Laplacian Spectra of Comaximal Graph of $\mathbb{Z}_n$

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Abstract

This article focuses on finding the eigenvalues of the Laplacian matrix of the comaximal graph $\Gamma(Z_n)$ of the ring $\mathbb{Z}_n$ for $n > 2$. We determine the eigenvalues of $\Gamma(Z_n)$ for various $n$ and also provide a procedure to find the eigenvalues of $\Gamma(Z_n)$ for any $n > 2$. We show that $\Gamma(Z_n)$ is Laplacian Integral for $n = p^\alpha q^\beta$ where $p, q$ are primes and $\alpha, \beta$ are non-negative integers. The algebraic and vertex connectivity of $\Gamma(Z_n)$ have been shown to be equal for all $n > 2$. An upper bound on the second largest eigenvalue of $\Gamma(Z_n)$ has been obtained and a necessary and sufficient condition for its equality has also been determined. Finally we discuss the multiplicity of the spectral radius and the multiplicity of the algebraic connectivity of $\Gamma(Z_n)$. Some problems have been discussed at the end of this article for further research.

Keywords: comaximal graph; laplacian eigenvalues; vertex connectivity; algebraic connectivity; laplacian spectral radius; finite ring

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1 Introduction

Let $G$ be a finite simple undirected graph of order $n$ with vertex set $V(G) = \{v_1, v_2, \ldots, v_n\}$. Note that $v_i \sim v_j$ denotes that $v_i$ adjacent to $v_j$ for $1 \leq i \neq j \leq n$. The adjacency matrix of $G$ denoted by $A(G) = (a_{ij})$ is an $n \times n$ matrix defined as $a_{ij} = 1$ when $v_i \sim v_j$ and 0 otherwise. The Laplacian matrix $L(G)$ of $G$ is defined as $L(G) = D(G) - A(G)$ where $A(G)$ is the adjacency matrix of $G$ and $D(G)$ is the diagonal matrix of vertex degrees. Since the matrix $L(G)$ is a real, symmetric and a positive semi-definite matrix, all its eigenvalues are real and non-negative. Also 0 is an eigenvalue of $L(G)$ with eigenvector $[1, 1, 1, \ldots, 1]^T$ whose multiplicity equals the number of connected components in the graph $G$. Let the eigenvalues of $L(G)$ be denoted by $\lambda_1 \geq \lambda_2 \geq \cdots \geq \lambda_{n-1} \geq \lambda_n = 0$. The largest eigenvalue $\lambda_1$ is known as the spectral radius of $G$ and the second smallest eigenvalue $\lambda_{n-1}$ is known as the algebraic connectivity of $G$. Also $\lambda_{n-1} > 0$ if and only if $G$ is connected. The term algebraic connectivity was given by Fiedler in [4]. A separating set in a connected graph $G$ is a set $S \subset V(G)$ such that $V(G) \setminus S$ has more than 1 connected component. The vertex connectivity of $G$
denoted by $\kappa(G)$ is defined as $\kappa(G) = \min\{|S| : S$ is a separating set of $G\}$. The papers [6] and [7] list several interesting properties of $\lambda_{n-1}$ and $\kappa$. Readers may refer to [5] for a survey on $L(G)$ of a graph $G$. A graph $G$ whose adjacency matrix has all its eigenvalues as integers is known as integral graph. Harary and Schwenk first posed the question "Which graphs have integral spectra" in [9]. Given a positive integer $n$, it is difficult to locate the graphs with integral spectra among the graphs having $n$ number of vertices. For a survey and more detailed information about integral graphs, the readers are referred to [14]. A graph $G$ is called Laplacian integral if all the eigenvalues of $L(G)$ are integers.

Let $R$ be a commutative ring with unity. The comaximal graph of a ring $R$ denoted by $\Gamma(R)$ was introduced by Sharma and Bhatwadekar in [1]. The vertices of the graph $\Gamma(R)$ are the elements of the ring $R$ and any two distinct vertices $x, y$ of $\Gamma(R)$ are adjacent if and only if $Rx + Ry = R$. They proved that $R$ is a finite ring if and only if the chromatic number of $\Gamma(R)$ denoted by $\chi(\Gamma(R))$ is finite. It was further shown that $\chi(\Gamma(R))$ satisfies $\chi(\Gamma(R)) = t + l$ where $t$ denotes the number of maximal ideals of $R$ and $l$ denotes the number of units of $R$. A lot of research has been done on the comaximal graph of a ring $R$ over the last few decades. For some literature on $\Gamma(R)$, readers may refer to the works [2], [3] and [4].

Let $\mathbb{Z}_n$ denote the ring of integers modulo $n$ where $n > 2$. Let the elements of $\mathbb{Z}_n$ be denoted by $0, 1, 2, \ldots, n - 2, n - 1$. The number of integers prime to $n$ and less than $n$ is denoted by Euler Totient function $\phi(n)$. We say that $d$ is a proper divisor of $n$ if $d$ divides $n$ and $d$ does not equal 1 or $n$. A subring $I$ of a ring $R$ is said to be an ideal of $R$ if for all $r \in R$ and $i \in I, ri, ir \in I$. The ideal generated by $r$ is defined to be the smallest ideal of $R$ containing $r$ and is denoted by $\langle r \rangle$. A ring $R$ is said to be a Principal Ideal Ring (PIR) if every ideal $I$ of $R$ is of the form $I = \langle r \rangle$ for some $r \in R$. A ring $R$ is said to be an integral domain if $xy = 0$ in $R$ implies either $x = 0$ or $y = 0$. A ring $R$ is said to be a Principal Ideal Domain (PID) if it is a PIR and an integral domain. For any given set $A$, $|A|$ denotes the number of elements in the set $A$.

By a null graph on $n$ vertices denoted by $\Theta_n$, we shall mean a graph having $n$ vertices and zero edges. An empty graph means a graph with no vertices and edges. The join of two graphs $G_1 = (V_1, E_1)$ and $G_2 = (V_2, E_2)$ denoted by $G_1 \vee G_2$ is a graph obtained from $G_1$ and $G_2$ by joining each vertex of $G_1$ to all vertices of $G_2$. The union of two graphs $G_1 = (V_1, E_1)$ and $G_2 = (V_2, E_2)$ denoted by $G_1 \cup G_2$ is the graph with vertex set $V = V_1 \cup V_2$ and edge set $E = E_1 \cup E_2$. For various standard terms related to ring theory and graph theory used in the article, the readers are referred to the texts [11] and [12] respectively.

Throughout this article by eigenvalues and characteristic polynomial of $\Gamma(\mathbb{Z}_n)$, we shall mean the eigenvalues and characteristic polynomial of $L(\Gamma(\mathbb{Z}_n))$. The set of all eigenvalues of $G$ is denoted by $\sigma(G)$. We denote the characteristic polynomial of $G$ by $\mu(G, x)$.

