THE ZETA FUNCTIONS OF THE FANO SURFACES OF CUBIC THREEFOLDS

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Abstract. We give an algorithm to compute the zeta function of the Fano surface of lines of a smooth cubic threefold $F \hookrightarrow \mathbb{P}^4$ defined over a finite field. We obtain some examples of Fano surfaces with supersingular reduction.

1. Introduction.

Let $k = \mathbb{F}_q$ be a finite field of characteristic $p \neq 2$ and let $F \hookrightarrow \mathbb{P}^4/k$ be a smooth cubic hypersurface defined over $k$. Through a generic point of $\bar{F}$ pass 6 lines of $\bar{F}$. The variety that parametrizes the lines on $F$ is a smooth projective surface defined over $k$ called the Fano surface of lines of $F$. This surface $S$ is minimal of general type.

In [15], we prove that the Tate conjecture holds for $S$. Recall that the zeta function $Z(X, T)$ of a smooth variety $X/k$ encodes the number of rational points of $X$ over the extensions $k_r = \mathbb{F}_{q^r}$. The Tate conjecture for $X$ predicts that the Picard number $\rho_X$ of $X/k$ equals to the order of pole of the zeta function $Z(X, q^{-s})$ at $s = 1$.

Knowing that the Tate conjecture holds for a Fano surface $S$, it is then natural to study its consequences and applications: this is the aim of this paper. Using the results of Bombieri and Swinnerton-Dyer [5], we give an algorithm that compute the zeta function $Z(S, T)$ for a Fano surface $S$ defined over $k$ containing a $k$-rational point. We are therefore able to obtain the Picard number $\rho_S$, which is an important invariant but usually difficult to compute.

The intermediate Jacobian $J(F)$ of the cubic $F$ is a certain 5-dimensional Abelian variety canonically associated to $F$. This variety $J(F)$ is isomorphic over $k$ to the Albanese variety of $S$, and it is also a Prym variety $Pr(C/\Gamma)$ associated to a degree 2 cover $C \rightarrow \Gamma$ of a certain plane quintic curve $\Gamma$. As we will see, it is equivalent to compute the zeta functions $Z(F, T)$, $Z(S, T)$ or $Z(J(F), T)$.

In [6], Kedlaya gives an algorithm to compute the zeta function of cubic threefolds $F$ that are triple abelian cover of $\mathbb{P}^3$ branched over a smooth cubic surface. This algorithm runs by computing the number of points on the cubic hypersurface $F \hookrightarrow \mathbb{P}^4$ and uses the order 3 symmetries of $F$ in order to reduce the computations. The algorithm we give (implemented in Sage, see [14]) runs for any cubics containing a $k$-rational line. To our knowledge, it seems
also to be the first algorithm proposed in order to compute the zeta function of a Prym variety.

We then study some interesting examples of cubic threefolds and compute the zeta functions of their Fano surfaces over fields of characteristic 5 and 7. We study also the Fano surface $S/Q$ of the Klein cubic threefold:

$$F/Q = \{x_1^2x_2 + x_2^2x_3 + x_3^2x_4 + x_4^2x_5 + x_5^2x_1 = 0\}.$$ 

The surface $S$ has good reduction $S_p$ at every prime $p \neq 11$.

**Corollary 1.** Let be $p \neq 2, 11$. If 11 is a square modulo $p$, then $S_p$ has geometric Picard number 25, otherwise $S_p$ is supersingular, i.e. its geometric Picard number equals $45 = b_2$.

To the knowledge of the author, there are only a few cases where the zeta function of surfaces have been computed: apart from the cases of Abelian surfaces and a product of two curves, the zeta function has been computed for some examples of $K3$, some elliptic surfaces, for Fermat and some Delsarte hypersurfaces (see [18] and the references therein).

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2. The zeta function of a cubic threefold and its Fano surface.

