Strongly representable atom structures

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Abstract. We show that the variety of representable algebras for many cylindric like algebras of finite dimension $> 2$ is not atom canonical. Our result works for any class $K$ between diagonal free algebras and polyadic equality algebras of finite dimension $> 2$. We also show, modifying a construction of Hirsch and Hodkinson, that for such classes of algebras (in the same dimensions) the class of strongly representable atom structures is not elementary. For finite dimensions $n > 2$ we also show that several proper subvarieties of the representable cylindric and polyadic equality algebras of dimension $n$, whose members have a neat embedding property, are not atom canonical. From this we infer that the omitting types theorem fails for $L_n$ (first order logic restricted to the first $n$ variables, $n$ finite $> 2$) in quite a strong sense, namely, if we consider clique guarded semantics, and first order definable expansions of $L_n$. Finally, we show that it is undecidable to tell whether a finite $CA_3$ is in $SR_3CA_{3+k}$ for $k \geq 3$, by reducing the problem to the analogous one for relation algebras.

1 Introduction

We follow the notation of [1] which is in conformity with that of [25]. Assume that we have a class of Boolean algebras with operators for which we have a semantical notion of representability (like Boolean set algebras or cylindric set algebras). A weakly representable atom structure is an atom structure such that at least one atomic algebra based on it is representable. It is strongly representable if all atomic algebras having this atom structure are representable. The former is equivalent to that the term algebra, that is, the algebra generated by the atoms, in the complex algebra is representable, while the latter is equivalent to that the complex algebra is representable.

Could an atom structure be possibly weakly representable but not strongly representable? Ian Hodkinson [10], showed that this can indeed happen for both cylindric algebras of finite dimension $\geq 3$, and relation algebras, in the
context of showing that the class of representable algebras, in both cases, is not closed under Dedekind-MacNeille completions. In fact, he showed that this can be witnessed on an atomic algebras, so that the variety of representable relation algebras algebras and cylindric algebras of finite dimension > 2 are not atom-canonical. (The complex algebra of an atom structure is the completion of the term algebra.) This construction is somewhat complicated using a rainbow atom structure. It has the striking consequence that there are two atomic algebras sharing the same atom structure, one is representable the other is not.

This construction was simplified and streamlined, by many authors, including the first author [12], but Hodkinson’s construction, as we indicate below, has a very large potential to prove analogous theorems on Dedekind-MacNeille completions, and atom-canonicity for several proper sub-varieties of the variety of representable cylindric-like algebras such as polyadic algebras with and without equality and Pinter’s substitution algebras.

Our first theorem extends this result (existence of weakly representable atom structures that are not strongly representable) to many cylindric like algebras of relations using three different constructions based on three different graphs. The first is due to Hokinson, the second was given in [12] and the third is new. The three constructions presented herein model theoretically gives a polyadic atomic equality algebra of dimension $n > 2$ such that the diagonal free reduct of its completion is not representable.

We also show (the stronger result) that there for $n > 2$ finite, there is is a polyadic equality atomic algebra $\mathfrak{A}$ such that $\mathfrak{A} \in C_{n+4}$ inferring that the varieties $\mathfrak{A} \in C_{n+k}$, for $n > 2$ finite for any $k \geq 4$, and for $K \in \{CA, PEA\}$ are not closed under Dedekind-MacNeille completions.

Now that we have two distinct classes, namely, the class of weakly atom structures and that of strongly atom structures; the most pressing need is to try to classify them. Venema proved (in a more general setting) that the former is elementary, while Hirsch and Hodkinson show that the latter is not elementary. Their proof is amazing depending on an ultraproduct of Erdos probabilistic graphs, witness theorem [12].

We know that there is a sequence of strongly representable atom structures whose ultraproduct is only weakly representable, it is not strongly representable. This gives that the class $K = \mathfrak{A} \in CA_n : \mathfrak{A}$ is atomic and $\text{CmAtA} \in RCA_n$ is not elementary, as well.

Here we extend Hirsch and Hodkinson’s result to many cylindric-like algebras, answering a question of Hodkinson’s for PA and PEA. The proof is based on the algebras constructed in [27] by noting that these algebras can be endowed with polyadic operations (in an obvious way) and that they are generated by elements whose dimension sets $< n$. The latter implies that an algebra is representable if and only if its diagonal free reduct is representable (One side is trivial, the other does not hold in general).
Lately, it has become fashionable in algebraic logic to study representations of abstract algebras that has a complete representation. A representation of $A$ is roughly an injective homomorphism from $f : A \to \wp(V)$ where $V$ is a set of $n$-ary sequences; $n$ is the dimension of $A$, and the operations on $\wp(V)$ are concrete and set theoretically defined, like the boolean intersection and cylindrifiers or projections. A complete representation is one that preserves arbitrary disjuncts carrying them to set theoretic unions. If $f : A \to \wp(V)$ is such a representation, then $A$ is necessarily atomic and $\bigcup_{x \in \text{At} A} f(x) = V$.

Let us focus on cylindric algebras. It is known that there are countable atomic RCA$_n$s when $n > 2$, that have no complete representations; in fact, the class of completely representable CA$_n$s when $n > 2$, is not even elementary [9]. Such a phenomena is also closely related to the algebraic notion of *atom canonicity*, as indicated, which is an important persistence property in modal logic and to the metalogical property of omitting types in finite variable fragments of first order logic. Recall that a variety $V$ of boolean algebras with operators is atom-canonical, if whenever $A \in V$, and $A$ is atomic, then $\text{CmAt} A \in V$.

If $\text{At}$ is a weakly representable but not strongly representable, then $\text{CmAt}$ is not representable; this gives that $\text{RCA}_n$ for $n > 2$ finite, is not atom canonical. Also $\text{CmAt} A$ is the Dedekind-MacNeille completion of $A$, and so obviously $\text{RCA}_n$ is not closed under Dedekind-MacNeille completions.

On the other hand, $A$ cannot be completely representable for, it can be shown without much ado, that a complete representation of $A$ induces a representation of $\text{CmAt} A$.

Finally, if $A$ is countable, atomic and has no complete representation then the set of co-atoms (a co-atom is the complement of an atom), viewed in the corresponding Tarski Lindenbaum algebra, $\mathfrak{sm}_T$, as a set of formulas, is a non principal-type that cannot be omitted in any model of $T$.

The reader is referred to [14] for an extensive discussion of such notions and more.

Our main new results are

Let $n > 2$ be finite.

(1) Showing that for any class $K$ between $\text{Df}_n$ and $\text{PEA}_n$, there is a weakly representable atom structure that is not strongly representable. In fact, we show that there is a relation algebra $\mathfrak{R}$, such that the set of all $n$ basic matrices form a cylindric bases, $\text{CmMat}_n \text{At} \mathfrak{R}$ is representable but $\mathfrak{R}_d \text{CmMat}_n \text{At} \mathfrak{R}$ is not. This gives the same result for $\mathfrak{R}$, cf. theorem 3.3.

(2) Showing that for $K \in \{\text{PEA}, \text{CA}\}$, the class $\mathfrak{m}_n K_{n+k}$ is not atom canonical for $k \geq 4$, theorem 3.11 extending a result in [15]. We give two entirely different proofs. This answers a question in [20, problem 12, p. 627] which was raised again in [24, problem 1, p. 131].
(3) Showing that the class of strongly representable atom structures for any \( K \) between \( Df_n \) and \( PEA_n \) is not elementary, theorem 4.12. This answers a question of Hodkinson’s [11, p.284].

(4) We show that the omitting types theorem fails for finite variable fragments of first order logic, as long as we have more than 2 variables, even if we consider clique guarded semantics, theorem 3.14. This is stronger, in a way, than the result reported in [13] and proved in detail in [7]. The latter result is also approached, and it is strengthened conditionally, cf. theorem 3.12. The condition is the existence of certain finite relation algebras.

(5) Studying the omitting types theorem in many multi-dimensional modal logics, that are cylindrifier free, but are endowed with the operations of finite substitutions or only some of them, as a sample witness 3.20. We also prove a new negative omitting types theorem addressing first order definable expansions of first order logic as defined in [5], theorem 3.17.

(6) Showing that for a finite algebra in \( CA_3 \) and \( PEA_3 \), it is undecidable whether \( \mathfrak{A} \in S\mathfrak{R}_3 CA_{3+k} \) or not, for \( k \geq 2 \), theorem 5.1. It follows that are finite algebras with no finite relativized representations. In contrast, we show that when we restrict taking subneat reducts to finite \( m \) dimensional algebras, then for any \( 2 \leq n < m \), every finite algebra has a finite relativized representation.

On the notation

The notation used is in conformity with [1] but the following list may help. For rainbow cylindric algebras, we deviate from the notation therein, we find it more convenient to write \( CA_{G,\Gamma} \) - treating the greens \( G \), as a parameter that can vary - for the rainbow algebra \( R(\Gamma) \). The latter is defined to be \( Cm(\rho(K)) \) where \( K \) is a class of models in the rainbow signature satisfying the \( L_{\omega_1,\omega} \) rainbow theory, and \( \rho(K) \) is the rainbow atom structure [9].

We may also add polyadic operations, obtaining polyadic equality rainbow algebras, denoted by \( PEA_{G,\Gamma} \). It is obvious how to add the polyadic operations.

But in all cases our view is the conventional (more restrictive) one adopted in [10]; we view these models as coloured graphs, that is complete graphs labelled by the rainbow colours. As usual, we interpret ‘an edge coloured by a green say’, to mean that the pair defining the edge satisfies the corresponding green binary relation. Our atom structures will consist of surjections from \( n \) (the dimension) to finite coloured graphs, roughly finite coloured graphs.

\( PEA \) denotes polyadic equality algebras, \( PA \) denotes polyadic algebras, \( Sc \) denotes Pinter’s algebras, and of course \( CA \) denotes cylindric algebras [1]. \( PEA_n \)
denotes PEAs of dimension $n$ and same for all other algebras. The following information is folklore: CAs and PA are proper reducts of PEAs and Scs are proper reducts of all. Finally, following the usual notation again, Df denotes diagonal free CAs, and these are proper reducts of all of the above. For an atomic $\mathfrak{A}$ we write $\text{At}\mathfrak{A}$ for its atom structure and for an atom structure $\mathfrak{A}$ we write $\text{CmAt}$ for its complex algebra. In particular $\text{CmAt}$ is the completion of $\mathfrak{A}$ in case our varieties are completely additive. (PA and Scs are not completely additive).

QEA stands for the class of quasi-polyadic equality algebras, these coincide with PEA for finite dimensions, but in the infinite dimensional case, the difference between these classes are substantial, for example the latter always has uncountably many operations, which is not the case with the former in case of countable dimensions.

Furthermore, QEA can be formulated in the context of a system of varieties definable by a schema in the infinite dimensional case; which is not the case with PA the same can be said. To avoid conflicting notation, we write QEA and QA only for infinite dimensions, meaning that we are only dealing with finite cylindrifiers and finite substitutions.

By a graph we will mean a pair $\Gamma = (G, E)$, where $G \neq \phi$ and $E \subseteq G \times G$ is a reflexive and symmetric binary relation on $G$. We will often use the same notation for $\Gamma$ and for its set of nodes ($G$ above). A pair $(x, y) \in E$ will be called an edge of $\Gamma$. See [29] for basic information (and a lot more) about graphs. For neat reducts we follow [24] and for omitting types we follow [14].

$L_n$ or $L^n$ will denote first order logic restricted to the first $n$ variables and $L_{\infty, \omega}^n$ will denote $L_{\infty, \omega}$ restricted to the first $n$ variables. (In the latter case infinitary conjunctions, hence also infinitary disjunctions, are allowed).

## 2 Preliminaries

The action of the non-boolean operators in a completely additive atomic Boolean algebra with operators is determined by their behavior over the atoms, and this in turn is encoded by the atom structure of the algebra.

**Definition 2.1. (Atom Structure)** Let $\mathfrak{A} = \langle A, +, -, 0, 1, \Omega_i : i \in I \rangle$ be an atomic boolean algebra with operators $\Omega_i : i \in I$. Let the rank of $\Omega_i$ be $\rho_i$. The *atom structure* $\text{At}\mathfrak{A}$ of $\mathfrak{A}$ is a relational structure

$$\langle \text{At}\mathfrak{A}, R_{\Omega_i} : i \in I \rangle$$

where $\text{At}\mathfrak{A}$ is the set of atoms of $\mathfrak{A}$ as before, and $R_{\Omega_i}$ is a $(\rho(i)+1)$-ary relation over $\text{At}\mathfrak{A}$ defined by

$$R_{\Omega_i}(a_0, \cdots, a_{\rho(i)}) \iff \Omega_i(a_1, \cdots, a_{\rho(i)}) \geq a_0.$$
Similar 'dual' structure arise in other ways, too. For any not necessarily atomic BAO $\mathfrak{A}$ as above, its ultrfilter frame is the structure

$$\mathfrak{A}_+ = \langle \text{Uf}(\mathfrak{A}), R_{\Omega} : i \in I \rangle,$$

where $\text{Uf}(\mathfrak{A})$ is the set of all ultrafilters of (the boolean reduct of) $\mathfrak{A}$, and for $\mu_0, \cdots, \mu_{\rho(i)} \in \text{Uf}(\mathfrak{A})$, we put $R_{\Omega_i}(\mu_0, \cdots, \mu_{\rho(i)})$ iff $\{\Omega(a_1, \cdots, a_{\rho(i)}) : a_j \in \mu_j$ for $0 < j \leq \rho(i)\} \subseteq \mu_0$.

**Definition 2.2. (Complex algebra)** Conversely, if we are given an arbitrary structure $\mathcal{S} = \langle S, r_i : i \in I \rangle$ where $r_i$ is a $(\rho(i) + 1)$-ary relation over $S$, we can define its complex algebra

$$\text{Cm}(\mathcal{S}) = \langle \phi(S), \cup, \setminus, \phi, S; \Omega_i \rangle_{i \in I},$$

where $\phi(S)$ is the power set of $S$, and $\Omega_i$ is the $\rho(i)$-ary operator defined by

$$\Omega_i(X_1, \cdots, X_{\rho(i)}) = \{s \in S : \exists s_1 \in X_1 \cdots \exists s_{\rho(i)} \in X_{\rho(i)} : r_i(s, s_1, \cdots, s_{\rho(i)})\},$$

for each $X_1, \cdots, X_{\rho(i)} \in \phi(S)$.

It is easy to check that, up to isomorphism, $\text{At}(\text{Cm}(\mathcal{S})) \cong \mathcal{S}$ always, and $\mathfrak{A} \subseteq \text{Cm}(\text{At}\mathfrak{A})$ for any completely additive atomic boolean algebra with operators $\mathfrak{A}$. If $\mathfrak{A}$ is finite then of course $\mathfrak{A} \cong \text{Cm}(\text{At}\mathfrak{A})$.

1. Atom structure of diagonal free-type algebra is $\mathcal{S} = \langle S, R_{c_i} : i < n \rangle$, where the $R_{c_i}$ is binary relation on $S$.

2. Atom structure of cylindric-type algebra is $\mathcal{S} = \langle S, R_{c_i}, R_{d_{ij}} : i, j < n \rangle$, where the $R_{d_{ij}}$, $R_{c_i}$ are unary and binary relations on $S$. The reduct $\mathfrak{Rd}_{df}\mathcal{S} = \langle S, R_{c_i} : i < n \rangle$ is an atom structure of diagonal free-type.

3. Atom structure of substitution-type algebra is $\mathcal{S} = \langle S, R_{c_i}, R_{s_{ij}} : i, j < n \rangle$, where the $R_{d_{ij}}$, $R_{s_{ij}}$ are unary and binary relations on $S$, respectively. The reduct $\mathfrak{Rd}_{df}\mathcal{S} = \langle S, R_{c_i} : i < n \rangle$ is an atom structure of diagonal free-type.

4. Atom structure of quasi polyadic-type algebra is $\mathcal{S} = \langle S, R_{c_i}, R_{s_{ij}} : i, j < n \rangle$, where the $R_{c_i}$, $R_{s_{ij}}$ and $R_{s_{ij}}$ are binary relations on $S$. The reducts $\mathfrak{Rd}_{df}\mathcal{S} = \langle S, R_{c_i} : i < n \rangle$ and $\mathfrak{Rd}_{df}\mathcal{S} = \langle S, R_{c_i}, R_{s_{ij}} : i, j < n \rangle$ are atom structures of diagonal free and substitution types, respectively.

5. The atom structure of quasi polyadic equality-type algebra is $\mathcal{S} = \langle S, R_{c_i}, R_{d_{ij}}, R_{s_{ij}} : i, j < n \rangle$, where the $R_{d_{ij}}$ is unary relation on $S$, and $R_{c_i}$, $R_{s_{ij}}$ are binary relations on $S$. 
(a) The reduct \( \mathfrak{R}_{df}S = \langle S, R_{c_i} : i \in I \rangle \) is an atom structure of diagonal free-type.

(b) The reduct \( \mathfrak{R}_{ca}S = \langle S, R_{c_i}, R_{d_{ij}} : i, j \in I \rangle \) is an atom structure of cylindric-type.

(c) The reduct \( \mathfrak{R}_{Sc}S = \langle S, R_{c_i}, R_{s_{ij}} : i, j \in I \rangle \) is an atom structure of substitution-type.

(d) The reduct \( \mathfrak{R}_{qa}S = \langle S, R_{c_i}, R_{s_{ij}}, R_{s_{ij}} : i, j \in I \rangle \) is an atom structure of quasi polyadic-type.

**Definition 2.3.** An algebra is said to be representable if and only if it is isomorphic to a subalgebra of a direct product of set algebras of the same type.

**Definition 2.4.** Let \( S \) be an \( n \)-dimensional algebra atom structure. \( S \) is strongly representable if every atomic \( n \)-dimensional algebra \( \mathfrak{A} \) with \( \text{At}\mathfrak{A} = S \) is representable. We write \( \text{SDF}_n, \text{SCS}_n, \text{SSCS}_n, \text{SQS}_n \) and \( \text{SQES}_n \) for the classes of strongly representable \( (n\text{-dimensional}) \) diagonal free, cylindric, substitution, quasi polyadic and quasi polyadic equality algebra atom structures, respectively.

Note that for any \( n \)-dimensional algebra \( \mathfrak{A} \) and atom structure \( S \), if \( \text{At}\mathfrak{A} = S \) then \( \mathfrak{A} \) embeds into \( \text{Clm}S \), and hence \( S \) is strongly representable iff \( \text{Clm}S \) is representable.

### 3 Weakly representable atom structures that are not strongly representable, and atom canonicity

Here we use fairly simple model theoretic arguments, to prove the non atom-canonicity of several classes consisting of subneat reducts. We need some standard model theoretic preparations. This proof unifies Rainbow and Monk like constructions used in [10] and [12], and introduces a new Monk-like algebra based on a new graph (different than those used in the later two references). Such Monk-like algebras will be abstracted later on, presented as atom structures of models of a first order theory as presented in [9]; however, the construction is different.

**Theorem 3.1.** Let \( \Theta \) be an \( n \)-back-and-forth system of partial isomorphism on a structure \( A \), let \( \bar{a}, \bar{b} \in \mathfrak{A} \), and suppose that \( \theta = (\bar{a} \mapsto \bar{b}) \) is a map in \( \Theta \). Then \( A \models \phi(\bar{a}) \) iff \( A \models \phi(\bar{b}) \), for any formula \( \phi \) of \( L_{\infty \omega}^n \).

**Proof.** By induction on the structure of \( \phi \). \( \square \)
Suppose that \( W \subseteq {}^n A \) is a given non-empty set. We can relativize quantifiers to \( W \), giving a new semantics \( \models_W \) for \( L_{\infty \omega}^n \), which has been intensively studied in recent times. If \( \bar{\alpha} \in W \):

- for atomic \( \phi \), \( A \models_W \phi(\bar{\alpha}) \) iff \( A \models \phi(\bar{\alpha}) \)
- the boolean clauses are as expected
- for \( i < n \), \( A \models_W \exists x_i \phi(\bar{\alpha}) \) iff \( A \models_W \phi(\bar{\alpha}') \) for some \( \bar{\alpha}' \in W \) with \( \bar{\alpha}' \equiv_i \bar{\alpha} \).

**Theorem 3.2.** If \( W \) is \( L_{\infty \omega}^n \) definable, \( \Theta \) is an \( n \)-back-and-forth system of partial isomorphisms on \( A, \bar{\alpha}, \bar{\beta} \in W \), and \( \bar{\alpha} \mapsto \bar{\beta} \in \Theta \), then \( A \models \phi(\bar{\alpha}) \) iff \( A \models \phi(\bar{\beta}) \) for any formula \( \phi \) of \( L_{\infty \omega}^n \).

**Proof.** Assume that \( W \) is definable by the \( L_{\infty \omega}^n \) formula \( \psi \), so that \( W = \{ \bar{a} \in {}^n A : A \models \psi(\bar{a}) \} \). We may relativize the quantifiers of \( L_{\infty \omega}^n \)-formulas to \( \psi \). For each \( L_{\infty \omega}^n \)-formula \( \phi \) we obtain a relativized one, \( \phi^\psi \), by induction, the main clause in the definition being:

- \( (\exists x_i \phi)^\psi = \exists x_i (\psi \land \phi^\psi) \).

Then clearly, \( A \models_W \phi(\bar{\alpha}) \) iff \( A \models \phi^\psi(\bar{\alpha}) \), for all \( \bar{\alpha} \in W \). \( \square \)

The following theorem unifies and generalizes the main theorem in [10] and in [12]. It shows that sometime Monk like algebras and rainbow algebras do the same thing. We shall see that each of the constructions has its assets and liabilities. In the rainbow case the construction can be refined by truncating the greens and reds to be finite, to give sharper results as shown in theorem .

While Monk’s algebra in one go gives the required result for both relation and cylindric like algebras, and it can be generalized to show that the class of strongly representable atom structures for both relation and cylindric algebras is not elementary reprofing a profound result of Hirsch and Hodkinson, using Erdos probabilistic graphs, or anti Monk ultraproducts [4,12].

First a piece of notation. For an atomic relation algebra \( R \), \( \text{Mat}_n \text{At} R \) denotes the set of basic matrices on \( R \). We say that \( \text{Mat}_n \text{At} R \) is a polyadic basis if it is a symmetric cylindric bases (closed under substitutions).

The idea of the proof, which is quite technical, is summarized in the following:

1. We construct a labelled hypergraph \( M \) that can be viewed as an \( n \) homogeneous model of a certain theory (in the rainbow case it is an \( L_{\omega_1 \omega} \) theory, in the Monk case it is first order.) This model gets its labels from a fixed in advance graph \( G \); that also determines the signature of \( M \). By \( n \) homogeneous we mean that every partial isomorphism of \( M \) of size \( \leq n \) can be extended to an automorphism of \( M \).
(2) We have finitely many shades of red, outside the signature; these are basically non principal ultrafilters, but they can be used as labels.

(3) We build a relativized set algebra based on $M$, by discarding all assignments whose edges are labelled by shades of red getting $W \subseteq nM$.

(4) $W$ is definable in $\forall M$ be an $L_{\infty, \omega}$ formula hence the semantics with respect to $W$ coincides with classical Tarskian semantics (when assignments are in $\forall M$). This is proved using certain $n$ back and forth systems.

(5) The set algebra based on $W$ (consisting of sets of sequences (without shades of reds labelling edges) satisfying formulas in $L^n$ in the given signature) will be an atomic algebra such that its completion is not representable. The completion will consist of interpretations of $L^n_{\infty, \omega}$ formulas; though represented over $W$, it will not be, and cannot be, representable in the classical sense.

**Theorem 3.3.** (1) There exists a polyadic equality atom structure $At$ of dimension $n$, such that $TmAt$ is representable as a $PEA_n$, but not strongly representable. In fact $Rd_CmAt$ is not representable for any signature $t$ between that of $Df$ and $Sc$.

(2) Furthermore, there exists an atomic relation algebra $R$ that that the set of all basic matrices forms an $n$ dimensional polyadic basis such that $TmMat_nAtR$ is representable as a polyadic equality algebra, while $Rd_CmMat_nAtR$ is not (as a diagonal free cylindric algebra of dimension $n$). In particular, $AtR$ is weakly but not strongly representable.

**Proof.** (1) Constructing an $n$ homogeneous model

$L^+$ is the rainbow signature consisting of the binary relation symbols $g_i : i < n - 1, g_0 : i < |G|, w, w_i : i < n - 2, r_{jk}^i : i < \omega, j < k < |R|$ and the $(n - 1)$ ary-relation symbols $y_S : S \subseteq G$ together with a shade of red $\rho$ which is outside the rainbow signature, but it is a binary relation, in the sense that it can label edges in coloured graphs. Here we take, like Hodkinson [10], $G = R = \omega$. graphs. In the following theorem we shall see that by varying these parameters, namely when $|G| = n + 2$ and $|R| = n + 1$ we get sharper results.

Let $\mathcal{G} \mathcal{G}$ be the class of all coloured graph in this rainbow signature. Let $T_r$ denote the rainbow $L_{\infty, \omega}$ theory [9]. Let $G$ be a countable disjoint union of cliques each of size $n(n - 1)/2$ or $N$ with edge relation defined by $(i, j) \in E$ iff $0 < |i - j| < N$. Let $L^+$ be the signature consisting of the binary relation symbols $(a, i)$, for each $a \in G \cup \{\rho\}$ and $i < n$. Let $T_m$ denote the following (Monk) theory:
$M \models T_m$ if for all $a, b \in M$, there is a unique $p \in G \cup \{\rho\} \times n$, such that $(a, b) \in p$ and if $M \models (a, i)(x, y) \land (b, j)(y, z) \land (c, l)(x, z)$, then $|\{i, j, l\}| > 1$, or $a, b, c \in G$ and $\{a, b, c\}$ has at least one edge of $G$, or exactly one of $a, b, c$ - say, $a$ - is $\rho$, and $bc$ is an edge of $G$, or two or more of $a, b, c$ are $\rho$. We denote the class of models of $T_m$ which can also be viewed as coloured graphs with labels coming from $G \cup \rho \times n$ also by $GG$.

We deliberately use this notation to emphasize the fact that the proof for all three cases considered is essentially the same; the difference is that our three term algebras, to be constructed are set algebras with domain a subset of $nM$, where $M$ is a model that embeds all coloured graphs and to be defined shortly, are just based on a different graph. So such a notation simplifies matters considerably.

