Time Fisher Information associated with Fluctuations in Quantum Geometry

Salman Sajad Wani
Canadian Quantum Research Center 204-3002, 32 Ave Vernon, BC V1T 2L7 Canada

James Q. Quach
Institute for Photonics and Advanced Sensing and School of Physical Sciences,
The University of Adelaide, South Australia 5005, Australia

Mir Faizal
Canadian Quantum Research Center 204-3002, 32 Ave Vernon, BC V1T 2L7 Canada
Irving K. Barber School of Arts and Sciences, University of British Columbia - Okanagan,
Kelowna, British Columbia V1V 1V7, Canada and
Department of Physics and Astronomy, University of Lethbridge, Lethbridge, AB T1K 3M4, Canada

As time is not an observable, we use Fisher information (FI) to address the problem of time. We show that the Hamiltonian constraint operator cannot be used to analyze any quantum process for quantum geometries that are associated with time-reparametrization invariant classical geometries. This is because the Hamiltonian constraint does not contain FI about time. We demonstrate that although the Hamiltonian operator is the generator of time, the Hamiltonian constraint operator cannot observe the change that arises through the passage of time. This means that the problem of time is inescapably problematic in the associated quantum gravitational theories. Although we explicitly derive these results on the world-sheet of bosonic strings, we argue that it holds in general. We also identify an operator on the world-sheet which contains FI about time in a string theoretical processes. Motivated by this observation, we propose that a criteria for a meaningful operator of any quantum gravitational process, is that it should contain non-vanishing FI about time.

It is known that for any classical geometry with time-reparametrization invariance, the Hamiltonian represents the temporal part of the diffeomorphism constraint. The Hamiltonian constraint is a generator of gauge transformation, due to the diffeomorphism invariance being a gauge degree of freedom. This leads to an absence of physical time in quantum gravity, and this absence of time is called the problem of time. This problem occurs in almost all approaches to quantum gravity, such as the Wheeler-DeWitt approach, loop quantum gravity, discrete quantum gravity, group field theory, quantized modified gravity, or in the quantization of both brane world theories and Kaluza–Klein geometries. It has been suggested that the problem of time can be resolved by novel interpretations of quantum gravity, such as in the frozen formalism, the use of suitable operators for Cauchy surfaces, the use of matter fields as time, the use of scale factors as time, the matrix formulation of quantum gravity, unimodular gravity in the Ashtekar formulations, the rigging map of group averaging, conditional probabilities, non-perturbative quantum gravity, third quantization, and even fourth quantization.

Part of the enigma of time is that it is not an observable in quantum mechanics. This motivates us to address the problem of time through an information-theoretic lens. Specifically, we consider time to be a hidden or unobservable variable which one may only indirectly probe through quantum operators. The amount of information that one may extract about unobservables through observables is given by the FI. To analyze the FI of time, we first observe that time is associated with changes in all the aforementioned proposed solutions to the problem of time. The quantity associated with changes in Hamiltonian is work. At the microscopic scale however, work is a notoriously subtle concept as quantum fluctuations are on the same
order of magnitude as expectations values \[39–41\]. As such there is no single definition of work distributions in quantum theory, and several schemes exists \[42\]. The two-point measurement (TPM) scheme is the most established, where the work distributions is obtained with two projective measurements of the system energy at the beginning and end of a process \[43\]–\[44\]. The TPM however can not be applied to relativistic systems, such as in quantum field theory (QFT), as the projective measurements may lead to locality violation and superluminal signals \[45–47\]. For this reason, the Ramsey scheme was developed to construct the work distribution for relativistic systems \[48–50\]. Here an auxiliary qubit is coupled to the system, and information about the system is transferred to the qubit, where measurements are done. Specifically, the qubit engages the system in an evolution conditional on whether the qubit is excited or not. By preparing the qubit in a superposition of ground and excited states, this process transfers the data about the characteristic function of the TPM work distribution to the state of the qubit. This non-invasive procedure acquires statistics which otherwise would require projective measurements. It has been shown that the work distribution obtained with the Ramsey scheme is well defined for QFT, even though project measurements may not be \[48\].

The notion that energetic change or work can reveal something about time can be formalised with the FI \[33–38\]. This is because the FI quantifies the amount of information that an observable random variable, in this case the work, provides about an unknown or hidden parameter, in this case time. In this letter, we will extend the Ramsey interferometric scheme to the world-sheet of bosonic strings, to show the impossibility of the Hamiltonian constraint operator to probe change and therefore time. We will show that the associated work distribution contains no FI with respects to time. We offer the mass-squared operator as an alternative operator that may probe time with, non-zero time FI.

