CANONICAL WEIERSTRASS REPRESENTATIONS FOR MINIMAL
SPACE-LIKE SURFACES IN $\mathbb{R}^4_1$

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Abstract. A space-like surface in Minkowski space-time is minimal if its mean curvature vector field is zero. Any minimal space-like surface of general type admits special isothermal parameters - canonical parameters. For any minimal surface of general type parameterized by canonical parameters we obtain Weierstrass representations - canonical Weierstrass representations via two holomorphic functions. We find the expressions of the Gauss curvature and the normal curvature of the surface with respect to this pair of holomorphic functions. We find the relation between two pairs of holomorphic functions generating one and the same minimal space-like surface of general type. The canonical Weierstrass formulas allow us to establish geometric correspondence between minimal space-like surfaces of general type and classes of pairs of holomorphic functions in the Gauss plane.

1. Introduction

A two-dimensional surface $\mathcal{M}$ in the four-dimensional Minkowski space-time $\mathbb{R}^4_1$ is said to be space-like if the induced metric on the tangential space at any point of $\mathcal{M}$ is positive definite. If $\mathcal{M}$ is a space-like surface in $\mathbb{R}^4_1$, we denote by $T_p(\mathcal{M})$ and $N_p(\mathcal{M})$ the tangential space and the normal space at a point $p \in \mathcal{M}$, respectively. The flat Levi-Civita connection on $\mathbb{R}^4_1$ is denoted by $\nabla$. Then the second fundamental tensor $\sigma$ of $\mathcal{M}$ is given by

$$\sigma(X, Y) = (\nabla_X Y)^{\perp}; \quad X, Y \text{ tangent vectors to } \mathcal{M} \text{ at a point } p \in \mathcal{M}.$$ 

The space-like surface $\mathcal{M}$ is minimal if its mean curvature vector field $H = \frac{1}{2} \text{trace } \sigma$ is zero, i.e. $H = 0$.

A general approach to Weierstrass representations of minimal space-like surfaces in $\mathbb{R}^4_1$ was given in [9] and [4].

In [1] minimal space-like surfaces in $\mathbb{R}^4_1$ were studied with respect to special isothermal parameters and a fundamental theorem of Bonnet type in terms of the Gauss curvature $K$ and the normal curvature $\kappa$ was proved.

The question when a complete minimal space-like surface is a plane was studied in [5].

In this paper we consider canonical Weierstrass representations for minimal space-like surfaces in $\mathbb{R}^4_1$.

A point $p \in \mathcal{M}$ is said to be degenerate, if the set $\{\sigma(X, Y); X \in T_p(\mathcal{M}), Y \in T_p(\mathcal{M})\}$, is contained in one of the two light-like one-dimensional subspaces of $N_p(\mathcal{M})$.

We call a minimal space-like surface, free of degenerate points, a minimal space-like surface of general type.

Let $(\mathcal{M}, x(u, v))$ be a space-like surface in $\mathbb{R}^4_1$, parameterized by isothermal coordinates $(u, v)$. In isothermal coordinates the space-like surface $\mathcal{M}$ is minimal if and only if the position vector function $x(u, v)$ is harmonic.

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We describe the properties of minimal surfaces in terms of the complex vector function \( \Phi(t) = x_u - ix_v, \; t = u + iv. \)

The standard Weierstrass representations for minimal space-like surfaces are in terms of three holomorphic functions.

Using special isothermal parameters (canonical parameters) on a minimal space-like surface of general type, we obtain canonical Weierstrass representations in terms of two holomorphic functions.

We call the isothermal parameters canonical of the first type (the second type) if \( \Phi' = \pm 1, \Phi'' = -1, \Phi'' = -1 \). The special parameters, used in [1] occur to be canonical of the first type.

In Theorem 6.6 we prove that:

Any minimal space-like surface in \( \mathbb{R}^4 \), free of degenerate points, admits locally canonical coordinates of both types.

In Theorem 9.3 we prove the following statement.

Any minimal space-like surface \( M \) of general type, parameterized by canonical coordinates of the first type, has the following Weierstrass representation:

\[
\begin{align*}
\phi_1 &= \frac{i}{2} \frac{g_1 g_2 + 1}{\sqrt{g_1' g_2}}, \\
\phi_2 &= \frac{1}{2} \frac{g_1 g_2 - 1}{\sqrt{g_1 g_2}'}, \\
\phi_3 &= \frac{1}{2} \frac{g_1 + g_2}{\sqrt{g_1 g_2}}, \\
\phi_4 &= \frac{1}{2} \frac{g_1 - g_2}{\sqrt{g_1 g_2}}.
\end{align*}
\]

(1.1)

where \((g_1, g_2)\) is a pair of holomorphic functions satisfying the conditions:

\[
(1.2) \quad g_1'g_2' \neq 0; \quad g_1g_2' \neq -1.
\]

Conversely, if \((g_1, g_2)\) is a pair of holomorphic functions satisfying the conditions (1.2), then the formulas (1.1) generate a minimal space-like surface of general type, parameterized by canonical coordinates of the first type.

We call the representation of \( \Phi \) in Theorem 9.3 canonical Weierstrass representation.

In terms of the above canonical representation the coefficients of the first fundamental form are given by

\[
E = G = \frac{|1 + g_1 g_2|^2}{4|g_1' g_2'|^2}.
\]

The Gauss curvature \( K \) and the curvature of the normal connection \( \kappa \) (the normal curvature) are given by

\[
K = \text{Re} \frac{-16|g_1' g_2| g_1' g_2'}{|1 + g_1 g_2|^2 (1 + g_1 g_2)^2}, \quad \kappa = \text{Im} \frac{-16|g_1' g_2| g_1' g_2'}{|1 + g_1 g_2|^2 (1 + g_1 g_2)^2}.
\]

Theorem 9.3 gives a representation of any minimal space-like surface of general type in terms of two holomorphic functions. The following question arises naturally:

If \((g_1, g_2)\) and \((\hat{g}_1, \hat{g}_2)\) are two pairs of holomorphic functions generating one and the same minimal space-like surface of general type, what is the relation between them?

We answer to this question in Theorem 12.2.
Let $\hat{\mathcal{M}}, \hat{x}$ and $(\mathcal{M}, x)$ be two minimal space-like surfaces of general type, given by the canonical Weierstrass representation of the type (1.1). The following conditions are equivalent:

1. $(\hat{\mathcal{M}}, \hat{x})$ and $(\mathcal{M}, x)$ are related by a transformation in $\mathbb{R}_1^4$ of the type: $\hat{x}(t) = A x(t) + b$, where $A \in \text{SO}(3, 1, \mathbb{R})$ and $b \in \mathbb{R}_1^4$.

2. The functions in the Weierstrass representations of $(\hat{\mathcal{M}}, \hat{x})$ and $(\mathcal{M}, x)$ are related by the following equalities:

$$
\begin{align*}
\hat{g}_1 &= \frac{ag_1 + b}{cg_1 + d}; \\
\hat{g}_2 &= \frac{\bar{d}g_2 - \bar{c}}{-bg_2 + \bar{a}},
\end{align*}
$$

$a, b, c, d \in \mathbb{C}$, $ad - bc \neq 0$.

2. Preliminaries

Let $\mathbb{R}_1^4$ denote the standard Minkowski space-time. This is a four-dimensional space endowed with the indefinite dot product:

$$
(2.1) \quad a \cdot b = a_1 b_1 + a_2 b_2 + a_3 b_3 - a_4 b_4.
$$

If $\mathcal{M}$ is a two-dimensional manifold and $x : \mathcal{M} \to \mathbb{R}_1^4$ is an immersion of $\mathcal{M}$ into $\mathbb{R}_1^4$, then we say that $\mathcal{M}$ is a (regular) surface in $\mathbb{R}_1^4$. We denote by $T_p(\mathcal{M})$ the tangential space to $\mathcal{M}$ at a point $p$ identifying $T_p(\mathcal{M})$ with the corresponding plane in $\mathbb{R}_1^4$. $N_p(\mathcal{M})$ will stand for the normal space to $\mathcal{M}$ at the point $p$, which is the orthogonal complement to $T_p(\mathcal{M})$ in $\mathbb{R}_1^4$. If the induced metric onto $T_p(\mathcal{M})$ is positive definite, the surface $\mathcal{M}$ is said to be space-like. Then the induced metric onto the normal space $N_p(\mathcal{M})$ is of signature $(1,1)$. The surface $\mathcal{M}$ with the induced metric becomes a two-dimensional Riemannian space.

Let $E = x_u^2$, $F = x_u \cdot x_v$ and $G = x_v^2$ be the coefficients of the first fundamental form on $\mathcal{M}$. The surface $\mathcal{M}$ admits locally around any point $p \in \mathcal{M}$ isothermal coordinates (parameters) $(u, v)$, which means that $E = G$ and $F = 0$. Together with the real coordinates $(u, v) \in \mathcal{D}$ we also consider the complex coordinate $t = u + iv$, identifying $\mathbb{R}^2$ with the complex plane $\mathbb{C}$. Thus all functions defined around $p$ can be considered as functions of the complex variable $t$. Throughout this paper we consider isothermal coordinates $(u, v)$ on $\mathcal{M}$.

We also consider the complexified tangential space $T_{p, \mathcal{C}}(\mathcal{M})$ and the complexified normal space $N_{p, \mathcal{C}}(\mathcal{M})$ at a point $p$ in $\mathcal{M}$ as the corresponding 2-planes in $\mathbb{C}^4$.

If $a$ and $b$ are two vectors in $\mathbb{C}^4$, then by $a \cdot b$ (or $ab$) we denote the bilinear product in $\mathbb{C}^4$, which is the natural extension of the product in $\mathbb{R}_1^4$ given by (2.1). Together with the bilinear product in $\mathbb{C}^4$ we also consider the indefinite Hermitian product of $a$ and $b$, given by

$$
a \cdot \bar{b} = a_1 \bar{b}_1 + a_2 \bar{b}_2 + a_3 \bar{b}_3 - a_4 \bar{b}_4.
$$

The square of $a$ with respect to the bilinear product is

$$
a^2 = a \cdot a = a_1^2 + a_2^2 + a_3^2 - a_4^2
$$

and the norm of $a$ with respect to the Hermitian product is

$$
\| a \|^2 = a \cdot \bar{a} = |a_1|^2 + |a_2|^2 + |a_3|^2 - |a_4|^2.
$$

The spaces $T_{p, \mathcal{C}}(\mathcal{M})$ and $N_{p, \mathcal{C}}(\mathcal{M})$ are closed with respect to the complex conjugation and are orthogonal with respect to both: bilinear and Hermitian product. Therefore we have the following orthogonal decomposition

$$
\mathbb{C}^4 = T_{p, \mathcal{C}}(\mathcal{M}) \oplus N_{p, \mathcal{C}}(\mathcal{M}).
$$
For a given vector \( a \in \mathbb{C}^4 \) we denote by \( a^\top \) and \( a^\perp \) the orthogonal projections of \( a \) into \( T_{p,C}(\mathcal{M}) \) and \( N_{p,C}(\mathcal{M}) \), respectively, i.e.

\[
a = a^\top + a^\perp.
\]

The above decomposition does not depend on the bilinear or the Hermitian dot product in \( \mathbb{C}^4 \).

The second fundamental form on \( \mathcal{M} \) is denoted by \( \sigma \). By definition we have:

\[
\sigma(X,Y) = (\nabla_X Y)^\perp,
\]

where \( X, Y \in T(M) \), and \( \nabla \) is the canonical flat connection in \( \mathbb{R}^4 \).

Let \( X_1 \) and \( X_2 \) denote the unit tangent vectors to \( \mathcal{M} \) at a point \( p \) having the same directions as the coordinate vectors \( x_u \) and \( x_v \), respectively, i.e.

\[
X_1 = \frac{x_u}{\|x_u\|} = \frac{x_u}{\sqrt{E}}; \quad X_2 = \frac{x_v}{\|x_v\|} = \frac{x_v}{\sqrt{G}} = \frac{x_v}{\sqrt{E}}.
\]

The mean curvature \( H \) of \( \mathcal{M} \) is the vector function

\[
H = \frac{1}{2} \text{trace} \sigma = \frac{1}{2}(\sigma(X_1, X_1) + \sigma(X_2, X_2)).
\]

A space-like surface \( \mathcal{M} \) in \( \mathbb{R}^4 \) is said to be minimal if \( H = 0 \) at any point of \( \mathcal{M} \).

3. THE COMPLEX FUNCTION \( \Phi(t) \).

Let \( \mathcal{M} \) be a space-like surface in \( \mathbb{R}^4 \), parameterized by isothermal coordinates. The complex-valued vector function \( \Phi(t) \) on \( \mathcal{M} \) with values in \( \mathbb{C}^4 \) is defined by

\[
(3.1) \quad \Phi(t) = 2\frac{\partial x}{\partial t} = x_u - ix_v.
\]

The defining equality (3.1) implies that:

\[
\Phi^2 = (x_u - ix_v)^2 = x_u^2 - x_v^2 - 2x_u x_v i.
\]

Then the following equalities are equivalent:

\[
\Phi^2 = 0 \iff x_u^2 - x_v^2 = 0 \iff \frac{E}{F} = \frac{x_u^2}{x_v^2} = G.
\]

Hence, the parameters \((u, v)\) are isothermal if and only if:

\[
(3.2) \quad \Phi^2 = 0.
\]

For the norm of \( \Phi \) we find

\[
\|\Phi\|^2 = \Phi \overline{\Phi} = x_u^2 + x_v^2 = 2E = 2G.
\]

Therefore

\[
(3.3) \quad E = G = \frac{1}{2} \|\Phi\|^2, \quad F = 0
\]

and

\[
(3.4) \quad I = \frac{1}{2} \|\Phi\|^2 (du^2 + dv^2) = \frac{1}{2} \|\Phi\|^2 |dt|^2.
\]

From the above it follows that \( \Phi \) satisfies the condition:

\[
(3.5) \quad \|\Phi\|^2 > 0.
\]
Differentiating equality \((3.1)\) and using that \(\frac{\partial}{\partial t} \frac{\partial}{\partial \bar{t}} = \frac{1}{4} \Delta\), we find

\[(3.6)\]

\[
\frac{\partial \Phi}{\partial \bar{t}} = \frac{\partial}{\partial \bar{t}} \left( 2 \frac{\partial \Phi}{\partial t} \right) = \frac{1}{2} \Delta x,
\]

where \(\Delta\) denotes the Laplace operator.

