Fermionic Vector Model Solitons in the Large $N$ Limit

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The large $N$ limit of fermionic vectors models is studied using bilocal variables, in the framework of a collective field theory approach. The large $N$ configuration is determined completely using only classical solutions of the model. Further, the Bethe-Salpeter equations of the model are cast as a Green’s function problem. One of the main results of this work is to show that this Green’s function is in fact the large $N$ bilocal itself.
1. Introduction

Systematic large $N$ expansions remain among the most promising tools that can be used to probe the non-perturbative aspects of quantum field theories. For matrix theories, the leading contribution is determined by summing all of the planar diagrams. Except for the simplest models, this is still not possible. In contrast, the situation for vector models is much better (some work on vector/matrix models includes, for instance, [1]). Indeed, the large $N$ limit can often be constructed explicitly. The AdS/CFT correspondence relates the large $N$ limit of matrix theories to a dual gravitational theory. One hopes, that with a better understanding of this correspondence, large $N$ expansions of matrix theories will become possible. It would be useful to have a simpler example with which the correspondence can be studied. Recently, it has been suggested that the large $N$ expansions of vector models can also be related to dual gravitational theories [2], the so-called higher spin theories. In much the same way that vector models were used to understand many of the features expected in QCD, it may be possible to use the vector model/higher spin theories correspondence as toy models for the AdS/CFT correspondence. Given this renewed interest in the large $N$ limit of vector models, it is appropriate to revisit the outstanding issues in the large $N$ expansions of these theories.

A concrete and very interesting proposal for a derivation of the vector model/higher spin theories correspondence was given in [3]. This article employed collective field theory to give a description of bosonic vector models in terms of bilocal variables. The collective field theory is a promising tool to attack this problem. Experience with the AdS/CFT correspondence suggests that supersymmetric examples may be the most amenable to analysis. For this reason, the construction of the collective field theory for fermionic vector models is well motivated. In particular, the descriptions of solitonic objects in the large $N$ limit of fermionic vector models is not well understood. This was the primary motivation for this work.

We discuss the description of the soliton of the Gross-Neveu model, both at the leading and next to leading orders. Surprisingly, both are determined largely by the classical solutions of the model. This is likely to provide useful insights into how one would approach similar questions for supersymmetric vector models which are relevant for the vector model/higher spin theories correspondence. Quite apart from the motivation provided by this correspondence, the fluctuations at subleading order are interesting in their own right. Indeed, they supply important information about the stability of the solution (absence or presence of tachyons) and about the space of solutions (via the zero modes).
This paper is organized as follows: in section 2 the bilocal formalism is discussed and some central results stated. Using the bilocal formalism, a leading order Ansatz is constructed in section 3, first for the perturbative vacuum of the Gross-Neveu model, and then for the soliton. Section 4 is concerned with the next to leading order fluctuations. In section 5, we discuss our results.

2. The bilocal formalism

We follow an approach based on collective field theory [4] to treat the large $N$ limit of vector models [5], [6]. In this approach a change in variables is implemented from the original fermionic field to time-ordered bilocals. The bilocal fields are singlets so that in terms of these variables, all $N$ dependence is explicit. Consequently, these are the correct variables in which to develop a systematic $1/N$ expansion. As usual, this change of variables gives rise to a Jacobian [8]. Incorporating the effects of this Jacobian produces an effective action that can be used to generate systematic corrections in $1/N$.

