ON THE PYTKEEV PROPERTY IN SPACES OF CONTINUOUS FUNCTIONS (II)

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Abstract. We prove that for each Polish space $X$, the space $C(X)$ of continuous real-valued functions on $X$ satisfies (a strong version of) the Pytkeev property, if endowed with the compact-open topology. We also consider the Pytkeev property in the case where $C(X)$ is endowed with the topology of pointwise convergence.

1. Introduction

For a topological space $X$, $C(X)$ is the family of all real-valued continuous functions on $X$. We consider two standard topologies on $C(X)$, which make it a topological group. Let $0$ denote the constant zero function on $X$.

$C_k(X)$ denotes $C(X)$, endowed with the compact-open topology. For a set $K \subseteq X$ and $n \in \mathbb{N}$, let

$$[K; n] = \left\{ f \in C_k(X) : \forall x \in K \ |f(x)| < \frac{1}{n}\right\}.$$

When $K$ ranges over the compact subsets of $X$ and $n$ ranges over $\mathbb{N}$, the sets $[K; n]$ form a local base at $0$.

$C_p(X)$ denotes $C(X)$, endowed with the topology of pointwise convergence. Here, a local base at $0$ is given by the sets $[F; n]$, where $n \in \mathbb{N}$, and $F$ ranges over the finite subsets of $X$.

$C_k(X)$ is metrizable if, and only if, $X$ is hemicompact (i.e., there is a countable family of compact sets such that each compact subset of $X$ is contained in some member of the family) [9]. In particular, $C_k(\mathbb{N}^\mathbb{N})$ is not metrizable. Restricting attention to first countable spaces $X$, McCoy [9] observed that for $C_k(X)$ to be metrizable, it suffices that it has the Fréchet-Urysohn property, that is, for each $A \subseteq C_k(X)$ with $0 \in \overline{A}$, there is a sequence of elements of $A$ converging to $0$. Despite

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the fact that $C_k(\mathbb{N}^\mathbb{N})$ does not have the Fréchet-Urysohn property, we show in Section 2 that it has the slightly weaker Pytkeev property.

As for $C_p(X)$, it is metrizable if, and only if, $X$ is countable. Here, the Fréchet-Urysohn property does not imply metrizability, and Sakai asked whether for $C_p(X)$, the Pytkeev property implies the Fréchet-Urysohn property. We establish several weaker assertions (Section 3).

2. The compact-open topology

Let $X$ be a topological space. $C_k(X)$ has the Pytkeev property if for each $A \subseteq C_k(X)$ with $0 \in \overline{A} \setminus A$, there are infinite sets $A_1, A_2, \ldots \subseteq A$ such that each neighborhood of $0$ contains some $A_n$.

The notion of a $k$-cover is central in the study of local properties of $C_k(X)$ (see 3 and references therein). A cover $\mathcal{U}$ of $X$ is a $k$-cover of $X$ if $X \not\in \mathcal{U}$, but for each compact $K \subseteq X$, there is $U \in \mathcal{U}$ such that $K \subseteq U$.

**Theorem 1.** $C_k(\mathbb{N}^\mathbb{N})$ has the Pytkeev property.

**Proof.** By a theorem of Pavlovic and Pansera, it suffices to prove that for each open $k$-cover $\mathcal{U}$ of $X$, there are infinite sets $\mathcal{U}_1, \mathcal{U}_2, \ldots \subseteq \mathcal{U}$ such that $\bigcap \mathcal{U}_n : n \in \mathbb{N}$ is a $k$-cover of $X$. We will show that $\mathbb{N}^\mathbb{N}$ has the mentioned covering property.

To this end, we set up some basic notation. For $s \in \mathbb{N}^{<\aleph_0}$, $[s] = \{f \in \mathbb{N}^\mathbb{N} : s \subseteq f\}$, and $|s|$ denotes the length of $s$. For $S \subseteq \mathbb{N}^{<\aleph_0}$, $[S] = \bigcup_{s \in S} [s]$. For an open $U \subseteq \mathbb{N}^\mathbb{N}$, $U(n) = \{s \in \mathbb{N}^n : [s] \subseteq U\}$. Note that for each $n$, $[U(n)] \subseteq [U(n+1)]$, and $U = \bigcup_n [U(n)]$.