The last formula implies that \(\frac{\partial \Phi}{\partial \bar{t}}\) is a real vector function, i.e.

\[(3.7)\]

\[
\frac{\partial \Phi}{\partial \bar{t}} = \frac{\partial \bar{\Phi}}{\partial t}.
\]

Thus, any space-like surface in \(\mathbb{R}^4\) parameterized by isothermal coordinates, determines a function \(\Phi\) given by \((3.1)\), which satisfies the conditions:

\[(3.8)\]

\[
\Phi^2 = 0, \quad \|\Phi\|^2 > 0, \quad \frac{\partial \Phi}{\partial t} = \frac{\partial \bar{\Phi}}{\partial t}.
\]

Conversely, any function \(\Phi\) satisfying these three conditions determines locally a space-like surface in isothermal coordinates up to a translation.

The last assertion follows immediately from the fact that \((3.7)\) is the integrability condition for the system

\[(3.9)\]

\[
x_u = \text{Re}(\Phi), \quad x_v = -\text{Im}(\Phi).
\]

Next we express the vectors \(x_u, x_v\) and the second fundamental form \(\sigma\) of \(M\) by means of \(\Phi\).

Taking into account \((3.1)\) we have:

\[(3.10)\]

\[
x_u = \text{Re}(\Phi) = \frac{1}{2}(\Phi + \bar{\Phi}), \quad x_v = -\text{Im}(\Phi) = \frac{i}{2}(\Phi - \bar{\Phi}).
\]

Equality \((3.6)\) implies that:

\[
\left( \frac{\partial \Phi}{\partial t} \right) \perp = \left( \frac{1}{2} \Delta x \right) \perp = \frac{1}{2} (x_u \perp + x_v \perp) = \frac{1}{2} (\nabla_{x_u} \perp x_u + \nabla_{x_v} \perp x_v) = \frac{1}{2} (\sigma(x_u, x_u) + \sigma(x_v, x_v)).
\]

Differentiating \((3.1)\) with respect to \(t\) we find

\[(3.11)\]

\[
\frac{\partial \Phi}{\partial t} = \frac{1}{2} (x_u \perp - x_v \perp) - ix_{uv}.
\]

and

\[(3.12)\]

\[
\left( \frac{\partial \Phi}{\partial t} \right) \perp = \frac{1}{2} (\sigma(x_u, x_u) - \sigma(x_v, x_v)) - i\sigma(x_u, x_v).
\]

Therefore

\[
\sigma(x_u, x_u) = \text{Re} \left( \frac{\partial \Phi}{\partial t} \right) \perp + \text{Re} \left( \frac{\partial \Phi}{\partial t} \right) \perp;
\]

\[(3.13)\]

\[
\sigma(x_v, x_u) = \text{Re} \left( \frac{\partial \Phi}{\partial t} \right) \perp - \text{Re} \left( \frac{\partial \Phi}{\partial t} \right) \perp;
\]

\[
\sigma(x_u, x_v) = -\text{Im} \left( \frac{\partial \Phi}{\partial t} \right) \perp.
\]
Finally we give transformation formulas for the function $\Phi$ under a change of the isothermal coordinates and under a motion in $\mathbb{R}^4_1$.

Let us consider the change of the isothermal coordinates given by $t = t(s)$. Since the transformation of the isothermal coordinates is conformal in $\mathbb{C}$, then the function $t(s)$ is either holomorphic or antiholomorphic. Denote by $\tilde{\Phi}(s)$ the corresponding function in the new coordinates.

First, let us consider the holomorphic case. Taking into account (3.1) we have:
\[
\tilde{\Phi}(s) = 2 \frac{\partial x}{\partial s} = 2 \frac{\partial x}{\partial t} \frac{\partial t}{\partial s} + 2 \frac{\partial x}{\partial \bar{t}} \frac{\partial \bar{t}}{\partial s} = 2 \frac{\partial x}{\partial t} \frac{\partial t}{\partial s}.
\]
Therefore, if the change $t = t(s)$ is holomorphic, then
\[
(3.14) \quad \tilde{\Phi}(s) = \Phi(t(s)) \frac{\partial t}{\partial s}.
\]
In the antiholomorphic case we have similarly
\[
\tilde{\Phi}(s) = 2 \frac{\partial x}{\partial s} = 2 \frac{\partial x}{\partial t} \frac{\partial t}{\partial s} + 2 \frac{\partial x}{\partial \bar{t}} \frac{\partial \bar{t}}{\partial s} = 2 \frac{\partial x}{\partial \bar{t}} \frac{\partial \bar{t}}{\partial s},
\]
i.e.
\[
(3.15) \quad \tilde{\Phi}(s) = \bar{\Phi}(t(s)) \frac{\partial \bar{t}}{\partial s}.
\]

Now let $(\mathcal{M}, x)$ and $(\tilde{\mathcal{M}}, \tilde{x})$ be two surfaces in $\mathbb{R}^4_1$, parameterized by isothermal coordinates $t = u + iv$ in one and the same domain $\mathcal{D} \subset \mathbb{C}$. Suppose that $(\tilde{\mathcal{M}}, \tilde{x})$ is obtained by $(\mathcal{M}, x)$ by means of a motion in $\mathbb{R}^4_1$:
\[
(3.16) \quad \dot{x}(t) = A x(t) + b; \quad A \in \text{O}(3, 1, \mathbb{R}), \ b \in \mathbb{R}^4_1.
\]
Differentiating (3.16) we find the relation between the corresponding functions $\Phi$ and $\tilde{\Phi}$:
\[
(3.17) \quad \dot{\Phi}(t) = A \Phi(t); \quad A \in \text{O}(3, 1, \mathbb{R}).
\]
Conversely, if $\Phi$ and $\tilde{\Phi}$ are connected by (3.17), then we have $\dot{x}_u = A x_u$ and $\dot{x}_v = A x_v$, which implies (3.16). Hence, the relations (3.16) and (3.17) are equivalent.

4. Characterizing of minimal space-like surfaces in $\mathbb{R}^4_1$ by means of $\Phi$

Let $\mathcal{M}$ be a surface in $\mathbb{R}^4_1$, parameterized by isothermal coordinates and let $\Phi$ be the function defined by (3.1).

Differentiating (3.2), we find:
\[
(4.1) \quad \Phi \cdot \frac{\partial \Phi}{\partial \bar{t}} = 0.
\]
In view of (3.7) the function $\frac{\partial \Phi}{\partial \bar{t}}$ is real and (4.1) after a complex conjugation implies that:
\[
(4.2) \quad \bar{\Phi} \cdot \frac{\partial \Phi}{\partial \bar{t}} = 0.
\]
Since $\Phi$ and $\bar{\Phi}$ form a basis for $T_C(M)$, then equalities (4.1) and (4.2) mean that $\frac{\partial \Phi}{\partial \bar{t}}$ is orthogonal to $T(\mathcal{M})$ and therefore
\[
(4.3) \quad \frac{\partial \Phi}{\partial \bar{t}} \in N(\mathcal{M}).
\]
In view of (3.6) we find successively:

\[
\frac{\partial \Phi}{\partial \bar{t}} = \left( \frac{\partial \Phi}{\partial \bar{t}} \right)^\perp = \frac{1}{2} (\Delta x)^\perp = \frac{1}{2} (\nabla_{x_u} x_u + \nabla_{x_v} x_v)^\perp = \frac{1}{2} (\sigma(x_u, x_u) + \sigma(x_v, x_v)) = E \frac{1}{2} (\sigma(X_1, X_1) + \sigma(X_2, X_2)) = EH .
\]

Finally we have:

(4.4)

\[
\frac{\partial \Phi}{\partial \bar{t}} = \frac{1}{2} \Delta x = EH .
\]

Equality (4.4) implies immediately the following statement.

**Theorem 4.1.** Let \((M, x)\) be a space-like surface in \(\mathbb{R}^4\) parameterized by isothermal coordinates \((u, v) \in D\) and \(\Phi(t)\) be the complex-valued vector function in \(D\) defined by:

\[
\Phi(t) = 2 \frac{\partial x}{\partial t} = x_u - ix_v, \quad t = u + iv,
\]

Then the following conditions are equivalent:

1. The function \(\Phi(t)\) is holomorphic \((\frac{\partial \Phi}{\partial \bar{t}} = 0)\);
2. The function \(x(u, v)\) is harmonic \((\Delta x = 0)\);
3. \((M, x)\) is minimal space-like surface in \(\mathbb{R}^4\) \((H = 0)\).

Let \((M, x)\) be a minimal space-like surface. Then the harmonic conjugate function \(y\) to the function \(x\) is determined by the Cauchy-Riemann equations

\[
y_u = -x_v; \quad y_v = x_u .
\]

Let us introduce the function \(\Psi\) by the equality

\[
\Psi = x + iy.
\]

The function \(\Psi\) is holomorphic and \(x, \Phi\) are expressed by \(\Psi\) in the following way:

\[
x = \text{Re} \Psi; \quad \Phi = x_u - ix_v = x_u + iy_u = \frac{\partial \Psi}{\partial u} = \Psi'.
\]

Since \(\frac{\partial \Phi}{\partial t} = 0\), then \(\frac{\partial \Phi}{\partial \bar{t}} = \Phi'\) and

(4.5)

\[
\sigma(X_2, X_2) = -\sigma(X_1, X_1).
\]

Therefore

\[
\sigma(x_v, x_v) = E \sigma(X_2, X_2) = -E \sigma(X_1, X_1) = -\sigma(x_u, x_u).
\]

Then formulas (3.11) and (3.12) get the following form:

(4.6)

\[
\Phi' = \frac{\partial \Phi}{\partial u} = x_{uu} - ix_{uv}; \quad \Phi'^\perp = x_{uu}^\perp - ix_{uv}^\perp = \sigma(x_u, x_u) - i\sigma(x_u, x_v).
\]

Formulas (3.13) become correspondingly

(4.7)

\[
\sigma(x_u, x_u) = \text{Re}(\Phi'^\perp) = \frac{1}{2} (\Phi'^\perp + \Phi'^\perp) = \frac{1}{2} (\Phi'^\perp + \overline{\Phi'^\perp})
\]

\[
\sigma(x_v, x_v) = -\text{Re}(\Phi'^\perp) = -\frac{1}{2} (\Phi'^\perp + \overline{\Phi'^\perp}) = -\frac{1}{2} (\Phi'^\perp + \overline{\Phi'^\perp})
\]

\[
\sigma(x_u, x_v) = -\text{Im}(\Phi'^\perp) = \frac{-1}{2i} (\Phi'^\perp - \overline{\Phi'^\perp}) = \frac{i}{2} (\Phi'^\perp - \overline{\Phi'^\perp}).
\]
5. Expressions for $K$ and $\kappa$ of a minimal space-like surface by means of $\Phi$

Let $(\mathcal{M}, x)$ be a minimal space-like surface in $\mathbb{R}^4$ parameterized by isothermal parameters. Choose a pair $n_1$ and $n_2$ of orthonormal vector functions in $N(\mathcal{M})$ of $\mathcal{M}$, such that $n_1^2 = 1$, $n_2^2 = -1$ and the quadruple $(X_1, X_2, n_1, n_2)$ is right oriented in $\mathbb{R}^4$.

For a given normal vector $n$ we denote by $A_n$ the Weingarten operator in $T(\mathcal{M})$. This operator and $\sigma$ are related by the equality $A_n X \cdot Y = \sigma(X, Y) \cdot n$. The condition $H = 0$ implies that for any $n$ trace $A_n = 0$. Then the operators $A_{n_1}$ and $A_{n_2}$ have the following representation

$$(5.1) \quad A_{n_1} = \begin{pmatrix} \nu & \lambda \\ \lambda & -\nu \end{pmatrix}; \quad A_{n_2} = \begin{pmatrix} \rho & \mu \\ \mu & -\rho \end{pmatrix}$$

and the components of $\sigma$ are as follows:

$$(5.2) \quad \sigma(X_1, X_1) = (\sigma(X_1, X_1) \cdot n_1)n_1 - (\sigma(X_1, X_1) \cdot n_2)n_2 = \nu n_1 - \rho n_2$$

Denote by $R$ the curvature tensor of $\mathcal{M}$. Then the Gauss equation and $(3.3)$ give:

$$(5.3) \quad K = R(X_1, X_2)X_2 \cdot X_1 = \sigma(X_1, X_1)\sigma(X_2, X_2) - \sigma^2(X_1, X_2) = -\sigma^2(X_1, X_1) - \sigma^2(X_1, X_2).$$

Now $(5.2)$ and $(5.3)$ imply

$$(5.4) \quad K = -(\nu^2 - \rho^2) - (\lambda^2 - \mu^2) = \det(A_{n_1}) - \det(A_{n_2}).$$

On the other hand we get from $(4.6)$:

$$(5.5) \quad \Phi' = \sigma(\sqrt{E}X_1, \sqrt{E}X_1) - i\sigma(\sqrt{E}X_1, \sqrt{E}X_2) = E(\sigma(X_1, X_1) - i\sigma(X_1, X_2)).$$

Calculating the norm $\|\Phi'\|$, we find

$$\|\Phi'\|^2 = \Phi' \cdot \Phi' = E(\sigma(X_1, X_1) - i\sigma(X_1, X_2))E(\sigma(X_1, X_1) + i\sigma(X_1, X_2)) = E^2(\sigma^2(X_1, X_1) + \sigma^2(X_1, X_2)).$$

Taking into account the last equality and $(3.3)$ we have:

$$(5.6) \quad \sigma^2(X_1, X_1) + \sigma^2(X_1, X_2) = \frac{\|\Phi'\|^2}{E^2} = \frac{4\|\Phi'\|^2}{\|\Phi\|^4}.$$