Now in all cases, there is a countable $n$ homogeneous coloured graph (model) $M \in GG$ of both theories with the following property:

- If $\Delta \subseteq \Delta' \in GG$, $|\Delta'| \leq n$, and $\theta : \Delta \rightarrow M$ is an embedding, then $\theta$ extends to an embedding $\theta' : \Delta' \rightarrow M$.

We proceed as follows. We use a simple game. Two players, $\forall$ and $\exists$, play a game to build a labelled graph $M$. They play by choosing a chain $\Gamma_0 \subseteq \Gamma_1 \subseteq \ldots$ of finite graphs in $GG$; the union of the chain will be the graph $M$. There are $\omega$ rounds. In each round, $\forall$ and $\exists$ do the following. Let $\Gamma \in GG$ be the graph constructed up to this point in the game. $\forall$ choose $\Delta \subseteq \Delta' \in GG$ of size $< n$, and an embedding $\theta : \Delta \rightarrow \Gamma$. He then chooses an extension $\Delta \subseteq \Delta^+ \in GG$, where $|\Delta^+\setminus\Delta| \leq 1$. These choices, $(\Delta, \theta, \Delta^+)$, constitute his move. $\exists$ must respond with an extension $\Gamma \subseteq \Gamma^+ \in GG$ such that $\theta$ extends to an embedding $\theta^+ : \Delta^+ \rightarrow \Gamma^+$. Her response ends the round. The starting graph $\Gamma_0 \in GG$ is arbitrary but we will take it to be the empty graph in $GG$. We claim that $\exists$ never gets stuck - she can always find a suitable extension $\Gamma^+ \in GG$. Let $\Gamma \in GG$ be the graph built at some stage, and let $\forall$ choose the graphs $\Delta \subseteq \Delta^+ \in GG$ and the embedding $\theta : \Delta \rightarrow \Gamma$. Thus, his move is $(\Delta, \theta, \Delta^+)$. We now describe $\exists$’s response. If $\Gamma$ is empty, she may simply plays $\Delta^+$, and if $\Delta = \Delta^+$, she plays $\Gamma$. Otherwise, let $F = \text{rng}(\theta) \subseteq \Gamma$. (So $|F| < n$.) Since $\Delta$ and $\Gamma \upharpoonright F$ are isomorphic labelled graphs (via $\theta$), and $GG$ is closed under isomorphism, we may assume with no loss of generality that $\forall$ actually played $(\Gamma \upharpoonright F, \text{Id}_F, \Delta^+)$, where $\Gamma \upharpoonright F \subseteq \Delta^+ \in GG$, $\Delta^+\setminus F = \{\delta\}$, and $\delta \notin \Gamma$. We may view $\forall$’s move as building a labelled graph $\Gamma^* \supseteq \Gamma$, whose nodes are those of $\Gamma$ together with $\delta$, and whose edges are the edges of $\Gamma$ together with edges from $\delta$ to every node of $F$. The labelled graph structure on $\Gamma^*$ is given by
• \( \Gamma \) is an induced subgraph of \( \Gamma^* \) (i.e., \( \Gamma \subseteq \Gamma^* \))

• \( \Gamma^* \upharpoonright (F \cup \{\delta\}) = \Delta^+ \). Now \( \exists \) must extend \( \Gamma^* \) to a complete graph on the same node and complete the colouring yielding a graph \( \Gamma^+ \in \mathcal{G}\mathcal{G} \). Thus, she has to define the colour \( \Gamma^+(\beta, \delta) \) for all nodes \( \beta \in \Gamma \setminus F \), in such a way as to meet the required conditions. For rainbow case \( \exists \) plays as follows:

1. If there is no \( f \in F \), such that \( \Gamma^*(\beta, f), \Gamma^*(\delta, f) \) are coloured \( g_t \) and \( g_u \) for some \( t, u \), then \( \exists \) defined \( \Gamma^+(\beta, \delta) \) to be \( w_0 \).
2. Otherwise, if for some \( i \) with \( 0 < i < n - 1 \), there is no \( f \in F \) such that \( \Gamma^*(\beta, f), \Gamma^*(\delta, f) \) are both coloured \( g_i \), then \( \exists \) defines the colour \( \Gamma^+(\beta, \delta) \) to be \( w_i \), say the least such.
3. Otherwise \( \delta \) and \( \beta \) are both the apexes on \( F \) in \( \Gamma^* \) that induce the same linear ordering on (there are nor green edges in \( F \) because \( \Delta^+ \in \mathcal{G}\mathcal{G} \), so it has no green triangles). Now \( \exists \) has no choice but to pick a red. The colour she chooses is \( \rho \).

This defines the colour of edges. Now for hyperedges, for each tuple of distinct elements \( \bar{a} = (a_0, \ldots, a_{n-2}) \in n^{-1}(\Gamma^+) \) such that \( \bar{a} \notin n^{-1}\Gamma \cup n^{-1}\Delta \) and with no edge \((a_i, a)\) coloured greed in \( \Gamma^+ \), \( \exists \) colours \( \bar{a} \) by \( y_S \) where \( S = \{i < \omega : \text{there is a } i \text{ cone with base } \bar{a} \} \). Notice that \( |S| \leq F \). This strategy works.

For the Monk case \( \exists \) plays as follows:

Now \( \exists \) must extend \( \Gamma^* \) to a complete graph on the same node and complete the colouring yielding a graph \( \Gamma^+ \in \mathcal{G}\mathcal{G} \). Thus, she has to define the colour \( \Gamma^+(\beta, \delta) \) for all nodes \( \beta \in \Gamma \setminus F \), in such a way as to meet the conditions of definition 1. She does this as follows. The set of colours of the labels in \( \{\Delta^+ (\delta, \phi) : \phi \in F \} \) has cardinality at most \( n - 1 \). Let \( i < n \) be a ”colour” not in this set. \( \exists \) labels \( (\delta, \beta) \) by \( (\rho, i) \) for every \( \beta \in \Gamma \setminus F \). This completes the definition of \( \Gamma^+ \).

It remains to check that this strategy works—that the conditions from the definition of \( \mathcal{G}\mathcal{G} \) are met. It is sufficient to note that

- if \( \phi \in F \) and \( \beta \in \Gamma \setminus F \), then the labels in \( \Gamma^+ \) on the edges of the triangle \( (\beta, \delta, \phi) \) are not all of the same colour (by choice of \( i \))
- if \( \beta, \gamma \in \Gamma \setminus F \), then two the labels in \( \Gamma^+ \) on the edges of the triangle \( (\beta, \gamma, \delta) \) are \( (\rho, i) \).

Now there are only countably many finite graphs in \( \mathcal{G}\mathcal{G} \) up to isomorphism, and each of the graphs built during the game is finite. Hence \( \forall \) may arrange to play every possible \( (\Delta, \theta, \Delta^+) \) (up to isomorphism)
at some round in the game. Suppose he does this, and let $M$ be the union of the graphs played in the game. We check that $M$ is as required. Certainly, $M \in \mathcal{G}\mathcal{G}$, since $\mathcal{G}\mathcal{G}$ is clearly closed under unions of chains. Also, let $\Delta \subseteq \Delta' \in \mathcal{G}\mathcal{G}$ with $|\Delta'| \leq n$, and $\theta : \Delta \to M$ be an embedding. We prove that $\theta$ extends to $\Delta'$, by induction on $d = |\Delta'\setminus \Delta|$. If this is 0, there is nothing to prove. Assume the result for smaller $d$. Choose $a \in \Delta' \setminus \Delta$ and let $\Delta^+ = \Delta' \cup \{a\} \in \mathcal{G}\mathcal{G}$. As, $|\Delta| < n$, at some round in the game, at which the graph built so far was $\Gamma$, say, $\forall$ would have played $(\Delta, \theta, \Delta^+)$ (or some isomorphic triple). Hence, if $\exists$ constructed $\Gamma^+$ in that round, there is an embedding $\theta^+ : \Delta^+ \to \Gamma^+$ extending $\theta$. As $\Gamma^+ \subseteq M, \theta^+$ is also an embedding: $\Delta^+ \to M$. Since $|\Delta'\setminus \Delta^+| < d, \theta^+$ extends inductively to an embedding $\theta' : \Delta' \to M$, as required.

(4) Relativization, back and forth systems ensuring that relativized semantics coincide with the classical semantics

For the rainbow algebra, let

$$ W_r = \{ \bar{a} \in {}^n M : M \models ( \bigwedge_{i<j<n,l<n} \neg \rho(x_i, x_j))(\bar{a}) \}, $$

and for the Monk like algebra, $W_m$ is defined exactly the same way by discarding assignments whose edges are coloured by one of the $n$ reds $(\rho, i), i < n$. In the former case we are discarding assignments who have a $\rho$ labelled edge, and in the latter we are discarding assignments that involve edges coloured by any of the $n$ reds $(\rho, i), i < n$. We denote both by $W$ relying on context.

The $n$-homogeneity built into $M$, in all three cases by its construction implies that the set of all partial isomorphisms of $M$ of cardinality at most $n$ forms an $n$-back-and-forth system. But we can even go further. We formulate our statement for the Monk algebra based on $\mathcal{G}$ whose underlying set is $\mathbb{N}$ since this is the new case. (For the other case the reader is referred to [10] and [12]).

A definition: Let $\chi$ be a permutation of the set $\omega \cup \{\rho\}$. Let $\Gamma, \Delta \in \mathcal{G}\mathcal{G}$ have the same size, and let $\theta : \Gamma \to \Delta$ be a bijection. We say that $\theta$ is a $\chi$-isomorphism from $\Gamma$ to $\Delta$ if for each distinct $x, y \in \Gamma$,

- If $\Gamma(x, y) = (a, j)$ with $a \in \mathbb{N}$, then there exist unique $l \in \mathbb{N}$ and $r$ with $0 \leq r < N$ such that $a = Nl + r$. 

$$ \Delta(\theta(x), \theta(y)) = \begin{cases} (N\chi(i) + r, j), & \text{if } \chi(i) \neq \rho \\ (\rho, j), & \text{otherwise.} \end{cases} $$
If $Γ(x, y) = (ρ, j)$, then

$$\triangle(θ(x), θ(y)) \in \begin{cases} 
\{(Nχ(ρ) + s, j) : 0 \leq s < N\}, & \text{if } χ(ρ) \neq ρ \\
\{(ρ, j)\}, & \text{otherwise.}
\end{cases}$$

We now have for any permutation $χ$ of $ω \cup \{ρ\}$, $Θ^χ$ is the set of partial one-to-one maps from $M$ to $M$ of size at most $n$ that are $χ$-isomorphisms on their domains. We write $Θ$ for $Θ^{Id_{ω∪\{ρ\}}}$.  

We claim that for any any permutation $χ$ of $ω \cup \{ρ\}$, $Θ^χ$ is the set of partial one-to-one maps from $M$ to $M$ of size at most $n$ that are $χ$-isomorphisms on their domains. We write $Θ$ for $Θ^{Id_{ω∪\{ρ\}}}$.  

We claim that for any any permutation $χ$ of $ω \cup \{ρ\}$, $Θ^χ$ is an $n$-back-and-forth system on $M$. 

Clearly, $Θ^χ$ is closed under restrictions. We check the “forth” property. Let $θ ∈ Θ^χ$ have size $t < n$. Enumerate $\text{dom}(θ), \text{rng}(θ)$, respectively as $\{a_0, \ldots, a_{t-1}\}, \{b_0, \ldots b_{t-1}\}$, with $θ(a_i) = b_i$ for $i < t$. Let $a_t ∈ M$ be arbitrary, let $b_t ∈ M$ be a new element, and define a complete labelled graph $Δ ⊇ M ↾ \{b_0, \ldots, b_{t-1}\}$ with nodes $\{b_0, \ldots, b_t\}$ as follows.

Choose distinct "nodes" $e_s < N$ for each $s < t$, such that no $(e_s, j)$ labels any edge in $M ↾ \{b_0, \ldots, b_{t-1}\}$. This is possible because $N ≥ n(n−1)/2$, which bounds the number of edges in $Δ$. We can now define the colour of edges $(b_s, b_t)$ of $Δ$ for $s = 0, \ldots, t − 1$.

- If $M(a_s, a_t) = (Ni + r, j)$, for some $i ∈ N$ and $0 ≤ r < N$, then

$$\triangle(b_s, b_t) = \begin{cases} 
(Nχ(i) + r, j), & \text{if } χ(i) \neq ρ \\
\{(ρ, j)\}, & \text{otherwise.}
\end{cases}$$

- If $M(a_s, a_t) = (ρ, j)$, then assuming that $e_s = Ni + r$, $i ∈ N$ and $0 ≤ r < N$,

$$\triangle(b_s, b_t) = \begin{cases} 
(Nχ(ρ) + r, j), & \text{if } χ(ρ) \neq ρ \\
\{(ρ, j)\}, & \text{otherwise.}
\end{cases}$$

This completes the definition of $Δ$. It is easy to check that $Δ ∈ ΕΕ$. Hence, there is a graph embedding $ϕ : Δ → M$ extending the map $Id_{\{b_0, \ldots, b_{t-1}\}}$. Note that $ϕ(b_t) \notin \text{rng}(θ)$. So the map $θ^+ = θ ∪ \{(a_t, ϕ(b_t))\}$ is injective, and it is easily seen to be a $χ$-isomorphism in $Θ^χ$ and defined on $a_t$. The converse, “back” property is similarly proved ( or by symmetry, using the fact that the inverse of maps in $Θ$ are $χ^{-1}$-isomorphisms).

We now derive a connection between classical and relativized semantics in $M$, over the set $W$:
Recall that $W$ is simply the set of tuples $\bar{a}$ in $^nM$ such that the edges between the elements of $\bar{a}$ don’t have a label involving $\rho$ (these are $(\rho, i)i < n$). Their labels are all of the form $(Ni + r, j)$. We can replace $\rho$-labels by suitable $(a, j)$-labels within an $n$-back-and-forth system. Thus, we may arrange that the system maps a tuple $\bar{b} \in ^nM \setminus W$ to a tuple $\bar{c} \in W$ and this will preserve any formula containing no relation symbols $(a, j)$ that are “moved” by the system.

Indeed, we show that the classical and $W$-relativized semantics agree. $M \models_W \varphi(\bar{a})$ iff $M \models \varphi(\bar{a})$, for all $\bar{a} \in W$ and all $L^n$-formulas $\varphi$.

The proof is by induction on $\varphi$. If $\varphi$ is atomic, the result is clear; and the boolean cases are simple. Let $i < n$ and consider $\exists x_i \varphi$. If $M \models_W \exists x_i \varphi(\bar{a})$, then there is $\bar{b} \in W$ with $\bar{b} = _i \bar{a}$ and $M \models_W \varphi(\bar{b})$. Inductively, $M \models \varphi(\bar{b})$, so clearly, $M \models_W \exists x_i \varphi(\bar{a})$. For the (more interesting) converse, suppose that $M \models_W \exists x_i \varphi(\bar{a})$. Then there is $\bar{b} \in ^nM$ with $\bar{b} = _i \bar{a}$ and $M \models \varphi(\bar{b})$. Take $L_{\varphi, \bar{b}}$ to be any finite subsystem of $L$ containing all the symbols from $L$ that occur in $\varphi$ or as a label in $M \models \text{rng}(\bar{b})$. (Here we use the fact that $\varphi$ is first-order. The result may fail for infinitary formulas with infinite signature.) Choose a permutation $\chi$ of $\omega \cup \{\rho\}$ fixing any $i'$ such that some $(Ni' + r, j)$ occurs in $L_{\varphi, \bar{b}}$ for some $r < N$, and moving $\rho$. Let $\theta = Id_{\{a_m : m \neq i\}}$. Take any distinct $l, m \in \omega \setminus \{i\}$. If $M(a_l, a_m) = (Ni' + r, j)$, then $M(b_l, b_m) = (Ni' + r, j)$ because $\bar{a} = _i \bar{b}$, so $(Ni' + r, j) \in L_{\varphi, \bar{b}}$ by definition of $L_{\varphi, \bar{b}}$. So, $\chi(i') = i'$ by definition of $\chi$. Also, $M(a_l, a_m) \neq (\rho, j)(any \ \text{any})$ because $\bar{a} \in W$. It now follows that $\theta$ is a $\chi$-isomorphism on its domain, so that $\theta \in \Theta^\chi$. Extend $\theta$ to $\theta' \in \Theta^\chi$ defined on $b_i$, using the “forth” property of $\Theta^\chi$. Let $\bar{c} = \theta'(\bar{b})$. Now by choice of of $\chi$, no labels on edges of the subgraph of $M$ with domain $\text{rng}(\bar{c})$ involve $\rho$. Hence, $\bar{c} \in W$. Moreover, each map in $\Theta^\chi$ is evidently a partial isomorphism of the reduct of $M$ to the signature $L_{\varphi, \bar{b}}$. Now $\varphi$ is an $L_{\varphi, \bar{b}}$-formula. We have $M \models \varphi(\bar{a})$ iff $M \models \varphi(\bar{c})$. So $M \models W \varphi(\bar{c})$. Inductively, $M \models_W \varphi(\bar{c})$. Since $\bar{c} = _i \bar{a}$, we have $M \models_W \exists x_i \varphi(\bar{a})$ by definition of the relativized semantics. This completes the induction.

(5) **The atoms, the required atomic algebra**

Now let $L$ be $L^+$ without the red relation symbols. The logics $L_n$ and $L^n_{\omega \omega}$ are taken in this signature.

For an $L^n_{\omega \omega}$-formula $\varphi$, define $\varphi^W$ to be the set $\{\bar{a} \in W : M \models_W \varphi(\bar{a})\}$.

Then the set algebra $\mA$ (actually in all three cases) is taken to be the relativized set algebra with domain

$$\{\varphi^W : \varphi \text{ a first-order } L_n - \text{formula}\}$$
and unit $W$, endowed with the algebraic operations $d_{ij}, c_i,$ etc., in the standard way, and of course formulas are taken in the suitable signature. The completion of $\mathfrak{A}$ is the algebra $\mathfrak{C}$ with universe $\{\phi^W : \phi \in L_{n,\omega}^n\}$ with operations defined as for $\mathfrak{A}$, namely, usual cylindrifiers and diagonal elements, reflecting existential quantifiers, polyadic operations and equality. Indeed, $\mathfrak{A}$ is a representable (countable) atomic polyadic algebra of dimension $n$.

Let $\mathcal{S}$ be the polyadic set algebra with domain $\wp(\mathcal{M})$ and unit $n\mathcal{M}$. Then the map $h: \mathfrak{A} \rightarrow \mathcal{S}$ given by $h: \phi^W \mapsto \{\bar{a} \in n\mathcal{M} : \mathcal{M} \models \phi(\bar{a})\}$ can be checked to be well-defined and one-one. It clearly respects the polyadic operations, also because relativized semantics and classical semantics coincide on $L_n$ formulas in the given signature, this is a representation of $\mathfrak{A}$.

(6) The atoms

A formula $\alpha$ of $L_n$ is said to be $MCA$ ("maximal conjunction of atomic formulas") if (i) $M \models \exists x_0 \ldots x_{n-1} \alpha$ and (ii) $\alpha$ is of the form

$$\bigwedge_{i \neq j < n} \alpha_{ij}(x_i, x_j),$$

where for each $i, j$, $\alpha_{ij}$ is either $x_i = x_j$ or $R(x_i, x_j)$ for some binary relation symbol $R$ of $L$.

Let $\varphi$ be any $L_{\infty,\omega}^n$-formula, and $\alpha$ any $MCA$-formula. If $\varphi^W \cap \alpha^W \neq \emptyset$, then $\alpha^W \subseteq \varphi^W$. Indeed, take $\bar{a} \in \varphi^W \cap \alpha^W$. Let $\bar{a} \in \alpha^W$ be arbitrary. Clearly, the map $(\bar{a} \mapsto \bar{b})$ is in $\Theta$. Also, $W$ is $L_{\infty,\omega}^n$-definable in $\mathcal{M}$, since we have

$$W = \{\bar{a} \in n\mathcal{M} : \mathcal{M} \models (\bigwedge_{i < j < n} (x_i = x_j \lor \bigvee_{R \in L} R(x_i, x_j)))(\bar{a})\}.$$ 

We have $M \models_W \varphi(\bar{a})$ iff $M \models_W \varphi(\bar{b})$. Since $M \models_W \varphi(\bar{a})$, we have $M \models_W \varphi(\bar{b})$. Since $\bar{b}$ was arbitrary, we see that $\alpha^W \subseteq \varphi^W$. Let

$$F = \{\alpha^W : \alpha \text{ an } MCA, L^n - \text{formula}\} \subseteq \mathfrak{A}.$$ 

Evidently, $W = \bigcup F$. We claim that $\mathfrak{A}$ is an atomic algebra, with $F$ as its set of atoms. First, we show that any non-empty element $\varphi^W$ of $\mathfrak{A}$ contains an element of $F$. Take $\bar{a} \in W$ with $M \models_W \varphi(\bar{a})$. Since $\bar{a} \in W$, there is an $MCA$-formula $\alpha$ such that $M \models_W \alpha(\bar{a})$. Then $\alpha^W \subseteq \varphi^W$. By definition, if $\alpha$ is an $MCA$ formula then $\alpha^W$ is non-empty. If $\varphi$ is an $L^n$-formula and $\emptyset \neq \varphi^W \subseteq \alpha^W$, then $\varphi^W = \alpha^W$. It follows that each $\alpha^W$ (for $MCA \alpha$) is an atom of $\mathfrak{A}$.
Now for the rainbow signature, a formula $\alpha$ of $MCA$ formulas of $L_n$ are adapted to the rainbow signature. $\alpha$ is such if $M \models \exists x_0 \ldots x_{n-1} \alpha$ and (ii) $\alpha$ is of the form

$$\bigwedge_{i \neq j < n} \alpha_{ij}(x_i, x_j) \land yS(x_0, \ldots x_{n-1}),$$

where for each $i, j$, $\alpha_{ij}$ is either $x_i = x_j$ or $R(x_i, x_j)$ for some binary relation symbol $R$ of the rainbow signature. In this case there is a one to one correspondence between coloured graphs whose edges do not involve the red shade and the $MCA$ formulas, and both are the atom of $\mathfrak{A}$, and for that matter its completion. (Here any $s$ satisfying an $MCA$ formula defines a coloured graph and for any other $s'$, the graph determined by it is isomorphic to that determined by $s$; and that’s why precisely $MCA$ formulas are atoms, namely, surjections from $n$ to coloured graphs.

(7) The complex algebra

Define $\mathfrak{C}$ to be the complex algebra over $\text{At}\mathfrak{A}$, the atom structure of $\mathfrak{A}$. Then $\mathfrak{C}$ is the completion of $\mathfrak{A}$. The domain of $\mathfrak{C}$ is $\phi(\text{At}\mathfrak{A})$. The diagonal $d_{ij}$ is interpreted as the set of all $S \in \text{At}\mathfrak{A}$ with $a_i = a_j$ for some $\bar{a} \in S$. The cylindrification $c_i$ is interpreted by $c_i X = \{ S \in \text{At}\mathfrak{A} : S \subseteq c_i^3(S') \text{ for some } S' \in X \}$, for $X \subseteq \text{At}\mathfrak{A}$. Finally $p_{ij} X = \{ S \in \text{At}\mathfrak{A} : S \subseteq p_{ij}^3(S') \text{ for some } S' \in X \}$. Let $\mathfrak{D}$ be the relativized set algebra with domain $\{ \phi^W : \phi \text{ an } L_{\omega}^\infty \text{ formula } \}$, unit $W$ and operations defined like those of $\mathfrak{A}$.

$\mathfrak{C} \cong \mathfrak{D}$, via the map $X \mapsto \bigcup X$.

The map is clearly injective. It is surjective, since

$$\phi^W = \bigcup \{ \alpha^W : \alpha \text{ an } MCA\text{-formula, } \alpha^W \subseteq \phi^W \}$$

for any $L_{\omega}^\infty$ formula $\phi$. Preservation of the Boolean operations and diagonals is clear. We check cylindrifications. We require that for any $X \subseteq \text{At}\mathfrak{A}$, we have $\bigcup c_i^3 X = c_i^3(\bigcup X)$ that is

$$\bigcup \{ S \in \text{At}\mathfrak{A} : S \subseteq c_i^3 S' \text{ for some } S' \in X \} =$$

$$\{ \bar{a} \in W : \bar{a} \equiv_i \bar{a}' \text{ for some } \bar{a}' \in \bigcup X \}. $$

Let $\bar{a} \in S \subseteq c_i S'$, where $S' \in X$. So there is $\bar{a}' \equiv_i \bar{a}$ with $\bar{a}' \in S'$, and so $\bar{a}' \in \bigcup X$.

Conversely, let $\bar{a} \in W$ with $\bar{a} \equiv_i \bar{a}'$ for some $\bar{a}' \in \bigcup X$. Let $S \in \text{At}\mathfrak{A}$, $S' \in X$ with $\bar{a} \in S$ and $\bar{a}' \in S'$. Choose $MCA$ formulas $\alpha$ and $\alpha'$ with
\( S = \alpha^W \) and \( S' = \alpha'^W \). then \( a \in \alpha^W \cap (\exists x;\alpha')^W \) so \( \alpha^W \subseteq (\exists x;\alpha')^W \), or \( S \subseteq \varepsilon_i(S') \). The required now follows. We leave the checking of substitutions to the reader.

The non representability of the complex algebra in the case of the rainbow algebra is in [10], witness also the argument in the coming Theorem 3.10.

(8) Using the Monk algebras

To prove item (2) we use Monk’s algebras. Any of the two will do just as well:

We show that their atom structure consists of the \( n \) basic matrices of a relation algebra. In more detail, we define a relation algebra atom structure \( \alpha(G) \) of the form \((\{1'\} \cup (G \times n), R_1, \hat{R}, R_i)\). The only identity atom is \( 1' \). All atoms are self converse, so \( \hat{R} = \{(a, a) : a \text{ an atom }\} \). The colour of an atom \((a, i) \in G \times n \) is \( i \). The identity \( 1' \) has no colour. A triple \((a, b, c)\) of atoms in \( \alpha(G) \) is consistent if \( R; (a, b, c) \) holds. Then the consistent triples are \((a, b, c)\) where

- one of \( a, b, c \) is \( 1' \) and the other two are equal, or
- none of \( a, b, c \) is \( 1' \) and they do not all have the same colour, or
- \( a = (a', i), b = (b', i) \) and \( c = (c', i) \) for some \( i < n \) and \( a', b', c' \in G \), and there exists at least one graph edge of \( G \) in \( \{a', b', c'\} \).

