Lyapunov-like functions for attitude control via feedback integrators

Tejaswi K. C.\textsuperscript{,3} Srikant Sukumar\textsuperscript{1} and Ravi Banavar\textsuperscript{2}

Abstract—The notion of feedback integrators permits Euclidean integration schemes for dynamical systems evolving on manifolds. Here, a constructive Lyapunov function for the attitude dynamics embedded in an ambient Euclidean space has been proposed. We then combine the notion of feedback integrators with the proposed Lyapunov function to obtain a feedback law for the attitude control system. The combination of the two techniques yields a domain of attraction for the closed loop dynamics, where earlier contributions were based on linearization ideas. Further, the analysis and synthesis of the feedback scheme is carried out entirely in Euclidean space. The proposed scheme is also shown to be robust to numerical errors.

I. INTRODUCTION

There are many established techniques for attitude control design employing parametrization of the set of rotational matrices [1]. A brief summary of the representations involved in description of the kinematics of motion is given in [2]. Simple control laws in terms of Euler parameters [3], Cayley-Rodrigues parameters [4] have been formulated. However, using such parametrization and hence local charts could cause undesirable unwinding behavior [5] and require switching between these local coordinate systems for control design.

On the other hand, in the recent past, coordinate-free techniques using geometric ideas have been used to design rigid body attitude controllers [6], [7], [8]. However, implementing such feedback laws from geometric control theory [9], [10] requires special variants of numerical integrators (e.g. variational integrator) to preserve the geometric structure of the manifold and yield reliable results.

While simple geometric PD controllers can be used to stabilize a rigid body [7], numerical integration errors quickly creep into the digital implementations of these schemes, thus resulting in the states not lying on the $SO(3)$ manifold, and being pushed into the ambient space of $3 \times 3$ real-matrices. In such situations can we still guarantee that these numerical schemes will recover and converge to the manifold? Feedback integrators [11] provide a positive answer to this question. Figure 1 illustrates this scenario with $\mathcal{M}$ being the set of tuples of $3 \times 3$ matrices and angular velocity vectors, while $\mathcal{M}$ is the set of tuples of valid rotation matrices and angular velocities. In [11] the authors have shown that if the rigid body dynamics is seen as the restriction of a special vector field in an ambient Euclidean space, then Euclidean numerical integration schemes also lead to convergence of states to the manifold. Further, for the case when trajectories starting from an ambient space converge to an embedded submanifold, [12] shows that the omega limit set lies in a unique connected component of the level sets corresponding to a Lyapunov-like function. Our work builds on these two techniques to design Euclidean controllers which guarantee that the rigid body converges to an equilibrium point $x_0$ on $\mathcal{M}$, even if at some instants the states do not lie on $\mathcal{M}$.

More recent work by [13] has addressed this problem by linearizing the ambient dynamics, thus is only valid in a small neighborhood around the desired set-point. We briefly introduce the same in section II. In this article (primarily in section III), we have developed a new procedure for nonlinear design using Lyapunov-like functions on the ambient system to guarantee asymptotic convergence to an equilibrium point in $\mathcal{M}$. Finally, to demonstrate the performance of the controller, numerical simulations are presented in section IV.

Notation:

- Euclidean inner product is used in this paper:
  \[
  \langle A, B \rangle = \sum_{i,j} A_{i,j} B_{i,j} = tr(A^T B)
  \]
  for matrices of identical dimensions. The norm induced by this inner product is used for vectors and matrices.
- $SO(3) = \{ R \in \mathbb{R}^{3 \times 3} \mid R^T R = I, \det(R) = 1 \}$ is the Lie group of all rotations and $so(3) = \{ A \in \mathbb{R}^{3 \times 3} \mid A = -A^T \}$ is the corresponding Lie algebra.
- Hat map $\land : \mathbb{R}^3 \rightarrow so(3)$,
  \[
  \hat{\Omega} = \begin{bmatrix}
  0 & -\Omega_3 & \Omega_2 \\
  \Omega_3 & 0 & -\Omega_1 \\
  -\Omega_2 & \Omega_1 & 0
  \end{bmatrix}
  \]
  for $\Omega \in \mathbb{R}^3$. The inverse map is the vee map, $\vee$, such that $(\hat{\Omega})^\vee = \Omega$ for all $\Omega \in \mathbb{R}^3$ and $(A^\vee) = A$ for all $A \in so(3)$.
- For a square matrix $A$, $A_s := (A + A^T)/2$ is the symmetric part and $A_k := (A - A^T)/2$ is the skew-symmetric part.

