Vector Bundles on a K3 Surface

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Abstract

A K3 surface is a quaternionic analogue of an elliptic curve from a viewpoint of moduli of vector bundles. We can prove the algebraicity of certain Hodge cycles and a rigidity of curve of genus eleven and gives two kind of descriptions of Fano threefolds as applications. In the final section we discuss a simplified construction of moduli spaces.

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1. Introduction

A locally free sheaf $E$ of $O_X$-modules is called a vector bundle on an algebraic variety $X$. As a natural generalization of line bundles vector bundles have two important roles in algebraic geometry. One is the linear system. If $E$ is generated by its global sections $H^0(X, E)$, then it gives rise to a morphism to a Grassmann variety, which we denote by $\Phi_E : X \rightarrow G(H^0(E), r)$, where $r$ is the rank of $E$. This morphism is related with the classical linear system by the following diagram:

\[
\begin{array}{c}
X \\
\Phi_E \\
\Phi_L \\
\rangle \\
P^* H^0(L) \cdots \rightarrow P^*(\wedge^r H^0(E)),
\end{array}
\]

where $L$ is the determinant line bundle of $E$ and $\Phi_L$ is the morphism associated to it.

The other role is the moduli. The moduli space of line bundles relates a (smooth complete) algebraic curve with an abelian variety called the Jacobian variety, which is crucial in the classical theory of algebraic functions in one variable. The moduli of vector bundles also gives connections among different types varieties, and often yields new varieties that are difficult to describe by other means.

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In higher rank case it is natural to consider the moduli problem of $E$ under the restriction that $\det E$ is unchanged. In view of the above diagram, vector bundles and their moduli reflect the geometry of the morphism $X \to \mathbb{P}^* H^0(L)$ via Grassmannians and Plücker relations. In this article we consider the case where $X$ is a K3 surface, which is one of two 2-dimensional analogues of an elliptic curve and seems an ideal place to see such reflection.

2. Curves of genus one

The moduli space of line bundles on an algebraic variety is called the Picard variety. The Picard variety $\text{Pic} C$ of an algebraic curve $C$ is decomposed into the disjoint union $\bigsqcup_{d \in \mathbb{Z}} \text{Pic} d C$ by the degree $d$ of line bundles. Here we consider the case of genus 1. All components $\text{Pic} d C$ are isomorphic to $\mathbb{C}$ if the ground field is algebraically closed. But this is no more true otherwise. For example the Jacobian $\text{Pic} 0 C$ has always a rational point but $C$ itself does not. We give other examples:

Example 1 Let $C_4$ be an intersection of two quadrics $q_1(x) = q_2(x) = 0$ in the projective space $\mathbb{P}^4$ and $P$ the pencil of defining quadrics. Then the Picard variety $\text{Pic} _2 C_4$ is the double cover of $P \simeq \mathbb{P}^1$ and the branch locus consists of 4 singular quadrics in $P$. Precisely speaking, its equation is given by $\tau^2 = \text{disc} (\lambda_1 q_1 + \lambda_2 q_2)$.

Let $G(2, 5) \subset \mathbb{P}^9$ be the 6-dimensional Grassmann variety embedded into $\mathbb{P}^9$ by the Plücker coordinate. Its projective dual is the dual Grassmannian $G(5, 2) \subset \hat{\mathbb{P}}^9$, where $G(2, 5)$ parameterizes 2-dimensional subspaces and $G(5, 2)$ quotient spaces.

Example 2 A transversal linear section $C = G(2, 5) \cap H_1 \cap \cdots \cap H_5$ is a curve genus 1 and of degree 5. Its Picard variety $\text{Pic} _2 C$ is isomorphic to the dual linear section $\hat{C} = G(5, 2) \cap \langle H_1, \ldots, H_5 \rangle$, the intersection with the linear subspace spanned by 5 points $H_1, \ldots, H_5 \in \hat{\mathbb{P}}^9$.

