Polygamy of Distributed Entanglement

Francesco Buscemi,1 Gilad Gour,2,3 and Jeong San Kim2

1 Statistical Laboratory, DPMMS, University of Cambridge, Wilberforce Road, CB3 0WB, UK
2 Institute for Quantum Information Science, University of Calgary, Alberta T2N 1N4, Canada
3 Department of Mathematics and Statistics, University of Calgary, Alberta T2N 1N4, Canada

(Dated: March 26, 2009)

While quantum entanglement is known to be monogamous (i.e. shared entanglement is restricted in multi-party settings), here we show that distributed entanglement (or the potential for entanglement) is by nature polygamous. By establishing the concept of one-way unlocalizable entanglement (UE) and investigating its properties, we provide a polygamy inequality of distributed entanglement in tripartite quantum systems of arbitrary dimension. We also provide a polygamy inequality in multi-qubit systems, and several trade offs between UE and other correlation measures.

PACS numbers: 03.67.-a, 03.67.Hk, 03.65.Ud,

I. INTRODUCTION

Quantum entanglement is a non-local quantum correlation providing a lot of useful applications in the field of quantum communications and computations such as quantum teleportation and quantum key distribution [1, 2, 3]. This important role of quantum entanglement has stimulated intensive study in both way of its quantification and qualification.

One of the essential differences of quantum correlations (especially, quantum entanglement) from other classical ones is that it cannot be freely shared among the parties in multipartite quantum systems. In particular, a pair of components that are maximally entangled cannot share entanglement [4, 5] nor classical correlations [6] with any part of the rest of the system, hence the term Monogamy of Entanglement (MoE) [7]. Monogamy of entanglement was shown to have a complete mathematical characterization for multi-qubit systems [5] using a certain entanglement measures, the concurrence [8].

Whereas MoE shows the restricted sharability of multiparty quantum entanglement, the distribution of entanglement, or Entanglement of Assistance (EoA) [9, 10] in multipartite quantum systems was shown to have a dually monogamous (or Polygamous) property. Using Concurrency of Assistance (CoA) [11] as the measure of distributed entanglement, it was also shown that whereas monogamy of entanglement inequalities provide an upper bound for bipartite sharability of entanglement in a multipartite system, the same quantity provides a lower bound for distribution of bipartite entanglement in a multipartite system [12]. In this paper, by introducing the concept of One-way Unlocalizable Entanglement (UE), we provide a polygamy inequality of entanglement in tripartite quantum systems of arbitrary dimension using entropic entanglement measure. Based on the functional relation between concurrence and entropic measure in two-qubit systems, we provide a polygamy inequality in multi-qubit systems. We also provide several trade offs between UE and other correlations such as EoA, and localizable entanglement.

The paper is organized as follows. In Sec. II we provide the definition of UE, and its basic properties. In Sec. III we provide a polygamy inequality of distributed entanglement in tripartite quantum systems in terms of entropy and EoA. In Sec. IV we generalize the polygamy inequality of entanglement into multi-qubit systems, and provide a more tight polygamy inequality for three-qubit systems. In Sec. V we provide several trade offs between UE and other correlations, and we summarize our results, in Sec. VI.

II. ONE-WAY UNLOCALIZABLE ENTANGLEMENT

A. Definition

For any bipartite quantum state $\rho_{AB}$, its one-way distillable common randomness [13] is defined as

$$C_D^-(\rho_{AB}) = \lim_{n \to \infty} \frac{1}{n} I^-(\rho_{AB}^\otimes n),$$

(1)

where, the function $I^-(\rho_{AB})$ [14] is

$$I^-(\rho_{AB}) = \max_{\{M_x\}} \left[ S(\rho_A) - \sum_x p_x S(\rho_A^x) \right],$$

(2)

and where the maximum is taken over all the measurements $\{M_x\}$ applied on system $B$. Here, $S(\rho_A)$ is the von Neumann entropy of $\rho_A = \text{tr}_B(\rho_{AB})$. $p_x = \text{tr}[(I_A \otimes M_x)\rho_{AB}]$ is the probability of the outcome $x$, and $\rho_A^x \equiv \text{tr}_B[(I_A \otimes M_x)\rho_{AB}]/p_x$ is the state of system $A$ when the outcome was $x$.

For a tripartite pure state $|\psi\rangle_{ABC}$ with $\rho_A = \text{tr}_C[|\psi\rangle_{ABC}\langle\psi|]$, $\rho_{AB} = \text{tr}_C[|\psi\rangle_{ABC}\langle\psi|]$, and $\rho_{AC} = \text{tr}_B[|\psi\rangle_{ABC}\langle\psi|]$, it was shown that [10]

$$S(\rho_A) = I^-(\rho_{AB}) + E_f(\rho_{AC}).$$

(3)
Here, \( E_f(\rho_{AC}) \) is the Entanglement of Formation (EoF) of \( \rho_{AC} \) defined as\(^{13}\)

\[
E_f(\rho_{AC}) = \min \sum_i p_i S(\rho^i_A), \tag{4}
\]

where the minimization is taken over all pure state decompositions of \( \rho_{AC} \) such that,

\[
\rho_{AC} = \sum_i p_i |\phi^i\rangle_{AC} \langle \phi^i|, \tag{5}
\]

with \( \text{tr}_C|\phi^i\rangle_{AC} \langle \phi^i| = \rho^i_A \).

As a dual quantity to EoF, EoA is defined by the maximum average entanglement of \( \rho_{AC} \),

\[
E_u(\rho_{AC}) = \max \sum_i p_i S(\rho^i_A), \tag{6}
\]

over all possible pure state decompositions of \( \rho_{AC} \).

**Definition 1.** The one-way unlocalizable entanglement (UE) of a bipartite state \( \rho_{AB} \) is defined as follows:

\[
E_u^-(\rho_{AB}) := S(\rho_A) - E_u(\rho_{AC}), \tag{7}
\]

where \( \rho_{AC} \) denotes the reduced state of a purification \( |\psi\rangle_{ABC} \) of \( \rho_{AB} \).

