Hodge structure on the Cohomology of the Moduli space of Higgs bundles

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Abstract

Let $C$ be a smooth projective curve over $\mathbb{C}$ of genus $g \geq 2$. Let $\mathcal{H}_n$ be the moduli space of stable Higgs bundles of rank $r$ and degree $d$ on $C$ with values in $K \otimes \mathcal{O}(np)$, where $K$ is the canonical bundle on $C$, $p$ any marked point in $C$, and $(r,d) = 1$. We prove that the natural Hodge structure on $H^k(\mathcal{H}_n, \mathbb{Q})$ is pure of weight $k$.

An object that has been studied in great depth for many years now is the moduli space of flat unitary connections on a compact Riemann surface. Due to the famous theorem of Narasimhan and Seshadri [10], this space has been investigated by algebraic geometers under the guise of the moduli space of vector bundles on a compact Riemann surface. More recently, several people have focussed their attention on the moduli space of all flat connections (as opposed to just unitary ones). There is a similar correspondence theorem here as well, which identifies (only topologically as in the former) this space with the moduli space of Higgs bundles on the compact surface.

We know that under suitable conditions the moduli space of vector bundles is a fine moduli space, so there exists a (holomorphic) universal bundle on this space. It was shown by Atiyah and Bott [1] that the Chern classes of this bundle can be decomposed appropriately into components known as universal classes, which then generate the cohomology ring of the moduli space. Recently, a similar result was proven in the case of Higgs bundles by Thaddeus and Hausel [16] in rank 2, and by Markman [9] in general rank. In this note, we use their results to compute the Hodge structure on the cohomology of the moduli space of Higgs bundles on a smooth projective curve. The result is somewhat surprising since the moduli space is non-compact yet all of its cohomology groups have pure weight.

Outline. This paper is divided into three parts. In section 1, we give a brief introduction to the moduli space of Higgs bundles and recall some of the results and constructions related to these objects. In section 2, we state the main result. Finally, section 3 contains the proof of the result.

Notation and Conventions. Throughout the paper, $C$ will denote the compact Riemann surface or smooth projective curve, of genus $g \geq 2$. We shall use $\mathcal{H}_n$ to denote moduli spaces over $C$ of Higgs bundles $(E, \phi)$ of rank $r$ and degree $d$ with values in $K(n) = K \otimes \mathcal{O}(np)$, where $p$ is a marked point in $C$, and $n$ is any non-negative integer. We will always assume that $r$ and $d$ are coprime. All cohomology is with rational coefficients unless otherwise stated.
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1 Higgs bundles

Higgs bundles were first studied in great detail by Hitchin and Simpson \cite{Hitchin87a, Simpson90}. Let $C$ be a smooth complex projective curve of genus $g \geq 2$. Let $L$ be any holomorphic line bundle on $C$. A Higgs bundle on $C$ with values in $L$ is a pair $(E, \phi)$, where $E$ is a holomorphic vector bundle on $C$, and $\phi$, often called a Higgs field, is an element of $H^0(C, \text{End}(E) \otimes L)$. We say that a Higgs bundle is semi-stable if for any $\phi$-invariant subbundle $F \subset E$, $\deg F / \text{rk} F \leq \deg E / \text{rk} E$. We say that the Higgs bundle is stable if this inequality is strict. We will denote by $r$ the rank of $E$ and by $d$ the degree of $E$.

We now specialize to the case when $L = K(n) = K \otimes \mathcal{O}(np)$, where $K$ is the canonical bundle on $C$, $p$ is a marked point in $C$, and $n \geq 0$. Then the following is known.

**Theorem (Hitchin, Simpson, Nitsure \cite{Hitchin87a, Simpson90, Nitsure91}).** For fixed rank $r$, and degree $d$ coprime to $r$, and for any $n \geq 0$, there exists a moduli space $\mathcal{H}_n$ of stable Higgs bundles with values in $K(n)$, which is a smooth quasi-projective variety of dimension $r^2(2g-2+n)+2$.

When $n = 0$, the space $\mathcal{H}_0$ of Higgs bundles is related to the space of flat connections (or connections of constant central curvature) by the following result.

**Theorem (Hitchin, Corlette, Donaldson \cite{Hitchin87a, Corlette87, Donaldson87}).** Let $r$ and $d$ be coprime. Denote by $\mathcal{H}$ the space of $\text{GL}(r, \mathbb{C})$-connections on $C$ of constant curvature $d\omega I$ up to gauge equivalence, where $\omega$ is a 2-form on $C$ chosen so that $\int_C \omega = 2\pi/r$, and $I$ is the $r \times r$ identity matrix. Then $\mathcal{H}$ is a smooth variety diffeomorphic to the moduli space $\mathcal{H}_0$.

