Optimal Portfolio Selection with Regime-Switching Hamilton-Jacobi-Bellman (HJB) Equation and Maximum Value-at-Risk (MVaR) Constraint

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Abstract. The forming of portfolio is necessary to determine the decision of the best investment so as to investors can identify the securities and determine the allocation of asset to obtain an optimal portfolio. The problem in forming optimal portfolio is the determination of the proportion which is allocated at investment assets in order to maximize expected return with certain risks. The model contains regime-switching market models, which states are interpreted as the states of economy. The risk measuring instrument used is VaR and MVaR is defined as the maximum value of the VaRs in all economy states. The optimal proportion formula is sought by using the stochastic optimal control theory with the aim of maximizing the discounted utility of consumption over a finite time horizon. We use regime-switching Hamilton-Jacobi-Bellman (HJB) equation and then derive a system of coupled HJB equation corresponding to the economy states. Lagrange multiplier method is used to solve the optimization problem with the constraint. We apply Kuhn-Tucker conditions due to MVaR is an inequality function, so that we derive the optimal investment and the optimal consumption. Finally, numerical examples are investigated, and the effect of parameter on the optimal investment and on the optimal consumption are studied.

1. Introduction
The optimal portfolio allocation problem is quite importance in finance. The problem is to search the best allocation of wealth among some kinds of securities. The pioneer Markowitz [1] provided a mathematical model to formulate the optimal portfolio allocation problem and developed the mean-variance approach for this problem. Furthermore, the pioneering development of the optimal portfolio allocation problem in a continuous-time framework is provided by Merton [2,3]. Afterwards, many researchers extended Merton’s model.

There could be substantial changes in the economic condition over a period of time. The introduction of regime-switching effect is important in economics and finance. The market parameters, such as interest rate of bank bond, the appreciation rate, and volatility rate of risky asset, are modulated by the financial market state which is often modeled by a Markov chain. Neglecting the regime-switching effect may lead to underestimation of the risk portfolio and suboptimal results. Hamilton [4] pioneered the econometric applications of regime-switching models. Some papers with regime-switching models in finance include [7] and [8] for consumption–investment problems. Usually, investment–consumption problems focus on maximizing utility from consumption. The investor wants to maximize his/her total discounted utility over period of time in which he/she invests. In this paper, the risk is measured by using Value-at-Risk (VaR). In each state regime-switching
model, we constrain a VaR value of the portfolio in a short time duration. Maximum Value-at-Risk (denoted by MVaR) is defined as the maximum value of the VaRs in all economy states. We consider the optimal investment-consumption problem subject to a maximum Value-at-Risk constraint.

This paper is structured as follows. Section 2 describes the price dynamics of the model. Section 3 determines the optimal portfolio problem and formulates the MVaR constraint. Section 4 derives the regime-switching HJB equation and present the method of the Lagrange. Section 5 determines the value function and derives the optimal solutions. In Section 6, we make numerical experiments to investigate the effect of many parameters on the optimal investment and on the optimal consumption. Finally, we conclude the paper in the last section.

2. Model formulation

In this section, we shall consider a continuous-time financial model in which two assets are traded continuously over a finite time horizon \( T = [0, T] \), where \( T \in (0, \infty) \), one is risk-free asset \( B \) and the other one is a risky asset \( P \). We shall consider the situation when there are two states in the Markov chain. Denote \( X = \{X(t)\}_{t \in T} \) as a continuous-time Markov chain with a finite state space \( x = (x_1, x_2) \). The state of \( X \) are interpreted as different states of an economy. Following [5], we shall represent the state \( x \) as a finite set of unit vectors \( \mathbf{e} = \{\mathbf{e}_1, \mathbf{e}_2\} \), where \( \mathbf{e}_1 = (1, 0)' \) and \( \mathbf{e}_2 = (0, 1)' \), where ‘’ represents the transpose of a matrix or a vector. Suppose that the Markov chain has a generator \( Q = (q_{ij})_{2 \times 2} \), \( i, j = 1, 2 \). The price process of risk-free asset \( B \) is governed by

