On a class of arithmetic convolutions involving arbitrary sets of integers

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Abstract

Let \( d, n \) be positive integers and \( S \) be an arbitrary set of positive integers. We say that \( d \) is an \( S \)-divisor of \( n \) if \( d \mid n \) and \(\gcd (d, n/d) \in S \). Consider the \( S \)-convolution of arithmetical functions given by (1.1), where the sum is extended over the \( S \)-divisors of \( n \).

We determine the sets \( S \) such that the \( S \)-convolution is associative and preserves the multiplicativity of functions, respectively, and discuss other basic properties of it. We give asymptotic formulae with error terms for the functions \( \sigma_S(n) \) and \( \tau_S(n) \), representing the sum and the number of \( S \)-divisors of \( n \), respectively, for an arbitrary \( S \). We improve the remainder terms of these formulae and find the maximal orders of \( \sigma_S(n) \) and \( \tau_S(n) \) assuming additional properties of \( S \). These results generalize, unify and sharpen previous ones.

We also pose some problems concerning these topics.

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1 Introduction

Let \( \mathbb{N} \) denote the set of positive integers and let \( S \) be an arbitrary subset of \( \mathbb{N} \). For \( n, d \in \mathbb{N} \) we say that \( d \) is an \( S \)-divisor of \( n \) if \( d \mid n \) and \( \gcd (d, n/d) \in S \), notation \( d \mid_S n \). Consider the \( S \)-convolution of arithmetical functions \( f \) and \( g \) defined by

\[
(f *_S g)(n) = \sum_{d \mid_S n} f(d)g(n/d) = \sum_{d \mid n} \rho_S((d, n/d))f(d)g(n/d),
\]

where \( \rho_S \) stands for the characteristic function of \( S \).

Let \( \tau_S(n) \) and \( \sigma_S(n) \) denote the number and the sum of \( S \)-divisors of \( n \), respectively.

For \( S = \mathbb{N} \) we obtain the Dirichlet convolution and the familiar functions \( \tau(n) \) and \( \sigma(n) \). For \( S = \{1\} \) we have the unitary convolution and the functions \( \tau^*(n) \) and \( \sigma^*(n) \). These have been studied extensively in the literature, see for example [3] and its bibliography.

Among other special cases we mention here the following ones.
Let $P$ be an arbitrary subset of the primes $p$ and $S$ be the multiplicative semigroup generated by $P \cup \{1\}$, i.e., $S = (P) \equiv \{1\} \cup \{n > 1 : p|n \Rightarrow p \in P\}$. Then the $(P)$-convolution is the concept of the cross-convolution, see [7], which is a special regular convolution of Narkiewicz-type [4].

If $S$ is the set of $k$-free integers, $k \geq 2$, i.e., $S = Q_k \equiv \{1\} \cup \{n > 1 : p|n \Rightarrow p^k \not| n\}$, then the $Q_k$-divisors are the $k$-ary divisors and (1.1) is the $k$-ary convolution, see [5], [6].

Let $L_k$ denote the set of $k$-full integers, i.e., $L_k \equiv \{1\} \cup \{n > 1 : p|n \Rightarrow p^k|n\}$, where $k \in \mathbb{N}, k \geq 2$. The $L_k$-convolution given by

$$
(f *_{L_k} g)(n) = \sum_{d|n} \frac{f(d)g(n/d)}{(d,n/d) \in L_k}
$$

seems to not have been investigated till now.

The aim of this note is to study some basic properties of the $S$-convolution, to give asymptotic formulae for the functions $\sigma_S(n)$ and $\tau_S(n)$ and to investigate the maximal orders of these functions.

Assuming that $1 \in S$ (then $1|sn$ and $n|sn$ for every $n \in \mathbb{N}$), we determine in Section 2 the subsets $S$ such that the $S$-convolution is associative and preserves the multiplicativity of functions, respectively.

The most interesting property is that of associativity. It turns out that, for example, the $Q_k$-convolution with $k \geq 2$ is not associative, but the $L_k$-convolution is associative.

The $L_k$-convolution has also other nice properties, which are analogous to those of the Dirichlet convolution and of the unitary convolution. For example, the set of all complex valued arithmetical functions $f$ with $f(1) \neq 0$ forms a commutative group under the $L_k$-convolution and the set of all nonzero multiplicative functions forms a subgroup of this group.

Furthermore, let $\mu_k$ denote the inverse with respect the $L_k$-convolution of the constant 1 function. We call it ”$k$-full M"obius function”, which is multiplicative and for every prime power $p^a$, $\mu_k(p^a) = -1$ for $1 \leq a < 2k$ and $\mu_k(p^a) = \mu_k(p^{a-1}) - \mu_k(p^{a-k})$ for $a \geq 2k$.

