SINGULAR QUASILINEAR ELLIPTIC SYSTEMS WITH SUB-, SUPER- AND HOMOGENEOUS CONDITIONS

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Abstract. In this paper we establish existence, nonexistence and regularity of positive solutions for a class of singular quasilinear elliptic systems subject to (sub-, super-) homogeneous condition. The approach is based on sub-supersolution methods for systems of quasilinear singular equations combined with perturbation arguments involving singular terms.

1. INTRODUCTION AND MAIN RESULTS

We consider the following system of quasilinear and singular elliptic equations:
\begin{equation}
\begin{cases}
-\Delta_p u = \lambda u^{\alpha_1} v^{\beta_1} & \text{in } \Omega, \\
-\Delta_q v = \lambda u^{\alpha_2} v^{\beta_2} & \text{in } \Omega, \\
u, v > 0 & \text{in } \Omega, \\
u, v = 0 & \text{on } \partial \Omega,
\end{cases}
\end{equation}
(1.1)

where \( \Omega \) is a bounded domain in \( \mathbb{R}^N \) (\( N \geq 2 \)) with \( C^1,\alpha \) boundary \( \partial \Omega \), \( \alpha \in (0, 1) \), \( \lambda \) is a positive parameter, \( \Delta_p \) and \( \Delta_q \), \( 1 < p, q < N \), are the \( p \)-Laplacian and \( q \)-Laplacian operators, respectively, that is, \( \Delta_p u = \text{div} \left( |\nabla u|^{p-2} \nabla u \right) \) and \( \Delta_q v = \text{div} \left( |\nabla v|^{q-2} \nabla v \right) \). We consider the system (1.1) in a singular case assuming that
\begin{equation}
(1.2) \quad \alpha_1, \beta_2 < 0 < \alpha_2, \beta_1.
\end{equation}

This assumption make system (1.1) be cooperative, that is, for \( u \) (resp. \( v \)) fixed the right term in the first (resp. second) equation of (1.1) is increasing in \( v \) (resp. \( u \)).

Problem (1.1) arises in several fields of application. For instance, it appears in the study of non-Newtonian fluid mechanics both for \( p, q > 2 \) (dilatants fluids) and for \( 1 < p, q < 2 \) (pseudoplastic fluids), see [2]. If \( p, q = 2 \) they are Newtonian fluids. It also arises in the study of population dynamics [21], quasiconformal mappings [11] and other topics in geometry [22].

Recently, singular system (1.1) with cooperative structure was mainly studied in [6,7,16]. In [10] existence and boundedness theorems for (1.1) was established by using the sub-supersolution method for systems combined with perturbation techniques. In [6] one gets existence, uniqueness, and regularity of a positive solution on the basis of an iterative scheme constructed through a sub-supersolution. In [7] an existence theorem involving sub-supersolution was obtained through a fixed point argument in a sub-supersolution setting. The semilinear case in (1.1)
(i.e. \( p = q = 2 \)) was considered in \([5, 10, 17]\) where the linearity of the principal part is essentialy used. In this context, the singular system \( (1.1) \) can be viewed as the elliptic counter-part of a class of Gierer-Meinhardt systems that models some biochemical processes (see, e.g. \([17]\)). It can be also given an astrophysical meaning since it generalize to system the well-known Lane-Emden equation, where all exponents are negative (see \([5]\)). The complementary situation for system \( (1.1) \) with respect to \( (1.2) \) is the so-called competitive system, which has recently attracted much interest. Relevant contributions regarding this topic can be found in \([7, 14, 15]\).

It is worth pointing out that the aforementioned works have examined only the subhomogeneous case \( \Theta > 0 \) of singular problem \( (1.1) \) with

\[
(1.3) \quad \Theta = (p - 1 - \alpha_1)(q - 1 - \beta_2) - \beta_1 \alpha_2.
\]

The constant \( \Theta \) is related to system stability \( (1.1) \) that behaves in a drastically different way, depending on the sign of \( \Theta \). For instance, for \( \Theta < 0 \) system \( (1.1) \) is not stable in the sense that possible solutions cannot be obtained by iterative methods (see \([4]\)).

Unlike in the above references, the novelty of this paper is to establish the existence, regularity and nonexistence of (positive) solutions for singular problem \( (1.1) \) by processing simultaneously the three cases: 'subhomogeneous' for \( \Theta > 0 \), 'homogeneous' when \( \Theta = 0 \) and 'superhomogeneous' if \( \Theta < 0 \). This seems to be the first work with regard to homogeneous and superhomogeneous cases for singular systems. However, we point out that even in the subhomogeneous case, our study completes those made in the above papers considering that \( \gamma > 1 \) in \((1.4)\) and the exponents \( \alpha_2, \beta_1 > 0 \) in \((1.2)\) are arbitrary.

Our results on existence for problem \( (1.1) \) are contained in the next theorem.

**Theorem 1.** Assume \((1.2)\) and \(|\Theta| > 0 \) (resp. \( \Theta = 0 \)) hold with

\[
(1.4) \quad \alpha_1, \beta_2 > -1 - \frac{1}{\gamma},
\]

for some constant \( \gamma > 1 \). Then problem \((1.1)\) possesses a (positive) solution \((u, v)\) in \((W^{1,p}_0(\Omega) \cap L^\infty(\Omega)) \times (W^{1,q}_0(\Omega) \cap L^\infty(\Omega))\) for each (resp. large) \( \lambda > 0 \).