Hamiltonian constraint operator. For open bosonic strings with Neumann boundary conditions, we can expand the operator corresponding to the world-sheet fields \(\hat{X}^\mu(\tau, \sigma)\) in terms of modes as

\[
\hat{X}^\mu(\tau, \sigma) = \hat{x}^\mu + 2\alpha' \tau \hat{p}^\mu + i\sqrt{2\alpha'} \sum_{n \neq 0} \frac{1}{n} \hat{a}_n^\mu e^{-in\tau} \cos n\sigma ,
\]

where \(\hat{x}^\mu\) is the center of mass, \(\hat{p}^\mu\) is the momentum of the center of mass, \(\hat{a}_n^\mu\) are the string oscillatory modes, and \(\alpha'\) is the string length scale. Here \(\sigma \in [0, \pi]\) and \(\tau\) are the spatial and temporal world-sheet coordinates, respectively.

We explicitly introduce change into the system by perturbing it with a dilaton field \(\phi(X)\), which couples to the world-sheet curvature \(R\) in the standard way. Expanding the dilaton field as \(\phi(X) \approx \phi_0 + (\partial_{\mu} \phi) \hat{X}^\mu\), setting \(\phi_0 = 0\), and absorbing the time dependence in \(\chi(\tau)\), we can write \(\phi(X) \approx \lambda \chi(\tau) c_\mu \hat{X}^\mu\), where \(c_\mu\) is a constant vector. Now for an interacting Hamiltonian, \(\chi(\tau)\) will act as a switching function which turns on the interaction for finite duration, \(\lambda\) will be viewed as a coupling constant. Thus, the Hamiltonian of the perturbed string can be written as

\[
\hat{H}_X(\tau) \approx \hat{H} + \lambda \chi(\tau) \int d\sigma R c_\mu \hat{X}^\mu = \hat{H} + \hat{H}_I(\tau)
\]

with \(\hat{H}\) as the free string Hamiltonian. The interaction with the dilaton field is chosen such that perturbation only exists between \(\tau > 0\) and \(\tau < t\). The unitary evolution operator resulting from the dilaton field perturbation is

\[
\hat{U}(t) = T \exp \left( -i\lambda \int_0^t d\tau \hat{H}_I(\tau) \right),
\]

where \(T\) denotes time ordering between \(0 < \tau < t\).
The change in energy of the string is characterised by the work probability distribution with distribution variable $\mathcal{H}$

$$P(\mathcal{H}) = \sum_{il} p_{il} \delta(\mathcal{H} - \Delta H_{il})$$

with the possible values of work $\Delta H_{il} \equiv E'_l - E_i$ being the difference between the initial and final eigenvalues of $\hat{H}(\tau)$, and $p_{il} = \langle H_i | \hat{\rho} | H_i \rangle (\hat{H}'_l | U | H'_l)^2$, where $| H_i \rangle$ are the initial eigenstates and $| H'_l \rangle$ the final eigenstates of the free string Hamiltonian. Associated with the probability distribution is the characteristic function of a real-valued variable $\theta$, expressed as

$$\tilde{P}(\theta) = \int d\mathcal{H} P(\mathcal{H}) e^{i\theta \mathcal{H}} = \langle e^{i\theta \mathcal{H}} \rangle$$

In the TPM scheme, the work probability distribution is obtained with projective measurements of $E_i$ and $E'_l$. As discussed above this is problematic for systems with a Lorentz structure, as it is incompatible with relativistic causality. To overcome this problem we follow the Ramsey interferometric scheme, and prepare a pure string state $\sum_n d_n | n; p \rangle$ (with $n$ representing the string oscillatory modes, and $p$ representing the momentum of the center of mass), and then using a combination of $n, m$ different string oscillatory states to write: $\hat{\rho} = N \sum_l \sum_k d_k d_l^* | k; p \rangle \langle l; p |$. We couple an auxiliary qubit to the string. The string and qubit (which is initially in the ground state) are prepared in a product state, $\hat{\rho}_{\text{tot}} = \hat{\rho} \otimes \hat{\rho}_{\text{aux}}$. A Hadamard operator is then applied to the qubit. The state of the total system is then dictated by the unitary evolution operator

$$\hat{C}_\theta(t) = \hat{U} e^{-i\theta \hat{H}(0)} \otimes |0\rangle \langle 0| + e^{-i\theta \hat{H}(t)} \hat{U} \otimes |1\rangle \langle 1|,$$

with the qubit state given by $\hat{\rho}_{\text{aux}} = \text{Tr}_X[\hat{C}_\theta(t) \hat{\rho}_{\text{tot}} \hat{C}_\theta(t)^\dagger]$, where $\text{Tr}_X$ is a trace over the string states. Applying a final Hadamard operation, the qubit state in the weak coupling limit is (see Appendix 1)

$$\hat{\rho}_{\text{aux}} = \frac{1}{2} \left( 1 + \hat{\sigma}_z \right).$$

We make the observation that in general

$$\hat{\rho}_{\text{aux}} = \frac{1}{2} \left\{ 1 + \text{Re}[\tilde{P}(\theta)] \hat{\sigma}_z + \text{Im}[\tilde{P}(\theta)] \hat{\sigma}_y \right\}.$$  