1.1 Arrangement of the Article

In Section 2 we provide the preliminary theorems that have been used throughout the article. In Section 3 we discuss the structure of $\Gamma(\mathbb{Z}_n)$ and express it in terms of its subgraphs.
Let $G$ be a graph. The subgraph of $G$ induced by a set of vertices is the graph $G[V]$ where $V$ is a subset of the vertex set of $G$. Theorem 2.1 states that in this section we will provide some preliminary theorems that will be required in our subsequent sections.

In this section we will provide some preliminary theorems that will be required in our subsequent sections.

Theorem 2.1 (Corollary 3.7 of \[8\]). Let $G_1 \cup G_2$ denote the join of two graphs $G_1$ and $G_2$. Then

$$\mu(G_1 \cup G_2, x) = \frac{x(x-n_1-n_2)}{(x-n_1)(x-n_2)} \mu(G_1, x-n_2) \mu(G_2, x-n_1)$$

where $n_1$ and $n_2$ are orders of $G_1$ and $G_2$ respectively.

Theorem 2.2 (Theorem 3.1 of \[8\]). Let $G$ be the disjoint union of the graphs $G_1, G_2, \ldots, G_k$. Then

$$\mu(G, x) = \prod_{i=1}^{k} \mu(G_i, x).$$

Theorem 2.3 (Theorem 2.2 of \[8\]). If $G$ is a simple graph on $n$ vertices then the largest eigenvalue $\lambda_1$ of $G$ satisfies $\lambda_1 \leq n$, where the equality holds if and only if its complement $G^c$ is disconnected.

Definition 2.4 (Definition 3.9.1 of \[13\]). Given a graph $G$ with vertex set $V(G)$, a partition $V(G) = V_1 \cup V_2 \cup \cdots \cup V_k$ is said to be an equitable partition of $G$ if every vertex in $V_i$ has the same number of neighbors $b_{ij}$ in $V_j$ where $1 \leq i, j \leq k$.

Theorem 2.5 (Theorem 2.1 of \[10\]). Let $G$ be a non-complete, connected graph on $n$ vertices. Then $\kappa(G) = \lambda_{n-1}(G)$ if and only if $G$ can be written as $G = G_1 \cup G_2$, where $G_1$ is a disconnected graph on $n - \kappa(G)$ vertices and $G_2$ is a graph on $\kappa(G)$ vertices with $\lambda_{n-1}(G_2) \geq 2\kappa(G) - n$.

Definition 2.6 (See Page 15 of \[16\]). Let $H$ be a graph with vertex set $V(H) = \{1, 2, \ldots, k\}$. Let $G_i$ be disjoint graphs of order $n_i$ with vertex sets $V(G_i)$ where $1 \leq i \leq k$. The $H$-join of graphs $G_1, G_2, \ldots, G_k$ denoted by $H[G_1, G_2, \ldots, G_k]$ is formed by taking the graphs $G_i$ and any two vertices $v_i \in G_i$ and $v_j \in G_j$ are adjacent if $i$ is adjacent to $j$ in $H$. 
**Theorem 2.7** (Theorem 8 of [15]). Let us consider a family of \( k \) graphs \( G_j \) of order \( n_j \), with \( j \in \{1,2,\ldots,k\} \) having Laplacian spectrum \( \sigma(G_j) \). If \( H \) is a graph such that \( V(H) = \{1,2,\ldots,k\} \), then the Laplacian spectrum of \( H[G_1,G_2,\ldots,G_k] \) is given by

\[
\sigma(H[G_1,G_2,\ldots,G_k]) = \left( \bigcup_{j=1}^{k}(N_j + \sigma(G_j) \setminus \{0\}) \right) \cup \sigma(M)
\]

where

\[
M = \begin{bmatrix}
N_1 & -\rho_{1,2} & \cdots & -\rho_{1,k} \\
-\rho_{1,k} & N_2 & \cdots & -\rho_{2,k} \\
\vdots & \vdots & \ddots & \vdots \\
-\rho_{1,k} & -\rho_{2,k} & \cdots & N_k
\end{bmatrix},
\]

\[
\rho_{a,b} = \rho_{b,a} = \begin{cases} 
\sqrt{n_an_b} & \text{if } lq \in E(H) \\
0 & \text{otherwise}
\end{cases}
\]

and \( N_j = \left\{ \sum_{i \in N_H(j)} n_i \right\} \) if \( N_H(j) \neq \emptyset \) otherwise.

Here \( N_H(j) = \{i : ij \in E(H)\} \).

**3 Structure of \( \Gamma(\mathbb{Z}_n) \) and its Laplacian Spectra**

In this section we first give a description of the structure of \( \Gamma(\mathbb{Z}_n) \). We show that \( \Gamma(\mathbb{Z}_n) \) can be expressed as the join and union of certain sub-graphs of \( \Gamma(\mathbb{Z}_n) \). We then investigate Laplacian Spectra of \( \Gamma(\mathbb{Z}_n) \) for various \( n \).

We first find an equivalent condition for adjacency of two vertices in \( \Gamma(\mathbb{Z}_n) \).

Using the adjacency criterion for any two vertices in \( \Gamma(R) \), we find that two vertices \( u, v \in \Gamma(\mathbb{Z}_n) \) are adjacent if and only if \( \mathbb{Z}_n u + \mathbb{Z}_n v = \mathbb{Z}_n \). Now since \( \mathbb{Z}_n \) is a **Principal Ideal Domain**, so \( \mathbb{Z}_n u = \langle u \rangle \). Thus the adjacency criterion in \( \Gamma(\mathbb{Z}_n) \) becomes the following:

\( u \) is adjacent to \( v \) in \( \Gamma(\mathbb{Z}_n) \) \( \iff \langle u \rangle + \langle v \rangle = \mathbb{Z}_n \) (2)

where \( \langle u \rangle, \langle v \rangle \) denote the ideal generated by \( u, v \) respectively.

If \( V(\Gamma(\mathbb{Z}_n)) \) denotes the set of vertices of \( \Gamma(\mathbb{Z}_n) \) then \( V(\Gamma(\mathbb{Z}_n)) = S \cup T \) where \( S = \{a : \gcd(a, n) = 1\} \) and \( T = \mathbb{Z}_n \setminus S \). Thus the vertex set of \( \Gamma(\mathbb{Z}_n) \) is \( S \cup T \). The following observations about \( \Gamma(\mathbb{Z}_n) \) can be easily made.

**Lemma 3.1.** If \( v \in S \) then \( v \) is adjacent to \( w \) for all \( w \in \Gamma(\mathbb{Z}_n) \) and hence \( \deg(v) = n - 1 \).

**Proof.** Since \( v \) is an unit in \( \mathbb{Z}_n \), so \( \langle v \rangle = \mathbb{Z}_n \). Thus for all \( w(\neq v) \in \Gamma(\mathbb{Z}_n) \), \( \langle v \rangle + \langle w \rangle = \mathbb{Z}_n + \langle w \rangle = \mathbb{Z}_n \) and hence \( \deg(v) = n - 1 \).

