Curvature distribution and autocorrelations in elliptic cylinders and cones

Cite as: AIP Advances 9, 085304 (2019); https://doi.org/10.1063/1.5106380
Submitted: 27 April 2019 . Accepted: 29 July 2019 . Published Online: 06 August 2019

Sanju Gupta  and Avadh Saxena

ARTICLES YOU MAY BE INTERESTED IN

Molecular dynamics of nanodroplet impact: The effect of particle resolution in the projectile model
AIP Advances 9, 085204 (2019); https://doi.org/10.1063/1.5100964

Low damping magnetic properties and perpendicular magnetic anisotropy in the Heusler alloy Fe$_{1.5}$CoGe
AIP Advances 9, 085205 (2019); https://doi.org/10.1063/1.5104313

Behavior of a liquid drop in a rounded corner: Different contact angles
AIP Advances 9, 085203 (2019); https://doi.org/10.1063/1.5100300
Curvature distribution and autocorrelations in elliptic cylinders and cones

Cite as: AIP Advances 9, 085304 (2019); doi: 10.1063/1.5106380
Submitted: 27 April 2019 • Accepted: 29 July 2019 • Published Online: 6 August 2019

Sanju Gupta1,a) and Avadh Saxena2

AFFILIATIONS
1 Department of Physics & Astronomy, Western Kentucky University, Bowling Green, Kentucky 42101, USA
2 Theoretical Division and Center for Nonlinear Studies, Los Alamos National Laboratory, Los Alamos, New Mexico 87545, USA

The author to whom correspondence should be made. E-mail: sanju.gupta@wku.edu

ABSTRACT
Not all micro-vessels (MV) are traditionally circular and there are examples of elliptic cylindrical MVs in life sciences, particularly if projected with a slant. Similarly, certain biological structures, ferroelectric liquid crystals, aluminum oxide clusters and witherite crystallites’ cross-section appear to be elliptical cones. We studied the mean curvature (H) distribution of these elliptic morphological structures with geometric parameter such as eccentricity; e (ratio of semi-minor to semi-major axes) and a measure of how much diagonal section deviates from circularity and height (h) in case of cones. By means of topographical cues, we defined the curvature-curvature autocorrelation function (gk) and applied this notion to mean curvature (H) of circular and elliptical cylinders and cones. The Fourier transform of correlation function, i.e. “curvature factor” is analogous to “structure factor (or Patterson function)” in X-ray and neutron scattering intensity. It elucidates critically important information related to surface curvature fluctuation relevant to shape (geometry), network and phase transformation. The latter is induced by cells under mechanical stress, occurring in many soft systems (polymeric liquid crystals, foams, bubbles) and biological tissues, particularly cell walls of primary and branched vessels bed in microvasculature that distributes blood within tissue during hypertension and migraines. This perspective is useful in a sustained release of angiogenic/vasculogenic factors and relevant for precision medicine and engineered microvessels and tissues in vitro and in vivo extended cellular processes. The quantitative analysis carried out in this work facilitates our understanding of the mechanical mechanisms associated with thrombosis during surgery that typically occur in bent or stretched MVs due to microenvironment such as localized shear stresses and biochemical factors.

© 2019 Author(s). All article content, except where otherwise noted, is licensed under a Creative Commons Attribution (CC BY) license (http://creativecommons.org/licenses/by/4.0/). https://doi.org/10.1063/1.5106380

