Algebraic Bethe Ansatz for $\mathfrak{gl}(2,1)$ Invariant 36-Vertex Models

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ABSTRACT

Four dimensional irreducible representations of the superalgebra $\mathfrak{gl}(2,1)$ carry a free parameter. We compute the spectra of the corresponding transfer matrices by means of the nested algebraic Bethe ansatz together with a generalized fusion procedure.

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1 Introduction

Recently vertex models built from representations of the superalgebra $gl(2,1)$ or $q$-deformations thereof have attracted increasing attention \[1, 2, 3, 4\]. One reason is their relation to integrable models of interacting electrons in one spatial dimension: for example, the supersymmetric $t–J$ model \[5, 6, 7\] is obtained in the hamiltonian limit of the transfer matrix for the vertex model based on the three dimensional fundamental representation \[12\] of $gl(2,1)$ \[3, 4\]. A special feature of this algebra is the existence of a family of four dimensional representations parametrized by a parameter $b \neq \pm \frac{1}{2}$. From the corresponding vertex model a one-parametric integrable model of interacting electrons can be derived \[1\]. This system of electrons with correlated hopping has been solved recently by means of the co-ordinate Bethe Ansatz \[8, 9\].

The relation to integrable vertex models provides an embedding of these models into the framework of the Quantum Inverse Scattering Method. However, up to now a direct solution by means of the nested algebraic Bethe Ansatz has been obtained only for the supersymmetric $t–J$ model \[3, 4\]. The fusion method used to solve vertex models corresponding to higher-dimensional representations of ordinary Lie algebras such as $sl_n$ is not applicable due to the peculiarities of the representation theory of a superalgebra. Only for the model with $b = \frac{3}{2}$ this method can be applied. Knowing the eigenvalues of the transfer matrix for this case Maassarani \[2\] has made a conjecture for the general case which is in agreement with the spectrum of the hamiltonian obtained from the co-ordinate Bethe Ansatz.

In this paper we extend the approach used by Maassarani to vertex models built from $R$-matrices defined on tensor products of two different four dimensional representations $[b_1, \frac{1}{2}] \otimes [b_2, \frac{1}{2}]$ to compute the eigenvalues of the corresponding transfer matrix $\tau^{b_1 b_2} (\mu)$: In the following section we give a short overview over the three and four dimensional representations of $gl(2,1)$ that are used together with the $R$-matrices acting on tensor products of these. In Section 3 and 4 we obtain two inequivalent sets of Bethe Ansatz equations for this model corresponding to a different choice of the reference state. Then we use the fusion procedure to find the eigenvalues of the transfer matrix for $b_1 = \frac{3}{2}$. Then, using the set of Yang–Baxter equations for the interwtners between the different representations in addition with the known analytic properties of the eigenvalues of the transfer matrix we can determine the eigenvalues of $\tau^{b_1 b_2} (\mu)$ up to an overall factor which is fixed by studying the model built from two vertices only.

2 $R$-matrices for $[\frac{1}{2}]_+$ and $[b, \frac{1}{2}]$ representations

In this section we present the $R$-matrices acting on tensor products of three dimensional $[\frac{1}{2}]_+$ and four dimensional $[b, \frac{1}{2}]$ representations of $gl(2,1)$ along with the corresponding Yang-Baxter equations. Before we discuss the particular form of the representations we introduce the notation $[x]$ for the grading of an object $x$:

\[
[x] = \begin{cases} 
0 & \text{if } x \text{ is bosonic (even)} \\
1 & \text{if } x \text{ is fermionic (odd)}
\end{cases}
\]  

(2.1)

The multiplication rule in graded tensor products differs from the ordinary one by the appearance of additional signs. For homogeneous elements $B$ and $v$ we get

\[
(A \otimes B)(v \otimes w) = (-1)^{[B][v]} (Av) \otimes (Bw)
\]  

(2.2)
Using homogeneous bases in the two vectorspaces this equation can be written in components:

\[(A \otimes B)_{i_2 j_2}^{i_1 j_1} = (-1)^{([i_2]+[j_2])/[j_1]} A_{i_1 j_1} B_{i_2 j_2} \] \tag{2.3}

The even part of the superalgebra \(gl(2, 1)\) consist of the direct sum of a \(su(2)\) and a \(u(1)\) Lie algebra. Thus the basis vectors of the irreducible representation can be labeled by the eigenvalue \(B\) of the \(u(1)\) operator, the total spin and the \(z\)-component of the spin: \(|B, S, S_z\rangle \begin{bmatrix} 0 \\ 1 \end{bmatrix} \).

The three dimensional representation \([\frac{1}{2}]_+\) contains a doublet with \(B = \frac{1}{2}\) and a singlet with \(B = 1\). We arrange the basis in the following order:

\[ |\frac{1}{2}, \frac{1}{2}, \frac{1}{2}\rangle, \quad |\frac{1}{2}, \frac{1}{2}, -\frac{1}{2}\rangle, \quad |1, 0, 0\rangle \] \tag{2.4}

The first two states are considered as bosonic (grading 0), the last one as fermionic (grading 1). For the supersymmetric \(t\)-\(J\) model they are identified with the electronic states with a spin up or a spin down electron and an empty site. In this case one should choose the opposite grading \([1,1,0]\).

In the four dimensional \([b, \frac{1}{2}]\) representation we find a doublet with \(B = b\) and two singlets with \(B = b \pm \frac{1}{2}\), respectively. The basis is ordered according to

\[ |b, \frac{1}{2}, \frac{1}{2}\rangle, \quad |b, \frac{1}{2}, -\frac{1}{2}\rangle, \quad |b - \frac{1}{2}, 0, 0\rangle, \quad |b + \frac{1}{2}, 0, 0\rangle \] \tag{2.5}

Here the grading of the basis vectors is \([1,1,0,0]\). For the model with correlated hopping they correspond to states with a single spin up or spin down electron, an empty site and a doubly occupied site. The bosonic ones may be exchanged leading to an equivalent model.

