ON THE RAMSEY NUMBERS FOR A COMBINATION OF PATHS AND JAHANGIRS

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ABSTRACT. For given graphs $G$ and $H$, the Ramsey number $R(G, H)$ is the least natural number $n$ such that for every graph $F$ of order $n$ the following condition holds: either $F$ contains $G$ or the complement of $F$ contains $H$. In this paper, we improve the Surahmat and Tomescu’s result [9] on the Ramsey number of paths versus Jahangirs. We also determine the Ramsey number $R(\cup G, H)$, where $G$ is a path and $H$ is a Jahangir graph.

1. Introduction

The study of Ramsey Numbers for (general) graphs have received tremendous efforts in the last two decades, see few related papers [1, 2, 3, 4, 6, 8] and a nice survey paper [7]. One of useful results on this is the establishment of a general lower bound by Chvátal and Harary [5], namely $R(G, H) \geq (\chi(G) - 1)(c(H) - 1) + 1$, where $\chi(G)$ is the chromatic number of $G$ and $c(H)$ is the number of vertices in the largest component of $H$.

Let $G(V, E)$ be a graph with the vertex-set $V(G)$ and edge-set $E(G)$. If $(x, y) \in E(G)$ then $x$ is called adjacent to $y$, and $y$ is a neighbor of $x$ and vice versa. For any $A \subseteq V(G)$, we use $N_A(x)$ to denote the set of all neighbors of $x$ in $A$, namely $N_A(x) = \{y \in A \mid (x, y) \in E(G)\}$. Let $P_n$ be a path with $n$ vertices, $C_n$ be a cycle with $n$ vertices, and $W_m$ be a wheel of $m + 1$ vertices, i.e., a graph consisting of a cycle $C_m$ with one additional vertex adjacent to all vertices of $C_m$. For $m \geq 2$, the Jahangir graph $J_{2m}$ is a graph consisting of a cycle $C_{2m}$ with one additional vertex adjacent alternatively to $m$ vertices of $C_{2m}$. For example, Figure 1.

Recently, Surahmat and Tomescu [9] studied the Ramsey number of a combination of $P_n$ versus a $J_{2m}$, and obtained the following result.

Theorem A. [9]

\[
R(P_n, J_{2m}) = \begin{cases} 
6 & \text{if } (n, m) = (4, 2), \\
 n + 1 & \text{if } m = 2 \text{ and } n \geq 5, \\
 n + m - 1 & \text{if } m \geq 3 \text{ and } n \geq (4m - 1)(m - 1) + 1.
\end{cases}
\]

\footnote{The figure $J_{16}$ appears on Jahangir’s tomb in his mausoleum, it lies in 5 km north-west of Lahore, Pakistan across the River Ravi. His tomb was built by his Queen Noor Jehan and his son Shah-Jehan (This was emperor who constructed one of the wonder of world Taj Mahal in India) around 1637 A.D. It has a majestic structure made of red sand-stone and marble.}
In this paper, we determine the Ramsey numbers involving paths and Jahangir graphs. For particular, we improve the Surahmat and Tomescu’s result for Jahangir graphs $J_6$, $J_8$ and $J_{10}$ as follows.

**Theorem 1.1.** $R(P_n, J_{2m}) = n + m - 1$ for $n \geq 2m + 1$ and $m = 3, 4$ or 5.

We are also able to determine the Ramsey number $R(kP_n, J_{2m})$, for any integer $k \geq 2$, $m \geq 2$. These results are stated in the following theorems.

**Theorem 1.2.** $R(kP_n, J_4) = kn + 1$, for $n \geq 4$, $k \geq 1$, except for $(n = 4, k = 1)$.

**Theorem 1.3.** $R(kP_n, J_{2m}) = kn + m - 1$, for any integer $n \geq 2m + 1$ if $m = 3, 4$ or 5; and for $n \geq (4m - 1)(m - 1) + 1$ if $m \geq 6$, where $k \geq 2$.

