Boundary Value Problem for
\[ r^2 \frac{d^2 f}{dr^2} + f = f^3 \] (II): Connection Formula

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Abstract
In this paper, we study the analytic expansion in a small neighbor-
hood of 0 in the complex plane for the solution to the equation
\[ p \frac{dp}{dz} - p = z(z - 1)(z - 2) \] satisfying \( p(z) = -z + O(z^2) \) as \( z \to 0 \).
We show that the expansion is valid for \( |z| \leq s_0 \), where \( s_0 > 1 \). Then we get an explicit formula for \( p(1) \) which is used to give the connection formula for the problem
\[ r^2 f'' + f = f^3, \quad f(1) = 0, \quad f(\infty) = 1. \]

1 Introduction
In this paper we continue the discussion in [3] for the boundary value prob-
lem
\[ r^2 f'' + f = f^3, \quad 0 < r < \infty, \quad (1.1) \]
\[ f(r) \to 0, \text{ as } r \to 0, \quad (1.2) \]
\[ f(\infty) = 1, \quad (1.3) \]
where \( f = f(r) \), and ‘ means the derivative. By the transformation \( r = e^x, f(r) = y(x) \), the equation is changed to
\[ y'' - y' + y = y^3. \] In [3], we proved that the following boundary value problem
\[ y'' - y' + y = y^3, \quad 0 < x < \infty, \quad (1.4) \]
\[ (P^+) \quad y(0) = 0, y(\infty) = 1, \quad (1.5) \]
\[ y(x) > 0, \quad 0 < x < \infty, \quad (1.6) \]

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has a unique solution $y^*(x)$, and we denote $a^* = y^{**}(0)$, which is a positive number. We will find the value of this number $a^*$, which is very important for this equation.

Since $y^*(x)$ is monotonically increasing on $x \in [-\epsilon^-, \infty)$ for some small number $\epsilon^- > 0$, and $y^*(0) = 0, y^*(\infty) = 1$, we have $z(x) = 1 - y^*(x)$ is ranged in $[0, 1 - y^*(-\epsilon^-)]$, where $y^*(-\epsilon^-)$ is a small negative number. We take $z'(x)$ as a function of $z$ for $z \in [0, 1 - y^*(-\epsilon^-)]$, say $z' = P(z)$. Then $P(z)$ satisfies a first order differential equation derived from the equation for $y^*$, and a corresponding asymptotics as $z \to 0$, derived from the asymptotics of $y^*$ as $x \to \infty$. This forms an initial value problem for $P(z)$. If we can get the analytic formula for $P(z)$, then we are at least able to get $a^* = y^*(0)$ by the property of $y^*(x)$ at $x = +\infty$, which is a way to solve the connection problem, and get the global behaviour of the solution.

To solve the initial value problem, we change the the first order equation into an integral equation combined with the initial condition

$$q(z)^2 = -2 \int_0^1 tq(zt) dt + 2 - 2z + \frac{z^2}{2}, \quad 0 \leq z \leq 1 + \delta_0,$$

$$q(z) = 1 + O(z), z \to 0,$$

where $\delta_0 = -y^*(-\epsilon^-) > 0$ is a small number. We first show that this problem has a unique solution. We construct the solution by an iteration sequence $\{q_n(z)\}$, which gives $P(z) = -z \lim_{n \to \infty} q_n(z)$ for $z \in [0, \epsilon]$ for a small $\epsilon > 0$.

We then want to analytically extend the solution to a neighborhood of 0 in the complex plane ($|z| \leq \epsilon$). We first extend $\{q_n(z)\}$ into $|z| \leq \epsilon$, which is seen by the construction of $\{q_n(z)\}$. The $\{q_n(z)\}$ form a uniformly bounded sequence on $|z| \leq \epsilon$. By the normal family theory [1], $\{q_n(z)\}$ has a convergent subsequence in $|z| \leq \epsilon$, which converges to an analytic function in $|z| \leq \epsilon$. Specially this analytic function is equal to $P(z)$ for $z \in [0, \epsilon]$. So we have extend $P(z)$ from the interval $[0, \epsilon]$ to $|z| \leq \epsilon$. Therefore we get the analytic expansion $P(z) = \sum_{n=1}^{\infty} b_n z^n$ for $|z| \leq \epsilon$, where the coefficients $b_n$ can be obtained from the equation of $P(z)$. We then show this series is convergent to $P(z)$ wherever $P(z)$ is bounded by the fact that this series is convergent to $P(z)$ on $[0, \epsilon]$, and $P(z)$ is bounded on $[0, 1 + \delta_0]$. We then obtain a formula for $a^* = y^*(0)$. The analytic continuity to a small neighborhood of 0 in complex plane will be done in Sect. 2, and the convergent radius will be discussed in Sect. 3. This method also works for other similar equations to solve the connection problem.
2 Analytic Continuity