Moreover, if

\[
(1.5) \quad \alpha_1, \beta_2 > -\frac{1}{\gamma},
\]

there exists \( \beta \in (0, 1) \) such that \((u, v) \in C^{1,\beta}(\overline{\Omega}) \times C^{1,\beta}(\overline{\Omega})\) for each (resp. large) \( \lambda > 0 \).

A solution of \((1.1)\) is understood in the weak sense, that is, a pair \((u, v) \in W^{1,p}_0(\Omega) \times W^{1,q}_0(\Omega)\), which are positive a.e. in \( \Omega \) and satisfying

\[
\begin{align*}
\int_{\Omega} |\nabla u|^{p-2} \nabla u \nabla \varphi \, dx &= \lambda \int_{\Omega} u^{\alpha_1} v^{\beta_1} \varphi \, dx, \\
\int_{\Omega} |\nabla v|^{q-2} \nabla v \nabla \psi \, dx &= \lambda \int_{\Omega} u^{\alpha_2} v^{\beta_2} \psi \, dx,
\end{align*}
\]

for all \((\varphi, \psi) \in W^{1,p}_0(\Omega) \times W^{1,q}_0(\Omega)\).

Our second result shows that problem \((1.1)\) with the homogeneous condition \( \Theta = 0 \) has no solutions for \( \lambda > 0 \) small.

**Theorem 2.** Assume \((1.2)\) and \( \Theta = 0 \) hold with

\[
(1.6) \quad \alpha_1, \beta_2 \in (-1, 0)
\]
and

\[ \beta_1 = \frac{q}{p} (p - 1 - \alpha_1) \quad \text{or} \quad \alpha_2 = \frac{q}{q} (q - 1 - \beta_2) \]

Then there exists a constant \( \lambda_* > 0 \) such that problem (1.1) has no solution for every \( \lambda \in (0, \lambda_*) \).

The main technical difficulty consists in the presence of singular terms in system (1.1) with (1.2), expressed through (sub, super) homogeneous condition. Our approach is chiefly based on the sub-supersolution method in its version for systems [3, section 5.5]. However, this method cannot be directly implemented due to the presence of singular terms in system (1.1). So, we first disturb system (1.1) by introducing a parameter \( \varepsilon > 0 \). This gives rise to a regularized system for (1.1) depending on \( \varepsilon \) whose study is relevant for our initial problem. By applying the sub-supersolution method, we show that the regularized system has a positive solution \((u_\varepsilon, v_\varepsilon)\) in \( C^{1,\beta}(\Omega) \times C^{1,\beta}(\Omega) \) for some \( \beta \in (0, 1) \). It is worth noting that the choice of suitable functions with an adjustment of adequate constants is crucial in order to construct the pair of a sub- supersolution, independent of \( \varepsilon \) small. In similar fashion, this choice enable us to process simultaneously all situations regarding the sign of the constant \( \Theta \). The (positive) solution \((u, v)\) of (1.1) is furnished by the regularity result in [9] under hypothesis (1.5). We emphasize that in the homogeneous case \( \Theta = 0 \) the existence of solutions requires \( \lambda > 0 \) to be large whereas the nonexistence of solutions is obtained for some additional restrictions on \( \alpha_2, \beta_1 \) provided that \( \lambda \) is small. A striking feature of our approach is the simplicity of the used techniques despite the serious difficulties raised by the presence of singularities in the system (1.1) under (sub-, super-) homogeneous condition.

The rest of the paper is organized as follows. Section 2 is devoted to the existence of solutions for the regularized system. Section 3 established the proof of theorems 1 and 2.

2. The regularized system

Given \( 1 < p < +\infty \), the space \( L^p(\Omega) \) and \( W^{1,p}_0(\Omega) \) are endowed with the usual norms \( \|u\|_p = (\int_{\Omega}|u|^p \, dx)^{1/p} \) and \( \|u\|_{1,p} = (\int_{\Omega}|\nabla u|^p \, dx)^{1/p} \), respectively. We will also utilize the space \( C^{1,\beta} \) = \( \{ u \in C^{1,\beta} : u = 0 \text{ on } \partial \Omega \} \) for a suitable \( \beta \in (0, 1) \). In what follows, we denote by \( \phi_{1,p} \) and \( \phi_{1,q} \) the normalized positive eigenfunctions associated with the principal eigenvalues \( \lambda_{1,p} \) and \( \lambda_{1,q} \) of \( -\Delta_p \) and \( -\Delta_q \), respectively:

\[ -\Delta_p \phi_{1,p} = \lambda_{1,p} |\phi_{1,p}|^{p-2} \phi_{1,p} \quad \text{in } \Omega, \quad \phi_{1,p} = 0 \quad \text{on } \partial \Omega, \quad \|\phi_{1,p}\|_p^p = 1 \]

and

\[ -\Delta_q \phi_{1,q} = \lambda_{1,q} |\phi_{1,q}|^{q-2} \phi_{1,q} \quad \text{in } \Omega, \quad \phi_{1,q} = 0 \quad \text{on } \partial \Omega, \quad \|\phi_{1,q}\|_q^q = 1. \]

The strong maximum principle ensures the existence of positive constants \( l_1 \) and \( l_2 \) such that

\[ l_1 \phi_{1,p}(x) \leq \phi_{1,q}(x) \leq l_2 \phi_{1,p}(x) \] for all \( x \in \Omega \).
For a later use we recall that there exists a constant $l > 0$ such that
\begin{equation}
\phi_{1,p}(x), \phi_{1,q}(x) \geq ld(x) \text{ for all } x \in \Omega,
\end{equation}
where $d(x) := \text{dist}(x, \partial \Omega)$ (see, e.g., [3]).

