ANALYSIS OF A DIFFUSIVE SIS EPIDEMIC MODEL WITH SPONTANEOUS INFECTION AND A LINEAR SOURCE IN SPATIALLY HETEROGENEOUS ENVIRONMENT

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ABSTRACT. In this paper, we investigate a diffusive SIS epidemic model with spontaneous infection and a linear source in spatially heterogeneous environment. We first prove that the solution of the model is bounded when the susceptible and infected individuals have same or distinct dispersal rates. The global stability of the constant endemic equilibrium is proved by constructing suitable Lyapunov functionals when all parameters are positive constants. We employ the topological degree argument to show the existence of positive steady state. Most importantly, we have also investigated the asymptotic profiles of the positive steady state as the dispersal rate of susceptible or infected individuals tends to zero or infinity. Our result reveals that a linear source and spontaneous infection can significantly enhance disease persistence no matter what dispersal rate of the susceptible or infected population is small or large, which leads to the situation that when total population number allows to vary, disease becomes more difficult to control.

1. Introduction. Since the work of Allen et al. [1], a diffusive susceptible-infected-susceptible (SIS) model have been developed and extended to investigate the joint effects of the spatial heterogeneity and the mobility of individuals on the spatial transmission of a disease. Some important questions such as distinct dispersal rate may have different impacts on disease dynamics have been investigated and answered. For example, the authors in [1] demonstrated that if low-risk site (the low-risk site contains the positions where the disease transmission rate is less than the recovery rate) is not empty, the disease can be controlled as the dispersal rate of the susceptible individuals approaches zero, while Peng [16] showed that disease persistent as the dispersal rate of the infected individuals approaches zero. Combining this result with the one in [1], one then obtain that limiting the movement of susceptible individuals is a better control strategy than limiting the movement of infected individuals. In a recent work of Wu and Zou [21], they used mass action transmission mechanism instead of saturation transmission mechanism and obtained that in strong contrast to [1, 16], disease can not be controlled by limiting the movement of susceptible individuals when the total population is large, while if the total population number is small in some extent, disease can be controlled as the dispersal rate of the susceptible individuals approaches zero.

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Note that the results in [1, 16, 21] are obtained under the assumption that total number of population remains constant. In fact, when using mass action transmission mechanism instead of saturation transmission mechanism in model studied [1, 16], the basic reproduction number is dependent on the population size, hence total number of population plays an import role in controlling the disease by population culling.

As pointed in [15], the combination of spatial heterogeneity and temporal periodicity can enhance the disease persistence. Recently, Li et al. [10] and Li et al. [11] studied SIS epidemic reaction-diffusion system with a linear source or logistic source in spatially heterogeneous environment. They concluded that a varying total population can enhance disease persistence. This leads to new challenges in disease control and prediction. Some recent works on asymptotical profiles of the endemic steady state for large and small diffusion rates, we refer interested readers to [18, 16, 17, 19, 4] and the references therein.

Spatial heterogeneity may have influence on disease transmission. As shown in [1, 16, 21], spatial domain $\Omega$ is divided into the high-risk domain and low-risk domain, respectively, which directly have influence on the existence and non-existence of an endemic equilibrium. In [21], the definition of basic reproduction number, high-risk domain and low-risk domain are dependent not only on the total population but also the size of the domain $|\Omega|$, which suggest that that it should be combined with the size of the region though when make disease control by using population culling. On the other hand, spatial heterogeneity may also leads to the occurrence of the concentration phenomenon. In [21], infected individuals may concentrate at a single point when the movement of infected individuals is limited. In above works, to capture the influence of spatial heterogeneity, the parameters, $\beta$ and $\gamma$ are all assumed to be positive Hölder continuous functions on $\overline{\Omega}$, but not positive constants.

In our current work, we still continuous to investigate the joint effect of the spatial heterogeneity and the mobility of individuals in a diffusive SIS model. We introduce a recent work developed in [19]. The authors in [19] extended the diffusive SIS model in [1], due to spontaneous factors (for example, emotions, behaviors, et al.), to incorporate the effect of spontaneous social infection in addition to disease transmission, which is a infection mechanism and differ from the transmission between susceptible and infected individuals. This spontaneous social infection stem form the recent work of [8], where the authors used SIS model with spontaneous social infection to confirm the relative importance of social transmission by quantitatively comparing rates of spontaneous versus contagious infection. By inserting an term hereby susceptible individuals become infected individuals at a rate $\eta(x)S$, the authors in [19] studied the following diffusive SIS model under homogeneous Neumann boundary condition:

$$\begin{align*}
\frac{\partial S}{\partial t} - dS\Delta S &= -\frac{\beta(x)SI}{S + I} - \eta(x)S + \gamma(x)I, \quad x \in \Omega, \, t > 0, \\
\frac{\partial I}{\partial t} - dI\Delta I &= \beta(x)SI - \eta(x)S - \gamma(x)I, \quad x \in \Omega, \, t > 0, \\
\frac{\partial S}{\partial n} &= \frac{\partial I}{\partial n} = 0, \quad x \in \partial \Omega, \, t > 0, \\
S(x, 0) &= S_0(x) \geq 0, \quad I(x, 0) = I_0(x) \geq 0, \neq 0,
\end{align*}$$

(1)

where $S$ and $I$ stand for the density of susceptible and infected individuals, respectively; $\beta$, $\eta$ and $\gamma$ are positive Hölder continuous functions on $\overline{\Omega}$ representing for the
transmission rate and recovery rate, respectively; $d_S$ and $d_I$ are positive constants measuring the dispersal rate of susceptible and infected individuals respectively; $S$ and $I$ are positive constants measuring the dispersal rate of susceptible and infected individuals respectively; The habitat $\Omega \subset \mathbb{R}^N$ is a bounded domain with smooth boundary $\partial \Omega$. The homogeneous Neumann boundary conditions mean that no population flux crosses the boundary $\partial \Omega$.

We note that in model (1), the rate of spontaneous infection is proportional to the number of susceptible individuals, which in turn contributes to the migration of infected individuals. As studied in above works [1, 16, 21], the basic reproductive number is defined by the fundamental quantity to describe disease dynamics. However, in model (1), due to the spontaneous infection term, social behaviors can be adopted independently of neighbors mean, which means that model (1) no longer have a obvious definition of the basic reproductive number for the behavior of disease persistence. We also note that even in the absence of disease transmission between susceptible and infected individuals there would be a non-zero steady state occurs. By investigating the asymptotic profile of endemic equilibrium if the dispersal rate of the susceptible or infected population is small or large, and combining this result with the one in [1], their theoretical results reveal that spontaneous infection can enhance disease persistence.

Motivated by these works, considering the fact that the population growth is ignored in model (1), and thus, it is natural to consider model with spontaneous infection and a linear source of susceptible individuals in spatially heterogeneous environment. This constitutes one motivation of the present paper. With these considerations, we consider the following diffusive SIS model with a linear external source:

$$\begin{cases}
\frac{\partial S}{\partial t} - d_S \Delta S = \Lambda(x) - b(x)S - \beta(x)SI + \eta(x)S + \gamma(x)I, & x \in \Omega, \ t > 0, \\
\frac{\partial I}{\partial t} - d_I \Delta I = \beta(x)SI + \eta(x)S - \gamma(x)I, & x \in \Omega, \ t > 0, \\
\frac{\partial S}{\partial n} = \frac{\partial I}{\partial n} = 0, & x \in \partial \Omega, \ t > 0,
\end{cases}$$

(2)

where $d_S, d_I, \beta(x), \gamma(x)$ and $\eta(x)$ have the same epidemiological interpretation as in (1). The linear function $\Lambda(x) - b(x)S$ has been widely used in the literature (see, for example [2, 7]) to represent the external source for the susceptible individuals. Throughout the paper, it is assumed that initially, $S_0$ and $I_0$ are nonnegative continuous functions on $\Omega$.

Due to (2), the equilibrium problem satisfies the following elliptic system

$$\begin{cases}
-d_S \Delta S = \Lambda(x) - b(x)S - \beta(x)SI + \eta(x)S + \gamma(x)I, & x \in \Omega, \\
-d_I \Delta I = \beta(x)SI + \eta(x)S - \gamma(x)I, & x \in \Omega, \\
\frac{\partial S}{\partial n} = \frac{\partial I}{\partial n} = 0, & x \in \partial \Omega.
\end{cases}$$

(3)

From the standard theory for parabolic equations, (2) admits a unique classical solution $(S, I) \in C^{2,1}(\bar{\Omega} \times \mathbb{R})$. The strong maximum principle and the Hopf boundary lemma for parabolic equations guarantee that both $S(x, t)$ and $I(x, t)$ are positive for $x \in \Omega$ and $t \in (0, \infty)$. We denote by $EE$ the componentwise positive
solution \((S, I) \in C^2(\Omega) \times C^2(\Omega)\) to (3). It follows from the well-known strong maximum principle and Hopf lemma for elliptic equations that \(S(x) > 0\) and \(I(x) > 0\) for all \(x \in \Omega\). Note that, due to the spontaneous infection term, model (2) no longer have the basic reproductive number. For convenience of notations, throughout the paper, we denote
\[
H^+ = \max_{x \in \Omega} H(x) \in (0, \infty) \quad \text{and} \quad H_- = \min_{x \in \Omega} H(x) \in (0, \infty),
\]
where \(H(x) = \Lambda(x), b(x), \beta(x), \eta(x), \gamma(x)\), respectively.