Comparing Eq. (7) with Eq. (5), we see that the characteristic function for the bosonic string is

$$\tilde{P}(\theta) = 1.$$  

Taking the first moment of Eq. (5), the average difference between the initial and final eigenvalues of $\hat{H}_X$ (work) is

$$\langle \mathcal{H} \rangle = -i \frac{d}{d\theta} \tilde{P}(\theta) |_{\theta=0} = 0 .$$

The absence of work occurs due to the inability of the Hamiltonian constraint to have any information about time. This is directly observed through the FI of the work distribution with respects to time

$$F(t) = \int P(\mathcal{H}) \frac{\partial}{\partial t} \log P(\mathcal{H}) |^2 d\mathcal{H} = 0 .$$
where \( P(\mathcal{H}) = \delta(\mathcal{H}) \) is the inverse Fourier transform of Eq. (9). Even though we had explicitly introduced change by perturbing the system, and indeed the string states have changed as dictated by \( \hat{U} \), Eq. (10) tells us that \( \hat{H} \) can have no information of this change. In other words, we have made the seemingly paradoxical observation that even though the Hamiltonian can evolve string states, it cannot be used to observe such evolution.

Even though this result was derived explicitly for string theory, it holds not only for string theory but any quantum geometry associated with a time-reparametrization invariant classical geometry, as the Hamiltonian is also a constraint in them. In contrast, the FI of time with respect to the Hamiltonian of a quantum field theory does not vanish, due to the non-vanishing quantum work distribution of quantum fields \[48\].

From another viewpoint, the FI is related to the Mandelstam-Tamm bound \( \tau \geq 1/F \). As the Mandelstam-Tamm provides the quantum speed limit of evolution between states, vanishing FI with respect to time, suggests either a static universe or that the Hamiltonian operator cannot probe time. We will show that is the latter, as we find an alternative operator that yields a finite Mandelstam-Tamm bound.

**Mass-squared operator.** If one considers the mass-energy dispersion relation in quantum field theory, the mass of a field can vary as it interacts with its environment. In contrast, the energy of a string is a constant, as the Hamiltonian in string theory is a constraint, and it is its mass that can be treated as a dynamical variable. As it is natural to classify states using string oscillatory modes, we can write the corresponding characteristic function for \( M \)

\[
\hat{M}^2 = \frac{1}{\alpha'} \left( \sum_{i=1}^{24} \sum_{n>0} \hat{\alpha}_n^i \hat{\alpha}_n^i - 1 \right) .
\]

Now following what was done for the Hamiltonian operator, we define a distribution variable for \( \hat{M}^2 \) as \( \mathcal{M}^2 \), such that \( P(\mathcal{M}^2) = \sum_{n,l} p_{nl} \delta(\mathcal{M}^2 - \Delta \mathcal{M}^2_{nl}) \), where \( p_{nl} = \langle M^2 | \hat{\beta} | M^2 \rangle \langle M^2 | \hat{U} | M^2 \rangle^2 \) is the associated joint probability distribution, and \( \Delta \mathcal{M}^2_{nl} \) is the difference between the final and initial eigenvalues of \( \hat{M}^2 \). Thus, we can write the corresponding characteristic function for \( \mathcal{M}^2 \) as

\[
\hat{P}(\theta) = \int d\mathcal{M}^2 P(\mathcal{M}^2) e^{i\theta \mathcal{M}^2} = \langle e^{i\theta \mathcal{M}^2} \rangle \]

Applying the Ramsey scheme, the unitary evolution operator is

\[
\hat{C}_\theta(t) = \hat{U} e^{-i\theta \hat{M}^2(0)} \otimes |0\rangle \langle 0| + e^{-i\theta \hat{M}^2(t)} \hat{U} \otimes |1\rangle \langle 1| .
\]

Here the systems evolution is governed by \( \hat{H}_f \), but importantly, the changes in states are probed with \( \hat{M}^2 \). Following the previous analysis (see Appendix 2), we observe that Eq. (14) leads to the characteristic function

\[
\hat{P}(\theta) = 1 + \lambda^2 N \alpha' \sum_{n=1}^{\infty} \sum_{k=0}^{\infty} \frac{d_k^2}{n^2} (n+2k) \hat{\chi}(n) \hat{\chi}(-n) \Phi(n)^2 \left[ \exp \left( -i \frac{\theta n}{\alpha'} \right) - 1 \right] ,
\]

where \( \hat{\chi}(n) \) is a Fourier transformation of the switching function \( \chi(\tau) \) and \( |\Phi(n)|^2 = \Phi(n)^\mu \Phi(n)_\mu \), with \( \Phi(n)_\mu \) as the cosine transformation of the \( Rc_\mu \) (for constant \( c_\mu \)). Taking the first moment of \( \hat{P}(\theta) = \langle e^{i\theta \mathcal{M}^2} \rangle \), the
average difference between the initial and final eigenvalues of $\hat{M}^2$ is

$$\langle M^2 \rangle = \mathcal{N}^2 N^2 \sum_{n=1}^{\infty} \sum_{k=0}^{\infty} \frac{d^2_{\chi}(n+2k)|\tilde{\chi}(n)|^2|\Phi(n)|^2}{n}$$ (16)

As this does not vanish in general, one can analyze the changes in string states with $\hat{M}^2$.

