Variable $G$ and $\Lambda$: scalar-tensor versus RG-improved cosmology

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Abstract

We study the consequences due to time varying $G$ and $\Lambda$ in scalar-tensor theories of gravity for cosmology, inspired by the modifications introduced by the Renormalization Group (RG) equations in the Quantum Einstein Gravity. We assume a power-law scale factor in presence contemporarily of both the scalar field and the matter components of the cosmic fluid, and analyze a special case and its generalization, also showing the possibility of a phantom cosmology. In both such situations we find a negative kinetic term for the scalar field $Q$ and, possibly, an equation-of-state parameter $w_Q < -1$. A violation of dominant energy condition (DEC) for $Q$ is also possible in both of them; but, while in the first special case the $Q$-energy density then remains positive, in the second one we find it negative.
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I. INTRODUCTION

In cosmology the Newton coupling term $G$ is usually assumed as constant, as well as the cosmological term $\Lambda$. As a matter of fact, it can be easily shown that, once we suppose a time varying $\Lambda$, if we do not want to lose covariance of the equations we have also to implement the time dependence of $G$. But there are also measurements which now offer the possibility of not excluding some variability of such important parameters; the experimental range of variation of $G$ seems in fact to be $10^{-12} \text{yr}^{-1} < \dot{G}/G < 10^{-10} \text{yr}^{-1}$. (For some recent considerations on this, based on white dwarf asteroseismology, see Ref. [1].) It therefore seems reasonable to take

$$\frac{\dot{G}}{G} = \sigma H_0,$$

$(1)$

$H_0$ being the Hubble term and $\sigma \sim 1$ [2]. (As a matter of fact, many current estimates give $|\sigma| \leq 0.1$, but they are usually made on solar system (or at most galactic) scale. As we are concerned here on cosmological scales, we assume a more conservative position.) This kind of behavior is in fact obtained in various nonstandard descriptions, for example, in scalar-tensor theories of gravity. On the other hand, $\Lambda$ has widely been inserted into equations as a time varying quantity (see Ref. [3] and references therein, for instance), often without any consideration on covariance and the contemporary variability of the $G$-term. Anyway, we can suppose [2, 3, 4, 5] that $\Lambda_0 \sim H_0^2$, also in order to explain current astrophysical observations, which lead, as known, to dark energy models of the universe.

Also, due to recent work on cosmology in the Planck era, soon after the initial Big Bang singularity, some models have been discussed [6], where the Newton constant $G$ and the cosmological constant $\Lambda$ may be dynamically coupled to the geometry of spacetime. For instance, the Einstein equations can be modified by the renormalization group (RG) equations for Quantum Einstein Gravity [7], solving horizon and flatness problems of the cosmological standard model, without need of introducing inflation. The use of a dark energy scalar field to solve the cosmic coincidence problem is also not needed when the large scale dynamics of the universe is studied in such a framework, since the vacuum energy density, depending on $G$, can be automatically equalled to the matter energy density, so that $\Omega_\Lambda = \Omega_m = 1/2$ [8]. In such a context it results $\sigma \equiv (3/2)(1 + w_m)$, $w_m$ being the ratio of the pressure to the energy density of the matter component [9].

Different approaches are possible. For example, one can try to set the improvement at the
level of the action \([\text{10}]\). (See comments about this approach in the concluding remarks.) It is also possible to treat \(G\) and \(\Lambda\) as dynamical variables in the action from the very beginning \([\text{11}]\).

Here, we want to show that some of what is gained in an approach like that in Refs. \([\text{6, 7, 8}]\) can also be obtained in a scalar-tensor theory of gravity, and comment on this. Let us first stress that in such a kind of models both a time-varying effective \(G\) and a \textit{natural} effective cosmological \(\Lambda\)-term, also varying with time, may be introduced \([\text{12, 13, 14}]\). Let us thus consider a nonminimally coupled theory for cosmology, where a scalar field \(Q\) couples to geometry via a function \(F = F(Q)\); in the energy content of the universe we also include nonrelativistic matter, which is supposed to evolve substantially decoupled from \(Q\). This kind of model gives the possibility to describe the cosmological situation at least starting from the decoupling period. As a matter of fact, therefore, we expect to recover some of the features characterizing the nowadays debate on present cosmology. In this paper we do not aim at developing our approach completely, but we anyway think we can at least grasp the most relevant aspects (like, for instance, the present cosmic acceleration) as dynamical results of the model. As a matter of fact, we also find the possibility to recover a phantom cosmology, with a negative kinetic term but positive definite energy density, and an equation of state \(w_Q < -1\).

Phantom matter is usually defined as that with equation-of-state parameter \(w_Q < -1\) \([\text{15}]\). It was introduced in the context of cosmology assuming a minimal coupling between scalar and gravitational fields, when such limits on \(w_Q\) are directly implied by taking \(\rho_Q > 0\), \(V(Q) > 0\), and \(\dot{Q}^2 < 0\). It always results \(p_Q < 0\), too. From this all, the dominant energy condition (DEC) \(\rho_Q + p_Q \geq 0\) \([\text{16}]\) results also violated \([\text{15, 17}]\), leading to a situation that has been not often considered in standard cosmology, with some difficulties in making the phantom model stable \([\text{17}]\). (See Ref. \([\text{18}]\) for a possible finite-time DEC violation, arising from a bulk viscous stress due to a particle production.) It is also interesting to note that scalar fields with negative kinetic energies \([\text{15, 17}]\) were not used in minimally coupled cosmology since the old Steady State theory \([\text{19, 20, 21, 22}]\), where a creation field with such a feature was in fact introduced \([\text{23}]\). (For some comments on \(\rho_Q < 0\), see Refs. \([\text{23, 24, 25}]\), for instance.)

Nonetheless, it has to be stressed that the observational limits on \(w_Q\) give \(-1.62 < w_Q < -0.74\) at the 95% confidence level \([\text{26}]\), or \(-1.18 < w_Q < -0.93\), using also the Wilkinson
Microwave Anisotropy Probe (WMAP) data \[27\], so allowing the possibility of what is now called superquintessence \[28\] or phantom energy \[15\]. There is also evidence for some consistency between the current cosmic age estimator data and phantom energies \[27, 29\]. A discussion on whether this component of the universe may lead to a big rip \[30, 31\] or not \[32, 33\] is still open, even if the occurrence of a possible singularity in the next future has recently been discussed also when $\rho_Q > 0$ and $\rho_Q + 3p_Q > 0$ \[34\].