**Lemma 2.** Assume that $\mathcal{U}$ is an open $k$-cover of $\mathbb{N}^\mathbb{N}$. Then:

1. $\mathcal{V} = \{[U(n)] : U \in \mathcal{U}, n \in \mathbb{N}\}$ is a $k$-cover of $\mathbb{N}^\mathbb{N}$.

2. There is $n$ such that $\{U(n) : U \in \mathcal{U}\}$ is infinite.

3. For each compact $K \subseteq \mathbb{N}^\mathbb{N}$, there is $n$ such that $\{U(n) : U \in \mathcal{U}, K \subseteq [U(n)]\}$ is infinite.

**Proof.** (1) For each compact $K \subseteq \mathbb{N}^\mathbb{N}$, there is $U \in \mathcal{U}$ such that $K \subseteq U$. As $U = \bigcup_n [U(n)]$ and $K$ is compact, there is $n$ such that $K \subseteq [U(n)] \in \mathcal{V}$.

(2) Assume that for each $n$, $\{U(n) : U \in \mathcal{U}\}$ is finite. Note that for each $U \in \mathcal{U}$ and each $n$, $[U(n)] \subseteq U \neq \mathbb{N}^\mathbb{N}$, and therefore $U(n) \neq \mathbb{N}^n$. Proceed by induction on $n$:

**Step 1.** As $\mathcal{U}(1) = \{U(1) : U \in \mathcal{U}\}$ is finite and $\mathbb{N} \not\in \mathcal{U}(1)$, there is a finite $F_1 \subseteq \mathbb{N}$ which is not contained in any member of $\mathcal{U}(1)$.

**Step n.** As $\mathcal{U}(n) = \{U(n) : U \in \mathcal{U}\}$ is finite and $F_{n-1} \times \mathbb{N}$ is not contained in any member of $\mathcal{U}(n)$, there is a finite $F_n \subseteq F_{n-1} \times \mathbb{N}$
which is not contained in any member of \( \mathcal{U}(n) \), and such that \( F_n \upharpoonright (n - 1) = F_{n-1} \).

Take \( K = \bigcap_n [F_n] \) (the set of all infinite branches through the finitely splitting tree \( \bigcup_n F_n \)). As \( K \) is compact, there is \( U \in \mathcal{U} \) such that \( K \subseteq U \). As \( U = \bigcup_n [U(n)] \) and \( K \) is compact, there is \( n \) such that \( K \subseteq [U(n)] \). But then \( F_n \subseteq U(n) \), a contradiction.

(3) By (1), \( \{[U(n)] : U \in \mathcal{U}, n \in \mathbb{N}, K \subseteq [U(n)] \} \) is a \( k \)-cover of \( \mathbb{N}^\mathbb{N} \). By (2), there is \( m \) such that

\[
\mathcal{V} = \{[[U(n)](m)] : U \in \mathcal{U}, n \in \mathbb{N}, K \subseteq [U(n)] \}
\]

is infinite. For all \( U \) and \( n \), \([U(n)](m) \) is equal to \([U(n)] \) when \( n \leq m \), and to \([U(m)] \) when \( m < n \). Thus, \( \mathcal{V} = \bigcup_{n \leq m} \{[U(n)] : U \in \mathcal{U}, K \subseteq [U(n)] \} \), and therefore there is \( n \leq m \) such that \( \{[U(n)] : U \in \mathcal{U}, K \subseteq [U(n)] \} \) is infinite. □

For each \( n \) and \( s \in \mathbb{N}^n \), let \( \leq s = \{t \in \mathbb{N}^n : t \leq s \} \), where \( \leq \) is pointwise. The following lemma gives more than what is needed in our theorem.

**Lemma 3.** Let \( \mathcal{U} \) be an open \( k \)-cover of \( \mathbb{N}^\mathbb{N} \). There is \( S \subseteq \mathbb{N}^{<\mathbb{N}_0} \) such that for each \( s \in S \), \( \mathcal{U}_s = \{U \in \mathcal{U} : \leq s \subseteq U \} \) is infinite, and \( \{\leq s : s \in S \} \) is a clopen \( k \)-cover of \( \mathbb{N}^\mathbb{N} \) (refining \( \bigcap \mathcal{U}_s : s \in S \)).

**Proof.** We actually prove the stronger result, that the statement in the lemma holds when

\[
\mathcal{U}_s = \{[U(|s|)] : U \in \mathcal{U}, \leq s \subseteq U \}
\]

for each \( s \in S \).

