Local randomness in Hardy’s correlations: implications from the information causality principle

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Abstract
Study of non-local correlations in terms of Hardy’s argument has been quite popular in quantum mechanics. Hardy’s non-locality argument depends on some kind of asymmetry, but a two-qubit maximally entangled state, being symmetric, does not exhibit this kind of non-locality. Here we ask the following question: can this feature be explained by some principle outside quantum mechanics? The no-signaling condition does not provide a solution. But, interestingly, the information causality principle (Pawlowski et al 2009 Nature \textbf{461} 1101) offers an explanation. It shows that any generalized probability theory which gives completely random results for local dichotomic observable, cannot provide Hardy’s non-local correlation if it is restricted by a necessary condition for respecting the information causality principle. In fact, the applied necessary condition imposes even more restrictions on the local randomness of measured observable. Still, there are some restrictions imposed by quantum mechanics that are not reproduced from the considered information causality condition.

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1. Introduction

Violation of the Bell-type inequalities [1] by quantum mechanics shows that nature is non-local. Nevertheless, quantum correlations respect the causality principle [2]. However, there are also other non-signaling post-quantum correlations [3] which cannot be distinguished from
quantum correlation by subjecting them to the causality principle. Though post-quantum correlations are not observed in experiments, but still we do not understand what underlying physical principle(s) completely distinguishes quantum correlations from non-physical post-quantum correlations.

Recent studies have shown that quantum features like violation of Bell-type inequalities [3], intrinsic randomness, no-cloning [4, 5], information-disturbance tradeoff [6], secure cryptography [7–9], teleportation [10] and entanglement swapping [11] are also enjoyed by other post-quantum no-signaling theories. On the other hand, for no-signaling correlations some implausible features have also been noticed: e.g. some no-signaling correlations would make certain distributed computational tasks trivial [12–15] and would have very limited dynamics [16]. So the study of the non-local correlations in the general no-signaling framework [4–17] leads us toward a deeper understanding of quantum correlations.

Very recently, non-violation of information causality (IC) [18] has been identified as one of the foundational principle of nature; it is compatible with experimentally observed quantum and classical correlations but rules out an unobserved class of non-local correlation as non-physical. The principle states that communication of \(m\) classical bits causes information gain of at most \(m\) bits. This is a generalization of the no-signaling principle; the case \(m = 0\) corresponds to no-signaling. Applying the IC principle to non-local correlations, we get Tsirelson’s bound [19] and all correlations that go beyond Tsirelson’s bound violate the principle of IC [18]. In [20], it was shown that though some part of the quantum boundary can be derived from a necessary condition (given in [18]) for violating IC, this condition is not sufficient for distinguishing quantum correlations from all post-quantum correlations which are below Tsirelson’s bound. So it remains interesting to see if the full power of IC (some other conditions derived from IC) can eliminate remaining post-quantum correlations below Tsirelson’s bound. Along with the research in the direction of completely distinguishing quantum correlations from rest of the non-local correlations, it would also be interesting to apply the known IC condition(s) for qualitative/quantitative study of certain specific features of non-local correlations. For instance, it was known that the maximum success probability of Hardy’s non-locality argument [21, 22] under the no-signaling restriction is 0.5 [23] and within quantum mechanics the maximum takes the value 0.09 [24]; then by applying the IC principle in [25] it was shown that the upper bound on the success probability reduces to 0.20717.