**Lemma 3.2.** The vertex \( 0 \in T \) is adjacent only to the members of \( S \) and hence \( \deg(0) = |S| = \phi(n) \).
Let $G_1$ denote the induced subgraph of $\Gamma(\mathbb{Z}_n)$ on the set $\mathcal{S}$ and $G'_2$ denote the induced subgraph of $\Gamma(\mathbb{Z}_n)$ on the set $\mathcal{T}$. Since $\mathcal{S}$ has $\phi(n)$ elements, using Lemma 3.1 we find that $G_1$ is the complete graph on $\phi(n)$ vertices and hence $G_1 \cong K_{\phi(n)}$. Also using Lemma 3.1 we find that every vertex of $\Gamma(\mathbb{Z}_n)$ in $\mathcal{S}$ is adjacent to all the vertices of $\mathcal{T}$, which makes us conclude that

$$\Gamma(\mathbb{Z}_n) = G_1 \vee G'_2 \cong K_{\phi(n)} \vee G'_2$$

where $G'_2$ is a graph on $n - \phi(n)$ vertices.

Again using Lemma 3.2 we find that $G'_2$ is the union of graphs $\Theta_1$ and $G_2$ where $G_2$ is a graph on $n - \phi(n) - 1$ vertices and $\Theta_1$ denotes the null graph on the set $\{0\}$. The graph $G_2$ is the induced sub-graph of $\Gamma(\mathbb{Z}_n)$ on the set $\mathcal{T} \setminus \{0\}$. Thus we have

$$\Gamma(\mathbb{Z}_n) \cong K_{\phi(n)} \vee G'_2 \cong K_{\phi(n)} \vee (G_2 \cup \Theta_1).$$

We shall use Equation 3 repeatedly in our results.

**Theorem 3.3.** The characteristic polynomial of $\Gamma(\mathbb{Z}_n)$ is $\mu(\Gamma(\mathbb{Z}_n)) = x(x - n)^{\phi(n)} \mu(G_2, x - \phi(n))$ where $G_2$ is given by Equation 3.

**Proof.** Using Equation 3 and Theorems 2.1 and 2.2 we obtain,

$$\mu(\Gamma(\mathbb{Z}_n)) = \mu(G_1 \vee G'_2) = \mu(K_{\phi(n)} \vee G'_2)$$

$$= \frac{x(x - n)}{(x - \phi(n))(x - (n - \phi(n)))} \mu(K_{\phi(n)}, x - (n - \phi(n))) \mu(G'_2, x - \phi(n))$$

$$= \frac{x(x - n)}{(x - \phi(n))(x - (n - \phi(n)))}(x - (n - \phi(n)))^{\phi(n) - 1} \mu(G'_2, x - \phi(n))$$

$$= \frac{x(x - n)^{\phi(n)}}{(x - \phi(n))} \mu(G'_2, x - \phi(n)) = x(x - n)^{\phi(n)} \mu(G_2, x - \phi(n)).$$

\[\square\]

The following observation about $\mu(\Gamma(\mathbb{Z}_n))$ can be easily made

**Corollary 3.4.** If $n > 2$, then $n$ is an eigenvalue of $\Gamma(\mathbb{Z}_n)$ with multiplicity atleast $\phi(n)$.

**Corollary 3.5.** If $n = p$ where $p$ is a prime number, then $p$ and 0 are eigenvalues of $\Gamma(\mathbb{Z}_n)$ with multiplicity $p - 1$ and 1 respectively.

**Proof.** When $n = p$ is a prime number, then $\phi(n) = n - 1$ and hence the set $\mathcal{T} \setminus \{0\}$ is empty which implies that the graph $G_2$ is the empty graph. Thus using Equation 4

$$\mu(\Gamma(\mathbb{Z}_n)) = x(x - n)^{n - 1} = x(x - p)^{p - 1}.$$  

\[\square\]

From Equation 4 of Theorem 3.3 we find that the eigenvalues of $\Gamma(\mathbb{Z}_n)$ are known if the spectra of the graph $G_2$ given in Equation 3 is completely determined. We thus proceed to study the graph $G_2$ in more detail.
3.1 Structure of $G_2$

Let $n = p_1^{\alpha_1} p_2^{\alpha_2} \cdots p_k^{\alpha_k}$ be a prime factorization of $n$ where $p_1 < p_2 < \cdots < p_k$ are primes and $\alpha_i$ are positive integers.

The total number of positive divisors of $n$ is given by $\sigma(n) = (\alpha_1 + 1)(\alpha_2 + 1) \cdots (\alpha_k + 1)$. The total number of proper positive divisors of $n$ will be given by $\sigma(n) - 2$. We denote $\sigma(n) - 2$ by $w$.

Let $d_1 < d_2 < \cdots < d_w$ be the set of all proper divisors of $n$ arranged in increasing order. For each $d_i$ where $1 \leq i \leq w$ we define

$$A_{d_i} = \{ x : \gcd(x, n) = d_i \}. \quad (5)$$

Any element of $A_{d_i}$ is of the form $zd_i$ where $\gcd(z, \frac{n}{d_i}) = 1$ and hence number of elements of $A_{d_i}$ is $\phi(\frac{n}{d_i})$. Thus $|A_{d_i}| = \phi(\frac{n}{d_i})$. Clearly $V(G_2) = \bigcup_{i=1}^{w} A_{d_i}$.

Lemma 3.6. $x_i \in A_{d_i}$ is adjacent to $x_j \in A_{d_j}$ if and only if $\gcd(d_i, d_j) = 1$.

Proof. Assume that $x_i \in A_{d_i}$ is adjacent to $x_j \in A_{d_j}$. Using Equation 2, $x_i$ adjacent to $x_j$ implies $\langle x_i \rangle + \langle x_j \rangle = \mathbb{Z}_n$ which in turn implies that either $\gcd(x_i, x_j) = 1$ or $\gcd(x_i, x_j)$ is an unit in $\mathbb{Z}_n$. We consider the following two cases:

Case 1: $\gcd(x_i, x_j) = 1$
Let $\gcd(d_i, d_j) = d$, then $d|d_i$ and $d|d_j$. Since $\gcd(x_i, n) = d_i$ and $\gcd(x_j, n) = d_j$, we have $d_i|x_i$ and $d_j|x_j$ which in turn implies that $d|x_i$ and $d|x_j$. Again since $\gcd(x_i, x_j) = 1$, $d = 1$ and hence $\gcd(d_i, d_j) = 1$.

Case 2: $\gcd(x_i, x_j)$ is an unit in $\mathbb{Z}_n$.
Let $\gcd(x_i, x_j) = a$ which is an unit in $\mathbb{Z}_n$ and hence $\gcd(a, n) = 1$. Let $\gcd(d_i, d_j) = d$, then $d|d_i$ and $d|d_j$. Since $\gcd(x_i, n) = d_i$ and $\gcd(x_j, n) = d_j$, we have $d_i|x_i$, $d_i|n$ and $d_j|x_j$, $d_j|n$ which in turn implies that $d|x_i$, $d|x_j$ and $d|n$. Since $\gcd(x_i, x_j) = a$, $d|a$. Since $\gcd(a, n) = 1$, from the facts that $d|a$ and $d|n$ it follows that $d = 1$. Hence $\gcd(d_i, d_j) = 1$.

