Generating Solutions to the Einstein Field Equations

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Abstract

New solutions to the Einstein field equations may be generated from already existing ones (seed solutions), that admit at least one Killing vector, through the use of symmetries in the potential space. In this framework, the set of the old potentials are transformed into a new set, either by continuous transformations or by discrete transformations. Accordingly, we demonstrate a method of obtaining new stationary axisymmetric solutions to the Einstein field equations in vacuum.

1 Introduction

From a known solution to the equations of General Relativity (GR) in vacuum that admits a congruence of Killing vectors, one may generate other solutions \cite{1}. The Killing vectors of the original solution are assumed to have a non-zero norm and a twist potential, both of which will be transformed to construct the Killing vectors of the resultant solution.

In the present article, we give several examples on how to use one of the methods of generating solutions.

In view of this method, any spacetime metric can be expressed in two equivalent ways; either in the form of the line element

\begin{equation}
\begin{aligned}
ds^2 &= \sigma \left( dt + A_i dx^i \right)^2 + h_{ij} dx^i dx^j,
\end{aligned}
\end{equation}

or as the symmetric tensor

\begin{equation}
\begin{aligned}
g_{\mu\nu} &= h_{\mu\nu} + \frac{1}{\sigma} k_\mu k_\nu .
\end{aligned}
\end{equation}
In Eqs. (1) and (2), Greek indices refer to the four-dimensional spacetime, Latin indices refer to the three-dimensional slice that is perpendicular to a particular Killing vector, $k_\mu$, and $h_{ij}$ is the three-metric of such a slice. Moreover, in Eq. (1), the three-vector $A_i$ is defined by

$$k_i = \sigma A_i ,$$  

(3)

where $\sigma$ is the negative norm of the Killing vector $k_\mu$, i.e.,

$$k_\mu k^\mu = -\sigma .$$  

(4)

There is also a twist potential, $\omega$, associated with $k_\mu$, defined by

$$\omega_{, \mu} = \eta_{\mu\nu\rho\sigma} k_\nu \partial_\rho k_\sigma .$$  

(5)

In Eq. (5), the comma denotes partial derivative, $\eta_{\mu\nu\rho\sigma} = \sqrt{|g|} \varepsilon_{\mu\nu\rho\sigma}$, where $g$ is the determinant of the metric tensor (2) and $\varepsilon_{\mu\nu\rho\sigma}$ is the completely antisymmetric symbol of four indices, with $\varepsilon_{0123} = 1$.

Upon consideration of $\sigma$ and $\omega$, a complex potential, $\epsilon$, and a conformal change of the three-metric are defined, as follows

$$\epsilon = \sigma + i\omega ,$$  

(6)

$$\tilde{h}_{ij} = \sigma h_{ij} .$$  

(7)

Using both of these quantities, the new solution is accordingly generated by

$$\epsilon' = \frac{\epsilon}{1 + i\alpha \epsilon} ,$$  

(8)

and

$$\tilde{h}'_{ij} = \tilde{h}_{ij} ,$$  

(9)

where $\alpha$ is a real constant. In Eqs. (8) and (9), the original (seed) metric is denoted without a prime, while the generated (new) metric is indicated with a prime. Assuming that the twist potential of the original solution is zero, the norm of the new Killing vector, $\sigma'$, the new twist potential, $\omega'$, and the new three-metric, $h'_{ij}$, are given by

$$\sigma' = \frac{\sigma}{1 + \alpha^2 \sigma^2} ,$$  

(10)

$$\omega' = -\frac{\alpha \sigma^2}{1 + \alpha^2 \sigma^2} ,$$  

(11)

and

$$h'_{ij} = \frac{\sigma'}{\sigma'} h_{ij} ,$$  

(12)

respectively. Now, the components of the Killing vector, $k'_{\mu}$, corresponding to the new metric, can be obtained by solving the definition equation of the new twist potential, $\omega'$ (cf. Eq. 5).

In what follows, we shall use this method, to generate twisting axisymmetric solutions of GR, that originate from any static axisymmetric solution given in...
Weyl coordinates. As we shall demonstrate, either the spacelike, \( \frac{\partial}{\partial \phi} \), or the timelike, \( \frac{\partial}{\partial t} \), Killing vector can be used, to generate a twist potential. More specifically:

In Sect. 2, we give the particulars in detail, using the spacelike Killing vector. We then demonstrate the method, by applying it to the Schwarzschild solution. In Sect. 3, the method is applied to any static axisymmetric solution using the timelike Killing vector. Accordingly, a new solution is generated, originating, once again, from the Schwarzschild solution. This new solution turns out to be the Taub-NUT metric [2], [3]. Finally, in Sect. 4, we apply the method of generating solutions (upon the action of the timelike Killing vector) to the \( \gamma \)-metric [4], [5] (also called the Zipoy-Voorhees solution).

2 Solutions generated from static axisymmetric ones, using the spacelike Killing vector

In Weyl coordinates, static axisymmetric solutions of GR are described by [6]

\[
\begin{align*}
  ds^2 &= e^{2\lambda} dt^2 - e^{-2\lambda} [e^{2\mu} (d\rho^2 + dz^2) + \rho^2 d\phi^2] , \\
\end{align*}
\]

(13)

where

\[
\begin{align*}
  \lambda,_{\rho\rho} + \frac{\lambda,_{\rho}}{\rho} + \lambda,_{zz} &= 0 , \\
  \mu,_{\rho} &= \rho (\lambda^2,_{\rho} - \lambda^2,_{z} ) , \\
  \mu,_{z} &= 2\rho \lambda,_{\rho} \lambda,_{z} .
\end{align*}
\]

(14)

The metric given by Eq. (13) admits two Killing vectors, one spacelike, \( \partial/\partial \phi \), and one timelike, \( \partial/\partial t \). Hence, originating from (13), we may generate new solutions to the Einstein field equations in vacuum, using either of these two Killing vectors.

