A SPINOR DESCRIPTION OF FLAT SURFACES IN $\mathbb{R}^4$

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Abstract. We describe the flat surfaces with flat normal bundle and regular Gauss map immersed in $\mathbb{R}^4$ using spinors and Lorentz numbers. We obtain a new proof of the local structure of these surfaces. We also study the flat tori in the sphere $S^3$ and obtain a new representation formula. We then deduce new proofs of their global structure, and of the global structure of their Gauss map image.

Introduction

In this paper we are interested in flat surfaces with flat normal bundle in $\mathbb{R}^4$ and in their description using spinors. We show how a general spinor representation formula of surfaces in $\mathbb{R}^4$ permits to obtain the following important results concerning their local and global structure:

1. locally a flat surface with flat normal bundle and regular Gauss map depends on four functions of one variable [6, 5];

2. a flat torus immersed in $S^3$ is a product of two horizontal closed curves in $S^3 \subset \mathbb{H}$ [3, 14, 15, 11].

With this formalism we also obtain the structure of the Gauss map image of a flat torus in $S^3$ [7, 18].

This approach permits a unified treatment of numerous questions concerning flat surfaces with flat normal bundle in $\mathbb{R}^4$; the main idea is to write the general spinor representation formula obtained in [2] in parallel frames adapted to the surfaces; with the spinor construction and the representation formula at hand, the proofs then appear to be quite simple.

In the paper, the principal object attached to a flat immersion with flat normal bundle in $\mathbb{R}^4$ is a map $g$, which represents a constant spinor field in a moving frame adapted to the immersion; as a consequence of the spinor representation formula, it appears that $g$, together with the metric of the surface, determines the immersion (with an explicit formula). Moreover, in the special case of a surface in $S^3$, $g$ also determines the metric of the surface and thus entirely determines the immersion. The structure of the flat immersions with flat normal bundle in $\mathbb{R}^4$ thus crucially relies on the structure of $g$. In the paper, the map $g$ is considered as a curve into a sphere, parameterized by the Lorentz numbers. We show that its arc length parametrization yields natural coordinates on the surface which generalize the asymptotic Tchebycheff net of the flat surfaces in $S^3$ (Theorem [2] and Remark 6.11); as a corollary of the spinor representation formula, we then obtain the local structure

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of the flat immersions with flat normal bundle in $\mathbb{R}^4$ (Theorem 3 and Corollary 1). When the surface is compact, we show that the arc length parametrization of $g$ gives in fact a global parametrization of the surface (Proposition 7.1), and, as a corollary, we obtain a new representation of the flat tori in $S^3$ (Theorem 4); we then observe that the two natural projections of $g$ on its positive and negative parts are in fact the two curves in the Kitagawa representation of the torus defined by $g$ (Theorem 5 and Section 7.5). Finally, a Hopf projection of $g$ gives the Gauss map of the surface, which permits to study easily the structure of the Gauss map image of the flat tori in $S^3$ (Corollary 2).

We previously used this approach in [1] to study flat surfaces with flat normal bundle and regular Gauss map in 4-dimensional Minkowski space $\mathbb{R}^{1,3}$. The present paper uses similar ideas, and gives applications of the spinor representation formula contained in [2].

We quote papers concerning flat surfaces with flat normal bundle in $\mathbb{R}^4$. In [3] Bianchi constructed flat surfaces in $S^3$ as a product, in the quaternions, of two special curves. Sasaki [14] and Spivak [15] classified the complete flat surfaces in $S^3$. In [11], Kitagawa gave a method to construct all the flat tori in $S^3$: a flat torus is a product of two curves in the unit quaternions; these curves are constructed as asymptotic lifts of periodic curves in $S^2$ (the Kitagawa representation). In [9], Gálvez and Mira constructed non-trivial flat tori with flat normal bundle in $\mathbb{R}^4$ which are not contained in any 3-sphere, which raises the question of the construction of all the flat tori with flat normal bundle in $\mathbb{R}^4$. Enomoto [7] and Weiner [18] described the flat tori in $S^3$ in terms of their Gauss map.

The outline of the paper is as follows: in Section 1 we describe the Clifford algebra of $\mathbb{R}^4$ and its spinor representation using quaternions and Lorentz numbers, in Section 2 we describe the spinor bundle twisted by a bundle of rank two on a Riemannian surface and in Section 3 the spinor representation of a surface in $\mathbb{R}^4$, rewriting the spinor representation formula of [2] using Lorentz numbers. Using the same formalism, we then describe in Section 4 the Gauss map of a surface in $\mathbb{R}^4$, together with its relation to the spinor representation formula of the surface. Section 5 is devoted to Lorentz surfaces and Lorentz numbers. We then give the local description of the flat surfaces with flat normal bundle and regular Gauss map in $\mathbb{R}^4$ in Section 6 and finally describe the flat tori in $S^3$ and their Gauss map in Section 7. An appendix on Lorentz numbers and quaternions ends the paper.

1. Clifford algebra of $\mathbb{R}^4$ and spin representations with Lorentz numbers

In this section, we describe the Clifford algebras and spinors of $\mathbb{R}^4$ using Lorentz numbers and quaternions.

1.1. Lorentz numbers and quaternions. The algebra of Lorentz numbers is the algebra

$$\mathcal{A} = \{ x + \sigma y, \ x, y \in \mathbb{R} \},$$
with the usual operations, where $\sigma$ is a formal element satisfying $\sigma^2 = 1$. We consider the quaternions with coefficients in $\mathbb{A}$,

$$\mathbb{H}^A := \{a_0 I + a_1 I + a_2 J + a_3 K, \ a_0, a_1, a_2, a_3 \in \mathbb{A}\},$$

where $I, J, K$ satisfy

$$I^2 = J^2 = K^2 = -1, \quad IJ = -JI = K.$$

For $\xi = a_0 I + a_1 I + a_2 J + a_3 K$ belonging to $\mathbb{H}^A$, we denote

$$\xi := a_0 I - a_1 I - a_2 J - a_3 K,$$

and we consider the following two inner products on $\mathbb{H}^A$:

$$\langle\langle , \rangle\rangle : \mathbb{H}^A \times \mathbb{H}^A \to \mathbb{H}^A \quad (\xi, \xi') \mapsto \bar{\xi} \xi$$

and

$$H : \mathbb{H}^A \times \mathbb{H}^A \to \mathbb{A} \quad (\xi, \xi') \mapsto a_0 a'_0 + a_1 a'_1 + a_2 a'_2 + a_3 a'_3$$

where $\xi = a_0 I + a_1 I + a_2 J + a_3 K$ and $\xi' = a'_0 I + a'_1 I + a'_2 J + a'_3 K$. We note that

$$H(\xi, \xi) = \langle\langle \xi, \xi \rangle\rangle$$

for all $\xi \in \mathbb{H}^A$.

### 1.2. Clifford map, Clifford algebra and Spin representations

We consider the Clifford map

$$\mathbb{R}^4 \to \mathbb{H}^A(2)$$

$$(x_0, x_1, x_2, x_3) \mapsto \begin{pmatrix} 0 & \sigma x_0 I + x_1 I + x_2 J + x_3 K \\ -\sigma x_0 I + x_1 I + x_2 J + x_3 K & 0 \end{pmatrix}$$

where $\mathbb{H}^A(2)$ stands for the space of $2 \times 2$ matrices with coefficients in $\mathbb{H}^A$; using this map, the Clifford algebra of $\mathbb{R}^4$ identifies to

$$Cl(4) = \left\{ \begin{pmatrix} a & b \\ \bar{b} & \bar{a} \end{pmatrix}, \ a, b \in \mathbb{H}^A \right\}.$$

Here and below, for $\xi = a_0 I + a_1 I + a_2 J + a_3 K \in \mathbb{H}^A$ we denote

$$\xi := \bar{a_0} I + \bar{a_1} I + \bar{a_2} J + \bar{a_3} K,$$

where, if $a_i = x_i + \sigma y_i$ belongs to $\mathbb{A}$, its conjugate is $\bar{a_i} := x_i - \sigma y_i$. Its subalgebra of even elements is

$$Cl_0(4) = \left\{ \begin{pmatrix} a & 0 \\ 0 & \bar{a} \end{pmatrix} \right\}, \ a \in \mathbb{H}^A \simeq \mathbb{H}^A,$$

and we have

$$Spin(4) = \{a \in \mathbb{H}^A : H(a, a) = 1\} =: S^3_A.$$

We note the identification

$$S^3_A \simeq S^3 \times S^3$$

when we identify $\mathbb{H}^A$ to $\mathbb{H} \oplus \mathbb{H}$ by the isomorphism

$$\begin{pmatrix} \xi_+ \xi_- \end{pmatrix}.$$
where \( \xi, \xi_+ \) and \( \xi_- \) are linked by
\[
\xi = \frac{1 + \sigma}{2} \xi_+ + \frac{1 - \sigma}{2} \xi_-. 
\]
Moreover, we have the double covering
\[
\Phi : Spin(4) \rightarrow SO(4) \quad q \mapsto (\xi \in \mathbb{R}^4 \mapsto q \xi q^{-1} \in \mathbb{R}^4),
\]
where \( \mathbb{R}^4 \subset \mathbb{H}^4 \) is defined by
\[
\mathbb{R}^4 := \{ \xi \in \mathbb{H}^4 : \hat{\xi} = -\xi \} = \{ \sigma x_0 I + x_1 I + x_2 J + x_3 K, (x_0, x_1, x_2, x_3) \in \mathbb{R}^4 \}. 
\]
We note that the euclidean metric on \( \mathbb{R}^4 \) is given by the restriction of the form \( H \).

Let \( \rho : Cl(4) \rightarrow End_{\mathbb{C}}(\mathbb{H}^4) \) be the complex representation of \( Cl(4) \) on \( \mathbb{H}^4 \) given by
\[
\rho \left( \begin{array}{cc} a & b \\ \bar{b} & \bar{a} \end{array} \right) : \xi \mapsto \left( \begin{array}{c} \xi \\ \hat{\xi} \end{array} \right) \mapsto \left( \begin{array}{c} a \xi + b \hat{\xi} \\ a \hat{\xi} + b \xi \end{array} \right)
\]
where the complex structure on \( \mathbb{H}^4 \) is given by the right multiplication by \( J \). The restriction of the representation \( \rho \) to \( Spin(4) \) gives
\[
\rho_{|Spin(4)} : Spin(4) \rightarrow End_{\mathbb{C}}(\mathbb{H}^4) \quad a \mapsto (\xi \in \mathbb{H}^4 \mapsto a \xi \in \mathbb{H}^4).
\]
This representation splits into
\[
\mathbb{H}^4 = \mathbb{H}^+ \oplus \mathbb{H}^-
\]
where \( \mathbb{H}^+ = \{ \xi \in \mathbb{H}^4 : \sigma \xi = \xi \} \) and \( \mathbb{H}^- = \{ \xi \in \mathbb{H}^4 : \sigma \xi = -\xi \} \); this decomposition corresponds to \( \mathbb{H} \), since
\[
\mathbb{H}^+ = \left\{ \frac{1 + \sigma}{2} \xi, \xi \in \mathbb{H} \right\} \simeq \mathbb{H} \quad \text{and} \quad \mathbb{H}^- = \left\{ \frac{1 - \sigma}{2} \xi, \xi \in \mathbb{H} \right\} \simeq \mathbb{H}.
\]

1.3. Spinors under the splitting \( \mathbb{R}^4 = \mathbb{R}^2 \times \mathbb{R}^2 \). We now consider the splitting \( \mathbb{R}^4 = \mathbb{R}^2 \times \mathbb{R}^2 \) and the corresponding inclusion
\[
SO(2) \times SO(2) \subset SO(4).
\]
Using the definition \( \Phi \) of \( \Phi \), we easily get
\[
\Phi^{-1}(SO(2) \times SO(2)) = \{ \cos \theta + \sin \theta I, \theta \in \mathbb{A} \} =: S^1_{\mathbb{A}} \subset Spin(4),
\]
where the cos and sin functions are naturally defined on the Lorentz numbers by
\[
\cos \theta = \sum_{n=0}^{\infty} \frac{(-1)^n}{2n!} \theta^{2n} \quad \text{and} \quad \sin \theta = \sum_{n=0}^{\infty} \frac{(-1)^n}{2n+1!} \theta^{2n+1}
\]
for all \( \theta \in \mathbb{A} \). Indeed, setting \( \theta = \frac{1 + \sigma}{2} s + \frac{1 - \sigma}{2} t \), we have in fact
\[
\cos \theta = \frac{1 + \sigma}{2} \cos s + \frac{1 - \sigma}{2} \cos t \quad \text{and} \quad \sin \theta = \frac{1 + \sigma}{2} \sin s + \frac{1 - \sigma}{2} \sin t,
\]
and the usual trigonometric formulas hold for these trigonometric functions; then, setting \( \theta = \theta_1 + \sigma \theta_2 \) with \( \theta_1, \theta_2 \in \mathbb{R} \), we get, in \( \mathbb{H}^4 \),
\[
\cos \theta + \sin \theta I = (\cos \theta_2 + \sigma \sin \theta_2 I)(\cos \theta_1 + \sin \theta_1 I),
\]
and $\Phi(\cos \theta + \sin \theta \, I)$ appears to be the rotation of $\mathbb{R}^4$ consisting in a rotation of angle $2\theta_1$ in $\{0\} \times \mathbb{R}^2$ and in a rotation of angle $2\theta_2$ in $\mathbb{R}^2 \times \{0\}$. Summing up the preceding results, we define

$$\text{Spin}(2) := \{ \cos \theta_1 + \sin \theta_1 \, I, \ \theta_1 \in \mathbb{R} \} \subset \text{Spin}(4)$$

and

$$\text{Spin}'(2) := \{ \cos \theta_2 + \sigma \sin \theta_2 \, I, \ \theta_2 \in \mathbb{R} \} \subset \text{Spin}(4),$$

and we have

$$S^1_A = \text{Spin}'(2).\text{Spin}(2) \simeq \text{Spin}(2) \times \text{Spin}(2)/\mathbb{Z}_2,$$

and the double covering

$$\Phi : \ S^1_A = \text{Spin}'(2).\text{Spin}(2) \rightarrow \ 	ext{SO}(2) \times \text{SO}(2).$$

If we now restrict the spin representation $\rho$ of $\text{Spin}(4)$ to $S^1_A \subset \text{Spin}(4)$ the representation in $\mathbb{H}^4 = \mathbb{H}^4_+ \oplus \mathbb{H}^4_-$ splits into four subspaces

$$\mathbb{H}^4_+ = S^{++} \oplus S^{--} \quad \text{and} \quad \mathbb{H}^4_- = S^{+-} \oplus S^{-+}$$

where

$$S^{++} = \frac{1 + \sigma}{2} (\mathbb{R} J \oplus \mathbb{R} K), \ S^{--} = \frac{1 + \sigma}{2} (\mathbb{R} I \oplus \mathbb{R} I), \ S^{+-} = \frac{1 - \sigma}{2} (\mathbb{R} I \oplus \mathbb{R} I)$$

and

$$S^{-+} = \frac{1 - \sigma}{2} (\mathbb{R} J \oplus \mathbb{R} K).$$

**Remark 1.1.** The representation

$$\rho : \text{Spin}'(2) \times \text{Spin}(2) \rightarrow \text{End}_\mathbb{C}(\mathbb{H}^4)$$

$$(g_1, g_2) \mapsto \rho(g) : \xi \mapsto g_1 \xi g_2,$$

where $g = g_1 g_2 \in S^1_A = \text{Spin}'(2).\text{Spin}(2) \subset \mathbb{H}^4$, is equivalent to the representation

$$\rho_1 \otimes \rho_2 = \rho_1^+ \otimes \rho_2^+ \oplus \rho_1^- \otimes \rho_2^- \oplus \rho_1^+ \otimes \rho_2^- \oplus \rho_1^- \otimes \rho_2^+$$

of $\text{Spin}(2) \times \text{Spin}(2)$, where $\rho_1 = \rho_1^+ + \rho_1^-$ and $\rho_2 = \rho_2^+ + \rho_2^-$ are two copies of the spinor representation of $\text{Spin}(2)$; moreover, the decomposition (11) corresponds to the decomposition (10); indeed, writing

$$\theta = \theta_1 + \sigma \theta_2 = \frac{1 + \sigma}{2} (\theta_1 + \theta_2) + \frac{1 - \sigma}{2} (\theta_1 - \theta_2),$$

it is not difficult to see that the restrictions of the representation (11) to the subspaces $S^{++}, S^{--}, S^{+-}$ and $S^{-+}$ are respectively equivalent to the multiplications by $e^{-i(\theta_1 + \theta_2)}$, $e^{i(\theta_1 + \theta_2)}$, $e^{i(\theta_1 - \theta_2)}$ and $e^{-i(\theta_1 + \theta_2)}$ on $\mathbb{C}$ where $\theta_2 \in \mathbb{R}/2\pi \mathbb{Z}$ describes the first factor and $\theta_1 \in \mathbb{R}/2\pi \mathbb{Z}$ the second factor of $\text{Spin}'(2) \times \text{Spin}(2)$, as in (7)-(8).

