Neutral delay and a generalization of electrodynamics

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The equations for the electromagnetic two-body problem are neutral-delay equations that for generic initial data have solutions with discontinuous derivatives. If one wants to use these neutral-delay equations with arbitrary initial data, solutions with discontinuous derivatives must be allowed. Surprisingly, this same neutrality is compatible with the recently developed variational method with mixed-type boundaries for the Wheeler-Feynman electrodynamics. We show that two-body electromagnetic orbits with discontinuous velocities are physically necessary by showing that orbits with vanishing far-fields almost everywhere must have some discontinuous velocities on a few points. We generalize the Wheeler-Feynman electrodynamics with the variational method to include all continuous trajectories, allowing piecewise-differentiable weak solutions represented by trajectories with fields defined almost everywhere (but on a set of points of zero measure where velocities jump). Along with this generalization we formulate the generalized absorber hypothesis that the far-fields vanish asymptotically almost everywhere and show that bounded two-body orbits satisfying the generalized absorber hypothesis need to have discontinuous derivatives on a few points. We also give the general solution for the family of bounded non-radiating two-body orbits. We discuss the physics of orbits with discontinuous derivatives and show that these conserve the physical momentum, stressing the differences to classical variational methods. Last, we discuss how the electromagnetic variational method with mixed-type boundaries is well-posed but lacks reversibility.

INTRODUCTION

Since the speed of light is constant in inertial frames, the equations of motion for point-charges are state-dependent-delay equations. More specifically, the Wheeler-Feynman electrodynamics has mixed-type-state-dependent-neutral-delay equations of motion for the two-body problem. The theory of delay equations is still on the make but it is already known that delay-only equations with generic continuous initial data have continuous solutions with a discontinuous derivative at the initial time. The derivative becomes continuous at the next breaking point and progresses from \( C^k \) to \( C^{k+1} \) at successive breaking points. On the other hand, for a neutral-delay equation with generic continuous initial data, solutions can have discontinuous derivatives at all breaking points. If one wants to use the electromagnetic neutral-delay equations with arbitrary initial data, solutions with discontinuous derivatives must be expected. Surprisingly, this same neutrality is compatible with the recently developed variational method with mixed-type boundaries for the Wheeler-Feynman electrodynamics. Two-body electromagnetic orbits with discontinuous velocities are not that odd physically, and in fact here we show that any sequence of orbits radiating less and less converges to orbits with discontinuous velocities. For these limiting orbits where the acceleration is not defined on a few points, the variational method offers a well-posed alternative to define weak trajectories beyond those satisfying a neutral-delay equation everywhere. The variational method describes naturally all continuous orbits with possibly discontinuous derivatives. The relation to Maxwell’s electrodynamics is of inclusion, i.e., this generalization contains the \( C^2 \) orbits of the Wheeler-Feynman theory. As shown in Ref. if boundary data are such that the extremum orbit turns out to be piecewise \( C^2 \) with continuous velocities, the equations of the Wheeler-Feynman electrodynamics hold everywhere with the exception of the breaking points (a set of measure zero). Here we show that if the velocity is discontinuous at these breaking points, further non-local momentum conservations are required by the variational method, another striking difference to the classical local two-body variational methods. Finally, if the extremum in not piecewise \( C^2 \) the variational method yields a generalized weak dynamics beyond description by the Wheeler-Feynman neutral-delay equations.

The variational equations of Ref. along piecewise \( C^2 \) orbits include the electromagnetic-fields as the farthest-reaching couplings to the other particle’s trajectory. In the original articles, Wheeler and Feynman attempted to derive an electrodynamics with retarded-only fields from the hypothesis that the universal far-fields vanish at all times, yielding a self-force depending on the derivative of the acceleration. Here we generalize the absorber hypothesis to trajectories with a discontinuous velocity and acceleration on a set of measure zero by defining the far-fields almost everywhere with the usual electromagnetic formulas while allowing...
the fields to be undefined on a set of points of measure zero, thus arriving at the generalized absorber hypothesis that the far-fields vanish almost everywhere, henceforth called G.A.H. The generalized flux of the Poynting vector is a physical property involving an integral, so that it can be evaluated even with fields undefined on a set of measure zero. Here we show that a bounded two-body G.A.H. orbit must have discontinuous derivatives on a few points, by giving the general solution to a simple neutral-delay equation.