\[
dB(t) = B(t) \, r_i \, dt, \quad B(0) = b_0, \quad i = 1, 2
\]

where \( r_i \) denotes the instantaneous market interest rate with \( r_1 > 0, r_2 > 0, i = 1, 2 \). Suppose \( \mu_i \) and \( \sigma_i \) denote the appreciation rate and the volatility rate of the risky asset with \( \mu_i > r_i \) and \( \sigma_i > 0 \) for each \( i = 1, 2 \).

Let \( W(t) \) denote a standard Brownian motion on \( (\Omega, \mathcal{F}, \mathcal{P}) \). The price process of risky asset \( P \) is assumed to be

\[
dP(t) = P(t) \left( \mu_i \, dt + \sigma_i \, dW(t) \right), \quad P(0) = p_0, \quad t \in T, \quad i = 1, 2
\]

We assume that the market agent can invest into the risk-free asset \( B \) and the risky asset \( P \). Let \( \pi_i(t) \) denotes the amount of wealth allocated to the risky asset \( P \) at time \( t \) and state \( i \). Let \( c_i(t) \) denotes the consumption rate of an economic agent at time \( t \) and state \( i \). Define \( u(t) = (c_i(t), \pi_i(t)) \) as our control process. Suppose \( \{S(t)\}_{t \in T} \) denotes the net wealth process of the economic agent with initial wealth \( S(0) = s_0 \) and initial state \( X(0) = x_0 \), then

\[
dS(t) = [\pi_i(t)(\mu_i - r_i) + S(t)r_i - c_i(t)] dt + \pi_i(t)\sigma_i dW(t)
\]

3. Determination of optimal portfolio problem and VaR constraint

In this section, we will discuss about the optimal portfolio problem and VaR constraint.

3.1. Forming the Objective Function Using Stochastic Optimal Control Theory

Let \( U(t, c_i) : \mathcal{T} \times [0, \infty) \rightarrow \mathbb{R} \) denote a utility function such that for each \( t \in \mathcal{T} \). \( U(t, c_i) \) is strictly increasing and strictly concave. Let \( \delta \) denotes a discount factor and a positive constant. Suppose the economic agent needs continuous consumption over the time horizon \( \mathcal{T} \) and he/she uses the portfolio process \( \pi_i(t) \) available at initial wealth \( S(0) = s_0 \) and initial state \( X(0) = x_0 \). So, the expected discounted utility of the agent is defined as

\[
J(s_0, x_0, c_i(t), \pi_i(t)) = E \left[ \int_{t}^{T} U(\tau, c_i(\tau)) d\tau \left| S_0 = s_0, X(0) = x_0 \right. \right]
\]

3.2. Value-at-Risk (VaR) Constraint

In this sequel, we shall present the MVaR constraint of the problem. We assume that the portfolio is adjusted frequently so that the interval from \( t \) to \( t + h \) is small, where \( h > 0 \). It is reasonable to assume that there is no trading in between constraint revaluation and that the consumption is approximately constant in the little time interval \([t, t + h] \) since in reality, one can only adjust the portfolio discretely over time, and the portfolio/consumption decision is made at the beginning of the
time horizon. That is, \( \pi(\mathbf{r}) = \pi(t) \) and \( c(\mathbf{r}) = c(t) \) for all \( \mathbf{r} \in [t, t + h] \). We also assume that there is no regime switching in the little time interval. In other words, \( X(\mathbf{r}) = X(t) \), for all \( \mathbf{r} \in [t, t + h] \).

