Abstract

Three dimensional topological field theories associated with the three dimensional version of Abelian and non-Abelian Seiberg-Witten monopoles are presented. These three dimensional monopole equations are obtained by a dimensional reduction of the four dimensional ones. The starting actions to be considered are Gaussian types with random auxiliary fields. As the local gauge symmetries with topological shifts are found to be first stage reducible, Batalin-Vilkovisky algorithm is suitable for quantization. Then BRST transformation rules are automatically obtained. Non-trivial observables associated with Chern classes are obtained from geometric sector and are found to correspond to those of the topological field theory of Bogomol’nyi monopoles.
1 Introduction

Topological field theories [1, 2, 3, 4] are often used to study topological nature of manifolds. In particular, three and four dimensional topological field theories are well developed. The most well-known three dimensional topological field theory would be the Chern-Simons theory whose partition function gives Ray-Singer torsion of three manifolds [3] and the other topological invariants can be obtained as gauge invariant observables i.e., Wilson loops. The correlation functions can be identified with knot or link invariants e.g., Jones polynomial or its generalizations. On the other hand, in four dimensions, a twisted $N=2$ supersymmetric Yang-Mills theory developed by Witten [4] also has a nature of topological field theory. This Yang-Mills theory can be interpreted as Donaldson theory [5] and the correlation functions are identified with Donaldson polynomials which classify smooth structures of topological four manifolds. However, a new topological field theory on four manifolds was discovered in recent studies of electric-magnetic duality of supersymmetric gauge theory. The story of this is described as follows.

Seiberg and Witten [6, 7] studied the electric-magnetic duality of $N=2$ supersymmetric $SU(2)$ Yang-Mills gauge theory (for reviews, see Refs. 8,9,10,11,12) by using a version of Montonen-Olive duality and obtained exact solutions. According to this result, the exact low energy effective action can be determined by a certain elliptic curve with a parameter $u = \langle \text{tr} \phi^2 \rangle$, where $\phi$ is a complex scalar field in the adjoint representation of the gauge group, describing the quantum moduli space. For large $u$, the theory is weakly coupled and semiclassical, but at $u = \pm \Lambda^2$ corresponding to strong coupling regime, where $\Lambda$ is the dynamically generated mass scale, the elliptic curve becomes singular and the situation of the theory changes drastically. At these singular points, magnetically charged particles become massless. Witten showed that at $u = \pm \Lambda^2$ the topological quantum field theory was related to the moduli problem of counting the solution of the (Abelian) “Seiberg-Witten monopole equations” [13] and it gave a dual description for the $SU(2)$ Donaldson theory. The particularly interesting fact is that the partition function of this $U(1)$ gauge theory produces a new topological invariant [13, 14, 15, 16, 17, 18, 19].
The topological field theory of the Seiberg-Witten monopoles was discussed by several authors. Labastida and Mariño [20] took the Mathai-Quillen formalism [21, 22, 23] and found that the resulting action was equivalent to that of the twisted $N = 2$ supersymmetric Maxwell coupled with a twisted $N = 2$ hypermultiplet. Furthermore, they generalized their results for non-Abelian cases [24, 25] and determined polynomial invariants for $SU(2)$ case corresponding to a generalization of Refs. 13,14 in Abelian case. In these studies, the topological field theories were formulated as Witten type. On the other hand, Hyun et al. [26, 27] discussed a non-Abelian topological field theory in view of twisting of $N = 2$ supersymmetric Yang-Mills coupled with $N = 2$ matters and obtained similar polynomial invariants. There are other approaches to obtain the topological action, in fact, Zang et al. [28] derived the topological action as a BRST variation of a certain gauge fermion and Gianvittorio et al. [29, 30] discussed in view of a covariant gauge fixing procedure.

In three dimensions, a topological field theory of Bogomol’nyi monopoles can be obtained from a dimensional reduction of Donaldson theory [31, 32] and the partition function of this theory gives the Casson invariant [22]. However, the three or two dimensional topological field theory of Seiberg-Witten monopoles does not seem to be fully discussed, although several authors point out its importance [11, 28, 33]. Zang et al. [28] performed a dimensional reduction of the Abelian Seiberg-Witten theory from four to three dimensions and found the reduced topological action. They also found in view of Mathai-Quillen formalism that the partition function of this three dimensional theory can be interpreted as a Seiberg-Witten version of the Casson invariant of three manifolds.

In this paper, we discuss the topological quantum field theories associated with the three dimensional version of Abelian and non-Abelian Seiberg-Witten monopoles by applying Batalin-Vilkovisky quantization. In particular, we construct the topological actions, topological observables and BRST transformation rules. In section 2, we briefly review the essence of topological quantum field theories both Witten type and Schwarz type. The reader who is interested in the results of this paper may neglect this section. In section 3, the dimensional reduction of the Abelian and non-Abelian Seiberg-Witten monopole equations are
explicitly performed and the three dimensional monopole (3-d monopole) equations are obtained. We also obtain quadratic actions which reproduce these three dimensional monopole equations as minimum. In section 4, we construct topological field theories of these three dimensional monopoles taking the actions including random auxiliary fields as a starting point. As the local gauge symmetries of them are classified as first stage reducible with on-shell reducibility, Batalin-Vilkovisky algorithm is suitable to quantize these theories. Then we can automatically obtain the BRST transformation rules by construction. It is shown that the observables in geometric sector can be obtained from a standard fashion, but those in matter sector are found to be trivial. The reader will find that our results for Abelian case are consistent with those of the dimensionally reduced version of the topological field theory of four dimensional Seiberg-Witten monopoles \cite{28}, while those for non-Abelian case are new results. It is interesting to compare our results with those of the topological field theory of Bogomol’nyi monopoles. Section 5 is a summary and we mention some open problems. In Appendix A and B, we summarize the result of the Batalin-Vilkovisky quantization for the four dimensional non-Abelian Seiberg-Witten monopoles, for the reader’s convenience.

**Notations**

We use following notations unless we mention especially. Let $X$ be a compact orientable spin four manifold with no boundary and $g_{\mu\nu}$ be its Riemannian metric tensor with $g = \det g_{\mu\nu}$. We use $x_{\mu}$ as the local coordinates on $X$. $\gamma_{\mu}$ are Dirac’s gamma matrices and $\sigma_{\mu\nu} = [\gamma_{\mu}, \gamma_{\nu}]/2$ with $\{\gamma_{\mu}, \gamma_{\nu}\} = g_{\mu\nu}$ (see also Appendix C). $M$ is a Weyl fermion and $\overline{M}$ is a complex conjugate of $M$. We suppress spinor indices. The Lie algebra $\mathfrak{g}$ is defined by $[T^a, T^b] = i f_{abc} T^c$, where $T^a$ is a generator normalized as $\text{tr } T^a T^b = \delta^{ab}$. The symbol $f_{abc}$ is a structure constant of $\mathfrak{g}$ and is anti-symmetric in its indices.

The Greek indices $\mu, \nu, \alpha$ etc run from 0 to 3. The Roman indices $a, b, c, \cdots$ are used for the Lie algebra indices running from 1 to dim $\mathfrak{g}$, whereas $i, j, k, \cdots$ are the indices for space coordinates. Space-time indices are raised and lowered with $g_{\mu\nu}$. The repeated indices are assumed to be summed. $\epsilon_{\mu\nu\rho\sigma}$ is an anti-symmetric tensor with $\epsilon_{0123} = 1$. We often use the
abbreviation of roman indices as $\theta = \theta^a T^a$ etc in order to suppress the summation over Lie algebra indices.

## 2 Quick tour to topological field theory

This section is devoted to a brief review of topological field theory. The reader who is interested in the details should refer to Refs. 1,2,3,4,9.

Let $\phi$ be any field content. For a local symmetry of $\phi$, we can construct a nilpotent BRST operator $Q_B$ ($Q_B^2 = 0$). The variation of any functional $O$ of $\phi$ is denoted by

$$\delta O = \{Q_B, O\},$$

(2.1)

where the blacket $\{*, *\}$ means a graded commutator, namely, if $O$ is bosonic the bracket means a commutator $[*, *]$ and otherwise it is an anti-bracket.

Then we can give the definition of topological field theory [1].

**Definition** A topological field theory consists of

1. a collection of Grassmann graded fields $\phi$ on an $n$-dimensional Riemannian manifold $X$ with a metric $g$,

2. a nilpotent Grassmann odd operator $Q$,

3. physical states to be $Q$-cohomology classes,

4. an energy-momentum tensor $T_{\alpha\beta}$ which is $Q$-exact for some functional $V_{\alpha\beta}$ such as

$$T_{\alpha\beta} = \{Q, V_{\alpha\beta}(\phi, g)\}.$$ 

(2.2)

In this definition, $Q$ is often identified with $Q_B$ and is in general independent of the metric. There are several examples of topological field theories which do not satisfy this definition, but this definition is useful in many cases.

There are two broad types of topological field theories satisfying this definition and they are classified into Witten type [4] or Schwarz type [3] (there are several non-standard Schwarz type theories, e.g., higher dimensional BF theories, but here we do not consider such cases).
For Witten type theory, the quantum action $S_q$ which comprises the classical action, ghost and gauge fixing terms, can be represented by $S_q = \{ Q_B, V \}$, for some function $V$ of metric and fields and BRST charge $Q_B$. Under the metric variation $\delta_g$ of the partition function $Z$, it is easy to see that

$$ \delta_g Z = \int D\phi \ e^{-S_q} \left( -\frac{1}{2} \int_X d^n x \sqrt{g} \delta g^{\alpha \beta} T_{\alpha \beta} \right) $$
$$ = \int D\phi \ e^{-S_q} \{ Q, \chi \} $$
$$ \equiv \langle \{ Q, \chi \} \rangle = 0, \quad (2.3) $$

where

$$ \chi = -\frac{1}{2} \int_X d^n x \sqrt{g} \delta g^{\alpha \beta} V_{\alpha \beta}. \quad (2.4) $$

The last equality in (2.3) follows from the BRST invariance of the vacuum and means that $Z$ is independent of the local structure of $X$, that is, $Z$ is a “topological invariant” of $X$.