Now $(5.6)$ and $(5.3)$ imply that

$$(5.7) \quad K = -4\|\Phi'\|^2 / \|\Phi\|^4.$$

In the last formula we can represent $\|\Phi'\|^2$ in a different way. Note that $\Phi^2 = 0$ means that $\Phi$ and $\Phi'$ are orthogonal with respect to the Hermitian dot product in $\mathbb{C}^4$. In view of $(3.1)$ and $(3.10)$ it follows that they form an orthogonal basis of the complexified tangential plane of $\mathcal{M}$. Therefore the tangential projection of $\Phi'$ is as follows:

$$\Phi'^T = \frac{\Phi'^T \cdot \Phi}{\|\Phi\|^2} \Phi + \frac{\Phi'^T \cdot \Phi}{\|\Phi\|^2} \Phi = \frac{\Phi' \cdot \Phi}{\|\Phi\|^2} \Phi + \frac{\Phi' \cdot \Phi}{\|\Phi\|^2} \Phi.$$
Differentiating $\Phi^2 = 0$, we find $\Phi \cdot \Phi' = 0$. Thus the projection of $\Phi'$ has the form:

$$\Phi'^\top = \frac{\Phi'}{||\Phi||^2} \Phi; \quad \Phi'\perp = \Phi' - \Phi'^\top = \Phi' - \frac{\Phi'}{||\Phi||^2} \Phi. \quad (5.8)$$

Using the second equality of (5.8) by means of complex conjugation we get:

$$||\Phi'^\perp||^2 = \Phi' \cdot \Phi'\perp = \left(\Phi' - \frac{\Phi'}{||\Phi||^2} \Phi\right) \left(\Phi' - \frac{\Phi'}{||\Phi||^2} \Phi\right)$$

$$= \Phi' \cdot \Phi' - \frac{\Phi'}{||\Phi||^2} \Phi \cdot \Phi' - \frac{\Phi'}{||\Phi||^2} \Phi \cdot \Phi' + \frac{||\Phi||^2}{||\Phi||^4} \Phi \cdot \Phi'$$

$$= ||\Phi'||^2 - \frac{||\Phi'||^2}{||\Phi||^2} - \frac{||\Phi'||^2}{||\Phi||^2} + \frac{||\Phi'||^2}{||\Phi||^4} = ||\Phi'||^2 - \frac{||\Phi'||^2}{||\Phi||^2}$$

Since the norm of the bi-vector $\Phi \wedge \Phi'$ is given by:

$$||\Phi \wedge \Phi'||^2 = ||\Phi||^2 ||\Phi'||^2 - |\Phi \cdot \Phi'|^2,$$

then we have

$$||\Phi'^\perp||^2 = \frac{||\Phi||^2 ||\Phi'||^2 - |\Phi \cdot \Phi'|^2}{||\Phi||^2} = \frac{||\Phi \wedge \Phi'||^2}{||\Phi||^2}$$

and

$$K = -\frac{4||\Phi'^\perp||^2}{||\Phi||^4} = -\frac{4||\Phi \wedge \Phi'||^2}{||\Phi||^6}. \quad (5.9)$$

In order to obtain formulas for the normal curvature $\nu$, let us denote by $R^N$ the curvature tensor of the normal connection of $\mathcal{M}$. The Ricci equation and $(5.1)$ imply

$$\nu = R^N(X_1, X_2, n_1, n_2) = R^N(X_1, X_2)n_2 \cdot n_1$$

$$= [A_{n_2}, A_{n_1}]X_1 \cdot X_2 = A_{n_1} X_1 \cdot A_{n_2} X_2 - A_{n_2} X_1 \cdot A_{n_1} X_2$$

$$= (\nu X_1 + \lambda X_2) \cdot (\mu X_1 - \rho X_2) - (\rho X_1 + \mu X_2) \cdot (\lambda X_1 - \nu X_2)$$

$$= \nu \mu - \nu \rho + \lambda \mu - \lambda \rho - (\rho \lambda - \rho \nu + \mu \lambda - \mu \nu)$$

$$= 2\nu \mu - 2\rho \lambda. \quad (5.10)$$

We denote by $\text{det}(a, b, c, d)$ the determinant of the vectors $a, b, c$ and $d$, with respect to the standard basis in $\mathbb{C}^4$. Taking into account $(5.2)$, we have

$$\text{det}(x_u, x_v, \sigma(x_u, x_u), \sigma(x_u, x_u)) = \text{det}(\sqrt{E}X_1, \sqrt{E}X_2, \sigma(\sqrt{E}X_1, \sqrt{E}X_1), \sigma(\sqrt{E}X_1, \sqrt{E}X_2))$$

$$= E^3 \text{det}(X_1, X_2, \sigma(X_1, X_1), \sigma(X_1, X_2))$$

$$= E^3 \text{det}(X_1, X_2, \nu n_1 - \rho n_2, \lambda n_1 - \mu n_2)$$

$$= -E^3 \text{det}(X_1, X_2, \nu n_1, \mu n_2) - E^3 \text{det}(X_1, X_2, \rho n_2, \lambda n_1)$$

$$= E^3(-\nu \mu + \rho \lambda) \text{det}(X_1, X_2, n_1, n_2) = E^3(-\nu \mu + \rho \lambda).$$

From the last equation it follows that

$$-\nu \mu + \rho \lambda = \frac{1}{E^3} \text{det}(x_u, x_v, \sigma(x_u, x_u), \sigma(x_u, x_v)). \quad (5.11)$$
Replacing $x_u$ and $x_v$ by (3.10) we find

\[
\begin{align*}
\det(x_u, x_v, \sigma(x_u, x_u), \sigma(x_u, x_v)) &= \frac{i}{4} \det(\Phi + \bar{\Phi}, \Phi - \bar{\Phi}, \sigma(x_u, x_u), \sigma(x_u, x_v)) \\
&= \frac{i}{4} \det(\Phi, \bar{\Phi}, \sigma(x_u, x_u), \sigma(x_u, x_v)) + \frac{i}{4} \det(\Phi, \bar{\Phi}, \sigma(x_u, x_u), \sigma(x_u, x_v)) \\
&= -\frac{i}{2} \det(\Phi, \bar{\Phi}, \sigma(x_u, x_u), \sigma(x_u, x_v)).
\end{align*}
\]

Similarly, using (4.7), we get:

\[
\begin{align*}
\det(\Phi, \bar{\Phi}, \sigma(x_u, x_u), \sigma(x_u, x_v)) &= -\frac{i}{2} \det(\Phi, \bar{\Phi}, \Phi' \perp, \Phi' \perp).
\end{align*}
\]

Now (5.13) and (5.12) give

\[
\begin{align*}
\det(x_u, x_v, \sigma(x_u, x_u), \sigma(x_u, x_v)) &= -\frac{i}{4} \det(\Phi, \bar{\Phi}, \Phi' \perp, \Phi' \perp).
\end{align*}
\]

In view of (5.10), (5.11) and (5.14) we have

\[
\kappa = 2\nu\mu - 2\rho\lambda = -\frac{2}{E^3} \det(x_u, x_v, \sigma(x_u, x_u), \sigma(x_u, x_v)) = \frac{1}{2E^3} \det(\Phi, \bar{\Phi}, \Phi', \bar{\Phi'}).
\]

Using (3.3) we find:

\[
\kappa = \frac{4}{\|\Phi\|} \det(\Phi, \Phi', \bar{\Phi'}).
\]

For any minimal space-like surface $(\mathcal{M}, x)$ in $\mathbb{R}^4$, parameterized by isothermal coordinates the Gauss curvature $K$ and the normal curvature $\kappa$ are given by the formulas:

\[
\begin{align*}
K &= -\nu^2 - \lambda^2 + \rho^2 + \mu^2, \quad \kappa = 2\nu\mu - 2\rho\lambda; \\
K &= \frac{-4\|\Phi' \perp\|^2}{\|\Phi\|^4} = -\frac{4\|\Phi \wedge \Phi'\|^2}{\|\Phi\|^6}, \quad \kappa = \frac{4}{\|\Phi\|^6} \det(\Phi, \Phi', \bar{\Phi'}).
\end{align*}
\]

6. Existence of canonical coordinates on a minimal space-like surface

Let $\mathcal{M}$ be a minimal space-like surface in $\mathbb{R}^4$.

**Definition 6.1.** A point $p \in \mathcal{M}$ is said to be degenerate if the set

\[\{\sigma(X, Y); \ X \in T_p(\mathcal{M}), Y \in T_p(\mathcal{M})\}\]

is contained into one of the light-like one-dimensional subspaces of $N_p(\mathcal{M})$.

Let $(\mathcal{M}, x = \text{Re } \Psi)$ be a minimal space-like surface in $\mathbb{R}^4$ parameterized by isothermal coordinates $(u, v)$.

**Theorem 6.2.** A point $p \in \mathcal{M}$ is degenerate if and only if $\Phi' \perp^2 = 0$. 
Proof. Let us consider again equality (5.3). Squaring both sides of the equality, we find

\[
\Phi'^\perp = E^2(\sigma^2(X_1, X_1) - i2\sigma(X_1, X_1)\sigma(X_1, X_2) - \sigma^2(X_1, X_2))
\]
\[
= E^2(\sigma^2(X_1, X_1) - \sigma^2(X_1, X_2)) - i2E^2\sigma(X_1, X_1)\sigma(X_1, X_2).
\]

The last equality implies the following equivalence:

\[
\sigma(X_1, X_1) \perp \sigma(X_1, X_2) \quad \Leftrightarrow \quad \Phi'^\perp = 0.
\]

Assuming that the point into consideration is degenerate, then it follows that \(\sigma^2(X_1, X_1) = 0\), \(\sigma^2(X_1, X_2) = 0\) and \(\sigma(X_1, X_1)\sigma(X_1, X_2) = 0\). Now (6.2) implies that \(\Phi'^\perp = 0\).

Let \(\Phi'^\perp = 0\). We have to prove that the vectors \(\sigma(X_1, X_1)\) and \(\sigma(X_1, X_2)\) lie in one and the same light-like one-dimensional subspace of \(\mathcal{N}_p(\mathcal{M})\). If we assume that \(\sigma^2(X_1, X_1) > 0\), then it follows that \(\sigma^2(X_1, X_2) > 0\) and \(\sigma(X_1, X_1) \perp \sigma(X_1, X_2)\), which is a contradiction.

Similarly, assuming that \(\sigma^2(X_1, X_1) < 0\), we obtain the metric on \(\mathcal{N}_p(\mathcal{M})\) is negative definite, which is a contradiction.

Thus \(\sigma^2(X_1, X_1) = 0\), \(\sigma^2(X_1, X_2) = 0\) and \(\sigma(X_1, X_1) \perp \sigma(X_1, X_2)\). Hence the vectors \(\sigma(X_1, X_1)\) and \(\sigma(X_1, X_2)\) are light-like and lie in one and the same one-dimensional subspace of \(\mathcal{N}_p(\mathcal{M})\). \(\square\)

Next we prove that \(\Phi'^\perp\) is a holomorphic function of \(t\). In general it doesn’t follow that the projection \(\Phi'^\perp\) is a holomorphic function, but we shall prove that \(\Phi'^\perp = \Phi^2\).

In order to prove the last equality, we square the second equality in (5.8) and get:

\[
\Phi'^\perp = \Phi^2 - 2\Phi\Phi'\cdot\Phi + \left(\frac{\Phi'\cdot\Phi}{\|\Phi\|^2}\right)^2 \Phi^2.
\]

Taking into account equalities \(\Phi^2 = 0\) and \(\Phi \cdot \Phi' = 0\), we find:

\[
\Phi'^\perp = \Phi^2.
\]

Thus we obtained that any degenerate point of \(\mathcal{M}\) is a zero of the holomorphic function \(\Phi^2\). This implies immediately the following characterization of the set of degenerate points of a minimal space-like surface:

**Theorem 6.3.** If \(\mathcal{M}\) is a connected minimal space-like surface in \(\mathbb{R}_t^4\), then: either it consists of degenerate points or the set of the degenerate points is countable without any limit points.

Further in this section we consider minimal space-like surfaces in \(\mathbb{R}_t^4\) without degenerate points.

We give the following definitions:

**Definition 6.4.** The isothermal coordinates \((u, v)\) on a minimal space-like surface are said to be canonical of the first type if

\[
\sigma(X_1, X_1) \perp \sigma(X_1, X_2),
\]
\[
E^2(\sigma^2(X_1, X_1) - \sigma^2(X_1, X_2)) = 1.
\]

Because of (6.1), the isothermal parameters \((u, v)\) are canonical of the first type if and only if

\[
\Phi^2 = \Phi'^\perp = 1.
\]
Definition 6.5. The isothermal coordinates \((u, v)\) on a minimal space-like surface are said to be canonical of the second type if
\[
\sigma(X_1, X_1) \perp \sigma(X_1, X_2), \\
E^2(\sigma^2(X_1, X_1) - \sigma^2(X_1, X_2)) = -1.
\]

The isothermal coordinates \((u, v)\) are canonical of the second type if and only if:
\[
\Phi'^2 = \Phi'^{-2} = -1.
\]

Theorem 6.6. Any minimal space-like surface in \(\mathbb{R}_4^1\), free of degenerate points, admits locally canonical coordinates of both types.

Proof. For arbitrary isothermal coordinates \((u, v)\) on the surface, denote \(t = u + vi\). Let us consider the change \(t = \tilde{t}(\hat{t})\) of the complex variable \(t\) by the new complex variable \(\hat{t}\). We shall find the conditions under which the new variable determines canonical coordinates. First, the new coordinates have to be isothermal, i.e. \(t = \tilde{t}(\hat{t})\) is a conformal map in \(\mathbb{C}\). Therefore the function \(t(\hat{t})\) is either holomorphic or antiholomorphic. The case of an antiholomorphic function is reduced to the case of a holomorphic function by means of the additional change \(\hat{t} = \bar{s}\).