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1\textsuperscript{Associate Professor, Systems and Control Engineering, Indian Institute of Technology, Bombay srikant@sc.iitb.ac.in}
2\textsuperscript{Professor, Systems and Control Engineering, Indian Institute of Technology, Bombay ravi.banavar@iitb.ac.in}
3\textsuperscript{Undergraduate Student, Aerospace Engineering, Indian Institute of Technology, Bombay kctejaswi1999@gmail.com}
II. STABILIZATION OF A RIGID BODY USING LINEARIZATION

This section summarizes the linearization procedure introduced in [13]. Consider a control system $\Sigma$ on $\mathbb{R}^n$,

$$\Sigma : \dot{x} = X(x, u), x \in \mathbb{R}^n, u \in \mathbb{R}^k$$

Assume that there is an m-dimensional submanifold $\mathcal{M}$ of $\mathbb{R}^n$ that is invariant under the flow of the system. So we can restrict the system to $\mathcal{M}$ as,

$$\Sigma|\mathcal{M} : \dot{x} = X(x, u), x \in \mathcal{M}, u \in \mathbb{R}^k$$

It is convenient to use the ambient control system $\Sigma$ and the Cartesian coordinates on the ambient space in order to design controllers for the system $\Sigma|\mathcal{M}$ on the manifold $\mathcal{M}$.

Let $\hat{V}$ be a non-negative function on the euclidean space such that $\mathcal{M} = \hat{V}^{-1}(0)$. At every point in $\mathcal{M}$ as $\hat{V}$ attains its minimum value of 0, $\nabla \hat{V}(x) = 0, \forall x \in \mathcal{M}$. We obtain a new ambient control system by subtracting $\nabla \hat{V}$ from the control vector field,

$$\tilde{\Sigma} : \dot{x} = \tilde{X}(x, u), x \in \mathbb{R}^n, u \in \mathbb{R}^k$$

with $\tilde{X}(x, u) = X(x, u) - \nabla \hat{V}(x)$. It is easily verified that $\tilde{\Sigma}|\mathcal{M} = \Sigma|\mathcal{M}$, meaning that the system dynamics is preserved on $\mathcal{M}$. The negative gradient of $\hat{V}$ helps in making $\mathcal{M}$ attractive for $\tilde{\Sigma}$ dynamics [11].

Now, let $(x_0, u_0) \in \mathcal{M} \times \mathbb{R}^k$ be an equilibrium point of $\Sigma|\mathcal{M}$ with $X(x_0, u_0) = 0$. Jacobian linearization can be carried out on the ambient system $\Sigma$ around the equilibrium point in the ambient space to come up with stabilizing controllers for the original system on the manifold. The linearization of $\tilde{\Sigma}$ is given by,

$$\tilde{\Sigma}_0 : \dot{x} = \frac{\partial \tilde{X}}{\partial x}(x_0, u_0)(x - x_0) + \frac{\partial \tilde{X}}{\partial u}(x_0, u_0)(u - u_0)$$

where $(x, u) \in \mathbb{R}^n \times \mathbb{R}^k$.

**Theorem 1. [13, Theorem II.3]** If a linear feedback controller $u : \mathbb{R}^n \to \mathbb{R}^k$ exponentially stabilizes the equilibrium point $x_0$ for the linearization $\tilde{\Sigma}_0$ of the ambient system $\Sigma$, then it also exponentially stabilizes the equilibrium point $x_0$ for $\Sigma|\mathcal{M}$.

We are concerned the application of theorem 1 to the rigid body system with full actuation,

$$\dot{R} = R\dot{\Omega}$$

$$\dot{\Omega} = u$$

where $(R, \Omega) \in \mathcal{M} \subset \mathbb{R}^{3 \times 3} \times \mathbb{R}^3$. $\mathcal{M} = SO(3) \times \mathbb{R}^3$ is the invariant manifold being considered. It is assumed that the control input is appropriately scaled and shifted to account for nonlinear terms in the dynamics.

