Cosmic String Loops Collapsing to Black Holes

R.N. Hansen, M. Christensen and A.L. Larsen

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Physics Department, University of Odense, Campusvej 55, 5230 Odense M, Denmark

Abstract

We examine the question of collapse of Turok’s two-parameter family of cosmic strings. We first perform a classification of the strings according to the specific time(s) at which the minimal string size is reached during one period. We then obtain an exact analytical expression for the probability of collapse to black holes for the Turok strings. Our result has the same general behavior as previously obtained in the literature but we find, in addition, a numerical prefactor that changes the result by three orders of magnitude. Finally we show that our careful computation of the prefactor helps to understand the discrepancy between previously obtained results and, in particular, that for “large” values of $G_{\mu}$, there may not even be a discrepancy. We also give a simple physical argument that can immediately rule out some of the previously obtained results.

*Electronic address: rnh@fysik.ou.dk
†Electronic address: inane@fysik.sdu.dk
‡Electronic address: all@fysik.ou.dk
1 Introduction

According to the hoop-conjecture [1], a cosmic string that contracts to a size smaller than its Schwarzschild radius will collapse and form a black hole. This process is of particular interest and importance in connection with primordial black holes, Hawking radiation, high energy cosmic gamma bursts etc.

In a network of cosmic strings, only a very small fraction, $f$, of strings are expected to collapse to black holes. Many attempts have been made to obtain a value for $f$ (see for instance [2]-[14]), but the results deviate wildly.

Interestingly enough, only in one of the pioneering papers on the subject, the one by Polnarev and Zembowicz [2], is the derivation of $f$ based on exact analytical expressions for the cosmic strings involved in the process. In all other discussions the derivation of $f$ is based on linearized expressions for the string configurations (for instance [3, 4]), rather general arguments and estimates (for instance [3, 1]), observational data concerning high energy cosmic rays (for instance [7]-[9]) or the derivation is purely numerical (for instance [10, 11]). And even in Ref. [2], the final computation of $f$ is actually numerical, although it could in fact have been performed analytically.

In the present paper we consider the question of collapse for the analytical two-parameter family of strings introduced by Turok [15]. This is the same family of strings that was considered in [2]. However, simple explicit examples show that the results obtained in [2] are correct only in part of the parameter-space. We will show that some essential points were missed in Ref. [2], and that the results obtained there are not completely correct.

First of all, we make a classification of the string configurations according to their general behavior during one period of oscillation. Together with some simple explicit examples, this analysis reveals that the corresponding result obtained in [2] is in fact only correct in approximately half of the two-dimensional parameter-space. Secondly, we then derive the exact analytical expression for the probability $f$ of string collapse to black holes. Our result for $f$ agrees partly with that of Ref. [2] in the sense that $f \propto \left( G\mu \right)^{5/2}$, where $\mu$ is the string tension and $G$ is Newton's constant. However, we find a numerical prefactor in the relation, expressed in terms of Euler’s gamma-function, which is of the order 2000. For comparison, in Ref. [2] the prefactor was found (numerically) to be of the order 1. Finally we show that our careful computation of the prefactor helps to understand the discrepancy between previously obtained results and, in particular, that for “large” values of $G\mu$, there may not even be a discrepancy. We also give a simple physical argument.
that can immediately rule out some of the previously obtained results.

The paper is organized as follows. In Section 2, we introduce the string configurations of Turok [13] and discuss the results obtained by Polnarev and Zembowicz [2]. In Section 3, we give the precise definition of the minimal string size $R$, and we make a complete classification of the two-dimensional parameter-space. The classification is in terms of the specific time(s) at which the value $R$ comes out for a particular string configuration during one period of “oscillation”. In Section 4, we derive an exact analytical expression for the probability of string collapse to black holes and we give the approximate result in the physically realistic limit $G_{\mu} << 1$. Finally in Section 5, we argue that our careful computation of the prefactor helps to understand the discrepancy between previously obtained results, and we give our concluding remarks. Some of the details of the computations used in sections 3 and 4 are presented in the Appendix.