Note that $\mu_1 \equiv \mu$ is the ordinary M"obius function. The function $\mu_2$ takes the values $-1, 0, 1$.

We pose the following problems: Which are the values taken by $\mu_k$? Investigate asymptotic properties of $\mu_k$.

Note that the $S$-convolution is contained in the concept of the $K$-convolution to be defined in Section 2. Although there exist characterizations of basic properties of $K$-convolutions, see [2] and [3], Chapter 4, no study of (1.1) has been made in the literature.

Section 3 contains certain identities showing that for every $S$ the $S$-convolution of two completely multiplicative functions can be expressed with the aid of their Dirichlet convolution and their unitary convolution, respectively.

Asymptotic formulae with error terms for the functions $\sigma_S(n)$ and $\tau_S(n)$, involving arbitrary subsets $S$, are given in Section 4. We show that the remainder terms can be sharpened assuming additional properties of $S$.

In Section 5 we determine the maximal order of $\sigma_S(n)$ assuming that $S$ is multiplicative, i.e. $1 \in S$ and $\rho_S$ is multiplicative, and give the maximal order of $\tau_S(n)$ for an arbitrary $S$ with $1 \in S$.

What can be said on the maximal order of $\sigma_S(n)$ for an arbitrary subset $S$?

The results of Sections 4 and 5 are obtained by elementary methods, they generalize, unify and improve the corresponding known results concerning the functions $\sigma(n)$, $\tau(n)$,
their unitary analogues $\sigma^*(n)$, $\tau^*(n)$, those involving $k$-ary divisors and the functions $\sigma_A(n)$, $\tau_A(n)$ associated with cross-convolutions, see [3], [5], [6], [7], [8].

## 2 Properties of the $S$-convolution

It is immediate that the $S$-convolution is commutative and distributive with respect ordinary addition for every $S$.

Assume in this section that $1 \in S$. Then $1|Sn$ and $n|Sn$ for every $n \in \mathbb{N}$ and denoting $\delta \equiv \rho_{\{1\}}$, i.e. $\delta(1) = 1$ and $\delta(n) = 0$ for $n > 1$, we have $f \ast_S \delta = f$ for every function $f$. This means that $\delta$ is the identity element for $\ast_S$.

We say that $S$ is multiplicative if $1 \in S$ and its characteristic function $\rho_S$ is multiplicative.

The $K$-convolution of arithmetical functions $f$ and $g$ is given by

$$ (f \ast_K g)(n) = \sum_{d|n} K(n,d) f(d) g(n/d), $$

where $K$ is a complex valued function defined on the set of all ordered pairs $(n,d)$ with $n,d \in \mathbb{N}$ and $d|n$.

For $K(n,d) = \rho_S((d,n/d))$ (2.1) becomes (1.1), therefore the $S$-convolution is a special $K$-convolution.

**Theorem 2.1** The $S$-convolution preserves the multiplicativity of functions if and only if $S$ is multiplicative.

**Proof.** It is known ([3], Chapter 4) that the $K$-convolution preserves the multiplicativity if and only if

$$ K(mn,de) = K(m,d)K(n,e) $$

holds for every $m,n,d,e \in \mathbb{N}$ such that $(m,n) = 1$ and $d|m, e|n$.

Hence the $S$-convolution has this property if and only if

$$ (2.2) \quad \rho_S((de,mn/de)) = \rho_S((d,m/d))\rho_S((e,n/e)) $$

for every $m,n,d,e \in \mathbb{N}$ with $(m,n) = 1$ and $d|m, e|n$.

If $S$ is multiplicative, then for every $m,n,d,e$ given as above $(d,m/d)$ and $(e,n/e)$ are relatively prime, $(de,mn/de) = (d,m/d)(e,n/e)$ and we obtain (2.2).

Conversely, if (2.2) holds and $M,N \in \mathbb{N}$, $(M,N) = 1$ are given integers, then taking $d = M, m = M^2, e = N, n = N^2$ we obtain

$$ \rho_S(MN) = \rho_S(M)\rho_S(N), $$

showing that $S$ is multiplicative. \Diamond

**Remark.** It follows that all the convolutions mentioned in the Introduction preserve the multiplicativity.

**Theorem 2.2** The $S$-convolution is associative if and only if the following conditions hold:

1. $S$ is multiplicative,
2. for every prime $p$ and for every $j \in \mathbb{N}$ if $p^j \in S$, then $p^\ell \in S$ for every $\ell > j$. 