In the axisymmetric solution given by Eq. (13), the norm of the spacelike Killing vector, \( k_\phi = \partial/\partial \phi \), is \( \sigma = \rho^2 \exp(-2\lambda) \), so that \( k_\phi k^\phi = -\sigma < 0 \) (as it should), and the twist potential vanishes (\( \omega = 0 \)); hence, the seed potential (cf. Eq. 6) reads

\[
\begin{align*}
  \epsilon &= \sigma = \rho^2 e^{-2\lambda} .
\end{align*}
\]

(15)

To generate a new solution originating from (13), we first use Eq. (12), to determine the new three-metric, \( h'_{ij} \), as

\[
\begin{align*}
  h'_{ij} &= \frac{\sigma}{\sigma'} \begin{pmatrix}
  e^{2\lambda} & 0 & 0 \\
  0 & -e^{2\mu-2\lambda} & 0 \\
  0 & 0 & -e^{2\mu-2\lambda}
\end{pmatrix} ,
\end{align*}
\]

(16)

with determinant \( h' = \det |h'_{ij}| = \left( \frac{\sigma}{\sigma'} \right)^3 e^{4\mu-2\lambda} \), where, in view of Eq. (10),

\[
\begin{align*}
  \frac{\sigma}{\sigma'} &= 1 + \alpha^2 \sigma^2 .
\end{align*}
\]

(17)
On the other hand, the new twist potential is defined by
\[ \omega' \cdot \mu = \eta^\mu_{\beta\gamma\delta} k'_\beta \partial_\gamma k'_\delta, \tag{18} \]
where \( k'_\beta \) is the new Killing vector and
\[ \omega' \cdot \mu = \eta^\mu_{\rho\rho} \omega'_{\cdot \rho}, \quad \eta^\mu_{\rho\rho} = (0, h'^{\rho\rho}, \omega'_{\cdot \rho}, h'^{zz}, \omega'_{\cdot z}, 0). \tag{19} \]
Now, all we have to do, is to determine the components of the vector \( A_i \) (i = t, \( \rho \), z), and, through those, the exact form of \( k'_i = \sigma' A_i \), as well. Since we are interested in axisymmetric solutions of GR, in what follows we assume that all functions depend only on \( \rho \) and \( z \). In this context, it can be shown that the components \( A_\rho = constant = A_z \) and may be chosen to vanish. Accordingly,
\[ k'_\beta = (\sigma' A_t, 0, 0, \sigma'). \tag{20} \]
Upon consideration of Eq. (20), Eq. (18) is written in the form
\[ \omega'_{\cdot \rho} = \eta^{\rho\phi\gamma} k'_\phi \partial_\gamma k'_\phi = \eta^{\rho\phi\gamma} \left[ k'_\phi \partial_\gamma k'_\phi - k'_\phi \partial_\gamma k'_\phi \right], \tag{21} \]
where
\[ -\eta^{\rho\phi\gamma} = \eta^{\rho\phi\gamma}, \quad \eta^{\rho\phi\gamma} = \eta^{\rho\phi\gamma} \quad \text{and} \quad \eta^{\rho\phi\gamma} \eta^{\rho\phi\gamma} = -1. \tag{22} \]
Now, we multiply both sides of Eq. (21) by \( \eta_{\rho\phi\gamma} \). The left hand side yields
\[ \eta_{\rho\phi\gamma} h'^{\rho\rho} \omega'_{\cdot \rho} = (|\sigma' h'|)^{1/2} h'^{\rho\rho} \omega'_{\cdot \rho}, \tag{23} \]
while the right hand side results in
\[ \eta_{\rho\phi\gamma} \eta^{\rho\phi\gamma} \left[ k'_\phi \partial_\gamma k'_\phi - k'_\phi \partial_\gamma k'_\phi \right] = - (\sigma')^2 \partial_z A_t, \tag{24} \]
where \( \eta_{\rho\phi\gamma} = \sqrt{|\sigma' h'|} \varepsilon_{\rho\phi\gamma} \varepsilon_{\rho\phi\gamma} = 1. \) Consequently, upon consideration of Eqs. (23) and (24), Eq. (21) leads to
\[ (|\sigma' h'|)^{1/2} h'^{\rho\rho} \omega'_{\cdot \rho} = - (\sigma')^2 \partial_z A_t \tag{25} \]
and
\[ (|\sigma' h'|)^{1/2} h'^{zz} \omega'_{\cdot z} = (\sigma')^2 \partial_\rho A_t. \tag{26} \]
Further manipulation of Eq. (25) yields
\[ (|\sigma' h'|)^{1/2} h'^{\rho\rho} \omega'_{\cdot \rho} = \left[ -\frac{\sigma'}{\sigma} e^{2\lambda - 2\mu} \right] (|\sigma' h'|)^{1/2} \omega'_{\cdot \rho} = - \rho \omega'_{\cdot \rho}, \tag{27} \]
and, similarly, from Eq. (26) we obtain
\[ (|\sigma' h'|)^{1/2} h'^{zz} \omega'_{\cdot z} = \left[ -\frac{\sigma'}{\sigma} e^{2\lambda - 2\mu} \right] (|\sigma' h'|)^{1/2} \omega'_{\cdot z} = - \rho \omega'_{\cdot z}. \tag{28} \]
However, in view of Eq. (11), and taking into account Eq. (15), we have

\[
\frac{\partial \omega'}{\partial \rho} = \frac{\partial \omega'}{\partial \epsilon} \frac{\partial \epsilon}{\partial \rho} = - \frac{4\alpha e^2}{(1 + \alpha^2\epsilon^2)^2} \left( \frac{1}{\rho} - \lambda, \rho \right) \tag{29}
\]

\[
\frac{\partial \omega'}{\partial z} = \frac{\partial \omega'}{\partial \epsilon} \frac{\partial \epsilon}{\partial z} = \frac{4\alpha e^2}{(1 + \alpha^2\epsilon^2)^2} \lambda, z. \tag{30}
\]

Inserting Eqs. (29) and (30) into Eqs. (27) and (28), respectively, we obtain

\[
4\alpha \left( 1 - \rho \frac{\partial \lambda}{\partial \rho} \right) = - \frac{\partial A_t}{\partial z}, \tag{31}
\]

\[
4\alpha \rho \frac{\partial \lambda}{\partial z} = - \frac{\partial A_t}{\partial \rho}. \tag{32}
\]

We observe that, as far as the system of Eqs. (31) and (32) is concerned, the integrability condition is satisfied, simply by reference to the definition equation of \( \lambda \) (the first one of Eqs. 14).

