Multi-time Lagrangian 1-forms for families of Bäcklund transformations. Relativistic Toda-type systems

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Received 11 August 2014, revised 24 November 2014
Accepted for publication 29 December 2014
Published 4 February 2015

Abstract
We establish the pluri-Lagrangian structure for families of Bäcklund transformations of relativistic Toda-type systems. The key idea is a novel embedding of these discrete-time (one-dimensional) systems into certain two-dimensional (2D) pluri-Lagrangian lattice systems. This embedding allows us to identify the corner equations (which are the main building blocks of the multi-time Euler–Lagrange equations) with local superposition formulae for Bäcklund transformations. These superposition formulae, in turn, are key ingredients necessary to understand and to prove commutativity of the multi-valued Bäcklund transformations. Furthermore, we discover a 2D generalization of the spectrality property known for families of Bäcklund transformations. This result produces a family of local conservation laws for 2D pluri-Lagrangian lattice systems, with densities being derivatives of the discrete 2-form with respect to the Bäcklund (spectral) parameter. Thus, a relation of the pluri-Lagrangian structure with more traditional integrability notions is established.

Keywords: Lagrangian mechanics, integrable systems, Euler–Lagrange equations, relativistic Toda systems

1. Introduction

This paper can be considered as a continuation of our recent paper [BPS13], where we gave an application of the general Lagrangian theory of discrete integrable systems of classical
mechanics, developed in [Sur13], to families of Bäcklund transformations for non-relativistic Toda-type systems. The development of the general theory in [Sur13] was prompted by an example of the discrete time Calogero–Moser system studied in [YKL11], and belongs to the line of research on Lagrangian theory of discrete integrable systems initiated by Lobb and Nijhoff in [LN09] and followed by a number publications [ALN12, BS10a, BS12, LN10, LQ09, BPS14b, BPS14a]. The notion of integrability of discrete systems, lying at the basis of this development, is that of the multi-dimensional consistency. This understanding of integrability has been a major breakthrough [Nij02, B02, ABS03] and stimulated an impressive activity boost in the area, see [BS08].

The original idea of Lobb and Nijhoff can be summarized as follows: solutions of integrable systems deliver critical points simultaneously for actions along all manifolds of the corresponding dimension in multi-time; the Lagrangian form is closed on solutions. This idea resembles the classical notion of pluriharmonic functions and, more generally, of pluriharmonic maps [BFPP93, OV90, Rud69], which are simultaneous extremals of the Dirichlet energy along all holomorphic curves in a multi-dimensional complex vector space. This motivated us in [BS14, BPS14a] to introduce a novel term for the new branch of the calculus of variations specific for integrable systems: we call the corresponding systems pluri-Lagrangian, and we argue that integrability of variational systems should be understood as the existence of the pluri-Lagrangian structure. In the present paper, we hope to provide an additional evidence in favor of this view.

We establish and investigate the pluri-Lagrangian structure for a more general class of systems than the one studied in [BPS13], namely for the so-called relativistic Toda-type systems of the following general form:

\[
\ddot{x}_k = r(\dot{x}_k) \left( f(x_{k+1} - x_k) - f(x_k - x_{k-1}) \right) + \dot{x}_{k+1} g(x_{k+1} - x_k) - \dot{x}_{k-1} g(x_k - x_{k-1}).
\]

The general form of a discrete-time relativistic Toda-type system is

\[
G(\ddot{x}_k - x_k) - G(x_k - \ddot{x}_k) = H(x_{k+1} - x_k) - H(x_k - x_{k-1}) + F(\dot{x}_{k+1} - x_k) - F(x_k - \dot{x}_{k+1}).
\]

A theory and an exhaustive list of integrable systems of this type can be found in [Ad99, AS97a, AS97b, Sur03, BS10b]. The most general relativistic Toda-type systems are the elliptic one and its hyperbolic and rational degenerations, which are similar to (1) but with the analogs of the functions \(f\) and \(g\) depending not only on differences \(x_{k+1} - x_k\), but also on sums \(x_{k+1} + x_k\). Their discrete-time versions share the same feature: they are similar to (2) but the analogs of the functions \(F\), \(G\) and \(H\) depend not only on differences but also on sums of the respective arguments. Actually, the discrete time elliptic relativistic Toda system and its degenerations are more symmetric in the sense that all three functions \(F\), \(G\) and \(H\) are essentially the same, differing only by certain parameters. For details about this more general class of relativistic Toda-type systems, see [AS04]. The methods and results of the present paper are perfectly valid for these more general systems, but would require a slight adjustment of notations, which we would like to leave to an interested reader.

We consider (1) as a system of ordinary differential equations with a single independent variable being the time \(t \in \mathbb{R}\). From this point of view, the lattice index \(k\) just enumerates the components of the dependent variable \(x\). The same interpretation is imposed on the discrete time system (2), where the tilde over and under \(x_k\) denotes the shift forward, resp. backward, in the discrete time \(t \in h\mathbb{Z}\). Therefore, we refer to both systems (1), (2) as to one-dimensional ones. However, it is one of the advantages of the time discretization that it makes apparent a
symmetry between the space and time variables which remains hidden in the continuous time situation. It turns out to be advantageous to re-interpret (2) as a particular case of two-dimensional (2D) lattice systems. Among other things, this interpretation will allow us to derive a pluri-Lagrangian structure for these (one-dimensional) systems, based on a general theory of discrete 2D pluri-Lagrangian systems developed in [BPS14a].

The structure and the main results of the present paper are as follows.

- In section 2, we provide the reader with an overview of the theory of pluri-Lagrangian systems in dimensions \( d = 1, 2 \), following mainly [Sur13, BPS14a]. The fundamental notion of consistent systems of 2D, resp. three-dimensional (3D) corner equations, which are main building blocks of pluri-Lagrangian systems, is reminded in detail.

- Then, in section 3, we present the construction of two mutually commuting families of symplectic maps (Bäcklund transformations) from a generic discrete 2D pluri-Lagrangian system generated by a discrete 3-point 2-form. The commutativity proof is based on the construction of the so called local superposition formulae which turn out to be nothing but the suitably interpreted 3D corner equations. Moreover, these superposition formulae enable us to handle the multi-valuedness of Bäcklund transformations in the case of periodic boundary conditions, by means of a precise description of the branching behavior of the multi-valued maps.

- In sections 4–8, these general results are applied to all systems of the relativistic Toda type, as listed in [Adl99, Sur03]. For each of the systems, we identify all the ingredients of the pluri-Lagrangian structure, which allows us to give unified proofs for commutativity of all maps in question. In particular, in all cases we prove the so-called closure relation, which expresses the fact that the Lagrangian 1-form on the multi-time (space of independent variables) is closed on solutions of variational equations, and turns out to be the main feature of the Lagrangian theory. These results generalize the ones from our recent paper [BPS13]: sending the relativistic parameter \( \alpha \) to 0, we recover the corresponding results for the non-relativistic case obtained in [BPS13]. This reinforces the observation already made in [BS10b]: the non-relativistic degeneration obscures the natural relations to 2D lattice systems.

- Finally, in section 9, we turn to the question of the relation of the pluri-Lagrangian structure to more traditional attributes and notions of integrability. In the one-dimensional context, such a relation is established through connecting the closure relation with the spectrality property, introduced by Kuznetsov and Sklyanin [KS98], which says that the derivative of the Lagrangian with respect to the parameter of a family of Bäcklund transformations is a generating function of common integrals of motion for the whole family. In the 2D context, we establish a new result which connects the closure relation with a parameter-dependent family of local conservation laws. Again, the densities of these conservation laws turn out to be composed of the derivatives of the Lagrangian 2-form with respect to the Bäcklund parameter. Moreover, in the framework of the relativistic Toda-type systems, these conservation laws turn out to constitute a local form of the integrals of motion provided by the spectrality property.

2. General theory of discrete pluri-Lagrangian systems

**Definition 2.1.** \((d\text{-dimensional pluri-Lagrangian problem})\). Let \( \mathcal{L} \) be a discrete \( d \)-form on \( \mathbb{Z}^m \), depending on some field \( x : \mathbb{Z}^m \to \mathcal{X} \), where \( \mathcal{X} \) is some vector space.
To an arbitrary oriented $d$-dimensional manifold $\Sigma$ in $\mathbb{Z}^m$, there corresponds the action functional, which assigns to $x \mid_{V(\Sigma)}$, i.e., to the fields at the vertices of $\Sigma$, the number

$$S_{\Sigma} = \sum_{\sigma \in \Sigma} \mathcal{L}(\sigma).$$

We say that the field $x : V(\Sigma) \to \mathcal{X}$ is a critical point of $S_{\Sigma}$, if at any interior point $n \in V(\Sigma)$, we have

$$\frac{dS_{\Sigma}}{dx(n)} = 0.$$

We say that the field $x : \mathbb{Z}^m \to \mathcal{X}$ solves the pluri-Lagrangian problem for the Lagrangian $d$-form $\mathcal{L}$ if, for any oriented $d$-dimensional manifold $\Sigma$ in $\mathbb{Z}^m$, the restriction $x \mid_{V(\Sigma)}$ is a critical point of the corresponding action $S_{\Sigma}$.

2.1. One-dimensional pluri-Lagrangian systems, $d = 1$

This section is based on [Sur13].

In the case $d = 1$, $\mathcal{L}$ is a function of directed edges $\sigma$ of $\mathbb{Z}^m$ with $\mathcal{L}(-\sigma) = -\mathcal{L}(\sigma)$. It is supposed that $\mathcal{L}(\sigma)$ depends on the values of the field $x : \mathbb{Z}^m \to \mathcal{X}$ at the endpoints of the edge $\sigma$. Thus, if the following notations are used: $x$ for $x(n)$ at a generic point $n \in \mathbb{Z}^m$, and

$$x_i = x(n + e_i), \quad x_{-i} = x(n - e_i), \quad i = 1, \ldots, m,$$

where $e_i$ is the unit vector of the $i$th coordinate direction, then we assume that

$$\mathcal{L}(\sigma_i) = \mathcal{L}(n, n + e_i) = \Lambda_i(x, x_i) \quad \Leftrightarrow \quad \mathcal{L}(-\sigma_i) = \mathcal{L}(n + e_i, n) = -\Lambda_i(x, x_i).$$

Here $\Lambda_i : \mathcal{X} \times \mathcal{X} \to \mathbb{R}$ are local Lagrangian functions corresponding to the edges of the $i$th coordinate direction.

