Precision electroweak shift of muonium hyperfine splitting

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ABSTRACT

Electroweak second order shifts of muonium ($\mu^+e^-$ bound state) energy levels are calculated for the first time. Calculation starts from on-shell one-loop elastic $\mu^+e^-$ scattering amplitudes in the center of mass frame, proceed to renormalization and to derivation of muonium matrix elements by using the momentum space wave functions. This is a reliable method unlike the unjustified four-Fermi approximation in the literature. Corrections of order $\alpha G_F$ (with $\alpha \sim 1/137$ the fine structure constant and $G_F$ the Fermi constant) and of order $\alpha G_F/(m_Z a_B)$ (with $m_Z$ the Z boson mass and $a_B$ the Bohr radius) are derived from three classes of Feynman diagrams, Z self-energy, vertex and box diagrams. The ground state muonium hyperfine splitting is given in terms of the only experimentally unknown parameter, the smallest neutrino mass. It is however found that the neutrino mass dependence is very weak, making its detection difficult.
**Introduction**  
Electroweak correction to atomic energy levels has been found an important tool to provide parity violation of atomic force, which was proved by atomic parity violation experiments\([1]\), giving the weak mixing angle consistent with the one determined by neutral current weak interaction phenomena in the high energy frontier. This correction is of the first order of the Fermi coupling constant \(G_F \sim 1.166 \times 10^{-5}\text{GeV}^{-2}\). Second order electroweak effects have often been stated to be negligible, presumably due to a misconception that it might be of order \(G_F^2\). But actually in the renormalizable electroweak theory second order correction is of order \(\alpha G_F\) for flavor diagonal parts. Flavor changing effects of order \(G_F^2 \delta m^2\) with \(\delta m^2\) mass differences of quarks or leptons\([2]\) do not give a good hint because this involves off-diagonal matrix elements unlike our interest in flavor-diagonal parts. The question then arises how large the coefficient of \(\alpha G_F\) order is. The answer to this question is given only after a full body of one-loop calculation, which we shall address to.

A different motivation for second order electroweak calculation is how the long range force mediated by nearly massless neutrino-pairs emerges in atomic energy shifts. The long range force caused by the neutrino pair exchange between charged fermions has been studied in the literature\([3]\),\([4]\),\([5]\) since the old days of four-Fermi theory of weak interaction. We shall demonstrate that our full-body electroweak calculation does not justify the four-Fermi approximation, and estimate\([6]\) of atomic energy shifts based on this approximation is dubious.

We concentrate on purely leptonic bound systems, since spectroscopy of hydrogen and muonic hydrogen both suffers from the proton structure effect, recently of much debate. But if strong interaction is well under control, applications to these systems should be possible.

We use the natural unit of \(\hbar = c = 1\) unless otherwise stated.

**First order electroweak effect and basic framework for one-loop calculation** \(SU(2) \times U(1)\) electroweak gauge theory\([7]\) is renormalizable, which means that their higher loop effects to muonium energy levels are calculable from the first principles. A non-trivial part of calculation from higher order loop diagrams is renormalization, whose method is however well established\([8]\). We work in the Feynman gauge within a more general framework of \(R_\xi\) gauge\([9]\). This considerably simplifies the burden of calculation.

We introduce finite neutrino masses in the electroweak theory which is however generated by physics beyond the standard electroweak theory. There are two kinds of neutrino masses, the Dirac and the Majorana types. Since distinction of these two cases is found difficult in muonium (Mu) spectroscopy, we shall present calculations in the easier case of Dirac neutrino. We regard the vanishing smallest neutrino mass is highly unlikely, and assume that all neutrinos are massive regardless of their small values.

There exists the first order electroweak shift of order \(G_F\). Contribution of Z boson exchange to HFS arises from the fact that Z boson has a coupling of the axial vector current along with the vector current. Unlike the particle velocity operator for the vector current the spatial part of axial vector current is the spin current. Z boson exchange thus gives rise to a spin-spin interaction \(\vec{S}_\mu \cdot \vec{S}_e\) between \(\mu^+\) and \(e^-\), which is the same operator form as the one responsible for ordinary magnetic HFS. The predicted HFS of 1s Mu is \(-\sqrt{2} \alpha^3 G_F m_e^3 (1 + m_e/m_\mu)^{-3}/\pi = -65\) Hz, which sets a scale of weak effects.

