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Landau-Zener transition driven by a slow noise

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The effect of a slow noise in non-diagonal matrix element, $J(t)$, that describes the diabatic level coupling, on the probability of the Landau-Zener transition is studied. For slow noise, the correlation time, $\tau_c$, of $J(t)$ is much longer than the characteristic time of the transition. Existing theory for this case suggests that the average transition probability is the result of averaging of the conventional Landau-Zener probability, calculated for a given constant $J$, over the distribution of $J$. We calculate a finite-$\tau_c$ correction to this classical result. Our main finding is that this correction is dominated by sparse realizations of noise for which $J(t)$ passes through zero within a narrow time interval near the level crossing. Two models of noise, random telegraph noise and gaussian noise, are considered. Naturally, in both models the average probability of transition decreases upon decreasing $\tau_c$. For gaussian noise we identify two domains of this fall-off with specific dependencies of average transition probability on $\tau_c$.

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I. INTRODUCTION

A standard expression \[1-4\] for the probability of the Landau-Zener transition between the two diabatic levels, $\pm vt/2$, reads

$$P_{LZ} = 1 - Q_{LZ}, \quad Q_{LZ} = \exp \left( -\frac{2\pi J^2}{v} \right), \quad (1)$$

where $J$ is the off-diagonal matrix element. Correspondingly, $Q_{LZ}$ is the probability to stay on the initial diabatic level.

The dynamics of the transition is governed by the following system

$$\begin{cases} 
\dot{a}_1 = \frac{vt}{2} a_1 + J a_2, \\
\dot{a}_2 = -\frac{vt}{2} a_2 + J a_1,
\end{cases} \quad (2)$$

where $a_1, a_2$ denote the amplitudes for the particle to reside on the diabatic levels 1 and 2, respectively.

The question about the value of $P_{LZ}$ when the transition is driven by noise, so that $J$ is a random function of time, was first put forward and answered in Refs. [5,6]. The answer depends on whether the noise is slow or fast. More precisely, on whether the noise correlation time, $\tau_c$, is longer or shorter than the time of the Landau-Zener transition. For large matrix element, $J \gg v^{1/2}$, when $P_{LZ}$ is close to 1, this transition time is given by

$$\tau_{LZ} = \frac{v}{J}, \quad (3)$$

Fast noise corresponds to $\tau_{LZ} \gg \tau_c$, i.e. the matrix element oscillates many times in the course of the transition. Then it is apparent that both outcomes of the transition are almost equally probable. In other words, $Q_{LZ}$ differs from 1/2 only slightly. It was demonstrated in Ref. [6] that, for the fast noise,

$$Q_{LZ} = \frac{1}{2} \left[ 1 + \exp \left( -\frac{4\pi J^2}{v} \right) \right], \quad (4)$$

where $\langle ... \rangle$ denotes the averaging over the realizations of $J(t)$, and $J^2$ is defined via the noise correlator

$$\langle J(t) J(t') \rangle = J^2 K(t - t') \quad (5)$$

at coinciding times, where $K(0) = 1$.

FIG. 1: Schematic illustration of the Landau-Zener transition driven by slow telegraph noise. For a typical noise realization the matrix element switches from $J_0$ to $-J_0$ at time $t_0 \sim \tau_c \gg \tau_{LZ}$. For such realizations the probability, $Q_{LZ}$, to stay on the same diabatic level is exponentially small. However, for sparse realizations with $t_0 \lesssim \tau_{LZ}$, the value $Q_{LZ}$ is close to 1. Thus, it is these sparse realizations are responsible for the finite-$\tau_c$ correction.

In the opposite limit of slow noise $\tau_c \gg \tau_{LZ}$, the result obtained in Ref. [6] is also physically transparent. Namely, one can neglect the change of $J(t)$ during the transition. This suggests that the expression Eq. [1] should be simply averaged over the distribution, $P(J)$, of the matrix element

$$Q_{LZ} = \int_{-\infty}^{\infty} dJ \ P(J) \exp \left\{ -\frac{2\pi J^2}{v} \right\}. \quad (6)$$

Further developments in the theory of the noise-driven Landau-Zener transition in Refs. [7-23] included: (i) consideration of specific microscopic models of the environment leading to random $J(t)$; (ii) extension of Eq. [4]
to the case when $J(t)$ has both constant and fluctuating components (Pokrovsky-Sinitsyn formula); and (iii) generalization of the theory of the fast noise to the case of three and more crossing levels.

With regard to the papers [7][23], we note that the results for the average transition probabilities were obtained only in two “extreme” cases: $\tau_c \to 0$ for the fast noise and $\tau_c \to \infty$ for the slow noise. The corrections in small parameters $\tau_c/\tau_{LZ}$ for the fast noise and $\tau_{LZ}/\tau_c$ for the slow noise were not found. There is actually a finite-$\tau$ noise as in Ref. [17], when $J$, the message, consider the particular model of telegraph but rather from sparse noise realizations. To illustrate effects, or in other words, not from typical, $\tau$ in small parameters between the two values $\pm \tau$, results for the average transition probabilities were obtained for the generalization of the theory of the fast noise to the case of components (Pokrovsky-Sinitsyn formula); and (iii) generalization to the case when $J$.

