Holomorphic isometric embeddings of the projective space into quadrics

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Abstract
We classify holomorphic isometric embeddings of the projective space into quadrics using the generalization of do Carmo–Wallach theory in Nagatomo [Harmonic maps into Grassmann manifolds, arXiv:mathDG/1408.1504[mathDG], Holomorphic maps into Grassmann manifolds (Harmonic maps into Grassmann manifolds III), Annals of Global analysis and Geometry 60, 33–63 (2021). An explicit description of their moduli spaces up to image and gauge equivalence is provided.

Keywords Moduli spaces · Holomorphic isomorphic embeddings · Projective space · Complex quadric · Vector bundles

Mathematics Subject Classification 2010 Primary 32H02; Secondly 53C07

1 Introduction
The purpose of the present paper is to obtain the explicit description of the moduli space $\mathcal{M}_k$ of full holomorphic isometric embeddings of the complex projective space $\mathbb{CP}^m$ into the Grassmann manifold $Gr_n(\mathbb{R}^{n+2})$ parametrizing codimension-2 oriented subspaces of $\mathbb{R}^{n+2}$ of degree $k$ modulo congruence. Main Theorem generalizes [6] and [8, Theorems 6.18 and 6.19], in which the domain manifold is the complex projective line. If we fix an orientation of $\mathbb{R}^{n+2}$, then, by standard arguments (cf. [5, pp. 278–282]), the target space can be regarded as the complex quadric hypersurface in $\mathbb{CP}^{n+1}$ and this is the reason for the title. Our strategy is to apply a generalized do Carmo and Wallach theory based on a generalization of theorem of Tsunero Takahashi developed in [8, 9]. The main difference of both generalizations from the original do Carmo and Wallach theory [2] and Takahashi’s theorem [10] is that we exploit differential geometry of vector bundles with connections.

To this aim we shall study first the moduli space $\mathcal{M}_k$ modulo gauge equivalence of maps which plays a more fundamental role in our generalization of do Carmo–Wallach theory. The definition of gauge equivalence, fullness and the degree of maps will be formulated in

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Sect. 2, where a summary of the main results in [8, 9] has been included for convenience of the reader.

In the next section, we interpret how general theory fits into our special case and introduce a general result on the moduli space $\mathcal{M}_k$ in [9]. From the viewpoint of representation theory, this enables us to consider only unitary representation and develop more transparent argument compared with [6], in which we also needed to take account of orthogonal representation.

In the fourth section, we exhibit our main result. (More precise statement will be given in Theorem 4.5.)

**Main Theorem.** If $f : \mathbb{C}^p \to Gr_n(\mathbb{R}^{n+2})$ is a full holomorphic isometric embedding of degree $k$, then $n \leq 2^m k - 2$.

Let $\mathcal{M}_k$ be the moduli space of full holomorphic isometric embeddings of degree $k$ of $\mathbb{C}^p$ into $Gr_N(\mathbb{R}^{N+2})$ modulo the gauge equivalence of maps, where $N = 2^m k - 2$. Then, $\mathcal{M}_k$ can be regarded as an open bounded convex body in a complex vector space $V_k$ (which is defined in Sect. 4).

We further concern with the compactification of the moduli space. Let $\overline{\mathcal{M}_k}$ be the closure of the moduli $\mathcal{M}_k$ by topology induced from the $L^2$-inner product. Every boundary point of $\overline{\mathcal{M}_k}$ distinguishes a subspace $\mathbb{R}^{k+2}$ of $\mathbb{R}^{N+2}$ and describes a full holomorphic isometric embedding into $Gr_p(\mathbb{R}^{p+2})$ which can be regarded as totally geodesic submanifold of $Gr_N(\mathbb{R}^{N+2})$. The appearance of the totally geodesic submanifold can be interpreted by geometry of vector bundles; the Euclidean space $\mathbb{R}^{N+2}$ can be regarded as a space of sections of the universal quotient bundle over $Gr_N(\mathbb{R}^{N+2})$. The inner product on $\mathbb{R}^{N+2}$ determines the orthogonal decomposition of $\mathbb{R}^{N+2} : \mathbb{R}^{p+2} = \mathbb{R}^{p+2} \oplus \mathbb{R}^{p+2}$. Then the totally geodesic submanifold $Gr_p(\mathbb{R}^{p+2})$ can be obtained as the common zero set of sections of the universal quotient bundle, which belongs to $\mathbb{R}^{p+2}$.

Eventually, we show that the moduli space up to congruence $\mathbb{M}_k$ can be described as an $S^1$-quotient of $\mathcal{M}_k$ (Theorem 5.1), where the group $S^1$ is the centralizer of the holonomy group of the canonical connection on $\mathcal{O}(k) \to \mathbb{C}^p$ in the structure group. The moment map reduction induces a foliation of $\mathbb{M}_k$ whose general leaf is a projective space.

**2 Preliminaries**

Throughout this paper, every manifold is supposed to be connected. For a vector bundle $V \to M$ over a manifold $M$, $\Gamma(V)$ denotes the space of sections of $V \to M$. Then the evaluation map $ev : M \times \Gamma(V) \to V$ is defined as $ev(x, t) = t(x)$ for $x \in M$ and $t \in \Gamma(V)$. If we have a subspace $W \subset \Gamma(V)$, then the restriction of $ev$ to $M \times W$ is also called the evaluation map. When $V \to M$ has a compatible connection with a fibre metric $h_V$, then the connection is denoted by $\nabla^V$. We abbreviate those to $(V, h_V, \nabla^V)$ or $(V \to M, h_V, \nabla^V)$.