I. INTRODUCTION
The nontrivial notions of curved geometry characterized by mean and Gaussian curvature and (local and global) topology are becoming prevalent and exist concomitantly with regard to many microscopic and macroscopic objects in physical and biophysical world.1–5 Materials science and engineering is definitely playing a central role in this endeavor through the discovery of advanced and complex materials spanning all length scales and heterogeneous integration with deformable layouts or conformable curved surfaces to facilitate interfacial compatibility for tissue engineering useful in human tissue and organ replacement.6–9 Moreover, the variety and structural complexity of forms displayed by mineralized inorganic matter in nature and organic structures of living organisms have always been a rich source of fascination and of great interest to academia, medicine and technological industry.10–11 Traditionally, the technological interest arises from the superior microscopic and macroscopic physical-chemical-biological properties displayed by organic-inorganic biominerals and biochemical structures produced in or by living systems. Biological systems are naturally occurring self-assembled and hierarchical complex networks that involve a multitude of geometrical shapes (i.e. curvature distributions) and topological property variation.12 However, unlike their inorganic solid counterparts, they are elastically soft and therefore easily deformable and conformable. As an example, take our own skin. Human skin is a complex remarkable organ that consists of many layers and an integrated, stretchable network of sensors that relay information about tactile and thermal stimuli to the brain, allowing us to maneuver within our environment safely and effectively.13–14 Of particular interest is the wrinkled...
swelling of skin after hydration when soaked in water, to bathing children and discerning adults alike. The swelling behavior is a consequence of the highly symmetric geometry of keratin fibers (the structural proteins) inside the cells. While it may appear as an unconventional perspective on a biochemical system, the ability of human skin to swell multifold when absorbing water is best understood as the geometric problem of packing helices and embedding the hyperbolic plane in Euclidean 3-space. The intricate structure inside skin cells touches on various branches of modern geometry, from triply-periodic minimal surfaces and symmetric hyperbolic patterns to entanglement and braids. Thus the latest advances at the interface of geometry, soft matter physics and biological materials are paving the way towards geometrically inspired new (bio)materials.

As another example of body organs are microvascular networks or microcirculatory system that support microvessels (MVs), blood and lymphatic vessels with radii in the range 40–120 μm and mediate the interaction between blood and tissues. This extensive, heterogeneous and complex organ plays a critical role in human physiology and disease prevention. It nourishes almost all living human cells and maintains a local microenvironment that is vital for tissue and organ function. In other words, it ensures selective supply of necessary substances to the cells, removes catabolic wastes and allows circulation of oxygen-rich blood from heart to the whole body (see Fig. 1, panel A). In the endocrine glands, MVs are additionally involved in hormone transfer to systemic circulation. It defines the biological and physical characteristics of the microenvironment within tissues and plays a central role in the initiation and progression of pathologies including cancer and cardiovascular diseases.

Similarly, physiologically speaking, microcirculation is so important that MVs are certainly a vital area of research in experimental biology and biomedicine such as tissue engineering research useful in human organ replacement, in surgical planning and quantitative diagnostics (see Fig. 1). However, challenges related to vascularization in vitro and in vivo remain. Electron microscopy and magnetic resonance imaging remain some of the main methods for studying MV ultrastructure (see Fig. 1). The native microvasculature can be visualized and observed in vivo under a fluorescence microscope (not an ordinary but a confocal laser microscope). During the formation of the vascular system, successively smaller blood vessels sprout from existing ones to form networks of capillaries uniquely adapted to the function of the organ they enter. This process, called angiogenesis, involves complex interactions between several molecular signaling pathways. With TEM, the size parameters of MVs and of their components are measured on section profiles. Stereological design-based methods do not require assumptions about object shapes, and they find an increasingly wide use in modern research. Currently available design-based methods are meant for unbiased estimation of the number and volume of particles of arbitrary shape. As for estimating sizes of section profiles of elongated objects including MVs, existing design-based methods cannot be applied. Specifically, correction coefficients are introduced to minimize the bias caused by non-perpendicular sectioning of MVs. This makes it possible to determine MV sizes from measurements of their section profiles, the shape of a MV on limited lengths of its basal surface being approximated by a straight circular cylinder. However, MVs cannot always be regarded as circular cylinders. They may have a significant ellipticity of perpendicular section profiles (see Figure 1, panel A). Although we have found only one article whose findings experimentally prove this point (Ref. 31): venular MVs in the pig heart are studied by optical sectioning, i.e. by changing the focal plane through the thickness of the histological section; their mean axial ratio is equal to 1.32 (postcapillary venules) or 1.38 (venules of the next order), and general considerations suggest that microcirculatory networks might be capable of acquiring an ellipticity in the regions of comparably low blood or lymph hydraulic pressure, as well as in places of increased outer pressure brought to bear by the surrounding tissue and localized shear stresses. Typically, vascular smooth muscle cells can induce vasodilation/vasoconstriction by changing their tone/texture and mechanical stresses induced on the wall are influenced by the cellular arrangement as well as the microvascular cross-sectional shape. Therefore, accurate information and understanding on the three-dimensional structure is essential for revealing the regulatory mechanism of blood flow in the vasculature network. The latter is hypothetically proposed to cause thrombosis leading to abnormalities, bending/stretching and narrowings (stenoses) of MVs which have been tested experimentally and computationally. Therefore, for MVs that carry blood and other biological fluids, any deformation in their shape/geometry may have important implications for precise medical diagnostics. Consequently, the revealed ellipticity of the coronary venules can hardly be a single case. The fact that there is no other relevant data on the ellipticity of MVs, it can be explained by challenges in determining MV sizes by a conventional method such as TEM which produces a properly magnified image. Taking this into account, an elliptical cylinder is undoubtedly not only preferable but also is the most appropriate description to a circular one in modeling the MV shape on limited lengths of MV basal surface. This model becomes indispensable in this particular study when a MV is estimated to have an ellipticity of its profile obtained by sectioning perpendicular to the MV longitudinal axis, hereafter, the cross-sectional profile (see Fig. 1). Additionally, since this elliptical cylinder model is more general, it should be chosen in case there is no information on the characteristic MV shape. The model treats MVs with circular profiles as elliptical cylinders of zero eccentricity. Nevertheless, despite all its benefits, the model practically finds no application, and the main reason for this seems to be the unknown relationships between the true (or 3-dimensional) and the section profile (or 2-dimensional) sizes in the elliptical cylinder model. As a result, it remains impossible to determine MV sizes or size distributions on the basis of the size distributions of randomly oriented MV profiles. As such, microvascular plays an important role in homeostatic processes as diverse as organ perfusion/oxygenation, angiogenesis, thrombosis, fluid and solute balance, blood pressure, and inflammation. Microvascular dysfunction can lead to dysregulation of these fundamental processes and is a central media-tor in a variety of human diseases. The aim of the present paper is to define these geometric relationships quantitatively and to make recommendations in terms of their potential applications for surgery and precision medicine as well as for development of engineered microvessels. The latter has advanced markedly in the decade following the development of early pre-vascularization techniques.

