THE FINITENESS OF SOLUTIONS OF DIOPHANTINE EQUATION AND PRIMES SPLITTING COMPLETELY

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ABSTRACT. We consider a Bertrand type estimate for primes splitting completely. As one of its applications, we show the finiteness of trivial solutions of Diophantine equation about the factorial function over number fields except for the case the rational number field.

1. Introduction

Let $K$ be a number field and let $\mathcal{O}_K$ be its ring of integers. We consider the factorial function generalized to number fields. This function $\Pi_K(x)$ is defined as

$$
\Pi_K(x) = \prod_{n \leq x} a_K(n),
$$

where $a_K(n)$ is the number of ideals with $\mathfrak{m}a = n$. It is well known that $a_K(n)$ has the multiplicative property

$$
a_K(mn) = a_K(m)a_K(n) \quad \text{if } \gcd(m, n) = 1.
$$

In the following, we will use abbreviation $a(n)$ for $a_K(n)$. If the product is empty then we assign it the value 1. We consider the equation

$$(1.1) \quad \Pi_K(l_1) \cdots \Pi_K(l_{m-1}) = \Pi_K(l_m)
$$

for $2 \leq l_1 \leq \cdots \leq l_{m-1} < l_m$.

In the case $K = \mathbb{Q}$, this equation has infinitely many solutions but most of them satisfy $l_m - l_{m-1} = 1$ and they are called trivial solutions. In this case, it is known that if the ABC conjecture holds, then there are only finitely many non-trivial solutions of (1.1) [Lu07]. The 3-tuple $(6, 7, 10)$ is one of such non-trivial solutions, but we do not know others. In [HPS18], they show that non-trivial 3-tuple solutions of (1.1) other than $(l_1, l_2, l_3) = (6, 7, 10)$ satisfy $l_3 < 5(l_2 - l_1)$ and if $l_2 - l_1 \leq 10^6$ then the only non-trivial solution to (1.1) is $(6, 7, 10)$.

In general case, a solution $(l_1, \ldots, l_m)$ is called trivial if there exists no ideals with $\mathfrak{m}a \in (l_{m-1}, l_m)$. For example, when $K = \mathbb{Q}(\sqrt{-3})$, we describe the splitting of prime ideals in $\mathcal{O}_K = \mathbb{Z}[\frac{1+\sqrt{-3}}{2}]$ as follows:

| prime $p$ in $\mathbb{Z}$ | how to split in $\mathcal{O}_K$ | $a(p^k)$ for $k \geq 1$ |
|---------------------------|-----------------------------|--------------------------|
| $p \equiv 1 \mod 3$      | $(p)$ splits completely in $\mathcal{O}_K$. | $a(p^k) = k + 1$ |
| $p \equiv 2 \mod 3$      | $(p)$ is also a prime ideal of $\mathcal{O}_K$. | $a(p^k) = 0, a(p^{2k}) = 1$ |
| $p = 3$                  | $(3)$ ramifies in $\mathcal{O}_K$. | $a(p^k) = 1$ |

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From the multiplicative property of \( a(n) \) we can calculate \( a(n) \) for all \( n \). One can check that two 3-tuples \((4, 9, 12)\) and \((12, 247, 252)\) are trivial solutions while the 3-tuple \((16, 111, 117)\) is a non-trivial solution. It is not known whether or not there exist infinitely many solutions of (1.1).

In this paper, we show the finiteness of trivial solutions.

**Theorem 1.2.** For any number field \( K \neq \mathbb{Q} \), there exist only finitely many trivial solutions of Diophantine equation (1.1).

As we remarked above, there are infinitely many trivial solutions in the case \( K = \mathbb{Q} \). On the other hand, this theorem asserts that there exist only finitely many trivial solutions for any number fields \( K \neq \mathbb{Q} \). That is, there is an essential difference between the case \( K = \mathbb{Q} \) and the general case.

2. Auxiliary lemmas

In this section we show and introduce some auxiliary lemmas to prove Theorem 1.2. The first lemma gives a necessary and sufficient condition for the existence of trivial solution.

