On the regularity of Ricci flows coming out of metric spaces
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Abstract
We consider smooth, not necessarily complete, Ricci flows, \((M, g(t))_{t \in (0,T)}\) with \(\text{Ric}(g(t)) \geq -1\) and \(|\text{Rm}(g(t))| \leq c/t\) for all \(t \in (0,T)\) coming out of metric spaces \((M, d_0)\) in the sense that \((M, d(g(t)), x_0) \to (M, d_0, x_0)\) as \(t \searrow 0\) in the pointed Gromov-Hausdorff sense. In the case that \(B_{g(t)}(x_0,1) \subseteq M\) for all \(t \in (0,T)\) and \(d_0\) is generated by a smooth Riemannian metric in distance coordinates, we show using Ricci-harmonic map heat flow, that there is a corresponding smooth solution \(\tilde{g}(t)_{t \in (0,T)}\) to the \(\delta\)-Ricci-DeTurck flow on an Euclidean ball \(B_{r_0}(p_0) \subseteq \mathbb{R}^n\), which can be extended to a smooth solution defined for \(t \in [0,T)\). We further show, that this implies that the original solution \(g\) can be extended to a smooth solution on \(B_{d_0}(x_0,r/2)\) for \(t \in [0,T)\), in view of the method of Hamilton.

Contents

1 Introduction  
   1.1 Overview .............................................. 2  
   1.2 Main results .......................................... 4  
   1.3 Metric space convergence and the condition (1.1) .......... 5  
   1.4 Related results and works ................................ 5  
   1.5 Outline of Paper ..................................... 6  
   1.6 An open problem ..................................... 7  
   1.7 Notation ............................................... 7  
   1.8 Acknowledgements .................................... 8  

2 Ricci-harmonic map heat flow for functions with bounded gradient  8

3 Almost isometries, distance coordinates and Ricci-harmonic map heat flow  13
   3.1 Almost isometries and Ricci-harmonic map heat flow .......... 15  
   3.2 Ricci-harmonic map heat flow of \((1 + \varepsilon_0)\)-Bi-Lipschitz maps and distance coordinates ............... 18
1 Introduction

1.1 Overview

In this paper, we investigate and answer in certain cases the following question.

Problem 1.1. Let \((M,g(t))_{t \in (0,T)}\) be a Ricci flow which satisfies

\[ |\text{Rm}(\cdot, t)| \leq \frac{c_0}{t}, \]

and for which \((M, d(g(t)))\) Gromov-Hausdorff converges to a metric space \((M, d_0)\) as \(t \searrow 0\). The Riemannian manifolds \((M, g(t))\) are not a priori assumed to be complete for each \(t \in (0, T)\).

What further assumptions on the regularity of \((M, d_0)\) and \((M, g(t))_{t \in (0, T)}\) guarantee that \((M, g(t))\) converges smoothly (or continuously) to a smooth (or continuous metric) as \(t\) approaches zero?

If we assume further that \((M, g(t))_{t \in (0, T)}\) is complete for each \(t \in (0, T)\) and that the Ricci curvature of the solution is uniformly bounded from below, that is we assume

\[ \text{Ric}(\cdot, t) \geq -1, \quad |\text{Rm}(\cdot, t)| \leq \frac{c_0}{t}, \]

then the existence of a unique metric \(d_0\) whose topology agrees with that of \((M, g(t))\) \(t \in (0, T)\) is guaranteed by the results in [17, 19]. More explicitly, defining \(d_t = d(g(t))\) to be the distance with respect to \(g(t)\) on \(M\), in Lemma 3.1 of [19] it is shown that \(d_t \to d_0\) for a metric \(d_0\) on \(M\), and that the estimates

\[ e^t d_0 \geq d_t \geq d_0 - \gamma(n) \sqrt{c_0 t} \quad \text{for all} \quad t \in (0, T), \]

hold in this case, which implies convergence of \(d_t\) in the \(C^0\) sense to \(d_0\), and is a stronger convergence than that of Gromov-Hausdorff convergence. In particular, this implies that the metric space \((M, d_0)\) has the same topology as \((M, d(g(t)))\) for all \(t \in (0, T)\), as is shown for example in the proof of Theorem 9.2 of [17], or can be seen directly using the estimate (3.4) of [19]. Since Gromov-Hausdorff limits are unique up to isometries, we see that the following is true in this setting: if \((M, g(t), p) \to (X, d_X, y)\) in the Gromov-Hausdorff sense for a sequence of times \(t_i > 0\) with \(t_i \searrow 0\) as \(i \to \infty\), then \((X, d_X, y)\) is isometric to \((M, d_0, p)\) which is a manifold. Hence it is not possible that complete solutions satisfying (1.1) come out of metric spaces which are not manifolds.

Note that if we have \(\text{Ric}(\cdot, t) \geq -C\), and \(|\text{Rm}(\cdot, t)| \leq c_0/t\) for all \(t \in (0, T)\) for some \(C \geq 1\), then if we scale once, \(\tilde{g}(t) = C g(t/C)\), then the new solution \(\tilde{g}(\cdot, t)\) satisfies (1.1).

In the papers [18, 19], see [19, Lemma 3.1], these estimates are given in a local setting and it is shown there that if \(\tilde{B}_{g(t)}(x_0, 1) \Subset M\), \(M\) connected but \((M, g(t))\) not necessarily
complete, for all \( t \in (0,1) \), then there is a well defined limit \( d(g(t)) \to d_0 \) as \( t \searrow 0 \) on \( X := (\cap_{\ell \in (0,1)} B_{g_\ell}(x_0, r)) \) for some \( r = r(c_0, n) > 0 \), and that
\[
e^t d_0 \geq d_1 \geq d_0 - \gamma(n) \sqrt{r_0} \quad \text{for all} \quad t \in [0, S(n, c_0)) \cap [0, T) \text{ on } B_{d_0}(x_0, r),
\]
and the topology of \((X, d_0)\) and \((X, d(g(t)))\) agrees for all \( t \in (0, T) \).

That is, assuming \( B_{g(t)}(x_0, \ell) \in M \) for all \( t \in (0, T) \) for some \( \ell > 0 \), we can always assume that the metric \( d_0 \) exists (locally) and that the convergence is in the above sense, (1.2) (which is stronger than Gromov-Hausdorff convergence) when examining the local behaviour of the metric near a point \( x_0 \in M \).

Examples of solutions satisfying the conditions (1.1) are as follows:

1. Expanding gradient Ricci solitons coming out of smooth cones \((M^n,d_X)\) where \((M^n,d_X)\) is the completion of \((\mathbb{R}^+ \times S^{n-1},dr^2 + r^2 \gamma)\), where \( \gamma \) is a Riemannian metric on the sphere, which is smooth and whose curvature operator has eigenvalues larger than one. ‘Coming out’ here means that \((M,d(g(t))) \to (M^n,d_X)\) in the pointed Gromov-Hausdorff sense as \( t \to 0 \). In the paper [14] of the last two named authors of the current paper, cones of this type which arise as smooth limits (away from the tip) of a blow-down of a non-compact manifold with non-negative, bounded curvature operator and asymptotically Euclidean volume growth were considered. They showed, in this case, that there is an expanding soliton coming out of the cone, with non-negative curvature operator, satisfying (1.1). The construction of the solution in the paper [14] guarantees, that the convergence as \( t \to 0 \) to the initial values is, in this case, in the \( C^\infty_{\text{loc}} \) sense away from the tip. Later, the first named author of the current paper, Deruelle, extended these results in [8] to show that there is always a soliton coming out of any smooth cone of the type considered at the beginning of this example. The construction of Deruelle also guaranteed that the convergence is in the \( C^\infty_{\text{loc}} \) sense away from the tip: this existence result is based on the Nash-Moser fixed point theorem. Problem 1.1 was partly motivated by the cost of using such a ”black box”. Indeed, the Nash-Moser fixed point theorem is not so sensitive to the nature of the non-linearities of the Ricci flow equation as long as the corresponding linearized operator satisfies the appropriate Fredholm properties. In particular, the use of the Nash-Moser fixed point theorem does not shed new light on the smoothing effect of the Ricci flow. Finally, we emphasize the fact that uniqueness of such solutions is unknown among the class of asymptotically conical gradient Ricci solitons with positive curvature operator. Indeed, the proof given in [8] only works if any such two expanding gradient Ricci solitons with the same tangent cone at infinity have a vanishing Bianchi gauge.

2. Let \((M^n_i(0),g_i(0),x_i))_{i \in \mathbb{N}}\) be a sequence of smooth Riemannian manifolds with bounded curvature, such that \( \mathcal{R}(g_i(0)) + c \cdot Id(g_i(0)) \in \mathcal{C}_\mathcal{K} \) and \( \text{Vol}(B_{g_i(0)}(x)) \geq v_0 \) for all \( x \in M_i \) for all \( i \in \mathbb{N} \), for some \( c, v_0 > 0 \) where \( \mathcal{R} \) is the curvature operator, \( Id \) is the identity operator of the sphere, \( Id(g)(ijkl) = g_{ij}g_{kl} - g_{ik}g_{jl}, \) and \( \mathcal{C}_\mathcal{K} \) is the cone of i) non-negative curvature operators, respectively ii) 2-non-negative curvature operators, respectively iii) weakly PIC1 curvature operators, respectively iv) weakly PIC2 curvature operators. Then [[17] for (i), (ii) in case \( n = 3 \), [2] for (i) -(iv) for general \( n \in \mathbb{N} \)] there are solutions \((M_i,g_i(t),x_i))_{t \in (0,T(n,v_0))}\) such that \( \mathcal{R}(g(t)) + C \cdot Id \in \mathcal{C}_\mathcal{K} \) (for some new \( C > 0 \)) which is stronger than, i.e. it implies, the condition \( \text{Ric}(g(t)) \geq -c(n)C \). After scaling each solution once by a large constant \( K = K(n,v_0)\), \( g_i(t) = Kg_i(t/K)\), we obtain a sequence of solutions satisfying (1.1). After taking a sub-sequence, we obtain a pointed Cheeger-Hamilton limit solution \((M^n, g(t), x_\infty))_{t \in (0,1)}\) which satisfies (1.1).

3. More generally, if we take a sequence of smooth complete solutions \((M_i,g_i(t),x_i))_{t \in (0,1)}\) satisfying (1.1), and \( \text{Vol}(B_{g_i(t)}(x_i)) \geq v_0 \) for all \( i \in \mathbb{N} \) for all \( t \in (0,1) \), then af-
ter taking a subsequence, we obtain a pointed Cheeger-Hamilton limit solution 
\((M^n, g(t), x_\infty)_{t \in (0, 1)}\) which satisfies (1.1).

Problem 1.1 can be considered locally in the context of the above examples as follows.

Assume that \((M, g(t))_{t \in [0, \infty)}\) is a smooth expanding soliton, with non-negative Ricci curvature (or curvature operator) coming out of a smooth (continuous) cone \((M^n, d_X) = (\mathbb{R}^+ \times S^{n-1}, dr^2 \oplus r^2 \gamma)\), where \(\gamma\) is a smooth (continuous) Riemannian metric. Does the solution \((M, g(t))_{t \in [0, 1]}\) come out smoothly (continuously)? That is, is the solution \((M\backslash \{p\}, g(t))_{t \in [0, 1]}\) smooth (continuous), where \(p\) is the tip of the cone, and \(g_0\) is the cone metric on \(M\backslash \{p\}\) at time zero?

If we replace the assumption that \(\gamma\) is smooth (continuous) on \(S^{n-1}\) to \(\gamma\) is smooth (continuous) on an open set \(V \subseteq S^n\), we ask the question: is \(((V \times \mathbb{R}^+), g(t)|_{V \times \mathbb{R}^+})_{t \in [0, 1]}\) smooth (continuous)?

In the setting of Example 3) let \((M, d_0, x_\infty)\) be the limit as \(t \to 0\) of \((M, d(g(t)), x_\infty)\). We note that \((M, d_0, x_\infty)\) is also isometric to the Gromov-Hausdorff limit of \((M_i, d(g_i(0)), x_i)\) as \(i \to \infty\), in view of the estimates (1.2). Let \(V \subseteq M\) be an open set such that \((V, d_0)\) is isometric to a smooth (continuous) Riemannian manifold. Can \((V, g(t))_{t \in [0, 1]}\) be extended smoothly (continuously) to \(t = 0\), that is, does there exist a smooth (continuous) \(g_0\) on \(V\) such that \((V, g(t))_{t \in [0, 1]}\) is smooth (continuous)?

We will see in Theorem 1.3 in the next section, that the answer to each of these questions in the smooth setting is yes, if we measure the smoothness of the initial metric space appropriately. The answer to each of these questions is also yes in the continuous setting, see Theorem 1.4 in the next section, if we measure the continuity of the initial metric appropriately and the convergence in the continuous setting is measured up to diffeomorphisms.

The smoothness (respectively continuity) of a metric space in this paper will be measured in the following way. In the following \(0 < \epsilon_0(n) < 1\) is a small fixed positive constant depending only on \(n\). We denote with \(\mathbb{B}_r(x) \subset \mathbb{R}^n\) the Euclidean ball with radius \(r\), centred at \(x\).

**Definition 1.2.** Let \((X, d_0)\) be a metric space and let \(V\) be a set in \(X\). We say \((V, d_0)\) is smoothly (respectively continuously) \(n\)-Riemannian if for all \(x_0 \in V\) there exist \(0 < \tilde{r}, r\) with \(\tilde{r} < \frac{1}{10}r\) and points \(a_1, \ldots, a_n \in B_{d_0}(x_0, r)\) such that the map

\[
F_0(x) := (d_0(x, a_1), \ldots, d_0(x, a_n)), \quad x \in B_{d_0}(x_0, r),
\]

is a \((1 + \epsilon_0)\) Bi-Lipschitz homeomorphism on \(B_{d_0}(x_0, 5\tilde{r})\) and the push-forward of \(d_0\) by \(F_0\) denoted by \(d_0\) and defined by \(d_0(\tilde{x}, \tilde{y}) := d_0((F_0)^{-1}(\tilde{x}), (F_0)^{-1}(\tilde{y}))\) on \(\mathbb{B}_{4\tilde{r}}(F_0(x_0))\) \(\subseteq F_0(B_{d_0}(x_0, 5\tilde{r}))\) is induced by a smooth Riemannian metric, that is, more specifically, there exists a smooth (respectively continuous) Riemannian metric \(\tilde{g}_0\) defined on \(\mathbb{B}_{4\tilde{r}}(F_0(x_0))\), such that \(\tilde{d}_0\) satisfies \(\tilde{d}_0 = d(\tilde{g}_0)\), when restricted to \(\mathbb{B}_{\tilde{r}}(F_0(x_0))\).

For a Riemannian metric \(g\) defined on an open set \(\Omega\) in \(\mathbb{R}^n\), we have used the notation \(d(g)\) to refer to the metric on \(\Omega\) defined by \(d(g)(x, y) = \inf\{L_g(\gamma) \mid \gamma\) a smooth path in \(\Omega\) from \(x\) to \(y\}\), for all \(x, y \in \Omega\), where \(L_g(\gamma)\) is the length of the curve \(\gamma\) with respect to \(g\).

### 1.2 Main results

The first theorem stated here gives a positive answer to the questions posed in the smooth setting in the previous subsection.
Theorem 1.3. Let \((M, g(t))_{t\in [0,T]}\) be a smooth solution to Ricci flow satisfying (1.1) and assume \(B_{g(t)}(x_0,1) \subset M\) for all \(0 < t \leq T\) and let \((B_{g_0}(x_0,r), d_0)\) be the \(C^0\) limit of \((B_{g(t)}(x_0,r), d(g(t)))\) as \(t \downarrow 0\) (which always exists in view of [19, Lemma 3.1]). Assume further that \((B_{g_0}(x_0,r), d_0)\) is smoothly n-Riemannian in the sense of Definition 1.2 near \(x_0\). Then there exists a smooth Riemannian metric \(g_0\) on \(B_{d_0}(x_0,s)\) for some \(s > 0\) such that we can extend the smooth solution \((B_{d_0}(x_0,s), g(t))_{t\in [0,T]}\) to a smooth solution \((B_{d_0}(x_0,s), g(t))_{t\in [0,T]}\) by defining \(g(0) = g_0\).

As noted in the paper [20] by Topping, this result was known to be correct in dimension two for compact manifolds without boundary, by results of Richard in [13].

The next theorem is concerned with the questions asked in the previous subsection in the continuous setting.

Theorem 1.4. Let \((M, g(t))_{t\in [0,T]}\) be a smooth solution to Ricci flow satisfying (1.1) and assume \(B_{g(t)}(x_0,1) \subset M\) for all \(0 < t \leq T\) and let \((B_{g_0}(x_0,r), d_0)\) be the \(C^0\) limit of \((B_{g(t)}(x_0,r), d(g(t)))\) as \(t \downarrow 0\) (which always exists in view of [19, Lemma 3.1]). Assume further that \((B_{d_0}(x_0,r), d_0)\) is continuously n-Riemannian in the sense of Definition 1.2. Then for any sequence \(t_i\) with \(0 < t_i \to 0\) as \(i \to \infty\), there exists a radius \(v > 0\) and a continuous Riemannian metric \(g_0\), defined on \(B_v(p)\), \(p \in \mathbb{R}^n\), and family of smooth diffeomorphisms \(Z_i : B_{d_0}(x_0,2v) \to \mathbb{R}^n\) such that \((Z_i)_*(g(t_i))\) converges in the \(C^0\) sense to \(g_0\) as \(t_i \downarrow 0\) on \(B_v(p)\).

1.3 Metric space convergence and the condition (1.1)

Assume we have a smooth complete solution to Ricci flow \((M^n, g(t))_{t\in (0,1)}\), satisfying \(|\text{Rm}(\cdot, t)| \leq c_0/t\) for all \(t \in (0, 1)\) for some \(c_0 \geq 1\), but that we don’t assume the condition \(\text{Ric}(g(t)) \geq -k\) for all \(t \in (0, 1)\) for some \(k \in \mathbb{R}^+\). Then there is no guarantee that a limit metric \(d_0 = \lim_{t \to 0} d(g(t))\) exists.

If we have a sequence of smooth complete solutions \((M^n, g_i(t), x_i)_{t\in [0,1]}\), satisfying \(|\text{Rm}(g_i(t))| \leq c_0/t\) and \(\text{Vol}(B_{g_i(t)}(x_i)) \geq v_0 > 0\) for all \(t \in (0,1)\) for all \(x \in M_i\) for some \(c_0, v_0 > 0\), for all \(i \in \mathbb{N}\), we obtain a limiting solution in the smooth Cheeger-Hamilton sense, \((M^n, g(t), p)_{t\in (0,1)}\), which satisfies \(|\text{Rm}(\cdot, t)| \leq c_0/t\) for all \(t \in (0, 1)\), but again, there is no guarantee that a limit metric \(d_0 = \lim_{t \to 0} d(g(t))\) exists. Furthermore, if a pointed Gromov-Hausdorff limit \((M, d_0, p), as \(t \to 0\), of \((M, d(g(t)), p)\) exists and if a Gromov-Hausdorff limit \((X, d_X, y)\) in \(i \in \mathbb{N}\) of \((M^n, d(g_i(0)), x_i)_{i\in \mathbb{N}}\) exists, then there is no guarantee that \((X, d_X, y)\) is isometric to \((M, d_0, p)\) or that \((M, d_0)\) has the same topology as \(d(g(t))\) for \(t > 0\).

An example which considers the metric behaviour under limits of solutions with no uniform bound from below on the Ricci curvature but with \(|\text{Rm}(\cdot, t)| \leq c_0/t\) is given in a recent work of Peter Topping [20]. There, he constructs examples of smooth solutions \((\mathbb{T}^2, g_i(t))_{t\in (0,1)}\) to Ricci flow, satisfying \((\mathbb{T}^2, d(g_i(0))) \to (\mathbb{T}^2, d(\delta))\), as \(i \to \infty\), where \(\delta\) is the standard flat metric on \(\mathbb{T}^2\), and \(|\text{Rm}(g_i(t))| \leq c/t\) for all \(t \in (0,1), i \in \mathbb{N}\), for some \(c > 0\), but so that the limiting solution \((\mathbb{T}^2, g(t))_{t\in (0,1)}\) satisfies \((\mathbb{T}^2, d(g(t)))_{t\in (0,1)} = (\mathbb{T}^2, d(\delta))\), where \((\mathbb{T}^2, d)\) is isometric to \((\mathbb{T}^2, d(2\delta))\). The initial smooth data \(g_i(0)\) do not satisfy \(\text{Ric}(g_i(0)) \geq -k\) for some fixed \(k > 0\) for all \(i \in \mathbb{N}\), and so the arguments used to show that the Gromov-Hausdorff limit of the initial data is the same as the limit as \(t \to 0\) of the limiting solution, are not valid.

1.4 Related results and works

We begin by considering the heat equation. If we have a smooth solution \(u : \mathbb{R}^n \times (0, 1) \to \mathbb{R}\), with \(\frac{\partial u}{\partial t} = \Delta u\), and we know that \(u(\cdot, t) \to u_0(\cdot)\) locally uniformly on \(\mathbb{B}_1(0)\), where \(u_0\)
is smooth on $B_1(0)$, then the solution can be locally extended to a smooth local solution $v : B_{1/2}(0) \times [0, 1) \to \mathbb{R}$, by defining $v(\cdot, 0) = u_0(\cdot)$ on $B_{1/2}(0)$, as we now explain. We know that the function $u : \mathbb{R}^n \to \mathbb{R}$ is continuous. Hence using the linear parabolic theory, see for example Theorem 2.1 of this paper with $M = \mathbb{R}^n$ and $g(t) = \delta$ for all $t \in [0, 1]$, we can find a map $z : \mathbb{R} \to \mathbb{R}$ such that $z \in C^\infty(B_{1/4}(0) \times [0, 1]) \cap C^\infty(B_{3/4}(0) \times [0, 1])$ such that $z = u$ on $P = \partial B_{1/4}(0) \times [0, 1] \cup B_{3/4}(0) \times \{0\}$ and such that $z$ solves $\frac{\partial}{\partial t} z = \Delta z$ on $B_{3/4}(0) \times [0, 1]$. In particular $\frac{\partial}{\partial t}(z - u) = \Delta(z - u)$ and $z - u = 0$ on $P$ and $(z - u)$ is continuous on $B_{1/4}(0) \times [0, 1]$. From the maximum principle, we see that $z - u = 0$ and hence $u = z$ is smooth on $\mathbb{R}^n \times [0, 1]$ as required. Here the linear theory simplifies the situation. We have also assumed that $u(\cdot, t) \to u_0$ locally uniformly. In the Ricci flow setting, assuming (1.1), we saw above that the initial values must be taken on uniformly, albeit for the distance, not necessarily the Riemannian metric.