The one-way unlocalizable entanglement can be equivalently characterized as follows:

**Lemma 1.** For any given bipartite state \( \rho_{AB} \), its one-way unlocalizable entanglement is given by

\[
E_u^-(\rho_{AB}) = \min_{\{M_x\}} \left[ S(\rho_A) - \sum_x p_x S(\rho^x_A) \right], \tag{8}
\]

where the minimum is taken over all possible rank-1 measurements \( \{M_x\} \) applied on subsystem \( B \).

**Proof.** Eq. (8) can be rewritten as

\[
E_u^-(\rho_{AB}) = S(\rho_A) - \max_{\{M_x\}} \sum_x p_x S(\rho^x_A), \tag{9}
\]

where the maximum is taken over all possible rank-1 measurements \( \{M_x\} \) applied on system \( B \).

Since \( |\psi\rangle_{ABC} \) is a pure state, all possible pure state decompositions of \( \rho_{AC} \) can be realized by rank-1 measurements of subsystem \( B \), and conversely, any rank-1 measurement can be induced from a pure state decomposition of \( \rho_{AC} \). Thus, the second term on the right hand side of Eq. (9) is the maximum average entanglement over all possible pure state decomposition of \( \rho_{AC} \), which is the definition of \( E_u(\rho_{AC}) \), and this completes the proof. \(\blacksquare\)

By definition, the UE of \( \rho_{AB} \) is the difference between \( S(\rho_A) \) and \( E_u(\rho_{AC}) \). Here, \( S(\rho_A) \) quantifies the entanglement of the pure state \( |\psi\rangle_{A|BC} \) with respect to the bipartite cut \( A-BC \), whereas \( E_u(\rho_{AC}) \) measures the maximum average entanglement that can be localized on the subsystem \( AC \) with the assistance of \( B \). The terminology used is then clear. Figure 1 graphically illustrates this separation.

**B. Properties**

1. **Subadditivity**

**Lemma 2.** For all bipartite states \( \rho_{AB} \) and \( \sigma_{A'B'} \),

\[
E_u^-(\rho_{AB} \otimes \sigma_{A'B'}) \leq E_u^-(\rho_{AB}) + E_u^-(\sigma_{A'B'}), \tag{10}
\]

where

\[
E_u^-(\rho_{AB} \otimes \sigma_{A'B'}) = \min_{\{L_z\}} \left[ S(\rho_A \otimes \sigma_{A'}) - \sum_z r_z S(\tau_{AA'}^z) \right], \tag{11}
\]

with \( r_z = \text{tr}[(I_{AA'} \otimes L_z)\rho_{AB} \otimes \sigma_{A'B'}], \tau_{AA'}^z = \text{tr}_{B'B'}[(I_{AA'} \otimes L_z)\rho_{AB} \otimes \sigma_{A'B'}]/r_z \), and the minimum is taken over all possible rank-1 measurements \( \{L_z\} \) applied on subsystem \( BB' \).

**Proof.** Let \( \{M_x\} \) and \( \{N_y\} \) be the optimal rank-1 measurements on subsystems \( B \) and \( B' \) for \( E_u^-(\rho_{AB}) \) and \( E_u^-(\sigma_{A'B'}) \) respectively, then, we have

\[
E_u^-(\rho_{AB}) + E_u^-(\sigma_{A'B'}) \geq E_u^-(\rho_{AB} \otimes \sigma_{A'B'}), \tag{12}
\]

where \( p_x \rho^x_A = \text{tr}_B[(I_A \otimes M_x)\rho_{AB}], q_y \sigma^y_{A'} = \text{tr}_{B'}[(I_A' \otimes N_y)\sigma_{A'B'}], \) and the second equality is due to the additivity of von Neumann entropy and the definition of \( E_u^-(\rho_{AB} \otimes \sigma_{A'B'}) \). \(\blacksquare\)

By Lemma 2 we can assure the existence of the regularized UE

\[
E_{u,\infty}^-(\rho_{AB}) := \lim_{n \to \infty} \frac{E_u^-(\rho_{AB}^\otimes n)}{n}, \tag{13}
\]

which satisfies

\[
E_{u,\infty}^-(\rho_{AB}) \leq E_u^-(\rho_{AB}). \tag{14}
\]
Lemma 3. For any bipartite state $\rho_{AB}$,

$$E_u^-(\rho_{AB}) \geq \max\{I_e^-(\rho_{AB}), 0\}, \tag{15}$$

where $I_e^-(\rho_{AB}) := S(\rho_A) - S(\rho_{AB})$ is the coherent information of $\rho_{AB}$.

Proof. Let $|\psi\rangle_{ABC}$ be a purification of $\rho_{AB}$, then due to the monotonicity of entanglement, we have

$$E_a(\rho_{AC}) \leq \min\{S(\rho_A), S(\rho_C)\}, \tag{16}$$

where $\rho_{AC} = \text{tr}_B|\psi\rangle_{ABC}\langle\psi|$. Thus, together with Lemma 1, we have

$$E_u^-(\rho_{AB}) = S(\rho_A) - E_a(\rho_{AC}) \geq \max\{S(\rho_A) - S(\rho_C), 0\} = \max\{I_e^-(\rho_{AB}), 0\}, \tag{17}$$

where the last equality is due to the purity of $|\psi\rangle_{ABC}$, that is, $S(\rho_C) = S(\rho_{AB})$.

Since $|\psi\rangle_{ABC}^\otimes n$ is a purification of both $\rho_{AB}^\otimes n$ and $\rho_{BC}^\otimes n$, we have

$$E_u^-(\rho_{AB}^\otimes n) + E_a(\rho_{AC}^\otimes n) = nS(\rho_A). \tag{18}$$

By taking the limit $n \to \infty$, and due to the relation $\lim_{n \to \infty} E_a(\rho_{AC}^\otimes n) = \min\{S(\rho_A), S(\rho_C)\}$, we have that

$$E_u^{-\infty}(\rho_{AB}) = \max\{I_e^{-}(\rho_{AB}), 0\}. \tag{20}$$

Eq. (20) implies that, in the asymptotic limit of many copies, separable states do not exhibit quantumness in their correlations, or their correlations are completely erasable. This is a strong evidence that the distinction between separable and entangled states is operational only in asymptotic sense, since separable states can exhibit non-zero UE in finite case.