**Lemma 2.1.** The following two statements are equivalent.

1. \( m \)-tuple \((l_1, \ldots, l_m)\) is a trivial solution.
2. Let \( l_m = \prod p^{r_p} \), then

\[
\Pi_K(l_1) \cdots \Pi_K(l_{m-2}) = l_m^{a(l_m)} = \left( \prod p^{a(p^{r_p})} \right).
\]

**Proof.** Let \((l_1, \ldots, l_m)\) be a trivial solution. Then the equation \( \Pi_K(l_1) \cdots \Pi_K(l_{m-1}) = \Pi_K(l_m) \) can be rewritten as

\[
\Pi_K(l_1) \cdots \Pi_K(l_{m-2}) = \prod_{l_{m-1} < \Re a \leq l_m} \Re a = l_m^{a(l_m)}.
\]

The second equality follows since there exists no ideal in the interval \((l_{m-1}, l_m)\).

Conversely, when \( \Pi_K(l_1) \cdots \Pi_K(l_{m-2}) = l_m^{a(l_m)} \), we define \( l_{m-1} = \max\{ \Re a \mid a : \text{ideal} \} \cap [l_{m-2}, l_m) \). Then

\[
\Pi_K(l_m) = l_m^{a(l_m)} \Pi_K(l_{m-1}) = \Pi_K(l_1) \cdots \Pi_K(l_{m-1}).
\]

Therefore, \((l_1, \ldots, l_m)\) be a solution. From the definition of \( l_{m-1} \), there exists no ideal with their ideal norm being in the interval \((l_{m-1}, l_m)\). Hence \((l_1, \ldots, l_m)\) is a trivial solution. This proves this equivalence. \( \square \)

The case \( K = \mathbb{Q} \), we know \( a(n) = 1 \) for all \( n \). This asserts that \((l_1, \ldots, l_m)\) is a trivial solution if and only if \( l_m - l_{m-1} = 1 \). This does not contradict the definition of trivial solution.

In 2017, Hulse and Murty gave one of generalizations of Bertrand’s postulate, or Chebyshev’s theorem, to number fields [HM17]. The original Bertrand postulate states that for any \( x > 1 \) there exists a prime number in the interval \([x, 2x]\). This is a result weaker than the prime number theorem but we can prove this without information about the zeros of the zeta function.
In the followings, we consider a Bertrand type estimate for primes splitting completely by following the way of [HM17]. In [HM17], they use the following effective version of prime ideal theorem given by Lagarias and Odlyzko. Let $L/K$ be a Galois extension. For each conjugacy class $C$ of $G$, we define $\pi_C(x)$ by

$$\pi_C(x) = |\{(p \subset O_K \mid p \text{ is unramified in } L, [(p, L/K)] = C, \Re p \leq x\}|,$$

where $[(p, L/K)]$ is the conjugacy class of Frobenius map corresponding to $p$.

**Lemma 2.2** (Theorem 1.3. of [LO77]). Let $L/K$ be a Galois extension of number fields with $[L:Q] = n$ and let $D_L$ be the absolute value of the discriminant of $L$. Then there exist effectively computable positive constants $c_1$ and $c_2$, such that if $x > \exp(10n(\log D_L)^2)$ then

$$\left| \pi_C(x) - \frac{|C|}{|G|} \log x + \frac{|C|}{|G|} (-1)^{c_1} \log x^\beta \right| \leq c_1 x \exp \left( -c_2 \sqrt{\frac{\log x}{n}} \right),$$

where $\log x^\beta$ only occurs if there exists an exceptional real zero $\beta$ of $\zeta_L(s)$ such that $1 - (4\log D_L)^{-1} < \beta < 1$. Also $\varepsilon_L = 0$ or 1 depending on $L$.

If $p$ splits completely in $L$ then the Frobenius map $(p, L/K)$ is trivial and $||(p, L/K)|| = 1$. From the definition of $\varepsilon_L$ in [LO77], we obtain $\varepsilon_L = 0$. The next theorem gives a Bertrand type estimate for primes splitting completely.

**Theorem 2.3.** Let $K$ be a number field and $K^{gal}$ be the Galois closure of $K/Q$ with $[K^{gal}:Q] = k$ and let $D$ be the absolute value of the discriminant of $K^{gal}$. For any $A > 1$ there exists an effectively computable constant $c_A > 0$ depending only on $A$ such that for $x > \exp(c_Ak(\log D)^2)$ there is a prime splitting completely with $p \in [x, Ax]$.