In general, for Witten type theory, $Q_B$ can be constructed by an introduction of a topological shift with other local gauge symmetry [34, 35]. For example, in order to obtain the topological Yang-Mills theory on four manifold $M^4$, we introduce the shift in the gauge transformation for the gauge field $A_\mu^a$ such as $\delta A_\mu^a = D_\mu \theta^a + \epsilon^a_\mu$, where $D_\mu$ is a covariant derivative, $\theta^a$ and $\epsilon^a_\mu$ are the (Lie algebra valued) usual gauge transformation parameter and topological shift parameter, respectively. In order to see the role of this shift, let us consider the first Pontrjagin class on $M^4$ given by

$$ S = \frac{1}{8} \int_{M^4} \epsilon^{\mu \nu \rho \sigma} F_{\mu \nu}^a F_{\rho \sigma}^a d^4 x, \quad (2.5) $$

where $F_{\mu \nu}^a$ is a field strength of the gauge field. We can easily check the invariance of (2.3) under the action of $\delta$. In this sense, (2.5) has a larger symmetry than the usual (Yang-Mills) gauge symmetry. Taking this into account, we can construct the topological Yang-Mills gauge theory [34, 35, 36]. We can also consider similar “topological” shifts for matter fields as will be shown in section 4.

In addition, in general, Witten type topological field theory can be obtained from the quantization of some Langevin equations [31]. This approach has been used for the construc-
tion of several topological field theories, e.g., supersymmetric quantum mechanics, topological sigma models or Donaldson theory [3, 37] (we will use this approach for the $N = 4$ theory [38] in next communication [39]).

On the other hand, Schwarz type theory [3] begins with any metric independent classical action $S_c$ as a starting point, but $S_c$ is assumed not to be a total derivative. Then the quantum action (up to gauge fixing term) can be written by

$$S_q = S_c + \{Q, V(\phi, g)\}$$

for some function $V$. For this quantum action, we can easily check the topological nature of the partition function, but note that the energy-momentum tensor contributes only from the second term in (2.6). One of the differences between Witten type and Schwarz type theories can be seen in this point. Namely, the energy-momentum tensor of the classical action for Schwarz type theory vanishes because it is derived as a result of metric variation.

Finally, we comment on the local symmetry of Schwarz type theory. Let us consider the Chern-Simons theory as an example. The classical action

$$S_{CS} = \int_{M^3} d^3 x \left( A \wedge dA + \frac{2}{3} A \wedge A \wedge A \right)$$

is a topological invariant which gives the second Chern class of three manifold $M^3$. As is easy to find, $S_{CS}$ is not invariant under the topological gauge transformation, although it is (Yang-Mills) gauge invariant. Therefore the quantization is proceeded by the standard BRST method. This is a general feature of Schwarz type theory.

### 3 Dimensional reduction

In this section, the dimensional reduction of the Abelian and non-Abelian Seiberg-Witten monopole equations is presented. For mathematical progresses on Seiberg-Witten monopoles, see Refs. 15,16 for Abelian case and Ref. 17 for non-Abelian case.

First, let us recall the Seiberg-Witten monopole equations in four dimensions. We assume that $X$ has Spin structure. Then there exist rank two positive and negative spinor bundles
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For Abelian gauge theory, we introduce a complex line bundle $L$ and a connection $A_\mu$ on $L$. The Weyl spinor $M(\overline{M})$ is a section of $S^+ \otimes L$ ($S^+ \otimes L^{-1}$), hence $M$ satisfies the positive chirality condition $\gamma^5 M = M$. If $X$ does not have Spin structure, we introduce Spin$^c$ structure and Spin$^c$ bundles $S^\pm \otimes L$, where $L^2$ is a line bundle. In this case, $M$ should be interpreted as a section of $S^+ \otimes L$. Below, we assume Spin structure. The reader who is interested in the physical implications of Spin and Spin$^c$ structures should refer to the excellent review Ref. 11 and references therein.

The Abelian Seiberg-Witten monopole equations [13] in four dimensions are the set of following differential equations

$$F^+_{\mu\nu} + \frac{i}{2} \overline{M} \sigma_{\mu\nu} M = 0,$$

$$i\gamma^\mu D_\mu M = 0,$$

(3.1)

where $F^+_{\mu\nu}$ is the self-dual part of the $U(1)$ curvature tensor

$$F_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu,$$

$$F^+_{\mu\nu} = P^+_{\mu\nu\rho\sigma} F^{\rho\sigma}$$

(3.2)

and $P^+_{\mu\nu\rho\sigma}$ is the self-dual projector defined by

$$P^+_{\mu\nu\rho\sigma} = \frac{1}{2} \left( \delta_{\mu\rho} \delta_{\nu\sigma} + \frac{\sqrt{g}}{2} \epsilon_{\mu\nu\rho\sigma} \right).$$

(3.3)

Note that the second term in the first equation of (3.1) is also self-dual [11]. On the other hand, the second equation in (3.1) is a twisted Dirac equation whose covariant derivative $D_\mu$ is given by

$$D_\mu = \partial_\mu + \omega_\mu - iA_\mu,$$

(3.4)

where

$$\omega_\mu = \frac{1}{4} \omega_{\mu}^{\alpha\beta} [\gamma_\alpha, \gamma_\beta]$$

(3.5)

is the spin connection 1-form on $X$.

In order to perform a reduction to three dimensions, let us first assume that $X$ is a product manifold of the form $X = Y \times [0,1]$, where $Y$ is a three dimensional compact
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A manifold which has Spin structure. We may identify $t \in [0, 1]$ as a “time” variable, or, we assume $t$ as the zero-th coordinate of $X$, whereas $x_i$ ($i = 1, 2, 3$) are the coordinates on (space manifold) $Y$. Then the metric is given by

$$ds^2 = dt^2 + g_{ij}dx^i dx^j. \quad (3.6)$$

The dimensional reduction is proceeded by assumning that all fields are independent of $t$. Below, we suppress the volume factor $\sqrt{g}$ of $Y$ for simplicity.

First, let us consider the Dirac equation. After the dimensional reduction, the Dirac equation will be

$$\gamma^i D_i M - i\gamma^0 A_0 M = 0. \quad (3.7)$$

As for the first monopole equation, using (3.2) we find that

$$F_{i0} + \frac{1}{2} \epsilon_{ijk} F^{jk} = -i\overline{M} \sigma_{i0} M, \quad (3.8)$$

$$F_{ij} + \epsilon_{ijk0} F^{k0} = -i\overline{M} \sigma_{ij} M.$$

Since the above two equations are dual each other, the first one, for instance, can be reduced to the second one by a contraction with the totally anti-symmetric tensor. Thus it is sufficient to consider one of them. Here, we take the first equation in (3.8).

After the dimensional reduction, (3.8) will be

$$\partial_i A_0 - \frac{1}{2} \epsilon_{ijk} F^{jk} = -i\overline{M} \sigma_{i0} M, \quad (3.9)$$

where we have set $\epsilon_{ijk} \equiv \epsilon_{0ijk}$.

Therefore, the three dimensional version of the Seiberg-Witten equations are given by

$$\partial_i b - \frac{1}{2} \epsilon_{ijk} F^{jk} + i\overline{M} \sigma_{i0} M = 0, \quad (3.10)$$

$$i(\gamma^i D_i - i\gamma^0 b) M = 0,$$

where $b \equiv A_0$. The factor $i$ of the Dirac equation is for later convenience.

It is now easy to establish the non-Abelian 3-d monopole equations (for the four dimensional version, see Refs. 9,24,25,26,27 and Appendix A) as

$$\partial_i b^a + f_{abc} A_i^b b^c - \frac{1}{2} \epsilon_{ijk} F^{ajk} + i\overline{M} \sigma_{i0} T^a M = 0,$$

$$i(\gamma^i D_i - i\gamma^0 b^a) M = 0. \quad (3.11)$$
where we have abbreviated $\mathcal{M}_{\mu\nu} T^a - \frac{1}{2} \sigma_{\mu\nu} (T^a)_{ij} M^j$, subscripts of $(T^a)_{ij}$ run 1 to dim $g$ and $b^a \equiv A^a_0$.

Next, let us find an action which produces (3.10). We can easily find that the simplest one is given by

$$S = \frac{1}{2} \int_Y \left[ \left( \partial_i b - \frac{1}{2} \epsilon_{ijk} F^{jk} + i \mathcal{M} \sigma_{i0} M \right)^2 + |i(\gamma^i D_i - i\gamma^0 b) M|^2 \right] d^3 x. \quad (3.12)$$

Note that the minimum of (3.12) is given by (3.10). In this sense, the 3-d monopole equations are not equations of motion but constraints. Furthermore, there is a constraint for $b$. To see this, let us rewrite (3.12) as

$$S = \int_Y d^3 x \left[ \frac{1}{2} \left( \frac{1}{2} \epsilon_{ijk} F^{jk} - i \mathcal{M} \sigma_{i0} M \right)^2 + \frac{1}{2} |\gamma^i D_i M|^2 + \frac{1}{2} (\partial_i b)^2 + \frac{1}{2} b^2 |M|^2 \right]. \quad (3.13)$$

The minimum of this action is clearly given by the 3-d monopole equations with $b = 0$, for non-trivial $A_i$ and $M$. However, for trivial $A_i$ and $M$, we may relax the condition $b = 0$ to $\partial_i b = 0$, i.e., $b$ is (in general) a non-zero constant. This can be also seen from (3.9).

Accordingly, we obtain

$$\frac{1}{2} \epsilon_{ijk} F^{jk} - i \mathcal{M} \sigma_{i0} M = 0, \quad i\gamma^i D_i M = 0, \quad b = 0 \text{ or } \partial_i b = 0, \quad (3.14)$$

as an equivalent expression to (3.10), but we will use (3.10) for convenience. The Gaussian action will be used in the next section in order to construct a topological field theory by Batalin-Vilkovisky quantization algorithm. The non-Abelian version of (3.12) and (3.14) would be obvious.

There is another action which can produce (3.14) as equations of motion. It is given by a Chern-Simons action coupled with a matter [16, 18, 28] which is analogous to the action in massive gauge theory [40], but we do not discuss the quantum field theory of this Chern-Simons action.
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4 Topological field theories of the 3-d monopoles

In this section, we construct topological field theories associated with the Abelian and non-Abelian 3-d monopoles by Batalin-Vilkovisky quantization algorithm.

4.1 Abelian case

A three dimensional action for the Abelian 3-d monopoles was found by the direct dimensional reduction of the four dimensional one [28, 33], but we show that the three dimensional topological action can be also directly constructed from the 3-d monopole equations.

4.1.1 Topological action

A topological Bogomol’nyi action was constructed by using Batalin-Vilkovisky quantization algorithm [31] (similar construction can be found in two dimensional version [11]) or quantization of a magnetic charge [32]. The former is based on the quantization of a certain Langevin equation (“Bogomol’nyi monopole equation”) and the classical action is quadratic, but the latter is based on the “quantization” of the pure topological invariant by using the Bogomol’nyi monopole equation as a gauge fixing condition.

In order to compare the action to be constructed with those of Bogomol’nyi monopoles [31, 32], we take Batalin-Vilkovisky procedure. The reader who is unfamiliar with this construction may consult the references [1, 31, 36, 37, 41, 42, 43, 44, 45].