This equivalence is only topological, since the complex structures on the two moduli spaces are different.

We now focus our attention on $\mathcal{H}_n$. This space can be constructed gauge-theoretically in the following manner (see \cite{Donoho84} for details). Let $\mathcal{V}$ be a Hermitian vector bundle on $C$ of rank $r$ and degree $d$. We denote by $\mathcal{A}$ the complex affine space of holomorphic structures on $\mathcal{V}$. Let $\hat{\Omega}^{p,q}$ be the Sobolev completion of the space of differential forms of Sobolev class $L^2_{k-p-q}$. For $n \geq 0$, we can then define a map

$$\bar{\partial} : \mathcal{A} \times \hat{\Omega}^{0,0}(\text{End } \mathcal{V} \otimes K(n)) \to \hat{\Omega}^{0,1}(\text{End } \mathcal{V} \otimes K(n))$$

by $\bar{\partial}(E, \phi) = \bar{\partial}_E \phi$, and let $\mathcal{B}_n = \bar{\partial}^{-1}(0)$. Let $\mathcal{G}$ be the gauge group of all complex automorphisms of $\mathcal{V}$, and $\overline{\mathcal{G}}$ be the quotient of $\mathcal{G}$ by the central subgroup $\mathbb{C}^* \subset \mathcal{G}$. Then we know that $\mathcal{G}$ acts on $\mathcal{A}$ and $\hat{\Omega}^{1,0}(\text{End } \mathcal{V} \otimes K(n))$ thus inducing an action on $\mathcal{B}_n$. Finally, if we denote by $\mathcal{B}_n^s \subset \mathcal{B}_n$ the open subset of stable Higgs bundles, then we have $\mathcal{H}_n = \mathcal{B}_n^s / \overline{\mathcal{G}}$.

We also know that for all $n \geq 0$, $\mathcal{H}_n \times C$ carries on it a universal family $(E_n, \Phi_n)$. Although the universal bundles $E_n$ are not canonical, the projective bundles $\mathbb{P}(E_n)$ are, and we recall briefly one method of constructing these bundles.
We start with the tautological rank \( r \) vector bundle \( E_A \) on \( A \times C \) with the constant scalars \( \mathbb{C}^\times \subset \mathcal{G} \) acting trivially on the base and as scalars in the fibre of \( E_A \). Using the projection map \( \pi : B_n^r \to A \), we pull back the bundle \( E_A \) to get a bundle we will call \( E_{B_n} \) on \( B_n^r \times C \). Next, we notice that the group \( \mathcal{G} \) acts on this bundle, thus inducing an action on the projective bundle \( \mathbb{P}(E_{B_n}) \) with the constant scalars \( \mathbb{C}^\times \subset \mathcal{G} \) acting trivially on the fibres. Hence the bundle \( \mathbb{P}(E_{B_n}) \) is a \( \mathcal{G} \)-equivariant bundle. Since \( \mathcal{G} \) also acts freely on \( B_n^r \), we see that \( \mathbb{P}(E_{B_n}) \) descends to a \( \mathbb{P}^{r-1} \)-bundle \( \mathbb{P}(E_n) \) on \( H_n \times C \). Any lift of this bundle is a universal bundle on \( H_n \times C \). In general of course such lifts don’t always exist. However, for the spaces \( H_n \), one can always find a lift and thereby construct a universal bundle. The crucial point to note here is that the projective bundle \( \mathbb{P}(E_n) \) on \( H_n \times C \) is constructed by pulling back the bundle \( E_A \) which is independent of the parameter \( n \). This allows us to do the following.

Thaddeus and Hausel [16] showed that the spaces \( H_n \) constructed in the above manner form a resolution tower, that is there are natural inclusions \( H_n \hookrightarrow H_{n+1} \), which allow us to construct the direct limit space which we shall denote by \( H_\infty \). They also showed that the \( B_n \)'s also have natural inclusions which allows us to construct the direct limit \( B_\infty \) and that \( B_\infty \) is a principal \( \mathcal{G} \)-bundle on \( H_\infty \) with total space contractible. Since the projections \( B_n \to A \) commute with the inclusions \( B_n \hookrightarrow B_{n+1} \), and the construction of the bundles \( \mathbb{P}(E_n) \) did not depend on \( n \), we can now take the direct limit of these bundles and thus obtain a projective bundle on \( H_\infty \) which we shall denote by \( \mathbb{P}(E_{\text{dim}}) \). We also note that due to the way it is constructed, \( \mathbb{P}(E_{\text{dim}}) \) is the canonical bundle \( \mathbb{P}(E) \) where \( E \) is any universal bundle on \( H_\infty \).