2.1. Notations, hypothesis. Let $F$ be a smooth cubic hypersurface defined over a field $k$ of characteristic not 2 and let $S$ be its Fano surface. The surface $S$ is a smooth geometrically connected variety defined over $k$ [2, Thm 1.16 i and (1.12)]. We will suppose that $S$ has a rational point $s_0$ i.e. that $F$ contains a line defined over $k$. The Albanese variety $A$ of $S$ is 5 dimensional and is defined over $k$ [11 Lem. 3.1]. Let $\vartheta : S \to A$ be the Albanese map such that $\vartheta s_0 = 0$. Let $\Theta$ be the (reduced) image of $S \times S$ under the map $(s_1, s_2) \to \vartheta s_1 - \vartheta s_2$. The variety $\Theta$ is a divisor on $A$ defined over $k$ and $(A, \Theta)$ is a principally polarized abelian variety [4] Prop. 5. In this paper, we call the pair $J(F) = (A, \Theta)$ the intermediate Jacobian of $F$.

2.2. Zeta functions, Weil polynomials. Let $X$ be a smooth projective $n$-dimensional variety over a finite field $k = \mathbb{F}_q$ such that $\bar{X}$ (the variety over an algebraic closure $\bar{k}$ of $k$) is smooth. Let $N_r$ be the number of rational points of $\bar{X}$ over $\mathbb{F}_q$. The zeta function of $X$ is defined by $Z(X, T) = exp(\sum_{r \geq 0} N_r \frac{T^r}{r!})$ and can be written:

$$Z(X, T) = \sum_{i=0}^{i=2n} P_i(X, T)^{(-1)^{i+1}}$$

where $P_i(X, T)$ is the Weil polynomial with integer coefficients:

$$P_i(X, T) = \det(1 - \pi^*|H^i(\bar{X}, \mathbb{Q}_\ell)|),$$

for $\pi$ the Frobenius endomorphism of $\bar{X}$. Suppose that $X$ is a surface. Let $Br(X)$ be the Brauer group of $X$ and let $NS_k(X)$ be its Néron-Severi group.
**Theorem 2.** (9, Artin-Tate Conjecture). Suppose that the Tate conjecture is satisfied by $X$ and the characteristic of $k$ is at least 3. The Brauer group of $X$ is finite and:

$$\lim_{s \to 1} \frac{P_2(X,q^{-s})}{(1 - q^{1-s})^2} = \frac{(-1)^{\rho_X - 1}|Br(X)|Disc(NS_k(X))}{q^{\alpha(X)}|NS_k(X)_{tor}|^2}$$

where $\alpha(X) = \chi(X,\mathcal{O}_X) - 1 + \dim PicVar(X)$. $NS_k(X)_{tor}$ is the torsion subgroup of $NS_k(X)$, $\rho_X$ and $Disc(NS_k(X))$ are respectively the rank and the absolute value of the discriminant of $NS_k(X)/NS_k(X)_{tor}$.

Note that the order of a finite Brauer group is always a square. Let $S$ be the Fano surface of a smooth cubic threefold $F \hookrightarrow \mathbb{P}^4$. The Picard variety $PicVar(S)$ of $S$ is reduced of dimension 5 [21, Lem. 1.1], and $\chi = 6$, therefore $\alpha(S) = 10$.

As $H^2(S,\mathbb{Q}_{\ell}) \cong \wedge^2 H^1(\bar{A},\mathbb{Q}_{\ell}) \cong H^2(\bar{A},\mathbb{Q}_{\ell})$, the zeta function of $S$ can be computed if we know the zeta function of $A$; in fact we just need to know the action of the Frobenius on $H^1(\bar{A},\mathbb{Q}_{\ell})$.

**Theorem 3.** Suppose characteristic $\neq 2$. The étale cohomology groups $H^3(F,\mathbb{Q}_{\ell})$ and $H^1(\bar{A},\mathbb{Q}_{\ell}(1))$ are isomorphic as Galois modules.

**Proof.** By [11] (14) and Thm 3, we have $H^3(F,\mathbb{Q}_{\ell}) \cong T_\ell(A) \otimes \mathbb{Q}_{\ell}(1)$ where $T_\ell(A)$ is the Tate module of $A$. But $T_\ell(A) \otimes \mathbb{Q}_{\ell}$ is isomorphic to $H^1(\bar{A},\mathbb{Q}_{\ell})$. □

**Remark 4.** It would be interesting to know a bound on the order of the torsion subgroup $NS(S)_{tor}$. In the examples we give in Section 6, we compute the values of $\lim_{s \to 1} \frac{P_2(S,q^{-s})}{(1 - q^{1-s})^2}$. For all the limits we computed, we get results on the form $\frac{N}{q^{2r}}$, $N \in \mathbb{Z}$, giving the idea that the group $NS(S)_{tor}$ should be trivial.