In order to obtain the topological action associated with 3-d monopoles, we introduce random Gaussian fields \( G_i \) and \( \nu(\vec{v}) \) and then start with the action

\[
S_c = \frac{1}{2} \int_Y \left[ \left( G_i - \partial_i b + \frac{1}{2} \epsilon_{ijk} F_{jk} - iM\sigma_0 M \right)^2 + \left| (\nu - i\gamma^i D_i M - \gamma^0 b M) \right|^2 \right] d^3x. \tag{4.1}
\]

Note that \( G_i \) and \( \nu(\vec{v}) \) are also regarded as auxiliary fields. This action reduces to (3.12) in the gauge

\[
G_i = 0, \quad \nu = 0. \tag{4.2}
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G_i = 0, \quad \nu = 0. \tag{4.2}
\]
Firstly, note that (4.1) is invariant under the topological gauge transformation

\[ \delta A_i = \partial_i \theta + \epsilon_i, \]
\[ \delta b = \tau, \]
\[ \delta M = i \theta M + \varphi, \]
\[ \delta G_i = \partial_i \tau - \epsilon_{ijk} \partial^j \epsilon^k + i (\varphi \sigma_{i0} M + \overline{M} \sigma_{i0} \varphi), \]
\[ \delta \nu = i \theta \nu + \gamma^i \epsilon_i M + i \gamma^i D_i \varphi + \gamma^0 b \varphi + \gamma^0 \tau M, \]

(4.3)

where \( \theta \) is the parameter of gauge transformation, \( \epsilon_i \) and \( \tau \equiv \epsilon_4 \) are parameters which represent the topological shifts and \( \varphi \) the shift on the spinor space. The brackets for indices means anti-symmetrization, i.e.,

\[ A_{[i} B_{j]} = A_i B_j - A_j B_i. \]

(4.4)

Here, let us classify the gauge algebra (4.3). This is necessary to use Batalin-Vilkovisky algorithm. Let us recall that the local symmetry for fields \( \phi_i \) can be written generally in the form

\[ \delta \phi_i = R^i_\alpha (\phi) \epsilon^\alpha, \]

(4.5)

where the indices mean the label of fields and \( \epsilon^\alpha \) is a some local parameter. When \( \delta \phi_i = 0 \) for non-zero \( \epsilon^\alpha \), this symmetry is called first stage reducible. In the reducible theory, we can find zero-eigenvectors \( Z^\alpha_a \) satisfying \( R^i_\alpha Z^\alpha_a = 0 \). Moreover, when the theory is on-shell reducible, we can find such eigenvectors by using equations of motion.

For the case at hand, under the identifications

\[ \theta = \Lambda, \ \epsilon_i = -\partial_i \Lambda, \ \varphi = -i \Lambda M \]

(4.6)

and

\[ \tau = 0, \]

(4.7)

(4.3) will be

\[ \delta A_i = 0, \]
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\[ \delta b = 0, \]
\[ \delta M = 0, \]
\[ \delta G_i = 0, \]
\[ \delta \nu = i\Lambda(\nu - i\gamma^i D_i \nu - \gamma^0 b M)|_{\text{on-shell}} = 0. \]  

(4.8)

Then for \( \delta A_i \), for example, the \( R \) coefficients and the zero-eigenvectors are derived from

\[ \delta A_i = R^A_i \theta^\theta + R^{A_i}_{\epsilon^j} \theta^\epsilon_j = 0, \]

(4.9)

that is

\[ R^A_i = \partial_i, \quad R^{A_i}_{\epsilon^j} = \delta_{ij}, \quad Z^\theta_{\Lambda} = 1, \quad Z^{\epsilon_j}_{\Lambda} = -\partial_j. \]  

(4.10)

Of course, similar relations hold for other fields. The reader may think that the choice (4.7) is not suitable as a first stage reducible theory, but note that the zero-eigenvectors appear on every point where the gauge equivalence and the topological shift happen to coincide. In this three dimensional theory, \( b(A_0) \) is invariant for the usual infinitesimal gauge transformation because of its “time” independence, so (4.7) means that the existence of the points on spinor space where the topological shift trivializes indicates the first stage reducibility.

If we carry out BRST quantization via Faddeev-Popov procedure in this situation, the Faddeev-Popov determinant will have zero modes. Therefore in order to fix the gauge further we need a ghost for ghost. This reflects on the second generation gauge invariance (4.8) realized on-shell. However, since \( b \) is irrelevant to \( \Lambda \), the ghost for \( \tau \) will not couple to the second generation ghost. With this in mind, we use Batalin-Vilkovisky algorithm in order to make BRST quantization.

Let us assign new ghosts carrying opposite statistics to the local parameters. The assortment is given by

\[ \theta \rightarrow c, \quad \epsilon_i \rightarrow \psi_i, \quad \tau \rightarrow \xi, \quad \varphi \rightarrow N, \]

(4.11)

and

\[ \Lambda \rightarrow \phi. \]

(4.12)
Ghosts in (4.11) are first generations, in particular, \( c \) is Faddeev-Popov ghost, whereas \( \phi \) is a second generation ghost. Their Grassmann parity and ghost number (\( U \) number) are given by
\[
\begin{array}{cccccc}
c & \psi_i & \xi & N & \phi \\
1^- & 1^- & 1^- & 1^- & 2^+
\end{array}
\]
where the superscript of ghost number denotes the Grassmann parity. Note that the ghost number counts the degree of differential form on the moduli space \( M \) of the solution to the 3-d monopole equations. The minimal set \( \Phi_{\text{min}} \) of fields consists of
\[
A_i^0 \quad b^0 \quad M^0 \quad G_i^0 \quad \nu^0
\]
and (4.13).

On the other hand, the set of anti-fields \( \Phi_{\text{min}}^* \) carrying opposite statistics to \( \Phi_{\text{min}} \) is given by
\[
\begin{array}{cccccc}
A_i^* & b^* & M^* & G_i^* & \nu^* & c^* & \psi_i^* & N^* & \phi^*
\\
-1^- & -1^- & -1^- & -1^- & -1^- & -2^- & -2^- & -2^- & -3^-
\end{array}
\]
Next step is to find a solution to the master equation with \( \Phi_{\text{min}} \) and \( \Phi_{\text{min}}^* \), given by
\[
\frac{\partial_r S}{\partial \Phi^A} \frac{\partial_l S}{\partial \Phi^*_A} - \frac{\partial_r S}{\partial \Phi^*_A} \frac{\partial_l S}{\partial \Phi^A} = 0,
\]
where \( r(l) \) denotes right (left) derivative.

The general solution for the first stage reducible theory at hand can be expressed by
\[
S = S_c + \Phi^i R^i_{\alpha} C_1^\alpha + C_1^\alpha (Z^{\alpha}_{\beta} C_2^\beta + T^{\alpha}_{\beta\gamma} C_1^\beta C_1^\gamma) + C_2^\gamma A^\gamma_\beta C_1^\alpha C_2^\beta + \Phi_i^\beta \Phi^j B^{ij}_\alpha C_2^\alpha + \cdots,
\]
where \( C_1^\alpha (C_2^\alpha) \) denotes generally the first (second) generation ghost and only relevant terms in our case are shown. We often use \( \Phi_{\text{min}}^A = (\phi^i, C_1^\alpha, C_2^\beta) \), where \( \phi^i \) denote generally the fields. In this expression, the indices should be interpreted as the label of fields. Do not confuse with space-time indices. The coefficients \( Z^{\alpha}_{\beta}, T^{\alpha}_{\beta\gamma} \), etc can be directly determined from the master equation. In fact, it is known that these coefficients satisfy the following relations
\[
R^i_{\alpha} Z^{\alpha}_{\beta} C_2^\beta - 2 \frac{\partial_r S_c}{\partial \phi^j} B^{\alpha\beta}_\gamma C_2^\alpha (-1)^{|i|} = 0,
\]
\[
\frac{\partial_r R^a_i C_1^\alpha}{\partial \phi^j} R^j_i C_1^\beta + R^i \alpha T^\alpha_{\beta \gamma} C_1^\gamma C_1^\beta = 0,
\]

\[
\frac{\partial_r Z^{\alpha}_2 C_2^\beta}{\partial \phi^j} R^j_1 C_1^\gamma + 2 T^\alpha_{\beta \gamma} C_1^\gamma C_2^\delta + Z^{\alpha}_2 A^\beta_{\beta \gamma} C_2^\delta C_1^\gamma C_1^\delta = 0,
\] (4.18)

where \(|i|\) means the Grassmann parity of the \(i\)-th field.

In these expansion coefficients, \(R^i_\alpha\) and \(Z^{\alpha}_2\) are related to the local symmetry (4.3). On the other hand, as \(T^\alpha_{\beta \gamma}\) is related to the structure constant of a given Lie algebra for a gauge theory, it is generally called as structure function. Of course if the theory is Abelian, such structure function does not appear. However, for a theory coupled with matters, all of the structure functions do not always vanish, even if the gauge group is Abelian. At first sight, this seems to be strange, but the expansion (4.17) obviously detects the coupling of matter fields and ghosts. In fact, the appearance of this type of structure function is required in order to make the action to be constructed being full BRST invariant.

After some algebraic works, we will find the solution to be

\[
S(\Phi_{\text{min}}, \Phi^*_{\text{min}}) = S_c + \int_Y \Delta S d^3x, \tag{4.19}
\]

where

\[
\Delta S = A^*_i (\partial^j c + \psi^j) + b^* \xi + M^* (i c M + N) + \overline{M^*} (-i c \overline{M} + \overline{N})
\]

\[
+ G^*_i \left[ \partial^j \xi - e^{ijk} \partial_j \psi_k + i (\overline{N} \sigma^0 M + \overline{M} \sigma^0 N) \right]
\]

\[
+ \nu^* (i c \nu + i \gamma^i D_i N + \gamma^i \psi_i M + \gamma^0 b N + \gamma^0 \xi M)
\]

\[
+ \nu^* (i c \nu + i \gamma^i D_i N + \gamma^i \psi_i M + \gamma^0 b N + \gamma^0 \xi M)
\]

\[
+ c^* \phi - \psi^*_i \partial^i \phi - i N^* (\phi M + c N) + i \overline{N}^* \left( \phi \overline{M} + c \overline{N} \right)
\]

\[
+ 2i \nu^* \overline{\nu}^* \phi. \tag{4.20}
\]

We augment \(\Phi_{\text{min}}\) by new fields \(\chi, d, \mu, \zeta, \lambda, \rho, \eta, e\) and the corresponding anti-fields. Their ghost number and Grassmann parity are given by

\[
\begin{array}{cccccccc}
\chi & d & \mu & \zeta & \lambda & \rho & \eta & e \\
-1^- & 0^+ & -1^- & 0^+ & -2^- & -1^- & -1^- & 0^+ \\
\end{array}
\] (4.21)
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and

\[
\begin{array}{cccc}
\chi^*_i & \mu^* & \lambda^* & \rho^* \\
0^+ & 0^+ & 1^- & 0^+
\end{array}
\]  

(4.22)

Then we look for the solution

\[
S' = S(\Phi_{\text{min}}, \Phi^*_{\text{min}}) + \int_Y (\chi^* i d_i + \mu^* \zeta + \mu^* \zeta + \rho^* e + \lambda^* \eta) d^3 x, 
\]

(4.23)

where \(d_i, \zeta, e, \eta\) are Lagrange multiplier fields.