A one-dimensional manifold $\Sigma$ is understood as an arbitrary discrete curve (path) which is a concatenation of a sequence of directed edges in $\mathbb{Z}^m$ such that the endpoint of any edge is the beginning of the next one. The action $S_{\Sigma}$ is understood as the sum of values of $\mathcal{L}$ evaluated over all directed edges of which $\Sigma$ consists.

Any interior point of any discrete curve $\Sigma$ in $\mathbb{Z}^m$ is of one of the four types shown on figure 1.

The pieces of discrete curves as on figures 1(b)–(d) will be called 2D corners. Observe that a straight piece of a discrete curve, as on figure 1(a) is a sum of 2D corners, as on figures 1(b) and (c). The whole variety of Euler–Lagrange equations for a pluri-Lagrangian system with $d = 1$ reduces to the following three types of 2D corner equations:

$$\frac{\partial \Lambda_i(x, x_i)}{\partial x} = \frac{\partial \Lambda_j(x, x_j)}{\partial x} = 0,$$

$$\frac{\partial \Lambda_i(x_{-i}, x)}{\partial x} + \frac{\partial \Lambda_j(x, x_j)}{\partial x} = 0,$$

$$\frac{\partial \Lambda_i(x_{-i}, x)}{\partial x} - \frac{\partial \Lambda_j(x-j, x)}{\partial x} = 0.$$
In particular, the standard single-time discrete Euler–Lagrange equation,
\[ \frac{\partial A_j(x_{-i}, x)}{\partial x} + \frac{\partial A_i(x, x_i)}{\partial x} = 0, \]
corresponding to a straight piece of a discrete curve as on figure 1(a) is a consequence of equations (4) and (5), corresponding to 2D corners as on figures 1(b) and (c).

To discuss the consistency of the system of 2D corner equations, it will be more convenient to re-write them with appropriate shifts, as
\[ \frac{\partial A_i(x, x_i)}{\partial x} - \frac{\partial A_j(x, x_j)}{\partial x} = 0, \quad (E) \]
\[ \frac{\partial A_j(x, x_j)}{\partial x_i} + \frac{\partial A_i(x_i, x_{ij})}{\partial x_j} = 0, \quad (E_j) \]
\[ \frac{\partial A_j(x, x_j)}{\partial x_j} + \frac{\partial A_i(x, x_{ij})}{\partial x_j} = 0, \quad (E_i) \]
In this form, 2D corner equations \((E)\)–\((E_0)\) correspond to the four vertices of an elementary square \(\sigma_{ij}\) of the lattice, as on figure 2(a). Consistency of the system of 2D corner equations \((E)\)–\((E_0)\) should be understood as follows: start with the fields \(x, x_i, x_j\) satisfying equation \((E)\). Then each of equations \((E_i), (E_j)\) can be solved for \(x_{ij}\). Thus, we obtain two alternative values for the latter field. Consistency takes place if these values coincide identically (with respect to the initial data), and, moreover, if the resulting field \(x_{ij}\) satisfies equation \((E_0)\). In other words:

**Definition 2.2.** The system of 2D corner equations \((E)\)–\((E_0)\) is called *consistent*, if it has the minimal possible rank 2, i.e., if exactly two of these four equations are independent.

Observe that 2D corner equations \((E)\)–\((E_0)\) can be put as

\[
\frac{\partial \Lambda_i(x_i, x_{ij})}{\partial x_{ij}} - \frac{\partial \Lambda_j(x_i, x_{ij})}{\partial x_{ij}} = 0.
\]

\((E_0)\)

where \(\Lambda_{ij}\) is the action along the boundary of an oriented elementary square \(\sigma_{ij}\) (this action can be identified with the discrete exterior derivative \(d\mathcal{L}\) evaluated at \(\sigma_{ij}\)),

\[
S_{ij} = d\mathcal{L}(\sigma_{ij}) = \Delta_i \mathcal{L}(\sigma_{ij}) - \Delta_j \mathcal{L}(\sigma_{ij}) = \Lambda_i(x_i, x_{ij}) + \Lambda_j(x_i, x_{ij}) - \Lambda_i(x_j, x_{ij}) - \Lambda_j(x_j, x_{ij}).
\]

Here and in what follows, \(\Delta_i = T_i - I\) is the difference operator, \(T_i\) being the shift operator in the \(i\)th coordinate, so that, e.g., \(T_i x = T_i x(n) = x(n + e_i) = x_i\), \(T_i x_j = T_i x(n + e_j) = x(n + e_i + e_j) = x_{ij}\).

The main feature of our definition is that the ‘almost closedness’ of the 1-form \(\mathcal{L}\) on solutions of the system of 2D corner equations is, so to say, built-in from the outset.

**Theorem 2.3.** For any pair of the coordinate directions \(i, j\), the action \(S_{ij}\) over the boundary of an elementary square of these coordinate directions is constant on solutions of the system of 2D corner equations (13):

\[
S_{ij}(x, x_i, x_{ij}, x_j) = \ell_{ij} = \text{const mod } 0, \ldots, \frac{\partial S_{ij}}{\partial x} = 0, \ldots, \frac{\partial S_{ij}}{\partial x_{ij}} = 0.
\]

In particular, if all these constants \(\ell_{ij}\) vanish, then the discrete 1-form \(\mathcal{L}\) is closed on solutions of the Euler–Lagrange equations, so that the critical value of the action functional \(S_{ij}\) does not depend on the choice of the curve \(\Sigma\) connecting two given points in \(\mathcal{P}\).

We now turn to the Hamiltonian part of the theory of one-dimensional pluri-Lagrangian systems. Consistency of the system of 2D corner equations (4)–(6) is equivalent to existence of a function \(p : \mathcal{P}_m \rightarrow \mathcal{X}\) satisfying all the relations

\[
p = \frac{\partial \Lambda_i(x_i)}{\partial x}, \quad i = 1, \ldots, m, \quad (8)
\]

\[
p = -\frac{\partial \Lambda_i(x_{ij})}{\partial x}, \quad i = 1, \ldots, m. \quad (9)
\]
We say that the multi-time discrete Lagrangian 1-form $\mathcal{L}$ is Legendre transformable, if all the equation (8) can be solved for $x_i$ in terms of $x, p$. In this case, equations

$$p = \frac{\partial \Lambda_i(x, x_i)}{\partial x}, \quad p_i = -\frac{\partial \Lambda_i(x, x_i)}{\partial x_i},$$

(10)
define a symplectic map $F_i : (x, p) \mapsto (x_i, p_i)$.

**Theorem 2.4.** For a consistent one-dimensional pluri-Lagrangian system with a Legendre-transformable 1-form $\mathcal{L}$, maps $F_i$ commute:

$$F_i \circ F_j = F_j \circ F_i,$$

(11)
see figure 2(b)). Conversely, for a given system of $m$ commuting symplectic maps $F_i$ admitting Lagrangians (generating functions) $\Lambda_i$, the 1-form $\mathcal{L}$ defined by $\mathcal{L}(\sigma_i) = \Lambda_i(x, x_i)$, generates a consistent one-dimensional pluri-Lagrangian system.

### 2.2. Two-dimensional pluri-Lagrangian systems, $d = 2$

This section is based on [BPS14a].

In the case $d = 2$, $\mathcal{L}$ is a function of oriented elementary squares

$$\sigma_{ij} = (n, n + e_i, n + e_i + e_j, n + e_j),$$

such that $\mathcal{L}(\sigma_{ij}) = -\mathcal{L}(\sigma_{ji})$. It is supposed that $\mathcal{L}(\sigma_{ij})$ depends on the values $x, x_i, x_j, x_{ij}$ of the field $x : \mathbb{Z}^m \to \mathcal{X}$ at the four vertices of the elementary square $\sigma_{ij}$. A 2D manifold $\Sigma$ is understood as a quad-surface, i.e., an oriented surface in $\mathbb{R}^m$ composed of elementary squares of $\mathbb{Z}^m$. The action $S_2$ is understood as the sum of the values of $\mathcal{L}$ evaluated over all oriented elementary squares of which $\Sigma$ is composed.

One can show that the flower of any interior vertex of an oriented quad-surface $\Sigma$ in $\mathbb{Z}^m$ can be represented as a sum of (oriented) 3D corners in $\mathbb{Z}^{m+1}$. Here, a 3D corner is a quad-surface consisting of three elementary squares adjacent to a vertex of valence 3. Examples of 3D corners are given in [BPS14a]. As a consequence, the action for any flower can be represented as a sum of actions for several 3D corners. Thus, Euler–Lagrange equation for any interior vertex $n$ of $\Sigma$ can be represented as a sum of several Euler–Lagrange equations for 3D corners. This justifies the following fundamental definition:

**Definition 2.5.** The system of 3D corner equations for a given discrete 2-form $\mathcal{L}$ consists of discrete Euler–Lagrange equations for all possible 3D corners in $\mathbb{Z}^m$. If the action for the surface of an oriented elementary cube $\sigma_{ijk}$ of the coordinate directions $i, j, k$ (which can be identified with the discrete exterior derivative $d\mathcal{L}$ evaluated at $\sigma_{ijk}$) is denoted by

$$S_{ijk} = d\mathcal{L}(\sigma_{ijk}) = \Delta_k \mathcal{L}(\sigma_{ij}) + \Delta_i \mathcal{L}(\sigma_{jk}) + \Delta_j \mathcal{L}(\sigma_{ki}),$$

(12)
then the system of 3D corner equations consists of the eight equations

$$\frac{\partial S_{ijk}}{\partial x} = 0, \quad \frac{\partial S_{ijk}}{\partial x_i} = 0, \quad \frac{\partial S_{ijk}}{\partial x_j} = 0, \quad \frac{\partial S_{ijk}}{\partial x_k} = 0,$$

$$\frac{\partial S_{ijk}}{\partial x_{ij}} = 0, \quad \frac{\partial S_{ijk}}{\partial x_{jk}} = 0, \quad \frac{\partial S_{ijk}}{\partial x_{ik}} = 0, \quad \frac{\partial S_{ijk}}{\partial x_{ijk}} = 0,$$

(13)
for each triple $i, j, k$. 7
Thus, the system of 3D corner equations encompasses all possible discrete Euler–Lagrange equations for all possible quad-surfaces $\Sigma$. In other words, solutions of a 2D pluri-Lagrangian problem as introduced in definition 2.1 are precisely solutions of the corresponding system of 3D corner equations.