### 2.3. Zeta function of a smooth cubic threefold

The upper half part of the Hodge diamond of a smooth cubic threefold $F$ over $\mathbb{C}$ is:

$$
\begin{array}{cccc}
1 & & & \\
& 0 & 0 & \\
& & 0 & 1 & 0 \\
& 0 & 5 & 5 & 0 \\
\end{array}
$$

Thus $P_0(F,T) = 1 - T$, $P_1(F,T) = 1 = P_5(F,T)$, $P_2(F,T) = 1 - qT$. The polynomials $P_4$ and $P_6$ are computed by Poincaré duality: $P_4 = 1 - q^2T$, $P_6(F,T) = 1 - q^3T$.

**Corollary 5.** We have:

$$Z(F,T) = \frac{P_3(F,T)}{(1 - T)(1 - qT)(1 - q^2T)(1 - q^3T)},$$

and

$$N_\pi(F) = (1 + q^r + q^{2r} + q^{3r}) - q^r Tr(\pi^r[H^1(\bar{A},\mathbb{Q}_{\ell})]),$$

where $\pi$ is the Frobenius endomorphism of $\bar{A}$ and $Tr$ denotes the trace.

The roots of $P_1(F,T)$ have absolute value $q^{-i/2}$. We have:

$$P_1(S,T) = P_1(A,T) = P_3(F,\frac{T}{q}).$$
Since by a computer it is possible to compute the number of points of \( F \), it is possible in theory to get the zeta function of \( F, A \) and \( S \). Kedlaya’s algorithm computes the number of points of cubics \( F \) that are cyclic triple cover of \( \mathbb{P}^3 \). Alternatively, we can also compute the number of points of \( A \), which is a Prym variety of a degree 2 étale cover of a certain plane quintic; this is the algorithm we have implemented.

2.4. Zeta function and Picard number of a Fano surface. Let \( S \) be the Fano surface of a smooth cubic \( F \hookrightarrow \mathbb{P}^4 \) over \( \mathbb{F}_q \). We have

\[
Z(S, T) = \frac{P_1(S, T)P_3(S, T)}{(1 - T)P_2(S, T)(1 - q^2T)}.
\]

We can write

\[
P_1 = \prod_{i=1}^{10}(1 - \omega_iT) \in \mathbb{Z}[T] \text{ with } |\omega_i| = q^{1/2}.
\]

As

\[
H^2(\tilde{S}, \mathbb{Q}_\ell) = \wedge^2 H^1(\tilde{S}, \mathbb{Q}_\ell)
\]

we get

\[
P_2(S, T) = \prod_{1 \leq i < j \leq 10}(1 - \omega_i\omega_jT).
\]

Moreover \( P_3(S, T) = q^{15}T^{10}P_1(S, \frac{1}{qT}) \).

Thus

\[
Z(S, T) = \frac{\prod_{i=1}^{10}(1 - \omega_iT) \prod_{i=1}^{10}(1 - \frac{q^2}{\omega_i^2}T)}{(1 - T)(1 - q^2T) \prod_{1 \leq i < j \leq 10}(1 - \omega_i\omega_jT)}.
\]

The order of the pole at 1 of \( Z(S, q^{-s}) \) equals the number of elements of the set \( \{(i, j) / 1 \leq i < j \leq 10 \text{ and } \omega_i\omega_j = q\} \); it is the multiplicity of the root \( 1/q \) in \( P_2(S, T) = P_2(A, T) \). The Tate conjecture for Fano surfaces proved in [15] predicts that the Picard number \( \rho_S \) of \( S \) is equal to the order of pole of the zeta function \( Z(S, q^{-s}) \) at \( s = 1 \), thus by [2.1] we get:

