CONGRUENCES MODULO POWERS OF 2 FOR THE NUMBER OF UNIQUE PATH PARTITIONS

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Abstract. We compute the congruence class modulo 16 of the number of unique path partitions of $n$ (as defined by Olsson), thus generalising previous results by Bessenrodt, Olsson and Sellers [Ann. Combin. 13 (2013), 591–602].

1. Introduction

Unique path partitions were introduced by Olsson in [3]. Their study is motivated from the Murnaghan–Nakayama rule for the calculation of the value of characters of the symmetric group. They were completely characterised by Bessenrodt, Olsson and Sellers in [1]. They used this characterisation to derive a formula for the generating function for the number $u(n)$ of all unique path partitions of $n$. This formula reads (cf. [1, Remark 3.6])

$$
\sum_{n \geq 1} u(n)q^n = 2 \sum_{i \geq 1} q^{2i-1}(1 + q^{2i-1}) \prod_{j=0}^{i-2} \frac{1}{1 - q^{2j}} = 2 \left( q(1 + q) + \sum_{i \geq 2} \frac{q^i + 1}{1 - q^2} \cdot \frac{q^{2i-2} (1 + q^{2i-1})}{\prod_{j=1}^{i-2} (1 - q^{2j})} \right).
$$

(1.1)

The final part in [1] concerns congruences modulo 8 for $u(n)$. The corresponding main result [1, Theorem 4.6] provides a complete description of the behaviour of $u(n)$ modulo 8 (in terms of the related sequence of numbers $w(n)$; see the next section for the definition of $w(n)$). The arguments to arrive at this result are mainly of a recursive nature.

The purpose of this note is to show that a more convenient and more powerful method to derive congruences (modulo powers of 2 is by an analysis of the generating function (1.1). Not only are we able to recover the result from [1], but in addition we succeed in determining the congruence class of $u(n)$ modulo 16, see (2.1) and Theorem 7 thus solving the problem left open in the last paragraph of [1]. We point out that the approach presented here is very much inspired by calculations in [2, Appendix], where expressions similar to the one on the right-hand side of (1.1) appear, with the role of the prime number 2 replaced by 3, though.

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2. AN EQUIVALENT EXPRESSION FOR THE GENERATING FUNCTION

We start with the observation (already made in [1]) that, first, all numbers $u(n)$ are divisible by $2$, and, second, we have $u(2n) = u(2n - 1)$ for all $n$. This is easy to see from the right-hand side of (1.1) since it has the form $2(1 + q) f(q^2)$, where $f(t)$ is a formal power series in $t$. We therefore divide the right-hand side of (1.1) by $2(1 + q^{-1})$, subsequently replace $q$ by $q^{1/2}$, and consider the “reduced” generating function

$$
\sum_{n \geq 2} w(n) q^n = \sum_{i \geq 2} q^{2i-1} (1 + q^{2i-2}) \frac{1}{(1 - q) \prod_{j=0}^{i-3} (1 - q^{2j})}.
$$

In other words, we have

$$
2w(n) = u(2n) = u(2n - 1)
$$

for all $n$.

Using the convention

$$
\sum_{k=M}^{N-1} \text{Expr}(k) = \begin{cases} 
\sum_{k=M}^{N-1} \text{Expr}(k), & N > M, \\
0, & N = M, \\
-\sum_{k=N}^{M-1} \text{Expr}(k), & N < M.
\end{cases}
$$

(2.2)

for sums, we rewrite the above equation in the following way:

