Bayesian Inference of High-density Nuclear Symmetry Energy from Radii of Canonical Neutron Stars

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Abstract

The radius $R_{1.4}$ of neutron stars (NSs) with a mass of 1.4 $M_\odot$ has been extracted consistently in many recent studies in the literature. Using representative $R_{1.4}$ data, we infer high-density nuclear symmetry energy $E_{\text{sym}}(\rho)$ and the associated nucleon specific energy $E_d(\rho)$ in symmetric nuclear matter (SNM) within a Bayesian statistical approach using an explicitly isospin-dependent parametric equation of state (EOS) for nucleonic matter. We found the following. (1) The available astrophysical data can already significantly improve our current knowledge about the EOS in the density range of $\rho_0 - 2.5 \rho_0$. In particular, the symmetry energy at twice the saturation density $\rho_0$ of nuclear matter is determined to be $E_{\text{sym}}(2 \rho_0) = 39.2^{+12.1}_{-6.1}$ MeV at a 68% confidence level. (2) A precise measurement of $R_{1.4}$ alone with a 4% 1σ statistical error but no systematic error will not greatly improve the constraints on the EOS of dense neutron-rich nucleonic matter compared to what we extracted from using the available radius data. (3) The $R_{1.4}$ radius data and other general conditions, such as the observed NS maximum mass and causality condition, introduce strong correlations for the high-order EOS parameters. Consequently, the high-density behavior of $E_{\text{sym}}(\rho)$ inferred depends strongly on how the high-density SNM EOS $E_d(\rho)$ is parameterized, and vice versa. (4) The value of the observed maximum NS mass and whether it is used as a sharp cutoff for the minimum maximum mass or through a Gaussian distribution significantly affects the lower boundaries of both $E_d(\rho)$ and $E_{\text{sym}}(\rho)$ only at densities higher than about 2.5$\rho_0$.

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1. Introduction

The energy per nucleon $E(\rho, \delta)$ (also referred as nucleon specific energy) in nuclear matter at a nucleon density $\rho$, isospin asymmetry $\delta \equiv (n - p)/n$, and zero temperature is the most basic input for calculating the equation of state (EOS) of cold neutron star (NS) matter regardless of the complexity of the models used. It can be well approximated by the isospin-parabolic expansion (Bombaci & Lombardo 1991),

$$E(\rho, \delta) = E_0(\rho) + E_{\text{sym}}(\rho) \cdot \delta^2 + O(\delta^4),$$

(1)

where $E_d(\rho)$ is the energy per nucleon in symmetric nuclear matter (SNM) having equal numbers of neutrons and protons, while the symmetry energy $E_{\text{sym}}(\rho)$ encodes the energy associated with the neutron-richness of the system. On one hand, much progress has been made over the last few decades in constraining the SNM EOS $E_d(\rho)$ not only around but also significantly above the saturation density of nuclear matter, $\rho_0 \approx 2.8 \times 10^{14}$ g cm$^{-3}$ (corresponding to $\rho_0 \approx 0.16$ nucleons fm$^{-3}$), by combining the knowledge gained from analyzing both astrophysical observations and terrestrial nuclear experiments (see, e.g., Danielewicz et al. 2002; Trautmann & Wolter 2012; Oertel et al. 2017; Burgio & Fantina 2018; Garg & Colò 2018; Vidaña 2018; Zhang & Li 2019c). On the other hand, while the symmetry energy $E_{\text{sym}}(\rho)$ mostly around and below $\rho_0$ has been relatively well constrained in recent years (Li et al. 2014), very little is known about the symmetry energy at suprasaturation densities. In fact, it has been broadly recognized that the high-density nuclear symmetry energy is presently among the most important but undetermined quantities reflecting the still-mysterious nature of dense neutron-rich nuclear matter (Baran et al. 2005; Steiner et al. 2005; Li et al. 2008; Lattimer 2012; Tsang et al. 2012; Horowitz et al. 2014; Baldo & Burgio 2016; Li 2017). Thus, not surprisingly, to determine the density dependence of nuclear symmetry energy was identified as a major scientific thrust for nuclear astrophysics in both the U.S. 2015 Long Range Plan for Nuclear Sciences (LRPNS 2015) and the Nuclear Physics European Collaboration Committee (NuPECC) 2017 Long Range Plan (NuPECC 2017).

Both the magnitude and slope of nuclear symmetry energy contribute to the pressure of NS matter. For example, the pressure of $npe$ matter in NSs at $\beta$ equilibrium at density $\rho$ and isospin asymmetry $\delta$ is explicitly

$$P(\rho, \delta) = \rho^2 \left[ \frac{dE_d(\rho)}{d\rho} + \frac{dE_{\text{sym}}(\rho)}{d\rho} \delta^2 \right] + \frac{1}{2} \delta(1 - \delta) \rho \cdot E_{\text{sym}}(\rho).$$

(2)

The first term is the SNM pressure $P_0(\rho) = \rho^2 \frac{dE_d(\rho)}{d\rho}$, while the last two terms are the isospin-asymmetric pressure $P_{\text{asy}}(\rho, \delta) = \rho^2 \frac{dE_{\text{sym}}(\rho)}{d\rho} \delta^2 + \frac{1}{2} \delta(1 - \delta) \rho \cdot E_{\text{sym}}(\rho)$ from nucleons and electrons, separately. At the saturation density $\rho_0$, $P_0$ vanishes, and the electron contribution is also negligible, leaving the total pressure completely determined by the slope of the symmetry energy. Both $P_0$ and $P_{\text{asy}}$ increase with density with rates determined separately by the respective density dependencies of the SNM EOS and the symmetry energy. In the region around $\rho_0 \sim 2.5 \rho_0$, $P_{\text{asy}}$ dominates over $P_0$ using most EOSs available. At higher densities, the SNM pressure $P_0$ dominates, while $P_{\text{asy}}$ also plays an important role, depending on the high-density behaviors of nuclear symmetry energy (Li & Steiner 2006).
The exact transition of dominance from $P_{\text{asy}}$ to $P_0$ depends on the stiffnesses of both the SNM EOS and the symmetry energy. It is also well known that the radius $R_{1.4}$ of canonical NSs is essentially determined by the pressure at densities around $\rho_0 \sim 2.5\rho_0$ (Lattimer & Prakash 2000), while the maximum mass of NSs is determined by the pressure at higher densities reached in the core. Thus, knowledge of the density dependence of nuclear symmetry energy is important for understanding measurements of the masses and especially the radii of NSs. Moreover, the critical densities for forming hyperons (Sumiyoshi & Toki 1994; Lee 1996; Kubis & Kutschera 2003; Providência et al. 2019), $\Delta(1232)$ resonances (Drago et al. 2014; Cai et al. 2015; Zhu et al. 2016; Sahoo et al. 2018; Ribes et al. 2019), kaon condensation (Odrzywolek & Kutschera 2009), and the quark phase (Di Toro et al. 2010; Wu & Shen 2019) are also known to depend sensitively on the high-density nuclear symmetry energy. Information about the latter is thus a prerequisite for exploring the evolution of the NS matter phase diagram in the isospin dimension. Once $E_{\text{sym}}(\rho)$ is better determined and, hopefully, with more astrophysical data, it will be interesting to introduce extra model parameters characterizing the physics associated with the exotic particles and/or new phases predicted to appear in superdense neutron-rich matter. With the very limited data available and the expensive computational costs of simultaneously inferring a lot more than the six EOS parameters we already have in the minimum NS model consisting of only nucleons and two leptons, our goals in this work are conservative and practical. However, inferring new physics parameters associated with the exotic degrees of freedom and new phases in superdense neutron-rich matter from astrophysical data by extending the model used in the present work are high on our working agenda.

To constrain the EOS of ultradense neutron-rich nuclear matter has been a long-standing goal of several branches of astrophysics and astronomy. It is a major science driver in building several new research facilities, such as various advanced X-ray observatories and Earth-based large telescopes, as well as gravitational wave detectors on Earth and in space. In particular, ongoing and planned observations (Bogdanov et al. 2019; Fonseca et al. 2019; Özel & Freire 2016; Watts et al. 2016; Watts 2019) with, e.g., Chandra, XMM-Newton, the Neutron Star Interior Composition Explorer (NICER), the upgraded LIGO, and the VIRGO gravitational wave detector, are all aiming at more precisely measuring the mass–radius correlations of NSs to help constrain the EOS of dense neutron-rich nuclear matter. In the near future, the Advanced Telescope for High-ENergy Astrophysics (Athena)\(^3\) enhanced X-ray Timing and Polarimetry (eXTP; Zhang et al. 2019), Large Observatory for X-ray Timing (LOFT-P; Wilson-Hodge et al. 2016), and Spectroscopic Time-Resolving Observatory for Broadband Energy X-rays (STROBE-X; Ray et al. 2019) will further improve the precision of the mass and/or radius measurements. On the other hand, terrestrial nuclear reactions induced by high-energy radioactive ion beams at several new facilities under construction in several countries are also expected to provide improved constraints on the EOS, especially its symmetry energy term, of dense neutron-rich nuclear matter (see, e.g., Balantekin et al. 2014; Hong et al. 2014; Tamii et al. 2014; Li 2017; Trautmann 2019).

