Stability of Abrikosov lattices under gauge-periodic perturbations

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Abstract
We consider Abrikosov-type vortex lattice solutions of the Ginzburg–Landau equations of superconductivity, consisting of single vortices, for magnetic fields close to the second critical magnetic field $H_{c2} = \kappa^2$ and for superconductors filling the entire $\mathbb{R}^2$. Here $\kappa$ is the Ginzburg–Landau parameter. The lattice shape, parametrized by $\tau$, is allowed to be arbitrary (not just triangular or rectangular). Within the context of the time-dependent Ginzburg–Landau equations, called the Gorkov–Eliashberg–Schmid equations, we prove that such lattices are asymptotically stable under gauge-periodic perturbations for $\kappa^2 > \frac{1}{2}(1 - \frac{1}{\beta(\tau)})$ and unstable for $\kappa^2 < \frac{1}{2}(1 - \frac{1}{\beta(\tau)})$, where $\beta(\tau)$ is the Abrikosov constant depending on the lattice shape $\tau$. This result goes against the common belief among physicists and mathematicians that Abrikosov-type vortex lattice solutions are stable only for triangular lattices and $\kappa^2 > \frac{1}{2}$. (There is no real contradiction though as we consider very special perturbations.)

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1. Introduction

1.1. Background

The macroscopic theory of superconductivity, by now a classical theory presented in any book on superconductivity and solid state or condensed matter physics (see e.g. [21, 31]), was developed by Ginzburg and Landau along the lines of Landau’s theory of the second order phase transitions before the microscopic theory was discovered. At the foundation of this theory lie the Ginzburg–Landau equations for the order parameter and magnetic potential. The time-dependent generalization of these equations was proposed by Schmid [28] and Gorkov and Eliashberg [14] and are known as the Gorkov–Eliashberg or Gorkov–Eliashberg–Schmid equations (as well as the time-dependent Ginzburg–Landau equations). The latter equations
have a much narrower range of applicability than the Ginzburg–Landau equations [31] and many refinements have been proposed, but even a slight improvement of these equations is, at least notationally, extremely cumbersome.

By far, the most important and celebrated solutions of the Ginzburg–Landau equations of superconductivity are vortices and vortex lattices, discovered by Abrikosov [1], and known as Abrikosov (vortex) lattice solutions. Among other things understanding these solutions is important for maintaining the superconducting current in type II superconductors, i.e. for \( \kappa > \frac{1}{\sqrt{2}} \).

Abrikosov lattice solutions have been extensively studied in the physics literature. Among many rigorous results, we mention that the existence of these solutions was proven rigorously in [5, 8, 12, 26, 32]. Moreover, important and fairly detailed results on asymptotic behaviour of solutions, for \( \kappa \to \infty \) and applied magnetic fields, \( h \), satisfying \( h \leq \frac{1}{2} \log \kappa + \text{const} \) (the London limit), were obtained in [6] (see this paper and the book [27] for references to earlier work). Further extensions to the Ginzburg–Landau equations for anisotropic and high temperature superconductors in the \( \kappa \to \infty \) regime can be found in [2, 3].

In this paper, we address the problem of the stability of Abrikosov vortex lattice solutions within the framework of the time-dependent Ginzburg–Landau equations, known as the Gorkov–Eliashberg–Schmid equations. We consider such solutions for lattices of arbitrary shape (in the extensive literature on the subject such solutions are considered only for triangular or rectangular lattices) and for magnetic fields close to the second critical magnetic field \( H_{c2} = \kappa^2 \) (the other case of magnetic fields larger than but close to the first critical magnetic field \( H_{c1} \) treated in the literature is not addressed here) and, as common, for superconductors filling all of \( \mathbb{R}^2 \).

We consider the simplest perturbations, namely those having the same (gauge-)periodicity as the underlying (stationary) Abrikosov vortex lattice solutions (we call such perturbations gauge-periodic) and prove for a lattice of arbitrary shape that, under gauge-periodic perturbations,

(i) Abrikosov vortex lattice solutions are asymptotically stable for \( \kappa^2 > \frac{1}{2}(1 - \frac{1}{\beta(\tau)}) \);

(ii) Abrikosov vortex lattice solutions are unstable for \( \kappa^2 < \frac{1}{2}(1 - \frac{1}{\beta(\tau)}) \).

Here \( \beta(\tau) \) is the Abrikosov constant depending on the lattice shape \( \tau \) (a complex number parametrizing the lattice shapes), defined in (11). (For the definitions of various stability notions see section 1.6.)

This result belies the common belief among physicists and mathematicians that Abrikosov-type vortex lattice solutions are stable only for triangular lattices and \( \kappa > \frac{1}{\sqrt{2}} \) and it seems this is the first time the threshold

\[
\kappa_c(\tau) := \frac{1}{\sqrt{2}} \sqrt{1 - \frac{1}{\beta(\tau)}}
\]

has been isolated.

Gauge-periodic perturbations are not a common type of perturbation occurring in superconductivity, but the methods we develop are fairly robust and can be extended—at the expense of significantly more technicalities—to substantially wider classes of perturbation, which will be done elsewhere. Moreover, the same techniques could be used in other problems of pattern formation, which are ubiquitous in applications.

To the best of our knowledge there have been no prior results on the asymptotic stability of Abrikosov lattices. Orbital stability can often be deduced if the solution in question is a minimizer of an appropriate energy functional (see [9]). Hence the orbital stability of Abrikosov lattices for \( \kappa^2 > \frac{1}{2} \) follows from the variational proof of [26] (see also [12])
that the single vortex Abrikosov lattices for $\kappa^2 > \frac{1}{2}$ are global minimizers of the Ginzburg–Landau energy functional on the fundamental cell. However, variational techniques do not give asymptotic stability.

In the rest of this section we present the basic equations involved, discuss their properties and related definitions, and present our result.

### 1.2. Gorkov–Eliashberg–Schmid equations

Macroscopically, the states of superconductors are described by the triples, $(\Psi, A, \Phi) : \mathbb{R}^+ \times \mathbb{R}^2 \to \mathbb{C} \times \mathbb{R}^2 \times \mathbb{R}$, where $\Psi$ is the (complex-valued) order parameter, $A$ is the vector potential, and $\Phi$ is the scalar potential. Physically, $|\Psi|^2$ gives the (local) density of electrons having formed Cooper pairs. $\text{curl}A$ is the magnetic field, and $-\partial_t A - \nabla \Phi$ is the electric field.

The equations are given by the Gorkov–Eliashberg–Schmid equations (or time-dependent Ginzburg–Landau equations) in $\mathbb{R}^2$, which can be written as

$$
\gamma \partial_{\Psi} \Psi = \Delta_A \Psi + \kappa^2 (1 - |\Psi|^2) \Psi,
$$

$$
\sigma \partial_{A} A = -\text{curl}^* \text{curl} A + \text{Im}(\overline{\Psi} \nabla \Psi).
$$

Here $\kappa$ is a positive constant, $\gamma$ a complex number with $\text{Re}\gamma > 0$, $\sigma$ a positive $2 \times 2$ matrix and $\nabla_A = \nabla - iA$ and $\Delta_A = \nabla_A \cdot \nabla_A$ are the covariant gradient and Laplacian, and $\partial_{\Psi} A$ is the covariant time derivative $\partial_{\Psi} A = (\partial_{A} + i\Phi) \Psi$ or $\partial_{A} A = \partial_{A} + \nabla \Phi$, $\text{curl} A := \partial_{x_1} A_2 - \partial_{x_2} A_1$ and $\text{curl}^* f = (\partial_{x_1} f, -\partial_{x_2} f)$. The second equation is Ampère’s law, $\text{curl}^* B = J_N + J_S$, with $J_N = -\sigma (\partial_A A + \nabla \Phi)$ (using Ohm’s law with $\sigma$ as the conductivity tensor) being the normal current associated with the electrons not having formed Cooper pairs, and $J_S = \text{Im}(\overline{\Psi} \nabla \Psi)$ being the supercurrent associated with the electrons having formed such pairs. Note that for a solution $(\Psi, A, \Phi)$ of (2) the pair $(\Psi, A)$ determines $\Phi$ through the equation

$$
\Delta \Phi = -\partial_t \text{div} A + \text{div} \sigma^{-1} \{\text{Re}(\overline{\Psi} \nabla \Psi) - \text{curl}^* \text{curl} A\}.
$$

The equations (2) have the structure of a gradient-flow equation for the Ginzburg–Landau energy functional given by

$$
\mathcal{E}_{\Omega}(\Psi, A) = \frac{1}{2} \int_{\Omega} \left( |\nabla_A \Psi|^2 + |\text{curl} A|^2 + \frac{\kappa^2}{2} (1 - |\Psi|^2)^2 \right),
$$

where $\Omega$ is any domain in $\mathbb{R}^2$. Indeed, (if we ignore for the moment any boundary terms) they can be put in the form

$$
\partial_{\Psi} \Psi = -\lambda \mathcal{E}_{\Omega}'(\Psi, A),
$$

where $\lambda$ is the block-diagonal matrix given by $\lambda := \text{diag}(\gamma^{-1}, \sigma^{-1})$, and $\mathcal{E}_{\Omega}'$ is given by

$$
\mathcal{E}_{\Omega}'(\Psi, A) = (-\Delta_A \Psi - \kappa^2 (1 - |\Psi|^2) \Psi, \text{curl}^* \text{curl} A - \text{Im}(\overline{\Psi} \nabla \Psi)),
$$

and is formally the $L^2$-gradient of $\mathcal{E}_{\Omega}(\Psi, A)$. Although (5) is not a standard form of the gradient flow, we show in lemma 14 below that the energy (4) decreases under the flow.

The local existence theory for (2) follows the lines of parabolic theory (see [22, 23]) and can be found in [10].

### 1.3. Ginzburg–Landau equations

The static solutions of the Gorkov–Eliashberg–Schmid equations (2) are triples $(\Psi, A, 0)$ independent of time. In this case $(\Psi, A)$ satisfies the well-known Ginzburg–Landau equations, which describe superconductors in thermodynamic equilibrium and are given by

$$
-\Delta_A \Psi = \kappa^2 (1 - |\Psi|^2) \Psi,
$$

$$
\text{curl}^* \text{curl} A = \text{Im}(\overline{\Psi} \nabla \Psi).
$$
Not surprisingly, they are the Euler–Lagrange equations for the energy functional (4): 
\[ E(\Psi, A) = 0. \]

1.4. Symmetries

The Gorkov–Eliashberg–Schmid equations (2) admit several continuous symmetries, that is, transformations which map solutions to solutions:

**Gauge symmetry.** for any sufficiently regular function \( \eta : \mathbb{R}^2 \to \mathbb{R} \),

\[ (\Psi(x, t), A(x, t), \Phi(x, t)) \mapsto (e^{i\eta(x,t)}\Psi(x, t), A(x, t) + \nabla \eta(x, t), \Phi(x, t) - \partial_t \eta(x, t)). \]

**Translation symmetry.** for any \( h \in \mathbb{R}^2 \),

\[ (\Psi(x, t), A(x, t), \Phi(x, t)) \mapsto (\Psi(x + h, t), A(x + h, t), \Phi(x + h, t)). \]

**Rotation and reflection symmetry.** for any \( R \in \text{O}(2) \) (including the reflections \( f(x) \mapsto f(-x) \))

\[ (\Psi(x, t), A(x, t), \Phi(x, t)) \mapsto (\Psi(R^{-1}x, t), RA(R^{-1}x, t), \Phi(Rx, t)). \]

These symmetries restrict to symmetries of the Ginzburg–Landau equations by considering time-independent transformations.

1.5. Abrikosov lattices

Let \( \mathcal{L} \) be a lattice in \( \mathbb{R}^2 \), i.e. a subset \( \mathcal{L} = \mathbb{Z}v_1 \oplus \mathbb{Z}v_2 \), where \( v_1, v_2 \in \mathbb{R}^2 \) are linearly independent. An Abrikosov (vortex) lattice is a pair, \( (\Psi, A) : \mathbb{R}^2 \to \mathbb{C} \times \mathbb{R}^2 \), for which the physical characteristics \( |\Psi|^2, B = \text{curl} A \) and \( J_S = \text{Im}(\overline{\Psi} \nabla \Psi) \) are doubly periodic with respect to a lattice \( \mathcal{L} \). One can show that a pair \( (\Psi, A) \) is an Abrikosov lattice if and only if it is gauge-periodic (with respect to a lattice \( \mathcal{L} \)) in the sense that there exist functions \( g_v : \mathbb{R}^2 \to \mathbb{R}, v \in \mathcal{L} \), such that for all \( v \in \mathcal{L} \) and \( x \in \mathbb{R}^2 \),

\[ \Psi(x + v) = e^{i\theta_v(x)}\Psi(x), \]
\[ A(x + v) = A(x) + \nabla g_v(x). \]

(In the terminology of [29] the pair \( (\Psi, A) \) is equivariant under the group of lattice translations.) An important property of gauge-periodic pairs is the quantization of magnetic flux. Let \( \Omega \) be any fundamental cell of \( \mathcal{L} \). Then the magnetic flux quantization property states that \( \int_\Omega \text{curl} A = 2\pi n \) for some integer \( n \). This can be written in terms of the average magnetic flux, \( b = \langle \text{curl} A \rangle_\mathcal{L} \) as

\[ b = \frac{2\pi n}{|\Omega|}, \]

where \( |\Omega| \) denotes the area of \( \Omega \), and \( \langle f \rangle_\mathcal{L} \) is the average of a function \( f \) over the lattice cell,

\[ \langle f \rangle_\mathcal{L} = \frac{1}{|\Omega|} \int_\Omega f. \]

Using the reflection symmetry of the problem, one can easily check that we can always assume \( n \geq 0 \).