Without switching near $t = 0$, the probability $Q_{LZ}$ is exponentially small. On the other hand, with switching, we will demonstrate below, $Q_{LZ}$ is close to 1. Since the probability that the switching takes place at $t_0 \lesssim \tau_{LZ}$ is $\sim \tau_{LZ}/\tau_c$, the correction to $Q_{LZ}$ can be estimated as $\sim \tau_{LZ}/\tau_c$. This correction becomes important even at large enough $\tau_c$, since in the limit $\tau_c \to \infty$ the value $Q_{LZ}$ is exponentially small. In the next Section we justify the above picture by a rigorous calculation. In Sect. III we study the case of a gaussian noise when $J(t)$ is a smooth function of time. We show that in this case the finite-$\tau_c$ correction to $Q_{LZ}$, again, originates from the sparse realizations of noise when $J(t)$ passes through zero near the moment of the level crossing. In addition to finite-$\tau_c$ correction, the realizations with $J(t_0) = 0$ for $t_0 \ll \tau_c$ determine the behavior of $Q_{LZ}$ upon decreasing of the correlation time. We identify two domains of $\tau_c$ with distinctly different $Q_{LZ}(\tau_c)$ dependence. In Sect. IV which concludes the paper, we speculate on the form of the finite-$\tau_c$ correction to $Q_{LZ}$ in the limit of fast noise $\tau_c \ll \tau_{LZ}$.

II. TELEGRAPH NOISE: SWITCHING NEAR $t = 0$

Consider the system Eq. (2) and assume that the coupling constant is equal to $J_0$ for $t < t_o$, while for $t > t_o$ it switches to $-J_0$, see Fig. 1. Solutions of the system for $t < t_o$ are expressed via the parabolic cylinder functions [24] as follows

$$\begin{align*}
a_1 &= D_\nu(z), \\
a_2 &= -i\sqrt{\nu}D_{\nu-1}(z),
\end{align*}$$

where the argument $z$ is defined as $z = \sqrt{\nu}e^{i\pi/4}t$ and the index $\nu$ is given by

$$\nu = -\frac{iJ^2}{v}.$$ 

The solution Eq. (7) ensures that $a_2(-\infty) = 0$, i.e. that the particle is in the state $a_1$ away from the crossing. For $t > t_o$ solution of the system represents a linear combination of two parabolic-cylinder functions

$$\begin{align*}
a_1 &= AD_\nu(z) + BD_\nu(-z), \\
a_2 &= i\sqrt{\nu}AD_{\nu-1}(z) - i\sqrt{\nu}BD_{\nu-1}(-z),
\end{align*}$$

The fact that the coefficient in front of $D_{\nu-1}(z)$ in $a_2$ is equal to $-A$, unlike Eq. (7), is the consequence of the switching. The coefficients $A$ and $B$ are found from continuity of $a_1$ and $a_2$ at $t = t_o$. The corresponding system reads

$$\begin{align*}
D_\nu(z_o) &= AD_\nu(z_o) + BD_\nu(-z_o), \\
D_{\nu-1}(z_o) &= -AD_{\nu-1}(z_o) + BD_{\nu-1}(-z_o),
\end{align*}$$

where $z_o = \sqrt{\nu}e^{i\pi/4}t_o$. From this system we readily derive the following expressions

$$A = \frac{D_{\nu-1}(-z_o)}{D_{\nu-1}(z_o)} - \frac{D_{\nu}(-z_o)}{D_{\nu}(z_o)}, \quad B = \frac{2}{\frac{D_{\nu-1}(-z_o)}{D_{\nu-1}(z_o)} + \frac{D_{\nu}(-z_o)}{D_{\nu}(z_o)}}. \quad (11)$$

Without switching, we would have $A = 1, B = 0$. Suppose now, that the switching took place at $t_0 = 0$. Then from Eq. (11) we have $A = 0$ and $B = 1$. This illustrates the dramatic effect of switching on the probability $Q_{LZ}$. Indeed, in the limit $z_o \to \infty$, the ratio $|D_\nu(z_o)/D_\nu(-z_o)|^2$ approaches a small value $\exp(-2\pi J_0^2/v)$, which is the value of $Q_{LZ}$ without switching. By contrast, with $B = 1$ as a result of switching, $Q_{LZ}$ is equal to 1.

