Universality in Percolation of arbitrary Uncorrelated Nested Subgraphs

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The study of percolation in so-called nested subgraphs implies a generalization of the concept of percolation since the results are not linked to specific graph process. Here the behavior of such graphs at criticality is studied for the case where the nesting operation is performed in an uncorrelated way. Specifically, I provide an analytic derivation for the percolation inequality showing that the cluster size distribution under a generalized process of uncorrelated nesting at criticality follows a power law with universal exponent $\gamma = 3/2$. The relevance of the result comes from the wide variety of processes responsible for the emergence of the giant component that fall within the category of nesting operations, whose outcome is a family of nested subgraphs.

I. INTRODUCTION

Random graph theory is based on an ensemble formalism [1, 2] close to statistical mechanics [3]. Under this framework, some purely mathematical phenomena, like the emergence of the giant connected component (hereafter, GCC) [4] can be understood as a phase transition in the sense of, for example, the transition from ferromagnetic to paramagnetic phase in the Ising model. In Landau’s theoretical hallmark to study phase transitions, one of the main features of a system at criticality is that some thermodynamical magnitude $m$ (the order parameter of the system) displays a singularity in one of its derivatives $\frac{\partial m}{\partial \epsilon}$. Under mild assumptions, one can show that $m$ approaches to the singularity as a power law of a control parameter $\epsilon$, i.e., $m(\epsilon) \propto \epsilon^{\beta}$. Following the statistical mechanics analogy, in this note I study the critical exponents related to the component size distribution at the critical point when the GCC of an arbitrary family of nested subgraphs emerges.

In a previous paper [5], we defined and characterized the so-called nested subgraphs, which are a collection of families of subgraphs of a given graph whose members can be ordered by inclusion. We assume that these subgraphs are obtained through an arbitrary algorithm whose outcome holds some probabilistic requirements. The main achievement of the developed formalism was that the results were not linked to a specific subgraph, but they were general to all subgraphs satisfying a small set of probabilistic constraints. Among others, we identified as nested subgraphs the families of $K$-cores [8], [9], the $K$-scaffolds [10, 11] or the subgraphs obtained through random deletion of nodes [12, 13]. It is worth noting that the study of critical thresholds for the emergence of the GCC of an arbitrary family of nested subgraphs implies a generalization of the concept of percolation based in a threshold in the probability of deletion of randomly chosen nodes. Furthermore, we showed that the degree distribution of a scale-free network with exponent higher than 2 was invariant under nesting operations [9].

In this work I study the emergence of cluster sizes at criticality when an uncorrelated nesting algorithm is applied, which implies that the probability of removal or survival of a given node can be expressed as a function of its connectivity and, in the extreme cases, of both its connectivity and a mean field approach of the connectivity of its first neighbors. We observe that the possible long range dependencies conditioning the emergence of the GCC automatically rules out such a subgraph from our study, even a probabilistic interpretation of the probability of removal can be defined [6].

The behavior of cluster sizes at criticality is studied following a methodology based on generating functions [7]. Generating functions to study the emergence of the GCC were introduced for the first time in [15]. However, in this work I use the proposal made in [12, 13, 16], also based on the generating function formalism, which revealed specially suitable to study network phenomena from the physical point of view. In this framework, the study of the emergence of the GCC resembles the study of phase transitions under Landau’s theoretical framework [5]. With this mathematical apparatus I show that the probability distribution for the size of components at criticality follows a power-law with universal exponent $\gamma = 3/2$, no matter the kind of subgraph is emerging. Previous work derived this exponent for ordinary percolation [18] and for the emergence of the giant $K$-core [17]. The relevance of this result comes from the wide variety of processes that lead to the emergence (or disappearance) of the GCC which can be included in the category of nesting operations.

II. UNCORRELATED NESTED SUBGRAPHS

Formally, a complex network is topologically described by a graph $G(V, \Gamma)$ where $V$ is the set of nodes and $\Gamma \in V \times V$ the set of edges connecting nodes of $V$. If $p_k$ is the probability that a randomly chosen node $e$ is connected to $k$ other nodes (noted $d(e) = k$), then the collection of $p_k$’s defines a sequence of real numbers $\{p_k\}_{k=1}^\infty$ (the so-called degree distribution) whose generating functions...
are \( g_0(z) = \sum_k p_k z^k; \quad g_1(z) = \frac{1}{\langle k \rangle} \frac{d}{dz} g_0(z) \)

where
\[
\langle k \rangle = \frac{d}{dz} g_0(z)|_{z=1} = \sum_k k p_k
\]
is the average connectivity of \( G \). We assume that our \( \langle p_k \rangle_{k=1}^\infty \) is, at least, 1-smooth, i.e., that \( \langle k \rangle < \infty \) \[18\].