We start with an action invariant under $U(N)$ transformations. These transformations act as ($\alpha$ is a spinor index)

$$
\psi_\alpha \rightarrow \psi'_\alpha = U \psi_\alpha, \quad \psi_\alpha^\dagger \rightarrow \psi'^\dagger_\alpha = \psi_\alpha^\dagger U^\dagger, \quad U \in U(N)
$$
on our fermionic variables. We change variables from the original fermionic fields to the invariant bilocals, given by

$$
\sigma_{\alpha\beta}(x, y) = \bar{\psi}_\alpha^a(x) \psi_\beta^a(y).
$$

Requiring that the Schwinger-Dyson equations for arbitrary singlet operators derived using the bilocal variables agree with the equations derived using the original variables yields a differential equation for the Jacobian $J$. The equation can be solved exactly to obtain [8]

$$
\log J = - \left[ N + mL^d \delta^d(0) \right] \text{Tr} \log \sigma,
$$

where the trace runs both over the spinor indices and over functional space. In the above expression, $m$ is the dimension of the Clifford algebra, $L^d$ is the volume of spacetime and the delta function is in momentum space. The leading term in this expression was also obtained in [7]. In terms of this Jacobian the effective action is given by
\[ S_{\text{eff}} = -i \log J + NS \]
\[ = iN \text{Tr} \log \sigma + NS + imL^d \delta^d(0) \text{Tr} \log \sigma \]
\[ = NS_0 + S_1. \]

It was shown in [6] that it is consistent to treat the last term as subleading in a systematic \(1/N\) expansion. In the above, the fields have been rescaled such that an overall factor \(N\) multiplies the action. Consequently, the leading large \(N\) configuration, \(\sigma^0\), is now obtained by solving

\[ \frac{\delta S_0}{\delta \sigma} \bigg|_{\sigma^0} = 0. \]

The systematic expansion is now obtained by expanding about this leading configuration

\[ \sigma_{\alpha\beta}(x, y) = \sigma^0_{\alpha\beta}(x, y) + \frac{1}{\sqrt{N}} \eta_{\alpha\beta}(x, y). \]

Performing a systematic expansion of the effective action we obtain

\[ S_{\text{eff}} = NS_0(\sigma^0) + S_1(\sigma^0) - \frac{i}{2} C_2 + \frac{1}{2} D_2 \]
\[ + \sum_{n=1}^{\infty} \frac{1}{\sqrt{N^n}} \left( \frac{1}{(n+2)!} \left[ C_{n+2} \frac{mL^d \delta^d(0) C_n}{n} \right] + \frac{D_{n+2}}{(n+2)!} \right). \]

(2.1)

where

\[ D_n = \int d^d x_1 \cdots \int d^d x_n \int d^d y_1 \cdots \int d^d y_n \frac{\delta^n S}{\delta \sigma_{\alpha_1\beta_1}(x_1, y_1) \cdots \delta \sigma_{\alpha_n\beta_n}(x_n, y_n)} \bigg|_{\sigma^0} \]
\[ \times \eta_{\alpha_1\beta_1}(x_1, y_1) \cdots \eta_{\alpha_n\beta_n}(x_n, y_n), \]

and

\[ C_n = \text{Tr}((\sigma^0)^{-1} \eta)^n. \]

The first corrections to the leading configuration will come from the (quadratic in \(\eta\)) terms \(C_2\) and \(D_2\). The use of these terms to generate corrections is one of the questions with which we will be concerned in this paper. In particular, we solve the Bethe-Salpeter which is obtained as the wave equation for the bilocal following from the quadratic action.
3. Large N and the classical solution

The results of the previous section (in particular, equation (2.1)) clearly show that the
effective action for the bilocal field is determined by the leading large \( N \) configuration \( \sigma^0 \).
In this section, our aim is to argue that this leading configuration can be constructed using
the classical solutions of the fermionic vector model. This extends earlier results obtained
for bosonic vector models [5], [8]. The result for fermionic models is based largely on the
unpublished works [9], [10]. It extends and completes the discussion appearing in [11].

As we mentioned in the introduction, our arguments are illustrated using the Gross-
Neveu (GN) model. However, we would like to emphasize that the methods developed in
this article are general. The Lagrangian of the GN model is given by [12]

\[
L = \bar{\psi}^a(x) (i\gamma^\mu \partial_\mu) \psi^a(x) + \frac{1}{2} g^2 (\bar{\psi}^a(x)\psi^a(x))^2,
\]

where \( a = 1, \ldots, N \) is a color index. Note the presence of a quartic interaction and absence
of a mass term. In the context of a path-integral quantization, one would treat \( \psi \) and \( \bar{\psi} \) as
Grassman valued fields. It is thus natural to suspect that the relevant classical solutions
are obtained by treating \( \psi \) as a classical Grassman field. We will argue that this is not the
case - one should treat \( \psi \) as a normal commuting function.

