THE FOCUSING NLS EQUATION WITH STEP-LIKE OSCILLATING BACKGROUND: SCENARIOS OF LONG-TIME ASYMPTOTICS

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Abstract. We consider the Cauchy problem for the focusing nonlinear Schrödinger equation with initial data approaching two different plane waves
\[ A_j e^{i\phi_j} e^{-2iB_j x}, \quad j = 1, 2 \] as \( x \to \pm\infty \).
Using Riemann–Hilbert techniques and Deift-Zhou steepest descent arguments, we study the long-time asymptotics of the solution. We detect that each of the cases \( B_1 < B_2 \), \( B_1 > B_2 \), and \( B_1 = B_2 \) deserves a separate analysis. Focusing mainly on the first case, the so-called shock case, we show that there is a wide range of possible asymptotic scenarios. We also propose a method for rigorously establishing the existence of certain higher-genus asymptotic sectors.

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1. Introduction

We consider the Cauchy problem for the focusing nonlinear Schrödinger (NLS) equation
\[ iq_t + q_{xx} + 2|q|^2 q = 0, \quad x \in \mathbb{R}, \quad t \geq 0, \]
\[ q(x, 0) = q_0(x), \quad x \in \mathbb{R}, \]
with initial data approaching oscillatory waves at plus and minus infinity:
\[ q_0(x) \sim \begin{cases} A_1 e^{i\phi_1} e^{-2iB_1 x}, & x \to -\infty, \\ A_2 e^{i\phi_2} e^{-2iB_2 x}, & x \to +\infty, \end{cases} \]
where \( \{A_j, B_j, \phi_j\}_1^2 \) are real constants such that \( A_j > 0 \). Our goal is to describe the long-time behavior of the solution \( q(x, t) \) for different choices of the parameters \( \{A_j, B_j, \phi_j\}_1^2 \). The tools we use are Riemann–Hilbert (RH) techniques and Deift-Zhou steepest descent arguments.

In order for the formulation of the Cauchy problem (1.1)-(1.2) to be complete, it has to be supplemented with boundary conditions for \( t > 0 \). These boundary conditions are the natural extensions of (1.2) to \( t > 0 \) and are given by
\[ \int_0^{(-1)^j \infty} |q(x, t) - q_{0j}(x, t)| dx < \infty \quad \text{for all } t \geq 0, \quad j = 1, 2, \]
where \( q_{0j}(x, t), \quad j = 1, 2 \) are the plane wave solutions of the NLS equation satisfying the initial conditions \( q_{0j}(x, 0) = A_j e^{i\phi_j} e^{-2iB_j x} \), that is,
\[ q_{0j}(x, t) = A_j e^{i\phi_j} e^{-2iB_j x + 2i\omega_j t}, \quad \omega_j := A_j^2 - 2B_j^2. \]
The RH formalism, which can be viewed as a version of the inverse scattering transform (IST) method, is well-developed for problems with “zero boundary conditions”, that is, for problems where...
the solution is assumed to decay to 0 as \( x \to \pm \infty \) for each \( t \geq 0 \). In particular, detailed asymptotic formulas can be derived by using the steepest descent method for RH problems introduced by Deift and Zhou [11]. The adaptation of the RH formalism and the Deift-Zhou approach to problems with "nonzero boundary conditions" has been the subject of more recent works.

1.1. Previous work on the focusing NLS with nonzero boundary conditions. The first studies of the focusing NLS equation with nonzero boundary conditions by the IST method were presented in [15,18], where initial profiles satisfying (1.2) with \( A_1 = A_2, \phi_1 = \phi_2, \) and \( B_1 = B_2 = 0 \) were considered. In particular, the Ma soliton [18] (also discovered in [15]) was introduced. It was also mentioned in [18] that a plane wave solution corresponds to a one-band potential in the spectrum of the Zakharov–Shabat scattering equations, whereas the cnoidal wave (elliptic function) and the multicnoidal wave (hyperelliptic function) solutions correspond to two-band and \( N \)-band potentials, respectively. A perturbation theory for the NLS equation with non-vanishing boundary conditions was put forward in [14], where particular attention was paid to the stability of the Ma soliton.

An IST approach for initial data satisfying (1.2) with \( A_1 = A_2, \phi_1, \phi_2 \in \mathbb{R}, \) and \( B_1 = B_2 = 0 \) was presented in [1], and was further developed in [2,3]. In particular, it was shown in [2,3] that for such initial data, the long-time behavior is described by three asymptotic sectors in the \((x,t)\) half-plane \( t > 0 \): two sectors adjacent to the half-axes \( x < 0, t = 0 \) and \( x > 0, t = 0 \) in which the solution asymptotes to modulated plane waves, and a middle sector in which the solution asymptotes to an elliptic (genus 1) modulated wave. An IST formalism for the case of asymmetric nonzero boundary conditions \((A_1 \neq A_2, \phi_1, \phi_2 \in \mathbb{R}, B_1 = B_2 = 0)\) was presented in [12].

In [7], the long-time asymptotics was studied for the symmetric shock case of \( A_1 = A_2, \phi_1 = \phi_2, \) \( B_1 = -B_2 < 0 \). In this case, the asymptotic picture is symmetric under \( x \mapsto -x \). Five asymptotic sectors were described in [7]: a central sector containing the half-axis \( x = 0, t > 0 \) in which the solution \( q(x,t) \) asymptotes to a modulated elliptic (genus 1) wave [7, Theorem 1.2], two contiguous sectors (the transition regions) in which the leading asymptotics is described by modulated hyperelliptic (genus 2) waves [7, Theorem 1.3], and two sectors adjacent to the \( x \)-axis in which \( q(x,t) \) asymptotes to modulated plane (genus 0) waves.

The long-time asymptotics in the case when the left background is zero (i.e., when \( A_1 = 0 \) and \( A_2 \neq 0 \)) was analyzed in [4]. It was shown that the asymptotic picture involves three sectors in this case: a slow decay sector (adjacent to the negative \( x \)-axis), a modulated plane wave sector (adjacent to the positive \( x \)-axis), and a modulated elliptic wave sector (between the first two).

1.2. Summary of results. The main takeaways of the present paper can be summarized as follows: (a) Whereas earlier studies focused on specific choices of the parameters \( A_j, B_j, \) and \( \phi_j \), we introduce a RH approach for the solution of (1.1) with initial data satisfying (1.2) for general values of \( \{A_j, B_j, \phi_j\} \). Since the case \( B_1 = B_2 \) has already been treated in the literature, we sometimes assume \( B_1 \neq B_2 \) for conciseness.

(b) We show that the panorama of asymptotic scenarios arising from (1.1)-(1.2) is surprisingly rich; in fact, we detect several new scenarios even in the symmetric shock case studied in [7]. More precisely, our analysis in Section 5 shows that the scenario presented in [7] is only one of five different possible scenarios in this case. Whereas the long-time behavior along the \( t \)-axis is always described by a genus 1 wave, the asymptotics along the lines \( x/t = c \), for small values of \( c \), can be either a genus 1 (as in [7]), a genus 2, or a genus 3 wave depending on the value of \( A_j/(B_2 - B_1) \). Asymmetric parameter choices give rise to an even wider range of possibilities.

(c) We propose an approach for rigorously establishing the existence of certain higher-genus asymptotic sectors. A sector in which the leading asymptotics of the solution can be expressed in terms of theta functions associated with a genus \( g \) Riemann surface is referred to as a genus \( g \) sector. At a technical level, such sectors arise when the definition of the so-called \( g \)-function involves a Riemann surface [10]. In order for the \( g \)-function to be suitable for the asymptotic analysis, certain parameters appearing in its definition need to satisfy a nonlinear system of equations. The relevant asymptotic sector exists only if this system has a solution. For example, the asymptotic analysis for the genus 2 sector carried out in [7] implicitly assumes that the system of equations [7, Eqs. (3.29)]
has a solution. In Section 6, we establish the existence of this genus 2 sector rigorously. Although we only provide details for this particular genus 2 sector, we expect that our approach can be used to show existence also of other genus g sectors appearing in this paper and elsewhere.

1.3. Organization of the paper. Our analysis is based on a RH formalism which is developed in Section 2. In Sections 3–5, we analyze the long-time behavior of the solution \( q(x, t) \) of (1.1)–(1.2). In Section 3, we show, for any choice of the parameters \( A_j, \phi_j \), and \( B_1 \neq B_2 \), that the leading behavior of \( q \) near the negative and positive halves of the \( x \)-axis is described by the plane waves \( q_{01} \) and \( q_{02} \), respectively.

Away from the \( x \)-axis, the asymptotic analysis turns out to be very different in the two cases \( B_1 > B_2 \) and \( B_1 < B_2 \). Section 4 is devoted to the case \( B_1 > B_2 \), called the rarefaction case. In this case, the asymptotic picture resembles two copies of that found in [4], namely, the solution is slowly decaying near the \( t \)-axis and in two transition sectors the asymptotics has the form of elliptic waves. Section 5 is devoted to the case \( B_1 < B_2 \), called the shock case. Restricting ourselves to the symmetric case of \( A_1 = A_2, \phi_1, \phi_2 \in \mathbb{R} \), and \( B_1 = -B_2 \) (the latter actually being no loss of generality), we describe all the possible asymptotic scenarios that can occur. Finally, in Section 6, we establish the existence of the genus 2 asymptotic sectors featured in [7]. Forthcoming papers will be devoted to a detailed analysis of the asymptotics in a genus 3 sector [5,6].

2. The Riemann–Hilbert formalism

2.1. Notation. As above, we let \( \{A_j, B_j, \phi_j\}_1^2 \) denote real constants such that \( A_j > 0 \). We let \( \Sigma_j = [E_j, E_j] \), where \( E_j := B_j + iA_j \), denote the vertical segment \( \Sigma_j = \{B_j + is \mid |s| \leq A_j\} \) oriented upwards; see Figure 2.1 in the cases \( B_1 < B_2 \) (rarefaction) and \( B_1 > B_2 \) (shock).

We let \( \mathbb{C}^+ = \{\text{Im}k > 0\} \) and \( \mathbb{C}^- = \{\text{Im}k < 0\} \) denote the open upper and lower halves of the complex plane. The Riemann sphere will be denoted by \( \hat{\mathbb{C}} = \mathbb{C} \cup \{\infty\} \). We write \( \ln k \) for the logarithm with the principal branch, that is, \( \ln k = \ln|k| + i\arg k \) where \( \arg k \in (-\pi, \pi] \). Unless specified otherwise, all complex powers will be defined using the principal branch, i.e., \( z^\alpha = e^{\alpha \ln z} \).

We let \( f^*(k) := \overline{f(k)} \) denote the Schwarz conjugate of a function \( f(k) \).

Given an open subset \( D \subset \mathbb{C} \) bounded by a piecewise smooth contour \( \Sigma \), we let \( \hat{E}_2(D) \) denote the Smirnoff class consisting of all functions \( f(k) \) analytic in \( D \) with the property that for each connected component \( D_j \) of \( D \) there exist curves \( \{C_n\}_1^{\infty} \subset D_j \) such that the \( C_n \) eventually surround each compact subset of \( D_j \) and \( \sup_{n \geq 2} ||f|_{L^2(C_n)} < \infty \). All RH problems in the paper are \( 2 \times 2 \) matrix-valued and are formulated in the \( L^2 \)-sense as follows (see [16,17]):

\[
\begin{align*}
\{ & m \in I + \hat{E}^2(\mathbb{C} \setminus \Sigma), \\
& m_+(k) = m_-(k)J(k) \quad \text{for a.e.} \; k \in \Sigma, \\
\}
\end{align*}
\tag{2.1}
\]

where \( m_+ \) and \( m_- \) denote the boundary values of the solution \( m \) from the left and right sides of the contour \( \Sigma \). All contours will be invariant under complex conjugation and the jump matrix \( J \equiv J(k) \) will always satisfy

\[
J = \begin{cases} 
\sigma_2 J^* \sigma_2, & k \in \Sigma \setminus \mathbb{R}, \\
\sigma_2 (J^*)^{-1} \sigma_2, & k \in \Sigma \cap \mathbb{R}, 
\end{cases} \quad \text{where} \quad \sigma_2 := \begin{pmatrix} 0 & -i \\
 i & 0 \end{pmatrix}.
\tag{2.2}
\]

Together with uniqueness of the solution of the RH problem (2.1), this implies the symmetry

\[
m = \sigma_2 m^* \sigma_2, \quad k \in \mathbb{C} \setminus \Sigma.
\tag{2.3}
\]

2.2. Reduction. The study of (1.1)–(1.2) can be reduced to one of the following three cases, depending on whether \( B_1 < B_2, B_1 > B_2 \), or \( B_1 = B_2 \):

(i) \( B_1 = -1, B_2 = 1, \) and \( \phi_2 = 0; \)
(ii) \( B_1 = 1, B_2 = -1, \) and \( \phi_2 = 0; \)
(iii) \( B_1 = B_2 = 0. \)
To see this, note that if $q(x, t)$ satisfies (1.1a), then so does the function $\tilde{q}(x, t)$ defined by
\[
\tilde{q}(x, t) := Aq(x + 4Bt), A^2t)e^{-2iB(x+2Bt)},
\]
for any choice of $A > 0$ and $B \in \mathbb{R}$. If $q_0$ satisfies (1.2), then $\tilde{q}_0$ satisfies
\[
\tilde{q}_0(x) \sim \begin{cases} A_1' e^{i\phi_1} e^{-2iB_1'x}, & x \to -\infty, \\
A_2' e^{i\phi_2} e^{-2iB_2'x}, & x \to +\infty, \end{cases}
\]
where
\[
A_1' = AA_1, \quad A_2' = AA_2, \quad B_1' = B_1A + B, \quad B_2' = B_2A + B.
\]
If $B_2 > B_1$, then, by choosing
\[
A = \frac{2}{B_2 - B_1} > 0, \quad B = \frac{B_1 + B_2}{B_1 - B_2},
\]
we can arrange so that $B_2' = -B_1' = 1$. Similarly, if $B_2 < B_1$, then, by choosing
\[
A = \frac{2}{B_1 - B_2} > 0, \quad B = \frac{B_1 + B_2}{B_2 - B_1},
\]
we can arrange so that $B_1' = -B_2' = 1$. In either of these cases, shifting $x$ by a constant if necessary, we may also assume that $\phi_2 = 0$ (and thus denote $\phi_1 = \phi$). On the other hand, if $B_1 = B_2$, then by choosing $A = 1$ and $B = -B_1 = -B_2$, we can arrange so that $B_1' = B_2' = 0$. Therefore we may, without loss of generality, restrict our attention to solutions whose initial data satisfy one of the following conditions:

- If $B_1 < B_2$, then
\[
q_0(x) \sim \begin{cases} A_1 e^{i\phi} e^{2ix}, & x \to -\infty, \\
A_2 e^{-2ix}, & x \to +\infty. \end{cases} \tag{2.4}
\]

- If $B_1 > B_2$, then
\[
q_0(x) \sim \begin{cases} A_1 e^{i\phi} e^{-2ix}, & x \to -\infty, \\
A_2 e^{2ix}, & x \to +\infty. \end{cases} \tag{2.5}
\]

- If $B_1 = B_2$, then
\[
q_0(x) \sim \begin{cases} A_1 e^{i\phi_1}, & x \to -\infty, \\
A_2 e^{i\phi_2}, & x \to +\infty. \end{cases} \tag{2.6}
\]

However, in what follows we often prefer to keep the setting with arbitrary $B_j$ and $\phi_j$.

2.3. Background solutions. The IST formalism in the form of a RH problem requires that the solution $q(x, t)$ can be represented in terms of the solution of a $2 \times 2$-matrix RH problem whose formulation (jump conditions and possible residue conditions) involves only spectral functions which are defined in terms of the initial data. In the adaptation of the IST to case of “nonzero backgrounds”, the first step is to find a convenient description of the background solutions of the Lax pair equations (see, e.g., [4, Eqs. (1.4)-(1.5)]), i.e., the solutions $\Phi_{0j}(x, t, k)$, $j = 1, 2$ of the equations
\[
\Phi_x(x, t, k) = U(x, t, k)\Phi(x, t, k), \quad \Phi_t(x, t, k) = V(x, t, k)\Phi(x, t, k),\]
with
\[
U = -ik\sigma_3 + \begin{pmatrix} 0 & q \\ -\bar{q} & 0 \end{pmatrix}, \quad V = -2ik^2\sigma_3 + 2k \begin{pmatrix} 0 & q \\ -\bar{q} & 0 \end{pmatrix} + i \left( \frac{|q|^2}{\bar{q}x} - \frac{q_x}{|q|^2} \right), \tag{2.7a,b}
\]
where $\sigma_3 := \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$ and $q(x, t) = q_{0j}(x, t)$ with $q_{0j}$ as in (1.3b). These solutions $\Phi_{0j}(x, t, k)$ of (2.7) will play the role that $e^{(\xi - 2izt + x^2)\tau_3}$ plays in the case of decaying initial data.

In view of the central role of the RH problem in the IST method, it is natural to try to characterize the background solutions in terms of the solutions of appropriate RH problems.
For $j = 1, 2$, we introduce the functions
\[ X_j(k) = \sqrt{(k - E_j)(k - E_j)}, \quad \Omega_j(k) = 2(k + B_j)X_j(k), \quad (2.8) \]
\[ \nu_j(k) = \left(\frac{k - E_j}{k - E_j}\right)^\frac{1}{2}, \quad \mathcal{E}_j(k) = \frac{1}{2} \begin{pmatrix} \nu_j + \nu_j^{-1} & \nu_j - \nu_j^{-1} \\ \nu_j - \nu_j^{-1} & \nu_j + \nu_j^{-1} \end{pmatrix}. \quad (2.9) \]
We choose the branches of the square and fourth roots so that these functions are analytic in $\mathbb{C} \setminus \Sigma_j$ and satisfy the large $k$ asymptotics
\[ X_j(k) = k - B_j + O(k^{-1}), \quad \Omega_j(k) = 2k^2 + \omega_j + O(k^{-1}), \]
\[ \nu_j(k) = 1 + O(k^{-1}), \quad \mathcal{E}_j(k) = I + O(k^{-1}). \]
We denote by $X_j \pm$, $\Omega_j \pm$, $\nu_j \pm$, and $\mathcal{E}_j \pm$ their boundary values from the left and right sides of $\Sigma_j$. Note that $X_j^\pm = X_j$, $\Omega_j^\pm = \Omega_j$, $\nu_j^\pm = \nu_j^{-1}$, and $\mathcal{E}_j^\pm = \sigma_2 \mathcal{E}_j \sigma_2$. The background solutions $\Phi_{0j}(x, t, k)$, $j = 1, 2$ are defined as follows:
\[ \Phi_{0j}(x, t, k) := e^{-iB_jx + i\omega_j t}\sigma_3 N_j(k)e^{-iX_j(k)x - i\Omega_j(k)t}\sigma_3, \quad (2.10a) \]
\[ N_j(k) := e^{i\phi_j} \mathcal{E}_j(k)e^{-i\phi_j}\sigma_3. \quad (2.10b) \]
The functions $N_j$ and $\Phi_{0j}$ are analytic in $k \in \mathbb{C} \setminus \Sigma_j$. They satisfy the relations $\det \Phi_{0j} = \det N_j = \det \mathcal{E}_j \equiv 1$ and the symmetry (2.3). Since $\Sigma_j$ is oriented upwards (see Figure 2.1), $\nu_{j+}(k) = i\nu_{j-}(k)$ for $k \in \Sigma_j$ and thus $N_j$, $j = 1, 2$ satisfies the RH problem
\[ N_j \in I + \hat{E}^2(\mathbb{C} \setminus \Sigma_j), \quad \left\{ \begin{array}{l} N_{j+}(k) = N_{j-}(k) \begin{pmatrix} 0 & i e^{-i\phi_j} \\ i e^{i\phi_j} & 0 \end{pmatrix}, & k \in \Sigma_j. \end{array} \right. \quad (2.11) \]

2.4. Jost solutions and spectral functions. Assuming that $q(x, t)$ satisfies the Cauchy problem defined by (1.1) and (1.3), define the Jost solutions $\Phi_j \equiv \Phi_j(x, t, k)$, $j = 1, 2$ of the Lax pair equations (2.7) by
\[ \Phi_j(x, t, k) := \mu_j(x, t, k)e^{-iX_j(k)x - i\Omega_j(k)t}\sigma_3, \quad (2.12) \]
where $X_j$, $\Omega_j$ are as in (2.8), and $\mu_j$, $j = 1, 2$ solve the Volterra integral equations
\[ \mu_j(x, t, k) = e^{iB_jx + i\omega_j t}\sigma_3 N_j(k) + \int_{(-1)^{j-\infty}}^x \Phi_{0j}(y, t, k)\Phi_{0j}^{-1}(y, t, k)[(Q - Q_{0j})(y, t)]\mu_j(y, t, k)e^{-iX_j(k)(y-x)}\sigma_3 \, dy \quad (2.13) \]
with $N_j$ as in (2.10b) and
\[ Q = \begin{pmatrix} 0 & q \\ -q & 0 \end{pmatrix}, \quad Q_{0j} = \begin{pmatrix} 0 & q_{0j} \\ -q_{0j} & 0 \end{pmatrix}. \]
The symmetry properties of $\Phi_{0j}$, $N_j$, $X_j$, and $\Omega_j$ imply that $\mu_j$ and $\Phi_j$ satisfy the symmetry (2.3). Observe that $\Phi_j$, $j = 1, 2$ solve the Volterra integral equations
\[ \Phi_j(x, t, k) = \Phi_{0j}(x, t, k) + \int_{(-1)^{j-\infty}}^x \Phi_{0j}(y, t, k)\Phi_{0j}^{-1}(y, t, k)[(Q - Q_{0j})(y, t)]\Phi_j(y, t, k) \, dy. \quad (2.14) \]
In what follows $\mu^{(i)}$ denotes the $i$-th column of a matrix $\mu$.

