Unifying exact scalar field cosmologies

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Abstract. We present a mechanism that generates the exact solutions of scalar field cosmologies in an unified way. This applies both to standard and phantom scalar fields in the framework of Einsteins General Relativity, and also to extended scenarios based on generalized gravity theories such as Scalar-Tensor theories. We provide a classification of the solutions in terms a single generating function. The existence of form-invariant dualities between the solutions is also discussed.

1. Introduction
During approximately the past three decades there has been considerable interest in finding exact solutions of the well known scalar field equations in a spatially flat Friedman-Robertson-Walker (FRW) space-time. Inflation has provided the underlying motivation [1]. Since it is very difficult to obtain exact solutions for specific potentials emerging from particle physics, the focus was directed to the solutions of particular cases which had remarkable properties, and which still shared a general particle physics motivation, e.g., exponential potentials [2]. Some methods to obtain exact solutions were devised such as the use of the scalar field, \(\phi\), as the independent variable, and the study of Lie symmetries of the equations [3]. A skillful combination of various, separate methods has since lead some authors to derive some classes of new solutions.

Here we are concerned with the question of unifying all the previous solutions in a single, and simple framework, which moreover permits the derivation of novel solutions\[4\]. Our procedure applies to models with self-interacting scalar fields in the framework of General Relativity (GR), encompassing both standard models of scalar field cosmologies, i.e., those in which the scalar field has a positive kinetic energy, phantom models where the latter is negative, and scalar-tensor models as well.

2. Equations and procedure
Although our procedure applies to spacetimes of \((N + 1)\) dimensions, here we restrict ourselves to GR, and consider a \((3 + 1)\)-dimensional spatially flat Friedmann spacetime and we assume that the matter content is a minimally coupled scalar field. The Einstein equations are (we adopt units in which \(8\pi G = c^2 = 1\))

\[
\dot{\phi} + 3H \dot{\phi} + \varepsilon V_{,\phi} = 0, \tag{1}
\]

\[
3H^2 = \varepsilon \frac{1}{2} \dot{\phi}^2 + V(\phi), \tag{2}
\]
where the overdot represents the time derivative, $a(t)$ is the scale factor, $H = \dot{a}/a$ is the Hubble parameter, and $\varepsilon$ parameter is a discriminator that takes the values $\varepsilon = \pm 1$ distinguishing standard scalar field models from phantom field models. The value $\varepsilon = +1$ corresponds to the former models and $\varepsilon = -1$ to the latter.

Introducing the new time variable $d\tau = Hdt$ and the dimensionless variable $x = \dot{\phi}/H$, we obtain

$$\frac{dx}{d\phi} = \frac{x'}{\phi'} = \left(3 - \varepsilon \frac{1}{2} x^2\right) \left(\frac{x + \varepsilon \frac{V}{V'}}{x}\right). \tag{3}$$

The latter Eq. (3) can be approached in two alternative ways. On the one hand, given the potential $V(\phi)$ we can solve it to obtain $x(\phi)$. On the other hand, we can instead arbitrarily choose $x = x(\phi)$, and derive the corresponding scalar field potential

$$V(\phi) = A \left(3 - \varepsilon \frac{1}{2} x^2(\phi)\right) e^{-\frac{2\varepsilon}{2} \int x(\phi)d\phi}. \tag{4}$$

where $A$ is an arbitrary integration constant, and also

$$H(\phi) = \pm \sqrt{A} e^{-\frac{\varepsilon}{2} \int x(\phi)d\phi}, \tag{5}$$

$$\int dt = \int \frac{d\phi}{\pm \sqrt{A} x(\phi) \exp\left(-\frac{\varepsilon}{N-1} \int x(\phi)d\phi\right)}, \tag{6}$$

which is the final quadrature.

Therefore, selecting a suitable $x = x(\phi)$, one easily integrates the equations for the scalar field model and derives an exact solution in parametric form, or in closed form whenever (6) is invertible. From the definition (2) we see that $x(\phi)$ is related to the usual barotropic index $\gamma = (\rho + p)/\rho$ in the following way

$$x(\phi) = \pm \sqrt{3\gamma} \tag{7}$$

and thus choosing $x(\phi)$ amounts to a specification of the equation of state. On the other hand $x(\phi)$ relates to the slow roll parameters [5] in the following, simple way

$$\epsilon(\phi) = \frac{x^2(\phi)}{2} = \frac{2H'(\phi)}{H(\phi)}, \tag{8}$$

$$\bar{\eta}(\phi) = x'(\phi) + \frac{x^2(\phi)}{2} = 2\frac{H''(\phi)}{H(\phi)}. \tag{9}$$

### 3. Exact GR Solutions

We illustrate our procedure by giving, for $\varepsilon = 1$ (i.e., standard models) a brief list of cases yielding well-known solutions, and also by deriving a novel exact solution. Known solutions arise from the following correspondences:

