WOLSTENHOLME’S THEOREM: ITS GENERALIZATIONS
AND EXTENSIONS IN THE LAST HUNDRED AND FIFTY
YEARS (1862–2012)

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Abstract. In 1862, 150 years ago, J. Wolstenholme proved that for
any prime \( p \geq 5 \) the numerator of the fraction

\[
1 + \frac{1}{2} + \frac{1}{3} + \cdots + \frac{1}{p-1}
\]

written in reduced form is divisible by \( p^2 \) and that the numerator of the
fraction

\[
1 + \frac{1}{2^2} + \frac{1}{3^2} + \cdots + \frac{1}{(p-1)^2}
\]

written in reduced form is divisible by \( p \).

The first of the above congruences, the so-called Wolstenholme’s the-
orem, is a fundamental congruence in Combinatorial Number Theory.
In this article, consisting of 11 sections, we provide a historical survey
of Wolstenholme’s type congruences, related problems and conjectures.
Namely, we present and compare several generalizations and extensions
of Wolstenholme’s theorem obtained in the last hundred and fifty years.
In particular, we present about 80 variations and generalizations of this
theorem including congruences for Wolstenholme primes. These con-
gruences are discussed here by 33 remarks.

1. Introduction

Congruences modulo primes have been widely investigated since the time
of Fermat. Let \( p \) be a prime. Then by Fermat little theorem, for each integer
\( a \) not divisible by \( p \)

\[
a^{p-1} \equiv 1 \pmod{p}.
\]

Furthermore, by Wilson theorem, for any prime \( p \)

\[
(p-1)! + 1 \equiv 0 \pmod{p}.
\]

2010 Mathematics Subject Classification. Primary 11B75; Secondary 11A07, 11B65,
11B68, 05A10.

Keywords and phrases: congruence modulo a prime (prime power), Wolstenholme’s
theorem, Bernoulli numbers, generalization of Wolstenholme’s theorem, Ljunggren’s
congruence, Jacobsthal(-Kazandzidis) congruence, Wolstenholme prime, Leudesdorf’s
theorem, converse of Wolstenholme’s theorem, \( q \)-analogues of Wolstenholme’s type
congruences.
From Wilson theorem it follows immediately that \((n - 1)! + 1\) is divisible by \(n\) if and only if \(n\) is a prime number.

"In attempting to discover some analogous expression which should be divisible by \(n^2\), whenever \(n\) is a prime, but not divisible if \(n\) is a composite number", in 1819 Charles Babbage [5] is led to the congruence

\[
\binom{2p - 1}{p - 1} \equiv 1 \pmod{p^2}
\]

for primes \(p \geq 3\). In 1862 J. Wolstenholme proved that the above congruence holds modulo \(p^3\) for any prime \(p \geq 5\).

As noticed in [36], many great mathematicians of the nineteenth century considered problems involving binomial coefficients modulo a prime power (for instance Babbage, Cauchy, Cayley, Gauss, Hensel, Hermite, Kummer, Legendre, Lucas, and Stickelberger). They discovered a variety of elegant and surprising theorems which are often easy to prove. For more information on these classical results, their extensions, and new results about this subject, see Dickson [23], Granville [36] and Guy [39].

Suppose that a prime \(p\) and pair of integers \(n \geq m \geq 0\) are given. A beautiful theorem of E. Kummer from year 1852 (see [51] and [23, p. 270]) states that if \(p^r\) is the highest power of \(p\) dividing \(\binom{n}{m}\), then \(r\) is equal to the number of carries when adding \(m\) and \(n - m\) in base \(p\) arithmetic. If \(n = n_0 + n_1 p + \ldots + n_s p^s\) and \(m = m_0 + m_1 p + \ldots + m_s p^s\) are the \(p\)-adic expansions of \(n\) and \(m\) (so that \(0 \leq m_i, n_i \leq p - 1\) for each \(i\)), then by Lucas’s theorem from year 1878 ([54]; also see [23, p. 271] and [36]),

\[
\binom{n}{m} \equiv \prod_{i=0}^{s} \binom{n_i}{m_i} \pmod{p}.
\]

This immediately yields

\[
\binom{np}{mp} \equiv \binom{n}{m} \pmod{p}
\]

since the same products of binomial coefficients are formed on the right side of Lucas’s theorem in both cases, other than an extra \(\binom{0}{0} = 1\).

**Remark 1.** A direct proof of the congruence (1), based on a polynomial method, is given in [66, Solution of Problem A-5, p. 173].

Notice that the congruence (1) with \(n = 2\) and \(m = 1\) becomes

\[
\binom{2p}{p} \equiv 2 \pmod{p},
\]

whence by the identity \(\binom{2p}{p} = 2\binom{2p-1}{p-1}\), it follows that for any prime \(p\)

\[
\binom{2p - 1}{p - 1} \equiv 1 \pmod{p}.
\]
As noticed above, in 1819 Babbage ([5]; also see [36, Introduction] or [23, page 271]) showed that the congruence (2) holds modulo $p^2$, that is, for a prime $p \geq 3$ holds

$$\binom{2p-1}{p-1} \equiv 1 \pmod{p^2}.$$  

The congruence (3) is generalized in 1862 by Joseph Wolstenholme [99] as presented in the next section. Namely, Wolstenholme’s theorem asserts that

$$\binom{2p-1}{p-1} \equiv 1 \pmod{p^3}$$

for all primes $p \geq 5$. Wolstenholme’s theorem plays a fundamental role in Combinatorial Number Theory. In this article, we provide a historical survey of Wolstenholme’s type congruences, related problems and conjectures concerning to the Wolstenholme primes. This article consists of 11 sections in which we present numerous generalizations and extensions of Wolstenholme’s theorem established in the last hundred and fifty years.

The article is organized as follows. In Section 2, we present extensions of Wolstenholme’s theorem up to modulus $p^9$. In the next section, many of these congruences are expressed in terms of Bernoulli numbers. Section 4 is devoted to the Wolstenholme’s type harmonic series congruences. Certain Wolstenholme’s type supercongruences are given in the next section. In Section 6, we present Ljunggren’s congruence and Jacobsthal-Kazandzidis congruence and their variations modulo higher prime powers. In the next section, we give several characterizations of Wolstenholmes primes and related conjectures. Wolstenholme’s type theorems for composite moduli are established in Section 8. The converse of Wolstenholme’s theorem is discussed in Section 9. In the next section, we present some recent congruences for binomial sums closely related to Wolstenholme’s theorem. Finally, some $q$-analogues of Wolstenholme’s type congruences are given in the last section of this survey article.

The Bibliography of this article contains 107 references consisting of 12 textbooks and monographs, 90 papers, 3 problems, Sloane’s On-Line Encyclopedia of Integer Sequences and one Private correspondence. In this article, some results of these references are cited as generalizations of certain Wolstenholme’s type congruences, but without the expositions of related congruences. The total number of citations given here is 197.

2. WOLSTENHOLME’S THEOREM AND ITS EXTENSIONS UP TO MODULO $p^7$

In 1862, 150 years ago, at the beginning of his celebrated paper ”On certain properties of prime numbers” [99, page 35], J. Wolstenholme wrote:
The properties I propose to prove in this article, for any prime number \( n > 3 \), are (1) that the numerator of the fraction

\[
1 + \frac{1}{2} + \frac{1}{3} + \cdots + \frac{1}{n-1}
\]

when reduced to its lowest terms is divisible by \( n^2 \), (2) the numerator of the fraction

\[
1 + \frac{1}{2^2} + \frac{1}{3^2} + \cdots + \frac{1}{(n-1)^2}
\]

is divisible by \( n \), and (3) that the number of combinations of \( 2n - 1 \) things, taken \( n - 1 \) together, diminished by 1, is divisible by \( n^3 \). I discovered the last to hold, for several cases, in testing numerically a result of certain investigations, and after some trouble succeeded in proving it to hold universally. The method I employed is somewhat laborious, and I should be glad if some of your readers would supply a more direct proof....