A non-linear setting closer to the one we consider is as follows. In the paper [1], Appleton considers (among other things) the $\delta$-Ricci-DeTurck flow of metrics $g_0$ on $\mathbb{R}^n$ which are close to the standard metric $\delta$, in the sense that $|g_0 - \delta| \leq \epsilon(n)$. In the work of Koch and Lamm, see [11, Theorem 4.3], it was shown that there always exists a weak solution $(\mathbb{R}^n, g(t))_{t \in [0, \infty)}$ in this case. Weak solutions defined on $[0, T)$ ($T = \infty$ is allowed) are smooth for all $t > 0$ and $h(t, x) := g(x, t) - \delta(x)$ has bounded $X_T$ norm, where

$$\|h\|_{X_T} := \sup_{0 < t < T} \|h(t)\|_{\delta} + \sup_{x \in \mathbb{R}^n} \sup_{0 < R^2 < T} \left( R^{-\frac{\delta}{2}} \|\nabla h\|_{L^2(B_R(x) \times (0, R^2))} + R^{\frac{\delta}{2}} \|\nabla h\|_{L^{\infty}(B_R(x) \times (0, R^2))} \right)$$

If the initial values $g_0$ are $C^0$ then the initial values are realised in the $C^0$ sense, that is $|g(t) - g_0|_{\delta} \to 0$ as $t \to 0$. Appleton showed, see [1, Theorem 4.5], that any weak solution $h(t) := g(t) - \delta$ which has $g_0 \in C^{2,\alpha}_loc(\mathbb{R}^n)$ and $|h_0|_{\delta} \leq \epsilon(n)$, must have $h(t) \in H^{2+\alpha, 1+\frac{\alpha}{2}}_loc(\mathbb{R}^n \times [0, \infty))$. In particular the zeroth, first and second spatial derivatives of $h(t)$ locally approach those of $h_0$ as $t \to 0$. That is, for classical initial data $h_0 \in C^{2,\alpha}_loc(\mathbb{R}^n)$, any weak solution $h(t)$ restricted to $\Omega$ approaches $h(0)$ in the $C^{2,\alpha}(\Omega)$ norm on $\Omega$ for any precompact, open set $\Omega$.

Raphaël Hochard has in his PhD-thesis [10] proven some results similar to, or the same as, some of those appearing in Sections 2 and 3 of this paper. We received a copy of Hochard’s thesis, after a pre-print version, including the relevant sections, of this paper was finished but not yet published. We have included references to the results of Hochard at the appropriate points throughout this paper. His approach, in studying the relevant objects, differs slightly, as we explain at the relevant points.

### 1.5 Outline of Paper

We outline the proof method of the main theorems, Theorem 1.3 and 1.4. The method is somewhat similar to the one we used above to show smoothness of solutions to the heat equation coming out of smooth initial data, which are a priori smooth for times larger than zero.

We show, in the setting we are considering, that there is a smooth, on the interior, solution to the Ricci-harmonic map heat flow, $Z : B_R(x_0, 1) \times [0, T) \to \mathbb{R}^n$, with initial and boundary values given by the map $F_0$, which represents distance coordinates at time zero. The a priori estimates we prove in Sections 2 and 3 help us to construct the solution, and from the Regularity Theorem 3.7, we see that the solution is $1+\alpha(n)$ Bi-Lipschitz at any time $t \in (0, S(n)) \cap [0, T)$. The explicit construction of $Z$ is carried out in Theorem 3.10.
Hence, we may consider the push forward \( \tilde{g}(t) := (Z_t)_*(g(t)) \), which is then, by construction a solution to the \( \delta \)-Ricci-DeTurck flow with background metric \( \delta \), and \( \tilde{g}(t) \) is \( \alpha(n) \) close to \( \delta \) in the \( C^0 \) sense. We then restrict to the case that the push forward of \( d_0 \) with respect to \( F_0 \) is generated locally by a continuous Riemannian metric \( \tilde{g}_0 \). A further application of the regularity theorem, Theorem 3.7 of Section 3, then shows that \( \tilde{g}(t) \) converges to the continuous metric \( \tilde{g}_0 \) (locally). This is explained in detail in Theorem 4.3 in Section 4. If we assume further that \( \tilde{g}_0 \) is smooth, and sufficiently close to \( \delta \), then we consider the Dirichlet Solution \( \ell \) to the \( \delta \)-Ricci-DeTurck flow on an Euclidean ball \( \mathbb{B}_r(0) \times [0, T] \), with parabolic boundary data given by \( \tilde{g} \). The existence of this solution is shown in Section 5, where Dirichlet solutions to the \( \delta \)-Ricci-DeTurck flow with given parabolic boundary values \( C^0 \) close to \( \delta \) are constructed. The \( L^2 \)-lemma, Lemma 6.1 of Section 6, tells us that the (weighted) spatial \( L^2 \) norm of the difference \( g_1 - g_2 \) of two solutions \( g_1, g_2 \) to the \( \delta \)-Ricci-DeTurck flow defined on an Euclidean ball is non-increasing, if \( g_1 \) and \( g_2 \) have the same values on the boundary of that ball, and are sufficiently close to \( \delta \) for all time \( t \in [0, T] \). An application of the \( L^2 \)-lemma then proves that \( \ell = \tilde{g} \). The construction of \( \ell \), carried out in Section 5, then guarantees that \( \ell \) is smooth on \( \mathbb{B}_r(0) \times [0, T] \). Hence \( \tilde{g} \) is smooth on \( \mathbb{B}_r(0) \times [0, T] \). Section 7 completes the proof of Theorem 1.3: The smoothness of \( \tilde{g} \) on \( \mathbb{B}_r(0) \times [0, T] \) implies that one can extend \( g \) smoothly (locally) to \( t = 0 \). In Section 7 we also discuss some of the consequences of Theorem 1.3 in the context of expanding gradient Ricci solitons with non-negative Ricci curvature.

1.6 An open problem

The lower bound on the Ricci curvature in (1.1) is used crucially to obtain the bound from above for \( d_t \) in (1.2). It is also used in Section 4, when showing that \( \tilde{g}(t) \) converges to \( \tilde{g}_0 \) in the \( C^0 \) norm.

Problem 1.5. Can the bound from below on the Ricci curvature in Section 3 and/or other sections be replaced by a weaker condition?

We comment on this at various points in the paper.

1.7 Notation

(1) For a connected Riemannian manifold \( (M, g), x, y \in M, r \in \mathbb{R}^+ \): 

(1a) \( (M, d(g)) \) refers to the associated metric space, \( d(g)(x, y) = \inf_{\gamma \in G_{x,y}}(L_g(\gamma)) \) where \( G_{x,y} \) refers to the set of smooth regular curves \( \gamma : [0, 1] \to M \), with \( \gamma(0) = x, \gamma(1) = y \), and \( L_g(\gamma) \) is the Length of \( \gamma \) with respect to \( g \).

(1b) \( B_g(x, r) := B_{d(g)}(x, r) := \{ y \in M \mid d(g)(y, x) < r \} \).

(1c) If \( g \) is locally in \( C^2 \): \( \text{Ric}(g) \) is the Ricci Tensor, \( \text{Rm}(g) \) is the Riemannian curvature tensor, and \( \text{R}(g) \) is the scalar curvature.

(2) For a one parameter family \( (g(t))_{t \in (0, T)} \) of Riemannian metrics on a manifold \( M \), the distance induced by the metric \( g(t) \) is denoted either by \( d(g(t)) \) or \( d_t \) for \( t \in (0, T) \).

(3) For a metric space \( (X, d), x \in M, r \in \mathbb{R}^+ \), \( B_d(x, r) := \{ y \in M \mid d(y, x) < r \} \).

(4) \( B_R(m) \) always refers to an Euclidean ball with radius \( R > 0 \) and centre \( m \in \mathbb{R}^n \).
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2 Ricci-harmonic map heat flow for functions with bounded gradient

We prove some local results about the Ricci-harmonic map heat flow. R. Hochard, in independent work, proved some results in his Ph.D.-thesis which are similar to or the same as those of this chapter, see [10, Section II.3.2]. Hochard uses blow up arguments (i.e. contradiction arguments combined with scaling arguments) to prove some of his estimates, whereas we use a more direct argument involving the maximum principle applied to various evolving quantities. The first theorem we present is a local version of a theorem of Hamilton, [9, p. 15] in the setting that the curvature of the Ricci flow is bounded by a constant times the inverse of time, and the Ricci curvature is bounded from below.

**Theorem 2.1.** Let \((M^n, g(t))_{t \in [0,T)}\) be a smooth solution to Ricci flow with

\[
\text{Ric}(g(t)) \geq -1, \quad |\text{Rm}(\cdot, t)| \leq \frac{c_0}{t},
\]

for all \(t \in [0,T]\) which satisfies \(B_{g(0)}(x_0, 2) \subset M\) and \(F_0 : B_{g(0)}(x_0, 1) \rightarrow \mathbb{R}^n\) be a smooth map such that

- \(|\nabla F_0|^2_{g_0} \leq c_1\)
- \(F_0(B_{g(0)}(x_0, 1)) \subset B_r(0)\) for some \(r \leq 2\).

Then there is a unique solution \(F \in C^\infty(B_{g(0)}(x_0, 1) \times [0,T]; \mathbb{R}^n) \cap C^0(B_{g(0)}(x_0, 1) \times [0,T]; \mathbb{R}^n)\), to the Dirichlet problem, for the Ricci-harmonic map heat flow

\[
\frac{\partial}{\partial t} F = \Delta_{g(t)} F, \quad F(\cdot, 0) = F_0,
\]

and,

\[
F(\cdot, t)|_{\partial B_{g_0}(x_0, 1)} = F_0|_{\partial B_{g_0}(x_0, 1)} \quad \text{for } t \in [0,T],
\]

\[
F_t(B_{g(t)}(x_0, 1/2)) \subset B_r(0), \quad \text{for all } t \leq T,
\]

\[
B_{g(t)}(x_0, 3/4) \subset B_{g_0}(x_0, 1), \quad \text{for all } t \leq \min(T, S(n, c_0)),
\]

\[
|\nabla F(\cdot, t)|^2_{g(t)} \leq c_0 c_1 n, \quad \text{on } B_{g(t)}(x_0, 1/2), \text{and for all } t \leq \min(T, S(n, c_0)).
\]

**Remark 2.2.** We are implicitly assuming in the statement of Theorem 2.1 that the geodesic sphere \(\partial B_{g(0)}(x_0, 1)\) is a smooth \((n-1)\)-dimensional manifold.

**Proof.** Since \(B_{g(0)}(x_0, 1) \subset M\) is a compact set, and the solution \((M, g(t))_{t \in [0,T]}\) is smooth, we see that we can find a finite collection of coordinate charts which cover \(B_{g(0)}(x_0, 1)\) such that the metric is uniformly equivalent to the standard metric \(\delta\) (in
boundary data is zero), and so we find a solution $S$ where $Z$.

The maximum principle applied to $|\nabla S|$ of why this is the case. We modify the boundary data slightly, in order to satisfy the compatibility conditions of $k$-th order, i.e. we set

$$\tilde{F}_\alpha(x,t) := (1 - \eta_\alpha(t))F_0(x) + \eta_\alpha(t) \left( F_0(x) + t\varphi_1(x) + \frac{t^2}{2!}\varphi_2(x) + \frac{t^3}{3!}\varphi_3(x) + \ldots + \frac{t^k}{k!}\varphi_k(x) \right),$$

where $\varphi_i := \partial^i_t|_{t=0}F$ is the right-hand side we obtain if we differentiate the equation (2.2) satisfied by $F$ in time $i$-times at time zero ($\varphi$ only depends on $F_0$ and $g$). Here $\eta_\alpha : [0,T] \to [0,1]$ is a smooth cut off function in time which satisfies $\eta_\alpha = 1$ on $[0,\alpha]$, $\eta_\alpha = 0$ on $[2\alpha, T]$. Then we consider the problem of finding solutions to the following Dirichlet problem:

$$\begin{align*}
\frac{\partial}{\partial t}H_\alpha &= \Delta_{g(t)}H_\alpha, \\
H_\alpha(\cdot, 0) &= F_0, \\
H_\alpha|_P &= \tilde{F}_\alpha,
\end{align*}$$

where $P$ is the parabolic boundary $P := \partial B_{g_0}(x_0, 1) \times [0, T] \cup B_{g_0}(x_0, 1) \times \{0\}$.

This problem satisfies the compatibility conditions of $k$-th order at $t = 0$. Furthermore, notice that $\tilde{F}_\alpha(\cdot, t) = F_0(\cdot)$ for $t \geq 2\alpha$, and $F_\alpha(\cdot, 0) = F_0(\cdot)$.

Next, we consider the related problem of finding a solution $S_\alpha \in C^2(B_{g(0)}(x_0, 1) \times [0, T], \mathbb{R}^n)$ to

$$\begin{align*}
\frac{\partial}{\partial t}S_\alpha &= \Delta_{g(t)}S_\alpha + Z_\alpha, \\
S_\alpha(\cdot, 0) &= 0, \\
S_\alpha|_P &= 0,
\end{align*}$$

where $Z_\alpha := -\frac{\partial}{\partial t}\tilde{F}_\alpha + \Delta_{g(t)}\tilde{F}_\alpha$.

Now $S_\alpha$ satisfies the compatibility conditions at least of order 1 at time $t = 0$ (which is equivalent to $Z_\alpha(\cdot, 0) = 0$ on the boundary of $B_{g(0)}(x_0, 1)$ in this case, since the parabolic boundary data is zero), and so we find a solution $S_\alpha \in H^{2+\beta, 1+\beta/2}(B_{g(0)}(x_0, 1) \times [0, T]) \cap C^\infty(B_{g(0)}(x_0, 1) \times (0, T))$ for $\beta \in (0, 1)$. Furthermore $|\nabla S_\alpha|(x, 0) = 0$, if $x \in B_{g_0}(x_0, 1)$, $|\frac{\partial}{\partial t}S_\alpha(t)| \leq c(g_0, F_0)$ for $t \leq 2\alpha$ (and hence for all $t \in [0, T]$), which means $|Z_\alpha| \leq c(g|_{[0,T]}, F_0, T)$ for all $t \leq T$, and so [12, Lemma 6.1, Chapter VII] is applicable, which then tells us that $|\nabla S_\alpha||_P \leq c(g|_{[0,T]}, F_0, T)$.

The maximum principle applied to $|\nabla S_\alpha|$ then shows us that $|\nabla S_\alpha| \leq c(g|_{[0,T]}, F_0, T)$. Hence, $|S_\alpha||_{P_\varepsilon} \leq c(\varepsilon)$ where $P_\varepsilon$ consists of the points which are in an $\varepsilon$-tubular neighbourhood of $P$ measured with respect to the parabolic distance and where here $c(\varepsilon) \to 0$ as $\varepsilon \to 0$. Going back to $H_\alpha$, that is setting $H_\alpha := S_\alpha + \tilde{F}_\alpha$, we see that $|H_\alpha - \tilde{F}_\alpha| \leq c(\varepsilon)$ on $P_\varepsilon$, where $c(\varepsilon) \to 0$ as $\varepsilon \to 0$, independent of $\alpha$. 

9
Interior estimates applied to the evolution equation for $S_a$ show that $|\nabla^i H_\alpha| \leq c(i, F_0, g_0, T, \sigma)$ for all $i \leq k - 2$ on $B_{g_0}(x_0, 1 - \sigma) \times [0, T]$; a maximum principle with cut off and the difference quotient method suffices to show this. By Arzelà-Ascoli’s Theorem, we now get the required solution by taking a limit as $\alpha$ goes to 0.

This finishes the brief explanation of the existence result. We now return to the proof of Theorem 2.1.

The statement (2.3) follows from the Maximum Principle and the evolution equation for $|F|^2 = \sum_{i=1}^n (F_i^2)$:

\[
(\frac{\partial}{\partial t} - \Delta_{g(t)}) |F|^2 = -2|\nabla F|_{g(t)}^2.
\]

Statement (2.4) follows from the distance estimates, (1.2), which hold on $B_{g(0)}(x_0, 1)$ for any solution to Ricci flow satisfying (2.1) and $B_{g(0)}(x_0, 2) \subseteq M$: see [18, Lemma 3.1].

Regarding (2.5), we first recall the following fundamental evolution equation satisfied by $|\nabla F|_{g(t)}^2$:

\[
(\frac{\partial}{\partial t} - \Delta_{g(t)}) |\nabla F|_{g(t)}^2 = -2|\nabla^2 F|_{g(t)}^2.
\]

Notice that the term $\text{Ric}(g(t))(\nabla F, \nabla F)$ showing up in the Bochner formula applied to $\nabla F$ cancels with the pointwise evolution equation of the squared norm of $\nabla F$ along the Ricci flow.

In case the underlying manifold is closed, the use of the maximum principle would give us the expected result.

In order to localize this argument, we construct a Perelman type cut-off function $\eta : M \to [0, 1]$ with $\eta(\cdot, t) = 0$ on $M \setminus B_{g(t)}(x_0, \frac{2}{3})$ and $\eta(\cdot, t) = e^{-k(n, c_0)} t$ on $B_{g(t)}(x_0, \frac{1}{2})$ such that $\frac{\partial}{\partial t} \eta(\cdot, t) \leq \Delta_{g(t)} \eta(\cdot, t)$ everywhere, and $|\nabla \eta|_{g(t)} \leq c_3(n, t) \eta$ everywhere, as long as $t \leq \min(S(n, c_0), T)$: see, for example, [18, Section 7] for details.

We consider the function $Z := \eta |\nabla F|_{g(t)}^2 + c_2 |F|^2$, with $c_2 = 10c(n)c_3(n)$. The quantity $Z$ is less than $c_1 + 4c_2$ everywhere at time zero. We consider a first time and point where $Z$ becomes equal to $c_1 + 5c_2$ on $B_{g(t)}(x_0, 1)$. This must happen in $B_{g(0)}(x_0, 1)$, since $\eta = 0$ on a small open set $U$ containing $\partial B_{g(0)}(x_0, 1)$ and $c_2 |F|^2 < 4c_2$ by (2.3). At such a point and time, we have by (2.6) and (2.7) together with the properties of $\eta$,

\[
0 \leq \left( \frac{\partial}{\partial t} - \Delta_{g(t)} \right) Z(x, t)
\leq -2c_2 |\nabla F|_{g(t)}^2 - 2\eta |\nabla^2 F|_{g(t)}^2 - 2g(t)(\nabla \eta, \nabla |\nabla F|_{g(t)}^2)
\leq -2c_2 |\nabla F|_{g(t)}^2 - 2\eta |\nabla^2 F|_{g(t)}^2 + 2c(n)|\nabla \eta|_{g(t)} |\nabla F|_{g(t)} |\nabla^2 F|_{g(t)}
\leq -2c_2 |\nabla F|_{g(t)}^2 - 2\eta |\nabla^2 F|_{g(t)}^2 + 4c(n) \frac{|\nabla \eta|_{g(t)}^2}{\eta} |\nabla F|_{g(t)}^2 + \eta |\nabla^2 F|_{g(t)}^2
< 0,
\]

by the choice of $c_2$, which is a contradiction. Hence $Z(x, t) \leq c_1 + 5c_2$ for all $t \leq S(n, c_0)$, which implies

\[
e^{-kt} |\nabla F|_{g(t)}^2(\cdot, t) \leq e^{-kt} |\nabla F|_{g(t)}^2(\cdot, t) + c_2 |F|^2(\cdot, t) \leq (c_1 + 5c_2),
\]

on $B_{g(t)}(x_0, \frac{1}{2})$ for all $t \leq \min(S(n, c_0), T)$, which implies

\[
|\nabla F|_{g(t)}^2(\cdot, t) \leq e^{kS} (c_1 + 3c_2),
\]

on $B_{g(t)}(x_0, \frac{1}{2})$ for all $t \leq \min(S(n, c_0), T)$ as required.
We want also to prove an estimate for the second covariant derivatives of a solution to the Ricci-harmonic map flow. In fact, once we have a solution to the Ricci-harmonic map heat flow with bounded gradient, the solution smooths out the second derivatives in a controlled way, as the following theorem shows.

**Theorem 2.3.** For all $c_1 > 0$ and $n \in \mathbb{N}$, there exists a positive number $\varepsilon_0(c_1,n)$ such that the following is true. Let $(M^n, g(t))_{t \in [0,T)}$ be a smooth solution to Ricci flow such that

$$\text{Ric}(g(t)) \geq -1, \quad |\text{Rm}(\cdot,t)| \leq \frac{\varepsilon_0^2}{t}, \quad t \in (0,T),$$

where $\varepsilon_0 \leq \varepsilon_0$. Assume furthermore that $B_{g(0)}(x_0,1) \subset M$, and $F : B_{g(0)}(x_0,1) \times [0,T] \to \mathbb{R}^n$ is a smooth solution to the Ricci-harmonic map heat flow

$$\frac{\partial}{\partial t} F(x,t) = \Delta_{g(t)} F(x,t),$$

for all $(x,t) \in B_{g(0)}(x_0,1) \times [0,T]$, such that $|\nabla g(t)(\cdot,t)|^2_{g(t)} \leq c_1$ on $B_{g(t)}(x_0,1)$ for all $t \in [0,T]$. Then

$$t|\nabla g(t)|^2_{g(t)} \leq c(n,c_1),$$

on $B_{g(t)}(x_0,1/4)$ for all $t \leq \min\{S(n), T\}$, where $S(n) > 0$ is a constant just depending on $n$.

**Remark 2.4.** The condition $|\text{Rm}(\cdot,t)| \leq \frac{\varepsilon_0^2}{t}$ where $\varepsilon_0 \leq \varepsilon_0(c_1,n)$ is sufficiently small is not necessary: $|\text{Rm}(\cdot,t)| \leq k/t$ with $k$ arbitrary is sufficient in our argument, as can be seen by examining the proof, but then the times $t$ for which the conclusions are valid, are required to satisfy $t \leq S(c_1,k,n)$, where $S(c_1,k,n) > 0$ is sufficiently small. We only consider small $k$, as this is sufficient for the setting of the following chapters. A version of this theorem, with $c_1 = c(n)$ and the condition $|\text{Rm}(\cdot,t)| \leq k/t$, $k$ arbitrary, was independently proved by Raphaël Hochard in his Ph.D.-thesis using a contradiction argument: see [10, Lemma II.3.9].