$$
\sum_{n \geq 2} w(n) q^n = \sum_{i \geq 2} q^{2i-1} \frac{1 + q^{2i-2}}{(1 - 2q + q^2) \prod_{j=1}^{i-3} (1 - q^{2j})}
\begin{align*}
&= \sum_{i \geq 2} q^{2i-1} \frac{1 + q^{2i-2}}{(1 - \frac{2q}{1+q^2})(1 + q^2) \prod_{j=1}^{i-3} (1 - q^{2j})} \\
&= \sum_{i \geq 2} q^{2i-1} \frac{1 + q^{2i-2}}{(1 - \frac{2q}{1+q^2})(1 - q^4) \prod_{j=2}^{i-3} (1 - q^{2j})} \\
&= \sum_{i \geq 2} q^{2i-1} \frac{1 + q^{2i-2}}{(1 - \frac{2q}{1+q^2})(1 - q^4) \prod_{j=3}^{i-3} (1 - q^{2j})} \\
&= \sum_{i \geq 1} q^{2i-1} \frac{1 + q^{2i-2}}{(1 - \frac{2q}{1+q^2})(1 - \frac{2q^4}{1+q^4}) \cdots (1 - \frac{2q^{2i-4}}{1+q^{2i-4}})(1 - q^{2i-2})} \\
&\quad + \sum_{i \geq 1} q^{2i} \frac{1}{(1 - \frac{2q}{1+q^2})(1 - \frac{2q^4}{1+q^4}) \cdots (1 - \frac{2q^{2i-4}}{1+q^{2i-4}})} \\
&= \sum_{i \geq 1} q^{2i-1} \frac{1 + q^{2i-2}}{\prod_{j=0}^{i-2} (1 - \frac{2q^{2j}}{1+q^{2j+1}})} + \sum_{i \geq 1} q^{2i} \frac{1}{\prod_{j=0}^{i-1} (1 - \frac{2q^{2j}}{1+q^{2j+1}})}.
\end{align*}
$$

(2.3)

From the last expression it is immediately obvious that $w(n)$ is odd if and only if $n$ is a power of $2$, thus recovering the first assertion of [1 Cor. 4.3]. The above mentioned mod-4 result [1 Theorem 4.6] for $w(n)$ — which, by (2.1), translates into a mod-8 result for $u(n)$ — can also be derived within a few lines from the above expression.
In the next section, we show how to obtain congruences modulo 8 for \( w(n) \), which, by [2.1], translate into congruences modulo 16 for the unique path partition numbers \( u(n) \).

3. Congruences modulo powers of 2

In what follows, we write

\[ f(q) = g(q) \mod 2^\gamma \]

to mean that the coefficients of \( q^i \) in \( f(q) \) and \( g(q) \) agree modulo \( 2^\gamma \) for all \( i \). We apply geometric series expansion in (2.3), and at the same time we neglect terms which are divisible by 8. For example, we expand

\[
\frac{1}{1 - \frac{2q}{1 + q^2}} = 1 + \frac{2q}{1 + q^2} + \frac{4q^2}{(1 + q^2)^2} \mod 8.
\]

In this manner, we obtain the congruence

\[
\sum_{n \geq 2} w(n)q^n = \sum_{i \geq 1} q^{2i-1} \left( 1 + \frac{2q^{2i-2}}{1 - q^{2i-2}} + 2 \sum_{j=0}^{i-2} \frac{q^{2j}}{1 + q^{2j+1}} \right)
\]

\[
+ 4 \sum_{j=0}^{i-2} \frac{q^{2j}}{1 - q^{2j-2}} \sum_{s \leq t \leq i-2} (1 + q^{2s+1})(1 + q^{2t+1})
\]

\[
+ \sum_{i \geq 1} q^{2i} \left( 1 + 2 \sum_{j=0}^{i-2} \frac{q^{2j}}{1 + q^{2j+1}} + 4 \sum_{0 \leq s \leq t \leq i-1} (1 + q^{2s+1})(1 + q^{2t+1}) \right)
\]

(modulo 8).

After rearrangement, this becomes

\[
\sum_{n \geq 2} w(n)q^n = \sum_{i \geq 1} q^{2i} + \frac{2q^3}{1 - q} + 2 \sum_{j \geq 1} \frac{1}{1 - q^{2j}} \left( q^{2j+2j+1} + q^{2j-2} \left( 1 - q^{2j-1} \right) \sum_{\ell \geq 2j} q^{2\ell} \right)
\]

\[
+ 4 \sum_{1 \leq s < t} \frac{q^{2s-2} + 2^{2t-2}}{(1 - q^{2s})(1 - q^{2t})} \left( q^{2t-1} \left( 1 + q^{2t-2} \right) + \sum_{\ell \geq 2t} q^{2\ell} \right)
\]

\[
+ 4 \sum_{s \geq 1} \frac{q^{2s-1}}{1 - q^{2s}} \sum_{\ell \geq 2s} q^{2\ell} \mod 8. \tag{3.1}
\]

We must now analyse the individual sums in (3.1).