Let $\bar{\Omega}$ be a bounded domain in $\mathbb{R}^N$ with $C^{1,\alpha}$ boundary $\partial \bar{\Omega}$, $\alpha \in (0,1)$, such that $\bar{\Omega} \subset \Omega$. We denote by $\lambda_{1,p}$ and $\lambda_{1,q}$ the first eigenvalue of $-\Delta_p$ on $W_0^{1,p}(\Omega)$ and of $-\Delta_q$ on $W_0^{1,q}(\Omega)$, respectively. Let $\phi_{1,p}$ be the normalized positive eigenfunction of $-\Delta_p$ corresponding to $\lambda_{1,p}$, that is
\begin{equation}
-\Delta_p \phi_{1,p} = \lambda_{1,p} \phi_{1,p}^{p-1} \text{ in } \bar{\Omega}, \quad \phi_{1,p} = 0 \text{ on } \partial \bar{\Omega}.
\end{equation}
Similarly, let $\phi_{1,q}$ be the normalized positive eigenfunction of $-\Delta_q$ corresponding to $\lambda_{1,q}$, that is
\begin{equation}
-\Delta_q \phi_{1,q} = \lambda_{1,q} \phi_{1,q}^{q-1} \text{ in } \bar{\Omega}, \quad \phi_{1,q} = 0 \text{ on } \partial \bar{\Omega}.
\end{equation}
By the definition of $\bar{\Omega}$ and the strong maximum principle, there exists a constant $\rho > 0$ sufficiently small such that
\begin{equation}
\phi_{1,p}(x), \phi_{1,q}(x) > \rho \text{ in } \bar{\Omega}.
\end{equation}
Without loss of generality we assume that
\begin{equation}
M = \max\{\max_{\bar{\Omega}} \phi_{1,p}(x), \max_{\bar{\Omega}} \phi_{1,q}(x), \max_{\bar{\Omega}} \phi_{2, p}(x), \max_{\bar{\Omega}} \phi_{2, q}(x)\}.
\end{equation}
In order to regularize the singular problem (1.1), we introduce for every $\varepsilon > 0$ the auxiliary problem
\begin{equation}
\begin{cases}
-\Delta_p u = \lambda(u + \varepsilon)^{\alpha_1} u^{\beta_1} & \text{ in } \Omega \\
-\Delta_q v = \lambda u^{\alpha_2} (v + \varepsilon)^{\beta_2} & \text{ in } \Omega \\
u, v > 0 & \text{ in } \Omega \\
u, v = 0 & \text{ on } \partial \Omega.
\end{cases}
\end{equation}
System (2.7) provides approximate solutions for our initial problem (1.1). In our approach, we need to construct the sub-supersolution pairs of (2.7).

