Complete Einstein equation from the generalized First Law of Entanglement

Eunseok Oh\textsuperscript{1}, I.Y. Park\textsuperscript{2} and Sang-Jin Sin\textsuperscript{3}

\textsuperscript{1} Department of Physics, Hanyang University, Seoul 04763, Korea.
\textsuperscript{2} Department of Applied Mathematics, Philander Smith College Little Rock, AR 72223, USA
(Dated: July 3, 2018)

Recently it was observed that the first law of Entanglement leads to the linearized Einstein equation. In this paper, we point out that the gravity dual of an relative entropy expression is equivalent to the full non-linear Einstein equation. We also construct an entanglement vector field $V_E$ whose flux is the entanglement entropy. The flow of the vector field looks like sewing two space regions along the interface.

PACS numbers: 11.25.Tq, 03.65.Ud, 04.25.dg

I. INTRODUCTION

One of most inspiring ideas in recent development of string theory is the suggestion \cite{1,2} that the classical spacetime is a consequence of the quantum entanglement without which two nearby regions of spacetime would take apart \cite{1,2} and moreover, the Einstein equation itself is coming from a relation of entanglement entropy at least in linearized level \cite{3}. The latter is a consequence of connecting two different descriptions of entanglement entropy (EE): one as the area of Ryu-Takayanagi surface \cite{4} and the other as the expectation value of the modular Hamiltonian \cite{5}. Later, it was pointed out \cite{6} that such relation between the first law of EE and linearized gravity equation are connected through the off-shell Noether theorem formulated by Wald \cite{7,11}.

Deriving the Einstein equation from the first law has much similarity to the activity of 90’s lead by the work of Jacobson\cite{12}: assuming the thermodynamic first law he got the gravity equation. The difference of the recent activity \cite{3,6} is that the entanglement first law and its gravity dual themselves are derived from the conformal field theory (CFT) although it gave only linearized equation. That is, recent activities aim to derive the Einstein equation of the dual gravity of a CFT assuming the presence of holography. In ref. \cite{13}, the authors extended the program to the non-linear second order in perturbative scheme. The major efforts of ref. \cite{13} is devoted to derive the ‘gravity dual expression of the relative entropy’ (GDERE) starting from CFT up to second order.

While proving the GDERE from the CFT to all order is yet to be done, we can still ask ”if we assume this part is done, does it imply the full non-linear Einstein equation?” The goal of this paper is to prove that the answer is yes. As we will see later, having the GDERE gives the ‘gravitational form of generalized first law of entanglement entropy’ and it is equivalent to the Einstein equation.

The other goal of this paper is to construct a vector field associated with the EE whose flux is the EE independent of the surface over which the vector field is integrated. The flux line, once the total flux quantized, is analogous to the microscopic worm-hole and concentrated along the boundary of the entangled regions.

II. EINSTEIN EQUATION FROM ENTANGLEMENT IN LINEAR ORDER

To set up notation, we start with a short review of relevant concepts. Given a physical state given by a density matrix $\rho$ and a ball like region $B$ of radius $R$, one can decompose the Hilbert space into tensor product $\mathcal{H} = \mathcal{H}_B \otimes \mathcal{H}_B$, where $\mathcal{H}_B$ is the Hilbert space of local fields over $B$. The reduced density operator $\rho_B = \text{Tr}_{\mathcal{H}_B} \rho$. The entanglement entropy is given by $S_B = -\text{Tr}_{B} \ln \rho_B$. From now on, we delete the subscript $B$ when there is no confusion. The modular Hamiltonian $H_0 = -\log \rho_0$ for a reference state $\rho_0$ which is normalized by $\text{Tr}_{\mathcal{H}_0} \rho_0 = 1$. If we call the expectation value of the modular Hamiltonian for the state $\rho$ as the ‘Energy’ of the state $\rho$, then we have $E = \langle H_0 \rangle = -\text{Tr}_{\mathcal{H}_0} \ln \rho_0$. Under finite variation of the state from $\rho_0$ to $\rho$, we have following identity

