Abstract

In multi-agent systems, intelligent agents are tasked with making decisions that have optimal outcomes when the actions of the other agents are as expected, whilst also being prepared for unexpected behaviour. In this work, we introduce a new risk-averse solution concept that allows the learner to accommodate unexpected actions by finding the minimum variance strategy given any level of expected return. We prove the existence of such a risk-averse equilibrium, and propose one fictitious-play type learning algorithm for smaller games that enjoys provable convergence guarantees in certain games classes (e.g., zero-sum or potential). Furthermore, we propose an approximation method for larger games based on iterative population-based training that generates a population of risk-averse agents. Empirically, our equilibrium is shown to be able to reduce the reward variance, specifically in the sense that off-equilibrium behaviour has a far smaller impact on our risk-averse agents in comparison to playing other equilibrium solutions. Importantly, we show that our population of agents that approximate a risk-averse equilibrium is particularly effective in the presence of unseen opposing populations, especially in the case of guaranteeing a minimal level of performance which is critical to safety-aware multi-agent systems.

1 Introduction

Game Theory (GT) has become an important analytical tool in solving Machine Learning (ML) problems; the idea of "gamification" has become popular in recent years [32, 18] particularly in multi-agent systems research. Despite the importance of risk-aversion in the single-agent decision making literature [35, 24, 7], it is surprising that this research area is not lively in the game theory domain. This paper aims to begin the exploration and fill the research gap in the multi-agent strategic decision-making literature based on the notion of risk-aversion.

One reason that risk-aversion is important is that multi-agent interaction is rife with strategic uncertainty; this is because performance doesn’t solely depend on ones own action. It is rarely the case that one will have complete certainty over the execution and the strategy of the opponent in situations ranging from board games to economic negotiations [5]. This presents a dilemma for autonomous decision-makers in human-AI interaction as one can no longer rely on perfect execution or strategy knowledge. Therefore, an important yet unaddressed issue is what happens in the case when the equilibrium play breaks down and there is a possibility of "off-equilibrium" play (i.e. actions executed by agents that deviate from what is expected). Off-equilibrium play can arise in an array of
Figure 1: Two cars are rewarded for reaching their destination quickly. They are stuck behind slow moving tractors but can stay in their lanes and arrive safely, but slowly. They can also overtake to arrive quickly, but if the other also overtakes they will crash, leading to large negative payoffs.

circumstances, from misunderstandings of reward structures to execution fatigue, but will lead to the execution of an unexpected pure strategy. Hedging against off-equilibrium play is important for the agents as otherwise it can lead to large costs. As demonstrated in Fig. (1), a breakdown in the pure-strategy Nash equilibrium (NE) could lead to both cars overtaking and crashing into each other, a negative result in a critical multi-agent system.

Traditional equilibrium solutions in GT (e.g. NE, Trembling Hand Perfect Equilibrium (THPE) [1]) lack the ability to handle this style of risk as either: 1) they assume strategies are executed perfectly, and/or, 2) large cost may be undervalued in the utility function if there is a low probability attached to them. We address these by introducing a new framework for studying risks in multi-agent systems through mean-variance analysis. In our framework, strategies are evaluated both in how they perform against the expected strategy of the opponent, but also how they perform in the off-equilibrium setting where either a low probability strategy or one with zero probability is selected by the other agent. This is taken into account by the performance variance of a strategy against all potential strategies of the other agent. For example, the driving example in Fig. (1) describes a simple scenario where, due to the critical nature of wanting to avoid crashing, the benefits of overtaking may be entirely redundant with the possibility of unintended play leading to crashes.

We summarise the contributions of our paper here:

1. We introduce a new solution (RAE) concept of risk-averse equilibrium based on mean-variance components of the available strategies. Our framework generalises the single-agent mean-variance decision framework to multi-agent settings.
2. We show that the RAE always exists in finite games, and that it is solvable in the class of games with the fictitious-play property. This, as we later show, unlocks a powerful array of computational methods for solving games.
3. We show that for any level of expected return the RAE solution concept always returns the lowest variance solution.
4. We provide two methods to solve for an RAE in both small and large game settings.
5. Empirically, we demonstrate that: 1) RAE is able to locate and play safe strategies when in the presence of appealing yet risky strategies 2) A by-product of RAE is that it can be used as a Nash equilibrium selection tool in the presence of a "risk-dominant" equilibrium 3) RAE is able to find a low risk strategy in a safety-sensitive autonomous driving environment.

2 Related Work

There exists two relevant bodies of work in the domain of risk-averse GT, those works that aim to empirically study the presence of risk-aversion in humans during laboratory experiments, and those works that aim to develop new frameworks that can help explain these empirical findings.

On the empirical side, the first paper to provide evidence that humans would prefer to bet on known probability devices, rather than on other human choices, suggesting strategic uncertainty is by Camerer [6]. Bohnet [3] similarly found that subjects are more trusting in an objective randomisation device rather than other humans. Eichberger [9] found that more trust is placed in game theorists than “grannies” as the latter is a source of strategic ambiguity. Similar practices are noted in the game setting which more closely model multi-agent interactions, especially in the form of ambiguity aversion, such as those games outside of 3-color Ellsberg Urn tasks [15], public goods and weakest link games [16], or in the presence of strategic complements and strategic substitutes [17]. For an extensive survey of the experimental evidence, we refer readers to [13].
The theoretical literature in this area can be divided into three distinct sections. Harsanyi [14] introduced the notion of risk-dominance as a Nash equilibrium (NE) refinement [28], which suggests that increasing levels of strategic ambiguity will lead to the equilibrium with the lowest deviation losses. Risk dominance is however limited in the fact that it is restricted to the set of NE strategies, and therefore may be risk-dominant in comparison to other NE but not particularly risk-averse at all. Selten [1] set out the THPE which deals with strategic risk by accounting for off-equilibrium play. However, this is sensitive to strictly dominated strategies and assumes all trembles happen with the same probability to any action which can hide the impact of large downside risk which is problematic for safety-sensitive systems (e.g., autonomous driving). Our work instead takes into account the probability of all other actions being played in the risk calculation, which allows for differing levels of importance being placed on strategies that may lead to costs. McKelvey [22] utilises the Quantal Response Equilibrium (QRE) to introduce errors into strategy selection but with lower percentages on big mistakes which also discounts the impact of large downside risk. We would argue that the aforementioned method undervalues big mistakes which can be particularly damaging in large-scale real-world settings where safety is critical, whereas our method hedges away from high cost risk. Yekkehkany [34] utilises a similar mean-variance equilibrium concept based on risk derived from one-shot play in the probabilistic setting rather than the expectation setting of this work. This does not apply more generally to the machine learning setting where reward probabilities are not known and is therefore more difficult to apply practically. Another approach to opponent uncertainty is opponent modelling, for example [33] which is a framework to detect differing types of policies. Our approach is different as it does not explicitly model the opponents, but attempts to learn a strategy that adapts to any opponent strategy in terms of risk.