Thus if $x_i \in A_{d_i}$ is adjacent to $x_j \in A_{d_j}$, then $\gcd(d_i, d_j) = 1$.

Conversely, we now assume that $\gcd(d_i, d_j) = 1$. Let $d = \gcd(x_i, x_j)$. We claim that either $d = 1$ or $d$ is an unit in $\mathbb{Z}_n$. Assume the contrary, then $d > 1$ and $d$ is not an unit in $\mathbb{Z}_n$ which implies $d(>1)$ divides $n$.

If $d = \gcd(x_i, x_j)$ then $d|x_i$, $d|x_j$

$$\implies (d|x_i, d|n) \implies d|d_i = \gcd(x_i, n) \implies d|d_j = \gcd(x_j, n)$$

$$\implies d|\gcd(d_i, d_j) = 1$$

which is a contradiction.

Thus either $d = 1$ or $d$ is an unit in $\mathbb{Z}_n$ and hence $\langle x_i \rangle + \langle x_j \rangle = \langle d \rangle = \mathbb{Z}_n$ which implies by Equation 2 that $x_i \in A_{d_i}$ is adjacent to $x_j$ in $A_{d_j}$.

Lemma 3.7. If $v_i \in A_{d_i}$ is adjacent to $v_j \in A_{d_j}$ for some $i \neq j$, then $v_i$ is adjacent to $v_j$ for all $v_j \in A_{d_j}$. 

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Proof. Let \( v_i \in A_{d_i} \) is adjacent to \( v_j \in A_{d_j} \) for some \( i \neq j \), then using Lemma 3.6, \( \gcd(d_i, d_j) = 1 \). Let \( v'_j \neq v_j \) be another member of \( V_j \), then \( \gcd(v'_j, n) = d_j \). Using the fact that \( \gcd(d_i, d_j) = 1 \) and Lemma 3.6, we conclude that \( v'_j \) is adjacent to \( v_i \). \( \square \)

Lemma 3.8. No two members of the set \( A_{d_i} \) are adjacent.

Proof. If \( v_i, v_j \in A_{d_i} \) then \( \gcd(v_i, n) = \gcd(v_j, n) = d_i \). Using Lemma 3.6, the proof follows. \( \square \)

If \( v_i \in A_{d_i} \), using Lemma 3.7, we observe that the number of neighbors of \( v_i \) in \( A_{d_j} \) where \( j \neq i \) is fixed, i.e. either the number of neighbors of \( v_i \) in \( A_{d_j} \) equals 0 or \( |A_{d_j}| \). Also using Lemma 3.8, the number of neighbors of \( v_i \) in \( A_{d_i} \) equals 0 for all \( 1 \leq i \leq w \). If we denote \( V_i = A_{d_i} \) where \( 1 \leq i \leq w \), then using Definition 2.4, we find that \( V_1 \cup V_2 \cup \ldots \cup V_w \) is an equitable partition of the graph \( G_2 \).

Thus we have the following theorem;

Theorem 3.9. For any \( n \geq 2 \), the induced subgraph \( G_2 \) of \( \Gamma(Z_n) \) with vertex set \( V(G_2) \) has an equitable partition as \( V(G_2) = \bigcup_{i=1}^{w} A_{d_i} \), where \( w \) denotes the total number of positive proper divisors of \( n \) and the sets \( A_{d_i} \) have been defined as in Equation 3.

3.2 Laplacian Spectra of \( \Gamma(Z_n) \) for any \( n > 2 \)

In this section we provide the procedure to find the eigenvalues of \( \Gamma(Z_n) \) for any \( n > 2 \). Using Equation 4 of Theorem 3.3, we find that determining the spectra of \( \Gamma(Z_n) \) for any \( n > 2 \) boils down to finding the spectra of induced subgraph \( G_2 \) of \( \Gamma(Z_n) \).

Using Theorems 3.9 and 2.6, it is evident that \( G_2 \) is the \( H \)-join of the graphs \( G_{d_i} \) where \( G_{d_i} \) is the induced subgraph of \( \Gamma(Z_n) \) on \( A_{d_i} \), and \( H \) can be obtained as follows:

Construction of \( H \): \( V(H) = \{d_i : 1 \leq i \leq w \text{ where } d_i \text{ is a positive proper divisor of } n\} \).

The vertices \( d_i, d_j \) are adjacent in \( H \) if and only if \( \gcd(d_i, d_j) = 1 \).

Thus \( E(H) = \{d_id_j : \gcd(d_i, d_j) = 1\} \).

We use Theorem 2.7 to determine the spectra of \( G_2 \). We find that \( G_2 \) is the \( H \)-join of \( G_{d_i} \), where \( G_{d_i} \) is a null graph on \( \phi(n/d_i) \) vertices. Hence \( \sigma(G_{d_i}) = \{0\} \).

Also, \( N_H(d_j) = \{d_i : \gcd(d_i, d_j) = 1\} \)

and hence \( N_{d_i} = \sum_{d_i \in N_H(d_j)} n_i = \sum_{d_i; \gcd(d_i, d_j) = 1} \phi(n/d_i) \).

Moreover, \( n_{d_i} = \phi(n/d_i) \), where \( 1 \leq i \leq w \).

Example 3.10. If \( n = pqr \) where \( p, q, r \) are primes with \( p < q < r \), then the proper positive divisors of \( n \) are \( p, q, r, pq, pr, qr \). Using the construction of \( H \) given above, we find that \( G_2 \) is the \( H \)-join of \( G_p, G_q, G_r, G_{pq}, G_{pr}, G_{qr} \) where \( H \) is given by(Figure 7).
Now we have,

\[ N_p = \phi\left(\frac{pqr}{q}\right) + \phi\left(\frac{pqr}{r}\right) + \phi\left(\frac{pqr}{qr}\right) \]

\[ = \phi(pr) + \phi(pq) + \phi(p) = (p-1)(r-1) + (p-1)(q-1) + (p-1) \]

\[ = (p-1)\{r-1 + q-1 + 1\} = (p-1)(q+r-1). \]  \hspace{1cm} (6)

Similarly, \( N_q = (q-1)(p+r-1), N_r = (r-1)(p+q-1) \)

\[ N_{pq} = (p-1)(q-1), N_{pr} = (p-1)(r-1) \text{ and } N_{qr} = (q-1)(r-1) \]

Also

\[ n_p = (q-1)(r-1), n_q = (p-1)(r-1), n_r = (p-1)(q-1) \]

\[ n_{pq} = r-1, n_{pr} = q-1, n_{qr} = p-1. \] \hspace{1cm} (7)

Using Theorem 2.4 we find that the eigenvalues of \( G_2 \) are \( (p-1)(q+r-1) \) with multiplicity \( qr - r - q, (q-1)(p+r-1) \) with multiplicity \( pr - r - p, (r-1)(p+q-1) \) with multiplicity \( pq - p - q, (p-1)(q-1) \) with multiplicity \( r-2, (p-1)(r-1) \) with multiplicity \( q-2 \) and \( (q-1)(r-1) \) with multiplicity \( p-2 \) and remaining eigenvalues are the eigenvalues of \( 6 \times 6 \) matrix \( M \) (Equation (1)) whose entries can be determined from Equations (6) and (7).