$$\Delta E - \Delta S = S(\rho|\rho_0),$$

where

$$\Delta E = -\text{Tr}(\rho - \rho_0) \ln \rho_0,$$

$$\Delta S = -\text{Tr}_B \ln \rho + \text{Tr}_{\mathcal{H}_0} \ln \rho_0,$$

$$S(\rho|\rho_0) = \text{Tr}_B(\ln \rho - \ln \rho_0).$$

Three important remarks are in order. First, \cite{1,2} and \cite{3} can be used as the definition and a result interchangeably. Second, $\Delta E$ is not a total variation while $\Delta S$ is, because the relative entropy, $S(\rho|\rho_0)$, can not be so. Similar phenomena will be observed in their gravitational versions. Finally, the relative entropy is alway positive \cite{14} and this is the origin of the entanglement first law: as a function of $\rho$, $S(\rho|\rho_0)$ is minimal at the reference state. Such extremality condition is the usual entanglement first law,

$$\delta E - \delta S = 0,$$

where $\delta f = \frac{df}{d\lambda}|_{\lambda=0}$ for $f$ which is a one parameter family of $\lambda \in [0,\varepsilon]$. The positivity of the relative entropy is also related to the positivity of energy \cite{15,16} and that of Fisher metric for information theory \cite{17}. Both terms of

\textsuperscript{*} lspk.lpg@gmail.com, inyongpark05@gmail.com, sangjin.sin@gmail.com

\textsuperscript{{1}} Eunseok Oh, lspk.lpg@gmail.com

\textsuperscript{{2}} I.Y. Park, inyongpark05@gmail.com

\textsuperscript{{3}} Sang-Jin Sin, sangjin.sin@gmail.com
the first law can be calculated in gravitational languages using the AdS/CFT and Ryu-Takayanagi formula and it turns out that the first law leads to the linearized Einstein equation as we will review below.

Suppose the density operator depends on parameters \( R^1, R^2, \ldots, R^M \) which we symbolically denote by a vector \( R \) and let \( \rho_0 = \rho(R_0) \) and \( \rho = \rho(R_1) \) for some \( R_0, R_1 \). Introducing the modular potential \( V = -\ln \rho \) and the modular force \( F_\alpha = -\nabla_\alpha V \) in the parameter space, we can express the relative entropy as

\[
S(\rho|\rho_0) = \left( \int_C dR \cdot F \right),
\]

which can be interpreted as the ‘work’, \( W \), done on the system by \( F \) to change the system from \( \rho_0 \) to \( \rho \). Notice that it is independent of the path \( C \) connecting \( \rho_0 \) and \( \rho \) of the integration. Then the identity \[1\] itself, although in a finite difference form, can be considered as a first law’,

\[
\Delta E - \Delta S = W = S(\rho|\rho_0),
\]

which we call ‘generalized entanglement first law’. In fact, it has a gravity version. Our claim is that while we get the linearized gravity equation by using \[5\], we will get the full non-linear equation if we use the gravity version of \[7\].

For any CFT vacuum \( \rho_0 = |0\rangle \langle 0| \), a conformal mapping can be constructed which maps the causal development of the ball \( B \) to a hyperbolic cylinder \( H^{d-1} \times \mathbb{R}_r \) and \( \rho_0 \) to a thermal density operator \( \exp(-\beta RH_x) \) of CFT on hyperbolic space. Namely, the vacuum state is mapped to a thermal state of temperature \( T = 1/(\beta RH_x) \) on the \( H^{d-1} \) and modular Hamiltonian actually generates the time evolution of CFT on hyperbolic space. According to the AdS/CFT the thermal state on \( H^{d-1} \) can be represented by a AdS black hole with temperature \( T = 1/(\beta RH_x) \), the AdS-Rindler space, which can be figured as a patch of AdS space with Poincare metric.

As described above, the Hamiltonian \( H_T = \int_{H^{d-1}} T_{tt} \) is equal to the Unitarily transformed modular hamiltonian of the original CFT in the flat space \[5\]: \( H_0 = 2\pi R U H_T U^{-1} \). Using this, the authors of \[5\] expressed the modular Hamiltonian \( H_0 \) in terms of energy momentum tensor of CFT

\[
H_0 = 2\pi \int_B d^{d-1}x \frac{R^2 - |\vec{x}|^2}{2R} T_{tt} = \int_B d\sigma^\alpha \zeta_B^\alpha T_{\mu\nu},
\]

where \( \vec{x} = 0 \) is located at the center of the ball of radius \( R \) and \( \zeta_B^\alpha \) is the pullback of the killing vector \( \zeta_B \) by the mapping that maps the causal development of \( B \) to the hyperbolic cylinder \( H^{d-1} \times \mathbb{R}_r \). It can be considered as the boundary restriction of a Killing vector \( \zeta \) of AdS which vanishes at \( B \). More explicitly