Taking the inverse Fourier transform of Eq. (15),

$$P(M^2) = \delta(M^2) + \lambda^2 \mathcal{N}^2 \alpha' \sum_{n=1}^{\infty} \sum_{k=0}^{\infty} (2 + 2 \cos nt)$$

$$|\Phi(n)|^2 \frac{d^2(k+n)}{n^3} \left[ \delta \left( \frac{n}{\alpha'} - M^2 \right) - \delta(M^2) \right],$$ (17)

we can write down the FI of work distribution of the $\hat{M}^2$ with respects to time

$$F(t) = \int P(M^2) \frac{\partial}{\partial t} \log P(M^2) |2dM^2|. \quad (18)$$

This integral can be expressed as

$$F(t) = \frac{\dot{K}^2}{1 - K} + (\lambda^2 \mathcal{N}^2 \alpha')^2 \frac{Y}{K}$$ (19)

where $\dot{K}$ is the time derivative of $K$, with $K = \lambda^2 \mathcal{N}^2 \alpha' \sum_{n=1}^{\infty} \sum_{k=0}^{\infty} d^2_{\Phi}(n+2k)n^{-3}[2 + 2 \cos nt]|\Phi(n)|^2$, and $Y = \sum_{n=1}^{\infty} \sum_{k=0}^{\infty} d_{\Phi}^2(k+n+2k)|\Phi(n)|^2$. As in general $|\Phi(n)|^2 \neq 0$ for a dilaton field, the FI of $\hat{M}^2$ with respects to time does not vanish, in stark constrast to $\hat{H}$. In fact, it is exactly for this reason that $\hat{M}^2$ has been used to obtain non-trivial results about string states rather than $\hat{H}$ [54, 55], even though it has never been explicitly explained from this information-theoretic viewpoint. Motivated by this observation, we propose that an essential property for an operator to analyze change in a quantum gravitational process, is that its FI with respects to time should not vanish from the distribution of the difference between eigenvalues of that operator, $F(t) \neq 0$.

**Outlook.** We demonstrated that the inability of the Hamiltonian constraint operator to probe change is related to the absence of FI with respect to time. We argue this holds for any quantum geometry associated with time-reparametrization invariant classical geometry. From this one may conclude that either the Hamiltonian constraint operator is an inappropriate operator for the probing of time, or that time-reparametrization invariance should be broken for time to exists. We showed that the mass-squared operator is a natural alternative that may probe time. That time-reparameterization invariance should be broken certainly is also an avenue worth pursuing, and our work provides the information-theoretic framework to consider the problem of time in universes with spontaneously broken time-reparametrization invariance.

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APPENDIX 1. HAMILTONIAN CONSTRAINT OPERATOR

The open bosonic strings with Neumann boundary conditions, can be expanded in terms of modes as

\[ \hat{X}^\mu(\tau, \sigma) = \hat{x}^\mu + 2\alpha' \tau \hat{p}^\mu + i\sqrt{2}\alpha' \sum_{n \neq 0} \frac{1}{n} \hat{\alpha}^\mu_n e^{-in\tau} \cos n\sigma, \]

where \( \hat{x}^\mu \) is the center of mass, \( \hat{p}^\mu \) is the momentum of the center of mass, \( \hat{\alpha}^\mu_n \) are the string oscillatory modes, and \( \alpha' \) is the string length scale. We start by preparing an initial string-qubit system as

\[ \sum_k d_k |k; p\rangle \]

(with \( k \) representing the string oscillatory modes, with \( p \) as the momentum of the center of mass). Using a combination of \( k, l \) different string oscillatory modes, we write

\[ \hat{\rho} = \sum_{k} \sum_{l} d_k d_l^* |k; p\rangle\langle l; p|, \]

where \( N = 1/\sum_k |d_k|^2 \). We couple this string state to an auxiliary qubit, and define \( \hat{\rho}_{\text{tot}} = \hat{\rho} \otimes \hat{\rho}_{\text{aux}} = \hat{\rho} \otimes |0\rangle\langle 0| \), where \( |0\rangle\langle 0| \) is the ground state of the string auxiliary qubit. Upon applying the first Hadamard gate on this string auxiliary qubit, we obtain \( \hat{\rho}_{\text{tot}} = \hat{\rho} \otimes |+\rangle\langle +| \). Now we perturb it by a dilaton field. The Hamiltonian for the system perturbed by the dilaton field can be written as

\[ \hat{H}_X(\tau) = \hat{H} + \lambda \chi(\tau) \int d\sigma R\hat{X}^\mu = \hat{H} + \hat{H}_I(\tau) \]