2. The Proof of Theorems

The proof of Theorem 1.

Consider graph $G \cong K_{m-1} \cup K_{n-1}$. Clearly, $G$ contains no $P_n$ and $\overline{G}$ contains no $J_{2m}$. Thus, $R(P_n, J_{2m}) \geq n + m - 1$. For $m = 3, 4$ or 5 and $n \geq 2m + 1$, we will show that $R(P_n, J_{2m}) \leq n + m - 1$. Let $F$ be a graph of $n + m - 1$ vertices containing no $P_n$. Take any longest path $L$ in $F$. Let $L$ be $(x_1, x_2, \ldots, x_k)$, and $Y = V(F) \setminus V(L)$. Since $k \leq n - 1$, then $|Y| \geq m$. Obviously, $yx_1, yx_k$ are not in $E(F)$, for any $y \in Y$. Now, consider the following two cases

**Case 1.** $2m \leq |L| \leq n - 1$.

Let $|L| = t$ and $A = \{x_2, x_3, \ldots, x_{2m-1}\}$ be the set of first $2m - 2$ vertices of $L$ after $x_1$. Take the set of any $m$ distinct vertices of $Y$ and denote it by $B = \{y_1, \ldots, y_m\}$. By the maximality of $L$, every vertex of $B$ has at most $m - 1$ neighbors in $A$. If there are two vertices of $B$ having $m - 1$ neighbors in $A$ then all the neighbors are intersected.

**Subcase 1.1** There exists $b \in B$, $|N_A(b)| = m - 1$.

Let $A_1 = A \setminus N_A(b)$ and take any vertex $v_1$ of $A_1$ whose the highest degree at $B$. Define $D_1 = \{x_1, x_t, b\} \cup A_1 \setminus \{v_1\}$, and $D_2 = \{v_1\} \cup B \setminus \{b\}$. By the maximality of $L$, $d_{D_1}(w) \leq 1$ for any vertex $w$ of $D_2$. In particular, $d_{D_1}(v_1) = 0$. Since $v_1$ has the highest degree then there are at most $m - 2$ edges connecting vertices between $D_1$ and $D_2$ in $F$. This implies that $D_1 \cup D_2$ will induces a $J_{2m}$ in $\overline{F}$.

**Subcase 1.2** All vertices $b \in B$, $|N_A(b)| \leq m - 2$.

If $m = 3$ then let $D_1 = \{\text{any two vertices of } A\}$. If $m = 4$ then by the pigeonhole principle there exists two vertices of $A$ has neighbors at most 1 in $B$. In this case let $D_1 = \{\text{three vertices of } A \text{ with two of degree at most one }\}$. If $m = 5$ then by the pigeonhole principle there exists three vertices of $A$ has neighbors at most 2 in $B$. In this case let $D_1 = \{\text{four vertices of } A \text{ with three of degree at most two}\}$. Therefore, $\{x_1, x_t\} \cup D_1 \cup B$ will induce a $J_{2m}$ in $\overline{F}$.
Case 2. $1 \leq |L| \leq 2m - 1$. 
We breakdown the proof into several subcases.

**Subcase 2.1.** $1 \leq |L| \leq 3$
In this case, the component of $F$ is either $K_1$, $P_2$, $C_3$ or a star. Therefore, $\overline{F}$ contains a $J_{2m}$, for $m = 3, 4$ or 5.

