A QUATERNIONIC NULLSTELLENSATZ

GIL ALON AND ELAD PARAN

ABSTRACT. We prove a Nullstellensatz for the ring of polynomial functions in $n$ non-commuting variables over Hamilton’s ring of real quaternions. We also characterize the generalized polynomial identities in $n$ variables which hold over the quaternions, and more generally, over any division algebra.

1. INTRODUCTION

1.1. The quaternionic Nullstellensatz. Let $F$ be a field, and consider the ring $R = F[x_1, \ldots, x_n]$. For any ideal $I \subseteq R$ one attaches the zero locus $Z(I) \subseteq F^n$, the set of points where all the polynomials in $I$ vanish. For any set $Z \subseteq F^n$ one attaches the ideal $I(Z)$ consisting of all the polynomials in $R$ which vanish on every point of $Z$.

In the case where $F$ is algebraically closed, Hilbert’s Nullstellensatz states that for any ideal $I \subseteq R$ we have $V(Z(I)) = \sqrt{I}$, the radical of $I$.

Similarly, in the case $K = \mathbb{R}$, the Real Nullstellensatz [2, Corollary 4.1.8] states that $I_R(Z_R(I)) = \sqrt{I}$, where $\sqrt{I}$ is the real radical of $I$, defined by

$$\sqrt{I} = \{ f \in R : \exists m \geq 0, l \geq 0, f_1, \ldots, f_l \in R, f^{2m} + \sum f_i^2 \in I \}$$

In the present work we prove an analogous theorem over Hamilton’s non-commutative algebra of real quaternions, $\mathbb{H}$. Let $R = \mathbb{H}[x_1, \ldots, x_n]$ be the ring of quaternionic polynomial functions, namely, the functions $f : \mathbb{H}^n \to \mathbb{H}$ expressible in the form

$$f(x_1, \ldots, x_n) = \sum_{i=1}^{r} a_{i,1}x_{\mu_{i,1}}, a_{i,2}x_{\mu_{i,2}} \ldots a_{i,k}x_{\mu_{i,k}} a_{i,k+1}$$

(1.1) Where $r \geq 0$, $k_l \geq 0$, $\mu_{i,p} \in \{1, \ldots, n\}$ and $a_{i,p} \in \mathbb{H}$. The ring operations are simply pointwise addition and multiplication of functions. By definition, we can evaluate each element of $R$ at every point $a = (a_1, \ldots, a_n) \in \mathbb{H}^n$. Moreover, for each fixed $a \in \mathbb{H}$, evaluation at $a$ is a ring homomorphism from $R$ to $\mathbb{H}$. Hence, the notions of $I(Z) \subseteq R$ for $Z \subseteq \mathbb{H}^n$ and $Z(I) \subseteq \mathbb{H}^n$ for an ideal $I \subseteq R$ can be defined as in the commutative case, and $I(Z)$ is an ideal of $\mathbb{H}$.

The conjugation map on $\mathbb{H}$ is defined by $a + bi + cj + dk = a - bi - cj - dk$ (for $a, b, c, d \in \mathbb{R}$). For any $f \in R$, let $\overline{f}$ be the function $\overline{f}(a) = \overline{f(a)}$. As we show below, we always have $\overline{f} \in R$ as well. Our main theorem states:

**Theorem 1.** Let $I$ be an ideal of $R$. Then $I(Z(I)) = \overline{\sqrt{I}}$, where

$$\overline{\sqrt{I}} = \{ f \in R : \exists m \geq 0, n \geq 0, f_1, \ldots, f_n \in R, (\overline{f})^{m} + \sum f_i \overline{I} \in I \}$$

All the ideals in this paper are two-sided.
Moreover, we characterize the ideals $I$ for which $I(Z(I)) = I$ (See Theorem 9 below for details).

1.2. Polynomial functions. Our proof of Theorem 11 relies on some algebraic properties of the ring of polynomial functions over a division algebra.