We henceforth adopt a unit system where the speed of light is \( c = 1 \) and apply the usual formulas of electrodynamics to piecewise-defined trajectories with the exception of the few points where the past/future velocities/accelerations are undefined, where we declare the fields are undefined (a set of measure zero for the flux integral). The far-electric field of a point charge in the Wheeler-Feynman electrodynamics involves a sum of advanced and retarded fields,

\[
E = \frac{1}{2}E^{\text{adv}} + \frac{1}{2}E^{\text{ret}},
\]

while the far-magnetic field is given by

\[
B = \frac{1}{2}n_+ \times E^{\text{adv}} - \frac{1}{2}n_- \times E^{\text{ret}},
\]

where unit vectors \( n_\pm \) point away from the advanced/retarded position of the charge, respectively. We consider a spatially bounded orbit enclosed by a sphere of radius \( R \) in an inertial frame, with \( R \) much larger than the maximum orbital excursion, so that \( n_+ = n_- \equiv n \). The space-time points \((t, Rn)\) on the sphere are specified by the time \( t \) and the unit vector \( n \) and the Poynting vector \( P = E \times B \) evaluated with Eqs. (1) and (2) at \((t, Rn)\) becomes

\[
P = \frac{1}{4} |E^{\text{adv}}|^2 - |E^{\text{ret}}|^2 n
\]

where single bars denote Euclidean modulus, and we have used the transversality of the far-fields, i.e., \( n \cdot E^{\text{adv}} = n \cdot E^{\text{ret}} = 0 \). Notice that the G.A.H. implies the vanishing of the flux \[3\] because \( |E^{\text{ret}}|^2 = |E^{\text{adv}}|^2 = 0 \) almost everywhere, while the vanishing of the surface integral of Eq. [3] alone does not imply the G.A.H. For example the circular two-body orbits \[7,8\] of the Wheeler-Feynman electrodynamics do not satisfy the G.A.H. We henceforth introduce an index \( k = 1,2 \) to label the charges. The far-magnetic field is proportional to \( E^{\text{ret}}(t,n) \) by

\[
B^{\text{ret}}(t,n) = n \times E^{\text{ret}}.
\]

For a continuous and piecewise \( C^1 \) trajectory \( x_k : [-\Lambda_k, T_kF] \rightarrow \mathbb{R}^3 \) the retarded far-electric field of charge \( q_k \) defined piecewise at the space-time point \((t, Rn)\) is given by the Lienard-Wiechert formula\[9\]

\[
E^{\text{ret}}_k(t,n) = q_k n \times \frac{\{n - v_{k-} \times a_{k-}\}}{(1 - n \cdot v_{k-})^3 R}.
\]

Because of Eq. (4) it suffices to study the vanishing of the electric far-retarded fields and we henceforth assume the orbit is time-reversible so that the vanishing of the retarded far-fields implies the vanishing of the advanced far-fields. In Eq. (1), unit vector \( n \) points from the charge’s retarded position \((t_{k-}, x_k(t_{k-}))\) to the space-time point \((t, Rn)\) while \( v_{k-} \equiv dx_k/dt|_{t_{k-}} \) and \( a_{k-} \equiv d^2x_k/dt^2|_{t_{k-}} \) are respectively the Cartesian velocity and Cartesian acceleration of the point charge, which along a generalized continuous-only orbits are defined almost everywhere in past light-cone of \((t, Rn)\). The time of particle \( k \) in lightcone with \((t, Rn)\) is defined implicitly by the retardation condition

\[
t_{k-} = t - |x_k(t_{k-}) - Rn|,
\]

where single bars stand for Cartesian distance. If the piecewise defined velocity \( v_{k-} \equiv dx_k/dt|_{t_{k-}} \) is lesser than light, Eq. (11) defines the retarded time \( t_{k-} \) as an implicit function of time \( t \) with a piecewise defined derivative

\[
\frac{dt_{k-}}{dt} = \frac{1}{(1 - n \cdot v_{k-})}. \tag{7}
\]