The situation $t_0 = 0$ is most favorable for enhancement of $Q_{LZ}$. In order to find the average enhancement, we should analyze the expressions for $A$ and $B$, given by Eq. (11), as a function of $z_o$. The fact which allows to carry out this analysis analytically is that the typical value of $Q_{LZ}$ is small. This is also equivalent to the condition $J_0^2 \gg v$, or $|\nu| \gg 1$. Conventionally, the behavior of the parabolic cylinder functions at large $|z_o| \gg 1$ is obtained from the semiclassics. The condition $|\nu| \gg 1$ justifies the semiclassical analysis even for small $z_o$. This analysis is carried out in Appendix A. Substituting Eqs. (A8) and (A10) into Eq. (11), we obtain the following expression for $B$:
The probability to stay on the same diabatic level in the presence of the switching is shown schematically versus the moment of switching, $t_0$. The probability $Q_{LZ}(t_0)$ is described by the Lorentzian Eq. (17) for $t_0 \sim \tau_{LZ}$ and approaches asymptotically to $\exp(-2\pi|\nu|)$ for $t_0$ much bigger than the time, $\tau_{LZ}$, of the Landau-Zener transition.

$$B = 2 \left[ \frac{\left(\frac{\sqrt{4\nu - z_0^2}}{\sqrt{4\nu + z_0^2} + i z_0}\right)^{\nu + 1/2} + e^{i\nu\pi + \frac{iz_0}{2}} \sqrt{4\nu - z_0^2}}{e^{iz_0} \sqrt{4\nu - z_0^2} + e^{i\nu\pi} \left(\frac{\sqrt{4\nu - z_0^2} - iz_0}{\sqrt{4\nu + z_0^2} + iz_0}\right)^{\nu + 1/2}} \right]^{-1} + \left[ \frac{\left(\frac{\sqrt{4\nu - z_0^2} - iz_0}{\sqrt{4\nu + z_0^2} + iz_0}\right)^{\nu - 1/2} - e^{i\nu\pi + \frac{iz_0}{2}} \sqrt{4\nu - z_0^2}}{e^{iz_0} \sqrt{4\nu - z_0^2} - e^{i\nu\pi} \left(\frac{\sqrt{4\nu - z_0^2} - iz_0}{\sqrt{4\nu + z_0^2} + iz_0}\right)^{\nu - 1/2}} \right].$$

(12)

The above expression for $B$ allows a dramatic simplification. It is achieved upon adding the two fractions and subsequently rewriting the result in the form of a single fraction. The result reads

$$B = \frac{i z_0 + i\sqrt{4\nu - z_0^2} \sin(\phi + \nu \pi)}{\sqrt{4\nu - z_0^2} \sinh(\nu \pi)},$$

(13)

where the phase $\phi$ is defined as

$$\phi = -\frac{z_0}{2} \sqrt{4\nu - z_0^2} - i\nu \ln \left(\frac{\sqrt{4\nu - z_0^2} - iz_0}{4\nu}\right)^2.$$  

(14)

In a similar way, from Eq. (11) we get a simplified expression for $A$

$$A = -\frac{i\sqrt{4\nu} \sin \phi + iz_0 \cosh(\nu \pi)}{\sqrt{4\nu - z_0^2} \sinh(\nu \pi)}.$$  

(15)

The exact expression for the probability to stay on the same diabatic level in the presence of the switching is $Q_{LZ} = |a_1(\infty)/a_1(-\infty)|^2$. From Eq. (9) we can express this probability in terms of $A$ and $B$ as follows

$$Q_{LZ} = |A|^2 e^{-2\pi|\nu|} + (A^* B + B^* A) e^{-\pi|\nu|} + |B|^2.$$  

(16)

Let us first check that for large $|z_0| \gg |\nu|^{1/2}$ Eq. (16) reproduces $Q_{LZ} = \exp(-2\pi|\nu|)$, as is expected on physical grounds. Indeed, in this limit, we can replace $A$ by $-1$ and $B$ by $2\exp(-\pi|\nu|)$. Then the second and the third terms in Eq. (16) cancel out.

To study the behavior of $Q_{LZ}$ in the domain of $|z_0| \sim |\nu|^{1/2}$, we notice that only the third term does not contain any exponentially small factor. Hence, for this term,

we can use the asymptotic expression in which corrections of the order $\exp(-\pi|\nu|)$ are neglected. Namely,

$$Q_{LZ} = \frac{4}{4 + \left(\frac{t_0}{\tau_{LZ}}\right)^2}.$$  

(17)

In Fig. [2] the dependence of $Q_{LZ}$ on the moment of switching is shown schematically. It is described by the Lorentzian Eq. (17) for $t_0 \sim \tau_{LZ}$ and approaches $\exp(-2\pi|\nu|)$ for $t_0$ much bigger than the time, $\tau_{LZ}$.

As we established above, the switching at $t_0 = 0$ leads to the probability $Q_{LZ} = 1$. Then Eq. (17) suggests that the enhancement of $Q_{LZ}$ takes place in the domain of $t_0 \sim \pm 2\tau_{LZ}$. Since $t_0$ is random, we average $Q_{LZ}$ over $t_0$ to get the net enhancement of $Q_{LZ}$, and present the final result in the form

$$1 - P_{LZ} = \exp \left\{ - \frac{2\pi J^2}{v} \right\} + \frac{1}{\tau_c} \int_{-\infty}^{\infty} dt_0 \frac{4}{4 + \left(\frac{t_0}{\tau_{LZ}}\right)^2}.$$

$$= \exp \left\{ - \frac{2\pi J^2}{v} \right\} + 2\pi \frac{\tau_{LZ}}{\tau_c}.$$  

(18)