Now any lattice \( \mathcal{L} \) can be given a basis \( \{v_1, v_2\} \) such that the complex number \( \tau = \frac{|v_2|}{|v_1|} e^{i\theta} \), where \( \theta \) is the angle between \( v_1 \) and \( v_2 \), satisfies the conditions that \( |\tau| \geq 1 \), \( \text{Im} \tau > 0 \), \(-\frac{1}{2} < \text{Re} \tau \leq \frac{1}{2} \), and \( \text{Re} \tau \geq 0 \), if \(|\tau| = 1 \). Although the basis is not unique, the value of \( \tau \) is, and we will call it the shape parameter of the lattice.
Note that $\tau$, $b$, and $n$ determine the lattice $\mathcal{L}$ up to rotation (which is a symmetry of the Ginzburg–Landau equations). We will say that a pair $(\Psi, A)$ is of type $(\tau, b, n)$, if the underlying lattice has shape parameter $\tau$, the average magnetic flux per lattice cell is equal to $b$, and there are $n$ quanta of magnetic flux per lattice cell. We also restrict ourselves to $C^\infty$ pairs $(\Psi, A)$ which suffices for us due to elliptic and parabolic regularity.

For $b = \kappa^2$ and on lattices, satisfying (10), we define a function, called the Abrikosov constant (we call it Abrikosov parameter), playing an important role in what follows, as

$$\beta(\tau) = \frac{\langle |\Psi(0)|^4 \rangle_{\mathcal{L}}}{\langle |\Psi(0)|^2 \rangle_{\mathcal{L}}}.$$  \hspace{1cm} (11)

Here $\Psi(0) \not= 0$ is the unique solution of the equation $-\Delta_b^0 \Psi(0) = b \Psi(0)$ on $\Omega$, satisfying the gauge-periodic boundary condition $\Psi(0)(x + \nu) = e^{i\frac{\kappa_0}{2} Jx} \Psi(0)(x)$, for any element $\nu$ of a basis of $\mathcal{L}$. Here $A_b^0(x) := \frac{b}{2} J x$, where $J$ is the symplectic matrix $J = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}$. ($b$ is the simple lowest eigenvalue of the operator $-\Delta_b$, defined on functions specified, see e.g. [32].) We have the following existence theorem (see \cite{8, 12, 26, 32, 33}).

**Theorem 1.** Let $\tau$ by any lattice shape parameter and let $b$ obey $|b - \kappa^2| \ll 1$ and $(\kappa - \kappa_0(\tau))(\kappa^2 - b) > 0$. Then there exists an Abrikosov lattice solution $u_{b,0}^\tau = (\Psi_{b,0}^\tau, A_{b,0}^\tau)$ of type $(\tau, b, 1)$.

**Remarks.**

(1) The most general condition $(\kappa - \kappa_0(\tau))(\kappa^2 - b) > 0$ has appeared implicitly in \cite{32} and explicitly, in \cite{33}.

(2) More detailed properties of these solutions are given in sections 2.1 and 3.4.

As we deal only with the case $n = 1$, we now assume that this is so and drop $n$ from the notation and write $(\tau, b)$ for $(\tau, b, 1)$.

### 1.6. Gauge periodic perturbations and stability

We now wish to study the stability of these Abrikosov lattice solutions under a class of perturbations that preserve the double-periodicity of the solution. More precisely, we consider perturbations $v = (\xi, \alpha)$ of the $(\tau, b)$-Abrikosov lattice $u_{b,0}^\tau$ s.t. $u_{b,0}^\tau + v$ is again of the type $(\tau, b)$. From the definition of the type $(\tau, b)$ pairs, it follows that $\xi$ and $\alpha$ satisfy the conditions

$$\xi(x + \nu) = e^{i\kappa_{0}(\tau)} \xi(x) \quad \text{and} \quad \alpha(x + \nu) = \alpha(x),$$  \hspace{1cm} (12)

for any element $\nu$ of $\mathcal{L}$. We introduce our space of perturbations $P_b^\tau$ to consist of all pairs $v = (\xi, \alpha) \in H^1_{\text{loc}}(\Omega_b^\tau)$ satisfying (12) and some fixed gauge condition. By the condition (12), it suffices to restrict the problem to any fundamental cell $\Omega_b^\tau$ of the lattice $\mathcal{L}_b$. Thus we identify $P_b^\tau$ with the space of pairs $v = (\xi, \alpha) \in H^1(\Omega_b^\tau \times \mathbb{R}^2)$ satisfying (12) for any element $\nu$ of a basis of $\mathcal{L}_b$. This space is naturally a (real) Hilbert space with the $H^1$ inner product of $v = (\xi, \alpha)$ and $v' = (\xi', \alpha')$ given by

$$\langle v, v' \rangle_{H^1} = \frac{1}{|\Omega_b^\tau|} \Re \int_{\Omega_b^\tau} \left( \overline{\xi'} \hat{\xi} + \nabla \overline{\xi'} \cdot \nabla \xi + \alpha \cdot \alpha' + \sum_{\xi=1}^2 \nabla \alpha_{k} \cdot \nabla \alpha_{k}' \right).$$

Here we use that the covariant gradient preserves the quasiperiodicity conditions and the dot product is in $\mathbb{R}^2$. We focus on solutions $(\Psi, A, \Phi)$ of (2) with the initial conditions of the form $u_0 \equiv (\Psi_0, A_0) = u_{b,0}^\tau + v_0$, $v_0 \in P_b^\tau$. 

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Let $T_{\gamma}$ be the transformation given by (8) and let $H^s_{\text{per}}$ be the Sobolev space of periodic functions on $\Omega$ of order $s$. We now consider the manifold $\mathcal{M} := \{T_{\gamma}u^b_{\tau} : \gamma \in H^2_{\text{per}}\}$ of Abrikosov lattices and its tubular neighbourhood $U_\delta$, given by

$$U_\delta = \{T_{\gamma}(u^b_{\tau} + v) : \gamma \in H^2_{\text{per}}, \ v \in \mathcal{T}^2_{\delta}, \ \|v\|_{H^1} < \delta\}. \quad (13)$$

We will say that $u^b_{\tau}$ is orbitally stable under gauge-periodic perturbations if for all $\epsilon > 0$, there exists $\delta > 0$ such that if a $C^1$ solution $(\Psi, A, \Phi)$ obeys $u = (\Psi, A) \in U_\delta$ when $t = 0$, then $u \in U_\epsilon$ for all $t \geq 0$, and asymptotically stable, if there exist $\gamma(t) \in H^2_{\text{per}}$, such that any $C^1$ solution $(\Psi, A, \Phi)$ obeys $u = (\Psi, A) \in U_\delta$ when $t = 0$, satisfies $u - T_{\gamma(t)}u^b_{\tau} \to 0$, as $t \to \infty$. We will say that $u^b_{\tau}$ is unstable under gauge-periodic perturbations if it is not orbitally stable.

The main result in this paper is the following.

**Theorem 2.** We assume for simplicity that $\gamma = \sigma = 1$. For all $b$ sufficiently close to $\kappa^2$, the Abrikosov lattice $u^b_{\tau}$ is asymptotically stable under gauge-periodic perturbations if $\kappa^2 > \frac{1}{2}(1 - \frac{1}{\beta(\tau)})$ and is unstable if $\kappa^2 < \frac{1}{2}(1 - \frac{1}{\beta(\tau)})$.

**Remarks.**

1. The Abrikosov parameter, $\beta(\tau)$, enters through the smallest eigenvalue of the Hessian of the energy functional (4) on the lattice cell under the gauge-periodicity conditions alluded to above.

2. We explain briefly at the end of section 2 how to extend the theorem above to $\gamma$ and $\sigma$ satisfying $\gamma > 0$ and $\sigma > 0$.

This paper is organized as follows. In the next section we prove the theorem above, modulo a statement about properties of the Hessian of the Ginzburg–Landau energy functional. The latter statement is proven in section 3. In the appendix we use energy methods to prove the weaker statement of orbital stability.

As we consider $\tau$ fixed, from now on we do not display it in the notation below.

For functions $A$ and $B$, we will use the notation $A \lesssim B$ signifying that $A \leq CB$ for some absolute (numerical) constant $0 < C < \infty$.

**2. Proof of theorem 2**

**2.1. Choosing the gauge**

Using the symmetries of the Ginzburg–Landau equations, it was shown in [8, 26, 30, 32] that any pair $(\Psi, A)$ of type $(\tau, b) \equiv (\tau, b, 1)$ is gauge equivalent to one satisfying the following conditions:

1. $(\Psi, A)$ is gauge-periodic with respect to the lattice $\mathcal{L}_b$ spanned by $r_b(1, 0)$ and $r_b(\text{Rer.}, \text{Im.})$, where $r_b = \sqrt{\frac{2}{\beta(\tau)}}$.

2. The function $g_b(x)$ in (9) can be chosen as $g_b(x) = \frac{b}{2}x \cdot Jx$, where, recall, $J$ is the symplectic matrix $J = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}$, and therefore, for any element $v$ of a basis of $\mathcal{L}_b$, $\Psi$ and $A$ satisfy the quasiperiodic boundary conditions

$$\Psi(x + v) = e^{\frac{b}{2}x \cdot Jx} \Psi(x) \quad \text{and} \quad A(x + v) = A(x) + \frac{b}{2}Jv, \quad (14)$$

3. $(A(x) - A^0_b(x))_{\mathcal{L}_b} = 0$, where $A^0_b(x) := \frac{b}{2}Jx$.

4. $\text{div}A(x) = 0$. 


We note that the only continuous symmetry that preserves these properties is the $U(1)$ symmetry
\[ (\Psi, A) \mapsto (e^{i\gamma} \Psi, A), \quad \forall \gamma \in \mathbb{R}. \]
From now on we assume that $u_b = (\Psi_b, A_b)$ satisfies (I)–(IV) and that perturbations $v = (\xi, \alpha)$ satisfy (I)–(III), i.e.
\[ \xi(x + v) = e^{i\frac{\gamma}{2} x \cdot J^\nu \xi(x)} \quad \text{and} \quad \alpha(x + v) = \alpha(x), \quad (15) \]
for any element $v$ of a basis of $L_b$, and
\[ \langle \alpha \rangle_{L_b} = 0. \quad (16) \]
Thus our space of perturbations $\mathcal{P}_b$ now is given by
\[ \mathcal{P}_b := \{ v = (\xi, \alpha) \in H^1(\Omega_b; \mathbb{C} \times \mathbb{R}^2) | \ (15)–(16) \text{ are satisfied} \}. \quad (17) \]
We fix the lattice $L_b$ spanned by $r_b(1,0)$ and $r_b(\text{Re} \tau, \text{Im} \tau)$, where $r_b = \sqrt{2\pi \text{Im} \tau}$.

We consider solutions, $(\Psi, A, \Phi)$, of (2) with initial conditions $(\Psi_0, A_0, \Phi_0)$ satisfying
\[ \Psi_0(x + v) = e^{i\gamma / 2} x \cdot J^\nu \Psi_0(x), \quad A(x + v, t) = A(x, t) + \frac{i}{2} J^\nu \quad \text{and} \quad \Phi(x + v, t) = \Phi(x, t). \]

Condition (i) breaks the translational symmetry and a part of the gauge symmetry (8), leaving the gauge symmetry with $\gamma \in H^2_\text{per}$. We use the latter with the gauge transformation $T_{\eta}$, with $\eta(x, t) = \int_0^t \Phi(x, s) \, ds \in H^2_\text{per}$, to achieve
\[ \Phi(x, t) = 0. \]

The latter gauge transformation also preserves (ii), as follows from the relation
\[ \int_{\Omega} dx \nabla \eta(x) = \int_0^t ds \int_{\Omega} dx \nabla \Phi(x, s) = 0, \]
obtained by using integration by parts and the fact that the boundary terms vanish due to the periodicity of $\Phi$.