Let $GL^+(3) = \{R \in \mathbb{R}^{3 \times 3} \mid \det R > 0\}$ and define a function $\hat{V}$ on $GL^+(3) \times \mathbb{R}^3$ by

$$\hat{V}(R, \Omega) = \frac{k_v}{4} ||R^T R - I||^2$$

with constant $k_v > 0$. One can verify that $\hat{V}^{-1}(0) = \mathcal{M}$ and

$$\nabla_R \hat{V} = -k_v R(R^T R - I), \nabla_{\Omega} \hat{V} = 0$$

So the modified rigid body system ($\tilde{\Sigma}$) in the ambient space is,

$$\dot{R} = R\dot{\Omega} - k_v R(R^T R - I)$$

$$\dot{\Omega} = u$$

To design a controller, the system (3) is linearized to get,

$$\dot{Z}_s = -2k_v Z_s$$

$$\dot{Z}_k = \Omega$$

$$\dot{\Omega} = u$$

with $Z = R_0^T \Delta R = R_0^T (R - R_0)$ being a transformation of $R$.

For $\mathbb{R} \ni k_p, k_d > 0$, the linear PD controller

$$u = -k_p Z_k - k_d \Omega$$

exponentially stabilizes the equilibrium point $(R_0, 0)$ for both the linearized system (4) and the rigid body system (1) on $\mathcal{M}$.

III. AMBIENT CONTROL FORMULATION USING LYAPUNOV-LIKE FUNCTIONS

The results of the previous section are obtained via linearization and therefore suffer from obvious drawbacks such as the inability to accurately estimate the region of convergence. In this section, we present a novel nonlinear design method based on Lyapunov techniques which is utilized to stabilize the rigid body using feedback integrators.

An important result on locating $\omega$-limit sets using height functions is employed. Given the bounded solution of an autonomous vector field on a Riemannian manifold, a finer estimate of the location of $\omega$-limit set can be obtained using results in [12]. We summarize the same here.

**A. Preliminaries**

The set-up in [12, Section 2] is restated while changing the notations from $\mathcal{M}$ to $\mathfrak{M}$, $\mathcal{S}$ to $\mathcal{M}$ and $\Omega$ to $\omega$ so that consistency with the rest of our paper is maintained:

- A Riemannian manifold $(\mathfrak{M}, g)$ of class $C^2$ on which a locally Lipschitz continuous vector field

$$\dot{x} = f(x)$$

is given.

- Consider a Cauchy problem for (5) with initial value $x(0)$ such that the corresponding solution $x(t, x(0))$ is bounded.

- Assume that the $\omega$-limit set $\omega(x(0))$, which is a compact and connected set, is contained in a closed embedded submanifold $\mathcal{M} \subset \mathfrak{M}$. Equivalently $\mathcal{M}$ is attracting for the solution of (5) starting at $x(0)$.

- Let $O$ be an open tubular neighborhood of $\mathcal{M}$ in $\mathfrak{M}$. Assume that there exists a real-valued $C^1$ function $\tilde{W} : O \to \mathbb{R}$ such that $\tilde{W}(x) \leq 0$ on $\mathcal{M}$, where $\tilde{W}(x)$ is
the derivative of \( W(x) \) along the flow (Lie derivative). Moreover, let \( E := \{ x \in \mathcal{M} \mid W(x) = 0 \} \) so that \( W(x) < 0 \) on \( \mathcal{M} \setminus E \).

The function \( W \) as described above is called a height function for the pair \( (\mathcal{M}, f) \).

**Definition 1.** [12, Definition 5] Let \( \{ E_i \}_{i \in I} \) be the connected components of \( E \), where \( I = \{ 1, 2, \ldots \} \subset \mathbb{Z}^+ \). Given a function \( W \) as in the assumptions, we say that the components \( \{ E_i \}_{i \in I} \) are contained in \( W \) if each \( E_i \) lies in a level set of \( W \), and the subset \( \{ W(E_i) \}_{i \in I} \subset \mathbb{R} \) has at most a finite number of accumulation points in \( \mathbb{R} \).