III. POLYGAMEY OF ENTANGLEMENT IN TRIPARTITE QUANTUM SYSTEMS

For any bipartite pure state $|\phi\rangle_{AB}$, its concurrence $C(|\phi\rangle_{AB})$ is defined as

$$C(|\phi\rangle_{AB}) = \sqrt{2(1 - \text{tr}\rho_A^2)}, \tag{21}$$

where $\rho_A = \text{tr}_B(|\phi\rangle_{AB}\langle\phi|)$. For any mixed state $\rho_{AB}$, its concurrence is defined via convex-roof extension, that is,

$$C(\rho_{AB}) = \min_k \sum p_k C(|\phi_k\rangle_{AB}), \tag{22}$$

where the minimum is taken over all possible pure state decompositions, $\rho_{AB} = \sum_k p_k |\phi_k\rangle_{AB}\langle\phi_k|$. As a dual value to concurrence, CoA of $\rho_{AB}$ is defined as

$$C^a(\rho_{AB}) = \max_k \sum p_k C(|\phi_k\rangle_{AB}), \tag{23}$$

where the maximum is taken over all possible pure state decompositions of $\rho_{AB}$.

By using concurrence and CoA as the quantification of bipartite entanglement, it was shown that there exists a polygamy relation of entanglement in multi-qubit systems. More precisely, for any pure state $|\psi\rangle_{A_1\cdots A_n}$ in an $n$-qubit system $\mathcal{H}_{A_1} \otimes \cdots \otimes \mathcal{H}_{A_n}$, where $\mathcal{H}_{A_i} \cong \mathbb{C}^2$ for $i = 1, \ldots, n$,

$$C^2_{A_1}(A_2\cdots A_n) \leq (C^2_{A_1}A_2)^2 + \cdots + (C^2_{A_1A_n})^2, \tag{24}$$

where $C_{A_1}(A_2\cdots A_n)$ is the concurrence of $|\psi\rangle_{A_1\cdots A_n}$ with respect to the bipartite cut $A_1$ and $A_2 \cdots A_n$, and $C_{A_iA_{i+1}}$ is the CoA of $\rho_{A_1A_i} = \text{tr}_{A_2\cdots A_{i-1}}(\rho_{A_1\cdots A_n} |\psi\rangle_{A_1\cdots A_n}\langle\psi|)$ for $i = 2, \ldots, n$. In this section, we provide an analytic upper bound of UE in Eq. (3), and derive a polygamy inequality of entanglement in terms of von-Neumann entropy and EoA for tripartite quantum systems of arbitrary dimension.

First, for an upper bound of UE, we have the following theorem.

Theorem 4. For any bipartite state $\rho_{AB}$ in a bipartite quantum system $\mathcal{H}_A \otimes \mathcal{H}_B$,

$$E_u^-(\rho_{AB}) \leq \frac{I(\rho_{AB})}{2}, \tag{25}$$

where $I(\rho_{AB}) = S(\rho_A) + S(\rho_B) - S(\rho_{AB})$ is the mutual information of $\rho_{AB}$.

Proof. Let $\rho_B = \sum_{i=0}^{d_B-1} \lambda_i |e_i\rangle_B\langle e_i|$, where $\{|e_i\rangle_B\rangle\rangle_B$ be a spectral decomposition of $\rho_B = \text{tr}_A(\rho_{AB})$ where $d_B$ is the dimension of the subsystem $\mathcal{H}_B$. The proof method follows the construction used in [17]. For any state $\sigma \in \mathcal{H}_B$, define the channels

$$M_0(\sigma) : = \sum_{i=0}^{d_B-1} |e_i\rangle\langle e_i|, \quad M_1(\sigma) : = \sum_{i=0}^{d_B-1} |\tilde{e}_i\rangle\langle \tilde{e}_i| \sigma |\tilde{e}_i\rangle\langle \tilde{e}_i|, \tag{26}$$

where $\{|\tilde{e}_j\rangle\rangle\rangle_B$ is the Fourier basis such that,

$$|\tilde{e}_j\rangle = \frac{1}{\sqrt{d}} \sum_{k=0}^{d_B-1} \omega_d^{jk} |e_k\rangle, \quad j = 0, \ldots, d_B - 1, \tag{27}$$

and $\omega_d = e^{2\pi i/d}$ is the $d$-th root of unity.
Notice that $M_0(\rho_B) = \rho_B$, and $M_1(\rho_B) = \frac{1}{d_B} I_B$, so that $M_1(M_0(\rho_B)) = M_0(M_1(\rho_B)) = \frac{1}{d_B} I_B$. We can also write

$$M_0(\sigma) = \frac{1}{d_B} \sum_{b=0}^{d_B-1} Z^b \sigma Z^{-b}, \quad M_1(\sigma) = \frac{1}{d_B} \sum_{a=0}^{d_B-1} X^a \sigma X^{-a},$$

where $Z$ and $X$ are generalized $d_B$-dimensional Pauli operators,

$$Z = \sum_{j=0}^{d_B-1} \omega^j_0 |e_j\rangle\langle e_j|,$$

$$X = \sum_{j=0}^{d_B-1} |e_{j+1}\rangle\langle e_j| = \sum_{j=0}^{d_B-1} \omega_-^j |\tilde{e}_j\rangle\langle \tilde{e}_j|.$$  \hspace{1cm} (28)

In the following, we will write

$$(I_A \otimes M_0)(\rho_{AB}) = \sum_{i=0}^{d_B-1} \sigma_i^A \otimes \lambda_i |e_i\rangle_B \langle e_i|,$$

$$(I_A \otimes M_1)(\rho_{AB}) = \sum_{j=0}^{d_B-1} \tau_j^A \otimes \frac{1}{d_B} |\tilde{e}_j\rangle_B \langle \tilde{e}_j|.$$  \hspace{1cm} (30)