### 2.2. Decomposition

The tangent space to the infinite-dimensional manifold of Abrikosov lattices $\mathcal{M} = \{ T_\gamma u_b : \gamma \in H^2_\text{per} \}$ at $u_b \in \mathcal{M}$ is spanned by the infinitesimal gauge transformation, $G_\gamma = \partial_\gamma T_\gamma u_b |_{|\gamma|=0}$, with $\gamma \in H^2_\text{per}$, given by
\[ G_\gamma = (i\gamma \Psi_b, \nabla \gamma). \quad (18) \]
We introduce the $L^2$ inner product
\[ \langle v, v' \rangle_{L^2} = \frac{1}{|\Omega_b|} \text{Re} \int_{\Omega_b} (\bar{\xi} \xi' + \alpha \cdot \alpha). \quad (19) \]
We now prove the following decomposition for $u$ close to the manifold $\mathcal{M}$.

**Proposition 3.** There exist $\delta_0 > 0$ and a map $\eta : U_{\delta_b} \to H^2_\text{per}$ such that $T_\eta(u) - u_b \perp G_\gamma$, $\forall \gamma \in H^2_\text{per}$ (with respect to the $L^2$ inner product).
Proof. Let $X$ be the affine space $X = u_b + \mathcal{P}_b$. The orthogonality condition $v \perp G_y, \forall y \in H^2_{\text{per}}$ is equivalent to the condition

$$\text{Im}(\bar{\Psi}_b \xi) - \text{div} \alpha = 0,$$

for $v = (\xi, \alpha)$. Hence, the desired $\chi = \eta(u)$ solves the equation

$$\text{Im}(\bar{\Psi}_b(e^{i\xi} \Psi - \Psi_b)) - \text{div}(A - A_b - \nabla \chi) = 0,$$

which can be rewritten as a nonlinear Poisson equation for $\chi$.

$$- \Delta \chi - \text{Im}(\bar{\Psi}_b e^{i\xi}(\Psi_b + \xi)) + \text{div} \alpha = 0,$$

where $u - u_b = (\xi, \alpha)$. For $\|u - u_b\|_{H^1}$ sufficiently small, this equation has a solution in $H^2_{\text{per}}$. (We can write $\chi = \eta + c$, where $\eta \in H^2_{\text{per}}$, $(\eta) = 0$, and $c = (\chi) \in \mathbb{R}$, which allows us to rewrite the above equation as the two equations $- \Delta \eta - \text{Im}(\bar{\Psi}_b e^{i\eta}(\Psi_b + \xi)) + \text{div} \alpha = 0$ and $\text{Im}(\bar{\Psi}_b e^{i\eta}(\Psi_b + \xi)) = 0$ for the unknowns $\eta$ and $c$ (the latter equation is the mean of the first one). These equations have a unique solution in $H^2_{\text{per}} \times \mathbb{R}$.) This gives us a neighbourhood $V$ of $u_b$ in $X$ and a map $h : V \to H^2_{\text{per}}$ such that (21) holds for $(\chi = h(u), u), u \in V$. We now define the map $\eta$ on $U_\delta$ for $\delta < \delta_0$ as $\eta(u) = \theta(u') - \theta$, where $u' := T_{-\theta} u$ and $\theta$ is such that $u' \in V$. It is easy to verify that $\eta$ is well defined and that $(\chi = \eta(u), u)$ satisfies (21). The proof is complete. \hfill $\square$

2.3. Hessian

The chief tool in the proof of the stability result is the analysis of the associated Hessian, i.e., the second derivative of the energy functional $E$. We first note that $E$ is a well-defined functional on $U_\delta$. We define the Hessian of $E$ at $u$ to be the operator $E''(u) := DE'(u)$, on the domain $D(E''(u)) := \mathcal{P}_b$. Here $D$ is the Gâteaux derivative and $E'$ is the $L^2$-gradient of $E$ defined by $DE(u)v = \langle E'(u), v \rangle_{L^2}$, for all $v \in \mathcal{P}_b$. Recall that the Gâteaux derivative, $D$, of a map or a functional, is a linear map or functional, defined by $DF(u)v = \frac{\partial}{\partial s} F(u + sv)|_{s=0}$. We denote the Hessian at $u_b$ by $L_b := E''(u_b)$. Explicitly, for $v = \left(\begin{array}{c} u \\ \alpha \end{array}\right)$, it is given by

$$L_b v = \left( -\Delta_b \xi + \kappa^2(2|\Psi_b|^2 - 1)\xi + \kappa^2\Psi_b^2 \xi + 2i(\nabla_b \Psi_b) \cdot \alpha + i\Psi_b \text{div} \alpha \right. \left. + \text{curl}^* \text{curl} \alpha + |\Psi_b|^2 \alpha - \text{Im}(\bar{\xi} \nabla_b \Psi_b + \Psi_b \nabla_b \xi) \right).$$

(23)

This is a real-linear operator on the space $L^2(\Omega, \mathbb{C}) \times L^2(\Omega, \mathbb{R}^2)$ with the domain $\mathcal{P}_b$. We define for it the notion of spectrum and of discrete spectrum in the usual way. We introduce the new parameter

$$\epsilon = \sqrt{\frac{\kappa^2 - b}{\kappa^2(2\kappa^2 - 1)\beta(\tau) + 1}}.$$

(24)

The term $(2\kappa^2 - 1)\beta(\tau) + 1$ in the denominator is necessary in order to have a positive expression under the square root and to calibrate the size of the perturbation domain. In fact, (24) is the unique, up to a trivial normalization, form of the bifurcation parameter for the Abrikosov lattices (see appendix B). The main result concerning this operator $L_b$ which we use in the proof of theorem 2 is the following theorem, whose proof we postpone until section 3.
Theorem 4. Suppose that \( \kappa^2 \neq \frac{1}{4} (1 - \frac{1}{\beta(t)}) \) and that \( b \) is sufficiently close to \( \kappa^2 \). Then we have the following statements:

1. The operator \( L_b \) has a real, discrete spectrum which includes the eigenvalue 0, with eigenfunctions \( G_\gamma, \forall \gamma \in H^2_{\text{per}} \).
   
   \[
   L_b G_\gamma = 0, \forall \gamma \in H^2_{\text{per}},
   \]

   while its lowest eigenvalue, \( \theta \), on the subspace

   \[
   \mathcal{P}^1_b := \{ v \in \mathcal{P}_b \mid v \perp G_\gamma, \forall \gamma \in H^2_{\text{per}} \}
   \]

   is of the form

   \[
   \theta := 2b \left[ (2\kappa^2 - 1) \beta(t) + 1 \right] \epsilon^2 + O(\epsilon^3).
   \]

   Consequently, if \( \kappa^2 < \frac{1}{2} (1 - \frac{1}{\beta(t)}) \), then \( L_b \) has a negative eigenvalue.

2. If \( \kappa^2 > \frac{1}{2} (1 - \frac{1}{\beta(t)}) \), then there is a uniform constant \( c > 0 \), such that for all \( v \in \mathcal{P}^1_b \),

   \[
   \langle v, L_b v \rangle_{L^2} \geq c \| v \|_{H^1}. \tag{29}
   \]

3. There exists a positive constant \( c > 0 \) such that, for all \( v \in \mathcal{P}_b \),

   \[
   \| \langle v, L_b v \rangle_{L^2} \| \leq c \| v \|_{H^1}. \tag{30}
   \]

A result in this direction was obtained in [12].

2.4. Asymptotic stability

We now assume that we have the case \( \kappa^2 > \frac{1}{2} (1 - \frac{1}{\beta(t)}) \) and prove asymptotic stability of the Abrikosov lattice \( u_b \). To this end we derive and use differential inequalities for the Lyapunov functional

\[
\Lambda(t) = \frac{1}{2} \langle v, L_b v \rangle_{L^2}. \tag{30}
\]

Lemma 5. Let \( u(t) := (\Psi(t), A(t)) \) be a solution of the Gorkov–Eliashberg–Schmid equations (2) on the time interval \([0, T]\) satisfying \( T_{\gamma(t)} u(t) = u_b + v(t), \gamma(t) \in H^2_{\text{per}}, v \in C^1([0, T]; \mathcal{P}^1_b) \) and \( \| v(0) \|_{H^1} \ll 1 \). Then \( \| v(t) \|_{H^1} \leq e^{-\delta t/2} \| v(0) \|_{H^1} \) for \( \delta < \theta \) for all \( t \in [0, T] \), where \( \theta \) is as in theorem 4.

Proof. We plug the decomposition \( u(t) = T_{\gamma(t)} (u_b + v(t)) \) into (2) and use the covariance of this equation with respect to the transformation \( T_{\gamma(t)} \) to obtain the equation

\[
\dot{v} = -L_b v + N(v) + G_\gamma,
\]

where \( N(v) \) is the nonlinearity given by \( N(v) = \tilde{N}(v) + \tilde{\nu} \), with \( \tilde{v} := \left( \begin{array}{c} \tilde{\nu} \\ \tilde{\gamma} \end{array} \right) \) and

\[
\tilde{N}(v) = \left( \begin{array}{c}
\left( \nabla \Lambda_0^2 + \alpha \cdot \nabla \Lambda_0^2 + |\alpha|^2 \right) \delta + |\alpha|^2 \Psi_b - \kappa^2 (2 \text{Re}(\bar{\Psi}_b \xi) - |\xi|^2) \xi + \kappa^2 (|\xi|^2) \right) \Psi_b \\
\text{Im}(\bar{\xi} \nabla \Lambda_0^2 + \text{Im}(\bar{\xi} \nabla \Lambda_0^2 \xi - i \alpha \xi \Psi_b + \bar{\Psi}_b \xi) - i \alpha |\xi|^2)
\end{array} \right)
\]

for \( v = \left( \begin{array}{c} \nu \\ \gamma \end{array} \right) \). Using equation (31) and \( \langle G_1, L_b v \rangle = 0 \), we obtain

\[
-\dot{\Lambda}(t) = \langle v, L_b^2 v \rangle_{L^2} - \langle N(v), L_b v \rangle_{L^2}.
\]

We estimate the nonlinearity, \( N(v) \). Multiplying (31) scalarly by \( G_\chi \) and using \( \langle G_\chi, v \rangle = 0, L_b G_\chi = 0 \) and \( \langle G_\chi, \tilde{\nu} \rangle = \text{Re}(\bar{\chi} \Psi_b \xi - \bar{\gamma} \Delta \gamma) \), we find \( \langle G_\chi, \tilde{N}(v) \rangle + \langle \chi, \gamma \text{Re}(\Psi_b \xi) - \bar{\gamma} \Delta \gamma \rangle = 0 \). Since \( \chi \in H^2_{\text{per}} \) is arbitrary, the latter equation implies...
as the two equations
\[ (-\Delta + \text{Re}(\tilde{\Psi}_b\xi))\bar{\eta} = -\text{Im}(\tilde{\Psi}_b\tilde{\eta}(v)) + \text{div} \tilde{N}_a(v) - c\text{Re}(\tilde{\Psi}_b\xi) \]
\[ \dot{c}(\text{Re}(\tilde{\Psi}_b\xi)) = -(\text{Im}(\tilde{\Psi}_b\tilde{\eta}(v)) + \bar{\eta}\text{Re}(\tilde{\Psi}_b\xi)) \]
for the two unknowns \( \eta \) and \( c \) (the latter equation is the mean of the first one). For \( \|\xi\|_{L^2} \) sufficiently small, the first equation has a unique solution
\[ \bar{\eta} = (-\Delta + \text{Re}(\tilde{\Psi}_b\xi))^{-1}[-\text{Im}(\tilde{\Psi}_b\tilde{\eta}(v)) + \text{div} \tilde{N}_a(v) - c\text{Re}(\tilde{\Psi}_b\xi)] \]
where, recall, \( H_{\text{per}}^2 := \{ \eta \in H_{\text{per}}^1, \langle \eta \rangle = 0 \}. \) Plugging this into the second equation, we find
\[ \dot{c}(\|(-\Delta + \text{Re}(\tilde{\Psi}_b\xi))^{-1}\|\text{Re}(\tilde{\Psi}_b\xi)) = -\langle \text{Im}(\tilde{\Psi}_b\tilde{\eta}(v)) + (-\Delta + \text{Re}(\tilde{\Psi}_b\xi))^{-1}\text{Im}(\tilde{\Psi}_b\tilde{\eta}(v)) - \text{div} \tilde{N}_a(v)\rangle. \]
The latter equation either has a unique solution for \( \dot{c} \) or \( \dot{c} \) drops out of the equation. In the latter case, we set \( c \) to zero.

