Abstract

In this paper, we study the $L(2,1)$-Labeling of the Mycielski and the iterated Mycielski of graphs in general. For a graph $G$ and all $t \geq 1$, we give sharp bounds for $\lambda(M^t(G))$ the $L(2,1)$-labeling number of the $t$-th iterated Mycielski in terms of the number of iterations $t$, the order $n$, the maximum degree $\Delta$, and $\lambda(G)$ the $L(2,1)$-labeling number of $G$. For $t = 1$, we present necessary and sufficient conditions between the 4-star matching number of the complement graph and $\lambda(M(G))$ the $L(2,1)$-labeling number of the Mycielski of a graph, with some applications to special graphs. For all $t \geq 2$, we prove that for any graph $G$ of order $n$, we have $2^{t-1}(n+2) - 2 \leq \lambda(M^t(G)) \leq 2^t(n+1) - 2$. Thereafter, we characterize the graphs achieving the upper bound $2^t(n+1) - 2$, then by using the Marriage Theorem and Tutte's characterization of graphs with a perfect 2-matching, we characterize all graphs without isolated vertices achieving the lower bound $2^{t-1}(n+2) - 2$. We determine the $L(2,1)$-labeling number for the Mycielski and the iterated Mycielski of some graph classes.

Keywords  Frequency assignment · $L(2,1)$-Labeling · Mycielski construction · Matching

1 Introduction

The graphs considered in this paper are finite, simple, and undirected. For graph terminology, we refer to [23].

In 1992, J.R. Griggs and R.K. Yeh [11] studied a variation of the frequency assignment problem [12], where close transmitters must receive different channels and closer transmitters must receive different channels at least two apart. This problem is known as the $L(2,1)$-Labeling problem, the main target is to come up with a frequency assignment with low-frequency bandwidth.

Formally, the $L(2,1)$-labeling of a graph $G = (V, E)$, is a function $f$ from the vertex set $V$ to the set of all nonnegative integers, such that $|f(x) - f(y)| \geq 2$ if $d_G(x, y) = 1$ and $|f(x) - f(y)| \geq 1$ if $d_G(x, y) = 2$, where $d_G(x, y)$ is the distance between the vertices $x$ and $y$ in $G$. The span of an $L(2,1)$-labeling $f$ is the difference between the largest and the smallest label used by $f$. We may always consider zero as the smallest label used, so that the span is the highest label assigned. A $k$-$L(2,1)$-labeling is an $L(2,1)$-labeling with no label
greater than $k$, the minimum $k$ so that $G$ has a $k$-$L(2,1)$-labeling is called the $L(2,1)$-labeling number or $\lambda$-number of $G$, and denoted by $\lambda(G)$. An $L(2,1)$-labeling with span $\lambda(G)$ is called a $\lambda$-labeling.

The $L(2,1)$-labeling has been extensively studied (see surveys [3, 24]). The determination of the exact value of $\lambda(G)$ is an NP-Hard problem for graphs in general, it is NP-Complete to determine whether a graph admits an $L(2,1)$-labeling with span at most $\lambda \geq 4$ [7], the problem remains NP-Complete even restricted to some graph families (see NP-completeness results references in [3]). Therefore, the aim of the research was to bound the $\lambda$-number for graphs. By using the greedy algorithm, Griggs and Yeh [11] proved that $\lambda(G) \leq \Delta^2 + 2\Delta$ for any graph $G$, where $\Delta$ is the maximum degree of $G$. This upper bound was later improved by Gonçalves in [10] to $\Delta^2 + \Delta - 2$, and it is the best known upper bound for $\lambda(G)$ in terms of the maximum degree for graphs in general. Griggs and Yeh [11] conjectured that $\lambda(G) \leq \Delta^2$, for any graph $G$ with $\Delta \geq 2$, it is called $\Delta^2$-conjecture and is one of the most captivating open problems about graph labeling with distance conditions. This conjecture was proven to be true by Havet et al. [13] for graphs with a large maximum degree. The $L(2,1)$-labeling number attracted attention not only for general graphs but also when considering specific graph classes. The decision version of the $L(2,1)$-labeling problem has been proven to be polynomial for complete graphs, paths, cycles, wheels, trees, complete $k$-partite graphs, among other few graph classes. For an overview on the subject of the $L(2,1)$-labeling (and its generalizations), we refer the reader to the surveys [3, 24].

In this paper, we investigate the $L(2,1)$-labeling of the Mycielski and the iterated Mycielski of graphs. In search of triangle-free graphs with a large chromatic number, Mycielski [19] used the following transformation.

**Definition 1.1.** For a given graph $G = (V,E)$ of order $n$ with $V = \{v_1, v_2, \ldots, v_n\}$. The Mycielski graph of $G$, denoted $M(G)$, is the graph with vertex set $V \cup V'$, where $V' = \{v'_i : v_i \in V\}$ and edge set $E \cup \{v_i v'_j : v_i, v_j \in E\} \cup \{v'_i u : v'_i \in V'\}$. The vertex $v'_i$ is called the copy of the vertex $v_i$ and $u$ is called the root of $M(G)$.

The $t$-th iterated Mycielski graph of $G$, denoted $M^t(G)$, is defined recursively with $M^0(G) = G$ and for $t \geq 1$ $M^t(G) = M(M^{t-1}(G))$. If $t = 1$, $M^1(G)$ is the Mycielski graph of $G$ and is denoted simply $M(G)$. It is known that $\chi(M(G)) = \chi(G) + 1$, and $\omega(M(G)) = \max(2, \omega(G))$, for any graph $G$, where $\chi(G)$ and $\omega(G)$ are respectively the chromatic number and the clique number of $G$. Many aspects and invariants of the Mycielski graphs have been studied (see for example [2, 4, 5, 8, 10, 17, 20]). Mycielski graphs are known to be hard-to-color instances and are used for testing coloring algorithms [4]. The $L(2,1)$-labeling of the Mycielski of graphs has been previously investigated in [17] and [20]. A 4-star matching $H$ of a graph $G$ is a subgraph such that $H$ is a collection of vertex disjoint star graphs $K_{1,1}$, $K_{1,2}$, $K_{1,3}$ or $K_{1,4}$. The 4-star matching number is the maximum order of a 4-star matching of $G$. In [17], W. Lin and P. Lam gave sufficient conditions on the 4-star matching number of the complement graph $\bar{G}$, so that $\lambda(M(\bar{G})) \leq 2n$ and $\lambda(M(G)) = 2n + k$, for any $k \geq 1$. This allows them to prove that $\lambda(M(G))$ can be computed in polynomial time for graphs with diameter at most 2, and then give the $\lambda$-number of the Mycielski of complete graph $K_n$, and the Mycielski of the graph join of complete graph and the empty graph. Z. Shao and R. Solis-Oba in [20], also studied the $L(2,1)$-labeling number of the Mycielski and the iterated Mycielski of graphs. The authors as well gave the $\lambda$-number of the Mycielski of complete graph, and depending on the number of iterations determine the exact value or give bounds for $\lambda(M^t(K_n))$, then provided bounds for $\lambda(M^t(G))$ for any graph $G$.

In this paper, we continue the work started by Lin and Lam [17], and Shao and Solis-Oba [20]. In Section 2, we give some preliminary results about the Mycielski and iterated Mycielski of graphs, and some previous results on the $L(2,1)$-labeling number of graphs.

Section 3 is dedicated to the $L(2,1)$-labeling number of $M(G)$. First, we provide bounds involving the order $n$, the maximum degree $\Delta$ and the $\lambda$-number of $G$. Then we complete the equivalence relationship between the 4-star matching number and the $L(2,1)$-labeling number of the Mycielski of a graph. Afterward, we give applications of this result to the $L(2,1)$-labeling number of the Mycielski of some particular graphs, not mentioned in [17]. The end of Section 3 is dedicated to graphs with a lower bound $\lambda(M(G)) = n + 1$, we give a condition for a graph implying that $\lambda(M(G)) = n + 1$. Then we determine the $L(2,1)$-labeling number of $M(P_n)$ and $M(C_n)$ the Mycielski graph of path and cycle respectively, which allow us to determine all the connected graphs realizing $\lambda(M(G))$ equal to 4, 5 and 6 respectively.

Section 4 is devoted to the $t$-th iterated Mycielski of graphs with $t \geq 2$. As in Section 3, we give bounds for $\lambda(M^t(G))$ in terms of the number of iterations $t$, the order, the maximum degree, and $\lambda(G)$. Then we show that for all $t \geq 2$, $\lambda(M^t(K_n)) = |M^t(K_n)| - 1 = 2^t(n+1) - 2$, then we characterize all graphs having $\lambda(M^t(G)) = |M^t(G)| - 1 = 2^t(n+1) - 2$. Later, we give a necessary and sufficient condition for any graph $G$. 


without isolated vertices achieving a lower bound $2^t - (n + 2) - 2$ for the $\lambda$-number of the iterated Mycielski of $G$, we apply that to get an upper bound that can be calculated in polynomial time for any graph $G$, then we determine $\lambda(M(P_n))$, and $\lambda(M'(C_n))$. Finally, we propose a weak version of the $\Delta^2$-conjecture for the $L(2, 1)$-labeling of the Mycielski and iterated Mycielski of graphs.

2 Preliminaries and previous results

For a graph $G$, let $\Delta_M$, $\text{deg}_M(x)$, and $d_M(x, y)$ denote respectively, the maximum degree, the degree of a vertex $x$, and the distance between the vertices $x$ and $y$ in $M^t(G)$. If $t = 1$, we denote simply $\Delta_M$, $\text{deg}_M(x)$, and $d_M(x, y)$. As a consequence of Definition 1.1, we have the following.

Lemma 2.1. If $G$ is a graph of order $n$, then $|M^t(G)| = 2^t(n + 1) - 1$.

Proof. From Definition 1.1, we have $|M(G)| = 2n + 1 = 2(n + 1) - 1$. By using induction, we can show that $|M^t(G)| = 2^t(n + 1) - 1$.

Observation 2.1. If $H$ is a subgraph of a graph $G$, then for any $t \geq 1$, $M^t(H)$ is a subgraph of $M^t(G)$.

Lemma 2.2. For a graph $G$ of order $n$ and maximum degree $\Delta$. For any $t \geq 1$, we have $\Delta_M = \max(2^{t-1}(n + 1) - 1, 2^t\Delta)$.

Proof. By Definition 1.1, we have $\text{deg}_M(u) = n$, $\text{deg}_M(x) = 2\text{deg}_G(x)$, and $\text{deg}_M(x') = \text{deg}_G(x) + 1$ for all $x \in V$, where $x'$ is the copy of the vertex $x$ in $M(G)$. Then $\Delta_M = \max(n, 2\Delta)$. Suppose that for $k \geq 1$, we have $\Delta_M = \max(2^{k-1}(n + 1) - 1, 2^k\Delta)$.

For $k + 1$, if $2^{k-1}(n + 1) - 1 \geq 2^k\Delta$, then $\Delta_M = 2^{k-1}(n + 1) - 1$. Let $v$ be a vertex of $M^k(G)$, such that $\text{deg}_M(v) = \Delta_M$. From Definition 1.1, $\text{deg}_M(v) = 2\text{deg}_M(v) = 2^n + 1 < 2^k\Delta$. Also $\text{deg}_M(x') = \text{deg}_M(x) + 1 < 2^{k+1}\Delta$. So $\Delta_M = \max(2^{k}(n + 1) - 1, 2^k\Delta)$.

