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CELLULAR BASIS AND DIMENSIONS OF CELL MODULES OF PARTY ALGEBRAS

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Abstract. In this paper, the basis for the cell modules of party algebras are obtained and its dimension formula is given explicitly.

1. Introduction
A basis for the centralizer algebra of tensor product of monomial representations of the complex reflection group \( G(m, p, n) \) is explicitly given in [13], which led to the definition of Tanabe algebra as a subalgebra of partition algebra over the complex field in [11]. The particular case of a Tanabe algebra is a party algebra, which was introduced by Kosuda in [5]. The ordinary representations of the party algebras over the field of characteristic zero have been studied in [7] and [11]. The cellularity of these algebras are studied by Kosuda in [8].

In this paper, we establish a basis and dimension for the cell modules of party algebras.

2. Preliminaries
For \( k \in \mathbb{N} \), let \( k = \{1, 2, \ldots, k\} \), \( k' = \{1', 2', \ldots, k'\} \) and \( E_k \) (\( E_{k'} \)) denotes the set of all set partitions of \( k(k') \) respectively. For \( d^+ \in E_k \), the elements of \( d^+ \) are called connected components. Any \( d^+ \in E_k \) can be represented as a simple graph (or diagram) consisting of \( k \) vertices arranged in a row and an edge can be drawn between two vertices if the vertices are in the same connected component of \( d^+ \) and the diagram need not be unique. The diagram \( d^{+'} \in E_{k'} \) is obtained from \( d^+ \) by re-indexing the vertices \( i \) as \( i' \). Similarly for \( d^- \in E_{k'} \), the elements of \( d^- \) are called connected components. Any \( d^- \in E_{k'} \) can be represented as a simple graph (or diagram) consisting of \( k \) vertices arranged in a row and an edge can be drawn between two vertices if the vertices are in the same connected component of \( d^- \) and the diagram need not be unique. The diagram \( d^- \in E_k \) is obtained from \( d^- \) by re-indexing the vertices \( i' \) as \( i \).

For example \( k = 10 \), \( d^+ = \{\{1, 7, 9\}, \{2, 4\}, \{3\}, \{5, 6\}, \{8, 10\}\} \) the diagrams \( d^+ \) and \( d^{+'} \) are
Definition 2.1. [12] A partition of a non-negative integer \( k \) is a sequence of non-negative integers \( \alpha = (\alpha_1, \alpha_2, \ldots, \alpha_s) \) such that \( \alpha_1 \geq \alpha_2 \geq \ldots \geq \alpha_s \geq 0 \) and \( |\alpha| = \alpha_1 + \alpha_2 + \ldots + \alpha_s = k \). The non-zero \( \alpha_i \)'s are called the parts of \( \alpha \) and the number of non-zero parts is called the length of \( \alpha \). The notation \( \alpha \vdash k \) denotes that \( \alpha \) is a partition of \( k \).

Let \( P_k = \{ \alpha \mid \alpha \vdash k \} \).

For example, \( P_5 = \{(5), (4, 1), (3, 2), (3, 1, 1), (2, 2, 1), (2, 1, 1, 1), (1, 1, 1, 1, 1)\} \).

Define \( \varphi : E_k \rightarrow P_k \) such that \( \varphi(d^+) = \lambda = (k^{r_k}, (k - 1)^{r_{k-1}}, \ldots, 2^{r_2}, 1^{r_1}) \) where \( i^{r_i} = i, i, i, \ldots, i \) \( r_i \)-times and \( r_i \) is the number of connected components in \( d^+ \) having exactly \( i \) elements.

Definition 2.2. [3] For \( k \in \mathbb{N} \), let \( A_k \) be the set of all set partitions of \( \{k \cup k'\} \). For \( d \in A_k \), the elements of \( d \) are called as connected components. Any \( d \in A_k \) can be represented as a simple graph (or diagram) consisting of \( 2k \) vertices arranged in two rows. The first (top) row vertices are indexed by \( 1, 2, \ldots, k \) and the second (bottom) row vertices are indexed by \( 1', 2', \ldots, k' \) respectively, an edge can be drawn between two vertices if the vertices are in same connected component of \( d \) and the diagram need not be unique.

For example \( k = 10 \),

\[
\begin{align*}
\text{d} & = \begin{array}{ccccccccccc}
1 & 2 & 3 & 4 & 5 & 6 & 7 & 8 & 9 & 10 \\
1' & 2' & 3' & 4' & 5' & 6' & 7' & 8' & 9' & 10'
\end{array}
\end{align*}
\]

Definition 2.3. [3] The composition \( d \circ d' \) of partition diagrams \( d, d' \in A_k \) to be the set partition \( d \circ d' \in A_k \) obtained by placing \( d \) above \( d' \) and identifying bottom row vertices of \( d \) with the top row vertices of \( d' \) removing any connected components that live entirely in the middle row.