Historically, the key challenge of computational GT is how to solve for a NE. For example, in two-player zero-sum games, it is theoretically possible to solve for an NE directly via linear programming (LP) [27]. Another approach to finding an equilibrium is the iterative method Fictitious Play (FP) [4], where players make best-responses to the time-average action of the opponent. However, in practice both the above approaches can be strictly intractable. This notion of intractability arises when the action space become prohibitively large (both solutions have complexity that scales with the action space) or even continuous, which makes LP inapplicable. Limitations due to action space lead to a general wave of methods that focus on starting with a "restricted" action space and iteratively expanding said space in order to approximate an equilibrium with the best possible strategies. Notably, Double Oracle (DO) [23, 8, 21] and Policy-Space Response Oracles (PSRO) [18, 20, 29, 10] methods are the two major frameworks in this area. These methods all follow iterative best-response dynamics where at each iteration, a best response policy is found against a previous aggregated policy (e.g., NE or the time-average strategy of the restricted action space). In this work we face a similar challenge in terms of the difficulty of solving for our own equilibrium. In this paper we demonstrate how FP and PSRO can be applied as a solver for our new equilibrium concept. In doing so, we provide a concrete methodology for obtaining solutions in our setting. However, we must adapt them as they are generally designed for risk-neutral equilibria which is not the case for this work.

3 Preliminaries & Notations

In this section, we introduce the preliminaries and notation required to understand our formulation. A normal-form game (NFG) is the standard representation of strategic interaction in GT. A finite $n$-person NFG is a tuple $(N, A, u)$, where $N$ is a finite set of $n$ players, $A = A^1, ..., A^n$ is the joint action profile, with $A^i$ being the strategies available to player $i$, and $u = (u^1, ..., u^n)$ where $u^i : A \rightarrow \mathbb{R}$ is the real-valued utility function for each player. A player plays a mixed-strategy, $\sigma^i \in \Delta_{A^i}$, which is a probability distribution over their possible actions.

We also study the setting where the action space is restricted (as in DO [23] and PSRO [18]) in games with large action spaces. One example of this is when we view actions at the policy level and instead model them as RL policies instead of atomic actions. The resulting matrix game is known as a meta-game (payoffs are obtained empirically through simulation). A meta-game is denoted by $\langle N, \Phi, M \rangle$, where $N$ is the finite set of players, $\Phi^i$ represents a population of policies (e.g. deep RL models) and $\Phi = \prod_{i \in N} \Phi^i$ is the space of joint policy profiles, the meta-game payoff table $M : \phi \rightarrow \mathbb{R}^{\Phi \times \Phi}$ is constructed by simulating games that cover different policy combinations. We use $\pi^i \in \Delta_{\Phi^i}$ to denote the meta-policy (e.g., player $i$ plays [RL-Model $1$, RL-Model $2$] with probability $[0.3, 0.7]$), and thus $\pi = (\pi^1, ..., \pi^n)$ is a joint meta-policy profile.
Without loss of generality, we assume that we provide definitions based on playing a symmetric game, where \( \gamma \) is a non-trivial risk-aversion parameter.

The weighted co-variance matrix for the utility matrix \( M \) is a \( |A| \times |A| \) matrix \( \Sigma_M = [c_{ij}] \) with entries

\[
c_{jk} = \frac{1}{1 - \sum_{i=1}^{|A|} \sigma_i^2 \sigma_j \sum_{i=1}^{|A|} \sigma_i (u(a_i, a_j) - \bar{M}_j) (u(a_i, a_k) - \bar{M}_k)},
\]

where \( \bar{M}_i = \frac{1}{|A|} \sum_{k=1}^{|A|} \sigma_k u(a_i, a_k) \), i.e. the weighted average utility for action \( i \). As we are trying to minimise variance with respect to the opponent strategy we used a weighted covariance matrix such that potential variance caused by each action is weighted by its probability of selection under the opponent strategy. As will be discussed later, all actions will receive positive probability under our framework and therefore will always provide some weight in the variance calculation. This allows us to define the mixed-strategy \( \sigma \) utility variance based as follows:

\[
\text{Var}(\sigma, M) = \sum_{k=1}^{|A|} \sum_{n=1}^{|A|} \sigma_k \sigma_n c_{kn} = \sum_{k=1}^{|A|} \sigma_k \sum_{i=1}^{|A|} \sigma_i (u(a_i, a_k) - \bar{M}_k) (u(a_i, a_k) - \bar{M}_k) = \sigma^T \cdot \Sigma_M \cdot \sigma.
\]

The final utility function \( r \) which considers both on- and off-equilibrium utility for strategy \( \sigma \) is,

\[
r(\sigma, \sigma) = \sigma^T \cdot \Sigma_M \cdot \sigma - \gamma \left( \sigma^T \cdot \Sigma_M \cdot \sigma \right),
\]

where \( \gamma \in \mathbb{R} \) is the risk-aversion parameter. Applying Eq. (5) to Fig. (1) we see how we arrive at a strategy profile that has our desired properties. Consider two joint strategy profiles, \( \sigma_1 = ((1, 0), (1, 0)) \) and a Nash equilibrium \( \sigma_2 = ((0, 1), (1, 0)) \) where \((1, 0)\) represents playing \textit{Stay in Lane} with probability 1. On equilibrium, the Nash profile receives a large utility, \( r(\sigma_1) = 5 \) and \( r(\sigma_2) = 20 \). However, \( \text{Var}(\sigma_1) = 12.5 \) and \( \text{Var}(\sigma_2) = 2500 \), i.e. the Nash strategy has huge variance for Player 1. Therefore, \( r(\sigma_1) = 5 - 12.5\gamma \) and \( r(\sigma_2) = 20 - 2500\gamma \) and we have for any non-trivial risk-aversion parameter \( \gamma \) that it is optimal to play the safe equilibrium.
4.2 Equilibrium Concept

We now define our new equilibrium concept based on the utility function (5). First start by defining the best-response map:

$$\sigma^*(\sigma) \in \arg \max_{\sigma} \sigma^T \cdot M \cdot \sigma - \gamma \left( \sigma^T \cdot \Sigma_M \cdot \sigma \right)$$

s.t. \( \sigma(a) \geq 0 \), \( \forall a \in A \)

\( \sigma^T 1 = 1 \),

(6)

where due to the quadratic term \( \sigma^T \cdot \Sigma_M \cdot \sigma \) and the constraints, we have a Quadratic Programme (QP). The above programme finds \( \sigma \) such that the mean-variance utility is maximised, whilst ensuring no strategies are assigned negative action probability, and that the action probabilities sum to one. We now show that a key property of the optimal solution found is that the strategy \( \sigma^* \) is the minimum-variance strategy given the level of expected return achieved by \( \sigma^* \).

**PROPOSITION 1** The solution to optimisation (6) provides the same solutions to the following:

$$\sigma^* \in \arg \min_{\sigma} \sigma^T \cdot \Sigma_M \cdot \sigma$$

s.t. \( \sigma^T \cdot M \cdot \sigma \geq \mu_b \)

\( \sigma(a) \geq 0 \), \( \forall a \in A \)

\( \sigma^T 1 = 1 \),

(7)

where \( \mu_b \in \mathbb{R} \) is the lowest level of expected return that the actor is willing to accept.

**PROOF 2 (Sketch of Proof)** Let \( \mathcal{M}_\gamma \) be the solution set of Eq. 6 and \( \mathcal{M} \) be the solution set of Eq. 7. If \( \sigma \) solves \( \mathcal{M}_\gamma \) and we set \( \mu_b = \sigma^T \cdot M \cdot \sigma \), then the solution sets of \( \mathcal{M}_\gamma \) and \( \mathcal{M} \) coincide. Refer to Appendix (A) for full proof.

This implies that, for any on-equilibrium utility a strategy receives, it is also the minimum variance solution that can attain that utility. The level of utility desired is controlled by the risk-aversion parameter \( \gamma \). Based on Eq. (6), we define the equilibrium for the strategy profile \( \sigma \).