It has been proved that the problem \((P^+)\) has a unique solution 
\[ y^*(x) = y(x, a^*). \]

We then want to extend the solution to the negative axis. The method we will use to extend the solution is tightly related to the value of \(a^*\). So we need more properties of the solution. The idea here is to represent \(y^*\) as a function of \(y^*\), and then to reduce (1.4) to a first-order equation. Then the first-order equation might help us to get the value of \(a^*\).

Since \(y^*(0) = a^* > 0\), we can choose a small \(\epsilon^- > 0\), such that \(y^*(x) > 0\), for \(x \in [-\epsilon^-, \infty)\) (Lemma 4 [3]). Let \(\delta_0 = -y^*(-\epsilon^-) > 0\), and

\[ z(x) = 1 - y^*(x), \quad (2.1) \]

for \(-\epsilon^- < x < \infty\). Then equation (1.4) becomes

\[ z'' - z' = z(z - 1)(z - 2), \quad (2.2) \]

where \(0 < z \leq 1 + \delta_0\), and \(z(x) \to 0\), as \(x \to \infty\). Because \(z'\) is always negative(Lemma 4 [3]), i.e., \(z(x)\) is strictly monotone, \(z'(x)\) can be thought as a function of \(z\). Let

\[ z'(x) = P(z(x)). \quad (2.3) \]

We then obtain the following lemma by Lemma 4 (iii) [3].

**Lemma 1** \(p = P(z)\) solves the following problem

\[ pp' - p = z(z - 1)(z - 2), 0 < z < 1 + \delta_0, \quad (2.4) \]

\[ p(z) = -z + O(z^2), z \to 0, \quad (2.5) \]

and

\[ a^* = -P(1), \quad (2.6) \]

\[ a^{*2} = \frac{1}{2} + 2 \int_0^1 P(z) \, dz. \quad (2.7) \]

**Proof.** In Lemma 4 (iii) [3], we obtained that \(\frac{y'(x)}{y(x)-1} = -1 + O(e^{-x})\), \(y(x) = 1 - ce^{-x} + O(e^{-2x})\), as \(x \to \infty\), which implies (2.3). The formula (2.4) is obtained from \(a^* = y^*(0)\), by using (2.1), (2.3) and the fact \(y^*(0) = 0\). And (2.7) follows from taking integral from 0 to 1 on both sides of the equation (2.4) and using (2.6). □
We next want to find analytic expression of $P(z)$, so that $a^* = -P(1)$ can be explicitly defined. The plan is as follows. First we use the integral equation obtained from (2.4) to express $P(x)$ as a limit of a function sequence in a neighborhood of 0. Each function of the sequence can be analytically extended to a neighborhood of 0 in the complex plane. Then by the normal family theory of analytic functions, this sequence has a convergent subsequence, which converges to $P(z)$ on the real axis part, and converges to an analytic function in the neighborhood of 0 in the complex plane. So $P(z)$ is analytically continued to a neighborhood of 0 in the complex plane.

Then we have the Taylor’s series expansion of $P(z)$ in this neighborhood. We then prove the convergent radius of this series is greater than 1, so that the expression of $a^*$ is found, which will be discussed in the next section.