We can use Eq. (7) to express the far-electric-field (5) as

\[
E^{\text{ret}}_k(t,n) = \frac{q_k n}{R} \times \frac{d^2}{dt^2} (x_k(t_{k-})), \tag{8}
\]

where \( x_k(t_{k-}) \) is the position of particle \( k \) at time \( t_{k-} \). Henceforth charge 1 is supposed positive and equal to \( q \) while charge 2 is negative and equal to \(-q\). The G.A.H. for piecewise differentiable orbits is expressed almost everywhere by

\[
E^{\text{ret}} = E^{\text{ret}}_1 + E^{\text{ret}}_2 = \frac{q n}{R} \times \frac{d^2}{dt^2} (x_1(t_{1-}) - x_2(t_{2-})) = 0. \tag{9}
\]

Assuming the \( x_i(t_i) \) piecewise \( C^2 \), condition (10) has a piecewise-linear continuous solution defined for \( t \) in each closed-open interval \( t \in (t_{\sigma}, t_{\sigma+1}) \) with \( \sigma \in \mathbb{Z} \) by

\[
x_1(t_{1-}) - x_2(t_{2-}) = D_\sigma(n) + n f_\sigma(t, n) + (t - t_\sigma)V_\sigma(n), \tag{10}
\]

where the \( D_\sigma(n) \) and \( V_\sigma(n) \) are arbitrary bounded functions and the \( f_\sigma(t, n) \) are bounded and piecewise \( C^2 \). It is possible to choose \( n \cdot D_\sigma(n) = 0 \) and adjust \( D_\sigma(n) \) to make the left-hand-side of Eq. (10) continuous.

Along a spatially bounded orbit Eq. (11) is approximated for large values of \( R \) by

