Maxwell’s equations in 4-dimensional Euclidean space

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Abstract. The paper formulates Maxwell’s equations in 4-dimensional Euclidean space by embedding the electromagnetic vector potential in the frame vector $g_0$. Relativistic electrodynamics is the first problem tackled; in spite of using a geometry radically different from that of special relativity, the paper derives relativistic electrodynamics from space curvature. Maxwell’s equations are then formulated and solved for free space providing solutions which rotate the vector potential on a plane; these solutions are shown equivalent to the usual spacetime formulation and are then discussed in terms of the hypersphere model of the Universe recently proposed by the author.

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1. Introduction

In recent years I’ve been proposing an extension of optics to 4-dimensional space, naturally calling this discipline 4-dimensional optics (4DO). Initially this was proposed as an alternative formulation for relativistic problems; two papers that use this view and provide a good introduction are Almeida [1, 2]. The latter of these references has some flaws in its approach to electromagnetism which will be corrected here; it also suggested a metric for vacuum which was superseded by a more appropriate one in [3]. In this very recent work I made some cosmological predictions arising from a development of 4DO, which had so far been impossible with a general relativity approach, showing that dark matter is really unnecessary for the explanation of observations if one accepts 4-space as being Euclidean with coordinate $x^0$ the radius of an hypersphere. The latter work left unexplained the fact that photons were constrained to great circles in a 4-dimensional hypersphere rather than following straight line geodesics.

The explanation of photon behaviour calls for a full exposition of Maxwell’s equations in 4DO context, which has not yet been done in a formal way. The present paper is a presentation of electromagnetism in 4DO space, introducing the electromagnetic vector potential as part of the space frame and deriving Maxwell’s equations in a natural way. The solution of Maxwell’s equations leading to electromagnetic waves is then shown to be bound to great circles on the 4D hypersphere, thus solving the difficulty in Almeida [3].

In the exposition I will make full use of an extraordinary and little known mathematical tool called geometric (Clifford) algebra, which received an important thrust with the works of David Hestenes [4]. A good introduction to geometric algebra can be found in Gull et al. [5] and in the following paragraphs I will use the notation and conventions of the latter. Expressing Maxwell’s equations in the formalism of geometric algebra is not new; the Cambridge Group responsible for the reference above uses this approach in one of their courses [6] for the relativistic formulation of those equations, which are then condensed in the extraordinarily compact equation

$$\nabla^2 A = J.$$  

Although this formulation is valid for Minkowski spacetime, with a different signature from 4DO, I shall follow closely that group’s exposition.

Before embarking into the transposition of electromagnetism to 4DO the paper makes a brief introduction to geometric algebra and makes a quick revision of 4DO’s principles, using the opportunity to express them in the former’s formalism.

2. Introduction to geometric algebra

The geometric algebra of Euclidean 4-space $\mathcal{G}_4$ is generated by the frame of orthonormal vectors $\{\sigma_\mu\}$, $\mu = 0 \ldots 3$, verifying the relation

$$\sigma_\mu \cdot \sigma_\nu = \frac{1}{2} (\sigma_\mu \sigma_\nu + \sigma_\nu \sigma_\mu) = \delta_{\mu\nu}. \quad (2)$$
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The algebra is 16-dimensional and is spanned by the basis

\[ 1, \{\sigma_\mu\}, \{\sigma_\mu \sigma_\nu\}, \{\sigma_\mu I\}, I, \]

1 scalar 4 vectors 6 bivectors 4 trivectors 1 fourvector

where \( I \equiv \sigma_0 \sigma_1 \sigma_2 \sigma_3 \) is also called the pseudoscalar. The elements of this basis are such that all vectors and the pseudoscalar square to unity

\[ (\sigma_\mu)^2 = 1, \quad I^2 = 1; \]

and all bivectors and trivectors square to -1

\[ (\sigma_\mu \sigma_\nu)^2 = -1, \quad (\sigma_\mu I)^2 = -1. \]

It will be convenient to shorten the product of basis vectors with a multi-index compact notation; for instance \( \sigma_\mu \sigma_\nu \equiv \sigma_{\mu\nu} \).

