The density theorem for projective representations via twisted group von Neumann algebras

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1. Introduction

Let \((\pi, \mathcal{H}_\pi)\) be a unitary representation of a locally compact group \(G\) and let \(\Gamma\) be a discrete subset of \(G\). The study of spanning and linear independence properties of \(\Gamma\)-indexed families of the form

\[
\pi(\Gamma)\eta = (\pi(\gamma)\eta)_{\gamma \in \Gamma}
\]

for \(\eta \in \mathcal{H}_\pi\) is fundamental in many areas of applied harmonic analysis, including time-frequency analysis and wavelet theory [13,17,21,23,25]. Under certain assumptions on \(G\) and \(\pi\), fundamental results known as density theorems provide basic obstructions to the spanning and linear independence properties of such families depending only on notions of density of \(\Gamma\) in \(G\) [18,33].

While there exist many different spanning properties for families in Hilbert spaces, we will focus on frames in the present paper. Unlike e.g. Schauder bases, frames provide unconditionally convergent expansions of...
every element in the Hilbert space, making them ideal for applied harmonic analysis. The dual notion is that of a Riesz sequence, which is a strong notion of linear independence (see Section 5.1 for definitions).

When $\Gamma$ is a lattice in $G$, i.e. a discrete subgroup with finite covolume $\text{vol}(G/\Gamma)$, the density of $\Gamma$ is appropriately measured by $1/\text{vol}(G/\Gamma)$. In [40], Romero and van Velthoven prove the following general density theorem for frames and Riesz sequences of the form (1):

**Theorem 1.1.** Let $(\pi, \mathcal{H}_\pi)$ be an irreducible, square-integrable, unitary representation of a second-countable, unimodular, locally compact group $G$ with formal dimension $d_\pi$, and let $\Gamma$ be a lattice in $G$. Then the following hold for $\eta \in \mathcal{H}_\pi$:

1. If $\pi(\Gamma)\eta$ is a frame for $\mathcal{H}_\pi$, then $d_\pi \text{vol}(G/\Gamma) \leq 1$.
2. If $\pi(\Gamma)\eta$ is a Riesz sequence for $\mathcal{H}_\pi$, then $d_\pi \text{vol}(G/\Gamma) \geq 1$.

In fact, Romero and van Velthoven prove Theorem 1.1 more generally for projective unitary representations [40, Theorem 7.4], that is, maps $\pi$ from $G$ into the unitary operators on $\mathcal{H}_\pi$ that satisfy a composition rule of the form

$$\pi(x)\pi(y) = \sigma(x,y)\pi(xy) \quad \text{for all } x, y \in G.$$ 

Here, $\sigma$ is an associated measurable function $G \times G \to \mathbb{T}$ called a 2-cocycle, and ordinary representations correspond to $\sigma = 1$. We will work with projective representations since one of our motivating examples is projective, namely the Weyl–Heisenberg representation that forms the basis of Gabor analysis. The density theorem has a long history in Gabor analysis, see [4,12,27,29,38]. We delay the definition of the Weyl–Heisenberg representation to Section 6.2.

In the present paper we consider converses to the lattice density theorem for projective representations, that is, the following problem:

**Problem 1.2.** Let $(\pi, \mathcal{H}_\pi)$ be a projective, irreducible, square-integrable, unitary representation of a second-countable, unimodular, locally compact group $G$ with formal dimension $d_\pi > 0$, and let $\Gamma$ be a lattice in $G$.

1. Does $d_\pi \text{vol}(G/\Gamma) \leq 1$ imply the existence of $\eta \in \mathcal{H}_\pi$ such that $\pi(\Gamma)\eta$ is a frame for $\mathcal{H}_\pi$?
2. Does $d_\pi \text{vol}(G/\Gamma) \geq 1$ imply the existence of $\eta \in \mathcal{H}_\pi$ such that $\pi(\Gamma)\eta$ is a Riesz sequence for $\mathcal{H}_\pi$?

The above problem has been considered several times in the literature. For Gabor frames in $L^2(\mathbb{R}^d)$ it was an open problem for several years with important partial progress made by Daubechies, Grossmann, and Morlet in the one-dimensional case $d = 1$ [13] and by Han and Wang for separable lattices in higher dimensions [26]. Bekka settled the problem in [10] where it is proved that for an arbitrary lattice $\Gamma$ in $\mathbb{R}^{2d}$, the condition $d_\pi \text{vol}(\mathbb{R}^{2d}/\Gamma) \leq 1$ is sufficient for the existence of a Gabor frame $\pi(\Gamma)\eta$. We mention that the result can also be inferred from a computation of Rieffel [39, Theorem 3.5]. However, the corresponding statement for Gabor Riesz sequences has been lacking so far.

A natural approach to the density theorem and its possible converses goes via the dimension theory for Hilbert modules over von Neumann algebras, as demonstrated by Bekka in [10]. The paper [10] concerns (nonprojective) square-integrable, irreducible, unitary representations $(\pi, \mathcal{H}_\pi)$ of second-countable, unimodular, locally compact groups $G$. It is shown that the existence of a frame $\pi(\Gamma)\eta$ for some $\eta \in \mathcal{H}_\pi$ and lattice $\Gamma \subseteq G$ is equivalent to $\text{cdim}_{L(\Gamma)} \mathcal{H}_\pi \leq 1$, where $\text{cdim}_{L(\Gamma)} \mathcal{H}_\pi$ is the center-valued von Neumann dimension of $\mathcal{H}_\pi$ as a Hilbert module over the group von Neumann algebra of $\Gamma$. Thus, in situations where $\text{cdim}_{L(\Gamma)} \mathcal{H}_\pi$ collapses to the scalar operator $d_\pi \text{vol}(G/\Gamma)I$, the first part of Problem 1.2 has a positive answer. However,
this collapsing does not always happen; it fails e.g. for discrete series representations of $SL(2, \mathbb{R})$ [10, Example 1] (see also Section 6.1). On the other hand, when $L(\Gamma)$ is a factor, i.e., when $\Gamma$ is an ICC group, $\text{cdim}_{L(\Gamma)} \mathcal{H}_\pi$ collapses and reduces to the famous formula
\[
\text{dim}_{L(\Gamma)} \mathcal{H}_\pi = d_\pi \text{vol}(G/\Gamma)
\]
which goes back to the work of Atiyah–Schmid [2]; see also [19, Theorem 3.1.1].

In [40], Romero and van Velthoven also consider the converse of Theorem 1.1 for projective representations. In particular, they show that when $(\Gamma, \sigma)$ satisfies Kleppner’s condition (see Section 3.1), then $d_\pi \text{vol}(G/\Gamma) \leq 1$ (resp. $d_\pi \text{vol}(G/\Gamma) \geq 1$) implies the existence of a frame (resp. Riesz sequence) of the form $\pi(\Gamma)\eta$. In the present paper we approach their result from the viewpoint of twisted group von Neumann algebras by adapting Bekka’s arguments to the setting of projective representations. Thus, the group von Neumann algebra $L(\Gamma)$ is replaced by the twisted group von Neumann algebra $L(\Gamma, \sigma)$. Twisted group operator algebras were introduced by Zeller-Meier in [42] and have been studied in e.g. [8,9,32,35,36,39]. We state two of our main results in the following theorem, which are adaptations of Bekka’s result [10, Theorem 1] (see Section 3 for definitions of $\sigma$-twisted convolution, $\sigma$-regularity and $\sigma$-positivity):

**Theorem 1.3.** Let $(\pi, \mathcal{H}_\pi)$ be a $\sigma$-projective, irreducible, square-integrable, unitary representation of a second-countable, unimodular, locally compact group $G$ and let $\Gamma$ be a lattice in $G$. Then the representation $\pi|_{\Gamma}$ of $\Gamma$ extends to give $\mathcal{H}_\pi$ the structure of a Hilbert $L(\Gamma, \sigma)$-module. The center-valued von Neumann dimension of $\mathcal{H}_\pi$ is the (possibly unbounded) operator on $\ell^2(\Gamma)$ given by $\sigma$-twisted convolution $f \mapsto \phi \ast \sigma f$ with the function $\phi$ on $\Gamma$ defined as follows:

\[
\phi(\gamma) = \begin{cases} 
\frac{d_\pi}{|C_\gamma|} \int_{G/\Gamma_\gamma} \overline{\sigma(\gamma, y)\sigma(y, y^{-1}\gamma y)} \langle \eta, \pi(y^{-1}\gamma y)\eta \rangle \, d\gamma(y) & \text{if } C_\gamma \text{ is } \sigma \text{-regular and finite,} \\
0 & \text{otherwise.}
\end{cases}
\]

Here, $C_\gamma$ denotes the conjugacy class of $\gamma$, $\Gamma_\gamma$ denotes the centralizer of $\gamma \in \Gamma$, and $\eta$ is any unit vector in $\mathcal{H}_\pi$.

Moreover, the following hold:

1. There exists a frame of the form $\pi(\Gamma)\eta$ for some $\eta \in \mathcal{H}_\pi$ if and only if $\delta_e - \phi$ is a $\sigma$-positive definite function.
2. There exists a Riesz sequence of the form $\pi(\Gamma)\eta$ for some $\eta \in \mathcal{H}_\pi$ if and only if $\phi - \delta_e$ is a $\sigma$-positive definite function.

Note that $\phi(e) = d_\pi \text{vol}(G/\Gamma)$, so that $d_\pi \text{vol}(G/\Gamma) \leq 1$ (resp. $d_\pi \text{vol}(G/\Gamma) \geq 1$) when $\delta_e - \phi$ (resp. $\phi - \delta_e$) is $\sigma$-positive definite; this recovers the density theorem (Theorem 1.1). When $\sigma$ is the trivial 2-cocycle, [10, Theorem 1] is recovered. A special case of Theorem 1.3 occurs when $(\Gamma, \sigma)$ satisfies Kleppner’s condition. This condition equivalent to the factoriality of $L(\Gamma, \sigma)$, in which case the center-valued von Neumann dimension of $\mathcal{H}_\pi$ reduces to the scalar operator $d_\pi \text{vol}(G/\Gamma)I$. Thus, we recover the result of Romero and van Velthoven [40, Theorem 1.1]. The scalar-valued von Neumann dimension in the projective setting was also computed for certain representations in [37].

The description of $\text{cdim}_{L(\Gamma, \sigma)} \mathcal{H}_\pi$ in Theorem 1.3 is obtained along the same lines as in the untwisted case in [10] with the necessary modifications needed to incorporate the 2-cocycle $\sigma$; see Section 4. An important ingredient in the proof is a description of the center-valued trace for twisted group von Neumann algebras, which does not seem to have appeared in the literature before; see Proposition 3.2. We also show in Corollary 4.6 that the situation is particularly simple when $G$ is abelian and the whole group $G$ satisfies
Kleppner’s condition with respect to $\sigma$: In this case, the center-valued von Neumann dimension collapses to the scalar operator $d_\pi \text{vol}(G/\Gamma)I$. Thus we get a complete converse to the density theorem in this setting, which we state here as a theorem:

**Theorem 1.4.** Let $(\pi, \mathcal{H}_\pi)$ be a $\sigma$-projective representation of a locally compact group satisfying the hypotheses of Theorem 1.3. Assume additionally that $G$ is abelian and that $(G, \sigma)$ satisfies Kleppner’s condition. Then for any lattice $\Gamma$ in $G$, we have that

$$\text{cdim}_{L(\Gamma, \sigma)} \mathcal{H}_\pi = d_\pi \text{vol}(G/\Gamma)I.$$  

Consequently, the following hold:

1. There exists a frame of the form $\pi(\Gamma)\eta$ for some $\eta \in \mathcal{H}_\pi$ if and only if $d_\pi \text{vol}(G/\Gamma) \leq 1$.
2. There exists a Riesz sequence of the form $\pi(\Gamma)\eta$ for some $\eta \in \mathcal{H}_\pi$ if and only if $d_\pi \text{vol}(G/\Gamma) \geq 1$.

Note that Kleppner’s condition for $(G, \sigma)$ in Theorem 1.4 is much weaker than Kleppner’s condition for $(\Gamma, \sigma)$, where $\Gamma$ is a particular lattice. In particular, the above result can be applied immediately to characterize the existence of Gabor frames and Gabor Riesz sequences over arbitrary lattices in the general locally compact abelian setting; see Theorem 6.1. For Riesz sequences, this result is new even in the case of Gabor frames for $L^2(\mathbb{R}^d)$. We also remark that the argument given by Bekka in [10] to characterize the existence of Gabor frames for $L^2(\mathbb{R}^d)$ relies on the fact that Heisenberg group is a nilpotent Lie group and thus cannot be applied directly to Gabor frames in the general locally compact abelian setting, where no such structure is present.