More precisely, the first mentioned result of J. Wolstenholme [99] asserts that if \( p \geq 5 \) is a prime, then the numerator of the fraction

\[
1 + \frac{1}{2} + \frac{1}{3} + \cdots + \frac{1}{p-1}
\]

written in the reduced form is divisible by \( p^2 \). For a proof, also see [40, p. 89], [4, p. 116] and External Links listed in Appendix A).

From this congruence it can be easily deduced that the binomial coefficient \( \binom{2p-1}{p-1} \) satisfies the congruence

\[
\binom{2p-1}{p-1} \equiv 1 \pmod{p^3}.
\]

for any prime \( p \geq 5 \) (see e.g., [40, p. 89], [4, p. 116] and [7]).

As usual in the literature, in this note the congruence (4) is also called Wolstenholme’s theorem. Notice also that from the identity \( \binom{2n}{n} = 2\binom{2n-1}{n-1} \), \( n = 1, 2, \ldots \), we see that (4) also may be written as

\[
\binom{2p}{p} \equiv 2 \pmod{p^3}.
\]

The congruence (4) is generalized by J.W.L. Glaisher in 1900. Namely, by a special case of Glaisher’s congruence ([32, p. 21], [33, p. 323]; also cf. [60, Theorem 2]), for any prime \( p \geq 5 \) we have

\[
\binom{2p-1}{p-1} \equiv 1 - 2p \sum_{k=1}^{p-1} \frac{1}{k} \pmod{p^4}.
\]
In 1995 R.J. McIntosh [60, p. 385] established a generalization of (5) modulo $p^5$; he showed that for any prime $p \geq 7$

$$\binom{2p-1}{p-1} \equiv 1 - p^2 \sum_{k=1}^{p-1} \frac{1}{k^2} \pmod{p^5}.$$  

On the other hand, as an immediate consequence of a result by J. Zhao in 2007 [103, Theorem 3 with $n = 2$ and $r = 1$], for any prime $p \geq 7$

$$\binom{2p-1}{p-1} \equiv 1 + 2p \sum_{k=1}^{p-1} \frac{1}{k} \pmod{p^5}.$$  

In 2010 R. Tauraso [94, Theorem 2.4] proved that for any prime $p \geq 7$

$$\binom{2p-1}{p-1} \equiv 1 - 2p \sum_{k=1}^{p-1} \frac{1}{k} - 2p^2 \sum_{k=1}^{p-1} \frac{1}{k^2} \pmod{p^6}.$$  

which also can be written as [55, Corollary 1.4]

$$\binom{2p-1}{p-1} \equiv 1 - 2p \sum_{k=1}^{p-1} \frac{1}{k} - 2p^2 \sum_{k=1}^{p-1} \frac{1}{k^2} \pmod{p^6}.$$  

**Remark 2.** Clearly, both congruences (8) and (9) can be considered as generalizations of (4) modulo $p^6$.

Quite recently, in 2011 R. Meštrović [55, Theorem 1.1] extended the congruence (9); he proved that for any prime $p \geq 11$

$$\binom{2p-1}{p-1} \equiv 1 - 2p \sum_{k=1}^{p-1} \frac{1}{k} + 4p^2 \sum_{1 \leq i < j \leq p-1} \frac{1}{ij} \pmod{p^7},$$

which by using the shuffle relation also can be written in terms of two power sums as

$$\binom{2p-1}{p-1} \equiv 1 - 2p \sum_{k=1}^{p-1} \frac{1}{k} + 2p^2 \left( \left( \sum_{k=1}^{p-1} \frac{1}{k} \right)^2 - \sum_{k=1}^{p-1} \frac{1}{k^2} \right) \pmod{p^7}.$$  

**Remark 3.** Note that the congruences (10) and (11) reduces to the identity, while for $p = 7$ (10) and (11) are satisfied modulo $7^6$.

Quite recently, R. Tauraso [95] informed the author that using a very similar method to the method applying in [55] to prove the above congruence (10), this congruence can be improved to the following result: for any prime
\( p \geq 7 \)

\[
\binom{2p - 1}{p - 1} \equiv 1 + 2p \sum_{k=1}^{p-1} \frac{1}{k} + 2 \frac{2^3}{3} \sum_{k=1}^{p-1} \frac{1}{k^3} + 2p^2 \left( \sum_{k=1}^{p-1} \frac{1}{k} \right)^2 \\
+ \frac{2}{5} p^5 \sum_{k=1}^{p-1} \frac{1}{k^5} + \frac{4}{3} p^4 \left( \sum_{k=1}^{p-1} \frac{1}{k} \right) \left( \sum_{k=1}^{p-1} \frac{1}{k^3} \right) \pmod{p^9}.
\]

Here we noticed that, using the method applied in [55, Lemmas 2.2-2.4], the term \( \sum_{k=1}^{p-1} \frac{1}{k^5} \) on the right of (12) can be eliminated to obtain that for any prime \( p \geq 7 \)

\[
\binom{2p - 1}{p - 1} \equiv 1 + p \sum_{k=1}^{p-1} \frac{1}{k} - \frac{p^2}{2} \left( 5 \left( \sum_{k=1}^{p-1} \frac{1}{k} \right)^2 + \sum_{k=1}^{p-1} \frac{1}{k^2} \right) \\
- \frac{p^3}{30} \left( 15 \left( \sum_{k=1}^{p-1} \frac{1}{k} \right) \left( \sum_{k=1}^{p-1} \frac{1}{k^2} \right) - 2 \sum_{k=1}^{p-1} \frac{1}{k^3} \right) \\
+ \frac{p^4}{40} \left( 35 \left( \sum_{k=1}^{p-1} \frac{1}{k^2} \right)^2 - 26 \sum_{k=1}^{p-1} \frac{1}{k^4} \right) \pmod{p^9}.
\]

**Remark 4.** A computation via Mathematica verifies that both congruences (12) and (13) hold.

3. **Wolstenholme’s Type Congruences in Terms of Bernoulli Numbers**

The Bernoulli numbers \( B_k \) \((k \in \mathbb{N})\) are defined by the generating function

\[
\sum_{k=0}^{\infty} B_k \frac{x^k}{k!} = \frac{x}{e^x - 1}.
\]

It is easy to find the values \( B_0 = 1, B_1 = -\frac{1}{2}, B_2 = \frac{1}{6}, B_4 = -\frac{1}{30}, \) and \( B_n = 0 \) for odd \( n \geq 3 \). Furthermore, \((-1)^{n-1} B_{2n} > 0\) for all \( n \geq 1 \). These and many other properties can be found, for instance, in [46] or [35].

The Glaisher’s congruence (5) involving Bernoulli number \( B_{p-3} \) may be written as

\[
\binom{2p - 1}{p - 1} \equiv 1 - \frac{2}{3} p^3 B_{p-3} \pmod{p^4}
\]

for all primes \( p \geq 7 \).
More generally, J.W.L. Glaisher (\[32\], p. 21), \([33\], p. 323\]) proved that for any positive integer \(n \geq 1\) and any prime \(p \geq 5\)
\[
\left(\frac{np - 1}{p - 1}\right) \equiv 1 - \frac{1}{3}n(n - 1)p^3B_{p - 3} \pmod{p^4}.
\]
(15)

Also, the congruence \([6]\) (cf. the congruence \([17]\) below) in terms of Bernoulli numbers may be written as
\[
\left(\frac{2p - 1}{p - 1}\right) \equiv 1 - p^3B_{p^3 - p^2 - 2} \pmod{p^5}.
\]
(16)

for each prime \(p \geq 7\).