Otherwise, if $2^k\Delta \geq 2^{k-1}(n + 1)$, then by the inductive hypothesis $\Delta_M = 2^{k}\Delta$. We have $\text{deg}_M(v) = 2\text{deg}_M(x) \leq 2^{k+1}\Delta$, for all $x \in V$. For $x' \in M^{k+1}$, $\text{deg}_M(x') = \text{deg}_M(x) + 1 < 2^{k+1}\Delta$. Also $\text{deg}_M(x') = 2^{k}(n + 1) - 1 < 2^{k+1}\Delta$. Thus, $\Delta_M = 2^{k+1}\Delta$. It follows that $\Delta_M = \max(2^{k+1}(n + 1) - 1, 2^{k+1}\Delta)$.

Notice that $M(G)$ is a connected graph if and only if $G$ has no isolated vertices. The diameter of a graph $diam(G)$, is the greatest distance between any pair of vertices in $G$. If $G$ is disconnected, then $diam(G)$ is considered to be infinite. In [8], D.C Fisher et al. proved that $diam(M(G)) = \min(\max(2, diam(G)), 4)$, for every graph $G$ without isolated vertices. The following lemmas are a consequence of the proof of this result and the definition of $M(G)$.

Lemma 2.3. [8] For $v_i$ and $v_j$ two non-isolated vertices in $G$. We have $d_M(u, v_i) = 1$, $d_M(u, v_j) = 2$, $d_M(v_i', v_j') = 2$, $d_M(v_i, v_j) = \min(3, d(v_i, v_j))$, and $d_M(v_i, v_j) = \min(4, d(v_i, v_j))$.

If $v_i$ is an isolated vertex in $G$, then $v_i$ is isolated in $M(G)$, and $v_i'$ is adjacent to the root $u$.

Lemma 2.4. If $G$ is a graph without isolated vertices. For $t \geq 1$, $diam(M^t(G)) = \min(\max(2, diam(G)), 4)$.

Proof. Based on [8], we have $diam(\text{M}(G)) = \min(\max(2, diam(G)), 4)$. Suppose that for $k \geq 1$, we have $diam(M^k(G)) = \min(\max(2, diam(G)), 4)$. We have $M^{k+1}(G) = M(M^k(G))$, so $diam(M^{k+1}(G)) = \min(\max(2, diam(M^k(G))) = \min(\max(2, diam(M^k(G)), 4)$. If $diam(G) = 1$ or 2, then by the inductive hypothesis $diam(M^k(G)) = 2$, it follows that $diam(M^{k+1}(G)) = 2$. If $diam(G) = 3$, by the inductive hypothesis $diam(M^k(G)) = 3$ and so $diam(M^{k+1}(G)) = 3$. By using the same argument if $diam(G) \geq 4$, we get that $diam(M^{k+1}(G)) = 4$.

By Lemma 2.4, if the diameter of a graph $G$ is 1 or 2, then the diameter of the $t$-th iterated Mycielski $M^t(G)$ is 2, for any $t \geq 1$. It is clear from the definition of the $L(2, 1)$-Labeling, that any vertices at distance less than or equal to 2 must be assigned distinct labels. So for any diameter two graph $G$, all the vertices must be assigned different labels $\lambda(G) \geq |G| - 1$. These arguments will also be used throughout the paper.

We recall some previous results on the $L(2, 1)$-labeling of graphs.
Lemma 2.5. [11] If $G$ is a graph of maximum degree $\Delta \geq 1$, then $\lambda(G) \geq \Delta + 1$. If $\lambda(G) = \Delta + 1$, then for every vertex $v$ of degree $\Delta$, $f(v) = 0$ or $\Delta + 1$ for any $\lambda$-labeling $f$.

For $t \geq 1$, from Lemma 2.5 and Lemma 2.2, an obvious lower bound for $\lambda(M^t(G))$ would be $\max(2^{t-1}(n+1), 2^t \Delta + 1)$.

Lemma 2.6. [6] If $H$ is a subgraph of a graph $G$, then $\lambda(H) \leq \lambda(G)$.

Theorem 2.1. [11] If $G$ is a diameter 2 graph with maximum degree $\Delta$, then $\lambda(G) \leq \Delta^2$.

In the proof of Theorem 2.1, Griggs and Yeh proved that for a graph $G$ of order $n$ and maximum degree $\Delta \geq (n-1)/2 \geq 3$, we have $\lambda(G) < \Delta^2$. Since $\Delta_M = \max(n, 2\Delta)$ and $|M(G)| = 2n+1$, it means the $\Delta^2$-conjecture is true for the Mycielski of any graph $G$ of order $n \geq 3$.

The path covering number of a graph $p_n(G)$, is the smallest number of vertex-disjoint paths needed to cover all the vertices of a graph $G$. The complement graph $\overline{G}$ of a graph $G$ is the graph whose vertex set is $V$ and where $xy \in E(\overline{G})$ if only if $xy \notin E(G)$. In [4], Georges et al. related the path covering number of the complement graph $\overline{G}$ to the $L(2, 1)$-labeling number of $G$, in the following.

Theorem 2.2. [6] For any graph $G$ of order $n$, we have

• $\lambda(G) \leq n - 1$ if and only if $p_n(G) = 1$.
• $\lambda(G) = n + r - 2$ if and only if $p_n(\overline{G}) = r \geq 2$.

3 The Mycielski of a graph $M(G)$

3.1 Bounds for the $L(2, 1)$-labeling number of $M(G)$

Theorem 3.1. Let $G$ be a graph of order $n \geq 1$, and maximum degree $\Delta \geq 0$, we have

$$\max(n+1, 2(\Delta+1)) \leq \lambda(M(G)) \leq (n+1) + \lambda(G).$$

Proof. According to the definition of the Mycielski of a graph, the degree of the root $\deg_M(u) = n$, then $\lambda(M(G)) \geq n + 1$. Otherwise, for $\Delta \geq 1$, we have the star graph $K_{1, \Delta}$ is a subgraph of $G$. Then by Observation 2.1 and Lemma 2.0, we have $\lambda(M(G)) \geq \lambda(M(K_{1, \Delta}))$. Since $\text{diam}(K_{1, \Delta}) = 2$ and $|K_{1, \Delta}| = \Delta + 1$, it follows that $\text{diam}(M(K_{1, \Delta})) = 2$, and $\lambda(M(K_{1, \Delta})) \geq 2(\Delta + 1) - 1 = 2(\Delta + 1)$. Thus, $\lambda(M(G)) \geq 2(\Delta + 1)$.

For the upper bound, let $h$ be a $\lambda$-labeling of $G$. We denote $M(G)$ the Mycielski graph of $G$, with vertex set $V(M(G)) = \{v_i, v'_i, u : 1 \leq i \leq n\}$, where $v'_i$ is the copy of $v_i$ in $M(G)$ and $u$ is the root. Since every $\lambda$-labeling must assign the label 0 to a vertex of $G$, we consider without loss of generality that $h(v_n) = 0$. We define the following labeling $f$ on $V(M(G))$.

$$f(x) = \begin{cases} i - 1 & \text{if } x = v'_i, 1 \leq i \leq n, \\ n + h(v_i) & \text{if } x = v_i, 1 \leq i \leq n, \\ (n+1) + \lambda(G) & \text{if } x = u. \end{cases}$$

Now we will check that $f$ is an $L(2, 1)$-labeling of $M(G)$, we get five cases.

• We have $|f(v'_i) - f(v'_j)| = |i - j| \geq 1$ and $d_M(v'_i, v'_j) = 2$, for all $1 \leq i, j \leq n$.

• By Lemma 2.5 if $d_M(v_i, v_j) = 1$ (respectively $= 2$), then $d_G(v_i, v_j) = 1$ (respectively $= 2$). We have $|f(v_i) - f(v_j)| = |h(v_i) - h(v_j)|$. This means $|f(v_i) - f(v_j)| \geq 2$, if $d_M(v_i, v_j) = 1$ and $|f(v_i) - f(v_j)| \geq 1$, if $d_M(v_i, v_j) = 2$.

• For all $1 \leq i, j \leq n$, we have $|f(v_i) - f(v_j)| = |n + h(v_i) - j + 1|$. The distance two conditions are respected for all the following cases.
  1. $\text{if } 1 \leq j \leq n - 1, \text{ then } |f(v_i) - f(v'_j)| \geq 2$.
  2. $\text{if } j = n \text{ and } i = 1, \text{ we have } |f(v_n) - f(v'_1)| = 1, \text{ and } d_M(v_n, v'_1) \geq 2$.
  3. $\text{if } j = n \text{ and } d_G(v_i, v_n) = 1, \text{ we have } |h(v_i) - h(v_n)| \geq 2$, so $h(v_i) \geq 2$. It follows that $|f(v_i) - f(v'_n)| \geq 2$.
  4. $\text{if } j = n \text{ and } d_G(v_i, v_n) \geq 2$, by Lemma 2.3 we have $d_M(v_i, v'_n) \geq 2$, and $|f(v_i) - f(v'_n)| \geq 1$.

• For all $1 \leq i \leq n$, $|f(u) - f(v'_i)| = |(n+1) + \lambda(G) - i + 1| \geq 2$. 

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• For all $1 \leq i \leq n$, $|f(u) - f(v_i)| = |(n + 1) + \lambda(G) - (n + h(v_i))| \geq 1$, and $d_M(u, v_i) \geq 2$.

So $f$ is an $L(2,1)$-labeling of $M(G)$ with span $(n + 1) + \lambda(G)$. Hence $\lambda(M(G)) \leq (n + 1) + \lambda(G)$. \hfill \Box

**Corollary 3.1.** If $G$ is a diameter 2 graph of maximum degree $\triangle$, then $\lambda(M(G)) \leq 2(\triangle^2 + 1)$.

**Proof.** By Theorem 2.3 for a diameter 2 graph, we have $\lambda(G) \leq \triangle^2$. Also, we have $|G| \leq \triangle^2 + 1$, known as the Moore bound due to Hoffman and Singleton [14]. By combining this with the upper bound of Theorem 3.1 we get that $\lambda(M(G)) \leq 2(\triangle^2 + 1)$. \hfill \Box

The bound $2(\triangle^2 + 1)$ in Corollary 3.1 can only be attained by the Mycielski of diameter two Moore graphs [14], since the diameter of the Mycielski of these graphs is two, and these are the only diameter two graphs with order $\triangle^2 + 1$ and $\lambda$-number equal to $\triangle^2$ [11]. The only known graphs achieving this bound are $C_5$ the cycle of order 5, the Petersen graph, and the Hoffman-Singleton graph.

### 3.2 $L(2,1)$-labeling number of the Mycielski and the star matching of the complement

By using the upper bound of Theorem 3.1 and Theorem 2.2, we can link the $\lambda$-number of $M(G)$ to the path covering of the complement graph $\overline{G}$. So if $p_v(\overline{G}) = 1$, i.e. $\overline{G}$ has a Hamiltonian path, then $\lambda(M(G)) \leq 2n$, the equality holds for diameter two graphs. Also if $p_v(\overline{G}) \geq 2$, then $\lambda(M(G)) \leq 2n + p_v(\overline{G}) - 1$. But for more relevant conditions, the study of the path covering of the complement of $M(G)$ is required.

We can see that for any graph $G$, $\overline{M(G)}$ the complement of the Mycielski graph of $G$ is a connected graph. The neighborhood of $u$ in $\overline{M(G)}$ is $V$. For all $1 \leq i \leq n$, $v_i v'_i \in E(\overline{M(G)})$. For $i \neq j$, $v_i v'_j \in E(\overline{M(G)})$. Also $v_i v'_j, v_i v'_j \in E(\overline{M(G)})$ if and only if $v_i v_j \notin E(G)$. The subgraph induced by the set $V$ is $\overline{G}$. The subgraph induced by the set $V'$, is the complete graph on $n$ vertices.