For \(1 \leq p \leq \infty\), let \(L_p(\Omega)\) denote the Banach space of functions \(u\) whose \(p\)-th power of the absolute value is integrable on \(\Omega\) with the normal norms
\[
\|u\|_{L^p} = \left(\int_\Omega |u|^p\right)^{\frac{1}{p}}, \quad 1 \leq p < \infty; \quad \|u\|_{L^\infty} = \text{ess sup}_{x \in \Omega} |u(x)|, \quad p = \infty.
\]

The remainder of this paper is organized as follows. In section 2, we shall establish the uniform bounds of solutions for cases of \(d_S = d_I\) and \(d_S \neq d_I\), which gives the explicit and implicit \(L^\infty\)-estimates of solutions, respectively. Hereafter, we investigate the local and global stability of the unique constant \(EE(S, I)\) of (2) when all parameters, \(\Lambda, b, \beta, \gamma\) and \(\eta\) are positive constants in section 3. In section 4, we obtain the existence of \(EE\) in spatially heterogeneous environment by using a topological degree arguments. Section 5 is devoted to investigating the asymptotic behavior of \(EE\) when dispersal rate of the susceptible or infected population is small or large.

2. Uniform boundedness of solutions to (2). We first introduce the following lemma, which will be used later.

**Lemma 2.1.** (see [13, Lemma 1]) For the scalar reaction-diffusion equation
\[
\begin{cases}
\frac{\partial W(x, t)}{\partial t} - d\Delta W(x, t) = \Lambda(x) - mW(x, t), & x \in \Omega, \quad t > 0, \\
\frac{\partial W(x, t)}{\partial v} = 0, & x \in \partial \Omega,
\end{cases}
\]
where \(d, m > 0, \Lambda(x)\) is a continuous and positive function on \(\Omega\). System (5) admits a unique positive steady state \(W^*\), which is globally attractive in \(C(\Omega, \mathbb{R})\). Moreover, if \(\Lambda(x) = \Lambda\) for \(\forall x \in \Omega\), then \(W^* = \frac{\Lambda}{m}\).

Note that susceptible and infected individuals disperse at different rate \(d_S, d_I\). It will brings some challenges to prove the eventual uniform boundedness of solutions. In fact, we can not apply the method of adding up the first two equations in (2) for the case \(d_S \neq d_I\). However, it is a special case for (2) with the assumption that both susceptible and infected individuals share the same diffusion rate.

2.1. Uniform bounds of solutions for case of \(d_S = d_I\).

**Proposition 2.2.** Suppose that \(d_S = d_I\). Let \(\epsilon\) be given positive constant such that \(b_- - \epsilon(\beta^+ + \eta^+) > 0\) and
\[
m = \min \left\{ b_- - \epsilon(\beta^+ + \eta^+), \frac{\epsilon\gamma - \epsilon}{1 + \epsilon} \right\}.
\]
Then, \(\forall x \in \bar{\Omega}, t \geq 0\), any solution of (2) satisfies
\[
S(x, t) + I(x, t) \leq \max \{\frac{\Lambda^+}{m}, \ S_0^+ + (1 + \epsilon)I_0^+\}.
\]
Proof. Let us define
\[ U(x, t) = S(x, t) + (1 + \epsilon)I(x, t). \]
Calculating the derivative of \( U(x, t) \) along time \( t \), we have
\[
\frac{\partial U}{\partial t} - d\Delta U = \Lambda(x) - b(x)S + \epsilon\beta(x)\frac{SI}{S + I} + \epsilon\eta(x)S - \epsilon\gamma(x)I
\leq \Lambda(x) - [b_0 - \epsilon(b^+ + \eta^+)]S - \frac{\epsilon\gamma^+}{1 + \epsilon}(1 + \epsilon)I
\leq \Lambda(x) - mU.
\]
Hence, \( U \) is a lower solution of (5), and \( \max\{\Lambda^+/m, S_0^+ + (1 + \epsilon)I_0^+\} \) is an upper solution of (5). Thus by the method of upper/lower solution and the comparison principle, we have
\[ U(x, t) \leq \max\{\Lambda^+/m, S_0^+ + (1 + \epsilon)I_0^+\}, \quad \forall x \in \Omega, t \geq 0, \]
which in turn implies that any solution of (2) satisfies \( L^\infty \)-estimates, that is, there exists a large enough positive constant \( M \) such that \( \|S\|_\infty, \|I\|_\infty < M \) holds. \( \square \)

Following the line of [20], we next apply the theory of invariant region to establish the upper bounds of solutions. We aim to construct a positive invariant rectangle \( \Xi \) such that any solution of (2) from \( \Xi \) will stay in \( \Xi \). To this end, we set
\[
\begin{align*}
\mathcal{V}_1(S, I) &= \Lambda(x) - b(x)S - \beta(x)\frac{SI}{S + I} - \eta(x)S + \gamma(x)I, \quad (x, t) \in \Omega \times [0, \infty), \\
\mathcal{V}_2(S, I) &= \beta(x)\frac{SI}{S + I} + \eta(x)S - \gamma(x)I, \quad (x, t) \in \Omega \times [0, \infty),
\end{align*}
\]
and consider the the vector field \( \mathcal{V}_1, \mathcal{V}_2 \) points inward on the boundary of \( \Xi \) in the \( (S, I) \) phase plane.

**Proposition 2.3.** Let \( \mathcal{K}^+ \) and \( \mathcal{K}^- \) be the larger root of equations
\[ -\gamma^+\mathcal{K}^2 + (\beta^- + b^- + \eta^- - \gamma^+)\mathcal{K} + b^- + \eta^- = 0, \quad (7) \]
and
\[ -\gamma^-\mathcal{K}^2 + (\beta^+ + \eta^+)\mathcal{K} + \eta^+ = 0, \quad (8) \]
respectively, and suppose that \( \mathcal{K}^- < \mathcal{K}^+ \). Then for any given positive constant \( \mathcal{K} \in (\mathcal{K}^-, \mathcal{K}^+) \),
we have
\[ S(x, t) \leq M, \quad I(x, t) \leq \mathcal{K}M, \quad \forall x \in \Omega, t \geq 0, \]
where
\[ M = \max \left\{ S_0^+, \frac{I_0^+}{\mathcal{K}}, \frac{\Lambda^+}{b^- + \eta^-}, \frac{\Lambda^+(\mathcal{K} + 1)}{-\gamma^+\mathcal{K}^2 + (\beta^- + b^- + \eta^- - \gamma^+)\mathcal{K} + b^- + \eta^-} \right\}. \]

**Proof.** In fact, we set \( \Xi = [0, M] \times [0, \mathcal{K}M] \), and initial condition \((S_0(x), I_0(x)) \in \Xi\), for all \( x \in \Omega \). It is easy to see that
\[
\begin{align*}
\mathcal{V}_1(0, I) &= \Lambda(x) + \gamma(x)I \geq 0, \quad \forall I \in [0, \mathcal{K}M], \\
\mathcal{V}_2(S, 0) &= \eta(x)S \geq 0, \quad \forall S \in [0, M].
\end{align*}
\] (9)
Thus, we only need to verify that the following inequalities:
\[
\begin{align*}
\mathcal{V}_1(M, I) &= \Lambda(x) - b(x)M - \beta(x)\frac{MI}{M + I} - \eta(x)M + \gamma(x)I \leq 0, \quad \forall I \in [0, \mathcal{K}M], \\
\mathcal{V}_2(S, \mathcal{K}M) &= \beta(x)\frac{KMS}{S + \mathcal{K}M} - \gamma(x)\mathcal{K}M + \eta(x)S \leq 0, \quad \forall S \in [0, M].
\end{align*}
\] (10)
It follows from $\mathcal{K} \in (\mathcal{K}_-, \mathcal{K}^+)$ that for all $S \in [0, M],$
\[ V_2(S, \mathcal{K}M) \leq V_2(M, \mathcal{K}M) \leq \beta^+ \frac{\mathcal{K}M}{1+\mathcal{K}} - \gamma_- \mathcal{K}M + \eta^+ M \]
\[ = \frac{M}{1+\mathcal{K}} \left[ -\gamma_- \mathcal{K}^2 + (\beta^+ + \eta^+ - \gamma_-) \mathcal{K} + \eta^+ \right] \leq 0. \]

Next we prove $V_1(M, I) \leq 0, \forall I \in [0, \mathcal{K}M],$
\[ V_1(M, I) \leq \Lambda^+ - b_- M - \frac{MI}{M+I} - \eta_- M + \gamma^+ I = -\mathcal{G}(I). \]

