Gravitational lensing in a non-uniform plasma

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Accepted 2010 January 5. Received 2009 December 24; in original form 2009 March 23

ABSTRACT

We develop a model of gravitational lensing in a non-uniform plasma. When a gravitating body is surrounded by a plasma, the lensing angle depends on the frequency of the electromagnetic wave, due to the dispersion properties of the plasma, in the presence of a plasma inhomogeneity, and of gravity. The second effect leads, even in a uniform plasma, to a difference of the gravitational photon deflection angle from the vacuum case, and to its dependence on the photon frequency. We take into account both effects, and derive the expression for the lensing angle in the case of a strongly non-uniform plasma in the presence of gravitation. The dependence of the lensing angle on the photon frequency in a homogeneous plasma resembles the properties of a refractive prism spectrometer, the strongest action of which is for very long radio waves. We discuss the observational appearance of this effect for the gravitational lens with a Schwarzschild metric, surrounded by a uniform plasma. We obtain formulae for the lensing angle and the magnification factors in this case and discuss the possibility of observation of this effect by the planned very long baseline interferometry space project RadioAstron. We also consider models with a non-uniform plasma distribution. For different gravitational lens models we compare the corrections to the vacuum lensing due to the gravitational effect in the plasma, and due to the plasma inhomogeneity. We show that the gravitational effect could be detected in the case of a hot gas in the gravitational field of a galaxy cluster.

Key words: gravitation – gravitational lensing: strong – gravitational lensing: weak – gravitational lensing: micro – plasmas.

1 INTRODUCTION

The photon deflection angle in a vacuum, in the Schwarzschild metric with a given mass $M$, is determined, for small deflection angles $\dot{\alpha} \ll 1$, by the formula

$$\dot{\alpha} = \frac{4GM}{c^2b} = \frac{2R_S}{b},$$

(1)

where $b$ is the impact parameter, and $b \gg R_S$, $R_S = 2GM/c^2$ is the Schwarzschild radius (Misner, Thorne & Wheeler 1973; Schneider, Ehlers & Falco 1992; Landau & Lifshitz 1993). When the impact parameter is close to its critical value, corresponding to the capture of the photon by the black hole, the expression for the deflection angle is more complicated (Darwin 1959; Misner et al. 1973; Virbhadra & Ellis 2000; Bozza et al. 2001; Bisnovatyi-Kogan & Tsupko 2008).

It is interesting to consider the gravitational lensing in a plasma, because in space the light rays mostly propagate through this medium. In an inhomogeneous plasma, photons move along curved trajectories, because a plasma is a dispersive medium with a permittivity tensor depending on its density (Landau & Lifshitz 1960).

In a dispersive non-uniform medium a photon trajectory depends also on the photon frequency, and this effect has no relation to the gravity. The photon deflection in a non-homogeneous plasma, in the presence of gravity, has been considered by Muhleman, Ekers & Fomalont (1970), Lightman et al. (1979) and Bliokh & Minakov (1989). The study was performed in a linear approximation, with the two effects (vacuum deflection due to gravitation, and deflection due to non-homogeneity of the medium) considered separately. The first effect is achromatic, while the second one depends on the photon frequency if the medium is dispersive, but equals zero if the medium is homogeneous.

A general theory of the geometrical optics in a curved space–time, in an arbitrary medium, is presented in the book by Synge (1960). On the basis of his general approach we have developed the model of gravitational lensing in a plasma. In our previous work (Bisnovatyi-Kogan & Tsupko 2009) we have shown that in a non-linear approach a new effect appears. Even in the homogeneous plasma, the gravitational deflection angle really depends on the frequency of the photon, and is different from the vacuum case. We have derived the expression for the deflection angle of the photon in a weak gravitational field, in a weakly inhomogeneous plasma.

In this work we use a more general approach and derive the deflection angle for the photon moving in a weak gravitational field, in
the Schwarzschild metric, in the arbitrary inhomogeneous plasma. Such an approach is more appropriate for propagation in a cosmic plasma, because the plasma density changes significantly, from the density of the interstellar medium to the density of a black hole neighbourhood. We consider here only the situation when the whole deflection angle, from the combined plasma and gravity effects, remains small. In Section 2 we derive a general expression for the deflection angle in an inhomogeneous plasma, in a curved space-time. In Section 3 we discuss in detail the important case of the weak Schwarzschild field with a spherically symmetric distribution of plasma, and derive the formula for the deflection in this situation. Our approach allows us to consider two effects simultaneously: the difference of the gravitational deflection in a plasma from the vacuum case; and the non-relativistic effect (refraction) connected with the plasma inhomogeneity. In the paper of Kulsrud & Loeb (1992) it was shown that in a homogeneous plasma the photon wave packet moves like a particle with a velocity equal to the group velocity of the wave packet, and with a mass equal to the plasma frequency. In Section 4 we show that our result for a homogeneous plasma follows also from this approach. In Section 5 we discuss the observational appearance for a Schwarzschild point-mass lens, surrounded by a uniform plasma (the effect of a gravitational radio spectrometer). In particular, we obtain formulae for the magnification factors in this case. We also estimate the possibility of observation of this effect by the planned very long baseline interferometry (VLBI) space project RadioAstron. In Section 6 we consider models with a non-uniform plasma distribution. For different gravitational lens models we compare the corrections to the vacuum lensing due to the gravity effect in a plasma and due to the plasma inhomogeneity.

After the publication of our previous work (Bisnovatyi-Kogan & Tsupko 2009), two papers concerning the effect of amplification of gravitational deflection in a dispersive medium were published (Dressel et al. 2009; Linet 2009). These two papers do not cover the topic of astrophysics: the authors calculate the vertical deflection of a light ray in a medium with strong frequency-dependent dispersion in a uniform gravitational field, and estimate such an effect for laboratory experiments with Earth’s gravity. In the paper by Linet (2009) the Syngy method of calculation is used.

## 2 Deflection Angle in an Inhomogeneous Plasma in the Presence of Gravity

Let us consider a static space-time with a metric

$$dx^2 = g_{ik} dx^i dx^k = g_{00} dx^0 dx^0 + g_{0i}(dx^0)^2,$$

\(i, k = 0, 1, 2, 3, \quad \alpha, \beta = 1, 2, 3.\) \hspace{1cm} (2)

Here \(g_{00}\) does not depend on time. Let us assume that the gravitational field is weak, so we can write

$$g_{ik} = \eta_{ik} + h_{ik}, \quad h_{ik} \ll 1, \quad h_{ik} \to 0 \quad \text{under} \quad x^\alpha \to \infty. \hspace{1cm} (3)$$

Here \(\eta_{ik}\) is the flat space metric (\(-1, 1, 1, 1\)), and \(h_{ik}\) is a small perturbation. Note (Landau & Lifshitz 1993) that

$$g^{ik} = \eta^{ik} - h^{ik}, \quad \eta^{ik} = \eta_{ik}, \quad h^{ik} = h_{ik}. \hspace{1cm} (4)$$

Let us consider, in this gravitational field, a static inhomogeneous plasma with a refraction index \(n\), which depends on the space location \(x^\alpha\) and the photon frequency \(\omega(x^\alpha)\):

$$n^2 = 1 - \frac{\omega^2}{\omega^2(x^\alpha)} = 1 - \frac{4\pi e^2 N(x^\alpha)}{mc} = K e N(x^\alpha). \hspace{1cm} (5)$$

Here \(\omega(x^\alpha)\) is the frequency of the photon, which depends on the space coordinates \(x^1, x^2, x^3\) due to the presence of the gravitational field (gravitational redshift). We denote \(\omega(\infty) = \omega, e\) is the charge of the electron, \(m\) is the electron mass, \(\omega_0\) is the electron plasma frequency, \(N(x^\alpha)\) is the electron concentration in an inhomogeneous plasma, and we do not assume that \(\omega(\infty) = 0\).