\(^{3}\) https://sci.esa.int/web/athena/

### Table 1

| $R_{1.4}$ (km) (90% CFL) | Source      | Reference               |
|-------------------------|-------------|-------------------------|
| 11.9 ± 0.8, 10.8 ± 0.8  | Imaginary case 1 | This work               |
| 11.7 ± 0.8              | Imaginary case 2 | This work               |
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Figure 1. Nuclear symmetry energy (upper) and energy per nucleon in SNM (lower) as functions of the reduced density $\rho/\rho_0$. The black, blue, and red regions represent the prior, as well as the 90% and 68% credible areas of the posterior functions inferred from the Bayesian analysis in this work using the three radius data listed in Table 1 and all known astrophysical constraints about the EOS of NS matter, respectively.

these three cases well serve our main purposes in this work. We would like to establish a generally useful reference for measuring the progress in determining the high-density nuclear symmetry to be brought to us hopefully soon by the new astrophysical observations and terrestrial experiments. Besides being useful for screening predictions of nuclear many-body theories, the obtained posterior probability distribution functions (PDFs) of six parameters involved in an explicitly isospin-dependent parametric EOS are used to construct the confidence boundaries of both the SNM EOS $E_{d}(\rho)$ and symmetry energy $E_{\text{sym}}(\rho)$.

For busy readers who are eager to quickly know the most important conclusions of our work, shown in Figure 1 is the inferred symmetry energy $E_{\text{sym}}(\rho)$ and SNM EOS $E_{d}(\rho)$ as functions of the reduced density $\rho/\rho_0$ using the real data set. The black, blue, and red regions represent the prior, 90%, and 68% credible areas, respectively. Compared to what we currently know (the prior functions) mostly based on nuclear experiments and theories, the refinement brought in by the astrophysical observations is very significant. More specifically, at densities between $\rho_0$ and 2.5$\rho_0$, both $E_{d}(\rho)$ and $E_{\text{sym}}(\rho)$ are constrained reasonably tightly within their respective narrow bands, approximately independent of whether we parameterize $E_{d}(\rho)$ with two or three terms and $E_{\text{sym}}(\rho)$ with three or four terms, respectively. In particular, the symmetry energy at 2$\rho_0$ is determined to be $E_{\text{sym}}(2\rho_0) = 39.2^{+12.1}_{-12.1}$ MeV at 68% CFL. However, at densities above about 2.5$\rho_0$, the 68% confidence boundaries for both $E_{d}(\rho)$ and $E_{\text{sym}}(\rho)$ diverge, depending strongly on the EOS parameterizations used.

The rest of the paper is organized as follows. In the next section, we outline the explicitly isospin-dependent parametric EOS for modeling NSs containing neutrons, protons, electrons, and muons (the npe$\mu$ matter). We then outline the steps we used within the Markov chain Monte Carlo (MCMC) approach to sample the posterior PDFs of six EOS parameters. In Section 3, we present and discuss the posterior PDFs and correlations of the EOS parameters under various conditions to explore possible model dependences, as well as effects of uncertain factors and data accuracies. We then construct the 68% confidence boundaries of both the SNM EOS $E_{d}(\rho)$ and symmetry energy $E_{\text{sym}}(\rho)$. Finally, we summarize our main findings.

2. Theoretical Framework

2.1. Explicitly Isospin-dependent Parametric EOS for the Cores of NSs

In Bayesian inferences of nuclear EOSs from astrophysical data, various parametric and nonparametric representations of the EOS at suprasaturation densities have been used in the literature (see, e.g., Steiner et al. 2010; Alvarez-Castillo et al. 2016; Özel et al. 2016; Raithel et al. 2017; Raaijmakers et al. 2018; Riley et al. 2018; Lim & Holt 2019; Greif et al. 2019; Miller et al. 2019; Landry & Essick 2019). There is relatively little disagreement about the EOSs of the low-density crust and nuclear matter near $\rho_0$. For a very recent review of different ways of parameterizing the EOSs and their advantages, drawbacks, technical applicabilities, and proposals of reducing the prior dependence of Bayesian inference of EOS parameters, we refer the reader to Section 2.4 of Baiotti (2019) and references therein. Because the pressure in NSs at $\beta$ equilibrium is a function only of the density, namely, the pressure $P(\rho)$ is barotropic, most studies parameterize the pressure directly as a function of density using piecewise polytropic parameterizations (Lindblom 2010), spectral parameterizations (Lindblom 2018), or parameterizations generated with a Gaussian process (Landry & Essick 2019). Indeed, the EOS given in terms of $P(\rho)$ is enough for solving the Tolman–Oppenheimer–Volkov (TOV) equations (Tolman 1934; Oppenheimer & Volkoff 1939) to obtain a mass–radius sequence. However, to obtain accurate information about the high-density nuclear symmetry energy and the corresponding density profile of proton fraction $x_p(\rho)$ in the cores of NSs at $\beta$ equilibrium, one has to construct the pressure $P(\rho)$ by directly parameterizing the underlying $E_{d}(\rho)$ and $E_{\text{sym}}(\rho)$ separately. Since $x_p(\rho)$ in NSs at $\beta$ equilibrium is uniquely determined by $E_{\text{sym}}(\rho)$ through the chemical equilibrium and charge neutrality conditions, the pressure $P(\rho)$ can be easily constructed from the parameterized $E_{d}(\rho)$ and $E_{\text{sym}}(\rho)$. Such a procedure has been used in a number of studies for several purposes in the literature (see, e.g., Oyamatsu & Iida 2007; Totani et al. 2012; Malik et al. 2018; Margueron et al. 2018a, 2018b; Zhang et al. 2018; Li & Sedrakian 2019; Lim & Holt 2019), albeit sometimes using different numbers of parameters for either one or both of $E_{d}(\rho)$ and $E_{\text{sym}}(\rho)$. For a recent review, see Li et al. (2019). In this work, we use this procedure in preparing the NS EOS for our Bayesian inference of the high-density nuclear symmetry energy.
previous work, however, we fixed the low-order EOS parameters at their currently known most probable values, then several NS observables were inverted in the three-dimensional high-density EOS parameter space. In the Bayesian analyses here, we infer the PDFs of all six EOS parameters from the $R_{1.4}$ data discussed in the previous section. Thus, in the remainder of this section, we only summarize the parts of our EOS parameterization most relevant for the Bayesian analyses.

Within the minimal model of NSs consisting of neutrons, protons, electrons, and muons at β equilibrium, the pressure

$$P(\rho, \delta) = \rho^2 \frac{d\epsilon(\rho, \delta)}{d\rho}$$

is determined by the energy density

$$\epsilon(\rho, \delta) = \rho [E(\rho, \delta) + M_N] + \epsilon(\rho, \delta),$$

where $M_N$ represents the average nucleon mass and $\epsilon(\rho, \delta)$ denotes the lepton energy density. As shown by Equation (1), the nucleon energy $E(\rho, \delta)$ is given by the SNM EOS $E_0(\rho)$ and the symmetry energy $E_{sym}(\rho)$, which are respectively parameterized according to

$$E_0(\rho) = E_0(\rho_0) + K_0 \left( \frac{\rho - \rho_0}{\rho_0} \right)^2 + J_0 \left( \frac{\rho - \rho_0}{3\rho_0} \right)^3,$$

where $E_0(\rho_0) = -15.9$ MeV at $\rho_0 = 0.16$ nucleons fm$^{-3}$ (both $E_0(\rho_0)$ and $\rho_0$ are fixed at these values in this work) and

$$E_{sym}(\rho) = E_{sym}(\rho_0) + L \left( \frac{\rho - \rho_0}{3\rho_0} \right)^2 + K_{sym} \left( \frac{\rho - \rho_0}{2\rho_0} \right)^2 + J_{sym} \left( \frac{\rho - \rho_0}{3\rho_0} \right)^3.$$