Though the procedure given in Subsection 3.2 allow us to determine the spectra of \( \Gamma(Z_n) \) for any \( n > 2 \), it is difficult to use it when number of divisors of \( n \) is large as otherwise we would have to deal with large matrices which would make the calculations difficult.

In the next subsection we calculate the spectra of \( \Gamma(Z_n) \) for some specific \( n \) which in turn helps us observe some interesting facts about \( \Gamma(Z_n) \).

3.3 \( \Gamma(Z_n) \) is Laplacian Integral for \( n = p^\alpha q^\beta \)

In the next two theorems namely Theorem 3.11 and 3.12 we find the spectra of \( \Gamma(Z_n) \) for \( n = p^\alpha q^\beta \) which in turn helps us conclude that \( \Gamma(Z_n) \) is Laplacian Integral for \( p, q \) are primes where \( p < q \) and \( \alpha, \beta \) are non-negative integers.
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**Theorem 3.11.** When \( n = p^m \) where \( p \) is a prime and \( m > 1 \) is a positive integer, then the eigenvalues of \( \Gamma(Z_n) \) are \( n \) with multiplicity \( \phi(n) \), \( \phi(n) \) with multiplicity \( n - \phi(n) - 1 \) and 0 with multiplicity 1.

*Proof.* When \( p \) is a prime and \( m > 1 \) is a positive integer, the proper divisors of \( p^m \) are \( p, p^2, p^3, \ldots, p^{m-2}, p^{m-1} \). We partition the vertex set \( V(G_2) \) of \( G_2 \) as \( V_1, V_2, \ldots, V_{m-2}, V_{m-1} \) where \( V_i = A_{p^i} = \{ x : \gcd(x, n) = p^i \} \).

Since \( \gcd(p^i, p^j) = p^{\min\{i,j\}} \neq 1 \), using Lemmas 3.6 and 3.8 we find that \( x_i \in V_i \) is not adjacent to \( x_j \in V_j \) for all \( 1 \leq i, j \leq m - 1 \). Thus no two vertices in the graph \( G_2 \) are adjacent and hence \( G_2 = \Theta_{n-\phi(n)-1} \). Using Equation 4 we obtain \( \mu(\Gamma(Z_n)) = x(x-n)^{\phi(n)}(x-\phi(n))^{n-\phi(n)-1} \).

**Theorem 3.12.** If \( n = p^\alpha q^\beta \) where \( p, q \) are primes with \( p < q \) and \( \alpha, \beta \) are positive integers, then the eigenvalues of \( \Gamma(Z_n) \) are \( n \) with multiplicity \( \phi(n) \), \( (t+1)(p-1)+\phi(n) \) with multiplicity \( (t+1)(q-1)-1, (t+1)(q-1)+\phi(n) \) with multiplicity \( (t+1)(p-1)-1, \phi(n) \) with multiplicity \( t+1 \) and \( (t+1)(p+q-2)+\phi(n) \), 0 each with multiplicity 1 where \( t = p^{\alpha-1}q^{\beta-1} - 1 \).

*Proof.* If \( n = p^\alpha q^\beta \) where \( p, q \) are primes with \( p < q \) and \( \alpha, \beta \) are positive integers, then the proper divisors of \( n \) are \( p^i q^j \) where \( 0 \leq i \leq \alpha, 0 \leq j \leq \beta \) with \( i+j \notin \{0, \alpha+\beta\} \). We partition the vertex set \( V(G_2) \) as

\[
V(G_2) = \left( A_p \cup A_{p^2} \cup \cdots \cup A_{p^\alpha} \right) \cup \left( A_q \cup A_{q^2} \cup \cdots \cup A_{q^\beta} \right) \\
\cup \left( \bigcup_{j=1}^{\beta} A_{p^j} \right) \cup \left( \bigcup_{j=1}^{\beta} A_{p^2 q^j} \right) \cup \cdots \cup \left( \bigcup_{j=1}^{\beta-1} A_{p^\alpha q^j} \right).
\]

(8)

If \( 1 \leq i \leq \alpha, 1 \leq j \leq \beta \), then \( \gcd(p^i, q^j) = 1 \). Using Lemma 3.6 and 3.7 we find that every vertex of \( A_{p^i} \) is adjacent to every vertex of \( A_{q^j} \).

Also Lemma 3.6 indicates that if \( 1 \leq i \leq \alpha, 1 \leq j \leq \beta \) with \( i+j \neq \alpha+\beta \), then no vertex of \( A_{p^i q^j} \) is adjacent to any other vertex of \( G_2 \). If we draw the graph \( G_2 \) with the vertex partitions as given in Equation 8, it looks like the following (Figure 2).
Figure 2: $G_2$ for $n = p^α q^β$

(A solid line in the figure indicates that each vertex of $A_{d_i}$ is adjacent to each vertex of $A_{d_j}$. No line between two nodes $A_{d_i}$ and $A_{d_j}$ indicates that no vertex of $A_{d_i}$ is adjacent to any vertex of $A_{d_j}$.)

Let $G_{21}$ be the induced subgraph of $G_2$ on the set $A_p \cup A_{p^2} \cup \cdots \cup A_{p^α}$ and $G_{22}$ be the induced subgraph of $G_2$ on the set $A_q \cup A_{q^2} \cup \cdots \cup A_{q^β}$. 

Now the number of elements in \( A_p \cup A_{p^2} \cup \cdots \cup A_{p^c} \) is \( \sum_{i=1}^{\alpha} |A_{p^i}| \). Hence

\[
\sum_{i=1}^{\alpha} |A_{p^i}| = |A_p| + |A_{p^2}| + \cdots + |A_{p^c}| = \phi(p^{\alpha-1}q^\beta) + \phi(p^{\alpha-2}q^\beta) + \cdots + \phi(q^\beta) = p^{\alpha-1}(1 - \frac{1}{p})q^\beta(1 \frac{1}{q}) + p^{\alpha-2}(1 - \frac{1}{p})q^\beta(1 \frac{1}{q}) + \cdots
\]

\[+ \cdots + p(1 - \frac{1}{p})q^\beta(1 \frac{1}{q}) + q^\beta(1 \frac{1}{q}) = q^\beta(1 - \frac{1}{q})(1 - \frac{1}{p})\{p + p^2 + \cdots + p^{\alpha-1}\} + q^\beta(1 \frac{1}{q})
\]

\[= q^\beta(1 - \frac{1}{q})(p^{\alpha-1} - 1) + 1 = q^\beta-1p^{\alpha-1}(q - 1).
\]

Again the number of elements in the set \( A_q \cup A_{q^2} \cup \cdots \cup A_{q^\beta} \) is \( \sum_{i=1}^{\beta} |A_{q^i}| \). Using similar calculations as in Equation (9) we find that

\[
\sum_{i=1}^{\beta} |A_{q^i}| = p^{\alpha-1}q^{\beta-1}(p - 1).
\]