For example,

\[
\begin{align*}
\text{d} & = \begin{array}{ccccccc}
1 & 2 & 3 & 4 & 5 & \ldots & \\
1' & 2' & 3' & 4' & 5' & & \\
\end{array} \\
\text{d'} & = \begin{array}{ccc}
1 & 2 & 3 \\
1' & 2' & 3'
\end{array}
\end{align*}
\]

the composition of \( d \circ d' \) is

\[
\begin{array}{ccccccc}
1 & 2 & 3 & 4 & 5 & \ldots & \\
1' & 2' & 3' & 4' & 5' & & \\
\end{array}
\]

The composition is independent of choice of representative of the diagrams \( d \) and \( d' \). Diagram multiplication make \( A_k \) into an associative monoid with identity, \( 1 = \{\{1, 1'\}, \{2, 2'\}, \{3, 3'\}, \ldots, \{k, k'\}\} \).
Definition 2.4. For \( k > 0 \), the partition algebra \( R(x)A_k \) is an associative algebra over the polynomial ring \( R(x) \) with basis \( A_k \), where \( R \) is a commutative ring with unity, \( R(x)A_k \) is the linear span of \( A_k \) over the polynomial ring \( R(x) \). The multiplication defined by \( dd' = x^{l_{dd'}} (d \circ d') \), where \( d, d' \in A_k \), \( d \circ d' \) is the product in the monoid \( A_k \) and \( l_{dd'} \) is the number of connected components removed from the middle row when constructing the composition \( d \circ d' \).

For example,

\[
\begin{align*}
\begin{array}{cccccc}
1 & 2 & 3 & 4 & 5 \\
1' & 2' & 3' & 4' & 5'
\end{array}
\end{align*}
\]

\[
\begin{align*}
\begin{array}{cccccc}
1 & 2 & 3 & 4 & 5 \\
1' & 2' & 3' & 4' & 5'
\end{array}
\end{align*}
\]

the composition of \( dd' \) is

\[
\begin{align*}
\begin{array}{cccccc}
1 & 2 & 3 & 4 & 5 \\
1' & 2' & 3' & 4' & 5'
\end{array}
\end{align*}
\]

since one connected component is removed from the middle row.

2.1. Party Algebras

Definition 2.5. For \( d = \{B_1, B_2, \ldots, B_s\} \in A_k \), let \( d^+ = \{B_i \cap k \mid 1 \leq i \leq s\} \) and \( d^- = \{B_i \cap k' \mid 1 \leq i \leq s\} \), \( d^+(d^-) \) is the top (bottom) row of the diagram \( d \).

Definition 2.6. For \( d = \{B_1, B_2, \ldots, B_s\} \in A_k \), let \( N(B_i) = \#\{B_i \cap k\} \) and \( M(B_i) = \#\{B_i \cap k'\} \), \( \#\{B_i \cap k\} (\#\{B_i \cap k'\}) \) is the number of elements in the connected component \( B_i \cap k \) (\( B_i \cap k' \)).

Definition 2.7. For \( k \in \mathbb{N} \), let

\[
L_k = \{d \in A_k \mid N(B_i) = M(B_i) \text{ for all } i\}.
\]

Theorem 2.8. For \( k \in \mathbb{N} \) the party algebra \( R(x)L_k \) is an associative subalgebra of \( R(x)A_k \) with basis \( L_k \).

Definition 2.9. Let \( d_i^+(d_i^-) \) and \( d_j^+(d_j^-) \) be the partitions of \( k(k') \), \( d_i^+(d_i^-) \) is said to be finer than \( d_j^+(d_j^-) \), if each connected component of \( d_i^+(d_i^-) \) is contained in some connected component of \( d_j^+(d_j^-) \).

Definition 2.10. For \( d \in L_k \), let \( l(d) \) be the number of connected components of \( d \).

Remark 2.11. For \( d_i, d_j \in L_k \), we have

i). \( t(d_i d_j) \leq \min(t(d_i), t(d_j)) \).
ii). \( t(d_i d_j) = \min(t(d_i), t(d_j)) \) if and only if either \( d_i^- \) is finer than \( d_j^+ \) or \( d_j^- \) is finer than \( d_i^+ \).

Remark 2.12. For \( d \in L_k \), we have \( \varphi(d) = \varphi(d') \).
2.2. Generators and Relations of party algebras
For \( k \in \mathbb{N} \), let
\[
s_i = \begin{pmatrix}
1 & 2 & 3 & \cdots & i-1 & i & i+1 & i+2 & k \\
1' & 2' & 3' & \cdots & (i-1)' & i' & (i+1)' & (i+2)' & k'
\end{pmatrix}
\]
\[
p_{i+\frac{1}{2}} = \begin{pmatrix}
1 & 2 & 3 & \cdots & i-1 & i & i+1 & i+2 & k \\
1' & 2' & 3' & \cdots & (i-1)' & i' & (i+1)' & (i+2)' & k'
\end{pmatrix}
\]

**Theorem 2.13.** [7] For \( k \in \mathbb{N} \), the party algebra \( R(x)L_k \) is generated by \( \{p_{1+\frac{1}{2}}, s_1, s_2, s_3, \ldots, s_{k-1}\} \) with relations:

i) \( s_i^2 = 1 \quad 1 \leq i \leq k - 1 \).

ii) \( s_is_{i+1}s_i = s_{i+1}s_is_{i+1} \quad 1 \leq i \leq k - 2 \).

iii) \( s_is_j = s_js_i \quad |i-j| \geq 2, \quad 1 \leq i, j \leq k - 1 \).

iv) \( (p_{1+\frac{1}{2}})^2 = p_{1+\frac{1}{2}} \).

v) \( p_{1+\frac{1}{2}}s_1 = s_1p_{1+\frac{1}{2}} = p_{1+\frac{1}{2}} \).

vi) \( p_{1+\frac{1}{2}}s_i = s_ip_{1+\frac{1}{2}} \quad 3 \leq i \leq k - 1 \).

vii) \( p_{1+\frac{1}{2}}s_2p_{1+\frac{1}{2}}s_2 = s_2p_{1+\frac{1}{2}}s_2p_{1+\frac{1}{2}} \).

viii) \( p_{1+\frac{1}{2}}s_2s_1s_3s_2p_{1+\frac{1}{2}}s_2s_1s_3s_2 = s_2s_1s_3s_2p_{1+\frac{1}{2}}s_2s_1s_3s_2p_{1+\frac{1}{2}} \).