We apply our results to characterize not only the existence of frames and Riesz sequences of the form $\pi(\Gamma)\eta$, but also more general multiwindow super systems associated to $\pi$. For wavelet and Gabor systems, multiwindow and super systems were introduced and studied systematically by Balan [6,7] and Han and Larson [25]. Density theorems for multiwindow and super systems in the Gabor case and beyond have been considered in [3,5,22,28]. See Section 5.2 for details.

Finally, we mention the related paper [24] which considers density theorems in the more general setting of a projective representation of a countable group, and the recent preprint [1] which connects affine density with von Neumann dimension.

### 1.1. Structure of paper

The paper is structured as follows: In Section 2 we cover the necessary background on Hilbert modules over finite von Neumann algebras and their scalar-valued and center-valued dimensions. In Section 3 we introduce twisted group von Neumann algebras of discrete groups and describe their canonical center-valued trace. In Section 4 we compute the center-valued von Neumann dimension of $\mathcal{H}_\pi$ as a Hilbert $L(\Gamma, \sigma)$-module. In Section 5 we apply the results to frame theory, proving density theorems for multiwindow super systems along with converses. Finally, in Section 6, we consider two examples, one of them being Gabor systems over locally compact abelian groups.

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2. Hilbert modules over von Neumann algebras

2.1. Center-valued traces

Let $M$ be a finite von Neumann algebra equipped with a faithful normal tracial state $\tau$. Denote by $L^2(M, \tau)$ the Hilbert space obtained from the GNS construction of $(M, \tau)$, and by $\Omega$ its cyclic vector. We will represent $M$ as operators on $L^2(M, \tau)$ unless otherwise stated.

Since $M$ is finite, it has a canonical center-valued trace, i.e., a normal bounded linear map $\text{Tr}: M \to \mathcal{Z}(M)$ uniquely determined by the following properties:

1. $\text{Tr}(ab) = \text{Tr}(ba)$ for all $a, b \in M$.
2. $\text{Tr}(ba) = b\text{Tr}(a)$ for all $a \in M$ and $b \in \mathcal{Z}(M)$.
3. $\text{Tr}(a) = a$ for all $a \in \mathcal{Z}(M)$.
4. $\text{Tr}(a^*a) = 0$ implies $a = 0$ for all $a \in M$.

Moreover the center-valued trace and $\tau$ relate in the following way [14, p. 278, Proposition 3]:

$$\tau(ab) = \tau(\text{Tr}(a)b) = \tau(a\text{Tr}(b)) = \tau(\text{Tr}(ab)) = \tau(\text{Tr}(a)\text{Tr}(b)).$$

(2)

If $\mathcal{K}$ is a separable, infinite-dimensional Hilbert space with orthonormal basis $(e_i)_{i=1}^\infty$ then the von Neumann algebra $M \otimes \mathcal{B}(\mathcal{K})$ is semifinite. By [41, p. 330, Theorem 2.34] it admits a faithful, semifinite, normal extended center-valued trace $\Phi$ which we can describe as follows: If $a$ is a positive element of $M \otimes \mathcal{B}(\mathcal{K})$ then $a$ can be decomposed into a matrix $(a_{ij})_{i,j=1}^\infty$ with entries in $M$ such that

$$\langle a(\xi \otimes e_i), \eta \otimes e_j \rangle = \langle a_{ij}\xi, \eta \rangle \quad \text{for} \quad a \in M \otimes \mathcal{B}(\mathcal{K}), \xi, \eta \in L^2(M, \tau) \text{ and } i, j \in \mathbb{N}.$$

Then $\Phi$ is defined as

$$\Phi(a) = \sum_i \text{Tr}(a_{ii}),$$

(3)

see [10, p. 332]. We also obtain a faithful, semifinite, normal extended scalar-valued trace $\tau \otimes \text{tr}$ on $M \otimes \mathcal{B}(\mathcal{K})$ given by

$$(\tau \otimes \text{tr})(a) = \sum_i \tau(a_{ii}).$$

(4)

The main property of $\Phi$ that we will need is the following proposition. We call projections $p$ and $q$ called Murray–von Neumann equivalent, written $p \sim q$, if there exists a partial isometry $u \in M$ such that $u^*u = p$ and $uu^* = q$. We also write $p \preceq q$ when $p$ is Murray–von Neumann equivalent to a subprojection of $q$.

Proposition 2.1. Let $\Phi$ be a normal, faithful, semifinite extended center-valued trace on a von Neumann algebra $M$. Let $p$ and $q$ be projections in $M$. Then the following hold:

1. If $p \preceq q$ then $\Phi(p) \leq \Phi(q)$, and if $p \sim q$ then $\Phi(p) = \Phi(q)$.
2. Suppose that $p$ and $q$ are finite projections. Then $p \preceq q$ if and only if $\Phi(p) \leq \Phi(q)$, and $p \sim q$ if and only if $\Phi(p) = \Phi(q)$.

Proof. The proof is essentially in [10, Proposition 2], but we give it here for completeness. Part (1) goes as follows: If $p \preceq q$ then we can find a partial isometry $u \in M$ such that $p = u^*u$ and $uu^* \leq q$. By positivity
and the tracial property of $\Phi$ this implies $\Phi(p) \leq \Phi(q)$. The implication from $p \sim q$ to $\Phi(p) = \Phi(q)$ now follows from anti-symmetry.

Before we prove part (2), we make the following observation: If $p, q$ are finite projections in $M$ with $p \leq q$ and $\Phi(p) = \Phi(q)$, then $p = q$. Indeed, we have

$$\Phi(p) = \Phi(q) = \Phi(p) + \Phi(q - p).$$

Since $p$ is finite, [41, p. 331, Proposition 2.35] implies that $\Phi(p)$ is finite almost everywhere as a function on the measure space $(X, \mu)$ such that $\mathcal{Z}(M) \cong L^\infty(X, \mu)$. Hence we may cancel $\Phi(p)$ in the above equation. The faithfulness of $\Phi$ now implies $q = p$.

We now prove (2). Suppose $p$ and $q$ are finite and that $\Phi(p) \leq \Phi(q)$. By [41, p. 293, Theorem 1.8] we can find a projection $z$ in the center of $M$ such that $(1 - z)p \preceq (1 - z)q$ and $zq \preceq zp$. By part (1) we get $\Phi(zq) \leq \Phi(zp)$. However, we also have that

$$\Phi(zp) = z\Phi(p) \leq z\Phi(q) = \Phi(zq).$$

Thus we have both $\Phi(zq) = \Phi(zp)$ and $zq \preceq zp$, say $zq \sim q' \leq zp$. By part (1), $\Phi(q') = \Phi(zq) = \Phi(zp)$, so our observation implies that $q' = zp$. Hence $zp \sim zq$, so

$$p = zp + (1 - z)p \sim zq + (1 - z)p \preceq zq + (1 - z)q = q.$$

The implication from $\Phi(p) = \Phi(q)$ to $p \sim q$ now follows from [41, p. 291, Proposition 1.3].

2.2. Hilbert modules

We continue to assume that $M$ is a finite von Neumann algebra equipped with a faithful normal tracial state $\tau$. A (left) Hilbert $M$-module is a Hilbert space $\mathcal{H}$ together with a unital normal $*$-homomorphism $\pi: M \to \mathcal{B}(\mathcal{H})$. We write $a\xi = \pi(a)\xi$ for $a \in M$ and $\xi \in \mathcal{H}$. We often write $M\mathcal{H}$ when we want to emphasize that $\mathcal{H}$ is a Hilbert module over $M$. A Hilbert module $\mathcal{H}$ is called separable if $\mathcal{H}$ is separable as a Hilbert space. Two Hilbert $M$-modules are isomorphic if there exists an $M$-linear unitary operator between them.

Fixing a separable infinite-dimensional Hilbert space $\mathcal{K}$, we can turn the tensor product $L^2(M, \tau) \otimes \mathcal{K}$ into a Hilbert $M$-module via

$$a(\xi \otimes \eta) = (a\xi) \otimes \eta \quad \text{for } a \in M, \xi \in L^2(M, \tau) \text{ and } \eta \in \mathcal{K}.$$ 

This is nothing but the countable direct sum $\bigoplus_{j=1}^\infty L^2(M, \tau)$ where $M$ acts diagonally. One of the basic facts about Hilbert modules over $M$ is that under certain separability assumptions they can all be realized as submodules of this direct sum [30, Proposition 2.1.2]:

**Proposition 2.2.** Let $M$ be a finite von Neumann algebra which has separable predual, let $\tau$ be a faithful normal tracial state on $M$ and let $\mathcal{K}$ be a separable infinite-dimensional Hilbert space. Then every separable Hilbert $M$-module is isomorphic to a submodule of $L^2(M, \tau) \otimes \mathcal{K}$.

Now let $M'$ be the commutant of $M$ in $\mathcal{B}(L^2(M, \tau))$ which is also finite. The trace $\tau$ extends to a state on $\mathcal{B}(L^2(M, \tau))$ via $a \mapsto \langle a\Omega, \Omega \rangle$ for $a \in \mathcal{B}(L^2(M, \tau))$ and the restriction to $M'$ is also a faithful normal tracial state. Let $\mathcal{H}$ be a separable Hilbert $M$-module, which we embed into $L^2(M, \tau) \otimes \mathcal{K}$ as in Proposition 2.2. Denote by $p$ the projection of $L^2(M, \tau) \otimes \mathcal{K}$ onto $\mathcal{H}$. Then this projection is in the commutant of $M$ on the Hilbert space $L^2(M, \tau) \otimes \mathcal{K}$. This commutant equals $M' \otimes \mathcal{B}(\mathcal{K})$. The latter is a semifinite von Neumann
algebra with center-valued trace $\Phi$ as defined in (3). We define the center-valued von Neumann dimension of $\mathcal{H}$ to be

$$\text{cdim}_M \mathcal{H} = \Phi(p) = \sum_i \text{Tr}(p_{ii})$$

(5)

where $\text{Tr}$ denotes the canonical center-valued trace on $M'$. It follows from part (1) of Proposition 2.1 that this definition is independent of the chosen projection. Moreover we define the scalar-valued von Neumann dimension of $\mathcal{H}$ to be

$$\dim_M \mathcal{H} = (\tau \otimes \text{tr})(p) = \sum_i \tau(p_{ii}).$$

(6)

Note that these two notions of dimension coincide precisely when $M$ is a factor.

**Proposition 2.3.** Let $M$ be a finite von Neumann algebra with separable predual equipped with a faithful normal tracial state $\tau$, and let $\mathcal{H}$ and $\mathcal{H}'$ be separable Hilbert $M$-modules.

1. If $\mathcal{H}$ is isomorphic to a submodule of $\mathcal{H}'$ then $\text{cdim}_M \mathcal{H} \leq \text{cdim}_M \mathcal{H}'$, and if $\mathcal{H} \cong \mathcal{H}'$ then $\text{cdim}_M \mathcal{H} = \text{cdim}_M \mathcal{H}'$.

2. Suppose that $\dim_M \mathcal{H}, \dim_M \mathcal{H}' < \infty$. Then $\mathcal{H}$ is isomorphic to a submodule of $\mathcal{H}'$ if and only if $\text{cdim}_M \mathcal{H} \leq \text{cdim}_M \mathcal{H}'$, and $\mathcal{H} \cong \mathcal{H}'$ if and only if $\text{cdim}_M \mathcal{H} = \text{cdim}_M \mathcal{H}'$.

3. $\text{cdim}_M (\mathcal{H} \oplus \mathcal{H}') = \text{cdim}_M \mathcal{H} + \text{cdim}_M \mathcal{H}'$.

**Proof.** Let $\text{cdim}_M \mathcal{H} = \Phi(p)$ for a projection $p \in M' \otimes \mathcal{B}(\mathcal{K})$. Since $\tau \otimes \text{id}$ is a faithful, semifinite, normal extended scalar-valued trace on $M \otimes \mathcal{B}(\mathcal{K})$, $(\tau \otimes \text{id})(p) < \infty$ implies that $p$ is a finite projection. Thus, since $\dim_M \mathcal{H} = (\tau \otimes \text{id})(p)$ both (1) and (2) follow directly from Proposition 2.1. (3) follows from the additivity of $\Phi$. □

3. Twisted group von Neumann algebras

3.1. 2-cocycles and projective representations

In this section $G$ denotes a locally compact group with identity $e$. A 2-cocycle on $G$ is a Borel measurable function $\sigma : G \times G \to T$ that satisfies the following properties:

$$\sigma(x,y)\sigma(xy,z) = \sigma(x,yz)\sigma(y,z) \quad \text{for } x,y,z \in G,$$

(7)

$$\sigma(e,e) = 1.$$  

(8)

If $\sigma$ is a 2-cocycle then the pointwise conjugate $\sigma^*$ is also a 2-cocycle.