**Proposition 2.1** (analyticity). The column $\mu_1^{(1)}$ is analytic in $\mathbb{C}^+ \setminus \Sigma_1$ with a jump across $\Sigma_1$. The column $\mu_2^{(2)}$ is analytic in $\mathbb{C}^+ \setminus \Sigma_2$ with a jump across $\Sigma_2$. The column $\mu_1^{(2)}$ is analytic in $\mathbb{C}^- \setminus \Sigma_1$ with a jump across $\Sigma_1$. The column $\mu_2^{(1)}$ is analytic in $\mathbb{C}^- \setminus \Sigma_2$ with a jump across $\Sigma_2$.

**Proof.** The first and second columns of (2.13) involve the exponentials $e^{-iX_j(k)(y-x)}$ and $e^{iX_j(k)(y-x)}$, respectively. Hence the domains of definition of the columns $\mu_1^{(1)}$ are determined by the sign of $\text{Im} X_j$. For example, since the Volterra equation of $\mu_1^{(1)}$ involves the exponential $e^{-2iX_1(k)(y-x)}$, $\mu_1^{(1)}$ is defined and analytic in the domain of $\mathbb{C}^+ \setminus \Sigma_1$ where $\text{Im} X_1(k) > 0$. \(\Box\)
For \( k \in \Sigma_1 \cup \Sigma_2 \), one can define the \( 2 \times 2 \) matrices \( \mu_{j \pm} \) as solutions of (2.13) with \( N_j, X_j, \Omega_j \), and \( \Phi_{0j} \) replaced by \( N_{j \pm}, X_{j \pm}, \Omega_{j \pm} \), and \( \Phi_{0j \pm} \), respectively. We also define

\[
\Phi_j(x, t, k) := \mu_{j \pm}(x, t, k)e^{(-iX_{j \pm}(k)x-i\Omega_{j \pm}(k)t)\sigma_3}.
\]

The symmetry properties of \( N_{j \pm}, X_{j \pm}, \Omega_{j \pm}, \) and \( \Phi_{0j \pm} \) imply that \( \mu_j \) and \( \Phi_j \) also satisfy (2.3).

For \( k \in \mathbb{R}, \Phi_2(x, t, k) \) and \( \Phi_1(x, t, k) \) are related by a scattering matrix \( S(k) \), which is independent of \((x, t)\) and has determinant 1. The symmetry (2.3) implies that \( S(k) \) has the same matrix structure as in the case of zero background:

\[
\Phi_2(x, t, k) = \Phi_1(x, t, k)S(k), \quad k \in \mathbb{R}, \quad k \neq B_1, B_2,
\]

\[
S(k) = \begin{pmatrix} a^*(k) & b(k) \\ -b^*(k) & a(k) \end{pmatrix}.
\]  

By Proposition 2.1, \( a(k) \) and \( b(k) \) are analytic in \( \mathbb{C}^+ \setminus (\Sigma_1 \cup \Sigma_2) \) and \( \mathbb{C}^- \setminus (\Sigma_1 \cup \Sigma_2) \), respectively, with jumps across \( \Sigma_1 \cup \Sigma_2 \). Moreover, \( a(k) = 1 + O(1/k) \) as \( k \to \infty \) in \( \mathbb{C}^+ \) and \( b(k) = O(1/k) \) as \( k \to \infty \) for \( k \in \mathbb{R} \). Setting \( t = 0 \) in (2.15), it follows that \( a(k) \) and \( b(k) \) are determined by \( q_0(x) \).

2.5. The basic RH problem. As in the case of zero background, the analytic and asymptotic properties of \( \Phi_{j \pm} \) suggest that we introduce the \( 2 \times 2 \) matrix-valued function \( m(x, t, k) \) by

\[
m(x, t, k) := \begin{pmatrix} \phi_{1}(x, t, k) \\ \phi_{2}(x, t, k) \end{pmatrix} e^{(ikx+2ik^2t)\sigma_3}, \quad k \in \mathbb{C}^+,
\]

\[
\begin{pmatrix} \phi_{1}(x, t, k) \\ \phi_{2}(x, t, k) \end{pmatrix} e^{e^{ikx+2ik^2t)\sigma_3}, \quad k \in \mathbb{C}^-,
\]

and that we characterize \( m(x, t, k) \) as the solution of a RH problem whose data are uniquely determined by \( q_0(x) \). Since \( \Phi_1 \) and \( \Phi_2 \) satisfy (2.3), so does \( m \).

![Figure 2.1](image)

Figure 2.1. The contour \( \Sigma = \mathbb{R} \cup \Sigma_1 \cup \Sigma_2 \) for the basic RH problem in the rarefaction case (left) and shock case (right).

The function \( m \) satisfies the RH problem

\[
\begin{cases}
m(x, t, \cdot) \in I + \hat{E}^2(\mathbb{C} \setminus \Sigma), \\
m_+(x, t, k) = m_-(x, t, k)J(x, t, k) \quad \text{for a.e. } k \in \Sigma,
\end{cases}
\]  

(2.17a)

where \( \Sigma := \mathbb{R} \cup \Sigma_1 \cup \Sigma_2 \) and

\[
J(x, t, k) = e^{-(ikx+2ik^2t)\sigma_3}J_0(k)e^{(ikx+2ik^2t)\sigma_3}
\]

(2.17b)

for some matrix \( J_0(k) \) yet to be specified. Since \( m \) obeys (2.3), the matrices \( J \) and \( J_0 \) satisfy the symmetries (2.3). Our next goal is to determine \( J_0(k) \) on each part of the contour \( \Sigma \).

2.5.1. Jump across \( \mathbb{R} \). Introduce the reflection coefficient \( r(k) \) by

\[
r(k) := \frac{b^*(k)}{a(k)}, \quad k \in \mathbb{R}, \quad k \neq B_1, B_2.
\]

(2.18)

The scattering relation (2.15) can be rewritten as a jump condition.

**Lemma 2.2.** For \( k \in \mathbb{R} \), \( J_0 \equiv J_0(k) \) is given by

\[
J_0 = \begin{pmatrix} 1 + rr^* & r^* \\ r & 1 \end{pmatrix} = \begin{pmatrix} 1 & r^* \\ 0 & 1 \end{pmatrix} \begin{pmatrix} 1 & 0 \\ r & 1 \end{pmatrix}, \quad k \in \mathbb{R}, \quad k \neq B_1, B_2.
\]

(2.19)
2.5.2. Jumps across $\Sigma_1$ and $\Sigma_2$. When determining the jump of $m(x,t,k)$ across $\Sigma_1$ and $\Sigma_2$, two cases are to be distinguished.

1. $\Sigma_1 \cap \Sigma_2 \neq \emptyset$, i.e. $B_1 = B_2$.
2. $\Sigma_1 \cap \Sigma_2 = \emptyset$, i.e. $B_1 \neq B_2$.

As we noticed in the Introduction, the first case has attracted more attention in the literature, see [1–3,12]. Henceforth we only consider the second case, that is, the case $B_1 \neq B_2$.

**Lemma 2.3.** Suppose $B_1 \neq B_2$. Then

$$J_0 = \begin{cases} 
\begin{pmatrix} 1 & 0 \\
\frac{1}{a^+a^-} & 1 \\
\frac{\partial}{\partial x} & \frac{\partial}{\partial x}
\end{pmatrix}, & k \in \Sigma_1 \cap \mathbb{C}^+, \\
\begin{pmatrix} 1 & 0 \\
\frac{1}{a^+a^-} & 1 \\
\frac{\partial}{\partial x} & \frac{\partial}{\partial x}
\end{pmatrix}, & k \in \Sigma_2 \cap \mathbb{C}^+,
\end{cases}$$

and

$$J_0 = \begin{cases} 
\begin{pmatrix} 1 & 0 \\
\frac{1}{a^+a^-} & 1 \\
\frac{\partial}{\partial x} & \frac{\partial}{\partial x}
\end{pmatrix}, & k \in \Sigma_1 \cap \mathbb{C}^-, \\
\begin{pmatrix} 1 & 0 \\
\frac{1}{a^+a^-} & 1 \\
\frac{\partial}{\partial x} & \frac{\partial}{\partial x}
\end{pmatrix}, & k \in \Sigma_2 \cap \mathbb{C}^-.
\end{cases}$$

(2.20)

**Proof.** For $k \in \Sigma_1 \cup \Sigma_2$, introduce the solutions $\Xi_j(x,t,k)$, $j = 1,2$ of the integral equations

$$\Xi_j(x,t,k) = I + \int_{(-1)^{j-1}t}^{x} \Phi_{0j}(x,t,k)\Phi_{0j}^{-1}(y,t,k) [(Q - Q_{0j})(y,t)] \Xi_j(y,t,k)\Phi_{0j}(y,t,k)\Phi_{0j}^{-1}(x,t,k) dy.$$ 

For each fixed $(y,t)$, the function $\Phi_{0j}(x,t,k)\Phi_{0j}^{-1}(y,t,k)$ is a solution of the $x$-part (2.7a) with $q$ replaced by $q_{0j}$. Since this solution equals the identity matrix at $x = y$ and the matrix $U$ in (2.7a) is a polynomial in $k$, we conclude that $\Phi_{0j}(x,t,k)\Phi_{0j}^{-1}(y,t,k)$ is an entire function of $k$, well defined for $k \in \Sigma_1 \cup \Sigma_2$. Thus, $\Phi_j$ and $\Xi_j\Phi_{0j}$ solve the same integral equation for $k \in \Sigma_j$, and $\Phi_j$ and $\Xi_j\Phi_{0j}$ solve the same integral equation for $k \in \Sigma_{j'}$, $j' \neq j$. Hence, $\Phi_{1\pm}(x,t,k)$ and $\Phi_{2\pm}(x,t,k)$ can be written as follows for $k \in \Sigma_1 \cup \Sigma_2$:

$$\Phi_{1\pm} = \Xi_1 \Phi_{01\pm} \quad \text{and} \quad \Phi_2 = \Xi_2 \Phi_{02}, \quad k \in \Sigma_1,$$

$$\Phi_{2\pm} = \Xi_2 \Phi_{02\pm} \quad \text{and} \quad \Phi_1 = \Xi_1 \Phi_{01}, \quad k \in \Sigma_2.$$  

(2.21a, b)

Next, introduce the scattering matrices $S_{\pm}(k)$ on $\Sigma_1 \cup \Sigma_2$:

$$\Phi_{2\pm}(x,t,k) = \Phi_{1\pm}(x,t,k)S_{\pm}(k), \quad k \in \Sigma_2,$$

$$\Phi_{2}(x,t,k) = \Phi_{1\pm}(x,t,k)S_{\pm}(k), \quad k \in \Sigma_1.$$  

(2.22a, b)

Notice that det $S_{\pm}(k) = 1$. Let us consider the two cases $k \in \Sigma_2$ and $k \in \Sigma_1$ separately.

1. For $k \in \Sigma_2$, we use (2.22a) and (2.21b) to write $S_{\pm}(k) = \Phi_{1\pm}^{-1}(x,t,k)\Xi_2(x,t,k)\Phi_{02\pm}(x,t,k)$. Setting $x = t = 0$ we have $S_{\pm}(k) = P_2(k) N_{\pm}(k)$, with some $P_2(k)$. Hence, using (2.11),

$$S_{+}(k) = S_{-}(k) \begin{pmatrix} 0 & \frac{\partial}{\partial x} \\
\frac{\partial}{\partial y} & 0
\end{pmatrix}, \quad k \in \Sigma_2.$$  

(2.23)

In particular,

$$S_{12+} = i e^{i \phi_2} S_{12-}, \quad S_{22+} = i e^{i \phi_2} S_{22-}.$$  

(2.24)

By (2.16) the jump relation across $\Sigma_2 \cap \mathbb{C}^+$ reads as follows for $x = t = 0$:

$$\begin{pmatrix} \Phi_{1+}^{(1)} \\
\Phi_{2+}^{(2)}
\end{pmatrix} = \begin{pmatrix} \Phi_{1-}^{(1)} \\
\Phi_{2-}^{(2)}
\end{pmatrix} \begin{pmatrix} a_+ \\
a_-
\end{pmatrix} + \begin{pmatrix} c_2 \\
c_2
\end{pmatrix} \begin{pmatrix} a_+ \\
a_-
\end{pmatrix}$$

for some function $c_2 \equiv c_2(k)$. Thus

$$\Phi_{2+}^{(2)} - \Phi_{2+}^{(1)} = \frac{c_2}{a_+a_-} \Phi_1^{(1)}.$$  

(2.25)

Let us calculate $c_2$. From the scattering relation (2.22a) we have

$$\Phi_{2+}^{(2)} = S_{12+} \Phi_{1+}^{(1)} + S_{22+} \Phi_{1+}^{(2)}.$$  

(2.26)
Since \( \det \Phi_1 = 1 \) we thus have \( \det \left( \Phi_1(1) \Phi_2(2) \right) = S_{22 \pm} \). Since \( a := (\Phi_1^{-1} \Phi_2)_{22} \) (see (2.15)), we also have \( \det(\Phi_1(1) \Phi_2(2)) = a \pm \). Therefore,

\[
S_{22 \pm} = a \pm, \quad k \in \Sigma_2 \cap \mathbb{C}^+.
\]  

(2.26)

From (2.25) and (2.26) we obtain

\[
\frac{\Phi_2^{(2)}}{a_+} - \frac{\Phi_2^{(2)}}{a_-} = \left( \frac{S_{12}^+}{S_{22}^+} - \frac{S_{12}^-}{S_{22}^-} \right) \Phi_1^{(1)}.
\]

Using (2.24) and the fact that \( \det S_- = 1 \) we have

\[
\frac{S_{12}^+}{S_{22}^+} - \frac{S_{12}^-}{S_{22}^-} = \frac{i e^{i \phi_2} S_{11}^- - S_{22}^+ - S_{12}^- - S_{21}^-}{S_{22}^+ S_{22}^-} = \frac{i e^{i \phi_2}}{a_+ a_-}
\]

and thus \( c_2 = i e^{i \phi_2} \).

(2) For \( k \in \Sigma_1 \), we use (2.22b) and (2.21a) to write \( S_\pm(k) = \Phi_{01 \pm}^{-1}(x, t, k) \Xi_1^{-1}(x, t, k) \Phi_2(x, t, k) \).

Setting \( x = t = 0 \), this relation reads \( S_\pm(k) = (N_{1 \pm}(k))^{-1} P_1(k) \) for some \( P_1(k) \). Hence, by (2.11),

\[
S_- S_-^{-1} = (N_{1 \pm}(k))^{-1} N_{1 +} = \begin{pmatrix} 0 & i e^{i \phi_1} \\ i e^{-i \phi_1} & 0 \end{pmatrix},
\]

so we have

\[
S_-(k) = \begin{pmatrix} 0 & i e^{i \phi_1} \\ i e^{-i \phi_1} & 0 \end{pmatrix} S_+(k), \quad k \in \Sigma_1.
\]  

(2.27)

In particular,

\[
S_{21}^- = i e^{-i \phi_1} S_{11}^+, \quad S_{22}^- = i e^{-i \phi_1} S_{12}^+, \quad k \in \Sigma_1.
\]  

(2.28)

By (2.16) the jump relation across \( \Sigma_1 \cap \mathbb{C}^+ \) has the form

\[
\begin{pmatrix} \Phi_1^{(1)}(1) \\ \Phi_2^{(2)} \end{pmatrix} = \begin{pmatrix} \Phi_1^{(1)}(1) & \Phi_2^{(2)} \end{pmatrix} \begin{pmatrix} 1 & 0 \\ c_1 & 1 \end{pmatrix},
\]

for some function \( c_1 = c_1(k) \). Thus,

\[
\frac{\Phi_1^{(1)}(1)}{a_+} - \frac{\Phi_1^{(1)}(1)}{a_-} = c_1 \Phi_2^{(2)}.
\]

On the other hand, from the scattering relation (2.22b) and \( \det S_\pm = 1 \), we get

\[
\Phi_1^{(1)}(1) = S_{22 \pm} \Phi_2^{(1)}(1) - S_{21 \pm} \Phi_2^{(2)}.
\]  

(2.29)

Since \( \Phi_2 = 1 \), this relation gives \( \det \left( \Phi_1^{(1)}(1) \Phi_2^{(2)} \right) = S_{22 \pm} \). Since \( \det \left( \Phi_1^{(1)}(1) \Phi_2^{(2)} \right) = a \pm \) we get

\[
S_{22 \pm} = a \pm, \quad k \in \Sigma_1 \cap \mathbb{C}^+.
\]  

(2.30)

By (2.30) and (2.29),

\[
\frac{\Phi_1^{(1)}(1)}{a_+} - \frac{\Phi_1^{(1)}(1)}{a_-} = \left( \frac{S_{21}^-}{S_{22}^+} - \frac{S_{21}^+}{S_{22}^-} \right) \Phi_2^{(2)}.
\]

As above, using (2.28) and the fact that \( \det S_+ = 1 \), we arrive at

\[
\frac{S_{21}^-}{S_{22}^+} - \frac{S_{21}^+}{S_{22}^-} = \frac{i e^{-i \phi_1}}{a_+ a_-}
\]

and thus \( c_1 = \frac{i e^{-i \phi_1}}{a_+ a_-} \). The expressions for \( k \in \Sigma_3 \cap \mathbb{C}^- \) follow from the symmetry (2.3). \( \square \)
2.5.3. Jumps across $\Sigma_1$ and $\Sigma_2$ when $a$ and $b$ have analytic continuation. The analytic properties of the eigenfunctions and spectral functions discussed in sections 2.4, 2.5.1, and 2.5.2 are satisfied if the initial data $q_0(x)$ approach the backgrounds in such a way that the difference is integrable (in $L^1(\pm\infty, 0)$), see (1.2) and (1.3). However, in the remainder of the paper, we make the following assumption on $q_0$ for simplicity.

Assumption (on $q_0$ and $B_j$). Henceforth, we will assume that $B_1 \neq B_2$, that $q_0$ is smooth and that

$$q_0(x) = \begin{cases} A_1 e^{i\phi_1 e^{-2iB_1 x}}, & x < -C, \\ A_2 e^{i\phi_2 e^{-2iB_2 x}}, & x > C, \end{cases}$$

(2.31)

for some $C > 0$, i.e., that $q_0(x) = q_{01}(x, 0)$ for $x < -C$ and $q_0(x) = q_{02}(x, 0)$ for $x > C$.