In 2008 C. Helou and G. Terjanian \([42\]) established many Wolstenholme’s type congruences modulo \(p^k\) with a prime \(p\) and \(k \in \mathbb{N}\) such that \(k \leq 6\). As an application, by \([42\] Corollary 2(2), p. 493 (also see Corollary 6(2), p. 495)), for any prime \(p \geq 5\) we have
\[
\left(\frac{2p - 1}{p - 1}\right) \equiv 1 - p^3B_{p^3 - p^2 - 2} + \frac{1}{3}p^5B_{p - 3} - \frac{6}{5}p^5B_{p - 5} \pmod{p^6}.
\]
(17)

Applying a technique of Helou and Terjanian \([42\]) based on Kummer type congruences, in 2011 R. Meštrović \([55\], Corollary 1.3\) proved that the congruence \([10]\) may be expressed in terms of Bernoulli numbers as
\[
\left(\frac{2p - 1}{p - 1}\right) \equiv 1 - p^3B_{p^3 - p^2 - 2} + p^5\left(\frac{1}{2}B_{p^2 - p - 4} - 2B_{p^3 - p^3 - 4}\right)
+ p^6\left(\frac{2}{9}B_{p - 3}^2 - \frac{1}{3}B_{p - 3} - \frac{1}{10}B_{p - 5}\right) \pmod{p^7}.
\]
(18)

for all primes \(p \geq 11\).

Remark 5. Note that reducing the moduli and using the Kummer congruences presented in \([42\], from \([18]\) may be easily deduced the congruence \([17]\)\).

\[
\]

4. WOLSTENHOLME’S TYPE HARMONIC SERIES CONGRUENCES

Here, as usually in the sequel, we consider the congruence relation modulo a prime power \(p^e\) extended to the ring of rational numbers with denominators not divisible by \(p\). For such fractions we put \(m/n \equiv r/s \pmod{p^e}\) if and only if \(ms \equiv nr \pmod{p^e}\), and the residue class of \(m/n\) is the residue class of \(mn'/n'\) where \(n'\) is the inverse of \(n\) modulo \(p^e\).

As noticed in Section 2, in 1862 J. Wolstenholme \([99]\) proved that for any prime \(p \geq 5\)
\[
1 + \frac{1}{2} + \frac{1}{3} + \cdots + \frac{1}{p - 1} \equiv 0 \pmod{p^2}.
\]
(19)
This is in fact an equivalent reformulation of Wolstenholme’s theorem given by the congruence (4). Wolstenholme [99] also proved that for any prime \( p \geq 5 \)

\[
1 + \frac{1}{2^2} + \frac{1}{3^2} + \cdots + \frac{1}{(p-1)^2} \equiv 0 \pmod{p}.
\]

E. Alkan [11, Theorem 2, p. 1001] in 1994 proved that for each prime \( p \geq 5 \) the numerator of the fraction

\[
\frac{1}{1(p-1)} + \frac{1}{2(p-2)} + \cdots + \frac{1}{(p-1)(p+1)}
\]

is divisible by \( p \) and also noticed that the congruence (19) can be deduced from it.

**Remark 6.** In 1999 W. Kimball and W. Webb [50] established the analogue of the congruence (19) in terms of Lucas sequences which in particular case reduces to (19). Their result is generalized in 2008 by H. Pan [64, Theorem 1.1].

Generalizations of (19) and (20) were established by M. Bayat [8, Theorem 3] in 1997 (also cf. [31] and [104, Lemma 2.2 and Remark 2.3]) as follows. If \( m \) is a positive integer and \( p \) a prime such that \( p \geq m + 3 \), then

\[
\sum_{k=1}^{p-1} \frac{1}{k^m} \equiv \begin{cases} 
0 \pmod{p} & \text{if } m \text{ is even} \\
0 \pmod{p^2} & \text{if } m \text{ is odd}.
\end{cases}
\]

**Remark 7.** For \( j = 1, 2, 3 \) the numerators of harmonic numbers

\[
H_j(n) := \sum_{k=1}^{n} \frac{1}{k^j}, \quad n = 1, 2, 3, \ldots
\]

written in reduced form are Sloane’s sequences [79] sequences A001008, A007406 and A007408], respectively.

**Remark 8.** For a given prime \( p \), in [26] and [10] the authors considered and investigated the set \( J(p) \) of \( n \) for which \( p \) divides the numerator of the harmonic sum \( H_n := \sum_{k=1}^{p-1} 1/k \). It is conjectured in [26, Conjecture A on page 250] that the set \( J(p) \) is finite for all primes \( p \). This conjecture is recently generalized by J. Zhao [105].

In 1900 J.W.L. Glaisher ([33, pp. 333-337]; also see [34, (v) and (vi) on page 271]) proved the following generalizations of the congruences (20) and (21) (also see [86, Theorem 5.1(a) and Corollary 5.1]). if \( m \) is a positive integer and \( p \) a prime such that \( p \geq m + 3 \), then

\[
\sum_{k=1}^{p-1} \frac{1}{k^m} \equiv \begin{cases} 
\frac{m}{m+1} \frac{p^m B_{p-1-m}}{2(m+1)} \pmod{p^2} & \text{if } m \text{ is even} \\
\frac{m(m+1)}{2(m+2)} \frac{p^2 B_{p-2-m}}{2} \pmod{p^3} & \text{if } m \text{ is odd}.
\end{cases}
\]
Remark 9. In 2000 Z.H. Sun \[86, Remark 5.1\] established generalizations modulo $p^4$ of both parts of the congruence (22). □

In particular, taking $m = 1, 2, 3$ into the congruence (22) we obtain that for each prime $p \geq 5$

\[
\sum_{k=1}^{p-1} \frac{1}{k} \equiv -\frac{1}{3}p^2 B_{p-3} \pmod{p^3},
\]

(23)

\[
\sum_{k=1}^{p-1} \frac{1}{k^2} \equiv \frac{2}{3}p B_{p-3} \pmod{p^2}
\]

(24)

and that for each prime $p \geq 7$

\[
\sum_{k=1}^{p-1} \frac{1}{k^3} \equiv -\frac{6}{5}p^2 B_{p-5} \pmod{p^3}.
\]

(25)

Remark 10. The congruence (22) was also proved in 1938 by E. Lehmer \[52\]. This congruence was generalized in 2007 by X. Zhou and T. Cai \[107, Lemma 3\] to multiple harmonic sums; also see \[104, Theorem 2.14\]. □

Another generalization of the congruence (19) is due in 1954 by L. Carlitz \[15\]: if $m$ is an arbitrary integer, then for each prime $p \geq 5$

\[
\frac{1}{mp + 1} + \frac{1}{mp + 2} + \frac{1}{mp + 3} + \cdots + \frac{1}{mp + (p-1)} \equiv 0 \pmod{p^2}.
\]

(26)

Remark 11. The congruence (26) was also proved in 1989 by S. Zhang \[102\]. □

Using $p$-adic $L$-functions, that is the Washington’s $p$-adic expansion of the sum $\sum_{k=1}^{np} 1/k^r$ \[98\], in 2000 S. Hong \[44, Theorem 1.1\] proved the following generalization of a Glaisher’s congruence (22). Let $p$ be an odd prime and let $n \geq 1$ and $r \geq 1$ be integers. Then

\[
\sum_{k=1}^{p-1} \frac{1}{(np + k)^r} \equiv \begin{cases} 
-\frac{(2n+1)(r+1)^3}{2(r+2)} p^2 B_{p-r-2} & \text{if } r \text{ is odd} \\
\frac{r}{r+1} p B_{p-r-1} & \text{if } r \text{ is even} \\
-(2n+1)p & 
\end{cases} \pmod{p^3}
\]

if $p \geq r + 4$ and $p \geq r + 3$, respectively. □

(27)

In 2002 Slavutskii \[78\] showed how a more general sums (i.e., the sum (27) with a power $p^t$, $t \in \mathbb{N}$, instead of $p$) may be studied by elementary methods without the help of $p$-adic $L$-functions. Namely, by \[78, Theorem...\]
1.2] if \( p \) is an odd prime, \( n \geq 1 \), \( r \geq 1 \) and \( l \geq 1 \) are integers, then
\[
\sum_{k=1 \atop (k,p)=1}^{p^{l-1}} \frac{1}{(np^l + k)^r} \equiv \begin{cases} 
\frac{-(2n+1)r^2}{2^{p-1}} B_{p^{l-1} - p^l - 1 - 1} & \text{mod } p^{3l} \text{ if } r \text{ is odd and } p \geq r + 4 \\
\frac{r}{2^{l+1}} B_{p^{2l-1} - p^{2l-2} - 1} & \text{mod } p^{3l-1} \text{ if } r \text{ is even and } p \geq r + 3 \\
-(2n+1)p^{2l-1} & \text{mod } p^{2l} \text{ if } r = p - 2.
\end{cases}
\]

Remark 12. It is obvious that taking \( l = 1 \) into (28), we immediately obtain (27). In [59] R. Meštrović proved the congruence (27) by using very simple and elementary number theory method.