Summarizing, the solution generated from the seed metric (13) by the action of the \( \phi \)-Killing vector, is given by

\[
d s^2 = \left( 1 + \alpha^2 \rho^4 e^{-4\lambda} \right) \left[ e^{2\lambda} d t^2 - e^{2\mu - 2\lambda} \left( d \rho^2 + dz^2 \right) \right] - \frac{\rho^2 e^{-2\lambda}}{1 + \alpha^2 \rho^4 e^{-4\lambda}} (d \phi + A_t dt)^2, \tag{33}
\]

together with the system of Eqs. (14), as well as Eqs. (31) and (32).

### 2.1 New solutions generated from the Schwarzchild solution

We shall apply the aforementioned mechanism of generating solutions, using as seed metric the Schwarzchild solution in cylindrical coordinates, i.e.,

\[
d s^2 = e^{2\lambda} d t^2 - e^{2\mu - 2\lambda} \left( d \rho^2 + dz^2 \right) - \rho^2 e^{-2\lambda} d \phi^2, \tag{34}
\]

where

\[
e^{2\lambda} = \frac{L - m}{L + m}, \]

\[
e^{2\mu - 2\lambda} = \frac{(L + m)^2}{l_+ l_-}, \tag{35}
\]

with \( m \) being the central mass, that is responsible for the spherically symmetric gravitational field, and

\[
L = \frac{1}{2} (l_+ + l_-), \]

\[
l_\pm^2 = \rho^2 + (z \pm m)^2. \tag{36}
\]
Using Eqs. (34) - (36), we can now integrate Eqs. (31) and (32), to obtain
\[ A_t = 2\alpha [(l_+ - l_-) - 2z] . \]  
(37)

Hence, in view of Eqs. (33) and (37), the new metric generated from the Schwarzschild solution reads
\[ ds^2 = \frac{1}{(L - m)^2} [(L - m)^2 + \alpha^2 \rho^4 (L + m)^2] \left[ \frac{L - m}{L + m} dt^2 - \frac{(L + m)^2}{l_+ l_-} (d\rho^2 + dz^2) \right] - \rho^2 \left( L - m \right) \left( L + m \right) \left( L + m \right)^2 \left( d\phi + 2\alpha [(l_+ - l_-) - 2z] dt \right)^2 . \]  
(38)

Furthermore, performing the coordinate transformation \((t, \rho, z, \phi) \Rightarrow (t, r, \theta, \phi)\),
\[ \rho = \sqrt{r^2 - 2mr \sin \theta} , \ z = (r - m) \cos \theta , \ L = r - m , \ l_+ - l_- = 2m \cos \theta , \]  
(39)
we find that, the metric given by Eq. (34) reduces to the Schwarzschild solution in spherical \((t, r, \theta, \phi)\) coordinates,
\[ ds^2 = \left( 1 - \frac{2m}{r} \right) dt^2 - \frac{dr^2}{1 - \frac{2m}{r}} - r^2 \left( d\theta^2 + \sin^2 \theta d\phi^2 \right) , \]  
(40)
as it should; while, the metric given by Eq. (38) results in
\[ ds^2 = \left( 1 + \alpha^2 r^4 \sin^4 \theta \right) \left[ \left( 1 - \frac{2m}{r} \right) dt^2 - \left( 1 - \frac{2m}{r} \right)^{-1} dr^2 - r^2 d\theta^2 \right] - \frac{r^2 \sin^2 \theta}{\left( 1 + \alpha^2 r^4 \sin^4 \theta \right)^2} \left[ d\phi - 4\alpha (r - 2m) \cos \theta dt \right]^2 . \]  
(41)

The Riemann-square invariant of the new solution (41), is given by
\[ I = \mathcal{R}_{\mu\nu\lambda\rho} \mathcal{R}^{\mu\nu\lambda\rho} \]
\[ = \frac{1}{r^6 \left[ 1 + \alpha^2 r^4 \sin^4 \theta \right]^6} \left\{ -12\alpha^8 r^{16} \sin^{12} \theta \left[ 12rr_s \sin^4 \theta \right. \right. \]
\[- 16r^2 - 21r_s^2 \sin^4 \theta + 24rr_s \sin^2 \theta \]
\[+ 48\alpha^6 r^{12} \sin^8 \theta \left[ -70r_s^2 \sin^4 \theta - 60r^2 + 27rr_s \sin^4 \theta + 102rr_s \sin^2 \theta \right] \]
\[+ 120\alpha^4 r^8 \sin^4 \theta \left[ 21r_s^2 \sin^4 \theta - 52rr_s \sin^2 \theta + 24r^2 + 6rr_s \sin^4 \theta \right] \]
\[- 96\alpha^2 r^5 \left[ -18r_s \sin^2 \theta + 15r_s \sin^4 \theta + 4r \right] + 12r_s^2 \right\} , \]
(42)
where \(r_s = 2m\). In the limit of vanishing \(m\), the invariant given by Eq. (42) does not vanish, and, therefore, we have generated a new solution from vacuum. On the other hand, for \(m \neq 0\) and \(\alpha = 0\), we obtain
\[ I = \mathcal{R}_{\mu\nu\lambda\rho} \mathcal{R}^{\mu\nu\lambda\rho} = \frac{12r_s^2}{r^6} = I_{Schw} , \]  
(43)
i.e., we recover the Riemann-square invariant associated with the Schwarzschild solution. At this point, we should note that, every difference in the metric given by Eq. (41), as compared to the original Schwarzschild solution (40), is included in the terms containing \( \alpha \). When \( \alpha \) is not equal to zero, these changes result in a dramatic variance on the \( z(= r \sin \theta) \) dependence. Furthermore, there is an additional rotation experienced by the coordinate system, which changes its direction at \( r = 2m \), as well as at \( \theta = \pi/2 \).

