On the Sequential Multiknapsack polytope

Paolo Detti *

Abstract

The Sequential Multiple Knapsack Problem is a special case of Multiple knapsack problem in which the items sizes are divisible. A characterization of the optimal solutions of the problem and a description of the convex hull of all the integer solutions are presented. More precisely, it is shown that a new formulation of the problem allows to generate a decomposition approach for enumerating all optimal solutions of the problem. Such a decomposition approach is used for finding the inequalities (defined by an inductive scheme) describing the Sequential Multiple Knapsack polytope.

Keywords: integer programming, sequential multiple knapsack problem, optimal solutions, polytope description.

1 Introduction

The Sequential Multiple Bounded Knapsack Problem (SMKP) can be stated as follows. There are a set $N$ of $n$ item types, $N = \{1, \ldots , n\}$, and a set $M$ of knapsacks, $M = \{1, \ldots , m\}$. Each item of type $j$ has a size $s_j \in \mathbb{Z}^+$, a value $v_j \in \mathbb{R}$ and an upper bound $b_j \in \mathbb{Z}^+$. Item sizes are divisible, i.e., $s_{j+1}/s_j \in \mathbb{N}$, for all $j = 1, \ldots , n - 1$. Each knapsack $i$ has a capacity $c_i \in \mathbb{Z}^+$. The problem is

---

* Dipartimento di Ingegneria dell’Informazione, Università di Siena, Via Roma 56, Italy, e-mail: detti@dii.unisi.it, tel. +39 0577-234850 (1022), fax +39 0577-233602.
to find the number $x_{i,j}$ of items of type $j$, for $j = 1, \ldots, n$, to be assigned to each knapsack $i$, such that: (1) The total value $\sum_{i=1}^{m} \sum_{j=1}^{n} v_j x_{i,j}$ of the assigned items is maximum; (2) $\sum_{j=1}^{n} s_j x_{i,j} \leq c_i$ for $i = 1, \ldots, m$ (i.e., the total size of items assigned to a knapsack does not exceed the capacity of the knapsack); (3) $\sum_{i=1}^{m} x_{i,j} \leq b_j$ for $j = 1, \ldots, n$ (i.e., the total number of the assigned items of type $j$ does not exceed the upper bound). Without loss of generality we will assume $s_1 = 1$. The sequential single knapsack problem (SKP) has been addressed in the literature by several authors. For the unbounded case (i.e., $b_j = \infty$, for all $j$), Marcotte [2] presents a linear time algorithm for SKP and Pochet and Wolsey [4] give an explicit description of the polytope. For the bounded case (where $b_j < \infty$, for all $j$), in [3], a description of the bounded SKP polytope is provided, and in [5] an $O(n^2 \log n)$ algorithm is proposed. In [1], the sequential multiple knapsack problem is addressed, and a polynomial $O(n^2 + nm)$ algorithm is presented.

In this paper, a new formulation for SMKP and a characterization of the optimal solutions of the new formulation is provided, leading to a description of the convex hull of all the integer solutions of SMKP. More precisely, first new formulations are proposed for SMKP. Some new formulations restrict the set of feasible solutions of the problem, but guarantee the existence of at least an optimal solution in the restricted feasible set. Then, a problem transformation from a new formulation to another problem is presented, such that SMKP and the transformed problem are equivalent in terms of optimization. This last result has been obtained generalizing the approach proposed in [3], and is used for finding an decomposition approach that allows the enumeration of all the optimal solutions of the transformed problem. The decomposition approach is then used, as in in [3], for finding a description of the SMKP polytope related to the new formulation.

Summarizing, the main contributions of this paper are: the presentation of new ILP formulations for SMKP; the proposal of an approach for decomposing and enumerating the optimal solutions of
a transformed problem equivalent to SMKP in terms of optimization; the definition of inductive inequalities for describing the convex hull of all integer solutions of the new formulations proposed for SMKP.

In Section 2, basic properties of feasible solutions and new formulations for SMKP are presented. In Section 3 a problem transformation is presented and in Section 4 a decomposition scheme is given for the optimal solutions of the transformed problem. In Section 5 a description of the convex hull of the feasible solutions of SMKP is presented.

2 A new formulation for SMKP

The two following proposition hold since the sizes of the items are divisible.

**Proposition 2.1** Given a set $A \subseteq N$, let $s$ be the biggest size of the items in $A$. Let $c$ be an integer such that: (1) $s$ divides $c$; (2) $\sum_{j \in A} s_j \geq c$. Then a set $B \subseteq A$ exists such that $\sum_{j \in B} s_j = c$.

The following proposition directly follows from Proposition 2.1.

**Proposition 2.2** Given a set $A \subseteq N$, let $s$ be the biggest size of the items in $A$, and let $b \geq s$ be an integer such that $s$ divides $b$. Then the minimum number of subsets, each of total size at most $b$, in which $A$ can be partitioned is $\lceil \sum_{j \in A} s_j / b \rceil$. Moreover, a partition exists in which $\lceil \sum_{j \in A} s_j / b \rceil - 1$ subsets have total size $b$.

The following considerations allow to define a new formulation for SMKP. Let $l$ be the number of different item sizes in $N$, i.e., $l = |\{s_j | j = 1, \ldots, n\}|$. Let these sizes be denoted by $d_1 < d_2 < \ldots < d_l$, and let $n_1, n_2, \ldots, n_l$ be the number of item types with sizes $d_1, d_2, \ldots, d_l$, respectively. Hence, $n_k = |\{j | s_j = d_k \ j = 1, \ldots, n\}|$. Given a size $d_k$, we re-index the item types of size $d_k$, say $\{j_1, j_2, \ldots, j_{n_k}\}$, in non-increasing order of values $v$, i.e., in such a way that $v_{j_1} \geq v_{j_2} \geq \ldots \geq v_{j_{n_k}}$, for all $k = 1, \ldots, l$. 


Let us consider the items with the smallest size \( d_1 = 1 \), and let \( r_1^i = (c_i \mod d_2) \). Note that the effective capacity of knapsack \( i \) that can be used for all items of size larger than \( d_1 \) is \( c_i - r_1^i \), for \( i = 1, \ldots, m \). Hence, the capacities \( r_1^i \), for \( i = 1, \ldots, m \), of each knapsack can be only used to assign items of size \( d_1 \). (Observe that the capacity of knapsack \( i \) that can be used to assign only items of size \( d_1 = 1 \) is \( \lfloor r_1^i/d_1 \rfloor d_1 = r_1^i \). Let us consider the items with the size \( d_2 \). If \( d_2 < d_1 \), let \( r_2^i = (c_i - r_1^i) \mod d_3 \), otherwise let \( r_2^i = (c_i - r_1^i) \). Since, by definition, \( d_2 \) divides \( c_i - r_1^i \) and \( d_3 \), it follows that \( d_2 \) divides \( r_2^i \). Note that, the capacities \( r_2^i \), for \( i = 1, \ldots, m \), can be used only to assign items of sizes \( d_1 \) and \( d_2 \). The above argument can be repeated for items of sizes \( d_3, \ldots, d_l-1 \), by defining the capacities \( r_h^i \), for \( h = 3, \ldots, l-1 \) and \( i = 1, \ldots, m \), and then setting \( r_i^i = c_i - \sum_{h=1}^{l-1} r_h^i \). By the above discussion, each knapsack \( i \) can be partitioned into \( l \) parts of capacities \( r_1^i, \ldots, r_l^i \), such that only items not bigger than \( d_h \) can be assigned to a part \( h \), with \( 1 \leq h \leq l \). Given \( j \in N \), we denote by \( g(j), 1 \leq g(j) \leq l \), the index such that \( d_{g(j)} = s_j \). Let \( x_{i,j}^h \) be the number of items of type \( j \) assigned to the part of knapsack \( i \) (of capacity \( r_h^i \)), with \( h \geq g(j) \). By the above discussion, it follows that SMKP can be also formulated as:

\[
\begin{align*}
\max \sum_{i=1}^{m} \sum_{j=1}^{n} \sum_{h=g(j)}^{l} v_j x_{i,j}^h \\
\sum_{j \in N : g(j) \leq h} s_j x_{i,j}^h &\leq r_h^i \quad \text{for } h = 1, \ldots, l \text{ and } i = 1, \ldots, m \\
\sum_{i=1}^{m} \sum_{h=g(j)}^{l} x_{i,j}^h &\leq b_j \quad \text{for } j = 1, \ldots, n \\
x_{i,j}^h &\in \mathbb{Z}^+ \cup \{0\} \quad \text{for } i = 1, \ldots, m, \ j = 1, \ldots, n \text{ and } h = g(j), \ldots, l.
\end{align*}
\]

The objective function accounts for the maximization of the value of the assigned items. The first set of constraints states that the total size of the items \( j \) such that \( s_j \leq d_h \) (or, equivalently \( g(j) \leq h \)) assigned to the part \( h \) of knapsack \( i \) does not exceed the capacity \( r_h^i \). The second set of constraints states that the total number of assigned items of type \( j \) cannot exceed the upper bound \( b_j \). From now on, we consider the SMKP formulation (\( \text{SMKP} \)).

Given an instance of SMKP and an integer \( h, 1 \leq h \leq l \), we denote by \( \text{SMKP}(h) \) the “restricted” SMKP instance, in which the item set is \( \{j \in N : s_j \leq d_h\} \) and each knapsack \( i \) has the reduced
capacity $c_i = \sum_{q=1}^{h} r_i^q$, for $i = 1, \ldots, m$. A formulation for SMKP($h$) reads as follows

$$\max \sum_{i=1}^{m} \sum_{j=1}^{n} \sum_{q=g(j)}^{h} v_j x_{i,j}^q$$

subject to

$$\sum_{j \in N: g(j) \leq q} s_j x_{i,j}^q \leq r_i^q \text{ for } q = 1, \ldots, h \text{ and } i = 1, \ldots, m$$

$$\sum_{i=1}^{m} \sum_{q=g(j)}^{h} x_{i,j}^q \leq b_j \text{ for } j \in \{N : s_j \leq d_h\}$$

$$x_{i,j}^q \in \mathbb{Z}^+ \cup \{0\} \text{ for } i = 1, \ldots, m, j \in \{N : s_j \leq d_h\} \text{ and } q = g(j), \ldots, h.$$

In [1], a polynomial algorithm, called A-OPT, for finding an optimal solution of SMKP is proposed. A-OPT is a recursive algorithm based on Formulation (1). The basic idea of the algorithm is reported in the following. Let $\alpha_1$ be the maximum number of items of size $d_1$ that can be assigned to the $m$ knapsacks, when each knapsack has a restricted capacity $r_i^1$, $i = 1, \ldots, m$. Since the total number of items of size $d_1$ is $t_1 = \sum_{j=1:s_j=d_1}^{n} b_j$, we have $\alpha_1 = \min\{t_1, \sum_{i=1}^{m} \lfloor \frac{r_i^1}{d_1} \rfloor\}$. Lemma 2.3 holds [1].

**Lemma 2.3** An optimal solution for SMKP exists in which the first $\alpha_1$ (i.e., with the biggest values) items of size $d_1$ are assigned to the knapsacks using at most a capacity $r_i^1$, for $i = 1, \ldots, m$.

According to Lemma 2.3, A-OPT recursively works as follows. First of all, the first $\alpha_1$ items of size $d_1$ are assigned to the knapsacks using at most a capacity $r_i^1$ for each knapsack $i = 1, \ldots, m$. The not assigned items of size $d_1$ are grouped by a procedure, called grouping procedure, described in the following. The not assigned items of size $d_1$ are lined up in non-increasing value order. Then, groups of $d_2/d_1$ items are replaced by items with size $d_2$. To each new item of size $d_2$, a value $v$ is assigned, given by the sum of the values of the grouped items. (The last new item built so far may not contain $d_2/d_1$ items of size $d_1$, but we assign to it a size of $d_2$ and a value equal to the sum of the values of the grouped items.) After the grouping procedure, we get an instance with $l-1$ different item sizes, where $d_1 = d_2, d_2 = d_3, \ldots, d_{l-1} = d_1$, and $m$ knapsacks of capacity $c_i - r_i^1$, for $i = 1, \ldots, m$. A-OPT can be recursively called on this instance.
Notation that will be used in the following is now introduced. For $a \in \mathbb{R}$, we denote $a^+ = \max\{0, a\}$. Given a feasible solution $x$ of SMKP and two integers $1 \leq h_1 \leq h_2 \leq l$, let $x^{h_1,h_2}$ be the "partial" solution of $x$ related to the parts of the knapsacks $\{h_1, \ldots, h_2\}$, i.e., $x^{h_1,h_2}$ is the vector containing the components $x_{i,j}^{q}$, for $q = h_1, \ldots, h_2$, $i = 1, \ldots, m$ and $j \in \{N : d_j \leq d_{h_2}\}$. If $h_1 > h_2$ then $y^{h_1,h_2}$ does not assign any item. To simplify the notation, if $h_1 = h_2$, $x^{h_1}$ is used instead of $x^{h_1,h_1}$. Let $S(x)$ and $S(x^{h_1,h_2})$ be the set of items assigned in $x$ and $x^{h_1,h_2}$ respectively. Given an item type $j$, let $x_{j}^{h_1,h_2}$ be the number of items of type $j$ assigned in $x$ to the parts $\{h_1, \ldots, h_2\}$ of knapsacks, and let $S(x_{j}^{h_1,h_2})$ be the corresponding set of items. Moreover, given a subset of items $A$, let $f(A)$ and $v(A)$ be the total size and value, respectively, of the items in $A$, i.e., $f(A) = \sum_{j \in A} s_j |A_j|$ and $v(A) = \sum_{j \in A} v_j |A_j|$, where $|A_j|$ denotes the number of items of type $j$ in $A$. For the sake of simplicity, $f(x^{h_1,h_2})$ ($v(x^{h_1,h_2})$) can be used instead of $f(S(x^{h_1,h_2}))$ ($v(S(x^{h_1,h_2}))$) to denote the total size (the total value) of the items assigned by $x$ to the knapsacks parts $\{h_1, \ldots, h_2\}$.

In the remaining part of this section, new formulations of SMKP are proposed based on restrictions of the set of feasible solutions of Problem (1). Such restrictions guarantee the existence of at least an optimal solution of the SMKP in the restricted sets.

### 2.1 Definition of OPT solution

In what follows, the definition of OPT solution is given.

**Definition 2.1** Let $x$ be a feasible solution for SMKP. The solution $x$ has the OPT property (or, equivalently, $x$ is an OPT solution), if for each feasible solution $\tilde{x}$ that assigns the same items of $x$, i.e., $S(x) = S(\tilde{x})$, we have $v(x^{1,h}) \geq v(\tilde{x}^{1,h})$, for $h = 1, \ldots, l$.

The following observation holds since algorithm A-OPT assigns to the $h$-th part of the knapsack, at the $h$-th iteration, as many (grouped) items of size $d_h$ as possible (according to their order).
Observation 2.4 A-OPT produces optimal OPT solutions.

In view of Definition 2.1 and Observation 2.4, it follows that for each feasible solution $x$, the solution $\bar{x}$, obtained by applying A-OPT on the restricted item set $S(x)$, is an OPT solution. Hence, given an SMKP instance, we can restrict to consider only OPT solutions and reformulate SMKP as follows. Let $P^{OPT}$ be the convex hull of the OPT solutions for SMKP, then SMKP can be also formulated as $\max \{ v(x) : x \in P^{OPT} \}$. Given an SMKP instance, let $O^{OPT}$ be the set of the OPT solutions that are optimal. Given an optimal OPT solution $x$ for SMKP, in the following (Lemma 2.6) we show that $x_{1,h}^{1,h}$ is optimal for SMKP($h$), for $h = 1, \ldots, l$. At this aim Lemma 2.5 is useful.

Lemma 2.5 Given an SMKP instance, let $x$ be an optimal solution for SMKP. Let $\bar{x}$ be the solution produced by A-OPT when applied to the set $S(x)$. Let $x_O$ be the optimal solution found by A-OPT when applied to the whole item set $N$. Then, if during A-OPT the (grouped) items with the same size and value are suitably ordered, $S(x_O^{1,h}) = S(\bar{x}_1^{1,h})$, for $h = 1, \ldots, l$.

Proof. The proof is reported in the Appendix. □

Lemma 2.6 Let $x$ be an optimal solution for SMKP satisfying the OPT property. Then $x_{1,h}^{1,h}$ is optimal for SMKP($h$), for $h = 1, \ldots, l$.

Proof. The thesis directly follows from Observation 2.4 and Lemma 2.5 □

2.2 Definition of ordered solution

In this section, the definition of ordered solution is given.
Definition 2.2  Given a feasible solution $x$ for SMKP, $x$ is called ordered if, for each knapsack part $h = 1, \ldots, l$, a set of items $\Gamma$ assigned to the part $h$ in $x$ with size $s(\Gamma) \leq d_h$ exists only if an item of size $s(\Gamma)$ and value $v(\Gamma)$ either assigned to a part bigger than $h$ or not assigned in $x$ does not exist.

Given a feasible solution $x$ for SMKP, it is always possible to build an ordered solution $\bar{x}$ such that $v(x^{1,h}) = v(\bar{x}^{1,h})$ and $s(x^{1,h}) = s(\bar{x}^{1,h})$, for $h = 1, \ldots, l$. In fact, let us suppose that a set $\Gamma$, with $|\Gamma| > 1$ and $f(\Gamma) \leq d_h$, is assigned to a part $h$ in $x$ and that an item $j$ exists, with $s_j = s(\Gamma)$ and value $v_j = v(\Gamma)$, either assigned to a part bigger than $h$ or not assigned in $x$. In the first case, let $\bar{x}$ be the solution obtained by swapping $\Gamma$ and $j$ in $x$, while in the second case let $\bar{x}$ be the solution obtained by assigning $j$ in place of the items in $\Gamma$. If $\bar{x}$ contains a new set $\Gamma$ as defined above, then the above argument can repeated, otherwise, $\bar{x}$ is an ordered solution.

2.3 A new formulation for SMKP

By the discussions presented in Sections 2.1 and 2.2 it follows that we can restrict to consider only OPT and ordered solutions. Let $P^{OO}$ be the convex hull of ordered and OPT solutions for SMKP (formulated as in (1)), and let $O^{OO}$ be the set of optimal ordered and OPT solutions. Then, SMKP can be formulated as the problem of finding a solution in $O^{OO}$, or equivalently, as

$$\max \{ v(x) : x \in P^{OO} \} \quad (3)$$

In the rest of the paper we will show how to find a description of polytope $P^{OO}$. It is important to observe that, by definition of OPT and ordered solution, $P^{OO}$ directly depends from the values of the objective coefficients $v_1, \ldots, v_n$.

3 A problem transformation

In this section, a transformation is presented from SMKP, formulated as in (1), to a new problem, called Modified-SMKP (M-SP). This transformation extends that presented in [3] (for the sequential knapsack
Definition 3.1 Given an MSKP instance, let \( B_w = \{w_1, \ldots, w_k\}, w_1 < \ldots < w_k \), be a subset of item types. \( B_w \) is called a block if, for every \( j \in \{2, \ldots, k\} \), \( s_{w_j} \leq s_{w_1} + \sum_{v=1}^{j-1} b_{w_v} s_{w_v} \). The number \( \tilde{b}_w = \sum_{v=1}^{k} b_{w_v} s_{w_v} / s_{w_1} \) is called the multiplicity of block \( B_w \).

The following property holds.

Proposition 3.1 Given a block \( B_w = \{w_1, \ldots, w_k\}, \) let

\[
\delta_q^h \in \{0, s_{w_1}, 2s_{w_1}, \ldots, b_{w_1} s_{w_1}, (b_{w_1} + 1)s_{w_1}, \ldots, b_{w_1} s_{w_1} + b_{w_2} s_{w_2}, \ldots, \sum_{v=1}^{k} b_{w_v} s_{w_v}\}, \quad q = 1, \ldots, l \text{ and } h = q, \ldots, l, \text{ such that:}
\]

\[
\sum_{h=q}^{l} \left\lfloor \frac{\delta_q^h}{d_q} \right\rfloor \leq \sum_{w \in B_w : g(w) = q} b_w \quad \text{for } q = 1, \ldots, l \text{ and } h = q, \ldots, l. \tag{4}
\]

Then a subset \( R \subseteq B_w \) and not negative integers \( \lambda_j^h \), for \( j \in R \) and \( h = g(j), \ldots, l \), exist such that:

\[
0 \leq \sum_{h=g(j)}^{l} \lambda_j^h b_j, \quad \text{for all } j \in R; \tag{5}
\]

\[
\sum_{j \in R : g(j) = q} \lambda_j^h = \left\lfloor \frac{\delta_q^h}{d_q} \right\rfloor, \quad \text{for } q = 1, \ldots, l \text{ and } h = q, \ldots, l. \tag{6}
\]

Proof. The proof is by induction on the number of different item sizes \( l \). Suppose that \( l = 1 \). Then, by Definition 3.1, all the items have size \( d_1 = s_{w_1} = 1 \) and belong to the same block \( B_w \). Let \( j' \) be the smallest index such that \( \sum_{j \in B_{w_j} \leq j'} b_j \geq \left\lfloor \frac{d_1}{d_1} \right\rfloor = d_1^1 \). Then, since \( d_1^1 \in \{0, s_{w_1}, 2s_{w_1}, \ldots, b_{w_1} s_{w_1}\} \) and \( d_1 = s_{w_1} = 1 \), the thesis follows by setting \( R = \{w_1, \ldots, j'\}, \lambda_j^1 = b_j \) if \( j < j' \) and \( \lambda_j^1 = d_1^1 - \sum_{j \in R : j < j'} \lambda_j^1 \).

Let us suppose that the thesis holds up to \( l = a - 1 \geq 1 \) and show it for \( l = a \).

Given a block \( B_w = \{w_1, \ldots, w_k\}, \) let \( \delta_q^h \in \{0, s_{w_1}, 2s_{w_1}, \ldots, b_{w_1} s_{w_1}, \ldots, \sum_{v=1}^{k} b_{w_v} s_{w_v}\}, \) for \( q = 1, \ldots, a \) and \( h = q, \ldots, a \), such that:

\[
\sum_{h=q}^{a-1} \left\lfloor \frac{\delta_q^h}{d_q} \right\rfloor + \left\lfloor \frac{\delta_q^a}{d_q} \right\rfloor \leq \sum_{w \in B_w : g(w) = q} b_w \quad \text{for } q = 1, \ldots, a. \tag{7}
\]

Observe that, by (7), we also have

\[
\sum_{h=q}^{a-1} \left\lfloor \frac{\delta_q^h}{d_q} \right\rfloor \leq \sum_{w \in B_w : g(w) = q} b_w \quad \text{for } q = 1, \ldots, a - 1. \tag{8}
\]
Let $A = \sum_{w \in B_w: g(w) < a} b_w s_w$. By (3), we have \( \sum_{h=q}^{a-1} [\delta_q^h/d_q] d_q \leq \sum_{w \in B_w: g(w) = q} b_w d_q \leq A \) for $q = 1, \ldots, a - 1$. Hence, $\delta_q^h \leq A$, i.e., $\delta_q^h \in \{0, s_{w_1}, 2s_{w_1}, \ldots, b_{w_1}s_{w_1}, \ldots, \sum_{w: g(w) < a} b_w s_w\}$, for $q = 1, \ldots, a - 1$ and $h = q, \ldots, a - 1$. Hence, the values $\delta_q^h$, for $q = 1, \ldots, a - 1$, satisfy the hypothesis of the proposition, and by induction, a subset $\tilde{R} \subseteq B_w$ such that $s_j < d_a$ for all $j \in \tilde{R}$, and integers $\lambda_j^h$, for $j \in \tilde{R}$ and $h = g(j), \ldots, a - 1$, exist such that

$$0 \leq \sum_{h=g(j)}^{a-1} \lambda_j^h \leq b_j,$$

for all $j \in \tilde{R}$, and

$$\sum_{j \in \tilde{R}: g(j) = q} \lambda_j^h = \lceil \delta_q^h/d_q \rceil,$$

for $q = 1, \ldots, a - 1$. Hence, Inequality (7) reads as

$$\sum_{j \in \tilde{R}: g(j) = q} \sum_{h=q}^{a-1} \lambda_j^h + [\delta_q^a/d_q] \leq \sum_{j \in \tilde{R}: g(j) = q} b_j + \sum_{w \in B_w \setminus \tilde{R}: g(w) = q} b_w \quad \text{for} \quad q = 1, \ldots, a,$$

that can be rewritten as

$$[\delta_q^a/d_q] \leq \sum_{j \in \tilde{R}: g(j) = q} (b_j - \sum_{h=q}^{a-1} \lambda_j^h) + \sum_{w \in B_w \setminus \tilde{R}: g(w) = q} b_w \quad \text{for} \quad q = 1, \ldots, a. \quad (9)$$

For each $q = 1, \ldots, a$, let $R_q^g$ be a subset of items in $B_w$ of size $d_q$ with minimum total size such that:

(i) $|R_q^g| \leq b_j - \sum_{h=q}^{a-1} \lambda_j^h$, for all $j \in \tilde{R} \cap R^g$, and $|R_q^g| \leq b_j$, for all $j \in R^g \setminus \tilde{R}$ (recall that $R_q^g$ is the set of items of type $j$ in $R^q$);

(ii) $\sum_{j \in R^g} |R_q^g| = [\delta_q^a/d_q].$

Observe that, by (2), the sets $R^1, \ldots, R^a$ exist and they are disjoint by definition. Then, the thesis follows by setting $R = \tilde{R} \cup \{R^1 \cup \ldots \cup R^a\}$, and $\lambda_j^q = |R_q^g|$ for all $j \in \{R^1 \cup \ldots \cup R^a\}$. \( \square \)

Given an SMKP instance, let $B_1, \ldots, B_t$ be a partition of $N$ into blocks. Under the block partition $B_1, \ldots, B_t$, an instance of the new problem M-SP is obtained from an instance of SMKP by modifying the item and the knapsack sets as explained in the following. All the item types belonging to a block $B_w = \{w_1, \ldots, w_k\}$ are replaced by items of size $s_{w_1}$ and profit $v_{w_1}$, for $w = 1, \ldots, t$. More precisely, an item $w \in B_w$, of size $s_{w_1}$, upper bound $b_{w_1}$, value $v_{w_1}$ is replaced by $b_{w_1} s_{w_1}/s_{w_1}$ items of type $w$ of size $f_w = s_{w_1}$ and value $p_w = v_{w_1}$. Such new items are denoted as items of type $w$. Let $N'$ be the set of all the new items produced by applying the above replacing procedure to all the blocks. Note that, $N'$ can be partitioned into sets $T_1, \ldots, T_l$, where $T_q$, $1 \leq q \leq l$, contains all the new items obtained by items in $N$ of size $d_q$. Items in $N'$ of different sizes may belong to the same set $T_q$, since they may belong to different blocks. In what follows, let $\tilde{b}_{w,q}$ be the total number of
items of type \( w \) belonging to \( T_q \), in M-SP, and let \( \tilde{b}_w = \sum_{q=1}^{l} \tilde{b}_{w,q} \). By definition, the following observation holds.

**Observation 3.2**  The number of items of type \( w \) belonging to a set \( T_q \), \( \tilde{b}_{w,q} \), is multiple of \( \frac{d_w}{f_w} \) and is equal to

\[
\tilde{b}_{w,q} = \sum_{w \in B_w : f(w) = d_q} \frac{b_w d_q}{f_w}.
\]  

(10)

Furthermore, in M-SP, the \( m \) knapsacks are replaced by a single knapsack composed of \( l \) parts. To each part \( h \) a capacity \( \bar{c}_h = \sum_{i=1}^{m} r_i^h \) is associated, for \( h = 1, \ldots, l \). Items of the set \( T_q \) can be only assigned to the parts \( q, q+1, \ldots, l \).

M-SP is the problem of finding an assignment of the items in \( N' \) that maximizes the total profit of the assigned items and that satisfies given conditions, described in the following. Let \( y_{w,q}^h \) be an integer variable denoting the number of items of type \( w \), belonging to the set \( T_q \) and assigned to the part \( h \) of the knapsack, for \( w = 1, \ldots, t \), \( q = 1, \ldots, l \) and \( h = 1, \ldots, l \). A solution \( y \) is feasible for M-SP if the following constraints are satisfied:

\[
\sum_{w=1}^{t} \sum_{q=1}^{h} \left\lfloor f_w y_{w,q}^h / d_q \right\rfloor d_q \leq \bar{c}_h \text{ for } h = 1, \ldots, l \tag{11}
\]

\[
\sum_{h=q}^{l} \left\lfloor f_w y_{w,q}^h / d_q \right\rfloor \leq \sum_{w \in B_w : f(w) = d_q} b_w = f_w \tilde{b}_{w,q} / d_q \text{ for } w = 1, \ldots, t \text{ and } q = 1, \ldots, l \tag{12}
\]

where the last equality in (12) follows from (10).