**Corollary 6.** The Fano surface has Picard number at least 5.

*Proof.* This follows from the more general fact that over a finite field, the rank of the Néron-Severi group of an Abelian variety \( A \) is always larger or equal to \( \dim A = q \). For the proof, one may assume that \( A \) is simple. Let \( \omega_1, ..., \omega_{2g} \) be the inverse of the roots of the characteristic polynomial \( P_1 \). We must prove that at least \( g \) among the products \( \omega_i\omega_j \), for \( i < j \), are equal to \( q \). Since \( q/\omega_i \) is also a root, the difficulty occurs only when there are real roots, ie roots equal to \( \pm q^{1/2} \). Since \( A \) is simple, the Honda-Tate Theorem ([13], Th. 4.7.2) implies that if \( q^{1/2} \not\in \mathbb{Z} \), then \( g = 2 \), \( P_1(T) = (T^2 - q)^2 \) and if \( q' = q^{1/2} \in \mathbb{Z} \), then \( P_1(T) = (T \pm q')^2 \), \( g = 1 \). In both cases the multiplicities of the roots are 2.

Let \( N_r(X) \) be the number of rational points of a variety \( X \) over \( \mathbb{F}_q^r \). For each Weil polynomial \( P_i(X, T) \), let \( \omega_{i,1}, ..., \omega_{i,b_i} \) be the reciprocal roots of \( P_i \). The formula expressing the zeta function gives:

\[
N_r(X) = \sum_{i=0}^{\dim X} (-1)^i \sum_{j=1}^{b_i} \omega_{i,j}^r.
\]

We will use in section 4 this formula for computing the numbers \( N_1(S), N_2(S) \) of the Fano surface \( S \) from the knowledge of the reciprocal roots \( \omega_{1,1}, ..., \omega_{1,10} \) of \( P_1(S, T) \).
3. The intermediate Jacobian as a Prym variety.

Let \( F \hookrightarrow \mathbb{P}^4 \) be a smooth cubic threefold defined over the finite field \( k = \mathbb{F}_q \) of characteristic not 2. We will suppose that \( F \) contains a line \( L \) defined over \( k \), or equivalently that the Fano surface contains a \( k \)-rational point.

The aim of this section is to give an algorithm that compute the zeta function of the cubic \( F \), the Fano surface \( S \) and the Albanese variety \( A \) by using the conic bundle structure on an appropriate blow-up of \( F \).

The algorithm will use two auxiliary curves: \( C_L \) the incidence curve parametrizing the lines on \( F \) meeting \( L \) and \( \Gamma_L \) the curve parametrizing planes \( Y \) containing \( L \) and such that the degree 3 plane curve \( Y \cap F \) is the union of three lines.

Let \( x_1, \ldots, x_5 \) be coordinates in \( \mathbb{P}^4 \) and let \( L \hookrightarrow F \) be a \( k \)-rational line on \( F \). We can suppose that \( L = \{ x_1 = x_2 = x_3 = 0 \} \) and \( F = \{ F_{eq} = 0 \} \) with

\[
F_{eq} = \ell_1 x_1^4 + 2\ell_2 x_4 x_5 + \ell_3 x_5^2 + 2q_1 x_4 + 2q_2 x_5 + f,
\]

where \( \ell_1, \ell_2, \ell_3 \) are linear, \( q_1, q_2 \) are quadratic and \( f \) is a cubic homogenous forms in the variables \( x_1, x_2, x_3 \). Let \( X \hookrightarrow \mathbb{P}^4 \) be the plane \( x_4 = x_5 = 0 \). Any plane \( Y \hookrightarrow \mathbb{P}^4 \) containing the line \( L \) meets \( X \) into a unique point. Thus the plane \( X \) parametrizes planes in \( \mathbb{P}^4 \) containing \( L \). For a \( k \)-point \( x = (x_1 : x_2 : x_3 : 0 : 0) \) of \( X = \mathbb{P}^2 \), we denote by \( Y_x \) the unique plane containing \( x \). The intersection of \( F \) and \( Y_x \) has equation \( F_{eq}(y_1 x_1 : y_1 x_2 : y_1 x_3 : y_2 : y_3) = 0 \), i.e.:

\[
y_1(\ell_1 y_2^2 + 2\ell_2 y_2 y_3 + \ell_3 y_3^2 + 2q_1 y_1 y_2 + 2q_2 y_1 y_3 + y_2^2 f) = 0
\]

in the plane \( Y_x \) with coordinates \( y_1, y_2, y_3 \). Therefore, the intersection \( Y \cap F \) equals \( L + Q_x \) where

\[
Q_x = \{ \ell_1 y_2^2 + 2\ell_2 y_2 y_3 + \ell_3 y_3^2 + 2q_1 y_1 y_2 + 2q_2 y_1 y_3 + y_2^2 f = 0 \}.
\]