Let $C > 1$ a real constant and let $\xi_1, \xi_2 \in C^1(\bar{\Omega})$ be the functions defined as follows:
\begin{equation}
\begin{cases}
-\Delta_p \xi_1 = C^{\delta(p-1)} \xi_1^{\theta_1} & \text{ in } \bar{\Omega} \\
\xi_1 = 0 & \text{ on } \partial \bar{\Omega}
\end{cases}
\quad \begin{cases}
-\Delta_q \xi_2 = C^{\delta(q-1)} \xi_2^{\theta_2} & \text{ in } \bar{\Omega} \\
\xi_2 = 0 & \text{ on } \partial \bar{\Omega}
\end{cases}
\end{equation}
with constants $\delta, \theta_1$ and $\theta_2$ satisfying
\begin{equation}
0 > \theta_1 > \max\{-1, \alpha_1\}, \quad 0 > \theta_2 > \max\{-1, \beta_2\}
\end{equation}
and
\begin{equation}
\delta < \min\{\frac{\theta_1 - 1}{\alpha_1}, \frac{\theta_2 - 1}{\beta_2}\} < 0
\end{equation}
where $k > 0$ is a constant to be chosen later on. Functions $\xi_1$ and $\xi_2$ satisfying
\begin{equation}
C^{\delta} c_{1,p} \xi_1 \leq \xi_1 \leq C^{\delta} c_{2,p} \phi_{1,p} \quad \text{and} \quad C^{\delta} c_{1,q} \phi_{1,q} \leq \xi_2 \leq C^{\delta} c_{2,q} \phi_{1,q},
\end{equation}
with constants $c_2 \geq c_1 > 0$ and $c'_2 \geq c'_1 > 0$ (see [8]).
Set
\begin{equation}
(\varpi, \varpi) = C^{-\delta}(\xi_1, \xi_2).
\end{equation}
and
\[(2.13) \quad (\omega, \psi) = (C^\sigma \phi_{1,p}^\gamma, C^\kappa \phi_{1,q}^\gamma),\]
where $\gamma > 1$ is a real constant and $\sigma$ is defined by
\[(2.14) \quad \sigma = -\text{sgn}(\Theta) = \begin{cases} 1 & \text{if } \Theta < 0 \\ 0 & \text{if } \Theta = 0 \\ -1 & \text{if } \Theta > 0. \end{cases}\]
We have $(\omega, \psi) \geq (\omega, \psi)$ in $\Omega$. Indeed, from (2.8), (2.12), (2.11), (2.6), (2.16), (2.9), we have
\[-\Delta_p \tilde{u} = C^{-\delta(p-1)} C^{\delta(p-1)} \xi_1^\theta = \xi_1^\theta \geq (C^\delta \tilde{c}_2 \phi_{1,p}^\theta)^\theta = (C^\delta \tilde{c}_2 M)^\theta \geq C^\sigma(p-1) \gamma p-1 \lambda_{1,p} M^\gamma(p-1) \geq C^\sigma(p-1) \gamma p-1 \lambda_{1,p} \phi_{1,p}^\gamma(p-1) = -\Delta_p \tilde{u} \text{ in } \Omega \]
and
\[-\Delta_q \tilde{v} = C^{-\delta(q-1)} C^{\delta(q-1)} \xi_2^\theta = \xi_2^\theta \geq (C^\delta \tilde{c}_2 \phi_{1,q}^\theta)^\theta = (C^\delta \tilde{c}_2 M)^\theta \geq C^\sigma(k-1) \gamma q-1 \lambda_{1,q} M^\gamma(q-1) \geq C^\sigma(k-1) \gamma q-1 \lambda_{1,q} \phi_{1,q}^\gamma(q-1) = -\Delta_q \tilde{v} \text{ in } \Omega,\]
provided that $C > 1$ is large enough. Then the monotonicity of the operators $-\Delta_p$ and $-\Delta_q$ leads to the conclusion.

We state the following result regarding the regularized system.

**Theorem 3.** Assume (1.2) holds. Then if $|\Theta| > 0$ (resp. $\Theta = 0$) there exists $\varepsilon_0 > 0$ such that for all $\varepsilon \in (0, \varepsilon_0)$, system (2.7) has a (positive) solution $(u_\varepsilon, v_\varepsilon) \in C^1(\Omega) \times C^1(\Omega)$, $\delta \in (0, 1)$, for each $\lambda > 0$ (resp. large $\lambda > 0$). In addition it hold
\[(2.15) \quad u_\varepsilon(x) \leq u(x) \leq \overline{u}(x) \quad \text{and} \quad v_\varepsilon(x) \leq v(x) \leq \overline{v}(x) \quad \text{for a.a. } x \in \Omega, \quad \text{for all } \varepsilon \in (0, \varepsilon_0).\]

**Proof.** We shall verify that $(\omega, \psi)$ in (2.13) is a subsolution for problem (2.7) for all $\varepsilon \in (0, \varepsilon_0)$. A direct computation gives
\[(2.16) \quad -\Delta_p \tilde{u} = C^\sigma(p-1) \gamma p-1 \phi_{1,p}^\gamma(p-1) - \gamma - 1 (p-1) |\nabla \phi_{1,p}|^p \]
and
\[(2.17) \quad -\Delta_q \tilde{v} = C^\sigma(k-1) \gamma q-1 \phi_{1,q}^\gamma(q-1) - \gamma - 1 (q-1) |\nabla \phi_{1,q}|^q .\]