It is important to note that the condition of applicability of Eq. (17) is $\tau_{LZ} \leq \tau_c$. Under this condition, the second term of Eq. (18) can greatly exceed the first term, which is the result corresponding to no switching.
III. GAUSSIAN NOISE

Suppose that $J(t)$ changes continuously as in Fig. 3 and that the typical $J$ is much bigger than $v/\sqrt{2}$, so that the typical $P_{LZ}$ is close to 1. As $J$ slowly changes with time, the maximum contribution to $Q_{LZ}$ will come from the time domain when it takes small values $J \lesssim v^{1/2}$. Since the portion of these domains is $\sim v^{1/2}/J$, the ratio $v^{1/2}/J$ determines the average $Q_{LZ}$. This is how the analytical result of Ref. 15 can be interpreted. This result applies for very long correlation times, $\tau_c \to \infty$.

As we are interested in a finite-\(\tau\) correction, we will study the domains of small $J$ in more detail. A nontrivial consequence of small $J$ values is that, when the noise takes these values, the Landau-Zener transition time is much shorter than the typical $\tau_{LZ}$. This, in turn, suggests that even when the condition of slow noise is violated for typical $J$, it can still be met for anomalously small $J$, which determines $Q_{LZ}$. Thus, one can expect a nontrivial behavior of $Q_{LZ}$ upon decreasing the correlation time.

Naturally, small $J$-values are realized in the vicinity of zeros of $J(t)$. Similar to the previous Section, the most relevant are the realizations of noise when a zero occurs in the vicinity of $t = 0$, when the levels cross. For these realizations we can linearize $J(t)$ as follows

$$J(t) = (t-t_0)J', \tag{19}$$

where $J'$ is the slope and $t_0$ is much smaller than the correlation time.

Our prime observation is that the system Eq. (2) can be solved exactly with $J(t)$ in the form Eq. (19). This is achieved by introducing new variables

$$ \begin{cases} b_1 = a_1 \cos \varphi + a_2 \sin \varphi, \\ b_2 = a_1 \sin \varphi - a_2 \cos \varphi, \end{cases} \tag{20}$$

where the angle $\varphi$ is defined as

$$\tan(2\varphi) = \frac{2J'}{v}. \tag{21}$$

It is straightforward to check that the system of equations for $b_1, b_2$ has the form

$$ \begin{cases} \dot{b}_1 = \frac{v}{2} \left( \frac{(t-t_0) \sin^2(2\varphi)}{\cos(2\varphi)} \right) b_1 + \frac{v}{2} t_0 \sin(2\varphi) b_2, \\ \dot{b}_2 = -\frac{v}{2} \left( \frac{(t-t_0) \sin^2(2\varphi)}{\cos(2\varphi)} \right) b_2 + \frac{v}{2} t_0 \sin(2\varphi) b_1. \tag{22} \end{cases}$$

We see that the system has reduced to the original system Eq. (2) with renormalized velocity $\tilde{v} = v/\cos(2\varphi)$ and the coupling $\tilde{J} = v t_0 \sin(2\varphi)/2$. The established mapping allows us to write the answer for $Q_{LZ}$ straightaway

$$Q_{LZ}(t_0) = \exp \left\{ 2\pi J'^2 t_0 \left| \cos^3(2\varphi) \right| \right\}. \tag{23}$$

Using Eq. (21), the above expression can be recast into the form

$$Q_{LZ} = \exp \left\{ -\frac{2\pi J'^2 t_0}{v} \left( \frac{v^2}{v^2 + 4J'^2} \right)^{3/2} \right\}. \tag{24}$$

The crucial assumption made in course of deriving Eq. (21) is that the linearization Eq. (19) is valid during the entire renormalized transition time, $\tilde{\tau}_{LZ}$. We are now in position to check this assumption. Indeed,

$$\tilde{\tau}_{LZ} = \frac{J'}{\tilde{v}} = \frac{J'}{v^2 + 4J'^2 t_0}. \tag{25}$$

Since the first factor does not exceed 1, we conclude that the condition $t_0 \ll \tau_c$ is the only condition necessary for Landau-Zener transition to be dominated by a local zero of $J(t)$.

Contribution of the realizations with $t_0 \ll \tau_{LZ}$ to the probability $Q_{LZ}$ is given by the average

$$\overline{Q}_{LZ} = \int_{-\infty}^{\infty} dt_0 Q_{LZ}(t_0) = \left( \frac{v}{\pi} \right)^{1/2} \left( \frac{1}{|J'| \tau_c} \right) \left( \frac{v^2 + 4J'^2}{v^2} \right)^{3/4}. \tag{26}$$

We are interested in $\tau_c$-dependence of the probability $Q_{LZ}$. The correlation time enters into Eq. (26) in two ways: firstly it is present directly in the denominator, and, secondly, via $J'$, since its typical value is $J' \sim J_c/\tau_c$. The remaining task is to average Eq. (26) over the random slopes, $J'$. We will perform this averaging in the domains of small and large $J'$ separately:

(i) Small $J'$. In this limit, we can replace the bracket in Eq. (26) by 1. The average over $J'$ does not diverge, since $J'$ cannot be smaller than the minimal value determined by the condition $t_0 \ll \tau_c$. Since the typical $t_0$ is $\sim v^{1/2}/J_c$, the estimate for the minimal $J'$ is $J' < v^{1/2}/\tau_c$. This minimal $J'$ fixes the lower limit in the integral

$$\langle \overline{Q}_{LZ} \rangle = \left( \frac{v}{\pi} \right)^{1/2} \left( \frac{1}{J_c^2 \tau_c} \right) \int_{v^{1/2}/\tau_c}^{\infty} \frac{dJ'}{\tau_c} \exp \left( -\frac{J'^2}{2J_c^2} \right). \tag{27}$$
where \( J'_c \) is the width of the Gaussian distribution of \( J' \),
\[
J'_c^2 = J_c^2 \frac{\partial^2 K}{\partial t^2} \mid_{t_1=t_2},
\]
and the correlator \( K \) is defined in Eq. (5). Within a number under the logarithm, the final result for the double average reads
\[
\langle Q_{LZ} \rangle = \left( \frac{v}{\pi} \right)^{1/2} \left( \frac{1}{J'_c} \right) \ln \left( \frac{J'_c}{v^{1/2}} \right).
\]  
(29)
Let us compare this result with a standard expression derived in Ref. [6]. In our limit, \( J_c \gg v^{1/2} \), the result of gaussian averaging in Eq. (6) reads
\[
\langle Q_{LZ} \rangle = \left( \frac{v}{4\pi} \right)^{1/2} \left( \frac{1}{J'_c} \right).
\]  
(30)
Since \( J'_c \sim J_c/\tau_c \), the prefactor in Eq. (29) is of the same order as in Eq. (30). In particular case of Gaussian correlator, we have \( J_c/\tau_c = 2^{1/2} J_c \), so that the prefactor in Eq. (29) is two times bigger than in Eq. (30). However, our result contains an additional big factor \( \ln(J_c/v^{1/2}) \). This factor originates from realizations of noise for which \( J(t) \) passes through zero near \( t = 0 \). To the best of our knowledge, the importance of these realizations leading to the enhancement of \( Q_{LZ} \) was not mentioned in the literature.

The result Eq. (29) yields the average \( Q_{LZ} \) in the limit \( \tau_c \to \infty \). Calculation of the finite-\( \tau_c \) correction is straightforward. Expanding the bracket in Eq. (26) to the first power in the small parameter \( J'^2/v^2 \), and averaging over \( J' \), we get
\[
\langle Q_{LZ} \rangle - \langle Q_{LZ} \rangle(\infty) = \frac{3}{(\pi v^3)^{1/2}} \int_0^\infty dJ' \frac{J'_c}{J'_c J_c} \exp \left( -\frac{J'^2}{2J_c^2} \right).
\]  
(31)
It follows from Eq. (31) that upon decreasing the correlation time the correction to \( Q_{LZ} \) grows as \( 1/\tau_c^2 \). It is instructive to rewrite this correction as
\[
\langle Q_{LZ} \rangle - \langle Q_{LZ} \rangle(\infty) = 3 \left( \frac{2}{\pi} \right)^{1/2} \left( \frac{J_c}{v^{1/2}} \right) \left( \frac{1}{v^{1/2} \tau_c} \right)^2.
\]  
(32)
Naturally, the applicability of Eq. (32) is terminated as \( \tau_c \) becomes smaller than \( J_c/v \).

(ii) Large \( J' \). In this limit the bracket in Eq. (26) should be replaced by \( (2J'/v)^{3/2} \). Then the averaging over \( J' \) yields
\[
\langle Q_{LZ} \rangle \approx \frac{2}{\pi} \Gamma \left( \frac{3}{4} \right) \left( \frac{J_c}{v^{1/2}} \right)^{1/2} \left( \frac{1}{v^{1/2} \tau_c} \right)^{3/2},
\]  
(33)
i.e. \( \langle Q_{LZ} \rangle \) grows as \( \tau_c^{-3/2} \) upon the decreasing of the correlation time. The lower boundary of the large \( J' \) domain is determined by the condition \( \langle Q_{LZ} \rangle \sim 1 \), which yields \( \tau_c \sim J_c^{1/3}/v^{2/3} \). Different behaviors of \( \langle Q_{LZ} \rangle \) are illustrated in Fig. 4.

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**IV. CONCLUDING REMARKS**

(i) The main results of the present paper are the expressions Eqs. (29), (32), and (33) for the probability, \( Q_{LZ} \), to stay on the same diabatic level after the transition. These results pertain to the domain where this probability is small, i.e. for \( J_c^2 \gg v \), or equivalently for \( |v| \gg 1 \), where the parameter \( v \) is defined by Eq. (8). Below we combine these results into a single expression for \( Q_{LZ}(\tilde{\tau}_c) \), where
\[
\tilde{\tau}_c = v^{1/2} \tau_c
\]  
(34)
is the dimensionless correlation time. Then one obtains
\[
\langle Q_{LZ} \rangle = \frac{1}{(\pi |v|)^{1/2}} \begin{cases} 
\ln |v|^{1/2} + 3 \cdot 2^{1/2} |v|^{1/2} \tilde{\tau}_c \gg |v|^{1/2} \\
\frac{\ln |v|}{\sqrt{|v|}} \left( \frac{|v|}{\tilde{\tau}_c} \right)^{3/4}, \ |v|^{1/2} \gg \tilde{\tau}_c \gg |v|^{1/6}
\end{cases}
\]  
(35)
The second line describes the crossover between the slow-noise and fast-noise regimes upon decreasing the correlation time.