**Proof.** In the following, we denote constants $C(\varepsilon_0,c_1,n)$ simply by $\varepsilon_0$ if $C(\varepsilon_0,c_1,n)$ goes to 0 as $\varepsilon_0$ tends to 0 and $c_1$ and $n$ remain fixed. For example $\epsilon_0^2 n^2 \varepsilon_0$ and $b(c_1,n) \sqrt{\varepsilon_0}$ are denoted by $\varepsilon_0$ if $b(c_1,n)$ is a constant depending on $c_1$ and $n$. Let

$$Z := t(a_0 + |\nabla F|_{g(t)}^2)|\nabla^2 F|_{g(t)}^2.$$

The evolution equation for $Z$ can be calculated as follows. Locally, $\nabla^2 F = \partial_i \partial_j F - \Gamma(g)_{ij}^k \partial_k F$, where $\partial_i$ stands for a partial derivative. Shi’s estimates and the distance estimates (1.2) guarantee that $|\nabla \text{Rm}(g(t))| \leq \varepsilon_0 t^{-3/2}$ for $t \leq S(n)$ on $B_{g(t)}(x_0,3/4)$: see for example Lemma 3.1 (after scaling once by 400). For the sake of clarity in the computation to follow, we use the notation $\nabla$ to denote $\nabla_{g(t)}$ at a time $t$, $\text{Rm}$ to denote $\text{Rm}(g(t))$ at a time $t$, et cetera, although the objects in question do indeed depend on the evolving metric $g$. On $B_{g(t)}(x_0,3/4)$, we see for $t \leq S(n)$, that

$$\frac{\partial}{\partial t} |\nabla^2 F|_{g(t)}^2 = \nabla^2 \left( \frac{\partial}{\partial t} F \right)^2_{ij} + (\nabla \text{Ric} \ast \nabla F)^k_{ij}$$

$$= \nabla^2 (\Delta F)_{ij}^k + (\nabla \text{Ric} \ast \nabla F)_{ij}^k$$

$$= g^{kl} \nabla_i \nabla_j \nabla_k \nabla_l F^k + (\nabla \text{Ric} \ast \nabla F)_{ij}^k$$

$$= g^{kl} \nabla_i (\nabla_s \nabla_j \nabla_l F^k + (\text{Rm} \ast \nabla F)_{ij}^k) + (\nabla \text{Ric} \ast \nabla F)_{ij}^k$$

$$= g^{kl} \nabla_i \nabla_s \nabla_j \nabla_l F^k + (\text{Rm} \ast \nabla^2 F)_{ij}^k + (\nabla \text{Rm} \ast \nabla F)_{ij}^k$$

$$= g^{kl} \nabla_s \nabla_i \nabla_j \nabla_l F^k + (\text{Rm} \ast \nabla^2 F)_{ij}^k + (\nabla \text{Rm} \ast \nabla F)_{ij}^k$$

$$= g^{kl} \nabla_s (\nabla \nabla_i \nabla_j \nabla_l F^k + (\text{Rm} \ast \nabla F)_{ij}^k) + (\nabla \text{Rm} \ast \nabla^2 F)_{ij}^k + (\nabla \text{Rm} \ast \nabla F)_{ij}^k$$

$$= \Delta (\nabla_i \nabla_j F^k) + (\text{Rm} \ast \nabla^2 F)_{ij}^k + (\nabla \text{Rm} \ast \nabla F)_{ij}^k.$$
Here we have used freely the formula for the commutation of the second covariant derivatives of a tensor $T$, $\nabla^2 T(V, W, \cdot) = \nabla^2 T(W, V, \cdot) + (\text{Rm} + T)(V, W, \cdot)$: see for example [21]. This means in particular that,

$$
\frac{\partial}{\partial t}[\nabla^2 F]_{g(t)}^2 \leq \Delta_{g(t)}([\nabla^2 F]_{g(t)}^2) - 2|\nabla^3 F|_{g(t)}^2 + c(n)|\text{Rm}(g(t))|_{g(t)}[\nabla^2 F]_{g(t)}^2 \\
+ c(n)|\nabla \text{Rm}(g(t))|_{g(t)}[\nabla^2 F]_{g(t)} |\nabla F|_{g(t)}
$$

\[(2.8)\]

Using (2.7) together with (2.8), one sees that the following evolution (in)equalities are satisfied by $Z$:

$$
\left(\frac{\partial}{\partial t} - \Delta_{g(t)}\right) Z = \left(\frac{\partial}{\partial t} - \Delta_{g(t)}\right) \left(t(a_0 + |\nabla F|_{g(t)}^2) |\nabla^2 F|_{g(t)}^2\right)
$$

\[
= \left(a_0 + |\nabla F|_{g(t)}^2\right) \left(|\nabla^2 F|_{g(t)}^2 + t \left( \partial_t - \Delta_{g(t)} \right) \left(|\nabla F|_{g(t)}^2\right) \right)
+ t(a_0 + |\nabla F|_{g(t)}^2) \left( \partial_t - \Delta_{g(t)} \right) |\nabla^2 F|_{g(t)}^2 - 2t(g(t)) \left(|\nabla |\nabla F|_{g(t)}^2, |\nabla^2 F|_{g(t)}^2\right)
\leq \left(a_0 + |\nabla F|_{g(t)}^2\right) |\nabla^2 F|_{g(t)}^2 - 2t|\nabla^2 F|_{g(t)}^2 - 2a_0 t |\nabla^3 F|_{g(t)}^2
+ (1 + a_0) \epsilon_0 |\nabla^2 F|_{g(t)}^2 + \left(\frac{1 + a_0}{t}\right) |\nabla^2 F|_{g(t)}^2
\]

\[
\leq \frac{Z}{t} - 2t|\nabla^2 F|_{g(t)}^2 - 2a_0 t |\nabla^3 F|_{g(t)}^2 + a_0 \epsilon_0 |\nabla^2 F|_{g(t)}^2
+ \frac{a_0 \epsilon_0}{t} + t(c(n, c_1) |\nabla^3 F|_{g(t)}^2 |\nabla^2 F|_{g(t)}^2
\]

\[
\leq \frac{Z}{t} + (c(n, c_1) - 2a_0 t |\nabla^3 F|_{g(t)}^2 - t|\nabla^2 F|_{g(t)}^2 + a_0 \epsilon_0 |\nabla^2 F|_{g(t)}^2 + \frac{a_0 \epsilon_0}{t} + \frac{t^2}{2} |\nabla^3 F|_{g(t)}^2 + \frac{a_0 \epsilon_0}{t}
\]

where we assume that $a_0 \geq 1$, $c(n, c_1)$ denotes a positive constant depending on the dimension $n$ and the Lipschitz constant $c_1$, that may vary from line to line and we have used Young’s inequality freely. Therefore, the function $Z$ satisfies the following:

$$
\partial_t Z \leq \Delta_{g(t)} Z + \frac{Z}{t} - \frac{Z^2}{2(a_0 + c_1)^2 t} + \frac{a_0^2 \epsilon_0}{t}
$$

\[(2.9)\]

if $a_0$ is chosen sufficiently large such that $a_0 \geq c(n, c_1)$.

In case the underlying manifold is closed, the use of the maximum principle would give us the expected result: if there is a first time and point $(x, t)$ where $Z(x, t) = 10a_0^2$ for example, we obtain a contradiction. Hence we must have $Z \leq 10a_0^2$.

In order to localize this argument, we introduce, as in the proof of Theorem 2.1, a Perelman type cut-off function $\eta : M \to [0, 1]$ with $\eta(\cdot, t) = 0$ on $B_{g(t)}(x_0, 2/3)$ and $\eta(\cdot, t) = e^{-t}$ on $B_{g(t)}(x_0, 1/2)$ such that $\frac{\partial}{\partial t}\eta(\cdot, t) \leq \Delta_{g(t)} \eta(\cdot, t)$ everywhere, and $|\nabla \eta|_{g(t)}^2 \leq c(n)\eta$ everywhere, as long as $t \leq S(n) \leq 1$: see for example [18, Section 7] for details.

We first derive the evolution equation of the function $\bar{Z} := \eta Z$ with the help of inequality (2.9) for $t \leq S(n)$:

$$
\partial_t \bar{Z} \leq \Delta_{g(t)} \bar{Z} - 2(\nabla \eta, \nabla \bar{Z})_{g(t)} + \frac{\bar{Z}}{t} - \eta \frac{Z^2}{4a_0^2 t} + \frac{a_0^2 \epsilon_0}{t}
$$

\[(2.10)\]
Multiplying the previous differential inequality (2.10) by the non-negative function $\eta$ gives:

$$\eta (\partial_t - \Delta_{g(t)}) \hat{Z} \leq -2\eta (\nabla\eta, \nabla \hat{Z})_{g(t)} + \frac{\eta \hat{Z}}{t} - \frac{\hat{Z}^2}{4a_t^2} + \eta^2 \frac{\epsilon_0^2}{t}$$

$$\leq -2(\nabla\eta, \nabla \hat{Z})_{g(t)} + c(n)\hat{Z} + \frac{\hat{Z}}{t} - \frac{\hat{Z}^2}{4a_t^2} + \frac{\epsilon_0^2}{t}$$

where again, $c(n)$ denotes a positive constant depending on the dimension only and which may vary from line to line. If the maximum of $\hat{Z}$ at any time is larger that $100\epsilon_0^2$ then this value must be achieved at some first time and point $(x, t)$ with $t = 0$, since $\hat{Z}(\cdot, 0) = 0$. This leads to a contradiction if $t \leq S(n) \leq 1$.

3 Almost isometries, distance coordinates and Ricci-harmonic map heat flow

In this section we are interested in smooth, not necessarily complete, solutions to Ricci flow $(M, g(t))_{t \in (0,T)}$ with $T \leq 1$ where $M$ is connected, satisfying the following: given $\epsilon_0 > 0$ and some fixed $R > 100\epsilon_0^2 + 200$ we assume that $B_{g(t)}(x_0, 200R) \subset M$ for all $t \in (0, T)$ as well as

(a) $|\text{Rm}(\cdot, t)| \leq \frac{\epsilon_0^2}{t}$ on $B_{g(t)}(x_0, 200R) \subset M$ for all $t \in (0, T)$

(b) $\text{Ric}(g(t)) \geq -1$ on $B_{g(t)}(x_0, 200R) \subset M$ for all $t \in (0, T)$

Using (a), (b) as well as $R > 100\epsilon_0^2 + 200$ together with [19, Lemma 3.1], we see that there exists a unique metric $d_0 : (\cap_{\tau \in (0,T)} B_{g(t)}(x_0, 50R)) \times (\cap_{\tau \in (0,T)} B_{g(t)}(x_0, 50R)) \to \mathbb{R}^n$ such that for all $T \geq r \geq 0$,

$$e^{t-r}d_t \geq d_t \geq d_x - c(n)\epsilon_0 \sqrt{t-r} \quad \forall t \in [r, T] \text{ on } B_{d_0}(x_0, 20R) \subset B_{g(t)}(x_0, 50R)$$

where $d_t = d(g(t))$ is the distance with respect to $g(t)$ on $M$: $d_t(x, y) = \inf, L_{g(t)}(\gamma)$ where $\gamma$ is any smooth curve between $x$ and $y$ in $M$ and $L_{g(t)}$ is the length of this curve with respect to $g(t)$. Furthermore $B_{g(t)}(x_0, r) = \{ x \in M \mid d_t(x, x_0) < r \}$, $B_{d_0}(x_0, r) = \{ x \in M \mid d_0(x, x_0) < r \}$. Hence, replacing $c(n)\epsilon_0$ by $\epsilon_0$, we have

$$d_t + \epsilon_0 \sqrt{t-r} \geq d_t \geq d_t - \epsilon_0 \sqrt{t-r} \quad \text{on } B_{d_0}(x_0, 20R) \subset M,$$

$$B_{g(t)}(x_0, 20R) \subset M$$

for all $0 \leq r < t, s \in [0, T]$.

By scaling the solution once by $\epsilon_0^2/R^2 \geq 1$ we may assume without loss of generality that the inequalities (3.1) and inclusion (3.2) above hold for all $0 \leq r < t, s \in [0, 1] \cap (0, T)$. We assume this in the following.

Also the estimates of Shi hold, as is explained in the following lemma.

**Theorem 3.1.** Let $(M, g(t))_{t \in (0,T)}$ be a smooth, not necessarily complete, solution to Ricci flow satisfying (a), (b), (3.1) and (3.2) for $0 \leq r < t, s \in [0, 1] \cap (0, T)$ for some $R \geq 100\epsilon_0^2 + 200$. Then

$$\sum_{i=0}^j |\nabla^i \text{Rm}(\cdot, t)|^2 \leq \frac{\beta(k, n, \epsilon_0)}{t^{j+1}}$$

for all $x \in B_{d_0}(x_0, 10R)$ for all $t < \min(1, T)$, where $\beta(k, n, \epsilon_0) \to 0$ for fixed $k$ and $n$, as $\epsilon_0 \to 0$. 

13
Proof. We scale the solution \( \tilde{g}(\cdot, \tilde{t}) := t^{-1}g(\cdot, \tilde{t}) \), so that time \( t \) in the original solution scales to time \( \tilde{t} \) equal to 1 in the new one. We now have \( |\text{Rm}(\cdot, s)| \leq 2 \varepsilon_0^2 \) on \( B_{\hat{g}(1/2)}(x, 1) \) for all \( s \in [1/2, 2] \), for all \( x \in B_{d_0}(x_0, 10R) \), in view of (a) and the (scaled) distance estimates (3.1). The estimates of Shi, see for example of [7, Theorem 6.5], give us that \( \sum_{i=0}^{n} |\nabla^{i}\text{Rm}(x, 1)|^2 \leq \beta(k, n, \varepsilon_0) \) where \( \beta(k, n, \varepsilon_0) \to 0 \) as \( \varepsilon_0 \to 0 \), as claimed. \( \square \)

In the main theorem of this chapter, Theorem 3.7, we consider maps \( F_t \), and \( \varepsilon_0 < 1 \) which satisfy the following property:

\[ F_t : B_{d_0}(x_0, 100) \to \mathbb{R}^n \]

(c) \[ |F_t(x) - F_t(y)| \leq \left( (1 - \varepsilon_0)d_t(x, y) - \varepsilon_0\sqrt{t}, (1 + \varepsilon_0)d_t(x, y) + \varepsilon_0\sqrt{t} \right). \]

If such a map \( F_t \) exists for all \( t \in (0, T) \), and we further assume that \( \sup_{t \in (0, T)} |F_t(x_0)| < \infty \), then for any sequence \( t_i > 0 \), \( t_i \to 0 \) with \( i \to \infty \), we can, after taking a subsequence, find a limiting map, \( F_0 \) which is the \( C^0 \) limit of \( F_t \), \( \sup_{x \in B_{d_0}(x_0, 100)} |F_0(x) - F_t(x)| \to 0 \), as \( i \to \infty \) which satisfies

\[ (1 - \varepsilon_0) \leq \frac{|F_0(x) - F_0(y)|}{d_0(x, y)} \leq (1 + \varepsilon_0) \]

on \( B_{d_0}(x_0, 100) \). Indeed, we first define \( F_0 \) on a dense, countable subset \( D \subset B_{d_0}(x_0, 100) \) using a diagonal subsequence and the theorem of Heine-Borel, and then we extend \( F_0 \) uniquely, continuously to all of \( B_{d_0}(x_0, 100) \), which is possible in view of the fact that the Bi-Lipschitz property (3.4) is satisfied on \( D \). The sequence \( (F_t)_t \) converges uniformly to \( F_0 \) in view of (3.1), (c) and (3.4).

Thus \( F_0 \) is a \( (1 + \varepsilon_0) \) Bi-Lipschitz map between the metric spaces \( (B_{d_0}(x_0, 100), d_0) \) and \( (F_0 : B_{d_0}(x_0, 100), \hat{d}) \). This is equivalent to \( H_0(\cdot) = F_0(\cdot) - F_0(x_0) \) being a \( (1 + \varepsilon_0) \) Bi-Lipschitz map between the metric spaces \( (B_{d_0}(x_0, 100), d_0) \) and \( H_0(B_{d_0}(x_0, 100)) \) where \( B_{100/\varepsilon(n)}(0) \subseteq H_0(B_{d_0}(x_0, 100)) \subseteq B_{100\varepsilon(n)}(0) \).

In Theorem 3.7, we see that if we consider a Ricci-harmonic map heat flow of one of the functions \( F_t \) (for small enough \( t \)), and we assume that the solution satisfies a gradient bound, \( |\nabla F_t|_{g(t)} \leq c_1 \), on some ball, then after flowing for a time \( t \), the resulting map will be a \( (1 + \alpha_0) \) Bi-Lipschitz map on a smaller ball, where \( \alpha_0 \) if \( \varepsilon_0 \leq \varepsilon_0(n, \alpha_0, c_1) \) is small enough. This property continues to hold if we flow for a time \( s \) where \( t \leq s \leq \min(S(n, \alpha_0, c_1), T) \).

For convenience, we introduce the following notation:

**Definition 3.2.** Let \( (W, d) \) be a metric space. We call \( F : (W, d) \to \mathbb{R}^n \) an \( \varepsilon_0 \) almost isometry if

\[ (1 - \varepsilon_0)d(x, y) - \varepsilon_0 \leq |F(x) - F(y)| \leq (1 + \varepsilon_0)d(x, y) + \varepsilon_0 \]

for all \( x, y \in W \). \( F : (W, d) \to \mathbb{R}^n \) is a \( 1 + \varepsilon_0 \) Bi-Lipschitz map, if

\[ (1 - \varepsilon_0)d(x, y) \leq |F(x) - F(y)| \leq (1 + \varepsilon_0)d(x, y) \]

for all \( x, y \in W \).

In the main application, we will assume:

\[ \text{there are points } a_1, \ldots, a_n \in B_{d_0}(x_0, R), \text{ such that the distance coordinates} \]

(c) \[ F_0 : B_{d_0}(x_0, R) \to \mathbb{R}^n, F_0(\cdot) := (d_0(a_1, \cdot), \ldots, d_0(a_n, \cdot)) =: ((F_0)_1(\cdot), \ldots, (F_0)_n(\cdot)), \]

is a \( (1 + \varepsilon_0) \) Bi-Lipschitz homeomorphism on \( B_{d_0}(x_0, 100) \).
As a consequence of this assumption and the distance estimates (3.1), we see that the corresponding distance coordinates at time $t$, $F_t : B_{d_0}(x_0, 50) \to \mathbb{R}^n$, given by $F_t(x) := (d_t(a_1, \cdot), \ldots, d_t(a_n, \cdot))$, are mappings satisfying property (c), for all $t \in (0, T)$.

**Remark 3.3.** R. Hochard also looked independently at some related objects in his Ph.D.-thesis, and some of the infinitesimal results he obtained there are similar to those of this section, c.f. [10, Theorem II.3.10], as we explained in the introduction. Hochard considers points $x_0$ which are so called $(m, \varepsilon)$ explosions at all scales less then $R$ (only $m = n$ is relevant in this discussion). The condition, for $m = n$, says that there exist points $p_1, \ldots, p_n$ such that for all $x$ in the ball $B_{d_0}(x_0, R)$ and all $r < R$, there exists an $\varepsilon r$ GH approximation $\psi : B_{d_0}(x, r) \to \mathbb{R}^n$ such that the components $\psi^i$ are each close to the components of distance coordinates $d(\cdot, p_i) - d(x, p_i)$ at the scale $r$, in the sense that $|\psi^i(\cdot) - (d(\cdot, p_i) - d(x, p_i))| \leq \varepsilon r$ on $B_{d_0}(x, r)$. Our approach and our main conclusion differ slightly to the approach and main conclusions of Hochard. The condition (c) we consider above looks at the closeness of the maps $F_t$ to being a Bi-Lipschitz homeomorphism, and this closeness is measured at time $t$ using the maps $F_t$, and our main conclusion, is that the map will be a $1 + \alpha_0$ Bi-Lipschitz homeomorphism after flowing for an appropriate time by Ricci-harmonic map heat flow, if $t > 0$ is small enough. We make the assumption on the evolving curvature, that it is close to that of $\mathbb{R}^n$, after scaling in time appropriately. Nevertheless, the proof of Theorem 3.7 below and of [10, Theorem II.3.10] have a number of similarities, as do some of the concepts.

The main application of this section is, assuming (a),(b), and that $F_0$ are distance coordinates which define a $1 + \varepsilon_0$ Bi-Lipschitz homeomorphism, to show that it is possible to define a Ricci-DeTurck flow $(\tilde{g}(s))_{s \in (0, T]}$ starting from the metric $\tilde{d}_0 := (F_0)_*, d_0$, on some Euclidean ball, which is obtained by pushing forward the solution $(g(s))_{s \in (0, T]}$ by diffeomorphisms. The sense in which this is to be understood, respectively this is true, will be explained in more detail in the next paragraph.

The strategy we adopt is as follows. Assuming (a), (b), we consider the distance maps $F_{t_i}$ at time $t_i$ defined above for a sequence of maps $t_i \to 0$ with $t_i \to 0$. We mollify each $F_{t_i}$ at a very small scale, so that we are in the smooth setting, but so that the essential property, (c), of the $F_{t_i}$ is not lost (at least up to a factor 2). Then we flow each of the $F_{t_i}$ on $B_{d_0}(x_0, 100)$ by Ricci-harmonic map heat flow, keeping the boundary values fixed. The existence of the solutions $Z_{t_i} : B_{d_0}(x_0, 100) \times [t_i, T) \to \mathbb{R}^n$, is guaranteed by Theorem 2.1. According to Theorem 3.7, the $Z_{t_i}(s)$ are then $1 + \alpha_0$ Bi-Lipschitz maps (on a smaller ball), for all $s \in [t_i, S(n, \varepsilon_0)] \cap (0, T]$, if $\varepsilon_0 = \varepsilon_0(\alpha_0, n)$ is small enough. If we take the push forward of $g(\cdot)$ with respect to $Z_{t_i}(\cdot)$ (on a smaller ball), then after taking a limit of a subsequence in $i$, we obtain a solution $\tilde{g}(s)_{s \in (0, S(n, \varepsilon_0)] \cap (0, T]}$ to the $\delta$-Ricci-DeTurck flow such that $(1 - \alpha_0)\delta \leq \tilde{g}(s) \leq (1 + \alpha_0)\delta$ for all $s \in (0, S(n, \varepsilon_0)] \cap (0, T]$ in view of the estimates of Theorem 3.7. The solution then satisfies $d(\tilde{g}(t)) \to \tilde{d}_0 := (F_0)_*, d_0$ as $t \to 0$ and hence may be though of as a solution to Ricci-DeTurck flow coming out of $\tilde{d}_0$. This is explained in Theorem 3.10. In the next section we examine the regularity properties of this solution, which depend on the regularity properties of $\tilde{d}_0$.

### 3.1 Almost isometries and Ricci-harmonic map heat flow

In this subsection we provide some technical lemmas giving insight into the evolution of almost isometries under Ricci-harmonic map heat flow. These results will be needed in the following subsection.

**Lemma 3.4.** For all $\sigma > 0$, there exists an $0 < \gamma(\sigma, c_1) \leq \sigma$ small with the following property: if $L : B_{\gamma^{-1}(0)} \to \mathbb{R}^n$ is a $\gamma$ almost isometry fixing 0, and $|dL|\in C^0(B_{\gamma^{-1}(0)}) \leq c_1$, then there exists an $S \in O(n)$ such that $|L - S|\in C^0(B_{\gamma^{-1}(0)}) \leq \sigma$. 