Then $(V_1, h_{V_1}, \nabla^{V_1})$ is said to be isomorphic to $(V_2, h_{V_2}, \nabla^{V_2})$ if there exists a bundle map $V_1 \to V_2$ preserving the fibre metrics and the connections, and such a bundle map is called a bundle isomorphism. We denote by $\underline{W}$ a trivial vector bundle $\underline{W} \to M$ with fibre $W$ over a manifold $M$.

We need to consider holomorphic vector bundles over Kähler manifolds. Let $(V, h)$ be a pair of holomorphic vector bundle $V \to M$ over a Kähler manifold $M$ and a Hermitian metric $h$ on $V \to M$. We call $(V, h)$ a Hermitian (vector) bundle. We have a unique connection $\nabla$ compatible with $h$ and the holomorphic vector bundle structure, which is called the *Hermitian connection* [4]. Then $(V_1, h_1)$ is called to be holomorphically isomorphic to $(V_2, h_2)$ if
\((V_1, h_1, \nabla^{V_1})\) is isomorphic to \((V_2, h_2, \nabla^{V_2})\), where \(\nabla^{V_i}\) are the Hermitian connections for \(i = 1, 2\).

From now on, we give a short account of results in [8] and [9] in case that the target is a quadric. Let \(W\) be a real \(N\)-dimensional vector space and \(Gr_p(W)\) the real Grassmannian parametrizing oriented \(p\)-planes in \(W\). We have a subbundle of \(W \to Gr_p(W)\) called the tautological vector bundle \(S \to Gr_p(W)\). Then the universal quotient bundle denoted by \(Q \to Gr_p(W)\) is defined as the quotient bundle of \(W\) by \(S \to Gr_p(W)\). We can use the natural projection \(\tilde{W} \to Q\) to regard \(W\) as a subspace of \(\Gamma(Q)\). Thus the natural projection is also recognized as the evaluation map. By fixing an inner product on \(W, S\) and \(Q \to Gr_p(W)\) inherit fibre-metrics and the canonical connections.

Let \(f : M \to Gr_p(W)\) be a map of a manifold \(M\) into a Grassmannian. We pull back the universal quotient bundle denoted by \(f^*Q \to M\). If the induced linear map \(W \to \Gamma(f^*Q)\) is an isomorphism, then the map \(f\) is called to be a full map [8, Definition 5.2]. This definition of fullness coincides with the one used in [2] when the target space is the sphere.

Suppose that \(V \to M\) is an oriented real vector bundle of rank \(q\) and consider an \(N\)-dimensional space of sections \(W \subset \Gamma(V)\). Then the vector bundle \(V \to M\) is said to be globally generated by \(W\) if the evaluation map is surjective. In addition, suppose that \(W\) is oriented. Then we have a map \(f : M \to Gr_p(W)\), where \(Gr_p(W)\) is an oriented real Grassmannian and \(p = N - q\), defined by

\[
f(x) = \text{Ker} \, ev_x = \{t \in W \mid t(x) = 0\}, \quad x \in M,
\]

where the orientation of \(\text{Ker} \, ev_x\) is inherited from those of \(V_x\) and \(W\). The map \(f\) is called to be the induced map by the pair \((V, W, f)\) (cf. [8]). From the definition, \(V \to M\) can be naturally identified with \(f^*Q \to M\). Therefore, any smooth map \(f : M \to Gr_p(W)\) can be recognized as the induced map determined by the pair \((f^*Q \to M, W)\).

Assume that \(V \to M\) has a fibre-metric. Since \(W\) is supposed to have an inner product, the adjoint of the evaluation map \(ev^* : V \to \tilde{W}\) gives a natural identification between \(V \to M\) and \(f^*Q \to M\), when the image of \(ev^*\) is restricted to \(f^*Q \to M\). Then \(ev^* : V \to f^*Q\) is also called a natural identification.

Let us introduce two equivalence relations of maps [8, Definitions 5.5, 5.6 and 5.7], one of which is well-known as congruence and the other is needed to state a generalization of do Carmo and Wallach theorem. Let \(f_1\) and \(f_2 : M \to Gr_p(W)\) be maps. Then \(f_1\) is called image equivalent to \(f_2\) if there exists an isometry \(\phi \) of \(Gr_p(W)\) such that \(f_2 = \phi \circ f_1\). Each isometry \(\phi \) of \(Gr_p(W)\) gives the bundle isomorphism of \(Q \to Gr_p(W)\) denoted by \(\phi \) which covers the isometry \(\phi \). Suppose that \(f_1\) and \(f_2 : M \to Gr_p(W)\) are regarded as induced maps by \((V, W)\). Then, \(f_1\) is said to be gauge equivalent to \(f_2\), if there exists an isometry \(\phi \) of \(Gr_p(W)\) such that \(f_2 = \phi \circ f_1\) and \(ev_x^* = \phi \circ ev_x^*,\) where \(ev_x^* : V \to f_i^*Q\) \((i = 1, 2)\) are natural identifications. In this case, the induced connections on \(V \to M\) by \(ev_x^*\) are in the same orbit under the gauge transformation.