(ii) In obtaining the results Eqs. (29), (32), and (33) we assumed that it is only the single zero in \( J(t) \) closest to the level crossing, that is responsible for the probability \( Q_{LZ} \). This \( Q_{LZ} \), which is the probability for the particle to stay on the diabatic level, can also be viewed as the probability to change the adiabatic level. In fact, in the domain \( J_c^{1/3}/v^{2/3} \ll \tau_c \ll J_c/v \), many (\( \sim J_c/v \)) zeros of \( J(t) \) will cause local minima(maxima) in the upper(lower) adiabatic energy level, as illustrated in Fig. 5. In other words, many “local” Landau-Zener transitions precede the transition near \( t = 0 \). For our results to apply it is necessary that, in course of each of this transition, the particle does not change the adiabatic level. Calculation of probability of
changing the levels is performed in full analogy to that described above for \( J(t_0) = 0 \) with \( t_0 \ll \tau_e \).

One has to linearize \( J(t) \) near actual zero and perform the rotation from \((a_1, a_2)\) to \((b_1, b_2)\). It is important that the time of the local transition turns out to be \( \sim \nu^2 \tau c / J_e \), which is much smaller than \( \tau_e \), thus confirming that the transition is indeed local. For the probability to stay on the same adiabatic level, the above procedure yields the result

\[
1 - \exp \left[ -2\pi^3 \left( \frac{\nu^2 \tau c^3}{J_e} \right) \left( 1 - \frac{\nu^2 \tau c^2}{J_e^2} \right) \right],
\]

(actual numerical factor in the exponent depends on the location of the transition point within \( \tau_\tau \). For the lower boundary \( \tau_\tau \sim J_e^{1/3}/\nu^{2/3} \), the number in exponent is close to 1 indicating that at this boundary the regime of the fast noise takes over. At the upper boundary \( \tau_\tau \sim J_e/\nu \) the number in the exponent is large (\( \sim |\nu| \)). Thus the particle does not change the adiabatic level at the moments when \( J(t) \) passes through zero at times \( \gtrsim \tau_\tau \), Fig. 5).

(iii) It is important to relate our results to the pioneering calculation in Ref. [6]. The expression for \( Q_{LZ} \) in the presence of noise in this paper was based on the expansion of \( Q_{LZ} \) in powers of \( J_e^2 \).

\[
Q_{LZ} = \sum_{n=0}^{\infty} (-1)^n J_e^{2n} L^{(n)},
\]

where the coefficients \( L^{(n)} \) are the \( 2n \)-fold integrals of the type

\[
L^{(n)} = \sum_{m=1}^{\infty} \frac{\int_{-\infty}^{\infty} d\tau_1 \int_{-\infty}^{\infty} d\tau_2 \cdots \int_{-\infty}^{\infty} d\tau_{2m} \cdots \int_{-\infty}^{\infty} d\tau_{2n}}{2m^{2n-2m-1}} F^{(n)}(\tau_1, \cdots, \tau_{2n}) \exp \left[ \frac{\nu}{2} \sum_{j=1}^{2n} (-1)^j \tau_j^2 \right],
\]

where \( F^{(n)} \) is the product of the correlators \( \exp \left( (\tau_i - \tau_j)/\tau_e \right) \) “forbidding” the variables \( \tau_i \) and \( \tau_j \) to differ more than \( \tau_e \). It is seen from Eq. (38) that typical values of \( \tau_i \) are \( \nu^{-1/2} \), Thus the restrictions imposed by the correlator start to matter for the terms with \( n \gtrsim \tau_e \nu^{1/2} \). For smaller \( n \) the restriction is not important and \( F^{(n)} \) manifests itself only in the form of combinatorial factor \( = (2n-1)!! \).

With this factor in front of \( L^{(n)} \) the sum Eq. (37), instead of \( \exp (-2\pi J_e^2/\nu) \), reduces to

\[
Q_{LZ} = \frac{1}{(1 + 4\pi J_e^2/\nu)^{1/2}}.
\]

The above result does not depend on the form of correlator, and is interpreted in Ref. [6] as the average of \( \exp (-2\pi J_e^2/\nu) \) with Gaussian distribution \( P(J) = (2\pi)^{-1/2} J e^{-J^2/2J_e^2} \). This already reveals an inconsistency, since the particular form, \( \exp (-|t_i - t_j|/\tau_e) \), of the correlator chosen in Ref. [6] corresponds to the telegraph noise with \( P(J) = \frac{1}{2} \delta(J - J_e) + \delta(J + J_e) \). Correspondingly, averaging over the distribution of \( J \) yields \( \exp (-2\pi J_e^2/\nu) \) instead of Eq. (39), correctly implying that for \( \tau_e \rightarrow \infty \) switchings of \( J \) do not affect the transition probability.