Let $m$ be an integer greater or equal to $2$. An $m$-star matching $H$ of $G$ is a subgraph of $G$, such that each component of $H$ is isomorphic to a star graph $K_{1,i}$, with $1 \leq i \leq m$. The $m$-star matching number, denoted $s_m(G)$, is the maximum order of an $m$-star matching of $G$, an $m$-star matching of order $s_m(G)$, is said to be maximum. If $s_m(G) = |G|$, we say that $G$ has a perfect $m$-star matching, a perfect $m$-star matching is known also as star-factor or $\{K_{1,1}, K_{1,2}, \ldots, K_{1,m}\}$-factor [22], the problem of finding whether or not a graph $G$ admits a perfect $m$-star matching can be solved in polynomial time [15]. In [17], Lin and Lam studied the $m$-star matching and the $m$-star matching number $s_m(G)$. They delivered an algorithm to compute $s_m(G)$ running in $O(|V||E|)$. Then they related the 4-star matching number of $\overline{G}$ to the path covering number of $\overline{M(G)}$. In the following we denote by $i_4(G)$ the number of vertices unmatched in a maximum 4-star matching of $G$, i.e. $i_4(G) = n - s_4(G)$.

**Theorem 3.2.** [17] For any graph $G$, we have

(i) if $i_4(G) \leq 4$, then $p_v(\overline{M(G)}) = 1$.
(ii) if $i_4(G) \geq 5$, then $p_v(\overline{M(G)}) = \lceil \frac{i_4(G)}{2} \rceil - 1$.

We show that the converse holds in both cases, similarly to Theorem 2.2 in [9].

**Theorem 3.3.** For any graph $G$, we have

(a) $i_4(G) \leq 4$ if and only if $p_v(\overline{M(G)}) = 1$.
(b) $\lceil \frac{i_4(G)}{2} \rceil = r \geq 3$ if and only if $p_v(\overline{M(G)}) = r - 1$.

**Proof.** (a) Considering (i) and the contraposition of (ii) in Theorem 3.2 we get the necessity and sufficiency.
(b) We use induction on $r$. Let $r = 3$.

**Claim 3.1.** If $p_v(\overline{M(G)}) = 2$, then the root $u$ is not an end-vertex of a path in a minimum path covering of $\overline{M(G)}$.

**Proof.** If $p_v(\overline{M(G)}) = 2$, let $P^1$ and $P^2$ be the two paths of a minimum path covering of $\overline{M(G)}$, suppose that $u$ is an end-vertex of $P^1$. Since $u$ is adjacent in $\overline{M(G)}$ to every vertex in $V$, a vertex in $V$ cannot be an end-vertex of $P^2$, otherwise $\overline{M(G)}$ has a Hamiltonian path. So both ends of $P^2$ are from $V'$. Since the subgraph induced by $V'$ is a complete graph, the other extremity of $P^1$ is in $V$. Let $z$ be the other end of $P^1$, $x'$ and $y'$ the ends of $P^2$. Since $u$ is adjacent to $z$, and $x'$ is adjacent to $y'$. If $z'$ the copy of $z$ belongs
to $P^1$, we have $z'$ is adjacent to $x'$ and $y'$, we can construct a Hamiltonian path of $\overrightarrow{M}(G)$. If $z'$ belongs to $P^2$, since $z$ is adjacent to $z'$, in this case also $\overrightarrow{M}(G)$ has a Hamiltonian path, a contradiction. \hfill \Box

If $p_v(\overrightarrow{M}(G)) = 2$, let $x, y \in V$, such that $x$ or its copy and $y$ or its copy are end-vertices of the two different paths in a minimum path covering of $\overrightarrow{M}(G)$. We consider the graph $H$ with vertex set $V$, and edge set of its complement $E(\overleftarrow{H}) = E(\overrightarrow{G}) \cup \{xy\}$. It is clear that $p_v(\overrightarrow{M}(H)) = 1$, and $i_4(\overleftarrow{H}) \geq i_4(G) - 2$. Since $p_v(\overrightarrow{M}(G)) = 2$, according to (a) we have $4 \geq i_4(H)$, and $i_4(G) \geq 5$. It follows that $\lceil \frac{i_4(G)}{2} \rceil = 3$. So from Theorem 3.3 (ii), we have Theorem 3.4 (b) is true for $r = 3$. We suppose that (b) is true for $3 \leq r \leq k$, and let $r = k + 1$.

If $p_v(\overrightarrow{M}(G)) = k$. Let $x, y \in V$, such that $x$ or its copy and $y$ or its copy are end-vertices of two different paths in a minimum path covering of $\overrightarrow{M}(G)$. We consider the graph $H$ with vertex set $V$, and edge set of its complement $E(\overleftarrow{H}) = E(\overrightarrow{G}) \cup \{xy\}$. We have $p_v(\overrightarrow{M}(H)) = k - 1$, and $i_4(\overleftarrow{H}) \geq i_4(G) - 2$. So by the inductive hypothesis $\lceil \frac{i_4(H)}{2} \rceil = k$, hence $2k + 2 \geq i_4(G)$. Since $p_v(\overrightarrow{M}(G)) = k$, by the inductive hypothesis $i_4(G) \geq 2k + 1$. It follows that $\lceil \frac{i_4(G)}{2} \rceil = k + 1$. Theorem 3.2 (ii) completes the equivalence. \hfill \Box

By combining Theorem 3.2 and Theorem 3.3 we get the following results.

**Theorem 3.4.** For any graph $G$ of order $n$, we have
(a) $\lambda(M(G)) \leq 2n$ if and only if $i_4(G) \leq 4$.
(b) For any positive integer $r$, we have
$$\lambda(M(G)) = 2n + r \text{ if and only if } \left\lceil \frac{i_4(G)}{2} \right\rceil = r + 2.$$ 

Next, we give applications of this previous theorem to the $\lambda$-number of the Mycielski of certain graphs.

If the diameter of $G$ is 1 or 2, then $diam(M(G)) = 2$, we can conclude from Theorem 3.4 that $\lambda(M(G)) = 2n + \max\{2, \left\lceil \frac{i_4(G)}{2} \right\rceil\} - 2$.

**Corollary 3.2.** Let $G$ be a graph of order $n$, if the clique number $\omega(G) \leq 4$, then $\lambda(M(G)) \leq 2n$.

**Proof.** By Theorem 3.3 (a) if $\lambda(M(G)) > 2n$, then $i_4(G) \geq 5$. This means that $\omega(G) \geq 5$. \hfill \Box

The graphs with clique number less or equal to 4 in Corollary 3.2 include trees, planar graphs, and subcubic graphs.

If $X$ is any subset of $V$, we denote $N_G(X)$ the set of all vertices in $V$ adjacent to at least one vertex from $X$ in $G$. In [17], a criterion for a graph to have a perfect $m$-star matching is given, this appeared also in [1 15 22].

**Theorem 3.5.** [1 15 17 22] A graph $G$ has a perfect $m$-star matching if and only if for any independent set $S$ in $G$, $|N_G(S)| \geq |S|/m$.

**Corollary 3.3.** For a graph $G$ of order $n$ and maximum degree $\Delta \leq n - 2$. If $3(n - 1) + \delta \geq 4\Delta$, then $\lambda(M(G)) \leq 2n$.

**Proof.** Let $\overline{\Delta}$ and $\overline{\delta}$ denote respectively the maximum and minimum degree of the complement graph $\overrightarrow{G}$. For any independent set $S$ in $G$, let $|E_{\overrightarrow{G}}(S)|$ denote the number of edges incident to the vertices of $S$ in $G$, we have
$$|N_{\overrightarrow{G}}(S)| \overline{\Delta} \geq |E_{\overrightarrow{G}}(S)| \geq \overline{\delta}|S| \quad (1)$$

If $3(n - 1) + \delta \geq 4\Delta$, since $\overline{\Delta} = (n - 1) - \delta$ and $\overline{\delta} = (n - 1) - \Delta$ means $\overline{\delta} \geq \overline{\Delta}$, therefore from Inequality (1) we get that $|N_{\overrightarrow{G}}(S)| \geq |S|/4$, for any $S$ independent set in $G$. Then by Theorem 3.5 $G$ has a perfect 4-star matching. Hence from Theorem 3.4 (a), we have $\lambda(M(G)) \leq 2n$. \hfill \Box

From Corollary 3.3 any regular graph $G$ of order $n$, except complete graphs, has $\lambda(M(G)) \leq 2n$. In [17], it is shown that for complete graph $\lambda(M(K_2)) = 4$ and $\lambda(M(K_n)) = 2n + \left\lceil \frac{n}{2} \right\rceil - 2$ for $n \geq 3$. Next, we determine the exact $\lambda$-number of the Mycielski of complete $k$-partite graphs.
Corollary 3.4. Let $G$ be a complete $k$-partition of order $n$, where the partition sets consist of $p$ sets of order greater or equal to $2$ and $q$ singletons.

- If $q \leq 4$, then $\lambda(M(G)) = 2n$.
- If $q \geq 5$, then $\lambda(M(G)) = 2n + \lceil \frac{q}{2} \rceil - 2$.

Proof. We have $\overline{G}$ is formed of $p$ connected components that are complete graphs of order greater or equal to $2$, and $q$ isolated vertices. Therefore $\lambda(\overline{G}) = q$. If $q \leq 4$, by Theorem 3.4 (a), $\lambda(M(G)) \leq 2n$. Since $\text{diam}(M(G)) = 2$, it follows that $\lambda(M(G)) = 2n$. If $q \geq 5$, then by Theorem 3.4 (b), $\lambda(M(G)) = 2n + \lceil \frac{q}{2} \rceil - 2$. □

Let $G_1, G_2$ be two disjoint graphs. The disjoint union of $G_1$ and $G_2$, denoted by $G_1 \cup G_2$, is the graph with vertex set $V(G_1) \cup V(G_2)$ and edge set $E(G_1) \cup E(G_2)$. The joint of $G_1$ and $G_2$ denoted $G_1 \vee G_2$ is the graph obtained from $G_1 \cup G_2$ by joining each vertex of $G_1$ to each vertex of $G_2$.

Corollary 3.5. Let $G_1, G_2, \ldots, G_k$ be a collection of disjoint graphs having respectively $n_1, n_2, \ldots, n_k$ vertices. Let $n = \sum_{i=1}^{k} n_i$, then $\lambda(M(G_1 \vee G_2 \vee \ldots \vee G_k)) = 2n + \max\{2, \lceil \frac{n}{2} \rceil \} - 2$, where $I = \sum_{i=1}^{k} \lambda(M(G_i))$.

Proof. Let $G = G_1 \vee G_2 \vee \ldots \vee G_k$, we have $\overline{G} = \overline{G_1} \cup \overline{G_2} \cup \ldots \cup \overline{G_k}$. It follows that $\lambda(M(G)) = \sum_{i=1}^{k} \lambda(M(G_i)) = I$. Thus, by Theorem 3.4 (a) if $I \leq 4$, then $\lambda(M(G)) \leq 2n$. Since $\text{diam}(M(G)) = 2$, it follows that $\lambda(M(G)) = 2n$. If $I \geq 5$, from Theorem 3.4 (b), $\lambda(M(G)) = 2n + \lceil \frac{I}{2} \rceil - 2$. □

3.3 Graphs with $\lambda(M(G)) = n + 1$

For $k \geq 1$, the $k$th power of a graph $G$ is the graph $G^k$ with vertex set $V$ and edge set $E(G^k) = \{v_i v_j : 1 \leq d_G(v_i, v_j) \leq k\}$. Then the square of a graph $G^2$ has the edge set of its complement graph $E(\overline{G^2}) = \{v_i v_j : d_G(v_i, v_j) \geq 3\}$. Next we give a condition, so that $\lambda(M(G)) = n + 1$.

Lemma 3.1. In a graph $G$ of order $n$, if the vertex set $V$ can be partitioned into $k \geq 1$ vertex-disjoint cliques in $G^2$, such that at least $k - 1$ cliques are of order greater or equal 3. Then $\lambda(M(G)) = n + 1$.