The conic \( Q_x \) is irreducible over \( \bar{k} \) if and only if it is smooth. The scheme parametrizing the planes \( Y_x \) such that \( Q_x \) decomposes (over an extension) as \( Q_x = L_1 + L_2 \) with \( L_1, L_2 \) two lines, is a plane reduced quintic curve \( \Gamma_L \hookrightarrow X = \mathbb{P}^2(x_1, x_2, x_3) \) defined over \( k \), of equation \( \det M_{\Theta} = 0 \), where:

\[
M_{\Theta} = \begin{pmatrix}
\ell_1 & \ell_2 & q_1 \\
\ell_2 & \ell_3 & q_2 \\
q_1 & q_2 & f
\end{pmatrix}.
\]

Let \( C_L \) be the incidence scheme parametrizing the lines that meet \( L \).

**Lemma 7.** ([15] Lem. 2, 4; [12] Pro. (1.25)). The plane quintic curve \( \Gamma_L \) has at most nodal singularities. A points \( x \) on \( \Gamma_L \) is singular if and only if \( Y_x \cap F = L + 2L' \), where \( L' \) is a line. The curve \( \Gamma_L \) is smooth for \( L \) generic. The scheme \( C_L \) is a reduced one dimensional subscheme of the Fano surface \( S \). There is a natural degree 2 map \( \mu: C_L \rightarrow \Gamma_L \) that is ramified precisely over the singularities of \( \Gamma \). The point \( p \) of \( C_L \) is smooth if and only if \( \mu(p) \) is smooth. If \( \Gamma_L \) is smooth, the Albanese variety of \( S \) is isomorphic to the Prym variety \( Pr(C_L/\Gamma_L) \).

Let \( \tilde{F} \rightarrow F \) be the blow-up of \( F \) along a line \( L \) on \( F \). The variety \( \tilde{F} \) has a natural structure of a conic bundle:

\[
\tilde{F} \rightarrow \mathbb{P}^2 = X
\]
where $X$ parametrizes the planes containing the line $L$. The fiber over the point $x$ is (isomorphic to) the quadric $Q_x$ such that $F \cap Y_x = Q + L$. For $x$ defined over $k$, the number of $k$-points of $Q_x$ is:

i) $q + 1$ if $Q_x$ is geometrically irreducible i.e. $x \notin \Gamma_L(k)$ or if $x \in \Gamma_L(k)$ is a singular point of $\Gamma$, in that case $Q_x = 2L'$ with $L'$ a $k$-rational line on $F$.

ii) $2q+1$ if $Q_x = L_1 + L_2$ with $L_1, L_2$ two different lines defined over $k$, i.e. the 2 points in $C_L$ over $x \in \Gamma_L(k)$ are in $C_L(k)$.

iii) 1 if the conic $Q_x$ degenerates to a union of two line over a (degree 2) extension of $k$, i.e. if the 2 points in $C_L$ over $x \in \Gamma_L(k)$ are not in $C_L(k)$.

Let $N_r(X)$ denotes the number of rational points of a variety $X$ over the degree $r$ extension $k_r = \mathbb{F}_{q^r}$ of $k$. The following Proposition is [5, Formula (18)], however we reproduce the proof here because it explains the algorithm computing $N_r(F)$ we describe below.