The second example we consider is that of elliptic cone like structures found in certain witherite (BaCO₃) crystals and some ferroelectric liquid crystals (FLCs). The former appear as
gel-grown biomorphic aggregates in optical micrographs of silica-witherites in a specific range of pH. The starting solution comprises Na$_2$SiO$_3$–BaCl$_2$–NaOH. The morphology depends on the aging period between mixing and aggregate formation. The elliptic cone shaped structures (in ~500 micron field of view) initially emerge from saddle-shaped sheets that grow from globular aggregates. There are preferred orientations of the optically anisotropic material within the witherite crystallites which impart birefringence to the whole aggregate. Similarly, infrared spectroscopy in conjunction with quantum chemical calculations of certain highly symmetric small aluminum oxide clusters has revealed exceptionally stable conical structures. A circular cone is a special case of an elliptic cone when the cross section is devoid of any eccentricity. As indicated above, from a geometric perspective, both the elliptic cone and the elliptic cylinder are quadric surfaces. However, elliptic cones are not as ubiquitous in nature as circular cones. Nonetheless at a macroscopic length scale, some volcanic mouths and lake basins are characterized as an elliptic cone. Elliptic cones are also invoked in the micro-geometric modeling of polished surfaces of certain alloys. Moreover, they are used in the analysis of the propagation direction.
of coronal mass ejection from solar disk halos. Diffraction optics by an elliptic cone is an interesting subject in its own right. In analogy with the elliptical cylinder, below we have explored curvature autocorrelations for the conical geometry as well.29 In the biological context conical structures occur in a variety of systems. A cone-shaped capsid, which is a protein structure, contains genome of the HIV-1 virus. The flexible nature of this structure not only accommodates interactions with host cell factors but also points to the possible ellipticity of the cone.30 Similarly, the medulla in human kidney and certain other mammals is divided into conical masses called renal pyramids.31 Their flexibility also indicates that the renal cones in general are elliptic in shape. For color vision, cone shaped photoreceptor cells in the eye of humans and certain vertebrates respond to light at different wavelengths in different ways.32 In contrast to rod cells they function better in bright light. Human retina contains about six to seven million densely packed cone cells which are most concentrated towards the macula area and these cells quickly reduce in number towards retinal periphery. Among the Earth’s oldest macroscopic fossils are stromatolites whose morphology may contain hints of biological processes. The ones forming in the absence of sedimentation are conical in shape and are likely to contain important records of biological processes, in particular, they may point to a photosynthetic origin.33 Notably, their morphology and organization provides a link between form and physiological function. There are many cone-forming microbes, specifically cyanobacteria, found in both hot springs and lithified carbonated rocks.34 The latter possibly contain the earliest records of such biological processes as phototaxis, oxygenation of the environment and photosynthesis. In the context of botany, a conical structure is called strobilus. In the majority of insect-pollinated flowers the petals have conical epidermal cells influencing pollination behavior.35 In a different botanical context, cone seed structures in conifers are well known where much of the cone ontology is determined by intercalary activity resulting in cone closure subsequent to pollination.36 The ovulate cone morphology thus has consequences for seed protection and dispersal.