As discussed earlier (Zhang et al. 2018; Zhang & Li 2019a), the above parameterizations naturally become the Taylor expansions in the limit of $\rho \to \rho_0$. Compared to the widely used multisegment polytropic EOSs that often have connecting densities/pressures of different density regions as parameters, the asymptotic boundary conditions of the above parameterizations near $\rho_0$ and $\delta = 0$ facilitate the use of prior knowledge of the EOS provided by terrestrial nuclear laboratory experiments and/or nuclear theories. Near $\rho_0$, when the above parameterizations become Taylor expansions of the nuclear energy density functionals, the EOS parameters start obtaining their asymptotic physical meanings. Namely, the $K_0$ parameter becomes the incompressibility of SNM $K_0 = 9\rho_0^2 \frac{\partial^2 E_0(\rho)}{\partial \rho^2} |_{\rho = \rho_0}$, the $J_0$ parameter becomes the skewness of SNM $J_0 = 27\rho_0^3 \frac{\partial^3 E_0(\rho)}{\partial \rho^3} |_{\rho = \rho_0}$ at saturation density, and the four parameters involved in the $E_{sym}(\rho)$ become the magnitude $E_{sym}(\rho_0)$, slope $L = 3\rho_0 \frac{\partial E_{sym}(\rho)}{\partial \rho} |_{\rho = \rho_0}$, curvature $K_{sym} = 9\rho_0^2 \frac{\partial^2 E_{sym}(\rho)}{\partial \rho^2} |_{\rho = \rho_0}$, and skewness $J_{sym} = 27\rho_0^3 \frac{\partial^3 E_{sym}(\rho)}{\partial \rho^3} |_{\rho = \rho_0}$ of nuclear symmetry energy at saturation density, respectively. These connections between the EOS parameters and the characteristics of nuclear matter and symmetry energy at $\rho_0$ naturally provide us with some useful information about the EOS parameters. Summarized in Table 2 are the currently known ranges of the asymptotic values of the EOS parameters near $\rho_0$ based on the systematics of terrestrial nuclear experiments and predictions of various nuclear theories. As indicated in Table 2, while $K_0$, $E_{sym}(\rho_0)$, and $L$ have been constrained to relatively small ranges (Danielewicz et al. 2002; Shlomo et al. 2006; Piekarewicz 2010; Li & Han 2013; Oertel et al. 2017), $J_0$, $K_{sym}$, and $J_{sym}$ describing the EOS of dense neutron-rich matter are only poorly known in a wide range (Tews et al. 2017; Zhang et al. 2017). In the Bayesian inferences of their PDFs from astrophysical data, we use these ranges as the prior ranges. We use uniform prior PDFs within these ranges for the six EOS parameters, as there is no known physical preference for any specific value of these parameters within the ranges specified. For example, the ranges of $E_{sym}(\rho_0)$ and $L$ were obtained from the systematics of 53 different analyses of some terrestrial experiments and astrophysical observations (Li & Han 2013; Oertel et al. 2017; Li et al. 2019). At present and to the best of our knowledge, one can only reasonably assume a priori that the EOS parameters can equally take any value within the ranges listed.

The two parameterizations of Equations (5) and (6) are combined using Equation (1) to obtain the pressure $P(\rho)$ as a function of density only once the density profile of the isospin asymmetry $\delta(\rho)$ (or the proton fraction $x_\rho(\rho)$) is calculated self-consistently from the β-equilibrium condition $\mu_n - \mu_p = \mu_e = \delta E_{sym}(\rho)$ and the charge neutrality condition $\rho_p = \rho_e + \rho_\nu$. The chemical potential for a particle $i$ is obtained from $\mu_i = \partial \epsilon(\rho, \delta) / \partial n_i$. As an illustration, shown in Figure 2 are samples of the $E_{sym}(\rho)$ and corresponding $\delta(\rho)$ values generated by individually varying the $L$, $K_{sym}$, and $J_{sym}$ parameters while all other parameters are fixed as indicated in the figure: (a) $L = 40, 50, 60, 70, \text{ and } 80$ MeV; (b) $K_{sym} = -400, -300, -200, -100, 0, \text{ and } 100$ MeV; and (c) $J_{sym} = -200, 0, 200, 400, 600, \text{ and } 800$ MeV. Clearly, very diverse behaviors of $E_{sym}(\rho)$ and the corresponding $\delta(\rho)$ can be sampled. As expected, $L$, $K_{sym}$, and $J_{sym}$ gradually affect the high-density behavior of the symmetry energy more significantly. Interestingly, because of the $E_{sym}(\rho) \cdot \delta^2$ term in the nuclear energy density functional, a higher value of $E_{sym}(\rho)$ will lead to a smaller $\delta(\rho)$ at β equilibrium. The ramifications of this effect, regarded as the isospin fractionation, are well known in nuclear physics and have been studied extensively in heavy-ion collisions at intermediate energies (see, e.g., Muller & Serot 1995; Xu et al. 2000; Li et al. 2008).

Theoretically, the parameterization of $E_{sym}(\rho)$ may not always approach zero at zero density when all of its parameters are randomly generated. While one can cure this completely by introducing additional parameters or using different forms at very low densities (see, e.g., Holt & Lim 2018; Margueron et al. 2018a), practically, we avoid this problem by adopting the NV EOS (Negele & Vautherin 1973) for the inner crust and the BPS EOS (Baym et al. 1971b) for the outer crust. The crust-core transition density and pressure are determined by investigating the thermodynamical instability of the uniform

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**Table 2**

Prior Ranges of the Six EOS Parameters Used

| Parameters | Lower Limit | Upper Limit (MeV) |
|------------|-------------|-------------------|
| $K_0$      | 220         | 260               |
| $J_0$      | -800        | 400               |
| $K_{sym}$  | -400        | 100               |
| $J_{sym}$  | -200        | 800               |
| $L$        | 30          | 90                |
| $E_{sym}(\rho_0)$ | 28.5 | 34.9               |
matter in the core (Zhang et al. 2018). When the incompressibility of npeμ matter (Kubis 2004, 2007; Lattimer & Prakash 2007),

\[
K_\mu = \rho^2 \frac{d^2 E_0}{d\rho^2} + 2 \rho \frac{dE_0}{d\rho} + \frac{\rho^2}{\delta^3} \left[ \frac{d^2 E_{sym}}{d\rho^2} + 2 \rho \frac{dE_{sym}}{d\rho} - 2E_{sym} \left( \frac{dE_{sym}}{d\rho} \right)^2 \right],
\]

becomes negative at low densities, the uniform matter becomes unstable against the formation of clusters. Numerical examples of the crust–core transition density and pressure can be found in Zhang et al. (2018). Clearly, \( K_\mu \) depends mainly on \( K_0, L, \) and \( K_{sym} \). For fast MCMC samplings, we have prepared a table of the crust–core transition density and pressure as functions of \( K_0, L, \) and \( K_{sym} \) with a bin size of 5 MeV. The latter defines the internal energy resolution of our Bayesian inference. This table is available upon request from the authors of this work.

### 2.2. Bayesian Inference Approach

As discussed above and in Zhang et al. (2018) and Zhang & Li (2019a), Equations (5) and (6) have dual meanings (either as parameterizations or Taylor expansions) near the saturation density, but when applied to high densities, they are simply parameterizations. In Zhang et al. (2018) and Zhang & Li (2019a), by fixing the three low-order parameters \( K_0, E_{sym}(\rho_0), \) and \( L \) at their most probable values currently known, only the effects of the three high-density (order) parameters \( J_0, K_{sym}, \) and \( J_{sym} \) were studied by individually inverting several observables directly in the three-dimensional high-density EOS parameter space. While the results are very useful, not only the effects of the uncertainties of the low-order parameters but also the correlations among the EOS parameters were not considered or treated on equal footing. In the present work, we treat the six parameters in Equations (5) and (6) as free parameters to be constrained simultaneously by the same astrophysical data and several known constraints using the Bayesian inference approach.