2. **Twisted spinor bundle**

We assume that $M$ is an oriented surface, with a metric, and that $E \rightarrow M$ is a vector bundle of rank two, oriented, with a fibre metric and a connection compatible with the metric. We set $\Sigma := \Sigma E \otimes \Sigma M$, the tensor product of spinor bundles constructed from $E$ and $TM$. We denote by $Q := Q_E \times_M Q_M$ the $\text{SO}(2) \times \text{SO}(2)$ principal bundle on $M$, product of the positive and orthonormal frame bundles on $E$ and $TM$. If $\tilde{Q} := \tilde{Q}_E \times_M \tilde{Q}_M$ is the $\text{Spin}(2) \times \text{Spin}(2)$ principal bundle product
of the spin structures $\tilde{Q}_E \to Q_E$ and $\tilde{Q}_M \to Q_M$, the bundle $\Sigma$ is associated to $\tilde{Q}$ and to the representation $[10]$, that is
\[ \Sigma \simeq \tilde{Q} \times \mathbb{H}^4/\rho. \]
This is because $\rho$ is equivalent to the representation $\rho_1 \otimes \rho_2$, tensor product of two copies of the spin representation of $Spin(2)$ (see Remark [11] above). Obviously, the maps $\xi \mapsto g\xi$ belong in fact to $End_{\mathbb{H}^4}(\mathbb{H}^4)$, the space of endomorphisms of $\mathbb{H}^4$ which are $\mathbb{H}^4$–linear, where the linear structure on $\mathbb{H}^4$ is given by the multiplication on the right: $\Sigma$ is thus naturally equipped with a linear right-action of $\mathbb{H}^4$. Since the products $\langle \langle ., . \rangle \rangle$ and $H(.,.)$ on $\mathbb{H}^4$ are preserved by the multiplication of $Spin(4)$ on the left, $\Sigma$ is moreover naturally equipped with the products
\[ \langle \langle , \rangle \rangle : \Sigma \times \Sigma \to \mathbb{H}^4 \]
\[ (\varphi, \varphi') \mapsto \langle \langle \xi, \xi' \rangle \rangle \]
and
\[ H : \Sigma \times \Sigma \to A \]
\[ (\varphi, \varphi') \mapsto H(\xi, \xi'), \]
where $\xi$ and $\xi' \in \mathbb{H}^4$ are the coordinates of $\varphi$ and $\varphi'$ in some spinorial frame.

We quote the following properties: for all $X \in E \oplus TM$ and $\varphi, \psi \in \Sigma$,
\[ \langle \langle X \cdot \varphi, \psi \rangle \rangle = -\langle \langle \varphi, X \cdot \psi \rangle \rangle \] (12)
and
\[ \langle \langle \varphi, \psi \rangle \rangle = \overline{\langle \langle \psi, \varphi \rangle \rangle.} \] (13)

**Notation.** Throughout the paper, we will use the following notation: if $\tilde{s} \in \tilde{Q}$ is a given frame, the brackets $[\cdot]$ will denote the coordinates $\in \mathbb{H}^4$ of the spinor fields in $\tilde{s}$, that is, for $\varphi \in \Sigma$,
\[ \varphi \simeq [\tilde{s}, [\varphi]] \in \Sigma \simeq \tilde{Q} \times \mathbb{H}^4/\rho. \]
We will also use the brackets to denote the coordinates in $\tilde{s}$ of the elements of the Clifford algebra $Cl(E \oplus TM)$: $X \in Cl_0(E \oplus TM)$ and $Y \in Cl_1(E \oplus TM)$ will be respectively represented by $\lfloor X \rfloor$ and $\lfloor Y \rfloor \in \mathbb{H}^4$ such that, in $\tilde{s}$,
\[ X \simeq \begin{pmatrix} [X] & 0 \\ 0 & [\tilde{X}] \end{pmatrix} \]
and
\[ Y \simeq \begin{pmatrix} 0 & [Y] \\ [\tilde{Y}] & 0 \end{pmatrix}. \]

We note that
\[ [X \cdot \varphi] = [X][\varphi] \quad \text{and} \quad [Y \cdot \varphi] = [Y][\tilde{\varphi}]. \]

If $(e_0, e_1)$ and $(e_2, e_3)$ are positively oriented and orthonormal frames of $E$ and $TM$, a frame $\tilde{s} \in \tilde{Q}$ such that $\pi(\tilde{s}) = (e_0, e_1, e_2, e_3)$, where $\pi : \tilde{Q} \to Q_E \times_M Q_M$ is the natural projection onto the bundle of the orthonormal frames of $E \oplus TM$, will be called adapted to the frame $(e_0, e_1, e_2, e_3)$; in such a frame, $e_0, e_1, e_2$ and $e_3 \in Cl_1(E \oplus TM)$ are respectively represented by $\sigma I, I, J$ and $K \in \mathbb{H}^4$. 
3. Spinor representation of surfaces in $\mathbb{R}^4$

The aim of this section is to formulate the main results of [2] using the formalism introduced in the previous sections. We keep the notation of the previous section, and recall that $\Sigma = \Sigma E \otimes \Sigma M$ is equipped with a natural connection
$$\nabla = \nabla^E \otimes \text{id}_{\Sigma M} + \text{id}_{\Sigma E} \otimes \nabla^M,$$
the tensor product of the spinor connections on $\Sigma E$ and on $\Sigma M$, and also with a natural action of the Clifford bundle
$$\text{Cl}(E \oplus TM) \cong \text{Cl}(E) \hat{\otimes} \text{Cl}(M);$$
see [2]. This permits to define the Dirac operator $D$ acting on $\Gamma(\Sigma)$ by
$$D\varphi = e_2 \cdot \nabla e_2 \varphi + e_3 \cdot \nabla e_3 \varphi,$$
where $(e_2, e_3)$ is an orthonormal basis of $TM$. We have the following:

**Proposition 3.1.** Let $\vec{H}$ be a section of $E$, and assume that $\varphi \in \Gamma(\Sigma)$ is such that
$$D\varphi = \vec{H} \cdot \varphi \quad \text{and} \quad H(\varphi, \varphi) = 1. \quad (14)$$
We define the $\mathbb{H}^A$-valued 1-form $\xi \in \Omega^1(M, \mathbb{H}^A)$ by
$$\xi(X) := \langle (X \cdot \varphi, \varphi) \rangle \in \mathbb{H}^A \quad (15)$$
for all $X \in TM$, where the pairing $\langle \cdot, \cdot \rangle : \Sigma \times \Sigma \to \mathbb{H}^A$ is defined in the previous section. Then

1. the form $\xi$ satisfies
$$\xi = -\tilde{\xi},$$
and thus takes its values in $\mathbb{R}^4 \subset \mathbb{H}^A$ (see (6));
2. the form $\xi$ is closed:
$$d\xi = 0.$$

**Proof.** The first part of the proposition is a consequence of (12) and (13), and the second part relies on the Dirac equation (14); we refer to [2] for the detailed proof of a very similar proposition. \hfill \Box

We may rewrite Theorem 1 of [2] for surfaces in $\mathbb{R}^4$ as follows:

**Theorem 1.** Suppose that $M$ is moreover simply connected. The following statements are equivalent.

1. There exists a spinor field $\varphi$ of $\Gamma(\Sigma)$ with $H(\varphi, \varphi) = 1$ solution of the Dirac equation
$$D\varphi = \vec{H} \cdot \varphi.$$  
2. There exists a spinor field $\varphi \in \Gamma(\Sigma)$ with $H(\varphi, \varphi) = 1$ solution of
$$\nabla_X \varphi = -\frac{1}{2} \sum_{j=2,3} e_j \cdot B(X, e_j) \cdot \varphi,$$
where $B : TM \times TM \to E$ is bilinear with $\frac{1}{2} \text{tr} (B) = \vec{H}$, and where $(e_2, e_3)$ is an orthonormal basis of $TM$ at every point.
3. There exists an isometric immersion $F$ of $M$ into $\mathbb{R}^4$ with normal bundle $E$ and mean curvature vector $\vec{H}$. 


Moreover, \( F = \int \xi \), where \( \xi \) is the closed 1-form on \( M \) with values in \( \mathbb{R}^4 \) defined by
\[
\xi(X) := \langle \langle X \cdot \varphi, \varphi \rangle \rangle \in \mathbb{R}^4 \subset H^4
\]
(16) for all \( X \in TM \).

The proof that (1) is equivalent to (3) is very simple: assuming first that \( M \) is immersed in \( \mathbb{R}^4 \), the spinor bundle of \( \mathbb{R}^4 \) restricted to \( M \) identifies to \( \Sigma = \Sigma E \otimes \Sigma M \) where \( E \) is the normal bundle of the surface in \( \mathbb{R}^4 \), and the restriction to \( M \) of the constant spinor field 1 of \( \mathbb{R}^4 \) satisfies (1) (by the spinor Gauss formula, see [2]); conversely, if \( \varphi \in \Gamma(\Sigma) \) satisfies (1), it is easy to check that the formula \( F = \int \xi \), where \( \xi \) is defined by (16), defines an isometric immersion with normal bundle \( E \) and mean curvature vector \( \vec{H} \); see [1, 2].

Remark 3.2. The map \( X \in E \mapsto \langle \langle X \cdot \varphi, \varphi \rangle \rangle \in \mathbb{R}^4 \) identifies \( E \) with the normal bundle of the immersion; it preserves the metrics, the connections and the fundamental forms. See [1, 2].

Remark 3.3. Applications of the spinor representation formula in Sections 6 and 7 will rely on the following simple observation: assume that \( F_o : M \hookrightarrow \mathbb{R}^4 \) is an isometric immersion and consider \( \varphi = 1|_M \) the restriction to \( M \) of the constant spinor field 1 of \( \mathbb{R}^4 \); if \( F = \int \xi \), \( \xi \) is the immersion given in the theorem, then \( F \simeq F_o \). This is in fact trivial since
\[
\xi(X) = \langle \langle X \cdot \varphi, \varphi \rangle \rangle \quad (17)
\]
is the immersion given in the theorem, then \( F \simeq F_o \). This is in fact trivial since
\[
\xi(X) = \langle \langle X \cdot \varphi, \varphi \rangle \rangle = \langle \langle X \rangle \rangle \simeq X \quad (18)
\]
in a spinorial frame \( \tilde{s} \) of \( \mathbb{R}^4 \) which is above the canonical basis (in such a frame \( \langle \varphi \rangle = \pm 1 \)). The representation formula (17), when written in moving frames adapted to the immersion, will give non trivial formulas.