On the tensorproduct of two \([\frac{1}{2}]_+\) representations we have an \(R\)-matrix \(r^{33}\):

\[ r^{33}(\lambda) = a(\lambda)id_9 + b(\lambda)\Pi_{33} \] \tag{2.6}

Here \(id_9\) denotes the \(9 \times 9\) identity matrix and \(\Pi_{33}\) is the graded permutation operator with matrix elements

\[ (\Pi_{33})_{i_2 j_2}^{i_1 j_1} = (-1)^{([i_2]+[j_2])} \delta_{i_1 j_1} \delta_{i_2 j_2}. \] 

The functions \(a\) and \(b\) are given by

\[ a(\lambda) = \frac{\lambda}{\lambda + 1}, \quad b(\lambda) = \frac{1}{\lambda + 1} \] \tag{2.7}

This \(R\)-matrix is a solution of the Yang-Baxter equation

\[ r^{33}_{12}(\lambda - \mu)r^{33}_{13}(\lambda)r^{33}_{23}(\mu) = r^{33}_{23}(\mu)r^{33}_{13}(\lambda)r^{33}_{12}(\lambda - \mu) \] \tag{2.8}

Here the lower indices denote the spaces in which the \(R\)-matrix acts.

The \(R\)-matrix \(R^{3b}\) on the tensorproduct \([\frac{1}{2}]_+ \otimes [b, \frac{1}{2}]\) can be constructed from the elementary intertwiners of the irreducible components of the tensorproduct \([\frac{1}{2}][2]\). The result is:

\[ R^{3b}(\mu) \sim It_1 + \frac{2\mu - 2b - 1}{2\mu + 2} It_2 \] \tag{2.9}

where \(It_1\) and \(It_2\) are the operators intertwining the eight dimensional \([b + 1, \frac{1}{2}]\) subrepresentation and the four dimensional \([b + 1, \frac{1}{2}]\) subrepresentation, respectively.

The tensorproduct \([b_1, \frac{1}{2}] \otimes [b_2, \frac{1}{2}]\) contains three irreducible components, namely \([b_1+b_2, 1]\) \((D=8)\), \([b_1+b_2+1, \frac{1}{2}]\) \((D=4)\) and \([b_1 + b_2 - 1, \frac{1}{2}]\) \((D=4)\). The \(R\)-matrix is given by the following combination of intertwiners:

\[ R^{b_1b_2}(\mu) \sim It_1 + \frac{4\mu - b_1 - b_2 - 1}{4\mu + b_1 + b_2 + 1} It_2 + \frac{4\mu + b_1 + b_2 - 1}{4\mu - b_1 - b_2 + 1} It_3 \] \tag{2.10}
where \(I_{t1}, I_{t2} \) and \(I_{t3} \) are the intertwiners for the eight and the two four dimensional irreducible components, respectively.

For the \( R \)-matrices defined in Eqs. (2.9) and (2.10) the following Yang-Baxter equations hold:

\[
\begin{align*}
    r_{312}^{33}(\lambda - \mu) R_{13}^{3b}(\lambda) R_{23}^{33}(\mu) &= R_{23}^{3b}(\mu) R_{13}^{3b}(\lambda) r_{12}^{33}(\lambda - \mu) \tag{2.11} \\
    R_{12}^{3b}(\lambda - \mu) R_{13}^{3b}(\lambda) R_{23}^{b_1b_2}(\mu) &= R_{23}^{b_1b_2}(\mu) R_{13}^{3b}(\lambda) R_{12}^{3b}(\lambda - \mu) \tag{2.12} \\
    R_{12}^{b_1b_2}(\lambda - \mu) R_{13}^{b_1b_2}(\lambda) R_{23}^{b_2b_3}(\mu) &= R_{23}^{b_2b_3}(\mu) R_{13}^{b_1b_2}(\lambda) R_{12}^{b_1b_2}(\lambda - \mu) \tag{2.13}
\end{align*}
\]

Writing the Yang-Baxter equations in components one has to include additional signs due to the grading:

\[
    r_{12}(\lambda - \mu)_{i_1,j_1}^{i_2,j_2} R_{13}(\lambda)_{j_1,j_2}^{j_3,j_3} R_{23}(\mu)_{j_3,j_3}^{j_2,j_2} (-1)^{[j_2][j_1]} = R_{23}(\mu)_{j_3,j_3}^{j_2,j_2} R_{13}(\lambda)_{j_2,j_2}^{j_1,j_1} r_{12}(\lambda - \mu)_{j_2,j_2}^{j_1,j_1} (-1)^{[j_2][j_1]} \tag{2.14}
\]

From the \( R \)-matrices for the different representations we can construct monodromy matrices by taking matrix products in one component of the tensor product – the auxiliary or matrix space.

\[
    T(\mu)^{a_1,\ldots,a_L}_{b_1,\beta_1,\ldots,\beta_L} = R(\mu)^{a_L,\ldots,a_1}_{\alpha_L,\ldots,\alpha_1} R(\mu)^{\alpha_L,-1,\ldots,\alpha_1}_{\beta_L,-1,\ldots,\beta_1} \cdots R(\mu)^{\alpha_2,\beta_2}_{\alpha_1,\beta_1} R(\mu)^{\alpha_1}_{\beta_1} (-1)^{\sum_{i=2}^L ([\alpha_i] + [\beta_i]) \sum_{i=1}^{L-1} [\alpha_i]} \tag{2.15}
\]

Again the grading gives rise to additional signs. As a consequence of Eqs. (2.8) and (2.11)-(2.13) the monodromy matrices satisfy the following Yang-Baxter equations:

\[
\begin{align*}
    r_{312}^{33}(\lambda - \mu) T_{13}^{33}(\lambda) T_{23}^{33}(\mu) &= T_{23}^{33}(\mu) T_{13}^{33}(\lambda) r_{12}^{33}(\lambda - \mu) \tag{2.16} \\
    r_{312}^{33}(\lambda - \mu) T_{13}^{3b}(\lambda) T_{23}^{3b}(\mu) &= T_{23}^{3b}(\mu) T_{13}^{3b}(\lambda) r_{12}^{3b}(\lambda - \mu) \tag{2.17} \\
    R_{12}^{b_1b_2}(\lambda - \mu) T_{13}^{b_1b_2}(\lambda) T_{23}^{b_2b_3}(\mu) &= T_{23}^{b_2b_3}(\mu) T_{13}^{b_1b_2}(\lambda) R_{12}^{b_1b_2}(\lambda - \mu) \tag{2.18} \\
    R_{12}^{b_1b_2}(\lambda - \mu) T_{13}^{b_2b_3}(\lambda) T_{23}^{b_3b_3}(\mu) &= T_{23}^{b_3b_3}(\mu) T_{13}^{b_2b_3}(\lambda) R_{12}^{b_1b_2}(\lambda - \mu) \tag{2.19}
\end{align*}
\]

