Self-similar Rayleigh–Taylor mixing with accelerations varying in time and space

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As a ubiquitous paradigm of instabilities and mixing that occur in instances as diverse as supernovae, plasma fusion, oil recovery, and nanofabrication, the Rayleigh–Taylor (RT) problem is rightly regarded as important. The acceleration of the fluid medium in these instances often depends on time and space, whereas most past studies assume it to be constant or impulsive. Here, we analyze the symmetries of RT mixing for variable accelerations and obtain the scaling of correlations and spectra for classes of self-similar dynamics. RT mixing is shown to retain the memory of deterministic conditions for all accelerations, with the dynamics ranging from superballistic to subdiffusive. These results contribute to our understanding and control of the RT phenomena and reveal specific conditions under which Kolmogorov turbulence might be realized in RT mixing.

Rayleigh–Taylor and Richtmyer–Meshkov instabilities | variable acceleration | interfacial mixing | self-similar solutions | group theory

Turbulence is an unfinished problem of classical physics. Its theoretical richness attracts physicists and mathematicians alike, while its practical importance demands the attention of engineers and practitioners. Isotropy, homogeneity, and localness of scale-to-scale interactions are some of the fundamental paradigms that have advanced our understanding of turbulent dynamics. However, realistic processes often depart from these ideal paradigms. An important class of such processes is linked to the Rayleigh–Taylor (RT) mixing encountered in a large variety of configurations (1, 2).

RT instability occurs when the fluid interface is perturbed near the equilibrium state and develops under an accelerating velocity field. When the acceleration is constant, the instability is referred to as the classical RT instability, and the impulsive (e.g., shock-driven) case is known as the Richtmyer–Meshkov (RM) instability (3, 4). In general, the flow transitions from an initial stage of quick perturbation growth to a nonlinear stage, in which the growth rate slows and the interface is transformed to a combination of the large-scale coherent structure and shear-driven small scales. The final stage of this instability is an intense interfacial mixing, whose dynamics is believed to be self-similar. For an appreciation of the problem in a rich variety of contexts, see refs. 5–22.

This work adopts continuum fluid equations and advances the significant success achieved via the group-theory approach analyzing symmetries and invariant properties of RT dynamics (19, 22, 23). We explore the special classes of self-similar dynamics of RT mixing driven by accelerations that obey power laws in time and in space, which have not been discussed in earlier work (22–25). Our results explain existing experiments, broaden the horizons of studies of RT-relevant processes, and outline conditions under which Kolmogorov turbulence may manifest in RT/RM mixing.

Theoretical Approach

Governing Equations. Each fluid obeys the conservation laws for mass, momentum, and energy (3, 6–8) as:

\[ \frac{\partial \rho}{\partial t} + \frac{\partial \rho v_i}{\partial x_i} = 0, \]
\[ \frac{\partial \rho v_i}{\partial t} + \frac{\partial \rho v_i v_j}{\partial x_j} + \frac{\partial P}{\partial x_i} = 0, \]
\[ \frac{\partial E}{\partial t} + \partial (E + P)v_i/\partial x_i = 0, \]

where \((x, t) = (x, y, z, t)\) are spatial coordinates and time, \(\rho, v_i, P\) and \(E = \rho(\vec{v}^2/2 + \epsilon)\) are the density, velocity, pressure, and energy density, \(\epsilon\) and \(W = \epsilon + p/\rho\) being specific internal energy and enthalpy. These nonlinear partial differential equations are augmented with boundary conditions at the unstable and freely evolving interface and by those at the outer boundaries of the domain: \(\vec{v} \cdot \vec{n} = 0\), \([P] = 0\), \(\vec{v} \to \pm \vec{\infty} = 0\), and \([\vec{v} \cdot \vec{r}]\) and \([W]\) assume arbitrary values; here, \([\cdot]\) marks the function jump at the interface, and \(\vec{n}\) and \(\vec{r}\) are the normal and tangential unit vectors of the interface. These boundary conditions are augmented by initial conditions on flow perturbations at the interface and...
in the bulk. The boundary-value problem is influenced by possible singularities and secondary instabilities, and the initial value problem is ill-posed. The gravity \( g = (0, 0, -g) \), and \( g = \|g\| \), is directed from the heavy to the light fluid. In spatially extended systems, the flow can be periodic in the \((x, y)\) plane normal to gravity. The governing equations for nonideal fluids can be modified as necessary; for example, with the inclusion of the kinematic viscosity \( \nu \), the momentum equation acquires the term \(-\nu \partial^2 u / \partial x_1 \partial x_1\); for constant acceleration, this yields a viscous scale \( \nu \sim \left(\nu^2 / g\right)^{1/3} \).

While RT dynamics is complex because of many coupled scales, it has some features of universality and order in the early, as well as late, stages of development. They enable us to study from principles based on the group-theory approach. The approach reveals that the scale-dependent RT dynamics has the multiscale and interfacial character and that, for constant acceleration, the self-similar RT mixing exhibits a certain degree of order. As already stated, RT dynamics is often driven by accelerations that vary in time and space, which we take to be in the form of power laws. In applications, the power laws can be obtained from fits to the time- and space-dependence of observed accelerations. We focus on the analysis of invariants, scaling laws, correlations, and fluctuations of RT mixing by employing the momentum model described immediately below.