Here we have expanded the dilaton field as

\[ \phi(X) \approx \phi_0 + (\partial_\mu \phi) \hat{X}^\mu = \lambda \chi(\tau) c_\mu \hat{X}^\mu, \]

where \( \phi_0 = 0 \), \( \hat{H} \) is the free Hamiltonian of the system, \( R \) is the scalar curvature on the world-sheet, and \( c_\mu \) is a constant vector. We have taken the standard coupling of the strings states to a dilaton field. Here the switching function is \( \chi(\tau) \) and \( R\hat{X}^\mu \) is the smearing function. This dilaton field \( \phi(X) \) interacts with the strings, and this perturbation can be expressed using a unitary \( \hat{U}(t) \), such that

\[ \hat{U}(t) = \mathcal{T} \exp \left( -i\lambda \int d\tau \chi(\tau) \int d\sigma R\hat{X}^\mu(\tau, \sigma) \right) \]

where \( \mathcal{T} \) denotes time ordering between \( 0 < \tau < t \). Using the Dyson expansion, we obtain

\[ \hat{U}(t) = 1 + \hat{U}^{(1)} + \hat{U}^{(2)} + \hat{O}(\lambda^3) \]
The state of the total system is then evolved by the unitary operator, and expressed using $\hat{C}_\theta(t)$ (with $\theta$ as the real-valued variable used in the definition of the characteristic function for this system)

$$\hat{C}_\theta(t) = \hat{U}(t) e^{-i\theta \hat{H}(0)} \otimes |0\rangle \langle 0| + e^{-i\theta \hat{H}(t)} \hat{U}(t) \otimes |1\rangle \langle 1|$$

The reduced states is a qubit state, and can be written as

$$\hat{\rho}_{\text{aux}}(t) = \text{Tr}_X[\hat{C}_\theta(t) \hat{\rho}_{\text{tot}} \hat{C}_\theta^\dagger(t)]$$

where $\text{Tr}_X$ is a trace over the $\hat{X}^\mu$. Now using the Dyson expansion of the unitary operator $\hat{U}(t)$, we can write

$$\hat{\rho}_{\text{aux}}(t) = \hat{\rho}_{\text{aux}}^0(t) + \hat{\rho}_{\text{aux}}^1(t) + \hat{\rho}_{\text{aux}}^2(t) + O(\lambda^3)$$

Now we will explicitly evaluate these terms. The first order term in the Dyson expansion of the qubit state is

$$\hat{\rho}_{\text{aux}}^0(t) = |+\rangle \langle +|$$

We can also write $\hat{\rho}_{\text{aux}}^1(t)$ as

$$\hat{\rho}_{\text{aux}}^1(t) = \text{Tr}_X \left[ (\hat{U}^{(1)}(t) e^{-i\theta \hat{H}(t)} \otimes |0\rangle \langle 0| + e^{-i\theta \hat{H}(t)} \hat{U}^{(1)}(t) \otimes |1\rangle \langle 1|) \hat{\rho}_{\text{tot}} \right.$$

$$\left. \times (e^{i\theta \hat{H}(t)} \otimes |0\rangle \langle 0| + e^{i\theta \hat{H}(t)} \otimes |1\rangle \langle 1|) \right]$$

$$+ \text{Tr}_X \left[ (e^{-i\theta \hat{H}(t)} \otimes |0\rangle \langle 0| + e^{-i\theta \hat{H}(t)} \otimes |1\rangle \langle 1|) \hat{\rho}_{\text{tot}} \right.$$

$$\left. \times (e^{i\theta \hat{H}(t)} \hat{U}^{(1)}(t) \otimes |0\rangle \langle 0| + \hat{U}^{(1)}(t) e^{i\theta \hat{H}(t)} \otimes |1\rangle \langle 1|) \right]$$

Using the property $\hat{U}^{(1)} = -\hat{U}^{(1)}$, the coefficient of $|0\rangle \langle 0|$ is given by

$$\text{Tr}_X \left[ \hat{U}^{(1)} e^{-i\theta \hat{H}(t)} \hat{\rho}_{\text{tot}} e^{i\theta \hat{H}(t)} + e^{-i\theta \hat{H}(t)} \hat{\rho}_{\text{tot}} e^{i\theta \hat{H}(t)} \hat{U}^{(1)} \right] = 0$$