In the above constraints, \( \left\lfloor f_w y_{w,q}^h / d_q \right\rfloor d_q \) represents the “occupancy” of the items of type \( w \) in \( T_q \) assigned to the part \( h \) of the knapsack, in terms of chunks of size \( d_q \). While \( \left\lfloor f_w y_{w,q}^h / d_q \right\rfloor \) is the minimum number of chunks of size at most \( d_q \) that can be obtained using the items of type \( w \) in \( T_q \), that are assigned to the part \( h \) of the knapsack. Hence, the first set of constraints state that the total “occupancy” in the part \( h \) cannot exceed the capacity \( \bar{c}_h \). Observe that, only items in \( T_q \) with \( q \leq h \) can be assigned to the part \( h \). constraints (12) limit the total number of chunks of size at most \( d_q \) that can be obtained using the items of type \( w \) in \( T_q \).

A formulation for M-SP is reported in the following.

\[
\max \sum_{h=1}^{l} \sum_{q=1}^{h} \sum_{w=1}^{t} p_w y_{w,q}^h
\]

subject to

constraints (11) and (12) \( y_{w,q}^h \in \mathbb{Z}^+ \cup \{0\} \) for \( w = 1, \ldots, t \), \( q = 1, \ldots, l \) and \( h = q, \ldots, l \).
Note that, in M-SP, the item sizes are divisible, too. Pochet and Weismantel [3] introduced a special partition of \( N \) into blocks, called \textit{maximal block partition} and showed its uniqueness. In this partition, a block contains only items \( j \in N \) with the same \textit{gain per unit} \( \frac{w}{s_j} \), and has the following property: given two blocks \( B_1 \) and \( B_2 \) containing items with the same gain per unit, the set \( B_1 \cup B_2 \) is not a block.

In the rest of the paper, we defines M-SP on the maximal block partition and assume the items in M-SP ordered in such a way that: \( f_1 \leq \ldots \leq f_t \) and that \( p_w \geq p_{w+1} \), if \( f_w = f_{w+1} \). By the procedure for the construction of the maximal blocks, the following relation holds [3]:

\[
f_w > \sum_{u \in \{1, \ldots, w-1\}} \frac{f_u}{f_u} \sum_{q=1}^{l_u} \tilde{b}_{u,q} = \sum_{u \in \{1, \ldots, w-1\}} \frac{f_u}{f_u} \tilde{b_u} \quad \text{for } w = 1, \ldots, t.
\] (13)

A notation similar to that introduced for SMKP is now introduced for M-SP. Given a feasible solution \( y \) of M-SP and two integers \( 1 \leq h_1 \leq h_2 \leq l \), let \( y^{h_1, h_2} \) be the "partial" solution of \( y \) related to the parts of the knapsack \( \{h_1, \ldots, h_2\} \), i.e., \( y^{h_1, h_2} \) is the vector containing the components \( y_{w,q}^{h_1, h_2} \) for \( w = 1, \ldots, t \), \( h = h_1, \ldots, h_2 \) and \( q = 1, \ldots, h_2 \). If \( h_1 > h_2 \), then \( y^{h_1, h_2} \) does not assign any item. To simplify the notation, if \( h_1 = h_2 \), \( y^{h_1} \) is used instead of \( y^{h_1, h_2} \). Let \( S(y) \) and \( S(y^{h_1, h_2}) \) be the set of items assigned in \( y \) and \( y^{h_1, h_2} \) respectively.

Given a block \( w \) and a set \( T_q \), let \( y_{w,q}^{h_1, h_2} \) be the number of items of type \( w \) belonging to the set \( T_q \) assigned, in \( y \), to the parts \( \{h_1, \ldots, h_2\} \) of the knapsack, and let \( S(y_{w,q}^{h_1, h_2}) \) be the corresponding set of items. Moreover, given a set of items \( A \subseteq N' \), let \( f(A) \) and \( p(A) \) be the total size (called in the following “total size”, too) and value, respectively, of the items in \( A \), i.e., \( f(A) = \sum_{w \in A} f_w |A_w| \) and \( p(A) = \sum_{w \in A} p_w |A_w| \), where \( |A_w| \) denotes the number of items of type \( w \) in \( A \). For the sake of simplicity, \( f(y^{h_1, h_2}) \) \( (p(y^{h_1, h_2})) \) is used instead of \( f(S(x^{h_1, h_2})) \) \( (p(S(x^{h_1, h_2}))) \) to denote the total size (the total profit) of the items assigned by \( y \) to the knapsacks parts \( \{h_1, \ldots, h_2\} \).

### 3.1 Correspondance between feasible solutions of SMKP and M-SP

Given an instance of M-SP and an integer \( h \), \( 1 \leq h \leq l \), we denote by M-SP(\( h \)) the “restricted” M-SP instance, obtained from the instance SMKP(\( h \)), as defined in [2]. Observe that, by Definition 3.1, if a set \( B \subseteq N \) is block in SMKP, then the set \( \{ j \in B : s_j \leq d_h \} \) is a block in SMKP(\( h \)), too. We show now that there is a one to many correspondence between feasible solutions of M-SP (M-SP(\( h \))) and feasible solutions of SMKP (SMKP(\( h \))).
Given a M-SP instance, corresponding to the maximal block partition of \( N \), let \( y \in \mathbb{Z}^{l \times 1 \times l} \) be a feasible solution. For each block \( B_w \), let \( \delta_w^{h} = f_w^{h/y_{w, q}} \) for \( q = 1, \ldots, l \) and \( h = q, \ldots, l \). By Proposition 5, a subset \( R_w \subseteq B_w \) and not negative integers \( \lambda_j^h \), for \( j \in R_w \), exist such that conditions (5) and (6) are satisfied.

A feasible solution of SMKP is built as follows. For each block \( B_w \), let \( \sum_{i=1}^{m} x_{i,j}^h = \lambda_j^h \), if \( j \in R_w \), for \( h = 1, \ldots, l \). By condition (5), it follows that \[ \sum_{h=g(j)}^{1} \sum_{i=1}^{m} x_{i,j} = \sum_{h=g(j)}^{1} \lambda_j^h \leq b_j, \] for each \( j \in B_w \) and \( w = 1, \ldots, t \) (i.e., \( x \) satisfies the second set of constraints in (1)). Let \( R = \bigcup_{w=1}^{t} R_w \). Since \( \delta_w^{h/y_{w, q}} = f_w^{h/y_{w, q}} \), and by conditions (6) and (11), we have:
\[
\sum_{w=1}^{t} \sum_{q=1}^{h} \left[ \frac{f_w^{h/y_{w, q}}}{d_q} \right] d_q = \sum_{w=1}^{t} \sum_{q=1}^{h} \left[ \frac{\delta_w^{h/y_{w, q}}}{d_q} \right] d_q = \sum_{q=1}^{h} \sum_{j \in R: g(j) \leq h} \lambda_j^h d_q = \sum_{j \in \Omega^h} \delta_j^h s_j \leq c_h = \sum_{i=1}^{m} r_i^h, \quad \text{for } h = 1, \ldots, l.
\]

Hence, since \( \sum_{i=1}^{m} x_{i,j}^h = \lambda_j^h \), the following inequality holds

\[
\sum_{j \in R: g(j) \leq h} \sum_{i=1}^{m} x_{i,j}^h s_j \leq c_h = \sum_{i=1}^{m} r_i^h \quad \text{for } h = 1, \ldots, l.
\]

(14)

Recall that the items in \( R \) that can be assigned to a part \( h \) must have sizes not greater than \( d_h \). By (14), since the item sizes are divisible and since \( d_h \) divides \( r_i^h \), for \( i = 1, \ldots, m \), it is always possible to partition the set \( \{ j \in R : g(j) \leq h \} \) into \( m \) subsets, say \( \Omega^h_1, \ldots, \Omega^h_m \), such that \( \sum_{j \in \Omega^h_i} s_j x_{i,j}^h \leq r_i^h \), for \( i = 1, \ldots, m \) (i.e., \( x \) satisfies the first set of constraints in (1)). Note that, several of such partitions may exist. Hence, \( x \) is feasible for SMKP. We show now that \( v(x) \geq p(y) \). In fact, since \[ \sum_{j \in R: g(j) = q} \lambda_j^h = \left[ \delta_w^{h/y_{w, q}}/d_q \right] = [f_w^{h/y_{w, q}}/d_q] \] (condition 6), then
\[
\sum_{j \in R: g(j) = q} \lambda_j^h \geq f_w^{h/y_{w, q}}/d_q, \quad \text{and} \quad \sum_{j \in R: g(j) \leq h} \lambda_j^h = \sum_{q=1}^{h} \sum_{j \in R: g(j) = q} \lambda_j^h \geq \sum_{q=1}^{h} f_w^{h/y_{w, q}}/d_q. \]

By the partition into maximal blocks, for each \( j \in B_w \) we have \( v_j / s_j = f_w^{h/y_{w, q}} / d_q \), \( p_w / f_w \), i.e., \( v_j = p_w d_q / f_w \). Hence, recalling that
\[ R = \bigcup_{w=1}^{t} R_w, \] we have \[ v(x) = \sum_{h=1}^{l} \sum_{j \in R: g(j) \leq h} v_j \sum_{i=1}^{m} x_{i,j}^h = \sum_{h=1}^{l} \sum_{j \in \Omega^h} \sum_{i=1}^{m} v_j \lambda_j^h \geq \sum_{i=1}^{m} \sum_{h=1}^{l} \sum_{q=1}^{h} v_j f_w^{h/y_{w, q}} / d_q = p(y).
\]

On the other hand, given a feasible solution of SMKP, \( x \in \mathbb{Z}^{n \times m \times l} \) and a partition into maximal blocks, we build a corresponding feasible solution of M-SP by setting \( y_{w, q} = x_{i,j} = \sum_{j \in B_w : g(j) = q} (d_q / f_w) \sum_{i=1}^{m} x_{i,j}^h \). First, observe that, since \( f_w^{h/y_{w, q}} / d_q = \sum_{j \in B_w : g(j) = q} (d_q / f_w) \sum_{i=1}^{m} x_{i,j}^h \), then \( f_w^{h/y_{w, q}} / d_q \) is integer.

The sum of the first set of constraints in (11) with respect to index \( i \) gives
\[
\sum_{j \in N: s_j \leq d_i} s_j x_{i,j} = \sum_{i=1}^{m} \sum_{q=1}^{h} \sum_{u=1}^{h} \sum_{w=1}^{t} \frac{f_w^{h/y_{w, q}}}{d_q} = \sum_{i=1}^{m} \sum_{q=1}^{h} \left[ \frac{f_w^{h/y_{w, q}}}{d_q} \right] d_q = \sum_{i=1}^{m} x_i^h = c_i \quad \text{(i.e., } y \text{ satisfies constraints (11))}.
\]

The sum of the second set of constraints in (11) with respect to the items \( j \) in a given block \( B_w \) gives
\[
\sum_{j \in B_w : g(j) = q} \sum_{h: h \geq q} \sum_{i=1}^{m} x_{i,j} = \sum_{h: h \geq q} \left[ \sum_{i=1}^{m} \frac{f_w^{h/y_{w, q}}}{d_q} \right] \leq \sum_{j \in B_w : g(j) = q} b_j, \quad \text{for } q = 1, \ldots, l \text{ and } w = 1, \ldots, t \quad \text{(i.e., } y \text{ satisfies constraints (12))}.
\]
Finally, recalling that by the partition into maximal blocks it follows that \( v_j = p_w d_{g(j)}/f_w \), then \( p_w y_{w,q}^h = \sum_{j \in B_{w,g(j) = q}} \sum_{i=1}^m v_j x_{i,j}^h \), i.e., \( p(y) = v(x) \)

It is straightforward to observe that, all the above arguments can be also used to state the correspondence between feasible solutions of SMKP\((h)\) and M-SP\((h)\), for \( h = 1, \ldots, l \). The two following propositions hold.

**Proposition 3.3** For a given knapsack’s part \( h \), a many to one correspondence exists between feasible solutions \( x^{1,h} \) of SMKP\((h)\) and feasible solutions \( y^{1,h} \) of M-SP\((h)\), and it holds that \( v(x^{1,h}) \geq p(y^{1,h}) \).

**Proposition 3.4** Given a feasible solution \( x^{1,h} \) of SMKP\((h)\), let \( y^{1,h} \) be the corresponding feasible solution of M-SP\((h)\). Then \( v(x^{1,h}) = p(y^{1,h}) \).

### 3.2 Valid inequalities for MSKP and M-SP

Suppose that the following inequality

\[
\sum_{q=1}^l \sum_{w=1}^t \nu_{w,q} \sum_{h=q}^l y_{w,q}^h \leq \nu_0.
\]

is valid for all feasible solutions of M-SP.

By Section 3.1 we have that if \( x \) is feasible for MSKP (formulated as in (1)) then the solution \( y \) in which \( y_{w,q}^h = \sum_{j \in B_{w,g(j) = q}} (d_q/f_w) \sum_{i=1}^m x_{i,j}^h \), for \( w = 1, \ldots, t \), \( q = 1, \ldots, l \) and \( h = q, \ldots, l \) is feasible for M-SP. Hence, by setting \( \mu_{j,q} = \nu_{w,q}(d_q/f_w) \) if item \( j \in B_w \) and \( s_j = d_q \), it follows that the inequality is valid for MSKP (formulated as in (1)). In fact

\[
\sum_{q=1}^l \sum_{w=1}^t \nu_{w,q} \sum_{h=q}^l (d_q/f_w) \sum_{i=1}^m x_{i,j}^h = \sum_{q=1}^l \sum_{w=1}^t \nu_{w,q} d_q/f_w \sum_{j \in B_{w,g(j) = q}} \sum_{h=q}^l \sum_{i=1}^m x_{i,j}^h = \sum_{q=1}^l \sum_{w=1}^t \sum_{h=q}^l \sum_{i=1}^m x_{i,j}^h \leq \nu_0.
\]

\[
(16)
\]

### 3.3 OPT and ordered solutions for M-SP

As for SMKP, the definition of OPT solution is now introduced for M-SP.

**Definition 3.2** Let \( y \) be a feasible solution for M-SP. The solution \( y \) has the OPT property, and \( y \) is called OPT solution, if for each feasible solution \( \bar{y} \) such that \( S(y) = S(\bar{y}) \), it holds that \( p(y^{1,h}) \geq p(\bar{y}^{1,h}) \), for \( h = 1, \ldots, l \).
Also in this case, observe that there are feasible solutions of M-SP that do not satisfy the OPT property, and that the items in a not OPT solutions can be always reallocated to the knapsack parts to get an OPT solutions.

Relations between optimal solutions of SMKP and M-SP are now established. By Propositions 3.3 and 3.4, Proposition 3.5 follows.

**Proposition 3.5** Let \( x \) and \( y \) be two optimal solutions for SMKP and M-SP, respectively. Then \( v(x) = p(y) \).

**Proposition 3.6** Let \( y \) be an optimal solution for M-SP that satisfies the OPT property. Then, for any objective function coefficients \( f_w \), with \( w = 1, \ldots, t \), \( d_q \) divides \( f_w y_{w,q}^h \), for \( q = 1, \ldots, l \), \( w = 1, \ldots, t \) and \( h = q, \ldots, l \).

*Proof.* By contradiction, let \( h \) be the first part, in \( y \), such that \( (f_w y_{w,q}^h \mod d_q) \neq 0 \) for given \( w \) and \( q \), with \( q \leq h \). By definition of \( h \), there exists a subset \( A \) of items of type \( w \) in \( T_q \), not assigned to parts \( q, \ldots, h-1 \) in \( y \), such that \( f(A) = \left[ \frac{f_w y_{w,q}^h}{d_q} \right] - f_w y_{w,q}^h > 0 \). Since \( y \) is feasible, by (11), it follows that it is feasible to assign to the part \( h \) in \( y \) the items in \( A \). Then, the new solution \( \bar{y}^{1,h} \) in which \( \bar{y}^{1,h-1} = y^{1,h-1}, y_{e,w}^h = y_{e,u}^h \) for \( e \neq w \) or \( u \neq q \) and \( y_{w,q}^h = y_{w,q}^h + f(A)/f_w \) is feasible, and we have \( p(\bar{y}^{1,h}) > p(y^{1,h}) \), contradicting the hypothesis. \( \square \)

**Lemma 3.7** An optimal solution \( x \) for SMKP satisfies the OPT property, if and only if the corresponding solution \( y \) of M-SP is optimal and satisfies the OPT property.

*Proof.* Let \( x \) be an optimal solution for SMKP satisfying the OPT property and let \( y \) be the correspondent M-SP solution. By construction, we have that \( v(x^{1,h}) = p(y^{1,h}) \), for \( h = 1, \ldots, l \). Since \( x \) satisfies the OPT property and by Proposition 3.3 it follows that \( y \) satisfies the OPT property, too.

On the other hand, let \( y \) be an optimal solution for M-SP that satisfies the OPT property. By Proposition 3.1 let \( x \) be a solution of SMKP corresponding to \( y \), in which, for each block \( B_w \), \( \sum_{i=1}^{m} x_{i,j}^h = \lambda_j^h = f_w y_{w,q}^h \), for \( h = 1, \ldots, l \) and \( j \in R_w \). Recall that, by Proposition 3.6 \( d_q \) divides \( f_w y_{w,q}^h \), and, by the partition into maximal blocks \( p_w/f_w = v_j/s_j \) for all \( j \in B_w \). Then by definition of \( x \), \( \sum_{j \in R_w : g(j) = q} \sum_{i=1}^{m} x_{i,j}^h = \sum_{j \in R_w : g(j) = q} \lambda_j^h = f_w y_{w,q}^h/d_q \), i.e., \( y_{w,q}^h = \sum_{j \in R_w : g(j) = q} \sum_{i=1}^{m} x_{i,j}^h / f_w \). Since, \( p_w = f_w v_j/s_j \), then \( p_w y_{w,q}^h = \sum_{j \in R_w : g(j) = q} \sum_{i=1}^{m} v_j x_{i,j}^h \), i.e., \( p(y^{1,h}) = v(x^{1,h}) \) for all \( h \). Observe that, \( x^{1,h} \) is optimal for SMKP(\( h \)) for all \( h \). Otherwise, if a solution \( \tilde{x}^{1,h} \)
such that \( v(x^{1,h}) < v(\bar{x}^{1,h}) \) exists, letting \( \bar{y}^{1,h} \) be the solution corresponding to \( \bar{x}^{1,h} \), by Proposition 3.4 we have \( p(y^{1,h}) = v(x^{1,h}) < v(\bar{x}^{1,h}) = p(\bar{y}^{1,h}) \). Contradicting the hypothesis on \( y \).

\[ \square \]

By Lemmas 2.6 and 3.7, Lemma 3.8 follows.

**Lemma 3.8** Let \( y \) be an optimal solution for M-SP satisfying the OPT property. Then \( y^{1,h} \) is optimal for \( M-SP(h) \), for \( h = 1, \ldots, l \).

In the following, as for SMKP, the definition of ordered solution for M-SP is introduced.

**Definition 3.3** Given a feasible solution \( y \) for M-SP, \( y \) is called ordered solution if, for a given knapsack part \( h = 1, \ldots, l \) and a positive index \( q \leq h \), a set of items \( \Gamma \subseteq \{ T_1 \cup \ldots \cup T_{q-1} \} \) assigned to the part \( h \) in \( y \) with size \( f(\Gamma) = d_q \) exists, only if a set of items \( \Gamma' \) belonging to the set \( T_q \), with total size \( f(\Gamma') = f(\Gamma) = d_q \) and total value \( p(\Gamma') = p(\Gamma) \), either assigned to a part bigger than \( h \) or not assigned in \( y \) does not exist.

Given a feasible solution \( y \) for M-SP, it is always possible to build an ordered solution \( \bar{y} \) such that \( p(\bar{y}^{1,h}) = p(y^{1,h}) \) and \( f(\bar{y}^{1,h}) = f(y^{1,h}) \), for \( h = 1, \ldots, l \), as showed in the following. In fact, let us suppose that a set \( \Gamma \subseteq \{ T_1 \cup \ldots \cup T_{q-1} \} \) with total size \( f(\Gamma) = d_q \) and total value \( p(\Gamma) \), is assigned to a part \( h \) in \( y \), and that a set of items \( \Gamma' \subseteq T_q \) with total size \( f(\Gamma') = f(\Gamma) = d_q \) and total value \( p(\Gamma') = p(\Gamma) \), exists, either (i) assigned to a part bigger than \( h \) or (ii) not assigned in \( y \). In case (i), let \( \bar{y} \) be the feasible solution obtained by swapping \( \Gamma \) and \( \Gamma' \) in \( y \), while in case (ii) let \( \bar{y} \) be the feasible solution obtained by assigning \( \Gamma' \) in place of \( \Gamma \). If \( \bar{x} \) contains a new set \( \Gamma \) as defined above, then the above argument can repeated, otherwise, \( \bar{x} \) is an ordered solution.

Proposition 3.9 directly follows by Proposition 3.6 and by the correspondence between feasible solutions of SMKP and M-SP (stated in Section 3.1).

**Proposition 3.9** If \( x \) is an optimal ordered OPT solution for SMKP then the corresponding solution \( y \) of M-SP is an optimal ordered OPT solution, too, and vice versa.

Let \( MP^{OO} \) be the set containing the OPT and ordered solutions of M-SP, and let \( MO^{OO} \) be the set of the optimal solutions of M-SP in \( MP^{OO} \).

M-SP can be formulated as: \( \max \{ \sum_{h=1}^{l} \sum_{q=1}^{h} \sum_{w=1}^{l} p_w y_{w,q}^{h} : y \in MP^{OO} \} \).
4 Computing the optimal solutions of M-SP and MSKP

In this section, we describe an inductive procedure for decomposing and enumerating the optimal solutions in \(MP_{OO}\), i.e., the solutions in \(MO_{OO}\). At this aim, the following notation is introduced.

Given a positive integer \(b \leq l\) and an item type \(k\), with \(1 \leq k \leq t\), let \(N(k)\) be the set of items of types \(1, \ldots, k\) belonging to the sets \(T_1, \ldots, T_b\). Let \(\delta_h = \min\{d_h, d_b\}\), for \(h = 1, \ldots, l\), and let \(F_h \leq \bar{c}_h\), for \(h = 1, \ldots, l\), be positive integers such that \(\delta_h\) divides \(F_h\). We denote by \(MP(k, b, \mathbf{F})\) the set of points \(y\) satisfying the following conditions:

\[
\sum_{w=1}^{k} \sum_{q=1}^{\min\{h, b\}} \left\lceil \frac{f_{w}y_{wq}^{h}}{d_q} \right\rceil d_q \leq F_h \text{ for } h = 1, \ldots, l \tag{17}
\]

\[
\sum_{h=q}^{l} \left\lceil \frac{f_{w}y_{wq}^{h}}{d_q} \right\rceil \leq \frac{f_{w}\tilde{b}_{wq}}{d_q} \text{ for } w = 1, \ldots, k \text{ and } q = 1, \ldots, b \tag{18}
\]

\[
y_{wq}^{h} \in \mathbb{Z}^+ \cup \{0\} \text{ for } w = 1, \ldots, k, q = 1, \ldots, b \text{ and } h = q, \ldots, l. \tag{19}
\]

Furthermore, we denote by \(MP_{OO}(k, b, \mathbf{F})\) the set of the points in \(MP(k, b, \mathbf{F})\) that are OPT and ordered solutions. Let \(MO_{OO}(k, b, \mathbf{F})\) be the set of the optimal solutions in \(MP_{OO}(k, b, \mathbf{F})\).

We denote by \(MS-P(k, b, \mathbf{F})\) the integer program

\[
\max \{\sum_{h=1}^{l} \sum_{q=1}^{\min\{h, b\}} \sum_{w=1}^{l} p_{w}y_{wq}^{h} : \ y \in MP_{OO}(k, b, \mathbf{F})\} \tag{20}
\]

Let \(T_1', \ldots, T_b'\) be the sets containing the items in \(N(k)\) belonging to the sets \(T_1, \ldots, T_b\), respectively, and let \(T_{b+1}' = T_{b+2}' = \ldots = T_l' = \emptyset\). W.l.o.g., in the rest of the paper we assume that \(N(k)\) contains at least an item of type \(k\). A consequence of this fact is that \(f_k\) divides \(d_b\).

In the following lemma we show that, if \(b = 1\), \(MO_{OO}(k, b, \mathbf{F})\) contains a single point.

**Lemma 4.1** \(MO_{OO}(k, 1, \mathbf{F})\) contains a single solution, \(\hat{y}\), in which

\[
\sum_{h=1}^{l} \hat{y}_{j,1}^{h} = \min\{\sum_{h=1}^{l} F_h - \sum_{w=1}^{j-1} \sum_{h=1}^{l} \hat{y}_{w1,1}^{h}, \tilde{b}_{j,1}\} \text{ for } j = 1, \ldots, k, \tag{21}
\]

and

\[
\hat{y}_{j,1}^{h} = \min\{F_h - \sum_{w=1}^{j-1} \hat{y}_{w1,1}^{h}, \tilde{b}_{j,1} - \sum_{p=1}^{h-1} \hat{y}_{p1,1}^{h}\} \text{ for } j = 1, \ldots, k, \text{ and } h = 1, \ldots, l. \tag{22}
\]
Proof. Observe that, in M-SP\((k, 1, \mathbf{F})\), all items have the same size \(\delta_1 = d_1 = f_1 = 1\), since they belong to the set \(T^1\), and can be assigned to all the parts of the knapsack. Hence, the \(l\) knapsack parts can be replaced by a single part of capacity \(\sum_{h=1}^{l} F_h\). By the ordering of the item types proposed in Section 3 since all items have size 1, we have \(p_k < p_{k-1} < \ldots < p_1\). Hence, an optimal solution \(\hat{y}\) must assign the following number of items of type 1:

\[
\sum_{h=1}^{l} \hat{y}^h_{1,1} = \min \left\{ \sum_{h=1}^{l} F_h; \tilde{b}_{1,1} \right\}.
\]

Moreover since \(\hat{y}\) satisfies the OPT property, it must be \(\hat{y}_{1,1}^1 = \min\{F_1; \tilde{b}_{1,1}\}\), \(\hat{y}_{2,1}^1 = \min\{F_2; \tilde{b}_{1,1} - \hat{y}_{1,1}^1\}\), \ldots, and finally \(\hat{y}_{1,1}^l = \min\{F_l; \tilde{b}_{1,1} - \sum_{h=1}^{l-1} \hat{y}_{h,1}^1\}\).

Let us consider now the items of type 2. Applying similar arguments to those used for items of type 1, we have

\[
\sum_{h=1}^{l} \hat{y}^h_{2,1} = \min \left\{ \sum_{h=1}^{l} F_h - \sum_{h=1}^{l-1} \hat{y}^h_{1,1}; \tilde{b}_{2,1} \right\}.
\]

Also in this case, since \(\hat{y}\) satisfies the OPT property, the unique values attained by \(\hat{y}_{2,1}^1, \hat{y}_{2,1}^2, \ldots, \hat{y}_{2,1}^l\) can be easily derived, i.e., \(\hat{y}_{2,1}^1 = \min\{F_1 - \hat{y}_{1,1}^1; \tilde{b}_{2,1}\}\), \(\hat{y}_{2,1}^2 = \min\{F_2 - \hat{y}_{1,1}^1; \tilde{b}_{2,1} - \hat{y}_{2,1}^1\}\), \ldots, \(\hat{y}_{2,1}^l = \min\{F_l; \tilde{b}_{2,1} - \sum_{h=1}^{l-1} \hat{y}_{h,1}^1\}\).