Using this, the expression for \( N(v) \) and Sobolev embedding theorems we obtain easily the following rough estimate:
\[ \|N(v)\|_{L^2} \lesssim \|v\|_{H^1}^2 (\|v\|_{H^1} + \|v\|_{H^1}^3), \] (34)
which implies that
\[ |\langle N(v), L_b v \rangle_{L^2} | \lesssim \|L_b v\|_{L^2} \|v\|_{H^1} (\|v\|_{H^1} + \|v\|_{H^1}^3). \] (35)
Since, by (29), \( \|v\|_{H^1} \lesssim \|L_b v\|_{L^2} + \|v\|_{L^2} \) and \( \|v\|_{H^1}^2 \lesssim \Lambda(t) \), the last inequality implies
\[ |\langle N(v), L_b v \rangle_{L^2} | \lesssim (\|L_b v\|_{L^2}^2 + \Lambda(t))^{\frac{1}{2}} (\Lambda(t)^{\frac{3}{2}} + \Lambda(t)^{\frac{5}{2}}). \] (36)
Next, we think of \( L_b \) as the restriction to the subspace \( \mathcal{T}_b \) and define \( L_b^\alpha \), for \( 0 < \alpha < 1 \), by the formula \( L_b^\alpha = C \int_0^\infty (\frac{1}{\omega} - \frac{1}{1 + \omega})^\alpha d\omega \), where \( C^{-1} := \int_0^\infty (\frac{1}{\omega} - \frac{1}{1 + \omega})^\alpha d\omega. \) (Another way to proceed is to use the complex-linear extension of \( L_b \) constructed in section 3.3.) Then writing \( \langle v, L_b^\alpha v \rangle_{L^2} = \langle L_b^{\frac{1}{2}} v, L_b^{\frac{1}{2}} L_b^\alpha v \rangle_{L^2} \) and using (28), we find \( \langle v, L_b^\alpha v \rangle_{L^2} \lesssim \theta \langle v, L_b v \rangle_{L^2} \approx 20 \Lambda(t). \) Using this, we obtain
\[ -\partial_t \Lambda(t) \gtrsim \theta \Lambda(t)^{\frac{5}{2}} + \left[ \frac{1}{2} - c(\Lambda(t)^{\frac{1}{2}} + \Lambda(t)) \right] \|L_b v\|_{L^2}^2 - \Lambda(t)^{\frac{3}{2}} - \Lambda^2(t). \] (37)
If we now assume that \( \Lambda(t) \ll 1 \), then this gives
\[ -\partial_t (e^{\delta t} \Lambda(t)) \gtrsim (\theta - \delta) e^{\delta t} \Lambda(t). \] (38)
Integrating the last inequality from 0 to \( t \), one finds
\[ \Lambda(0) \gtrsim e^{\delta t} \Lambda(t) + (\theta - \delta) \int_0^t e^{\delta s} \Lambda(s) \, ds \] (39)
and in particular \( \Lambda(t) \leq e^{-\delta t} \Lambda(0) \) for \( \delta < \theta \). Taking \( \Lambda(0) \ll 1 \), we see that our assumption \( \Lambda(t) \ll 1 \) is justified, which completes the argument. Finally, appealing again to (29) shows that \( \|v\|_{H^1} \leq e^{-\delta t/2} \|v(0)\|_{H^1} \) for \( \delta < \theta \).

To complete the proof of asymptotic stability, let \( \delta_0 \) be given by proposition 3 and let \( v(0) \in \mathcal{T}_b \). By standard parabolic existence theory (see e.g. [10, 22, 23]) for equation (31), written for the real and imaginary parts of \( v \), has a unique solution \( u(t) \in U_{\delta_0} \) for \( t \leq T' \), for some \( T \gg 0 \). Let \( T_* \) be the supremum of such \( T \). If \( T_* < \infty \), then \( u(T_*) \in \partial U_{\delta_0} \). Then by proposition 3, there is \( \gamma(t) \in \mathbb{R} \) so that \( T_{\gamma(t)} u(t) = u_b + v(t) \), with \( v(t) \in C^1([0, T]; \mathcal{T}_b) \). By lemma 5, \( \|v(t)\|_{H^1} \leq e^{-\delta t/2} \|v(0)\|_{H^1} \approx \frac{1}{2} \delta_0 \) for all \( t \in [0, T_*] \), which contradicts
Clearly, \( L_b \) is the negative eigenvector of \( \lambda \eta \). We normalize \( \eta \) so that \( \| \eta \|_{L^2} = 1 \).

For \( \delta > 0 \) we now define \( u(t) \) to be the solution with the initial datum \( u_{0,0} = u_b + \delta \eta \).

We write this solution as \( u(t) = u_b + v_\delta(t) \). Then \( v_\delta(t) \) satisfies equation (31) with the initial condition \( v_{\delta,0} = \delta \eta \). Using the Duhamel principle and the fact that \( e^{-\delta \eta} \eta = e^{-\delta \eta} \eta \) we rewrite the latter equation in the form
\[
v_\delta(t) = \delta e^{\delta t} \eta + \int_0^t e^{-\delta \eta(s)} N(v_\delta(s)) \, ds,
\]
where \( N(v) \) is the nonlinearity given in the line above (32). It satisfies for any \( \epsilon > 0 \) the following estimate:
\[
\| N(v) \|_{H^{-\epsilon}} \lesssim \| v \|^2_{H^1} + \| v \|^4_{H^3}.
\]
Next, for an appropriate large constant \( c > \lambda \), we have by the standard elliptic theory, similarly to (29), \( \| w \|^2_{H^1} \lesssim \langle w, (L_b + c)w \rangle_{L^2} \). The self-adjoint complex-linear extension of the operator \( L_b \) obtained in the next section and the invariance of the image of \( \mathcal{P}_b \) under this extension imply the spectral decomposition for \( L_b \), which gives that \( \langle w, (L_b + c)e^{-\delta \eta} \rangle \lesssim \langle t^2 + c \rangle e^{\delta \eta} \| w \|^2_{L^2} \). The last two estimates imply the bound \( \| e^{-\delta \eta} \|_{H^1} \lesssim (1 + t^2 + c^2) \| w \|_{H^{-\epsilon}} \).

Using the latter bound and (42) and writing in the rest of the proof \( \| v \| \) for \( \| v \|_{H^1} \), we obtain
\[
\| v_\delta(t) - \delta e^{\delta t} \eta \| \lesssim \int_0^t \| e^{-\delta \eta(s)} N(v_\delta(s)) \| \, ds
\]
\[
\lesssim \int_0^t (1 + (t - s)^{\frac{1}{2}}) e^{\delta(s-t)} \| N(v_\delta(s)) \|_{L^2} \, ds
\]
\[
\lesssim \int_0^t e^{\delta(s-t)} (\| v_\delta(s) \|^2 + \| v_\delta(s) \|^3) \, ds.
\]

Now, let \( M \) be a constant satisfying \( 0 < M < \| \eta \| = 1 \) and define
\[
T_1 := \sup \{ s : \| v_\delta(s) - \delta e^{\delta s} \eta \| \leq M \delta e^{\delta s} \}.
\]
Clearly, \( T_1 > 0 \), and so for \( 0 \leq t \leq T_1 \), we have by triangle inequality
\[
\| v_\delta(s) \| \| \leq \| v_\delta(s) - \delta e^{\delta s} \eta \| + \delta e^{\delta s} \| \eta \| \leq (M + 1) \delta e^{\delta s}.
\]
Therefore, by (43) and (47), we have for \( 0 \leq t \leq T_1 \),
\[
\| v_\delta(t) - \delta e^{\delta t} \eta \| \lesssim \int_0^t e^{\delta(s-t)} (\delta^2 e^{2\delta s} + \delta^3 e^{3\delta s}) \, ds
\]
\[
= \delta^2 e^{2\delta} \int_0^t e^{\delta s} (1 + \delta e^{\delta s}) \, ds.
\]
We choose \( T_2 \) to satisfy \( \delta e^{\delta T_2} = 1 \). Then
\[
\| v_\delta(t) - \delta e^{\delta t} \eta \| \leq C \delta^2 e^{2\delta t}, \quad \text{for } 0 \leq t \leq \min(T_1, T_2).
\]
Pick $C' \geq M$. $C$ and define the constant $T^\delta > 0$ by the relation

$$C' \delta e^{\lambda T^\delta} = M. \quad (49)$$

Note that, since $C' \geq M$, we have $T^\delta < T_2$. We claim that $T^\delta \leq T_1$. If not, then $T_1 < T^\delta \leq T_2$, and by (48) and (49),

$$\|v_\delta(T_1) - \delta e^{\lambda T_1} \eta\| < M \delta e^{\lambda T_1}.$$  

But this result contradicts the definition of $T_1$ in (46). Hence $T^\delta \leq T_1$, and therefore $T^\delta \leq \min(T_1, T_2)$. Now we have by the triangle inequality, (48), (49) and the condition $\|\eta\| = 1$,

$$\|v_\delta(T^\delta)\| \geq \|\delta e^{\lambda T^\delta} \eta\| - \|v_\delta(T^\delta) - \delta e^{\lambda T^\delta} \eta\| \geq \delta e^{\lambda T^\delta} - \delta e^{\lambda T^\delta} M.$$  

If we set $\nu:=(1-M)M > 0$, then the last equation, together with (49), implies $\|v_\delta(T^\delta)\| \geq \nu$, which can be rewritten as

$$\|u(T^\delta) - u_b\| \geq \nu. \quad (50)$$

Now we note that for $\delta$ sufficiently small, the unique minimizer $\gamma_*$ of $\|u - T_\gamma u_b\|^2$ satisfies $\gamma_* = O(\delta)$ and therefore (50) implies

$$\inf_{\gamma} \|u(T^\delta) - T_\gamma u_b\| \geq \nu - O(\delta). \quad (51)$$

In other words, $\forall \delta > 0$ sufficiently small, there is $u_0 \in U_\delta$ such that $u(T^\delta) \notin U_{\frac{\nu}{2}}$, $\forall t \geq 0$, for a fixed $\nu > 0$ independent of $\delta$. This implies instability.

Remark 6. To extend the proofs of stability and instability to arbitrary positive $\gamma$ and $\sigma$, we change definition (30) of the Lyapunov functional to

$$\Lambda_1(t) = \frac{1}{2} \langle v, \lambda^{-1} L_b v \rangle_{L^2}. \quad (52)$$

and note that plugging the decomposition $u(t) = T_{-\gamma(t)}(u_b + v(t))$ into (2) and using the covariance of this equation with respect to the transformation $T_{\gamma(t)}$ gives the equation

$$\partial_t v = -\lambda L_b v + \lambda N(v). \quad (53)$$

After this one proceeds as above, but in addition to $L_b$ one would also have to estimate the operator $L_b + \lambda^{-1} L_b \lambda$.

3. Estimates on Hessian

In this section we prove theorem 4 concerning the positivity of the Hessian $L_b$. In what follows we omit the subindex $L^2$ in the inner products and norms.

Note that in principle we do not have to work in the same gauge as was chosen in section 2.1, but can change the gauge using the formula

$$E''(T^{-1} u) = T^* E''(u) T, \quad (54)$$

where, recall, $E''$ denotes the Hessian of $E$ and $T$ is a symmetry of $E$, i.e. $E(T u) = E(u)$. For example, if we did not have the property (ii) of section 2.1, we could have achieved it by choosing an appropriate gauge (this would change the gauge of $u_b$ in (14)).
3.1. Shifted Hessian

First, we recall that the orthogonality condition \( v \perp G_\gamma \), \( \forall \gamma \in H^2_{\text{per}} \), which defines \( P^\perp_\gamma \), is equivalent to the condition (20), for \( v = (\xi, \alpha) \). Now, we pass to a shifted Hessian \( \tilde{L}_b \) which induces the same quadratic form as \( L_b \) on \( P^\perp_\gamma \), but is somewhat simpler in the complexified form (see section 3.3):

\[
\langle \tilde{L}_b v, v \rangle_{L^2_b} = \langle L_b v, v \rangle_{L^2_b} + \int (\Im(\Psi_b \xi) - \div \alpha)^2.
\]

(55)

It follows from (20) that the two induced quadratic forms do indeed agree on the subspace \( P^\perp_\gamma \).