**Remark 1.** We formulated the system of 3D corner equations for a generic 2-form $\mathcal{L}$. In particular cases the quantity $S^{ijk}$ could be independent on some of the fields at the corners of the cube. Then the system of 3D corner equations (13) could contain less equations.

Of course, in order that the above definition be meaningful, the system of 3D corner equations has to be consistent:

**Definition 2.6.** The system (13) is called consistent, if it has the minimal possible rank 2, i.e., if exactly two of these equations are independent.

Again, the ‘almost closedness’ of the 2-form $\mathcal{L}$ on solutions of the system of 3D corner equations is built-in from the outset.

**Theorem 2.7.** For any triple of the coordinate directions $i, j, k$, the action $S^{ijk}$ over an elementary cube of these coordinate directions is constant on solutions of the system of 3D corner equations (13):

$$S^{ijk}(x, \ldots, x_{ijk}) = e^{ijk} = \text{const} \left( \partial S^{ijk}/\partial x = 0, \ldots, \partial S^{ijk}/\partial x_{ijk} = 0 \right).$$

The most interesting case is, of course, when all $e^{ijk} = 0$. Then $d\mathcal{L} = 0$, that is, the discrete 2-form $\mathcal{L}$ is closed on solutions of the system of 3D corner equations, so that the critical value of the action $S_\Sigma$ does not change under perturbations of the quad-surface $\Sigma$ in $\mathbb{Z}^m$ fixing its boundary.

**3. From 2D pluri-Lagrangian systems to relativistic Toda type systems**

We start with a general 3-point 2-form

$$\mathcal{L}(\sigma_{ij}) = L_i(x_i - x) - L_j(x_j - x) - A_{ij}(x_j - x_i),$$

where the Lagrangians $L_i$ and $A_{ij}$ only depend on the differences of the fields at the end points, and the diagonal Lagrangians are skew-symmetric in the sense that $A_{ij}(x) = -A_{ji}(-x)$.

For a 3-point 2-form, expression (12) specializes to

$$S^{ijk} = L_i(x_{ik} - x_k) + L_j(x_{ij} - x_i) + L_k(x_{jk} - x_j) - L_i(x_{ij} - x_j) - L_j(x_{jk} - x_k) - L_k(x_{ik} - x_i) - A_{ij}(x_{ik} - x_k) - A_{jk}(x_{jk} - x_{ij}) + A_{ij}(x_j - x_i) + A_{jk}(x_k - x_j) + A_{ki}(x_i - x_k).$$

Thus, $S^{ijk}$ depends on neither $x$ nor $x_{ijk}$, and its domain of definition is better visualized as an octahedron shown in figure 3.
Accordingly, the system of corner equations consists of six equations per elementary 3D cube, which we denote by \( \psi_i \), \( \psi_j \), \( \psi_k \), \( \psi_{ij} \), \( \psi_{ik} \), and \( \psi_{jk} \). Our main assumption is that the system of corner equations is consistent. To write them down, we set

\[
\psi_i(x) = \frac{\partial L_i(x)}{\partial x}, \quad \phi_{ij}(x) = \frac{\partial L_{ij}(x)}{\partial x}.
\]  

(16)

In particular, we have: \( \phi_{ij}(x) = \phi_{ji}(-x) \). In terms of these functions, corner equations read:

\[
\psi_i(x_{ij} - x_i) + \phi_{ij}(x_j - x_i) = \psi_j(x_{jk} - x_i) + \phi_{ik}(x_k - x_i), \quad (E_i)
\]

\[
\psi_j(x_{ij} - x_j) + \phi_{jk}(x_{ij} - x_k) = \psi_i(x_{ij} - x_j) + \phi_{ij}(x_{ij} - x_{jk}). \quad (E_{ij})
\]

In what follows, one of the coordinate directions (which we denote as the 0th one) plays a distinguished role, it enumerates the sites of the relativistic Toda chains. We will use the index \( n \) for this coordinate direction only. Accordingly, we will only consider surfaces in \( \mathbb{Z}^m \) which contain, along with any point, the whole line through this point parallel to the 0th coordinate axis. One can call such surfaces cylindrical. The set of values of \( x \) along such a line, \( x = \{ x_n : n \in \mathbb{Z} \} \), or, upon a finite-dimensional reduction, \( x = \{ x_n : 1 \leq n \leq N \} \), is an element of the configuration space of the relativistic Toda lattice. We use the accents \( \sim \) and \( \overset{\sim}{\sim} \) to denote the shift in the discrete times corresponding to all other coordinate directions.

**Definition 3.1.** The map \( F_i : (x, p) \mapsto (\tilde{x}_i, \tilde{p}_n) \) is the symplectic map with the generating function

\[
\mathcal{L}_i(x, \tilde{x}) = \sum_{n=1}^{N} L_i(\tilde{x}_n - x_n) - \sum_{n=1}^{N} L_0(x_{n+1} - x_n) - \sum_{n=1}^{N} A_0(x_{n+1} - \tilde{x}_n),
\]  

(17)

thus its equations of motion \( p_n = -\partial \mathcal{L}_i/\partial x_n \), \( \tilde{p}_n = \partial \mathcal{L}_i/\partial \tilde{x}_n \) read:

\[
F_i : \begin{cases} 
    p_n = \psi_i(\tilde{x}_n - x_n) + \phi_{i0}(x_n - \tilde{x}_{n-1}) - \psi_0(x_{n+1} - x_n) + \psi_0(x_n - x_{n-1}), \\
    \tilde{p}_n = \psi_i(\tilde{x}_n - x_n) + \phi_{i0}(x_{n+1} - \tilde{x}_n).
\end{cases}
\]  

(18)
The Euler–Lagrange equations read

\[ \sum_{n=1}^{N} A_n \frac{\partial F}{\partial x_n} - \sum_{n=1}^{N} L \left( x_n - x_{n-1} \right) = \psi_0 \left( y_{n+1} - y_n \right) - \psi_0 \left( y_n - y_{n-1} \right) + \phi_0 \left( x_{n+1} - x_n \right) - \psi_0 \left( x_n - x_{n-1} \right). \] (19)

This map corresponds to the edges \((x, \bar{x}) = (x, x_i)\) of the \(i\)th coordinate direction, to which the strip supporting \(\mathbb{L}\) projects along the 0th coordinate axis. See the identifications of variables on figure 4. We denote the index set of the maps \(F_i\) by \(I = \{i, j, \ldots\}\).

**Definition 3.2.** The map \(G_k : (x, p) \mapsto (\bar{x}, \bar{p})\) is the symplectic map with the generating function

\[ \mathbb{M}_k(x, \bar{x}) = \sum_{n=1}^{N} A_k \left( x_n - x_{n-1} \right) + \sum_{n=1}^{N} L_0 \left( x_n - x_{n-1} \right) - \sum_{n=1}^{N} L_k \left( x_n - x_{n-1} \right), \] (20)

thus its equations of motion \(p_k = -\partial \mathbb{M}_k / \partial x_n, \bar{p}_k = \partial \mathbb{M}_k / \partial \bar{x}_n\) read:

\[ G_k : \begin{cases} p_k = \phi_k \left( x_n - x_{n-1} \right) + \psi_k \left( x_n - x_{n-1} \right), \\ \bar{p}_k = \phi_k \left( \bar{x}_n - x_{n-1} \right) + \psi_k \left( \bar{x}_n - x_{n-1} \right) - \psi_k \left( \bar{x}_n - \bar{x}_{n-1} \right). \end{cases} \] (21)

The Euler–Lagrange equations read

\[ \phi_k \left( \bar{x}_n - x_{n-1} \right) + \phi_k \left( x_n - x_{n-1} \right) = -\psi_k \left( x_{n+1} - x_n \right) + \psi_k \left( x_n - x_{n-1} \right) + \psi_k \left( \bar{x}_{n+1} - \bar{x}_n \right) - \psi_k \left( \bar{x}_n - \bar{x}_{n-1} \right). \] (22)

This map corresponds to the negatively directed edges \((x, \bar{x}) = (x, x_k)\) of the \(k\)th coordinate direction, to which the strip supporting \(\mathbb{M}_k\) projects along the 0th coordinate axis. See the identifications of variables on figure 5. We denote the index set of the maps \(G_k\) by \(K = \{k, \ell, \ldots\}\), and assume it to be disjoint from \(I = \{i, j, \ldots\}\).

In the present paper, we consider maps \(F_i, G_k\) with finitely many degrees of freedom \((1 \leq n \leq N)\). This requires to specify certain boundary conditions. We will consider either the so-called open-end or periodic boundary conditions.
Open-end boundary conditions correspond to letting the second and the third sums in the 
Lagrangian functions (17), (20) extend over $1 \leq n \leq N - 1$ only. Effectively, this 
amounts to omitting terms containing $x_0$ or $\tilde{x}_0$ from the expressions for $p_1$ and $\tilde{p}_1$, and 
likewise omitting terms containing $x_{N+1}$ or $\tilde{x}_{N+1}$ from the expressions for $p_N$ and $\tilde{p}_N$. In 
this case, maps $F_i$ and $G_k$ are single-valued functions of $(x, p)$.

Periodic boundary conditions correspond to letting all indices be taken mod $N$, so that 
$x_0 = x_N, x_{N+1} = x_1$. In this case, these maps are double-valued, so that the very notion of 
their commutativity has to be clarified. We achieve this along the same lines as in the 
previous work [BPS13].

**Theorem 3.3.** Let $i, j, k$, and $\ell$ be four different indices from $I = \{i, j, \ldots\}$ and 
$K = \{k, \ell, \ldots\}$. Then any two of the maps $F_i, F_j, G_k$, and $G_\ell$ commute.

The next three subsections are devoted to the proof of this theorem. We prove separately 
the commutativity of $F_i$ and $F_j$, of $G_k$ and $G_\ell$, and of $F_i$ and $G_\ell$. As explained in section 2.1, 
each such statement is equivalent to consistency of the corresponding system of 2D corner equations.