The main result is stated below.

**Theorem 2.** [12, Theorem 6] If the components \( \{ E_i \}_{i \in I} \) are contained in \( W \) according to Definition 1, then \( \omega(x(0)) \subset E_i \) for a unique \( i \in I \).

**B. Rigid body stabilization**

Using the above result, we propose an ambient nonlinear controller for rigid body stabilization. Consider again the feedback integrator form of the rigid body dynamics (3),

\[
\dot{R} = R\Omega - k_e R(R^T R - I)
\]

\[
\dot{\Omega} = u
\]

(6)

For this system, \( \mathcal{M} = SO(3) \times \mathbb{R}^3 \) is the invariant manifold being considered which is embedded in the ambient space \( \mathfrak{R} = \mathbb{R}^{3 \times 3} \times \mathbb{R}^3 \). Also, consider again the function \( V \) defined in (2).

**Theorem 3.** The control law given by,

\[
u = -k_p Z_k^\vee - k_d \Omega, \quad \mathbb{R} \ni k_p, k_d > 0
\]

(7)

asymptotically stabilizes an equilibrium point \((R_0, 0) \in \mathcal{M} \) of the system (6) for almost all initial conditions starting from \( \dot{V}^{-1}(0, c) \) and some \( c > 0 \).

The corresponding closed-loop system employing (6) and (7) is,

\[
\dot{R} = R\hat{\Omega} - k_e R(R^T R - I)
\]

\[
\dot{\hat{\Omega}} = -k_p Z_k^\vee - k_d \Omega
\]

(8)

with \((R, \Omega)\) as its states.

**Proof.** We first verify the assumptions corresponding to the set-up in section III-A,

- We have a Riemannian manifold \( (\mathbb{R}^{3 \times 3} \times \mathbb{R}^3, \cdot) \) on which a locally Lipschitz continuous vector field (8) is given.
- It can be directly claimed from [14, Theorem 2] that every trajectory of (6) starting from a point in \( \dot{V}^{-1}(0, c) \), for some \( c > 0 \), stays in \( \dot{V}^{-1}(0, c) \) for all \( t \geq 0 \) and asymptotically converges to the set \( \mathcal{M} = \dot{V}^{-1}(0) \) as \( t \to \infty \). Since \( \dot{V}^{-1}(0, c) \) is compact and positively invariant, the first state \( \hat{R} \) in (8) is bounded if initial states \((R(0), \Omega(0)) \in \dot{V}^{-1}(0, c) \). Now, consider a function \( V_2 = \frac{1}{2}||\Omega||^2 \) whose derivative is evaluated using (8):

\[
\dot{V}_2 = -k_d||\Omega||^2 - k_p \Omega^T Z_k^\vee
\]

\[
\leq -k_d||\Omega||^2 - k_p \frac{||\Omega||}{k_d}||Z_k^\vee||^2 \leq -k_d||\Omega||^2 \leq 0
\]

if \( ||\Omega|| \geq \frac{k_p}{k_d(1 - \epsilon)} ||Z_k^\vee|| \). We know that \( Z_k^\vee \) is bounded because \( R \) is already shown to be bounded. Hence, either \( ||\Omega|| \) is bounded by a fraction of \( ||Z_k^\vee|| \) or \( \dot{V}_2 \leq 0 \) implying that \( ||\Omega|| \) is non-increasing. So, \( \Omega \) is bounded. Therefore we have a Cauchy problem for (8) with initial value \((R(0), \Omega(0))\) such that the solution is bounded.

- From the previous point, we know that the \( \omega \)-limit set \( \omega(R(0), \Omega(0)) \), which is a compact and connected set, is contained in a closed embedded submanifold \( \mathcal{M} = SO(3) \times \mathbb{R}^3 \subset \mathbb{R}^{3 \times 3} \times \mathbb{R}^3 \).