2.3. Cellular Basis of symmetric group algebra \( R(x)S_k \)

**Definition 2.14.** For \( k > 0 \), we define, the symmetric group \( S_k \) generated by \( s_1, s_2, \ldots, s_{k-1} \) with relations:

i) \( s_i^2 = 1 \quad 1 \leq i \leq k - 1 \).

ii) \( s_is_{i+1}s_i = s_{i+1}s_is_{i+1} \quad 1 \leq i \leq k - 2 \).

iii) \( s_is_j = s_js_i \quad |i-j| \geq 2, \quad 1 \leq i, j \leq k - 1 \).

**Definition 2.15.** For \( k > 0, m > 0 \) we define \( R(x)S_k \) is the subalgebra of \( R(x)L_k \).

**Definition 2.16.** [2] A cellular algebra over \( R \) is an associative (unital) algebra \( A \) together with cell datum \( (\Lambda, M, c, i) \) where

i) \( \Lambda \) is a partially ordered set (poset) and for each \( \alpha \in \Lambda \), \( M(\alpha) \) is a finite set (the set of \( \text{“tableaux of type } \alpha \text{”} \) such that \( c : \bigcup_{\alpha \in \Lambda} (M(\alpha) \times M(\alpha)) \to A \) is an injective map with image an \( R \)–basis of \( A \).

ii) If \( \alpha \in \Lambda \) and \( s, t \in M(\alpha) \) write \( c(s, t) = m_{st}^\alpha \in A \). Then \( i \) is an \( R \)–linear anti-automorphism of \( A \) such that \( i(m_{st}^\alpha) = m_{ts}^\alpha \).
iii) If $\alpha \in \Lambda$ and $s, t \in M(\alpha)$ then for any element $a \in A$ we have

$$am^\alpha_{st} \equiv \sum_{s' \in M(\alpha)} r_a(s', s)m^\alpha_{s't} \pmod{A(< \alpha)}$$

where $r_a(s', s) \in R$ is independent of $t$ and where $A(< \alpha)$ is the $R$–submodule of $A$ generated by $\{m^\beta_{s't'} \mid \beta < \alpha; s', t' \in M(\beta)\}$.

**Definition 2.17.** [12] For $\alpha, \beta \in P_k$, we say that $\alpha$ dominates $\beta$, and write $\alpha \succeq \beta$, if $\sum_{i=1}^j \alpha_i \geq \sum_{i=1}^j \beta_i$ for all $j$.

For example, $(3, 3, 2) \succeq (3, 2, 2, 1)$.

**Lemma 2.18.** [10] For $k \in \mathbb{N}$, $(P_k, \succeq)$ is a partially ordered set.

**Lemma 2.19.** [10] Let $i_k$ be the map defined by $i_k : \sigma \mapsto \sigma^{-1}$ for each $\sigma \in S_k$, extended to $R(x)S_k$ by linearity. Then $i_k$ is an $R(x)$–algebra anti-automorphism of $R(x)S_k$.

**Definition 2.20.** [10] For $\alpha \vdash k$, the $t$ is said to be standard if the entries of each column of $t$ increase when read from top to bottom and entries of each row of $t$ increase from left to right.

For example, $\alpha = (3, 2, 2, 1)$ we have

$$t = \begin{array}{ccc}
1 & 2 & 4 \\
3 & 5 & \\
6 & 8 & \\
7 & 
\end{array}$$

**Definition 2.21.** [10] For $\alpha \vdash k$, the super standard tableau $t^\alpha$ is the unique $\alpha$–tableau in which has as its entries the integers $1, 2, \cdots, k$ appearing in increasing sequence from left to right and top to bottom.

For example, $\alpha = (3, 2, 2, 1)$ we have

$$t^\alpha = \begin{array}{ccc}
1 & 2 & 3 \\
4 & 5 & \\
6 & 7 & \\
8 & 
\end{array}$$

Let $\alpha \vdash k$ be a partition. The symmetric group $S_k$ acts from the left on the set of $\alpha$–tableaux by permutating the entries.

If $t$ is a $\alpha$–tableau, then $d(t) \in S_k$ is the permutation defined by the equation $d(t)t^\alpha = t$.

For example, $\alpha = (3, 2, 2, 1)$, $t$ and $t^\alpha$ are in above example then we have,

$$(3, 4)(7, 8) \begin{array}{ccc}
1 & 2 & 3 \\
4 & 5 & \\
6 & 7 & \\
8 & 
\end{array} = \begin{array}{ccc}
1 & 2 & 4 \\
3 & 5 & \\
6 & 8 & \\
7 & 
\end{array}$$
3. Cellular Basis and Dimensions of the cell modules of Party algebras

In this section, we establish a basis and dimension for the cell modules of the party algebra.

