Some Examples of Bicrossed Products with the Rapid Decay Property

Hua Wang

Abstract

We consider bicrossed products obtained by twisting compact semi-direct products by a suitable finite subgroup. Under some restriction, we give a practical criterion for the discrete dual of such bicrossed products to have the rapid decay property (property (RD)). Using this theory, we construct some examples of discrete quantum groups with (RD) but do not grow polynomially. Further examples that do not satisfy the hypothesis of our main result are also constructed.

1 Introduction

The rapid decay property, which shall be referred as property (RD), or simply (RD) in this paper, be it for discrete groups (Jolissaint [7]) or discrete quantum groups (Vergnioux [13], see also Bhowmick, Voigt & Zacharias [1] for a non-unimodular variation), is an interesting approximation property, partly due to its connection with $K$-theory, $L^2$-homology et cetera. On the other hand, from its introduction by Kac ([8]) in the 1960s to the culmination work of Vaes & Vainerman ([12]) in the general setting of locally compact groups, the bicrossed product construction, as a systematic procedure to produce large classes of non-commutative and non-cocommutative quantum groups, plays an important role in the development of the still rapidly growing field of quantum groups. In the work of Fima & the author [5], we give a systematic theoretical study of property (RD) for (the discrete dual of) bicrossed products $\Gamma \bowtie \bowtie G$ of a matched pair $(\Gamma \bowtie G)$ of a compact group $G$ with a discrete one $\Gamma$. The main results there are that $\hat{\Gamma} \bowtie \bowtie G$ has (RD) if and only if both $\Gamma$ and $\hat{G}$ have (RD) in a compatible manner (Theorem 2.8 below has a more precise formulation). One key component in formulating these results is the notion of a matched pair of length functions, which is intimately related to the fusion structure of irreducible representations of the compact quantum group $\Gamma \bowtie \bowtie G$. We point out that one major drawback of these theoretical results is that, in practice, due to the complexity of this notion, it can be quite difficult to verify that the witnessing length functions for the relevant approximation properties like (RD) or polynomial growth are matched or not. This is the main reason for the lack of concrete examples in [5].

This paper picks up where [5] left off, namely, the construction of examples, to support the theory developed there. We work in a particular class of bicrossed products already considered in Fima, Mukherjee & Patri [4], which we now briefly describe and refer to § 3 for more details. We start from a nontrivial semidirect product $G \rtimes \Gamma$ with $G$ a compact group and $\Gamma$ a discrete one, as well as a finite subgroup $\Lambda$ of $\Gamma$ that does not lie entirely in the center of $\Gamma$. One can then show that the pair $(\Gamma, G \rtimes \Lambda)$ is matched in a natural way (again see § 3), where the underlying left action $\alpha^\Lambda : \Gamma \curvearrowleft G \rtimes \Lambda$ and right action $\beta^\Lambda : \Gamma \curvearrowright G \rtimes \Lambda$ are both nontrivial, hence yields a non-commutative, non-cocommutative bicrossed product $\Gamma \bowtie \bowtie (G \rtimes \Lambda)$. We restrict our attention to the study of (RD) and the closely related notion of polynomial growth of the dual $\Gamma \bowtie \bowtie (G \rtimes \Lambda)$. 1
This has the advantage that the underlying representation theory can be divided into two well-understood parts: the first part is the representation theory of the bicrossed products studied in [5], which, modulo the relations determined by the matched pair \((\Gamma, G \rtimes \Lambda)\), reduces to the second part concerning the representation theory of the semidirect product \(G \rtimes \Lambda\), which is a special case studied in [14]. Moreover, the finiteness of the group \(\Lambda\) provides a way to control the length functions, which eventually leads to a characterization result that gets rid of the technically involved notion of matched pair of length functions (see Definition 2.6), this is the contents of Theorem 1.2.

We now state the main theoretical results of this paper consisting of the following two theorems.

**Theorem 1.1.** In the above settings. If there is a \(\Gamma\)-invariant length function \(l_{\hat{G}}\) on \(\hat{G}\), and a \(\beta\Lambda\)-invariant length function \(l_\Gamma\) on \(\Gamma\), such that both \((\hat{G}, l_{\hat{G}})\) and \((\Gamma, l_\Gamma)\) have polynomial growth (resp. (RD)), then the dual of the bicrossed product, namely \(\hat{\Gamma} \bowtie \bowtie (G \rtimes \Lambda)\), also has polynomial growth (resp. (RD)).

**Theorem 1.2.** Let \(\tilde{\tau} : \Gamma \to \text{Out}(G)\) be the composition of \(\tau : \Gamma \to \text{Aut}(G)\) with the canonical projection \(\text{Aut}(G) \to \text{Out}(G)\). If \(\text{Image}(\tilde{\tau})\) is finite, then the following are equivalent:

1. \(\hat{\Gamma} \bowtie \bowtie (G \rtimes \Lambda)\) has polynomial growth (resp. (RD));
2. both \(\Gamma\) and \(\hat{G}\) have polynomial growth (resp. (RD)).

Theorem 1.1 provides a sufficient condition for \(\hat{\Gamma} \bowtie \bowtie (G \rtimes \Lambda)\) to have (RD) (resp. polynomial growth). Albeit still non-trivial, this condition is much easier to check in some concrete cases than the condition of matched pair of length functions used in Theorem 2.7 and Theorem 2.8, which we shall exploit in § 8.

Theorem 1.2 is a stronger result, provided that the assumption there is satisfied, as it completely gets rid of the difficult problem of checking the compatibility of the relevant length functions. This leads to some concrete examples of \(\Gamma \bowtie \bowtie (G \rtimes \Lambda)\) having (RD) but not polynomial growth (§ 7). However, it has the limitation of not covering many other interesting examples, as will be demonstrated again in § 8. It is in the opinion of the author that a more unified approach is desired that relaxes the assumption of Theorem 1.2 while still at least includes the examples in § 8.

This paper is organized as follows. In the long preliminary § 2, we provide all the needed background on the relevant representation theory, which can be rather involved. Besides the purpose of making the paper reasonably self-contained, it also serves to fix the notation for our later use. In § 3, we briefly give a conceptual and still elementary explanation of how one can obtain non-trivial matched pair by twisting semidirect products by a finite subgroup, followed by § 4 which fixes more notation to help us deal with the arising subtleties. After the stage is set, we prove the main theorems in § 5 and § 6. Finally, using these tools, in § 7, we illustrate how to construct the our desired examples using Theorem 1.2, and further examples are considered in § 8, showing the limitation of the procedure in § 7.

We finish this introduction by making some convention.

### 1.1 Convention and notation

Regarding the terminologies and notation of compact quantum groups, our choices are mainly in consistent with the works of Neshveyev & Tuset [11] and Woronowicz [17].
All representations and projective representations of (quantum) groups are assumed to be unitary and finite dimensional unless otherwise stated. We view discrete quantum groups as the dual of compact ones.

When $G$ is a compact (quantum) group, the set of equivalence classes of irreducible representations of $G$ is denoted by $\text{Irr}(G)$, and the trivial representation of $G$ by $\varepsilon_G$, or simply by $\varepsilon$ when $G$ is clear from context. If $u$ is a representation of $G$, then the equivalence class of $u$ is denoted by $[u]$. The dual of $G$, as a discrete quantum group, is denoted by $\hat{G}$. The tensor product of two representations $u$ and $v$ of $G$ is denoted by $u \times v$, and the tensor product of classes $x, y \in \text{Irr}(G)$ by $x \otimes y$. The space of operators that intertwine $u$ and $v$ is denoted by $\text{Mor}_G(u, v)$.

When $H, K$ are Hilbert spaces, the notation $B(H, K)$ means the Banach space of all bounded linear operators from $H$ to $K$. The space $B(H, H)$ is denoted by $B(H)$.

For a topological group $G$, we use $\text{Inn}(G)$ to denote the normal subgroup of $\text{Aut}(G)$ consisting of inner automorphisms, where $\text{Aut}(G)$ is the group of topological automorphisms of $G$. The outer automorphism group $\text{Out}(G)$ is of course the quotient group $\text{Aut}(G)/\text{Inn}(G)$.

1.2 Acknowledgement

This work is supported by the ANR project ANCG (No. ANR-19-CE40-0002). The author would also like to thank Professor Pierre Fima for several useful discussions.

2 Preliminaries

As preliminaries, we briefly describe the relevant results of ([5]) and ([14]), which form the theoretical framework of this paper. To make the presentation more self-contained, we also briefly recall the bicrossed product construction as presented in [4], and property (RD) in the discrete quantum group setting ([13], see also [1]).

2.1 Rapid decay and polynomial growth for discrete quantum groups of Kac type

As the bicrossed products of matched pair of groups considered here are always of Kac type, we only recall the relevant formulation in this simplified setting.

Let $H$ be a compact quantum group of Kac type and $\hat{H}$ its dual (as a discrete quantum group of Kac type), $\ell^\infty(\hat{H})$ be the $\ell^\infty$-direct sum $\bigoplus_{x \in \text{Irr}(H)} B(H_x)$, and $c_0(\hat{H})$ the ideal of the algebraic direct sum, whose $C^*$-completion is denoted by $c_0(\hat{H})$. A length function on $\hat{H}$ is a map $l : \text{Irr}(H) \to [0, \infty]$ such that (i) $l([\varepsilon_H]) = 0$; (ii) $l(x) = l(\overline{x})$ for all $x \in \text{Irr}(H)$; (iii) $l(z) \leq l(x) + l(y)$ if $z \subseteq x \otimes y$. In the following, $p_x$ denotes the central projection of $c_0(\hat{H})$ that corresponds to the block $B(H_x)$. Take $a \in c_c(\hat{H})$, the Fourier transform $F_{\hat{H}}(a)$ of $a$ is defined to be

$$F_{\hat{H}}(a) = \sum_{x \in \text{Irr}(H)} (\dim x) \left[\text{Tr}_{H_x} \otimes \text{id}(u^x(a^* p_x \otimes 1))\right] \in \text{Pol}(\hat{H})$$

where $u^x$ is a unitary representation on $H_x$, chosen (and fixed) in the class $x$. The Sobolev-0-norm $\|a\|_{\hat{H}, 0}$ is determined by

$$\|a\|_{\hat{H}, 0}^2 = \sum_{x \in \text{Irr}(H)} (\dim x) \text{Tr}_{H_x}(a^* a^* p_x).$$
Given a length function $l$ on $\hat{H}$, consider the element $L = \sum_{x \in \text{Irr}(\hat{H})} l(x) p_x$ that is affiliated (in the sense of Woronowicz [16]) to $c_0(\hat{H})$. Denote the spectral projection of $L$ corresponding to the interval $[n, n + 1]$ by $q_n$. The pair $\left(\hat{H}, l\right)$ is said to have

- the rapid decay property (RD), if there exists a polynomial $P \in \mathbb{R}[X]$, such that for all $k \in \mathbb{N}$ and $a \in q_k c_0(\hat{H})$, we have $\|a\|_{C(\hat{H})} \leq P(k)\|a\|_{H,0}$, where $C(\hat{H})$ is the $C^*$-algebra of the reduced version of $\hat{H}$;

- polynomial growth, if there exists a polynomial $P \in \mathbb{R}[X]$, such that for all $k \in \mathbb{N}$,

\[ \sum_{x \in \text{Irr}(\hat{H}), k \leq l(x) < k + 1} (\dim x)^2 \leq P(k). \]

We say that $\hat{H}$ has (RD) (resp. polynomial growth), if there exists a length function $l$ on $\hat{H}$ such that the pair $\left(\hat{H}, l\right)$ has (RD) (resp. polynomial growth).

From the work [13], it is known that the pair $\left(\hat{H}, l\right)$ having polynomial growth implies it having (RD), and the converse holds if $\hat{H}$ is co-amenable; it is also shown there that the duals of compact connected real Lie groups have polynomial growth.

### 2.2 Bicrossed products and (RD)

#### 2.2.1 Construction of the bicrossed product

We briefly describe the bicrossed product construction associated to a matched pair of classical groups, which is the type of examples to be considered in this work, and refer to [4] or [15, Chapter I] for details and more background.