Equation (2.4) with the condition (2.5) is equivalent to the following integral equation

$$\frac{1}{2}p^2 = \int_0^z p(s) \, ds + z^2 - z^3 + \frac{z^4}{4}, \quad 0 \leq z \leq 1 + \delta_0. \tag{2.8}$$

Let

$$p(z) = -zq(z).$$

Then (2.8) becomes

$$q(z)^2 = -2 \int_0^1 t \, q(zt) \, dt + 2 - 2z + \frac{z^2}{2}, \quad 0 \leq z \leq 1 + \delta_0. \tag{2.9}$$

**Lemma 2** The following problem

$$q(z)^2 = -2 \int_0^1 t \, q(zt) \, dt + 2 - 2z + \frac{z^2}{2}, \quad 0 \leq z \leq 1 + \delta_0, \tag{2.10}$$

$$q(z) = 1 + O(z), z \to 0, \tag{2.11}$$

has a unique solution. And then we obtain that equation (2.4) with (2.3) has unique solution $P(z)$.

**Proof.** We have seen that this problem has a solution by Lemma 1. Now suppose there are two solutions $q(z)$ and $\bar{q}(z)$ to the problem. Then by (2.11) there exist $M_1 > 0$, $\epsilon_1 > 0$, satisfying $M_1 \epsilon_1 < \frac{2}{3}$, so that if $0 \leq z \leq \epsilon_1$,

$$|q(z) - 1| \leq M_1 z \leq M_1 \epsilon_1 < \frac{2}{3}, \tag{2.12}$$

$$|\bar{q}(z) - 1| \leq M_1 z \leq M_1 \epsilon_1 < \frac{2}{3}. \tag{2.13}$$
which imply that 
\[ q(z), q'(z) > 0. \]

Since \( q, \bar{q} \) satisfy equation (2.10), there is
\[ (q(z) + \bar{q}(z))|q(z) - \bar{q}(z)| \leq 2 \int_0^1 t \left(|q(zt) - 1| + |\bar{q}(zt) - 1|\right) dt \leq \frac{4}{3} M_1 |z|. \]

By (2.12) and (2.13), we have
\[ q(z) + \bar{q}(z) \geq 2(1 - M_1 \epsilon_1). \]

Therefore we arrive at
\[ |q(z) - \bar{q}(z)| \leq 2 M_1 \gamma_1 |z|, \]
for 0 \( \leq z \leq \epsilon_1 \), where \( \gamma_1 = \frac{1}{3(1-M_1 \epsilon_1)} \). By induction method (see the proof of Lemma 4), we can prove that
\[ |q(z) - \bar{q}(z)| \leq 2 M_1 \gamma_1^n |z|, \]
for any integer \( n > 0 \), and 0 \( \leq z \leq \epsilon_1 \). Letting \( n \to \infty \), we see that \( q(z) = \bar{q}(z) \) for 0 \( \leq z \leq \epsilon_1 \).

When \( z > \epsilon_1 \), consider equation (2.4). Since \( P(z) < 0 \), for \( z \in [\epsilon_1, 1+\delta_0] \), the Lipschitz condition is satisfied. Then the general uniqueness theorem [2] of solution implies \( q = \bar{q} \). \( \square \)

Now let us make an iteration sequence
\[ q_0(z) = 1, \quad (2.14) \]
\[ q_{n+1}(z) = -2 \int_0^1 t q_n(zt) dt + 2 - 2z + \frac{z^2}{2}, \quad n \geq 0, \quad (2.15) \]
in order to give a limit representation for the solution \( q(z) \). Choose \( \epsilon_0, M > 0 \) satisfying
\[ M \epsilon_0 < \frac{2}{3}, \]
\[ (2 + \epsilon_0)(1 + M \epsilon_0) \leq M, \]
\[ (1 + M \epsilon_0)(\frac{2}{3} M + 2 + \frac{\epsilon_0}{2}) \leq M. \]

For example we can choose \( M = 10 \). Then a sufficiently small \( \epsilon_0 \) would satisfies all of the three conditions. We have the following results for the iteration sequence.