- Let \( O \) be an open tubular neighborhood of \( \mathcal{M} \) in \( \mathfrak{R} \). This set is being used in our context to help determine the region of convergence of \( \mathfrak{R} \). There is a real-valued \( C^1 \) function \( W : O \to \mathbb{R} \) such that \( W \leq 0 \) on \( \mathcal{M} \), defined as below:

\[
W(R, \Omega) = \frac{k_p}{4} (||Z_\Omega||^2 + ||Z_k^\vee||^2) + \frac{1}{2} ||\Omega||^2 + \epsilon \langle Z_k^\vee, \Omega \rangle
\]

(9)

which serves as the height function with \( Z = R_0^T (R - R_0) \). The derivative of \( \dot{W}(R, \Omega) \) along the flow on \( \mathcal{M} \) is (section VI-B),

\[
\dot{W}|_{\mathcal{M}}(R, \Omega) \leq -(k_d - \epsilon)||\Omega||^2 - \epsilon k_d ||Z_k^\vee||^2 \leq 0
\]

(10)

for \( 0 < \epsilon < \frac{4k_p k_d}{4k_p + k_d^2} \).

\( E \) is defined as \( \{(R, \Omega) \in \mathcal{M} \mid \dot{W}(R, \Omega) = 0\} \):

\[
E = \{(R, \Omega) \in SO(3) \times \mathbb{R}^3 \mid Z_k^\vee = 0, \Omega = 0\}
\]

(11)

so that \( \dot{W}(R, \Omega) < 0 \) on \( \mathcal{M} \setminus E \).

Thus the main assumptions required for theorem 2 are satisfied.

We observe that the height function can be re-written as

\[
W(R, \Omega) = \frac{k_p}{4} tr((I - R_0^T R)^T (I - R_0^T R)) + \frac{1}{2} ||\Omega||^2 + \epsilon \langle Z_k^\vee, \Omega \rangle
\]

which is different from standard Lyapunov functions used for rigid body stabilization like \( V(R, \Omega) = \frac{k_p}{4} tr(I - R_0^T R) + \frac{1}{2} ||\Omega||^2 \). Among other changes, it has an additional cross term \( \langle Z_k^\vee, \Omega \rangle \) which helps us to identify the equilibrium point as one of the connected components of \( E \) and then employ theorem 2 to prove asymptotic convergence.

**Remark 1.** The maximum value that the real number \( c \) can take is less than \( k_c/12 \) (see section VI-A) and this ensures that there exists an open tubular neighborhood \( O \) which is the superset of \( \dot{V}^{-1}(0, c) \).

On the set \( E \) we know that,

\[
Z_k^\vee = 0 \Rightarrow R_0^T R - R_0^T R_0 = 0
\]

With \( \bar{R} = R_0^T R \),

\[
\dot{\bar{R}} = \bar{R}^T \Rightarrow \dot{\bar{R}}^2 = I
\]
Using the axis-angle representation of rotation matrices [15],
\[ \hat{R} = \exp(\theta \xi) = I + \sin \theta \xi + (1 - \cos \theta) \xi^2, \xi \in \mathfrak{so}(3), \]
\[ \hat{R}^2 = I \Rightarrow e^{2\theta \xi} = I = e^{2\pi k}, k, \xi \in \mathbb{R}^3, \|k\| = 1 \]
\[ \Rightarrow \theta = n\pi, \xi = k \]

The set of all such matrices, \( \hat{R} \), can be divided into two sets as follows,
\[ \theta = 2m\pi, m \in \mathbb{Z} \Rightarrow \hat{R} = I \Rightarrow tr(\hat{R}) = 3 \quad (12) \]
\[ \theta = (2m + 1)\pi, m \in \mathbb{Z} \]
\[ \Rightarrow \hat{R} = \exp(\pi \xi) \neq I \Rightarrow tr(\hat{R}) = 1 + 2\cos \theta = -1 \quad (13) \]