**Subcase 2.2.** $4 \leq |L| \leq m + 1$.
Let $L = (x_1, x_2, \cdots, x_t)$, where $t \leq m + 1$, and so $|Y| = |V(F)\setminus V(L)| \geq 2m - 1$. Now, consider the set $N_Y(x_2)$ of vertices in $Y$ adjacent to $x_2$. Note that any vertex of $N_Y(x_2)$ is nonadjacent to any other vertices of $Y$. If $|N_Y(x_2)| \geq m - 2$ then form two sets $D_1$ and $D_2$ as follows. The set $D_1$ consists of $x_1, x_t$ and any $m - 2$ vertices of $N_Y(x_2)$. The set $D_2$ consists of the other vertices of $Y$ not selected in $D_1$. Thus, $|D_1| = m$ and $|D_2| = m + 1$. By the maximality of $L$, there is no edge connecting any vertex of $D_1$ to any vertex of $D_2$. Thus, the set $D_1 \cup D_2$ induces $K_{m,m+1} \supseteq J_{2m}$ in $\overline{F}$. If $|N_Y(x_2)| = m - 3$ then take $D_1 = \{x_1, x_t, x_2\} \cup N_Y(x_2)$, and $D_2$ as the set of the remaining vertices of $Y$. Now, consider the set of the remaining vertices of $Y$. Then, $D_1 \cup D_2$ again contains $K_{m,m+1} \supseteq J_{2m}$ in $\overline{F}$. Now, if $|N_Y(x_2)| = m - 4$ (for $m = 4$ or 5) then in showing $\overline{F} \supseteq J_{2m}$ take $D_1 = \{x_1, x_t, x_2, x_{t-1}\} \cup N_Y(x_2)$, and $D_2$ as the set of the remaining vertices of $Y$ not adjacent to $x_{t-1}$. This is true since $|N_Y(x_{t-1})| \leq 1$ (by symmetrical argument). If $|N_Y(x_2)| = m - 5$ (for $m = 5$ only), then $D_1 = \{x_1, x_2, x_{t-1}, x_t, b\}$ where $b$ is a vertex at distance two from $x_3$ or $b$ is any vertex of $Y$ with a smallest degree, and $D_2$ as the set of the remaining vertices of $Y$. Thus, $D_1 \cup D_2$ will induce $J_{10}$ in $\overline{F}$.

**Subcase 2.3.** $|L| = m + 2$.
Let $L = (x_1, x_2, \cdots, x_t)$ where $t = m + 2$, then $|Y| = |V(F)\setminus V(L)| \geq 2m - 2$. Now, consider the set $N_Y(x_2)$ of vertices in $Y$ adjacent to $x_2$. Note that any vertex of $N_Y(x_2)$ is nonadjacent to any other vertices of $Y$. If $|N_Y(x_2)| \geq m - 2$ then form two sets $D_1$ and $D_2$ as follows. If $x_3$ is nonadjacent to $x_{m+2}$ then $D_1 = \{x_1, x_{m+2}\} \cup \{any \ m - 2 \ vertices \ of \ N_Y(x_2)\}$ and $D_2$ consists of $x_3$ together with the remaining vertices of $Y$. Otherwise (if $x_3 \sim x_{m+2}$), take $D_1 = \{x_1, x_{m+2}, x_4\} \cup \{any \ m - 2 \ vertices \ of \ N_Y(x_2)\}$ and $D_2$ consists of any $m$ remaining vertices of $Y$. By the maximality of $L$, there is no edge connecting any vertex of $D_1$ to any vertex of $D_2$. Thus, the set $D_1 \cup D_2$ induces $K_{m,m+1} \supseteq J_{2m}$ in $\overline{F}$. If $|N_Y(x_2)| = m - 3$ then take $D_1 = \{x_1, x_t, x_2\} \cup N_Y(x_2)$, and $D_2$ as the set of the remaining vertices of $Y$. Then, $D_1 \cup D_2$ again contains $K_{m,m+1} \supseteq J_{2m}$ in $\overline{F}$. Now, if $|N_Y(x_2)| = m - 4$ (for $m = 4$ or 5) then in showing $\overline{F} \supseteq J_{2m}$ take $D_1 = \{x_1, x_t, x_2, x_{t-1}\} \cup N_Y(x_2)$, and $D_2$ as the set of the remaining vertices of $Y$ not adjacent to $x_{t-1}$. This is true since $|N_Y(x_{t-1})| \leq 1$ (by symmetrical argument). If $|N_Y(x_2)| = m - 5$ (for $m = 5$ only), then $D_1 = \{x_1, x_2, x_{t-1}, x_t, b\}$ where $b$ is a vertex at distance two from $x_3$ or $b$ is any vertex of $Y$ with a smallest degree, and $D_2$ as the set of the remaining vertices of $Y$. Thus, $D_1 \cup D_2$ will induce $J_{10}$ in $\overline{F}$.
Subcase 2.4. $|L| = m + 3$ (or $2m - 1, 2m - 2$ if $m = 4, 5$ respectively).
Let $L$ be $(x_1, x_2, \cdots, x_t)$ where $t = m + 3$, then $|Y| = |V(F)\setminus V(L)| \geq 2m - 3$. Now, consider the set $N_Y(x_2)$ of vertices in $Y$ adjacent to $x_2$. Note that any vertex of $N_Y(x_2)$ is nonadjacent to any other vertices of $Y$. If $|N_Y(x_2)| \geq m - 1$ then form two sets $D_1$ and $D_2$ as follows. If $x_{t-1}$ is adjacent to some vertex of $N_Y(x_2)$ then by the maximality of $L$, $x_{t-2}$ is nonadjacent to $x_1$ and any vertex of $N_Y(x_2)$. In this case set $b = x_{t-1}$. If $x_{t-1}$ is nonadjacent to any vertex of $N_Y(x_2)$, then take $b = x_{t-2}$ provided $x_{t-1} \neq x_1$. Otherwise (if $x_{t-1} \sim x_1$), by the maximality of $L$ we have that $x_{t-2}$ is nonadjacent to $x_1$ and to any vertex of $N_Y(x_2)$. In this case, again take $b = x_{t-2}$.
Now, define $D_1 = \{x_1\} \cup \{\text{any } m - 1 \text{ vertices of } N_Y(x_2)\}$ and $D_2 = \{x_3, x_t, b\} \cup \{\text{ any } m - 2 \text{ other vertices of } Y\}$. By the maximality of $L$, there is no edge connecting any vertex of $D_1$ to any vertex of $D_2$. Thus, the set $D_1 \cup D_2$ induces $K_{m,m+1} \supseteq J_{2m}$ in $\overline{F}$.