On the other hand, when there is a nonminimal coupling between $Q$ and geometry, we can have negative kinetic energies for $Q$ without having $w_Q < -1$ or $\rho_Q + p_Q < 0$, even assuming $\rho_Q > 0$ and $V(Q) > 0$. This should be connected with the fact that now $\rho_Q$ and $p_Q$ have more complicated expressions, strongly depending on the time variation of the coupling function $F(Q(t))$ \[14\]. As a matter of fact, phantom energy has been already considered in the context of scalar-tensor theories \[31, 35, 36\]. (In Ref. \[29\] it is shown that a time dependent $w_Q$ is more favourable to the existence of epochs with $w_Q < -1$, while in Ref. \[31\] it is also shown that a big rip in the future evolution of the universe can be avoided for some range of the parameters involved once $w_Q = w_Q(t)$ is assumed, and it probably results unavoidable with a constant $w_Q$ parameter.)

Here, our aim is to assume $G$ and $\Lambda$ variations inspired by the RG improved gravity and comment on results deriving from plugging them into the general framework of a scalar-tensor theory with Lagrangian density

$$L = \sqrt{-g} \left[ F(Q)R + \frac{1}{2} g^{\mu\nu} Q_{,\mu} Q_{,\nu} - V(Q) \right] + L_m ,$$

where $F(Q)$ and $V(Q)$ are the generic functions respectively describing the coupling and the potential of the scalar field $Q$; $R$ is the scalar curvature and the metric is the Friedmann-Lemaitre-Robertson-Walker (FLRW) one. We represent with $L_m$ the usual matter contribution, seen as decoupled from the scalar sector, and the standard minimally coupled situation is recovered when $F = -1/2$, with units $8\pi G = c = 1$.

The use of power-law solutions for the scale factor $a = a(t)$ makes it clear that it is possible to have a phantom behavior only for special choices of the parameters involved. This has already been shown elsewhere \[31\]. But we can here show that we cannot simply neglect ordinary matter. Even if a superacceleration $\ddot{a} > 0$ implied by an equation-of-state parameter $w_Q < -1$ is supposed to lead to a regime where the scalar field $Q$ dominates \[27, 31\], it in fact appears interesting to better investigate the situation, since the range of
parameters allowing a phantom cosmology results tightly connected with the present dust content of the universe.

In the following, in Sec. II we give the basics of the nonminimally coupled theory used in our discussion, while in Sec. III we investigate a special (and motivated) choice of $H = H(t)$. Through a suitable generalization of some parameters, in Sec. IV we try to discover new and more general features of the model. Since we know that there must be a connection between our Jordan-frame formulation in Sec. III and the usual minimally coupled equations of cosmology in presence of a scalar field (i.e., the Einstein-frame formulation), Sec. V is dedicated to investigate the expression of the particular conformal transformation that realizes this. In Sec. VI, in the end, we trace the final comments and conclusions.

II. NONMINIMALLY COUPLED THEORY

In presence of a nonminimally coupled scalar field $Q$ in the universe, as already said, we have to consider an effective gravitational coupling, taking into account the fact that, now, geometry (and, thus, the gravitational field) is sensibly coupled to this new component. In what follows we only take two components filling the universe, i.e., matter and scalar field. By “matter” we here mean “dust” (cold noninteractive matter, i.e., baryons and dark matter together), so that the period involved by our description is placed substantially after the decoupling era, when the radiation component decreases much more than the matter one. On its side, the scalar field behavior is ruled by a potential $V(Q)$, which heavily enters into the equations, complicating their solutions. As to that, in what follows we only limit ourselves to write down the equations needed for our discussion. Further details can be found, for example, in Ref. [14].

Let us start from the Einstein and Klein-Gordon equations

$$3H^2 = G (\rho_m + \rho_Q),$$

$$\ddot{Q} + 3H \dot{Q} + 12H^2 F' + 6\dot{H}F' + V' = 0,$$

where dot stands for time derivative, the prime indicates derivative with respect to $Q$, the function $H = H(t)$ is the Hubble parameter, $\rho_m$ is the matter energy density, and

$$\rho_Q = \frac{1}{2} \dot{Q}^2 + V + 6\dot{H}F;$$

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$$\rho_Q = \frac{1}{2} \dot{Q}^2 + V + 6\dot{H}F;$$
we can also define

\[ p_Q = \frac{1}{2} \dot{Q}^2 - V - 2 \left( \ddot{F} + 2H \dot{F} \right). \]  

(6)

In these equations we have set as usual \( 8\pi G = c = 1 \) and introduced an effective gravitational coupling \( G \) such that

\[ G = -(2F)^{-1}, \]  

(7)

\( F = F(Q) \) being the coupling parameter of the theory; it can be shown that an effective \( \Lambda \)-term can also be introduced \[14\] such that

\[ \Lambda_{\text{eff}} = G \rho_Q. \]  

(8)

This, of course, implies that Eqs. (3) and (4) become

\[ 3H^2 = G \rho_m + \Lambda_{\text{eff}}, \]  

(9)

\[ \dot{Q}(\ddot{Q} + 3H \dot{Q} + V') + 3 \frac{G'}{G^2}(2H^2 + \dot{H}) = 0. \]  

(10)

We can easily see that

\[ \dot{\rho}_Q + 3H (p_Q + \rho_Q) = -6H^2 \dot{F}, \]  

(11)

and

\[ \dot{\Lambda}_{\text{eff}} + G \dot{\rho}_m = -3HG (p_Q + \rho_Q). \]  

(12)

This last equation has to be compared with the additional equation of the RG improved cosmology

\[ \dot{\Lambda}_{\text{eff}} + G \dot{\rho}_m = 0. \]  

(13)

We see that, if we want to interpret \( \Lambda_{\text{eff}} \) strongly, as a varying cosmological term, we have to make the additional assumption

\[ p_Q = -\rho_Q, \]  

(14)

which still fits the dominant energy condition \( p_Q + \rho_Q \geq 0 \) on the scalar field sector. This point marks a relevant difference with the minimal coupling case. There, in fact, Eq. (14) just leads to a constant potential, i.e., a genuinely constant \( \Lambda \). In our case, the situation is clearly different, and from Eqs. (5), (6), and (14) it is easy to derive

\[ (\dot{Q})^2 = 2(\ddot{F} - H \dot{F}). \]  

(15)
In the following, therefore, we should distinguish between \( \Lambda_{\text{weak}} \) and \( \Lambda_{\text{strong}} \), as we may name the different expressions taken by the \( \Lambda \)-term in the two possible situations.