By defining a four-partite quantum state $\Omega_{X Y A B}$ in $E\left(\mathbb{C}^{d_B} \otimes \mathbb{C}^{d_B} \otimes \mathbb{C}^{d_A} \otimes \mathbb{C}^{d_B}\right)$ such that

$$\Omega_{X Y A B} := \frac{1}{d_B} \sum_{x,y=0}^{d_B-1} |x\rangle_X \langle x| \otimes |y\rangle_Y \langle y| \otimes (I_A \otimes X^x_B Z^y_B) \rho_{AB} (I_A \otimes Z^{-y}_B X^{-x}_B),$$

we have

$$\Omega_{X A B} = \frac{1}{d_B} \sum_{x=0}^{d_B-1} |x\rangle_X \langle x| \otimes X^x_B \left(\sum_{i=0}^{d_B-1} \sigma_i^A \otimes \lambda_i |e_i\rangle_B \langle e_i|\right) X^{-x}_B,$$

$$\Omega_{Y A B} = \frac{1}{d_B} \sum_{y=0}^{d_B-1} |y\rangle_Y \langle y| \otimes Z^y_B \left(\sum_{j=0}^{d_B-1} \tau_j^A \otimes \frac{1}{d_B} |\tilde{e}_j\rangle_B \langle \tilde{e}_j|\right) Z^{-y}_B.$$  \hspace{1cm} (33)

and

$$\Omega_{A B} = \rho_A \otimes \frac{I_B}{d_B}.$$  \hspace{1cm} (34)

By straightforward calculation, we can obtain

$$I(\Omega_{X(AB)}) = S(\Omega_X) + S(\rho_{AB}) - S(\Omega_{X A B})$$

$$= \log d_B + \log d_B + S(\rho_A) - \log d_B$$

$$- S\left(\sum_i \sigma^i_A \otimes \lambda_i |e_i\rangle_B \langle e_i|\right)$$

$$= \log d_B + S(\rho_A) - H(\lambda) - \sum_i \lambda_i S(\sigma^i_A)$$

$$= \log d_B - S(\rho_B) + \chi(\mathcal{E}_0),$$

where $\lambda_i \sigma^i_A = \text{tr}_B[(I_A \otimes |e_i\rangle_B \langle e_i|) \rho_{AB}]$, and $\tau_j^A / d_B = \text{tr}_B[(I_A \otimes |\tilde{e}_j\rangle_B \langle \tilde{e}_j|) \rho_{AB}]$ for $i$, $j \in \{0, \ldots, d_B - 1\}$.

The induced ensembles on $A$ by the channels $M_0$ and $M_1$ will be denoted by $\mathcal{E}_0 := \{\lambda_i, \sigma^i_A\}$ and $\mathcal{E}_1 := \{\tau_j^A\}$, and the entropy defects of the induced ensembles on $A$ will be denoted as

$$\chi(\mathcal{E}_0) = S(\rho_A) - \sum_i \lambda_i S(\sigma^i_A),$$

$$\chi(\mathcal{E}_1) = S(\rho_A) - \frac{1}{d_B} \sum_i S(\tau^i_A).$$  \hspace{1cm} (31)

Analogously, we have

$$I(\Omega_{Y(AB)}) = \chi(\mathcal{E}_1),$$

$$I(\Omega_{XY(AB)}) = \log d_B + S(\rho_A) - S(\rho_{AB}) = \log d_B + I^r_c(\rho_{AB}).$$  \hspace{1cm} (36)

Due to the independence of subsystems $X$ and $Y$, we have $I(\Omega_{XY(AB)}) \geq I(\Omega_{X(AB)}) + I(\Omega_{Y(AB)})$, which implies

$$\chi(\mathcal{E}_0) + \chi(\mathcal{E}_1) \leq I(\rho_{AB}).$$  \hspace{1cm} (37)

Since $\chi(\mathcal{E}_0)$ and $\chi(\mathcal{E}_1)$ of Eq. (37) can be ob-
tained, respectively, from \( \rho_{AB} \) by rank-1 measurements \( \{|e_i\}_B \langle e_i | \} \) and \( \{|\tilde{e}_j\}_B \langle \tilde{e}_j | \} \) of subsystem \( B \), by defining a rank-1 measurement
\[
\left\{ \frac{|e_i\rangle_B \langle e_i |}{2}, \frac{|\tilde{e}_j\rangle_B \langle \tilde{e}_j |}{2} \right\}_{i,j},
\]
(38)
we have
\[
E_u^-(\rho_{AB}) \leq \frac{\lambda(\mathcal{E}_0)}{2} + \frac{\lambda(\mathcal{E}_1)}{2} \leq \frac{I(\rho_{AB})}{2},
\]
(39)
which completes the proof. \( \square \)

**Corollary 1.** For any tripartite pure state \( |\psi\rangle_{ABC} \), we have
\[
S(\rho_A) \leq E_a(\rho_{AB}) + E_a(\rho_{AC}).
\]
(40)

**Proof.** By Lemma 7 we have
\[
E_a(\rho_{AC}) = S(\rho_A) - E_u^-(\rho_{AB}),
E_a(\rho_{AB}) = S(\rho_A) - E_u^-(\rho_{AC}),
\]
and thus,
\[
E_a(\rho_{AC}) + E_a(\rho_{AB}) = 2S(\rho_A) - E_u^-(\rho_{AB}) - E_u^-(\rho_{AC}).
\]
(41)
Now, by Theorem 4 we have
\[
E_a(\rho_{AC}) + E_a(\rho_{AB}) \geq 2S(\rho_A) - \frac{I(\rho_{AB})}{2} - \frac{I(\rho_{AC})}{2}
= 2S(\rho_A) - S(\rho_A)/2 - S(\rho_B)/2 + S(\rho_{AB})/2
- S(\rho_A)/2 - S(\rho_C)/2 + S(\rho_{AC})/2
= S(\rho_A).
\]
(42)

Corollary 1 tells us that for a tripartite pure state of arbitrary dimension, there exists a polygamy relation of entanglement in terms of entropy of entanglement and EoA. Furthermore, this is, we believe, the first result of the polygamous (or dually monogamous) property of distribution of entanglement in multipartite higher-dimensional quantum systems rather than qubits.

**IV. POLYGAMY RELATION OF ENTANGLEMENT IN MULTI-QUBIT QUANTUM SYSTEMS**

In this section, we show that the polygamy inequality of entanglement in Corollary 1 can be generalized into multipartite quantum systems for the case when each subsystem is a two-level quantum system. By investigating the functional relation between concurrence and EoF in two-qubit systems, we show that there exists a polygamy inequality of entanglement in terms of entropy and EoA in \( n \)-qubit systems. We also show that, in three-qubit systems, we have a more tight polygamy inequality than Eq. (10) in Corollary 1.