Proof. Let $V = \bigcup_{r=1}^{k} S_r$, such that $S_r$ are vertex-disjoint cliques in $G^2$ of order $|S_r| = n_r \geq 3$ for $1 \leq r \leq k - 1$, and $|S_k| = n_k \geq 1$, where $\sum_{r=1}^{k} n_r = n$. For $1 \leq r \leq k$, let us denote $S_r = \{v_{i,r} : 1 \leq i \leq n_r\}$, $v_{i,r}$ is the copy of the vertex $v_i$, and $n_r$ is the root of $M(G)$. We have $d_G(v_{i,r}, v_{j,r}) \geq 3$ for any two distinct vertices in $S_r$, so a vertex in $S_{r+1}$ can be adjacent to at most one vertex in $S_r$. For $1 \leq r \leq k - 1$, the cliques $S_r$ in $G^2$ are symmetric of order greater or equal 3, we suppose without loss of generality that $d_G(v_{i,r}, v_{i+1,r}) \geq 2$, for $1 \leq r \leq k - 1$. Let $v_1 = 0$ and for $r \geq 2$, $v_r = v_{j,r}$. With respect to the previous assumption, we label the vertices of $M(G)$ as following.

- For $1 \leq r \leq k - 1$, define $f(v_{i,r}) = v_r$. For $2 \leq i \leq n_r$, $f(v_{i,r}) = v_r + 1$. Also $f(v'_{1,r}) = v_r + 1$, and $f(v'_{2,r}) = v_r + i - 1$.
- If $|S_k| = 1$, let $f(v_{1,k}) = n$, and $f(v'_{1,k}) = n - 1$.
- If $|S_k| = 2$, let $f(v_{1,k}) = n - 2$, $f(v'_{1,k}) = n - 1$, $f(v_{2,k}) = n - 1$, and $f(v'_{2,k}) = n - 2$.
- If $|S_k| \geq 3$, define $f(v_{1,k}) = v_k$. For $2 \leq i \leq n_k$, $f(v_{i,k}) = v_k + 1$. Also $f(v'_{1,k}) = v_k + 1$, and $f(v'_{2,k}) = v_k + i - 1$.

Finally, label the root $f(u) = n + 1$. We have $d_G(v_{i,r}, v_{j,r+1}) \geq 3$, and for $1 \leq r \leq k - 1$ we have $d_G(v_{i,r}, v_{i+1,r}) \geq 3$. This means by Lemma 2.4 that $d_M(v_{i,r}, v_{j,r+1}) \geq 3$, $d_M(v_{i,r}, v_{j,r}) = 3$, and $d_M(v'_{i,r}, v_{i+1,r+1}) \geq 2$. The labeling $f$ is an $L(2,1)$-labeling of $M(G)$ with span $n + 1$. Hence $\lambda(M(G)) = n + 1$. □

In the case of the empty graph $\overline{K_n}$, we have $\lambda(M(\overline{K_n})) \cong K_{1,n} \cup \overline{K_n}$. Since $\lambda(K_{1,n}) = n + 1$, we have $\lambda(M(\overline{K_n})) = n + 1$, we can get the same result using Lemma 3.1. We are now interested in some connected graphs, we consider the graph path $P_n$ and cycle $C_n$.  

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Let $P_n$ denote the graph path of order $n \geq 3$, with vertex set $V(P_n) = \{v_1, \ldots, v_n\}$ and edge set $E(P_n) = \{v_iv_{i+1} : 1 \leq i \leq n - 1\}$. Denote $V(M(P_n)) = V(P_n) \cup \{v'_i : 1 \leq i \leq n\} \cup \{u\}$, where $v'_i$ is the copy of the vertex $v_i$, and $u$ is the root of $M(P_n)$.

**Proposition 3.1.**

$$\lambda(M(P_n)) = \begin{cases} 6 & \text{if } n = 3, 4, \\ 7 & \text{if } n = 5, \\ n + 1 & \text{if } n \geq 6. \end{cases}$$

**Proof.**

- For $n = 3$, we have $\text{diam}(P_3) = 2$. So from Theorem 3.1, $\lambda(M(P_3)) = 6$.
- For $n = 4$, we have a $6$-$L(2,1)$-labeling of $M(P_4)$ shown in Figure 1. Hence $\lambda(M(P_4)) \leq 6$. Also we have $M(P_4)$ is a subgraph of $M(P_5)$. By Lemma 2.6, it follows that $\lambda(M(P_4)) \geq \lambda(M(P_3)) = 6$. Thus, $\lambda(M(P_4)) = 6$.
- For $n = 5$, Figure 2 illustrates a $7$-$L(2,1)$-labeling of $M(P_5)$. This implies also by Theorem 3.1 that $6 \leq \lambda(M(P_3)) \leq 7$.

![Figure 1: A 6-$L(2,1)$-labeling of $M(P_4)$](image1)

![Figure 2: A 7-$L(2,1)$-labeling of $M(P_5)$](image2)

Suppose that $\lambda(M(P_3)) = 6$. Then there is an $L(2,1)$-labeling $f$ of $M(P_5)$ using labels in the set $L = \{0, 1, 2, 3, 4, 5, 6\}$. Since $\text{deg}_M(u) = 5$, by Lemma 2.6, $f(u) = 0$ or $6$, without loss of generality, we suppose that $f(u) = 0$. We denote $N(v)$ the open neighborhood of a vertex $v$, and $N^2(v)$ the set of all vertices at distance at most 2 from a vertex $v$ in $M(P_5)$. We have $N(u) = \{v'_1, v'_2, v'_3, v'_4, v'_5\}$, and $d_M(v'_i, v'_j) = 2$, for $1 \leq i, j \leq 5$. So each vertex $v'_i$ receives a distinct label from the set $\{2, 3, 4, 5, 6\}$. We have $N^2(v_3) = \{u, v'_1, v'_2, v'_3, v'_4, v'_5\}$, and each vertex in $N^2(v_3)$ having a distinct label in $\{0, 2, 3, 4, 5, 6\}$, which leaves only the label 1 from $L$ available for $v_3$. Then $f(v_3) = 1$.

We have $N^2(v_2) = \{u, v_3, v'_1, v'_2, v'_3, v'_4, v'_5\}$, each vertex in $N^2(v_2)$ having a distinct label from $L$. So $f(v_2)$ is a distinct label for $v_2$. Also $N^2(v_4) = \{u, v_3, v'_2, v'_3, v'_4, v'_5\}$, so $f(v_4) = f(v'_1)$. Therefore $f(v_4)$ is a distinct label for $v_4$. Also $N^2(v_1) = \{u, v_2, v_3, v'_1, v'_2, v'_3, v'_4\}$, each vertex in $N^2(v_1)$ having a distinct label from $L$.

Suppose that $\lambda(M(P_3)) = 7$. Then we define a labeling $f$ on $V(M(P_n))$ as follows.

- For $n \geq 6$, we define a labeling $f$ on $V(M(P_n))$ as follows.
  - $f(u) = 0$, $f(v'_1) = 6$, $f(v'_2) = 5$, $f(v'_3) = 4$, $f(v'_4) = 7$, $f(v'_5) = 2$, $f(v'_6) = 3$, and $f(v'_i) = i + 1$ if $i \geq 7$.

- For $n \geq 6$, we define a labeling $f$ on $V(M(P_n))$ as follows.
  - $f(u) = 0$, $f(v'_1) = 6$, $f(v'_2) = 5$, $f(v'_3) = 4$, $f(v'_4) = 7$, $f(v'_5) = 2$, $f(v'_6) = 3$, and $f(v'_i) = i + 1$ if $i \geq 7$.

The idea is to come up with a $7$-$L(2,1)$-labeling of the subgraph induced by $H = \{u, v_i, v'_i : 1 \leq i \leq n\}$ isomorphic to $M(P_6)$. Then it follows from Theorem 3.1 that $\lambda(M(P_n)) = n + 1$, for $n \geq 6$. It follows from Theorem 3.1 that $\lambda(M(P_n)) = n + 1$, for $n \geq 6$. 

$\square$
Let \( C_n \) be the graph cycle, with vertex set \( V(C_n) = \{v_0, v_1, \ldots, v_{n-1}\} \) and edge set \( E(C_n) = \{v_i v_{i+1(\mod n)} : 0 \leq i \leq n-1\} \), where the indices are taken modulo \( n \). We denote \( V(M(C_n)) = V(C_n) \cup \{v'_i : 1 \leq i \leq n\} \cup \{u\} \), we have \( E(M(C_n)) = \{v_i v_{i+1(\mod n)}, v_i v'_{i+1(\mod n)} : 0 \leq i \leq n-1\} \cup \{v'_i u : 0 \leq i \leq n-1\} \).

**Proposition 3.2.**

\[
\lambda(M(C_n)) = \begin{cases} 
6 & \text{if } n = 3, \\
8 & \text{if } n = 4, \\
10 & \text{if } n = 5, \\
n+1 & \text{if } n \geq 6.
\end{cases}
\]

**Proof.**

- For \( 3 \leq n \leq 5 \), since diam\((C_3) = 1\), diam\((C_4) = \text{diam}(C_5) = 2\), from Lemma 2.3 diam\((M(C_3)) = \text{diam}(M(C_4)) = \text{diam}(M(C_5)) = 2\). By applying Theorem 3.4 we get that \( \lambda(M(C_3)) = 6 \), \( \lambda(M(C_4)) = 8 \), and \( \lambda(M(C_5)) = 10 \).

- For \( n \geq 6 \), we have Figure 3, Figure 4, and Figure 5 respectively present an \( L(2,1) \)-Labeling for \( M(C_6) \), \( M(C_7) \), and \( M(C_8) \), respectively with span 7, 8, and 9. It follows from the lower bound in Theorem 3.1 that \( \lambda(M(C_6)) = 7 \), \( \lambda(M(C_7)) = 8 \), and \( \lambda(M(C_8)) = 9 \).

![Figure 3: A 7-L(2,1)-labeling of \( M(C_6) \)](image)

![Figure 4: A 8-L(2,1)-labeling of \( M(C_7) \)](image)

![Figure 5: A 9-L(2,1)-labeling of \( M(C_8) \)](image)

For \( n \geq 9 \), we partition the vertex set \( V(C_n) \) into cliques in \( C_n^2 \) as following.

If \( n \equiv 0 \pmod{3} \), for \( 0 \leq i \leq \frac{n}{3} - 1 \), the sets \( S_i = \{v_{i}, v_{i+\frac{n}{3}}, v_{i+2\frac{n}{3}}\} \) form disjoint cliques of order 3 in \( C_n^2 \). We have \( V(C_n) = \bigcup_{i=0}^{\frac{n}{3}-1} S_i \).

If \( n \equiv 1 \pmod{3} \), for \( 0 \leq i \leq \frac{n}{3} \}, the sets \( S_i = \{v_{i}, v_{i+\frac{n}{3}}, v_{i+2\frac{n}{3}}\} \) form disjoint cliques of order 3 in \( C_n^2 \). We have \( V(C_n) = \bigcup_{i=0}^{\frac{n}{3}-1} S_i \cup \{v_{n-1}\} \).

If \( n \equiv 2 \pmod{3} \), for \( 1 \leq i \leq \frac{n}{3} \}, the sets \( S_i = \{v_{i}, v_{i+\frac{n}{3}}, v_{i+2\frac{n}{3}}\} \) form disjoint cliques of order 3 in \( C_n^2 \), and \( v_0 v_{\frac{n}{3}} \) is an edge in \( C_n^2 \). We have \( V(C_n) = \bigcup_{i=1}^{\frac{n}{3}-1} S_i \cup \{v_0, v_{\frac{n}{3}}\} \).