We will say that \( S(A, \Gamma_A) \) is an induced subgraph \[19\] of \( G(V, \Gamma) \) if \( A \subseteq V \) and \( \Gamma_A \subseteq \Gamma \) being \( \Gamma_A \in A \times A \). A \( K \)-nested family of subgraphs \( N \) \[10\] is a collection of subgraphs of a given graph \( G \) whose members can be ordered by inclusion \[21\]:
\[
\ldots S_{K+1}(G) \subseteq S_K(G) \subseteq S_{K-1}(G) \ldots
\]

Let the graph \( S_K = S_K(V_{S_K}, E_{S_K}) \) be a member of a nested family of subgraphs of a given graph \( G \). For every family of \( K \)-nested subgraphs we associate a *nesting function*, \( \varphi_K(k) \), namely the probability for a randomly chosen node \( e \in V \) with degree \( d(e) = k \) to belong to \( S_K \):
\[
\varphi_K(k) = \mathbb{P}(e \in V_{S_K} | d(e) = k)
\]
Since \( \varphi_K(k) \) is a probability, we can express it like a function,
\[
\varphi_K(k) : U \times \mathbb{N} \to [0, 1],
\]
where \( U \subseteq \mathbb{R} \) is a set that depends on the nature of the nesting. We need our nesting functions to fulfill the following conditions:

1. fixed \( K \), \( \varphi_K(k) \) is a non-decreasing function on \( k \)
2. fixed \( k \), \( \varphi_K(k) \) is a non-increasing function on \( K \)
3. \((\forall k)[(\exists \lambda_{S_K} \in (0, 1)](\lim_{k \to \infty} \varphi_K(k)) = \lambda_{S_K}]\), where \( \lambda_{S_K} \) is a scalar whose value will depend on the explicit form of the nesting algorithm.

To be consistent with the characterization, we make explicit an underlying assumption: Let \( S_K^\varphi, S_K^\varphi \) be a pair of subgraphs of some subgraph \( G \) with associated nesting functions \( \phi_K \) and \( \varphi_K \), respectively. Then we assume that
\[
(\forall k)(\phi_K(k) > \varphi_K(k)) \to (S_K^\varphi \subseteq S_K^\varphi).
\]

We observe that a nesting function takes into account all the nodes satisfying the imposed conditions: our subgraphs are maximal under the conditions imposed by the nesting function. Furthermore, from the properties of the nesting functions, we can conclude that the sequence
\[
\{\varphi_K(k)\}_{k=1}^\infty = \varphi_K(1), \varphi_K(2), \ldots, \varphi_K(i), \ldots
\]
is a Cauchy sequence.

Let us define the generating functions for an arbitrary \( K \)-nested subgraph with an associated nesting function \( \varphi_K(k) \) defined on a graph \( G \) with arbitrary degree distribution \( \{p_k\}_{k=1}^\infty \). To be precise, we are talking about the generating functions associated with the sequence \( \{\varphi_K(k)p_k\}_{k=1}^\infty \) of real numbers:
\[
f_0(z) = \sum_k p_k \varphi_K(k) z^k
\]
\[
f_1(z) = \frac{1}{\langle k \rangle} \frac{d}{dz} f_0(z) = \frac{1}{\langle k \rangle} \sum_k k p_k \varphi_K(k) z^{k-1}
\]
Notice that, generally, \( f_0(1), f_1(1) < 1 \). For the sake of completeness, the section ends with the asymptotic expression that accounts for the degree distribution of the nested subgraphs, \( p_{S_K} \), as found in \[6\]:
\[
p_{S_K}(k) = \frac{1}{f_0(1)} \sum_{i \geq k} \binom{i}{k} (f_1(1))^{k}(1 - f_1(1))^{i-k} p_i
\]
\[
\approx \frac{\lambda_{S_K}}{f_0(1)} \sum_{i \geq k} \binom{i}{k} (f_1(1))^{k}(1 - f_1(1))^{i-k} p_i
\]
\[
= \frac{\lambda_{S_K}}{f_0(1)} \frac{(f_1(1))^k}{k!} \frac{d^k}{dz^k} g_0(z) \bigg|_{z=1-f_1(1)}.
\]

We observe that the second step is valid from the fact that \( \{\varphi_K(k)\}_{k=1}^\infty \) is a Cauchy sequence.