### The Momentum Model

The model \((24, 26)\) stipulates that the dynamics of a parcel of fluid undergoing RT mixing is governed by the specific (per unit mass) balance of the rate of momentum gain, \( \mu \), and the rate of momentum loss, \( \bar{\mu} \), as

\[
\dot{h} = v, \quad \bar{\dot{\mu}} = \bar{\mu} - \mu.
\]

Here, \( h \) is the length scale along \( g \) and \( v \) the corresponding velocity. The rate of gain \((\text{loss})\) of specific momentum is \( \bar{\mu} = \bar{v} / \bar{v} \) \((\mu = \varepsilon / v)\), with \( \varepsilon \) \((v)\) as the rate of gain \((\text{loss})\) of specific energy. Relations \( \bar{v} = v \bar{\mu} \) and \( \varepsilon = \nu \bar{v} \) between the rates of change of specific momentum and energy are the standard relations between power and force \((\text{per unit mass})\) \((5)\). The rate of energy gain is \( \varepsilon = gBv \), where \( B > 0 \) is the buoyancy; we rescale \( gB \rightarrow g \) with no loss of generality. The rate of energy loss is \( \varepsilon = C\nu^3 / L \), where \( L \) is the length scale of the energy dissipation, and the constant \( C > 0 \) is related to the drag on the flow.\(^*\) The linear, nonlinear, and mixing dynamics have \((\text{for any} g)\) their own values of \( B \) and \( C \) due to their distinct characteristics. These considerations lead to \( \bar{\mu} = \nu \) and \( \mu = C\nu^3 / L \). In RT mixing, the boundary condition \( |v \cdot n| = 0 \) does not depend on \( \rho \).

The power-law accelerations that we consider vary in time as \( g = Gt^\alpha \), \( t > t_0 > 0 \), with the exponent \( \alpha \in (-\infty, \infty) \) and the prefactor \( G > 0 \) with dimension \( m/s^{2+\alpha} \). They also vary in space, \( g = G_h \), \( h > h_0 > 0 \), where the exponent \( \beta \in (1, \infty) \) and the prefactor \( G > 0 \) has the dimension \( m^{1-\beta}/s^2 \). We seek asymptotic solutions for \((t/t_0, h/h_0) \rightarrow \infty\), given some initial values at \((t_0, h_0)\). The acceleration effects are important for all times in RT dynamics, but asymptotically negligible in RM.

In RT flows, two independent macroscopic length scales are dynamically relevant: the perturbation amplitude \( h \) in the direction of acceleration and the wavelength \( \lambda \) in the normal plane—the vertical and the horizontal scales, respectively. The former is regarded as an integral scale, believed to grow self-similarly in the mixing regime. The latter is set by some deterministic conditions—e.g., by the wavelength of the initial perturbation. Both are the characteristic energy-containing macroscales. The case \( L \sim \lambda \) corresponds to the scale-dependent linear and nonlinear RT/RM dynamics. The case \( L \sim h \) for energy dissipation \( \varepsilon = C\nu^3 / L \) resembles the self-similar RT and RM mixing and is the focus of our work.

### Asymptotic Dynamics

The momentum model has the same symmetries and scaling transformations as the governing equations and can thus capture the invariance and scaling properties of the asymptotic solutions (to within prefactors). For time-varying acceleration \( g = Gt^\alpha \), in the mixing regime with \( L = h \), the dynamics is of the RT type for \( \alpha > \alpha_{cr} \) with the asymptotic solution given by \( h_{RT} = B_{RT} t^{2+\alpha} \), where the exponent is set by \( \alpha \), and the prefactor \( B_{RT} \) is set by the acceleration parameters and the drag \((22-24)\). For \( \alpha < \alpha_{cr} \), the dynamics is of the RM type and the asymptotic solution is \( h_{RM} = B_{RM} t^{2+\alpha_{cr}} \), where the critical exponent is set by the drag, \( \alpha_{cr} = -2 + (1 + C)^{-1} \), and the prefactor \( B_{RM} \) by the drag and deterministic conditions.

For space-varying acceleration \( g = Gt^{\beta} \), in the mixing regime with \( L = h \), the momentum model has the exact solution \( v^2 = 2Gh^{(1+\beta)} / (1 + \beta + 2C) + V^2 h(h/h_0)^{-2C} \), where the constant values \( V_0 \) and \( h_0 \) (with dimensions \( m/s \) and \( m \), respectively) are set by determinstic conditions. The dynamics is of the RT type for \( \beta > \beta_{cr} \) with the asymptotic solution given by \( v_{RT}^2 = 2Gh^{(1+\beta)} / (1 + \beta + 2C) \). The dynamics is of the RM type for \( \beta < \beta_{cr} \) with \( v_{RM}^2 = V_0^2(h/h_0)^{-2C} \), where the critical exponent \( \beta_{cr} = -1 + (2 + C) \) is set by the drag.

For variable accelerations \( g = Gt^\alpha \) and \( g = G\nu^3 \), in RT-type mixing, the rate of momentum loss is \( |\mu| \sim |\bar{\mu}| \) with small \( |\bar{\mu}|/\mu \) \((1)\), whereas, in RM-type mixing, the rate of momentum loss is \( |\mu| \sim |\bar{\mu}| \) with asymptotically vanishing \( |\mu|/\mu \) \(\rightarrow 0 \). Self-similar mixing transitions occur from the RT to the RM type for \( \alpha \sim \alpha_{cr} \) and for \( \beta \sim \beta_{cr} \). The critical exponents behave as \( \alpha_{cr} \rightarrow -1 \) and \( \beta_{cr} \rightarrow -1 \) for \( C \rightarrow 0 \) and as \( \alpha_{cr} \rightarrow -2 \) and \( \beta_{cr} \rightarrow -\infty \) for \( C \rightarrow \infty \). While the RT and RM solutions are coupled as particular and homogeneous solutions, they are effectively decoupled because of their distinct symmetries.

For both time- and space-varying accelerations, \( g = Gt^\alpha \) and \( g = G\nu^3 \), the asymptotic solutions for self-similar mixing are the same, when the exponents and prefactors relate as \( \beta = \alpha/(2 + \alpha) \) and \( G = (2G(\alpha - \alpha_{cr})(2 + \alpha)/(2 + \alpha_{cr}))^{1/(2 + \alpha_{cr})} \).

### Scaling Laws and Fluctuations with Variable Acceleration

In this section, we analyze the scaling laws, correlations, fluctuations, and spectra of self-similar mixing.