The geometric product of any two vectors \( a = a^\mu \sigma_\mu \) and \( b = b^\nu \sigma_\nu \) can be decomposed into a symmetric part, a scalar called the inner product, and an anti-symmetric part, a bivector called the exterior product.

\[ ab = a \cdot b + a \wedge b, \quad ba = a \cdot b - a \wedge b. \]

Reversing the definition one can write internal and exterior products as

\[ a \cdot b = \frac{1}{2} (ab + ba), \quad a \wedge b = \frac{1}{2} (ab - ba). \]

The exponential of bivectors is especially important and deserves an explanation here. If \( u \) is a bivector or trivector such that \( u^2 = -1 \) and \( \theta \) is a scalar

\[ e^{u\theta} = 1 + u\theta - \frac{\theta^2}{2!} - u \frac{\theta^3}{3!} + \frac{\theta^4}{4!} + \ldots \]

\[ = 1 - \frac{\theta^2}{2!} + \frac{\theta^4}{4!} - \ldots \{= \cos \theta \} + u\theta - u \frac{\theta^3}{3!} + \ldots \{= u \sin \theta \} \]

\[ = \cos \theta + u \sin \theta. \]

Although less important in the present work, the exponential of vectors and fourvector can also be defined; if \( h \) is a vector or fourvector such that \( h^2 = 1 \)

\[ e^{h\theta} = 1 + h\theta + \frac{\theta^2}{2!} + h \frac{\theta^3}{3!} + \frac{\theta^4}{4!} + \ldots \]

\[ = 1 + \frac{\theta^2}{2!} + \frac{\theta^4}{4!} + \ldots \{= \cosh \theta \} + h\theta + h \frac{\theta^3}{3!} + \ldots \{= h \sinh \theta \} \]

\[ = \cosh \theta + h \sinh \theta. \]

The exponential of bivectors is useful for defining rotations; a rotation of vector \( a \) by angle \( \theta \) on the \( \sigma_{12} \) plane is performed by

\[ a' = e^{-\sigma_{12}\theta/2} a e^{\sigma_{12}\theta/2} = Ra\bar{R}; \]
the tilde denotes reversion, reversing the order of all products. As a check we make
\( a = \sigma_1 \)

\[
e^{-\sigma_1 \theta/2} \sigma_1 e^{\sigma_1 \theta/2} = \left( \cos \frac{\theta}{2} - \sigma_{12} \sin \frac{\theta}{2} \right) \sigma_1 \left( \cos \frac{\theta}{2} + \sigma_{12} \sin \frac{\theta}{2} \right) = \cos \theta \sigma_1 + \sin \theta \sigma_2. \tag{11}
\]
Similarly, if we had made \( a = \sigma_2 \), the result would have been \(-\sin \theta \sigma_1 + \cos \theta \sigma_2\).

If we use \( B \) to represent a bivector and define its norm by \(|B| = (B \tilde{B})^{1/2}\), a general rotation in 4-space is represented by the rotor

\[
R \equiv e^{-B/2} = \cos(|B|/2) - \frac{B}{|B|} \sin(|B|/2). \tag{12}
\]

The rotation angle is \(|B|\) and it is performed on the plane defined by \( B \). A rotor is defined as a unitary even multivector (a multivector with even grade components only); we are particularly interested in rotors with scalar and bivector components. It is more general to define a rotation by a plane (bivector) then by an axis (vector) because the latter only works in 3D while the former is applicable in any dimension.

In a general situation the frame may not be orthonormed and we will generally define this frame by expressing its vectors in the fiducial frame \( \sigma_{\mu} \)

\[
g_{\mu} = h^\alpha_{\mu} \sigma_\alpha; \tag{13}
\]

where \( h^\alpha_{\mu} \) is called the fiducial tensor. The metric tensor is then defined by the inner products of the frame vectors

\[
g_{\mu\nu} = g_\mu \cdot g_\nu. \tag{14}
\]

Complementary we can then define the reciprocal frame by the relation

\[
g^{\mu} \cdot g_\nu = \delta^\mu_{\nu}. \tag{15}
\]

This method for frame and metric definition is absolutely general and it is signature preserving. Since our fiducial frame has signature \((++++)\) all the frames defined with reference to it will preserve that signature. Hestenes discusses curvature in geometric algebra formulation but he uses a Minkowski fiducial frame with signature \((+---+)\).

