega^{n-p} \). Recall that \( \pi_1^*A \wedge \pi_1^*\vartheta_1 = 0 \) and \( dA = 0 \). Then:

\[
\begin{align*}
    dB &= (n-p)\pi_1^*A \wedge \omega^{n-p} \wedge (\pi_1^*\vartheta_1 + \pi_2^*\vartheta_2) \\
    &= (n-p)\pi_1^*A \wedge \omega^{n-p} \wedge \pi_2^*\vartheta_2 = (n-p)B \wedge \pi_2^*\vartheta_2.
\end{align*}
\]

Consider the pushforward (i.e. the fibrewise integral) \((\pi_2)_*B \in C^\infty(Y)\) of \( B \) to \( Y \). Since \( B \) is of type \((n, n)\), it follows that

\[
(\pi_2)_*B(y) = \left. \left( A \wedge \omega^{n-p} \right) \right|_{\pi_2^{-1}(y)} = \left. \left( \int_{\pi_2^{-1}(y)} A \wedge (\omega^{n-p}) \right) \right|_{\pi_2^{-1}(y)}.
\]

Now, from the weak positivity of the \((p, p)\)-form \( A \), together with Equation (2.1), we infer that the function \((\pi_2)_*B\) is positive and nowhere vanishing. For all \( \eta \in \Lambda^*M \), we have

\[
(\pi_2)_*(\eta \wedge \pi_2^*\vartheta_2) = (\pi_2)_*(\eta) \wedge \vartheta_2.
\]

Therefore,

\[
\begin{align*}
    d(\pi_2)_*B &= (\pi_2)_*(dB) = (n-p)(\pi_2)_*(B \wedge \pi_2^*\vartheta_2) = (n-p)((\pi_2)_*B) \cdot \vartheta_2.
\end{align*}
\]

This implies that \( \vartheta_2 = \frac{1}{n-p}d \log((\pi_2)_*B) \) is exact, contradicting the assumption that \( Y \) is strictly LCK.

\[\square\]

5 | PRODUCTS WITH ONE FACTOR HAVING RATIONAL CURVES ARE NOT LCK

Theorem 5.1. Let \( X \) be a compact strict LCK manifold and \( C \subset X \) a rational curve (i.e. a closed analytic subspace whose normalisation is the projective line \( \mathbb{P}^1 \)) and \( Y \) a compact complex manifold, \( \dim Y > 0 \). Then the product \( M := X \times Y \) does not admit a strict LCK structure.
Proof. Let \((\omega, \theta)\) be an LCK structure on \(M\) and \(S\) the singular locus (possibly empty) of \(C\). Blowing-up (possibly iterated) \(M\) along \(S \times Y\), we get a new manifold \(\hat{M}\) (which is, in fact, isomorphic to \(\hat{X} \times Y\), where \(\hat{X}\) is an embedded resolution of \(C \subset X\)). Let \(E \subset \hat{M}\) be the exceptional divisor of the blow-up \(\sigma: \hat{M} \to M\) and let \(\hat{C} \subset \hat{X}\) be the embedded resolution of \(C \subset X\). Then \(\hat{M}\) is WLCK, with structure given by \((\sigma^*(\omega), \sigma^*(\theta))\); notice that \(B(\sigma^*(\omega)) = E\).

Let \(N := \hat{C} \times Y \subset \hat{X} \times Y\); then \(N\) is WLCK with the structure \((\sigma^*(\omega)|_N, \sigma^*(\theta)|_N)\). Notice that the bad locus (see Remark 2.12) \(B(\sigma^*(\omega)|_N)\) is just \(N \cap E\).

The projection \(p_Y: N \to Y\) has simply connected fibres, and thus, induces an isomorphism \(H^1(Y) \to H^1(N)\). It follows that \(\sigma^*(\theta)|_N\) is cohomologically a pullback. Since \(B(\sigma^*(\omega)|_N)\) is just \(N \cap E\), hence intersecting the fibres of \(p_Y\) in finitely many points, we see that Theorem 2.11 applies, and hence, \(\sigma^*(\theta)|_N\) is cohomologically zero, henceforth \(\theta|_Y\) is cohomologically trivial. Applying once again Theorem 2.11 to \(p_X: X \times Y \to X\), we get that \(\theta\) is cohomologically trivial: this means that \(M\) is GCK, hence \(X\) is GCK too, a contradiction, since we assumed \(X\) to be strict LCK.

\[\square\]

Remark 5.2. Owing to Theorem 2.9, the above Theorem 5.1 and its proof also apply to the case when instead of \(C\), we consider any analytic subspace \(Z \subset X\) of \(\mathcal{O}(Z) > 1\) and such that \(Z\) has a desingularisation of Kähler type.

6 | PRODUCTS OF COMPACT COMPLEX SURFACES ARE NOT LCK

Theorem 6.1. Let \(S\) be any compact complex surface. Assuming the GSS conjecture, then for any compact complex manifold \(Y\), the product \(M := S \times Y\) has no LCK metric.

Proof. By absurd, \(M\) is LCK. Then, by Theorem 5.1, \(S\) must be minimal and not of Kato type. Next, if \(S\) is elliptic or Hopf, it would admit an LCK metric with potential, so this case is ruled out by Theorem 3.3. Eventually, assuming true the GSS conjecture, we are left with the case when \(S\) is an Inoue surface, but this case is ruled out by Theorem 4.1.

\[\square\]

ACKNOWLEDGEMENTS

We thank the anonymous referee for her or his extremely useful remarks.