\[
t_{k-} = t - R + n \cdot x_k(t_{k-}). \tag{11}
\]
Notice that Eqs. (11) yield an implicit relation between $t_{1-}$ and $t_{2-}$,
\[
t_{1-} - t_{2-} = n \cdot (x_1(t_{1-}) - x_2(t_{2-})).
\] (12)
It is instructive to use the retardation conditions (6) with $k = 1, 2$ and the implicit function theorem to express $t_2$ as a function of $t_1$ and $n$. We define the influence interval of point $(t_{2-}, x_2(t_{2-}))$ by the interval containing $t_{1-}$ when $n$ varies arbitrarily in Equation (12), i.e.,
\[
t_{2-} - |x_1(t_{1-}) - x_2(t_{2-})| < t_{1-} < t_{2-} + |x_1(t_{1-}) - x_2(t_{2-})|.
\] (13)
The time span (13) is from the retarded light-cone time of $(t_{2-}, x_2(t_{2-}))$ to the advanced light-cone time of $(t_{2-}, x_2(t_{2-}))$, with the mixed-type boundary conditions of Ref. [5]. Notice that the future light-cone appeared naturally in the two-particle problem, even though we were dealing only with the retardation conditions (11). It follows from Eqs. (12) and (10) that
\[
f_\sigma(t, n) = (t_{1-} - t_{2-}) - (t - t_\sigma)n \cdot V_\sigma(n).
\] (14)
and we can re-write Eq. (10) as
\[
x_1(t_{1-}) - x_2(t_{2-}) = D_\sigma(n) + (t_{1-} - t_{2-})n \cdot (n \times V_\sigma(n)).
\] (15)
Since linear growth is unbounded, the only globally $C^2$ orbit must have $V_\sigma(n) = 0 \forall \sigma$ and it follows from Eq. (15) with $V_\sigma(n) = 0$ that
\[
x_1(t_{1-}) - x_2(t_{2-}) = D_\sigma(n) + (t_{1-} - t_{2-})n.
\] (16)
The derivative of Eq. (10) respect to time yields
\[
\frac{v_{1-}}{(1 - n \cdot v_{1-})} - \frac{v_{2-}}{(1 - n \cdot v_{2-})} = K_{12}n,
\] (17)
where
\[
K_{12} = \frac{1}{(1 - n \cdot v_{1-})} - \frac{1}{(1 - n \cdot v_{2-})}.
\] (18)
Equation (12) allow us to move $n$ in a cone with axis along $x_1(t_{1-}) - x_2(t_{2-}) \neq 0$ in a way that fixes $t_{1-}$ and $t_{2-}$ while changing $t$ with Eqs. (11). On the other hand, for fixed $t_{1-}$ and $t_{2-}$ the left-hand-side of Eq. (17) spans a plane of the fixed vectors $v_{1-}$ and $v_{2-}$, so that Eq. (17) hold only if $K_{12} = 0$, which combined with Eqs. (17) and (18) yields
\[
v_1(t_{1-}) = v_2(t_{2-}).
\] (19)
Equation (19) defines piecewise-constant velocities, and the only bounded choice is $v_1 = v_2 = 0$, as discussed in [10]. Nontrivial alternatives necessitate the introduction of a few discontinuities, i.e., use Eq. (15) with $V_\sigma \neq 0$, in which case the piecewise derivative of Eq. (15) respect to time yields
\[
\frac{v_{1-}}{(1 - n \cdot v_{1-})} - \frac{v_{2-}}{(1 - n \cdot v_{2-})} = K_{12}n - n \times (n \times V_\sigma(n)).
\] (20)
Notice that $K_{12}$ is still given by Eq. (18) and with nonzero $V_\sigma(n)$ the right-hand-side of Eq. (20) forms a complete 3-dimensional basis to express any vector (inside or outside the plane of $v_1(t_{1-})$ and $v_2(t_{2-})$). Equation (12) still allows one to move $n$ in a cone with axis along $x_1(t_{1-}) - x_2(t_{2-}) \neq 0$ in a way that fixes $t_{1-}$ and $t_{2-}$ while $t$ changes with Eqs. (11). By choosing $K_{12}$ and a nonzero $V_\sigma(n)$ for each $t \in (t_1, t_{1+1})$ we can describe any vector on the left-hand-side of Eq. (20), so that there is no inconsistency. Example time-reversible orbits satisfying Eq. (20) are piecewise-constant-velocity orbits generated by jumping one velocity at a given time while the other velocity jumps either in the backward or forward light-cone times symmetrically, as well as at every time in the forward and backward light-cones of a discontinuity time (the sewing chain illustrated in Fig. 2 of Ref. [5]). These piecewise-linear polygonal orbits can be checked to satisfy Eq. (9) by direct substitution and use of Eq. (7). To find the most general solution of Eq. (15) we notice that for any given piecewise defined trajectory $x_2(t_{2-})$ and given $D_\sigma(n)$ and $V_\sigma(n)$ Eq. (15) determines in general only a function $x_1(t_{1-}, n)$ of the two variables $(t_{1-}, n)$ by
\[
x_1(t_{1-}, n) = x_2(t_{2-}) + D_\sigma(n) + (t_{1-} - t_{2-})n - (t - t_\sigma)n \times (n \times V_\sigma(n)).
\] (21)
The implicit function theorem further determines $t_{2-}$ and $t$ as functions of $t_{1-}$ and $n$ by Eqs. (11), (12) and (21). A physical trajectory must satisfy the extra consistency requirement that the $x_1(t_{1-}, n)$ determined by Eq. (21) is a function of $t_{1-}$ only, i.e.,
\[
\frac{\partial x_1(t_{1-}, n)}{\partial n} = 0.
\] (22)
Condition (22) applied to the right-hand side of Eq. (21) is the extra condition determining a consistent trajectory. Since (22) must hold for all values of $t_{1-}$ inside each interval of the piecewise defined orbit, we must also have inside each piecewise interval that
\[
\frac{\partial^2 x_1(t_{1-}, n)}{\partial t_{1-} \partial n} = \frac{\partial}{\partial n} \left[ \frac{\partial x_1(t_{1-}, n)}{\partial t_{1-}} \right] = 0,
\] (23)
which can be expressed as
\[
\frac{\partial}{\partial n}[v_{2-} - n \frac{\partial x_2(t_{1-}, n)}{\partial t_{1-}}] + n \frac{\partial t_{1-}}{\partial t_{1-}} n \times (n \times V_\sigma(n)) = 0.
\] (24)
A symmetric compatibility condition follows by exchanging 1 and 2 in Eq. (24), determining a set of equation whose general solution is
\[
[v_{2-} - n \frac{\partial x_2(t_{1-}, n)}{\partial t_{1-}} + n \frac{\partial t_{1-}}{\partial t_{1-}} n \times (n \times V_\sigma(n)) = A_\sigma(t_{1-}),
\]
\[
[v_{1-} - n \frac{\partial x_2(t_{2-}, n)}{\partial t_{2-}} + n \frac{\partial t_{2-}}{\partial t_{2-}} n \times (n \times V_\sigma(n)) = B_\sigma(t_{2-}),
\] (25)
for arbitrary functions \( A_\sigma(t_{1-}) \) and \( B_{\sigma}(t_{2-}) \). We stress that the discontinuities introduced by the \( V_\sigma(n) \) play an essential part in the solution; Trying to solve either one of Eqs. (24) with \( V_\sigma(n) = 0 \) yields the former glitch generated by the rotation of the \( A_\sigma(t_{1-}) \) and \( v_{2-} \) or the plane of \( B_{\sigma}(t_{2-}) \) and \( v_{1-} \). Otherwise with nonzero \( V_\sigma(n) \) there is a complete basis to endow rotational invariance to Eqs. (25), which define the most general nonradiating bounded orbit. It can be seen that piecewise-linear polygonal orbit are a special case for constant \( A_\sigma \) and \( B_\sigma \). Next in line would be to find the extremum bounded orbits of the variational method [3] that also satisfy Eqs. (25), which should give an even better approximation for the orbits studied in Ref. [5] (even more astonishing hit at the spectroscopic lines!). The physical need of trajectories with discontinuous velocities is the justified as limitind orbits defined by sequences of bounded two-body orbits radiating less and less.