A matched pair of groups $(\Gamma, G)$ consists of a discrete group $\Gamma$, a compact (compact means quasi-compact and Hausdorff in this paper) group $G$, together with two actions $\alpha$ and $\beta$, where $\alpha : \Gamma \curvearrowright G$ is a right action of the group $\Gamma$ on the underlying set of $G$, and $\beta : \Gamma \curvearrowright G$ a right action of the group $G$ on the underlying set of $\Gamma$, such that

\[ \gamma \in \Gamma, g, h \in G, \implies \alpha_\gamma(gh) = \alpha_\gamma(g)\alpha_{\beta_\gamma(h)}(h), \quad (2.1a) \]

\[ r, s \in \Gamma, g \in G, \implies \beta_\gamma(rs) = \beta_\gamma(r)\beta_{\alpha_\gamma(g)}(s), \quad (2.1b) \]

\[ \gamma \in \Gamma, g \in G, \implies \alpha_\gamma(e_G) = e_G, \beta_\gamma(e_G) = e_\Gamma, \quad (2.1c) \]

where $e_\Gamma$ (resp. $e_G$) denotes the neutral element in the group $\Gamma$ (resp. $G$). It is well-known that $(\Gamma, G)$ is a matched pair of groups if and only if $\Gamma$ and $G$ can be simultaneously identified with (topological) subgroups of a locally compact group $H$ in such a way that $H = \Gamma G$, $\Gamma \cap G = \{e_H\}$. In this identification, the underlying actions $\alpha, \beta$ are characterized by the relation $\gamma g = \alpha_\gamma(g)\beta_\gamma(\gamma)$. One also proves that the actions $\alpha, \beta$ are automatically continuous, and $\alpha$ preserves the Haar measure of $G$.

Given a matched pair of groups $(\Gamma, G)$ with underlying actions $\alpha$ and $\beta$. To simplify notation, we often denote the right action $\beta$ by a dot, thus writing $\gamma \cdot g$ instead of $\beta_\gamma(g)$. In this spirit, the orbit of $\gamma \in \Gamma$ under $\beta$ is written as $\gamma \cdot G$. By the continuity of the action $\beta$, we see that the set $\mathcal{G}_{r,s} := \{g \in G \mid r \cdot g = s\}$ is clopen, thus its characteristic function $\nu_{r,s}$ is an element of $C(G)$. It is obvious that $\mathcal{G}_{r,s} \neq \emptyset$ if and only if $r, s$ belong to the same orbit, and $\mathcal{G}_\gamma := \mathcal{G}_{\gamma, \gamma}$ is the isotropy subgroup of $G$ fixing $\gamma \in \Gamma$. As $\mathcal{G}_\gamma$ is of finite index in $G$ (since $G$ compact and $G_\gamma$ open), each $\beta$-orbit is finite, and we denote the set of all $\beta$-orbits by $\text{Orb}_\beta$. 

4
The action \( \alpha : \Gamma \curvearrowright G \) induces a \( C^* \)-dynamical system \((C(G), \Gamma, \hat{\alpha})\), where \( C(G) \) is the algebra of continuous functions on \( G \), and \( \hat{\alpha} : \Gamma \to \text{Aut}(C(G)) \) the group homomorphism \( \gamma \mapsto \left\{ \alpha^*_\gamma \right\} : \varphi \mapsto \varphi \circ \alpha^*_\gamma \). On the reduced crossed product \( \Gamma \ltimes C(G) \), there exists a unique compact quantum group structure whose comultiplication \( \Delta \) satisfies the following conditions:

- \( \Delta \) restricts to the comultiplication on \( C(G) \) that is induced by the multiplication on \( G \);
- \( \Delta(u) = \sum_{\mu, \gamma \in \Gamma} u_{\gamma, \mu} \otimes u_\mu \), where we use \( u_r \) to denote the copy of \( r \in \Gamma \) in the crossed product \( \Gamma \ltimes C(G) \).

We denote the compact quantum group \((\Gamma \ltimes C(G), \Delta)\) by \( \Gamma \bowtie G \), or simply \( \Gamma \bowtie G \), dropping the actions \( \alpha \) and \( \beta \) when the context makes them clear, and call this compact quantum group the bicrossed product of the matched pair of groups \((\Gamma, G)\). The quantum group \( \Gamma \bowtie G \) is always of Kac type, so \( \S 2.1 \) applies to \( \Gamma \bowtie G \).

Throughout the rest of \( \S 2.2 \), we fix a matched pair of groups \((\Gamma, G)\). We denote the underlying left action by \( \alpha : \Gamma \curvearrowright G \), and the right action by \( \beta : \Gamma \curvearrowright G \).

### 2.2.2 Representation theory of the bicrossed product

The representation theory of \( \Gamma \bowtie G \) that we will need includes a classification of irreducible representations, the description of the conjugate representation (which is the contragredient representation since \( \Gamma \bowtie G \) is of Kac type) in terms of this classification, and most importantly, the fusion rules. The results summarized here are first obtained in \([5] \), with the exception of the simplified version of the fusion rules, Theorem 2.4, which is given in \([15, \text{Theorem I.4.9}] \). We refer the reader to \([5, \S 3] \) and \([15, \S 1.2 \& \S 1.4] \) for complete proofs of the results below. Motivational considerations of the classification of irreducible representations can be found in \([15, \S 1.3] \).

We will formulate our results using the notion of \( \theta \)-representations, which we now briefly describe. Take an orbit \( \theta \in \text{Orb}_\beta \). By an \( \theta \)-representation of \( G \), we mean a unitary operator \( U \) in \( B(\ell^2(\theta)) \otimes B(H) \otimes C(G) \), where \( H \) is some finite dimensional Hilbert space, such that if we write

\[
U = \sum_{r,s \in \theta} e_{r,s} \otimes u_{r,s}
\]

with \( (e_{r,s}) \) the canonical matrix units of \( B(\ell^2(\theta)) \) and \( u_{r,s} \in B(H) \otimes C(G) \), then (recall that \( v_{r,s} \) is the characteristic function of \( G_{r,s} \))

\[
\forall r, s \in \theta, \quad u^*_{r,s} u_{r,s} = u_{r,s} u^*_{r,s} = \text{id}_H \otimes v_{r,s} \in B(H) \otimes C(G).
\]

Viewing each \( u_{r,s} \) as a map from \( G \) into \( B(H) \), then the restriction \( u_{r,r G_r} \) is a unitary representation of \( G_r \) on \( H \). We say that \( U \) is \( \theta \)-irreducible if \( u_{r,r G_r} \) is irreducible for some \( r \in \theta \), which is equivalent to \( u_{r,r G_r} \) being irreducible for all \( r \in \theta \). If \( W = \sum_{r,s \in \theta} v_{r,s} \otimes w_{r,s} \) is another \( \theta \)-representation of \( G_r \), say on \( \ell^2(\theta) \otimes K \), we say that \( U \) and \( W \) are \( \theta \)-equivalent if there exists a unitary operator \( T \in B(H, K) \) such that \( T \) intertwines \( u_{r,r G_r} \) and \( v_{r,r G_r} \), for some \( r \in \theta \), which is also equivalent to \( T \) intertwines \( u_{r,r G_r} \) and \( v_{r,r G_r} \), for all \( r \in \theta \).

The \( \theta \)-representations can be used to produce representations of the bicrossed product \( \Gamma \bowtie G \) in the following way. If \( U \) is an \( \theta \)-representation as above, then the unitary operator

\[
\mathcal{R}_\theta(U) := \sum_{\gamma \in \theta} \left( \sum_{\gamma \in \theta} e_{r,\gamma} \otimes \text{id}_H \otimes u_\gamma \right) U
\]

is a unitary representation of \( \Gamma \bowtie G \) on \( \ell^2(\theta) \otimes H \). For \( \theta \in \text{Orb}_\beta \), denote the set of \( \theta \)-equivalence classes of \( \theta \)-irreducible representations by \( \text{Irr}_\theta(\Gamma \bowtie G) \).
Theorem 2.1 (Classification of irreducible representations). The map
\[ R: \coprod_{\sigma \in \text{Orb}_\beta} \text{Irr}_\sigma(G) \to \text{Irr}(\Gamma \rtimes G) \]
\[ [U] \in \text{Irr}_\sigma(G) \mapsto [R_\sigma(U)] \tag{2.3} \]
is a well-defined bijection.

The \( \sigma \)-irreducible representations are in fact just suitable copies of induced representations of irreducible representations of \( G \), for any \( \gamma \in \sigma \). The precise formulation of this correspondence is given by the following proposition.

Proposition 2.2. Let \( \sigma \) be a \( \beta \)-orbit, \( \gamma \in \sigma \), and \( u: G_\gamma \to B(H) \) a finite dimensional unitary representation of \( G_\gamma \). Take \( \sigma_\mu \in G_{\gamma, \mu} \) for each \( \mu \in \sigma \) with \( \sigma_\mu = e_G \) and define
\[ u_{r,s}(g) = \begin{cases} u(\sigma_\mu g \sigma_s^{-1}), & \text{if } g \in G_{r,s}; \\ 0, & \text{if } g \notin G_{r,s}. \end{cases} \]
Then the operator \( U = \sum_{r,s \in \sigma} e_{r,s} \otimes u_{r,s} \) is an \( \sigma \)-representation on \( \ell^2(\sigma) \otimes H \), and \( U \simeq \text{Ind}_{G_{\gamma}}^{G}(u) \).

Moreover, for any \( \gamma \in \sigma \), the map
\[ \Phi_\gamma: \text{Irr}(G_\gamma) \to \text{Irr}_\sigma(G) \]
\[ [u] \mapsto \sum_{r,s \in \sigma} e_{r,s} \otimes u_{r,s} \tag{2.4} \]
where \( u_{r,s} \) is defined above, is a well-defined (i.e. independent of the representative \( u \) of the class \( [u] \) and independent of the choices of \( \sigma_\mu \in G_{\gamma, \mu} \)) bijection.

For an \( \sigma \)-representation \( U = \sum_{r,s \in \sigma} e_{r,s} \otimes u_{r,s} \) on \( \ell^2(\sigma) \otimes H \), one can show that in fact \( u_{r,s} \in B(H) \otimes \text{Pol}(G) \). Let \( j: B(H) \to B(H) \) be the canonical conjugate linear anti-isomorphism, where \( H \) is the conjugate Hilbert space of \( H \), and let \( S \) be the antipode of the Hopf-\( \ast \)-algebra \( \text{Pol}(G) \). The matched pair relations (2.1a), (2.1b) and (2.1c) imply that \( \sigma^{-1} = \{ \gamma^{-1} \mid \gamma \in \sigma \} \) is also a \( \beta \)-orbit. The unitary
\[ U^\dagger := \sum_{r,s \in \sigma} e_{s^{-1}, r^{-1}} \otimes (\text{id} \otimes \alpha_s^{-1}) (j \otimes S) u_{r,s} \in B(\ell^2(\sigma^{-1})) \otimes B(\overline{H}) \otimes C(G) \]
is an \( \sigma^{-1} \)-representation of \( G \) on \( \ell^2(\sigma^{-1}) \otimes \overline{H} \), which is equivalent to the conjugate representation of \( U \).