Recall that for a vector model, the large \( N \) limit is reached by holding \( \lambda = g^2 N \) fixed
and taking \( N \to \infty \). The leading bilocal configuration is determined by the Schwinger-
Dyson equations. The simplification of the theory in the large \( N \) limit is reflected in the
factorization of expectation values of singlets. The Schwinger-Dyson equations

\[
0 = \int D\bar{\psi} D\psi \frac{\delta}{\delta \psi^a_\alpha(x)} \left[ \bar{\psi}^a_\beta(y) e^{iS} \right],
\]

for the GN model are

\[
N\delta_{\alpha \beta} \delta^2(x - y) \langle \bar{1} \rangle = i \langle \bar{\psi}^a_\beta(y) i(\gamma^\mu)^{\alpha \gamma} \partial_\mu \psi^a_\gamma(x) + g^2 \bar{\psi}^a_\beta(y) \psi^a_\alpha(x) \bar{\psi}^b_\rho(x) \psi^b_\rho(x) \rangle.
\]

In terms of the bilocal

\[
\sigma_{\alpha \beta}(x, y) = \bar{\psi}^a_\alpha(x) \psi^a_\beta(y),
\]

the large \( N \) Schwinger-Dyson equation\footnote{That is, the Schwinger-Dyson equation obtained after using factorization in the large N limit.} reads
\[
0 = iN\delta_{\alpha\beta}\delta^2(x-y) + i (\gamma^\mu)^{\alpha\nu} \partial_{x\mu} \sigma_{\beta\nu}(y,x) + g^2 \sigma_{\beta\alpha}(y,x)\sigma_{\rho\rho}(x,x),
\]
where the partial derivative is acting on \(x\). To interpret this equation, note that the \textit{classical} equation of motion reads
\[
0 = i (\gamma^{\mu})^{\alpha\nu} \partial_{x\mu} \sigma_{\beta\nu}(y,x) + g^2 \sigma_{\beta\alpha}(y,x)\sigma_{\rho\rho}(x,x).
\]
Thus, the large \(N\) Schwinger-Dyson equation gives the classical dynamics in the presence of a point source. We would like to emphasize that this result is obtained \textit{by treating} \(\psi\) \textit{as an ordinary commuting function} and not as a Grassman valued field. Further, the point source does \textit{not} arise from the specific action used. It is a general result that the large \(N\) configuration of the bilocal singlet field is simply the classical bilocal singlet field, obtained in the presence of a point source with strength proportional to \(N\) \[9\].

We now turn to the problem of explicitly determining the leading bilocal configuration in terms of the classical solutions. It is simplest to illustrate our method on the perturbative vacuum (no soliton) of the theory. Towards this end, note that there are two linearly independent classical solutions for any given energy
\[
\sigma_{\alpha\beta}(x,y) = (\gamma^\mu p_\mu + m)_{\alpha\beta} A_i^2 e^{-i(\omega_i(t_x-t_y)-k_i(x-y))},
\]
\[
\sigma_{\alpha\beta}(x,y) = (\gamma^\mu p_\mu - m)_{\alpha\beta} A_i^2 e^{i(\omega_i(t_x-t_y)-k_i(x-y))},
\]
where
\[
\omega_i^2 = k_i^2 + (g^2\sigma_{\alpha\alpha}(x,x))^2 = k_i^2 + m^2.
\]
Using these two solutions, it is natural to make the Ansatz
\[
\sigma_{\alpha\beta}(x,y) = \theta(t_x-t_y) (\gamma^\mu p_\mu + m)_{\alpha\beta} A_i^2 e^{-i(\omega_i(t_x-t_y)-k_i(x-y))} \\
- \theta(t_y-t_x) (\gamma^\mu p_\mu - m)_{\alpha\beta} A_i^2 e^{i(\omega_i(t_x-t_y)-k_i(x-y))}.
\]
It is clear that, for unequal times, this Ansatz solves the SD equation, since then the SD equation reduces to the classical equation of motion.