\[
\zeta_B = \frac{\pi}{R} \left( R^2 - z^2 - t^2 - x^i x_i \right) \partial_t - \frac{2\pi}{R} \left( z \partial_z + x^i \partial_i \right),
\]

and \( \zeta_B = \lim_{x \to 0} \zeta_B \). The entanglement energy \( E_B \) is given by \( E_B = \int_B \zeta_B^\alpha (T_{\mu\nu}) d\sigma^\alpha \). Now, the gravitational dual of \( \delta E_B \) is readily given since AdS/CFT dictionary gives the relation between the expectation value of energy momentum tensor and the metric variation, \( \langle T_{\mu\nu} \rangle \sim z^{d-2} \delta g_{\mu\nu} \). The gravitational dual of \( \delta S_B \) can be given using the Ryu-Takayanagi prescription \( S_B = \text{Area}[\tilde{B}] / 4G_N \) \[4\]. The crucial observation of \[6\] is that there exists a \( d - 1 \) form \( \chi \) in asymptotic AdS\(_{d+1}\) such that

\[
\int_B \chi = \delta E_B^{grav}, \quad \text{and} \quad \int_B \chi = \delta S_B^{grav}
\]

based on the formalism of Iyer-Wald \[8, 9\]:

\[
\delta E_B^{grav} - \delta S_B^{grav} = \int_{\Sigma} \chi = \int_{\partial \Sigma} d\chi,
\]

where \( \Sigma \) is \( t = 0 \) slice whose boundaries are \( B \) and \( \tilde{B} \). Since it turns out to be

\[
d\chi = -2\xi^a \delta E_{ab} \epsilon^b,
\]

the entanglement first law implies the linearised Einstein equation \( \delta E_{ab} = 0 \).

Since understanding Wald’s formalism is essential for later formalism, we describe it below shortly. Start from the Lagrangian written in differentiable form notation: \( L \equiv L[\phi] \epsilon \), where \( \epsilon \) is a collective representation of the bulk fields including the metric and \( \epsilon \) is the volume form. The general variation of \( L \) can be written as

\[
\delta L[\phi] = E^a \delta \phi + d\Theta[\delta \phi],
\]

where \( E^a \) denotes field equations and \( \Theta \) the symplectic potential current that contains Gibbons-Hawking term. When the variation is a diffeomorphism generated by a vector field \( \xi \), \( \delta L = d(\xi \cdot L) \) since \( \delta \zeta = i \xi d + d\xi \) and \( L \) is the top form. In terms of the Noether current codimension 1 form

\[
J_\xi = \Theta[\delta \zeta] - \zeta \cdot L.
\]

Eq. \[13\] for the diffeomorphic variation is

\[
dJ_\xi = -E^a \epsilon \cdot \delta \phi \epsilon^b,
\]

so that \( J \) is the closed form for the fields at on-shell. Therefore \( J_\xi = dQ_\xi \) at on-shell. For off-shell, one can show \[9\] that

\[
J_\xi = dQ_\xi + \xi^a C_a,
\]

where \( C_a \)’s are constraints which vanish for metric satisfying the equation of motion \[5\]:

\[
\delta Q = \frac{1}{16\pi G_N} \nabla^a \epsilon^b e_{ab}, \quad C_a = 2E^a_{ab} \epsilon^b,
\]

with

\[
E^a_{ab} = \frac{1}{8\pi G_N} (R_{ab} - \frac{1}{2} g_{ab} R) - T_{ab}^m.
\]

On the other hand, if we introduce \( \omega \), a 2 form in phase space but codimension 1 form in spacetime, by

\[
\omega(\delta_1 \phi, \delta_2 \phi) = \delta_1 \Theta(\delta_2 \phi) - \delta_2 \Theta(\delta_1 \phi),
\]
we can express $J_\xi$ in terms of $\omega$ as follows

$$\delta J_\xi = \omega(\delta \phi, \delta g) + d(\xi \cdot \Theta(\delta \phi)) - \xi \cdot E^\phi \delta \phi \tag{19}$$

Using Eqs. (16) and (19), we get an off-shell relation

$$d\chi = \omega(\delta \phi, \delta g) - \xi \cdot (\delta C + E^\phi \delta \phi),$$

with

$$\chi = \delta Q_\xi - \xi \cdot \Theta(\delta \phi). \tag{20}$$

So far $\delta$ is infinitesimal variation defined by $\delta \phi = d\phi(\lambda)/d\lambda |_{\lambda=0}$. The point of Holland and Wald [11] is that we should work in Holland-Wald off-shell identity by imposing the relative entropy expression. We will see that the answer is positive.