Theorem 2.3 (Conjugate of irreducible representations). The operation \( U \mapsto U^\dagger \) preserves \( \sigma \)-equivalence classes, and induces a well defined involution on \( \coprod_{\sigma \in \text{Orb}_\beta} \text{Irr}_\sigma(G) \), and the classification map \( R: \coprod_{\sigma \in \text{Orb}_\beta} \text{Irr}_\sigma(G) \to \text{Irr}(\Gamma \rtimes G) \) preserves involution, where the involution on the latter set is given by conjugation. Moreover, if \( \gamma \in \sigma \in \text{Orb}_\beta \), and \( U \) is an \( \sigma \)-representation such that \( U \simeq \text{Ind}(u) \) for some irreducible representation \( u \) of \( G_{\gamma} \), then \( \alpha_{\gamma^{-1}} \) restricts to an isomorphism of compact groups from \( G_{\gamma^{-1}} \) onto \( G_{\gamma} \), and \( U^\dagger \simeq \text{Ind}_{G_{\gamma^{-1}}}^{G_{\gamma}}(u \circ \alpha_{\gamma^{-1}}) \).
To describe the fusion rules of $\Gamma \bowtie G$, we need a certain twisted tensor product which we now describe. For $i = 1, 2, 3$, let $\mathcal{O}_i \in \text{Orb}_\beta$, suppose $U_i = \sum_{r,s \in \mathcal{O}_i} e_{r,s} \otimes u_{r,s}^{(i)}$ is an $\mathcal{O}_i$-representation on $\ell^2(\mathcal{O}_i) \otimes H_i$ and $W_i = \mathbf{R}_\mathcal{O}_i(U_i)$. We will describe $\dim \text{Mor}_{\Gamma \bowtie G}(W_3, W_1 \times W_2)$, which is slightly more general than the fusion rules (for which we only need the case where $U_i$ is $\mathcal{O}_i$-irreducible for all $i$).

In $\Gamma$, the product set
\[ \mathcal{O}_1 \mathcal{O}_2 = \{ \gamma_1 \gamma_2 \mid (\gamma_1, \gamma_2) \in \mathcal{O}_1 \times \mathcal{O}_2 \} \]
is a disjoint union of $\beta$-orbits. For $\gamma \in \mathcal{O}_1 \mathcal{O}_2$, put
\[ K_{\mathcal{O}_1, \mathcal{O}_2}^\gamma = \text{Vect}\{ \delta_{\gamma_1} \otimes \delta_{\gamma_2} \mid (\gamma_1, \gamma_2) \in \mathcal{O}_1 \times \mathcal{O}_2 \text{ with } \gamma_1 \gamma_2 = \gamma \} \subseteq \ell^2(\mathcal{O}_1) \otimes \ell^2(\mathcal{O}_2), \]
and define a map $U_1 \times_\gamma U_2 : G_\gamma \to B(K_{\mathcal{O}_1, \mathcal{O}_2}^\gamma)$ by
\[ (U_1 \times_\gamma U_2)(g) := \sum_{r_1,s_1 = r_2,s_2 = \gamma} (e_{r_1,s_1} \otimes e_{r_2,s_2}) \otimes u_{r_1,s_1}^{(1)}(\alpha_{r_2}(g)) \otimes u_{r_2,s_2}^{(2)}(g). \]

One checks that $U_1 \times_\gamma U_2$ is a unitary representation of $G_\gamma$, called the tensor product of $U_1$ and $U_2$ twisted by $\gamma$. We then have the following theorem which includes the fusion rules of $\Gamma \bowtie G$ as a special case.

**Theorem 2.4 (Fusion rules).** Using the above notations, the following hold.

(i) If $\mathcal{O}_3 \cap \mathcal{O}_1 \mathcal{O}_2 = \emptyset$, then $\dim \text{Mor}(W_3, W_1 \times W_2) = 0$;

(ii) otherwise $\mathcal{O}_3 \subseteq \mathcal{O}_1 \mathcal{O}_2$, then
\[ \dim \text{Mor}(W_3, W_1 \times W_2) = \dim \text{Mor}_{\mathcal{O}_3} \left( u_{\gamma \gamma,1\gamma}^{(3)} | G_\gamma, U_1 \times_\gamma U_2 \right) \]
\[ = \dim \text{Mor}_{G_\mu} \left( u_{\mu,\mu}^{(3)} | G_\mu, U_1 \times_\mu U_2 \right) \]
for all $\gamma, \mu \in \mathcal{O}_3$.

**Remark 2.5.** As mentioned at the beginning of § 2.2.2, Theorem 2.4 simplifies our previous result [5, Theorem 3.2].

### 2.2.3 Permanence of (RD) and polynomial growth

Intuitively speaking, the (dual of) the bicrossed product $\Gamma \bowtie G$ has (RD) (resp. polynomial growth) if and only both $\Gamma$ and $\hat{G}$ have the same property in a compatible manner. Here the precise formulation of this compatibility requires the notion of matched pair of length functions, which arise naturally from the representation theory of $\Gamma \bowtie G$ as described in § 2.2.2. Again, the following results are first obtained in [5] (see also [15, Chapter I] for a more polished treatment).

**Definition 2.6.** Let $l_\Gamma$ be a length function on $\Gamma$, $l_{\hat{G}}$ a length function on $\hat{G}$. The pair $(l_\Gamma, l_{\hat{G}})$ is said to be matched, if there are a family of maps
\[ \{ l_\sigma : \text{Irr}_\sigma(G) \to \mathbb{R}_{\geq 0} \mid \sigma \in \text{Orb}_\beta \} \]
indexed by $\text{Orb}_\beta$, such that the following conditions are satisfied:

- $l_{(\sigma \gamma)}([e_G]) = 0$;
• for all \( O \in \text{Orb}_\beta \) and \([U] \in \text{Irr}_\sigma(G)\), we have \( l_\sigma([U]) = l_{\sigma^{-1}}([U^\dagger])\);

• for \( i = 1, 2, 3 \), let \( O_i \in \text{Orb}_\beta \), and \([U_i] \in \text{Irr}_{O_i}(G)\), with \( U_i = \sum_{r,s \in O_i} e_{r,s} \otimes u_{r,s}^{(i)} \) being an \( O_i \)-irreducible \( O_i \)-representation of \( G \) on \( \ell^2(O_i) \otimes \mathcal{H}_i \), if

\[
\dim \text{Mor}_{G,\gamma} \left( u_{\gamma,\gamma}^{(3)}, U_1 \times \gamma, U_2 \right) \neq 0
\]

for some (hence for all, by Theorem 2.4) \( \gamma \in O_3 \), then

\[
l_{O_3}([U_3]) \leq l_{O_1}([U_1]) + l_{O_2}([U_2])
\]

• for all \([U] \in \text{Irr}_{\{e\}}(G) = \text{Irr}(G)\), we have \( l_\hat{G}([U]) = l_{\{e\}}([U])\);

• for all \( O \in \text{Orb}_\beta \), the image \( l_\Gamma(O) \) is the singleton \( l_\sigma([\varepsilon_O]) \), where \( \varepsilon_O := \sum_{r,s \in O} e_{r,s} \otimes v_{r,s} \) is the trivial \( O \)-representation (in particular, \( l_\Gamma \) is \( \beta \)-invariant).

If this is the case, we say that the family \( \{l_O \mid O \in \text{Orb}_\beta\} \) is **affording** for the matched pair \((\Gamma, l_\hat{G})\).

**Theorem 2.7** (Permanence of polynomial growth). The following are equivalent:

(i) \( \hat{\Gamma} \triangleleft \hat{G} \) has polynomial growth;

(ii) there exists a matched pair of length functions \((l_\Gamma, l_{\hat{G}})\), such that both \((\hat{\Gamma}, l_{\hat{G}})\) and \((\Gamma, l_\Gamma)\) have polynomial growth.

**Theorem 2.8** (Permanence of (RD)). The following are equivalent:

(i) \( \hat{\Gamma} \triangleleft \hat{G} \) has (RD);

(ii) there exists a matched pair of length functions \((l_\Gamma, l_{\hat{G}})\), such that \((\hat{\Gamma}, l_{\hat{G}})\) has polynomial growth and \((\Gamma, l_\Gamma)\) has (RD);

(iii) there exists a matched pair of length functions \((l_\Gamma, l_{\hat{G}})\), such that both \((\hat{\Gamma}, l_{\hat{G}})\) and \((\Gamma, l_\Gamma)\) have (RD).

**Remark 2.9.** When trying to construct concrete examples, the theoretical characterizations given in Theorems 2.7 and 2.8 have a serious drawback: it is in general very hard to check that the length functions on \( \Gamma \) and \( \hat{G} \) that one wants to use are actually matched (to determine the existence of an affording family is quite tricky).

### 2.3 Representation theory of some semi-direct products

For our purposes of constructing length functions on the dual, we need a good understanding of representations of semi-direct products of the form \( G \rtimes \Lambda \), where \( G \) is a compact (meaning quasi-compact and Hausdorff) group and \( \Lambda \) a finite group acting on \( G \) by topological automorphisms.

The more general results in [14] apply here by taking the compact quantum group \( G \) studied there to be the classical compact group \( G \). In this case, the irreducible representations can already be described by Mackey’s method [9], [10]. But even in this special case, the result on fusion rules that we will present seems fairly recent to the best of our knowledge. We refer our
reader to [1:4] for more details and complete proofs. We will also freely use the Peter-Weyl theory of projective representations of finite groups (see e.g. [3]).

Throughout the rest of § 2.3, we fix a compact group G, a finite group Λ, and a group homomorphism α : Λ → Aut(G), seen as Λ acting on G via topological automorphisms, and use G × Λ to denote the semi-direct with respect to this action. As a topological space, G × Λ is the Cartesian product G × Λ where Λ is equipped with the discrete topology. The multiplication on G × Λ is given by (g, v)(h, s) = (ga_v(h), rs), which makes G × Λ a compact group.

2.3.1 Classification of irreducible representations

The action α : Λ ↪ G by topological automorphisms induces an action ˜α : Λ ↪ Irr(G) via

\[ \lambda \cdot [u] := [u \circ \alpha_{\lambda}^{-1}] \]

which comes from the action of Λ on the (proper) class of all representations of G by \( \lambda \cdot u := u \circ \alpha_{\lambda}^{-1} \). For each \( x \in \text{Irr}(G) \), one can associate a (unitary) projective representation \( V \) of the isotropy subgroup \( \Lambda_x \), fixing \( x \), as follows. Take any representation \( u \in x \) on some Hilbert space \( H \). By definition, \( \lambda \cdot u \simeq u \) if and only if \( \lambda \in \Lambda_x \). Hence for any \( \lambda \in \Lambda_x \), there exists a unitary operator, uniquely determined up to a constant in the circle group \( \mathbb{T} = \{ z \in \mathbb{C} \mid |z| = 1 \} \), such that \( V(\lambda) \in \text{Mor}_G(\lambda \cdot u, u) \). As \( \text{Mor}_G(\lambda \cdot u, u) = \text{Mor}_G(\mu \cdot u, \mu \cdot u) \) for all \( \lambda, \mu \in \Lambda_x \), one checks that \( V(\lambda \mu) V(\mu)^* V(\lambda)^* \in \text{Mor}(u, u) \), hence is a multiple of \( \text{id}_H \) by a scalar in \( \mathbb{T} \).

Thus \( V : \Lambda_x \to B(H) \), \( \lambda \mapsto V(\lambda) \) is a projective representation of \( \Lambda_x \) on \( H \). The projective representation \( V \) is uniquely determined by the class \( x \in \text{Irr}(G) \) in the following sense: if \( V' \) is another projective representation of \( \Lambda_x \) such that \( V'(\lambda) \in \text{Mor}(\lambda \cdot v, v) \) for some \( v \in x \) and all \( \lambda \in \Lambda_x \), then for any unitary equivalence \( U \in \text{Mor}(u, v) \), there exists a unique map \( b : \Lambda_x \to \mathbb{T} \), such that for all \( \lambda \in \Lambda_x \), we have \( V(\lambda) = b(\lambda) U^* V'(\lambda) U \). The cohomology class \([\omega]\) in \( H^2(\Lambda_x, \mathbb{T}) \) (the second group cohomology of \( \Lambda_x \) with coefficients in \( \mathbb{T} \)) as a trivial \( \Lambda_x \)-module) of the cocycle \( \omega \) of the projective representation \( V \) depends only on \( x \in \text{Irr}(G) \). Obviously, we can replace \( \Lambda_x \) by any of its subgroup in the above discussion.

Notation 2.10. By \( \mathcal{G}_{\text{iso}}(\Lambda) \), we mean the set of finite intersections of the isotropy subgroups \( \Lambda_x \), \( x \in \text{Irr}(G) \).