### Dynamical Properties

In RT mixing with \( g = Gt^\alpha \) and \( \alpha > \alpha_{cr} \), the length scale varies with time as \( L \sim t^{(2 + \alpha)} \) and the velocity scale as \( v \sim t^{(\alpha + 1)} \). The length scale increases with time for all \( \alpha > \alpha_{cr} \), whereas the velocity scale increases with time for \( \alpha > -1 \), remains constant for \( \alpha = -1 \), and decreases for \( \alpha < -1 \). For the case with \( g = G\nu^3 \) and \( \beta > \beta_{cr} \), the velocity scale increases with the length scale for \( \beta > 1 \), decreases for \( \beta < -1 \), and is scale-independent for \( \beta = -1 \).

We compare these relations with scaling laws for diffusion \( L \sim t^{1/2} \), \( v \sim L^{-1} \), and ballistics \( L \sim t^2 \), \( v \sim L^{1/2} \), and denote similar/quicker/slower processes by the prefixes quasi/super/sub. We find (for \( \alpha > \alpha_{cr} \), \( \beta > \beta_{cr} \)) that RT mixing

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*While the rate of energy dissipation is \( \varepsilon = C\nu^3 / L \) in both RT mixing and Kolmogorov turbulence, the momentum and energy in RT mixing are gained by buoyancy, while the energy input in Kolmogorov turbulence arises from an external source operating at the large scale.*
is superballistic for $\alpha > 0, \beta > 0$; ballistic for $\alpha = 0, \beta = 0$; superdiffusive for $\alpha > -3/2, \beta > -3$; quasidiffusive for $\alpha = -3/2, \beta = -3$; and subdiffusive for $\alpha < -3/2, \beta < -3$. For $\alpha = -1, \beta = -1$, a transition occurs from dynamics with larger velocities at larger scales to one with larger velocities at smaller scales. In RM mixing $\alpha < \alpha_c$, $\beta < \beta_c$, the length scale increases with time as $L \sim t^{1/(1+\alpha)}$, whereas the velocity decreases $v \sim t^{-C/(1+\alpha)}$, and larger velocities correspond to smaller scales $v \sim L^{-C}$; the dynamics is subdiffusive for $C > 1$. Kolmogorov turbulence is superdiffusive, and larger velocities are associated with larger length scales.

Symmetries and Invariant Forms. RT mixing and homogeneous and isotropic turbulence have different symmetries. For the latter, they are Galilean-like, translations in time, as well as rotations and reflections in space. Turbulence is also invariant under the scaling transformation $L \rightarrow L K$, $v \rightarrow v K^n$, and $t \rightarrow t K^{1-n}$, conditional on $\nu \rightarrow \nu K^{1+n}$. For vanishing viscosity, $\nu/vL \rightarrow 0$, the scaling exponent is $n = 1/3$, and the associated invariant quantity is the rate of energy dissipation $\varepsilon = -(\partial v_i/\partial x_j)^2$, with $\varepsilon \rightarrow \varepsilon K^{3n-1}$.

RT mixing is noninertial and invariant under translations, rotations, and reflection only in the plane normal to the acceleration $g$. For variable accelerations ($\alpha > \alpha_c, \beta > \beta_c$), the dynamics is also invariant with respect to scaling transformation $L \rightarrow L K$, $v \rightarrow v K^n$, and $t \rightarrow t K^{1-n}$, conditional on $\nu \rightarrow \nu K^{1+n}$ and on $G \rightarrow G K^{(n+2)/(\alpha+1)}$ for $g = G t^n$, and $G \rightarrow G K^{2n/(\beta+2)}$ for $g = G h^\beta$. For vanishing viscosity $\nu/vL \rightarrow 0$, in the case $g = G t^n$, the scaling exponent $n = (\alpha + 1)/(\alpha + 2)$ leads to the invariance of the quantities that we refer to as modified rate of momentum loss and gain, given by $M \sim \nu^\alpha L^{1+\alpha}$ and $\tilde{M} = \alpha$, respectively. For $g = G h^\beta$, the scaling exponent is $n = (\beta + 1)/2$, and the invariant quantities are $\tilde{M} \sim \nu^\beta L^{\beta+1}$ and $\tilde{M} \supseteq \beta$. The invariant quantities in the two cases relate as $\tilde{M} = M^{(1-\beta)}$ and $\tilde{M} \sim \tilde{M}^{(1-\beta)}$, with $\beta = \alpha/(2 + \alpha)$.

In RT mixing with variable accelerations ($\alpha < \alpha_c, \beta < \beta_c$), the invariant quantities are the critical values of the modified rate of momentum loss $M_{cr} \sim \nu^\alpha L^{1+\alpha}$ and $\tilde{M}_{cr} \sim \nu^\beta L^{\beta+1}$, whereas the scaling exponent is $n = (\alpha_c + 1)/(\alpha_c + 2) = (\beta + 1)/2 = C$.

One can view the quantities $M$ and $\tilde{M}$ (as well as $\tilde{M}$ and $\tilde{M}$) as fractional derivatives $d^\alpha/dt^\alpha$ (or $d^\beta/dx^\beta$) of the loss and gain of specific momentum in the direction of $g$. The quantity $M_{cr}$ can be considered the $\alpha_c$-th order $t$-derivative of the momentum loss in the direction of $g$, while $\tilde{M}$ denotes the $\beta_c$-th derivative of the momentum loss in direction $g$.

In RT (RM) mixing with $\alpha > \alpha_c$ (or $\alpha < \alpha_c$) for time-varying acceleration $g = G t^n$ and with $\beta > \beta_c$ ($\beta < \beta_c$) for space-varying acceleration $g = G h^\beta$, the invariance of $M_{cr}$ and $M(\tilde{M}_{cr})$ implies that there is a nondissipative interscale transport (8, 16, 17, 23). The situation is similar to isotropic turbulence, where the invariance of energy-dissipation rate $\varepsilon$ is compatible with the existence of an inertial range of one to four.