On the other hand, another important Einstein equation holds

\[
2\dot{H} + 3H^2 = -\mathcal{G} (p_m + p_Q),
\]

which implies that

\[
\ddot{a} > 0 \iff (\rho_m + 3p_m) + (\rho_Q + 3p_Q) < 0.
\]

Since we introduce the equations of state

\[
p_m = w_m \rho_m, \quad p_Q = w_Q \rho_Q,
\]

with \( w_m = 0 \), it is then customary to deduce that the universe lives in an acceleration stage when

\[
w_Q < -\frac{1}{3}.
\]

As it seems generally accepted, observational results accounting for the most recent history of the universe indicate that this is just the case \[37, 38, 39\] and that right now, therefore, we are in a sort of soft inflationary epoch.

III. A SPECIAL CASE

The equations introduced above are hard to deal with, but can anyway be solved exactly in some cases \[14\]. For our purposes we here limit ourselves to postulate a given special solution (certainly leading to acceleration) and speculate on consequences. To choose a suitable one, even in a scalar-tensor theory, we again refer to RG improved cosmology \[8\], with which some of our considerations have to be naturally compared, and especially to the fact that the infrared (IR) fixed point model for cosmology fits the high redshift observations on Type Ia Supernovae \[40\].

Thus, we investigate the possibilities involved by setting

\[
H = \frac{4}{3} t^{-1},
\]

together with

\[
\mathcal{G} = \alpha t^2
\]
(α being an unknown parameter). Referring to Eq. 11 we see that this sets \( \sigma = 3/2 \). (As said, this rather high value cannot be excluded by present observations. See also Ref. 9 for a deeper discussion on this point.)

Due to Eq. (8), we can also easily see that it fits the behavior there recalled for Λ. As a matter of fact, Eqs. (7) and (20) give

\[
F = -\frac{1}{2\alpha} t^{-2},
\]

and

\[
a(t) \sim t^{\frac{2}{3}} \implies \rho_m \sim a^{-3} \sim t^{-4},
\]

since we are assuming dust as matter. Let us, therefore, pose

\[
\rho_m = Mt^{-4},
\]

with \( M \equiv \rho_{m0}t_0^{-4} \) a positive constant determined by the present values of matter energy density and time. From Eq. (3) we easily get

\[
\rho_Q = \left( \frac{16}{3\alpha} - M \right) t^{-4},
\]

which is positive when

\[
M < \frac{16}{3\alpha}.
\]

So, we find that, due to Eqs. (21) and (25), the two time behaviors for \( \rho_m \) and \( \rho_Q \) are equally time scaling. Allowing values of the parameters such that \( M > 16/(3\alpha) \) would imply \( \rho_Q < 0 \), leading to a situation that we choose not to discuss here. Also, from Eq. (8) we soon have

\[
\Lambda_{\text{eff}} = 3 \left( 1 - \frac{3M\alpha}{16} \right) H^2,
\]

that is, \( \Lambda_{\text{eff}} = \lambda H^2 \), with \( 0 < \lambda < 3 \), which is an acceptable order of magnitude (see Ref. 2, for example).

Substituting Eqs. (5) and (22) into Eq. (25), we find an equation which can be differentiated, so that, taking also Eq. (4) into account, we finally get

\[
\dot{Q}^2 = -\left( \frac{6}{\alpha} + M \right) t^{-4}.
\]

This means that, even if the energy density associated to \( Q \) is positive, the kinetic energy term is negative and \( \dot{Q} \) is imaginary; all the other quantities used in the following, and which are more significant in the development of our considerations, anyway result real.
On the other hand, let us note that in Ref. [8] there is already the reference to some solutions in a Brans-Dicke (BD) model [41], which also present something similar but disguised, since in that context a negative $\omega$ BD parameter is found to be necessary, in order to get an accelerated phase of expansion.

Let us go on, in our own context, underlining that, due to Eq. (5) there is no need to have $V > 0$ in order to get $\rho_Q > 0$. We in fact find (see Eq. (32) below) that it can be negative. From Eq. (28) we thus set

$$Q = Q_0 - i\sqrt{\frac{6}{\alpha}} + Mt^{-1},$$

where $Q_0$ being an integration constant that we can safely set to zero.

In a minimally coupled situation, this classical rotation of $Q$ to imaginary values can, for example, be considered as typical of an axionic component for vacuum phantom energy, which violates the DEC and has an increasing energy density (so allowing the possibility of a big rip), at least if one wants to keep the weak energy condition (WEC) [42]. Here, anyway, it is important to note that, due to Eq. (25), the nonminimally coupled situation implies always decreasing energy density with time, meaning that caution is needed when simply importing features from the minimally coupled case. (See Ref. [31] for a special nonminimally coupled model also exhibiting decreasing phantom energy.)

We can of course extract $F(Q)$ from Eq. (22)

$$F(Q) = \frac{1}{2(6 + M\alpha)}Q^2 < 0$$

and $V(Q)$ from Eq. (5). We have first to take into account that

$$V(t) = \left(\frac{1}{3\alpha} - \frac{M}{2}\right)t^{-4},$$

which is non negative when $M \leq 2/(3\alpha)$ and negative for $2/(3\alpha) < M < 16/(3\alpha)$ (due to Eq. (20)); thus, we finally find

$$V(Q) = \frac{\alpha (2 - 3M\alpha)}{6(6 + M\alpha)^2}Q^4.$$  

Let us stress that our initial guess in Eq. (20) has clearly to be meant as expressing an asymptotic regime. Thus, the above expressions for $F$ and $V$ are strictly valid only in this approximation. This explains why we do not recover the ordinary general relativistic theory.
with $F = -1/2$. (In order to obtain this, we should improve the approximation, which will be done in a forthcoming paper.)