**DEFINITION 3 (Risk-Averse Equilibrium (RAE))** A strategy profile \( \sigma \) is a risk-averse equilibrium if, for all agents, \( \sigma \) is a risk-averse Best Response to \( \sigma \), i.e. \( \sigma \in \max \{ r(a, \sigma) : a \in \Delta \} \).

Finally, a necessary property of many game-theoretic equilibrium concepts is whether a solution exists, at least in the finite game setting in order that they have practical use. For our equilibrium, we note the following result in mixed-strategies:

**THEOREM 4** For any finite N-player game where each player \( i \) has a finite \( k \) number of pure strategies, \( A^i = \{ a^i_1, ..., a^i_k \} \), an RAE exists.

We defer the proof of the result to Appendix (A). Importantly, Theorem 4 establishes the existence of solutions providing practical relevance for our equilibrium concept.

5 Equilibrium Learning via Stochastic Fictitious Play

We start by introducing how to solve for RAE in small NFGs, by showing that our utility function can be used as a form of the stochastic fictitious play (SFP) [11] learning rule. SFP convergence guarantees in a selection of games, most notably potential games [25, 26] and finite two-player zero-sum games [31]. Furthermore, SFP is noted to be robust to games outside of the above game classes [12], and we extend these empirical observations of SFP robustness in Appendix (B).

SFP describes a learning process where each player chooses a best response to their opponents’ time-average strategies. In SFP, a group of \( n \geq 2 \) players repeatedly play a \( n - \)player NFG. The state variable is \( Z_t \in \Delta_S \), whose components \( Z^i_t \) describe the time averages of each player’s behaviour, 

$$Z^i_t = \frac{1}{t} \sum_{u=1}^{t} \sigma^i_u$$
As mentioned earlier, SFP does not necessarily converge to an equilibrium in all game classes (but is with regards to our utility function (Proofs are deferred to Appendix (A)).

Algorithm 1 Iterative RAE Solver

1: Initialise: the “high-level” policy set \( \Phi = \prod_{i \in N} \Phi_i \)
2: for iteration \( t \in \{1, 2, \ldots \} \) do
3: \hspace{1em} for each player \( i \in N \) do:
4: \hspace{2em} Compute meta-policy \( \pi_i \) by SFP (Eq.8).
5: \hspace{2em} Find new policy by Oracle: \( \phi_i^t = O^i(\pi_i^{-t}) \).
6: \hspace{1em} Expand \( \Phi_{i+1}^t \leftarrow \Phi_i^t \cup \{ \phi_i^t \} \).
7: \hspace{1em} Update meta-payoff \( M_{i+1} \).
8: Return: \( \pi \) and \( \Phi \).

where \( \sigma^t_i \in \Delta_{S^i} \) represents the observed strategy of player \( i \) at time-step \( t \). A SFP process is one where each player best responds to the time-average action of their opponent, \( Z_i^{-t} \) such that,

\[
\sigma^t_{i+1} \in \arg \max_{\sigma} v^i(\sigma, Z_i^{-t}) - \lambda v^i(\sigma) \tag{8}
\]

\( v^i(\sigma) : \Delta_{S^i} \rightarrow \mathbb{R} \) is a strictly convex function, and the gradient of \( v^i(\sigma) \) becomes arbitrarily large near the boundary of the strategy simplex, i.e. \( \lim_{\sigma \rightarrow \partial(S^i)} |v^i(\sigma)| = \infty \). We propose the following with regards to our utility function (Proofs are deferred to Appendix (A)).

THEOREM 5 Our utility function Eq. 5 satisfies the necessary properties of a perturbation function to be an instance of SFP, therefore realising convergence guarantees in games that are solved by SFP.

As mentioned earlier, SFP does not necessarily converge to an equilibrium in all game classes (but is found to be robust empirically). Therefore, we show that if the SFP process does at least converge to a strategy then that strategy is guaranteed to be an RAE.

PROPOSITION 6 Suppose the SFP sequence \( \{ Z_t \} \) converges to \( \sigma \) in the observed strategy sense \(^2\), then \( \sigma \) is a risk-averse equilibrium.

Note that in the case of SFP we require a stronger notion of convergence in observed strategies \( \sigma^t_i \) rather than in beliefs \( Z_t^i \), but the usage of a converged final \( \sigma^t_i \) guarantees a risk-averse equilibrium.

6 Equilibrium Learning via Iterative Agent Generation

For games that can’t be tractably displayed in the normal-form, we instead use iterative solution frameworks, which make use of reinforcement learning (RL) agents as proxies for pure strategies. This approach aims to approximate an equilibrium in large games by finding a small representative collection of risk-averse strategies which can instead be selected over by RAE.

Consider two-player stochastic games \( G \) defined by the tuple \( \{ S, A, P, R \} \), where \( S \) is the set of states, \( A = A^1 \times A^2 \) is the joint action space, \( P : S \times A \times S \rightarrow [0, 1] \) is the state-transition function and \( R^i : S \times A \rightarrow \mathbb{R} \) is the reward function for player \( i \). An agent is a policy \( \phi \), where a policy is a mapping \( \phi : S \times A \rightarrow [0, 1] \) which can be described in both a tabular form or as a neural network. The final utility between two agents is defined to be \( M(\phi_i, \phi_j) \) (i.e., in the same manner defined for NFGs in Sec. 4.1), and represents the utility to agent \( \phi_i \) against opponent \( \phi_j \).

Our iterative framework revolves around \( T \in \mathbb{N}^+ \) iterative updates on a meta-game \( M \) (an NFG made up of RL agents as pure strategies) following the framework of PSRO [18]. At every iteration \( t \leq T \), a player is defined by a population of fixed agents \( \Phi_t = \Phi_0 \cup \{ \phi_1, \phi_2, \ldots, \phi_t \} \), where \( \Phi_0 \) is the initial random agent pool. For the sake of notation convenience, we will only consider the single-population case where players share the same \( \Phi_t \). As such, the population will generate a meta-game \( M_t \), a payoff matrix between all of the agents in the population, with individual entries being \( M(\phi_i, \phi_j) \) \( \forall \phi_i, \phi_j \in \Phi_t \).

To make use of a population \( \Phi_t \), we require a way to select which agents \( \phi_t \in \Phi_t \) will be utilised for training. This function \( f \) is a mapping \( f : M_t \rightarrow [0, 1] \) which takes as input a meta-game \( M_t \), and

\(^2\) Convergence in the time-average \( Z_t \) does not imply convergence in the actual strategy taken at each \( t \), but may for example imply cyclic actual behaviour that results in average behaviour converging.
outputs a meta-distribution $\pi_t = f(\Phi_t)$. The output $\pi_t$ is a probability assignment to each agent in the population $\Phi_t$ and, as we are in the single-population setting (i.e., symmetric play), we do not distinguish between populations. This is the equivalent of a mixed-strategy in a NFG, except now the actions are RL agents. We apply our risk-averse equilibrium concept (Def. 3) as the meta-solver. As $\phi$ are RL agents then the agents are sampled by their respective probability in $\pi_t$.

At each epoch the population $\Phi_t$ is augmented with a new agent that is a best-response to the meta-distribution $\pi_t$. Generally, this will be purely in terms of the environment reward as follows, and can be found with any Oracle process such as, Reinforcement Learning or an Evolutionary approach. This framework is displayed in Alg. (1). In our setting there are two properties that we are concerned with when adding a new agent to the population. Notably, how it impacts the environment return but also how it impacts the risk-profile of the population $\pi_{t+1} \cdot \Sigma M_{t+1} \cdot \pi_{t+1}$.