From the monodromy matrix the transfer matrix is obtained by taking the supertrace in the auxiliary space: \( \tau(\mu) = \sum_a (-1)^{[\alpha]} T(\mu)^a_a \). The Yang-Baxter equations (2.11) and (2.12)-(2.13) imply that transfer matrices acting in the same quantum space commute:

\[
\begin{align*}
    [\tau^{33}(\lambda), \tau^{33}(\mu)] &= 0 \\
    [\tau^{3b}(\lambda), \tau^{3b}(\mu)] &= 0 \\
    [\tau^{b_1b_2}(\lambda), \tau^{b_1b_2}(\mu)] &= 0 \\
    [\tau^{b_1b_3}(\lambda), \tau^{b_2b_3}(\mu)] &= 0
\end{align*}
\]

These equations imply that \( \tau^{3b} \), \( \tau^{b_1b} \) and \( \tau^{b_2b} \) share a system of common eigenvectors and thus can be diagonalized simultaneously.

### 3 Nested algebraic Bethe ansatz for \( \tau^{3b} \)

The transfer matrix \( \tau^{3b} \) can be diagonalized directly by means of a nested algebraic Bethe ansatz. The calculations can be performed in close analogy to the NABA for the supersymmetric \( t-J \)-model [3]. Thus we omit the details here and present only the most relevant steps.
We can represent $T^{3b}$ as a $3 \times 3$ matrix in the auxiliary space with entries being operators in the $N$-fold tensor product of four dimensional quantum spaces

$$T^{3b}(\lambda) = \begin{pmatrix} A_{11}(\lambda) & A_{12}(\lambda) & B_1(\lambda) \\ A_{21}(\lambda) & A_{22}(\lambda) & B_2(\lambda) \\ C_1(\lambda) & C_2(\lambda) & D(\lambda) \end{pmatrix} \tag{3.1}$$

From the Yang-Baxter equation (2.17) we can derive commutation relations for these quantum operators. The ones needed in the sequel are:

$$D(\lambda)B_i(\mu) = \frac{1}{a(\mu - \lambda)} B_i(\mu) D(\lambda) + \frac{b(\lambda - \mu)}{a(\lambda - \mu)} B_i(\lambda) D(\mu) \tag{3.2}$$

$$A_{12k_2}(\mu)B_1(\lambda) = \frac{1}{a(\lambda - \mu)} r(\lambda - \mu)^{i_1, i_2} B_1(\lambda) A_{12k_2}(\mu) - \frac{b(\lambda - \mu)}{a(\lambda - \mu)} B_{i_2}(\mu) A_{12k_2}(\lambda) \tag{3.3}$$

$$B_{i_2}(\mu)B_1(\lambda) = \frac{1}{b(\lambda - \mu) - a(\lambda - \mu)} r(\lambda - \mu)^{i_1, i_2} B_1(\lambda) B_{i_2}(\mu) \tag{3.4}$$

Here $r(\mu)$ is the $R$-matrix of the rational six vertex model, both states being bosonic:

$$r(\mu) = a(\mu) \text{id}_4 + b(\mu) \Pi_{22} \tag{3.5}$$

We choose the state

$$|0\rangle = \otimes b - \frac{i}{2}, 0, 0 \rangle \tag{3.6}$$

as reference state. The action of the monodromy matrix on this state is:

$$T^{3b}(\mu)|0\rangle = \begin{pmatrix} 1 & 0 & B_1(\mu) \\ 0 & 1 & B_2(\mu) \\ 0 & 0 & \left(\frac{4\mu - 2b + 5}{4\mu + 2b + 3}\right)^L \end{pmatrix}|0\rangle \tag{3.7}$$

hence it is an eigenstate of $\tau^{3b}(\mu) = -D(\mu) + A_{11}(\mu) + A_{22}(\mu)$. Starting from $|0\rangle$ we make the following ansatz for the Bethe vectors:

$$|\lambda_1, \ldots, \lambda_n\rangle = B_{a_1}(\lambda_1) \ldots B_{a_n}(\lambda_n)|0\rangle F^{a_n \ldots a_1} \tag{3.8}$$

where summation over repeated indices is implied and the amplitudes $F^{a_n \ldots a_1}$ are functions of the spectral parameters $\lambda_1, \ldots, \lambda_n$. In order to calculate the action of the transfer matrix $\tau^{3b}(\mu)$ on such a state, we use relations (3.2) and (3.3) to commute the operators $D$ and $A$ through all $B$’s until they hit the vacuum.

$$D(\mu)|\lambda_1, \ldots, \lambda_n\rangle = \left(\frac{4\mu - 2b + 5}{4\mu + 2b + 3}\right)^L \prod_{i=1}^{n} \frac{1}{a(\lambda_i - \mu)} |\lambda_1, \ldots, \lambda_n\rangle \tag{3.9}$$

$$[A_{11}(\mu) + A_{22}(\mu)]|\lambda_1, \ldots, \lambda_n\rangle = \prod_{i=1}^{n} \frac{1}{a(\lambda_i - \mu)} \tau^{(1)}(\mu)^{b_1, \ldots, b_n} F^{a_n \ldots a_1} \prod_{j=1}^{n} B_{b_j}(\lambda_j)|0\rangle \tag{3.10}$$

Here $\tau^{(1)}$ is the transfer matrix of an inhomogeneous six vertex model with $n$ sites that is constructed from the $R$-matrix (3.5).

$$\tau^{(1)}(\mu)^{b_1, \ldots, b_n} = r(\lambda_n - \mu)^{d_{n, d_n}} r(\lambda_{n-1} - \mu)^{d_{n-1, d_{n-1}}} \ldots r(\lambda_2 - \mu)^{d_2, d_2} r(\lambda_1 - \mu)^{d_1, d_1} \tag{3.11}$$
The amplitudes $F^{a_{n}...a_{1}}$ can now be identified with the components of a vector $F$ in the state space of this $n$ site model. As can be seen from Eqs. (3.9) and (3.10) a sufficient condition for $|\lambda_{1},...,\lambda_{n}\rangle$ to be an eigenvector of $\tau(\mu)$ is that the unwanted terms $\tilde{\Lambda}_{k}$ and $\Lambda_{k}$ cancel and that the vector $F$ is an eigenvector of the nested transfer matrix $\tau^{(1)}(\mu)$.