Since the complex projective space is a compact Hermitian symmetric space, suppose that \((G, K)\) is a Hermitian symmetric pair of compact type, where \(G\) is a simply-connected compact Lie group and \(K\) is a closed subgroup of \(G\), with the standard decomposition of Lie algebra \(\mathfrak{g} = \mathfrak{k} \oplus \mathfrak{m}\), where \(\mathfrak{g}\) and \(\mathfrak{k}\) are Lie algebras of \(G\) and \(K\), respectively. If we regard \(G \to G/K\) as a principal \(K\)-bundle, then the canonical connection is defined in such a way that the horizontal subspace of \(G\) is given by the left translation of \(\mathfrak{m}\). Let \(L \to G/K\) be a complex line bundle which is an associated bundle : \(L = G \times_K V_0\), where \(V_0\) is a complex 1-dimensional \(K\)-module. Then \(L \to G/K\) acquires a holomorphic vector bundle structure by the canonical connection. For an invariant Hermitian metric \(h\) on \(L\) which is unique up
to a positive constant multiple, the canonical connection is the Hermitian one for \((L, h)\) (for example, see [4, p. 121 Proposition 6.2]).

If \(L \to G/K\) has a non-trivial holomorphic section, then the Bott-Borel-Weil theorem implies that the space of holomorphic sections of \(L \to G/K\) denoted by \(W\) is an irreducible complex \(G\)-module and globally generates the bundle. From the Riemannian structure on \(G/K\) and \((L, h)\), \(W\) is equipped with a \(G\)-invariant \(L^2\)-Hermitian inner product. Then, the Kodaira embedding by \((L \to G/K, W)\) to a complex projective space is said to be a standard map. The key point is that \(W\) has a \(G\)-invariant Hermitian inner product, since we can discuss the pull-back connection. We restrict the coefficient field of the complex vector \(G\)-module and globally generates the bundle. From the Riemannian structure on \(G/K\) and \((L, h)\), \(W\) is equipped with a \(G\)-invariant \(L^2\)-Hermitian inner product. Then, the Kodaira embedding by \((L \to G/K, W)\) to a complex projective space is said to be a standard map. The key point is that \(W\) has a \(G\)-invariant Hermitian inner product, since we can discuss the pull-back connection. We restrict the coefficient field of the complex vector bundle to \(\mathbb{R}\) to obtain an oriented real vector bundle denoted by the same symbol \(L \to G/K\) with a fibre-metric induced by the real part of \(h\), where the orientation is given by the complex structure. In a similar way, we regard \(W\) as an oriented real vector space with \(G\)-invariant \(L^2\)-inner product denoted by \((\cdot, \cdot)_W\). Then, we obtain the induced map into an oriented real Grassmannian, which is also said to be standard. With these understood, from [8, Lemma 5.17], we derive

**Lemma 2.1** The \(K\)-module \(V_0\) can be regarded as a complex subspace of \(W\).

Denote by \(U_0\) the orthogonal complement of \(V_0\) in \(W\). Then, the standard map denoted by \(f_0 : M \to Gr_p(W)\) is explicitly written down as

\[
f_0([g]) = gU_0 \subset W, \quad \text{for all } [g] \in G/K, \quad g \in G,
\]

where \(p = \dim U_0\) and it is an \(G\)-equivariant map.

Next, \(S(W)\) denotes the set of symmetric endomorphisms of \(W\). We equip \(S(W)\) with a \(G\)-invariant inner product \((A, B)_S = \text{trace } AB\), for \(A, B \in S(W)\). Define a symmetric transformation \(S(u, v)\) for \(u, v \in W\) as

\[
S(u, v) := \frac{1}{2} \left\{ u \otimes (\cdot, v)_W + v \otimes (\cdot, u)_W \right\}.
\]

If \(U\) and \(V\) are subspaces of \(W\), we define a real subspace \(S(U, V) \subset S(W)\) spanned by \(S(u, v)\) where \(u \in U\) and \(v \in V\). In a similar fashion, \(GS(U, V)\) denotes the subspace of \(S(W)\) spanned by \(gS(u, v)\), where \(g \in G\).

With these understood, we can state a version of the generalization of the theorem of do Carmo–Wallach for holomorphic maps of \(G/K\) into a quadric:

**Theorem 2.2** Let \(f : G/K \to Gr_p(\mathbb{R}^{n+2})\) be a full holomorphic map satisfying the following condition:

(i) The pull-back \(f^*Q \to G/K\) of the universal quotient bundle \(Q \to Gr_p(\mathbb{R}^{n+2})\) with the pull-back metric is holomorphically isomorphic to \((L, h)\).

Then, there exists a positive semi-definite symmetric endomorphism \(T \in S(W)\) such that the pair \((W, T)\) satisfies the following three conditions:

(I) The vector space \(\mathbb{R}^{n+2}\) is a subspace of \(W\) with the inclusion \(\iota : \mathbb{R}^{n+2} \to W\) preserving the inner products, and \(L \to M\) is globally generated by \(\mathbb{R}^{n+2}\).

(II) As a subspace, \(\mathbb{R}^{n+2} = \ker T^\perp\) and the restriction of \(T\) is a positive symmetric endomorphism of \(\mathbb{R}^{n+2}\).