5. Wolstenholme’s Type Supercongruences

A. Granville [36] established broader generalizations of Wolstenholme’s theorem. As an application, it is obtained in [36] that for a prime \( p \geq 5 \) there holds
\[
\left( \frac{2p-1}{p-1} \right) / \left( \frac{2p}{p} \right)^3 \equiv \left( \frac{3}{2} \right) / \left( \frac{2}{1} \right)^3 \text{ (mod } p^5) .
\]
Moreover, by studying Fleck’s quotients, in 2007 Z.W. Sun and D. Wan [92, Corollary 1.5] discovered a new extension of Wolstenholme’s congruences. In particular, their result yields Wolstenholme’s theorem and for a prime \( p \geq 7 \) the following new curious congruence
\[
\left( \frac{4p-1}{2p-1} \right) \equiv \left( \frac{4p}{p} \right) - 1 \text{ (mod } p^5) .
\]
Further, the congruence (37) given in the next section (also cf. the congruences [62 7.1.10 and 7.1.11]) immediately implies that for a prime \( p \geq 5 \)
\[
\left( \frac{2p^2}{p^2} \right) \equiv \left( \frac{2p}{p} \right) \text{ (mod } p^6) \]
and
\[
\left( \frac{2p^3}{p^3} \right) \equiv \left( \frac{2p^2}{p^2} \right) \text{ (mod } p^9) .
\]
If \( p \) is a prime, \( k, n \) and \( m \) are positive integers such that \( n \geq m \), \( \binom{n}{m} \) is not divisible by \( p \) and \( m \equiv n \text{ (mod } p^k) \), then [62 7.1.16]
\[
\left( \frac{n}{m} \right) \equiv \left[ \frac{n}{p} \right] \left[ \frac{m}{p} \right] \text{ (mod } p^k) ,
\]
where \([a]\) denotes the integer part of a real number \( a\).
A harmonic type Wolstenholme’s supercongruence is established in 2011 by Y. Su, J. Yang and S. Li in [83, Theorem, p. 500] as follows: if \(m\) is any integer, \(r\) is a non-negative integer and \(p \geq 5\) is a prime such that \(p^r | 2m + 1\), then

\[
\sum_{k=1}^{p-1} \frac{1}{mp + k} \equiv 0 \pmod{p^{r+2}}.
\]

**Remark 13.** Note that the congruence (34) with \(r = 0\) reduces to the congruence (26).

6. Ljunggren’s and Jacobsthal’s Binomial Congruences

By Glaisher’s congruence (15) ([32, p. 21], [33, p. 323]), for any positive integer \(n\) and a prime \(p \geq 5\)

\[
\left( \frac{np - 1}{p - 1} \right) \equiv 1 \pmod{p^3},
\]

which by the identity \(\left( \frac{np}{p} \right) = n\left( \frac{np-1}{p-1} \right)\) yields [62, the congruence 7.1.5]

\[
\left( \frac{np}{p} \right) \equiv n \pmod{p^3}.
\]

In 1952 Ljunggren generalized the above congruence as follows ([12]; also see [6, Theorem 4], [36] and [81, Problem 1.6 (d)]): if \(p \geq 5\) is a prime, \(n\) and \(m\) are positive integers with \(m \leq n\), then

\[
\left( \frac{np}{mp} \right) \equiv \left( \frac{n}{m} \right) \pmod{p^3}.
\]

**Remark 14.** Note that the congruence (36) with \(m = 1\) and \(n = 2\) reduces to the Wolstenholme’s congruence (4). Furthermore, the combinatorial proof of (36) regarding modulo \(p^2\) can be found in [81, Exercise 14(c) on page 118]. □

Further, the congruence (36) is refined in 1952 by E. Jacobsthal ([12]; also see [36] and [22, Section 11.6, p. 380]) as follows: if \(p \geq 5\) is a prime, \(n\) and \(m\) are positive integers with \(m \leq n\), then

\[
\left( \frac{np}{mp} \right) \equiv \left( \frac{n}{m} \right) \pmod{p^t},
\]

where \(t\) is the power of \(p\) dividing \(p^3nm(n - m)\) (this exponent \(t\) can only be increased if \(p\) divides \(B_{p-3}\), the \((p - 3)rd\) Bernoulli number).

**Remark 15.** In the literature, the congruence (37) is often called Jacobsthal-Kazandzidis congruence (see e.g., [22, Section 11.6, p. 380]). □
In particular, the congruence (37) implies that for all nonnegative integers \( n, m, a, b \) and \( c \) with \( c \leq b \leq a \), and any prime \( p \geq 5 \)
\[
\binom{np^n}{mp^b} \equiv \binom{np^{a-c}}{mp^{b-c}} \pmod{p^{3+a+2b-3c}}.
\]

Moreover, taking \( a = b \) and \( c = 1 \) into (38) (cf. [30, Section 2, Lemma A], or for a direct proof see [37, Lemma 19 of Appendix]), we find that for any prime \( p \geq 5 \)
\[
\binom{np^{a}}{mp^{a}} \equiv \binom{np^{a-1}}{mp^{a-1}} \pmod{p^{3a}}.
\]

Using elementary method, in 1988 N. Robbins [70, Theorem 2.1] proved the following result. Let \( p \geq 3 \) be a prime and let \( n, m, a, b \) be nonnegative integers with \( 0 \leq b \leq a \), \( 0 < m < np^{a-b} \) and \( nm \not\equiv 0 \pmod{p} \). Then
\[
\binom{np^{a}}{mp^{b}} \equiv \binom{np^{a-b}}{m} \pmod{p^s}.
\]

Remark 16. Because the original source [12] is not easily accessible, the congruence (37) was rediscovered by various authors, including G.S. Kazandzidis ([48] and [49]) in 1968 (its proof is based on \( p \)-adic method) and Yu.A. Trakhtman [96] in 1974. Furthermore, in 1995 A. Robert and M. Zuber [71] (see also [72, Chapter 7, Section 1.6]) proposed a simple proof of the congruence (37) based on well-known properties of the \( p \)-adic Morita gamma function \( \Gamma_p \).

In 2008 Helou and Terjanian [42] (1) of Corollary on page 490] proved that if \( p \geq 5 \) is a prime, \( n \) and \( m \) are positive integers with \( m \leq n \), then
\[
\binom{np}{mp} \equiv \binom{n}{m} \pmod{p^s},
\]
where \( s \) is the power of \( p \) dividing \( p^3m(n - m) \binom{n}{m} \).

Remark 17. It is pointed out in [42, Remark 6 on page 490] that for a prime \( p \geq 5 \) using \( p \)-adic methods, the modulus \( p^s \) in the congruence (41) can be improved to \( p^f \), where \( f \) is the power of \( p \) dividing \( p^3mn(n - m) \binom{n}{m} \). Clearly, this result would be an improvement of Jacobsthal-Kazandzidis congruence given by (37). Notice also that Z.W. Sun and D.M. Davis [91, Lemma 3.2] via elementary method proved the congruence (37) modulo \( p^s \), where \( p \geq 3 \) is a prime and \( s \) is the power of \( p \) dividing \( p^2n^2 \).