The condition that the unwanted terms $\tilde{\Lambda}_{k}$ and $\Lambda_{k}$ ought to cancel leads to a set of equations for the spectral parameters $\lambda_{k}$:

$$\left(\frac{4\lambda_{k} - 2b + 5}{4\lambda_{k} + 2b + 3}\right)^{L} F^{a_{n}...a_{1}} = \tau^{(1)}(\lambda_{k})^{L} F^{a_{n}...a_{1}}, \quad k = 1,...,n \quad (3.12)$$

In a second step (nesting) we have to diagonalize the transfer matrix $\tau^{(1)}$. This goal is achieved by another Bethe ansatz which gives the well known results for the inhomogeneous, rational six vertex model. The amplitudes $F^{a_{n}...a_{1}}$ are the components of the corresponding eigenvectors. The eigenvalues are found to be:

$$\Lambda^{(1)}(\mu|\lambda_{1},...,\lambda_{n}|\nu_{1},...,\nu_{n_{1}}) = \prod_{i=1}^{n} a(\lambda_{i} - \mu) \prod_{j=1}^{n_{1}} \frac{1}{a(\mu - \nu_{j})} \prod_{j=1}^{n_{1}} \frac{1}{a(\nu_{j} - \mu)} \quad (3.13)$$

The spectral parameters $\nu_{j}$ are subject to the following set of Bethe equations:

$$\prod_{i=1}^{n} a(\lambda_{i} - \nu_{j}) = \prod_{k \neq j}^{n_{1}} \frac{a(\nu_{k} - \nu_{j})}{a(\nu_{j} - \nu_{k})}, \quad j = 1,...,n_{1} \quad (3.14)$$

If we insert the eigenvalues (3.13) of the nested transfer matrix into Eq. (3.12) we obtain the first set of Bethe equations:

$$\left(\frac{4\lambda_{k} - 2b + 5}{4\lambda_{k} + 2b + 3}\right)^{L} = \prod_{j=1}^{n_{1}} \frac{1}{a(\nu_{j} - \lambda_{k})}, \quad k = 1,...,n \quad (3.15)$$

From Eqs. (3.9) and (3.10) we can read of the eigenvalues of $\tau^{3b}(\mu)$ as now the eigenvalues of $\tau^{(1)}(\mu)$ are known:

$$\Lambda^{3b}(\mu|\lambda_{1},...,\lambda_{n}|\nu_{1},...,\nu_{n_{1}}) = - \left(\frac{4\mu - 2b + 5}{4\mu + 2b + 3}\right)^{L} \prod_{i=1}^{n} a(\lambda_{i} - \mu) \prod_{j=1}^{n_{1}} \frac{1}{a(\nu_{j} - \lambda_{i})} + \prod_{j=1}^{n_{1}} \frac{1}{a(\nu_{j} - \mu)} + \prod_{j=1}^{n_{1}} \frac{1}{a(\mu - \nu_{j})} \prod_{i=1}^{n} \frac{1}{a(\lambda_{i} - \mu)} \quad (3.16)$$

The eigenvalues and Bethe equations should be compared to the rational limit of the corresponding equations in Ref. [2]. We find complete agreement.

## 4 A second Bethe ansatz

In general the specific form of the eigenvalues and the Bethe equations depends on the particular choice of the reference state from which the Bethe vectors are built. For the transfer matrix $\tau^{3b}$ there exist a second possibility besides (3.6), namely the state:

$$|0\rangle = \otimes^{L}|b + \frac{1}{2}, 0, 0\rangle \quad (4.1)$$

The action of the monodromy matrix on this state is

$$T^{3b}(\mu)|0\rangle = \begin{pmatrix} \left(\frac{4\mu+2b+1}{4\mu+2b+3}\right)^{L} & 0 & 0 \\ 0 & \left(\frac{4\mu+2b+1}{4\mu+2b+3}\right)^{L} & 0 \\ C_{1}(\mu) & C_{2}(\mu) & \left(\frac{4\mu-2b-3}{4\mu+2b+3}\right)^{L} \end{pmatrix} |0\rangle \quad (4.2)$$
For the Bethe ansatz we need the following commutation relations between these operators, which are derived from the Yang Baxter equation (2.17).

\[
D(\mu)C_i(\lambda) = \frac{1}{a(\mu - \lambda)}C_i(\lambda)D(\mu) + \frac{b(\lambda - \mu)}{a(\lambda - \mu)}C_i(\mu)D(\lambda) \tag{4.3}
\]

\[
A_{i,k_1}(\lambda)C_{k_2}(\mu) = \frac{1}{a(\mu - \lambda)}r(\lambda - \mu)^{k_1,i_1}C_{i_2}(\mu)A_{i_1,k_2}(\lambda) - \frac{b(\lambda - \mu)}{a(\lambda - \mu)}C_{k_1}(\lambda)A_{i_1,k_2}(\mu) \tag{4.4}
\]

\[
C_{k_1}(\lambda)C_{k_2}(\mu) = \frac{1}{b(\lambda - \mu) - a(\lambda - \mu)}r(\lambda - \mu)^{k_1,i_1}C_{i_2}(\mu)B_{i_1}(\lambda) \tag{4.5}
\]

Here \(r(\mu)\) is again the \(R\)-matrix \((3.5)\) of the rational six vertex state model, with two bosonic states. Now we use the operators \(C_i\) to to build the Bethe vectors starting from the new reference state:

\[
|\lambda_1, \ldots, \lambda_n\rangle = C_{a_1}(\lambda_1) \ldots C_{a_n}(\lambda_n)|0\rangle F^{a_{n-1} \ldots a_1} \tag{4.6}
\]