An element $x \in G$ is called $\sigma$-regular if $\sigma(x,y) = \sigma(y,x)$ whenever $y$ commutes with $x$. If $x$ is $\sigma$-regular then every element in the conjugacy class $C_x$ of $x$ is $\sigma$-regular, hence it makes sense to talk about $\sigma$-regular conjugacy classes. We say that $(G,\sigma)$ satisfies Kleppner’s condition if the only $\sigma$-regular finite conjugacy class is the trivial one.

Denote by $\mathcal{U}(\mathcal{H})$ the set of unitary operators on $\mathcal{H}$. A $\sigma$-projective unitary representation of $G$ on a Hilbert space $\mathcal{H}$ is a map $\pi : G \to \mathcal{U}(\mathcal{H})$ that satisfies the following property:

$$\pi(x)\pi(y) = \sigma(x,y)\pi(xy) \quad \text{for all } x,y \in G.$$
We also require that \( \pi \) is strongly measurable: That is, the map \( G \to \mathcal{H} \) given by \( x \mapsto \pi(x)\xi \) is measurable for every \( \xi \in \mathcal{H} \).

Every 2-cocycle \( \sigma \) on \( G \) comes with two natural \( \sigma \)-projective representations: The \( \sigma \)-projective left regular representation of \( G \) is the representation \( \lambda_{\sigma} \) of \( G \) on \( L^2(G) \) given by

\[
\lambda_{\sigma}(x)f(y) = \sigma(x, x^{-1}y)f(x^{-1}y) \quad \text{for } x, y \in G \text{ and } f \in L^2(G),
\]

while the \( \sigma \)-projective right regular representation of \( G \) is the representation \( \rho_{\sigma} \) of \( G \) on \( L^2(G) \) given by

\[
\rho_{\sigma}(x)f(y) = \sigma(y, x)f(yx) \quad \text{for } x, y \in G \text{ and } f \in L^2(G).
\]

Using the 2-cocycle identity (7) one shows that the following conjugation identity holds for projective representations:

\[
\pi(y)^*\pi(x)\pi(y) = \sigma(x, y)\sigma(y, y^{-1}xy)\pi(y^{-1}xy) \quad \text{for all } x, y \in G. \tag{9}
\]

The associated function \( \tilde{\sigma} : G \times G \to \mathbb{T} \) given by

\[
\tilde{\sigma}(x, y) = \sigma(x, y)\sigma(y, y^{-1}xy) \quad \text{for } x, y \in G, \tag{10}
\]

has some important properties that form the basis for later 2-cocycle computations:

**Lemma 3.1.** Let \( \sigma \) be a 2-cocycle on a locally compact group \( G \). The following hold for all \( x, y, z \in G \) where \( \tilde{\sigma} \) is as defined in (10):

\[
\tilde{\sigma}(x, yz) = \tilde{\sigma}(x, y)\tilde{\sigma}(y^{-1}xy, z), \tag{11}
\]

\[
\tilde{\sigma}(x, y^{-1}) = \tilde{\sigma}(yx^{-1}, y), \tag{12}
\]

\[
\tilde{\sigma}(x, y) = \tilde{\sigma}(x, y') \quad \text{if } x \text{ is } \sigma \text{-regular and } y^{-1}xy = y'^{-1}xy'. \tag{13}
\]

**Proof.** Using the 2-cocycle identity (7) repeatedly, we obtain

\[
\tilde{\sigma}(x, y)\tilde{\sigma}(y^{-1}xy, z)
= \sigma(x, y)\sigma(y, y^{-1}xy)\sigma(y^{-1}xy, z)\sigma(z, z^{-1}y^{-1}xyz)
= \sigma(x, y)\sigma(xy, z)\sigma(y^{-1}xy, z)\sigma(y, y^{-1}xy)\sigma(xy, z)\sigma(z, z^{-1}y^{-1}xyz)
= \sigma(x, yz)\sigma(y, z)\sigma(y^{-1}xy, z)\sigma(y, y^{-1}xyz)\sigma(y^{-1}xy, z)\sigma(z, z^{-1}y^{-1}xyz)
= \sigma(x, yz)\sigma(y, z)\sigma(y, y^{-1}xyz)\sigma(z, z^{-1}y^{-1}xyz)
= \sigma(x, yz)\sigma(yz, z^{-1}y^{-1}xyz)
= \tilde{\sigma}(x, yz).
\]

This proves (11). Now (12) follows from the following special case of (11):

\[
1 = \tilde{\sigma}(x, y^{-1}y) = \tilde{\sigma}(x, y^{-1})\tilde{\sigma}(yx^{-1}, y).
\]

To prove (13), note that \( y^{-1}xy = y'^{-1}xy' \) implies that \( yy'^{-1}xy'y^{-1} = x \), hence \( x \) commutes with \( yy'^{-1} \).

Since \( x \) is assumed to be \( \sigma \)-regular, we obtain
\[ \tilde{\sigma}(x, y'y^{-1}) = \sigma(x, y'y^{-1})\overline{\sigma(y'y^{-1}, y'y^{-1}xy'y^{-1})} = \sigma(x, y'y^{-1})\overline{\sigma(y'y^{-1}, x)} = 1. \]

Applying (11) and (12) we get
\[ \tilde{\sigma}(x, y')\overline{\tilde{\sigma}(x, y)} = \tilde{\sigma}(x, y')\overline{\tilde{\sigma}(yy'y^{-1}, y)} = \tilde{\sigma}(x, y')\overline{\tilde{\sigma}(y'y^{-1}, y)} = \tilde{\sigma}(x, y'y^{-1}) = 1. \]

Hence \( \tilde{\sigma}(x, y) = \tilde{\sigma}(x, y'). \)

Note that when \( G \) is abelian, the identity (11) of Lemma 3.1 reduces to the fact that \( (x, y) \mapsto \sigma(x, y)\overline{\sigma(y, x)} \)

is a bicharacter on \( G \), which is well-known (see e.g. [31, Lemma 7.1]).

3.2. The center-valued trace on twisted group von Neumann algebras

Let \( \Gamma \) be a discrete group with a 2-cocycle \( \sigma \) and denote by \( \lambda_\sigma \) (resp. \( \rho_\sigma \)) the \( \sigma \)-twisted left (resp. right) regular representation of \( \Gamma \). We then define the following two associated von Neumann algebras on \( \ell^2(\Gamma) \):

\[ L(\Gamma, \sigma) = \lambda_\sigma(\Gamma)'' \quad \text{and} \quad R(\Gamma, \sigma) = \rho_\sigma(\Gamma)''. \]

The von Neumann algebra \( L(\Gamma, \sigma) \) is called the \( \sigma \)-twisted group von Neumann algebra of \( \Gamma \). It is well-known that \( R(\Gamma, \sigma) \) is the commutant of \( L(\Gamma, \sigma) \) on \( \ell^2(\Gamma) \) (see [32, Theorem 1]).

For the remainder of the section we set \( M = L(\Gamma, \sigma) \) and \( N = R(\Gamma, \sigma) \). Let \( \{ \delta_\gamma : \gamma \in \Gamma \} \) denote the usual orthonormal basis for \( \ell^2(\Gamma) \). The map \( \tau : B(\ell^2(\Gamma)) \to \mathbb{C} \) given by

\[ \tau(a) = \langle a\delta_\gamma, \delta_\gamma \rangle \quad \text{for} \quad a \in M \]

restricts to a faithful normal tracial state on both \( M \) and \( N \), which shows that these von Neumann algebras are finite. Moreover, the GNS construction \( L^2(M, \tau) \) with respect to \( \tau \) is canonically isomorphic to \( \ell^2(\Gamma) \) with cyclic, separating vector \( \Omega = \delta_\epsilon \).

**Proposition 3.2.** The center-valued trace on \( L(\Gamma, \sigma) \) is given as follows: If \( \gamma \in \Gamma \) is such that the conjugacy class \( C_\gamma \) is \( \sigma \)-regular and finite, say \( C_\gamma = \{ \beta_1^{-1}\gamma\beta_1, \ldots, \beta_k^{-1}\gamma\beta_k \} \), then

\[ \text{Tr}(\lambda_\sigma(\gamma)) = |C_\gamma|^{-1} \sum_{j=1}^{k} \sigma(\gamma, \beta_j) \overline{\sigma(\beta_j, \beta_j^{-1}\gamma\beta_j)} \lambda_\sigma(\beta_j^{-1}\gamma\beta_j). \]

Otherwise \( \text{Tr}(\lambda_\sigma(\gamma)) = 0 \).

**Proof.** To ease notation set \( \lambda = \lambda_\sigma \). Let \( \gamma \in \Gamma \). For any \( \beta \in \Gamma \) we have by (9) that

\[ \lambda(\beta)^*\lambda(\gamma)\lambda(\beta) = \tilde{\sigma}(\gamma, \beta)\lambda(\beta^{-1}\gamma\beta). \]

Taking center-valued traces, we obtain

\[ \text{Tr}(\lambda(\gamma)) = \tilde{\sigma}(\gamma, \beta) \text{Tr}(\lambda(\beta^{-1}\gamma\beta)). \]

If \( C_\gamma \) is infinite then \( \text{Tr}(\lambda(\gamma)) = 0 \) by [35, Lemma 2.2]. If \( C_\gamma \) is not \( \sigma \)-regular then there exists \( \beta \in \Gamma \) that commutes with \( \lambda \) yet \( \sigma(\gamma, \beta) \neq \sigma(\beta, \gamma) \). But then (15) gives \( \text{Tr}(\lambda(\gamma)) = \sigma(\gamma, \beta)\overline{\sigma(\beta, \gamma)} \text{Tr}(\lambda(\gamma)) \) which implies that \( \text{Tr}(\lambda(\gamma)) = 0 \).

Suppose now that \( C_\gamma \) is both \( \sigma \)-regular and finite, say \( C_\gamma = \{ \beta_1^{-1}\gamma\beta_1, \ldots, \beta_k^{-1}\gamma\beta_k \} \). Summing (14) over all \( \beta \in C_\gamma \) we get
Denote the right side of the above equality by $S$. We claim that $S$ is in the center of $L(\Gamma, \sigma)$. Indeed, if $\beta \in \Gamma$ we use Lemma 3.1 (11) to compute that

$$\lambda(\beta)^* S \lambda(\beta) = \sum_{j=1}^{k} \tilde{\sigma}(\gamma, \beta_j) \lambda(\beta)^* \lambda(\beta_j^{-1}) \gamma \beta_j \lambda(\beta)$$

$$= \sum_{j=1}^{k} \tilde{\sigma}(\gamma, \beta_j) \tilde{\sigma}(\beta_j^{-1} \gamma \beta_j, \beta) \lambda(\beta_j^{-1} \gamma \beta_j \beta)$$

$$= \sum_{j=1}^{k} \tilde{\sigma}(\gamma, \beta_j \beta) \lambda(\beta_j^{-1} \gamma \beta_j \beta).$$

The set \{${\beta_j^{-1} \gamma \beta_j \beta : 1 \leq j \leq k}$\} is equal to $C_\gamma$, say

$$\beta_j^{-1} \gamma \beta_j \beta = \beta_{m_j} ^{-1} \gamma \beta_{m_j}$$

for each $1 \leq j \leq k$, where \{${m_1, \ldots, m_k}$\} = \{${1, \ldots, k}$\}. Since $\gamma$ is $\sigma$-regular, Lemma 3.1 (3) implies that $\tilde{\sigma}(\gamma, \beta_j \beta) = \tilde{\sigma}(\gamma, \beta_{m_j})$ for each $j$, hence

$$\lambda(\beta)^* S \lambda(\beta) = \sum_{j=1}^{k} \tilde{\sigma}(\gamma, \beta_{m_j}) \lambda(\beta_{m_j}^{-1} \gamma \beta_{m_j}) = S.$$ 

We conclude that $S$ is in the center of $L(\Gamma, \sigma)$. Taking the center-valued trace on both sides of (16) we now obtain

$$|C_\gamma| \text{Tr}(\lambda(\gamma)) = \sum_{j=1}^{k} \text{Tr} (\lambda(\beta_j)^* \lambda(\beta_j)) = \sum_{j=1}^{k} \tilde{\sigma}(\gamma, \beta_j) \lambda(\beta_j^{-1} \gamma \beta_j).$$

Dividing by $|C_\gamma|$ on both sides finishes the proof. □

As a corollary, we obtain the following well-known characterization of when $L(\Gamma, \sigma)$ is a factor.