2 Two-Parameter Turok-Strings

The string equations of motion in flat Minkowski space take the form:

$$\ddot{X}^\mu - X'^{\mu} = 0 \quad (2.1)$$

supplemented by the constraints:

$$\eta_{\mu\nu}\dot{X}^{\mu}X^{\nu} = \eta_{\mu\nu}\left(\dot{X}^{\mu}\dot{X}^{\nu} + X'^{\mu}X'^{\nu}\right) = 0 \quad (2.2)$$

It is convenient to take $X^0 = A\tau$, where $A$ is a constant with dimension of length. Then Eqs. (2.1)-(2.2) become:

$$\ddot{\vec{X}} = \ddot{\vec{X}}$$

$$\dot{\vec{X}} \cdot \dot{\vec{X}} = 0$$

$$\dot{\vec{X}}^2 + \dot{\vec{X}}^2 = A^2 \quad (2.3)$$

A two-parameter solution to Eqs. (2.3) was first introduced by Turok [13]:

$$X(\tau, \sigma) = \frac{A}{2}\left[(1 - \alpha)\sin(\sigma - \tau) + \frac{\alpha}{3}\sin(3(\sigma - \tau) + \sin(\sigma + \tau)\right]$$

$$Y(\tau, \sigma) = \frac{A}{2}\left[(1 - \alpha)\cos(\sigma - \tau) + \frac{\alpha}{3}\cos(3(\sigma - \tau) + (1 - 2\beta)\cos(\sigma + \tau)\right]$$

$$Z(\tau, \sigma) = \frac{A}{2}\left[2\sqrt{\alpha(1 - \alpha)}\cos(\sigma - \tau) + 2\sqrt{\beta(1 - 2\beta)}\cos(\sigma + \tau)\right] \quad (2.4)$$
where $\alpha \in [0; 1]$, $\beta \in [0; 1]$. This family of solutions generalizes the solutions considered in Ref. [16].

The total mass-energy of the Turok-string is:

$$\text{Energy} = - \int d^3 \vec{X} \ T^0_0$$  \hspace{1cm} (2.5)

where:

$$T_{\mu\nu} \equiv \frac{2}{\sqrt{-g}} \frac{\delta S}{\delta g^{\mu\nu}} = \mu \int d\tau d\sigma \eta_{\mu\lambda} \eta_{\nu\delta} \left( \dot{X}^{\lambda} \dot{X}^{\delta} - X'^{\lambda} X'^{\delta} \right) \delta^4 \left( \vec{X} - \vec{X}(\tau, \sigma) \right)$$  \hspace{1cm} (2.6)

It follows that:

$$\text{Energy} = 2\pi A\mu$$  \hspace{1cm} (2.7)

which (by construction) is independent of the parameters $(\alpha, \beta)$. Similarly, one finds for the momenta:

$$P_i = - \int d^3 \vec{X} \ T^i_0 = 0$$  \hspace{1cm} (2.8)

In fact, the string center of mass is located at $\vec{X}_{cm} = (0, 0, 0)$ at all times, and the strings are symmetric under reflection in origo. The Schwarzschild radius corresponding to the energy (2.7) is

$$R_S = 4\pi A G \mu$$  \hspace{1cm} (2.9)

For realistic cosmic strings, the dimensionless parameter $G\mu$ is quite small [17, 18], say $G\mu \sim 10^{-6}$. It is then clear that a Turok-string (2.4) will typically be far outside its Schwarzschild radius (as follows since typically $X \sim A$, $Y \sim A$, $Z \sim A$). However, for certain particular values $(\alpha, \beta)$, a string might at some instant during its evolution be completely inside its Schwarzschild radius. Such a string will collapse and eventually form a black hole, according to the so-called hoop-conjecture [1]. To determine whether this happens or not, we must first find the minimal 3-sphere, that completely encloses the string, as a function of time. After minimization over time, we then get the minimal 3-sphere that can ever enclose the whole string. Let the radius of this sphere be $R$ (it will be defined more stringently in the next section). Then the condition for collapse is:

$$R \leq R_S$$  \hspace{1cm} (2.10)
That is to say,
\[ \frac{R}{A} \leq 4\pi G\mu \quad (2.11) \]

In a pioneering paper, Polnarev and Zembowicz [2] considered, among other things, the question of collapse of the Turok-strings (2.4). In connection with the minimal string size \( R \) they found:

- The strings have their minimal size \( R \) at
  \[ \tau = \frac{\pi}{2} \quad (2.12) \]

- For generic parameters \( (\alpha, \beta) \):
  \[ \frac{R^2}{A^2} = \left( \sqrt{\alpha(1-\alpha)} - \sqrt{\beta(1-\beta)} \right)^2 + \left( \frac{\alpha}{3} - \beta \right)^2 \quad (2.13) \]

The result (2.12) would be expected for a string experiencing a monopole-like oscillation, i.e. starting from maximal size at \( \tau = 0 \), then contracting isotropically to minimal size at \( \tau = \pi/2 \), and then re-expanding to its original maximal size at \( \tau = \pi \). As for the result (2.13) it was simply stated [2] without proof or derivation.