The action of \(\tau^{3b}\) on such a Bethe state is given by:

\[
D(\mu)|\lambda_1, \ldots, \lambda_n\rangle = \left(\frac{4\mu - 2b - 3}{4\mu + 2b + 3}\right)^L \prod_{i=1}^{n} \frac{1}{a(\mu - \lambda_i)}|\lambda_1, \ldots, \lambda_n\rangle \tag{4.7}
\]

\[
+ \sum_{k=1}^{n} (\tilde{A}_k)_{a_1, \ldots, a_n}^b \lambda F_{a_2, \ldots, a_{n-1}}^{a_1, \ldots, a_1} b_{b\mu} B_{b_j}(\lambda_j)|0\rangle \tag{4.8}
\]

\[
[A_{11}(\mu) + A_{22}(\mu)]|\lambda_1, \ldots, \lambda_n\rangle = \left(\frac{4\mu + 2b - 1}{4\mu + 2b + 3}\right)^L \prod_{i=1}^{n} \frac{1}{a(\mu - \lambda_i)}\tau^{(1)}(\mu)^{b_1, \ldots, b_n}_{a_1, \ldots, a_n} F_{a_2, \ldots, a_{n-1}}^{a_1, \ldots, a_1} \prod_{j=1}^{n} B_{b_j}(\lambda_j)|0\rangle \tag{4.9}
\]

Here \(\tau^{(1)}\) is the transfermatrix of an inhomogeneous \(n\) site model that is constructed from the \(R\)-matrix \((3.5)\).

\[
\tau^{(1)}(\mu)_{a_1, \ldots, a_n}^{b_1, \ldots, b_n} = r(\mu - \lambda_n)^{d_{n-1} d_n} r(\mu - \lambda_{n-1})^{d_{n-2} d_{n-1}} \ldots r(\mu - \lambda_2)^{d_2 d_1} r(\mu - \lambda_1)^{d_1 d} \tag{4.10}
\]

The condition that the unwanted terms ought to cancel leads to the following set of equations:

\[
\left(\frac{4\lambda_k - 2b - 3}{4\lambda_k + 2b - 1}\right)^L F_{p_n, \ldots, p_1}^{p_n, \ldots, p_1} F_{a_2, \ldots, a_{n-1}}^{a_1, \ldots, a_1}, \quad k = 1, \ldots, n \tag{4.11}
\]

As before the nested transfer matrix \(\tau^{(1)}\) is diagonalized by a second Bethe ansatz. The corresponding eigenvalues are found to be:

\[
\Lambda^{(1)}(\mu|\lambda_1, \ldots, \lambda_n|\nu_1, \ldots, \nu_{n_1}) = \prod_{i=1}^{n} a(\mu - \lambda_i) \prod_{j=1}^{n_1} \frac{1}{a(\mu - \nu_j)} + \prod_{j=1}^{n_1} \frac{1}{a(\nu_j - \mu)} \tag{4.12}
\]

where \(\nu_j\) are solutions of the following set of Bethe equations

\[
\prod_{j=1}^{n} a(\nu_j - \lambda_i) = \prod_{k \neq j}^{n_1} \frac{a(\nu_j - \nu_k)}{a(\nu_k - \nu_j)}, \quad j = 1, \ldots, n_1 \tag{4.13}
\]

If we insert the eigenvalues \(\lambda^{(1)}\) into Eqs. (4.10) we obtain the first level Bethe equations:

\[
\left(\frac{4\lambda_k - 2b - 3}{4\lambda_k + 2b - 1}\right)^L = \prod_{j=1}^{n_1} \frac{1}{a(\nu_j - \lambda_k)}, \quad k = 1, \ldots, n \tag{4.14}
\]
From Eqs. (4.7) and (4.8) we can now determine the eigenvalues of the transfer matrix \(\tau^{3b}(\mu)\):

\[
\Lambda^{3b}(\mu|\lambda_1, \ldots, \lambda_n|\nu_1, \ldots, \nu_n) = -\left(\begin{array}{c}
\frac{4\mu - 2b - 3}{4\mu + 2b + 3}
\end{array}\right)^L \prod_{i=1}^n \frac{1}{a(\mu - \lambda_i)}
+ \left(\begin{array}{c}
\frac{4\mu + 2b - 1}{4\mu + 2b + 3}
\end{array}\right)^L \left(\prod_{j=1}^{n_1} \frac{1}{a(\mu - \nu_j)} + \prod_{j=1}^{n_2} \frac{1}{a(\nu_j - \mu)} \prod_{i=1}^n \frac{1}{a(\mu - \lambda_i)}\right)
\]

(4.14)

This completes the second Bethe ansatz. We postpone the discussion of the relation between the two Bethe ansätze to Section 6.

## 5 Fusion

The tensor product of two \([\frac{3}{2}, 1]_+\) representations contains a four dimensional \([b = \frac{3}{2}, \frac{1}{2}]\) and five dimensional \([1]_+\) representation. We construct the fused \(R\)-matrix for the “scattering” of two composite “\([\frac{1}{2}, \frac{1}{2}]_+\)-particles” with a third “\([b, \frac{1}{2}]\) particle” from

\[
R_{12,3}(\lambda) = R_{13}^{3b}(\lambda + \lambda_0)R_{23}^{3b}(\lambda - \lambda_0)
\]

(5.1)

In general \([\frac{3}{2}, \frac{1}{2}]\) and \([1]_+\) states are mixed by the scattering. However the \([\frac{3}{2}, \frac{1}{2}]\) state will not be destroyed if the triangularity condition holds:

\[
P_{12}^5 R_{12,3} P_{12}^5 = 0
\]

(5.2)

Here \(P_{12}^5\) and \(P_{12}^\frac{3}{2}\) denote the projectors onto the five dimensional representation \([1]_+\) and the four dimensional \([\frac{3}{2}, \frac{1}{2}]\) representation, respectively. \(\tau^{33}\) becomes proportional to a projector onto the five dimensional representation for the special value 1 of the spectral parameter

\[
r^{33}_{12}(1) \sim P_{12}^5
\]

(5.3)

From the Yang-Baxter equation (2.11) we obtain

\[
P_{12}^5 R_{13}^{34}(\lambda + 1/2)R_{23}^{3b}(\lambda - 1/2) = R_{23}^{3b}(\lambda - 1/2)R_{13}^{3b}(\lambda + 1/2)P_{12}^5
\]