In order to obtain the quantum action we must fix the gauge. After a little thought, the best choice for the gauge fixing condition which can reproduce the action obtained from the dimensional reduction of the four dimensional one is found to be

\[
\begin{align*}
G_i &= 0, \\
\nu &= 0, \\
\partial^i A_i &= 0, \\
-\partial^i \psi_i + \frac{i}{2} (NM - MN) &= 0.
\end{align*}
\]

(4.24)

Thus we can obtain the gauge fermion carrying the ghost number \(-1\) and odd Grassmann parity,

\[
\Psi = -\chi^i G_i - \mu^\nu - \mu^\nu A_i - \lambda \left[ -\partial^i \psi_i + \frac{i}{2} (NM - MN) \right].
\]

(4.25)

The quantum action \(S_q\) can be obtained by eliminating anti-fields and are restricted to lie on the gauge surface

\[
\Phi^* = \frac{\partial_r \Psi}{\partial \Phi}.
\]

(4.26)

Therefore the anti-fields will be

\[
\begin{align*}
G_i^* &= -\chi_i, \quad \chi_i^* = -G_i, \quad \nu^* = -\mu, \quad \mu^* = -\nu, \quad \rho^* = -\nu, \\
M^* &= -\frac{i}{2} \lambda N, \quad N^* = \frac{i}{2} \lambda M, \quad N^* = -\frac{i}{2} \lambda M, \\
\rho^* &= \partial^i A_i, \quad A_i^* = -\partial_i \rho, \quad \psi_i^* = -\partial_i \psi_i, \\
\lambda^* &= \left[ -\partial^i \psi_i + \frac{i}{2} (NM - MN) \right], \quad c^* = \phi^* = b^* = \zeta^* (\zeta^*) = 0.
\end{align*}
\]

(4.27)

Then the quantum action \(S_q\) is given by

\[
S_q = S' (\Phi, \Phi^* = \partial_r \Psi / \partial \Phi),
\]

(4.28)
Substituting (4.27) into (4.28), we find that

\[ S_q = S_c + \int_Y \tilde{\Delta} S d^3 x, \]  

(4.29)

where

\[ \tilde{\Delta} S = \left( -\Delta \phi + \phi \overline{M} M - i \overline{N} N \right) \lambda - \left[ -\partial^i \psi_i + \frac{i}{2} (\overline{N} M - \overline{M} N) \right] \eta \]

\[ -\overline{\pi} (i \gamma^i D_i N + \gamma^i \psi_i M + \gamma^0 b N + \gamma^0 \xi M) \]

\[ + (i \gamma^i D_i N + \gamma^i \psi_i M + \gamma^0 b N + \gamma^0 \xi M) \mu + 2i \phi \overline{\mu} \mu \]

\[ - \chi^i \left[ \partial_i \xi - \epsilon_{ijk} \partial^j \psi^k + i (\overline{N} \sigma_{i0} M + \overline{M} \sigma_{i0} N) \right] \]

\[ + \rho (\Delta c + \partial^i \psi_i) - d^i G_i - \overline{\nu} \zeta - \overline{\nu} + e \partial^i A_i. \]  

(4.30)

Using the condition (4.2) with \( c = 0 \), we can arrive at

\[ S'_q = S_c|_{G_i = \nu(\sigma) = 0} + \int_Y \tilde{\Delta} S|_{c=0} d^3 x, \]  

(4.31)

where

\[ \tilde{\Delta} S|_{c=0} = \left( -\Delta \phi + \phi \overline{M} M - i \overline{N} N \right) \lambda - \left[ -\partial^i \psi_i + \frac{i}{2} (\overline{N} M - \overline{M} N) \right] \eta \]

\[ -\overline{\pi} (i \gamma^i D_i N + \gamma^i \psi_i M + \gamma^0 b N + \gamma^0 \xi M) \]

\[ + (i \gamma^i D_i N + \gamma^i \psi_i M + \gamma^0 b N + \gamma^0 \xi M) \mu + 2i \phi \overline{\mu} \mu \]

\[ - \chi^i \left[ \partial_i \xi - \epsilon_{ijk} \partial^j \psi^k + i (\overline{N} \sigma_{i0} M + \overline{M} \sigma_{i0} N) \right] \]

\[ + \rho \partial^i \psi_i + e \partial^i A_i. \]  

(4.32)

It is easy to find that (4.31) is consistent with the action found by the dimensional reduction of the four dimensional topological action [28].

### 4.1.2 BRST transformation

The Batalin-Vilkovisky algorithm also facilitates to construct BRST transformation rule. The BRST transformation rule for a field \( \Phi \) is defined by

\[ \delta_B \Phi = \epsilon \left. \frac{\partial_i S'}{\partial \Phi^*} \right|_{\Phi^* = \frac{\delta \Phi}{\delta \Phi^*}}, \]  

(4.33)
where $\epsilon$ is a constant Grassmann odd parameter. With this definition for \((4.30)\), we obtain

$$
\begin{align*}
\delta_B A_i &= -\epsilon (\partial_i c + \psi_i), \\
\delta_B b &= -\epsilon \xi, \\
\delta_B M &= -\epsilon (icM + N), \\
\delta_B G_i &= -\epsilon \left[ \partial_i \xi - \epsilon_{ijk} \partial^j \psi^k + i(\overline{N}\sigma_{i0}M + \overline{M}\sigma_{i0}N) \right], \\
\delta_B \nu &= -\epsilon (ic\nu + i\gamma^i D_i N + \gamma^i \psi_i M + \gamma^0 bN + \gamma^0 \xi M - i\mu \phi), \\
\delta_B c &= \epsilon \phi, \\
\delta_B \psi_i &= -\epsilon \partial_i \phi, \\
\delta_B \rho &= \epsilon e, \\
\delta_B \lambda &= -\epsilon \eta, \\
\delta_B \mu &= \epsilon \zeta, \\
\delta_B N &= -i\epsilon (\phi M + cN), \\
\delta_B \chi_i &= \epsilon d_i, \\
\delta_B \phi &= \delta_B \xi = \delta_B d_i = \delta_B e = \delta_B \zeta = \delta_B \eta = 0. 
\end{align*}
$$

(4.34)

It is clear at this stage that \((4.34)\) has on-shell nilpotency, i.e., the quantum equation of motion for $\nu$ must be used in order to have $\delta_B^2 = 0$. This is due to the fact that the gauge algebra has on-shell reducibility. Accordingly, the Batalin-Vilkovisky algorithm gives a BRST invariant action and on-shell nilpotent BRST transformation. Note that the equations

$$
\begin{align*}
\partial_i \xi - \epsilon_{ijk} \partial^j \psi^k + i(\overline{N}\sigma_{i0}M + \overline{M}\sigma_{i0}N) &= 0, \\
i\gamma^i D_i N + \gamma^i \psi_i M + \gamma^0 bN + \gamma^0 \xi M &= 0 
\end{align*}
$$

(4.35)

can be recognized as linearizations of the 3-d monopole equations and the number of linearly independent solutions gives the dimension of $\mathcal{M}$.

It is now easy to show that the global supersymmetry can be recovered from \((4.34)\). In Witten type theory, $Q_B$ can be interpreted as a supersymmetric BRST charge. We define
the supersymmetry transformation as
\[ \delta_S \Phi := \delta_B \Phi|_{c=0}. \] (4.36)
We can easily find that the result is consistent with the supersymmetry algebra of Ref. 28.

### 4.1.3 Off-shell action

As was mentioned before, the quantum action of Witten type topological field theory can be represented by BRST commutator with nilpotent BRST charge \( Q_B \). However, since our BRST transformation rule is on-shell nilpotent, we should integrate out \( \nu \) and \( G_i \) in order to obtain off-shell BRST transformation and off-shell quantum action.

For this purpose, let us consider the following terms in (4.30),
\[ \frac{1}{2} (G_i - X_i)^2 + \frac{1}{2} |\nu - A|^2 - i\pi c\nu + i\nu B - \zeta \nu - \bar{\nu} \zeta - d^* G_i, \] (4.37)
where
\[ X_i = \partial_i b - \frac{1}{2} \epsilon_{ijk} F^{jk} + i\overline{M}_i \sigma_0 \overline{M}, \quad A = i\gamma^i D_i M + \gamma^0 b M. \] (4.38)

Here, let us define
\[ \nu' = \nu - A, \quad B = -ic\mu - \zeta. \] (4.39)

\( \nu'(\overline{\nu}) \) and \( G_i \) can be integrated out and then (4.37) will be
\[ -\frac{1}{2} d_i d^i - d_i X^i - 2|B|^2 + \overline{B}\overline{A} + B\overline{A}. \] (4.40)

Consequently, we obtain the off-shell quantum action
\[ S_q = \{ Q, \tilde{\Psi} \}, \] (4.41)
where
\[ \tilde{\Psi} = -\chi^i \left( X_i + \frac{\alpha}{2} d_i \right) - \mu (i\gamma^i D_i M + \gamma^0 b M - \beta B) - \bar{\pi} (i\gamma^i D_i M + \gamma^0 b M - \beta B) + \rho \partial^i A_i 
- \lambda \left[ -\partial^i \psi_i + \frac{i}{2} (N M - \overline{M} N) \right]. \] (4.42)

\( \alpha \) and \( \beta \) are arbitrary gauge fixing paramaters. Convenience choice for them is \( \alpha = \beta = 1 \).