It may be noted that by repeating such calculations for all the other coefficients of Eq. (30), we can observe that all those coefficients vanish, and so we can write

$$\hat{\rho}_{\text{aux}}^1(t) = 0.$$
Now we can write the second order term as
\[
\hat{\rho}_{aux}^2(t) = \text{Tr}_{X} \left[ (\hat{U}^{(1)}(t)e^{-i\theta \hat{H}(t)} \otimes |0\rangle\langle 0| + e^{-i\theta \hat{H}(t)} \hat{U}^{(1)}(t) \otimes |1\rangle\langle 1|) \right.
\times \hat{\rho}_{tot} \left( e^{i\theta \hat{H}(t)} \hat{U}(t)^{(1)} \otimes |0\rangle\langle 0| + \hat{U}(t)^{(1)} \otimes e^{i\theta \hat{H}(t)} \otimes |1\rangle\langle 1| \right)
+ \text{Tr}_{X} \left[ (\hat{U}^{(2)}(t)e^{-i\theta \hat{H}(t)} \otimes |0\rangle\langle 0| + e^{-i\theta \hat{H}(t)} \hat{U}^{(2)}(t) \otimes |1\rangle\langle 1|) \right.
\times \hat{\rho}_{tot} \left( e^{i\theta \hat{H}(t)} \hat{U}(t)^{(2)} \otimes |0\rangle\langle 0| + \hat{U}(t)^{(2)} \otimes e^{i\theta \hat{H}(t)} \otimes |1\rangle\langle 1| \right) \right]
\]

This equation is of the form
\[
\hat{\rho}_{aux}^2(t) = a_0 |0\rangle\langle 0| + a_1 |1\rangle\langle 1| + a_2 |0\rangle\langle 0| + a_3 |1\rangle\langle 1| \tag{34}
\]

To obtain \((a_0, a_1, a_2, a_3)\), we first observe \(\hat{U}^{(1)}(t) = -\hat{U}^{(1)}(t)^\dagger\), and \(\hat{U}^{(2)}(t) = \hat{U}^{(2)}(t)^\dagger\). Then performing the integration, we can write
\[
\text{Tr}_{X} [\hat{U}^{(1)} \hat{\rho} \hat{U}^{(1)}]^\dagger] = -2 \text{Tr}_{X} [\hat{U}^{(2)} \hat{\rho}] \tag{35}
\]

Now using these expressions, we observe that \(a_0 = a_1 = 0\). To evaluate \(a_2\), we observe that as \(\hat{H}(t)\) is a constraint, it commutes with every operator, including \(\hat{U}\), and hence we can write
\[
a_2 = \text{Tr}_{X} \left[ \hat{U}^{(1)}(t)e^{-i\theta \hat{H}(t)} \hat{\rho} \hat{U}^{(1)}(t)e^{i\theta \hat{H}(t)} + \hat{U}^{(2)}(t)e^{-i\theta \hat{H}(t)} \hat{\rho} e^{i\theta \hat{H}(t)} \right.
+ e^{-i\theta \hat{H}(t)} \hat{\rho} e^{i\theta \hat{H}(t)} \hat{U}^{(2)}(t) \right] = 0 \tag{36}
\]

Similarly, we can demonstrate that \(a_3 = 0\). Using all these terms, we observe
\[
\hat{\rho}_{aux}^2(t) = 0. \tag{37}
\]

So the qubit state can be written as \(\hat{\rho}_{aux}(t) = |+\rangle\langle +|\) After perturbing the system by the dilaton field we now apply the second Hadamard gate, such that auxiliary qubit is given by
\[
\hat{\rho}_{aux}(t) = |0\rangle\langle 0| = \frac{1}{2}(1 + \sigma_z). \tag{38}
\]

It is known in general \[49, 50\]
\[
\hat{\rho}_{aux} = \frac{1}{2} \left[ I + \text{Re}[\hat{P}(\theta)] \sigma_z + \text{Im}[\hat{P}(\theta)] \sigma_y \right]. \tag{39}
\]

Thus, we observe that the characteristic function is
\[
\hat{P}(\theta) = 1 \tag{40}
\]

Using this characteristic function, we can calculate the average difference between the initial and final
eigenvalues of the Hamiltonian, and observe that it also vanishes

\[ \langle H \rangle = -i \frac{d}{d\theta} \tilde{P}(\theta)|_{\theta=0} = 0, \]  

(41)

where \( \mathcal{H} \) is the distribution variable, which is identified with quantum work \( \langle \mathcal{H} \rangle \). So, the quantum work of the world-sheet Hamiltonian vanishes for such string theoretical processes.

### APPENDIX 2. MASS-SQUARED OPERATOR

Now we will probe the system by an alternative operator, i.e., \( \hat{M}^2 \). Here we will still perturb the system with a dilaton field, and evolve it by \( \hat{H}_1 \). Thus, the system still evolves by the unitary operator, which was defined in Eq. (23), and then a Dyson expansion will be used to expand it. However, we will probe the effect of this perturbation using \( \hat{M}^2 \).