(III) The endomorphism \(T\) satisfies

\[
(T^2 - \text{Id}_W, GS(V_0, V_0))_S = 0, \quad (T^2, GS(mV_0, V_0))_S = 0. \tag{2.1}
\]
If \( \iota^* : W \to \mathbb{R}^{n+2} \) denotes the adjoint linear map of \( \iota : \mathbb{R}^{n+2} \to W \), then \( f : G/K \to Gr_n(\mathbb{R}^{n+2}) \) is realized as
\[
f([g]) = \left((\iota^* T_\iota)^{-1}(f_0([g]) \cap \text{Ker } T^\perp)\right),
\]
where the orientation of \( (\iota^* T_\iota)^{-1}(f_0([g]) \cap \text{Ker } T^\perp) \) is given by ones of \( L_{[g]} \) and \( \mathbb{R}^{n+2} \). Moreover, if the orientation of \( \text{Ker } T \) is fixed, then we have a unique holomorphic totally geodesic embedding of \( Gr_n(\mathbb{R}^{n+2}) \) into \( Gr_n(W) \) by \( \iota(R^{n+2}) = \text{Ker } T^\perp \), where \( n' = n + \dim \text{Ker } T \) and a bundle isomorphism \( (ev \circ (\iota^* T_\iota))^* : L \to f^* Q \) as the natural identification by \( f \).

Such two maps \( f_i \), \( (i = 1, 2) \) are gauge equivalent if and only if \( \iota^* T_1 = \iota^* T_2 \), where \( T_i \) and \( \iota \) correspond to \( f_i \) in (2.2), respectively.

Conversely, suppose that a vector space \( \mathbb{R}^{n+2} \) with an inner product and an orientation, and a positive semi-definite symmetric endomorphism \( T \in \text{End } (W) \) satisfying conditions (I), (II) and (III) are given. Then we have a unique holomorphic totally geodesic embedding of \( Gr_n(\mathbb{R}^{n+2}) \) into \( Gr_n(W) \) after fixing the orientation of \( \text{Ker } T \) and the map \( f : G/K \to Gr_p(\mathbb{R}^{n+2}) \) defined by (2.2) is a full holomorphic map into \( Gr_p(\mathbb{R}^{n+2}) \) satisfying the condition (i) with bundle isomorphism \( L \cong f^* Q \) as the natural identification.

**Proof** This is obtained by a combination of Theorem 5.24 in [8] and Theorem 5.9 in [9]. \( \square \)

**Remark 1** If \( f \) satisfies the condition (i) in the theorem, then \( f \) is said to satisfy the gauge condition for \( (V, h) \).

### 3 Holomorphic isometric embeddings into quadrics

The aim of this section is to introduce holomorphic isometric embeddings from \( \mathbb{C}P^m \) into \( Gr_n(\mathbb{R}^{n+2}) \) and we will show that they satisfy the hypothesis of Theorem 2.2. Thus the construction of the moduli space reduces to finding symmetric endomorphisms satisfying the condition (2.1) in Theorem 2.2. Then a general property of the moduli space obtained in [9] is exhibited, which gives a substantial restriction to symmetric endomorphisms to be considered.

The universal quotient bundle on \( Gr_n(\mathbb{R}^{n+2}) \) has a holomorphic bundle structure by the canonical connection. Notice that the curvature two-form \( R \) of the canonical connection on the universal quotient bundle is the fundamental two-form \( \omega_Q \) on \( Gr_n(\mathbb{R}^{n+2}) \) up to a constant multiple:
\[
R = -\sqrt{-1}\omega_Q.
\]
Denote by \( \omega_0 \) the fundamental two-form on \( \mathbb{C}P^m \). When \( R_1 \) denotes the curvature form of the canonical connection on the hyperplane bundle \( \mathcal{O}(1) \to \mathbb{C}P^m \), we also have \( R_1 = -\sqrt{-1}\omega_0 \). In what follows, we will denote by \( \mathcal{O}(k) \to \mathbb{C}P^m \) the \( k \)-th tensor power of the hyperplane bundle and by \( h_k \) the standard Einstein-Hermitian metric on \( \mathcal{O}(k) \to \mathbb{C}P^m \).

If \( f : \mathbb{C}P^m \to Gr_p(\mathbb{R}^{n+2}) \) is a holomorphic map, then the pull-back of the universal quotient bundle is also a holomorphic line bundle on \( \mathbb{C}P^m \).

**Definition 1** Let \( f : \mathbb{C}P^m \to Gr_n(\mathbb{R}^{n+2}) \) be a holomorphic embedding. Then \( f \) is called an isometric embedding of degree \( k \) if \( f^* \omega_Q = k\omega_0 \) (and so, \( k \) must be a positive integer).
By definition, \( f : \mathbf{C}P^m \to Gr_n(\mathbf{R}^{n+2}) \) is a holomorphic isometric embedding of degree \( k \) if and only if the pull-back of the canonical connection on the universal quotient bundle on \( Gr_n(\mathbf{R}^{n+2}) \) is also the canonical connection on \( \mathcal{O}(k) \to \mathbf{C}P^m \). Thus \( f : \mathbf{C}P^m \to Gr_n(\mathbf{R}^{n+2}) \) is a holomorphic isometric embedding of degree \( k \) if and only if \( f \) satisfies the gauge condition for \( (\mathcal{O}(k), h_k) \). Since \( (\mathcal{O}(k), h_k) \) has the canonical connection as the Hermitian connection, we can apply Theorem 2.2 to obtain the moduli space \( \mathcal{M}_k \) of holomorphic isometric embeddings of degree \( k \) modulo the gauge equivalence of maps.