It is enough to consider only the case of a holomorphic function \(t(\hat{t})\). Let \(\tilde{\Psi}\) be the holomorphic function representing \(\mathcal{M}\) with respect to the new coordinates and \(\tilde{\Phi}\) be its derivative. Then we have
\[
\tilde{\Phi} = \tilde{\Psi}' = \tilde{\Psi}'t' = \tilde{\Phi}t'.
\]
The derivative of \(\tilde{\Phi}\) with respect to \(\hat{t}\) is given by \(\tilde{\Phi}' = \tilde{\Phi}_t t'^2 + \tilde{\Phi} t''\). Since \(\tilde{\Phi}\) is tangent to the surface \(\mathcal{M}\), then \(\tilde{\Phi}^\perp = 0\) and consequently
\[
\tilde{\Phi}^\perp = (\tilde{\Phi}_t t'^2 + \tilde{\Phi} t''^2) = \tilde{\Phi}^\perp t'^2;
\]
\[
\tilde{\Phi}^\perp = \tilde{\Phi}^\perp t'^4.
\]
According to (6.4) and (6.6) the new complex variable \(\hat{t}\) determines canonical coordinates if \(\tilde{\Phi}^\perp = \pm 1\). If \(\tilde{\Phi}^\perp = 0\), then by virtue of (6.9) it follows that \(\tilde{\Phi}^\perp = 0\). The last condition means that the point is degenerate, which is impossible. Hence \(\tilde{\Phi}^\perp \neq 0\). Then the function \(\hat{t}\) determines canonical coordinates if and only if \(\tilde{\Phi}^\perp t'^4 = \pm 1\), i.e. \(t(\hat{t})\) satisfies the following ordinary complex first order differential equation:
\[
\sqrt[4]{\pm \tilde{\Phi}^\perp} dt = d\hat{t}.
\]
Integrating (6.10) and taking into account that the left side of the equality is holomorphic, we obtain \(\hat{t}\) as a holomorphic function of \(t\). The condition \(\tilde{\Phi}^\perp \neq 0\) means that \(\hat{t} \neq 0\) and the correspondence between \(\hat{t}\) and \(t\) is one-to-one. Consequently \(\hat{t}\) determines isothermal coordinates satisfying the condition \(\tilde{\Phi}^\perp = \pm 1\), which implies that they are canonical. 

Next we consider the question of uniqueness of canonical coordinates. Suppose that \(t\) and \(\hat{t}\) are canonical of one and the same type. Then \(t = \tilde{t}(\hat{t})\) is either holomorphic or antiholomorphic. According to (6.4) and (6.6) equality (6.9) implies that
\[
\pm 1 = \tilde{\Phi}^\perp = \tilde{\Phi}^\perp t'^4 = \pm 1t'^4 = \pm t'^4.
\]
Therefore \(t'^4 = 1\) and \(t' = \pm 1; \pm i\). We get from here that \(t\) and \(\hat{t}\) are related by one of the following equalities: \(t = \pm \hat{t} + c; \pm i\hat{t} + c\), where \(c = \text{const}\).
The anti-holomorphic case is reduced to the holomorphic one by the change \( \tilde{t} = \bar{s} \) and we get: \( t = \pm \tilde{t} + c; \pm i \tilde{t} + c \).

Thus we obtain eight possible relations between \( t \) and \( \tilde{t} \). Under the natural initial condition \( c = 0 \), these relations mean that:

The canonical coordinates of one and the same type are unique up to a direction and numbering of the coordinate lines.

Finally, we consider the relations between canonical coordinates of different type. Let \( t = u + vi \) be canonical coordinates of the first type and introduce new coordinates by means of the formula \( t = e^{\frac{\pi}{4} i \tilde{t}} \). Then \( t^{\prime 4} = -1, \bar{\Phi}^{(1)}_{\tilde{t}} = -1 \), and consequently \( \tilde{t} \) determines canonical coordinates of the second type. This construction shows that the canonical coordinates of both types are obtained from each other by a rotation of the angle \( \frac{\pi}{4} \) in the coordinate plane \((u, v)\).

Let \((\mathcal{M}, x)\) be a minimal space-like surface in \( \mathbb{R}^4 \) free of degenerate points, parameterized by canonical coordinates of the first type. We can precise the choice of the orthonormal pair \( n_1, n_2 \) in \( N(\mathcal{M}) \). Since \( \sigma(X_1, X_1) \perp \sigma(X_1, X_2) \), then we can choose \( n_1 \) and \( n_2 \) to be collinear with \( \sigma(X_1, X_1) \) and \( \sigma(X_1, X_2) \). More precisely, if at a point we have \( \sigma(X_1, X_1) \neq 0 \), then we choose \( n_1 \) with the same direction as \( \sigma(X_1, X_1) \), and \( n_2 \) so that the quadruple \((X_1, X_2, n_1, n_2)\) is a positive oriented basis in \( \mathbb{R}^4 \). Then \( n_2 \) is collinear with \( \sigma(X_1, X_2) \). Under these conditions formulas \((6.12)\) get the form:

\[
\begin{align*}
\sigma(X_1, X_1) &= \nu n_1, \\
\sigma(X_1, X_2) &= -\mu n_2, & \nu > 0, \\
\sigma(X_2, X_2) &= -\nu n_1;
\end{align*}
\]

(6.11)

Therefore we have \( \lambda = 0 \) and \( \rho = 0 \) and formulas \((5.1)\) become as follows:

\[
A_{n_1} = \begin{pmatrix} \nu & 0 \\ 0 & -\nu \end{pmatrix}; \quad A_{n_2} = \begin{pmatrix} 0 & \mu \\ \mu & 0 \end{pmatrix}.
\]

(6.12)

If at a given non-degenerate point \( \sigma(X_1, X_1) = 0 \), then \( \sigma(X_1, X_2) \neq 0 \). In this case we can choose first \( n_2 \) collinear with the same direction with \( -\sigma(X_1, X_2) \), and then \( n_1 \) so that the quadruple \((X_1, X_2, n_1, n_2)\) forms a positive oriented basis in \( \mathbb{R}^4 \).

The functions \( \nu \) and \( \mu \) also satisfy the following relations:

\[
\begin{align*}
\nu^2 &= \sigma^2(X_1, X_1), \\
\mu^2 &= -\sigma^2(X_1, X_2).
\end{align*}
\]

(6.13)

The functions \( \nu \) and \( \mu \) are a pair of scalar invariants of a minimal space-like surface, free of degenerate points. These invariants completely determine the second fundamental form via \((6.11)\). The second condition in \((6.4)\) implies that the first fundamental form is completely determined by the formula:

\[
E = \frac{1}{\sqrt{\nu^2 + \mu^2}}.
\]

(6.14)

The relations between the pairs \((\nu, \mu)\) and \((K, \varkappa)\) are as follows:

\[
K = -\nu^2 + \mu^2, \quad \varkappa = 2\nu\mu.
\]

(6.15)

\[
\mu^2 = \frac{\sqrt{K^2 + \varkappa^2} + K}{2}, \quad \nu^2 = \frac{\sqrt{K^2 + \varkappa^2} - K}{2}.
\]

(6.16)
Using the above formulas we can characterize the degenerate points of $\mathcal{M}$ in terms of $K$ and $\kappa$.

**Theorem 6.7.** Let $\mathcal{M}$ be a minimal space-like surface with Gaussian curvature $K$ and normal curvature $\kappa$. A point $p \in \mathcal{M}$ is degenerate if and only if $K = 0$ and $\kappa = 0$.

Proof. If $p$ is not a degenerate point in $\mathcal{M}$, then we can introduce canonical coordinates of the first type in a neighborhood of $p$. Formulas (6.13) imply that at least one of $(\nu, \mu)$ is different from 0. Applying (6.15) we obtain that at least one of $(K, \kappa)$ is also different from 0.

If $p$ is a degenerate point, then $\sigma(X_1, X_1)$ and $\sigma(X_1, X_2)$ are lightlike. Then (5.3) implies that $K = 0$. Further it follows that $\sigma(X_1, X_1)$ and $\sigma(X_1, X_2)$ are collinear. Therefore the determinant of the four vectors $x_u, x_v, \sigma(x_u, x_u)$ and $\sigma(x_u, x_v)$ is zero. Hence, in view of (5.14) and (5.15) it follows that $\kappa = 0$.

Finally we add some formulas for $\nu$, $\mu$ and $\kappa$ in canonical coordinates of the first type. Equalities (6.4) and (5.3) imply that

$$\sigma^2(X_1, X_1) - \sigma^2(X_1, X_2) = \frac{1}{E^2} = \frac{4}{\|\Phi\|^4}.$$  

By virtue of (6.17), (5.6) and (6.13) we find

$$\nu^2 = \frac{2(1 + \|\Phi'^\perp\|^2)}{\|\Phi\|^4}, \quad \mu^2 = \frac{2(1 - \|\Phi'^\parallel\|^2)}{\|\Phi\|^4}.$$  

Hence

$$|\kappa| = |2\nu\mu| = \frac{4\sqrt{1 - \|\Phi'^\parallel\|^4}}{\|\Phi\|^4}.$$  

7. **General Weierstrass representations for minimal space-like surfaces.**

In this section we give several types of general Weierstrass representations for minimal space-like surfaces in $\mathbb{R}^4$. In $\mathbb{R}^4$ such formulas were considered in [8], [7]. In $\mathbb{R}^4$ general Weierstrass representations were used in [2], [3].

Let $(\mathcal{M}, x)$: $x = \text{Re} \Psi$ be a minimal space-like surface in $\mathbb{R}^4$, parameterized by isothermal coordinates, and let $\Phi = \Psi'$. If $\Phi = (\phi_1, \phi_2, \phi_3, \phi_4)$, then the condition $\Phi^2 = 0$ is equivalent to the condition that the coordinates are

$$\phi_1^2 + \phi_2^2 + \phi_3^2 - \phi_4^2 = 0.$$  

The relation (7.1) can be parameterized in different ways by means of three holomorphic functions.

First we shall find a representation of $\Phi$ by means of trigonometric functions. We write (7.1) in the following forms:

$$\phi_1^2 + \phi_2^2 = -\phi_3^2 + \phi_4^2, \quad \phi_1^2 + \phi_3^2 = -\phi_2^2 + \phi_4^2, \quad \phi_2^2 + \phi_3^2 = -\phi_1^2 + \phi_4^2.$$  

At least one of these three quantities $\phi_1^2 + \phi_2^2$, $\phi_1^2 + \phi_3^2$ and $\phi_2^2 + \phi_3^2$ is different from zero. (The opposite leads to $\phi_1^2 + \phi_2^2 + \phi_3^2 = 0$ and $\phi_1^2 = \phi_2^2 = \phi_3^2 = \phi_4^2 = 0$, which contradicts to the condition $\mathcal{M}$ is regular.) Without loss of generality we can assume that $\phi_1^2 + \phi_2^2 \neq 0$, which means that there exists a holomorphic function $f \neq 0$, such that:

$$f^2 = \phi_1^2 + \phi_2^2 = -\phi_3^2 + \phi_4^2.$$  

The last equality is equivalent to the next one:

\begin{equation}
(\phi_1/f)^2 + (\phi_2/f)^2 = (\phi_3/1f)^2 + (\phi_4/f)^2 = 1.
\end{equation}

Hence, there exist holomorphic functions \( h_1 \) and \( h_2 \), such that:

\[
\frac{\phi_1}{f} = \cos h_1, \quad \frac{\phi_2}{f} = \sin h_1, \quad \frac{\phi_3}{f} = \cos h_2, \quad \frac{\phi_4}{f} = \sin h_2.
\]

Thus we found the following representation of the function \( \Phi \):

\[
\Phi : \begin{align*}
\phi_1 &= f \cos h_1, \\
\phi_2 &= f \sin h_1, \\
\phi_3 &= if \cos h_2, \\
\phi_4 &= f \sin h_2; \\
f &\neq 0.
\end{align*}
\]

Next we have to express the condition \( \|\Phi\|^2 > 0 \) in terms of the triple \((f, h_1, h_2)\). Equality (7.3) implies that

\begin{equation}
\|\Phi\|^2 = \Phi\bar{\Phi} = |f|^2(\cos h_1 \cos \bar{h}_1 + \sin h_1 \sin \bar{h}_1 + \cos h_2 \cos \bar{h}_2 - \sin h_2 \sin \bar{h}_2)
\end{equation}

\begin{align*}
(7.5) &= |f|^2(\cos(h_1 - \bar{h}_1) + \cos(h_2 + \bar{h}_2)) \\
&= |f|^2(\cos(2i \text{Im} h_1) + \cos(2 \text{Re} h_2)) \\
&= |f|^2(\cos(2\text{Im} h_1) + \cos(2 \text{Re} h_2)).
\end{align*}

Since \( \cosh(2 \text{Im} h_1) \geq 1 \geq |\cos(2 \text{Re} h_2)| \), then it follows from (7.5) that \( \|\Phi\|^2 \geq 0 \). The equality is equivalent to \( \cosh(2 \text{Re} h_2) = 1 \) and \( \cos(2 \text{Re} h_2) = -1 \), i.e. \( \text{Im} h_1 = 0 \) and \( \text{Re} h_2 = \frac{\pi}{2} + k\pi; \ k \in \mathbb{Z} \). Thus we obtained that the triple \((f, h_1, h_2)\) in the representation (7.4) satisfies the conditions

\begin{equation}
(7.6) \quad f \neq 0; \quad \text{Im} h_1 \neq 0 \text{ or } \text{Re} h_2 \neq \frac{\pi}{2} + k\pi; \ k \in \mathbb{Z}.
\end{equation}

Hence, any minimal space-like surface \( \mathcal{M} \) in \( \mathbb{R}^4 \), parameterized by isothermal coordinates, admits Weierstrass representation of the type (7.4), where the triple \((f, h_1, h_2)\) satisfies the conditions (7.6).

Conversely, any triple \((f, h_1, h_2)\) of holomorphic functions, defined in a domain in \( \mathbb{C} \) and satisfying the conditions (7.4), determines by (7.4) a holomorphic \( \mathbb{C}^4 \)-valued function \( \Phi \). It follows from (7.6) that \( \|\Phi\|^2 > 0 \). By direct computations we get \( \Phi^2 = 0 \). Then the surface \( \mathcal{M} : x = \text{Re}(\Psi) \), where \( \Psi \) is determined by the equality \( \Psi' = \Phi \), is a minimal space-like surface in \( \mathbb{R}^4 \), parameterized by isothermal coordinates.

So, we proved the following statement.

Any triple of holomorphic functions \((f, h_1, h_2)\) satisfying (7.6), generates by means of formulas (7.4) a minimal space-like surface in \( \mathbb{R}^4 \).

Finally let us establish to what extent the function \( \Phi \) determines the functions \((f, h_1, h_2)\).

Suppose that one and the same function \( \Phi \) is represented by two different triples \((f, h_1, h_2)\) and \((\hat{f}, \hat{h}_1, \hat{h}_2)\). Then (7.2) and (7.4) imply the following relations between both triples:

\[
\begin{align*}
\hat{f} &= f \\
\hat{h}_1 &= h_1 + 2k_1\pi \quad &\text{or} &\quad \hat{h}_1 &= h_1 + (2k_1 + 1)\pi; \quad k_1 \in \mathbb{Z} \\
\hat{h}_2 &= h_2 + 2k_2\pi \quad &\hat{h}_2 &= h_2 + (2k_2 + 1)\pi; \quad k_2 \in \mathbb{Z}
\end{align*}
\]
With the aid of different substitutions in (7.4) we can obtain other Weierstrass representations for minimal space-like surfaces in \( \mathbb{R}^4_1 \).