The key in this approach is the calculation of the posterior PDFs of the model EOS parameters through the MCMC sampling. For completeness, we recall here the Bayes theorem,

\[
P(M|D) = \frac{P(D|M)P(M)}{\int P(D|M)P(M) dM},
\]

where \( P(M|D) \) is the posterior probability for the model \( M \) given the data set \( D \), which is what we are seeking here; \( P(D|M) \) is the likelihood function or conditional probability for a given theoretical model \( M \) to correctly predict the data \( D \); and \( P(M) \) denotes the prior probability of the model \( M \) before being confronted with the data. The denominator in Equation (8) is the normalization constant. We uniformly sample each EOS parameter \( p_i \) randomly between its minimum \( p_{min,i} \) and maximum \( p_{max,i} \) values given in Table 2 according to \( p_i = p_{min,i} + (p_{max,i} - p_{min,i})x \), with the random number \( x \) between zero and 1. Using the generated EOS parameters, \( p_{1,2,\ldots,6} \) one can construct the NS EOS model \( M(p_{1,2,\ldots,6}) \) as we outlined in the previous subsection. The NS mass–radius sequence is then obtained by solving the TOV equations. The resulting theoretical radius \( R_{th,j} \) is subsequently used to calculate the likelihood of this EOS parameter set with respect to the observed radius \( R_{obs,j} \) with \( j = 1, 2, 3 \) in the data set \( D(R_{1,2,3}) \) given in Table 1 according to

\[
P[D(R_{1,2,3})|M(p_{1,2,\ldots,6})] = \prod_{j=1}^{3} \frac{1}{\sqrt{2\pi} \sigma_{obs,j}} \exp\left[ -\frac{(R_{th,j} - R_{obs,j})^2}{2\sigma_{obs,j}^2} \right],
\]

where \( \sigma_{obs,j} \) is the 1σ error bar of the observation \( j \). For the radius data given in Table 1, the upper/lower error bar in radius at 90% CFL is 1.645σ. For the asymmetric case with different upper and lower error bars, we took their average before calculating the \( \sigma \) value. In applying the approach to canonical NSs with the same mass of 1.4 \( M_\odot \), \( R_{th,j} \equiv R_{1,2,3}^j \) is independent of the index \( j \); i.e., we treat the three radii in the data set the same as those from independent observations of the same NS. We will calculate both the marginalized PDFs of all individual EOS parameters and the two-parameter correlations by

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**Figure 2.** Symmetry energy \( E_{sym}(\rho) \) and isospin asymmetry \( \delta(\rho) \) in NS matter at equilibrium as a function of the reduced density \( \rho/\rho_0 \) for (a) \( L = 40, 50, 60, 70, \) and 80 MeV; (b) \( K_{sym} = -400, -300, -200, -100, 0, \) and 100 MeV; and (c) \( J_{sym} = -200, 0, 200, 400, 600, \) and 800 MeV taken from Zhang et al. (2018).
integrating over all other parameters. In particular, the PDF of a model parameter \( p_i \) is analytically given by

\[
P(p_i | D) = \frac{\int P(D | M) dp_1 dp_2 \cdots dp_{i-1} dp_{i+1} \cdots dp_6}{\int P(D | M) P(M) dp_1 dp_2 \cdots dp_6},
\]

while the correlation function between any two parameters \( p_i \) and \( p_j \) can be similarly written as an integral of \( P(D | M) \) over all parameters except \( p_i \) and \( p_j \) themselves. Numerically, \( P(D(R_{1.4}) | M(p_{1,2,\ldots,6})) \) is simulated using the MCMC sampling approach. The integrations are also done naturally using the Monte Carlo approach. Namely, the PDF of a model parameter \( p_i \) is obtained by summing up the accepted \( P(D | M) \) at all steps in the MCMC process, and the sum is binned only according to the value of \( p_i \), regardless of all other parameters. The two-dimensional correlation function between the two parameters \( p_i \) and \( p_j \) is obtained by binning the sum according to both \( p_i \) and \( p_j \) regardless of the values of all other parameters. While it is necessary to normalize the PDFs of single EOS parameters individually, unnormalized correlation functions are sufficient for all practical purposes.

Many MCMC techniques are available in the literature; in this work, we adopted the Metropolis–Hastings algorithm (Metropolis et al. 1953; Hastings 1970). In developing our MCMC package in Fortran 90, we used as an example the MCMC code developed in Keating & Cherry (2009) following the procedure given in Zhang et al. (2006). It is well known that because the MCMC process does not start from an equilibrium distribution, the initial samples in the so-called burn-in period have to be thrown away (Trotta 2017). The adequate burn-in step numbers are those after which either the posterior densities or means of the model parameters remain approximately constant on their trace plots during the MCMC sampling. Shown in Figure 3 are the mean values for our six EOS parameters as functions of the step number. It is seen that after about 40,000 steps, the mean values for all of the parameters stay approximately constant. Thus, we take 40,000 burn-in steps in all calculations performed in this work.

3. Results and Discussions

All EOSs we constructed satisfy the following default conditions: (i) the crust–core transition pressure stays positive; (ii) the thermodynamical stability condition, \( dP/d\rho \geq 0 \), is satisfied at all densities; (iii) the causality condition is enforced at all densities; and (iv) the generated NS EOS should be stiff enough to support the currently observed maximum mass of NSs. We consider the maximum mass as a variable and discuss its effects on what we infer about the EOS parameters by using either a sharp cutoff at 1.97, 2.01, or 2.17 \( M_{\odot} \) or a Gaussian distribution for the maximum mass centered around 2.01 or 2.17 \( M_{\odot} \) with a 1\( \sigma \) error of 0.04 or 0.11 \( M_{\odot} \), respectively. In presenting and discussing our results in the following, we use as a baseline the default result obtained by using the sharp cutoff at 1.97 \( M_{\odot} \), namely, all EOSs have to be able to support NSs at least as massive as 1.97 \( M_{\odot} \) and the three real \( R_{1.4} \) radius data given in Table 1. Based on all the information we have so far, the conclusions based on the default results are the most realistic and conservative. Results from variations of the physics conditions and/or using the imagined data sets will then be compared with the default results. We focus on the marginalized PDFs of the model parameters and their correlations induced by the physics conditions and/or the radius data. The \( E_0(\rho) \) and \( E_{\text{sym}}(\rho) \) at various CFLs in relevant cases will be constructed. Since the PDFs of EOS parameters are asymmetric in most cases, in presenting our results, we use the highest
3.1. The Posterior PDFs and Correlations of EOS Parameters

Shown in Figures 4 and 5 are the marginalized posterior PDFs of the six EOS parameters and their correlations. The blue and red regions outline the 90% and 68% credible intervals of the PDFs, respectively. The most probable values of the six EOS parameters and their 68% and 90% confidence boundaries are summarized in Table 3. Several interesting observations can be made.

1. The $J_0$ and $K_{\text{sym}}$ parameters are relatively well constrained, and larger values of $J_{\text{sym}}$ but smaller values of $L$ are preferred, while the saturation parameters $K_0$ and $E_{\text{sym}}(\rho_0)$, as shown in Figure 5, are essentially not affected by the radius data and the default physics conditions used in this baseline calculation compared to their prior ranges given in Table 2. These features are
Figure 5. Posterior PDFs of low-density EOS parameters and their correlations with the high-density EOS parameters.
Table 3
Most Probable Values of the EOS Parameters and Their 68% and 90% Confidence Boundaries

| Parameter (MeV) | 68% Boundaries | 90% Boundaries |
|----------------|----------------|----------------|
| $J_0$          | $-190^{+40}_{-20}$ | $-190^{+40}_{-30}$ |
| $K_0$          | 222^{+26}_{-20}   | 222^{+25}_{-35}   |
| $J_{sym}$      | $800^{+50}_{-60}$ | $800^{+100}_{-160}$ |
| $K_{sym}$      | $-230^{+50}_{-70}$ | $-230^{+160}_{-70}$ |
| $L$            | $39^{+8}_{-7}$    | $39^{+8}_{-7}$    |
| $E_{sym}(\rho_0)$ | $34^{+1}_{-8}$   | $34^{+1}_{-2}$    |

3. Some of the PDFs have extended tails and/or shoulders due to the correlations among the six EOS parameters. For example, the PDF of $J_0$ has an appreciable tail extending to large positive $J_0$ values. Much effort has been devoted to predicting the value of $J_0$ over the years. For example, in Farine et al. (1997), $J_0 = -700 \pm 500$ MeV was obtained by analyzing nuclear giant monopole resonances. From an earlier Bayesian analysis of some NS data, $J_0 = -280^{+70}_{-40}$ MeV was inferred by assuming $r_{ph} \gg R$, where $r_{ph}$ and $R$ represent the photospheric and stellar radii, respectively, but $J_0 = -500^{+170}_{-290}$ MeV was obtained with $r_{ph} = R$ (Steiner et al. 2010). By using a correlation analysis method based on the Skyrme Hartree–Fock energy density functional (Chen 2011), a value of $J_0 = -355 \pm 95$ MeV was estimated. Within a nonlinear relativistic mean field model (Cai & Chen 2017), by combining the constraints imposed by the flow data in heavy-ion collisions (Danielewicz et al. 2002) and the mass of PSR J0348+0432 (Antoniadis et al. 2013), a range of $-494 \text{ MeV} \leq J_0 \leq -10 \text{ MeV}$ for $J_0$ was inferred.