**3.1. Proof of commutativity of the maps $F_i, F_j$**

The 2D corner equations for the pluri-Lagrangian system corresponding to two maps $F_i$ and 
$F_j$ read:

$$
\psi_i(x_n - x_n) + \phi_{j0}(x_n - x_{n-1}) = \psi_j(x_n - x_n) + \phi_{i0}(x_n - x_{n-1}), \quad (E)
$$

$$
\psi_i(\tilde{x}_n - \tilde{x}_n) + \phi_{j0}(\tilde{x}_n - \tilde{x}_{n-1}) = \psi_j(\tilde{x}_n - \tilde{x}_n) + \phi_{i0}(\tilde{x}_n - \tilde{x}_{n-1})
- \psi_0(\tilde{x}_{n+1} - \tilde{x}_n) + \psi_0(\tilde{x}_n - \tilde{x}_{n-1}), \quad (E_i)
$$

$$
\psi_j(\tilde{x}_n - \tilde{x}_n) + \phi_{i0}(\tilde{x}_{n+1} - \tilde{x}_n) = \psi_i(\tilde{x}_n - \tilde{x}_n) + \phi_{j0}(\tilde{x}_n - \tilde{x}_{n+1})
- \psi_0(\tilde{x}_{n+1} - \tilde{x}_n) + \psi_0(\tilde{x}_n - \tilde{x}_{n-1}), \quad (E_j)
$$

$$
\psi_i(\tilde{x}_n - \tilde{x}_n) + \phi_{j0}(\tilde{x}_{n+1} - \tilde{x}_n) = \psi_j(\tilde{x}_n - \tilde{x}_n) + \phi_{i0}(\tilde{x}_{n+1} - \tilde{x}_n), \quad (E_0)
$$

A visualization of the 2D corner equations embedded in $\mathbb{Z}^3$ is given in figure 6. Discrete 
curves in the multi-time plane $\mathbb{Z}^2$ are in a one-to-one correspondence with cylindrical surfaces
in $\mathbb{Z}^3$, via the projection along the first coordinate direction of $\mathbb{Z}^3$. For 2D corners, this is illustrated in figure 6.

Consistency of the above system of 2D corner equations is proven with the help of the following statement.

**Theorem 3.4.** Suppose that the fields $\tilde{x}, \hat{x},$ and $\tilde{\hat{x}}$ satisfy 2D corner equations $(E)$. Define the fields $\hat{\tilde{x}}, \tilde{\hat{x}}$ by any of the following four formulae, which are equivalent by virtue of $(E)$:

$$
\psi_j \left( \tilde{\hat{x}}_n - \tilde{x}_n \right) + \phi_j \left( \tilde{\hat{x}}_{n+1} - \tilde{x}_{n+1} \right) = \psi_0 \left( \tilde{x}_{n+1} - \tilde{x}_n \right) + \phi_0 \left( x_{n+1} - \tilde{x}_n \right),
$$

**(S1)**

$$
\psi_j \left( \tilde{x}_n - \tilde{x}_n \right) + \phi_j \left( \tilde{x}_{n+1} - \tilde{x}_{n+1} \right) = \psi_0 \left( \tilde{x}_{n+1} - \tilde{x}_n \right) + \phi_0 \left( x_{n+1} - \tilde{x}_n \right),
$$

**(S2)**

$$
\psi_i \left( \tilde{x}_{n+1} - \tilde{x}_n \right) + \phi_i \left( \tilde{x}_{n+1} - \tilde{x}_{n+1} \right) = \psi_0 \left( \tilde{x}_{n+1} - \tilde{x}_n \right) + \phi_0 \left( x_{n+1} - \tilde{x}_n \right),
$$

**(S3)**

$$
\psi_j \left( \tilde{x}_{n+1} - \tilde{x}_n \right) + \phi_j \left( \tilde{x}_{n+1} - \tilde{x}_{n+1} \right) = \psi_0 \left( \tilde{x}_{n+1} - \tilde{x}_n \right) + \phi_0 \left( x_{n+1} - \tilde{x}_n \right),
$$

**(S4)**

called superposition formulae (note that each one of these formulae is local with respect to $\tilde{x}$). Then the 2D corner equations $(E_i)$, $(E_j)$, and $(E_{ij})$ are satisfied, as well.

**Proof.** Consider the sublattice $\mathbb{Z}^3$ spanned by the coordinate directions 0 (indexed by the letter $n$), $i$, corresponding to the map $F_i$ (shift in this direction being denoted by $\sim$), and $j$, corresponding to the map $F_j$ (shift in this direction being denoted by $\vee$). See figure 7.

One easily checks that, upon the identifications as on figure 7, the two corner equations $(E_i)$, $(E_{ij})$ and the four superposition formulae $(S1)$–$(S4)$ build nothing but the
system of 3D corner equations \((E_i), \ (E_j)\). Due to consistency of the latter system, as formulated in theorem 3.4, if equation \((E)\) and one of equations \((S_1)\)–\((S_4)\) hold, then equation \((E_{ij})\) and the remaining three of equations \((S_1)\)–\((S_4)\) are satisfied, as well. Furthermore, equation \((E_i)\) is the difference of \((S_1)\) and the downshifted version of \((S_3)\), while equation \((E_j)\) is the difference of \((S_2)\) and the downshifted version of \((S_4)\). This completes the proof. □

This theorem provides us with an exhaustive understanding of commutativity of double-valued Bäcklund transformations in the periodic case:

- in the Lagrangian picture, suppose that we are given fields \(x, \tilde{x}, \hat{x}\) satisfying the 2D corner equation \((E)\). Each of equations \((E_i), \ (E_j)\) produces two values for \(\tilde{x}\). Consistency is reflected in the following fact: one of the values for \(\tilde{x}\) obtained from \((E_i)\) coincides with one of the values for \(\tilde{x}\) obtained from \((E_j)\). Indeed, this common value is nothing but \(\tilde{x}\) obtained from the superposition formulae \((S_1), \ (S_2), \ (S_3)\) or \((S_4)\), as in theorem 3.4.

- In the symplectic maps picture, each of the compositions \(F_i \circ F_j\) and \(F_j \circ F_i\) applied to a point \((x, p)\) produces four different branches for \((\tilde{x}, \hat{x})\). Commutativity is reflected in the following fact: each of the branches of \(F_i \circ F_j\) coincides with one of the branches of \(F_j \circ F_i\). Indeed, theorem 3.4 delivers four possible values for \((\tilde{x}, \hat{x})\) satisfying all 2D corner equations \((E)\)–\((E_{ij})\), namely one \(\tilde{x}\) for each of the four possible combinations of \((\tilde{x}, \hat{x})\). The reader is referred to [BPS13] for a graphical illustration and more details.

3.2. Proof of commutativity of the maps \(G_k, G_\ell\)

The 2D corner equations for the pluri-Lagrangian system corresponding to the two maps \(G_k\) and \(G_\ell\) read:

\[
\begin{align*}
\phi_{k0}(\tilde{x}_n - x_n) + \psi_k(x_n - \tilde{x}_{n-1}) &= \phi_{\ell0}(\tilde{x}_n - x_n) + \psi_\ell(x_n - \tilde{x}_{n-1}), \quad (E) \\
\phi_{k0}(\tilde{x}_n - x_n) + \psi_k(x_{n+1} - \tilde{x}_n) - \psi_0(\tilde{x}_{n+1} - \tilde{x}_n) + \psi_0(\tilde{x}_n - \tilde{x}_{n-1}) &= \phi_{\ell0}(\tilde{x}_n - \tilde{x}_{n}) + \psi_\ell(\tilde{x}_n - \tilde{x}_{n-1}). \quad (E_k)
\end{align*}
\]
Theorem 3.5. Suppose that the fields $x$, $\tilde{x}$, and $\hat{x}$ satisfy 2D corner equations (E). Define the fields $\vec{x}$ by any of the following four formulae, which are equivalent by virtue of (E):

\[
\psi_0\left(\hat{x}_n - x_n\right) + \phi_0\left(x_{n+1} - \hat{x}_n\right) = \psi_0\left(\hat{x}_{n+1} - \hat{x}_n\right) + \phi_0\left(\hat{x}_n - \hat{x}_{n-1}\right),
\]

\[
\psi_1\left(x_{n+1} - \hat{x}_n\right) + \phi_1\left(\hat{x}_n - \hat{x}_{n+1}\right) = \psi_0\left(\hat{x}_{n+1} - \hat{x}_n\right) + \phi_0\left(\hat{x}_n - \hat{x}_{n-1}\right),
\]

\[
\psi_0\left(\hat{x}_n - \hat{x}_{n-1}\right) + \phi_0\left(\hat{x}_{n+1} - \hat{x}_n\right) = \psi_0\left(\hat{x}_{n-1} - \hat{x}_n\right) + \phi_0\left(\hat{x}_n - \hat{x}_{n-1}\right),
\]

\[
\psi_1\left(x_{n+1} - \hat{x}_n\right) + \phi_1\left(\hat{x}_n - \hat{x}_{n-1}\right) = \psi_0\left(\hat{x}_{n-1} - \hat{x}_n\right) + \phi_0\left(\hat{x}_n - \hat{x}_{n-1}\right).
\]

A visualization of the 2D corner equations embedded in $\mathbb{Z}^3$ is given in figure 8.
Proof. We identify the fields as on figure 9. Then equations \((E)\), \((Ek\ell)\) and \((S1)–(S3)\) build the system of consistent 3D corner equations \((E_i)\), \((E_k)\). More precisely, the correspondence is as follows:

- The downshifted version of \((E)\) is \((E_0)\).
- The downshifted version of \((Ek\ell)\) is \((E_0)0\).
- The downshifted versions of \((S1)\) and \((S2)\) are \((E_0)\ell\) and \((E_0)k\), respectively.
- Equations \((S4)\) and \((S3)\) are \((E_0)\ell\) and \((E_0)k\), respectively.