Remark. Condition (2) is equivalent with the following: for every prime \( p \) one of the next statements is true:
(i) \( p^j \in S \) for every \( j \in \mathbb{N} \),
(ii) \( p^j \notin S \) for every \( j \in \mathbb{N} \),
(iii) there exists \( e = e(p) \in \mathbb{N} \) depending on \( p \) such that \( p^j \notin S \) for every \( 1 \leq j < e \) and \( p^j \in S \) for every \( j \geq e \).

Proof. It is known ([3], Chapter 4) that the \( K \)-convolution is associative if and only if
\[
K(n,d)K(d,e) = K(n,e)K(n/e,d/e)
\]
holds for every \( n, d, e \in \mathbb{N} \) with \( d|n, e|d \).

Therefore the \( S \)-convolution is associative if and only if
\[
(2.3) \quad \rho_S((d,n/d))\rho_S((e,d/e)) = \rho_S((e,n/e))\rho_S((d/e,n/d))
\]
for every \( n, d, e \in \mathbb{N} \) with \( d|n, e|d \).

First we show that if *\( S \) is associative, then \( \rho_S \) is multiplicative. Suppose that (2.3) is satisfied, let \( M, N \in \mathbb{N} \), \( (M,N) = 1 \) and take \( n = M^2N^2, d = MN, e = M \). Then we have
\[
\rho_S((MN, MN))\rho_S((M, N)) = \rho_S((M, MN^2))\rho_S((N, MN)),
\]
hence
\[
\rho_S(MN) = \rho_S(M)\rho_S(N).
\]
Assume now that \( S \) is multiplicative. Then, taking \( n = p^c, d = p^b, e = p^a \), (2.3) is equivalent to
\[
(2.4) \quad \rho_S((p^b,p^{a-b}))\rho_S((p^c,p^{b-c})) = \rho_S((p^c,p^{a-c}))\rho_S((p^{b-c},p^{a-b}))
\]
for every prime \( p \) and for every \( 0 \leq c \leq b \leq a \). Note that it is sufficient to require (2.4) for every \( 0 < c < b < a \).

Suppose that \( p^j \in S \), where \( j \in \mathbb{N} \) and let \( \ell > j \). We show that \( p^\ell \in S \).

Case 1. \( \ell < 2j \). Take \( a = \ell + 2j, b = \ell + j, c = \ell \). From (2.4) we obtain
\[
\rho_S((p^{\ell+j}, p^j))\rho_S((p^\ell, p^j)) = \rho_S((p^\ell, p^{2j}))\rho_S((p^j, p^j)),
\]
\[
\rho_S(p^\ell)p_S(p^j) = \rho_S(p^\ell)p_S(p^j),
\]
giving \( \rho_S(p^\ell) = 1 \).

Case 2. \( \ell \geq 2j \). Now let \( a = 2\ell, b = \ell, c = \ell - j \). From (2.4) we have
\[
\rho_S((p^\ell, p^\ell))\rho_S((p^{\ell-j}, p^j)) = \rho_S((p^{\ell-j}, p^{\ell+j}))\rho_S((p^j, p^\ell)),
\]
\[
\rho_S(p^\ell)p_S(p^j) = \rho_S(p^{\ell-j})p_S(p^j),
\]
thus
\[
(2.5) \quad \rho_S(p^\ell) = \rho_S(p^{\ell-j}).
\]

If \( \ell = kj + r \), where \( k \geq 2 \) and \( 0 \leq r < j \), then applying (2.5) we have
\[
\rho_S(p^\ell) = \rho_S(p^{\ell-j}) = \rho_S(p^{\ell-2j}) = ... = \rho_S(p^{j+r}) = 1,
\]

4
where \( j \leq j + r < 2j \) and we use the result of Case 1.

In order to complete the proof we show that if \( S \) is multiplicative and condition (2) holds, then we have (2.4) for every \( 0 < c < b < a \).

Consider the cases of the Remark of above. For (i) and (ii) (2.4) holds trivially. In case (iii) if \( p^j \not\in S \) for every \( 1 \leq j \leq e - 1 \) and \( p^j \in S \) for every \( j \geq e \), then (2.4) means that the statements "\([b \geq e \text{ and } a - b \geq e]\) and \((c \geq e \text{ and } b - c \geq e)\)" and "\((c \geq e \text{ and } a - c \geq e)\) and \((b - c \geq e \text{ and } a - b \geq e)\)" are equivalent. A quick check shows that this is true.  

**Remark.** From Theorem 2.2 we obtain that the \( Q_k \)-convolution \((k \geq 2)\) is not associative, but the \( L_k \)-convolution and the (P)-convolution defined in the Introduction are associative.