First, we define the following two quantities:

\[
\theta_i(\mathbf{r}) = \theta_i(t) = \frac{\pi_i(t)\mu_i - r_i - c_i(t)}{-r_i} \quad \text{and} \quad Y(\mathbf{r}) = \exp[-r_i \tau]S(\tau), i = 1, 2, \ldots, N, \mathbf{r} \in [t, t + h].
\]

To define VaR constraints in the sequel, we regulate (3) and get \( dS(t) - r_i S(t)dt = -r_i \theta_i(t)dt + \pi_i(t) \sigma_i dW(t) \). Put \( Y(t) = \exp[-r_i \tau]S(t) \), then we get

\[
dY(t) = -\exp[-r_i \tau]r_i \theta_i(t)dt + \exp[-r_i \tau] \pi_i(t) \sigma_i dW(t)
\]

\[
\Leftrightarrow \int_t^{t+h} dY(\tau) = \int_t^{t+h} -\exp[-r_i \tau]r_i \theta_i(\tau)d\tau + \int_t^{t+h} \exp[-r_i \tau] \pi_i(t) \sigma_i dW(\tau)
\]

\[
\Leftrightarrow \frac{dS(t + h)}{S(t)} = \exp[-r_i h][S(t) - \theta_i(t)] + \theta_i(t) + \pi_i(t) \sigma_i \exp[-r_i (t + h)] \int_t^{t+h} \exp[-r_i \tau] dW(\tau)
\]

Mean and variance of \( dW(t) \) are \( E(dW(t)) = 0 \) and \( Var(dW(t)) = dt \). We define the discounted net loss of the portfolio over \( [t, t + h] \) as \( \Delta_i S(t, h) = S(t) - \exp[-r_i h]S(t + h), \forall i = 1, 2 \). It can be viewed as a function of the random variable \( S(t + h) \) with mean and variance:

\[
E[\Delta_i S(t, h)] = \theta_i(t)(1 - \exp[-r_i h]) \quad \text{and} \quad Var[\Delta_i S(t, h)] = \sigma_i^2 \pi_i^2(t)(2r_i)^{-1}(1 - \exp[-2r_i h])
\]

Now, we define the VaR value of \( \Delta_i S(t, h) \) with the probability level \( \beta \) as \( VaR(t, h, i, \beta) \). Since \( \Delta_i S(t, h) \) is normally distributed under the condition \( X(t) = \epsilon_i \) and we assume that there is no short selling, then we derive

\[
P[\Delta_i S(t, h) \geq VaR(t, h, i, \beta)|X(t) = \epsilon_i] = \beta
\]

\[
\Leftrightarrow VaR(t, h, i, \beta) = \theta_i(t)(1 - \exp[-r_i h]) \quad \text{and} \quad \Phi^{-1}(\beta) \sqrt{\frac{\sigma_i^2 \pi_i^2(t)}{2r_i}} (1 - \exp[-2r_i h]), i = 1, 2
\]

(5)

We define the MVaR of the portfolio with the probability level \( \beta \) over the time horizon \( [t, t + h] \) as

\[
MVar(t, h, \beta) = \max_{i=1,2} VaR(t, h, i, \beta)
\]

In other words, MVaR is the maximum value of the VaRs of the portfolio in a short time duration over different states of economy. Then, the constraint of restricting the MVaR at level \( R \) is

\[
MVar(t, h, \beta) \leq R
\]

This is equivalent to the following constraint:

\[
\theta_i(t)(1 - \exp[-r_i h]) \quad \text{and} \quad \Phi^{-1}(\beta) \sqrt{\frac{\sigma_i^2 \pi_i^2(t)}{2r_i}} (1 - \exp[-2r_i h]) \leq R
\]

(6)