Since the system of 3D corner equations is consistent and, therefore, has rank 2, the following argumentation works: if \((E)\) and one of equations \((S1)–(S3)\) are satisfied, then equation \((Ek\ell)\) and the remaining three equations of \((S1)–(S3)\) are fulfilled, as well. Furthermore, equation \((E_k)\) is a difference of \((S1)\) and \((S3)\), and equation \((E_\ell)\) is a difference of \((S2)\) and \((S4)\). This completes the proof. □

3.3. Proof of commutativity of the maps \(F_\ell, G_\ell\)

The 2D corner equations for the pluri-Lagrangian system corresponding to the maps \(F_\ell\) and \(G_\ell\) (whose actions are encoded by \(\psi\) and \(\psi\), respectively) are given by:

\[
\begin{align*}
\psi_0(\tilde{x}_n - x_n) + \phi_0(\tilde{x}_n - \tilde{x}_{n-1}) - \psi_0(x_{n+1} - x_n) + \psi_0(x_n - x_{n-1}) & = \phi_0(\tilde{x}_n - x_n) + \psi_\ell(x_n - \tilde{x}_{n-1}), \\
\psi_0(\tilde{x}_n - x_n) + \phi_0(\tilde{x}_{n+1} - \tilde{x}_n) & = \phi_0(\tilde{x}_n - x_n) + \psi_\ell(x_{n+1} - \tilde{x}_n), \\
\psi_0(\tilde{x}_n - \tilde{x}_n) + \phi_0(\tilde{x}_n - \tilde{x}_{n-1}) & = \phi_0(\tilde{x}_n - x_n) + \psi_\ell(x_{n+1} - \tilde{x}_n), \\
\psi_0(\tilde{x}_n - \tilde{x}_n) + \phi_0(\tilde{x}_{n+1} - \tilde{x}_n) & = \phi_0(\tilde{x}_n - x_n) + \psi_\ell(x_{n+1} - \tilde{x}_n) - \psi_0(\tilde{x}_{n+1} - \tilde{x}_n).
\end{align*}
\]

A visualization of the 2D corner equations embedded in \(Z^3\) is given in figure 10.

Theorem 3.6. Suppose that the fields \(x, \tilde{x}\), and \(\tilde{x}\) satisfy 2D corner equations \((E)\). Define the fields \(\tilde{\tilde{x}}\) by any of the following two formulae, which are equivalent by virtue of \((E)\):

\[
\begin{align*}
\psi_0(\tilde{\tilde{x}}_n - x_n) + \phi_0(\tilde{\tilde{x}}_n - \tilde{x}_{n-1}) & = \phi_0(\tilde{x}_n - x_n) + \psi_0(x_{n+1} - x_n), \\
\end{align*}
\]

Figure 9. Identification of fields: the 0th coordinate direction enumerates the lattice cites, the coordinate directions \(k, \ell\) correspond to the maps \(G_k, G_\ell\).
called superposition formulae. Then the 2D corner equations \((E_\ell), (E_i)\) and the following two equations are satisfied, as well:

\[
\psi_0\left(\tilde{x}_{n+1} - \tilde{x}_n\right) + \phi_0\left(\tilde{x}_{n+1} - \tilde{x}_n\right) = \psi_\ell\left(x_{n+1} - x_n\right) + \phi_\ell\left(x_{n+1} - \tilde{x}_n\right),
\]  \(S2\)

\[
\psi_0\left(\tilde{x}_n - \tilde{x}_{n-1}\right) + \phi_0\left(\tilde{x}_n - \tilde{x}_{n-1}\right) = \psi_\ell\left(x_n - \tilde{x}_{n-1}\right) + \phi_\ell\left(x_n - \tilde{x}_{n-1}\right),
\]  \(S3\)

\[
\psi_\ell\left(\tilde{x}_{n+1} - \tilde{x}_n\right) + \phi_\ell\left(x_{n+1} - \tilde{x}_n\right) = \phi_{i0}\left(\tilde{x}_{n+1} - \tilde{x}_n\right) + \psi_0\left(\tilde{x}_{n+1} - \tilde{x}_n\right).
\]  \(S4\)

**Proof.** We see that the present theorem works in the same manner as theorems 3.4 and 3.5, with the only difference that now we only have two local superposition formulas \((S1)\) and \((S2)\). We identify the fields in the way which is in figure 11. Then equations \((E_\ell), (E_i)\) and \((S1)-(S4)\) build the system of consistent 3D corner equations \((E_\ell), (E_0)\). More precisely, the correspondence is as follows:

- Equation \((E_\ell)\) is \((E_0)\).
- Equation \((E_i)\) is \((E_\ell)\).
- Equation \((S1)\) is \((E_\ell)\).
- Equation \((S2)\) is \((E_{i0})\).
- Equation \((S3)\) is \((E_\ell)\).
- Equation \((S4)\) is \((E_{i0})\).
First, we observe that equation \((E)\) is a sum of equations \((S1)\) and \((Ei)\). Therefore, if \((E)\) and one of equations \((S1), (S2)\) hold, the remaining equations of \((S1), (S2)\) holds, too.

Due to the fact that the system of 3D corner equations has rank 2, we can claim the following: if \((E)\) and one of equations \((S1), (S2)\) is satisfied, then equations \((E\ell), (Ei), (S3)\) and \((S4)\) are fulfilled, as well.

Finally, we observe that equation \((Ei)\) is a sum of (an upshifted version of) equations \((S3)\) and \((S4)\). This completes the proof. □

4. Bäcklund transformations for symmetric systems of the relativistic Toda type

We start with the pluri-Lagrangian systems related to the quad-equation \(Q_{1}^{0}, Q_{1}^{1},\) and \(Q_{3}^{0}\) from the ABS list (see [BPS14b] for further information). Each of these systems is constructed with the help of just one fundamental function \(\phi(x)\), which is given for these three cases by

\[
\phi(x; \alpha) = \frac{\alpha}{x}, \quad \phi(x; \alpha) = \frac{1}{2} \log \frac{x + \alpha}{x - \alpha}, \quad \text{and} \quad \phi(x; \alpha) = \frac{1}{2} \log \frac{\sinh (x + \alpha)}{\sinh (x - \alpha)},
\]  

(23)

respectively. The corner equations are given by \((E_{ij})\), \((E_{ij})\) with the leg functions

\[
\psi_{i}(x) = \phi(x; \alpha_{i}), \quad \phi_{ij}(x) = \phi(x; \alpha_{i} - \alpha_{j}),
\]

as well as the following choice of parameters:

\[
\alpha_{0} = \alpha, \quad \alpha_{i} = \lambda, \quad \alpha_{j} = \mu, \quad \alpha_{k} = \lambda + \alpha, \quad \alpha_{t} = \mu + \alpha.
\]

Thus, the two mutually commuting families of Bäcklund transformations are given by

\[
F_{ij} : \begin{cases} 
\tilde{p}_{n} &= \phi(\tilde{x}_{n} - x_{n}; \lambda) + \phi(x_{n} - \tilde{x}_{n-1}; \lambda - \alpha) \\
&- \phi(x_{n+1} - x_{n}; \alpha) + \phi(x_{n} - x_{n-1}; \alpha), \\
\tilde{p}_{n} &= \phi(\tilde{x}_{n} - x_{n}; \lambda) + \phi(x_{n+1} - \tilde{x}_{n}; \lambda - \alpha),
\end{cases}
\]

(24)
The functions \( \phi_{ij}(x) \), \( \phi_{k\ell}(x) \), and \( \phi_{i\ell}(x) \) used in theorems 3.4, 3.5, and 3.6 to prove commutativity of any two of the maps \( F_i \), \( F_j \), \( G_k \), and \( G_{\ell} \), are given by

\[
\phi_{ij}(x) = \phi(x; \lambda - \mu), \quad \phi_{k\ell}(x) = \phi(x; \lambda - \mu), \quad \text{and} \quad \phi_{i\ell}(x) = \phi(x; \lambda - \mu - \alpha).
\]

5. Bäcklund transformations for the modified exponential system of the relativistic Toda type

The pluri-Lagrangian system playing the ‘master’ role of the system for all the asymmetric systems of the relativistic Toda type is described in the following proposition.

**Proposition 5.1.** The pluri-Lagrangian system consisting of the 3D corner equations \( (\mathcal{E}_i), (\mathcal{E}_j) \) with the leg functions

\[
\psi_j(x) = \log \frac{\alpha_j + e^x}{\beta_j}, \quad \phi_{ij}(x) = \log \frac{\beta_j - \beta_i e^x}{\alpha_j - \alpha_i e^x}
\]

is consistent, and the corresponding discrete Lagrangian 2-form \( \mathcal{L} \) is closed on its solutions.

**Proof.** The system in question is a slight generalization of a similar pluri-Lagrangian system for which the analogous statements were proven in [BPS14a], and which consists of the 3D corner equations \( (\mathcal{E}_i), (\mathcal{E}_j) \) with the leg functions

\[
\bar{\psi}_j(x) = \log \left( \gamma_j - e^x \right), \quad \bar{\phi}_{ij}(x) = \log \frac{\gamma_i - \gamma_j e^x}{\gamma_j - \gamma_i e^x}.
\]

Setting

\[
x_i = \bar{x}_i + \log \left( -\gamma_i \beta_i \right), \quad x_{ij} = \bar{x}_{ij} + \log \left( \gamma_i \gamma_j \beta_i \beta_j \right), \quad \text{and} \quad \alpha_i = \gamma_i^2 \beta_i,
\]

we find:

\[
\psi_j(x_{ij} - x_i) = \psi_j(x_{ij} - x_i) + \log \gamma_j, \quad \phi_{ij}(x_j - x_i) = \phi_{ij}(x_j - x_i) - \log \left( \gamma_i \gamma_j \right).
\]

This yields, up to additive constants,

\[
L_j(x_{ij} - x_i) = \mathcal{L}_j \left( \bar{x}_{ij} - \bar{x}_i \right) + \log \gamma_j \left( \bar{x}_{ij} - \bar{x}_i \right),
\]

\[
\Lambda_{ij}(x_j - x_i) = \mathcal{\Lambda}_{ij} \left( \bar{x}_j - \bar{x}_i \right) - \log \left( \gamma_i \gamma_j \right) \left( x_j - x_i \right).
\]

Now, the closure relation for the system with the leg functions (26) is immediately seen to be equivalent to the closure relation for the system with the leg functions (27). \( \square \)
In identifying the leg functions, we will often silently perform shifts similar to (29),
$$\psi_j \rightarrow \psi_j + \epsilon_j, \quad \phi_{ij} \rightarrow \phi_{ij} - \epsilon_i - \epsilon_j,$$
which affect neither the 3D-corner equations nor the closedness of the pluri-Lagrangian 2-form $\mathcal{L}$.