The optics in a curved space–time, in a medium, were developed by Syngy (1960). It was shown that, for the static case, the connection between the phase velocity \(u\) and a 4-vector of the photon momentum \(p^j\), using the refraction index of the medium \(n, n = c/u\), is written as

$$\frac{c^2}{u^2} = n^2 = 1 + \frac{p^1 p^1}{(p^0 - \omega_0)^2}. \hspace{1cm} (6)$$

Here \(c\) is the light velocity in a vacuum. The refraction index \(n\), defined for a plasma in (5), is a function of \(x^\alpha\) and \(\omega(x^\alpha)\). In the vacuum \(n = 1\), and we can obtain from (6) the usual relation for the square of the photon 4-vector: \(p p^j = 0\). In the medium, the square of the photon 4-vector is not equal to zero. For the medium in a flat space–time we have

$$g_{00} = -1, \quad g_{0i} = 1, \quad p^0 = -p_0, \quad p^\alpha = p^\alpha, \quad n^2 = 1 + \frac{(p^0)^2 + (p^\alpha)^2}{(p^0)^2}. \hspace{1cm} (7)$$

and obtain the usual relation between the space and time components of the 4-vector of the photon (Landau & Lifshitz 1960; Ginzburg 1970; Zheleznjakov 1977),

$$\frac{(p^0)^2}{n^2} = \frac{{(p^0)}^2}{(p^0)^2}. \hspace{1cm} (8)$$

For a static medium in a static gravitational field, we have (Syngy 1960, see also Møller 1972)

$$p_0 \sqrt{-g_{00}} = -p_0 \sqrt{-g_{00}} = -\frac{1}{c} \omega(x^\alpha), \hspace{1cm} (9)$$

where \(h\) is the Planck constant. A zero component of the 4-momentum is the energy divided by \(c\) (see Landau & Lifshitz 1993), so in a flat space–time we have

$$p_0 = -p_0 = -\frac{1}{c} \omega(x^\alpha), \hspace{1cm} (10)$$

where \(\omega = \omega(\infty)\). The components of the 4-vector \(p^j\), during an arbitrary motion in a non-homogeneous medium, in a flat space are written as

$$p^j = \left( p^0, p^\alpha \right) = \left( \frac{\omega_0}{c}, \frac{n \omega_0}{c}, \frac{n \omega_0 e}{c} \right),$$

$$p^\alpha = \left( p_0, p^\alpha \right) = \left( \frac{\omega_0}{c}, \frac{n \omega_0}{c}, \frac{n \omega_0 e}{c} \right), \quad n^2 = 1 - \frac{\omega_0^2}{\omega^2}, \hspace{1cm} (11)$$

where \(e^\alpha \equiv p^\alpha/p\) and \(e_\alpha = p_\alpha/p = (p = \sqrt{p^0 + p^0})\) are the unit 3-vectors in the direction of the 3-vector \(p^\alpha\) and \(p_\alpha\) correspondingly. In the flat space we have \(e^\alpha = e_\alpha\). We see that for the photons moving in the plasma \(p^\alpha p_\alpha = -m_{\text{eff}}^2\), with \(m_{\text{eff}} = \frac{\omega_0}{c} \sqrt{1 - n^2}\).

Using (6) and (9), we see that this relation is valid also for any static gravitational field, with \(m_{\text{eff}} = \frac{\omega_0}{c} \sqrt{1 - n^2}\). We have then, using (5), that the effective photon mass \(m_{\text{eff}}\) and effective velocity \(v_{\text{eff}}\) in a plasma are written as (Kulsrud & Loeb 1992)

$$m_{\text{eff}} = \frac{\omega_0(x^\alpha)}{c^2} \sqrt{1 - n^2} = \frac{\omega_0}{c^2},$$

$$v_{\text{eff}}^2 = \frac{p^2 c^2}{m_{\text{eff}}^2 c^2} = 1 - \frac{\omega_0^2}{\omega^2(x^\alpha)} c^2 = \frac{n^2 c^2}{c^2}. \hspace{1cm} (12)$$
Thus the effective photon velocity equals the group velocity of the photon in a plasma:

\[ v_g = \left( \frac{\partial (\omega_c)}{\partial \omega} \right)^{-1} c = n_c. \]  

(13)

The energy of the photon in a plasma \( E_{\text{eff}} \), and the relation between the energy and the effective momentum \( p_{\text{eff}} \) are also the same, as for the massive particle (Misner et al. 1973)

\[ E_{\text{eff}} = h\omega(x^\alpha) = \sqrt{-g_{\alpha\beta} \dot{\epsilon}^\alpha \dot{c}}, \]

\[ p_{\text{eff}} = \sqrt{E_{\text{eff}}^2 - m_{\text{eff}}^2 c^2} = \frac{n h\omega(x^\alpha)}{c}. \]  

(14)

The trajectories of photons, in the presence of the gravitational field, may be obtained from the variational principle (Synge 1960)

\[ \delta \left( \int p_i \, dx^i \right) = 0, \]  

(15)

with the restriction (6), which may be written in the form

\[ W(x^i, p_i) = \frac{1}{2} \left( g^{ij} p_i p_j - (n^2 - 1)  \left( p_0 \sqrt{-g_{\alpha\beta}} \right)^2 \right) = 0. \]  

(16)

Here we define the scalar function \( W(x^i, p_i) \) of \( x^i \) and \( p_i \). The variational principle (15), with the restriction \( \delta W(x^i, p_i) = 0 \), leads to the following system of differential equations (Synge 1960):

\[ \frac{dx^i}{dz} = \frac{\partial W}{\partial p_i}, \quad \frac{dp_i}{dz} = -\frac{\partial W}{\partial x^i}, \]  

(17)

with the parameter \( \lambda \) changing along the light trajectory. In the case of a plasma with the refraction index \( n \), the restriction (16) can be reduced, with the use of (5) and (9), to the form

\[ W(x^i, p_i) = \frac{1}{2} \left( g^{ij} p_i p_j + \frac{\alpha h^2}{c^2} \right) = 0. \]  

(18)

From (17) we obtain the system of equations for the space components \( x^i, p_i \):

\[ \frac{dx^\alpha}{dz} = \dot{g}^{\alpha\beta} p_\beta, \quad \frac{dp_\beta}{dz} = -\frac{1}{2} \dot{g}^{\alpha\beta} p_\beta p_\gamma - \frac{1}{2} \frac{h^2}{c^2} (\alpha \dot{c})_\beta, \]  

or

\[ \frac{dx^\alpha}{dz} = \dot{g}^{\alpha\beta} p_\beta, \quad \frac{dp_\alpha}{dz} = -\frac{1}{2} \dot{g}^{\alpha\beta} p_\beta p_\gamma - \frac{1}{2} \frac{h^2}{c^2} \frac{4\pi e^2}{m} N \alpha. \]  

(19)

(20)

It follows from (17) that in the static field the component \( p_0 \) is constant along the trajectory. Let us consider a photon moving along the z-axis in a curved space–time, in an inhomogeneous plasma. Because of the curved space–time metric, and plasma inhomogeneity, the photon will move along the curved trajectory. We use the approximation in which deviations of the photon trajectory from the straight line are small. Therefore for the null approximation we use the 4-vector in a flat space

\[ p^\mu = \left( \frac{h\omega}{c}, 0, 0, \frac{n h\omega}{c} \right), \quad p_i = \left( -\frac{h\omega}{c}, 0, 0, \frac{n h\omega}{c} \right). \]  

(21)

The unit 3-vector in the direction of the photon momentum is written in the null approximation as \( e^\mu = e_\mu = (0, 0, 1) \). We integrate the equations (20), calculating the right-hand side, by using the null approximation in the trajectory of the photon, with \( p^\mu \) from (21). We obtain then from the first equation in (20)

\[ \frac{dz}{\lambda c} = \frac{n h\omega}{c}, \quad \lambda_c = c \frac{dz}{n h\omega}. \]  

(22)