Then, $a(k)$ and $b(k)$ are both analytic in $\mathbb{C} \setminus (\Sigma_1 \cup \Sigma_2)$, and the scattering matrices $S_{\pm} \equiv S_{\pm}(k)$ on $\Sigma_1 \cup \Sigma_2$ can be written as

$$S_{\pm} = \begin{pmatrix} a_{\pm}^* & b_{\pm} \\ -b_{\pm}^* & a_{\pm} \end{pmatrix}.$$  

(2.32)

Accordingly, the relations (2.23) and (2.27) between $S_+$ and $S_-$ imply relations amongst $a_{\pm}(k)$ and $b_{\pm}(k)$:

$$\begin{cases} a_+ = -ie^{-i\phi_2} b_-, \\
 b_+ = -ie^{i\phi_1} a_- \end{cases}, \quad k \in \Sigma_1,$$

(2.33)

$$\begin{cases} a_+ = -ie^{i\phi_1} b_-, \\
 b_+ = ie^{i\phi_2} a_- \end{cases}, \quad k \in \Sigma_2,$$

Moreover, in this case, using that $\det S_-=1$,

$$r_+(k) - r_-(k) = \frac{ie^{-i\phi_2}}{a_+(k)a_-(k)}, \quad k \in \Sigma_1,$$

(2.34)

$$\tilde{r}_+(k) - \tilde{r}_-(k) = \frac{ie^{i\phi_1}}{a_+(k)a_-(k)}, \quad k \in \Sigma_2,$$

(2.35)

where

$$r(k) := \frac{b^*(k)}{a(k)}, \quad \tilde{r}(k) := \frac{b(k)}{a(k)},$$

(2.36)

so that the jump matrix $J_0 \equiv J_0(k)$ can be written as follows for $k \in \Sigma_1 \cup \Sigma_2$:

$$J_0 = \begin{pmatrix} 1 & 0 \\ r_+ - r_- & 1 \end{pmatrix}, \quad k \in \Sigma_1 \cap \mathbb{C}^+, \quad J_0 = \begin{pmatrix} a_+ & (\tilde{r}_+ - \tilde{r}_-)a_- \\ 0 & a_- \end{pmatrix}, \quad k \in \Sigma_2 \cap \mathbb{C}^+, \quad J_0 = \begin{pmatrix} 1 & r_+ - r_- \\ 0 & 1 \end{pmatrix}, \quad k \in \Sigma_1 \cap \mathbb{C}^-,$$

(2.37)

and

$$J_0 = \begin{pmatrix} a_+ & (\tilde{r}_+ - \tilde{r}_+)a_+ - a_- \\ 0 & a_- \end{pmatrix}, \quad k \in \Sigma_2 \cap \mathbb{C}^-.$$

(2.38)

2.5.4. Behavior at infinity. Since $q_0$ is smooth, then, as for the problem with zero background [13, Part one, Chapter I, §6],

$$a(k) = 1 + O(k^{-1}), \quad k \in \mathbb{C}^+ \cup \mathbb{R}, \quad k \to \infty,$$

(2.39)

$$b(k) = O(k^{-1}), \quad k \in \mathbb{R}, \quad k \to \infty.$$

Thus,

$$r(k) = O(k^{-1}), \quad k \in \mathbb{R}, \quad k \to \infty.$$
Lemma 2.4. Under assumption (2.31) on $g_0$,
\[
a(k) = 1 + O \left( \frac{e^{\alpha C|\text{Im} k|}}{k} \right) \quad \text{and} \quad b(k) = O \left( \frac{e^{\alpha C|\text{Im} k|}}{k} \right), \quad k \in \mathbb{C}, \quad k \to \infty.
\] (2.40)

Moreover,
\[
r(k) = O \left( \frac{e^{\alpha C|\text{Im} k|}}{k} \right), \quad k \in \mathbb{C}^+ \cup \mathbb{R}, \quad k \to \infty.
\] (2.41)

Proof. We first estimate $\Phi_1(x, 0, k)$. Introduce
\[
\hat{\Phi}_1(x, k) := e^{iB_1 x \sigma_3} \Phi_1(x, 0, k), \quad \hat{\Phi}_{01}(x, k) := e^{iB_1 x \sigma_3} \Phi_{01}(x, 0, k),
\]

$G_1(\tau, k) := N_i(k) e^{-iX_1(k) \sigma_3} N_i^{-1}(k), \quad \hat{Q}_1(x) := e^{iB_1 x \sigma_3} (Q(x, 0) - Q_{01}(x, 0)) e^{-iB_1 x \sigma_3}$.

Then, under assumption (2.31), the integral equation (2.13) can be written for $t = 0$ as a Volterra integral equation for $\hat{\Phi}_1$:
\[
\hat{\Phi}_1(x, k) = \hat{\Phi}_{01}(x, k) + \int_{-C}^{x} G_1(x - y, k) \hat{Q}_1(y) \hat{\Phi}_1(y, k) dy,
\] or, in operator form,
\[
\hat{\Phi}_1 = \hat{\Phi}_{01} + K_1 \hat{\Phi}_1,
\] (2.42)

where $K_1$ is an integral operator acting on $\mathcal{C}(\mathbb{R})$ as follows:
\[
(K_1 f)(x) = \begin{cases} \int_{-C}^{x} G_1(x - y, k) \hat{Q}_1(y) f(y) dy, & x \geq -C, \\ 0, & \text{otherwise.} \end{cases}
\]

Let $\| \|$ denote some $2 \times 2$ matrix norm and $\mu_1 := |\text{Im} X_1(k)|$. We have the estimate
\[
\| G_1(\tau, k) \| \leq D e^{\mu_1 \tau}, \quad \tau \geq 0
\]
for some positive constant $D$. Moreover, from (2.10), enlarging $D$ if necessary, we get
\[
\| \hat{\Phi}_{01}(x, k) \| \leq D e^{\mu_1 |x|} \leq \begin{cases} D e^{\mu_1 C}, & -C \leq x < 0, \\ D e^{\mu_1 x}, & x \geq 0, \end{cases}
\]
provided $k$ is far from $E_1$ and $\tilde{E}_1$. Equation (2.42) can be solved by the Neumann series
\[
\hat{\Phi}_1 = \sum_{n=0}^{\infty} K_1^n \hat{\Phi}_{01}.
\] (2.43)

We will now prove the estimate
\[
\| K_1^n \hat{\Phi}_{01}(x, k) \| \leq D^{n+1} e^{\mu_1 (x+2C)} p_{11}^{n+1}(x), \quad x \geq -C,
\] (2.44)

where $p_{11}(x) := \int_{-C}^{x} \| \hat{Q}_1(y) \| dy$. For $n = 1$ and $-C \leq x < 0$ we indeed have
\[
\| K_1 \hat{\Phi}_{01}(x, k) \| \leq \int_{-C}^{x} D e^{\mu_1 (x-y)} \| \hat{Q}_1(y) \| D e^{\mu_1 C} \leq D^2 e^{\mu_1 (x+2C)} p_1(x).
\]

Moreover, for $x \geq 0$,
\[
\| K_1 \hat{\Phi}_{01}(x, k) \| \leq \int_{-C}^{0} D e^{\mu_1 (x-y)} \| \hat{Q}_1(y) \| D e^{\mu_1 C} dy + \int_{0}^{x} D e^{\mu_1 (x-y)} \| \hat{Q}_1(y) \| D e^{\mu_1 y} dy
\]
\[
\leq D^2 e^{\mu_1 (x+2C)} p_1(x).
\]

Thus, we are done for $n = 1$. Then, using (2.44) for $n - 1$ we get the estimate for $n$:
\[
\| K_1 \left( K_1^{n-1} \hat{\Phi}_{01}(x, k) \right) \| \leq \int_{-C}^{x} \| G_1(x - y, k) \| \| \hat{Q}_1(y) \| \| K_1^{n-1} \hat{\Phi}_{01}(x, k) \| dy
\]
\[
\leq \int_{-C}^{x} D e^{\mu_1 (x-y)} p_1(y) D^n e^{\mu_1 (y+2C)} p_{11}^{n-1}(y) dy
\]
\[
= D^{n+1} e^{\mu_1 (x+2C)} p_{11}^{n}(x) / n!.
\]
Hence, the solution $\hat{\Phi}_1$ of \((2.42)\) satisfies $\|\hat{\Phi}_1(x, k)\| \leq De^{O_p(x)}e^{\mu_1(x+2C)}$ for $x > -C$, and thus
\[
\|\Phi_1(x, 0, k)\| \leq De^{O_p(x)}e^{\mu_1(x+2C)}, \quad x > -C.
\] (2.45a)

Since $\det \Phi_1 = 1$ we have the same estimate for $\|\Phi_1^{-1}(x, 0, k)\|$. Similarly, we get the estimate
\[
\|\Phi_2(x, 0, k)\| \leq De^{O_p(x)}e^{\mu_1(2C-x)}, \quad x < C,
\] (2.45b)

where $p_2(x) := \int_x^C \|\hat{Q}_2(y)\| dy$ and $\mu_2 := |\text{Im} X_2(k)|$. Now, setting $x = t = 0$ in (2.15) and using (2.45) we arrive at the estimates
\[
|a(k)| \leq De^{4C\mu}, \quad |b(k)| \leq De^{4C\mu}, \quad k \in \mathbb{C},
\] (2.46)

where $\mu := \max(\mu_1, \mu_2) = |\text{Im} k| + O\left(\frac{1}{k}\right)$. Further, taking into account the estimates
\[
G_1(\tau, k) = e^{-iX_1(k)\tau\sigma_3} + O\left(\frac{e^{\phi_1}}{k}\right), \quad k \to \infty,
\]
\[
\hat{\Phi}_{01}(x, k) = e^{-iX_1(k)x\sigma_3} + O\left(\frac{e^{\phi_1}}{k}\right), \quad k \to \infty,
\]

one can estimate $(K_1\hat{\Phi}_{01})(x, k)$ for $x > -C$ as follows:
\[
(K_1\hat{\Phi}_{01})(x, k) = \int_{-C}^x e^{iX_1(k)(y-x)\sigma_3}\hat{Q}_1(y)e^{-iX_1(k)y\sigma_3}dy + O\left(\frac{e^{\phi_1(x+2C)}}{k}\right), \quad k \to \infty.
\]

Since $\hat{Q}_1$ is off-diagonal, integrating by parts in the integral produces a factor $\frac{1}{X_1(k)} \sim \frac{1}{k}$, then, the total estimate for $(K_1\hat{\Phi}_{01})(x, k)$ takes the form $O\left(\frac{e^{\phi_1(x+2C)}}{k}\right)$. Hence, writing the series (2.43) as
\[
\hat{\Phi}_1 = \hat{\Phi}_{01} + \sum_{n=1}^{\infty} K^n_1 \hat{\Phi}_{01},
\]

we get
\[
\Phi_1(x, 0, k) = \Phi_{01}(x, 0, k) + O\left(\frac{e^{\phi_1(x+2C)}}{k}\right), \quad k \to \infty.
\]

By similar arguments,
\[
\Phi_2(x, 0, k) = \Phi_{02}(x, 0, k) + O\left(\frac{e^{\phi_2(2C-x)}}{k}\right), \quad k \to \infty.
\]

Using these estimates at $x = 0$ we get
\[
\Phi_1^{-1}(0, 0, k)\Phi_2(0, 0, k) = \left(I + O\left(\frac{1}{k}\right) + O\left(\frac{e^{\phi_1}}{k}\right)\right)\left(I + O\left(\frac{1}{k}\right) + O\left(\frac{e^{\phi_2}}{k}\right)\right) = I + O\left(\frac{e^{\phi_1}}{k}\right).
\]

Thus, estimates (2.46) can be improved to
\[
a(k) = 1 + O\left(\frac{e^{\phi_1}}{k}\right), \quad b(k) = O\left(\frac{e^{\phi_1}}{k}\right), \quad k \to \infty.
\]

This proves (2.40). Using (2.39), the estimate (2.41) follows. \$
\Box$

2.5.5. Behavior at the ends of $\Sigma_1$ and $\Sigma_2$. We have shown (see the proof of Lemma 2.3) that the scattering matrices on $\Sigma_1$ and $\Sigma_2$ can be represented as follows:
\begin{itemize}
  \item for $k \in \Sigma_2$, $S_\pm(k) = P_2(k)e^{\frac{16\pi^2}{\sigma_3}E_{2\pm}(k)}e^{-\frac{16\pi^2}{\sigma_3}}$, where $P_2(k)$ is non-singular at $k = E_2$ and $k = \bar{E}_2$ with $\det P_2(k) \equiv 1$;
  \item for $k \in \Sigma_1$, $S_\pm(k) = e^{\frac{16\pi^2}{\sigma_3}E_{1\pm}^{-1}(k)}e^{-\frac{16\pi^2}{\sigma_3}}P_1(k)$, where $P_1(k)$ is non-singular at $k = E_1$ and $k = \bar{E}_1$ with $\det P_1(k) \equiv 1$.
\end{itemize}

Thus the behavior of the spectral functions $a(k)$ and $b(k)$ as $k \to E_j$ and as $k \to \bar{E}_j$, $j = 1, 2$ is determined by the behavior of $\nu_j(k)$ involved in $E_j(k)$. It follows that we have one of the two possibilities:
\begin{itemize}
  \item[(i)] (generically) $a(k)$ and $b(k)$ have singularities of order $O((k - E_j)^{-\frac{1}{4}})$ and $O((k - \bar{E}_j)^{-\frac{1}{4}})$;
  \item[(ii)] they decay to zero at order $O((k - E_j)^{\frac{1}{4}})$ or $O((k - \bar{E}_j)^{\frac{1}{4}})$.
\end{itemize}

Moreover, the singularities $\kappa_l$ (if any) of the entries of $m(x, t, k)$ defined at the beginning of Section 2.5 are of order $O((k - \kappa_l)^{-\frac{1}{4}})$. \$
\Box$

2.5.6. Spectral functions for pure step initial conditions. For pure step initial conditions, i.e.,

\[ q_0(x) := \begin{cases} A_1 e^{i\phi_1} e^{-2iB_1 x}, & x < 0, \\ A_2 e^{i\phi_2} e^{-2iB_2 x}, & x > 0, \end{cases} \tag{2.47} \]

the spectral functions can be calculated explicitly. In this case, \( (2.15) \) evaluated at \( x = t = 0 \) gives

\[ S(k) := \begin{pmatrix} a^*(k) & b(k) \\ -b^*(k) & a(k) \end{pmatrix} = N_1^{-1}(k)N_2(k) = e^{i\phi_1 - \sigma_1} E_1^{-1}(k) e^{-i\phi_2 - \sigma_2} E_2(k) e^{-i\sigma_3}, \]

where \( \phi := \phi_1 - \phi_2 \). Thus \( a \equiv a(k) \) and \( b \equiv b(k) \) are explicitly given by

\[
\begin{align*}
a &= \frac{1}{4} \left[ e^{-i\phi} \left( \nu_1 - \nu_1^{-1} \right) \left( \nu_2 - \nu_2^{-1} \right) + \left( \nu_1 + \nu_1^{-1} \right) \left( \nu_2 + \nu_2^{-1} \right) \right], \\
b &= \frac{1}{4} \left[ e^{i\phi_2} \left( \nu_1 + \nu_1^{-1} \right) \left( \nu_2 - \nu_2^{-1} \right) - e^{i\phi_1} \left( \nu_1 - \nu_1^{-1} \right) \left( \nu_2 + \nu_2^{-1} \right) \right],
\end{align*}
\]

where \( \nu_j \equiv \nu_j(k), j = 1, 2 \) are given by \( (2.9) \).

2.5.7. Summary. The basic RH problem is the RH problem defined by \( (2.17) \) with jump \( J_0 \) given by \( (2.19) \) and \( (2.20) \). Recall that the scattering data \( a(k), b(k), \) and \( r(k) \) are uniquely determined by \( q_0(x) \), and we assumed \( a(k) \neq 0 \) for all \( k \in \mathbb{C}^+ \).

**Basic RH-problem.** Given \( r(k) \) for \( k \in \mathbb{R}, a_+(k) \) and \( a_-(k) \) for \( k \in (\Sigma_1 \cup \Sigma_2) \cap \mathbb{C}^+ \), find \( m(x,t,k) \) analytic in \( k \in \mathbb{C} \setminus \Sigma \) that satisfies \( (2.17) \) completed by \( (2.19) \) and \( (2.20) \).

**Proposition 2.5.** Let \( m(x,t,k) \) be the solution of the basic RH problem. Then, the solution \( q(x,t) \) of the Cauchy problem \( (1.1)-(1.2) \) is given by

\[ q(x,t) = 2i \lim_{k \to \infty} km_{12}(x,t,k). \]

3. Asymptotics: the plane wave region

For simplicity, we make the following assumption:

**Assumption.** We assume that \( a(k) \neq 0 \) for all \( k \).

3.1. Preliminaries. The representation of the solution of the Cauchy problem for a nonlinear integrable equation in terms of the solution of an associated RH problem makes it possible to analyze the long-time asymptotics via the Deift-Zhou steepest descent method. Originally, this method was proposed for problems with zero background [11]. Its adaptation to problems with non-zero background has required the development of the so-called \( g \)-function mechanism [10]. This mechanism is relevant when some entries of the jump matrix grow exponentially as \( t \to +\infty \).

![Figure 3.1. Signature table of Im \( \theta(\xi, k) \) for \( \xi \gg 0 \): rarefaction (left), shock (right)](image-url)
The general idea consists in replacing the original “phase function"
\[ \theta(\xi, k) := 2k^2 + \xi k, \quad \xi := \frac{x}{t} \]
(3.1)
in the jump matrix (see (2.17b))
\[ J(x, t, k) = e^{-it\theta(\xi, k)\sigma_3}J_0(k)e^{it\theta(\xi, k)\sigma_3} \]
by another analytic (up to jumps across certain arcs) function \( g(\xi, k) \) chosen in such a way that, after appropriate triangular factorizations of the jump matrices and associated re-definitions ("deformations") of the original RH problem, the jumps containing, originally, exponentially growing entries, become (piecewise) constant matrices (independent of \( k \), but dependent, in general, on \( x \) and \( t \)) of special structure whereas the other jumps decay exponentially to the identity matrix. The structure of the “limiting” RH problem is such that the problem can be solved explicitly in terms of Riemann theta functions and Abel integrals on Riemann surfaces associated with the limiting RH problem. For different ranges of the parameter \( \xi = x/t \), different Riemann surfaces (with different genera) may appear \[3,4,7\].

According to the values of the parameters \( A_j, B_j \), there are different scenarios. Each of them is characterized by the set of appropriate \( g \)-functions that we are led to introduce to perform the asymptotic analysis. All these \( g \)-functions have two properties in common:

(i) the symmetry \( g = g^* \),

(ii) the asymptotics
\[ g'(\xi, k) = \theta'(\xi, k) + O(k^{-2}) = 4k + \xi + O(k^{-2}), \quad k \to \infty, \]
where \( g' \) and \( \theta' \) denote the derivatives of \( g \) and \( \theta \) with respect to \( k \).

These properties imply that the level set \( \text{Im} g(\xi, k) = 0 \) has two infinite branches: the real axis and another branch which asymptotes to the vertical line \( \text{Re} k = -\xi/4 \). In what follows the term “infinite branch” always refers to this last branch and we call the intersection points of the real axis with the other branches of the level set \( \text{Im} g = 0 \) “real zeros” of \( \text{Im} g \).

3.2. Asymptotics for large \( |\xi| \): Plane waves. A common fact concerning the long-time asymptotics (that holds for any relationships amongst \( B_j \) and \( A_j \)) for problems with backgrounds satisfying (1.2) is that for \( \xi < C_1 \) and for \( \xi > C_2 \), with some \( C_j \) that can be expressed in terms of \( B_j \) and \( A_j \), the solution asymptotes to the corresponding plane waves, with additional phase factors depending on \( \xi \). See Figure 3.3.

Indeed, the “signature table” (the distribution of signs of \( \text{Im} \theta(\xi, k) \) in the \( k \)-plane) shows that \( J(\xi, k) \) contains exponentially growing entries if \( |\xi| \gg 0 \). More precisely, for \( \xi \ll 0 \), the jump across \( \Sigma_1 \) is growing whereas the jump across the complementary arc \( \Sigma_2 \) is bounded, and for \( \xi \gg 0 \), the
jump across $\Sigma_2$ is growing whereas the jump across the complementary arc $\Sigma_1$ is bounded (see Figure 3.1). For such values of $\xi$, we introduce the $g$-functions

$$g_j(\xi, k) := \Omega_j(k) + \xi X_j(k),$$

with $j = 1$ for $\xi \ll 0$ and $j = 2$ for $\xi \gg 0$. These $g$-functions satisfy the above properties $g = g^*$ and (3.2). Thus, besides $\mathbb{R}$, the level set $\text{Im} \, g_j = 0$ has another infinite branch asymptotic to the line $\text{Re} \, k = -\frac{\xi}{2}$. It also has a finite branch $\hat{\Sigma}_j$ connecting $E_j$ and $\bar{E}_j$ (see Figure 3.2). We consider $m^{(1)}$ defined by

$$m^{(1)}(x, t, k) := e^{-i\hat{g}_j^{(0)}(\xi)\sigma_3} m(x, t, k) e^{i\hat{g}_j(\xi, k) - \theta(\xi, k)\sigma_3},$$

where $g_j^{(0)}(\xi) := \omega_j - \xi B_j = A_j^2 - 2B_j^2 - 2B_j$ is defined in such a way that

$$g_j(\xi, k) = 2k^2 + \xi k + g_j^{(0)}(\xi) + O(k^{-1}), \quad k \to \infty.$$  

(3.4)

In terms of $m^{(1)}$, the jump relation becomes

$$m^{(1)}_+(x, t, k) = m^{(1)}_-(x, t, k) J^{(1)}(x, t, k)$$

across $\mathbb{R} \cup \hat{\Sigma}_1 \cup \Sigma_2$ for $\xi \ll 0$ and $\mathbb{R} \cup \Sigma_1 \cup \hat{\Sigma}_2$ for $\xi \gg 0$, with $J^{(1)}(x, t, k)$ bounded as $t \to +\infty$. For instance, for $\xi \gg 0$,

$$J^{(1)}(x, t, k) = \begin{pmatrix} a_{++}(k) e^{i\hat{g}_j(\xi, k) - g_{2+}(\xi, k)} & i e^{i\phi_j} \\ 0 & a_{--}(k) e^{-i\hat{g}_j(\xi, k) - g_{2-}(\xi, k)} \end{pmatrix}, \quad k \in \hat{\Sigma}_2 \cap \mathbb{C}^+,$$

which is bounded in $t$ since $\text{Im} \, g_{2+}(\xi, k) = 0$ for $k \in \hat{\Sigma}_2$.