There are three classes of Feynman diagrams at one loop level as shown in Fig[1]~Fig[3] (and similar ones) that contribute to atomic force between two charged fermions like \(\mu^+ e^-\): they are called Z-boson self-energy, vertex diagrams and box diagrams. Our calculation starts from low energy elastic scattering amplitudes in the center of mass frame, proceed to renormalization for self-energy and vertex, derive the force potential by Fourier-transforming the momentum transfer \(\vec{q}\) to the position vector \(\vec{r}\), and finally calculate relevant atomic matrix elements, using relevant wave functions. The procedure is standard and there seems no other rigorous way to extract atomic energy shifts. Convergence of integrals is found better if one uses wave functions in the momentum space, hence we bypass derivation of the force potential which is much more complicated than those shown in \([3]\)~\([5]\), valid only in the four-Fermi approximation. In the four-Fermi approximation all heavy internal boson (\(Z, W\)) lines are shrunk to a point, and there is no structure difference in all of one-loop Feynman diagrams. We demonstrate explicitly below that this approximation is wrong and three classes of Feynman diagrams give different dependences on relevant parameters such as
$m_Z, m_W, \alpha$ and neutrino masses.

![Figure 1](image1.png)

Figure 1: $Z$ boson self-energy diagrams of neutrino $\nu$, charged lepton $l$ and $WH$ pairs.

![Figure 2](image2.png)

Figure 2: Vertex diagrams with $Z$ attached to $e^-$ line containing $W^+W^-Z$ coupling in the left. The triangle vertex may also be attached to electron line.

![Figure 3](image3.png)

Figure 3: $t$-channel exchange consisting of pairs $(W^+W^-), (Z,Z), (Z,\gamma)$

The ghost contribution is suppressed by a mass factor when it is coupled to fermions.

**Vacuum polarization and self-energy diagram of $Z$ boson** The force potential may be derived by Fourier-transforming the elastic scattering amplitude $A_{EW}(\vec{q}) = \Pi^R(\vec{q})/(\vec{q}^2 + m_Z^2)^2$. Here $\Pi^R$ is space-space component of vacuum polarization tensor. The space-space component gives in the Feynman gauge of $Z$ propagator the required structure $\propto \vec{S}_\mu \cdot \vec{S}_e$ of HFS. It is found that the Fourier transform is a complicated function of position. We shall bypass this derivation and calculate diagonal matrix elements using the momentum-space wave function of muonium. According to [11], the Schrödinger equation including the electroweak correction may be written down as an integral equation in the momentum space, and one derives
the energy shift caused by the electroweak term $A_{EW}(\vec{q})$:

$$
\Delta_{1sHFS}^{\nu p} = -\frac{(g^2 + (g')^2)^2}{32} J_i, \quad J_i = \frac{1}{(2\pi)^3} \int d^3 p d^3 q \psi^*_c(\vec{p} + \vec{q}) A^i_{EW}(\vec{q}) \psi_c(\vec{p}),
$$

(1)

$$
A^i_{EW}(\vec{q}) = c_i \frac{\Pi^R(q^2)}{(q^2 + m^2_Z)^2}, \quad c_\nu = 1, \quad c_i = 1 - 4 \sin^2 \theta_w + 8 \sin^4 \theta_w \sim 0.50 c_\nu,
$$

(2)

where $\psi_c(\vec{p})$ is the momentum-space wave function under the Coulomb potential. $g$, $g'$ are two gauge coupling constants and $\theta_w = \arctan g'/g$ is the weak mixing angle with $\sin^2 \theta_w \sim 0.239$ at low energies.

We renormalize the transverse polarization tensor at a space-like momentum transfer $q^2 = q_0^2 - \vec{q}^2 \equiv -\mu^2 < 0$ to derive $\Pi^R_i(q^2)$, and impose the mass shell condition at $q^2 = m^2_Z$. The renormalization point $\mu^2$ should be chosen to match physics in question, and for our atomic calculation the appropriate point is $\mu^2 = O(\alpha^2 m^2_e)$, the inverse of atomic size squared. For the neutrino-pair exchange and the lepton-pair and quark-pair exchange this results in, to a good approximation,

$$
\Pi^R_i(q^2) = -\frac{2}{3(4\pi)^2 q^2 \ln(m_\mu)^2} = \frac{2}{15(4\pi)^2 \frac{q^4}{m^4_{1\nu}}},
$$

(3)

The massless neutrino limit is well defined. The ground state muonium HFS is calculated as

$$
\Delta_{1sHFS}^{\nu p} = -\frac{\alpha G_F}{\sqrt{2\pi^2 \sin^2 \theta_w \cos^2 \theta_w}} a_\mu^{-3} c_i K_i \sim -3.8 \text{ Hz } c_i K_i, \quad a_\mu = \frac{1 + m_\mu}{\alpha m_e},
$$

(4)

$$
K_\nu = \frac{1}{4 m_Z a_\nu} (\ln(m_Z a_\mu) - 1), \quad K_i = -\frac{0.50}{15 m_Z a_\nu (m_\mu^2)^2}.
$$

(5)

The contribution is suppressed by an extra large factor $m_Z a_\mu$ from $O(\alpha G_F)$, and its value is insensitive to neutrino mass. Numerically, we find $\Delta^{\nu p} = -2.8 \times 10^{-2} \text{ mHz}$ adding three neutrino pairs, and the lepton-pair and the quark-pair contributions are more than six orders of magnitudes smaller.