The Jacobsthal’s congruence (37) is refined in 2007 by J. Zhao [104, Theorem 3.2] as follows. For a prime \( p \geq 7 \) define \( w_p < p^2 \) to be the unique nonnegative integer such that
\[
w_p \equiv \left( \sum_{k=1}^{p-1} 1/k \right) / p^2 \pmod{p^2}.
\]
Then for
all positive integers $n$ and $m$ with $n \geq m$

\[(42) \quad \binom{np}{mp} / \binom{n}{m} \equiv 1 + w_p n m (n - m) p^3 \pmod{p^5}.\]

**Remark 18.** Since $\frac{1}{2} \binom{2p}{p} = \binom{2p-1}{p-1}$, taking $n = 2$ and $m = 1$ into (42), it becomes
\[
\binom{2p - 1}{p - 1} \equiv 1 + 2 w_p p^3 \pmod{p^5},
\]
which is actually (7). □

Further, in 2008 Helou and Terjanian [42, Proposition 2 (1)] generalized the congruence (42) modulo $p^6$ in the form involving the Bernoulli numbers as follows. Let $p \geq 5$ be a prime. Then for all positive integers $n$ and $m$ with $n \geq m$

\[(43) \quad \binom{np}{mp} / \binom{n}{m} \equiv 1 - m n (n - m) \left( \frac{p^3}{2} B_{p^3 - p - 2} - \frac{p^5}{6} B_{p - 3} + \frac{1}{5} (m^2 - mn + n^2) p^5 B_{p - 5} \right) \pmod{p^6}.\]

As an application of (43), it can be obtained [42, the congruence of Remark 4 on page 489] that for each prime $p \geq 5$

\[(44) \quad \binom{np}{mp} / \binom{n}{m} \equiv 1 - \frac{1}{3} m n (n - m) p^3 B_{p - 3} \pmod{p^4}.\]

**Remark 19.** Taking $m = 1$ into (44), it immediately reduces to the Glaisher’s congruence (15) which for $n = 2$ becomes (14). □

**Remark 20.** In [42, Sections 3 and 4] C. Helou and G. Terjanian established numerous Wolstenholme’s type congruences of the form $\binom{np}{mp} \equiv \binom{n}{m} P(n, m, p) \pmod{p^k}$ and $\binom{mnp}{np} \equiv \binom{mn}{n} P(n, m, p) \pmod{p^k}$, where $p$ is a prime, $k \in \{4, 5, 6\}$, $m, n, \in \mathbb{N}$ with $m \leq n$, and $P$ is a polynomial of $m, n$ and $p$ involving Bernoulli numbers as its coefficients. □

**Remark 21.** In 2011 R. Meštrović [57] discussed the following type congruences:
\[
\binom{np^k}{mp^k} \equiv \binom{n}{m} \pmod{p^r},
\]
where $p$ is a prime, $n, m, k$ and $r$ are various positive integers. □
7. WOLSTENHOLME PRIMES

A prime \( p \) is said to be a Wolstenholme prime (see [60] p. 385 or [56]; this is the Sloane’s sequence A088164 from [79]) if it satisfies the congruence

\[
\binom{2p - 1}{p - 1} \equiv 1 \pmod{p^4}.
\]

Clearly, this is equivalent to the fact that the Wolstenholme quotient \( W_p \) defined as

\[
W_p = \frac{(2p - 1) - 1}{p^3}, \quad p \geq 5
\]

is divisible by \( p \) (\( W_p \) is the Sloane’s sequence A034602 from [79]; also cf. a related sequence A177783).

Notice that by a special case of Glaisher’s congruence ([32, p. 21], [33, p. 323]; also cf. [60, Corollary, p. 386]) given by (14) it follows that

\[
W_p \equiv -2/3 B_{p-3} \pmod{p}.
\]

From the congruence (45) we see that \( p \) is a Wolstenholme prime if and only if \( p \) divides the numerator of the Bernoulli number \( B_{p-3} \). This is by (5) also equivalent with the fact (e.g., see [27] or [104, Theorem 2.8]) that the numerator of the fraction \( \sum_{k=1}^{p-1} 1/k \) written in reduced form is divisible by \( p^3 \), and also by (6) with the fact that the numerator of the fraction \( \sum_{k=1}^{p-1} 1/k^2 \) written in reduced form is divisible by \( p^2 \).

In other words, a Wolstenholme prime is a prime \( p \) such that \( (p, p - 3) \) is an irregular pair (see [47] and [13]). The Wolstenholme primes therefore form a subset of the irregular primes (see e.g., [60, p. 387]).

In 1995 McIntosh [60] observed that by a particular case of a result of Stafford and Vandiver [80] in 1930 and Fermat’s Little Theorem, for any prime \( p \geq 11 \)

\[
B_{p-3} \equiv 1/21 \sum_{k=[p/6]+1}^{[p/4]} 1/k^3 \pmod{p}.
\]

Remark 22. The congruence (46) is very useful in the computer search of Wolstenholme primes (see [61]).

Only two Wolstenholme primes are known today: 16843 and 2124679. The first was found (though not explicitly reported) by Selfridge and Pollack in 1964 (Notices Amer. Math. Soc. 11 (1964), 97), and later confirmed by W. Johnson [47] and S.S. Wagstaff (Notices Amer. Math. Soc. 23 (1976), A-53). The second was discovered by Buhler, Crandall, Ernvall and Metsänkylä in 1993 [13]. In 1995, McIntosh [60] determined by calculation that there is no other Wolstenholme prime \( p < 5 \cdot 10^8 \). In 2007 R.J.
McIntosh and E.L. Roettger [61] reported that these primes are only two Wolstenholme primes less than $10^9$. However, using the argument based on the prime number theorem, in 1995 McIntosh [60, p. 387] conjectured that there are infinitely many Wolstenholme primes. It seems that the proof of this assertion would be very difficult.

In 2007 J. Zhao [103] defined a Wolstenholme prime via harmonic numbers; namely, a prime $p$ is a Wolstenholme prime if

$$\sum_{k=1}^{p-1} \frac{1}{k} \equiv 0 \pmod{p^3}.$$

In 2011 R. Meštrović [56, Proposition 1] proved that if $p$ is a Wolstenholme prime, then

$$\binom{2p-1}{p-1} \equiv 1 + p \sum_{k=1}^{p-1} \frac{1}{k} - \frac{p^2}{2} \sum_{k=1}^{p-1} \frac{1}{k^2} + \frac{p^3}{3} \sum_{k=1}^{p-1} \frac{1}{k^3} - \frac{p^4}{4} \sum_{k=1}^{p-1} \frac{1}{k^4} + \frac{p^5}{5} \sum_{k=1}^{p-1} \frac{1}{k^5} - \frac{p^6}{6} \sum_{k=1}^{p-1} \frac{1}{k^6} \pmod{p^8}.$$

The above congruence can be simplified as follows [56, Proposition 2]:

$$\binom{2p-1}{p-1} \equiv 1 + \frac{3p}{2} \sum_{k=1}^{p-1} \frac{1}{k} - \frac{p^2}{4} \sum_{k=1}^{p-1} \frac{1}{k^2} + \frac{7p^3}{12} \sum_{k=1}^{p-1} \frac{1}{k^3} + \frac{5p^5}{12} \sum_{k=1}^{p-1} \frac{1}{k^5} \pmod{p^8}.$$