In Sections 3,4 we shall show that (2.12) and (2.13) are not completely correct. In fact, they are correct in part of the two-dimensional parameter-space but incorrect in other parts. A simple example showing that (2.13) cannot be correct, is provided by the case \( (\alpha, \beta) = (3/4, 1/4) \). In that case the result (2.13) would actually give \( R = 0 \), which would mean that the whole string had collapsed to a point at some instant. This would imply that at some instant \( X = Y = Z = 0 \), but that is certainly impossible for \( (\alpha, \beta) = (3/4, 1/4) \). The problem is that (2.13) gives the minimal distance from origo to the string, but it is the maximal distance which is relevant for the minimal 3-sphere.

3 Minimal String Size and Classification

For a given pair of parameters \( (\alpha, \beta) \), we define the minimal string size \( R^2 \) as the (square of the) radius of the minimal 3-sphere that can ever enclose
the string completely. More precisely:

\[ R^2 \equiv \min_{\tau \in [0; \pi]} \max_{\sigma \in [0; \pi]} (R^2(\tau, \sigma)) \]  

(3.1)

where:

\[ R^2(\tau, \sigma) = X^2(\tau, \sigma) + Y^2(\tau, \sigma) + Z^2(\tau, \sigma) \]  

(3.2)

as obtained using Eq. (2.4). Thus, for a fixed time \( \tau \), we first compute the maximal distance from origo (i.e. the string center of mass) to the string. This gives the minimal 3-sphere that encloses the string at that particular instant. We then minimize this maximal distance over all times. This gives altogether the minimal string size \( R^2 \). And this is obviously the quantity that must be compared with the (square of the) Schwarzschild radius \( R_S^2 \). Notice that we need only maximize over \( \sigma \in [0; \pi] \) and minimize over \( \tau \in [0; \pi] \) in Eq. (3.1). This is due to the reflection symmetry and time periodicity of the Turok-strings (2.4).

We now outline the computation of \( R^2 \). The details can be found in the Appendix, and some analytical results are given also in Section 4.

We first solve the equation:

\[ \frac{\partial R^2(\tau, \sigma)}{\partial \sigma} = 0 \]  

(3.3)

for fixed time \( \tau \). This leads to a quartic equation in \( \cos(2\sigma) \):

\[ \cos^4(2\sigma) + a \cos^3(2\sigma) + b \cos^2(2\sigma) + c \cos(2\sigma) + d = 0 \]  

(3.4)

where the coefficients \( (a, b, c, d) \) depend on time \( \tau \) as well as on the parameters \( (\alpha, \beta) \). The explicit expressions are given in the Appendix. The solutions to equation (3.4) are explicitly known, leading to \( \sigma = \sigma(\tau) \) for given values of \( (\alpha, \beta) \). By insertion of these solutions \( \sigma = \sigma(\tau) \) into \( R^2(\tau, \sigma) \), it is then straightforward to obtain the maximal distance in the square bracket of Eq. (3.1). This is now a function of \( \tau \), which finally has to be minimized over \( \tau \in [0; \pi] \). For more details, see the Appendix and Section 4.

The result is shown in Fig. 1, that is, the minimal string size \( R^2 \) as a function of \( (\alpha, \beta) \). Not surprisingly, the minimal string size is generally of the order 1 (in units of \( A \); see Eq. (2.4)). The exception is in the vicinity of \( (\alpha, \beta) = (0, 0) \), where the minimal string size is close to zero. This was also
to be expected since the vicinity of (0, 0) describes the near-spherical strings, and those are the only ones expected to have a chance to collapse, due to their relatively small angular momentum. These strings will be considered in detail in Section 4.