Further deformations of the RH problem (see [3, 4, 7] for details) lead finally to two model RH problems ($j = 1, 2$) of the form (2.11):

$$m^{\text{mod-}j} \in \mathcal{I} + \hat{E}^2(\mathbb{C} \setminus \hat{\Sigma}_j),$$

$$m^{\text{mod-}j}_+(k) = m^{\text{mod-}j}_-(k) \begin{pmatrix} 0 & i e^{i\phi_j} \\ i e^{-i\phi_j} & 0 \end{pmatrix}, \quad k \in \hat{\Sigma}_j,$$

(3.5)

which apply for $(-1)^j \xi \gg 0$ and are explicitly solvable. Returning to $m(x, t, k)$, one obtains the large $t$ asymptotics for

$$q(x, t) = 2i \lim_{k \to \infty} km_{12}(x, t, k)$$

in the form

$$q(x, t) = A_j e^{-2B_j x + 2i\omega_j t + i\psi_j(\xi)} + O(t^{-\frac{1}{2}}), \quad (-1)^j \xi \gg 0, \ j = 1, 2,$$

(3.6)

where $\psi_1(-\infty) = \phi_1$ and $\psi_2(\infty) = \phi_2$.

**Figure 3.3.** The large $|\xi|$ sectors
3.3. Asymptotics in other domains. The $g$-function presented above is inappropriate in the region between the plane wave sectors $\xi < C_1$ and $\xi > C_2$. The asymptotic picture in this region is sharply different for the two cases
- $B_1 > B_2$, rarefaction case,
- $B_1 < B_2$, shock case.

In the following two sections, we study these two cases separately.

4. Asymptotics: the rarefaction case

In the rarefaction case $B_1 > B_2$, the asymptotic picture does not qualitatively depend on the values of the amplitudes $A_1$ and $A_2$ and is actually a doubling of that found in the case where one of the backgrounds is zero, see [4]. The asymptotic picture in the half-plane $t > 0$ consists of five sectors: two modulated plane wave sectors, a slow decay sector, and two modulated elliptic wave sectors (also known as transition regions). See Figure 4.1.

![Figure 4.1. The different sectors in the rarefaction case](image)

4.1. Plane waves: $\xi < -4B_1 - 4\sqrt{2}A_1$ and $\xi > -4B_2 + 4\sqrt{2}A_2$. We already know that the asymptotics has the form of plane waves for $\xi < C_1$ and $\xi > C_2$, see section 3.2. Here $C_1$ and $C_2$ are given by the same expressions as when one of the backgrounds is zero [4]:

$$C_1 = -4B_1 - 4\sqrt{2}A_1, \quad C_2 = -4B_2 + 4\sqrt{2}A_2.$$  

Indeed, suppose first that $\xi \gg 0$. Let $g \equiv g_2(\xi, k)$ be the plane wave $g$-function given by (3.3) and let $g'$ be its derivative with respect to $k$. In this case,

$$g'(\xi, k) = 4 \frac{(k - \mu_1(\xi))(k - \mu_2(\xi))}{\sqrt{(k - E_2)(k - E_2)}},$$  

where $\mu_j \equiv \mu_j(\xi)$, $j = 1, 2$, are the two self-intersections of the curve $\text{Im } g_2(\xi, k) = 0$:

$$\mu_1 = \frac{B_2}{2} - \frac{\xi}{8} - \frac{1}{8} \sqrt{(\xi + 4B_2)^2 - 32A_2^2}, \quad \mu_2 = \frac{B_2}{2} - \frac{\xi}{8} + \frac{1}{8} \sqrt{(\xi + 4B_2)^2 - 32A_2^2}.$$  

Therefore, $-\frac{\xi}{4} < \mu_1 < \mu_2 < B_2$. As $\xi$ decreases, the infinite branch of $\text{Im } g$ moves to the right and $g$ remains an appropriate $g$-function until the infinite branch hits the finite branch, i.e., until the zeros $\mu_1$ and $\mu_2$ merge, which happens at $\xi = \xi_{\text{merge}} = -4B_2 + 4\sqrt{2}A_2 = C_2$ (see Figure 4.2). This indicates the end of the right plane wave sector and that a new $g$-function is required for the asymptotic analysis when $\xi < C_2$. A similar analysis for $\xi \ll 0$ shows that $C_1 = -4B_1 - 4\sqrt{2}A_1$. 
4.2. Elliptic waves: $-4B_1 - 4\sqrt{2}A_1 < \xi < -4B_1$ and $-4B_2 < \xi < -4B_2 + 4\sqrt{2}A_2$. As \( \xi \) decreases from \( C_2 \), a new \( g \)-function \( g \equiv \tilde{g}_2 \) is needed. The transition from the right plane wave sector to the contiguous sector is reflected in the derivative \( g' \) by the emergence of two complex conjugate zeros \( \beta \) and \( \bar{\beta} \), and the merging of the two real zeros \( \mu_1 \) and \( \mu_2 \) into a single real zero \( \mu \):

\[
g'(\xi, k) = 4 \frac{ (k - \mu(\xi))(k - \beta(\xi))(k - \bar{\beta}(\xi)) }{ \sqrt{(k - E_2)(k - \bar{E}_2)(k - \beta(\xi))(k - \bar{\beta}(\xi))} },
\]

where the parameters \( \mu(\xi) \) and \( \beta(\xi) \) are subject to the conditions:

(i) Behavior at \( k = \infty \):

\[
g'(\xi, k) = \theta'(\xi, k) + O(k^{-2}) = 4k + \xi + O(k^{-2}), \quad k \to \infty.
\]

(ii) Normalization:

\[
\int_{E_2}^{\bar{E}_2} \mathrm{d}g = 0.
\]

The existence of such a \( g \)-function can be proved using the arguments in [4, Section 4.3.1]. This new \( g \)-function is appropriate for the analysis of the long-time asymptotics in the sector \( \xi \in (-4B_2, -4B_2 + 4\sqrt{2}A_2) \). Further deformations of the RH problem (see [4, Section 4.3]) lead to the model RH problem:

\[
\begin{cases}
  m_{\text{mod}} \in I + \hat{E}^2(\mathbb{C} \setminus (\tilde{\Sigma}_1 \cup \tilde{\Sigma}_2)), \\
  m_+^{\text{mod}}(k) = m_-^{\text{mod}}(k) \begin{pmatrix}
    0 & i e^{i(D_l x + iG_l t - \phi_l)} \\
    i e^{-iD_l x - iG_l t + \phi_l} & 0
  \end{pmatrix}, \quad k \in \tilde{\Sigma}_l, \quad l = 1, 2.
\end{cases}
\]

Thus, the leading term of the asymptotics is given in terms of modulated elliptic waves attached to the genus 1 Riemann surface \( w^2 = (k - E_2)(k - \bar{E}_2)(k - \beta(\xi))(k - \bar{\beta}(\xi)) \) (see [4, Theorem 3]):

\[
q(x, t) = \hat{A}_2 \frac{\Theta(\beta_2 t + \gamma_2)}{\Theta(\bar{\beta}_2 t + \bar{\gamma}_2)} e^{i\nu_2 t} + O(t^{-1/2}).
\]

A similar analysis applies to the transition from the left plane wave sector to the contiguous sector $-4B_1 - 4\sqrt{2}A_1 < \xi < -4B_1$. 

---

**Figure 4.2. Rarefaction: \( \xi > \xi_{\text{merge}} \) (left), \( \xi = \xi_{\text{merge}} \) (right)**

---

\[
\begin{align*}
  E_2 & \quad \Sigma_2 \\
  \mu_1 & \quad B_2 \\
  E_1 & \quad \Sigma_1 \\
  \hat{\Sigma}_2 & \quad \bar{\Sigma}_2 \\
  E_2 & \quad \Sigma_1 \\
  \mu_2 & \quad B_2 \\
  E_1 & \quad \Sigma_1 \\
  \end{align*}
\]
4.3. **Slow decay: \(-4B_1 < \xi < -4B_2\).** As \(\xi \downarrow -4B_2\), the zero \(\beta(\xi)\) approaches \(E_2\) and \(\mu(\xi)\) approaches \(-\xi\). As a result, at \(\xi = -4B_2\) the derivative of the \(g\)-function \(g \equiv \tilde{g}_2\) takes the form

\[
g'(\xi, k) = 4k + \xi = \theta'(\xi, k).
\]

This is consistent with the fact that for \(-4B_1 < \xi < -4B_2\), the original phase function \(\theta(\xi, k)\) is such that the off-diagonal entries of the jump matrices in the original RH problem (2.17) across both arcs \(\Sigma_1\) and \(\Sigma_2\) decay (exponentially) to 0 as \(t \to +\infty\). This suggests keeping \(g(\xi, k) = \theta(\xi, k)\) for this range (see Figure 4.3), which implies that the asymptotics for \(\xi \in (-4B_1, -4B_2)\) is essentially the same as in the case of zero background, i.e., \(q(x, t) = O(t^{-1/2})\) and this estimate can be made more precise by detailing the main contribution from the critical point \(k = -\xi/4 \in \mathbb{R}\) (see [9]).

**Proposition 4.1** (slow decay). For \(-4B_1 < \xi < -4B_2\), the long time asymptotics of \(q(x, t)\) has the form of slow decaying oscillations of Zakharov–Manakov type:

\[
q(x, t) = \frac{c_0(\xi)}{\sqrt{t}}e^{i\left(c_1(\xi)t + c_2(\xi)\log t + c_3(\xi)\right)} + o(t^{-1/2}),
\]

where the coefficients \(c_j(\xi)\) are determined in terms of the spectral functions \(a(k)\) and \(b(k)\) associated with the initial data \(q(x, 0)\), see (4.14).

**Proof.** The proof is similar to the analogous proof in the case of zero background [9]; it is based on deformations of the original RH problem by “opening lenses” (from \(-\infty\) to \(-\frac{\xi}{4}\) and from \(-\frac{\xi}{4}\) to \(+\infty\), which leads to a RH problem on a cross centered at \(k = -\frac{\xi}{4}\) with jump matrices decaying to the identity matrix uniformly outside any vicinity of \(-\frac{\xi}{4}\). A specific feature of the present case of nonzero background is that one also needs to take care of the jumps across \(\Sigma_1\) and \(\Sigma_2\). To deal with these jumps, we first introduce the function \(d(k) \equiv d(\xi, k)\) which solves the scalar RH problem relative to the contour \(\Sigma_d := (-\infty, -\frac{\xi}{2}) \cup \Sigma_2\) with the jump condition \(d_+(k) = d_-(k)J_d(k)\), where

\[
J_d = \begin{cases} 
1 + |r|^2, & k \in (-\infty, -\frac{\xi}{4}), \\
\frac{a_+}{a_-}, & k \in \Sigma_2 \cap \mathbb{C}^+, \\
\frac{a_-}{a_+}, & k \in \Sigma_2 \cap \mathbb{C}^-,
\end{cases}
\]

and the normalization condition \(d(k) \to 1\) as \(k \to \infty\). Its solution is given by the Cauchy integral

\[
d(k) = \exp \left\{ \frac{1}{2\pi i} \int_{\Sigma_d} \frac{\log J_d(s)}{s-k} \, ds \right\} = d_0(k)d_1(k)d_2(k),
\]

where

\[
\begin{align*}
d_0(k) &= \frac{\sqrt{a_+}}{\sqrt{a_-}}, \\
d_1(k) &= \begin{cases} 
\frac{r}{\sqrt{|r|^2 - 1}} & k \in (-\infty, -\frac{\xi}{4}), \\
1 & k \in \Sigma_2 \cap \mathbb{C}^+, \\
1 & k \in \Sigma_2 \cap \mathbb{C}^-,
\end{cases}
\end{align*}
\]

\[
\begin{align*}
d_2(k) &= \frac{\sqrt{a_+}}{\sqrt{a_-}} - 1, \\
&= \frac{1}{\sqrt{1 - |r|^2}} - 1, \quad k \in \Sigma_2 \cap \mathbb{C}^+,
\end{align*}
\]

\[
\begin{align*}
d_2(k) &= \frac{\sqrt{a_+}}{\sqrt{a_-}} - 1, \\
&= \frac{1}{\sqrt{1 - |r|^2}} - 1, \quad k \in \Sigma_2 \cap \mathbb{C}^-.
\end{align*}
\]
where

\[
d_0(k) = \exp \left\{ \frac{1}{2\pi i} \int_{-\infty}^{\frac{-\xi}{4}} \log(1 + |r(s)|^2) \frac{ds}{s - k} \right\},
\]

\[
d_1(k) = \exp \left\{ \frac{1}{2\pi i} \int_{0}^{\frac{\pi}{4}} \log \frac{a_+(s)}{a_-(s)} \frac{ds}{s - k} \right\}, \quad d_2(k) = \exp \left\{ \frac{1}{2\pi i} \int_{-i\infty}^{0} \log \frac{a_+(s)}{a_-(s)} \frac{ds}{s - k} \right\}.
\]

The behavior of \(d_0(k)\) as \(k \to -\frac{\xi}{4}\) is the same as in the case of zero background:

\[
d_0(k) = \left( k + \frac{\xi}{4} \right)^{i\nu(-\xi/4)} e^{\chi(k)},
\]

where

\[
\nu(-\xi/4) = -\frac{1}{2\pi} \log \left( 1 + |r(-\xi/4)|^2 \right) \in \mathbb{R},
\]

\[
\chi(k) = -\frac{1}{2\pi} \int_{-\frac{\xi}{4}}^{\frac{-\xi}{4}} \log(k - s) \log(1 + |r(s)|^2) d(s \in i\mathbb{R}).
\]

On the other hand,

\[
d_1 \left( -\frac{\xi}{4} \right) d_2 \left( -\frac{\xi}{4} \right) = \exp \left\{ \frac{i}{\pi} \Im \left[ \int_{0}^{\frac{\pi}{4}} \log \frac{a_+(i\tau)}{a_-(i\tau)} \frac{d\tau}{i\tau + \frac{\xi}{4}} \right] \right\}.
\]

Then, introducing \(m^{(1)}(x, t, k) := m(x, t, k)d(\xi, k)^{-\sigma_3}\), \(k \in \mathbb{C} \setminus \Sigma\) we have

\[
m^+(x, t, k) = m^-(x, t, k)e^{-i\theta(\xi, k)\sigma_3} J_{0}^{(1)}(k)e^{i\theta(\xi, k)\sigma_3}, \quad k \in \Sigma, \tag{4.9}
\]

where the jump \(J_{0}^{(1)} = d_{-}^{-\sigma_3} J_0 d_{+}^{-\sigma_3}\) has the form of either triangular matrices whose diagonal part is the identity matrix (for \(k \in \Sigma_1 \cup \Sigma_2\)), or products of such matrices (for \(k \in \mathbb{R}\)):

\[
J_{0}^{(1)} = \begin{cases}
\begin{pmatrix}
1 & r^2 d & 0 \\
0 & 1 & 0 \\
r & -d & 1 \\
\end{pmatrix}, & k \in (-\xi/4, +\infty), \\
\begin{pmatrix}
1 & r^2 d & 0 \\
0 & 1 & 0 \\
r & -d & 1 \\
\end{pmatrix}, & k \in (-\infty, -\xi/4), \\
\begin{pmatrix}
1 & 0 \\
\frac{r}{a^+_1 - a^-_1} & 1 \\
0 & 1 \\
\end{pmatrix}, & k \in \Sigma_1 \cap \mathbb{C}^+, \\
\begin{pmatrix}
1 & 0 \\
\frac{r^{-i\theta}}{a^+_1 - a^-_1} & 1 \\
0 & 1 \\
\end{pmatrix}, & k \in \Sigma_2 \cap \mathbb{C}^+, \\
\sigma_2 d_0^{(1)} \sigma_1, & k \in (\Sigma_1 \cup \Sigma_2) \cap \mathbb{C}^-.
\end{cases}
\tag{4.10}
\]

The second transformation reduces the jump to the cross \(\Sigma_{cr} = \cup_{j=1}^{4} L_j\) centered at \(k = -\frac{\xi}{4}\), see Figure 4.4. Introduce

\[
m^{(2)}(x, t, k) := m^{(1)}(x, t, k)e^{-i\theta(\xi, k)\sigma_3} G(k)e^{i\theta(\xi, k)\sigma_3}, \tag{4.11}
\]
where $G \equiv G(k)$ is chosen as follows:

$$
G = \begin{cases} 
\begin{pmatrix} 1 & 0 \\ -rd^{-2} & 1 \end{pmatrix}, & k \in D_1, \\
\begin{pmatrix} 1 & -\tilde{r}a^2d^2 \\ 0 & 1 \end{pmatrix}, & k \in D_3, \\
\begin{pmatrix} 1 & 0 \\ \tilde{r}^*a^2d^{-2} & 1 \end{pmatrix}, & k \in D_4, \\
\begin{pmatrix} 1 & r^*d^2 \\ 0 & 1 \end{pmatrix}, & k \in D_6, \\
I, & k \in D_2 \cup D_5.
\end{cases}
$$

(4.12)

Recall that $\tilde{r}(k) := \frac{b(k)}{a(k)}$. Then

$$
m^{(2)}_+(x,t,k) = n^{(2)}_-(x,t,k)e^{-i\theta(\xi,k)\sigma_3}J^{(2)}_0(k)e^{i\theta(\xi,k)\sigma_3}, \quad k \in \Sigma \cup \Sigma_{cr},
$$

(4.13)

where $J^{(2)}_0 = G^{-1}J^{(1)}_0G_+$ is as follows:

(1) For $k \in \mathbb{R}$, $J^{(2)}_0 = I$ by the very construction of $G$.

(2) For $k \in \Sigma_1 \cup \Sigma_2$, we also have $J^{(2)}_0 = I$. Indeed, it follows from (4.10) and (4.12) that for $k \in \Sigma_1 \cap \mathbb{C}^+$, $J^{(2)}_0 = (\frac{1}{\#} 0)$ with

$$
\# := d^{-2} \left( r_+ - r_+ - \frac{ie^{-i\phi_1}}{a_+a_-} \right) = 0,
$$

in view of (2.34). Similarly, it follows from (4.8) and (2.35) that for $k \in \Sigma_2 \cap \mathbb{C}^+$, $J^{(2)}_0 = (1 \frac{1}{\#} 0)$ with

$$
\#: = \tilde{r}_-a_-^2d_-^2 - \tilde{r}_+a_+^2d_+^2 + ie^{i\phi_2}d_+d_- = a_+a_-d_+d_+ \left( \tilde{r}_+ - \tilde{r}_+ + \frac{ie^{i\phi_2}}{a_+a_-} \right) = 0.
$$

Then, by symmetry, $J^{(2)}_0 = I$ also for $k \in (\Sigma_1 \cup \Sigma_2) \cap \mathbb{C}^-$.  

**Figure 4.4.** Contour deformation for $-4B_1 < \xi < -4B_2$
(3) For $k \in \Sigma_{cr}$,

$$
J^{(2)}_0 = \begin{cases} 
\begin{pmatrix} 1 & 0 \\ rd^{-2} & 1 \end{pmatrix}, & k \in L_1, \\
\begin{pmatrix} 1 & \tilde{r}a^2d^2 \\ 0 & 1 \end{pmatrix}, & k \in L_2, \\
\begin{pmatrix} 1 & 0 \\ -\tilde{r}^*a^2d^{-2} & 1 \end{pmatrix}, & k \in L_3, \\
\begin{pmatrix} 1 & -r^*d^2 \\ 0 & 1 \end{pmatrix}, & k \in L_4.
\end{cases}
$$

The RH problem for $m^{(2)}$ is the same as in the case of zero background (see [9]), the only difference being an additional factor (depending on $\xi$ only) in the approximation

$$
d(k) \sim \left( k + \frac{\xi}{4} \right)^{i\nu(-\xi/4)} e^{i\hat{\chi}(-\xi/4)}, \quad k \to -\frac{\xi}{4},
$$

where

$$
\hat{\chi}(-\xi/4) = \chi(-\xi/4) + \frac{i}{\pi} \text{Im} \int_0^{+\infty} \log \frac{a_{\nu}(i\tau)}{a_{\nu}(i\tau)} \frac{\xi}{\tau + \frac{\xi}{4}} d\tau.
$$

It follows that the asymptotics of $q(x,t)$ has the form (4.7) with $c_0$, $c_1$, $c_2$, and $c_3$ given by

$$
c_0(\xi) = \left( \frac{1}{4\pi} \log(1 + |r(-\xi/4)|^2) \right)^{1/2},
\quad c_1(\xi) = \frac{\xi^2}{4},
\quad c_2(\xi) = -\nu(-\xi/4),
\quad c_3(\xi) = -3\log 2 \nu(-\xi/4) + \frac{\pi}{4} + \arg \Gamma(i\nu(-\xi/4)) - \arg r(-\xi/4) - 2i\hat{\chi}(-\xi/4).
$$

4.4. Summary. In the rarefaction case there is only one asymptotic scenario.

**Theorem 4.2** (rarefaction). Suppose $B_1 > B_2$. The long-time asymptotics is then as follows.

(i) Plane wave region: $\xi < -4B_1 + 4\sqrt{2}A_1$ ($j = 1$) and $\xi > -4B_2 + 4\sqrt{2}A_2$ ($j = 2$). The leading term is a plane wave of constant amplitude:

$$
q(x,t) = A_j e^{-2iB_j x + 2i\omega_j t + iv_j(\xi)} + O(t^{-\frac{1}{2}}), \quad j = 1, 2.
$$

(ii) Elliptic wave region: $-4B_1 - 4\sqrt{2}A_1 < \xi < -4B_1$ ($j = 1$) and $-4B_2 < \xi < -4B_2 + 4\sqrt{2}A_2$ ($j = 2$). The leading term is a modulated elliptic wave:

$$
q(x,t) = \hat{A}_j \Theta(\beta_j t + \gamma_j) e^{i\nu_j t} + O(t^{-1/2}),
$$

where all coefficients depend on $\xi$. Moreover, $\hat{A}_j$ is of the order of $A_j$ and $\Theta(z) \equiv \sum_{m \in \mathbb{Z}} e^{2\pi i (\frac{1}{2} \tau m^2 + mz)}$ is an elliptic function with modular invariant $\tau \equiv \tau(\xi)$.