Let \( S \) be the set of all \( s \in \mathbb{N}^{<\mathbb{N}_0} \) such that \( \mathcal{U}_s \) is infinite. If \( K \subseteq \mathbb{N}^\mathbb{N} \) is compact, take \( f \in \mathbb{N}^\mathbb{N} \) such that the compact set \( K(f) = \{g \in \mathbb{N}^\mathbb{N} : g \leq f \} \) contains \( K \). By Lemma 2 there is \( n \) such that there are infinitely many sets \( U(n), U \in \mathcal{U} \), with \( K(f) \subseteq [U(n)] \), that is, \( \leq f \upharpoonright n \subseteq U \). Thus, \( f \upharpoonright n \in S \). Clearly, \( K \subseteq K(f) \subseteq \leq f \upharpoonright n \). □

This completes the proof of Theorem 1.

**Definition 4.** For shortness, we say that a topological space \( X \) is **nice** if there is a countable family \( \mathcal{C} \) of open subsets of \( X \), such that for each open \( k \)-cover \( \mathcal{U} \) of \( X \), \( \mathcal{S} = \{V \in \mathcal{C} : (\exists \mathcal{U}_n \subseteq \mathcal{U}) V \subseteq U \} \) is a \( k \)-cover of \( X \).

By Lemma 3, \( \mathbb{N}^\mathbb{N} \) is nice.

**Definition 5.** A topological space \( Y \) has the **strong Pytkeev property** if for each \( y \in Y \), there is a countable family \( \mathcal{N} \) of subsets of \( Y \), such
that for each neighborhood \( U \) of \( y \) and each \( A \subseteq Y \) with \( y \in \overline{A} \setminus A \), there is \( N \in \mathcal{N} \) such that \( N \subseteq U \) and \( N \cap A \) is infinite.

If \( Y \) is first countable, then it has the strong Pytkeev property. The converse fails, even in the realm of \( C_k(X) \). Indeed, \( C_k(\mathbb{N}^\mathbb{N}) \) is not first countable (since it is a non-metrizable topological group), and we have the following.

**Theorem 6.** \( C_k(\mathbb{N}^\mathbb{N}) \) has the strong Pytkeev property.

Theorem 6 follows from the following.

**Lemma 7.** If \( X \) is nice, then \( C_k(X) \) has the strong Pytkeev property.

**Proof.** Let \( \mathcal{C} \) be as in the definition of niceness for \( X \). It suffices to verify the strong Pytkeev property of \( C_k(X) \) at \( 0 \). Set

\[
\mathcal{N} = \{ [V; n] : V \in \mathcal{C}, n \in \mathbb{N} \}.
\]

Assume that \( A \subseteq C_k(X) \) and \( 0 \in \overline{A} \setminus A \). There are two cases to consider.

Case 1: For each \( n \), there is \( f_n \in A \cap [X; n] \) (equivalently, there are infinitely many such \( n \)). Given any neighborhood \([K; m] \) of \( 0 \), take \( V \in \mathcal{C} \) with \( K \subseteq V \). Then \([V; m] \subseteq [K; m]\), and \([V; m] \cap A \supseteq \{ f_n : n \geq m \} \) is infinite.

Case 2: There is \( N \) such that for each \( n \geq N \), \( A \cap [X; n] = \emptyset \). Fix \( n \geq N \). \( U_n = \{ f^{-1}([-1/n, 1/n]) : f \in A \} \) is a \( k \)-cover of \( X \). Thus,

\[
S_n = \{ V \in \mathcal{C} : (\exists \infty U \in U_n) V \subseteq U \} \subseteq \{ V \in \mathcal{C} : (\exists \infty f \in A) V \subseteq f^{-1}([-1/n, 1/n]) \} = \{ V \in \mathcal{C} : [V; n] \cap A \text{ is infinite} \}
\]

is a \( k \)-cover of \( X \).

Consider any (basic) open neighborhood \([K; n] \) of \( 0 \). Take \( V \in S_n \) such that \( K \subseteq V \). Then \([V; n] \in \mathcal{N} \), \([V; n] \subseteq [K; n] \), and \([V; n] \cap A \) is infinite. \( \square \)

A function \( f : X \to Y \) is **compact-covering** if for each compact \( K \subseteq Y \), there is a compact \( C \subseteq X \) such that \( K \subseteq f[C] \). Hereditary local properties of a space \( C_k(X) \) are clearly preserved when transforming \( X \) by a continuous compact-covering functions. (Indeed, if \( f : X \to Y \) is a continuous compact-covering surjection, then \( g \mapsto g \circ f \) is an embedding of \( C_k(Y) \) into \( C_k(X) \).)