First, let us consider the functional relation of concurrence with EoF in two-qubit systems. For a 2-qubit mixed state \( \rho_{AB} \) (or a pure state \( |\psi\rangle_{AB} \in \mathbb{C}^2 \otimes \mathbb{C}^d \)), the relation between its concurrence, \( C_{AB} \) and \( E_f(\rho_{AB}) \) can be given as a monotone increasing, convex function \( E \) such that
\[
E_f(\rho_{AB}) = E(C_{AB}),
\]
(44)
where
\[
E(x) = H\left(\frac{1}{2} + \frac{1}{2} \sqrt{1 - x^2}\right), \quad \text{for } 0 \leq x \leq 1,
\]
(45)
and \( H(\cdot) \) is the binary entropy function \( H(x) = -[x \log_2 x + (1-x) \log_2 (1-x)] \). The same function \( E(x) \) relates also the EoA of a bipartite state \( \rho_{AB} \) with its CoA via the equation
\[
E_a(\rho_{AB}) \geq E(C_{AB}),
\]
(46)
which is due to the convexity of \( E \) and the definition of EoA. The following lemma shows an important property of the function \( E(x) \).

**Lemma 5.**
\[
E(\sqrt{x^2 + y^2}) \leq E(x) + E(y),
\]
(47)
for \( 0 \leq x, y \leq 1 \) such that \( 0 \leq x^2 + y^2 \leq 1 \).

**Proof.** By considering
\[
f(x,y) = E(x) + E(y) - E(\sqrt{x^2 + y^2}),
\]
(48)
as a two-variable real-valued function on the domain \( D = \{(x,y)|0 \leq x, y \leq 1, 0 \leq x^2 + y^2 \leq 1 \} \), it is enough to show that \( f(x,y) \geq 0 \) in \( D \).

Since \( D \) is a compact subset in \( \mathbb{R}^2 \), whereas \( f \) is analytic on the interior of \( D \), and continuous on \( D \), the minimum value of \( f \) arises only on the critical points or on the boundary of \( D \). It can be directly checked that \( f \) does not have any vanishing gradient on the interior of \( D \), and has non-negative function values on the boundary of \( D \). Thus, \( f \) is non-negative on the domain \( D \). \( \square \)

**A. Three-qubit systems**

A direct observation from Eq. (14) shows that, for a 3-qubit pure state \( |\psi\rangle_{ABC} \),
\[
C_{A(BC)}^2 = C_{AB}^2 + (C_{AC}^a)^2,
\]
(49)
where \( C_{AB} \) and \( C_{AC}^a \) are the concurrence and concurrence of assistance of \( \rho_{AB} \) and \( \rho_{AC} \), respectively. (Later, Eq. (49) was formally shown in \( [19] \).) From Eq. (49) together with Lemma 5 we have the following theorem.
Theorem 6. For a three-qubit pure state $|\psi\rangle_{ABC}$, 
\[
S(\rho_A) \leq F_f(\rho_{AB}) + E_v(\rho_{AC}).
\] (50)

Proof. Since $|\psi\rangle_{ABC}$ is a bipartite pure state in $\mathbb{C}^2 \otimes \mathbb{C}^4$ with respect to the bipartite cut $A$ and $BC$, we have, 
\[
S(\rho_A) = F_f(\rho_{A|BC}) = E(C_{A|BC}).
\] (51)

Thus, 
\[
S(\rho_A) = E(C_{A|BC}) = E(C_{A|BC}) \leq E(F(\rho_{AB}) + E_v(\rho_{AC}), (52)
\]
where the first inequality is by Lemma 3, and the second inequality is by Eq. (46).

Thus, the polygamy relation of distributed entanglement in tripartite quantum systems obtained in Corollary 1 can have a more tight form in three-qubit systems. Furthermore, the result of Theorem 6 together with Eqs. (53) and (54) give us the following corollary.

Corollary 2. For any two-qubit mixed state $\rho_{AB}$ with rank less than or equal to two, 
\[
I^-(\rho_{AB}) \leq E_v(\rho_{AB}),
\] (53)
\[
E_v^-(\rho_{AB}) \leq F_f(\rho_{AB}).
\] (54)

Remark 1. Eq. (51) of Corollary 3 implies that any two-qubit separable state $\rho_{AB}$ of rank less than or equal to two has zero UE, $E_v(\rho_{AB}) = 0$. However, this is not generally true for two-qubit separable states of rank larger than two. Here, we provide an example of a two-qubit rank-three separable state with non-zero UE.

Example: Let us consider the following state in $\mathbb{C}^2 \otimes \mathbb{C}^2 \otimes \mathbb{C}^3$ quantum system [20], 
\[
|\Psi\rangle_{ABC} = \frac{1}{\sqrt{2}} |x\rangle_{AC} |0\rangle_B + \frac{1}{\sqrt{2}} |y\rangle_{AC} |1\rangle_B.
\] (55)

where $|x\rangle$ and $|y\rangle$ are two orthogonal states in the $\mathbb{C}^2 \otimes \mathbb{C}^3$ such that 
\[
|x\rangle = (|02\rangle + \sqrt{2}|10\rangle)/\sqrt{3},
\]
\[
|y\rangle = (|12\rangle + \sqrt{2}|01\rangle)/\sqrt{3}.
\] (56)

First, since $\rho_A = (|0\rangle_A \langle 0| + |1\rangle_A \langle 1|)/2$, it is clear that $C_{A|BC} = \sqrt{2}(1 - \text{tr}\rho_A^2) = 1$, therefore we have $S(\rho_A) = 1$.

Since $\rho_{AC} = (|x\rangle_{AC} \langle x| + |y\rangle_{AC} \langle y|)/2$, Hugston-Jozsa-Wootters (HJW) theorem [21] says that for any decompositions of $\rho_{AC} = \sum p_i |\phi_i\rangle_{AC} \langle \phi_i|$, there exists an unitary operator $u_i$ such that $\sqrt{p_i} |\phi_i\rangle_{AC} = (u_i |x\rangle_{AC} + u_i |y\rangle_{AC})/\sqrt{2}$ with $2p_i = |u_i|^2_x + |u_i|^2_y$. Thus,
\[
\text{tr}_C(|\phi_i\rangle_{AC} \langle \phi_i|) = \frac{1}{6p_i} \left( |u_i|^2_x + 2|u_i|^2_y \right)
\]
\[
= \frac{1}{3} I_A + \frac{1}{3} |\psi_i\rangle \langle \psi_i|,
\] (57)

with $|\psi_i\rangle = (u_i^* |0\rangle + u_i^* |1\rangle)/\sqrt{2p_i}$, and we obtain that 
\[
C(|\phi_i\rangle_{AC}) = \frac{2\sqrt{p_i}}{3}
\]
for any pure state $|\phi_i\rangle_{AC}$ in any pure state decomposition of $\rho_{AC}$.