**Remark 2** Since we identify the complex quadric with an oriented real Grassmannian \( Gr_n(\mathbf{R}^{n+2}) \), we need to fix an orientation of \( \mathbf{R}^{n+2} \) to define a complex structure of \( Gr_n(\mathbf{R}^{n+2}) \). We denote by \( \mathbf{R}^{n+2} \) the Euclidean space with the fixed orientation. The Euclidean space with the reversed orientation is denoted by \( \mathbf{R}^{n+2} \). Define \( \tau : Gr_n(\mathbf{R}^{n+2}) \to Gr_n(\mathbf{R}^{n+2}) \) to be the map obtained by switching the orientation of \( n \)-dimensional subspaces of \( \mathbf{R}^{n+2} \). Then \( \tau : Gr_n(\mathbf{R}^{n+2}) \to Gr_n(\mathbf{R}^{n+2}) \) is a holomorphic isometry. In the sequel, we do not distinguish a map \( f : M \to Gr_n(\mathbf{R}^{n+2}) \) from a map \( \tau \circ f : M \to Gr_n(\mathbf{R}^{n+2}) \).

We need a property of the moduli space \( \mathcal{M}_k \) stated in [9]. Let \((L, h) \) be a Hermitian holomorphic line bundle over a Kähler manifold \( M \). We denote by \( J_L \) the complex structure on \( L \to M \). The complex structure \( J \) of \( H^0(M, L) \) is induced by \( J_L \) in such a way that \( J_Lev = evJ \), where \( ev : H^0(M, L) \to L \) is the evaluation map.

Suppose that \( C^{l+1} \) is a complex subspace of \( H^0(M, L) \). We also regard \( C^{l+1} \) as a real vector space with the complex structure denoted by \( (\mathbf{R}^{n+2}, J) \) where \( n = 2l \). Suppose that the inner product on \( \mathbf{R}^{n+2} \) is induced by taking the real part of the \( L^2 \)-Hermitian inner product on \( H^0(M, L) \). When the induced map \( f : M \to Gr_n(\mathbf{R}^{n+2}) \) by \( (L \to M, \mathbf{R}^{n+2}) \) is a holomorphic isometric immersion, it follows from \( J_Lev = evJ \) that \( f \) can also be regarded as a holomorphic isometric immersion into \( \mathbf{C}P^l \) which is a totally geodesic complex submanifold of \( Gr_n(\mathbf{R}^{n+2}) \).

We denote by \( H(\mathbf{R}^{n+2}) \) the space of Hermitian endomorphisms on \( (\mathbf{R}^{n+2}, J) \):

\[
H(\mathbf{R}^{n+2}) := \{ A \in S(\mathbf{R}^{n+2}) \mid JA = AJ \},
\]

where \( S(\mathbf{R}^{n+2}) \) denotes the space of symmetric endomorphisms on \( \mathbf{R}^{n+2} \). Since \( \mathbf{R}^{n+2} \) is now supposed to have a \( U(l+1) \)-structure, we have the orthogonal decomposition: \( S(\mathbf{R}^{n+2}) = H(\mathbf{R}^{n+2}) \oplus S_I(\mathbf{R}^{n+2}) \). The orthogonal complement \( S_I(\mathbf{R}^{n+2}) \) is identified with \( S^2C^{l+1} \) which is the symmetric power of \( C^{l+1} \) of degree 2. The subspace \( S_I(\mathbf{R}^{n+2}) \) is characterized as

\[
S_I(\mathbf{R}^{n+2}) = \{ A \in S(\mathbf{R}^{n+2}) \mid JA = -AJ \}.
\]

**Theorem 3.1** [9, Theorem 6.10] Let \((L, h) \) be a Hermitian holomorphic line bundle over a Kähler manifold \( M \). Let \( \mathcal{M} \) be the moduli space of full holomorphic isometric immersions of \( M \) into a quadric \( Gr_n(\mathbf{R}^{n+2}) \) with the gauge condition for \((L, h) \) modulo gauge equivalence relation of maps.

Suppose that \( \mathbf{R}^{n+2} \) is a complex subspace of \( H^0(M, L) \) and the inner product on \( \mathbf{R}^{n+2} \) is compatible with the complex structure. If there exists \( f \in \mathcal{M} \) such that the evaluation map \( ev : \mathbf{R}^{n+2} \to L \) by \( f \) satisfies \( J_Lev = evJ \), then \( \mathcal{M} \) has the induced complex structure and is an open submanifold of a complex subspace of \( S_I(\mathbf{R}^{n+2}) \).