Definition 2.11. Take any \( \Lambda_0 \in \mathcal{G}_{\text{iso}} \), a generalized representation parameter (abbreviated as GRP later) associated with \( \Lambda_0 \) (for the semidirect product \( G \rtimes \Lambda \)) is a triple \((u, V, v)\), where \( u \) is a representation of \( G \) on some finite dimensional Hilbert space \( H \), \( V \) is a projective representation of \( \Lambda_0 \) on the same \( H \) with \( V(\lambda) \in \text{Mor}_G(\lambda \cdot u, u) \) for each \( \lambda \in \Lambda_0 \), and \( v \) is a projective representation of \( \Lambda_0 \) on a possibly different finite dimensional Hilbert space \( K \), such that the cocycle of \( v \) is the opposite of that of \( V \). The GRP \((u, V, v)\) is called a representation parameter (abbreviated as RP later) if \( u \) is irreducible; and if in addition, \( \Lambda_0 = \Lambda_{[u]} \), we say that \((u, V, v)\) is a distinguished representation parameter (abbreviated as DRP in the following).

Given a RP \((u, V, v)\) associated with \( \Lambda_0 \) as in the definition above, one can associate an irreducible representation \( \mathcal{R}_{\Lambda_0}(u, V, v) \) of \( G \rtimes \Lambda_0 \) (which is a subgroup of \( G \rtimes \Lambda \) in which we are interested), called the representation of \( G \rtimes \Lambda_0 \) parameterized by \((u, V, v)\), as follows: the carrier space of this representation is \( H \otimes K \), and as a map from \( G \rtimes \Lambda_0 \) to \( \text{U}(H \otimes K) \), the representation \( \mathcal{R}_{\Lambda_0}(u, V, v) \) sends \((g, \lambda) \in G \rtimes \Lambda_0 \) to \( u(g) V(\lambda) \otimes v(\lambda) \). Now the induced representation \( \mathcal{R}(u, V, v) := \text{Ind}_{G \rtimes \Lambda_0}^{G \rtimes \Lambda} (\mathcal{R}_{\Lambda_0}(u, V, v)) \) is called the representation of \( G \rtimes \Lambda \) parameterized by \((u, V, v)\). If \((u, V, v)\) is distinguished, i.e. if \((u, V, v)\) is a DRP, then \( \mathcal{R}(u, V, v) \) is irreducible.

Fix a \( \Lambda_0 \in \mathcal{G}_{\text{iso}} \), we say two DRP associated with \( \Lambda_0 \), namely \((u_1, V_1, v_1)\) and \((u_2, V_2, v_2)\), are equivalent, if \( \mathcal{R}_{\Lambda_0}(u_1, V_1, v_1) \) and \( \mathcal{R}_{\Lambda_0}(u_2, V_2, v_2) \) are equivalent. This equivalence relation can be characterized more concretely as the equivalence of the following conditions:
We give a description of the fusion rules for $\Lambda \rtimes D$.

2.3.2 The fusion rules

For each $\Lambda_0 \in \mathcal{G}_{\text{iso}}$, we denote the set of equivalence classes of DRPs associated with $\Lambda_0$ by $\mathcal{D}_{\Lambda_0}$, and we denote by $\mathcal{D}$ the union of $\mathcal{D}_{\Lambda_0}$, $x \in \text{Irr}(G)$.

Given a GRP (resp. RP, resp. DRP) $\mathcal{D} := (u, V, v)$, the componentwise contragredient $(u^c, V^c, v^c)$ is still a GRP (resp. RP, resp. DRP), called the contragredient of $\mathcal{D}$, and is denoted by $\mathcal{D}^c$. Moreover, for each $r \in \Lambda$, denote by $\text{Ad}_r$ the inner automorphisms $x \mapsto r x r^{-1}$ of $\Lambda$, then the triple

$$r \cdot \mathcal{D} := (r \cdot u, r \cdot V, r \cdot v) = (u \circ \alpha^{-1}_r, V \circ \text{Ad}_{r^{-1}}|_{r \Lambda_0 r^{-1}}, v \circ \text{Ad}_{r^{-1}}|_{r \Lambda_0 r^{-1}})$$

is a GRP (resp. RP, resp. DRP) associated with the subgroup $r \Lambda_0 r^{-1} \in \mathcal{G}_{\text{iso}}$. One checks easily that $r \cdot [\mathcal{D}] := [r \cdot \mathcal{D}]$ gives an action of $\Lambda$ on $\mathcal{D}$, and this action preserves contragredients.

Theorem 2.13 (Classification of irreducible representation of $G \rtimes \Lambda$). The mapping

$$\Psi : \mathcal{D} \to \text{Irr}(G \rtimes \Lambda)$$

$$\mathcal{D} := [(u, V, v)] \in \mathcal{D}_{\Lambda_0} \mapsto \Psi_{\Lambda_0}(\mathcal{D}) := [\mathcal{D}(u, V, v)]$$

is a well-defined surjection whose fibers are exactly the orbits of the action $\Lambda \rtimes \mathcal{D}$. Moreover, $\Psi_{\Lambda_0}$, hence in particular $\Psi$, preserves contragredients.

2.3.2 The fusion rules

We give a description of the fusion rules for $G \rtimes \Lambda$ with the help of a reduction process.

For $i = 1, 2, 3$, consider an irreducible representation $W_i$ of $G \rtimes \Lambda$ parameterized by some DRP $\mathcal{D}_i := (u_i, V_i, v_i)$ associated with the isotropy subgroup $\Lambda_i := \Lambda[u_i]$. By Theorem 2.13, one only needs to find a more or less explicit formula for

$$\dim \text{Mor}_{G \rtimes \Lambda}(W_1, W_2 \times W_3)$$

in order to obtain a description of the desired fusion rules.

We now describe the reduction procedure mentioned above. One checks easily that $(u_2 \times u_3, V_2 \times V_3, v_2 \times v_3)$ is a GRP associated with the subgroup $\Lambda_0 := \cap_{i=1}^3 \Lambda_i \in \mathcal{G}_{\text{iso}}$. To fix notation, let $H_i$ (resp. $K_i$) be the carrier space of $u_i$ (resp. $v_i$). Then there exists a unique subrepresentation $u_p$ of $u_2 \times u_3$ determined by a unique subspace $H_p$ of $H_1 \otimes H_2$, such that $u_p$ is maximal among the subrepresentations of $u_2 \times u_3$ that are equivalent to multiples of $u_1$, and we denote by $n$ the multiplicity $\dim \text{Mor}_{G}(u_1, u_p)$ of $u_1$ in $u_p$ (which of course could be 0).

Lemma 2.14. The following hold.

1. The subspace $H_p$ is invariant under $V$, and determines a projective subrepresentation $V_p$ of $V$, such that $V_p(\lambda) \in \text{Mor}(\lambda \cdot u_p, u_p)$ for all $\lambda \in \Lambda_0$. 

10
(2) There exists a unique projective representation \( V'_p \) of \( \mathbb{C}^n \), such that \( V_p \) is equivalent to \( V_1 \times V'_p \), with the projective representations \( V'_p \times v_2 \times v_3 \) and \( v_1 \) of \( \Lambda_0 \) sharing the same cocycle.

Definition 2.15. We call \( \dim \text{Mor}_\Lambda(v_1, V'_p \times v_2 \times v_3) \) the incidence number of the triple \((\mathcal{P}_1, \mathcal{P}_2, \mathcal{P}_3)\) of DRPs.

Notation 2.16. For all \( r_i \in \Lambda, i = 1, 2, 3 \), we denote by \( m(r_1, r_2, r_3) \) the incidence number of \((\mathcal{P}_1, \mathcal{P}_2, \mathcal{P}_3)\). It is easy to check that \( m(r_1, r_2, r_3) = m(s_1, s_2, s_3) \) if \( r_1 \Lambda_i = s_1 \Lambda_i \), i.e. the incidence numbers \( m(z_1, z_2, z_3) := m(r_1, r_2, r_3) \) with \( r_i \) in the left coset \( z_i \in \Lambda/\Lambda_i \) is well-defined, i.e. independent of the choice of the representative \( r_i \in z_i, i = 1, 2, 3 \). Moreover, denote by \( \Lambda(z_1, z_2, z_3) \) the intersection \( \cap_{i=1}^3 r_i \Lambda r_i^{-1} \), which is again independent of the choices \( r_i \in z_i, i = 1, 2, 3 \).

Theorem 2.17. The fusion rules for \( G \rtimes \Lambda \) is given by the following formula

\[
\dim \text{Mor}(W_1, W_2 \times W_3) = \sum_{z_1 \in \Lambda/\Lambda_1} \sum_{z_2 \in \Lambda/\Lambda_2} \sum_{z_3 \in \Lambda/\Lambda_3} \frac{m(z_1, z_2, z_3)}{[\Lambda : \Lambda(z_1, z_2, z_3)]},
\]

where \([\Lambda : \Lambda_0]\) denotes the index of the subgroup \( \Lambda_0 \) of \( \Lambda \).

3 Nontrivial bicrossed product from semidirect product

Let \( G \) be a compact group, \( \Gamma \) a discrete group acting on \( G \) via topological automorphisms given by a group morphism \( \tau : \Gamma \to \text{Aut}(G) \). Using these data, one can form the semidirect product \( G \rtimes \Gamma \), which is a locally compact group whose underlying topological space is the topological product \( G \times \Gamma \), and whose group law is given by

\[
(g_1, \gamma_1)(g_2, \gamma_2) = (g_1 \tau_{\gamma_1}(g_2), \gamma_1 \gamma_2).
\]  \hspace{1cm} (3.1)

Note that the insertion \( \iota : \Gamma \to G \rtimes \Gamma, \gamma \mapsto (e_G, \gamma) \) \((e_G \text{ denotes the identity in } G)\) is a group morphism. In particular, the mapping

\[
\theta : \Gamma \to \text{Aut}(G \rtimes \Gamma)
\]

\[
\gamma \mapsto (\text{Ad}_{\iota \tau}(\gamma) = \text{Ad}_{(e_G, \gamma)}
\]

is a group morphism. For all \((g, r) \in G \rtimes \Gamma \) and \( \gamma \in \Gamma \), we have

\[
(e_G, \gamma)(g, r)(e_G, \gamma)^{-1} = (\tau_{\gamma}(g), \gamma r)(e_G, \gamma^{-1}) = (\tau_{\gamma}(g), \gamma r \gamma^{-1})
\]  \hspace{1cm} (3.3)

Thus as a map from the set \( G \times \Gamma \) to itself, we have

\[
\theta_\gamma := \theta(\gamma) = \tau_\gamma \times \text{Ad}_{\theta} : G \times \Gamma \to G \times \Gamma.
\]  \hspace{1cm} (3.4)

Now consider any finite subgroup \( \Lambda \) of \( \Gamma \). The group morphism \( \theta \) defined in (3.2) restricts to the subgroup \( \Lambda \) to give an action \( \Lambda \cap G \rtimes \Gamma \) by topological automorphisms. This allows us to form yet another semidirect product \((G \rtimes_\theta \Gamma) \times_\theta \Lambda \), whose underlying topological space is \( G \times \Gamma \times \Lambda \). It is clear that the group law on \((G \rtimes_\theta \Gamma) \times_\theta \Lambda \) is given by

\[
(g_1, \gamma_1, r_1)(g_2, \gamma_2, r_2) = ((g_1, \gamma_1) \theta_{\gamma_1}(g_2, \gamma_2), r_1 r_2)
\]

\[
= (g_1 \tau_{\gamma_1}(g_2), \gamma_1 r_1 \gamma_2 r_1^{-1}, r_1 r_2)
\]

\[
= (g_1 \tau_{\gamma_1 r_1}(g_2), \gamma_1 r_1 \gamma_2 r_1^{-1}, r_1 r_2).
\]  \hspace{1cm} (3.5)
By (3.5), both the mapping
\[ \iota_{1,3} : G \rtimes_\tau \Lambda \to (G \rtimes_\tau \Lambda) \rtimes_\theta \Lambda \]
\[ (g, r) \mapsto (g, e, r), \] (3.6)
and
\[ \iota_2 : \Gamma \to (G \rtimes_\tau \Gamma) \rtimes_\theta \Lambda \]
\[ \gamma \mapsto (e_G, \gamma, e) \] (3.7)
are injective group morphisms, such that for all \( \gamma \in \Gamma, (g, r) \in G \rtimes_\tau \Lambda \), we have
\[ \forall g \in G, \gamma, r \in \Gamma, \quad \iota_{1,3}(g, r) \iota_2(r^{-1}\gamma r) = (g, e, r)(e_G, r^{-1}\gamma r, e) = (g, \gamma, r), \] (3.8)
which implies that
\[ \iota_{1,3}(G \rtimes_\tau \Lambda) \iota_2(\Gamma) = (G \rtimes_\tau \Gamma) \rtimes_\theta \Lambda. \] (3.9)