Integrating the SD equation with respect to \(t_x\) from \(t_y - \varepsilon\) to \(t_y + \varepsilon\) \[9\]
\[
-\frac{i}{\varepsilon} \int_{t_y-\varepsilon}^{t_y+\varepsilon} dt_x N\delta_{\alpha\beta}\delta(x-y)\delta(t_x-t_y)
\]
\[
= \int_{t_y-\varepsilon}^{t_y+\varepsilon} dt_x \left[ (i(\gamma^0)^{\alpha\nu} \partial_{\nu} - i(\gamma^i)^{\alpha\nu} \partial_{i\nu}) \sigma_{\beta\nu}(y,x) + g^2 \sigma_{\beta\alpha}(y,x)\sigma_{\rho\rho}(x,x) \right].
\]
The expression on the left is an integral over a delta function and is trivial to perform. Consider now the right side. It is only the first term in the above expression, the time derivative, which contributes. We obtain

\[-iN\delta_{\alpha\beta}\delta(x - y) = i(\gamma^0)^{\alpha\nu}\sigma_{\beta\nu}(y, x)\bigg|_{y \to x}^{|y - x|}\]

\[= \sum_i \left[ \theta(t_x - t_y)i(\gamma^0)^{\alpha\nu}(\gamma^\mu p_\mu + m)_{\beta\nu} A_i^2 e^{-i(k_i(y - x))} - \theta(t_y - t_x)i(\gamma^0)^{\alpha\nu}(\gamma^\mu p_\mu - m)_{\beta\nu} A_i^2 e^{i(k_i(y - x))} \right].\]

This last expression is rearranged to become

\[-iN\delta_{\alpha\beta}\delta(x - y) = 2\delta_{\alpha\beta}\sum_i A_i^2 \omega_i e^{-i(k_i(y - x))},\]

which can be solved to give

\[A_i = \sqrt{\frac{N}{2L\omega_i}}.\]

Substitution of \(\omega_i^2 = k_i^2 + (g^2\sigma_{\alpha\alpha}(x, x))^2\) and the normalization condition \(\sum_i A_i = 1\) gives the constraint

\[1 = \frac{N}{L} g^2 \sum_i \frac{1}{\sqrt{k_i^2 + (g^2\sigma_{\alpha\alpha}(x, x))^2}}\]

which is the gap equation for this theory \[9\]. Although we have considered the perturbative vacuum of the field theory the argument is general. Indeed, a completely parallel argument allows one to construct the large N bilocal corresponding to the kink. Using the classical solution of \[13\] we easily obtain

\[\sigma_{\beta\theta}(y, x) = \sum_k [N\theta(t_x - t_y)\tilde{\psi}_{\beta}(y, k)\psi_{\theta}(x, k) - N\theta(t_y - t_x)\tilde{\psi}_{c\beta}(y, k)\psi_{c\theta}(x, k)],\]

where \(\psi_{\theta}(x, k)\) is the classical solution of \[13\]

\[
\psi = \begin{bmatrix} \psi_1(x) \\ \psi_2(x) \end{bmatrix} = \begin{bmatrix} V_1(x) + iV_2(x) \\ U_1(x) + iU_2(x) \end{bmatrix}
\]