For \( \alpha = (\alpha_1, \alpha_2, \cdots, \alpha_l) \in P_k \), the number of connected components \( r_{ij} \) is equal to the number of \( \alpha_i \)'s equal to \( i_j \), and so the dimension of the cell module \( [\alpha] \) is

\[
\prod_{i=1}^{k!} \alpha_i
\]

Definition 2.22. \([10]\) For \( \alpha = (\alpha_1, \alpha_2, \cdots, \alpha_l) \vdash k \), the Young subgroup \( S_\alpha = \{ \sigma \in S_k \mid \sigma \text{ permutes elements row wise} \} \simeq S_{\alpha_1} \times S_{\alpha_2} \times \cdots \times S_{\alpha_l} \) will be the row stabilizer of \( t^\alpha \) in \( S_k \).

For example, \( \alpha = (3, 2, 2, 1) \) then \( S_\alpha = S_{[1,2,3]} \times S_{[4,5]} \times S_{[6,7]} \times S_{[8]} \simeq S_3 \times S_2 \times S_2 \times S_1 \). For a partition \( \alpha \vdash k \), Murphy defines the element \( m_\alpha \in R(x)S_k \) by \( m_\alpha = \sum_{\alpha \in S_\alpha} \sigma \), and associates to each pair \( s, t \) of standard \( \alpha \)-tableaux the element \( m^\alpha_{st} = i_k(d(s))m_\alpha(d(t)) \).

Definition 2.23. \([10]\) For \( \alpha \vdash k \), let \( M_\alpha = \{ t \mid t \text{ is a standard tableau} \} \) and \( M_k = \bigcup_{\alpha \in P_k} M_\alpha \).

Definition 2.24. \([10]\) For \( k \in \mathbb{N} \), let \( c_k : \bigcup_{\alpha \in P_k} (M_\alpha \times M_\alpha) \to R(x)S_k \) is defined as \( c_k(s, t) = m^\alpha_{st} \).

Lemma 2.25. \([10]\) For \( (s, t) \in \bigcup_{\alpha \in P_k} (M_\alpha \times M_\alpha) \), we have \( i_k(m^\alpha_{st}) = m^\alpha_{ts} \).

Lemma 2.26. \([10]\) For \( k \in \mathbb{N} \), \( M_k \) is a cellular basis for \( R(x)S_k \), i.e.

\[
am^\alpha_{st} = \sum_{s' \in M_\alpha} a(s, s')m^\alpha_{s't} \pmod{R(x)S_k(<\alpha)}
\]

where \( a(s', s) \in R(x) \) is independent of \( t \) and \( R(x)S_k(<\alpha) \) is the \( R(x) \)-submodule of \( R(x)S_k \) generated by \( \{ m^{\beta \cdot \rho} \mid \beta \triangleleft \alpha; s', \rho \in M_\beta \} \).

Theorem 2.27. \([10]\) The algebra \( R(x)S_k \) is cellular with cell datum \( (P_k, M_k, c_k, i_k) \).

Theorem 2.28. \([12]\) For \( \alpha = (\alpha_1, \alpha_2, \cdots, \alpha_k) \in P_k \), the dimension of the cell module \( [\alpha] \) is

\[
\prod_{i=1}^{k!} \alpha_i
\]

\[d^\alpha = \begin{array}{cccccccccccc}
1 & 2 & 3 & 4 & 5 & 6 & 7 & 8 & 9 & 10 & 11 \\
\end{array}
\]

Definition 3.1. For \( \alpha = (i_1^{r_1}, i_2^{r_2}, \cdots, i_l^{r_l}) \in P_k \), we index the connected components of \( d^\alpha \) as follows \((1, i_1), (2, i_2), \cdots, (r_{ij}, i_{ij}) \) for \( j = 1 \) to \( l \), \( i_j \) denotes the size of the connected components and \( r_{ij} \) is the number of connected components of size \( i_j \).

For example \( \alpha = (1^2, 2^3, 3^1) \),

\[d^\alpha = \begin{array}{cccccccccccc}
1 & 2 & 3 & 4 & 5 & 6 & 7 & 8 & 9 & 10 & 11 \\
\end{array}
\]

Definition 3.2. For \( \alpha = (i_1^{r_1}, i_2^{r_2}, \cdots, i_l^{r_l}) \in P_k \), the diagram \( d^\alpha \) is obtained from \( d^\alpha \) by re-indexing the vertices \( 1, 2, \cdots, k \) as \( 1', 2', \cdots, k' \) and the connected components \((m, i_j)\) as \((m', i_j)\) where \( m = 1 \) to \( r_{ij} \) and \( j = 1 \) to \( l \).

Definition 3.3. For \( \alpha = (i_1^{r_1}, i_2^{r_2}, \cdots, i_l^{r_l}) \in P_k \), the diagram \( U^\alpha_{d^\alpha} \) is defined as drawing \( d^\alpha \) below \( d^\alpha \) and joining the connected components \((m, i_j)\) and \((m', i_j)\) where \( m = 1 \) to \( r_{ij} \) and \( j = 1 \) to \( l \).

For example \( \alpha = (1^2, 2^3, 3^1) \),

\[d^\alpha = \begin{array}{cccccccccccc}
1 & 2 & 3 & 4 & 5 & 6 & 7 & 8 & 9 & 10 & 11 \\
\end{array}
\]
Proof. For $\sigma \in S_k$ acts on the vertices of $d^\alpha$ by permuting the vertices $1, 2, \ldots, k$ and the diagram is denoted by $\sigma(d^\alpha)$.