We can also get the expression of pressure from Eq. (6)

$$p_Q = -\frac{8}{3\alpha} t^{-4},$$

so that it always results $p_Q < 0$. This soon gives

$$w_Q \equiv \frac{p_Q}{\rho_Q} = -\frac{8}{16 - 3M\alpha} = {\text{const.}} < 0,$$

due to Eq. (26), which represents a necessary constraint in our treatment. The position $-1 \leq w_Q < 0$, on the other hand, implies $M \leq 8/(3\alpha)$. But, of course, what can be most interesting here is the possibility that $w_Q < -1$, a range of values usually involved by the negative kinetic term for the scalar field [15, 13]. As a matter of fact, such a feature is realized when $M > 8/(3\alpha)$. Thus, due to Eqs. (31) and (32), we find $w_Q < -1$ only for negative values of $V$, i.e., when $8/(3\alpha) < M < 16/(3\alpha)$.

We can schematically summarize the most important informations in Table I. The third case there recorded (characterized by a constant negative potential) gives a strong $\Lambda_{eff}$, according to the above discussion, and is strictly equivalent to the RG improved environment (see below), which is at the very origin of this treatment. But, as should have been expected, we can see that the situation here is richer. For example, we find a phantom energy presence in the fourth situation. We also see that, remarkably, the presence of matter ($M \neq 0$) is crucial in understanding what happens. Situations differs from one another depending on suitable ranges of the $M$ parameter, which is peculiar to characterize the amount of ordinary matter in the cosmic content. So, we can find DEC violation, but only for a range of $M$ values depending on the $\alpha$-parameter characterizing the scale of time variation of the running gravitational $G$-term, or we should rather say that such a violation is obtained for a range of $\alpha$-values constrained by the present-day matter content in $M$.

A final interesting insight is given by the calculation of the matter density parameter. We find a constant matter density universe

$$\Omega_m \equiv \frac{G\rho_m}{3H^2} = \frac{3\alpha M}{16} < 1,$$

due to Eq. (26). When $M = 8/(3\alpha)$ it is $\Omega_m = 1/2$; such a value is in fact produced not using the Newtonian coupling parameter $G_N$, but through an effective scale dependent $G$,
as discussed in Ref. [8]. As a matter of fact, we do find a complete correspondence with the RG-improved solutions in Ref. [8], once we pose $\alpha = 3\pi g_\ast \lambda_\ast$. However, our case is somewhat more general, since we also consider $w_Q \neq -1$ and always get a negative $\dot{Q}^2$-term.

IV. A FAMILY OF MORE GENERAL CASES

We want now to show that the ansatz in Sec. III, although inspired by a natural guess in a very different context, reveals to have very nice features also in this situation. In order to see this we try to release conditions in Eqs. (20) and (21), and set more generally

$$H = nt^{-1}, \quad G = \alpha t^m,$$

where $m > 0$ and $n > 1$. What we discussed in the previous section is then relative to the special values $n = 4/3$ and $m = 2$. Here, we want to generalize the behaviors analyzed before and see what happens, which features are retained and which are not, with a position that corresponds to releasing the constraint $\Lambda G = \text{const.}$ of the RG improved approach.

As a first thing, let us note that Eq. (36) is consistent with what one can expect for $\dot{G}/G$ (that is, its proportionality to $H$, through a factor $\sigma = m/n$). Later on we will anyway see that something more careful has to be said about $\Lambda$.

From Eq. (3) we now find

$$\rho_Q = \left( \frac{3n^2}{\alpha t} t^{3n-m-2} - M \right) t^{-3n},$$

so that

$$\rho_Q > 0 \implies t^{3n-m-2} > \frac{\alpha M}{3n^2}. \quad (38)$$

When $m = 3n - 2$, this condition on scalar field energy density leads to $3n^2 > \alpha M$, i.e., to a generalization of the constraint in Eq. (26) found for $n = 4/3$ (and $m = 2$). If $m < 3n - 2$, it gives

$$t > \left( \frac{\alpha M}{3n^2} \right)^{\frac{1}{3n-m-2}}, \quad (39)$$

so that it results $\rho_Q > 0$ only when $t > \bar{t} \equiv \left( \frac{\alpha M}{3n^2} \right)^{\frac{1}{3n-m-2}}$. If instead $m > 3n - 2$, we get

$$t < \left( \frac{3n^2}{\alpha M} \right)^{-\frac{1}{3n+m+2}}, \quad (40)$$

which does not allow to have $\rho_Q > 0$ after the time $\tilde{t} \equiv \left( \frac{3n^2}{\alpha M} \right)^{-\frac{1}{3n+m+2}}$. 

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Only the first two situations \((m \leq 3n - 2)\) will be considered here, since the third one \((m > 3n - 2)\) seems to be possibly appropriate only for an earlier epoch (but then we should consider radiation + matter + scalar field). Therefore, in the following we shall assume

\[3n \geq m + 2, \quad t > \tilde{t}. \tag{41}\]

Performing exactly the same steps as in the previous section, we can now see that

\[\dot{Q}^2 = -\{\frac{1}{\alpha} \left[ m^2 + (n+1)m - 2n \right] t^{3n-m-2} + M \} t^{-3n}. \tag{42}\]

In the case when \(3n = m + 2\), and taking into account \(n > 1\) and \(M > 0\), it is easy to check that \(\dot{Q}^2 < 0\) again. This case is illustrated below (see subsection A3). When \(3n > m + 2\), the first term dominates more and more at increasing times. As to the sign of the coefficient of such a term in \(t^{-m-2}\), since \(\alpha > 0\) it is easy to see that it is non negative when

\[m \geq \tilde{m} \equiv -\frac{1}{2} \left( n + 1 - \sqrt{(n+1)^2 + 8n} \right). \tag{43}\]

This is in agreement with Eq. (41), implying \(n > 2/3\), since we have chosen \(m > 0\) from the very beginning of our considerations; this means that (being \(n > 1\)) it has to be \(m \geq \tilde{m}\) (with \(\sqrt{3} - 1 < \tilde{m} < 2\)), in order to get a negative kinetic term for the scalar field. In particular, we may note that the special value \(m = \tilde{m}\) leads to \(\dot{Q}^2 = -Mt^{-3n} < 0\), that is, to a kind of behavior of the function \(\dot{Q}^2\), which is clearly like the one previously found when \(n = 4/3\) and \(m = 2\). This means that the generalization we are examining here for \(m \geq \tilde{m}\) can still lead to a phantom cosmology. It remains to see whether the other related features we found before are also reproduced in this more generalized context.