The portfolio optimization problem with MVaR constraint is then formulated as

\[
\max_{\pi_i(t), c_i(t)} \mathbb{E} \left[ \int_t^T U(\tau, c_i(\tau))d\tau \right]
\]

subject to

\[
dS(t) = [\pi_i(t)(\mu_i - r_i) + S(t)r_i - c_i(t)]dt + \pi_i(t) \sigma_i dW(t)
\]

\[
\theta_i(t)(1 - \exp[-r_i h]) \quad \text{and} \quad \Phi^{-1}(\beta) \sqrt{\frac{\sigma_i^2 \pi_i^2(t)}{2r_i}} (1 - \exp[-2r_i h]) \leq R
\]

4. Regime-switching HJB equation

In this section, we shall derive a regime-switching HJB equation for the value function described in (7). We shall also derive a system of coupled HJB equations corresponding to the regime-switching HJB equation. Then, we shall derive the optimally conditions for solving the optimal problem.

First, we consider the following optimal value function as
Let $V_i = V(t, s, \epsilon_i)$ for each $i = 1, 2$ and let $V = (V_1, V_2)'$. Hence, the vector $V$ of the value function as different regimes satisfies the following systems of coupled HJB equations:

\[
\sup_{\pi_1, c_1} \left[ U(t, c_1(t)) + \frac{\partial V_1}{\partial t} + [\pi_1(t)(\mu_1 - r_1) + S(t)r_1 - c_1(t)] \frac{\partial V_1}{\partial s} + \frac{1}{2} \left[ (\pi_1(t)\sigma_1)^2 \right] \frac{\partial^2 V_1}{\partial s^2} + \langle V, Q \epsilon_1 \rangle \right] = 0
\]

\[
\sup_{\pi_2, c_2} \left[ U(t, c_2(t)) + \frac{\partial V_2}{\partial t} + [\pi_2(t)(\mu_2 - r_2) + S(t)r_2 - c_2(t)] \frac{\partial V_2}{\partial s} + \frac{1}{2} \left[ (\pi_2(t)\sigma_2)^2 \right] \frac{\partial^2 V_2}{\partial s^2} + \langle V, Q \epsilon_2 \rangle \right] = 0
\]

with terminal condition $V(T, s, \epsilon_i) = 0$ for $i = 1, 2$ and subject to constraint:

\[
\theta_i(t)(1 - \exp[-r_i h]) - \Phi^{-1}(\beta) \sqrt{\frac{\sigma_i^2 \pi_i^2(t)}{2r_i}} (1 - \exp[-2r_i h]) \leq R
\]

In order to derive the optimal investment strategy and consumption rate, we introduce the Lagrange functions and tackle the optimization problem (9) and (10) as follow for $i = 1, 2$:

\[
\mathcal{V}_i(\pi_i(t), c_i(t), \lambda(t)) = \left( U(t, c_i(t)) + \frac{\partial V_i}{\partial t} + [\pi_i(t)(\mu_i - r_i) + S(t)r_i - c_i(t)] \frac{\partial V_i}{\partial s} + \frac{1}{2} \left[ (\pi_i(t)\sigma_i)^2 \right] \frac{\partial^2 V_i}{\partial s^2} + \langle V, Q \epsilon_i \rangle \right) + \lambda \left(\left( \frac{\pi_i(t)(\mu_i - r_i) - c_i(t)}{-r_i} \right)(1 - \exp[-r_i h]) - \Phi^{-1}(\beta) \sqrt{\frac{\sigma_i^2 \pi_i^2(t)}{2r_i}} (1 - \exp[-2r_i h]) \right) - R
\]

These are the three conditions that must be fulfill in order to achieve the optimal process for $i = 1, 2$:

\[
\mathcal{V}_{\pi_i(t), c_i(t)} \mathcal{V}_i(\pi_i(t), c_i(t), \lambda(t)) = 0
\]

\[
\left( \frac{\pi_i(t)(\mu_i - r_i) - c_i(t)}{-r_i} \right)(1 - \exp[-r_i h]) - \Phi^{-1}(\beta) \sqrt{\frac{\sigma_i^2 \pi_i^2(t)}{2r_i}} (1 - \exp[-2r_i h]) \leq R
\]

\[
\lambda(t) \left(\left( \frac{\pi_i(t)(\mu_i - r_i) - c_i(t)}{-r_i} \right)(1 - \exp[-r_i h]) - \Phi^{-1}(\beta) \sqrt{\frac{\sigma_i^2 \pi_i^2(t)}{2r_i}} (1 - \exp[-2r_i h]) \right) - R = 0
\]