For gaussian noise, our result Eq. (29) does not coincide with Eq. (39). The reason is that the logarithmic function in Eq. (29), originating from sparse realizations, is a singular function of \( J_e^2 \) and cannot be captured by expansion in powers of \( J_e^2 \). Equally, for the telegraph noise, the finite-\( \tau_e \) correction Eq. (31), \( \sim \tau_{LZ}/\tau_e \), is proportional to \( |J_e| \), and, thus, is also singular function of \( J_e^2 \). In general, the effects due to sparse realizations cannot be captured within the perturbative expansion.

(iv) Throughout the paper we assumed that the noise is slow and searched for finite - \( \tau_e^{-1} \) corrections to the probability, \( 1 - P_{LZ} \), to stay on the same diabatic level. We argued that these corrections are dominated by sparse realizations of noise. For the fast noise the probability to stay exceeds the probability to make a transition by \( \frac{1}{2} \exp (-4\pi J_e^2/\nu) \).

The small parameter \( \tau_e/\tau_{LZ} \) insures that the noise is fast. Below we argue qualitatively that sparse
realizations of noise can dominate the finite-\(\tau_c\) correction to the standard result.

Consider the case of the fast telegraph noise. During the time \(\tau_{LZ}\), the off-diagonal matrix element switches from \(J_0\) to \(-J_0\) approximately \(\tau_{LZ}/\tau_c\) times. However, for certain sparse realizations, the switching happens only once at the moment \(t_0 \sim \tau_c\). The probability of this realization can be estimated as \(t_0/\tau_c\exp(-\tau_{LZ}/\tau_c)\). However, as was demonstrated in Sect. III for such realizations, the system will stay on the initial diabatic level. This suggests the following form of \(P_{LZ}\) in the limit of fast noise

\[
\frac{1}{2} - P_{LZ} = \frac{1}{2} \exp\left\{-\frac{4\pi J_z^2}{\nu}\right\} + \exp\left\{-\frac{J_z}{\nu\tau_c}\right\},
\]

where the second term accounts for the sparse realizations. Note that this term dominates the first term for \(\tau_c \gg 1/4\pi J_z\). Remarkably, this \(\tau_c\) belongs to the domain of fast noise. Indeed, for \(\tau_c = 1/4\pi J_z\), the ratio \(\tau_c/\tau_{LZ}\) is equal to \(v/4\pi J_z^2\), i.e. it is small. Consequently, as in the case of the slow telegraph noise, we again come to the conclusion that finite-\(\tau_c\) correction can dominate the result.

Appendix A: Asymptotic behavior of \(D_{\nu}(z)/D_{\nu}(-z)\)

Since we are interested in the domain \(z \sim |\nu|\), standard \(z \to \pm \infty\) asymptotes of the parabolic cylinder function \(\text{D}_\nu\) are insufficient. Therefore, we start from the following integral representation

\[
D_{\nu}(z) = \sqrt{\frac{2}{\pi}} e^{z^2/4} \int_0^\infty e^{-t^2/2} t^\nu \cos\left(zt - \frac{\nu \pi}{2}\right) dt. \tag{A1}
\]

It is convenient to divide the integral Eq. (A1) into two parts

\[
D_{\nu}(z) = I_+(z)e^{-i\nu \pi/2} + I_-(z)e^{i\nu \pi/2}, \tag{A2}
\]

where the functions \(I_+(z)\) and \(I_-(z)\) are defined as

\[
I_+(z) = \sqrt{\frac{1}{2\pi}} e^{z^2/4} \int_0^\infty e^{-t^2/2} t^\nu e^{itz} dt. \tag{A3}
\]

The advantage of introducing \(I_+\) and \(I_-\) is that they are suited for evaluation using the steepest descent method. To apply this method, we rewrite \(I_+\) in the form

\[
I_+(z) = \frac{1}{2\pi} e^{z^2/4} \int_0^\infty e^{tf(t)} dt, \quad f(t) = -t^2/2 + izt + \nu \ln t. \tag{A4}
\]

The extrema of \(f(t)\) correspond to \(t_\pm = \frac{iz}{2} \pm \sqrt{\frac{4\nu - z^2}{2}}\).

The real part of \(t_+\) is positive, and thus lies within the domain of integration. By contrast, \(\text{Re} t_-\) is negative and \(t_-\) should therefore be excluded.