It is straightforward to show that \( \tilde{L}_b \) is given by

\[
\tilde{L}_b v = \left( -\Delta_{A_b} - \kappa^2 + (2\kappa^2 + \frac{1}{2})|\Psi_b|^2 \right) \xi + (\kappa^2 - \frac{1}{2})|\Psi_b|^2 \xi + 2i(\nabla_{A_b} \Psi_b) \cdot \alpha
\]

\[-\Delta \alpha + |\Psi_b|^2 \alpha - 2 \text{Im}(\xi \nabla_{A_b} \Psi_b).\]

(56)

Let \( \mathcal{P}_{b,1} := \{ v \in P^\perp_b | \|v\| = 1 \} \), \( \mathcal{P}_{b,2} := \{ v \in P_b | \|v\| = 1 \} \) and \( \Gamma_1 := \{ \gamma \in H^2_{\text{per}}, \langle \gamma, (\Delta + |\Psi_b|^2)\gamma \rangle = 1 \} \). Define

\[
\theta' := \inf_{v \in \mathcal{P}_{b,1}} \langle \tilde{L}_b v, v \rangle \quad \text{and} \quad \mu := \inf_{\gamma \in \Gamma_1} \|(\Delta + |\Psi_b|^2)\gamma\|^2.
\]

(57)

Lemma 7.

(i) The zero modes \( G_\gamma \) of the operator \( L_b \) are not zero modes of \( \tilde{L}_b \).

(ii) The space \( P^\perp_\gamma \) is invariant under \( \tilde{L}_b \).

(iii) Let \( \mathcal{P}_{b,1} := \{ v \in P_b | \|v\| = 1 \} \). Then

\[
\inf_{v \in \mathcal{P}_{b,1}} \langle \tilde{L}_b v, v \rangle = \min(\theta', \mu).
\]

(58)

Proof. Plugging \( G_\gamma \) into (55) and using (25), we arrive at

\[
\langle G_\gamma, \tilde{L}_b G_\gamma \rangle = \langle \gamma, (\Delta + |\Psi_b|^2)\gamma \rangle,
\]

(59)

which is 0, if and only if \( (\Delta + |\Psi_b|^2)\gamma = 0 \), which implies \( \gamma = 0 \). This shows (i).

We now show that if \( v \) satisfies the orthogonality condition (20), then so does \( \tilde{L}_b v \).

Using \( \div A_b = 0 \), the fact that \( v = (\xi, \alpha) \) satisfies the orthogonality condition (20), so that \( \div \alpha = \Im(\Psi_b \xi) \), and eliminate quantities that are real we find

\[
\begin{align*}
\text{Im} & \left[ -\bar{\Psi}_b \Delta_{A_b} \xi - \kappa^2 \bar{\Psi}_b \xi + (2\kappa^2 + \frac{1}{2})|\Psi_b|^2 \bar{\Psi}_b \xi + (\kappa^2 - \frac{1}{2})|\Psi_b|^2 \bar{\Psi}_b \xi + 2i(\bar{\Psi}_b \nabla_{A_b} \Psi_b) \cdot \alpha \right] \\
- & \div \left[ -\Delta \alpha + |\Psi_b|^2 \alpha - 2 \text{Im}(\xi \nabla_{A_b} \Psi_b) \right] \\
= & \text{Im} \left[ -\bar{\Psi}_b \Delta \xi + 2iA_b \cdot \bar{\Psi}_b \nabla \xi + |A_b|^2 \bar{\Psi}_b \xi - \kappa^2 \bar{\Psi}_b \xi + (\kappa^2 + 1)|\Psi_b|^2 \bar{\Psi}_b \xi + 2i(\bar{\Psi}_b \nabla \Psi_b) \cdot \alpha \right. \\
+ & 2\bar{\Psi}_b \cdot \nabla \Psi_b + 2\bar{\xi} \nabla \Psi_b - 2iA_b \cdot \xi \nabla \Psi_b - 2iA_b \cdot \Psi_b \nabla \xi \\
+ & \left. |A_b|^2 \bar{\Psi}_b \xi - \kappa^2 |\Psi_b|^2 \bar{\Psi}_b \xi + 2i(\bar{\Psi}_b \nabla \Psi_b) \cdot \alpha \right]
\end{align*}
\]

Now, we use that \( \Psi_b \) is a solution of the Ginzburg–Landau equations to conclude that the rhs is 0. This shows (ii).

(iii) follows from (ii), but it is instructive to prove it directly. Let \( v \in \mathcal{P}_{b,1} \). Write it as \( v = s_1 v_1 + s_2 v_2 \), where \( v_1 \in P^\perp_\gamma \), \( v_2 = G_\gamma \), for some \( \gamma \in H^2_{\text{per}} \) s.t. \( \|G_\gamma\| = 1 \) and \( \|v_1\| = 1 \), and \( |s_1|^2 + |s_2|^2 = 1 \). Since \( L_b G_\gamma = 0 \) and, by (20), \( \text{Im}(\Psi_b \xi_1) - \div v_1 = 0 \), where

\[
\langle \tilde{L}_b v, v \rangle = |s_1|^2 \langle L_b v_1, v_1 \rangle + |s_2|^2 \langle (\Delta + |\Psi_b|^2)\gamma \rangle.
\]

(60)
follows from this relation and
\[ \|G_y\| = (\gamma, (-\Delta + |\Psi_b|^2)\gamma). \]  
(61)
This shows (iii). \qed

3.2. Rescaling

We now rescale the problem in order to be able to use freely the results of [32]. Given a pair \((\Psi, A)\) of type \((\tau, b)\), we set \(\sigma := \sqrt{1/b} = r_b^{1/4} \ell^{1/2} \), and introduce the rescaling \(U_\sigma : (\Psi(x), A(x)) \mapsto (\sigma \Psi(\sigma x), \sigma A(\sigma x))\). This has the effect that the rescaled state, \((\psi, a) := U_\sigma(\Psi, A)\), is of type \((\tau, 1)\). It is easy to verify that \(U_\sigma\) is a linear unitary bijection between \(\mathcal{P}_b\) and \(\mathcal{P}_1\). In particular, it preserves the \(L^2\) inner product, i.e., \((U_\sigma v, U_\sigma v') = (v, v')\). Moreover it preserves the orthogonality condition and therefore for \(v \in \mathcal{P}_b\), \(v \perp G_y\) if and only if \(U_\sigma v \perp G_y^{\text{resc}}\).

We note that the rescaled Abrikosov lattice solution \((\psi_b, a_b) := U_\sigma(\Psi_b, A_b)\) satisfies the rescaled Ginzburg–Landau equations
\[ (-\Delta - \lambda_b)\psi + \kappa^2 |\psi|^2 \psi = 0, \]
\[ \text{curl}\text{curl} a - \text{Im}(\overline{\nabla} \nabla \psi) = 0, \]  
(62)
where \(\lambda_b = \frac{\kappa^2}{b}\), and the quasiperiodic boundary conditions
\[ \psi_b(x + v) = e^{\pm i Jy} \psi_b(x) \quad \text{and} \quad a_b(x + v) = a_b(x) + \frac{i}{2} Jv, \]  
(63)
where \(v\) is either of the basis vectors of \(L_1\), as well as \(\text{div} a_b = 0\) and \(\langle a_b - \frac{1}{2} Jx \rangle_{L_1} = 0\). Thus \((\psi_b, a_b)\) are of type \((\tau, 1)\).

We now define the rescaled Hessian to be \(L_b^{\text{resc}} := \sigma^2 U_\sigma \hat{L}_b U_\sigma^{-1}\). With \(v = \left(\begin{array}{c} \xi \\ \alpha \end{array}\right)\), it is explicitly given by
\[ L_b^{\text{resc}} v = \left( \begin{array}{l} -\Delta \xi - \lambda_b \xi + (2\kappa^2 + \frac{1}{2})|\psi_b|^2 \xi + (\kappa^2 - \frac{1}{2}) \psi_b^* \xi + 2i\alpha \cdot \nabla \psi_b \\
-\Delta \alpha + |\psi_b|^2 \alpha - 2\text{Im}(\overline{\xi} \nabla \psi_b) \end{array} \right). \]  
(64)
As with \(\hat{L}_b\), this operator has no gauge zero modes.

For the rest of this section we write \(\mathcal{L}\), \(\Omega\), \(\mathcal{P}\) for \(\mathcal{L}_b\), \(\Omega_1\), \(\mathcal{P}_1\).

3.3. Complexification

In order to freely use the spectral theory, it is convenient to pass from the real-linear operator \(L_b^{\text{resc}}\) to a complex-linear one. To this end we complexify the space \(\mathcal{P}\) and extend the operator \(L_b^{\text{resc}}\) to the new spaces. We first identify \(\alpha : \mathbb{R}^2 \to \mathbb{R}^2\) with the function \(\overline{\alpha} : \mathbb{R}^2 \to \mathbb{C}\). (Whenever it does not cause confusion we drop the \(\overline{\cdot}\) superscript from the notation.) We note that \(\alpha \cdot \alpha^* = \text{Re}(\overline{\alpha} \alpha^*)\). We also introduce the differential operator \(\tilde{\delta} = \delta_{\xi} - i \delta_{\alpha}t\). We note that \(\tilde{\delta} \overline{\alpha} = \text{div} \alpha - i \text{curl} \alpha\), where the \(\tilde{\cdot}\) denotes complex conjugate operator. In general, for an operator \(A\), we write \(\tilde{A} := \mathcal{C} A \mathcal{C}\), where \(\mathcal{C}\) denotes complex conjugation.

We now consider the complex Hilbert space \(L^2(\Omega, \mathbb{C}^4)\) of vectors \((\xi, \phi, \alpha, \omega)\), with the usual \(L^2\) inner product
\[ \langle v, v' \rangle = \langle \xi \xi' + \tilde{\phi} \phi' + \tilde{\alpha} \alpha' + \tilde{\omega} \omega' \rangle_{\mathbb{C}}. \]
The original space \(L^2(\Omega, \mathbb{C}) \times L^2(\Omega, \mathbb{R}^2)\), on which \(L_b^{\text{resc}}\) is defined, is embedded in \(L^2(\Omega, \mathbb{C}^4)\) via the injections
\[ \pi_b : (\xi, \alpha) \mapsto \frac{1}{\sqrt{2}} (\xi, \pm \tilde{\xi}, \alpha, \pm \tilde{\alpha}), \]  
(65)
with inverses, $\pi_{\pm}^{-1}$, given by the obvious projection. $V_{\pm} := \text{Ran} \pi_{\pm}$ are real spaces spanning $L^2(\Omega, \mathbb{C}^4)$:

$$(\xi, \phi, \alpha, \omega) = \frac{1}{2}(\xi + \bar{\phi}, \xi + \phi, \alpha + \bar{\omega}, \bar{\alpha} + \omega) + \frac{1}{2}(\xi - \bar{\phi}, -\bar{\xi} + \phi, \alpha - \bar{\omega}, -\bar{\alpha} + \omega).$$

(66)

Moreover, the map $I : (\xi, \phi, \alpha, \omega) \to (i\xi, i\phi, i\alpha, i\omega)$ acts between $V_{\pm} : V_{\pm} \to V_{\bar{\pi}}$. This embedding transfers the operator $L_{\text{res}}$ to the range, $V_{\pm} := \text{Ran} \pi_{\pm}$, of this injection. We denote the resulting operator by $L_{\text{trans}}$. Its domain is $\pi^P$. We want to extend it to $L^2(\Omega, \mathbb{C}^4)$. To this end it is convenient to rewrite the operator $L_{\text{trans}}^P$ in complex notation. We introduce the notation $\partial_\alpha^\pm = \partial - ia^\pm$. Straightforward calculations show that

$$2\alpha \cdot \nabla_\alpha \psi_\beta = -i(\partial_{a_\beta}^\pm \psi_\beta)\alpha^\pm + i(\partial_{a_\beta}^\pm \psi_\beta)\bar{\alpha}^\pm,$$

and that

$$-\text{Im}(\bar{\xi}(\partial_\alpha^\pm \psi_\beta)^\pm) = \frac{i}{2}(\partial_{a_\beta}^\pm \psi_\beta)\bar{\xi} + \frac{i}{2}(\partial_{a_\beta}^\pm \psi_\beta)\bar{\xi}.$$  

(67)

Using the above relations we rewrite the operator $L_{\text{trans}}^P$ and then define its complex-linear extension, denoted by $K_b$, by the resulting matrix

$$K_b = \begin{pmatrix}
-\Delta_0 - \lambda_b + (2\kappa^2 + \frac{1}{2})|\psi_\beta|^2 & (\kappa^2 - \frac{1}{2})\psi_\beta^2 & -i(\partial_{a_\beta}^\pm \psi_\beta) & i(\partial_{a_\beta}^\pm \psi_\beta) \\
(\kappa^2 - \frac{1}{2})\psi_\beta^2 & -\Delta_0 - \lambda_b + (2\kappa^2 + \frac{1}{2})|\psi_\beta|^2 & -i(\partial_{a_\beta}^\pm \psi_\beta) & i(\partial_{a_\beta}^\pm \psi_\beta) \\
i(\partial_{a_\beta}^\pm \psi_\beta) & i(\partial_{a_\beta}^\pm \psi_\beta) & -\Delta + |\psi_\beta|^2 & 0 \\
i(\partial_{a_\beta}^\pm \psi_\beta) & i(\partial_{a_\beta}^\pm \psi_\beta) & 0 & -\Delta + |\psi_\beta|^2
\end{pmatrix}$$

(68)

on the domain which consists of all $v = (\xi, \phi, \alpha, \omega) \in H^2(\Omega, \mathbb{C}^4)$, with $\xi, \bar{\phi}, \alpha, \bar{\omega}$ satisfying the quasiperiodic boundary conditions

$$\chi(x + v) = e^{ix \cdot j_0} \chi(x)$$ and $$\sigma(x + v) = \sigma(x),$$

(69)

where, as above, $v$ is either of the basis vectors of $L$. (Note that similarly to the Riesz–Fischer $L^2$-space on a torus (see e.g. [25]), we could have used results of [32] to introduce the $L^2$-space on $\Omega$ satisfying the quasiperiodic conditions (69), rather than periodic ones.) Moreover, we restrict $K_b$ to the subspace determined by the conditions $(\alpha)_{\mathbb{C}} = 0$ and $(\omega)_{\mathbb{C}} = 0$.