To do this, we follow the PPO approach of [35] that optimises both performance and per-step reward variance, and as shown by [2] the variance of the per-step reward bounds the variance of the total reward from above and can be used as a proxy measure for our variance measure. To achieve this, an augmented MDP is used where the MDP reward, $r_i$ is replaced as follows:

$$r_i^* = r_i + \lambda (r_i)^2 + 2\lambda r_i y_i$$

where $y_i = \sum_{t=1}^{T} r_i^*$ is the average of the rewards during the data collection phase. Notably, as this variance is also with respect to the sampling probability defined by $\pi_t$ this optimises the correct co-variance matrix which is similarly weighted by $\pi_t$.

7 Experiments

We validate the effectiveness of our RAE on three environments that all display some risk component.

1. Randomly generated coordination games where some strategies that provide a high payoff if the other player selects the same strategy, but have large costs if not. There also exist strategies that have lower coordinated payoff but lower costs. We conduct experiments testing SFP on games with 100 actions, and utilising our iterative approach on games with 500 actions. Vanilla policy gradient RL agents are used for the iterative approach.

2. A generalised grid-world stag-hunt [30] game that boasts the properties of a payoff-dominant and risk-dominant equilibrium. In this game it is not possible tractably to list out all pure-strategies and therefore our iterative approach is applied. PPO RL agents are used.

3. An autonomous driving environment [19] based on Fig. (1). To test whether our desired equilibrium in Fig. (1) is in fact possible to attain in an RL setting. PPO RL agents are used.

For our SFP results we select the baselines to be NE, and the two GT risk equilibrium concepts of THPE [1] and QRE [22]. For our iterative experiments, we select the baselines to be PSRO-[Nash,
We first note that there is a stark contrast in the outcome of RAE versus the baselines. Dependent on the risk-aversion parameter ($\gamma$) RAE is progressively less likely to play a high-risk actions. This limits its reward in equilibrium but also improves risk-based performance (for example the minimum reward remains high). RAE is noticeably good at finding a strategy that satisfies a reasonable equilibrium return whilst also maintaining low strategy variance and strong minimum performance.

**Question 1. In the presence of appealing yet risky actions, is RAE able to locate safer options?**

We start by investigating performance in randomly generated coordination NFGs. These NFGs are designed so that there are actions in the games that perform incredibly well if your opponent follows the same strategy but have large negative payoffs if your opponent follows a different strategy. There also exist strategies that perform worse (but still positively) when coordinated on, but also maintain better performance (albeit worse than coordination) when the opponent plays a different action. We provide an example of game like this in Appendix D.

The design of these games is meant to draw out potential pitfalls in current GT solution concepts (e.g. Nash) that focus on appealing coordination utility without considering the off-equilibrium behaviour. We present our results in Fig. (2) where (a) represents games with 100 actions solved using SFP, and (b) represents games with 500 actions solved using our iterative framework.

We first note that there is a stark contrast in the outcome of RAE versus the baselines. Dependent on the risk-aversion parameter ($\gamma$) RAE is progressively less likely to play a high-risk actions. This limits its reward in equilibrium but also improves risk-based performance (for example the minimum reward remains high). RAE is noticeably good at finding a strategy that satisfies a reasonable equilibrium return whilst also maintaining low strategy variance and strong minimum performance.

**Question 2. Can RAE act as a NE selection method?**

A potential by-product of RAE is that it can be used as a NE selection tool. We evaluate this in a generalised stag-hunt grid world environment [30] where there exist both 'payoff-dominant' and 'risk-dominant' NEs. We provide full environment details in Appendix (D) and provide a visualisation in Fig. (3a). There are two differing goals in the environment: 1) Collect plants on your own and receive a low payoff that is safe or 2) Hunt the stag and receive a large positive payoff if the agents catch the stag together, and a large negative payoff if only one agent hunts the stag and is 'gored'. These can be seen as the 'risk-dominant' safe strategy and the 'payoff-dominant' risky strategy.

In Fig. (3) we demonstrate how RAE can effectively act as a NE selection method. In Fig. (3b) we present the on-equilibrium performance where each algorithm is training against itself. Notably, all the baselines focus on capturing the stag, which is the risky payoff-dominant strategy. However, our RAE finds the risk-dominant safe strategy in which it focuses on gathering plants and not going after the stag.
| Method       | Eqm Reward       | Eqm Variance   | Worst-Case       | Num. Crashes | Num. Arrivals |
|-------------|------------------|---------------|------------------|-------------|--------------|
| PSRO-Nash   | 0.85 ± 2.64      | 1.51 ± 0.24   | -4.84 ± 5.76     | 39.5 ± 2.12 | 10.5 ± 2.12  |
| PSRO-Uniform| -0.69 ± 0.87     | 1.70 ± 0.09   | -7.00 ± 2.01     | 42 ± 2.81   | 8.00 ± 2.83  |
| PSRO-THPE   | 0.34 ± 1.29      | 1.60 ± 0.14   | -5.32 ± 3.40     | 41.5 ± 2.16 | 8.50 ± 2.12  |
| PSRO-QRE    | 1.60 ± 0.97      | 1.44 ± 0.13   | -2.84 ± 0.94     | 43 ± 2.85   | 7.00 ± 2.85  |
| Self-Play   | 0.97 ± 2.14      | 1.53 ± 0.12   | -4.80 ± 5.92     | 38.5 ± 4.95 | 11.50 ± 4.95 |
| PSRO-RAE (Ours) | 4.36 ± 2.07  | 0.33 ± 0.004  | 0.10 ± 2.68      | 5.5 ± 0.71  | 46.00 ± 1.41 |

Figure 4: Results on autonomous driving environment. a) Results on 100 episodes over 5 seeds b) Position heat-map for RAE solution c) Position heat-map for Nash solution.

Question 3. **In safety-sensitive environments what sort of strategy does RAE learn to follow?**

Our final challenge concerns how RAE acts in an environment where avoiding any large downside possibility is important, for example autonomous driving where crashing is an undesirable outcome. Our environment is modelled on the example in Fig. (1) where there exists two-way traffic with slow-moving vehicles and potentially faster moving agents behind that may be interested in overtaking. From a game-theoretic standpoint this is a surprisingly difficult problem. A pure-strategy NE would prescribe that one agent would overtake and the other would wait, which is a strategy that is exposed to errors in execution and off-equilibrium play. We provide full environment details in Appendix (D).

In Fig. (4) we provide our results. In Table (a) we provide a collection of environment metrics where the average value is based off of 100 episodes in the environment and the standard deviation is based over 5 different training seeds. Firstly, we note that in terms of pure reward and reward variance RAE outperforms the baselines considerably, whilst also maintaining strong worst-case performance. More importantly, however, is the number of crashes and number of arrivals metrics. Notably, RAE arrives at a strategy that very rarely crashes, and nearly always arrives at the final destination. The same conclusion can not be drawn from any of the provided baselines.

To provide an understanding of why this is happening, in Fig. (4b) and (4c) we provide position heat-maps of the cars utilising the RAE strategy and the Nash strategy respectively. In the RAE heat-map one can see that the strategy taken is the safe strategy, i.e. follow behind until all vehicles in the on-coming lane have passed and then proceed to overtake. This strategy provides little reward for the vast majority of the episode, but remains sensitive to the risk-element of the environment which is our desired outcome. On the other hand, the Nash heat-map shows that the strategy is to overtake straight away and nearly always ends up in a crash due to car congestion in the middle of the episode.