Direct calculating the derivative of $\mathcal{G}(I)$ yields
\[ \mathcal{G}'(I) = -\frac{\gamma^+}{(M+I)^2} \left[ I + (\sqrt{\beta_-/\gamma^+} + 1)M \right] \left[ I - (\sqrt{\beta_-/\gamma^+} - 1)M \right]. \]

Note that if $\beta_-/\gamma^+ \leq 1$, then the function $\mathcal{G}(I)$ is non-increasing on the interval $[0, \infty).$ If $\beta_-/\gamma^+ > 1$, then $\mathcal{G}(I)$ is increasing on the interval $[0, (\sqrt{\beta_-/\gamma^+} - 1)M],$ while $\mathcal{G}(I)$ is decreasing on the interval $[(\sqrt{\beta_-/\gamma^+} - 1)M, \infty).$ Hence, we need the conditions
\[ \mathcal{G}(0) = (b_- + \eta_-)M - \Lambda^+ \geq 0, \quad \mathcal{G}(\mathcal{K}M) \geq 0, \quad (11) \]
to obtain the desired results. Clearly, due to the choice of $\mathcal{K}, M$, we can obtain that
\[ \mathcal{G}(\mathcal{K}M) = -\Lambda^+ + b_- M + \beta^- \frac{\mathcal{K}M}{1+\mathcal{K}} + \eta_- M - \gamma^+ \mathcal{K}M \]
\[ = \frac{1}{1+\mathcal{K}} \left\{ [-\gamma^+ \mathcal{K}^2 + (\beta^- + b_- + \eta_- - \gamma^+) \mathcal{K} + b_- + \eta_-]M - (1+\mathcal{K})\Lambda^+ \right\} \geq 0. \]

Hence, $\Xi = [0, M] \times [0, \mathcal{K}M]$ is a positive invariant rectangle of (2), this completes the proof. \hfill \Box

2.2. Uniform bounds of solutions for case of $d_S \neq d_I$. This subsection is devoted to giving an implicit bounds of solutions to (2) in the general case of $d_S \neq d_I.$ We introduce the useful lemma.

**Lemma 2.4.** (see [9, Theorem 1] or [5, Lemma 2.1]) Consider the following parabolic system
\[
\begin{cases}
\frac{\partial u_i}{\partial t} - d_i \Delta u_i = f_i(x, t, u), & x \in \Omega, \ t > 0, \ i = 1, \ldots, l, \\
\frac{\partial u_i}{\partial n} = 0, & x \in \partial \Omega, \ t > 0, \\
u_i(x, 0) = u_i^0(x) \in C(\overline{\Omega}), & x \in \Omega,
\end{cases}
\]
where $d_i > 0$, $(i = 1, \ldots, l)$ are constants, and suppose that, for each $k = 1, \ldots, l$, there exist some nonnegative constants $c_1$ and $c_2$, and $q > 0$, such that the functions $f_k$ satisfy the polynomial growth condition:
\[ |f_k(x, t, u)| \leq c_1 \sum_{i=1}^l |u_i|^q + c_2. \]

Let $p_0$ be a positive constant such that $p_0 > \frac{n}{2} \max\{0, (q - 1)\}$ and $\tau(u^0)$ be the maximal existence time of the solution $u = (u_1, \ldots, u_l)$ corresponding to the initial data $u^0$. The following statements hold:
(i) Suppose that there exists a positive constant $C_p(u^0)$ such that $\|u(\cdot, t)\|_{L^p(\Omega)} \leq C_p(u^0)$, $\forall t \in (0, \tau(u^0))$, then the solution $u$ exists for all time and there is a positive constant $C_{\infty}$ such that $\|u(\cdot, t)\|_{L^\infty(\Omega)} \leq C_{\infty}(u^0)$, $\forall t \in (0, \infty)$. 

(ii) If there exist a positive number $C$ and $K$ independent of initial data such that $\|u(\cdot, t)\|_{L^\infty(\Omega)} \leq K(u^0)$, $\forall t \in [0, \infty)$, then there is a positive number $K(\rho)$ independent of initial data such that $\|u(\cdot, t)\|_{L^\infty(\Omega)} \leq K(\rho)$, $\forall t \in [\rho, \infty)$.

We are now in a position to state the main result of $L^\infty$ estimates of solutions to (2).

**Proposition 2.5.** Let $(S, I)$ be the unique solution of (2). There exists a positive constant $C_1$ dependent to $(S_0, I_0) \in X^+$ such that the solution satisfies the following estimate

$$\|S(\cdot, t)\|_\infty + \|I(\cdot, t)\|_\infty \leq C_1, \quad \forall t \geq 0. \tag{12}$$

Furthermore, there exists a positive constant $C_2$ independent of $(S_0, I_0) \in X^+$ such that the solution satisfies the following estimate

$$\|S(\cdot, t)\|_\infty + \|I(\cdot, t)\|_\infty \leq C_2, \quad \forall t \geq T, \tag{13}$$

for some large time $T > 0$.

**Proof.** Denote by $\epsilon_0$ the any given positive constant such that $b_+ - \epsilon_0(\beta^+ + \eta^+) > 0$ and

$$m = \min \left\{ b_+ - \epsilon_0(\beta^+ + \eta^+), \frac{\epsilon_0\gamma_-}{1 + \epsilon_0} \right\}.$$

Let us define

$$U_1(t) = \int_\Omega [S(x, t) + (1 + \epsilon_0)I(x, t)]dx.$$

Simple calculation reveals that

$$\frac{dU_1(t)}{dt} = \int_\Omega \Lambda(x)dx - \int_\Omega b(x)Sdx + \epsilon_0 \int_\Omega \beta(x)\frac{SI}{S + I}dx + \epsilon_0 \int_\Omega \eta(x)Sdx - \epsilon_0 \int_\Omega \gamma(x)Idx$$

$$\leq \int_\Omega \Lambda(x)dx - b_- \int_\Omega Sdx + \epsilon_0 \beta^+ \int_\Omega \frac{SI}{S + I}dx + \epsilon_0 \eta^+ \int_\Omega Sdx - \epsilon_0 \gamma_- \int_\Omega Idx$$

$$\leq \int_\Omega \Lambda(x)dx - [b_+ - \epsilon_0(\beta^+ + \eta^+)] \int_\Omega Sdx - \epsilon_0 \gamma_- \int_\Omega (1 + \epsilon_0)Idx$$

$$\leq |\Omega|\Lambda^+ - mU_1(t).$$

It follows that

$$U_1(t) \leq U_1(0)e^{-mt} + \frac{|\Omega|\Lambda^+}{m}(1 - e^{-mt}), \tag{14}$$

that is, $(S, I)$ satisfies $L_1$-estimate. Further, from the positivity of $S$ and $I$, and with the help of (i) of Lemma 2.4 with $q = p_0 = 1$, we can conclude that

$$\limsup_{t \to \infty} U_1(t) \leq \frac{|\Omega|\Lambda^+}{m},$$

which is independent of initial data. By applying (ii) of Lemma 2.4 to (2), we have that (13) holds.
3. Global stability of the EE in homogeneous case. In this section, we aim to investigate the global stability of the EE of (2) when all parameters, $\Lambda, b, \beta, \gamma,$ and $\eta$ are positive constants. Elementary computation gives that (2) admits a unique constant equilibrium EE, denoted by $(\hat{S}, \hat{I})$, where

$$\hat{S} = \frac{\Lambda}{b}, \quad \hat{I} = \frac{(\beta + \eta - \gamma)\Lambda b + \sqrt{[(\beta + \eta - \gamma)\Lambda b]^2 + 4\gamma\eta\Lambda^2b^2}}{2\gamma b^2}.$$ 

We first study the linear stability of EE.