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Lemma 3

\[ q_n(z) > 0, \quad n \geq 0, \quad (2.16) \]
\[ |q_n(z) - 1| \leq M|z| < 1, \quad n \geq 1, \quad (2.17) \]

for \( 0 \leq z \leq \epsilon \), where
\[ \epsilon = \min \left( \frac{1}{4}, \epsilon_0 \right). \]

Proof. By (2.14) and (2.15), there is
\[ q_1^2 = 1 - 2z + \frac{z^2}{2}. \quad (2.18) \]

Let
\[ f_1(x) = -2z + \frac{z^2}{2}, \]
which satisfies
\[ |f_1| \leq 2\epsilon + \frac{\epsilon^2}{2} < 1, \]
for \( 0 \leq z \leq \epsilon \), since \( \epsilon < 1/4 \). We then have
\[
|q_1 - 1| = |(1 + f_1) - 1| = \left| \sum_{m=0}^{\infty} \frac{\frac{1}{2}(\frac{1}{2} - 1) \cdots (\frac{1}{2} - m + 1)}{m!} f_1^m - 1 \right| \\
\leq \sum_{m=1}^{\infty} |f_1|^m = \frac{|f_1|}{1 - |f_1|},
\]

where we have used \( |\frac{1}{2} - j| = j - \frac{1}{2} < j \) (\( j > 0 \)), and \( |(\frac{1}{2} - 1) \cdots (\frac{1}{2} - m + 1)| < (m - 1)! \) (\( m > 1 \)). Since \( 0 < \epsilon < \epsilon_0 \), by the definition of \( M \) and \( \epsilon_0 \) above, we have
\[ (2 + \frac{\epsilon}{2})(1 + M\epsilon) \leq M, \]

which implies \( |f_1(z)|(1 + M|z|) \leq (2|z| + \frac{|z|^2}{2})(1 + M|z|) \leq |z|(2 + \frac{\epsilon}{2})(1 + M\epsilon) \leq M|z| \), or \( |f_1(z)| \leq (1 - |f_1(z)|)M|z| \), and then
\[ \frac{|f_1|}{1 - |f_1|} \leq M|z|. \]

Hence \( |q_1 - 1| \leq M|z| \).

Now suppose \( |q_n - 1| \leq M|z| \). Let us show \( |q_{n+1} - 1| \leq M|z| \).
Let 

\[ f_{n+1} = -2 \int_0^1 t(q_n(zt) - 1)dt - 2z + \frac{z^2}{2}. \]

Then we have 

\[ q_{n+1}^2 = 1 + f_{n+1}. \]

We want to show 

\[ |f_{n+1}| \leq \frac{M|z|}{1 + M|z|}. \] (2.19)

In fact, there is 

\[
|f_{n+1}| \leq 2 \int_0^1 tM|z|tdt + 2|z| + \frac{|z|^2}{2} \\
= \frac{2}{3}M|z| + 2|z| + \frac{|z|^2}{2} \\
\leq \left( \frac{2}{3}M + 2 + \frac{\epsilon}{2} \right)|z|.
\]

Now because \(0 < \epsilon \leq \epsilon_0\), by the definition of \(\epsilon_0\), we deduce 

\[
(1 + M|z|)|f_{n+1}| \leq (1 + M\epsilon)|f_{n+1}| \\
\leq (1 + M\epsilon)(\frac{2}{3}M + 2 + \frac{\epsilon}{2})|z| \\
\leq (1 + M\epsilon_0)(\frac{2}{3}M + 2 + \frac{\epsilon_0}{2})|z| \\
\leq M|z|,
\]

which implies (2.19), and then 

\[
\frac{|f_{n+1}|}{1 - |f_{n+1}|} \leq M|z|. \] (2.20)

So we get 

\[
|q_{n+1} - 1| = |(1 + f_{n+1})^{\frac{1}{2}} - 1| \\
= \left| \sum_{m=0}^{\infty} \frac{1}{m!} \left( \frac{1}{2} - 1 \right) \cdots \left( \frac{1}{2} - m + 1 \right) (f_{n+1}^m - 1) \right| \\
\leq \frac{|f_{n+1}|}{1 - |f_{n+1}|} \leq M|z|.
\]

The lemma is proved. \(\square\)
Lemma 4 If $0 \leq z \leq \epsilon$, where $\epsilon$ is defined in Lemma 3, then there is

$$|q_{n+1} - q_n| \leq M^n|z|,$$

for $n \geq 0$, where

$$\gamma = \frac{1}{3(1 - M\epsilon)} < 1.$$

Proof. Since $q_0 = 1$, by Lemma 3, we see that

$$|q_1 - q_0| \leq M|z|.$$

Now suppose

$$|q_n - q_{n-1}| \leq M^{n-1}|z|,$$

let us show equation (2.21) is true.