While our previous work in this field focused on the notion of topology and geometry for a range of nanocarbons that are a subject of intense practical interest,37 especially graphene and curved nanocarbons including graphic cone as topological and quantum materials, the present work extends the importance of curvature distribution to elliptical cylinders, elliptical cones and curvature autocorrelation to medicine as well as to curved inorganic solid-state nanoparticles, respectively.

II. PARAMETERIZATION AND SIGNIFICANCE OF CURVATURE-CURVATURE CORRELATIONS

There are many diagnostic tools, such as small to medium angle X-ray (SAXS) and neutron scattering (SANS) and corresponding reactivity modes as well as various other scattering probes, which have been developed to interrogate the structural features and microstructure of non-curved or flat, Euclidean entities. Beyond structural characterization, these techniques have been extended to several other functionalities and topological structures, e.g. polarization, magnetization, vorticity, networks, knots and so forth. However, a generalization of this extensive suite of techniques is largely missing for curved geometries, which is highly desirable given the ubiquity of such objects in the physical world. This emerging perspective is one of our main motivations here. Thus, the present paper represents an attempt toward achieving this grand objective. Some examples of these objects include MVs in the shape of an elliptic cylinder (as discussed above) and certain crystallites which exist both as a circular and as an elliptic cone (see Fig. 1). Spheroidal and ellipsoidal vesicles are also known. More complex geometries, e.g. branched MVs with overlapping ellipses (Ref. 21) and double-toroid, also exist. Among the simplest of objects is a circular cylinder, whose cross-section is a circle with a constant radius (or curvature). From curvature correlation perspective it is a trivial case. The next simplest albeit non-trivial object with variable curvature is an elliptic cylinder, whose cross-section is an ellipse (see Fig. 1). We therefore use an elliptic cylinder as our workhorse example to illustrate the mathematical curvature-curvature correlation function (g_k), how to calculate it and what physical information can be extracted from this quantity. The procedure outlined here can be generalized (but not always analytically) to more complex geometries and surfaces including elliptical cones. In the latter case (in general) one calculates both the mean (H or κ) and Gaussian (K) curvature-curvature correlation functions. Beyond structure, if the surface is magnetic, polar or has some other functionality, then the interplay of this functionality with curvature provides extremely useful information about the properties of the surface including when an external electric, magnetic, stress or some other field is present in terms of the underlying curvature-curvature correlation functions. In particular, how the curved geometry influences the blood flow behavior and magnetic or polar interaction in functional materials could also be quite insightful for specific applications.