Rearranging (12), (13), and (14), we derive $\pi_{i_{\text{opt}}}(t)$ and $c_{i_{\text{opt}}}(t)$ for $i = 1, 2$.

\[
\pi_{i_{\text{opt}}}(t) = \begin{cases} 
- \frac{\mu_i - r_i}{\sigma_i^2 \frac{\partial^2 V_i}{\partial s^2}}, & \text{unconstrained MVaR} \\
R - \frac{c_i(t)}{r_i} (1 - \exp[-r_i h]), & \text{constrained MVaR}
\end{cases}
\]

and

\[
\frac{\partial V_i}{\partial s} = \begin{cases} 
- \frac{(\mu_i - r_i)(1 - \exp[-r_i h]) - \Phi^{-1}(\beta) \sqrt{\frac{\sigma_i^2 \pi_i^2(t)}{2r_i}} (1 - \exp[-2r_i h])}{r_i}, & \text{constrained MVaR}
\end{cases}
\]
\[
\frac{\partial U}{\partial c} = \frac{\partial V_1}{\partial s}, \frac{\partial U}{\partial s} = \frac{\partial V_1}{\partial c}
\]  

(16)

5. Determination of value function and optimal solutions

In this section, we shall consider the situation when there are two states in the Markov chain. The purpose of this portfolio is to maximize return, the return function that will be used is in the form of a utility function \(U(t, c_i(t))\) and the economic agent has a power utility function for consumption:

\[
U(t, c_i(t)) = \exp[-\delta t]c_i(t)^\gamma, \quad \delta > 0, \quad 0 < \gamma < 1, i = 1, 2
\]

(17)

Here \(\delta\) represents a discount factor for consumption, \(\gamma\) represents the investor’s preference, and it is assumed to be a positive constant. Moreover, we assume that the rate matrix \(Q\) of the Markov chain is

\[
Q = \begin{pmatrix} -p & p \\ p & -p \end{pmatrix}
\]

where \(p\) is a positive real constant. In this case, the value functions for two economics states and by substituting \(\pi_{i_{\text{opt}}}(t)\) and \(c_{i_{\text{opt}}}(t)\) into (9) and (10) satisfy the following coupled HJB equations:

\[
\frac{\partial V_1}{\partial t} + \exp[-\delta t]c_{1_{\text{opt}}}(t) + \left[\pi_{1_{\text{opt}}}(t)(\mu_1 - r_1) + sr_1 - c_{1_{\text{opt}}}(t)\right] \frac{\partial V_1}{\partial s} + \frac{1}{2}\pi_{1_{\text{opt}}}(t)s^2 \frac{\partial^2 V_1}{\partial s^2} + p(V_2 - V_1) = 0
\]

(18)

and

\[
\frac{\partial V_2}{\partial t} + \exp[-\delta t]c_{2_{\text{opt}}}(t) + \left[\pi_{2_{\text{opt}}}(t)(\mu_2 - r_2) + sr_2 - c_{2_{\text{opt}}}(t)\right] \frac{\partial V_2}{\partial s} + \frac{1}{2}\pi_{2_{\text{opt}}}(t)s^2 \frac{\partial^2 V_2}{\partial s^2} + p(V_1 - V_2) = 0
\]

(19)

Following the approach in [3], we assume that the value function is of the following form:

\[
V_i = V(t, s, e_i) = \exp[-\delta t]h_i(t, s)s^\gamma, \quad i = 1, 2
\]