The BRST transformation rule for \( X_i \) and \( B \) fields can be easily obtained, although we do not write down here.
4.1.4 Observables

We can now discuss the observables. For this purpose, let us define

\[\mathcal{A} = A + c,\]
\[\mathcal{F} = F + \psi - \phi,\]
\[\mathcal{K} = db + \xi,\]

(4.43)

where we have introduced differential form notations, but their meanings would be obvious. \(A\) and \(c\) are considered as a \((1, 0)\) and \((0, 1)\) part of 1-form on \((Y, \mathcal{M})\). Similarly, \(F, \psi\) and \(\phi\) are \((2, 0), (1, 1)\) and \((0, 2)\) part of the 2-form \(\mathcal{F}\), and \(db\) and \(\xi\) are \((1, 0)\) and \((0, 1)\) part of the 1-form \(\mathcal{K}\). Thus \(\mathcal{A}\) defines a connection 1-form on \((Y, \mathcal{M})\) and \(\mathcal{F}\) is a curvature 2-form. Note that the exterior derivative \(d\) maps any \((p_1, p_2)\)-form \(X_p\) of total degree \(p = p_1 + p_2\) to \((p_1 + 1, p_2)\)-form, but \(\delta_B\) maps any \((p_1, p_2)\)-form to \((p_1, p_2 + 1)\)-form. Also note that

\[X_p X_q = (-1)^{pq} X_q X_p.\]

(4.44)

Then the action of \(\delta_B\) is

\[(d + \delta_B)\mathcal{A} = \mathcal{F},\]
\[(d + \delta_B)b = \mathcal{K}.\]

(4.45)

\(\mathcal{F}\) and \(\mathcal{K}\) also satisfy

\[(d + \delta_B)\mathcal{F} = 0,\]
\[(d + \delta_B)\mathcal{K} = 0.\]

(4.46)

Eq. (4.46) can be interpreted as Bianchi identities in Abelian theory. Eqs. (4.45) and (4.46) mean anti-commuting property between the BRST variation \(\delta_B\) and the exterior differential \(d\), i.e., \(\{\delta_B, d\} = 0\).

The BRST transformation rule in geometric sector can be easily read from (4.34), i.e., \(\delta_B A, \delta_B \psi, \delta_B c\) and \(\delta_B \phi\). Eq. (4.46) implies

\[(d + \delta_B)\mathcal{F}^n = 0\]

(4.47)
and expanding the above expression by ghost number and form degree, we obtain the following \((i, 2n - i)\)-form \(W_{n,i}\),

\[
\begin{align*}
W_{n,0} &= \frac{\phi^n}{n!}, \\
W_{n,1} &= \frac{\phi^{n-1}}{(n-1)!}\psi, \\
W_{n,2} &= \frac{\phi^{n-2}}{2(n-2)!}\psi \wedge \psi - \frac{\phi^{n-1}}{(n-1)!}F, \\
W_{n,3} &= \frac{\phi^{n-3}}{6(n-3)!}\psi \wedge \psi \wedge \psi - \frac{\phi^{n-2}}{(n-2)!}F \wedge \psi,
\end{align*}
\]

where

\[
0 = \delta_B W_{n,0},
\]

\[
dW_{n,0} = \delta_B W_{n,1},
\]

\[
dW_{n,1} = \delta_B W_{n,2},
\]

\[
dW_{n,2} = \delta_B W_{n,3},
\]

\[
dW_{n,3} = 0.
\] (4.49)

Picking a certain \(k\)-cycle \(\gamma\) as a representative and defining the integral

\[
W_{n,k}(\gamma) = \int_\gamma W_{n,k},
\] (4.50)

we can easily prove

\[
\delta_B W_{n,k}(\gamma) = -\int_\gamma dW_{n,k-1}
\]

\[
= -\int_{\partial\gamma} W_{n,k-1}
\]

\[
= 0,
\] (4.51)

as a consequence of (4.49). Note that the last equality follows from the fact that the cycle \(\gamma\) is a simplex without boundary, i.e., \(\partial\gamma = 0\). Therefore, \(W_{n,k}(\gamma)\) indeed gives a topological invariant associated with \(n\)-th Chern class on \(Y \times M\).

On the other hand, since we have a scalar field \(b\) and its ghosts, we may construct topological observables associated with them. Therefore, the observables can be obtained
from the ghost expansion of

\[(d + \delta_B)\mathcal{F}^n \wedge \mathcal{K}^m = 0.\]  \hfill (4.52)

Explicitly, for \(m = 1\), for example, we obtain

\[0 = \delta_B W_{n,1,0},\]

\[dW_{n,1,0} = \delta_B W_{n,1,1},\]

\[dW_{n,1,1} = \delta_B W_{n,1,2},\]

\[dW_{n,1,2} = \delta_B W_{n,1,3},\]

\[dW_{n,1,3} = 0,\]  \hfill (4.53)

where

\[W_{n,1,0} = \frac{\phi^n}{n!} \xi,\]

\[W_{n,1,1} = \frac{\phi^{n-1}}{(n-1)!} \psi \xi - \frac{\phi^n}{n!} db,\]

\[W_{n,1,2} = \frac{\phi^{n-2}}{2(n-2)!} \psi \wedge \psi \xi - \frac{\phi^{n-1}}{(n-1)!} F \xi - \frac{\phi^{n-1}}{(n-1)!} \psi \wedge db,\]

\[W_{n,1,3} = \frac{\phi^{n-3}}{6(n-3)!} \psi \wedge \psi \wedge \psi \xi + \frac{\phi^{n-1}}{(n-1)!} F \wedge db + \frac{\phi^{n-2}}{2(n-2)!} (2\psi \wedge F \xi + \psi \wedge \psi \wedge db).\]  \hfill (4.54)

These corresponds to the cocycles \([32]\) in \(U(1)\) case.

Next, let us look for the observables for matter sector. The BRST transformation rules in this sector is given by \(\delta_B, \delta_B N, \delta_B c\) and \(\delta_B \phi\). At first sight, the matter sector does not have any observable, but we can find the combined form

\[\tilde{W} = i\phi \overline{M} + \overline{N}\]  \hfill (4.55)

is an observable. However, unfortunately, as \(\tilde{W}\) is cohomologically trivial because \(\delta_B \tilde{W} = 0\) but \(d\tilde{W} \neq \delta_B \tilde{W}'\) for some \(\tilde{W}'\). Accordingly, \(\tilde{W}\) does not give any new topological invariant. Hyun \textit{et al.} \([26, 27]\) identified \(\tilde{W}\) as a part of the bare mass term of the hypermultiplet in their twisting construction of topological QCD in four dimensions.
In topological Bogomol’nyi theory, there is a sequence of observables associated with a magnetic charge. For the Abelian case, it is given by

\[ W = \int_Y F \wedge db. \]  

(4.56)

As is pointed out for the case of Bogomol’nyi monopoles [31], we can not obtain the observables related with this magnetic charge by the action of \( \delta_B \) as well, but we can construct those observables by anti-BRST variation \( \bar{\delta}_B \) which maps \((m, n)\)-form to \((m, n - 1)\)-form. \( \bar{\delta}_B \) can be obtained by a discrete symmetry which is realized as “time reversal symmetry” in four dimensions. In our three dimensional theory, the discrete symmetry is given by

\[ \begin{align*}
\phi &\rightarrow -\lambda, \quad \lambda \rightarrow -\phi, \quad N \rightarrow i\sqrt{2}\mu, \quad \mu \rightarrow \frac{i}{\sqrt{2}}N, \\
\psi_i &\rightarrow \frac{\chi_i}{\sqrt{2}}, \quad \chi_i \rightarrow \sqrt{2}\psi_i, \quad \eta \rightarrow \sqrt{2}\xi, \quad \xi \rightarrow -\frac{\eta}{\sqrt{2}}
\end{align*} \]  

(4.57)

with

\[ b \rightarrow -b. \]  

(4.58)

Eq. (4.58) is an additional symmetry [31]. Note that we must also change \( N \) and \( \mu \) (and their conjugates). The positive chirality condition for \( M \) should be used in order to check the invariance of the action. In this way, we can obtain anti-BRST transformation rule by substituting (4.57) and (4.58) into (4.34) and then we can obtain the observables associated with the magnetic charge by using the action of this anti-BRST variation [31].

The topological observables available in this theory are the same with those of topological Bogomol’nyi monopoles.

Finally, let us briefly comment on our three dimensional theory. First note that Lagrangian \( L \) and Hamiltonian \( H \) in dimensional reduction can be considered as equivalent. This is because the relation between them is defined by

\[ H = p\dot{q} - L, \]  

(4.59)

where \( q \) is any field, the overdot means time derivative and \( p \) is a canonical conjugate momentum of \( q \), and the dimensional reduction requires the time independence of all fields,
thus $H = -L$ in this sense. Though we have constructed the three dimensional action directly from the 3-d monopole equations, our action may be interpreted essentially as the Hamiltonian of the four dimensional Seiberg-Witten theory. In this sense \[4\], the ground states may correspond to the “Floer groups ?” of $Y$, but we do not know the precise correspondence.

### 4.2 Non-Abelian case

It is easy to extend the results obtained in the previous subsection to non-Abelian case. In this subsection, we summarize the results for the non-Abelian 3-d monopoles.

#### 4.2.1 Non-Abelian topological action

With the auxiliary fields $G_{\mu\nu}^a$ and $\nu$, we consider

$$S_c = \frac{1}{2} \int_Y d^3x \left[ (G_i^a - K_i^a)^2 + |\nu - i\gamma^i D_i M - \gamma^0 b M|^2 \right],$$

where

$$K_i^a = \partial_i b^a + f_{abc} A^b_i \theta^c - \frac{1}{2} \epsilon_{ijk} F_{jk}^a + i M \sigma_{i0} T^a M.$$  

Note that the minimum of (4.60) with the gauge

$$G_i^a = \nu = 0$$

are given by the non-Abelian 3-d monopoles. We take the generator of Lie algebra in the fundamental representation, e.g., for $SU(n)$,

$$(T_a)_{ij}(T^a)_{kl} = \delta_{il}\delta_{jk} - \frac{1}{n} \delta_{ij}\delta_{kl}.$$  

Extension to other Lie algebra and representation is straightforward.

The gauge transformation rule for (4.60) is given by

$$\delta A_i^a = \partial_i \theta^a + f_{abc} A^b_i \theta^c + \epsilon_i^a,$$

$$\delta b^a = f_{abc} b^b \theta^c + \tau^a,$$
\[ \delta M = i \theta M + \varphi, \]
\[ \delta G_i^a = f_{abc} G_i^b \theta^c + \left[ -\epsilon_{ijk} (\partial^j \epsilon^{ak} + f_{abc} \epsilon^b \Lambda^c) ight. \]
\[ + \partial_i \tau^a + f_{abc} (\epsilon^b \phi^c - \tau^b \Lambda_i^c) + i (\overline{\varphi} \sigma_0 T^a M + \overline{M} \sigma_0 T^a \varphi) \left], \]
\[ \delta \nu = i \gamma^i \partial_i \phi + \gamma^i \epsilon_i M + \gamma^0 b \varphi + \gamma^0 \tau M + i \theta \nu. \tag{4.64} \]

Note that we have a \( G_i^a \) term in the transformation of \( G_i^a \), while it did not appear in Abelian theory.