Linearizing (2) around unique constant equilibrium $(\hat{S}, \hat{I})$ by using $\xi = S - \hat{S}$ and $\delta = I - \hat{I}$ yields

$$\begin{cases} \frac{\partial \xi}{\partial t} = d_S \Delta \xi - (A + b)\xi - B\delta, & x \in \Omega, \ t > 0, \\ \frac{\partial \delta}{\partial t} = d_I \Delta \delta + A\xi + B\delta, & x \in \Omega, \ t > 0, \end{cases}$$

(15)

where $A = \frac{\beta (\hat{S})^2}{(\hat{S} + \hat{I})^2} + \eta > 0, \ B = \frac{\beta (\hat{S})^2}{(\hat{S} + \hat{I})^2} - \gamma < 0.$ Inserting $(\xi, \delta) = (e^{\sigma t}\phi, e^{\sigma t}\psi)$ into (15), we have

$$\begin{cases} \sigma \phi = d_S \Delta \phi - (A + b)\phi - B\psi, & x \in \Omega, \\ \sigma \psi = d_I \Delta \psi + A\phi + B\psi, & x \in \Omega, \end{cases}$$

(16)

with boundary condition $\frac{\partial \phi}{\partial n} = \frac{\partial \psi}{\partial n} = 0, \ x \in \partial \Omega.$

**Theorem 3.1.** Let $(\hat{S}, \hat{I})$ be the unique constant EE, then it is uniformly asymptotically stable.

**Proof.** Denote

$$\mathbb{A} := \begin{pmatrix} d_S \Delta - (A + b) & -B \\ A & d_I \Delta + B \end{pmatrix}.$$ 

It is well known that the a sequence of eigenvalues $0 = \mu_1 < \mu_2 \leq \cdots$ denote all the eigenvalues of $-\Delta$ with zero Neumann boundary condition on $\partial \Omega$. Denote by $E(\mu_i)$ the subspace generated by the eigenfunctions corresponding to $\mu_i$ $(i = 0, 1, 2, \cdots)$. Let $N_i = \dim E(\mu_i)$ be the algebraic multiplicity of $\mu_i$ and $\phi_{i1}, \phi_{i2}, \cdots, \phi_{iN_i}$ be an orthonormal basis of $E(\mu_i)$, that is, $\{\phi_{ij}\}^N_i$ constitute a complete set of linearly independent eigenfunctions corresponding to $\mu_i$. Define

$$X = \left\{(u, w, v) \in [C^1(\Omega)]^3 : \frac{\partial u}{\partial n} = 0 \text{ on } \partial \Omega \right\}, \ X_{ij} = \{c\phi_{ij} | c \in \mathbb{R}^3\}, \ X_i = \bigoplus_{j=1}^{N_i} X_{ij}.$$ 

Thus $X = \bigoplus_{i=0}^{\infty} X_i$. It is well-known that $X_i(i = 1, 2, \ldots)$ is invariant under the linearization. $\sigma$ is an eigenvalue of $\mathbb{A}$ on $X_i$ if and only if $\sigma$ is an eigenvalue of the matrix

$$\mathbb{A}_i := \begin{pmatrix} -d_S \mu_i - (A + b) & -B \\ A & -d_I \mu_i + B \end{pmatrix}.$$ 

Hence the determinant and trace of $A_i$ can be calculated as,

$$\det \mathbb{A}_i = d_S d_I \mu_i^2 + \mu_i[(A + b)d_I - B d_S] - b B > 0,$$

and

$$\text{Tr} \mathbb{A}_i = -(d_S + d_I)\mu_i - (A + b) + B < 0.$$ 

It follows that the two eigenvalues $\sigma_i^+$ and $\sigma_i^-$ of $\mathbb{A}_i$ have negative real parts. The local asymptotic stability of EE directly follows. $\square$
We next investigate the global stability of the EE of (2) when all parameters, \( \Lambda, b, \beta, \gamma, \) and \( \eta \) are positive constants by constructing Lyapunov functional. Here, Proposition 2.5 remains true.

We state the main result of this section.

**Theorem 3.2.** Let \((\hat{S}, \hat{I})\) be the unique constant EE, then it is globally attractive.

**Proof.** Let us define

\[
U_2(t) := \int_{\Omega} \left[ \left( 1 - \frac{\hat{S}^2}{S^2} \right) dS + \left( 1 - \frac{\hat{I}^2}{I^2} \right) dI \right] dx.
\]

Thus elementary calculating yields

\[
\frac{dU_2(t)}{dt} = \int_{\Omega} \left[ \frac{S^2 - \hat{S}^2}{S^2} dS \Delta S + \frac{I^2 - \hat{I}^2}{I^2} dI \Delta I \right] dx + \int_{\Omega} \frac{S^2 - \hat{S}^2}{S^2} (\Lambda - bS)dx
\]

\[+
\int_{\Omega} \left[ \frac{S^2 - \hat{S}^2}{S^2} \left( -\frac{\beta SI}{S + I} - \eta S + \gamma I \right) + \frac{I^2 - \hat{I}^2}{I^2} \left( \frac{\beta SI}{S + I} + \eta S - \gamma I \right) \right] dx
\]

\[=
- \int_{\Omega} \left[ \frac{2\hat{S}^2}{S^2} |\nabla S|^2 + \frac{2\hat{I}^2}{I^2} |\nabla I|^2 \right] dx + \int_{\Omega} \frac{(S + \hat{S})(S - \hat{S})}{S^2} b(S - \hat{S})dx
\]

\[+
\int_{\Omega} \left[ \frac{\beta SI}{S + I} + \eta S - \gamma I \right] \left[ \frac{I^2 - \hat{I}^2}{I^2} - \frac{S^2 - \hat{S}^2}{S^2} \right] dx
\]

\[=
- \int_{\Omega} \left[ \frac{\beta SI^2}{(S + I)(S + I)} \left( \frac{\hat{I}}{I} - \frac{\hat{S}}{S} \right)^2 - \frac{\hat{I}}{I} + \frac{\hat{S}}{S} \right] dx - \int_{\Omega} \eta \frac{SI}{I} \left( \frac{\hat{I}}{I} - \frac{\hat{S}}{S} \right)^2 \left( \frac{\hat{I}}{I} + \frac{\hat{S}}{S} \right) dx
\]

\[\leq 0, \ \forall t > 0.
\]

Here we used the following equalities, \( \Lambda = b\hat{S}, \) and \( \gamma = \frac{\beta \hat{S}}{S + I} + \frac{\eta \hat{S}}{I}. \) Thus, \( U_2(t) \) is a Lyapunov function for system (2) with all parameters, \( \Lambda, b, \beta, \gamma, \) and \( \eta \) are positive constants. Obviously, \( \frac{dU_2(t)}{dt} = 0 \) if and only if \((S, I) = (\hat{S}, \hat{I}).\)

By using some standard arguments, we can see that

\[
(S(x,t), I(x,t)) \rightarrow (\hat{S}, \hat{I}) \text{ in } [L^2(\Omega)]^2, \text{ as } t \rightarrow \infty.
\]

It follows from Proposition 2.5 that for some positive constant \( C_0, \)

\[
\|S(\cdot, t)\|_{C^2(\Omega)} + \|I(\cdot, t)\|_{C^2(\Omega)} \leq C_0, \ \forall t \geq 1.
\]

Hence, the Sobolev embedding theorem allows one to claim

\[
(S(x,t), I(x,t)) \rightarrow (\hat{S}, \hat{I}) \text{ in } [L^\infty(\Omega)]^2, \text{ as } t \rightarrow \infty.
\]

We can use LaSalle's invariance principle to show that the system (1) admits a connected global attractor on \( \mathbb{R}^+ \) and

\[\lim_{t \rightarrow \infty} (S(\cdot, t), I(\cdot, t)) = (\hat{S}, \hat{I}).\]

Combined with Theorem 3.1, EE is globally asymptotically stable. This completes the proof. \( \square \)
4. Existence of the $EE$. In this section, we aim to investigate the existence of $EE$ when $\Lambda(x), b(x), \beta(x), \gamma(x)$ and $\eta(x)$ are positive Hölder continuous functions on $\Omega$. Before going into details, we introduce a useful lemma from [18, Lemma 2.1], which gives local result for weak super solution of linear elliptic equations (also see, for example, [12]).

Lemma 4.1. (see [18, Lemma 2.1] or [12]) Let $\Omega$ be a bounded Lipschitz domain in $\mathbb{R}^n$. Let $\Lambda$ be a non-negative constant and suppose that $z \in W^{1,2}(\Omega)$ is a non-negative weak solution of the inequalities

\[
0 \leq -\Delta z + \Lambda z \quad \text{in} \quad \Omega, \quad \frac{\partial z}{\partial n} \leq 0 \quad \text{on} \quad \partial \Omega.
\]

Then, for any $q \in [1, n/(n-2))$, there exists a positive constant $C_0$, depending only on $q$, $\Lambda$ and $\Omega$, such that

\[
\|z\|_q \leq C_0 \inf_{\Omega} z.
\]

We will resort to topological degree approach to establish the existence $EE$ of (2). The main result in this section are described as follows.

Theorem 4.2. The steady state problem (3) admits at least one positive solution.

Proof. Let us introduce the following auxiliary system:

\[
\begin{cases}
-d_S \Delta S = (\theta \Lambda(x) + (1 - \theta)\Lambda_0) - (\theta b(x) + (1 - \theta)b_0) S - (\theta \beta(x) + (1 - \theta)\beta_0) \frac{S I}{S + I} - (\theta \eta(x) + (1 - \theta)\eta_0) S + (\theta \gamma(x) + (1 - \theta)\gamma_0) I, \quad x \in \partial \Omega, \\
-d_I \Delta I = (\theta \beta(x) + (1 - \theta)\beta_0) \frac{S I}{S + I} + (\theta \eta(x) + (1 - \theta)\eta_0) S - (\theta \gamma(x) + (1 - \theta)\gamma_0) I, \quad x \in \partial \Omega, \\
\frac{\partial S}{\partial n} = \frac{\partial I}{\partial n} = 0, \quad x \in \partial \Omega,
\end{cases}
\]

(17)

where $\Lambda_0, b_0, \beta_0, \gamma_0, \eta_0$ are positive constants and the parameter $\theta \in [0, 1]$. Note that problem (17) with $\theta = 1$ becomes steady state problem (3).