In order to obtain Weierstrass representation by means of hyperbolic functions, we make the following substitution in (7.4):

\[
f \rightarrow if; \quad h_1 \rightarrow -ih_1; \quad h_2 \rightarrow \pi + ih_2.
\]

Thus we obtain the following Weierstrass representation:

\[
\Phi : \quad \phi_1 = if \cosh h_1, \quad \phi_2 = f \sinh h_1, \quad \phi_3 = f \cosh h_2, \quad \phi_4 = f \sinh h_2.
\]

(7.7)

Taking into account (7.6), it follows that the functions \((f, h_1, h_2)\) satisfy the conditions:

\[
f \neq 0; \quad \text{Re} h_1 \neq 0 \text{ or } \text{Im} h_2 \neq \frac{\pi}{2} + k\pi; \quad k \in \mathbb{Z}.
\]

(7.8)

Further, let us change the functions \(h_1\) and \(h_2\) in (7.7) by \(w_1\) and \(w_2\) in the following way:

\[
w_1 = h_1 + h_2, \quad w_2 = h_1 - h_2
\]

(7.9)

Then we obtain the following representation of the surface:

\[
\Phi : \quad \phi_1 = if \cosh \frac{w_1 + w_2}{2}, \quad \phi_2 = f \sinh \frac{w_1 + w_2}{2}, \quad \phi_3 = f \cosh \frac{w_1 - w_2}{2}, \quad \phi_4 = f \sinh \frac{w_1 - w_2}{2}
\]

(7.10)

It follows from (7.8) that \((f, w_1, w_2)\) satisfy the conditions:

\[
f \neq 0; \quad \text{Re}(w_1 + w_2) \neq 0 \text{ or } \text{Im}(w_1 - w_2) \neq (2k + 1)\pi; \quad k \in \mathbb{Z}.
\]

(7.11)

The last conditions can be written in the form:

\[
f \neq 0; \quad \text{Re}(w_1 + \bar{w}_2) \neq 0 \text{ or } \text{Im}(w_1 + \bar{w}_2) \neq (2k + 1)\pi; \quad k \in \mathbb{Z}.
\]

(7.12)

Thus we obtained the following more simple form for the conditions (7.11):

\[
f \neq 0; \quad w_1 + \bar{w}_2 \neq (2k + 1)\pi i; \quad k \in \mathbb{Z}.
\]

(7.13)

Next we introduce the functions \(g_1\) and \(g_2\) by the equalities:

\[
g_1 = e^{w_1}; \quad g_2 = e^{w_2}.
\]

(7.14)

Using these functions, we obtain from (7.10) the Weierstrass representation, which is the analogue of the classical Weierstrass representation for minimal surfaces in \( \mathbb{R}^3 \). Consequently we calculate the coordinate functions:

\[
\phi_1 = \frac{if}{2}(e^{\frac{w_1 + w_2}{2}} + e^{-\frac{w_1 + w_2}{2}}) = \frac{if}{2\sqrt{g_1g_2}}(g_1g_2 + 1),
\]

\[
\phi_2 = \frac{f}{2}(e^{\frac{w_1 + w_2}{2}} - e^{-\frac{w_1 + w_2}{2}}) = \frac{f}{2\sqrt{g_1g_2}}(g_1g_2 - 1).
\]
\[
\phi_3 = \frac{f}{2} \left( e^{\frac{w_1-w_2}{2}} + e^{-\frac{w_1-w_2}{2}} \right) = \frac{f}{2 \sqrt{g_1g_2}} (g_1 + g_2),
\]
\[
\phi_4 = \frac{f}{2} \left( e^{\frac{w_1+w_2}{2}} - e^{-\frac{w_1+w_2}{2}} \right) = \frac{f}{2 \sqrt{g_1g_2}} (g_1 - g_2).
\]

In the last equalities we make the substitution
\[
(7.14) \quad f \rightarrow f2\sqrt{g_1g_2}
\]
and obtain the following ‘polynomial’ Weierstrass representation:
\[
\Phi : \quad \phi_1 = \imath f(g_1g_2 + 1), \quad \phi_2 = f(g_1g_2 - 1), \quad \phi_3 = f(g_1 + g_2), \quad \phi_4 = f(g_1 - g_2).
\]

Now we shall determine the conditions which satisfy the functions \((f, g_1, g_2)\). It follows from \((7.13)\) that the condition \(w_1 + \bar{w}_2 \neq (2k + 1)\pi; \quad k \in \mathbb{Z}\) is equivalent to the condition \(e^{w_1+\bar{w}_2} \neq e^{(2k+1)\pi} = -1; \quad k \in \mathbb{Z}\), which gives \(g_1g_2 \neq -1\). Therefore we obtained from \((7.12)\) the following conditions:
\[
(7.16) \quad f \neq 0; \quad g_1g_2 \neq -1.
\]

Conversely, if \((f, g_1, g_2)\) are three holomorphic functions defined in a domain in \(\mathbb{C}\) and satisfying \((7.16)\), then formulas \((7.15)\) determine a holomorphic function \(\Phi\) with values in \(\mathbb{C}^4\). Equalities \((7.16)\) imply that \(\|\Phi\|^2 > 0\). By direct computations we get from \((7.15)\) equality \((7.1)\), which is \(\Phi^2 = 0\). If we determine the function \(\Psi\) by the equality \(\Psi^f = \Phi\) and define \(\mathcal{M} : x = \text{Re}(\Psi)\), then \(\mathcal{M}\) is a minimal space-like surface in \(\mathbb{R}_4^1\), parameterized by isothermal coordinates.

Thus we obtained:

Any three holomorphic functions \((f, g_1, g_2)\) satisfying \((7.16)\), generates via \((7.15)\) a minimal space-like surface in \(\mathbb{R}_4^1\).

Remark 7.1. We obtained the representation \((7.15)\) using \((7.13)\), which implies that the functions \(g_1\) and \(g_2\) are different from zero at any point. This follows from the fact that we chose \(\phi_1^2 + \phi_2^2 \neq 0\). If any of the functions \((g_1, g_2)\) is zero at a fixed point, then it follows directly from \((7.15)\) \(\Phi^2 = 0\) and \(\|\Phi\|^2 = 2|f|\). This means that \((7.15)\) again determine a minimal space-like surface in \(\mathbb{R}_4^1\). Therefore there is no need to add new conditions for \(g_1\) and \(g_2\) other than these from \((7.16)\).

In the end we show that the functions \((f, g_1, g_2)\) can be expressed by the components of the vector function \(\Phi\). Directly from \((7.15)\) we get:
\[
i\phi_1 + \phi_2 = -f(g_1g_2 + 1) + f(g_1g_2 - 1) = -2f,
\]
\[
\phi_3 + \phi_4 = f(g_1 + g_2) + f(g_1 - g_2) = 2fg_1,
\]
\[
\phi_3 - \phi_4 = f(g_1 + g_2) - f(g_1 - g_2) = 2fg_2.
\]

Hence, the functions \(f\), \(g_1\) and \(g_2\) are expressed as follows:
\[
(7.17) \quad f = \frac{1}{2}(i\phi_1 + \phi_2), \quad g_1 = \frac{\phi_3 + \phi_4}{i\phi_1 + \phi_2}, \quad g_2 = \frac{\phi_3 - \phi_4}{i\phi_1 + \phi_2}.
\]
8. SOME FORMULAS, RELATED TO WEIERSTRASS REPRESENTATIONS

In this section we use the Weierstrass representation (7.7) for minimal space-like surfaces by means of hyperbolic functions. Using the functions \( f, h_1, h_2 \), respectively \( f, w_1, w_2 \), we obtain some formulas, which we use further.

First we introduce some subsidiary functions and denotations.

The holomorphic vector function \( a \) is defined by the equality:

\[
(8.1) \quad a = \frac{\Phi}{f}.
\]

Next we introduce the following denotations:

\[
(8.2) \quad \alpha = \text{Re}(h_1), \quad \beta = \text{Im}(h_2).
\]

These functions determine the function \( \theta \), given by:

\[
(8.3) \quad \theta = \text{Re} h_1 + i \text{Im} h_2 = \alpha + i \beta.
\]

The function \( \theta \) is a complex harmonic function, which in general is not holomorphic.

Under these denotations applying the Cauchy-Riemann equations, we have:

\[
(8.4) \quad h'_1 = \text{Re} (h_1)'_u + i \text{Im} (h_1)'_u = \text{Re} (h_1)'_v - i \text{Re} (h_1)'_v = \alpha' - i \beta',
\]

\[
\bar{h}'_2 = \text{Re} (h_2)'_u + i \text{Im} (h_2)'_u = \text{Im} (h_2)'_v + i \text{Im} (h_2)'_v = \beta' + i \alpha' .
\]

For \( w'_1 \) and \( w'_2 \) we find, respectively:

\[
(8.5) \quad w'_1 = (\alpha'_u + \beta'_v) - i(\alpha'_v - \beta'_u),
\]

\[
\bar{w}'_2 = (\alpha'_u - \beta'_v) - i(\alpha'_v + \beta'_u).
\]

Using (7.7) and (8.1), we get the following formulas for \( a, \bar{a}, a' \) and \( \bar{a}' \):

\[
(8.6) \quad a = (i \cosh h_1, \sinh h_1, \cosh h_2, \sinh h_2),
\]

\[
\bar{a} = (-i \cosh h_1, \sinh h_1, \cosh h_2, \sinh h_2).
\]

\[
\bar{a}' = (i h'_1 \sinh h_1, h'_1 \cosh h_1, h'_2 \sinh h_2, h'_2 \cosh h_2),
\]

\[
\bar{\bar{a}}' = (i h'_1 \sinh h_1, h'_1 \cosh h_1, h'_2 \sinh h_2, h'_2 \cosh h_2).
\]

Further we find the scalar products between the functions \( a, \bar{a}, a' \) and \( \bar{a}' \). Differentiating the equality \( a^2 = 0 \), we have:

\[
(8.7) \quad a^2 = aa' = \bar{a}^2 = \bar{a}a' = 0
\]

Taking scalar multiplications in (8.6), we also obtain:

\[
\|a\|^2 = a\bar{a} = \cosh h_1 \cosh h_1 + \sinh h_1 \sinh h_1 + \cosh h_2 \cosh h_2 - \sinh h_2 \sinh h_2
\]

\[
= \cosh(h_1 + h_1) + \cosh(h_2 - h_2)
\]

\[
= \cosh(2 \text{Re} h_1) + \cosh(2i \text{Im} h_2)
\]

\[
= 2 \cosh(\text{Re} h_1 + i \text{Im} h_2) \cosh(\text{Re} h_1 - i \text{Im} h_2)
\]

\[
= 2 \cosh(\theta) \cosh(\bar{\theta}) = 2|\cosh(\theta)|^2;
\]

\[
(8.9) \quad a\bar{a}' = h'_1 \cosh h_1 \sinh h_1 + h'_1 \sinh h_1 \cosh h_1 \sinh h_1 + \cosh h_2 \sinh h_2 - \sinh h_2 \sinh h_2 \sinh h_2
\]

\[
= h'_1 \sinh(h_1 + h_1) - h'_2 \sinh(h_2 - h_2)
\]

\[
= h'_1 \sinh(2 \text{Re} h_1) - h'_2 \sinh(2i \text{Im} h_2);
\]

\[
(8.10) \quad \bar{a}a' = \bar{a}\bar{a}' = h'_1 \sinh(2 \text{Re} h_1) + h'_2 \sinh(2i \text{Im} h_2);
\]

\[
(8.11) \quad a'^2 = -h'^2_1 \sinh^2 h_1 + h'^2_1 \cosh^2 h_1 + h'^2_2 \sinh^2 h_2 - h'^2_2 \cosh^2 h_2
\]

\[
= h'^2_1 - h'^2_2 = w'_1 w'_2;
\]
Now by virtue of (8.11) we find

\begin{align}
\|a\|^2 &= a \cdot a' = |h_1'|^2 \sinh h_1 \sinh \bar{h}_1 + |h_2'|^2 \cosh h_1 \cosh \bar{h}_1 \\
&\quad + |h_2|^2 \sinh h_2 \sinh \bar{h}_2 - |h_2|^2 \cosh h_2 \cosh \bar{h}_2 \\
&= |h_1|^2 \cosh(h_1 + \bar{h}_1) - |h_2|^2 \cosh(h_2 - \bar{h}_2) \\
&= |h_1|^2 \cosh(2Re h_1) - |h_2|^2 \cosh(2i Im h_2).
\end{align}

Further we obtain formulas for \(a'^\perp\), \(a'^\perp^2\) and \(\|a'^\perp\|^2\) expressed by means of \(h_1, h_2\) and \(w_1, w_2\), respectively. For \(a'^\perp\) we have \(a'^\perp = a' - a'^\top\). The equality \(a^2 = 0\) means that the vectors \(a\) and \(\bar{a}\) are orthogonal with respect to the Hermitian dot product in \(\mathbb{C}^4\). Therefore the tangential vector \(a'^\top\) is decomposed as follows:

\[
a'^\top = \frac{a'^\top \cdot \bar{a}}{\|a\|^2} a \quad \|a'^\top \|^2 = \left(\frac{a'^\top \cdot \bar{a}}{\|a\|^2} a\right) \cdot \left(\frac{a'^\top \cdot \bar{a}}{\|a\|^2} a\right) = \frac{a'^\top \cdot \bar{a}}{\|a\|^2} a \cdot \bar{a} = \frac{a'^\top \cdot \bar{a}}{\|a\|^2} a.
\]

Equality (8.7) implies that \(a' \cdot a = 0\). Thus we obtained:

\[
a'^\top = \frac{a'}{\|a\|^2} a; \quad a'^\perp = a' - a'^\top = a' - \frac{a'}{\|a\|^2} a.
\]

Taking square in both sides of (8.13), we get:

\[
a'^\perp^2 = a'^2 = 2a \frac{a'}{\|a\|^2} a + \left(\frac{a'}{\|a\|^2} a\right)^2 = a'^2 = a'^2.
\]

Taking again into account (8.7), we have \(a' \cdot a = 0\) and \(a^2 = 0\). Consequently \(a'^\perp^2 = a^2\).