The gauge algebra (4.64) possesses on-shell zero modes as in the Abelian case. Setting
\[ \theta^a = \Lambda^a, \epsilon_i^a = -\partial_i \Lambda^a - f_{abc} A_i^b \Lambda^c, \tau^a = -f_{abc} b^b \Lambda^c, \varphi = -i \Lambda M, \tag{4.65} \]
we can easily find that (4.64) closes
\[ \delta A_i^a = 0, \]
\[ \delta b^a = 0, \]
\[ \delta M = 0, \]
\[ \delta G_i^a = f_{abc} \Lambda^c [G_i^b - K_i^b]_{\text{on-shell}} = 0, \]
\[ \delta \nu = i \Lambda [\nu - i (\gamma^i \partial_i - i \gamma^0 b) M]_{\text{on-shell}} = 0, \tag{4.66} \]
when the equations of motion of \( G_i^a \) and \( \nu \) are used. Note that we must use both equations of motion of \( G_i^a \) and \( \nu \) in the non-Abelian case, while only “\( \nu \)” was needed for the Abelian theory. Furthermore, as \( \varphi \) is a parameter in the spinor space, \( \varphi \) is not \( \mathbf{g} \)-valued, in other words, \( \varphi \neq \varphi^a T^a \). Eq.(4.64) is first stage reducible.

The assortment of ghost fields, the minimal set \( \Phi_{\text{min}} \) of the fields and the ghost number and the Grassmann parity, furthermore those for \( \Phi_{\text{min}}^* \) would be obvious.

Then the solution to the master equation will be
\[ S(\Phi_{\text{min}}, \Phi_{\text{min}}^*) = S_c + \int_Y \text{tr} \Delta S_n d^3x, \tag{4.67} \]
where
\[ \Delta S_n = A_i^a (D^i c + \psi^i) + b^a (i [b, c] + \xi) \]
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\[ +M^* (i c M + N) + \overline{M}^* (-i c \overline{M} + \overline{N}) \]
\[ +G_i^* G^i - i N^* (\phi M + c N) + i \overline{N}^* (\phi \overline{M} + c \overline{N}) \]
\[ +\nu^* (i c \nu + i \gamma^i D_i N + \gamma^i \psi_i M + \gamma^0 b N + \gamma^0 \xi M) \]
\[ +\overline{\nu}^* (i c \overline{\nu} + i \gamma^i D_i N + \gamma^i \psi_i M + \gamma^0 b N + \gamma^0 \xi M) \]
\[ +2i \nu^* \tau^i \phi + \psi_i^* (-D^i \phi - i \{ \psi^i, c \}) + c^* \left( \phi - \frac{i}{2} \{ c, c \} \right) - i \phi^* [\phi, c] \]
\[ -\frac{i}{2} \{ G_i^*, G^{*i} \} \phi + i \xi^* ([b, \phi] - \{ \xi, c \}). \]

Here

\[ \tilde{G}_i = i [c, G_i] - \epsilon_{ijk} D^j \psi^k + D_i \xi + [\psi_i, \xi] + i (\overline{N} \sigma_{i0} T_a T^a M + \overline{M} \sigma_{i0} T_a T^a N). \]  

(4.69)

The equations

\[-\epsilon_{ijk} D^j \psi^k + D_i \xi + [\psi_i, \xi] + i (\overline{N} \sigma_{i0} T_a T^a M + \overline{M} \sigma_{i0} T_a T^a N) = 0, \]
\[ i \gamma^i D_i N + \gamma^i \psi_i M + \gamma^0 b N + \gamma^0 \xi M = 0, \]  

(4.70)

can be seen as linearizations of non-Abelian 3-d monopoles.

We augment \( \Phi_{\text{min}} \) by new fields \( \chi_i^a, d_i^a, \mu(\overline{\nu}), \zeta(\overline{\zeta}), \lambda, \rho, \eta, e \) and the corresponding anti-fields, but Lagrange multipliers fields \( d_i^a, \zeta(\overline{\zeta}), e, \eta, \) are assumed not to have anti-fields for simplicity and therefore their BRST transformation rules are set to zero. This simplification means that we do not take into account of BRST exact terms. In this sense, the result to be obtained will correspond to those of the dimensionally reduced version of the four dimensional theory \([24, 25, 26, 27]\) up to these terms, i.e., topological numbers.

From the gauge fixing condition

\[ G_i^a = 0, \]
\[ \nu = 0, \]
\[ \partial^i A_i = 0, \]
\[ -D^i \psi_i + \frac{i}{2} (\overline{N} M - \overline{M} N) = 0, \]

(4.71)

the gauge fermion will be

\[ \Psi = -\chi^i G_i - \overline{\nu} \nu - \mu \overline{\nu} + \rho \partial^i A_i - \lambda \left[ -D^i \psi_i + \frac{i}{2} (\overline{N} M - \overline{M} N) \right]. \]  

(4.72)
The anti-fields are then given by

\[ G_i^* = -\chi_i, \quad \chi_i^* = -G_i, \quad \nu^* = -\mu, \quad \mu^* = -\nu, \quad \mu = -\nu, \quad \lambda^* = -\frac{i}{2}\lambda, \quad \lambda = -\frac{i}{2}\lambda \]

\[ M^* = -\frac{i}{2}\lambda N, \quad M^* = \frac{i}{2}\lambda M, \quad N^* = -\frac{i}{2}\lambda M, \quad N^* = -\frac{i}{2}\lambda M \]

\[ \rho^* = \partial_i A_i, \quad A_i^* = -\partial_i \rho + i[\lambda, \psi_i], \quad \psi_i^* = -\partial_i \lambda, \quad \lambda^* = -\left[ -D_i \psi_i + [b, \xi] + \frac{i}{2}(NM-MN) \right], \quad b^* = c^* = \xi^* = \phi^* = \zeta^*(\zeta^*) = 0. \]

Therefore we find the quantum action

\[ S_q = S_c + \int_Y \text{tr} \left( \Delta S_n d^3 x \right), \] \hspace{1cm} (4.74)

where

\[ \Delta S_n = -\left[ -D_i \psi_i + [b, \xi] + \frac{i}{2}(NM-MN) \right] \eta - \lambda(D_i D^i \phi + iD_i \{\psi_i, c\}) + i\lambda \{\psi_i, D^i c + \psi_i^i\} + (\phi MM - iNN)\lambda - \chi^i \left[ i[c, G_i] + \epsilon_{ijk} D^j \psi^k + D_k \xi + [\psi_k, \xi] + \frac{i}{2}(N\sigma^{ij} T^a M + M\sigma^{ij} T^a N) \right]
- \pi \left[ i\gamma^\mu D_\mu N + \gamma^\mu \psi_\mu M + ic\nu \right] + (i\gamma^i D_i N + \gamma^\mu \psi_\mu M + ic\nu)\mu + 2i\phi \pi \mu - \frac{i}{2} \{\chi_i, \chi^i\} \phi + \rho(\partial_i D^i c + \partial_i \psi_i)
- d^i G_i - \bar{\zeta} \nu - \bar{\nu} \zeta + e\partial_i A_i. \] \hspace{1cm} (4.75)

In this quantum action, setting

\[ M(M) = N(N) = \mu(\overline{\mu}) = \nu(\overline{\nu}) = 0, \] \hspace{1cm} (4.76)

we can find that the resulting action coincides with that of Bogomol’nyi monopoles [31].

Finally, in order to obtain the off-shell quantum action, both the auxiliary fields should be integrated out by the similar technique presented in Abelian case, but we remain it the reader’s exercise.

### 4.2.2 BRST transformation

The BRST transformation rule is given by

\[ \delta_B A_i = -\epsilon(D_i c + \psi_i), \]
\begin{align*}
\delta_B b &= -\epsilon (i[c,b] + \xi), \\
\delta_B \xi &= i\epsilon ([b,\phi] - \{\xi,c\}), \\
\delta_B M &= -\epsilon (icM + N), \\
\delta_B G_i &= -\epsilon (G_i - i[x_i,\phi]), \\
\delta_B \nu &= -\epsilon (ic\nu + \gamma^\mu D_\mu N + \gamma^\mu \psi_\mu M - i\mu\phi), \\
\delta_B c &= \epsilon \left(\phi - i\frac{1}{2}\{c,c\}\right), \\
\delta_B \psi_i &= -\epsilon (D_i\phi + i\{\psi_i,c\}), \\
\delta_B \rho &= \epsilon e, \\
\delta_B \lambda &= -\epsilon \eta, \\
\delta_B \mu &= \epsilon \zeta, \\
\delta_B N &= -i\epsilon (\phi M + cN), \\
\delta_B \chi_i &= \epsilon d_i, \\
\delta_B \phi &= i\epsilon [\phi,c], \\
\delta_B d_i &= \delta_B e = \delta_B \zeta = \delta_B \eta = 0. 
\end{align*}

(4.77)

It is easy to obtain supersymmetry also in this case. However, as we have omitted the BRST exact terms, the supersymmetry in our construction does not detect them.

### 4.2.3 Observables

We have already constructed the topological observables for Abelian case. Also in non-Abelian case, the construction of observables is basically the same. But the relation (4.43) and (4.46) are required to modify

\begin{align*}
(d + \delta_B) A - \frac{i}{2} [A, A] &= \mathcal{F}, \\
(d + \delta_B) b - i [A, b] &= \mathcal{K}
\end{align*}

(4.78)

and

\begin{align*}
(d + \delta_B) \mathcal{F} - i [A, \mathcal{F}] &= 0,
\end{align*}
respectively, where $[\ast, \ast]$ is a graded commutator. The observables in geometric and matter sector are the same as before, but we should replace $db$ by $d_A b$ in (4.54) as well as (4.78) and (4.79), where $d_A$ is a exterior covariant derivative and trace is required. In addition, the magnetic charge observables are again obtained by anti-BRST variation as outlined before.

The observables in geometric sector are those in (4.48) and follow the cohomological relation (4.49). In this way, the topological observables available in this three dimensional theory are precisely the Bogomol’nyi monopole cocycles [32].

5 Summary

We have presented the existence of the topological field theories which describe the moduli space of Abelian and non-Abelian three dimensional Seiberg-Witten monopole equations by using the Batalin-Vilkovisky quantization procedure. In the Abelian case, our topological action with a certain gauge condition is found to be consistent with that of the dimensionally reduced version of the four dimensional one. We have also established the three dimensional non-Abelian action. The interesting point is that this non-Abelian action can be viewed as the Bogomol’nyi monopole topological action including matter and its associated ghost. We have easily obtained the BRST and anti-BRST transformation rules. The topological observables related to the Chern classes can be found by the standard fashion. We have found that they are precisely the cocycles of Bogomol’nyi monopole topological field theory.