We next proceed the following three claims concern with upper and lower bounds for any positive solution $(S,I)$ to (17) and the existence of positive solution to (17).

Claim 1. Upper bounds for any positive solution $(S,I)$ to (17). We integrate both sides of the first two equations of (17) and add up to obtain

\[
\int_{\Omega} (\theta \Lambda(x) + (1 - \theta)\Lambda_0) dx = \int_{\Omega} (\theta b(x) + (1 - \theta)b_0) S dx.
\]

(18)

From the second equation of (17), we have

\[
\int_{\Omega} \left[ (\theta \beta(x) + (1 - \theta)\beta_0) \frac{S I}{S + I} + (\theta \eta(x) + (1 - \theta)\eta_0) S \right] dx = \int_{\Omega} (\theta \gamma(x) + (1 - \theta)\gamma_0) I dx.
\]

(19)
and
\[
\int_{\Omega} I dx \leq \frac{\int_{\Omega} \left[ (\theta \beta(x) + (1 - \theta) \beta_0) S + (\theta \eta(x) + (1 - \theta) \eta_0) S \right] dx}{\theta \gamma_+ + (1 - \theta) \gamma_0} \\
\leq \frac{\max \{ \beta^+, \beta_0 \} + \max \{ \eta^+, \eta_0 \}}{\min \{ \gamma_-, \gamma_0 \}} \frac{\max \{ \Lambda^+, \Lambda_0 \}}{\min \{ b^-, b_0 \}} |\Omega|,
\]
where |\Omega| is the volume of \( \Omega \). It follows that any positive solution \((S, I)\) to (17) satisfies \( L^1 \)-estimate.

Further, we can verify that there exists a positive constant \( C \) independent of \( \theta \in [0, 1] \) such that
\[
\int_{\Omega} \left[ (\theta \Lambda(x) + (1 - \theta) \Lambda_0) + (\theta \gamma(x) + (1 - \theta) \gamma_0) I \right] dx \\
\leq \max \{ \Lambda^+, \Lambda_0 \} |\Omega| + \max \{ \gamma^+, \gamma_0 \} \int_{\Omega} I dx \leq C
\]
and
\[
\int_{\Omega} \left( \frac{(\theta \beta(x) + (1 - \theta) \beta_0) SI}{S + I} + (\theta \eta(x) + (1 - \theta) \eta_0) S \right) dx \\
\leq \max \{ \beta^+, \beta_0 \} \int_{\Omega} S dx + \max \{ \eta^+, \eta_0 \} \int_{\Omega} S dx \leq C.
\]

Next we allow the positive constant \( C \) to be different from place to place in the forthcoming discussion, but it does not depend on \( \theta \in [0, 1] \).

By applying the well known \( L^1 \)-estimate theory for elliptic equation [3, Theorem 8], we obtain
\[
\|S\|_{W^{1,1}(\Omega)} \leq C \quad \text{and} \quad \|I\|_{W^{1,1}(\Omega)} \leq C.
\]
From the Sobolev embedding theorem, we have
\[
W^{1,1}(\Omega) \hookrightarrow L^p(\Omega), \quad \forall 1 \leq p \leq \frac{n}{n-1} \quad \text{or} \quad 1 \leq p < \infty \quad \text{if} \quad n = 1.
\]
It follows that
\[
\|S\|_{L^p(\Omega)}, \quad \|I\|_{L^p(\Omega)} \leq C, \quad \forall 1 \leq p \leq \frac{n}{n-1} \quad \text{or} \quad 1 \leq p < \infty \quad \text{if} \quad n = 1.
\]
By applying the well-known \( L^p \)-estimate for elliptic equations (see [6]) leads to
\[
\|S\|_{W^{2,p}(\Omega)}, \quad \|I\|_{W^{2,p}(\Omega)} \leq C, \quad \forall 1 \leq p \leq \frac{n}{n-1} \quad \text{or} \quad 1 \leq p < \infty \quad \text{if} \quad n = 1.
\]
Further using the Sobolev embedding theorem again, we have
\[
\|S\|_{L^{p^*}(\Omega)}, \quad \|I\|_{L^{p^*}(\Omega)} \leq C, \quad \forall 1 \leq p^* \leq \frac{n}{n-3} \quad \text{or} \quad 1 \leq p^* < \infty \quad \text{if} \quad n = 3.
\]
Repeating the above procedure finitely many times, we can obtain that
\[
\|S\|_{L^{p^{**}}(\Omega)} \leq C, \quad \|I\|_{L^{p^{**}}(\Omega)} \leq C, \quad \forall 1 \leq p^{**} < \infty.
\]
Further applying the \( L^p \)-estimate again, we claim that
\[
\|S\|_{W^{2,p^{**}}(\Omega)}, \quad \|I\|_{W^{2,p^{**}}(\Omega)} \leq C, \quad \forall 1 \leq p^{**} < \infty.
\]
Picking $p^{**}$ sufficiently large such that $p^{**} > n$, and using the Sobolev embedding theorem again yield
\[ W^{2,p^{**}}(\Omega) \hookrightarrow C^{1+\alpha}(\Omega), \quad \alpha = 1 - \frac{n}{p^{**}}. \]
Consequently, we have
\[ \|S\|_{L^{\infty}(\Omega)} \leq C, \quad \|I\|_{L^{\infty}(\Omega)} \leq C. \]

**Claim 2.** Lower bounds for any positive solution $(S, I)$ to (17). We integrate the first equation of (17) over $\Omega$ to obtain
\[
\int_{\Omega} \left( (\theta b(x) + (1 - \theta)b_0)S + (\theta \beta(x) + (1 - \theta)\beta_0) \frac{SI}{S + I} + (\theta \eta(x) + (1 - \theta)\eta_0)S \right) dx
\]
\[ = \int_{\Omega} \left[ (\theta \Lambda(x) + (1 - \theta)\Lambda_0) + (\theta \gamma(x) + (1 - \theta)\gamma_0)I \right] dx. \] (20)
It follows that
\[
\int_{\Omega} Sdx \geq \frac{\min\{\Lambda_-, \Lambda_0\}}{\max\{b^+, b_0\} + \max\{\beta^+, \beta_0\} + \max\{\eta^+, \eta_0\}} |\Omega|. \quad (21)
\]
From first equation of (17), we have
\[-\Delta S + \frac{1}{d_S} \max\{b^+, b_0\} S \geq \frac{1}{d_I} \max\{\beta^+, \beta_0\} S > 0, \quad \forall x \in \Omega. \]
Hence, by using Lemma 4.1 with $q = 1$, together with (21), we get
\[ S(x) \geq C, \quad \forall x \in \Omega. \quad (22) \]

We next estimate lower bounds for $I$. From (19), we have
\[
\int_{\Omega} I dx \geq \frac{\min\{\eta_-, \eta_0\}}{\max\{\gamma^+, \gamma_0\}} \int_{\Omega} S dx \geq C. \quad (23)
\]
From second equation of (17), we have
\[-\Delta I + \frac{1}{d_I} \max\{\gamma^+, \gamma_0\} I \geq \frac{1}{d_I} \max\{\eta^+, \eta_0\} S > 0, \quad \forall x \in \Omega. \]
Hence, by using Lemma 4.1 with $q = 1$, together with (22), we get
\[ I(x) \geq C, \quad \forall x \in \Omega. \quad (24) \]
Consequently, from Claims 1 and 2, we arrive at the conclusion that there exists a positive constant $C_* > 1$ such that any positive solution $(S, I)$ to (17) satisfies
\[ \frac{1}{C_*} < S(x), I(x) < C_*, \quad \forall x \in \Omega, \quad (25) \]
where $C_*$ is independent of $\theta \in [0, 1]$.

**Claim 3.** Existence of positive solution to (17). From Claim 2, we define
\[ \Theta = \{(S, I) \in C(\Omega) \times C(\Omega) : \frac{1}{C_*} < S(x), I(x) < C_*, \quad \forall x \in \Omega\}. \]
It is easy to see that (17) has no positive solution $(S, I) \in \partial \Theta$. For $\theta \in [0, 1]$, we define the following operator
\[ \mathcal{A}(\theta, (S, I)) = (-\Delta + I)^{-1} (\mathcal{F}_1(\theta, (S, I)), \mathcal{F}_2(\theta, (S, I))). \]
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where
\[
F_1(\theta, (S,I)) = S + d_S^{-1} \left\{ [(1 - \theta)\Lambda_0 + \theta \Lambda] - [(1 - \theta)b_0 + \theta \eta]S - \frac{[(1 - \theta)\beta_0 + \theta \beta]SI}{S + I} \right. \\
+ \left. [(1 - \theta)\gamma_0 + \theta \gamma]I - [(1 - \theta)\eta_0 + \theta \eta]S \right\},
\]
\[
F_2(\theta, (S,I)) = I + d_I^{-1} \left\{ \frac{[(1 - \theta)\beta_0 + \theta \beta]SI}{S + I} - [(1 - \theta)\gamma_0 + \theta \gamma]I + [(1 - \theta)\eta_0 + \theta \eta]S \right\}.
\]

We next investigate the existence of fixed point of the operator \( A(1, \cdot) \) in \( \Theta \) which is equivalent to the existence of positive solutions to (17). In fact, \( A \) is a compact operator from \([0,1] \times \Theta \) to \( C(\bar{\Omega}) \times C(\bar{\Omega}) \) follows from standard elliptic regularity theory. Here \( A \) subject to Neumann boundary condition over \( \Omega \). On the other hand, we can see that \( (S,I) \neq A(\theta, (S,I)), \forall \theta \in [0,1] \) and \((S,I) \in \partial \Theta \). Hence, the topological degree \( \text{deg}(I - A(\theta, \cdot), \Theta) \) is well-defined, which is also independent of \( \theta \in [0,1] \).