This form is in line with the form of the power utility function. As in [6], we neglect the derivatives of \(h_i\), \(i = 1, 2\) with respect to \(v\) and obtain

\[
\frac{\partial V_i}{\partial s} = \gamma \exp[-\delta t]h_i(t, s)s^{\gamma - 1}, \quad \frac{\partial^2 V_i}{\partial s^2} = \gamma (\gamma - 1) \exp[-\delta t]h_i(t, s)s^{\gamma - 2},
\]

(20)

where \(h_i^\gamma\) represent the derivative of \(h_i\) with respect to \(t\). Substitute (20) to (18) and (19), also substitute (20) to (15), we derive the optimal proportion of risky asset (\(\pi_{i_{\text{opt}}}(t)\)) and the optimal consumption (\(c_{i_{\text{opt}}}(t)\)) for \(i = 1, 2\)

\[
\pi_{i_{\text{opt}}}(t) = \begin{cases} 
\left(\frac{\mu_i - r_i}{\sigma_i^2(1 - \gamma)}\right), \text{unconstrained MVaR} \\
\frac{R - c_i(t)}{r_i} \left(1 - \exp[-r_i h]\right) \frac{\sigma_i^2}{\sqrt{2r_i}} \left(1 - \exp[-2r_i h]\right), \text{constrained MVaR}
\end{cases}
\]

(21)

and

\[
c_{i_{\text{opt}}}(t) = s \exp \left[ - \int_0^t \frac{1}{2} \frac{(\mu_i - r_i)^2}{\sigma_i^2(1 - \gamma)^2} + \frac{r_i\gamma - \delta}{(1 - \gamma)} ds \right] \int_t^T \exp \left[ \int_t^s \frac{1}{2} \frac{(\mu_i - r_i)^2}{\sigma_i^2(1 - \gamma)^2} + \frac{r_i\gamma - \delta}{(1 - \gamma)} ds \right] ds
\]

6. Simulation of proportion and consumption calculation in optimal portfolio

In this section, we shall conduct numerical experiments to provide many special effects between parameters as follows. Suppose the investor has some specific values for the model parameters: \(T = 25\), \(R = 5\), \(\beta = 0.99\), \(\delta = 0.2\), \(\gamma = 0.5\), \(r_1 = 0.15\), \(r_2 = 0.1\), \(\mu_1 = 0.2\), \(\mu_2 = 0.15\), \(\sigma_1 = 0.4\), \(\sigma_2 = 0.6\). We substitute all of those value to (21). In this example, we assume that there are two regimes in
the economy, where regime 1 represents a bull market, a market in which share prices are rising or encouraging buying and regime 2 represents a bear market, a market in which prices are falling or encouraging selling. According to [9], the stock returns are higher in a bull market, so $\mu_1 > \mu_2$. [10] found that stock volatility is higher in a bear market; thus $\sigma_1 < \sigma_2$.

![Figure 1. $\pi_{1,opt}$ and $\pi_{2,opt}$ for $R = 5, t = 5$](image1)

![Figure 2. $\pi_{1,opt}$ and $\pi_{2,opt}$ for $R = 5, t = 12$](image2)

From the graph, we conclude that for the same consumption rate, the allocation of wealth for risky asset in a bull market is bigger than the allocation of wealth for risky asset in a bear market. But, in a bear market, the amount of consumption will be bigger than the amount of consumption in bull market.

7. Conclusions
We consider the optimal portfolio selection problem subject to a maximum Value-at-Risk constraint when the price dynamics of risky asset are governed by a Markov-modulated GBM. The model also contains regime-switching market modes. The optimal selection problem is formulated as a constrained utility maximization problem over a finite time horizon. We derive a system of coupled HJB equations for the problem. Moreover, with the help of Lagrange multiplier method, we derive the optimal investment and the optimal consumption. By numerical simulation, we conclude that the allocation of wealth for risky asset in bull market is bigger rather than the bear market.

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