The cycle \( C_n \) in the three cases verifies the condition in Lemma 3.1. Hence \( \lambda(M(C_n)) = n+1 \) for \( n \geq 6 \).
For a connected graph $G$ of order $n$, in Theorem 3.1 we have $\lambda(M(G)) \geq n + 1$. It means that for any fixed positive integer $k$, there are finitely many connected graphs having $\lambda(M(G)) = k$. In the following we characterize the connected graphs with $\lambda(M(G))$ equal to 4, 6 and 7, these are the smallest possible values for the $\lambda$-number of the Mycielski of any non-trivial connected graph.

**Corollary 3.6.** For a connected graph $G$, we have

1. $\lambda(M(G)) = 4$ if and only if $G$ is $K_2$,
2. $\lambda(M(G)) = 6$ if and only if $G \in \{P_5, P_6, C_6\}$,
3. $\lambda(M(G)) = 7$ if and only if $G \in \{P_6, P_6, C_6\}$.

**Proof.** From Theorem 3.1 for a connected graph $G$ of order $n$ and maximum degree $\Delta$, we have $\lambda(M(G)) \geq \max(n + 1, 2(\Delta + 1))$. This means if $\Delta \geq 3$, then $\lambda(M(G)) \geq 8$. The only connected graph with $\Delta = 1$ is $K_2$, and we have from Theorem 3.1, $\lambda(M(K_2)) = 4$. If $\Delta = 2$, then $G$ is either a path or a cycle, from Theorem 3.1 we have $\lambda(M(G)) \geq 6$. By using Proposition 3.1 and Proposition 3.2 we can conclude the results.

4 The iterated Mycielski of a graph $M^t(G)$

4.1 Bounds for $\lambda(M^t(G))$

**Theorem 4.1.** If $G$ is a graph of order $n \geq 2$ and maximum degree $\Delta \geq 0$. For $t \geq 2$, we have

$$2^{t-1}\max(n + 2, 2(\Delta + 2)) - 2 \leq \lambda(M^t(G)) \leq (2^t - 1)(n + 1) + \lambda(G).$$

**Proof.** For a graph $G$ of order $n \geq 2$ from Definition 2.1, we have $K_{1,n}$ is a subgraph of $M(G)$. Then by Observation 2.1 $M^{t-1}(K_{1,n})$ is a subgraph of $M^t(G)$. Since $\text{diam}(K_{1,n}) = 2$, it follows from Lemma 2.1 and Lemma 2.6 that $\lambda(M^t(G)) \geq \lambda(M^{t-1}(K_{1,n})) \geq |M^{t-1}(K_{1,n})| - 1$. By Lemma 2.7 $|M^{t-1}(K_{1,n})| = 2^{t-1}(n + 2) - 1$, hence $\lambda(M^t(G)) \geq 2^{t-1}(n + 2) - 2$, for $t \geq 2$. If $\Delta \geq 1$, we have $K_{1,\Delta}$ is a subgraph of $G$. By using the same arguments as preceding, we get that $\lambda(M^t(G)) \geq 2^t(\Delta + 2) - 2$.

On the other hand, for $t \geq 2$, we have $M^t(G) = M(M^{t-1}(G))$. So by the upper bound of Theorem 3.1, $\lambda(M^t(G)) \leq (|M^{t-1}(G)| + 1) + \lambda(M^{t-1}(G)) = 2^{t-1}(n + 1) + \lambda(M^{t-1}(G))$. Recursively we get that $\lambda(M^t(G)) \leq \sum_{i=0}^{t-1} 2^i(n + 1) + \lambda(G) = (2^t - 1)(n + 1) + \lambda(G)$.

Notice that the lower bound $2^{t-1}(n + 2) - 2$ and the upper bound of Theorem 4.1 are true even for the trivial graph $K_1$. The upper bound coincides with the upper bound in Theorem 3.1 for $t = 1$. As a consequence we make the following observation.

**Observation 4.1.** If a graph $G$ of order $n$ has $\lambda(G) \leq n - 1$, then for any $t \geq 1$, $\lambda(M^t(G)) \leq |M^t(G)| - 1 = 2^t(n + 1) - 2$, there is equality if $G$ is of diameter two.

Further, we denote $V^t = \{v_i^k : 1 \leq i \leq n$ and $0 \leq k \leq 2^t - 1\}$, the set composed of the vertices of $V$ and all their copies in $M^t(G)$, where $v_i^0$ is the copy of $v_i^0$ in $M(G)$. $v_i^2$ and $v_i^3$ are respectively the copies of $v_i^0$ and $v_i^1$ in $M^2(G)$. $v_i^4, v_i^5, v_i^6, v_i^7$ are respectively the copies of $v_i^0, v_i^1, v_i^2, v_i^3$ in $M^3(G)$ and so forth. In $M^t(G)$ for $0 \leq k \leq 2^{t-1} - 1$, we have $v_i^{t-1+k}$ is the exact copy of the vertex $v_i^k$ from $M^{t-1}(G)$. For $t \geq 2$, let $U_i$ be the set of all the roots (i.e. roots and their consecutive copies in all levels) in $M^t(G)$. Recursively $U_i = U_{i-1} \cup U_{i-1} \cup \{u_{t,0}\}$ and $|U_i| = 2^t - 1$. We denote the set of roots $U_i = \{u_{i,j} : 1 \leq i \leq t$ and $0 \leq j \leq 2^{t-1} - 1\}$, such that for example in $M^3(G)$, $u_{1,0}$ is the root of $M^3(G)$, $u_{1,1}$ the copy of $u_{1,0}$, and $u_{2,0}$ the root of $M^2(G)$. $u_{1,2}, u_{1,3}, u_{2,1}$ are respectively the copies of $u_{1,0}, u_{1,1}, u_{2,0}$, and $u_{3,0}$ is the root in $M^3(G)$, and so forth. Figure 3 illustrate an adjacency of a vertex and its copies $v_i^k$ in $M^2(G)$, with respect to the above ordering.

**Lemma 4.1.** If $d_G(v_i^0, v_j^0) \leq 2$, then for any $t \geq 1$ and all $0 \leq k, m \leq 2^t - 1$, we have $d_{M^t}(v_i^k, v_j^m) \leq 2$, and if $v_i^k$ is not an isolated vertex for $k \neq m$, we have $d_{M^t}(v_i^k, v_j^m) = 2$.

**Proof.** By using Lemma 2.3 inductively, we get the results.

The *eccentricity* of a vertex $v$ in a graph $G$, being the greatest distance between $v$ and any other vertex in $G$. By Lemma 4.1 if a vertex has eccentricity 1 or 2 in $G$, then the vertex and all its copies are of eccentricity 2.
in $M^t(G)$. In a graph $G$ without isolated vertices, we have from the definition of the Mycielski construction, the eccentricity of the root in $M(G)$ is 2, so from above the eccentricity of all the roots and their copies is 2 in $M^t(G)$, for any $t \geq 1$.

**Proposition 4.1.** If $G$ is a graph without isolated vertices of order $n$, with $k$ vertices of eccentricity 2, for $t \geq 1$, we have $\lambda(M^t(G)) \geq 2^{t-1}(n + k + 2) - 2$.

**Proof.** For $t \geq 1$, let $v_0^0, v_0^1, \ldots, v_0^k$ be the vertices of eccentricity 2 in $G$. Let $V_i^{t-1}$ be the set composed of a vertex $v_i^t$ and all its copies in $M^{t-1}(G)$. In $M^t(G)$, by Lemma 4.1 and Definition 1.1 the vertices in $\bigcup_{i=1}^k V_i^{t-1} \cup V_i^t \cup U_{t-1} \cup \{u_{t,0}\}$ are all within distance two, where $U_{t-1}$ is the set of roots and their copies in $M^{t-1}(G)$, $V_i^{t-1}$ is the set of copies of the vertices of $M^{t-1}(G)$ in $M^t(G)$, and $u_{t,0}$ is the root of $M^t(G)$. Hence $\lambda(M^t(G)) \geq \sum_{i=1}^k |V_i^{t-1}| + |V_i^t| + |U_{t-1}| = k2^{t-1} + 2^{t-1}(n + 1) - 1 + 2^{t-1} - 1 = 2^{t-1}(n + k + 2) - 2$. \hfill $\square$

For a graph $G$ of order $n$, by Proposition 4.1 if $\lambda(M(G)) = n+1$, then $G$ has at most one vertex of eccentricity 2. Also for $t \geq 2$, if $\lambda(M^t(G)) = 2^{t-1}(n + 2) - 2$, then no vertex in $G$ has eccentricity 2. There exist graphs with one vertex of eccentricity 2 and $\lambda(M(G)) = n + 1$, Figure 7 illustrate a tree graph $T$ of order 9 with one vertex of eccentricity 2, having $\lambda(M(T)) = 10$. Therefore from Proposition 4.1 we have $\lambda(M(G)) = n + 1$ does not mean necessary that $\lambda(M^t(G)) = 2^{t-1}(n + 2) - 2$, for $t \geq 2$.

4.2 Graphs with $\lambda(M^t(G)) = 2^t(n + 1) - 2$

Shao and Solis-Oba in [20], gave bounds for the $\lambda$-number of some iterated Mycielski of complete graph $K_n$. In the following, we give the exact value of the $\lambda$-number of $M^t(K_n)$, for any $t \geq 2$.

**Theorem 4.2.** For any $t \geq 2$ and $n \geq 2$, we have $\lambda(M^t(K_n)) = 2^t(n + 1) - 2$.

**Proof.** For $n \geq 2$, we have $\text{diam}(K_n) = 1$, so by Lemma 2.4 for any $t \geq 2$, we have $\text{diam}(M^t(K_n)) = 2$. Let $V^2 = \{v_i^k : 0 \leq k \leq 3$ and $1 \leq i \leq n\}$ be the set composed of the vertex of $V$ and all their consecutive copies in $M^2(K_n)$. Let $\chi_i$ with $1 \leq i \leq n$, be a sequence of vertices in $M^2(K_n)$, where $\chi_i = v_i^1v_i^0v_i^1$ if $i$ is odd and $\chi_i = v_i^1v_i^0v_i^2$ if $i$ is even. We label the vertices of $M^2(K_n)$ using consecutive labels beginning with 0, in the following order $\chi_1\chi_2 \cdots \chi_nv_n^3v_{n-1}^3 \cdots v_1^3u_{11}u_{10}u_{20}$. 

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**Figure 7:** A 10-L(2,1)-Labeling of the Mycielski graph of a tree $T$ of order 9.
This does not violate the distance two conditions, since two consecutive vertices are either a vertex and its copy, or two vertices from the same level, which are successively at distance two. This leads to an $L(2,1)$-labeling of $M^2(K_n)$ with span $|M^2(K_n)| - 1$. Since the diameter is 2, then $\lambda(M^2(K_n)) = |M^2(K_n)| - 1$. From Observation 4.1 and Lemma 2.1 we get $\lambda(M^t(K_n)) = |M^t(K_n)| - 1 = 2^t(n + 1) - 2$, for any $t \geq 2$.

Since any graph $G$ of order $n \geq 2$ is a subgraph of the complete graph $K_n$, we can conclude that for $t \geq 2$, we have $\lambda(M^t(G)) \leq |M^t(G)| - 1 = 2^t(n + 1) - 2$. This could also be proven using Theorem 4.2 by showing that for any graph $G$, the complement of the Mycielski $\overline{M}(G)$ has a perfect 4-star matching, which means by Theorem 4.2(a) that $\lambda(M^t(G)) \leq |M^t(G)| - 1$, then the result follows from Observation 4.1 for any $t \geq 2$.

**Corollary 4.1.** Let $G_1$ and $G_2$ be two graphs of the same order $|G_1| = |G_2| \geq 2$. For any $t \geq 2$, we have $\lambda(M^t(G_1)) + 2^t \leq \lambda(M^{t+1}(G_2))$.