Let $D$ be a division algebra, $F = C(D)$ its center, and let $m = [D : F]$. Let $R = D[x_1, ..., x_n]$ be the ring of polynomial functions in $x_1, ..., x_n$, namely, functions $f : D^n \to D$ expressible in the form (1.1) with coefficients $a_{i,p} \in D$. For any $k \geq 1$, let $D_c[x_1, ..., x_k]$ be the ring of polynomials in $k$ central variables: These polynomials take the classical form

$$\sum a_I x^I = \sum_{0 \leq i_1, ..., i_k \leq N} a_{i_1, ..., i_k} \prod_{l=1}^{k} x_i^{i_l}$$

for some $N \geq 0$ and $a_I \in D$. Their multiplication is defined by

$$(\sum a_I x^I) \left( \sum b_J x^J \right) = \sum a_I b_J x^{I+J}$$

We will denote the resulting ring by $D_c[x_1, ..., x_k]$ (the subscript $c$ stands for the variables being central: Indeed, the variables $x_1, ..., x_k$ are in the center of this ring). For any $k$-tuple $a = (a_1, ..., a_k) \in D^k$, there is an evaluation map $e_a : D_c[x_1, ..., x_k] \to D$, defined by $e_a(f) = f(a)$. However, unlike the case of $D[x_1, ..., x_k]$, $e_a$ is a ring homomorphism if and only if $a \in F^k$. We will only evaluate elements of $D_c[x_1, ..., x_k]$ at such points.

It was proven by Wilczyński [6, Theorem 4.1] that there is, in fact, a ring isomorphism

$$D[x_1, ..., x_n] \cong D_c[y_1, y_2, ..., y_{mn}]$$

In our proof of Theorem 11 we use an explicit form of the above isomorphism, for which we prove a connection between substitution of points in $D^n$ in elements of $D[x_1, ..., x_n]$ and substitution of points in $F^{mn}$ in the corresponding elements of $D_c[y_1, y_2, ..., y_{mn}]$. See Theorem 11.2 below.

1.3. Generalized polynomial identities. The final aspect of this work concerns generalized polynomial identities. These have been introduced by Amitsur in [1], as a generalization of the much studied polynomial identities. We consider the algebra of formal non-commutative polynomials $A = D_F \langle x_1, ..., x_n \rangle := D \ast_F F \langle x_1, ..., x_n \rangle$ (this follows the notation in [8]). Recall that each element of $A$ is of the form (1.1) with coefficients in $D$, and that the scalars in $F$ are in the center of $A$. We have a canonical homomorphism $A \to D[x_1, ..., x_n]$. The kernel of the above homomorphism is the set of generalized polynomial identities in $n$ variables over $K$. Let us denote this ideal by $\text{GPI}(K, n)$. In theorem 8 we prove that $\text{GPI}(K, n)$ is finitely generated as an ideal of $A$, and describe explicitly a set of $3 \binom{n}{2} + 1$ generators of this ideal.

1.4. The case $D = \mathbb{H}$. To further explain our main ideas, let us write the isomorphism $\mathbb{H}[x] \cong \mathbb{H}_c[y_1, y_2, y_3, y_4]$ explicitly. Consider the following elements of $\mathbb{H}[x]$:
\[
Y_1 = \frac{1}{4}(x - ix - jx - kxk) \\
Y_2 = \frac{1}{4}(jxk - xi - ix - kxj) \\
Y_3 = \frac{1}{4}(kxi - xj - jx - ixxk) \\
Y_4 = \frac{1}{4}(ixj - xk - kx - jxk)
\]

It is a matter of direct verification to see that for any quaternion \(q = a_1 + a_2i + a_3j + a_4k\) (with \(a_1, a_2, a_3, a_4 \in \mathbb{R}\)) we have \(Y_i(q) = a_i\) for all \(i\). So each \(Y_i\) as a polynomial function, takes only real values, and in particular we have \(Y_iY_j = Y_jY_i\) for all \(i, j\). Moreover, we have the identity \(x = Y_1 + iY_2 + jY_3 + kY_4\). Define a homomorphism \(\phi : \mathbb{H}[x] \to \mathbb{H}[y_1, ..., y_4]\) by \(\phi(x) = y_1 + iy_2 + jy_3 + ky_4\). Then \(\phi\) is an isomorphism, as an inverse homomorphism can be defined by \(y_i \mapsto Y_i\).

As we show below, the ideal of generalized identities in one variable over \(\mathbb{H}\), \(\text{GPI}(\mathbb{H}, 1)\) is generated by the following 19 elements: \(Y_sY_t - Y_tY_s\) (for \(1 \leq s < t \leq 4\)), \(Y_s^a - aY_s\) (for \(1 \leq s \leq 4\) and \(a \in \{i, j, k\}\)) and \(x - (Y_1 + iY_2 + jY_3 + kY_4)\).