In general we have

\[
\sum_{h=1}^{l} \hat{y}^h_{j,1} = \min \left\{ \sum_{h=1}^{l} F_h - \sum_{w=1}^{j-1} \sum_{h=1}^{l} \hat{y}^h_{w,1}; \tilde{b}_{j,1} \right\} \text{ for } j = 1, \ldots, k,
\]

and the unique values attained by \(\hat{y}_{j,1}^1, \hat{y}_{j,1}^2, \ldots, \hat{y}_{j,1}^l\) are \(\min\{F_1 - \sum_{w=1}^{j-1} \hat{y}_{w,1}^1; \tilde{b}_{j,1}\}\), \(\min\{F_2 - \sum_{w=1}^{j-1} \hat{y}_{w,1}^2; \tilde{b}_{j,1} - \hat{y}_{1,1}^1\}\), \ldots, \(\min\{F_l - \sum_{w=1}^{j-1} \hat{y}_{w,1}^l; \tilde{b}_{j,1} - \sum_{h=1}^{l-1} \hat{y}_{h,1}^1\}\), respectively. \(\square\)

In what follows a characterization of the optimal solutions in \(\text{MP}^{OO}(k, b, \mathbf{F})\) when \(b > 1\) is given. Given an item type \(j\), let \(H_j\) be the set of items in MS-P\((k, b, \mathbf{F})\) having unit gain value strictly bigger than \(\frac{p_j}{f_j}\), for \(j = 1, \ldots, k\), i.e.,

\[
H_j = \{ t \in N(k) : \frac{p_t}{f_t} > \frac{p_j}{f_j} \}.
\]

Given a part \(g = 1, \ldots, l\) and an item type \(j = 1, \ldots, k\), let \(\tilde{H}^g_j\) be the subset of items of \(H_j\) defined as follows

\[
\tilde{H}^g_j = \{ H_j \setminus \{ u \in T^g : u > j \} \} \cap (T^1 \cup T^2 \ldots \cup T^l).
\]
Note that, since \( k \) is the biggest item type in MS-P\((k, b, \mathbf{F})\), we have

\[
\bar{H}_k^g = H_k \cap (T_1^g \cup T_2^g \ldots \cup T_g^g).
\]  

(28)

In what follows, we introduce and (recursively) define the sets \( H_j^g \) for \( j = k \) and \( g = 1, \ldots, l \). We define \( H_1^1 = \bar{H}_1^1 \). Let \( v^1 = \min\{\lfloor f(\bar{H}_1^1)/\delta_1 \rfloor \delta_1; F_1\} \) and let \( H_1^1(min) \) be a subset of \( H_1^1 \) of total size \( v^1 \) (such a set exists, since all items in \( \bar{H}_1^1 \) have size \( d_1 = \delta_1 = 1 \)). Moreover, let \( H_2^1 = \bar{H}_2^1 \setminus H_1^1(min) \) and let \( H_2^1(min) \) be a subset of \( H_2^1 \) of total size \( v^2 = \min\{\lfloor f(H_2^1)/\delta_2 \rfloor \delta_2; F_2\} \). By Proposition 2.1 since \( \delta_2 \) divides \( v^2 \) and \( F_2 \), and since \( H_2^1 \) contains items with size that are divisible and not bigger than \( \delta_2 \), the set \( H_2^1(min) \) always exists.

In general, let \( H_j^g \) and \( H_k^g \), for \( g = 1, \ldots, l \), be recursively defined as

\[
H_j^g = H_j^g \setminus \bigcup_{h=1}^{g-1} H_h^j(min)
\]

(29)

in which \( H_h^j(min) \), for \( h = 1, \ldots, g - 1 \), is a subset of \( H_h^j \) of total size \( v^h \), where

\[
v^h = \min\{\lfloor f(H_h^j)/\delta_h \rfloor \delta_h; F_h\}.
\]

(30)

By Proposition 2.1 it easy to see that the sets \( H_h^j(min) \), for \( g = 3, \ldots, l - 1 \), always exist, too.

**Definition 4.1** Given a solution \( \hat{y} \) in \( MP^O(k, b, \mathbf{F}) \) and a part \( \hat{g} \in \{1, \ldots, b\} \) we call the set of items in

\[
\bar{H}_k^g \setminus S(\hat{y}^{1: \hat{g} - 1})
\]

as the set of items of \( H_k \) available to be assigned to the part \( \hat{g} \) in \( \hat{y} \).

Two key Lemmas are now introduced useful for characterizing the optimal solutions in \( MP^O(k, b, \mathbf{F}) \).

**Lemma 4.2** Given an optimal solution \( \hat{y} \) in \( MP^O(k, b, \mathbf{F}) \), \( b > 1 \), for each \( g = 1, \ldots, l \), the total size of the items of \( H_k \) available to be assigned to the part \( g \) in \( \hat{y} \) (i.e., \( f(\bar{H}_k^g \setminus S(\hat{y}^{1:g - 1})) \)) is at least \( \lfloor f(H_k^g)/\delta_g \rfloor \delta_g \).

**Proof.**

The quantity \( f(\bar{H}_k^g \setminus S(\hat{y}_k^{1:g - 1})) \) is minimum when the total size of the items in \( H_k \) assigned to the parts \( 1, \ldots, g - 1 \) in \( \hat{y} \) is maximum. The minimum value of \( f(\bar{H}_k^g \setminus S(\hat{y}_k^{1:g - 1})) \) can be determined by an iterative procedure described in the following and proved later by induction. Starting from the first part of the knapsack,
the procedure assigns items of \( H_k \) with a total size as bigger as possible to each part. Hence, recalling that all items in \( \bar{H}^1_k \) have size \( \delta_1 = 1 \), the maximum total size of items in \( \bar{H}^1_k \) that can be assigned to the part 1 is

\[
\bar{v}^1 = \min \{ f(\bar{H}^1_k); F_1 \}. \tag{31}
\]

Hence, we must have \( f(\bar{H}^1_k \cap S(\bar{y}^1_k)) \leq \bar{v}^1 \). Let \( H^1_k(max) \) be a set of items of \( H^1_k \) of total size \( \bar{v}^1 \). Note that, since \( d_1 = 1 \) such a set exists, and, recalling the definition of \( H^1_k(min) \), we have \( f(H^1_k(max)) = f(H^1_k(min)) \).

We define \( H^2_k = \bar{H}^2_k \setminus H^1_k(max) \). In general, for a given knapsack part \( h \), we define

\[
H^h_k = \bar{H}^h_k \setminus \bigcup_{i=1}^{h-1} H^i_k(max) \tag{32}
\]

where \( H^h_k(max) \) is a subset of \( H^h_k \) of total size \( \bar{v}^h = \min \{ F_h, f(H^h_k) \} \). Observe that, a set of total size \( \bar{v}^h \) always exists by Proposition 2.1 such a fact holds, since the biggest size of the items in \( H^h_k \) is \( \delta_h \), the item sizes are divisible, and \( \delta_h \) divides \( F_h \). Then, the procedure assigns to the part \( h \) the set \( H^h_k(max) \), for \( h = 1, \ldots, g - 1 \).

We now show, by induction on the knapsack parts, that the above procedure is correct. Namely, that a set of the items of \( H_k \) available to be assigned to the part \( g \) with minimum total size in \( \bar{y} \) is

\[
H^{g^h}_k = \bar{H}^{g^h}_k \setminus \bigcup_{h=1}^{g-1} H^h_k(max) \tag{33}
\]

i.e.,

\[
f(H^{g^h}_k) \leq f(\bar{H}^{g^h}_k \setminus S(\bar{y}^{1,g-1}_k)). \tag{34}
\]

When \( g = 1 \), relation (34) follows by (31) and by definition of \( H^1_k(max) \). By induction, suppose that

\[
f(H^{g^h-1}_k) \leq f(\bar{H}^{g^h-1}_k \setminus S(\bar{y}^{1,g-2}_k)), \tag{35}
\]

and let \( \bar{y} \) be a feasible solution such that \( f(\bar{H}^{g^h-1}_k \setminus S(\bar{y}^{1,g-2}_k)) = f(H^{g^h-1}_k) \). In \( \bar{y} \) and \( \bar{y} \), the maximum total size of the items that can be assigned to the part \( g - 1 \) respectively are \( \min \{ F_{g-1}; f(H^{g^h-1}_k) \} = f(H^{g^h-1}_k(max)) \) and \( \min \{ F_{g-1}; f(\bar{H}^{g^h-1}_k \setminus S(\bar{y}^{1,g-2}_k)) \} \). Recall that, by definition of \( \bar{H}^{g}_k \) and since \( k \) is the biggest item type, given a feasible solution \( y \), it holds that \( \bar{H}^{g}_k = \bar{H}^{g^h-1}_k \cup (H_k \cap T^g) \).

Two cases can be distinguished: a) \( F_{g-1} \leq f(H^{g^h-1}_k) \); b) \( F_{g-1} > f(H^{g^h-1}_k) \). In Case a), the maximum total size of the items that can be assigned to the part \( g \) in \( \bar{y} \) and \( \bar{y} \), respectively, are
\[ f(\tilde{H}_k^{g-1} \setminus S(\tilde{y}_k^{1,g-2})) + f(H_k \cap T^g) - F_{g-1} = f(\tilde{H}_k^{g-1}) + f(H_k \cap T^g) - f(H_k^{g-1}(max)) = f(H_k^g) \]

and \( f(\tilde{H}_k^{g-1} \setminus S(\tilde{y}_k^{1,g-2})) + f(H_k \cap T^g) - F_{g-1} \). And the thesis easily follows by (35). In Case b), if \( F_{g-1} \geq f(\tilde{H}_k^{g-1} \setminus S(\tilde{y}_k^{1,g-2})) \), then the minimum total size of the items that remains to assign to the part \( g \) in \( \tilde{y} \) and \( \tilde{y} \) is \( f(H_k \cap T^g) = f(H_k^g) \). (Recall, that, by definition, \( H_k \cap T^g = \emptyset \) if \( g > b \).) Hence, the (35) follows. Otherwise, if \( f(\tilde{H}_k^{g-1} \setminus S(\tilde{y}_k^{1,g-2})) - F_{g-1} = \alpha > 0 \), we have \( f(H_k^g) = f(H_k \cap T^g) \) and \( f(\tilde{H}_k^{g-1} \setminus S(\tilde{y}_k^{1,g-2})) \geq f(H_k \cap T^g) + \alpha \), and relation (34) holds.

By the definition of \( \mathcal{H}_k^g \) (see (32)), by (34) and by definition of \( \mathcal{H}_k^g \) (see (29) and (30)) we have

\[ f(\mathcal{H}_k^g) \leq f(\mathcal{H}_k^g). \] (36)

The thesis of the lemma is proved by showing that

\[ f(\mathcal{H}_k^g) \geq \left| f(\mathcal{H}_k^g) / \delta_g \right| \delta_g = \left| f(\mathcal{H}_k^g) / \delta_g \right| \delta_g, \] (37)

implying that, by (34), \( f(\tilde{H}_k^g \setminus S(\tilde{y}_k^{1,g-1})) \geq \left| f(\mathcal{H}_k^g) / \delta_g \right| \delta_g \). Relation (37) is proved by induction. When \( g = 1 \), recalling the definitions of \( \mathcal{H}_k^g \) (29) and \( \mathcal{H}_k^g \) (32), the thesis trivially holds since \( \mathcal{H}_k^0(max) = \mathcal{H}_k^0(min) = \emptyset \).

Assume that Relation (37) holds for \( g - 1 \) and show it for \( g \). By induction, we have

\[ \left| f(\mathcal{H}_k^{g-1}) / \delta_{g-1} \right| \delta_{g-1} = \left| f(\mathcal{H}_k^{g-1}) / \delta_{g-1} \right| \delta_{g-1} \] (38)

and hence \( f(\mathcal{H}_k^{g-1}) + \delta_{g-1} > f(\mathcal{H}_k^{g-1}) \). Recall that, by definition, \( \tilde{H}_k^g = \tilde{H}_k^{g-1} \cup (T'_g \cap H_k) \) where, by definition of M-SP, \( \delta_g \) divides \( f(T'_g \cap H_k) \). Hence, by definition,

\[ \mathcal{H}_k^g = \tilde{H}_k^g \setminus \bigcup_{h=1}^{g-1} \mathcal{H}_k^h(max) = \tilde{H}_k^{g-1} \cup (T'_g \cap H_k) \setminus \bigcup_{h=1}^{g-1} \mathcal{H}_k^h(max) = \mathcal{H}_k^{g-1} \cup (T'_g \cap H_k) \setminus \mathcal{H}_k^{g-1}(max) \] (39)

and

\[ \mathcal{H}_k^g = \tilde{H}_k^g \setminus \bigcup_{h=1}^{g-1} \mathcal{H}_k^h(min) = \tilde{H}_k^{g-1} \cup (T'_g \cap H_k) \setminus \bigcup_{h=1}^{g-1} \mathcal{H}_k^h(min) = \mathcal{H}_k^{g-1} \cup (T'_g \cap H_k) \setminus \mathcal{H}_k^{g-1}(min) \] (40)

where the last equality of (39) and (40) respectively follow since the sets \( \mathcal{H}_k^h(max) \) and \( \mathcal{H}_k^h(min) \), for \( h = 1, \ldots, g - 2 \), are disjoint subsets of \( \tilde{H}_k^{g-1} \), and since, by definition, \( \mathcal{H}_k^{g-1} = \tilde{H}_k^{g-1} \setminus \bigcup_{h=1}^{g-2} \mathcal{H}_k^h(max) \) and \( \mathcal{H}_k^{g-1} = \tilde{H}_k^{g-1} \setminus \bigcup_{h=1}^{g-2} \mathcal{H}_k^h(min) \). From (39) and (40) we respectively have

\[ f(\mathcal{H}_k^g) = f(\mathcal{H}_k^{g-1}) + f(T'_g \cap H_k) - f(\mathcal{H}_k^{g-1}(max)) \] (41)
and

\[ f(\mathcal{H}_k^g) = f(\mathcal{H}_k^{g-1}) + f(T_g \cap H_k) - f(\mathcal{H}_k^{g-1}(\min)). \]  \hspace{1cm} (42)

Two cases are possible: (1) \( f(\mathcal{H}_k^{g-1}(\min)) = \min\{f(\mathcal{H}_k^{g-1}), F_{g-1}\} = F_{g-1} \); and (2) \( f(\mathcal{H}_k^{g-1}(\max)) = \min\{f(\mathcal{H}_k^{g-1}), F_{g-1}\} = f(\mathcal{H}_k^{g-1}). \)

In Case (1), since by induction \( |f(\mathcal{H}_k^{g-1})|/\delta_{g-1} \delta_{g-1} = |f(\mathcal{H}_k^{g-1})|/\delta_{g-1} \delta_{g-1} \) (see (38)) and since \( \delta_{g-1} \) divides \( F_{g-1} \), we have \( \min\{|f(\mathcal{H}_k^{g-1})|/\delta_{g-1} |\delta_{g-1}, F_{g-1}| = F_{g-1}, \) too, i.e., \( f(\mathcal{H}_k^{g-1}(\min)) = f(\mathcal{H}_k^{g-1}(\max)) = F_{g-1}. \) Hence, since \( \delta_{g-1} \) divides \( f(T_g \cap H_k) - f(\mathcal{H}_k^{g-1}(\min)) = f(T_g \cap H_k) - f(\mathcal{H}_k^{g-1}(\max)) \) and \( \delta_g \), and from (11), we have:

\[
\begin{align*}
\delta_g &= \left\lfloor \frac{f(\mathcal{H}_k^{g-1}) + f(T_g \cap H_k) - f(\mathcal{H}_k^{g-1}(\min))}{\delta_{g-1}} \right\rfloor \delta_{g-1} \\
&\geq \left\lfloor \frac{f(\mathcal{H}_k^{g-1}) + f(T_g \cap H_k) - f(\mathcal{H}_k^{g-1})(\min)}{\delta_g} \right\rfloor \delta_g
\end{align*}
\]

showing the thesis.

In Case (2), relation (41) becomes \( f(\mathcal{H}_k^{g-1}) = f(T_g \cap H_k). \) By definition of \( \mathcal{H}_k^{g-1}(\min) \) and by (38) we have

\[
\begin{align*}
f(\mathcal{H}_k^{g-1}(\min)) &= \min\{\left\lfloor \frac{f(\mathcal{H}_k^{g-1})}{\delta_{g-1}} \right\rfloor \delta_{g-1}, F_{g-1}\} = \min\{\left\lfloor \frac{f(\mathcal{H}_k^{g-1})}{\delta_{g-1}} \right\rfloor \delta_{g-1}, F_{g-1}\} = \left\lfloor \frac{f(\mathcal{H}_k^{g-1})}{\delta_{g-1}} \right\rfloor \delta_{g-1}.
\end{align*}
\]

Hence, Equation (42) becomes: \( f(\mathcal{H}_k^{g-1}) = f(\mathcal{H}_k^{g-1}) + f(T_g \cap H_k) - \left\lfloor \frac{f(\mathcal{H}_k^{g-1})}{\delta_{g-1}} \right\rfloor \delta_{g-1} \). Since \( f(\mathcal{H}_k^{g-1}) - \left\lfloor \frac{f(\mathcal{H}_k^{g-1})}{\delta_{g-1}} \right\rfloor \delta_{g-1} < \delta_{g-1} \) and \( \delta_g \geq \delta_{g-1} \), relation (37) holds, i.e., the thesis.

\[ \square \]

In Lemma 4.3 an upper bound on the total size of the items of \( H_k \) available to be assigned to the part \( h \) in an optimal solution \( \hat{y} \) is given.

**Lemma 4.3** Given an optimal solution \( \hat{y} \) in \( MP^O(k,b,F) \), with \( b > 1 \), the total size of the items of \( H_k \), available to be assigned to the part \( h \) in \( \hat{y} \) (i.e., \( f(\mathcal{H}_k^h) \setminus S(\hat{y}^{1:h-1})) \) is smaller than or equal to \( f(\mathcal{H}_k^h) + \delta_h \), for \( h = 1, \ldots, l. \)
Proof. The proof is by induction on the knapsack part \( h \). If \( h = 1 \) then \( S(\hat{y}^{1,0}) = \emptyset \), and \( \tilde{H}_k^1 \setminus S(\hat{y}^{1,0}) = \tilde{H}_k^1 \). Since, by definition, \( H_k^1 = \tilde{H}_k^1 \), the thesis trivially follows. Assume the thesis holds up to part \( h < l \), and let us consider the part \( h + 1 \). By induction, the total size of the items in the set \( \tilde{H}_k^h \setminus S(\hat{y}^{1,h-1}) \), in the following denoted as \( A_h \) (i.e., \( A_h = f(\tilde{H}_k^h \setminus S(\hat{y}^{1,h-1})) \)), is smaller than or equal to \( f(\mathcal{H}_k^h) + \delta_h \). Observe that, \( A_{h+1} = f(\tilde{H}_k^{h+1} \setminus S(\hat{y}^{1,h})) \) is maximum when \( A_h \) is maximum and \( S(\hat{y}) \cap \tilde{H}_k^h \) has minimum total size. By definition, \( \tilde{H}_k^{h+1} \setminus S(\hat{y}^{1,h}) = \tilde{H}_k^h \setminus (S(\hat{y}^{1,h-1}) \cup S(\hat{y})) \cup (H_k \cap T_{h+1}') \). (Recall that, by definition of \( T_h' \), \( H_k \cap T_{h+1}' = \emptyset \) if \( h > b \).) Observe that, \( S(\hat{y}^{1,h-1}) \) and \( S(\hat{y}) \) are disjoint sets, and \( \tilde{H}_k^h \setminus (S(\hat{y}^{1,h-1}) \cup S(\hat{y})) \) and \( (H_k \cap T_{h+1}') \) are disjoint sets, too. Hence, in general, we have

\[
f(\tilde{H}_k^{h+1} \setminus S(\hat{y}^{1,h})) = A_{h+1} = A_h - f(\tilde{H}_k^h \setminus S(\hat{y})) + f(H_k \cap T_{h+1}').
\] (43)

Observe that, if \( \tilde{H}_k^h = \emptyset \), then by definition \( A_h = 0, \tilde{H}_k^{h+1} = H_k \cap T_{h+1}' \) and \( f(\mathcal{H}_k^{h+1}) = f(H_k \cap T_{h+1}') \). Hence, by (43) and since \( \delta_{h+1} \neq 0 \), the Lemma trivially holds. In the following, let us suppose that \( \tilde{H}_k^h \neq \emptyset \).

Two cases can be considered: 1) \( F_h > [A_h/\delta_h]\delta_h \); 2) \( F_h \leq [A_h/\delta_h]\delta_h \).

Case 1. Since, by definition, \( \delta_h \) divides \( F_h \), then \( F_h \geq [A_h/\delta_h]\delta_h + \delta_h > A_h \). Let \( \gamma \) be the total size of the items of \( H_k \setminus S(\hat{y}^{1,h-1}) \) assigned by \( \hat{y} \) to \( h \), i.e., \( \gamma = f(\hat{y} \cap H_k) = \sum_{r \in H_k} f_r \hat{y}^h_r \). In the following we show that \( \gamma > [A_h/\delta_h]\delta_h - \delta_h \). By contradiction, let us suppose that \( \gamma \leq [A_h/\delta_h]\delta_h - \delta_h \), and let \( \Omega = \tilde{H}_k^h \setminus S(\hat{y}^{1,h}) \). Hence, \( f(\Omega) = A_h - \gamma \geq A_h - [A_h/\delta_h]\delta_h + \delta_h \geq \delta_h \).

Let \( \Psi = S(\hat{y}^h) \setminus H_k \), i.e., the set of items in \( N(k) \setminus H_k \) assigned to part \( h \) in \( \hat{y} \), and let \( \sigma = f(\Psi) = \sum_{r \in N(k) \setminus H_k} f_r \hat{y}^h_r \), i.e., the total size of the items in \( N(k) \setminus H_k \) assigned to the part \( h \) in \( \hat{y} \). If \( \sigma \leq F_h - A_h = F_h - \gamma - f(\Omega) \), then it is feasible to assign all items in \( \Omega \) to the part \( h \) in \( \hat{y} \). Namely, the new solution \( \tilde{y}^{1,h} \) assigns all items assigned in \( S(\hat{y}^{1,h}) \) to the parts \( 1, \ldots, h \), and all items in \( \tilde{H}_k^h \setminus S(\hat{y}^{1,h-1}) \) to the part \( h \) belongs to \( MP^{OO}(k, h, F) \) and has a strictly better objective function than \( \tilde{y}^{1,h} \) (since \( S(\hat{y}^{1,h}) \) is a subset strictly contained in \( S(\hat{y}^{1,h}) \)), contradicting Lemma 3.3.

Suppose now that \( f(\Psi) = \sigma > F_h - A_h \). Since \( \delta_h \) is the biggest size of the items that can be assigned to the part \( h \), the item sizes are divisible and \( \hat{y} \) is feasible (i.e., satisfies constraints (11)), \( \Psi \) can be partitioned into two subsets \( \Psi^1 \) and \( \Psi^2 \), such that: (i) \( f(\Psi^1) = F_h - [A_h/\delta_h]\delta_h \), (ii) \( f(\Psi^2) = \sigma - f(\Psi^1) > 0 \), and (iii)
constraints (11) are satisfied, i.e., \( \min\{h, b\} \leq \sum_{q=1}^{\min\{h, b\}} \left( \sum_{w \in \Psi \cap T_q} f_w y_w^q \right) = \sigma. \)

Observe that \( \delta_h \) divides \( f(\Psi^1) \) and that \( f(\Psi^2) \leq |A_h/\delta_h| \delta_h - \gamma \). This last inequality holds since otherwise \( f(\hat{y}^h) = \gamma + f(\Psi^1) + f(\Psi^2) > \gamma + F_h - |A_h/\delta_h| \delta_h - |A_h/\delta_h| \delta_h - \gamma = F_h \), i.e., \( \hat{y} \notin MP^{OO}(k, b, F) \). Hence, since \( \gamma \leq |A_h/\delta_h| \delta_h - \delta_h \) (by the contradiction hypothesis) then \( f(\Psi^2) \leq \delta_h \). The total size of the items assigned to the part \( h \) is

\[
f(\hat{y}^h) = \gamma + f(\Psi^1) + f(\Psi^2) \leq F_h - |A_h/\delta_h| \delta_h + |A_h/\delta_h| \delta_h - \delta_h + f(\Psi^2) = F_h - \delta_h + f(\Psi^2). \tag{44}
\]

Recall that \( f(\Omega) \geq \delta_h \). Hence, since \( \delta_h \) is the biggest size of the items that can be assigned to the part \( h \), the item sizes are divisible, and \( \hat{y} \) is feasible, a subset \( \Omega^1 \subseteq \Omega \) exists such that \( f(\Omega^1) = \delta_h \) and \( \sum_{q=1}^{\min\{h, b\}} \sum_{w \in \Omega^1 \cap T_q} f_w y_w^q = \delta_h. \)

By (44), it is feasible to replace \( \Psi^2 \) by \( \Omega^1 \) in \( \hat{y} \). Hence, the new solution \( \hat{y} \) with \( \hat{y}^{1, h-1}_w = \hat{y}^{1, h-1}_w, \hat{y}^h_w = \hat{y}^h_w \) for \( r \in \Psi^1, \hat{y}^h_r = 0 \) for \( r \in \Psi^2 \) and \( \hat{y}^h_w = |\Omega^1|, \) for \( w \in \Omega^1 \), belongs to \( MP^{OO}(k, b, F) \) and has a strictly better objective function than \( \hat{y}^{1, h} \) because \( p_w/f_w > p_r/f_r \) for all \( w \in \Omega^1 \subseteq H_k \) and \( r \in \Psi^1 \subseteq N(k) \setminus H_k \), contradicting Lemma 3.8.

Hence, we have proved that \( \gamma \geq |A_h/\delta_h| \delta_h. \)

Recalling that \( f(\hat{H}_k^h \cap S(\hat{y}^h)) = \gamma \), relation (43) reads as

\[
A_{h+1} = A_h - \gamma + f(\hat{H}_k \cap T'_{h+1}). \tag{45}
\]

By the above discussion we have \( \gamma > |A_h/\delta_h| \delta_h - \delta_h. \) Moreover, by Lemma 1.2 \( A_h \geq |f(\hat{H}_k^h)/\delta_h| \delta_h \) and, hence, \( \gamma > |f(\hat{H}_k^h)/\delta_h| \delta_h - \delta_h, \) too.

As a consequence and since, by induction, \( A_h \leq f(\hat{H}_k^h) + \delta_h, \) relation (43) becomes

\[
A_{h+1} < f(\hat{H}_k^h) + \delta_h - |f(\hat{H}_k^h)/\delta_h| \delta_h + f(\hat{H}_k \cap T'_{h+1})
\]

that by the hypothesis of Case 1 is equal to

\[
f(\hat{H}_k^h) + 2\delta_h - \min\{F_h; |f(\hat{H}_k^h)/\delta_h| \delta_h \} + f(\hat{H}_k \cap T'_{h+1})) = f(\hat{H}_k^h) + 2\delta_h \leq f(\hat{H}_k^h) + \delta_{h+1},
\]

where the last inequality holds since \( \delta_{h+1} \geq 2\delta_h. \) Hence, the thesis follows.

Case 2. Again, let \( \gamma = f(\hat{y}^h \cap H_k) = \sum_{r \in H_k} f_r \hat{y}_r^h. \) If \( F_h = 0, \) then \( \gamma = 0 \) and \( \min\{F_h; |f(\hat{H}_k^h)/\delta_h| \delta_h \} = 0, \) too. Since by induction, \( A_h \leq f(\hat{H}_k^h) + \delta_h, \) relation (43) becomes

\[
A_{h+1} = A_h - \gamma + f(\hat{H}_k \cap T'_{h+1}) \leq
\]

24
\[ f(H_k^b) + \delta_h - \min\{F_h; |f(H_k^b)/\delta_h|\delta_h\} + f(H_k \cap T'_{h+1}) \leq f(H_k^{b+1}) + \delta_h < f(H_k^{b+1}) + \delta_{h+1}. \]

Hence, w.l.o.g., let us assume \( F_h > 0 \). In the following, we show that \( \gamma > F_h - \delta_h \). By contradiction, let us suppose \( \gamma \leq F_h - \delta_h \). Hence, since \( |A_h/\delta_h|\delta_h \geq F_h \), we have \( A_h = f(\hat{H}_k^b \setminus S(\hat{y}^{1,b}) \geq \delta_h \). Then, by denoting \( \rho = \sum_{r \in \mathbb{N}(k) \setminus H_k} f, \hat{y}_r^b \), arguments similar to those applied to Case 1 can be used to get a contradiction. Hence, \( \gamma > F_h - \delta_h \) and we have \( \gamma > F_h - \delta_h \geq \min\{|A_h/\delta_h|\delta_h; F_h\} = \delta_h \geq \min\{|f(H_k^b)/\delta_h|\delta_h; F_h\} - \delta_h \), where the second inequality follows by Lemma 4.2. Hence, by relation 4.3 and since by induction \( A_h \leq f(H_k^b) + \delta_h \) we have

\[ A_{h+1} = A_h - \gamma + f(H_k \cap T'_{h+1}) < f(H_k^b) + \delta_h - \min\{F_h; |f(H_k^b)/\delta_h|\delta_h\} + \delta_h + f(H_k \cap T'_{h+1}) \]

that, by the definition of \( H_k^{b+1} \), is equal to \( f(H_k^b) + 2\delta_h \leq f(H_k^{b+1}) + \delta_{h+1} \), where the last inequality holds since \( \delta_{h+1} \geq 2\delta_h \). Hence, \( A_{h+1} < f(H_k^{b+1}) + \delta_{h+1} \), i.e., the thesis.