3. Moduli K3 surfaces

A compact complex 2-dimensional manifold $S$ is a K3 surface if the canonical bundle is trivial and the irregularity vanishes, that is, $K_S = H^1(O_S) = 0$. A smooth quartic surface $S_4 \subset \mathbb{P}^3$ is the most familiar example. Let us first look at the 2-dimensional generalization of Example 1:

Example 3 Let $S_8$ be an intersection of three general quadrics in $\mathbb{P}^5$ and $N$ the net of defining quadrics. Then the moduli space $M_S(2, O_S(1), 2)$ is a double cover of $N \simeq \mathbb{P}^2$ and the branch locus, which is of degree 6, consists of singular quadrics in $N$.

Here $M_S(r, L, s)$, $L$ being a line bundle, is the moduli space of stable sheaves $E$ on a K3 surface $S$ with rank $r$, $\det E \simeq L$ and $\chi(E) = r + s$. Surprisingly two

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1 More precisely, this holds true if $C$ has a rational point.
2 Two components $\text{Pic} 0 C$ and $\text{Pic}_{g-1} C$ deserve the name Jacobian. They coincide in our case $g = 1$. 
surfaces $S_8 = (2) \cap (2) \cap (2) \subset P^5$ and $M_S(2, O_S(1), 2) \rightarrow P^2$ in this example are both K3 surfaces. This is not an accident. In respect of moduli space, vector bundles a K3 surface look like Picard varieties in the preceding section.

**Theorem 1** ([10], [11]) The moduli space $M_S(r, L, s)$ is smooth of dimension $(L^2) - 2rs + 2$. $M_S(r, L, s)$ is again a K3 surface if it is compact and of dimension 2.

A K3 surface $S$ and a moduli K3 surface appearing as $M_S(r, L, s)$ are not isomorphic in general but their polarized Hodge structures, or periods, are isomorphic to each other over $Q$ ([11]). The moduli is not always fine but there always exists a universal $P_{r-1}$-bundle over the product $S \times M$. Let $A$ be the associated sheaf of Azumaya algebras, which is of rank $r^2$ and locally isomorphic to the matrix algebra $Mat_r(O_{S \times M})$. $A$ is isomorphic to $\text{End} E$ if a universal family $E$ exists. The Hodge isometry between $H^2(S, Q)$ and $H^2(M_S(r, L, s), Q)$ is given by $c_2(A)/2r \in H^4(S \times M, Q) \simeq H^2(S, Q) \otimes H^2(M, Q)$.

Example 2 has also a K3 analogue. Let $\Sigma_{12} \subset P_{15}$ be a 10-dimensional spinor variety $SO(10)/U(5)$, that is, the orbit of a highest weight vector in the projectivization of the 16-dimensional spinor representation. The anti-canonical class is 8 times the hyperplane section and a transversal linear section $S = \Sigma_{12} \cap H_1 \cap \cdots \cap H_8$ is a K3 surface (of degree 12). As is similar to $G(2, 5)$ the projective dual $\hat{\Sigma}_{12} \subset \hat{P}_{15}$ of $\Sigma_{12}$ is again a 10-dimensional spinor variety.

**Example 4** The moduli space $M_S(2, O_S(1), 3)$ is isomorphic to the dual linear section $\hat{S} = \hat{\Sigma}_{12} \cap \langle H_1, \ldots, H_8 \rangle$.

In this case, moduli is fine and the relation between $S$ and the moduli K3 are deeper. The universal vector bundle $E$ on the product gives an equivalence between the derived categories $D(S)$ and $D(\hat{S})$ of coherent sheaves, the duality $\hat{S} \simeq S$ holds (cf. [17]) and moreover the Hilbert schemes $\text{Hilb}^2 S$ and $\text{Hilb}^2 \hat{S}$ are isomorphic to each other.

**Remark** (1) Theorem 1 is generalized to the non-compact case by Abe [11].

(2) If a universal family $E$ exists, the derived functor with kernel $E$ gives an equivalence of derived categories of coherent sheaves on $S$ and the moduli K3 (Bridgeland [4]). In even non-fine case the derived category $D(S)$ is equivalent to that of the moduli K3 $M$ twisted by a certain element $\alpha \in H^2(M, Z/rZ)$ (Căldăraru [5]).