4. The Gauss Map of a Surface in \( \mathbb{R}^4 \)

We assume in this section that \( M \) is an oriented surface immersed in \( \mathbb{R}^4 \). We consider \( \Lambda^2 \mathbb{R}^4 \), the vector space of bivectors of \( \mathbb{R}^4 \) endowed with its natural metric \( \langle ., . \rangle \). The Grassmannian of the oriented 2-planes in \( \mathbb{R}^4 \) identifies with the submanifold
\[
Q = \{ \eta \in \Lambda^2 \mathbb{R}^4 : \langle \eta, \eta \rangle = 1, \ \eta \wedge \eta = 0 \},
\]
and the oriented Gauss map of \( M \) with the map
\[
G : M \to Q, \ p \mapsto G(p) = u_1 \wedge u_2,
\]
where \( (u_1, u_2) \) is a positively oriented and orthonormal basis of \( T_p M \). The Hodge \( * \) operator \( \Lambda^2 \mathbb{R}^4 \to \Lambda^2 \mathbb{R}^4 \) is defined by the relation
\[
(* \eta, \eta') = \eta \wedge \eta' \quad (19)
\]
for all \( \eta, \eta' \in \Lambda^2 \mathbb{R}^4 \), where we identify \( \Lambda^4 \mathbb{R}^4 \) to \( \mathbb{R} \) using the canonical volume element \( e_0^0 \wedge e_1^1 \wedge e_2^2 \wedge e_3^3 \) on \( \mathbb{R}^4 \); here and below \( (e_0^0, e_1^1, e_2^2, e_3^3) \) stands for the canonical basis of \( \mathbb{R}^4 \). It satisfies \( *^2 = id_{\Lambda^2 \mathbb{R}^4} \) and thus \( \sigma = * \) naturally defines a \( \mathcal{A} \)-module structure on \( \Lambda^2 \mathbb{R}^4 \) : it is such that
\[
a \eta = a_1 \eta + a_2 * \eta
\]
for all $a = a_1 + \sigma a_2 \in \mathcal{A}$ and $\eta \in \Lambda^2 \mathbb{R}^4$. We also define
\[ H(\eta, \eta') = \langle \eta, \eta' \rangle + \sigma \eta \wedge \eta' \in \mathcal{A} \] (20)
for all $\eta, \eta' \in \Lambda^2 \mathbb{R}^4$. This is a $\mathcal{A}$-bilinear map on $\Lambda^2 \mathbb{R}^4$ since, by (19),
\[ H(\sigma \eta, \eta') = H(\eta, \sigma \eta') = \sigma H(\eta, \eta') \]
for all $\eta, \eta' \in \Lambda^2 \mathbb{R}^4$, and we have
\[ Q = \{ \eta \in \Lambda^2 \mathbb{R}^4 : H(\eta, \eta) = 1 \} \] (21)
The bivectors
\[ E_1 = e_2 \wedge e_3, \quad E_2 = e_3 \wedge e_1, \quad E_3 = e_1 \wedge e_2 \]
form a basis of $\Lambda^2 \mathbb{R}^4$ as a module over $\mathcal{A}$; this basis is such that
\[ H(E_i, E_j) = \delta_{ij} \]
for all $i, j$. Using the Clifford map defined Section 1.2, and identifying $\Lambda^2 \mathbb{R}^4$ with the elements of order 2 of $Cl_0(4) \subset Cl(4) \subset H^A(2)$ (see (1)), we get
\[ E_1 = I, \quad E_2 = J, \quad E_3 = K \] (22)
and
\[ \Lambda^2 \mathbb{R}^4 = \{ aI + bJ + cK \in H^A : (a, b, c) \in \mathcal{A}^3, \quad a^2 + b^2 + c^2 = 1 \} =: S_A^2. \] (23)
We define the cross product of two vectors $\xi, \xi' \in \Im \mathbb{H}^A := \mathcal{A}I \oplus \mathcal{A}J \oplus \mathcal{A}K$ by
\[ \xi \times \xi' := \frac{1}{2} (\xi \xi' - \xi' \xi) \in \Im \mathbb{H}^A. \]
It is such that
\[ \langle\langle \xi, \xi' \rangle\rangle = H(\xi, \xi') I + \xi \times \xi'. \]
Lemma 4.1. If $\xi \times \xi' = 0$ where $\xi$ is invertible in $\mathbb{H}^A$, then
\[ \xi' = \lambda \xi \]
for some $\lambda \in \mathcal{A}$.
Proof. Writing
\[ \xi = \frac{1 + \sigma}{2} \xi_+ + \frac{1 - \sigma}{2} \xi_- \quad \text{and} \quad \xi' = \frac{1 + \sigma}{2} \xi'_+ + \frac{1 - \sigma}{2} \xi'_- \]
where $\xi_+, \xi_-$, $\xi'_+$ and $\xi'_-$ belong to $\Im \mathbb{H}$, the condition $\xi \times \xi' = 0$ is equivalent to the two conditions $\xi_+ \times \xi'_+ = 0$ and $\xi_- \times \xi'_- = 0$, where the cross product is here the usual cross product in $\Im \mathbb{H} \simeq \mathbb{R}^3$; this is thus also equivalent to the fact that both $\xi_+$, $\xi'_+$ and $\xi_-$, $\xi'_-$ are linearly dependent in $\mathbb{R}^3$. If $\xi$ is moreover invertible in $\mathbb{H}^A$, then $\xi_+$ and $\xi_-$ are not zero (see Lemma A.1 in the Appendix), and the result easily follows.
We also define the \textit{mixed product} of three vectors \( \xi, \xi', \xi'' \in \mathbb{H}^A \) by
\[
[\xi, \xi', \xi''] := H(\xi \times \xi', \xi'') \in A.
\]
The mixed product is a \( A \)-valued volume form on \( \mathbb{H}^A \); it induces a natural \( A \)-valued area form \( \omega_Q \) on \( Q \) by
\[
\omega_Q(p, \xi, \xi') := [\xi, \xi', p]
\]
for all \( p \in Q \) and all \( \xi, \xi' \in T_p Q \). We now compute the pull-back by the Gauss map of the area form \( \omega_Q \):

**Proposition 4.2.** We have
\[
G^* \omega_Q = (K + \sigma K_N) \omega_M,
\]
where \( \omega_M \) is the area form, \( K \) is the Gauss curvature and \( K_N \) is the normal curvature of \( M \mapsto \mathbb{R}^4 \). Assuming moreover that
\[
dG_{x_0} : T_{x_0} M \to T_{G(x_0)} Q
\]
is one-to-one at some point \( x_0 \in M \), then \( K = K_N = 0 \) at \( x_0 \) if and only if the linear space \( dG_{x_0}(T_{x_0} M) \) is some \( A \)-line in \( T_{G(x_0)} Q \), i.e.
\[
dG_{x_0}(T_{x_0} M) = \{ a U, a \in A \},
\]
where \( U \in T_{G(x_0)} Q \subset \mathbb{H}^A \) is such that \( H(U, U) = 1 \).

**Proof.** Let \((e_2, e_3)\) be a positively oriented and orthonormal frame tangent to \( M \). By definition, we have
\[
G^* \omega_Q(e_2, e_3) = H(\mathbf{d}G(e_2) \times \mathbf{d}G(e_3), G).
\]
Since \( G = e_2 \wedge e_3 \), we have
\[
dG = B(e_2,.) \wedge e_3 + e_2 \wedge B(e_3,.) \simeq B(e_2,.) \cdot e_3 + e_2 \cdot B(e_3,.)
\]
where \( B : TM \times TM \to E \) is the second fundamental form of \( M \mapsto \mathbb{R}^4 \), and straightforward computations give
\[
dG(e_2) \times dG(e_3) = \frac{1}{2}(dG(e_2) \cdot dG(e_3) - dG(e_3) \cdot dG(e_2)) = \alpha e_2 \cdot e_3 + \beta
\]
with
\[
\alpha = B(e_2, e_3) \cdot B(e_2, e_3) - \frac{1}{2}(B(e_2, e_2) \cdot B(e_3, e_3) + B(e_3, e_3) \cdot B(e_2, e_2))
\]
and
\[
\beta = (B(e_2, e_2) - B(e_3, e_3)) \cdot B(e_2, e_3) - B(e_2, e_3) \cdot (B(e_2, e_2) - B(e_3, e_3)).
\]
Here the dot \( . \) stands for the Clifford product in \( Cl(4) \), which, for elements in \( Cl_0(4) \simeq \mathbb{H}^A \), coincides with the product in \( \mathbb{H}^A \). Writing
\[
B = \begin{pmatrix} a & c \\ c & b \end{pmatrix} e_0 + \begin{pmatrix} e & g \\ g & f \end{pmatrix} e_1
\]
in the basis \((e_2, e_3)\), we then easily obtain
\[
\alpha = ab + ef - c^2 - g^2 \quad \text{and} \quad \beta = ((a - b)g - (c - f)c) e_0 \cdot e_1.
\]
The first term is \( \alpha = K \) and the second term is
\[
\beta = -(S_{e_0} \circ S_{e_1} - S_{e_1} \circ S_{e_0})(e_2, e_3) e_0 \cdot e_1 = K_N e_0 \cdot e_1.
\]
where, for $\nu \in E$, $S_{\nu} : TM \to TM$ is such that
\[
\langle S_{\nu}(X), Y \rangle = \langle B(X, Y), \nu \rangle
\]
for all $X, Y \in TM$. This finally gives
\[
dG(e_2) \times dG(e_3) = K e_2 \cdot e_3 + K_N e_0 \cdot e_1. \tag{26}
\]
Since $e_2 \cdot e_3 = G$ and $e_0 \cdot e_1 = \sigma e_2 \cdot e_3 = \sigma G$, \([24]\) and $H(G, G) = 1$ imply \([24]\).

We finally prove the last claim of the proposition: if $K = K_N = 0$, then formula \([24]\) implies that $dG(e_2) \times dG(e_3) = 0$; writing
\[
dG(e_2) = \frac{1 + \sigma}{2} \xi_+ + \frac{1 - \sigma}{2} \xi_- \quad \text{and} \quad dG(e_3) = \frac{1 + \sigma}{2} \xi'_+ + \frac{1 - \sigma}{2} \xi'_-
\]
with $\xi_+, \xi'_+, \xi_-, \xi'_- \in \Im H$, this implies that $\xi_+, \xi'_+$ and $\xi_-, \xi'_-$ are linearly dependent (Lemma \([4.1]\) and its proof). We note that $\xi_+ = \xi'_+ = 0$ or $\xi_- = \xi'_- = 0$ are not possible since it would contradict that $dG$ is one-to-one. We deduce that $dG(T_{x_0,M})$ contains an element invertible in $H^A$; assuming that $dG(e_2)$ and $dG(e_3)$ are not invertible, if $\xi_+ \neq 0$ we get $\xi_- = 0$ and thus $\xi'_- \neq 0$, which in turn implies $\xi'_+ = 0$; thus
\[
\lambda dG(e_2) + (1 - \lambda) dG(e_3) = \frac{1 + \sigma}{2} \lambda \xi_+ + \frac{1 - \sigma}{2} (1 - \lambda) \xi'_-
\]
yields an invertible element in $H^A$ for $\lambda \neq 0, 1$ (see Lemma \([4.1]\) in the appendix).

We denote by $u \in T_{x_0,M}$ a vector such that $dG(u)$ is invertible in $H^A$. If $v$ is another vector belonging to $T_{x_0,M}$, \([24]\) implies that $dG(u) \times dG(v) = 0$ and thus that
\[
dG(v) = \lambda dG(u) \tag{27}
\]
for some $\lambda \in A$ (Lemma \([4.1]\)). Thus $dG(T_{x_0,M})$ belongs to \(\{a U', a \in A\}\), where $U' := dG(u) \in H^A$ is invertible. Finally, considering $\mu \in A$ such that $\mu^2 = H(U', U')$ (Lemma \([A.2]\)) and setting $U := \mu^{-1} U'$, we get $H(U, U) = 1$ and $dG(T_{x_0,M}) \subset \{a U, a \in A\}$. This is an equality since $dG$ is one-to-one.

Conversely, if $dG(T_{x_0,M})$ is some $A$-line in $T_{G(x_0)}Q$, we obviously get $G^* \omega_Q = 0$ at $x_0$, and \([24]\) implies that $K = K_N = 0$.

**Remark 4.3.** As a corollary of the proposition, we easily obtain the Gauss-Bonnet and the Whitney formulas: integrating \([24]\), we get
\[
\int_M G^* \omega_Q = \int_M (K + \sigma K_N) \omega_M. \tag{28}
\]
Writing $\omega_Q = \frac{1 + \sigma}{2} \omega_1 + \frac{1 - \sigma}{2} \omega_2$ and
\[
G = \frac{1 + \sigma}{2} G_1 + \frac{1 - \sigma}{2} G_2,
\]
we easily get
\[
G^* \omega_Q = \frac{1 + \sigma}{2} G_1^* \omega_1 + \frac{1 - \sigma}{2} G_2^* \omega_2.
\]
Since the real forms $\omega_1, \omega_2$ are in fact the usual area forms on each one of the two factors $S^2$ of the splitting \([24]\), we get $\int_M G_1^* \omega_1 = 4\pi d_1$ and $\int_M G_2^* \omega_2 = 4\pi d_2$ where $d_1$ and $d_2$ are the degrees of $G_1 : M \to S^2$ and $G_2 : M \to S^2$ respectively, and formula \([28]\) thus yields
\[
\int_M (K + \sigma K_N) \omega_M = \frac{\pi}{2} \left( \frac{1 + \sigma}{2} d_1 + \frac{1 - \sigma}{2} d_2 \right),
\]
which gives
\[
\int_M K \omega_M = 2\pi(d_1 + d_2) \quad \text{and} \quad \int_M K_N \omega_M = 2\pi(d_1 - d_2).
\]
See \cite{10} and \cite{17} for other proofs and further consequences of these formulas.

We finally give the expression of the Gauss map when the immersion is given by a spinor field $\varphi \in \Gamma(\Sigma)$, as in Theorem \[1\].

**Lemma 4.4.** The Gauss map of the immersion defined by $\varphi$ is given by

$$ G = g^{-1}Ig $$

(29)

where $g = [\varphi]$ in some local section of $\tilde{Q}$. In this formula

$$ G : M \to S^3_\mathbb{A} \subset S^3 \mathbb{H} $$

and

$$ g : M \to S^3_\mathbb{A} \subset S^3 \mathbb{H} $$

are viewed as maps with values in the quaternions $\mathbb{H}$; see \cite{21} and \cite{2}.

**Proof.** We assume that $\varphi = [\tilde{s}, g \in \Sigma \simeq \tilde{Q} \times S^3 \mathbb{H}$ (see Section \[2\]), and we denote by $(e_2, e_3)$ the positively oriented and orthonormal basis tangent to the immersion which is associated to $\tilde{s}$. We first note that $G = \langle e_2 \cdot e_3 \cdot \varphi, \varphi \rangle$. Indeed,

$$ v_2 := \langle e_2 \cdot \varphi, \varphi \rangle \quad \text{and} \quad v_3 := \langle e_3 \cdot \varphi, \varphi \rangle $$

form a positive and orthonormal basis tangent to the immersion (see Theorem \[1\]),

and the Gauss map $G = v_2 \wedge v_3$ identifies to

$$ G = v_2 v_3 = [\varphi][e_2][e_3] \simeq (x, [\varphi_x]^{-1} g^{-1} \varphi_x) $$

(30)

(identification of $\Lambda^2 \mathbb{R}^4$ with the elements of order 2 of $Cl_o(\mathbb{R}^4) \subset \mathbb{H}^4$ (2), as in the beginning of the section). Since $v_2 = [\tilde{s}, e_2]$, we get

$$ G = \overline{\varphi} I \overline{\varphi} \varphi $$

(31)

since $[\varphi] = g$ and $[e_2 \cdot e_3] = I$ in $\tilde{s}$; this gives (29) since $\overline{\varphi} \varphi = g^{-1}$ ($g \in Spin(4)$). \[\square\]

**Remark 4.5.** The lemma implies that the diagram

$$ \begin{array}{ccc}
\Sigma E \otimes \Sigma M & \xrightarrow{\varphi} & \Sigma E \otimes \Sigma M \\
p \downarrow & & p \downarrow \\
M & \xrightarrow{G} & M \times Q
\end{array} $$

is commutative, where the projection $p$ is defined by

$$ p : \Sigma E \otimes \Sigma M \to M \times Q, \\
\varphi_x \mapsto (x, [\varphi_x]^{-1} I [\varphi_x]) $$

(here $\varphi_x$ belongs to the fibre above the point $x \in M$, and $[\varphi_x]$ denotes its component in some frame belonging to $\tilde{Q}_x$), and where the horizontal map is $x \mapsto (x, G(x))$. Thus $\varphi$ is a lift of the Gauss map (in fact a horizontal lift, if we consider the connection $\nabla \varphi - \eta \cdot \varphi$ with $\eta := -1/2 \sum_{j=2,3} e_j \cdot B(\cdot, e_j)$). Theorem \[4\] thus roughly says that some lift of the Gauss map permits to write the immersion intrinsically.

We will get a nicer picture in the case of a flat immersion with flat normal bundle; see Lemma \[6\] and Proposition \[7\] below.
5. Lorentz Surfaces and Lorentz Numbers

In this section we present elementary results concerning Lorentz surfaces and Lorentz numbers; we refer to [4, 12] for an exposition of the theory in the framework of para-complex geometry.