The vertices of \( G_2 \) which are not adjacent to any other vertex in \( G_2 \) are the members of the set

\[
\left( \bigcup_{j=1}^{\beta} A_{pq^j} \right) \cup \left( \bigcup_{j=1}^{\beta} A_{p^2q^j} \right) \cup \cdots \cup \left( \bigcup_{j=1}^{\beta-1} A_{p^{\alpha}q^j} \right)
\]

Using Equations (9) and (10), the number of such vertices denoted by \( t \) equals

\[
t = p^\alpha q^\beta - \phi(p^\alpha q^\beta) = 1 - p^{\alpha-1}q^{\beta-1}(p - 1) - p^{\alpha-1}q^{\beta-1}(q - 1)
\]

\[= p^\alpha q^\beta - 1 - p^{\alpha-1}q^{\beta-1}\{(p - 1)(q - 1) + p - 1 + q - 1\}
\]

\[= p^\alpha q^\beta - 1 - p^{\alpha-1}q^{\beta-1}(pq - 1) = p^{\alpha-1}q^{\beta-1} - 1.
\]

Clearly the induced subgraph of \( G_2 \) on \( p^{\alpha-1}q^{\beta-1} - 1 \) vertices is a null graph.

Since every vertex of the graph \( G_{21} \) is adjacent to every vertex of the graph \( G_{22} \) and the remaining vertices of \( G_2 \) are not adjacent to any other vertex, the following is evident

\[
G_2 = (G_{21} \lor G_{22}) \cup \Theta_t.
\]

Using Equations (9) and (10) and theorem 2.1 we obtain

\[
\mu((G_{21} \lor G_{22}), x) = x \left( x - p^{\alpha-1}q^{\beta-1}(q - 1) \right)^{p^{\alpha-1}q^{\beta-1}(p - 1) - 1}
\]

\[\times \left( x - p^{\alpha-1}q^{\beta-1}(p - 1) \right)^{p^{\alpha-1}q^{\beta-1}(q - 1) - 1} \left( x - (p^{\alpha-1}q^{\beta-1}(p + q - 2)) \right).
\]
Using Theorem 2.2, Equation 11 and Equation 12 we obtain,

\[
\mu(G_2, x) = x^t \times \mu((G_{21} \lor G_{22}), x) = x^{t+1} \left(x - p^{\alpha-1} q^{\beta-1}(q-1) \right)^p a^{q^{\beta-1}(p-1)-1} \\
\times \left(x - p^{\alpha-1} q^{\beta-1}(p-1) \right)^{q^{\beta-1}(q-1)-1} \left(x - (p^{\alpha-1} q^{\beta-1}(p+q-2)) \right).
\]

(13)

Using Equation 13 in Equation 4 we have

\[
\mu(\Gamma(Z_n), x) = x(x - n)^{\phi(n)} \mu(G_2, x - \phi(n)) \\
= x(x - n)^{\phi(n)} \left(x - (p^{\alpha-1} q^{\beta-1}(q-1) - \phi(n)) \right)^p a^{q^{\beta-1}(p-1)-1} \\
\times \left(x - p^{\alpha-1} q^{\beta-1}(p-1) - \phi(n) \right)^{q^{\beta-1}(q-1)-1} \left(x - (p^{\alpha-1} q^{\beta-1}(p+q-2) - \phi(n)) \right).
\]

Thus the eigenvalues of \( \Gamma(Z_n) \) are \( n \) with multiplicity \( \phi(n) \), \( (t+1)(p-1) + \phi(n) \) with multiplicity \( (t+1)(q-1) - 1 \), \( (t+1)(q-1) + \phi(n) \) with multiplicity \( (t+1)(p-1) - 1 \), \( \phi(n) \) with multiplicity \( t+1 \) and \( (t+1)(p+q-2) + \phi(n) \), 0 each with multiplicity 1.

\( \square \)

Using Corollary 3.5 and Theorems 3.11 and 3.12 the following is evident,

**Theorem 3.13.** If \( n = p^\alpha q^\beta \) where \( p, q \) are primes and \( \alpha, \beta \) are non-negative integers, then \( \Gamma(Z_n) \) is Laplacian Integral.

4 Algebraic Connectivity and Vertex Connectivity of \( \Gamma(Z_n) \)

In this section we investigate the algebraic connectivity(\( \lambda_{n-1} \)) and vertex connectivity(\( \kappa \)) of \( \Gamma(Z_n) \) for any \( n > 2 \). We also show that \( \lambda_{n-1} \) and \( \kappa \) are equal for any \( n > 2 \).

**Lemma 4.1.** If \( n > 2 \), then \( \phi(n) \) is an eigenvalue of \( \Gamma(Z_n) \) with multiplicity atleast 1.

**Proof.** Since 0 is always an eigenvalue of the Laplacian matrix of a given graph \( G \), so the Laplacian matrix of the graph \( G_2 \) also has 0 as an eigenvalue. Using Equation 4, \( x - \phi(n) \) is a factor of \( \mu(G_2, x - \phi(n)) \) which in turn implies

\[
\mu(\Gamma(Z_n), x) = x(x - n)^{\phi(n)} \mu(G_2, x - \phi(n)) \\
= x(x - n)^{\phi(n)}(x - \phi(n))g(x - \phi(n)) \text{ where}
\]

\( g(x) \) is a polynomial of degree \( n - \phi(n) - 2 \). Hence \( \phi(n) \) is an eigenvalue of \( \Gamma(Z_n) \) with multiplicity atleast 1.

**Theorem 4.2.** \( \lambda_{n-1}(\Gamma(Z_n)) = \phi(n) \).
Proof. Using Lemma 4.1, \( \phi(n) \) is an eigenvalue of \( \Gamma(\mathbb{Z}_n) \). Since the smallest root of the polynomial \( g(x - \phi(n)) \) in Equation (14) is \( \phi(n) \) and \( 0 < \phi(n) < n \), we conclude that the second smallest root of \( \mu(\Gamma(\mathbb{Z}_n), x) \) is \( \phi(n) \) which implies that \( \lambda_{n-1}(\Gamma(\mathbb{Z}_n)) = \phi(n) \).

\[ \square \]

Theorem 4.3. For all \( n > 2 \), \( \kappa(\Gamma(\mathbb{Z}_n)) = \lambda_{n-1}(\Gamma(\mathbb{Z}_n)) = \phi(n) \).

Proof. Using Equation (3) we find that \( \Gamma(\mathbb{Z}_n) = (G_2 \cup \Theta_1) \cup K_{\phi(n)} \). If we take \( G_1 = G_2 \cup \Theta_1 \) and \( G_2 = K_{\phi(n)} \), we find that \( G_1 \) is a disconnected graph on \( n - \phi(n) \) vertices and \( G_2 \) is a graph on \( \phi(n) \) vertices. Clearly \( \lambda_{n-1}(G_2) = \lambda_{n-1}(K_{\phi(n)}) = \phi(n) \). We find that if we assume \( \kappa(\Gamma(\mathbb{Z}_n)) = \phi(n) \), then all the conditions of Theorem 2.5 along with the inequality \( \lambda_{n-1}(G_2) \geq 2\kappa(G) - n \) is satisfied. Hence we conclude that \( \kappa(\Gamma(\mathbb{Z}_n)) = \lambda_{n-1}(\Gamma(\mathbb{Z}_n)) = \phi(n) \).

\[ \square \]

5 Largest & Second Largest Eigenvalue of \( \Gamma(\mathbb{Z}_n) \)

In this section we discuss about the second largest eigenvalue \( \lambda_2 \) of \( \Gamma(\mathbb{Z}_n) \) which in turn helps us to find certain information about the largest eigenvalue \( \lambda_1 \) of \( \Gamma(\mathbb{Z}_n) \).