3. Moving frames in 4DO

In this section we will be looking at the frame applicable to a moving observer and making a parallel to Lorentz transformations in special relativity. This is a reformulation of the presentation in [1] using the formalism of geometric algebra. We must recall that 4DO is characterised by time being evaluated as geodesic arc length in 4-dimensional space

\[
c^2 dt^2 = g_{\mu\nu} dx^\mu dx^\nu. \tag{16}
\]

From this point onwards we avoid all problems of dimensional homogeneity by using normalising factors listed in Table I for all units, defined with recourse to the
fundamental constants: Planck constant divided by $2\pi$ ($\hbar$), gravitational constant ($G$), speed of light ($c$) and proton charge ($e$). This normalisation defines a system of *non-dimensional units* with important consequences, namely: 1) All the fundamental constants, $\hbar$, $G$, $c$, $e$, become unity; 2) a particle’s Compton frequency, defined by $\nu = mc^2/\hbar$, becomes equal to the particle’s mass; 3) the frequent term $GM/(c^2r)$ is simplified to $M/r$.

In non-dimensional units equation (16) above can be obtained from the equivalent vector definition \[ ds = g_{\mu\nu}dx^\nu, \] where $ds$ is the displacement vector and $dt^2 = ds \cdot ds$. Consequently the velocity vector is defined by \[ v = g_{\mu\nu}\dot{x}^\mu, \] where “dot” over a variable means time derivative.

As an introductory example to moving observer frames we will suppose an observer $\bar{O}$ with velocity $v = \cos \theta \sigma_0 + \sin \theta \sigma_1$, Fig. (1). The moving observer is obviously stationary in his own frame and so his $\bar{x}^0$ axis must be aligned with $v$; hence the corresponding frame vector must be obtained from $v$ by product with a scalar: $g_0 = \lambda v$.

In 4DO light is characterised by geodesics with $dx^0 = 0$, a necessary condition for the 3-space velocity to have unitary norm. These displacements must be evaluated similarly by the fixed and moving observers so their three frame vectors corresponding to 3-space must remain unaltered: $g_i = \sigma_i$.

Any coordinate change cannot alter a displacement so that the latter can be equally expressed in either coordinate system \[ ds = \sigma_\mu dx^\mu = g_\mu d\bar{x}^\mu. \] Applying to the case under study \[ ds = \sigma_0 dx^0 + \sigma_1 dx^1 = \lambda(\cos \theta \sigma_0 + \sin \theta \sigma_1) d\bar{x}^0. \]

As last argument special relativity imposes the invariance of $(dx^0)^2 = dt^2 - \sum(dx^i)^2$; in 4DO we have, from Eq. (16), $dt^2 = (dx^0)^2 + \sum(dx^i)^2$. Consequently, for compatibility with special relativity, we must require also that $\ddagger$

\[ d\bar{x}^0 = dx^0. \]

$\ddagger$ In Almeida [1] a different argument is used to justify this invariance.
Figure 1. In his own frame observer \( \bar{O} \) is fixed and \( g_0 \) must be aligned with the velocity; time intervals must be the same in both frames; \( dx^0 \) and time intervals must be preserved when the coordinates are transformed; the displacement \( g_0 dx^0 \) can be decomposed into \( \sigma_0 dx^0 + \sigma_1 dx^1 \).

Combining Eqs. (20) and (21) it must be \( \lambda = \sec \theta \) and

\[
g_0 = \sigma_0 + \tan \theta \sigma_1 = \sigma_0 (1 + \tan \theta \sigma_0_1). \tag{22}
\]

The metric tensor components can be evaluated by Eq. (14)

\[
g_{00} = 1 + \tan^2 \theta, \quad g_{01} = g_{10} = \tan \theta. \tag{23}
\]

The evaluation of \( dt^2 \) is unaltered because this is still the fixed observer’s time measurement performed in the moving observer’s coordinates; we call this a coordinate change, which is different from a metric change. Using Eq. (17) a general displacement is evaluated in the two frames as

\[
d s = \sigma_0 dx^0 + \sigma_1 dx^1 = (\sigma_0 + \tan \theta \sigma_1) dx^0 + \sigma_1 dx^1; \tag{24}
\]

and the coordinate conversion is immediate

\[
dx^1 = dx^1 - \tan \theta dx^0. \tag{25}
\]