4. **Shafarevich conjecture**

Let $S$ and $T$ be algebraic K3 surfaces and $f$ a Hodge isometry between $H^2(S, Z)$ and $H^2(T, Z)$. Then the associated cycle $Z_f \in H^4(S \times T, Z) \simeq H^2(S, Z)^\vee \otimes H^2(T, Q)$ on the product $S \times T$ is a Hodge cycle. This is algebraic by virtue of the Torelli type theorem of Shafarevich and Piatetskij-Shapiro. Shafarevich conjectured in [23] a generalization to Hodge isometries over $Q$. Our moduli theory is able to answer this affirmatively.

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3We take the complex number field $C$ as ground field except for sections 2 and 7.
Theorem 2  Let \( f : H^2(S, \mathbb{Q}) \to H^2(T, \mathbb{Q}) \) be a Hodge isometry. Then the associated (Hodge) cycle \( Z_f \in H^4(S \times T, \mathbb{Q}) \) is algebraic.

In [11], we already proved this partially using Theorem 1 (cf. [21] also). What we need further is the moduli space of projective bundles. Let \( P \to S \) be a \( \mathbb{P}^{r-1} \)-bundle over \( S \). The cohomology class \([P] \in H^1(S, PGL(r, \mathcal{O}_S))\) and the natural exact sequence (in the classical topology)

\[
0 \to \mathbb{Z}/r\mathbb{Z} \to SL(r, \mathcal{O}_S) \to PGL(r, \mathcal{O}_S) \to 1
\]

define an element of \( H^2(S, \mathbb{Z}/r\mathbb{Z}) \), which we denote by \( w(P) \).

Fix \( \alpha \in H^2(S, \mathbb{Z}) \) and \( r \), we consider the moduli of \( \mathbb{P}^{r-1} \)-bundles \( P \) over \( S \) with \( w(P) = \alpha \mod r \) which are stable in a certain sense. If the self intersection number \((\alpha \cdot \alpha)\) is divisible by \( 2r \), then the moduli space contains a 2-dimensional component, which we denote by \( M_S(\alpha/r) \). The following, a honest generalization of computations in [11], is the key of our proof:

Proposition 1  Assume that \((\alpha \cdot \alpha)\) is divisible by \( 2r^2 \). Then \( H^2(M_S(\alpha/r), \mathbb{Z}) \) is isomorphic to \( L_0 + \mathbb{Z}\alpha/r \subset H^2(S, \mathbb{Q}) \) as polarized Hodge structure, where \( L_0 \) is the submodule of \( H^2(S, \mathbb{Z}) \) consisting of \( \beta \) such that the intersection number \((\beta \cdot \alpha)\) is divisible by \( r \).

For example let \( S_2 \) be a double cover of \( \mathbb{P}^2 \) with branch sextic. If \( \alpha \in H^2(S, \mathbb{Z}) \) satisfies \((\alpha \cdot h) \equiv 1 \mod 2\) and \((\alpha \cdot \alpha) \equiv 0 \mod 4\), then \( M_S(\alpha/2) \) is a K3 surface of degree 8. This is the inverse correspondence of Example 1 (cf. [20, 24]). Details will be published elsewhere.

5. Non-Abelian Brill-Noether locus

Let \( C \) be a smooth complete algebraic curve. As a set a Brill-Noether locus of \( C \) is a stratum of the Picard variety \( \text{Pic}^d C \) defined by \( h^0(L) \), the number of global sections of a line bundle \( L \). The standard notation is

\[
W^r_d = \{ [L] | h^0(L) \geq r + 1 \} \subset \text{Pic}^d C,
\]

for which we refer [2]. Non-Abelian analogues are defined in the moduli space \( \mathcal{U}_C(2) \) of stable 2-bundles on \( C \) similarly. The non-Abelian Brill-Noether locus of type III is

\[
\mathcal{S}\mathcal{U}_C(2, K : n) = \{ F | \det F \simeq \mathcal{O}_C(K_C), h^0(F) \geq n \} \subset \mathcal{U}_C(2)
\]

for a non-positive integer \( n \), and type II is

\[
\mathcal{S}\mathcal{U}_C(2, K : nG) = \{ F | \det F \simeq \det G \otimes \mathcal{O}_C(K_C), \dim \text{Hom}(G, F) \geq n \} \subset \mathcal{U}_C(2)
\]

for a vector bundle \( G \) of rank 2 and \( n \equiv \deg G \mod 2 \). By virtue of the (Serre) duality, these have very special determinantal descriptions. We give them scheme structures using these descriptions ([19]).
Assume that $C$ lies on a K3 surface $S$. If $E$ belongs to $M_S(r, L, s)$, then the restriction $E|_C$ is of canonical determinant and we have $h^0(E|_C) \geq h^0(E) \geq \chi(E) = r + s$. So $E|_C$ belongs to $SU_C(2, K : r + s)$ if it is stable. This is one motivation of the above definition. The case of genus 11, the gap value of genera where Fano 3-folds of the next section do not exist, is the most interesting.