Lemma 1. Let \( n \geq 2 \), and write \( n = \sum_{i=a}^{e} n_i \cdot 2^i \), with \( 0 \leq n_i \leq 1 \) for all \( i \) and \( n_a \neq 0 \neq n_e \). Then the coefficient of \( q^n \) in

\[
\sum_{j \geq 1} q^{2j+2j+1} \left( 1 - q^{2j} \right)
\]

is equal to \( \lfloor a/2 \rfloor \) if \( n \) is not a power of 2, and it is equal to \( \max\{\lfloor a/2 \rfloor - 1, 0\} \) otherwise.

Proof. By geometric series expansion, we see that the coefficient of \( q^n \) in (3.2) is equal to the number of possibilities to write \( n = (k + 3)2^j \) for some \( j \geq 1 \) and \( k \geq 0 \). For fixed \( j \), we can find a suitable \( k \) if and only if \( n \geq 3 \cdot 2^j \). If \( n \) is not a power of 2, this is equivalent to the condition that \( 2j \leq a \). The claim follows immediately. \( \square \)
Lemma 2. Let $n \geq 2$, and write $n = \sum_{i=a}^{e} n_i \cdot 2^i$ as in Lemma 1. Then the coefficient of $q^n$ in
\[
\sum_{j \geq 1} q^{2j-2} \frac{1}{1 - q^{2j}} \sum_{\ell \geq 2j} q^{2\ell}
\] (3.3)
is equal to $e - 2j + 1$ if $a = 2j - 2$, $n_{a+1} = n_{2j-1} = 0$, and $n$ is not a power of 2, and it is equal to 0 otherwise.

Proof. By geometric series expansion, we see that the coefficient of $q^n$ in (3.3) is equal to the number of possibilities to write $n = 2^{2j-2} + k \cdot 2^{2j} + 2^\ell$ for some $j \geq 1$, $\ell \geq 2j$, and $k \geq 0$. The claim follows immediately. \hfill \Box

Lemma 3. Let $n \geq 2$, and write $n = \sum_{i=a}^{e} n_i \cdot 2^i$ as in Lemma 2. Then the coefficient of $q^n$ in
\[
\sum_{j \geq 1} q^{2j-2+2^{j-1}} \frac{1}{1 - q^{2j}} \sum_{\ell \geq 2j} q^{2\ell}
\] (3.4)
is equal to $e - 2j + 1$ if $a = 2j - 2$, $n_{a+1} = n_{2j-1} = 1$, and it is equal to 0 otherwise.

Proof. By geometric series expansion, we see that the coefficient of $q^n$ in (3.4) is equal to the number of possibilities to write $n = 2^{2j-2} + 2^{2j-1} + k \cdot 2^{2j} + 2^\ell$ for some $j \geq 1$, $\ell \geq 2j$, and $k \geq 0$. The claim follows immediately. \hfill \Box

Lemma 4. Let $n \geq 2$, and write $n = \sum_{i=a}^{e} n_i \cdot 2^i$ as in Lemma 1. Then the coefficient of $q^n$ in
\[
\sum_{s \geq 1} q^{2s-1} \frac{1}{1 - q^{2s}} \sum_{\ell \geq 2s} q^{2\ell}
\] (3.5)
is equal to $e - 2s + 1$ if $a = 2s - 1$ and $n$ is not a power of 2, and it is equal to 0 otherwise.

Proof. By geometric series expansion, we see that the coefficient of $q^n$ in (3.5) is equal to the number of possibilities to write $n = 2^{2s-1} + k \cdot 2^{2s} + 2^\ell$ for some $s \geq 1$, $\ell \geq 2s$, and $k \geq 0$. The claim follows immediately. \hfill \Box

Lemma 5. Let $n \geq 2$, and write $n = \sum_{i=a}^{e} n_i \cdot 2^i$ as in Lemma 1. Then the coefficient of $q^n$ in
\[
\sum_{1 \leq s < t} q^{2s-2+2^{t-2}} \frac{1}{1 - q^{2s-1}} \frac{1}{1 - q^{2t-1}} \sum_{\ell \geq 2t-1} q^{2\ell}
\] (3.6)
is congruent to
\[
e - \chi(e \text{ even}) \sum_{i=a+2}^{e} n_i - a \cdot n_{a+2} + \left\lfloor \frac{1}{2} (e - a - 1) \right\rfloor \pmod{2},
\] (3.7)
where $\chi(S) = 1$ if $S$ is true and $\chi(S) = 0$ otherwise.