15
Proof. If not, then for some $\sigma > 0$, we have a sequence of maps $L_i : B_{\sigma^{-1}}(0) \to \mathbb{R}^n$ such that $L_i$ is a weak $\frac{1}{2}$ isometry, but $L_i$ is not $\sigma$ close in the $C^0$ sense to any element $S \in O(n)$ on $B_{\sigma^{-1}}(0)$. By taking a subsequence we obtain convergence in the $C^0$ sense of $L_i$ to an element $S \in O(n)$, since the maps are Lipschitz, and hence we can find a subsequence which converges in the $C^{0,\alpha}$ norms on compact subsets. This leads to a contradiction. \[ \square \]

**Lemma 3.5.** For all $c_1, \alpha_0 > 0$, $n \in \mathbb{N}$ there exists $0 < \alpha(c_1, n, \alpha_0) \leq \alpha_0$ such that the following is true. Let $F : B_{\alpha^{-1}}(0) \times [0, 1] \to \mathbb{R}^n$ be a smooth solution to the harmonic map heat flow with an evolving background metric,

$$\frac{\partial}{\partial t} F(\cdot, t) = \Delta_{h(t)} F(\cdot, t),$$

where $h(\cdot, t) \in [0, 1]$ is smooth, and $F_0(0) = 0$. We assume that $F_0$ is an $\alpha$ almost isometry with respect to $h(0)$, and further that:

$$\begin{align*}
|h_{ij} - \delta_{ij}| &\leq C(\alpha) \|\partial_t^2 h(\cdot, t)\|_{C^0(B_{\alpha^{-1}}(0))}, \\
|\nabla F(\cdot, t)|_{h(t)} &\leq c_1 \quad \text{on } B_{\alpha^{-1}}(0), \\
|\nabla^2 F(\cdot, t)|_{h(t)} &\leq \frac{c_1}{\sqrt{t}} \quad \text{on } B_{\alpha^{-1}}(0)
\end{align*}$$

for all $t \in [0, 1]$. Then

$$|d F(\cdot, s)(v)| \in (1 - \alpha_0, 1 + \alpha_0) |v|_{h(s)}$$

(3.5) $$|F(x, s) - F(y, s)| \leq (1 - \alpha_0) d(h(s))(x, y) + (1 + \alpha_0) d(h(s))(x, y)$$

$$|F_0(x) - F(x, t)| \leq \alpha_0,$$

for all $s \in [1/2, 1]$, for all $t \in [0, 1]$, for all $x, y \in B_{\alpha^{-1}}(0)$ and $v \in T_x B_{\alpha^{-1}}(0)$.

**Proof.** From Lemma 3.4, we know that there exists an $S \in O(n)$ such that

$$|F(\cdot, 0) - S(\cdot)|_{C^0(B_{\alpha^{-1}}(0))} \leq \beta$$

where $0 < \alpha \leq \beta = \beta(n, c_1, \alpha)$, but still $\beta(n, c_1, \alpha) \to 0$ as $\alpha \to 0$.

We also know that $|\Delta h(t) F(x, t)| \leq c_1 / \sqrt{t}$ and hence $|F(x, t) - F(x, 0)| \leq 2c_1$ for all $t \in [0, 1]$ for all $x \in B_{\beta^{-1}}(0)$ which implies

$$|F(x, t) - S(x)| \leq |F(x, t) - F(x, 0)| + |F(x, 0) - S(x)| \leq 3c_1$$

(w.l.o.g. $\beta \leq c_1$) for all $t \in [0, 1]$ for all $x \in B_{\beta^{-1}}(0)$.

Let $\eta : B_{\beta^{-1}}(0) \to [0, 1]$ be a smooth cut off function such that $\eta(\cdot) = 1$ on $B_{\beta^{-1}/2}(0)$, $\eta(\cdot) = 0$ on $\mathbb{R}^n \setminus B_{\beta^{-1}}(0)$, $|D^2 \eta| + (|D \eta|^2 / \eta) \leq c(n) \beta^2$, $|D \eta| \leq c(n) \beta$. Due to the fact that $h$ is a close to $\delta$ in the $C^2$ norm, we see $|\Delta h(S)(x, t)| = |h^{ij}(x, t)(\partial_i \partial_j S(x) - \Gamma(h)^{ij}(x, t) \partial_k S)| \leq c(n) \alpha$. Hence, we have

$$\frac{\partial}{\partial t} (F - S)(\cdot, t) = \Delta_{h(t)} (F - S)(\cdot, t) + E,$$

where $|E| \leq c(n) \alpha$. But then, using the cut off function $\eta$ on $B_{\beta^{-1}}(0)$, we get

$$\begin{align*}
\frac{\partial}{\partial t} (|F - S|^2 \eta) &\leq \Delta_{h(t)} (|F - S|^2 \eta) - 2\eta h(t) \nabla (F - S)^2 \partial_t |F - S|^2 \eta - \eta (F - S|^2 \partial_t \Delta h(t) \eta \\
&\leq \Delta_{h(t)} (|F - S|^2 \eta) + \eta (c(n) \beta |F - S|^2 - \eta h(t) \nabla (F - S)^2 \eta) + c(n) c_1 \alpha \leq \Delta_{h(t)} (|F - S|^2 \eta) + c(n, c_1) \beta.
\end{align*}$$
Hence, by the maximum principle, \( |F - S|^2(t) \leq \alpha^2 + c(n, c_1)\beta \leq c(n, c_1)\beta \) for all \( t \in [0, 1] \) on \( \mathbb{B}_{-1/2}(0) \).

Since \( |\nabla(F - S)|^2 + |\nabla^2(F - S)|^2 \leq c(n)c_1 \) for \( s \in [1/2, 1] \) on \( \mathbb{B}_{-1/2}(0) \) we can use interpolation inequalities as in [6, Lemma B.1] to deduce that on \( \mathbb{B}_{-1/4}(0) \)

\[
|D(F - S)|_t \leq c(n, c_1)\beta^{\frac{1}{2}}.
\]

Again, since \( h(t) \) is \( \alpha \)-close to \( \delta \) this implies that for \( \alpha \) sufficiently small

\[
dF(\cdot, s)(v) \in (1 - \alpha_0, 1 + \alpha_0)
\]

for all \( v \in T_y\mathbb{R}^n \) of length one with respect to \( h(s) \) for \( s \in [1/2, 1] \) and \( y \in \mathbb{B}_{-1/4}(0) \). We also see that

\[
|F(x, s) - F(y, s)| = |DF(p, s)(v_p)||x - y| \in (d(h(s))(x, y)(1 - \alpha_0), d(h(s))(x, y)(1 + \alpha_0))
\]

where by the mean value theorem \( p \) is some point on the unit speed line between \( x \) and \( y \), and \( v_p \) is a vector of length one with respect to \( \delta \) and

\[
|F(x, t) - F_0(x)| \leq |F(x, t) - S| + |F_0(x) - S| \leq c(n, c_1)\beta^{\frac{1}{2}} \leq \alpha_0
\]

for all \( x, y \in \mathbb{B}_{-1/4}(0) \), for all \( s \in [1/2, 1] \), \( t \in [0, 1] \) for \( \alpha \) sufficiently small.

\[\Box\]

**Lemma 3.6.** For all \( n, k \in \mathbb{N}, L > 0 \) there exists an \( \varepsilon_0 = \varepsilon_0(n, k, L) > 0 \) such that the following holds. Let \( M^n \) be a connected smooth manifold, and \( g \) and \( h \) be smooth Riemannian metrics on \( M \) with \( B_h(y_0, L) \subseteq M \) and \( |d_h - d_g|_{C^0(B_h(y_0, L))} \leq \varepsilon_0 \). Assume further that

\[
\sup_{B_h(y_0, L)} (|Rm(g)|_g + \ldots + |^g\nabla^{k+2}Rm(g)|_g) \leq \varepsilon_0
\]

and that there exists a map \( F : B_h(y_0, L) \rightarrow \mathbb{R}^n \) which is an \( \varepsilon_0 \) almost isometry with respect to \( h \), that is

\[
(1 - \varepsilon_0)d_h(z, y) - \varepsilon_0 \leq |F(z) - F(y)| \leq (1 + \varepsilon_0)d_h(z, y) + \varepsilon_0
\]

for all \( z, y \in B_h(y_0, L) \), and \( F(y_0) = 0 \). Then \( (B_h(y_0, L/2), g) \) is \( 1/L \)-close to the Euclidean ball \( (\mathbb{B}_{L/2}(0), \delta) \) in the \( C^k \)-Cheeger-Gromov sense.

**Proof.** Assume it is not the case. Then there is an \( L > 0 \) for which the theorem fails. Then we can find sequences \( g(i), h(i), M(i), F(i) : B_h(i)(y_0, L) \rightarrow \mathbb{R}^n \) satisfying the above conditions with \( \varepsilon_0 := 1/i \) but so that the conclusion of the theorem is not correct. Using the almost isometry, we see that for any \( \varepsilon > 0 \) we can cover \( B_h(i)(y_0(i), 6L/7) \) by \( N(\varepsilon) \) balls (with respect to \( h \)) of radius \( \varepsilon \), for all \( i \). Hence, using \( |d_h(i) - d_g(i)|_{C^0(B_h(y_0(i), L))} \leq 1/i \), we see that the same is true for \( B_g(i)(y_0(i), 5L/6) \subseteq B_h(i)(y_0(i), 6L/7) \) with respect to \( g(i) \): we can cover \( B_g(i)(y_0(i), 5L/6) \) by \( N(\varepsilon) \) balls (with respect to \( g(i) \)) of radius \( \varepsilon \), for all \( i \). Hence, due to the compactness theorem of Gromov (see for example [3, Theorem 8.1.10]), there is a Gromov-Hausdorff Limit

\[
(X, d) = \lim_{i \rightarrow \infty} (B_g(i)(y_0(i), 4L/5), g(i)) = \lim_{i \rightarrow \infty} (B_h(i)(y_0(i), 4L/5), h(i)).
\]

In particular, there must exist

\[
G(i) : B_d \left( y_0, \frac{3L}{4} \right) \rightarrow B_h(i) \left( y_0(i), \frac{4L}{5} \right),
\]

which are \( \varepsilon(i) \) Gromov-Hausdorff approximations, where \( \varepsilon(i) \rightarrow 0 \) as \( i \rightarrow \infty \). Using the maps \( G(i) \) and the \( 1/4 \) almost isometries \( F(i) \), we see that there is a pointwise limit map, \( H := \lim_{i \rightarrow \infty} F(i) \circ G(i) \),

\[
H : \left( B_d \left( y_0, \frac{3L}{4} \right), d \right) \rightarrow \left( \mathbb{B}_{\frac{4}{5}}(0), \delta \right),
\]

17
which is an isometry. Hence the volume of \( B_{g(i)}(y_0(i), 2L/3) \) converges to \( \omega_n(2L/3)^n \) (in particular the sequence is non-collapsing) as \( i \to \infty \), since volume is convergent for spaces of bounded curvature (which are for example Aleksandrov spaces and spaces with Ricci curvature bounded from below). Hence \( (B_{g(i)}(y_0(i), L/2), g(i)) \) converges to \((\mathbb{B}_{L/2}(0), \delta)\) in the \( C^k \) norm in the Cheeger-Gromov sense, which leads to a contradiction if \( i \) is large enough.

\[ \square \]

### 3.2 Ricci-harmonic map heat flow of \((1 + \varepsilon_0)\)-Bi-Lipschitz maps and distance coordinates

We begin with the Regularity Theorem, Theorem 3.7, for solutions to the Ricci-harmonic map heat flow, whose initial values are sufficiently close to those of a \((1 + \varepsilon_0)\) Bi-Lipschitz map.

**Theorem 3.7.** For all \( \alpha_0 \in (0, 1), n \in \mathbb{N} \) and \( c_1 \in \mathbb{R}^+ \), there exists \( \varepsilon_0(n, c_1, \alpha_0) > 0 \) and \( S(n, c_1, \alpha_0) > 0 \) such that the following holds. Let \((M, g(t))_{t \in (0, T)}\) be a smooth solution to Ricci flow satisfying the condition (a), (3.1) and (3.2) for \( 0 \leq r < t, s \in [0, T] \), where \( d_0 \) is the metric appearing in (3.1). If \( m_0 \in B_{d_0}(x_0, 15R) \) and \( Z : B_{d_0}(m_0, 10) \times [t, T] \to \mathbb{R}^n \) is a solution to the Ricci-harmonic map heat flow

\[
\frac{\partial}{\partial s} Z = \Delta_{g(s)} Z
\]

for some \( t \in [0, T] \) on \( B_{d_0}(m_0, 10) \) for all \( s \in [t, T] \), which satisfies \( |g(s)\nabla Z|_{g(s)} \leq \alpha_1 \) on \( B_{d_0}(m_0, 10) \) for all \( s \in [t, T] \), and the initial values of \( Z \) satisfy (c), that is

\[
(3.6) \quad |Z(x, t) - Z(y, t)| = ((1 - \varepsilon_0)d_0(x, y) - \varepsilon_0\sqrt{t}, (1 + \varepsilon_0)d_0(x, y) + \varepsilon_0\sqrt{t})
\]

for all \( x, y \in B_{d_0}(m_0, 10) \), then

\[
(3.7) \quad |g(s)\nabla^2 Z(x, s)|_{g(s)}^2 \leq \alpha_0(s - t)^{-1}
\]

\[
(3.8) \quad |Z(x, s) - Z(x, t)| \leq c(n, 1, n)\sqrt{s - t}
\]

\[
(3.9) \quad |dZ(x, s)(v)| \in (1 - \alpha_0, 1 + \alpha_0)|v|_{g(s)},
\]

\[
(3.10) \quad |Z(x, s) - Z(y, s)| \in ((1 - \alpha_0)d_0(x, y), (1 + \alpha_0)d_0(x, y))
\]

for all \( x, y \in B_{d_0}(m_0, 2), v \in T_xB_{d_0}(m_0, 2), s \in [2t, S(n, c_1, \alpha_0)] \cap [0, T/2] \). Furthermore, the maps \( Z(s) : B_{d_0}(m_0, 3/2) \to D_s := Z(s)(B_{d_0}(m_0, 3/2)) \subseteq \mathbb{R}^n \) for \( s \in [2t, S(n, c_1, \alpha_0)] \cap [0, T/2] \) are homeomorphisms and their image satisfies \( B_{5/4}(Z(s)(m_0)) \subseteq D_s \subseteq B_2(Z(s)(m_0)) \), for \( Z(s) := Z_s(\cdot) \).

A direct consequence of (3.9) is the following corollary.

**Corollary 3.8.** Assuming the set up of Theorem 3.7, the following metric inequalities hold

\[
(3.11) \quad (1 - \alpha_0)^2g(x, s) \leq (Z(s))^*\delta \leq (1 + \alpha_0)^2g(x, s)
\]

\[
(3.12) \quad (1 - \alpha_0)^2(Z(s))_*g(y, s) \leq \delta \leq (1 + \alpha_0)^2(Z(s))_*g(y, s)
\]

for all \( x \in B_{d_0}(m_0, 2) \), for all \( y \in B_{5/4}(Z(s)(m_0)) \subseteq Z(s)(B_{d_0}(m_0, 2)) \subseteq \mathbb{R}^n \) and for all \( s \in [2t, S(n, c_1, \alpha_0)] \cap [0, T/2] \).

**Proof of Theorem 3.7.** From Theorem 2.3, we know that \( |\nabla^2 Z(x, s)|_{g(s)} \leq c(n, s)^{-1/2} \) and hence

\[
(3.13) \quad |Z(x, s) - Z(x, t)| \leq c(n, s)^{-1/2} \sqrt{s - t}
\]
for all $s \in [t, S(n, c_1)] \cap [0, T]$, $x \in B_{d_0}(m_0, 8) \subseteq B_{g(\alpha)}(m_0, 9) \subseteq B_{d_0}(m_0, 10)$. We show the rest of the estimates hold for arbitrary $r \in [2t, S(n, c_1, \alpha_0)] \cap [0, T/2]$, $x \in B_{d_0}(m_0, 4)$, if $S(n, c_1, \alpha_0)$ is chosen to be small enough. First we scale the solution to the Ricci-harmonic map heat flow and the metric $g(t)$ by $1/\sqrt{r}$ respectively $1/r$:

$$
\tilde{Z}(z, s) := \frac{1}{\sqrt{r}} Z(z, sr) \quad \text{and} \quad \tilde{g}(\cdot, s) = \frac{1}{r} g(\cdot, rs)
$$

for $z \in B_{d_0}(m_0, 8)$. Then the solution is defined for $s \in [\tilde{t} := t/r, \tilde{T} := T/r]$ on $B_{d_0}(x, 1/\sqrt{r})$ for any $x \in B_{d_0}(m_0, 7)$ where $\tilde{t} = t/r \leq 1/2$, since $r \geq 2t$ and $\tilde{T} \geq 2$, since $r \leq T/2$, and the radius $V := 1/\sqrt{r}$ satisfies $V \geq 1/\sqrt{S(n, c_1, \alpha_0)}$. Since before scaling, we have $|\text{Rm}(\cdot, s)| \leq \varepsilon_0^2/s$, after scaling we still have $|\tilde{\text{Rm}}(\cdot, s)| \leq \varepsilon_0^2$ on $B_{d_0}(x, V)$ for all $s \in [\tilde{t}, \tilde{T}]$. The time $r$ has scaled to the time 1. The property (3.6) scales to

$$
|\tilde{Z}(z, \tilde{t}) - \tilde{Z}(y, \tilde{t})| \in \left((1 - \varepsilon_0) \tilde{d}_\epsilon(z, y) - \varepsilon_0 \sqrt{\tilde{t}}, (1 + \varepsilon_0) \tilde{d}_\epsilon(z, y) + \varepsilon_0 \sqrt{\tilde{t}}\right)
$$

for all $z, y \in B_{d_0}(x, V)$ since $\tilde{t} = t/r \leq 1/2$. The inequality (3.13) scales to

$$
|\tilde{Z}(z, \tilde{s}) - \tilde{Z}(y, \tilde{s})| \leq c(1, n) |\tilde{s} - \tilde{t}|
$$

for all $\tilde{s} \in [\tilde{t}, \tilde{T}]$, $z \in B_{d_0}(x, V)$, and the gradient estimate, is also scale invariant: $|\nabla \tilde{Z}(\cdot, \tilde{s})|_{\tilde{g}(\cdot)} \leq c_1$ still holds on $B_{d_0}(x, V)$ for all $\tilde{s} \in [\tilde{t}, \tilde{T}]$. We also have

$$
|\tilde{Z}(\cdot, \tilde{s}) - \tilde{Z}(\cdot, \tilde{t})| \leq c(1, n) \sqrt{\tilde{s} - \tilde{t}}
$$

on $B_{d_0}(x, V)$, due to (3.15), for $\tilde{\sigma} := \sigma + \tilde{t}$, $\sigma \in (0, 1)$. Hence

$$
|\tilde{Z}(z, \tilde{\sigma}) - \tilde{Z}(y, \tilde{\sigma})| \geq |\tilde{Z}(z, \tilde{t}) - \tilde{Z}(y, \tilde{t})| - |\tilde{Z}(z, \tilde{t}) - \tilde{Z}(\tilde{z}, \tilde{\sigma})| - |\tilde{Z}(\tilde{y}, \tilde{t}) - \tilde{Z}(y, \tilde{\sigma})|
$$

$$
\geq (1 - \varepsilon_0) \tilde{d}_\epsilon(z, y) - \varepsilon_0 - c(1, n) \sqrt{\tilde{t}}
$$

$$
\geq (1 - \sigma) \tilde{d}_\epsilon(z, y) - 2c(1, n) \sqrt{\tilde{t}}
$$

for $\sigma$ fixed and $\varepsilon_0 \leq \sigma^2$, and similarly,

$$
|\tilde{Z}(z, \tilde{\sigma}) - \tilde{Z}(y, \tilde{\sigma})| \leq (1 + \sigma) \tilde{d}_\epsilon(z, y) + 2c(1, n) \sqrt{\tilde{t}}
$$

for $z, y \in B_{d_0}(x, V)$ if $\varepsilon_0 \leq \sigma^2$. That is $\tilde{Z}(\cdot, \tilde{\sigma})$ is an $\alpha^2$ almost isometry on $B_{d_0}(x, V)$ if we choose $\sigma = \alpha^k$. At this point we fix $\alpha := \alpha(n, c_1, \alpha_0)$ where $\alpha$ is the function appearing in the statement of Lemma 3.5 and we set $\sigma := \alpha^k$. Without loss of generality, $\alpha \leq \alpha_0 < c(1, n)$, and $(\alpha_0)^{-1} \geq 2c(1, n)$ for any given $c(1, n) \geq 1$. We also still assume $\varepsilon_0 \leq \sigma^2 = \alpha^{16}$, so that the previous conclusion, $\tilde{Z}(\cdot, \tilde{\sigma})$ is an $\alpha^2$-almost isometry, and hence certainly an $\alpha$-almost isometry, on $B_{d_0}(x, V)$, holds, as explained above.

The curvature estimate, $|\tilde{\text{Rm}}(\cdot, \tilde{s})| \leq \varepsilon_0^2/s$ for all $s \in [0, 2] \supseteq [\tilde{t}, \tilde{T}]$, holds, as do the scaled distance estimates,

$$
\tilde{d}_\epsilon + \varepsilon_0 \sqrt{s - \tilde{t}} \geq \tilde{d}_\sigma \geq \tilde{d}_\epsilon - \varepsilon_0 \sqrt{s - \tilde{t}}
$$

on $B_{d_0}(x, V) \subseteq M$, for all $0 \leq \ell \leq s \in [0, 2]$. The estimates of Shi imply, as explained in Lemma 3.1, that at time $\tilde{\sigma} := \tilde{t} + \sigma = \tilde{t} + \alpha^k$,

$$
|\tilde{\text{Rm}}(\cdot, \tilde{\sigma}) + |\nabla \tilde{\text{Rm}}|^2(\cdot, \tilde{\sigma}) + \ldots + |\nabla^k \tilde{\text{Rm}}|(\cdot, \tilde{\sigma}) \leq \frac{\beta(k, n, \varepsilon_0)}{\sigma^{k+2}} = \frac{\beta(k, n, \varepsilon_0)}{\alpha^{8k+16}}
$$

on $B_{d_0}(x, 2V/3)$ where $\beta(k, n, \varepsilon_0) \to 0$ as $\varepsilon_0 \to 0$ for fixed $k$ and $n$. In particular

$$
\sup_{B_{d_0}(x, 2V/3)} |\tilde{\text{Rm}}(\cdot, \tilde{\sigma}) + |\nabla \tilde{\text{Rm}}|^2(\cdot, \tilde{\sigma}) + \ldots + |\nabla^k \tilde{\text{Rm}}|(\cdot, \tilde{\sigma}) \to 0
$$
as $\varepsilon_0 \to 0$ for fixed $c_1, k, n, \alpha_0$. Without loss of generality, $L = 1/\sqrt{S(n,c_1,\alpha_0)} \geq 10/\alpha$ and hence $B_{\delta_0}(x, 2\alpha^{-1}) \subseteq B_{\delta_0}(x, L/2) \subseteq B_{\delta_0}(x, 2V/3)$ for all $s \in [0, 2]$.