We further note that substituting a quaternion \(q = a_1 + a_2i + a_3j + a_4k\) (with \(a_1, a_2, a_3, a_4 \in \mathbb{R}\)) in any polynomial \(f \in \mathbb{H}[x]\) has the same effect as substituting the real numbers \(a_1, a_2, a_3, a_4\) in \(\phi(f)\). We can therefore establish a dictionary between systems of equations in quaternionic variables and such systems in real variables. Via this correspondence, the Real Nullstellensatz translates to a quaternionic one, which is theorem 9.

1.5. Outline. Our paper is organized as follows: In section 2, we collect standard results from the theory of central simple algebras to prove that an analog of the functions \(Y_1, ..., Y_4\) exist for any division algebra. In section 3, we discuss the isomorphism [12] and describe a set of generators for \(\text{GPI}(D, n)\). In section 4 we turn our attention to the case \(D = \mathbb{H}\) and prove the Nullstellensatz for \(\mathbb{H}[x_1, ..., x_n]\).

2. Coordinate functions as noncommutative polynomials

We start with the following lemmas, which use standard arguments on central simple algebras.

Lemma 2. Let \(D\) be a division algebra. If \(a_1, ..., a_n \in D\) are linearly independent over the center \(F = C(D)\), and \(b_1, ..., b_n \in D\) are not all equal to 0, then there exists \(x \in D\) such that \(\sum a_ixb_i \neq 0\).

Proof. For \(n = 1\) the claim is trivial. Assume the contrary of the claim, and that \(b_1, ..., b_n\) is a shortest counterexample, i.e. \(n\) is minimal. Then all \(b_i\) are nonzero. We may assume without loss of generality that \(a_1 = 1\). By our assumption, we have \(\sum a_ixb_i = 0\) for all \(x \in D\). Let \(a \in D\). By substituting \(ax\) for \(x\) in the above equality we get \(\sum a_ixb_i = 0\). Multiplying from the left by \(a\), we get \(\sum aa_ixb_i = 0\). Subtracting, we get \(\sum (a_1 - a)x_i = 0\). Since \(aa_1 - a_1a = 0\), we must have by the minimality assumption \(aa_1 - a_1a = 0\) for all \(i\). Hence \(a_i \in F\) for all \(i\), contrary to the independence assumption. ∎
Lemma 3. Let $D$ be a division algebra, and let $F = C(D)$. Then any $F$-linear endomorphism of $D$ can be expressed in the following form: $f(x) = \sum_{i=1}^{m} a_i x b_i$ for some fixed $a_i, b_i \in D$.

Proof. Let $m = [D : F]$. Let $U$ be the $F$-vector space of maps $f : D \to D$ of the form $f(x) = \sum_{i=1}^{m} a_i x b_i$. We have $U \subseteq \text{End}_F(D)$. Clearly, $\dim_F \text{End}_F(D) = m^2$. Let $v_1, \ldots, v_m$ be an $F$-linear basis of $D$. For any $1 \leq i, j \leq m$ let $f_{ij} \in U$ be the function $f_{ij}(x) = v_i x v_j$. By Lemma 2, the functions $f_{ij}$ are $F$-linearly independent. As there are $m^2$ such functions, we have $U = \text{End}_F(D)$. \hfill $\square$

Corollary 4. Let $D$ be a division algebra of dimension $m$ over its center $F = C(D)$. Let $v_1, \ldots, v_m$ be an $F$-linear basis of $D$. Then there exist $b_{st}^i \in F$ (for $1 \leq i, s, t \leq m$) such that for any $x = \sum_{i=1}^{m} c_i v_i$ (with $c_i \in F$), we have

$$c_i = \sum_{s=1}^{m} \sum_{t=1}^{m} b_{st}^i v_s x v_t$$

Moreover, the elements $b_{st}^i$ are unique.