We first study the connectivity of \( G_2 \).

Theorem 5.1. The graph \( G_2 \) is connected if and only if \( n \) is a product of distinct primes.

Proof. Let \( n = p_1^{\alpha_1}p_2^{\alpha_2} \cdots p_m^{\alpha_m} \) where \( p_i \) are distinct primes and \( \alpha_i \) are positive integers, \( 1 \leq i \leq m \).

We first assume that \( G_2 \) is connected. In order to show that \( n \) is a product of distinct primes, we prove that \( \alpha_i = 1 \) for all \( 1 \leq i \leq m \). Assume the contrary that \( \alpha_i > 1 \) for at least one \( i \). Without loss of generality we take \( \alpha_1 > 1 \). We consider the vertex \( a = p_1p_2p_3 \cdots p_m \) of \( G_2 \).

Clearly \( a \neq 0 \) as \( \alpha_1 > 1 \). Consider any other vertex of \( G_2 \) say \( w \). Since \( V(G_2) = \cup_{i=1}^m A_{d_i} \), where \( A_{d_i} \) has been defined in Equation (3) \( w \in A_{d_i} \) for some positive proper divisor \( d_i \) of \( n \). Thus \( \gcd(w, n) = d_i \). Also \( a \in A_{p_1p_2p_3 \cdots p_m} \). Since \( \gcd(d_i, p_1p_2p_3 \cdots p_m) \neq 1 \), using Lemma 3.6 we conclude that \( w \) is not adjacent to \( a \). Since \( w \) is arbitrary, we find that the vertex \( a \in G_2 \) is not adjacent to any other vertex of \( G_2 \) which contradicts the fact that \( G_2 \) is connected.

Hence our assumption that \( \alpha_1 > 1 \) is false. Thus \( \alpha_i = 1 \) for all \( 1 \leq i \leq m \) which proves that \( n \) is a product of distinct primes.

Conversely we assume that \( n \) is a product of distinct primes. In order to show that \( G_2 \) is connected we choose two arbitrary distinct vertices \( x_i, x_j \in G_2 \). Then \( x_i \in A_{d_i} \) and \( x_j \in A_{d_j} \) for some proper positive divisor \( d_i, d_j \) of \( n \). We consider the following two cases which may arise.

1. \( \gcd(d_i, d_j) = 1 \) Using Lemma 3.6, \( x_i \) and \( x_j \) are adjacent in \( G_2 \).

2. \( \gcd(d_i, d_j) \neq 1 \)

Since \( \gcd(d_i, d_j) \neq 1 \), \( d_i, d_j \) have a prime factor in common. Since \( n \) is a product of distinct primes, so there exists a prime factor \( p_1 \) of \( n \) such that \( \gcd(d_i, p_1) = 1 \). Also it is possible to choose another prime factor \( p_2 \neq p_1 \) of \( n \) such that \( \gcd(d_j, p_2) = 1 \). Since
\[ \gcd(p_1, p_2) = 1, \text{ if we choose } x_{p_1} \in A_{p_1} \text{ and } x_{p_2} \in A_{p_2} \text{ then } x_{p_1} \text{ is adjacent to } x_{p_2}. \] Thus using Lemma 3.6 we obtain a path of length 3 from \( x_i \) to \( x_j \) given by
\[
 x_i \rightarrow x_{p_1} \rightarrow x_{p_2} \rightarrow x_j
\]
Combining cases 1 and 2 we find that any two vertices of \( G_2 \) are either adjacent or there exists a path between them which implies that \( G_2 \) is connected when \( n \) is a product of distinct primes.
Thus \( G_2 \) is connected if and only if \( n \) is a product of distinct primes.

Now we investigate the connectivity of the complement of the graph \( G_2 \) (denoted by \( G_2^c \)) when \( n \) is a product of distinct primes.

When \( n \) is a product of two distinct primes, i.e. \( n = pq \), then \( n \) has only two distinct proper positive divisors namely \( p \) and \( q \). Thus \( V(G_2) = A_p \cup A_q \). Since \( \gcd(p, q) = 1 \), using Lemma 3.6 \( G_2 \) becomes

(Here the solid line indicates that each vertex of \( A_p \) is adjacent to each vertex of \( A_q \)).

Clearly \( G_2^c \) is disconnected when \( n = pq \).
In the next theorem, we investigate the connectivity of \( G_2^c \) when \( n \) is a product of more than 2 distinct primes.

**Theorem 5.2.** If \( n \) is a product of more than two distinct primes, then \( G_2^c \) is connected.

**Proof.** Let \( n = p_1p_2p_3\cdots p_m \) where \( p_i \) are distinct primes and \( m > 2 \). Let \( x_i, x_j \) be two distinct vertices of \( G_2^c \). Then \( x_i \in A_{d_i} \) and \( x_j \in A_{d_j} \) where \( d_i, d_j \) are positive proper divisors of \( n \). We consider the following two cases:

1. \( \gcd(d_i, d_j) \neq 1 \)
   Using Lemma 3.6 \( x_i \in A_{d_i} \) is not adjacent to \( x_j \in A_{d_j} \) in \( G_2 \) which implies that \( x_i \in A_{d_i} \) is adjacent to \( x_j \in A_{d_j} \) in \( G_2^c \).

2. \( \gcd(d_i, d_j) = 1 \)
   Using Lemma 3.6 \( x_i \in A_{d_i} \) is not adjacent to \( x_j \in A_{d_j} \) in \( G_2 \). Let \( p_1 \) be a prime factor of \( d_i \) and \( p_2 \) be a prime factor of \( d_j \). Since \( n \) is a product of more than two distinct primes, so \( p_1p_2 \) is a positive proper divisor of \( n \). Hence using Lemma 3.6 there exists \( y \in A_{p_1p_2} \) such that \( x_i, x_j \) are not adjacent to \( y \) in \( G_2 \). Thus \( y \) is adjacent to both \( x_i \) and \( x_j \) in \( G_2^c \) and hence there exists a path of length 2 given by \( x_i \rightarrow y \rightarrow x_j \) from \( x_i \) to \( x_j \) in \( G_2^c \).
Combining cases 1 and 2 we find that any two vertices of \( G_2^c \) are either adjacent or there exists a path between them which implies that \( G_2^c \) is connected when \( n \) is a product of more than two distinct primes.

**Theorem 5.3.** \( \lambda_2(\Gamma(Z_n)) \leq n - 1 \) where equality holds if and only if \( n \) is a product of two distinct primes.