By (3.16), (3.17), and (3.14), we see using Lemma 3.6 with $h = \tilde{g}(\tilde{t})$, and $g = \tilde{g}(\tilde{\sigma})$, $L = \alpha^{-1}$, that $(B_{\delta_0}(x, \alpha^{-1}), \tilde{g}(\tilde{\sigma}))$ is close in the $C^k$-norm to a Euclidean ball with the standard metric in the Cheeger-Gromov sense (that is up to smooth diffeomorphisms), if $\varepsilon_0$ is small enough.

Hence there are geodesic coordinates $\varphi$ on the ball $(B_{\delta_0}(x, \alpha^{-1}), \tilde{g}(\tilde{\sigma}))$ such that the metric $\tilde{g}(\tilde{\sigma})$ written in these coordinates is $\alpha$ close to $\delta$ in the $C^k$ norm, if we keep $\sigma$ fixed and choose $\varepsilon_0$ small enough. Using (3.17), and the evolution equation $\frac{\partial}{\partial \tilde{t}} \tilde{g} = -2\text{Ric}(\tilde{g})$ in the coordinates $\varphi$, we see that the evolving metric $h(\cdot) = \varphi^* (\tilde{g}(\cdot))$ in these coordinates is also, without loss of generality, $\alpha$ close to $\delta$ for $t \in [\tilde{t}, 2]$ in the $C^2$ norm.

Using the above, we see that $G(\cdot, t) := \tilde{Z}(\cdot, t+\tilde{\sigma}) - \tilde{Z}(\cdot, \sigma)) \circ (\varphi)^{-1}(\cdot)$ defined on $B_{\alpha^{-1}}(0) \times [0, 3/2]$ sends 0 to 0 at $t = 0$, is Lipschitz with respect to $\delta$ with Lipschitz constant $2c(1, n)$ and is an $\alpha$ almost isometry at time 0. Lemma 3.5 is then applicable to the function $G$ and tells us $G(\cdot, s)$ is an $\alpha_0^3$ almost isometry at $s = 1 - \tilde{\sigma}$ on a ball of radius $(\alpha_0^{-3})$ and that the inequalities (3.5) hold. Hence,

$$|d\tilde{Z}(v)(1)| \leq \left(1 - \alpha_0^3\right) |v|_{\tilde{g}(1)},$$

$$|\tilde{Z}(z, 1) - \tilde{Z}(y, 1)| \leq \left(1 - \alpha_0^3\right) d_{\tilde{g}(1)}(z, y),$$

and

$$|\tilde{Z}(z) - \tilde{Z}(y)| \leq \alpha_0^3 \sqrt{1 - t}$$

for all $z, y \in B_{\delta_0}(x, \alpha^{-3})$ for all $v \in T_z M$. This scales back to

$$|dZ(v)(r)| \leq \left(1 - \alpha_0^3\right) |v|_{g(r)},$$

$$|Z(z, r) - Z(y, r)| \leq \left(1 - \alpha_0^3\right) d_g(z, y),$$

and

$$|Z(z, t) - Z(y, t)| \leq \alpha_0^3 \sqrt{r - t}$$

for all $z, y \in B_{\delta_0}(x, \sqrt{r}/\alpha_0^3)$ for all $v \in T_z M$.

For $z \in B_{\delta_0}(x, \sqrt{r}/(2\alpha_0^3))$ and $y \in (B_{\delta_0}(x, \sqrt{r}/\alpha_0^3))^c \cap (B_{\delta_0}(m_0, 6))$, we show that the property (3.10) also holds. For such $z, y$, we have, $d_0(z, y) \geq \sqrt{r}/(2\alpha_0^3)$ and hence

$$|Z(z, r) - Z(y, r)| \geq |Z(z, t) - Z(y, t)| - c(1, n) \sqrt{r - t},$$

$$\geq (1 - \varepsilon_0) d_t(z, y) - \varepsilon_0 \sqrt{r - t} - c(1, n) \sqrt{r},$$

$$\geq (1 - \varepsilon_0) d_0(z, y) + \varepsilon_0 d_t(z, y) - 2c(1, n) \sqrt{r},$$

$$\geq (1 - \varepsilon_0) d_0(z, y) + \varepsilon_0 \sqrt{r} - 2c(1, n) \sqrt{r},$$

$$\geq (1 - \alpha_0^3) d_0(z, y) - 3c(1, n) \sqrt{r},$$

$$\geq (1 - 10\alpha_0^3) d_0(z, y) + 9\alpha_0^3 d_0(z, y) - 3c(1, n) \sqrt{r},$$

$$\geq (1 - 10\alpha_0^3) d_0(z, y) + 9\alpha_0^3 \sqrt{r} - 3c(1, n) \sqrt{r},$$

$$\geq (1 - 10\alpha_0^3) d_0(z, y) + 9c(1, n) \sqrt{r} - 3c(1, n) \sqrt{r},$$

$$\geq (1 - 10\alpha_0^3) d_0(z, y) + \sqrt{r} + \sqrt{r},$$

$$\geq (1 - 10\alpha_0^3) d_t(z, y),$$

$$\geq (1 - \alpha_0) d_t(z, y)\),$$

as required, where the first inequality follows from (3.13), the second from the condition (c), the seventh and eighth from the fact that $d_0(z, y) \geq \sqrt{r}/(2\alpha_0^3) \geq c(1, n) \sqrt{r}/\alpha_0^3$ and
we have used $\varepsilon_0 < \alpha_0^4$, $c(1,n) \geq 1$, the distance estimates (3.1), and $\alpha_0 \leq 1/c(1,n)$ freely.

It remains to show the property that $B_2(Z(s)(m_0)) \subseteq D_s \subseteq B_2(Z(s)(m_0))$ for $D_s := Z(s)(B_{d_0}(m_0, 3/2)) \subseteq \mathbb{R}^n$, for $s \in (2t, s]$, $s \leq T/2$, $Z(s) := Z(t, s)$. Observe that $Z(s) : B_{d_0}(m_0, 3/2) \to \mathbb{R}^n$ is smooth and satisfies (3.9). In particular, $Z(s)$ is a local diffeomorphism, due to theInverse Function Theorem, and hence $B_2(Z(s)(m_0)) \subseteq D_s$ for some maximal $r > 0$. Let $p_i := Z(s)(x_i) \in B_2(Z(s)(m_0)) \cap D_s$, such that $p_i \to p \in \partial B_2(Z(s)(m_0))$ where $p \notin D_s$. Assume that $r \leq 5/4$. Then $x_i \in B_{d_0}(m_0, 5/4 + C(n)\alpha_i)$. After taking a subsequence, if necessary, $x_i \to x \in B_{d_0}(m_0, 5/4 + C(n)\alpha_i)$, and consequently, $p_i = Z(s)(x_i) \to Z(s)(x) = p \in D_s$, as $i \to \infty$, which is a contradiction to the definition of $r > 0$, if $r \leq 5/4$. Hence $r \geq 5/4$, which implies $B_{5/4}(Z(s)(m_0)) \subseteq D_s$, as claimed (the second inclusion follows immediately from the Bi-Lipschitz property).

With the help of the previous theorem, we now show that it is possible to construct a solution to the $-\Delta$-Ricci-DeTurck flow coming out of $d_0 := (F_0), d_0$ using the harmonic map heat flow, if we assume that (c) is satisfied. First we show that by slightly mollifying the distance coordinates at time $t$, we obtain maps which satisfy (c).

**Lemma 3.9.** Let $(M, g(t))_{t \in [0,T]}$ be a solution to Ricci flow satisfying (a),(b),(c), and let $F_t : B_{d_0}(m_0, R) \to \mathbb{R}^n$ be given by

$$F_t(x) = (d_t(a_1, x), \ldots, d_t(a_n, x))$$

using the (fixed) points $a_1, \ldots, a_n$ from (c) (which define distance coordinates at $t = 0$). Then by slightly mollifying $F_t$ we obtain a map $\tilde{F}_t : B_{d_0}(m_0, R) \to \mathbb{R}^n$ which is smooth and satisfies $|\nabla \tilde{F}_t|_{g(t)} \leq c(n)$ as well as (c) (with $\varepsilon_0$ replaced by $2\varepsilon_0$), provided $t \leq T(\varepsilon_0, R)$.

**Proof.** As already noted, $F_t|_{B_{d_0}(x_0, 5)} : B_{d_0}(x_0, 50) \to \mathbb{R}^n$ satisfies (c) in view of the distance estimates (3.1), if $t \leq T(\varepsilon_0, R)$. Also, it is well known, that the Lipschitz norm of any map $F_t$ as defined above may be estimated by a constant depending only on $n$: $\sup_{x \neq y \in B_{d_0}(x_0, R)} \frac{|F_t(x) - F_t(y)|}{d_t(x, y)} \leq c(n)$, in view of the triangle inequality. Hence, by slightly mollifying the map $F_t$, we obtain a map $\tilde{F}_t : B_{d_0}(x_0, 50) \to \mathbb{R}^n$ which is smooth and satisfies (c) (with $\varepsilon_0$ replaced by $2\varepsilon_0$) and $|\nabla \tilde{F}_t|_{g(t)} \leq c(n)$.

**Theorem 3.10.** Let conditions (a),(b),(c) be satisfied and $\tilde{F}_t$ be the slightly mollified functions from Lemma 3.9 for any sequence of times $t_i > 0$ with $t_i \to 0$ as $i \to \infty$. Let $Z_{t_i} : B_{d_0}(x_0, 100) \times [t_i, \tilde{S} := \min(S(n, \alpha_0), T/2)] \to \mathbb{R}^n$ be the Dirichlet solution to the Ricci-harmonic map heat flow with boundary and initial values given by $Z_{t_i}(\cdot, s)|_{\partial B_{d_0}(x_0, 100)} = \tilde{F}_t(\cdot)$, for all $s \in [t_i, \tilde{S}]$ and $Z_{t_i}(\cdot, t_i) = \tilde{F}_t(\cdot)$. Then, after taking a subsequence in $i$, the maps $Z_{t_i} : B_{d_0}(x_0, 3/2) \to D_{s,i} := Z_{t_i}(s)(B_{d_0}(x_0, 3/2)) \subseteq \mathbb{R}^n$ are homeomorphisms for all $s \in [2t_i, \tilde{S}]$, with $\mathbb{B}(F_0(x_0)) \subseteq D_{s,i}$ and $(Z_{t_i})_* g \to \bar{g}$ smoothly, as $i \to \infty$ on compact subsets of $\mathbb{B}(F_0(x_0)) \times (0, \tilde{S}]$, where $\bar{g}(s) s \in (0, \tilde{S}]$ is a smooth family of metrics which solve the $\Delta$-Ricci-DeTurck flow and

$$1 - \alpha_0 \delta \leq \bar{g}(s) \leq (1 + \alpha_0) \delta$$

for all $s \in (0, \tilde{S})$, if $\varepsilon_0 = \varepsilon_0(\alpha_0, n) > 0$ from (a), and (c) are small enough. The metric $\bar{d}(t) := d(\bar{g}(t))$ satisfies, $\bar{d}(t) \to d_0 := (F_0), d_0$ uniformly on $\mathbb{B}(F_0(x_0))$ as $t \to 0$.

**Remark 3.11.** Examination of the proof of Theorem 3.10 shows that:

(i) We can remove condition (b) if we assume that the estimates (3.1) are satisfied.
(ii) If we remove condition (c) and replace it by the assumption: there exists a sequence of times \( t_i \to 0 \) with \( t_i \to \infty \), and maps \( \tilde{F}_{t_i} : B_{d_{\theta}}(x_0, 100) \to \mathbb{R}^n \) each of which satisfies (c), \( \sup_{t \in \mathbb{N}} |\tilde{F}_{t_i}(x_0)| < \infty \), and \( |\nabla \tilde{F}_{t_i}|_{g(t_i)} \leq c_1 \) then we can use these \( \tilde{F}_{t_i} \) in the above, instead of the slightly mollified distance functions, and the conclusions of the theorem still hold for \( s \leq \hat{S} := \min\{S(n, c_1, \alpha_0)\}, T/2 \) if the \( \varepsilon_0 = \varepsilon_0(n, c_1, \alpha_0) \) appearing in (a) and (c) is small enough. In this case, \( F_0 \) is the uniform \( C^0 \) limit of a subsequence of the \( F_{t_i} \) as \( i \to \infty \) and satisfies (3.4). The existence of such an \( F_0 \) is always guaranteed in this setting, as explained directly after the introduction of the condition (c).

**Proof.** Theorem 3.7 tells us that the maps \( Z_{t_i}(s) : B_{d_{\theta}}(x_0, 3/2) \to D_{s,i} \) are homeomorphisms with \( B_{5/4}(Z_{t_i}(s)(x_0)) \subseteq D_{s,i} \) for \( s \in [2t_i, \hat{S}] \). Hence

\[
\mathbb{B}_1(F_0(x_0)) \subseteq B_{5/4}(Z_{t_i}(s)(x_0)) \subseteq D_{s,i},
\]

for \( s \in [2t_i, \hat{S}] \), in view of (3.8). We define \( \tilde{g}_i(s) := (Z_{t_i})_* \tilde{g}(s) \) for \( s \in [2t_i, \hat{S}] \) on \( \mathbb{B}_2(F_0(x_0)) \). This is well defined in view of Theorem 3.7. Then \( \tilde{g}_i \) is a solution to the \( \delta \)-Ricci-DeTurck flow on \( \mathbb{B}_{3/2}(F_0(x_0)) \) (see [9, Chapter 6] for instance) and satisfies the metric inequalities (3.18) for all \( s \in (2t_i, \hat{S}) \) in view of Corollary 3.8. Using [16, Lemma 4.2] we see that

\[
|D^j \tilde{g}_i(s)| \leq \frac{c(j, n)}{(s - 2t_i)^{p(j)}}
\]

for all \( j \in \mathbb{N} \), for all \( s \in (2t_i, \hat{S}) \) on \( \mathbb{B}_1(F_0(0)) \). Taking a subsequence in \( i \) we obtain the desired solution \( \tilde{g}(s)_{s \in (0, \hat{S})} \) on \( \mathbb{B}_1(F_0(0)) \) with

\[
|D^j \tilde{g}(s)| \leq \frac{c(j, n)}{s^{p(j)}}.
\]

The \( Z_{t_i} \) all satisfy the estimates stated in the conclusions of Theorem 3.7, and so there is a uniform \( C^{1, \alpha} \) limit map \( Z : B_{d_{\theta}}(x_0, 2) \times (0, \hat{S}) \to \mathbb{R}^n \), in view of the Theorem of Arzelà-Ascoli. Furthermore, \( |Z(s, \cdot) - F_0(\cdot)| \leq c_1 \sqrt{s} \) for \( s \in (0, \hat{S}) \) in view of the estimate (3.8). Let \( v, w \in \mathbb{B}_1(0) \) be arbitrary, and \( x, y \) the corresponding points in \( B_{d_{\theta}}(x_0, 2) \) at time \( s \), that is the unique points \( x, y \) with \( Z_s(x) = v, Z_s(y) = w \). Then

\[
\tilde{d}_s(v, w) = d_s(x, y)
\]

\[
\leq d_0(x, y) + \varepsilon_0 \sqrt{s} = \tilde{d}_0(F_0(x), F_0(y)) + \varepsilon_0 \sqrt{s} \leq \tilde{d}_0(Z_s(x), Z_s(y)) + \beta(s) + \varepsilon_0 \sqrt{s} = \tilde{d}_0(v, w) + \beta(s) + \varepsilon_0 \sqrt{s},
\]

where \( \beta(s) \to 0 \) as \( s \to 0 \). Here, we have used the fact that \( \tilde{d}_0 := (F_0)_* d_0 \) is continuous, and hence uniformly continuous on \( \mathbb{B}_4(0) \), and that \( \sup_{B_{d_{\theta}}(x_0, 2)} |Z_s(\cdot) - F_0(\cdot)| \leq c_1 \sqrt{s} \) for all \( s \in (0, \hat{S}) \) in the above. The continuity of \( \tilde{d}_0 := (F_0)_* d_0 \) with respect to the norm in \( \mathbb{R}^n \) follows from the fact that \( d_0 \) is a metric, equivalent to the standard metric on \( \mathbb{R}^n \) in view of the property 3.4. Similarly, \( \tilde{d}_s(v, w) \geq \tilde{d}_0(v, w) - \beta(s) - \varepsilon_0 \sqrt{s} \), as required. 

\[\square\]

## 4 Ricci-harmonic map heat flow in the continuous setting

We now assume, in addition to the assumptions (a),(b) and (c) of the previous chapter, more regularity on \( d_0 \) and \( d_0 \). Namely, we assume that \( \tilde{d}_0 \) is generated by a continuous Riemannian metric \( \tilde{g}_0 \) on \( \mathbb{B}_1(0) \). This assumption will guarantee for all \( \varepsilon > 0 \) the existence of local maps defined on balls of radius \( r(\varepsilon) \), which are \( 1 + \varepsilon \) Bi-Lipschitz maps at \( t = 0 \). We explain this in the following Lemma.
**Lemma 4.1.** Let $F_0 : B_{d_0}(x_0, 1) \to \mathbb{R}^n$ be a $1 + \varepsilon_0$ Bi-Lipschitz homeomorphism with $F_0(x_0) = 0$, and assume that $d_0 = (F_0)_* d_0$ is generated on $\mathbb{S}_{\mathbb{S}}(4) \setminus 0$ by a continuous Riemannian metric $\hat{g}_0$, which is defined on $\mathbb{B}_{1}(0)$. Then for all $r > 0$ such that for all $p \in B_{d_0}(x_0, 1/8)$ there exists a linear transformation, $A = A(p)$, $A : \mathbb{B}_r(F_0(p)) \to \mathbb{R}^n$ with $|A - Id|_{C^0} \leq 2$ such that $F_0 := A \circ F_0$ satisfies

$$(1 - \varepsilon)|\hat{F}_0(y) - \hat{F}_0(q)| \leq d_0(y, q) \leq (1 + \varepsilon)|\hat{F}_0(y) - \hat{F}_0(q)|$$

for all $y, q \in B_{d_0}(p, r/2)$, and $\text{Vol}(B_{d_0}(p, s)) \in (|1 - \varepsilon|^n \omega_n s^n, (1 + \varepsilon)|\omega_n s^n|)$ for all $s \leq r$, for all such $p$.

**Proof.** The continuity of $\hat{g}_0$ means: for any $\varepsilon > 0$ and any $x \in \mathbb{B}_1(0)$ we can find an $r > 0$ and a linear transformation, $A : \mathbb{B}_r(x) \to A(B_r(x))$ with $|A - Id|_{C^0(B_r(x))} \leq c(n)\varepsilon$ so that $\hat{g}_0 := A_*(\hat{g}_0)$ satisfies $|\hat{g}_0 - \delta|_{C^0(B_r(x))} \leq \varepsilon$, and $\hat{g}_0(x) = \delta$. For the distance $\hat{d}_0 := A_* \hat{d}_0$ this means

$$(1 - \varepsilon)|z - w| \leq \hat{d}_0(z, w) \leq (1 + \varepsilon)|z - w|$$

for all $z, w \in \mathbb{B}_{r/2}(A(x))$. Returning to the original domain, we see that this means

$$(1 - \varepsilon)|\hat{F}_0(y) - \hat{F}_0(q)| \leq d_0(y, q) \leq (1 + \varepsilon)|\hat{F}_0(y) - \hat{F}_0(q)|$$

for all $y, q \in B_{d_0}(p, r/2)$ where $\hat{F}_0(p) = A(x)$, and $\hat{F}_0 = A \circ F_0$. This means in particular in view of the existence of the $(1 + \varepsilon)$ Bi-Lipschitz map $\hat{F}$, that $(1 + c(n)\varepsilon)^n \omega_n s^n \geq \text{Vol}(B_{d_0}(p, s)) \geq (1 - c(n)\varepsilon)^n \omega_n s^n$ for all $s \leq r$.

This implies that we can replace condition (a) by condition

$$(\hat{a}) \quad |\text{Rm}(\cdot, t)| \leq \varepsilon \frac{|t|}{t} \quad \text{on} \quad B_{d_0}(x_0, \frac{1}{100}) \quad \text{for all} \quad t \in (0, 1)$$

where $\varepsilon : [0, 1] \to [0, 1]$ is a continuous non-decreasing function with $\varepsilon(0) = 0$

as we show in the following Lemma.

**Lemma 4.2.** Assume $(M, g(t))_{t \in (0, T)}$ is a solution to Ricci flow satisfying (a) and (b), and assume $F_0$ is Bi-Lipschitz and that $\bar{d}_0 = (F_0)_* d_0$ satisfies the assumptions of Lemma 4.1. Then (a) holds.

**Proof.** Let $\sigma > 0$ be given, and assume that there are $t_i \to 0$ and $p_i \in B_{d_0}(x_0, 1/100)$ with $|\text{Rm}(p_i, t_i)| = \sigma/t_i$. We scale the solution $(g(t))_{t \in (0, T)}$ so that the time $t_i$ scales to time 1, i.e. we define a sequence of solutions to Ricci flow as follows: $g_i(s) := t_i^{-1}g(t_is)$ for $s \in (0, T/t_i)$.