However, in contrast to the mostly negative values of $J_0$ mentioned above, positive $J_0$ values with much lower probabilities are allowed, as indicated in the top left panel of Figure 4. This is due to the correlations among $J_0$, $J_{sym}$, and $K_{sym}$, especially the correlation between $J_0$ and $J_{sym}$. As discussed in Zhang et al. (2018) and illustrated in Figure 2, extremely small values of $J_{sym}$ (e.g., negative values of $J_{sym}$) lead to supersoft symmetry energies. The system then becomes very neutron-rich, with $\delta \approx 1$ at high densities. In this case, as shown in Figure 2, the symmetry energy decreases with increasing density, giving a negative contribution (the second term in Equation (2)) to the total

easily understandable. Since the average density in canonical NSs with $1.4 M_\odot$ is not so high, it is difficult to constrain the high-density symmetry energy parameter $J_{sym}$ by using the $R_{1.4}$ data. As discussed in Lattimer & Prakash (2000, 2001), the radius of canonical NSs is most sensitive to the pressure around $2 \rho_0$. One thus expects that the $R_{1.4}$ data are most useful for constraining the EOS parameters having the largest influence on the pressure around $2 \rho_0$. This appears to be the case for the $K_{sym}$ parameter. It plays a pivotal role in determining the pressure at the intermediate densities reached in canonical NSs and thus is narrowed down to a small range by the $R_{1.4}$ data used in this analysis. The parameter $J_0$ characterizing the high-density behavior of the SNM EOS $E_0(\rho)$ is sensitive to the NS maximum mass but not the radius of canonical NSs (Zhang et al. 2018; Zhang & Li 2019a). The condition that all EOSs should be stiff enough to support NSs at least as massive as $1.97 M_\odot$ narrowed down the $J_0$ parameter range. We shall discuss in more detail the individual roles of $J_0$ and $J_{sym}$ in Sections 4.2 and 4.3, respectively, as they are not always considered simultaneously in Bayesian inferences of the nuclear EOS in the literature.

2. Generally speaking, mathematically, one expects that a given parameter is most likely to correlate with its nearest neighbors used in parameterizing a specific function, e.g., $K_{sym}$ with its left neighbor $L$ and right neighbor $J_{sym}$ in parameterizing $E_{sym}(\rho)$. All six of the EOS parameters started as independent ones; they become correlated basically by satisfying several energy conservation and chemical and mechanical equilibrium conditions when the radii and masses from solving the TOV equations were required to reproduce the radius data within certain ranges under the conditions specified. Two nearby parameters in a given function can easily compensate for each other to reproduce the same data under the same conditions. It is thus not surprising that there are stronger anticorrelations between $L$ and $K_{sym}$ and between $J_{sym}$ and $K_{sym}$ but only very weak correlations between the saturation-density parameters and the high-density parameters.

Two parameters used in parameterizing $E_0(\rho)$ and $E_{sym}(\rho)$ may also be correlated through the total nucleon specific energy $E(\rho, \delta)$ in Equation (1). Depending on the resulting value of $\delta$, two parameters from the two parameterizations generally have to be at the same order in density to have strong correlations. For example, $J_0$ and $J_{sym}$, both at the third order in density, are expected to be strongly correlated when $\delta$ is close to 1, achieved when the symmetry energy is supersoft, especially with negative $J_{sym}$ as shown in Figure 2. However, correlations between low-order (e.g., $K_0$) and high-order (e.g., $J_{sym}$) parameters are expected to be very weak. In fact, a strong anticorrelation between $J_0$ and $J_{sym}$ is necessary in order to support the same observed NS maximum mass. Namely, increasing $J_0$ is needed when $J_{sym}$ is decreasing to keep the high-density pressure strong enough to support the same massive NSs. In addition, an anticorrelation between $L$ and $K_{sym}$ exists independent of the values $J_{sym}$ takes to keep the same radius $R_{1.4}$. These features are consistent with those found from studying the constant surfaces of radii, speed of sound, and/or the maximum mass in the EOS parameter spaces (Zhang & Li 2019a, 2019b). It is also interesting to note that there is a positive correlation between $J_0$ and $K_{sym}$. This is understandable from the anticorrelations between $J_0$ and $J_{sym}$ and between $K_{sym}$ and $K_{sym}$. However, this finding is different from that found in Baillot d’Etivaux et al. (2019), where a strong anticorrelation between $J_0$ and $K_{sym}$ was observed, while a similarly strong anticorrelation between $L$ and $K_{sym}$ was also found, as in our present work. We found that this is due to the fact that $J_{sym}$ was set to zero in Baillot d’Etivaux et al. (2019). In fact, results very similar to theirs are obtained when we set $J_{sym} = 0$ in the present work, as shown in Figure 6 and discussed in more detail in Section 4.3. It is seen that an anticorrelation between $J_0$ and $K_{sym}$ now appears, while that between $L$ and $K_{sym}$ stays the same.
pressure. To support the same NSs as massive as 1.97 M\(_\odot\) or even 1.4 M\(_\odot\), a much larger contribution to the total pressure from the symmetric part (E\(_0(\rho)\) term) is thus required. Because of the weaker correlation between K\(_0\) and the data/conditions used, as shown in Figure 5, it is J\(_0\) that dominates the contribution to the high-density pressure from the E\(_0(\rho)\) term. Therefore, the value of J\(_0\) has to be positive when J\(_{\text{sym}}\) is very small. This conclusion can also be seen in Figure 10 of Zhang & Li (2019a), where the constant surface of the maximum mass 2.01 M\(_\odot\) was examined in the three-dimensional J\(_0\)–K\(_{\text{sym}}\)–J\(_{\text{sym}}\) high-density EOS parameter space. We note here that such features were not seen in other Bayesian analyses because E\(_{\text{sym}}(\rho)\) was not allowed to become supersoft a priori by setting J\(_{\text{sym}}\) = 0, or often the isospin-independent polytropic EOSs are used at high densities.

It is also interesting to see that there is a small shoulder for the PDF of K\(_{\text{sym}}\) around K\(_{\text{sym}}\) \approx -100 MeV. This is also because of the correlations among J\(_0\), J\(_{\text{sym}}\), and K\(_{\text{sym}}\). As shown in Figure 4, when J\(_0\) has values in the region of 0 \sim 200 MeV, J\(_{\text{sym}}\) has negative values, while K\(_{\text{sym}}\) has values of about -100 MeV with appreciably higher probabilities than the surrounding area on the contour plots of the correlations.

### 3.2. The Bayesian Inferred High-density EOS Confidence Intervals

After obtaining the most probable values and the quantified uncertainties for the EOS parameters, one can derive the corresponding EOS according to Equations (5) and (6). Show in Figure 1 are the symmetry energy E\(_{\text{sym}}(\rho)\) and the nucleon specific energy E\(_0(\rho)\) in SNM as functions of the reduced density \(\rho/\rho_0\). They are also tabulated with their corresponding 90% and 68% confidence ranges in Tables 4 and 5, respectively.

In Figure 1, the black, blue, and red regions represent the prior, 90%, and 68% credible areas, respectively. As mentioned in the Introduction, both E\(_{\text{sym}}(\rho)\) and E\(_0(\rho)\) are constrained into narrow bands at densities smaller than about 2.5\(\rho_0\). However, the symmetry energy diverges broadly at higher densities because of the poor constraint on J\(_{\text{sym}}\). More quantitatively, the symmetry energy at 2\(\rho_0\) is found to be E\(_{\text{sym}}(2\rho_0) = 39.2^{+12.4}_{-8.2}\) MeV at 68% CFL, as shown in Table 4. This value is consistent with the values extracted very recently from several other studies, albeit not always with quantified uncertainties. For example, E\(_{\text{sym}}(2\rho_0) = 46.9 \pm 10.1\) MeV at 100% CFL was found in Zhang & Li (2019a) by inverting several NS observables in the three-dimensional J\(_0\)–K\(_{\text{sym}}\)–J\(_{\text{sym}}\) high-density EOS parameter space, while all other low-order

Figure 6. Posterior PDFs and correlations of high-density EOS parameters by setting J\(_{\text{sym}}\) = 0.
parameters are fixed at their currently known most probable values. While this comparison is somewhat unfair because of the different assumptions and methods used, the general agreement is encouraging. Interestingly, an upper bound of 53.2 MeV was derived very recently in Tong et al. (2019) by studying the radii of neutron drops using state-of-the-art nuclear energy density functional theories. In addition, $E_{\text{sym}}(2\rho_0) \in [39.4, 54.5, 54.5, 77.3, 97.1]$ MeV was found from combined analyses of several NS observables and terrestrial laboratory data (Zhou et al. 2019). The value of $E_{\text{sym}}(2\rho_0)$ that we inferred also falls well within the range of about $26 - 66$ MeV from the Bayesian analyses of Bailiot d’Etivaux et al. (2019). Moreover, in a recent study of cooling timescales of proto-NSs (Nakazato & Suzuki 2019), $E_{\text{sym}}(2\rho_0)$ was used as a controlling parameter in constructing a series of phenomenological EOS models. It was found that EOS modes with $E_{\text{sym}}(2\rho_0) = 40 - 60$ MeV can account for the NS radius and tidal deformability indicated by GW170817. Overall, the results from all of these studies are in general agreement within the still relatively large uncertainties.