As a first step, we can use Eq. (5) to find

\[V (Q(t)) = \frac{1}{2} \left[ \frac{1}{\alpha} (6n^2 + m^2 - 5nm - 2n + m) t^{3n-m-2} - M \right] t^{-3n}. \tag{44}\]

Note that if we want the potential to be positive after some time \(t^*\), then it is necessary that

\[6n^2 + m^2 - 5nm - 2n + m > 0, \tag{45}\]

implying

\[m < m_- \equiv 2n, \quad \text{or} \quad m > m_+ \equiv 3n - 1; \tag{46}\]

thus, for \(n > 1\), we find

\[m < 2n. \tag{47}\]
This result is more restrictive than the one in Eq. (41), even if certainly does not contradict the one in Eq. (43).

More difficult is here to get \( V = V(Q) \), since it results (suitably choosing the sign)

\[
Q(t) = -\frac{2t\sqrt{At^{-m-2} - Mt^{-3n}}}{3n-2} + \left[ \frac{2A(2 + m - 3n)t^{-m-1}}{(3n - 2)(2 + 2m - 3n)} \frac{M - At^{3n-m-2}}{M(At^{-m-2} - Mt^{-3n})} \Phi \right],
\]

(48)

where \( A = A(n, m) \) is a constant depending on the values for \( \alpha, n, \) and \( m \)

\[
A(n, m) \equiv -\frac{1}{\alpha}(m^2 + nm - 2n + m),
\]

(49)

and \( \Phi = \Phi(n, m) \) is a hypergeometric function \[44\] depending on time and all the parameters introduced

\[
\Phi(n, m) \equiv _2F_1\left[ \frac{2 + 2m - 3n}{4 + 2m - 6n}, \frac{1}{2} \frac{6 + 4m - 9n}{4 + 2m - 6n}; \frac{At^{3n-m-2}}{M} \right].
\]

(50)

This means, in fact, that it is in general impossible to find the analytical form of the function \( t = t(Q) \), that should be substituted into Eq. (44), so as to deduce \( V = V(Q) \). However, we can easily choose some suitable special values and see what happens at least in the related situations.

**A. Some specific choices**

In the following, we will in fact deal with three subcases, since they soon result analytically treatable and/or physically meaningful. We choose to describe only the essentials we can easily deduce from them, being our main purposes purely indicative here with respect to the still possible full treatment we could perform with numerical techniques, within a context that should certainly be much more refined.

1. **\( M = 0 \) (i.e., complete scalar field dominance)**

When the dust component can be neglected (for example, in a late time superaccelerating stage of complete scalar field dominance), we have that \( M = 0 \). So, Eq. (37) gives an always positive scalar-field energy density for any values of \( n \) and \( m \). Also, due to Eq. (43), we see that \( \dot{Q}^2 < 0 \) when \( m \geq \bar{m} \), which implies the possibility to have parameter values leading to a phantom behavior of the scalar field cosmology.
More in detail, with \( M = 0 \), from Eq. (42) it results that \( \dot{Q}^2 = At^{-m-2} \). Note also that \( A < 0 \) when \( m > \tilde{m} \). (The situation with \( m = \tilde{m} \) is equivalent to choosing \( A = 0 \) and leads to a constant scalar field; since we have also posed \( M = 0 \), we will not discuss it here.) We have \( \dot{Q} = \pm i \sqrt{|A| t^{-m-2}} \) and find an imaginary scalar field as a function of time (not being important the precise sign we choose). It can be soon inverted to

\[
t(Q) = i^{2/m} \left[ \frac{2 \sqrt{-A}}{m} \right]^{2/m} Q^{-2/m}.
\]  
(51)

Inserting Eq. (51) into \( F(t) = -t^{-m}/(2\alpha) \), we can now find \( F = F(Q) \) as a quadratic function

\[
F(Q) = -\frac{m^2}{8\alpha A} Q^2 < 0
\]  
(52)

(being \( Q^2 \) and \( A \) negative), and use Eq. (44) to write the potential proportionally to some power of the scalar field

\[
V(Q) = -i^{-4/m} \left\{ 2^{-3-4/m} m^{2+4/m} (-\alpha A)^{-1-2/m} [-\alpha A - 6n(m - n)] \alpha^{2/m} \right\} Q^{2+4/m}.
\]  
(53)

Of course, we want \( V \) to be a real function of \( Q \), so that \( 4/m \) has to be an integer. If it is an even number, so is also the exponent of \( Q \), and \( i^{-4/m} Q^{2+4/m} \) is the product of two real quantities. When \( 4/m \) is an odd number, such a product is still real, being both the factors purely imaginary. We here choose to deal only with even powers of \( Q \), so neglecting the second possible situation and taking \( 4/m \) even from now on.

Being \( A < 0 \), it results \( n < m(m+1)/(2-m) \) (including \( m = 2 \)), which soon leads to values of the exponent \( p \equiv 2 + 4/m \neq 4 \). Moreover, due to the condition \( m \geq \tilde{m} \) (where \( \sqrt{3} - 1 < \tilde{m} < 2 \)), we find \( p > 4 \). This means that we have only one possible even value for \( p \), namely, \( p_1 \equiv 6 \); this value, of course, leads to a positive power law potential

\[
V(Q) = \frac{\alpha^2(-6n^2 + 7n - 2)}{2^7(2-n)^3} Q^6,
\]  
(54)

where \( 1 < n < 2 \), and we have \( \sigma = n/m \sim 1 \), as required.