The distance estimates (3.1), the estimates of Shi, (3.3), hold, and $\text{Vol}(B_{\bar{d}_0}(p_i, 1)) \geq \omega_n/2$ for all $s \in (0, 100)$ in view of Corollary 3.8. Hence, after taking a subsequence, we obtain a smooth solution $(\Omega, \ell(t), p_0)_{t \in (0, 10)}$ with $|\text{Rm}(p_0, 1)| = \sigma$, $\text{Ric} \geq 0$ everywhere, and $|\text{Rm}(\cdot, t)| \leq \varepsilon_0/t$ everywhere. Furthermore, writing $d_0(i) := t_i^{-1/2}d_0$, we see, in view of the distance estimates, that $d(\ell(t)) \to \bar{d}_0$ with $t \to 0$, where $(B_{\bar{d}_0}(p_i, 1))_{t \in (0, 10)}$ is in the Gromov-Hausdorff sense, as $i \to \infty$. But $(\Omega, \bar{d}_0)$ must be isometric to $(\mathbb{R}^n, \delta)$ since there are $1 + \varepsilon$ Bi-Lipschitz maps $F(i) : (B_{\bar{d}_0}(p_i, r\varepsilon)/\sqrt{t_i}), d_0(i)) \to \mathbb{R}^n$ for all $\varepsilon > 0$, in view of Lemma 4.1. Using a similar argument to the one used in [22] and [14], we see that the asymptotic volume ratio of $(\Omega, \ell(t))$ must be $\omega_n$, as we now explain. Take a sequence $\hat{R}_i \to \infty$ at time $s > 0$. Scale $\ell_i(s) := R_i^{-2}d(s, R_i^2)$. Since $(\Omega, R_i^{-1}\bar{d}_0)$ is isometric to $(\mathbb{R}^n, \delta)$ for all $i \in \mathbb{N}$, we must have $(B_{\bar{d}_0}(x_0, 1), \ell_i) \to (\mathbb{B}_{1}(0), \delta)$ in the Gromov-Hausdorff sense as $i \to \infty$, for $\ell_i := \ell_i(1/R_i^2) = (R_i^{-4}t_i(1))$ in view of the
scaled distance estimates (3.1). Hence, Vol(B_{\tilde{t}}(x_0,1),\tilde{t}) \to \omega_n as i \to \infty, in view of
the Theorem of Cheeger-Colding on volume convergence (Theorem 5.9 of [5]). But this
means that the asymptotic volume ratio of (\Omega, \ell(s), p_0) is \omega_n, and hence, (\Omega, \ell(s), p_0)
is isometric to (\mathbb{R}^n, \delta), in view of the Bishop-Gromov comparison principle (the case of
equality). This contradicts the fact that |\text{Rm}(p_0,1)| = \sigma.

Note that we obtain the better distance estimates in the setting of this Theorem,

\begin{equation}
|d_r + \varepsilon(t)\sqrt{t-r} - d_t| \geq d_t - \varepsilon(t)\sqrt{t-r} \text{ for all } t \in [r,1)
\end{equation}
on B_{d_0}(x_0, 20R) \subseteq B_{\tilde{g}(s)}(x_0, 50R)

for all \( r \geq 0 \), where \( \varepsilon \) is without loss of generality, the same function appearing in the
condition (\text{iii}).

**Theorem 4.3.** Assume \((M, g(t))_{t \in \mathbb{R}}\) is a solution to Ricci flow satisfying (a) and (b),
and assume \( F_0 \) is a \( 1 + \varepsilon_0 \) Bi-Lipschitz map and \( d_0 = (F_0)_* d_0 \) satisfies the
assumptions of Lemma 4.1. Then the solution \((B_{1/2}(0), \tilde{g}(s))_{s \in [0, \min(\tau, S(n, a_0, c_1))]}\) to \( \delta \)-Ricci-DeTurck
flow constructed in Theorem 3.10 satisfies \(|\tilde{g}(s) - \tilde{g_0}|_{C^0(\mathbb{B}_{1/20}(0))} \to 0\) as \( s \to 0 \).

**Proof.** Using Lemma 4.1 we see the following: for any \( \varepsilon > 0 \) and any \( p_0 \in B_{d_0}(x_0, 1/20) \)
we can find an \( r > 0 \) and a linear transformation \( A : \mathbb{R}^n \to \mathbb{R}^n \) with \( x = F_0(p_0) \), such that

\begin{equation}
(1 - \varepsilon)|\tilde{F}_0(y) - \tilde{F}_0(y)| \leq d_0(y, q) \leq (1 + \varepsilon)|\tilde{F}_0(y) - \tilde{F}_0(q)|
\end{equation}

for all \( y, q \in B_{d_0}(p_0, r) \), for \( \tilde{F}_0 = A \circ F_0 \). Now since \( A \) is a linear transformation with
\( |A - I|_{C^0} \leq 2 \), and (4.1) holds, we have \( |\tilde{F}_t - \tilde{F}_0|_{C^0(B_{d_0}(x_0, 1/2))} \leq \varepsilon(t)\sqrt{t} \text{ for } \tilde{F}_t = A \circ F_t \),
and hence

\begin{equation}
(1 - \varepsilon)d_t(v, q) - \varepsilon\sqrt{t} \leq |\tilde{F}_t(y) - \tilde{F}_t(q)| \leq (1 + \varepsilon)d_t(v, q) \varepsilon\sqrt{t}
\end{equation}
on \( B_{d_0}(p_0, r) \) for all \( t \leq T(\varepsilon) \), where we have also used (4.2). Let \( Z_{t_i} : B_{d_0}(x_0, 1/2) \times [t_i, S(n, a_0)] \to \mathbb{R}^n \) be the solutions to Ricci-harmonic map heat flow constructed
in Theorem 3.10. Then \( \tilde{Z}_{t_i} = A \circ Z_{t_i} \) is also a solution to Ricci-harmonic map heat flow.
Using the regularity theorem, Theorem 3.7, we see that we must have

\begin{equation}
(1 - \sigma(\varepsilon))d_0(z, w) \leq |\tilde{Z}_{t_i}(z, s) - \tilde{Z}_{t_i}(w, s)| \leq (1 + \sigma(\varepsilon))d_0(z, w)
\end{equation}

\begin{equation}
|\tilde{g}(s)\nabla Z_{t_i}(v)| \in ((1 - \sigma(\varepsilon))|v|_{\tilde{g}(s)}, (1 + \sigma(\varepsilon))|v|_{\tilde{g}(s)})
\end{equation}

for all \( z, w \in B_{d_0}(x, r/5) \) for all \( v \in T_zB_{d_0}(x, r/5) \), for all \( s \in (2t_i, S(n, \varepsilon)) \) where
(4.2) \( \varepsilon \) \( \to 0 \) as \( \varepsilon \) \( \to 0 \). Hence Corollary 3.8 tells us that the push forward \( \tilde{g}_i(s) := (\tilde{Z}_{t_i})*(g(s))_{s \in (2t_i, S(n, \varepsilon))} \) satisfies \( |\tilde{g}_i(s) - \delta|_{C^0(\mathbb{B}_{1/20}(\varepsilon))} \leq \sigma(\varepsilon) \). Transforming back with \( A^{-1} \)
we see that this means \( |\tilde{g}_i(s) - \tilde{g}_0|_{C^0(\mathbb{B}_{1/20}(s))} \leq \sigma(\varepsilon) \) for all \( s \in (2t_i, S(n, \varepsilon)) \), and hence
(4.2) \( \varepsilon \) \( \to 0 \) as \( \varepsilon \) \( \to 0 \) for all \( s \in (0, S(n, \varepsilon)) \). As \( x \in B_{1/20}(0) \) was arbitrary, we see
by letting \( \varepsilon \) \( \to 0 \), that \( |\tilde{g}(s) - \tilde{g}_0|_{C^0(\mathbb{B}_{1/20}(0))} \to 0 \) as \( s \to 0 \), as required.

The estimates of the previous theorem allow us to give a proof of the first main theorem:

**Proof of Theorem 1.4.** Let \( a_1, \ldots, a_n \in B_{d_0}(x_0, r) \) be as in the statement of Theorem
1.4. That is, \( F_0(s) := (d_0(a_1), \ldots, d_0(a_n)) \) is \( 1 + \varepsilon_0 \) Bi-Lipschitz on \( B_{d_0}(x_0, 5r) \)
for some \( \tilde{r} \leq r/5 \). The metric \( d_0(\tilde{x}, \tilde{y}) := d_0((F_0)^{-1}(\tilde{x}), (F_0)^{-1}(\tilde{y})) \) defined on \( B_\tau(F_0(x_0)) \)
is generated by a continuous (with respect to the standard topology on \( \mathbb{R}^n \)) Riemannian
metric \( \tilde{g}_0 \) defined on \( B_\tau(F_0(x_0)) \subseteq \mathbb{R}^n \). By scaling everything once, that is \( \tilde{g}(t) = \tilde{r}^{-1}\tilde{g}(t) \)
(4.2) \( \varepsilon \) \( \to 0 \) as \( \varepsilon \) \( \to 0 \) for all \( s \in (0, S(n, \varepsilon)) \). As \( x \in B_{1/20}(0) \) was arbitrary, we see
by letting \( \varepsilon \) \( \to 0 \), that \( |\tilde{g}(s) - \tilde{g}_0|_{C^0(\mathbb{B}_{1/20}(0))} \to 0 \) as \( s \to 0 \), as required.

The conclusions of that theorem, when scaled back, imply the conclusions of Theorem 1.4.
5 Existence and estimates for the Ricci-DeTurck Flow with $C^0$ boundary data

In this section we construct solutions $\ell$ to the Dirichlet problem for the $\delta$-Ricci-DeTurck flow on a Euclidean ball, which are smooth up to the boundary at time zero, and have $C^0$ parabolic boundary values. These solutions are constructed as a limit of smooth solutions $\ell_n$ whose parabolic boundary values converge to those of $\ell$.

Recall that the $\delta$-Ricci-DeTurck flow equation for a smooth family of metrics $\ell$ is given by (see [9, p. 15] and/or [15, Lemma 2.1])

$$
\frac{\partial}{\partial t} \ell_{ij} = \ell^{ab} \partial_a \partial_b \ell_{ij} + \frac{1}{2} \ell^{ab} \rho_{ab}(\partial_t \ell_{ij} + 2\partial_a \ell_{jp} \partial_q \ell_{ib} - 2\partial_a \ell_{jq} \partial_p \ell_{ib} - 2\partial_j \ell_{pa} \partial_b \ell_{iq} - 2\partial_j \ell_{qa} \partial_p \ell_{ib})
$$

(5.1)

First we prove an estimate about the closeness of smooth solutions to $\delta$ in the $C^0$ norm, assuming $C^0$ closeness on the spatial boundary and a bound on the $C^2$ norm at time zero.

**Lemma 5.1.** Let $\ell$ be a $H^{2+\alpha, 1+\alpha}(B_R(0) \times [0, T]) \cap C^0(B_R(0) \times [0, T])$ solution to the $\delta$-Ricci-DeTurck flow such that $\langle \cdot, 0 \rangle = \ell_0$, $\|\ell(\cdot, t) - \ell_0\|_{L^\infty(B_R(0))} \leq \beta$ for $t \in [0, T]$ and $\|\ell_0 - \delta\|_{C^2(B_R(0))} \leq \varepsilon(n)$, where $\beta \leq \varepsilon(n)$. Then,

$$
\|\ell(\cdot, t) - \delta\|_{L^\infty(B_R(0))} \leq \varepsilon(n) \varepsilon(n), \text{ for all } t \in [0, T],
$$

$$
\|\ell(\cdot, t) - \ell_0\|_{L^\infty(B_R(0))} \leq \beta + \varepsilon(n)t, \text{ for all } t \in [0, T].
$$

**Proof.** We will denote in the following $|\cdot|$ all norms induced by the metric $\delta$. From smoothness and the boundary conditions, we know that $\ell$ is a smooth invertible metric for a small time interval $[0, \tau]$ with $|\ell - \delta|^2 \leq \varepsilon(n)$ during this time interval. By (5.1) we can compute

$$
\frac{\partial}{\partial t} |\ell - \delta|^2 \leq 2(\ell - \delta)^i j \ell^{ab} \partial_a \partial_b (\ell - \delta)_{ij} + c(n)|\ell - \delta|D\ell|^2
$$

$$
\leq \ell^{ab} \partial_a \partial_b |\ell - \delta|^2 - 2(D\ell, D\ell)_{\ell} + c(n)|\ell - \delta|D\ell|^2
$$

$$
\leq \ell^{ab} \partial_a \partial_b |\ell - \delta|^2 - |D\ell|^2 + c(n)\varepsilon(n)|\ell|D\ell|^2
$$

$$
\leq \ell^{ab} \partial_a \partial_b |\ell - \delta|^2
$$

for all $t \in [0, \tau]$ if $\varepsilon(n)$ is sufficiently small. Thus by the maximum principle $|\ell - \delta|^2 \leq \varepsilon(n)$ remains true as long as this is true on the boundary. Thus we can take $\tau = T$.

We perform a similar calculation for $|\ell - \ell_0|^2$. By the above estimate, we can freely use that $\frac{1}{2} \delta \leq \ell \leq 2\delta$, for all $t \in [0, T]$. We also use, that $|D\ell_0| + |D^2\ell_0| \leq \varepsilon(n)$ due to the assumptions.

$$
\frac{\partial}{\partial t} |\ell - \ell_0|^2 \leq 2(\ell - \ell_0)^i j \ell^{ab} \partial_a \partial_b (\ell - \ell_0)_{ij} + c(n)|\ell - \ell_0||D\ell|^2
$$

$$
= 2(\ell - \ell_0)^i j \ell^{ab} \partial_a \partial_b (\ell - \ell_0)_{ij} + 2(\ell - \ell_0)^i j \ell^{ab} \partial_a \partial_b (\ell_0)_{ij} + c(n)\varepsilon(n)|D\ell|^2
$$

$$
\leq \ell^{ab} \partial_a \partial_b |\ell - \ell_0|^2 - |D(\ell - \ell_0)|^2 + \varepsilon(n) + c(n)\varepsilon(n)|D\ell|^2
$$

$$
\leq \ell^{ab} \partial_a \partial_b |\ell - \ell_0|^2 - \frac{1}{2}|D\ell|^2 + c(n)\varepsilon(n)
$$

if $\varepsilon(n)$ is small enough. Hence,

$$
\frac{\partial}{\partial t} (|\ell - \ell_0|^2 - c(n)\varepsilon(n)t) \leq \ell^{ab} \partial_a \partial_b (|\ell - \ell_0|^2 - \varepsilon(n)c(n)t),
$$

25
and, consequently,

\[ |\ell - \ell_0|^2 \leq \beta + \epsilon(n)c(n)t \]

for \( t \leq T \), in view of the fact that \( |\ell - \ell_0|^2 \leq \beta + \epsilon(n)c(n)t \) on \( \partial B_R(0) \times \{t\} \) for \( t \leq T \), and \( |\ell - \ell_0|^2 = 0 \) for \( t = 0 \) on \( \partial B_R(0) \).

We now consider the problem of constructing solutions to the Dirichlet problem for the \( \delta\)-Ricci-DeTurck flow, with boundary data \( h \) given on the parabolic boundary \( P \) of \( B_R(0) \times [0,T] \). We will assume that the boundary data is given as the restriction of \( h \in C^\infty(\overline{B_R(0)} \times [0,T]) \) which is \( \epsilon(n) \) close in the \( C^0 \) norm to \( \delta \) on \( \overline{B_R(0)} \times [0,T] \). We now explain how to construct a solution to this Dirichlet problem if the compatibility conditions of the first type are satisfied.

**Definition 5.2.** Let \( h \in C^\infty(\overline{B_R(0)} \times [0,T]) \) be a smooth family of Riemannian metrics on \( \overline{B_R(0)} \). We say \( h \) satisfies \( \text{Comp}_k \), or \( h \) satisfies the compatibility conditions of the \( k \)-th order, if

\[
\frac{\partial}{\partial t} h(x,0) = L_l(h(x,0)) \quad \text{for } l = 1, \ldots, k
\]

for all \( x \in \partial B_R(0) \), where \( L_l \) is the differential operator of order \( 2l \) which one obtains by differentiating (5.1) \( l \)-times with respect to \( t \), and inserting iteratively the already obtained formulas for the \( m \)-th derivative in time for \( m = 1, \ldots, l - 1 \). For example

\[
L_1(h)_{ij} = h^{ab} \partial_a \partial_b h_{ij} + \frac{1}{2} h^{ab} h^{pq} (\partial_i h_{pa} \partial_j h_{qb} + 2 \partial_i h_{pa} \partial_j h_{pb} \partial_q h_{jb} - 2 \partial_j h_{pa} \partial_i h_{qb} + 2 \partial_j h_{pa} \partial_b h_{iq} - 2 \partial_j h_{pa} \partial_b h_{iq} - 2 \partial_j h_{pa} \partial_b h_{jq}).
\]

and

\[
L_2(h)_{ij} = -h^{ab} h^{pq} L_1(h)_{im} \partial_a \partial_b h_{jq} + h^{ab} \partial_a \partial_b L_1(h)_{ij} + \ldots.
\]

We are now prepared to derive the following existence result.

**Theorem 5.3.** Let \( h \in C^\infty(\overline{B_R(0)} \times [0,T]) \), \( T \leq 1 \) and assume \( |h(t) - \delta| \leq \epsilon(n) \) for all \( t \in [0,T] \), and \( |h_0 - \delta|_{C^2(\overline{B_R(0)})} \leq \epsilon(n) \) and \( h \) satisfies \( \text{Comp}_1 \). Then there exists a solution \( \ell \in C^\infty(\overline{B_R(0)} \times [0,T]) \cap H^{2+\alpha,1+\alpha}(\overline{B_R(0)} \times [0,T]) \) to the \( \delta\)-Ricci-DeTurck flow such that \( \ell|_P = h|_P \) on the parabolic boundary \( P \). Furthermore,

\[
|\ell(x,s) - h(x,s)| \leq C(n,R,\alpha) K(s^{-1}, \|h\|_{C^2(\overline{B_R(0)} \times [s,T])}) (R - |x|)^\alpha
\]

for all \( s \in (0,T) \), \( x \in B_R(0) \), for any given \( \alpha \in (0,1) \), where \( K : \mathbb{R}^+ \times \mathbb{R}^+ \to \mathbb{R} \) is a monotone increasing function with respect to each of its argument.

**Proof.** The first part of the proof follows closely the proof of the existence result given in [15, Chapter 3]. Assume for the moment that a solution \( \ell \) exists, and set \( S := \ell - h \). Then \( S = 0 \) on the parabolic boundary, and the evolution equation for \( S \) is:

\[
\frac{\partial}{\partial t} S = \frac{\partial}{\partial t} \ell - \frac{\partial}{\partial t} h
\]

\[
= \ell^{kl} \partial_k \partial_l \ell + \ell^{-1} \ast \ell^{-1} \ast D \ell \ast D \ell - \frac{\partial}{\partial t} h
\]

\[
= \ell^{kl} \partial_k \partial_l S + \ell^{kl} \partial_k \partial_l h + \ell^{-1} \ast \ell^{-1} \ast D \ell \ast D \ell - \frac{\partial}{\partial t} h
\]

\[
= a^{kl}(S(x,t),x,t) \partial_k \partial_l h + b(S(x,t),DS(x,t),x,t)
\]

where \( a^{kl}(z,t) := (h(x,t) + z)^{kl} \) is the inverse of \( h(x,t) + z \) (which is well defined as long as \( h(x,t) + z \) is invertible) and \( b \) is defined similarly

\[
b(z,p,x,t) := (h(x,t) + z)^{-1} \ast (h(x,t) + z)^{-1} \ast (p + Dh(x,t)) \ast (p + Dh(x,t))
\]

\[
- \frac{\partial}{\partial t} h(x,t) + (h(x,t) + z)^{kl} \partial_k \partial_l h(x,t),
\]
where this again is well defined as long as \( z + h(x, t) \) is invertible. Note that if \( |z| \) is sufficiently small then \( z + h(x, t) \) is invertible. Since \( \ell \) is assumed to be a solution, and \( \ell = h \) on \( P \), where \( |h - \ell|_{C^\infty} \leq \varepsilon(n) \), and \( |f_0 - \ell|_{C^2} = |h_0 - \ell|_{C^2} \leq \varepsilon(n) \), we obtain that \( |(f(\cdot, t) - \ell_0)|_{C^0(\mathbb{R}^n(0))} \leq \varepsilon(n) \) for all \( t \in [0, T] \) in view of Lemma 5.1, and hence \( |S(t)|_{C^0(\mathbb{R}^n(0))} \leq \varepsilon(n) \) for all \( t \in [0, T] \).

We divide \( S \) by a small number \( \delta(n) > 0 \), and call it \( \tilde{S} \), i.e.

\[
\tilde{S} := \delta^{-1}(n)S,
\]

where we assume \( \varepsilon(n) \ll \delta(n) \), for example we choose \( \varepsilon(n) = \delta^3(n) \). Hence \( |\tilde{S}(t)|_{C^0(\mathbb{R}^n(0))} \) is still small for all times \( t \in [0, T] \), and we have \( |\tilde{S}| \leq \sqrt{\varepsilon(n)} \). The evolution of \( \tilde{S} \) may be written as

\[
(5.3) \quad \frac{\partial}{\partial t} \tilde{S}(x, t) = \tilde{a}^{ij}(\tilde{S}(x, t), x, t)\partial_i\partial_j \tilde{S} + \tilde{b}(\tilde{S}(x, t), D\tilde{S}(x, t), x, t),
\]

where \( \tilde{a}^{ij}(\tilde{z}, x, t) = (h(x, t) + \delta \tilde{z})^{ij} \) is the inverse of \( h(x, t) + \delta \tilde{z} \), and

\[
\tilde{b}(\tilde{z}, \tilde{p}, x, t) := \delta h(x, t) + \delta \tilde{z})^{-1} * (h(x, t) + \delta \tilde{z})^{-1} * (\tilde{p} + \delta^{-1} Dh(x, t)) * (\tilde{p} + \delta^{-1} Dh(x, t)) - \frac{1}{\delta} \frac{\partial}{\partial t} h(x, t) + (h(x, t) + \delta \tilde{z})^{ij} \delta^{-1} \partial_k \partial_i h(x, t),
\]

In the setting we are examining, we see, defining \( C_1(h, n) := 10|h|_{C^2,1(\mathbb{R}^n(0) \times [0, 1])} \), that

\[
|\tilde{b}(\tilde{z}, \tilde{p}, x, t)| \leq \delta(n)(C_1(h)|\delta^{-2}(n) + |\tilde{p}|^2)
\]

for the \( \tilde{z} = \tilde{S}(x, t) \) we are considering, since \( |\tilde{S}|_{C^0} \leq \sqrt{\varepsilon(n)} \). As long as \( |\tilde{S}(\cdot, t)|_{C^0} \leq \sqrt{\varepsilon(n)} \) for \( t \in [0, 1] \), we have \( 2\delta \tilde{z} \geq \tilde{a}^{ij}(\tilde{S}(x, t), x, t) \geq \frac{1}{2} \delta \tilde{z} \) and \( |\tilde{b}(\tilde{S}(x, t), D\tilde{S}(x, t), x, t)| \leq \delta(n)(C_1(h)|\delta^{-2}(n) + |D\tilde{S}(x, t)|^2) \) in this case. We write this as \( |\tilde{b}(\tilde{S}(x, t), D\tilde{S}(x, t), x, t)| \leq Q(|D\tilde{S}(x, t)|, |\tilde{S}(x, t)|)(1 + |D\tilde{S}(x, t)|^2) \) where \( Q(\rho, 0) = 2\delta(n)Q(0) + 2\delta(n)(1 - \rho(0))(1 + 100C_1(h)|\delta^{-2}(n)) \) where \( \rho \) is a smooth cut-off function with \( \rho(0) = 0 \) for all \( r \leq 100C_1(h)|\delta^{-2}(n) \) and \( \rho(1) = 1 \) for all \( r \geq 200C_1(h)|\delta^{-2}(n) \).