We will say that a surface \( M \) is a Lorentz surface if there is a covering by open subsets \( M = \cup_{\alpha \in S} U_{\alpha} \) and charts \( \varphi_{\alpha} : U_{\alpha} \rightarrow A, \quad \alpha \in S \) such that the transition functions
\[
\varphi_{\beta} \circ \varphi_{\alpha}^{-1} : \varphi_{\alpha}(U_{\alpha} \cap U_{\beta}) \subset A \rightarrow \varphi_{\beta}(U_{\alpha} \cap U_{\beta}) \subset A, \quad \alpha, \beta \in S
\]
are conformal maps in the following sense: for all \( a \in \varphi_{\alpha}(U_{\alpha} \cap U_{\beta}) \) and \( h \in A \),
\[
d(\varphi_{\beta} \circ \varphi_{\alpha}^{-1})_a (\sigma h) = \sigma d(\varphi_{\beta} \circ \varphi_{\alpha}^{-1})_a (h).
\]
A Lorentz structure is also equivalent to a smooth family of maps
\[
\sigma_x : T_x M \rightarrow T_x M, \quad \text{with} \quad \sigma_x^2 = id_{T_x M}, \quad \sigma_x \neq \pm id_{T_x M}.
\]
This definition coincide with the definition of a Lorentz surface given in [19]: a Lorentz structure is equivalent to a conformal class of Lorentz metrics on the surface, that is to a smooth family of cones in every tangent space of the surface, with distinguished lines. Indeed, the cone at \( x \in M \) is
\[
\text{Ker} (\sigma_x - id_{T_x M}) \cup \text{Ker} (\sigma_x + id_{T_x M})
\]
where the sign of the eigenvalues \( \pm 1 \) permits to distinguish one of the lines from the other.

If \( M \) is moreover oriented, we will say that the Lorentz structure is compatible with the orientation of \( M \) if the charts \( \varphi_{\alpha} : U_{\alpha} \rightarrow A, \quad \alpha \in S \) preserve the orientations (the positive orientation in \( A = \{ x + \sigma y, \ x, y \in \mathbb{R} \} \) is naturally given by \( (\partial_x, \partial_y) \)). In that case, the transition functions are conformal maps \( A \rightarrow A \) preserving orientation.

If \( M \) is a Lorentz surface, a smooth map \( \psi : M \rightarrow A \), or \( A^n \), or a Lorentz surface) will be said to be a conformal map if \( d\psi \) preserves the Lorentz structures, that is if
\[
d\psi_x(\sigma_x u) = \sigma_{\psi(x)} (d\psi_x(u))
\]
for all \( x \in M \) and \( u \in T_x M \). In a chart \( A = \{ x + \sigma y, \ x, y \in \mathbb{R} \} \), a conformal map satisfies
\[
\frac{\partial \psi}{\partial y} = \sigma \frac{\partial \psi}{\partial x} \tag{32}
\]
Defining the coordinates \((s, t)\) such that
\[
x + \sigma y = 1 + \frac{\sigma}{2} s + 1 - \frac{\sigma}{2} t \tag{33}
\]
\((s \text{ and } t \text{ are parameters along the distinguished lines})\) and writing
\[
\psi = 1 + \frac{\sigma}{2} \psi_1 + 1 - \frac{\sigma}{2} \psi_2
\]
with \( \psi_1, \psi_2 \in \mathbb{R} \), \( \psi_1 \) reads
\[
\partial_t \psi_1 = \partial_s \psi_2 = 0,
\]
and we get
\[ \psi_1 = \psi_1(s) \quad \text{and} \quad \psi_2 = \psi_2(t). \]
A conformal map is thus equivalent to two functions of one variable. We finally note that if \( \psi : M \to A^n \) is a conformal map, we have, in a chart \( a : \mathcal{U} \subset A \to M \),
\[ d\psi = \psi' da, \]
where \( da = dx + \sigma dy \) and \( \psi' \) belongs to \( A^n \); this is a direct consequence of (32).

As a consequence of Proposition 4.2, if \( K = K_N = 0 \) and if \( G : M \to Q \) is a regular map (i.e. if \( dG_x \) is injective at every point \( x \) of \( M \)), there is a unique Lorentz structure \( \sigma \) on \( M \) such that
\[ dG_x (\sigma u) = \sigma dG_x (u) \quad (34) \]
for all \( x \in M \) and all \( u \in T_x M \). This is because
\[ dG_x (T_x M) = \{ a U, a \in A \} \]
for some \( U \in T_{G(x)} Q \subset H^4 \) in that case, which implies that \( dG_x (T_x M) \) is stable by multiplication by \( \sigma \). This Lorentz structure is the Lorentz structure introduced in [5] .

6. LOCAL DESCRIPTION OF FLAT SURFACES WITH FLAT NORMAL BUNDLE IN \( \mathbb{R}^4 \)

In this section we suppose that \( M \) is simply connected and that the bundles \( TM \) and \( E \) are flat \((K = K_N = 0)\). We recall that the bundle \( \Sigma := \Sigma E \otimes \Sigma M \) is associated to the principal bundle \( \tilde{Q} \) and to the representation \( \rho \) of the structure group \( Spin(2) \times Spin(2) \) in \( H^4 \) given by (10). Since the curvatures \( K \) and \( K_N \) are zero, the spinorial connection on the bundle \( \tilde{Q} \) is flat, and \( \tilde{Q} \) admits a parallel local section \( \tilde{s} \); since \( M \) is simply connected, the section \( \tilde{s} \) is in fact globally defined.

We consider \( \varphi \in \Gamma (\Sigma) \) a solution of
\[ D \varphi = \tilde{\bar{H}} \cdot \varphi \quad (35) \]
such that \( H(\varphi, \varphi) = 1 \), and \( g = [\varphi] : M \to Spin(4) \) the coordinates of \( \varphi \) in \( \tilde{s} : \)
\[ \varphi = [\tilde{s}, g] \quad \in \quad \Sigma = \tilde{Q} \times H^4 / \rho. \]
Note that, by Theorem 1 \( \varphi \) also satisfies
\[ \nabla_X \varphi = \eta(X) \cdot \varphi \]
for all \( X \in TM \), where \( \eta(X) = -1/2 \sum_{j=2,3} e_j \cdot B(X, e_j) \) for some bilinear map \( B : TM \times TM \to E \).

In the following, we will denote by \((e_0, e_1)\) and \((e_2, e_3)\) the parallel, orthonormal and positively oriented frames, respectively normal, and tangent to \( M \), corresponding to \( \tilde{s} \) (i.e. such that \( \pi(\tilde{s}) = (e_0, e_1, e_2, e_3) \) where \( \pi : \tilde{Q} \to Q_E \times Q_M \) is the natural projection).

We moreover assume that the Gauss map \( G \) of the immersion defined by \( \varphi \) is regular, and consider the Lorentz structure \( \sigma \) induced on \( M \) by \( G \), defined by (34).
A spinor description of flat surfaces in $\mathbb{R}^4$

We now show that $g$ is in fact a conformal map admitting a special parametrization, its arc length, and that, in such a special parametrization, $g$ depends on a single conformal map $\psi : U \subset \mathcal{A} \to \mathcal{A}$. To state the theorem, we define
\[ \mathbb{G} := \{ a \mapsto \pm a + b, \ b \in \mathcal{A} \}, \]
subgroup of transformations of $\mathcal{A}$.

**Theorem 2.** Under the hypotheses above, we have the following:

1. The map $g : M \to S^3_\mathcal{A} \subset \mathbb{H}^4_\mathcal{A}$ is a conformal map, and, at each point of $M$, there is a local chart $a : U \subset \mathcal{A} \to M$, unique up to the action of $\mathbb{G}$, which is compatible with the orientation of $M$ and such that $g : U \subset \mathcal{A} \to S^3_\mathcal{A}$ satisfies
\[ H(g', g') \equiv 1, \]
where $g' : U \subset \mathcal{A} \to \mathbb{H}^4_\mathcal{A}$ is such that $dg = g' da$.

2. There exists a conformal map $\psi : U \subset \mathcal{A} \to \mathcal{A}$ such that
\[ g'g^{-1} = \cos \psi \ J + \sin \psi \ K, \tag{36} \]
where $a : U \subset \mathcal{A} \to M$ is a chart defined in 1-.

**Remark 6.1.** The local chart $a$ may be interpreted as the arc length of $g$, and the function $\psi'$ as the geodesic curvature of $g : U \subset \mathcal{A} \to S^3_\mathcal{A}$: indeed (36) expresses that $\psi$ is the angle of the derivative of the "curve" $g$ in the trivialization $TS^3_\mathcal{A} \simeq S^3_\mathcal{A} \times \mathcal{A}^3$ with respect to the fixed basis $J, K$.

For the proof of the theorem, we will need the following lemmas:

**Lemma 6.2.** Denoting by $[\eta] \in \Omega^1(M, \mathbb{H}^4_\mathcal{A})$ the 1-form which represents $\eta$ in $\tilde{s}$, we have
\[ [\eta] = dg \ g^{-1} = \eta_1 J + \eta_2 K, \tag{37} \]
where $\eta_1$ and $\eta_2$ are 1-forms on $M$ with values in $\mathcal{A}$.

**Proof.** Since $\varphi = [\tilde{s}, g]$ and $\nabla \varphi = [\tilde{s}, dg]$ ($\tilde{s}$ is a parallel section of $\tilde{Q}$), the equation $\nabla \varphi = \eta \cdot \varphi$ reads $[\eta] = dg \ g^{-1}$, where $[\eta] \in \Omega^1(M, \mathbb{H}^4_\mathcal{A})$ represents $\eta$ in $\tilde{s}$. Since $\eta(X) = -\frac{1}{2} \sum_{j=2,3} \langle e_j, B(X, e_j) \rangle$, we have
\[ [\eta(X)] = -\frac{1}{2} \sum_{j=2,3} [e_j][B(\tilde{X}, e_j)] \]
with $[e_2] = J, [e_3] = K$ and $[B(X, e_j)] \in \mathbb{R} \sigma 1 \oplus \mathbb{R} I$, $j = 2, 3$ (since $B(X, e_j)$ is normal to the surface (see the end of Section 2)), and (37) follows. \hfill \Box

**Lemma 6.3.** The form $\tilde{\eta} = \langle (\eta \cdot \varphi), \varphi \rangle \in \Omega^1(M, \Im \mathbb{H}^4_\mathcal{A})$ is given by
\[ \tilde{\eta} = \frac{1}{2} G^{-1} dG = g^{-1} dg. \tag{38} \]

**Proof of Lemma 6.3.** Recalling Lemma 4.4, we have
\[ G = g^{-1} I g = \overline{\varphi} I g. \tag{39} \]
Differentiating this identity, we get
\[ dG = d\overline{\varphi} I g + \overline{\varphi} I dg. \]
Since \( G^{-1} = \overline{G} = -G \) (\( G \) belongs to \( S^3_A = S^3_A \cap \Im m \mathbb{H}^4 \)), we get
\[
G^{-1}dG = -\overline{g}Ig dgIg - \overline{g}Ig dg.
\] (40)

The second term is \( \overline{g}dg \). To analyze the first term, we note that \( gdg = \overline{dg}g \) is a linear combination of \( J \) and \( K \) (Lemma 6.2) and thus anticommutes with \( I \); the first term in (40) is thus
\[
\frac{1}{2}G^{-1}dG = \overline{g}dg = g^{-1}dg.
\]
which gives the second identity in (38). Finally, this is the form \( \tilde{\eta} \) since
\[
\tilde{\eta} = \langle \eta \cdot \varphi, \varphi \rangle = g^{-1} \eta g \quad \text{(41)}
\]
with \( \eta = dg g^{-1} \) (Lemma 6.2). □

Formula (29) in Lemma 4.4 together with the special form of (37) may be rewritten as follows:

**Lemma 6.4.** Consider the projection
\[
p: \text{Spin}(4) \subset \mathbb{H}^4 \rightarrow Q \subset \Im m \mathbb{H}^4
\]
\[
g \mapsto g^{-1}Ig
\]
as a \( S^1_A \) principal bundle, where the action of \( S^1_A \) on \( \text{Spin}(4) \) is given by the multiplication on the left. This fibration is formally analogous to the classical Hopf fibration \( S^3 \subset \mathbb{H} \rightarrow S^2 \subset \Im m \mathbb{H}, g \mapsto g^{-1}Ig \). It is equipped with the horizontal distribution given at every \( g \in \text{Spin}(4) \) by
\[
\mathcal{H}_g := d(R_{g^{-1}})_{g^{-1}}(AJ \oplus AK) \subset T_g \text{Spin}(4)
\]
where \( R_{g^{-1}} \) stands for the right-multiplication by \( g^{-1} \) on \( \text{Spin}(4) \). The distribution \( (\mathcal{H}_g)_{g \in \text{Spin}(4)} \) is \( H \)-orthogonal to the fibers of \( p \), and, for all \( g \in \text{Spin}(4), dp_g : \mathcal{H}_g \rightarrow T_{p(g)}Q \) is an isomorphism which preserves \( \sigma \) and such that
\[
H(dp_g(u), dp_g(v)) = 4H(u, v)
\]
for all \( u \in \mathcal{H}_g \). With these notations, we have
\[
G = p \circ g,
\] (42)
and the map \( g : M \rightarrow \text{Spin}(4) \) appears to be a horizontal lift to \( \text{Spin}(4) \) of the Gauss map \( G : M \rightarrow Q \) (formulas (29) and (37)).