**Corollary 3.3.** The von Neumann algebra $L(\Gamma, \sigma)$ is a factor if and only if $(\Gamma, \sigma)$ satisfies Kleppner’s condition.

**Proof.** $L(\Gamma, \sigma)$ is a factor if and only if the center-valued trace reduces to the canonical faithful tracial state on $L(\Gamma, \sigma)$. By the expression in Proposition 3.2 this happens if and only if the only $\sigma$-regular finite conjugacy class is \{${\epsilon}$\}. □

An analogous computation to that of the proof of Proposition 3.2 shows that the center-valued trace on $R(\Gamma, \sigma)$ is given by

$$\text{Tr}(\rho_{\sigma}(\gamma)) = |C_\gamma|^{-1} \sum_{j=1}^{k} \sigma(\gamma, \beta_j) \overline{\sigma(\beta_j, \beta_j^{-1} \gamma \beta_j)} \rho_{\sigma}(\beta_j^{-1} \gamma \beta_j)$$

(17)

if $C_\gamma$ is finite and $\sigma$-regular and $\text{Tr}(\rho_{\sigma}(\gamma)) = 0$ otherwise. Note that for $R(\Gamma, \overline{\sigma}) = L(\Gamma, \sigma)'$ one needs to conjugate $\sigma$ in the above equation.
3.3. Fourier coefficients and positivity

We can express the vectors \( \delta_\gamma \) for \( \gamma \in \Gamma \) in terms of \( \lambda_\sigma \) and \( \rho_\sigma \) as

\[
\delta_\gamma = \lambda_\sigma(\gamma)\delta_e = \rho_\sigma(\gamma)^*\delta_e.
\]  

(18)

The \( \sigma \)-twisted convolution of two functions \( f, g : \Gamma \to \mathbb{C} \) is given by

\[
(f *_\sigma g)(\gamma) = \sum_{\gamma' \in \Gamma} \sigma(\gamma', \gamma'^{-1}\gamma)f(\gamma')g(\gamma'^{-1}\gamma) = \sum_{\gamma' \in \Gamma} f(\gamma')\lambda_\sigma(\gamma')g(\gamma) \text{ for } \gamma \in \Gamma.
\]

Given \( a \in L(\Gamma, \sigma) \), then the Fourier coefficient of \( a \) is the element \( \hat{a} = a\delta_e \) of \( \ell^2(\Gamma) \). Since \( \delta_e \) is a separating vector for \( L(\Gamma, \sigma) \) on \( \ell^2(\Gamma) \), it follows that \( a \) is uniquely determined by \( \hat{a} \). Using twisted convolution, we can describe how \( a \) acts on \( \ell^2(\Gamma) \) via its Fourier coefficient; see [8, p. 343]:

\[
a f = \hat{a} *_\sigma f \text{ for all } f \in \ell^2(\Gamma).
\]

In particular \( \hat{a} *_\sigma f \in \ell^2(\Gamma) \) for all \( f \in \ell^2(\Gamma) \). Using (18), the values of \( \hat{a} \) can be expressed as

\[
\hat{a}(\gamma) = \langle \hat{a}, \delta_\gamma \rangle = \tau(\lambda_\sigma(\gamma)^*a) = \tau(\rho_\sigma(\gamma)a).
\]

A function \( \phi \in \ell^\infty(\Gamma) \) is called \( \sigma \)-positive definite if for all \( \gamma_1, \ldots, \gamma_n \in \Gamma \) and \( c_1, \ldots, c_n \in \mathbb{C} \) we have

\[
\sum_{i,j} \sigma(\gamma_j\gamma_i^{-1}, \gamma_i)c_i\overline{c_j}\phi(\gamma_j\gamma_i^{-1}) \geq 0.
\]

Equivalently, the matrix \( (\sigma(\gamma_j\gamma_i^{-1}, \gamma_i)\phi(\gamma_j\gamma_i^{-1}))_{i,j=1} \) is positive semidefinite.

**Proposition 3.4.** A function \( \phi \in \ell^\infty(\Gamma) \) is \( \sigma \)-positive definite if and only if

\[
\langle \phi *_\sigma f, f \rangle \geq 0
\]

for all functions \( f : \Gamma \to \mathbb{C} \) with finite support. In particular, \( a \in L(\Gamma, \sigma) \) is positive if and only if \( \hat{a} \) is a \( \sigma \)-positive definite function.

**Proof.** For a function \( f \) on \( \Gamma \) of finite support we have that

\[
\langle \phi *_\sigma f, f \rangle = \sum_{\gamma \in \Gamma} (\phi *_\sigma f)(\gamma)\overline{f(\gamma)}
\]

\[
= \sum_{\gamma, \gamma' \in \Gamma} \phi(\gamma')f(\gamma'^{-1}\gamma)\sigma(\gamma', \gamma'^{-1}\gamma)\overline{f(\gamma)}
\]

\[
= \sum_{\gamma, \gamma' \in \Gamma} \sigma(\gamma\gamma'^{-1}, \gamma')f(\gamma')\overline{f(\gamma)}\phi(\gamma\gamma'^{-1}).
\]

Letting the support of \( f \) be \( \{\gamma_1, \ldots, \gamma_n\} \) and setting \( c_i = f(\gamma_i) \) for each \( 1 \leq i \leq n \), the above expression becomes exactly the expression in the definition of \( \sigma \)-positivity.

The positivity of \( a \in L(\Gamma, \sigma) \) is equivalent to \( \langle af, f \rangle = \langle \hat{a} *_\sigma f, f \rangle \geq 0 \) for all \( f \in \ell^2(\Gamma) \). By writing \( f \) as a limit of functions on \( \Gamma \) of finite support the last condition is equivalent to \( \hat{a} \) being \( \sigma \)-positive definite as shown above. \( \Box \)
4. Hilbert modules from square-integrable representations

Throughout this section $G$ denotes a second-countable, unimodular, locally compact group, $\sigma$ denotes a 2-cocycle on $G$, and $\Gamma$ denotes a lattice in $G$. We fix a Haar measure on $G$ and simply write $dz$ when we integrate with respect to it. Soon $(\pi, \mathcal{H}_\pi)$ will denote a $\sigma$-projective unitary representation of $G$ which is irreducible and square-integrable as defined in Proposition 4.2. We will denote by $\lambda^{C}_\sigma$ (resp. $\lambda^{F}_\sigma$) the $\sigma$-projective left regular representation of $G$ (resp. of $\Gamma$). We will also let $M = L(\Gamma, \sigma)$ and $N = R(\Gamma, \sigma)$.

4.1. Lattices in locally compact groups

Suppose $\Gamma$ is a lattice in $G$. Then there exists a Borel measurable set $B \subseteq G$ such that the collection $\{\gamma B : \gamma \in \Gamma\}$ forms a partition of $G$ [34]. Such a set $B$ is called a fundamental domain for $\Gamma$ in $G$. Weil’s formula relates integration over $G$ to integration over $\Gamma$ and $B$:

$$\int_G f(x) \, dx = \sum_{\gamma \in \Gamma \backslash B} \int f(\gamma y) \, dy \quad \text{for all } f \in L^1(G).$$

In general, there are many fundamental domains for $\Gamma$ in $G$, but they all have the same measure. This number is called the covolume of $\Gamma$ in $G$ and is denoted by $\text{vol}(G/\Gamma)$. If $B$ has finite measure then $\Gamma$ is called a lattice in $G$.

Now set $\mathcal{K} = L^2(B)$ and fix a 2-cocycle $\sigma$ on $G$. Let $M = L(\Gamma, \sigma)$ be the $\sigma$-twisted group von Neumann algebra of $\Gamma$ with its canonical trace $\tau$. Then according to Section 3.2, the GNS construction of $(M, \tau)$ can be naturally identified with $L^2(\Gamma)$. Because of our assumptions on $G$, the Hilbert space $\mathcal{K}$ is separable and infinite-dimensional if $G$ is infinite. Consequently the Hilbert $M$-module $\ell^2(\Gamma) \otimes \mathcal{K}$ is exactly the module into which every separable Hilbert $M$-module embeds according to Proposition 2.2. If $G$ is finite then $L(\Gamma, \sigma)$ is finite-dimensional and in this case every Hilbert $M$-module also embeds into $\ell^2(\Gamma) \otimes \mathcal{K}$. The following proposition shows that $\ell^2(\Gamma) \otimes \mathcal{K}$ can be naturally identified with $L^2(G)$.

**Proposition 4.1.** The von Neumann algebra $\lambda^{C}_\sigma(\Gamma)''$ on $L^2(G)$ is isomorphic to $L(\Gamma, \sigma)$. Moreover, the Hilbert $L(\Gamma, \sigma)$-module $L^2(G)$ is isomorphic to $\ell^2(\Gamma) \otimes \mathcal{K}$ via the map $U : \ell^2(\Gamma) \otimes \mathcal{K} \to L^2(G)$ given by

$$U(f \otimes g)(\gamma y) = \sigma(\gamma, y)f(\gamma)g(y) \quad \text{for } \gamma \in \Gamma \text{ and } y \in B.$$  

**Proof.** That $U$ is a well-defined unitary operator follows from the fact that $B$ is a fundamental domain for $\Gamma$ in $G$.

Let $f \in \ell^2(\Gamma)$ and $g \in \mathcal{K}$. The following computation shows that $U$ intertwines $\lambda^{C}_\sigma \otimes I$ and $\lambda^{F}_\sigma$, where $\lambda^{C}_\sigma$ denotes the $\sigma$-twisted left regular representation of $\Gamma$:

$$\lambda^{C}_\sigma(\gamma)U(f \otimes g)(\gamma y) = \sigma(\gamma, \gamma^{-1} \gamma' y)U(f \otimes g)(\gamma^{-1} \gamma' y)$$

$$= \sigma(\gamma, \gamma^{-1} \gamma' y)\sigma(\gamma^{-1} \gamma', y)f(\gamma^{-1} \gamma' y)g(y)$$

$$= \sigma(\gamma, \gamma^{-1} \gamma')\sigma(\gamma^{-1} \gamma', y)f(\gamma^{-1} \gamma' y)g(y)$$

$$= \sigma(\gamma', y)(\lambda^{C}_\sigma(\gamma)f \otimes g)(\gamma y)$$

$$= U(\lambda^{C}_\sigma(\gamma)f \otimes g)(\gamma y).$$

Thus the map $L(\Gamma, \sigma) \to \lambda^{C}_\sigma(\Gamma)''$ given by $a \mapsto U(a \otimes I)U^*$ is an isomorphism of von Neumann algebras and the Hilbert $L(\Gamma, \sigma)$-modules $\ell^2(\Gamma) \otimes \mathcal{K}$ and $L^2(G)$ are isomorphic. \(\square\)
Let $(e_i)_{i \in \mathbb{N}}$ be an orthonormal basis for $\mathcal{K}$, and denote by $\tilde{e}_i$ the extension of $e_i$ to the whole of $G$ by zero outside of $B$. Then $U(\delta_e \otimes e_i) = \tilde{e}_i$. Since $(\delta_\gamma)_{\gamma \in \Gamma}$ is an orthonormal basis for $\ell^2(\Gamma)$, it follows that $(\delta_\gamma \otimes e_i)_{\gamma \in \Gamma, i \in \mathbb{N}}$ is an orthonormal basis for $\ell^2(\Gamma) \otimes \mathcal{K}$. Since

$$U(\delta_\gamma \otimes e_i) = U(\lambda^\Gamma_\gamma(\delta_e \otimes e_i)) = \lambda^G_\gamma(\delta_e \otimes e_i) = \lambda^G_{\sigma}(\gamma)\tilde{e}_i$$

it follows from Proposition 4.1 that $(\lambda^G_{\sigma}(\gamma)\tilde{e}_i)_{\gamma \in \Gamma, i \in \mathbb{N}}$ is an orthonormal basis for $L^2(G)$.

4.2. Orthogonality relations

The following orthogonality relation for the matrix coefficients of irreducible square-integrable representations forms the basis for our considerations, see [37, Definition/Proposition 3.1] for a proof.

**Proposition 4.2.** Let $\pi$ be a $\sigma$-projective irreducible unitary representation of a unimodular locally compact group $G$. The following are equivalent:

1. There exist nonzero vectors $\xi, \eta \in \mathcal{H}_\pi$ such that $\int_{\mathcal{G}} |\langle \xi, \pi(x)\eta \rangle|^2 \, dx < \infty$.
2. For every $\xi, \eta \in \mathcal{H}_\pi$ we have that $\int_{\mathcal{G}} |\langle \xi, \pi(x)\eta \rangle|^2 \, dx < \infty$.
3. $\pi$ is a subrepresentation of the $\sigma$-twisted left regular representation of $G$.