**Proof.** It is well known that a prime $p$ splits completely in $K$ if and only if it splits completely in the smallest Galois extension $K^{gal}$ of $Q$ containing $K$. Without loss of generality, we assume that $K/Q$ is a Galois extension. Let $\pi_{s.c.}(x)$ be the number of primes $p \leq x$ splitting completely in $K$. From Lemma 2.2 and the above remark, we get

$$\pi_{s.c.}(Ax) - \pi_{s.c.}(x)$$

$$> \frac{1}{k} (\log(Ax) - \log(x)) - \frac{1}{k} \left( \log((Ax)^\beta) - \log(x^\beta) \right) - 2Ac_1x \exp \left( -c_2 \sqrt{\frac{\log x}{k}} \right).$$

It suffices to show that the right hand side is positive for $x > \exp(c_Ak(\log D)^2)$.

Stark [St74] showed that if $K/Q$ is a Galois extension and $\beta$ exists then

$$1 - \frac{1}{4\log D} < \beta < 1 - \frac{6}{\pi} \frac{1}{D^{\frac{1}{2}}}.$$

One can check that $\log((Ax)^\beta) - \log(x^\beta)$ is a monotonically increasing function in $\beta$ for fixed $x > \exp(10k(\log D)^2)$ and $A > 1$, so we put $\beta_0 = 1 - \frac{6}{\pi}D^{-\frac{1}{2}}$. By
integration by parts we find that it suffices to show

\[ \frac{Ax}{\log Ax} \frac{(Ax)^\beta_0}{\beta_0 \log Ax} + \int_{(Ax)^{\beta_0}} \frac{dt}{(\log t)^2} > \frac{x}{\log x} \frac{x^{\beta_0}}{\beta_0 \log x} + \int_{x^{\beta_0}}^{x} \frac{dt}{(\log t)^2} + 2Ak_c x \exp \left( -c_2 \sqrt{\frac{\log x}{k}} \right) \]

for \( x > \exp(10k(\log D)^2) \) and \( A > 1 \). The function \( \int_{x^{\beta_0}}^{x} \frac{dt}{(\log t)^2} \) also increases as \( x \) increases for \( x > \exp(10k(\log D)^2) \), so our goal is to show

\[ \frac{Ax^{\beta_0} - (Ax)^{\beta_0}}{\beta_0 \log Ax} > \frac{x^{\beta_0} - x^{\beta_0}}{\beta_0 \log x} + 2Ak_c x \exp \left( -c_2 \sqrt{\frac{\log x}{k}} \right). \]

Divided by \( \frac{x^{\beta_0} - x^{\beta_0}}{\beta_0 \log x} \) this is equivalent to

\[ A^{\beta_0} - (Ax)^{\beta_0 - 1} \log x \frac{x}{\log Ax} > 1 + \frac{2A^{\beta_0} k_c \log x}{\beta_0 - x^{\beta_0 - 1}} \exp \left( -c_2 \sqrt{\frac{\log x}{k}} \right). \]

Now we denote \( x = \exp(c_A k(\log D)^2) \). Then the right hand side is equal to

\[ \frac{2A^{\beta_0} 4k^2 c_A (\log D)^2 D^{-c_2 \sqrt{\pi^2}}}{\beta_0 - x^{\beta_0 - 1}}. \]

The denominator of (2.5) is monotonically increasing and positive for \( x > 10 \). Moreover the Minkowski bound \( \frac{k}{D} \leq \left( \frac{4}{\pi} \right)^k \left( \frac{k!}{2^{k-1}} \right) \) [La94] leads

\[ \frac{k}{D} \leq \left( \frac{4}{\pi} \right)^k \left( \frac{k!}{2^{k-1}} \right) \leq \frac{8}{\pi^2}. \]

The numerator of (2.5) is equal to

\[ 2A^{\beta_0} 4k^2 c_A \frac{\log D^2}{D^{-c_2 \sqrt{\pi^2}}}. \]

For \( c_A > 4c_2^{-2} \) this function is monotonically decreasing and from (2.6) we can choose \( c_A \) independent of \( K \). Therefore, the both sides of (2.4) is decreasing for \( x > \exp(c_A k(\log D)^2) \). Also the left hand side converges to \( A \) and the right does 1 as \( x \) tends to infinity. Thus there exists \( c_A \), independent of \( K \), such that if \( x > \exp(c_A k(\log D)^2) \), then inequality (2.4) holds, that is, \( \pi_{s,c}(Ax) - \pi_{s,c}(x) > 0 \). This proves the theorem.