Recall that when \( \theta = 0 \), the unique positive constant steady state of (2) becomes
\[
\hat{S}_0 = \frac{\Lambda_0}{b_0}, \quad \hat{I}_0 = \frac{(\beta_0 + \eta_0 - \gamma_0)\Lambda_0 b_0 + \sqrt{[(\beta_0 + \eta_0 - \gamma_0)\Lambda_0 b_0]^2 + 4\gamma_0\eta_0\Lambda_0^2b_0^2}}{2\gamma_0 b_0^2}.
\]

The linearly stability of \((\hat{S}_0, \hat{I}_0)\) directly follows from Theorem 3.1. From the well-known Leray-Schauder degree index formula (see, for example, [14, Theorem 2.8.1]), we have
\[
\text{deg}(I - A(0, \cdot), \Theta) = \text{index}(I - A(0, \cdot), (S^*, I^*)) = 1.
\]

Further, from the homotopy invariance of the Leray-Schauder degree, we can conclude that
\[
\text{deg}(I - A(1, \cdot), \Theta) = \text{deg}(I - A(0, \cdot), \Theta) = 1,
\]
i.e., \( A(1, \cdot) \) has at least one fixed point in \( \Theta \). Consequently, (3) has at least one positive solution. The completes the proof.

5. Asymptotic profiles of the EE. Note that Theorem 4.2 guarantees that (3) has at least one positive solution. In this section, we aim to investigate the asymptotic behavior of the EE of (3) when the diffusion rate \( d_S \) or \( d_I \) tends to zero or infinity.

5.1. Prior estimation of solution of (3). We begin with the prior estimation for solutions to (3). We can obtain the \( L^\infty \)-estimates for \((S,I)\) of (3) and \( L^1 \) lower bounds estimates for \((S,I)\) of (3).

We integrate both sides of the first two equations of (3) over \( \Omega \) and add up to obtain
\[
\int_\Omega b(x)Sdx = \int_\Omega \Lambda(x)dx. \tag{26}
\]
It follows that
\[
\int_\Omega Sdx \leq \frac{1}{b_-} \int_\Omega \Lambda(x)dx \leq \frac{\Lambda^+}{b_-} |\Omega|. \tag{27}
\]
From the second equation of (3) and (27), we obtain
\[
\gamma - \int_{\Omega} Idx \leq \int_{\Omega} \gamma(x)Idx = \int_{\Omega} \beta(x) \frac{SI}{S + I} dx + \int_{\Omega} \eta(x)Sdx \\
\leq (\beta^+ + \eta^+) \int_{\Omega} Sdx \leq (\beta^+ + \eta^+) \frac{\Lambda^+}{b_-} |\Omega|,
\]
which in turn yields
\[
\int_{\Omega} Idx \leq \frac{(\beta^+ + \eta^+) \Lambda^+}{b_- \gamma_-} |\Omega|. \tag{28}
\]
By using the similar arguments as in Claim 1 of Theorem 4.2, we can obtain
\[
\|S\|_{L^\infty(\Omega)} \leq C \quad \text{and} \quad \|I\|_{L^\infty(\Omega)} \leq C. \tag{29}
\]
On the other hand, from the first equation of (3), we have
\[
\int_{\Omega} (b(x) + \eta(x))Sdx + \int_{\Omega} \beta(x) \frac{SI}{S + I} dx \geq \int_{\Omega} \Lambda(x)dx.
\]
It follows that
\[
\int_{\Omega} (b(x) + \eta(x) + \beta(x))Sdx \geq \Lambda_- |\Omega|,
\]
which implies that
\[
\int_{\Omega} Sdx \geq \frac{\Lambda_-}{b^+ + \beta^+ + \eta^+} |\Omega|. \tag{30}
\]
From the second equation of (3), we obtain
\[
\int_{\Omega} \gamma(x)Idx \geq \int_{\Omega} \eta(x)Sdx.
\]
This together with (30), we have
\[
\int_{\Omega} Idx \geq \frac{\Lambda_- \eta_-}{\gamma^+(b^+ + \beta^+ + \eta^+)} |\Omega|. \tag{31}
\]
5.2. The case of \(d_S \to 0\).

**Theorem 5.1.** Assume that EE exists. Fix \(d_I > 0\), and let \(d_S \to 0\), then every positive solution \((S_{d_S}, I_{d_S})\) of (3) satisfies (up to a subsequence of \(d_S \to 0\))
\[
(S_{d_S}, I_{d_S}) \to (S^{**}, I^{**}) \quad \text{uniformly on} \quad \overline{\Omega},
\]
where
\[
S^{**}(x) = G(x, I^{**}(x)) = \frac{\Lambda + (\gamma - b - \eta - \beta)I^{**} + \sqrt{[\Lambda + (\gamma - b - \eta - \beta)I^{**}]^2 + 4(b + \eta)(\Lambda + \gamma I^{**})I^{**}}}{2(b + \eta)}
\]
and \(I^{**}\) is a positive solution to
\[
\begin{cases}
-d_I \Delta I^{**} = \frac{\beta(x)G(x, I^{**}(x))I^{**}}{G(x, I^{**}(x)) + I^{**}} + \eta(x)G(x, I^{**}(x)) - \gamma(x)I^{**}, & x \in \Omega, \\
\frac{\partial I^{**}}{\partial n} = 0, & x \in \partial \Omega.
\end{cases}
\]
Proof. We divide the proof into two steps.

Step 1. Convergence of $I$. Note that $I$ satisfies

\[
\begin{aligned}
-d_n \Delta I &= \beta(x)\frac{SI}{S+I} + \eta(x)S - \gamma(x)I, & x \in \Omega, \\
\frac{\partial I}{\partial n} &= 0, & x \in \partial \Omega.
\end{aligned}
\]  

(33)

From (29) and the standard $L^p$-estimate for elliptic equations, we can obtain that

\[
\|I\|_{W^{2,p}(\Omega)} \leq C, \quad \forall p \geq 1.
\]

Further using the Sobolev embedding theorem, we have

\[
\|I\|_{C^{1+\alpha}(\Omega)} \leq C \quad \text{for} \quad \alpha = 1 - \frac{n}{p},
\]

where $p$ is large enough ($p > n$) such that $0 < \alpha < 1$. It follows that there exists a subsequence of $d_n \to 0$, say $d_n := d_{\tilde{n}}$, satisfying $d_{\tilde{n}} \to 0$ as $n \to \infty$, and a corresponding positive solution $(S_{\tilde{n}}, I_{\tilde{n}}) = (S_{d_{\tilde{n}}}, I_{d_{\tilde{n}}})$ of (3), such that

\[
I_{\tilde{n}} \to I^{**}(x) \quad \text{uniformly on} \quad \overline{\Omega}, \quad \text{as} \quad n \to \infty,
\]

(34)

where $I^{**} \in C^1(\Omega)$ and $I^{**} \geq 0$.