Since $|\phi_i\rangle_{AC}$ is a 2 \otimes 3 pure state, we have 
\[
E_f(\rho_{AC}) = E_v(\rho_{AC}) = \log_2 3 - \frac{2}{3}.
\] (58)

and thus $E_f(\rho_{AC}) = E_v(\rho_{AC}) = \log_2 3 - \frac{2}{3}$.

Now, we have $E_v(\rho_{AB}) = S(\rho_A) - E_v(\rho_{AC}) = \frac{1}{3} \log_2 3 > 0$, whereas, it can be easily seen that $\rho_{AB}$ has a Positive Partial Transposition (PPT) which is equivalent to separability for two-qubit states [22]. Thus, $\rho_{AB}$ is a two-qubit, rank-three separable state with non-zero UE.

B. $n$-qubit systems

The polygamy inequality of entanglement in $n$-qubit systems in Eq. (49) gives us an inequality 
\[
C_{A_1(A_2 \cdots A_n)} \leq (C_{A_1 A_2})^2 + \cdots + (C_{A_1 A_n})^2.
\] (59)

Thus, together with Lemma 5 we have the following theorem.

Theorem 7. For any $n$-qubit pure state $|\psi\rangle_{A_1 \cdots A_n}$, 
\[
S(\rho_{A_1}) \leq E_v(\rho_{A_1 A_2}) + \cdots + E_v(\rho_{A_1 A_n}).
\] (60)

Proof. First, let us assume that $(C_{A_1 A_2}^2)^2 + \cdots + (C_{A_1 A_n}^2)^2 \leq 1$, then we have 
\[
S(\rho_{A_1}) = E(C_{A_1(A_2 \cdots A_n)}) \leq E\left(\sqrt{(C_{A_1 A_2})^2 + \cdots + (C_{A_1 A_n})^2}\right)
\]
\[
\leq E\left(C_{A_1 A_2}\right) + E\left(\sqrt{(C_{A_1 A_3})^2 + \cdots + (C_{A_1 A_n})^2}\right)
\]
\[
\leq E\left(C_{A_1 A_2}\right) + E\left(C_{A_1 A_3}\right) + \cdots + E\left(C_{A_1 A_n}\right)
\]
\[
\leq E_v(\rho_{A_1 A_2}) + \cdots + E_v(\rho_{A_1 A_n}),
\] (61)
where the first inequality is due to the monotonicity of the function $E$, the second and third inequalities are obtained by iterating Lemma 5 and the last inequality is by Eq. (60).

Now, assume that $(C^a_{A_1A_2})^2 + \cdots + (C^a_{A_1A_n})^2 > 1$. Since $S(\rho_{A_1A_2}) \leq 1$ for any $n$-qubit pure state $|\psi\rangle_{A_1\cdots A_n}$, it is enough to show that $E_{\rho_{A_1A_2}} + \cdots + E_{\rho_{A_1A_n}} \geq 1$.

Let us first note that there exist $k \in \{2, \ldots, n-1\}$ that satisfies

$$(C^a_{A_1A_2})^2 + \cdots + (C^a_{A_1A_k})^2 \leq 1,
(C^a_{A_1A_2})^2 + \cdots + (C^a_{A_1A_{k+1}})^2 > 1, \quad (62)$$

and let

$$T = (C^a_{A_1A_2})^2 + \cdots + (C^a_{A_1A_{k+1}})^2 - 1. \quad (63)$$

Since, $(C^a_{A_1A_{k+1}})^2 - T = 1 - (C^a_{A_1A_2})^2 - \cdots - (C^a_{A_1A_k})^2$, we have

$$0 \leq (C^a_{A_1A_{k+1}})^2 - T \leq 1, \quad (64)$$

and

$$1 = E\left(\sqrt{(C^a_{A_1A_2})^2 + \cdots + (C^a_{A_1A_k})^2 - T}\right)
\leq E\left(\sqrt{(C^a_{A_1A_2})^2 + \cdots + (C^a_{A_1A_k})^2}\right)
+ E\left(\sqrt{(C^a_{A_1A_{k+1}})^2 - T}\right)
\leq E\left(C^a_{A_1A_2}\right) + \cdots + E\left(C^a_{A_1A_k}\right) + E\left(C^a_{A_1A_{k+1}}\right)
\leq E_{\rho_{A_1A_2}} + \cdots + E_{\rho_{A_1A_{k+1}}}, \quad (65)$$

where the first and second inequalities are by Lemma 5 and by monotonicity of $E$, and the last inequality is by Eq. (60).

**V. UNLOCALIZABLE ENTANGLEMENT VERSUS OTHER MEASURES OF CORRELATION**

In this section, we provide some properties of UE concerned with several other correlation measures. By investigating the relation between UE and EoF in $2 \otimes 2 \otimes d$ quantum system, we show that any two-qubit state with zero UE is a separable state. We also provide a quantitative relation among entropy, localizable entanglement, and UE for tripartite mixed states.

**A. $2 \otimes 2 \otimes d$ pure state**

Let $|\psi\rangle_{ABC}$ be a tripartite pure state in $\mathbb{C}^2 \otimes \mathbb{C}^2 \otimes \mathbb{C}^d$.

**Theorem 8.** For any 2-qubit state $\rho_{AB}$,

$$E_u(\rho_{AB}) = 0 \implies \rho_{AB} \text{ separable}, \quad (66)$$

or, equivalently, if $S(\rho_A) = E_{\rho_{AC}}$, then $E_{\rho_{AB}} = 0$.

