Active Brownian particles climb up activity gradient to reach the fuel source

Hidde Vuijk\textsuperscript{1}, Abhinav Sharma\textsuperscript{1}, Debasish Mondal\textsuperscript{1,2}, Jens-Uwe Sommer\textsuperscript{1,3}, Holger Merlitz\textsuperscript{1}

\textsuperscript{1}Leibniz-Institut für Polymerforschung Dresden, Institut Theorie der Polymere, 01069 Dresden, Germany
\textsuperscript{2} Indian Institute of Technology, Department of Chemistry, 175056 Tirupati, India
\textsuperscript{3}Technische Universität Dresden, Institute of Theoretical Physics, 01069 Dresden, Germany

We study dynamical properties of confined, self-propelled Brownian particles in an inhomogeneous activity profile. Using Brownian dynamics simulations, we show that an inhomogeneous activity strongly biases the probability of an active particle to reach a target which is located at higher activities, bearing strong resemblance to chemotaxis. Similarly, the first passage time is strongly reduced when the target is located at higher activities. We present a theoretical approach which is based on a coarse-grained equation of motion of an active particle in which the orientational degrees of freedom are integrated out. We further derive an approximate Fokker-Planck equation and show that the theoretical predictions are in very good agreement with the Brownian dynamics simulations.

INTRODUCTION

Active matter is ubiquitous in Biology. Examples include cytoskeletal molecular motors performing directed motion on filaments inside a cell \cite{1}, nucleic acid motors involved in transcription process inside nucleus \cite{2}, and even microscopic living objects such as the \textit{Escherichia coli} bacteria which generates motion using helical flagella \cite{3}. The defining characteristic of active matter is that it is intrinsically nonequilibrium. The constituents of active matter generate motion by consuming energy from their local environment. A model system of active matter, which has been a subject of considerable attention, is an assembly of active Brownian particles (ABPs). In addition to the solvent-induced Brownian motion, active particles undergo self-propulsion resulting in persistent character of particle trajectories. It is the self-propulsion feature of ABPs, generally termed as the activity, which captures the nonequilibrium nature of active matter.

There exist synthetic \cite{4-7} and living systems \cite{8-11} for which the activity is not uniform but dependent on the spatial location of the particles. In a recent experimental study of synthetic microswimmers \cite{12} position-dependent activity was implemented using an inhomogeneous laser field, resulting in phototaxis. Chemotaxis has been observed in a system of self-propelled supramolecular nanomotors exhibiting a drift towards increasing fuel concentrations \cite{13}. Theoretical studies of inhomogeneous active systems have also revealed some very interesting phenomena such as directional transport of colloids inside a bath of active walkers \cite{14}, torque-free polarization of ABPs \cite{15}, pseudochemotaxis \cite{16} in an activity profile and activity density waves \cite{17}. Position-dependent activity also features in the energy depot model \cite{18}.

The previous studies pose a fundamental question: Is active motion directly related with chemotaxis by physical principles? A first step towards an answer is the inspection of a most simple system which features both activity and a very primitive concept for a "food-source" namely an increasing activity towards the source caused by a gradient of available food in the proximity of the source. Motivated by these considerations, we consider in this work systems for which the activity varies in space, focusing on two dynamical properties of an active system: target finding probability and the mean first passage time (MFPT) to target. The target finding probability is the probability that a particle introduced at a given location, exits through a specified boundary representing the target. The mean first passage time to target is a measure of average reaction time in finite domains \cite{19}.

In the context of active systems, the average reaction time has been previously studied in homogeneous systems \cite{20,21}. Here we show that both these dynamical properties are strongly dependent on the spatial distribution of activity. In particular, we find that the insights gained from steady-state inhomogeneous active systems, such as preferential accumulation in the low-activity regions and orientation of particles antiparallel to the activity gradient, cannot be used to understand the dynamical properties of an inhomogeneous active system. Using Brownian dynamics simulations, we show that an inhomogeneous activity strongly biases the probability of an active particle to reach a target which is located at higher activities, bearing strong resemblance to chemotaxis. While activity always decreases the first passage time, the effect is most pronounced when the target is located at higher activities.