**Proposition 8.** We have

$$N_r(F) = q^{3r} + q^{2r} + q^r + 1 + q^r(N_r(C_L) - N_r(\Gamma_L)).$$

**Proof.** Let $a$ be the number of $\mathbb{F}_{q^r}$-rational singularities of $\Gamma_L$ (and $C_L$).

Taking care of the three above possibilities i), ii) and iii), we get:

$$N_1(F) = (q + 1)(N_1(\mathbb{P}^2) - N_1(\Gamma_L) + a) + (2q + 1)\left(\frac{1}{2}(N_1(C_L) - a) + (N_1(\Gamma_L) - a - \frac{1}{2}(N_1(C_L) - a))\right)$$

thus

$$N_1(F) = q^3 + 2q^2 + 2q + 1 + q(N_1(C_L) - N_1(\Gamma_L)).$$

As each point on $L \hookrightarrow F$ is replaced by a $\mathbb{P}^1$ on $\bar{F}$, we have moreover:

$$N_1(F) = N_1(F) - (q + 1) + (q + 1)^2,$$

thus $N_1(F) = q^3 + q^2 + q + 1 + q(N_1(C_L) - N_1(\Gamma_L)).$ \hfill \Box

Let $\mu : C_L \rightarrow \Gamma_L$ be the degree 2 map and let $x$ be a $k$-rational smooth point on $\Gamma$. In order to compute the numbers $N_r(C_L) - N_r(\Gamma_L)$, we need to understand when the two $k$-rational points in $\mu^{-1}x$ are $k$-rational. For $1 \leq i \leq 3$, let $\delta_i \in H^0(\Gamma_L, \mathcal{O}(a))$, $a = 2$ or $4$ be the $(i, i)$-minor of the matrix $M_0$.

**Proposition 9.** Let $x$ be a $k$-rational smooth point of $\Gamma$. There exists an integer $1 \leq i = i(x) \leq 3$ such that $\delta_i(x) \neq 0$. The curve $C_L$ has two rational points over $x \in \Gamma_L(k)$ if and only if $-\delta_i(x) \in (k^*)^2$.

**Proof.** Let $x$ a $k$-rational smooth point of $\Gamma$ and let $Q = Q_x$ such that $F \cap Y_x = L + Q$. The line $L = \{y_1 = 0\}$ meets $Q_x \hookrightarrow Y_x$ in the points such that:

$$y_1 = \ell_1 y_2^2 + 2\ell_2 y_2 y_3 + \ell_3 y_3^2 = 0.$$

Therefore, if $-\delta_3(x) = \ell_3^2 - \ell_1 \ell_2(x)$ is nonzero, the curve $C_L$ has two rational points over $x \in \Gamma_L(k)$ if and only if $-\delta_3(x) \in (k^*)^2$.

For $\delta_3(x) = 0$, we only sketch the proof ; see also [3, Lemme 1.6] and its proof. In that case, we have $Y_x \cdot F = L + Q_x = L + L_1 + L_2$ with $L_1, L_2$ defined over $\bar{k}$ and meeting in a $k$-rational point $p$ of $L$. It is possible to explicitly compute a model in $\mathbb{P}^3$ of (the degree 6) scheme $X_p$ of lines in the cubic $F$ going through $p$. By knowing $X_p$, we can determine whether the two points on $X_p$ corresponding to the lines $L_1, L_2$ are $k$-rational or not and this is so if and only if $-\delta_1$ is a nonzero square or $-\delta_2$ is a nonzero square. \hfill \Box
Let us describe the algorithm for the computation of the numbers \( N_r(F) \) and \( N_r(C_L) - N_r(\Gamma_L) \).

The input data is a cubic threefold \( F \) over \( \mathbb{F}_q \) containing a \( \mathbb{F}_q \)-rational line \( L \). To this data is associated the matrix \( M_\Theta \) defined above whose determinant is the equation of the quintic \( \Gamma_L \leftrightarrow X = \mathbb{P}^2 \) (maybe singular). Then we compute \( N_r = N_r(C_L) - N_r(\Gamma_L) \) as follows:

Initiate \( N_r := 0 \). For \( x \in \mathbb{P}^2(\mathbb{F}_q) \), if \( \det(M_\Theta)(x) = 0 \) then if \(-\delta_3(x) \in (\mathbb{F}_q^*)^2\) then \( N_r := N_r + 1 \), else if \(-\delta_3(x) \neq 0 \), then \( N_r := N_r - 1 \), otherwise if \(-\delta_1(x) \in (\mathbb{F}_q^*)^2\) then \( N_r := N_r + 1 \) else if \(-\delta_1(x) \neq 0 \), then \( N_r := N_r - 1 \), otherwise if \(-\delta_2(x) \in (\mathbb{F}_q^*)^2\) then \( N_r := N_r + 1 \) else if \(-\delta_2(x) \neq 0 \) then \( N_r := N_r - 1 \) end if, end for.