In the following, given an item type \( w \) and a knapsack part \( h \), we denote by \( T'_h \) the set of items of type \( w \) in \( T_h \). The following theorem states the maximum and minimum value assumed by \( y_{k,b}^b \), for \( \hat{y} \) in \( MOO(k,b,F) \).

**Theorem 4.4** Given an optimal solution \( \hat{y} \) in \( MP^O(k,b,F) \), with \( b > 1 \), the number of items of type \( k \) of the set \( T'_h \) assigned to the part \( b \), \( y_{k,b}^b \), takes the following values:

\[
y_{k,b}^b \geq \min\left\{ \left( F_b - \left[ |f(H_k^b)/\delta_b|\delta_b /\delta_b \right] \right) + \frac{\delta_b}{f_k}, \hat{b}_{k,b} \right\}
\]

and

\[
y_{k,b}^b \leq \min\left\{ \left( F_b - \left[ |f(H_k^b)/\delta_b|\delta_b /\delta_b \right] \right) + \frac{\delta_b}{f_k}, \hat{b}_{k,b} \right\}.
\]

**Proof.** Obviously, if \( \hat{b}_{k,b} = 0 \) then \( y_{k,b}^b = 0 \) in any optimal solution of \( MP^O(k,b,F) \), and the theorem holds. Hence, suppose that \( \hat{b}_{k,b} > 0 \), i.e., items of type \( k \) exist in \( T'_h \).

Observe that, by definition of \( b \), \( \delta_b = d_b \). Let \( \hat{H}_k^b \) be the set of items of \( H_k \) not assigned by \( \hat{y} \) to \( 1, \ldots, b-1 \) and that can be assigned to the part \( b \) (i.e., the items contained in \( \hat{H}_k^b = H_k \cap \{T'_1 \cup \ldots T'_b\} \setminus S(\hat{y}^{1,b-1}) \equiv \hat{H}_k^b \setminus S(\hat{y}^{1,b-1}) \)).

As in 3, several cases are considered.

Case 1. \( F_b \leq f(H_k^b) \). Since \( \delta_b \) divides \( F_b \), it follows that \( F_b \leq |f(H_k^b)/\delta_b|\delta_b \). In this case, the Theorem states
that $\hat{y}_{b,k}^{b} = 0$. Observe that, by Lemma 4.2, $f(\hat{H}_k^b) \geq |f(H_k^b) / \delta_b| \delta_b$. Hence, $F_b \leq f(\hat{H}_k^b)$.

By contradiction, suppose that $\hat{y}_{k,b}^b > 0$. Since $F_b \leq f(\hat{H}_k^b)$, we have $\sum_{q=1}^{b} \sum_{r \in H_k^b} f_r \hat{y}_{r,q}^b + f_k \hat{y}_{k,b}^b \leq F_b \leq f(\hat{H}_k^b) = \sum_{r \in H_k^b} f_r \sum_{q=1}^{b} (\hat{y}_{r,q}^b - \hat{y}_{r,q}^{b-1})$, and, since $\hat{y}_{k,b}^b > 0$, $\sum_{r \in H_k^b} f_r \sum_{q=1}^{b} (\hat{y}_{r,q}^b - \hat{y}_{r,q}^{b-1} - \hat{y}_{r,q}^b) \geq f_k \hat{y}_{k,b}^b > 0$. Observe that, by Lemma 3.6, $\delta_b$ divides $f_k \hat{y}_{k,b}^b$. Moreover, observe that: (i) $\delta_b$ is the biggest item size; (ii) $\delta_q$ divides $f_r \hat{y}_{r,q}^b$, for all $q$ and $r$ (by definition); (iii) $\delta_q$ divides $f_r \hat{y}_{r,q}^b$, for all $r = 1, \ldots, k$, $q = 1, \ldots, b$ and $h = q, \ldots, l$ (by Lemma 3.6). Hence, since $\delta_1, \delta_2, \ldots, \delta_b$ are divisible, integers $\lambda_{r,q} \in \{0, \ldots, \hat{b}_{r,q} - \hat{y}_{r,q}^b \}$ for all items types $r$ in $\hat{H}_k^b$ and $q = 1, \ldots, b$, exist, such that $\delta_q$ divides $f_r \lambda_{r,q}$ and $\sum_{q=1}^{b} \sum_{r \in H_k^b} f_r \lambda_{r,q} = \sum_{q=1}^{b} \sum_{r \in H_k^b} [f_r \lambda_{r,q} / \delta_q] \delta_q = f_k \hat{y}_{k,b}^b$.

Consider now a new solution $\tilde{y}^{1,b}$ such that: $\tilde{y}_{r,q}^{1,b-1} = \hat{y}_{r,q}^{1,b-1}$, $\tilde{y}_{r,q}^{1,b} = \lambda_{r,q}$ for $r \in \hat{H}_k^b$ and $q = 1, \ldots, b$, and $\tilde{y}_{k,b}^b = 0$. Note that, by construction and by the above observations, $\tilde{y}^{1,b}$ is feasible and, by definition of $\hat{H}_k^b$, has objective function strictly better than $\hat{y}^{1,b}$, i.e., $\tilde{y}^{1,b}$ is not an OPT solution, a contradiction.

Case 2. $F_b - [f(H_k^b) / \delta_b] \delta_b \geq f_k \hat{b}_{k,b}$. Then by (40) and (47), $\hat{y}_{k,b}^b = \hat{b}_{k,b}$. Note that, since by Lemma 4.3, $f(\hat{H}_k^b) \leq f(H_k^b) + \delta_b$, it follows that $F_b - f(\hat{H}_k^b) \geq f_k \hat{b}_{k,b}$, too.

By contradiction, suppose that $\hat{y}_{k,b}^b < \hat{b}_{k,b}$. By Lemma 3.6, $\delta_b$ divides $f_k \hat{y}_{k,b}^b$, and since $\delta_b$ divides $f_k \hat{b}_{k,b}$ (by definition), it follows that $f_k \hat{b}_{k,b} - f_k \hat{y}_{k,b}^b = \alpha \delta_b$, with $\alpha$ positive integer. Hence, there exists a set of items of type $k$ that are not assigned in $\hat{y}^b$, belonging to $T_k^b$ and having total size $\alpha \delta_b$. Since $\hat{y}$ is an ordered solution, a set $A$ of items belonging to $T_k^b \cup \ldots \cup T_{k-1}^b$, of total size $f(A) \geq \delta_b$ and assigned to the part $b$ in $\hat{y}$ does not exist. (Otherwise, in $\hat{y}$, an unassigned set of items in $T_k^b$, with total size $\delta_b$, can be feasibly allocated (by Lemma 4.2) to the part $b$ in place of a subset of $A$ of total size $\delta_b$, i.e., $\hat{y}$ is not an ordered solution.) Let $\Omega$ be the set of items (of type $k$) belonging to $T_1^b \cup \ldots \cup T_k^b - 1$ and assigned to part $b$ in $\hat{y}$ (i.e., $\Omega = \{S(\hat{y}_{k1}^b) \cup \ldots \cup S(\hat{y}_{kb-1}^b)\}$).

By the above discussion, it follows that $f(\Omega) < \delta_b$. Moreover, let $\Gamma = \{T_1^b \cup \ldots \cup T_k^b\} \setminus \{T_k^b \cup H_k\}$, i.e., the set of items in $\{T_1^b \cup \ldots \cup T_k^b\}$ that neither belong to $T_k^b$ nor to $H_k$. Let $\rho = \sum_{q=1}^{b} \sum_{r \in \Gamma} f_r \hat{y}_{r,q}^b$ (i.e., $\rho$ is the total size of the items in $\Gamma$ assigned to the part $b$ in $\hat{y}$). If $\rho < \alpha \delta_b$, since $F_b - f(\hat{H}_k^b) \geq f_k \hat{b}_{k,b}$, the solution $\tilde{y}^{1,b}$ with $\tilde{y}_{r,q}^{1,b-1} = \hat{y}_{r,q}^{1,b-1}$, $\tilde{y}_{r,q}^{1,b} = 0$ for $r \in \Gamma$ and $q = 1, \ldots, b$, $\tilde{y}_{r,q}^{1,b} = \hat{y}_{r,q}^b$ for $r \in \hat{H}_k^b$ and $q = 1, \ldots, b$, and $\tilde{y}_{k,b}^b = \hat{b}_{k,b}$ is feasible and yields a bigger objective function value than $\hat{y}^{1,b}$, since $\frac{\rho_k}{\rho} \geq \frac{\rho_k}{\rho}$ for all $r \in \Gamma$ (recall that the items in $\Gamma$ do not belong to $H_k$). Hence, $\tilde{y}^{1,b}$ is not an OPT solution. A contradiction. Hence, it must be $\rho \geq \alpha \delta_b$.

Recall that, by Lemma 3.6, $\delta_q$ divides $f_r \hat{y}_{r,q}^b$, for all item types $r$ and $q = 1, \ldots, b$. Observe that, since $\hat{y}$
is an ordered solution, a subset $\Theta$ of $\Gamma \cap \{T'_1 \cup \ldots \cup T'_{b-1}\}$ such that
$$\sum_{q=1}^{b} \sum_{r \in \Theta} [f_{r} \tilde{y}^b_{r,q}/\delta_q] \delta_q = \sum_{q=1}^{b} \sum_{r \in \Theta} f_{r} \tilde{y}^b_{r,q} = \delta_b$$
and $p(\Theta) = p_k \delta_b/f_k$ does not exist. (Otherwise, a set of items belonging to $T^k_b$ and having total size $\delta_b$ that are not assigned or that are assigned to the parts $b + 1, \ldots, l$ in $\hat{y}$ (such a set exists in this case) can be assigned in place of $\Theta$, i.e., $\hat{y}$ is not an ordered solution.)

Let us consider the items in $r \in \Gamma \cap \{T'_1 \cup \ldots \cup T'_{b-1}\}$ such that $\frac{\tilde{y}^b_{r,q}}{\delta_q} = \frac{\tilde{y}^b_{r,q}}{\delta_q}$, then
$$\sum_{r \in \Gamma \cap \{T'_1 \cup \ldots \cup T'_{b-1}\}} \frac{\tilde{y}^b_{r,q}}{\delta_q} < \delta_b$$
(48)

Let $\Delta$ be the set containing all the items of types $1, \ldots, k - 1$ in $\{T'_1 \cup \ldots \cup T'_b\}$. By the partition into maximal blocks (see (13)) and since $\Delta$ is strictly contained in the set of the items of types $1, \ldots, k - 1$ in $N(k)$, we have $\delta_b \geq f_k > \sum_{r \in \Delta \setminus H_k} \sum_{q=1}^{b} \tilde{y}^b_{r,q}$. Obviously, since $\Delta \setminus H_k \subseteq \Delta$, we have $\delta_b \geq f_k > \sum_{r \in \Delta \setminus H_k} \sum_{q=1}^{b} \tilde{y}^b_{r,q}$, too. Furthermore, since by definition $\Gamma \cap \Delta \subseteq \Delta \setminus H_k$, it also holds that
$$\delta_b \geq f_k > \sum_{r \in \Gamma \cap \Delta} \sum_{q=1}^{b} \tilde{y}^b_{r,q}.$$ (49)

Recall that (by Proposition 3.6) if an item $r \in T'_b$ is assigned to the part $b$ in $\hat{y}$, then $f_{r} \tilde{y}^b_{r,b} = \beta \delta_b$, with $\beta$ positive integer. As a consequence and by (19), items in $T'_b \cap \Delta \cap \Gamma$ such that $\frac{\tilde{y}^b_{r,q}}{\delta_q} = \frac{\tilde{y}^b_{r,q}}{\delta_q}$ do not exist. By (48) and (49), it follows that
$$\sum_{q=1}^{b} \sum_{r \in \Gamma} \frac{f_{r} \tilde{y}^b_{r,q}}{\delta_q} \delta_q = \sum_{q=1}^{b} \sum_{r \in \Gamma} f_{r} \tilde{y}^b_{r,q} < \delta_b.$$ (50)

By the divisibility of the item sizes, since $\rho \geq \alpha \delta_b$ and all the items have size not bigger than $\delta_b$, and by Lemma 3.6 integers $\lambda_{r,q} \in \{0, \ldots, \tilde{y}^b_{r,q}\}$ exist for all $r \in \Gamma$ and $q = 1, \ldots, b$, such that $\sum_{q=1}^{b} \sum_{r \in \Gamma} f_{r} \lambda_{r,q} = \sum_{q=1}^{b} \sum_{r \in \Gamma} \frac{f_{r} \tilde{y}^b_{r,q}}{\delta_q} \delta_q = f_k (\tilde{b}_{k,b} - \tilde{y}^b_{r,q})$ (where $\alpha \geq 1$ and integer). As a consequence and by (50), it follows that $\sum_{q=1}^{b} \sum_{r \in \Gamma} p_r \lambda_{r,q} < p_k (\tilde{b}_{k,b} - \tilde{y}^b_{r,q}).$ Hence, the solution $\tilde{y}^{1,b}$ with $\tilde{y}^{1,b-1} = \tilde{y}^{1,b-1}$, $\tilde{y}^{b}_{r,q} = \tilde{y}^{b}_{r,q} - \lambda_{r,q}$ for $r \in \Gamma$ and $q = 1, \ldots, b$, $\tilde{y}^{b}_{r,q} = \tilde{y}^{b}_{r,q}$ for $r \in \tilde{H}_k$ and $q = 1, \ldots, b$, and $\tilde{y}^{b}_{r,q} = \tilde{b}_{k,b}$ is feasible, and yields a bigger objective function value than $\tilde{y}^{1,b}$, i.e., $\tilde{y}^{1,b}$ is not an OPT solution, a contradiction.

Case 3. If Cases 1 and 2 do not hold, then $F_b > f(H^b_k)$ and $F_b - (f(H^b_k) + \delta_b) \delta_b < f_k \tilde{b}_{k,b}$. Hence, the theorem
states that \( \hat{y}_{k,b}^b \geq \left( \frac{F_b - [(f(H_k^b) + \delta_b)/\delta_b] \delta_b}{\delta_b} \right)^+ \tilde{b}_{k,b} \) and \( \hat{y}_{k,b}^b \leq \min \{ \left( \frac{F_b - [(f(H_k^b) + \delta_b)/\delta_b] \delta_b}{\delta_b} \right)^+ \tilde{b}_{k,b} \} \). Also in this case, by contradiction, suppose that the theorem does not hold. Hence, either (a) \( \hat{y}_{k,b}^b < \left( \frac{F_b - [(f(H_k^b) + \delta_b)/\delta_b] \delta_b}{\delta_b} \right)^+ \tilde{b}_{k,b} \) or (b) \( \hat{y}_{k,b}^b > \min \{ \left( \frac{F_b - [(f(H_k^b) + \delta_b)/\delta_b] \delta_b}{\delta_b} \right)^+ \tilde{b}_{k,b} \} \).

In Case (a), since \( F_b - [(f(H_k^b) + \delta_b)/\delta_b] \delta_b < f_k \tilde{b}_{k,b} \), we have \( [(f(H_k^b) + \delta_b)/\delta_b] \delta_b + \left( \frac{F_b - [(f(H_k^b) + \delta_b)/\delta_b] \delta_b}{\delta_b} \right)^+ \delta_b \leq F_b \). Hence, an argument similar to that used in Case 2 can be applied to show that there exists a feasible solution \( \tilde{y} \) with \( \hat{y}_{k,b}^b = \left( \frac{F_b - [(f(H_k^b) + \delta_b)/\delta_b] \delta_b}{\delta_b} \right)^+ \tilde{b}_{k,b} \).

In Case (b), we obviously have \( \hat{y}_{k,b}^b \leq \tilde{b}_{k,b} \) and the theorem is not satisfied only if \( \left( \frac{F_b - [(f(H_k^b) + \delta_b)/\delta_b] \delta_b}{\delta_b} \right)^+ \tilde{b}_{k,b} < \hat{y}_{k,b}^b \) and \( \hat{y}_{k,b}^b > \left( \frac{F_b - [(f(H_k^b) + \delta_b)/\delta_b] \delta_b}{\delta_b} \right)^+ \tilde{b}_{k,b} \). Since, by Lemma 4.2, \( f(\hat{H}_k^b) \geq [f(H_k^b)/\delta_b] \delta_b \), it follows that \( \hat{y}_{k,b}^b > \left( \frac{F_b - [(f(H_k^b) + \delta_b)/\delta_b] \delta_b}{\delta_b} \right)^+ \tilde{b}_{k,b} \), too. Hence, since \( \tilde{y} \) is feasible and by algebra, we have \( \sum_{q=1}^{b} \sum_{r \in H_k^b} f_r \hat{y}_{r,q}^b + f_k \hat{y}_{k,b}^b \leq F_b \leq f(\hat{H}_k^b) + F_b - [f(\hat{H}_k^b)/\delta_b] \delta_b \) (the last inequality follows since \( f(\hat{H}_k^b) - [f(\hat{H}_k^b)/\delta_b] \delta_b \geq 0 \). Then a similar argument to that of Case 1 can be applied to get a contradiction. \( \square \)

Recalling Lemma 3.6 (stating that \( \delta_b \) divides \( f_k \tilde{y}_{k,b}^b \)), Theorem 4.4 provides the at most three possible values assumed by \( \hat{y}_{k,b}^b \), for \( \hat{y} \) in \( MO^O(k,b,F) \). Recalling that \( T_{b+1} \cup T_{b+2} \cup \ldots \cup T_l = \emptyset \), Theorem 4.4 can be also applied to determine the possible values assumed by \( \hat{y}_{k,b}^{b+1} \), \ldots, \( \hat{y}_{k,b}^l \), as explained in the following. For each value of \( \hat{y}_{k,b}^b \) specified by Theorem 4.4 (such that \( \delta_b \) divides \( f_k \tilde{y}_{k,b}^b \)) we can generate a new instance in which we set \( \tilde{b}_{k,b} = b_{k,b} - \hat{y}_{k,b}^b \) and \( F_b = F_b - f_k \hat{y}_{k,b}^b \), \( T_b = T_b^k \setminus S(\hat{y}_{k,b}^b) \), i.e., \( N(k) = N(k) \setminus S(\hat{y}_{k,b}^b) \). Observe that, by Lemma 3.6, \( \delta_b \) still divides \( F_b \). Theorem 4.4 can be now applied to the new instance to determine all the possible values assumed by \( \hat{y}_{k,b}^{b+1} \). Applying recursively the above argument, we can compute all the possible values assumed by \( \hat{y}_{k,b}^{b+2} \), \ldots, \( \hat{y}_{k,b}^l \). In the following theorem, we show that the values assumed by \( \sum_{h=b}^{l} \hat{y}_{k,b}^h \) are at most three.

**Theorem 4.5** Given an optimal solution \( \hat{y} \) in \( MO^O(k,b,F) \), with \( b > 1 \), and an item type \( k \), we have

\[
\sum_{h=b}^{l} \hat{y}_{k,b}^h \geq \min \left\{ \left( \frac{\sum_{h=b}^{l} F_h - [(f(H_k^h) + \delta_b)/\delta_b] \delta_b}{\delta_b} \right)^+ \tilde{b}_{k,b} \right\}
\]  

and

\[
\sum_{h=b}^{l} \hat{y}_{k,b}^h \leq \min \left\{ \left( \frac{\sum_{h=b}^{l} F_h - [(f(H_k^h)/\delta_b)] \delta_b}{\delta_b} \right)^+ \tilde{b}_{k,b} \right\}.
\]
Proof. Since, by Lemma 3.6, \( \delta_b \) divides \( \delta_b \hat{y}_{k,b}^h \), it follows that \( \delta_b \) divides \( \delta_b \sum_{h=b}^l \hat{y}_{k,b}^h \), too. Hence, if (51) and (52) holds, we have that \( \sum_{h=b}^l \hat{y}_{k,b}^h \) takes at most three values.

To prove conditions (51) and (52), the following observations are in order:

1) All items in \( \text{M-SP}(k, b, \mathbf{F}) \) belong to \( T'_1 \cup \cdots \cup T'_{l'} \) (i.e., \( T'_{b+1} = \cdots = T'_{l'} = \emptyset \)).

2) Each item in \( \text{M-SP}(k, b, \mathbf{F}) \) can be assigned to all the knapsack parts \( h \) in \( \{b, b+1, \ldots, l\} \).

3) \( \delta_b \) divides the part capacities \( F_b, F_{b+1}, \ldots, F_l \).

Hence, w.l.o.g, we can replace the knapsack parts \( \{b, b+1, \ldots, l\} \) by a single part, say \( b \), with capacity \( \sum_{h=b}^l F_h \).

Note that, Lemmas 1.2 and 1.3 still hold providing that the minimum and maximum total size of items in \( H_k \) available to be assigned to the (new) part \( b \) in \( \hat{y} \) are \([f(H^b_k)/\delta_b] \delta_b \) and \( f(H^b_k) + \delta_b \), respectively. Hence, the thesis follows by applying Theorem 4.4. \( \square \)

Now, we show how the possible values assumed by \( \hat{y}_{k-1b}^b \) (i.e., the number of items of type \( k-1 \) in \( T'_k \) assigned to part \( b \) in \( \hat{y} \)) can be determined, too. For each value of \( \hat{y}_{k,b}^h \), for \( h = b, \ldots, l \), specified by Theorem 4.4 we consider a new \( \text{M-SP}(k, b, \mathbf{F}) \) instance in which \( F_h = F_h - f_k \hat{y}_{k,b}^h \), for \( h = b, \ldots, l \), \( (F_h = F_h, \) for \( h = 1, \ldots, b-1) \)

and \( N(k) = N(k) \setminus T^k_k \). Note that, by Lemma 3.6, \( F_h \), for \( h = b, \ldots, l \), is still multiple of \( \delta_b \), and since \( \delta_b \) is the biggest item size of the instance, \( F_b, F_{b+1}, \ldots, F_l \) are multiple of \( \delta_{b-1}, \delta_{b-1}, \ldots, \delta_1 \), too. Furthermore, observe that \( \tilde{H}^b_{k-1} \), defined as in (27), with \( k-1 \) in place of \( k \), may contain items of type \( k \) (belonging to \( T'_1 \cup \cdots \cup T'_{b-1} \) only), but this fact does not affect the definition of \( \mathcal{H}^k_{k-1} \) (defined as in (29), with \( k-1 \) in place of \( k \)), since all items in the instance have size not bigger than \( \delta_b \). Hence, Lemmas 4.2 and 4.3 still hold, and allow to determine the minimum and maximum total size of items in \( \tilde{H}^b_{k-1} \setminus S(\hat{y}^{1,b-1}) \). Theorem 4.6 sets the possible values taken by \( \hat{y}_{k-1b}^b \).

**Theorem 4.6** Given an optimal solution \( \hat{y} \) in \( \text{MP}^{OO}(k, b, \mathbf{F}) \), \( b > 1 \), let \( F_h = F_h - f_k \hat{y}_{k,b}^h \), for \( h = b, \ldots, l \), and \( N(k) = N(k) \setminus T^k_k \). Then, \( \hat{y}_{k-1b}^b \) takes the following values.

\[
\hat{y}_{k-1b}^b \geq \min \left\{ \frac{F_b - [(f(H^b_{k-1}) + \delta_b)/\delta_b] \delta_b}{\delta_b} \right\} + \frac{\delta_b}{f_{k-1}} \hat{b}_{k-1b}^b
\]

(53)
and
\[ y_{k-1b}^b \leq \min \left\{ \frac{F_b - |f(H_{k-1}^b)|/\delta_b}{\delta_b} \right\} + \frac{\delta_b}{f_{k-1} - b_{k-1b}} \] (54)

Proof. The same cases considered in the proof of Theorem 4.3 are used. Observe that, by Lemma 3.6, \( \delta_b \) still divides \( F_b = F_b - f_k \hat{y}_{k,b}^b \). Let \( \hat{H}_{k-1}^b \) be the set of items of \( H_{k-1} \) (defined in (20)) not assigned by \( \hat{y} \) to the parts \( 1, \ldots, b-1 \) and that can be assigned to the part \( b \) (i.e., \( \hat{H}_{k-1}^b \equiv H_{k-1} \setminus \{T_1 \cup \ldots T_b\} \setminus S(\hat{y}^1_{1,b-1}) \equiv H_{k-1} \setminus S(\hat{y}^1_{1,b-1})) \).

Obviously, since by the hypothesis \( N(k) \) does not contain \( T_b^k \), by definition, \( H_{k-1}, \hat{H}_{k-1}^b \) and \( \hat{H}_{k-1}^b \) do not contain items in \( T_b^k \), too.

Case 1. \( F_b \leq f(H_{k-1}^b) \). Theorem states that \( \hat{y}_{k-1b}^b = 0 \). The same argument employed in Case 1 of Theorem 4.3 can be used to show the thesis.

Case 2. \( F_b - |f(H_{k-1}^b) + \delta_b|/\delta_b \geq f_k b_{k-1b} \). Then by (20) and (17), \( \hat{y}_{k-1b}^b = \tilde{b}_{k-1b} \). Also in this case, the same argument employed in Case 2 of Theorem 4.3 can be used to show the thesis. In particular, note that, the set \( \Gamma \), now defined as \( \Gamma = N(k) \setminus \{T_b^{k-1} \cup H_{k-1}\} \), may contain items of type \( k \) belonging \( \{T_1 \cup \ldots T_{b-1}\} \). However, since \( \hat{y} \) is an ordered solution, a subset of \( \Theta \) of \( \Gamma \cap \{T_1 \cup \ldots T_{b-1}\} \) such that \( \sum_{q=1}^b \sum_{r \in \Theta} [f_r y_{r,q}^b/\delta_q] \delta_q = \sum_{q=1}^b \sum_{r \in \Theta} f_r y_{r,q}^b = \delta_b \) and \( p(\Theta) = p_{k-1} \delta_b / f_{k-1} \) does not exist also in this case. A consequence of this fact is that Inequality (30) still holds. Hence, the thesis follows by applying the same arguments employed in Case 2 of Theorem 4.3.

Case 3. See Case 3 of Theorem 4.3.

Hence, recursively applying Theorems 4.4 and 4.6, all the possible values assumed by \( \hat{y}_{w,b}^b \) for \( h = b, \ldots, l \) and \( w = 1, \ldots, k \) can be detected. Moreover, by recursively applying Theorem 4.5, it follows that the values assumed by \( \sum_{h=b}^l \hat{y}_{w,b}^b \) are at most three, for \( w = 1, \ldots, k \).

Given an optimal solution \( \hat{y} \) in \( MPOO(k, b-1, F) \), let \( G \) be the vector with components \( G_h = F_h - \sum_{w=1}^k f_k \hat{y}_{w,b}^b \), for \( h = b, \ldots, l, G_h = F_h \), for \( h = 1, \ldots, b-1 \), and let \( N(k) = N(k) \setminus T_b^l \). Now, Theorems 4.4 and 4.6 can be applied on the instance \( M-SP(k, b-1, G) \) for finding the values attained by \( \hat{y}_{w,b-1}^b \), for \( h = b-1, \ldots, l \) and \( w = 1, \ldots, k \) (with \( \hat{y} \) optimal solution in \( MPOO(k, b-1, G) \)). And so on.

Example
Consider the following instance of MKSP with $|N| = 6$ item types and $|M| = 3$ knapsacks.

$$\max \sum_{i=1}^{3} (4x_{i,1} + 28x_{i,2} + 15x_{i,3} + 14x_{i,4} + 28x_{i,5} + 32x_{i,6})$$

$$x_{1,1} + 2x_{1,2} + 2x_{1,3} + 2x_{1,4} + 4x_{1,5} + 4x_{1,6} \leq 7$$

$$x_{2,1} + 2x_{2,2} + 2x_{2,3} + 2x_{2,4} + 4x_{2,5} + 4x_{2,6} \leq 2$$

$$x_{3,1} + 2x_{3,2} + 2x_{3,3} + 2x_{3,4} + 4x_{3,5} + 4x_{3,6} \leq 6$$

$$\sum_{i=1}^{3} x_{i,1} \leq 2; \sum_{i=1}^{3} x_{i,2} \leq 4; \sum_{i=1}^{3} x_{i,3} \leq 8; \sum_{i=1}^{3} x_{i,4} \leq 7; \sum_{i=1}^{3} x_{i,5} \leq 2; \sum_{i=1}^{3} x_{i,6} \leq 1$$

$$x \in \mathbb{Z}^+ \cup \{0\} \text{ for } i = 1, \ldots, 3, j = 1, \ldots, 6$$

In the instance, 3 six different sizes exist (i.e., 1, 2, 4), hence $l = 3$. To formulate the problem as in (1), we have to compute the quantities $r_i^h$, for $h = 1, 2$ and $i = 1, 2, 3$. Hence, we have $\sum_{i=1}^{3} r_i^1 = 1+0+0$, $\sum_{i=1}^{3} r_i^2 = 2+2+2 = 6$, and $\sum_{i=1}^{3} (c_i - \sum_{h=1}^{3} r_i^h) = 4 + 0 + 4 = 8$.