The operator $K_b$ is clearly complex-linear, self-adjoint, has purely discrete spectrum and, as it is not hard to check, satisfies

$$V_{\pm} \text{ are invariant under } K_b \text{ and } K_b|_{V_{\pm}} = L_{\text{trans}}^P.$$  

(70)

$$K_b = K_b|_{V_+} + K_b|_{V_-}, \quad \sigma(K_b|_{V_+}) = \sigma(K_b|_{V_-}) \quad \sigma(K_b) = \sigma(K_b|_{V_+}) \cup \sigma(K_b|_{V_-}).$$  

(71)

For the second equation, we used (66) and that $K_b$ obviously commutes with $I$. (70) implies

$$(v, L_{\text{res}}^P v) = (\pi_+ v, L_{\text{trans}}^P \pi_+ v) = (\pi_+ v, K_b \pi_+ v) = (\pi_+ v, K_{b+} \pi_+ v),$$

(72)

where $K_{b+} := K_b|_{V_+}$. Thus we want to find the ground state energy of $K_{b+}$.

### 3.4. Perturbation theory

It is shown in [32] (see equations (5.1)–(5.5), (6.3), (9.1) and (10.1), or [33], equations (5.1)–(5.6), (6.3), (9.3) and (9.9), and see appendix B) that for each $\tau$ there is $\epsilon_0 > 0$, such that the solutions $(\psi_\beta, a_\beta, \lambda_\beta)$ form a real-analytic branch of solutions in the (bifurcation) parameter $\epsilon \in [0, \epsilon_0]$ defined in (24) and have the following expansions

$$\psi_\beta = \epsilon \psi^0 + \epsilon^3 \psi^1 + O(\epsilon^2),$$

$$a_\beta = a^0 + \epsilon^2 a^1 + O(\epsilon^4),$$

$$\lambda_\beta = 1 + \epsilon^2 \lambda^1 + O(\epsilon^4),$$

(73)
where \( a^0 := \frac{1}{2} J x, \psi^0, a^1 \) and \( \lambda^1 \) satisfy the following relations:

\[
\partial^* a_0 \psi_0 = 0, \quad \langle |\psi_0|^2 \rangle_L = 2,
\]

\[
i \tilde{a} a^1 = \frac{1}{2} (\langle |\psi_0|^2 \rangle_L - |\psi_0|^2),
\]

\[
\Delta a^1 = \frac{i}{2} \bar{\psi}^0 (\partial_{x^0} \psi_0).
\]

(74)

(75)

(76)

\[
\lambda^1 = [1 + (2\kappa^2 - 1) \beta(\tau)].
\]

(77)

**Remark.** Of course, the third expansion in (73), i.e. \( \lambda_b = 1 + \epsilon^2 \lambda^1 + O(\epsilon^4) \), with \( \lambda^1 \) given by (77), follows from the definition of \( \lambda_b \) and \( \epsilon \). However, as shown in [32] (see also [33] and appendix B), (73) and the normalization \( \langle |\psi_0|^2 \rangle_L = 2 \) determine \( \epsilon \) as in (24).

We now prove the following proposition.

**Proposition 8.** For \( \epsilon > 0 \) sufficiently small, the lowest eigenvalue, \( \theta^0 \), of \( K_b := K_b|_{V_b^+} \equiv L_b^\text{transf} \) is of the form

\[
\theta^0 = 2 \left[ (2\kappa^2 - 1) \beta(\tau) + 1 \right] \epsilon^2 + O(\epsilon^3).
\]

(78)

Moreover, the corresponding eigenfunction satisfies the orthogonality condition (20), at least to the order \( O(\epsilon^4) \).

**Proof.** We use the expansions above to expand \( K_b \) in powers of \( \epsilon \) and the relation \( \partial^* a_0 \psi_0 = 0, \) to simplify the resulting terms to obtain

\[
K_b = K_0^0 + \epsilon W_1 + \epsilon^2 W_2 + o(\epsilon^3),
\]

(79)

where

\[
K_0^0 = \begin{pmatrix}
-\Delta_{x^0} - 1 & 0 & 0 & 0 \\
0 & -\Delta_{x^0} - 1 & 0 & 0 \\
0 & 0 & -\Delta & 0 \\
0 & 0 & 0 & -\Delta
\end{pmatrix},
\]

(80)

\[
W_1 = \begin{pmatrix}
0 & 0 & 0 & 0 \\
0 & 0 & -i(\partial_{x^0} \psi_0) & 0 \\
0 & i(\partial_{x^0} \psi_0) & 0 & 0 \\
-\bar{\psi}_0 (\partial_{x^0} \psi_0) & 0 & 0 & 0
\end{pmatrix},
\]

\[
W_2 = \begin{pmatrix}
B^0 & (\kappa^2 - \frac{1}{2}) |\psi_0|^2 & 0 & 0 \\
(\kappa^2 - \frac{1}{2}) |\psi_0|^2 & B^0 & 0 & 0 \\
0 & 0 & |\psi_0|^2 & 0 \\
0 & 0 & 0 & |\psi_0|^2
\end{pmatrix},
\]

(81)

where

\[
B^0 = -\lambda^1 + (2\kappa^2 + \frac{1}{2}) |\psi_0|^2 - i a^1 \partial_{x^0}^* + i \bar{a} a^1 \partial_{x^0}.
\]

The unperturbed operator \( K_0^0 \) reduces to the operators studied previously and whose spectra were described in detail (see e.g. in [32], where these operators are denoted by \( L_n^p \) with \( n = 1 \) and \( M \)). In particular it is shown there that, in general, \( K_0^0 \) has the eigenvectors \( (\psi_0, \pm \bar{\psi}_0, 0, 0) \) and \( (0, 0, 1, \pm 1) \). The latter two are ruled out by the condition that \( \langle \alpha \rangle_L = 0 \) and \( \langle \omega \rangle_L = 0 \). We summarize the properties of \( K_0^0 \) in the following lemma.

**Lemma 9.** \( K_0^0 \) is a nonnegative self-adjoint operator with discrete spectrum. It has a zero eigenvalue of multiplicity 2 and the kernel is spanned by the elements

\[
v_{0\pm} = (\psi_0, \pm \bar{\psi}_0, 0, 0).
\]

(82)

(83)
Note that \( v_{0b} \in V_+ \). Hence the operator \( K_+^0 := K^0|_{V_+} \) which is the zero-order approximation of the operator \( K_+ := K_b|_{V_+} \) that we are interested in, has the simple lowest eigenvalue 0 with the eigenfunction \( v_{0b} \).

By standard perturbation theory (see e.g. \([17, 19]\)), the spectrum of \( K_b \) consists of eigenvalues which cluster in \( \epsilon \)-neighbourhoods of the eigenvalues of \( K^0 \) and each cluster has the same total multiplicity as the eigenvalue of \( K^0 \) it originates from. Of course, the same is true for its restriction \( K_+ := K_b|_{V_+} \). Namely, the spectrum of \( K_+ \) consists of eigenvalues which cluster in \( \epsilon \)-neighbourhoods of the eigenvalues of \( K_+^0 := K^0|_{V_+} \) and each cluster has the same total multiplicity as the corresponding eigenvalue of \( K_+^0 \). Thus \( K_+ \) has the simple eigenvector \( v_{0b} \), which is a perturbation of the simple eigenvector \( v_{0b} \) of \( K_+^0 \) with the smallest eigenvalue 0. It suffices to determine the corresponding eigenvalue of \( K_+ \), which we denote \( \theta \).

To find \( \theta \) we use the Feshbach–Schur map argument (see e.g. \([7, 17]\)). This argument says that given a projection \( P \),

\[
\lambda \in \sigma(K_+) \quad \text{if and only if} \quad \lambda \in \sigma(F_P(\lambda)),
\]

where, with \( P = 1 - P \),

\[
F_P(\lambda) := \left[ PK_+P - PK_+\hat{P}(\hat{P}K_+\hat{P} - \lambda)^{-1}\hat{P}K_+P\right]_{\text{Ran}P},
\]

provided the operator \( \hat{P}K_+\hat{P} - \lambda \) is invertible on \( \text{Ran}\hat{P} \) and the operators \( \hat{P}K_+P \) and \( PK_+\hat{P} \) are bounded. (The latter conditions suffice for the right-hand side of (85) to be well defined. The proof of the above statement is elementary and is given in appendix C.) We use this argument with the projection \( P \) given by the orthogonal projection onto \( v_{0b} \in \text{Null}K_+^0 \). Due to the relation \( \hat{P}K_+P = \hat{P}(K_+ - K_+^0)P \) and the straightforward estimate

\[
\|K_+ - K_+^0\| \lesssim \epsilon,
\]

we see that the operator \( \hat{P}K_+P \) (and therefore also its adjoint \( PK_+\hat{P} \)) is bounded. To show the invertibility of the operator \( \hat{P}K_+\hat{P} - \lambda \) on \( \text{Ran}\hat{P} \), we note that it is the restriction of the operator \( \hat{Q}K_+\hat{Q} - \lambda \), where \( \hat{Q} \) is the orthogonal projection onto \( \text{Null}K^0 \) and \( \hat{Q} := I - Q \), to the subspace \( V_+ \). We know that \( \sigma(QK^0\hat{Q}) \subset [v_0, \infty) \) for some \( v_0 > 0 \) and therefore, by standard perturbation theory we have that

\[
\sigma(QK^0\hat{Q})_{\text{Ran}\hat{Q}} \subset [c, \infty),
\]

with \( c = v_0 + O(\epsilon) \). Hence the self-adjoint operator \( \hat{Q}K_+\hat{Q} - \lambda \) is invertible on \( \text{Ran}\hat{Q} \), provided \( \lambda < c \), and therefore its restriction \( \hat{P}K_+\hat{P} - \lambda \) (to real-linear subspace \( V_+ \)) is invertible on \( \text{Ran}\hat{P} \) and \( \|\hat{P}K_+\hat{P} - \lambda\|^{-1} \leq \epsilon^{-1} \) (again provided \( \lambda < c \)). Hence (85) is well defined for \( \lambda < c \).