(iii) Slow decay region: $-4B_1 < \xi < -4B_2$. The leading term is a modulated plane wave whose amplitude is slowly decaying:

$$
q(x,t) = \frac{c_0(\xi)}{\sqrt{t}} e^{(c_1(\xi)t + c_2(\xi) \log t + c_3(\xi))} + O(t^{-1/2}) = O(t^{-1/2}).
$$
5. Asymptotics: the shock case

The shock case $B_1 < B_2$ turns out to be much richer than the rarefaction case. There are several asymptotic scenarios depending on the values of $A_1/(B_2 - B_1)$ and $A_2/(B_2 - B_1)$, see Section 2.2.

**Assumption.** Henceforth, for simplicity, we assume we are in the symmetric shock case, i.e.,

$$A_1 = A_2 = A > 0 \quad \text{and} \quad B_2 = -B_1 = B > 0. \quad (5.1)$$

Asymptotic scenarios then depend only on the ratio $A/B$.

5.1. Plane waves: $|\xi| \gg 0$. As already seen in Section 3, and as in the rarefaction case, appropriate $g$-functions for $|\xi| \gg 0$ are still $g_j$, $j = 1, 2$ given by (3.3), and the asymptotics are plane waves of type (3.6). The asymptotics is characterized by two properties:

(i) the infinite and finite branches of $\text{Im} \ g_j(\xi, k) = 0$ defined by (3.3) cross the real axis at two distinct points, $\mu_1(\xi)$ and $\mu_2(\xi)$, respectively;

(ii) the points $E_1$ and $E_2$ are located on the same side from the infinite branch of $\text{Im} \ g_j(\xi, k) = 0$.

As $|\xi|$ decreases, the end of the plane wave asymptotic region is associated with the violation of either (i) or (ii).

For large positive values of $\xi$, let $g \equiv g_2(\xi, k)$ be the plane wave $g$-function given by (3.3). The two real zeros $\mu_j \equiv \mu_j(\xi)$, $j = 1, 2$ of $\text{Im} \ g(\xi, k)$ are given by (see (4.2))

$$\mu_1 = \frac{B}{2} - \frac{\xi}{8} - \frac{1}{8}\sqrt{(\xi + 4B)^2 - 32A^2}, \quad \mu_2 = \frac{B}{2} - \frac{\xi}{8} + \frac{1}{8}\sqrt{(\xi + 4B)^2 - 32A^2}, \quad (5.2)$$

and

$$g'(\xi, k) = 4 \frac{(k - \mu_1(\xi))(k - \mu_2(\xi))}{\sqrt{(k - E_1)(k - E_2)}}. \quad (5.3)$$

As $\xi$ decreases, the infinite branch of the curve $\text{Im} \ g = 0$ moves to the right. In contrast with

the rarefaction case where there was only one possibility, there are now three possibilities (see Figures 5.1 and 5.2):

Case 1. The infinite branch hits $E_1$ and $\bar{E}_1$ before the two real zeros $\mu_1$ and $\mu_2$ merge.

Case 2. The two real zeros $\mu_1$ and $\mu_2$ merge before the infinite branch hits $E_1$ and $\bar{E}_1$.

Case 3. The infinite branch hits $E_1$ and $\bar{E}_1$ at the same time as the two real zeros $\mu_1$ and $\mu_2$ merge.

The infinite branch hits $E_1$ and $\bar{E}_1$ for $\xi = \xi_{E_1}$ where

$$\xi_{E_1} = 2(B + \sqrt{A^2 + B^2}). \quad (5.4)$$

On the other hand, the two real zeros of $g'$ merge for $\xi = \xi_{\text{merge}}$ where

$$\xi_{\text{merge}} = 4(-B + \sqrt{2}A). \quad (5.5)$$

![Figure 5.1](image-url)
Hence the infinite branch of \( \text{Im} \ g = 0 \) hits \( E_1 \) and \( \bar{E}_1 \) before the zeros merge if \( \xi_{E_1} > \xi_{\text{merge}} \), i.e., if
\[
\frac{A}{B} < \frac{2}{7}(2 + 3\sqrt{2}) \approx 1.7836.
\]
Thus:
- Case 1 occurs if \( \frac{A}{B} < \frac{2}{7}(2 + 3\sqrt{2}) \),
- Case 2 occurs if \( \frac{A}{B} > \frac{2}{7}(2 + 3\sqrt{2}) \),
- Case 3 occurs for \( \frac{A}{B} = \frac{2}{7}(2 + 3\sqrt{2}) \).

![Figure 5.2. Shock, case 2, \( \xi_{\text{merge}} > \xi_{E_1} : \xi > \xi_{\text{merge}} \) (left), \( \xi = \xi_{\text{merge}} \) (right)](image)

Each of these cases signifies the ending of the plane wave sector, because the \( g \)-function \( g_2(\xi, k) \) from (3.3) stops to provide a signature table appropriate for subsequent deformations (see, e.g., [4,7] for details) and thus a more complicated \( g \)-function is required. In particular, Case 1 was addressed in [7, Section 4], where a genus 2 region adjacent to the plane wave region was specified. In [7], this region was characterized as the values of \( \xi \) for which a system of nonlinear equations [7, Eqs. (4.12)–(4.15)] is solvable, giving the parameters of the asymptotics in this region. The solvability issue for this system was not addressed in [7]. The value of \( \xi \) separating the plane wave sector from the genus 2 sector was given, in our notation, as the value for which \( \mu_1(\xi) = -B \), i.e., the value at which the infinite branch of \( \text{Im} \ g_2 = 0 \) touches the vertical segment \((E_1, \bar{E}_1)\). This value, which in our notation is \( 4B + \frac{A^2}{B} \), is strictly greater than the correct value \( \xi_{E_1} \) given by (5.4). Also notice that the other two possibilities were not considered in [7]. One can show that in Case 2, the asymptotics in the adjacent sector is given in terms of a genus 1 elliptic wave (here the transition is similar to that occurring in the rarefaction case, see [4]), whereas in Case 3, the asymptotics in the adjacent sector is given in terms of a genus 3 hyperelliptic wave.

A similar analysis applies to the left plane wave sector.

5.2. Asymptotics for small \( |\xi| \). We next analyze the possible asymptotic scenarios in the “middle” domain. The distribution of the asymptotic sectors is expected to be symmetric under \( \xi \mapsto -\xi \), and thus special attention will be paid to the case \( \xi = 0 \), i.e., to the asymptotics along the \( t \)-axis.

5.2.1. Case \( \frac{A}{B} < 1 \) and \( |\xi| < \xi_0 \). In [7, Section 3], the asymptotics in the sector \( \{ \xi : |\xi| < \xi_0 \} \) (for some \( \xi_0 > 0 \)) was actually discussed under the assumption that the signature table of \( \text{Im} \ g \) for the associated \( g \)-function was as in [7, Figure 3.3 (a)]; see also Figure 5.3 (right). In terms of the derivative \( g' = dg/dk \) of the associated \( g \)-function, it means that \( g'(\xi, k) \) has the form
\[
g'(\xi, k) = 4 \frac{(k - \mu_1(\xi))(k - \mu_0(\xi))(k - \mu_2(\xi))}{\sqrt{(k - E_1)(k - E_1)(k - E_2)(k - E_2)}},
\] (5.6)
where $\mu_1(\xi) < \mu_0(\xi) < \mu_2(\xi)$ are all real: they are the self-intersection points of the curve $\text{Im} \, g(\xi, k) = 0$. In [7, Formula (3.27)] the associated $g$-function is of the form $f + 2G$, with $f(\xi, k) = g_2(\xi/2, k)$, and $G = O(1)$ as $k \to \infty$.

Let us check the validity of this assumption considering $\xi = 0$. In this case, the symmetry implies that $\mu_0(0) = 0$ whereas $\mu_2(0) = -\mu_1(0) > 0$, and the signature table has the form indicated in Figure 5.3 (left). Then, as $k \to \infty$, from (5.6) we have

$$g'(0, k) = 4k \left(1 + \frac{1}{k^2} [-\mu_2^2(0) + B^2 - A^2] + O(k^{-3}) \right).$$

Comparing this with

$$g'(\xi, k) = 4k + \xi + O(k^{-2}),$$

which follows from (3.4) (we indeed have $g = g_2 + O(1)$ as $k \to \infty$), we obtain that

$$\mu_2^2(0) = B^2 - A^2,$$

which can only be valid in the case $A < B$ (recall that $\mu_2(0)$ is real and nonzero).

The signature table for small enough $|\xi|$ has a similar structure, see Figure 5.3 (right), and, as it was shown in [7, Section 3], a $g$-function with derivative of the form (5.6) is indeed suitable for the asymptotic analysis in the sector $|\xi| < \xi_0$, leading to genus 1 asymptotics in this sector.

On the other hand, in the case $A > B$, the situation is different.

5.2.2. Case $\frac{A}{B} \geq 1$ and $\xi = 0$.

**Proposition 5.1.** Assume that (5.1) holds with $\frac{A}{B} \geq 1$. Then, for $\xi = 0$ an appropriate $g$-function has a derivative of the form

$$g'(0, k) = 4 \frac{k(k^2 + \alpha_0^2)}{\sqrt{(k - E_1)(k - E_2)(k - E_1)(k - E_2)}}.$$

where $\alpha_0 = \sqrt{A^2 - B^2}$, generating genus 1 asymptotics for $x = 0$.

**Comment.** The proof of Proposition 5.1 consists in performing the asymptotic analysis for $\xi = 0$ using the $g$-function (5.9) and showing that it leads to genus 1 asymptotics, expressed in terms of elliptic functions attached to the Riemann surface $w^2 = (k - E_1)(k - E_1)(k - E_2)(k - E_2)$. Details will be given elsewhere.

The signature tables are shown in Figure 5.4 in the cases $\frac{A}{B} = 1$ (left) and $\frac{A}{B} > 1$ (right).
5.2.3. Case $\frac{A}{B} > 1$ and $0 < \varepsilon < |\xi| < \xi_0$. The form of the derivative of a $g$-function given by (5.9) is unstable with respect to $\xi$. In particular, in the case $A > B$ we have the following.

**Proposition 5.2.** Assume that (5.1) holds with $\frac{A}{B} > 1$. Then, for all $\xi$ with $\varepsilon < |\xi| < \xi_0$, for some $\xi_0 > 0$ and any $\varepsilon \in (0, \xi_0)$, an appropriate $g$-function has a derivative of the following form, generating genus 3 asymptotics (see Figure 5.5 (right)):

$$g'(\xi, k) = 4 \frac{(k - \mu(\xi))(k - \alpha(\xi))(k - \tilde{\alpha}(\xi))(k - \beta(\xi))(k - \tilde{\beta}(\xi))}{w(\xi, k)},$$  \hspace{1cm} (5.10)

where $w^2 = (k - E_1)(k - \bar{E}_1)(k - E_2)(k - \bar{E}_2)(k - \alpha(\xi))(k - \tilde{\alpha}(\xi))(k - \beta(\xi))(k - \tilde{\beta}(\xi))$.

As $\xi \to 0$, $\alpha(\xi)$ and $\beta(\xi)$ approach $\alpha(0) = \beta(0) = i\alpha_0 \equiv i\sqrt{A^2 - B^2}$.

Thus, the long-time asymptotics of $q$ is given in terms of hyperelliptic functions attached to the genus 3 Riemann surface $M \equiv M(\xi)$ defined by $w^2 = (k - E_1)(k - \bar{E}_1)(k - E_2)(k - \bar{E}_2)(k - \alpha(\xi))(k - \tilde{\alpha}(\xi))(k - \beta(\xi))(k - \tilde{\beta}(\xi))$.

**Comment.** The proof of Proposition 5.2 relies on the solvability of a system of equations which characterize genus 3 asymptotics (see [5]):

\[\int_{a_1} \dd g = \int_{a_2} \dd g = \int_{a_3} \dd g = 0,\]

\[\lim_{k \to \infty} \left( \frac{\dd g}{\dd k} - 4k \right) = \xi, \quad \lim_{k \to \infty} k \left( \frac{\dd g}{\dd k} - 4k - \xi \right) = 0,\] 
\hspace{1cm} (5.11a)

\hspace{1cm} (5.11b)
where \( \tilde{g} \) denotes the differential on \( M \) given by \( dg \) on the upper sheet and \(-dg\) on the lower sheet, and \( a_1, a_2, a_3 \) are certain paths on \( M \). The definition (5.10) of \( g' \) depends on five real parameters \( \alpha_1 = \text{Re}\alpha, \alpha_2 = \text{Im}\alpha, \beta_1 = \text{Re}\beta, \beta_2 = \text{Im}\beta, \) and \( \mu \), and (5.11) is actually a system of five equations. The proof of solvability reduces to the application of the implicit function theorem for the vector function \( g(\xi) = \{\alpha_1(\xi), \alpha_2(\xi), \beta_1(\xi), \beta_2(\xi), \mu(\xi)\} \). Details are given in [3].

Proposition 5.2 justifies the importance of studying the genus 3 sector as well as the merging of \( \alpha(\xi) \) and \( \beta(\xi) \) characterizing a transition zone (smaller than any sector \(|\xi| < \varepsilon \) for any \( \varepsilon > 0 \)) connecting the axis \( \xi = 0 \), where the asymptotics is genus 1, to the genus 3 sector \( \varepsilon < \xi < \xi_0 \) (similarly for the negative values of \( \xi \)). Details are given in [6].

5.3. Overview of scenarios in the symmetric shock case. In this subsection, we describe the five possible asymptotic scenarios that may arise in the symmetric shock case. The first three scenarios correspond to Case 1, the fourth to Case 3, and the fifth to Case 2. There are two “bifurcation values” of \( \frac{A}{B} \): the first, \( \frac{A}{B} = \frac{2}{3}(2 + 3\sqrt{2}) \), determines the three cases 1, 2, and 3, the second, \( \frac{A}{B} = 1 \), determines the three subcases of Case 1. By symmetry, it is enough to consider \( \xi \geq 0 \).

5.3.1. 1st Scenario. This is the scenario developed by Buckingham and Venakides in [7].

\[
0 < \frac{A}{B} < 1
\]

| \( 0 < \xi < \xi_0 \) | \( \xi = \xi_0 \) | \( \xi_0 \leq \xi < \xi_{E_1} \) | \( \xi = \xi_{E_1} \) | \( \xi > \xi_{E_1} \) |
|-----------------------------|-----------------|-----------------|-----------------|-----------------|
| genus 1                     | genus 2         | infinite branch | hits \( E_1, \bar{E}_1 \) | wave plane       |
| residual region             | transition region|                 |                 |                 |
| \( \alpha, \bar{\alpha} \) merge into a third real zero |                 |                 |                 |                 |

We are in Case 1. As \( \xi \) decreases from \( +\infty \), the \( g \)-function \( g_2 \) can be used to carry out the asymptotic analysis until the infinite branch of \( \text{Im} g_2 = 0 \) hits \( E_1 \) and \( \bar{E}_1 \), i.e., as long as \( \xi > \xi_{E_1} \). For \( \xi < \xi_{E_1} \), a new \( g \)-function is needed, whose existence is established in Section 6. The derivative of this \( g \)-function has two real zeros \( \mu_1 \) and \( \mu_2 \), and two nonreal zeros \( \alpha \) and \( \bar{\alpha} \) which emerge from \( E_1 \) and \( \bar{E}_1 \) at \( \xi = \xi_{E_1} \):

\[
g'(\xi, k) = 4 \frac{(k - \mu_1(\xi))(k - \mu_2(\xi))(k - \alpha(\xi))(k - \bar{\alpha}(\xi))}{\sqrt{(k - \bar{E}_1)(k - E_1)(k - \bar{E}_2)(k - E_2)(k - \alpha(\xi))(k - \bar{\alpha}(\xi))}}. \tag{5.12}
\]

The asymptotic analysis associated with (5.12) is developed in [7], assuming implicitly that the system of associated equations [7, Eqs. (3.29)] determining the parameters involved in (5.12) has a solution. It leads to genus 2 asymptotics for \( g(x, t) \), in terms of functions attached to the hyperelliptic Riemann surface \( M(\xi) \) defined by \( w^2 = (k - E_1)(k - \bar{E}_1)(k - E_2)(k - \bar{E}_2)(k - \alpha(\xi))(k - \bar{\alpha}(\xi)) \).

This new \( g \)-function remains appropriate until the nonreal zeros \( \alpha(\xi) \) and \( \bar{\alpha}(\xi) \) merge into a third real zero \( \mu_0(\xi) \), which happens for \( \xi = \xi_\alpha \). For \( 0 \leq \xi \leq \xi_\alpha \), the asymptotic analysis can be carried out as in [7, Section 3], using a \( g \)-function whose derivative is as in (5.6):

\[
g'(\xi, k) = 4 \frac{(k - \mu_1(\xi))(k - \mu_2(\xi))(k - \mu_0(\xi))}{\sqrt{(k - E_1)(k - \bar{E}_1)(k - E_2)(k - \bar{E}_2)}}. \tag{5.13}
\]

It is this scenario, with the \( g \)-functions (5.12) and (5.13), that is presented in detail in [7].

5.3.2. 2nd Scenario.

\[
\frac{A}{B} = 1
\]

| \( \xi = 0 \) | \( 0 < \xi < \xi_{E_1} \) | \( \xi = \xi_{E_1} \) | \( \xi > \xi_{E_1} \) |
|-----------------|-----------------|-----------------|-----------------|
| genus 1         | genus 2         | infinite branch | hits \( E_1, \bar{E}_1 \) |
| \( \alpha, \bar{\alpha}, \mu_1 \) all merge at the origin |                 |                 |                 |
This is a limit case of the first scenario. In this case, $\xi_\alpha$ becomes 0 and thus the genus 1 range from the previous case shrinks to the single value $\xi = 0$, with $g'(0, k)$ given by (5.9) and $\alpha(0) = \bar{\alpha}(0) = \mu_1(0) = 0$.

5.3.3. 3rd Scenario.

$1 < \frac{A}{B} < \frac{2}{3}(2 + 3\sqrt{2})$

| $\xi = 0$ | $0 < \xi < \xi_\mu$ | $\xi = \xi_\mu$ | $\xi_\mu < \xi < \xi_{E_1}$ | $\xi = \xi_{E_1}$ | $\xi > \xi_{E_1}$ |
|-----------|---------------------|-----------------|--------------------------|-----------------|-----------------|
| genus 1   | genus 3             | $\alpha, \beta$ merge | the real zeros $\mu_1, \mu_2$ merge | genus 2         | genus 0         |

We are still in Case 1. As $\xi$ decreases from $+\infty$, the $g$-function $g_2$ is appropriate as long as $\xi > \xi_{E_1}$. Then, a new $g$-function is required whose derivative $g'$ has the form (5.12) and thus the asymptotics can be done as in [7, Section 4]. This $g$-function remains appropriate until the two real zeros $\mu_1(\xi)$ and $\mu_2(\xi)$ of $g'$ merge, which happens for $\xi = \xi_\mu$. Finally, for $0 < \xi < \xi_\mu$, a third $g$-function is to be considered with derivative of the form (5.10), that is,

$$g'(\xi, k) = 4 \frac{(k - \mu(\xi))(k - \alpha(\xi))(k - \bar{\alpha}(\xi))(k - \beta(\xi))(k - \bar{\beta}(\xi))}{w(\xi, k)},$$

with

$$w^2 = (k - E_1)(k - \bar{E_1})(k - E_2)(k - \bar{E_2})(k - \alpha(\xi))(k - \bar{\alpha}(\xi))(k - \beta(\xi))(k - \bar{\beta}(\xi)).$$

(5.14)

This leads to a genus 3 asymptotic formula, expressed in terms of hyperelliptic functions attached to the Riemann surface defined by (5.14). All details are given in [5].