Thus \( E = E_1 \cup E_2 \) is described below,
- \( E_1 = \{(R_0, 0)\} \). As this subset contains only one point, it is trivially connected. Value of \( W \) in \( E_1 \) evaluated with \((Z, \Omega) = (0, 0)\) gives \( W(R, \Omega) = 0 \).
- \( E_2 = \{(R, \Omega) \in SO(3) \times \mathbb{R}^3 \mid tr(R^T R) = tr(\hat{R}) - 1, \hat{R} = \hat{R}^T, \Omega = 0\} \). A point in the set \( E_2 \) has the form \((R_0, \hat{R}, 0)\) where \( \hat{R} = e^{\pi \xi} \) for an unit vector \( \xi \) as in (13). Consider two points \( x_1 = (R_0 e^{\pi \xi_1}, 0) \) and \( x_2 = (R_0 e^{\pi \xi_2}, 0) \) in \( E_2 \) and the corresponding axis vectors \( \xi_1, \xi_2 \in \mathbb{R}^3 \) with unit magnitudes. Define a path variable \( \xi(\alpha) = \frac{(1 - \alpha)\xi_1 + \alpha \xi_2}{\| (1 - \alpha)\xi_1 + \alpha \xi_2 \|}, \alpha \in [0, 1] \) such that \( \xi(0) = \xi_1 \) and \( \xi(1) = \xi_2 \). The corresponding path in \( M \) connecting \( x_1 \) and \( x_2 \) is \( \{x(\alpha) = (R(\alpha), \Omega(\alpha)) \in SO(3) \times \mathbb{R}^3 \mid R(\alpha) = R_0 e^{\pi \xi(\alpha)} \Omega = 0\} \).

Any point \( x(\alpha) \) in this path connecting \( x_1, x_2 \) also belongs to \( E_2 \), meaning the set \( E_2 \) is path connected and hence connected. Evaluating \( W \) in \( E_2 \) with \((Z_k, \Omega) = (0, 0), W(R, \Omega) = \frac{k_p}{4}\|Z_s\|^2 = \frac{k_p}{4} tr((R^T R - I)(R^T R - I)) = \frac{k_p}{4} tr(R^2 - 2\hat{R} + I) = \frac{k_p}{4} tr(2I - 2\hat{R}) = 2k_p \)
as \( \hat{R} = R_0^T R = \hat{R}^T \) and \( tr(R_0^T R) = -1 \) on \( E_2 \).

Figure 2 illustrates the basic components involved in this proof. \( \mathfrak{R} \) is the set of tuples of \( 3 \times 3 \) matrices and angular velocity vectors, while \( \mathcal{M} \) is the set of tuples of valid rotation matrices and angular velocities. Further, \( O \) is the tubular neighborhood of \( \mathcal{M} \) in which the height function \( W \) is defined. On the y-axis, the value of \( W(R, \Omega) \) for any \((R, \Omega) \in O \) is shown. We have already proved that \( W(R, \Omega) < 0 \) on \( \mathcal{M} \setminus (E_1 \cup E_2) \). From (9) and (11), any subset of \( E \) which lies in a level set of \( W \) will have the structure,
\[ W^{-1}(\frac{k_p}{4}\|Z_s\|^2 = c) \]
where \( c \geq 0 \). As shown earlier, the connected components of \( E \) lie in level sets of \( W \).

Finally completing the arguments of the proof,
- \( E_1 \subset W^{-1}(0), E_2 \subset W^{-1}(2k_p) \) which implies that \( \{W(E_i)\}_{i=1,2} = \{0, 2k_p\} \) has no accumulation point in \( \mathbb{R} \). Hence, we can say that \( \{E_i\}_{i=1,2} \) are contained in \( W \) using definition 1.

\[ \mathcal{M} = SO(3) \times \mathbb{R}^3 \]

\[ \mathfrak{R} = \mathbb{R}^{3 \times 3} \times \mathbb{R}^3 \]

Fig. 2. Illustration of the components involved for the case of rigid body stabilization. A general version can be found in [12].

- Employing theorem 2, \( \omega(R(0), \Omega(0)) \subset E_i \) for a unique \( i \in \{1, 2\} \).
- From the dynamics (8) we know that \( E_2 \) is forward invariant, that is,
\[ Z_k^T = 0, \Omega = 0, R^T R = I \]
\[ \Rightarrow \hat{R} = 0, \Omega = 0 \]
on \( E_2 \). In addition, \( E_2 \) is an unstable set for the system dynamics on \( \mathcal{M} \) [8].

This proves that \( \omega(R(0), \Omega(0)) \subset E_1 = \{(R_0, 0)\} \) for almost all \((R(0), \Omega(0)) \) in \( V^{-1}([0, c]) \).

\[ \square \]

IV. SIMULATIONS

In this section we look at a few numerical examples to illustrate the strategies previously presented.