In this paper, we have not include the mass term for the Weyl spinor, but the introduction of the mass term may connect the Bogomol’nyi and the 3-d Seiberg-Witten monopole topological field theory, as was shown that the mass term interpolates Donaldson theory and Seiberg-Witten theory in four dimensions [27]. This point of view should be further studied.

We have already known various progress on the self-dual Yang-Mills equation, but there remain several tasks for Seiberg-Witten equations, so let us briefly comment on them as open problems.

1. Integrability of Seiberg-Witten equations.
As is well-known, the self-dual Yang-Mills equation can be reduced to some solitonic equations such as non-linear Schrödinger equation or KdV equation after suitable choice for the gauge fields [46, 47], although there is no proof that the self-dual Yang-Mills equation is indeed integrable. On the other hand, as for the Seiberg-Witten equations, they can not be viewed as integrable equations at first sight, but it was found that the Seiberg-Witten equations on $\mathbb{R}^2$ could be realized as Liouville vortex equations which are manifestly integrable [48] (as for a solution on $\mathbb{R}^3$, there is a Freund’s solution [49]). This fact seems to connect integrable systems and Seiberg-Witten monopoles, but unfortunately we do not know whether there exist another examples of integrable systems related to the Seiberg-Witten monopoles. Furthermore, as explicit solutions to non-Abelian Seiberg-Witten equations have not ever been found, we can not pursue the integrability. For this direction, twistor program [50] may be available, as is often used for the self-dual Yang-Mills equation [46, 51, 52].

2. Reduction to two dimensional surfaces (Riemann surfaces $\Sigma$).

We can dimensionally reduce the Seiberg-Witten equations onto two dimensional surfaces. As have been mentioned before, the operation of dimensional reduction connects the theories between four and three, and three and two dimensions, the two dimensonal theory may be regarded as a dual $U(1)$ theory for the $SU(2)$ Hitchin equations [53] (usually, the Lie group for Hitchin equations is taken $SO(3)$ rather than $SU(2)$), i.e., two dimensional Yang-Mills-Higgs equations. One approach to study this observation is to construct solutions to the Seiberg-Witten equations on Riemann surfaces and compare their properties with those of the Hitchin equations. Recently, the reduced Seiberg-Witten equations were studied and it was pointed out that the set of equations had an extremely similar structure to the Hitchin equations, except the distinction of Higgs field and Weyl spinor [54]. We would like to interprete the relationship between these two theories in the context of topological quantum field theory, but any progress such as the study of the topological field theory associated with the two dimensional Seiberg-Witten monopoles has not ever been made (a topological action is obtained
by a dimensional reduction [33], although Yang-Mills theory on Riemann surfaces are well discussed (see e.g., Ref. 2 and references therein). We know that the Yang-Mills-Higgs theory in two dimensions is closely related to a conformal field theory [55], but is it true also in two dimensional Seiberg-Witten theory?

There are other problems such as supersymmetric extension [56, 57], twistor description [58, 59], and so on, but a lot of (topological) field theoretical techniques to study these problems have been developed. Nevertheless, the Seiberg-Witten theory does not seem to be fully discussed even in four dimensions as well as lower dimensions in contrast with the Donaldson theory in the context of topological quantum field theory. Filling the gap may be an attract problem, but much efforts will be required.

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Appendix A. Non-Abelian Seiberg-Witten theory

In this Appendix, we summarize on the construction of the topological action for non-Abelian Seiberg-Witten monopoles in four dimensions by Batalin-Vilkovisky algorithm.

Let $P$ be a principal bundle with a compact simple Lie group $G$ and by $E$ we mean the associated bundle to $P$. Then the gauge field is the connection on $E$ and $M$ is a section of $S^+ \otimes E$.

Then the non-Abelian Seiberg-Witten monopole equations in four dimensions [24, 25, 26, 27] are defined by

\[ F_{\mu\nu}^{a+} + \frac{i}{2} M \sigma_{\mu\nu} T^a M = 0, \]
\[ \bar{i} \gamma^\mu D_\mu M = 0, \]  \hspace{1cm} (A1)

where

\[ F_{\mu\nu}^a = P^{+}_{\mu\nu\rho\sigma} F^\rho\sigma^a, \]
\[ F_{\mu\nu}^a = \partial_\mu A_\nu^a - \partial_\nu A_\mu^a + f_{abc} A_\mu^b A_\nu^c. \]  \hspace{1cm} (A2)
We take the classical action
\[
S_c = \frac{1}{4} \int_X d^4x \left[ \left( G_{\mu \nu}^a + F_{\mu \nu}^a - \frac{i}{2} M_{\sigma \mu} T^a M \right)^2 + 2 \nu - i \gamma^\mu D_\mu M \right]^2,
\]
(A3)
where \( G_{\mu \nu}^a \) is a self-dual auxiliary field satisfying
\[
G_{\mu \nu}^a = P_{\mu \nu \rho \sigma}^+ G_{\rho \sigma}^a.
\]
(A4)
The minimum of (A3) is then given by
\[
G_{\mu \nu}^a - F_{\mu \nu}^a - \frac{i}{2} M_{\sigma \mu} T^a M = 0,
\]
\[
\nu - i \gamma^\mu D_\mu M = 0.
\]
(A5)
The gauge symmetry of (A3) is
\[
\delta A_\mu^a = \partial_\mu \theta^a + f_{abc} A_\mu^b \theta^c + \epsilon_\mu^a,
\]
\[
\delta M = i \theta M + \varphi,
\]
\[
\delta G_{\mu \nu}^a = f_{abc} G_{\mu \nu}^b \theta^c + P_{\mu \nu \rho \sigma}^+ \left[ \partial_\rho \epsilon_\sigma^{|a} + f_{abc} \epsilon_\sigma^{|b} A_\sigma^{|c} + \frac{i}{2} (\sigma_\rho \sigma_\tau T^a M + M_{\sigma \rho \sigma} T^a \varphi) \right],
\]
\[
\delta \nu = i \gamma^\mu D_\mu \varphi + \gamma^\mu \epsilon_\mu M + i \theta \nu.
\]
(A6)
First stage reducibility of (A6) can be seen from the identification
\[
\theta^a = \Lambda^a, \quad \epsilon_\mu^a = - \partial_\mu \Lambda^a - f_{abc} A_\mu^b \Lambda^c, \quad \varphi = - i \Lambda M.
\]
(A7)
In fact, (A6) closes on-shell
\[
\delta A_\mu^a = 0,
\]
\[
\delta M = 0,
\]
\[
\delta G_{\mu \nu}^a = f_{abc} \Lambda^c \left[ G_{\mu \nu}^b - P_{\mu \nu \rho \sigma}^+ \left( F_{\rho \sigma}^b + \frac{i}{2} M_{\sigma \rho \sigma} T^b M \right) \right]_{\text{on-shell}} = 0,
\]
\[
\delta \nu = i \Lambda (\nu - i \gamma^\mu D_\mu M)_{\text{on-shell}} = 0.
\]
(A8)
In the \( R \) coefficient and zero-eigenvector notation, we obtain
\[
R_{\mu \nu}^A = \partial_\mu \delta_{ab} + f_{acb} A_\mu^c,
\]
Topological field theories associated with 3-d SW monopoles

\[ R^A_{\mu} = \delta_{ab} \delta_{\mu\nu}, \]
\[ R^M = i T^a M, \]
\[ R^\phi = 1, \]
\[ R^{G^a_{\mu\nu}} = f_{acb} G^c_{\mu\nu}, \]
\[ R^{C^a_{\mu\nu}} = \frac{i}{2} P^+_{\mu\nu\rho\sigma} \mathcal{M} \sigma^{\rho\sigma} T^a, \]
\[ R^{q_a} = i T^a \nu, \]
\[ R^{\nu} = i \gamma^\mu D^\mu, \]
\[ R^{\nu} = \gamma^\mu T^a M, \]

(A9)

and

\[ Z^{q_a}_{\lambda^b} = \delta_{ab}, \]
\[ Z^{q_a}_{\lambda^b} = - \partial_\mu \delta_{ab} - f_{acb} A^c_{\mu}, \]
\[ Z^{\phi}_{\lambda} = - i M. \]

(A10)

The assortment of ghost fields would be obvious again. Then we obtain the minimal solution to the master equation (see Appendix B)

\[ S(\Phi_{\text{min}}, \Phi^*_{\text{min}}) = S_c + \int_X \text{tr} \Delta S d^4x, \]

(A11)

where

\[ \Delta S = A^*_\mu (D^\mu c + \psi^\mu) + M^*(icM + N) + M^*(-ic\overline{M} + \overline{N}) \]
\[ + G^*_\mu \overline{G}^{\mu\nu} - i N^*(\phi M + cN) + i \overline{N}^*(\phi \overline{M} + c\overline{N}) \]
\[ + \nu^* (i \gamma^\mu D^\mu N + \gamma^\mu \psi^\mu M + ic\nu) + \overline{\nu}^* (i \gamma^\mu D^\mu N + \gamma^\mu \psi^\mu M + ic\nu) \]
\[ + 2i \nu^* \overline{\nu}^* \phi + \psi^*_\mu (-D^\mu \phi - i \{\psi^\mu, c\}) + c^* \left( \phi - \frac{i}{2} \{c, c\} \right) - i \phi^*[\phi, c] \]
\[ - \frac{i}{2} \{G^*_\mu, G^{*\mu\nu}\} \phi, \]

(A12)

where

\[ \overline{G}^{\mu\nu} = i [c, G_{\mu\nu}] + P^+_{\mu\nu\rho\sigma} \left[ D_{\mu\nu} \psi^{\rho\sigma} + \frac{i}{2} (\overline{N} \sigma^{\rho\sigma} T^a M + \overline{M} \sigma^{\rho\sigma} T^a T^a N) \right]. \]

(A13)

The equations

\[ D_{[\mu} \psi_{\nu]} + \frac{i}{2} (\overline{N} \sigma_{\mu\nu} T^a M + \overline{M} \sigma_{\mu\nu} T^a T^a N) = 0, \]
\[ i \gamma^\mu D^\mu N + \gamma^\mu \psi^\mu M = 0, \]

(A14)
can be seen as linearizations of the non-Abelian monopole equations and the number of linearly independent $\psi_\mu$ and $N$ gives the dimension of the moduli space $\mathcal{M}$ of solutions of (A1). The dimension $d(\mathcal{M})$ was found to be (for $SU(n)$ case)

$$d(\mathcal{M}) = (4n - 2)c_2(E) - \frac{n^2 - 1}{2}(\chi(X) + \sigma(X)) - \frac{d_R}{4}\sigma(X),$$

(A15)

where $\chi$ and $\sigma$ are Euler number and signature of $X$, respectively, $c_2(E)$ is the second Chern class of the representation bundle and $d_R$ is the dimension of the representation $R$ of the Lie algebra, but as we take $R$ to be fundamental representation, $d_R$ is identified with $n$.