**Proof.** For $t \geq 2$, let $G_1$ and $G_2$ be two graphs such that $|G_1| = |G_2| = n \geq 2$. By Theorem 4.1 and Theorem 4.2 we have $\lambda(M^t(G_1)) \leq 2^t(n + 1) - 2$ and $\lambda(M^{t+1}(G_2)) \geq 2^t(n + 2) - 2$. Hence $\lambda(M^t(G_1)) + 2^t \leq \lambda(M^{t+1}(G_2))$.

Let us denote $\overline{M}(G)$ the complement graph of $M^t(G)$, the close relation between Hamiltonicity and the $L(2,1)$-Labeling allow us to prove the following.

**Corollary 4.2.** For any graph $G$ and any $t \geq 2$, $\overline{M}(G)$ is a Hamiltonian graph.

**Proof.** Let $G$ be a graph of order $n$, first we show that $\overline{M}(G)$ is Hamiltonian.

Let $\chi_i$ with $2 \leq i \leq n$, be a sequence of vertices in $\overline{M}(G)$, where $\chi_i = v_i^0v_i^0v_i$ if $i$ is odd, and $\chi_i = v_i^1v_i^0v_i^2$ if $i$ is even. Take the vertices of $\overline{M}(G)$ in the following order, $v_i^0v_i^1v_i^2v_i^3\ldots v_i^{n-1}v_i^n\ldots v_i^3v_i^2v_i^1$. Notice that this is similar to the order proposed in Theorem 4.2 for labeling $M^2(K_n)$. Since every two consecutive vertices are non-adjacent in $M^2(G)$, then the vertices of $\overline{M}(G)$ taken in the above order form a Hamiltonian cycle. Thus, for any graph $G$ we have $\overline{M}(G)$ is Hamiltonian. For $t \geq 2$, since $M^t(G) \equiv M^2(M^{t-2}(G))$, then $\overline{M}(G)$ is a Hamiltonian graph for any $t \geq 2$.

Next we characterize the graphs with $\lambda(M^t(G)) = 2^t(n + 1) - 2$, for $t \geq 2$.

**Theorem 4.3.** Let $G$ be a graph of order $n \geq 2$. For $t \geq 2$, we have $\lambda(M^t(G)) = 2^t(n + 1) - 2$ and only if $G \cong K_n$ or $\text{diam}(G) = 2$.

**Proof.** For $t \geq 2$, if $G \cong K_n$ by Theorem 4.2 we have $\lambda(M^t(G)) = 2^t(n + 1) - 2$. If $\text{diam}(G) = 2$, from Theorem 4.2 we have $\lambda(M^t(G)) \leq 2^t(n + 1) - 2$. By Lemma 2.4 $\text{diam}(M^t(G)) = 2$, the vertices must be assigned distinct labels, hence $\lambda(M^t(G)) = 2^t(n + 1) - 2$.

The converse, suppose that $G$ is a graph of order $n \geq 2$, with $\text{diam}(G) \geq 3$. So there are at least two vertices at distance greater or equal to 3, one from another. Without loss of generality, we suppose that $d_G(v_i^0, v_i^0) \geq 3$. For $t = 2$, let $\chi_i$ with $2 \leq i \leq n - 1$, be a sequence of vertices in $M^2(G)$, where $\chi_i = v_i^0v_i^0v_i^1$ if $i$ is odd, and $\chi_i = v_i^1v_i^0v_i^2$ if $i$ is even. The labeling $f$ assigns consecutive labels to the vertices beginning with 0 in the following order, $v_i^0v_i^1v_i^2v_i^3\ldots v_{i-1}^0v_i^3\ldots v_n^0v_{n-1}^0\ldots v_i^0v_i^1v_i^2$. This is similar to the order in Theorem 4.2. The maximum label assigned is $f(v_i^2) = 4n - 5$. We have $d_G(v_i^0, v_i^0) \geq 3$, so by Lemma 2.4 we have $d_{M^3}(v_i^0, v_i^0) \geq 3$, and $d_{M^2}(v_i^0, v_i^0) = 3$. We label $f(v_i^0) = f(v_i^2) = 4n - 5$, $f(v_i^1) = 4n - 4$, $f(v_i^0) = 4n - 3$, $f(v_i^2) = 4n - 2$, $f(u_{i+1}) = 4n - 1$, $f(u_{i+1}) = 4n$, $f(u_{i+2}) = 4n + 1$. This is a valid $L(2,1)$-Labeling of $M^2(G)$ with span $4n + 1$. Hence $\lambda(M^2(G)) \leq 4n + 1 = 4(n + 1) - 3$. From the upper bound of Theorem 4.1 and Theorem 4.1 for all $t \geq 3$, we have $\lambda(M^t(G)) \leq (2^{t-2} - 1)(|M^2(G)| + 1) + \lambda(M^2(G))$, since $|M^2(G)| = 4(n + 1) - 1$, it follows that for all $t \geq 2$, $\lambda(M^t(G)) \leq 2^t(n + 1) - 3$.

**4.3 Graphs with $\lambda(M^t(G)) = 2^{t-1}(n + 2) - 2$**

**Lemma 4.2.** Let $t \geq 2$ and $1 \leq i, j \leq n$, for $1 \leq k \leq 2^{t-1} - 1$, we have $d_{M^t}(v_i^k, v_j^{2^{t-1}+k}) = 2$, and for $2^{t-1} + 1 \leq k \leq 2^t - 1$, we have $d_{M^t}(v_i^k, v_j^{2^{t-1}-1}) = 2$.  

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Proof. For $1 \leq k \leq 2t^i - 1$, we have $v_j^{2t^i + k}$ is the copy of $v_j^k$ in $M^t(G)$. Since $d_{M^t-1}(v_j^k, v_j^{2t^i + k}) = 2$, by Lemma 2.3 we have $d_{M^t}(v_j^k, v_j^{2t^i + k}) = 2$.

For $t \geq 2$, $v_i^0$ is the copy of $v_i^1$. So by Lemma 2.3 $d_{M^t}(v_i^0, v_i^1) = 2$. Since $d_{M^2}(v_i^0, v_i^1) = 2$, by using Lemma 2.3 inductively, we can show that for $2t^i - 1 \leq k \leq 2t^i - 1$, we have $d_{M^t}(v_i^k, v_i^{2t^i - 1}) = 2$.

**Lemma 4.3.** If $v_i^0$ and $v_i^j$ are not isolated vertices, for $0 \leq k \leq 2t^i - 1$, we have $d_{M^t}(v_i^k, v_i^{2t^i - k - 1}) = \min(3, d_G(v_i^0, v_i^j))$.

Proof. We have $v_i^{2t^i - k - 1}$ is the copy of $v_i^{2t^i - k - 1}$ in $M^t(G)$, by Lemma 2.3 we have $d_{M^t}(v_i^k, v_i^{2t^i - k - 1}) = \min(3, d_{M^t-1}(v_i^k, v_i^{2t^i - k - 1}))$. If $0 \leq k \leq 2t^i - 2$, we have $d_{M^t-1}(v_i^k, v_i^{2t^i - k - 1}) = \min(3, d_{M^t-2}(v_i^k, v_i^{2t^i - 2})$). Otherwise, if $2t^i - 2 \leq k \leq 2t^i - 1$, by symmetry $k = 2t^i - 1 - m - 1$ where $0 \leq m \leq 2t^i - 2$, so $d_{M^t-1}(v_i^k, v_i^{2t^i - k - 1}) = d_{M^t-1}(v_i^{2t^i - m - 1}, v_j^m) = \min(3, d_{M^t-2}(v_i^{2t^i - m - 1}, v_j^m))$. By recursively using Lemma 2.3 we get $d_{M^t}(v_i^k, v_i^{2t^i - k - 1}) = \min(3, d_G(v_i^0, v_i^j))$.

In the case where $v_i^0$ or $v_i^j$ are isolated vertices, for $1 \leq k \leq 2t^i - 1$, we have $d_{M^t}(v_i^k, v_i^{2t^i - k - 1}) = 3$.

The direct product $G \times K_2$, called the canonical double cover (or Kronecker double cover) is a bipartite graph with two partition sets $X = V \times \{x\}$ and $Y = V \times \{y\}$, where $(v_i, x)(v_j, y) \in E(G \times K_2)$ if and only if $v_i v_j \in E(G)$.

From Lemma 4.3, $v_i^{2t^i - 1}, v_i^{2t^i - 1} \in E(M^t(G))$ if and only if $v_i^0 v_j^0 \in E(G)$. Since two copies of the same vertex or copies from the same level are non-adjacent, we have

**Observation 4.2.** For $t \geq 2$, let $S = \{v_i^{2t^i - 1}, v_i^{2t^i - 1} : 1 \leq i \leq n\}$. In $M^t(G)$, the subgraph induced by the vertices in $S$ is isomorphic to $G \times K_2$.

A matching in a graph $G$ is a collection of vertex-disjoint edges in $G$, a perfect matching is a matching that covers all the vertices of $G$. The following theorem known as the Marriage Theorem, gives a criterion for any bipartite graph $G = (X, Y)$ to have a perfect matching.

**Theorem 4.4.** (The Marriage Theorem). Let $G = (X, Y)$ be a bipartite graph, then $G$ has a perfect matching if and only if $|X| = |Y|$ and for any $S \subseteq X$, $|N_G(S)| \geq |S|$.

A 2-matching of a graph $G$ is an assignment of weights $0$, $1$, or $2$ to the edges of $G$, such that the sum of weights of edges incident to any vertex in $G$ is less or equal to 2 (see Chapter 6. in [13]). A 2-matching of a graph $G$ can be seen as components with degree vertex at most 2. The sum of weights in a 2-matching is called the size. The maximum size of a 2-matching is denoted by $\nu_2(G)$, which can be computed in polynomial time [21]. A perfect 2-matching is a 2-matching where the sum of weights incident to any vertex in $G$ is exactly 2. W. Tutte in [21], provides a characterization for the existence of perfect 2-matching of a graph.

**Theorem 4.5.** [21] A graph $G$ has a perfect 2-matching if and only if for any independent set $S \subseteq V$, $|N_G(S)| \geq |S|$.

A perfect 2-matching can be seen as a spanning subgraph in which each component is a single edge $K_2$ or a cycle, since every even cycle has a perfect matching, a graph with a perfect 2-matching has a spanning subgraph in which each component is a single edge or an odd cycle. It is easy to see from the two preceding Theorem 4.3 and Theorem 4.3, that the existence of perfect 2-matching in a graph $G$ is equivalent to that $G \times K_2$ admits a perfect matching.

**Theorem 4.6.** Let $G$ be a graph without isolated vertices of order $n \geq 2$. For $t \geq 2$, $\lambda(M^t(G)) = 2t^i - 1(n + 2) - 2$ if and only if for any $S \subseteq V$ $|D_2(S)| \geq |S|$, where $D_2(S) = \{x \in V : \exists v \in S, d_G(x, v) > 2\}$.