Thus, after perturbing the system by the dilaton field, we can write the unitary evolution of the system (string and auxiliary qubit) for \( \hat{M}^2 \) as

\[ \hat{C}_{\theta}(t) = \hat{U}(t)e^{-i\hat{M}^2(0)} \otimes |0\rangle\langle 0| + e^{-i\hat{M}^2(t)}\hat{U}(t) \otimes |1\rangle\langle 1| \]  

(42)

Now we can write the auxiliary qubit at time \( \tau = t \) as

\[ \hat{\rho}_{\text{aux}}(t) = \text{Tr}_X[\hat{C}_{\theta}(t) \hat{\rho}_{\text{tot}} \hat{C}_{\theta}^\dagger(t)] \]  

(43)

Using the Dyson expansion, we can again express this reduced state of the qubit as

\[ \hat{\rho}_{\text{aux}}(t) = \hat{\rho}^0_{\text{aux}}(t) + \hat{\rho}^1_{\text{aux}}(t) + \hat{\rho}^2_{\text{aux}}(t) + \mathcal{O}(\lambda^3) \]  

(44)

We can now evaluate these terms for the qubit. The zeroth-order term is

\[ \hat{\rho}^0_{\text{aux}}(t) = |+\rangle\langle +| \]  

(45)

The first-order term can be expressed as

\[ \hat{\rho}^1_{\text{aux}}(t) = \text{Tr}_X \left[ (\hat{U}^{(1)}(t)e^{-i\hat{M}^2(0)} \otimes |0\rangle\langle 0| + e^{-i\hat{M}^2(t)}\hat{U}^{(1)}(t) \otimes |1\rangle\langle 1|) \right. \]

\[ \times \hat{\rho}_{\text{tot}}(e^{i\theta}\hat{M}^2(0) \otimes |0\rangle\langle 0| + e^{i\theta}\hat{M}^2(t) \otimes |1\rangle\langle 1|) \]

\[ + \text{Tr}_X \left[ (e^{-i\theta}\hat{M}^2(0) \otimes |0\rangle\langle 0| + e^{-i\theta}\hat{M}^2(t) \otimes |1\rangle\langle 1|)\hat{\rho}_{\text{tot}} \right. \]

\[ \times (e^{i\theta}\hat{M}^2(0)\hat{U}^{(1)}(t) \otimes |0\rangle\langle 0| + \hat{U}^{(1)}(t)e^{i\theta}\hat{M}^2(t) \otimes |1\rangle\langle 1|) \]  

(46)

The coefficient of \( |0\rangle\langle 0| \) term is

\[ \text{Tr}_X[\hat{U}^{(1)}e^{-i\hat{M}^2} \hat{\rho}_{\text{tot}} e^{i\theta}\hat{M}^2 + e^{-i\theta}\hat{M}^2 \hat{\rho}_{\text{tot}} e^{i\theta}\hat{M}^2 \hat{U}^{(1)}]\]  

(47)

Here we have again used the cyclical property of trace and \( \hat{U}^{(1)} = -\hat{U}^{(1)}\dagger \). Using the same argument for all the coefficients in \( \hat{\rho}^1_{\text{aux}}(t) \), we observe that they all vanish, and thus

\[ \hat{\rho}^1_{\text{aux}}(t) = 0 \]  

(48)
Now we can write $\hat{\rho}_{\text{aux}}^2(t)$ as
\[
\hat{\rho}_{\text{aux}}^2(t) = \text{Tr}_X \left[ \hat{\rho}_{\text{tot}} (e^{i\hat{M}^2(t)} \hat{U}^{(1)}(t) \otimes |0\rangle \langle 0| + e^{i\hat{M}^2(t)} \hat{U}^{(1)}(t) \otimes |1\rangle \langle 1|) \right] 
+ \text{Tr}_X \left[ \hat{\rho}_{\text{tot}} (e^{i\hat{M}^2(t)} \hat{U}^{(2)}(t) \otimes |0\rangle \langle 0| + e^{i\hat{M}^2(t)} \hat{U}^{(2)}(t) \otimes |1\rangle \langle 1|) \right] 
+ \text{Tr}_X \left[ \hat{\rho}_{\text{tot}} (e^{i\hat{M}^2(t)} \otimes |0\rangle \langle 0| + e^{i\hat{M}^2(t)} \otimes |1\rangle \langle 1|) \right] 
\]
\[
(49)
\]
This expression is again of the form
\[
\hat{\rho}_{\text{aux}}^2(t) = a_0 \otimes |0\rangle \langle 0| + a_1 \otimes |1\rangle \langle 1| + a_2 \otimes |0\rangle \langle 0| + a_3 \otimes |0\rangle \langle 1|
\]
(50)
Here we can write $a_2 = a_{21} + a_{22}$, where $a_{21}$ is the contribution from $U^{(1)}$ and $U^{(1)\dagger}$ and $a_{22}$ is the contribution from $U^{(2)}$ and $U^{(2)\dagger}$. Using Eq. (51), we observe that coefficients of diagonal terms $|0\rangle \langle 0|$ and $|1\rangle \langle 1|$ vanish, i.e., $a_0 = a_1 = 0$. Using the property of $\hat{M}^2$ operator, i.e., the eigenvalue of $\hat{M}^2$ operator on the $n$ excited state is $(n - 1)/\alpha'$, the coefficient $a_{21}$ can be expressed as
\[
a_{21} = \frac{1}{2} \text{Tr}_X \left[ \hat{U}^{(1)}(t) e^{-i\hat{M}^2} \hat{U}^{(1)\dagger}(t) e^{i\hat{M}^2} \right] 
= N \text{Tr}_X \left[ \chi(t) \sum_{n \neq 0, m \neq 0} \frac{1}{nm} \int d\tau \chi(\tau) \int d\tau' \chi(\tau') \int R_{\mu} \cos(n\sigma) d\sigma 
\times \int R^{c}_{\nu} d\sigma' e^{-i\mu \sigma} \sum_{k} \sum_{l} d_{k} d_{l} e^{-i(\theta_{k-\mu}^{\nu} - \frac{k}{\alpha'})} (\hat{\alpha}_{-n}^\mu |k; p \rangle \langle l; p| \hat{\alpha}_{\nu m}) 
\right] 
\]
(51)
Now observing that $\langle k; p| \hat{\alpha}_{\nu m} \hat{\alpha}_{-n}^\mu |l; p\rangle = \delta_{\nu m} \delta_{l}^{k} \sqrt{n + l} \sqrt{n + k}$, we can write
\[
a_{21} = N \chi(n) (n + 2k) |\hat{\chi}(n)|^2 \Phi(n)^{\mu} \Phi_{\mu}(n) e^{-i\frac{\alpha}{\alpha'}}
\]
(52)
Here $\hat{\chi}(n)$ is a Fourier transformation of $\chi(\tau)$ and $|\Phi(n)|$ is the cosine transformation of $R_{\mu}$. The coefficient $a_{22}$ can be expressed as
\[
a_{22} = -N \chi(n) (n + 2k) |\hat{\chi}(n)|^2 \Phi(n)^{\mu} \Phi_{\mu}(n)
\]
(53)
We can also write the coefficient $a_3 = a_{31} + a_{32}$, with $a_{31}$ being the coefficient of $|0\rangle \langle 1|$ from $U^{(1)}$ and $U^{(1)\dagger}$, and $a_{31}$ is the coefficient of $|1\rangle \langle 0|$ from $U^{(2)}$ and $U^{(2)\dagger}$. Now we observe that $a_{31}$ is the complex conjugate of
\( a_{21} = a_{21} \), and \( a_{32} = a_{22} \). Using these expression, the qubit can be write as