The corresponding $M$-SP is defined as follows. The set of items in $N$ can be partitioned into the 5 maximal blocks $B_1 = \{1\}$, $B_2 = \{2\}$, $B_3 = \{3\}$, $B_4 = \{4,5\}$ and $B_5 = \{6\}$, with multiplicity $\tilde{b}_{1,1} = 2$, $\tilde{b}_{2,2} = 4$, $\tilde{b}_{3,1} = 2$, $\tilde{b}_{4,2} = 7$ and $\tilde{b}_{5,3} = 4$, and, finally, $\tilde{b}_{5,3} = 2$, respectively (all other $\tilde{b}_{p,q}$s are 0). Accordingly, the sets $T_q'$ contain the following item types: $T_1' = \{1\}$, $T_2' = \{2,3,4\}$ and $T_3' = \{4,5\}$.

Hence, in M-SP, the item set is $N' = \{1,2,3,4,5\}$, the profits $p_1, \ldots, p_5$ are equal to 4, 28, 15, 14 and 32, respectively, and the sizes $f_1, \ldots, f_5$ are equal to 1, 2, 2, 2 and 4, respectively. The ratios $p_i / f_i$, for $i = 1, \ldots, 5$, are 4, 14, 15/2, 7 and 8 respectively.

The knapsack has 3 parts of capacity $\bar{c}_1 = \sum_{i=1}^{3} r_i^1 = 1$, $\bar{c}_2 = \sum_{i=1}^{3} r_i^2 = 6$ and $\bar{c}_3 = \sum_{i=1}^{3} (c_i - \sum_{h=1}^{3} r_i^h) = 8$. In what follows, $MO(k,b,(F_1,F_2,F_3))$ denotes the set $MO(k,b,F)$, where $F$ has components $F_1$, $F_2$, and $F_3$.

$MO(5,3,(1,6,8))$: Since, $T_3' = \{4,5\}$, by Theorems 4.4 and 4.6 we can compute $y_{5,3}^3$ and $y_{4,3}^4$. At this aim, we need to calculate $f(H_3^k)$ and $f(H_4^k)$. We have $\tilde{H}_3^k = \emptyset$ and $\tilde{H}_4^k = \tilde{H}_5^k = \{2\}$. Hence, $f(H_3^k) = 0$, $f(H_5^k) = f(\tilde{H}_5^k) = 8$, $f(H_3^k(\text{min})) = \min\{f(H_3^k)/2 \mid 2; F_2\} = 6$, $f(H_5^k) = f(\tilde{H}_5^k) - f(H_5^k(\text{min})) = 2$.

We have $H_4 = \{2,3,5\}$ and $\tilde{H}_4^k = \emptyset$, $\tilde{H}_4^k = \tilde{H}_5^k = \{2,3\}$. Hence, $f(H_4^k) = 0$, $f(H_5^k) = f(H_5^k) = 24$, $f(H_4^k(\text{min})) = \min\{f(H_4^k)/2 \mid 2; F_2\} = 6$, $f(H_5^k) = f(\tilde{H}_5^k) - f(H_5^k(\text{min})) = 18$.

By Theorem 4.4 since $F_3 = 8$ and $f(H_3^k) = 2$, we have $y_{5,3}^3 \in \{0,1,2\}$. By Theorem 4.6 since $f(H_4^k) = 18$,}

31
it follows that \( y_{3,3}^3 = 0 \) in every optimal solution. Since no other item type exists in the set \( T'_3 \), the new sets to consider are \( MO(5,2,(1,6,8)) \), \( MO(5,2,(1,6,4)) \) and \( MO(5,2,(1,6,0)) \).

\[ \text{MO}(5,2,(1,6, F_3)) : \ T'_2 = \{2,3,4\} \]. We have \( H_3 = \{2,5\} \), \( \tilde{H}_3^1 = \emptyset \), \( \tilde{H}_3^2 = \{2\} \) and \( f(\tilde{H}_3^2) = f(\tilde{H}_3^2) = 8 \).

Recalling that \( f(\tilde{H}_3^1) = 24 \) and \( F_2 = 6 \), since \( f(\tilde{H}_3^1) = 8 \), we have that \( y_{3,2}^2 = y_{3,2}^3 = 0 \) in every optimal solution. Moreover, by Theorem 4.5 and since \( f(\tilde{H}_3^2) = 24 \) and \( F_2 + F_3 \leq 14 \), it follows that \( y_{3,2}^1 + y_{3,2}^3 = 0 \), i.e., \( y_{3,2}^3 = 0 \) too. By Theorem 4.5 since \( 6 \leq F_2 + F_3 \leq 14 \) (with \( F_2 = 6 \)), \( f(\tilde{H}_3^2) = 8 \) and \( \tilde{b}_{3,2} = 8 \), we also have \( y_{3,2}^3 \in \{(F_3 - 4)^+/2, (F_3 - 2)^+/2\} \) (recall that \( y_{3,2}^3 = 0 \)).

Since \( H_2 = \emptyset \) and \( f(\tilde{H}_2^2) = 0 \), \( y_{2,2}^2 \in \{2,3\} \).

When \( y_{3,2}^3 = (F_3 - 4)^+/2 \), then, by Theorem 4.5 \( y_{2,2}^2 + y_{3,2}^3 = 4 \). This leads to the new set \( \text{MO}(5,1,(1,F_2,F_3)) \), with \( F_2 + F_3 = 3 \). When \( y_{3,2}^3 = (F_3 - 2)^+/2 \), then, by Theorem 4.5 \( y_{2,2}^2 + y_{3,2}^3 \in \{3,4\} \). When \( y_{2,2}^3 + y_{3,2}^3 = 3 \), we have the new set \( \text{MO}(5,1,(1,F_2,F_3)) \), with \( F_2 + F_3 = 2 \). When \( y_{2,2}^3 + y_{3,2}^3 = 4 \) we have to consider the set \( \text{MO}(5,1,(1,0,0)) \).

\[ \text{MO}(5,1,(1,F_2,F_3)) \], with \( F_2 + F_3 = 2; \text{MO}(5,1,(1,0,0)) : \text{Since } T'_1 = \{1\} \text{ and } \tilde{b}_{1,1} = 2, \text{ by Lemma 4.1 we have } y_{1,1}^3 = 1 \text{ and } y_{1,1}^2 + y_{1,1}^3 = 1 \text{ in } \text{MO}(5,1,(1,F_2,F_3)) \), and \( y_{1,1}^3 = 1 \) and \( y_{1,1}^2 + y_{1,1}^3 = 0 \) in \( \text{MO}(5,1,(1,0,0)) \).

5 Description of the convex hull of the feasible solutions of SMKP

In this section, a description of the convex hull of the solutions of MSKP that are ordered and that satisfy the OPT property, i.e., \( P^{OO} \), is provided. At this aim, we use the same approach employed in \([3]\) and first consider the set \( MP^{OO}(k,b,F) \).

Given M-SP\((k,b,F)\), let \( I(k,b,F) \) be a set of inequalities satisfying the three following conditions:

1) If \( F'_h = F_h \), for \( h = 1, \ldots, b - 1 \), and if \( d_b \) divides \( F_h \) and \( F'_h \) for \( h = b, \ldots, l \), then for each inequality \( I \) in \( I(k,b,F) \) there exists one in \( I(k,b,F') \) with the same left-hand side of \( I \).

2) The inequalities in \( I(k,b,F) \) are valid for \( MP^{OO}(k,b,F) \).

3) Each solution in \( MP^{OO}(k,b,F) \) is contained in (at least) a face induced by an inequality in \( I(k,b,F) \).

We first show that, when \( b = 1 \), \( I(k,1,F) \) contains a single inequality. By Lemma 4.1 the following lemma directly follows.
Lemma 5.1  The following inequality satisfies conditions 1), 2) and 3) for MP$O^O(k, 1, F)$.

\[
\sum_{w=1}^{k} \sum_{h=1}^{l} y_{w,1}^h \leq \sum_{w=1}^{k} \min \{ \sum_{h=1}^{l} (F_h - \sum_{j=1}^{w-1} \hat{y}_{j,1}^h); \hat{b}_{w,1} \} \tag{55}
\]

where \( \sum_{h=1}^{l} \hat{y}_{w,1}^h \), for \( w = 1, \ldots, k \), are defined as in \([24]\).

Now, we the inequalities \( I(k, b, F) \) for \( b > 1 \), by a double induction on \( b \) and on the item types contained in the set \( T'_b \). Namely, we assume by induction that there exists a set of inequalities \( I(k, b - 1, F) \) that satisfies conditions 1), 2) and 3), where \( F \) is a vector such that \( d_h \) divides \( F_h \), for \( h = 1, \ldots, b - 2 \), and \( d_{b-1} \) divides \( F_{b-1}, \ldots, F_l \). The basic step of the induction is for \( b = 1 \) and is given by Lemma 5.1. Then, we define the set \( I(k, b, F) \) and prove that it satisfies conditions 1), 2) and 3) by induction on the item types contained in the set \( T'_b \). Let the inequalities in \( I(k, b - 1, F) \) be defined as

\[
\sum_{q=1}^{b-1} \sum_{w=1}^{k} \frac{a_{w,q} f_w}{d_q} \sum_{h=q}^{l} y_{w,q}^h \leq g_{k,b-1} (\sum_{h=b-1}^{l} F_h). 
\]

The inequalities in \( I(k, b, F) \) for \( b > 1 \) are of the form

\[
\sum_{q=1}^{b} \sum_{w=1}^{k} \frac{a_{w,q} f_w}{d_q} \sum_{h=q}^{l} y_{w,q}^h \leq g_{k,b} (\sum_{h=b}^{l} F_h) 
\]

where for each inequality in \( I(k, b, F) \) there exists an inequality in \( I(k, b - 1, F) \) having the same coefficients \( a_{w,q} \), for \( w = 1, \ldots, k \) and \( q = 1, \ldots, b - 1 \).

In what follows, notation and definitions are introduced in order to define the coefficient \( a_{k,b} \). In particular, the coefficient \( a_{k,b} \) can assume two values. (The coefficients \( a_{w,b} \), for \( w = 1, \ldots, k - 1 \), and \( a_{w,q} \), for \( w = 1, \ldots, k \), \( q = 1, \ldots, b - 1 \) are then recursively defined.) Recall that \( \delta_b = d_b \).

We have

\[
a_{k,b} = \begin{cases} 
\alpha_{k,b} \\
\beta_{k,b}
\end{cases}
\tag{56}
\]

where, if \( k > 1 \),

\[
\alpha_{k,b} = g_{k-1,b} (\lfloor f(\mathcal{H}_k^b) / \delta_b \rfloor \delta_b + \delta_b) - g_{k-1,b} (\lfloor f(\mathcal{H}_k^b) / \delta_b \rfloor \delta_b) 
\tag{57}
\]

\[
\beta_{k,b} = g_{k-1,b} (\lfloor f(\mathcal{H}_k^b) / \delta_b \rfloor \delta_b + 2 \delta_b) - g_{k-1,b} (\lfloor f(\mathcal{H}_k^b) / \delta_b \rfloor \delta_b + \delta_b). 
\tag{58}
\]
and, if \( k = 1 \),
\[
\alpha_{1,b} = g_{k,b-1}(F_{b-1} + \lfloor f(H_k^b)/\delta_b \rfloor \delta_b + \delta_b) - g_{k,b-1}(F_{b-1} + \lfloor f(H_k^b)/\delta_b \rfloor \delta_b)
\]  
(59)

\[
\beta_{1,b} = g_{k,b-1}(F_{b-1} + \lfloor f(H_k^b)/\delta_b \rfloor \delta_b + 2\delta_b) - g_{k,b-1}(F_{b-1} + \lfloor f(H_k^b)/\delta_b \rfloor \delta_b + \delta_b).
\]  
(60)

Function \( g_{k,b}(\sum_{h=b}^l F_h) \) will be defined later. In the following, we define the quantities \( s_{k,b}, \sigma_{k,b} \) and \( U \) as:

\[
s_{k,b} = \begin{cases} 
\frac{\sum_{h=b}^l F_h - \lfloor f(H_k^b)/\delta_b \rfloor \delta_b}{\delta_b} & \text{if } a_{k,b} = \alpha_{k,b} \\
\frac{\sum_{h=b}^l F_h - (\lfloor f(H_k^b)/\delta_b \rfloor \delta_b + \delta_b)}{\delta_b} & \text{if } a_{k,b} = \beta_{k,b}
\end{cases}
\]  
(61)

\[
\sigma_{k,b} = \begin{cases} 
0 & \text{if } s_{k,b} < 0 \\
s_{k,b} & \text{if } 0 \leq s_{k,b} \leq \frac{f_k b_{k,b}}{d_b} \\
\frac{f_k b_{k,b}}{d_b} & \text{if } s_{k,b} > \frac{f_k b_{k,b}}{d_b}
\end{cases}
\]  
(62)

\[
U = \begin{cases} 
\sum_{h=b}^l F_h & \text{if } s_{k,b} < 0 \\
\lfloor f(H_k^b)/\delta_b \rfloor \delta_b & \text{if } a_{k,b} = \alpha_{k,b} \text{ and } 0 \leq s_{k,b} \leq \frac{f_k b_{k,b}}{d_b} \\
\lfloor f(H_k^b)/\delta_b \rfloor \delta_b + \delta_b & \text{if } a_{k,b} = \beta_{k,b} \text{ and } 0 \leq s_{k,b} \leq \frac{f_k b_{k,b}}{d_b} \\
\sum_{h=b}^l F_h - f_k \delta_{k,b} & \text{if } s_{k,b} > \frac{f_k b_{k,b}}{d_b}
\end{cases}
\]  
(63)

Then, if \( k = 1 \), \( g_{1,b}(\sum_{h=b}^l F_h) \) is defined as
\[
g_{1,b}(\sum_{h=b}^l F_h) = g_{k,b-1}(F_{b-1} + U) + a_{1,b} \sigma_{1,b}
\]  
(64)

and, if \( k > 1 \),
\[
g_{k,b}(\sum_{h=b}^l F_h) = g_{k-1,b}(U) + a_{k,b} \sigma_{k,b}.
\]  
(65)

The following two lemmas establish properties of function \( g_{k,b}(\cdot) \).
Lemma 5.2 For any \( b = 1, \ldots, l \), \( k = 1, \ldots, t \) and integers \( F_b, \ldots, F_t \) multiple of \( d_b \), the function \( g_{k,b}(\sum_{h=b}^{l} F_h + s_{k,b}d_b) \) is a concave non-decreasing function of \( s_{k,b} \) (where the function is defined over all integers \( s_{k,b} \) such that \( \sum_{h=b}^{l} F_h + s_{k,b}d_b \geq 0 \).

Proof. The thesis is proved by induction on \( b \). When \( b = 1 \), by (55), function \( g_{k,1}(\sum_{h=b}^{l} F_h) \) is

\[
g_{k,1}(\sum_{h=b}^{l} F_h) = \sum_{w=1}^{k} \min\{ \sum_{h=1}^{l} (F_h - \sum_{j=1}^{w-1} \tilde{y}_{h,1}^j; \tilde{b}_{w,1}) \} = \sum_{w=1}^{k} \min\{ \sum_{h=1}^{l} F_h + \sum_{h=b}^{l} \sum_{j=1}^{w-1} \tilde{y}_{h,1}^j; \tilde{b}_{w,1} \},
\]

where \( \tilde{y}_{w,1}^h \), for \( w = 1, \ldots, k \) and \( h = 1, \ldots, l \), are constants whose values are specified by Lemma 4.1. Observe that, each function \( \min\{ \sum_{h=1}^{l} (F_h - \sum_{j=1}^{w-1} \tilde{y}_{h,1}^j) + d_s_{k,b}; \tilde{b}_{w,1} \} \) is a concave non-decreasing function of \( s_{k,b} \), for \( w = 1, \ldots, k \). Hence, \( g_{k,1}(\sum_{h=b}^{l} F_h + d_s_{k,b}) \) is a concave non-decreasing function of \( s_{k,b} \), too.

Let us assume that the statement is true for \( b - 1 \) and show it for \( b \). We prove the thesis by induction on the item types contained in the set \( T'_b \). Namely, we first show the thesis when \( T'_b \) only contains items of type 1, then we assume that the thesis holds when \( T'_b \) contains items of types \( 1, \ldots, k-1 \) and show it when \( T'_b \) contains items of types \( 1, \ldots, k \). We use the following notation (recall that \( \delta_b = d_b \))

\[
A = \begin{cases} 
[f(H_k^b)/\delta_b] \delta_b & \text{if } a_{k,b} = \alpha_{k,b} \\
[f(H_k^b)/\delta_b] \delta_b & \text{if } a_{k,b} = \beta_{k,b} 
\end{cases} \tag{66}
\]

and \( G_{k,b}(s_{k,b}) = g_{k,b}(A + (s_{k,b} + 1)d_b) - g_{k,b}(A + s_{k,b}d_b) \).

Recalling (63), we prove the thesis by showing that the function \( G_{k,b}(s_{k,b}) \) is a non-increasing function of \( s_{k,b} \). When \( T'_b \) only contains items of type 1, we have \( G_{1,b}(s_{1,b}) = g_{1,b}(A + (s_{1,b} + 1)d_b) - g_{1,b}(A + s_{1,b}d_b) \) and by definition of function \( g_{1,b}(\cdot) \) (see (63)) we obtain

\[
G_{1,b}(s_{1,b}) = \begin{cases} 
g_{1,b-1}(F_{b-1} + A + (s_{1,b} + 1)d_b) - g_{1,b-1}(F_{b-1} + A + s_{1,b}d_b) & \text{if } s_{1,b} < 0 \\
a_{1b} = g_{1,b-1}(F_{b-1} + A + d_b) - g_{1,b-1}(F_{b-1} + A) & \text{if } 0 \leq s_{1,b} < \tilde{b}_{1,b} \\
g_{1,b-1}(F_{b-1} + A + (s_{1,b} + 1 - \tilde{b}_{1,b}/d_b)d_b) - g_{1,b-1}(F_{b-1} + A + (s_{1,b} - \tilde{b}_{1,b}/d_b)d_b) & \text{otherwise.}
\end{cases}
\]

The thesis follows by induction, since \( g_{1,b-1}(\sum_{h=b-1}^{l} P_h + p d_b) = g_{1,b-1}(\sum_{h=b-1}^{l} P_h + p(d_b/d_{b-1})d_b) \) is a concave non-decreasing function of \( p \), for any integers \( P_{b-1}, \ldots, P_l \) multiple of \( d_{b-1} \).
Assume that the thesis holds when \( T'_h \) contains items of types 1, \ldots, \( k - 1 \) and show it when \( T'_h \) contains items of types 1, \ldots, \( k \). We have \( G_{k,b}(s_{k,b}) = g_{k,b}(A + (s_{k,b} + 1)d_b) - g_{k,b}(A + s_{k,b}d_b) \), and by definition of function \( g_{k,b}(\cdot) \) (see \( \text{(65)} \)) we obtain

\[
G_{k,b}(s_{k,b}) = \begin{cases} 
  g_{k-1,b}(A + (s_{k,b} + 1)d_b) - g_{k-1,b}(A + s_{k,b}d_b) & \text{if } s_{k,b} < 0 \\
  a_{k,b} = g_{k-1,b}(A + d_b) - g_{k-1,b}(A) & \text{if } 0 \leq s_{k,b} < \tilde{b}_{k,b} \\
  g_{k-1,b}(A + (s_{k,b} + 1 - \tilde{b}_{k,b}/d_b)d_b) - g_{k-1,b}(A + (s_{k,b} - \tilde{b}_{k,b}/d_d)d_b) & \text{otherwise.}
\end{cases}
\]

Also in this case the thesis follows by induction.

\[\square\]

In Lemma \[5.3\] two properties of function \( g_{k,b}(\cdot) \) are given, directly following from Lemma \[5.2\]

**Lemma 5.3** Let \( F \) and \( G \) be two natural numbers such that \( F \leq G \) and \( d_b \) divides \( F \) and \( G \). Then,

1. \( g_{k,b}(G) + \sigma(g_{k,b}(F + d_b) - g_{k,b}(F)) \geq g_{k,b}(G + \sigma d_b) \);
2. \( g_{k,b}(F - \sigma d_b) + \sigma(g_{k,b}(G) - g_{k,b}(G - d_b)) \leq g_{k,b}(F) \);

for every \( \sigma \in \mathbb{N} \) such that \( F - \sigma d_b \geq 0 \).

To show that the inequalities in \( I(k, b, F) \) satisfy conditions 1), 2) and 3) we use an inductive argument on the item types contained in the item set \( T'_h \). Namely, we first prove the thesis (in Theorem \[5.4\]) when \( T'_h \) only contains items of type 1 (with size \( s_1 = d_1 = f_1 = 1 \)), then we assume the thesis holds when \( T'_h \) contains items of types 1, \ldots, \( k - 1 \) and prove it when \( T'_h \) contains items of types 1, \ldots, \( k \) (in Theorem \[5.5\]).

**Theorem 5.4** Suppose that \( T'_h \) only contains items of type 1. If the inequalities in \( I(k, b - 1, F) \) satisfy conditions 1), 2) and 3), for all \( F \) such that \( d_h \) divides \( F_h \), for \( h = 1, \ldots, b - 1 \), and \( d_{b-1} \) divides \( F_{b-1}, \ldots, F_1 \), then the inequalities in \( I(k, b, F) \) satisfy conditions 1), 2) and 3) for all \( F \) such that \( d_h \) divides \( F_h \), for \( h = 1, \ldots, b - 1 \), and \( d_b \) divides \( F_b, \ldots, F_1 \).

**Proof.** Since \( T'_h \) only contains items of type 1, the set \( I(k, b, F) \) contains inequalities of the type:

\[
\sum_{q=1}^{b-1} \sum_{w=1}^{k} \frac{a_{w,q}f_w}{d_q} \sum_{h=q}^{l} y_{w,q}^h + \frac{a_{1,b}f_1}{d_b} \sum_{h=b}^{l} y_{1,b}^h \leq g_{1,b} \left( \sum_{h=b}^{l} F_h \right),
\]

where \( f_1 = 1 \) and the coefficients \( a_{w,q} \), for \( w = 1, \ldots, k \) and \( q = 1, \ldots, b - 1 \), and \( a_{1,b} \) are defined as in \( \text{(66)} \).
Recall that, by definition, $d_b = d_b$. In what follows $d_b$ and $\delta_b$ will be used indifferently.

(a)

Let $F$ and $F'$ be two vectors with $l$ components such that $F_h = F'_h$ for $h = 1, \ldots, b-1$ and $d_b$ divides $F_h$ and $F'_h$ for $h = b, \ldots, l$. Then condition 1) follows by the inductive hypothesis on $I(k, b-1, F)$ and by definitions (57)–(60). In fact, since, by definition, $f(H_q^k)$ depends by the values of $F_1, \ldots, F_{q-1}, a_{wq}$, for $w = 1, \ldots, k$ and for $q = 1, \ldots, b - 1$, attains the same value both for $F$ and for $F'$.

(b)

We show that the inequalities are valid for $M^{PO}(k, b, F)$ (i.e., condition 2) holds). Let $y \in M^{PO}(k, b, F)$.

We show that the following inequality holds in $y$:

$$\sum_{q=1}^{b-1} \sum_{w=1}^{k} \frac{a_{wq}}{d_q} \sum_{h=q}^{l} \frac{y_h^{wq}}{d_b} + \frac{\alpha_{1,b}}{d_b} \sum_{h=b}^{l} \frac{y_h}{d_b} \leq g_{k,b-1}(F_{b-1} + \sum_{h=b}^{l} F_h - \sum_{h=b}^{l} \left[ \frac{f_1 y_{h,b}}{d_b} \right] d_b) + \frac{\alpha_{1,b}}{d_b} \sum_{h=b}^{l} \left[ \frac{f_1 y_{h,b}}{d_b} \right] d_b \right). \quad (68)$$

Recall that, by the hypothesis, $d_b$ divides $F_h$ for $h = b, \ldots, l$. Obviously, $\frac{a_{1,b}}{d_b} \sum_{h=b}^{l} y_h^{bq} \leq \frac{\alpha_{1,b}}{d_b} \sum_{h=b}^{l} \left[ \frac{f_1 y_{h,b}}{d_b} \right] d_b$. Hence, inequality (68) directly holds since, by condition 1) and induction, the inequality:

$$\sum_{q=1}^{b-1} \sum_{w=1}^{k} \frac{a_{wq}}{d_q} \sum_{h=q}^{l} y_h^{wq} \leq g_{k,b-1}(G)$$

is valid for $I(k, b-1, F')$, for each vector $F'$ such that: (i) $F'_h = F_h$ for $h = 1, \ldots, b-2$; (ii) $F'_{b-1}, \ldots, F'_l$ are multiple of $d_{b-1}$ such that $\sum_{h=b-1}^{l} F'_h = G$.

To show that all the inequalities in $I(k, b, F)$, of the form (67), are valid, different cases are considered (corresponding to the different values attained by $s_{1,b}$, as defined in (61)).

If $\sum_{h=b}^{l} F_h \leq \lfloor f(H_1^t)/d_b \rfloor d_b$, then, by (61) and (62), $s_{1,b} = 0$ and $\sigma_{1,b} = 0$, both when $a_{1,b} = \alpha_{1,b}$ and $a_{1,b} = \beta_{1,b}$. If $\sum_{h=b}^{l} y_{1,b}^h = 0$. Hence, from definition (63), i.e., $g_{1,b}(\sum_{h=b}^{l} F_h) = g_{k,b-1}(F_{b-1} + \sum_{h=b}^{l} F_h)$, and by (68) it follows that the inequalities (67) are valid. Suppose that $\sum_{h=b}^{l} y_{1,b}^h > 0$. Then if $a_{1,b} = \alpha_{1,b}$ we have

$$\sum_{q=1}^{b-1} \sum_{w=1}^{k} \frac{a_{wq}}{d_q} \sum_{h=q}^{l} y_h^{wq} + \frac{\alpha_{1,b}}{d_b} \sum_{h=b}^{l} y_h^{bq} \leq g_{k,b-1}(F_{b-1} + \sum_{h=b}^{l} F_h - \sum_{h=b}^{l} \left[ \frac{f_1 y_{h,b}}{d_b} \right] d_b) + \frac{\alpha_{1,b}}{d_b} \sum_{h=b}^{l} \left[ \frac{f_1 y_{h,b}}{d_b} \right] d_b \right) = (69)$$

$$g_{k,b-1}(F_{b-1} + \sum_{h=b}^{l} F_h - \sum_{h=b}^{l} \left[ \frac{f_1 y_{h,b}}{d_b} \right] d_b) + \sum_{h=b}^{l} \left[ \frac{f_1 y_{h,b}}{d_b} \right] (g_{k,b-1}(F_{b-1} + \lfloor f(H_1^t)/d_b \rfloor d_b + \delta_b) - g_{k,b-1}(F_{b-1} + \lfloor f(H_1^t)/d_b \rfloor d_b)) \leq (70)$$

$$g_{k,b-1}(F_{b-1} + \sum_{h=b}^{l} F_h) = g_{1,b}(\sum_{h=b}^{l} F_h) \quad (71)$$
where the equality (69) follows from the definition of \( \alpha_{1,b} \) (see (69)), the inequality (70) follows from condition (ii) of Lemma 5.3 and equality (71) from (64). Hence, the inequality (67) in which \( a_{1,b} = \alpha_{1,b} \) is valid. The above argument can be applied by replacing in (69), \( a_{1,b} \) with \( \beta_{1,b} \), and, in (70) the expression (60) in place of (59). Hence, the thesis follows.