It is clear that
\[ \iota_{1,3}(G \rtimes_\tau \Lambda) \cap \iota_2(\Gamma) = \{(e_G, e, e)\}. \] (3.10)
Moreover,
\[ \forall g \in G, \gamma, r \in \Gamma, \quad \iota_2(\gamma) \iota_{1,3}(g, r) = (e_G, \gamma, e)(g, e, r) \]
\[ = (\tau_\gamma(g), \gamma, r) = (\tau_\gamma(g), e, r)(e_G, r^{-1}\gamma r, e) \]
\[ = \iota_{1,3}(g, r) \iota_2(r^{-1}\gamma r), \] (3.11)

**Proposition 3.1.** Let \( \Gamma \) be a discrete group, \( G \) a compact group, and \( \tau : \Gamma \to \operatorname{Aut}(G) \) a left action of \( \Gamma \) on \( G \) by topological automorphisms. If \( \Lambda \) is a finite subgroup of \( \Gamma \), then \( (\Gamma, G \rtimes_\tau \Lambda) \) is a matched pair of groups with left action
\[ \alpha^\Lambda : \Gamma \times (G \rtimes_\tau \Lambda) \to G \rtimes_\tau \Lambda \]
\[ (\gamma, (g, r)) \mapsto (\tau_\gamma(g), r), \] (3.12)
and right action
\[ \beta^\Lambda : \Gamma \times (G \rtimes_\tau \Lambda) \to \Gamma \]
\[ (\gamma, (g, r)) \mapsto r^{-1}\gamma r. \] (3.13)
Moreover, the following hold.

\begin{enumerate}
\item The action \( \alpha^\Lambda \) is trivial if and only if \( \tau \) is trivial;
\item The action \( \beta^\Lambda \) is trivial if and only if \( \Lambda \subseteq Z(\Gamma) \), where \( Z(\Gamma) \) is the centre of \( \Gamma \).
\end{enumerate}

**Proof.** That \( (\Gamma, G \rtimes_\tau \Lambda) \) is a matched pair with the actions \( \alpha^\Lambda \) and \( \beta^\Lambda \) follows from (3.9), (3.10) and (3.11). (1) and (2) are direct consequences of the definition of \( \alpha^\Lambda \) and \( \beta^\Lambda \). \( \square \)

4 More notation

For the convenience of our discussion, we now introduce and fix some notation related to the bicrossed product of the matched pair \( (\Gamma, G \rtimes_\tau \Lambda) \) with the actions \( \alpha^\Lambda \) and \( \beta^\Lambda \), as described in § 3.
The bicrossed product of the matched pair \((\Gamma, G \rtimes \Lambda)\) is denoted by \(\Gamma_{\alpha \Lambda} \bowtie_\beta \Lambda (G \rtimes \Lambda)\). When there is no risk of confusion, we often omit the actions and simply write \(G \rtimes \Lambda\) and \(\Gamma \bowtie (G \rtimes \Lambda)\). Moreover, Aut\((G)\) denotes the group of topological automorphisms of \(G\).

The isotropy subgroup of \(G \rtimes \Lambda\) fixing \(\gamma \in \Gamma\) with respect to the action \(\beta^\Lambda\) is easily seen to be \(G \rtimes \Lambda_\gamma\), where

\[
\Lambda_\gamma := \{ r \in \Lambda \mid \gamma r = r \gamma \},
\]

i.e. \(\Lambda_\gamma\) is the centralizer of \(\gamma\) in \(\Lambda\).

For \(x \in \text{Irr}(G)\), \(\gamma \in \Gamma\), we denote the isotropy subgroup of \(\Lambda_\gamma\) fixing \(x\) with respect to the action \(\Lambda_\gamma \rtimes \text{Irr}(G)\), \(\gamma, [u] \mapsto [u \circ \tau]\) by \(\Lambda_{\gamma,x}\), i.e.

\[
\Lambda_{\gamma,x} := \{ r \in \Lambda \mid r \cdot x = x \}.
\]

We also need to fix some notation concerning the representation theory of \(\Gamma \bowtie (G \rtimes \Lambda)\), this is where the preliminaries described in § 2 come into play.

Let \(\gamma \in \Gamma\). Suppose \(\Lambda_0\) is an isotropy subgroup of \(\Lambda_\gamma\) with respect to the action \(\Lambda_\gamma \rtimes \text{Irr}(G)\). Let \(\mathcal{D}_{\gamma,\Lambda_0}\) denote the set of equivalent DRPs (distinguished representation parameters) (see § 2.3.1) associated with \(\Lambda_0\), and let

\[
\Psi_{\gamma,\Lambda_0} : \mathcal{D}_{\gamma,\Lambda_0} \to \text{Irr}(G \rtimes \Lambda_\gamma)
\]

be the injection used to classify irreducible unitary representations of \(G \rtimes \Lambda_\gamma\) as in Theorem 2.13. Let \(\mathcal{D}_\gamma\) be the set of equivalence classes of all DRPs for the semidirect product \(G \rtimes \Lambda_\gamma\), we then have an action of \(\Lambda_\gamma\) on the class of all DRPs for \(G \rtimes \Lambda_\gamma\), which passes to the quotient and yields an action \(\Lambda_\gamma \rtimes \mathcal{D}_\gamma\) as described in § 2.3.1. We thus have the classification surjection

\[
\Psi_{\gamma} : \mathcal{D}_\gamma \to \text{Irr}(G \rtimes \Lambda_\gamma)
\]

\[
[(u,V,v)] \in \mathcal{D}_{\gamma,\Lambda_0} \mapsto \Psi_{\gamma,\Lambda_0} \left( [(u,V,v)] \right),
\]

whose fibers are exactly the orbits for the action \(\Lambda_\gamma \rtimes \mathcal{D}_\gamma\).

When \(\gamma = e_\Gamma\), we then have \(\Lambda_\gamma = \Lambda\) and we write \(\Psi_{e_\Gamma}\) simply as \(\Psi\).

We use \(\text{Orb}_{\beta^\Lambda}\) to denote the set of \(\beta^\Lambda\)-orbits. For each \(\mathcal{O} \in \text{Orb}_{\beta^\Lambda}\), let \(\mathcal{R}_\mathcal{O}\) be the mapping from the class of \(\mathcal{O}\)-representations of \(G \rtimes \Lambda\) to the class of finite dimensional unitary representations of the bicrossed product \(\Gamma \bowtie (G \rtimes \Lambda)\) as in § 2.2.2, and let \(\text{Irr}_{\mathcal{O}}(G \rtimes \Lambda)\) denote the set of equivalence classes of \(\mathcal{O}\)-irreducible \(\mathcal{O}\)-representations. We thus have the classification bijection

\[
\mathcal{R} : \bigoplus_{\mathcal{O} \in \text{Orb}_{\beta^\Lambda}} \text{Irr}_\mathcal{O}(G \rtimes \Lambda) \to \text{Irr}(\Gamma \bowtie (G \rtimes \Lambda))
\]

\[
[U] \in \text{Irr}_\mathcal{O}(G \rtimes \Lambda) \mapsto [\mathcal{R}_\mathcal{O}(U)].
\]

\section{Proof of Theorem 1.1}

The purpose of this section is to establish Theorem 1.1 which gives a sufficient condition for the (dual of) bicrossed product \(\Gamma \bowtie (G \rtimes \Lambda)\) to have polynomial growth (resp. property (RD)).

We first establish the following technical result.

\textbf{Lemma 5.1.} Suppose \(l_{\tilde{G}} : \text{Irr}(G) \to \mathbb{R}_{\geq 0}\) is a \(\Gamma\)-invariant length function on \(\tilde{G}\), i.e. \(l_{\tilde{G}}([u^x \circ \tau]) = l_{\tilde{G}}(x)\) whenever \(\gamma \in \Gamma\), \(u^x \in x \in \text{Irr}(G)\), and \(l_{\Gamma}\) is a \(\beta^\Lambda\)-invariant length function on \(\Gamma\). Then

\[
l_{\tilde{G} \rtimes \Lambda} : \text{Irr}(G \rtimes \Lambda) \to \mathbb{R}_{\geq 0},
\]

\[
\Psi([u,V,v]) \mapsto l_{\tilde{G}}([u])
\]
is a well-defined length function on $G \rtimes \Lambda$ such that the pair $(l_{\Gamma}, l_{G \rtimes \Lambda})$ is matched.

Proof. The fact that $l_{G \rtimes \Lambda}$ is well-defined (i.e. does not depend on the choice of the DRP $(u, V, v)$) follows from Theorem 2.13 and the $\Lambda$-invariance of $l_{G\rtimes\Lambda}$. We now show that the pair $(l_{\Gamma}, l_{G \rtimes \Lambda})$ of length functions is matched.

For all $\mathcal{O} \in \text{Orb}_{\beta, \gamma}$, define $l_\mathcal{O} : \text{Irr}(G \rtimes \Lambda) \to \mathbb{R}_{\geq 0}$ via the following procedure. Take any $\gamma \in \mathcal{O}$, and let $\Phi_\gamma : \text{Irr}(G \rtimes \Lambda, \gamma) \to \text{Irr}_\mathcal{O}(G \rtimes \Lambda)$ be the canonical bijection as in Proposition 2.2. To avoid over-complication of our notation, we often implicitly identify $\text{Irr}(G \rtimes \Lambda, \gamma)$ with $\text{Irr}_\mathcal{O}(G \rtimes \Lambda)$ via the bijection $\Phi_\gamma$, when doing so won’t cause a risk of confusion. For each DRP $(u, V, v)$ of $G \rtimes \Lambda$,

$$l_\mathcal{O} \left( \Psi, \left( ([u, V, v]) \right) \right) := l_G([u]) + l_\mathcal{O}(\gamma).$$

By Theorem 2.13 again, we see that (5.1) yields a well-defined mapping $l_\mathcal{O} : \text{Irr}(G \rtimes \Lambda) \to \mathbb{R}_{\geq 0}$. It is clear that $l_{\{e\}} = l_G$ via the identification of $\text{Irr}_{\text{aff}}(G \rtimes \Lambda)$ with $\text{Irr}(G \rtimes \Lambda, \gamma)$ by $\Phi_{\text{aff}}$. Moreover, we have $[\varepsilon_\mathcal{O}] = \Psi_\gamma([\varepsilon_G, \varepsilon_{\Lambda_{\text{aff}}}, \varepsilon_{\Lambda_{\gamma}}])$, so that

$$l_\mathcal{O}(\gamma) = l_{G\rtimes\Lambda}([\varepsilon_G]) + l_\mathcal{O}(\gamma) = l_\mathcal{O}(\varepsilon_\mathcal{O}).$$

Therefore, to finish the proof, it remains to show that $(l_\mathcal{O})_{\mathcal{O} \in \text{Orb}_{\beta, \gamma}}$ is an affording family in the sense of Definition 2.6. By definition, it is clear that

$$l_{\{e\}}([\varepsilon_{G \rtimes \Lambda}]) = l_G([\varepsilon_G]) + l_\mathcal{O}(\varepsilon_{\mathcal{O}}) = 0. $$

The condition $l_\mathcal{O}(\varepsilon_\mathcal{O}) = l_G([\varepsilon_G]) + l_\mathcal{O}(\varepsilon_{\mathcal{O}}) = 0$ can also be easily checked. Indeed, if $[U]$ is given by $\Psi_\gamma([u, V, v])$, then $[U']$ is given by $\Psi_{\gamma^{-1}}([u, V', v'])$ (see Theorem 2.3). By (5.1), we now have

$$l_\mathcal{O}(\varepsilon_\mathcal{O}) = l_{G\rtimes\Lambda}([\varepsilon_G]) + l_\mathcal{O}(\gamma) = l_{G\rtimes\Lambda}([\varepsilon_G]) + l_\mathcal{O}(\gamma^{-1}) = l_{G\rtimes\Lambda}([\varepsilon_G]),$$

(5.2)

By Definition 2.6, it remains only to establish the following claim.