Then
\[(2.18) \quad (\tilde{u} + \varepsilon)^{-\alpha_1} \tilde{u} - \beta_1 (-\Delta_p \tilde{u}) = C^\sigma(p-1-k_1) \gamma p-1 (C^\sigma \tilde{\phi}_{1,p}^\gamma + \varepsilon)^{-\alpha_1} \gamma p-1 \tilde{\phi}_{1,p}^\gamma - \gamma - 1 (p-1) |\nabla \phi_{1,p}|^p \]
and
\[(2.19) \quad (\tilde{v} + \varepsilon)^{-\alpha_2} \tilde{v} - \beta_2 (-\Delta_q \tilde{v}) = C^\sigma(k-1-\alpha_2) \gamma q-1 \tilde{\phi}_{1,q}^\gamma - \gamma - 1 (q-1) |\nabla \phi_{1,q}|^q .\]
For a fixed $\delta > 0$ sufficiently small we denote
\[(2.20) \quad \Omega_\delta = \{x \in \Omega : \text{dist} (x, \partial \Omega) < \delta\}.\]
Since
\[(2.21) \quad \phi_{1,p}, \phi_{1,q} = 0 \quad \text{and} \quad |\nabla \phi_{1,p}|, |\nabla \phi_{1,q}| > 0 \quad \text{on } \partial \Omega,\]
we have
\begin{equation}
\lambda_{1,p} \phi_{1,p}(x)^p - |\nabla \phi_{1,p}(x)|^p \leq 0 \quad \text{for all } x \in \Omega_\delta
\end{equation}
and
\begin{equation}
\lambda_{1,q} \phi_{1,q}(x)^q - |\nabla \phi_{1,q}(x)|^q \leq 0 \quad \text{for all } x \in \Omega_\delta.
\end{equation}
We recall that there exists a constant \( \mu = \mu(\delta) > 0 \) such that
\begin{equation}
\phi_{1,p}(x), \phi_{1,q}(x) \geq \mu \quad \text{in } \Omega \setminus \Omega_\delta.
\end{equation}
Fix
\[ \varepsilon_0 = \min\{C^\alpha, C^{\sigma k}\}. \]
We first deal with the nonhomogeneous condition \(|\Theta| > 0\). Let us choose the constant \( k > 0 \) as follows:
\begin{equation}
\begin{array}{ll}
p - \frac{\alpha_1}{\beta_1} < k < \frac{\alpha_2}{\beta_1} & \text{if } \Theta < 0 \\
p - \frac{\alpha_1}{\beta_1} > k > \frac{\alpha_2}{\gamma - \beta_2} & \text{if } \Theta > 0.
\end{array}
\end{equation}
This is possible in view of (2.3). By (2.24), (2.13) and (2.14) observe that
\[ \sigma(p - 1 - \alpha_1 - k\beta_1), \sigma(k(q - 1 - \beta_2) - \alpha_2) < 0. \]
Then, since \( \gamma > 1 \), using (2.23), (2.16), (2.24) and (2.23), for all \( \varepsilon \in (0, \varepsilon_0) \) and for each \( \lambda > 0 \), it follows that
\begin{equation}
\begin{aligned}
C^{\sigma(p-1-k\beta_1)} & \gamma^{p-1} \lambda_{1,p} (C^{\sigma} \phi_{1,p} + \varepsilon)^{-\alpha_1} \phi_{1,q}^{-\gamma \beta_1} \\
& \leq C^{\sigma(p-1-k\beta_1)} \gamma^{p-1} \lambda_{1,p} (C^{\sigma} \phi_{1,p} + \varepsilon_0)^{-\alpha_1} \phi_{1,q}^{\gamma \beta_1 - 1}(M \gamma + 1)^{-\alpha_1} \phi_{1,q}^{-\gamma \beta_1} \\
& \leq C^{\sigma(p-1-\alpha_1-k\beta_1)} \gamma^{p-1} \lambda_{1,p} \gamma^{\gamma \beta_1 - 1}(M \gamma + 1)^{-\alpha_1} \phi_{1,q}^{-\gamma \beta_1} \\
& \leq \lambda \quad \text{in } \Omega \setminus \Omega_\delta,
\end{aligned}
\end{equation}
provided \( C > 1 \) is sufficiently large. Proceeding in the same way, for each \( \lambda > 0 \) and for all \( \varepsilon \in (0, \varepsilon_0) \), we see from (2.3), (2.16), (2.24) and (2.23) that
\begin{equation}
\begin{aligned}
C^{\sigma(k(q-1)-\alpha_2)} & \gamma^{q-1} \lambda_{1,q} \phi_{1,q}^{-\gamma \alpha_2} \phi_{1,q}^{\gamma(q-1)} (C^{\sigma} \phi_{1,q}^2 + \varepsilon)^{-\beta_2} \\
& \leq C^{\sigma(k(q-1)-\alpha_2)} \gamma^{q-1} \lambda_{1,q} (l \phi_{1,q}(x))^{-\gamma \alpha_2} \phi_{1,q}^{\gamma(q-1)} (C^{\sigma} \phi_{1,q}^2 + \varepsilon_0)^{-\beta_2} \\
& \leq C^{\sigma(k(q-1)-\alpha_2)} \gamma^{q-1} \lambda_{1,q} \gamma^{\gamma \alpha_2 - 1}(M \gamma + 1)^{-\beta_2} \phi_{1,q}^{-\gamma \alpha_2} \\
& \leq \lambda \quad \text{in } \Omega \setminus \Omega_\delta,
\end{aligned}
\end{equation}
provided \( C > 1 \) is sufficiently large.
Now, we examine the case \( \Theta = 0 \). By (2.23), (2.6) and (2.23) and recalling that
\[ \sigma = 0 \]
\begin{equation}
\begin{aligned}
\gamma^{p-1} \lambda_{1,p} \phi_{1,p}^{\gamma \beta_1 + \varepsilon} - \alpha_1 \phi_{1,p}^{\gamma \beta_1} \\
& \leq \gamma^{p-1} \lambda_{1,p} \gamma^{\gamma \beta_1 - 1}(M \gamma + 1)^{-\alpha_1} \phi_{1,q}^{-\gamma \beta_1} \\
& \leq \gamma^{p-1} \lambda_{1,p} \gamma^{\gamma \beta_1 - 1}(M \gamma + 1)^{-\alpha_1} \\
& \leq \lambda \quad \text{in } \Omega \setminus \Omega_\delta,
\end{aligned}
\end{equation}
provided λ > 0 is sufficiently large. Similarly,
\[
\gamma^{-1} \lambda_1 \phi_1^\gamma (\phi_1^\gamma + \varepsilon)^{-\beta_2} \phi_1^{\gamma(p-1)} \phi_1^{-\gamma \alpha_2} \\
\leq \gamma^{-1} \lambda_1 l_1^{\alpha_2} (M \gamma + 1)^{-\beta_2} \phi_1^{1-q-\alpha_2} \\
\leq \gamma^{-1} \lambda_1 l_2^{\alpha_2} (M \gamma + 1)^{-\beta_2} \begin{cases} 
M^{\gamma(q-1-\alpha_2)} & \text{if } \alpha_2 \leq q-1 \\
\mu^{\gamma(q-1-\alpha_2)} & \text{if } \alpha_2 \geq q-1
\end{cases} \\
\leq \lambda \text{ in } \Omega \backslash \Omega_\delta,
\]
provided λ > 0 is sufficiently large.