In this communication we apply the IC condition in order to study the property of local randomness for a bipartite probability distribution which exhibits Hardy’s non-locality [21, 22]. Our motivation for this study came from the fact that Hardy’s non-locality argument in quantum mechanics does not work for a maximally entangled state [22, 26], and at the same time for a maximally entangled state, the local density matrix being completely random, both the results for any dichotomic observable are equally probable. Keeping this in mind, we asked a more general question: for two two-level systems, how many observables and in which way, out of four entering in the Hardy’s non-locality argument, can be locally random? We want to study this question in the context of probability distribution which respects an IC condition as well as in the context of quantum mechanics. We see that the applied IC condition itself imposes powerful restriction but still it does not reproduce all the restrictions imposed by quantum mechanics. In this context, it is to be mentioned that no-signaling condition does not impose any such restriction. Interestingly we observed that the applied necessary condition for respecting IC allows at most two observables, one on each side, chosen in a restricted way to be completely random, and quantum mechanics allows only one of them to be completely random.
This communication is organized as follows. In section 2 we discuss the general structure of the set of a bipartite two input–two output non-signaling correlations. In section 3 we restrict the type of correlations in section 2 by Hardy’s non-locality conditions. In section 4 we study the property of local randomness in Hardy’s correlation; in section 4.1 we make this study for no-signaling correlations; in section 4.2 we study for correlations respecting an IC condition and in section 4.3 we work for quantum correlations. We give our conclusions in section 5.

2. Bipartite non-signaling correlations

Let us consider a bipartite black box shared between two parties, Alice and Bob. Alice and Bob input variables \( x \) and \( y \) at their end of the box, respectively, and receive outputs \( a \) and \( b \). For fixed input variables there can be different outcomes with certain probabilities. The behavior of these correlation boxes is fully described by a set of joint probabilities \( P(ab|xy) \).

In this communication, we focus on the case of binary inputs and outputs \((a, b, x, y \in \{0, 1\})\). Then we have a set of 16 joint probabilities defining a bipartite binary input–binary output correlation box. These types of correlations can be represented by a \(4 \times 4\) correlation matrix

\[
\begin{pmatrix}
P(00|00) & P(01|00) & P(10|00) & P(11|00) \\
P(00|01) & P(01|01) & P(10|01) & P(11|01) \\
P(00|10) & P(01|10) & P(10|10) & P(11|10) \\
P(00|11) & P(01|11) & P(10|11) & P(11|11)
\end{pmatrix}
\]

We note that since \(P(ab|xy)\) are probabilities, they satisfy positivity, \(P(ab|xy) \geq 0 \forall a, b, x, y,\) and normalization \(\sum_{a,b} P(ab|xy) = 1 \forall x, y.\) Since we have to study non-signaling boxes, i.e. we require that Alice cannot signal to Bob by her choice \(x\) and vice versa, the marginal probabilities \(P_a|_x\) and \(P_b|_y\) must be independent of \(y\) and \(x\), respectively. The full set of non-signaling boxes forms an eight-dimensional polytope [17] which has 24 vertices, eight extremal non-local boxes and local deterministic boxes. The extremal non-local correlations have the form

\[
P_{\alpha\beta\gamma}^{NL} = \begin{cases} 
\frac{1}{4} & \text{if } a \oplus b = XY \oplus \alpha X \oplus \beta Y \oplus \gamma, \\
0 & \text{otherwise},
\end{cases}
\]

where \(\alpha, \beta, \gamma \in \{0, 1\}\) and \(\oplus\) denotes addition modulo 2. Similarly, the local deterministic boxes are described by

\[
P_{\alpha\beta\gamma\delta}^{L} = \begin{cases} 
1 & \text{if } a = \alpha X \oplus \beta, \\
0 & \text{otherwise},
\end{cases}
\]

where \(\alpha, \beta, \gamma, \delta \in \{0, 1\}\) and \(\oplus\) denotes addition modulo 2.

Thus we can see that any bipartite two input–two output non-signaling correlation box can be expressed as a convex combination of the above 24 local/non-local vertices.