However, the computation of the minimal string size $R^2$ gives some more information about the Turok-strings, namely the time(s) $\tau$ at which the strings are minimal. This gives rise to a precise classification of the Turok-strings, and a subsequent subdivision into 3 different families (see Fig. 2):

I. These strings have their minimal size at $\tau = \pi/2$. That is, starting from their original size at $\tau = 0$, they generally contract to their minimal size at $\tau = \pi/2$, and then generally expand back to their original size at $\tau = \pi$.

II. These strings start from their minimal size at $\tau = 0$. Then they generally expand towards their maximal size and then recontract towards their minimal size at $\tau = \pi$.

III. These strings have their minimal size at two values of $\tau$ symmetrically around $\pi/2$. That is, they first generally contract and reach the minimal size at some $\tau_0 \in [0; \pi/2]$. Then they expand for a while, and then recontract and reach the minimal size again at $\tau = \pi - \tau_0$. Then they expand again towards the original size at $\tau = \pi$. In this family of strings, the value of $\tau_0$ depends on $(\alpha, \beta)$.

It must be stressed that the strings in most cases do not expand or contract isotropically. They typically expand in some directions while contracting in other directions. This is why we use the expressions “generally expand” and “generally contract”, which refer to the minimal string size as a function of time (the radius of the minimal 3-sphere enclosing the string, as a function of time).

Notice that besides the three above-mentioned families of strings, there are a number of degenerate cases at the different boundaries. For instance, at the boundary $\alpha = 1$, the strings have their minimal size at $\tau = 0$, $\tau = \pi/3$, $\tau = 2\pi/3$ and $\tau = \pi$. On the other hand, at the boundary between regions I and II, the strings have their minimal size at $\tau = 0$, $\tau = \pi/2$ and $\tau = \pi$. Notice also that the 3 points $(\alpha, \beta) = (1, 0)$, $(\alpha, \beta) = (1, 1)$ and $(\alpha, \beta) = (0, 1)$ correspond to rigidly rotating strings, thus they have their
Let us close this section with a comparison with the result (2.12) of Ref. [2]. We see that the result (2.12) is correct in the region I of parameter-space, but incorrect in regions II and III.

4 Probability of String Collapse

In this section we consider the question of collapse of the Turok-strings (2.4). As already discussed in the previous section, the only strings with a chance to collapse are those corresponding to parameters \((\alpha, \beta)\) located in the vicinity of \((0, 0)\). That is, we need only consider strings in the family I of Fig. 2.

Using the results of the Appendix, it is then straightforward to show that the minimal string size \(R^2\), as defined in Eq. (3.1), is given by:

\[
R^2 = \text{Max} (R^2_1, R^2_2) = \text{Max} \left( \frac{R^2_1}{A^2}, \frac{R^2_2}{A^2} \right)
\]

(4.1)

where:

\[
\frac{R^2_1}{A^2} = \frac{4\alpha^2}{9}
\]

(4.2)

\[
\frac{R^2_2}{A^2} = \left( \sqrt{\alpha (1 - \alpha)} - \sqrt{\beta (1 - \beta)} \right)^2 + \left( \frac{\alpha}{3} - \beta \right)^2
\]

(4.3)

Notice that Eq. (4.3) is precisely the result (2.13) of Polnarev and Zembowicz [2]. However, in Ref. [2], the other solution (4.2) was completely missed, and this is actually the relevant solution in Eq. (4.1) in approximately half of the parameter-space \((\alpha, \beta)\).

According to the hoop-conjecture (see for instance [1]), the condition for collapse to a black hole is then given by Eq. (2.11), with \(R\) given by Eqs. (4.1)-(4.3):

\[
\text{Max} \left( R^2_1, R^2_2 \right) \leq (4\pi AG\mu)^2
\]

(4.4)

which should be solved for \((\alpha, \beta)\) as a function of \(G\mu\). This can be easily done analytically. The result is shown in Fig. 3: the part of parameter-space fulfilling inequality (4.4) is bounded by the \(\alpha\)-axis, the \(\beta\)-axis, the straight line \(\alpha = 3R_S/2A\) and the two curves:

\[
\beta_{\pm}(\alpha) = \frac{-16\alpha^3 - 12\alpha^2 + 27\alpha - 9 (2\alpha - 3) R^2_S / A^2 \pm 6\sqrt{D}}{3 (-32\alpha^2 + 24\alpha + 9)}
\]