Gathering (2.18), (2.19), (2.21), (2.22), (2.25), (2.27), (2.28) and (2.28) together and bearing in mind that γ > 1 yields
\[- \Delta_p \Omega \leq \lambda (u + \varepsilon) \alpha_2 \beta_1 \text{ in } \Omega \text{ and } - \Delta_p \Omega \leq \lambda \alpha_2 (u + \varepsilon) \beta_2 \text{ in } \Omega,
\]
proving that (u, v) in (2.13) is a subsolution pair for problem (2.7) for all ε ∈ (0, ε_0).

Next, we construct a supersolution for problem (2.7). Taking into account (12), (2.12), (2.14), (2.15) and (2.16), for all ε ∈ (0, ε_0), we derive that in Ω one has
\[
(\overline{u} + \varepsilon)^{-\alpha_1 \beta_1} (- \Delta_p \overline{u}) \geq \nabla^{-\alpha_1 \beta_1} (- \Delta_p \overline{u}) \\
= C^{-\delta} (p-1-\alpha_2-\beta_2) C^{\delta(q-1-\alpha_2-\beta_2)} (C^\delta l_1^{-1} \phi_1^\gamma)^{-\alpha_2} \geq C^{\delta(q-1-\alpha_2-\beta_2)} (C^\delta l_1^{-1} \phi_1^\gamma)^{-\alpha_2} \\
\geq C^{\delta q} (c^\delta \phi_1^\gamma)^{\beta_2-\beta_2} \phi_1^{-\beta_2} \phi_1^{\gamma} (C^\delta l_1^{-1} \phi_1^\gamma)^{-\alpha_2} \\
\geq C^{\delta q} (c^\delta \phi_1^\gamma)^{\beta_2-\beta_2} \phi_1^{-\beta_2} \phi_1^{\gamma} (C^\delta l_1^{-1} \phi_1^\gamma)^{-\alpha_2} \\
\geq \lambda \text{ in } \Omega,
\]
provided C > 0 is sufficiently large. Similarly, we have
\[
(\overline{v} + \varepsilon)^{-\alpha_2 \beta_2} (\overline{\Delta_p} \overline{v}) \geq \nabla^{-\alpha_2 \beta_2} (\overline{\Delta_p} \overline{v}) \\
= C^{-\delta} (q-1-\alpha_2-\alpha_2) C^{\delta(q-1-\alpha_2-\alpha_2)} (C^{\alpha_2+\beta_2} (c^\delta \phi_1^\gamma)^{\alpha_2-\alpha_2} \\
\geq C^{\delta q} (c^\delta \phi_1^\gamma)^{\beta_2-\beta_2} \phi_1^{-\beta_2} \phi_1^{\gamma} (C^{\alpha_2+\beta_2} (c^\delta \phi_1^\gamma)^{\alpha_2-\alpha_2} \\
\geq C^{\delta q} (c^\delta \phi_1^\gamma)^{\beta_2-\beta_2} \phi_1^{-\beta_2} \phi_1^{\gamma} (C^{\alpha_2+\beta_2} (c^\delta \phi_1^\gamma)^{\alpha_2-\alpha_2} \\
\geq \lambda \text{ in } \Omega,
\]
This shows that (u, v) is a subsolution pair for problem (2.7) for all ε ∈ (0, ε_0).

Then we may apply the general theory of sub-supersolutions for systems of quasilinear equations (see [3] section 5.5), which implies the existence of a solution (u_ε, v_ε) of problem (2.7), for all ε ∈ (0, ε_0). Moreover, applying the regularity theory (see [12]), we infer that (u_ε, v_ε) ∈ C^1_0 (\overline{\Omega}) \times C^1_0 (\overline{\Omega}) for a suitable β ∈ (0, 1) and for all ε ∈ (0, ε_0). This completes the proof. \(\square\)

**Remark 1.** A careful inspection of the proof of Theorem 3 shows that the constants C > 1 when |Ω| > 0 and λ > 0 for Ω = 0 can be precisely estimated.

### 3. PROOF OF THE MAIN RESULTS

This section is devoted to the proof of Theorems 1 and 2.