At this point, it is useful to note that constraining $E_{\text{sym}}(\rho)$ at supersaturation densities has been a long-standing goal of the low-intermediate-energy heavy-ion reaction community. However, the current results on $E_{\text{sym}}(2\rho_0)$ based on existing data from heavy-ion reactions with stable beams are still inconclusive (Li et al. 2019; Trautmann 2019). Quantitatively, the 68% confidence upper boundary of $E_{\text{sym}}(2\rho_0)$ inferred in this work overlaps with the 100% confidence lower boundary of $E_{\text{sym}}(2\rho_0)$ extracted by the ASY-EOS Collaboration from analyzing their data on the relative flows of neutrons with respect to protons and tritons with respect to $^{3}$He and yields ratios of light isobars (Russotto et al. 2011, 2016). Our results, however, are significantly above the suprsoft $E_{\text{sym}}(2\rho_0)$ preferred by some transport model analyses of the earlier data on the charged pion ratio (Xiao et al. 2009). Hopefully, ongoing and planned new experiments, especially with high-energy radioactive beams at several advanced rare isotope beam facilities, together with improved analysis tools more systematically tested by the broad reaction community, will provide more rigorous terrestrial constraints on $E_{\text{sym}}(\rho)$ at supersaturation densities. Then, a much more meaningful comparison of the high-density nuclear symmetry energy functionals extracted from astrophysical observations and terrestrial experiments can be made.

As discussed earlier, the $J_0$ parameter has the strongest influence on the maximum mass of NSs. Because of the sharp cutoff at 1.97 $M_\odot$ used in the default calculations, the PDF of $J_0$ is rather focused. Consequently, as shown in the lower panel of Figure 1, the $E_{\text{sym}}(\rho)$ in SNM is well constrained up to about $4\rho_0$. More quantitatively, $E_{\text{sym}}(4\rho_0) = 63.5\pm0.6$ MeV at 68% CPL, as shown in Table 5. In a recent study (Zhang & Li 2019c), the influence of the recently reported mass $M = 2.17^{+0.11}_{-0.13} M_\odot$ of PSR J0740+6620 on the EOS of neutron-rich nuclear matter was analyzed. This preliminary mass was revised to $2.14^{+0.07}_{-0.05} M_\odot$ in the final publication by Cromartie et al. (2019). This revision will not change any of our conclusions in this work but will slightly affect the numerical results by Zhang & Li (2019c). Zhang & Li (2019c) found that the preliminary masses of this NS can raise the lower limit of $J_0$ from about $-220$ to $-150$ MeV. It provides a tighter constraint on the EOS of neutron-rich nucleonic matter, especially its
4. Individual Roles of Key Model Ingredients and NS Data in Bayesian Inference of the High-density EOS Parameters

As we discussed earlier, different ways of parameterizing the EOS have been used in the literature. In particular, not all isospin-dependent parameterizations simultaneously use both the $J_0$ and $J_{\text{sym}}$ terms. How does this different treatment affect what we learn about the high-density behavior of nuclear EOS from the same radius data set? Thanks to the continuing great efforts in astronomy, a new record of the maximum mass of NSs may be set at any time. How does the variation of the NS maximum mass affect what we learn about the high-density behavior of nuclear EOS from the same radius data set? In addition, it has been pointed out that whether one uses a sharp cutoff or a Gaussian function for the NS maximum mass may significantly affect what one can infer from the Bayesian analyses (Miller et al. 2019). How does this affect what we learn about the high-density behavior of the nuclear EOS? Moreover, we have a dream: the $R_{1.4}$ of canonical NSs has just been measured to better than 5% 1σ statistical accuracy with no disagreement from different observations and/or analyses. How does this dream case help improve our knowledge of the EOS compared to the default calculations in the previous section? In the following, we discuss the results of our studies on these questions.

4.1. The Role of the Causality Condition

The causality condition requires that the speed of sound, defined by $v_s^2 = \frac{dP}{d\rho}$, should not exceed the speed of light, namely, $0 \leq v_s^2 < c^2$. This condition naturally limits some of the EOS parameters through the pressure $P$ and the energy density $\epsilon$. While to the best of our knowledge, there is no disagreement about whether the causality condition should be enforced or not, it is educational and interesting to compare the results obtained with and without enforcing the causality condition. At least, it allows us to check whether our Bayesian inference code does what it is supposed to do. Moreover, we can learn which parameters are most affected by the causality condition.

Shown in Figure 7 are the posterior PDFs of high-density EOS parameters with or without considering the causality condition. The correlation contours are obtained without using the causality condition. Compared to the PDFs obtained with causality, a wider range extending to large positive values of $J_0$ with higher probabilities is now allowed. This is simply because the causality condition sets the absolute upper limit of the NS maximum mass and, consequently, the upper limit of $J_0$. Without enforcing the causality condition, the most stringent constraint on $J_0$ now comes from the requirement to support NSs at least as massive as 1.97 $M_\odot$. However, this only sets a lower limit for $J_0$, while its upper limit has been removed. Because of its anticorrelation with $J_0$, the parameter $J_{\text{sym}}$ now has a higher probability of being small when $J_0$ has a higher probability of being large. It is also worth noting that the small peak for the PDF of $K_{\text{sym}}$ at about $-100$ MeV disappears now. Different from the correlations among $J_{\text{sym}}$, $K_{\text{sym}}$, and $J_0$ shown in Figure 4, due to the larger and almost the same probability for $J_0$ in the range of $0$ MeV $\leq J_0 \leq 400$ MeV, the corresponding $J_{\text{sym}}$ has the same probability in the range of $-200$ MeV $\leq J_0 \leq 0$ MeV. This makes the small shoulder in the PDF of $K_{\text{sym}}$ disappear when the causality condition is switched off. As shown by the blue curves in Figure 7, both $J_{\text{sym}}$ and $K_{\text{sym}}$ obtain higher probabilities of having higher values when the causality condition is turned on. Consequently, the high-density symmetry energy becomes stiffer when the causality condition is enforced. We also notice that the causality as a high-density condition has little effect on the PDFs of $L$ and the low-order parameters, as one expects.

4.2. The Role of the $J_0$ Parameter of the High-density SNM EOS

The high-order parameter $J_0$ characterizes the high-density behavior of the SNM EOS $E_0(\rho)$. To examine its role, shown in the left panels of Figure 8 are the PDFs of $J_{\text{sym}}$, $K_{\text{sym}}$, and $L$ when the parameter $J_0$ is fixed at 0, $-180$, and $-220$ MeV, respectively. For comparison, the PDFs (gray lines) for them in the default case with $J_0$ in the range of $-800$ to $400$ MeV shown in Figure 4 are also included. It is seen that $J_0$ has significant effects on the PDFs of $J_{\text{sym}}$ and $K_{\text{sym}}$ but less influence on the PDF of $L$. The effects of $J_0$ on the parameters $K_0$ and $E_{\text{sym}}(\rho)$ are very weak and thus not shown here. As shown in Figure 4, because of the negative (positive) correlations between $J_0$ and $J_{\text{sym}}$ ($K_{\text{sym}}$), an increase of $J_0$ leads to a decrease of $J_{\text{sym}}$ but an increase of $K_{\text{sym}}$. It is interesting to note that $J_{\text{sym}}$ becomes completely negative when $J_0$ is set to zero. This is understandable by noticing that the most probable value of $J_0$ is $-190$ MeV in the default case, as shown in Table 3. Artificially setting $J_0 = 0$ is equivalent to making the SNM EOS super-stiff with a much higher contribution to the total pressure. Then, to meet the same conditions, especially the maximum mass and causality constraints, the contribution to the pressure from the symmetry energy has to be significantly reduced by making $J_{\text{sym}}$ largely negative to cancel out the increase of the pressure due to the effectively large increase in $J_0$. On the other hand, it is known that the radius of canonical NSs is almost independent of $J_{\text{sym}}$ while to maintain the same radius, $K_{\text{sym}}$ and $L$ have to be anticorrelated, as shown in Figure 7 of Zhang & Li (2019b). It is thus also easy to understand that while the peak of the PDF of $K_{\text{sym}}$ is shifted from about $-230$ to $-50$ MeV, the PDF of $L$ shifts appreciably toward smaller $L$ values.