Reducing the modulus in the previous congruence, we can obtain the following simpler congruences for Wolstenholme prime $p$ [56, Corollary 1]:

$$\binom{2p-1}{p-1} \equiv 1 - 2p \sum_{k=1}^{p-1} \frac{1}{k} - 2p^2 \sum_{k=1}^{p-1} \frac{1}{k^2} \pmod{p^7}$$

$$\equiv 1 + 2p \sum_{k=1}^{p-1} \frac{1}{k} + \frac{2p^3}{3} \sum_{k=1}^{p-1} \frac{1}{k^3} \pmod{p^7}.$$  

In terms of the Bernoulli numbers the congruence (49) may be written as [56, Corollary 2]

$$\binom{2p-1}{p-1} \equiv 1 - p^3 B_{p-2} - \frac{3}{2} p^5 B_{p-4} + \frac{3}{10} p^6 B_{p-5} \pmod{p^7}.$$
The congruence (50) can be given by the following expression involving lower order Bernoulli numbers [56, Corollary 2]:

\[
\binom{2p - 1}{p - 1} \equiv 1 - p^3 \left( \frac{8}{3} B_{p-3} - 3 B_{2p-4} + \frac{8}{5} B_{3p-5} - \frac{1}{3} B_{4p-6} \right) \\
- p^4 \left( \frac{8}{9} B_{p-3} - \frac{3}{2} B_{2p-4} + \frac{24}{25} B_{3p-5} - \frac{2}{9} B_{4p-6} \right) \\
- p^5 \left( \frac{8}{27} B_{p-3} - \frac{3}{4} B_{2p-4} + \frac{72}{125} B_{3p-5} - \frac{4}{27} B_{4p-6} + \frac{12}{5} B_{5p-6} \right) \\
- B_{2p-6} - \frac{2}{25} p^6 B_{p-5} \pmod{p^7}.
\] (51)

Remark 23. Combining the first congruence of (49) and a recent result of the author in [55, Theorem 1.1] given by the congruence (11), we obtain a new characterization of Wolstenholme primes as follows [55, Remark 1.6]. □

A prime \( p \) is a Wolstenholme prime if and only if

\[
\binom{2p - 1}{p - 1} \equiv 1 - 2p \sum_{k=1}^{p-1} \frac{1}{k} - 2p^2 \sum_{k=1}^{p-1} \frac{1}{k^2} \pmod{p^7}.
\]

Remark 24 ([55, Remark 1]). A computation via Mathematica shows that no prime \( p < 10^5 \) satisfies the second congruence from (49), except the Wolstenholme prime 16843. Accordingly, an interesting question is as follows: Is it true that the second congruence from (49) yields that a prime \( p \) is necessarily a Wolstenholme prime? We conjecture that this is true. □

8. Wolstenholme’s Type Theorems for Composite Moduli

For positive integers \( k \) and \( n \) let \( (k, n) \) denotes the greatest common divisor of \( k \) and \( n \). In 1889 C. Leudesdorf [53] (also see [40, Ch. VIII] or [73, Chapter 3, the congruence (15) on page 244]) was proved that for any positive integer \( n \) such that \( (n, 6) = 1 \)

\[
\sum_{k=1, (k, n)=1}^{n-1} \frac{1}{k} \equiv 0 \pmod{n^2},
\] (52)

where the summation ranges over all \( k \) with \( (k, n) = 1 \).

Remark 25. Observe that when \( n = p \geq 5 \) is a prime, then the congruence (52) reduces to the Wolstenholme’s congruence (19). □

Remark 26. In 1934 S. Chowla [20] gave a very short and elegant proof of the congruence (52). This was in another way also proved by H. Gupta [38, the congruence (1)]. Furthermore, in 1933 Chowla [19] proved Leudesdorf’s theorem when \( n \) is a power of a prime \( p \geq 5 \). □
The binomial coefficient’s analogue of (52) was established in 1995 by R.J. McIntosh. Namely, in [60, Section 2] for positive integers \( n \), the author defined the modified binomial coefficients

\[
\binom{2n-1}{n-1}' = \prod_{k=1 \atop (k,n)=1}^{n} \frac{2n-k}{k}
\]

and observed that for primes \( p \),

\[
\binom{2p-1}{p-1}' = \binom{2p-1}{p-1}.
\]

Then by [60, Theorem 1], for \( n \geq 3 \)

\[
\binom{2n-1}{n-1}' \equiv 1 + n^2 \varepsilon_n \pmod{n^3},
\]

where

\[
\varepsilon_n = \begin{cases} 
\frac{n}{2} & \text{if } n \text{ is a power of } 2, \\
(-1)^{r+1} \frac{n}{3} & \text{if } n \equiv 0 \pmod{3} \text{ and } n \text{ has exactly } \ r \text{ distinct prime factors, each } \not\equiv 1 \pmod{6}, \\
0 & \text{otherwise}.
\end{cases}
\]

Remark 27. McIntosh [60, page 384] also noticed that in 1941 H.W. Brinkman [74] in his partial solution to David Segal’s conjecture observed the following relation between the ordinary binomial coefficient and the modified binomial coefficient:

\[
\binom{2n-1}{n-1} = \prod_{d|n} \binom{2d-1}{d-1}',
\]

which for example, for \( n = p^2 \) with a prime \( p \) becomes

\[
\binom{2p^2-1}{p^2-1} = \binom{2p-1}{p-1}' \binom{2p-1}{p-1}.
\]

Remark 28. Some extensions of (52) can be found in the monograph of G.H. Hardy and E.M. Wright [40, Ch. VIII], [41] and [68].

Further generalizations of Leudesdorf’s congruence (52) were obtained by L. Carlitz [15] in 1954 and by H.J.A. Duparc and W. Peremans [25 Theorem 2] in 1955. Their result (i.e., [25, Theorem 2]) was also established in 1982 by I. Gessel [29, Theorems 1 and 2] and by I.Sh. Slavutskii [76, Corollary 1] in 1999.

Namely, by [76, Corollary 1(a)], \( s \) is an even positive integer and \( n \) a positive integer such that \((p, n) = 1\) for all primes \( p \) such that \((p - 1) \mid s\), then

\[
\sum_{k=1 \atop (k,n)=1}^{n-1} \frac{1}{k^s} \equiv 0 \pmod{n}.
\]
On the other hand, by [76, Corollary 1(b)], if \( s \) is an odd positive integer and \( n \) a positive integer such that 1) \( p - 1 \) don’t divide \( s + 1 \) for every prime \( p \) with \( p \mid n \); or 2) \( p \mid s \) for all primes \( p \) such that \( (p - 1) \mid (s + 1) \) and \( p \mid n \), then

\[
\sum_{k=1}^{n-1} \frac{1}{k^s} \equiv 0 \pmod{n^2}.
\]

Both congruences (54) and (55) are immediate consequences of Theorem 2 in [76] (also see [77, Theorem 1] and [73, Chapter 3, the congruence (15’) on page 244): if \( s \) and \( n \) are positive integers such that \( (n, 6) = 1 \) and \( t = (\varphi(n^2) - 1)s \), where \( \varphi(\cdot) \) denotes the Euler’s totient function, then

\[
\sum_{k=1}^{n-1} \frac{1}{k^s} \equiv \begin{cases} 
    n \prod_{p \mid n} (1 - p^{t-1})B_t & \text{for even } s \\
    \frac{L}{2} n^2 \prod_{p \mid n} (1 - p^{t-2})B_{t-1} & \text{for odd } s.
\end{cases}
\]

Notice that by the congruence (56) (cf. (6) and (7) in [76]), it follows that for \( n = p^l \) with \( l \in \mathbb{N} \) and a prime \( p \geq 5 \)

\[
\sum_{k=1}^{p^l-1} \frac{1}{k^s} \equiv \begin{cases} 
    p^l B_t \pmod{p^{2l}} & \text{for even } s, t = (\varphi(p^{2l}) - 1)s \\
    \frac{L}{2} p^{2l}B_{t-1} \pmod{p^{2l}} & \text{for odd } s, t = (\varphi(p^{2l}) - 1)s.
\end{cases}
\]