(5.4)

This allows to prove the triangularity condition (5.2) for \(R_{12,3}\) for \(\lambda_0 = \frac{1}{2}\). Thus the matrix

\[
R_{(12),3} = P_{12}^\frac{3}{2} R_{12,3} P_{12}^\frac{3}{2}
\]

(5.5)

indeed describes the “scattering” of a \([\frac{3}{2}, \frac{1}{2}]\) particle with a \([b, \frac{1}{2}]\) particle. A Yang-Baxter equation holds for this \(R\)-matrix:

\[
R_{(12),3}(\lambda - \mu)R_{(12),4}(\lambda)R_{32}^{3b}(\mu) = R_{32}^{3b}(\mu)R_{(12),4}(\lambda)R_{(12),3}(\lambda - \mu)
\]

(5.6)

The triangularity condition (5.2) states that \(R_{12,3}\) becomes an upper block-triangular matrix if we change the basis in the tensor product \(1 \otimes 2\) such that the first 4 vectors form a basis for \([\frac{3}{2}, \frac{1}{2}]\) and the next five a basis for \([1]_+\). Let \(B\) be the corresponding transformation matrix. Then we have:

\[
B^{-1} R_{12,3} B = \begin{pmatrix}
R_{3}^{3b} & * \\
0 & R_{5}^{3b}
\end{pmatrix}
\]

(5.7)
We can replicate this $R$-matrix to a chain of $L$ sites (the auxillary space is the tensor product $1 \otimes 2$). Taking the trace over spaces 1 and 2 leads to the following relation between the transfer matrices of the various models:

$$\tau^{3b}(\lambda + \frac{1}{2}) \tau^{3b}(\lambda - \frac{1}{2}) = \tau^{5b}(\lambda) + \tau^{5b}(\lambda)$$  \hspace{1cm} (5.8)

Thus for the eigenvalues of the transfer matrices we find the relation

$$\Lambda^{3b}(\lambda + \frac{1}{2}) \Lambda^{3b}(\lambda - \frac{1}{2}) = \Lambda^{5b}(\lambda) + \Lambda^{5b}(\lambda)$$  \hspace{1cm} (5.9)

Unfortunately there are two unknowns in this equation, namely $\Lambda^{5b}(\lambda)$ and $\Lambda^{5b}(\lambda)$. We expect the eigenvalues corresponding to Bethe vectors (3.8) to be a sum of the vacuum expectation values of the diagonal operators of the transfer matrix dressed by some factors that depend on the spectral parameters of the Bethe ansatz state. However this is not sufficient to assign the various parts of the LHS of (5.9) to $\Lambda^{5b}(\lambda)$ and $\Lambda^{5b}(\lambda)$ because the transfer matrices have common vacuum expectation values. As second argument we use the analyticity of the eigenvalues. The Bethe ansatz equations (3.14) and (3.15) are exactly the conditions that the eigenvalues are analytic functions of $\mu$. The diagonal parts of monodromy matrix $T^{5b}$ have the following four eigenvalues on the reference state $|0\rangle$ (5.6):

$$T^{5b}_{11}(\mu)|0\rangle = T^{5b}_{22}(\mu)|0\rangle = \left(\frac{2\mu - b + \frac{3}{2}}{2\mu + b + \frac{3}{2}}\right)^L |0\rangle$$

$$T^{5b}_{33}(\mu)|0\rangle = T^{5b}_{44}(\mu)|0\rangle = \left(\frac{2\mu - b + \frac{3}{2}}{2\mu + b + \frac{3}{2}}\right)^L |0\rangle$$  \hspace{1cm} (5.10)

For $T^{54}$ we find

$$T^{54}_{11}(\mu)|0\rangle = T^{54}_{22}(\mu)|0\rangle = T^{54}_{33}(\mu)|0\rangle = T^{54}_{44}(\mu)|0\rangle = 1|0\rangle$$

Inserting the eigenvalues (5.10) into Eq. (5.9) we obtain:

$$\Lambda^{3b}\left(\mu - \frac{1}{2}\right) \Lambda^{3b}\left(\mu + \frac{1}{2}\right) = - \left(\frac{2\mu - b + \frac{3}{2}}{2\mu + b + \frac{3}{2}}\right)^L \left\{ \prod_{i=1}^{n} \frac{\mu - \lambda_i - \frac{3}{2}}{\mu - \lambda_i + \frac{1}{2}} \frac{n_{1}}{n_{1}} \prod_{j=1}^{n_{1}} \frac{\mu - \nu_j - \frac{3}{2}}{\mu - \nu_j + \frac{1}{2}} + \prod_{i=1}^{n} \frac{\mu - \lambda_i + \frac{1}{2}}{\mu - \lambda_i - \frac{3}{2}} \frac{n_{1}}{n_{1}} \prod_{j=1}^{n_{1}} \frac{\mu - \nu_j + \frac{1}{2}}{\mu - \nu_j - \frac{3}{2}} \right\}$$

$$\quad \quad \quad + \left(\frac{2\mu - b + \frac{3}{2}}{2\mu + b + \frac{3}{2}}\right)^L \prod_{i=1}^{n} \frac{\mu - \lambda_i - \frac{3}{2}}{\mu - \lambda_i + \frac{1}{2}}$$

$$\quad \quad \quad - \left(\frac{2\mu - b + \frac{7}{2}}{2\mu + b + \frac{7}{2}}\right)^L \left\{ \prod_{i=1}^{n} \frac{\mu - \lambda_i - \frac{7}{2}}{\mu - \lambda_i + \frac{1}{2}} \frac{n_{1}}{n_{1}} \prod_{j=1}^{n_{1}} \frac{\mu - \nu_j - \frac{7}{2}}{\mu - \nu_j + \frac{1}{2}} + \prod_{i=1}^{n} \frac{\mu - \lambda_i + \frac{1}{2}}{\mu - \lambda_i - \frac{7}{2}} \frac{n_{1}}{n_{1}} \prod_{j=1}^{n_{1}} \frac{\mu - \nu_j + \frac{1}{2}}{\mu - \nu_j - \frac{7}{2}} \right\}$$