**Proof of Theorem 1** The proof relies on Theorem 3. Set ε = \(\frac{1}{n}\) with any positive integer n > 1/ε_0. From Theorem 3 with ε = \(\frac{1}{n}\), we know that there exist u_n := u_{\frac{1}{n}} and v_n := v_{\frac{1}{n}} such that
\[
\begin{cases}
\langle - \Delta_p u_n, \varphi \rangle = \lambda \int_{\Omega} (u_n + \frac{1}{n})^{\alpha_1} v_n^{\beta_1} \varphi \, dx \\
\langle - \Delta_q v_n, \psi \rangle = \lambda \int_{\Omega} u_n^{\alpha_2} (v_n + \frac{1}{n})^{\beta_2} \psi \, dx
\end{cases}
\]
for all \((\varphi, \psi) \in W_0^{1,p}(\Omega) \times W_0^{1,q}(\Omega)\). By taking \(\varphi = u_n\) in (3.1) and since \(\alpha_1 < 0\), we get
\[
\|u_n\|_{1,p}^p = \lambda \int_\Omega (u_n + \frac{1}{n})^\alpha v_n^\beta u_n \, dx \leq \lambda \int_\Omega u_n^{\alpha_1+1} v_n^\beta \, dx.
\]
If \(-1 \leq \alpha_1 < 0\) (see (3.2)), on the basis of (3.2) with \(\epsilon = \frac{1}{n}\) and (2.15), it follows directly that \(\{u_n\}\) is bounded in \(W_0^{1,p}(\Omega)\). It remains to argue when the exponent \(\alpha_1\) verifies 
\(-1 - \frac{1}{\gamma} < \alpha_1 < -1\) (see (12, Lemma in page 726)). By virtue of the strong maximum principle for the negative Dirichlet
\(p\)-Laplacian (see (12)), we can find an eigenfunction \(\phi_1\) corresponding to the first eigenvalue of \((-\Delta, H_0^1(\Omega))\) such that \(\phi_1 \geq \varphi_1\) in \(\Omega\). Then, on account of (3.3) with \(\epsilon = \frac{1}{n}\), (2.15), (2.12), and (2.13) it holds the estimate
\[
\|u_n\|_{1,p}^p \leq C_1 \int_\Omega \phi_1^{\gamma(\alpha_1+1)} \, dx,
\]
where \(C_1 = \lambda C^{\alpha_1+1-\delta \beta_1} \|\xi^2\|_{\beta_1}\). Thanks to (12) Lemma in page 726], we know that \(\phi_1^{\gamma(\alpha_1+1)} \in L^1(\Omega)\). Then the above inequality yields \(\{u_n\}\) is bounded in 
\(W_0^{1,p}(\Omega)\). In quite similar way, we show that the sequence \(\{v_n\}\) is bounded in 
\(W_0^{1,q}(\Omega)\). We thus allowed to extract subsequences (still denoted by \(\{u_n\}\) and 
\(\{v_n\}\)) such that
\[
(u_n, v_n) \rightharpoonup (u, v) \text{ in } W_0^{1,p}(\Omega) \times W_0^{1,q}(\Omega).
\]
The convergence in (3.3) combined with Rellich embedding theorem and (2.15) enable us to get
\[
0 < \underline{u} \leq u \leq \overline{u} \quad \text{and} \quad 0 < \underline{v} \leq v \leq \overline{v} \quad \text{in } \Omega.
\]
Inserting \((\varphi, \psi) = (u_n - u, v_n - v)\) in (3.1) yields
\[
\begin{align*}
\begin{cases}
(-\Delta_p u_n, u_n - u) = \lambda \int_\Omega (u_n + \frac{1}{n})^\alpha v_n^\beta (u_n - u) \, dx \\
(-\Delta_q v_n, v_n - v) = \lambda \int_\Omega u_n^{\alpha_2} (u_n + \frac{1}{n})^{\beta_2} (v_n - v) \, dx.
\end{cases}
\end{align*}
\]
We claim that
\[
\lim_{n \to \infty} (-\Delta_p u_n, u_n - u) = \lim_{n \to \infty} (-\Delta_q v_n, v_n - v) \leq 0.
\]
Indeed, from (3.3), we have
\[
(u_n + \frac{1}{n})^\alpha v_n^\beta (u_n - u) \leq u_n^{\alpha_1} v_n^\beta (u_n - u) \leq u_n^{\alpha_1+1} v_n^\beta
\]
\[
\leq C_{-\delta_1} \|\xi_1\|_{\alpha_1+1} \|\xi_2\|_{\beta_1} \quad \text{if } \alpha_1 \geq -1,
\]
\[
C_{\alpha_1+1-\delta_1} \phi_1^{\gamma(\alpha_1+1)} \|\xi_2\|_{\beta_1} \quad \text{if } -1 - \frac{1}{\gamma} < \alpha_1 < -1.
\]
Then, using again the Lemma in page 726], the function
\[
h(x) = \begin{cases}
C_{-\delta_1} \|\xi_1\|_{\alpha_1+1} \|\xi_2\|_{\beta_1} & \text{if } \alpha_1 \geq -1, \\
C_{\alpha_1+1-\delta_1} \phi_1^{\gamma(\alpha_1+1)} \|\xi_2\|_{\beta_1} & \text{if } -1 - \frac{1}{\gamma} < \alpha_1 < -1.
\end{cases}
\]
belongs to \(L^1(\Omega)\). This way,
\[
(u_n + \frac{1}{n})^\alpha v_n^\beta (u_n - u) \in L^1(\Omega) \quad \text{and} \quad (u_n + \frac{1}{n})^\alpha v_n^\beta (u_n - u) \leq h \text{ a.e. in } \Omega.
\]
Using (3.3), (3.5), and applying Fatou’s lemma, it follows that
\[
\lim_{n \to \infty} \int_\Omega (u_n + \frac{1}{n})^\alpha v_n^\beta (u_n - u) \, dx
\]
\[
\leq \int_\Omega \lim_{n \to \infty} \sup \left( (u_n + \frac{1}{n})^\alpha v_n^\beta (u_n - u) \right) \, dx \to 0 \text{ as } n \to +\infty.
\]
Similarly, we prove that  \( \lim_{n \to \infty} (-\Delta_p v_n, v_n - v) \leq 0 \). Then the \( S_+ \)-property of \(-\Delta_p\) and \(-\Delta_q\) on \( W^{1,p}_0(\Omega) \) and \( W^{1,q}_0(\Omega) \), respectively, guarantees that

\[
(u_n, v_n) \to (u, v) \text{ in } W^{1,p}_0(\Omega) \times W^{1,q}_0(\Omega).
\]