3. Hardy’s correlations under the no-signaling condition

A bipartite two input–two output Hardy’s correlation puts simple restrictions on a certain choice of 4 out of 16 joint probabilities in the correlation matrix. One such choice is \(P(11|11) > 0, P(11|01) = 0, P(11|10) = 0, P(00|00) = 0\) and it is easy to argue that these correlations are non-local. To show this, let us suppose that these correlations are local, i.e. they can be simulated by non-communicating observers with only shared randomness as a resource. Now
consider the subset of those random variables $\lambda$ shared between two observers such that for $\lambda$'s belonging to this subset input $x = 1, y = 1$ give output $a = 1, b = 1$ (this subset is nonempty since $P(11|11) > 0$), now conditions $P(11|01) = 0$ and $P(11|10) = 0$ tell that within this subset input $x = 0, y = 0$ would give output $a = 0, b = 0$, this would imply that $P(00|00) > 0$, but it contradicts the condition $P(00|00) = 0$. Hence, these correlations are non-local. If we further restrict these correlations by the no-signaling condition we get Hardy's non-signaling boxes. It is easy to check that these boxes can be written as a convex combination of 5 of the 16 local vertices $P_{L}^{0001}, P_{L}^{0011}, P_{L}^{1000}, P_{L}^{1100}, P_{L}^{1111}$ and 1 of the 8 non-local vertices $P_{NL}^{001}$. Then,
\[
P_{ab|XY}^{NL} = c_{1}P_{L}^{0001} + c_{2}P_{L}^{0011} + c_{3}P_{L}^{1000} + c_{4}P_{L}^{1100} + c_{5}P_{L}^{1111} + c_{6}P_{NL}^{001},
\]
where $\sum_{j=1}^{6} c_{j} = 1$. From here the correlation matrix for these Hardy's non-signaling boxes can be written as
\[
\left(\begin{array}{cccc}
0 & c_{1} + c_{2} + \frac{c_{6}}{2} & c_{3} + c_{4} + \frac{c_{6}}{2} & c_{5} \\
c_{2} & c_{1} + \frac{c_{6}}{2} & c_{3} + c_{4} + \frac{c_{6}}{2} & 0 \\
c_{4} & c_{1} + c_{2} + c_{5} + \frac{c_{6}}{2} & c_{3} + \frac{c_{6}}{2} & 0 \\
c_{2} + c_{4} + c_{5} + \frac{c_{6}}{2} & c_{1} & c_{3} & \frac{c_{6}}{2}
\end{array}\right).
\]

4. Property of local randomness in Hardy's correlations

For the most general bipartite correlation an input $x$ on Alice's side is locally random if the marginal probabilities of all possible outcomes on Alice's side for this input are equal and similarly for Bob. In the case of two input–two output bipartite correlations, an input $x$ on Alice’s side is locally random if $P(0|x) = P(1|x) = \frac{1}{2}$ in terms of joint probabilities this would mean that for any choice of Bob’s input $y$, $P(00|xy) + P(01|xy) = P(10|xy) + P(11|xy) = \frac{1}{2}$. Similarly an input $y$ on Bob’s side is locally random if $P(0|y) = P(1|y) = \frac{1}{2}$ in terms of joint probabilities this can be expressed as, for any choice of Alice’s input $x$, $P(00|xy) + P(10|xy) = P(01|xy) + P(11|xy) = \frac{1}{2}$. Let us denote the 0 and 1 inputs on Alice’s (Bob’s) side as $0_{A}(0_{B})$ and $1_{A}(1_{B})$, respectively. We would now like to see what choices of inputs from the set $\{0_{A}, 1_{A}, 0_{B}, 1_{B}\}$ can be locally random for a given class of Hardy’s correlations.

4.1. Hardy’s correlations respecting no-signaling

In the case of Hardy’s correlations which respects no-signaling, the condition of local randomness for each of the possible inputs is given in table 1. Now let us see that for Hardy’s correlations respecting no-signaling, what choices of inputs can be locally random. We give the results for every case in table 2. We can read from here that although in order to show the property of local randomness, Hardy’s correlations become much restricted, yet we get solutions for each case. If we get solutions for the case 1, it is obvious that there are solutions in all the remaining cases 2–15, nevertheless we write the complete table giving the form of solutions in each case for later reference.