Now by virtue of (8.11) we find

\[
a'^\perp^2 = a'^2 = h_1'^2 - h_2'^2 = w_1 w_2
\]

Using (8.13) and applying complex conjugation, we calculate \(\|a'^\perp\|^2\):

\[
\begin{align}
\|a'^\perp\|^2 &= a'^\perp \cdot a'^\perp = \left(\frac{a'}{\|a\|^2} a\right) \left(\frac{\bar{a}'}{\|\bar{a}\|^2} \bar{a}\right) \\
&= a' \cdot \bar{a}' - \frac{\bar{a}}{\|a\|^2} a' \cdot \bar{a} - \frac{\bar{a}}{\|a\|^2} a' \cdot \bar{a} + \frac{(a' \cdot \bar{a})(\bar{a}' \cdot a)}{\|a\|^4} a \cdot \bar{a} \\
&= \frac{\|a'^2 - \|a\|^2 \|\bar{a}\|^2 + \|a'\|^2 \|\bar{a}'\|^2\|a\|^2}{\|a\|^4} = \frac{\|a'^2 - \|\bar{a}'\|^2\|a\|^2}{\|a\|^2}.
\end{align}

Let us denote the numerator in (8.15) by \(k_1\). Applying equalities (8.8), (8.10) and (8.12) we find:

\[
k_1 = \|a\|^2 \|a'^2 - \|\bar{a}'\|^2\|a\|^2 = (|h_1|^2 - |h_2|^2)(1 + \cosh(2Re h_1) \cos(2i Im h_2)) \\
+ 2 \Im(h_1 h_2) \sinh(2Re h_1) \sin(2i Im h_2) \\
= (\alpha_1'^2 + \alpha_2'^2 - \beta_1'^2 - \beta_2'^2)(1 + \cosh(2\alpha) \cos(2\beta)) \\
+ 2(\alpha_1 \beta_1' + \alpha_2 \beta_2') \sinh(2\alpha) \sin(2\beta).
\]
Denote the determinant of the vectors $a, \tilde{a}, a'$ and $\tilde{a}'$ by $k_2$. Applying formulas (8.6), we find:

$$k_2 = \det(a, \tilde{a}, a', \tilde{a}')$$

$$= -2 \Im(h_1^* h_2)(1 + \cosh(2 \Re h_1) \cos(2 \Im h_2))$$

$$+ (|h_1'|^2 - |h_2'|^2) \sinh(2 \Re h_1) \sin(2 \Im h_2)$$

$$= -2(\alpha'_u \beta'_u + \alpha'_v \beta'_v)(1 + \cosh(2\alpha) \cos(2\beta))$$

$$+ (\alpha'_u^2 + \alpha'_v^2 - \beta'_u^2 - \beta'_v^2) \sinh(2\alpha) \sin(2\beta) \ .$$

Next we simplify the expressions for $k_1$ and $k_2$ calculating the complex quantity $-k_1 + ik_2$:

$$-k_1 + ik_2 = -\left(\alpha'_u^2 + \alpha'_v^2 - \beta'_u^2 - \beta'_v^2 + 2i(\alpha'_u \beta'_u + \alpha'_v \beta'_v)\right)(1 + \cosh(2\alpha) \cos(2\beta))$$

$$+ (2i(\alpha'_u \beta'_u + \alpha'_v \beta'_v) + \alpha'_u^2 + \alpha'_v^2 - \beta'_u^2 - \beta'_v^2) \sinh(2\alpha) \sin(2\beta)$$

$$= -2((\alpha'_u + i\beta'_u)^2 + (\alpha'_v + i\beta'_v)^2)$$

$$= (1 + \cosh(2\alpha) \cosh(2i\beta) - \sinh(2\alpha) \sin(2\beta))$$

$$-2((\alpha'_u + i\beta'_u)^2 + (\alpha'_v + i\beta'_v)^2) \cosh^2(\alpha - i\beta) \ .$$

Using the function $\theta$, defined by (8.3), we obtain another form of $-k_1 + ik_2$:

$$-k_1 + ik_2 = -2(\theta_u^2 + \theta_v^2) \cosh^2(\theta) \ .$$

Further we express $-k_1 + ik_2$ in terms of $w_1$ and $w_2$. For the first factor in (8.19) we have:

$$\theta_u^2 + \theta_v^2 = (\Re w'_2 + i(-\Im w'_2))(\Re w'_1 + i \Im w'_1) \ .$$

The above formulas imply that:

$$\theta_u^2 + \theta_v^2 = w'_1 \bar{w}'_2 \ .$$

In order to find the second factor in (8.19), first we find $\theta$:

$$\theta = \alpha + i\beta = \Re h_1 + i\Im h_2$$

$$= \frac{1}{2}(h_1 + \bar{h}_1) + i\frac{21}{2}(h_2 - \bar{h}_2)$$

$$= \frac{1}{2}(h_1 + h_2) + \frac{1}{2}(\bar{h}_1 - \bar{h}_2) \ .$$

Taking into account the above equality and (7.9), we find:

$$\theta = \frac{w_1 + \bar{w}_2}{2} \ .$$

Consequently

$$\cosh(\theta) = \frac{1}{2}(e^{w_1/2} + e^{-w_1/2}) = \frac{1}{2}e^{-w_1/2}(e^{w_1/2} + 1)$$

$$= \frac{1}{2}e^{-w_1/2}e^{-\frac{w_1}{2}}(1 + e^{w_1}e^{\bar{w}_2}) \ .$$

Finally we have

$$\cosh^2(\theta) = \frac{1}{4}e^{-w_1}e^{-w_2}(1 + e^{w_1}e^{\bar{w}_2})^2 \ .$$

Now we replace (8.20) and (8.22) into (8.19) and obtain:

$$-k_1 + ik_2 = -\frac{1}{2}w'_1 \bar{w}'_2 e^{-\bar{w}_1} e^{-w_2}(1 + e^{w_1}e^{\bar{w}_2})^2 \ .$$
9. Canonical Weierstrass representation for minimal space-like surfaces of general type

In this section we introduce canonical Weierstrass representations for minimal space-like surfaces of general type in $\mathbb{R}^4_1$. Weierstrass representations with respect to canonical coordinates were obtained in [6] for $\mathbb{R}^3_1$ and in [7] for $\mathbb{R}^4_1$.

Definition 9.1. A minimal space-like surface in $\mathbb{R}^4_1$ is said to be of general type if it is free of degenerate points in the sense of 6.1.

Let the minimal space-like surface $M$ of general type be parameterized by canonical coordinates of the first type. Consider the Weierstrass representation (7.7) by means of hyperbolic functions. The condition (6.4) leads to a relation between the three functions $f$, $h_1$, and $h_2$. In order to obtain this relation, we express the condition $\Phi' \perp^2 = 1$ via $f$, $h_1$, and $h_2$. By virtue of (8.1) we have $\Phi = f a$ and therefore $\Phi' = f a' + f a''$. Since the vector $a$ is tangential to $M$, then we get:

$$\Phi' \perp^2 = (f a + f a') \perp = f a' \perp; \quad \Phi' \perp^2 = f^2 a' \perp^2.$$  

Because of (8.14) we have $a' \perp^2 = h_1' \perp^2 - h_2' \perp^2$ and consequently $\Phi' \perp^2 = f^2 (h_1' \perp^2 - h_2' \perp^2)$. Taking into account the last equality and (6.3), we obtain that the minimal space-like surface $M$ given by (7.7) is parameterized by canonical coordinates of the first type if and only if:

$$\Phi' = f^2 (h_1' \perp^2 - h_2' \perp^2) = 1.$$  

The last formula and (7.7) imply the following statement.

Theorem 9.2. Any minimal space-like surface $M$ of general type, parameterized by canonical coordinates of the first type, has the following Weierstrass representation:

$$\Phi : \begin{align*}
\phi_1 &= i \frac{\cosh h_1}{\sqrt{h_1'^2 - h_2'^2}}, \\
\phi_2 &= \frac{\sinh h_1}{\sqrt{h_1'^2 - h_2'^2}}, \\
\phi_3 &= \frac{\cosh h_2}{\sqrt{h_1'^2 - h_2'^2}}, \\
\phi_4 &= \frac{\sinh h_2}{\sqrt{h_1'^2 - h_2'^2}},
\end{align*}$$

where $(h_1, h_2)$ are holomorphic functions satisfying the conditions:

$$h_1'^2 \neq h_2'^2; \quad \text{Re} \ h_1 \neq 0 \text{ or } \text{Im} \ h_2 \neq \frac{\pi}{2} + k\pi; \ k \in \mathbb{Z}.$$  

Conversely, if $(h_1, h_2)$ is a pair of holomorphic functions satisfying the conditions (9.4), then formulas (9.3) give a minimal space-like surface of general type, parameterized by canonical coordinates of the first type.

We call the representation of $\Phi$ in Theorem 9.2 canonical Weierstrass representation. Using the functions $w_1$ and $w_2$, given by (7.9), then the condition (9.2) gets the form:

$$\Phi'^2 = f^2 w_1' w_2' = 1$$

where $(w_1, w_2)$ are holomorphic functions satisfying the conditions:

$$w_1'^2 \neq w_2'^2; \quad \text{Re} \ w_1 \neq 0 \text{ or } \text{Im} \ w_2 \neq \frac{\pi}{2} + k\pi; \ k \in \mathbb{Z}.$$  

Conversely, if $(w_1, w_2)$ is a pair of holomorphic functions satisfying the conditions (9.4), then formulas (9.3) give a minimal space-like surface of general type, parameterized by canonical coordinates of the first type.
If we replace \( h_1 \) and \( h_2 \) with \( w_1 \) and \( w_2 \) into (9.3), then we obtain the following canonical Weierstrass representation for \( M \):

\[
\Phi : \begin{align*}
\phi_1 &= \frac{i}{\sqrt{w_1'w_2'}} \cosh \frac{w_1 + w_2}{2}, \\
\phi_2 &= \frac{1}{\sqrt{w_1'w_2'}} \sinh \frac{w_1 + w_2}{2}, \\
\phi_3 &= \frac{1}{\sqrt{w_1'w_2'}} \cosh \frac{w_1 - w_2}{2}, \\
\phi_4 &= \frac{1}{\sqrt{w_1'w_2'}} \sinh \frac{w_1 - w_2}{2}.
\end{align*}
\]

(9.6)

According to (7.12), the functions \((w_1, w_2)\) satisfy the conditions:

\[
w_1'w_2' \neq 0; \quad w_1 + \bar{w}_2 \neq (2k + 1)\pi i; \quad k \in \mathbb{Z}.
\]

(9.7)

Conversely, if \((w_1, w_2)\) is a pair of holomorphic functions, satisfying the conditions (9.7), then the formulas (9.6) generate a minimal space-like surface of general type, parameterized by canonical coordinates of the first type.

Finally, using the functions \(g_1\) and \(g_2\), given by (7.13), we obtain a canonical Weierstrass representation of the type (7.15). Differentiating (7.13), we get:

\[
g_1' = e^{w_1}w_1' = g_1w_1'; \quad g_2' = e^{w_2}w_2' = g_2w_2'.
\]

(9.8)

From here we have:

\[
w_1' = \frac{g_1'}{g_1}, \quad w_2' = \frac{g_2'}{g_2}.
\]

(9.9)

Applying (7.14) and (9.9) to the condition (9.5), we get

\[
(f^2 \sqrt{g_1g_2})^2 \frac{g_1'g_2'}{g_1g_2} = 1.
\]

(9.10)

Consequently the isothermal coordinates are canonical of the first type if and only if

\[
\Phi'^2 = 4f^2g_1g_2 = 1.
\]

Next we express \( f \) from the last equality of (9.10) and replace it into (7.15). Thus we obtain the following statement.

**Theorem 9.3.** Any minimal space-like surface \( M \) of general type, parameterized by canonical coordinates of the first type, has the following Weierstrass representation:

\[
\Phi : \begin{align*}
\phi_1 &= \frac{1}{2} \frac{g_1g_2 + 1}{\sqrt{g_1'g_2'}}, \\
\phi_2 &= \frac{1}{2} \frac{g_1g_2 - 1}{\sqrt{g_1'g_2'}}, \\
\phi_3 &= \frac{1}{2} \frac{g_1 + g_2}{\sqrt{g_1'g_2'}}, \\
\phi_4 &= \frac{1}{2} \frac{g_1 - g_2}{\sqrt{g_1'g_2'}}.
\end{align*}
\]

(9.11)

According to (7.16) the functions \((g_1, g_2)\) in this representation satisfy the conditions:

\[
g_1'g_2' \neq 0; \quad g_1g_2 \neq -1.
\]

(9.12)
Conversely, if \((g_1, g_2)\) is a pair of holomorphic functions satisfying the conditions (9.12), then formulas (9.11) generate a minimal space-like surface of general type, parameterized by canonical coordinates of the first type.

The above canonical Weierstrass representation seems to be the most useful and applicable representation.

10. The First Fundamental Form and the Curvatures \(K, \kappa\) in a General Weierstrass Representation

Let \(\mathcal{M}\) be a minimal space-like surface in \(\mathbb{R}^4\), parameterized by isothermal coordinates. First we consider the case, when \(\mathcal{M}\) is given by (7.7). In order to obtain a formula for \(E\), we use equalities (3.3), (8.1) and (8.8). Thus we get:

\[
E = \frac{1}{2} \|\Phi\|^2 = \frac{1}{2} \|\mathfrak{a}\|^2 = |f|^2 \cosh \theta^2,
\]

where \(\theta\) is the function (8.3).