**Theorem 2.3** If conditions (1) and (2) of Theorem 2.2 hold, then the set of all complex valued arithmetical functions forms a commutative (and associative) ring with identity with respect to ordinary addition and \( S \)-convolution (in particular \( L_k \)-convolution).

This ring has no divisors of zero if and only if \( S = \mathbb{N} \), i.e. \(*_S\) is the Dirichlet convolution.

**Proof.** The first part of this result follows at once from Theorem 2.2 and from the previous remarks.

Furthermore, it is well-known that for the Dirichlet convolution there are no divisors of zero. Conversely, suppose that \( S \not= \mathbb{N} \) satisfies conditions (1) and (2) of Theorem 2.2. Then there exists a prime \( p \) such that \( p \not\in S \) and the following functions are divisors of zero:

\[
 f(n) = g(n) = \begin{cases} 1, & \text{if } n = p, \\ 0, & \text{otherwise.} \end{cases}
\]

**Theorem 2.4** If conditions (1) and (2) of Theorem 2.2 hold, then the set of all complex valued arithmetical functions \( f \) with \( f(1) \neq 0 \) forms a commutative group under \( S \)-convolution (in particular \( L_k \)-convolution) and the set of all nonzero multiplicative functions forms a subgroup of this group.

**Proof.** This yields in a similar manner as in case of the Dirichlet convolution and unitary convolution or in general for certain \( K \)-convolutions, see [3], Ch.4. 

Consider now the "\( k \)-full"-convolution corresponding to \( S = L_k \), the set of \( k \)-full numbers. Let \( \mu_k \) denote the "\( k \)-full Möbius function", representing the inverse of the function \( I(n) = 1, n \in \mathbb{N} \) with respect to this convolution. According to Theorem 2.4 \( \mu_k \) is multiplicative and a short computation shows that for every prime power \( p^a \),

\[
 \mu_k(p^a) = -1, \quad 1 \leq a < 2k \quad \text{and} \quad \mu_k(p^a) = \mu_k(p^{a-1}) - \mu_k(p^{a-k}), \quad a \geq 2k.
\]

Observe that \( \mu_1 \equiv \mu \) is the ordinary Möbius function.

For the "squarefull Möbius function" \( \mu_2 \) (case \( k = 2 \)) we have \( \mu_2(p) = \mu_2(p^2) = \mu_2(p^3) = -1 \) and

\[
 \mu_2(p^a) = \mu_2(p^{a-1}) - \mu_2(p^{a-2}), \quad a \geq 4.
\]

Therefore, \( \mu_2(p) = \mu_2(p^2) = \mu_2(p^3) = -1, \mu_2(p^4) = 0, \mu_2(p^5) = \mu_2(p^6) = 1, \mu_2(p^7) = 0, \mu_2(p^8) = \mu_2(p^9) = -1, \mu_2(p^{10}) = 0, \ldots \).
The values taken by $\mu_2$ are $-1, 0, 1$. This is not true for $\mu_3$, since $\mu_3(p^a) = -1$ for $1 \leq a \leq 5$, $\mu_3(p^6) = 0, \mu_3(p^7) = 1, \mu_3(p^8) = \mu_3(p^9) = 2, \mu_3(p^{10}) = 1, \mu_3(p^{11}) = -1, \mu_3(p^{12}) = -3, \mu_3(p^{13}) = -4, \ldots$

We pose the following problems: Which are the values taken by $\mu_k$? Investigate asymptotic properties of $\mu_k$. Does it posses a mean value?

## 3 Identities

For an arbitrary $S \subseteq \mathbb{N}$ let $\mu_S$ be the Möbius function of $S$ defined by

$$
\sum_{d|n} \mu_S(n) = \rho_S(n), \quad n \in \mathbb{N},
$$

see [1], therefore, by Möbius inversion,

$$
\mu_S(n) = \sum_{d|n} \rho_S(d) \mu(n/d), \quad n \in \mathbb{N},
$$

where $\mu \equiv \mu_{\{1\}}$ is the ordinary Möbius function.