We present a theoretical approach which is based on a coarse-grained equation of motion of an active particle in which the orientational degrees of freedom are integrated out. Using the coarse-grained equation of motion, we derive a Fokker-Planck equation and show that the theoretical predictions are in very good agreement with the Brownian dynamics simulations.
The stochastic vectors $\mathbf{D}$ and $\mathbf{D}'$ have zero mean and time correlations theoretically tractable model of active particles evolving according to the Langevin equations \[ \mathbf{\dot{r}} = \mathbf{\xi} + \mathbf{\chi}(\mathbf{r}). \] Here, the stochastic force $\mathbf{\chi}(\mathbf{r})$ is position-dependent and has the time correlation \[ \langle \mathbf{\chi}(\mathbf{r}, t)\mathbf{\chi}(\mathbf{r}, t') \rangle = \frac{D_a(r)}{\tau_a} e^{-|t-t'|/\tau_a}, \] where $D_a(r) = v_0^2(r) \tau_a/3$ denotes a position-dependent coefficient and $\tau_a = 1/(2D_t)$ is the persistence time of the orientation of the active particle. Due to the presence of colored noise in Eq. (2), an exact Fokker-Planck equation for the time evolution of probability density cannot be obtained. However, there exist different schemes \[25\, 27\, 28\], following which an approximate Fokker-Planck equation can be derived. Here, we use the simplest approximation of reducing $\mathbf{\chi}(\mathbf{r})$ to a white noise with time correlation \[ \langle \mathbf{\chi}(\mathbf{r}, t)\mathbf{\chi}(\mathbf{r}, t') \rangle = D_a(r) \delta(t-t'). \] This assumption is valid in the limit of vanishing persistence time of the active particle. Further assuming that $v_0(r)$ depends only on the radial distance $r$, the following Fokker-Planck equation can be written for the probability density $P(r, t)$:

\[ \partial_t P(r, t) = \partial_r \left[ \frac{-\partial_r D_a(r)}{2} P(r, t) + \partial_r (D_t + D_a(r)) P(r, t) \right], \]

where $P(r, t)$ is the probability density of finding an active particle at position $r$ at time $t$, $\partial_t = \partial/\partial t$ is the time derivative and $\partial_r = \partial/\partial r$ is the spatial derivative. In Eq. (4), on the right hand side, the first term in square brackets is the drift term that has its origin in the multiplicative nature of the noise $\mathbf{\chi}(\mathbf{r})$ in Eq. (2) \[29\]. The second term can be interpreted as diffusion with an effective diffusion coefficient $(D_t + D_a(r))$. We note that $D_a(r)$ can be much larger than $D_t$ and hence, the diffusion of a particle may be governed predominantly by the activity. For noninteracting particles, the enhanced diffusivity of active particles is reminiscent of Brownian particles at a vastly increased effective temperature \[30\].

To calculate the MFPT to target, we confine the particle to a spherical geometry as shown in Fig. 1. The inner boundary $r = r_0$ represents the target and is therefore considered as an absorbing boundary whereas the outer boundary at $r = R$ is reflecting. At time $t = 0$, an ABP is introduced at a position $r$ such that $r_0 \leq r \leq R$. The MFPT to target of a particle, $\tau(r)$, is the average time taken by a particle starting at $r$ to reach the inner boundary $r_0$. This can be calculated from the Fokker-Planck Eq. (4) as \[29\]

\[ \tau(r) = \int_{r_0}^r dz \frac{z^{-2}}{\sqrt{D_t + D_a(z)}} \int_z^R dy \frac{y^2}{\sqrt{D_t + D_a(y)}}. \]
Another dynamical quantity that is strongly dependent on the spatial distribution of the activity is the target finding probability \( \pi_{r_0}(r) \) when both \( r = r_0 \) and \( r = R \) are absorbing boundaries. \( \pi_{r_0}(r) \) is the probability that a particle that is introduced at \( r \) at time \( t = 0 \) reaches the specified target, i.e., exits through the inner boundary \( r_0 \) before it vanishes through the outer boundary. This can be calculated from the Fokker-Planck Eq. (4) as \[ \pi_{r_0}(r) = \frac{1}{2} \int_{r_0}^{R} dz z^{-2} (D_t + D_A(z))^{-\frac{1}{2}}. \] (6)

The activity profile is assumed to be spherically symmetric and is taken to be of the form
\[ v_0(r) = \frac{c}{4\pi(R - r_0)r^2}. \] (7)

In the equation above, \( c \) can be interpreted as the ‘total activity’ available in the spherical geometry between \( r_0 \) and \( R \). The \( 1/r^2 \) decay of the activity is motivated from the following consideration: The region \( 0 < r \leq r_0 \) may represent a source of fuel for the active particles \[13]. Under the assumption that there is a constant flux of fuel from the source towards the outer boundary, the \( 1/r^2 \)-form of the activity corresponds to the decay of the stationary concentration profile in 3D. When the same amount of ‘total activity’ is distributed uniformly instead, the activity profile \( v_0^u(r) \) is given as
\[ v_0^u(r) = \frac{3c}{4\pi(R^3 - r_0^3)}. \] (8)

Equations (7) and (8) are used to obtain \( D_A(r) = \frac{v_0^u}{6D_r} \), which in turn is used as input to Eqs. (5) and (6) to calculate MFPT and target finding probability.