**Corollary 8.** For each Polish space \( X \), \( C_k(X) \) has the strong Pytkeev property.
Proof. $X$ is the image of $\mathbb{N}^\mathbb{N}$ under a continuous compact-covering function. Indeed [7]: There is a closed $C \subseteq \mathbb{N}^\mathbb{N}$ such that $X$ is the image of $C$ under a perfect (thus compact-covering) function. As $C$ is closed, it is a retract of $\mathbb{N}^\mathbb{N}$, and the retraction is clearly compact covering.

3. The topology of pointwise convergence

There is a very rich local-to-global theory, due to Arhangel’skiǐ and his followers, which studies local properties of $C_p(X)$ by translating them into covering properties. An elegant and uniform treatment of covering properties was given by Scheepers [16, 6]. We recall a part of this theory that puts the results of the present section in their proper context.

Let $X$ be a topological space. $U$ is a cover of $X$ if $X = \bigcup U$ but $X \not\in U$. A cover $U$ of $X$ is an $\omega$-cover of $X$ if for each finite subset $F$ of $X$, there is $U \in U$ such that $F \subseteq U$. $U$ is a $\gamma$-cover of $X$ if it is infinite and for each $x$ in $X$, $x \in U$ for all but finitely many $U \in U$. Let $\mathcal{O}$, $\Omega$, and $\Gamma$ denote the collections of all open covers, $\omega$-covers, and $\gamma$-covers of $X$, respectively. Let $\mathcal{A}$ and $\mathcal{B}$ be collections of covers of a space $X$. Following are selection hypotheses which $X$ may satisfy or not satisfy [16].

\begin{align*}
S_1(\mathcal{A}, \mathcal{B}): \text{For all } U_1, U_2, \ldots \in \mathcal{A}, \text{ there are } U_1 \in U_1, U_2 \in U_2, \ldots, \text{ such that } \{U_1, U_2, \ldots \} \in \mathcal{B}.
S_{fin}(\mathcal{A}, \mathcal{B}): \text{For all } U_1, U_2, \ldots \in \mathcal{A}, \text{ there are finite } F_1 \subseteq U_1, F_2 \subseteq U_2, \ldots, \text{ such that } \bigcup_{n \in \mathbb{N}} F_n \in \mathcal{B}.
U_{fin}(\mathcal{A}, \mathcal{B}): \text{For all } U_1, U_2, \ldots \in \mathcal{A}, \text{ there are finite } F_1 \subseteq U_1, F_2 \subseteq U_2, \ldots, \text{ such that } \{\bigcup F_1, \bigcup F_2, \ldots \} \in \mathcal{B}.
\end{align*}

Some of the properties defined in this manner were studied earlier by Hurewicz ($U_{fin}(\mathcal{O}, \Gamma)$), Menger ($S_{fin}(\mathcal{O}, \mathcal{O})$), Rothberger ($S_1(\mathcal{O}, \mathcal{O})$, traditionally known as the $C''$ property), Gerlits and Nagy ($S_1(\Omega, \Gamma)$, traditionally known as the $\gamma$-property), and others. Each of these properties is either trivial, or equivalent to one in Figure 1 (where an arrow denotes implication) [6].

In the remainder of this paper, all spaces $X$ are assumed to be Tychonoff. A space $X$ satisfies $S_1(\Omega, \Gamma)$ if, and only if, $C_p(X)$ has the Fréchet-Urysohn property [5]. In particular, if $X$ satisfies $S_1(\Omega, \Gamma)$, then $C_p(X)$ has the Pytkeev property.

**Problem 9** (Sakai [14]). Assume that $C_p(X)$ has the Pytkeev property. Must $X$ satisfy $S_1(\Omega, \Gamma)$?

For metric spaces $X$ which are countable unions of totally bounded subspaces, Miller proved that consistently, $X$ is countable whenever
\( C_p(X) \) has the Pytkeev property (this is essentially proved in Theorem 18 of [18]). It follows that a positive answer to Sakai’s Problem [9] is consistent in this realm. However, we suspect that the following holds.