Taking into account (8.21), we find the following formula for \(E\) with respect to the representation (7.10):

\[
E = |f|^2 \left| \cosh \frac{w_1 + \bar{w}_2}{2} \right|^2.
\]

In order to obtain a formula with respect to the representation (7.15), we express \(\cosh(\theta)\) by means of \(g_j\), \((j = 1; 2)\), given by (7.13). Then (8.22) gives that

\[
\cosh^2(\theta) = \frac{(1 + g_1 \bar{g}_2)^2}{4g_1 \bar{g}_2}.
\]

Passing from the representation (7.10) by means of \(w_j\) to the representation (7.15) by means of \(g_j\), \((j = 1; 2)\), as a consequence of (7.14) \(|f|^2\) has to be replaced by:

\[
|f|^2 \rightarrow 4|f|^2|g_1 \bar{g}_2|.
\]

Now applying (10.3) and (10.4) to (10.1), we find the following formula for the coefficient \(E\) in the representation (7.15):

\[
E = |f|^2 \left| 1 + g_1 \bar{g}_2 \right|^2.
\]

Further we find the corresponding formulas for \(K\) and \(\kappa\). In the formula (5.17) we replace \(\Phi'\) by (9.11) and get:

\[
K = \frac{-4\|\Phi'\|^2}{\|\Phi\|^4} = \frac{-4\|\mathfrak{a}'\|^2}{\|\mathfrak{a}\|^4} = \frac{-4|f|^2\|\mathfrak{a}'\|^2}{|f|^4\|\mathfrak{a}\|^4} = \frac{-4\|\mathfrak{a}'\|^2}{|f|^4\|\mathfrak{a}\|^4}.
\]

Using (8.15) and taking into account (8.16), we find:

\[
K = \frac{-4(|\mathfrak{a}|^2 |\mathfrak{a}'|^2 - |\mathfrak{a} \cdot \mathfrak{a}'|^2)}{|f|^2|\mathfrak{a}|^6} = \frac{-4k_1}{|f|^2|\mathfrak{a}|^6}.
\]

A similar formula for \(\kappa\) can be derived using the second equality in (5.17). We find consecutively:

\[
\kappa = \frac{4}{|f|^6\|\mathfrak{a}\|^6} \det(fa, \bar{f}a, fa + \bar{f}a, \bar{f}'a + \bar{f}'\bar{a}) = \frac{4|f|^4}{|f|^6\|\mathfrak{a}\|^6} \det(a, \bar{a}, a', \bar{a}') = \frac{4k_2}{|f|^2|\mathfrak{a}|^6}.
\]
Thus we have:

\[ K = \frac{-4k_1}{|f|^2|a|^6}; \quad \varkappa = \frac{4k_2}{|f|^2|a|^6}. \]

It is useful to unite \( K \) and \( \varkappa \) in one formula by the complex quantity \( K + i\varkappa \). It follows from (10.6) that:

\[ K + i\varkappa = \frac{4(-k_1 + ik_2)}{|f|^2|a|^6}. \]

Replacing \( |a|^2 \) and \(-k_1 + ik_2\) respectively by (8.8) and (8.19) we get:

\[ K + i\varkappa = \frac{4(-2(\theta_u^2 + \theta_v^2) \cosh^2(\hat{\theta}))}{|f|^2|\cosh(\theta)|^6}. \]

Finally we obtained:

\[ K + i\varkappa = \frac{-\theta_u^2 + \theta_v^2}{|f|^2|\cosh(\theta)|^2 \cosh^2(\theta)}, \]

where \( \theta \) is the function (8.3). Thus the formulas for \( K \) and \( \varkappa \) related to the representation (7.7) are:

\[ K = \text{Re} \left( \frac{-\theta_u^2 + \theta_v^2}{|f|^2|\cosh(\theta)|^2 \cosh^2(\theta)} \right), \]

\[ \varkappa = \text{Im} \left( \frac{-\theta_u^2 + \theta_v^2}{|f|^2|\cosh(\theta)|^2 \cosh^2(\theta)} \right). \]

In order to express \( K \) and \( \varkappa \) by means of the functions \( w_j \), \( (j = 1; 2) \) in the representation (7.10), we use (8.20) and (8.21). Applying them to (10.8) we find:

\[ K + i\varkappa = \frac{-w_1'w_2'}{|f|^2|\cosh \frac{w_1 + w_2}{2}|^2 \cosh^2 \frac{w_1 + w_2}{2}}. \]

The corresponding formulas in terms of \( g_j \), \( (j = 1; 2) \) in the representation (7.15) follow by using (8.20), (10.4) and (8.22):

\[ K + i\varkappa = \frac{-4g_1'g_2'}{|f|^2|1 + g_1\bar{g}_2|^2(1 + g_1\bar{g}_2)^2}. \]

By virtue of (7.13) and (9.9), the last formula takes the form:

\[ K + i\varkappa = \frac{-4g_1'g_2'}{|f|^2|1 + g_1\bar{g}_2|^2(1 + g_1\bar{g}_2)^2}. \]

Applying (10.5), we get:

\[ K + i\varkappa = \frac{-4g_1'g_2'}{E(1 + g_1\bar{g}_2)^2}. \]

The corresponding formulas for \( K \) and \( \varkappa \), related to the representation (7.15) are:

\[ K = \text{Re} \left( \frac{-4g_1'g_2'}{|f|^2|1 + g_1\bar{g}_2|^2(1 + g_1\bar{g}_2)^2} \right) = \text{Re} \left( \frac{-4g_1'g_2'}{E(1 + g_1\bar{g}_2)^2} \right), \]

\[ \varkappa = \text{Im} \left( \frac{-4g_1'g_2'}{|f|^2|1 + g_1\bar{g}_2|^2(1 + g_1\bar{g}_2)^2} \right) = \text{Im} \left( \frac{-4g_1'g_2'}{E(1 + g_1\bar{g}_2)^2} \right). \]
The above formulas have been found by Asperti A. and Vilhena J. in [2].

11. THE FIRST FUNDAMENTAL FORM AND THE CURVATURES $K$, $\kappa$, WITH RESPECT TO A CANONICAL WEIERSTRASS REPRESENTATION.

Let $\mathcal{M}$ be a minimal space-like surface of general type, parameterized by canonical coordinates of the first type.

First we obtain a formula for the coefficient $E$ with respect to the canonical Weierstrass representation (9.3). Applying (9.5) and (8.20), we find the following formula for $|f|^2$:

$$|f|^2 = \frac{1}{|w_1'w_2'|} = \frac{1}{|\theta_u^2 + \theta_v^2|}.$$  

Replacing into the general formula (10.1), we get:

$$E = \frac{|\cosh(\theta)|^2}{|\theta_u^2 + \theta_v^2|}.$$  

To obtain a formula for $E$ with respect to the canonical Weierstrass representation (9.6), we use (10.2) and (9.5). Thus we find a formula for $E$ in terms of $w_1$ and $w_2$:

$$E = \frac{1}{|w_1'w_2'|}. $$

In a similar way, if $\mathcal{M}$ is given by (9.11), we replace $f$ into the general formula (10.5) by the help of (9.10) and obtain:

$$E = \frac{|1 + \overline{g_2}g_2|^2}{4|g_1'g_2'|}. $$

Next we find formulas for the curvatures $K$ and $\kappa$. Using the representation (9.3), we replace $f$ into the general formula (10.8) by means of (11.1), and get:

$$K + i\kappa = \frac{-|\theta_u^2 + \theta_v^2| (\theta_u'^2 + \theta_v'^2)}{|\cosh(\theta)|^2 \cosh^2(\theta)}. $$

To obtain a formula for $K + i\kappa$, when $\mathcal{M}$ is given by (9.6), we replace into the general formula (10.10) the function $f$ by means of (9.5). Thus we have:

$$K + i\kappa = \frac{-|w_1'w_2'| w_1'w_2'}{\cosh^2 \frac{w_1 + \overline{w_2}}{2} \cosh^2 \frac{w_1' + \overline{w_2'}}{2}}. $$

To obtain a formula for $K + i\kappa$, when $\mathcal{M}$ is represented by (9.11), we replace into the general formula (10.11) the function $f$ by the help of (9.10). Hence, we have:

$$K + i\kappa = \frac{-16|g_1'g_2'| g_1'' g_2''}{|1 + \overline{g_2}g_2|^2 (1 + \overline{g_2}g_2)^2}. $$

The formulas for the curvatures $K$ and $\kappa$ with respect to the representation (9.11) are as follows:

$$K = \text{Re} \frac{-16|g_1'g_2'| g_1'' g_2''}{|1 + \overline{g_2}g_2|^2 (1 + \overline{g_2}g_2)^2}, $$

$$\kappa = \text{Im} \frac{-16|g_1'g_2'| g_1'' g_2''}{|1 + \overline{g_2}g_2|^2 (1 + \overline{g_2}g_2)^2}. $$
12. Change of the functions \((g_1, g_2)\) under some basic geometric transformations of the minimal space-like surface

Let \(\mathcal{M}\) be a minimal space-like surface of general type, parameterized by canonical coordinates \((u, v)\) of the first type. The complex variable \(t\) is given by \(t = u + iv\). We suppose that \(\mathcal{M}\) is given by the canonical representation \((g_1(t), g_2(t))\) of holomorphic functions. The aim of this section is to study the changes of the pair \((g_1, g_2)\) under geometric transformations of the surface.

First we consider the case of a motion of the surface \(\mathcal{M}\) in \(\mathbb{R}^4_1\). We shall use some basic formulas and facts about the spinors in \(\mathbb{R}^4_1\). Let us recall some of these formulas in a form useful for an application to the theory of minimal space-like surfaces. To any vector \(x\) in \(\mathbb{R}^4_1\) we associate a Hermitian 2 \(\times\) 2-matrix \(S\) as follows:

\[
S = \begin{pmatrix}
x_3 + x_4 & ix_1 + x_2 \\
-ix_1 + x_2 & -x_3 + x_4
\end{pmatrix}
\]

(12.1)

This correspondence is a linear isomorphism between \(\mathbb{R}^4_1\) and the space of Hermitian 2 \(\times\) 2-matrices. This correspondence has the following property: \(\det S = -x^2\). The last property means that from any linear operator acting in the space of Hermitian 2 \(\times\) 2-matrices and preserving the determinant, can be obtained an orthogonal operator in \(\mathbb{R}^4_1\).

If \(\tilde{A}\) is a complex 2 \(\times\) 2-matrix, then \(\tilde{A}S\tilde{A}^*\) is a Hermitian matrix, where \(\tilde{A}^*\) is the Hermitian conjugate of \(\tilde{A}\). What is more, if \(\det \tilde{A} = 1\), then \(\det \tilde{A}S\tilde{A}^* = \det S\). It follows from the above that to any matrix \(\tilde{A}\) in \(\text{SL}(2, \mathbb{C})\) corresponds an orthogonal matrix \(A\) in \(\text{SO}^+(3, 1, \mathbb{R})\). Therefore we have a group homomorphism \(\tilde{A} \rightarrow A\), which can be written as follows:

\[
\tilde{S} = \tilde{A}S\tilde{A}^* \rightarrow \tilde{x} = Ax.
\]

(12.2)

The so obtained homomorphism from \(\text{SL}(2, \mathbb{C})\) into \(\text{SO}^+(3, 1, \mathbb{R})\) is called spinor map. It is proved in the theory of spinors that the kernel of the spinor map consists of two elements: \(\pm I\), where \(I\) is the unitary matrix. Further, the image of this map is the connected component of the unity element in \(\text{SO}^+(3, 1, \mathbb{R})\), which is denoted in a standard way by \(\text{SO}^+(3, 1, \mathbb{R})\). Briefly speaking, this is the group of the matrices determining those transformations in \(\mathbb{R}^4_1\), preserving not only the orientation of \(\mathbb{R}^4_1\), but also preserve both: the direction of time and the orientation of the three-dimensional Euclidean subspace of \(\mathbb{R}^4_1\). These transformations of \(\mathbb{R}^4_1\) are called orthochronous transformations. The type of the kernel and the image of the spinor map \((12.2)\) implies that the spinor map induces the following group isomorphism:

\[
\text{SL}(2, \mathbb{C})/\{\pm I\} \cong \text{SO}^+(3, 1, \mathbb{R}).
\]

(12.3)

This means that \(\text{SL}(2, \mathbb{C})\) appears to be a two-sheeted covering of \(\text{SO}^+(3, 1, \mathbb{R})\) and hence we can identify it with the spin group \(\text{Spin}(3, 1)\) of \(\text{SO}^+(3, 1, \mathbb{R})\). In other words, \((12.3)\) gives a representation of \(\text{Spin}(3, 1)\) as \(\text{SL}(2, \mathbb{C})\), which is called the spinor representation. Note that the group \(\text{SL}(2, \mathbb{C})\) is connected and simply connected, and consequently it follows from the isomorphisms \((12.2)\) and \((12.3)\) that \(\text{SL}(2, \mathbb{C})\) also appears to be universal covering group for \(\text{SO}^+(3, 1, \mathbb{R})\).

Now, let \(x\) be an arbitrary complex vector in \(\mathbb{C}^4\). Up to now, considering different correspondences, we restricted \(x\) to be a real vector in \(\mathbb{R}^4_1\). It is an easy verification that the relations \((12.1)\) and \((12.2)\) are linear with respect to \(x\). Therefore, they are also valid when \(x\) is an arbitrary complex vector in \(\mathbb{C}^4\). The only difference is that \(S\) can be an arbitrary (not necessarily Hermitian) complex matrix. Under a motion of the complex vector \(x\) with a matrix in \(\text{SO}^+(3, 1, \mathbb{R})\), the matrix \(S\) is transformed in the same way, as it is described in \((12.2)\).
Let us return to minimal space-like surfaces. With the help of the above formulas we shall find how the functions giving the Weierstrass representation of a minimal space-like surface are transformed under a motion of the surface in $\mathbb{R}^4$.