In RT mixing ($\alpha > \alpha_c, \beta > \beta_c$), the invariance of $M$ and $\tilde{M}$ leads to $\varepsilon \sim t^{2(\alpha+1)}$ and $\tilde{\varepsilon} \sim L^{3(3\beta+1)/2}$ and to $\mu \sim \nu^\alpha$ and $\mu \sim L^\beta$. The rate of energy dissipation is scale-independent only for $\alpha = -1/2$ and $\beta = -1/3$, whereas the rate of momentum loss is constant only for $\alpha = \beta = 0$. In RM mixing, $\alpha < \alpha_c$ and $\beta < \beta_c$, rates of energy dissipation and momentum loss decay with time and with length scale for any $C$ (always positive), as $\varepsilon \sim t^{-3(1+3C)/(1+C)} \sim L^{-1+3C}$ and $\mu \sim t^{-3(1+2C)/(1+C)} \sim L^{-1+2C}$.

Enstrophy and helicity are the other invariants in classical turbulence (27, 28). In RT/RM mixing with variable accelerations, these quantities are, in general, scale-dependent.

Scaling Laws and Correlations. In RT mixing for which $\alpha > \alpha_c$ and $\beta > \beta_c$, the invariant properties of $M$ and $\tilde{M}$ lead to $v_{in}^{a+2}/L^{(\alpha+2)} \sim v_{in}^{a+1}$ and $v_{in}^{2}/L^{(\alpha+1)} \sim v_{in}^{1}/L^{(1+\alpha)}$, where $v_{in}$ is the velocity at the length scale $L$. This results in the scaling for the velocity as $v_{in}/v \sim (L^{(\alpha+2)}/(a+2)$ and $v_{in}/L \sim (L^{(\beta+1)}/\alpha+2)$ and in the N-th order structure functions $\sim (M^{(a+1)})^{N/(\alpha+2)}$ and $\sim (M^{(\beta+1)})^{N}/2$. In Kolmogorov turbulence, the velocity scaling is $v_{in}/v \sim (L^{\beta+1})$, and the N-th order structure function is $(\tilde{y})^{N}/3$ because of the invariant form of $\varepsilon \sim v^{3}/L$. Thus, we find that, when compared to Kolmogorov turbulence, the velocity correlations in RT mixing are stronger (weaker) for $\alpha > -1/2$ and $\beta > -1/3$ ($\alpha < -1/2$ and $\beta < -1/3$), and the two are the same for $\alpha = -1/2$ and $\beta = -1/3$. In RM mixing, $\alpha < \alpha_c$ and $\beta < \beta_c$, the invariance of $M_{cr}$ and $\tilde{M}_{cr}$, leads to the velocity scaling $v_{in}/v \sim (L^{(\alpha+2})$ and to the exponent $(-N)/C$ for the N-th order structure function, with larger velocities at larger length scales.

In RT mixing, $\alpha > \alpha_c$ and $\beta > \beta_c$, the Reynolds number $Re = vL/\nu$ is, in general, scale-dependent and is given by $Re \sim (\nu^{(2n+3)/\nu} + \nu \sim (L^{(2n+3)})^{\nu} / \nu$. Defining the local Reynolds number as $Re_l \sim vL/\nu$, we find the Reynolds-number scaling to be $Re_l \sim (L^{(2n+3)})^{\nu}$ and $Re_l \sim (L^{(\beta+3)}/\nu$ and the viscous scale to be $L_v \sim (\nu^{(\alpha+2)/M^{(\beta+3)})$ and $L_v \sim (\nu^{(\beta+1)/M^{(\beta+3)})$. In RT mixing, the viscous scale remains finite, is set by the viscosity and the acceleration strength, and is comparable to the fastest-growing wavelength $(\nu^{\alpha+2}/G)^{1/(\alpha+2)}$ and $(\nu^{\beta+1}/G)^{1/(\beta+3)}$. In RM mixing, $\alpha > \alpha_c$ and $\beta < \beta_c$, the scaling for the Reynolds number is $Re_l \sim (L^{(2n+3)})^{\nu} + \nu \sim (L^{(\beta+3)})$ and the viscous scale is $L_v \sim (\nu^{(\alpha+2)/M^{(\beta+3)})$ and $L_v \sim (\nu^{(\beta+1)/M^{(\beta+3)})$. Respectively, while the viscous scale is finite for late times in RM mixing, it is distinct from that at early times $\nu/v_{in}$, where $v_{in}$ is some initial growth rate. For reference, we recall that the relations in Kolmogorov turbulence are $Re_l \sim (L^{4/3}$ and $L_v \sim (\nu^{3/4})$.

Fluctuations and Spectra. Fluctuations with strengths far exceeding the noise from deterministic conditions are essential for systems to be regarded as turbulent.

To compare the strengths of velocity fluctuations caused by the dynamical process and by deterministic conditions, we consider two parcels of fluid involved in the flow with a time delay $\tau$ and an initial displacement $\lambda$ (5). In Kolmogorov turbulence (29, 30), which is a stochastic process with self-generated fluctuations, the fluctuations $\sim (\varepsilon v F)^{1/3}$ and $\sim (\varepsilon v \lambda^{1/4})$ are substantially stronger than the relative velocity of the fluid parcels $\sim (\varepsilon v)^{1/2}$ and $\sim (\varepsilon \lambda)^{1/3}$.