**Theorem 3** ([15]) If $C$ is a general curve of genus 11, then the Brill-Noether locus $T = SU_C(2, K : 7)$ of type III is a K3 surface and the restriction $L$ of the determinant line bundle is of degree 20.

There exists a universal family $E$ on $C \times T$. We moreover have the following:

- the restriction $E|_{x \times T}$ is is stable and belongs to $M_T(2, L, 5)$, for every $x \in C$, and
- the classification morphism $C \rightarrow \hat{T} = M_T(2, L, 5)$ is an embedding.

This embedding is a non-Abelian analogue of the Albanese morphism $X \rightarrow \text{Pic}_0(\text{Pic}_0 X)$ and we have the following:

**Corollary** A general curve of genus eleven has a unique embedding to a K3 surface.

In [14], we studied the forgetful map $\varphi_g$ from the moduli space $P_g$ of pairs of a curve $C$ of genus $g$ and a K3 surface $S$ with $C \subset S$ to the moduli space $M_g$ of curves of genus $g$ and the generically finiteness of $\varphi_{11}$. The above correspondence $C \mapsto \hat{T}$ gives the inverse rational map of $\varphi_{11}$. We recall the fact that $\varphi_{10}$ is not dominant in spite of the inequality $\dim P_{10} = 29 > \dim M_{10} = 27$ ([12]).

### 6. Fano 3-folds

A smooth 3-dimensional projective variety is called a Fano 3-fold if the anticanonical class $-K_X$ is ample. In this section we assume that the Picard group $\text{Pic} X$ is generated by $-K_X$. The self intersection number $(-K_X)^3 = 2g - 2$ is always even and the integer $g \geq 2$ is called the genus, by which the Fano 3-folds are classified into 10 deformation types. The values of $g$ is equal to 2, . . . , 10 and 12. A Fano 3-folds of genus $g \leq 5$ is a complete intersection of hypersurfaces in a suitable weighted projective space.

By Shokurov [25], the anticanonical linear system $|-K_X|$ always contain a smooth member $S$, which is a K3 surface. In [13] we classified Fano 3-folds $X$ of Picard number one using rigid bundles, that is, $E \in M_S(r, L, s)$ with $(L^2) - 2rs = -2$. For example $X$ is isomorphic to a linear section of the 10-dimensional spinor variety, that is,

$$X \simeq \Sigma_{12} \cap H_1 \cap \cdots \cap H_7,$$

in the case of genus 7 and a linear section

$$X \simeq \Sigma_{16} \cap H_1 \cap H_2 \cap H_3,$$  \( \text{(2)} \)

of the 6-dimensional symplectic, or Lagrangian, Grassmann variety $\Sigma_{16} = SP(6)/U(3) \subset P^{13}$ in the case of genus 9. The non-Abelian Brill-Noether loci shed new light on this classification.
\textbf{Theorem 4} A Fano 3-fold \(X\) of genus 7 is isomorphic to the Brill–Noether locus \(SU_C(2, K : 5)\) of Type III for a smooth curve \(C\) of genus 7.

This description is dual to the description \(2\) in the following sense. First two ambient spaces of \(X\), the moduli \(U_C(2)\) and the Grassmannian \(G(5, 10) \supset \Sigma_{12}\) are of the same dimension. Secondly let \(N_1\) and \(N_2\) be the normal bundles of \(X\) in these ambient spaces. Then we have \(N_1 \simeq N_2 \otimes \mathcal{O}_X(-K_X)\), that is, two normal bundles are twisted dual to each other.