In case any (and hence all) of the above assumptions hold, then there exists a number $d_\pi > 0$ called the formal dimension of $\pi$ such that

$$\int_{\mathcal{G}} \langle \xi, \pi(x)\eta \rangle \overline{\langle \xi', \pi(x)\eta' \rangle} \, dx = d_\pi^{-1} \langle \xi, \xi' \rangle \overline{\langle \eta, \eta' \rangle} \quad (20)$$

for all $\xi, \eta \in \mathcal{H}_\pi$.

Representations satisfying any of the equivalent conditions in Proposition 4.2 are called square-integrable. We now fix a square-integrable, $\sigma$-projective, irreducible, unitary representation $\pi$ of $G$.

We detail the passage from (2) to (3) in Proposition 4.2 as it will be relevant in this section. For $\xi, \eta \in \mathcal{H}_\pi$ we define the (generalized) wavelet transform $V_\eta \xi : G \to \mathbb{C}$ by

$$V_\eta \xi(x) = \langle \xi, \pi(x)\eta \rangle \quad \text{for all } x \in G.$$ 

From the assumption that (2) in Proposition 4.2 holds it follows that $V_\eta$ maps $\mathcal{H}_\pi$ into $L^2(G)$. Moreover, $V_\eta \xi$ intertwines $\pi$ and the $\sigma$-twisted left regular representation as can be seen from the following calculation for $x, y \in G$:

$$V_\eta(\pi(x)\xi)(y) = \langle \pi(x)\xi, \pi(y)\eta \rangle$$

$$= \langle \xi, \pi(x)^*\pi(y)\eta \rangle$$

$$= \langle \xi, \sigma(x, x^{-1})\sigma(x^{-1}, y)\pi(x^{-1}y) \rangle$$

$$= \sigma(x, x^{-1}y)V_\eta \xi(x^{-1}y)$$

$$= \lambda_\sigma(x)V_\eta \xi(y).$$

If one sets $\eta = \eta'$ to be a unit vector in (20) one obtains

$$\langle V_\eta \xi, V_\eta \xi' \rangle = d_\pi^{-1} \langle \xi, \xi' \rangle \quad \text{for all } \xi, \xi' \in \mathcal{H}_\pi.$$
It follows that the map $d_{\pi}^{1/2}V_\eta$ is an isometry from $\mathcal{H}_\pi$ to $L^2(G)$. Since $V_\eta$ is also an intertwiner, $\pi$ is a subrepresentation of $\lambda_\pi$. This establishes (3) of Proposition 4.2.

Now using the fact that $d_{\pi}^{1/2}V_\eta$ is an isometric intertwiner between $\pi$ and $\lambda_\pi$, $V_\eta\mathcal{H}_\pi$ is a submodule of the Hilbert $L(\Gamma, \sigma)$-module $L^2(G)$ from Proposition 4.1. Thus $\mathcal{H}_\pi$ becomes a Hilbert $L(\Gamma, \sigma)$-module isomorphic to $V_\eta\mathcal{H}_\pi$ via the action

$$a\xi := V_\eta^*aV_\eta\xi \quad \text{for } a \in L(\Gamma, \sigma) \text{ and } \xi \in \mathcal{H}_\pi.$$  

4.3. Computing the center-valued von Neumann dimension

In this subsection we will compute the center-valued von Neumann dimension of $\mathcal{H}_\pi$ as a Hilbert $L(\Gamma, \sigma)$-module. Our approach will be a modification of the approach in [10] with necessary changes needed to incorporate the 2-cocycle $\sigma$.

As before let $M = L(\Gamma, \sigma)$ and $N = R(\Gamma, \sigma)$. Let $\eta$ be any unit vector in $\mathcal{H}_\pi$. By the discussion in the previous section, $\mathcal{H}_\pi$ has the structure of a Hilbert $L(\Gamma, \sigma)$-module defined using the wavelet transform $V_\eta$.

Denoting by $U$ the intertwiner of Proposition 4.1, we have that $U^*V_\eta\mathcal{H}_\pi \subseteq \ell^2(\Gamma) \otimes K$, so let $p$ denote the projection of $\ell^2(\Gamma) \otimes K$ onto $U^*V_\eta\mathcal{H}_\pi$. Then $\text{cdim}_M \mathcal{H}_\pi = \Phi(p)$ where $\Phi$ is the faithful, semi-finite, normal extended center-valued trace of $N \otimes B(K)$ as defined in (3). Set $\tilde{p} = UpU^*$, i.e. $\tilde{p}$ is the orthogonal projection of $L^2(G)$ onto $V_\eta\mathcal{H}_\pi$.

Now $\Phi(p) = \sum_i \text{Tr}(p_{ii})$ where each $\text{Tr}(p_{ii})$ is a positive operator in $\mathcal{Z}(N) = \mathcal{Z}(M)$. By Section 3.3 they are all given by $\text{Tr}(p_{ii})f = \phi_i*\sigma f$ for $f \in \ell^2(\Gamma)$ where $\phi_i$ is the Fourier coefficient of $\text{Tr}(p_{ii})$. The values of $\phi_i$ can be expressed as $\phi_i(\gamma) = \tau(\rho_\sigma(\gamma)\text{Tr}(p_{ii}))$ for $\gamma \in \Gamma$. Summing these values over $i$ we obtain

$$\phi(\gamma) := \sum_i \tau(\rho_\sigma(\gamma)\text{Tr}(p_{ii})) = \tau(\rho_\sigma(\gamma)\sum_i \text{Tr}(p_{ii})) = \tau(\rho_\sigma(\gamma)\Phi(p)).$$

Note that for $a \in M$, $a \geq 0$, we have that

$$|\tau(a \Phi(p))| \leq \sum_i |\tau(a \text{Tr}(p_{ii}))| \leq \sum_i ||a||\tau(\text{Tr}(p_{ii})) = ||a||\dim_M \mathcal{H}_\pi.$$ 

Hence, if $\dim_M \mathcal{H}_\pi < \infty$, then the function $\phi$ is well-defined and in $\ell^\infty(\Gamma)$. Moreover, as a (possibly unbounded) operator $\Phi(p)$ acts as

$$\Phi(p)f = \sum_i \text{Tr}(p_{ii})f = \sum_i \phi_i*\sigma f = \phi*\sigma f$$ 

for $f \in \ell^2(\Gamma)$. Thus we can describe $\text{cdim}_M \mathcal{H}_\pi$ by describing $\phi$, and that is the content of the following theorem:

**Theorem 4.3.** The Hilbert $L(\Gamma, \sigma)$-module $\mathcal{H}_\pi$ has scalar-valued von Neumann dimension equal to

$$\dim_{L(\Gamma, \sigma)} \mathcal{H}_\pi = d_\pi \text{vol}(G/\Gamma)$$

which is finite since $\Gamma$ is a lattice in $G$. Furthermore, the center-valued von Neumann dimension of $\mathcal{H}_\pi$ is the (possibly unbounded) operator on $\ell^2(\Gamma)$ given by $f \mapsto f*\sigma \phi$ where $\phi \in \ell^\infty(\Gamma)$ is the function

$$\phi(\gamma) = \begin{cases} 
\frac{d_\pi}{|C_\gamma|} \int_{G/\Gamma_\gamma} \sigma(\gamma, y)\sigma(y, y^{-1}\gamma y)V_\eta(y^{-1}\gamma y)\text{d}(y\Gamma_\gamma) & \text{if } C_\gamma \text{ is } \sigma\text{-regular and finite}, \\
0 & \text{otherwise}. 
\end{cases}$$
Thus, \( V \) is an orthonormal set.

**Remark.** An implicit part of Theorem 4.3 is that for \( \sigma \)-regular \( \gamma \in \Gamma \), the function \( G \rightarrow \mathbb{C} \) given by \( y \mapsto \bar{\sigma}(\gamma, y)\sigma(y, y^{-1}\gamma y)V_\eta(y^{-1}\gamma y) \) is left \( \Gamma, \gamma \)-invariant, so that it makes sense to integrate this function over \( G/\Gamma, \gamma \).

Before proving Theorem 4.3 we need two lemmas. The proof of the first lemma contains only minor modifications of the proof in [10], but we include it for completeness.

**Lemma 4.4.** The scalar-valued von Neumann dimension of \( M\mathcal{H}_\pi \) is given by \( \dim_M \mathcal{H}_\pi = d_\pi \text{vol}(G/\Gamma) \). Moreover, if \( \Gamma \) is a lattice in \( G \) and \( \gamma \in \Gamma \) then

\[
\sum_i \tau(p_\gamma^i(\gamma)p_{\gamma i}) = d_\pi \int_B \bar{\sigma}(\gamma, y)\sigma(y, y^{-1}\gamma y)V_\eta(y^{-1}\gamma y) \, dy,
\]

where \( B \) is a fundamental domain for \( \Gamma \) in \( G \).

**Proof.** We use the notation of Proposition 4.1 and the following discussion. Thus we let \((e_i)_i\) be an orthonormal basis for \( K = L^2(B) \), and we denote by \( \tilde{e}_i \) the extension by zero of \( e_i \) to all of \( G \). Furthermore, let \((\eta_j)_j\) be an orthonormal basis for \( \mathcal{H}_\pi \). Let \( \eta \in \mathcal{H}_\pi \) have unit norm and set \( g_j = d_\pi^{1/2}V_\eta \eta_j \). Then \((g_j)_j\) is an orthonormal basis for \( V_\eta \mathcal{H}_\pi \) since \( d_\pi^{1/2}V_\eta \) is an isometry. As before \( \bar{p} \) denotes the projection of \( L^2(G) \) onto \( V_\eta \mathcal{H}_\pi \). We also let \( q \) denote the projection of \( L^2(G) \) onto \( U(\delta _\gamma \otimes K) \), which has orthonormal basis \((\tilde{e}_i)_i \).

We claim that the series

\[
\sum_{i, j} \langle \bar{e}_i, g_j \rangle \langle g_j, \lambda_\sigma^G(\gamma)\tilde{e}_i \rangle
\]

is absolutely convergent when \( \Gamma \) is a lattice in \( G \). Indeed, using Cauchy–Schwarz we obtain

\[
\sum_{i, j} |\langle \bar{e}_i, g_j \rangle \langle g_j, \lambda_\sigma^G(\gamma)\tilde{e}_i \rangle| \leq \sum_i \left( \sum_j |\langle \bar{e}_i, g_j \rangle|^2 \right)^{1/2} \left( \sum_j |\langle g_j, \lambda_\sigma^G(\gamma)\tilde{e}_i \rangle|^2 \right)^{1/2}
\]

\[
= \sum_i \|\bar{p}\tilde{e}_i\| \|\tilde{p}\lambda_\sigma^G(\gamma)\tilde{e}_i\| = \sum_i \|\bar{p}\tilde{e}_i\|^2 = \sum_i \sum_j |\langle \bar{e}_i, g_j \rangle|^2
\]

\[
= \sum_j \|q(g_j)\|^2 = \sum_j \int_B |g_j(y)|^2 \, dy = d_\pi \sum_j \int_B |\langle \eta_j, \pi(y)\eta \rangle|^2 \, dy
\]

\[
= d_\pi \int_B \sum_j |\langle \eta_j, \pi(y)\eta \rangle|^2 \, dy = d_\pi \int_B \|\pi(y)\|^2 \, dy
\]

\[
= d_\pi \text{vol}(G/\Gamma) < \infty.
\]

Thus, we can compute the sum in which order we like. Summing over \( j \) first we obtain

\[
\sum_{i, j} \langle \bar{e}_i, g_j \rangle \langle g_j, \lambda_\sigma^G(\gamma)\tilde{e}_i \rangle = \sum_i \left( \sum_j \langle \bar{e}_i, g_j \rangle g_j, \lambda_\sigma^G(\gamma)\tilde{e}_i \right) = \sum_i \langle \bar{p}\tilde{e}_i, \lambda_\sigma^G(\gamma)\tilde{e}_i \rangle
\]

\[
= \sum_i \langle p(\delta_e \otimes e_i), \delta_\gamma \otimes e_i \rangle = \sum_i \langle p_i, \delta_e, \delta_\gamma \rangle = \sum_i \tau(p_\gamma^i(\gamma)p_{\gamma i}).
\]
If we sum over $i$ first, instead, we obtain

$$\sum_{i,j} \langle \hat{e}_i, g_j \rangle \langle g_j, \lambda^G_\sigma(\gamma) \hat{e}_i \rangle = \sum_j \left( \lambda^G_\sigma(\gamma)^* g_j \sum_i \langle g_j, \hat{e}_i \rangle \hat{e}_i \right) = \sum_j \lambda^G_\sigma(\gamma)^* g_j, q(g_j)$$

$$= \sum_j \int_B \lambda^G_\sigma(\gamma)^* g_j(y) \overline{g_j(y)} \, dy = d_\pi \sum_j \int_B \langle \pi(\gamma)^* \eta_j, \pi(y) \eta \rangle \overline{\langle \eta_j, \pi(y) \eta \rangle} \, dy$$

$$= d_\pi \int_B \left( \sum_j \langle \pi(y) \eta, \eta_j \rangle \eta_j, \pi(\gamma) \pi(y) \eta \rangle \right) \, dy$$

$$= d_\pi \int_B \langle \pi(x) \eta, \sigma(\gamma, y) \pi(\gamma) \eta \rangle \, dy$$

$$= d_\pi \int_B \overline{\sigma(\gamma, y) \sigma(y, y^{-1} \gamma y)} \Pi_0 \eta \gamma^{-1} \gamma y \, dy.$$

This proves (21). In particular, when $\gamma = e$ we get

$$\dim_M H_\pi = \sum_i \tau(p_{ii}) = d_\pi \int_B \sigma(\gamma, e) \sigma(y, y^{-1} e y) \Pi_0 \eta (y^{-1} e y) \, dy = d_\pi \|\eta\| \int_B \, dy = d_\pi \text{ vol}(G/\Gamma). \qed$$

**Lemma 4.5.** Let $B$ be a fundamental domain for $\Gamma$ in $G$ and suppose $\gamma \in \Gamma$ is such that the conjugacy class $C_\gamma$ is $\sigma$-regular and finite, say $C_\gamma = \{ \beta_1^{-1} \gamma \beta_1, \ldots, \beta_k^{-1} \gamma \beta_k \}$. Then

$$\tilde{B} = \bigcup_{j=1}^k \beta_j B$$

is a fundamental domain for the $\Gamma_\gamma$ (the centralizer of $\gamma$ in $\Gamma$) in $G$.