In fact, when $n \geq 1$, there is

$$(q_{n+1} + q_n)||q_{n+1} - q_n|| = |(q_{n+1} + q_n)(q_{n+1} - q_n)|$$

$$= \left| -2 \int_0^1 t(q_n(zt) - q_{n-1}(zt))dt \right|$$

$$\leq 2 \int_0^1 t |q_n(zt) - q_{n-1}(zt)| dt$$

$$\leq \frac{2}{3}M^{n-1}|z|.$$

Still by Lemma 3, we have

$$q_{n+1} + q_n \geq 2 - 2M|z| \geq 2 - 2M\epsilon.$$

Thus

$$|q_{n+1} - q_n| \leq \frac{2}{6(1 - M\epsilon)}M^{n-1}|z|$$

$$= M\gamma^{n-1}|z|.$$

So we have proved the lemma. \qed

Lemma 5 For the $P(z)$ defined by (2.3), we have

$$P(z) = -z \lim_{n \to \infty} q_n(z),$$

for $0 \leq z \leq \epsilon$. 

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Proof. Since
\[ q_{n+1}(z) = q_0(z) + \sum_{k=0}^{n} (q_{k+1}(z) - q_k(z)), \]
by Lemma 4, we see that the sequence \( q_n(z) \) is uniformly convergent to a function, say \( q(z) \) on \([0, \epsilon] \). \( q(z) \) satisfies (2.10) and (2.11) for \( 0 \leq z \leq \epsilon \). By Lemma 2, \( q(z) \) is the unique solution. Thus \( P(z) = -zq(z) = -z \lim_{n \to \infty} q_n(z) \). \( \square \)

Define \( B_\epsilon = \{ z \in \mathbb{C} \mid |z| \leq \epsilon \} \), where \( \mathbb{C} \) is the complex plane.

**Theorem 1** There is unique analytic continuation of \( P(z) \) in \( B_\epsilon \), and then \( P(z) \) has the series expansion
\[ P(z) = \sum_{n=1}^{\infty} b_n z^n, \quad (2.23) \]
for \( z \in B_\epsilon \).

**Proof.** Let us go back to the sequence \( \{ q_n(z) \} \) defined by (2.14) and (2.15) for \( 0 \leq z \leq \epsilon \). We claim that the sequence can be analytically continued to \( z \in B_\epsilon \). In fact, if we review the proof of Lemma 3, it not hard to find that (2.17) is true for \( z \in B_\epsilon \). So each \( q_n(z) \) is well defined and analytic in \( B_\epsilon \). And the sequence \( \{ q_n(z) \} \) is uniformly bounded on \( B_\epsilon \). Thus \( \{ q_n(z) \} \) is a normal sequence (or normal family)[1]. Therefore \( \{ q_n(z) \} \) contains an uniformly convergent subsequence, say \( \{ q_{n_k}(z) \}_{k=1}^{\infty} \). Let
\[ \hat{q}(z) = \lim_{k \to \infty} q_{n_k}(z), \quad z \in B_\epsilon. \]
Specifically by Lemma 5
\[ \hat{q}(z) = \lim_{k \to \infty} q_{n_k}(z) = \lim_{n \to \infty} q_n(z) = q(z), \quad 0 \leq z \leq \epsilon. \]
Hence \( P(z) \) is analytically extended into \( B_\epsilon \), and \( P(z) \) has the Taylor expansion (2.23), where the coefficients \( b_n \) are uniquely defined by the equation (2.4) and (2.5). If \( P(z) \) has another extension, then the corresponding series expansion restricted on \([0, \epsilon] \) is the series (2.23) since \( P(z) \) must satisfy the equation (2.4) and (2.5) on \([0, \epsilon] \). That means it is still the same series. \( \square \)

By Theorem 1 and (2.3) we see that \( y^*(x) \) satisfies the following first-order equation
\[ y' + b_1(1 - y) + b_2(1 - y)^2 + \cdots + b_n(1 - y)^n + \cdots = 0, \]
at least for \( x > 0 \).
3 The Value of $a^*$

In the last section we have seen that $P(z)$, as an analytic function in the complex domain, has the series expansion. In this section, we investigate this series, and further get the formula for $a^*$.