We start with the following choice of parameters:
$$\alpha_0 = \frac{1}{\alpha}, \quad \alpha_i = -1, \quad \alpha_j = -1, \quad \alpha_k = \frac{1}{\lambda + \alpha}, \quad \alpha_l = \frac{1}{\mu + \alpha},$$
$$\beta_0 = -1, \quad \beta_i = \lambda, \quad \beta_j = \mu, \quad \beta_k = -1, \quad \beta_l = -1.$$

With this choice of parameters, the relevant leg functions for the map $F_i$ are
$$\psi_{0i}(x) = \log \left(1 + \alpha e^x\right), \quad \psi_i(x) = \log \frac{e^x - 1}{\lambda}, \quad \phi_{i0}(x) = \log \frac{1 + \lambda e^x}{1 + \alpha e^x},$$
and those relevant for the map $G_k$ are
$$\psi_k(x) = \log \left(1 + (\lambda + \alpha)e^x\right), \quad \phi_{k0}(x) = \log \frac{\lambda^{-1}(e^x - 1)}{1 - \lambda^{-1}a(e^x - 1)}.$$

Moreover, we always assume that the functions $\psi_j$, $\phi_{j0}$ are obtained from $\psi_i$, $\phi_{i0}$ by replacing $\lambda$ by $\mu$, and, the functions $\psi_i$, $\phi_{i0}$ are obtained from $\psi_x$, $\phi_{x0}$ in the same way. Thus, we are dealing with the following two families of Bäcklund transformations:

$$F_i: \begin{cases}
e^\mathcal{R}_i = \frac{e^{x_i-x_{i-1}} - 1}{\lambda} \cdot \frac{1 + \lambda e^{x_{i-1}-x_i}}{1 + \alpha e^{x_{i-1}-x_i}}, \\
e^\mathcal{R}_i = \frac{e^{x_i-x_{i-1}} - 1}{\lambda} \cdot \frac{1 + \lambda e^{x_{i+1}-x_i}}{1 + \alpha e^{x_{i+1}-x_i}},
\end{cases} \quad (31)$$

and

$$G_k: \begin{cases}
e^\mathcal{R}_k = \frac{\lambda^{-1}(e^{x_k-x_{k-1}} - 1)}{1 - \lambda^{-1}a(e^{x_k-x_{k-1}} - 1)} \cdot \left(1 + (\lambda + \alpha)e^{x_{k-1}-x_k}\right), \\
e^\mathcal{R}_k = \frac{\lambda^{-1}(e^{x_k-x_{k-1}} - 1)}{1 - \lambda^{-1}a(e^{x_k-x_{k-1}} - 1)} \cdot \left(1 + (\lambda + \alpha)e^{x_{k+1}-x_k}\right),
\end{cases} \quad (32)$$

The functions $\phi_{ij}(x)$, $\phi_{ik}(x)$, and $\phi_{il}(x)$ used in theorems 3.4, 3.5, and 3.6 to prove commutativity of any two of the maps $F_i$, $F_j$, $G_k$, and $G_l$, are given by
$$\phi_{ij}(x) = \log \frac{\lambda e^x - \mu}{e^x - 1}, \quad \phi_{ik}(x) = \log \frac{e^x - 1}{(\mu + \alpha)e^x - (\lambda + \alpha)},$$
$$\phi_{il}(x) = \log \frac{1 + \lambda e^x}{1 + (\mu + \alpha)e^x}.$$
6. Bäcklund transformations for the ‘master’ exponential system of the relativistic Toda type

The following system is algebraically similar to the modified exponential one, but provides us with more freedom in the choice of parameters. We set in the system of proposition 5.1:

\[
\begin{align*}
\alpha_0 &= \frac{1}{e\alpha}, & \alpha_i &= \frac{\lambda - \epsilon}{e}, & \alpha_j &= \frac{\mu - \epsilon}{e}, & \alpha_k &= \frac{1}{e(\alpha + \lambda)}, & \alpha_l &= \frac{1}{e(\alpha + \mu)}, \\
\beta_0 &= \frac{1}{\alpha - \epsilon}, & \beta_i &= \lambda, & \beta_j &= \mu, & \beta_k &= \frac{1}{\alpha + \lambda - \epsilon}, & \beta_l &= \frac{1}{\alpha + \mu - \epsilon}.
\end{align*}
\]

Up to the shifts of the type (30), we find:

\[
\begin{align*}
\psi_0(x) &= \log \left(1 + e\alpha e^x \right), & \psi_i(x) &= \log \left(1 + \frac{e}{\lambda}(e^x - 1)\right), \\
\phi_{i0}(x) &= \log \frac{1 - \lambda(\alpha - \epsilon)e^x}{1 - \alpha(\lambda - \epsilon)e^x},
\end{align*}
\]

and

\[
\begin{align*}
\psi_k(x) &= \log \left(1 + \epsilon(\lambda + \alpha)e^x \right), & \phi_{k0}(x) &= \log \frac{1 - \lambda^{-1}(\alpha - \epsilon)(e^x - 1)}{1 - \lambda^{-1}\lambda(e^x - 1)},
\end{align*}
\]

so that we are dealing with the following two families of Bäcklund transformations:

\[
F_i: \quad \begin{cases} 
 e^{\tau L_i} = \left(1 + e\lambda^{-1}(e^{x_{n-1}} - 1)\right) \cdot \frac{1 - \lambda(\alpha - \epsilon)e^{x_{n-1}}}{1 - \alpha(\lambda - \epsilon)e^{x_{n-1}}} \cdot \frac{1 + e\alpha e^{x_{n-1}}}{1 + e\alpha e^{x_{n-1}}}, \\
 e^{\tau R_i} = \left(1 + e\lambda^{-1}(e^{x_{n-2}} - 1)\right) \cdot \frac{1 - \lambda(\alpha - \epsilon)e^{x_{n-2}}}{1 - \alpha(\lambda - \epsilon)e^{x_{n-2}}},
\end{cases}
\]

and

\[
G_i: \quad \begin{cases} 
 e^{\tau L_i} = \frac{1 - (\alpha - \epsilon)\lambda^{-1}(e^{x_{n-1}} - 1)}{1 + \alpha\lambda^{-1}(e^{x_{n-1}} - 1)} \cdot \left(1 + e(\alpha + \lambda)e^{x_{n-1}}\right), \\
 e^{\tau R_i} = \frac{1 - (\alpha - \epsilon)\lambda^{-1}(e^{x_{n-2}} - 1)}{1 + \alpha\lambda^{-1}(e^{x_{n-2}} - 1)} \cdot \frac{1 + e\alpha e^{x_{n-2}}}{1 + e\alpha e^{x_{n-2}}},
\end{cases}
\]

The functions \(\phi_{ij}(x)\), \(\phi_{ik}(x)\), and \(\phi_{il}(x)\) used in theorems 3.4, 3.5, and 3.6 to prove commutativity of any two of the maps \(F_i\), \(F_j\), \(G_k\), and \(G_l\), are given by

\[
\begin{align*}
\phi_{ij}(x) &= \log \frac{\lambda e^x - \mu}{(\lambda - \epsilon)e^x - (\mu - \epsilon)}, & \phi_{ik}(x) &= \log \frac{(\mu + \alpha - \epsilon)e^x - (\lambda + \alpha - \epsilon)}{(\mu + \alpha)e^x - (\lambda + \alpha)}, \\
\phi_{il}(x) &= \log \frac{1 - \lambda(\mu + \alpha - \epsilon)e^x}{1 - (\lambda - \epsilon)(\mu + \alpha)e^x}.
\end{align*}
\]
7. Bäcklund transformations for the additive exponential system of the relativistic Toda type

Performing the limit $\epsilon \to 0$ in the pluri-Lagrangian system of section 6, we arrive at the following leg functions:

$$\psi_0(x) = ae^x, \quad \psi_i(x) = \frac{1}{\lambda}(e^x - 1), \quad \phi_{i0}(x) = \frac{(\lambda - \alpha)e^x}{1 - \lambda ae^x},$$

and

$$\psi_k(x) = (\lambda + \alpha)e^x, \quad \phi_{k0}(x) = \frac{\lambda^{-1}(e^x - 1)}{1 - \lambda^{-1}\alpha(e^x - 1)}.$$ 

Thus, we are dealing with the following two families of Bäcklund transformations:

$$F_i : \begin{cases} p_n = e^{\tilde{\tilde{x}}_n-x_n} - 1 + \frac{(\lambda - \alpha)e^{x_n-x_{n-1}}}{1 - \lambda ae^{x_n-x_{n-1}}} + ae^{x_n-x_{n-1}} - ae^{x_{n+1}-x_n}, \\ \tilde{p}_n = e^{\tilde{\tilde{x}}_n-x_n} - 1 + \frac{(\lambda - \alpha)e^{x_n-x_{n-1}}}{1 - \lambda ae^{x_n-x_{n-1}}}; \end{cases} \quad (35)$$

and

$$G_k : \begin{cases} p_n = \frac{\lambda^{-1}(e^{x_n-x_0} - 1)}{1 - \alpha^{-1}e^{x_n-x_0}} + (\lambda + \alpha)e^{x_0-x_{n-1}}, \\ \tilde{p}_n = \frac{\lambda^{-1}(e^{x_n-x_0} - 1)}{1 - \alpha^{-1}e^{x_n-x_0}} + (\lambda + \alpha)e^{x_{n+1}-x_n} - ae^{x_{n+1}-x_n} + ae^{x_n-x_{n-1}}. \end{cases} \quad (36)$$

Any two of the symplectic maps $F_i$, $F_j$, $G_k$, and $G_L$ commute, which is demonstrated in theorems 3.4, 3.5, and 3.6, with the following functions $\phi_{ij}$, $\phi_{kL}$, and $\phi_L$:

$$\phi_{ij}(x) = \frac{e^x - 1}{\lambda e^x - \mu}, \quad \phi_{kL}(x) = -\frac{e^x - 1}{(\mu + \alpha)e^x - (\lambda + \alpha)}, \quad \phi_L(x) = \frac{(\lambda - \mu - \alpha)e^x}{1 - \lambda(\mu + \alpha)e^x}.$$  

8. Bäcklund transformations for the asymmetric rational system of the relativistic Toda type

The last pluri-Lagrangian system we consider in the present paper is described in the following proposition.

Proposition 8.1. The system of 3D-corner equations $(\mathcal{E}_j)$, $(\mathcal{E}_j)$ with the leg functions

$$\psi_j(x) = \log (x + \alpha_j), \quad \phi_{ij}(x) = \frac{x + \beta_j - \beta_i}{x + \alpha_j - \alpha_i}, \quad \phi_{ij}(x) = 1 - \frac{(\lambda - \mu - \alpha)e^x}{1 - \lambda(\mu + \alpha)e^x}.$$  

is consistent, and the corresponding discrete Lagrangian 2-form $\mathcal{L}$ is closed on its solutions.