Proof. By geometric series expansion, we see that the coefficient of $q^n$ in (3.6) is equal to the number of possibilities to write
\[
n = (2k_1 + 1)^{2s-2} + (2k_2 + 1)^{2t-2} + 2^{2t-1+k_3}
\] (3.8)
for some $s$ and $t$ with $1 \leq s < t$ and $k_1, k_2, k_3 \geq 0$. Clearly, we need $a$ to be even in order that the number of these possibilities be non-zero. Given that $a = 2s - 2$, we just
have to count the number of possible triples \((t, k_2, k_3)\) in (3.8), since the appropriate \(k_1\) can certainly be found. If we fix \(t\) and \(k_3\), the number of possible \(k_2\)’s is

\[
\left\lfloor \frac{1}{2} \cdot \frac{n - 2^{2t-1} + k_3}{2^{2t-2}} + \frac{1}{2} \right\rfloor = \left\lfloor \frac{n}{2^{2t-1}} + \frac{1}{2} \right\rfloor - 2^{k_3}.
\]

This needs to be summed over all \(t\) and \(k_3\) with \(\frac{1}{2}(a + 2) = s < t \leq \frac{1}{2}(e + 1)\) and \(0 \leq k_3 \leq e - 2t + 1\). We obtain

\[
\sum_{t=s+1}^{\frac{1}{2}(e+1)} \sum_{k_3=0}^{e-2t+1} \left( \left\lfloor \frac{n}{2^{2t-1}} + \frac{1}{2} \right\rfloor - 2^{k_3} \right)
\]

\[
= \sum_{t=\frac{1}{2}(a+4)}^{\frac{1}{2}(e+1)} \sum_{k_3=0}^{e-2t+1} \left[ n_a \cdot 2^{a-2t+1} + \cdots + (n_{2t-2} + 1) \cdot 2^{-1} + n_{2t-1} + n_{2t} \cdot 2 + \cdots + n_e \cdot 2^{e-2t+1} \right] - \left\lfloor \frac{1}{2}(e - a - 1) \right\rfloor
\]

\[
= e \sum_{t=\frac{1}{2}(a+4)}^{\frac{1}{2}(e+1)} n_{2t-1} + e \sum_{t=\frac{1}{2}(a+4)}^{\frac{1}{2}(e-1)} n_{2t} + \left\lfloor \frac{1}{2}(e - a - 1) \right\rfloor \quad \text{(mod 2)}.
\]

**Lemma 6.** Let \(n \geq 2\), and write \(n = \sum_{i=a}^{e} n_i \cdot 2^i\) as in Lemma 7. Then the coefficient of \(q^n\) in

\[
\sum_{1 \leq s < t} q^{a2^{s-2} + 2^{s+t}} \frac{q^{2^{s-1}}}{(1 - q^{2^{s-1}})(1 - q^{2^{t-1}})} \quad \text{(3.9)}
\]

is congruent to

\[
\sum_{t=\frac{1}{2}(a+4)}^{\frac{1}{2}(e+1)} n_{2t-1} + \left\lfloor \frac{1}{2}(e - a - 1) \right\rfloor \quad \text{(mod 2).} \quad \text{(3.10)}
\]

**Proof.** By geometric series expansion, we see that the coefficient of \(q^n\) in (3.9) is equal to the number of possibilities to write

\[
n = (2k_1 + 1)2^{2s-2} + (k_2 + 2)2^{2t-1} \quad \text{(3.11)}
\]

for some \(s\) and \(t\) with \(1 \leq s < t\) and \(k_1, k_2 \geq 0\). Clearly again, we need \(a\) to be even in order that the number of these possibilities be non-zero. Given that \(a = 2s - 2\), we just have to count the number of possible pairs \((t, k_2)\) in (3.11), since the appropriate \(k_1\) can certainly be found. If we fix \(t\), the number of possible \(k_2\)’s is

\[
\left\lfloor \frac{n - 2t}{2^{2t-1}} + 1 \right\rfloor = \left\lfloor \frac{n}{2^{2t-1}} \right\rfloor - 1.
\]
This needs to be summed over all \( t \) with \( \frac{1}{2} (a + 2) = s < t \leq \frac{1}{2} (e + 1) \). We obtain