**Proof.** Let \( \lambda_1(G_2) \) denote the largest eigenvalue of the Laplacian matrix of \( G_2 \). Using Equation 4 of theorem 3.3 it is evident that the second largest eigenvalue of \( \Gamma(Z_n) \) is the largest eigenvalue of the Laplacian matrix of \( G_2 \) which implies

\[
\lambda_2(\Gamma(Z_n)) = \lambda_1(G_2) + \phi(n).
\]

Since \( G_2 \) is a graph on \( n - \phi(n) - 1 \) vertices, using Theorem 2.3 we have \( \lambda_1(G_2) \leq n - \phi(n) - 1 \) where equality holds if and only if \( G \) is connected and \( G_2^c \) is disconnected.

Using Theorems 5.1 and 5.2 we find that \( G_2 \) is connected if and only if \( n \) is a product of distinct primes and \( G_2^c \) is disconnected if \( n \) is a product of two primes. Thus

\[
\lambda_2(\Gamma(Z_n)) = \lambda_1(G_2) + \phi(n) \leq (n - \phi(n) - 1) + \phi(n) = n - 1
\]

where equality holds if and only if \( n \) is a product of two primes.

**Theorem 5.4.** For any \( n > 2 \), \( \lambda_1(\Gamma(Z_n)) = n \) has multiplicity exactly \( \phi(n) \).

**Proof.** Using Theorem 5.3, \( \lambda_2(\Gamma(Z_n)) \leq n - 1 \). Thus from Equation 4 of Theorem 3.3 we conclude that \( \lambda_1 = n \) has multiplicity exactly \( \phi(n) \).

**Theorem 5.5.** If \( n = \prod_{i=1}^{m} p_i^{\alpha_i} \) where \( p_i \) are distinct primes and \( \alpha_i \) are positive integers, then \( \phi(n) \) is an eigenvalue of \( \Gamma(Z_n) \) with multiplicity \( \frac{n}{\prod_{i=1}^{m} p_i} \).

**Proof.** Let us first assume that \( n \) is a product of distinct primes i.e. \( n = p_1 p_2 \cdots p_m \). Using Lemma 5.1 \( G_2 \) is connected and hence 0 is an eigenvalue of \( L(G_2) \) with multiplicity 1 which in turn using Equation 4 implies that \( \phi(n) \) is an eigenvalue of \( \Gamma(Z_n) \) with multiplicity 1. Since \( \frac{n}{\prod_{i=1}^{m} p_i} = 1 \), the theorem holds true.

We now assume that \( n \) is not a product of distinct primes, i.e. \( \alpha_i > 1 \) for at least one \( 1 \leq i \leq m \). The set of vertices of \( G_2 \) in \( \langle p_1 p_2 \cdots p_m \rangle \setminus \{0\} \) are not adjacent to any other vertex in \( G_2 \). Since the set \( \langle p_1 p_2 \cdots p_m \rangle \setminus \{0\} \) has \( \frac{n}{\prod_{i=1}^{m} p_i} - 1 \) elements, the graph \( G_2 \) has \( \frac{n}{\prod_{i=1}^{m} p_i} \) connected components. Hence 0 is an eigenvalue of \( L(G_2) \) with multiplicity \( \frac{n}{\prod_{i=1}^{m} p_i} \) which in turn using Equation 4 implies that \( \phi(n) \) is an eigenvalue of \( \Gamma(Z_n) \) with multiplicity \( \frac{n}{\prod_{i=1}^{m} p_i} \).
5.1 Vertex connectivity of $G_2$

This section deals with the vertex connectivity of $G_2$. Since $G_2$ is connected if and only if $n$ is a product of distinct primes, we discuss $\kappa(G_2)$ when $n = p_1p_2 \cdots p_m$ where $p_i$, $1 \leq i \leq m$ are distinct primes.

We first give an example to illustrate $\kappa(G_2)$.

**Example 5.6.** Suppose $n = 3 \times 5 \times 7$. Consider the vertex 15 in $G_2$. Consider the set $\langle 7 \rangle \setminus \{0\} = \{7k : 1 \leq k \leq 14\}$. We notice that 15 is adjacent only to the following vertices $\{7, 14, 28, 49, 56, 77, 91, 98\}$. Thus the set $\{7, 14, 28, 49, 56, 77, 91, 98\}$ is a separating set of $G_2$. The elements of the set $\{7, 14, 28, 49, 56, 77, 91, 98\}$ are of the form $\{7k : 1 \leq k \leq 14, \gcd(k, 14) = 1\}$. We find that $\kappa(G_2) \leq 8 = \phi(15)$.

We prove the above formally in the following theorem:

**Theorem 5.7.** If $n = p_1p_2 \cdots p_m$, then $\kappa(G_2) \leq \phi(p_1p_2p_3 \cdots p_{m-1})$.

**Proof.** We first verify the result when $n$ is a product of two distinct primes. The graph of $G_2$ when $n$ is a product of two distinct primes has been shown in Figure 3. If $n = p_1p_2$, then $G_2$ is the join of two disconnected graphs having vertex sets as $A_{p_1}$ and $A_{p_2}$ and hence $\kappa(G_2) = \min\{|V_1|, |V_2|\} = \min\{p_1 - 1, p_2 - 1\} = p_1 - 1 = \phi(p_1)$ and hence our result holds.

When $n = \prod_{i=1}^{m} p_i$ where $m > 2$, then the vertex $p_1p_2 \cdots p_{m-1}$ of the graph $G_2$ is adjacent only to those members $a$ of the set $\langle p_m \rangle \setminus \{0\}$ such that $\gcd(a, p_1p_2 \cdots p_{m-1}) = 1$. The number of those elements $a$ such that $\gcd(a, p_1p_2 \cdots p_{k-1}) = 1$ equals $\phi(p_1p_2 \cdots p_{m-1})$. Since the vertices $a$ for which $\gcd(a, p_1p_2 \cdots p_{m-1}) = 1$ becomes a separating set of the graph $G_2$, the result follows. \hfill \square

6 Problems

In this section we pose some problems for further research.

Using Theorem 3.13 we observe that $\Gamma(Z_n)$ is Laplacian Integral for $p^\alpha q^\beta$ where $p, q$ are primes and $\alpha, \beta$ are non-negative integers. Since it is quite motivating to find those graphs which are Laplacian Integral, we pose the following problem:

**Problem 6.1.** Is it true that $\Gamma(Z_n)$ is Laplacian Integral if and only if $n = p^\alpha q^\beta$ where $p, q$ are primes and $\alpha, \beta$ are non-negative integers? If not, then find all $n$ such that $\Gamma(Z_n)$ is Laplacian Integral.

Again, in subsection 5.1 we have provided an upper bound on the vertex connectivity of the graph $G_2$ which is an induced subgraph of $\Gamma(Z_n)$. Though we have provided an upper bound on $\kappa(G_2)$, the readers are encouraged to calculate the exact value of $\kappa(G_2)$ if possible. Thus we ask the following:

**Problem 6.2.** If $n = p_1p_2 \cdots p_m$ where $p_1 < p_2 < \cdots < p_m$ are primes, find $\kappa(G_2)$. 
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