The deflection angle is determined by a change of 3-vector \( e_\mu \), so let us express the second equation in (20) through the \( e_\mu \). We obtain successively

\[ \frac{n h\omega}{c^2} \frac{d(n h\omega e_\mu)}{dz} = -\frac{1}{2} \dot{g}^{\mu\nu} p_\mu p_\nu - \frac{1}{2} \frac{h^2}{c^2} K e N \alpha, \]  

\[ \frac{d(n h\omega e_\mu)}{dz} = e_\mu \frac{dz}{dz} + \frac{n h\omega}{c} \frac{de_\mu}{dz}, \]  

\[ \frac{n h\omega}{c^2} \frac{de_\mu}{dz} = n e_\mu \frac{dz}{dz} - \frac{1}{2} \frac{c^2}{n h^2 \omega^2} \times \left( \dot{g}^{\alpha\beta} p_\beta p_\gamma + \frac{g^{00} p_0^2}{\epsilon} + \frac{h^2}{c^2} K e N \alpha \right). \]  

(23)

On the right-hand side of this equation we use the components from (21). We are interested in the components of the 3-vector \( e_\mu, e_3 = 1 \), which are orthogonal to the initial direction of the propagation, at \( \alpha = 1, 2 \). On the right-hand side of (25) we use the null approximation with \( e_\mu = 0 \). Using (4), we obtain

\[ \frac{de_\mu}{dz} = \frac{1}{2 n^2 h^2 \omega^2} \times \left( h_{33,\alpha} + \frac{h_{00,\alpha}}{c^2} \right) \frac{dz}{dz} + \frac{K e N \alpha}{c^2} \frac{dz}{dz}. \]  

(24)

\[ \frac{de_\mu}{dz} = \frac{1}{2} h_{33,\alpha} + \frac{h_{00, \alpha}}{c^2} \frac{dz}{dz} \frac{K e N \alpha}{c^2} \]  

(25)

\[ \frac{de_\mu}{dz} = \frac{1}{2} h_{33,\alpha} + \frac{h_{00, \alpha}}{c^2} \frac{dz}{dz} \frac{K e N \alpha}{c^2} \]  

(26)

\[ \frac{de_\mu}{dz} = \frac{1}{2} h_{33,\alpha} + \frac{h_{00, \alpha}}{c^2} \frac{dz}{dz} \frac{K e N \alpha}{c^2} \]  

(27)

\[ \frac{de_\mu}{dz} = \frac{1}{2} h_{33,\alpha} + \frac{h_{00, \alpha}}{c^2} \frac{dz}{dz} \frac{K e N \alpha}{c^2} \]  

(28)

\[ \frac{de_\mu}{dz} = \frac{1}{2} h_{33,\alpha} + \frac{h_{00, \alpha}}{c^2} \frac{dz}{dz} \frac{K e N \alpha}{c^2} \]  

(29)

\[ \frac{de_\mu}{dz} = \frac{1}{2} h_{33,\alpha} + \frac{h_{00, \alpha}}{c^2} \frac{dz}{dz} \frac{K e N \alpha}{c^2} \]  

(30)

Note that \( \hat{a}_0 < 0 \) corresponds to bending of the light trajectory towards the gravitational centre, and \( \hat{a}_0 > 0 \) corresponds to the opposite deflection.

In our previous paper (Bisnovatyi-Kogan & Tsupko 2009) we considered a weakly inhomogeneous plasma with \( N(x^\mu) = N_0 + N_1(x^\mu) \), \( N_0 = \) constant, \( N_1 \ll N_0 \). Here we assume that the deflection angle is small, but we do not assume that \( N_1 \) is much smaller than \( N_0 \), so the representation of \( N(x^\mu) \) as a sum is not required.
3 LENSMING IN THE SCHWARZSCHILD METRIC

Let us calculate the deflection angle for a photon moving in an inhomogeneous plasma, in the Schwarzschild metric of the point mass \( M \), with

\[
d s^2 = -c^2(1 - R_s/r)\,dt^2 + \frac{dr^2}{1 - R_s/r} + r^2(d\theta^2 + \sin^2\theta\,d\varphi^2).
\]

The second term is an additive correction to the gravitational deflection, due to the presence of the plasma. This term is present in the deflection angle both in the inhomogeneous and in the homogeneous plasma, and depends on the photon frequency. The third term is a non-relativistic deflection due to the plasma inhomogeneity (the refraction). This term depends on the frequency, but it is absent if the plasma is homogeneous. The fourth term is a small additive correction to the refraction deflection in the inhomogeneous plasma. Calculation of the refraction deflection for a power-law concentration was given in our previous work (Bisnovatyi-Kogan & Tsupko 2009) [see also Muhllemann et al. (1970), Lightman et al. (1979), Bliokh & Minakov (1989) and Thompson et al. (1994)]. Note that the refraction in the inhomogeneous plasma with \( N(r) = N_0(R_t/r)^h \), where \( N_0 \) is constant, \( R_t = constant, \ h = constant \neq 0 \), leads to the refraction deflection angle \( \alpha_r \), which is opposite to the gravitational deflection (Bliokh & Minakov 1989; Bisnovatyi-Kogan & Tsupko 2009).

For \( \omega \gg \omega_c \) we have

\[
\alpha_r = \frac{1}{\omega^2} \frac{4\pi e^2}{m} \frac{N_0}{b} \left( \frac{R_t}{b} \right)^h \sqrt{\pi} \Gamma \left( \frac{h}{2} + \frac{1}{2} \right) \Gamma \left( \frac{h}{2} \right),
\]

(36)

For the arbitrary \( n \) one needs to use the expression (34) which is valid in a general case. The main approximation used here is the smallness of the deflection angle, which can be satisfied even if the concentration \( N \) changes significantly, or if the refraction index \( n \) is not close to unity. The most interesting result following from our calculation is that even in the case of a homogeneous plasma the photon deflection angle differs from the vacuum case, and depends on the plasma and photon frequency. Indeed, for \( \omega_c = constant \) we obtain from (34)

\[
\hat{\alpha}_b = -\frac{R_s}{b} \left( 1 + \frac{1}{\omega^2 - \omega_c^2} \right).
\]

(37)

This formula is valid only for \( \omega > \omega_c \), because the waves with \( \omega < \omega_c \) do not propagate in the plasma (Ginzburg 1970). Here \( \hat{\alpha}_b < 0 \). This means that the light ray is bent in the direction of the gravitation centre, as occurs in a vacuum. Thus the presence of a plasma increases the gravitational deflection angle. This formula is valid under the condition of smallness of \( \hat{\alpha}_b \), but this condition allows the second term in brackets to be much larger than the first one. So the gravitational deflection in a plasma can be significantly larger than in a vacuum. This effect has a general relativistic nature, in combination with the dispersive properties of the plasma. Such an effect may happen only for radio frequency photons, because optical frequencies are much higher than the plasma frequency \( \omega_c \), so the effect would be negligible.

In the literature on gravitational lensing theory the deflection angle is determined usually as a positive one (equation 1), and is defined as the difference between the initial and the final ray directions \( \hat{\alpha} = \varepsilon_m - \varepsilon_o, \) where \( \varepsilon \) is the unit tangent vector of a ray (Schneider et al. 1992). Therefore, if we use this definition, we will have an expression with the opposite sign:

\[
\hat{\alpha} = \frac{R_s}{b} \left( 1 + \frac{1}{1 - (\omega^2_c/\omega^2)} \right),
\]

(38)

which turns into the deflection angle for a vacuum \( 2R_s/b \), when \( \omega \rightarrow \infty \).