**Definition 3.5.** For $\alpha \in P_k$, $\sigma \in S_k$, define $\sigma(d^\alpha) \equiv d^\alpha$ if the diagram $d^\alpha$ and $\sigma(d^\alpha)$ are same irrespective of index of the vertices, (i.e. permutation of vertices within the connected components or permutation of connected components of same size.)

**Definition 3.6.** For $\alpha \in P_k$, define $St_l(d^\alpha) = \{ \sigma \in S_k | \sigma(d^\alpha) \equiv d^\alpha \}$.

**Remark 3.7.** For $\alpha \in P_k$, $St_l(d^\alpha)$ is a subgroup of $S_k$.

Proof. Since $S_k$ is a finite group, it is enough to verify the closure property. For $\sigma, \tau \in St_l(d^\alpha)$ we have

$$\sigma \circ \tau(d^\alpha) = \sigma(\tau(d^\alpha)) \equiv \sigma(d^\alpha) \equiv d^\alpha.$$

**Theorem 3.8.** For $\alpha = (i_1^{r_1}, i_2^{r_2}, \ldots, i_l^{r_l}) \in P_k$, the cardinality of $St_l(d^\alpha)$ is $\prod_{j=1}^l (r_j!)r_{i_j}(i_j!)$.

Proof. $\sigma \in St_l(d^\alpha)$ if and only if $\sigma(d^\alpha) \equiv d^\alpha$ i.e. for $j = 1 \to l$, $\sigma$ permutes the $r_{i_j}$ connected components in $r_{i_j}$! ways and each $r_{i_j}$ connected components has $i_j$ vertices and are permutes in $i_j$! ways, therefore the number of permutations which fixes $i_j^{r_j}$ is $(r_{i_j}!)r_{i_j}(i_j)!$, hence the cardinality of $St_l(d^\alpha)$ is $\prod_{j=1}^l (r_{i_j}!)r_{i_j}(i_j!)$.

**Corollary 3.9.** For $\alpha = (i_1^{r_1}, i_2^{r_2}, \ldots, i_l^{r_l}) \in P_k$, we have $\left| \frac{S_k}{St_l(d^\alpha)} \right| = \frac{k!}{\prod_{j=1}^l (r_{i_j}!)r_{i_j}(i_j!)}$.

**Definition 3.10.** For $\alpha = (i_1^{r_1}, i_2^{r_2}, \ldots, i_l^{r_l}) \in P_k$, $\sigma_j \in S_{r_{i_j}}, j = 1 \to l$ the diagram $(\sigma_1, \sigma_2, \ldots, \sigma_l)U_{d^\alpha}^{(\alpha)}$ is defined as drawing $d^\alpha$ below $d^\alpha$ and joining the connected components $(m, i_j)$ and $(\sigma_j(m), i_j)$ where $m = 1 \to r_{i_j}$ and $j = 1 \to l$.

**Definition 3.11.** For $\alpha = (i_1^{r_1}, i_2^{r_2}, \ldots, i_l^{r_l}) \in P_k$, $\sigma_j \in S_{r_{i_j}}, j = 1 \to l$, $\sigma, \tau \in \frac{S_k}{St_l(d^\alpha)}$, the diagram $\sigma[(\sigma_1, \sigma_2, \ldots, \sigma_l)U_{d^\alpha}^{(\alpha)}] \tau$ is obtained from $(\sigma_1, \sigma_2, \ldots, \sigma_l)U_{d^\alpha}^{(\alpha)}$ by permuting the top row vertices by $\sigma$ (i.e. $\sigma(d^\alpha)$) and bottom row vertices by $\tau$ (i.e. $\tau(d^\alpha)$).

**Theorem 3.12.** For $k > 0$, the set

$$\left\{ \sigma[(\sigma_1, \sigma_2, \ldots, \sigma_l)U_{d^\alpha}^{(\alpha)}] \tau | \sigma_j \in S_{r_{i_j}}, j = 1 \to l, \sigma, \tau \in \frac{S_k}{St_l(d^\alpha)} \right\}$$

is the basis for the party algebra $R(x)L_k$. 

$$U_{d^\alpha}^{(l)} = \begin{array}{cccccccccccc}
1 & 2 & 3 & 4 & 5 & 6 & 7 & 8 & 9 & 10 & 11

de & 1' & 2' & 3' & 4' & 5' & 6' & 7' & 8' & 9' & 10' & 11'
\end{array}$$
Proof. From the construction, it is clear that $\sigma[\sigma_1, \sigma_2, \ldots, \sigma_l]U_{d_0}^{\alpha} \tau \in L_k$ and any element in $L_k$ can be written in the form of $\sigma[\sigma_1, \sigma_2, \ldots, \sigma_l]U_{d_0}^{\alpha} \tau$. Therefore, $L_k = \{\sigma[(\sigma_1, \sigma_2, \ldots, \sigma_l)U_{d_0}^{\alpha} \tau | \sigma_j \in S_{r_{ij}}, j = 1 to l, \sigma, \tau \in S_{ki} \}$. 

Lemma 3.13. For $d_i = \sigma[(\sigma_1, \sigma_2, \ldots, \sigma_l)U_{d_0}^{\alpha} \tau$, $d_j = \sigma'[(\sigma_1', \sigma_2', \ldots, \sigma_l')U_{d_0}^{\alpha} \tau' \in L_k$, $t(d_i) = t(d_j) = t(\sigma'(d_0))$, and $\alpha \neq \beta$, then $t(d_i) < t(d_j)$. 