**Proof.** First suppose that $\tilde{B} \cap \gamma' \tilde{B} \neq \emptyset$ for some $\gamma' \in \Gamma_\gamma$. Then there exist $1 \leq i, j \leq k$ such that $\beta_i B \cap \gamma' \beta_j B \neq \emptyset$. Using that $B$ is a fundamental domain for $\Gamma$ in $G$ this implies that $\beta_i = \gamma' \beta_j$. But then

$$\beta_i^{-1} \gamma \beta_i = \beta_j^{-1} \gamma' \beta_j = \beta_j^{-1} \gamma \beta_j.$$

This forces $i = j$ which gives $B \cap \gamma' B \neq \emptyset$. This can only happen when $\gamma' = e$.

Let $x \in G$. Using that $B$ is a fundamental domain for $\Gamma$ in $G$ we can write $x = \gamma'' y$ where $\gamma'' \in \Gamma$ and $y \in B$. There exists $1 \leq j \leq k$ such that $\gamma''^{-1} \gamma \gamma'' = \beta_j^{-1} \gamma \beta_j$. But then $\gamma'' \beta_j^{-1} \in \Gamma_\gamma$ so $\gamma'' = \gamma' \beta_j$ for some $\gamma' \in \Gamma_\gamma$. Hence $x = \gamma'(\beta_j y) \in \gamma' \tilde{B}$. \qed

Now that we have Lemma 4.4 as well as the formula for the center-valued trace in Proposition 3.2, we are ready to prove Theorem 4.3.

**Proof of Theorem 4.3.** By Lemma 4.4 we know that the scalar-valued von Neumann dimension of $M H_\pi$ is equal to $d_\pi \text{ vol}(G/\Gamma)$. Using the relation between $\tau$ and the center-valued trace in (2) we have that

$$\phi(\gamma) = \tau(\rho_\sigma(\gamma) \Phi(p)) = \sum_i \tau(\rho_\sigma(\gamma) \text{ Tr}(p_{ii})) = \sum_i \tau(\text{ Tr}(\rho_\sigma(\gamma))) p_{ii}.$$
Using the formula for the center-valued trace on $N$ \((17)\) (and making sure to conjugate the 2-cocycle) as well as Lemma 4.4 we obtain $\phi(\gamma) = 0$ when $C_{\gamma}$ is infinite or not $\sigma$-regular. When $C_{\gamma}$ is both finite and $\sigma$-regular we obtain
\[
\phi(\gamma) = |C_{\gamma}|^{-1} \sum_{\iota} \sum_{j=1}^{k} \bar{\sigma}(\gamma, \beta_j) \tau(\rho_{\beta_j}(\beta_j^{-1}\gamma\beta_j)p_{\iota})
\]
\[
= d_{\pi}|C_{\gamma}|^{-1} \sum_{j=1}^{k} \int_{B} \bar{\sigma}(\beta_j^{-1}\gamma\beta_j, y)V_{\eta}\eta(y^{-1}\beta_j^{-1}\gamma\beta_jy)dy
\]
\[
= d_{\pi}|C_{\gamma}|^{-1} \sum_{j=1}^{k} \int \bar{\sigma}(\beta_j y)V_{\eta}\eta(y^{-1}\beta_j^{-1}\gamma\beta_jy)dy
\]
\[
= d_{\pi}|C_{\gamma}|^{-1} \sum_{j=1}^{k} \int \bar{\sigma}(\gamma, y)V_{\eta}\eta(y^{-1}\gamma y)dy.
\]
Hence, using the definition of $\hat{B}$ from Lemma 4.5, we obtain
\[
\phi(\gamma) = d_{\pi}|C_{\gamma}|^{-1} \int_{\hat{B}} \bar{\sigma}(\gamma, y)V_{\eta}\eta(y^{-1}\gamma y)dy.
\]
Note that the function $y \mapsto \bar{\sigma}(\gamma, y)V_{\eta}\eta(y^{-1}\gamma y)$ is left $\Gamma_{\gamma}$-invariant. Indeed, if $\gamma' \in \Gamma_{\gamma}$ then $\gamma'^{-1}\gamma\gamma' = e^{-1}\gamma e$ so Lemma 3.1 \((3)\) gives that $\bar{\sigma}(\gamma, \gamma') = \bar{\sigma}(\gamma, e) = 1$ since $\gamma$ is $\sigma$-regular. Hence, using Lemma 3.1 \((1)\), we get
\[
\bar{\sigma}(\gamma, \gamma')V_{\eta}\eta(y^{-1}\gamma^{-1}\gamma' y) = \bar{\sigma}(\gamma, \gamma')\bar{\sigma}(\gamma^{-1}\gamma', y)V_{\eta}\eta(y^{-1}\gamma y) = \bar{\sigma}(\gamma, y)V_{\eta}\eta(y^{-1}\gamma y).
\]
Thus, since $\hat{B}$ is a fundamental domain for $\Gamma_{\gamma}$ in $G$ by Lemma 4.5, we can integrate over $G/\Gamma_{\gamma}$ instead of $\hat{B}$. This leaves us with the formula in Theorem 4.3, finishing the proof. \(\square\)

For 2-cocycles on abelian groups satisfying Kleppner’s condition the center-valued von Neumann dimension takes a particularly simple form:

**Corollary 4.6.** Suppose that $G$ is abelian and that $(G, \sigma)$ satisfies Kleppner’s condition. Then the center-valued von Neumann dimension of $L(\Gamma, \sigma)H_{\pi}$ is given by
\[
cdim_{L(\Gamma, \sigma)} H_{\pi} = d_{\pi} \text{vol}(G/\Gamma)I.
\]

**Proof.** When $G$ is abelian we have that $C_{\gamma} = \{\gamma\}$ and $\Gamma_{\gamma} = \Gamma$ for every $\gamma \in \Gamma$. Hence the expression in Theorem 4.3 collapses to
\[
\phi(\gamma) = d_{\pi} \int_{G/\Gamma} \bar{\sigma}(\gamma, y)V_{\eta}\eta(y^{-1}\gamma y)dy(\gamma y)
\]
\[
= d_{\pi} \{\eta, \pi(\gamma)\eta\} \int_{G/\Gamma} \bar{\sigma}(\gamma, y)\eta(y, \gamma)dy(\gamma y).
\]
The map $y \Gamma \mapsto \overline{\sigma(\gamma,y)\sigma(y,\gamma)}$ is a character on $G/\Gamma$ by Lemma 3.1. Since $(G,\sigma)$ is assumed to satisfy Kleppner’s condition, this character is trivial if and only if $\gamma = e$. Hence

$$\phi(\gamma) = d_\pi \langle \eta, \pi(\gamma)\eta \rangle \text{vol}(G/\Gamma)\delta_{\gamma,e} = d_\pi \text{vol}(G/\Gamma)\delta_{\gamma,e}. $$

Since $\text{cdim}_M \mathcal{H}_\pi$ is uniquely determined by $\phi$ it follows that

$$\text{cdim}_{L(\Gamma,\sigma)} \mathcal{H}_\pi = d_\pi \text{vol}(G/\Gamma)I. \quad \square$$

5. Applications to frame theory

5.1. Frames and Riesz sequences

Let $\mathcal{H}$ be a (complex) Hilbert space and $J$ an index set. A family $(e_j)_{j \in J}$ in $\mathcal{H}$ is a frame for $\mathcal{H}$ if there exist constants $A, B > 0$ such that

$$A \|\xi\|^2 \leq \sum_{j \in J} |\langle \xi, e_j \rangle|^2 \leq B \|\xi\|^2 \text{ for all } \xi \in \mathcal{H}. $$

The numbers $A$ and $B$ are called lower and upper frame bounds, respectively. If one can choose $A = B = 1$ in the above equation, the frame $(e_j)_{j \in J}$ is called Parseval.

Associated to a frame $(e_j)_{j \in J}$ is the analysis operator, which is the injective bounded linear operator $C : \mathcal{H} \to \ell^2(\Gamma)$ given by

$$C \xi = ((\xi, e_j))_{j \in J} \text{ for } \xi \in \mathcal{H}. $$

When the frame is Parseval, the analysis operator is an isometry. The frame operator is the positive invertible operator $S = C^* C \in B(\mathcal{H})$ and the associated family $S^{-1/2} e_j$ is a Parseval frame. Conversely, if $C : \mathcal{H} \to \ell^2(J)$ is an isometry, then we obtain a Parseval frame $(e_j)_{j \in J}$ in $\mathcal{H}$ where $Ce_j$ is the orthogonal projection of $\delta_j \in \ell^2(J)$ onto the subspace $C \mathcal{H} \subseteq \ell^2(J)$. Thus, the existence of a frame in $\mathcal{H}$ indexed by $J$ is equivalent to the existence of an isometry $\mathcal{H} \to \ell^2(J)$.

The dual notion to a frame is that of a Riesz sequence. A family $(e_j)_{j \in J}$ is called a Riesz sequence for $\mathcal{H}$ if there exist constants $A, B > 0$ such that

$$A \|c\|^2 \leq \sum_{j \in J} |c_j e_j|^2 \leq B \|c\|^2 \text{ for all } c = (c_j)_{j \in J} \in \ell^2(J).$$

The numbers $A$ and $B$ are called lower and upper Riesz bounds, respectively. Note that an orthonormal family is precisely a Riesz sequence for which one can choose $A = B = 1$. A family $(e_j)_j$ that is both a frame and a Riesz sequence is called a Riesz basis.

Associated to a Riesz sequence is the synthesis operator $D : \ell^2(J) \to \mathcal{H}$ given by

$$D(c_j)_j = \sum_j c_j e_j \text{ for } (c_j)_{j \in J} \in \ell^2(J),$$

which is an injective bounded linear operator. It is isometric when $(e_j)_j$ is orthonormal. A Riesz sequence is always a Riesz basis (in particular a frame) for its closed linear span $K = \text{span} \{e_j : j \in J\}$, so the restriction $S|_K$ of its frame operator $S$ to $K$ is invertible. The associated family $(S^{-1/2} e_j)_{j \in J}$ is then orthonormal. Conversely, if $D : \ell^2(J) \to \mathcal{H}$ is an isometry, then $(D\delta_j)_{j \in J}$ is orthonormal. This shows that the existence of a Riesz sequence in $\mathcal{H}$ indexed by $J$ is equivalent to the existence of an isometry $\ell^2(J) \to \mathcal{H}$.
5.2. Multiwindow super systems

Let \((\pi, \mathcal{H}_\pi)\) be a \(\sigma\)-projective unitary representation of a locally compact group \(G\) and let \(\Gamma\) be a lattice in \(G\). We will be interested in frames and Riesz sequences for \(\mathcal{H}_\pi\) of the form

\[
\pi(\Gamma)\eta = (\pi(\gamma)\eta)_{\gamma \in \Gamma}
\]

for vectors \(\eta \in \mathcal{H}_\pi\). More generally, we define the \(n\)-multiwindow \(d\)-super system associated to a matrix \((\eta_{i,j})_{i,j=1}^{n,d}\) of vectors in \(\mathcal{H}_\pi\) to be the \(\Gamma \times \{1, \ldots, n\}\)-indexed family

\[
\left((\pi(\gamma)\eta_{i,1}, \ldots, \pi(\gamma)\eta_{i,d})\right)_{\gamma \in \Gamma, 1 \leq i \leq n}
\]

in \(\mathcal{H}_\pi^d\). If an \(n\)-multiwindow \(d\)-super system is a frame for \(\mathcal{H}_\pi^d\) we call it an \(n\)-multiwindow \(d\)-super frame. We will say that \((\pi, \Gamma)\) admits an \(n\)-multiwindow \(d\)-super frame if there exists an \(n\)-multiwindow \(d\)-super frame of the form (23) for some \((\eta_{i,j})_{i,j=1}^{n,d}\), and we call \((\eta_{i,j})_{i,j=1}^{n,d}\) the generators of the frame. We make analogous definitions for Riesz sequences and Riesz bases.