4 ANALOGY BETWEEN A PHOTON IN PLASMA AND A MASSIVE PARTICLE

Using the analogy between the motion of a photon in a homogeneous plasma, and a massive particle in the gravitational field (12) and (14), we may easily find the deflection angle of the photon in the plasma, produced by a point mass, in a weak field approximation. A test massive particle passing with velocity \( v \) near a spherical body with a mass \( M \), having impact parameter \( b \gg R_s \), deflects to the angle \( \alpha_m \) defined as (Misner et al. 1973; Lightman et al. 1979)

\[
\alpha_m = \frac{R_s}{b} \left( 1 + \frac{1}{\beta^2} \right), \quad \beta = \frac{v}{c}.
\]

(39)

If we use in this formula \( v = \gamma v_0 = c \gamma [1 - (\omega^2_c/\omega^2)]^{1/2} \) we obtain the same formula (38) for the photon deflection in a homogeneous plasma. Note that this analogy is valid for any static gravitational field, not only for a weak field.
5 OBSERVATIONAL EFFECTS: THE PROJECT RADIOASTRON

We see from (38) that photons of smaller frequency, and larger wavelength, are deflected by a larger angle by the gravitation centre. The effect of a difference in the gravitational deflection angles is significant for longer wavelengths, when $\omega$ is approaching $\omega_\iota$. That is possible only for radio waves. Therefore, a gravitational lens in a plasma acts as a radio spectrometer (Bisnovatyi-Kogan & Tsupko 2009).

The observational effect of the frequency dependence may be represented in an example with off-line lensing by the Schwarzschild point-mass lens [see fig. 1 in the paper of Bisnovatyi-Kogan & Tsupko (2009)]. This lens gives two images of the source, on opposite sides of the lens. The angular positions and forms of the images depend on the Schwarzschild radius of the lens, and on the relative positions of the source, the lens and the observer. The dependence of the deflection angle on the frequency in a plasma leads to smearing of the images, and different parts of extended images have different spectra. In the case when the source, the lens and the observer are on the same straight line, the lensing image in a vacuum is a thin circle (Bisnovatyi-Kogan & Tsupko 2008). In a plasma this circle should have a finite thickness, with the radio spectra depending on the radius. The inner parts of the circle consist of more energetic photons than its outer parts [see fig. 2 in the paper by Bisnovatyi-Kogan & Tsupko (2009)]. In reality the gravitational lens is not a point mass; it has a complicated structure, and the position of the images differs from that of the point-mass lens. We should also note that the source must be radio-loud.

The standard model of gravitational lensing is based on the Einstein deflection angle (1), which should be replaced in the presence of a plasma by our formula (38). The angular half-separation due to gravitational lensing between the images of the source in a vacuum (Schneider et al. 1992) is of the order of

$$\theta_0 = \sqrt{\frac{2R_s D_{ls}}{D_s D_{ls}}}.$$

where $D_0$ is the distance between the observer and the lens, $D_s$ is the distance between the observer and the source, and $D_{ls}$ is the distance between the lens and the source. In the case of a perfect alignment of the source, the lens and the observer, the image of the source is called an Einstein ring, and its radius has an angular size $\theta_0$. The observed angular separation of quasar images is usually around 1 arcsec for lenses by a galaxy. Lensing in the presence of a homogeneous plasma (38) leads to an angular half-separation between images as

$$\theta_0^p = \sqrt{\left(1 + \frac{1}{\omega^2/\omega_0^2} \right) R_s \frac{D_{ls}}{D_s D_{ls}}}$$

which may be called a plasma Einstein ring. For $\omega^2/\omega_0^2 \ll 1$ we obtain

$$\theta_0^p = \left(1 + \frac{\omega_0^2}{4 \omega^2} \right) \theta_0.$$

The difference between angular separations of images in a vacuum and in a plasma $\Delta \theta_0$, produced by the same lens configuration, is equal to

$$\Delta \theta_0 = \theta_0^p - \theta_0 = \frac{\omega_0^2}{4 \omega^2} \approx 2 \times 10^7 \frac{N_e}{v^2}.$$

where $v$ is the photon frequency in Hz, $\omega = 2\pi v$. The formula (43) gives the difference between the deviation angle of the radio wave with a frequency $\nu$ and the optical image, which may be described by the vacuum formula (40).

Let us estimate the possibility of observation of this effect by the planned project RadioAstron (see the RadioAstron web page at http://www.asc.rssi.ru/radioastron/index.html). RadioAstron is a VLBI space project led by the Astro Space Center of Lebedev Physical Institute in Moscow. The payload is the Space Radio Telescope, based on the spacecraft Spektr-R. For the lowest frequency of RadioAstron, $\nu = 327 \times 10^6$ Hz, the angular difference between the vacuum and the plasma images is about $10^{-3}$ arcsec, when the plasma density on the photon trajectory in the vicinity of the gravitational lens is of the order of $N_e \sim 5 \times 10^4$ cm$^{-3}$. This angular resolution is supposed to be reached by the RadioAstron project.

The magnification of the image increases with increasing deflection angle, therefore different images may have different spectra in the radio band, when the light propagates in regions with different plasma density. The magnification is determined by the deflection law (Schneider et al. 1992). Let us demonstrate this in the example of point-mass lensing. The magnification factors of the primary image $\mu_+$, located at the same side as the source relative to the lens, and of the secondary image $\mu_-$, located at the opposite side, depend on the angular position of the source. Corresponding formulae can be found in Schneider et al. (1992):

$$\mu_+ = \frac{1}{4} \left[ \frac{y}{\sqrt{y^2 + 4}} + \frac{\sqrt{y^2 + 4}}{y} + 2 \right],$$

$$\mu_- = \frac{1}{4} \left[ \frac{y}{\sqrt{y^2 + 4}} + \frac{\sqrt{y^2 + 4}}{y} - 2 \right].$$

Here $y = \beta/\theta_0$, where $\beta$ is the angular position of the source relative to the line passing through the observer and the lens. The total magnification of the source is equal to

$$\mu_{tot} = \mu_+ + \mu_- = \frac{y^2 + 2}{y \sqrt{y^2 + 4}}.$$

Consideration of the total magnification factor is important for microlensing events when the separated images are not resolved, and the only observable effect is a change of the flux from the source due to lensing.

In the case of lensing in a plasma we can rewrite these formulae, using $\tilde{y} = \beta/\theta_0^p$ instead of $y$:

$$\mu_{pl}^+ = \frac{1}{4} \left[ \frac{\tilde{y}}{\sqrt{\tilde{y}^2 + 4}} + \frac{\sqrt{\tilde{y}^2 + 4}}{\tilde{y}} + 2 \right],$$

$$\mu_{pl}^- = \frac{1}{4} \left[ \frac{\tilde{y}}{\sqrt{\tilde{y}^2 + 4}} + \frac{\sqrt{\tilde{y}^2 + 4}}{\tilde{y}} - 2 \right],$$

$$\mu_{pl}^* = \mu_{pl}^+ + \mu_{pl}^- = \frac{\tilde{y}^2 + 2}{\tilde{y} \sqrt{\tilde{y}^2 + 4}}.$$

where

$$\tilde{y} = \frac{\beta}{\theta_0^p} = \frac{\beta}{\theta_0} \left[ \frac{1}{2} \left( 1 + \frac{\omega_0^2}{1 - (\omega^2/\omega_0^2)} \right) \right]^{-1/2}$$

$$= y \left[ \frac{1}{2} \left( 1 + \frac{1}{1 - (\omega^2/\omega_0^2)} \right) \right]^{-1/2}.$$
At large $\beta$ the total amplification factor goes to unity, because the influence of the lens on the light propagation becomes negligible. For the Schwarzschild lens at a small angle $\beta$, the amplification is inversely proportional to the angle $\beta$, and is proportional to the angular radius of the Einstein ring, which increases with decreasing frequency, approaching infinity at the plasma frequency. As $\beta$ goes to zero, the amplification increases, formally unrestricted for the point source.