\( \alpha(G) \) can be checked to be a relation atom structure. It is exactly the same as that used by Hirsch and Hodkinson in [20], except that we use \( n \) colours, instead of just 3, so that it a Monk algebra not a rainbow one. However, some monochromatic triangles are allowed namely the ‘dependent’ ones. This allows the relation algebra to have an \( n \) dimensional cylindric basis and, in fact, the atom structure of of \( M(G) \) is isomorphic (as a cylindric algebra atom structure) to the atom structure \( \text{Mat}_n \) of all \( n \)-dimensional basic matrices over the relation algebra atom structure \( \alpha(G) \).

Indeed, for each \( m \in \text{Mat}_n \), let \( \alpha_m = \bigwedge_{i,j<n} \alpha_{ij}. \) Here \( \alpha_{ij} \) is \( x_i = x_j \) if \( m_{ij} = 1' \) and \( R(x_i, x_j) \) otherwise, where \( \hat{R} = m_{ij} \in L \). Then the map \( (m \mapsto \alpha_m)^W \) is a well-defined isomorphism of \( n \)-dimensional cylindric algebra atom structures.

We show \( C\alpha(G) \) is not in \( \text{RRA} \). Hence the full complex cylindric algebra over the set of \( n \) by \( n \) basic matrices - which is isomorphic to \( \mathcal{C} \) is not in \( \text{RCA}_n \) for we have a relation algebra embedding of \( C\alpha(G) \) onto \( \text{RaCm}\mathcal{M}_n \).
Assume for contradiction that \(Cm\alpha(G) \in RCA_n\) for \(Y \subseteq \mathbb{N}\) and \(s < n\), set

\[ [Y, s] = \{(l, s) : l \in Y\}. \]

For \(r \in \{0, \ldots, N - 1\}\), \(NN + r\) denotes the set \(\{Nq + r : q \in \mathbb{N}\}\). Let

\[ J = \{1', [NN + r, s] : r < N, s < n\}. \]

Then \(\sum J = 1\) in \(Cm\alpha(G)\). As \(J\) is finite, we have for any \(x, y \in X\) there is a \(P \in J\) with \((x, y) \in h(P)\). Since \(Cm\alpha(G)\) is infinite then \(X\) is infinite. By Ramsey’s Theorem, there are distinct \(x_i \in X (i < \omega)\), \(J \subseteq \omega \times \omega\) infinite and \(P \in J\) such that \((x_i, x_j) \in h(P)\) for \((i, j) \in J, i \neq j\). Then \(P \neq 1'\). Also \((P; P) \neq 0\). If \(x, y, z \in M, a, b, c \in Cm\alpha(G), (x, y) \in h(a), (y, z) \in h(b), \) and \((x, z) \in h(c)\), then \((a; b) \cdot c \neq 0\). A non-zero element \(a\) of \(Cm\alpha(G)\) is monochromatic, if \(a \leq 1', a \leq [N, s]\) for some \(s < n\). Now \(P\) is monochromatic, it follows from the definition of \(\alpha\) that \((P; P) \cdot P = 0\).

This contradiction shows that \(Cm\alpha(G)\) is not representable and so \(CmMat_{n\alpha}(G)\) is non representable

(9) **Summarizing**

We have in all cases a labelled graph defined \(M\) as a model of a first order theory in the Monk case and of \(L\omega_1\omega\) in the rainbow case. We have also relativized \(^nM\) to \(W \subseteq \_nM\) by deleting assignments whose edges involve reds, and we defined an algebra containing the term algebra, namely, \(\mathfrak{A}\) as the atomic relativized set algebra with unit \(W\). In the case of Monk’s algebra we defined the relation algebra \(R\), such that \(At\mathfrak{A} \cong \mathfrak{Mat}_{n}At\mathfrak{A}\), and in all cases the complex algebra \(\mathfrak{C}\) of the atom structure is isomorphic to the set algebra \(\{\phi^M : \phi \in L^\omega_{\infty}\}\). Finally, we the \(Df\) reduct of this algebra, namely, \(\mathfrak{C} = CmAt\mathfrak{A}\), cannot be representable for else this induces a complete representation of \(\mathfrak{R}_{\alpha} \mathfrak{A}\), hence a complete representation of \(\mathfrak{A}\), which in turn induces a representation of \(\mathfrak{C}\), but we have shown that the latter cannot be representable.

**Corollary 3.4.** Let \(L\) be any signature between \(Df\) and \(PEA\).

1. There is an \(L\) atom structure that is weakly representable, but not strongly representable. In particular, there is an \(L\) atom structure, that carries two (atomic) algebras, one is representable, the other is not.

2. The class of \(L\) representable algebras is not atom-canonical, it is not closed under Dedekind-MacNeille completions and cannot be axiomatized by Sahlqvist equations.
The same holds for the class of representable relation algebras

Proof. A special case of theorem 5.3 below.

In contrast, we have:

Theorem 3.5. Let $\mathcal{R}$ and $M$ be as above. If $\mathcal{R}$ is finitely generated and $\text{Th}M$ is $\omega$ categorial or $M$ is ultrahomogeneous, then $\mathcal{M} \text{Mat}_{\omega} \text{At}\mathcal{R}$ is strongly representable.

Proof. Assume that $M$ is ultrahomogeneous and that $\mathcal{R}$ is finitely generated. Then $M$ is a countable model in binary relational finite signature, hence it has quantifier elimination. Since the relativized semantics for $L^n$ formulas coincide with the classical one, we have that $\mathfrak{A} \in \mathfrak{M}_n \mathcal{C}A_\omega$. (Remember that semantics is perturbed at formulas in $L^n_{\omega,\infty}$.) Now we show that $\mathfrak{A}$ is completely representable, then we build from this complete representation a representation of the complex algebra. Let $X = \text{At}\mathfrak{A}$. Let $\mathfrak{B} \in \mathfrak{L}_\omega$ be such that $\mathfrak{A} = \mathfrak{M}_n \mathfrak{B}$ and $A$ generates $\mathfrak{B}$, hence $\sum X = 1$ in $\mathfrak{B}$. Let $Y$ be the set of co-atoms, then $Y \subseteq \mathfrak{M}_n \mathfrak{B}$ is a non-principal type. By the usual Orey Henkin omitting types theorem for first order logic, we get a model, that is a representation that omits $Y$. That is for every non-zero $b \in \mathfrak{B}$, we get a set algebra $\mathfrak{C}$ having countable base, and $f : \mathfrak{B} \to \mathfrak{C}$ such that $f(\alpha) \neq 0$ and $\bigcap_{y \in Y} f(y) = \emptyset$. Define $g(x) = \{ s \in U^n : s^+ \in f(x) \}$, where $s^+ = s$ on $n$ and fixes all other points in $\omega \sim n$, then $g$ gives a complete simple representation such that $g(\alpha) \neq 0$. Taking disjoint unions over the non zero elements in $\mathfrak{A}$ as is common practise in algebraic logic, we get the desired complete representation, call it $f$.

Next we represent the complex algebra $\mathfrak{C}$. Recall that $\mathfrak{C}$ is the completion of $\mathfrak{A}$. Let $c \in C$, let $M$ be the base of the complete representation of $\mathfrak{A}$, and let $c^* = \{ x \in \mathfrak{A} : x \leq c \}$. Define $g(c) = \bigcup_{x \in c^*} f(x)$, then clearly $g$ defines a representation of $\mathfrak{C}$ into $\wp(nM)$, and we are done.

Recall that we dealt with relativized algebras in our previous proof. For an algebra $\mathfrak{A}$ and a logic $L$, that is an extension of $n$ variable first order logic, $L(A)$ denotes the signature (taken in this logic) obtained by adding an $m$ relation symbol for every element of $\mathfrak{A}$. The notion of relativized representations is indeed important in algebraic logic. $M$ is a relativized representation of an abstract algebra $\mathfrak{A}$, if there exists $V \subseteq mM$, and an injective homomorphism $f : \mathfrak{A} \to \wp(V)$. We write $M \models 1(\bar{s})$, if $\bar{s} \in V$.

Definition 3.6. In the next definition $m$ denotes the dimension.

(1) Let $M$ be a relativized representation of a $\mathcal{CA}_m$. A clique in $M$ is a subset $C$ of $M$ such that $M \models 1(\bar{s})$ for all $\bar{s} \in mC$. For $n > m$, let $C^n(M) = \{ \bar{a} \in nM : \text{rng}(\bar{a}) \text{ is a clique in } M \}$.
(2) Let $\mathfrak{A} \in \mathbf{CA}_n$, and $M$ be a relativized representation of $\mathfrak{A}$. $M$ is said to be $n$ square, $n > m$, if whenever $\bar{s} \in C^n(M)$, $a \in A$, and $M \models c_ia(\bar{s})$, then there is a $t \in C^n(M)$ with $i \equiv \bar{s}$, and $M \models a(t)$.

(3) $M$ is infinitary $n$ flat if for all $\phi \in L(A)^{\omega,\omega}$, for all $\vec{a} \in C^n(M)$, for all $i,j < n$, we have

$$M \models \exists x_1 \exists x_2 \phi \iff \exists x_1 \exists x_2 \phi(\vec{a}).$$

$M$ is just $n$ flat, if the above holds for first order formulas using $n$ variables, namely, for $\phi \in L(A)^{n}$.

(4) $M$ is said to be $n$ square if it is $n$ square, and there is an equivalence relations $E^t$, for $t = 1, \ldots, n$ on $C^n(M)$ such that $\Theta = \{(\vec{x} \to \vec{y}) : (\vec{x}, \vec{y}) \in \bigcup_{1 \leq i \leq n} E^i\}$ (here $\vec{x} \to \vec{y}$ is the map $x_i \mapsto y_i$) is an $n$ back and for system of partial isomorphisms on $M$.

Note that all of the above are a local form of representation where roughly cylindrifiers have witnesses only on $< m$ cliques. Closely related to relativized representations are the notion of hyperbasis. This is only formulated for relation algebras [20, 12.2.2]. We extend it to the polyadic case, and all its reducts down to $\mathbf{Scs}$.

**Definition 3.7.** Let $\mathfrak{A} \in \mathbf{PEA}_j$. Let $j + 1 \leq m \leq n \leq k < \omega$, and let $\Lambda$ be a non-empty set. An $n$ wide $m$ dimensional $\Lambda$ hypernetwork over $\mathfrak{A}$ is a map $N : \leq m \to \Lambda \cup \mathbf{At}\mathfrak{A}$ such that $N(\vec{x}) \in \mathbf{At}\mathfrak{A}$ if $|\vec{x}| = j$ and $N(\vec{x}) \in \Lambda$ if $|\vec{x}| \neq j$, (so that $j$ edges are labelled by atoms, and other edges with labels from $\Lambda$, with the following properties for each $i,k < j$, $\delta \in \leq m$, $\vec{z} \in \leq m - 2$ and $d \in m$:

- $N(\delta^j_i) \leq d_i$
- $N(x, x, \vec{z}) \leq d_{01}$
- $N(\delta[i \to d]) \leq c_i N(\delta)$
- $N(\delta \circ [i, j]) = s_{[i,j]} N(\delta)$.
- If $\vec{x}, \vec{y} \in \leq m$, $|\vec{x}| = |\vec{y}|$ and $\exists \vec{z}$, such that $N(x_i, y_i, \vec{z}) \leq d_{01}$, for all $i < |\vec{x}|$, then $N(\vec{x}) = N(\vec{y})$
- when $n = m$, then $N$ is called an $n$ dimensional $\Lambda$ hypernetwork.

Then $n$ wide $m$ dimensional hyperbasis can be defined like the relation algebra case [20], where the amalgamation property is the most important (it corresponds to commutativity of cylindrifiers).

For an $n$ wide $m$ dimensional $\Lambda$ hypernetworks, and $\vec{x} \in \leq m$, we define $N \equiv_x M$ if $N(\vec{y}) = M(\vec{y})$ for all $\vec{y} \in \leq m$, for $\vec{y} \in \leq m$. In more detail:
Definition 3.8. The set of all \( n \) wide \( m \) dimensional hypernetworks will be denoted by \( H^n_m(\mathfrak{A}, \Lambda) \). An \( n \) wide \( m \) dimensional \( \Lambda \), with \( j + 1 \leq m \leq n \) hyperbasis for \( \mathfrak{A} \) is a set \( H \subseteq H^n_m(\mathfrak{A}, \Lambda) \) with the following properties:

- For all \( a \in \text{At}\mathfrak{A} \), there is an \( N \in H \) such that \( N(0, 1, \ldots, j) = a \)
- For all \( N \in H \) all \( \bar{x} \in \text{nodes}(N) \), for all \( i < j \) for all \( a \in \text{At}\mathfrak{A} \) such that \( N(\bar{x}) \leq c_a \), there exists \( \bar{y} \equiv_i \bar{x} \) such that \( N(\bar{y}) = a \)
- For all \( M, N \in H \) and \( x, y < n \), with \( M \equiv_x y N \), there is \( L \in H \) such that \( M \equiv_x L \equiv_y N \). Here \( M \equiv_S N \), means that \( M \) and \( N \) agree off of \( S \).
- Assume that \( j + 1 \leq m \leq n \leq k \). For a \( k \) wide \( n \) dimensional hypernetwork \( N \), we let \( N|_m^k \) the restriction of the map \( N \) to \( \leq^k m \). So we restrict the first \( m \) nodes but keep all hyperlabels on them. For \( H \subseteq H^k_n(\mathfrak{A}, \Lambda) \) we let \( H|_m^n = \{ N|_m^k : N \in H \} \).
- When \( n = m \), \( H^n_m(\mathfrak{A}, \Lambda) \) is called an \( n \) dimensional hyperbases.

We say that \( H \) is symmetric, if whenever \( N \in H \) and \( \sigma : m \to m \), then \( N \circ \sigma \in H \), the latter, which we denote simply by \( N\sigma \) is defined by \( N\sigma(\bar{x}) = N(\sigma(x_0), \ldots, \sigma(x_n)) \), for \( \bar{x} \in \text{nodes}(N) \).

When \( n = m \), then \( H^n_m(\mathfrak{A}, \Lambda) \) denoted simply by \( H^n_m(\mathfrak{A}, \Lambda) \) is called an \( n \) dimensional hyperbasis. If an addition, \( |\Lambda| = 1 \), then \( H^n_m(\mathfrak{A}, \Lambda) \) is called an \( n \) dimensional cylindric basis. (These were defined by Maddux, and later generalized by Hirsch and Hodkinson to hyperbasis).

Lemma 3.9. If \( \mathfrak{A} \in \text{PEA}_m \) has an \( n \) dimensional hyperbasis then \( \mathfrak{A} \) has a relativized \( n \) smooth representation.

Proof. We build a relativized representation \( M \) in a step- by-step fashion; it will be a labelled hypergraph satisfying the the following properties

1. Each hyperedge is labelled by an atom of \( \mathfrak{A} \)
2. \( M(\bar{x}) \leq d_{ij} \) iff \( x_i = x_j \). (In this case, we say that \( M \) is strict).
3. For any clique \( \{x_0, \ldots, x_{n-1}\} \subseteq M \), there is a unique \( N \in H \) such that \( (x_0, \ldots, x_{n-1}) \) is labelled by \( N \) and we write this as \( M(x_0, \ldots, x_{n-1}) = N \)
4. If \( x_0, \ldots, x_{n-1} \in M \) and \( M(\bar{x}) = N \in H \), then for all \( i_0 \ldots i_{m-1} < n \), \( (x_{i_0}, \ldots, x_{i_{m-1}}) \) is a hyperedge and \( M(x_{i_0}, x_{i_1}, \ldots, x_{i_{m-1}}) = N(i_0, \ldots, i_{m-1}) \in \text{At}\mathfrak{A} \).
(5) $M$ is symmetric; it closed under substitutions. That is, if $x_0, \ldots, x_{n-1} \in M$ are such that $M(x_0, \ldots, x_{n-1}) = N$, and $\sigma : n \to n$ is any map, then we have $M(x_\sigma(0), \ldots, x_\sigma(n-1)) = N\sigma$.

(6) If $\bar{x}$ is a clique, $k < n$ and $N \in H$, then $M(\bar{x}) \equiv_k N$ if and only if there is $y \in M$ such that $M(x_0, \ldots, x_{k-1}, y, x_{k+1}, \ldots, x_{n-1}) = N$.

(7) For every $N \in H$, there are $x_0, \ldots, x_{n-1} \in M$, $M(\bar{x}) = N$.

We build a chain of hypergraphs $M_t : t < \omega$ their limit (defined in a precise sense) will as required. Each $M_t$ will have hyperedges of length $m$ labelled by atoms of $\mathfrak{A}$, the labelling of other hyperedges will be done later.

To avoid confusion we call hyperedges that are to be labelled by atoms atom-hyperedges, these have length exactly $m$, and those that have $\leq n$ length not equal to $m$ (recall that $3 \leq m < n$) that are labelled by elements in $n$ simply by hyperedges.

We proceed by a step by step manner, where the defects are treated one by one, and they are diffused at the limit obtaining the required hypergraph, which also consists of two parts, namely, atom-hyperedges and hyperedges.

This limiting hypergraph will be compete (all atom hypergraphs and hypergraphs will be labelled; which might not be the case with $M_t$ for $t < \omega$.)

Every atom-hyperedge will indeed be labelled by an atom of $\mathfrak{A}$ and hyperedges with length $\neq m$ will also be labelled, but by indices from $n$.

We require inductively that $M_t$ also satisfies:

Any clique in $M_t$ is contained in $\text{rng}(v)$ for some $N \in H$ and some embedding $v : N \in M_t$, such that $v : n \to \text{dom}(M_t)$ and this embedding satisfies the following two conditions:

(a) if $(v(i_0), \ldots, v(i_{m-1}))$ is a an atom hyperedge of $M_t$, then $M_t(v) = N(i_0, \ldots, i_{m-1})$.

(b) Whenever $a \in \leq^a n$ with $|a| \neq m$, then $v(a)$ is a hyperedge of $M_t$ and is labelled by $N(\bar{a})$.

(Note that an embedding might not be injective). A network is strict if $N(\bar{x}) \leq d_{ij}$, then $x_i = x_j$. For the base of the induction we construct $M_0$. We proceed as follows. Let $N$ be a hypernetwork and let $S \subseteq n$. $N|S$ is maximal strict if it is strict and any for $T \subseteq n$, such that $T$ contains $S$, $N|T$ is not strict.

Let $M_0$ be the disjoint union of all maximal strict labelled hypergraph $N|S$ where $N \in H$ and $S \subseteq N$ and the atom-hyperedges and hyperedges are induced (the natural way) from $N$. For all $i < n$, there is a unique $s_i \in S$, and $\bar{z}$, such that $N(i, s_i, \bar{z}) \leq d_{01}$, then set $v(i) = s_i$, for $i < n$.

Now assume inductively that $M_t$ has been defined for all $t < \omega$.

We now define $M_{t+1}$ such that for every quadruple $(N, v, k, N')$ where $N, N' \in H$, $k < n$ and $M \equiv_k N'$ and $v : N \to M_t$ is an embedding, then the restriction $v \upharpoonright n \sim \{k\}$ extends to an embedding $v' ; N' \to M_{t+1}$.
For each such \((N, v, k, N')\) we add just one new node \(\pi\), and we add the following atom hyperedges \((\pi, v(i_1)\ldots v(i_{m-2})\) for each \(i_1, \ldots, i_{m-2} \in n \sim k\) that are pairwise distinct. Such new atom-hyperedges are labelled by \(N'(k, i_0, \ldots, i_{m-2})\). This is well defined. We extend \(v\) by defining \(v'(k) = \pi\).

We add a new hyperedge \(v'(\bar{a})\) for every \(\bar{a} \in {}^n n\) of length \(\neq m\), with \(k \in \text{rng}(\bar{a})\) labelled by \(N'(\bar{a})\).

Then \(M_{t+1}\) will be \(M_t\) with its old labels, atom hyperedges, labelled hyperedges together with the new ones define as above.

It is easy to check that the inductive hypothesis are preserved.

Now define a labelled hypergraph as follows: \(\text{nodes}(M) = \bigcup \text{nodes}(M_i)\), for any \(\bar{x}\) that is an atom-hyperedge; then it is a one in some \(M_t\) and its label is defined in \(M\) by \(M_t(\bar{x})\).

The hyperedges are \(n\) tuples \((x_0, \ldots, x_{n-1})\). For each such tuple, we let \(t < \omega\), such that \(\{x_0 \ldots x_{n-1}\} \subseteq M_t\), and we set \(M(x_0, \ldots, x_{n-1})\) to be the unique \(N \in H\) such that there is an embedding \(v : N \to M\) with \(\bar{x} \subseteq \text{rng}(v)\). This can be easily checked to be well defined. We check existence and uniqueness.

The latter is clear from the definition of embedding. Note that there is an \(N \in H\) and an embedding \(v : N \to M_i\) with \((x_0, \ldots, x_{n-1}) \subseteq \text{rng}(v)\). So take \(\tau : n \to n\), such that \(x_i = v(\tau(i))\) for each \(i < n\). As \(H\) is symmetric, \(N\tau \in H\), and clearly \(v \circ \tau : N\tau \to M_i\) is also an embedding. But \(x_i = v \circ \tau(u)\) for each \(i < n\), then let \(M(x_0, \ldots, x_{n-1}) = N\tau\).

Now we show that \(M\) is an \(n\) square relativized representation. Let \(L(A)\) be the signature obtained by adding an \(n\) ary relation symbol for each element of \(A\). Define \(M \models r(\bar{x})\) if \(\bar{x}\) is an atom hyperedge and \(M(\bar{x}) \leq r\).

Now let \(\bar{x} \in C^n(M)\), \(k < m\) and \(M \models r(\bar{x})\). We require that there exists \(y \in C^n(M)\) \(y \equiv_k x\) and \(M \models r(\bar{y})\). Take \(i_0, \ldots, i_{m-1} < n\) different from \(k\). Now \(M(\bar{x}) = N\) for some \(N \in H\). By properties of hyperbasis, there is \(P \equiv H\) with \(P \equiv_k N\). Hence \(P(i_0, \ldots, i_{k-1}, k, i_{k+1} \ldots) \leq r\). But by properties of \(M\), there is an \(n\) tuple \(\bar{y} \equiv k\bar{x}\) such that \(M(y) = P\). Then \(\bar{y} \in C^n(M)\) is as required.

Finally, for \(n\) smoothness we need to define equivalence relations \(E^n\) on \(C^n(M)\), satisfying the definition, for \(0 \leq m \leq n\). For each such \(m\) and \(\bar{a} \in C^n(M)\), define \(a^* = (a_0, a_1, \ldots, a_{m-1}, a_0, \ldots, a_0) \in C^n(M)\), where we have \(n\) copies of \(a_0\) after \(a_0, a_1, \ldots, a_{m-1}\). Now for \(\bar{a}, \bar{b} \in C^n(M)\), set

\[
E^n(\bar{a}, \bar{b}) \iff M(a^*) = M(b^*).
\]

This is as required.

\[\square\]

**Theorem 3.10.** For every \(n \geq 3\) there exists a countable atomic \(\text{PEA}_n\), such that the \(\text{CA}\) reduct of its completion does not have an \(n + 4\) smooth representation, in particular, it is not representable. Furthermore, its \(\text{DF}\) reduct is not representable.
Proof. Here we closely follow [10]; but our reds and greens are finite, so we obtain a stronger result. We take $|G| = n + 2$, and $R = n + 1$. Let $L^+$ be the rainbow signature consisting of the binary relation symbols $g_i : i < n - 1, g_0^i : i < n + 2, w, w_i : i < n - 2, r_j^i (i < n + 1, j < k < n)$ and the $(n - 1)$ ary-relation symbols $y_S : S \subseteq n + 2$), together with a shade of red $\rho$ that is outside the rainbow signature but is a binary relation in the sense that it can label edges of coloured graphs. Let $\mathcal{G}$ be the class of corresponding rainbow coloured graphs. By the same methods as above, there is a countable model $M \in \mathcal{G}$ with the following property:

- If $\triangle \subseteq \triangle' \in \mathcal{G}$, $|\triangle'| \leq n$, and $\theta : \triangle \to M$ is an embedding, then $\theta$ extends to an embedding $\theta' : \triangle' \to M$. Now let $W = \{ \bar{a} \in {}^n M : M \models (\bigwedge_{i < j < k < n} - \rho(x_i, x_j))(\bar{a}) \}$. Then $\mathfrak{A}$ with universe $\{ \phi^W : \phi \in L_n \}$ and operations defined the usual way, is representable, and its completion, the complex algebra over the above rainbow atom structure, $\mathcal{C}$ has universe $\{ \phi^W : \phi \in L^w_n \}$.