A. Ideal case

Fig. 3. Nonlinear rigid body stabilization using height function

\[ W < 0 \text{ on } \mathcal{M} \setminus E \]
We demonstrate the performance of the rigid body system (8) with initial conditions in the manifold, $M$. Consider $(R_0, 0)$ as the desired equilibrium point $(R_0 = diag \{-1, -1, 1\})$ along with the initial conditions,

$$R(0) = \exp\left(\frac{2\pi}{3} e_2\right), \quad \Omega(0) = [0, 1, 1]^T$$

where $e_2 = [0, 1, 0]^T$ and the parameters being,

$$k_c = 1, k_p = 4, k_d = 2$$

The chosen value of $\epsilon$ is $0.99 \times \frac{4k_p k_d}{2\pi} = 1.584$ which satisfies the constraint needed in (10).

Figure 3 depicts the magnitudes of orientation and attitude errors along with the control magnitude. We recover the expected ideal performance in this case.

**B. Numerical Robustness**

To illustrate the strength of the proposed control, consider an initial state of the body not on $M$:

$$R(0) = 1.1 \times \exp\left(\frac{2\pi}{3} e_2\right) \in \mathbb{R}^{3 \times 3} \setminus SO(3)$$

with all the other conditions and parameters being identical to section IV-A. One can verify that,

$$||R(0)^T R(0) - I|| = 0.3637 < \sqrt{1/3}$$

implying that the initial condition is in the permitted set (section VI-A).

Since in practical applications randomness could seep into the system, we check robustness to measurement noise. To emulate measurement noise, white noise of relative magnitude, $10^{-3}$ is added to both the states ($R, \Omega$).

In this case too, convergence is observed to the desired equilibrium within the range of the measurement noise (fig. 4). We also notice that the state $R$ is outside $SO(3)$ initially, but soon converges to the manifold (modulo noise).

**V. Conclusions**

We initially introduced an existing linearization procedure for attitude control design in Euclidean space. Then, we proved that a single height function defined on the ambient Euclidean space, can be used to derive stabilizing nonlinear control for the attitude of a rigid body with a prescribed region of attraction. This is also illustrated through exemplary simulations. The algorithm is robust to measurement noise and numerical computation errors arising from digital implementation.

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**VI. APPENDIX**

**A. Calculation of the parameter, $c$**

We want to consider initial conditions for which the first state value does not lie on $SO(3)$. To exactly verify the existence of such $R(0)$ values, we need to evaluate the value of $c$ in $V^{-1}(0, c)$ of theorem 3.

We proceed by utilizing part of the proof of [14, Lemma 2]. Define $f : GL^+(3) \to \mathbb{R}_{\geq 0}$,

$$f(R) = \frac{k_c}{4} ||R^T R - I||^2$$

Take a small $\delta > 0$ such that every $A \in \mathbb{R}^{n \times n}$ with $||A - I|| \leq \delta$ is invertible. And let $c = k_c \delta^2/4$. Then, if $R \in f^{-1}((0, c])$, $||R^T R - I|| \leq \delta$, meaning $R^T R$ and $R$ are
invertible. Hence $f^{-1}([0,c]) \subset GL^+(3)$. For a value of $\chi$ close to $\delta$,
\[
\|A - I\| \leq \delta < \chi \Rightarrow \sum_{i,j}(A_{ij} - 1)^2 + \sum_{i \neq j}^3 A_{ij}^2 < \chi^2
\]
\[
\Rightarrow (A_{ii} - 1)^2 + \sum_{j \neq i, j=1}^3 A_{ij}^2 < \chi^2
\]
for any $i = 1, 2, 3$. So,
\[
0 > 2(A_{ii}^2 - 2A_{ii} + 1 - \chi^2 + \sum_{j \neq i}^3 A_{ij}^2)
\]
\[
\Rightarrow A_{ii}^2 - 2\sum_{j \neq i}^3 A_{ij}^2 > 0
\]
\[
\Rightarrow A_{ii}^2 > \sum_{j \neq i}^3 A_{ij}^2
\]
\[
\Rightarrow |A_{ii}| > \sum_{j \neq i}^3 |A_{ij}|
\]
which means that the matrix $A$ is strictly diagonally dominant. In summary, if $\delta < \chi < \sqrt{1/3}$, RHS of the above equation is always positive. Hence,
\[
A_{ii}^2 - 2\sum_{j \neq i}^3 A_{ij}^2 > 0
\]
\[
\Rightarrow |A_{ii}| > \sum_{j \neq i}^3 |A_{ij}|
\]
So its Lie derivative,
\[
\dot{W}|_{\mathcal{M}}(R, \Omega) = \frac{k_p}{2} \left( \left( Z_s, \frac{1}{2}(Z_s\Omega - \Omega Z_s) + Z_k\Omega - \frac{1}{2}Z_k^x \times \Omega \right) + \left( Z_k, \frac{1}{2}(Z_k\Omega + \Omega Z_k) + \Omega + \frac{1}{2}Z_k^x \times \Omega \right) \right) + \langle \Omega, -k_pZ_k^y - k_d\Omega \rangle + \epsilon \left( \frac{1}{2}(Z_s\hat{\Omega} + \hat{\Omega} Z_s)^y + \Omega + \frac{1}{2}Z_k^x \times \Omega, \Omega \right)
\]