If \( \frac{l}{h=b} F_h - ([f(H^b_1)/d_b] d_b + d_b) \geq f_1 \tilde{b}_{1,b} \), then \( s_{1,b} > \frac{f_1 \tilde{b}_{1,b}}{d_b} \) and \( \sigma_{1,b} = \frac{f_1 \tilde{b}_{1,b}}{d_b} \), both when \( a_{1,b} = \alpha_{1,b} \) and \( a_{1,b} = \beta_{1,b} \). If \( \frac{l}{h=b} y^b_{1,b} = \tilde{b}_{1,b} \), then from (64) and by induction we have that the inequalities (67) are valid. Suppose that \( \frac{l}{h=b} y^b_{1,b} < \tilde{b}_{1,b} \). If \( a_{1,b} = \beta_{1,b} \), by (68) and the definition of \( \beta_{1,b} \) (see (60)) we have

\[
\frac{b-1}{q=1} \sum_{w=1}^{k} \frac{a_{w,q} f_w}{d_q} \sum_{h=q}^{l} y^h_{w,q} + \frac{\beta_{1,b} f_1}{d_b} \sum_{h=b}^{l} y^h_{1,b} \leq \frac{b}{h=b} \sum_{h=b}^{l} F_h - \sum_{h=b}^{l} \left[ \frac{f_1 y^h_{1,b}}{d_b} \right] d_b \bigg) + \frac{\beta_{1,b}}{d_b} \sum_{h=b}^{l} \left[ \frac{f_1 y^h_{1,b}}{d_b} \right] d_b \bigg]
\]

and by condition (i) of Lemma 5.3 we have

\[
Q \leq g_{k,b-1}(F_{b-1} + \sum_{h=b}^{l} F_h - f_1 \tilde{b}_{1,b}) + \frac{f_1 \tilde{b}_{1,b}}{d_b} - \left( \sum_{h=b}^{l} F_h - \sum_{h=b}^{l} \left[ \frac{f_1 y^h_{1,b}}{d_b} \right] d_b \right) \bigg) + \frac{f_1 \tilde{b}_{1,b}}{d_b} \beta_{1,b} = g_{1,b} \left( \sum_{h=b}^{l} F_h \right).
\]

Hence, the thesis follows. If \( a_{1,b} = \alpha_{1,b} \), since \( \frac{l}{h=b} F_h - ([f(H^b_1)/d_b] d_b + d_b) \geq f_1 \tilde{b}_{1,b} \) implies \( \frac{l}{h=b} F_h - ([f(H^b_1)/d_b] d_b + d_b) \geq f_1 \tilde{b}_{1,b} \), the above arguments can be repeated and easily adapted to show the thesis.

Finally, consider the case in which \( \frac{l}{h=b} F_h - ([f(H^b_1)/d_b] d_b + d_b) < f_1 \tilde{b}_{1,b} \) and \( \frac{l}{h=b} F_h > [f(H^b_1)/d_b] d_b \) (i.e., \( \frac{l}{h=b} F_h \geq [f(H^b_1)/d_b] d_b + d_b \), since \( d_b \) divides \( \frac{l}{h=b} F_h \)). Two subcases can be considered: either \( \frac{l}{h=b} F_h -
\[ f(\mathcal{H}_1^l)/d_b \] \[ \leq f_1\beta_{1,b} \] or \[ \sum_{h=b}^{l} F_h - f(\mathcal{H}_1^l)/d_b \] \[ > f_1\beta_{1,b} \]. In the first subcase, obviously, \[ \sum_{h=b}^{l} F_h - f(\mathcal{H}_1^l)/d_b \] \[ \leq f_1\beta_{1,b} \] holds, too. Hence, we have \[ 0 \leq s_{1,b} \leq f_1\beta_{1,b} \]. Then, by (61), (63) and (64), and since \( \sigma_{1,b} = s_{1,b} \) (by (62)), inequality (68) becomes

\[
\sum_{q=1}^{b-k} \sum_{w=1}^{l} a_{w,q} f_w \sum_{h=q} y_{w,q} + \frac{a_{1,b} f_1}{d_b} \sum_{h=b} \leq g_{k,b-1}(F_{b-1} + \sum_{h=b}^{l} F_h - \sum_{h=b}^{l} f_1 y_{1,b} \frac{d_b}{d_b}) + a_{1,b} \sum_{h=b} \left[ f_1 y_{1,b} \frac{d_b}{d_b} \right] =
\]

\[
g_{k,b-1}(F_{b-1} + U - d_b(\sum_{h=b}^{l} f_1 y_{1,b} \frac{d_b}{d_b}) - s_{1,b})) + a_{1,b} \left( \sum_{h=b} \left[ f_1 y_{1,b} \frac{d_b}{d_b} \right] - s_{1,b} \right) + a_{1,b}s_{1,b}. \tag{72}
\]

If \[ \sum_{h=b}^{l} \left[ f_1 y_{1,b} \frac{d_b}{d_b} \right] = s_{1,b} \], by (68) and by recalling that \( g_{1,b}(\sum_{h=b}^{l} F_h) = g_{k,b-1}(F_{b-1} + U) + a_{1,b}\sigma_{1,b} \), the inequalities (67) are satisfied for \( a_{1,b} \in \{ \alpha_{1,b} ; \beta_{1,b} \} \).

Observe that since \( 0 \leq s_{1,b} \leq f_1\beta_{1,b} \) then

\[ a_{1,b} = g_{k,b-1}(F_{b-1} + U + d_b) - g_{k,b-1}(F_{b-1} + U). \tag{73} \]

If \[ \sum_{h=b}^{l} \left[ f_1 y_{1,b} \frac{d_b}{d_b} \right] > s_{1,b} \], then by condition (ii) of Lemma 5.3 and by (73) we have that

\[ g_{k,b-1}(F_{b-1} + U - d_b(\sum_{h=b}^{l} f_1 y_{1,b} \frac{d_b}{d_b}) - s_{1,b})) + a_{1,b} \left( \sum_{h=b} \left[ f_1 y_{1,b} \frac{d_b}{d_b} \right] - s_{1,b} \right) \leq g_{k,b-1}(F_{b-1} + U). \]

Then, by (72) and since \( g_{1,b}(\sum_{h=b}^{l} F_h) = g_{k,b-1}(F_{b-1} + U) + a_{1,b}s_{1,b} \):

\[
\sum_{q=1}^{b-k} \sum_{w=1}^{l} a_{w,q} f_w \sum_{h=q} y_{w,q} + \frac{a_{1,b} f_1}{d_b} \sum_{h=b} \leq g_{1,b}(\sum_{h=b}^{l} F_h)
\]

i.e., the thesis. Finally, if \[ \sum_{h=b}^{l} \left[ f_1 y_{1,b} \frac{d_b}{d_b} \right] < s_{1,b} \], by condition (i) of Lemma 5.3 and by (73) it follows that

\[
g_{k,b-1}(F_{b-1} + U - d_b(\sum_{h=b}^{l} f_1 y_{1,b} \frac{d_b}{d_b}) - s_{1,b})) + a_{1,b} \left( \sum_{h=b} \left[ f_1 y_{1,b} \frac{d_b}{d_b} \right] - s_{1,b} \right) =
\]

\[
g_{k,b-1}(F_{b-1} + U + d_b(s_{1,b} - \sum_{h=b}^{l} f_1 y_{1,b} \frac{d_b}{d_b})) - a_{1,b}(s_{1,b} - \sum_{h=b} \left[ f_1 y_{1,b} \frac{d_b}{d_b} \right]) \leq g_{k,b-1}(F_{b-1} + U).
\]

Hence, by (72) and since \( g_{1,b}(\sum_{h=b}^{l} F_h) = g_{k,b-1}(F_{b-1} + U) + a_{1,b}s_{1,b} \), it follows that the inequalities (67) are valid.
Let us consider the second subcase, i.e., \( \sum_{h=b}^{l} F_h - \lfloor f(H_1^h) \rfloor d_b > f_1 \tilde{b}_{1,b} \). Since \( d_b \) divides \( f_1 \tilde{b}_{1,b} \) and \( \sum_{h=b}^{l} F_h \), then \( \sum_{h=b}^{l} F_h - \lfloor f(H_1^h) \rfloor d_b + d_b \geq f_1 \tilde{b}_{1,b} \), too. Hence, we have \( 0 \leq s_{1,b} \leq \frac{f_1 \tilde{b}_{1,b}}{d_b} + 1 \). Hence, if \( 0 \leq s_{1,b} \leq \frac{f_1 \tilde{b}_{1,b}}{d_b} \), the same arguments of the first subcase can be used to show the thesis. If \( s_{1,b} = \frac{f_1 \tilde{b}_{1,b}}{d_b} + 1 > \frac{f_2 \tilde{b}_{1,b}}{d_b} \), the same arguments applied in the case \( \sum_{h=b}^{l} F_h - \lfloor f(H_1^h) \rfloor d_b + d_b \geq f_1 \tilde{b}_{1,b} \) can be used to prove that \( \text{(b)} \) holds.

\[ (c) \]

We show now that condition 3) holds, i.e., \( MO^O(k, b, F) \) is contained in the faces induced by the inequalities in \( I(k, b, F) \). According to the Definition of \( U \) and \( \sigma_{1,b} \), we can write \( \sum_{h=b}^{l} F_h = U + \sigma_{1,b} d_b \). Let \( y \in MO^O(k, b, F) \) be an optimal solution and let

\[
\frac{\sum_{h=b}^{l} F_h - \lfloor f(H_1^h) \rfloor d_b}{d_b}
\]

Since \( d_b = \delta_b \), Theorem 4.5 and Proposition 3.6 imply that \( \sum_{h=b}^{l} \frac{f_1 \tilde{b}_{1,b}}{d_b} \) can attain one of the following values:

\[
\min\{c^+; f_1 \tilde{b}_{1,b}/d_b\} \text{ or } \min\{c - 1^+; f_1 \tilde{b}_{1,b}/d_b\} \text{ or } \min\{c - 2^+; f_1 \tilde{b}_{1,b}/d_b\}.
\]

In the case \( c \leq 0 \) then \( \sum_{h=b}^{l} y_{1,b}^h = 0 \) in every optimal solution. Hence, by induction, for each optimal solution \( y \in MO^O(k, b, F) \) there exists an inequality in \( I(k, b - 1, F) \) such that

\[
\sum_{q=1}^{b-1} \sum_{w=1}^{k} \frac{a_{w,q} f_w}{d_q} \sum_{h=q}^{l} y_{w,q}^h = g_{k,b-1}(F_{b-1} + \sum_{h=b}^{l} F_h).
\]

Since in this case \( s_{1,b} \leq 0 \), then \( \sigma_{1,b} = 0 \) and \( g_{k,b-1}(F_{b-1} + \sum_{h=b}^{l} F_h) = g_{k,b}(\sum_{h=b}^{l} F_h) \). Hence, the thesis follows.

In the case \( c - 2 \geq f_1 \tilde{b}_{1,b}/d_b \), then Theorem 4.5 and Proposition 3.6 imply that \( \sum_{h=b}^{l} y_{1,b}^h = \tilde{b}_{1,b} \) in every optimal solution. Hence, by induction, for each optimal solution in \( y \in MO^O(k, b, F) \) there exists an inequality such that

\[
\sum_{q=1}^{b-1} \sum_{w=1}^{k} \frac{a_{w,q} f_w}{d_q} \sum_{h=q}^{l} y_{w,q}^h = g_{k,b-1}(F_{b-1} + \sum_{h=b}^{l} F_h - f_1 \tilde{b}_{1,b}).
\]

In this case \( s_{1,b} \geq f_1 \tilde{b}_{1,b}/d_b \), both when \( a_{1,b} = \alpha_{1,b} \) and \( a_{1,b} = \beta_{1,b} \). Hence, by definition, \( \sigma_{1,b} = f_1 \tilde{b}_{1,b}/d_b \). As \( \sum_{h=b}^{l} y_{1,b}^h = \tilde{b}_{1,b} \) in every optimal solution, we obtain

\[
\sum_{q=1}^{b-1} \sum_{w=1}^{k} \frac{a_{w,q} f_w}{d_q} \sum_{h=q}^{l} y_{w,q}^h + \frac{a_{1,b} f_1}{d_b} \sum_{h=b}^{l} y_{1,b}^h = g_{k,b-1}(F_{b-1} + \sum_{h=b}^{l} F_h - f_1 \tilde{b}_{1,b}) + a_{1,b} f_1 \tilde{b}_{1,b}/d_b = g_{k,b}(\sum_{h=b}^{l} F_h).
\]

And, the thesis holds both if \( a_{1,b} = \alpha_{1,b} \) and \( a_{1,b} = \beta_{1,b} \).
In the case \( c = f_1b_{1,b}/d_b + 1 \), two subcases can be considered: (I) \( [f(\mathcal{H}_1^1)/d_b] = [f(\mathcal{H}_1^2)/d_b] \); (II) \( [f(\mathcal{H}_1^1)/d_b] \neq [f(\mathcal{H}_1^2)/d_b] \). Theorem 1.3 and Proposition 3.6 imply that \( \sum_{h=b}^l f_1y^h_{1,b}/d_b = c - 1 = f_1b_{1,b}/d_b \) in Case (I), while \( \sum_{h=b}^l f_1y^h_{1,b}/d_b \in \{c - 1; c - 2\} \) in Case (II), in every optimal solution. (Note that, since w.l.o.g. \( c = f_1b_{1,b}/d_b + 1 \geq 2 \), then \( c - 2 \geq 0 \).) In Case (I), by induction, for each optimal solution \( y \in MO\omega(k, b, F) \) there exists an inequality such that: 
\[
\sum_{q=1}^{b-1} \sum_{w=1}^k \frac{a_{w,q}f_w}{d_q} \sum_{h=q}^l y^h_{w,q} = g_{k,b-1}(F_{b-1} + \sum_{h=b}^l F_h - (c - 1)d_b) = g_{k,b-1}(F_{b-1} + \sum_{h=b}^l F_h - (c - 1)d_b) = g_{k,b-1}(F_{b-1} + \sum_{h=b}^l F_h) + d_b.
\]

Let \( a_{1,b} = \alpha_1,b \), then we have \( s_{1,b} = \sigma_1,b = c - 1 \) and we obtain
\[
\sum_{q=1}^{b-1} \sum_{w=1}^k \frac{a_{w,q}f_w}{d_q} \sum_{h=q}^l y^h_{w,q} + \frac{\alpha_1,b}{d_b} \sum_{h=b}^l y^h_{b,b} = g_{k,b-1}(F_{b-1} + [f(\mathcal{H}_1^1)/d_b] + d_b) + (c - 1)\alpha_1,b =
\]
\[
g_{k,b-1}(F_{b-1} + [f(\mathcal{H}_1^1)/d_b]) + (g_{k,b-1}(F_{b-1} + [f(\mathcal{H}_1^2)/d_b] + d_b) - g_{k,b-1}(F_{b-1} + [f(\mathcal{H}_1^1)/d_b])) + (c - 1)\alpha_1,b =
\]
\[
g_{k,b-1}(F_{b-1} + [f(\mathcal{H}_1^1)/d_b]) + \alpha_1,b \sigma_1,b = g_{1,b}(\sum_{h=b}^l F_h).
\]

Note that, since \( [f(\mathcal{H}_1^1)/d_b] = [f(\mathcal{H}_1^2)/d_b] \), the above relations hold even if \( \alpha_1,b \) and \( [f(\mathcal{H}_1^2)/d_b] \) are replaced by \( \beta_1,b \) and \( [f(\mathcal{H}_1^1)/d_b] \), respectively. Hence, in this case, two inequalities exist that hold with equality at \( y \). In Case (II), by induction, for each optimal solution \( y \in MO\omega(k, b, F) \), two sub cases hold: (II.a) \( \sum_{h=b}^l f_1y^h_{1,b}/d_b = c - 1; \)

(II.b) \( \sum_{h=b}^l f_1y^h_{1,b}/d_b = c - 2 \). Then, by induction, there exist inequalities such that

Case (II.a)
\[
\sum_{q=1}^{b-1} \sum_{w=1}^k \frac{a_{w,q}f_w}{d_q} \sum_{h=q}^l y^h_{w,q} = g_{k,b-1}(F_{b-1} + \sum_{h=b}^l F_h - (c - 1)d_b) = g_{k,b-1}(F_{b-1} + [f(\mathcal{H}_1^1)/d_b] + d_b)
\]
or

Case (II.b)
\[
\sum_{q=1}^{b-1} \sum_{w=1}^k \frac{a_{w,q}f_w}{d_q} \sum_{h=q}^l y^h_{w,q} = g_{k,b-1}(F_{b-1} + \sum_{h=b}^l F_h - (c - 2)d_b) = g_{k,b-1}(F_{b-1} + [f(\mathcal{H}_1^1)/d_b] + 2d_b).
\]

Note that, since \( [f(\mathcal{H}_1^1)/d_b] \neq [f(\mathcal{H}_1^2)/d_b] \), when \( a_{1,b} = \beta_1,b \) we have \( s_{1,b} = \sigma_1,b = c - 1 \).

In Case (II.a) we obtain
\[
\sum_{q=1}^{b-1} \sum_{w=1}^k \frac{a_{w,q}f_w}{d_q} \sum_{h=q}^l y^h_{w,q} + \frac{\beta_1,b}{d_b} \sum_{h=b}^l y^h_{b,b} = g_{k,b-1}(F_{b-1} + [f(\mathcal{H}_1^1)/d_b] + d_b) + \beta_1,b(c - 1) =
\]
\[ g_{k,b-1}(F_{b-1} + |f(H^b_1)/db|) + db) + \beta_{1,b}\sigma_{1,b} = g_{1,b}(\sum_{h=b}^{l} F_h). \]

In Case (II.b) we have

\[
\sum_{q=1}^{b-1} \sum_{w=1}^{k} \frac{a_{w,q}f_w}{dq} \sum_{h=q}^{l} y_{w,q}^h + \frac{\beta_{1,b}f_1}{db} \sum_{h=b}^{l} y_{1,b}^h = g_{k,b-1}(F_{b-1} + |f(H^b_1)/db| + 2db) + \beta_{1,b}(c-2) = \\
g_{k,b-1}(F_{b-1} + |f(H^b_1)/db| + db) + (g_{k,b-1}(F_{b-1} + |f(H^b_1)/db| + 2db) - g_{k,b-1}(F_{b-1} + |f(H^b_1)/db| + db)) + \beta_{1,b}(c-2) = \\
g_{k,b-1}(F_{b-1} + |f(H^b_1)/db| + db) + \beta_{1,b}(c-1) = g_{k,b-1}(F_{b-1} + |f(H^b_1)/db| + db) + \beta_{1,b}\sigma_{1,b} = g_{1,b}(\sum_{h=b}^{l} F_h).
\]

Finally, we have the case \(1 \leq c \leq f_1\tilde{b}_{1,b}/db\). Two subcases are considered: \(2 \leq c \leq f_1\tilde{b}_{1,b}/db\) and \(c = 1 \leq f_1\tilde{b}_{1,b}/db\). In the subcase \(2 \leq c \leq f_1\tilde{b}_{1,b}/db\), Theorem 4.5 and Proposition 3.6 imply that \(\sum_{h=b}^{l} f_1 y_{1,b}^h/\{c-2; c-1; c\}\) in every optimal solution. If \(\sum_{h=b}^{l} f_1 y_{1,b}^h = c - 2\) or \(\sum_{h=b}^{l} f_1 y_{1,b}^h = c - 1\) then the thesis follows by using the same arguments of Cases (II.a) and (II.b). If \(\sum_{h=b}^{l} f_1 y_{1,b}^h = c,\) by the induction hypothesis, for each optimal solution in \(y \in MO^{O_{k,b,F}}\) there exists an inequality such that:

\[
\sum_{q=1}^{b-1} \sum_{w=1}^{k} \frac{a_{w,q}f_w}{dq} \sum_{h=q}^{l} y_{w,q}^h = g_{k,b-1}(F_{b-1} + \sum_{h=b}^{l} F_h - cd_b) = g_{k,b-1}(F_{b-1} + |f(H^b_1)/db|).
\] (74)

Hence, we obtain

\[
\sum_{q=1}^{b-1} \sum_{w=1}^{k} \frac{a_{w,q}f_w}{dq} \sum_{h=q}^{l} y_{w,q}^h + \frac{\alpha_{1,b}f_1}{db} \sum_{h=b}^{l} y_{1,b}^h = g_{k,b-1}(F_{b-1} + |f(H^b_1)/db|) + \alpha_{1,b}c = \\
g_{k,b-1}(F_{b-1} + |f(H^b_1)/db|) + \alpha_{1,b}\sigma_{1,b} = g_{1,b}(\sum_{h=b}^{l} F_h),
\] (75)

and the thesis follows. In the subcase \(c = 1 \leq f_1\tilde{b}_{1,b}/db\), Theorem 4.5 and Proposition 3.6 imply that \(\sum_{h=b}^{l} f_1 y_{1,b}^h/\{c-1 = 0; c\}\) in every optimal solution. If \(\sum_{h=b}^{l} f_1 y_{1,b}^h = c - 1\) the thesis follows by the above Case (I). If \(\sum_{h=b}^{l} f_1 y_{1,b}^h = c\) then the thesis follows from (74) and (75). \(\square\)

We are now in the position of proving the main theorem.

**Theorem 5.5** If the inequalities in \(I(k,b-1,F)\) satisfy conditions 1), 2) and 3), for all \(F\) such that \(d_h\) divides \(F_h\), for \(h = 1, \ldots, b-2,\) and \(d_{b-1}\) divides \(F_{b-1}, \ldots, F_1,\) then the inequalities in \(I(k,b,F)\) satisfy conditions 1), 2) and 3) for all \(F\) such that \(d_h\) divides \(F_h\), for \(h = 1, \ldots, b-1,\) and \(d_b\) divides \(F_b, \ldots, F_1.\)
Proof. The thesis is proved by induction on the item types contained in the set $T'_b$ (i.e., the items that can be assigned to the parts $b, b + 1, \ldots, l$). If $T'_b$ only contains items of type 1, the thesis follows by Theorem 5.4. Let us assume that the thesis holds when $T'_b$ contains items of type 1, $k - 1$ and show it when $T'_b$ contains items of type 1, $k$. Again, observe that $\delta_b = d_b$. Hence, in what follows we use $d_b$ and $\delta_b$ indifferently.

The set $I(k, b, F)$ contains inequalities of the type (where each coefficient $a_{wq} \in \{\alpha_{wq}, \beta_{wq}\}$):

$$
\sum_{q=1}^{b-1} \sum_{w=1}^{k} \frac{a_{wq} f_w}{d_q} \sum_{h=q}^{l} y_{w,q} + \sum_{w=1}^{k-1} \frac{a_{w,b} f_w}{d_b} \sum_{h=b}^{l} y_{w,b} + \frac{a_{k,b} f_k}{d_b} \sum_{h=b}^{l} y_{k,b,} \leq g_{k,b}(\sum_{h=b}^{l} F_h). \quad (76)
$$

(a)

Let $F$ and $F'$ be two vectors with $l$ components such that $F_h = F'_h$ for $h = 1, \ldots, b - 1$ and $d_b$ divides $F_h$ and $F'_h$ for $h = b, \ldots, l$. The condition 1) directly follows by induction, by definition of the coefficients $a_{k,b}$ (see (57) and (58)), and since $f(H'_k)$ only depends by the values of $F_1, \ldots, F_{b-1}$.

(b)

We show that the inequalities in $I(k, b, F)$ are valid for $MP^{OOG}(k, b, F)$ (i.e., condition 2) holds). Let $y \in MP^{OOG}(k, b, F)$. First we prove that the following inequality holds in $y$:

$$
\sum_{q=1}^{b-1} \sum_{w=1}^{k} \frac{a_{wq} f_w}{d_q} \sum_{h=q}^{l} y_{w,q} + \sum_{w=1}^{k-1} \frac{a_{w,b} f_w}{d_b} \sum_{h=b}^{l} y_{w,b} + \frac{a_{k,b} f_k}{d_b} \sum_{h=b}^{l} y_{k,b} \leq

\left[ g_{k-1,b} \left( \sum_{h=b}^{l} F_h - \sum_{h=b}^{l} \left[ \frac{f_k y_{k,b}}{d_b} \right] \right) \right] + a_{k,b} \sum_{h=b}^{l} \left[ \frac{f_k y_{k,b}}{d_b} \right] \quad (77)
$$

In fact, by induction and condition 1) we have that

$$
\sum_{q=1}^{b-1} \sum_{w=1}^{k} \frac{a_{wq} f_w}{d_q} \sum_{h=q}^{l} y_{w,q} + \sum_{w=1}^{k-1} \frac{a_{w,b} f_w}{d_b} \sum_{h=b}^{l} y_{w,b} \leq g_{k-1,b}(G)
$$

is a valid inequality for $MP(k, b, F')$, when $T'_b$ contains items of type 1, $k - 1$, for each vector $F'$ such that: (i) $F'_h = F_h$, for $h = 1, \ldots, b - 1$; (ii) $F'_h, \ldots, F'_l$ are integers multiple of $d_b$ such that $\sum_{h=b}^{l} F'_h = G$. As a consequence and since, by the hypothesis, $d_b$ divides $F_h$ for $h = b, \ldots, l$, the inequality (77) follows.

To show that all the inequalities in $I(k, b, F)$ are valid, different cases are considered, as in Theorem 5.4.

If $\sum_{h=b}^{l} F_h \leq f(H'_k)$, then $s_{k,b} \leq 0$ (and $\sigma_{k,b} = 0$) both when $a_{k,b} = \alpha_{k,b}$ and $a_{k,b} = \beta_{k,b}$. If $\sum_{h=b}^{l} y_{k,b} = 0$, then, by definition (35), $g_{k,b}(\sum_{h=b}^{l} F_h) = g_{k-1,b}(\sum_{h=b}^{l} F_h)$. Hence, by (77), the inequalities (76) are valid. Suppose that

$\sum_{h=b}^{l} y_{k,b} > 0$. Then, if $a_{k,b} = \alpha_{k,b}$ by (77) we have
\[
\sum_{q=1}^{b-1} \sum_{w=1}^{k} \frac{a_{w,q} f_w}{d_q} \sum_{h=q}^{l} y_{w,q}^h + \sum_{w=1}^{k-1} \sum_{w=1}^{b} \frac{a_{w,b} f_w}{d_b} \sum_{h=b}^{l} y_{w,b}^h + \frac{\alpha_{k,b} f_k}{d_b} \sum_{h=b}^{l} y_{k,b}^h \leq g_{k-1,b} \left( \sum_{h=b}^{l} F_h - \sum_{h=b}^{l} \left[ \frac{f_k y_{k,b}^h}{d_b} \right] d_b \right) + \alpha_{k,b} \sum_{h=b}^{l} \left[ \frac{f_k y_{k,b}^h}{d_b} \right] d_b + \frac{\beta_{k,b} f_k}{d_b} \sum_{h=b}^{l} y_{k,b}^h \leq g_{k-1,b} \left( \sum_{h=b}^{l} F_h - \sum_{h=b}^{l} \left[ \frac{f_k y_{k,b}^h}{d_b} \right] d_b \right) + \beta_{k,b} \sum_{h=b}^{l} \left[ \frac{f_k y_{k,b}^h}{d_b} \right] d_b + \frac{\beta_{k,b} f_k}{d_b} \sum_{h=b}^{l} y_{k,b}^h
\]

where the last inequality follows from condition (ii) of Lemma 5.3 and the last two equalities from (63) and (65) (and since \( \sigma_{k,b} = 0 \)). Hence, the inequalities (76) in which \( \alpha_{k,b} = \alpha_{k,b} \) are valid. The above argument holds even if \( \alpha_{k,b} \) is replaced with \( \beta_{k,b} \), and if \( \beta_{k,b} \) is replaced by the expression (58). Hence, the thesis follows.

If \( \sum_{h=b}^{l} F_h - ([f(H^b_k)/\delta_b]d_b + \delta_b) \geq f_k \tilde{b}_{k,b} \), then \( s_{k,b} = f_k \tilde{b}_{k,b} \) (and \( \sigma_{k,b} = \frac{f_k \tilde{b}_{k,b}}{d_b} \)), both when \( \alpha_{k,b} = \alpha_{k,b} \) and \( \alpha_{k,b} = \beta_{k,b} \). If \( \sum_{h=b}^{l} y_{k,b}^h = \tilde{b}_{k,b} \) then from (65) and by induction we have that all the inequalities (76) are valid.