(4.5)
where:

\[
D \equiv -\alpha (1 - \alpha) \left[8\alpha^2 - 6\alpha + 9 \left(\frac{R_S}{A} - 1\right) \frac{R_S}{A}\right] \left[8\alpha^2 - 6\alpha + 9 \left(\frac{R_S}{A} + 1\right) \frac{R_S}{A}\right]
\]

Notice also that,

\[
\beta_+(0) = \frac{R_S^2}{A^2}
\]

and

\[
\beta_-(\alpha_0) = 0 \quad \text{for} \quad \alpha_0 = \frac{9}{16} \left(1 - \sqrt{1 - \frac{32R_S^2}{9A^2}}\right)
\]

The probability for collapse into black holes is then given by the fraction \( f \):

\[
f = \int_{R \leq R_S} d\alpha d\beta = \int_0^{\alpha_0} d\alpha \int_0^{\beta_+(\alpha)} d\beta + \int_{\alpha_0}^{\beta_+(\alpha)} d\alpha \int_{\beta_-(\alpha)}^{\beta_+(\alpha)} d\beta
\]

\[
= \int_0^{\alpha_0} \beta_+(\alpha) d\alpha + \int_{\alpha_0}^{\beta_+(\alpha)} [\beta_+(\alpha) - \beta_-(\alpha)] d\alpha
\]

(4.8)

This equation represents the exact analytical result for the probability of collapse of the Turok-strings (2.4), for a given value of \( R_S/A = 4\pi G\mu \). The integrals in (4.8) are of hyper-elliptic type \[19\], and not very enlightening in the general case. However, using that typically \( G\mu \ll 1 \) (see \[17, 18\]), a simple approximation is obtained by keeping only the leading order terms:

\[
f = \frac{12\sqrt{6}}{5} (4\pi G\mu)^{\frac{5}{2}} \int_0^1 \frac{t^2 dt}{\sqrt{1 - t^4}} + \mathcal{O}\left((G\mu)^\frac{7}{2}\right)
\]

\[
= \frac{3\frac{3}{2}}{5 \Gamma^2\left(\frac{1}{4}\right)} (G\mu)^{\frac{5}{2}} + \mathcal{O}\left((G\mu)^{\frac{7}{2}}\right)
\]

(4.9)

The result (4.9) is a very good approximation for \( G\mu < 10^{-2} \), thus for any “realistic” cosmic strings we conclude:

\[
f \approx 2 \cdot 10^3 \cdot (G\mu)^{\frac{5}{2}}
\]

(4.10)

which is our final result of this section.

It should be stressed that we have been using the simplest and most naive version of the hoop-conjecture: namely, we did not take into account the angular momentum of the strings. However, numerical studies \[11\] of
other families of strings showed that inclusion of the angular momentum only leads to minor changes in the final result, so we expect the same will happen here. It should also be stressed that we have neglected a number of other physical effects that might change the probability of collapse. These include the gravitational field of the string and gravitational radiation.

Finally, as in all other discussions of the probability of string collapse, we are faced with the problem that we do not know the measure of integration in parameter-space. Thus using another measure in Eq. (4.8) would generally give a different result (see also Ref. [2]).

5 Conclusion

In this paper we examined, using purely analytical methods, the question of collapse of Turok’s two-parameter family of cosmic strings [15]. We made a complete classification of the strings according to the specific time(s) the minimal string size is reached during one period. This revealed that the previously obtained results [2] were only correct in part of the two-dimensional parameter-space.

We then obtained an exact analytical expression for the probability $f$ of collapse into black holes for the Turok strings, which partly agrees with that of Ref. [2] in the sense that $f \propto (G\mu)^{5/2}$. However, we showed that there is a large numerical prefactor in the relation. This factor is of the order 2000, and not, as previously stated [2], of the order 1.