The zeta function $\zeta_S$ is defined by

$$
\zeta_S(z) = \sum_{n=1}^{\infty} \frac{\rho_S(n)}{n^z}.
$$

It follows that $\zeta_N \equiv \zeta$ is the Riemann zeta function and

$$
\sum_{n=1}^{\infty} \frac{\mu_S(n)}{n^z} = \frac{\zeta_S(z)}{\zeta(z)} \quad (z > 1).
$$

**Theorem 3.1** If $S \subseteq \mathbb{N}$ and $f$ and $g$ are completely multiplicative functions, then for every $n \in \mathbb{N}$,

$$
(f \ast_S g)(n) = \sum_{d^2|n} \mu_S(d)f(d)g(d)(f \ast g)(n/d^2),
$$

where $\ast \equiv \ast_{\{1\}}$ is the Dirichlet convolution and

$$
(f \ast_S g)(n) = \sum_{d^2|n} \rho_S(d)f(d)g(d)(f \times g)(n/d^2),
$$

where $\times \equiv \ast_{\{1\}}$ is the unitary convolution.

**Proof.** Using (3.1) we have for every $n \in \mathbb{N}$,

$$
(f \ast_S g)(n) = \sum_{d|n} \rho_S((d,e))f(d)g(e) = \sum_{d|n} \left( \sum_{(d,e)} \mu_S(j) \right) f(d)g(e).
$$

Hence with $d = ja, e = jb$,

$$
(f \ast_S g)(n) = \sum_{j^2ab|n} \mu_S(j)f(ja)g(jb) = \sum_{j^2ab|n} \mu_S(j)f(j)f(a)g(j)g(b) =
$$
Theorem 4.1
If $\sum_{j^2 \ell = n} \mu_S(j) f(j) g(j) \sum_{ab = \ell} f(a) g(b) = \sum_{j^2 \ell = n} \mu_S(j) f(j) g(j) (f \ast g)(\ell)$, which is (3.4).

Furthermore,

$$(f \ast S g)(n) = \sum_{de = n} \rho_S((d, e)) f(d) g(e) = \sum_{a \in S} \sum_{de = n} f(d) g(e) =$$

$$= \sum_{a} \rho_S(a) \sum_{de = n} f(d) g(e).$$

With $d = ai, e = bj$ we get

$$(f \ast S g)(n) = \sum_{a^2b = n} \rho_S(a) f(a) g(a) f(i) g(j) = \sum_{a^2b = n} \rho_S(a) f(a) g(a) \sum_{ij = b} f(i) g(j) =$$

$$= \sum_{a^2b = n} \rho_S(a) f(a) g(a)(f \times g)(b),$$

giving (3.5). $\Diamond$

**Theorem 3.2** If $S \subseteq N$, then for every $n \in N$,

$$\tau_S(n) = \sum_{d^2|n} \mu_S(d) \tau(n/d^2) = \sum_{d^2|n} \rho_S(d) \tau^*(n/d^2),$$

$$\sigma_S(n) = \sum_{d^2|n} \mu_S(d) d\sigma(n/d^2) = \sum_{d^2|n} \rho_S(d) d\sigma^*(n/d^2).$$

**Proof.** This yields at once from Theorem 3.1 applied for $f(n) = g(n) = 1$ and $f(n) = n, g(n) = 1$, respectively. $\Diamond$

Note that if $S$ is multiplicative, then the functions $\tau_S(n)$ and $\sigma_S(n)$ are also multiplicative.

The generalized Euler function $\phi_S(n) = \#\{k \in N : k \leq n, (k, n) \in S\}$ was considered in [1] and one has $\phi_S = \mu_S * E = \rho_S * \phi$, where $E(n) = n, n \in N$ and $\phi \equiv \phi_{\{1\}}$ is the ordinary Euler function, see also [7].

**4 Asymptotic formulae**

The following asymptotic formulae generalize and improve the known formulae concerning the functions $\sigma(n)$, $\tau(n)$, their unitary analogues, those involving $k$-ary divisors and the functions $\sigma_A(n)$, $\tau_A(n)$ associated with cross-convolutions, cf. [3], Ch. 6; [5], Corollary 3.1.1; [6], Corollary 3.1; [7], Theorem 12; [8], Theorem 2; see also [9], Corollary 1.

**Theorem 4.1** If $S \subseteq N$, then

$$\sum_{n \leq x} \sigma_S(n) = \frac{\zeta(2) \zeta_S(3)}{2\zeta(3)} x^2 + R_S(x),$$

where the remainder term can be evaluated as follows:

1. \( R_S(x) = O(x \log^{8/3} x) \) for an arbitrary \( S \),
2. \( R_S(x) = O(x \log^{5/3} x) \) for an \( S \) such that \( \sum_{n \in S} \frac{1}{n} < \infty \) (in particular for every finite \( S \) and for every multiplicative \( S \)),
3. \( R_S(x) = O(x \log^{2/3} x) \) for every multiplicative \( S \) such that \( \sum_{p \notin S} \frac{1}{p} < \infty \) (in particular if the set \( \{ p : p \notin S \} \) is finite).