**Proof of Theorem 2.** Let \( a : U \subset A \rightarrow M \) be a chart given by the Lorentz structure induced by \( G \) and compatible with the orientation of \( M \). By Lemma 6.4, \( g : U \subset A \rightarrow S^3_A \) is a conformal map (since so are \( G \) and \( p \) in (32)). We consider \( g' \) such that \( dg = g' da \). If \( \mu : A \rightarrow A \) is a conformal map, we have \( H((g \circ \mu)', (g \circ \mu)') = \mu^2 H(g', g') \). We observe that we may find \( \mu \), unique up to the action of \( G \), such that \( \mu^2 H(g', g') \equiv 1 \). To this end, we first note the following

**Lemma 6.5.** The map \( g : M \rightarrow S^3_A \subset \mathbb{H}^4 \) is an immersion and \( g' \) is thus invertible in \( \mathbb{H}^4 \).
Proof. By (38), $g$ is an immersion since so is $G$. Assume by contradiction that $g'$ is not invertible in $\mathbb{H}^4$; $g'$ would belong to $\mathbb{H}^4 \cup H^4$ (Lemma A.1), and we would have (since $\partial_y = \sigma \partial_x$ and $g$ is conformal)

$$dg(h_g) = dg(\sigma \partial_x) = \sigma dg(\partial_x) = \sigma g' = \pm g' = \pm dg(\partial_x),$$

which contradicts that $dg$ is injective. □

Thus $H(g', g')$ is invertible in $A$, and its inverse is of the form $H(\xi, \xi)$ for some $\xi$ invertible in $\mathbb{H}^4$. Lemma A.2 thus gives $\mu'$, invertible in $A$, such that $\mu'^2 = H(g', g')^{-1}$. There are four solutions, $\pm \mu', \pm \sigma \mu'$, but only two of them define after integration a conformal map $\mu : A \to A$ preserving orientation. We then obtain $\mu$ by integration, unique up to the action of the group $G$. We finally note that $\mu$ is a diffeomorphism ($\mu'$ is invertible in $A$, which implies that $dg$ is an isomorphism), and, considering $g \circ \mu$ instead of $g$, we may thus assume that $H(g', g') = 1$, as claimed in the theorem. Writing

$$g = \frac{1 + \sigma}{2} g_1 + \frac{1 - \sigma}{2} g_2,$$

with $g_1 = g_1(s) \in \mathbb{H}$ and $g_2 = g_2(t) \in \mathbb{H}$ ($g$ is a conformal map), we get

$$g'g^{-1} = \frac{1 + \sigma}{2} g_1'g_1^{-1} + \frac{1 - \sigma}{2} g_2'g_2^{-1},$$

with $H(g_1'g_1^{-1}, g_1'g_1^{-1}) = H(g_2'g_2^{-1}, g_2'g_2^{-1}) = 1$. Since $g_1'g_1^{-1}$ and $g_2'g_2^{-1}$ belong to $\mathbb{R}J \oplus \mathbb{R}K$, we deduce that

$$g_1'g_1^{-1} = \cos \psi_1 J + \sin \psi_1 K \quad \text{and} \quad g_2'g_2^{-1} = \cos \psi_2 J + \sin \psi_2 K$$

(43)

for $\psi_1 = \psi_1(s)$ and $\psi_2 = \psi_2(t) \in \mathbb{R}$. The function

$$\psi = \frac{1 + \sigma}{2} \psi_1(s) + \frac{1 - \sigma}{2} \psi_2(t)$$

(44)

satisfies (36). □

Remark 6.6. Similarly to Remark 6.1, the functions $\psi_1'$ and $\psi_2'$, where $\psi_1$ and $\psi_2$ are defined by (44), may be interpreted as the geodesic curvatures of $g_1 : (s_1, s_2) \to S^3$ and $g_2 : (t_1, t_2) \to S^3$.

The aim is now to study the metric of the surface in the special chart $a = x + \sigma y$ adapted to $g$, given by Theorem 2. We recall that $(e_0, e_1)$ and $(e_2, e_3)$ are the parallel, orthonormal and positively oriented frames, respectively normal, and tangent to $M$, corresponding to $s$. Let us write

$$\tilde{H} = \kappa_0 e_0 + \kappa_1 e_1.$$

We suppose that $\psi : \mathcal{U} \subset A \to A$ is the conformal map defined above, and we write

$$\psi = \theta_1 + \sigma \theta_2$$

with $\theta_1$ and $\theta_2 \in \mathbb{R}$.

Lemma 6.7. We have

$$\begin{cases}
e_2 = \sin \theta_1 \frac{1}{\mu} \partial_x + \cos \theta_1 \frac{1}{\mu} \partial_y \\
e_3 = -\cos \theta_1 \frac{1}{\mu} \partial_x + \sin \theta_1 \frac{1}{\mu} \partial_y
\end{cases}$$

(45)
where $\lambda, \mu \in \mathbb{R}^*$ satisfy
\[
\begin{pmatrix}
\frac{1}{\mu} \\
\frac{1}{\lambda}
\end{pmatrix} = \begin{pmatrix}
\cos \theta_2 & \sin \theta_2 \\
-\sin \theta_2 & \cos \theta_2
\end{pmatrix} \begin{pmatrix}
h_0 \\
h_1
\end{pmatrix},
\]
(46)
Moreover, we have $\lambda \mu > 0$.

Proof. In the chart $a : U \subset A \rightarrow M$ introduced above, $e_2, e_3$ are represented by two functions $e_2, e_3 : U \subset A \rightarrow A$. In $\mathfrak{s}$, the Dirac equation (35) reads
\[
[e_2] \left[ \nabla_{e_2} \varphi \right] + [e_3] \left[ \nabla_{e_3} \varphi \right] = \left[ \mathcal{H} \right] \left[ \varphi \right],
\]
that is
\[
J \, dg(e_2) + K \, dg(e_3) = (-\sigma h_0 I + h_1 I) \, g;
\]
since $dg(e_2)g^{-1} = g'g^{-1}e_2$ and $dg(e_3)g^{-1} = g'g^{-1}e_3$, using (36) this may be written
\[
\begin{pmatrix}
\cos \psi & \sin \psi \\
\sin \psi & -\cos \psi
\end{pmatrix} \begin{pmatrix}
\sigma h_0 \\
h_1
\end{pmatrix} = \begin{pmatrix}
e_2 \\
e_3
\end{pmatrix}.
\]
(47)
Setting $c := h_0 \cos \theta_2 + h_1 \sin \theta_2$ and $d := -h_0 \sin \theta_2 + h_1 \cos \theta_2$, (47) reads
\[
e_2 = d \sin \theta_1 + \sigma \cos \theta_1 \quad \text{and} \quad e_3 = -d \cos \theta_1 + \sigma \sin \theta_1.
\]
Since $e_2$ and $e_3$ represent the independent vectors $e_2, e_3$, we have $cd \neq 0$; setting $\mu = 1/c$ and $\lambda = 1/d$, we get (45) and (46). Moreover, $\lambda \mu > 0$ since the bases $(e_2, e_3)$ and $(\partial_x, \partial_y)$ are both positively oriented.

\[\square\]

Proposition 6.8. In the chart $a = x + \sigma y$ of Theorem 2, the metric reads
\[
\lambda^2 \, dx^2 + \mu^2 \, dy^2;
\]
moreover, $\lambda$ and $\mu$ are solutions of the hyperbolic system
\[
\begin{align*}
\partial_x \mu &= \lambda \, \partial_y \theta_2 \\
\partial_y \lambda &= -\mu \, \partial_x \theta_2.
\end{align*}
\]
(49)

Remark 6.9. Note that
\[
\partial_x \theta_2 = \partial_y \theta_1 \quad \text{and} \quad \partial_y \theta_2 = \partial_x \theta_1
\]
(since $\psi = \theta_1 + \sigma \theta_2$ is a conformal map), and thus that (47) is equivalent to
\[
\begin{align*}
\partial_x \mu &= \lambda \, \partial_y \theta_1 \\
\partial_y \lambda &= -\mu \, \partial_x \theta_1.
\end{align*}
\]
(50)

Hyperbolic systems similar to (49) and (50) appear in [9].

Proof. We just write that the basis $(e_2, e_3)$, given in $(\partial_x, \partial_y)$ by (45), is orthonormal and parallel: since the basis $(e_2, e_3)$ is orthonormal, we get $I = P^t \, AP$, where $A$ is the matrix of the metric in $(\partial_x, \partial_y)$, and where
\[
P = \begin{pmatrix}
\frac{1}{\mu} \sin \theta_1 & -\frac{1}{\mu} \cos \theta_1 \\
\frac{1}{\mu} \cos \theta_1 & \frac{1}{\mu} \sin \theta_1
\end{pmatrix}
\]
is the matrix of change of bases, and thus
\[
A = \begin{pmatrix}
\lambda^2 & 0 \\
0 & \mu^2
\end{pmatrix},
\]
which is \[18\]. We then compute the Christoffel symbols using the Christoffel formulas, and easily get

\[
\Gamma^{x}_{xx} = \frac{1}{\lambda} \partial_x \lambda, \quad \Gamma^{x}_{yx} = \frac{1}{\mu} \partial_y \lambda, \quad \Gamma^{y}_{xy} = \frac{1}{\mu} \partial_x \mu, \quad \Gamma^{y}_{yy} = \frac{1}{\mu} \partial_y \mu.
\]

and

\[
\Gamma^{y}_{xx} = -\frac{\lambda}{\mu^2} \partial_y \lambda, \quad \Gamma^{x}_{yy} = -\frac{\mu}{\lambda^2} \partial_x \mu.
\]

Writing that \((e_2, e_3)\), given by \[18\], is parallel with respect to the metric \[18\], we get \[50\], and thus also \[49\].

**Remark 6.10.** (Relation to the second fundamental form). If we consider the orthonormal basis normal to the surface

\[ u_0 = \cos \theta_2 \ e_0 + \sin \theta_2 \ e_1 \quad \text{and} \quad u_1 = -\sin \theta_2 \ e_0 + \cos \theta_2 \ e_1, \tag{51} \]

and the orthonormal basis tangent to the surface

\[ u_2 = \frac{1}{\lambda} \partial_x \quad \text{and} \quad u_3 = \frac{1}{\mu} \partial_y, \]

the second fundamental form \(B\) is given by

\[ B(u_2, u_2) = \frac{2}{\lambda} u_1, \quad B(u_3, u_3) = \frac{2}{\mu} u_0, \quad B(u_2, u_3) = 0. \tag{52} \]

This may be proved by direct computations using the relation

\[ B(X, Y) = X \cdot \eta(Y) - \eta(Y) \cdot X \]

(see \[2\]) together with \[30\], \[37\] and \[44\].

**Remark 6.11.** The special chart in Theorem \[4\] generalizes the asymptotic Tchebychef net of a flat surface in \(S^3\). Indeed, we may easily obtain that the metric of the surface is given by

\[
\frac{\lambda^2 + \mu^2}{4} (ds^2 + dt^2) + \frac{\lambda^2 - \mu^2}{2} ds dt
\]

and the second fundamental form by

\[ B = \frac{1}{2} (\lambda u_1 + \mu u_0) (ds^2 + dt^2) + (\lambda u_1 - \mu u_0) ds dt
\]

in the coordinates \((s, t)\) defined in \[33\]. For a flat surface in \(S^3\) these formulas reduce in fact to

\[ ds^2 + dt^2 - 2 \cos 2\theta_2 \ ds dt \quad \text{and} \quad <B, e_1> = -2 \sin 2\theta_2 \ ds dt
\]

respectively, where, if \(e_0\) stands for the outer unit normal vector of \(S^3\), the vector \(e_1\) is such that \((e_0, e_1)\) is a positively oriented and orthonormal basis normal to \(M\); this is because \(\lambda = -2 \sin \theta_2\) and \(\mu = 2 \cos \theta_2\) in that case (see Remark \[7\] below). These last formulas are characteristic of an asymptotic Tchebychef net \[1\] \[11\].

We finally obtain the local structure of surfaces with \(K = K_N = 0\) and regular Gauss map (see \[5\] \[6\] for the first description, and \[9\] for another representation):
Theorem 3. Let \( \psi : \mathcal{U} \subset A \to A \) be a conformal map, and \( \theta_1, \theta_2 : \mathcal{U} \to \mathbb{R} \) be such that
\[
\psi = \theta_1 + \sigma \theta_2;
\]
suppose that \( \lambda, \mu \) are solutions of \((49)\) such that \( \lambda \mu > 0 \), and define
\[
e_2 = \sin \theta_1 \frac{1}{\lambda} + \sigma \cos \theta_1 \frac{1}{\mu} \quad \text{and} \quad e_3 = -\cos \theta_1 \frac{1}{\lambda} + \sigma \sin \theta_1 \frac{1}{\mu}.
\]
Then, if \( g : \mathcal{U} \to Spin(4) \subset \mathbb{H}^4 \) is a conformal map solving
\[
g'g^{-1} = \cos \psi J + \sin \psi K,
\]
and if we set
\[
\xi := g^{-1}(\omega_2 J + \omega_3 K)\hat{g}
\]
where \( \omega_2, \omega_3 : T\mathcal{U} \to \mathbb{R} \) are the dual forms of \( e_2, e_3 \in \Gamma(T\mathcal{U}) \), the function \( F = \int \xi \)
defines an immersion \( \mathcal{U} \to \mathbb{R}^4 \) with \( K = K_N = 0 \). Reciprocally, the immersions of \( M \) such that \( K = K_N = 0 \) and with regular Gauss map are locally of this form.

Proof. We first prove the direct statement. We consider the metric \( \lambda^2 dx^2 + \mu^2 dy^2 \) on \( \mathcal{U} \); it is such that \( e_2 \simeq e_2, e_3 \simeq e_3 \in \Gamma(T\mathcal{U}) \) defined by \((53)\) form an orthonormal frame of \( T\mathcal{U} \). Since \( (\lambda, \mu) \) is a solution of \((49)\), the frame \( (e_2, e_3) \) is parallel (see Proposition \((6.8)\) and its proof), and the metric is flat. We also consider the trivial bundle \( E := \mathbb{R}^2 \times \mathcal{U} \), with its trivial metric and its trivial connection; the canonical basis \( (e_0, e_1) \) of \( \mathbb{R}^2 \) thus defines orthonormal and parallel sections \( \in \Gamma(E) \). We set \( s := (e_0, e_1, e_2, e_3) \in Q = (SO(2) \times SO(2)) \times \mathcal{U}, \) and \( \hat{s} \in \hat{Q} = (Spin(2) \times Spin(2)) \times \mathcal{U} \) such that \( \pi(\hat{s}) = s \), where \( \pi : \hat{Q} \to Q \) is the natural covering (Section \((1.3)\) ). We then consider \( \varphi \in \Sigma = \hat{Q} \times \mathbb{H}^4 / \rho(\varphi) \) such that \( [\varphi] = g \) in \( \hat{s} \). The form \( \xi = \langle X \cdot \varphi, \varphi \rangle \) is a closed 1-form (since \( \varphi \) is a solution of the Dirac equation \( D\varphi = \bar{H} \cdot \varphi \), where \( \bar{H} = h_{0e_0} + h_{1e_1} \) is defined by \((10)\); see the proof of Lemma \((6.7)\) and Proposition \((3.1)\), and \( F = \int \xi \) is an isometric immersion of \( M \) in \( \mathbb{R}^4 \) whose normal bundle identifies to \( E \). Thus it is a flat immersion in \( \mathbb{R}^4 \), with flat normal bundle.

Reciprocally, if \( F : M \to \mathbb{R}^4 \) is the immersion of a flat surface with flat normal bundle and regular Gauss map, we have
\[
F = \int \xi, \quad \text{with} \quad \xi(X) = \langle X \cdot \varphi, \varphi \rangle,
\]
where \( \varphi \) is the restriction to \( M \) of the constant spinor field \( J \) of \( \mathbb{R}^4 \) (Remark \((3.3)\) ). In a parallel frame \( \hat{s} \), we have \( \varphi = [\hat{s}, g] \), where \( g : M \to Spin(4) \subset \mathbb{H}^4 \) is an horizontal and conformal map (Lemma \((6.3)\) ). In a chart compatible with the Lorentz structure induced by the Gauss map and adapted to \( g \) (Theorem \((2)\), \( \xi \) is of the form \((54)\) where \( (\omega_1, \omega_2) \) is the dual basis of the basis defined by \((53)\) and where in this last expression \( \lambda, \mu \) are solutions of \((19)\) (Proposition \((6.8)\).