We now use \( K_+ = K_+^0 + \epsilon W^1 + \epsilon^2 W^2 + o(\epsilon^3) \), the relation \( K_+^0 P = PK_+^0 = 0 \) and the facts \( \|PK_+\hat{P}\| = O(\epsilon) \) (by (86)) and \( \|\hat{P}K_+\hat{P} - \lambda\|^{-1} \lesssim 1 \), provided \( \lambda < c \) (by (87)). Since we are studying the eigenvalue in \( O(\epsilon) \)-neighbourhood of 0, we have that \( \lambda = O(\epsilon) \). Using this, we obtain

\[
F(\lambda) = \epsilon F_1 + \epsilon^2 F_2 + O(\epsilon^3),
\]

where

\[
F_1 := \langle v_{0b}, W^1 v_{0b} \rangle/\langle |v_{0b}|^2 \rangle \xi, \quad F_2 := \langle v_{0b}, [W^2 - W^1 \hat{P}(\hat{P}K_+^0\hat{P})^{-1}\hat{P}W^1]v_{0b} \rangle/\langle |v_{0b}|^2 \rangle \xi.
\]

The operator, \( W^1 \) is explicitly given by (80). This expression implies that \( W^1 \) flips the blocks. Since \( v_{0b} \) belongs to the first block, we have

\[
F_1 := \langle v_{0b}, W^1 v_{0b} \rangle = 0.
\]

We now turn to the \( \epsilon^2 \) order operator, \( F_2 \).
Lemma 10.
\[ \mathcal{F}_2 = 2[(2\kappa^2 - 1)\beta(x) + 1]. \]

**Proof.** We begin with \( \{v_{0+}, W^2 v_{0+}\}. \) We first note that \( W^2 \) and \( v_{0+} \) are explicitly given by (81) and (83). Using that \( W^2 \) preserves the blocks, while \( v_{0+} \) belongs to the first block and using the fact that \( \partial^* \psi^0 = 0 \), we calculate that
\[
(v_{0+}, W^2 v_{0+}) = 2(-\lambda \langle |\psi^0|^2\rangle + (2\kappa^2 + \frac{1}{2}) \langle |\psi^0|^4\rangle + \Re(\bar{\psi}^0 \partial^1 \partial^1 \psi^0) + (\kappa^2 - \frac{1}{2}) \langle |\psi^0|^4\rangle\).
\]
where \( \lambda \) is given in (77). We note that \( -\Delta = \bar{\partial}^* \bar{\partial} \). Using the identities (76) and (75), we obtain
\[
2\Re(\bar{a}^1 \bar{\psi}^0 \partial^0 \psi^0) = 4\Re(\bar{a}^1 \Delta \bar{a}^1) \psi^0 = -4\Re(\bar{a}^1 \bar{a}^1 \psi^0) = -\Re(\bar{a}^1 \bar{a}^1 \psi^0) + \Re(\bar{a}^1 \bar{a}^1 \psi^0).
\]
The equations (92) and (93) and relation (77) give
\[
\langle v_{0+}, W^2 v_{0+} \rangle = 4\kappa^2 \langle |\psi^0|^4\rangle.
\]
To compute the second term in \( \mathcal{F}_2 \) we note that \( \bar{P} W^1 P = W^1 P \), and use that \( W^1 \) flips the blocks, \( \bar{K}^0 \) preserves the blocks and \( v_{0+} \) belongs to the first block, we calculate
\[
\langle v_{0+}, W^1 \bar{P} (\bar{P} K^0 \bar{P})^{-1} \bar{P} W^1 v_{0+} \rangle = 2\Re(\bar{a}^0 \partial^0 \psi^0) = 8\Re(\bar{a}^0 \Delta \bar{a}^0) = 8\Re(\bar{a}^0 \bar{a}^0) = 8\Re(\bar{a}^0 \bar{a}^0) - 2\Re(\bar{a}^0 \bar{a}^0).
\]
Now, definition (89) and equations (94)–(96), the observation that definition (11) for the rescaled functions reads
\[
\beta(x) = \frac{\Re(\bar{a}^0 \bar{a}^0)}{\Re(\bar{a}^0 \bar{a}^0)}
\]
and the fact that \( \langle |v_{0+}|^2\rangle = 2\langle |\psi^0|^2\rangle \) and the normalization \( \langle |\psi^0|^2\rangle \) give (91). \( \square \)

Equations (84), (88), (90) and (91) imply the first part of proposition 8. To prove the second statement, we note that by theorem 18 (see the line preceding (114)), the eigenfunction corresponding to \( \theta^4 \) is given by \( Q v_{0+} \), where \( Q \) is defined in (114). It is a simple computation to check that it satisfies the orthogonality condition (20) to the order \( O(\epsilon^4) \). \( \square \)

Now we are ready for:

**Proof of theorem 4.** We restore the subindex \( L^2 \) in the inner products and norms. Proposition 8 and (72) imply that \( \theta^4 \) is also the lowest eigenvalue of \( L^\text{resc}_b \) on \( \mathcal{P}^\perp_b \) and therefore, by the formula \( L^\text{resc}_b := \sigma^2 U_\sigma L_b U_\sigma^{-1}, \sigma := \sqrt{\frac{1}{b}}, \) relating \( L^\text{resc}_b \) to \( L_b \), we see that \( \theta := \theta^4 \) is the smallest eigenvalue of \( L_b \). By lemma 7, this gives that \( \theta \) is the lowest eigenvalue of \( L_b \) on the subspace \( \mathcal{P}^\perp_b \) and therefore, for all \( v \in D(L_b) \cap \mathcal{P}^\perp_b \),
\[
\langle v, L_b v \rangle_L^2 \geq \frac{1}{2} \theta \|v\|^2_L.
\]
We upgrade now the lower bound on \( \langle v, L_b v \rangle_L^2 \) to that on \( \langle v, L_b v \rangle_H \). (We could have done this with the operator \( K_b \) as well.) We begin with
Lemma 11. For all \( v \in \calP_b \cap \calP_{b+} \), we have
\[
\frac{1}{2} \|v\|_{H^1}^2 - C\|v\|_{L^2}^2 \leq \langle v, L_b v \rangle_{L^2} \lesssim \|v\|_{H^1}^2,
\]  
for some positive constant \( C \).

Proof. We write \( v = (\xi, \alpha) \) and also \( A_b = A_0^b + P \). For convenience we simplify the notation. Integrating by parts we obtain
\[
\langle v, L_b v \rangle_{L^2} = \frac{1}{|\Omega|} \text{Re} \int_{\Omega} -\tilde{\xi} \Delta \xi^\alpha + 2iP \cdot \tilde{\xi} \nabla \xi^\alpha + |\xi|^2|\xi|^2 - \kappa^2|\xi|^2 + (2\kappa^2 + \frac{1}{2})|\Phi|^2|\xi|^2
\]
\[
+ \frac{\kappa}{2} \left( \xi^\alpha + 2i \alpha \cdot (\xi \nabla \xi^\alpha) \right) + 2(\alpha \cdot P) \Psi_b \tilde{\xi} + \frac{\kappa}{2} \Delta \alpha + |\Phi|^2|\alpha|^2
\]
\[
- 2\alpha \cdot \text{Im}(\xi \nabla \xi^\alpha) - (\alpha \cdot P) \text{Re}(\xi \nabla \xi^\alpha)
\]
\[
= \frac{1}{|\Omega|} \int_{\Omega} |\nabla \xi^\alpha|^2 + |\nabla \alpha|^2 + |\Phi|^2|\xi|^2 - \kappa^2|\xi|^2 + (2\kappa^2 + \frac{1}{2})|\Phi|^2|\xi|^2
\]
\[
+ \kappa \left( \xi^\alpha + 2i \alpha \cdot (\xi \nabla \xi^\alpha) \right) - 2P \cdot \text{Im}(\xi \nabla \xi^\alpha) - 4\alpha \cdot \text{Im}(\xi \nabla \xi^\alpha) \Psi_b.
\]

Using this expression and the estimate obtained with the help of the Schwarz inequality
\[
\left| \int_{\Omega} P \cdot \text{Im}(\xi \nabla \xi^\alpha) \right| \lesssim \|P\|_{L^\infty} \|\xi\|_{L^1} \|\nabla \xi^\alpha\|_{L^2} \lesssim \frac{1}{2} \|P\|_{L^\infty} \left( \frac{1}{r} \|\xi\|_{L^2}^2 + r \|\nabla \xi^\alpha\|_{L^2}^2 \right),
\]
for any \( r > 0 \), we obtain
\[
\langle v, L_b v \rangle_{L^2} \geq \|\nabla \xi^\alpha\|_{L^2}^2 + \|\nabla \alpha\|_{L^2}^2 - \kappa^2\|\xi\|_{L^2}^2
\]
\[
- C r \|\xi\|_{L^2}^2 - r C \|\nabla \xi^\alpha\|_{L^2}^2 - C \|\alpha\|_{L^2}^2 - C \|\xi\|_{L^2}^2.
\]

Now we choose \( r = \frac{1}{\kappa} \) so that we arrive at the lower bound in \( (99) \). To obtain the upper bound we use \( (100) \) and \( (101) \) again and the fact that \( \|\Phi\|_{L^\infty}, \|P\|_{L^\infty} \ll \infty. \)

Let now \( \delta \in [0, 1] \) be arbitrary. Then using \( (98) \) and \( (99) \)
\[
\langle v, L_b v \rangle_{L^2} = (1 - \delta) \langle v, L_b v \rangle_{L^2} + \delta \langle v, L_b v \rangle_{L^2}
\]
\[
\geq (1 - \delta) \theta \|v\|_{L^2}^2 + \delta \left( \frac{1}{2} \|v\|_{H^1}^2 - C \|v\|_{L^2}^2 \right)
\]
\[
= ((1 - \delta) \theta - \delta C) \|v\|_{L^2}^2 + \frac{\delta}{2} \|v\|_{H^1}^2.
\]

\( (28) \) now follows by choosing \( \delta = \frac{\theta}{2 + \theta C}. \) Next, estimate \( (29) \) follows from the upper bound in \( (99) \).

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Appendix A. Proof of orbital stability

We now assume that we have the case \( \kappa^2 > \frac{1}{2} (1 - \frac{1}{|\Omega|^2}) \) and prove orbital stability of the Abrikosov lattice \( A^\alpha_b \). This is a weaker statement than asymptotic stability, which we have already proven but which requires rougher analysis. We follow [15].

For orbital stability, it is more convenient to work in a gauge where \( \text{div} A = 0 \) rather than \( \Phi = 0 \). In this case, one can show that the only remaining continuous symmetry is the global
gauge symmetry \((\Psi, A, \Phi) \mapsto (e^{i\alpha} \Psi, A, \Phi)\) for \(\alpha \in \mathbb{R}\). Thus the manifold of stationary solutions is a one-dimensional manifold and the tangent space at \(u_b^\pm\) is spanned by the single vector
\[
\Gamma_b = (i\Psi_b, 0).
\]
It is straightforward to adapt proposition 3 to this case and prove an analogous decomposition for sufficiently small tubular neighbourhoods.

**Theorem 12.** For all \(b\) sufficiently close to \(\kappa^2\), the Abrikosov lattice \(u_b\) is orbitally stable under gauge-periodic perturbations if \(\kappa^2 > \frac{1}{2}(1 - \frac{1}{p(\kappa)}\).

**Proof.** As in the main text, we consider \(\tau\) fixed and do not display it in the notation. We will require a series of lemmas.

**Lemma 13.** There exists positive constants \(c\) and \(C\), such that for all \(v \in \mathcal{P}_b\), if \(v \perp \Gamma_b\), then for any \(\theta' < \theta\),
\[
\theta'\|v\|_{H_1}^2 - c\|v\|_{H_1}^3 - c\|v\|_{H_1}^4 \leq \mathcal{E}(u_b + v) - \mathcal{E}(u_b) \leq C(\|v\|_{H_1}^2 + \|v\|_{H_1}^3 + \|v\|_{H_1}^4). \tag{102}
\]

**Proof.** Using the Taylor expansion of \(\mathcal{E}\), together with the fact that \(\mathcal{E}'(u_b) = 0\) and \(v \perp \Gamma_b\), we have
\[
\mathcal{E}(u_b + v) - \mathcal{E}(u_b) = \frac{1}{2}(v, L_b v) + R(v),
\]
where the remainder \(R(v)\) is given by, setting \(v = (\xi, \alpha),
\[
R(v) = \int_{\mathcal{O}_b} |\alpha|^2(\text{Re}(\bar{\Psi}_b \xi) + \frac{\kappa}{2} |\xi|^2) - \alpha \cdot \text{Im}(\bar{\Psi}_b \xi) + \kappa^2 |\xi|^2(\text{Re}(\bar{\Psi}_b \xi) + \frac{1}{2}|\xi|^2).
\]
Using the Cauchy–Schwarz and Sobolev inequalities it is straightforward to show \(|R(v)| \leq c(\|v\|_{H_1}^2 + \|v\|_{H_1}^3).\) Using (28), this gives
\[
\mathcal{E}(u_b + v) - \mathcal{E}(u_b) \geq \theta\|v\|_{H_1}^2 - c\|v\|_{H_1}^3 - c\|v\|_{H_1}^4.
\]
On the other hand, by definition (23) of \(L_b\) and the boundedness of \(u_b^\pm\) together with its derivatives, we have \((v, L_b v) \lesssim \|v\|_{H_1}^3.\) This estimate together with the above estimate of \(|R(v)|\) gives the upper bound in (102) and this completes the proof. \(\square\)

**Lemma 14.** Suppose that \((\Psi, A, \Phi)\) is a solution of the Gorkov–Eliashberg–Schmid equations (2) on the time interval \([0, T]\) satisfying (I)–(III) and \(u(t) = (\Psi(t), A(t)) \in C^4([0, T]; U_b)\) for any \(\delta > 0\) and \(T > 0.\) Then the energy function \(\mathcal{E}(u)\) is a nonincreasing function in time.