In the Brownian dynamics simulations, we consider noninteracting point-particles with translational diffusion constant \( D_t = 1/30 \). The rotational diffusion constant \( D_r = 1/2 \). This corresponds to a quick rotation of the particle as compared to translation in the spirit of our mathematical approximation of short reorientation times made above. The total activity \( c \) is a free parameter. The trajectory of each particle is generated by integrating the Langevin equations in Eq. (1) using a time step \( dt = 10^{-2}\tau_D \), where \( \tau_D = 1/D_t \) is the time scale of translational diffusion over a unit length. We fix the boundaries of the spherical geometry as \( r_0 = 10 \) and \( R = 50 \). The MFPT to target and the target finding probability are calculated by averaging over \( 10^4 \) particle trajectories. We have also performed molecular dynamics simulations using the open source LAMMPS package \[31]. With the parameters chosen such that the momentum relaxation time scale is much smaller than the MPFT, the Brownian dynamics results are practically indistinguishable from the molecular dynamics results.

![FIG. 2. Target finding probability (a) and the mean first passage time to target (b) of active particles as a function of the total activity \( c \). Simulation data are shown as symbols. Squares and circles correspond to the inhomogeneous active system with activity distributed as in Eq. (7) and uniformly active system as in Eq. (8), respectively. The lines (dashed and solid) correspond to the theoretical prediction of Eq. (9) in (a) and Eq. (5) in (b). The target finding probability \( \pi_{r_0}(r) \) is calculated for \( r = 50/3 \), which yields equiprobable exit from \( r = r_0 \) or \( r = R \). \( \pi_{r_0}(r) \) does not change when the system is uniformly active (circles). However, when the same amount of total activity is distributed such that it increases towards the target \( r = r_0 \), the probability is strongly biased. The mean first passage time to target in (b) is normalized to its corresponding value \( \tau_0 \) in a passive system. Inhomogeneously distributed activity leads to a larger decrease in MFPT in comparison to a uniformly active system.](https://example.com/fig2.png)

**TARGET FINDING PROBABILITY**

We present the target finding probability \( \pi_{r_0}(r) \) in Fig. 2(a) as a function of the total activity \( c \). The two scenarios considered are (i) the activity is spatially distributed as in Eq. (7) such that it increases towards the inner boundary \( r_0 \) (target) and (ii) the activity is uni-
formally distributed as in Eq. (8). The target finding probability does not change in the case of uniform activity, see Fig. 2(a).

It is only when the activity is inhomogeneously distributed, the probability is strongly biased to reach the target $r_0$ located at higher activities. As can be seen in Fig. 2(a), the theoretical predictions are in good agreement with the simulation data. The starting location $r = 50/3$ is chosen as it corresponds to an equally likely exit from either of the two boundaries in a passive or a uniformly active system. We note that the qualitative behavior remains the same for any other starting location, i.e., probabilities to exit from either of the boundaries do not change in presence of uniform activity whereas in the case of inhomogeneous activity, the probability increases at the end where the activity increases.

The reason for the increase in target finding probability in an inhomogeneous active system is the noise-induced drift in Eq. (4) which is a fundamental feature of systems with multiplicative noise. Our coarse graining approach in which the orientational degrees of freedom are integrated out results in the equation of motion where inhomogeneous activity is mapped to a multiplicative noise (Eqs. (2) and (3)). Due to this term, an active particle, immediately after introduction, drifts towards the end where activity increases and is absorbed on arriving at the target.

The increase in likelihood of escaping through the target boundary is reminiscent of the chemotaxis phenomenon [3]. Chemotaxis is a fundamental sensory mechanism by which bacteria and other single- or multicellular organisms monitor the concentration gradients of specific chemicals, translating the information into motion either uphill or downhill to the gradient. The increased likelihood of escaping from the inner boundary, where activity increases, can be likened to an active particle climbing up the fuel gradient. This chemotactic behavior has been recently realised in experiments on supramolecular nanomotors which climb up the hydrogen peroxide concentration gradient [13].