8 Conclusion

We introduce a new risk-averse equilibrium concept, RAE, based on mean-variance analysis of strategies. To the best of our knowledge, we are the first to generalise the single-agent mean-variance decision framework to the multi-agent setting. Theoretically, we prove the existence and solvability of RAE and provide methods for arriving at a RAE in both small scale and large scale game settings. Empirically, we show that our RAE is able to locate safe strategies in the presence of risky strategies, act as a NE selection method in the presence of risk-dominant equilibria and is effective at finding a safe equilibrium in a safety-sensitive autonomous driving environment. Avenues for future work...
should focus on the limitations of the current RAE approach, namely non-convergence guarantees in certain classes of games and the fact that RAE minimises upside and downside variance, where minimising downside variance only would be a desirable property.
References

[1] R Selten Bielefeld. Reexamination of the perfectness concept for equilibrium points in extensive games. In Models of Strategic Rationality, pages 1–31. Springer, 1988.

[2] Lorenzo Bisi, Luca Sabbioni, Edoardo Vittori, Matteo Papini, and Marcello Restelli. Risk-averse trust region optimization for reward-volatility reduction. arXiv preprint arXiv:1912.03193, 2019.

[3] Iris Bohnet and Richard Zeckhauser. Trust, risk and betrayal. Journal of Economic Behavior & Organization, 55(4):467–484, 2004.

[4] George W Brown. Iterative solution of games by fictitious play. Activity analysis of production and allocation, 13(1):374–376, 1951.

[5] Evan M. Calford. Uncertainty aversion in game theory: Experimental evidence. Journal of Economic Behavior and Organization, 176:720–734, 2020.

[6] Colin F Camerer and Risto Karjalainen. Ambiguity-aversion and non-additive beliefs in non-cooperative games: experimental evidence. In Models and experiments in risk and rationality, pages 325–358. Springer, 1994.

[7] Yinlam Chow, Mohammad Ghavamzadeh, Lucas Janson, and Marco Pavone. Risk-constrained reinforcement learning with percentile risk criteria. The Journal of Machine Learning Research, 18(1):6070–6120, 2017.

[8] Le Cong Dinh, Yaodong Yang, Zheng Tian, Nicolas Perez Nieves, Oliver Slumbers, David Henry Mguni, and Jun Wang. Online double oracle. arXiv preprint arXiv:2103.07780, 2021.

[9] Jürgen Eichberger, David Kelsey, and Burkhard C Schipper. Granny versus game theorist: Ambiguity in experimental games. Theory and Decision, 64(2-3):333–362, 2008.

[10] Xidong Feng, Oliver Slumbers, Ziyu Wan, Bo Liu, Stephen McAleer, Ying Wen, Jun Wang, and Yaodong Yang. Neural auto-curricula in two-player zero-sum games. Advances in Neural Information Processing Systems, 34, 2021.

[11] Drew Fudenberg and David M Kreps. Learning mixed equilibria. Games and economic behavior, 5(3):320–367, 1993.

[12] Sam Ganzfried. Fictitious play outperforms counterfactual regret minimization. arXiv preprint arXiv:2001.11165, 2020.

[13] Glenn W Harrison and E Elisabet Rutström. Risk aversion in the laboratory. In Risk aversion in experiments. Emerald Group Publishing Limited, 2008.

[14] John C Harsanyi, Reinhard Selten, et al. A general theory of equilibrium selection in games. MIT Press Books, 1, 1988.

[15] David Kelsey and Sara Le Roux. An experimental study on the effect of ambiguity in a coordination game. Theory and Decision, 79(4):667–688, 2015.

[16] David Kelsey and Sara Le Roux. Dragon slaying with ambiguity: theory and experiments. Journal of Public Economic Theory, 19(1):178–197, 2017.

[17] David Kelsey and Sara le Roux. Strategic ambiguity and decision-making: an experimental study. Theory and Decision, 84(3):387–404, 2018.

[18] Marc Lanctot, Vinicius Zambaldi, Aurdünas Gruslys, Angeliki Lazaridou, Karl Tuyls, Julien Pérolat, David Silver, and Thore Graepel. A unified game-theoretic approach to multiagent reinforcement learning. In Proceedings of the 31st International Conference on Neural Information Processing Systems, pages 4193–4206, 2017.

[19] Edouard Leurent. An environment for autonomous driving decision-making. https://github.com/eleurent/highway-env, 2018.

[20] Stephen McAleer, John Lanier, Roy Fox, and Pierre Baldi. Pipeline PSRO: A scalable approach for finding approximate nash equilibria in large games. In Advances in Neural Information Processing Systems (NeurIPS), 2020.

[21] Stephen McAleer, John Lanier, Kevin Wang, Pierre Baldi, and Roy Fox. XDO: A double oracle algorithm for extensive-form games. Advances in Neural Information Processing Systems (NeurIPS), 2021.
[22] Richard D McKelvey and Thomas R Palfrey. Quantal response equilibria for normal form games. Games and economic behavior, 10(1):6–38, 1995.

[23] H Brendan McMahan, Geoffrey J Gordon, and Avrim Blum. Planning in the presence of cost functions controlled by an adversary. In Proceedings of the 20th International Conference on Machine Learning (ICML-03), pages 536–543, 2003.

[24] Oliver Mihatsch and Ralph Neuneier. Risk-sensitive reinforcement learning. Machine learning, 49(2):267–290, 2002.

[25] Dov Monderer and Lloyd S Shapley. Fictitious play property for games with identical interests. Journal of economic theory, 68(1):258–265, 1996.

[26] Dov Monderer and Lloyd S Shapley. Potential games. Games and economic behavior, 14(1):124–143, 1996.

[27] Oskar Morgenstern and John Von Neumann. Theory of games and economic behavior. Princeton university press, 1953.

[28] John Nash. Non-cooperative games. Annals of mathematics, pages 286–295, 1951.

[29] Nicolas Perez-Nieves, Yaodong Yang, Oliver Slumbers, David H Mguni, Ying Wen, and Jun Wang. Modelling behavioural diversity for learning in open-ended games. In Marina Meila and Tong Zhang, editors, Proceedings of the 38th International Conference on Machine Learning, volume 139 of Proceedings of Machine Learning Research, pages 8514–8524. PMLR, 18–24 Jul 2021.

[30] Alexander Peysakhovich and Adam Lerer. Prosocial learning agents solve generalized stag hunts better than selfish ones, 2017.

[31] Julia Robinson. An iterative method of solving a game. Annals of Mathematics, 54(2):296–301, 1951.

[32] Michael P Wellman. Methods for empirical game-theoretic analysis. In AAAI, pages 1552–1556, 2006.

[33] Tianpei Yang, Zhaopeng Meng, Jianye Hao, Chongjie Zhang, Yan Zheng, and Ze Zheng. Towards efficient detection and optimal response against sophisticated opponents, 2018.

[34] Ali Yekkehkhany, Timothy Murray, and Rakesh Nagi. Risk-averse equilibrium for games, 2020.