As an application of the congruence (57), we obtain the following result obtained in 1955 by H.J.A. Duparc and W. Peremans [25, Theorem 1] (cf. [76, Corollary 2]). Let \( n = p^l \) be a power of a prime \( p \geq 3 \). Then

\[
\sum_{k=1}^{p^l-1} \frac{1}{k^s} \equiv \begin{cases} 
    0 \pmod{p^{2l-1}} & \text{for odd } s \text{ with } p - 1 \mid s + 1 \text{ and } s \neq 0 \pmod{p} \\
    0 \pmod{p^{2l}} & \text{for odd } s \text{ with } s + 1 \neq 0 \pmod{p - 1} \text{ or } p \mid s \\
    0 \pmod{p^{l-1}} & \text{for even } s \text{ with } p - 1 \mid s \\
    0 \pmod{p^l} & \text{for even } s \text{ with } s \neq 0 \pmod{p - 1}.
\end{cases}
\]

Remark 29. Notice that the second part and the fourth part of the congruence (58) under the condition \( s \leq p - 3 \) were also proved in 1997 by M. Bayat [8, Theorem 4]. Moreover, the second part and the fourth part of the congruence (58) with \( n = ap^l \) \((a, l \in \mathbb{N})\) instead of \( p^l \), \( s = 1 \) and \( s = 2 \) were also proved in 1998 by D. Berend and J.E. Harmse [9, Proposition 2.2]. □

Wolstenholme type congruence for product of distinct primes was established in 2007 by S. Hong [45]: if \( m \) and \( n \) are integers with \( m \geq 0, n \geq 1, \langle n \rangle := \{1, \ldots, n\}, p_1, \ldots, p_n \) are distinct primes and all greater than 3,
then
\[
\sum_{\substack{k=1 \\
\forall i \in \{n\geq 5\} \land (k, p_i) = 1}}^{p_1 \cdots p_n} \frac{1}{mp_1 \cdots p_n + k} \equiv 0 \pmod{(p_1 \cdots p_n)^2}.
\]

9. ON THE CONVERSE OF WOLSTENHOLME’S THEOREM

If \( n \geq 5 \) is a prime, then by Wolstenholme’s theorem
\[
\binom{2n-1}{n-1} \equiv 1 \pmod{n^3}.
\]

Is the converse true? This question, still unanswered today, has been asked by J.P. Jones for many years (see [69, Chapter 2, p. 23], [36] and [39, B31, p. 131]).

In 2001 V. Trevisan and K.E. Weber [97, Theorem 1] proved that if \( n \) is an even positive integer, then
\[
\binom{2n-1}{n-1} \not\equiv 1 \pmod{n^3}.
\]

Following [69, Chapter 2, p. 23], the mentioned problem leads naturally to the following concepts and questions. Let \( n \geq 5 \) be odd, and let
\[
A(n) := \binom{2n-1}{n-1}.
\]

For each \( k \geq 1 \) we may consider the set
\[
W_k = \{ n \text{ odd} \, , \, n \geq 5 \, | \, A(n) \equiv 1 \pmod{n^k} \}.
\]

Obviously, \( W_1 \supset W_2 \supset W_3 \supset W_4 \supset \ldots \). From Wolstenholme’s theorem every prime number greater than 3 belongs to \( W_3 \). Jones’ question is whether \( W_3 \) is just the set of these prime numbers.

Notice that the set \( W_4 \) coincides with the set of all Wolstenholme primes defined in Section 7. The set of composite integers in \( W_2 \) contains the squares of Wolstenholme primes. McIntosh [60, p. 385] conjectured that these sets coincide and verified that this is true up to \( 10^9 \); the only composite number \( n \in W_2 \) with \( n < 10^9 \), is \( n = 283686649 = 16843^2 \). Furthermore, using the argument based on the prime number theorem, McIntosh ([60, p. 387]) conjectured that the set \( W_5 \) is empty; this means that no prime satisfies the congruence
\[
\binom{2p-1}{p-1} \equiv 1 \pmod{p^5}.
\]

Recall also that in 2010 K.A. Broughan, F. Luca and I.E. Shparlinski [11] investigated the subset \( W_1' \) consisting of all composite positive integers \( n \) belonging to the set \( W_1 \). They proved [11, Theorem 1] that the set
$W'_1$ is of asymptotic density zero. More precisely, if $W(x)$ is defined to be the number of composite positive integers $n \leq x$ which satisfy $\left(\frac{2n-1}{n-1}\right) \equiv 1 \pmod{n}$, then $\lim_{x \to \infty} W(x)/x = 0$. McIntosh \cite[385]{60} reported that the only elements of the set $W_1$ less than $10^9$ that are not primes nor prime powers are $29 \times 937$ and $787 \times 2543$, but none of these satisfy Wolstenholme’s congruence.

**Remark 30.** The converse of Wolstenholme’s theorem for particular classes of composite integers $n$ was discussed and proved in 2001 by Trevisan and Weber \cite{97}. Further, in 2008 Helou and Terjanian \cite[Section 5, Propositions 5-7]{42} deduced that this converse holds for many infinite families of composite integers $n$. □

10. **Binomial sums related to Wolstenholme’s theorem**

In 2006 M. Chamberland and K. Dilcher \cite{16} studied a class of binomial sums of the form

$$u^{\varepsilon}_{a,b}(n) := \sum_{k=0}^{n} (-1)^{\varepsilon k} \binom{n}{k}^a \binom{2n}{k}^b,$$

for nonnegative integers $a, b, n$ and $\varepsilon \in \{0, 1\}$, and showed that these sums are closely related to Wolstenholme’s theorem. Namely, they proved \cite[Theorem 3.1]{16} that for any prime $p \geq 5$ holds

$$(60) \quad u^{\varepsilon}_{a,b}(p) := \sum_{k=0}^{p} (-1)^{\varepsilon k} \binom{p}{k}^a \binom{2p}{k}^b \equiv 1 + (-1)^{\varepsilon}2^b \pmod{p^3},$$

except when $(\varepsilon, a, b) = (0, 0, 1)$.

In a subsequent paper in 2009 M. Chamberland and K. Dilcher \cite{17} studied the above sum for $(\varepsilon, a, b) = (1, 1, 1)$, that is, with the simplified notation, the sum

$$u(n) := \sum_{k=0}^{n} (-1)^k \binom{n}{k} \binom{2n}{k}.$$

Under this notation, the authors proved \cite[Theorem 2.1]{17} that for all primes $p \geq 5$ and integers $m \geq 1$ we have

$$(61) \quad u(mp) \equiv u(m) \pmod{p^3}.$$

In 2002 T.X. Cai and A. Granville \cite[Theorem 6]{14} proved the following result. If $p \geq 5$ is a prime and $n$ a positive integer, then

$$(62) \quad \sum_{k=0}^{p-1} (-1)^k \binom{p-1}{k}^n \equiv \begin{cases} \binom{np-2}{p-1} \pmod{p^4} & \text{if } n \text{ is odd} \\ 2^n(p-1) \pmod{p^3} & \text{if } n \text{ is even} \end{cases}$$
and

\[(63) \quad \sum_{k=0}^{p-1} \binom{p-1}{k}^n \equiv \begin{cases} \binom{n}{p-1}^{2^{n(p-1)}} \pmod{p^4} & \text{if } n \text{ is even} \\ \binom{n}{p-1} \pmod{p^3} & \text{if } n \text{ is odd} \end{cases} \]

In 2009 H. Pan [65, Theorem 1.1] generalized the second parts of congruences (62) and (63) as follows. Let \( p \geq 3 \) be a prime and let \( n \) be a positive integer. Then

\[(64) \quad \sum_{k=0}^{p-1} (-1)^{n-k} \binom{p-1}{k}^a \equiv 2^{a(p-1)} + \frac{n(n-1)(3n-4)}{48} p^3 B_{p-3} \pmod{p^4}. \]