Hence, from the general theory of non-linear parabolic equations of second order, see for example [12, Theorem 7.1, Chapter VII], we see that the equation (5.3) with zero parabolic boundary values has a solution \( \ell \in H^{2+r,1+n/2}(\mathbb{R}) \) for all times \( t \in [0, T] \), as long as \( |\tilde{S}(\cdot, t)|_{C^0} \leq \sqrt{\varepsilon(n)} \) for \( t \in [0, T] \).

Writing \( \ell = h + \delta \tilde{S} \) and using the arguments above, we see that this will not be violated for \( t \in [0, T] \), \( T \leq 1 \), and that \( \ell \) solves the \( \delta \)-Ricci-DeTurck equation and \( \ell|_P = h|_P \).

This proves the existence of the solution. It remains to prove the Hölder boundary estimate, (5.2). For ease of reading, we assume \( R = 1 \).

Consider for \( q, r \) fixed the function \( \varphi_- := ((\ell q - h q) + \lambda|\ell - h|^2) \), \( \varphi_- := -((\ell q - h q) + \lambda|\ell - h|^2) \), where \( \lambda = \lambda(n) \) is a sufficiently large constant such that \( \partial_k \varphi_- - \ell^{ij} \partial_i \partial_j \varphi_- \leq C(n, h)|\varphi_-|_{[s, T]} \), for \( 0 < \alpha < 1 \), where \( \varphi(x) = (1 - |x|) \), and \( \eta \) is a non-negative cut off function in time with \( \eta(t) = 0 \) for \( 0 \leq t \leq s \) and \( \eta(t) = 1 \) for \( 3s/2 \leq t \) such that \( |\eta'(t)|^2 \leq cs^{-2}\eta(t) \) for some
positive constant c. A direct calculation shows

\[ \ell^i \partial_i \partial_j \rho^\alpha = \ell^i \partial_i \left( \frac{\alpha \rho^{\alpha-1}(x_j)}{|x|} \right) \]

\[ = (\alpha - 1) \alpha \rho^{\alpha-2} \ell^i x^i x_j - \frac{\alpha \rho^{\alpha-1} \ell^i}{|x|} + \alpha \rho^{\alpha-3} \]

\[ \leq - \frac{\alpha(n-1)}{2} \rho^{\alpha-2} = \frac{\alpha |1 - \alpha|}{2} \rho^{\alpha-2} \]

\[- \leq -1 - \frac{\alpha |1 - \alpha|}{2} \rho^{\alpha-2} \]

for all \( |x| \in (1 - \delta_0(\alpha, n), 1) \). We note that \( \psi_{\pm} \) cannot be zero very close to \( \partial \mathbb{B}_R(0) \), since \( |D(\ell - h)| \) is bounded by some constant according to \([12, \text{Theorem 7.1, Chapter VII}]\), and \( \ell - h = 0 \) on \( \partial \mathbb{B}_R(0) \). Also, by choosing \( M = M(\alpha) \) large enough, we have without loss of generality, that \( \psi_{\pm}(x, \cdot) < 0 \) for \( |x| = 1 - \delta_0(\alpha, n) \). That is \( \psi_{\pm}(x, \cdot) < 0 \) for \( |x| = 1 - \delta_0(\alpha, n) \) and \( |x| = 1 - \varepsilon \) for all \( \varepsilon > 0 \), \( \varepsilon \ll \delta_0 \) sufficiently small. We also observe that \( \psi_{\pm}(\cdot, s) < 0 \) for all \( |x| \in [1 - \delta_0(\alpha, n), 1 - \varepsilon] \) for all \( t \leq s \). Hence if \( \psi_{\pm}(x, t) = 0 \) for some \( |x| \in [1 - \delta_0(\alpha, n), 1 - \varepsilon] \) for some \( t \geq s \), there must be a first time \( t \) for which this happens and this must happen at an interior point \( x \) of \( \mathbb{B}_1 - \varepsilon(0) \backslash \overline{\mathbb{B}_{1-\delta_0}(0)} \). We calculate at such a point \((x, t)\),

\[ \frac{\partial}{\partial t} \psi_{\pm} \leq \ell^i \partial_i \psi_{\pm} + (C(n, h) + \eta' \varphi_{\pm} - 2M - M\alpha |1 - \alpha| \rho^{\alpha-2} \]

\[ \leq \ell^i \partial_i \psi_{\pm} + (C(n, h) + \frac{c(n)}{s}) \eta \varphi_{\pm} - 2M - M\alpha |1 - \alpha| \rho^{\alpha-2} \]

\[ \leq \ell^i \partial_i \psi_{\pm} + (C(n, h) - 2M + \frac{c(n)}{s} (2M)^2 \rho^{\alpha-2} - M\alpha |1 - \alpha| \rho^{\alpha-2} < 0 , \]

for \( |x| \in (1 - \delta_0, 1 - \varepsilon) \), if \( M > C(C(n, h), \alpha, s) \) also holds. A contradiction. This leads to the desired estimate close to the boundary. For points \( |x| \in (0, 1 - \delta_0) \) the estimate follows immediately from the fact that \( \ell \) and \( \mathbb{h} \) are \( \varepsilon(n) \) close to \( \delta \) and hence bounded, and \( (1 - |x|)^\alpha \geq \delta_0^\alpha \) for \( |x| \in (0, 1 - \delta_0) \). \( \square \)

If we assume that higher order compatibility conditions are satisfied, then we obtain more regularity of the solution.

**Theorem 5.4.** Let \( h \in C^\infty(\overline{\mathbb{B}_R(0)} \times [0, T]) \), \( T \leq 1 \) and assume \( |h(\cdot, t) - \delta| \leq \varepsilon(n) \) for all \( t \in [0, T] \), and \( |h_0 - \delta|_{C^{2,\alpha}(\mathbb{B}_R(0))} \leq \varepsilon(n) \) and \( h \) satisfies Comp. Then there exists a solution \( \ell \in C^0(\overline{\mathbb{B}_R(0)} \times [0, T]) \cap H^{k+\alpha, \overline{\| \cdot \|_{C^2,\alpha(\mathbb{B}_R(0) \times [s,T])}}}(\mathbb{B}_R(0) \times [0, T]) \) to the \( \delta \)-Ricci-DeTurck flow and the values given by \( h \) on the parabolic boundary, that is \( \ell |_{\mathbb{B}_R(0)} = h |_{\mathbb{B}_R(0)} \). Furthermore,

\[ |f(x, s) - h(x, s)| \leq C(n, R, \alpha)K(s^{-1}, \|h\|_{C^2,\alpha(\mathbb{B}_R(0) \times [s,T])})(R - |x|)^\alpha \]

for all \( s \in (0, T) \), \( x \in \mathbb{B}_R(0) \), for any given \( \alpha \in (0, 1) \), where \( K : \mathbb{R}^+ \times \mathbb{R}^+ \rightarrow \mathbb{R} \) is a monotone increasing function with respect to each of its argument.

**Proof.** The proof is the same, except at the step where we used \([12, \text{Theorem 7.1, Chapter VII}]\) to obtain a solution in \( H^{2+\alpha, 1+\frac{3}{2}} \), we now obtain a solution \( \ell \in H^{k+\alpha, k+1+\frac{3}{2}} \), in view of the fact that the \( S \) satisfies the compatibility condition of \( k \)-th order. \( \square \)

We now explain how to construct a \( \delta \)-Ricci-DeTurck flow for parabolic boundary values given by \( h \) which do not necessarily satisfy compatibility conditions of the first order, but are smooth at \( t = 0 \), smooth on \( \overline{\mathbb{B}_R(0)} \times (0, T) \) and continuous on \( \mathbb{B}_R(0) \times [0, T] \). This is done by modifying the boundary values, so that the first (or higher order compatibility conditions) are satisfied, and then taking a limit. 28
Theorem 5.5. Let $h \in C^\infty(\overline{B}_R(0) \times (0,T]) \cap C^0(\overline{B}_R(0) \times [0,T])$, $T \leq 1$, such that $h(\cdot,0) \in C^\infty(\overline{B}_R(0))$ and assume $|h(\cdot,t) - \delta| \leq \varepsilon(n)$ for all $t \in [0,T]$, and $|h_0 - \delta|_{C^2(\overline{B}_R(0))} \leq \varepsilon(n)$. Then there exists a solution $\ell \in C^0(\overline{B}_R(0) \times [0,T]) \cap C^\infty(\overline{B}_R(0) \times (0,T])$ to the $\delta$-Ricci-DeTurck flow and the values given by $h$ on the parabolic boundary $P$, that is $\ell|_P = h|_P$. Furthermore $|\ell|_{C^\infty(\overline{B}_R(0) \times [0,T])} \leq c(R',|h_0|_{C^\infty(\overline{B}_R(0))})$ for any $R' < R$ and any $s \in \mathbb{N}$.

Proof. Let $\xi : \mathbb{R} \to \mathbb{R}$ be a monotone non-increasing smooth function whose image is contained in $[0,1]$, such so that $\xi$ is equal to 1 on $[0,\frac{\varepsilon}{4}]$ and equal to 0 on $[1,\infty)$.

For each $\tau \in [0,1]$, let $h(\tau)$ be the smooth Riemannian metric defined as follows:

$$h(\tau)|_{\overline{B}_R(0) \times [\tau,T]} := h,$$

$$h(\tau)|_{\overline{B}_R(0) \times [0,\tau]} := \xi \left(\frac{\tau}{\varepsilon}\right) h_0(x) + \left(1 - \xi \left(\frac{\tau}{\varepsilon}\right)\right) h(x,t) + t \xi \left(\frac{\tau}{\varepsilon}\right) L_1(h_0)(x),$$

where $L_1$ is as in Definition 5.2, i.e.

$$L_1(h_0) := (h_0)^{ij} \partial_i \partial_j h_0 + (h_0)^{-1} \ast (h_0)^{-1} \ast D h_0 \ast D h_0.$$

Using $|h(t) - \delta| \leq \varepsilon(n)$ and $|L_1(h_0)| \leq C(n)$, we see that $|h(\tau)(t) - \delta| \leq 2\varepsilon(n) = \varepsilon(n)$ if $\tau$ is sufficiently small. Furthermore, $\frac{\partial}{\partial t} h(\tau)(x,0) = L_1(h_0)(x,0) = L_1(h(\tau))(x,0)$, that is $h(\tau)$ satisfies $\operatorname{Comp}_1$. We may hence use Theorem 5.3 to obtain a solution $\ell(\tau) \in H^{2+\alpha,1}(\overline{B}_R(0) \times [0,T])$ to the $\delta$-Ricci-DeTurck flow with parabolic boundary data given by $h(\tau)$. From the definition of $h(\tau)$, we have on $\partial B_R(0)$ that

$$|\ell(\tau)(\cdot,t) - h_0)| = |h(\tau)(\cdot,t) - h_0(\cdot)| \leq |h(\cdot,t) - h_0(\cdot)| + t \varepsilon(n) \leq C(t,h,n)$$

where $C(t,h,n)$ is a function (independent of $\tau$) such that $C(t,h,n) \leq \varepsilon(n)$ and $C(t,h,n) \to 0$ as $t \to 0$ for $n$ and $h$ fixed. Lemma 5.1 then tells us that $|\ell(\tau)(\cdot,t) - h_0| \leq \tilde{C}(t,h,n)$ on all of $\overline{B}_R(0)$ for all $t \in [0,T]$ where $\tilde{C}(t,h,n) \to 0$ as $t \to 0$ for $n$ and $h$ fixed (independent of $\tau$). The boundary Hölder estimate (5.2) of Theorem 5.3, and the smoothness of $h = h(\tau)$ for $t \geq 2\tau$, also tells us, for any $\varepsilon > 0$ such that $\varepsilon(\cdot,2\tau,s)$, there is $a > 0$ such that $|\ell(\tau)(x,t) - h(x,t)| \leq \varepsilon$ for all $x \in \partial B_R(0) \setminus B_{R-a}(0)$ for all $t \in [s,T]$, where $\sigma = \sigma(\varepsilon,s,h,n)$ is independent of $\tau$ if $\tau$ is sufficiently small. These two $C^0$ estimates implies that we have the following uniform (in $\tau$) $C^0$ bounds

$$|\ell(\tau)(\cdot,t) - h_0| \leq C(\varepsilon,h,n)$$

where $C(\varepsilon,h,n) \to 0$ as $\varepsilon \to 0$ (for fixed $h$ and $n$) and $P = \overline{B}_R(0) \setminus \overline{B}_{R-\varepsilon}(0) \times [0,1] \cup \overline{B}_R(0) \times [0,\varepsilon]$ for $\tau$ sufficiently small.

If we define $h(\tau) := (\xi(t/\tau)h_0)(x) + (1 - \xi(t/\tau)) h(x,t) + t(\xi(t/\tau)L_1(h_0)(x) + \frac{\varepsilon}{4} L_2(h_0))$, then we still have $|\ell(\tau)(\cdot,t) - h_0| \leq C(t,h,n)$ and $|\ell(\tau)(\cdot,t) - h_0| \leq \tilde{C}(t,h,n)$ on all of $\overline{B}_R(0)$ for all $t \in [0,T]$ where $\tilde{C}(t,h,n) \to 0$ independent of $\tau$ if $\tau$ is sufficiently small, in view of Lemma 5.1. The Hölder estimates still hold: for any $\varepsilon > 0$ and $s > 0$,

$$|\ell(\tau)(x,t) - h(x,t)| \leq \varepsilon$$

for all $x \in \overline{B}_R(0) \setminus \overline{B}_{R-a}(0)$ for all $t \in [s,T]$, where $\sigma = \sigma(\varepsilon,s,h,n)$ is independent of $\tau$ if $\tau$ is sufficiently small. Thus, the uniform $C^0$ estimates hold $|\ell(\tau)(x,t) - h_0| \leq C(\varepsilon,h,n)$ where $C(\varepsilon,h,n) \to 0$ as $\varepsilon \to 0$ (for fixed $h$ and $n$) still hold. Continuing in this way, we can assume $\ell(\tau) \in H^{k+\alpha,1}(\overline{B}_R(0) \times [0,T])$ and $|\ell(\tau)(\cdot) - \delta|_{C^0(\overline{B}_R(0) \times [0,T])} \leq \varepsilon(n)$ and $\ell(\tau)(\cdot,0) = h_0$, and the uniform $C^0$ estimates hold $|\ell(\tau)(\cdot,t) - h_0| \leq C(\varepsilon)$ where $C(\varepsilon) \to 0$ as $\varepsilon \to 0$ and $P = \overline{B}_R(0) \setminus \overline{B}_{R-a}(0) \times [0,T] \cup \overline{B}_R(0) \times [0,\varepsilon]$. The proof of the interior estimates, explained in [16, Section 4] can be used here to show that $|\ell(\tau)(x,t) - C_{\varepsilon}(\overline{B}_R(0))| \leq C(\varepsilon - R')$ for all $t \in [0,T]$, for all $R' < R$. By Arzelà-Ascoli’s Theorem, one is able to take a limit: up to a subsequence, we obtain a solution $\ell$, with the desired properties.

\[\square\]
6 An $L^2$ estimate for the Ricci-DeTurck flow, and applications thereof

In this chapter we prove a lemma which estimates the change in the $L^2$ distance between two solutions to the $\delta$-Ricci-DeTurck flow. Lemma 6.1 considers smooth solutions which are $\varepsilon(n)$ close in the $L^2$ norm at time zero and agree at all times on the boundary. If we weigh the $L^2$ distance at time $t$ of two smooth solutions appropriately, then this quantity is decreasing. The weight has the property that it is uniformly bounded between 1 and 2, and hence the unweighted $L^2$ distance at time $t$ of the two solutions can only increase by a factor of at most 2. With the help of the $L^2$-Lemma, we prove some uniqueness theorems for solutions to the $\delta$-Ricci-DeTurck flow.

**Lemma 6.1 (L$^2$-Lemma).** Let $g_1, g_2$ be two smooth solutions to the $\delta$-Ricci-DeTurck flow on $\mathbb{B}_R(0) \times (S, T)$ such that $g_l \in C^\infty(\mathbb{B}_R(0) \times (S, T))$ for $l = 1, 2$ and that $g_1 = g_2$ on $\partial \mathbb{B}_R(0) \times (S, T)$. Let $h := g_1 - g_2$ and

$$v := |h|^2 \left( 1 + \lambda(|g_1 - \delta|^2 + |g_2 - \delta|^2) \right).$$

We assume that $\|g_1(\cdot, t) - \delta\| + \|g_1(\cdot, t) - \delta\| \leq \varepsilon(n)$ for all $t \in (S, T)$. Then for $\lambda \geq \hat{\lambda}(n)$ and $\varepsilon \leq \hat{\varepsilon}(n)$, where $\hat{\varepsilon}(n)$ is sufficiently small and $\hat{\lambda}(n)$ sufficiently large, it holds for $t \in (S, T)$ that

$$\frac{\partial}{\partial t} \int_{\mathbb{B}_R(0)} v \, dx \leq 0.$$

Before proving the Lemma, we state and prove two corollaries of this estimate.

**Corollary 6.2.** Let $g_1, g_2$ be two smooth solutions to the $\delta$-Ricci-DeTurck flow on $\mathbb{B}_R(0) \times (0, T)$ such that $g_l \in C^\infty(\mathbb{B}_R(0) \times (0, T)) \cap C^0(\mathbb{B}_R(0) \times [0, T))$ for $l = 1, 2$ and that $g_1 = g_2$ on $\partial \mathbb{B}_R(0) \times (0, T)$ and $g_1(\cdot, 0) = g_2(\cdot, 0)$. We assume further that $\|g_1(\cdot, t) - \delta\| + \|g_2(\cdot, t) - \delta\| \leq \varepsilon(n)$ for all $t \in (0, T)$. Then $g_1 = g_2$ on $\mathbb{B}_R(0) \times (0, T)$.

**Remark 6.3.** [7, Proposition 7.51] shows a uniqueness result of the Ricci-DeTurck flow with a background metric $g$ with bounded curvature for solutions $(g(t))_{t \in (0, 1)}$ that behave as follows: there exists a positive constant $A$ such that $A^{-1} g \leq g(t) \leq A g$ and $|\nabla^2 g(t)| + \sqrt{g} |\nabla g(t)| \leq A$ for all $t \in (0, 1)$. Its proof is based on the maximum principle. Corollary 6.2 assumes a stronger assumption on the closeness to the background Euclidean metric but it does not assume any a priori bounds on the first and second covariant derivatives: the proof is based on energy estimates.

**Proof of Corollary 6.2.** Corollary 6.2 is proved as follows. Let $h := g_1 - g_2$ and

$$v := |h|^2 \left( 1 + \lambda(|g_1 - \delta|^2 + |g_2 - \delta|^2) \right).$$

From the assumptions, we know that $|v(\cdot, 0)| = 0$ on $\mathbb{B}_R(0)$ and hence $\int_{\mathbb{B}_R(0)} v(x, \tau) dx \leq \sigma(\tau)$, where $\sigma(\tau)$ tends to 0 with $\tau \searrow 0$, in view of the continuity of $v$. We also have $g_1(\cdot, t) = g_2(\cdot, t)$ on $\partial \mathbb{B}_R(0)$ for all $t \in [0, T]$, and so Lemma 6.1 implies $\int_{\mathbb{B}_R(0)} v(x, t) dx \leq \sigma(\tau)$ for all $t \in (\tau, T)$. Taking a limit $\tau \searrow 0$, we see $\int_{\mathbb{B}_R(0)} v(x, t) dx = 0$ for all $t \in (0, T)$. This implies $g_1(\cdot, t) = g_2(\cdot, t)$ for all $t \in [0, T)$ as required.

By slightly modifying the previous proof, we can also show the following uniqueness statement.

**Corollary 6.4.** Let $h$ be a smooth solution to the $\delta$-Ricci-DeTurck flow on $\mathbb{B}_R(0) \times (0, T)$ such that $h \in C^\infty(\mathbb{B}_R(0) \times (0, T)) \cap C^0(\mathbb{B}_R(0) \times [0, T))$ and assume $h_0 \in C^\infty(\mathbb{B}_R(0))$ and let $\ell$ be the solution constructed in Theorem 5.5, with parabolic boundary data defined by $h|_{\partial \mathbb{B}_R(0)}$. We assume further that $|h(\cdot, t) - \delta| \leq \varepsilon(n)$ for all $t \in [0, T)$. Then $\ell = h$.
Proof. Let \( h(\tau) \) be the modified metric defined in the proof of Theorem 5.5, and \( \ell(\tau) \) the solutions defined there. Let \( v(\tau) := |h(\tau) - \ell(\tau)|^2 \left( 1 + \lambda(|h(\tau) - \delta|^2 + |\ell(\tau) - \delta|^2) \right) \). From the construction of \( h(\tau) \) we know that \( |v(\tau)(\cdot, \tau)| \leq \sigma(\tau) \) on \( \mathbb{B}_R(0) \), where \( \sigma(\tau) \to 0 \) with \( \tau \searrow 0 \). This implies

\[
\int_{\mathbb{B}_R(0)} v(\tau)(x, \tau) dx \leq \sigma(\tau),
\]

where \( \sigma(\tau) \to 0 \) with \( \tau \searrow 0 \). Since \( h = h(\tau) = \ell(\tau) \) on \( \partial \mathbb{B}_R(0) \) for all \( t \in [\tau, T] \) and the solution \( \ell(\tau) \) is \( C^k \) (in space and time) up to and including the boundary, we can use Lemma 6.1 to conclude \( \int_{\mathbb{B}_R(0)} v(\tau)(x, t) dx \leq \sigma(\tau) \) for all \( t \in (\tau, T) \). Taking a limit \( \tau \searrow 0 \), we see \( \int_{\mathbb{B}_R(0)} v(x, t) dx \leq 0 \) for all \( t \in (0, T) \), with \( v := |h - \ell|^2 \left( 1 + \lambda(|h - \delta|^2 + |\ell - \delta|^2) \right) \). This implies \( h(\cdot, t) = \ell(\cdot, t) \) for all \( t \in [0, T] \) as required. \( \square \)

Proof of Lemma 6.1. In the following, we will assume that \( \varepsilon(n) \) is a small positive constant, and \( \lambda(n) = 1/\sqrt{\varepsilon(n)} \) is a large constant, which satisfies \( \lambda(n)\varepsilon(n) = \sqrt{\varepsilon(n)} =: \sigma(n) \) per definition.