3 Solutions generated from static axisymmetric ones, using the timelike Killing vector

As far as the metric (13) is concerned, the timelike Killing vector associated with it, \( k_t = \partial_t \), has the norm \( \sigma = -\exp(2\lambda) \), so that \( k_t k^t = -\sigma > 0 \), as it should. In this case, since (once again) \( \omega = 0 \), the seed potential is given by

\[
\epsilon = \sigma = -\exp(2\lambda). \tag{44}
\]

Now, according to Eq. (12), the new three-metric tensor is written in the form

\[
h'_{ij} = \frac{-\sigma}{\sigma'} \begin{pmatrix}
  e^{2\mu-2\lambda} & 0 & 0 \\
  0 & e^{2\mu-2\lambda} & 0 \\
  0 & 0 & \rho^2 e^{2\lambda}
\end{pmatrix}, \tag{45}
\]

with determinant \( h' = \det |h'_{ij}| = -\left(\frac{\sigma}{\sigma'}\right)^3 \rho^2 e^{4\mu-6\lambda} \), where \( \left(\frac{\sigma}{\sigma'}\right) \) is given by Eq. (17).

In cylindrical coordinates, the new Killing vector reads

\[
k'_i = (\sigma', \sigma'A_i), \quad i = \rho, z, \phi, \tag{46}
\]

and, therefore, once again, we need to determine \( A_i \). To do so, we note that, the twist potential corresponding to the new solution, \( \omega' \), as given by Eq. (18), yields

\[
\eta_{\rho z \phi \omega'} k'_{\rho} = k'_z \left( \partial_\phi k'_\rho - \partial_\rho k'_\phi \right) + k'_\phi \left( \partial_\rho k'_z - \partial_z k'_\rho \right) + k'_z \left( \partial_z k'_\rho - \partial_\rho k'_z \right). \tag{47}
\]

Once again, axial symmetry implies that all functions depend only on \( \rho \) and \( z \) and, therefore, \( A_\rho = 0 = A_z \). Accordingly, by virtue Eq. (46), Eq. (47) results in

\[
(\sigma' h')^{1/2} h^{\rho \rho} \omega' = (\sigma')^2 \partial_z A_\phi, \tag{48}
\]

and

\[
(\sigma' h')^{1/2} h^{zz} \omega' = - (\sigma')^2 \partial_\rho A_\phi. \tag{49}
\]

In this case, in view of Eq. (11), and taking into account Eq. (44), we have

\[
\frac{\partial \omega'}{\partial \rho} = \frac{\partial \omega'}{\partial \epsilon} \frac{\partial \epsilon}{\partial \rho} = - \frac{4\alpha \epsilon^2}{(1 + \alpha^2 \epsilon^2)^2} \lambda, \rho, \tag{50}
\]

\[
\frac{\partial \omega'}{\partial z} = - \frac{4\alpha \epsilon^2}{(1 + \alpha^2 \epsilon^2)^2} \lambda, z. \tag{51}
\]
Now, inserting Eqs. (50) and (51) into Eqs. (48) and (49), respectively, we obtain

\[ 4\alpha\rho \frac{\partial \lambda}{\partial \rho} = \frac{\partial A_\phi}{\partial z}, \]

(52)

\[ 4\alpha\rho \frac{\partial \lambda}{\partial z} = -\frac{\partial A_\phi}{\partial \rho}, \]

(53)

where, once again, integrability of Eqs. (52) and (53) is guaranteed by the definition equation of \( \lambda \) (the first one of Eqs. 14).

Summarizing, the solution generated from the seed metric (13) by the action of the \( t \)-Killing vector, is written in the form

\[ ds^2 = e^{2\lambda} \left[ (dt + A_\phi d\phi)^2 - \frac{1 + \alpha^2 e^{4\lambda}}{\rho^2} \left[ e^{2\mu} (\rho^2 + d\theta^2) + \rho^2 d\phi^2 \right] \right], \]

(54)

together with the system of Eqs. (14), as well as Eqs. (52) and (53). Gautreau and Hoffman [7] have also shown how to generate new twisting solutions from a seed metric in Weyl coordinates, using another method, one that was pioneered by Papapetrou [8]. It is worth noting that, although originating from the same seed metrics, their method generates other solutions than those we find in the present article.