One might say that it is perhaps not so important whether the prefactor is 1 or 2000 since the exponent will more or less kill the probability of collapse anyway. This may very well be true for “small” values of $G\mu$ (say, $G\mu \sim 10^{-6}$), but for “large” values of $G\mu$ (say, $G\mu \sim 10^{-2}$) the situation is completely different. In fact, we shall now argue why it is so important to carefully compute the prefactor: We find that when using our result, a clear picture is beginning to emerge. Different computations based on different families of strings (not surprisingly) produce slightly different exponents, but they also produce completely different prefactors. Importantly, the two go in the same direction: An exponent larger by 1 is followed by a prefactor larger by a factor 100 (roughly speaking). For instance, the one-parameter family of Kibble and Turok [16] gives $f \approx 16\pi^2(G\mu)^2$, our computation for the two-parameter family gives $f \approx 2000(G\mu)^{5/2}$ and the Caldwell-Casper
computation \[11\] gives \( f \approx 10^5 (G\mu)^4 \). We find it extremely interesting that for \( G\mu \sim 10^{-2} \) (which is the range where the Caldwell-Casper computation is valid), the three computations basically agree giving \( f \approx 10^{-3} - 10^{-2} \). We therefore find that our careful computation of the prefactor is very important and has helped to understand the discrepancy between previously obtained results. And in particular, for "large" values of \( G\mu \), there may not even be a discrepancy since the different exponents and prefactors of the different computations actually produce more or less the same number for the probability of collapse. For "small" values of \( G\mu \), the picture is unfortunately not so clear, and more detailed investigations seem necessary.

It is actually possible to give a physical argument showing that the results of different computations (if they are done correctly) should merge for "large" values of \( G\mu \). Consider the rigidly rotating straight string, corresponding to \( \alpha = 0 \) and \( \beta = 1 \). It is easy to show that it is precisely inside its Schwarzschild radius for \( G\mu = (4\pi)^{-1} \). Now, it is well-known that the rigidly rotating string has the maximal angular momentum per energy (it is exactly on the leading Regge trajectory), and therefore is expected to be the most difficult string to collapse into a black hole. Therefore, for \( G\mu \approx (4\pi)^{-1} \) all other strings have already collapsed, and we should expect \( f \approx 1 \). This is indeed the case for the three above mentioned computations, while it does not hold for the computation of Ref.\[2\]. More generally, the condition \( f((4\pi)^{-1}) \approx 1 \) can be considered as a physical boundary condition, and therefore can be used to immediately rule out some of the previously obtained results for \( f \); for instance those of \[2\] and \[3\].

As a possible continuation of our work, it would be very interesting to consider more general multi-parameter families of strings, to see how general our result for \( f \) actually is. Such families of strings have been constructed and considered for instance in \[20, 21\], and more general ones can be obtained along the lines of \[22\].

Unfortunately, there are also still some open questions, as we discussed at the end of Section \[4\]. The main problem seems to be that we still do not know exactly what is the measure of integration in parameter-space \[4\].
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Appendix: The Minimal String Size

In this appendix, we give some details of the results presented in the sections \[3\] and \[4\].

The distance from origo to the string, as a function of $\tau$ and $\sigma$, is conveniently written as a polynomial in $\sin (2\sigma)$ and $\cos (2\sigma)$. From (3.2) and (2.4):

$$R^2(\tau, \sigma) = \frac{A^2}{4} \left\{ C_0 + \frac{4\alpha\beta}{3} [C_1 \sin (2\sigma) + C_2 \cos (2\sigma)] - \frac{4\alpha\beta}{3} [\cos (2\tau) \cos (2\sigma) + \sin (2\tau) \sin (2\sigma)] \cos (2\sigma) \right\} \quad \text{(A.1)}$$

where:

$$C_0 = 2 - \frac{8\alpha^2}{9} + 2 \left[ 2\sqrt{\alpha (1 - \alpha)} \sqrt{\beta (1 - \beta)} + (1 - \alpha) (1 - \beta) + \frac{\alpha\beta}{3} \right] \cos (2\tau)$$

$$C_1 = \frac{3}{4\alpha\beta} \left[ \frac{2\alpha}{3} (1 - \beta) \sin (4\tau) + \frac{8\alpha}{3} (1 - \alpha) \sin (2\tau) \right] \quad \text{(A.2)}$$

$$C_2 = \frac{3}{4\alpha\beta} \left[ \frac{2\alpha}{3} (1 - \beta) \cos (4\tau) + \frac{8\alpha}{3} (1 - \alpha) \cos (2\tau) + 2 \left( 2\sqrt{\alpha (1 - \alpha)} \sqrt{\beta (1 - \beta)} - \beta (1 - \alpha) \right) \right]$$

Then it is straightforward to show that the condition (3.3) leads to

$$x^4 + ax^3 + bx^2 + cx + d = 0 \quad \text{(A.3)}$$
where \( x \equiv \cos (2\sigma) \) and:

\[
\begin{align*}
 a &= -C_1 \sin (2\tau) - C_2 \cos (2\tau) \\
 b &= \frac{C_1^2 + C_2^2}{4} - 1 \\
 c &= \frac{C_1}{2} \sin (2\tau) + C_2 \cos (2\tau) \\
 d &= \frac{\sin^2 (2\tau) - C_2^2}{4}
\end{align*}
\]