If $|N_Y(x_2)| = m - 2$ then take $D_1 = \{x_1, x_2\} \cup N_Y(x_2)$, and $D_2 = \{x_3, x_t\} \cup \{\text{ any } m - 1 \text{ other vertices of } Y\}$. Then, $D_1 \cup D_2$ contains $K_{m,m+1}$ minus at most two edges $(x_2, x_3)$ and $(x_2, x_t)$ in $\overline{F}$. Therefore, $\overline{F} \supseteq J_{2m}$. Now, if $|N_Y(x_2)| = m - 3$ then in showing $\overline{F} \supseteq J_{2m}$ take $D_1 = \{x_1, x_2, x_t\} \cup N_Y(x_2)$, and $D_2 = \{x_3\} \cup \{\text{ any } m \text{ other vertices of } Y\}$. This is true since $D_1 \cup D_2$ contains $K_{m,m+1}$ minus at most two edges $(x_2, x_3)$ and $(x_2, x_t)$ in $\overline{F}$. If $|N_Y(x_2)| = m - 4$, then $D_1 = \{x_1, x_2, x_{t-1}, x_t\} \cup N_Y(x_2) \cup N_Y(x_{t-1})$ and $D_2$ as the set of the remaining vertices of $Y$. Thus, $D_1 \cup D_2$ will induce $K_{m,m+1}$ in $\overline{F}$. If $|N_Y(x_2)| = m - 5$ (only for $m = 5$), then $D_1 = \{x_1, x_2, x_{t-1}, x_t, b\}$, where $b$ is either $x_3$, a neighbor of $x_3$ in $Y$ or a vertex of $Y$ at distance two from $x_3$ and $D_2$ as the set of the remaining vertices of $Y$. Thus, $D_1 \cup D_2$ will induce $K_{m,m+1}$ minus at most one edge in $\overline{F}$.