### 3.1 New solutions generated from the Schwarzschild solution

Once again, we shall apply the method presented in Sect. 3, using as seed metric the Schwarzschild solution in cylindrical coordinates. By virtue of Eqs. (34) - (36), the system of Eqs. (52) and (53) is directly integrated, yielding

\[ A_\phi = \pm 2\alpha (l_+ - l_-). \]

(55)

Accordingly, in view of Eqs. (54) and (55), a new solution, that is generated from the seed metric (34) by the action of the \( t \)-Killing vector, arises, namely,

\[ ds^2 = e^{2\lambda} \left[ (dt - 2\alpha (l_+ + l_-) d\phi)^2 - \frac{1 + \alpha^2 e^{4\lambda}}{\rho^2} \left[ e^{2\mu} (\rho^2 + d\theta^2) + \rho^2 d\phi^2 \right] \right]. \]

(56)

To further scrutinize this result, we first perform the transformation given by Eq. (39), in order to obtain the solution (56) in spherical coordinates. Accordingly, Eq. (55) yields

\[ A_\phi = -4\alpha m \cos \theta \]

(57)

and the metric (56) is written in the form

\[ ds^2 = \frac{r^2 - 2mr}{r^2 + \alpha^2 (r - 2m)^2} (dt - 4\alpha m \cos \theta d\phi)^2 \]

\[ - \frac{r^2 + \alpha^2 (r - 2m)^2}{r^2 - 2mr} dr^2 - \left[ r^2 + \alpha^2 (r - 2m)^2 \right] d\Omega^2, \]

(58)
where \( d\Omega^2 = r^2 \left( d\theta^2 + \sin^2 \theta d\phi^2 \right) \). The question that arises now, is whether we can recover Eq. (58) using as seed metric the Schwarzschild solution in spherical coordinates, given by Eq. (40), which admits the timelike Killing vector \( \xi_t = \partial_t \) with norm \( \sigma = -\left(1 - \frac{2m}{r}\right) \). In this case,

\[
\omega = 0 \Rightarrow \epsilon = -\sigma = \left(1 - \frac{2m}{r}\right)
\]

(59)

and

\[
h_{ij} = \begin{pmatrix}
\frac{1-\frac{2m}{r}}{1-\frac{2m}{r}} & 0 & 0 \\
0 & r^2 & 0 \\
0 & 0 & r^2 \sin^2 \theta
\end{pmatrix}.
\]

(60)

Now, the combination of Eqs. (12) and (60) yield the three-metric of the new solution, as

\[
h'_{ij} = -\frac{\sigma}{\sigma'} \begin{pmatrix}
\frac{1-\frac{2m}{r}}{1-\frac{2m}{r}} & 0 & 0 \\
0 & r^2 & 0 \\
0 & 0 & r^2 \sin^2 \theta
\end{pmatrix},
\]

(61)

where, once again, \( \left(\frac{\sigma}{\sigma'}\right) \) is given by Eq. (17). Furthermore, upon consideration of Eq. (46), Eq. (18) results in the following system of equations

\[
\eta_{rr\phi} h'''' \omega', r = k'_{\phi} \partial_0 k'_{\phi} = (\sigma')^2 \partial_0 A_{\phi},
\]

(62)

\[
\eta_{r\theta\phi} h''' \omega', \theta = k'_{\phi} \partial_0 k'_{\theta} - k'_{\phi} \partial_\theta k'_{\phi} = - (\sigma')^2 \partial_\theta A_{\phi},
\]

(63)

where, by virtue of Eq. (59), from Eq. (11) we obtain

\[
\frac{\partial \omega'}{\partial r} = \frac{\partial \omega'}{\partial \epsilon} \frac{\partial \epsilon}{\partial r} = \frac{4\alpha m \epsilon}{r^2 \left(1 + \alpha^2 \epsilon^2\right)^2}
\]

(64)

and

\[
\frac{\partial \omega'}{\partial \theta} = \frac{\partial \omega'}{\partial \epsilon} \frac{\partial \epsilon}{\partial \theta} = 0.
\]

(65)

In view of Eqs. (64) and (65), Eqs. (62) and (63) yield

\[
-\sqrt{|\sigma|} \sin \theta \sqrt{1 - \frac{2m}{r}} \frac{4\alpha m \sigma}{\left(1 + \alpha^2 \sigma^2\right)^2} = \frac{\sigma^2}{\left(1 + \alpha^2 \sigma^2\right)^2} \partial_\theta A_{\phi},
\]

(66)

and, eventually,

\[
\partial_\theta A_{\phi} = \frac{4m}{2m} \sin \theta \sqrt{1 - \frac{2m}{r}},
\]

\[
\partial_r A_{\phi} = 0.
\]

(67)

Upon consideration of Eq. (59), Eqs. (67) yield

\[
A_{\phi} = -4m \cos \theta
\]

(68)
and, therefore, the metric generated from Schwarzschild by the action of the $t$-Killing vector is written in the form

\[ ds^2 = \frac{r^2 - 2mr}{r^2 + \alpha^2 (r - 2m)^2} (dt - 4\alpha m \cos \theta d\phi)^2 \]

\[ - \frac{r^2 + \alpha^2 (r - 2m)^2}{r^2 - 2mr} dr^2 - \frac{r^2 + \alpha^2 (r - 2m)^2}{r^2 - 2mr} d\Omega^2 , \tag{69} \]

which, indeed, coincides to Eq. (58).

The metric given by Eq. (69) admits four Killing vectors, namely,

\[ \xi_0 = [1, 0, 0, 0] , \]

\[ \xi_1 = \left[ -\frac{2\alpha r \sin \phi}{\sin \theta}, 0, \cos \phi, -\sin \phi \cot \theta \right] , \]

\[ \xi_2 = \left[ -\frac{2\alpha r \cos \phi}{\sin \theta}, 0, -\sin \phi, -\cos \phi \cot \theta \right] , \]

\[ \xi_3 = [-2\alpha r, 0, 0, 1] , \tag{70} \]

for which, the following commutation relations hold

\[ [\xi_i , \xi_j] = -\varepsilon_{ijk} \xi_k , \]

\[ [\xi_i , \xi_0] = 0 , \tag{71} \]

where $\varepsilon_{ijk}$ is the completely antisymmetric symbol of three indices, with $\varepsilon_{123} = 1 = \varepsilon_{r\theta\phi}$.