**Claim.** For $i = 1, 2, 3$, let $\mathcal{O}_i \in \text{Orb}_{\beta, \gamma}$, and $[U_i] \in \text{Irr}_{\mathcal{O}_i}(G \rtimes \Lambda)$, with

$$U_i = \sum_{r, s \in \mathcal{O}_i} e_{r, s} \otimes u_{r, s}^{(i)},$$

being an $\mathcal{O}_i$-irreducible $\mathcal{O}_i$-representation of $G \rtimes \Lambda$ on $\ell^2(\mathcal{O}_i) \otimes \mathcal{H}_i$. If

$$\dim \text{Mor}_{G \rtimes \Lambda} \left( u_{r, s}^{(i)} | G \rtimes \Lambda, U_1 \times \gamma U_2 \right) \neq 0$$

(5.3)

for some (hence for all) $\gamma \in \mathcal{O}_i$, then

$$l_{\mathcal{O}_3}([U_3]) \leq l_{\mathcal{O}_1}([U_1]) + l_{\mathcal{O}_2}([U_2]).$$

(5.4)

Before proving the claim, we remark that until now, only the $\Lambda$-invariance of $l_{G\rtimes\Lambda}$ is needed. The hypothesis that $l_{G\rtimes\Lambda}$ is $\Gamma$-invariant will play an important role in the proof of the claim as we will presently see.

We now prove the claim. Suppose $[U_i]$ is given by some $\Psi_{\gamma_i}([u_i, V_i, v_i])$ and let $\Lambda_i := \Lambda_{\gamma_i}([u_i])$ for each $i = 1, 2, 3$. Define $\mu \cdot u := u \circ \tau_\mu$ to be the left action of $\Gamma$ on the class of finite dimensional unitary representation of $G$, and let $M(u)$ denote the vector space of matrix coefficients of $u$. Using the character formulae for $U_1 \times \gamma U_2$ and for $\Psi_{\gamma_3}((u_3, V_3, v_3))$, as well as the construction of $\Psi_{\gamma_3}$, we see that as elements in $\text{Pol}(G) \otimes C(\Lambda_3)$, we have

$$\chi \left( U_{\gamma_3 | G \rtimes \Lambda_3} \right) \in \text{Vect} \left( \bigcup_{r \in \Lambda} M(r \cdot u_3) \right) \otimes C(\Lambda_3) \subseteq \text{Pol}(G) \otimes C(\Lambda_3),$$

(5.5)
and
\[ \chi(U_1 \times \gamma U_2) \in \text{Vect} \left( [\Gamma \cdot M(u_1)] [\Gamma \cdot M(u_2)] \right) \otimes C(\Lambda_\gamma), \tag{5.6} \]
where
\[ \forall i = 1, 2, \quad \Gamma \cdot M(u_i) := \bigcup_{r_i \in \Gamma} M(r_i \cdot u_i) \]
and \([\Gamma \cdot M(u_1)] [\Gamma \cdot M(u_2)]\) denotes product of form \(\varphi_1 \varphi_2 \in \text{Pol}(G)\) where \(\varphi_i \in \Gamma \cdot M(u_i)\) for \(i = 1, 2\). By (5.5), (5.6) and a simple calculation using the Haar state on \(C(G) \otimes C(\Lambda_\gamma) = \text{C}(G \ltimes \Lambda_\gamma)\), it is clear that (5.3) implies the existence of \(r \in \Lambda, r_1, r_2 \in \Gamma\), such that \(M(r \cdot u_3)\) and \(M(r_1 \cdot u_1) \cdot M(r_2 \cdot u_2)\) are not orthogonal with respect to the Haar measure on \(G\). Since the representation \(r \cdot u_3\) of \(G\) is irreducible, this forces that
\[ \dim \text{Mor}_G(r \cdot u_3, (r_1 \cdot u_1) \times (r_2 \cdot u_2)) \neq 0. \]
Hence
\[ r \cdot [u_3] \subseteq (r_1 \cdot [u_1]) \otimes (r_2 \cdot [u_2]). \tag{5.7} \]
Since \(l_G\) is a \(\Gamma\)-invariant length function, by (5.7), we have
\[ l_G([u_3]) = l_G(r \cdot [u_3]) \leq l_G(r_1 \cdot [u_1]) + l_G(r_2 \cdot [u_2]) = l_G([u_1]) + l_G([u_2]). \tag{5.8} \]
On the other hand, (5.3) also implies that \(\gamma_3 \in \mathcal{O}_3 \subseteq \mathcal{O}_1 \mathcal{O}_2\), so there is \(s_i \in \mathcal{O}_1, i = 1, 2\), such that \(s_1 s_2 = \gamma_3\). Using the fact that \(l_\Gamma\) is a \(\beta^\Lambda\)-invariant length function, we have
\[ l_\Gamma(\gamma_3) = l_\Gamma(s_1 s_2) \leq l_\Gamma(s_1) + l_\Gamma(s_2) = l_\Gamma(\gamma_1) + l_\Gamma(\gamma_2). \tag{5.9} \]
By (5.8), (5.9) and (5.1) again, we have
\[
\begin{align*}
l_{\mathcal{O}_3}([U_3]) &= l_G([u_3]) + l_\Gamma(\gamma_3) \\
&\leq l_G([u_1]) + l_G([u_2]) + l_\Gamma(\gamma_1) + l_\Gamma(\gamma_2) \\
&= l_{\mathcal{O}_1}([U_1]) + l_{\mathcal{O}_2}([U_2]).
\end{align*}
\]
This finishes the proof of the claim, and hence the lemma.

Now Theorem 1.1 follows from Lemma 5.1, Theorem 2.7 and Theorem 2.8.

Remark 5.2. It is however, unknown to the author that whether the polynomial growth (resp. \((\text{RD})\)) of the dual \(\Gamma \bowtie (G \rtimes \Lambda)\) implies the existence of a \(\Gamma\)-invariant length function \(l_\Gamma\) on \(\tilde{G}\) witnessing the polynomial growth (resp. \((\text{RD})\)) of \(\tilde{G}\). Later we will show that if the composition \(\Gamma \twoheadrightarrow \text{Aut}(G) \rightarrow \text{Out}(G)\) has finite image, then the converse of Theorem 1.1 also holds (Theorem 1.2).

6 Invariance of length functions and proof of Theorem 1.2

In this section, we partially treat the difficulty of the technical assumption on the \(\Gamma\)-invariance of the length function \(l_\Gamma\) on \(\tilde{G}\) that witnesses the polynomial growth or \((\text{RD})\) of \(\tilde{G}\), as presented in Theorem 1.1. The results here will be used in § 7 in which we give some concrete examples of bicrossed products whose duals have \((\text{RD})\) but do not have polynomial growth. We also point out that the examples that will be given in § 8 do not fit into this framework, thus we only have a partial understanding of the situation. In particular, the tools developed here lead to a natural proof of Theorem 1.2.

We begin by considering a technical lemma on the Fourier transform and the Sobolev-0-norm in the context of compact quantum groups of Kac type.
Lemma 6.1. Let $\mathbb{H}$ be a compact quantum group of Kac type. Suppose $\theta : C(\mathbb{H}) \to C(\mathbb{H})$ is an automorphism of $C^*$-algebras that intertwines the comultiplication $\Delta$ of $\mathbb{H}$ ($\theta$ is an automorphism of the quantum group $\mathbb{H}$). Then there exists an automorphism $\hat{\theta}$ of the involutive algebra $c_c(\widehat{\mathbb{H}})$, such that
\[
\forall a \in c_c(\widehat{\mathbb{H}}), \quad F_{\mathbb{H}}(\hat{\theta}(a)) = \theta(F_{\mathbb{H}}(a)) \quad \text{and} \quad \|\hat{\theta}(a)\|_{\mathbb{H},0} = \|a\|_{\mathbb{H},0}. \tag{6.1}
\]

Proof. Choose a complete set of representatives $\{u^x \mid x \in \text{Irr}(\mathbb{H})\}$ for $\text{Irr}(\mathbb{H})$, and denote the finite dimensional Hilbert space underlying the unitary representation $u^x$ by $\mathcal{H}_x$, so that
\[
c_c(\widehat{\mathbb{H}}) = \bigoplus_{x \in \text{Irr}(\mathbb{H})} \text{alg} B(\mathcal{H}_x).
\]
For each finite dimensional unitary representation $u \in B(\mathcal{H}) \otimes \text{Pol}(\mathbb{H})$ of $\mathbb{H}$ on $\mathcal{H}$, since $\theta$ is an automorphism of $\mathbb{H}$, the unitary operator
\[
\theta_*(u) := (\text{id} \otimes \theta)(u) \in B(\mathcal{H}) \otimes \text{Pol}(\mathbb{H}) \tag{6.2}
\]
remains a unitary representation of $\mathbb{H}$ on the same space $\mathcal{H}$. It is clear that $\theta_*$ also passes to a bijection of the set $\text{Irr}(\mathbb{H})$ to itself, which we still denote by $\theta_*$ by abuse of notation, via $\theta_*([u]) = [\theta_*(u)]$. In particular, for each $x \in \text{Irr}(\mathbb{H})$, we have $[u^{\theta_*(x)}] = \theta_*(x) = [\theta_*(u^x)]$, thus there exists a unitary
\[
T_x \in \text{Mor}_{\mathcal{H}}(u^{\theta_*(x)}, \theta_*(u^x)) \subseteq B(\mathcal{H}_{\theta_*(x)}),
\]
which is uniquely determined up to a multiple of a scalar in $\mathbb{T}$.

Take any
\[
a = (a_x)_{x \in \text{Irr}(\mathbb{H})} = \sum_{x \in \text{Irr}(\mathbb{H})} a_x \in c_c(\widehat{\mathbb{H}}), \tag{6.3}
\]
where the sum is finite (meaning all but finitely many $a_x \in B(\mathcal{H}_x)$ is 0). For each $x \in \text{Irr}(\mathbb{H})$, we set
\[
b_{\theta_*(x)} := T_x a_x T_x \in B(\mathcal{H}_{\theta_*(x)}). \tag{6.4}
\]
Then
\[
\dim(\theta_*(x)) = \dim x. \tag{6.5}
\]
By the choice of $T_x$, we have
\[
(\text{Tr}_{\mathcal{H}_{\theta_*(x)}} \otimes \text{id}) \left( u^{\theta_*(x)} (b_{\theta_*(x)} \otimes 1) \right) = (\text{Tr}_{\mathcal{H}_{\theta_*(x)}} \otimes \text{id}) \left( (T_x^* \otimes 1) [\theta_*(u^x)] (a_x \otimes 1) (T_x \otimes 1) \right) = (\text{Tr}_{\mathcal{H}_x} \otimes \text{id}) \left( [\theta_*(u^x)] (a_x \otimes 1) \right) = \theta \left( (\text{Tr}_{\mathcal{H}_x} \otimes \text{id}) \left( [u^x] (a_x \otimes 1) \right) \right), \tag{6.6}
\]
and
\[
(\text{Tr}_{\mathcal{H}_{\theta_*(x)}} \otimes \text{id}) \left( b_{\theta_*(x)} b_{\theta_*(x)} \right) = (\text{Tr}_{\mathcal{H}_{\theta_*(x)}} \otimes \text{id}) \left( T_x^* a_x^* a_x T_x \right) = (\text{Tr}_{\mathcal{H}_{\theta_*(x)}} \otimes \text{id}) \left( a_x^* a_x \right). \tag{6.7}
\]
We now define
\[ \hat{\theta}(a) := \sum_{x \in \text{Irr}(\hat{H})} b_{\theta \ast(x)} \]  
(6.8)

Since \( \theta \ast : \text{Irr}(H) \to \text{Irr}(H) \) is a bijection, it is clear that (6.8) defines an automorphism \( \hat{\theta} \) of the involutive algebra \( c_c(\hat{H}) \). Finally, (6.1) follows from (6.6) and (6.7).