[35] Shangtong Zhang, Bo Liu, and Shimon Whiteson. Mean-variance policy iteration for risk-averse reinforcement learning. arXiv preprint arXiv:2004.10888, 2020.
Supplementary Material for Learning Risk-Averse Equilibria in Multi-Agent Systems
## Contents

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A Full Proofs

A.1 Proposition 1 [Minimum Variance Solution]

PROPOSITION 1. The solution to the optimisation (6) provides the same solutions to the following:

\[ \sigma^* \in \arg \min_{\sigma} \sigma^T \cdot \Sigma \cdot \sigma \]
\[ \text{s.t. } \sigma^T \cdot M \cdot \sigma^2 \geq \mu_b \]
\[ \sigma(a) \geq 0 \forall a \in A \]
\[ \sigma^T 1 = 1 \]

where \( \mu_b \in \mathbb{R} \) is the lowest level of expected return that the actor is willing to accept.

Proof. [2] shows by a Lagrange multiplier argument that the optimisation problem,

\[ \sigma^* \in \arg \min_{\sigma} \sigma^T \cdot \Sigma \cdot \sigma \]
\[ \text{s.t. } \sigma^T \cdot M \cdot \sigma^2 \geq \mu_b \]
\[ \sigma(a) \geq 0 \forall a \in A \]
\[ \sigma^T 1 = 1 \]

can be rewritten as

\[ \sigma^* \in \arg \min_{\sigma} \sigma^T \cdot \Sigma_M \cdot \sigma - \tau \left( \sigma^T \cdot M \cdot \sigma^2 \right) \]
\[ \text{s.t. } \sigma(a) \geq 0 \forall a \in A \]
\[ \sigma^T 1 = 1 \]

which can be equivalently expressed as,

\[ \sigma^* \in \arg \min_{\sigma} - \left( \sigma^T \cdot M \cdot \sigma^2 - \lambda \sigma^T \cdot \Sigma_M \cdot \sigma \right) \]
\[ \text{s.t. } \sigma(a) \geq 0 \forall a \in A \]
\[ \sigma^T 1 = 1 \]

where \( \lambda = \frac{1}{\tau} \).

A.2 Theorem 4 [RAE Existence]

THEOREM 4. For any finite N-player game where each player \( i \) has a finite \( k \) number of pure strategies, \( A^i = \{ a^i_1, ..., a^i_k \} \), an RAE exists

Proof. We base our proof on Kakutani’s Fixed Point Theorem

Lemma (Kakutani Fixed Point Theorem). Let \( A \) be a non-empty subset of a finite dimensional Euclidean space. Let \( f : A \to A \) be a correspondence, with \( x \in A \mapsto f(x) \subseteq A \), satisfying the following conditions:

1. \( A \) is a compact and convex set.
2. \( f(x) \) is non-empty for all \( x \in A \).
3. \( f(x) \) is a convex-valued correspondence: for all \( x \in A \), \( f(x) \) is a convex set.
4. \( f(x) \) has a closed graph: that is, if \( \{ x^n, y^n \} \to \{ x, y \} \) with \( y^n \in f(x^n) \), then \( y \in f(x) \).
Then, \( f \) has a fixed point, that is, there exists some \( x \in A \), such that \( x \in f(x) \).

We define our best-response function as \( B_i(\sigma_{-i}) = \arg \max_{a \in \Delta_i} r^i(a, \sigma_{-i}) \) where \( u_i \) is defined as in Eq. (5) and by definition \( s \) must satisfy all of the properties of a proper mixed-strategy, and the best-response correspondence is \( B : \Delta \rightarrow \Delta \) such that for all \( \sigma \in \Delta \), we have:

\[
B(\sigma) = [B_i(\sigma_{-i})]_{i \in N}
\]  \hspace{1cm} (5)

We show that \( B(\sigma) \) satisfies the conditions of Kakutani’s Fixed Point Theorem

1. \( \Delta \) is compact, convex and non-empty.
   By definition \( \Delta = \Pi_{i \in N} \Delta_i \)
   where each \( \Delta_i = \{ a | \sum_j a_j = 1 \} \) is a simplex of dimension \( |A_i| - 1 \), thus each \( \Delta_i \) is closed and bounded, and thus compact. Their product set is also compact.

2. \( B(\sigma) \) is non-empty.
   By the definition of \( B_i(\sigma_{-i}) \) where \( \Delta_i \) is non-empty and compact, and \( r^i \) is a quadratic and hence a polynomial function in \( a \). It is known that all polynomial functions are continuous, we can invoke Weirstrass’s Extreme Value Theorem which states

   **Lemma.** If a real valued-function \( f \) is continuous on the closed interval \( [a, b] \), then \( f \) must attain a maximum and a minimum, each at least once. That is, there exist numbers \( c \) and \( d \) in \( [a, b] \) such that:
   \[
f(c) \geq f(x) \geq f(d) \quad \forall x \in [a, b]
   \]

   Therefore, as \( \Delta_i \) is non-empty and compact and \( r^i \) is continuous in \( a \), \( B_i(\sigma_{-i}) \) is non-empty, and therefore \( B(\sigma) \) is also non-empty.

3. \( B(\sigma) \) is a convex-valued correspondence.
   Equivalently, \( B(\sigma) \subset \Delta \) is convex if and only if \( B_i(\sigma_{-i}) \) is convex for all \( i \).

   In order to show that \( B_i(\sigma_{-i}) \) is convex for all \( i \), we instead show that the Quadratic Programme defined by Eq. (6) is a special case of convex optimisation under certain conditions, and therefore by definition has a feasible set which is a convex set.

   A convex optimisation problem is one of the form,

   \[
   \begin{align*}
   \text{minimize} & \quad f_0(x) \\
   \text{s.t.} & \quad f_i(x) < 0, \ i = 1, \ldots, m \\
   & \quad a_i^T x = b_i, \ i = 1, \ldots, p
   \end{align*}
   \]  \hspace{1cm} (7)

   where \( f_0, \ldots, f_m \) are convex functions. The requirements for a problem to be a convex optimisation problem are:

   (a) the objective function must be convex
   (b) the inequality constraint functions must be convex
   (c) the equality constraint functions \( h_i(x) = a_i^T x = b_i \) must be affine

   We note that a quadratic form \( x^T A x \) is convex if \( A \) is positive semi-definite, and strictly convex if \( A \) is positive definite. In our constrained optimisation, the quadratic term \( \sigma^T \Sigma \sigma \) is always guaranteed to be at least convex as \( \Sigma \), the covariance matrix, is always at least PSD. Therefore, our objective function is convex. Additionally, it is easy to see that our inequality constraint functions are also convex and that our equality constraint function is affine. Therefore, our Quadratic Programme is an instance of a convex optimisation problem.

   Importantly, the feasible set of a convex optimisation problem is convex, since it is the intersection of the domain of the problem.
\[ D = \bigcap_{i=0}^{m} \text{dom} f_i, \]  

(8)

which itself is a convex set.

Therefore, for all members of the feasible set \( x, y \in B_{\lambda}(\sigma_{-i}) \) and all \( \theta \in [0, 1] \) we have that \( \theta x + (1 - \theta)y \in S \) and we have a convex-valued correspondence.

4. \( B(\sigma) \) has a closed graph.