We augment $\Phi_{\text{min}}$ by new fields $\chi^a_{\mu\nu}, d^a_{\mu\nu}, \mu(\overline{\mu}), \zeta, \lambda, \rho, \eta, e$ and the corresponding antifields,

$$\begin{array}{cccccccc}
\chi^a_{\mu\nu} & d^a_{\mu\nu} & -1^- & 0^+ & -1^- & 0^+ & \zeta & \lambda & \rho & \eta & e
\end{array}$$

(A16)

with

$$\begin{array}{cccccc}
\chi^* & \mu^* & \lambda^* & \rho^* & \eta^* & e^*
\end{array}$$

(A17)

where $\chi^a_{\mu\nu}(\chi^*_{\mu\nu})$ and $d^a_{\mu\nu}$ are self-dual. If we choose

$$G_{\mu\nu} = 0,$$

$$\nu = 0,$$

$$\partial^\mu A_\mu = 0,$$

$$-D^\mu \psi_\mu + \frac{i}{2}(\overline{N}M - \overline{M}N) = 0,$$

(A18)

we can obtain the non-Abelian four dimensional action compatible with that of Labastida and Mariño [20]. Then the gauge fermion is chosen to be

$$\Psi = -\chi^{\mu\nu}G_{\mu\nu} - \overline{\mu}\nu - \mu\overline{\nu} + \rho\partial^\mu A_\mu - \lambda \left[-D^\mu \psi_\mu + \frac{i}{2}(\overline{N}M - \overline{M}N)\right].$$

(A19)

After similar manipulations as have been done in the text, we find the quantum action to be

$$S_q = S_c + \int_X \text{tr} \Delta S d^4x.$$
where

\[
\tilde{\Delta} S = - \left[ - D^\mu \psi_\mu + \frac{i}{2} (\overline{\mathcal{N}} M - \overline{M} N) \right] \eta + \lambda \left[ i \{ \psi_\mu, D^\mu c + \psi_\mu \} - D_\mu (D^\mu \phi + i \{ \psi_\mu, c \}) \right] \\
+ (\phi \overline{M} M - i \overline{N} N) \lambda \\
- \chi^{\mu \nu} P^+_{\mu \nu \rho \sigma} \left[ 0 \sigma^\rho T_a T^a M + \overline{M} \sigma^{\rho \sigma} T_a T^a N \right] \\
- \overline{\mathcal{N}} (i \gamma^\mu D_\mu N + \gamma^\mu \psi_\mu M + ic\nu) + (i \gamma^\mu D_\mu N + \gamma^\mu \psi_\mu M + ic\nu) \mu \\
+ 2i \phi \overline{\mathcal{N}} - \frac{i}{2} \{ \chi_{\mu \nu}, \chi^{\mu \nu} \} \phi + \rho (\partial^\mu D_\mu c + \partial^\mu \psi_\mu) \\
- d^\mu \nu G_{\mu \nu} - \overline{\zeta} \nu - \overline{\nu} \zeta + e \partial^\mu A_\mu. \tag{A21}
\]

In this quantum action, setting

\[
M(\overline{M}) = N(\overline{N}) = \nu(\overline{\mathcal{N}}) = \mu(\overline{\mathcal{P}}) = 0, \tag{A22}
\]

we can find that the resulting action coincides with that of Donaldson theory \[1, 37\] up to BRST-exact terms.

The BRST transformation will be

\[
\delta_B A_\mu = - \epsilon (D_\mu c + \psi_\mu), \\
\delta_B M = - \epsilon (icM + N), \\
\delta_B G_{\mu \nu} = - \epsilon \left[ i [c, G_{\mu \nu}] - i [\chi_{\mu \nu}, \phi] \\
+ P^+_{\mu \nu \rho \sigma} \left[ 0 \sigma^\rho T_a T^a M + \overline{M} \sigma^{\rho \sigma} T_a T^a N \right] \right], \\
\delta_B \nu = - \epsilon (ic \nu + \gamma^\mu D_\mu N + \gamma^\mu \psi_\mu M - i \mu \phi), \\
\delta_B c = \epsilon \left( \phi - \frac{i}{2} \{ c, c \} \right), \\
\delta_B \psi_\mu = - \epsilon (D_\mu \phi + i \{ \psi_\mu, c \}), \\
\delta_B \rho = \epsilon e, \\
\delta_B \lambda = - \epsilon \eta, \\
\delta_B \mu = \epsilon \zeta, \\
\delta_B N = - i \epsilon (\phi M + c N), \\
\delta_B \chi_{\mu \nu} = \epsilon d_{\mu \nu},
\]
\[
\delta_B \phi = i \epsilon [\phi, c], \\
\delta_B d_{\mu\nu} = \delta_B e = \delta_B \zeta = \delta_B \eta = 0.
\]

(A23)

Note that we have enough fields to obtain precisely the Donaldson polynomials. They are the multiplet \((A_\mu, \psi_\mu, \phi)\). However, again, we do not obtain any topological invariants associated with matter sector.

**Appendix B. Solution to master equation**

In this Appendix, the expansion coefficients in (4.18) for the non-Abelian case in four dimensions are determined. Since \(B^{ij}_\alpha\) can be easily obtained, let us determine the structure functions \(T^{\alpha \beta \gamma}_{\theta \beta \gamma} C^\alpha_1 C^\beta_1\). They are determined from the relation

\[
\frac{\partial}{\partial \phi^k} R^G_{\alpha \beta \gamma} C^i_1 R^{\phi^k}_{j} C^j_1 + \left[ R^G_{\theta \mu \nu} T^{\theta \gamma}_{\beta \gamma} + R^G_{e \alpha \nu} T^{e \alpha}_{\beta \gamma} + R^G_{\phi \mu \nu} T^{\phi \gamma}_{\beta \gamma} + R^G_{\psi \mu \nu} T^{\psi \gamma}_{\beta \gamma} \right] C^\gamma_1 C^\beta_1 = 0.
\]

(B1)

For example, \(T^{\theta \beta \gamma}_{\beta \gamma}\) are obtained from

\[
f_{abc} f_{cej} G^e_{\mu \nu} c^f + f_{acb} G^e_{\mu \nu} T^{\phi b}_{\beta \gamma} C^\gamma_1 C^\beta_1 = 0.
\]

(B2)

The first term will be

\[
f_{abc} f_{cej} G^e_{\mu \nu} c^b c^f = f_{ade} f_{deb} G^e_{\mu \nu} c^e c^b
\]

\[
= - \left( f_{abd} f_{cde} + f_{bcd} f_{ade} \right) G^e_{\mu \nu} c^e c^b
\]

\[
= - f_{acb} f_{bed} G^e_{\mu \nu} c^b c^e - f_{bed} f_{ade} G^e_{\mu \nu} c^e c^b
\]

\[
= f_{adec} f_{bd} G^e_{\mu \nu} c^b c^e - f_{bed} f_{ade} G^e_{\mu \nu} c^e c^b
\]

\[
= - f_{adec} f_{bd} G^e_{\mu \nu} c^b c^e + f_{deb} f_{ace} G^e_{\mu \nu} c^e c^b,
\]

(B3)

thus we obtain

\[
f_{adec} f_{deb} G^e_{\mu \nu} c^e c^b = \frac{1}{2} f_{bed} f_{ace} G^e_{\mu \nu} c^e c^d.
\]

(B4)

Therefore,

\[
T^{\theta \beta \gamma}_{\beta \gamma} C^\gamma_1 C^\beta_1 = \frac{1}{2} f_{bed} c^e c^d.
\]

(B5)
Since the other expansion coefficients can be obtained from similar calculations, we remain them as the reader’s exercise. Final result will be

\[ B_\alpha^{G_\mu} G_\nu C_2^\alpha = \frac{f_{abc}}{2} \delta_{\mu \rho} \delta_{\nu \sigma} \phi^c, \quad B_\alpha^{\bar{\tau}} C_2^\alpha = -i \phi, \quad B_\alpha^{\bar{\tau}} C_2^\alpha = i \phi, \]

\[ T_{\beta \gamma} C_1^{\alpha} C_1^\beta = -i cN, \quad T_{\beta \gamma} C_1^{\alpha} C_1^\beta = i cN, \quad T_{\beta \gamma} C_1^{\alpha} C_1^\beta = f_{abc} \psi^b \phi^c, \]

\[ A_{\beta \gamma} C_1^\alpha C_1^\gamma = f_{abc} \phi^b \phi^c. \quad (B6) \]

**Appendix C. Convention for gamma matrix**

The convention for gamma matrix is as follows. Let \( \sigma^i \) be Pauli matrices

\[ \sigma^1 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \quad \sigma^2 = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}, \quad \sigma^3 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}. \quad (C1) \]

On \( \mathbb{R}^4 \) we define four gamma matrices

\[ \gamma^0 = \begin{pmatrix} 0 & I \\ I & 0 \end{pmatrix}, \quad \gamma^j = \begin{pmatrix} 0 & i\sigma^j \\ -i\sigma^j & 0 \end{pmatrix}. \quad (C2) \]

and

\[ \gamma^5 = \gamma^0 \gamma^1 \gamma^2 \gamma^3 \]

\[ = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}, \quad (C3) \]

where \( I \) is a \( 2 \times 2 \) unit matrix.

As can be easily seen from (C2), they satisfy

\[ \{\gamma^\mu, \gamma^\nu\} = 2 \delta_{\mu \nu}. \quad (C4) \]

It is often useful to define

\[ \sigma_{\mu \nu} = \frac{1}{2} [\gamma^\mu, \gamma^\nu]. \quad (C5) \]

Then

\[ \sigma_{ij} = i \epsilon_{ijk} \begin{pmatrix} \sigma^k & 0 \\ 0 & \sigma^k \end{pmatrix}, \quad \sigma_{k0} = i \begin{pmatrix} \sigma^k & 0 \\ 0 & -\sigma^k \end{pmatrix}. \quad (C6) \]

On curved manifolds, we multiply viervein to these gamma matrices (except \( \gamma^5 \)). On \( Y \times [0, 1], \gamma^0 \) is a constant matrix, while the others are not in general.
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