**Proof.** We have from (3.7),

\[
\sum_{n \leq x} \sigma_S(n) = \sum_{d \leq \sqrt{x}} \mu_S(d) \sum_{e \leq x/d^2} \sigma(e).
\]

Applying now the well-known result of Walfisz [10],

\[
\sum_{n \leq x} \sigma(n) = \frac{\zeta(2)}{2} x^2 + O(x \log^{2/3} x)
\]

we obtain

\[
\sum_{n \leq x} \sigma_S(n) = \sum_{d \leq \sqrt{x}} \mu_S(d) \left( \frac{\zeta(2)}{2d^2} x^2 + O \left( \frac{x}{d^2} (\log x)^{8/3} \right) \right) = \frac{\zeta(2)}{2} \sum_{d=1}^{\infty} \frac{\mu_S(d)}{d^3} x^2 + O \left( \frac{x^2}{\sqrt{x}} \sum_{d \leq \sqrt{x}} \frac{|\mu_S(d)|}{d^3} \right) + O \left( x(\log x)^{2/3} \sum_{d \leq \sqrt{x}} \frac{|\mu_S(d)|}{d} \right).
\]

For the main term apply (3.3) and the given error term yields from the next statements:

(a) For an arbitrary \( S \subseteq \mathbb{N} \), \( |\mu_S(n)| \leq \sum_{d|n} \rho_S(d) \leq \tau(n) \) for every \( n \in \mathbb{N} \) and

\[
\sum_{n \leq x} \frac{|\mu_S(n)|}{n} \leq \sum_{d \leq x} \frac{\rho_S(d)}{d} \sum_{e \leq x/d^2} \frac{1}{e} = O \left( \log x \sum_{d \leq x} \frac{\rho_S(d)}{d} \right) = \begin{cases} O(\log x), & \text{if } \sum_{n=1}^{\infty} \frac{\rho_S(n)}{n} < \infty, \\ O(\log^2 x), & \text{otherwise.} \end{cases}
\]

(b) If \( S \) is multiplicative, then \( \mu_S \) is multiplicative too, \( \mu_S(p^a) = \rho_S(p^a) - \rho_S(p^{a-1}) \) for every prime power \( p^a \) \( (a \geq 1) \) and \( \mu_S(n) \in \{-1, 0, 1\} \) for each \( n \in \mathbb{N} \).

(c) Suppose \( S \) is multiplicative. Then

\[
\sum_{p} \sum_{k=1}^{\infty} \frac{|\mu_S(p^k)|}{p^k} \leq \sum_{p} \left( \frac{|\rho_S(p) - 1|}{p} + \sum_{k=2}^{\infty} \frac{1}{p^k} \right) = \sum_{p \in S} \frac{1}{p(p-1)} + \sum_{p \notin S} \frac{1}{p-1} \leq 2 \left( \sum_{p \in S} \frac{1}{p^2} + \sum_{p \notin S} \frac{1}{p} \right) < \infty \text{ if } \sum_{p \notin S} \frac{1}{p} < \infty.
\]

It follows that in this case the series \( \sum_{n=1}^{\infty} \frac{|\mu_S(n)|}{n} \) is convergent. 

**Theorem 4.2** If \( S \) is an arbitrary subset of \( \mathbb{N} \), then

\[
(4.2) \quad \sum_{n \leq x} \tau_S(n) = \frac{\zeta_S(2)}{\zeta(2)} x \log x + 2\gamma - 1 + \frac{2\zeta_S(2)}{\zeta(2)} - \frac{2\zeta'(2)}{\zeta(2)} + O(\sqrt{x} \log^2 x),
\]

where \( \gamma \) is the Euler constant and \( \zeta_S'(z) \) is the derivative of \( \zeta_S(z) \).
This result follows applying the first identity of (3.6) and using Dirichlet’s formula
\[ \sum_{n \leq x} \tau(n) = x(\log x + 2\gamma - 1) + O(x^{\alpha}). \]

The remainder term of (4.2) can be improved assuming further properties of \( S \). For example, if \( S \) is multiplicative, then the error term is \( O(\sqrt{x}\log x) \) and if \( S \) (i.e. \( \rho_S \)) is completely multiplicative and \( \{p : p \notin S\} \) is a finite set, then the error term is \( O(x^{\alpha}) \). We do not go into details.

5 Maximal orders

Generalizing the result of Gronwall concerning the function \( \sigma(n) \) we prove the following theorem.