We show that $\mathfrak{C}$ is as desired. Assume, for contradiction, that $g : \mathfrak{C} \to \varphi(V)$ induces a relativized $n + 4$ flat representation. Then $V \subseteq {}^n N$ and we can assume that $g$ is injective because $\mathfrak{C}$ is simple. First there are $b_0, \ldots, b_{n - 1} \in N$ such $\bar{b} \in h(y_{n + 2}(x_0, \ldots, x_{n - 1}))^W$, cf [10] lemma 5.7. This tuple will be the base of finitely many cones, that will be used to force an inconsistent triple of reds. This is because $y_{n + 2}(\bar{x})^W \neq \emptyset$. For any $t < n + 3$, there is a $c_t \in N$, such that $\bar{b}_t = (b_0, \ldots, b_{n - 2}, c_t)$ lies in $h(g_0^t(x_0, x_{n - 1}))^W$ and in $h(g_i(x_i, x_{n - 1}))^W$ for each $i$ with $1 \leq i \leq n - 2$, cf [10] lemma 5.8. The $c_t$'s are the apexes of the cones with base $y_{n + 2}$. Take the formula

$$\phi_t = y_{n + 2}(x_0, \ldots, x_{n - 2}) \to \exists x_{n - 1} (g^i_0(x_0, x_{n - 1})) \land \bigwedge_{1 \leq i \leq n - 2} g_i(x_i, x_{n - 1}),$$

then $\phi_t^W = W$. Pick $c_t$ and $\bar{b}_t$ as above, and define for each $s < t < n + 3$, $c_{st}$ to be $(c_s, b_1, \ldots, b_{n - 2}, c_t) \in {}^n N$. Then $\bar{c}_{st} \notin h((x_0, \ldots, x_{n - 1})^W$. Let $\mu$ be the formula

$$x_0 = x_{n - 1} \lor w_0(x_0, x_{n - 1}) \lor \bigvee g(x_0, x_{n - 1}),$$

the latter formula is a first order formula consisting of the disjunction of the (finitely many ) greens. For $j < k < n$, let $R_{jk}$ be the $L_n$-formula

$$\bigvee_{i < o} r_j^i(x_0, x_{n - 1}).$$

Then $\bar{c}_{st} \notin h(\mu^W)$, now for each $s < t < n + 3$, there are $j < k < n$ with $c_{st} \in h(R_{jk})^W$. By the pigeon- hole principle, there are $s < t < n + 3$ and $j < k < n$ with $\bar{c}_{os}, \bar{c}_{ot} \in h(R_{jk})^W$. We have also $\bar{c}_{st} \in h(R_{j,k'}^W)$ for some $j', k'$ then the sequence $(c_0, c_s, b_1, \ldots, b_{n - 2}, \ldots, c_t) \in h(\chi^W)$ where

$$\chi = (\exists \bar{R}_{jk})(\exists x_{n - 1} (x_{n - 1} = x_1 \land \exists x_{j} R_{jk}) \land (\exists x_0 (x_0 = x_1) \land \exists x_{j} R_{j,k'})),$$

so $\chi^W \neq \emptyset$. Let $\bar{a} \in \chi^W$. Then $M \models W R_{jk}(a_0, a_{n - 1}) \land R_{jk}(a_0, a_1) \land R_{j,k'}(a_1, a_{n - 1})$. Hence there are $i, i'$ and $i'' < \omega$ such that

$$M \models W r_{jk}(a_0, a_{n - 1}) \land r_{jk}^{i''}(a_0, a_1) \land r_{j,k'}^{i''}(a_1, a_{n - 1}),$$

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cf. [10, lemma 5.12]. But this triangle is inconsistent. Note that this inconsistent red was forced by an $n + 4$ red clique labelling edges between apexes of the same cone, with base labelled by $y_{n+2}$.

For the last part, if its $Df$ reduct is representable, then $N_{d,y}A$ will be completely representable, hence $A$ itself will be completely representable because it is generated by elements whose dimension set $< n$, which is a contradiction.

**Corollary 3.11.** We have $C \notin S\mathfrak{A}_nCA_{n+4}$. In particular for any $k \geq 4$, the variety $S\mathfrak{A}_nCA_{n+k}$ is not atom canonical.

**Proof.** The first part. Assume, for contradiction, that $C \in S\mathfrak{A}_nCA_{n+4}$; let $C \subseteq \mathfrak{A}_nD$. Then $C^+ \in S\mathfrak{A}_nD^+$, and $D^+$ is of course atomic. We show that $C^+$ has an $n+4$ dimensional hyperbasis, then that it has an $n+4$ smooth representation, which contradicts the previous theorem. Here again hyperbasis are defined for cylindric algebras by a straightforward lifting from the relation algebra case.

First note that for every $n \leq l \leq m$, $\mathfrak{A}_lD^+$ is atomic. Indeed, if $x$ is an atom in $D^+$, and and $n \leq l < m$, then $c_l \ldots c_{m-l+1}x$ is an atom in $\mathfrak{A}_lD^+$, so if $c \neq 0$ in the latter, then there exists $0 \neq a \in \mathfrak{A}D^+$, such that $a \leq c$, and so $c_l \ldots c_{m-1+1}a \leq c_l \ldots c_{m-1+1}c = c$.

Let $\Lambda = \bigcup_{k<n+3} \mathfrak{A}_k\mathfrak{D}^+$, and let $\lambda \in \Lambda$. In this proof we follow closely section 13.4 in [20]. The details are omitted because they are identical to the corresponding ones in op.cit. For each atom $x$ of $D$, define $N_x$, easily checked to be an $m$ dimensional $\Lambda$ hypernetwork, as follows. Let $\bar{a} \in n^{+4}$ then if $|a| = n$, $N_x(a)$ is the unique atom $r \in \mathfrak{A}D$ such that $x \leq s_ar$. Here substitutions are defined as above. If $n \neq |\bar{a}| < n+3$, $N_x(\bar{a})$ the unique atom $r \in \mathfrak{A}_a\mathfrak{D}$ such that $x \leq s_ar$. $\mathfrak{A}_a\mathfrak{D}$ is easily checked to be atomic, so this is well defined.

Otherwise, put $N_x(a) = \lambda$. Then $N_x$ as an $n+4$ dimensional $\Lambda$ hypernetwork, for each such chosen $x$ and $\{N_x : x \in \mathfrak{A}C\}$ is an $n+4$ dimensional $\Lambda$ hyperbasis. Then viewing those as a saturated set of mosaics, one can can construct a complete $n+4$ smooth representation of $M$ of $C$, see [20, proposition 13.37]. But this contradicts the previous theorem 3.10.

**3.0.1 A different view, blowing up and blurring a finite rainbow cylindric algebra**

In the following we denote the rainbow algebra $R(\Gamma)$ defined in [9] by $CA_{G,R}$ where $R = \Gamma$ is the graph of reds, which will be a complete irreflexive graph, and $G$ the indices greens with subscript 0.

The idea used here is a typical instance of a blow up and blur construction. Let $L \subseteq RCA_n$ be closed under forming subalgebras. Start with a finite (atomic)
algebra $\mathcal{C}$ such that $\mathcal{C} \notin \mathcal{L}$. Then blow up and blur its atom structure by splitting each of its atoms into infinitely many. This way we get a new infinite atom structure $\mathbb{A}$ which has a finite set of blurs (non principal ultrafilters). These blurs play a double role. They blur $\mathbb{A}$ at this level, so it does not embed in $\mathbb{A}$, and viewed as colours they are also used to represent $\mathbb{A}$. 

But $\mathcal{C}$ is still there on the global level, meaning that it embeds into $\mathbb{C}$ by sending each atom to the infinite disjunct of its copies, the latter is complete, so these joins exist. (They do not exist in the term algebra, for otherwise it would also be non representable.)

This implies that $\mathbb{C}$ is also outside $\mathcal{L}$ because $\mathcal{C} \notin \mathcal{L}$, and $\mathcal{L}$ is closed under forming subalgebras.

Let $\mathbb{A}$ be the rainbow atom structure in [10] except that we have $n + 2$ greens and $n + 1$ reds, that is the rainbow atom structure dealt with in 3.10. The rainbow signature now consists of $g_i : i < n - 1$, $g_{i1} : i, j \in n + 2$, $r_{kl} : k, l \in n + 1$, $t \in \omega$, binary relations and $y S \subseteq \mathbb{Z}$, $S$ finite and a shade of red $\rho$; the latter is outside the rainbow signature, but it labels coloured graphs during the game, and in fact $\exists$ can win the $\omega$ rounded game and build the $n$ homogeneous model $\mathbb{M}$ by using $\rho$ when she is forced a red.

Then $\mathbb{A}$ is representable; this can be proved exactly as in [10]. The atoms of $\mathbb{A}$ are coloured graphs whose edges are not labelled by the one shade of red $\rho$; it can also be viewed as a set algebra based on $\mathbb{M}$ by relativizing semantics discarding assignments whose edges are labelled by $\rho$. A coloured graph (an atom) in $\mathcal{CA}_{n + 2, n + 1}$ is one such that at least one of its edges is labelled red. Now $\mathcal{CA}_{n + 2, n + 1}$ embeds into $\mathbb{C}$ by taking every red graph to the join of its copies, which exists because $\mathbb{C}$ is complete (these joins do not exist in the (not complete) term algebra; only joins of finite or cofinitely many reds do, hence it serves non representability.) A copy of a red graph is one that is isomorphic to this graph, modulo removing superscripts of reds. Another way to do this is to take every coloured graph to the interpretation of an infinite disjunct of the $MCA$ formulas (as defined in [10]), and to be dealt with below; such formulas define coloured graphs whose edges are not labelled by the shade of red, hence the atoms, corresponding to its copies, in the relativized semantics; this defines an embedding, because $\mathbb{C}$ is isomorphic to the set algebra based on the same relativized semantics using $L_{n, \omega}^{\infty}$ formulas in the rainbow signature. Here again $\mathbb{M}$ is the $n$ homogeneous model constructed in the new rainbow signature, though the construction is the same. But $\forall$ can win a certain finite rounded game on $\mathcal{CA}_{n + 1, n + 2}$, hence it is outside $\mathbb{M}$, $\mathcal{CA}_{n + 4}$ and so is $\mathbb{C}$, because the former is embeddable in the latter and $\mathcal{M}$ is a variety; in particular, it is closed under forming subalgebras.

For the definition of $n + k$ complex blur the reader is referred to [7, definition 3.1]; this involves a set $J$ of blurs and a ternary relation $E$. 

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**Theorem 3.12.** Let \( k \geq 1 \). Assume that there exists a finite relation algebra \( \mathcal{A} \notin S\mathfrak{RCA}_{n+k+1} \) that has \( n + k \) complex blur \((J, E)\). Let \( \mathcal{A} \mathcal{T} \) be the infinite atom structure obtained by blowing up and blurring \( \mathcal{A} \), in the sense of [7, p.72]. Then \( \text{Mat}_{n+k} \mathcal{A} \mathcal{T} \) is an \( n + k \) dimensional cylindric basis. Furthermore there exists representable algebras \( \mathcal{C}_n \) and \( \mathcal{C}_{n+k} \) such that \( \text{ImMat}_n \mathcal{A} \mathcal{T} \subseteq \mathcal{C}_n \) and \( \text{ImMat}_{n+k} \mathcal{A} \mathcal{T} \subseteq \mathcal{C}_{n+k} \), \( \mathcal{C}_n = \mathcal{N} \mathcal{T}_n \mathcal{C}_{n+k} \) and finally and \( \text{CmAt} \notin S\mathfrak{RCA}_{n+k+1} \).

**Proof.** Exactly like the proof in [7] by replacing the Maddux algebra defined on p.84 denoted by \( \mathcal{M} \) by \( \mathcal{A} \). \( \square \)

We note that \( k \) cannot be equal to 0, because Andréka provided a Sahlqvist axiomatization of \( S\mathfrak{RCA}_{n+1} \), for any finite \( n \), hence the latter is necessarily atom canonical.

**Theorem 3.13.** Any class \( \mathcal{K} \), such that \( \mathcal{K} \) contains the class of completely representable algebras and is contained in \( S\mathfrak{RCA}_{n+3} \) is not elementary.

**Proof.** Let \( G_k \) be the usual atomic game played on networks with \( k \) rounds [9, definition 3.3.2]. Let \( \mathcal{A} \) be the rainbow algebra \( \mathcal{C}A_{\mathbb{Z} \times \mathbb{N}} \), then \( \exists \) has a winning strategy in \( G_k \) for all finite \( k \geq n \) hence it is elementary equivalent to a countable completely representable algebra \( \mathcal{B} \) [18]. Let \( F^m \) be the \( \omega \) rounded atomic game except that \( \forall \)‘s moves are limited to \( m \) pebbles. Then it can be shown that if \( \mathcal{A} \in S\mathfrak{RCA}_n \), then \( \exists \) has a winning strategy in \( F^m \), as follows: In the initial round \( \forall \) plays a graph \( \Gamma \) with nodes \( 0, 1, \ldots, n-1 \) such that \( \Gamma(i, j) = w \) for \( i < j < n-1 \) and \( \Gamma(i, n-1) = g_i \) \((i = 1, \ldots, n-2)\), \( \Gamma(0, n-1) = g_0 \) and \( \Gamma(0, 1 \ldots n-2) = y_B \). In the following move \( \forall \) chooses the face \((0, \ldots, n-2)\) and demands a node \( n \) with \( \Gamma_2(i, n) = g_i \) \((i = 1, \ldots, n-2)\), and \( \Gamma_2(0, n) = g_0^{-1} \). \( \exists \) must choose a label for the edge \((n+1, n)\) of \( \Gamma_2 \). It must be a red atom \( r_m \). Since \(-1 < 0 \) we have \( m < n \). In the next move \( \forall \) plays the face \((0, \ldots, n-2)\) and demands a node \( n+1 \), with \( \Gamma_3(i, n) = g_i \) \((i = 1, \ldots, n-2)\), such that \( \Gamma_3(0, n+2) = g_0^{-2} \). Then \( \Gamma_3(n+1, n) \) \( \Gamma_3(n+1, n-1) \) both being red, the indices must match. \( \Gamma_3(n+1, n) = r_{m} \) and \( \Gamma_3(n+1, n-1) = r_{l}m \) with \( l < m \). In the next round \( \forall \) plays \((0, 1 \ldots n-2)\) and reuses the node \( 2 \) such that \( \Gamma_4(0, 2) = g_0^{-3} \). This time we have \( \Gamma_4(n, n-1) = r_{j} \) for some \( j < l \in \mathbb{N} \).

Continuing in this manner leads to a decreasing sequence in \( \mathbb{N} \).

But it can also be shown that \( \exists \) has a winning strategy in \( F^{n+3} \) [23, theorem 33, lemma 41]. It follows that \( \mathcal{A} \notin S\mathfrak{RCA}_{n+3} \), but it is elementary equivalent to a countable completely representable algebra. Indeed, using ultrapowers and an elementary chain argument, we obtain \( \mathcal{B} \) such \( \mathcal{A} \equiv \mathcal{B} \) [23, lemma 44], and \( \exists \) has a winning strategy in \( G_\omega \), so by [9, theorem 3.3.3], \( \mathcal{B} \) is completely representable. So if \( K \) is as above, then \( \mathcal{A} \notin K \) but \( \mathcal{B} \) is in \( K \), since \( \mathcal{A} \equiv \mathcal{B} \), it readily follows that \( K \) is not elementary. \( \square \)
3.1 Omitting types in clique guarded semantics, and in other contexts

We now formulate, and prove, three (negative) new omitting types theorems, that are consequences of the algebraic results formulated in theorems 3.11, 3.13 and 3.12.

Let $T$ be a countable first order theory and $\Gamma$ be a type that is realized in every model of $\Gamma$. Then the usual Orey Henkin theorem tells us that this type is necessarily principal, that is, it is isolated by a formula $\phi$. We call such a $\phi$ an $m$ witness, if $\phi$ is built up of $m$ variables.

**Theorem 3.14.**

(1) There is a countable $L_n$ theory $T$ a type $\Gamma$, realized in every smooth $n+4$ model but has no witness.

(2) Assume the hypothesis of theorem 3.12. Then there is a countable theory $T$, a type realized in every $n+k+1$ smooth model, but there is no $n+k$ witness.

(3) There is a countable $L_n$ theory $T$ and a type $\Gamma$ such that $\Gamma$ is realized in every smooth $n+3$ model, but does not have a witness.

**Proof.**

(1) Let $\mathfrak{A}$ be as in corollary 3.11 and let $\Gamma$ be the set of atoms. Then we claim that $\Gamma$ is realized in all $n+4$ models. Towards proving this claim consider a model $\mathfrak{M}$ of $T$. If $\Gamma$ is not realized in $\mathfrak{M}$, then this gives an $n+4$ complete representation of $\mathfrak{A} = \mathfrak{Fm}_T$, which is impossible, because $\text{CmAr}\mathfrak{A}$ is not in $\mathfrak{S}\text{Nr}_n \mathfrak{CA}_{n+4}$.

Assume that $\phi$ is witness. Then $\mathfrak{A}$ is simple, and so we can assume without loss of generality, that it is set algebra with a countable base. Let $\mathfrak{M} = (M, R)$ be the corresponding model to this set algebra in the sense of [25] sec 4.3. Then $\mathfrak{M} \models T$ and $\phi^{m} \in \mathfrak{A}$. But $T \models \exists x \phi$, hence $\phi^{m} \neq 0$, from which it follows that $\phi^{m}$ must intersect an atom $\alpha \in \mathfrak{A}$ (recall that the latter is atomic). Let $\psi$ be the formula, such that $\psi^{m} = \alpha$. Then it cannot be the case that that $T \models \phi \rightarrow \neg \psi$, hence $\phi$ is not a $k$ witness, and we are done.

(2) The same argument exactly using instead the statement in 3.12.

(3) No we use 3.13. Let $\mathfrak{A}$ be the term algebra of $\mathfrak{CA}_{\mathbb{Z}, \mathbb{N}}$. Then $\mathfrak{A}$ is an atomic countable representable algebra that is not in $\mathfrak{S}\mathfrak{Nr}_n \mathfrak{CA}_{n+3}$ (for the same reasons as above, namely, $\exists$ still can win all finite rounded games, while $\forall$ can win the game $F^{n+3}$, because $\mathfrak{A}$ and the term algebra have the same atom structure).

Assume that $\mathfrak{A} = \mathfrak{Fm}_T$, and let $\Gamma = \{ \phi : \neg \phi_T \text{ is an atom} \}$. Then $\Gamma$ is a non-principal type, because $\mathfrak{A}$ is atomic, but it has no $n+3$ flat
An unpublished result of Andrêka and Németi shows that the omitting types theorem fails for $L_2$ though Vaught’s theorem on existence of atomic models for atomic theories hold, [14], [20], [13]. In the next example we show that even Vaught’s theorem, hence $OTT$, fails when we consider logics without equality reflected algebraically by many reducts of polyadic algebras.

We first start with algebras that are cylindrifier free reducts of polyadic algebras. In this case set algebras are defined exactly like polyadic set algebras by discarding cylindrifiers. Such algebras are expansions of Pinter’s algebras studied by Sági, and explicitly mentioned by Hodkinson [11] in the context of searching for algebras, where atomicity coincides with complete representability.

Assem showed that for any ordinal $\alpha > 1$, and any infinite cardinal $\kappa$, there is an atomic set algebra (having as extra Boolean operations only finite substitutions) with $|A| = \kappa$, that is not completely representable. In particular, $A$ can be countable, and so the omitting types theorem, and for that matter Vaught’s theorem fail. This works for all dimensions, except that in the infinite dimensional case, semantics is relativized to weak set algebras. Do we have an analogous result, concerning failure of the omitting types theorem for fragments of $L_n$ without equality, but with quantifiers. The answer is yes.

This theorem holds for Pinter’s algebras and polyadic algebras, let $K$ denote either class. It suffices to show that there exists $B \in RK_n \cap \mathfrak{N}r_n K_{n+k}$ that is not completely representable. But this is not hard to show. Let $A$ be the cylindric algebra of dimension $n \geq 3$, $n$ finite, provided by theorem 1.1 in [7]. Then first we can expand $A$ to a polyadic equality algebra because it is a subalgebra of the complex algebra based on the atom structure of basic matrices. This new algebra will also be in $RPEA_n \cap \mathfrak{N}r_n PEA_{n+k}$. Its reduct, whether the polyadic or the Pinter, will be as desired.

Indeed consider the $PA$ case, with the other case completely analogous, this follows from the fact that $\mathfrak{N}r_n K_n \subseteq \mathfrak{N}d\mathfrak{N}r_n PEA_{n+k} = \mathfrak{N}r_n \mathfrak{R}dPEA_{n+k} \subseteq \mathfrak{N}r_n PA_{n+k}$, and that $A$ is completely representable if and only if its diagonal free reduct is. (This is proved by Hodkinson in [11], the proof depends essentially on the fact that algebras considered are binary generated).

Now what if we only have cylindrifiers, that is dealing with $Df_n$, $n \geq 3$. Let $A$ be the cylindric algebra as in the previous paragraph. Assume that there a type $\Gamma$, that is realized in every representation of $A$ but has no witness using extra $k$ variables. Let $B = \mathfrak{R}d_{df} A$.

Let $f : B \to C$ be a diagonal free representation of $B$. The point is that though $\Gamma$ is realized in every cylindric representation of $A$, there might be a
representation of its diagonal free reduct that omits \( \Gamma \), these are more, because we do not require preservation of the diagonal elements. This case definitely needs further research, and we are tempted to think that it is not easy.

### 3.2 Omitting types for finite first order expansions of \( L_n \)

Such formalisms were studied in [5] and [4]. First we recall what we mean by first order definable operation on the algebra level. These will be logical connectives in the corresponding logic, thus expanding \( L_n \).

**Definition 3.15.** Let \( \Lambda \) be a first order language with countably many relation symbols, \( R_0, \ldots, R_i, \ldots : i \in \omega \) each of arity \( n \). Let \( C_{s_n,t} \) denote the following class of similarity type \( t \):

(i) \( t \) is an expansion of \( t_{CA_n} \).

(ii) \( \mathcal{M}o_{ca}C_{s_n,t} = C_{s_n} \). In particular, every algebra in \( C_{s_n,t} \) is a boolean field of sets with unit \( ^nU \) say, that is closed under cylindrifications and contains diagonal elements.

(iii) For any \( m \)-ary operation \( f \) of \( t \), there exists a first order formula \( \phi \) with free variables among \( \{ x_0, \ldots, x_n \} \) and having exactly \( m, n \)-ary relation symbols \( R_0, \ldots, R_{m-1} \) such that, for any set algebra \( \mathfrak{A} \in C_{s_n,t} \) with base \( U \), and \( X_0, \ldots, X_{m-1} \in \mathfrak{A} \), we have:

\[
\langle a_0, \ldots, a_{n-1} \rangle \in f(X_0, \ldots X_{m-1})
\]

if and only if

\[
\mathcal{M} = \langle U, X_0, \ldots, X_{n-1} \rangle \models \phi[a_0, \ldots, a_{n-1}].
\]

Here \( \mathcal{M} \) is the first order structure in which for each \( i \leq m \), \( R_i \) is interpreted as \( X_i \), and \( \models \) is the usual satisfiability relation. Note that cylindrifications and diagonal elements are so definable. Indeed for \( i, j < n \), \( \exists x_i R_0(x_0 \ldots x_{n-1}) \) defines \( c_i \) and \( x_i = x_j \) defines Dedekind–MacNeille\( i_{ij} \).

(iv) With \( f \) and \( \phi \) as above, \( f \) is said to be a first order definable operation with \( \phi \) defining \( f \), or simply a first order definable operation, and \( C_{s_n,t} \) is said to be a first order definable expansion of \( C_{s_n} \).

(v) \( \text{RCA}_{n,t} \) denotes the class \( \text{SPC}_{s_{n,t}} \), i.e. the class of all subdirect products of algebras in \( C_{s_n,t} \). We also refer to \( \text{RCA}_{n,t} \) as a first order definable expansion of \( \text{RCA}_n \).
Theorem 3.16. Let $t$ be a finite expansion of the CA type. Then there are atomic algebras in $\text{RCA}_{n,t}$ that are not completely representable.

Proof. Let $k \in \omega$ such that all first order definable operations are built up using at most $n + k$ variables. This $k$ exists, because we have only finitely many of those operations. Let $\mathfrak{A} \in \text{RCA}_n \cap \text{Nr}_{n} \text{CA}_{n+k}$ be an atomic cylindric algebra that is not completely representable. Exists by [7] Then $\mathfrak{A}$ is closed under all the first order definable operations. But the cylindric reduct of $\mathfrak{A}$, is not completely representable, then $\mathfrak{A}$ itself is not.

$\text{RCA}_{n,t}$ is a variety, this can be proved exactly like the $\text{RCA}_n$ case, and in fact it is a discriminator variety with discriminator term $c_0 \ldots c_{n-1}$. (So it is enough to show that it is closed under ultraproducts.) A result of Biro says that it is not finitely axiomatizable.

The logic corresponding to $\text{RCA}_{n,t}$ has $n$ variables, and it has the usual first order connectives; (here we view cylindrifiers as unary connectives) and it has one connective for each first order definable operation. In atomic formulas the variables only appear in their natural order; so that they are restricted, in the sense of [?] sec 4.3.

Semantics are defined as follows: For simplicity we assume that we have an at most countable collection of $n$-ary relation symbols $\{R_i : i \in \omega\}$. (We will be considering countable languages anyway, to violate the omitting types theorem).

The inductive definition for the first order (usual) connectives is the usual. Now given a model $\mathfrak{M} = (M, R_i)_{i \in I}$ and a formula $\psi$ of $L_n^+$; with a corresponding connective $f_\phi$ which we assume is unary to simplify matters, and $s \in {}^n M$, then for any formula $\psi$:

$$\mathfrak{M} \models f_\phi(\psi)[s] \iff (\mathfrak{M}, \phi^\mathfrak{M}) \models \psi[s].$$

$L_n$ is the logic corresponding to $\text{CA}_n$, which has only restricted formulas, and $L_n$ is that corresponding to $\text{RPEA}_n$; the latter of course is a first order extension of the former because the substitutions corresponding to transpositions are first order definable.

Corollary 3.17. No first order finite extension of $L_n$ enjoys an omitting types theorem. This holds for languages with just one binary relation.

Proof. Let $\mathfrak{A}$ be as above, and let $\mathfrak{F}_T$ be the corresponding Tarski-Lindenbaum algebra. We assume that $\mathfrak{A} = \mathfrak{F}_T$ (not just isomorphic) Let $X = \text{At}\mathfrak{A}$ and let $\Gamma = \{\phi : \phi/ \equiv \in X\}$. Then $\Gamma$ cannot be omitted. \qed
3.3 Omitting types and complete representations for the multi modal logic of substitutions

This section is based on joint work with Assem [2]. We are certain that the ordinary omitting types theorem fails for $L_n$ without equality (that is for $Df_n$) for $n \geq 3$. One way, among many other, is to construct a representable countable atomic algebra $A \in RDf_n$, that is not completely representable. The diagonal free reduct of the cylindric algebra constructed in [7] is such.

Now what about $Df_2$? We do not know. But if we have only one replacement then it fails. For higher dimensions, the result follows from the following example from [6].

We give the a sketch of the proof, the interested reader can work out the details himself or either directly consult [6].

Example 3.18. Let $\mathfrak{B}$ be an atomless Boolean set algebra with unit $U$, that has the following property: For any distinct $u, v \in U$, there is $X \in B$ such that $u \in X$ and $v \in \sim X$. For example $\mathfrak{B}$ can be taken to be the Stone representation of some atomless Boolean algebra. The cardinality of our constructed algebra will be the same as $|B|$. Let $R = \{X \times Y : X, Y \in \mathfrak{B}\}$ and $A = \{\bigcup S : S \subseteq R : |S| < \omega\}$. Then $|R| = |A| = |B|$ and $\mathfrak{A}$ is a subalgebra of $\wp(2U)$. Also the only subset of $D_{01}$ in $\mathfrak{A}$ is the empty set. Let $S = \{X \times \sim X : X \in B\}$, $\bigcup S = \sim D_{01}$, and $\sum S = U \times U$. But $S_0(X \times \sim X) = (X \cap \sim X) \times U = \emptyset$ for every $X \in B$. Thus $S_0(\sum S) = U \times U$ and $\sum \{S_0(Z) : Z \in S\} = \emptyset$.