An important remark is that we should work in Holland-Wald offshell identity by imposing the relative entropy expression. We will see that the answer is positive.

III. NON-LINEAR EINSTEIN EQUATION FROM ENTANGLEMENT

The issue of full Einstein equation was discussed earlier in [20,23] and most notably in [13], where the program of getting gravity equation starting from CFT is extended perturbatively to second order. Essential part of above paper is to derive the gravity expression of relative entropy starting from the CFT up to the second order. Similar efforts have been made in [22]. Given the fact that completing this program to all order is certainly non-trivial, one may ask that if this part is assumed to be proven to all order, then can we actually show that the full-non-linear Einstein equation can be implied from there. This question can be addressed purely in gravitational context, because as we will see shortly, the gravity expression of relative entropy entropy can be derived from the Holland-Wald offshell identity by imposing the Einstein equation. One can ask the reverse question, namely, can we derive the Einstein equation from the relative entropy expression. We will see that the answer is positive.

To simplify the setting we consider only pure gravity so that $\phi(x;\lambda)$ is replaced by metric $g(x;\lambda)$, and choose $\xi$ as the Killing vector of AdS given in Eq. (9).

Integrating both side of Eq. (20) over $\Sigma$ whose boundary is $\bar{B}$ and $\tilde{B}$, we get Eq. (11) and (12). By integrating (21) over $\Sigma$, the region between $B$ and $\tilde{B}$ at time slice $\ell = 0$, we have

$$\int_B \chi - \int_{\tilde{B}} \chi = \int_\Sigma \omega(\lambda; \delta \phi, \delta g) + \int_\Sigma (\hat{E} + \hat{\mathcal{C}}) \tag{22}$$

where $\hat{E} = -\xi_B \xi_a E^{a}_{bc} g_{\lambda} \frac{d}{d\lambda} E^{bc} \hat{\mathcal{C}} = -\xi_B \frac{d}{d\lambda} C^a g_{\lambda}$. First consider only physical metrics which satisfy equations of motion, then $\hat{E} = \hat{\mathcal{C}} = 0$ so that

$$\int_B \chi - \int_{\tilde{B}} \chi = \int_\Sigma \omega(\lambda; \delta \phi, \delta g) \tag{23}$$

Notice that the right hand side is not zero since $\xi_B$ is Killing vector of the background metric $g_0$ not that of $g_\lambda$. One should also notice that the first term of (23) is not a total variation as one can see in (21) and therefore can not be written in general as $\frac{d}{d\lambda} S_{grav}$, while the second term is always a total variation so that it can be written as $\frac{d}{d\lambda} S_{grav}$. Integrating the Eq. (23) by $\int_0^\Sigma d\lambda$, we have

$$\Delta E^g_{B_{\Sigma}} \Delta S^g_{B_{\Sigma}} = \int_0^\Sigma d\lambda \int_\Sigma \omega(\lambda; \delta g, \delta_{\xi_B} g) \tag{24}$$

where

$$\Delta E^g_{B_{\Sigma}} = \int_0^\Sigma d\lambda \int_B \chi, \quad \Delta S^g_{B_{\Sigma}} = \int_0^\Sigma d\lambda \int_B \chi.$$ 

Since one can ‘define’ the relative entropy as the difference of $\Delta E$ and $\Delta S$ as we noted earlier, Eq. (24) can be used to identify the gravity version of relative entropy

$$S_{grav}(\rho|\rho_0) = \int_0^\Sigma d\lambda \int_\Sigma \omega(\lambda; \delta g, \delta_{\xi_B} g). \tag{25}$$

Then, Eq. (24) becomes

$$\Delta E^{grav}_{B_{\Sigma}} \Delta S^{grav}_{B_{\Sigma}} = S^{grav}(\rho|\rho_0), \tag{26}$$

which is nothing but the gravity dual of the generalized first law (7).