\[= \begin{bmatrix} \frac{\sqrt{\omega_k - k}}{g} \left[ -\tanh(mx) + \frac{\omega_k - k + im}{2\omega_k} (1 + \tanh(mx)) \right] e^{i(k(x - x))} \\ \frac{\sqrt{\omega_k + k}}{g} \left[ 1 - \frac{\omega_k - k + im}{2\omega_k} (1 + \tanh(mx)) \right] e^{i(k(x - x))} \end{bmatrix}.
\]
and $\psi_c$ is related to $\psi$ by charge conjugation. One can once again verify that this bilocal solves the large $N$ SD equation [9], [10]. Notice that this is in fact the solution to the classical Gross-Neveu model equations with $N = 1$. However, in the construction of the bilocal, it is clear that the sum over the spectral parameter $k$ is playing the role of a sum over color. This is reminiscent of the quenched momentum prescription for matrix models [14]. Exactly the same behaviour was noticed even earlier in the context of bosonic vector models [5], [8].

4. Fluctuations about the Leading Kink Solution

In the previous two sections we have provided a construction for the bilocal representing the leading large $N$ configuration. At finite $N$, corrections to the leading bilocal will become important. In this section, we address the problem of computing these corrections. The leading correction is determined by the terms in the effective action that are quadratic in $\eta$. These terms, when varied with respect to $\eta$ give the Bethe-Salpeter equation of the theory, in the one soliton background. The result of this section is that we are able to solve this Bethe-Salpeter equation. Once again, all that is needed for this construction is a knowledge of the classical solutions of the theory! It is primarily with the analysis of this section in mind that we chose to study the Gross-Neveu model. Using the fact that the model is integrable, we will be able to argue that we have constructed a complete set of solutions to the Bethe-Salpeter equation. Although our construction is completely general, the proof of the completeness of the solution set makes crucial use of the integrability of the model.

Our first observation is that the Bethe-Salpeter equation follows by minimizing the quadratic action or by linearizing the Schwinger-Dyson equations. Our use of the integrability of the Gross-Neveu model is to use the scattering data as suitable parameters for this linearization. Since the scattering data are a complete set, we are sure to obtain a complete set of solutions to the Bethe-Salpeter equation. Our approach is motivated by the stability analysis of [15].

The scalar field

\[ \text{That is, as long as the leading configuration is known, our construction can be carried out. Integrability of the model is not used; once the leading large } N \text{ configuration is known the construction is (in principle) mechanical.} \]
\[ \sigma(x, t) = \bar{\psi}\psi, \]
can be used to define a new field \( u(x, t) \) as
\[ \sigma(x, t) = e^{i u(x, t)}. \]

Using the equations of motion for the Gross-Neveu model, one finds that \( u(x, t) \) satisfies the Sine-Gordon equation. It is possible to explicitly obtain the dependence of \( u(x, t) \) (and hence \( \sigma(x, t) \)) on the scattering data, using the methods developed in [16]. The field \( \sigma(x) \) appears in the large \( N \) Schwinger-Dyson equations as a “potential”
\[ 0 = iN\delta_{\alpha\beta}\delta^2(x - y) + i(\gamma^\mu)^{\alpha\nu}\partial_{x\mu}\sigma_{\beta\nu}(y, x) + g^2\sigma_{\beta\alpha}(y, x)\sigma(x). \]

Denote the generic scattering data parameter by \( \Lambda \). Under \( \Lambda \to \Lambda + \Delta\Lambda \), we obtain a second solution to the Schwinger-Dyson equations. This follows as a consequence of the fact that the Schwinger-Dyson equations have no explicit dependence on \( \Lambda \). It is a simple task to show, by expanding,
\[ \sigma_{\alpha\beta}(x, y; \Lambda + \Delta\Lambda) = \sigma_{\alpha\beta}(x, y; \Lambda) + \frac{\delta\sigma_{\alpha\beta}(x, y; \Lambda)}{\delta\Lambda}\Delta\Lambda, \]
and using the fact that both \( \sigma_{\alpha\beta}(x, y; \Lambda + \Delta\Lambda) \) and \( \sigma_{\alpha\beta}(x, y; \Lambda) \) satisfy the Schwinger-Dyson equations, that \( \frac{\delta\sigma_{\alpha\beta}(x, y; \Lambda)}{\delta\Lambda} \) satisfies the Bethe-Salpeter equation. Now, for the problem considered here, since we know the dependence of \( \sigma(x) \) on \( \Lambda \) (thanks to the connection to the Sine-Gordon model) we can compute the variation of \( \psi_\alpha^a(x) \) using the (linearized) equation of motion
\[ (\partial_t + \partial_x)\delta\psi_2 = -g^2(\sigma\delta\psi_1 + \delta\sigma\psi_1) \]
\[ (\partial_t - \partial_x)\delta\psi_1 = g^2(\sigma\delta\psi_2 + \delta\sigma\psi_2). \] (4.1)