Hence we may pass to the limit in (3.1) to conclude that \((u, v)\) is a solution of problem (1.1). On account of (3.4), the weak solution \((u, v)\) of (1.1) is positive.

Furthermore, using (3.4), (2.12), (2.13) and (2.4), we get

\[
u^{\alpha_1} v^{\beta_1} \leq C_1 d(x)^{\gamma_{\alpha_1}} \quad \text{for all } x \in \Omega
\]

and

\[
u^{\alpha_2} v^{\beta_2} \leq C_2 d(x)^{\gamma_{\beta_2}} \quad \text{for all } x \in \Omega,
\]

where \(C_1\) and \(C_2\) are constants given by

\[
C_1 = (C^{\sigma_1} \gamma_{\alpha_1}) (C^{\delta_1} \|x\|_\infty)^{\beta_1} \quad \text{and} \quad C_2 = (C^{\sigma_2} \gamma_{\beta_2}) (C^{\delta_2} \|x\|_\infty)^{\beta_2}.
\]

Then (1.7) enable us to apply the regularity theory (see [9, Lemma 3.1]) to infer that \((u, v)\) belongs to \( C^{1,\beta}(\overline{\Omega}) \times C^{1,\beta}(\overline{\Omega}) \) for some \(\beta \in (0, 1)\). This complete the proof.

**Proof of Theorem 2.** Arguing by contradiction and suppose that \((u, v)\) is a positive solution of problem (1.1). Multiplying the first and the second equation in (1.1) by \(\nu\) and \(v\), respectively, it follows from (1.6) and Young inequality that

\[
\int_{\Omega} |\nabla u|^{p} \, dx = \lambda \int_{\Omega} \nu^{\alpha+1} v \, dx \leq \lambda \int_{\Omega} (\frac{\alpha+1}{p} \nu^{\alpha+1} v + \frac{p-1-\alpha}{p} \nu^{\beta_1}) \, dx
\]

and

\[
\int_{\Omega} |\nabla v|^{p} \, dx = \lambda \int_{\Omega} u^{\alpha} \nu^{\beta_2+1} \, dx \leq \lambda \int_{\Omega} (\frac{\beta_2+1}{q} u^{\beta_2+1} v^{1-p} \nu^{\beta_1} + \frac{p-1-\alpha}{p} v^{p-1}) \, dx.
\]

Adding (3.6) with (3.7), according to (1.7), this is equivalent to

\[
\|\nabla u\|_p^p + \|\nabla v\|_q^q \leq \lambda \left( (\frac{\alpha+1}{p} + \frac{q-1-\beta_2}{q}) \|u\|_p^p + (\frac{\beta_2+1}{q} + \frac{p-1-\alpha}{p}) \|v\|_q^q \right).
\]

Since \(\Theta = 0\), observe from (1.7) that

\[
\left\{ \begin{array}{c}
\frac{\alpha+1}{p} + \frac{q-1-\beta_2}{q} = \frac{\alpha+1+q-1-\beta_2}{p} \\
\frac{\beta_2+1}{q} + \frac{p-1-\alpha}{p} = \frac{\beta_2+1+p-1-\alpha}{q}
\end{array} \right.
\]

Recalling that the eigenvalues \(\lambda_{1,p}\) and \(\lambda_{1,q}\) introduced in (2.1) and (2.2) can be characterized by the minimum of Rayleigh quotient

\[
\lambda_{1,p} = \inf_{u \in W^{1,p}_0(\Omega) \setminus \{0\}} \frac{\|\nabla u\|_p^p}{\|u\|_p^p} \quad \text{and} \quad \lambda_{1,q} = \inf_{v \in W^{1,q}_0(\Omega) \setminus \{0\}} \frac{\|\nabla v\|_q^q}{\|v\|_q^q}.
\]

Then gathering (3.8), (3.9) and (3.10) together yields

\[
(\lambda_{1,p} - \frac{\alpha+1+q-1-\beta_2}{p} \lambda) \|u\|_p^p + (\lambda_{1,q} - \frac{\beta_2+1+p-1-\alpha}{q} \lambda) \|v\|_q^q \leq 0
\]

which is a contradiction for

\[
0 < \lambda < \lambda_* = \min \left\{ \frac{p}{\alpha+1+q-1-\beta_2} \lambda_{1,p}, \frac{q}{\beta_2+1+p-1-\alpha} \lambda_{1,q} \right\}.
\]

Thus, problem (1.1) has no solution for \(\lambda < \lambda_*\), which ends the proof of Theorem 2.

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