From (5.1) we have for \( g_l, l \in \{1, 2\} \):

\[
\frac{\partial}{\partial t} g_l = g_l^{ab} \partial_a \partial_b g_l + g_l^{-1} * g_l^{-1} * Dg_l * Dg_l,
\]

and

\[
\frac{\partial}{\partial t} |g_l - \delta|^2 \leq g_l^{ab} \partial_a \partial_b |g_l - \delta|^2 - \frac{2}{1 + \varepsilon} |Dg_l|^2.
\]

Summing over \( l = 1, 2 \), and writing \( \tilde{h}^{ab} := \frac{1}{2} (g_1^{ab} + g_2^{ab}) \) and \( \tilde{h}^{ab} := \frac{1}{2} (g_1^{ab} - g_2^{ab}) \), we get

\[
\frac{\partial}{\partial t} (|g_1 - \delta|^2 + |g_2 - \delta|^2) \\
\leq g_1^{ab} \partial_a \partial_b |g_1 - \delta|^2 + g_2^{ab} \partial_a \partial_b |g_2 - \delta|^2 - 2(1 - \varepsilon(n)) |Dg_1|^2 - 2(1 - \varepsilon(n)) |Dg_2|^2 \\
= \tilde{h}^{ab} \partial_a \partial_b (|g_1 - \delta|^2 + |g_2 - \delta|^2) + \tilde{h}^{ab} \partial_a \partial_b (|g_1 - \delta|^2 - |g_2 - \delta|^2) \\
- 2(1 - \varepsilon(n)) |Dg_1|^2 - 2(1 - \varepsilon(n)) |Dg_2|^2 \\
= \tilde{h}^{ab} \partial_a \partial_b (|g_1 - \delta|^2 + |g_2 - \delta|^2) + \sum_{l=1}^2 (\tilde{h} * (g_l - \delta) * D^2 g_l + \tilde{h} * Dg_l * Dg_l) \\
- 2(1 - \varepsilon(n)) |Dg_1|^2 - 2(1 - \varepsilon(n)) |Dg_2|^2 \\
\leq \tilde{h}^{ab} \partial_a \partial_b (|g_1 - \delta|^2 + |g_2 - \delta|^2) + \sum_{l=1}^2 (\tilde{h} * (g_l - \delta) * D^2 g_l) \\
- 2(1 - 2\varepsilon(n)) |Dg_1|^2 - 2(1 - 2\varepsilon(n)) |Dg_2|^2,
\]

in view of the fact that \( g_l \) is \( \varepsilon \) close to \( \delta \) for \( l = 1, 2 \).

We obtain for the difference \( h = g_1 - g_2 \) that

\[
\frac{\partial}{\partial t} h = g_1^{ab} \partial_a \partial_b g_1 + g_1^{-1} * g_1^{-1} * Dg_1 * Dg_1 \\
- g_2^{ab} \partial_a \partial_b g_2 - g_2^{-1} * g_2^{-1} * Dg_2 * Dg_2 \\
= \frac{1}{2} (g_1^{ab} + g_2^{ab}) \partial_a \partial_b h + \frac{1}{2} (g_1^{ab} - g_2^{ab}) \partial_a \partial_b (g_1 + g_2) \\
+ (g_1^{-1} - g_2^{-1}) * g_1^{-1} * Dg_1 * Dg_1 + g_2^{-1} * (g_1^{-1} - g_2^{-1}) * Dg_1 * Dg_1 \\
+ g_2^{-1} * g_2^{-1} * Dh * Dg_1 + g_2^{-1} * g_2^{-1} * Dg_2 * Dh,
\]

31
which we can write as
\[
\frac{\partial}{\partial t} h = \tilde{h}^{ab} \partial_a \partial_b h + \tilde{h}^{ab} \partial_a \partial_b (g_1 + g_2) + \tilde{h} \ast g_1^{-1} \ast Dg_1 \ast Dg_1 + g_2^{-1} \ast Dg_1 + g_2^{-1} \ast Dg_1 + g_2^{-1} \ast Dg_2 + Dh.
\]
This implies that
\[
\frac{\partial}{\partial t} |h|^2 \leq \tilde{h}^{ab} \partial_a \partial_b |h|^2 - \frac{2}{1 + \varepsilon} |Dh|^2 + h \ast \tilde{h} \ast D^2 (g_1 + g_2) + h \ast \tilde{h} \ast g_1^{-1} \ast Dg_1 \ast Dg_1 + h \ast g_2^{-1} \ast Dg_1 \ast Dg_1 + h \ast g_2^{-1} \ast g_2^{-1} \ast Dg_1 \ast Dg_1 + h \ast g_2^{-1} \ast g_2^{-1} \ast Dg_2 \ast Dg_2 + Dh,
\]
in view of the fact that \( \tilde{h} \) is \( \varepsilon \) close to \( \delta \). We now consider the test-function
\[
v := |h|^2 \left( 1 + \lambda(|g_1 - \delta|^2 + |g_2 - \delta|^2) \right)
\]
where \( \lambda(n) = 1/\sqrt{n} \). We have \( \lambda(n) \varepsilon(n) \leq \sigma = \sqrt{\varepsilon} \). We obtain
\[
\frac{\partial}{\partial t} v \leq \tilde{h}^{ab} \partial_a \partial_b v - 2\tilde{h}^{ab} \partial_a |h|^2 \partial_b (|g_1 - \delta|^2 + |g_2 - \delta|^2)
\]
\[
- 2\lambda(1 - 2\varepsilon(n)) |h|^2 \left( |Dg_1|^2 + |Dg_2|^2 \right)
- 2(1 - 2\varepsilon(n)) \left( 1 + \lambda(|g_1 - \delta|^2 + |g_2 - \delta|^2) \right) |Dh|^2
+ (1 + \lambda(|g_1 - \delta|^2 + |g_2 - \delta|^2)) \left( h \ast \tilde{h} \ast D^2 (g_1 + g_2) + h \ast \tilde{h} \ast g_1^{-1} \ast Dg_1 \ast Dg_1
+ h \ast g_2^{-1} \ast Dg_1 \ast Dg_1 + h \ast g_2^{-1} \ast g_2^{-1} \ast Dg_1 \ast Dg_1 + h \ast g_2^{-1} \ast g_2^{-1} \ast Dg_2 \ast Dg_2 + Dh \right)
+ \lambda |h|^2 \sum_{l=1}^2 (h \ast (g_l - \delta) \ast D^2 g_l).
\]
Using Young’s inequality and the fact that \( g_l \) is \( \varepsilon \) close to \( \delta \), for \( l = 1, 2 \), as well as \( |h \ast \tilde{h}| \leq c(n) |h|^2 \), we see that the first order terms appearing in the large brackets may be absorbed by the two negative first order gradient terms which appear just before the large brackets, if \( \lambda \geq \Lambda(n) \), where \( \Lambda(n) \) is sufficiently large. That is, we have
\[
\frac{\partial}{\partial t} v \leq \tilde{h}^{ab} \partial_a \partial_b v - 2\tilde{h}^{ab} \partial_a |h|^2 \partial_b (|g_1 - \delta|^2 + |g_2 - \delta|^2)
- 3\lambda |h|^2 \left( |Dg_1|^2 + |Dg_2|^2 \right) - \frac{3}{2} |Dh|^2
+ \left( 1 + \lambda(|g_1 - \delta|^2 + |g_2 - \delta|^2) \right) \left( h \ast \tilde{h} \ast D^2 (g_1 + g_2) \right)
+ \lambda |h|^2 \sum_{l=1}^2 (h \ast (g_l - \delta) \ast D^2 g_l).
\]
This second term on the right-hand side of (6.1) can be estimated as follows:
\[
|2\tilde{h}^{ab} \partial_a |h|^2 \partial_b (|g_1 - \delta|^2 + |g_2 - \delta|^2)| \leq \lambda |h| \ast Dh \ast h \ast (|g_1 - \delta| + |g_2 - \delta|)
* (Dg_1 + Dg_2)
\]
\[
\leq c(n) \varepsilon(n) \lambda|\varepsilon|||Dh||(|Dg_1| + |Dg_2|).
\]
Therefore, it can also be absorbed by the negative terms just before the big brackets, in view of the fact that \( \varepsilon(n) \lambda \leq \sqrt{\varepsilon(n)} \). This leads to
\[
\frac{\partial}{\partial t} v \leq \tilde{h}^{ab} \partial_a \partial_b v - \lambda |h|^2 \left( |Dg_1|^2 + |Dg_2|^2 \right) - |Dh|^2
+ \left( 1 + \lambda(|g_1 - \delta|^2 + |g_2 - \delta|^2) \right) \left( h \ast \tilde{h} \ast D^2 (g_1 + g_2) \right)
+ \lambda |h|^2 \sum_{l=1}^2 (h \ast (g_l - \delta) \ast D^2 g_l).
\]
In order to estimate the second order terms appearing in the equation (6.1), we integrate over $\mathcal{B} := \mathcal{B}_R(0)$:

$$
\frac{\partial}{\partial t} \int_{\mathcal{B}} v \leq \int_{\mathcal{B}} \tilde{h}^{ab}\partial_a \partial_b v - \lambda \int_{\mathcal{B}} |h|^2 (|Dg_1|^2 + |Dg_2|^2) - \int_{\mathcal{B}} |Dh|^2 \\
+ \int_{\mathcal{B}} (1 + \lambda(|g_1 - \delta|^2 + |g_2 - \delta|^2)) (h \ast \tilde{h} \ast D^2(g_1 + g_2)) \\
+ \int_{\mathcal{B}} \lambda \sum_{l=1}^{2} |h|^2 (\tilde{h} \ast (g_l - \delta) \ast D^2g_l).
$$

Since $Dv$ and $h$ are zero on the boundary of $\mathcal{B}$, we obtain no boundary terms when integrating the first and last two terms on the right hand side of the above by parts. Doing so, we get

$$
\frac{\partial}{\partial t} \int_{\mathcal{B}} v \leq - \int_{\mathcal{B}} \partial_a \tilde{h}^{ab} \partial_b v - \lambda \int_{\mathcal{B}} |h|^2 (|Dg_1|^2 + |Dg_2|^2) - \int_{\mathcal{B}} |Dh|^2 \\
+ \int_{\mathcal{B}} D \left( (1 + \lambda(|g_1 - \delta|^2 + |g_2 - \delta|^2)) \ast h \ast \tilde{h} \right) \ast D(g_1 + g_2) \\
+ \int_{\mathcal{B}} \sum_{l=1}^{2} \lambda D \left( |h|^2 (\tilde{h} \ast (g_l - \delta)) \right) \ast Dg_l \\
=: A + B + C + D + E.
$$

Note also that $|\tilde{h}| \leq c(n)|h|$. We estimate the integrand of $A$ as follows:

$$
|\partial_a (\tilde{h})^{ab} \partial_b v| \leq c(n)|Dg_1| + |Dg_2||Dv| \\
= c(n)(|Dg_1| + |Dg_2|) |D(|h|^2(1 + \lambda(|g_1 - \delta|^2 + |g_2 - \delta|^2)))| \\
\leq c(n)(|Dg_1| + |Dg_2|) \left( 2|h||Dh| + |h|^2 \lambda c(n)(|Dg_1| + |Dg_2|) \right) \\
\leq c(n)(|Dg_1| + |Dg_2|)|h||Dh| + |h|^2 \lambda c(n)(|Dg_1| + |Dg_2|)^2
$$

and hence the integral $A$ can be absorbed by the integrals $B$ and $C$. In estimating the integral of $D$, we will use

$$
|D\tilde{h}| \leq c(n)|Dh| + |h||Dg_1| + |h||Dg_2|,
$$

the validity of which can be seen by writing, $\tilde{h} = \frac{1}{2}((g_1)^{-1} - (g_2)^{-1}) = \frac{1}{2}g_1^{-1}(g_2 - g_1)g_2^{-1}$, differentiating, and keeping in mind that $g_1$ and $g_2$ are $\varepsilon$ close to $\delta$.

We estimate the integrand of $D$ as follows:

$$
D \left( (1 + \lambda(|g_1 - \delta|^2 + |g_2 - \delta|^2)) \ast h \ast \tilde{h} \right) \ast D(g_1 + g_2) \\
= D \left( (1 + \lambda(|g_1 - \delta|^2 + |g_2 - \delta|^2)) \right) \ast h \ast \tilde{h} \ast D(g_1 + g_2) \\
+ \left( (1 + \lambda(|g_1 - \delta|^2 + |g_2 - \delta|^2))D(h \ast \tilde{h}) \ast D(g_1 + g_2) \\
\leq c(n)\varepsilon(n)\lambda|h|^2(|Dg_1|^2 + |Dg_2|^2) \\
+ c(n) \left( |Dh||h| + |h|^2(|Dg_1| + |Dg_2|) \right) (|Dg_1| + |Dg_2|),
$$

and hence the integral $D$ can also be absorbed by the integrals $B$ and $C$. We estimate the final integral of $E$ in a similar way: the integrand of $E$ can be estimated by

$$
\sum_{l=1}^{2} \lambda D \left( |h|^2 (\tilde{h} \ast (g_l - \delta)) \right) \ast Dg_l \leq \lambda c(n)|h|^2|Dh|\varepsilon(n)(|Dg_1| + |Dg_2|) \\
+ \lambda c(n)|h|^3(|Dg_1|^2 + |Dg_2|^2)
$$

33
Recall that an expanding gradient Ricci soliton is a triple \((\mathcal{R}, \mathcal{M}, \mathcal{G})\) which comes out of a metric cone. Such solitons is unknown. Below, we make precise the meaning of an expanding gradient Ricci solitons: the initial condition can be interpreted as the asymptotic cone at spatial infinity and vice versa in case the potential function has a critical point.

Returning to the general setting, we assume further that we are in the setting of Lemma 1.4.1. That is, that \(d_0\) written in distance coordinates \(F_0\) near a point \(x_0\) is generated by a continuous Riemannian metric \(g_0\) on an Euclidean ball. Then, using Lemma 4.1.2, we see that we may assume that \(|\mathcal{R}(\cdot, t)| \leq \frac{d_1}{t}\) for all \(t \in (0, T)\), where \(d_1 : [0, 1] \to \mathbb{R}^+_0\) is a non-decreasing function with \(d(0) = 0\), and that the improved distance estimates

\[
d_r + \varepsilon(t)\sqrt{t - r} \geq d_r \geq d_r - \varepsilon(t)\sqrt{t - r} \quad \text{for all } t \in [r, T]
\]

\[
on B_{d_0}(x_0, \varepsilon) \subseteq B_{g(s)}(x_0, 2\varepsilon) \quad \text{for all } s \in [0, T]
\]

7 Smoothness of solutions coming out of smooth metric spaces

Let \((M, g(t))_{t \in (0, T)}\) be a smooth solution to Ricci flow satisfying

\[
|\mathcal{R}(\cdot, t)| \leq \frac{d_1}{t} \\
\mathcal{R}(g(t)) \geq -1
\]

for all \(t \in (0, T)\). As explained in the introduction, examples of solutions which satisfy these estimates are expanding gradient Ricci solitons coming out of smooth cones \((\mathbb{R}^+ \times S^{n-1}, d_r)\), where \(\gamma\) is a Riemannian metric on the sphere, which is smooth and whose curvature operator has eigenvalues larger than one. ‘Coming out’ here means that \((M, d(g(t)), p) \to (\mathbb{R}^n, d_X, o)\) in the pointed Gromov-Hausdorff sense as \(t \to 0\) for some point \(p\) in \(M\). In the papers [14] examples are constructed, and in the paper [8] it is shown that there is always a solution which comes out smoothly, in the sense that the convergence is in the \(C^\infty_{loc}\) sense away from the tip. The uniqueness of such solitons is unknown. Below, we make precise the meaning of an expanding gradient Ricci soliton which comes out of a metric cone.

Recall that an expanding gradient Ricci soliton is a triple \((\mathcal{R}^n, g, \nabla^g f)\) where \(\mathcal{R}\) is an \(n\)-dimensional Riemannian manifold with a complete Riemannian metric \(g\) and a smooth potential function \(f : M \to \mathbb{R}\) satisfying the equation

\[
\text{Ric}(g) - \text{Hess} f = -\frac{g}{2}
\]

Also, to each expanding gradient Ricci soliton, one may associate a self-similar solution of the Ricci flow. Indeed, let \((\psi_t)_{t > 0}\) be the flow generated by \(-\nabla^g f/t\) such that \(\psi_1 = \text{Id}_M\) and define \(g(t) := \psi_t^* g\) for \(t > 0\). Then \((M, g(t))_{t > 0}\) defines a Ricci flow thanks to (7.2). Next, we notice that if an expanding gradient Ricci soliton \((\mathcal{R}^n, g(t), p)_{t > 0}\) admits a limit as \(t\) tends to 0 in the pointed Gromov-Hausdorff sense for some point \(p\) that lies in the critical set of the potential function \(f\) then this limit must be the asymptotic cone in the sense of Gromov since \((\mathcal{R}^n, g(t), p)\) is isometric to \((\mathcal{R}^n, tg, \psi_t(p) = p)\) for \(t > 0\) as pointed metric spaces. Therefore, there is a space-time dictionary for expanding gradient Ricci solitons: the initial condition can be interpreted as the asymptotic cone at spatial infinity and vice versa in case the potential function has a critical point.
hold on some fixed ball.

We now make the further restriction, that the the metric \( \tilde{g}_0 \) is smooth on some Euclidean ball containing \( F_0(x_0) \) in the sense of Definition 1.2.

Theorem 1.3 shows in this case that the original Ricci flow solution comes out smoothly from some smooth initial data, if we restrict to a small enough neighbourhood of \( x_0 \).

**Proof of Theorem 1.3.** We consider \( (B_r(0), \tilde{g}(t))_{t \in [0, T] \cap [0, S(n, r, \varepsilon_0)]} \), the solution to \( \delta \)-Ricci-DeTurck flow of Theorem 4.3, which is smooth on \( B_r(0) \times (0, T) \) and continuous on \( B_r(0) \times [0, T] \cap [0, S(n, r, \varepsilon_0)] \) and satisfies \( \tilde{g}(\cdot, 0) = \tilde{g}_0 \in C^\infty(B_r(0)) \). Let \( t \in C^\infty(B_r(0) \times [0, 1]) \cap C^0(B_r(0) \times [0, 1]) \) be the solution to the \( \delta \)-Ricci-DeTurck flow that we obtain from Theorem 5.5, if we use \( h := \tilde{g} \) to define the parabolic boundary values. Corollary 6.4 tells us that \( t = \tilde{g} \) and hence \( \tilde{g} \in C^\infty(B_r(0) \times [0, 1]) \).

By the smoothness of \( \tilde{g} \), we see that we have

\[
\sup_{B_{3r/4}(0) \times (0, 1]} |\text{Rm}(\tilde{g}(t))|^2 + |\nabla\text{Rm}(\tilde{g}(t))|^2 + \ldots + |\nabla^k\text{Rm}(\tilde{g}(t))|^2 \leq C_k,
\]

for all \( t \in [0, 1] \).

By the original construction of \( \tilde{g} \), we have \( \tilde{g}(t) = \lim_{r \to \infty} Z_i(t)_i(g(t)) \) for all \( t > 0 \) where the \( Z_i(t) \) are smooth diffeomorphisms for all \( i \in \mathbb{N} \), and the limit is in the smooth sense on any compact subset of \( B_r(0) \times (0, 1) \). Hence, we must have

\[
\sup_{B_{3d_0}(x_0, \tilde{r}/2) \times (0, 1]} |\text{Rm}(g(t))|^2 + |\nabla\text{Rm}(g(t))|^2 + \ldots + |\nabla^k\text{Rm}(g(t))|^2 \leq C_k.
\]

Using the method of Hamilton, see [9, Section 6], we see that we can extend the solution smoothly back to time 0: there exists a smooth Riemannian metric \( g_0 \) defined on \( B_{d_0}(x_0, \tilde{r}/4) \) such that \( (B_{d_0}(x_0, \tilde{r}/4), g(t))_{t \in [0, 1]} \) with \( g(0) = g_0 \) is smooth. \( \square \)

We return to the expanding gradient Ricci soliton examples provided by [14] and [8] discussed at the beginning of this section. By construction, they have non-negative Ricci curvature and bounded curvature at time \( t = 1 \) which amounts to saying that the corresponding Ricci flows satisfy (7.1).

We make a small digression to show that if an expanding gradient Ricci soliton satisfies (7.1) then it must have non-negative Ricci curvature. Indeed, let \( (M, g(t) = t \varphi_t^* g)_{t \in (0, \infty)} \) be an expanding gradient Ricci soliton, satisfying 7.1 for all \( t \in (0, \infty) \). This clearly means that \( \text{Ric}(g(t)) \geq 0 \): if this were not the case, say \( \text{Ric}(g(x)(v, v) = -L \leq 0 \) for some \( x \in M \) and some vector \( v \in T_x M \) of unit length with respect to \( g \), then we must have \( \text{Ric}(g(t)(x_t))(v_t, v_t) = -L \) for all \( t > 0 \) where \( x_t := \varphi_t^{-1}(x) \) and \( v_t := (d_x \varphi_t)^{-1}(v) \). Consequently, \( \text{Ric}(g(t))(x_t) < -1 \) for \( t > 0 \) small enough, a contradiction. So without loss of generality, \( \text{Ric}(g(t)) \geq 0 \) and hence the asymptotic volume ratio

\[
\text{AVR}(g(t)) := \lim_{r \to \infty} \frac{\text{Vol}(B(t)(x, r))}{r^n}
\]

is well defined for all time \( t > 0 \) and all points \( x \in M \) by Bishop-Gromov’s Theorem. Moreover, Hamilton, [7, Proposition 9.46], has shown that \( \text{AVR}(g(t)) \) is positive for all positive times \( t \). Using the non-negativity of the Ricci curvature together with the soliton equation (7.2), one can show that the potential function is a proper strictly convex function. In particular, it admits a unique critical point \( p \) in \( M \) which is a global minimum. Since we are considering expanding gradient Ricci solitons, we know that \( (M, g(t), p) \) is isometric to \( (M, t g(1, p)) \) as pointed metric spaces, and hence the asymptotic volume ratio \( \text{AVR}(g(t)) \) is a constant independent of time \( t > 0 \). Let \( (M, d_0, o) \) be the well defined
limit of \((M, d(g(t)), p)\) as \(t \to 0\), the existence of which is explained in the introduction and guaranteed by [19, Lemma 3.1]. The theorem of Cheeger-Colding on volume convergence, now guarantees that the asymptotic volume ratio of \((M, d_0, o)\) is also AVR\(g(1)\) and that \((M, d_0, o)\) is a volume cone. In fact it is also a metric cone, due to [4, Theorem 7.6] and the fact that \((M, d_0, o)\) is the Gromov-Hausdorff limit of \((M, td(g(1)), p)\) for any sequence \(t \to 0\). If \(x_0 \in M\) is a point where \(d_0\) is locally smooth, in the sense explained in Definition 1.2, then \((B_{g(0)}(x_0, r), g(t)) \to (B_{g_0}(x_0, r), g_0)\) smoothly for some small \(r > 0\) as \(t \to 0\).

In particular, if \((M, d_0, o)\) is a smooth cone, away from the tip \(o\), in the sense that locally distance coordinates introduce a smooth structure near \(x_0\) for any \(x_0\) in \(M\) not in the tip of the cone, then the solution comes out smoothly from the cone away from the tip.

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36
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