In RT mixing with time- and space-varying accelerations ($\alpha > \alpha_c, \beta > \beta_c$), the fluctuations $\sim M^{(\alpha+1)}$ and $\sim M^{(\beta+1)}$ are comparable to the relative velocities of the fluid parcels, $M^{(\alpha+1)}$ and $\sim M^{(\beta+1)}$. Furthermore, the ratio of fluctuating and mean velocities is $\sim (\varepsilon v \lambda^{1/4})$ and $\sim (\lambda L)^{1/3}$. Similar consideration is applied to RM mixing, conditional on $\alpha \rightarrow \alpha_c$ with $M \rightarrow M_{cr}$ and on $\beta \rightarrow \beta_c$, with $M \rightarrow M_{cr}$. Hence, in both RT and RM mixing for both time- and space-varying accelerations, the fluctuations retain the memory of the deterministic
conditions. In RT mixing, the strength of fluctuations decays with time and length for $\alpha, \beta > -1$, is scale-independent for $\alpha, \beta = -1$, and increases with time and length for $\alpha_{cr} < \alpha < -1$ and $\beta_{cr} < \beta < -1$; it always increases with time and length in RM mixing.

In Kolmogorov turbulence, the invariance of the energy-dissipation rate leads to the kinetic energy spectrum of the form $E(k) \sim \varepsilon^{2/3} k^{-5/3}$, where $E(k)$ is the spectral density and $k$ is the wavenumber, $k \sim l^{-1}$. In RT mixing with time- and space-varying accelerations ($\alpha > \alpha_{cr}, \beta > \beta_{cr}$), due to the invariance of their respective modified rates of momentum $M$ and $\dot{M}$, the spectral density of the specific kinetic energy has the forms $E(k) \sim M^{2}(\alpha+2) k^{-(3\alpha+4)/(\alpha+2)}$ and $E(k) \sim \dot{M} k^{-(\beta+2)}$. The spectrum is steeper than the Kolmogorov form for $\alpha > -1/2$ and $\beta > -1/3$ and is less so for $\alpha \in (\alpha_{cr}, -1/2)$ and $\beta \in (\beta_{cr}, -1/3)$. The scaling exponents in $E(k)$ may change sign for either time- or space-varying accelerations since larger velocities correspond to smaller length scales for $\alpha \in (\alpha_{cr}, -1)$ and $\beta \in (\beta_{cr}, -1)$; this is in contrast to superdiffusive turbulence having larger velocities at larger scales. In RM mixing ($\alpha < \alpha_{cr}$ and $\beta < \beta_{cr}$), the spectral density is $E(k) \sim \dot{M}^{2}(\alpha_{cr}+2) k^{-(3\alpha_{cr}+4)/(\alpha_{cr}+2)}$ and $E(k) \sim M_{cr} k^{-(\beta_{cr}+2)}$. The signs of the scaling exponents are consistent with the subdiffusive character of RM mixing.

While spectral properties of kinetic energy fluctuations in RT and RM mixing differ from those of Kolmogorov turbulence, those of density fluctuations remain the same. This is because their dynamic balances are achieved per unit mass, and their respective invariant quantities contain no fluid density. The spectra for density fluctuations are given by $E(k) \sim \rho_{0} e_{0}^{2} k^{-1}$ for Kolmogorov turbulence; $E(k) \sim \rho_{0} M_{0}^{2} k^{-1}$ and $E(k) \sim \dot{\rho}_{0} M_{0}^{2} k^{-1}$ for RT mixing with $\alpha > \alpha_{cr}$ and $\beta > \beta_{cr}$; and $E(k) \sim \rho_{0} M_{cr}^{2} k^{-1}$ and $E(k) \sim \dot{\rho}_{0} M_{cr}^{2} k^{-1}$ for RM mixing with $\alpha < \alpha_{cr}$ and $\beta < \beta_{cr}$; here, $\rho_{0}$ is the heavy fluid density. Note the same scaling exponent $-1$ in all these cases.

The results of this section are summarized in Table 1.

### Special Self-Similar Class

Here, we consider the change in the characteristics of self-similar RT/RM mixing for a broad range of acceleration parameters, when the acceleration varies (in time or space).

**Superballistics.** For $\alpha > 0$ and $\beta \in (0, 1)$, RT mixing is superballistic, and the velocity correlations are strong. As $\alpha \to \infty$ and $\beta \to 1$, the invariant quantities are $M \sim v_{l} L$ and $\dot{M} \sim v_{l}^{2} L^{2}$; the velocity and Reynolds number scale as $v_{l} / v \sim (l / L)$ and $Re_{l} \sim Re \sim (l / L)^{2}$; and the spectral density is given by $E(k) \sim k^{-3}$.

**Ballistics.** For $\alpha = 0$ and $\beta = 0$, the invariant quantities are $M = \mu \sim v^{2} / L$ with $\dot{M} \sim M^{2}$. The velocity and the Reynolds number scale as $v_{l} / v \sim (l / L)^{1/2}$ and $Re_{l} / Re \sim (l / L)^{3/2}$, and the spectral density is $E(k) \sim k^{-2}$. This dynamics is ballistic with strong correlations and weak fluctuations, and their strength is set by deterministic conditions, decaying as $t^{-1}$ and $L^{-1/2}$.

**Super-Kolmogorov Case.** For $\alpha \in (-1/2, 0)$ and $\beta \in (-1/3, 0)$, RT mixing is subballistic and super-Kolmogorov; that is, its correlations are stronger and spectra steeper than in Kolmogorov turbulence.

**Quasi-Kolmogorov.** For $\alpha = -1/2$ and $\beta = -1/3$, RT mixing is quasi-Kolmogorov since it has the same scaling properties, including the invariance of $\varepsilon$, the velocity and Reynolds-number scaling, and the spectral density is $E(k) \sim k^{-5/3}$ (29, 30). In contrast to Kolmogorov turbulence, however, fluctuations in RT mixing are sensitive to deterministic conditions, with their strength decaying as $t^{-1/2}$ and $L^{-1/3}$.

**Sub-Kolmogorov.** For $\alpha < -1/2$ and $\beta < -1/3$ in RT mixing, the correlations are weaker, and the spectral roll-off is more gradual than in Kolmogorov turbulence, even though, for $\alpha > -1$ and $\beta > -1$, RT mixing has larger velocities at larger scales.