Proof. Let us define, for each $1 \leq i \leq m$, $\phi_i : D \to F$ by $\phi_i(\sum_{j=1}^{m} c_j v_j) = c_i$. Then $\phi_i$ is $F$-linear, so by Lemma 3 it has a representation of the form $\phi_i(x) = \sum a_j x b_j$. The existence of $b_{st}^i$ follows by expressing each of the elements $a_j, b_j$ as an $F$-linear combination of $v_1, \ldots, v_m$. The uniqueness follows from Lemma 2. \hfill $\square$

3. The Ring Isomorphism

We are now ready to prove our version of the isomorphism (1.2):

Theorem 5. Let $D$ be a division algebra of dimension $m > 1$ over its center $F = C(D)$. Let $v_1, \ldots, v_m$ be an $F$-linear basis of $D$. Let $b_{st}^i \in F$ be the elements determined by $v_1, \ldots, v_m$ as in Corollary 4. Consider the following elements $Y_{ij} \in D[x_1, \ldots, x_n]$ (for $1 \leq i \leq n$, $1 \leq j \leq m$):

$$Y_{ij} = \sum b_{st}^i v_s x_i v_t$$

Then there exists a unique isomorphism

$$\phi : D[x_1, \ldots, x_n] \xrightarrow{\sim} D_c[y_{ij} : 1 \leq i \leq n, 1 \leq j \leq m]$$

satisfying $\phi(Y_{ij}) = y_{ij}$ for all $i, j$. Moreover, we have for any $(a_{ij}) \in M_{n \times m}(F)$, and any $f \in D[x_1, \ldots, x_n]$,

$$(3.1) \quad f \left( \sum a_{1j} v_j, \ldots, \sum a_{nj} v_j \right) = \phi(f)((a_{ij}))$$

Proof. By Corollary 4, any substitution of $a_1, \ldots, a_n \in K$ in $Y_{ij}$ yields an element of $F$. Hence, the elements $Y_{ij}$ commute with the elements of $D$ and with each other. Moreover, by the same corollary, for any $1 \leq i \leq n$ the following identity holds:

$$(3.2) \quad x_i = \sum_j Y_{ij} v_j$$

Let us now define a homomorphism $\psi : D_c[y_{ij}] \to D[x_1, \ldots, x_n]$ by $\psi(y_{ij}) = Y_{ij}$ for all $i, j$, and generally,

$$\psi(\sum a_I (y_{ij})^I) = \sum a_I (Y_{ij})^I$$

Since the elements $Y_{ij}$ commute with the elements of $D$ and with each other, $\psi$ is indeed a homomorphism.
For any \((a_{ij}) \in M_{n \times m}(F)\), let \(a_i = \sum a_{ij}v_j\). By corollary \([4]\) we have
\[Y_{ij}(a_1, \ldots, a_n) = a_{ij}\]
Therefore, for any \(g = \sum a_I(y_{ij})^I \in D_c[y_{ij}]\) we have (since substitution of \(a_1, \ldots, a_n\)
\[\text{is a homomorphism from } D[x_1, \ldots, x_n] \to D\]
\[g(a_{ij}) = \sum a_I(a_{ij})^I \]
\[= \sum a_I(Y_{ij}(a_1, \ldots, a_n))^I \]
\[= \left(\sum a_I(Y_{ij})^I\right)(a_1, \ldots, a_n) \]
This implies that \(\psi\) is injective: Indeed, for \(0 \neq g \in D_c[y_{ij}]\) there exists a substitution \((a_{ij}) \in M_{n \times m}(F)\) such that \(g(a_{ij}) \neq 0\) (note that by Wedderburn’s theorem, since \(m > 1, k\) must be infinite). By the above identity \(\psi(g)\) attains a nonzero value, so \(\psi(g) \neq 0\). \(\psi\) is also surjective since by \([6]\), \(x_i = \psi(\sum v_jy_{ij})\). Hence \(\psi\) is an isomorphism. Let \(\phi\) be the inverse of \(\psi\), then \(\phi\) satisfies all the desired properties. \(\square\)

As a consequence, we get a set of generators for the identities over \(D\).

**Theorem 6.** Under the same notation as in Theorem \([3]\) the ideal of identities \(GPI(D,n) = \ker(D(x_1, \ldots, x_n) \to D[x_1, \ldots, x_n])\) is generated by the following elements:

- \(v_kY_{ij} - Y_{ij}v_k\) for \(1 \leq i \leq n, 1 \leq j, k \leq m\)
- \(Y_{ij}Y_{i'j'} - Y_{i'j}Y_{ij}\) for \(1 \leq i, i' \leq n, 1 \leq j, j' \leq m\)
- \(x_i - \sum_j Y_{ij}v_j\) for \(1 \leq j \leq m\)