Suppose that \( \sum_{h=b}^{l} y_{k,b}^h < \tilde{b}_{k,b} \). If \( \alpha_{k,b} = \beta_{k,b} \), by (77) we have

\[
\sum_{q=1}^{b-1} \sum_{w=1}^{k} \frac{a_{w,q} f_w}{d_q} \sum_{h=q}^{l} y_{w,q}^h + \sum_{w=1}^{k-1} \sum_{w=1}^{b} \frac{a_{w,b} f_w}{d_b} \sum_{h=b}^{l} y_{w,b}^h + \frac{\beta_{k,b} f_k}{d_b} \sum_{h=b}^{l} y_{k,b}^h \leq g_{k-1,b} \left( \sum_{h=b}^{l} F_h - \sum_{h=b}^{l} \left[ \frac{f_k y_{k,b}^h}{d_b} \right] d_b \right) + \beta_{k,b} \sum_{h=b}^{l} \left[ \frac{f_k y_{k,b}^h}{d_b} \right] d_b + \frac{\beta_{k,b} f_k}{d_b} \sum_{h=b}^{l} y_{k,b}^h
\]

and by condition (i) of Lemma 5.3 we have

\[
Q \leq g_{k-1,b} \left( \sum_{h=b}^{l} F_h - f_k \tilde{b}_{k,b} \right) + \left( \frac{f_k}{d_b} \tilde{b}_{k,b} - \sum_{h=b}^{l} \left[ \frac{f_k y_{k,b}^h}{d_b} \right] d_b \right) (g_{k-1,b}(\lceil f(H^b_k)/\delta_b \rceil \delta_b + 2\delta_b) - g_{k-1,b}(\lceil f(H^b_k)/\delta_b \rceil \delta_b + \delta_b)) = Q,
\]

where

\[
Q \leq g_{k-1,b} \left( \sum_{h=b}^{l} F_h - f_k \tilde{b}_{k,b} \right) + \left( \frac{f_k}{d_b} \tilde{b}_{k,b} - \sum_{h=b}^{l} \left[ \frac{f_k y_{k,b}^h}{d_b} \right] d_b \right) (g_{k-1,b}(\lceil f(H^b_k)/\delta_b \rceil \delta_b + 2\delta_b) - g_{k-1,b}(\lceil f(H^b_k)/\delta_b \rceil \delta_b + \delta_b)) + \frac{f_k}{d_b} \tilde{b}_{k,b} \beta_{k,b} = g_{k-1,b}(U) + \frac{f_k}{d_b} \tilde{b}_{k,b} \beta_{k,b} = g_{k-1,b} \left( \sum_{h=b}^{l} F_h \right),
\]

and

\[
Q \leq g_{k-1,b} \left( \sum_{h=b}^{l} F_h - f_k \tilde{b}_{k,b} \right) + \left( \frac{f_k}{d_b} \tilde{b}_{k,b} - \sum_{h=b}^{l} \left[ \frac{f_k y_{k,b}^h}{d_b} \right] d_b \right) (g_{k-1,b}(\lceil f(H^b_k)/\delta_b \rceil \delta_b + 2\delta_b) - g_{k-1,b}(\lceil f(H^b_k)/\delta_b \rceil \delta_b + \delta_b)) + \frac{f_k}{d_b} \tilde{b}_{k,b} \beta_{k,b} = g_{k-1,b}(U) + \frac{f_k}{d_b} \tilde{b}_{k,b} \beta_{k,b} = g_{k-1,b} \left( \sum_{h=b}^{l} F_h \right),
\]

44
where the last two equalities follow from (63) and (65). If \( a_{k,b} = \alpha_{k,b} \), the above arguments can be repeated and easily adapted to show the thesis, too.

Finally, consider the case in which \( \sum_{h=b}^{l} F_h - [f(H_h^k)/d_b]d_b < f_k b_{k,b} \) and \( \sum_{h=b}^{l} F_h > [f(H_h^k)/d_b]d_b \). Two subcases can be considered: either \( \sum_{h=b}^{l} F_h < f_k b_{k,b} \) or \( \sum_{h=b}^{l} F_h > f_k b_{k,b} \). In the first subcase, we have \( 0 \leq s_{k,b} \leq \frac{f_k b_{k,b}}{d_b} \). Then, by (61), (63) and (65), and since \( \sigma_{k,b} = s_{k,b} \) (see (62)), inequality (77) becomes

\[
\sum_{q=1}^{b-1} \sum_{w=1}^{k} \frac{a_{w,q}f_w}{d_q} \sum_{h=q}^{l} y_{w,q}^h + \sum_{w=1}^{k-1} \frac{a_{w,b}f_w}{d_b} \sum_{h=b}^{l} y_{w,b}^h + \frac{a_{k,b}f_k}{d_b} \sum_{h=b}^{l} y_{k,b}^h \leq
\]

\[
g_{k-1,b}(\sum_{h=b}^{l} \left\lfloor \frac{f_k y_{k,b}^h}{d_b} \right\rfloor) + a_{k,b}(\sum_{h=b}^{l} \left\lfloor \frac{f_k y_{k,b}^h}{d_b} \right\rfloor) - s_{kb} + a_{k,b}s_{k,b}. \tag{78}
\]

If \( \sum_{h=b}^{l} \left\lfloor \frac{f_k y_{k,b}^h}{d_b} \right\rfloor = s_{k,b} \), by (77) and by recalling that \( g_{k,b}(\sum_{h=b}^{l} F_h) = g_{k-1,b}(U) + a_{k,b}\sigma_{k,b} \), the inequalities (76) are satisfied for \( a_{k,b} \in \{a_{k,b}; \beta_{k,b}\} \). Observe that since \( 0 \leq s_{k,b} \leq \frac{f_k b_{k,b}}{d_b} \), then

\[
a_{k,b} = g_{k-1,b}(U + d_b) - g_{k-1,b}(U). \tag{79}
\]

If \( \sum_{h=b}^{l} \left\lfloor \frac{f_k y_{k,b}^h}{d_b} \right\rfloor > s_{k,b} \), then by condition (ii) of Lemma 5.3 and by (78) we have that

\[
g_{k-1,b}(U - d_b(\sum_{h=b}^{l} \left\lfloor \frac{f_k y_{k,b}^h}{d_b} \right\rfloor) - s_{kb}) + a_{k,b}(\sum_{h=b}^{l} \left\lfloor \frac{f_k y_{k,b}^h}{d_b} \right\rfloor) - s_{kb} \leq g_{k-1,b}(U).
\]

Then by (78) and since \( g_{k,b}(\sum_{h=b}^{l} F_h) = g_{k-1,b}(U) + a_{k,b}s_{k,b} \), we have that the inequality (76) holds. Finally, if

\[
\sum_{h=b}^{l} \left\lfloor \frac{f_k y_{k,b}^h}{d_b} \right\rfloor < s_{k,b}, \text{ by condition (i) of Lemma 5.3 and by (79) it follows that}
\]

\[
g_{k-1,b}(U - d_b(\sum_{h=b}^{l} \left\lfloor \frac{f_k y_{k,b}^h}{d_b} \right\rfloor) - s_{kb}) + a_{k,b}(\sum_{h=b}^{l} \left\lfloor \frac{f_k y_{k,b}^h}{d_b} \right\rfloor) - s_{kb} =
\]

\[
g_{k-1,b}(U + d_b(s_{k,b} - \sum_{h=b}^{l} \left\lfloor \frac{f_k y_{k,b}^h}{d_b} \right\rfloor)) - a_{k,b}(s_{k,b} - \sum_{h=b}^{l} \left\lfloor \frac{f_k y_{k,b}^h}{d_b} \right\rfloor) \leq g_{k-1,b}(U).
\]

Hence, by (78), the inequalities (76) are valid.
In the second subcase, i.e., \( \sum_{h=b}^l F_h - [f(\mathcal{H}_k^h)/d_b]d_b > f_k \tilde{b}_{k,b} \), we have \( 0 \leq s_{k,b} \leq \frac{f_k \tilde{b}_{k,b}}{d_b} + 1 \). If \( 0 \leq s_{k,b} \leq \frac{f_k \tilde{b}_{k,b}}{d_b} \) the arguments of the first subcase can be used to show the thesis. If \( s_{k,b} = \frac{f_k \tilde{b}_{k,b}}{d_b} + 1 > \frac{f_k \tilde{b}_{k,b}}{d_b} \), the arguments applied in the case \( \sum_{h=b}^l F_h - ([f(\mathcal{H}_k^h)/d_b]d_b + d_b) \geq f_k \tilde{b}_{k,b} \) prove that the inequalities (70) are valid.

(c)

We show now that condition 3) holds, i.e., \( MO^O(k, b, F) \) is contained in the faces induced by the inequalities in \( I(k, b, F) \). Recall that, by induction, condition 3) holds when \( T_k^i \) does no contain items of type \( k \). According to the Definition of \( U \) and \( s_{k,b} \), we can write \( \sum_{h=b}^l F_h = U + \sigma_{k,b}d_b \). Let \( y \in MO^O(k, b, F) \) be an optimal solution and let

\[
c = \sum_{h=b}^l F_h - [f(\mathcal{H}_k^h)/d_b]d_b \sum_{h=b}^l d_b.
\]

Recalling that \( \delta_b = d_b \), Theorem 4.5 and Proposition 3.6 imply that \( \sum_{h=b}^l \frac{f_k y_{k,b}}{d_b} \) can attain one of the following values: \( \min\{c^+; f_k \tilde{b}_{k,b}/d_b\} \) or \( \min\{(c-1)^+; f_k \tilde{b}_{k,b}/d_b\} \) or \( \min\{(c-2)^+; f_k \tilde{b}_{k,b}/d_b\} \).

In the case \( c \leq 0 \) then \( \sum_{h=b}^l y_{k,b}^h = 0 \) in every optimal solution. Hence, by induction, for each optimal solution in \( y \in MO^O(k, b, F) \) there exists a choice of \( a_{w,q} \in \{\alpha_{w,q}, \beta_{w,q}\} \) and \( a_{w,b} \in \{\alpha_{w,b}, \beta_{w,b}\} \), for \( q = 1, \ldots, b-1 \) and \( w = 1, \ldots, k \), such that

\[
\sum_{q=1}^{b-1} \sum_{w=1}^k a_{w,q} f_w d_q \sum_{h=q}^l y_{w,q}^h + \sum_{w=1}^{k-1} a_{w,b} f_w d_b \sum_{h=b}^l y_{w,b}^h = g_{k-1,b}(\sum_{h=b}^l F_h).
\]

Since in this case \( s_{k,b} \leq 0 \), then \( \sigma_{k,b} = 0 \) and \( g_{k-1,b}(\sum_{h=b}^l F_h) = g_{k,b}(\sum_{h=b}^l F_h) \). Hence, the thesis follows.

In the case \( c - 2 \geq f_k \tilde{b}_{k,b}/d_b \), then Theorem 4.5 and Proposition 3.6 imply that \( \sum_{h=b}^l y_{k,b}^h = \tilde{b}_{k,b} \) in every optimal solution. Hence, by induction, for each optimal solution in \( y \in MO^O(k, b, F) \) there exists an inequality such that

\[
\sum_{q=1}^{b-1} \sum_{w=1}^k a_{w,q} f_w d_q \sum_{h=q}^l y_{w,q}^h + \sum_{w=1}^{k-1} a_{w,b} f_w d_b \sum_{h=b}^l y_{w,b}^h = g_{k-1,b}(\sum_{h=b}^l F_h - f_k \tilde{b}_{k,b}).
\]

In this case \( s_{k,b} \geq f_k \tilde{b}_{k,b}/d_b \), both when \( a_{k,b} = \alpha_{k,b} \) and \( a_{k,b} = \beta_{k,b} \). Hence, by definition, \( \sigma_{k,b} = f_k \tilde{b}_{k,b}/d_b \), as \( \sum_{h=b}^l y_{k,b}^h = \tilde{b}_{k,b} \) in every optimal solution, we obtain

\[
\sum_{q=1}^{b-1} \sum_{w=1}^k a_{w,q} f_w d_q \sum_{h=q}^l y_{w,q}^h + \sum_{w=1}^{k-1} a_{w,b} f_w d_b \sum_{h=b}^l y_{w,b}^h + a_{k,b} \tilde{b}_{k,b} d_b \sum_{h=b}^l y_{k,b}^h = g_{k-1,b}(\sum_{h=b}^l F_h - f_k \tilde{b}_{k,b}) + a_{k,b} f_k \tilde{b}_{k,b}/d_b = g_{k,b}(\sum_{h=b}^l F_h).
\]
And, the thesis holds both if \( a_{k,b} = \alpha_{k,b} \) and \( a_{k,b} = \beta_{k,b} \).

In the case \( c = \frac{f_k b_{k,b}}{d_b} + 1 \), two subcases can be considered: (I) \( [f(H_k^b)/d_b] = [f(H_k^b)/d_b] \); (II) \( [f(H_k^b)/d_b] \neq [f(H_k^b)/d_b] \).

Theorem 4.5 and Proposition 4.6 imply that \( \sum_{h=b}^{l} f_k y_{h,k,b}/d_b = c-1 = f_k b_{k,b}/d_b \) in Case (I), while \( \sum_{h=b}^{l} f_k y_{h,k,b}/d_b \in \{c-1; c-2\} \) in Case (II) (note that since w.l.o.g. \( f_k b_{k,b}/d_b \geq 0 \) then \( c-2 \geq 0 \), in this case), in every optimal solution.

In Case (I), by induction, for each optimal solution in \( y \in MOOO(k,b,F) \) there exists an inequality such that

\[
\sum_{q=1}^{b-1} \sum_{w=1}^{k} \frac{a_{w,q} f_w}{d_q} \sum_{h=q}^{l} y_{h,w,q} + \sum_{w=1}^{k} \frac{a_{w,b} f_w}{d_b} \sum_{h=b}^{l} y_{h,w,b} = g_{k-1,b} \left( \sum_{h=b}^{l} F_h - (c-1) d_b \right) = g_{k-1,b} \left( [f(H_k^b)/d_b] + d_b \right). 
\]

Hence, setting \( a_{k,b} = \alpha_{k,b} \) and consequently \( s_{k,b} = \sigma_{k,b} = c - 1 \) we obtain

\[
\sum_{q=1}^{b-1} \sum_{w=1}^{k} \frac{a_{w,q} f_w}{d_q} \sum_{h=q}^{l} y_{h,w,q} + \sum_{w=1}^{k} \frac{a_{w,b} f_w}{d_b} \sum_{h=b}^{l} y_{h,w,b} + \frac{\alpha_{k,b} f_k}{d_b} \sum_{h=b}^{l} y_{h,k,b} = g_{k-1,b} \left( [f(H_k^b)/d_b] + d_b \right) + (c-1) \alpha_{k,b} = \\
g_{k-1,b} \left( [f(H_k^b)/d_b] \right) + \left( g_{k-1,b} \left( [f(H_k^b)/d_b] + d_b \right) - g_{k-1,b} \left( [f(H_k^b)/d_b] \right) \right) + (c-1) \alpha_{k,b} = \\
g_{k-1,b} \left( [f(H_k^b)/d_b] \right) + g_{k,b} \left( \sum_{h=b}^{l} F_h \right).
\]

Note that, since \( [f(H_k^b)/d_b] = [f(H_k^b)/d_b] \), the above relations hold even if \( \alpha_{k,b} \) and \( [f(H_k^b)/d_b] \) are replaced by \( \beta_{k,b} \) and \( [f(H_k^b)/d_b] \), respectively.

In Case (II), by induction, for each optimal solution in \( y \in MOOO(k,b,F) \) there exists an inequality such that:

(II.a)

\[
\sum_{q=1}^{b-1} \sum_{w=1}^{k} \frac{a_{w,q} f_w}{d_q} \sum_{h=q}^{l} y_{h,w,q} + \sum_{w=1}^{k} \frac{a_{w,b} f_w}{d_b} \sum_{h=b}^{l} y_{h,w,b} = g_{k-1,b} \left( \sum_{h=b}^{l} F_h - (c-1) d_b \right) = g_{k-1,b} \left( [f(H_k^b)/d_b] + d_b \right)
\]

if \( \sum_{h=b}^{l} f_k y_{h,k,b}/d_b = c-1 \); and

(II.b)

\[
\sum_{q=1}^{b-1} \sum_{w=1}^{k} \frac{a_{w,q} f_w}{d_q} \sum_{h=q}^{l} y_{h,w,q} + \sum_{w=1}^{k} \frac{a_{w,b} f_w}{d_b} \sum_{h=b}^{l} y_{h,w,b} + \frac{\beta_{k,b} f_k}{d_b} \sum_{h=b}^{l} y_{h,k,b} = g_{k-1,b} \left( \sum_{h=b}^{l} F_h - (c-2) d_b \right) = g_{k-1,b} \left( [f(H_k^b)/d_b] + 2d_b \right)
\]
if \( \sum_{h=b}^{l} f_k y_{k,h}/d_b = c - 2 \).

Note that, since \( f(H_k^b)/d_b \neq f(H_k^h)/d_b \), when \( a_{k,b} = \beta_{k,b} \) we have \( s_{k,b} = \sigma_{k,b} = c - 1 \).

In Case (II.a) we obtain

\[
\sum_{q=1}^{b-1} \sum_{w=1}^{k} \frac{a_{w,q} f_w}{d_q} \sum_{h=q}^{l} y_{w,q} + \frac{a_{w,b} f_w}{d_b} \sum_{h=q}^{l} y_{w,b} + \frac{\beta_{k,b}}{d_b} \sum_{h=q}^{l} y_{k,b} = g_{k-1,b}(\lfloor f(H_k^b)/d_b \rfloor + d_b) + \beta_{k,b}(c-1) =
\]

\[
g_{k-1,b}(\lfloor f(H_k^b)/d_b \rfloor + d_b) + \beta_{k,b}(c-1) = g_{k-1,b}(\lfloor f(H_k^b)/d_b \rfloor + d_b) + \beta_{k,b}(c-1) =
\]

In Case (II.b) we have

\[
\sum_{q=1}^{b-1} \sum_{w=1}^{k} \frac{a_{w,q} f_w}{d_q} \sum_{h=q}^{l} y_{w,q} + \frac{a_{w,b} f_w}{d_b} \sum_{h=q}^{l} y_{w,b} + \frac{\beta_{k,b}}{d_b} \sum_{h=q}^{l} y_{k,b} = g_{k-1,b}(\lfloor f(H_k^b)/d_b \rfloor + d_b) + \beta_{k,b}(c-2) =
\]

\[
g_{k-1,b}(\lfloor f(H_k^b)/d_b \rfloor + d_b) + \beta_{k,b}(c-2) = g_{k-1,b}(\lfloor f(H_k^b)/d_b \rfloor + d_b) + \beta_{k,b}(c-2) =
\]

Finally, we have the case \( 1 \leq c \leq f_k b_{k,b}/d_b \). Two subcases are considered: \( 2 \leq c \leq f_k b_{k,b}/d_b \) and \( c = 1 \leq f_k b_{k,b}/d_b \). In the subcase \( 2 \leq c \leq f_k b_{k,b}/d_b \), Theorem 4.5 and Proposition 3.6 imply that \( \sum_{h=b}^{l} f_k y_{k,h}/d_b \in \{c-2; c-1; c\} \) in every optimal solution. If \( \sum_{h=b}^{l} f_k y_{k,h}/d_b = c-2 \) or \( \sum_{h=b}^{l} f_k y_{k,h}/d_b = c-1 \) then the thesis follows by using the same arguments of Cases (II.a) and (II.b). If \( \sum_{h=b}^{l} f_k y_{k,h}/d_b = c \), by the induction hypothesis, for each optimal solution in \( y \in MOOO(k, b, F) \) there exists a choice of \( a_{w,q} \in \{\alpha_{w,q}, \beta_{w,q}\} \) and \( a_{w,b} \in \{\alpha_{w,b}, \beta_{w,b}\} \), for \( q = 1, \ldots, b-1 \) and \( w = 1, \ldots, k \), such that

\[
\sum_{q=1}^{b-1} \sum_{w=1}^{k} \frac{a_{w,q} f_w}{d_q} \sum_{h=q}^{l} y_{w,q} + \frac{a_{w,b} f_w}{d_b} \sum_{h=q}^{l} y_{w,b} = g_{k-1,b} \left( \sum_{h=b}^{l} F_h - c d_b \right) = g_{k-1,b}(\lfloor f(H_k^b)/d_b \rfloor).
\]

(80)

Hence, we obtain

\[
\sum_{q=1}^{b-1} \sum_{w=1}^{k} \frac{a_{w,q} f_w}{d_q} \sum_{h=q}^{l} y_{w,q} + \frac{a_{w,b} f_w}{d_b} \sum_{h=q}^{l} y_{w,b} + \frac{\alpha_{k,b}}{d_b} \sum_{h=q}^{l} y_{k,b} =
\]

\[
g_{k-1,b}(\lfloor f(H_k^b)/d_b \rfloor) + \alpha_{k,b} c = g_{k-1,b}(\lfloor f(H_k^b)/d_b \rfloor) + \alpha_{k,b} \sigma_{k,b} = g_{k,b} \left( \sum_{h=b}^{l} F_h \right),
\]

(81)

and the thesis follows. In the subcase \( c = 1 \leq f_k b_{k,b}/d_b \), Theorem 4.5 and Proposition 3.6 imply that \( \sum_{h=b}^{l} f_k y_{k,h}/d_b \in \{c - 1 = 0; c\} \) in every optimal solution. If \( \sum_{h=b}^{l} f_k y_{k,h}/d_b = c - 1 \) the thesis follows by the above Case (I). If \( \sum_{h=b}^{l} f_k y_{k,h}/d_b = c \) then the thesis follows from (80) and (81).

48
Example (continued)

We continue the example reported at the end of the Section 4. For the sake of simplicity we derive only the set of inequalities \( I(5, 2, (1, 6, 8)) \) (also in this case \( I(k, b, (F_1, F_2, F_3)) \) denotes the set \( I(k, b, F) \), where \( F \) has components \( F_1, F_2 \) and \( F_3 \) related to the solution set \( MO(5, 2, (1, 6, 8)) \equiv MO(4, 2, (1, 6, 8)) \). At this aim, we first consider the sets \( I(1, 1, (F_2, F_3)) \), \( I(2, 2, (1, 6, F_3)) \), \( I(3, 2, (1, 6, 8)) \), and, finally, \( I(5, 2, (1, 6, 8)) \equiv I(4, 2, (1, 6, 8)) \).

\[
I(1, 1, (F_2, F_3)) \equiv I(5, 1, (F_2, F_3))
\]

By (10), since \( \tilde{b}_{1,1} = 2 \), we have that \( I(1, 1, (F_2, F_3)) \) contains the inequality \( \sum_{h=1}^{4} y_{1,1}^h \leq 2 \) when \( F_2 + F_3 > 0 \), and \( \sum_{h=1}^{4} y_{1,1}^h \leq 1 \) if \( F_2 + F_3 = 0 \). Recall that, given \( \gamma \) in \( MO(1, 1, (F_2, F_3)) \), we have \( \gamma_{1,1}^1 = 1 \) and \( \gamma_{1,1}^2 + \gamma_{1,1}^3 = 1 \) if \( F_2 + F_3 = 2 \), and \( \gamma_{1,1}^1 = 1 \) and \( \gamma_{1,1}^2 + \gamma_{1,1}^3 = 0 \) if \( F_2 + F_3 = 0 \). Hence, \( I(1, 1, (F_2, F_3)) \) contains valid inequalities and \( MO^{oo}(1, 1, (F_2, F_3)) \) is contained in the faces induced by \( I(1, 1, (F_2, F_3)) \).

\[
I(2, 2, (1, 6, F_3))
\]

Computation of \( \alpha_{2,2} \) and \( \beta_{2,2} \): Recall that \( H_2 = \emptyset \) and hence \( f(H_2^2) = 0 \). By definition, we have \( \alpha_{2,2} = g_{1,2}(d_2) - g_{1,2}(0) = g_{1,2}(2) - g_{1,2}(0) = 1 \) and \( \beta_{2,2} = g_{1,2}(2d_2) - g_{1,2}(d_2) = g_{1,2}(4) - g_{1,2}(2) = g_{1,1}(1 + 4) - g_{1,1}(1 + 2) = 0 \). Hence, \( I(2, 2, (1, 6, F_3)) \) contains the two inequalities:

\[
\sum_{h=1}^{3} y_{1,1}^h + \alpha_{2,2} \sum_{h=2}^{3} y_{2,2}^h = \sum_{h=1}^{3} y_{1,1}^h + \sum_{h=2}^{3} y_{2,2}^h \leq g_{2,2}(6 + F_3)
\]

and

\[
\sum_{h=1}^{3} y_{1,1}^h + \beta_{2,2} \sum_{h=2}^{3} y_{2,2}^h = \sum_{h=1}^{3} y_{1,1}^h \leq g_{2,2}(6 + F_3).
\]

\[
I(3, 2, (1, 6, 8))
\]

Computation of \( \alpha_{3,2} \) and \( \beta_{3,2} \): Since \( f(H_3^2) = 8 \), we have \( \alpha_{3,2} = g_{2,2}(8 + d_2) - g_{2,2}(8) = g_{2,2}(10) - g_{2,2}(8) \) and \( \beta_{3,2} = g_{2,2}(8 + 2d_2) - g_{2,2}(8 + d_2) = g_{2,2}(12) - g_{2,2}(10) \).

Computation of \( \alpha_{3,2} \). Two cases hold: 1) \( \alpha_{2,2} = \alpha_{2,2} = 1 \); 2) \( \alpha_{2,2} = \beta_{2,2} = 0 \).

Case 1)
\[
g_{2,2}(10) \rightarrow s_{2,2} = 10/2 = 5, \sigma_{2,2} = 4, \text{ then } g_{2,2}(10) = g_{1,2}(2) + \alpha_{2,2}4 = g_{1,1}(1 + 2) + 4 = 6.
\]

Case 2)
$g_{2,2}(10) \rightarrow s_{2,2} = (10 - 2)/2 = 4, \sigma_{2,2} = 4$, then $g_{2,2}(10) = g_{1,2}(2) + \beta_{2,2}4 = g_{1,1}(1 + 2) = 2$.

$g_{2,2}(8) \rightarrow s_{2,2} = (8 - 2)/2 = 3, \sigma_{3,2} = 3$, then $g_{2,2}(8) = g_{1,2}(2) + \beta_{2,2}3 = g_{1,1}(1 + 2) = 2$. Hence, $\alpha_{3,2} = 0$.

Computation of $\beta_{3,2}$. Two cases hold: 1) $a_{2,2} = \alpha_{2,2} = 1$; 2) $a_{2,2} = \beta_{2,2} = 0$.

Case 1)

$g_{2,2}(12) \rightarrow s_{2,2} = 12/2 = 6, \sigma_{2,2} = 4$, then $g_{2,2}(12) = g_{1,2}(4) + \alpha_{2,2}3 = g_{1,1}(1 + 4) + 4 = 6$.

$g_{2,2}(10) \rightarrow s_{2,2} = 10/2 = 5, \sigma_{2,2} = 4$, then $g_{2,2}(10) = g_{1,2}(2) + \alpha_{2,2}4 = g_{1,1}(1 + 2) + 4 = 6$. Hence, $\beta_{3,2} = 0$.

Case 2)

$g_{2,2}(12) \rightarrow s_{2,2} = (12 - 2)/2 = 5, \sigma_{2,2} = 4$, then $g_{2,2}(12) = g_{1,2}(4) + \beta_{2,2}4 = g_{1,1}(1 + 4) = 2$.

$g_{2,2}(10) \rightarrow s_{2,2} = (10 - 2)/2 = 4, \sigma_{2,2} = 4$, then $g_{2,2}(10) = g_{1,2}(2) + \beta_{2,2}4 = g_{1,1}(1 + 2) = 2$. Hence, $\beta_{3,2} = 0$.

The inequalities in $I(3, 2, (1, 6, 8))$ will be given later.

$I(4, 2, (1, 6, 8)) \equiv I(5, 2, (1, 6, 8))$

Computation of $\alpha_{4,2}$ and $\beta_{4,2}$: Since $f(H^2_4) = 24$ we have $\alpha_{4,2} = g_{3,2}(24 + d_2) - g_{3,2}(24) = g_{3,2}(26) - g_{3,2}(24)$ and $\beta_{4,2} = g_{3,2}(24 + 2d_2) - g_{3,2}(24 + d_2) = g_{3,2}(28) - g_{3,2}(26)$.

Computation of $\alpha_{4,2}$. Two cases hold: 1) $a_{3,2} = \alpha_{3,2}$; 2) $a_{3,2} = \beta_{3,2} = 0$.