We will apply Theorem 3.1 to the case that the domain is the complex projective space \( \mathbf{C}P^m \). By Calabi’s rigidity theorem [1], the standard map into a complex projective space
by \((\mathcal{O}(k) \to \mathbb{C} P^m, H^0(\mathbb{C} P^m, \mathcal{O}(k)))\) is a unique holomorphic isometric embedding up to congruence. When \(\mathcal{O}(k) \to \mathbb{C} P^m\) is recognized as real vector bundle with an orientation and \(H^0(\mathbb{C} P^m, \mathcal{O}(k))\) as an oriented real vector space, then the standard map is also regarded as a holomorphic isometric embedding into a quadric. Then Theorem 3.1 yields that the moduli space of full holomorphic isometric embeddings of \(\mathbb{C} P^m\) into a quadric \(\text{Gr}_n(\mathbb{R}^{m+2})\) with the gauge condition for \((\mathcal{O}(k) \to \mathbb{C} P^m, h_k)\) modulo gauge equivalence relation of maps is an open submanifold of a complex subspace of \(S_I(\mathbb{R}^{m+2})\), where \(\mathbb{R}^{m+2} = H^0(\mathbb{C} P^m, \mathcal{O}(k))\).

4 Moduli space by gauge equivalence

In order to apply the generalized do Carmo–Wallach theory, we focus our attention on the space of symmetric endomorphisms of the space of holomorphic sections of the line bundles on the projective spaces. We consider a symmetric pair \((\text{SU}(m+1), \text{U}(m))\), where an element in the isotropy subgroup \(\text{U}(m)\) is of the form:

\[
\begin{pmatrix}
A & O \\
O & |A|^{-1}
\end{pmatrix}, \quad A \in \text{U}(m),
\]

and \(|A|\) denotes the determinant of \(A\). The corresponding symmetric space is the projective space \(\mathbb{C} P^m\) which is expressed as \(\text{SU}(m+1)/\text{U}(m)\).

Let \(\mathcal{H}_{n,k}^l\) be the irreducible representation space of \(\text{U}(n)\) which consists of harmonic polynomials on \(\mathbb{C}^n\) of bi-degree \((k, l)\). (Hence, \(\mathcal{H}_{n,k}^l\) is a subspace of \(S^k \mathbb{C}^n \otimes S^l \mathbb{C}^n\)). We also consider representations of \(\text{U}(n)\) and take the diagonal maximal torus of \(\text{U}(n)\). We denote by \(V_n(\lambda_1, \lambda_2, \cdots, \lambda_n)\) an irreducible complex representation of \(\text{U}(n)\) with the highest weight \((\lambda_1, \lambda_2, \cdots, \lambda_n)\), where \(\lambda_1 \geq \lambda_2 \geq \cdots \geq \lambda_n\) are integers. We can regard \(\mathcal{H}_{n,k}^l\) as \(\text{U}(n)\)-module, which is equivalent to \(V_n(k, 0, \cdots, 0, -l)\) as representation.

If \(\mathcal{C}_k\) denotes the one-dimensional representation \(V_m(k, k, \cdots, k)\) of \(\text{U}(m)\), then the homogeneous bundle \(\text{SU}(m+1) \times \text{U}(m) \mathcal{C}_{-k}\) is the complex line bundle \(\mathcal{O}(k) \to \mathbb{C} P^m\) of degree \(k\). We abbreviate a tensor product of vector bundles \(V\) and \(\mathcal{O}(k)\) as \((V \to \mathbb{C} P^m)\).

Next, \(\mathcal{C}^m\) denotes the standard representation of \(\text{U}(m)\): \(\mathcal{C}^m = V_m(1, 0, \cdots, 0)\). We have an exact sequence of vector bundles:

\[
0 \to S \to \mathbb{C}^m+1 \to \mathcal{O}(1) \to 0.
\]

Since the tautological vector bundle \(S\) can now be identified with \(\text{SU}(m+1) \times \text{U}(m) \mathcal{C}^m\) as homogeneous bundle, the holomorphic tangent bundle \(T \to \mathbb{C} P^m\) is a homogeneous vector bundle associated with the representation \(\mathcal{C}^m+1\). The holomorphic cotangent bundle \(T^* \to \mathbb{C} P^m\) is also a homogeneous vector bundle associated with the representation \(\mathcal{C}^m \otimes \mathcal{C}_1\) of \(\text{U}(m)\).

The \(\text{SU}(m+1)\)-representation space \(\mathcal{H}_{m+1}^{k,0} = S^k \mathcal{C}^{m+1}\) of holomorphic polynomials has the following irreducible decomposition as \(\text{U}(m)\)-module:

\[
\mathcal{H}_{m+1}^{k,0} = \bigoplus_{p=0}^k \mathcal{C}_{-k+p} \otimes \mathcal{H}_{m}^{p,0}.
\]

Let \(W\) be the space of holomorphic sections of \(\mathcal{O}(k) \to \mathbb{C} P^m\) which, by Bott-Borel–Weil theorem, is identified with the \(\text{SU}(m+1)\)-module \(S^k \mathcal{C}^{m+1}\). We use the same notation as in §3. The representation space \(H(W)\), which is, by definition, the space of Hermitian endomorphisms on \(W\), is irreducibly decomposed as \(\text{SU}(m+1)\)-modules:

\[
H(W) = \sum_{l=0}^k \mathcal{H}_{n+1}^{k-i,k-i}.
\]

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We will give an irreducible decomposition of $S_1(W)$ which is equivalent to $S^2W$ as representation.