**Proof.** Let \(f : D(x_1, \ldots, x_n) \to D[x_1, \ldots, x_n]\) be the canonical map, and let \(K \subseteq D(x_1, \ldots, x_n)\) be the ideal generated by the above elements. We shall prove that \(K = \ker f\). We have seen in the proof of Theorem \([4]\) that the elements \(Y_{ij}\) have all their values (as functions from \(D^n\) to \(D\)) in \(F\), and that the identities \(x_i = \sum_j Y_{ij}v_j\) hold in \(D\), so all the elements in the above list are indeed in the kernel of \(f\), i.e \(K \subseteq \ker f\). Now let \(p \in \ker f\). Using the above relations we can transform \(p\) to an element of the form \(p' = \sum a_I(Y_{ij})^I\) where the sum is over multi-indices \(I\) which are multisets of pairs \(i, j\), and \(a_I \in D\). We therefore have (since \(p - p' \in K\))
\[0 = \phi(f(p)) = \phi(f(p')) = \sum a_I(Y_{ij})^I\]
Therefore \(a_I = 0\) for all \(I\). We conclude that \(p' = 0\), hence \(p \in K\). \(\square\)

4. THE NULLSTELLENSATZ

We now focus on the case \(K = \mathbb{H}\), and set \(R = \mathbb{H}[x_1, \ldots, x_n]\). As mentioned in
the introduction, there is a conjugation map on \(\mathbb{H}\), defined by \(a + bi + cj + dk = a - bi - cj - dk\). Accordingly, for any \(f \in R\) the conjugate function \(\bar{f} : \mathbb{H}^n \to \mathbb{H}\) is defined by \(\bar{f}(a) = f(\bar{a})\).

**Lemma 7.** For any \(f \in R\) we have \(\bar{f} \in R\).

**Proof.** It follows from the formula of \(Y_1\) in the introduction that the following holds for any \(a \in H\):
\[
\bar{a} = -\frac{1}{2}(a + iai + jaj + kak)
\]
Hence, for any \( f \in R \), we have an equality of functions
\[
\bar{f} = -\frac{1}{2}(f + ifi + jfj + kfk)
\]

For any set \( I \subseteq R \), we define the zero locus of \( I \) as in the commutative case:
\[
Z_R(I) = \{ a = (a_1,\ldots,a_n) \in \mathbb{H}^n : f(a) = 0 \text{ for all } f \in I \}
\]
Similarly, for any set \( A \subseteq \mathbb{H}^n \) we define the ideal of \( A \) by
\[
I_R(A) = \{ f \in R : f(a) = 0 \text{ for all } a \in A \}
\]

**Definition 8.**

(1) We will call an ideal \( I \subseteq R \) **quaternionic** if the following condition holds:
   
   For any \( f_1,\ldots,f_k \in R \), if \( \sum f_iT_i \in I \) then \( f_i \in I \) for all \( i \).

(2) For any ideal \( I \subseteq R \), the **quaternionic radical** of \( I \), \( \sqrt[\nu]{I} \), is
\[
\sqrt[\nu]{I} = \left\{ f \in R : \exists m \geq 1, \ k \geq 0, \ f_1,\ldots,f_k \in R, \ (fT)^m + \sum f_iT_i \in I \right\}
\]

**Theorem 9.** (Quaternionic Nullstellensatz). Let \( I \subseteq R \) be an ideal.

(1) \( I = I_R(Z_R(I)) \) if and only if \( I \) is quaternionic.

(2) We always have \( I_R(Z_R(I)) = \sqrt[\nu]{I} \).

**Proof.** (1) Consider the ring
\[
S = \mathbb{H}[y_{ij} : 1 \leq i \leq n, 1 \leq j \leq 4]
\]
and its subring
\[
S' = \mathbb{R}[y_{ij} : 1 \leq i \leq n, 1 \leq j \leq 4]
\]

By Theorem\footnote{5} we have an isomorphism \( \phi : R \xrightarrow{\sim} S \). By Hilbert’s basis theorem, \( S \) is noetherian, and hence, so is \( R \). By \footnote{4} Proposition 17.5], \( S' \) is the center of \( S \), any ideal in \( S \) is generated by elements in \( S' \), and there is a one-to-one correspondence between ideals in \( S \) and ideals in \( S' \) given by the following operations: An ideal in \( E \subseteq S \) corresponds to its intersection with \( S' \), and an ideal \( E' \subseteq S' \) corresponds to its extension of scalars,
\[
E' \otimes \mathbb{H} = \{ a + bi + cj + dk : a, b, c, d \in E' \} \subseteq S
\]
Combining the isomorphism \( \phi \) with the above correspondence, we obtain a one-to-one correspondence between the ideals of \( R \) and those of \( S' \). Let \( I' = \phi(I) \cap S' \), the ideal corresponding to \( I \).