Finally, we will denote by \( c_1^n, \ldots, c_r^n \) the Chern classes of the universal bundle \( E_n \) on \( H_n \), and by \( \bar{c}_2^n, \ldots, \bar{c}_r^n \) the Chern classes of the projective bundle \( \mathbb{P}(E_n) \) on \( H_n \). The \( \bar{c}_k \) are elements of rational cohomology, and may be thought of as Chern classes of the universal bundle \( E \) twisted by a formal \( r \)-th root of \( \Lambda^r E^* \), so that the first Chern class vanishes. Similarly, \( \bar{c}_2, \ldots, \bar{c}_r \) will denote the Chern classes of the projective bundle \( \mathbb{P}(E_{\text{dim}}) \) on \( H_\infty \). One consequence of the above construction is that the Chern classes of \( \mathbb{P}(E_{\text{dim}}) \) restrict to their counterparts on \( H_n \) for \( n \geq 0 \). The bundle \( \mathbb{P}(E_{\text{dim}}) \) will play a key role in our proof in section 3.

## 2 Statement of the Result

**Theorem 1.** The natural Hodge structure on \( H^k(H_n, \mathbb{Q}) \) is pure of weight \( k \).

This is obviously known to be true in the case of smooth projective varieties over \( \mathbb{C} \). However, if the variety is singular or non-compact, we typically expect the natural Hodge structure on \( H^k \) to have some **mixing**, i.e., we expect the \( k \)-th cohomology group to contain elements of weight other than \( k \). So it is interesting and perhaps a little surprising that although \( H_n \) is non-compact, there is indeed no such **mixing** in its cohomology.

A simple consequence of the above result is the following. If \( X \) is any smooth compactification of \( H_n \), then the natural maps \( H^k(X, \mathbb{Q}) \to H^k(H_n, \mathbb{Q}) \) are surjective. This follows by first considering the case when the compactification divisor is a normal crossings divisor, and then using Hironaka’s resolution method to always reduce to this case.
The proof of the above theorem occupies the rest of the paper. We use the approach of Thaddeus and Hausel [10] to get a grip on the spaces \( \mathcal{H}_n \). More specifically, we prove first that the \( k \)-th cohomology space \( H^k(\mathcal{H}_\infty, \mathbb{Q}) \) of the direct limit \( \mathcal{H}_\infty \) of the spaces \( \mathcal{H}_n \) carries a natural Hodge structure. We then show that this in fact is pure, using the fact that the cohomology is generated by the universal classes of the direct limit bundle \( \mathbb{P}(E_{\text{lim}}) \). This finally allows us to use the surjective map from \( H^*(\mathcal{H}_\infty) \to H^*(\mathcal{H}_n) \) to complete the proof.

3 Proof

Lemma 1. The cohomology ring \( H^*(\mathcal{H}_\infty, \mathbb{Q}) \) carries a natural mixed Hodge structure (MHS).

Proof. Since each \( \mathcal{H}_n \) is a smooth quasi-projective variety, it carries a natural MHS [5]. Let \( W_{n,*} \) and \( F^{n,*} \) denote the weight and Hodge filtrations respectively on \( H^*(\mathcal{H}_n, \mathbb{Q}) \). Define filtrations \( W_* \) and \( F^* \) on \( H^*(\mathcal{H}_\infty, \mathbb{Q}) \) by setting \( W_* = \lim_{\rightarrow} W_{n,*} \) and \( F^* = \lim_{\leftarrow} F^{n,*} \). It is not difficult to verify that these filtrations satisfy the following properties:

(i) \( W_* \) defines a weight filtration on \( H^*(\mathcal{H}_\infty, \mathbb{Q}) \)

(ii) \( F^* \) defines a Hodge filtration on \( H^*(\mathcal{H}_\infty, \mathbb{Q}) \otimes \mathbb{C} \)

(iii) \( F^* \) induces a \( \mathbb{Q} \)-Hodge filtration of weight \( k \) on each \( \text{Gr}_k(W^*(H^*(\mathcal{H}_\infty, \mathbb{Q}))) = W_k/W_{k-1} \)

Consequently, we see that the filtrations \( F^* \) and \( W_* \) define a MHS on \( H^*(\mathcal{H}_\infty, \mathbb{Q}) \) in the sense of Deligne, although this is an infinite-dimensional vector space.