4.2. Hardy’s correlation respecting information causality

Let us first briefly discuss the principle of IC [18]; then we would apply it in our study of the property of local randomness for two input–two output Hardy’s non-signaling correlations. The IC principle states that for two parties Alice and Bob, who are separated in space, the
The principle can be well formulated in terms of a generic information processing task in which using all his local resources and information gain that Bob can reach about a previously unknown to him data set of Alice, by using all his local resources and $m$ classical bit communicated by Alice, is at most $m$ bits. This principle can be well formulated in terms of a generic information processing task in which

### Table 1.

For the no-signaling bipartite Hardy’s correlation with two dichotomic observables on either side, here each row gives the conditions which coefficients $c_i$s must satisfy for the corresponding input to be locally random.

| Input | Conditions for local randomness |
|-------|---------------------------------|
| $0_A$ | $c_1 + c_2 + \frac{c_6}{2} = \frac{1}{2}$ |
|       | $c_3 + c_4 + c_5 + \frac{c_6}{2} = \frac{1}{2}$ |
| $1_A$ | $c_1 + c_2 + c_4 + c_5 + \frac{c_6}{2} = \frac{1}{2}$ |
|       | $c_3 + \frac{c_6}{2} = \frac{1}{2}$ |
| $0_B$ | $c_3 + c_4 + \frac{c_6}{2} = \frac{1}{2}$ |
|       | $c_1 + c_2 + c_5 + \frac{c_6}{2} = \frac{1}{2}$ |
| $1_B$ | $c_2 + c_3 + c_4 + c_5 + \frac{c_6}{2} = \frac{1}{2}$ |
|       | $c_1 + \frac{c_6}{2} = \frac{1}{2}$ |

### Table 2.

For the no-signaling bipartite Hardy’s correlation with two dichotomic observables on either side, here each row gives the form of solutions for the corresponding choice of inputs to be locally random.

| Cases | Locally random inputs | $c_1$ | $c_2$ | $c_3$ | $c_4$ | $c_5$ | $c_6$ |
|-------|-----------------------|-------|-------|-------|-------|-------|-------|
| 1.    | $\{0_A, 1_A, 0_B, 1_B\}$ | $\frac{1}{2}(1 - c_6)$ | 0     | $\frac{1}{2}(1 - c_6)$ | 0     | 0     | $c_6$ |
| 2.    | $\{0_A, 1_A, 0_B\}$   | $c_1$ | $\frac{1}{2}(1 - c_6) - c_1$ | $\frac{1}{2}(1 - c_6)$ | 0     | 0     | $c_6$ |
| 3.    | $\{0_A, 1_A, 1_B\}$   | $\frac{1}{2}(1 - c_6)$ | 0     | $\frac{1}{2}(1 - c_6)$ | 0     | 0     | $c_6$ |
| 4.    | $\{0_A, 0_B, 1_B\}$   | $\frac{1}{2}(1 - c_6)$ | 0     | $\frac{1}{2}(1 - c_6)$ | 0     | 0     | $c_6$ |
| 5.    | $\{1_A, 0_B, 1_B\}$   | $\frac{1}{2}(1 - c_6)$ | 0     | $\frac{1}{2}(1 - c_6)$ | 0     | 0     | $c_6$ |
| 6.    | $\{0_A, 1_A\}$        | $c_1$ | $\frac{1}{2}(1 - c_6) - c_1$ | $\frac{1}{2}(1 - c_6)$ | 0     | 0     | $c_6$ |
| 7.    | $\{0_B, 1_B\}$        | $\frac{1}{2}(1 - c_6)$ | 0     | $\frac{1}{2}(1 - c_6)$ | 0     | 0     | $c_6$ |
| 8.    | $\{1_A, 1_B\}$        | $\frac{1}{2}(1 - c_6)$ | 0     | $\frac{1}{2}(1 - c_6)$ | 0     | 0     | $c_6$ |
| 9.    | $\{0_A, 0_B\}$        | $c_1$ | $\frac{1}{2}(1 - c_6) - c_1$ | $\frac{1}{2}(1 - c_6)$ | 0     | 0     | $c_6$ |
| 10.   | $\{0_A, 1_B\}$        | $\frac{1}{2}(1 - c_6)$ | 0     | $\frac{1}{2}(1 - c_6)$ | 0     | 0     | $c_6$ |
| 11.   | $\{1_A, 0_B\}$        | $c_1$ | $c_2$ | $\frac{1}{2}(1 - c_6)$ | 0     | $\frac{1}{2}(1 - c_6)$ | 0     | $c_6$ |
| 12.   | $\{0_A\}$             | $c_1$ | $\frac{1}{2}(1 - c_6) - c_1$ | $\frac{1}{2}(1 - c_6)$ | $c_3$ | $c_4$ | $\frac{1}{2}(1 - c_6)$ | $c_6$ |
| 13.   | $\{1_A\}$             | $c_1$ | $\frac{1}{2}(1 - c_6) - c_1$ | $\frac{1}{2}(1 - c_6)$ | $c_3$ | $c_4$ | $c_5$ | $c_6$ |
| 14.   | $\{0_B\}$             | $c_1$ | $\frac{1}{2}(1 - c_6) - c_1$ | $\frac{1}{2}(1 - c_6)$ | $c_3$ | $c_4$ | $c_5$ | $c_6$ |
| 15.   | $\{1_B\}$             | $\frac{1}{2}(1 - c_6)$ | $\frac{1}{2}(1 - c_6)$ | $\frac{1}{2}(1 - c_6)$ | $c_3$ | $c_4$ | $c_5$ | $c_6$ |
Alice is provided with $N$ random bits $\vec{a} = (a_1, a_2, \ldots, a_N)$ while Bob receives a random variable $b \in \{1, 2, \ldots, N\}$. Alice then sends $m$ classical bits to Bob, who must output a single bit $\beta$ with the aim of guessing the value of Alice’s $b$th bit $a_b$. Their degree of success at this task is measured by