Consider the bijection \( \rho : \mathbb{H}^n \xrightarrow{\sim} \mathbb{R}^{4n} \) defined by
\[
\rho((a_{s1} + a_{s2}i + a_{s3}j + a_{s4}k)_{1 \leq s \leq n}) = (a_{st})_{1 \leq s \leq n, 1 \leq t \leq 4}
\]
By the substitution formula \footnote{3} \footnote{11}, we have for any \( f \in R \) and \( a \in \mathbb{H}^n \),
\[
(4.1) \quad f(a) = \phi(f)(\rho(a))
\]
For an ideal \( I \) in \( S \) or \( S' \), let us denote the zero locus of \( I \) in \( \mathbb{R}^{4n} \) by \( Z_\mathbb{R}(I) \).
By (1.11) there is a one-to-one correspondence between $Z_{\mathbb{H}}(I)$ and the real zero locus of $\phi(I)$:

$$\rho(Z_{\mathbb{H}}(I)) = Z_{\mathbb{R}}(\phi(I)) = Z_{\mathbb{R}}(I')$$

The last equality follows from $I = I' \otimes \mathbb{H}$.

Let $\psi$ be (as previously denoted) the inverse of $\phi$. For any $B \subseteq \mathbb{R}^4$ let $I_{S'}(B)$ and $I_S(B)$ be the set of elements of $S$ (resp. $S'$) which vanish on all the points of $B$. By (1.11) we have, for any $A \subseteq \mathbb{H}^n$:

$$I_{\mathbb{H}}(A) = \psi(I_{S'}(\rho(A)))$$
$$= \psi(I_{S'}(\rho(A)) \otimes \mathbb{H})$$
$$= \psi(I_{S'}(\rho(A))) \otimes \mathbb{H}$$

We now apply results from real algebraic geometry. Recall [2, Section 4.1] that an ideal $E$ in $T = \mathbb{R}[x_1, ..., x_n]$ is called real if whenever $\sum f_i^2 \in E$ for some $f_i \in T$, we have $f_i \in E$ for all $i$. It is proved (ibid.) that for any ideal $E$ in $T$, we have $I_{\mathbb{R}}(Z_{\mathbb{R}}(E)) = E$ if and only if $E$ is real (here $Z_{\mathbb{R}}$ and $I_{\mathbb{R}}$ mean real zero locus and real ideal determined by a set, respectively). It is also proved that in general,

$$I_{\mathbb{R}}(Z_{\mathbb{R}}(E)) = \sqrt[2]{E}$$

(4.4)

$$= \{ f \in T : \exists m \geq 1, k \geq 0, f, ..., f_k \in T, f^{2m} + \sum f_i^2 \in E \}$$

Let us return to the ideal $I \subseteq R$. If $I = I_{\mathbb{H}}(Z_{\mathbb{H}}(I))$ then by the definition of $I_{\mathbb{H}}$, $I$ is quaternionic. On the other hand, if $I$ is quaternionic then $I'$ is a real ideal of $S'$: Indeed, if $f_i \in S'$ and $\sum f_i^2 \in I'$ then $\sum \psi(f_i)^2 \in I$. Since $\psi(y_{ij}) = Y_{ij}$, $\psi(f_i)$ is a polynomial with real coefficients in the elements $Y_{ij}$, so $\psi(f_i)$ attains only real values. Hence, $\psi(f_i) = \bar{\psi(f_i)}$. We get $\sum \psi(f_i)\bar{\psi(f_i)} \in I$, hence $\psi(f_i) \in I$ for all $I$, hence $f_i \in I'$ for all $i$. By the Real Nullstellensatz, we conclude that $I_{S'}(Z_{\mathbb{R}}(I')) = I'$. Hence, by (4.3) and (4.2),

$$I_{\mathbb{H}}(Z_{\mathbb{H}}(I)) = \psi(I_{S'}(\rho(Z_{\mathbb{H}}(I)))) \otimes \mathbb{H}$$
$$= \psi(I_{S'}(Z_{\mathbb{R}}(I'))) \otimes \mathbb{H}$$

(4.3)
$$= \psi(I') \otimes \mathbb{H} = I$$

as desired.