Overall, when the NS radii data and the maximum mass are used to infer nuclear symmetry energy, whether to keep a $J_0$ term and what its uncertainty range is all play particularly important roles in determining the PDFs of $J_{\text{sym}}$ and $K_{\text{sym}}$, namely, the high-density behavior of nuclear symmetry energy. Shown in the right panels of Figure 8 are the 68% CFL boundaries of the symmetry energy and nucleon specific energy $E_0(\rho)$ in SNM as a function of reduced density with $J_0 = 0$, $-180$, and $-220$ MeV. For comparison, the default results from Figure 1 are also included. The following observations can be made. (1) The symmetry energy obtained with a constant of $J_0 = -180$ MeV (which is very close to the most probable value of $-190$ MeV in the default case, as shown in Table 3) is close to the one in the default calculation using $J_0 = -800$ to $400$ MeV. (2) Below about 2.5 $\rho_0$, except for the case with $J_0 = 0$ MeV, both $E_{\text{sym}}(\rho)$ and $E_0(\rho)$ are almost independent of the $J_0$ value. (3) The value of $J_0$ is important for understanding the EOS at densities higher than about $3\rho_0$. 

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4.3. The Role of the $J_{\text{sym}}$ Parameter of High-density Symmetry Energy

The parameter $J_{\text{sym}}$ describes the behavior of nuclear symmetry energy at densities higher than about $2\rho_0$, as shown in the right panels of Figure 2. As noted above, the parameter $J_{\text{sym}}$ is less constrained by the present $R_{1.4}$ radius data because the average density reached in canonical NSs is not so high, and it is known that the radii of canonical NSs are most sensitive to the pressure around $2\rho_0$. As mentioned earlier, in some studies for various purposes in the literature, the parameter $J_{\text{sym}}$ is often set to zero, namely, parameterizing the symmetry energy up to $\rho^2$ only. Therefore, it is useful to study the effects of varying $J_{\text{sym}}$ on the PDFs of high-density EOS parameters within the framework of this work. As we shall see next, the effects of $J_{\text{sym}}$ can be explained similarly to those of $J_0$ using the same correlations among the high-density EOS parameters.

Shown in the left panels of Figure 9 are the PDFs for $J_0$, $K_{\text{sym}}$, and $L$ with $J_{\text{sym}} = 0$, 300, and 800 MeV, respectively. For comparison, the PDFs for them from the default calculation (gray lines) using $J_{\text{sym}}$ between $-200$ and 800 MeV are also included. The PDFs of $K_0$ and $E_{\text{sym}}(\rho_0)$ are not given here because of their weaker correlations with $J_{\text{sym}}$. It is seen that $J_0$...
is always negative when positive values of $J_{\text{sym}}$ are taken. It implies that the contribution to the pressure from the $E_{\text{sym}}\delta^2$ term in Equation (2) with positive $J_{\text{sym}}$ values is high enough that the required contribution from the SNM EOS $E_0(\rho)$ term is smaller. Both $J_0$ and $K_{\text{sym}}$ become larger as $J_{\text{sym}}$ decreases because of the negative correlations between $J_{\text{sym}}$ and $J_0$, as well as between $J_{\text{sym}}$ and $K_{\text{sym}}$, while the variation of $J_{\text{sym}}$ has less influence on the PDF of $L$.

We notice again that the most probable values of $J_0$ and $J_{\text{sym}}$ are $-190$ and $800$ MeV, respectively, in the default calculation. Setting $J_{\text{sym}}$ to zero requires a significant increase of $J_0$, leading to a much stiffer EOS for SNM at densities higher than about $2.5\rho_0$. This is clearly seen in the lower right panel of Figure 9, where the 68% CFL boundary of the SNM EOS with fixed $J_{\text{sym}}$ values is shown. The default results from Figure 1 are also shown for comparisons. The EOS of SNM becomes softer at...
densities higher than $2.5\rho_0$ as $J_{\text{sym}}$ increases. This is due to the fact that $J_0$ becomes small with increasing values of $J_{\text{sym}}$, while $K_0$ is weakly correlated with $J_{\text{sym}}$. Except for the extreme case with $J_{\text{sym}} = 800$ MeV, both $E_{\text{sym}}(\rho)$ and $E_0(\rho)$ are almost independent of $J_{\text{sym}}$ at densities below about $2.5\rho_0$. For $E_{\text{sym}}(\rho)$, the default calculation and the calculations with the fixed $J_{\text{sym}}$ values largely overlap up to about $5\rho_0$. In particular, the symmetry energy at high densities does not become much stiffer when the fixed value of $J_{\text{sym}}$ increases. This is because the value of $K_{\text{sym}}$ automatically becomes smaller as $J_{\text{sym}}$ increases due to their anticorrelation.

More comments on the case with $J_{\text{sym}} = 0$ are necessary. This option has been used in many studies in the literature (see, e.g., Alam et al. 2014; Baillot d’Etivaux et al. 2019). The PDFs for $J_0$, $K_{\text{sym}}$, and $L$, as well as their correlations in this case, have already been shown in Figure 6. As we noticed earlier, setting $J_{\text{sym}} = 0$ leads to the anticorrelation between $J_0$ and $K_{\text{sym}}$ consistent with the result in Baillot d’Etivaux et al. (2019) but contrary to our default results shown in Figure 4. This difference has some important consequences. For example, the value of $J_0 = -50^{+30}_{-30}$ MeV at 68% CFL is inferred from the present study, while $J_0 = 318^{+673}_{-368}$ MeV was obtained in

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**Figure 9.** Left: posterior PDFs of high-density EOS parameters by fixing $J_{\text{sym}}$ at 0, 300, and 800 MeV. Right: corresponding effects of inferring the high-density symmetry energy and nucleon specific energy $E_0(\rho)$ in SNM.
Baillot d’Etivaux et al. (2019). On the other hand, the values of $K_{\text{sym}} = -50^{+40}_{-20}$ MeV and $L = 44^{+16}_{-6}$ MeV at the 68% CIL, we inferred by setting $J_{\text{syn}} = 0$ are consistent with the results in Baillot d’Etivaux et al. (2019). Consistent with our conclusions about the role of $J_0$ in inferring the EOS of dense neutron-rich matter, while setting $J_0 = 0$ significantly affects the accuracy of inferring the high-density symmetry energy, setting $J_{\text{sym}} = 0$ significantly affects the accuracy of inferring the high-density EOS of SNM because of the strong anticorrelation between the $J_0$ and $J_{\text{sym}}$ parameters.

4.4. The Role of the Observed Maximum Mass of NSs in Constraining the High-density EOS

As mentioned earlier, the maximum mass of NSs is considered as an observable in this work. Since the maximum mass is still changing as more extensive and accurate observations are being made, it is interesting to examine how the value and the way the NS maximum mass information is used may affect what we learn about the high-density EOS from Bayesian inferences. For this purpose, we compare the results of the following calculations with the default results: (i) without requiring a lower limit for the

![Figure 10. Posterior PDFs of EOS parameters from calculations without a lower limit and with a sharp cutoff at 1.97, 2.01, and 2.17 $M_\odot$ for the NS maximum mass.](image-url)
maximum mass (referred as the NS minimum maximum mass in the following) the EOSs have to support; (ii) with a sharp cutoff at 1.97, 2.01, and 2.17 $M_\odot$, respectively, for the minimum maximum mass; and (iii) requiring a minimum maximum mass that is a Gaussian distribution centered at 2.01 and 2.17 $M_\odot$ with a 1$\sigma$ width of 0.04 and 0.11 $M_\odot$, respectively.

Shown in Figure 10 are the PDFs of the EOS parameters from cases (i) and (ii). Obviously and easily understood, the lower limit of $J_0$ has to go up to support gradually more massive NSs, while its upper limit remains the same. Consequently, the most probable value of $J_0$ increases, leading to a stiffer SNM EOS, as shown in Figure 12. It is interesting to see that the PDF of $J_{\text{sym}}$ shifts to favor higher $J_{\text{sym}}$ values, leading to an increased mean value of the latter when the NS minimum maximum mass increases from 1.97 to 2.17 $M_\odot$. Namely, the mean values of both $J_0$ and $J_{\text{sym}}$ now increase seemingly in contradiction with their anticorrelation observed in the default calculation where the NS minimum maximum mass is fixed at 1.97 $M_\odot$. This is easily understandable because the maximum pressure is no longer the same in calculations with different NS minimum maximum masses. Contributions to the increased pressure necessary to support more massive NSs can come from both the high-density SNM EOS and symmetry energy. As a result, the mean values of both $J_0$ and $J_{\text{sym}}$ increase as the NS minimum maximum mass increases, while the effects on other EOS parameters are weak. Thus, as
demonstrated recently in Zhang & Li (2019c), more precise measurements of the NS maximum mass will improve our knowledge of the high-density behavior of both the SNM EOS and nuclear symmetry energy. The same phenomena are observed when the Gaussian distributions are used for the minimum maximum mass, as shown in Figure 11 for case (iii). As indicated, one minimum maximum mass distribution centers at 2.01 $M_\odot$ with a 1σ width of 0.04 $M_\odot$, while the other one centers at 2.17 $M_\odot$ with a 1σ width of 0.11 $M_\odot$. The PDFs of both $J_0$ and $J_{\text{sym}}$ shift toward higher $J_0$ and $J_{\text{sym}}$ values as the central mass increases from 2.01 to 2.17 $M_\odot$. Again, this is easily understood.

