Fourier knots

Christoph Lamm

Rückertstr. 3,
65187 Wiesbaden, Germany,
e-mail: christoph.lamm@web.de

Abstract
We show that every knot has a checkerboard diagram and that every knot is the closure of a rosette braid. We define Fourier knots of type \((n_1, n_2, n_3)\) as knots which have parametrizations where each coordinate function \(x_i(t)\) is a finite Fourier series of length \(n_i\), and conclude that every knot is a Fourier knot of type \((1, 1, n)\) for some natural number \(n\).

1 Rosette braids

Let \(B_n\) be the braid group on \(n\) strings, \(\pi : B_n \to S_n\) be the map to the symmetric group on \(n\) letters and \(P_n = \ker(\pi)\) be the pure braid group on \(n\) strings. The generators of \(P_n\) are denoted by \(A_{i,j}\). If \(\pi_0\) is a permutation, then \(k(\pi_0)\) denotes its number of cycles.

Definition 1.1
A braid of the form

\[
\prod_{i=1}^{n} \left[ \prod_{j=dd}^{\sigma^{e_{i,j}}_j} \prod_{j=even}^{\sigma^{f_{i,j}}_j} \right], \quad \varepsilon_{i,j} \in \{\pm 1\}
\]

is called a rosette braid of type \((s, n)\). The set of rosette braids of type \((s, n)\) is denoted by \(\mathcal{R}(s, n)\).

Lemma 1.2

a) i) \(\alpha \in \mathcal{R}(s, 1) \Rightarrow k(\pi(\alpha)) = 1\).

ii) \(\alpha \in \mathcal{R}(s, s) \Rightarrow \pi(\alpha) = \text{id} \).

iii) \(\alpha \in \mathcal{R}(s, ns + 1) \Rightarrow k(\pi(\alpha)) = 1\).

b) For each generator \(A_{i,j}\) of \(P_n\) there is an \(\alpha \in \mathcal{R}(s, s)\), so that \(A_{i,j} = \alpha\).
Figure 1: A rosette braid of type (4, 3) and a checkerboard diagram of type (4, 3)

**Proof:**  

a) Proposition i) is true for \( s = 2 \), because then the braid word has the form \( \sigma \pm \). An element of \( \mathcal{R}(s, 1) \) is built out of an element of \( \mathcal{R}(s - 1, 1) \) by a Markov-II-move (insertion of \( \sigma \pm \)). Because the number of components is unchanged by a Markov-II-move, we conclude by induction that the proposition holds for all \( s \).

ii) As shown in part i), the permutation \( \pi(\alpha) \) of a braid \( \alpha \in \mathcal{R}(s, 1) \) consists of one cycle. Hence the permutation of a braid in \( \mathcal{R}(s, s) \) is the trivial permutation on \( s \) letters. Part iii) is an immediate consequence.

b) We consider the two strings \( i \) and \( j \) (\( i < j \)) of the braid \( \alpha_{i,j} \). If as above \( \pi_1 = \pi(\alpha) \) is the permutation of a braid \( \alpha \in \mathcal{R}(s, 1) \), then \( \pi_1 \) is a cycle of length \( s \). Hence there is a \( k \) with \( 1 \leq k < s \), so that \( \pi_1^k(i) > \pi_1^k(j) \), and thus the strings \( i \) and \( j \) cross each other. It is possible to arrange the strings in such a way that all strings but \( i \) and \( j \) can be pulled tight and the strings \( i, j \) form the generators \( \alpha_{i,j} \) or \( \alpha_{i,j}^{-1} \).

\[ \blacksquare \]

**Theorem 1.3**

Let \( \alpha \in B_s \) be a braid, with closure a knot. Then \( \alpha \) is conjugate to a rosette braid of type \((s, ns + 1)\) for a suitable \( n \).

**Proof:** Let \( \alpha \in B_s \) be a braid, so that \( \alpha \) is a knot. Let \( \delta \) be an arbitrary braid in \( \mathcal{R}(s, 1) \) and \( \pi_1 \) its permutation. The permutations \( \pi(\alpha) \) and \( \pi_1 \) are conjugate in the symmetric group because both consist of one cycle. Let \( \beta \in B_s \) be a braid, so that \( \pi(\beta)^{-1} \pi(\alpha) \pi(\beta) = \pi_1 \). Then \( \delta^{-1} \beta^{-1} \alpha \beta \) is a pure braid and because of Lemma 1.2 we can write it as an element of \( \mathcal{R}(s, ns) \) for a suitable \( n \). Multiplication with \( \delta \) yields \( \beta^{-1} \alpha \beta \) as an element of \( \mathcal{R}(s, ns + 1) \). Hence we have shown that \( \alpha \) is conjugate to a rosette braid of type \((s, ns + 1)\).

\[ \blacksquare \]

The braid index of a knot \( K \) is denoted by \( br(K) \).