5.3.4. 4th Scenario.
Figure 5.8. 3rd scenario (symmetric shock case): \( 1 < \frac{A}{T} < \frac{2}{7}(2 + 3\sqrt{2}) \)

\[
\frac{A}{T} = \frac{2}{7}(2 + 3\sqrt{2})
\]

| \( \xi = 0 \) | \( 0 < \xi < \xi_{E_1} \) | \( \xi = \xi_{E_1} = \xi_{\text{merge}} \) | \( \xi > \xi_{\text{merge}} \) |
|---|---|---|---|
| genus 1 | genus 3 | genus 0 |

\( \alpha, \beta \) merge
the infinite branch hits \( E_1, E_1 \)
and the real zeros \( \mu_1, \mu_2 \) merge

We are in Case 3, where \( \xi_{E_1} = \xi_{\text{merge}} \). This is a limiting case of the third scenario when \( \xi_{\mu} = \xi_{E_1} \). Thus, the genus 2 sector collapses and the genus 3 sector \( 0 < \xi < \xi_{\mu} \) becomes directly adjacent to the plane wave sector.

Figure 5.9. 4th scenario (symmetric shock case): \( \frac{A}{T} = \frac{2}{7}(2 + 3\sqrt{2}) \)

5.3.5. 5th Scenario.

\[
\frac{A}{T} > \frac{2}{7}(2 + 3\sqrt{2})
\]

| \( \xi = 0 \) | \( 0 < \xi < \xi_{\text{new}} \) | \( \xi = \xi_{E_1} \) | \( \xi_{\text{new}} < \xi < \xi_{\text{merge}} \) | \( \xi = \xi_{\text{merge}} \) | \( \xi > \xi_{\text{merge}} \) |
|---|---|---|---|---|---|
| genus 1 | genus 3 | genus 1 | genus 0 |

\( \alpha, \beta \) merge
the infinite branch hits \( E_1, \bar{E}_1 \)
the real zeros \( \mu_1, \mu_2 \) merge

We are in Case 2. As \( \xi \) goes down from \( +\infty \), the \( g \)-function \( g_2 \) is appropriate until the two real zeros \( \mu_1 \) and \( \mu_2 \) of \( g'_2 \) (see (4.1)) merge, that is, as long as \( \xi > \xi_{\text{merge}} \). Then, a new \( g \)-function \( g \equiv g_{\text{new}} \) is required whose derivative \( g' \) has the same form as in the rarefaction case:

\[
g'(\xi, k) = 4 \frac{(k - \mu(\xi))(k - \beta(\xi))(k - \bar{\beta}(\xi))}{\sqrt{(k - E_2)(k - E_2)(k - \beta(\xi))(k - \bar{\beta}(\xi))}}, \tag{5.15}
\]

and thus the asymptotics is given in terms of elliptic functions, as in [4]. This new \( g \)-function remains appropriate until the infinite branch of \( \text{Im} g_{\text{new}} \) hits \( E_1 \) and \( \bar{E}_1 \), which happens for \( \xi = \xi_{E_1} \). Finally, for \( 0 < \xi < \xi_{\text{new}} \), a third \( g \)-function is to be considered with derivative of the
form (5.10):

\[ g'(\xi, k) = 4 \frac{(k - \mu(\xi))(k - \alpha(\xi))(k - \bar{\alpha}(\xi))(k - \beta(\xi))(k - \bar{\beta}(\xi))}{w(\xi, k)}, \]

(5.16)

where

\[ w^2 = (k - E_1)(k - \bar{E}_1)(k - E_2)(k - \bar{E}_2)(k - \alpha(\xi))(k - \bar{\alpha}(\xi))(k - \beta(\xi))(k - \bar{\beta}(\xi)), \]

and where \( \alpha(\xi) \) emerges from \( E_1 \) at \( \xi = \xi_{E_1}^{\text{new}} \). As above, the parameters \( \mu(\xi), \alpha(\xi), \) and \( \beta(\xi) \) of this genus 3 sector are determined by the system of equations (5.11). The left end of the range characterized by (5.16) is \( \xi = 0 \). As \( \xi \to 0 \), \( \alpha(\xi) \) and \( \beta(\xi) \) both approach a single point \( \imath \alpha_0 \) with \( \alpha_0 = \sqrt{A^2 - B^2} \) whereas \( \mu(\xi) \to 0 \). At \( \xi = 0 \) the \( g \)-function takes the genus 1 form (5.9):

\[ g'(0, k) = 4 \frac{k(k^2 + \alpha_0^2)}{(k - E_1)(k - \bar{E}_1)(k - E_2)(k - \bar{E}_2)}. \]

6. Existence of a genus 2 sector

The first three scenarios in the symmetric shock case presented in the previous section include genus 2 sectors. We arrived upon these sectors by studying the dependence of the \( g \)-function on \( \xi \), and their existence is clearly confirmed by numerical computations. However, to actually prove that these sectors exist, it is necessary to show that the system of equations characterizing the parameters (see (6.1)) has a solution. In this section, we show that these genus 2 sectors actually exist by establishing solvability of this system. Even though we restrict attention to these particular sectors for definiteness, it seems clear that our approach can be used to show existence also of other similar higher-genus sectors. A key point in the approach is the introduction of an appropriate local diffeomorphism (see (6.34)) which makes it possible to apply the implicit function theorem.

6.1. Genus 2 Riemann surface and associated \( g \)-function. We consider the Cauchy problem for NLS defined by (1.1) and (2.31) for parameters satisfying (5.1) and \( \frac{A}{\pi} < \frac{2}{7}(2 + 3\sqrt{2}) \). In particular, we have \( E_1 = -B + iA \) and \( E_2 = B + iA \) with \( B > 0 \) and \( A > 0 \). These assumptions correspond to the first three scenarios of the symmetric shock case.

Let \( \Sigma_\alpha \) be the genus 2 hyperelliptic Riemann surface with branch points at \( E_1, \bar{E}_1, E_2, \bar{E}_2, \alpha, \bar{\alpha} \) for some nonreal complex number \( \alpha \) with \( \imath \alpha > 0 \). Let \( C \subset \mathbb{C} \) be the union of the cuts \([E_1, \bar{E}_1], [E_2, \bar{E}_2] \), and \([\alpha, \bar{\alpha}] \) (see Figure 6.1):

\[ C := \Sigma_1 \cup \Sigma_2 \cup [\alpha, \bar{\alpha}]. \]

Define the meromorphic differential \( \text{d}g \) on \( \Sigma_\alpha \) as follows:

\[ \text{d}g(k) := 4(k - \mu_1)(k - \mu_2)(k - \alpha)(k - \bar{\alpha}) \frac{\text{d}k}{w(k)}, \]

where \( \mu_1, \mu_2 \in \mathbb{R}, \mu_1 < \mu_2 \), and

\[ w(k) := \sqrt{(k - E_1)(k - \bar{E}_1)(k - E_2)(k - \bar{E}_2)(k - \alpha)(k - \bar{\alpha})}. \]
We view $\Sigma_\alpha$ as a two-sheeted cover of the complex plane such that $w(k^+) \sim k^3$ as $k \to \infty$, where $k^\pm$ denote the points on the upper and lower sheets which project onto $k$.

The definition of $dg$ depends on the four real numbers $\mu_1, \mu_2, \alpha_1, \alpha_2$, where $\alpha_1$ and $\alpha_2$ denote the real and imaginary parts of $\alpha$:

$$\alpha = \alpha_1 + i\alpha_2, \quad \alpha_2 > 0.$$ 

These four real numbers are determined by the four conditions

$$\int_{a_1} dg = \int_{a_2} dg = 0,$$  \quad (6.1a)$$

$$\lim_{k \to \infty} \left( \frac{dg}{dk} - 4k \right) = \xi, \quad \lim_{k \to \infty} k \left( \frac{dg}{dk} - 4k - \xi \right) = 0,$$  \quad (6.1b)$$

where we let $a_j, j = 1, 2,$ be a counterclockwise loop on the upper sheet enclosing $[\bar{E}_j, E_j]$ and no other branch points, see Figure 6.1. We let $\zeta = (\zeta_1, \zeta_2)$ be the normalized basis of $H^1(\Sigma_\alpha)$ which is dual to the canonical homology basis $\{a_j, b_j\}_{j=1}^2$ in the sense that $\{\zeta_j\}_{j=1}^2$ are holomorphic differentials such that

$$\int_{a_i} \zeta_j = \delta_{ij}, \quad i, j = 1, 2.$$

The basis $\{\zeta_j\}_{j=1}^2$ is explicitly given by $\zeta_j = \sum_{l=1}^2 A_{jl} \hat{\zeta}_l$, where

$$\hat{\zeta}_l = \frac{k^{l-1}}{w} dk$$  \quad (6.2)$$

and the invertible matrix $A$ is given by

$$(A^{-1})_{jl} = \int_{a_j} \hat{\zeta}_l.$$  \quad (6.3)$$

Note that $A$, $\zeta$, and $\hat{\zeta}$ depend on $\alpha$.

![Figure 6.1. The homology basis $\{a_j, b_j\}_{j=1}^2$ on the genus 2 Riemann surface $\Sigma_\alpha$.](image)

The conditions in (6.1b) can be formulated as

$$\frac{dg}{dk}(k^+) = 4k + \xi + O(k^{-2}), \quad k \to \infty.$$  \quad (6.4)
The solvability of the system of equations (6.1) characterizes the genus 2 sector. Since
\[ \frac{d g}{d k}(k) = \frac{d g}{d \bar{k}}(\bar{k}), \quad k \in \mathbb{C} \setminus \mathcal{C}, \] (6.5)
we have \( \int_A^B d g = \int_{\bar{A}}^{\bar{B}} d g \) where the contour in the second integral is the complex conjugate of the contour in the first integral. This implies that
\[ \int_{a_j} d g \in i \mathbb{R}, \quad j = 1, 2, \] (6.6)
so the conditions in (6.1a) are two real conditions.

As \( \xi \) decreases from \( +\infty \), the infinite branch hits \( E_1 \) and \( \bar{E}_1 \) when \( \xi = \xi_{E_1} \), where
\[ \xi_{E_1} = 2(B + |E_1|). \] (6.7)
For \( \xi > \xi_{E_1} \), we are in the genus 0 sector and the \( g \)-function is given by (see (5.3))
\[ d g = \frac{4(k - \mu_1)(k - \mu_2)}{\sqrt{(k - E_2)(k - E_1)}} d k, \]
where \( \mu_1 < \mu_2 \) are given by (5.2). For \( \xi = \xi_{E_1} \), we have
\[ \mu_1(\xi_{E_1}) = \frac{B - |E_1| - \sqrt{2B(3|E_1| + 5B) - 7A^2}}{4}, \]
\[ \mu_2(\xi_{E_1}) = \frac{B - |E_1| + \sqrt{2B(3|E_1| + 5B) - 7A^2}}{4}. \] (6.8)

As \( \xi \) decreases below \( \xi_{E_1} \), we expect to see a genus 2 sector. We will show that the system (6.1) indeed has a unique solution for \( \xi \in (\xi_{E_1} - \delta, \xi_{E_1}) \) for some \( \delta > 0 \) and that this solution can be extended until the qualitative structure of the \( g \)-function changes (see item (f) below).

**Theorem 6.1 (Existence of genus 2 sector).** Suppose \( 0 < \frac{A}{B} < \frac{2}{7}(2 + 3\sqrt{2}) \). Then there exists a \( \xi_m < \xi_{E_1} \) and a smooth curve
\[ \xi \mapsto (\alpha_1(\xi), \alpha_2(\xi), \mu_1(\xi), \mu_2(\xi)) \in \mathbb{R}^4 \]
defined for \( \xi \in (\xi_m, \xi_{E_1}) \) such that the following hold:
(a) For each \( \xi \in (\xi_m, \xi_{E_1}) \), \( (\xi, \alpha_1(\xi), \alpha_2(\xi), \mu_1(\xi), \mu_2(\xi)) \) is a solution of the system of equations (6.1).
(b) The curve \( \xi \mapsto (\mu_1(\xi), \mu_2(\xi)) \) is a smooth map \( (\xi_m, \xi_{E_1}) \to \mathbb{R}^2 \) such that
\[ \mu_1(\xi) < \mu_2(\xi) \quad \text{for} \quad \xi \in (\xi_m, \xi_{E_1}). \]
(c) The curve \( \xi \mapsto \alpha(\xi) = \alpha_1(\xi) + i\alpha_2(\xi) \) is a smooth map \( (\xi_m, \xi_{E_1}) \to \mathbb{C}^+ \setminus \{E_1, E_2\} \).
(d) As \( \xi \uparrow \xi_{E_1} \), we have
\[ \alpha(\xi) \to E_1, \quad \mu_1(\xi) \to \mu_1(\xi_{E_1}), \quad \mu_2(\xi) \to \mu_2(\xi_{E_1}), \] (6.9)
where \( \mu_1(\xi_{E_1}) \) and \( \mu_2(\xi_{E_1}) \) are given by (6.8), i.e., there is a continuous transition from the genus 0 sector \( \xi > \xi_{E_1} \) to the genus 2 sector at \( \xi = \xi_{E_1} \).
(e) For all \( \xi \in (\xi_m, \xi_{E_1}) \) sufficiently close to \( \xi_{E_1} \), we have \( \alpha_1(\xi) > \text{Re} E_1 \) so that the branch cut \([\alpha, \alpha]\) lies to the right of the cut \([E_1, E_1]\). In fact, as \( \xi \uparrow \xi_{E_1} \),
\[ \alpha(\xi) = E_1 + c_1 \frac{\xi_{E_1} - \xi}{\ln(|\xi_{E_1} - \xi|)} + o\left(\frac{\xi_{E_1} - \xi}{\ln(|\xi_{E_1} - \xi|)}\right), \] (6.10)
where
\[ c_1 := \frac{2BE_1}{A^2 + 4iAB - 3B^2 - B|E_2|} \]
has strictly positive real and imaginary parts.
(f) As \( \xi \downarrow \xi_m \), at least one of the following occurs:
(i) the zeros \( \mu_1 \) and \( \mu_2 \) merge,
(ii) \( \alpha(\xi) \) and \( \bar{\alpha}(\xi) \) merge at a point on the real axis, i.e., \( \alpha_2(\xi) \downarrow 0 \),
(iii) \( \alpha(\xi) \) approaches \( E_1 \) or \( E_2 \).
(iv) \( \xi_m = -\infty \).
(g) \( \alpha(\xi) = \alpha_1(\xi) + i \alpha_2(\xi) \) satisfies the following nonlinear ODE for \( \xi \in (\xi_m, \xi_E) \):

\[
\begin{pmatrix}
\alpha_1'(\xi) \\
\alpha_2'(\xi)
\end{pmatrix} = -P^{-1} G - P^{-1} A \left( \frac{\int_{a_1}^{k_1(k-\alpha_1)} dk}{\int_{a_2}^{k_2(k-\alpha_2)} dk} \right),
\]

where
- The matrix \( P(\xi, \alpha_1, \alpha_2) \) and the vector \( G(\alpha_1, \alpha_2) \) are defined by

\[
P = \begin{pmatrix}
P_{11} & P_{12} \\
P_{21} & P_{22}
\end{pmatrix}, \quad G = \begin{pmatrix}
G_1 \\
G_2
\end{pmatrix},
\]

where the entries \( \{P_{ij}(\xi, \alpha_1, \alpha_2)\}_{i,j=1}^2 \) and \( \{G_j(\alpha_1, \alpha_2)\}_{j=1}^2 \) are polynomials given by

\[
P_{11} = 12 \alpha_1^3 + 2 \alpha_2^2 \xi + 6 \alpha_1 \alpha_2^2 + 4 \alpha_1 A^2 + \alpha_2^2 \xi - 4 \alpha_1 B^2,
\]

\[
P_{21} = -12 \alpha_1^3 - 2 \alpha_1 \xi - 4 A^2 + 6 \alpha_2^2 + 4 B^2,
\]

\[
P_{12} = \alpha_2 (\alpha_1 \xi + 4 A^2 - 6 \alpha_2^2 - 4 B^2),
\]

\[
P_{22} = \alpha_2 (12 \alpha_1 + \xi),
\]

and

\[
G_1(\alpha_1, \alpha_2) = \alpha_1 (\alpha_1^2 + \alpha_2^2), \quad G_2(\alpha_1, \alpha_2) = \alpha_2^2 - \alpha_1^2.
\]

- The zeros \( \mu_j = \mu_j(\xi) \) are expressed in terms of \( \xi \) and \( \alpha_j = \alpha_j(\xi) \) by

\[
\mu_1 = \frac{1}{8} \left( -4 \alpha_1 - \xi - \sqrt{-48 \alpha_1^2 - 8 \alpha_1 \xi - 64 A^2 + 32 \alpha_2^2 + 64 B^2 + \xi^2} \right),
\]

\[
\mu_2 = \frac{1}{8} \left( -4 \alpha_1 - \xi + \sqrt{-48 \alpha_1^2 - 8 \alpha_1 \xi - 64 A^2 + 32 \alpha_2^2 + 64 B^2 + \xi^2} \right).
\]

**Remark 6.2.** Numerical simulations strongly suggest that as \( \xi \downarrow \xi_m \) (see item (f))
- case (i) (the zeros \( \mu_1 \) and \( \mu_2 \) merge) occurs if \( 1 < \frac{A}{B} < \frac{2}{3} (2 + 3 \sqrt{2}) \),
- case (ii) \( \alpha(\xi) \) and \( \alpha(\overline{\xi}) \) merge at a point on the real axis occurs if \( 0 < \frac{A}{B} < 1 \),
- whereas we expect both (i) and (ii) to occur for \( \frac{A}{B} = 1 \).

6.2. **Proof of Theorem 6.1.** The conditions in (6.1b) can be written more explicitly as

\[
4(\alpha_1 + \mu_1 + \mu_2) = -\xi,
\]

\[
2 \mu_2 (\alpha_1 + \mu_1) + 2 \alpha_1 \mu_1 - 2 A^2 + \alpha_2^2 + 2 B^2 = 0.
\]

Solving these two equations for \( \mu_1 \) and \( \mu_2 \), we find (6.15).

We write \( \alpha = \alpha_1 + i \alpha_2 \) and let \( x = (\xi, \alpha_1, \alpha_2) \in \mathbb{R}^3 \) denote the vector with coordinates \( (\xi, \alpha_1, \alpha_2) \). Let \( W \) denote the open subset of \( \mathbb{R}^3 \) consisting of all points \( x = (\xi, \alpha_1, \alpha_2) \in \mathbb{R}^3 \) such that \( \alpha_2 > 0 \), \( \alpha \notin \{E_1, E_2\} \), and the expression under the square roots in (6.15) is strictly positive. If we want to emphasize the dependence on \( x = (\xi, \alpha_1, \alpha_2) \), we will write \( dg \equiv dg(k; x), \mu_1 \equiv \mu_1(x), \) and \( \mu_2 \equiv \mu_2(x) \), where \( dg(k; x) \) is evaluated with \( \mu_1, \mu_2 \) given by (6.15).

We define the map \( F: W \rightarrow \mathbb{R}^2 \) by (see (6.6))

\[
F(x) = \frac{1}{i} \left( \int_{a_1}^{k_1} dg(k; x) \right)
\]

and let \( D_\alpha F \) denote the Jacobian matrix

\[
D_\alpha F(x) = \begin{pmatrix}
\frac{\partial}{\partial \alpha_1} F_1 & \frac{\partial}{\partial \alpha_2} F_1 \\
\frac{\partial}{\partial \alpha_1} F_2 & \frac{\partial}{\partial \alpha_2} F_2
\end{pmatrix} = \frac{1}{i} \left( \int_{a_1}^{k_1} \frac{\partial}{\partial \alpha_1} dg \int_{a_2}^{k_2} \frac{\partial}{\partial \alpha_2} dg \right),
\]

where \( \partial_{\alpha_j} := \frac{\partial}{\partial \alpha_j}, j = 1, 2 \).
Remark. The function $F$ is in general multivalued on $W$, because of a monodromy as $\alpha$ encircles $E_1$ or $E_2$. Strictly speaking, we should therefore define $F : W \to \mathbb{R}^2$, where $W$ denotes the universal cover of $W$. However, it can be proved using (6.4) that $F \mapsto MF$ for some matrix $M$ under such a monodromy transformation. In particular, the zero locus of $F$ is a well-defined subset of $W$. Thus, this distinction is of no consequence for us and will be suppressed from the notation.

Lemma 6.3. $F : W \to \mathbb{R}^2$ is a smooth map such that $\det D_\alpha F \neq 0$ at each point of $W$.

Proof. Smoothness follows directly from the definitions. We will prove that $\det D_\alpha F \neq 0$. For each $x \in W$, $dg(k; x)$ is a meromorphic differential on $\Sigma_\alpha$ whose only poles lie at $\infty^\pm$ and whose singular behavior at $\infty^\pm$ (which is prescribed by (6.4)) is independent of $\alpha_1$ and $\alpha_2$. It follows that $\partial_{\alpha_1} dg$ and $\partial_{\alpha_2} dg$ are holomorphic differentials on $\Sigma_\alpha$. More precisely, a direct computation gives

$$\partial_{\alpha_1} dg = \frac{P_{11} + kP_{21}}{w(k)} dk, \quad \partial_{\alpha_2} dg = \frac{P_{12} + kP_{22}}{w(k)} dk,$$

where $\{P_{ij}(\xi, \alpha_1, \alpha_2)\}_{i,j=1}^2$ are the polynomials defined in (6.13).