Recently, in 2011 R. Meštrović [58, Theorem 3] extended Pan’s congruence (64) for \( n = -1 \) by proving the following congruence for the sum of the reciprocals of binomial coefficients. Let \( p \geq 3 \) be a prime. Then

\[(65) \quad \sum_{k=0}^{p-1} \binom{p-1}{k}^{-1} \equiv 2^{1-p} - \frac{7}{24} p^3 B_{p-3} \pmod{p^4}. \]

In particular, we have

\[(66) \quad \sum_{k=0}^{p-1} \binom{p-1}{k}^{-1} \equiv 2^{1-p} \pmod{p^3}. \]

Many interesting congruences modulo \( p^k \) with \( k \geq 3 \) are in relation to the Apéry numbers \( A_n \) defined in 1979 by R. Apéry [3] as

\[ A_n := \sum_{k=0}^{n} \binom{n}{k}^2 \binom{n+k}{k}^2 = \sum_{k=0}^{n} \binom{n+k}{2k}^2 \binom{2k}{k}^2, \quad n = 0, 1, 2, \ldots. \]

For example, in 1982 I. Gessel [29] proved that for any prime \( p \geq 5 \)

\[(67) \quad A_{pn} \equiv A_n \pmod{p^3}, \quad n = 0, 1, 2, \ldots. \]

Remark 31. Z.W. Sun in [88], [90, pp. 48-49] and [89] also made many interesting conjectures on congruences involving the Apéry numbers \( A_n \).

Finally, we present here an interesting congruence proposed as a problem on W.L. Putnam Mathematical Competition [67]. If \( p \geq 5 \) is a prime and \( k = [2p/3] \), then by [67] Problem A5 (1996)

\[(68) \quad \binom{p}{1} + \binom{p}{2} + \cdots + \binom{p}{k} \equiv 0 \pmod{p^2}. \]
11. $q$-analogues of Wolstenholme’s type congruences

Recall that the generalized harmonic numbers $H_n^{(m)}$, $n, m = 0, 1, 2, \ldots$ are defined by

$$H_n^{(m)} = \sum_{k=1}^{n} \frac{1}{k^m}$$

(we assume that $H_0^{(m)} = 0$ for all $m$). Notice that

$$H_n^{(1)} := H_n = \sum_{k=1}^{n} \frac{1}{k}$$

is the harmonic number. A $q$-analog of $H_n$ is given by the $q$-harmonic numbers

$$H_n(q) := \sum_{k=1}^{n} \frac{1}{[k]_q}, \quad n \geq 0, \ |q| < 1,$$

where

$$[k]_q := \frac{1 - q^k}{1 - q} = 1 + q + \cdots + q^{k-1}.$$

A different $q$-analog of $H_n$ is

$$\tilde{H}_n(q) := \sum_{k=1}^{n} \frac{q^k}{[k]_q}, \quad n \geq 0, \ |q| < 1,$$

In 1999 G.E. Andrews [2, Theorem 4] proved a $q$-analogue of the weaker version (modulo $p$) of the congruence (19); namely, for primes $p \geq 3$

(69) $$H_{p-1}(q) \equiv \frac{p-1}{2}(1 - q) \quad (\text{mod } [p]_q).$$

Andrews also proved that for primes $p \geq 3$

(70) $$\tilde{H}_{p-1}(q) \equiv -\frac{p-1}{2}(1 - q) \quad (\text{mod } [p]_q).$$

In 2007 L.L. Shi and H. Pan [75, Theorem 1] (also see [63, the congruence (1.3)]) extended (69) to

(71) $$H_{p-1}(q) \equiv \frac{p-1}{2}(1 - q) + \frac{p^2-1}{24}(1 - q)^2[p]_q \quad (\text{mod } [p]_q^2)$$

for each prime $p \geq 5$.

In 2007 L.L. Shi and H. Pan [75, Lemma 2 (5) and (4)] also showed that for each prime $p \geq 5$

(72) $$\sum_{k=1}^{p-1} \frac{1}{[k]_q^2} \equiv -\frac{(p-1)(p-5)}{12}(1 - q)^2 \quad (\text{mod } [p]_q).$$
and

\[
\sum_{k=1}^{p-1} \frac{q^k}{[k]_q^2} \equiv -\frac{p^2 - 1}{12}(1 - q)^2 \pmod{[p]_q}.
\]

Recently, in 2011 A. Straub [82] proved a \(q\)-analogue of a classical binomial congruence (36) due to Ljunggren. If under the above notation we set

\[ [n]_q! := [n]_q[n - 1]_q \cdots [1]_q \]

and

\[ \binom{n}{k}_q := \frac{[n]_q!}{[k]_q![n - k]_q!} \]

(this is a polynomial in \(q\) with integer coefficients) then Straub [82, Theorem 1] proved that if \(p \geq 5\) is a prime, \(n\) and \(m\) are nonnegative integers with \(m \leq n\), then

\[
\binom{np}{mp}_q \equiv \binom{n}{m}_q \pmod{[p]_q^3},
\]

Notice that the congruence (74) reduced modulo \([p]_q^2\) becomes

\[
\binom{np}{mp}_q \equiv \binom{n}{m}_q \pmod{[p]_q^2},
\]

which was proved in 1995 by W.E. Clark [21] the congruence (2) on page 197.

Notice also that in 1999 G.E. Andrews [2] proved a similar result; e.g.:

\[
\binom{np}{mp}_q \equiv q^{(n-m)m(n)} \binom{n}{m}_q \pmod{[p]_q^2}.
\]

Furthermore, taking \(m = 2\) and \(n = 1\) into the congruence (74), we obtain a \(q\)-analogue of Wolstenholme’s theorem as:

\[
\binom{2p}{p}_q \equiv [2]_q^{p^2} - \frac{p^2 - 1}{12}(q^p - 1)^2 \pmod{[p]_q^3}.
\]

Remark 32. The congruences in this section are to be understood as congruences in the polynomial ring \(\mathbb{Z}[q]\). Note that it is clear that \([p]_q := 1 + q + \cdots + q^{p-1}\), as the \(p\)th cyclotomic polynomial is irreducible; hence the denominator of \(H_{p-1}(q)\), seen as a rational function of \(q\), is relatively prime to \([p]_q\).
Remark 33. In 2008 K. Dilcher [24, Theorems 1 and 2] generalized the congruences (69) and (70) deriving congruences \((\mod{p^q})\) for the generalized (or higher-order) \(q\)-harmonic numbers. His results are in fact \(q\)-analogues of the congruences \(H_{p-1}^{(k)} \equiv 0 \mod{p}\) which follow from (21).

\[\square\]

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Note. I found References [18], [84], [85], [93], [100], [101] and [106] on the Internet, but they are not cited in this article because of they are not accessible to the author.

APPENDIX

A) External Links on Wolstenholme’s theorem and Wolstenholme primes

Eric Weisstein World of Mathematics, Wolstenholme prime, http://mathworld.wolfram.com/Wolstenholme prime.html, from MathWorld.

Wikipedia http://en.wikipedia.org/wiki/Wolstenholme_prime
http://planetmath.org/encyclopedia/WolstenholmesTheorem.html
B) Sloane’s sequences related to Wolstenholme’s theorem and Wolstenholme primes

Sloane, N.J.A. Sequences A001008, A007406, A007408, A088164, A034602, A177783, in ”The On-Line Encyclopedia of Integer Sequences.” (published electronically at www.research.att.com/njas/sequences/).

C) List of papers/authors ordered by years of publications

28 1801] C.F. Gauss
[5 1819] C. Babbage
[51 1851] E.E. Kummer
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[83] 2011] Y. Su, J. Yang and S. Li
[88] 2011] Z.W. Sun
[89] 2011] Z.W. Sun
[90] 2011] Z.W. Sun
[95] 2011] R. Tauraso
[100] 2011] J. Yang

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