A. Parameterization of the curvature for an elliptic cylinder

An elliptic cylinder is parameterized by x = a cos φ, y = b sin φ and z = z. Here the variables a and b denote the semi-major and semi-minor radii of an elliptical cross-section of an elliptic cylinder (see Fig. 1). Here φ is the azimuthal angle (0 < φ < 2π) and 0 < b < a. Since parallel to the cylindrical axis it is flat on the surface of the elliptic cylinder, the second principal curvature κ_2 = 0 and the first principal curvature is given by κ_1 = 1/a^2 (or vice versa). Thus, the Gaussian curvature K = κ_1 κ_2 = 0 and the mean curvature H = 1/2 (κ_1 + κ_2) = 1/a^2 b. For an ellipse we define elliptic eccentricity (0 < e < 1) as e = \sqrt{1 - \frac{b^2}{a^2}}. The average mean curvature of an elliptic cylinder is given by integrating: \bar{H} = \frac{1}{2\pi} \int_0^{2\pi} h(\phi) d\phi = 4 \int_0^{\pi/2} h(\phi) d\phi = \frac{4}{a} E(e) \frac{b}{\sqrt{1-e^2}} \frac{1}{\sqrt{1-e^2}}. The average mean curvature of an elliptic cone is given by integrating: \bar{h}(\phi) = \int_0^{2\pi} h(\phi) d\phi = \frac{1}{a} E(e) \frac{b}{\sqrt{1-e^2}}.
apply to mean curvature except for a scaling factor. Thus for the elliptic cylinder we define curvature-curvature correlation function as \( g_k(\phi, e) = \frac{g(\phi)}{\kappa(e) \kappa(\phi)} = \frac{1}{4\kappa(e)} \int_{\phi}^{\pi} \frac{(1-e^2)^{\frac{1}{2}}}{\kappa(e)} d\phi \), where \( I(\phi, e) \) is numerically evaluated for different \( e \). Moreover, the integral \( I(\phi, e) \) function can be expressed analytically in terms of the incomplete elliptic integral of the second kind, that is \( I(a, e) \). However, here in order to compute it we opt to evaluate the integral \( I(\phi, e) \). The two-point curvature correlation is thus given by \( g_k(\phi, e) = \int_{\phi}^{\pi} k(\phi) k(\phi + \psi) d\psi = \frac{4(1-e^2)^{\frac{1}{2}}}{\kappa(1-e^2)} \int_{\phi}^{\pi} \frac{d\psi}{(1-e^2)\kappa(\phi + \psi)^{\frac{3}{2}}} \).

\[ = \frac{4(1-e^2)^{\frac{1}{2}}}{\kappa(1-e^2)\kappa(\phi + \psi)^{\frac{3}{2}}} \left[ E(\alpha_1, e) - E(\alpha_2, e) \right] = \left( \frac{\pi}{2} \right)^2 \left\{ \frac{4(1-e^2)^{\frac{1}{2}}}{\kappa(1-e^2)\kappa(\phi + \psi)^{\frac{3}{2}}} \left[ E(\alpha_1, e) - E(\alpha_2, e) \right] \right\} I(\phi, e) = \text{incomplete elliptic integral of second kind with modulus } e, \alpha_1 = \sin^{-1}\left( \frac{\cos \phi}{(1-e^2)\kappa(\phi)^{\frac{3}{2}}} \right), \alpha_2 = \sin^{-1}\left( \frac{\sin \phi}{(1-e^2)\kappa(\phi)^{\frac{3}{2}}} \right). \]

B. Parameterization of the curvature for an elliptic cone

An elliptic cone has an elliptical cross section with height \( h \), semi-major axis \( a \) and semi-minor axis \( b \). It is parameterized by \( x = \frac{h}{b} \cos \phi, \ y = \frac{h}{b} \sin \phi \) and \( z = u \), where \( 0 \leq \phi \leq 2\pi \), \( 0 \leq u \leq h \), and base eccentricity \( e \) is \( 0 \leq e \leq 1 \), \( e = \sqrt{1 - \frac{b^2}{a^2}} \), \( e = 0 \) (circular cone), \( e = 1 \) (plane) (see Fig. 1). Similar to an elliptical cylinder, the Gaussian curvature \( K \) is zero for elliptical cones (as a straight line can be drawn on the surface of a cone) and the mean curvature is given by: \( H(\phi, e, u) = \frac{(a^2 - b^2) e \sin^2 \phi}{(1-e^2)\kappa(\phi)^{\frac{3}{2}}}. \)

For a fixed height \( h = h_0 \), the two-point curvature correlation is evaluated similar to the case of an elliptic cylinder except that now the expression for the mean curvature of an elliptic cone \( H(\phi, e, u) \) should be used, followed by multiplication with a shifted function \( H(\phi + \psi, e, u) \) and integration. The result is analytically unwieldy and thus not provided here. For a general two-point curvature correlation, i.e. two points lying on the cone surface at different heights \( h_1 \) and \( h_2 \), we perform a double integral over the height and angle parameters following: \( g_k(\phi, u) = \frac{1}{[H(\phi, e, u)]^2} \int_0^u \int_0^u H(\phi, e, u) H(\phi + \psi, e, u + \nu) d\phi d\psi \), where \( h \) represents the height of the cone and \( H(\phi, e, u) \) implies the average value of mean curvature over the surface of the cone. This integral can only be evaluated numerically and we defer the discussion of results to a future study. Nonetheless, the notion of computing the general correlation function is conceptually similar to an elliptic cylinder but more complex even numerically.