Proof. For $\alpha \neq \beta$ and $t(d_i) = t(d_j) = t(\sigma'(d_0))$, we have $\sigma(d_0) \neq \sigma'(d_0)$, therefore, $\tau(d_0) \neq \tau'(d_0)$, hence $t(d_i) < t(d_j)$. 

Lemma 3.14. For $d_i = \sigma[(\sigma_1, \sigma_2, \ldots, \sigma_l)U_{d_0}^{\alpha} \tau$, $d_j = \sigma'[(\sigma_1', \sigma_2', \ldots, \sigma_l')U_{d_0}^{\alpha} \tau' \in L_k$, $t(d_i) = t(d_j) = t(\sigma'(d_0))$, and $\alpha \neq \beta$, then $t(d_i) < t(d_j)$. 

Proof. It is easy to see, from the composition of diagram $d_i \circ d_j$ the number of connected components in $d_i \circ d_j$ is less than $r$ for $\tau(d_0) \neq \tau'(d_0)$, also $d_i \circ d_j = \sigma[(\sigma_1 \circ \sigma_1', \sigma_2 \circ \sigma_2', \ldots, \sigma_l \circ \sigma_l')U_{d_0}^{\alpha} \tau' \neq \tau'(d_0)$. 

Lemma 3.15. For $d_i = \sigma[(\sigma_1, \sigma_2, \ldots, \sigma_l)U_{d_0}^{\alpha} \tau$, $d_j = \sigma'[(\sigma_1', \sigma_2', \ldots, \sigma_l')U_{d_0}^{\alpha} \tau' \in L_k$, $t(d_i) = t(d_j) = t(\sigma'(d_0))$, and $\alpha \neq \beta$, then $t(d_i) < t(d_j)$. 

Proof. It is easy to see, from the composition of diagram $d_i \circ d_j$ the number of connected components in $d_i \circ d_j$ is less than $r$, for $\sigma'(d_0)$ is not finer than $\tau(d_0')$. In the composition of the diagram $d_i \circ d_j$, the connected components of top row of $d_j$ (i.e.$\sigma'(d_0)$) are joined and the diagram is equal to the diagram of bottom row of $d_i$ (i.e.$\tau(d_0)$), for $\sigma'(d_0)$ is finer than $\tau(d_0')$, therefore there exist $\sigma'_i \in S_{r_{ij}}$ for $j = 1 to l$, $\tau'' \in S_{ki} \cup S_{kl} \cup S_{kl'}$, such that, $d_i \circ d_j = \sigma[(\sigma_1 \circ \sigma_1', \sigma_2 \circ \sigma_2', \ldots, \sigma_l \circ \sigma_l')U_{d_0}^{\alpha} \tau' \neq \tau''$. 

Definition 3.16. For $k \in \mathbb{N}$, let $A'_k = \{(\sigma_1, \ldots, \sigma_k) \mid \sigma = (\sigma_1, \ldots, \sigma_k) \in P_k, \sum_{j=1}^l i_j r_{ij} = k \}$. 

Definition 3.17. For $(r, \alpha, \lambda, q, \beta, \mu) \in A'_k$, we say that $(r, \alpha, \lambda) \geq (q, \beta, \mu)$ if $q > \mu$ and $\alpha > \beta$ or $\alpha = \beta$ and $\mu > \lambda$ for all $i = 1 to k$. 

Definition 3.18. For $k \in \mathbb{N}$, $(A'_k, \geq)$ is a partially ordered set. 

Proof. The proof is trivial.
Definition 3.20. For \( k \in \mathbb{N} \), let \( c'_k : \bigcup_{(r,\alpha,\overrightarrow{x}) \in \mathcal{A}'_k} \left( M'_k(r,\alpha,\overrightarrow{x}) \times M'_k(r,\alpha,\overrightarrow{x}) \right) \rightarrow \mathbb{K}L_k \) be defined as

\[
c'_k((s_{i_1}, s_{i_2}, \ldots, s_{i_l}, \sigma(d^\alpha)), (t_{i_1}, t_{i_2}, \ldots, t_{i_l}, \tau(d^\alpha))) = \sigma([m_{s_{i_1}t_{i_1}}, \ldots, m_{s_{i_l}t_{i_l}}]_{U_{d^\alpha}^L}) \tau.
\]

Lemma 3.21. For \( k \in \mathbb{N} \), the map \( c'_k \) is injective and the image of \( c'_k \) is a basis for \( \mathbb{K}L_k \).

Proof. From Theorem 2.27, the set \( \{m_{s_{i_1}t_{i_1}}^{\lambda_{i_1}}\} \) is a cellular basis for \( \mathbb{K}S_{r_{i_1}} \), for \( j = 1 \) to \( l \), hence \( c'_k \) is an injective map.

Definition 3.22. For \( k \in \mathbb{N} \), we define a map \( i'_k : \mathbb{K}L_k \rightarrow \mathbb{K}L_k \) as

\[
i'_k(\sigma([\sigma_1, \sigma_2, \ldots, \sigma_l]_{U_{d^\alpha}^L}) \tau) = \tau([\sigma_1^{-1}, \sigma_2^{-1}, \ldots, \sigma_l^{-1}]_{U_{d^\alpha}^L}) \sigma
\]

and extended linearly.