For $n > 2$, one takes $R = \{X_1 \times \ldots \times X_n : X_i \in \mathfrak{B}\}$ and the definition of $\mathfrak{A}$ is the same. Then, in this case, one takes $S$ to be $X \times \sim X \times U \times \ldots \times U$ such that $X \in B$. The proof survives verbatim. By taking $\mathfrak{B}$ to be countable, then $\mathfrak{A}$ can be countable, and so it violates the omitting types theorem.

Example 3.19. We consider a very simple case, when we have only transpositions. In this case omitting types theorems holds for countable languages and atomic theories have atomic models. Here all substitutions corresponding to bijective maps are definable. This class is defined by translating a finite presentation of $S_n$, the symmetric group on $n$ to equations, and postulating in addition that the substitution operators are Boolean endomorphisms. In this case, given an abstract algebra $\mathfrak{A}$ satisfying these equations and $a \in A$, non zero, and $F$ any Boolean ultrafilter containing $a$, then the map $f : \mathfrak{A} \rightarrow \wp(S_n)$ defined by $\{\tau \in S_n : s_{\tau}a \in F\}$ defines a Boolean endomorphism such that $f(a) \neq 0$.

(1) Now we show that the omitting types theorem holds. We use a fairly standard Baire category argument. Each $\eta \in S_n$ is a composition of transpositions, so that $s_\eta$, a composition of complete endomorphisms,
is itself complete. Therefore $\prod s_\eta x = 0$ for all $\eta \in S_n$. Then for all $\eta \in S_n$, $B_\eta = \bigcap_{x \in X} N_{s_\eta x}$ is nowhere dense in the Stone topology and $B = \bigcup_{\eta \in S_n} B_\eta$ is of the first category (in fact, $B$ is also nowhere dense, because it is only a finite union of nowhere dense sets).

Let $F$ be an ultrafilter that contains $a$ and is outside $B$. This ultrafilter exists by the celebrated Baire category theorem, since the complement of $B$ is dense. (Stone spaces are compact and Hausdorff). Then for all $\eta \in S_n$, there exists $x \in X$ such that $s_\tau x \notin F$. Let $h : A \to \wp(S_n)$ be the usual representation function; $h(x) = \{ \eta \in S_n : s_\eta x \in F\}$. Then clearly $\bigcap_{x \in X} h(x) = \emptyset$.

(2) Now further, with no restriction on cardinalities, every atomic algebra is completely representable. Indeed, let $B$ be an atomic transposition algebra, let $X$ be the set of atoms, and let $c \in B$ be non-zero. Let $S$ be the Stone space of $B$, whose underlying set consists of all Boolean ultrafilters of $B$. Let $X^*$ be the set of principal ultrafilters of $B$ (those generated by the atoms). These are isolated points in the Stone topology, and they form a dense set in the Stone topology since $B$ is atomic. So we have $X^* \cap T = \emptyset$ for every nowhere dense set $T$ (since principal ultrafilters, which are isolated points in the Stone topology, lie outside nowhere dense sets). Recall that for $a \in B$, $N_a$ denotes the set of all Boolean ultrafilters containing $a$.

Now for all $\tau \in S_n$, we have $G_{X,\tau} = S \sim \bigcup_{x \in X} N_{s_\tau x}$ is nowhere dense. Let $F$ be a principal ultrafilter of $S$ containing $c$. This is possible since $B$ is atomic, so there is an atom $x$ below $c$; just take the ultrafilter generated by $x$. Also $F$ lies outside the $G_{X,\tau}$'s, for all $\tau \in S_n$ Define, as we did before, $f_c$ by $f_c(b) = \{ \tau \in S_n : s_\tau b \notin F\}$. Then clearly for every $\tau \in S_n$ there exists an atom $x$ such that $\tau \in f_c(x)$, so that $S_n = \bigcup_{x \in A} f_c(x)$. Now for each $a \in A$, let $V_a = S_n$ and let $V$ be the disjoint union of the $V_a$'s. Then $\prod_{a \in A} \wp(V_a) \cong \wp(V)$. Define $f : A \to \wp(V)$ by $f(x) = g[(f_a x : a \in A)]$. Then $f : A \to \wp(V)$ is an embedding such that $\bigcup_{x \in A} f(x) = V$. Hence $f$ is a complete representation.

Let us consider algebras when both substitutions corresponding to replacements and transpositions are available. For any $\alpha \geq 2$ (infinite included) we denote this class by $SA\alpha$. Here we have a strong completeness theorem, namely, there is a finite schema of equations $\Sigma$ such that if $A \models \Sigma$, then $A$ is representable (in the infinite dimensional case we use weak units) but the class of subdirect products of set algebras is a variety. When $\alpha$ is finite the finite schema is simply a finite set of equations. Here is an example that these algebras may not be completely additive (idea borrowed from [6]):
Example 3.20. First it is easy to show that complete representability forces that the non-boolean operations are completely additive. Therefore it suffices to construct an atomic algebra such that \( \sum_{x \in \text{Ar} \mathfrak{A}^1} 1 \neq 1 \).

In what follows we produce such an algebra. (This algebra will be uncountable, due to the fact that it is infinite and complete, so it cannot be countable. In particular, it cannot be used to violate the omitting types theorem, the most it can say is that the omitting types theorem fails for uncountable languages, which is not too much of a surprise).

Let \( \mathbb{Z}^+ \) denote the set of positive integers. Let \( U \) be an infinite set. Let \( Q_n, n \in \omega, \) be a family of \( n \)-ary relations that form partition of \( nU \) such that \( Q_0 = D_{01} = \{ s \in nU : s_0 = s_1 \} \). And assume also that each \( Q_n \) is symmetric; for any \( i,j \in n, S_{ij}Q_n = Q_n \). Clearly such a partition exists. Now fix \( F \) a non-principal ultrafilter on \( \mathcal{P}(\mathbb{Z}^+) \). For each \( X \subseteq \mathbb{Z}^+ \), define

\[
R_X = \begin{cases} 
\bigcup \{Q_k : k \in X\} & \text{if } X \notin F, \\
\bigcup \{Q_k : k \in X \cup \{0\}\} & \text{if } X \in F
\end{cases}
\]

Let \( \mathfrak{A} = \{R_X : X \subseteq \mathbb{Z}^+\} \).

Notice that \( \mathfrak{A} \) is uncountable. Then \( \mathfrak{A} \) is an atomic set algebra with unit \( R_{\mathbb{Z}^+} \), and its atoms are \( R_{\{k\}} = Q_k \) for \( k \in \mathbb{Z}^+ \). (Since \( F \) is non-principal, so \( \{k\} \notin F \) for every \( k \)). We check that \( \mathfrak{A} \) is indeed closed under the operations. Let \( X, Y \) be subsets of \( \mathbb{Z}^+ \). If either \( X \) or \( Y \) is in \( F \), then so is \( X \cup Y \), because \( F \) is a filter. Hence

\[
R_X \cup R_Y = \bigcup \{Q_k : k \in X\} \cup \bigcup \{Q_k : k \in Y\} \cup Q_0 = R_{X \cup Y}
\]

If neither \( X \) nor \( Y \) is in \( F \), then \( X \cup Y \) is not in \( F \), because \( F \) is an ultrafilter.

\[
R_X \cup R_Y = \bigcup \{Q_k : k \in X\} \cup \bigcup \{Q_k : k \in Y\} = R_{X \cup Y}
\]

Thus \( \mathfrak{A} \) is closed under finite unions. Now suppose that \( X \) is the complement of \( Y \) in \( \mathbb{Z}^+ \). Since \( F \) is an ultrafilter exactly one of them, say \( X \) is in \( F \). Hence,

\[
\sim R_X = \sim \bigcup \{Q_k : k \in X \cup \{0\}\} = \bigcup \{Q_k : k \in Y\} = R_Y
\]

so that \( \mathfrak{A} \) is closed under complementation (w.r.t. \( R_{\mathbb{Z}^+} \)). We check substitutions. Transpositions are clear, so we check only replacements. It is not too hard to show that

\[
S^1_0(R_X) = \begin{cases} 
\emptyset & \text{if } X \notin F, \\
R_{\mathbb{Z}^+} & \text{if } X \in F
\end{cases}
\]

Now

\[
\sum \{S^1_0(R_k) : k \in \mathbb{Z}^+\} = \emptyset.
\]
and
\[ S_0^1(R_{Z^+}) = R_{Z^+} \]
\[ \sum \{ R_{\{k\}} : k \in Z^+ \} = R_{Z^+} = \bigcup \{ Q_k : k \in Z^+ \}. \]

Thus
\[ S_0^1(\sum \{ R_{\{k\}} : k \in Z^+ \}) \neq \bigcup \{ S_0^1(R_{\{k\}}) : k \in Z^+ \}. \]

Our next theorem gives a plethora of algebras that are not completely representable. Any algebra which shares the atom structure of $\mathfrak{A}$ constructed above cannot have a complete representation. Formally:

**Theorem 3.21.** Let $\mathfrak{A}$ be as in the previous example. Let $\mathfrak{B}$ be an atomic algebra in $\mathcal{SA}_n$ such that $\text{At}\mathfrak{A} \cong \text{At}\mathfrak{B}$. Then $\mathfrak{B}$ is not completely representable.

**Proof.** Let such a $\mathfrak{B}$ be given. Let $\psi : \text{At}\mathfrak{A} \to \text{At}\mathfrak{B}$ be an isomorphism of the atom structures (viewed as first order structures). Assume for contradiction that $\mathfrak{B}$ is completely representable, via $f$ say; $f : \mathfrak{B} \to \wp(V)$ is an injective homomorphism such that $\bigcup_{x \in \text{At}\mathfrak{B}} f(x) = V$. Define $g : \mathfrak{A} \to \wp(V)$ by $g(a) = \bigcup_{x \in \text{At}\mathfrak{A}, x \leq a} f(\psi(x))$. Then, it can be easily checked that $f$ establishes a complete representation of $\mathfrak{A}$. \qed

We notice that in this latter case, if we take the subalgebra generated by the atoms, then the complex algebra of its atom structure is not its completion. For, the former is not completely additive while the latter is. The complex algebra is always completely additive, in particular, if $\mathfrak{A} \in \mathcal{SA}_n$, then its canonical extension, since complex algebras are always completely additive, is completely representable; this holds for the infinite dimensional case when we consider weak models, as well. Thus we have:

**Theorem 3.22.** Let $\alpha > 1$ be an arbitrary ordinal. Then $\mathfrak{A} \in \mathcal{SA}_\alpha$ is representable if and only if its canonical extension is completely representable on weak units.

The above is an open problem for cylindric algebras. It is not known whether for infinite dimensions the canonical extension of a representable algebra is completely representable or not. This is true for finite dimensions though a classical result of Monk.

In completely additive varieties, the minimal completion of an atomic algebra $\mathfrak{B}$ is just the complex algebra of its atom structure, namely, $\text{CmAt}\mathfrak{B}$. $\text{BAO}$ denotes the class of Boolean algebras with operators. More generally, we have:
Theorem 3.23. Let \( \mathcal{B} \in \text{BAO} \) of type \( t \) be atomic, and let \( f \in t \setminus t_{\text{BA}} \) be completely additive on \( \mathcal{B} \). Then the map \( * : \mathcal{B} \to \text{CmAt}\mathcal{B} \), defined via \( a \mapsto a^* = \{ x \in \text{At}\mathcal{B} : x \leq a \} \) is a Boolean algebra embedding that respects \( f \). If \( \mathcal{B} \) is furthermore complete, then \( * \) is surjective.

Proof. It is clear that the map is a Boolean embedding. We check that it preserves \( f \). Suppose that \( f(b_0) = b \). We need to show that \( \text{CmAt}\mathcal{B} \models f(b_0^*) = b^* \). Let \( a \in \text{At}\mathcal{B} \). Then by definition of \( \text{CmAt}\mathcal{B} \), we have \( \text{CmAt}\mathcal{B} \models \{ a \} \leq f(b_0) \) iff \( \text{At}\mathcal{A} \models R_f(a_0, a) \) for some \( a_0 \in b_0^* \). By definition, this is iff \( \mathcal{B} \models a \leq f(a_0) \) for some \( a_0 \in b_0^* \). But because \( f \) is additive, this happens iff \( \mathcal{B} \models a \leq f(b_0) \), that is, iff \( a \in b^* \) iff \( \text{CmAt}\mathcal{B} \models \{ a \} \leq b^* \), and we are done.

Now assume that \( \mathcal{B} \) is complete and let \( X \in \text{CmAt}\mathcal{B} \), then \( X \subset \text{At}\mathcal{B} \). Let \( a = \sum X \), then it is easy to show that \( a^* = X \). This shows that the map \( * \) is surjective.

In what follows, we use the following notation: Let \( \mathcal{B} \) be an algebra having signature \( t_1 \). If \( t \subset t_1 \), then \( \text{Rd}_t \mathcal{B} \) is the algebra obtained from \( \mathcal{B} \) by restricting operations to \( t \), that is, discarding the operations in \( t \setminus t_1 \).

Corollary 3.24. The algebra \( \mathfrak{A} \), constructed in the above example, is complete. Furthermore, if \( t \) denotes the signature obtained from that of \( \text{SA}_n \) by discarding replacements, then \( \text{Rd}_t \text{CmAt}\mathfrak{A} \cong \text{Rd}_t \mathfrak{A} \). On the other hand, if we discard transpositions, then we get an example of an atomic Pinter’s algebra with no complete representation.

Proof. We first show that \( \mathfrak{A} \) is complete. Consider the family \( \{ R_{X_i} : i \in I \} \) where \( \{ X_i : i \in I \} \) is a given sequence of subsets of \( J \). Let \( X = \bigcup \{ X_i : i \in I \} \). We will show that

\[
R_X = \sum \{ R_{X_i} : i \in I \}.
\]

Clearly \( R_X \) is an upper bound. Also any upper bound must be of the form \( R_Y \) where \( R_{X_i} \subset R_Y \), so \( X_i \subset Y \) for all \( i \in I \). It follows that \( X \subset Y \). In particular, if \( X \in F \), then \( Y \in F \), because \( F \) is a filter. Therefore, \( R_X \subset R_Y \), and we are done. The second part follows from that transpositions are completely additive, and the third follows immediately from 3.23.

However, this is not the case for \( \text{PA}_n \), but all the same the class of strongly representable algebras for \( \text{PA}_n \), \( n > 2 \) is not elementary, as we proceed to show:

## 4 Graphs and Strong representability

Throughout this section, \( n \) is a finite ordinal \( > 2 \).

Definition 4.1. Let \( \Gamma = (G, E) \) be a graph.
1. A set $X \subset G$ is said to be independent if $E \cap (X \times X) = \emptyset$.

2. The chromatic number $\chi(\Gamma)$ of $\Gamma$ is the smallest $\kappa < \omega$ such that $G$ can be partitioned into $\kappa$ independent sets, and $\infty$ if there is no such $\kappa$.

**Definition 4.2.**

1. For an equivalence relation $\sim$ on a set $X$, and $Y \subseteq X$, we write $\sim \upharpoonright Y$ for $\sim \cap (Y \times Y)$. For a partial map $K : n \to \Gamma \times n$ and $i, j < n$, we write $K(i) = K(j)$ to mean that either $K(i), K(j)$ are both undefined, or they are both defined and are equal.

2. For any two relations $\sim$ and $\approx$. The composition of $\sim$ and $\approx$ is the set

$$\sim \circ \approx = \{(a, b) : \exists c(a \sim c \land c \approx b)\}.$$ 

**Definition 4.3.** Let $\Gamma$ be a graph. We define an atom structure $\eta(\Gamma) = \langle H, D_{ij}, \equiv_i, \equiv_{ij} : i, j < n \rangle$ as follows:

1. $H$ is the set of all pairs $(K, \sim)$ where $K : n \to \Gamma \times n$ is a partial map and $\sim$ is an equivalent relation on $n$ satisfying the following conditions

   (a) If $|n/\sim| = n$, then $\text{dom}(K) = n$ and $\text{rng}(K)$ is not independent subset of $n$.

   (b) If $|n/\sim| = n - 1$, then $K$ is defined only on the unique $\sim$ class $\{i, j\}$ say of size 2 and $K(i) = K(j)$.

   (c) If $|n/\sim| \leq n - 2$, then $K$ is nowhere defined.

2. $D_{ij} = \{(K, \sim) \in H : i \sim j\}$.

3. $(K, \sim) \equiv_i (K', \sim')$ iff $K(i) = K'(i)$ and $\sim \upharpoonright (n \setminus \{i\}) = \sim' \upharpoonright (n \setminus \{i\})$.

4. $(K, \sim) \equiv_{ij} (K', \sim')$ iff $K(i) = K'(j)$, $K(j) = K'(i)$, and $K(\kappa) = K'(\kappa)(\forall \kappa \in n \setminus \{i, j\})$ and if $i \sim j$ then $\sim = \sim'$, if not, then $\sim' = \sim \circ \sim[i, j]$.

It may help to think of $K(i)$ as assigning the nodes $K(i)$ of $\Gamma \times n$ not to $i$ but to the set $n \setminus \{i\}$, so long as its elements are pairwise non-equivalent via $\sim$.

For a set $X$, $\mathcal{B}(X)$ denotes the boolean algebra $\langle \wp(X), \cup, \setminus \rangle$. We write $a \cap b$ for $-(-a \cup -b)$.

**Definition 4.4.** Let $\mathfrak{B}(\Gamma) = (\mathfrak{B}(\eta(\Gamma)), c_i, s_{ij}, s_{ij}, d_{ij})_{i,j<n}$ be the algebra, with extra non-Boolean operations defined as follows:

$$d_{ij} = D_{ij}, \ c_i X = \{c : \exists a \in X, a \equiv_i c\}, \ s_{ij} X = \{c : \exists a \in X, a \equiv_{ij} c\},$$

$$s_{ij} X = \begin{cases} c_i (X \cap D_{ij}), & \text{if } i \neq j, \\ X, & \text{if } i = j. \end{cases}$$

For all $X \subseteq \eta(\Gamma)$. 

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Definition 4.5. For any \( \tau \in \{ \pi \in n^n : \pi \text{ is a bijection} \} \), and any \((K, \sim) \in \eta(\Gamma)\). We define \( \tau(K, \sim) = (K \circ \tau, \sim \circ \tau) \).

The proof of the following two Lemmas is straightforward.

Lemma 4.6.
For any \( \tau \in \{ \pi \in n^n : \pi \text{ is a bijection} \} \), and any \((K, \sim) \in \eta(\Gamma)\). \( \tau(K, \sim) \in \eta(\Gamma) \).

Lemma 4.7.
For any \((K, \sim), (K', \sim'), (K'', \sim'') \in \eta(\Gamma)\), and \(i, j \in n\):

1. \((K, \sim) \equiv_{ii} (K', \sim') \iff (K, \sim) = (K', \sim') \).
2. \((K, \sim) \equiv_{ij} (K', \sim') \iff (K, \sim) \equiv_{ji} (K', \sim') \).
3. If \((K, \sim) \equiv_{ij} (K', \sim'), (K, \sim) \equiv_{ij} (K'', \sim'')\), then \((K', \sim') = (K'', \sim'')\).
4. If \((K, \sim) \in D_{ij}\), then 
   \((K, \sim) \equiv_i (K', \sim') \iff \exists (K_1, \sim_1) \in \eta(\Gamma) : (K, \sim) \equiv_j (K_1, \sim_1) \wedge (K', \sim') \equiv_{ij} (K_1, \sim_1)\).
5. \(s_{ij}(\eta(\Gamma)) = \eta(\Gamma)\).

The proof of the next lemma is tedious but not too hard.

Theorem 4.8. For any graph \( \Gamma \), \( \mathfrak{B}(\Gamma) \) is a simple PEA\(_n\).

Proof. We follow the axiomatization in [26] except renaming the items by \( Q_i \). Let \( X \subseteq \eta(\Gamma) \), and \( i, j, \kappa \in n\):

1. \( s_i^i = ID \) by definition 4.4. \( s_iX = \{ c : \exists a \in X, a \equiv_{ii} c \} = \{ c : \exists a \in X, a = c \} = X \) (by Lemma 4.7(1));  
   \( s_{ij}X = \{ c : \exists a \in X, a \equiv_{ij} c \} = \{ c : \exists a \in X, a \equiv_{ji} c \} = s_{ji}X \) (by Lemma 4.7(2)).

2. Axioms \( Q_1, Q_2 \) follow directly from the fact that the reduct \( \mathbb{M}_{ca}\mathfrak{B}(\Gamma) = \langle \mathcal{B}(\eta(\Gamma)), c_i, d_{ij} \rangle_{i,j < n} \) is a cylindric algebra which is proved in [27].

3. Axioms \( Q_3, Q_4, Q_5 \) follow from the fact that the reduct \( \mathbb{M}_{ca}\mathfrak{B}(\Gamma) \) is a cylindric algebra, and from [25] (Theorem 1.5.8(i), Theorem 1.5.9(ii), Theorem 1.5.8(ii)).
4. \( s^i_j \) is a boolean endomorphism by [25] (Theorem 1.5.3).

\[
\begin{align*}
\mathbf{s}_{ij}(X \cup Y) &= \{ c : \exists a \in (X \cup Y), a \equiv_{ij} c \} \\
&= \{ c : (\exists a \in X \lor \exists a \in Y), a \equiv_{ij} c \} \\
&= \{ c : \exists a \in X, a \equiv_{ij} c \} \cup \{ c : \exists a \in Y, a \equiv_{ij} c \} \\
&= s^i_j X \cup s^i_j Y.
\end{align*}
\]

\( s^i_j (-X) = \{ c : \exists a \in (-X), a \equiv_{ij} c \} \), and \( s^i_j X = \{ c : \exists a \in X, a \equiv_{ij} c \} \) are disjoint. For, let \( c \in (s^i_j (X) \cap s^i_j (-X)) \), then \( \exists a \in X \land b \in (-X) \), such that \( a \equiv_{ij} c \), and \( b \equiv_{ij} c \). Then \( a = b \) (by Lemma 4.7 (3)), which is a contradiction. Also,

\[
\begin{align*}
\mathbf{s}_{ij} X \cup \mathbf{s}_{ij} (-X) &= \{ c : \exists a \in X, a \equiv_{ij} c \} \cup \{ c : \exists a \in (-X), a \equiv_{ij} c \} \\
&= \{ c : \exists a \in (X \cup -X), a \equiv_{ij} c \} \\
&= \mathbf{s}_{ij} \eta(\Gamma) \\
&= \eta(\Gamma). \text{ (by Lemma 4.7 (5))}
\end{align*}
\]

therefore, \( s^i_j \) is a boolean endomorphism.

5.

\[
\begin{align*}
\mathbf{s}_{ij} \mathbf{s}_{ij} X &= \mathbf{s}_{ij}\{ c : \exists a \in X, a \equiv_{ij} c \} \\
&= \{ b : (\exists a \in X \land c \in \eta(\Gamma)), a \equiv_{ij} c, \text{ and } c \equiv_{ij} b \} \\
&= \{ b : \exists a \in X, a = b \} \\
&= X.
\end{align*}
\]

6.

\[
\begin{align*}
\mathbf{s}_{ij} s^j_i X &= \{ c : \exists a \in s^j_i X, a \equiv_{ij} c \} \\
&= \{ c : \exists b \in (X \cap \mathbf{d}ij), a \equiv_i b \land a \equiv_{ij} c \} \\
&= \{ c : \exists b \in (X \cap \mathbf{d}ij), c \equiv_j b \} \text{ (by Lemma 4.7 (4))} \\
&= s^j_i X.
\end{align*}
\]

7. We need to prove that \( s^i_j s^i_k X = s^j_k s^i_j X \) if \( |\{i, j, k\}| = 3 \). Let \((K, \sim) \in s^i_j s^i_k X \) then \( \exists (K', \sim') \in \eta(\Gamma) \), and \( \exists (K'', \sim'') \in X \) such that \((K'', \sim'') \equiv_{i_k} (K', \sim') \) and \((K', \sim') \equiv_{ij} (K, \sim) \).

Define \( \tau : n \to n \) as follows:

\[
\begin{align*}
\tau(i) &= j \\
\tau(j) &= k \\
\tau(k) &= i, \text{ and} \\
\tau(l) &= l \text{ for every } l \in (n \setminus \{i, j, k\}).
\end{align*}
\]
Now, it is easy to verify that \( \tau(K', \sim') \equiv_{ij} (K'', \sim'') \), and \( \tau(K', \sim') \equiv_{jk} (K, \sim) \). Therefore, \((K, \sim) \in s_{jk} s_{ij} X\), i.e., \( s_{ij} s_{jk} X \subseteq s_{jk} s_{ij} X \). Similarly, we can show that \( s_{ij} s_{jk} X \subseteq s_{jk} s_{ij} X \).

8. Axiom \( Q_{10} \) follows from [25] (Theorem 1.5.7).

9. Axiom \( Q_{11} \) follows from axiom 2, and the definition of \( s_{ij} \).

Since \( R_{ca} B \) is a simple CA, by [27], then \( B \) is a simple PEA. This follows from the fact that ideals \( I \) is an ideal in \( R_{ca} B \) if and only if it is an ideal in \( B \).

Definition 4.9. Let \( C(\Gamma) \) be the subalgebra of \( B(\Gamma) \) generated by the set of atoms.

Note that the cylindric algebra constructed in [27] is \( R_{ca} B(\Gamma) \) not \( R_{ca} C(\Gamma) \), but all results in [27] can be applied to \( R_{ca} C(\Gamma) \). Therefore, since our results depends basically on [27], we will refer to [27] directly when we apply it to get any result on \( R_{ca} C(\Gamma) \).

Theorem 4.10. \( C(\Gamma) \) is a simple PEA generated by the set of the \( n - 1 \) dimensional elements.

Proof. \( C(\Gamma) \) is a simple QEA from Theorem 4.8. It remains to show that \( \{(K, \sim)\} = \prod \{c_i((K, \sim)) : i < n\} \) for any \((K, \sim) \in H\). Let \((K, \sim) \in H\), clearly \( \{(K, \sim)\} \leq \prod \{c_i((K, \sim)) : i < n\} \). For the other direction assume that \((K', \sim') \in H \) and \((K, \sim) \neq (K', \sim')\). We show that \((K', \sim') \not\in \prod \{c_i((K, \sim)) : i < n\} \). Assume toward a contradiction that \((K', \sim') \in \prod \{c_i((K, \sim)) : i < n\} \), then \((K', \sim') \in c_i((K, \sim)) \) for all \( i < n \), i.e., \( K'(i) = K(i) \) and \( \sim' \upharpoonright (n \setminus \{i\}) = \sim \upharpoonright (n \setminus \{i\}) \) for all \( i < n \). Therefore, \((K, \sim) = (K', \sim')\) which makes a contradiction, and hence we get the other direction.