Proof. The leg functions (37) are obtained from those in (26) via the following changes of variables and transformations of parameters:
\[ x \sim hx, \quad \alpha_i \sim -1 + \hbar \alpha_i, \quad \beta_i \sim -1 + \hbar \beta_i, \]

and then sending \( h \to 0 \).

We make the following choice of parameters:
\[
\alpha_0 = \frac{1}{\alpha}, \quad \alpha_i = 0, \quad \alpha_j = 0, \quad \alpha_k = \frac{1}{\alpha + \lambda}, \quad \alpha_\ell = \frac{1}{\alpha + \mu}, \\
\beta_0 = 0, \quad \beta_i = -\frac{1}{\lambda}, \quad \beta_j = -\frac{1}{\mu}, \quad \beta_k = 0, \quad \beta_\ell = 0.
\]

With this choice of parameters, we find the following leg functions:
\[
\psi_0(x) = \log (1 + \alpha x), \quad \psi_1(x) = \log \frac{x}{\lambda}, \quad \phi_{i0}(x) = \log \frac{1 + \lambda x}{1 + \alpha x},
\]

and
\[
\psi_2(x) = \log (1 + (\lambda + \alpha)x), \quad \phi_{i2}(x) = \log \frac{x}{\lambda + (\lambda + \alpha)x}.
\]

This corresponds to the following two families of Bäcklund transformations:
\[
F_i : \begin{cases}
\tilde{x}_i = \frac{x_i - x_{i+1}}{\lambda} \cdot \left( 1 + \frac{\lambda (x_i - x_{i+1})}{1 + \alpha (x_i - x_{i+1})} \right) \\
\tilde{\xi}_i = \frac{x_i - x_{i+1}}{\lambda} \cdot \left( 1 + \frac{\lambda (x_i - x_{i+1})}{1 + \alpha (x_i - x_{i+1})} \right)
\end{cases}
\]

and
\[
G_k : \begin{cases}
\tilde{\xi}_k = \frac{x_k - x_{k+1}}{\lambda} \cdot \left( 1 + \frac{\lambda (x_k - x_{k+1})}{1 + \alpha (x_k - x_{k+1})} \right) \\
\tilde{\xi}_k = \frac{x_k - x_{k+1}}{\lambda} \cdot \left( 1 + \frac{\lambda (x_k - x_{k+1})}{1 + \alpha (x_k - x_{k+1})} \right)
\end{cases}
\]

Any two of the symplectic maps \( F_i, F_j, G_k, \) and \( G_\ell \) commute, as follows from theorems \( 3.4, \ 3.5, \) and \( 3.6 \) with the functions
\[
\phi_{ij}(x) = \log \frac{x - \lambda + \lambda \mu x}{x}, \quad \phi_{i\ell}(x) = \log \frac{x}{\lambda - \mu + (\lambda + \alpha)(\mu + \alpha)x}, \\
\phi_{ij}(x) = \log \frac{1 + \lambda x}{1 + (\mu + \alpha)x}.
\]

9. Spectrality, integrals of motion, and conservation laws

In all our examples the maps \( F_i \) and \( G_k \) as introduced in definitions \( 3.1 \) and \( 3.2 \) are instances of Bäcklund transformations, i.e., one-parameter families of commuting symplectic maps characterized by parameter-dependent Lagrange functions \( \mathcal{L}(x, \tilde{x}; \lambda) \), resp. \( \mathcal{M}(x, \tilde{x}; \lambda) \). For such families, the following remarkable property was established in [Sur03]:

**Theorem 9.1.** For the pluri-Lagrangian system built by two commuting maps \( F_i, F_j \), from one family of Bäcklund transformations (whose action is denoted by \( \sim \) and \( \tilde{\sim} \), respectively),
the discrete multi-time Lagrangian 1-form is closed on solutions if and only if the quantity
\[ P_f(x, \tilde{x}; \lambda) := \partial \mathcal{L}(x, \tilde{x}; \lambda)/\partial \lambda \] is a common integral of motion for all \( F_j \).

This is a re-formulation of the mysterious ‘spectrality property’ of Bäcklund transformations discovered by Kuznetsov and Sklyanin [KS98]. In the specific context of relativistic Toda-type systems, we are actually dealing with two mutually commuting families of Bäcklund transformations. In this situation, the following weaker statement can be made.

**Theorem 9.2.** For the pluri-Lagrangian system built by two commuting maps \( G_k \) and \( F_j \) (whose action is denoted by \( \sim \) and \( \\sim\sim \), respectively) from two mutually commuting families of Bäcklund transformations, if the discrete multi-time Lagrangian 1-form is closed on solutions, then the quantity
\[ M_{\lambda\lambda\lambda} = \partial \mathcal{P}(x, \tilde{x}; \lambda)/\partial \lambda \] is a common integral of motion for all \( F_j \).

**Proof.** The closedness of the discrete multi-time Lagrangian 1-form is expressed by the following formula:
\[ \mathcal{M}(x, \tilde{x}; \lambda) + \mathcal{L}(\tilde{x}, \tilde{x}; \mu) - \mathcal{M}(\tilde{x}, \tilde{x}; \lambda) - \mathcal{L}(x, \tilde{x}; \mu) = 0. \]

Differentiating with respect to \( \lambda \) and taking into account that the terms with \( \partial \tilde{x}/\partial \lambda \) etc vanish due to the corresponding corner equations, we arrive at
\[ \frac{\partial \mathcal{M}(\tilde{x}, \tilde{x}; \lambda)}{\partial \lambda} - \frac{\partial \mathcal{M}(x, \tilde{x}; \lambda)}{\partial \lambda} = 0, \]
which is the required property.

Thus, for the maps \( F_i \) and \( G_k \) as introduced in definitions 3.1 and 3.2, with the understanding that the corresponding Lagrangian depend on the parameters, as in the examples discussed above, i.e.,
\[ \mathcal{L}(x, \tilde{x}; \lambda) = \sum_n L_1(\tilde{x}_n - x_n; \lambda) - \sum_n L_0(x_{n+1} - x_n; \alpha) - \sum_n A_{\alpha}(x_{n+1} - \tilde{x}_n; \lambda, \alpha), \] (40)
\[ \mathcal{M}(x, \tilde{x}; \lambda) = \sum_n A_{\alpha}(\tilde{x}_n - x_n; \lambda, \alpha) + \sum_n L_0(x_n - \tilde{x}_{n-1}; \alpha) - \sum_n L_k(x_n - \tilde{x}_{n-1}; \lambda), \] (41)
we arrive at the following generating functions of common integrals of all the maps \( F_j, G_k \):
\[ P_f(x, \tilde{x}; \lambda) = \sum_n \frac{\partial L_1(\tilde{x}_n - x_n; \lambda)}{\partial \lambda} - \sum_n \frac{\partial A_{\alpha}(x_{n+1} - \tilde{x}_n; \lambda, \alpha)}{\partial \lambda}, \] (42)
\[ P_G(x, \tilde{x}; \lambda) = \sum_n \frac{\partial L_0(\tilde{x}_n - x_n; \lambda, \alpha)}{\partial \lambda} - \sum_n \frac{\partial L_k(x_n - \tilde{x}_{n-1}; \lambda)}{\partial \lambda}. \] (43)

We are going to demonstrate that the 2D pluri-Lagrangian interpretation allows us to get additional important information. Namely, we can derive the local form of the fact that \( P_f \) and \( P_G \) are integrals of motion. For the sake of simplicity, we restrict ourselves to the local form of the statement that \( P_f(x, \tilde{x}; \lambda) \) is an integral of motion of the maps \( F_j \).

**Theorem 9.3.** If the 3-point discrete 2-form \( \mathcal{L} \) is closed on solutions of the system of 3D corner equations corresponding to the maps \( F_i \) and \( F_j \), then the latter system admits the
conservation law

\[ \Delta_j P_0 = \Delta_0 P_j, \quad (44) \]

with the densities

\[ P_0 = \frac{\partial L_i(\tilde{x}_n - x_n; \lambda)}{\partial \lambda} = \frac{\partial \Lambda_{ij}(x_{n+1} - \tilde{x}_n; \lambda, \alpha)}{\partial \lambda}, \quad (45) \]

\[ P_j = \frac{\partial L_i(\tilde{x}_n - x_n; \lambda)}{\partial \lambda} - \frac{\partial \Lambda_{ij}(\tilde{x}_n - \tilde{x}_n; \lambda, \mu)}{\partial \lambda}. \quad (46) \]

Observe that \( P_{i0} \) is the summand in (42).

(Recall that \( \Delta_j = T_j - I, \Delta_0 = T_0 - I \), where \( T_j \) is the shift corresponding to the map \( F_j \) and denoted by \( \wedge \), while \( T_0 \) is the shift \( n \rightarrow n + 1 \).)

**Proof.** We start by re-writing expression (15) for \( dL \) in the form specific for our present context:

\[ dL = S^{ij0} = L_i(\tilde{x}_{n+1} - x_{n+1}; \lambda) + L_j(\tilde{x}_n - \tilde{x}_n; \mu) + L_0(\tilde{x}_{n+1} - \tilde{x}_n; \alpha) \]

\[ - L_i(\tilde{x}_n - \tilde{x}_n; \lambda) - L_j(\tilde{x}_{n+1} - x_{n+1}; \mu) - L_0(\tilde{x}_{n+1} - \tilde{x}_n; \alpha) \]

\[ - \Lambda_{ij}(\tilde{x}_{n+1} - \tilde{x}_n; \lambda, \mu) - \Lambda_{ij0}(\tilde{x}_{n+1} - \tilde{x}_n; \mu, \alpha) + \Lambda_{ij0}(\tilde{x}_{n+1} - \tilde{x}_n; \lambda, \alpha) \]

\[ + \Lambda_{ij}(\tilde{x}_n - \tilde{x}_n; \lambda, \mu) + \Lambda_{ij0}(x_{n+1} - \tilde{x}_n; \mu, \alpha) - \Lambda_{ij0}(x_{n+1} - \tilde{x}_n; \lambda, \alpha) = 0. \quad (47) \]