Now that we know that bounded G.A.H. orbits must involve a discontinuous derivative on a few points, as compatible also with neutral-delay equations, let us examine the variational method of [3]. We henceforth assume that Let a continuous trajectories \( x_1(t_1) \) and \( x_2(t_2) \in \mathbb{R}^3 \) and piecewise \( C^1 \) in the above defined \((t_\sigma, t_{\sigma+1})\) intervals. The mixed-type boundary conditions for the variational method [3] as illustrated in Figure 1 are the initial point \( O_1 \) for trajectory 1 plus the segment of trajectory 2 inside the lightcone of \( O_A \), and the endpoint \( L_B \) for the trajectory 2 plus the segment of trajectory of particle 1 inside the lightcone of \( L_B \). For variances of trajectory 1 the variational method has a Lagrangian [3]

\[
S_1 = \int_0^{T_2} L_1(x_1, v_1, x_2, v_2) dt_1
\]

\[
= \int_0^{T_2} -m_1 \sqrt{1 - v_1^2} dt_1 + \int_0^{T_2} \frac{(1 - v_1 \cdot v_{2+})}{2r_{12+}(1 - n_{12-} \cdot v_{2+})} dt_1 + \int_0^{T_2} \frac{(1 - v_1 \cdot v_{2-})}{2r_{12-}(1 - n_{12-} \cdot v_{2-})} dt_1,
\]