Let us consider the magnification in a plasma when $\omega_0^2 / \omega^2 \ll 1$. Carrying out the expansion of the expressions (47)–(50), we obtain the additional terms that arise in the case of a plasma, as compared to the case of a vacuum:

$$
\mu^\text{pl}_+ = \mu_+ + \frac{1}{y(2y^2 + 4)^{1/2}} \frac{\omega_0^2}{\omega^2}, \quad (51)
$$

$$
\mu^\text{pl}_- = \mu_- + \frac{1}{y(2y^2 + 4)^{1/2}} \frac{\omega_0^2}{\omega^2}, \quad (52)
$$

$$
\mu^\text{pl}_\text{tot} = \mu_\text{tot} + \frac{2}{y(2y^2 + 4)^{1/2}} \frac{\omega_0^2}{\omega^2}. \quad (53)
$$

We see that the presence of a homogeneous plasma increases the magnification.

The ratio of the magnification of the primary image in the presence of a plasma $\mu^\text{pl}_+$ to the same value in a vacuum $\mu_+$ is given in Fig. 1, for $\omega = \sqrt{2} \omega_0$, according to (44) and (47). The upper curve in Fig. 1 is the ratio of the magnification of the secondary image in the presence of a plasma $\mu^\text{pl}_-$ to the same value in a vacuum $\mu_-$, according to (45) and (48). As $y$ approaches zero, the ratio of the magnifications $\mu^\text{pl}_+/\mu_+$ and $\mu^\text{pl}_-/-\mu_-$ becomes the constant $V_+$, depending on the frequency:

$$
V_+ = \frac{\mu^\text{pl}_+}{\mu_+} = \frac{\mu^\text{pl}_-}{\mu_-} = \frac{\theta_0^\text{pl}}{\theta_0} \approx \frac{1}{2} \left( 1 + \frac{1}{1 - (\omega_0^2/\omega^2)} \right). \quad (54)
$$

This value is equal to $\sqrt{3}/2$ at $\omega = \sqrt{2} \omega_0$. At large $y$ the ratio $\mu^\text{pl}_+/\mu_+$ tends to unity; the ratio $\mu^\text{pl}_-/\mu_-$ tends to the constant $V_-$:

$$
V_- \approx \left( \frac{\theta_0^\text{pl}}{\theta_0} \right)^4 = \left[ \frac{1}{2} \left( 1 + \frac{1}{1 - (\omega_0^2/\omega^2)} \right) \right]^2. \quad (55)
$$

This value is equal to $9/4$ at $\omega = \sqrt{2} \omega_0$. Note that both $\mu_-$ and $\mu^\text{pl}_-$ tend to zero at this limit.

The deflection angle in the presence of a plasma is larger than in a vacuum, so the amplification for lower frequencies is larger, and in this situation the image spectrum differs from the original spectrum of the source, having a more intensive low-frequency part. The light in two lensing images is propagating through different media with different plasma densities. Therefore different images of the same source may have different spectra in the radio band.

In the ideal case, when two images of the same source are formed by rays propagating through the uniform plasma with different concentrations, the spectra of two images should be different. In achromatic lensing the ratio of fluxes should be the same at all frequencies: $\mu^\text{opt}_+ / \mu^\text{opt}_- = \mu^\text{rad}_+ / \mu^\text{rad}_-$, so that $\mu^\text{opt}_+ / \mu^\text{opt}_- = \mu^\text{rad}_+ / \mu^\text{rad}_-$. The presence of a plasma leads to larger amplification at lower frequencies, so if the ratios of fluxes in the optical and the lower radio band of two lensing images are different, $\mu^\text{opt}_+ / \mu^\text{opt}_- \neq \mu^\text{rad}_+ / \mu^\text{rad}_-$, it may be related to the influence of the plasma, so that the image with larger relative radio flux propagates through the plasma with larger density. In Fig. 2 two cases are shown for the quantity $(\mu^\text{pl}_+/\mu_+)/(\mu^\text{pl}_-/-\mu_-)$, as a function of $\omega/\omega_0$, for different fixed $y$. In the first case the plasma density is a factor of 2 larger for rays forming the main image, and in the second case the ratio of densities is the opposite. While the vacuum value $\mu_+/\mu_-$ does not depend on the frequency, one of these two behaviours is expected for the ratio of fluxes of two images as a function of the frequency. In the considered ideal case this plot should give information about the ratio of plasma densities, and when the image with lower relative radio flux is formed by vacuum lensing, it should be possible to obtain the absolute value of the plasma density through which propagate rays from the image with higher relative radio flux.

From Fig. 2 we see that the spectral dependences can have different forms for different $y$: while the dashed line is less than unity, the solid line can be both larger and less than unity. In Fig. 2(a) in the case where the light rays corresponding to the primary image go through the denser plasma, we see that for smaller frequencies the primary image is more magnified, relative to the vacuum magnification, because of the plasma presence than the secondary image – see the solid line. In the opposite case, when the light rays corresponding to the secondary image pass through a greater density, we see that for the small frequencies the secondary image is more magnified, therefore the dashed line is less than unity. For bigger $y$ both curves are less than unity. In Figs 2(b) and (c) we see that with increasing $y$ the solid line goes to the region that is lower than unity. In Fig. 3 we plot the same dependence for the case where the plasma density is 10 times larger for rays forming the main image, and the second curve corresponds to the opposite ratio in densities. In the case where the difference between densities is larger (compared with the situation in Fig. 2), effects connected with the plasma are greater. Here the solid line remains above unity for greater values of $y$ than for the situation described in Fig. 2. We should note that for $y \geq 1$ the secondary image is very demagnified (Schneider et al. 1992; Schneider, Kochanek & Wambsganss 2006), so in a real situation we will have behaviour like in Figs 2(a) and 3(a), and the ratio of densities can be uniquely defined. More detail can be found in Appendix A.
Figure 2. The value $(\mu_{+}^{pl}/\mu_{+})/(\mu_{-}^{pl}/\mu_{-})$, as a function of $\omega/\omega_{e0}$, for different fixed $y$. $\omega_{e0}$ is the plasma frequency corresponding to the plasma concentration $N_{+}$, through which rays forming the primary image pass. $\omega_{e}$ is the plasma frequency corresponding to the plasma concentration $N_{-}$, through which rays forming the primary image pass. The solid curve is plotted for $\omega_{e0} = \omega_{00} \sqrt{2}$; it corresponds to $N_{+}/N_{-} = 2$. The dashed curve is plotted for $\omega_{e0} = \omega_{00}$; it corresponds to $N_{+}/N_{-} = 1/2$.

Figure 3. The value $(\mu_{+}^{pl}/\mu_{+})/(\mu_{-}^{pl}/\mu_{-})$, as a function of $\omega/\omega_{e0}$, for different fixed $y$. The solid curve is plotted for $\omega_{e0} = \omega_{00} \sqrt{10}$; it corresponds to $N_{+}/N_{-} = 10$. The dashed curve is plotted for $\omega_{e0} = \omega_{00}$; it corresponds to $N_{+}/N_{-} = 1/10$.

In this section we have discussed in detail new observational effects, connected with the gravitational deflection in a homogeneous plasma. In reality the distribution of plasma around the gravitating objects is non-homogeneous, and there is a deflection connected with the inhomogeneity of the medium. For the plasma density profile decreasing with distance from the centre, the refraction deflection has a sign opposite to the gravitational one. Therefore the refraction and gravitational deflections in the non-uniform plasma partially cancel each other, and the value and the sign of the
resulting angle depend on the particular configuration of the gravitational lens. In the following section we consider situations of lensing with a non-uniform plasma distribution.