If \(d = 1\) we obtain the \(n\)-multiwindow system \((\pi(\gamma)\eta_i)_{\gamma \in \Gamma, 1 \leq i \leq n}\) and if \(n = 1\) we obtain the \(d\)-super system \((\pi(\gamma)\eta_1, \ldots, \pi(\gamma)\eta_n)\) \(\gamma \in \Gamma\). If both \(n = d = 1\) we recover the system \(\pi(\Gamma)\eta\).

We need the following representation-theoretic characterizations of the existence of multiwindow super frames and Riesz sequences.

**Proposition 5.1.** The following are equivalent:

1. \((\pi, \Gamma)\) admits an \(n\)-multiwindow \(d\)-super frame (resp. \(n\)-multiwindow \(d\)-super Riesz sequence) (resp. \(n\)-multiwindow \(d\)-super Riesz basis).
2. \((\pi, \Gamma)\) admits a \(n\)-multiwindow \(d\)-super Parseval frame (resp. \(n\)-multiwindow \(d\)-super orthonormal sequence) (resp. \(n\)-multiwindow \(d\)-super orthonormal basis).
3. There exists \(\Gamma\)-invariant isometry \(\mathcal{H}_\pi^d \rightarrow \ell^2(\Gamma)^n\) (resp. \(\Gamma\)-invariant isometry \(\ell^2(\Gamma)^n \rightarrow \mathcal{H}_\pi^d\)) (resp. \(\Gamma\)-invariant unitary map \(\mathcal{H}_\pi^d \rightarrow \ell^2(\Gamma)^n\)).

**Proof.** Let \((\eta_{i,j})_{i,j=1}^{n,d}\) be the generators of an \(n\)-multiwindow \(d\)-super frame associated to \((\pi, \Gamma)\). Let \(C\) be the associated analysis operator. Then \(C\pi(\gamma) = \lambda_d(\gamma)C\) for all \(\gamma \in \Gamma\). Consequently, the frame operator \(S = C^*C\) commutes with \(\pi(\gamma)\) for \(\gamma \in \Gamma\), so the elements of the associated Parseval frame are of the form

\[
S^{-1/2}(\pi(\gamma)\eta_{i,1}, \ldots, \pi(\gamma)\eta_{i,d}) = (\pi(\gamma)S^{-1/2}\eta_{i,1}, \ldots, \pi(\gamma)S^{-1/2}\eta_{i,d})
\]

for \(\gamma \in \Gamma\) and \(1 \leq i \leq n\). In other words, the matrix of vectors \((S^{-1/2}\eta_{i,j})_{i,j=1}^{n,d}\) generates a \(n\)-multiwindow \(d\)-super Parseval frame.

If \((\eta_{i,j})_{i,j=1}^{n,d}\) are the generators of a \(n\)-multiwindow \(d\)-super Parseval frame then the coefficient operator \(C\) is a \(\Gamma\)-invariant isometry \(\mathcal{H}_\pi^d \rightarrow \ell^2(\Gamma)^n\). Conversely, suppose \(C: \mathcal{H}_\pi^d \rightarrow \ell^2(\Gamma)^n\) is a \(\Gamma\)-invariant isometry. Let \(e_i\) be the vector \((0, \ldots, 0, 0, \ldots, 0)\) \(\in \ell^2(\Gamma)^n\) where \(0\) is in the \(i\)th position. Then \(\{\lambda^G_\sigma(\gamma)e_i : \gamma \in \Gamma, 1 \leq i \leq n\}\) is an orthonormal basis for \(\ell^2(\Gamma)^n\). Consequently, if \(P\) denotes the projection of \(\ell^2(\Gamma)^n\) onto \(C(\mathcal{H}_\pi^d)\), then the vectors \(P\lambda^G_\sigma(\gamma)e_i = \pi(\gamma)Pe_i\) for \(\gamma \in \Gamma\) and \(1 \leq i \leq n\) form a Parseval frame for \(\mathcal{H}_\pi^d\). Letting \(Pe_i = (\eta_{i,1}, \ldots, \eta_{i,d})\), the matrix \((\eta_{i,j})_{i,j=1}^{n,d}\) generates a \(n\)-multiwindow \(d\)-super Parseval frame.

The arguments for Riesz sequences are similar. If \((\eta_{i,j})_{i,j=1}^{n,d}\) are the generators of an \(n\)-multiwindow \(d\)-super Riesz sequence then the frame operator \(S\) restricted to the closed linear span of the Riesz sequence is \(\Gamma\)-invariant, so the vectors \((S^{-1/2}\eta_{i,j})_{i,j=1}^{n,d}\) are the generators of an \(n\)-multiwindow \(d\)-super orthonormal
family. The corresponding analysis operator is a $\Gamma$-invariant isometry $\ell^2(\Gamma)^n \to \mathcal{H}^d_\pi$. Conversely, for a $\Gamma$-invariant isometry $D: \ell^2(\Gamma)^n \to \mathcal{H}^d_\pi$, the vectors $(\eta_{i,j})_{i,j=1}^{n,d}$ are the generators of an $n$-multiwindow $d$-super orthonormal family, where $D\delta_i = (\eta_{i,1}, \ldots, \eta_{i,d})$ for $1 \leq i \leq n$. □

We will also need the following generalization of [40, Proposition 7.6]. The strategy of the proof is the same.

**Proposition 5.2.** Suppose that $(\pi, \Gamma)$ admits an $n$-multiwindow $d$-super frame. If $d_\pi \operatorname{vol}(G/\Gamma) = n/d$, then $(\pi, \Gamma)$ admits an $n$-multiwindow $d$-super Riesz basis.

**Proof.** By Proposition 5.1, we can assume that there exist $(\eta_{i,j})_{i,j=1}^{n,d}$ that are generators of an $n$-multiwindow $d$-super Parseval frame. Let $B$ be a fundamental domain for $\Gamma$ in $G$, so that $\{B\gamma : \gamma \in \Gamma\}$ is a partition of $G$. Then for any $y \in B$ and $(\xi_j)_j^{d_1=1} \in \mathcal{H}_\pi$, we have that

$$\sum_{i=1}^{n} \sum_{\gamma \in \Gamma} |\langle (\eta_{i,j})_{i=1}^{n}, (\pi(y)\eta_{i,j})_{j=1}^{d}\rangle|^2 = \sum_{i=1}^{n} \sum_{\gamma \in \Gamma} |\langle (\pi(y)^*\xi_j)_{j=1}^{d}, (\pi(y)\eta_{i,j})_{j=1}^{d}\rangle|^2 = \|\pi(y)^*\xi_j\|^2 = \|\xi_j\|^2.$$ Integrating this equality over $y \in B$ and using the orthogonality relation in Proposition 4.2, we have that

$$\|\xi_j\|^2 \operatorname{vol}(G/\Gamma) = \int_B \|\xi_j\|^2 \, dx$$

$$= \sum_{i=1}^{n} \int_B \sum_{\gamma \in \Gamma} |\langle (\xi_j)_{j}, (\pi(y)\eta_{i,j})_{j}\rangle|^2 \, dx$$

$$= \int_G \sum_{i=1}^{n} |\langle (\xi_j)_{j}, (\pi(x)\eta_{i,j})_{j}\rangle|^2 \, dx$$

$$= \int_G \sum_{i=1}^{n} \sum_{j,j'=1}^{d} \langle \xi_j, (\pi(x)\eta_{i,j'}) \rangle \langle \xi_j, (\pi(x)\eta_{i,j}) \rangle \, dx$$

$$= d_\pi^{-1} \sum_{i=1}^{n} \sum_{j,j'=1}^{d} \langle \xi_j, \xi_j' \rangle \langle \pi(x)\eta_{i,j'}, \pi(x)\eta_{i,j} \rangle.$$ Picking $\xi_1, \ldots, \xi_n$ so that $\langle \xi_j, \xi_j' \rangle = \delta_{j,j'}$ for $1 \leq j, j' \leq d$, we get

$$d \operatorname{vol}(G/\Gamma) = \|\xi_j\|^2 \operatorname{vol}(G/\Gamma) = d_\pi^{-1} \sum_{i=1}^{n} \sum_{j=1}^{d} \|\pi(x)\eta_{i,j}\|^2 = d_\pi^{-1} \sum_{i=1}^{n} \sum_{j=1}^{d} \|\eta_{i,j}\|^2.$$ Since we assume that $d_\pi \operatorname{vol}(G/\Gamma) = n/d$, we get

$$\sum_{i=1}^{n} \sum_{j=1}^{d} \|\eta_{i,j}\|^2 = n. \quad (24)$$ For each $1 \leq i \leq n$, the vector $(\eta_{i,j})_{j=1}^{d} \in \mathcal{H}_\pi^d$ is a member of a Parseval frame, hence $\|\eta_{i,j}\|^2 = \sum_{j=1}^{d} \|\eta_{i,j}\|^2 \leq 1$. Combining this with (24), we must have $\|\eta_{i,j}\| = 1$ for each $1 \leq i \leq n$. But then every
5.3. The density theorem and converses

Using the results on the center-valued von Neumann dimension of $\mathcal{H}_\pi$ as a Hilbert module over the $\sigma$-twisted group von Neumann algebra $L(\Gamma, \sigma)$, we can now characterize the existence of $n$-multiwindow $d$-super frames in terms of the function $\phi$ from Theorem 4.3.

**Theorem 5.3.** Let $G$ be a second-countable, unimodular, locally compact group, let $\sigma$ be a 2-cocycle on $G$, and let $(\pi, \mathcal{H}_\pi)$ be a $\sigma$-projective, irreducible, square-integrable, unitary representation of $G$. Let $\Gamma$ be a lattice in $G$. Let $\phi$ be as in Theorem 4.3. Then the following hold:

1. $(\pi, \Gamma)$ admits an $n$-multiwindow $d$-super frame if and only if $(n/d)\delta_e - \phi$ is a $\sigma$-positive definite function.
2. $(\pi, \Gamma)$ admits an $n$-multiwindow $d$-super Riesz sequence if and only if $\phi - (n/d)\delta_e$ is a $\sigma$-positive definite function.
3. $(\pi, \Gamma)$ admits an $n$-multiwindow $d$-super Riesz basis if and only if $\phi = (n/d)\delta_e$.

**Proof.** Set $M = L(\Gamma, \sigma)$. By Proposition 5.1, $(\pi, \Gamma)$ admits an $n$-multiwindow $d$-super frame if and only if there is an $\pi|_\Gamma$-invariant isometry $\mathcal{H}_\pi^d \rightarrow \ell^2(\Gamma)^n$. This is the case if and only if $M \mathcal{H}_\pi^d$ is a submodule of $M \ell^2(\Gamma)^n$. By Proposition 2.3 this is the case if and only if

$$d \cdim_M \mathcal{H}_\pi = \cdim_M \mathcal{H}_\pi^d \leq \cdim_M \ell^2(\Gamma)^n = nI.$$

By Theorem 4.3 the center-valued von Neumann dimension $T := \cdim_M \mathcal{H}_\pi$ is given by convolution with the function $\phi$, hence determined by the values $\phi(\gamma) = \tau(\rho_\sigma(\gamma)T)$ for $\gamma \in \Gamma$. The condition $T \leq (n/d)I$ is equivalent to $\langle((n/d)\delta_e - \phi)*_\sigma f, f\rangle = \langle((n/d)I - T)f, f\rangle \geq 0$ for all finitely supported functions $f$ on $\Gamma$. By Proposition 3.4 this happens exactly when $(n/d)\delta_e - \phi$ is $\sigma$-positive definite.