In fact, the metric given by Eq. (69) allows of the same symmetries as the Taub-NUT metric \[[2, 3],

\[ ds^2 = \frac{R^2 - 2\mu R - l^2}{R^2 + l^2} (dT - 2l \cos \theta d\phi)^2 - \frac{R^2 + l^2}{R^2 - 2\mu R - l^2} dR^2 - (R^2 + l^2) d\Omega^2 . \tag{72} \]

In particular,

- both of them have no curvature singularities;
- provided that the values $\theta = 0$ and $\theta = \pi$ represent lines on the manifold, the time coordinate in both metrics is cyclic;
- both metrics admit three Killing vectors, in addition to $\partial/\partial t$, and all of them obey the commutation relations given by Eqs. (71).

In fact, the metrics given by Eqs. (69) and (72) represent the same line-element, as it can be seen by performing in Eq. (72) the following transformation on the radial coordinate,

\[ R = \left( 1 + \alpha^2 \right)^{1/2} r - \frac{2m\alpha^2}{(1 + \alpha^2)^{1/2}} , \tag{73} \]
together with a rescaling of the time coordinate in both Eqs. (69) and (72),

\[ T = 2l\tau, \quad t = 4\alpha m\tau, \]  

(74)

where

\[ l = \frac{2m\alpha}{(1 + \alpha^2)^{1/2}}, \quad \mu = \frac{m(\alpha^2 - 1)}{(1 + \alpha^2)^{1/2}}. \]  

(75)

Notice that, in this case, there is an additional relation between the various constants involved, namely,

\[ \frac{\mu}{l} = \frac{\alpha^2 - 1}{2\alpha}, \]  

(76)

i.e., \( \alpha \) controls the ratio \( \mu/l \).

4 Using the \( \gamma \)-metric as seed solution

The \( \gamma \)-metric [4] generalizes the Schwarzschild solution and it can be used also as a seed metric to generate new solutions. The \( \gamma \)-metric is given by [9], [10]

\[
\begin{align*}
    ds^2 &= \left(1 - \frac{2m}{r}\right)^\gamma dt^2 - \left(1 - \frac{2m}{r}\right)^{-\gamma} \left[\left(\frac{r^2 - 2mr + m^2 \sin^2 \theta}{r^2 - 2mr + m^2 \sin^2 \theta}\right)^{\gamma^2 - 1} dr^2 \\
         &+ \frac{(r^2 - 2mr)^{\gamma^2}}{(r^2 - 2mr + m^2 \sin^2 \theta)^{\gamma^2 - 1}} d\theta^2 + (r^2 - 2mr) \sin^2 \theta d\phi^2\right].
\end{align*}
\]  

(77)

The curved spacetime represented by the line element (77) possesses some very interesting features (see, e.g., [9]). In particular,

- if \( \gamma \neq 0 \) and \( m = 0 \), it corresponds to the Minkowski spacetime;

- for \( \gamma = 1 \) and \( m \neq 0 \), it represents the Schwarzschild spacetime;

- for \( \gamma \to \infty \) and \( m \to 0 \), but \( m\gamma \to \text{constant} \), it represents a Curzon spacetime;

- the \( \gamma \)-solution is the static limit of the Tomimatsu-Sato (TS) family of solutions (see, e.g., [1]). In this case, if the rotation parameter, \( q \), vanishes, the TS solution becomes identical to the \( \gamma \)-solution, with the TS deformation parameter, \( \delta \), being equal to the value of \( \gamma \).

The metric given by Eq. (77), is often referred to as the Voorhees metric or Zipoy-Voorhees (ZV) metric [5], [9] (for more details see, e.g., [1]). Upon consideration of the quaternionic version of the Ernst formalism, M. Hallilsoy [11] found a generalized version of the ZV metric. On the other hand, by treating the \( \gamma \)-solution as a seed (vacuum) metric, Richterek et al. [12], [13] obtained two new classes of solutions to the Einstein - Maxwell equations, the main properties of which are extensively discussed in [12] and [13].
Here, upon consideration of the mechanism of generating solutions presented in Sect. 3, we also find a new solution to the Einstein field equations in vacuum, which is written in the form

\[
ds^2 = \frac{(r^2 - 2mr)^\gamma}{r^{2\gamma} + \alpha^2(r - 2m)^{2\gamma}} [dt - 4\alpha \gamma m \cos \theta d\phi]^2 - \frac{r^{2\gamma} + \alpha^2(r - 2m)^{2\gamma}}{(r^2 - 2m)^{2\gamma}} \left[ \left( \frac{r^2 - 2mr}{r^2 - 2mr + m^2 \sin^2 \theta} \right)^{\gamma^2 - 1} dr^2 + \frac{(r^2 - 2mr)^\gamma}{(r^2 - 2mr + m^2 \sin^2 \theta)^{\gamma^2 - 1}} d\theta^2 + (r^2 - 2mr) \sin^2 \theta d\phi^2 \right]. \tag{78}
\]

A null tetrad (see, e.g., [14]) and the Newmann-Penrose invariants (see, e.g., [15]) of this new solution are given in the Appendix A.

5 Summary - Conclusions

In this article, we have demonstrated a method of generating new stationary axisymmetric solutions to the Einstein field equations in vacuum, that originate from already existing ones (seed solutions), admitting at least one Killing vector (either \(\partial/\partial\phi\) or \(\partial/\partial t\)).

Applying this method to the Schwarzschild solution, upon the action of the spacelike \(\left(\frac{\partial}{\partial \phi}\right)\) Killing vector, we have found a new family of vacuum solutions to the Einstein field equations, given by Eq. (38) (or Eq. 41). In the same reasoning, but this time upon the action of the timelike \(\left(\frac{\partial}{\partial t}\right)\) Killing vector, the Schwarzschild solution generates the new family of solutions given by Eq. (58) (or Eq. 69), which, upon the transformation given by Eqs. (73) - (75), coincides to the Taub-NUT metric, given by Eq. (72). Finally, using as seed solution the \(\gamma\)-metric (77), and upon the action of the timelike Killing vector, we have arrived at a new family of solutions to the Einstein field equations, given by Eq. (78), which depends on the parameters \(m, \alpha\) and \(\gamma\).