Proof. Let $G$ be a graph without isolated vertices of order $n \geq 2$, such that for $t \geq 2$, $\lambda(M^t(G)) = 2t^i - 1(n + 2) - 2$. Let $f$ be a $\lambda$-labeling of $M^t(G)$, using labels from the set $L = \{0, \ldots, 2t^i - 1(n + 2) - 2\}$. From Lemma 4.4, we have $d_{M^t}(v_i^k, u) \leq 2$ and $d_{M^t}(u, u') \leq 2$, for all $v_i^k \in V^t$ and all $u, u' \in U_t$. The roots are assigned distinct labels, different from the labels assigned to the vertices in $V^t$. So for $2t^i - 1 \leq k \leq 2t^i - 1$, we have $f(v_i^k) \in L \setminus f(U_i)$ and $|L \setminus f(U_i)| = 2t^i - 1n$. For $1 \leq i, j \leq n$, we have $d_{M^t}(v_i^k, v_j^{m^t}) = 2$, where
$2^{t-1} \leq k, m \leq 2^t - 1$. It follows that the $2^{t-1}n$ vertices $v_i^k$ where $2^{t-1} \leq k \leq 2^t - 1$, and $1 \leq i \leq n$, have distinct labels and use all the labels in $L \setminus f(U_i)$. By Lemma 12 we have $d_M(v_i^k, v_j^{2^{t-1} - 1}) = 2$, for $2^{t-1} + 1 \leq k \leq 2^t - 1$. The only labels remaining in $L \setminus f(U_i)$, for the vertices $v_j^{2^{t-1} - 1}$, are those assigned to the vertices $v_i^k$. Since $d_M(v_i^{2^{t-1} - 1}, v_j^{2^{t-1} - 1}) = 2$ and $d_M(v_i^{2^{t-1} - 1}, v_j^{2^{t-1} - 1}) = 2$, then $f(v_i^{2^{t-1} - 1}) \neq f(v_j^{2^{t-1} - 1})$ and $f(v_j^{2^{t-1} - 1}) \neq f(v_i^{2^{t-1} - 1})$. It follows that for any vertex $v_j^{2^{t-1} - 1}$, there is one and only one vertex $v_i^k$, such that $f(v_i^{2^{t-1} - 1}) = f(v_j^{2^{t-1} - 1})$. Let $(v_i, x)$ and $(v_j, y)$, $1 \leq i, j \leq n$ denote the vertices of $G \times K_2$, where $(v_i, x)(v_j, y) \in E(G \times K_2)$ if and only if $v_i^k v_j^m \in E(G)$. Let $M = \{(v_i, x)(v_j, y) : f(v_i^{2^{t-1} - 1}) = f(v_j^{2^{t-1} - 1})\}$. Since $f(v_i^{2^{t-1} - 1}) = f(v_j^{2^{t-1} - 1})$ means by Lemma 13 that $d_G(v_i^0, v_j^0) \geq 3$. From Observation 14 $M$ is a perfect matching of the graph $\overline{G^2} \times K_2$, then by Theorem 4.4 we get the necessity.

The converse, suppose that for any $S \subseteq V$, we have $\left| D_2(S) \right| \geq |S|$. This means by Theorem 4.5 that the graph $\overline{G^2}$ has a perfect 2-matching, which means that $\overline{G^2}$ has a spanning subgraph $H$, whose connected components are vertex-disjoint edges or odd cycles. Let $E^1, E^2, \ldots, E^r$ be the $K_2$ components, and $C^1, C^2, \ldots, C^s$ the odd cycle components of $H$. Let us denote the vertices of $V$ as $x_0^0, y_0^0$ is the edge $E^1$ and $c_{1,i}, c_{2,i}, \ldots, c_{m,i}$ is the odd cycle $C_i$, where $n_i = |C_i|$. We define an $L(2,1)$-Labeling $f$ to the vertices of $M^t(G)$ as follows.

Suppose that $r \geq 2$, first we label the vertices $x_1^1, y_1^1$ with $0 \leq k \leq 2^t - 1$, where $x_1^1$ and $y_1^1$ are the vertices $x_1^1$ and $y_1^1$ and their consecutive copies. The labeling $f$ assigns in descending order the labels $2^{t-1} - 1, 2^{t-1} - 2, \ldots, 0$ respectively to $x_1^1, x_1^1, \ldots, x_1^1$ and the labels $2^t - 1, 2^t - 2, \ldots, 2^t - 1$ respectively to $x_1^1, x_1^1, \ldots, x_1^1$. Then assign the same list of consecutive labels, now in ascending order $0, 1, 2^t - 1$ respectively to the vertices $y_1^1, y_1^1, \ldots, y_1^1$ and the labels $2^t - 1, 2^t - 1, \ldots, 2^t - 1$ respectively to $y_1^1, y_1^1, \ldots, y_1^1$.

- For $0 \leq k \leq 2^{t-1} - 1$, $f(x_1^1) = 2^{t-1} - k - 1$, and for $2^{t-1} \leq k \leq 2^t - 1$, $f(x_1^1) = 3 \times 2^{t-1} - 1$.
- For $0 \leq k \leq 2^{t-1} - 1$, $f(y_1^1) = k + 2^{t-1}$, and for $2^{t-1} \leq k \leq 2^t - 1$, $f(y_1^1) = k - 2^{t-1}$.

We have $f(x_1^1) = f(y_1^1)$ if $m = 2^t - k - 1$. Since $x_0^0 y_0^0 \in E(G^2)$, then $d_G(x_0^0, y_0^0) \geq 3$, so by Lemma 13 $d_M(x_1^1, y_1^1) = 3$. Otherwise $f(x_1^1) \neq f(y_1^1)$, since $x_0^0$ and $y_0^0$ are not adjacent in $G$ we have $d_M(x_1^1, y_1^1) \geq 2$, for all $0 \leq k, m \leq 2^t - 1$. Also $d_M(x_1^1, y_1^1) = d_M(y_1^1, x_1^1) = 2$, $f(x_1^1) \neq f(y_1^1)$ and $f(y_1^1) \neq f(y_1^1)$. The smallest label is $f(x_1^1 - 1) = f(y_1^1 - 1) = 0$, the maximum label is $f(x_1^1) = f(y_1^1) = 2^t - 1$.

For $2 \leq i \leq r$, we have $d_G(x_i^1, y_i^1) \geq 3$, so a vertex in $E_i$ cannot be adjacent in $G$ to both $x_i^0$ and $y_i^0$. Since in every $E_i$ the vertices $x_i^1$ and $y_i^1$ are symmetric, we rearrange the vertices of each $E_i$ depending on the cases:

i) If $x_1^1$ is adjacent in $G$ to a vertex in $E_i$, we consider without loss of generality that $x_i^0$ is adjacent to $y_i^0$.

ii) If $x_0^1$ is not adjacent to $E_i$ and $y_0^1$ is adjacent, we let $d_G(y_0^1, x_1^0) = 1$. Otherwise the vertices in $E_i^1$ and $E_i^0$ are mutually non-adjacent. This means that $d_G(x_0^1, x_1^0) \geq 2$, and $d_G(y_0^1, y_1^0) \geq 2$, for all $2 \leq i \leq r$.

With respect to the above assumptions, we label the vertices $x_i^1$ and $y_i^1$ with $2 \leq i \leq r$, as following.

- For $2 \leq i \leq r - 1$, and $0 \leq k \leq 2^t - 1$, $f(x_i^1) = (i - 1)2^t + f(x_1^1)$, and $f(y_i^1) = (i - 1)2^t + f(y_1^1)$.
- For $0 \leq k \leq 2^t - 1$, $f(x_i^1) = (r - 1)2^t + f(x_1^1)$, and for $2^t - 1 \leq k \leq 2^t - 1$, $f(x_i^1) = (r - 1)2^t + k$.
- For $0 \leq k \leq 2^t - 1$, $f(y_i^1) = r2^t - k - 1$, and for $2^t - 1 \leq k \leq 2^t - 1$, $f(y_i^1) = (r - 1)2^t + f(y_1^1)$.

The labeling $f$ uses distinct labels from $(i - 1)2^t, \ldots, i2^t - 1$, for every pair of $x_i^1, y_m^m$, where $m = 2^t - k - 1$, by using the same pattern for $x_i^1, y_m^m$ (except for $x_i^1, y_1^1$). In the case where $r = 1$, let for $0 \leq k \leq 2^t - 1$, $f(x_i^1) = 2^t - k - 1$, for $2^t - 1 \leq k \leq 2^t - 1$, $f(x_1^1) = k$, for $0 \leq k \leq 2^t - 1 - 1$, $f(x_1^1) = 2^t - k - 1$, and for
$2^{t-1} \leq k \leq 2^t - 1$, $f(y^t_k) = k - 2^{t-1}$. The only vertices from two different components, with the difference between the labels equal to 1, are for $x_{i-1}^t$ and $y_{i-1}^t$, with both $x_{i-1}^{2t-1}$ and $y_{i-1}^{2t-1}$. This does not violate the distance two conditions, since $d_G(x_{i-1}^0, y_{i}^0) \geq 2$, and $d_G(y_{i-1}^0, y_{i}^0) \geq 2$, for all $2 \leq i \leq r$. The maximum label assigned is $f(x_{i+1}^t) = f(y_{i+1}^t) = r2^t - 1$.

If $s \geq 1$, next we label the vertices of the odd cycle components $C^i$. We make the following claim.

**Claim 4.1.** For a vertex $v$ in $G$ not in the odd cycle component $C^i = c_{0,i}^0 c_{2,i}^0 \ldots c_{n_i,i}^0$, there is at least one edge $c_{0,i}^0 c_{q,i}^0 \in C^i$, such that $v$ is not adjacent in $G$ to both $c_{p,i}^0$ and $c_{q,i}^0$.

**Proof.** We prove this by using contradiction, we suppose that $v$ is adjacent to at least one endpoint of any $c_{p,i}^0 c_{q,i}^0 \in C^i$. We may assume that $v$ is adjacent to $c_{1,i}^0$. Since $d_G(c_{1,i}^0, c_{2,i}^0) \geq 3$, $v$ is not adjacent to $c_{2,i}^0$, so $v$ is adjacent to $c_{1,i}^0$, and so forth. Hence, if $j$ is odd $v$ is adjacent to $c_{j,i}^0$, and if $j$ is even $v$ is not adjacent to $c_{j,i}^0$. Since $v$ is adjacent to $c_{1,i}^0$, then $v$ is not adjacent to $c_{n_i,i}^0$. It follows that $n_i$ is even, a contradiction. □

Since the cycles $C^i$ are symmetric, we may consider that $d_G(y_r, c_{1,1}^0) \geq 2$, and $d_G(y_r, c_{n_1,1}^0) \geq 2$, and for $1 \leq i \leq s - 1$, $d_G(y_i, c_{n_i,1}^0, c_{n_i+1,1}^0) \geq 2$, and $d_G(y_i, c_{n_i,1}^0, c_{n_i+1,1}^0) \geq 2$. We label the vertices $c_{j,i}^k$ where $1 \leq j \leq n_i$, $1 \leq i \leq s$ and $0 \leq k \leq 2^t - 1$, with respect to the above assumptions.

- For $0 \leq k \leq 2^{t-1} - 1$, $f(c_{1,i}^1) = r2^t + 2^{t-1} - k - 1$, and for $2^{t-1} \leq k \leq 2^t - 1$, $f(c_{1,i}^1) = r2^t + k$.
- For $2 \leq j \leq n_i - 1$ and all $0 \leq k \leq 2^t - 1$, $f(c_{j,i}^1) = f(c_{1,i}^1) + (j-1)2^{t-1}$.
- For $0 \leq k \leq 2^{t-1} - 1$, $f(c_{n_i,1}^1) = f(c_{1,i}^1) + (n_i - 1)2^{t-1}$, and for $2^{t-1} \leq k \leq 2^t - 1$, $f(c_{n_i,1}^1) = f(c_{1,i}^1 - k)$. The smallest label for the vertices $c_{1,i}^1$ is $f(c_{1,i}^{2t-1}) = f(c_{n_i,1}^{2t-1}) = r2^t$ and the maximum is $f(c_{n_i,1}^{2t-1}) = f(c_{1,i}^{2t-1}) = r2^t + n_22^{t-1} - 1$. Now let $\varphi_i = r2^t + \sum_{j=1}^{i-1} n_j2^{t-1}$. For $2 \leq i \leq s$, we label $f(c_{1,i}^{2t-1}) = f(c_{n_i,1}^{2t-1}) = \varphi_i$, then

- For $0 \leq k \leq 2^{t-1} - 1$, $f(c_{1,i}^1) = \varphi_i + 2^{t-1} - k - 1$, and for $2^{t-1} \leq k \leq 2^t - 1$, $f(c_{1,i}^1) = \varphi_i + k$.
- For $2 \leq j \leq n_i - 1$ and all $0 \leq k \leq 2^t - 1$, $f(c_{j,i}^1) = f(c_{j,i}^1) + (j-1)2^{t-1}$.
- For $0 \leq k \leq 2^{t-1} - 1$, $f(c_{n_i,1}^1) = f(c_{1,i}^1) + (n_i - 1)2^{t-1}$, and for $2^{t-1} \leq k \leq 2^t - 1$, $f(c_{n_i,1}^1) = f(c_{1,i}^1 - k)$.