\textbf{Theorem 5} A Fano 3-fold of genus 9 is isomorphic to the Brill–Noether locus \(SU_C(2, K : 3G)\) of Type II for a nonsingular plane quartic curve \(C\) and \(G\) a rank 2 stable vector bundle over \(C\) of odd degree.

This description is also dual to \(3\) in the above sense: The moduli \(U_C(2)\) and the Grassmannian \(G(3, 6) \supset \Sigma_{16}\) are of the same dimension and the two normal bundles of \(X\) are twisted dual to each other. Each Fano 3-fold of genus 8, 12 and conjecturally 10 has also such a pair of descriptions.

7. Elementary construction

The four examples in sections 1 and 2 are very simple and invite us to a simplification of moduli construction. Let \(C_4\) be as in Example 1 and \(\text{Mat}_2\) the affine space associated to the 16-dimensional vector space \(\bigoplus_{i=0}^{3}(\mathbb{C}^2 \otimes \mathbb{C}^2)x_i\), where \((x_i)\) is the homogeneous coordinate of \(\mathbb{P}^3\). Let \(\text{Mat}_{2,1}\) be the closed subscheme defined by the condition that

\[
A(x) = \sum_{i=0}^{3} A_i x_i \in \text{Mat}_2 \text{ is of rank } \leq 1 \text{ everywhere on } C_4
\]

and \(R\) its coordinate ring. Then the Picard variety \(\text{Pic}_2 C_4\) is the projective spectrum \(\text{Proj} R^{SL(2) \times SL(2)}\) of the invariant ring by construction. (See \[15\] for details.) The above condition is equivalent to that \(\det A(x)\) is a linear combination of \(q_1(x)\) and \(q_2(x)\). The invariant ring is generated by three elements by Theorem 2.9.A of Weyl \[23\]. Two of them, say \(B_1, B_2\), are of degree 2 and correspond to \(q_1(x)\) and \(q_2(x)\), respectively. The rest, say \(T\) of degree 4, is the determinant of 4 by 4 matrix obtained from the four coefficients \(A_0, \ldots, A_3 \in \mathbb{C}^2 \otimes \mathbb{C}^2\) of \(A(x)\). There is one relation \(T^2 = f_4(B_1, B_2)\). Hence \(\text{Proj} R^{SL(2) \times SL(2)}\) is a double cover of \(\mathbb{P}^1\) as desired.

The moduli space \(M_S(2, \mathcal{O}_S(1), 2)\) in Example 3 is constructed similarly. Let \(\text{Alt}_4\) be the affine space associated to the vector space \(\bigoplus_{i=0}^{5}(\wedge^2 \mathbb{C}^4)x_i\) and \(\text{Alt}_{4,2}\) the subscheme defined by the condition that \(\sum_{i=0}^{5} A_i x_i \in \text{Alt}_4\) is of rank \(\leq 2\) everywhere on \(S_6\). Then the invariant ring of the action of \(SL(4)\) on \(\text{Alt}_{4,2}\) is generated by four elements \(B_1, B_2, B_3, T\) of degree 2, 2, 2, 6. There is one relation \(T^2 = f_6(B_1, B_2, B_3)\) and \(M_S(2, \mathcal{O}_S(1), 2)\), the projective spectrum \(\text{Proj} R^{SL(4)}\), is a double cover of \(\mathbb{P}^2\) as described.

The moduli space of vector bundles on a surface was first constructed by Gieseker \[6\]. He took the Mumford’s GIT quotient \[19\] of Grothendieck’s Quot scheme \[7\] by \(PGL\) and used the Gieseker matrix to measure the stability of the
action. In the above construction, we take the quotient of $\text{Alt}_{4,2}$, which is nothing but the affine variety of Gieseker matrices of suitable rank 2 vector bundles, by a general linear group $GL(4)$.

The Jacobian, or the Picard variety, of a curve is more fundamental. Weil [27] constructed $\text{Pic}_g C$ as an algebraic variety using the symmetric product $\text{Sym}^g C$ and showed its projectivity by Lefschetz’ 3Θ theorem. Later Seshadri and Oda [21] constructed $\text{Pic}_d C$ for arbitrary $d$ (over the same ground field as $C$) by also taking the GIT quotient of Quot schemes. The above constructions eliminate Quot schemes and the concept of linearization from those of Gieseker, Seshadri and Oda.

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