Subcase 2.5. $|L| = m + 4 = 2m - 1$ (only for $m = 5$).
Let $L$ be $(x_1, x_2, \cdots, x_t)$ where $t = 2m - 1$, then $|Y| = |V(F)\setminus V(L)| \geq 2m - 4$. Now, consider the set $N_Y(x_2)$ of vertices in $Y$ adjacent to $x_2$. Note that any vertex of $N_Y(x_2)$ is nonadjacent to any other vertices of $Y$. If $|N_Y(x_2)| \geq m - 2$ then form two sets $D_1$ and $D_2$ as follows. By the maximality of $L$, one element in each pair $\{x_4, x_5\}$ and $\{x_6, x_7\}$ is nonadjacent to all vertices of $N_Y(x_2)$. Call these two vertices by $b$ and $c$. Therefore, there are at most four edges connecting from $\{x_1, x_t\}$ to $\{x_3, b, c\}$ in $F$. Now, define $D_1 = \{x_1, x_t\} \cup \{\text{any } m - 2 \text{ vertices of } N_Y(x_2)\}$ and $D_2 = \{x_3, b, c\} \cup \{\text{any } m - 2 \text{ other vertices of } Y\}$. Thus, the set $D_1 \cup D_2$ induces $K_{5,6}$ minus four edges in $\overline{F}$, and so $\overline{F} \supseteq J_{10}$.

If $|N_Y(x_2)| = m - 3$ then By the maximality of $L$, one vertex in $\{x_4, x_5\}$ is nonadjacent to all vertices of $N_Y(x_2)$. Call this vertex by $b$. Therefore, there are at most four edges connecting from $\{x_1, x_2, x_t\}$ to $\{x_3, b\}$ in $F$. Now, take $D_1 = \{x_1, x_2, x_t\} \cup N_Y(x_2)$, and $D_2 = \{x_3, b\} \cup \{\text{any } m - 1 \text{ other vertices of } Y\}$. Then, $D_1 \cup D_2$ contains $K_{5,6}$ minus at most four edges in $\overline{F}$. Therefore, $\overline{F} \supseteq J_{10}$.

If $|N_Y(x_2)| = m - 4$ then take $D_1 = \{x_1, x_2, x_{t-1}, x_t\} \cup N_Y(x_2) \cup N_Y(x_{t-1})$, and $D_2 = \{x_3\} \cup \{\text{all the remaining vertices of } Y\}$. Then, $D_1 \cup D_2$ contains $K_{5,6}$ minus
possibly two edges \((x_3, x_{t-1})\) and \((x_3, x_t)\) in \(\overline{F}\). Therefore, \(\overline{F} \supseteq J_{10}\).

If \(|N_Y(x_2)| = m - 5\) then take \(D_1 = \{x_1, x_2, b, x_{t-1}, x_t\}\) where \(b\) is either \(x_3\) or \(x_4\) whose the smallest number of neighbors in \(Y\), and \(D_2 = Y\). Then, \(D_1 \cup D_2\) contains \(K_{5,6}\) minus at most three edges in \(\overline{F}\). Therefore, \(\overline{F} \supseteq J_{10}\). \(\square\)

The proof of Theorem 2.

For \(n = 4\) and \(k = 2\), consider graph \(G = K_1 \cup K_7\). Clearly \(G\) contains no \(2P_n\) and \(\overline{G}\) contains no \(J_4\). Hence \(R(2P_4, J_4) \geq 9\). To prove the upper bound, consider now graph \(F\) of order 9 containing no \(2P_4\). Take a longest path in \(F\) and call it \(L\).