Lemma 2. Let \( \Gamma_{k,i} \) denote the composition of the following maps.

\[
\begin{array}{ccc}
H^i(C \times \mathcal{H}_\infty, \mathbb{Q})(-k + 1) & \xrightarrow{\cup_{\pi_k}} & H^{i+2k}(C \times \mathcal{H}_\infty, \mathbb{Q})(1) \\
\uparrow {\rho_i} & & \downarrow f_C \\
H^i(C, \mathbb{Q})(-k + 1) & & H^{i+2k-2}(\mathcal{H}_\infty, \mathbb{Q})
\end{array}
\]

Then the cohomology \( H^*(\mathcal{H}_\infty, \mathbb{Q}) \) is generated as a ring by the images of the maps \( \Gamma_{k,i} \), \( 2 \leq k \leq r, 0 \leq i \leq 2 \).

Proof. See Thaddeus and Hausel [10], 10.1.

Now we focus our attention on \( \mathcal{H}_n \). Since \( \mathcal{H}_n \) is smooth quasi-projective, we may assume that \( \mathcal{H}_n = X_n \setminus D_n \) where \( X_n \) is smooth projective and \( D_n \) is a normal crossings divisor [4]. Let \( \Omega^p(*D_n) \) denote the sheaf on \( X_n \) of meromorphic \( p \)-forms that are holomorphic on \( \mathcal{H}_n \) and have poles of arbitrary (finite) order on \( D_n \). Similarly, let \( \mathcal{A}^p(*D_n) \) denote the sheaf on \( X_n \) associated to the presheaf \( U \to \Omega^p(U \setminus U \cap D_n) \), where \( \Omega^p \) is the space of smooth \( p \)-forms. Both of these fit into complexes of sheaves \( (\Omega^*(\mathcal{H}) \setminus \mathcal{D}) \) and \( (\mathcal{A}^*(\mathcal{H}) \setminus \mathcal{D}) \) on \( X_n \). Next, we let \( \Omega^p(\log D) \) be the subsheaf of \( \Omega^p(*D) \) generated locally by the holomorphic forms and the logarithmic differentials \( dz_i/z_i \), \( 1 \leq i \leq k \), where \( D_n \) can be written locally as \( \{ z_1 \cdots z_k = 0 \} \). Intrinsically, if \( f \) is a local defining equation of \( D_n \), then \( \Omega^p(\log D) \)
is given by those meromorphic $p$-forms $\psi$ such that both $f\psi$ and $fd\psi$ are holomorphic. We recall that both the inclusions $\Omega^{*}(\log D) \subset \mathcal{A}^{p}(\star D_{n})$ and $\Omega^{*}(\star D) \subset \mathcal{A}^{p}(\star D_{n})$ are quasi-isomorphisms. Since the sheaves $\mathcal{A}^{p}(\star D_{n})$ are fine, $H^{q}(X_{n}, \mathcal{A}^{p}(\star D_{n})) = 0$ for $q > 0$, using the spectral sequence for hypercohomology, we see that $\mathbb{H}^{*}(X_{n}, \Omega^{*}(\star D)) \cong \mathbb{H}^{*}(X_{n}, \mathcal{A}^{p}(\star D_{n})) \cong H_{d}^{*}(H^{0}(X_{n}, \mathcal{A}^{p}(\star D_{n}))) = H^{*}_{\text{DR}}(\mathcal{H}_{n})$.

Hence we see that the cohomology of $\mathcal{H}_{n}$ can be computed by using meromorphic forms on $X_{n}$ that are holomorphic on $\mathcal{H}_{n}$ and have poles along $D_{n}$. This enables us to define the weight of a $(p, q)$ form $\gamma$ in the cohomology of $\mathcal{H}_{n}$ (see [3] for more details). We say that the form $\gamma$ has weight $p + q + d$ where $d$ is the order of the pole along $D_{n}$ of the associated meromorphic form on $X_{n}$. In particular if $\gamma$ is the restriction of a form holomorphic on all of $X_{n}$, then it has weight $p + q$ (i.e., $d = 0$).

Before proving the next lemma, we discuss briefly the notion of Chern classes for coherent sheaves on a smooth projective variety. Let $X$ be a smooth projective variety. Given a coherent sheaf $\mathcal{F}$ on $X$, we can consider $\mathcal{F}$ as an element of the Grothendieck group $G(X)$. We consider $K(X)$, defined in a similar manner as $G(X)$ except using locally free sheaves, as the quotient of the free abelian group generated by all locally free (coherent) sheaves on $X$, by the subgroup generated by all expressions of the form $\mathcal{E} - \mathcal{E}' - \mathcal{E}''$, whenever $0 \to \mathcal{E}' \to \mathcal{E} \to \mathcal{E}'' \to 0$ is a short exact sequence of locally free sheaves. We recall that the definition of Chern classes extends naturally to $K(X)$ (see for instance Hartshorne, *Algebraic Geometry*). It can be shown using resolutions of coherent sheaves that the natural map from $K(X) \to G(X)$ is an isomorphism. This allows us to consider the image of the coherent sheaf $\mathcal{F}$ in $K(X)$, and consequently define Chern classes of $\mathcal{F}$. This turns out to be a good definition in that it satisfies all the defining properties of Chern classes. We now prove the following.