Case 1)

$g_{3,2}(26) \rightarrow s_{3,2} = (26 - 8)/2 = 9, \sigma_{3,2} = 8$, then $g_{3,2}(26) = g_{2,2}(10) + \alpha_{3,2}8$. By Cases 1 and 2 of $\alpha_{3,2}$, we have that $g_{2,2}(10) = 6$ when $a_{2,2} = \alpha_{2,2} = 1$, and $g_{2,2}(10) = 2$ when $a_{2,2} = \beta_{2,2} = 0$. Hence, $g_{3,2}(26) = 6 + \alpha_{3,2}8$ when $a_{2,2} = \alpha_{2,2} = 1$, and $g_{3,2}(26) = 2 + \alpha_{3,2}8$ when $a_{2,2} = \beta_{2,2} = 0$.

$g_{3,2}(24) \rightarrow s_{3,2} = (24 - 8)/2 = 8, \sigma_{3,2} = 8$, then $g_{3,2}(24) = g_{2,2}(8) + \alpha_{3,2}8$. By Cases 1 and 2 of $\alpha_{3,2}$, we have that $g_{2,2}(8) = 5$ when $a_{2,2} = \alpha_{2,2} = 1$, and $g_{2,2}(8) = 2$ when $a_{2,2} = \beta_{2,2} = 0$. Hence, $g_{3,2}(24) = 5 + \alpha_{3,2}8$ when $a_{2,2} = \alpha_{2,2} = 1$, and $g_{3,2}(24) = 2 + \alpha_{3,2}8$ when $a_{2,2} = \beta_{2,2} = 0$.

Hence, $\alpha_{4,2} = 6 + \alpha_{3,2}8 - (5 + \alpha_{3,2}8) = 1$ when $a_{2,2} = \alpha_{2,2} = 1$, and $\alpha_{4,2} = 2 + \alpha_{3,2}8 - (2 + \alpha_{3,2}8) = 0$ when $a_{2,2} = \beta_{2,2} = 0$.

Case 2)

$g_{3,2}(26) \rightarrow s_{3,2} = (26 - 10)/2 = 8, \sigma_{3,2} = 8$, then $g_{3,2}(26) = g_{2,2}(10) + \beta_{3,2}8 = g_{2,2}(10)$. By Cases 1 and 2 of $\beta_{3,2}$, we have that $g_{2,2}(10) = 6$ when $a_{2,2} = \alpha_{2,2} = 1$, and $g_{2,2}(10) = 2$ when $a_{2,2} = \beta_{2,2} = 0$. Hence, $g_{3,2}(26) = 6$
when $a_{2,2} = \alpha_{2,2} = 1$, and $g_{3,2}(26) = 2$ when $a_{2,2} = \beta_{2,2} = 0$.

$g_{3,2}(24) \to s_{3,2} = (24 - 10)/2 = 7$, $\sigma_{3,2} = 7$, then $g_{3,2}(24) = g_{2,2}(10) + \beta_{3,2} = g_{2,2}(10)$. Again, by Cases 1 and 2 of $\beta_{3,2}$, we have that $g_{2,2}(10) = 6$ when $a_{2,2} = \alpha_{2,2} = 1$, and $g_{2,2}(10) = 2$ when $a_{2,2} = \beta_{2,2} = 0$. Hence, $g_{3,2}(24) = 6$ when $a_{2,2} = \alpha_{2,2} = 1$, and $g_{3,2}(24) = 2$ when $a_{2,2} = \beta_{2,2} = 0$.

Hence, $\alpha_{4,2} = 0$ both when $a_{2,2} = \alpha_{2,2} = 1$, and when $a_{2,2} = \beta_{2,2} = 0$.

Computation of $\beta_{4,2}$. Two cases hold: 1) $a_{3,2} = \alpha_{3,2}$; 2) $a_{3,2} = \beta_{3,2} = 0$.

Case 1)

$g_{3,2}(28) \to s_{3,2} = (28 - 8)/2 = 10$, $\sigma_{3,2} = 8$, then $g_{3,2}(28) = g_{2,2}(12) + \alpha_{3,2}8$. By Cases 1 and 2 of $\alpha_{3,2}$, we have that $g_{2,2}(12) = 6$ when $a_{2,2} = \alpha_{2,2} = 1$, and $g_{2,2}(12) = 2$ when $a_{2,2} = \beta_{2,2} = 0$. Hence, $g_{3,2}(28) = 6 + \alpha_{3,2}8$ when $a_{2,2} = \alpha_{2,2} = 1$, and $g_{3,2}(28) = 2 + \alpha_{3,2}8$ when $a_{2,2} = \beta_{2,2} = 0$.

$g_{3,2}(26) \to s_{3,2} = (26 - 8)/2 = 9$, $\sigma_{3,2} = 8$, then $g_{3,2}(26) = g_{2,2}(10) + \alpha_{3,2}8$. By Cases 1 and 2 of $\alpha_{3,2}$, we have that $g_{2,2}(10) = 6$ when $a_{2,2} = \alpha_{2,2} = 1$, and $g_{2,2}(10) = 2$ when $a_{2,2} = \beta_{2,2} = 0$. Hence, $g_{3,2}(26) = 6 + \alpha_{3,2}8$ when $a_{2,2} = \alpha_{2,2} = 1$, and $g_{3,2}(26) = 2 + \alpha_{3,2}8$ when $a_{2,2} = \beta_{2,2} = 0$.

Hence, $\beta_{4,2} = 0$ both when $a_{2,2} = \alpha_{2,2} = 1$, and when $a_{2,2} = \beta_{2,2} = 0$.

Case 2)

$g_{3,2}(28) \to s_{3,2} = (28 - 10)/2 = 9$, $\sigma_{3,2} = 8$, then $g_{3,2}(28) = g_{2,2}(12) + \beta_{3,2}8 = g_{2,2}(12)$. By Cases 1 and 2 of $\beta_{3,2}$, we have that $g_{2,2}(12) = 6$ when $a_{2,2} = \alpha_{2,2} = 1$, and $g_{2,2}(12) = 2$ when $a_{2,2} = \beta_{2,2} = 0$. Hence, $g_{3,2}(28) = 6$ when $a_{2,2} = \alpha_{2,2} = 1$, and $g_{3,2}(28) = 2$ when $a_{2,2} = \beta_{2,2} = 0$.

$g_{3,2}(26) \to s_{3,2} = (26 - 10)/2 = 8$, $\sigma_{3,2} = 8$, then $g_{3,2}(26) = g_{2,2}(10) + \beta_{3,2}8 = g_{2,2}(10)$. By Cases 1 and 2 of $\beta_{3,2}$, we have that $g_{2,2}(10) = 6$ when $a_{2,2} = \alpha_{2,2} = 1$, and $g_{2,2}(10) = 2$ when $a_{2,2} = \beta_{2,2} = 0$. Hence, $g_{3,2}(26) = 6$ when $a_{2,2} = \alpha_{2,2} = 1$, and $g_{3,2}(26) = 2$ when $a_{2,2} = \beta_{2,2} = 0$.

Hence, also in this case, $\beta_{4,2} = 0$ both when $a_{2,2} = \alpha_{2,2} = 1$, and when $a_{2,2} = \beta_{2,2} = 0$.

Now, the inequalities in $I(3, 2, (1, 6, 8))$ and $I(4, 2, (1, 6, 8))$ are derived. By the above discussion, we have
that \( I(3, 2, (1, 6, 8)) \) contains the following inequalities:

\[
\sum_{h=1}^{3} y_{1,1}^h + \alpha_{2,2} \sum_{h=2}^{3} y_{2,2}^h + \alpha_{3,2} \sum_{h=2}^{3} y_{3,2}^h = \sum_{h=1}^{3} y_{1,1}^h + \sum_{h=2}^{3} y_{2,2}^h + \sum_{h=2}^{3} y_{3,2}^h \leq g_{3,2}(14) = 8 \quad (82)
\]

\[
\sum_{h=1}^{3} y_{1,1}^h + \beta_{2,2} \sum_{h=2}^{3} y_{2,2}^h + \alpha_{3,2} \sum_{h=2}^{3} y_{3,2}^h = \sum_{h=1}^{3} y_{1,1}^h + \sum_{h=2}^{3} y_{2,2}^h + \sum_{h=2}^{3} y_{3,2}^h \leq g_{3,2}(14) = 5 \quad (83)
\]

\[
\sum_{h=1}^{3} y_{1,1}^h + \alpha_{2,2} \sum_{h=2}^{3} y_{2,2}^h + \beta_{3,2} \sum_{h=2}^{3} y_{3,2}^h = \sum_{h=1}^{3} y_{1,1}^h + \sum_{h=2}^{3} y_{2,2}^h + \sum_{h=2}^{3} y_{3,2}^h \leq g_{3,2}(14) = 6 \quad (84)
\]

\[
\sum_{h=1}^{3} y_{1,1}^h + \beta_{2,2} \sum_{h=2}^{3} y_{2,2}^h + \beta_{3,2} \sum_{h=2}^{3} y_{3,2}^h = \sum_{h=1}^{3} y_{1,1}^h \leq g_{3,2}(14) = 2 \quad (85)
\]

Observe that, the right-hand side of the inequalities are different, since they are computed by (64) and (65) and depend on the variable coefficients of the left-hand sides. For instance, in (82), we have \( s_{3,2} = (14 - 8)/2 = 3 \), then \( \sigma_{3,2} = 3 \) and \( g_{3,2}(14) = g_{2,2}(8) + 3\alpha_{3,2} = g_{2,2}(8) + 3 \). Then, \( \sigma_{2,2} = (8)/2 = 4 \), \( \sigma_{2,2} = 4 \), and, hence, the right-hand side is \( g_{3,2}(14) = g_{1,2}(0) + 4\alpha_{2,2} + 3\alpha_{3,2} = g_{1,2}(1) + 4 + 3 = 8 \). While, in (83), we have \( s_{3,2} = (14 - 8)/2 = 3 \), then \( \sigma_{3,2} = 3 \) and \( g_{3,2}(14) = g_{2,2}(8) + 3\alpha_{3,2} \). Then, \( \sigma_{2,2} = (8 - 2)/2 = 3 \), \( \sigma_{2,2} = 3 \), and \( g_{3,2}(14) = g_{1,2}(2) + 3\beta_{2,2} + 3\alpha_{3,2} = g_{1,2}(1) + 3 = 5 \) (recall that \( \beta_{2,2} = 0 \)).

According to the values of \( \alpha_{4,2} \) and \( \beta_{4,2} \) computed above, \( I(4, 2, (1, 6, 8)) \) contains the following inequalities:

\[
\sum_{h=1}^{3} y_{1,1}^h + \alpha_{2,2} \sum_{h=2}^{3} y_{2,2}^h + \alpha_{3,2} \sum_{h=2}^{3} y_{3,2}^h + \alpha_{4,2} \sum_{h=2}^{3} y_{4,2}^h = \sum_{h=1}^{3} y_{1,1}^h + \sum_{h=2}^{3} y_{2,2}^h + \sum_{h=2}^{3} y_{3,2}^h + \sum_{h=2}^{3} y_{4,2}^h \leq g_{4,2}(14) = 8 \quad (86)
\]

\[
\sum_{h=1}^{3} y_{1,1}^h + \beta_{2,2} \sum_{h=2}^{3} y_{2,2}^h + \alpha_{3,2} \sum_{h=2}^{3} y_{3,2}^h + \alpha_{4,2} \sum_{h=2}^{3} y_{4,2}^h = \sum_{h=1}^{3} y_{1,1}^h + \sum_{h=2}^{3} y_{2,2}^h + \sum_{h=2}^{3} y_{3,2}^h \leq g_{4,2}(14) = 5 \quad (87)
\]

\[
\sum_{h=1}^{3} y_{1,1}^h + \alpha_{2,2} \sum_{h=2}^{3} y_{2,2}^h + \beta_{3,2} \sum_{h=2}^{3} y_{3,2}^h + \alpha_{4,2} \sum_{h=2}^{3} y_{4,2}^h = \sum_{h=1}^{3} y_{1,1}^h + \sum_{h=2}^{3} y_{2,2}^h + \sum_{h=2}^{3} y_{3,2}^h \leq g_{4,2}(14) = 6 \quad (88)
\]

\[
\sum_{h=1}^{3} y_{1,1}^h + \beta_{2,2} \sum_{h=2}^{3} y_{2,2}^h + \beta_{3,2} \sum_{h=2}^{3} y_{3,2}^h + \alpha_{4,2} \sum_{h=2}^{3} y_{4,2}^h = \sum_{h=1}^{3} y_{1,1}^h \leq g_{4,2}(14) = 2 \quad (89)
\]

\[
\sum_{h=1}^{3} y_{1,1}^h + \alpha_{2,2} \sum_{h=2}^{3} y_{2,2}^h + \alpha_{3,2} \sum_{h=2}^{3} y_{3,2}^h + \beta_{4,2} \sum_{h=2}^{3} y_{4,2}^h = \sum_{h=1}^{3} y_{1,1}^h + \sum_{h=2}^{3} y_{2,2}^h + \sum_{h=2}^{3} y_{3,2}^h \leq g_{4,2}(14) = 8 \quad (90)
\]

\[
\sum_{h=1}^{3} y_{1,1}^h + \beta_{2,2} \sum_{h=2}^{3} y_{2,2}^h + \alpha_{3,2} \sum_{h=2}^{3} y_{3,2}^h + \beta_{4,2} \sum_{h=2}^{3} y_{4,2}^h = \sum_{h=1}^{3} y_{1,1}^h + \sum_{h=2}^{3} y_{2,2}^h + \sum_{h=2}^{3} y_{3,2}^h \leq g_{4,2}(14) = 5 \quad (91)
\]

\[
\sum_{h=1}^{3} y_{1,1}^h + \alpha_{2,2} \sum_{h=2}^{3} y_{2,2}^h + \beta_{3,2} \sum_{h=2}^{3} y_{3,2}^h + \beta_{4,2} \sum_{h=2}^{3} y_{4,2}^h = \sum_{h=1}^{3} y_{1,1}^h + \sum_{h=2}^{3} y_{2,2}^h + \sum_{h=2}^{3} y_{3,2}^h \leq g_{4,2}(14) = 6 \quad (92)
\]

\[
\sum_{h=1}^{3} y_{1,1}^h + \beta_{2,2} \sum_{h=2}^{3} y_{2,2}^h + \beta_{3,2} \sum_{h=2}^{3} y_{3,2}^h + \beta_{4,2} \sum_{h=2}^{3} y_{4,2}^h = \sum_{h=1}^{3} y_{1,1}^h \leq g_{4,2}(14) = 2 \quad (93)
\]
Note that, the inequalities (91)–(93) are the same of (83)–(85) and that, since $f(H_2^4) = 24 > 14$, we have $g_{4,2}(14) = g_{3,2}(14)$ in all above inequalities. Recalling that, in the example reported at the end of Section 4, $MO(4, 2, (1, 6, 8))$ contains points $\hat{y}$ with the following characteristics:

1. $\sum_{h=1}^{3} \hat{y}_{1,1}^{h} = 2; \hat{y}_{2,2}^{2} + \hat{y}_{2,2}^{3} = 4; \hat{y}_{3,2}^{3} = 2; \hat{y}_{4,2}^{3} = 0$;
2. $\sum_{h=1}^{3} \hat{y}_{1,1}^{h} = 2; \hat{y}_{2,2}^{2} + \hat{y}_{2,2}^{3} = 3; \hat{y}_{3,2}^{3} = 3; \hat{y}_{4,2}^{3} = 0$;
3. $\sum_{h=1}^{3} \hat{y}_{1,1}^{h} = 1; \hat{y}_{2,2}^{2} + \hat{y}_{2,2}^{3} = 4; \hat{y}_{3,2}^{3} = 3; \hat{y}_{4,2}^{3} = 0$.

It is easy to verify that the inequalities in $I(4, 2, (1, 6, 8))$ are a valid for $MO(4, 2, (1, 6, 8))$, and that each point in $MO(4, 2, (1, 6, 8))$ is contained by (at least) a face induced by the inequalities in $I(4, 2, (1, 6, 8))$.

5.1 Description of $POO$

In this section, a description of $POO$ is provided as a system of inequalities. Similar arguments to those used in [2] are employed. Let $W \subseteq N$ be a subset of items, let $B = \{B_1, \ldots, B_t\}$ be the maximal block partition of $W$ into blocks. Let $r = \arg \min_{j \in B_q} \{s_j\}$, and let $f_q' = s_r$ and $p_q = p_r$ be the weight and the profit of the block $q$, respectively. Let $\hat{b}_q = \sum_{j \in B_q} \frac{s_j}{f_j}$ be the multiplicity of block $B_q$. Suppose that $f_1' \leq f_2' \leq \ldots \leq f_t'$ and let $f_q = f_q' / f_1'$, for $q = 1, \ldots, t$.

Let $F$ be a vector of $l$ components such that $F_h = \sum_{i=1}^{m} r_i^h$, for $h = 1, \ldots, l$, and let $MP^{W}_t(F/f_1')$ be the set containing all the OPT and ordered solutions of M-SP defined on the maximal block partition $B$ of $W$, weights $f_1, \ldots, f_t$, multiplicities $\hat{b}_1, \ldots, \hat{b}_t$, knapsack part capacities $F_h = \sum_{i=1}^{m} r_i^h$, for $h = 1, \ldots, l$, and objective coefficients $p_1, \ldots, p_t$. Hence, by (11) and (12), $MP^{W}_t(F/f_1')$ is defined as

$$MP^{W}_t(F/f_1') = \text{conv} \left\{ y \in (\mathbb{Z}^+ \cup \{0\})^{t \times (l \times t)}, \sum_{w=1}^{t} \sum_{q=1}^{h} \left[ \frac{f_w y_{wq}^{h}}{d_q} \right] d_q \leq F_h / f_1' \quad \text{for } h = 1, \ldots, l \right\} \quad \left( y \text{ is an OPT and an ordered solution} \right)$$

Observe that, since $MP^{W}_t(F/f_1')$ contains OPT and ordered solutions, then $MP^{W}_t(F/f_1')$ depends on the objective function coefficients. Let $I_W(t, l, F/f_1')$ be the set of inequalities, introduced in the previous Section,
satisfying conditions 1), 2) and 3) for \( MP^{OO,W}_i(F/f'_1) \), written as

\[
\sum_{w=1}^{l} \sum_{q=1}^{l} \frac{a_{w,q} f_w}{d_q} \sum_{h=q}^{l} y_{w,q}^h \leq g_{i,l}(F_l/f'_1).
\]

(95)

and let \( I(W) \) be the set containing inequalities of the form

\[
\sum_{w=1}^{l} \sum_{q=1}^{l} \frac{a_{w,q} f_w}{d_q} \sum_{h=q}^{l} x_{i,j}^h = \sum_{w=1}^{l} \sum_{q=1}^{l} \frac{a_{w,q} f_w}{d_q} \sum_{h=q}^{l} x_{i,j}^h \leq g_{i,l}(F_l/f'_1).
\]

(96)

**Theorem 5.6** The polytope \( P^{OO} \) is described by the following system

\[
I(W) for all \ W \subseteq N
\]

\[x \in (\mathbb{R}^+ \cup \{0\})^{n \times m \times l}.
\]

**Proof.** By Theorem 5.5, the inequalities in \( I_W(t, l, F/f'_1) \) are valid for all points in \( MP^{OO,W}_i(F/f'_1) \). Then, by the correspondence between feasible solutions of SMKP and M-SP (see Section 3.1) and by (16), we have that the inequalities in \( I(W) \) are valid for the polytope

\[
P^{OO}(W) = \text{conv} \left( \begin{array}{c}
\sum_{j \in W: s_{j, h} \leq j} s_{j, h} / f'_1 x_{i,j}^h \leq r_{i, h} / f'_1 \text{ for } h = 1, \ldots, l \text{ and } i = 1, \ldots, m \\
\sum_{i=1}^{m} \sum_{h: h \geq g(j)} x_{i,j}^h \leq b_j \text{ for } j \in W \\
x \text{ is an OPT and an ordered solution} \\
x \in (\mathbb{Z}^+ \cup \{0\})^{n \times m \times l}
\end{array} \right)
\]

(97)

i.e., \( P^{OO} \) restricted to the item set \( W \).

In the following, an inequality that is satisfied as an equation by all optimal solution of MSKP, formulated as in (1), is built. As in [3], if an objective function coefficient \( v_j < 0 \) exists for some \( j \in N \), then \( x_{i,j}^h = 0 \) for all \( h \) and \( i \) in any optimal solution. If \( v_j \geq 0 \) for all \( j \in N \) we set \( W = \{ j \in N : v_j > 0 \} \). Let \( B = \{ B_1, \ldots, B_t \} \) be the maximal block partition of \( W \), and let us consider the transformation into M-SP when only items in \( W \) are considered. By the arguments used in Section 3.1 on the equivalence of optimal solutions and by (16), inequalities in \( I(W) \) are satisfied at equality by all the optimal solution of problem max\{\( v^T x : x \in P^{OO}(W) \)\}.
As \( v_j = 0 \) for all \( j \in N \setminus W \), a solution \( x \) for MSKP defined on the set \( N \) can be optimal only if the components of \( x \) corresponding to itsms in \( W \) satisfy the inequalities in \( I(W) \) at equality.

\[ \square \]

References

[1] P. Detti, A polynomial algorithm for the multiple knapsack problem with divisible item sizes, *Information Processing Letters*, Vol. 109 (2009) 582–584.

[2] O. Marcotte, The cutting stock problem and integer rounding, *Mathematical Programming* 33 (1985) 82–92.

[3] Y. Pochet, R. Weismantel, The sequential knapsack polytope, *SIAM Journal on Optimization* 8 (1998) 248–264.

[4] Y. Pochet and L. A. Wolsey, Integer knapsack and flow covers with divisible coefficients: polyhedra, optimization and separation, *Discrete Applied Mathematics* 59 (1995) 57–74.

[5] W.F.J. Verhaegh and E.H.L. Aarts, A polynomial-time algorithm for knapsack with divisible item sizes, *Information Processing Letters* 62 (1997) 217–221.
6 Appendix

6.1 Proof of Lemma 2.5

By contradiction, assume that the lemma does not hold. By the optimality of A-OPT, one has $S(\bar{x}) = S(x)$, i.e., A-OPT is able to assign all items assigned in $x$. Let $x_O$ be the solution found by A-OPT applied to the whole item set $N$, let $q \leq h$ be the first index such that $S(\bar{x}^q) \neq S(x_O^q)$.

Let us consider the execution of A-OPT applied to the restricted set of items $S(\bar{x})$. At iteration $q$, after the grouping procedure, let $\bar{L}$ be the list (ordered in non-increasing order of value) of, eventually grouped, items of sizes $d_q$. Let us consider now, the execution of A-OPT applied to the whole item set. At iteration $q$, after the grouping procedure, let $L_O$ be the list (ordered in non-increasing order of value) of, eventually grouped, items of sizes $d_q$. In what follows, by abusing notation, given an item $t$ generated by the grouping procedure of A-OPT, we denote by $S(t)$ the set of items that have been used to generate it. If $t$ corresponds to a single item, let $S(t) = \{t\}$. To simplify the notation, $v(t)$ is used in place of $v(S(t))$. According to these two lists, let $j$ and $j_O$ be the first two assigned (eventually grouped) items by A-OPT in $\bar{L}$ and $L_O$, respectively, such that $S(j) \neq S(j_O)$. Observe that, by definition, before the assignment of $j$ and $j_O$, the set of items assigned by A-OPT (during the production of $\bar{x}$ and $x_O$, respectively) is the same. By the above observation and by definition of $\bar{L}$ and $L_O$, it follows that $v(j) \leq v(j_O)$. In the following, we show that $v(j) = v(j_O)$. By contradiction, suppose that $v(j) < v(j_O)$. Let $\{j, j^1, j^2, \ldots, j^z\}$ and $\{j_O, j_O^1, j_O^2, \ldots, j_O^y\}$ be the last parts of the ordered lists $\bar{L}$ and $L_O$, resulting after the assignment of $j$ and $j_O$, respectively. By definition of $j$ and $j_O$, observe that $S(j) \cup S(j^1) \cup S(j^2) \cup \ldots S(j^z)$ is a subset of $S(j_O) \cup S(j_O^1) \cup S(j_O^2) \cup \ldots S(j_O^y)$ (since $L_O$ is the list obtained by applying A-OPT on the whole item set $N$). Moreover, since we are considering the iteration $q$ of A-OPT, $f(S(t)) \leq d_q$ for all $t \in \{j, j^1, j^2, \ldots, j^z\} \cup \{j_O, j_O^1, j_O^2, \ldots, j_O^y\}$. Recall that, a size equal to $d_q$ is assigned by A-OPT to each item in $\{j, j^1, j^2, \ldots, j^z\} \cup \{j_O, j_O^1, j_O^2, \ldots, j_O^y\}$. Let $\{j, t^1, t^2, \ldots, t^g\}$ be the set containing all the items in $\{j, j^1, j^2, \ldots, j^z\}$ that are assigned in $\bar{x}$. Let $A = (S(j) \cup S(t^1) \cup S(t^2) \cup \ldots S(t^g)) \cap S(j_O)$. Two cases are considered: $A = \emptyset$ and $A \neq \emptyset$. If $A = \emptyset$ then it is feasible to replace the item $j$ with $j_O$ in $\bar{x}$, obtaining a solution $\bar{x}'$ such that $v(\bar{x}') > v(\bar{x})$. A contradiction of the optimality of $\bar{x}$. 

56
Suppose that $A \neq \emptyset$. Note that, since $f(S(j_o)) \leq d_q$, then $f(A) \leq d_q$, too. In the following, we show that the items in $S(j) \cup S(t^1) \cup S(t^2) \cup \ldots S(t^g)$ can be rearranged into $g + 1$ chunks (i.e., subsets) each of size at most $d_q$, in such a way that $A$ is contained in exactly one chunk. First observe that, by definition, the grouping procedure cannot arrange the items in $S(j) \cup S(t^1) \cup S(t^2) \cup \ldots S(t^g)$ in less than $g + 1$ chunks of size $d_q$. By Proposition 2.2, the items in $B = \{(S(j) \cup S(t^1) \cup S(t^2) \cup \ldots S(t^g)) \setminus A\}$ can be rearranged in $\lceil \frac{f(B)}{d_q} \rceil - 1$ chunks of total size $d_q$ and one chunk, say $C$, of total size at most $d_q$. Hence, $f(A \cup C) \leq 2d_q$. Now, if $f(A \cup C) > d_q$, then the items in the sets $C$ and $A$ can be arranged into two chunks, while, if $f(A \cup C) \leq d_q$, the items in $A \cup C$ are arranged in a single chunk. By Proposition 2.2 and the optimality of $A$-OPT, it follows that the number of chunks obtained so far is exactly $g + 1$. (Otherwise, the production of more than $g + 1$ chunks by the grouping procedure could lead to a not optimal solution, contradicting the optimality of $A$-OPT.) Let $\{\bar{j}, \bar{t}^1, \ldots, \bar{t}^2, \ldots, \bar{t}^g\}$ be the new $g + 1$ chunks, obtained so far. Observe that, $v(t) \leq v(j) < v(j_o)$ for each $t$ in $\{\bar{j}, \bar{t}^1, \ldots, \bar{t}^2, \ldots, \bar{t}^g\}$. Without loss of generality, let $A \subseteq \bar{j}$. Hence, $v(\bar{j}) < v(j_o)$. Since $A$-OPT has assigned a size $d_q$ to each of the (grouped) items in $\{j, t^1, t^2, \ldots, t^g\}$ before assigning them in $\bar{x}$, it is feasible replace items $j, t^1, t^2, \ldots, t^g$ with items $\bar{j}, \bar{t}^1, \bar{t}^2, \ldots, \bar{t}^g$. Let $\tilde{x}$ be the new feasible solution obtained after this replacement. Note that, $v(\tilde{x}) = v(\bar{x})$. Moreover, observe that, by replacing in $\tilde{x}$ the item $\bar{j}$ with $j_o$, a new feasible solution is obtained with objective function value strictly greater than $v(\bar{x}) = v(\tilde{x})$. A contradiction of the optimality of $\bar{x}$.

By the above discussion, it follows that $v(j) = v(j_o)$. Recall that, the items not yet assigned at the beginning of iteration $q$ by $A$-OPT, when $A$-OPT is applied to $S(\bar{x})$, are a subset of the items not yet assigned at the beginning of iteration $q$ by $A$-OPT, when $A$-OPT applied to the whole item set. As a consequence, when $A$-OPT is applied to the whole item set, during the grouping procedure of iteration $q$, an ordering of the items not yet assigned exists, such that, after the grouping procedure, $j$ appears in place of $j_o$ in the list $L_O$. And, in this case, $A$-OPT will assign $j$ instead of $j_o$ in $x_O$. \qed