$$I \equiv \sum_{K=1}^{N} I(a_K : \beta | b = K),$$

where $I(a_K : \beta | b = K)$ is Shannon mutual information between $a_K$ and $\beta$. Then the principle of IC says that physically allowed theories must have $I \leq m$. The result that both classical and quantum correlations satisfy this condition was proved in [18]. It was further shown there that, if Alice and Bob share arbitrary two input–two output non-signaling correlations corresponding to conditional probabilities $P(ab|x,y)$, then by applying a protocol by van Dam [12] and Wolf and Wullschleger [27], one can derive a necessary condition for respecting the IC principle. This necessary condition reads

$$E_2^2 + 2E_1^2 \leq 1,$$  \hspace{1cm} (4)

where $E_j = 2P_j - 1$ ($j = 1, 2$) and $P_1, P_2$ are defined by

$$P_1 = \frac{1}{2}[P_{a=b|00} + P_{a=b|10}] = \frac{1}{2}[P_{00|00} + P_{11|00} + P_{00|10} + P_{11|10}]$$

$$P_2 = \frac{1}{2}[P_{a=b|01} + P_{a\neq b|11}] = \frac{1}{2}[P_{00|01} + P_{11|01} + P_{01|11} + P_{10|11}].$$  \hspace{1cm} (5)

Here it is important to note that condition (4) is only a necessary condition (based on the protocol give in [18]) for respecting the IC principle. So a violation of (4) implies a violation of IC but the converse may not be true. In fact, it is shown in [20] that there are examples where condition (4) is satisfied but not IC. We now derive some one-way implications about the property of local randomness for two input–two output Hardy’s non-signaling correlations. It is easy to verify that restricting Hardy’s non-signaling correlations by condition (4) and interchanging the roles of Alice and Bob we get

$$c_2^2 + 2(c_4 + c_5)c_6 + 2(c_4 + c_5)(c_4 + c_5 - 1) \leq 0$$  \hspace{1cm} (6)

$$c_2^2 + 2(c_2 + c_5)c_6 + 2(c_2 + c_5)(c_2 + c_5 - 1) \leq 0.$$  \hspace{1cm} (7)

By applying these conditions for all possible choices of inputs that can be locally random for Hardy’s non-signaling correlations (table 2), we find that at least one of the above two conditions is violated for cases 1–8 but for cases 9–15 we can find $c$s satisfying the above two conditions. Thus for cases 1–8 we can conclude that IC is violated; hence they cannot be true in quantum mechanics also. Now we shall study cases 9–15 in the context of quantum mechanics in the following subsection.