Lemma 3.23. For \( k > 0 \), \( i'_k \) is an anti-automorphism.

Proof. For \( d = \sigma([\sigma_1, \sigma_2, \ldots, \sigma_l]_{U_{d^\alpha}^L}) \tau \in L_k \), we have

\[
i'_k(d) = i'_k(\sigma([\sigma_1, \sigma_2, \ldots, \sigma_l]_{U_{d^\alpha}^L}) \tau) = \tau([\sigma_1^{-1}, \sigma_2^{-1}, \ldots, \sigma_l^{-1}]_{U_{d^\alpha}^L}) \sigma = (d)^f
\]

where \((d)^f\) is the flip of the diagram \( d \).

For \( d_i, d_j \in L_k \), we have

\[
i'_k(d_i, d_j) = (d_i, d_j)^f = (d_j)^f(d_i)^f = i'_k(d_j) i'_k(d_i)
\]

hence, \( i'_k \) is an anti-automorphism.

Lemma 3.24. For \( k \in \mathbb{N} \), \( M'_k \) is a cellular basis for the party algebra \( \mathbb{K}L_k \).

Proof. For \( \sigma([\sigma_1, \sigma_2, \ldots, \sigma_m]_{U_{d^\beta}^L}) \tau, \sigma'[\sigma([m_{s_{i_1}t_{i_1}}^{\lambda_{i_1}}, \ldots, m_{s_{i_l}t_{i_l}}^{\lambda_{i_l}}]_{U_{d^\alpha}^L}) \tau] \in L_k \), \( t \left( U_{d^\beta}^L \right) = r', \alpha = (i_{11}^{r_{i_1}}, i_{21}^{r_{i_2}}, \ldots, i_{l1}^{r_{i_l}}), \beta = (j_{1j_1}^{r_{j_1}}, j_{2j_2}^{r_{j_2}}, \ldots, j_{lj_l}^{r_{j_l}}), \sum_{j=1}^l r_{i_j} = r \) and \( \sum_{i=1}^m r'_{i_j} = r' \).

Case 1. For \( r = r' \).

Subcase 1.1. For \( \alpha \neq \beta \), By Lemma 3.13, we have

\[
\left( \sigma([\sigma_1, \sigma_2, \ldots, \sigma_m]_{U_{d^\beta}^L}) \tau \right) \left( \sigma'[\sigma([m_{s_{i_1}t_{i_1}}^{\lambda_{i_1}}, \ldots, m_{s_{i_l}t_{i_l}}^{\lambda_{i_l}}]_{U_{d^\alpha}^L}) \tau] \right) \equiv 0 \mod \mathbb{K}L_k \leq (r,\alpha,\overrightarrow{x}).
\]

Subcase 1.2 For \( \alpha = \beta \).

Subcase 1.2.1. For \( \tau'(d^\beta) \neq \sigma(d^\alpha) \). By Lemma 3.14, we have

\[
\left( \sigma([\sigma_1, \sigma_2, \ldots, \sigma_m]_{U_{d^\beta}^L}) \tau \right) \left( \sigma'[\sigma([m_{s_{i_1}t_{i_1}}^{\lambda_{i_1}}, \ldots, m_{s_{i_l}t_{i_l}}^{\lambda_{i_l}}]_{U_{d^\alpha}^L}) \tau] \right) \equiv 0
\]
(mod $KL_k < \sim (r, \alpha, \lambda)).$

Subcase 2.2. For $\tau'(d^{\beta}) \equiv \sigma(d^{\alpha})$. By Lemma 3.14 we have

$$(\sigma([\sigma_1, \sigma_2, \ldots, \sigma_l]U_{d^{\alpha}}^{d^{\beta}})[\tau'])^t \equiv \sigma'((m_{s_{i_1}t_{i_1}}, \ldots, m_{s_{i_l}t_{i_l}})^{U_{d^{\alpha}}^{d^{\beta}}})^t'$$

By using Theorem 2.27 we have,

$$\sigma((\sigma_1m_{s_{i_1}t_{i_1}}, \ldots, \sigma_l^m_{s_{i_l}t_{i_l}})^{U_{d^{\alpha}}^{d^{\beta}}})^t'$$

which implies

$$(\mod KL_k < \sim (r, \alpha, \lambda)).$$

Subcase 2.1. For $\tau(d^{\beta})$ is not finer than $\sigma'(d^{\alpha})$. By Lemma 3.15 we have

$$\left(\sigma([\sigma_1, \sigma_2, \ldots, \sigma_l]U_{d^{\alpha}}^{d^{\beta}})[\tau'] \equiv \sigma'((m_{s_{i_1}t_{i_1}}, \ldots, m_{s_{i_l}t_{i_l}})^{U_{d^{\alpha}}^{d^{\beta}}})^t' \equiv 0 \right.$$  

$$(\mod KL_k < \sim (r, \alpha, \lambda)).$$

Subcase 2.2. For $\tau(d^{\beta})$ is finer than $\sigma'(d^{\alpha})$. By Lemma 3.15 there exist $\sigma_j \in S_{r_{ij}}$ for $j = 1$ to $l$ and $\sigma'' \in S_{i_k}$ we have

$$\left(\sigma([\sigma_1, \sigma_2, \ldots, \sigma_l]U_{d^{\alpha}}^{d^{\beta}})[\tau'] \equiv \sigma'((m_{s_{i_1}t_{i_1}}, \ldots, m_{s_{i_l}t_{i_l}})^{U_{d^{\alpha}}^{d^{\beta}}})^t' \equiv \sigma''((\sigma_1m_{s_{i_1}t_{i_1}}, \ldots, \sigma_l^m_{s_{i_l}t_{i_l}})^{U_{d^{\alpha}}^{d^{\beta}}})^t' \right.$$  