**Proposition 4.1** For an irreducible representation $S^kC^{m+1}$ of $U(m+1)$, we have an irreducible decomposition of $S^2(S^kC^{m+1})$ as unitary representation such that

$$S^2(S^kC^{m+1}) = igoplus_{i=0}^{2i \leq k} V(2k - 2i, 2i, 0, \cdots, 0).$$

**Proof** By a method in [3], we obtain

$$S^kC^{m+1} \otimes S^kC^{m+1} = \bigoplus_{i=0}^{k} V(2k - i, i, 0, \cdots, 0).$$

If $m = 1$, we may apply the method in the proof of [6,Corollary 4.3 and Proposition 4.7] or in [7,Proposition 4.2] to obtain the result. We take a subgroup $U(2)$ of $U(m + 1)$ in such a way that

$$\begin{pmatrix} A & O \\ O & I_{m-1} \end{pmatrix}, \quad A \in U(2).$$

Then any highest weight of $U(m + 1)$ restricted to the subgroup $U(2)$ gives the highest weight of $U(2)$. Thus we obtain the result. 

Since we should regard $W$ as orthogonal $SU(m+1)$-modules when applying a generalization of do Carmo–Wallach theory, we must consider an orthogonal irreducible decomposition of the space of symmetric endomorphisms of $W$ as $SU(m + 1)$-module. However, since the complex structure of $S_1(W)$ is invariant, Theorem 3.1 implies that we may decompose $S_1(W) = S^2(S^kC^{m+1})$ irreducibly as unitary representation. We therefore apply the decomposition in Proposition 4.1 to construct moduli spaces by Theorem 2.2.

We now describe the moduli space of holomorphic isometric embeddings of $CP^m$ into $Gr_p(W)$ of degree $k$ up to gauge equivalence. Following the generalization of do Carmo–Wallach theory, we must specify the intersection between $GS(mV_0, V_0)$ and $S_1(W)$.

**Lemma 4.2** $mV_0 = C_{-k+1} \otimes C^m$.

**Proof** Equation (4.1) gives a weight decomposition of $W$ with respect to $U(m)$. Since we have seen that the holomorphic tangent bundle of $CP^m$ is associated with the representation $C^{m'} \otimes C_{-1}$, $m$ is the real subspace of the direct sum $C^{m'} \otimes C_{-1} \oplus C^m \otimes C_1$.

**Lemma 4.3** We have that $GS(mV_0, V_0) \cap S_1(W) = V(2k, 0, \cdots, 0)$ appeared in Proposition 4.1.

**Proof** If $u_{-k}$ belongs to $V_0 = C_{-k}$ and $u_{-k+1}$ belongs to $mV_0 = C_{-k+1} \otimes C^m$, then $S(u_{-k}, u_{-k+1})$ has $-2k + 1$ as weight with respect to the center of $U(m)$. However, from Corollary 4.1 we see that the only component in the decomposition of $S_1(W)$ which can have a vector with $-2k + 1$ as weight is the top term $V(2k, 0, \cdots, 0)$. Therefore

$$GS(mV_0, V_0) \cap S_1(W) = V(2k, 0, \cdots, 0).$$

In other words, we obtain
Corollary 4.4  The orthogonal complement to $G S ( m V_0, V_0 )$ in $S_I ( W )$ is

$$V_k = \bigoplus_{i=1}^{2i \leq k} V(2k - 2i, 2i, 0, \ldots, 0).$$

This follows from applying the previous lemma to the decomposition of $S_I ( W )$ described in Proposition 4.1.

Remark 3  It follows from Corollary 5.18 in [9] that the first condition in (2.1) is automatically satisfied. In this case, we can also explain as follows. Let $G S_0 ( V_0, V_0 )$ be the orthogonal complement of the $G$-invariant, irreducible subrepresentation generated by the identity in $G S ( V_0, V_0 )$. Then, replacing the weight $-2k + 1$ by $-2k$ in the proof of Lemma 4.3, we have that

$$G S_0 ( V_0, V_0 ) \cap S_I ( W ) = V(2k, 0, \ldots, 0).$$

Theorem 4.5  If $f : C P^m \to G r_N ( R^{n+2} )$ is a full holomorphic isometric embedding of degree $k$, then $n \leq 2 \left( \frac{m+k}{k} \right) - 2$.

Let $M_k$ be the moduli space of full holomorphic isometric embeddings of degree $k$ of $C P^m$ into $G r_N ( R^{n+2} )$ by the gauge equivalence of maps, where $N = 2 \left( \frac{m+k}{k} \right) - 2$. Then, $M_k$ can be regarded as an open bounded convex body in $V_k$.

Let $\overline{M}_k$ be the closure of the moduli $M_k$ by topology induced from the inner product. Every boundary point of $\overline{M}_k$ distinguishes a subspace $R^{p+2}$ of $R^{n+2}$ and describes a full holomorphic isometric embedding into $G r_p ( R^{n+2} )$ which can be regarded as totally geodesic submanifold of $G r_N ( R^{n+2} )$. The inner product on $R^{n+2}$ determines the orthogonal decomposition of $R^{n+2} : R^{n+2} = R^{p+2} \oplus R^{p+2 \perp}$. Then the totally geodesic submanifold $G r_p ( R^{p+2} )$ can be obtained as the common zero set of sections of $Q \to G r_N ( R^{n+2} )$, which belongs to $R^{p+2 \perp}$.

Proof  The restriction $n \leq N$ follows from (I) in Theorem 2.2 and Bott–Borel–Weil theorem.