III. RESULTS AND DISCUSSION

We describe the physical interpretation and implications of these calculations in Fig. 2 and Fig. 3. In Fig. 2 we have plotted the mean curvature \( H \) as a function of the azimuthal angle \( \phi \) in the range \( 0 < \phi < \pi \) for different values of eccentricity for an elliptic cylinder. The curves are symmetric around \( \phi = \pi/2 \) as expected from the symmetry of an ellipse. For \( e = 1 \) there is no curvature (because it is a straight line) and we have \( H = 0 \). As \( e \) is decreased from 1 to 0, the mean curvature distribution becomes flatter because the elliptic cylinder becomes more circular and at \( e = 0 \) the mean curvature \( H \) is constant, given by \( 1/2a \), where \( a \) now denotes the radius of the

![FIG. 2. Mean curvature (H) distribution for MVs with different projected angle (ϕ) and with different shaped cross-section (circular and elliptical) via varying eccentricity (e).](image-url)

![FIG. 3. Mean curvature (H) distribution for elliptical cones (a) at u = 0, with varying eccentricity (e) and (b) along 2-axis (u) at eccentricity e = 0.7.](image-url)
resulting circle. This observation can serve as a quantitative indicator of the ellipticity of MVs and as a concomitant diagnostic measure of the presence of any pathology.

Figure 3a plots H distribution for an elliptic cone with fixed \( u = 0 \) and different eccentricity values keeping the semi-major axis \( a = 1 \) and \( h = \) twice the base semi-major axis = 2. Likewise, in Fig. 3b we have plotted the mean curvature distribution \( (H) \) as a function of \( \phi \) for fixed eccentricity \( e = 0.7 \) as \( u \) (which denotes the cone height parameter, \( 0 < u < h \)) is varied in the range \( 0 < u < 1.5 \), keeping the semi-major axis \( a = 1 \) and \( h = \) twice the base semi-major axis = 2. With decreasing \( u \), the \( H \) distribution becomes progressively flatter. Again, with decreasing eccentricity the mean curvature distribution becomes flatter.

In usual scattering of X-ray or neutrons from crystalline (or amorphous) materials the measured quantity is the structure factor which is the Fourier transform of the two-point atomic displacement correlation function (also known as Patterson function). It would be highly desirable to have an analogous quantity to characterize scattering from curved entities such as microvessels with elliptical cross-section, spheroidal or conical objects found in materials science and biophysical entities. To this end, the first objective would be to calculate the two-point curvature correlation function provided above. Its Fourier transform, we call it the curvature factor, should thus provide valuable information about curved objects. This framework is illustrated in Fig. 4 as a flow diagram and the results are shown in Fig. 5 as a function of the azimuthal angle \( \phi \) (0 < \( \phi < \pi/2 \)) for various values of eccentricity \( e \). For \( e = 1 \) it vanishes identically, as expected. For \( e = 0 \) (or circular cylinder) it is a constant, as expected. For other values of \( e \) the correlation function decreases monotonically reaching a fixed value at \( \phi = \pi \), which means at larger distances on an ellipse the curvature correlation becomes progressively weaker. The variation is more pronounced for larger values of \( e \). The maximum (at \( \phi = 0 \)) decreases monotonically as \( e \) increases which means the more eccentric the ellipse, the weaker its curvature auto-correlation. The Fourier transform of this function will then relate to the scattering data from experiments (X-ray, neutrons) on elliptic cylinders. If the elliptic cylinder has additional functionality, e.g. if it consists of a lipid bilayer with relative lipid concentration \( \phi \), then the local curvature will couple to \( \phi \) and there will be a preferential lipid phase separation.18 Both for this purpose and in general, the Helfrich-Canham energy penalty is proportional to just \( H \). Plotting \( H \) in this case will also have two maxima and two minima which will indicate the lipid concentration.