The effects of using different NS minimum maximum masses on the high-density symmetry energy and nucleon specific energy $E_{\text{sym}}(\rho)$ in SNM at 68% CFL are shown in Figure 12. The following observations can be made. (1) The maximum mass condition plays an increasingly more appreciable role in determining the symmetry energy only at densities higher than about 2.5$\rho_0$. A higher value for the NS minimum maximum mass raises the lower limit for the high-density symmetry energy mainly due to the increased mean value of $J_{\text{sym}}$, as we just discussed above, while the upper limit does not change much. (2) The maximum mass condition has a significant impact on the SNM EOS at $\rho > 2\rho_0$. The increased NS maximum mass requires higher values of $J_0$, while the saturation parameters are basically not affected, stiffening the SNM EOS above 2$\rho_0$. (3) As shown in the right panels of Figure 12, using the two different Gaussian distributions for the NS minimum maximum mass does not cause any obvious change in the high-density EOS compared to the calculations with the sharp cutoffs. Of course, the two different central masses have some obvious and easily understood effects. We emphasize that this is probably due to the fact that the radius data we used are only for a single NS of mass 1.4 $M_\odot$, instead of a group of NSs involving more massive ones.

4.5. The Role of $R_{1.4}$ Measurement Accuracy in Inferring Symmetry Energy Parameters

Here we study how much better we can infer the EOS parameters by using the two sets of imaginary data given in Table 1. Imaginary case 1 contains three data points with mean radii different from each other by about 1 km (i.e., a 10% systematic error) but the same absolute error bar of 0.8 km at 90% CFL. The average radius in this case is 11.5 km. Imaginary case 2 has a single radius of 11.9 km and the same statistical error bar of 0.8 km as in case 1. The PDFs of the EOS parameters from these two imaginary cases are compared with those from our default calculations in Figure 13. It is seen that case 1 and our default calculations give essentially the same PDFs for all six EOS parameters. However, in case 1, the statistical error is smaller compared to the default case, and since the mean radii of the three points are already different by about 1 km, it is not surprising that the PDFs of case 1 are not much different from the default case. The PDFs of imaginary case 2 are appreciably different from those of the default case, especially for $K_{\text{sym}}$ and $L$. More quantitatively, as listed in Table 6, the most likely values of both $K_{\text{sym}}$ and $L$ increase significantly, while their 68% CFL ranges remain approximately the same as in imaginary case 1 or the default case. In addition, the mean value of $J_0$ slightly increases, while the mean value of $J_{\text{sym}}$ decreases slightly.

Without changing any other conditions, the increase of mean radius from about 11.5 km in the default case to 11.9 km in imaginary case 2 requires an increase in pressure around 2$\rho_0$, while keeping the pressures at high density approximately the same to satisfy the same condition on the maximum mass. The increase in pressure around 2$\rho_0$ can be achieved by increasing $K_0$, $J_{\text{sym}}$, and/or $L$ and $K_{\text{sym}}$. An increase in $J_0$ normally leads to a decrease in $J_{\text{sym}}$, as we discussed earlier, due to the maximum mass constraint at high densities. Thus, the observed variations of the PDFs can all be qualitatively understood. Quantitatively, however, reducing the statistical
and removing the systematic error bars in measuring $R_{1.4}$ does not seem to help narrow down the PDFs of the EOS parameters compared to the default calculation using the real data. While we probably just made up bad numbers in our otherwise very good dream, this finding may be disappointing but not surprising. Our study here using only the radius data of a single canonical NS sets a useful reference. In reality, joint PDFs of the mass–radius measurements are normally inferred from Bayesian analyses of the raw data from observations. A collection of such data extending to heavy mass regions or the high-precision radius data of several NSs with different masses will certainly help put much tighter bounds on the PDFs of the high-density EOS parameters. Our preliminary studies using several different hypothetical mass–radius correlations between 1.2 and 2 $M_\odot$ indicate that they lead to very different PDFs of EOS parameters. The results of this study will be reported elsewhere.

Figure 13. Effects of different precision in measuring $R_{1.4}$ shown in Table 1 on the PDFs of EOS parameters.

### Table 6

| Quantity | Imaginary Case 1 | Imaginary Case 2 |
|----------|------------------|------------------|
| $J_0$    | $-190^{+30}_{-40}$ | $-170^{+50}_{-30}$ |
| $K_0$    | $232^{+14}_{-12}$  | $236^{+16}_{-12}$  |
| $J_{\text{sym}}$ | $800^{+20}_{-30}$  | $800^{+10}_{-40}$  |
| $K_{\text{sym}}$ | $-240^{+20}_{-30}$ | $-180^{+40}_{-30}$ |
| $L$      | $40^{+16}_{-16}$   | $44^{+18}_{-18}$   |
| $E_{\text{sym}}(\rho_0)$ | $33.8^{+1.2}_{-1.2}$ | $34.6^{+1.0}_{-1.0}$ |


5. Summary and Outlook

In summary, using an explicitly isospin-dependent parametric EOS of nucleonic matter, we carried out a Bayesian inference of high-density nuclear symmetry energy $E_{\text{sym}}(\rho)$ and the associated nucleon specific energy $E_0(\rho)$ in SNM using the latest $R_{1.4}$ radius data available in the literature, under several general conditions required for all NS models. The most important physics findings from this study are as follows.

1. The available astrophysical data can already significantly improve our knowledge of $E_0(\rho)$ and $E_{\text{sym}}(\rho)$ in the density range of $\rho_0 - 2.5\rho_0$ compared to what we currently know about them based mostly on terrestrial nuclear experiments and predictions of nuclear many-body theories. In particular, the symmetry energy at $2\rho_0$ is determined to be $E_{\text{sym}}(2\rho_0) = 39.2^{+12.2}_{-8.2}$ MeV at 68% CFL, approximately independent of the EOS parameterizations used and the uncertainties of the absolutely maximum mass of NSs. However, at higher densities, the 68% confidence boundaries for both $E_0(\rho)$ and $E_{\text{sym}}(\rho)$ diverge, depending strongly on the EOS parameterizations used and several uncertainties.

2. A precise measurement of $R_{1.4}$ alone with less than 5% 1σ statistical error and no systematic error will not much improve the constraint on the EOS of neutron-rich nucleonic matter at densities below about $2.5\rho_0$ compared to the constraints extracted from using the available radius data. We hope that high-precision joint PDFs from simultaneous mass–radius measurements extending to heavy mass regions will significantly narrow down the EOS both below and above $2.5\rho_0$.

3. The radius data and other general conditions, such as the observed NS maximum mass and causality condition, introduce strong correlations for the high-order parameters used in parameterizing $E_0(\rho)$ and $E_{\text{sym}}(\rho)$. Reflected clearly in the PDFs of the high-density EOS parameters, the inferred high-density behavior of $E_{\text{sym}}(\rho)$ depends strongly on how the high-density $E_0(\rho)$ is parameterized, and vice versa. This is particularly true for densities higher than about $2.5\rho_0$, where the third-order parameters $J_0$ and $J_{\text{sym}}$ play the dominating role. Since these two parameters are not always used simultaneously in parameterizing $E_0(\rho)$ and $E_{\text{sym}}(\rho)$ in the literature, different correlations among the PDFs of the EOS parameters and thus different high-density behaviors of the symmetry energy may be inferred from the same set of NS observational data.

4. The value of the observed NS maximum mass and whether it is used as a sharp cutoff for the minimum maximum mass or through a Gaussian distribution in the Bayesian analyses significantly affects the lower boundaries of $E_0(\rho)$ and $E_{\text{sym}}(\rho)$ only at densities higher than about $2.5\rho_0$. The EOS constraints extracted in the density region of $\rho_0 - 2.5\rho_0$ are not influenced by the remaining uncertainties about the NS absolute maximum mass.

Finally, we speculate that the radii of more massive NSs and additional messengers, especially those directly from NS cores or emitted during collisions between two NSs or heavy nuclei, will be useful to further constrain the EOS of dense neutron-rich nuclear matter, especially at densities higher than about $2.5\rho_0$.

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