Theorem 4.11. Let \( \Gamma \) be a graph.

1. Suppose that \( \chi(\Gamma) = \infty \). Then \( C(\Gamma) \) is representable.

2. If \( \Gamma \) is infinite and \( \chi(\Gamma) < \infty \) then \( R_{ca} \mathcal{C} \) is not representable.

Proof. 1. We have \( R_{ca} \mathcal{C} \) is representable (c.f., [27]). Let \( X = \{x \in \mathcal{C} : \Delta x \neq n\} \). Call \( J \subseteq \mathcal{C} \) inductive if \( X \subseteq J \) and \( J \) is closed under infinite unions and complementation. Then \( \mathcal{C} \) is the smallest inductive subset of \( C \). Let \( f \) be an isomorphism of \( R_{ca} \mathcal{C} \) onto a cylindric set algebra with base \( U \). Clearly, by definition, \( f \) preserves \( s_i^j \) for each \( i, j < n \). It remains to show that \( f \) preserves \( s_{ij} \) for every \( i, j < n \). Let \( i, j < n \), since \( s_{ij} \) is boolean endomorphism and completely additive, it suffices to show that
$$fs_{ij}x = s_{ij}fx$$ for all $$x \in \text{At}C$$. Let $$x \in \text{At}C$$ and $$\mu \in n \setminus \Delta x$$. If $$\kappa = \mu$$ or $$l = \mu$$, say $$\kappa = \mu$$, then

$$fs_{nl}x = fs_{nl}c_{\kappa}x = fs_{l}^{\kappa}x = s_{l}^{\kappa}fx = s_{nl}fx.$$

If $$\mu \not\in \{\kappa, l\}$$ then

$$fs_{nl}x = fs_{l}^{\kappa}s_{\mu}^{\kappa}c_{\mu}x = s_{l}^{\kappa}s_{\mu}^{\kappa}c_{\mu}fx = s_{nl}fx.$$

2. Assume toward a contradiction that $$\mathcal{R}_{df}C$$ is representable. Since $$\mathcal{R}_{ca}C$$ is generated by $$n - 1$$ dimensional elements then $$\mathcal{R}_{ca}C$$ is representable. But this contradicts Proposition 5.4 in [27].

**Theorem 4.12.** Let $$2 < n < \omega$$ and $$T$$ be any signature between $$Df_n$$ and $$\text{PEA}_n$$. Then the class of strongly representable atom structures of type $$T$$ is not elementary.

**Proof.** By Erdős’s famous 1959 Theorem [28], for each finite $$\kappa$$ there is a finite graph $$G_{\kappa}$$ with $$\chi(G_{\kappa}) > \kappa$$ and with no cycles of length $$< \kappa$$. Let $$\Gamma_{\kappa}$$ be the disjoint union of the $$G_l$$ for $$l > \kappa$$. Clearly, $$\chi(\Gamma_{\kappa}) = \infty$$. So by Theorem 4.11 (1), $$\mathcal{C}(\Gamma_{\kappa}) = \mathcal{C}(\Gamma_{\kappa})^+$$ is representable.

Now let $$\Gamma$$ be a non-principal ultraproduct $$\prod_D \Gamma_{\kappa}$$ for the $$\Gamma_{\kappa}$$. It is certainly infinite. For $$\kappa < \omega$$, let $$\sigma_{\kappa}$$ be a first-order sentence of the signature of the graphs, stating that there are no cycles of length less than $$\kappa$$. Then $$\Gamma_l \models \sigma_{\kappa}$$ for all $$l \geq \kappa$$. By Loś’s Theorem, $$\Gamma \models \sigma_{\kappa}$$ for all $$\kappa$$. So $$\Gamma$$ has no cycles, and hence by, [27] Lemma 3.2, $$\chi(\Gamma) \leq 2$$. By Theorem 4.11 (2), $$\mathcal{R}_{df}C$$ is not representable. It is easy to show (e.g., because $$\mathcal{C}(\Gamma)$$ is first-order interpretable in $$\Gamma$$, for any $$\Gamma$$) that

$$\prod_D \mathcal{C}(\Gamma_{\kappa}) \cong \mathcal{C}(\prod_D \Gamma_{\kappa}).$$

Combining this with the fact that: for any $$n$$-dimensional atom structure $$S$$

$$S$$ is strongly representable $$\iff$$ $$\mathcal{CmS}$$ is representable, the desired follows. □

### 4.0.1 The good and the bad

Here we give a different approach to Hirsch Hodkinson’s result. We use the notation and the general ideas in [9, lemmas 3.6.4, 3.6.6]. An important difference is that our cylindric algebras are binary generated, and they their atom structures are the set of basic matrices on relation algebras satisfying the same properties. We abstract the two Monk-like algebras dealt with in the proof of theorem 3.3.
Let \( G \) be a graph. One can define a family of first order structures (labelled graphs) in the signature \( G \times n \), denote it by \( I(G) \) as follows: For all \( a, b \in M \), there is a unique \( p \in G \times n \), such that \((a, b) \in p\). If \( M \models (a, i)(x, y) \land (b, j)(y, z) \land (c, l)(x, z) \), then \( \{|i, j, l| > 1 \text{ or } a, b, c \in G \text{ and } \{a, b, c\} \text{ has at least one edge of } G \} \). For any graph \( \Gamma \), let \( \rho(\Gamma) \) be the atom structure defined from the class of models satisfying the above, these are maps from \( n \to M \), \( M \in I(G) \), endowed with an obvious equivalence relation, with cylindrifiers and diagonal elements defined as Hirsch and Hodkinson define atom structures from classes of models, and let \( \mathcal{M}(\Gamma) \) be the complex algebra of this atom structure.

We define a relation algebra atom structure \( \alpha(G) \) of the form \((\{1'\} \cup (G \times n), R_{1'}, R, R_1)\). The only identity atom is \( 1' \). All atoms are self converse, so \( R = \{(a, a) : a \text{ an atom}\} \). The colour of an atom \((a, i) \in G \times n\) is \( i \). The identity \( 1' \) has no colour. A triple \((a, b, c)\) of atoms in \( \alpha(G) \) is consistent if \( R_1; (a, b, c) \) holds. Then the consistent triples are \((a, b, c)\) where

- one of \( a, b, c \) is \( 1' \) and the other two are equal, or
- none of \( a, b, c \) is \( 1' \) and they do not all have the same colour, or
- \( a = (a', i), b = (b', i) \) and \( c = (c', i) \) for some \( i < n \) and \( a', b', c' \in G \), and there exists at least one graph edge of \( G \) in \( \{a', b', c'\} \).

Note that some monochromatic triangles are allowed namely the ‘dependent’ ones. This allows the relation algebra to have an \( n \) dimensional cylindric basis and, in fact, the atom structure of \( \mathcal{M}(G) \) is isomorphic (as a cylindric algebra atom structure) to the atom structure \( \text{Mat}_n \) of all \( n \)-dimensional basic matrices over the relation algebra atom structure \( \alpha(G) \).

We show that \( \alpha(G) \) is strongly representable iff \( \mathcal{M}(G) \) is representable iff \( G \) has infinite chromatic number. This will give the result that strongly representable atom structures of both relation algebras and cylindric algebras of finite dimension \( > 2 \) in one go, using Erdos’ graphs of large chromatic number and girth.

The idea here is that the shades of red (addressed in the proof of theorem 3.3 for the two Monk-like algebras) will appear in the ultrafilter extension of \( G \), if it has infinite chromatic number as a reflexive node, [9, definition 3.6.5]. and its \( n \) copies, can be used to completely represent \( \mathcal{M}(G)^* \) (the canonical extension of \( \mathcal{M}(G) \)). Let \( \mathcal{M}(G)_+ \) be the ultrafilter atom structure of \( \mathcal{M}(G) \).

Fix \( G \), and let \( G^* \) be the ultrafilter extension of \( G \). First define a strong bounded morphism \( \Theta \) form \( \mathcal{M}(G)_+ \) to \( \rho(I(G^*)) \), as follows: For any \( x_0, x_1 < n \) and \( X \subseteq G^* \times n \), define the following element of \( \mathcal{M}(G^*) \):

\[
X^{(x_0, x_1)} = \{[f] \in \rho(I(G^*)) : \exists p \in X[M_f \models p(f(x_0), f(x_1))]\}.
\]
Let \( \mu \) be an ultrafilter in \( \mathcal{M}(G) \). Define \( \sim \) on \( n \) by \( i \sim j \) iff \( d_{ij} \in \mu \). Let \( q \) be the projection map from \( n \) to \( n/\sim \). Define a \( G^* \times n \) coloured graph with domain \( n/\sim \) as follows. For each \( v \in \Gamma^* \times n \) and \( x_0, x_1 < n \), we let

\[
M_\mu \models v(g(x_0), g(x_1)) \iff X^{(x_0, x_1)} \in \mu.
\]

Hence, any ultrafilter \( \mu \in \mathcal{M}(G) \) defines \( M_\mu \) which is a \( G^* \) structure. If \( \Gamma \) has infinite chromatic number, then \( G^* \) has a reflexive node, and this can be used to completely represent \( \mathcal{M}(G)^\sigma \), hence represent \( \mathcal{M}(G) \) as follows: To do this one tries to show \( \exists \) has a winning strategy in the usual \( \omega \) rounded atomic game on networks \([9]\), that test complete representability.

Her strategy can be implemented using the following argument. Let \( N \) be a given \( \mathcal{M}(G)^\sigma \) network. Let \( z \not\in N \) and let \( y = x[j]_z \in n(N \cup \{z\}) = nM \). Write \( Y = \{y_0, \ldots, y_{n-1}\} \). We need to complete the labelling of edges of \( M \).

We have a fixed \( i \in n \). Defines \( q_j : j \in n \sim \{i\} \), the unique label of any two distinct elements in \( Y \sim y_j \), if the latter elements are pairwise distinct, and arbitrarily otherwise. Let \( d \in G^* \) be a reflexive node in the copy that does not contain any of the \( q_j \)'s (there number is \( n-1 \)), and define \( M \models d(t_0, t_1) \) if \( z \in \{t_0, t_1\} \not\subseteq Y \). Labelling the other edges are like \( N \). The rest of the proof is similar to \([9]\).

The idea above is essentially due to Hirsch and Hodkinson, it also works for relation and cylindric algebras, and this is the essence. For each graph \( \Gamma \), they associate a cylindric algebra atom structure of dimension \( n \), \( \mathcal{M}(\Gamma) \) such that \( C_{\mathcal{M}}(\mathcal{M}(\Gamma)) \) is representable if and only if the chromatic number of \( \Gamma \), in symbols \( \chi(\Gamma) \), which is the least number of colours needed, \( \chi(\Gamma) \) is infinite. Using a famous theorem of Erdos as we did above, they construct a sequence \( \Gamma_r \) with infinite chromatic number and finite girth, whose limit is just 2 colourable, they show that the class of strongly representable algebras is not elementary. This is a reverse process of Monk-like constructions, which gives a sequence of graphs of finite chromatic number whose limit (ultraproduct) has infinite chromatic number.

In more detail, some statement fails in \( \mathfrak{A} \) iff \( \text{At} \mathfrak{A} \) be partitioned into finitely many \( \mathfrak{A} \)-definable sets with certain ‘bad’ properties. Call this a bad partition. A bad partition of a graph is a finite colouring. So Monk’s result finds a sequence of badly partitioned atom structures, converging to one that is not. This boils down, to finding graphs of finite chromatic numbers \( \Gamma_i \), having an ultraproduct \( \Gamma \) with infinite chromatic number.

Then an atom structure is strongly representable iff it has no bad partition using any sets at all. So, here, the idea is to find atom structures, with no bad partitions with an ultraproduct that does have a bad partition. From a graph Hirsch and Hodkinson constructed an atom structure that is strongly representable iff the graph has no finite colouring. So the problem that remains is to find a sequence of graphs with no finite colouring, with an ultraproduct
that does have a finite colouring, that is, graphs of infinite chromatic numbers, having an ultraproduct with finite chromatic number.

It is not obvious, a priori, that such graphs actually exist. And here is where Erdos’ methods offer solace. Indeed, graphs like are found using the probabilistic methods of Erdos, for those methods render finite graphs of arbitrarily large chromatic number and girth. By taking disjoint unions as above, one can get graphs of infinite chromatic number (no bad partitions) and arbitrarily large girth. A non principal ultraproduct of these has no cycles, so has chromatic number 2 (bad partition). This motivates:

**Definition 4.13.**

1. A Monks algebra is good if \( \chi(\Gamma) = \infty \)
2. A Monk’s algebra is bad if \( \chi(\Gamma) < \infty \)

It is easy to construct a good Monks algebra as an ultraproduct (limit) of bad Monk algebras. Monk’s original algebras can be viewed this way. The converse is, as illustrated above, is much harder. It took Erdos probabilistic graphs, to get a sequence of good graphs converging to a bad one.

Using our modified Monk-like algebras we obtain the result formulated in theorem 4.12. An immediate corollary is: The following corollary answers a question of Hodkinson’s in [11] see p. 284.

**Corollary 4.14.** For \( K \) any class between \( Df \) and \( PEA \) and any finite \( n > 2 \), the class \( K_s = \{ A \in K_n : CmAtA \in RK_n \} \), is not elementary.

## 5 Completions and decidability

Throughout this section \( m > 2 \) will denote the dimension and \( n \) will be always finite \( > m \). We address the decidability of the problem as to whether a finite \( CA_m \) is in \( S\Pi_mCA_n \) We obtain a negative result the lowest value of \( m \) namely when \( m = 3 \) and \( n \geq m + 3 \). This generalizes the result of Hodkinson in [11] proved for \( RCA_n \) but only for \( n = 3 \). For higher dimensions, the problem to the best of our knowledge remains unsettled. We also connect algebras having a (complete) neat embedding property to (complete) relativized representations, generalizing results proved by Hirsch and Hodkinson for relation algebras, witness [20, theorems 13.45, 13. 46].

**Theorem 5.1.** Let \( m \geq 3 \). Assume that for any simple atomic relation algebra \( A \) with atom structure \( S \), there is a cylindric atom structure \( H \), constructed effectively from \( AtA \), such that:

1. If \( \Upsilon mS \in RRA \), then \( \Upsilon mH \in RCA_m \).
2. If \( S \) is finite, then \( H \) is finite.
Then for all \( k \geq 3 \), \( \text{SNCA}_{m+k} \) is not closed under completions. and it is undecidable whether a finite cylindric algebra is in and \( \text{SNCA}_{m+k} \), and

**Proof.** For the first part. Let \( S \) be a relation atom structure such that \( \text{Tm} S \) is representable while \( \text{Cm} S \not\in RA_6 \). Such an atom structure exists [20, lemmas 17.34-35-36-37]. It follows that \( \text{Cm} S \not\in \text{SNr} \text{CA}_m \). Let \( H \) be the \( \text{CA}_m \) atom structure provided by the hypothesis of the previous theorem. Then \( \text{Tm} H \in RCA_m \). We claim that \( \text{Cm} H \not\in \text{SNr} \text{CA}_{m+k}, k \geq 3 \). For assume not, i.e. assume that \( \text{Cm} H \in \text{SNr} \text{CA}_{m+k}, k \geq 3 \). We have \( \text{Cm} S \) is embeddable in \( \text{RaCm} H \). But then the latter is in \( \text{SNrCA}_6 \) and so is \( \text{Cm} S \), which is not the case.

For the second part, suppose for contradiction that there is an algorithm \( \text{Al} \) to determine whether a finite algebra is in \( \text{SNr} \text{CA}_{m+k} \). We claim that we can now decide effectively whether a finite simple atomic relation algebra is in \( \text{SNrCA}_5 \) by constructing the \( \text{CA}_m \), \( \mathcal{C} = \text{Tm} H \) which is finite, and returning the answer \( \text{Al}(\mathcal{C}) \). This is the correct answer. However, this contradicts the result that it is undecidable to tell whether a finite relation algebra is representable or not [20, theorem 18.13].

**Corollary 5.2.** Let \( n \geq 6 \). Then the following hold:

1. The set of isomorphism types of algebras in \( \text{SNr}_{3}\text{CA}_n \) with infinite flat representations is not recursively enumerable
2. The equational theory of \( \text{SNr}_{3}\text{CA}_n \) is undecidable
3. The variety \( \text{SNr}_{3}\text{CA}_n \) is not finitely axiomatizable even in \( n \) order logic.
4. For every \( n \geq 6 \), there exists a finite algebra in \( \text{SNr}_{3}\text{CA}_6 \) that does not have a finite \( n \) dimensional hyperbasis

**Proof.**

1. Direct
2. Any such axiomatization will give a decision procedure for the class of finite algebras
3. Let \( K = \text{SNr}_{3}\text{CA}_n \). We reduce the problem of telling if a finite simple algebra is not in \( K \) to the problem of telling if an equation in the language of \( \text{CA}_3 \) is valid in \( K \). Let \( \mathfrak{A} \) be a finite relation algebra. Form \( \Delta(\mathfrak{A}) \), the diagram of \( \mathfrak{A} \), but using variables instead of constants; define its conjunction. Consider \( g = \exists a \in \Delta(\mathfrak{A}) \) relative to some enumeration \( \bar{a} \) of \( A \). Then \( \mathfrak{B} \models g \) iff \( A \not\in \mathfrak{B} \). Since \( K \) is a discriminator variety, the
quantifier free \( \neg \Delta(\mathfrak{A}) \) is equivalent over simple algebras to an equation \( e \), which is effectively constructed from \( \mathfrak{A} \). So, in fact, we have \( \mathfrak{B} \models e \) iff \( \mathfrak{A} \not\subseteq \mathfrak{B} \). So \( e \) is valid over simple \( K \) algebra iff for all \( \mathfrak{B} \in K, \mathfrak{A} \not\subseteq \mathfrak{B} \). Since \( \mathfrak{A} \) is simple, this happens if and only if \( \mathfrak{A} \not\in K \).

(4) Assume for contradiction that every finite algebra in \( S\aleph_3\mathbf{CA}_n \) has a finite \( n \) dimensional hyperbasis. We claim that there is an algorithm that decides membership in \( S\aleph_3\mathbf{CA}_6 \) for finite algebras:

- Using a recursive axiomatization of \( S\aleph_3\mathbf{CA}_n \) (exists), recursively enumerate all isomorphism types of finite \( \mathbf{CA}_3 \)s that are not in \( S\aleph_3\mathbf{CA}_n \).
- Recursively enumerate all finite algebras in \( S\aleph_3\mathbf{CA}_n \). For each such algebra, enumerate all finite sets of \( n \) dimensional hypernetworks over \( \mathfrak{A} \), using \( N \) as hyperlabels, and check to see if it is a hyperbasis. When a hyperbasis is located specify \( \mathfrak{A} \). This recursively enumerates all and only the finite algebras in \( S\aleph_3\mathbf{CA}_n \). Since any finite \( \mathbf{CA}_3 \) is in exactly one of these enumerations, the process will decide whether or not it is in \( S\aleph_3\mathbf{CA}_6 \) in a finite time.

Concerning theorem 5.1 we note that Monk and Maddux constructs such an \( H \) for \( n = 3 \) and Hodkinson constructs an \( H \) for arbitrary dimensions > 2, but \( H \) unfortunately the relation algebra does not embed into \( \mathbf{CmH} \).

**Corollary 5.3.** Assume the hypothesis in 3.12; let \( k \geq 1 \) and \( n \) be finite with \( n > 2 \). Then the following hold; in particular, when \( k = 3 \) we know, by theorem 3.10, that the following indeed hold.

1. There exist two atomic cylindric algebras of dimension \( m \) with the same atom structure, one representable and the other is not in \( S\aleph_m\mathbf{CA}_{m+k+1} \).
2. For \( n \geq 3 \) and \( k \geq 3 \), \( S\aleph_m\mathbf{CA}_{m+k+1} \) is not closed under completions and is not atom-canonical. In particular, \( \mathbf{RCA}_m \) is not atom-canonical.
3. There exists an algebra in \( S\aleph_m\mathbf{CA}_{m+k+1} \) with a dense representable subalgebra.
4. For \( m \geq 3 \) and \( k \geq 3 \), \( S\aleph_n\mathbf{CA}_{m+k+1} \) is not Sahlqvist axiomatizable. In particular, \( \mathbf{RCA}_m \) is not Sahlqvist axiomatizable.
5. There exists an atomic representable \( \mathbf{CA}_n \) with no \( m + k + 1 \) smooth complete representation; in particular it has no complete representation.
(6) The omitting types theorem fails for clique guarded semantics, when size of cliques are $< m + k + 1$.

Proof. We use the cylindric algebra, proved to exist conditionally, namely, $\mathfrak{A} = \mathfrak{C}_n$ in 3.12 in conformity with our notation we switch its dimension to $m$. The term algebra, which is contained in $\mathfrak{C}_m$ also can be used.

(1) $\mathfrak{A}$ and $\text{CmAt}\mathfrak{A}$ are such.

(2) $\text{CmAt}\mathfrak{A}$ is the Dedekind-MacNeille completion of $\mathfrak{A}$ (even in the PA and Sc cases, because $\mathfrak{A}$, hence its Sc and PA reducts are completely additive), hence $S\text{Mr}_n \mathfrak{C}A_{m+k+1}$ is not atom canonical [20, proposition 2.88, theorem, 2.96].

(3) $\mathfrak{A}$ is dense in $\text{CmAt}\mathfrak{A}$.

(4) Completely additive varieties defined by Sahlqvist equations are closed under Dedekind-MacNeille completions [20, theorem 2.96].

(5) Assume that $\mathfrak{A}$ has an $n = m + k + 1$ smooth complete representation $\mathfrak{M}$. $L(A)$ denotes the signature that contains an $n$ ary predicate for every $a \in A$. For $\phi \in L(A)_{\omega,\omega}^n$, let $\phi^M = \{ \bar{a} \in C^n(M) : M \models C \phi(\bar{a}) \}$, and let $\mathfrak{D}$ be the algebra with universe $\{ \phi^M : \phi \in L_{\omega,\omega}^n \}$ with usual Boolean operations, cylindrifiers and diagonal elements, cf. theorem 13.20 in [20]. The polyadic operations are defined by swapping variables. Define $\mathfrak{D}_0$ be the algebra consisting of those $\phi^M$ where $\phi$ comes from $L^n$. Assume that $M$ is $n$ square, then certainly $\mathfrak{D}_0$ is a subalgebra of the $\mathfrak{C}r_s_n$ (the class of algebras whose units are arbitrary sets of $n$ ary sequences) with domain $\phi(C^n(M))$ so $\mathfrak{D}_0 \in \mathfrak{C}r_s_n$. The unit $C^n(M)$ of $\mathfrak{D}_0$ is symmetric, closed under substitutions, so $\mathfrak{D}_0 \in \mathfrak{G}_n$ (these are relativized set algebras whose units are locally cube, they are closed under substitutions.) Since $M$ is $n$ flat we have that cylindrifiers commute by definition, hence $\mathfrak{D}_0 \in \mathfrak{C}A_n$.

Now since $M$ is infinitary $n$ smooth then it is infinitary $n$ flat. Then one proves that $\mathfrak{D} \in \mathfrak{C}A_n$ in exactly the same way. Clearly $\mathfrak{D}$ is complete. We claim that $\mathfrak{D}$ is atomic. Let $\phi^M$ be a non zero element. Choose $\bar{a} \in \phi^M$, and consider the infinitary conjunction $\tau = \bigwedge\{ \psi \in L_\infty : M \models C \psi(\bar{a}) \}$. Then $\tau \in L_\infty$, and $\tau^M$ is an atom, as required.

Now defined the neat embedding by $\theta(r) = r(\bar{x})^M$. Preservation of operations is straightforward. We show that $\theta$ is injective. Let $r \in A$ be non-zero. But $M$ is a relativized representation, so there $\bar{a} \in M$ with $r(\bar{a})$ hence $\bar{a}$ is a clique in $M$, and so $M \models r(\bar{x})(\bar{a})$, and $\bar{a} \in \theta(r)$. proving the required.
We check that it is a complete embedding under the assumption that $M$ is a complete relativized representation. Recall that $\mathfrak{A}$ is atomic. Let $\phi \in L^\infty$ be such that $\phi^M \neq 0$. Let $\bar{a} \in \phi^M$. Since $M$ is complete and $\bar{a} \in C^n(M)$ there is $\alpha \in \text{At}\mathfrak{A}$, such that $M \models \alpha(\bar{a})$, then $\theta(\alpha).\phi^C \neq 0$, and we are done. Now $\mathfrak{A} \in S_c \mathfrak{N}_m \mathcal{C}A_{m+k}$; it embeds completely into $\mathfrak{N}_n \mathcal{D}$, $\mathcal{D}$ is complete, then so is $\mathfrak{N}_n \mathcal{D}$, and consequently $\text{CmAt}\mathfrak{A} \subseteq \mathfrak{N}_n \mathcal{D}$, which is impossible, because we know that $\text{CmAt}\mathfrak{A} \notin S \mathfrak{N}_m \mathcal{C}A_{m+k}$.

(6) By theorem 3.14

\[ \square \]

**Corollary 5.4.** Let $2 < m < n$. Then the class of algebras having an $n+1$ flat representation is a variety, and it is not finitely axiomatizable over the class having $n$ flat representations. The class of algebras having complete $n$ smooth representations, when $n \geq m+3$ is not even elementary.