Applying Lemma 4.1 with $q = 1$ to (31), we obtain

\[
\|I\|_{L^1(\Omega)} \leq C \inf_{\Omega} I,
\]

which in turn implies that

\[
I(x) \geq C, \quad \forall x \in \overline{\Omega}.
\]

(35)

Hence,

\[
I_{\tilde{n}} \to I^{**}(x) > 0 \quad \text{uniformly on} \quad \overline{\Omega}, \quad \text{as} \quad n \to \infty.
\]

(36)

Step 2. Convergence of $S$. From the first equation of (3), $S_n$ satisfies the following equation

\[
\begin{aligned}
-d_n \Delta S_n &= \Lambda(x) - b(x)S_n - \beta(x)\frac{S_n I_n}{S_n + I_n} - \eta(x)S_n + \gamma(x)I_n, & x \in \Omega, \\
\frac{\partial S_n}{\partial n} &= 0, & x \in \partial \Omega.
\end{aligned}
\]

In light of (36), given any small $\epsilon > 0$, choosing sufficiently large $n$, we yield

\[
\begin{aligned}
\Lambda(x) - b(x)S_n - \beta(x)\frac{S_n I_n}{S_n + I_n} - \eta(x)S_n + \gamma(x)I_n \\
&\leq \Lambda(x) - b(x)S_n - \beta(x)\frac{S_n I^{**}(x) - \epsilon}{S_n + (I^{**}(x) - \epsilon)} - \eta(x)S_n + \gamma(x)(I^{**}(x) + \epsilon) \\
&= \frac{(H^1 \epsilon(x, I^{**}(x)) - S_n)(S_n - H^1 \epsilon(x, I^{**}(x)))}{S_n + (I^{**}(x) - \epsilon)}(b(x) + \eta(x)).
\end{aligned}
\]

where

\[
H^1 \epsilon(x, I^{**}(x)) = \frac{1}{2(b + \eta)} \left\{ \Lambda - (b + \beta + \eta)(I^{**} - \epsilon) + \gamma(I^{**} + \epsilon) \pm \sqrt{\Upsilon_1} \right\},
\]

\[
\Upsilon_1 = [\Lambda - (b + \beta + \eta)(I^{**} - \epsilon) + \gamma(I^{**} + \epsilon)]^2 + 4(b + \eta)(\Lambda(I^{**} - \epsilon) + \gamma(I^{**} - \epsilon)(I^{**} + \epsilon))
\]

and $H^1 \epsilon(x, I^{**}(x)) > 0$, $H^1 \epsilon(x, I^{**}(x)) < 0$. For given large $n$, we next consider
the following auxiliary problem
\[
\begin{cases}
-d_n \Delta \omega = \frac{(\mathcal{H}^{1,\epsilon}_{\pm}(x, I^{**}(x)) - \omega)(\omega - \mathcal{H}^{1,\epsilon}_{\pm}(x, I^{**}(x)))}{\omega + (I^{**} - \epsilon)}(b(x) + \eta(x)), & x \in \Omega, \\
\frac{\partial \omega}{\partial n} = 0, & x \in \partial \Omega.
\end{cases}
\]
(37)

It is easy to verify that \( S_n \) is a lower solution of (37). Let \( M \) being a sufficiently large positive constant. Then \( M \) is an upper solution of (37). Thus by the method of upper/lower solution, \( (37) \) has at least one solution \( \bar{\omega}_n \) satisfying \( S_n \leq \bar{\omega}_n \leq M \) on \( \bar{\Omega} \). Together with the fact that \( \mathcal{H}^{1,\epsilon}_{\pm}(x, I^{**}) > 0 \) and \( \mathcal{H}^{1,\epsilon}_{\pm}(x, I^{**}) < 0 \) on \( \bar{\Omega} \), we know that any positive solution \( \tilde{\omega}_n \) of (37) satisfies
\[
\tilde{\omega}_n \to \mathcal{H}^{1,\epsilon}_{\pm}(x, I^{**}(x)) \quad \text{uniformly on } \bar{\Omega}, \quad \text{as } n \to \infty,
\]
and that
\[
\limsup_{n \to \infty} S_n(x) \leq \mathcal{H}^{1,\epsilon}_{\pm}(x, I^{**}(x)) \quad \text{uniformly on } \bar{\Omega}.
\] (38)

In light of (36), given any small \( \epsilon > 0 \), choosing sufficiently large \( n \), we yield
\[
\Lambda(x) - b(x)S_n - \beta(x)\frac{S_n I_n}{S_n + I_n} - \eta(x)S_n + \gamma(x)I_n \\
\geq \Lambda(x) - b(x)S_n - \beta(x)\frac{S_n(I^{**}(x) + \epsilon)}{S_n + (I^{**} + \epsilon)} - \eta(x)S_n + \gamma(x)(I^{**}(x) - \epsilon) \\
= \frac{(\mathcal{H}^{2,\epsilon}_{\pm}(x, I^{**}(x)) - S_n)(S_n - \mathcal{H}^{2,\epsilon}_{\pm}(x, I^{**}(x)))}{S_n + (I^{**} + \epsilon)}(b(x) + \eta(x)),
\]
where
\[
\mathcal{H}^{2,\epsilon}_{\pm}(x, I^{**}(x)) = \frac{1}{2(b + \eta)} \left\{ \Lambda - (b + \beta + \eta)(I^{**} + \epsilon) + \gamma(I^{**} - \epsilon) \sqrt{\Upsilon_2} \right\},
\]
\[
\Upsilon_2 = \frac{[\Lambda - (b + \beta + \eta)(I^{**} + \epsilon)]^2 + 4(b + \eta)(\Lambda(I^{**} + \epsilon) + \gamma(I^{**} - \epsilon))}{\sqrt{\Upsilon_2}}
\]
and \( \mathcal{H}^{2,\epsilon}_{\pm}(x, I^{**}(x)) > 0 \), \( \mathcal{H}^{2,\epsilon}_{\pm}(x, I^{**}(x)) < 0 \). For given large \( n \), we next consider the following auxiliary problem
\[
\begin{cases}
-d_n \Delta \omega = \frac{(\mathcal{H}^{2,\epsilon}_{\pm}(x, I^{**}(x)) - \omega)(\omega - \mathcal{H}^{2,\epsilon}_{\pm}(x, I^{**}(x)))}{\omega + (I^{**} + \epsilon)}(b(x) + \eta(x)), & x \in \Omega, \\
\frac{\partial \omega}{\partial n} = 0, & x \in \partial \Omega.
\end{cases}
\] (39)

It is easy to verify that \( 0 \) is a lower solution of (39) and \( S_n \) is a upper solution of (39). Let \( M \) being a sufficiently large positive constant. Then \( M \) is an upper solution of (37). Thus by the method of upper/lower solution, (39) admits at least one positive solution, denote by \( \tilde{\omega}_n \), such that \( 0 \leq \tilde{\omega}_n \leq S_n \). Further, we have
\[
\liminf_{n \to \infty} S_n(x) \geq \mathcal{H}^{1,\epsilon}_{\pm}(x, I^{**}(x)) \quad \text{uniformly on } \bar{\Omega}.
\] (40)

By the arbitrariness of \( \epsilon \), we can obtain that
\[
\mathcal{H}^{1,0}_{\pm}(x, I^{**}(x)) = \mathcal{H}^{2,0}_{\pm}(x, I^{**}(x)) = G(x, I^{**}(x)).
\]
This, together with (38) and (40), we have
\[
S_n(x) \to G(x, I^{**}(x)) \quad \text{uniformly for } x \in \bar{\Omega}, \quad \text{as } n \to \infty.
\]
Further, from the second equation of (3), \( I^{**} \) satisfies (32). This completes the proof.

5.3. The case of \( d_I \to 0 \). This subsection is devoted to investigating of the asymptotic behavior of positive solutions of (3) with \( d_S > 0 \) being fixed and \( d_I \to 0 \). Our main result reads as follows.

**Theorem 5.2.** Fix \( d_S > 0 \) and let \( d_I \to 0 \). Then every positive solution of \((S, I)\) of (3) satisfies

\[
(S, I) \to (S^{**}, I^{**}) \text{ uniformly on } \overline{\Omega},
\]

where

\[
I^{**} = \frac{(\beta + \eta - \gamma) + \sqrt{(\beta + \eta - \gamma)^2 + 4\gamma \eta}}{2\gamma} S^{**}
\]  

(41)

and \( S^{**} \) is the unique positive solution of

\[
\begin{cases}
-d_S \Delta S^{**} = \Lambda(x) - b(x)S^{**} - \beta(x)\frac{S^{**}I^{**}}{S^{**} + I^{**}} - \eta(x)S^{**} + \gamma(x)I^{**}, & x \in \Omega, \\
\frac{\partial S^{**}}{\partial n} = 0, & x \in \partial \Omega.
\end{cases}
\]

(42)

**Proof.** It follows from (29) that

\[
S(x), I(x) \leq C, \quad \forall x \in \overline{\Omega}.
\]

Recall that \( S \) solves

\[
\begin{cases}
-d_S \Delta S + b(x)S + \eta(x)S + \beta(x)\frac{SI}{S + I} = \Lambda(x) + \gamma(x)I, & x \in \Omega, \\
\frac{\partial S}{\partial n} = 0, & x \in \partial \Omega.
\end{cases}
\]

(43)

From (29) and the standard \( L^p \)-estimate for elliptic equations, we can obtain that

\[
\|S\|_{W^{2,p}(\Omega)} \leq C, \quad \forall 1 < p < \infty.
\]

Further using the Sobolev embedding theorem, we have

\[
\|S\|_{C^{1+\alpha}(\Omega)} \leq C \quad \text{for} \quad \alpha = 1 - \frac{n}{p},
\]

where \( p \) is large enough (\( p > n \)) such that \( 0 < \alpha < 1 \). It follows that there exists a subsequence of \( d_I \to 0 \), say \( d_n := d_{I_n} \to 0 \) with \( d_n \to 0 \) as \( n \to \infty \), the corresponding positive solution sequence \((S_n, I_n) := (S_{I_n}, I_{I_n})\) of (3) satisfies

\[
S_n \to S^{**} \text{ uniformly on } \overline{\Omega}, \quad \text{as} \quad n \to \infty,
\]

(44)

where \( S^{**} \in C^1(\Omega) \). Applying Lemma 4.1 with \( q = 1 \) to (30), we obtain

\[
\|S\|_{L^1(\Omega)} \leq C \inf_{\overline{\Omega}} S,
\]

which in turn implies that

\[
S(x) \geq C, \quad \forall x \in \overline{\Omega}.
\]

(45)

Hence, \( S^{**} > 0 \) on \( \overline{\Omega} \). Note that \( I_n \) satisfies

\[
\begin{cases}
-d_I \Delta I_n = -\gamma I_n^2 + (\beta + \eta - \gamma)S_n I_n + \eta S_n^2, & x \in \Omega, \\
\frac{\partial I_n}{\partial n} = 0, & x \in \partial \Omega.
\end{cases}
\]

(46)
Thus by the method of upper/lower solution as the step 2 in the proof of Theorem 5.1 and making use of (44) and (46), we have
\[ I_n \to I^* \text{ in } C^1(\Omega), \text{ as } n \to \infty, \]
where \( I^* \) is given by (41). Clearly, \( S^* \) satisfies (42). This completes the proof. \( \square \)

5.4. The case of \( d_S \to \infty \). We now determine the asymptotic behavior of positive solutions of (3) when \( d_S \to \infty \), and we have the following statement.