Let \(L\) be \(x_1, x_2, \ldots, x_k\). Clearly, \(k \leq 7\), since \(F \not\supseteq 2P_4\). If \(A = V(F) \setminus V(L)\), then \(|A| \geq 2\). Any vertex of \(A\) is nonadjacent to \(x_1\) and \(x_k\). Thus, the number vertices in \(A\) must be exactly 2 and so \(k = 7\), since otherwise \(A\) together with \(\{x_1, x_k\}\) will form a \(K_{2,3} = J_4\) in \(\overline{F}\). Let \(A = \{y, z\}\), and consider the following two cases:

**Case 1.** Vertices \(y\) and \(z\) has a common neighbor in \(L\).

Let \(x_i\) be the common neighbor of \(y\) and \(z\) in \(L\), for some \(i \in \{2, 3, \ldots, 6\}\). Then, \(y, z\) are nonadjacent to \(x_i-1\) and \(x_{i+1}\), since otherwise the maximality of \(L\) will suffer. At least one of the last two vertices must differ with \(x_1\) and \(x_7\), call it \(w\). So, we have a \(J_4\) in \(\overline{F}\) formed by \(\{x_1, x_7, y, z, w\}\).

**Case 2.** Vertices \(y\) and \(z\) has no common neighbor in \(L\).

If there exists a vertex \(x_i, 2 \leq i \leq 6\), is nonadjacent to \(y\) and \(z\), then \(\{x_i, x_1, x_7, y, z\}\) forms a \(J_4\) in \(\overline{F}\). Thus, every \(x_i\) is adjacent to at least one of \(\{y, z\}\). Now, since \(y\) and \(z\) has no common neighbor in \(L\), without loss of generality we can assume that \(x_2y \in E(F)\), and so \(x_2z \notin E(F)\), \(x_3y \notin E(F)\), \(x_3z \in E(F)\), \(x_4z \notin E(F)\), \(x_4y \in E(F)\), \(x_5y \notin E(F)\) and \(x_5z \in E(F)\). Therefore, the path \(x_1, x_2, y, x_4, x_3, z, x_5, x_6, x_7\) is Hamiltonian, which contradicts the maximality of path \(L\) in \(F\).

Now, let \(n \geq 5\). Consider graph \(G = K_1 \cup K_{kn-1}\). Clearly \(G\) contains no \(kP_n\) and \(\overline{G}\) contains no \(J_4\). Hence \(R(kP_n, J_4) \geq kn-1+1+1 = kn+1\). For the upper bound, let \(F\) be a graph of order \(kn+1\) such that \(\overline{F}\) does not contain \(J_4\). By induction on \(k\), we will show that \(F\) contains \(kP_n\). By Theorem A gives a verification of the result for \(k = 1\). Assume the theorem is true for any \(s \leq k-1\), namely \(R(sP_n, J_4) = sn+1\), for \(n \geq 5\). Now consider graph \(F\) of \(kn+1\) vertices such that \(\overline{F} \not\supseteq J_4\). By the induction hypothesis, \(F\) will contain \((k-1)P_n\). Let \(Y = V(F) \setminus V((k-1)P_n)\). Then, \(|Y| = n+1 = R(P_n, J_4)\) and hence \(F[Y]\) contains a \(P_n\). In total, \(F\) will contain \(kP_n\). \(\square\)

The proof of Theorem 3.

Since graph \(G = K_{m-1} \cup K_{kn-1}\) contains no \(kP_n\) and \(\overline{G}\) contains no \(J_{2m}\), then \(R(kP_n, J_{2m}) \geq kn+m-1\). For proving the upper bound, let \(F\) be a graph of order \(kn+m-1\) such that \(\overline{F}\) contains no a \(J_{2m}\). We will show that \(F\) contains \(kP_n\). We use an induction on \(k\). For \(k = 1\) it is true from Theorem A. Now, let assume that the
theorem is true for all $s \leq k - 1$. Take any graph $F$ of $kn + m - 1$ vertices such that its complement contains no $J_{2m}$. By the hypothesis, $F$ must contain $(k-1)$ disjoint copies of $P_n$. Remove these copies from $F$, then the remaining vertices will induce another $P_n$ in $F$ since $\overline{F} \not\subseteq J_{2m}$. Therefore $F \supseteq kP_n$. 

The proof is complete. □

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