2. \( A = 0 \) (i.e., \( m = \tilde{m} \))

We have already noted that \( m = \tilde{m} \) leads to a complete dependence of the scalar-field time behavior on the matter content through the \( M \) parameter, since it is \( \dot{Q}^2 = -Mt^{-3n} \).
From $A = 0$ (i.e., $m = \bar{m}$), we get
\[ n = \frac{m(1 + m)}{2 - m}, \tag{55} \]
with $\sqrt{3} - 1 < m < 2$ (see above, after Eq. 43). We also have $\sigma = (1 + m)/(2 - m) \sim 1.36$ for $m = \sqrt{3} - 1$; thus, $m$ must stay very near to this value.

Let us derive the form of the potential, firstly finding the form of the imaginary $Q(t)$ and then writing time as (suitably choosing the sign)
\[ t(Q) = \left( \frac{3n - 2}{2i\sqrt{M}} \right)^{2/m} Q^{2/m} , \tag{56} \]
(with $i$ the imaginary unit) which is a real and positive quantity, being $Q^2 < 0$. Thus, we have
\[ F(Q) = -\frac{1}{2\alpha} (\frac{3n - 2}{2i\sqrt{M}})^{2/m} Q^{2/m} \tag{57} \]
and, substituting $n$ from Eq. 55, the potential in Eq. 44 becomes
\[ V(Q) = \left[ \frac{i(2+m)}{C(m)} \sqrt{M} \right]^{C(m)} D(m, \alpha, M)Q^{C(m)} , \tag{58} \]
where we have defined
\[ C(m) \equiv \frac{8 - 2m^2}{3m^2 + 5m - 4}, \tag{59} \]
\[ D(m, \alpha, M) \equiv [3m^3 + 6m^4 - 4M\alpha + 4mM\alpha - m^2(3 + M\alpha)] . \tag{60} \]
Again, choosing only even integer exponents $C(m)$ for $Q$ gives real values of the potential in Eq. 58. This gives the rather strange values $m = \{(3\sqrt{17} - 5)/8, (\sqrt{109} - 5)/7, (\sqrt{865} - 15)/20\}$. Moreover, we find that it is impossible to get both $V$ and $F$ real. We conclude that this case is not only little appealing, but also rather uninteresting from the physical point of view.

3. $m = 3n - 2$ (i.e., same powers of time)

When $m = 3n - 2$, it can be shown that taking $n > 1$ always leads to $A < 0$. Eliminating $m$ from the expression of $A$, we can get the expression of $Q(t)$ and find that time is again a real and positive function of $Q$
\[ t(Q) = \left[ -\frac{m}{2} \alpha^{1/2}(2 - 13n + 12n^2 + M\alpha)^{-1/2} \right]^{-2/m} Q^{-2/m} . \tag{61} \]
The coupling function is

$$F(Q) = -\frac{1}{2} \left[ \frac{m}{2} \alpha^{(m+2)/6m} \left( \frac{4m^2 + 3m + 3M \alpha - 4}{3} \right)^{-1/2} \right]^2 Q^2,$$

(62)

and Eq. (44) gives the real potential

$$V(Q) = -\left\{ \frac{m + 6M \alpha - 4}{6} \left[ \frac{m}{2} \alpha^{1/(m+2)} \left( \frac{4m^2 + 3m + 3M \alpha - 4}{3} \right)^{-1/2} \right]^{2(m+2)/m} \right\} Q^{2(m+2)/m},$$

(63)

where $m = 3n - 2$ is left as it is in order to have manageable expressions. Although not dictated by the reality condition, we can anyway ask for the exponent being an even integer $2K$. This gives $m = 2/(K - 1)$ and $n = 2K/[3(K - 1)]$, so that $K \neq 1, K < 3$ (for $n > 1$). Moreover, we have $\sigma = K/3$, which limits $K$ to stay in the interval $(1, 3)$. The allowed values for $n$ are thus only 3 and 4/3. While the first gives clearly nothing else but the case already treated in Sec. III, the second seems to be untenable on observational basis. Also, we have to be careful about the fact that this asymptotic behavior in the neighbourhood of $\infty$ is not necessarily the same as nowadays.

**B. Further general considerations**

There is something left to say in general in the case we are facing in this section. Let us go back to our first general considerations and, starting again from Eq. (6) and taking Eq. (41) into account, we can get the function $p_Q = p_Q(t)$ in its full generality

$$p_Q(t) = -\frac{n(3n - 2)}{\alpha} t^{-m-2} < 0.$$

(64)

We can soon see that such a result directly generalizes the one we already found when $n = 4/3$ and $m = 2$. Also note that, again, we always find $p_Q < 0$. As to the dominant energy condition for $Q$, it is noteworthy that it holds also in this more general situation, since we always find

$$\rho_Q + p_Q = \left( \frac{2n}{\alpha} t^{3n-m-2} - M \right) t^{-3n} \geq 0,$$

(65)

due to previous results on physically allowed times. As before, we are assuming $m \geq (m + 1)/3 > 1$, so that $m > 1$, too. Of course, we could also get a DEC violation, once we accept $t^{3n-m-2} < M \alpha/(2n)$. But, as a matter of fact, due to Eq. (38) and being
\( M \alpha/(2n) < M \alpha/(3n^2) \), such a violation could result only together with the violation of the condition \( \rho_Q > 0 \) on the \( Q \)-energy density. If we do not want to deal with such a negative energy density, therefore, we have to conclude that the situation we find here forbids a phantom cosmological behavior.

Let us compute, then, what happens to the scalar field equation of state. We get

\[
\frac{w_Q}{\rho_Q} = \frac{n(2 - 3n)t^{-m-2}}{3n^2t^{-m-2} - \alpha M^{-3n}},
\]

which is negative only at sufficiently late times, due again to Eq. (41). Note also that, differently from the special case treated in Sec. III, this expression of the equation of state is now in general non constant and, as \( t \to \infty \), it is such that \( w_Q \to -1 + 2(3n)^{-1} \); since \( n > 1 \), this means that the greater is \( n \) the more such an asymptotic value tends to \(-1\), always staying, however, \( > -1 \) and \( < 0 \). On the other hand, Eq. (66) yields

\[
w_Q = -1 \iff t^{3n-m-2} = \frac{\alpha M}{2n},
\]

that is, \( w_Q = -1 \) only at the well defined time

\[
t^* \equiv \left( \frac{\alpha M}{2n} \right)^{\frac{1}{3n-m-2}},
\]

such that \( t^* > \bar{t} \) (the time from which \( \rho_Q > 0 \)), as it could already be expected. Also, it is \( -1 < w_Q < 0 \) when \( t > t^* \), and then we recover an ordinary scalar field behavior in an ordinary cosmological context with dark energy after the decoupling epoch. Of course, however, a family of \( w < -1 \) models is still viable, since we can have a DEC violation when \( \bar{t} < t < t^* \), that is, for a limited period of time, during which it is anyway \( \rho_Q < 0 \).