Suppose to obtain a contradiction, that \( B(\sigma) \) does not have a closed graph. Then, there exists a sequence \((\sigma^n, \hat{\sigma}^n) \rightarrow (\sigma, \bar{\sigma})\) with \( \hat{\sigma}^n \in B(\sigma^n) \), but \( \bar{\sigma} \notin B(\sigma) \), i.e. there exists some \( i \) such that \( \hat{\sigma}_i \notin B_i(\sigma_{-i}) \). This implies that there exists some \( \sigma_i^\prime \in \Delta_i \) and some \( \epsilon > 0 \) such that

\[ r_i(\sigma_i^\prime, \sigma_{-i}) > r_i(\sigma_i, \sigma_{-i}) + 3\epsilon. \]

By the continuity of \( r_i \) and the fact that \( \sigma_{-i}^n \rightarrow \sigma_{-i} \), we have for sufficiently large \( n \),

\[ r_i(\sigma_i^\prime, \sigma_{-i}^n) \geq r_i(\sigma_i^\prime, \sigma_{-i}) - \epsilon. \]

and combining the preceding two relations we obtain

\[ r_i(\sigma_i^\prime, \sigma_{-i}^n) > r_i(\sigma_i, \sigma_{-i}) + 2\epsilon \geq r_i(\hat{\sigma}_i^n, \sigma_{-i}^n) + \epsilon \]

(11)

where the second relation follows from the continuity of \( r_i \). This contradicts the assumption that \( \hat{\sigma}_i^n \in B(\sigma_{-i}^n) \) and completes the proof.

Therefore, \( B(\sigma) \) satisfies the conditions of Kakutani’s Fixed Point Theorem, and therefore if \( \sigma^* \in B(\sigma^*) \) then \( \sigma^* \) is an equilibrium. \( \square \)

A.3 Theorem 5 [SFP Convergence]

THEOREM 5. Our utility function Eq. 5 satisfies the necessary properties of a perturbation function to be an instance of SFP, therefore realising convergence guarantees in games that are solved by SFP.

Proof. We show that our utility measure can be embedded as a version of stochastic fictitious play and therefore can be used to find equilibrium in two-player zero-sum games and potential games.

A smooth fictitious play procedure is one in which the best-response, \( B(\sigma) \), is derived from maximising a function of the form \( r_i(\sigma) - \lambda v_i(\sigma_i) \) where,

1. \( v_i(\sigma_i) : A_i \rightarrow \mathbb{R} \) is a strictly convex function.

2. The gradient of \( v_i(\sigma_i) \) becomes arbitrarily large near the boundary of the strategy simplex, i.e. \( \lim_{\sigma_i \rightarrow \partial A_i} |v_i(\sigma_i)| = \infty \)

which ensures that there exists a unique solution to the best-response, and that all pure strategies receive strictly positive probability in the best-response.

We have shown that our variance measure is a strictly convex objective under the assumption that \( \Sigma_i \) is positive-definite. Therefore, we need to show that the gradient satisfies the boundary condition.

We start by showing that \( \lim ||x_n|| = ||x_n|| \) if \( \lim x_n = x \),

**Theorem.** Let \( X \) and \( Y \) be normed spaces. If \( \lim x_n = x \) in \( X \) and \( T : X \rightarrow Y \) is continuous, then

\[ \lim T(x_n) = T(\lim x_n) \]
Proof. Let $\epsilon > 0$. As $T$ is continuous, by the epsilon-delta definition of continuous functions, there exists $\delta > 0$ such that,

$$||x - y|| < \delta \Rightarrow ||T(x) - T(y)|| < \epsilon$$

As $\lim x_n = x$, there exists $n_0 \in \mathbb{N}$ such that,

$$n > n_0 \Rightarrow ||x_n - x|| < \delta$$

and it follows that,

$$n > n_0 \Rightarrow ||T(x_n) - T(x)|| < \epsilon$$

and thus

$$\lim T(x_n) = T(x) = T\lim(x_n)$$

Since,

$$T : X \to \mathbb{R}$$

$$x \mapsto ||x||$$

is continuous, we have $\lim ||x_n|| = ||\lim x_n||$.

Next, we show that our gradient has a lower bound that satisfies the boundary condition. Note for this proof we replace $\sigma$ with $x$ and $\Sigma$ with $\text{Cov}$ as the proof relies upon the singular value decomposition and notation may become confusing.

Due to the symmetry of $\text{Cov}$, $\nabla x \text{Cov} x = 2 \text{Cov} x = W$, and we show that as $x \to \partial A$, $\lim_{x \to \partial A} ||Wx|| > +\infty$.

$$\lim_{x \to \partial A} ||Wx|| = \lim_{x \to \partial A} ||U\Sigma V^T x|| = \lim_{x \to \partial A} ||\Sigma V^T x|| \quad \text{as } U \text{ is orthogonal}$$

$$= \lim_{x \to \partial A} ||\Sigma(V^T x)|| = \lim_{x \to \partial A} \sum_i \sigma_i |(V^T x)_i|$$

where $\sigma_i$ is the $i$-th singular value

$$\geq \lim_{x \to \partial A} \sigma_{\min} \sum_i |(V^T x)_i| = \lim_{x \to \partial A} \sigma_{\min} ||V^T x|| = \lim_{x \to \partial A} \sigma_{\min} ||x|| \quad \text{as } V \text{ is orthogonal}$$

$$= \sigma_{\min} \lim_{x \to \partial S} ||x||$$

due to Theorem 4

At the boundary of the simplex, i.e. utilising a pure strategy, this is the specific case of a mixed-strategy where only Dirac probability distributions can be used. Therefore, in the limit there is infinite density upon the pure strategy at the edge of the simplex and we have that $\lim_{x \to \partial A} x = +\infty$. We can replace this in the above,

$$\lim_{x \to \partial A} ||Wx|| \geq \sigma_{\min} \lim_{x \to \partial A} ||x|| = \sigma_{\min}(+\infty)$$
as Cov is restricted to positive-definiteness, all singular values are strictly positive and we have the
desired result

\[
\lim_{x \to \partial A} ||Wx|| \geq +\infty
\]  \hspace{1cm} (12)

Therefore, our variance function is admissible as the perturbation function \( \nu_i(\sigma_i) \) in stochastic
fictitious play, and retains convergence guarantees.

A.4 Proposition 6 [SFP is RAE]

**Proposition 6.** Suppose the SFP sequence \( \{Z_t\} \) converges to \( \sigma \) in the observed strategy sense
\(^1\), then \( \sigma \) is a Risk-Averse equilibrium.

**Proof.** Assume the observed strategy has converged to \( \sigma = (\sigma^1, \sigma^2) \) and that the strategy is not an
RAE. This implies there exists some \( \sigma_i', \sigma_i \) such that:

\[
r_i(\sigma_i', \sigma_i^-) > r_i(\sigma^i, \sigma^i)
\]  \hspace{1cm} (13)

However, because \( \sigma \) has converged then the SFP sequence \( \{Z_t\} \) will also converge such that:

\[
r_i(\sigma^i, \sigma^{-i}) > r_i(\sigma^i, \sigma^{-i}) \quad \forall \sigma_i', \sigma_i \in \Delta^i
\]  \hspace{1cm} (14)

and therefore \( \sigma_i' \) can not be a best response to \( \sigma^{-i} \).

\(^1\)Convergence in the time-average \( Z_t \) does not imply convergence in the actual strategy taken at each \( t \), but
may for example imply cyclic actual behaviour that results in average behaviour converging.
B  SFP Robustness

Figure 1: Euclidean distance between observed actions after each iteration on randomly generated anti-coordination games. A distance of 0 implies that the process has converged.