**Conjecture 10 (CH).** There is \( X \subseteq \mathbb{N}^\mathbb{N} \) such that \( C_p(X) \) has the Pytkeev property, but \( X \) does not even satisfy Menger’s property \( S_{fin}(\mathcal{O}, \mathcal{O}) \).

It is therefore natural to consider the conjunction of “\( C_p(X) \) has the Pytkeev property” with properties in the Scheepers Diagram [11].

A combination of results of Kočinac and Scheepers [8] and Sakai [14] gives that if \( C_p(X) \) has the Pytkeev property and \( X \) satisfies \( U_{fin}(\mathcal{O}, \mathcal{O}) \), then all finite powers of \( X \) satisfy \( U_{fin}(\mathcal{O}, \Gamma) \) as well as \( S_1(\mathcal{O}, \mathcal{O}) \). We will prove several results of a similar flavor.

The combinatorial terminology in the remainder of the paper is as follows: For \( f, g \in \mathbb{N}^\mathbb{N} \), \( f \leq^* g \) means \( f(n) \leq g(n) \) for all but finitely many \( n \). \( B \subseteq \mathbb{N}^\mathbb{N} \) is bounded if there is \( g \in \mathbb{N}^\mathbb{N} \) such that for each \( f \in B \), \( f \leq^* g \). \( D \subseteq \mathbb{N}^\mathbb{N} \) is finitely dominating if its closure under pointwise maxima of finite subsets is dominating.

**Theorem 11.** If \( C_p(X) \) has the Pytkeev property and \( X \) satisfies \( U_{fin}(\mathcal{O}, \Omega) \), then \( X \) satisfies \( U_{fin}(\mathcal{O}, \Gamma) \) as well as \( S_1(\mathcal{O}, \mathcal{O}) \).

**Proof.** As \( C_p(X) \) has the Pytkeev property, \( X \) is Lindelöf and zero-dimensional [13]. This is needed for the application of the quoted combinatorial theorems below.

We first prove that \( X \) satisfies \( U_{fin}(\mathcal{O}, \Gamma) \). By [12], it suffices to prove the following.

**Lemma 12.** If \( C_p(X) \) has the Pytkeev property and \( X \) satisfies \( U_{fin}(\mathcal{O}, \Omega) \), then each continuous image \( Y \) of \( X \) in \( \mathbb{N}^\mathbb{N} \) is bounded.

**Proof.** Let \( Y \) be a continuous image of \( X \) in \( \mathbb{N}^\mathbb{N} \). Since we can transform \( Y \) continuously by \( f(n) \mapsto f(0) + f(1) + \cdots + f(n) + n \), we may assume
that all elements of $Y$ are increasing. If there is an infinite $I \subseteq \mathbb{N}$ such that \( \{ f \upharpoonright I : f \in Y \} \) is bounded, then $Y$ is bounded. We therefore assume that there is $N$ such that for each $n \geq N$, \( \{ f(n) : f \in Y \} \) is infinite.

As $Y$ satisfies $U_{\text{fin}}(\mathcal{O}, \Omega)$, $Y$ is not finitely dominating \[19\], that is, there is $g \in \mathbb{N}^\mathbb{N}$ such that the clopen sets $U_n = \{ f \in Y : f(n) \leq g(n) \}$, $n \geq N$, form an $\omega$-cover of $Y$. As $C_p(Y)$ has the Pytkeev property, there are infinite $I_1, I_2, \ldots \subseteq \mathbb{N} \setminus \{0, \ldots, N - 1\}$ such that \( \bigcap_{k \in I_n} U_k : n \in \mathbb{N} \) is an $\omega$-cover of $Y$ \[13\]. For each $n$, \( \{ f \upharpoonright I_n : f \in \bigcap_{k \in I_n} U_k \} \) is bounded, and therefore \( \bigcap_{k \in I_n} U_k \) is bounded. Thus, $Y = \bigcup_n \bigcap_{k \in I_n} U_k$ is bounded.

We now show that $X$ satisfies $S_1(\mathcal{O}, \mathcal{O})$. It suffices to prove that each continuous image $Y$ of $X$ in $\mathbb{N}^\mathbb{N}$ has strong measure zero with respect to the standard metric of $\mathbb{N}^\mathbb{N}$ \[4\]. Indeed, by Lemma 12 such an image $Y$ is bounded, and thus is a countable union of totally bounded subspaces of $\mathbb{N}^\mathbb{N}$. By a theorem of Miller \[18\], if $C_p(Y)$ has the Pytkeev property and $Y$ is a countable union of totally bounded subspaces, then $Y$ has strong measure zero.