**Theorem 5.1** Let \( S \) be an arbitrary multiplicative subset. Denote by \( P \) the set of primes \( p \) such that \( p^j \in S \) for every \( j \in \mathbb{N} \). For every \( p \notin P \) let \( s(p) \in \mathbb{N} \) denote the least exponent \( j \) such that \( p^j \notin S \) (i.e. \( p^j \in S \) for every \( 1 \leq j < s(p) \) and \( p^k(p) \notin S \)).

Then
\[ \limsup_{n \to \infty} \frac{\sigma_S(n)}{n \log \log n} = e^\gamma \prod_{p \notin P} \left( 1 - \frac{1}{p^{2s(p)}} \right). \]

**Proof.** For every \( p \in P, a \in \mathbb{N} \) and for every \( p \notin P, a < 2s(p) \) the \( S \)-divisors of \( p^a \) are all divisors \( 1, p, p^2, \ldots, p^a \). Hence \( \sigma_S(p^a) = \sigma(p^a) = 1 + p + p^2 + \ldots + p^a \).

For every \( p \notin P \) and \( a \geq 2s(p) \) the numbers \( p^a(p) \) and \( p^{a-s(p)} \) are certainly not \( S \)-divisors of \( p^a \), since \( (p^{a-s(p)}, p^{s(p)}) = p^{s(p)} \notin S \). Therefore \( \sigma_S(p^a) < (1 + p + p^2 + \ldots + p^{a-s(p)-1}) + (p^{a-s(p)+1} + \ldots + p^a) < p^{a-s(p)} + p^{a-s(p)+1} + \ldots + p^a \leq p^{a-2s(p)+1} + p^{a-2s(p)+2} + \ldots + p^a \).

We obtain that
\[ \frac{\sigma_S(p^a)}{p^a} \leq 1 + \frac{1}{p} + \frac{1}{p^2} + \ldots + \frac{1}{p^{2s(p)+1}} \]
holds for every prime power \( p^a \) with \( p \notin P \) with equality for \( a = 2s(p) - 1 \).

Also, for every \( p \in P, a \in \mathbb{N} \),
\[ \frac{\sigma_S(p^a)}{p^a} < \left( 1 - \frac{1}{p} \right)^{-1}. \]

We show that
\[ \frac{\sigma_S(n)}{n} \leq e^\gamma \prod_{p \notin P} \left( 1 - \frac{1}{p^{2s(p)}} \right) \log \log n(1 + o(1)) \quad \text{as} \quad n \to \infty. \]

Using (4.3) and (4.4) we have for every \( n \geq 1 \),
\[ \frac{\sigma_S(n)}{n} \leq \prod_{p^l(p)/p^l} \left( 1 - \frac{1}{p} \right)^{-1} \prod_{p^{l/p}/p} \left( 1 + \frac{1}{p} + \frac{1}{p^2} + \ldots + \frac{1}{p^{2s(p)-1}} \right) = \prod_{p^{l/p}/p} \left( 1 - \frac{1}{p} \right)^{-1} \prod_{p^{l/p}/p} \left( 1 + \frac{1}{p} + \frac{1}{p^2} + \ldots + \frac{1}{p^{2s(p)-1}} \right) \times \]
\[
\prod_{p \leq \log n \atop p \not\in P} \left( 1 + \frac{1}{p} + \frac{1}{p^2} + \ldots + \frac{1}{p^{2s(p)-1}} \right) \leq \\
\prod_{p \leq \log n \atop p \not\in P} \left( 1 - \frac{1}{p} \right)^{-1} \prod_{p \leq \log n \atop p \not\in P} \left( 1 + \frac{1}{p} + \frac{1}{p^2} + \ldots + \frac{1}{p^{2s(p)-1}} \right) \times \\
\prod_{p \leq \log n \atop p \not\in P} \left( 1 - \frac{1}{p} \right)^{-1} \prod_{p \leq \log n \atop p \not\in P} \left( 1 - \frac{1}{p} \right)^{-1} = \\
\prod_{p \leq \log n \atop p \not\in P} \left( 1 - \frac{1}{p^{2s(p)}} \right) \prod_{p \leq \log n \atop p \not\in P} \left( 1 - \frac{1}{p} \right)^{-1} \prod_{p \leq \log n \atop p \not\in P} \left( 1 - \frac{1}{p} \right)^{-1} \leq \\
\prod_{p \leq \log n \atop p \not\in P} \left( 1 - \frac{1}{p^{2s(p)}} \right) \prod_{p \leq \log n \atop p \not\in P} \left( 1 - \frac{1}{p} \right)^{-1} \prod_{p \leq \log n \atop p \not\in P} \left( 1 - \frac{1}{\log n} \right)^{-1} = \\
\left( 1 - \frac{1}{\log x} \right) \log \log n (1 + o(1)),
\]

applying Mertens’ theorem \( \prod_{p \leq x} (1 - \frac{1}{p}) = \frac{e^{-\gamma}}{\log x} (1 + o(1)) \) as \( x \to \infty \), and the fact that \( \# \{ p : p|n, p > \log n \} \leq \frac{\log n}{\log \log n} \).