So far, we have seen that the on-shell expression of Holland-Wald identity gives the gravitational version of the generalized first law. This has been known [13,16-18]. The authors of [13] proved differential version of (24) from the CFT up to second order, which enabled them to prove the Einstein equation to the corresponding order.

What we want to do is the reverse direction: if a metric satisfies the gravity version of generalized entanglement first law, it should satisfy Einstein equation. In other words, we want to prove that the gravity expression of the relative entropy, eq.(25) or its consequence (24), is
equivalent to the equation of motion. This is not a topology. Notice that deriving (25) from CFT is not our goal.

Namely, we want to derive the full Einstein equation, starting from Eq. (24). This is our main goal.

By integrating Eq. (22) in $\lambda$ over $[0, \varepsilon]$, we first rewrite it as

$$\Delta E_B^{\text{grav}} - \Delta S_B^{\text{grav}} - S_B^{\text{grav}}(\rho|\rho_0) = \int_0^\varepsilon d\lambda \int_\Sigma (\hat{E} + \hat{C}) \tag{27}$$

Now if we impose Eq. (24) or (26), which is the gravity dual of the generalized first law of entanglement, the right hand side of above equation vanishes. Taking the derivative of equation with respect to $\varepsilon$, we get

$$\hat{E}[g(\varepsilon)] + \hat{C}[g(\varepsilon)] = 0. \tag{28}$$

Using the explicit form of the constraint given in (17), we have

$$\xi_b E^{(cd)}[g(\varepsilon)]g(f) + 2\xi^a E^{(a)}_b[g(\varepsilon)] = 0. \tag{29}$$

where the prime denote $\frac{\partial}{\partial \varepsilon}$ and we deleted the subscript/superscript $g_b, B$ from the $E$ to simplify the notation. We expand the $E_{ab}[g(\varepsilon)]$ and $g_{ab}(\varepsilon)$ in $\varepsilon$:

$$E[g(\varepsilon)] = \sum_{n=0}^\infty \varepsilon^n E^{(n)} \quad \text{and} \quad g(\varepsilon) = \sum_{n=0}^\infty \varepsilon^n g^{(n)}. \tag{30}$$

Then Eq. (29) becomes

$$\sum_{n=1}^\infty \varepsilon^{n-1} \left[ \xi_b \sum_{k=1}^n kE^{(n-k)}[g_0] \cdot g^{(k)} + 2\xi^a nE^{(a)}_{ab} \right] = 0, \tag{31}$$

where $\cdot$ is for the full contraction. Requesting the analyticity in $\varepsilon$, each coefficient of above equation should be zero. It is useful to write the first few terms explicitly to see the structure:

$$\xi_b E^{(0)}[g_0] + 2\xi^a E^{(1)}_{ab} = 0, \quad \xi_b (2E^{(0)} \cdot g^{(1)} + E^{(1)} \cdot g^{(1)}) + 4\xi^a E^{(2)}_{ab} = 0, \quad \xi_b (3E^{(0)} \cdot g^{(2)} + 2E^{(1)} \cdot g^{(2)} + E^{(2)} \cdot g^{(1)} + 6\xi^a E^{(3)}_{ab} = 0, \quad \cdots. \tag{32}$$

Notice that this is the expansion around the AdS metric $g_0$, so that $E^{(0)}[g_0] = 0$, which implies $E^{(1)}[g_0] = 0$ by the first equation, which in turn implies $E^{(2)}[g_0] = 0$ by the second equation. In this way, all $E^{(n)}[g_0] = 0$ by the lower ones progressively, proving the whole non-linear Einstein equation

$$E[g(\varepsilon)] = 0, \tag{33}$$

for all order in $\varepsilon$. Therefore, the metric $g(\varepsilon)$ near $g_0$ satisfies full Einstein equation.

Summarizing, the full Einstein equation holds iff the generalized entanglement first law does, thanks to the geometric identity Eq. (26). In other words, the metric $g$ dual to the state $\rho$ compatible with the generalized first law satisfies the non-linear Einstein equation. Although (24) is derived using Einstein Equation, it is special so that it can imply the Einstein equation itself through the geometric identity. What is the implication of all this? It just means that the relative entropy (RE) expression or the generalized entanglement entropy contains on-shell information. This is clear from the linearized level. There, first law implies on-shell condition. The same should be true here. In fact, in CFT side, the RE can be evaluated only for physical configuration. Therefore on-shell information is hidden in the entanglement relationship. From gravity side, the Einstein equation is the criterion to judge whether a given metric configuration is physical. Therefore it is not surprising expression of RE encodes the information of on-shell-ness.