The coordinate transformation for a moving observer seems incompatible with the corresponding transformation in special relativity, which is described by a Lorentz transformation and produces time dilation. This incompatibility is only apparent since the moving observer has no reason to choose a skew frame and evaluates displacements with a standard orthonormed frame

\[
d\tilde{s} = \sigma_0 dx^0 + \sigma_1 (dx^1 - \tan \theta dx^0) = ds - \sigma_1 \tan \theta dx^0. \tag{26}
\]
Notice that the special relativity invariant is preserved in the moving observer’s evaluation of time
\[(ds)^2_{SR} = (dt)^2 - (dx^1)^2 = (\bar{dt})^2 - (\bar{dx}^1)^2 = (dx^0)^2.\]  
(27)

The previous example restrained the analysis to movement along the $\sigma_1$ direction, described by a velocity vector with $\sigma_0$ and $\sigma_1$ components. In general a coordinate change defines a skew frame whose vectors result from the transformation applied to the element $\sigma_0$
\[g_0 = \sigma_0 \left(1 + \frac{\sigma_0 v}{v_0} \right) = \sigma_0 \frac{\sigma_0 v}{\sigma_0 \cdot v} = \frac{v}{\sigma_0 \cdot v},\]  
(28)
where the bold $v$ represents the 3 spatial velocity components or the velocity vector in the non-relativistic sense. The coordinate conversion preserves $dx^0$ and for the remaining coordinates we have
\[dx^k = dx^k - \frac{v^k}{v_0} dx^0.\]  
(29)

The interval evaluated by the moving observer in his orthonormed frame is
\[d\bar{s} = d\bar{x}^\mu \sigma_\mu = ds - \frac{v}{v_0}.\]  
(30)

In conclusion, the 4DO counterpart of a Lorentz transformation in special relativity is a two-step process.

- A transformation of the spatial coordinates $x^i$, preserving the fixed observer’s time measurement; the corresponding frame is called the skew frame and the transformation is called a coordinate change.
- A jump into the orthonormed moving frame with the consequent time dilation. This is mathematically a metric change because the length of displacements is not preserved.

A concrete situation where a moving frame had to be considered appeared in Almeida \[3\, Eq. (14)\], an equation with the same form of Eq. (24), equivalent to a frame vector
\[g_0 = \sigma_0 + \frac{x^k}{x^0} \sigma_k.\]  
(31)

In this case the movement has a geometric cause and results from the natural (geometric) expansion of the Universe when the hypersphere model is assumed.

4. Electrodynamics as space curvature

Relativistic dynamics, for cases of isotropic media, is modeled in 4DO by the frame
\[g_0 = n_0 \sigma_0, \quad g_j = n_r \sigma_j,\]  
(32)
where $n_0$ and $n_r$ are scalar functions of the coordinates called refractive indices. Application of Eq. (16) with $c = 1$ leads to the time interval
\[dt^2 = (n_0 dx^0)^2 + (n_r)^2 \sum(dx^j)^2.\]  
(33)
If the refractive indices are not functions of $x^0$ the geodesics of the space so defined can be mapped to the geodesics of the relativistic space defined by the metric
\[ ds^2 = \left( \frac{dt}{n_0} \right)^2 - \left( \frac{n_r}{n_0} \right)^2 \sum (dx^j)^2; \] (34)
as is fully demonstrated in Almeida [3, Sec. 4]. The equivalence between 4DO and GR spaces stops when the metric is non-static, meaning that the refractive indices are functions of $x^0$ in 4DO or $t$ in GR. The two spaces are not equivalent either, for any displacements which involve parallel transport.

The first question we would like to answer in this section is whether a frame of the type defined by Eqs. (32) is adequate to model the dynamics of a charged particle under an electric field; for this we will use the refractive indices
\[ n_0 = 1 + \frac{qV}{m}, \quad n_r = 1, \] (35)
where $q$ is the charge of the moving particle, $m$ is its mass and $V$ is the electric potential, including the fine structure constant $\alpha$. So, for instance, the electric field of a stationary particle with charge $Q$ is
\[ V = \frac{\alpha Q}{r}. \] (36)
Notice that the refractive indices are defined for an interaction between two charges, the same happening with the space metric. The space that we are defining exists only for the interaction under study and it is not a pre-existing arena where the dynamics is played.