$\square$

$D_{\text{fin}}$ is the family of all subsets of $\mathbb{N}^\mathbb{N}$ which are not finitely dominating, and $\text{cov}(D_{\text{fin}}) = \min\{|\mathcal{F}| : \mathcal{F} \subseteq D_{\text{fin}} \text{ and } \bigcup\mathcal{F} = \mathbb{N}^\mathbb{N}\}$. The hypothesis $\text{cov}(D_{\text{fin}}) < \diamond$ holds, e.g., in the Cohen reals model, or if $\diamond$ is singular \[17\].

**Theorem 13** (cov($D_{\text{fin}}$) < $\diamond$). Assume that for each $Y \subseteq X$, $C_p(Y)$ has the Pytkeev property. Then $X$ satisfies $U_{\text{fin}}(\mathcal{O}, \Omega)$ as well as $S_1(\mathcal{O}, \mathcal{O})$.

**Proof.** By Theorem 11 it suffices to prove that $X$ satisfies $U_{\text{fin}}(\mathcal{O}, \Omega)$, or equivalently, that no continuous image $Y$ of $X$ in $\mathbb{N}^\mathbb{N}$ is finitely dominating.

Assume that $Y$ is a continuous image of $X$ in $\mathbb{N}^\mathbb{N}$. We may assume that all elements of $Y$ are increasing. Let $\kappa = \text{cov}(D_{\text{fin}}) < \diamond$, and $Y_\alpha \subseteq \mathbb{N}^\mathbb{N}$, $\alpha < \kappa$, be not finitely dominating and such that $\bigcup_{\alpha < \kappa} Y_\alpha = \mathbb{N}^\mathbb{N}$. For each $\alpha < \kappa$, $Y \cap Y_\alpha$ is not finitely dominating, and since it is a continuous image of a subset of $X$, $C_p(Y \cap Y_\alpha)$ has the Pytkeev property. The proof of Lemma 12 shows the following.

**Lemma 14.** Assume that $Z \subseteq \mathbb{N}^\mathbb{N}$, all elements of $Z$ are increasing, $Z$ is not finitely dominating, and $C_p(Z)$ has the Pytkeev property. Then $Z$ is bounded.

It follows that $Y \cap Y_\alpha$ is bounded for all $\alpha < \kappa$, and as $\kappa < \diamond$, $Y = \bigcup_{\alpha < \kappa} Y \cap Y_\alpha$ is not finitely dominating.
We now consider the strong Pytkeev property of $C_p(X)$. A space $Y$ has a countable cs*-character \cite{2} if for each $y \in Y$, there is a countable family $N$ of subsets of $Y$, such that for each sequence in $Y$ converging to $y$ (but not eventually equal to $y$) and each neighborhood $U$ of $y$, there is $N \in N$ such that $N \subseteq U$ and $N$ contains infinitely many elements of that sequence. Clearly, the strong Pytkeev property implies countable cs*-character. For topological groups, the conjunction of countable cs*-character and the Fréchet-Urysohn property implies metrizability \cite{2}. As $C_p(X)$ is a topological group, we have the following.

**Corollary 15.** If $C_p(X)$ has the Fréchet-Urysohn property as well as the strong Pytkeev property, then $X$ is countable. \hfill $\Box$

As the Pytkeev property follows from the Fréchet-Urysohn property, we have the following.

**Corollary 16.** The Pytkeev property for $C_p(X)$ does not imply the strong Pytkeev property for $C_p(X)$. \hfill $\Box$

If, consistently, there is an uncountable $X$ such that $C_p(X)$ has the strong Pytkeev property, then the answer to Sakai’s Problem \cite{9} is negative: By corollary \cite{15} in this case $C_p(X)$ cannot have the Fréchet-Urysohn property.\footnote{Unfortunately, this strategy does not work: Sakai has recently proved that if $C_p(X)$ has the strong Pytkeev property (or even just countable cs*-character), then $X$ is countable \cite{15}. This extends Corollary \cite{15} and can be contrasted with Theorem \cite{15}.}

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