**Proof.** Note that we have \(\partial_t u \in \mathcal{P}_b.\) Using the gradient-flow form of the equations (5), we see that
\[
\partial_t \mathcal{E}(u) = \langle \mathcal{E}'(u), \partial_t u \rangle_{L^2} = -\langle \mathcal{E}'(u), i(\Phi \Psi, \nabla \Phi) \rangle_{L^2} - \|\mathcal{E}'(u)\|_{L^2}^2
\]
\[
\leq \int_{\mathcal{O}_b} \text{Im}(\bar{\Phi} \nabla \Psi) \cdot \nabla \Phi - \text{Re}(\text{Im}(\bar{\Phi} \nabla \Psi)^2) - \text{curl}^* \text{curl} A \cdot \nabla \Phi + \text{Im}(\bar{\Psi} \nabla \Psi) \cdot \nabla \Phi \leq \int_{\mathcal{O}_b} -\text{Im}(\bar{\Psi} \nabla \Psi) \cdot \nabla \Phi - \text{Im}(\Phi \nabla \Psi)^2 + \text{Im}(\bar{\Psi} \nabla \Psi) \cdot \nabla \Phi = 0,
\]
where we used the fact that \(\Phi\) is real-valued and \(\text{div} \text{curl}^* = 0.\) \(\square\)
Lemma 15. Let \( (\Psi(t), A(t), \Phi(t)) \) be a solution of the Gorkov–Eliashberg–Schmid equations (2) on the time interval \([0, T]\) and denote \( u(t) = (\Psi(t), A(t)) \). Given \( \epsilon > 0 \) sufficiently small, there exists \( \delta > 0 \) such that if \( T > 0 \) satisfies \( T > \gamma(t) \) (with the first and second derivatives of \( \gamma(t) \)), \( \gamma(t) \in \mathbb{R} \), \( v \in C^1([0, T]; \mathbb{P}_b) \), \( v(t) \perp \Gamma_b \), and \( \| v(t) \| \leq \delta \), then \( \| v(t) \|_H^1 \leq \epsilon \) for all \( t \in [0, T] \).

Proof. We set \( \gamma(t) = \| v(t) \|_H^1 \). Using the inequalities (102) and lemma 14 we have

\[
C_1 N(\tau)^2 - C_2 N(\tau)^3 - C_3 N(\tau)^4 \leq E(u_b + v(t)) - E(u_b)
= E(u(t)) - E(u_b) \leq \| e(t) \|_H^1 \leq \epsilon \|

\]

Now there exists \( \delta_0 > 0 \) such that if the left-hand side \( C_1 N(\tau)^2 - C_2 N(\tau)^3 - C_3 N(\tau)^4 \leq \delta_0 \), then either \( 0 \leq N(\tau) \leq \epsilon \) or \( N(t) \geq \epsilon' \) for some \( \epsilon' > \epsilon \). We can choose \( \delta \) sufficiently small so that the right-hand side \( C_4 N(\tau)^2 + C_5 N(\tau)^3 + C_6 N(\tau)^4 \leq \delta_0 \). The result then follows from the continuity of \( N(t) \).

We can now prove the following proposition, which implies theorem 12.

Proposition 16. For any \( \epsilon > 0 \), there exists \( \delta > 0 \) such that if \( (\Psi, A, \Phi) \) is a \( C^1 \) solution of the Gorkov–Eliashberg–Schmid equations (2) and \( u(t) = (\Psi(t), A(t)) \) satisfies \( u(0) \in U_{\delta_0} \), then \( u(t) \in U_{\delta} \) for all \( t \geq 0 \).

Proof. Let \( \delta_0 \) be given by proposition 3. If \( u(0) \in U_{\frac{1}{4}\delta_0} \), there is \( T > 0 \) such that \( u(t) \in U_{\delta_0} \) for \( t \leq T \). Then by proposition 3, there is \( \gamma(t) \in \mathbb{R} \) so that \( T > \gamma(t) \) (with the first and second derivatives of \( \gamma(t) \), of Abrikosov lattice solutions).

Appendix B. Asymptotics of Abrikosov lattice solutions

In this appendix we derive the properties (74)–(77) of Abrikosov lattice solutions.

Proposition 17. Assume the equations (62) have a family, \( (\psi_b, a_b, \lambda_b) \), \( b \rightarrow k^2 \), of Abrikosov lattice solutions, with \( \langle \text{curl} \ a_b \rangle = 1 \), satisfying (73) (with the first and second derivatives of the remainders obeying similar estimates). Then \( \psi_0, a^1 \) and \( \lambda_1 \) satisfy equations (74)–(77).

Proof. Plugging (73) into (62) and taking \( \epsilon \rightarrow 0 \) gives \( (-\Delta_{\psi^0} - 1)\psi^0 = 0 \) and

\[
\text{curl}^\ast\text{curl}^\ast a^1 = \text{Im}(\psi_0^\ast \nabla^\ast \psi_0^0).
\]  

(Recall, that for a scalar function, \( f(x) \in \mathbb{R} \), \( \text{curl}^\ast f = (\partial_2 f, -\partial_1 f) \) is a vector.) The condition \( \text{div} \ a_b = 0 \) implies that \( \text{div} \ a^1 = 0 \), which together with (67), gives (76).

The operators

\[
\partial_{x_i} := \partial_i + \frac{1}{2} x_1 + \frac{i}{2} x_2 \quad \text{and} \quad \partial_{x_i}^\ast := -\partial_i + \frac{1}{2} x_1 - \frac{i}{2} x_2
\]

satisfy the following relations:

1. \([\partial_{x_i}, \partial_{x_j}] = 2\text{curl} a^0 = 2;\)
2. \(\Delta_{\partial_{x_i}}^\ast - 1 = \partial_{x_i}^\ast \partial_{x_i}^\ast.\)
(These operators are the harmonic oscillator annihilation and creation operators.) As for the harmonic oscillator, the above properties imply

\[ \text{Null}(-\Delta_0 - 1) = \text{Null} \delta^*_0. \tag{105} \]

It follows from (105) that \( \psi^0 \) satisfies \( \delta^*_0 \psi^0 = 0 \), which is the first equation in (74).

Now, multiplying the relation \( \delta^*_0 \psi^0 = 0 \) by \( \bar{\psi}_0 \) and taking imaginary and real parts of the result and rearranging terms gives

\[ \text{Im}(\overline{\psi^0} \nabla_a \psi^0) = -\frac{1}{2} \text{curl}^* |\psi^0|^2. \tag{106} \]

The equations (103) and (106) give \( \text{curl} a^1 = H - \frac{1}{2} |\psi^0|^2 \), with \( H \) a constant of integration. \( H \) has to be chosen so that \( \int_\Omega \text{curl} a_1 = 0 \), which yields \( H = \frac{1}{2} (|\psi^0|^2) \). This and the relation \( \text{div} a^1 = 0 \), which implies that \( \text{curl}(a^1)_C = (\text{curl} a^1)_C \), give equation (75).

Now we prove (77). We multiply equation (62) scalarly (in \( L^2(\Omega) \)) by \( \psi^0 \), use that the operator \(-\Delta_0 \) is self-adjoint and \((-\Delta_0 - n)\psi^0 = 0 \), substitute the expansions (73) and take \( \epsilon = 0 \), to obtain

\[ -\lambda_1 \int_\Omega |\psi_0|^2 + 2i \int_\Omega \bar{\psi}_0 a_1 \cdot \nabla a_1 \psi^0 + \kappa^2 \int_\Omega |\psi^0|^4 = 0. \tag{107} \]

This expression implies that the imaginary part of the second term on the left-hand side of (107) is zero. (We arrive at the same conclusion by integrating by parts and using that \( \text{div} a_1 = 0 \).) Therefore

\[ 2i \int_\Omega \bar{\psi}_0 a_1 \cdot \nabla a \psi^0 = -2 \int_\Omega a_1 \cdot \text{Im}(\bar{\psi}_0 \nabla a \psi^0) = -2 \int_\Omega a_1 \cdot \text{curl}^* \text{curl} a^1. \]

Integrating the last term by parts, we obtain

\[ 2i \int_\Omega \bar{\psi}_0 a_1 \cdot \nabla a \psi^0 = - \frac{1}{2} \int_\Omega |\psi_0|^4 + \frac{1}{2} (|\psi^0|^2) \int_\Omega |\psi_0|^2. \tag{108} \]

This equation together with (107) and the definition (97) gives (77).

This proposition implies that, under the assumption (73), the parameter \( \epsilon \) is fixed uniquely up to the normalization of \( \psi_0 \). Indeed, we observe that the third equation in (73) implies

\[ \epsilon^2 = \frac{\lambda_1}{\kappa^2} + O((\lambda - 1)^2), \]

which, together with the definition \( \lambda = \frac{\kappa^2}{\beta} \) and (77), yields

\[ \epsilon^2 = \frac{n(k^2 - b)}{\kappa^2[(k^2 - \frac{1}{2})\beta(\tau) + \frac{1}{2}(|\psi^0|^2)]} + O((k^2 - b)^2). \tag{109} \]

This equation implies the following necessary condition on existence of the solutions:

\[ b \leq k^2 \text{ if } (k^2 - \frac{1}{2})\beta(\tau) + \frac{1}{2} \geq 0 \quad \text{ and } \quad b > k^2 \text{ if } (k^2 - \frac{1}{2})\beta(\tau) + \frac{1}{2} < 0. \tag{110} \]

Appendix C. Proof of theorem 18

In this appendix we present for the reader’s convenience the main result of the Feshbach–Schur perturbation theory. Let \( P \) and \( \overline{P} \) be orthogonal projections on a separable Hilbert space \( X \), satisfying \( P + \overline{P} = 1 \). Let \( H \) be a self-adjoint operator on \( X \). We assume that \( \text{Ran} P \subset D(H) \), that \( H_{\overline{P}} := \overline{P} H \overline{P} \mid_{\text{Ran} \overline{P}} \) is invertible, and

\[ \| R_P \| < \infty, \quad \| P H R_P \| < \infty \quad \text{and} \quad \| R_P H P \| < \infty, \tag{111} \]

where \( R_P = \overline{P} H_{\overline{P}}^{-1} \overline{P} \). We define the operator

\[ F_P(H) := P(H - H R_P H)P \mid_{\text{Ran} P}. \tag{112} \]
The key result for us is the following:

**Theorem 18.** Assume (111) hold. Then

\[ Hψ = 0 ⇔ FP (H)φ = 0, \]  

where ψ and φ are related by \( φ = Pψ \) and \( ψ = Qφ \), with the (bounded) operator \( Q \) given by

\[ Q = Q(H) := P - R_{H}H P. \]  

**Proof.** First, in addition to (114), we define the operator

\[ Q^h = Q^h(H) := P - PHR_{H}. \]  

The operators \( P, Q \) and \( Q^h \) satisfy

\[ HQ = HP - H\bar{P}H^{-1}\bar{P}HP, \]

\[ Q^hH = H'P, \]

where \( H' = F_{P}(H) \). Indeed, using the definition of \( Q \), we transform

\[ HQ = HP - H\bar{P}H^{-1}\bar{P}HP = PHP - PPH\bar{P}H^{-1}\bar{P}HP = PHP - PPH^{-1}\bar{P}HP \]

\[ = PHP - PPHP = F_{P}(H). \]  

Next, we have

\[ Q^hH = PH - PH\bar{P}H^{-1}\bar{P}HP \]

\[ = PHP - PH\bar{P}H^{-1}\bar{P}HP = PHP - PPH\bar{P}H^{-1}\bar{P}HP \]

\[ = F_{P}(H). \]  

This completes the proof of (116).

Now, we show

\[ \text{Null } Q \cap \text{Null } H' = \{0\} \quad \text{and} \quad \text{Null } P \cap \text{Null } H = \{0\}. \]  

The first relation in (118) follows from the fact that the projections \( P \) and \( \bar{P} \) are orthogonal, which implies the inequality

\[ \|Qu\|^2 = \|Pu\|^2 + \|R_{H}Pu\|^2 \geq \|Pu\|^2, \]

and the relation \( \text{Null } P \subset \text{Null } H' \), which follows from the definition of \( H' \). To prove the second relation in (118) we use the equation \( P + \bar{P} = 1 \) and the definitions \( H_{P} = \bar{P}HP \bar{P} \) and (114) to obtain

\[ 1 = QP + R_{H}H, \]

which, in turn, implies the second relation in (118). Indeed, applying (119) to a vector \( φ \in \text{Null } P \cap \text{Null } H \), we obtain \( φ = QPφ + R_{H}Hφ = 0. \)

Now the statement (113) follows from relations (116) and (118). \( \square \)

**Remark.** Similarly, it is easy to show that

\[ 0 \in \sigma(H) ⇔ 0 \in \sigma(F_{P}(H)). \]  

\[ (120) \]
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