**Proof.** Suppose $S(\rho_A) = E_{\rho_{AC}}$, and let $\rho_{AC} = \sum_i p_i |\phi_i\rangle_{AC} \langle \phi_i|$ be an optimal decomposition such that,

$$E_{\rho_{AC}} = \sum_i p_i E(|\phi_i\rangle_{AC})$$

$$= \sum_i p_i S(\rho_{A_i}), \quad (67)$$

where $\rho_{A_i} = \text{tr}_C(|\phi_i\rangle_{AC} \langle \phi_i|)$ and $\rho_A = \sum_i p_i \rho_{A_i}$.

The concavity of von Neumann entropy says, $S(\rho_A) \geq \sum_i p_i S(\rho_{A_i})$ and the equality holds if and only if $\rho_{A_i}$ are identical for all $i$. So, by the assumption, $\rho_{A_i}$ are identical for all $i$.

Since $|\phi_i\rangle_{AC}$ is a pure state in $2 \otimes d$ system and its concurrence is $C(|\phi_i\rangle_{AC}) = 2 \sqrt{\text{det} \rho_{A_i}}$, we also have that $C(|\phi_i\rangle_{AC})$ are identical for all $i$, say $C(|\phi_i\rangle_{AC}) = C^*_A$. Now, we have

$$S(\rho_{A_i}) = E_f(|\phi_i\rangle_{AC}) = E(C(|\phi_i\rangle_{AC})) = E(C^*_A), \quad (68)$$

for all $i$, and

$$E(C_{A(BC)}) = E_f(|\psi_{A(BC)}\rangle)$$

$$= S(\rho_A)$$

$$= \sum_i p_i S(\rho_{A_i})$$

$$= \sum_i p_i E(C(|\phi_i\rangle_{AC}))$$

$$= E(C^*_A), \quad (69)$$

where $C_{A(BC)}$ is the concurrence of $|\psi_{A(BC)}\rangle$ between subsystems $A$ and $BC$, and $E(\cdot)$ is the function in Eq. (15).

Since $E(\cdot)$ is strictly monotone increasing, (the first derivative $\frac{d}{dC}E(C)$ is 0 at $C = 0$ and positive elsewhere), we have

$$C_{A(BC)} = C^*_A, \quad (70)$$

therefore

$$C_{A(BC)} \geq C^*_A \geq \sum_i p_i C(|\phi_i\rangle_{AC}) = C^*_A = C_{A(BC)}, \quad (71)$$

and thus,

$$C_{A(BC)} = C^2_A. \quad (72)$$

Now, by the Theorem 3 in [20], we have $C(\rho_{AB}) = 0$ where $\rho_{AB}$ is a 2-qubit state, which implies $E_f(\rho_{AB}) = 0$.

Any two-qubit state with zero UE is separable by Theorem 8 and any two-qubit separable state with rank less than or equal to two has zero UE by Corollary 2. However, the converse of Theorem 5 is not generally true, since Remark 6 provides us a two-qubit separable state with non-zero UE.
B. Tripartite Mixed State

Since it is known that the EoA is not a bipartite measure nor an entanglement monotone [22], it is not clear yet if there is any quantitative relation between $E_a(\rho_{AB})$ and $E_a(\rho_{A(BD)})$ for a tripartite mixed state $\rho_{ABD}$. In fact, this is equivalent to the quantitative relation between $E_u^-(\rho_{AC})$ and $E_u^-(\rho_{A(CD)})$. This is because, if we consider a purification $|\psi\rangle_{ABCD}$ of $\rho_{ABD}$, then any direction of a quantitative relation between $E_a(\rho_{AB})$ and $E_a(\rho_{A(BD)})$, say $E_a(\rho_{AB}) \leq E_a(\rho_{A(BD)})$, would give us

$$S(\rho_A) = E_u^-(\rho_{A(CD)}) + E_a(\rho_{AB}) = E_u^-(\rho_{AC}) + E_a(\rho_{A(BD)}) \geq E_u^-(\rho_{AC}) + E_a(\rho_{AB}),$$

(73)

which implies $E_u^-(\rho_{AC}) \leq E_u^-(\rho_{A(CD)})$.

In this section, we pay our attention only to local rank-1 measurements of each subsystems, and we derive a quantitative relation between localizable entanglement, and UE for tripartite mixed states.

For $\rho_{ACD} = \text{tr}_B|\psi\rangle_{ABCD}\langle\psi|$, let us define

$$\tilde{E}_u^-(\rho_{A(CD)}) = \min_{\{M_x \otimes N_y\}} \left[ S(\rho_A) - \sum_{x,y} p_{xy} S(\rho_A^{xy}) \right],$$

(74)

where $p_{xy} \equiv \text{tr}[(I_A \otimes M_x \otimes N_y)\rho_{ACD}]$ is the probability of the outcome $x$ and $y$ on subsystems $C$ and $D$ respectively, and $\rho_A^{xy} \equiv \text{tr}_C[(I_A \otimes M_x \otimes N_y)\rho_{ACD}] / p_{xy}$ is the state of system $A$ when the outcome were $x$ and $y$. The minimum in Eq. (74) is taken over all possible rank-1 measurements $\{M_x\}$ and $\{N_y\}$ on subsystems $C$ and $D$ respectively. By definition, we have

$$\tilde{E}_u^-(\rho_{A(CD)}) \geq E_u^-(\rho_{AC}).$$

(75)

Furthermore, we have the following lemma.

**Lemma 9.** For any tripartite state $\rho_{ACD}$,

$$\tilde{E}_u^-(\rho_{A(CD)}) \geq E_u^-(\rho_{AC}).$$

(76)

**Proof.** For $\rho_{ACD}$, let $\{M_x\}$ and $\{N_y\}$ are the optimal rank-1 measurements of $C$ and $D$ respectively, such that

$$\tilde{E}_u^-(\rho_{A(CD)}) = S(\rho_A) - \sum_{x,y} p_{xy} S(\rho_A^{xy}).$$

(77)

Due to the concavity of von Neumann entropy, we have

$$\tilde{E}_u^-(\rho_{A(CD)}) = S(\rho_A) - \sum_{x,y} p_{xy} S(\rho_A^{xy}) \geq S(\rho_A) - \sum_{x,y} p_x S(\rho_A^{xy}) \geq E_u^-(\rho_{AC}),$$

(78)

where

$$p_x = \text{tr}_C[(I_A \otimes M_x)\rho_{AC}] = \sum_y \text{tr}_C[(I_A \otimes M_x \otimes N_y)\rho_{ACD}] = \sum_y p_{xy},$$

(79)

and the second inequality is due to the definition of $E_u^-(\rho_{AC})$. □

Now, we are ready to have the following theorem.