As a corollary, we obtain a new proof of the following result \((6)\):

Corollary 1. Locally, a flat surface with flat normal bundle and regular Gauss map depends on 4 functions of one variable.

Proof. We first observe that the function \( \psi \) depends on two functions of one variable: since \( \psi : A \to A \) is a conformal map, writing
\[
\psi = \frac{1 + \sigma}{2} \psi_1 + \frac{1 - \sigma}{2} \psi_2,
\]

we have

\[ \psi_1 = \psi_1(s) \quad \text{and} \quad \psi_2 = \psi_2(t) \]

where the coordinates \((s, t)\) are defined by (33). Now, writing the system (49) in the coordinates \((s, t)\) we get

\[
\partial_s \begin{pmatrix} \lambda \\ \mu \end{pmatrix} = \begin{pmatrix} +1 & 0 \\ 0 & -1 \end{pmatrix} \partial_t \begin{pmatrix} \lambda \\ \mu \end{pmatrix} - \frac{1}{2} \begin{pmatrix} 0 & \psi_1' + \psi_2' \\ \psi_2' - \psi_1' & 0 \end{pmatrix} \begin{pmatrix} \lambda \\ \mu \end{pmatrix}; \tag{55}
\]

we may solve a Cauchy problem for this system: once \(\psi_1\) and \(\psi_2\) are given, a solution of (55) depends on two functions \(\mu(0, t), \lambda(0, t)\) of the variable \(t\). By Theorem 3 the surface depends on \(\psi_1(s), \psi_2(t), \mu(0, t)\) and \(\lambda(0, t)\).

**Remark 6.12.** We may try to solve the hyperbolic system (49), prescribing additionally the conformal class of the metric: setting

\[ \lambda^2 dx^2 + \mu^2 dy^2 = e^{2f} (\lambda_o^2 dx^2 + \mu_o^2 dy^2) \]

where \(\lambda_o\) and \(\mu_o\) are given functions, (49), in the form (33), easily gives

\[ \partial_s f = A \quad \text{and} \quad \partial_t f = B \tag{56} \]

where \(A\) and \(B\) depend on the functions \(\lambda_o, \mu_o, \psi_1, \psi_2\) and their first derivatives. The existence of a solution thus relies on the compatibility condition \(\partial_s A = \partial_t B\), which depends on the Gauss map \(G\) up to its third derivatives (since the second derivatives of \(\psi\) depend on \(g\) up to its third derivatives, and \(g\) is a horizontal lift of \(G\)). If this condition holds, the solution \(f\) is unique, up to a constant: in the given conformal class, the metric is thus unique up to homothety. This is a special case of the general problem concerning the prescription of the Gauss map, whose solvability relies on a condition of order 3 on the prescribed Gauss map; see [10, 17].

### 7. The structure of the flat tori in \(S^3\)

In this section we obtain a spinorial proof of the description of the flat tori in \(S^3\) [3, 14, 15, 11]: a flat torus in \(S^3\) is necessarily a product of two horizontal curves in \(S^3 \subset \mathbb{H}\). We determine in the first subsection the global Lorentz structure induced on a flat torus in \(S^3\), then study the global structure of the map which represents a constant spinor field in a parallel frame adapted to the torus, we then give a representation formula for the flat tori in \(S^3\) and finally prove that they are always products of two horizontal closed curves in \(S^3\); in the last two subsections, we link this description to the classical Kitagawa representation of flat tori in \(S^3\), and also deduce the structure of the Gauss map image.

#### 7.1. The Lorentz structure induced by the Gauss map

We suppose here that \(M\) is an oriented flat torus immersed in \(S^3\); its normal bundle \(E \rightarrow M\) is trivial, and has a natural orientation. We note that the bundles of frames \(Q_E\) and \(Q_M\) are trivial, since there exist globally defined positively oriented and orthonormal frames \((e_0, e_1)\) and \((e_2, e_3)\), which are respectively normal and tangent to \(M\); in the following, we moreover assume that these frames are parallel \((K = K_N = 0)\). We then consider two spin structures

\[ \hat{Q}_E \rightarrow Q_E \quad \text{and} \quad \hat{Q}_M \rightarrow Q_M \]

such that, setting \(\hat{Q} = \hat{Q}_E \times_M \hat{Q}_M\), the spinor bundle \(\Sigma = \Sigma E \otimes \Sigma M\) identifies to the bundle \(\Sigma = \Sigma E \otimes \Sigma M = \hat{Q} \times \mathbb{H}^4 / \rho\) where \(\rho\) is defined by (31). We note that the Gauss map of \(M\) is regular (since \(M\) is in \(S^3\)), and we suppose that \(M\) is endowed with
the Lorentz structure induced by its Gauss map (see Section 5) and compatible with its orientation.

**Proposition 7.1.** Let us denote by \( \pi : \tilde{M} \to M \) the universal covering of \( M \), and consider the Lorentz structure on \( \tilde{M} \) such that \( \pi \) is a conformal map. Then \( \tilde{M} \) is conformal to \( A \), and \( M \) is conformal to \( A/\Gamma \), where \( \Gamma \) is a subgroup of translations of \( A \cong \mathbb{R}^2 \) generated by two elements

\[
\Gamma = \langle \tau_1, \tau_2 \rangle,
\]

where the translations \( \tau_1 = a \mapsto a + (s_1, t_1) \) and \( \tau_2 = a \mapsto a + (s_2, t_2) \) are such that

\[
s_1 \mathbb{Z} \oplus s_2 \mathbb{Z} = S \mathbb{Z} \quad \text{and} \quad t_1 \mathbb{Z} \oplus t_2 \mathbb{Z} = T \mathbb{Z}
\]

for some positive numbers \( S, T \) (the coordinates \( (s, t) \) are defined in (53)). Moreover, \( \text{rank} \; \Gamma = 2 \) since the quotient \( A/\Gamma \) is complete since \( M \) is compact (Prop. 3.4.10 in [16]). We will implicitly do these identifications below.

**Proof.** We first consider a local section \( \tilde{s} \in \Gamma(\tilde{Q}) \) which is a local lift of the parallel frame \( s = (e_0, e_1, e_2, e_3) \in \Gamma(Q) \), and the locally defined map \( g : M \to S^3_A \) such that

\[
\varphi = [\tilde{s}, g] \in \Sigma = \tilde{Q} \times \mathbb{H}^4/\rho
\]

where \( \varphi = I|_M \in \Gamma(\Sigma) \). By Theorem 2, there exists a chart \( \psi : \mathcal{U} \subset A \to M \) (the arc length of \( g \)) compatible with the orientation and such that

\[
g'g^{-1} = \cos \psi \, J + \sin \psi \, K
\]

for some conformal map \( \psi : \mathcal{U} \subset A \to A \); this chart is moreover unique up to the action of \( G = \{ a \mapsto \pm a + b, \ b \in A \} \). We note that in fact this chart does not depend on the choice of the local section \( \tilde{s} \), since, if we choose the other local lift of \( s \), the spinor field \( \varphi \) is represented by \(-g\) instead of \( g \). Using these very special charts, we may consider the Lorentz structure on \( M \) as a \((G, X)\)-structure, with \( X = A \) and \( G = \{ a \mapsto \pm a + b, \ b \in A \} \). We consider its developing map

\[
D : \tilde{M} \to X = A,
\]

its holonomy representation \( h : \gamma \in \pi_1(M) \to g_\gamma \in G \) and its holonomy group \( \Gamma = h(\pi_1(M)) \); from the general theory of the \((G, X)\)-structures (see [16]), there is a map \( \overline{D} : M \to X/\Gamma = A/\Gamma \) such that the diagram

\[
\begin{array}{ccc}
\tilde{M} & \xrightarrow{D} & A \\
\pi \downarrow & & \downarrow \\
M & \xrightarrow{\overline{D}} & A/\Gamma
\end{array}
\]

commutes. Moreover, if \( \tilde{M} \) is endowed with the Lorentz structure induced by \( \pi \), the developing map \( D \) is by construction a conformal map. We note that the \((G, X)\)-structure is complete since \( M \) is compact (Prop. 3.4.10 in [16]). This implies that \( D \) identifies \( \tilde{M} \) to \( A \) and that \( \overline{D} \) identifies \( M \) to \( A/\Gamma \) in the diagram above (p. 142 and Prop. 3.4.5 in [16]). We will implicitly do these identifications below.

The group \( \Gamma \) is commutative and generated by two elements, since so is \( \pi_1(M) \); moreover, \( \text{rank} \; \Gamma = 2 \) since the quotient \( A/\Gamma \) is compact. We then easily deduce that \( \Gamma \) is generated by \( \tau_1 = a \mapsto a + b_1, \tau_2 = a \mapsto a + b_2 \), where \( b_1 = (s_1, t_1) \) and \( b_2 = (s_2, t_2) \) (in the coordinates \( (s, t) \)) are independent vectors of \( A \cong \mathbb{R}^2 \).
We now consider the Gauss map \( G : \mathcal{A}/\Gamma \to S^2_G \) and its lift to the universal covering \( \tilde{G} : \mathcal{A} \to S^2_A \). Since \( \tilde{G} \) is a conformal map, it is of the form
\[
\tilde{G} = \frac{1+\sigma}{2} \tilde{G}_1 + \frac{1-\sigma}{2} \tilde{G}_2
\]
with \( \tilde{G}_1 = \tilde{G}_1(s) \) and \( \tilde{G}_2 = \tilde{G}_2(t) \in S^2 \). Moreover, \( \tilde{G} \) is \( \Gamma \)-invariant and thus satisfies: \( \forall \ a \in \mathcal{A}, \ \forall \ p,q \in \mathbb{Z}, \)
\[
\tilde{G}(a + p(s_1,t_1) + q(s_2,t_2)) = \tilde{G}(a),
\]
that is
\[
\tilde{G}_1(s + ps_1 + qs_2) = \tilde{G}_1(s) \quad \text{and} \quad \tilde{G}_2(t + pt_1 + qt_2) = \tilde{G}_2(t);
\]
thus the subgroups \( s_1\mathbb{Z} \oplus s_2\mathbb{Z} \) and \( t_1\mathbb{Z} \oplus t_2\mathbb{Z} \) are not dense in \( \mathbb{R} \) (since the maps \( \tilde{G}_1 \) and \( \tilde{G}_2 \) are not constant (\( G \) is an immersion, and so is \( \tilde{G} \)) and thus are of the form \([58]\) for some positive numbers \( S \) and \( T \). We now show that the line \( s \) is closed in the quotient \( \mathcal{A}/\Gamma \). We consider \( (p,q) \in \mathbb{Z}^2 \backslash \{(0,0)\} \) such that \( 0 = pt_1 + qt_2 \) (if \( t_2 \neq 0 \), \( [58] \) implies that \( t_1/t_2 \) belongs to \( \mathbb{Q} \). Since \( ps_1 + qs_2 \) belongs to \( S\mathbb{Z} \), there exists \( m \in \mathbb{Z} \backslash \{0\} \) such that \( ps_1 + qs_2 = mS \) (note that \( m \neq 0 \) since \( (s_1,t_1) \) and \( (s_2,t_2) \) are linearly independent). Finally,
\[
m(S,0) = p(s_1,t_1) + q(s_2,t_2)
\]
belongs to \( \Gamma \), and the result follows. Similarly, the line \( t \) is closed in \( \mathcal{A}/\Gamma \). \( \square \)

**Remark 7.2.** The curves \( \tilde{G}_1 \) and \( \tilde{G}_2 \) in \([59]\) are periodic, with period \( S \) and \( T \) respectively (see \([60]\), together with the definition \([58]\)).

**Remark 7.3.** The form of condition \([58]\) exactly means that the lines \( s \) and \( t \) are closed in the quotient \( \mathcal{A}/\Gamma \). Indeed, let us suppose that the line \( s \) is closed in \( \mathcal{A}/\Gamma \), and consider \( (p,q) \in \mathbb{Z}^2 \backslash \{(0,0)\} \) such that
\[
ps_1 + qs_2 = S' \quad \text{and} \quad pt_1 + qt_2 = 0,
\]
for some \( S' \in \mathbb{R} \backslash \{0\} \). We will prove the second condition in \([58]\). We may assume without loss of generality that \( p \) and \( q \) are relatively prime numbers, or equivalently that
\[
\alpha p + \beta q = 1
\]
for some \( \alpha, \beta \in \mathbb{Z} \). Setting \( T = \frac{1}{p} = -\frac{1}{q} \) (if \( p \) or \( q \) is 0, then \( t_1 \) or \( t_2 \) is 0 (by \([61]\)) and the second condition in \([58]\) is trivial), we have \( t_1 = qT \) and \( t_2 = -pT \), and thus \( t_1\mathbb{Z} + t_2\mathbb{Z} \subset T\mathbb{Z} \); moreover,
\[
T = (\alpha p + \beta q)T = -\alpha t_2 + \beta t_1 \in t_1\mathbb{Z} + t_2\mathbb{Z},
\]
and thus \( t_1\mathbb{Z} + t_2\mathbb{Z} = T\mathbb{Z} \), which is the second condition in \([58]\).

7.2. The structure of the map \( g \). Since the bundle \( \tilde{Q} \to M \) introduced at the beginning of the previous section is maybe not trivial, we consider the universal covering \( \pi : M \simeq \mathcal{A} \to M \simeq \mathcal{A}/\Gamma \), and the pull-backs
\[
\pi^* \tilde{Q} \simeq \pi^* \tilde{Q}_E \times_M \pi^* \tilde{Q}_M \quad \text{and} \quad \pi^* \Sigma \simeq \pi^* \tilde{Q} \times \mathbb{H}^A / \rho.
\]
The bundle \( \pi^* \tilde{Q} \to M \) is trivial and admits a global section \( \tilde{s} \) which is a lift of the parallel frame \((e_0,e_1,e_2,e_3)\) in \( \pi^* \tilde{Q}_E \times_M \pi^* \tilde{Q}_M \) (since \( M \) is simply connected). We then define the map \( g : \mathcal{A} \to S^2_A \) such that
\[
\varphi = [\tilde{s},g] \in \pi^* \Sigma \simeq \pi^* \tilde{Q} \times \mathbb{H}^A / \rho
\]
where \( \varphi = \pi^* I_M \in \Gamma(\pi^* \Sigma) \) is the constant spinor field \( I \) of \( \mathbb{R}^4 \), restricted to \( M \) and pulled-back to the universal covering of \( M \). It appears that the map \( g \) is globally an horizontal lift of the Gauss map, with a very simple structure. We consider

\[
G : A/\Gamma \to S^2_A \quad \text{and} \quad \tilde{G} : A \to S^2_A,
\]

the Gauss map of \( M \cong A/\Gamma \), and its lift to the universal covering; they are conformal maps by construction. We also consider the Hopf fibration

\[
p : S^3_A \subset \mathbb{H}^4 \to S^2_A \subset \mathbb{H}^4 \quad \text{and} \quad g \mapsto g^{-1}Ig,
\]

with its natural horizontal distribution (see Lemma 6.4).