The frame of Eqs. (35) produces the time definition
\[ dt^2 = \left( 1 + \frac{qV}{m} \right)^2 (dx^0)^2 + \sum (dx^j)^2. \] (37)
Following the procedure in Almeida [3] we will find the geodesic equations by first dividing both members by $dt^2$ and defining a constant Lagrangian $L = 1/2$
\[ 1 = 2L = \left( 1 + \frac{qV}{m} \right)^2 (\dot{x}^0)^2 + \sum (\dot{x}^j)^2. \] (38)
Noting that the Lagrangian is independent from $x^0$, there must be a conserved quantity
\[ \left( 1 + \frac{qV}{m} \right)^2 \dot{x}^0 = \frac{1}{\gamma}. \] (39)
The remaining Euler-Lagrange equations for the geodesics are
\[ \ddot{x}^j = \frac{q}{m} \left( 1 + \frac{qV}{m} \right) \partial_j V(\dot{x}^0)^2; \] (40)
Replacing with $\dot{x}^0$ from Eq. (39)
\[ \ddot{x}^j = \frac{q}{m\gamma^2} \left( 1 + \frac{qV}{m} \right)^{-3} \partial_j V. \] (41)
In the limit of speeds much smaller than the speed of light $\gamma \to 1$. As long as $qV/m \ll 1$ the equation represents the classical dynamics of a particle with mass $m$ and charge $q$
under the electric potential $V$. Electric potentials typically decrease with $1/r$ and thus the parenthesis can be taken as unity for large distances from electric field sources; for the interaction between two electrons, considering the non-dimensional units’ normalising factors, this condition means distances considerably larger than $2.8 \times 10^{-15}$ m.

Having shown that dynamics under an electric field can be modeled by a suitably chosen frame, we need to investigate if the same applies when a magnetic field is present. Magnetic fields are originated by moving charges and we have seen how a moving frame can be obtained from a stationary one by a transformation applied to its zeroth vector. Since electric field dynamics only implies the consideration of refractive index $n_0$, the other refractive index $n_r$ remaining unity, it is natural to admit that an electromagnetic interaction could be modeled by the following frame

$$g_0 = \sigma_0 + \frac{qA^\mu\sigma_\mu}{m}, \quad g_j = \sigma_j,$$  \hspace{1cm} (42)

where $A = A^\mu\sigma_\mu$ represents the vector potential; note that the vector potential is referred to the orthonormed frame $\sigma_\mu$ and not to the skew frame $g_\mu$. The equation above is equivalent to the definition of a fiducial tensor but the vector potential approach is more convenient for the derivations that follow.

The metric tensor elements are obtained, as usual, by Eq. (14)

$$g_{00} = \left(1 + \frac{qA^0}{m}\right)^2, \quad g_{0j} = g_{j0} = \frac{qA^j}{m}, \quad g_{jk} = \delta_{jk}.$$  \hspace{1cm} (43)

Instead of using the geodesic Lagrangian to find its Euler-Lagrange equations, we will follow a different procedure, starting with the velocity vector from Eq. (18) with the frame vectors from Eqs. (42) with the frame vectors from Eqs. (12)

$$v = \frac{qA}{m} \dot{x}^0 + \sigma_\mu \ddot{x}^\mu.$$  \hspace{1cm} (44)

We will now define the vector derivative

$$\nabla = \sigma_\mu \partial_\mu.$$  \hspace{1cm} (45)

This is used for the derivation of an identity relative to acceleration

$$\dot{v} = \dot{v}^\mu \sigma_\mu$$

$$= \partial_\nu v^\mu \dot{x}^\nu \sigma_\mu$$

$$= \partial_\nu v^\mu v^\nu [\sigma_\nu \cdot (\sigma_\nu \wedge \sigma_\mu)]$$

$$= v \cdot (\nabla \wedge v).$$  \hspace{1cm} (46)

Applying to Eq. (44)

$$\dot{v} = v \cdot \left[\nabla \wedge \left(\frac{qA}{m} \dot{x}^0 + \sigma_\mu \ddot{x}^\mu\right)\right].$$  \hspace{1cm} (47)

The exterior product with the second term inside the parenthesis is necessarily null because it implies deriving the coordinates with respect to other coordinates and not to themselves; so we have finally

$$\dot{v} = \frac{qA}{m} v \cdot (\nabla \wedge A).$$  \hspace{1cm} (48)
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It is now convenient to define the Faraday bivector $F = \nabla \wedge A$. $F$ is necessarily a bivector because it is defined as an exterior product of two vectors; we can separate it into electric and magnetic components

$$F = E^j \sigma_0 + B^k i \sigma_k.$$  \hfill (49)