Proposition 6.2. Let \( \hat{H} = (C(\hat{H}), \Delta) \) be a compact quantum group of Kac type. Suppose \( \Theta \) is a finite subgroup of \( \text{Aut}(C(\hat{H}), \Delta) \). The following are equivalent.

1. There exists a length function \( l \) on \( \hat{H} \) and \( P(X) \in \mathbb{R}[X] \), such that
   \[
   \forall k \in \mathbb{N}, \ a \in Q_{l,k}c_c(\hat{H}) \implies \|F_{\hat{H}}(a)\| \leq P(k)\|a\|_{\hat{H},0},
   \]  
   (6.9)

   where \( Q_{l,n} := \sum_{x \in \text{Irr}(\hat{H}), l(x) < n+1} P_x \in \ell^\infty(\hat{H}) \).

2. There exists a \( \Theta \)-invariant length function \( l_\Theta \) on \( \hat{H} \) and \( Q(X) \in \mathbb{R}[X] \), such that
   \[
   \forall k \in \mathbb{N}, \ a \in Q_{l_\Theta,k}c_c(\hat{H}) \implies \|F_{\hat{H}}(a)\| \leq Q(k)\|a\|_{\hat{H},0}.
   \]  
   (6.10)

Proof. Obviously (2) implies (1).

Now suppose (1) holds and let’s prove (2). Let \( n = |\Theta| \) and suppose \( \theta_1, \ldots, \theta_n \) form an enumeration of all elements of \( \Theta \). Let \( l_i \) denote the length function \( l \circ (\theta_i)_\ast \) on \( \hat{H} \) (see the discussion after (6.2) in the proof of Lemma 6.1). Put

\[ l_\Theta := \frac{1}{|\Theta|} \sum_{i=1}^n l_i, \]  
(6.11)

then it is clear that \( l_\Theta \) is a \( \Theta \)-invariant length function on \( \hat{H} \). For each \( k \in \mathbb{N} \), define

\[ F_{\Theta,k} := \{ x \in \text{Irr}(\hat{H}) \mid l_\Theta(x) < k + 1 \}, \]  
(6.12)

and for \( i = 1, \ldots, n \), put

\[ F_{i,k} := \{ x \in \text{Irr}(\hat{H}) \mid l_{\theta_i}(x) < k + 1 \}. \]  
(6.13)

By (6.9), (6.13) and (6.11), we have

\[ F_{\Theta,k} \subseteq \bigcup_{i=1}^n F_{i,k}. \]  
(6.14)

Define

\[ \xi : F_{\Theta,k} \to \{1, \ldots, n\}, \quad x \mapsto \inf\{i \mid x \in F_{i,k}\}. \]  
(6.15)

Note that (6.14) guarantees that \( \xi \) is well-defined.

We now prove (6.10) holds for some suitable polynomial \( Q(X) \in \mathbb{R}[X] \), which will finish the proof. Since \( a \in Q_{l_\Theta,k}c_c(\hat{H}) \), there exists a finite subset \( F \) of \( F_{\Theta,k} \), such that

\[ a = \sum_{x \in F} a_x = \sum_{i=1}^n a_i, \]  
(6.16)
where for each $i$,

$$a_i := \sum_{x \in F^{\setminus \xi^{-1}(i)}} a_x \in Q_{l_i,k}.$$  

(6.17)

By Lemma 6.1 and (1), we have

$$\forall i = 1, \ldots, n, \quad \|F_{\hat{H}}(a_i)\| \leq P(k)\|a_i\|_{H,0}.$$  

(6.18)

hence

$$\|F_{\hat{H}}(a)\|^2 \leq \left(\sum_{i=1}^{n} \|F_{\hat{H}}(a_i)\|\right)^2 \leq n \left(\sum_{i=1}^{n} \|F_{\hat{H}}(a_i)\|^2\right) \leq n[P(k)]^2 \left(\sum_{i=1}^{n} \|a_i\|_{H,0}^2\right).$$  

(6.19)

Thus taking $Q(X) = \sqrt{\|\Theta\|P(X)} \in \mathbb{R}[X]$, we have (6.10).

**Corollary 6.3.** The following are equivalent:

1. $\Gamma$ has polynomial growth (resp. (RD));
2. there exists a $\beta^\Lambda$-invariant length function $l_\Gamma$ on $\Gamma$, such that $(\Gamma, l_\Gamma)$ has polynomial growth (resp. (RD)).

**Proof.** This follows from Proposition 6.2 by letting $\Theta = \{\text{Ad}_r \in \text{Aut}(\Gamma) \mid r \in \Lambda\}$ and $\mathbb{H} = \hat{\Gamma}$.  

**Proof of Theorem 1.2.** We begin by observing more closely the action $\Gamma \curvearrowright \text{Irr}(G)$. It is clear that this action is actually given by $\text{Aut}(G)$ acting on $G$, composed with the group morphism $\tau : \Gamma \to \text{Aut}(G)$ with respect to which we form the semidirect product (see the beginning of §3). More precisely, there is a natural action $\text{Aut}(G) \curvearrowright \text{Irr}(G)$ by letting $(\theta, [u]) \mapsto [\theta_* (u)]$, and the action $\Gamma \curvearrowright \text{Irr}(G)$ is given by $(\gamma, x) \mapsto \tau(\gamma) \cdot x$. By definition, one has

$$\text{Inn}(G) \subseteq \bigcap_{x \in \text{Irr}(G)} [\text{Aut}(G)]_x,$$  

(6.20)

where

$$[\text{Aut}(G)]_x := \{\theta \in \text{Aut}(G) \mid \theta \cdot x = x\}.$$

Thus passing to the quotient, it is in fact $\text{Out}(G) = \text{Aut}(G) / \text{Inn}(G)$ that acts on $\text{Irr}(G)$ (inner automorphisms fixes equivalence classes of representations by definition). Thus to talk about the $\Gamma$ invariance of a given length function $l$ on $\hat{G}$, it suffices to consider the invariance of $l$ under the image of the composition of group morphisms $\tau : \Gamma \to \text{Aut}(G)$ and the canonical projection $\text{Aut}(G) \to \text{Out}(G)$.

With the above consideration in mind, Theorem 1.2 now follows from Theorem 1.1, Proposition 6.2 (posing $\mathbb{H} = G$ and $\Theta = \text{Image(}\hat{\tau}\text{)}) and Corollary 6.3.  

**7 Examples of bicrossed products with rapid decay but not polynomial growth–part I**

In this section, we construct some examples of bicrossed products whose duals have property (RD) but do not have polynomial growth, using the framework developed in §6. We shall frequently use Jolissaint’s theorem on rapid decay of amalgamated product of groups, which we record here for convenience of the reader.
Theorem 7.1 (Jolissaint). Suppose $\Gamma_1, \Gamma_2$ are two discrete groups with property (RD), $\Lambda$ is a finite group, $j_i : \Lambda \to \Gamma_i$ is an injective group morphism for $i = 1, 2$, then the amalgamated product $\Gamma_1 \ast_{\Lambda} \Gamma_2$ with respect to $j_1, j_2$ also has property (RD).

Proof. This is part of [7, Theorem 2.2.2]. □

We will refer Theorem 7.1 as Jolissaint’s theorem hereafter.

Example 7.2. Take $\Gamma = \text{PSL}_2(\mathbb{Z}) \simeq (\mathbb{Z}/2\mathbb{Z}) \ast (\mathbb{Z}/3\mathbb{Z})$, with the isomorphism determined by identifying $\mathbb{Z}/2\mathbb{Z}$ with the cyclic group generated by $s \in \Gamma$, and $\mathbb{Z}/3\mathbb{Z}$ with the cyclic group generated by $t \in \Gamma$, where

$$s = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} \quad \text{and} \quad t = \begin{pmatrix} 0 & -1 \\ 1 & 1 \end{pmatrix}.$$ 

(see e.g. [2, Example E.10 on page 476]). Let $G$ be any compact connected real Lie group that admits an element $x \in G$ of order 2, and an element $y$ of order 3, such that $\{x, y\} \not\subseteq Z(G)$ (e.g. $G = \text{SO}(3, \mathbb{R})$, $x$ is any rotation by $\pi$, $y$ any rotation by $2\pi/3$), where $Z(G)$ is the center of $G$.

Now the mapping $s \mapsto \text{Ad}_x$, $t \mapsto \text{Ad}_y$ determines a unique group morphism

$$\tau : \Gamma \to \text{Out}(G) \subseteq \text{Aut}(G)$$

so $\tilde{\tau} : \Gamma \to \text{Out}(G)$ is trivial (hence of finite image). Put $\Lambda < \Gamma$ to be $\langle s \rangle$ or $\langle t \rangle$ (the subgroup generated by $s$ or $t$ respectively). Since $\Lambda \not\subseteq Z(\Gamma)$, it follows from the choice of $x$ and $y$ that the resulting bicrossed product $\tilde{G} := \Gamma_\Lambda \rtimes_{\phi \Lambda} Z(\Gamma)$ is nontrivial (Proposition 3.1).

By Jolissaint’s theorem, $\text{PSL}_2(\mathbb{Z})$ has (RD), but $\text{PSL}_2(\mathbb{Z})$ does not have polynomial growth since it is not virtually nilpotent (by Gromov’s famous theorem on polynomial growth, see [6]), and by Vergnioux [13], $\tilde{G}$ has polynomial growth, thus Theorem 1.2 applies and we see that $\tilde{G}$ has (RD) but not polynomial growth.

Example 7.3. Let $G$ be any compact group with $\tilde{G}$ having polynomial growth (e.g. all connected compact real Lie group), and $\Lambda$ a finite subgroup of $\text{Aut}(G)$. Take $\Gamma$ to be a nontrivial semidirect product of the free group $F_2$ on two generators (here $F_2$ can be replaced by any discrete group with (RD) but without polynomial growth) with $\Lambda$ (in particular, $\Lambda$ is nontrivial). Then the obvious action of $\Lambda$ on $G$ and the canonical projection $F_2 \times \Lambda \to \Lambda$ together yield a nontrivial left action $\tau$ of $\Gamma$ on $G$ by topological automorphisms. The same reasoning as in Example 7.2 shows that $\Gamma \rtimes (G \rtimes \Lambda)$ is also a bicrossed product whose dual has (RD) but not polynomial growth.

Using Theorem 1.2, many more examples can be constructed in the same spirit.

8 Examples of bicrossed products with rapid decay but not polynomial growth—part II

Despite of the fact that Theorem 1.2 yields many interesting concrete examples of bicrossed products with property (RD) as shown in § 7, it is worth pointing out that the restriction the finiteness of the image $\text{Image}(\tilde{\tau})$ is too strong to include many other interesting examples, which we will now show in this section. To make the contrast even more dramatic, we show how to construct examples of nontrivial bicrossed product of the form $\Gamma \rtimes (G \rtimes \Lambda)$ whose dual has (RD) but not polynomial growth, while $\text{Image}(\tilde{\tau})$ as in Theorem 1.2 is infinite (hence Theorem 1.2 no longer applies).

We begin with a simple result in finite group theory.
Lemma 8.1. If $A$ is a finite abelian group, then there exists infinitely many non-isomorphic finite abelian groups $B$, such that $A$ is isomorphic to a subgroup of $\text{Aut}(B)$.

Proof. Since $A$ is a direct sum of finite cyclic groups, without loss of generality, we may assume $A$ is cyclic of order $n$, with $a$ as a generator. Set $B$ to be the $n$-fold direct sum of any nontrivial finite abelian group $C$, and define $\sigma(a) \in \text{Aut}(B)$ to be the permutation

$$(c_1, \ldots, c_n) \mapsto (c_2, \ldots, c_n, c_1).$$

Then it is clear that

$$\sigma : A \to \text{Aut}(B)$$

$$a^m \mapsto [\sigma(a)]^m$$

is a well-defined injective group morphism. \qed

As we will see later, Theorem 1.2 no longer applies for the examples constructed below in this section due to the violation of the finiteness of $\text{Image}(\tilde{\tau})$. We will thus have to resort to Theorem 1.1 to prove the (RD) of the dual of the bicrossed product $\Gamma \bowtie \triangleleft (G \rtimes \Lambda)$. Here, the $\beta^\Lambda$-invariance of the length function on $\Gamma$ is not a problem thanks to Corollary 6.3. But the $\Gamma$-invariance of the length function on $\hat{G}$ requires some work.