**Theorem 5.3.** Fix \( d_I > 0 \) and let \( d_S \to \infty \), then every positive solution \((S, I)\) of (3), up to a subsequence of \( d_S \), satisfies
\[ (S, I) \to (S^\infty, I^\infty) \text{ uniformly on } \Omega, \]
where \( S^\infty \) is a positive constant and \( I^\infty > 0 \) on \( \Omega \), and \((S^\infty, I^\infty)\) solves
\[
\begin{aligned}
\int_{\Omega} \left[ \Lambda(x) - b(x)S^\infty - \eta(x)S^\infty - \frac{\beta(x)S^\infty I^\infty}{S^\infty + I^\infty} + \gamma(x)I^\infty \right] \, dx &= 0, \\
-d_I \Delta I^\infty &= \frac{\beta(x)S^\infty I^\infty}{S^\infty + I^\infty} + \eta(x)S^\infty - \gamma(x)I^\infty, \quad x \in \Omega, \\
\frac{\partial I^\infty}{\partial n} &= 0, \quad x \in \partial \Omega.
\end{aligned}
\]

**Proof.** From (29), (35) and (45), we can obtain the upper and lower bonds for \((S, I)\) of (3) that
\[ \frac{1}{C} \leq S(x), I(x) \leq C, \quad \forall x \in \Omega. \]

Note that the first equation of (3) satisfies
\[
\begin{aligned}
-\Delta S &= \frac{1}{d_S} \left[ \frac{\Lambda(x)}{S} - \frac{\beta(x)I}{S + I} - b(x) - \eta(x) + \frac{\gamma(x)I}{S} \right] S, \quad x \in \Omega, \\
\frac{\partial S}{\partial n} &= 0 \quad x \in \partial \Omega.
\end{aligned}
\]

It follows from a standard compactness argument to (48), there exists a subsequence of \( d_S \), labeled by \( d_n := d_{S,n} \) with \( d_n \to \infty \) as \( n \to \infty \) such that the corresponding positive solution \((S_n, I_n) := (S_{d_n}, I_{d_n})\) of (3) satisfies \( S_n \to S^\infty \) in \( C^1(\Omega) \) as \( n \to \infty \), where \( S^\infty > 0 \) on \( \Omega \). On the other hand, \( S^\infty \) satisfies
\[ -\Delta S^\infty = 0, \quad x \in \Omega; \quad \frac{\partial S^\infty}{\partial n} = 0, \quad x \in \partial \Omega. \]

Clearly, \( S^\infty > 0 \) on \( \Omega \) must be a positive constant.

Similarly, by the second equation of (3), we can confirm that
\[ I_n \to I^\infty \text{ in } C^1(\Omega), \text{ as } n \to \infty, \]
where the nonnegative function \( I^\infty \in C^1(\Omega) \). Furthermore, \( I^\infty > 0 \) on \( \Omega \) directly follows from \( \frac{1}{C} \leq S(x), I(x) \leq C \). Finally, from the standard elliptic regularity theory and the second equation of (3), it is easily checked that \((S^\infty, I^\infty) \in C^2(\Omega) \times C^2(\Omega) \) solves (47). \( \square \)
5.5. **The case of** $d_I \to \infty$. This subsection is devoted to the asymptotic behavior of positive solutions of (3) when $d_I \to \infty$. Our result reads as follows.

**Theorem 5.4.** Fix $d_S > 0$ and let $d_I \to \infty$, then every positive solution $(S, I)$ of (3), up to a subsequence of $d_I$, satisfies

$$(S, I) \to (S_\infty, I_\infty) \text{ uniformly on } \overline{\Omega},$$

where $I_\infty$ is a positive constant and $S_\infty > 0$ on $\overline{\Omega}$, and $(S_\infty, I_\infty)$ solves

$$
\begin{aligned}
-d_S \Delta S_\infty &= \Lambda(x) - b(x)S_\infty - \eta(x)S_\infty - \frac{\beta(x)S_\infty I_\infty}{S_\infty + I_\infty} + \gamma(x)I_\infty, \quad x \in \Omega, \\
\frac{\partial S_\infty}{\partial n} &= 0, \quad x \in \partial \Omega, \\
\int_{\Omega} \left[ \frac{\beta(x)S_\infty I_\infty}{S_\infty + I_\infty} + \eta(x)S_\infty - \gamma(x)I_\infty \right] dx &= 0.
\end{aligned}
$$

(49)

**Proof.** From (29), (35) and (45), we can obtain the upper and lower bonds for $(S, I)$ of (3) that

$$\frac{1}{C} \leq S(x), I(x) \leq C, \quad \forall x \in \overline{\Omega}.$$ 

Thus, a standard compactness argument as in the proof of Theorem 5.3 concludes that there exists a subsequence of $d_I \to \infty$, labeled by $d_n := d_{I,n}$, satisfying $d_n \to \infty$ as $n \to \infty$, such that the corresponding positive solution sequence $(S_{d_n}, I_{d_n})$ of (3) satisfies

$$(S_{d_n}, I_{d_n}) \to (S_\infty, I_\infty) \text{ in } C^1(\overline{\Omega}), \quad \text{as } d_I \to \infty,$$

where $I_\infty$ is a positive constant and $S_\infty > 0$ on $\overline{\Omega}$. Further, it is easily seen from the first equation of (3) that $(S_\infty, I_\infty)$ satisfies

$$\int_{\Omega} \left[ \frac{\beta(x)S_\infty I_\infty}{S_\infty + I_\infty} + \eta(x)S_\infty - \gamma(x)I_\infty \right] dx = 0,$$

this proves (49). \(\square\)

6. **Conclusion and discussion.** In this paper, we have investigated a diffusive SIS model with heterogeneous coefficients, spontaneous infection and a linear source in spatially heterogeneous environment. We have identified uniform bounds of solutions for cases of $d_S = d_I$ and $d_S \neq d_I$, which gives the explicit and implicit $L^\infty$-estimates of solutions, respectively (see section 2). Due to the lack of basic reproduction number, the unique constant $EE(S, I)$ of (2) when all parameters, $\Lambda, b, \beta, \gamma$ and $\eta$ are positive constants does not depend on it. We further investigated the global stability of the unique constant $EE(S, I)$ by constructing suitable Lyapunov functional (see Theorem 3.2).

We have also explored the asymptotic profiles of $EE$ as the dispersal rate of susceptible or infected hosts tends to zero or $\infty$. Our result in Theorems 5.1 and 5.2 reveal that as the dispersal rate of susceptible or infected hosts tends to zero, susceptible and infected individuals will persist on the $\overline{\Omega}$ at a positive level. This forms a strong contrast to the model studied in [1, 16, 21]. Our result in Theorems 5.3 and 5.4 reveal that as the dispersal rate of susceptible or infected hosts tends to $\infty$, susceptible and infected individuals will persist on the $\overline{\Omega}$ at a positive level. Specifically, $S_\infty > 0$ on $\overline{\Omega}$ must be a positive constant as $d_S \to \infty$ and $I_\infty$ is a positive constant as $d_I \to \infty$. Thus, in this scenario, slow or fast movement of susceptible and infected individuals can not eradicate the disease at all. Our result
also consistent with those in [11, 19], that is, when total population number allows to vary, disease becomes more difficult to control. Combined with the results in [19], a linear source and spontaneous infection can significantly enhance disease persistence no matter what dispersal rate of the susceptible or infected population is small or large. We also mentioned that in Theorems 5.1 and 5.2, we can not determine the final size of $I^{**}$ and $S^{**}$, while in the Theorems 4.1 and 4.2 of [19], they can be computed as positive constant. The essential reason lie in the fact that the total population size is a positive constant in [19] without varying total population setting. Such results can help us better understand the role of linear source and spontaneous infection in designing the elimination measures in order to eradicate the disease.

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