It is worth noting that, upon consideration of this new method of generating stationary axisymmetric solutions to the Einstein field equations in vacuum, a novel parameter, namely, \(\alpha\), should be imposed, which, throughout our analysis, has been considered to be real. Nevertheless, in the case where this parameter is complex, our method reduces to the Kinnersley V transformation [16], which mixes gravity with electromagnetism, while, if \(\alpha\) is purely imaginary, it represents the potential of the electromagnetic field and there is no twist.

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Appendix A

In this Appendix, we present the Newmann-Penrose scalars [15] of the metric given by Eq. (78), which is generated from the $\gamma$-metric (77). In the metric given by Eq. (78), we set

$$f = \frac{\epsilon}{1 + \alpha^2 \epsilon^2} = \frac{(r^2 - 2mr)^\gamma}{r^{2\gamma} + \alpha^2 (r - 2m)^{2\gamma}} \tag{A1}$$

Accordingly, we consider the following set of orthonormal vectors (see, e.g., [14])

$$A_\mu = \left[ \sqrt{f}, 0, 0, -4\alpha m \cos \theta \sqrt{f} \right]$$

$$B_\mu = \left[ 0, -\frac{1}{\sqrt{f}} \left( \frac{r^2 - 2mr}{r^2 - 2mr + m^2 \sin^2 \theta} \right)^{\frac{2}{2-1}}, 0, 0 \right]$$

$$P_\mu = \left[ 0, 0, \frac{1}{\sqrt{f}} \left( \frac{r^2 - 2mr}{r^2 - 2mr + m^2 \sin^2 \theta} \right)^{\frac{2}{2-1}}, 0 \right]$$

$$Q_\mu = \left[ 0, 0, 0, \frac{\sin \theta}{\sqrt{f}} \sqrt{r^2 - 2mr} \right] \tag{A2}$$

from which, a null tetrad for the metric (78) can be determined, as follows

$$l_\mu = \frac{B_\mu + A_\mu}{\sqrt{2}}, \quad k_\mu = \frac{B_\mu - A_\mu}{\sqrt{2}}$$

$$m_\mu = \frac{P_\mu + iQ_\mu}{\sqrt{2}}, \quad m_\mu = \frac{P_\mu - iQ_\mu}{\sqrt{2}} \tag{A3}$$

resulting in

$$l_\mu = \frac{1}{\sqrt{2}} \left[ \sqrt{f}, -\frac{1}{\sqrt{f}} \left( \frac{r^2 - 2mr}{r^2 - 2mr + m^2 \sin^2 \theta} \right)^{\frac{2}{2-1}}, 0, -4\alpha m \cos \theta \sqrt{f} \right]$$

$$l^\mu = \frac{1}{\sqrt{2}} \left[ -\frac{1}{\sqrt{f}} \left( \frac{r^2 - 2mr + m^2 \sin^2 \theta}{r^2 - 2mr} \right)^{\frac{2}{2-1}}, \sqrt{f}, 0, 0 \right]$$

$$k_\mu = \frac{1}{\sqrt{2}} \left[ -\sqrt{f}, -\frac{1}{\sqrt{f}} \left( \frac{r^2 - 2mr + m^2 \sin^2 \theta}{r^2 - 2mr + m^2 \sin^2 \theta} \right)^{\frac{2}{2-1}}, 0, 4\alpha m \cos \theta \sqrt{f} \right] \tag{A4}$$

$$k^\mu = \frac{1}{\sqrt{2}} \left[ \frac{1}{\sqrt{f}} \sqrt{f} \left( \frac{r^2 - 2mr + m^2 \sin^2 \theta}{r^2 - 2mr} \right)^{\frac{2}{2-1}}, 0, 0 \right]$$
\[ m_\mu = \frac{1}{\sqrt{2}} \begin{bmatrix} 0, 0, 1 \sqrt{\frac{r^2 - 2mr}{r^2 - 2mr + m^2 \sin^2 \theta}} \frac{i \sin \theta}{\sqrt{r^2 - 2mr}} \end{bmatrix} \]

\[ m^\nu = \frac{\sqrt{2}}{\sqrt{2} \sqrt{r^2 - 2mr}} \begin{bmatrix} 4i \alpha m \cos \theta, 0, \left( \frac{r^2 - 2mr + m^2 \sin^2 \theta}{r^2 - 2mr} \right) \frac{2 \gamma - 1}{\sin \theta}, -i \end{bmatrix} \]

The tetrad given by Eqs. (A4) satisfies the orthonormality conditions

\[ l^\mu l_\mu = k^\mu k_\mu = m^\mu m_\mu = 0 \]
\[ l^\mu m_\mu = k^\mu m_\mu = 0 \]
\[ l^\mu k_\mu = 1, \quad m^\mu m_\mu = 1. \quad (A5) \]

Furthermore, in view of Eqs. (A4), we verify that the components of the metric tensor given by Eq. (78) may obtained from the relation [13]

\[ g_{\mu \nu} = l_\mu k_\nu + l_\mu m_\nu + m_\mu n_\nu + m_\mu m_\nu. \quad (A6) \]