The output \( N_r \) equals \( N_r(C_L) - N_r(\Gamma_L) \). Remark that the \(-\delta_i\) are transition functions of an invertible sheaf \( \mathcal{L} \) on \( \Gamma_L \) such that \( \mathcal{L} \otimes^2 = \omega_T \). The data of \( \mathcal{L} \) corresponds to the degree 2 cover \( \mu : C_L \rightarrow \Gamma_L \) and a point on \( \Gamma_L \) is singular if and only if \( \forall 1 \leq i \leq 3, \delta_i(x) = 0 \).

The knowledge of \( N_1, \ldots, N_5 \) is enough to get the 5 first coefficients of the degree 10 polynomial \( P_1(Pr(C_L/\Gamma_L), T) \in \mathbb{Z}[T] \) and the remaining 5 ones are determined by the symmetries of \( P_1 \).

Remark 10. This algorithm for computing the action of the Frobenius on the Prym variety \( Pr(C_L/\Gamma_L) \) is generalizable to other plane curves occurring as discriminant locus of other quadric bundles, see e.g. [3].

4. Examples

4.1. Reduction in characteristic 5 and 7 of a cubic threefold. Let \( F \leftrightarrow \mathbb{P}^4 \) be the cubic threefold with equation:

\[
F_{eq} = x_1x_4^3 + 2x_2x_4x_5 + x_3x_5^2 + 2q_1x_4 + 2q_2x_5 + f, 
\]

where:

\[
q_1 = x_1^2 + 2x_2^2 + x_2x_3 + x_3^2 \\
q_2 = x_1x_2 + 4x_2x_3 + x_3^2 \\
f = x_2^2x_3 - (x_1^3 + 4x_1x_2^2 + 2x_3^2).
\]

The cubic \( F \) is smooth in characteristic 5, 7, 11 and 13 ; it is singular in characteristic 2, 3. The associated quintic curve \( \Gamma = \Gamma_L \) is smooth in characteristic 3, 7, 11, 13, but singular in characteristic 2, 5.

Remark 11. The cubic \( F/Q \) and its Fano surface \( S/Q \) have bad reduction at the place 3, however the intermediate Jacobian \( J(F) \simeq Pr(C_L/\Gamma_L) \) has good reduction. This is the same phenomena as for the curves and their Jacobian. We remark also that the curves \( \Gamma_L \) and \( C_L \) both have bad reduction at the place 5, but the associated Prym variety has good reduction.

We have implemented the algorithm in Sage. Using a personal laptop, it takes 5 minutes to obtain \( P_1(S, T) \) for \( S \) over \( \mathbb{F}_7 \).

Over \( \mathbb{F}_5 \), we get (see [14]) :

\[
P_1(S_{/\mathbb{F}_5}, T) = (5T^2 + 1)(625T^8 + 50T^6 + 40T^5 - 6T^4 + 8T^3 + 2T^2 + 1).
\]

The Fano surface \( S_{/\mathbb{F}_5} \) has Picard number 5 and contains 33 \( \mathbb{F}_3 \)-rational points. We have:

\[
A_5 := \lim_{s \to 1} \frac{P_2(S_{/\mathbb{F}_5}, 5^{-s})}{(1 - 5^{1-s})^5} = \frac{218 \cdot 3^5 \cdot 157}{5^{10}}.
\]
Over $\mathbb{F}_7$, we get:

$$P_1(S/\mathbb{F}_7, T) = 1 + 4T + 15T^2 + 46T^3 + 159T^4 + 460T^5 + 1113T^6 + 2254T^7 + 5145T^8 + 9604T^9 + 16807T^{10}.$$ 

It is an irreducible polynomial over $\mathbb{Q}$, therefore by the Honda-Tate Theorem \cite{10} Theorems 2-3, App. 1, the intermediate jacobian $J(F)$ of $F$ is simple. The Fano surface $S/\mathbb{F}_7$ has Picard number 5 and 97 $\mathbb{F}_7$-rational points. We obtain:

$$A_7 := \lim_{s \to 1} \frac{P_2(S/\mathbb{F}_7, 7^{-s})}{(1 - 7^{1-s})^5} = \frac{2^4 \cdot 83^2 \cdot 557 \cdot 5737}{7^{10}}.$$ 