Finally, the matter density parameter becomes

\[
\Omega_m \equiv \frac{G \rho_m}{3H^2} = \frac{\alpha M}{3n^2t^{m+2-3n}} < 1,
\]

implying that \( \Omega_m = 1/2 \) only for \( n = 4/3 \). It is interesting to note here that \( \Omega_m \) is now a decreasing function of time.

As a last remark, consider the expression of \( \Lambda_{eff} \) due to Eq. (8)

\[
\Lambda_{eff} = \alpha \left( \frac{3n^2}{\alpha}t^{3n-m-2} - M \right)t^{m-3n}
\]

and compare it to the expected dependence \( \Lambda_{eff} \sim H^2 \), which is in fact given by the situation described in the previous section, as seen. Now, since \( H^2 \sim t^{-2} \) this means we
have to compare $t^{-2}$ and $t^{m-3n}$. As a matter of fact, taking Eq. (31) gives $m - 3n \leq -2$, so that we can soon assert that we are actually facing a sort of generalization of what is generally expected and which we refer to in the Introduction. Also, note that we should probably choose values of the parameters such that $3n \simeq -m - 2$. But this would again imply constant values of $w_Q$ and $\Omega_m$, being $p_Q \sim \rho_Q \sim \rho_m$.

Let us finally note that, due to Eq. (36), it is $t \sim a^{1/n}$ and

$$\Lambda_{\text{eff}}(a) = \alpha \left( \frac{3n^2}{\alpha} a^{\frac{3n-m-2}{n}} - M \right) a^{\frac{m-3n}{n}},$$

so that

$$\frac{d \ln \Lambda_{\text{eff}}}{d \ln a} = -\frac{2}{n}$$

when $3n = m + 2$. (This means that for $n = 4/3$ and $m = 2$, for example, we find a decaying behavior of the cosmological constant as clearly described in Eq. (71) with $n = 4/3$, i.e., $\Lambda_{\text{eff}}(a) = \Lambda_0 a^{-3/2}$, $\Lambda_0$ being a constant.)

V. CONFORMAL TRANSFORMATION

In this section we make a sort of final comment. Our purpose is to find the particular conformal transformation that connects the nonminimally coupled model discussed in Sec. III with the usual minimally coupled one we could write down in presence of both the scalar field and dust. As a matter of fact, it is well known that such a connection between, respectively, the \textit{Jordan frame} and the \textit{Einstein frame} is indeed possible [45]. We here will limit our considerations to the special transformation needed to go from one framework to the other one.

Performing on the cosmological metric $g$ (in the Jordan frame) the transformation

$$\hat{g} = f(Q)^2 g$$

($\hat{g}$ denoting corresponding quantities in the Einstein frame), with a generic function $f$, our limited aim is then to determine explicitly the form of this function. It is easy to see that $d\hat{t} = fdt$, $\hat{a} = fa$, and the equations become [45]

$$3\hat{H}^2 = \hat{\rho}_m + \hat{\rho}_Q,$$

$$\hat{Q''} + 3\hat{H}\hat{Q'} - \frac{d\hat{V}}{d\hat{Q}} - \frac{df/d\hat{Q}}{\sqrt{1 + 6df/d\hat{Q}}} = 0,$$

19
where $8\pi G = 1$ and prime denotes derivatives with respect to $\hat{t}$ and

$$\hat{\rho}_m = M f^{-1} \hat{a}^{-3}, \quad d\hat{Q} = \sqrt{\frac{3(dF)^2 - F}{2F^2}} dQ, \quad \dot{V} = \frac{V}{4F^2},$$

being

$$f(Q) = \sqrt{-2F(Q)}.$$  \hspace{1cm} (76)

Note also that, as usual, in the Klein-Gordon equation for $Q(\hat{t})$ an interaction term appears between the scalar field and the matter density (through $f$).

If we use $Q(t)$ as defined in the Jordan frame (i.e., in Sec. III through Eq. (29)), with $F = F(Q)$ as given in Eq. (30), we soon find $f$ as a function of time

$$f(t) = \frac{t^{-1}}{\sqrt{\alpha}},$$

which in turn implies that the potential results constant

$$\dot{V} = \frac{\alpha(2 - 3M\alpha)}{6}.$$  \hspace{1cm} (77)

On the other hand, we can easily check the reality of the root term for $d\hat{Q}$ in Eq. (76), since it is $[3(dF/dQ)^2 - F]/(2F^2) = -M\alpha Q^{-2} > 0$, being $Q$ an imaginary quantity. We can also get the relation between the transformed time $\hat{t}$ and $t$

$$\hat{t} = \frac{\ln t}{\sqrt{\alpha}},$$

and find a de Sitter-like behavior of the scale factor in the Einstein frame

$$a(\hat{t}) = \frac{1}{\sqrt{\alpha}} \exp \left( \frac{\sqrt{\alpha} \hat{t}}{3} \right) \rightarrow \dot{H} = \frac{\sqrt{\alpha}}{3} = const.;$$  \hspace{1cm} (80)

it is also

$$\dot{\hat{Q}} = i\alpha \sqrt{M} \hat{t},$$  \hspace{1cm} (81)

from which

$$\dot{\hat{Q}}' = i\alpha \sqrt{M} \hat{t}, \dot{\hat{Q}}'' = 0.$$  \hspace{1cm} (82)

As to the energy density of the scalar field in the Einstein frame, it also results constant

$$\dot{\hat{\rho}}_{\hat{Q}} \equiv \frac{1}{2} (\dot{\hat{Q}}')^2 + \dot{\hat{V}} = \alpha \left( \frac{1}{3} - M\alpha \right).$$  \hspace{1cm} (83)

Such a definition of the $\hat{Q}$-energy density is, of course, different from the one we have used in the Jordan frame, since there is no coupling now between geometry and $\hat{Q}$. The result
shown in Eq. (84) soon gives a positive energy density for the $\hat{Q}$-field when $M < (3\alpha)^{-1}$, which is allowed by the constraint in Eq. (26) for the positivity of the $Q$ energy density. Note that, for $1/(3\alpha) < M < 16/(3\alpha)$, negative values of $\hat{\rho}_Q$ can also be given.