Liviu Ornea partially supported by Romanian Ministry of Education and Research, Program PN-III, Project number PN-III-P4-ID-PCE-2020-0025, Contract 30/04.02.2021.

Misha Verbitsky partially supported by the HSE University Basic Research Program, FAPERJ E-26/202.912/2018 and CNPq - Process 310952/2021-2.

Victor Vuletescu partially supported by Romanian Ministry of Education and Research, Program PN-III, Project number PN-III-P4-ID-PCE-2020-0025, Contract 30/04.02.2021.

JOURNAL INFORMATION

The Bulletin of the London Mathematical Society is wholly owned and managed by the London Mathematical Society, a not-for-profit Charity registered with the UK Charity Commission. All surplus income from its publishing programme is used to support mathematicians and mathematics research in the form of research grants, conference grants, prizes, initiatives, early career researchers and the promotion of mathematics.
REFERENCES

1. D. Angella, M. Parton, and V. Vuletescu, On locally conformally Kähler threefolds with algebraic dimension two, Int. Math. Res. Not. 5 (2023), 3948–3969.

2. V. Apostolov and G. Dloussky, On the Lee classes of locally conformally symplectic complex surfaces, J. Sympl. Geom. 16 (2018), 931–958.

3. F. A. Belgun, On the metric structure of non-Kähler complex surfaces, Math. Ann. 317 (2000), 1–40.

4. M. Brunella, Locally conformally Kähler metrics on Kato surfaces, Nagoya Math. J. 202 (2011), 77–81.

5. P. Gauduchon and L. Ornea, Locally conformally Kähler metrics on Hopf surfaces, Ann. Inst. Fourier 48 (1998), 1107–1128.

6. M. Inoue, On surfaces of class VII$_0$, Invent. Math. 24 (1974), 269–310.

7. N. Istrati, Existence criteria for special locally conformally Kähler metrics, Ann. Mat. Pura Appl. 198 (2019), 335–353.

8. N. Istrati and A. Otiman, De Rham and twisted cohomology of Oeljeklaus-Toma manifolds, Ann. Inst. Fourier (Grenoble) 69 (2019), no. 5, 2037–2066.

9. N. Istrati, A. Otiman, and M. Pontecorvo, On a class of Kato manifolds, Int. Math. Res. Not. 7 (2021), 5366–5412. arXiv:1905.03224.

10. N. Istrati, A. Otiman, M. Pontecorvo, and M. Ruggiero, Toric Kato manifolds, J. Ecole Polytechnique 9 (2022), 1347–1395.

11. K. Oeljeklaus and M. Toma, Non-Kähler compact complex manifolds associated to number fields, Ann. Inst. Fourier 55 (2005), no. 1, 1291–1300.

12. L. Ornea, M. Parton, and V. Vuletescu, Holomorphic submersions of locally conformally Kähler manifolds, Ann. Mat. Pura Appl. (4) 193 (2014), no. 5, 1345–1351.

13. L. Ornea and M. Verbitsky, Locally conformal Kähler manifolds with potential, Math. Ann. 348 (2010), 25–33.

14. L. Ornea and M. Verbitsky, Locally conformally Kähler metrics obtained from pseudoconvex shells, Proc. Amer. Math. Soc. 144 (2016), 325–335.

15. L. Ornea and M. Verbitsky, Lee classes on LCK manifolds with potential, Tohoku Math. J., to appear, arXiv:2112.03363.

16. L. Ornea and M. Verbitsky, Non linear Hopf manifolds are locally conformally Kähler, J. Geom. Anal. 33 (2023), Article number: 201. arXiv:2202.12398.

17. L. Ornea and M. Verbitsky, Principles of locally conformally Kähler geometry, arXiv:2208.07188.

18. L. Ornea, M. Verbitsky, and V. Vuletescu, Blow-ups of locally conformally Kähler manifolds, Int. Math. Res. Not. IMRN 2013, no. 12, 2809–2821.

19. A. Otiman, Morse-Novikov cohomology of locally conformally Kähler surfaces, Math. Z. 289 (2018), no. 1–2, 605–628. arXiv:1609.07675.

20. F. Tricerri, Some examples of locally conformal Kähler manifolds, Rend. Sem. Mat. Torino 40 (1982), no. 1, 81–92.

21. K. Tsukada, Holomorphic maps of compact generalized Hopf manifolds, Geom. Dedicata 68 (1997), 61–71.

22. K. Tsukada, The canonical foliation of a compact generalized Hopf manifold, Differential Geom. Appl. 11 (1999), no. 1, 13–28.

23. I. Vaisman, On locally and globally conformal Kähler manifolds, Trans. Amer. Math. Soc. 262 (1980), 533–542.

24. I. Vaisman, Generalized Hopf manifolds, Geom. Dedicata 13 (1982), 231–255.

25. V. Vuletescu, Blowing-up points on l.c.K. manifolds, Bull. Math. Soc. Sci. Math. Roumanie (N.S.) 52(100) (2009), no. 3, 387–390.

26. M. Verbitsky, V. Vuletescu, and L. Ornea, Classification of non-Kähler surfaces and locally conformally Kähler geometry, Russian Math. Surveys 76 (2021), 261–290. arxiv:1810.05768.