$E^j$ and $B^k$ are the components of electric and magnetic field vectors, respectively; $i = \sigma_{123}$ is a special trivector incorporating the 3 spatial frame vectors, so that $i \sigma_k$ is a spatial bivector where the component $\sigma_k$ is not present. Using boldface letters to represent 3-vectors, the Faraday bivector can be rewritten as

$$F = E \sigma_0 + i B.$$  \hfill (50)

And the acceleration becomes

$$\dot{v} = \frac{\dot{x}^0 q}{m} v \cdot (E \sigma_0 + i B).$$  \hfill (51)

This form of the Lorentz acceleration is relativistic because the norm of the velocity vector will be kept equal to unity at all times.

5. Maxwell’s equations

Using the vector derivative defined in Eq. (45) we write the derivative of Faraday bivector

$$\nabla F = \nabla \cdot F + \nabla \wedge F,$$  \hfill (52)

The internal product in the second member is expanded as

$$\nabla \cdot F = \partial_k E^k \sigma_0 - \partial_0 E^k \sigma_k + (\partial_3 B^2 - \partial_2 B^3) \sigma_1$$
$$+ (\partial_1 B^3 - \partial_3 B^1) \sigma_2 + (\partial_2 B^1 - \partial_1 B^2) \sigma_3.$$  \hfill (53)

A careful look at the second member shows that the first term is the divergence of $E$ multiplied by $\sigma_0$. The second term is $-\partial_0 E$, but we must take into account that in a stationary frame $dt = dx^0$, so this term can be taken as the negative of the time derivative in such frame. The last 3 terms represent the 3-dimensional cross product, for which we will use the symbol ”×”. If the bold symbol ”\nabla” represents the usual 3-dimensional nabla operator, the equation can be written

$$\nabla \cdot F = \nabla \cdot E \sigma_0 + \nabla \times B - \partial_0 E.$$  \hfill (55)

Defining the current vector $J = \rho \sigma_0 + J$, for a stationary frame, the first two Maxwell’s equations are expressed by

$$\nabla \cdot F = J.$$  \hfill (56)

For the remaining two equations we have to look at the exterior product $\nabla \wedge F$

$$\nabla \wedge F = (\partial_3 E^2 - \partial_2 E^3) \sigma_{023} + (\partial_1 E^3 - \partial_3 E^1) \sigma_{031}$$
$$+ (\partial_2 E^1 - \partial_1 E^2) \sigma_{012} + i \partial_k B^k + \partial_0 B^k I \sigma_k$$
$$= i (\nabla \times E \sigma_0 + \nabla \cdot B + \partial_0 B \sigma_0).$$  \hfill (57)
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The remaining two equations are expressed as $\nabla \wedge F = 0$ and the four equations can then be condensed in the very compact form

$$\nabla F = J.$$  \hfill (60)

Additionally, it is possible to choose $A$ such that $\nabla \cdot A = 0$, so that an equivalent form of the four equations is

$$\nabla^2 A = J, \quad (\nabla \cdot A = 0.)$$  \hfill (61)

Solutions of special interest are those where the second member is null ($J = 0$). Some solutions can be found easily by splitting $\nabla^2 A$ in its 0th and spatial components

$$\nabla^2 A = \partial_{00} A + \nabla^2 A,$$  \hfill (62)

where the second term is a standard 3D Laplacian. One possible solution arises immediately when $\partial_{00} A = \omega^2 A$ with $\omega$ real;

$$\nabla^2 A = -\omega^2 A,$$  \hfill (63)

results in a well known Helmholtz equation. Looking for a particular solution we assume only $x^3$ and $x^0$ dependence to find

$$A = A_0 e^{-\omega x^0} e^{i\sigma_3 \omega x^3}.$$  \hfill (64)

Where $u^2 = -1$ and $u$ must commute with $\sigma_3$. The two possibilities are either $u = i$ or $u = i\sigma_3$; we will choose the latter. Since we know that $A$ is a vector it must be

$$A_0 = \alpha \sigma_1 + \beta \sigma_2;$$  \hfill (65)

an eventual component aligned with $\sigma_0$ has been ignored.