$$\quad \quad \quad + 1 \left\{ \prod_{i=1}^{n} \frac{\mu - \lambda_i - \frac{3}{2}}{\mu - \lambda_i + \frac{1}{2}} + \prod_{j=1}^{n_{1}} \frac{\mu - \nu_j - \frac{3}{2}}{\mu - \nu_j + \frac{1}{2}} + \prod_{i=1}^{n} \frac{\mu - \lambda_i - \frac{7}{2}}{\mu - \lambda_i + \frac{1}{2}} \frac{n_{1}}{n_{1}} \prod_{j=1}^{n_{1}} \frac{\mu - \nu_j + \frac{3}{2}}{\mu - \nu_j - \frac{3}{2}} + \prod_{i=1}^{n} \frac{\mu - \lambda_i + \frac{1}{2}}{\mu - \lambda_i - \frac{7}{2}} \frac{n_{1}}{n_{1}} \prod_{j=1}^{n_{1}} \frac{\mu - \nu_j - \frac{3}{2}}{\mu - \nu_j + \frac{3}{2}} \right\}$$  \hspace{1cm} (5.12)
The first two terms clearly belong to $\Lambda_T^b(\mu)$ and the third one to $\lambda^{b}(\mu)$. Checking the residues at the poles $\mu = \lambda_i + \frac{1}{2}, \mu = \lambda_i - \frac{1}{2}$ and $\mu = \nu_j + \frac{1}{2}$ it can be easily seen that $\prod_{i=1}^n \frac{\mu - \lambda_i - \frac{3}{2}}{\mu - \lambda_i + \frac{1}{2}}$ is the missing part of $\Lambda_T^b$. Thus we have

$$
\Lambda_T^b(\mu) = - \left( \frac{2\mu - b + \frac{3}{2}}{2\mu + b + \frac{3}{2}} \right)^L \left\{ \prod_{i=1}^n \frac{\mu - \lambda_i - \frac{3}{2} \mu - \nu_j - \frac{1}{2}}{\mu - \lambda_i + \frac{1}{2} \mu - \nu_j + \frac{1}{2}} + \prod_{i=1}^n \frac{\mu - \lambda_i - \frac{3}{2} \mu - \nu_j + \frac{3}{2}}{\mu - \lambda_i + \frac{1}{2} \mu - \nu_j - \frac{1}{2}} \right\} + \left( \frac{2\mu - b + \frac{3}{2} \mu - b + \frac{7}{2}}{2\mu + b + \frac{3}{2} \mu + b + \frac{7}{2}} \right)^L \left\{ \prod_{i=1}^n \frac{\mu - \lambda_i - \frac{3}{2} \mu - \nu_j - \frac{1}{2}}{\mu - \lambda_i + \frac{1}{2} \mu - \nu_j + \frac{1}{2}} + \prod_{i=1}^n \frac{\mu - \lambda_i - \frac{3}{2} \mu - \nu_j + \frac{3}{2}}{\mu - \lambda_i + \frac{1}{2} \mu - \nu_j - \frac{1}{2}} \right\} \right.
$$

(5.13)

The situation encountered so far is not yet satisfactory as one wishes to diagonalize the transfer matrix for a $[b_1, \frac{1}{2}]$ representation in the auxiliary space and a $[b_2, \frac{1}{2}]$ representation in the quantum space. With the current results we are limited to the case $b_1 = \frac{3}{4}$. Especially we cannot handle the situation of $[b, \frac{1}{2}] \otimes [b, \frac{1}{2}]$, which gives rise to models of correlated electrons with an additional free parameter.

From the fact that $\tau_T^{b_2}(\mu)$ and $\tau_T^{b_2}(\lambda)$ commute and thus share a system of common eigenvectors, we conclude that the Bethe ansatz equations (3.14) and (3.15) must be preserved.

We modify the eigenvalues (5.13) by replacing the vacuum expectation values of the diagonal elements of $T_T^{b_2}$ (5.10) by those of $T_T^{b_2}$ and then modify the “dressing factors” such that the Bethe ansatz equations still ensure the analyticity.

Examining the second term of (5.13), we see that the singularities of the “dressing factor” must be at values of $\mu$ such that $2\mu + b_1 - b_2 + 1 = 2\lambda_i - b_1 + \frac{1}{2}$ in order to lead to the Bethe ansatz equation (3.13). This fixes the denominator of the “dressing factor” to be $2\mu - 2\lambda_i + b_1 - \frac{1}{2}$. A similar factor comes with the vacuum expectation value $2\mu + b_1 - b_2 + 1$. The right hand side of (3.13) then is used to fix the “nested dressing factor”. A similar argument works for the last term, where the singularity has to be such that $2\mu + b_1 - b_2 = 2\lambda_i - b_1 + \frac{3}{2}$.

Only the numerator of the product $\prod_{i=1}^n$ still remains unkown. We use the first non trivial solution of the BAEq’s for a two site model, namely $n = 1, n_1 = 0, \lambda_1 = -1$ and determine the corresponding eigenvalue by operating with the transfer matrix on the corresponding Bethe vector. Evaluating the result at $\mu = (b_2 - b_1)/2$ isolates the last term. This leads to the final result for the eigenvalues of $\tau_T^{b_2}$:

$$
\Lambda_T^{b_2}(\mu) = - \left( \frac{2\mu + b_1 - b_2}{2\mu + b_1 + b_2 - 1} \right)^L \left\{ \prod_{i=1}^n \frac{2\mu - 2\lambda_i - b_1 - \frac{3}{2} \mu - 2\nu_j + b_1 + \frac{3}{2}}{2\mu - 2\lambda_i + b_1 + \frac{3}{2} \mu - 2\nu_j - b_1 - \frac{1}{2}} \right\} + \left( \frac{2\mu + b_1 - b_2}{2\mu + b_1 + b_2 - 1} \right)^L \left\{ \prod_{i=1}^n \frac{2\mu - 2\lambda_i - b_1 - \frac{3}{2} \mu - 2\nu_j + b_1 + \frac{3}{2}}{2\mu - 2\lambda_i + b_1 + \frac{3}{2} \mu - 2\nu_j - b_1 - \frac{1}{2}} \right\} \right.
$$

(5.14)

We checked the result numerically. For $b_1 = b_2$ the equations are equivalent to the rational limit of the conjectures in Ref. 2. Taking the logarithmic derivative of the eigenvalues we find the energies that where calculated in Ref. 3 by means of co-ordinate Bethe ansatz.