**Corollary 1.4**

Every knot \( K \) is the closure of a rosette braid with \( br(K) \) strings.
Figure 2: \( \delta \) and \( \gamma \) to modify the permutation of a braid which constitutes a plat. The \( \delta' \) and \( \gamma' \) are the mirrored operations for the lower plat closure.

### 2 Checkerboard diagrams

**Definition 2.1**

A knot diagram is called a **checkerboard diagram of type** \((2b, n)\), if it is the plat closure of a braid \( \sigma_2^{\varepsilon_2} \ldots \sigma_{2b-2}^{\varepsilon_{2b-2}} \cdot \alpha \) with \( \alpha \in \mathcal{R}(2b, n) \) and \( \varepsilon_2, \ldots, \varepsilon_{2b-2} \in \{\pm 1\} \).

Let \( \pi_0 \) be the permutation of the braid \( \sigma_2^{\varepsilon_2} \ldots \sigma_{2b-2}^{\varepsilon_{2b-2}} \cdot \alpha \in B_{2b} \). The plat-operations \( \delta, \gamma, \delta' \) and \( \gamma' \) which we need for Lemma 2.2 are defined in Figure 2.4

**Lemma 2.2**

Let \( K \) be a knot which is given as a plat closure of a braid \( \alpha \in B_s \). Then there is a sequence of operations \( \delta, \delta', \gamma \) and \( \gamma' \) which transforms the plat \( \pi \) to a plat \( \beta \) with \( \pi(\beta) = \pi_0 \).

**Proof:** Using the operations of Figure 2, the permutation \( \pi_0 \) can be produced step by step. We start with string 1 and move its end-position to the position \( \pi_0(1) = 1 \). Then, travelling along the knot we can successively adjust the end-positions at the upper and lower plat-closure of the braid. The result is a plat with permutation \( \pi_0 \). \( \square \)

**Theorem 2.3**

Every knot with bridge number \( b \) has a checkerboard diagram of type \((2b, 2nb)\) for a suitable \( n \).

**Proof:** In Lemma 2.2 we succeeded to represent the knot \( K \) as a plat \( \bar{\alpha} \) with \( \pi(\alpha) = \pi_0 \). We consider the pure braid \( \beta = \sigma_2 \sigma_4 \ldots \sigma_{s-2} \cdot \alpha \). By Lemma 1.2 the braid \( \beta \) can be written as a rosette braid \( \beta' \) of type \((s, ns)\) for some natural number \( n \). Hence \( \sigma_2^{-1} \sigma_4^{-1} \ldots \sigma_{s-2}^{-1} \cdot \beta' \) is a checkerboard diagram for \( K \). If we
choose the plat representative of $K$ with $2b(K)$ strings, then the checkerboard diagram is of type $(2b(K), 2nb(K))$. \qed

3 Fourier knots

We call a series of the form

$$\sum_{i=1}^{n} \alpha_i \cos(2\pi m_i t + \varphi_i)$$

with $t, \alpha_i, \varphi_i \in \mathbb{R}, m_i \in \mathbb{N} \ (i = 1, \ldots, n)$ a finite Fourier series of length $n$.

Definition 3.1

A knot is a Fourier knot of type $(n_1, n_2, n_3)$ if it can be parametrized by coordinate functions $x_1, x_2, x_3 : [0, 1] \to \mathbb{R}$ which are finite Fourier series of length $n_1 \leq n_2 \leq n_3$.

Remark 3.2

The Fourier knots of type $(1, 1, 1)$ are the Lissajous knots. They were studied in the articles [1], [2] and [6]. Not all knots are Lissajous knots, but the next theorem proclaims that every knot is a Fourier knot of an especially simple type. Fourier knots were also defined in [3] and [8].

Theorem 3.3

Every knot $K$ is a Fourier knot of type $(1, 1, n_K)$ for some $n_K \in \mathbb{N}$.

Proof: We consider a Fourier knot of type $(1, 1, n)$ and its projection on the $x$-$y$-plane. By [6] the knot diagram is a checkerboard diagram. Conversely, by Theorem 2.3 every knot has a checkerboard diagram. The height-function in $z$-direction can be approximated by a finite Fourier series. \qed

Remark 3.4

The trefoil knots and the figure-eight knot are not Lissajous knots. They are Fourier knots of type $(1, 1, 2)$. The parametrizations of [1] (with a correction of misprints) are

$$x_1(t) = \cos(2t + 6), \quad x_2(t) = \cos(3t + 0.15), \quad x_3(t) = \cos(4t + 1) + \cos(5t)$$

for a trefoil and
\[ x_1(t) = \cos(2t + 0.8), \ x_2(t) = \cos(3t + 0.15), \ x_3(t) = \cos(4t + 1) + \cos(5t) \]
for the figure-eight knot. Here we use the interval \( t \in [0, 2\pi] \), in order to have the same parametrization as in [1].

**Definition 3.5**

If \( K \) is a knot, the *Fourier index* of \( K \) is the smallest number \( n \) for which \( K \) is a Fourier knot of type \((1, 1, n)\).

It is not known if there are knots with arbitrarily high Fourier index.

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