Differentiating equation (47) with respect to \( \lambda \) and taking into account that the terms containing \( \partial \tilde{x}_n/\partial \lambda \) etc vanish by virtue of the corresponding 3D corner equations, we arrive at

\[ \frac{\partial L_j(\tilde{x}_{n+1} - x_{n+1}; \lambda)}{\partial \lambda} - \frac{\partial L_i(\tilde{x}_n - \tilde{x}_n; \lambda)}{\partial \lambda} \]

\[ - \frac{\partial \Lambda_{ij}(\tilde{x}_{n+1} - \tilde{x}_n; \lambda, \mu)}{\partial \lambda} + \frac{\partial \Lambda_{ij}(\tilde{x}_{n+1} - \tilde{x}_n; \lambda, \alpha)}{\partial \lambda} \]

\[ + \frac{\partial \Lambda_{ij}(\tilde{x}_n - \tilde{x}_n; \lambda, \mu)}{\partial \lambda} - \frac{\partial \Lambda_{ij0}(x_{n+1} - \tilde{x}_n; \lambda, \alpha)}{\partial \lambda} = 0. \]

This is equivalent to the statement of the theorem. \( \square \)

**Example:** Modified exponential system. We start with the maps \( F_i \). An easy computation shows:

\[ \frac{\partial L_i(x; \lambda)}{\partial \lambda} = -\frac{x}{\lambda}, \quad \frac{\partial \Lambda_{i0}(x; \lambda, \alpha)}{\partial \lambda} = \frac{1}{\lambda} \log (1 + \lambda e^x). \]

Thus, theorems 9.1 and 9.2 lead to the following generating function of integrals of motion for all maps \( F_j, G_j \):

\[ \frac{\partial \Sigma(x, \tilde{x}; \lambda)}{\partial \lambda} = \log P_{ij}(x, \tilde{x}; \lambda), \]

\[ \Sigma(x, \tilde{x}; \lambda) = \log P_{ij}(x, \tilde{x}; \lambda), \]

\[ \Sigma(x, \tilde{x}; \lambda) = \log P_{ij}(x, \tilde{x}; \lambda), \]

\[ \Sigma(x, \tilde{x}; \lambda) = \log P_{ij}(x, \tilde{x}; \lambda), \]

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\[ \Sigma(x, \tilde{x}; \lambda) = \log P_{ij}(x, \tilde{x}; \lambda), \]

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\[ \Sigma(x, \tilde{x}; \lambda) = \log P_{ij}(x, \tilde{x}; \lambda), \]

\[ \Sigma(x, \tilde{x}; \lambda) = \log P_{ij}(x, \tilde{x}; \lambda), \]

\[ \Sigma(x, \tilde{x}; \lambda) = \log P_{ij}(x, \tilde{x}; \lambda), \]

\[ \Sigma(x, \tilde{x}; \lambda) = \log P_{ij}(x, \tilde{x}; \lambda), \]

\[ \Sigma(x, \tilde{x}; \lambda) = \log P_{ij}(x, \tilde{x}; \lambda), \]
where
\[
P_L(x, \vec{x}; \lambda) = \prod_{n=1}^{N} e^{x_n-x_1} \left(1 + \lambda e^{x_{n+1}-x_n} \right).
\] (48)

It is instructive to have a look at the local form of this result provided by theorem 9.3. We compute:
\[
\frac{\partial A_{ij}(x; \lambda, \mu)}{\partial \lambda} = \frac{1}{\lambda} \log(\lambda e^x - \mu),
\]
and end up with the conservation law (44) with the densities
\[
P_0 = \log e^{x_n-x_1} \left(1 + \lambda e^{x_{n+1}-x_n} \right), \quad P_{ij} = \log e^{x_n-x_1} \left(\lambda e^{x_n-x_1} - \mu \right).
\] (49)

We can give a nice expressions for the generating function of integrals \(P_L(x, \vec{x}; \lambda)\) in terms of the canonically conjugate variables \(x, p\).

**Theorem 9.4.** Set
\[
U_n(x, p; \lambda) = \left(1 + \lambda \left( e^p + e^{x_n-x_1} \right) \right) \begin{pmatrix}
\log(\lambda e^x - \mu) \\
0
\end{pmatrix},
\] (50)

and further
\[
T_n(x, p; \lambda) = U_N(x, p; \lambda)\ldots U_2(x, p; \lambda)U_1(x, p; \lambda).
\]

Then, in the periodic case \(P_L(x, \vec{x}; \lambda)\) is an eigenvalue of the matrix \(T_N(x, p; \lambda)\), while in the open-end case \(P_L(x, \vec{x}; \lambda)\) is the \((1,1)\)-entry of the matrix \(T_N(x, p; \lambda)\).

**Proof.** Set
\[
\gamma_n = e^{x_n-x_1} \left(1 + \lambda e^{x_{n+1}-x_n} \right) \begin{pmatrix}
1 + a e^{x_{n+1}-x_n} \\
1 + a e^{x_{n+1}-x_n}
\end{pmatrix},
\]
so that \(\gamma_1\cdots\gamma_2\gamma_3 = P_L(x, \vec{x}; \lambda)\) in both the periodic and the open-end cases. By a straightforward computation based on the first formula in (31) one checks that
\[
U_n(x, p; \lambda) [\gamma_{n-1}] = \gamma_n,
\]
where we write
\[
\begin{pmatrix}
a \\
\end{pmatrix} \begin{pmatrix}
z \\
\end{pmatrix} := \frac{az + b}{cz + d}.
\]
Equivalently,
\[
U_n(x, p; \lambda) \begin{pmatrix}
\gamma_{n-1} \\
1
\end{pmatrix} \sim \begin{pmatrix}
\gamma_n \\
1
\end{pmatrix}.
\]
The proportionality coefficient can be determined by comparing the second components of these vectors, which results in
\[
U_n(x, p; \lambda) \begin{pmatrix}
\gamma_{n-1} \\
1
\end{pmatrix} = \gamma_{n-1} \begin{pmatrix}
\gamma_n \\
1
\end{pmatrix}.
\] (51)

Thus, in the periodic case \(\gamma_n\cdots\gamma_2\gamma_1\) is the eigenvalue of \(T_N(x, p; \lambda)\) corresponding to the eigenvector \(\begin{pmatrix}
\gamma_n \\
1
\end{pmatrix}\). In the open end case, equation (51) holds true for \(2 \leq n \leq N\), and has to be
supplemented with

\[ U_1(x, p; \lambda) \begin{pmatrix} 1 \\ 0 \end{pmatrix} = \begin{pmatrix} y \\ 1 \end{pmatrix} \quad \text{and} \quad (1 \ 0) \begin{pmatrix} y_N \\ 1 \end{pmatrix} = y_N. \]

This yields

\[ (1 \ 0) T_N(x, p; \lambda) \begin{pmatrix} 1 \\ 0 \end{pmatrix} = y_N \cdots y_1. \]

Turning now to the maps \( G_k \), we find:

\[ \frac{\partial L_k(x; \lambda)}{\partial \lambda} = -\frac{1}{\lambda + \alpha} \log \left( 1 - \frac{\alpha}{\lambda} (e^\lambda - 1) \right), \quad (52) \]

\[ \frac{\partial M(x; \bar{x}; \lambda)}{\partial \lambda} = -\frac{1}{\lambda + \alpha} \log \left( 1 + (\lambda + \alpha) e^\lambda \right). \quad (53) \]

According to theorems 9.1 and 9.2, we compute:

\[ \frac{\partial P_k(x, x; \lambda)}{\partial \lambda} = -\frac{1}{\lambda + \alpha} \log P_k(x, x; \lambda), \]

where

\[ P_k(x, x; \lambda) = \prod_{n=1}^{N} \frac{e^{x_n-x_0}}{1 - \frac{\alpha}{\lambda} (e^{x_n-x_0} - 1)} \left( 1 + (\lambda + \alpha) e^{x_n-x_{n-1}} \right). \quad (54) \]

Again, \( P_k(x, x; \lambda) \) is a generating function of common integrals of motion for all the maps \( F_j, G \ell \). Remarkably, in terms of the canonically conjugate variables \( x, p \) this is essentially the same function as \( \lambda \sim P_k(x, x; \lambda) \).

**Theorem 9.5.** In the periodic case, \( P_k(x, x; \lambda) \) is an eigenvalue of the matrix \( T_N(x, p; \lambda + \alpha) \), while in the open-end case \( P_k(x, x; \lambda) \) is the (1,1)-entry of the matrix \( T_N(x, p; \lambda + \alpha) \).

**Proof.** Set

\[ \beta_n = \frac{e^{x_n-x_0}}{1 - \frac{\alpha}{\lambda} (e^{x_n-x_0} - 1)} \left( 1 + (\lambda + \alpha) e^{x_n-x_{n-1}} \right). \]

Then one easily checks with the help of the first formula in (32) that

\[ U_k(x, p; \lambda + \alpha) \left[ \beta_{n+1} \right] = \beta_n, \]

Then, the proof goes along the same lines as the proof of theorem 9.4. \( \square \)

**10. Conclusions**

In the present paper, we applied the general theory of 2D pluri-Lagrangian systems, developed in [BPS14a], to the analysis of Bäcklund transformations for relativistic Toda-type systems. It was possible due to a novel way to embed the one-dimensional relativistic Toda-
type systems into certain two-dimensional lattice systems. This embedding is well suited for a proof of commutativity of Bäcklund transformations, as well as for the proof of the closure relation of the corresponding action functional. A different relation of this kind between discrete Toda-type systems and 3D consistent systems of quad-equations was discussed previously, see [AS04, BS02, BS10b], and led to a systematic derivation of zero-curvature representations for (discrete and continuous) relativistic and non-relativistic Toda-type systems. That relation can be described as follows: solutions of discrete Toda-type systems can be regarded as restrictions of solutions of quad-equations to an even sublattice

\[ Z_{\text{even}}^m = \left\{ n = (n_1, \ldots, n_m) \in Z^m : [n] = n_1 + \cdots + n_m \equiv 0 \pmod{2} \right\}. \]

In this interpretation, solutions of discrete Toda type systems can be said to forget about the values of solutions of quad-equations at all points of the odd sublattice. In our present approach, solutions to both systems live on one and the same lattice \( Z^m \). Moreover, solutions of discrete Toda type systems are not derived from solutions of quad-equations, since the former are more general than the latter (each solution of quad-equations satisfies the corresponding discrete Toda type system, but not vice versa). A connection between these two approaches is presently unclear and remains to be worked out in detail.

**Acknowledgments**

This research was supported by the DFG Collaborative Research Center TRR 109 ‘Discretization in Geometry and Dynamics’.

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