The labeling $f$ uses $n_i2^{t-1}$ distinct labels for the $n_i2^t$ vertices of each component $C_i$ and their copies. For $0 \leq k \leq 2^{t-1} - 1$, we have $f(c_{1,i}^1) = f(c_{1,i}^{2t-k-1})$, and for $2 \leq j \leq n_i$, $f(c_{j,i}^1) = f(c_{j-i}^{2t-k-1})$. It is possible, since $d_G(c_{j,i}^1, c_{j+1,i}^1) \geq 3$, which means by Lemma 1.3 that $d_M(c_{j,i}^1, c_{j+1,i}^{2t-k-1}) = 3$. For two vertices $c_{j,i}^1, c_{j+1,i}^1$, the difference between the labels is equal to 1 in the following cases: i) The vertices are copies of the same vertex, or if $2^{t-1} \leq k, m \leq 2^t - 1$, in those two cases $d_M(c_{j,i}^1, c_{j+1,i}^{2t-k-1}) = 2$. ii) For $l = j + 1$, we have $d_G(c_{j+1,i}^1, c_{j+1,i}^{2t-k-1}) \geq 3$, then $d_M(c_{j,i}^1, c_{j+1,i}^{2t-k-1}) \geq 2$. iii) If $l = j+2, k = 2^t - 1$ and $m = 2^{t-1} - 1$, we have $d_M(c_{j,i}^1, c_{j+1,i}^{2t-k-1}) = 2$. The difference between the labels is equal to 1 for vertices from two different odd cycle components, only occur for $c_{n_i,i}^0$ and $c_{n_i,i}^{2t-1}$ with $c_{n_i,i}^{2t-1}$ and $c_{n_i,i}^{2t-1}$, for $1 \leq i \leq s - 1$, we have $d_G(c_{n_i,i}^0, c_{n_i,i}^{2t-1}) \geq 2$ and $d_G(c_{n_i,i}^0, c_{n_i,i}^{2t-1}) \geq 2$, from Lemma 1.2 the vertices are at distance greater or equal 2 in $M'(G)$. The maximum label assigned is $f(c_{n_i,i}^0) = f(c_{n_i,i}^{2t-1}) = r2^t + \sum_{j=1}^{i-1} n_j2^{t-1} - 1 = n2^{t-1} - 1$.

We finally label the remaining $2^t - 1$ roots with consecutive labels beginning with the label $n2^{t-1}$ in the following order

$u_{1,2^{t-1} - 1} u_{1,2^{t-1} - 2} \ldots u_{1,0} u_{2,2^{t-2} - 1} u_{2,2^{t-2} - 2} \ldots u_{2,0} u_{3,2^{t-3} - 1} \ldots u_{t,0}$
This produces an $L(2,1)$-labeling with the same schema for $M'(G)$ implying the maximum size of a 2-matching of $G^2$.

The labeling defined in Theorem 4.6 is a valid $L(2,1)$-labeling for any graph $G$ of order $n \geq 2$, if $G^2$ has a perfect 2-matching, then we can label the vertices of $M'(G)$ with a labeling having span $2^{t-1}(n+2)-2$. Next, we give an upper bound for $\lambda(M'(G))$ implying the maximum size of a 2-matching of $G^2$.

**Theorem 4.7.** Let $G$ be a graph of order $n \geq 2$, with $\nu_2(G^2) = p$. For $t \geq 2$, we have $\lambda(M'(G)) \leq 2^{t-1}(2n-p - 2)$.

**Proof.** Let $G$ be a graph with $\nu_2(G^2) = p$. So there is an induced subgraph $H$ of $G^2$ of order $p$, such that $H$ has a perfect 2-matching. Let $V_H$ be the set of vertices of $H$, from Theorem 4.6 we can label the vertices of $M'(G[V_H])$ with an $L(2,1)$-Labeling $f$ with span $2^{t-1}(p+2)-2$, where $f(u,v) = 2^{t-1}(p+2)-2$.

Now in $M'(G)$, if $p < n$ the vertices remaining unlabeled by $f$ are the vertices in $V \setminus V_H$ and their copies. Let us denote $v^k_i$, where $1 \leq i \leq q$, and $0 \leq k \leq 2^{t-1} - 1$, such that $p + q = n$, the vertices of $V \setminus V_H$ and their consecutive copies. Let $\chi_i$ with $2 \leq i \leq q$, be a sequence of vertices in $M'(G)$, where $\chi_i = v^i_0v^i_1$ if $i$ is odd, and $\chi_i = v^i_1v^i_0$ if $i$ is even. The only vertex labeled $2^{t-1}(p+2)$ by $f$ is used, using consecutive labels we label the vertices $v^k_i$, with $1 \leq i \leq q$ beginning with the label $2^{t-1}(p+2)-1$, in the following order

$v^0_1v^1_1\chi_2v^3_1v^3_3\chi_4v^5_1\ldots v^a_0v^a_1\ldots v^3_0v^5_1\ldots v^{2^{t-1}-1}_1$.

This produces an $L(2,1)$-labeling with span $2^{t-1}(p+2)-2 + 2^{t-1}(n-p) = 2^{t-1}(2n-p-2)$.

Similarly to Subsection 3.3 we put interest in connected graphs, the path $P_n$ and cycle $C_n$, which we use to determine some connected graphs with the smallest $\lambda(M'(G))$.

**Corollary 4.3.** For $t \geq 2$,

$$
\lambda(M'(P_n)) = \begin{cases} 
4 \times 2^t - 2 & \text{if } n = 3, 4, 5, \\
2^{t-1}(n+2)-2 & \text{if } n \geq 6.
\end{cases}
$$

**Proof.** For $n = 3$, we have $\text{diam}(P_3) = 2$, by Theorem 4.3 for $t \geq 2$ we have $\lambda(M'(P_3)) = 4 \times 2^t - 2$.

For $n = 4$, $P_4$ consists of a single edge and 2 isolated vertices. So $\nu_2(P_4) = 2$, it follows from Theorem 4.7 that $\lambda(M'(P_4)) \leq 4 \times 2^2 - 2$. Since $M'(P_3)$ is a subgraph of $M'(P_4)$, from above $\lambda(M'(P_4)) = 4 \times 2^2 - 2$.

For $n = 5$, $P_5$ consists of 2 independent edges and one isolated vertex. Hence $\nu_2(P_5) = 4$, so from Theorem 4.7 $\lambda(M'(P_5)) \leq 4 \times 2^2 - 2$. Also $M'(P_3)$ is a subgraph of $M'(P_5)$, then $\lambda(M'(P_5)) = 4 \times 2^2 - 2$.  

![Figure 8: An $L(2,1)$-Labeling of $M^2(G)$ as in Theorem 4.6, where $G^2$ has a perfect 2-matching with two $K_2$ components and two cycles of order 3 and 5, here the edges represent a perfect matching of $G^2 \times K_2$.](image-url)
For $n \geq 6$, it is easy to see that the path $P_n$ verifies the condition of Theorem 4.6 thus $\lambda(M^t(P_n)) = 2^{t-1}(n+2) - 2$.

**Corollary 4.4.** For $t \geq 2$,

$$\lambda(M^t(C_n)) = \begin{cases} 
4 \times 2^t - 2 & \text{if } n = 3, \\
5 \times 2^t - 2 & \text{if } n = 4, \\
6 \times 2^t - 2 & \text{if } n = 5, \\
2^{t-1}(n + 2) - 2 & \text{if } n \geq 6.
\end{cases}$$

**Proof.** We have $diam(C_3) = 1$, and $diam(C_4) = diam(C_5) = 2$. So by Theorem 4.4 for $t \geq 2$, we have $\lambda(M^t(C_3)) = 4 \times 2^t - 2$, $\lambda(M^t(C_4)) = 5 \times 2^t - 2$, and $\lambda(M^t(C_5)) = 6 \times 2^t - 2$. If $n \geq 6$, the cycle $C_n$ satisfies the condition of Theorem 4.6 then $\lambda(M^t(C_n)) = 2^{t-1}(n + 2) - 2$.

**Corollary 4.5.** Let $G$ be a connected graph, for $t \geq 2$ we have
1) $\lambda(M^t(G)) = 3 \times 2^t - 2$ if and only if $G$ is $K_2$,
2) $\lambda(M^t(G)) = 4 \times 2^t - 2$ if and only if $G \in \{P_3, P_4, P_5, P_6, C_3\}$,
3) $\lambda(M^t(G)) = 9 \times 2^t - 2$ if and only if $G \in \{P_7, C_7\}$.

**Proof.** We have $K_2$ is the only graph with $\Delta = 1$, by Theorem 4.2 $\lambda(M^t(K_2)) = 3 \times 2^t - 2$. From the lower bound of Theorem 4.1, for $t \geq 2$, $\lambda(M^t(G)) \geq 2^{t-1}\max(n + 2, 2(\Delta + 2)) - 2$. Then if $\Delta \geq 2$ we have $\lambda(M^t(G)) \geq 4 \times 2^t - 2$. Also if $\Delta \geq 3$, we have $\lambda(M^t(G)) \geq 5 \times 2^t - 2$. Since the graphs in Corollary 4.3 and Corollary 4.4 are the only connected graphs with $\Delta = 2$, then we can conclude.

For any other non-trivial connected graph $G$ not mentioned in Corollary 4.5 for $t \geq 2$, we have $\lambda(M^t(G)) \geq 5 \times 2^t - 2$.

5 Open problems

From the statement of the $\Delta^2$-conjecture, and the upper bound of Theorem 3.1 and Corollary 4.1, we propose a weaker conjecture for the $L(2, 1)$-labeling number of the Mycielski and the iterated Mycielski of graphs.

**Conjecture 5.1.** For any graph $G$ of order $n \geq 1$, with maximum degree $\Delta$, and for all $t \geq 1$, we have $\lambda(M^t(G)) \leq (2^t - 1)(n + 1) + \Delta^2$.

It is clear from Theorem 3.1 and Corollary 4.1 that if $\lambda(G) \leq \Delta^2$, then for any $t \geq 1$, $\lambda(M^t(G)) \leq (2^t - 1)(n + 1) + \Delta^2$, also if it is true for an iteration $t$ then it is for any iteration greater. From our study, for any $t \geq 1$, the only graphs with at least one edge that we know having $\lambda(M^t(G)) = (2^t - 1)(n + 1) + \Delta^2$, are the graph $K_2$, and the graphs achieving the bound in Corollary 3.1 which are the cycle $C_5$, the Petersen graph, the Hoffman-Singleton graph, and possibly a diameter two Moore graph of maximum degree 57, and order $57^2 + 1$ if such graph exists.

The complexity of the $L(2, 1)$-labeling problem for the Mycielski of graphs should be more investigated, whether for general graphs or the Mycielski of graphs not still studied. For instance, trees since the $L(2, 1)$-labeling number can be determined in polynomial time for trees [6], we may ask if it is also the case for the Mycielski graphs generated from trees?

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