**Theorem 10.** For any tripartite mixed state $\rho_{ABC}$ with a purification $|\psi\rangle_{ABC}$,

$$S(\rho_A) \geq \tilde{E}_a(\rho_{AB}) + E_u^-(\rho_{AC}),$$

(81)

where $\tilde{E}_a(\rho_{AB})$ is the localizable entanglement [24] of $\rho_{AB}$, defined by

$$\tilde{E}_a(\rho_{AB}) = \max_{\{M_x \otimes N_y\}} \sum_{x,y} p_{xy} S(\rho_A^{xy})$$

(82)

over all possible rank-1 measurements $\{M_x\}$ and $\{N_y\}$ on subsystems $C$ and $D$ respectively.

**Proof.** Eq. (74) can be rewritten as

$$\tilde{E}_u^-(\rho_{A(CD)}) = S(\rho_A) - \max_{\{M_x \otimes N_y\}} \sum_{x,y} p_{xy} S(\rho_A^{xy}) = S(\rho_A) - \tilde{E}_a(\rho_{AB}),$$

(83)

and Lemma 9 completes the proof. □
Theorem 10 can be considered as an alternative of Lemma 1 for mixed states case. Furthermore, Theorem 10 together with Lemma 1 give us the following simple corollary.

**Corollary 3.** For any tripartite mixed state $\rho_{ABC}$ with a purification $|\psi\rangle_{ABCD}$,

$$E_a(\rho_{A(BC)}) \geq \tilde{E}_a(\rho_{AB}).$$  \hfill (84)

**Proof.** By Theorem 10, we have

$$S(\rho_A) \geq \tilde{E}_a(\rho_{AB}) + E_a^- (\rho_{AD})$$  \hfill (85)

for any pure state $|\psi\rangle_{ABCD}$, whereas

$$S(\rho_A) = E_a(\rho_{A(BC)}) + E_a^- (\rho_{AD})$$  \hfill (86)

for the tripartite partition $A - BC - D$. \hfill \Box

---

**VI. CONCLUSION**

In this paper, we have proposed the concept of UE, and shown that the polygamous nature of distributed quantum entanglement in multipartite systems is strongly due to this unlocalizable character. As the mathematical interpretation for this polygamous nature of quantum entanglement, we have established polygamy inequalities of entanglement in tripartite quantum systems with arbitrary dimension, and multi-qubit systems. We have also provided several trade offs between UE and other correlations such as EoA, and localizable entanglement.

This is the first result where polygamous property of quantum entanglement in multipartite higher-dimensional quantum systems is provided. Furthermore, the proposed inequalities are in terms of the entropic entanglement measures such as entropy of entanglement for pure states and EoA. In other words, the proposed polygamy inequalities of distributed entanglement have been shown in terms of the actual quantification of entanglement with operational meanings, rather than using other entanglement measures such as concurrence.

---

**Acknowledgments**

JSK would like to thank Soojoon Lee for useful discussion, and acknowledges the support from iCORE, MITACS (QIP project) and US Army. GG acknowledges financial support from NSERC and MITACS-QIP.

---

[1] C. H. Bennett, G. Brassard, C. Crepeau, R. Jozsa, A. Peres and W. K. Wootters, Phys. Rev. Lett. 70, 1895 (1993).

[2] C. Bennett and G. Brassard, in Proceedings of IEEE International Conference on Computers, Systems, and Signal Processing (IEEE Press, New York, Bangalore, India, 1984), p. 175-179.

[3] C. H. Bennett, Phys. Rev. Lett. 68, 3121 (1992).

[4] V. Coffman, J. Kundu and W. K. Wootters, Phys. Rev. A 61, 052306 (2000).

[5] T. Osborne and F. Verstraete, Phys. Rev. Lett. 96, 220503 (2006).

[6] M. Koashi and A. Winter, Phys. Rev. A 69(2) 022309 (2004).

[7] B. M. Terhal, IBM J. Research and Development 48, 71 (2004).

[8] W. K. Wootters, Phys. Rev. Lett. 80, 2245 (1998).

[9] D. P. DiVincenzo em et al., Lect. Notes Comput. Sci. 1509, 247 (1999).

[10] O. Cohen, Phys. Rev. Lett. 80, 2493 (1998).

[11] T. Laustsen, F. Verstraeete and S. J. van Enk, Quantum Inf. Comput. 3, 64 (2003).

[12] G. Gour, S. Bandyopadhyay and B. C. Sanders, J. Math. Phys. 48, 012108 (2007).

[13] I. Devetak, A. Winter, IEEE Transactions on Information Theory 50(12) pp. 3183-3196 (2004).

[14] L. Henderson and V. Vedral, J. Phys. A: Math. Gen. 34, 6899 (2001).

[15] C. H. Bennett, D. P. DiVincenzo, J. A. Smolin and W. K. Wootters, Phys. Rev. A 54, 3824 (1996).

[16] J. A. Smolin, F. Verstraeete and A. Winter, Phys. Rev. A 72, 052317 (2005).

[17] M. Christandl and A. Winter, IEEE Trans. Inf. Theory 51, 3159–3165 (2005).

[18] M. A. Nielsen and I. L. Chuang, Quantum Computation and Quantum Information (Cambridge University Press, Cambridge, U.K., 2000).

[19] C-s. Yu and H-s. Song, Phys. Rev. A 76, 022324 (2007).

[20] D. P. Chi, J. W. Choi, K. Jeong, J. S. Kim, T. Kim and S. Lee, J. Math. Phys. 49, 112102 (2008).

[21] L. P. Hughston, R. Jozsa and W. K. Wootters, Phys. Lett. A 183, 14 (1993).

[22] P. Horodecki, Phys. Lett. A 232, 333 (1997).

[23] G. Gour and R. W. Spekkens, Phys. Rev. A 73, 062331 (2006).

[24] M. Popp, F Verstraeete, M. A. Martin-Delgado and J. I. Cirac, Phys. Rev. A 71, 042306 (2005).