Let us finally note that, at a first glance, things may appear simpler in the Einstein frame, since there we get a constant potential for the scalar field. However, there also appears a very uncomfortable interaction term between $\hat{Q}$ and $M$ in the energy density term $\hat{\rho}_m$ (together with an additive term in the Klein-Gordon equation for the scalar field). There is no apparent physical justification for such a situation. This fact confirms, in our opinion, the nonequivalent settings of the theory in the two different frameworks.

VI. CONCLUSIVE REMARKS

In the sections above we have examined the consequences of some assumptions on a standard nonminimally coupled cosmology, once we get some inspiration for such assumptions from the modifications introduced by the Renormalization Group (RG) equations in the Quantum Einstein Gravity, with reference to works by M. Reuter and collaborators [6, 7, 8, 40, 46]. This means that we have assumed some power law behaviors for the scale factor and effective $G$ and $\Lambda$ (according to the results of those papers), and we have investigated their use inside our different situation. Both the special case with a $4/3$-exponent and a generalized one have been treated, showing that there is the possibility to get a phantom cosmology, with an equation of state $w_Q < -1$. Among other things, it seems interesting to note that this is not only connected to negative kinetic energy terms for the scalar field, but also that, in some situations, weak energy condition (WEC) violations become possible.

Most of all, we want to underline that the phantom behavior is generally recovered for some ranges of values of the parameters used in the models. This is already known as common in cosmology with nonminimal coupling [31], but it is usually derived neglecting the presence of other ingredients in the cosmic fluid besides the scalar field. Here, we have instead found it with both the scalar and the matter fields, also showing that the parameter values allowing a phantom behavior are strictly constrained by the presence of the parameter $M$, i.e., the present-day matter content. Due to the still wide range of indeterminacy in the measures of the dimensionless matter density parameter $\Omega_m$ [47], this in fact leads to a degeneration.
A comparison of Secs. III and IV shows that the case directly imported from the RG improved setting is by far simpler and more elegant. The exponent $4/3$ has also been tested against SNIa results (in fact the older release) [40]. We want to mention here that, in the context of the usual minimal coupling setting, it is in fact possible to find a general exact solution, which again shows an asymptotic $t^{4/3}$ behavior [48]. This solution has been tested with a full set of data [48, 49, 50].

In other words, it appears like there is a sort of conspiracy in favour of this value. The meaning of this, if any, is clearly still obscure and deserves further investigation.

Also, some comments are in order about the work done in Ref. [10]. It deals with the RG-improved gravity by replacing $G_N$ and $\Lambda$ with scalar functions in the Einstein-Hilbert action. The comparison with our work can be made posing $g_0\lambda_0 = -\frac{5}{2\pi M}$ and $\alpha = -\frac{6}{M}$, so implying $\dot{Q}^2 = 0$, as can be easily seen from our Eq. (28). This, of course, leads to an impossibility of reconstructing the functions $F = F(Q)$ and $V = V(Q)$. This suggests that the comparison is not trivial and deserves further investigation.

Another aspect that, in our opinion, has to be emphasized is the always decreasing behavior of $\rho_Q(t)$ in the Jordan frame, versus the constant $\dot{\rho}_Q$ in the Einstein frame, both in full contradiction with what is commonly expected in minimally coupled cosmologies (where usually a big rip is expected). It seems to us that one should probably be cautious in studying approximated situations in the phase around the present-day evolutionary stage of the universe, simply importing results and expectations from minimally coupled cosmologies into those like the ones we have treated here and, probably, in all kinds of scalar-tensor cosmologies.

On the other hand, let us stress again that the models treated in this paper include only dust (ordinary nonrelativistic matter plus cold dark matter) and scalar-field components, so that they can suitably describe the cosmological situation only starting from the decoupling period. In this epoch, the radiation component decreases much more than the matter one, and has been completely neglected in our treatment. But what does it change in the considerations above when we simply replace dust with radiation? The situation so described is probably interesting only when the scalar field can be better interpreted as an inflaton. (It could also be investigated soon after inflation ends, when the scalar field could resemble what remains of the inflaton plus what eventually can be guessed as added to it in order to give the dark energy field much later emerging and dominating the cosmic fluid content.)
The problem is anyway analytically treatable, at least for a situation similar to that dealt with in Sec. III, but we postpone it to another forthcoming work. We can here simply note that the coupling function $F(Q)$ now results always positive, so indicating that $G_{\text{eff}}$ gives a repulsive force, while the potential $V(Q)$ is always a negative quartic function of $Q$. We are in a regime where both the kinetic energy and the pressure of the scalar field are always negative, but the dominant energy condition can still be recovered. Finally, let us note that we can get a sort of phantom cosmological behavior also in this situation.

| $0 < M < \frac{2}{3\alpha}$ | $-1 < w_Q < 0$ | $Q^2 < 0$ | $p_Q > 0$ | $\rho_Q + p_Q > 0$ | $V > 0$ |
|---------------------------|----------------|----------|----------|----------------|-------|
| $\frac{2}{3\alpha} < M < \frac{8}{3\alpha}$ | $-1 < w_Q < 0$ | $Q^2 < 0$ | $p_Q > 0$ | $\rho_Q + p_Q > 0$ | $V < 0$ |
| $M = \frac{8}{3\alpha}$ | $w_Q = -1$ | $Q^2 < 0$ | $p_Q > 0$ | $\rho_Q + p_Q = 0$ | $V < 0$ |
| $\frac{8}{3\alpha} < M < \frac{16}{3\alpha}$ | $w_Q < -1$ | $Q^2 < 0$ | $p_Q > 0$ | $\rho_Q + p_Q < 0$ | $V < 0$ |

TABLE I: Summary of the possible situations recovered in Sec. III. They differ from one another depending on suitable ranges of the $M$ parameter, characterizing the amount of ordinary matter in the universe.

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