\[
\sum_{t = s + 1}^{\left\lfloor \frac{1}{2} (e + 1) \right\rfloor} \left( \left\lfloor \frac{n}{2^{t - 1}} \right\rfloor - 1 \right) \equiv \sum_{t = \frac{1}{2} (a + 4)}^{\left\lfloor \frac{1}{2} (e + 1) \right\rfloor} \left[ n_a \cdot 2^{a - 2t + 1} + \cdots + n_{2t - 2} \cdot 2^{1 - t} + \right.
\]

\[
\left. + (n_{2t - 1} - 1) + n_{2t} \cdot 2 + \cdots + n_e \cdot 2^{e - 2t + 1} \right] \equiv \sum_{t = \frac{1}{2} (a + 4)}^{\left\lfloor \frac{1}{2} (e + 1) \right\rfloor} n_{2t - 1} - \frac{1}{2} (e - a - 1) \pmod{2}. \quad \square
\]

We are finally in the position to state and prove our main result. It expresses the congruence class of \( w(n) \) modulo 8 — and thus, by \( 2\cdot 11 \), the congruence class of the unique path partition number \( u(n) \) modulo 16 — in terms of the binary digits of \( n \). We point out that the assertion \( 3.12 \) already appeared in [1, Prop. 4.5].

**Theorem 7.** Let \( n \geq 2 \), and write \( n = \sum_{i = a}^{e} n_i \cdot 2^i \) as in Lemma 7. Then, if \( a = e \) (i.e., if \( n \) is a power of 2), the number \( w(n) \) is congruent to

\[
2 \left\lfloor a / 2 \right\rfloor + 1 \pmod{8}, \quad (3.12)
\]

while it is congruent to

\[
2 + 2 \left\lfloor a / 2 \right\rfloor + 2 \chi(a \text{ even})(1 - 2n_{a + 1})(e - a - 1) + 4 \chi(a \text{ odd})(e - a)
\]

\[
+ 4 \chi(a \text{ even}) \left( e \sum_{i = a + 2}^{e - \chi(e \text{ even})} n_i + a \cdot n_{a + 2} + \sum_{t = \frac{1}{2} (a + 4)}^{\left\lfloor \frac{1}{2} (e + 1) \right\rfloor} n_{2t - 1} \right) \pmod{8} \quad (3.13)
\]

otherwise.

**Proof.** Let first \( n = 2^a \). We must then read the coefficient of \( q^n \) on the right-hand side of \( 3.1 \) and reduce the result modulo 8. Non-zero contributions come from the very first sum, from the series \( 2q^3 / (1 - q^2) \), and from the series which is discussed in Lemma 7. Altogether, we obtain

\[
1 + 2 \chi(a \geq 2) + 2 \max \{ \left\lfloor a / 2 \right\rfloor - 1, 0 \},
\]

which can be simplified to \( 3.12 \).

Now let \( n \) be different from a power of 2. The non-zero contributions when reading the coefficient of \( q^n \) on the right-hand side of \( 3.1 \) come again from the series \( 2q^3 / (1 - q^2) \), and from the series discussed in Lemmas 7.16. These contributions add up to

\[
2 \chi(n \geq 3) + 2 \left\lfloor a / 2 \right\rfloor + 2 \chi(a \text{ even}, n_{a + 1} = 0)(e - a - 1)
\]

\[
+ 2 \chi(a \text{ even}, n_{a + 1} = 1)(e - a - 1) + 4 \chi(a \text{ odd})(e - a)
\]

\[
+ 4 \chi(a \text{ even}) \left( e \sum_{i = a + 2}^{e - \chi(e \text{ even})} n_i - a \cdot n_{a + 2} + \left\lfloor \frac{1}{2} (e - a - 1) \right\rfloor + \sum_{t = \frac{1}{2} (a + 4)}^{\left\lfloor \frac{1}{2} (e + 1) \right\rfloor} n_{2t - 1} + \left\lfloor \frac{1}{2} (e - a - 1) \right\rfloor \right).
\]

This expression can be simplified to result in \( 3.13 \). \quad \square

It is clear that, in the same way, one could also derive a result for \( w(n) \) modulo 16, 32, \ldots, albeit at the cost of considerably more work.
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