Using Proposition 5.1 in a similar manner shows that $(\pi, \Gamma)$ admits an $n$-multiwindow $d$-super Riesz sequence if and only if $\phi - (n/d)\delta_e$ is $\sigma$-positive definite. Combining the statements for frames and Riesz sequences, we see that if $(\pi, \Gamma)$ admits an $n$-multiwindow $d$-super Riesz basis, then $\phi = (n/d)\delta_e$. Conversely, suppose $\phi = (n/d)\delta_e$. Then in particular, $d_\pi \vol(G/\Gamma) = n/d$. Since $\phi$ is $\sigma$-positive definite, $(\pi, \Gamma)$ admits an $n$-multiwindow $d$-super frame by what we already proved. By Proposition 5.2, $(\pi, \Gamma)$ then admits an $n$-multiwindow $d$-super Riesz basis. $\square$

As a corollary we get the following generalization of the density theorem from [40] to $n$-multiwindow $d$-super systems:

**Theorem 5.4.** Let $G$ be a second-countable, unimodular, locally compact group, let $\sigma$ be a 2-cocycle on $G$, and let $(\pi, \mathcal{H}_\pi)$ be a $\sigma$-projective, irreducible, square-integrable, unitary representation of $G$. Let $\Gamma$ be a lattice in $G$. Then the following hold:

1. If $(\pi, \Gamma)$ admits an $n$-multiwindow $d$-super frame, then $d_\pi \vol(G/\Gamma) \leq n/d$.
2. If $(\pi, \Gamma)$ admits an $n$-multiwindow $d$-super Riesz sequence, then $d_\pi \vol(G/\Gamma) \geq n/d$.
3. If $(\pi, \Gamma)$ admits an $n$-multiwindow $d$-super Riesz basis, then $d_\pi \vol(G/\Gamma) = n/d$.

**Proof.** Follows from Theorem 5.3 and the fact that $d_\pi \vol(G/\Gamma) = \cdim_M \mathcal{H}_\pi = \tau(\cdim_M \mathcal{H}_\pi)$. $\square$
Combining Corollary 4.6 and Theorem 5.3 we get another immediate corollary, which gives a complete converse to the density theorem when \( G \) is abelian and \((G, \sigma)\) satisfies Kleppner’s condition.

**Theorem 5.5.** Let \( G \) be a second-countable, abelian, locally compact group, let \( \sigma \) be a 2-cocycle on \( G \), and let \((\pi, \mathcal{H}_\pi)\) be a \( \sigma \)-projective, irreducible, square-integrable, unitary representation of \( G \). Suppose that \((G, \sigma)\) satisfies Kleppner’s condition. Let \( \Gamma \) be a lattice in \( G \). Then the following hold:

1. \((\pi, \Gamma)\) admits an \( n \)-multiwindow \( d \)-super frame if and only if \( d_\pi \text{vol}(G/\Gamma) \leq n/d \).
2. \((\pi, \Gamma)\) admits an \( n \)-multiwindow \( d \)-super Riesz sequence if and only if \( d_\pi \text{vol}(G/\Gamma) \geq n/d \).
3. \((\pi, \Gamma)\) admits an \( n \)-multiwindow \( d \)-super Riesz basis if and only if \( d_\pi \text{vol}(G/\Gamma) = n/d \).

### 6. Examples

#### 6.1. Holomorphic discrete series representations

We present an example of a projective representation for which the associated center-valued von Neumann dimension does not collapse to the scalar operator \( d_\pi \text{vol}(G/\Gamma) \).

The connected, simple Lie group \( G = SL(2, \mathbb{R}) \) acts on the complex upper half-plane \( \mathbb{C}^+ = \{ x + iy \in \mathbb{C} : y > 0 \} \) via

\[
A \cdot z = \frac{az + b}{cz + d}, \quad A = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in SL(2, \mathbb{R}), \ z \in \mathbb{C}^+.
\]

For \( \alpha > 1 \), let \( A_{\alpha}^2(\mathbb{C}^+) \) be the Bergman space of holomorphic functions \( \xi \) on \( \mathbb{C}^+ \) satisfying

\[
\|\xi\|_{A_{\alpha}^2}^2 := \int_{\mathbb{C}^+} |\xi(x + iy)|^2 y^{\alpha - 2} \, dx \, dy < \infty.
\]

The representation of \( G \) on \( A_{\alpha}^2(\mathbb{C}^+) \) given by

\[
\pi_\alpha(A)\xi(z) = (cz + d)^{-\alpha}\xi(A^{-1}z), \quad A = \begin{pmatrix} a & b \\ c & d \end{pmatrix}, \ \xi \in A_{\alpha}^2(\mathbb{C}^+),
\]

is known as a *holomorphic discrete series representation* of \( G \), cf. [37] or [40, Example 9.2] for details. Because of the need to choose a branch of the logarithm function to define \((cz + d)^{-\alpha}\), \( \pi_\alpha \) is in general projective when \( \alpha \notin \mathbb{Z} \). With Haar measure on \( G \) normalized as in [40, Example 9.2] the formal dimension of \( \pi \) becomes \( d_\pi = (\alpha - 1)/(8\pi) \).

Let \( \Gamma \) be a lattice in \( G \). We compute \( \phi \) from Theorem 1.3 with respect to \( \pi_\alpha \) and \( \Gamma \) (for \( \alpha \in \mathbb{Z} \), \( \phi \) was computed in [10, Example 1]). Suppose \( A \in \Gamma \) is \( \sigma \)-regular in \( \Gamma \) and has finite conjugacy class. By [10, Theorem 2], \( A \) belongs to the center \( \{-I, I\} \) of \( G \). We know that \( \phi(I) = d_\pi \text{vol}(G/\Gamma) \), so it remains to check \( A = -I \). A direct computation shows that \( \pi_\alpha(-I) \) commutes with \( \pi_\alpha(B) \) for all \( B \in G \), hence

\[
\pi_\alpha(A)\pi_\alpha(-I) = \sigma(B, -I)\pi_\alpha(-I)\pi_\alpha(B) = \sigma(B, -I)\pi_\alpha(-I)\pi_\alpha(B) = \sigma(B, -I)\pi_\alpha(-I)\pi_\alpha(B).
\]

It follows that \( \sigma(B, -I) = \sigma(-B, B) \) for all \( B \in G \). Since \( \pi(-I) = (-1)^{-\alpha} \), we get

\[
\phi(-I) = d_\pi \int_{G/\Gamma} \overline{\sigma(-B, B)\sigma(B, -I)\eta, \pi_\alpha(-I)\eta} \, d(B\Gamma) = (-1)^{\alpha} d_\pi \text{vol}(G/\Gamma).
\]
Hence
\[
\phi(A) = \begin{cases} 
    d_{\pi_\alpha} \text{vol}(G/\Gamma), & \text{if } A = I, \\
    (-1)^\alpha d_{\pi_\alpha} \text{vol}(G/\Gamma), & \text{if } A = -I, \\
    0, & \text{otherwise}.
\end{cases}
\]

Let us draw conclusions about the existence of frames and Riesz sequences of the form \(\pi_\alpha(\Gamma)\eta\). The function \(\delta_e - \phi\) on \(\Gamma\) is \(\sigma\)-positive definite if and only if \(I - M\) is a positive semidefinite matrix, where
\[
M = (\sigma(BA^{-1}, A)\phi(BA^{-1}))_{A,B\in\{I,-I\}} = \begin{pmatrix} d_{\pi_\alpha} \text{vol}(G/\Gamma) & (-1)^\alpha d_{\pi_\alpha} \text{vol}(G/\Gamma) \\ (-1)^{-\alpha} d_{\pi_\alpha} \text{vol}(G/\Gamma) & d_{\pi_\alpha} \text{vol}(G/\Gamma) \end{pmatrix}.
\]

This happens if and only if both \(\text{tr}(I - M) \geq 0\) and \(\det(I - M) \geq 0\), which translates into the condition \(d_{\pi_\alpha} \text{vol}(G/\Gamma) \leq 1/2\). Hence by Theorem 1.3, there exists a frame of the form \(\pi_\alpha(\Gamma)\eta\) if and only if \(d_{\pi_\alpha} \text{vol}(G/\Gamma) \leq 1/2\), which is stronger than the condition \(d_{\pi_\alpha} \text{vol}(G/\Gamma) \leq 1\). This shows that Problem 1.2 does not have a positive answer in general.

By a similar matrix computation or the use of [11, Proposition C.1.2], \(\phi - \delta_e\) can never be \(\sigma\)-positive definite as this would imply
\[
d_{\pi_\alpha} \text{vol}(G/\Gamma) = |(\phi - \delta_e)(-I)| \leq (\phi - \delta_e)(I) = d_{\pi_\alpha} \text{vol}(G/\Gamma) - 1,
\]

a contradiction. Hence there are no Riesz sequences of the form \(\pi_\alpha(\Gamma)\eta\). Another way to see this is to note that \(\pi_\alpha(\Gamma)\eta\) is always linearly independent since \(\pi_\alpha(-I)\eta = (-1)^{-\alpha} \eta = (-1)^{-\alpha} \pi_\alpha(I)\eta\), cf. [40, Example 9.2].

### 6.2. Gabor analysis

We end with an application to Gabor analysis on locally compact abelian groups [20]. Let \(A\) be a second-countable, locally compact abelian group with Pontryagin dual \(\hat{A}\) and set \(G = A \times \hat{A}\). There is a natural choice of normalization of the Haar measure on \(G\) given as follows: Pick any Haar measure \(\mu\) on \(A\) and the corresponding dual measure \(\hat{\mu}\) on \(\hat{A}\) such that Plancherel’s formula holds:
\[
\int_A |f(x)|^2 \, d\mu(x) = \int_{\hat{A}} |\hat{f}(\omega)|^2 \, d\hat{\mu}(\omega), \quad f \in L^1(A) \cap L^2(A).
\]

The product measure on \(G = A \times \hat{A}\) associated with \(\mu\) and \(\hat{\mu}\) is independent of the choice \(\mu\) of Haar measure on \(A\), and we will assume this measure on \(G\).

The Weyl–Heisenberg 2-cocycle of \(G\) is given by
\[
\sigma((x,\omega),(x',\omega')) = \overline{\omega}(x) \text{ for } (x,\omega), (x',\omega') \in G.
\]

Note that \((G,\sigma)\) satisfies Kleppner’s condition: Indeed, suppose \((x,\omega)\in G\) is such that \(\sigma((x,\omega),(x',\omega')) = \sigma((x',\omega'),(x,\omega))\) for all \((x',\omega')\in G\). Then \(\omega'(x) = \omega(x')\) for all \(x'\in A\) and \(\omega'\in \hat{A}\). Setting \(\omega' = 1\) gives that \(\omega\) is the trivial character and setting \(x' = 1\) gives that \(\omega'(x) = 1\) for all \(\omega'\in \hat{A}\) which implies that \(x = e\) by Pontryagin duality.

The Weyl–Heisenberg representation is the square-integrable, irreducible, \(\sigma\)-projective representation of \(G\) on \(L^2(A)\) given by
\[
\pi(x,\omega)\xi(t) = \omega(t)\xi(x^{-1}t) \text{ for } (x,\omega)\in G, \xi \in L^2(A) \text{ and } t \in A.
\]
The orthogonality relations for the short-time Fourier transform ([20]) yield that \( d_s = 1 \). In this setting a system of the form \( \pi(\Gamma)\eta \) for some \( \eta \in L^2(A) \) and \( \Gamma \) a lattice in \( G = A \times \hat{A} \) is called a Gabor system. If it has the frame property in \( L^2(A) \) we call it a Gabor frame, and similarly for Riesz sequences and Riesz bases. We also speak of \( n \)-multiwindow \( d \)-super Gabor frames and Gabor Riesz bases where the definitions are according to Section 5.2. The following theorem is an immediate consequence of Theorem 5.5.

**Theorem 6.1.** Let \( A \) be a second-countable, locally compact abelian group, and let \( \Gamma \) be a lattice in \( G = A \times \hat{A} \). Then the following hold:

1. There exists an \( n \)-multiwindow \( d \)-super Gabor frame over \( \Gamma \) if and only if \( \text{vol}(G/\Gamma) \leq n/d \).
2. There exists an \( n \)-multiwindow \( d \)-super Gabor Riesz sequence over \( \Gamma \) if and only if \( \text{vol}(G/\Gamma) \geq n/d \).
3. There exists an \( n \)-multiwindow \( d \)-super Gabor Riesz basis over \( \Gamma \) if and only if \( \text{vol}(G/\Gamma) = n/d \).

The above theorem applies e.g. to the case where \( A \) is the adele group of a global field which was studied in [15,16].

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