4.3. Hardy’s correlation in quantum mechanics

Violation of IC for cases 1–8 implies that there are no quantum solution for these cases. To resolve the remaining cases (9–15), we consider a two-qubit pure quantum state. It is to be mentioned that for two qubits, Hardy’s argument runs only for pure entangled state [28]. So without loss of any generality, we consider the following two-qubit state:

$$|\Psi\rangle = \cos \beta |0\rangle_A |0\rangle_B + \exp(i\gamma) \sin \beta |1\rangle_A |1\rangle_B.$$  \hspace{1cm} (8)
Then the density matrix $\rho_{AB} = |\Psi\rangle\langle\Psi|$ can be written in terms of Pauli matrices as

$$\rho_{AB} = \frac{1}{4} \left[ I^A \otimes I^B + (\cos^2 \beta - \sin^2 \beta) I^A \otimes \sigma_z^B + (\cos^2 \beta - \sin^2 \beta) \sigma_z^A \otimes I^B + (2 \cos \beta \sin \beta) \sigma_x^A \otimes \sigma_x^B \right. $$

$$\left. + (2 \cos \beta \sin \beta) \sigma_y^A \otimes \sigma_y^B \right] .$$

(9)

The reduced density matrices $\rho_A$ and $\rho_B$ are

$$\rho_A = \frac{1}{4} \left[ I + (\cos^2 \beta - \sin^2 \beta) \sigma_z^A \right] .$$

(10)

$$\rho_B = \frac{1}{4} \left[ I + (\cos^2 \beta - \sin^2 \beta) \sigma_z^B \right] .$$

(11)

In general an observable on a single qubit can be written as $\hat{n} \cdot \sigma$ where $\hat{n} = (\sin \theta \cos \phi, \sin \theta \sin \phi, \cos \theta)$ is any unit vector in $\mathbb{R}^3$ and $\sigma = (\sigma_x, \sigma_y, \sigma_z)$. Then the projectors on the eignestates of these observables are

$$P^\pm = \frac{1}{2} [ I \pm \hat{n} \cdot \sigma ] .$$

(12)

For observable on Alice’s side to be locally random,

$$\text{Tr}(\rho_A P^+) = \text{Tr}(\rho_A P^-) .$$

(13)

Similarly for observable on Bob’s side to be locally random,

$$\text{Tr}(\rho_B P^+) = \text{Tr}(\rho_B P^-) .$$

(14)

On simplifying this we find that for a non-maximally entangled state an observable is locally random if and only if $\theta = \frac{\pi}{2}$, i.e. $\hat{n}$ is of the form $(\cos \phi, \sin \phi, 0)$. Here we would also like to mention that for a maximally entangled state any observable shows the property of local randomness, but we know that Hardy’s argument does not run for a maximally entangled state. This also follows from the IC principle, as for a maximally entangled state any four arbitrary observables (two on Alice’s side and two on Bob’s side) are locally random and we saw that if so, it violates the IC principle.

Now suppose $A$ ($0_A$) and $A'$ ($1_A$) are the observables on Alice’s side and $B$ ($0_B$) and $B'$ ($1_B$) are the observables on Bob’s side. Here outputs 0 and 1 will correspond to outcomes $+1$ and $-1$, respectively. Then Hardy’s correlation can be written as

$$P(A = +1, B = +1) = \cos^2 \beta \cos^2 \frac{\theta_A}{2} \cos^2 \frac{\theta_B}{2} \sin^2 \beta \sin^2 \frac{\theta_A}{2} \sin^2 \frac{\theta_B}{2} + 2 \cos \beta \sin \beta \sin \frac{\theta_A}{2} \sin \frac{\theta_B}{2} \cos \frac{\theta_A}{2} \cos \frac{\theta_B}{2} \cos(\phi_A + \phi_B - \gamma) = 0$$