Thw weak boson pair contribution is calculated as

$$
\Delta_{1sHFS}^{wp} = -\frac{3 \alpha G_F}{80 \sqrt{2\pi^2 m_Z a_\mu \cos^2 \theta_w}} a_\mu^{-3} \sim -6.2 \times 10^{-6} \text{ mHz}.
$$

(6)

**Vertex and box diagrams**

The relevant integral for vertex operator is, after dimensional regularization and renormalization, $B^R(q) = B(q) - B(0)$

$$
B(q) = i \int \frac{d^4 k}{(2\pi)^4} \frac{N}{k^2 - \Delta(q)^3} = -\frac{1}{(4\pi)^2} \int_0^1 dx \int_0^{1-x} dy \left( \ln \Delta(q) + \frac{m_i^2}{\Delta(q)} y^2 \right),
$$

(7)

$$
\Delta = M^2 y + \vec{q}^2(1 - x - y) - m_\mu^2 y(1 - y) + m_i^2(1 - y).
$$

(8)

To leading $M^2 = m_W^2, m_Z^2$ order,

$$
B^R(q) \simeq -\frac{q^2}{M^2(4\pi)^2} \frac{1}{3} \ln \frac{M}{q} + \frac{5}{36}.
$$

(9)

The ground state Mu HFS is calculated:

$$
\Delta_{1sHFS}^{\nu \nu} = \frac{\alpha G_F}{48 \sqrt{2\pi^2 \sin^2 \theta_w \cos^2 \theta_w}} a_\mu^{-3} \frac{m_\mu}{m_Z a_\mu} \left( \frac{5}{12} - \ln \frac{m_Z}{m_W} \right),
$$

(10)

$$
\Delta_{1sHFS}^{Zl} = -\frac{\alpha G_F}{96 \sqrt{2\pi^2 \sin^2 \theta_w \cos^2 \theta_w}} a_\mu^{-3} \frac{5}{12}.
$$

(11)
These vertex contributions are doubled adding the vertex attached to another side of lepton. Numerically, it is of order, $1.4 \times 10^{-5}$ mHz, for $W - \nu$ vertex, and $-1.0 \times 10^{-5}$ mHz for $Zl$ vertex.

The vertex contribution of triangle $Z - \gamma - \gamma$ has infrared divergence problem, which shall be discussed elsewhere along with other diagrams containing photon.

The box diagram is simplest to calculate, since renormalization is not required, resulting in

$$\Delta_{\text{HFS},b}^{(1s)} = -\frac{G_F a^3_n}{2\sqrt{2\pi}} \frac{1}{\sin^2 \theta_w} \left( 1 + \frac{20}{\cos^2 \theta_w} (\sin^2 \theta_w - \frac{1}{4})^2 \right) \sim -160 \text{ mHz}.$$ \hspace{1cm} (12)

This contribution of order $\alpha G_F$ is the largest among three classes of Feynman diagrams.

**Summary for the ground state Mu HFS**

Our result of 1s Mu HFS is $(-160 + (-2.8 + 0.4) \times 10^{-2}) \text{ mHz}$. Three numbers refer to major contributions from box, self-energy and vertex diagrams. Neutrino mass dependence appears with a factor $m^2_\nu \ln m^2_\nu$ besides $\alpha G_F$, hence this is a very small correction to HFS.

The present status of 1s Mu HFS is as follows: latest and best experimentally measured 1s Mu HFS is 4 463 302 776(51) Hz, \cite{12}, while theoretical prediction is 4 463 302 891(272) Hz \cite{13}, \cite{14}. A goal of QED higher order calculations is around 10 Hz \cite{15}. According to \cite{16}, the level of HFS accuracy of order 10 Hz is feasible in a forthcoming experiment. These inputs are sufficient to discover the major electroweak effect $-65 \text{ Hz}$.

Since our work indicates that the second order electroweak shift to 1s Mu HFS is around 400 smaller than the first order shift, the next goal beyond $-65 \text{ Hz}$ is to look for new physics. Any discrepancy may be attributed to new physics beyond the standard electroweak theory since second order effects are negligible, provided QED higher order corrections are better understood. Assuming a comparable coupling with electroweak theory, new physics may be searched to energy scale of a few TeV.

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