6 MODELS WITH A NON-UNIFORM PLASMA DISTRIBUTION

6.1 Singular isothermal sphere

Let us consider a simple model of a singular isothermal sphere (Chandrasekhar 1939; Binney & Tremaine 1987) which is often used in the lens modelling of galaxies and clusters of galaxies (Schneider et al. 1992, 2006; Wambgsans 1998; Bartelmann & Schneider 2001). The density distribution is written as

\[ \rho(r) = \frac{\sigma_v^2}{2\pi G r^2}. \]  

(56)

Here \( \sigma_v \) is a one-dimensional velocity dispersion (for stars in galaxies or for galaxies in clusters of galaxies). The projected surface mass density of a singular isothermal sphere to the lens plane, perpendicular to the light ray, is equal to

\[ \Sigma(b) = \frac{\sigma_v^2}{2Gb}. \]  

(57)

The deflection due to an axisymmetric mass distribution with impact parameter \( b \) is equal to the Einstein angle for the mass \( M(b) \), where \( M(b) \) is the projected mass enclosed by the circle of radius \( b \). In other words it is the mass inside the cylinder with radius \( b \) (Clark 1972; Bloikh & Minakov 1989; Schneider et al. 1992, 2006). So for this case we should rewrite formula (35), substituting \( M(b) \):

\[ \hat{\alpha}_0 = \frac{4GM(b)}{c^2b} + \frac{2GM(b)b}{c^2\omega^2} \int_0^\infty \frac{\omega^2}{r^3} \frac{dN(r)}{dr} dr + \frac{K_v b}{\omega^2} \int_0^\infty \frac{\omega^2}{r} \frac{dN(r)}{dr} dr. \]

(58)

Here we rewrite the formula in such a way that the gravitational deflection due to the inhomogeneity of the plasma, and the term \( \hat{\alpha}_3 \) is a correction to the gravitational deflection due to the presence of the plasma, the term \( \hat{\alpha}_3 \) is the refraction deflection due to the inhomogeneity of the plasma, and the term \( \hat{\alpha}_4 \) is a correction to the third term. We are interested mainly in the effects described by the terms \( \hat{\alpha}_1, \hat{\alpha}_2 \) and \( \hat{\alpha}_3 \).

For the singular isothermal sphere the projected mass \( M(b) \) is

\[ M(b) = \int_0^b \Sigma(b') 2\pi b' db' = \frac{\sigma_v^2}{G} \pi b. \]

(59)

So the vacuum gravitational deflection due to the singular isothermal sphere is constant:

\[ \hat{\alpha}_1 = 4\pi \frac{\sigma_v^2}{c^2}. \]

(60)

For the concentration of the plasma we have

\[ N(r) = \frac{\rho(r)}{k m_p}. \]

(61)

where \( m_p \) is the proton mass, and \( \kappa \) is a non-dimensional coefficient responsible for the dark matter contribution, and is approximately equal to \( \kappa \simeq 6 \). We assume here that \( \rho(r) \), which is given by the formula (56), is the density of all kinds of matter, not only plasma particles. Thus the plasma frequency is equal to

\[ \omega_p^2 = K_e N(r) = \frac{K_e}{k m_p} \rho(r). \]

(62)

The correction to the gravitational deflection due to the presence of the plasma is

\[ \hat{\alpha}_2 = \frac{2\pi b^2 \sigma_v^2 K_v}{\omega^2 c^2 k m_p} \int_0^\infty \frac{\rho(r)}{r^3} \frac{dr}{r^3}. \]

(63)

Integration of such expressions can be performed using Gradshteyn & Ryzhik (1965), and the properties of the \( \Gamma \)-function:

\[ \int_0^\infty \frac{dz}{(z^2 + b^2)^{3/2}} = \frac{1}{\hbar b^{1/2}} \frac{\sqrt{\pi} \Gamma\left(\frac{5}{2}\right)}{\Gamma\left(\frac{3}{2}\right)}, \]

(64)

where

\[ \Gamma(x) = \int_0^\infty t^{x-1} e^{-t} dt. \]

Using (64), we obtain the deflection angles \( \hat{\alpha}_2 \) and \( \hat{\alpha}_3 \):

\[ \hat{\alpha}_2 = \frac{2 \sigma_v^2 \sigma_v^2}{3 \omega^2 c^2 k m_p} \frac{K_v}{G\omega^2 b^2}. \]

(65)

\[ \hat{\alpha}_3 = -\frac{K_v b}{\omega^2 k m_p} \frac{\pi G}{2} \int_0^\infty \frac{dz}{(z^2 + b^2)^2} = \frac{1}{4} \frac{K_v}{k m_p} \frac{\sigma_v^2}{G\omega^2 b^2}. \]

(66)

For the ratio of the angles \( \hat{\alpha}_2 \) and \( \hat{\alpha}_3 \) we have

\[ \left| \frac{\hat{\alpha}_2}{\hat{\alpha}_3} \right| = \frac{8 \sigma_v^2}{3 c^2}. \]

(67)

6.2 Non-singular isothermal gas sphere

Let us consider a gravitational lens model of an isothermal sphere, in which the singularity at the origin is replaced by a finite core (Hinshaw & Krauss 1987; Wu 1996):

\[ \rho(r) = \frac{\sigma_v^2}{2\pi G (r^2 + r_c^2)} = \frac{\rho_0}{\left(1 + \frac{r^2}{r_c^2}\right)^3}, \quad \rho_0 = \frac{\sigma_v^2}{2\pi G r_c^3}. \]

(68)

where \( r_c \) is the core radius. The corresponding projected surface mass density for this model is

\[ \Sigma(b) = \frac{\sigma_v^2}{2G \sqrt{b^2 + r_c^2}}. \]

(69)

The total projected mass within \( b \) and the vacuum gravitational deflection angle are:

\[ M(b) = \frac{\pi \sigma_v^2}{G} \left( \sqrt{b^2 + r_c^2} - r_c \right). \]

(70)

\[ \hat{\alpha}_1 = 4\pi \frac{\sigma_v^2}{c^2} \frac{\sqrt{b^2 + r_c^2} - r_c}{b}. \]

(71)

Analogously to the previous subsection, we reduce the angle \( \hat{\alpha}_2 \) to the form

\[ \hat{\alpha}_2 = \frac{2GM(b)b}{c^2\omega^2} \frac{K_v}{k m_p} \frac{\sigma_v^2}{2\pi G} \int_0^\infty \frac{dz}{z^3 (r^2 + r_c^2)} \]

\[ = \frac{2GM(b)b}{c^2\omega^2} \frac{K_v}{k m_p} \frac{\sigma_v^2}{2\pi G} \int_0^\infty \frac{dz}{(z^2 + b^2)^{3/2} (z^2 + b^2 + r_c^2)}. \]