Figure 2: Euclidean distance between observed actions after each iteration on randomly generated coordination games. A distance of 0 implies that the process has converged.
Figure 3: Euclidean distance between observed actions after each iteration on randomly generated games. A distance of 0 implies that the process has converged.
## C Hyperparameter Settings

Table 1: Hyper-parameter settings for our experiments.

| SETTINGS                  | VALUE | DESCRIPTION                                              |
|---------------------------|-------|----------------------------------------------------------|
| **SFP COORDINATION GAMES** |       |                                                          |
| ACTION DIMENSION          | 100   | NUMBER OF PURE STRATEGIES AVAILABLE                      |
| FP ITERATIONS             | 100   | NUMBER OF FP BELIEF UPDATES                              |
| TREMBLE PROBABILITY       | 0.001 | PROBABILITY OF TREMBLING TO ANOTHER STRATEGY             |
| QUANTAL TYPE              | SOFTMAX | TYPE OF QUANTAL RESPONSE EQUILIBRIUM                |
| # OF SEEDS                | 50    | # TRIALS                                                  |
| **PSRO NFG COORDINATION GAMES** |       |                                                          |
| ORACLE METHOD             | REINFORCE | SUBROUTINE OF GETTING ORACLES       |
| PSRO ITERATIONS           | 15    | NUMBER OF PSRO ITERATIONS                              |
| ACTION DIMENSION          | 500   | NUMBER OF PURE STRATEGIES AVAILABLE                      |
| LEARNING RATE             | 0.005 | ORACLE LEARNING RATE                                    |
| ORACLE EPOCHS             | 2000  | ORACLE TOTAL EPOCHS                                     |
| ORACLE EPOCH TIMESTEPS    | 100   | TIMESTEPS PER ORACLE EPOCH                              |
| RAE Gamma                 | 0.1, 0.5 | VARIANCE AVERSION PARAMETER                     |
| METASOLVER                | RAE SFP | METASOLVER METHOD                                      |
| METASOLVER ITERATIONS     | 100   | METASOLVER ITERATIONS                                  |
| # OF SEEDS                | 20    | # OF TRIALS                                              |
| **STAG-HUNT GRID-WORLD**  |       |                                                          |
| ORACLE METHOD             | MV-PPO [5] | SUBROUTINE OF GETTING ORACLES       |
| PSRO ITERATIONS           | 10    | NUMBER OF PSRO ITERATIONS                              |
| GOR cost                  | 2     | COST FOR GETTING CAUGHT BY STAG                         |
| PPO HYPERPARAMS           | DEFAULT SB3 [4] | PPO HYPERPARAMETER VALUES   |
| MV-PPO VARIANCE AVSION    | 0.15  | PPO VARIANCE AVSION PARAMETER                          |
| RAE Gamma                 | 0.15  | VARIANCE AVSION PARAMETER                              |
| METASOLVER                | RAE SFP | METASOLVER METHOD                                      |
| METASOLVER ITERATIONS     | 100   | METASOLVER ITERATIONS                                  |
| # OF SEEDS                | 5     | # OF TRIALS                                              |
| **TWO-WAY ENVIRONMENT**   |       |                                                          |
| ORACLE METHOD             | MV-PPO [5] | SUBROUTINE OF GETTING ORACLES       |
| PSRO ITERATIONS           | 7     | NUMBER OF PSRO ITERATIONS                              |
| PPO HYPERPARAMS           | DEFAULT SB3 [4] | PPO HYPERPARAMETER VALUES   |
| MV-PPO VARIANCE AVSION    | 0.5   | PPO VARIANCE AVSION PARAMETER                          |
| RAE Gamma                 | 0.5   | VARIANCE AVSION PARAMETER                              |
| METASOLVER                | RAE SFP | METASOLVER METHOD                                      |
| METASOLVER ITERATIONS     | 100   | METASOLVER ITERATIONS                                  |
| # OF SEEDS                | 5     | # OF TRIALS                                              |
D Environments

D.1 Randomly Generated NFGs

We randomly generate coordination games with $N$ actions in the following way:

**Algorithm 1 Iterative RAE Solver**

1: **Initialise:** Empty $N \times N$ payoff matrix $P$
2: for each action $i$ do:
3: Sample coordination element, $p_{ii} \sim \mathcal{U}(5, 15)$
4: Set Payoff matrix element $P_{ii} = |p_{ii}|$
5: if $P(X \leq p_{ii}) > 0.9$ do
6: for all other actions $j$ do
7: Sample anti-coordination element $p_{ij} \sim \mathcal{U}(-10, 15)$
8: Set Payoff matrix element $P_{ij} = P_{ji} = p_{ij}$
9: else do
10: for all other actions $j$ do
11: Sample anti-coordination element $p_{ij} \sim \mathcal{U}(0, 10)$
12: Set Payoff matrix element $P_{ij} = P_{ji} = p_{ij}$
13: **Return:** $P$.

A simple 3 action example of a NFG generated following the above would be:

![Payoff Matrix Example](image)

Figure 4: Game where one strategy (dotted outline) provides a high return assuming successful coordination but high variance in case the opponent does not coordinate correctly.

D.2 Stag Hunt Grid World

Our stag-hunt environment is taken from [3] where we slightly alter the parameters of the game. A $5 \times 5$ grid is used with 2 players, 1 stag and 2 plants randomly spawned in. The action set of the players is $A = \{\text{left, right, up, down}\}$. The stag at every time-step will move one grid space closer to the closest player on the grid, the plants do not move.

There are 3 different rewards signals in the game:

1. If a player moves over a plant they get $r = 2$ and the plant respawns elsewhere on the grid.
2. If both players move over the stag at the same time both receive $r = 5$ and the stag respawns elsewhere on the grid.
3. If a player moves over the stag on their own, or the stag moves over them on their own the player receives $r = -2$ and the stag respawns elsewhere on the grid.
4. Otherwise $r = 0$.

D.3 Autonomous Driving Environment

Our driving environment is based on the two-way environment from [1] where we make modifications to the reward function to introduce a larger factor of risk-aversion into the game. The goal of the controlled drivers is to reach the end of the road (the destination) whilst avoiding crashing and coming into too close contact with other vehicles. Slow moving drivers populate the roads moving at a constant speed of 20.

There are four reward signals in the environment:

1. If the car crashes $r = -2$.
2. If the car arrives at the destination $r = 2$.
3. If the car is travelling at a good speed ([25,30]), $r = 0.2$.
4. If the car comes very close to another car $r = -0.1$
5. Otherwise $r = 0$. 
E Compute Architecture

All experiments run on one machine with:

- AMD Ryzen Threadripper 3960X 24 Core
- 1 x NVIDIA GeForce RTX 3090
References

[1] Edouard Leurent. An environment for autonomous driving decision-making. https://github.com/eleurent/highway-env, 2018.

[2] Robert C. Merton. An analytic derivation of the efficient portfolio frontier. Journal of Financial and Quantitative Analysis, 7(4):1851–1872, 1972.

[3] Alexander Peysakhovich and Adam Lerer. Prosocial learning agents solve generalized stag hunts better than selfish ones, 2017.

[4] Antonin Raffin, Ashley Hill, Adam Gleave, Anssi Kanervisto, Maximilian Ernestus, and Noah Dormann. Stable-baselines3: Reliable reinforcement learning implementations. Journal of Machine Learning Research, 22(268):1–8, 2021.

[5] Shangtong Zhang, Bo Liu, and Shimon Whiteson. Mean-variance policy iteration for risk-averse reinforcement learning. arXiv preprint arXiv:2004.10888, 2020.