It follows from (III) in Theorem 2.2 that each of these endomorphisms defines a full holomorphic isometric embedding $C P^m \to G r_N ( R^{n+2} )$ of degree $k$. Since the standard map into $G r_N ( R^{n+2} )$ is the composite of the standard map $C P^m \to C P^N$ and the totally geodesic embedding $C P^N \hookrightarrow G r_N ( R^{n+2} )$, we can apply Theorem 3.1 and Corollary 4.4 to conclude that $M_k$ is a bounded connected open convex body in $V_k$ with the topology induced by the $L^2$ scalar product.

Under the natural compactification in the $L^2$-topology, the boundary points correspond to endomorphisms $T$ which are not positive definite, but positive semi-definite. It follows from Theorem 2.2 that each of these endomorphisms defines a full holomorphic isometric embedding $C P^m \to G r_p ( R^{p+2} )$, of degree $k$ with $p = 2k - \dim \ker T$, whose target embeds in $G r_N ( R^{n+2} )$ as a totally geodesic submanifold. The image of the embedding $G r_p ( R^{p+2} ) \hookrightarrow G r_N ( R^{n+2} )$ is determined by the common zero set of sections in $\ker T$. (See also the Remark after Proposition 5.14 in [8] for the geometric meaning of the compactification of the moduli space.)

5 Moduli space by image equivalence

The moduli space $M_k$ has a natural complex structure inherited from the $C$-vector space $V_k$ [9]. We can show that the centralizer of the holonomy group acts on $M_k$. For a general theory, see [9].
Theorem 5.1 Let $M_k$ be the moduli space of holomorphic isometric embeddings of $\mathbb{C}P^m$ into $\text{Gr}_N(\mathbb{R}^{N+2})$ of degree $k$ by the image equivalence of maps, where $N = 2\binom{m+k}{k} - 2$. Then we have $M_k = M_k / S^1$.

Proof Suppose that two full holomorphic isometric embeddings $f_1, f_2 : \mathbb{C}P^m \to \text{Gr}_N(\mathbb{R}^{N+2})$ of degree $k$ to be image equivalent. They may represent distinct points in $M_k$. By definition of image equivalence, there is an isometry $\psi$ of $\text{Gr}_{2k}(\mathbb{R}^{2k+2})$ such that $f_2 = \psi \circ f_1$, then $f_2^*Q = f_1^*\psi^*Q$ as sets. Using the natural identifications $ev_1^*$ and $ev_2^*$, we introduce new bundle isomorphisms $\mathcal{O}(k) \to f_2^*Q$ defined by $\tilde{\psi} \circ ev_1^*$ and $ev_2^*$. Hence, we have a gauge transformation $ev_2^{*-1}\tilde{\psi}ev_1^*$ on the line bundle $\mathcal{O}(k) \to \mathbb{C}P^m$ preserving the metric and the connection. By connectedness of $\mathbb{C}P^m$, $ev_2^{*-1}\tilde{\psi}ev_1^*$ is regarded as an element, say $c$, of the centralizer of the holonomy group of the connection in the structure group of $\mathcal{O}(k) \to \mathbb{C}P^m$. Since $\tilde{\psi}ev_1^* = cev_2^*$, $(f_1, ev_1^*)$ and $(f_2, ev_2^*)$ are in the same orbit of the $S^1$-action on $M_k$. □

Remark 4 The moduli space $M_k$ has a complex structure (see remark in §4) and a metric induced by the inner product, both of which are preserved by the $S^1$-action. Hence, $M_k$ is a Kähler manifold with an $S^1$-action preserving the Kähler structure. We can find a moment map $\mu : M_k \to \mathbb{R}$ which is expressed as $\mu = |Id - T^2|^2$. By the moment map reduction, $M_k$ has a foliation whose general leaves are the complex projective spaces.

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References

1. Calabi, E.: Isometric imbedding of complex manifolds. Ann. Math. 58, 1–23 (1953)
2. do Carmo, M. P., Wallach, N. R.: Minimal immersions of spheres into spheres. Ann. Math 93, 43–62 (1971)
3. Fulton, W.: Young Tableaux with Applications to Representation Theory and Geometry. London Mathematical Society Student Texts (35), Cambridge University Press, Cambridge (1997)
4. Kobayashi, S.: Differential Geometry of Complex Vector Bundles. Iwanami Shoten and Princeton University, Tokyo (1987)
5. Kobayashi, S., Nomizu, K.: Foundations of Differential Geometry, vol. II. Interscience Publishers (1969)
6. Macia, O., Nagatomo, Y., Takahashi, M.: Holomorphic isometric embeddings of projective lines into quadrics. Tohoku Math. J. 69, 525–545 (2017)
7. Macia, O., Nagatomo, Y.: Moduli of Einstein-Hermitian harmonic mappings of the projective lines into quadrics. Ann. Glob. Anal. Geom. 53, 503–520 (2018)
8. Nagatomo, Y.: Harmonic maps into Grassmann manifolds, arXiv:mathDG/1408.1504 [mathDG]
9. Nagatomo, Y.: Holomorphic maps into Grassmann manifolds (Harmonic maps into Grassmann manifolds III), Annals of Global analysis and Geometry 60, 33–63 (2021)
10. Takahashi, T.: Minimal immersions of Riemannian manifolds. J. Math. Soc. Japan 18, 380–385 (1966)

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