(72)
By substitution $z^2/(z^2 + b^2) = x^2$ the integral
\[ I_1 = \int_0^\infty \frac{dz}{(z^2 + b^2)^{3/2}(z^2 + b^2 + r_c^2)} \]
is reduced to
\[ I_1 = \int_0^1 \left( \frac{1}{b^2 + r_c^2} - \frac{1}{r_c^2(b^2 + r_c^2)(1 - x^2))} \right) dx. \quad (73) \]
Integration can be performed with the use of Dwight (1961):
\[ I_1 = \frac{1}{b^2 r_c^2} - \frac{1}{r_c^2 \sqrt{b^2 + r_c^2}} \arctan \left( \frac{r_c}{\sqrt{b^2 + r_c^2}} \right). \quad (74) \]
For $\hat{\alpha}_2$ we obtain
\[ \hat{\alpha}_2 = \frac{\sigma_b^2}{c^2 \omega^2 G} \frac{K_e}{\kappa m_p} b \left( \sqrt{b^2 + r_c^2} - r_c \right) \times \left[ \frac{1}{b^2 r_c^2} - \frac{1}{r_c^2 \sqrt{b^2 + r_c^2}} \arctan \left( \frac{r_c}{\sqrt{b^2 + r_c^2}} \right) \right]. \quad (75) \]
Angle $\hat{\alpha}_3$ can be calculated using formula (64):
\[ \hat{\alpha}_3 = -\frac{K_e b}{c^2 \omega^2 G} \frac{1}{2} \frac{\sigma_b^2}{\kappa m_p} b \left( \frac{\int_0^\infty dz}{(z^2 + b^2 + r_c^2)^2} \right) \]
\[ = -\frac{\pi \sigma_b^2}{4 \omega^2 G} \frac{K_e}{\kappa m_p} b, \quad (76) \]
Let us express angles $\hat{\alpha}_2$ and $\hat{\alpha}_3$ in terms of the central density $\rho_0$, using (68). We obtain, for $r_c \gg b$,
\[ \hat{\alpha}_2 = \frac{2 \pi^2 G \rho_0^2}{c^2 \omega^2} \frac{K_e}{\kappa m_p} b r_c, \quad \hat{\alpha}_3 = -\frac{\pi \rho_0}{2 \omega^2 \kappa m_p} \frac{b}{r_c}; \quad (77) \]
and for $b \gg r_c$, similar to the case of the singular isothermal sphere, we have
\[ \hat{\alpha}_2 = \frac{8 \pi^2 G \rho_0^2}{3 c^2 \omega^2} \frac{K_e}{\kappa m_p} r_c^3, \quad \hat{\alpha}_3 = -\frac{\pi \rho_0}{2 \omega^2 \kappa m_p} \frac{r_c^2}{b^2}. \quad (78) \]
Introducing the mass of the uniform core $M_c = \frac{4 \pi}{3} \rho_0 r_c^3$, and its gravitational radius $R_{sc} = \frac{2 G M_c}{c^2},$ we obtain the ratio of these angles as
\[ \frac{\hat{\alpha}_2}{\hat{\alpha}_3} = \frac{3 R_{sc}}{2} \left( r_c \gg b, \hat{\alpha}_2 = \frac{2 R_{sc}}{r_c} \right). \quad (79) \]
In the realistic cases $\frac{\hat{\alpha}_2}{\hat{\alpha}_3} \ll 1$ in (67) and (79), because spheres have, respectively, $\sigma_\epsilon \ll c$, and $R_{sc} \ll r_c$. Besides, these relations are needed for the stability of isothermal spheres (see Bisnovatyi-Kogan & Zeldovich 1969). Therefore in this configuration the non-uniform plasma deflection effects are much stronger than the gravitational plasma effects, and have the opposite direction.

### 6.3 Plasma sphere around a black hole

Let us consider a black hole of mass $M_0$, surrounded by an electron–proton plasma. We will consider a case where we can neglect the self-gravitation of the plasma particles, compared to the gravity of a central black hole. Let us find, in the Newtonian approximation, the density distribution of the plasma in the gravitational field of a central point mass $M_0$. The equation of hydrostatic equilibrium for a spherically symmetric mass distribution of an isothermal gas with the equation of state $P = \rho \Theta T$ in the field of the central mass is (Chandrasekhar 1939; Binney & Tremaine 1987)
\[ \frac{\partial}{\partial r} \frac{\rho}{\rho} = -\frac{GM_0}{r^2}, \quad (80) \]
Here $\rho$, $P$ and $T$ are the density, the pressure and the temperature of the plasma, $\Theta = k_B/\rho m_p$ is the gas constant, $k_B$ is the Boltzmann constant and $m_p$ is the proton mass. To obtain a deflection angle we need to find the plasma density distribution from equation (80) and calculate the integrals with this density in formula (58). However, it is interesting that we can find the ratio of angles $|\hat{\alpha}_2/\hat{\alpha}_3|$ without solving equation (80) and calculation of the integrals in $\hat{\alpha}_2$ and $\hat{\alpha}_3$. Let us rewrite equation (80) in the form
\[ \frac{K_e b}{\rho_0^2} \frac{\Theta}{T} \frac{1}{r} \frac{dN(r)}{dr} = -\frac{K_e b}{\rho_0^2} \frac{GM_0}{r^3} N(r), \quad (81) \]
where $N(r) = \rho(r)/m_p$ is the plasma concentration. If we compare it with expressions for $\hat{\alpha}_2$ and $\hat{\alpha}_3$ in (58) and take into account that $\omega^2 = \frac{4 \pi e^2}{m} \rho_0 = K_e \rho_0$, we obtain that
\[ \frac{\hat{\alpha}_2}{\hat{\alpha}_3} = \frac{2 \Theta T}{c^2}. \quad (82) \]
We see that this ratio does not depend on the value of the central mass $M_0$. For the boundary condition
\[ \rho(r_0) = \rho_0 \]
we obtain from equation (80) the density distribution
\[ \rho(r) = \rho_0 \exp \left[ \frac{GM_0}{\rho_0^2} \frac{1}{\Theta T} \left( \frac{1}{r} - \frac{1}{r_0} \right) \right] = \rho_0 \exp \left[ \frac{B}{r} \left( \frac{1}{r} - \frac{1}{r_0} \right) \right], \quad (83) \]
where
\[ B = \frac{GM_0}{\rho_0^2} \Theta T \]
For angles $\hat{\alpha}_2$ and $\hat{\alpha}_3$ we obtain
\[ \hat{\alpha}_2 = \frac{2 \pi^2 G \rho_0^2}{c^2 \omega^2} \frac{K_e}{\kappa m_p} \rho_0 e^{-B/r_0} \text{Int} 2, \quad (85) \]
\[ \hat{\alpha}_3 = -\frac{K_e b}{\rho_0^2 \kappa m_p} \frac{B}{r} \rho_0 e^{-B/r_0} \text{Int} 2, \quad (86) \]
where
\[ \text{Int} 2 = \int_0^{\infty} \frac{1}{(b^2 + z^2)^{3/2}} \exp \left( \frac{B}{\sqrt{b^2 + z^2}} \right) dz. \quad (87) \]
Calculation of Int 2 can be found in Appendix B.

While the temperature of the non-self-gravitating sphere may have arbitrary values, the plasma effects may be comparable to and even less than the general relativistic plasma effects. This is due to the fact that with increasing temperature the plasma density can become arbitrarily uniform, with corresponding decrease of the non-uniform plasma effect for refraction. We have used a Newtonian non-relativistic description of the gas sphere, but from the arguments listed above it is clear that this conclusion remains valid also with the correct relativistic considerations.

### 6.4 Plasma in a galaxy cluster

In a galaxy cluster the electron distribution may be more homogeneous due to the large temperature of the electrons. An appropriate approach for this case is to consider a singular isothermal sphere as a model for the distribution of the gravitating matter, neglecting the mass of the plasma, and to find a plasma density distribution from the solution of the equation of hydrostatic equilibrium of a plasma in the gravitational field of a singular isothermal sphere. In this approximation the density distribution of the gravitating matter has the form
\[ \rho_{pl}(r) = \frac{\sigma^2}{2 \pi Gr^2}, \quad (88) \]

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with the vacuum gravitational deflection angle
\[ \hat{\alpha}_1 = 4\pi \frac{\sigma_e^2}{c^2}. \]  

For the mass inside a sphere with a radius \( r \) we have
\[ M(r) = \frac{2\sigma_e^2}{G} r, \]
and the equation of hydrostatic equilibrium of the plasma is
\[ \frac{\mathcal{W}T}{\rho} \frac{d\rho}{dr} = -\frac{2\sigma_e^2}{r}. \]

For the boundary condition \( \rho(r_0) = \rho_0 \) we obtain the plasma density from equation (91) as
\[ \rho(r) = \rho_0 \left( \frac{r}{r_0} \right)^{-s}, \quad s = 2\sigma_e^2 \frac{\mathcal{W}T}{\rho}. \]

The angles \( \hat{\alpha}_2 \) and \( \hat{\alpha}_3 \) are calculated using formula (64), so we have
\[ \hat{\alpha}_2 = \frac{2\pi^2 \sigma_e^2 c^2}{c^2 \omega^2(s+1) m_p} K_e \Gamma \left( \frac{s+1}{2} \right) \rho_0 \left( \frac{r_0}{b} \right)^{s}, \]
\[ \hat{\alpha}_3 = -\frac{K_e}{\omega^2 m_p} \rho_0 \frac{r_0}{b} \int_0^\infty \frac{dz}{\left( z^2 + b^2 \right)^{s/2+1}} \]
\[ = -\frac{K_e \sqrt{\pi}}{\omega^2 m_p} \Gamma \left( \frac{s+1}{2} \right) \rho_0 \left( \frac{r_0}{b} \right)^{s}. \]