3. PROOF OF THE MAIN THEOREM

In this section, we show Theorem 1.2. We write the main theorem again.

**Theorem 1.2.** For any number field \( K \neq \mathbb{Q} \), there exists only finitely many trivial solutions of Diophantine equation (1.1).

**Proof of the main theorem.** Let \( p_1 \) denote \( \min \{ \mathfrak{a} : \text{ideal of } \mathcal{O}_K \cap \mathbb{Z}_{>1} \} \). From Theorem 2.3, there exists \( c_{p_1} \) such that there is a prime splitting completely in \( [x, p_1 x] \) for \( x \geq \exp(c_{p_1} k(\log D)^2) \). Let \( P_0(x) \) be the set of all primes \( p \) splitting completely in \( K \) with \( p \leq x \). Let \( q \) be a prime splitting completely such that \( q \geq \exp(c_{p_1} k(\log D)^2) \) and \( k_{P_0(q)} > k(m-2) \). For \( q \leq l_{m-2} < p_1 q \), if \( m \)-tuple
(l_1, \ldots, l_m) is trivial then from Lemma 2.1 we obtain the following prime factorization of \( \Pi_K(l_1) \cdots \Pi_K(l_{m-2}) \):

\[
\Pi_K(l_1) \cdots \Pi_K(l_{m-2}) = \left( q^{r_q} \prod_{p \neq q} p^{r_p} \right)^{\Pi_p a(p^{r_p})},
\]

where \( r_p \geq 0 \) for all \( p \) and \( r_q \geq 0 \).

Since \( r_q \geq 1 \) and \( a(p^{r_p}) \geq k \) for all \( p \in P_0(q) \), the product of factorial function \( \Pi_K(l_1) \cdots \Pi_K(l_{m-2}) \) needs to be divided by at least \( q^{k[P_0(q)]} \). The second smallest \( q \)-factor appears at \( p_1 q \), so this product \( \Pi_K(l_1) \cdots \Pi_K(l_{m-2}) \) is divided by \( q^{k(m-2)} \) at most for \( q \leq l_{m-2} < p_1 q \). From the assumption \( k(m-2) < k[P_0(q)] \), \( m \)-tuple \((l_1, \ldots, l_m)\) is not trivial for all \( q \leq l_{m-2} < p_1 q \). On the other hand, from Theorem 2.3 there exists a new prime splitting completely \( q_1 \) with \( q_1 \in [q + \frac{1}{p_1 q}, p_1 q + \frac{1}{2}] \) and \( \Pi_K(l_1) \cdots \Pi_K(l_{m-2}) \) also needs to be divided by \( q_1^{k[P_0(q)]} \) for \( l_{m-2} > p_1 q \).

From the same argument above, there exists no trivial solutions \((l_1, \ldots, l_m)\) for all \( q_1 \leq l_{m-2} < p_1 q_1 \) and there exists a new prime splitting completely \( q_2 \) with \( q_2 \in [q_1 + \frac{1}{2p_1 q_1}, p_1 q_1 + \frac{1}{2}] \).

By induction, there exists no trivial solution \((l_1, \ldots, l_m)\) for \( l_{m-2} \geq q \). This shows the finiteness. \( \square \)

4. THE UPPER BOUND OF TRIVIAL SOLUTIONS

Our main theorem implies the finiteness of trivial solutions for any \( K \neq \mathbb{Q} \). Therefore, it is a new problem to find all trivial solutions of equation (1.1). We know that the constant \( c_A \) in Theorem 2.3 is effective, so one can give an explicit upper bound for \( l_{m-2} \).

Since the constant \( c_A \) depends on \( c_1 \) and \( c_2 \), we can calculate \( c_A \) explicitly by the proof of Theorem 2.3. We obtain \( c_1 = 199 \) and \( c_2 = \frac{7}{200} \) from the same computations as Lagarias and Odlyzko [LO77] and it is enough to choose \( c_A = 7 \times 10^7 \). The details of them are omitted, because our computations are very messy.

From the proof of Theorem 1.2, we conclude that when \( K \neq \mathbb{Q} \) is a number field, there exists no trivial solutions of \( \Pi_K(l_1) \cdots \Pi_K(l_{m-1}) = \Pi_K(l_m) \) for \( l_{m-2} > \exp(7 \times 10^7 k (\log D)^2) \).

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