One important remark is that while $\chi$ is a total derivative $\lambda$ on $B$ due to the vanishing of $\xi$ on $B$, it is not so on $B$. Therefore $\Delta S_B^{\text{grav}}$ is a total variation but $\Delta E_B^{\text{grav}}$ is not so in general. This is exactly the same property of $\Delta E, \Delta S$ in CFT side as we emphasized earlier. However, for an integrable case where $\int_\Sigma \xi \cdot \omega = 0$, the situation is better, because there exist $\hat{K}$ and $W_\xi$ such that $\xi_B \cdot \Theta(\frac{d}{d\lambda}g) = \frac{d}{d\lambda}(\xi_B \cdot K)$ and $W_\xi = \int_{B\setminus\hat{B}} (Q - \xi \cdot K)$ respectively [10], so that we can rewrite (23) as [10] [17] [19]

$$\frac{d}{d\lambda}W_\xi = \int_\Sigma \omega(\xi \cdot \frac{d}{d\lambda}g, \delta_\xi g). \tag{35}$$

This can be integrated over $\lambda$ to give

$$\Delta E_B^{\text{grav}} - \Delta S_B^{\text{grav}} = \Delta W_\xi. \tag{36}$$

where $\Delta$ is a variation from $\rho_0$ to $\rho$ whose dual geometries are $g_0, g$ respectively. This means that, for an integrable case, the relative entropy is a total variation and it can be interpreted as the work done on the system to change it from $\rho_0$ to $\rho$.

Our method can be easily generalized to the case with inclusion of matter or higher derivatives. For the reference states other than the AdS vacuum, the barrier is the proof of the existence of the Killing vector and its Holland-Wald gauge condition. We leave these matter to the future works.

**IV. ENTANGLEMENT VECTOR FIELD**

In ref. [24], the authors tried to reformulate entanglement as the a flux of vector field $V$. Consider a surface $B'$ in $t = 0$ slice whose boundary is the same as that of $B$. Our goal is to construct a vector field $V$ such that

$$\int_{B'} V_E d\sigma_a = \int_B V^a_E d\sigma_a = S_B. \tag{37}$$
Such vector field should be divergenceless in the subspace of $t = 0$ slice. Also it must be a codimension 2 form to produce an one form upon restriction. Natural candidate is $\pi Q$ restricted to the constant time slice and we start from the observation

$$\int_{\tilde{B}} Q = S_B, \quad dQ = -\xi \cdot \epsilon L \neq 0, \quad (38)$$
on shell, where we used Eq. (10) and the fact that $\xi$ is the Killing vector of $g$. Now we can construct a vector field $V$ by restricting the codimension 2 form $Q$ to the $t = 0$ slice. Noticing that among the components of $\xi$, only $\xi^t$ is non-zero, we have

$$16\pi G_N Q = \nabla^a \xi^b \epsilon_{ab} = -2\nabla_a \xi^t \sqrt{-g} \epsilon^a := V_0 \epsilon^a. \quad (39)$$

In one form notation, the $V_0$ is give by

$$V = \frac{4\pi}{R^2} \left[ \left( \frac{R^2 - z^2 - \bar{z}^2}{2z} \right) dz + x^i dx^i \right]. \quad (40)$$

It is easy to check that $\int_{\tilde{B}} V_0 \epsilon^a = 4\pi \text{Area} \tilde{B}$. Therefore it is tempting to call $V_0$ as entanglement vector field. However, for a vector field to be interpreted as a flux, it should be divergenceless so that the flux on arbitrary surface $B'$ is equal to $S_B$. Unfortunately, $V$ is not divergence free. In fact, in $t = 0$ slice of $AdS_{d+1}$,

$$\nabla_a V^a = \frac{2\pi d}{R^2} (z^2 + \bar{z}^2 - R^2) = (-2d) n \cdot \xi, \quad (41)$$

where $n$ is the normal vector of the hypersurface $\Sigma$. Furthermore, while we expect that the entanglement vector’s flux is highly concentrated at the boundary of the region $B$, the flux of $V$, as one can see in the Fig.1, is almost uniformly distributed over $\tilde{B}$.