- $x(\phi) = \lambda$ with $\lambda$ = constant, which gives the power-law solutions associated with the exponential potential $V(\phi) = V_0 e^{-\lambda \phi}[2]$;
- $x(\phi) = \lambda \phi$ yields the Easther solution [6]for which $V(\phi) = A \left(3 - \phi^2/2\right) e^{-\phi^2/2}$, and which gives a constant scalar spectral index equal to 3;
• \( x(\phi) = \beta/\phi \) yields Barrow’s intermediate inflation solutions \([7]\) associated with \( V(\phi) = \left(3 - \frac{\beta^2}{2\phi^2}\right)\phi^{-\beta} \);
• \( x(\phi) = (N - 1)\frac{G(\phi)}{\dot{G}(\phi)} \) with \( G(\phi) = AH(\phi) + L(\phi) \), where \( H \) are linear generating functions and \( L(\phi) = c_1\phi + c_2\phi^2 \), where \( c_1, c_2 \) are real constants, corresponds to Chimento’s solutions\([8]\);
• More cases can be found in Ref. \([4]\).

A new solution can be derived from the prescription \( x(\phi) = -2W'(\phi) \), where \( W(\phi) \) is the Lambert function \([9]\) defined as
\[
W(\phi) e^{W(\phi)} = \phi.
\]

When \( \phi \) is real, for \(-1/e \leq \phi < 0\) there are two possible real values of \( W(\phi) \). We denote just by \( W(\phi) \) the branch satisfying \(-1 \leq W(\phi)\). In this case one gets for the self-interacting potential \( V(\phi) \) the expression
\[
V(\phi) = A \left(3 - \frac{1}{2}\frac{W(\phi)^2}{(1 + W(\phi))^2}\right)\phi^2/W^2(\phi).
\]

The Hubble parameter and the velocity of the \( \phi \) field are
\[
H(\phi) = \pm\sqrt{A}e^{W(\phi)},
\]
\[
\dot{\phi} = \pm\sqrt{A}\frac{W'W}{W} = \pm\sqrt{A}\frac{1}{1 + W(\phi)},
\]

and thus one gets
\[
\pm\sqrt{A}(t - t_0) = \phi \left(W(\phi) + \frac{1}{W(\phi)}\right).
\]

The examples given here show not only that diverse solutions correspond to different choices for \( x(\phi) \), but also that this provides the basis for a classification of the solutions. To conclude this section, we refer that for \( \varepsilon = -1 \) it is possible to obtain phantom exact solutions as counterparts for all choices of \( x(\phi) \) yielding standard solutions. To the best of our knowledge this gives a plethora of new phantom exact solutions.

### 4. Form-invariance dualities

Recently Chimento and Lazkoz have discussed \([10]\) invertible maps that preserve the form of the equations of motion of flat FRW models. We generalise their results for the case of general scalar field models.

In our present case the transformations arise from \((x(\phi), \phi) \rightarrow (\bar{x}(\bar{\phi}), \bar{\phi})\), satisfying the two conditions
\[
\frac{\bar{x}^2}{x^2} = \pm\left(\frac{H}{\dot{H}}\right)^2 \frac{d\bar{H}}{d\dot{H}}, \quad \left(\frac{d\bar{\phi}}{d\phi}\right)^2 = \frac{d\bar{H}}{d\dot{H}}
\]

which leave invariant the equations of motion. In parallel, we also have
\[
\frac{d\bar{\phi}}{xH} = \frac{d\phi}{xH}
\]
which translates the fact that the form invariance transformation preserves the time variable (6).

According to our procedure, the choices of $x(\phi)$ and $\bar{x}(\bar{\phi})$ determine the corresponding potentials. If we choose for instance $x(\phi) = \lambda$ and $\bar{x}(\bar{\phi}) = \bar{\lambda}$ we obtain

$$\bar{\phi} = \frac{\lambda}{\bar{\lambda}} \phi + \phi_0 \quad \text{and} \quad \bar{H} = \left(\frac{\lambda}{\bar{\lambda}}\right)^2 H,$$

and thus we recover the results of Refs. [10]. If $x = \lambda$, and $\bar{x} = \bar{\lambda}/\bar{\phi}$ we derive

$$\bar{\phi}^\beta = A \left(\exp\left(\frac{\lambda}{N-1}\phi\right) + \nu\right)$$

where

$$\beta = 2 + \frac{\bar{\lambda}}{N-1}, \quad A = \frac{(N-1)^\beta}{\lambda} \left(\frac{H_0 \bar{\lambda}}{H_0 \lambda}\right)$$

and $\nu$ is once again an integration constant. When $\nu = 0$, $\bar{H}(H)$ is simply

$$\bar{H} = \frac{H_0}{H_0} A^{\beta-2} H^{1-\beta}.$$

The multiplying factor on the right-hand side can be negative in which case the transformation can map standard scalar field cosmology into a generalised phantom one.

5. Discussion

The procedure investigated here permits to recover all known exact scalar field solutions, and allows the derivation of new exact solutions, such as exact solutions of phantom cosmologies. Everything depends on the choice of a single generating function $x(\phi)$ that hence sets the basis for a classification of the solutions. It also permits us to establish form-invariance dualities by means of a simple mapping between different $x(\phi)$.

The details of the work reported here, and of a follow up devoted to scalar-tensor theories can be found in[4, 11].

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