**Proposition 7.4.** The function \( g : A \to S^3_A \) is a horizontal lift of \( G \) and \( \tilde{G} : \) the diagram

\[
\begin{array}{ccc}
A & \xrightarrow{g} & S^3_A \\
\downarrow & & \downarrow \\
A/\Gamma & \xrightarrow{g} & S^2_A
\end{array}
\]

commutes. Moreover, there exists a conformal map \( \psi : A \to A \) such that

\[
g'g^{-1} = \cos \psi \, J + \sin \psi \, K.
\]

**Proof.** We have \( \tilde{G} = g^{-1}Ig \) where \( g'g^{-1} \) belongs to \( AJ \oplus AK \) (see Lemmas 4.4 and 6.2). This is the first part of the proposition. Moreover, the existence of \( \psi : A \to A \) may be proved exactly as in the proof of (36) in Theorem 2. \( \square \)

We finally give the structure of \( g \) (recall Proposition 7.1 and the definition (58) of \( S \) and \( T \):

**Proposition 7.5.** We have

\[
g = \frac{1+\sigma}{2} g_1 + \frac{1-\sigma}{2} g_2
\]

with \( g_1 = g_1(s) \) and \( g_2 = g_2(t) \in S^3 \). Moreover, one of the following two situations occurs:

1. \( g_1 \) and \( g_2 \) are periodic curves with period \( S \) and \( T \) respectively;
2. \( g_1 \) and \( g_2 \) satisfy

\[
g_1(s + kS) = (-1)^k g_1(s) \quad \text{and} \quad g_2(t + kT) = (-1)^k g_2(t)
\]

for all \( s, t \in \mathbb{R} \) and \( k \in \mathbb{Z} \); in that case the subgroup \( \Gamma \) is such that

\[
\Gamma \subset \{(mS, nT), \ m, n \in \mathbb{Z}, \ m \equiv n [2]\}.
\]

**Proof.** Identity (62) is a consequence of the fact that \( g \) is a conformal map. Since \( \varphi = [\tilde{s}, g] \) is \( \Gamma \)-invariant (\( \varphi \) is the pull-back of a section of a bundle on \( A/\Gamma \) and since \( \tilde{s} \) is a lift of the \( \Gamma \)-invariant frame \( (e_0, e_1, e_2, e_3) \), we get

\[
g(a + \gamma) = \varepsilon \gamma \, g(a)
\]

for all \( a \in A \) and all \( \gamma \in \Gamma \), where \( \varepsilon : \Gamma \to \{\pm 1\} \). In view of the definition (58) of \( S \) and \( T \), this easily gives

\[
g_1(s + S) = \pm g_1(s) \quad \text{and} \quad g_2(t + T) = \pm g_2(t)
\]
Remark 7.6. that form. ψ the function ϕ and for all $S$ defines a flat torus immersed in $\Gamma$-invariant, we then deduce $kS, k$ (64): $\Gamma$ is exactly the set of elements of the form $(q_0 I + q_1 J + q_2 I + q_3 K)$. Since its Gauss map is represented by $\sigma_0 I + q_1 J + q_2 I + q_3 K$ is in fact $\pi/2$ orientation-preserving, we have

$$\sigma_0 I + q_1 J + q_2 I + q_3 K \subset A$$

which identifies $\frac{1}{2} (q + \bar{q}) + \frac{1}{2} (q - \bar{q}) \in \mathbb{R}^4 \subset \mathbb{H}^4$ to $q \in \mathbb{H}$. Writing that $\xi(e_0)$ is $\Gamma$-invariant, we then deduce

$$\Gamma(s + ps_1 + qs_2) g_2(t + pt_1 + qt_2) = \overline{(s)} g_2(t)$$

for all $s, t \in \mathbb{R}$ and all $p, q \in \mathbb{Z}$, and thus

$$\overline{\Gamma}(s + kS) g_2(t + k'T) = \overline{\Gamma}(s) g_2(t)$$

for all $s, t \in \mathbb{R}$ and all $k, k' \in \mathbb{Z}$ s.t. $kS = ps_1 + qs_2$ and $k'T = pt_1 + qt_2$ for some $p, q \in \mathbb{Z}$. Taking (65) into account, the only possibilities are then the two cases in the statement of the proposition; finally, in the second case, $\Gamma$ necessarily satisfies (61): $\Gamma$ is exactly the set of elements of the form $(kS, k'T)$ where $kS = ps_1 + qs_2$ and $k'T = pt_1 + qt_2$, and (63), together with (63), implies that $k \equiv k'$ [2].

7.3. Spinor representation of the flat tori in $S^3$. We obtain here an explicit description of the flat tori in $S^3$; it follows from the general spinor representation formula of Theorem 11 written in frames adapted to the tori:

**Theorem 4.** Let $\Gamma$ be a subgroup of translations of $A \simeq \mathbb{R}^2$ satisfying the conditions (57) and (58), and $g : A \to S^3_A$ be a conformal map such that
g' = g^{-1} = \cos \psi J + \sin \psi K.

(69)

for some conformal map $\psi : A \to A$, satisfying one of the two conditions in Proposition 7.2. Writing

$$\psi = \theta_1 + \sigma \theta_2,$$

(70)

we moreover assume that $\theta_2$ belongs to $(\pi/2, \pi)$ mod. $\pi$. Then the formula

$$F = \sigma \overline{\psi} \hat{g}$$

(71)

defines a flat torus immersed in $S^3$. Conversely, a flat torus immersed in $S^3$ is of that form.

**Remark 7.6.** By (62), the functions $\cos \psi$ and $\sin \psi$ appear to be $\Gamma$-periodic, and the function $\psi$ induces a map $A/\Gamma \to A/\Gamma_{2\pi}$ where $\Gamma_{2\pi} = 2\pi \mathbb{Z} \oplus \sigma 2\pi \mathbb{Z} \subset A$. Moreover, since $\theta_2 : A \to \mathbb{R}$ induces a map $A/\Gamma \to \mathbb{R}/2\pi \mathbb{Z}$ and is assumed to belong to $(\pi/2, \pi)$ mod. $\pi$, $\theta_2$ is in fact $\Gamma$-periodic.

**Proof.** We first assume that $F : M \to S^3$ is the immersion of a flat torus, and we fix an orientation on $M$. Since its Gauss map $G : M \to \mathbb{Q} \subset \Lambda^2 \mathbb{R}^4$ is regular (as it is for every surface in $S^3$), we may consider the Lorentz structure induced on $M$ by $G$ and compatible with the orientation; we have $M \simeq A/\Gamma$, where $\Gamma$ is a subgroup of translations of $A \simeq \mathbb{R}^2$ which satisfies (57) and (58). We now consider
\( \varphi \in \Gamma(\Sigma) \) the restriction to \( M \) of the constant spinor field \( J \in \mathbb{H}^A \) of \( \mathbb{R}^4 \). As above, we consider the pull-backs of the bundles and sections to the universal covering \( \tilde{M} \simeq A \to M \simeq A/\Gamma \). We fix \( \tilde{s} \), a global section of \( \pi^*\tilde{Q} \) which is a lift of a parallel frame \( (e_0, e_1, e_2, e_3) \), where \( e_0 \) is here the unit outer normal vector of the sphere and \( (e_0, e_1) \) and \( (e_2, e_3) \) are respectively normal and tangent to the torus; we have

\[
F \simeq \langle \langle e_0 \cdot \varphi, \varphi \rangle \rangle,
\]
since, for a surface in \( S^3 \), the position vector \( F \) coincides with the unit outer normal vector of the sphere (see (18)), and thus

\[
F \simeq [\varphi] [e_0] [\varphi] = \sigma \tilde{g} \hat{g}
\]
where \( g : \tilde{M} \simeq A \to S^3_A \subset \mathbb{H}^A \) represents the spinor field \( \varphi \) in \( \tilde{s} \). Easy computations using (69) and (70) then yield

\[
\partial_x F = 2 \sin \theta_2 \cos \theta_2 \quad \text{and} \quad \partial_y F = -2 \sin \theta_2 \cos \theta_2.
\]

Using these formulas, the basis \((\partial_x F, \partial_y F)\) has the orientation of the basis \((dF(e_2), dF(e_3))\) if and only if the determinant

\[
\begin{vmatrix}
\sin \theta_1 & \cos \theta_1 \\
-\cos \theta_1 & \sin \theta_1 \\
\end{vmatrix} = -4 \sin \theta_2 \cos \theta_2.
\]

is positive \((dF(e_2) \simeq \langle \langle e_2 \cdot \varphi, \varphi \rangle \rangle \) is \( g^{-1}J\hat{g} \) and \( dF(e_3) \simeq \langle \langle e_3 \cdot \varphi, \varphi \rangle \rangle \) is \( g^{-1}K\hat{g} \) (see (18)), which reads

\[
\theta_2 \in (\pi/2, \pi) \mod \pi.
\]

We now prove the direct statement: we assume that \( F \) is given by (71); using (72) and (73) we get

\[
\bar{\partial}_x F \bar{\partial}_x F = 4 \sin^2 \theta_2, \quad \bar{\partial}_y F \bar{\partial}_y F = 4 \cos^2 \theta_2
\]

and

\[
\frac{1}{2} \left( \bar{\partial}_x F \bar{\partial}_y F + \bar{\partial}_y F \bar{\partial}_x F \right) = 0.
\]

This implies that \( F \) is an immersion if \( \theta_2 \neq 0 [\pi/2] \) and that

\[
\frac{1}{2 \sin \theta_2} \partial_x F \quad \text{and} \quad -\frac{1}{2 \cos \theta_2} \partial_y F
\]

form an orthonormal basis tangent to the immersion in that case, which is moreover positively oriented if \( \theta_2 \) belongs to \((\pi/2, \pi) \mod \pi \) (on the torus, we choose the orientation induced by \( F \), i.e. such that \((\partial_x F, \partial_y F)\) is positively oriented). Thus the Gauss map is given by

\[
\tilde{G} = \frac{1}{2 \sin \theta_2} \partial_x F \wedge -\frac{1}{2 \cos \theta_2} \partial_y F \simeq -\frac{1}{4 \sin \theta_2 \cos \theta_2} \partial_x F \partial_y \hat{F}
\]

(see (30)), which, by (72) and (73), easily gives \( \tilde{G} = g^{-1}g \). Since \( g \) is a conformal map, so is \( \tilde{G} \), which implies that the immersion is flat with flat normal bundle (see Proposition 4.2). \(\Box\)

**Remark 7.7.** By (77)-(78), the metric of a flat torus in \( S^3 \) is given by

\[
4(\sin^2 \theta_2 \, dx^2 + \cos^2 \theta_2 \, dy^2)
\]

in the special chart \( a = x + \sigma y \).
7.4. **Consequence: the description of the flat tori in $S^3$.** We consider here the Hopf fibration

$$ S^3 \subset \mathbb{H} \to S^2 \subset \mathbb{H} $$

we recall that a unit speed curve $\gamma : I \to S^3$ is said to be horizontal if

$$ \gamma' \gamma^{-1} = \cos \psi J + \sin \psi K $$

for some function $\psi : I \to \mathbb{R}$. As a corollary of Theorem 4, we obtain the following description of the flat tori in $S^3$ [3, 14, 15, 11]:

**Theorem 5.** A flat torus immersed in $S^3$ may be written as a product

$$ F = g_1 g_2 $$

in the quaternions, where $g_1$ and $g_2$ are two horizontal closed curves in $S^3 \subset \mathbb{H}$.

**Proof.** By Theorem 4 the torus is represented by a map $g : A \to S^3$. Writing

$$ g = \frac{1 + \sigma}{2} g_1 + \frac{1 - \sigma}{2} g_2 $$

with $g_1, g_2 \in S^3 \subset \mathbb{H}$, formula (71) gives

$$ F = \frac{\sigma}{2} (\overline{g_1}g_2 + \overline{g_2}g_1) + \frac{1}{2} (\overline{g_1}g_2 - \overline{g_2}g_1) \simeq \overline{g_1}g_2, $$

where we use the identification (67). Since $g$ is a conformal and horizontal map, the maps $g_1$ and $g_2$ are curves

$$ g_1 : \mathbb{R} \to S^3, \quad s \mapsto g_1(s) \quad \text{and} \quad g_2 : \mathbb{R} \to S^3, \quad t \mapsto g_2(t) $$

which are horizontal with respect to the Hopf fibration (77). They are respectively periodic with period $S$ and $T$, or $2S$ and $2T$.

**Remark 7.8.** The curves $g_1$ and $g_2$ in (78) appear to be the components of the constant spinor field $\varphi = 1_M \in \mathbb{H}^4$ of $\mathbb{R}^4$ in a frame adapted to the torus in $S^3$.

**Remark 7.9.** In the Kitagawa representation [11], the curves $g_1$ and $g_2$ are constructed as asymptotic curves in $S^3$; here, they appear as horizontal curves. See Section 7.5 for the precise relation to the Kitagawa representation.

**Remark 7.10.** The converse statement of the theorem is true: (78) defines a flat torus in $S^3$ if it is an immersion, which is guaranteed if $\theta_2 \neq 0 [\pi/2]$ in the statement of Theorem 4 assuming that $\theta_2 \in (\pi/2, \pi)$ mod. $\pi$, and since $\theta_2 = (\psi_1 - \psi_2)/2$ (recall the definition (43) of $\psi_1$ and $\psi_2$), we get equivalently

$$ \psi_1(s) - \psi_2(t) \in (\pi, 2\pi) \mod. 2\pi $$

for all $s, t \in \mathbb{R}$, that is, the functions $\psi_1$ and $\psi_2$ take their values in intervals of length $< \pi$ such that

$$ (2k + 1)\pi < \min \psi_1 - \max \psi_2 \leq \max \psi_1 - \min \psi_2 < 2(k + 1)\pi \quad (79) $$

for some $k \in \mathbb{Z}$. The Kitagawa representation gives a nice interpretation of this condition; see Section 7.5 below.
Remark 7.11. It may be proved that a horizontal curve $\gamma$ in $S^3$, parameterized by arc length and biregular, has constant torsion $= \pm 1$, where the sign depends on the orientation of $S^3$. Moreover, the curve $\gamma$ has the opposite torsion. Thus, if $g_1, g_2$ are biregular curves, they have constant torsion $\pm 1$, and the surface $F = \overline{g}_1 g_2$ is the product of a curve with torsion 1 by a curve with torsion $-1$; this is the form of the flat tori in $S^3$ constructed in [3, 14, 15].