**III. BEHAVIOR AT CRITICALITY**

Once the operation of nesting is accomplished, the obtained subgraph can display many components of several sizes, including, in some cases, one component of infinite size containing a finite fraction of all nodes, the GCC. Let \( \pi_s \) be the probability for a randomly chosen node \( e \) to belong to a component with \( s \) nodes. We observe that the collection of \( \pi_s \)’s form a sequence of real numbers \( \{\pi_s\}_{s=1}^\infty \) whose associated generating functions are:
\[
h_0(z) = \sum_s \pi_s z^s
\]
\[
h_1(z) = \frac{1}{\langle s \rangle} \frac{d}{dz} h_0(z) = \frac{1}{\langle s \rangle} \sum_s s \pi_s z^{s-1}
\]
being
\[
\langle s \rangle = \frac{d}{dz} h_0(z) \bigg|_{z=1}.
\]
the average size of components other than the GCC. If \( h_0(1) \) is the probability that a randomly chosen node \( e \) is not in the GCC, then the probability for such a node to belong to the GCC, noted \( \pi_\infty \), will be \( \pi_\infty = 1 - h_0(1) \). However, this formulation does not help us to understand
the problem. Following techniques close to the ones
developed to study branching processes, we can find an
alternative form for \( h_0 \) and \( h_1 \). Indeed, it can be shown
that \( h_1 \) displays a Dyson-like recurrence relation [16,
[12], [13]:

\[
\begin{align*}
h_1(z) &= 1 - f_1(1) + z \sum_{k=1}^{\infty} \frac{p_k}{(k)} \varphi_k(1) + 2 z^2 \sum_{k=1}^{\infty} \frac{p_k}{(k)} \varphi_k(2) h_1(z) + \\
&\quad + 3 z^3 \varphi_k(3) h_1^2(z) + ... \\
&= 1 - f_1(1) + z \sum_{k=1}^{\infty} \frac{p_k}{(k)} h_1^{k-1}(z) \\
&= 1 - f_1(1) + z [ f_1(h_1(z)) ] \\
\end{align*}
\]

and that the generating function for the size of the com-
ponent to which a randomly chosen node belongs to is:

\[
h_0(z) = 1 - f_0(1) + z f_0(h_1(z)).
\]

With this formulation,

\[
\pi_\infty = f_0(1) - f_0(u),
\]

where \( u \) is the first, non-trivial solution of the self-
consistent equation \( u = 1 - f_1(1) + f_1(u) \) [12], [13]. Fur-
thermore, from the above definition of \( h_0 \) we can obtain a
useful expression of \( \langle s \rangle \):

\[
\langle s \rangle = \left. \frac{d}{dz} h_0(z) \right|_{z=1} = f_0(1) + \left. \frac{d}{dz} f_0(z) \right|_{z=1} - 1 - \left. \frac{d}{dz} f_1(z) \right|_{z=1}.
\]

As in modern theory of phase transitions, the main
feature of this phase transition is the existence of a singu-
larity in some thermodynamic/statistical magnitude [8]. In
our case, the phase transition can be identified with the
singularity we find in the component size distribution,
\( \langle s \rangle \) (eq. 6), at:

\[
\left. \frac{d}{dz} f_1(z) \right|_{z=1} = 1.
\]

Before the transition, \( \pi_\infty = 0 \), being all components of
finite size and, after the transition, \( \pi_\infty > 0 \), and the
remaining components display still finite size. Specifically,
from eqs. [5] [7], [13], [18], [6] it can be shown that if:

\[
\sum_k k(k-2)p_k > \sum_k k(k-1)(1 - \varphi_k(k))p_k
\]
then there exists a single component of infinite size con-
taining a finite fraction of nodes, i.e., the GCC.
The phase transition referred also as the percolation thresh-
old, is located at the point where:

\[
\sum_k k(k-2)p_k = \sum_k k(k-1)(1 - \varphi_K(k))p_k.
\]

The critical region is located near the percolation thresh-
old (if it exists), i.e., in the region near \( \langle s \rangle \). To study the
behavior of the cluster size distribution near the singu-
larity, we look at the expression of \( h_0(z) \), both depending
on \( f_0 \) and \( h_1 \). Nevertheless, we assume that \( \frac{d}{dz} f_0(z) \) con-
verges for any \( |z| \leq 1 \) (i.e. \( k < \infty \) and well defined,
being \( \{ \varphi_k(k)p_k \}_k \geq 1 \) at least 1-smooth). Thus we must
look for the singularity in \( h_1 \). To study \( h_1 \) near the tran-
sition, we define its functional inverse, \( h_1^{-1}(\tau) = z \):

\[
h_1^{-1}(\tau) = \frac{\tau - 1 + f_1(1)}{f_1(1)}
\]

(8)

Note that, due to the fact that all the members of the se-
cquence \( \{ \varphi_k(k)p_k \}_k \geq 1 \) are not negative, we can be sure
that the of zeros of \( f_1 \) fall outside the statistically relevant
region -and, hence the poles of \( h_1^{-1} \). Thus we assume,
without any loss of generality, that \( f_1(z) \neq 0 \). Consis-
tently, we expect to find the singularity at the point
where:

\[
\left. \frac{d}{d\tau} h_1^{-1}(\tau) \right|_{\tau=\tau^*} = 0.
\]