Expanding \(f(t)\) near \(t_+\), we get from Eq. (A4)

\[
I_+(z) \approx \sqrt{\frac{1}{2\pi}} e^{z^2/4} e^{f(t_+)} \int_{t_-}^{\infty} d(t - t_+) \\
\times \exp \left[-\frac{1}{2} \left(1 + \frac{\nu}{t_+} \right)(t - t_+)^2\right]. \tag{A5}
\]

Upon performing the gaussian integration, the result can be cast in the form

\[
I_+(z) \approx \exp \left(\frac{iz\sqrt{4\nu - z^2}}{4} - \frac{\nu}{2}\right) (4\nu - z^2)^{-1/4} \\
\times \left(\frac{iz + \sqrt{4\nu - z^2}}{2}\right)^{\nu+1/2}. \tag{A6}
\]

The condition of applicability of the steepest descent method is that typical \((t - t_+)\) contributing to the integral Eq. (A5) is much smaller than \(t_+\). It is easy to see that for \(z \sim |\nu|\) this condition is satisfied when \(|\nu| \gg 1\), i.e. in the case we are interested in. Calculation of \(I_-(z)\) is similar to the above calculation. The saddle points are \(-t_\pm\), and only \(t = -t_-\) contributes to the integral. The result reads

\[
I_-(z) \approx \exp \left(-\frac{iz\sqrt{4\nu - z^2}}{4} - \frac{\nu}{2}\right) (4\nu - z^2)^{-1/4} \\
\times \left(-\frac{iz + \sqrt{4\nu - z^2}}{2}\right)^{\nu+1/2}. \tag{A7}
\]

Naturally, the result Eq. (A7) is consistent with the relation \(I_+(z) = I_-(z)\), which follows from Eq. (A3). If one takes a limit \(z \gg |\nu|\) in Eqs. (A6), (A7), and substitute the result into Eq. (A2), one would recover the textbook asymptotes of \(D_{\nu}(z)\). We are interested in the ratio \(D_{\nu}(-z)/D_{\nu}(z)\). Using Eqs. (A5), (A6), this ratio can be presented in a concise form

\[
\frac{D_{\nu}(-z)}{D_{\nu}(z)} \approx \frac{I_-(z) + e^{i\nu \pi} I_+(-z)}{I_+(z) + e^{i\nu \pi} I_-(z)} \\
\approx \exp\left(\frac{\sqrt{4\nu - z^2} - iz}{\sqrt{4\nu - z^2} + iz}\right) \nu^{\nu+1/2} + e^{i\nu \pi + \frac{iz}{2}} \sqrt{4\nu - z^2} \nu^{\nu+1/2}. \tag{A8}
\]

To verify that Eq. (A8) has the right limit we recall that for \(z \to \infty\) it should reproduce \(Q_{LZ}^{-1/2}\). Indeed, in this limit, the denominator in the fraction turns to \(\exp(\frac{t_+}{2} \sqrt{4\nu - z^2})\), while the numerator turns to \(\exp(\frac{t_+}{2} \sqrt{4\nu - z^2} + izt)\). Consequently, Eq. (A8) yields \(\exp(\pi|\nu|)\). Furthermore, for \(z \to 0\), the ratio becomes 1, as expected.

Next we turn to the calculation of \(D_{\nu-1}(-z)/D_{\nu-1}(z)\). Since the integral representation of \(D_{\nu}(z)\) contains \(\nu\)
in the form of $t''$ in the integrand, evaluation of the asymptote of $D_{\nu-1}(-z)$ with the steepest descent simply amounts to dividing the result for $I_+(z)$ by $t_+$, and the result for $I_-(z)$ by $-t_-$. This yields

$$D_{\nu-1}(z) \approx \frac{2e^{-i(\nu-1)\pi/2}}{iz + \sqrt{4\nu - z^2}} I_+(z) + \frac{2e^{i(\nu-1)\pi/2}}{-iz + \sqrt{4\nu - z^2}} I_-(z).$$

Then the ratio $D_{\nu-1}(-z)/D_{\nu-1}(z)$ can be cast in the form similar to Eq. (A8)

$$\frac{D_{\nu-1}(-z)}{D_{\nu-1}(z)} = \frac{I_+(z) + e^{i\nu\pi} I_+(z)}{I_+(z) + e^{i\nu\pi} I_-(z)} \approx \frac{\left( \frac{\sqrt{4\nu - z^2} - iz}{\sqrt{4\nu - z^2} + iz} \right)^{\nu-1/2} - e^{i\nu\pi + i\frac{\pi}{2}} \sqrt{4\nu - z^2}}{e^{i\frac{\pi}{2}} \sqrt{4\nu - z^2} + e^{i\nu\pi} \left( \frac{\sqrt{4\nu - z^2} - iz}{\sqrt{4\nu - z^2} + iz} \right)^{\nu-1/2}}.$$  

(A10)

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25. Strictly speaking, Eq. (23) relates the values of $|b_1|^2$ at $t \to \pm \infty$, provided that $b_2(-\infty) \to 0$. We are interested in the transition probability from the particular initial state $|a_1(-\infty)|^2 = 1$, with $a_1$ being a linear combination of $b_1$ and $b_2$. The applicability of Eq. (23) is ensured by the fact that Eq. (27) determines two values of $\varphi$, which differ by $\pi/2$. Transition probabilities for these two values are the same. The proper initial state can thus be “arranged” by combining the solutions for $b_1, b_2$ corresponding to the two $\varphi$-values.