(A.4)

The solutions to Eq. (A.3) can be written down in closed form. Define

\[
\begin{align*}
 X &\equiv b^2 - 3ac + 12d \\
 Y &\equiv 2b^3 - 9abc + 27c^2 + 27a^2d - 72bd \\
 Z &\equiv \left[ Y + \sqrt{-4X^3 + Y^2} \right]^{\frac{1}{3}} \\
 W &\equiv 2^{\frac{1}{3}}X^\frac{2}{3} + \frac{Z}{3}^{2\frac{1}{3}}
\end{align*}
\]

(A.5)

Then the 4 solutions are:

\[
\begin{align*}
 x_{1,2} &= \frac{-a}{4} - \frac{1}{2} \sqrt{\frac{a^2}{4} - \frac{2b}{3} + W} \pm \frac{1}{2} \sqrt{\frac{a^2}{2} - \frac{4b}{3} - W - \frac{(-a^3 + 4ab - 8c)}{4\sqrt{\frac{a^2}{4} - \frac{2b}{3} + W}}} \\
 x_{3,4} &= \frac{-a}{4} + \frac{1}{2} \sqrt{\frac{a^2}{4} - \frac{2b}{3} + W} \pm \frac{1}{2} \sqrt{\frac{a^2}{2} - \frac{4b}{3} - W + \frac{(-a^3 + 4ab - 8c)}{4\sqrt{\frac{a^2}{4} - \frac{2b}{3} + W}}}
\end{align*}
\]

(A.6)

which give \( \sigma = \sigma (\tau) \) for given values of \((\alpha, \beta)\). These solutions are inserted into (A.1) and then the result of the square bracket in Eq. (3.1) is determined. Finally this function of \( \tau \) must be minimized.

As an example, consider strings in the region I of parameter-space; see Fig. 5. Using the above formulas, one finds:

\[
\begin{align*}
 x_{1,2} &= -\frac{C_2}{2} \\
 x_{3,4} &= \mp 1
\end{align*}
\]

(A.7)
with
\[ C_2 = \frac{3}{2\alpha\beta} \left[ \left( \frac{\alpha}{3} + \beta \right) (\alpha - \beta) - \left( \sqrt{\alpha (1 - \alpha)} - \sqrt{\beta (1 - \beta)} \right)^2 \right] \] (A.8)

Insertion into Eq. (A.1) then leads directly to the result for the minimal string size in the region I,
\[ R^2 = \text{Max} \left( R_1^2, R_2^2 \right) \] (A.9)

where
\[ R_1^2 = \frac{4A^2\alpha^2}{9} \]
\[ R_2^2 = A^2 \left[ \left( \sqrt{\alpha (1 - \alpha)} - \sqrt{\beta (1 - \beta)} \right)^2 + \left( \frac{\alpha}{3} - \beta \right)^2 \right] \] (A.10)
c.f. Eqs. (4.2)-(4.3).
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CAPTIONS FOR FIGURES

Figure 1: The radius of the minimal 3-sphere completely enclosing a string with parameters \((\alpha, \beta)\) plotted for all parameter-space. Notice that \(R^2\) is close to zero only near \((\alpha, \beta) = (0, 0)\), that is for the near-spherical string configurations.

Figure 2: The considered strings fall into three families. The ones that reach their minimal size \(R\) at \(\tau = \pi/2\) (I), at \(\tau = 0, \pi\) (II) and at \(\tau = \tau_0, \pi - \tau_0\) for \(\tau_0 \in ]0, \pi/2[\) (III).

Figure 3: The region of parameter-space which contains the strings falling inside their own Schwarzschild radius is bounded by \(\alpha = 0, \beta = 0, \alpha = 3R_S/2A\) and the two curves \(\beta_{\pm}\). Here \(G\mu = 10^{-2}\) is chosen in order to illustrate the general form of the region. As \(G\mu\) decreases, the region is relatively stretched out and becomes quite narrow.
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Cosmic String Loops Collapsing to Black Holes

Figure 1
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Cosmic String Loops Collapsing to Black Holes

Figure 2
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Figure 3