We know that,
\[
\begin{align*}
\langle Z_s, Z_s\hat{\Omega} \rangle &= tr(Z_s^T Z_s\hat{\Omega}) = tr((Z_s^T Z_s)\hat{\Omega}) = 0 \\
\langle Z_s, Z_k\hat{\Omega} \rangle &= tr(Z_s^T Z_k\hat{\Omega}) = tr(\hat{\Omega}(Z_s Z_k^T)) = 0 \\
\langle Z_k, Z_k^x \times \Omega \rangle &= 2\langle Z_k^x, Z_k^x \times \Omega \rangle = 2\langle Z_k, Z_k^x \times Z_k^x \rangle = 0 \\
\langle Z_s, Z_k \hat{\Omega} \rangle &= 2\langle Z_s^T Z_k, \hat{\Omega} \rangle = 2tr(Z_s^T Z_k\hat{\Omega}) = -(Z_s, Z_k\hat{\Omega})
\end{align*}
\]

\[
\begin{align*}
\langle Z_s, (Z_s\hat{\Omega} + \hat{\Omega} Z_s)^y, \Omega \rangle &= \frac{1}{2} \langle \hat{\Omega}, (Z_s\hat{\Omega} + \hat{\Omega} Z_s) \rangle \\
&= \frac{1}{2} tr(\hat{\Omega}^T Z_s\hat{\Omega} + \hat{\Omega}^T \hat{\Omega} Z_s) = tr(\hat{\Omega}^T Z_s\hat{\Omega})
\end{align*}
\]

So we have the simplification,
\[
\dot{W}|_{\mathcal{M}}(R, \Omega) = -(k_d - \epsilon)||\Omega||^2 - ek_d||Z_k^y||^2 + \epsilon^2 tr(\hat{\Omega}^T Z_s\hat{\Omega})
\]

Now with $\hat{R} = R_0^T R$,
\[
x^T Z_s x = x^T ((R_0^T R - I)s) x = x^T \left( \frac{\hat{R} + \hat{R}^T}{2} - I \right) x = \frac{1}{2} (x^T \hat{R} x + x^T \hat{R}^T x) - ||x||^2 \leq 0
\]
as $\hat{R}$ is a rotation matrix and $||\hat{R}x|| = ||x||$. So $Z_s$ is negative semi-definite implying,
\[
tr(\hat{\Omega}^T Z_s\hat{\Omega}) = \sum_{i=1}^3 (\Omega \times e_i)^T Z_s(\Omega \times e_i) \leq 0
\]
with $(e_i)_{ij} = \delta_{i,j}, j = 1, 2, 3$. Hence,
\[
\dot{W}|_{\mathcal{M}}(R, \Omega) \leq -(k_d - \epsilon)||\Omega||^2 - ek_d||Z_k^y||^2 - ek_p||Z_k^x||^2 \leq 0
\]
is negative definite if $0 < \epsilon < \frac{4k_p k_d}{4k_p + k_d}$.