Considering that the stationary distribution of active particles in an inhomogeneous activity profile tends to accumulate in the low activity region, the chemotactic behavior of active particles appears paradoxical. Recently, Ghosh et al. [15] addressed this paradox by emphasizing the distinction between the dynamical and stationary behavior of inhomogeneous active systems. The stationary distribution is obtained under the assumption that the active particle is trapped between two reflecting boundaries. The target finding probability, in contrast, is the likelihood of reaching a target boundary. In a stationary scenario, the drift of the particle towards the end where activity increases, is only a transient effect and does not impact the stationary distribution.

The MFPT of an active particle starting at the outer boundary $r = R$ to reach the target at $r = r_0$ is shown in Fig. 2(b). We normalize the MFPT with its corresponding value in a passive system ($\tau_0$). Increasing activity always decreases the MFPT. However, the decrease in MFPT is much more pronounced when the activity is spatially distributed according to Eq. (7). As can be seen in Fig. 2(b), the theoretical predictions are in good agreement with the simulation data. In our coarse grained approach, a uniformly active system is equivalent to a passive system with an effective diffusion constant ($D_t + D_a$). The decrease in the MFPT can thus be simply attributed to the increased diffusivity of the particle. However, when the activity is inhomogeneously distributed, the particle drifts in the direction of the activity gradient towards the absorbing boundary. This drift which is in addition to the increased diffusivity of the particle leads to the pronounced decrease in the MFPT when the activity is inhomogeneously distributed.

A particularly simple but instructive case that illustrates the role of the spatial distribution of activity can be constructed as follows. We consider two scenarios, called the forward and the backward scenario. In the forward scenario, the region $r_0 < r < r_s$ is uniformly active and $r_s < r < R$ passive. In the backward scenario, the situation is reversed with the active region becoming passive and vice versa. The intermediate distance
\[ r_s = \sqrt{(r_0^2 + R^2)/2} \] is chosen such that the total activity in both scenarios is the same (see inset of Fig. 3). The activity gradient is everywhere zero except at \( r = r_s \) implying that the noise-induced drift of the particle occurs only at \( r = r_s \) towards \( r_0 \) in the forward scenario and \( R \) in the backward scenario. In Fig. 3, we plot the MFPT as a function of the total activity \( c \) for the forward \( (\tau_f(R)) \) and backward \( (\tau_b(R)) \) scenarios. Clearly, the forward scenario yields a much faster passage to the target. One can show in a straightforward calculation that the difference between the MFPTs of the backward and the forward scenario, \( \delta \tau = \tau_b(R) - \tau_f(R) \) is given as

\[
\delta \tau = \frac{D_a}{6(D_t + D_a)} \left[ \frac{2r_s^3}{r_0} - \frac{2R^3}{r_s} + 3R^2 - 4r_s^2 + r_0^2 \right], \quad (9)
\]

where \( D_a = c^2/(6D_a) \). It can be easily shown that \( \delta \tau \) is always positive. This simple case serves to illustrate the strong influence of the spatial distribution of activity on the MFPT. With the active region closer to the absorbing boundary, the MFPT in the forward scenario is significantly smaller than in the backward scenario.

We consider how the MFPT changes as a function of the distance \( r \) from the target \( r = r_0 \). In Fig. 4(a) we plot the MFPT as a function of \( r \) for different values of \( c \) when activity is distributed inhomogeneously as in Eq. (7). As can be seen in the Fig. 4(a), the MFPT decreases monotonically with \( c \) for any given \( r \). For a fixed \( c \), the MFPT increases with increasing \( r \). This seems to be a trivial observation as one expects \( \tau(r) \) to increase with increasing distance from the target. However, there is a subtle geometrical aspect that becomes evident when one considers \( \tau'(r) \), the rate of change of \( \tau(r) \) with respect to \( r \) as shown in Fig. 4(b). For a passive system \( (c = 0) \), \( \tau'(r) \) is largest for \( r = r_0 \) and then decreases monotonically with increasing \( r \). However, in an active system, \( \tau'(r) \) shows a qualitatively different behavior with \( r \). With increasing \( c \), the decay of \( \tau'(r) \) near \( r_0 \) becomes increasingly slow, approaching an almost flat profile for large \( c \).