where unit vector \( n_{12\pm} \) points respectively from the advanced/retarded position \((t_{2\pm}, x_2(t_{2\pm}))\) to the space-time point \((t_1, x_1(t_1)) \) and \( v_{2\pm} \equiv d x_2/d t_2 |_{t_2\pm} \) while \( r_{12\pm} \equiv |x_1(t_1) - x_2(t_{2\pm})| \) are the distances in light-cone to trajectory 2. The variational method [3] is defined in the space of \( C^1 \) orbital variations of trajectory 1 with fixed endpoints, i.e.,

\[
\delta x_1(t_1 = 0) = 0,
\]

\[
\delta x_1(t_1 = T_{2-}) = 0.
\]

The linear expansion of \( S_1 \) about the orbit defines the Frechét derivative, i.e.,

\[
\delta S_1 = \int_0^{T_2} \left[ \frac{\partial L_1}{\partial x_1} \cdot \delta x_1 + \frac{\partial L_1}{\partial v_1} \cdot \delta v_1 \right] dt_1 + O(\delta x_1^2),
\]

where \((\cdot, \cdot)\) is the scalar product of \( \mathbb{R}^3 \). In particular when \( x_1 : [0, T_2] \rightarrow \mathbb{R}^3 \) is piecewise \( C^2 \) we can integrate Eq. (28) by parts in each interval and re-arrange yielding

\[
\delta S_1 = \int_0^{T_2} \left( \delta x_1 \cdot \frac{\partial L_1}{\partial x_1} - \frac{d}{dt} \left( \frac{\partial L_1}{\partial v_1} \right) \right) dt_1 - \sum_{\sigma = N-1} \left( \delta x_1(t_\sigma) \cdot \frac{\partial L_1}{\partial v_1} \right)_{t_{\sigma+1}}^{t_{\sigma+1}}.
\]

The extremal orbits in the class of continuous orbital variations must (i) Satisfy the Euler-Lagrange equation to vanish the first line of Eq. (24) with arbitrary variations and (ii) Conserve the momentum currents

\[
\frac{\partial L_1}{\partial v_1} |_{t_{\sigma+1}}^{t_{\sigma+1}} = \frac{\partial L_1}{\partial v_1} (x_1, v_{1,\sigma+1}) - \frac{\partial L_1}{\partial v_1} (x_1, v_{1,\sigma}) = 0.
\]

For local Lagrangians usually one has \( \partial L/\partial v_1 = G(x_1, v_1, x_2) \) such that Eq. (30) requires the velocity to be continuous. For piecewise \( C^2 \) orbits continuity of velocity implies that only generally \( C^2 \) are possible, a popular result of classical mechanics. On the contrary, for the Lagrangian given in Eq. (14) of Ref. [3] the current is

\[
\frac{\partial L_1}{\partial v_1} = \frac{m_1 v_1}{\sqrt{1 - v_1^2}} + \frac{m_2 v_2}{\sqrt{1 - v_2^2}} + \frac{v_2^2}{2r_{12-}(1 - n_{12-} \cdot v_{2-})} + \frac{v_2^2}{2r_{12+}(1 - n_{12+} \cdot v_{2+})},
\]