\[
\hat{\rho}_{\text{aux}}(t) = \mathcal{N} \lambda^2 \alpha' \left[ \sum_{n=1}^{\infty} \frac{d_2^2}{n^2} (n + 2k) |\tilde{\chi}(n)|^2 \Phi(n)^\mu \Phi_{\mu}(n) (e^{-i \frac{\theta_n}{\alpha'}} - 1) \right] \frac{1}{2} |1\rangle\langle 0|
\]

\[
+ i \mathcal{N} \lambda^2 \alpha' \left[ \sum_{n=1}^{\infty} \frac{d_2^2}{n^2} (n + 2k) |\tilde{\chi}(n)|^2 \Phi(n)^\mu \Phi_{\mu}(n) (e^{i \frac{\theta_n}{\alpha'}} - 1) \right] \frac{1}{2} |0\rangle\langle 1|
\]

\[
+ |+\rangle\langle +| \quad (54)
\]

Now we apply the second Hadamard, to obtain the final state of the qubit. Using this state of the qubit, and Eq. (39), we can explicitly write the characteristic function as

\[
\tilde{P}(\theta) = \mathcal{N} \lambda^2 \alpha' \sum_{n=1}^{\infty} \sum_{k=0}^{\infty} \frac{d_2^2}{n^2} (n + 2k) |\tilde{\chi}(n)|^2 \Phi(n)^\mu \Phi_{\mu}(n) \left( \cos \left( \frac{\theta_n}{\alpha'} \right) - 1 \right)
\]

\[
+ i \mathcal{N} \lambda^2 \alpha' \sum_{n=1}^{\infty} \sum_{k=0}^{\infty} \frac{d_2^2}{n^2} (n + 2k) |\tilde{\chi}(n)|^2 \Phi(n)^\mu \Phi_{\mu}(n) \left( \sin \left( \frac{\theta_n}{\alpha'} \right) \right)
\]

\[
+ 1 \quad (55)
\]

We can write the \( \hat{M}^2 \) analog for quantum work as

\[
\langle M^2 \rangle = -i \frac{d}{d\theta} \tilde{P}(\theta)|_{\theta=0}, \quad (56)
\]

where \( M^2 \) is the distribution variable. This measures the average of the difference between the initial and final eigenvalues of \( M^2 \), and can be explicitly written as

\[
\langle M^2 \rangle = \mathcal{N} \lambda^2 \sum_{n=1}^{\infty} \sum_{k=0}^{\infty} \frac{d_2^2}{n^2} (n + 2k) |\tilde{\chi}(n)|^2 \Phi(n)^\mu \Phi_{\mu}(n). \quad (57)
\]

Now we observe that this does not generally vanish, as \( |\tilde{\chi}(n)|^2 \neq 0 \) and \( \Phi(n)^\mu \Phi_{\mu}(n) \neq 0 \).