**Steady Flex.** We call self-similar RT mixing with $\alpha, \beta = -1$ as “steady flex.” This is because the mixing has larger velocities at larger (smaller) time and length scales for $\alpha, \beta > -1$ ($< -1$), with $\alpha, \beta = -1$ indicating an inflection point (23). For $\alpha, \beta = -1$, RT mixing has $M \sim l$, its velocities are scale-independent ($\eta_{l} \sim 0$), the Reynolds number scales as $Re_{l} / Re \sim l / L$, and the velocity fluctuations are set by deterministic conditions and, hence, have constant strength, with the spectral density of the form $E(k) \sim v_{l}^{-1}$.

**Superdiffusion.** For $\alpha > -3/2, \beta > -3$, RT mixing is superdiffusive, even though it has larger velocities at smaller length scales for $\alpha \in (-3/2, 1), \beta \in (-3, -1)$.

**Quasidiffusion.** For $\alpha = -3/2, \beta = -3$, RT mixing is quasidiffusive, with the diffusion scaling law given by $M \sim t^{1/2}$ and $v_{l} / v \sim (l / L)$ and with the scale-invariant Reynolds number $Re_{l} \sim Re$. At $\alpha = -3/2, \beta = -3$, the fluctuations are set by deterministic conditions; they increase (decrease) in strength with time (length scale) leading to the spectral density $E(k) \sim k$. The sign of the spectral exponent is consistent with the quasidiffusive character of the dynamics having larger velocities at smaller scales. For $\alpha < -3/2, \beta < -3$, RT mixing is subdiffusive, and its dynamics is strongly influenced by deterministic conditions for smaller acceleration exponents.

**Subdiffusive RM Mixing.** For $\alpha \sim \alpha_{cr}, \beta \sim \beta_{cr}$, the mixing is of the RM type. For $\alpha < \alpha_{cr}, \beta < \beta_{cr}$, RM mixing is independent of the acceleration and is set by the drag and the deterministic conditions; and, as a subdiffusive process, it has larger and more intense motions at small scales. The velocity and the Reynolds
number scale as \( v_t / v \sim (L/l)^C \) and \( Re_t / Re = (L/l)^{(C-1)} \); the strength of deterministic fluctuations increases as \( t^C/(C+1) \) and \( L^C \); and the spectral density for the specific kinetic energy is \( E(k) \sim k^{2(C-1)} \).

**RT Mixing in Laboratory Experiments**

**Spectral Properties.** In a special self-similar class of RT/RM mixing with variable accelerations in time and space, theoretical spectra are power laws, expected to manifest over scales \( l \in (l_{\text{min}}, l_{\text{max}}) \), which are far from the largest scale \( \sim L \) and the smallest scale \( \sim l_v \). These conditions, easily stated in theory, are a challenge to achieve in experiments, where we expect the dynamics to be scale-invariant for scales \( l \gg l_v \) and be scale-dependent for \( l \sim l_v \). For a spectral density \( E(k) \) with \( k \sim l^{-1} \), the function behaving as a scale-invariant power law \( \sim k^a \) for \( k \ll k_v \), and a scale-dependent exponential \( \sim e^{bk} \) for \( k \sim k_v \), we may expect \( E(k) \) to be a compound function of the form \( k^a e^{bk} \).

As an illustration, this function will be applied below to available experimental data for the accurate evaluation of spectral shapes and scaling.

To analyze available experimental data on spectra in RT mixing, we apply a rigorous statistical technique for fitting parameters of a given spectral model to data from ref. 31. The method constructs the maximum-likelihood estimator of the fitting parameters, numerically solves the optimization problem, estimates statistical errors, verifies goodness of fit, and analyzes the dependence of the fitting parameters for the fitting range with the left and right cut-offs \( k_l, k_r \) (31).

Figure 1 presents the analysis of data (13) by the method of ref. 31. The fit parameters have small relative errors and an acceptably high goodness-of-fit score; it is outlined within the ellipse on the left. The best fit for the density fluctuations \( \rho \) is achieved for \( k \in (55, 3500) \) marked in red on the right.

**Table 2 presents the analysis of raw data of fluctuations spectra in RT mixing with constant \( g \) in fluid experiments (13) (velocity \( v^2 \), density \( \rho \)); the data analysis has been conducted by Mr. K.C. Williams (32). These results are available in the supporting information and will be published separately. In the domain \( k \in (k_l, k_r) \), where relative errors of the fit parameters are the smallest and the goodness-of-fit score is high, the experimental data agree with our theory. Particularly, the power-law exponents for the velocity and density fluctuations conform to their theoretical values in the \( k \)-spectrum at \( \alpha = \beta = 0 \), to within about 3%. Furthermore, the scale set by the exponential decay rate \( |b|^{-1} \) for velocity fluctuations \( v^2 \) is comparable within \( \sim 30\% \) to the scale \( k_v \sim l_v \). Note that for density fluctuations \( \rho \), the scale set by the exponential decay rate \( |b|^{-1} \) differs from \( k_v \) by a factor of four. This indicates that RT mixing remains heterogeneous at small scales.

**Sensitivity to Deterministic Conditions.** We can similarly analyze the data in plasma experiments (33). Remarkably, in plasma experiments (33) with \( Re \sim 7 \times 10^9 \), the power-law spectrum spans a decade or so, similar to those (13) in fluids with \( Re \sim 3.4 \times 10^4 \). In Kolmogorov turbulence, the ratio between the integral and viscous scales is \( Re^{3/4} \), leading technically to \( \sim 6.8 \) and \( \sim 4.5 \) decades in the two experiments, respectively. The relatively short and decade-wide range of scales of the inertial range found in the experiments (13, 33) indicates, to some extent, the deterministic rather than stochastic character of RT mixing, in agreement with our analysis.