The qualitative change in the behavior of \( \tau'(r) \) is due to an underlying competing effect between the available phase-space and activity which can be understood as following. For a particle introduced at \( r \) such that \( r_0 < r < R \), the passage to \( r = r_0 \) includes trajectories spanning the volume between \( r \) and the reflecting boundary at \( R \). The smaller \( r - r_0 \) is, the larger is this extra volume (phase-space) between \( r \) and \( R \) in which the particle can ‘wander’ before being absorbed at \( r = r_0 \). This phase-space effect gives rise to the observed strong decrease of \( \tau'(r) \) in a passive system. However, in an inhomogeneous active system, there is an additional competing effect due to the activity which tends to prevent the excursion of the particle away from \( r_0 \). These two competing effects give rise to the observed slower decay of \( \tau'(r) \). We emphasize that even for \( \tau'(r) \), the emergence of this subtle competing effect between the available phase-space and activity is due to the inhomogeneous distribution of the activity. In a system with uniform activity, \( \tau'(r) \) decays strongly away from the absorbing boundary as for the passive system.

The behavior of \( \tau'(r) \) is strongly dependent on the activity profile. The motivation for the \( 1/r^2 \)-form of activity in three-dimensions \( (1/r \) in two-dimensions) is based on the physical consideration of constant flux of fuel from the inner boundary towards the outer boundary. Interestingly, when the activity decays faster than \( 1/r^3 \) in three-dimensions, \( \tau'(r) \) becomes nonmonotonic. In the inset of Fig. 4(b), we plot \( \tau'(r) \) for the activity distributed as \( 1/r^3 \) corresponding to a total activity of \( c = 16 \times 10^4 \). With this choice of activity profile, one.
obtains a maximum in $\tau'(r)$ which shifts away from $r_0$ towards $R$ with increasing $c$. Such rapidly decaying activity profile would naturally arise in systems in which the fuel molecules injected by the source, have a finite lifetime. Fuel molecules may bind to species other than the active particles resulting in an attenuation of freely available fuel. An activity profile that decays faster than $r^{-(d-1)}$ for $d \geq 2$ does not only compensate for the competing phase-space effect but dominates it near $r_0$ for larger $c$ giving rise to the observed nonmonotonicity of $\tau'(r)$. This can be easily demonstrated quantitatively by differentiating Eq. 5 twice with respect to $r$ for the chosen activity profile. We note that the phase-space related effects as discussed here are absent inside one-dimensional systems such as linear channels, but are of significance in majority of natural scenarios.

It is important to recall that active particles exhibit non-isotropic distribution of their orientation vectors in presence of an activity gradient [15]. In the current setup, however, these gradients are insufficiently steep to generate significant effects on the walker’s orientations. In fact, our Fokker-Planck equation reproduces the chemotactical behavior in complete absence of any orientational inhomogeneities, which is also in agreement with earlier study by Ghosh et al. [10]. It is therefore safe to conclude that orientational effects are of no relevance for the emergence of chemotaxis in the context of our work. In our simulations of active particles we find that $\tau(r)$ exhibits a small but finite discontinuity at $r = r_0$. This is a consequence of the finite persistence time of active particles. An active particle starting very close to $r = r_0$ may move away or into the absorbing boundary, depending on whether its orientation points away or towards the absorbing boundary, respectively. If the orientation is towards the inner boundary, the particle would be quickly absorbed. However, when the orientation is away from the boundary, the particle will move away on the time scale of its persistence before it reorients towards the inner boundary and eventually get absorbed. In our theoretical approach, the equation of motion of active particle is reduced to a Markovian process which completely ignores this ‘memory effect’ due to the finite persistence time of active particles. Therefore, there is a slight mismatch between the theoretical predictions and the simulations. Nevertheless, except in the immediate neighborhood of $r_0$, the theoretical predictions are in very good agreement with the simulations.

CONCLUSIONS

We studied the dynamical properties of noninteracting active particles in an inhomogeneous activity profile. Using Brownian dynamics simulations, we computed the target finding probability and mean first passage time to target of active particles confined to a spherical geometry. We showed that an inhomogeneous activity strongly biases the target finding probability of an active particle, bearing strong resemblance to chemotaxis [16, 17]. In contrast, in a uniformly active system, the probabilities remain the same as in a passive system. The first passage time to target is also strongly dependent on the spatial distribution of the activity. While the activity always decreases the first passage time, the effect is most pronounced when the target is located at higher activities. We further showed that inhomogeneous activity can give rise to subtle effects such as the nonmonotonic behavior of $\tau'(r)$, which are absent in uniformly active systems or linear channels.