Differentiating eq. (8), we see that:

\[
f_1(\tau^*) - (\tau^* - 1 + f_1(1)) \left. \frac{d}{d\tau} f_1(z) \right|_{z=\tau^*} = 0
\]

As we argued above, we expect the phase transition of
the system to occur at \( \frac{d}{dz} f_1(z) \mid_{z=1} = 1 \). Thus, if \( \tau^* = 1 \),
all the terms are cancelled. Furthermore, from (8) we can see
that, if \( \tau^* = 1 \), then, \( \tau^* = 1 \). Collecting the
above ingredients, and assuming that \( h_1^{-1} \) is analytical
near the singularity of \( h_1 \), we can perform the power
series expansion of \( h_1^{-1} \) about 1:

\[
h_1^{-1}(z) = 1 + \sum_{i=1}^{\infty} \frac{1}{i!} \frac{d^i}{dz^i} h_1^{-1}(z) \mid_{z=1} (1-z)^i
\]

\[
= 1 - \frac{1}{2 f_1^{2}(1)} \frac{d^2}{dz^2} f_1(z) \mid_{z=1} (1-z)^2 + \mathcal{O}(1-z)^3
\]

(recall that that \( \frac{d}{d\tau} h_1^{-1}(\tau=1) = 0 \)). We can assume without
any loss of generality that:

\[
\left. \frac{1}{2 f_1^{2}(1)} \frac{d^2}{dz^2} f_1(z) \right|_{z=1} \neq 0.
\]

Thus, knowing that \( h_1^{-1}(h_1(z)) = z \), we are legitimated
to say that, near \( z = 1 \):

\[
z \approx 1 - \frac{1}{2 f_1^{2}(1)} \frac{d^2}{dz^2} f_1(z) \mid_{z=1} (1-h_1(z))^2
\]

This enables us to find the exponent \( \beta \), indicating the
power-law behavior of \( h_1(z) \) near the singularity. Specif-
ically,

\[
h_1(z) \approx 1 - c \sqrt{1-z}
\]
being \( c \) a constant depending on the values of both \( f_1(1) \) and \( \frac{d^2}{dz^2} f_1(1) \). Thus, near the transition, \( h_1(z) \propto (1 - z)^\beta \), with \( \beta = 1/2 \), the standard mean field exponent. We observe that \( h_0(z) \) behaves identically near the singularity. Indeed, if we are close to \( z = 1 \):

\[
\lim_{z \to 1} h_0(z) \propto (1 - z)^\beta + O(1 - z).
\]

However, we did not end the job, since we are also interested in the cluster size probability distribution \( \{ \pi_s \}_{s=1}^\infty \). We attack the problem by expanding in power series the leading term of \( h_0(z) \) when \( z \) is close to 1:

\[
\pi_s = \frac{1}{s!} \frac{d^s}{dz^s} h_0(z)|_{z=0} 
\propto \frac{s!}{s!} (1 - z)^s = \frac{s!}{s!} (s - \beta - 1) = \frac{\Gamma(s - \beta)}{\Gamma(s + 1)} \approx \frac{1}{\Gamma(-s)} \left( \frac{s - \beta - 1}{e} \right)^{s - \beta - 1} \sqrt{2\pi(s - \beta - 1)} 
\times \left[ \frac{1}{s!} \sqrt{2\pi s} \right]^{-1} 
\approx \frac{(es)^{-1(1+\beta)}}{\Gamma(-s)}.
\]

where, in this case \( \Gamma \), refers to the ordinary Gamma Function and the last step is obtained assuming \( s \to \infty \) and, hence, applying Stirling’s approach \[20\]. Since \( \beta = 1/2 \), in the limit of large \( s \):

\[
\pi_s \propto s^{-\gamma}; \quad \gamma = 1 + \beta = \frac{3}{2}.
\]

IV. DISCUSSION

In this short note I demonstrated that a wide variety of graph processes display the same behaviour at criticality. Specifically, given any iterative nesting operation, we expect the cluster size distribution to follow a power law with universal exponent \( \gamma = 3/2 \) at the critical region where the giant component emerges. As pointed out concerning the \( K \)-core in \[17\], the emerging components do respect the connectivity requirements imposed by the nesting algorithm, a feature that goes far from ordinary percolation, where only to be connected is required. Beyond its intrinsic theoretical interest, the broad class of mechanisms that can be described through a nesting algorithm makes the universality of this result potentially powerful to understand natural phenomena at criticality where some kind of non-correlated pruning/addition process is at work.

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