Sensitivity of RT/RM mixing to deterministic conditions for all accelerations and Reynolds numbers is an important outcome of
group-theory analysis. It explores the experiments (8–10) on RT mixing conducted for a broad range of parameters and conditions, including steady and variable accelerations, and periodic and localized perturbations of the interface and flow fields. In the experiments (8–10), RT mixing remains interfacial, heterogeneous, and anisotropic, and it retains the memory of deterministic and initial conditions at the Reynolds numbers \( Re \sim 3.2 \times 10^5 \), in conformity with our results.

**Laminarization of Accelerated Flows.** Our analysis finds that for accelerations \( g \sim t^{\alpha}, \alpha > -1/2 \) and \( g \sim h^{\beta}, \beta > -1/3 \), the correlations in RT mixing are stronger and spectra are steeper than in Kolmogorov turbulence. Hence, strongly accelerated RT mixing may laminarize. Our results are consistent with experiments in accelerated boundary layers (34–37), which laminarize when rapidly accelerated. In our analysis, rapid accelerations correspond to large acceleration exponents.

**Interplay between Acceleration and Turbulence.** Our theory explains experiments and simulations of RT mixing with constant acceleration, which found the kinetic energy spectra to be steeper than Kolmogorov’s (18). It also explains the flattening of initially steep spectra of the kinetic energy in simulations of RM mixing (16). It further clarifies why spectra of density fluctuations in RT/RM mixing and Kolmogorov turbulence obey the same scaling: In all these cases, the dynamic balances are achieved per unit mass. Hence, our results apply to a study of the interplay of turbulence with acceleration. The theory says that RT mixing is quasi-Kolmogorov for the acceleration exponent \( \alpha = -1/2 \). Does this imply that variable acceleration \( g \sim t^{-1/2} \) produces and maintains turbulence the same way as Kolmogorov turbulence? Our theory suggests that it is possible, conditional on the existence of a source supplying energy at a constant rate and the flow becoming homogeneous and isotropic. While the acceleration may quickly set appropriate scaling laws, one needs a source of turbulent energy, as well as a high Reynolds number (i.e., large span of scales) to maintain strong and stochastic fluctuations.

**Anomalous Scaling.** The departures from the Kolmogorov scaling laws appear to be clear in realistic turbulent processes (38–41). They are often referred to as “anomalous scaling.” Our analysis finds that RT/RM mixing with variable accelerations (in time and in space) belong to a special self-similar class and exhibit significant departures in terms of invariant, scaling, spectral, and correlation properties from Kolmogorov turbulence. This suggests that these types of anomalies can be rather normal and that one can systematically incorporate realistic environments into a self-consistent format (11, 12).

**Discussion**

It is amazing that RT mixing, whose simple paradigm is that of an overturned cup of water (8), is encountered in many processes in nature and applications, from supernovae and fusion to fossil-fuel recovery. This paper has focused on the effect of variable accelerations on self-similar RT mixing—particularly on its scaling, correlations, and spectra.

We have studied properties of RT/RM dynamics with accelerations varying in both time and space. Within the group-theory approach (20, 22), we elaborated on the concept of the invariance of the modified rate of momentum loss and identified special classes of self-similarity. We found that self-similar dynamics of RT/RM mixing can vary from superballistic to subdiffusive, depending on the accelerations, and retains the memory of the deterministic conditions for any acceleration. We have analyzed available experimental data and found excellent quantitative and qualitative agreement with a broad range of experiments and simulations in fluids and plasmas (8–10, 13, 19, 33). These rich properties of RT/RM dynamics with variable accelerations emphasize the need for further experimental and numerical studies of RT/RM dynamics, achieving substantial spans of spatial and temporal scales, higher precision, and larger amounts of data, in order to accurately quantify the interplay of accelerations, unstable interfaces, and deterministic conditions in realistic environments (38). Our results help the understanding of natural phenomena for which RT/RM mixing is critically important, such as supernovae and fusion (20, 21). In summary, the motion of water from an overturned cup remains an inspiration for researchers in science, mathematics and engineering and is still a source of open problems to a curious mind.