Remark 12. We prove in \cite{10} that a generic Fano surface over $\mathbb{C}$ has Picard number $\rho = 1$. One would like to exhibit an example of a Fano surface over $\mathbb{Q}$ with $\rho = 1$. By reducing the above Fano surface $S/\mathbb{Q}$ modulo a prime we obtain the bound $\rho_S \leq 5$. Since $A_5/A_7$ is not a square in $\mathbb{Q}$, we can apply the van Luijk method \cite{8} and obtain the inequality $\rho_S \leq 4$.

Over the field $\mathbb{F}_{11}$, the computation becomes difficult: it needs 4 minutes to get $N_5$ but more that 24 hours to get $N_5$. By [6, Lem. 1.2.3], for positive integers $q, d, j$ and complex numbers $a_1, \ldots, a_{j-1}$, there exists a certain disk of radius $\frac{q}{j} q^{j/2}$ which contains every $a_j$ for which we can choose $a_{j+1}, \ldots, a_d \in \mathbb{C}$ so that the polynomial

$$R(T) = 1 + \sum_{j=1}^{j=d} a_j T^j$$

has all roots on the circle $|T| = q^{-1/2}$. In our case, $\frac{q}{j} q^{j/2} = \frac{10}{9} 11^{5/2} = 802.623...$ and we obtain that $P_1(S/\mathbb{F}_{11}, T) = Q_a$, where $a$ is an integer in $\{80, \ldots, 332\}$ and

$$Q_a = 1 - T + 13T^2 + T^3 - 28T^4 + aT^5 - 11 \cdot 28T^6 + 112T^7 + 13T^8 - 11^4 T^9 + 11^5 T^{10}.$$ 

For all the consecutive values $a \in \{80, \ldots, 332\}$, the polynomial $Q_a$ has its roots equal to $11^{-1/2}$ (with error at most $10^{-10}$) and we cannot distinguish the $a$ corresponding to our Fano surface $S$.

4.2. The Klein cubic threefold. Let $S/\mathbb{Q}$ be the Fano surface of lines of the Klein cubic threefold:

$$F/\mathbb{Q} = \{x_1^2 x_2 + x_2^2 x_3 + x_3^2 x_4 + x_4^2 x_5 + x_5^2 x_1 = 0\}.$$ 

It is easy to check that $F$ (hence $S$) has good reduction at every prime $p \neq 11$.

Proposition 13. Let us suppose $p \neq 2$. If 11 is not a square modulo $p$, then $S_p$ the reduction mod $p$ is supersingular i.e. its geometric Picard number equals 45 = $b_2$, otherwise $S_p$ has geometric Picard number 25.

Proof. Let be $\nu = \frac{-1 + \sqrt{-11}}{2}$, $O = \mathbb{Z}[\nu]$ and $E = \mathbb{C}/\mathbb{Z}[\nu]$. The intermediate jacobian $J(F)/\mathbb{C}$ is isomorphic to $E^5$ (see \cite{17}). By \cite{19} App. A3, the elliptic curve $E$ has the following model over $\mathbb{Q}$:

$$y^2 + y = x^3 - x^2 - 7x + 10,$$

which we still denote by $E$. The curve $E$ has good reduction for prime $p \neq 11$ and it has complex multiplication by $\mathcal{O}$ (over a certain extension).
We use the criteria of Deuring [7, Chap. 13, Thm 12]: for odd \( p \neq 11 \), the reduction of \( E \) modulo \( p \) is a supersingular if and only if \( p \) is inert or ramified in \( \mathcal{O} \). By classical results on number theory, an odd prime \( p \neq 11 \) is inert or ramified in \( \mathcal{O} \) if and only if 11 is not a square modulo \( p \).

Over an extension, the intermediate Jacobian \( J(F) \) is isogenous to \( E^5 \).

By [10], the geometric Picard number of the reduction modulo \( p \) of \( J(F) \) is therefore 45 if 11 is not a square modulo \( p \), and 25 otherwise. \( \square \)

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