In flat geometries (or usual crystals) the Fourier transform of the two-point correlation function is called ‘structure factor’ which provides useful information about the distribution of points, i.e. the crystal symmetry, microstructure, etc. In an analogous way, we defined the Fourier transform of the curvature-curvature correlation function, which would correspond to “curvature factor” (Fig. 5). The latter can therefore serve as a valuable diagnostic tool in terms of relating various pathologies to the distortion of micro-vessels, etc. We can extend this correlation function to condensed materials where for electrons residing on an elliptic cylinder, the quantum potential experiences charge “localization” near high curvature points reflected in the wavefunctions and the corresponding eigenvalues tend to have bound states. Likewise, curvature induced geometric frustration has also been studied on the magnetic surface of an elliptic cylinder.25 Here the geometric frustration refers to a mismatch of the characteristic length scale of the elliptic cylinder and the size of the magnetic domain wall, i.e. soliton. Cylindrical vesicles containing magnetorheological fluids and anisotropic microtubules can also serve as examples of elliptic cylinders.

The above two examples of elliptic cylinder and elliptic cone belong to a class of surfaces with zero Gaussian curvature. On the other hand, a vast variety of surfaces such as spheroids and ellipsoids possess positive Gaussian curvature. However, there is also a class of triply (periodic) minimal surfaces with negative Gaussian curvature. Gyroid P and D surfaces belong to this class and many physical systems including block co-polymers, micro-emulsions as a result of self-organization at a supramolecular level and nanocarbon allotropes also exhibit them. Bio-membranes consisting of lipid mixtures and bio-inspired photonic bandgap materials represent...
two other distinct examples of self-assembled triply periodic minimal surfaces. Interestingly these surfaces are minimal surfaces which means that there mean curvature $H=0$ and the Gaussian curvature is a constant, $K = -\text{const}$. The overall morphology is determined by the shape of internal interfaces that separate different domains. In this case, one could calculate the autocorrelation or the Gaussian curvature autocorrelation function, which would be complementary to the mean curvature autocorrelation we calculated for elliptic cylinders and cones. A distinct feature in this case would be to measure/calculate curvature along one axis of the gyroid structure and compute the curvature (auto- or cross-) correlation function along another axis, and then determine correlative index which subsequently can be associated with physical-chemical properties. Moreover, since surface free energy is a dominant physical factor governing the shape of these interfaces, SANS/SAXS are a versatile tool to study properties of some of these structures. Similarly, gyroid photonic band gap materials naturally provide a frequency gap for electromagnetic wave propagation. By varying process parameters such as lattice constants and refractive index contrast, morphologies of such gyroids can be manipulated as desired to achieve applications in specific wavelength range. In particular, a network or a multilayer dielectric composite structure with periodicity matching that of the wavelength of light can greatly influence the optical properties, e.g. iridescence.

IV. CONCLUSION

In summary, it is hoped that our approach presented here will allow for a better and quantitative understanding of the microvessels behavior under hypertension and the ability to better predict their behavior for surgical planning, angiogenesis and during thrombosis. The same holds true for the elliptic cone structures in witherite crystallites and more generally for the inorganic as well as soft (e.g. ferroelectric liquid crystals) materials and biological systems (e.g. cyanobacteria, retinal cone cells, etc.). In this context, Fig. 4 captures the framework within which to understand curvature correlations and the curvature factor. For flat systems, typically the displacement autocorrelation or the Patterson function is calculated in Cartesian coordinates. Following Fourier transform, it provides the structure factor $S(q)$, which is compared with the scattering intensity obtained from experiments. Analogously, for curved geometries, we first calculate the curvature autocorrelation function and its Fourier transform provides the “curvature factor” $K(k)$, the structure factor analog for curved systems. Scattering intensity measurements on curved systems can be compared with the curvature factor, as calculated here in the special cases of an ellipse and a cone. The methodology developed here has some limitation. For instance, for size distributions of blood vessels, there can be a distribution of elliptic cross sections that all have different eccentricities. One could average over the eccentricities to have an effective ellipse with an average eccentricity and the corresponding curvature autocorrelation function, which would be an approximate approach. If there are local distortions of a micro-vessel such that the cross section is not an exact ellipse or along the length of the micro-vessel there is a variation in the cross section then in addition to mean curvature correlations, Gaussian curvature correlations would need to be computed which is beyond the scope of the current manuscript. We defer such more intricate models to future research. Nevertheless, this study offers a perspective and a platform for development of engineered microvessels that can be studied in the laboratory setting of a multitude of biophysical and biochemical stimuli in a precise manner. Similarly, the problem of a more general shaped elliptic cone (i.e. with local distortion) calculations are not analytically tractable and defer them for future study. However, we expect to generalize the curvature correlation procedure numerically to both the circular and elliptic cones in the near future as well as for gyroid-like structures. Lastly, we envision creating patient-specific microvessel models could lead to personalized treatment and tested against various therapies.