Now we show that this upper bound is asymptotically attained.

For a given \( \varepsilon > 0 \) choose \( t \) so large such that

\[
\prod_{p > t} \left( 1 - \frac{1}{p^a} \right) \geq 1 - \varepsilon.
\]

For this \( t \) choose an exponent \( a \geq 1 \) such that

\[
\prod_{p \leq t} \left( 1 - \frac{1}{p^a} \right) \geq 1 - \varepsilon.
\]

Consider the sequence \( (n_k)_{k \geq 1} \) given by

\[
n_k = \prod_{p \leq t} p^{a-1} \prod_{p \leq e^k} p^{2s(p)-1} \prod_{t < p \leq e^k} p.
\]

We obtain

\[
\sigma_S(n_k) / n_k = \prod_{p \leq t} \left( 1 + \frac{1}{p} + \frac{1}{p^2} + \ldots + \frac{1}{p^{a-1}} \right) \times \\
\prod_{p \leq t} \left( 1 + \frac{1}{p} + \frac{1}{p^2} + \ldots + \frac{1}{p^{2s(p)-1}} \right) \prod_{t < p \leq e^k} \left( 1 + \frac{1}{p} \right) \geq \\
\prod_{p \leq t} \left( 1 - \frac{1}{p^a} \right) \prod_{p \leq e^k} \left( 1 - \frac{1}{p^{2s(p)}} \right) \prod_{p > t} \left( 1 - \frac{1}{p^2} \right) \prod_{p \leq e^k} \left( 1 - \frac{1}{p} \right)^{-1} \geq 
\]
\[
\geq (1 - \varepsilon)^2 \prod_{p \not \in P} \left( 1 - \frac{1}{p^{2s(p)}} \right) e^\gamma k(1 + o(1)) \quad \text{as} \quad k \to \infty,
\]

applying Mertens’ theorem again.

Furthermore, considering the Chebysev function \( \theta(x) = \sum_{p \leq x} \log p \) and using the elementary estimate \( \theta(x) = O(x) \), we get

\[
\log n_k \leq O(1) + \theta(e^k) = O(e^k).
\]

Hence, for sufficiently large \( k \),

\[
\log \log n_k \leq O(1) + k < (1 + \varepsilon)k.
\]

Therefore

\[
\limsup_{k \to \infty} \frac{\sigma_S(n_k)}{n_k \log \log n_k} \geq \frac{(1 - \varepsilon)^2}{1 + \varepsilon} e^\gamma \prod_{p \not \in P} \left( 1 - \frac{1}{p^{2s(p)}} \right),
\]

and the proof is complete. ♦

A direct consequence of Theorem 5.1 is the following result.

**Theorem 5.2** Let \( S \) be an arbitrary multiplicative subset and suppose that there exists \( s \in \mathbb{N} \) such that for every prime \( p \), \( p^j \in S \) for every \( 1 \leq j < s \) and \( p^s \not \in S \). Then

\[
\limsup_{n \to \infty} \frac{\sigma_S(n)}{n \log \log n} = \frac{e^\gamma}{\zeta(2s)}.
\]

This result can be applied for \( S = Q_k \) (case \( s = k \geq 1 \)), for \( S = L_k \) (case \( s = 1 \)).

What is the maximal order of \( \sigma_S(n) \) for an arbitrary subset \( S \) ?

**Theorem 5.3** Let \( S \) be an arbitrary subset such that \( 1 \in S \). Then

(4.5)

\[
\limsup_{n \to \infty} \frac{\log \tau_S(n) \log \log n}{\log n} = \log 2.
\]

**Proof.** It is well-known that this result holds for the function \( \tau(n) \) (case \( S = \mathbb{N} \)) and that for the sequence \( n_k = p_1p_2...p_k \), where \( p_i \) is the \( i \)-th prime,

\[
\lim_{k \to \infty} \frac{\log \tau(n_k) \log \log n_k}{\log n_k} = \log 2.
\]

Taking into account that if \( 1 \in S \), then \( \tau_S(n) = \tau(n) \) for every squarefree \( n \) and \( \tau_S(n) \leq \tau(n) \) for every \( n \in \mathbb{N} \), (4.5) follows at once. ♦

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