By using Theorem 2.27 we have,

$$\sigma''((\sigma_1m_{s_{i_1}t_{i_1}}, \ldots, \sigma_l^m_{s_{i_l}t_{i_l}})^{U_{d^{\alpha}}^{d^{\beta}}})^t'$$

which implies

$$(\mod KL_k < \sim (r, \alpha, \lambda)).$$

Subcase 2.2. For $\tau(d^{\beta})$ is finer than $\sigma'(d^{\alpha})$. By Lemma 3.15 there exist $\sigma_j \in S_{r_{ij}}$ for $j = 1$ to $l$ and $\sigma'' \in S_{i_k}$ we have

$$\left(\sigma([\sigma_1, \sigma_2, \ldots, \sigma_l]U_{d^{\alpha}}^{d^{\beta}})[\tau'] \equiv \sigma'((m_{s_{i_1}t_{i_1}}, \ldots, m_{s_{i_l}t_{i_l}})^{U_{d^{\alpha}}^{d^{\beta}}})^t' \equiv \sigma''((\sigma_1m_{s_{i_1}t_{i_1}}, \ldots, \sigma_l^m_{s_{i_l}t_{i_l}})^{U_{d^{\alpha}}^{d^{\beta}}})^t' \right.$$  

By using Theorem 2.27 we have,

$$\sigma''((\sigma_1m_{s_{i_1}t_{i_1}}, \ldots, \sigma_l^m_{s_{i_l}t_{i_l}})^{U_{d^{\alpha}}^{d^{\beta}}})^t'$$

which implies

$$(\mod KL_k < \sim (r, \alpha, \lambda)).$$

The above summation is independent of $(t_{i_1}, t_{i_2}, \ldots, t_{i_l}, \tau'(d^{\alpha}))$. 

\[ \Box \]
**Theorem 3.25.** For $k \in \mathbb{N}$, the party algebra $\mathbb{K}L_k$ is cellular with cell datum $(N_k', M_k', e_k', i_k')$.

**Proof.** The proof follows from the Definition 2.24, Lemma 3.18, Lemma 3.21, Lemma 3.23 and Lemma 3.24.

**Definition 3.26.** For $(r, \alpha, \vec{\lambda}) \in N_k', \alpha = (i_1^{r_1}, i_2^{r_2}, \ldots, i_l^{r_l})$ and $\vec{\lambda} = (\lambda^{i_1}, \lambda^{i_2}, \ldots, \lambda^{i_l})$, we define

$$L_{(r, \alpha, \vec{\lambda})} = \left\{ \sigma[(m_{s_1 t_1}^{\lambda_1}, \ldots, m_{s_l t_l}^{\lambda_l})U^\alpha_{d^\alpha}] \bigg| m_{s_l t_l}^{\lambda_l} \in \mathbb{K}S_{r_l}, \sigma \in \frac{S_k}{St_l(d^\alpha)} \right\}.$$ 

**Theorem 3.27.** The set $L_{(r, \alpha, \vec{\lambda})}$ is a basis for the cell module $(r, \alpha, \vec{\lambda}) \in N_k'$.

**Proof.** From the Definition of cellular basis, it is clear that $L_{(r, \alpha, \vec{\lambda})}$ is a basis for the cell module $(r, \alpha, \vec{\lambda}) \in N_k'$.

**Theorem 3.28.** For $(r, \alpha, \vec{\lambda}) \in N_k', \alpha = (i_1^{r_1}, i_2^{r_2}, \ldots, i_l^{r_l})$ and $\vec{\lambda} = (\lambda^{i_1}, \lambda^{i_2}, \ldots, \lambda^{i_l})$, the cardinality of $L_{(r, \alpha, \vec{\lambda})}$ is

$$\left( \frac{k!}{\prod_{j=1}^l (r_j!)^j} \right)^2 \prod_{j=1}^l |\lambda^j|.$$ 

**Proof.** For $(r, \alpha, \vec{\lambda}) \in N_k', \alpha = (i_1^{r_1}, i_2^{r_2}, \ldots, i_l^{r_l})$ and $\vec{\lambda} = (\lambda^{i_1}, \lambda^{i_2}, \ldots, \lambda^{i_l})$. From Corollary 3.39 the cardinality of the set $\left\{ \sigma(d^\alpha) \big| \sigma \in \frac{S_k}{St_l(d^\alpha)} \right\}$ is $\frac{k!}{\prod_{j=1}^l (r_j!)^j}$, the cardinality of the set $\left\{ \sigma(U^\alpha_{d^\alpha}) \big| \sigma \in \frac{S_k}{St_l(d^\alpha)} \right\}$ is $\left( \frac{k!}{\prod_{j=1}^l (r_j!)^j} \right)^2$. From Theorem 2.28 the cardinality of $L_{(r, \alpha, \vec{\lambda})}$ is

$$\left( \frac{k!}{\prod_{j=1}^l (r_j!)^j} \right)^2 \prod_{j=1}^l |\lambda^j|.$$ 

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