We can proceed with results from the second Bethe ansatz in a similar manner. To keep the presentation short we will only give the corresponding results for the more general case of inhomogeneous chains in Sec. 6.
6 Inhomogeneous chains

A straightforward generalisation is now the construct a monodromy matrix from $R^{b_k}$ matrices, i.e. we choose the representation $[b, \frac{1}{2}]$ in the auxiliary space and the representation $[b_i, \frac{1}{2}]$ in the quantum space at site $i$. This model is also integrable by construction because of the Yang-Baxter equation (2.13). Modifying the vacuum expectation values, the eigenvalues and Bethe ansatz equations can be derived from the ones obtained in the previous section. It is convenient to rescale and shift the spectral parameters according to $\lambda_k \rightarrow -i\lambda_k - 1$, $\nu_j \rightarrow -i\nu_j + \frac{i}{2}$:

\[
A^{b_k}\mu) = -\prod_{k=1}^L \frac{2\mu + b - b_k}{2\mu + b + b_k - 1} \left\{ \prod_{i=1}^n \frac{2\mu + 2i\lambda_i - b + 1}{2\mu + 2i\lambda_i - b + \frac{5}{2}} \prod_{j=1}^{n_1} \frac{2\mu + 2i\nu_j - b + \frac{3}{2}}{2\mu + 2i\nu_j + b + \frac{3}{2}} \right\}
\]

The Bethe equations are

\[
\prod_{k=1}^L \frac{\lambda_l - i(\frac{b_k}{2} - \frac{1}{4})}{\lambda_l + i(\frac{b_k}{2} - \frac{5}{4})} = \prod_{j=1}^{n_1} \frac{\lambda_l - \nu_j + \frac{i}{2}}{\lambda_l - \nu_j - \frac{i}{2}}, \quad l = 1, \ldots, n (6.2)
\]

\[
\prod_{i=1}^n \frac{\lambda_l - \nu_i + \frac{i}{2}}{\lambda_l - \nu_i - \frac{i}{2}} = \prod_{j=1}^{n_1} \frac{\nu_j - \nu_i - i}{\nu_j - \nu_i + i}, \quad i = 1, \ldots, n_1 (6.3)
\]

In a similar fashion we can proceed with the results from the second Bethe ansatz. Finally we obtain:

\[
A^{b_k}\mu) = -\prod_{k=1}^L \frac{2\mu + b - b_k}{2\mu + b + b_k - 1} \left\{ \prod_{i=1}^n \frac{2\mu + 2i\lambda_i - b + 1}{2\mu + 2i\lambda_i - b + \frac{5}{2}} \prod_{j=1}^{n_1} \frac{2\mu + 2i\nu_j - b + \frac{3}{2}}{2\mu + 2i\nu_j + b + \frac{3}{2}} \right\}
\]

The Bethe equations are

\[
\prod_{k=1}^L \frac{\lambda_l - i(\frac{b_k}{2} + \frac{1}{4})}{\lambda_l + i(\frac{b_k}{2} + \frac{5}{4})} = \prod_{j=1}^{n_1} \frac{\lambda_l - \nu_j + \frac{i}{2}}{\lambda_l - \nu_j - \frac{i}{2}}, \quad l = 1, \ldots, n (6.5)
\]

\[
\prod_{i=1}^n \frac{\lambda_l - \nu_i + \frac{i}{2}}{\lambda_l - \nu_i - \frac{i}{2}} = -\prod_{j=1}^{n_1} \frac{\nu_j - \nu_i - i}{\nu_j - \nu_i + i}, \quad i = 1, \ldots, n_1 (6.6)
\]

This second set of Bethe equations can be obtained from the first set (6.2, 6.3) by replacing $b_k$ by $-b_k$. If in addition $b$ is replaced by $-b$ the eigenvalues (6.1) and (6.4) are found to be equal up to an overall factor. This behaviour is explained by the fact that there exist an automorphism of the superalgebra $gl(2,1)$ which maps the $u(1)$ operator $R$ onto $-B$. 

10
7 Discussion

In this paper we have computed the spectrum of vertex models invariant under the action of the superalgebra $gl(2,1)$ by means of the Bethe Ansatz. Depending on the choice of the reference state in the four dimensional quantum space of the local vertices two different Bethe Ansätze are possible which are related to the automorphism of the superalgebra which maps the corresponding states onto each other. The solutions start from one of the bosonic highest weight states. This is different from the situation in the three state model corresponding to the supersymmetric $t$-$J$ model for which three Bethe Ansätze corresponding to the various possibilities of ordering of the basis (2.4)—namely FFB, FBF and BFF—can be constructed \[3\].

The fact that there exists a family of four dimensional representations for this superalgebra allows to introduce a new type of inhomogenous four-state vertex models by allowing the parameter $b$ to take different values in different quantum spaces. Studying the Hamiltonian limit of this class of inhomogenous vertex models leads to systems of electrons with correlated hopping with a spatially varying parameter. It should be noted however, that the $R$ matrix $R^{b_1 b_2}(\mu)$ becomes proportional to a (graded) permutation operator for some values of $\mu$ only if $b_1 = b_2 = b$. The existence of such a shift point is necessary for the construction of a local Hamiltonian from the transfer matrix. Thus to limit the range of interaction one should consider a model with a sufficient number of sites carrying the same representation as the auxiliary space of the monodromy matrix. A possible example is a single $b'$ “impurity” in a chain built from $R^{bb}$ otherwise. We shall study the effect of such an impurity on the thermodynamic properties of an correlated electronic system in a forthcoming paper.

Note added: After completion of this work we received a preprint by P. B. Ramos and M. J. Martins \[13\] who obtain the spectrum (5.14) of the transfer matrix $\tau^{bb}(\mu)$ by applying the algebraic Bethe Ansatz to the $4 \times 4$ monodromy matrix $T^{bb}(\mu)$ directly. Their results coincide with ours, it should be noted though that their discussion of the unwanted terms arising in this procedure is not complete.

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