(2) Let $I$ be any ideal of $R$. If $f, f_i \in R$, $(f\overline{f})^m + \sum f_i\overline{f_i} \in I$, and $a \in Z_{\mathbb{H}}(I)$, then we have $(f(a)\overline{f(a)})^m + \sum f_i(a)\overline{f_i(a)} = 0$, hence $f(a) = 0$. This shows that $\sqrt{T} \subseteq I_{\mathbb{H}}(Z_{\mathbb{H}}(I))$. On the other hand, using (1.3), (1.2) and the Real Nullstellensatz, we have

$$I_{\mathbb{H}}(Z_{\mathbb{H}}(I)) = \psi(I_{S'}(\rho(Z_{\mathbb{H}}(I)))) \otimes \mathbb{H}$$
$$= \psi(I_{S'}(Z_{\mathbb{R}}(I'))) \otimes \mathbb{H}$$
$$= \psi(\sqrt{T}) \otimes \mathbb{H}$$

so any element of $I_{\mathbb{H}}(Z_{\mathbb{H}}(I))$ is of the form

$$f = f_1 + f_2i + f_3j + f_4k$$

where $f_i \in \psi(\sqrt{T})$ for all $1 \leq i \leq 4$. The elements $f_i$, being isomorphic images of elements of the center of $S$, are in the center of $R$. By the definition of the real
radical, we can find some \( m_i \geq 1 \) and \( f_{ij} \in \psi(S') \) such that \( f_i^{2m_i} + \sum_j f_{ij}^2 \in \psi(I') \). From this follows that for any \( y \in \psi(S') \) we have \( y^2 f_i^{2m_i} + \sum_j (y f_{ij})^2 \in \psi(I') \). Let us consider, for any \( m \geq 1 \),

\[
(fI)^m = \left( \sum_{i=1}^{4} f_i^2 \right)^m
\]

for \( m \) sufficiently large, when we expand the right-hand side by the multinomial formula (keeping in mind that the summands commute), we get that each summand is of the form \( y^2 f_i^{2m_i} \) for some \( y \in \psi(S') \) and \( 1 \leq i \leq 4 \), hence can be complemented by a sum of squares of elements in \( \psi(S') \) to an element of \( \psi(I') \). We conclude that for some \( m \geq 1 \) there are elements \( g_i \in \psi(S') \), such that

\[
(fI)^m + \sum g_i^2 \in \psi(I') \subseteq I
\]

but \( g_i = \bar{g_i} \), so \( f \in \sqrt[\angle]{I} \). Hence, \( I_{\mathbb{H}}(Z_{\mathbb{H}}(I)) \subseteq \sqrt[\angle]{I} \) and we have inclusions in both directions.

**Corollary 10.** The quaternionic radical of an ideal in \( R \) is the minimal quaternionic ideal containing it.

**Proof.** Indeed, let \( I \) be an ideal of \( R \). By Theorem 9 \( \sqrt[\angle]{I} = I_{\mathbb{H}}(Z_{\mathbb{H}}(I)) \) is a quaternionic ideal. If \( J \) is a quaternionic ideal containing \( I_{\mathbb{H}}(Z_{\mathbb{H}}(I)) \) then \( Z_{\mathbb{H}}(J) \subseteq Z_{\mathbb{H}}(I) \), so \( J = I_{\mathbb{H}}(Z_{\mathbb{H}}(J)) \supseteq I_{\mathbb{H}}(Z_{\mathbb{H}}(I)) \).

**Acknowledgement.** We are thankful to Bruno Deschamps for suggesting us to prove some of the results in this paper for any division algebra.

**References**

[1] Amitsur, S. A. (1965). Generalized polynomial identities and pivotal monomials. Transactions of the American Mathematical Society, 114(1), 210-226.

[2] Jacek Bochnak, Michel Coste and Marie-Francoise Roy, *Real Algebraic Geometry*, Springer, 1998

[3] Cohn, P. M. (1977). Skew field constructions (Vol. 27). CUP Archive.

[4] K. R. Goodearl and R. B. Warfield, *An introduction to noncommutative Noetherian rings*, Cambridge, 1989.

[5] Lawrence, J., & Simons, G. E. (1989). Equations in division rings—a survey. The American mathematical monthly, 96(3), 220-232.

[6] Wilczyński, D. M. (2014). On the fundamental theorem of algebra for polynomial equations over real composition algebras. Journal of Pure and Applied Algebra, 218(7), 1195-1205.