The ratio of these angles is equal to
\[ \frac{\hat{\alpha}_2}{\hat{\alpha}_3} = \frac{\pi^2}{c^2} \frac{\Gamma \left( \frac{s+1}{2} \right) \Gamma \left( \frac{s+1}{2} \right)}{\Gamma \left( \frac{s+1}{2} \right) \Gamma \left( \frac{s+1}{2} \right)}. \]

Under the condition \( s \ll 1 \) which corresponds to \( 2\sigma_e^2 \ll \mathcal{W}T \), the expressions are simplified to
\[ \hat{\alpha}_2 = 2\pi^2 \frac{\sigma_e^2}{c^2 \omega^2 m_p} K_e \rho_0 \left( \frac{r_0}{b} \right)^{s}, \]
\[ \hat{\alpha}_3 = -\frac{1}{\mathcal{W}T \omega^2 m_p} \rho_0 \left( \frac{r_0}{b} \right)^{s}, \]
\[ \frac{\hat{\alpha}_2}{\hat{\alpha}_3} = \frac{2\mathcal{W}T}{c^2}. \]

If relativistic plasma is present in a galaxy cluster, for example in jets from active galactic nuclei, the plasma general relativistic effects may be larger than the effects of the non-uniform plasma. If the distribution of plasma is not spherically symmetric, there may be a distribution of plasma with density gradient opposite to the direction of the gravitational force, for example, in the presence of rotation. In this situation the angles \( \hat{\alpha}_2 \) and \( \hat{\alpha}_3 \) may be of the same sign.

Let us estimate the Thomson optical depth for Thomson scattering of photons in the case of gravitational lensing in a plasma, and the depth of the bremsstrahlung absorption. For the Thomson optical depth \( \tau \) we have
\[ \tau \approx N \sigma_e L, \]
where \( N \) is the electron concentration, \( \sigma_e = 6.65 \times 10^{-25} \text{cm}^2 \) is the Thomson scattering cross-section, and \( L \) is the characteristic distance for scattering. So the optical depth will be \( \tau < 1 \) for \( N \leq 0.5 \times 10^4 (10^2 \text{pc}/L) \text{cm}^{-3} \).

For the optical depth for the absorption of radiation in the case \( \omega \gg \omega_e \) we have (Ginzburg 1970)
\[ \tau = \frac{10^{-2} N^2}{T^{3/2} \nu} \left[ 17.7 + \ln \frac{T^{3/2}}{\nu} \right] L. \]

So the optical depth will be \( \tau < 1 \) for
\[ N \simeq 0.2 \times 10^3 \left( \frac{T}{10^8 \text{K}} \right)^{3/4} \nu \left( \frac{10^2 \text{pc}}{L} \right)^{1/2}. \]

7 CONCLUSIONS

In an observation of images of a lensed source in the presence of a plasma, the following features may be observed.

(1) The spectra of the two images may be different on the long-wave side because of different plasma properties along the trajectories of light rays forming the images.

(2) The extended image may have different spectra in different parts of the image, with a maximum of the spectrum shifting to the long-wave side in regions with a larger deflection angle.

(3) The presence of plasma may influence the timing effects in binary relativistic systems, similar to the double pulsar system J0737-3039A–J0737-3039B, and may induce a spectral dependence of the properties of fluctuations of the microwave background radiation.

In summary, we also make the following points.

(i) We have carried out calculations for models with a non-uniform plasma distribution: singular and non-singular isothermal spheres; hot gas inside the gravitational field of a black hole; and a cluster of galaxies.

(ii) For different gravitational lens models we have compared the corrections to the vacuum lensing due to the effect of gravity in the plasma, and due to plasma inhomogeneity. We have shown that the gravitational effect could be detected in the case of a hot gas in the gravitational field of a galaxy cluster.

(iii) We have made estimates of the optical depth due to Thomson scattering and free–free absorption during the process of gravitational lensing in a plasma.

ACKNOWLEDGMENTS

This work was partially supported by RFBR grants 08-02-00491 and 08-02-90106, the RAN Program ‘Formation and evolution of stars and galaxies’ and Russian Federation President Grant for Support of Leading Scientific Schools NSh-3458.2010.2. The work of OYuT was also partially supported by the Dynasty Foundation and Russian Federation President Grant for Support of Young Scientists MK-8696.2010.2.

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APPENDIX A: INVESTIGATION OF SPECTRAL BEHAVIOUR IN FIGS 2 AND 3

The quantity \((\mu^{\varphi}_{+,\varphi}/\mu_+)/(\mu^{\varphi}_{-,\varphi}/\mu_-)\) under \(\omega \gg \omega_{\text{e}}\) has the following form:

\[
\left(\frac{\mu^{\varphi}_{+,\varphi}}{\mu_+}\right) / \left(\frac{\mu^{\varphi}_{-,\varphi}}{\mu_-}\right) = 1 + \frac{1}{2\omega^2(y^2 + 4)} \times \left[\omega_{\text{e}}^2(y^2 + 2 - y\sqrt{y^2 + 4}) - \omega_{\text{e}}^2(y^2 + 2 + y\sqrt{y^2 + 4})\right].
\]  

(A1)

We see the following.

(1) If the rays forming the primary image go through a greater density than the rays forming the secondary image, that is \(\omega_{\text{e}}^+ > \omega_{\text{e}}^\text{crit}\), then \((\mu^{\varphi}_{+,\varphi}/\mu_+)/(\mu^{\varphi}_{-,\varphi}/\mu_-)\) can be both larger and less than unity (for large \(\omega\)) depending on the angular position of the source \(\gamma\). In Figs 2 and 3 we see both situations (the solid lines).

(2) If the rays forming the secondary image go through a greater density than the rays forming the primary image, that is \(\omega_{\text{e}}^+ < \omega_{\text{e}}^\text{crit}\), then \((\mu^{\varphi}_{+,\varphi}/\mu_+)/(\mu^{\varphi}_{-,\varphi}/\mu_-)\) can only be less than unity (see Figs 2 and 3, the dashed lines).

The critical value of \(\gamma\) separating the two cases of the location of the solid line (below unity and above unity) equals

\[
\gamma_{\text{crit}} = \frac{\sqrt{-8 + 6\sqrt{2}}}{2} \approx 0.348
\]  

(A2)

for \(\omega_{\text{e}}^+ = 2\omega_{\text{e}}^\text{crit}\) (the solid lines in Fig. 2), and

\[
\gamma_{\text{crit}} = \frac{\sqrt{-200 + 110\sqrt{10}}}{10} \approx 1.216
\]  

(A3)

for \(\omega_{\text{e}}^+ = 10\omega_{\text{e}}^\text{crit}\) (the solid lines in Fig. 3).

APPENDIX B: CALCULATION OF THE INTEGRAL \(\text{Int}2\)

By substituting \(1/\sqrt{x^2 + b^2} = x\) the integral \(\text{Int}2\) is reduced to

\[
\text{Int}2 = \frac{1}{b} \int_0^1 \frac{x}{\sqrt{1 - x^2}} e^{bx} \, dx.
\]  

(B1)

Integration can be performed with the use of Gradshteyn & Ryzhik (1965):

\[
\text{Int}2 = \frac{1}{b^2} \left(1 + \frac{\pi}{2} \left[I_1(B/b) + L_1(B/b)\right]\right).
\]  

(B2)

where \(I_1\) is the Bessel function of the first kind and \(L_1\) is the Struve function.

Under condition \(B/b \ll 1\) which corresponds to \(GM_0/b \ll \Re T\), the integral \(\text{Int}2\) is reduced to

\[
\text{Int}2 = \frac{1}{b^2} \left(1 + \frac{\pi}{4} \frac{GM_0}{b^2\Re T}\right).
\]  

(B3)

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