**Lemma 3.** Let $\Gamma_{k,i}^{n}$ denote the composition of the following maps.

\[
\begin{array}{ccc}
H^{i}(C \times \mathcal{H}_{n}, \mathbb{Q})(-k + 1) & \xrightarrow{\iota_{\mathcal{E}_{n}}} & H^{i+2k}(C \times \mathcal{H}_{n}, \mathbb{Q})(1) \\
\uparrow \rho_{1}^{*} & & \downarrow J_{C} \\
H^{i}(C, \mathbb{Q})(-k + 1) & & H^{i+2k-2}(\mathcal{H}_{n}, \mathbb{Q})
\end{array}
\]

Then for each $k$, $i$, $\Gamma_{k,i}^{n}$ is a morphism of Hodge structures.

**Proof.** We note that $\Gamma_{k,i}^{n}$ is a composition of three maps: a pullback $\rho_{1}^{*}$, cupping with the Chern class $c_{k}^{n}$, followed by integration along the curve $C$. Pulling back a form $\omega \in H^{i}(C, \mathbb{Q})$ to $H^{i}(C \times \mathcal{H}_{n}, \mathbb{Q})$ does not change the Hodge type or weight of the form (since $\rho_{1}^{*}\omega$ will not have any poles along $D_{n}$), so this is a morphism of Hodge structures. Next we observe that the universal bundle $\mathcal{E}_{n}$ on $\mathcal{H}_{n}$ can be extended to a coherent sheaf $\mathcal{E}_{n}$ on $X_{n}$. Then the $k$-th Chern class of $\mathcal{E}_{n}$ restricts to give the Chern class $c_{k}^{n}$ of $\mathcal{E}_{n}$. Hence $\iota_{\mathcal{E}_{n}}^{n}$ (which we already know to be of pure Hodge type $(k, k)$) does not have any poles along $D_{n}$ and therefore has weight $2k$. Consequently, cupping the pullback $\rho_{1}^{*}\omega$ of any $\omega \in H^{i}(C, \mathbb{Q})$ with $\iota_{\mathcal{E}_{n}}^{n}$ will affect the Hodge type as expected by $(k, k)$, but will also shift weight exactly by $2k$. As a result, this is also a morphism of Hodge structures. Finally, integrating a form $\rho_{1}^{*}\omega \cup c_{k}^{n}$ along $C$ is also a morphism of Hodge structures since the Hodge type will shift down by $(1, 1)$ and weight will always shift exactly by $2$ (since $\rho_{1}^{*}\omega \cup c_{k}^{n}$ has no poles along $D_{n}$). 

\[\square\]
Thus each $\Gamma_{k,i}^n$ is a morphism of Hodge structures. We can now finish the proof.

Proof of Theorem 1. We now take the limit of these maps as $n \to \infty$ to get a map from $H^i(C, \mathbb{Q}) \to H^{i+2k-2}(\mathcal{H}_\infty, \mathbb{Q})$, which is exactly the map $\Gamma_{k,i}$ constructed before. Since the MHS on $H^*(\mathcal{H}_\infty, \mathbb{Q})$ was obtained as a limit of the MHS on $H^*(\mathcal{H}_n, \mathbb{Q})$, $\Gamma_{k,i}$ is also a morphism of Hodge structures. By Lemma 2, we know that the image of $\Gamma_{k,i}$ generates the cohomology ring $H^*(\mathcal{H}_\infty, \mathbb{Q})$. Since $\Gamma_{k,i}$ is a morphism of Hodge structures, any $\omega \in H^k(\mathcal{H}_\infty, \mathbb{Q})$ has weight $k$. Next we recall that the natural map $H^*(\mathcal{H}_\infty, \mathbb{Q}) \to H^*(\mathcal{H}_n, \mathbb{Q})$ is surjective and because of the manner in which the MHS on $H^*(\mathcal{H}_\infty, \mathbb{Q})$ was induced, this is also a morphism of Hodge structures. Hence, we see that for any $k$, all elements of $H^k(\mathcal{H}_n, \mathbb{Q})$ are of weight $k$. This completes the proof. □

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