ACKNOWLEDGMENTS

This work was supported in part by KY NSF EPSCoR, KY NASA EPSCoR and in part by the U.S. Department of Energy.

REFERENCES

1. S. Gupta and A. Saxena, in The Role of Topology in Materials, Vol. 189, Ch. 1 (Eds.) S. Gupta and A. Saxena, Springer Series in Solid-State Sciences, Springer International Publishing, Germany (2018).
2. S. Gupta and A. Saxena, J. Appl. Phys. 109, 074316 (2011) [Virtual Journal of Nanoscale Science & Technology April issue (2011)].
3. S. Gupta and A. Saxena, Mater. Res. Bull. 39, 265 (2014).
4. M. I. Monastyrskii, Topology of Gauge Fields and Condensed Matter (Plenum Press, New York, 1993).
5. J. M. Seddon and R. H. Templer, Elsevier Science B. V., Handbook of Biological Physics, Vol. 1, Ed, E. Sackmann 97 (1995).
6. A. Cavalcanti, B. Shrinizadeh, R. A. Freitas, Jr., and T. Hogg, Nanotechnology 19, 015103 (2008).
7. S. V. Murphy and A. Atala, Nat. Biotechnol. 32, 773 (2014).
8. J. Rouwkema and A. Khademhosseini, Trends in Biotechnol. 34, 733 (2016).
9. L. Y. Chiu, M. Montgomery, Y. Liang, H. Liu, and M. Radisic, Proc. Nat. Acad. Sci. 109, E3413 (2012).
10. Y. Zheng, J. Chen, M. Craven, N. W. Choi, S. Totorica, A. Diaz-Santana, P. Kermani, R. Hempstead, C. Fischbach-Teschi, J. A. Lopez, and A. D. Stroock, Proc. Nat. Acad. Sci. 109, 9342 (2012).
11. M. Kellemeier, H. Colfen, and J. M. Garcia-Ruiz, Eur. J. Inorg. Chem. 2012, 5123–5144.
12. S. Gupta and A. Saxena, J. Appl. Phys. 112, 114316 (2012).
13. M. Evans and R. Roth, Phys. Rev. Lett. 112, 038102 (2014) and references therein.
14. M. Garcia-Ruiz, A. Carnerup, A. G. Christy, N. J. Welham, and S. T. Hyde, Astrobiology 2, 353 (2002).
15. G. T. Vickers, Powder Technol. 86, 195 (1996).
16. R. A. Krasnoperov and A. N. Gerasimov, Expt. Biol. Med. 228, 84 (2003).
17. D. Hanahan and R. A. Weinberg, Cell 100, 57 (2000).
18. E. J. Ratteguy, J. Mol. Med. 73, 333 (1995) and references therein.
19. J. Fessler and A. Macovski, Proc. SPIE 1092, 22 (1989); J. Fessler, D. Nishimura, and A. Macovski, in IEEE Engineering in Medicine 7 Biology Society 11th Annual International Conference -0561 CH2770-6/89/0000-0561 (1989).
20. K. Kitamura, J. Tobis, and J. Sklansky, IEEE Transactions on Medical Imaging 7, 173 (1988).
21. G. Zhang, J. Morales, and J. M. Garcia-Ruiz, J. Mater. Chem. 9, 1568 (2017).
22. E. Rivkin, S. M. Almeida, D. F. Ceccarelli, Y.-C. Jiang, T. A. MacLean, T. Srikrumar, H. Huang, W. H. Dunham, R. Fukumura, G. Xie, Y. Gondo, B. Raught, A. C. Gingras, F. Sicieri, and S. P. Cordes, Nature 498, 318 (2013).
23. C. V. Howard and M. G. Reed, RMS Handbook 41 (BIOS Scientific Publishers, Oxford, UK, 1998).