We found that the insights gained from steady-state inhomogeneous active systems, such as preferential accumulation in the low-activity regions and orientation of particles antiparallel to the activity gradient, cannot be used to understand the dynamical properties of an inhomogeneous active system. Active particles drift in the direction of increasing activity, a purely transient effect which strongly influences the target finding probability and the first passage time to target. In contrast, the stationary distribution, which is obtained under the condition of the particle trapped between two reflecting boundaries is not impacted by the the transient drift.

Finally, considering activity profiles in general the following interesting questions arise: Which activity profile, for a given total activity $c$, yields the minimum mean first passage time? Which profile yields the maximum target finding probability? In the near future, we will include interaction potential between the source ($r_0$) and active particle. It will be particularly interesting to investigate under what conditions an active particle exhibits chemotactical behavior when it interacts via a repulsive interaction with the source of fuel.

[1] F. Jülicher, A. Ajdari, and J. Prost, Reviews of Modern Physics 69, 1269 (1997).
[2] M. R. Singleton, M. S. Dillingham, and D. B. Wigley, Annu. Rev. Biochem. 76, 23 (2007).
[3] H. C. Berg, E. coli in Motion (Springer Science & Business Media, 2008).
[4] S. Saha, R. Golestanian, and S. Ramaswamy, Physical Review E 89, 062316 (2014).
[5] Y. Hong, N. M. Blackman, N. D. Kopp, A. Sen, and D. Velegol, Physical review letters 99, 178103 (2007).
[6] M. P. Magiera and L. Brendel, Physical Review E 92, 012304 (2015).
[7] J. Stenhammar, R. Wittkowski, D. Marenduzzo, and M. E. Cates, Science advances 2, e1501850 (2016).
[8] H. C. Berg, D. A. Brown, et al., Nature 239, 500 (1972).
[9] G. Jékely, J. Colombelli, H. Hausen, K. Guy, E. Stelzer, F. Nédélec, and D. Arendt, Nature 456, 395 (2008).
[10] W. D. Hoff, M. A. van der Horst, C. B. Nudel, and K. J. Hellingwerf, Chemotaxis: Methods and Protocols pp. 25–
49 (2009).

[11] J. F. Jikeli, L. Alvarez, B. M. Friedrich, L. G. Wilson, R. Pascal, R. Colin, M. Pichlo, A. Rennhack, C. Brenker, and U. B. Kaupp, Nature communications 6 (2015).

[12] C. Lozano, B. Ten Hagen, H. Löwen, and C. Bechinger, Nature communications 7, 12828 (2016).

[13] F. Peng, Y. Tu, J. van Hest, and D. A. Wilson, Angewandte Chemie 127, 11828 (2015).

[14] H. Merlitz, C. Wu, and J.-U. Sommer, Soft Matter 13, 3726 (2017).

[15] A. Sharma and J. Brader, Physical review E 96, 032604 (2017).

[16] P. K. Ghosh, Y. Li, F. Marchesi, and F. Nori, Physical Review E 92, 012114 (2015).

[17] A. Geiseler, P. Hänggi, F. Marchesi, C. Mulhern, and S. Savel’ev, Physical Review E 94, 012613 (2016).

[18] F. Schweitzer, W. Ebeling, and B. Tilch, Physical Review Letters 80, 5044 (1998).

[19] A. Szabo, K. Schulten, and Z. Schulten, The Journal of chemical physics 72, 4350 (1980).

[20] A. Sharma, R. Wittmann, and J. M. Brader, Physical Review E 95, 012115 (2017).

[21] A. Scacchi and A. Sharma, arXiv preprint arXiv:1708.05591 (2017).

[22] Y. Fily and M. C. Marchetti, Physical review letters 108, 235702 (2012).

[23] T. F. Farage, P. Krinninger, and J. M. Brader, Physical Review E 91, 042310 (2015).

[24] R. Wittmann and J. M. Brader, EPL (Europhysics Letters) 114, 68004 (2016).

[25] C. Maggi, U. M. B. Marconi, N. Gnan, and R. Di Leonardo, Scientific reports 5 (2015).

[26] U. M. B. Marconi and C. Maggi, Soft matter 11, 8768 (2015).

[27] R. F. Fox, Physical Review A 33, 467 (1986).

[28] R. F. Fox, Physical Review A 34, 4525 (1986).

[29] C. W. Gardiner et al., Handbook of stochastic methods, vol. 3 (Springer Berlin, 1985).

[30] M. Cates and J. Tailleur, EPL (Europhysics Letters) 101, 20010 (2013).

[31] S. Plimpton, P. Crozier, and A. Thompson, Sandia National Laboratories 18 (2007).