7.5. The Kitagawa representation of the flat tori in $S^3$. We now explain how the previous results are related to the Kitagawa representation of the flat tori in $S^3$ [11]. We assume that $g : A \to S^3_A$ satisfies the hypothesis of Theorem 4: formula (71) (or equivalently (78)) defines a flat torus immersed in $S^3$. We moreover consider $\psi : A \to A$ such that (69) holds, and we set $\psi_1, \psi_2 \in \mathbb{R}$ such that
\[
\psi = 1 + \sigma 2 \psi_1 + 1 - \sigma 2 \psi_2.
\]
We begin with a simple lemma:

Lemma 7.12. There exists $\alpha \in \mathbb{R}$ such that
\[
\sin(\psi_1(s) - \alpha) > 0 \quad \text{and} \quad \sin(\psi_2(t) - \alpha) > 0
\]
for all $s, t \in \mathbb{R}$. In particular we have
\[
\alpha \notin \psi_1(\mathbb{R}) \cup \psi_2(\mathbb{R}) \mod. \pi.
\]

Proof. From (79) we get
\[
\max \psi_2 + (2k + 1)\pi < \min \psi_1 \quad \text{and} \quad \max \psi_1 < \min \psi_2 + 2(k + 1)\pi.
\]
We take $\alpha$ such that
\[
\max \psi_2 + (2k + 1)\pi < \alpha < \min \psi_1.
\]
We have
\[
0 < \psi_1 - \alpha < \max \psi_1 - (\max \psi_2 + (2k + 1)\pi),
\]
and since
\[
\max \psi_1 - (\max \psi_2 + (2k + 1)\pi) < (\min \psi_2 + 2(k + 1)\pi) - (\max \psi_2 + (2k + 1)\pi) \leq \pi,
\]
we get $\sin(\psi_1 - \alpha) > 0$. Similarly, we have
\[
\psi_2 - \alpha \leq \max \psi_2 - \alpha < \max \psi_2 - (\max \psi_2 + (2k + 1)\pi) = -(2k + 1)\pi
\]
and
\[
\psi_2 - \alpha \geq \min \psi_2 - \alpha > \min \psi_2 - \min \psi_1 \geq \min \psi_2 - \max \psi_1 > -2(k + 1)\pi,
\]
and thus $\sin(\psi_2 - \alpha) > 0$. \hfill \Box

For such a number $\alpha$, we set
\[
J_\alpha = \cos \alpha J + \sin \alpha K \in \mathbb{H}
\]
and consider the Hopf fibration
\[
h_\alpha : S^3 \subset \mathbb{H} \to S^2 \subset \Im \mathbb{H}
\]
\[
\quad u \mapsto u^{-1} J_\alpha u.
\]
We then consider the curves
\[
\gamma_1 = h_\alpha(g_1) \quad \text{and} \quad \gamma_2 = h_\alpha(g_2),
\]
where \( g_1 \) and \( g_2 \) are such that
\[
g = \frac{1+\sigma}{2} g_1 + \frac{1-\sigma}{2} g_2.
\]

**Lemma 7.13.** The curves \( \gamma_1, \gamma_2 : S^1 \to S^2 \) are immersions with geodesic curvatures
\[
k_{\gamma_1} = \cotan(\psi_1 - \alpha) \quad \text{and} \quad k_{\gamma_2} = \cotan(\psi_2 - \alpha).
\]
They moreover satisfy
\[
k_{\gamma_1}(S^1) \cap k_{\gamma_2}(S^1) = \emptyset.
\]

We assume here that \( S^2 \) is oriented w.r.t. its inner unit normal vector.

**Proof.** We have
\[
\gamma'_1 = \overline{g_1 J_\alpha g_1} + \overline{g_1 J_\alpha g_1}'
= -\overline{g_1 (cos \psi_1 J + sin \psi_1 K) J_\alpha g_1} + \overline{g_1 J_\alpha (cos \psi_1 J + sin \psi_1 K) g_1}
= 2 \sin(\psi_1 - \alpha) \overline{g_1 J_\alpha g_1},
\]
which is not 0, since \( \alpha \notin \psi_1(\mathbb{R}) \setminus [\pi] \). Thus (and since \( \sin(\psi_1 - \alpha) > 0 \))
\[
\frac{\gamma'_1}{|\gamma'_1|} = \overline{g_1 I_\alpha g_1}.
\]

By an analogous computation, we get
\[
\left( \frac{\gamma'_1}{|\gamma'_1|} \right)' = (\overline{g_1 I_\alpha g_1})' = 2 \cos \psi_1 \overline{g_1 K_\alpha g_1} - 2 \sin \psi_1 \overline{g_1 J_\alpha g_1};
\]
inserting finally
\[
J_\alpha = \cos \alpha J + \sin \alpha K \quad \text{and} \quad K_\alpha = -\sin \alpha J + \cos \alpha K
\]
in the last formula, we get
\[
\left( \frac{\gamma'_1}{|\gamma'_1|} \right)' = 2 \cos(\psi_1 - \alpha) \overline{g_1 K_\alpha g_1} - 2 \sin(\psi_1 - \alpha) \overline{g_1 J_\alpha g_1}.
\]

The last term is normal to \( S^2 \) at the point \( \gamma_1 = \overline{g_1 J_\alpha g_1} \), and the tangent frame \( \overline{g_1 I_\alpha g_1}, \overline{g_1 K_\alpha g_1} \) is positively oriented if \( S^2 \) is oriented by the inner normal vector \( -\overline{g_1 J_\alpha g_1} \). The geodesic curvature of \( \gamma_1 \) in \( S^3 \) is thus given by the coefficient of the first term, divided by \( 2 \sin(\psi_1 - \alpha) \) (the derivative in (82) has to be taken with respect to arc length), which is the first formula in (80). Finally, (81) is a consequence of the two formulas in (80) and of the condition
\[
2\theta_2 = \psi_1 - \psi_2 \in (\pi, 2\pi) \setminus [2\pi]
\]
and of the property
\[
cotan(a) = \cotan(b) \iff a - b = 0 \setminus [\pi]
\]
for all \( a, b \neq 0 \setminus [\pi] \). \( \square \)

We now consider the double covering
\[
p_\alpha : S^3 \to US^2
u \mapsto (u^{-1} J_\alpha u, u^{-1} I u),
\]
where \( US^2 \) denotes the unit tangent bundle of \( S^2 \). If \( c : \mathbb{R} \to S^2 \) is an immersion, a curve \( \hat{c} : \mathbb{R} \to S^3 \) such that \( p \circ \hat{c} = (c, c'/|c'|) \) is said to be an asymptotic lift of \( c \), which means that the curve \( \hat{c} \) belongs to the Hopf cylinder \( h_{\alpha}^{-1}(c) \subset S^3 \), and is
such that $B(\hat{e}', \hat{e}') = 0$, where $B$ is the second fundamental form of $h^{-1}(c)$ in $S^3$; we refer to [11] for more information concerning asymptotic lifts.

We finally note that the curves $g_1$ and $g_2$ are asymptotic lifts of $\gamma_1$ and $\gamma_2$:

**Lemma 7.14.** We have

$$p_\alpha(g_1) = \left( \frac{\gamma'_1}{|\gamma_1|}, \frac{\gamma'_2}{|\gamma_2|} \right) \quad \text{and} \quad p_\alpha(g_2) = \left( \frac{\gamma'_1}{|\gamma_1|}, \frac{\gamma'_2}{|\gamma_2|} \right).$$

**Proof.** We already noticed that

$$\gamma_1 = g_1 J_\alpha g_1 \quad \text{and} \quad \gamma'_1 = g_1 I g_1.$$

$$\Box$$

The results in Theorems 4 and 5 may thus be interpreted as follows: a flat torus immersed in $S^3$ is a product of the form (78) in $S^3$, where $g_1$ and $g_2$ are asymptotic lifts of two curves $\gamma_1$, $\gamma_2$ satisfying (81). This is the Kitagawa representation of the flat tori in $S^3$; see [11], and also [9,18].

**7.6. The Gauss map image of a flat torus in $S^3$.** Since the map $g$ appears to be a lift of the Gauss map (Proposition 7.4), we easily deduce the structure of the Gauss map image of the flat tori in $S^3$ [7,18]:

**Corollary 2.** Let us consider the Gauss map

$$G : \mathcal{A}/\Gamma \to S^2_A \simeq S^2 \times S^2$$

of a flat torus $F : \mathcal{A}/\Gamma \to S^3$. Its image is a product of closed curves $\gamma_1 \times \gamma_2$ whose geodesic curvatures $k_{\gamma_1}$ and $k_{\gamma_2}$ satisfy

$$-\pi < \int_I k_{\gamma_1} - \int_J k_{\gamma_2} < \pi \quad \text{(83)}$$

for all subintervals $I, J$ of $\mathbb{R}$, and

$$\int_{\gamma_1} k_{\gamma_1} = \int_{\gamma_2} k_{\gamma_2} = 0. \quad \text{(84)}$$

**Proof.** By Theorem [11] $F = \sigma\tilde{g}$, where $g : \mathcal{A} \to S^3_A$ is a horizontal and conformal map; moreover, the map $g$ is of the form $g = (g_1, g_2)$ where $s \mapsto g_1(s)$ and $t \mapsto g_2(t)$ are two closed curves, respectively with period $S$ and $T$, or $2S$ and $2T$ (Proposition 7.5). Since $g$ is a lift of the Gauss map $G$, the image of $G$ is also a product of closed curves $\gamma_1 \times \gamma_2$ (precisely, $\gamma_1$ is the projection of $g_1$ by the Hopf fibration (77), $i = 1, 2$). Moreover, writing

$$\psi(a_2) - \psi(a_1) = \int_{(a_1,a_2)} d\psi \quad \text{(85)}$$

for all $a_1, a_2 \in \mathcal{A}$, and $\psi(s,t) = \frac{1+\sigma}{2} \psi_1(s) + \frac{1-\sigma}{2} \psi_2(t)$, we get

$$d\psi = \partial_s \psi ds + \partial_t \psi dt \quad = \frac{1}{2} (\psi'_1 ds + \psi'_2 dt) + \frac{\sigma}{2} (\psi'_1 ds - \psi'_2 dt),$$
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and thus, taking the \( \sigma \)-component of (85),

\[
\theta_2(a_2) - \theta_2(a_1) = \frac{1}{2} \left( \int_{(s_1, s_2)} \psi'_1 \, ds - \int_{(t_1, t_2)} \psi'_2 \, dt \right) = \frac{1}{2} \left( \int_I k_{\gamma_1} - \int_J k_{\gamma_2} \right),
\]

(86)

where \( \gamma_1 : I \to S^2 \) (resp. \( \gamma_2 : J \to S^2 \)) is the projection of \( g_1 : (s_1, s_2) \to S^3 \) (resp. \( g_2 : (t_1, t_2) \to S^3 \)) parameterized by arc length. The last equality is a consequence of the fact that \( \psi'_1 \) and \( \psi'_2 \) are the geodesic curvatures of \( g_1 \) and \( g_2 \) (Remark 6.6) and that the integrals with respect to arc length of the geodesic curvatures of \( g_1 \) and \( g_2 \) and of their projections on \( S^2 \) coincide.

Since \( \theta_2 \) belongs to \( (\pi/2, \pi) \) mod. \( \pi \), \( \theta_2(a_2) - \theta_2(a_1) \) belongs to \( (-\pi/2, \pi/2) \) and (83) follows. Equation (84) is also a consequence of (86) together with the fact that \( \theta_2 \) is \( \Gamma \)-periodic (Remark 7.6).

\[\Box\]

Appendix A. Auxiliary results on Lorentz numbers and quaternions

A.1. Invertible elements in the algebra \( \mathbb{H}^A \). We describe here the invertible elements in the algebra of quaternions with coefficients in \( \mathbb{A} \). We first note that the set of invertible elements of \( \mathbb{A} \) is

\[ \mathbb{A}^* = \mathbb{A} \setminus (1 \pm \sigma)\mathbb{R}. \]  

(87)

Lemma A.1. Let us define

\[ \mathbb{H}^A_+ = \{ \xi \in \mathbb{H}^A : \sigma \xi = \xi \} \quad \text{and} \quad \mathbb{H}^A_- = \{ \xi \in \mathbb{H}^A : \sigma \xi = -\xi \}. \]

The set of invertible elements of \( \mathbb{H}^A \) is

\[ \mathbb{H}^A^* = \mathbb{H}^A \setminus (\mathbb{H}^A_+ \cup \mathbb{H}^A_-). \]

Proof. Let \( \xi = a_0 \mathbbm{1} + a_1 \mathbbm{I} + a_2 \mathbbm{J} + a_3 \mathbbm{K} \in \mathbb{H}^A \), with \( a_i = u_i + \sigma v_i, \, u_i, v_i \in \mathbb{R} \). A straightforward computation yields

\[ H(\xi, \xi) = \overline{\xi} \xi = \sum_{i=0}^{3} (u_i^2 + v_i^2) + 2\sigma \sum_{i=0}^{3} u_i v_i. \]  

(88)

We note that \( \xi \) is invertible in \( \mathbb{H}^A \) if and only if \( H(\xi, \xi) \) is invertible in \( \mathbb{A} \) (the inverse of \( \xi \) is then \( \overline{\xi}/H(\xi, \xi) \)). Thus \( \xi \) is not invertible in \( \mathbb{H}^A \) if and only if

\[ \sum_{i=0}^{3} (u_i^2 + v_i^2) = 2\varepsilon \sum_{i=0}^{3} u_i v_i, \]

with \( \varepsilon = \pm 1 \) (by (87)). This gives \( \sum_i (u_i - \varepsilon v_i)^2 = 0 \), and thus \( \xi \in \mathbb{H}^A_+ \cup \mathbb{H}^A_- \), since \( \mathbb{H}^A_+ \) and \( \mathbb{H}^A_- \) are explicitly given by

\[ \mathbb{H}^A_+ = \{ (1 + \sigma)q, \, q \in \mathbb{H} \} \quad \text{and} \quad \mathbb{H}^A_- = \{ (1 - \sigma)q, \, q \in \mathbb{H} \}. \]
A.2. Square roots in the Lorentz numbers.

Lemma A.2. Let $b \in \mathcal{A}$. There exists $a \in \mathcal{A}$ such that

$$a^2 = b$$

if and only if $b$ belongs to the cone

$$\mathcal{C} = \{ u + \sigma v \in \mathcal{A}, \ u \geq 0, \ v \in \mathbb{R} : -u \leq v \leq u \};$$

moreover, equation (89) has exactly four solutions if $b$ belongs to the interior of $\mathcal{C}$. In particular, if $\xi \in \mathbb{H}^4$ is invertible, then equation

$$a^2 = H(\xi, \xi)$$

has four solutions in $\mathcal{A}$, which are moreover invertible in $\mathcal{A}$.

Proof. Setting $a = x + \sigma y$ and $b = u + \sigma v$, equation (89) reads

$$x^2 + y^2 = u \quad \text{and} \quad 2xy = v,$$

which is solvable if and only if $u \geq 0$ and $-u \leq v \leq u$. If these conditions hold, the solutions are

$$x = \frac{1}{2} \left( \varepsilon_1 \sqrt{u + v} + \varepsilon_2 \sqrt{u - v} \right) \quad \text{and} \quad y = \frac{1}{2} \left( \varepsilon_1 \sqrt{u + v} - \varepsilon_2 \sqrt{u - v} \right),$$

where $\varepsilon_1, \varepsilon_2 = \pm 1$, and the first part of the lemma follows. For the last claim, we note that formula (88) implies that $H(\xi, \xi)$ belongs to the interior of the cone $\mathcal{C}$ whenever it is invertible; the solutions are then moreover clearly invertible in $\mathcal{A}$. \[\square\]

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