**Proof.** For brevity, we denote the class of algebras having an $n$ flat representation by $\text{RCA}_{n,f}$, and that of having $n$ smooth representation by $\text{RCA}_{n,s}$. The dimension is $m$ and $n > m > 2$. For the first part. Assume that $\mathfrak{A}$ has an $n$ flat representation $\mathfrak{M}$. As in the above proof, take $\mathfrak{D}_0$ be the algebra consisting of those $\phi^M$ where $\phi$ comes from $L^n$. Since $M$ is $n$ flat we have that cylindrifiers commute by definition, hence $\mathfrak{D}_0 \in \text{CA}_n$. Now as above the map $\theta(r) = r(\bar{x})^M$ is a neat embedding, hence $\mathfrak{A} \subseteq \mathfrak{N}_m \mathcal{C}A_n$, so that $\mathfrak{A} \in S \mathfrak{N}_m \mathcal{C}A_n$. Conversely, if $\mathfrak{A} \in S \mathfrak{N}_m \mathcal{C}A_n$, then one can build an $n$ flat representation by showing that $\mathfrak{A}$ has an $n$ dimensional hyperbasis, see theorem 3.10. Hence $\text{RCA}_{n,f} \subseteq S \mathfrak{N}_m \mathcal{C}A_n$. But by the celebrated result of Hirsch and Hodkinson [20] theorem 15.1(4)], we have that for $k \geq 1$, $S \mathfrak{N}_m \mathcal{C}A_{m+k+1}$ is not finitely axiomatizable over $S \mathfrak{N}_m \mathcal{C}A_{m+k}$ and we are done.

For the second part, clearly, every complete representation is $n$ smooth. By theorem 3.13 it suffices to show that $\text{RCA}_{m+3,s} \subseteq S_c \mathfrak{N}_m \mathcal{C}A_{m+3}$. Now assume that $\mathfrak{A}$ has a complete $m$ flat representation. But the above, using the same notation, the neat embedding is a complete embedding under the assumption that $M$ is a complete $n$ smooth relativized representation into $\mathcal{D}$, and we are done. \[ \square \]

**Theorem 5.5.** Let $1 < n < m$. Then the following are equivalent:

(1) $\mathfrak{A} \in S \mathfrak{N}_n \mathcal{C}A_m$

(2) $\mathfrak{A}$ has an smooth representation

(3) $\mathfrak{A}$ has $n$ infinitary $n$ flat representation

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Proof. (1) (1) to (2). Assume that $A \in S\mathfrak{M}_n^CA_m$. Then $\mathfrak{A}^+ \in S\mathfrak{M}_n^CA_m$ has a hyperbasis, hence $\mathfrak{A}$ has an $n$ smooth relativized representation, which is infinitary $n$ flat, witness theorem 3.10.

(2) (2) to (3). Here we use the elementary view to relativized representation. We translate the existence of an $n$ smooth representation to a first order theory then we use saturation, that is the base of the relativized representation will be an $\omega$ saturated model of this theory.

Assume that $A$ has an $n$ smooth representation. Let $\text{Clique}(\bar{x})$ be the formula

$$\bigwedge_{i_0, \ldots, i_{m-1} < n} 1(x_{i_0}, \ldots, x_{i_{m-1}})$$

We extend the theory $T(A)$ by

$$\forall \bar{x}(\text{Clique}(\bar{x}) \land c_k a(\bar{y}) \implies \exists x_{i_k} (\text{Clique}(\bar{x}) \land a(\bar{z}) \land (\bar{z})_k = x_{i_k} \land \bar{z} \equiv_k \bar{y})$$

Here $\bar{x}$ is of length $n$ and $\bar{y}$ is of length $m$. This is for all $i_0, \ldots, i_{m-1}, i_k < n$ and all $a \in A$, $i_k \notin \{i_0, \ldots, i_{m-1}\}$. Now extend $L(\mathfrak{A})$ to the language $L(\mathfrak{A}, E)$ by adding $2l$ many ary predicates for $0 < l \leq n$ and extend the theory $Sq^n(\mathfrak{A})$ to

$$\forall \bar{x} \bar{y} \bar{z}(\text{Clique}(\bar{x}) \implies [E^l(x, y) \land E^l(y, z) \rightarrow E^l(x, z)]$$

and the other axioms are defined the obvious way:

$$\forall xy(E^l(x, y) \rightarrow E^j(x \circ \theta, y \circ \theta)), \: j, l < n, \theta : j \rightarrow l$$

$$\forall xy(E^m(x, y) \land r(\bar{x}) \rightarrow r(\bar{y})), \: r \in A$$

$$\forall xy(E^{n-2}(x, y) \land \text{Clique}(xx) \land \text{Clique}(yy) \rightarrow \exists z(E^{n-1}(xx, yz) \land \text{Clique}(yyz))$$

Assume that $Sq^n(\mathfrak{A})$ has an $n$ smooth relativization, then it is consistent. Let $M$ be an $\omega$ saturated model, of this theory, then we show that it is a complete $n$ smooth relativized representation. We will define an injective homomorphism $h : \mathfrak{A}^+ \rightarrow \varphi(\mathfrak{A}^+)$

First note that the set $f_\bar{x} = \{a \in A : a(\bar{x})\}$ is an ultrafilter in $\mathfrak{A}$, whenever $\bar{x} \in M$ and $M \models 1(\bar{x})$ Now define

$$h(S) = \{\bar{x} \in 1 : f_\bar{x} \in S\}.$$

We check only injectivity uses saturation. The rest is straightforward. It suffices to show that for any ultrafilter $F$ of $\mathfrak{A}$ which is an atom in $\mathfrak{A}^+$, we have $h(\{F\}) \neq 0$. Let $p(\bar{x}) = \{a(\bar{x}) : a \in F\}$. Then this type is finitely satisfiable. Hence by $\omega$ saturation $p$ is realized in $M$ by $\bar{y}$.
say. Now $M \models 1(\bar{y})$ and $F \subseteq f_{\bar{x}}$, since these are both ultrafilters, equality holds. Note that any partial isomorphism of $M$ is also a partial isomorphism of $M$ regraded as a complete representation of $\mathfrak{A}^+$. Hence $\mathfrak{A}^+$ has an infinitary $n$ flat, hence $\mathfrak{A}$ has infinitary $n$ flat.

(3) (3) to (1) From above.

Remark 5.6. For any cardinal $\kappa$, $K_\kappa$ will denote the complete irreflexive graph with $\kappa$ nodes. Let $p < \omega$, and $I$ a linearly irreflexive ordered set, viewed as model to a signature containing a binary relation $<$. $M[p, I]$ is the disjoint union of $I$ and the complete graph $K_p$ with $p$ nodes. $<$ is interpreted in this structure as follows $<^I \cup <^{K_p} \cup I \times K_p) \cup (K_p \times I)$ where the order on $K_p$ is the edge relation.

$\text{CRA}_{m,n}$ denotes the class of $\text{PEA}_{m,n}$s with $n$ smooth relativized representations. For $n \geq m + 3$, the class $\text{CRA}_{m,n}$ is not elementary. A rainbow argument can be used lifting winning strategies from the Ehrenfeucht–Fraïssé pebble game to rainbow atom structures. Let $A = M[n - 4, \mathbb{Z}]$ and $B = M[n - 4, \mathbb{N}]$, then it can be shown $\exists$ has a winning strategy for all finite rounded games (with $n$ nodes) on $\mathcal{CA}_{A,B}$, namely, in $G^n_r$ for all $r > n$, so she has a winning strategy in $G^n_\omega$ on any non trivial ultrapower, from which an elementary countable subalgebra $\mathcal{B}$ can be extracted (using an elementary chain argument) in which $\exists$ also has winning strategy in $G^n_\omega$, so that $\mathcal{B}$ has an $n$ smooth complete representation.

But it can also be shown that $\forall$ can the $\omega$ rounded game on $\mathcal{CA}_{A,B}$, also with $n$ nodes, hence the latter does not have an $n$ complete relativized representation, but is elementary equivalent to one that does. Let $\text{EFF}^p_r[A, B]$ denote the Ehrenfeucht–Fraïssé pebble forth game defined in [20] between two structures $A$ and $B$ with $p$ pebbles and $r$ rounds, which each player will use as a private game to guide her/him in the rainbow game on coloured graphs. In his private game, $\forall$ always places the pebbles on distinct elements of $\mathbb{Z}$. She uses rounds $0, \ldots n - 3$, to cover $n - 4$ and first two elements of $\mathbb{Z}$. Because at least two out of three distinct colours are related by $<$, $\exists$ must respond by pebbling $n - 4 \cup \{e, e'\}$ for some $e, e' \in \mathbb{N}$. Assuming that $\forall$ has not won, then he has at least arranged that two elements of $\mathbb{Z}$ are pebbled, the corresponding pebbles in $B$ being in $\mathbb{N}$. Then $\forall$ can force $\exists$ to play a two pebble game of length $\omega$ on $\mathbb{Z}, \mathbb{N}$ which he can win, bombarding her with cones with green tints, in the graph game. In her private game, $\forall$ picks up a spare pebble pair and place the first pebble of it on $a \in A$. By the rules of the game, $a$ is not currently occupied by a pebble. $\exists$ has to choose which element of $B$ to put the pebble on. $\exists$ chooses an unoccupied element in $n - 4$, if
possible. If they are all already occupied, she chooses \( b \) to be an arbitrary element \( x \in \mathbb{N} \). Because there are only \( n - 3 \) pebble pairs, \( \exists \) can always implement this strategy and win.

We can now lift her winning strategy of the same game but now played on coloured graphs, the atoms of \( CA_{A,B} \), as before. In this game \( \exists \) has a winning strategy in all finite rounded games, but \( \forall \) can win the \( \omega \) rounded game.

(1) This example is a modification of [9, exercise 1, p. 485], by lifting the relation algebra construction therein to the cylindric case. Let \( A(n) \) be an infinite atomic atomic relation algebra; the atoms of \( A(n) \) are \( \text{Id} \) and \( a^k(i,j) \) for each \( i < n - 2, j < \omega \) \( k < \omega_1 \).

(2) All atoms are self converse.

(3) We list the forbidden triples \( (a,b,c) \) of atoms of \( A(n) \)- those such that \( a.(b;c) = 0 \). Those triples that are not forbidden are the consistent ones. This defines composition: for \( x, y \in A(n) \) we have

\[
x; y = \{ a \in \text{At}(A(n)); \exists b, c \in \text{At}A : b \leq x, c \leq y, (a, b, c) \text{ is consistent } \}
\]

Now all permutations of the triple \( (Id,s,t) \) will be inconsistent unless \( t = s \). Also, all permutations of the following triples are inconsistent:

\[
(a^k(i,j), a^{k'}(i,j), a^{k''}(i,j'))
\]

if \( j \leq j' < \omega \) and \( i < n - 1 \) and \( k, k', k'' < \omega_1 \). All other triples are consistent.

Then for any \( r \geq 1, A(n - 1, r) \) embeds completely in \( A(n) \) the obvious way, hence, the latter has no \( n \) dimensional hyperbasis, because the former does not. Indeed \( A(n - 1, r) = \mathfrak{A}^{\tau}(n - 1, r) \subseteq \mathfrak{A}(n) \in S_c\mathfrak{RnCA}_n \) which is a contradiction.

Then \( A(n) \) has an \( m \) dimensional hyperbasis for each \( m < n - 1 \), by proving that \( \exists \) has winning strategy in the hyperbasis game \( G_{r,n}^m(A(n), \omega) \), for any \( r < \omega \), that is, for any finite rounded game.

Now write \( T \) for \( G_{r,n}^m \). Consider \( \mathfrak{M} = \mathfrak{M}(A(n), m, n, \omega, \mathfrak{C}) \) as a 5 sorted structure with sorts \( A(n), \omega, H_n^m(A), \omega \) and \( \mathfrak{C} \). Then \( \exists \) has a winning strategy in \( G(T \upharpoonright 1 + 2r, \mathfrak{M}) \). \( \exists \) has a winning strategy in \( G(T \upharpoonright r, \mathfrak{M}) \) for all finite \( r > 0 \). So \( \exists \) has a winning strategy in \( G(T, \prod_D \mathfrak{M}) \), for any non principal ultrafilter on \( \omega \). Hence there is a countable elementary subalgebra \( \mathfrak{C} \) of \( \prod_D \mathfrak{M} \) such that \( \exists \) has a winning strategy in \( G(T, \mathfrak{C}) \)

Hence \( \mathfrak{C} \) has the form \( \mathfrak{M}(\mathfrak{B}, m, n, \Lambda, \mathfrak{A}) \) for some atomic \( \mathfrak{B} \in RA \) countable set \( \Lambda \) and countable atomic \( m \) dimensional \( \mathfrak{A} \in CA_m \) such that
At\mathfrak{A} \cong \text{At\mathfrak{C}}(H_m^n(\mathfrak{B},\Lambda)). Furthermore, we have \mathfrak{B} < \prod_D \mathfrak{A}(n) and \mathfrak{A} < \prod_D \mathfrak{A}(n). Thus \exists \text{ has a winning strategy in } G(T, \mathfrak{M}, m, n, \Lambda, \mathfrak{A}) and she also has a winning strategy in G_{\omega}^{m,n}(\mathfrak{B}, \Lambda). So \mathfrak{B} \in S\mathfrak{R}aCA_n and \mathfrak{A} embeds into \mathfrak{C}(H_m^n(\mathfrak{B},\Lambda) \in \mathfrak{N}t_m CA_n. and we are done. In fact, one can show that both \mathfrak{B} and \mathfrak{A} are actually representable, by finding a representation of \prod \mathfrak{A}(n)/\mathcal{F} (or an elementary countable subalgebra of it) embedding every \( m \) hypernetwork in such a way so that the embedding respects \equiv_i for every \( i < m \), but we do not need that much.

(4) Our next \( \text{CA}_m \) is \( \text{A}^n_r \), the rainbow cylindric algebra based on \( A = M[n - 3, 2^{r-1}], \) and \( B = M[n - 3, 2^{r-1} - 1] \), as defined in [20] lemma 17.15, 17.16, 17.17]. These structures were used by Hirsch and Hodkinson to show that \( \text{RA}_n \) is not finitely axiomatizable over \( \text{RA}_{n+1} \); here \( \text{RA}_m \) is the variety of relation algebras with \( m \) dimensional relational basis. Now we put them to a different use:

**Definition 5.7.** Let \( K \subseteq \mathcal{L} \) be classes of algebras. We say that the distance between \( K \) and \( \mathcal{L} \) is infinite if there exists a sequence \( \mathfrak{A}_r \in \mathcal{L} \sim K \) such that \( \prod_{r \in \mathcal{F}} \mathfrak{A}_r \in K \), for any non principal ultrafilter \( \mathcal{F} \).

Note that if \( K \) and \( \mathcal{L} \) are varieties then this means that the former is not finitely axiomatizable over the latter, like the class of algebras having \( n+1 \) smooth relativized representations and that of those algebras having \( n \) smooth relativized representations. But even if the classes are not even elementary, like \( \text{CRA}_{m,n+1} = S\text{N}t_n \text{CA}_{n+k} \), then this definition also makes sense as we proceed to show.

(1) \exists \text{ has a winning strategy in the game } G_{\omega}^n(\mathfrak{A}_r^n)
(2) \exists \text{ has a winning strategy in } G_{\omega}^n(\mathfrak{A}_n^n).
(3) \forall \text{ has a winning strategy in } G_{\omega}^{n+1}(\mathfrak{A}_n^n)
(4) The distance between \( \text{CRA}_{m,n+1} \) and \( \text{CRA}_{m,n} \) is infinite.

We have \( \mathfrak{A}_r^n \in \text{CRA}_n \sim \text{CRA}_{n+1} \). \exists \text{ has a winning strategy in } G_{\omega}^{n+1}(\mathfrak{A}_r^n) for all finite \( r \), then \exists \text{ has a winning strategy in } G_{\omega}^{n+1}(\prod_r \mathfrak{A}_r^n/D), \text{ for any non principal ultrafilter } D, \text{ so the latter is in } \text{CRA}_{n+1} \text{.}

### 5.1 More on atom structures

**Definition 5.8.** A class \( K \) is gripped by its atom structures, if whenever \( \mathfrak{A} \in K \cap \text{At} \), and \( \mathfrak{B} \) is atomic such that \( \text{At\mathfrak{B}} = \text{At\mathfrak{A}} \), then \( \mathfrak{B} \in K \).

(1) A class \( K \) is strongly gripped by its atom structures, if whenever \( \mathfrak{A} \in K \cap \text{At} \), and \( \mathfrak{B} \) is atomic such that \( \text{At\mathfrak{B}} \equiv \text{At\mathfrak{A}} \), then \( \mathfrak{B} \in K \).
(2) A class $K$ of atom structures is infinitary gripped if whenever $A \in K \cap \text{At}$ and $B$ is atomic, such that $\text{At}B \equiv_{\infty, \omega} \text{At}B$, then $B \in K$.

(3) An atomic game is strongly gripping for $K$ if whenever $\exists$ has a winning strategy for all finite rounded games on $\text{At}A$, then $A \in K$.

(4) An atomic game is gripping if $\exists$ has a winning strategy in the $\omega$ rounded game on $\text{At}A$, then $A \in K$.

Notice that infinitary gripped implies strongly gripped implies gripped (by its atom structures). For the sake of brevity, we write only (strongly) gripped, without referring to atom structures. In the next theorem, all items except the first applies to all algebras considered. The first applies to any class $K$ between Sc and PEA (where the notion of neat reducts is not trivial). The $n$th Lyndon condition is a first order sentences that codes a winning strategy for $\exists$ in $n$ rounds.

The elementary class satisfying al such sentences is denoted by $\text{LCA}_m$. It is not hard to show that $\text{UpUrCRA}_m = \text{LCA}_m$ for any $n > 2$. This follows from the simple observation that if $\exists$ has a winning strategy in all finite rounded atomic games on an atom structure of a $\text{CA}_m$, then this algebra is necessarily elementary equivalent to a countable completely representable algebra.

**Theorem 5.9.**

(1) The class of neat reducts for any dimension is not gripped, hence is neither strongly gripped nor infinitary gripped.

(2) The class of completely representable algebras is gripped but not strongly gripped.

(3) The class of algebras satisfying Lyndon conditions is gripped and strongly gripped.

(4) The class of representable algebras is not gripped.

(5) The Lyndon usual atomic game is gripping but not strongly gripping for completely representable algebras, it is strongly gripping for $\text{LCA}_m$, when $m > 2$.

**Proof.** (1) This example is an adaptation of an example used in [16] to show that the class of neat reducts is not closed under forming subalgebras, and also used other contexts proving negative results on various amalgamation properties for cylindric-like algebras [21], [24].

Here we slightly generalize the example by allowing an arbitrary field to rather than $\mathbb{Q}$ to show that there is an atom structure that carries simultaneously an algebra in $\text{At}_{\omega} \text{CA}_{\omega+\omega}$ and an algebra not in $\text{At}_{\omega} \text{CA}_{\omega+1}$. 

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This works for all $\alpha > 1$ (infinite included) and other cylindric-like algebras as will be clear from the proof. Indeed, the proof works for any class of algebras whose signature is between $\mathcal{Sc}$ and $\mathcal{QEA}$. (Here we are using the notation $\mathcal{QEA}$ instead of $\mathcal{PEA}$ because we are allowing infinite dimensions).

Let $\alpha$ be an ordinal $> 1$; could be infinite. Let $\mathfrak{F}$ be field of characteristic $0$.

$$V = \{s \in ^\alpha \mathfrak{F} : |\{i \in \alpha : s_i \neq 0\}| < \omega\},$$

$$\mathcal{E} = (\wp(V), \cup, \cap, \sim, \emptyset, V, c_i, d_{i,j}, s_{\tau})_{i,j \in \alpha, \tau \in FT_{\alpha}}.$$  

Then clearly $\wp(V) \in \mathcal{Nr}_\alpha \mathcal{QPEA}_{\alpha+\omega}$. Indeed let $W = \alpha + \omega \mathfrak{F}(0)$. Then

$$\psi : \wp(V) \rightarrow \mathcal{Nr}_\alpha \wp(W)$$

defined via

$$X \mapsto \{s \in W : s \upharpoonright \alpha \in X\}$$

is an isomorphism from $\wp(V)$ to $\wp(W)$. We shall construct an algebra $\mathfrak{A}$, $\mathfrak{A} \notin \mathcal{Nr}_\alpha \mathcal{QPEA}_{\alpha+1}$. Let $y$ denote the following $\alpha$-ary relation:

$$y = \{s \in V : s_0 + 1 = \sum_{i>0} s_i\}.$$  

Let $y_s$ be the singleton containing $s$, i.e. $y_s = \{s\}$. Define as before $\mathfrak{A} \in \mathcal{QPEA}_\alpha$ as follows:

$$\mathfrak{A} = Sg^\mathcal{E}\{y, y_s : s \in y\}.$$  

Now clearly $\mathfrak{A}$ and $\wp(V)$ share the same atom structure, namely, the singletons. Then we claim that $\mathfrak{A} \notin \mathcal{Nr}_\alpha \mathcal{QPEA}_\beta$ for any $\beta > \alpha$. The first order sentence that codes the idea of the proof says that $\mathfrak{A}$ is neither an elementary nor complete subalgebra of $\wp(V)$. Let $\mathcal{At}(x)$ be the first order formula asserting that $x$ is an atom. Let

$$\tau(x, y) = c_1(c_0 x \cdot s_1^0 c_1 y) \cdot c_1 x \cdot c_0 y.$$  

Let

$$\mathcal{Rc}(x) := c_0 x \cap c_1 x = x,$$

$$\phi := \forall x (x \neq 0 \rightarrow \exists y (\mathcal{At}(y) \wedge y \leq x)) \wedge \forall x (\mathcal{At}(x) \rightarrow \mathcal{Rc}(x)),$$

$$\alpha(x, y) := \mathcal{At}(x) \wedge x \leq y,$$

and $\psi(y_0, y_1)$ be the following first order formula

$$\forall z (\forall x (\alpha(x, y_0) \rightarrow x \leq z) \rightarrow y_0 \leq z) \wedge \forall x (\mathcal{At}(x) \rightarrow \mathcal{At}(c_0 x \cap y_0) \wedge \mathcal{At}(c_1 x \cap y_0))$$

$$\rightarrow [\forall x_1 \forall x_2 (\alpha(x_1, y_0) \wedge \alpha(x_2, y_0) \rightarrow \tau(x_1, x_2) \leq y_1)].$$
\[ \forall z (\forall x_1 \forall x_2 (\alpha(x_1, y_0) \land \alpha(x_2, y_0) \rightarrow \tau(x_1, x_2) \leq z) \rightarrow y_1 \leq z) \].

Then

\[ \mathfrak{M}_\alpha \mathcal{QEA}_\beta \models \phi \rightarrow \forall y_0 \exists y_1 \psi(y_0, y_1). \]

But this formula does not hold in \( \mathfrak{A} \). We have \( \mathfrak{A} \models \phi \) and not \( \mathfrak{A} \models \forall y_0 \exists y_1 \psi(y_0, y_1) \). In words: we have a set \( X = \{ y_s : s \in V \} \) of atoms such that \( \sum^\mathfrak{A} X = y \), and \( \mathfrak{A} \) models \( \phi \) in the sense that below any non zero element there is a rectangular atom, namely a singleton.

Let \( Y = \{ \tau(y_r, y_s) : r, s \in V \} \), then \( Y \subseteq \mathfrak{A} \), but it does have one in any full neat reduct \( \mathfrak{B} \) containing \( \mathfrak{A} \), and this is \( \tau_\alpha(y, y) = (s_0^1 c_\alpha x \cdot s_0^0 c_\alpha y) \).

In \( \varphi(V) \) this last is \( w = \{ s \in \alpha \hat{S}^{(0)} : s_0 + 2 = s_1 + 2 \sum_{i > 1} s_i \} \), and \( w \notin \mathfrak{A} \).

The proof of this can be easily distilled from [16, main theorem]. For \( y_0 = y \), there is no \( y_1 \in \mathfrak{A} \) satisfying \( \psi(y_0, y_1) \). Actually the above proof proves more. It proves that there is a \( \mathfrak{C} \in \mathfrak{M}_\alpha \mathcal{QEA}_\beta \) for every \( \beta > \alpha \) (equivalently \( \mathfrak{C} \in \mathfrak{M}_\alpha \mathcal{QEA}_\omega \)), and \( \mathfrak{A} \subseteq \mathfrak{C} \), such that \( \mathfrak{M}_{\mathfrak{S}_c} \mathfrak{A} \notin \mathfrak{M}_\alpha \mathcal{S}_c^{\alpha+1} \).

See [24, theorems 5.1.4-5.1.5] for an entirely different example.

(2) The algebra \( \mathbf{PEA}_{Z, N} \), and its various reducts down to \( \mathbf{Scs} \), shows that the class of completely representable algebras is not strongly gripped. Indeed, it can be shown that \( \exists \) can win all finite rounded atomic games but \( \forall \) can win the \( \omega \) rounded game. It is known that this class is gripped.

An atom structure is completely representable iff one, equivalently, all atomic algebras sharing this atom structure are completely representable [9], [18].

(3) This is straightforward from the definition of Lyndon conditions.

(4) Any weakly representable atom structure that is not strongly representable detects this, see e.g. [10], [12], [7, theorems 1.1, 1.2] and theorems 3.11 and theorem 3.3. For a potential stronger result, see theorem 3.12 above.

(5) Follows directly from the definition.

Recall the definition of relativized representation, definition 5.11. Here \( m < \omega \) denotes the dimension and \( \mathbf{CRA}_{m,n} \) denotes the class of \( \mathbf{CA}_{m,s} \) with an \( n \) relativized smooth representation. There is no restriction whatsoever on \( n \) except that it is \( > m \). In particular \( n \) can be infinite.

**Theorem 5.10.** Regardless of cardinalities, \( \mathfrak{A} \in \mathbf{CRA}_{m, \omega} \) iff \( \exists \) has a winning strategy in \( G_\omega \).
Proof. One side is obvious. Now assume that \( \exists \) has a winning strategy in the \( \omega \) rounded game, using \( \omega \) many pebbles. We need to build an \( \omega \) relativized complete representation. The proof goes as follows. First the atomic networks are finite, so we need to convert them into \( \omega \) dimensional atomic networks. For a network \( N \), and a map \( v : \omega \rightarrow N \), let \( Nv \) be the network induced by \( v \), that is \( Nv(s) = N(v \circ s) \). Let \( J \) be the set of all such \( Nv \), where \( N \) occurs in some play of \( G_\omega^\omega(\mathcal{A}) \) in which \( \exists \) uses his winning strategy and \( v : \omega \rightarrow N \) (so via these maps we are climbing up \( \omega \)).

This can be checked to be an \( \omega \) dimensional hyperbasis (extended to the cylindric case the obvious way). So \( \mathcal{A} \in S_c \mathfrak{It}_n \mathcal{CA}_\omega \). We can use that the basis consists of \( \omega \) dimensional atomic networks, such that for each such network, there is a finite bound on the size of its strict networks. Then a complete \( \omega \) relativized representation can be obtained in a step by step way, requiring inductively in step \( t \), there for any finite clique \( C \) of \( M_t \), \( |C| < \omega \), there is a network in the base, and an embedding \( v : N \rightarrow M_t \) such that \( \text{rng} v \subseteq C \).