![Diagram](a)

**FIG. 1.** (a) Entanglement wedge and flow of vector field $\xi$ and $V$. (b) Flow of vector field $V$ within $\Sigma$. The red circle is the Ryu-Takayanagi surface

Therefore we look for a balancing vector field $V_0$ such that $\nabla_a (V^a - V_0^a) = 0$ and flux of $V_0$ over $\tilde{B}$ is zero. We take ansatz $V_0 = V_0 r dr$ and boundary condition $V_0|_{r = R} = 0$. One remark is that when we take the divergence of $V$, we should consider $\xi^t$ as a scalar once we restrict $Q$ to $t = 0$ slice. In $AdS_{d+1}$, it can be given by

$$V_0 = \frac{2\pi d}{R} \frac{(r - R)^2}{r^2 \cos^2 \theta} dr, \quad (42)$$

where $r^2 = z^2 + \bar{z}^2$ and $\cos \theta = z/r$. The final form of the entanglement vector field is give by $V_E = V - V_0$ whose explicit form in polar coordinate is

$$V_E = \frac{2\pi}{R} \left[ \frac{R^2 + R^2 \tan^2 \theta dr - \frac{(R^2 - r^2)}{r} \cot \theta d\theta} \right] - V_0 \quad (43)$$

which is divergence free vector field whose flux over any $B'$ is $S_B$ if $B'$ is homologous to $B$. One can easily verify that $V_E$ satisfies Eq. (37), and for $AdS_3$ the flux of each vector fields are

$$\int_B V_0 \epsilon^a = \frac{c}{9 \epsilon^2}, \quad \int_B V_0 \epsilon^a = \frac{c}{9 \epsilon^2} - \frac{c}{3} \ln \frac{2R}{\epsilon}, \quad (44)$$

where $\epsilon$ is the UV cut-off of $z$ and $c = \frac{3L}{2G_N}$ with $L$ the AdS radius and $c$ is the central charge of the dual CFT$_2$.

![Diagram](b)

**FIG. 2.** (a) Flow of the Entanglement vector field $V_E$. (b) Cartoon of 3d version of left figure where it is rotated around z-axis.

Our goal here is to explicitly construct the thread vector of ref. [24], where the authors suggested to replace minimal surface by a divergenceless vector. Notice however, the flux line in fig. 15 of ref.[21] is similar to our vector field $V$ in FIG. 1 which is not divergenceless. If we impose zero divergence condition, the resulting vector field $V_E$ has the flux lines concentrated at the boundary of the two regions, which reveal quite interesting phenomena: entanglement is done mostly at the boundary of the two entangled regions. As a consequence, the flux of $V_E$, as one can see in the Fig.2, look like sewing the two regions $B$ and $\tilde{B}$ along their interface through the holographic direction, which is an anticipated feature for the entanglement entropy vector field but was not expected from the general argument of ref. [24].

V. DISCUSSION

We have shown that the generalized entanglement first law implies the full Einstein equation. It would be interesting to study the case in the presence of matter fields or higher curvature term. We also constructed a vector field $V$ in AdS space whose flux on arbitrary surface homologous to $B$ is equal to the entanglement entropy. It would be interesting if we can utilize the entanglement vector field to discuss the black hole information problem.
Acknowledgments SJS want to thank Kimyeong Lee, Sungjae Lee and Piljin Yi for discussion. This work is supported by Mid-career Researcher Program through the National Research Foundation of Korea grant No. NRF-2016R1A2B3007687. We appreciate the hospitality of APCTP during the focus workshop “Geometry and Holography of Quantum critical point”.