Now, following [13], the Newmann-Penrose (NP) invariants corresponding to the solution (78), that is generated from the \( \gamma \)-metric by the action of the t-Killing vector, are given by

\[ \Psi_0 = 2 C_{\text{arsq}} l^\mu m^\nu m^\gamma \Rightarrow \]
\[ \Psi_0 = \frac{\gamma (2 - 1) m^3 (r - m)}{2 \left( r^2 - 2mr + m^2 \sin^2 \theta \right)^{\gamma^2 - 2}} \times \frac{\left( r^2 - 2mr + m^2 \sin^2 \theta \right)^{\gamma^2 - 2}}{\left( r^2 - 2mr + m^2 \sin^2 \theta \right)^{\gamma^2 - 2}} \times \left\{ \left[ (r^2 - 2mr + m^2 \sin^2 \theta)^{2 - 1} \right] + 2i \left[ (r^2 - 2mr + m^2 \sin^2 \theta)^{2 - 1} \right] \right\} \quad (A7) \]

\[ \Psi_1 = -2 C_{\text{arsq}} l^\mu m^\nu (k^\mu l^\nu + m^\mu m^\nu) = 0 \quad (A8) \]

\[ \Psi_2 = 2 C_{\text{arsq}} l^\mu m^\nu n^\gamma \]

\[ = -\frac{Z^{\gamma^2 - 3}}{8 \sin^2 \theta Y^2 X^{\gamma^2 - 2}} \begin{bmatrix} -4m^2 (2 - 1) Z Y^2 X^2 \sin^2 \theta \cos 2 \theta \\
-4(\gamma - 1) Z Y^2 X^2 \sin^2 \theta \\
+8\gamma (2 - 1) Z Y^2 X^2 (X Y^2 \sin^2 \theta + 8 \alpha^2 m^2 \gamma^2 \cos^2 \theta X Y^2) \\
+4Z^2 Y^2 X (\gamma - 1) Y^2 \sin^2 \theta + 16 \alpha^2 m^2 \gamma^2 \cos^2 \theta X Y^2) \\
+8(\gamma - 1) m^4 Y^2 X \sin^4 \theta \cos^2 \theta \\
-4(\gamma - 1) m^2 Z^2 Y^2 X \sin^2 \theta \cos^2 \theta X Y^2 + 16 \alpha^2 m^2 \gamma^2 X Y^2 \\
-8\gamma^2 Z^2 Y^2 (2 X + X - \gamma - \gamma^2 + 1) \sin^2 \theta + 32 \alpha^2 m^2 \gamma^2 \cos^2 \theta X Y^2 \sin^2 \theta \\
-4 \gamma (r - m) Z^2 Y^3 X \sin^2 \theta [(\gamma^2 - 3) \gamma + (\gamma^2 - 2 \gamma + 1) X - \gamma^2 + \gamma + 1] \sin^2 \theta + 16 \alpha^2 m^2 \gamma^2 \cos^2 \theta X Y^2 \sin^2 \theta \\
+16 \alpha^2 m^2 \gamma^2 \cos^2 \theta X Y^2 \sin^2 \theta + 16 \alpha^2 m^2 \gamma^2 \cos^2 \theta X Y^2 \sin^2 \theta \\
-4 \gamma (\gamma - 1) (r - m) Z Y^2 X [X - X - \gamma^2 + 1] \sin^2 \theta + 16 \alpha^2 m^2 \gamma^2 \cos^2 \theta X Y^2 \sin^2 \theta \\
+4 \gamma (\gamma - 1) (r - m) Z Y^2 X [X - X - \gamma^2 + 1] \sin^2 \theta + 16 \alpha^2 m^2 \gamma^2 \cos^2 \theta X Y^2 \sin^2 \theta \\
+4 \gamma (\gamma - 1) (r - m)^2 Z Y^2 X [X - X - \gamma^2 + 1] \sin^2 \theta + 16 \alpha^2 m^2 \gamma^2 \cos^2 \theta X Y^2 \sin^2 \theta \end{bmatrix} \]
\[ + 8(\gamma^2 - 1)(r-m)^2Y^4X^3 \sin^2 \theta + 32\alpha^2 m^2 \gamma^2 Z^2 Y^2 X^{-\gamma^2 + 3\gamma + 1} \sin^2 \theta \}, \]

\[(A9)\]

where \( C_{arsq} \) is the Weyl tensor, and we have set

\[ X = r^2 - 2mr, \quad Y = r^{2\gamma} + \alpha^2 (r-2m)^{2\gamma}, \quad \text{and} \quad Z = r^2 - 2mr + m^2 \sin^2 \theta. \quad (A10) \]

Finally,

\[ \Psi_3 = -C_{arsq} k^a \bar{m}^r (k^s l^q + \bar{m}^a m^q) = 0 \quad (A11) \]

and

\[ \Psi_4 = 2C_{arsq} n^a n^s \bar{m}^r \bar{m}^q = -\Psi_0 \quad (A12) \]

In view of Eqs. (A7) - (A12), we conclude the following:

In the case where \( \alpha = 0 \),

- for every \( \gamma \neq 0, 1 \), the scalars coincide with those of the \( \gamma \)-solution and, as it is known, the resulting metric is type I in Petrov’s classification \[18, 19\];
- for \( \gamma = 1 \), only \( \Psi_2 \) is non zero, representing the corresponding quantity of the Schwarzschild solution.

In the case where \( \alpha \neq 0 \),

- for \( \gamma \neq 0 \) and \( \theta = 0 \) or \( \pi \), \( \Psi_0 = 0 = \Psi_4 \) and \( \Psi_2 \) blows up;
- for \( \gamma \neq 0, 1 \) and \( \theta = \frac{\pi}{2} \), there exist singularities at \( r = 0 \) and \( r = 2m \);
- for \( \gamma = 1 \) and \( \theta \in (0, \pi) \), the metric is type D in Petrov’s classification;
- for \( 0 < \gamma < \sqrt{3} \) and \( \theta \neq 0, \pi \), there exists a singularity at \( r^2 - 2mr + m^2 \sin^2 \theta = 0 \).

In fact, if \( \gamma \neq 0, 1 \) and \( \theta \neq 0, \pi \), the metric (78) is type I in Petrov’s classification, exhibiting singularities at \( r = 0 \) and \( r = 2m \). Singularities also exist at \( r_{\pm} = m (1 \pm \cos \theta) \), if \( \gamma \in (1, \sqrt{3}) \).

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