Lemma 8.2. Let $\Xi_1, \Xi_2, \ldots$ be a sequence of finite discrete (hence compact) groups. The product group $\prod_{i=1}^{\infty} \text{Aut}(\Xi_i)$ naturally acts pointwise on the direct sum $\oplus_{i=1}^{\infty} \Xi_i$, giving the canonical inclusion $\prod_{i=1}^{\infty} \text{Aut}(\Xi_i) \subseteq \text{Aut}(\oplus_{i=1}^{\infty} \Xi_i)$. Under this setting, there exists a $\prod_{i=1}^{\infty} \text{Aut}(\Xi_i)$-invariant length function $l$ on the discrete group $\oplus_{i=1}^{\infty} \Xi_i$, such that the pair $(\oplus_{i=1}^{\infty} \Xi_i, l)$ has polynomial growth.

Proof. Let $N_i = |\Xi_i|$ for all $i \in \mathbb{N}_{>0}$ and take $M_k = \prod_{i=1}^{k} N_i$ for all $k \in \mathbb{N}$ (we make the convention that $M_0 = 1$). Let $e_i$ be the neutral element of the group $\Xi_i$, and denote the characteristic function of $\Xi_i \setminus \{e_i\}$ by $\chi_i$. Define

$$l : \bigoplus_{i=1}^{\infty} \Xi_i \to \mathbb{R}_{\geq 0}$$

$$(\xi_i) \mapsto \sum_{i=1}^{\infty} \chi_i(\xi_i) M_i.$$

Then it is clear that $l$ is a $\prod_{i=1}^{\infty} \text{Aut}(G_i)$-invariant length function on $\bigoplus_{i=1}^{\infty} \Xi_i$. Moreover, for all $n \in \mathbb{N}_{>0}$, there exists a unique $k \geq 1$, such that $M_{k-1} \leq n < M_k$. Then, by the definition of $l$, we have

$$\{\xi = (\xi_i) \in \bigoplus_{i=1}^{\infty} \Xi_i \mid l(\xi) < n\} \subseteq \{\{\xi_i \in \bigoplus_{i=1}^{\infty} \Xi_i \mid \forall i \geq k, \xi_i = e_i\}\}.$$

Thus

$$|\{\xi = (\xi_i) \in \bigoplus_{i=1}^{\infty} \Xi_i \mid l(\xi) < n\}| \leq \prod_{i=1}^{k-1} N_i = M_{k-1} \leq n.$$

In particular, $(\bigoplus_{i=1}^{\infty} \Xi_i, l)$ has polynomial growth. \qed

We are now prepared to construct new examples of bicrossed product of the form $\Gamma \bowtie (G \rtimes \Lambda)$ that don’t fit into the framework of Theorem 1.2.
Example 8.3. Let \( \Lambda \) be any nontrivial finite abelian group. By Lemma 8.1, one can take a sequence of finite abelian groups \( (G_i)_{i=1}^{\infty} \), such that \( \Lambda \) is isomorphic to a subgroup of \( \text{Aut}(G_i) \) for each \( i = 1, 2, \ldots \) via an injective group morphism \( j_i : \Lambda \hookrightarrow \text{Aut}(G_i) \). Equip each \( G_i \) with the discrete topology, and \( G := \prod_{i=1}^{\infty} G_i \) the product topology. Then \( G \) is a compact abelian group. In particular, the character group \( \chi(G) \) of \( G \), as a set of one-dimensional representations, is a complete set of representatives of \( \text{Irr}(G) \). By Pontryagin’s duality, we have \( \chi(G) \cong \bigoplus_{i=1}^{\infty} \chi(G_i) \), and it is clear that length functions on \( \hat{G} \) become exactly length functions on the discrete group \( \chi(G) \) of continuous characters of \( G \). But as a finite abelian group, each \( G_i \) is isomorphic to \( \chi(G_i) \) (albeit the isomorphism is not natural in the categorical sense). Thus Lemma 8.2 shows that there exists a \( \bigoplus_{i=1}^{\infty} \text{Aut}(G_i) \)-invariant length function \( l_G \) on \( \hat{G} \), such that \( (\hat{G}, l_G) \) has polynomial growth, where we’ve used the canonical inclusion \( \prod_{i=1}^{\infty} \text{Aut}(G_i) \subset \text{Aut}(G) \).

The construction of \( \Gamma \) takes some more work which we now explain. First we take \( \Lambda' \) to be any nontrivial finite group and set \( \Gamma_1 \) to be the free product \( \Lambda \ast \Lambda' \). It follows from Jolissaint’s theorem and Gromov’s theorem that \( \Gamma_1 \) has (RD) but not polynomial growth. Define \( j : \Lambda \hookrightarrow \prod_{i=1}^{\infty} \text{Aut}(G_i) \) to be the mapping \( \lambda \mapsto (j_1(\lambda), j_2(\lambda), \cdots) \). Take any infinite discrete subgroup \( \Gamma_2' \) of \( \bigoplus_{i=1}^{\infty} \text{Aut}(G_i) \subset \prod_{i=1}^{\infty} \text{Aut}(G_i) \) such that \( j(\Lambda) \) is contained in the normalizer of \( \Gamma_2' \) in \( \prod_{i=1}^{\infty} \text{Aut}(G_i) \). Obviously \( j(\Lambda) \) and \( \Gamma_2' \) intersect trivially, thus the subgroup of \( \prod_{i=1}^{\infty} \text{Aut}(G_i) \) generated by \( j(\Lambda) \) and \( \Gamma_2' \) is the (internal) semidirect product of \( \Gamma_2' \) with \( j(\Lambda) \), which we denote by \( \Gamma_2 \). Since \( \bigoplus_{i=1}^{\infty} \text{Aut}(G_i) \) has polynomial growth by Lemma 8.2, it follows that \( \Gamma_2' \), hence \( \Gamma_2 \), (note that \( [\Gamma_2' : \Gamma_2] = |\Lambda| \) is finite) has polynomial growth. In particular, \( \Gamma_2 \) has (RD), and \( j : \Lambda \hookrightarrow \prod_{i=1}^{\infty} \text{Aut}(G_i) \) restricts an injective group morphism, which we still denote by \( j \), from \( \Lambda \) into \( \Gamma_2 \). To facilitate our discussion, we identify \( \Lambda \) with its copy in \( \Gamma_1 = \Lambda \ast \Lambda' \) and in \( \Gamma \) via \( j \). This allows us to form the amalgamated product of \( \Gamma_1 \) and \( \Gamma_2 \) over \( \Lambda \), which we denote by \( \Gamma \). Jolissaint’s theorem applies again and proves that \( \Gamma \) has (RD). Moreover, \( \Gamma \) does not have polynomial growth since its subgroup \( \Gamma_1 \) does not. We also make the obvious identification of \( \Lambda \) with \( j(\Lambda) \) in \( \Gamma \). By Corollary 6.3, there exists a \( \Lambda \)-invariant length function \( l_{\Gamma_1} \) on \( \Gamma_1 \), meaning \( l_{\Gamma_1} = l_{\Gamma_1} \circ \text{Ad}_r \) for all \( r \in \Lambda \), such that \( (\Gamma, l_{\Gamma_1}) \) has (RD).

Finally, let’s explain how the action, which is a group morphism \( \tau : \Gamma \rightarrow \text{Aut}(G) \), is defined. The trivial group morphism \( \Lambda' \rightarrow \text{Aut}(G) \), together with \( j : \Lambda \rightarrow \prod_{i=1}^{\infty} \text{Aut}(G_i) \subseteq \text{Aut}(G) \) and the universal property of free products, yields a group morphism \( \tau_1 : \Gamma_1 \rightarrow \text{Aut}(G) \). Let \( \tau_2 \) be the simple inclusion \( \Gamma_2 \hookrightarrow \prod_{i=1}^{\infty} \text{Aut}(G_i) \subseteq \text{Aut}(G) \). It is clear that \( \tau_1 \) and \( \tau_2 \) agree on \( \Lambda \), thus the universal property of the amalgamated product \( \Gamma_1 \ast_{\Lambda} \Gamma_2 \) applies and determines a unique group morphism \( \tau : \Gamma \rightarrow \text{Aut}(G) \) that restricts to \( \tau_1 \) on the corresponding copy of \( \Gamma_1 \). We can finally construct the bicrossed product \( \Gamma \bowtie (G \rtimes \Lambda) \), and we conclude by Theorem 1.1 that the dual of \( \Gamma \bowtie (G \rtimes \Lambda) \) has (RD) (it does not have polynomial growth because of Theorem 2.7 and the fact that \( \Gamma \) does not have polynomial growth).

It is clear by our construction that \( \text{Image}(\tau) = \Gamma_2 \subseteq \text{Aut}(G) = \text{Out}(G) \) is infinite, thus Theorem 1.2 does not apply.

References

[1] Jyotishman Bhowmick, Christian Voigt, and Joachim Zacharias, *Compact quantum metric spaces from quantum groups of rapid decay*, J. Noncommut. Geom. 9 (2015), no. 4, 1175–1200.

[2] Nathanial P. Brown and Narutaka Ozawa, *C*-algebras and finite-dimensional approximations*, Graduate Studies in Mathematics, vol. 88, American Mathematical Society, Providence, RI, 2008.

[3] Chuangxun Cheng, *A character theory for projective representations of finite groups*, Linear Algebra Appl. 469 (2015), 230–242.

[4] Pierre Fima, Kunal Mukherjee, and Issan Patri, *On compact bicrossed products*, J. Noncommut. Geom. 11 (2017), no. 4, 1521–1591.
[5] Pierre Fima and Hua Wang, *Rapid decay and polynomial growth for bicrossed products*, J. Noncommut. Geom. 15 (2021), no. 3, 1105–1128.

[6] Mikhael Gromov, *Groups of polynomial growth and expanding maps*, Inst. Hautes Études Sci. Publ. Math. 53 (1981), 53–73.

[7] Paul Jolissaint, *Rapidly decreasing functions in reduced $C^*$-algebras of groups*, Trans. Amer. Math. Soc. 317 (1990), no. 1, 167–196.

[8] G. I. Kac, *Group extensions which are ring groups*, Mat. Sb. (N.S.) 76 (118) (1968), 473–496.

[9] George W. Mackey, *Imprimitivity for representations of locally compact groups. I*, Proc. Nat. Acad. Sci. U. S. A. 35 (1949), 537–545.

[10] ———, *Unitary representations of group extensions. I*, Acta Math. 99 (1958), 265–311.

[11] Sergey Neshveyev and Lars Tuset, *Compact quantum groups and their representation categories*, Cours Spécialisés [Specialized Courses], vol. 20, Société Mathématique de France, Paris, 2013.

[12] Stefaan Vaes and Leonid Vainerman, *Extensions of locally compact quantum groups and the bicrossed product construction*, Adv. Math. 175 (2003), no. 1, 1–101.

[13] Roland Vergnioux, *The property of rapid decay for discrete quantum groups*, J. Operator Theory 57 (2007), no. 2, 303–324.

[14] Hua Wang, *On representations of semidirect products of a compact quantum group with a finite group*, arXiv preprint arXiv:1909.02359 (2019).

[15] ———, *Rapid decay of bicrossed products and representation theory of some semidirect products*, Ph.D. Thesis, 2020.

[16] S. L. Woronowicz, *Unbounded elements affiliated with $C^*$-algebras and noncompact quantum groups*, Comm. Math. Phys. 136 (1991), no. 2, 399–432.

[17] ———, *Compact quantum groups*, Symétries quantiques (Les Houches, 1995), 1998, pp. 845–884.