**Lemma 6** There is the following recursion relation for the coefficients $b_n$ in (2.23)

\[
\begin{align*}
  b_1 &= -1, \\
  b_2 &= \frac{3}{4}, \\
  b_3 &= \frac{1}{40}, \\
  b_n &= \frac{1}{n + 2} \sum_{k=2}^{n-1} k b_k b_{n-k+1},
\end{align*}
\]

where $n \geq 4$. And

\[b_n > 0\]

for $n \geq 2$.

**Proof.** Because (2.23) is the solution to (2.4), substitution of the series in (2.23) into the equation (2.4) we then get the recursion relation by comparing the coefficients of $z^n$ on both sides of the equation. And it’s easy to see that $b_n > 0$ for $n \geq 2$. \qed

**Lemma 7** Let $r_0$ be the convergent radius of series (2.23). We have

\[r_0 \geq 1 + \delta_0.\]  

**Proof.** By Theorem 1, we see that

\[r_0 > 0.\]

If $r_0 < 1 + \delta_0$, then there are two possibilities.

(i) $\sum_{n=1}^{\infty} b_n r_0^n$ is not convergent. Because $b_n > 0 (n \geq 2)$, we then have

\[\sum_{n=1}^{\infty} b_n r_0^n = +\infty.\]  

Let

\[R(z) = z + P(z).\]
Since $P(z)$ is bounded, $R(z)$ is also bounded. Then $b_n > 0 (n \geq 2)$ implies

$$0 < \sup_{[0,r_0]} R(z) < \infty.$$ 

Now choose a positive number

$$R_0 > \sup_{[0,r_0]} R(z).$$ 

Set

$$s_N(z) = \sum_{n=2}^{N} b_n z^n.$$ 

By (3.6), there is a $N > 0$, such that

$$s_N(r_0) > R_0.$$ 

Since $s_N(z)$ is a polynomial, which is continuous at $z = r_0$, there is $\delta_1 \in (0, r_0)$, such that if $\delta \in [0, \delta_1]$, there is

$$s_N(r_0 - \delta) > R_0,$$ 

which implies

$$R(r_0 - \delta) > s_N(r_0 - \delta) > R_0 > \sup_{[0,r_0]} R(z).$$ 

This is a contradiction.

(ii) $\sum_{n=1}^{\infty} b_n r_0^n$ is convergent, but $\sum_{n=1}^{\infty} b_n (r_0 + \delta)^n$ is not convergent for $\delta > 0$, i.e.

$$\sum_{n=1}^{\infty} b_n (r_0 + \delta)^n = \infty.$$ 

Then there is $N > 0$, and a positive sequence $\{\delta_k\}$, with $\delta_k \to 0$, as $k \to \infty$, such that

$$s_N(r_0 + \delta_k) > R_1$$ 

for some positive number

$$R_1 > \sup_{[0,r_0]} R(z).$$ 

Because $s_N(z)$ is continuous, letting $k \to \infty$, we get

$$R(r_0) > s_N(r_0) > \sup_{[0,r_0]} R(z),$$ 

which is a contradiction. So we must have $r_0 \geq 1 + \delta_0$. $\Box$
Theorem 2  We have the following formulas for $a^*$,

(i)  \[ a^* = - \sum_{n=1}^{\infty} b_n, \quad (3.7) \]

(ii) \[ a^* = \left( \frac{1}{2} + 2 \sum_{n=1}^{\infty} \frac{b_n}{n+1} \right)^\frac{1}{2}, \quad (3.8) \]

(iii) \[ \left( \frac{1}{2} + 2 \sum_{n=1}^{N_1} \frac{b_n}{n+1} \right)^{\frac{1}{2}} < a^* < - \sum_{n=1}^{N_2} b_n, \quad (3.9) \]

for any $N_1 \geq 2$, and $N_2 \geq 1$. And specially

\[ a^* < \frac{1}{4}. \quad (3.10) \]

Proof. By Lemma 1 and Lemma 7, we obtain (3.7) and (3.8). Since $b_n > 0$, for $n \geq 2$, (3.9) is true. Choose $N_2 = 2$, (3.10) is derived. \(\Box\)

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