In terms of $P_{ij}$, $i, j = 1, 2$ and $\hat{\zeta}_l$, $l = 1, 2$ (defined in (6.2)), we can write (6.17) as

$$\begin{pmatrix} \partial_{\alpha_1} dg \\ \partial_{\alpha_2} dg \end{pmatrix} = \begin{pmatrix} P_{11} \hat{\zeta}_1 + P_{21} \hat{\zeta}_2 \\ P_{12} \hat{\zeta}_1 + P_{22} \hat{\zeta}_2 \end{pmatrix}.$$

Substitution into (6.16) yields

$$D_\alpha F(x) = \frac{1}{i} \begin{pmatrix} P_{11} \int_{a_1} \hat{\zeta}_1 + P_{21} \int_{a_1} \hat{\zeta}_2 \\ P_{12} \int_{a_1} \hat{\zeta}_1 + P_{22} \int_{a_2} \hat{\zeta}_2 \end{pmatrix}$$

$$= \frac{1}{i} \begin{pmatrix} P_{11} (A^{-1})_{11} + P_{21} (A^{-1})_{12} \\ P_{12} (A^{-1})_{21} + P_{22} (A^{-1})_{22} \end{pmatrix}$$

$$= -iA^{-1} P.$$

We conclude that $D_\alpha F$ is invertible if and only if the matrix $P$ is invertible. A straightforward computation using (6.13) gives

$$\det P = \alpha_2 \left[ 16A^4 + 16A^2 \left( \alpha_1 (6\alpha_1 + \xi) - 3\alpha_2^2 - 2B^2 \right) + 36 \left( 4\alpha_1^4 + \alpha_2^4 \right) + 48\alpha_1^2 \xi \\
+ \xi^2 (4\alpha_1^2 + \alpha_2^2) + 16B^4 - 16B^2 (\alpha_1 (6\alpha_1 + \xi) - 3\alpha_2^2) \right].$$

Recalling the expressions (6.15) for $\mu_1, \mu_2$, this can be rewritten more concisely as

$$\det P = 16\alpha_2 \left( (\alpha_1 - \mu_1)^2 + \alpha_2 \right) \left( (\alpha_1 - \mu_2)^2 + \alpha_2 \right)$$

$$= 16\alpha_2 |\alpha - \mu_1|^2 |\alpha - \mu_2|^2.$$

In particular, $\det P > 0$ on $W$ (on which $\alpha_2 > 0$).

If $x = (\xi, \alpha_1, \alpha_2) \in W$ is a solution of $F(x) = 0$, then Lemma 6.3 and the implicit function theorem implies that the level set $F = 0$ locally near $x$ can be parametrized by a smooth curve $\xi \mapsto (\xi, \alpha_1(\xi), \alpha_2(\xi))$ such that

$$\begin{pmatrix} \alpha_1'(\xi) \\ \alpha_2'(\xi) \end{pmatrix} = -D_\alpha F(\xi, \alpha_1(\xi), \alpha_2(\xi))^{-1} \frac{\partial F_1}{\partial \xi} \bigg|_{(\xi, \alpha_1(\xi), \alpha_2(\xi))},$$

where $\partial_\xi := \frac{\partial}{\partial \xi}$. A computation shows that

$$\partial_\xi dg = \frac{(k + \alpha_1)(k - \alpha_2)}{w(k)} dk, \quad G_1 \hat{\zeta}_1 + G_2 \hat{\zeta}_2 + \frac{k^2(k - \alpha_1)}{w(k)} dk,$$
where the polynomials \( \{ G_j(\alpha_1, \alpha_2) \}_j^2 \) are given by (6.14). Thus

\[
\left( \frac{\partial_t F_1}{\partial \xi} \right) = \frac{1}{i} \left( \int_{a_1} \frac{\partial_t \xi}{\partial d} - \int_{a_2} \frac{\partial_t \xi}{\partial d} \right) = \frac{1}{i} \left( G_1 \int_{a_1} \hat{\xi}_1 + G_2 \int_{a_2} \hat{\xi}_2 + f_{a_1} \frac{k^2(k-\alpha_1)}{w(k)} dk \right)
\]

\[
= \frac{1}{i} \left( G_1(A^{-1})_{11} + G_2(A^{-1})_{12} + \int_{a_1} \frac{k^2(k-\alpha_1)}{w(k)} dk \right)
\]

\[
= -i A^{-1} G - i \left( \int_{a_1} \frac{k^2(k-\alpha_1)}{w(k)} dk \right),
\]

where

\[
G = \begin{pmatrix} G_1 \\ G_2 \end{pmatrix}.
\]

Note that \( \frac{k^2(k-\alpha_1)}{w(k)} \) is a meromorphic differential on \( \Sigma_\alpha \) of the second kind (i.e., all residues are zero) which is holomorphic except for two double poles at \( \infty^\pm \) such that

\[
\frac{k^2(k-\alpha_1)}{w(k^\pm)} = \pm 1 + O(k^{-2}), \quad k \to \infty.
\]

Substituting (6.18) and (6.20) into (6.19), we find

\[
\left( \frac{\alpha_1'(\xi)}{\alpha_2'(\xi)} \right) = -iP^{-1} A \left( -i A^{-1} G - i \left( \int_{a_1} \frac{k^2(k-\alpha_1)}{w(k)} dk \right) \right)
\]

\[
= -iP^{-1} \left( \int_{a_1} \frac{k^2(k-\alpha_1)}{w(k)} \right),
\]

which is the ODE in (6.11).

We have shown that the nonlinear ODE (6.11) describes the solution curves of \( F = 0 \) whenever such curves exist. By Lemma 6.3, each solution curve can be continued as long as it stays in \( W \) and the zeros \( \{ \mu_j \}_j^2 \) remain bounded. We will show in the next lemma that \( \alpha, \mu_1, \mu_2 \) remain bounded on the zero set of \( F \) unless \( |\xi| \to \infty \). Therefore, the solution curve can either be extended indefinitely to all \( \xi \in (-\infty, \xi_{E_1}) \) or it ends at a point \( \xi = \xi_m \) where at least one of the following must occur:

(i) the zeros \( \mu_1 \) and \( \mu_2 \) merge,
(ii) \( \alpha_2 \downarrow 0 \) (i.e., \( \alpha \) and \( \bar{\alpha} \) merge),
(iii) \( \alpha \) hits one of the branch points \( E_1 \) or \( E_2 \).

**Lemma 6.4.** As \( \alpha = \alpha_1 + io_2 \to \infty \), the function \( F \) satisfies

\[
|F(\xi, \alpha_1, \alpha_2)| \to \infty,
\]

uniformly for \( \xi \) in bounded subsets of \( \mathbb{R} \) and \( \arg \alpha \in [0, \pi] \). In particular, if \( F(\xi, \alpha_1(\xi), \alpha_2(\xi)) = 0 \), then \( \alpha(\xi), \mu_1(\xi), \) and \( \mu_2(\xi) \) remain bounded whenever \( \xi \) does.

**Proof.** Let \( w_1(k) = \sqrt{(k - E_1)(k - E_1)(k - E_2)(k - E_2)} \) with branch cuts along \( [E_1, E_1] \) and \( [E_2, E_2] \) and the branch fixed by the condition that \( w_1(k) \sim k^2 \) as \( k \to \infty \). As \( \alpha \to \infty \) along the ray \( \alpha_1 = qo_2, q \in \mathbb{R} \), we have

\[
dg(k; x) = \left[ 2(2q^2 - 1)\sqrt{1 + q^2} \alpha_2^2 + \left( \frac{6qk}{\sqrt{1 + q^2}} + \xi q \sqrt{1 + q^2} \right) \alpha_2 \right] \frac{dk}{w_1(k)} + O(\alpha_2),
\]

uniformly for \( q \) and \( \xi \) in bounded subsets of \( \mathbb{R} \) and for \( k \) in compact subsets of \( \mathbb{C} \setminus \{ E_1, E_1, E_2, E_2 \} \).

Letting

\[
J_j = \frac{1}{i} \int_{a_j} \frac{dk}{w_1(k)}, \quad K_j = \frac{1}{i} \int_{a_j} \frac{kdk}{w_1(k)}, \quad j = 1, 2,
\]

then

\[
\hat{\xi}_1 G \to (6.20)
\]

However, it is not clear how to proceed from here.
we find, for \( j = 1, 2 \),
\[
F_j(x) = 2(2q^2 - 1)\sqrt{1 + q^2}\alpha_j^2 J_j + \left( \frac{6qK_j}{\sqrt{1 + q^2}} + \xi q\sqrt{1 + q^2} J_j \right) \alpha_j^2 + O(\alpha^2),
\]
uniformly for \( q \) and \( \xi \) in bounded subsets of \( \mathbb{R} \). Using that
\[
J_1 = -J_2 \neq 0, \quad K_1 = K_2 \neq 0,
\]
we infer that
\[
F_1(x) - F_2(x) = 4(2q^2 - 1)\sqrt{1 + q^2}\alpha_1 J_1 + 2\xi q\sqrt{1 + q^2} J_1 \alpha_2^2 + O(\alpha^2),
\]
\[
F_1(x) + F_2(x) = \frac{12qK_1}{\sqrt{1 + q^2}} \alpha_2^2 + O(\alpha^2),
\]
uniformly for \( q \) and \( \xi \) in bounded subsets of \( \mathbb{R} \). Equation (6.21a) implies that \( |F(\xi, \alpha_1, \alpha_2)| \to \infty \) as \( \alpha \to \infty \), uniformly for \( q \) in compact subsets of \( \mathbb{R} \setminus \{ \pm 1/\sqrt{2} \} \) and \( \xi \) in bounded subsets of \( \mathbb{R} \). Equation (6.21b) implies that \( |F(\xi, \alpha_1, \alpha_2)| \to \infty \) as \( \alpha \to \infty \), uniformly for \( q \) in compact subsets of \( \mathbb{R} \setminus \{ 0 \} \) and \( \xi \) in bounded subsets of \( \mathbb{R} \). Combining these two conclusions, we find that
\[
|F(\xi, \alpha_1, \alpha_2)| \to \infty \quad \text{as} \quad \alpha \to \infty \quad \text{for} \quad \arg \alpha \in (\epsilon, \pi - \epsilon).
\]

To show that \( |F(\xi, \alpha_1, \alpha_2)| \to \infty \) as \( \alpha \to \infty \) also for \( \arg \alpha \in [0, \epsilon] \cup [\pi - \epsilon, \pi] \), we instead use the fact that, as \( \alpha \to \infty \) along the ray \( \alpha_2 = q\alpha_1, q \in \mathbb{R} \), we have
\[
d_{g}(k; x) = 2(2 - q^2)\sqrt{1 + q^2}|\alpha_1|^3 \frac{dk}{w_1(k)} + O(\alpha^2),
\]
uniformly for \( q \) and \( \xi \) in bounded subsets of \( \mathbb{R} \) and \( k \) in compact subsets of \( \mathbb{C} \setminus \{ E_1, E_1, E_2, E_2 \} \).

We conclude that \( F(\xi, \alpha_1, \alpha_2) \to \infty \) as \( \alpha \to \infty \), uniformly for \( \xi \) in bounded subsets of \( \mathbb{R} \) and \( \arg \alpha \in [0, \pi] \). The second statement follows because, by (6.15), \( \mu_1 \) and \( \mu_2 \) remain bounded whenever \( \alpha \) and \( \xi \) stay bounded. □

It remains to show that the zero set \( F = 0 \) contains a curve which satisfies (6.9) and (6.10) as \( \xi \to \xi_{E_1} \). The limits \( \lim_{\xi \to \xi_{E_1}} \mu_j(\xi) = \mu_j(\xi_{E_1}), j = 1, 2 \), are a consequence of (6.15) if we can show that the zero set of \( F \) contains a smooth curve \( (\xi, \alpha_1(\xi), \alpha_2(\xi)) \) which approaches the point
\[
x_0 := (\xi_{E_1}, \Re E_1, \Im E_1) \in \partial\mathcal{W}
\]
as \( \xi \to \xi_{E_1} \). To prove this, we will first show that \( F \) has a continuous extension to \( x_0 \) such that \( F(x_0) = 0 \) and then apply a boundary version of the implicit function theorem at the point \( x_0 \). The proof is complicated by the fact that the Riemann surface \( \Sigma_\alpha \) degenerates to a genus zero surface as \( \alpha \) approaches \( E_1 \). This implies that the partial derivatives \( \partial_{\alpha_1} F_1 \) and \( \partial_{\alpha_2} F_1 \) blow up like \( \ln |\alpha - E_1| \) in this limit. Therefore, we cannot apply the implicit function theorem at the point \( x_0 \in \partial\mathcal{W} \) directly to \( F \); instead we will introduce a function \( \tilde{F} \), which is a modified version of \( F \), and apply the implicit function theorem to this modified function.

We begin by establishing the behavior of \( F \) and its first order partial derivatives as \( \alpha \to E_1 \). The analysis of the second component \( F_2 \) is easier than the analysis of \( F_1 \), because \( F_2 \) is nonsingular at \( \alpha = E_1 \). We therefore begin with \( F_2 \).

Let \( B_R \subset \mathbb{R}^3 \) denote the open ball of radius \( R > 0 \) centered at \( x_0 \). Let \( L \subset \mathbb{R}^3 \) denote the line on which \( \alpha = E_1 \):
\[
L = \{ (\xi, \Re E_1, \Im E_1) \mid \xi \in \mathbb{R} \}.
\]
Let \( x_L = (\xi, \Re E_1, \Im E_1) \) denote the orthogonal projection of \( x = (\xi, \alpha_1, \alpha_2) \) onto \( L \). Note that \( \text{dist}(x, L) = |\alpha - E_1| \). By choosing \( R > 0 \) sufficiently small, we may assume that \( B_R \setminus L \subset \mathcal{W} \) and, say, \( R < \min\{A, B, 1\} / 2 \).

Let \( \Sigma_0 \) denote the genus 0 Riemann surface with a single cut from \( E_2 \) to \( E_2 \) defined by
\[
w_0^2 = (k - E_2)(k - \bar{E_2}).
\]
We view this as a two-sheeted cover of the complex plane such that \( w_0(k) = \sqrt{(k - E_2)(k - \bar{E_2})} \sim k \) as \( k \to \infty \) on the upper sheet.
Lemma 6.5 (Behavior of $F_2(x)$ as $\alpha \to E_1$). The function $F_2: B_R \setminus L \to \mathbb{R}$ extends to a smooth function $B_R \to \mathbb{R}$. Moreover, the following estimates hold uniformly for $x \in B_R$:

\begin{align*}
F_2(x) &= O(|\alpha - E_1|), \\
\partial_k F_2(x) &= O(|\alpha - E_1|), \\
\partial_{\alpha_j} F_2(x) &= q_j(\xi) + O(|\alpha - E_1|), \quad j = 1, 2,
\end{align*}

(6.22a, 6.22b, 6.22c)

where $\{q_j(\xi)\}_1^2$ are linear functions of $\xi \in \mathbb{R}$ given by

\begin{align*}
q_1(\xi) &= -\pi \Im \left\{ \frac{Q(\xi)}{\sqrt{B \sqrt{E_2}}} \right\}, \\
q_2(\xi) &= \pi \Re \left\{ \frac{Q(\xi)}{\sqrt{B \sqrt{E_2}}} \right\},
\end{align*}

with

\begin{equation}
Q(\xi) := -2iA^2 + A(\xi - 12B) + 2iB(4B - \xi) \quad (6.23)
\end{equation}

and the principal branch is used for $\sqrt{E_2}$. For $\xi = \xi_{E_1}$, it holds that $q_1(\xi_{E_1}) \neq 0$ and $q_2(\xi_{E_1}) \neq 0$.

Proof. In the limit as $x \in B_R \setminus L$ approaches $L$, we have $\alpha \to E_1$ and $\bar{\alpha} \to \bar{E}_1$, so that the Riemann surface $\Sigma_\alpha$ degenerates to the genus zero surface $\Sigma_0$. With appropriate choices of the branches, we have

\[ F_2(x) = \frac{1}{i} \int_{\alpha_2} \frac{4(k - \mu_1(x))(k - \mu_2(x))\sqrt{(k - \alpha)(k - \bar{\alpha})}}{(k - E_1)(k - E_2)(k - E_2)} \, dk. \]

We see that the integrand is smooth as a function of $x \in B_R$ and analytic as a function of $k$ for $k$ in a neighborhood of the contour $a_2$. This shows that $F_2: B_R \setminus L \to \mathbb{R}$ extends to a smooth function $B_R \to \mathbb{R}$.

To prove (6.22a), we note that a Taylor expansion gives

\begin{equation}
\frac{dg}{dk}(k; x) = \frac{4(k - \mu_1(x_L))(k - \mu_2(x_L))}{w_0(k)} \left(1 + O(|\alpha - E_1|)\right), \quad (6.24a)
\end{equation}

uniformly for $x \in B_R$ and $k$ on $a_2$. Similarly we also have

\begin{equation}
\frac{dg}{dk}(k; x) = \frac{4(k - \mu_1(x))(k - \mu_2(x))}{w_0(k)} \left\{1 - \frac{\alpha - E_1}{2(k - E_1)} - \frac{\bar{\alpha} - \bar{E}_1}{2(k - E_1)} + O(|\alpha - E_1|^2)\right\}, \quad (6.24b)
\end{equation}

uniformly for $x \in B_R$ and $k$ on $a_2$. It follows from (6.24a) that

\[ F_2(x) = \frac{1}{i} \int_{a_2} dg(k; x) = \frac{1}{i} \int_{a_2} \frac{4(k - \mu_1(x_L))(k - \mu_2(x_L))}{w_0(k)} \, dk + O(|\alpha - E_1|). \]

Deforming the contour to infinity and using that

\[ \frac{4(k - \mu_1(x_L))(k - \mu_2(x_L))}{w_0(k)} = 4k - 4(\mu_1(x_L) + \mu_2(x_L) - B) \]

\[ + \frac{4(\mu_1(x_L) - B)(\mu_2(x_L) - B) - 2A^2}{k} + O(k^{-2}) \]

we find that the integral over $a_2$ vanishes. This proves (6.22a).

To derive the expansions of the first-order partial derivatives, we use (6.24) to compute

\[ \partial_k \frac{dg}{dk}(k; x) = \partial_k \frac{4(k - \mu_1(x))(k - \mu_2(x))(k - \alpha)(k - \bar{\alpha})}{w(k; x)} = X_0 + O(|\alpha - E_1|) \]

and, similarly,

\[ \partial_{\alpha_j} \frac{dg}{dk}(k; x) = X_j + O(|\alpha - E_1|), \quad j = 1, 2, \]
where the error terms are uniform with respect to $k \in a_2$ and $\{X_j\}_{0}^{2}$ are short-hand notations for the expressions

$$X_0 := \frac{k - B}{w_0(k)},$$

$$X_1 := \frac{A^2(-10B + 2k + \xi) + 2B(B + k)(\xi - 4B)}{(k - E_1)(k - \bar{E}_1)w_0(k)},$$

$$X_2 := -A \left(\frac{2A^2 + 4B^2 + B(12k + \xi) - k\xi}{(k - E_1)(k - \bar{E}_1)w_0(k)}\right).$$

Consequently, deforming the contour to infinity and noting that the residue of $X_j$ at $k = \infty$ vanishes for each $j$, we obtain

$$\partial_\xi F_2(x) = \frac{1}{i} \int_{a_2} X_0 dk + O(|\alpha - E_1|) = O(|\alpha - E_1|),$$

$$\partial_\alpha F_2(x) = \frac{1}{i} \int_{a_2} X_1 dk + O(|\alpha - E_1|) = -2\pi \left(\text{Res}_{k=E_1} + \text{Res}_{k=\bar{E}_1}\right)X_1 + O(|\alpha - E_1|)$$

$$= q_1(\xi) + O(|\alpha - E_1|),$$

$$\partial_{\alpha_2} F_2(x) = \frac{1}{i} \int_{a_2} X_2 dk + O(|\alpha - E_1|) = -2\pi \left(\text{Res}_{k=E_1} + \text{Res}_{k=\bar{E}_1}\right)X_2 + O(|\alpha - E_1|)$$

$$= q_2(\xi) + O(|\alpha - E_1|),$$

uniformly for $x \in \bar{B}_R$. This proves (6.22b) and (6.22c).

In order to prove that $q_1(\xi_{E_1}) \neq 0$ and $q_2(\xi_{E_1}) \neq 0$, it is sufficient to verify that $\frac{Q^2}{BB_2} \notin \mathbb{R}$. But evaluation at $\xi = \xi_{E_1}$ gives

$$Q(\xi_{E_1}) := 2A(|E_2| - 5B) - 4iB(|E_2| - B) - 2iA^2$$

and then a computation yields

$$\text{Im}\{Q^2 \bar{E}_2\} = 16\left[4AB^3(|E_2| - B) + A^3B(3|E_2| - 5B)\right].$$

The right-hand side is strictly positive for $A, B > 0$. This proves that $q_j(\xi_{E_1}) \neq 0$ for $j = 1, 2$ and completes the proof of the lemma. \qed

We next consider the first component $F_1(x)$ for $(\xi, \alpha)$ near $(\xi_{E_1}, E_1)$. Since it is enough for our purposes, we will for simplicity restrict attention to $\alpha$ such that $\alpha_1 \geq \text{Re } E_1$; this will simplify the specification of some branches of square roots. As above, we let $R > 0$ be small. We recall that $x_0 = (\xi_{E_1}, \text{Re } E_1, \text{Im } E_1) \in L$ and let $S_R \subset \mathbb{R}^3$ denote the open half-ball

$$S_R = B_R \cap \{\alpha_1 > \text{Re } E_1\}.$$  

Square roots and logarithms are defined using the principal branch unless specified otherwise.