In order to make interpretation of Eq. (64) easier we shall now write it in a slightly modified form

$$A = e^{-\omega x^0} e^{i\sigma_3 \omega x^3/2}(\alpha \sigma_1 + \beta \sigma_2) e^{-i\sigma_3 \omega x^3/2}.$$  \hfill (66)

It is now apparent that the first exponential factor is evanescent in the positive $x^0$ direction and grows to infinity in the negative direction; this is an uncomfortable situation that will be resolved below. The remaining factors represent a vector that rotates in the $i\sigma_3$ plane, along the $x^3$ direction, with angular frequency $\omega$. With an adequate choice for the origin of $x^3$ it is possible to make $\beta = 0$.

Before we proceed to the analysis of evanescence along $x^0$ let us consider consider the expression for $A$ when $x^0$ is constant; setting $\alpha' = \alpha \exp(-\omega x^0)$ it is

$$A = \alpha' \sigma_1 e^{-i\sigma_3 \omega x^3}.$$  \hfill (67)

Along the direction normal to the rotation plane we can define time to be equal to the distance traveled, $t = x^3$ and so $A$ can also be expressed as a time function

$$A = \alpha' \sigma_1 e^{-i\sigma_3 \omega t}.$$  \hfill (68)

In spacetime formulation time is a coordinate and we express the fact that $t = x^3$ by giving $A$ opposite dependencies in the two variables

$$A = \alpha' \sigma_1 e^{-i\sigma_3 \omega(x^3 - t)}.$$  \hfill (69)
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This is the spacetime formulation for a circularly polarised electromagnetic wave. In Almeida, the Universe is modeled as an hypersphere whose radius is the coordinate \( x^0 \). For displacements that are small compared to the hypersphere radius it is possible to use length coordinates rather than angles, resulting in the \( g_0 \) frame vector of Eq. 11. A comparison between this equation and Eq. 12 shows that the Universe’s expansion is the source of a vector potential \( A = x^k \sigma_k / x^0 \); the derivations in the previous section show that there are solutions to Maxwell’s equations that force the vector potential to rotate and the frame vector \( g_0 \) with it. These solutions force \( x^0 \) to stay constant and so they are constrained to hyperspherical surfaces of constant radius. This solves the unanswered question raised in Almeida whereby photons were constrained to great circles on the hypersphere, rather than to geodesic straight lines.

We can now turn our attention to the evanescence problem or rather to the problem of an amplitude growing exponentially in the negative \( x^0 \) direction. This is the direction pointing towards the hypersphere centre and suggests that the approximations made in order to use flat space are the source of the inconsistency. Recalling that in the hypersphere model the Universe is expanding at the speed of light, we can expect that there are resonating modes inside the hypersphere, which are evanescent to its outside due to continuity on the hypersphere border. A correct solution of Maxwell’s equations should not ignore this fact and will be pursued in forthcoming work. The resonating modes are so closely spaced with the current size of the Universe that we can actually consider a continuous distribution, but they manifest themselves as electromagnetic waves when space is artificially flattened.

6. Conclusion

The formalism of geometric algebra was used in this paper as useful mathematical tool for writing complex equations in a compact form and facilitate their geometrical interpretation. Although most readers will not be familiar with this algebra, we think that the simplifications achieved through its use are well worth the extra effort of learning some of its rudiments. A reformulation of Maxwell’s equations in 4-dimensional Euclidean space is the goal of this work but an introduction to the discipline of 4-dimensional optics was felt necessary, namely to explain the new transformation for moving frames and its relation to Lorentz transformations in special relativity.

Through the consideration of an electromagnetic vector potential embedded in one of the frame vectors, the paper shows that relativistic electrodynamics can be derived from curved space geodesics. In this way the Lorentz force acquires the characteristic of an inertial force, which is best described by inertial movement in curved space. Maxwell’s equations are then established in this space, with geometric algebra achieving the feat of their condensation in the equation \( \nabla^2 A = J \). Solving the equations for free space leads to rotation of frame vector \( g_0 \) on a plane lying on 3-space, with rotation progressing at arbitrary frequency along the direction normal to the rotation plane. These solutions are evanescent on the positive \( x^0 \) direction and grow to infinity in the opposite direction.
The inconsistency is attributed to artificial flattening of the hypersphere space proposed in previous work \cite{3} and further work is suggested to fully clarify this point.

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