First, let the minimal space-like surface $(\mathcal{M}, \xi)$ be parameterized by arbitrary isothermal coordinates. If the surface $(\mathcal{M}, \xi)$ is obtained from $(\mathcal{M}, x)$ by means of orthochronous transformation in $\mathbb{R}^4$, then we have $\dot{x}(t) = A \dot{x}(t) + b$, where $A \in \text{SO}^+(3,1)$ and $b \in \mathbb{R}^4$. The function $\Phi$ defined by (3.1), as we noted by the formula (3.17), is transformed by: $\hat{\Phi} = A \Phi$. Next we introduce the complex matrix $S_\Phi$, which is obtained by $\Phi$ according to the rule (12.1):

$$S_\Phi = \begin{pmatrix} \phi_3 + \phi_4 & i\phi_1 + i\phi_2 \\ -i\phi_1 + \phi_2 & -\phi_3 + \phi_4 \end{pmatrix}.$$  

We denote by $\hat{\Phi}$ any of the two matrices in $\text{SL}(2, \mathbb{C})$, corresponding to $\Phi$ by means of the homomorphism (12.2). If $S_\hat{\Phi}$ is the matrix obtained from $(\mathcal{M}, \xi)$, according to (12.2) it is related to $S_\Phi$ as follows:

$$S_\hat{\Phi} = \hat{\Phi} S_\Phi \hat{\Phi}^*.$$

Now, suppose that $\mathcal{M}$ is given by a Weierstrass representation of the type (7.15). By direct calculations we find:

$$\begin{align*}
i\phi_1 + \phi_2 &= -f(g_1 g_2 + 1) + f(g_1 g_2 - 1) = -2f, \\
-i\phi_1 + \phi_2 &= f(g_1 g_2 + 1) + f(g_1 g_2 - 1) = 2f g_1 g_2, \\
\phi_3 + \phi_4 &= f(g_1 + g_2) + f(g_1 - g_2) = 2f g_1, \\
-\phi_3 + \phi_4 &= -f(g_1 + g_2) + f(g_1 - g_2) = -2f g_2.
\end{align*}$$

Consequently, the matrix $S_\Phi$ is represented by means of $f$, $g_1$ and $g_2$ as follows:

$$S_\Phi = \begin{pmatrix} 2f g_1 & -2f \\ 2f g_1 g_2 & -2f g_2 \end{pmatrix}.$$  

Denoting the elements of $S_\Phi$ by $s_{ij}$, then we have the following expressions for $f$, $g_1$ and $g_2$:

$$\begin{align*}
f &= -\frac{1}{2} s_{12}, \\
g_1 &= -\frac{s_{11}}{s_{12}}, \\
g_2 &= \frac{s_{22}}{s_{12}}.
\end{align*}$$

Since $S_\Phi$ is transformed according the rule (12.5), then via (12.7) we shall find the transformation formulas for the functions $\hat{f}$, $\hat{g}_1$ and $\hat{g}_2$. For that purpose we denote the elements of $\hat{A}$ in the following way:

$$\hat{A} = \begin{pmatrix} \tilde{a} & \tilde{b} \\ -\bar{c} & \bar{d} \end{pmatrix}; \quad a, b, c, d \in \mathbb{C}; \quad ad - bc = 1.$$  

After multiplying the matrices in (12.5) and simplifying, we get:

$$S_\hat{\Phi} = \begin{pmatrix} 2f(a g_1 + b)(-b g_2 + \bar{a}) & 2f(c g_1 + \bar{d})(\tilde{b} g_2 - \bar{a}) \\ 2f(a g_1 + \bar{b})(\bar{d} g_2 - \bar{c}) & 2f(c g_1 + d)(-\tilde{b} g_2 + \tilde{c}) \end{pmatrix}.$$  

Applying (12.7) to $\hat{f}$, $\hat{g}_1$ and $\hat{g}_2$, we find the transformation formulas of the functions in the Weierstrass representation of the type (7.15) under an orthochronous transformation of $\mathcal{M}$ in $\mathbb{R}^4$:

$$\begin{align*}
\hat{f} &= f(c g_1 + d)(-\tilde{b} g_2 + \bar{a}), \\
\hat{g}_1 &= a g_1 + b; \quad \hat{g}_2 = \frac{\bar{d} g_2 - \bar{c}}{-b g_2 + \tilde{a}}.
\end{align*}$$
Now, let us consider the inverse statement. Suppose that \((\hat{M}, \hat{x})\) and \((M, x)\) are two minimal space-like surfaces, given by the Weierstrass representation of the type (7.15), related by means of (12.10). We shall show that they can be obtained one from the other by an orthochronous transformation in \(\mathbb{R}^4_1\). For that purpose, we introduce \(\hat{A}\) by means of (12.8). Let \(A\) be the corresponding to \(\hat{A}\) matrix under the homomorphism (12.2). With the help of \(A\) we obtain a third surface \((\hat{M}, \hat{x})\) given by the formula: \(\hat{x} = Ax\). We proved that \(\hat{M}\) has a Weierstrass representation with functions also satisfying (12.10). Therefore, \(\hat{M}\) and \(\hat{M}\) are generated by one and the same functions by means of formulas (7.15) and consequently they are obtained one from the other by a translation in \(\mathbb{R}^4_1\). Since \(\hat{M}\) is obtained from \(M\) by an orthochronous transformation, then \(\hat{M}\) is also obtained from \(M\) by orthochronous transformation. Summarizing we obtain the following statement:

**Theorem 12.1.** Let \((\hat{M}, \hat{x})\) and \((M, x)\) be two minimal space-like surfaces in \(\mathbb{R}^4_1\), given by Weierstrass representations of the type (7.15). The following conditions are equivalent:

1. \((\hat{M}, \hat{x})\) and \((M, x)\) are related by an orthochronous transformation in \(\mathbb{R}^4_1\) of the type: \(\hat{x}(t) = Ax(t) + b, \) where \(A \in SO^+(3, 1, \mathbb{R})\) and \(b \in \mathbb{R}^4_1\).

2. The functions in the Weierstrass representations of \((\hat{M}, \hat{x})\) and \((M, x)\) are related by equalities (12.10), where \(a, b, c, d \in \mathbb{C}, ad - bc = 1\).

Up to now we considered only the case of a motion from the connected component of the identity in \(O(3, 1, \mathbb{R})\). Next we show that any of the three remaining cases can be reduced to the considered one. Let us consider the case of a transformation, which is not orthochronous. Such a concrete transformation can be obtained by a change of the signs of the four coordinates: \(\hat{x}(t) = -x(t)\). This implies the change of the sign of the function \(f\) in the Weierstrass representation (7.15), while the functions \(g_1\) and \(g_2\) remain the same. Any non-orthochronous transformation can be obtained as a composition of this concrete transformation and an orthochronous transformation in \(\mathbb{R}^4_1\). Therefore, if two minimal space-like surfaces are obtained one from the other by a non-orthochronous transformation, then the functions in the Weierstrass representation are related by formulas, which are similar to (12.10) with the only difference in the sign of the formula for \(\hat{f}\). Further, we consider the case of a non-orthochronous improper transformation. An example of such a transformation is the symmetry with respect to the hyperplane \(x_4 = 0\) which is given by the change of the sign of \(x_4\). This implies a change of the places of both functions \(g_1\) and \(g_2\) in the Weierstrass representation, while the function \(f\) remains the same. Any non-orthochronous improper transformation can be obtained as a composition of this symmetry and an orthochronous transformation. Therefore the functions in the Weierstrass representation are changed similarly to (12.10), but this time the formulas for \(\hat{g}_1\) and \(\hat{g}_2\) change their places, while the formula for \(\hat{f}\) is the same. Finally, we consider the case of an orthochronous improper transformation. Such a transformation can be obtained as a combination of the last two cases. Therefore, the transformation formulas for the functions in the Weierstrass representation are obtained from (12.10), by the change of the sign of \(\hat{f}\) and the change of the places of the formulas for \(\hat{g}_1\) and \(\hat{g}_2\).

Now, let \(\hat{M}\) and \(M\) be two minimal space-like surfaces, parameterized by canonical coordinates and the surface \(\hat{M}\) is obtained from \(M\) by a motion in \(\mathbb{R}^4_1\). Suppose that \(\hat{M}\) is given by a canonical Weierstrass representation of the type (9.11). Since \(\hat{x} = Ax + b\) implies that \(\hat{\Phi}' = A\Phi'\), then we have \(\hat{\Phi}'^2 = \Phi'^2 = 1\). Consequently the canonical coordinates of \(\hat{M}\) appear to be also canonical coordinates for \(\hat{M}\). Taking into account that the canonical
Weierstrass representation (9.11) is a special case of the representation (7.15), then the pair \((g_1, g_2)\) is transformed by the formulas (12.10). Note that these formulas can be applied in the cases of an orthochronous or a non-orthochronous transformation. This is so because the two cases differ from each other only the formula for the function \(f\). Further, we see that it only remain the linear fractional functions from (12.10), which allow us to replace the condition \(ad - bc = 1\) with the more general condition \(ad - bc \neq 0\). This is possible, because the linear fractional function does not change if its matrix is multiplied by a non zero factor.

Summarizing the above remarks, in view of Theorem 12.1 we obtain the following statement.

**Theorem 12.2.** Let \((\hat{\mathcal{M}}, \hat{x})\) and \((\mathcal{M}, x)\) be two minimal space-like surfaces of general type, given by the canonical Weierstrass representation of the type (9.11). The following conditions are equivalent:

1. \((\hat{\mathcal{M}}, \hat{x})\) and \((\mathcal{M}, x)\) are related by a transformation in \(\mathbb{R}^4_1\) of the type:
   \[ \hat{x}(t) = Ax(t) + b, \quad \text{where} \quad A \in \text{SO}(3, 1, \mathbb{R}) \quad \text{and} \quad b \in \mathbb{R}^4. \]
2. The functions in the Weierstrass representations of \((\hat{\mathcal{M}}, \hat{x})\) and \((\mathcal{M}, x)\) are related by the following equalities:

   \begin{align*}
   \hat{g}_1 &= \frac{ag_1 + b}{cg_1 + d}; \\
   \hat{g}_2 &= \frac{dg_2 - \bar{c}}{-bg_2 + \bar{a}},
   \end{align*}

   where \(a, b, c, d \in \mathbb{C}, \ ad - bc \neq 0\).

If \((\hat{\mathcal{M}}, \hat{x})\) and \((\mathcal{M}, x)\) are related by an improper transformation in \(\mathbb{R}^4_1\), then in the formulas (12.11) one has to change the places of \(\hat{g}_1\) and \(\hat{g}_2\).

In the end, we write down (12.11) in a form, which is useful for applications. For that purpose, let us denote by \(Gz\), where \(G \in \text{GL}(2, \mathbb{C})\) and \(z \in \mathbb{C}\), the standard action of the group \(\text{GL}(2, \mathbb{C})\) in the complex plane by means of linear fractional transformations. Denoting by \(B\) the matrix of the linear fractional function for \(\hat{g}_1\) in (12.11), by direct computations we see that the matrix of \(\hat{g}_2\) is up to a factor the matrix \(B^{-1}\). Hence, the formulas (12.11) can be written briefly as follows:

\begin{align*}
\hat{g}_1 &= Bg_1; \\
\hat{g}_2 &= B^{-1}g_2; \quad B \in \text{GL}(2, \mathbb{C}).
\end{align*}

Finally we give a natural approach to the family of the minimal space-like surfaces of general type, associated with a given one. Let \((g_1(t), g_2(t))\) be a pair of holomorphic functions defined in a disc \(\mathcal{D}\), centered at \((0, 0)\) in the parametric plane \(\mathbb{C}\). Consider the minimal space-like surface \((\mathcal{M}, x)\), generated by the pair \((g_1(t), g_2(t))\) by means of (9.11). For any complex number \(a, |a| = 1\) we introduce the pair of holomorphic functions

\begin{align*}
(\tilde{g}_1(s), \tilde{g}_2(s)) &= (g_1(as), g_2(as)); \quad s \in \mathcal{D}
\end{align*}

and denote by \(\tilde{\mathcal{M}}\) the minimal space-like surface, generated by the pair \((\tilde{g}_1, \tilde{g}_2)\) by means of (9.11). Further we denote by \(\Phi, \Psi\) and \(\hat{\Phi}, \hat{\Psi}\) the corresponding vector holomorphic functions on \(\mathcal{M}\) and \(\tilde{\mathcal{M}}\). Replacing (12.13) into the representation (9.11), we get the following relation between \(\hat{\Phi}\) and \(\Phi\):

\begin{align*}
\hat{\Phi}(s) &= \frac{1}{a} \Phi(as).
\end{align*}

Since \(\hat{\Phi}^2 = 1\), then \(s = \frac{1}{a}\) determines canonical coordinates on \(\tilde{\mathcal{M}}\).

After an integration we obtain the corresponding formula for \(\hat{\Psi}\):

\begin{align*}
\hat{\Psi}(s) &= \frac{1}{a^2} \Psi(as).
\end{align*}
Denoting \( a = e^{i\frac{\pi}{2}} \) and \( \tilde{\mathcal{M}} = \mathcal{M}_\varphi \), we have:

Any minimal space-like surface of general type \( \mathcal{M} : x = x(t); \ t \in \mathcal{D} \) generates a one-parameter family \( \{ \mathcal{M}_\varphi \} \) of minimal space-like surfaces, given by the formula

\[
\mathcal{M}_\varphi : x_\varphi(s) = \text{Re}(e^{-i\varphi} \Psi(e^{i\frac{\pi}{2}} s)); \quad \varphi \in [0, \frac{\pi}{2}], \quad s \in \mathcal{D},
\]

where \( s = e^{-i\frac{\pi}{2}} t \) determines canonical coordinates on \( \mathcal{M}_\varphi \).

The surfaces of the family \( \{ \mathcal{M}_\varphi \} \) are said to be associated with the given surface \( \mathcal{M} \).

Since the generating holomorphic functions of the family of the associated surfaces are given by (12.13), taking into account formulas (11.4) and (11.8), we observe that the transformation \( \mathcal{M} \rightarrow \mathcal{M}_\varphi \) given by \( s \rightarrow e^{i\frac{\pi}{2}} s \) preserves \( E, K \) and \( \kappa \), i.e. it is a special isometry between \( \mathcal{M} \) and \( \mathcal{M}_\varphi \) preserving the normal curvature \( \kappa \).

Denote by \( \bar{\mathcal{M}} \) the minimal space-like surface conjugate to \( \mathcal{M} \), which is given by the formula

\[
y = \text{Im}(\Psi) = \text{Re}(-i\Psi).
\]

Then \( \bar{\mathcal{M}} \) is the associated with \( \mathcal{M} \) surface \( \mathcal{M}_\varphi \), \( \varphi = \frac{\pi}{2} \).

Thus we have:

If the minimal space-like surface \( \mathcal{M} \), parameterized by canonical coordinates, is generated by the pair \( (g_1(t), g_2(t)) \), then the minimal space-like surface \( \mathcal{M} \), conjugate to \( \mathcal{M} \), is generated by the pair \( (g_1(e^{i\frac{\pi}{2}} s), g_2(e^{i\frac{\pi}{2}} s)) \) with canonical parameter \( s = e^{-i\frac{\pi}{2}} t \).

If \( t \) determines canonical coordinates of the first type on \( \mathcal{M} \), then \( e^{-i\frac{\pi}{2}} t \) gives canonical coordinates of the second type on \( \mathcal{M} \) and vice versa.

Hence:

The canonical coordinates of the second type on \( \mathcal{M} \) are canonical coordinates of the first type on \( \mathcal{M} \) and the canonical coordinates of the first type on \( \mathcal{M} \) are canonical coordinates of the second type on \( \mathcal{M} \).

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