Here we consider finite sequences of arbitrarily large length, rather than fixed length \( n \) tuples. This is because an \( \omega \) relativized representation only requires cylindrifier witnesses over finite sized cliques, not necessarily cliques that are uniformly bounded.

**Theorem 5.11.** \( \text{CRA}_m \subset \text{CRA}_{m,\omega} \), the strict inclusion can be only witnessed on uncountable algebras. Furthermore, the class \( \text{CRA}_{m,\omega} \) is not elementary. The classes \( S_c \mathfrak{It}_n \mathcal{CA}_\omega \), \( \text{CRA}_m \), and \( \text{CRA}_{m,\omega} \), coincide on atomic countable algebras.

**Proof.** That \( \text{CRAS}_{m,\omega} \) is not elementary is witnessed by the rainbow algebra \( \text{PEA}_{K_n,K} \), where the latter is a disjoint union of \( K_n \), \( n \in \omega \). \( \exists \) has a winning strategy for all finite length games, but \( \forall \) can win the infinite rounded game. Hence \( \exists \) can win the transfinite game on an uncountable non-trivial ultrapower of \( \mathcal{A} \), and using elementary chains one can find an elementary countable subalgebra \( \mathcal{B} \) of this ultrapower such that \( \exists \) has a winning strategy in the \( \omega \) rounded game. This \( \mathcal{B} \) will have an \( \omega \) square representation hence will be in \( \text{CRA}_{m,\omega} \), and \( \mathcal{A} \) is not in the latter class. The three classes coincide on countable atomic algebras with the class of completely representable algebras [24, theorem 5.3.6].

**Theorem 5.12.** If \( \mathcal{A} \in \mathcal{CA}_3 \) is finite, and has an \( n \) square relativized representation, with \( 3 < n \leq \omega \), then it has a finite \( n \) square relativized representation. If \( n \geq 6 \), this is not true for \( n \) smooth relativized representations.

**Proof.** The first part follows from the fact that the first order theory coding the existence of square representations can be coded in the clique guarded fragment of first order logic, and indeed in the loosely guarded fragment of first order logic which has the finite base property [20, corollary 19.7]. The
second part follows from the fact that the problem of deciding whether a finite \( CA_3 \) is in \( S\forall\forall_3 CA_n \), when \( n \geq 6 \), is undecidable, from which one can conclude that there are finite algebras in \( S\forall\forall_3 CA_n \), \( n \geq 6 \) that do not have a finite \( n \) dimensional hyperbasis [20, corollary 18.4] and these cannot possibly have finite representations.

A contrasting result is:

**Theorem 5.13.** Let \( \mathfrak{A} \) be a finite \( CA_m \). Then the following are equivalent

1. \( \mathfrak{A} \) has a finite \( n \) smooth relativized representation
2. \( \mathfrak{A} \in S\forall\forall_m CA_n \)
3. \( \mathfrak{A} \) has a finite \( n \) dimensional hyperbasis.

**Proof.** (1) to (2) to (3) is exactly as above, by noting that if \( \mathfrak{A} \) is finite then \( \mathcal{D} \) as defined in corollary [5] is also finite, and that this gives necessarily a finite hyperbasis, witness the proof of theorem [3.10].

It remains to show that (1) implies (3). Let \( H \) be a finite \( n \) dimensional hyperbasis; we can assume that is symmetric, that is, closed under substitutions. This does not affect finiteness. Let \( L \) be the finite signature consisting of an \( m \)-ary relation symbol for every element of \( \mathfrak{A} \), together with an \( n \)-ary relation symbol \( R_N \) for each \( N \in H \). Define an \( L \) structure by

\[
M \models a(\bar{x}) \text{ iff } 1(\bar{x}) \text{ and } M(\bar{x}) \leq a,
\]

and

\[
M \models R_N(x_0, \ldots, x_{n-1}) \text{ iff } M(x_0, \ldots, x_n-1) = N
\]

for all \( a \in A \) and \( N \in H \). Then it is not hard to show that \( M \) satisfies the axioms postulated in theorem [3.9] and these can be coded as a fragment of the loosely guarded fragment. \( \square \)

### 6 Neat atom structures

Next we introduce several definitions an atom structures concerning neat embeddings: Here we denote the dimension by \( n \), where \( n \) is finite. \( n \) will be always \( > 1 \) and often greater than 2.

**Definition 6.1.**

1. Let \( 1 \leq k \leq \omega \). Call an atom structure \( \alpha \) weakly \( k \) neat representable, if the term algebra is in \( RCA_n \cap \forall\forall_n CA_{n+k} \), but the complex algebra is not representable.

2. Call an atom structure \( k \) neat, \( k > n \), if there is an atomic algebra \( \mathfrak{A} \), such that \( At\mathfrak{A} = \alpha \) and \( \mathfrak{A} \in \forall\forall_n CA_k \).
(3) Let $k \leq \omega$. Call an atom structure $\alpha$ $k$ complete, if there exists $\mathfrak{A}$ such that $At\mathfrak{A} = \alpha$ and $\mathfrak{A} \in S_c\mathfrak{N}_n\mathfrak{C}A_{n+k}$.

**Definition 6.2.** Let $K \subseteq \mathfrak{C}A_n$, and $\mathcal{L}$ be an extension of first order logic. $K$ is detectable in $\mathcal{L}$, if for any $\mathfrak{A} \in K$, $\mathfrak{A}$ atomic, and for any atom structure $\beta$ such that $At\mathfrak{A} \equiv_\mathcal{L} \beta$, if $\mathfrak{B}$ is an atomic algebra such that $At\mathfrak{B} = \beta$, then $\mathfrak{B} \in K$.

Roughly speaking, $K$ is detectable in $\mathcal{L}$ if whenever an atomic algebra is not in $K$ then $\mathcal{L}$ can witness this. In particular, a class that is not detectable in first order logic is simply elementary. A class that is not witnessed by quasi (equations) is a (quasi) variety.

We investigate the existence of such structures, and the interconnections. Note that if $\mathcal{L}_1$ is weaker than $\mathcal{L}_2$ and $K$ is not detectable in $\mathcal{L}_2$, then it is not detectable in $\mathcal{L}_1$. We also present several $K$s and $\mathcal{L}$s as in the second definition. All our results extend to Pinter’s algebras and quasi polyadic algebras with and without equality. But first another definition.

We now prove:

**Theorem 6.3.**

(1) Let $n$ be finite $n \geq 3$. Then there exists a countable weakly $k$ neat atom structure of dimension $n$ if and only if $k < \omega$.

(2) There is an $\omega$ rounded game that determines neat atom structures.

(3) The class of completely representable algebras, and strongly representable ones of dimension $> 2$, is not detectable in $L_{\omega,\omega}$, while the class $\mathfrak{N}_n\mathfrak{C}A_m$ for any ordinals $1 < n < m < \omega$, is not detectable even in $L_{\infty,\omega}$. For for infinite $n$, $\mathfrak{N}_n\mathfrak{C}A_m$ is not detectable in first order logic nor in the quantifier free reduct of $L_{\infty,\omega}$.

(4) There is an atom structure that is not $n+3$ complete but is elementary equivalent to one that is $\omega$ neat.

**Proof.** (1) Follows from [7], see also theorem 3.12 above. Here $k$ cannot be infinite, for else the term algebra will be in $\mathfrak{N}_n\mathfrak{C}A_\omega$, hence by [24, 5.3.6] would be completely representable, which makes its atom structure $At\mathfrak{A}$ strongly representable [9, 3.5.1], but then $\mathfrak{C}m\mathfrak{A}t$ will be representable.

(2) For the definition of a network and atomic games on networks, we refer to [9] definitions 3.3.1, 3.3.2, 3.3.3]. For an atomic network and for $x, y \in \text{nodes}(N)$, we set $x \sim y$ if there exists $\bar{z}$ such that $N(x, y, \bar{z}) \leq d_01$. The equivalence relation $\sim$ over the set of all finite sequences over $\text{nodes}(N)$ is defined by $\bar{x} \sim \bar{y}$ iff $|\bar{x}| = |\bar{y}|$ and $x_i \sim y_i$ for all $i < |\bar{x}|$ (It can be checked that this indeed an equivalence relation.)
A hypernetwork $N = (N^a, N^h)$ over an atomic polyadic equality algebra $\mathfrak{C}$ consists of a network $N^a$ together with a labelling function for hyperlabels $N^h : \subseteq \text{nodes}(N) \rightarrow \Lambda$ (some arbitrary set of hyperlabels $\Lambda$) such that for $\bar{x}, \bar{y} \in \subseteq \text{nodes}(N)

\text{IV. } \bar{x} \sim \bar{y} \Rightarrow N^h(\bar{x}) = N^h(\bar{y}).

If $|\bar{x}| = k \in \mathbb{N}$ and $N^h(\bar{x}) = \lambda$ then we say that $\lambda$ is a $k$-ary hyperlabel. $(\bar{x})$ is referred to a a $k$-ary hyperedge, or simply a hyperedge. (Note that we have atomic hyperedges and hyperedges) When there is no risk of ambiguity we may drop the superscripts $a,h$. There are short hyperedges and long hyperedges (to be defined in a while). The short hyperedges are constantly labelled. The idea (that will be revealed during the proof), is that the atoms in the neat reduct are no smaller than the atoms in the dilation. (When $\mathfrak{A} = 9\mathfrak{r}_{\omega, n} \mathfrak{B}$, it is common to call $\mathfrak{B}$ a dilation of $\mathfrak{A}$.) We know that there is a one to one correspondence between networks and coloured graphs. If $\Gamma$ is a coloured graph, then by $N_{\Gamma}$ we mean the corresponding network defined on $n - 1$ tuples of the nodes of $\Gamma$ to to coloured graphs of size $\leq n$.

(1) A hyperedge $\bar{x} \in \subseteq \text{nodes}(\Gamma)$ of length $m$ is short, if there are $y_0, \ldots, y_{n-1} \in \text{nodes}(N)$, such that $N_{\Gamma}(x_i, y_0, \bar{z}) \leq d_{01}$, or $N(\Gamma(x_i, y_1, \bar{z}) \ldots$ or $N(x_i, y_{n-1}, \bar{z}) \leq d_{01}$ for all $i < |x|$, for some (equivalently for all) $\bar{z}$. Otherwise, it is called long.

(2) A hypergraph $(\Gamma, l)$ is called $\lambda$ neat if $N_{\Gamma}(\bar{x}) = \lambda$ for all short hyperedges.

This game is similar to the games devised by Robin Hirsch in [23, definition 28], played on relation algebras. However, lifting it to cylindric algebras is not straightforward, for in this new context the moves involve hyperedges of length $n$ (the dimension), rather than edges. In the $\omega$ rounded game $J$, $\forall$ has three moves.

The first is the normal cylindrifier move. There is no polyadic move. The next two are amalgamation moves. But the games are not played on hypernetworks, they are played on coloured hypergraphs, consisting of two parts, the graph part that can be viewed as an $L_{\omega_1, \omega}$ model for the rainbow signature, and the part dealing with hyperedges with a labelling function. The amalgamation moves roughly reflect the fact, in case $\exists$ wins, then for every $k \geq n$ there is a $k$ dimensional hyperbasis, so that the small algebra embeds into cylindric algebras of arbitrary large dimensions. The game is played on $\lambda$ neat hypernetworks, translated to $\lambda$ neat hypergraphs, where $\lambda$ is a label for the short hyperedges.
For networks $M, N$ and any set $S$, we write $M \equiv^S N$ if $N|_S = M|_S$, and we write $M \equiv_S N$ if the symmetric difference

$$\Delta(\text{nodes}(M), \text{nodes}(N)) \subseteq S$$

and $M \equiv^{(\text{nodes}(M) \cup \text{nodes}(N)) \setminus S} N$. We write $M \equiv_k N$ for $M \equiv \{k\} N$.

Let $N$ be a network and let $\theta$ be any function. The network $N\theta$ is a complete labelled graph with nodes $\theta^{-1}(\text{nodes}(N)) = \{x \in \text{dom}(\theta) : \theta(x) \in \text{nodes}(N)\}$, and labelling defined by

$$(N\theta)(i_0, \ldots i_{\mu-1}) = N(\theta(i_0), \theta(i_1), \theta(i_{\mu-1})),$$

for $i_0, \ldots i_{\mu-1} \in \theta^{-1}(\text{nodes}(N))$. We call this game $H$. It is $\omega$ rounded. Its first move by $\forall$ is the usual cylindrifier move (equivalently) $\forall s$ move in $F^\omega$, but $\forall$ has more moves which makes it harder for $\exists$ to win. These notions apply equally well to hypernetworks.

$\forall$ can play a transformation move by picking a previously played hypernetwork $N$ and a partial, finite surjection $\theta : \omega \rightarrow \text{nodes}(N)$, this move is denoted $(N, \theta)$. $\exists$ must respond with $N\theta$.

Finally, $\forall$ can play an amalgamation move by picking previously played hypernetworks $M, N$ such that $M \equiv^{\text{nodes}(M) \cap \text{nodes}(N)} N$ and $\text{nodes}(M) \cap \text{nodes}(N) \neq \emptyset$. This move is denoted $(M, N)$. Here, unlike $H$, there is no restriction on the number of overlapping nodes, so in principal the game is harder for $\exists$ to win.

To make a legal response, $\exists$ must play a $\lambda_0$-neat hypernetwork $L$ extending $M$ and $N$, where $\text{nodes}(L) = \text{nodes}(M) \cup \text{nodes}(N)$.

The next theorem is a cylindric-like variation on [23, theorem 39], formulated for polyadic equality algebras, but it works for many cylindric-like algebras like $\text{Sc}$, $\text{PA}$ and $\text{CA}$. But it does not apply to Dfs because the notion of neat reducts for Dfs is trivial.

Let $\mathfrak{A}$ be an atomic polyadic equality algebra with a countable atom structure $\alpha$. If $\exists$ can win the $\omega$ rounded game $J$ on $\alpha$, then there exists a locally finite $\text{PEA}_\omega$ such that $\text{At}\mathfrak{A} \cong \text{At}\mathfrak{N}_\alpha \mathfrak{C}$. Furthermore, $\mathfrak{C}$ can be chosen to be complete, and $\text{CmAt}\mathfrak{A} = \mathfrak{N}_\alpha \mathfrak{C}$.

For the first part. Fix some $a \in \alpha$. Using $\exists$ s winning strategy in the game of neat hypernetworks, one defines a nested sequence $N_0 \subseteq N_1 \ldots$ of neat hypernetworks where $N_0$ is $\exists$’s response to the initial $\forall$-move $a$, such that

1. If $N_r$ is in the sequence and $b \leq c_i N_r(f_0, \ldots, x, \ldots f_{n-2})$, then there is $s \geq r$ and $d \in \text{nodes}(N_s)$ such that $N_s(f_0, f_{i-1}, d, f_{i+1}, \ldots f_{n-2}) = b$. 

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2. If $N_r$ is in the sequence and $\theta$ is any partial isomorphism of $N_r$, then there is $s \geq r$ and a partial isomorphism $\theta^+$ of $N_s$ extending $\theta$ such that $\text{rng}(\theta^+) \supseteq \text{nodes}(N_r)$.

Now let $N_a$ be the limit of this sequence, that is $N_a = \bigcup N_i$, the labelling of $n-1$ tuples of nodes by atoms, and the hyperedges by hyperlabels done in the obvious way. This limit is well-defined since the hypernetworks are nested. We shall show that $N_a$ is the base of a weak set algebra having unit $V = \omega N_a^{(p)}$, for some fixed sequence $p \in \omega N_a$.

We can make $U_a$ into the universe an $L$ relativized structure $N_a$; here relativized means that we are only taking those assignments agreeing cofinitely with $f_a$, we are not taking the standard square model. However, satisfiability for $L$ formulas at assignments $f \in U_a$ is defined the usual Tarskian way, except that we use the modal notation, with restricted assignments on the left: For $r \in \mathfrak{A}l_{0,\ldots,l_{n-1},i_0,\ldots,i_{k-1}} < \omega$, $k$-ary hyperlabels $\lambda$, and all $L$-formulas $\phi, \psi$, let $\text{We can make $U_a$ into the base of an $L$-structure } N_a \text{ and evaluate $L$-formulas at } f \in U_a$ as follow. For $b \in \alpha$, $l_0,\ldots,l_{\mu-1},i_0,\ldots,i_{k-1} < \omega$, $k$-ary hyperlabels $\lambda$, and all $L$-formulas $\phi, \psi$, let

$$N_a, f \models b(x_{l_0} \ldots x_{l_n-1}) \iff N_a(f(l_0),\ldots,f(l_{n-1})) = b$$

$$N_a, f \models \lambda(x_{i_0},\ldots,x_{i_{k-1}}) \iff N_a(f(i_0),\ldots,f(i_{k-1})) = \lambda$$

$$N_a, f \models \neg \phi \iff N_a, f \not\models \phi$$

$$N_a, f \models (\phi \lor \psi) \iff N_a, f \models \phi \text{ or } N_a, f \models \psi$$

$$N_a, f \models \exists x_i \phi \iff N_a, f[i/m] \models \phi, \text{ some } m \in \text{nodes}(N_a)$$

For any $L$-formula $\phi$, write $\phi^{N_a}$ for the set of all $n$-ary assignments satisfying it; that is $\{f \in \text{nodes}(N_a) : N_a, f \models \phi \}$. Let $D_a = \{\phi^{N_a} : \phi \text{ is an } L\text{-formula}\}$. Then this is the universe of the following weak set algebra

$$\mathfrak{D}_a = (D_a, \cup, \sim, D_{ij}, C_i)_{i,j<\omega}$$

then $\mathfrak{D}_a \in \text{RCA}_\omega$. (Weak set algebras are representable).

For any $L$-formula $\phi$, write $\phi^{N_a}$ for $\{f \in \text{nodes}(N_a) : N_a, f \models \phi \}$. Let $\text{Form}^{N_a} = \{\phi^{N_a} : \phi \text{ is an } L\text{-formula}\}$ and define a cylindric algebra

$$\mathfrak{D}_a = (\text{Form}^{N_a}, \cup, \sim, D_{ij}, C_i, i, j < \omega)$$

where $D_{ij} = (x_i = x_j)^{N_a}, C_i(\phi^{N_a}) = (\exists x_i \phi)^{N_a}$. Observe that $\top^{N_a} = U_a$, $(\phi \lor \psi)^{N_a} = \phi^{\ast N_a} \cup \psi^{N_a}$, etc. Note also that $\mathfrak{D}$ is a subalgebra of the $\omega$-dimensional cylindric set algebra on the base $\text{nodes}(N_a)$, hence $\mathfrak{D}_a \in \text{Lf}_\omega \cap \text{Ws}_\omega$, for each atom $a \in \alpha$, and is clearly complete.
Let $C = \prod_{a \in \alpha} D_a$. (This is not necessarily locally finite). Then $C \in \text{RCA}_\omega$, and $C$ is also complete, will be shown to be is the desired generalized weak set algebra, that is the desired dilation. Note that unit of $C$ is the disjoint union of the weak spaces. Then $\text{At}_n C$ is atomic and $\alpha \cong \text{At}_n C$ — the isomorphism is $b \mapsto (b(x_0, x_1, \ldots, x_{n-1})^D_a : a \in A)$.

Now we can work in $L_{\infty, \omega}$ so that $C$ is complete by changing the defining clause for infinitary disjunctions to

$$N_a, f \models (\bigvee_{i \in I} \phi_i) \text{ iff } (\exists i \in I)(N_a, f \models \phi_i)$$

By working in $L_{\infty, \omega}$, we assume that arbitrary joins hence meets exist, so $C_a$ is complete, hence so is $C$. But $\text{CmAt} A \subseteq \text{At}_n C$ is dense and complete, so $\text{CmAt} A = \text{At}_n C$.

(3) Follows from [27], generalized in theorem 4.12 above, [9] theorem 3.6.11, corollary 3.7.1, and [24] theorems 5.1.4, 5.1.5.

(4) see 3.13

(5) $\exists$ has a winning strategy in the game devised below on the rainbow algebra $CA_{Z,N}$ which is not completely representable but is elementary equivalent to an algebra in $\text{At}_n CA_{\omega}$. Since atom structures are interpretable in their algebras, we are done.

Remark 6.4. (1) Note that if we use the following game, then we get a much stronger result (witness the next item), namely, that $A = \text{At}_n C$.

Here $\forall$ has even more moves and this allows is to remove $\text{At}$ from both sides of the above equation, obtaining a much stronger result. The main play of the stronger game $K(A)$ is a play of the game $J(A)$.

The base of the main board at a certain point will be the atomic network $X$ and we write $X(\bar{x})$ for the atom that labels the edge $\bar{x}$ on the main board. But $\forall$ can make other moves too, which makes it harder for $\exists$ to win and so a winning strategy for $\exists$ will give a stronger result. An $n$ network is a finite complete graph with nodes including $n$ with all edges labelled by elements of $A$. No consistency properties are assumed.

$\forall$ can play an arbitrary $n$ network $N$, $\exists$ must replace $N(n)$ by some element $a \in A$. The idea, is that the constraints represented by $N$ correspond to an element of the $\text{RCA}_\omega$ being constructed on $X$, generated by $A$. This network is placed on the side of the main board. $N$ asserts that whenever it appears in $X$ you can never have an atom not below holding between the embedded images of $n$. But it also asserts that whenever
an atom below $a$ holds in $X$, there are also points in $X$ witnessing all the nodes of $N$. The final move is that $\forall$ can pick a previously played $n$ network $N$ and pick any tuple $\bar{x}$ on the main board whose atomic label is below $N(\bar{n})$.

$\exists$ must respond by extending the main board from $X$ to $X'$ such that there is an embedding $\theta$ of $N$ into $X'$ such that $\theta(0) = x_0, \ldots, \theta(n-1) = y_{n-1}$ and for all $i \in N$ we have $X(\theta(i_0), \ldots, \theta(i_{n-1})) \subseteq N(i)$. This ensures that in the limit, the constraints in $N$ really define $a$. If $\exists$ has a winning strategy in $K(A)$ then the extra moves mean that every $n$ dimensional element generated by $\mathfrak{A}$ in the $\text{RCA}_\omega$ constructed in the play is an element of $\mathfrak{A}$.

(2) The example in [16] shows that there is a neat atom structure that carries algebra that is not in $\text{RCA}_n\mathbb{C}A_{n+1}$

**Theorem 6.5.** If there exists an atomic polyadic equality algebra with countably many atoms such that $\exists$ has a winning strategy for $J_n$ for every $n$, and $\forall$ has a winning strategy in $F^{n+k}$ on its $\text{Sc}$ reduct, then $\text{Rd}_{\text{sc}}\mathfrak{A} \not\equiv S_c\text{Rt}_n\text{Sc}_{n+k}$ but $\mathfrak{A} \not\in \text{UpUr}\text{Rt}_n\text{QPEA}_\omega$. In fact, there is a countable atomic algebra $\mathfrak{D} \in \text{Rt}_n\text{QEA}_\omega$, such that $\mathfrak{A} \equiv \mathfrak{D}$. In particular, for any class $K$ of Pinter’s algebras, cylindric algebras and polyadic equality algebras any $m \geq n+k$, and any class $L$ such that $\text{Rt}_nK \subseteq L \subseteq \text{Rt}_n\text{K}_{m+k}$, $L$ is not elementary. In particular, $\text{CRK}_n$, and $\text{Rt}_n\text{CA}_{n+k}$ are not elementary.

**Proof.** Assume $\forall$ can win the game $F^{n+k}$ on $\text{Rd}_{\text{sc}}\mathfrak{A}$. Hence $\text{Rd}_{\text{sc}}\mathfrak{A} \not\equiv S_c\text{Rt}_n\text{Sc}_{n+3}$. For $n < \omega$, assume that $\exists$ has a winning strategy $\sigma_n$ in $J_n(\mathfrak{A})$. We can assume that $\sigma_n$ is deterministic. Let $\mathfrak{B}$ be a non-principal ultrapower of $\mathfrak{A}$. Then $\exists$ has a winning strategy $\sigma$ in $J(\mathfrak{B})$ — essentially she uses $\sigma_n$ in the $n$’th component of the ultraproduct so that at each round of $J(\mathfrak{B})$, $\exists$ is still winning in co-finitely many components, this suffices to show she has still not lost. Now use an elementary chain argument to construct countable elementary subalgebras $\mathfrak{A} = \mathfrak{A}_0 \leq \mathfrak{A}_1 \leq \ldots \leq \mathfrak{B}$. For this, let $\mathfrak{A}_{i+1}$ be a countable elementary subalgebra of $\mathfrak{B}$ containing $\mathfrak{A}_i$, and all elements of $\mathfrak{B}$ that $\sigma$ selects in a play of $J(\mathfrak{B})$ in which $\forall$ only chooses elements from $\mathfrak{A}_i$. Now let $\mathfrak{A}' = \bigcup_{i<\omega} \mathfrak{A}_i$. This is a countable elementary subalgebra of $\mathfrak{B}$ and $\exists$ has a winning strategy in $H(\mathfrak{A}')$. Hence by the elementary chain argument there is a countable $\mathfrak{A}'$ such that $\exists$ can win the $\omega$ rounded game on its atom structure, hence $\mathfrak{A}' \equiv \mathfrak{A}$ but the former is in $\text{Rt}_n\text{PEA}_\omega$.

**Theorem 6.6.** When $k \leq 3$ the above statement is strictly stronger than the statement in theorem 3.15.

**Proof.** This follows from the fact that the inclusion $\text{Rt}_nK_\omega \subset S_c\text{Rt}_nK_\omega$ is strict. In fact, for any $n > 1$, the inclusion $\text{Rt}_n\text{CA}_{n+k} \subset S_c\text{Rt}_n\text{CA}_{n+k}$ is strict.

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for every \( k > n \), witness \([24]\) and also first item of \([5,9]\). Also in the first statement we have \( \mathcal{A} \cong \mathcal{D} \) and \( \text{At}\mathcal{D} \cong \text{At}\mathcal{M}_n \mathcal{C} \), for some \( \mathcal{C} \in \text{Lf}_\omega \), while in the present case we can remove \( \text{At} \) from the two sides of the equation, namely, we have \( \mathcal{D} \cong \mathcal{M}_n \mathcal{C} \). In view of example \([5.9]\) this is obviously much stronger. \( \square \)

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