The variation of the bilocal then follows
\[ \delta\sigma_{\alpha\beta}(x, y) = \delta\bar{\psi}_\alpha^a(x)\psi_\beta^a(y) + \bar{\psi}_\alpha^a(x)\delta\psi_\beta^a(y). \]

The solution to the system of equations (4.1) can be written as
\[ \begin{bmatrix} \delta\psi_1(x) \\ \delta\psi_2(x) \end{bmatrix} = \int_{-\infty}^{\infty} d^2y \begin{bmatrix} G_{11}(x, y) & G_{12}(x, y) \\ G_{21}(x, y) & G_{22}(x, y) \end{bmatrix} \begin{bmatrix} \delta\psi_1(y) \\ \delta\psi_2(y) \end{bmatrix}. \]
where the Green’s function $G_{\alpha\beta}(x, y)$ satisfies

$$i\gamma^\mu \partial_\mu G(x, y) + g^2 \sigma(x) G(x, y) = \delta^{(2)}(x - y).$$

The central observation of this section is that the above equation is nothing but the equation for the large $N$ bilocal and as a result, the above Green’s function is equal to the large $N$ bilocal configuration.

For the case of the Gross-Neveu model, it is possible to explicitly implement this construction. Further, the construction of a complete set of solutions to the Bethe-Salpeter equation can be used to compute the propagator for the fluctuation $\eta$ in the background of the soliton. This allows one to compute (for example) the leading $1/N$ correction to the soliton mass which is known \[17\]. The specific form of the solutions to the Bethe-Salpeter equation, the use of these equation to determine the $\eta$ propagator and corrections computed using this propagator will be given in \[18\].

If the model was not integrable, the form of $\delta \sigma$ would have been unknown. In this case the large $N$ bilocal configuration continues to furnish a suitable Green’s function. Using this Green’s function one obtains an integral equation that can be iterated in much the same way that one treats scattering problems in non-relativistic quantum mechanics. In this case, one obtains the analog of a Born approximation for the solutions of the Bethe-Salpeter equation.

5. Conclusion

We have studied the large $N$ limit of fermionic vectors models using a collective field theory approach. The large $N$ configuration was determined completely using only classical solutions of the equations of motion, obtained by treating the fermionic fields as ordinary commuting functions. For the case of the Gross-Neveu model, we have demonstrated that a complete set of solutions to the Bethe-Salpeter equation can be constructed. The central insight is that the solutions can be determined using a Green’s function method, and further that this Green’s function is in fact the large $N$ bilocal configuration itself! For more general theories, which will not be integrable, this insight will provide an iterative approximation scheme for the solutions to the Bethe-Salpeter equation. It is surprising that so much of the dynamics of a fermionic quantum field theory can be captured by well chosen classical solutions.
Our analysis of the Gross-Neveu model may now be extended to construct the propagator for the soliton, and compute, for example, corrections to quantities of interest which can be read from the (known) exact S-matrix of the model [17]. This work is in progress [18]. It would be interesting to extend this analysis to other completely integrable models.

For models which are not integrable, we have outlined a procedure to determine an integral equation for the solutions to the Bethe-Salpeter equation. The kernel of this integral equation is the large $N$ bilocal. This integral equation could be iterated to obtain approximate solutions. The accuracy and convergence of this iterative procedure is an interesting question which deserves to be explored.

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