(15)

$$P(A = -1, B' = -1) = \cos^2 \beta \sin^2 \frac{\theta_A}{2} \sin^2 \frac{\theta_B}{2} + \sin^2 \beta \cos^2 \frac{\theta_A}{2} \cos^2 \frac{\theta_B}{2} + 2 \cos \beta \sin \beta \sin \frac{\theta_A}{2} \sin \frac{\theta_B}{2} \cos \frac{\theta_A}{2} \cos \frac{\theta_B}{2} \cos(\phi_A + \phi_B - \gamma) = 0$$

(16)

$$P(A' = -1, B = -1) = \cos^2 \beta \sin^2 \frac{\theta_A}{2} \sin^2 \frac{\theta_B}{2} + \sin^2 \beta \cos^2 \frac{\theta_A}{2} \cos^2 \frac{\theta_B}{2} + 2 \cos \beta \sin \beta \sin \frac{\theta_A}{2} \sin \frac{\theta_B}{2} \cos \frac{\theta_A}{2} \cos \frac{\theta_B}{2} \cos(\phi_A + \phi_B - \gamma) = 0$$

(17)

$$P(A' = -1, B' = -1) = \cos^2 \beta \sin^2 \frac{\theta_A}{2} \sin^2 \frac{\theta_B}{2} + \sin^2 \beta \cos^2 \frac{\theta_A}{2} \cos^2 \frac{\theta_B}{2} + 2 \cos \beta \sin \beta \sin \frac{\theta_A}{2} \sin \frac{\theta_B}{2} \cos \frac{\theta_A}{2} \cos \frac{\theta_B}{2} \cos(\phi_A + \phi_B - \gamma) \neq 0.$$  

(18)
For these Hardy’s correlation if observables \( A \) and \( B \) (\( 0_A \) and \( 0_B \)) are locally random, then \( \theta_A = \theta_B = \frac{\pi}{2} \); then from equation (15) we get

\[
1 + \sin 2\beta \cos(\phi_A + \phi_B - \gamma) = 0.
\]  

(19)

This equation is satisfied only if \( \sin 2\beta \) takes the value +1 or −1; in either case the corresponding state has to be a maximally entangled state, but this cannot be the case. Therefore, we conclude that observables \( A \) and \( B \) cannot be locally random in quantum mechanics. Similarly we can see that local randomness of two observables in the cases \( A' \) and \( B \) (\( 1_A \) and \( 0_B \)) and \( A \) and \( B' \) (\( 0_A \) and \( 1_B \)) is also not possible.

Now we consider the case of just one observable—say \( A \) (\( 0_A \)) from the set \( \{A, A', B, B'\} \) to be locally random (and similarly for the cases \( A', B, B' \)). Then we find that there are non-maximally entangled states and choices of observables \( A, A', B, B' \) such that one of the observable is locally random. We give an example, consider the state \( \beta = \frac{\pi}{6} \), and \( \gamma = \frac{\pi}{2} \), and choose the observable \( A \) as \( \theta_A = \frac{\pi}{2} \) and \( \phi_A = \pi \), \( A' \) as \( \theta_{A'} = 2\tan^{-1}\left(\tan\frac{\pi}{6}\right) \) and \( \phi_{A'} = -\pi \), \( B \) as \( \theta_B = \frac{\pi}{2} \) and \( \phi_B = \pi \), and \( B' \) as \( \theta_{B'} = \frac{\pi}{2} \) and \( \phi_{B'} = -\pi \); then it can be easily checked that for this choice of state and observable, Hardy’s argument runs and the observable \( A \) is locally random. Thus by analyzing the remaining cases (9–15) within quantum mechanics, we can now conclude that for a quantum mechanical state showing Hardy’s non-locality, at most one out of the four observable can be locally random.

5. Conclusion

The maximally entangled state in quantum mechanics does not reproduce Hardy’s correlation, whereas generalized non-signaling theory put no such restriction on the local randomness of the observable for Hardy’s correlation. We study all the possibilities of local randomness in Hardy’s correlation in the context of IC condition. We observe that not only in terms of the value of the maximal probability of success [25], but also in terms of local randomness there is a gap between quantum mechanics and IC condition. It remains to see, whether some stronger necessary condition for IC can fill this gap.

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