[1] M. Van Raamsdonk, “Building up spacetime with quantum entanglement,” Gen. Rel. Grav. 42, 2323 (2010) [Int. J. Mod. Phys. D 19, 2429 (2010)] [arXiv:1005.3035]
[2] J. Maldacena and L. Susskind, “Cool horizons for entangled black holes,” Fortsch. Phys. 61, 781 (2013) [arXiv:1306.0533 [hep-th]].
[3] N. Lashkari, M. B. McDermott and M. Van Raamsdonk, “Gravitational dynamics from entanglement ‘thermodynamics’,” JHEP 1404, 195 (2014) doi:10.1007/JHEP04(2014)195 [arXiv:1308.3716].
[4] S. Ryu and T. Takayanagi, “Aspects of Holographic Entanglement Entropy,” JHEP 1404, 045 (2016) [arXiv:1308.3716].
[5] H. Casini, M. Huerta and R. C. Myers, “Towards a derivation of holographic entanglement entropy,” JHEP 1105, 036 (2011) [arXiv:1102.0440 [hep-th]].
[6] T. Faulkner, M. Guica, T. Hartman, R. C. Myers and M. Van Raamsdonk, “Gravitation from Entanglement in Holographic CFTs,” JHEP 1403, 051 (2014) [arXiv:1312.7856 [hep-th]].
[7] R. M. Wald, “Black hole entropy is the Noether charge,” Phys. Rev. D 48, no. 8, R3427 (1993) doi:10.1103/PhysRevD.48.R3427 [gr-qc/9307038].
[8] V. Iyer and R. M. Wald, “Some properties of Noether charge and a proposal for dynamical black hole entropy,” Phys. Rev. D 50, 846 (1994) doi:10.1103/PhysRevD.50.846 [gr-qc/9403028].
[9] V. Iyer and R. M. Wald, “A Comparison of Noether charge and Euclidean methods for computing the entropy of stationary black holes,” Phys. Rev. D 52, 4430 (1995) doi:10.1103/PhysRevD.52.4430 [gr-qc/9503052].
[10] R. M. Wald and A. Zoupas, “A General definition of ‘conserved quantities’ in general relativity and other theories of gravity,” Phys. Rev. D 61, 084027 (2000) doi:10.1103/PhysRevD.61.084027 [gr-qc/9911095].
[11] S. Hollands and R. M. Wald, “Stability of Black Holes and Black Branes,” Commun. Math. Phys. 321, 629 (2013) doi:10.1007/s00220-012-1638-1 [arXiv:1201.0463].
[12] T. Jacobson, Phys. Rev. Lett. 75, 1260 (1995) doi:10.1103/PhysRevLett.75.1260 [gr-qc/9404004].
[13] T. Faulkner, F. M. Haehl, E. Hijano, O. Parrilkar, C. Rabideau and M. Van Raamsdonk, “Nonlinear Gravity from Entanglement in Conformal Field Theories,” JHEP 1708, 057 (2017) doi:10.1007/JHEP08(2017)057 [arXiv:1705.03026 [hep-th]].
[14] D. D. Bianco, H. Casini, L. Y. Hung and R. C. Myers, “Relative Entropy and Holography,” JHEP 1308, 060 (2013) [arXiv:1305.3182].
[15] J. Lin, M. Marcolli, H. Ooguri and B. Stoica, “Locality of Gravitational Systems from Entanglement of Conformal Field Theories,” Phys. Rev. Lett. 114, 221601 (2015) [arXiv:1412.1879].
[16] N. Lashkari, J. Lin, H. Ooguri, B. Stoica and M. Van Raamsdonk, “Gravitational positive energy theorems from information inequalities,” PTEP 2016, no. 12, 12C109 (2016) [arXiv:1605.01075 [hep-th]].
[17] N. Lashkari and M. Van Raamsdonk, “Canonical Energy is Quantum Fisher Information,” JHEP 1604, 153 (2016) doi:10.1007/JHEP04(2016)153 [arXiv:1508.00897].
[18] M. Van Raamsdonk, “Lectures on Gravity and Entanglement,” doi:10.1142/9789813149441-0005 [arXiv:1609.00026 [hep-th]].
[19] D. L. Jafferis, A. Lewkowycz, J. Maldacena and S. J. Suh, JHEP 1606, 004 (2016) doi:10.1007/JHEP06(2016)004 [arXiv:1512.06431 [hep-th]].
[20] B. Swingle and M. Van Raamsdonk, “Universality of Gravity from Entanglement,” JHEP 1405, 2933 [arXiv:1405.2933].
[21] T. Jacobson, “Entanglement Equilibrium and the Einstein Equation,” Phys. Rev. Lett. 116, no. 20, 201101 (2016) doi:10.1103/PhysRevLett.116.201101 [arXiv:1505.04755 [gr-qc]].
[22] G. Srosi and T. Ugajin, “Modular Hamiltonians of excited states, OPE blocks and emergent bulk fields,” arXiv:1705.01486 [hep-th].
[23] X. Dong and A. Lewkowycz, “Entropy, Extremality, Euclidean Variations, and the Equations of Motion,” arXiv:1705.08453 [hep-th].
[24] M. Freedman and M. Headrick, “Bit threads and holographic entanglement,” arXiv:1604.00354 [hep-th].