**Data Availability.** There are no data underlying this work.

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1. R. Lord, Investigation of the character of the equilibrium of an incompressible heavy fluid of variable density. Proc. Lond. Math. Soc. 14, 170–177 (1883).
2. R. M. Davies, G. I. Taylor, The mechanics of large bubbles rising through extended liquids and through liquids in tubes. Proc. R. Soc. Lond. A Math. Phys. Sci. 200, 375–390 (1950).
3. R. D. Richtmyer, Taylor instability in shock acceleration of compressible fluids. Commun. Pure Appl. Math. 13, 297–319 (1960).
4. E. E. Meshkov, Instability of the interface of two gases accelerated by a shock. Sov. Phys. Fluid Dyn. 4, 101–104 (1969).
5. L. D. Landau, E. M. Lifshitz, Course of Theoretical Physics XXX (Pergamon Press, New York, 1987).
6. Y. B. Zel’dovich, Y. P. Raizer, Physics of Shock Waves and High-Temperature Hydrodynamic Phenomena (Dover, New York, English ed. 2, 2002).
7. D. Arnett, Supernovae and Nucleosynthesis: An Investigation of the History of Matter, from the Big Bang to the Present (Philips Trans.- Royal Soc., Math. Phys. Eng. Sci. 358, 619–660 (2017).
8. S. I. Anisimov, R. P. Drake, S. Gauthier, E. E. Meshkov, S. I. Abarzh, What is certain and what is not so certain in our knowledge of Rayleigh-Taylor mixing? Philos. Trans.- Royal Soc., Math. Phys. Eng. Sci. 371, 20130266 (2013).
9. E. E. Meshkov, Some peculiar features of hydrodynamic instability development. Philos. Trans.- Royal Soc. Math. Phys. Eng. Sci. 371, 20120288 (2013).
10. E. E. Meshkov, S. I. Abarzh, On Rayleigh-Taylor interface mixing, Fluid Dyn. Res. 51, 065502 (2019).
11. S. I. Abarzh, W. A. Goddard, Interfaces and mixing: Non-equilibrium transport across the scales. Proc. Natl. Acad. Sci. USA 116, 18171–18174 (2019).
12. S. I. Abarzh, S. Gauthier, K. R. Sreenivasan, Turbulent mixing and beyond: Non-equilibrium processes from atomistic to astrophysical scales. Philos. Trans.- Royal Soc., Math. Phys. Eng. Sci. 371, 20120435 (2013).
13. B. Akula, P. Suchandra, M. Mikhaeil, D. Ranjan, Dynamics of unstably stratified free shear flows: An experimental investigation of coupled Kelvin-Helmholtz and Rayleigh-Taylor instability. J. Fluid Mech. 816, 619–660 (2017).
14. J. Glimm, D. H. Sharp, T. Kaman, H. Lim, New directions for Rayleigh-Taylor mixing. Philos. Trans.- Royal Soc., Math. Phys. Eng. Sci. 371, 20120183 (2013).
15. K. Kadlau, J. L. Barber, T. C. Gemmell, B. L. Holian, B. J. Alder, Atomistic methods in fluid simulation. Philos. Trans.- Royal Soc., Math. Phys. Eng. Sci. 368, 1547–1560 (2010).
16. B. Thumber et al., Late-time growth rate, mixing, and anisotropy in the multimode narrowband Richtmyer-Meshkov instability. The θ-group collaboration. Phys. Fluids 29, 105107 (2017).
17. V. E. Neuzhava, Theory of turbulent mixing. Sov. Phys. Dokl. 20, 398–400 (1975).
18. J. R. Ristorelli, I. I. Clark, Rayleigh-Taylor turbulence: Self-similar analysis and direct numerical simulations. J. Fluid Mech. 507, 213–253 (2004).
19. N. C. Sawin et al., Rayleigh-Taylor mixing in supernova experiments. Phys. Plasmas 22, 102707 (2015).
20. S. I. Abarzh et al., Supernova, nuclear synthesis, fluid instabilities, and interfacial mixing. Proc. Natl. Acad. Sci. USA 116, 18184–18192 (2019).
21. B. R. Remington et al., Rayleigh-Taylor instabilities in high-energy density settings on the National Ignition Facility. Proc. Natl. Acad. Sci. USA 116, 18233–18238 (2019).
22. S. I. Abarzh, Self-similar interfacial mixing with variable acceleration. Phys. Fluids 33, 122110 (2021).
23. S. I. Abarzh, On fundamentals of Rayleigh-Taylor turbulent mixing. Europhys. Lett. 91, 35001 (2010).
24. S. I. Abarzh, Review of theoretical modelling approaches of Rayleigh-Taylor instabilities and turbulent mixing. Philos. Trans.- Royal Soc., Math. Phys. Eng. Sci. 368, 1809–1828 (2010).
25. I. Sedov, Similarity and Dimensional Methods in Mechanics (CRC Press, Boca Raton, FL, ed. 10, 1993).
26. K. R. Sreenivasan, S. I. Abarzh, Acceleration and turbulence in Rayleigh-Taylor mixing. Philos. Trans.- Royal Soc. A Math. Phys. Eng. Sci. 371, 20130267 (2013).
27. J. J. Moreau, Constantes d’un flot tourbillonnaire en fluid parfait barbotant. C. R. Heb. Séances Acad. Sci Paris 252, 2810–2812 (1961).
28. H. K. Moffatt, The degree of knottedness of tangled vortex lines. J. Fluid Mech. 35, 117–129 (1969).
29. A. N. Kolmogorov, Local structure of turbulence in an incompressible fluid for very large Reynolds numbers. Dokl. Akad. Nauk SSSR 30, 301–305 (1941).
30. A. N. Kolmogorov, Energy dissipation in locally isotropic turbulence. Dokl. Akad. Nauk SSSR 32, 19 (1941).

31. D. Pfefferlé, S. I. Abarchi, Whittle maximum likelihood estimate of spectral properties of Rayleigh-Taylor interfacial mixing using hot-wire anemometry experimental data. Phys. Rev. E 102, 053107 (2020).

32. K. C. Williams, "Fluctuations Spectra in Rayleigh-Taylor Interfacial Mixing" in 2022 March Meeting of the American Physical Society, Chicago, IL. https://meetings.aps.org/Meeting/MAR22/Session/G00.289 (Accessed 31 October 2022).

33. G. Rigon et al., Micron-scale phenomena observed in a turbulent laser-produced plasma. Nat. Commun. 12, 2679 (2021).

34. R. Narasimha, K. R. Sreenivasan, Relaminarization in highly accelerated turbulent boundary layers. J. Fluid. Mech. 61, 417–447 (1973).

35. R. Narasimha, K. R. Sreenivasan, Relaminarization of fluid flows. Adv. Appl. Mech. 19, 221–309 (1979).

36. K. R. Sreenivasan, P. J. Strykowski, Stabilization effects in flow through helically coiled pipes. Exp. Fluids 1, 31–36 (1983).

37. K. R. Sreenivasan, Laminarescent, relaminarizing and retransitional flows. Acta Mech. 44, 1–48 (1982).

38. K. R. Sreenivasan, R. A. Antonia, The phenomenology of small-scale turbulence. Annu. Rev. Fluid Mech. 28, 433–472 (1997).

39. A. Pouquet, P. D. Mininni, The interplay between helicity and rotation in turbulence: Implications for scaling laws and small-scale dynamics. Phil. Trans. R. Soc. A 368, 1635–1662 (2010).

40. V. Yakhot, D. Donzis, Emergence of multiscaling in a random-force stirred fluid. Phys. Rev. Lett. 119, 044501 (2017).

41. J. C. Klewicki, G. P. Chini, J. F. Gibson, Prospectus: Towards the development of high-fidelity models of wall turbulence at large Reynolds number. Philos. Trans. - Royal Soc., Math. Phys. Eng. Sc. 375, 20160092 (2017).