with an analogous continuity along the trajectory of particle 2. The Noether two-body-momentum of Ref. [3], formula (A23), involves an integral that is insensitive to velocity jumps plus two non-local currents given by Eq. (31), (see Eqs. (A25) and (A26) of Ref. [3]) that are sensitive to jumps. Therefore even the physical Noether two-body momentum is conserved as long as Eq. (31) is continuous across the jumps. The physical justification to generalize electrodynamics with the variational method of Ref. [3] is precisely to include the non-radiating orbits of Ref. [3] in the family of bounded G.A.H. orbits. We have here synthesized three different reasons for orbits with a discontinuous derivative: (i) inclusion of bounded G.A.H. orbits, (ii) the conservation of Noether’s momentum (iii) compatibility with the neutrality of the equations and (iv) the fact that the variational method is natural in a space completed to contain orbits with discontinuous velocities. The linearization of the neutral-delay equations of motion used in Ref. [3] about circular orbits can produce piecewise-defined orbits with discontinuous derivatives. Moreover, these orbits can be approximated by a series of stiff approximants having continuous derivatives and an increasingly sharp edges to approximate the
velocity jumps, i.e., the stiff perturbations in Ref. [3] that predicted orbits in the atomic magnitude with a surprising precision from a non-radiation condition involving a flux integral. We notice that this surprising difference to the popular variational principles of classical mechanics required a minimum of two bodies and a non-local Lagrangian.

Electromagnetism was originally formulated with the integral laws of Ampere, Gauss and Faraday, and only much later differential equations holding everywhere were introduced by Maxwell. The requirement of a second derivative existing everywhere is not needed for particle dynamics, where one is concerned only with an integral of the force along trajectories. The variational method with mixed-type boundaries of the Wheeler-Feynman electrodynamics [4] allows for such generalization, a step back from Maxwell’s equations in the sense of weak solutions.

The bounded two-body G.A.H. orbit is a seed to start a perturbation theory for a three-body motion by extending the variational method of [5] to three charges and placing the third charge at a large distance. For the three-body variational method the interaction terms fall sufficiently fast (i.e., the far-fields of the bounded orbit) almost everywhere along the third trajectory. We see that a bounded G.A.H. orbit is special in the sense that it does not "disturb" the universe. Moreover, there is an stability provided by the G.A.H. when the flux of the semi-sum field vanishes in a large universe containing many charges. The G.A.H. is much better than having the (non-zero) fluxes of both retarded and advanced fields cancel each other. If the G.A.H. holds, an offending perturbation of size \( \varepsilon \) in the retarded far-field (or in the advanced far-field) perturbs the flux Eq. (3) at \( O(\varepsilon^2) \), otherwise the flux (3) is perturbed at \( O(\varepsilon) \).

The dependence on the mixed-type boundaries must be investigated for scattering trajectories with discontinuous velocities and accelerations at the boundaries; these are likely to have future continuations involving stiffer jumps at later times, so that particles collide with laboratory boundaries, which can be regarded as a generalized type of radiative loss. This is to be contrasted with the fate of \( C^\infty \) trajectories determined by continuation of \( C^\infty \) initial data as discussed in Ref. [11]; for these it is shown in [11] that the equations of motion with advanced fields can be solved for the most advanced acceleration of the other particle, yielding delay-only equations, Eq. (22) of Ref. [12], so that no future information is needed to define solutions. On the contrary, the future data in general is only part of the mixed-boundary data used in [5]. The variational method [2] with generic mixed-type boundary data determines orbits with discontinuous velocities in a way that backward continuation is not unique. The lack of well-posed reversibility solves a long-standing debate for the absorber hypothesis with \( C^\infty \) trajectories; The generalization of Ref. [12] allows no unique backward continuation because there are no equations of motion to run backwards. To continue the trajectories into the past with the variational method one needs to guess pre-historical boundaries for one particle and solve another mixed-type-boundary-value problem with that postulated segment of trajectory plus the other particle’s earlier past data. This construction is not unique because there is not a unique pre-history. The lack of a unique past for the other particle does not violate causality if the predicted future of one particle is far enough from the past data of the other particle because the advanced/retarded fields involve respectively a future or a past position measured at another spatial point by another clock synchronized a la Einstein. In the theory of relativity one is not allowed to compare times not measured at the same point so that to falsify the predicted future of the other spatial point one would have to travel to that spatial point. For mixed-type boundary data we with a large enough time-separation between past data and future data, any piecewise-subluminal orbit arrives after the predicted future happens, so that no contradiction is involved, only ill-posed (non-unique) backward continuation.

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