**Lemma 6.6** (Behavior of $F_1(x)$ as $\alpha \to E_1$). As $x \in \bar{S}_R \setminus L$ approaches the line $L$ (in other words, as $\alpha \to E_1$), $F_1(x)$ admits an asymptotic expansion to all orders of the form

$$F_1(x) \sim \text{Im}\left\{\sum_{n,m=0}^{\infty} \left[c_{nm}(\xi) + d_{nm}(\xi)(\alpha - E_1)\ln(\alpha - E_1)\right](\alpha - E_1)^n(\bar{\alpha} - \bar{E}_1)^m\right\}, \quad (6.25)$$

where $\{c_{nm}(\xi), d_{nm}(\xi)\}_{n,m=0}^{\infty}$ are smooth complex-valued functions of $\xi$. Moreover, the expansion (6.25) can be differentiated termwise with respect to $\alpha_1, \alpha_2$, and $\xi$. In particular, the following estimates are valid uniformly for $x = (\xi, \alpha_1, \alpha_2) \in \bar{S}_R \setminus L$:

$$F_1(x) = f_0(\xi) + O(|\alpha - E_1|\ln|\alpha - E_1|), \quad (6.26a)$$

$$\partial_\xi F_1(x) = f_0'(\xi) + O(|\alpha - E_1|\ln|\alpha - E_1|),$$

$$\partial_\alpha F_1(x) = \text{Im}\{d_{00}(\xi)\ln(\alpha - E_1)\} + f_1(\xi) + O(|\alpha - E_1|\ln|\alpha - E_1|), \quad (6.26b)$$

$$\partial_{\alpha_1} F_1(x) = \text{Im}\{id_{00}(\xi)\ln(\alpha - E_1)\} + f_2(\xi) + O(|\alpha - E_1|\ln|\alpha - E_1|), \quad (6.26c)$$

$$\partial_{\alpha_2} F_1(x) = \text{Im}\{i\theta_{00}(\xi)\ln(\alpha - E_1)\} + f_3(\xi) + O(|\alpha - E_1|\ln|\alpha - E_1|), \quad (6.26d)$$

$$\partial_\bar{\alpha} F_1(x) = \text{Im}\{i\theta_{00}(\xi)\ln(\alpha - E_1)\} + f_4(\xi) + O(|\alpha - E_1|\ln|\alpha - E_1|), \quad (6.26e)$$

$$\partial_{\bar{\alpha}_1} F_1(x) = \text{Im}\{i\theta_{00}(\xi)\ln(\alpha - E_1)\} + f_5(\xi) + O(|\alpha - E_1|\ln|\alpha - E_1|), \quad (6.26f)$$

$$\partial_{\bar{\alpha}_2} F_1(x) = \text{Im}\{i\theta_{00}(\xi)\ln(\alpha - E_1)\} + f_6(\xi) + O(|\alpha - E_1|\ln|\alpha - E_1|), \quad (6.26g)$$
where

• $f_0(\xi)$ is the linear real-valued function defined by
  \[ f_0(\xi) = -8\sqrt{B}(\text{Im} \sqrt{E_2})(\xi - \xi_{E_1}). \]  

(6.27)

• $d_{00}(\xi)$ is the linear function of $\xi \in \mathbb{R}$ given by
  \[ d_{00}(\xi) = \frac{-iQ(\xi)}{\sqrt{B} \sqrt{E_2}} \]  

(6.28)

with $Q(\xi)$ defined in (6.23).

• $(f_j(\xi))_{1}^{\infty}$ are smooth real-valued functions of $\xi \in \mathbb{R}$.

Proof. In order to derive (6.25), we fix a large negative number $p < 0$. For $z_0, z_1 \in \mathbb{C}$, we let $[z_0, z_1]$ denote the straight line segment from $z_0$ to $z_1$, and we let $[z_0, z_1]^{+}$ denotes its preimage in the upper sheet under the natural projection $\Sigma_\alpha \to \mathbb{C}$. Deforming the contour and using the symmetry $dg(k) = \overline{dg(k)}$, we see that, for $x \in S_R \setminus L$,

\[ F_1(x) = \frac{1}{i} \int_{z_1} dg = -\frac{2}{i} \left( \int_{[p,E_1]^+} dg + \int_{[\bar{E}_1,p]^{+}} dg \right) = \text{Im} \left\{-4 \int_{[p,E_1]} dg \right\}. \]  

(6.29)

Defining the function $h(k;x)$ for $k$ in a neighborhood of $[p, E_1]$ by

\[ h(k;x) = -\frac{4(k - \mu_1(x))(k - \mu_2(x)\sqrt{\alpha - k}}{\sqrt{E_1 - k\sqrt{(E_2 - k)(E_2 - k)}}}, \]

we have

\[ h(k;x) = \frac{\sqrt{E_1 - k}}{\sqrt{\alpha - k}} \frac{dg}{dk} (k^+; x) \quad \text{for } k \in [p, E_1]. \]

Here and elsewhere in the proof, the principal branch is adopted for all square roots and logarithms. The function $h$ depends smoothly on $x \in S_R$ and is analytic for $k$ in a neighborhood of $[p, E_1]$. Defining $I_l(\alpha)$ by

\[ I_l(\alpha) \coloneqq \int_{[p_1,E_1]} (E_1 - k)^{l - \frac{1}{2}} \sqrt{\alpha - k} \, dk, \quad l = 0, 1, \ldots, \]

and employing the expansion

\[ h(k;x) \sim \sum_{n,m,l \geq 0} h_{nm}(\xi)(\alpha - E_1)^n(\bar{\alpha} - \bar{E}_1)^m(E_1 - k)^l, \]

where $h_{nm}(\xi)$ are smooth functions, we infer that if $p_1 \in [p, E_1]$ is a point sufficiently close to $E_1$, then we have the expansion

\[ \int_{[p_1,E_1]} dg(k;x) = \int_{[p_1,E_1]} h(k;x) \frac{\sqrt{\alpha - k}}{\sqrt{E_1 - k}} \, dk \sim \sum_{n,m,l \geq 0} h_{nm}(\xi)(\alpha - E_1)^n(\bar{\alpha} - \bar{E}_1)^mI_l(\alpha) \]  

(6.30)

and this expansion can be differentiated termwise with respect to $\alpha_1$, $\alpha_2$, and $\xi$.

We claim that there exist complex coefficients $\{q_l\}_{l \geq 0}$ and $\{r_{lj}\}_{l,j \geq 0}$ such that

\[ I_l(\alpha) \sim q_l(\alpha - E_1)^{l+1} \ln(\alpha - E_1) + \sum_{j=0}^{\infty} r_{lj}(\alpha - E_1)^j \]  

(6.31)

for each integer $l \geq 0$ as $\alpha \to E_1$. Indeed, the statement is true for $l = 0$ by direct computation. Moreover, an integration by parts gives, for $l \geq 1$,

\[ I_l(\alpha) = -\frac{2}{3}(E_1 - p_1)^{l - \frac{1}{2}}(\alpha - p_1)^{\frac{3}{2}} - \frac{2(l - \frac{1}{2})}{3} \int_{[p_1,E_1]} (E_1 - k)^{l - \frac{1}{2}}(\alpha - k)^{\frac{3}{2}} \, dk \]

\[ = -\frac{2}{3}(E_1 - p_1)^{l - \frac{1}{2}}(\alpha - p_1)^{\frac{3}{2}} - \frac{2(l - \frac{1}{2})}{3} \{ (\alpha - E_1)I_{l-1}(\alpha) + I_l(\alpha) \}. \]
Solving for \( I_l(\alpha) \), we obtain
\[
I_l(\alpha) = \frac{1}{1 + \frac{2}{3}(l - \frac{1}{2})} \left\{ -\frac{2}{3}(E_1 - p_1)^{\frac{1}{2}}(\alpha - p_1)^{\frac{1}{2}} - \frac{2}{3}(l - \frac{1}{2})(\alpha - E_1)I_{l-1}(\alpha) \right\},
\]
and hence (6.31) follows for all integers \( l \geq 0 \) by induction.

Equations (6.30) and (6.31) imply that, as \( \alpha \to E_1 \),
\[
-4 \int_{[p, E_1]} d(g(k; x)) = \sum_{n, m \geq 0} [c_{nm}(\xi) + d_{nm}(\xi)(\alpha - E_1) \ln(\alpha - E_1)](\alpha - E_1)^n(\bar{\alpha} - \bar{E}_1)^m,
\]
where \( c_{nm}(\xi), d_{nm}(\xi) \) are smooth complex-valued functions of \( \xi \) which are independent of \( \alpha_1 \) and \( \alpha_2 \), and the expansion can be differentiated termwise with respect to \( \alpha_1, \alpha_2, \) and \( \xi \). The existence of the expansion (6.25) now follows from (6.29).

The rest of the lemma follows from (6.25) if we can verify the expressions (6.27) and (6.28) for \( f_0 \) and \( d_{00} \). To derive the expression (6.27) for \( f_0 \), we note that by (6.29) (see also (6.24a))
\[
f_0(\xi) = \lim_{\alpha \to E_1} \frac{F_1(x)}{\alpha - E_1} = 2i \lim_{\alpha \to E_1} \left( \int_{[p, E_1]} + \int_{[\bar{E}_1, p]} \right) d(g(k; x))
\]
\[
= 2i \left( \int_{[p, E_1]} + \int_{[\bar{E}_1, p]} \right) d(g(k; x)) = -2i \int_{[\bar{E}_1, E_1]} \frac{4(k - \mu_1(x_{L}) + \mu_2(x_{L}))}{\sqrt{(E_2 - k)(E_2 - k)}} dk.
\]
Substituting in the expressions for \( \mu_1(x_{L}) \) and \( \mu_2(x_{L}) \) and integrating, we find
\[
f_0(\xi) = -2i \int_{[\bar{E}_1, E_1]} \frac{2A^2 - (B - k)(4k + \xi)}{\sqrt{A^2 + (B - k)^2}} dk
\]
\[
= -2i \left[ (2B + 2k + \xi) \sqrt{A^2 + (B - k)^2} \right]_{k=\bar{E}_1}^{E_1}
\]
\[
= 16A\bar{B} Re \sqrt{E_2} - 8\sqrt{B} Im \sqrt{E_2}.
\]
Observing that the definition (6.7) of \( \xi_{E_1} \) can be rewritten as
\[
\xi_{E_1} = \tan\left( \frac{1}{2} \arctan \frac{A}{2} \right) = 2A \frac{Re \sqrt{E_2}}{Im \sqrt{E_2}},
\]
the expression for \( f_0 \) in (6.27) follows.

We finally derive the expression (6.28) for \( d_{00} \). Using (6.32) and then (6.17), we see that
\[
d_{00}(\xi) = \lim_{\alpha \to E_1} \frac{-4 \int_{[p, E_1]} \partial_{\alpha} d(g(k; x))}{\ln(\alpha - E_1)} = \lim_{\alpha \to E_1} \frac{-4 \int_{[p, E_1]} \frac{P(x_k) + P_1(x_k)}{\alpha - E_1} dk}{\ln(\alpha - E_1)}.
\]
Consequently,
\[
d_{00}(\xi) = \lim_{\alpha \to E_1} \frac{4 \int_{[p, E_1]} \frac{P_1(x_{L}) + P_2(x_{L})}{\sqrt{E_1 - k} \sqrt{E_1 - k} \sqrt{E_2 - k} \sqrt{E_2 - k} \sqrt{E_2 - k}} dk}{\ln(\alpha - E_1)}
\]
\[
= \lim_{\alpha \to E_1} \frac{4 \int_{[p, E_1]} \frac{P_1(x_{L}) + P_2(x_{L})}{\sqrt{E_1 - k} \sqrt{E_2 - E_2} \sqrt{E_2 - E_1} \sqrt{E_2 - E_1} \sqrt{E_1 - k} \sqrt{E_2 - k}} dk}{\ln(\alpha - E_1)}.
\]
Using that
\[
\int_{[p, E_1]} \frac{dk}{\sqrt{E_1 - k} \sqrt{\alpha - k}} = -2 \ln \left( \sqrt{E_1 - k} + \sqrt{\alpha - k} \right) \bigg|_{k=p}^{E_1}
\]
\[
= -2 \ln \left( \frac{\sqrt{\alpha - E_1}}{\sqrt{E_1 - p} + \sqrt{\alpha - p}} \right),
\]
we find
\[
d_{00}(\xi) = -4 \frac{P_1(x_{L}) + P_2(x_{L})}{(E_1 - E_1) \sqrt{E_2 - E_1} \sqrt{E_2 - E_1}} = \frac{2A^2 + iA(12B - \xi) + 2B(\xi - 4B)}{\sqrt{B} \sqrt{E_2}},
\]
which proves (6.28).

Lemma 6.5 and 6.6 show that the smooth map $F: \bar{S}_R \setminus L \to \mathbb{R}^2$ extends continuously to a map $\bar{S}_R \to \mathbb{R}^2$ (i.e., $F$ can be continuously extended to the set where $\alpha = E_1$) and that on the line $L$ where $\alpha = E_1$ this extension is given by

$$F(\xi, \text{Re} \ E_1, \text{Im} \ E_1) = \left( -8\sqrt{B} \left( \text{Im} \sqrt{E_2} \right) (\xi - \xi_1), 0 \right).$$

In particular, $F(\xi, \text{Re} \ E_1, \text{Im} \ E_1)$ vanishes if and only if $\xi = \xi_{E_1}$. This suggests that the zero set of $F$ indeed contains a curve starting at the point $x_0 = (\xi_{E_1}, \text{Re} \ E_1, \text{Im} \ E_1)$. However, Lemma 6.6 also implies that the extension of $F$ to $\bar{S}_R$ is not $C^1$, because the partial derivatives $\partial_\alpha F_1$, $j = 1, 2$, are singular as $\alpha \to E_1$. Thus, in order to apply the implicit function theorem, we will define a modification $\tilde{F}$ of $F$. The singular behavior of $\partial_\alpha F_1$ stems from the existence of a term proportional to $(\alpha - E_1) \ln(\alpha - E_1)$ in the expansion (6.25) of $F$. As motivation for the definition of $\tilde{F}$, we therefore consider the following simple example.

**Example 6.7.** Consider the function $f: (0, 1) \to \mathbb{R}$ defined by $f(x) = x \ln x$. Although $f(x)$ has a continuous extension to $x = 0$, the derivative $f'(x) = 1 + \ln x$ is singular at $x = 0$. However, the modified function $\tilde{f}: (0, 1) \to \mathbb{R}$ defined by

$$\tilde{f}(x) = f\left(\frac{x}{\ln x}\right) = -x + \frac{x \ln(|\ln x|)}{\ln x}$$

is such that both $\tilde{f}(x)$ and its derivative $\tilde{f}'(x) = -1 + \frac{1 + (\ln x - 1) \ln(|\ln x|)}{\ln^2 x}$ extend continuously to $x = 0$.

Employing the standard identification of $\mathbb{C}$ with $\mathbb{R}^2$, we can write $F(\xi, \alpha) \equiv F(\xi, \alpha_1, \alpha_2)$. Let $R > 0$ be small. We define the modified function $\tilde{F}: \bar{S}_R \setminus L \to \mathbb{R}^2$ by

$$\tilde{F}(\xi, \alpha) = \begin{pmatrix} F_1(\xi, \varphi(\alpha)) \\ F_2(\xi, \varphi(\alpha)) \ln|\alpha - E_1| \end{pmatrix},$$

where

$$\varphi(\alpha) = E_1 + \frac{\alpha - E_1}{|\ln|\alpha - E_1||}. \quad (6.34)$$

There is an $r \in (0, R)$ such that $(\xi, \alpha) \mapsto (\xi, \varphi(\alpha))$ is a diffeomorphism from $\bar{S}_r \setminus L$ onto a subset of $\bar{S}_R \setminus L$. Then, since $F: \bar{S}_R \setminus L \to \mathbb{R}^2$ is smooth, $\tilde{F}: \bar{S}_r \setminus L \to \mathbb{R}^2$ is also smooth. The next lemma shows that $\tilde{F}$ extends to a $C^1$ map $\bar{S}_r \to \mathbb{R}^2$.

**Remark 6.8.** In addition to incorporating the dilation defined by $\varphi$, the definition of $\tilde{F}$ also includes a factor of $|\ln|\alpha - E_1||$ in the second component. This factor has been included in order to make the partial derivative $\partial_\alpha \tilde{F}/\partial_{\alpha_2}$ nonzero at $x_0$ (so that we later can apply the implicit function theorem at $x_0$).

**Lemma 6.9.** The map $\tilde{F}: \bar{S}_r \setminus L \to \mathbb{R}^2$ and its Jacobian matrix of first order partial derivatives

$$D\tilde{F}(x) = \begin{pmatrix} \partial_\xi F_1 & \partial_{\alpha_1} F_1 & \partial_{\alpha_2} F_1 \\ \partial_\xi F_2 & \partial_{\alpha_1} F_2 & \partial_{\alpha_2} F_2 \end{pmatrix}$$

can be extended to continuous maps on $\bar{S}_r$. Moreover, this extension satisfies

$$\tilde{F}(x_0) = 0, \quad D\tilde{F}(x_0) = \begin{pmatrix} f_0'(\xi_{E_1}) & -\text{Im} \ d_{00}(\xi_{E_1}) & -\text{Re} \ d_{00}(\xi_{E_1}) \\ 0 & q_1(\xi_{E_1}) & q_2(\xi_{E_1}) \end{pmatrix}.$$

**Proof.** The proof consists of long but straightforward computations using the Taylor expansions of Lemma 6.5 and Lemma 6.6. Since

$$\varphi(\alpha) - E_1 = \frac{\alpha - E_1}{|\ln|\alpha - E_1||},$$
we find from the Taylor expansions (6.22) and (6.26) that

\[
\tilde{F}_1(x) = f_0(\xi) + O\left(\frac{\alpha - E_1}{\ln |\alpha - E_1|} \left(1 + \ln \left|\frac{\alpha - E_1}{\ln |\alpha - E_1|}\right\right)\right),
\]

\[
\partial_\xi \tilde{F}_1(x) = f'_0(\xi) + O\left(\frac{\alpha - E_1}{\ln |\alpha - E_1|} \left(1 + \ln \left|\frac{\alpha - E_1}{\ln |\alpha - E_1|}\right\right)\right),
\]

\[
\tilde{F}_2(x) = O(\alpha - E_1), \quad \partial_\xi \tilde{F}_2(x) = O(\alpha - E_1),
\]

which shows that these functions have continuous extensions to \(\tilde{S}_c\). Write \(\varphi(\alpha) = \varphi_1(\alpha) + i\varphi_2(\alpha)\). Using that

\[
\partial_{\alpha_1} \ln |\alpha - E_1| = \frac{\alpha_1 - \text{Re} E_1}{|\alpha - E_1|^2}, \quad \partial_{\alpha_2} \ln |\alpha - E_1| = \frac{\alpha_2 - \text{Im} E_1}{|\alpha - E_1|^2},
\]

we find

\[
\partial_{\alpha_1} \varphi_1(\alpha) = \frac{1}{|\ln |\alpha - E_1||} + \left(\frac{|\alpha_1 - \text{Re} E_1|^2}{|\ln |\alpha - E_1||^2 |\alpha - E_1|^2}\right) = \frac{1}{|\ln |\alpha - E_1||} + O\left(\frac{1}{|\ln |\alpha - E_1||^2 |\alpha - E_1|^2}\right),
\]

\[
\partial_{\alpha_2} \varphi_2(\alpha) = \frac{1}{|\ln |\alpha - E_1||} + \left(\frac{|\alpha_2 - \text{Im} E_1|^2}{|\ln |\alpha - E_1||^2 |\alpha - E_1|^2}\right) = \frac{1}{|\ln |\alpha - E_1||} + O\left(\frac{1}{|\ln |\alpha - E_1||^2 |\alpha - E_1|^2}\right),
\]

\[
\partial_{\alpha_2} \varphi_1(\alpha) = \partial_{\alpha_1} \varphi_2(\alpha) = \frac{(\alpha_1 - \text{Re} E_1)(\alpha_2 - \text{Im} E_1)}{|\ln |\alpha - E_1||^2 |\alpha - E_1|^2} = O\left(\frac{1}{|\ln |\alpha - E_1||^2 |\alpha - E_1|^2}\right).
\]

Hence, by (6.26),

\[
\partial_{\alpha_1} \tilde{F}_1(\xi, \alpha) = \partial_{\alpha_1} F_1(\xi, \varphi(\alpha))\partial_{\alpha_1} \varphi_1(\alpha) + \partial_{\alpha_2} F_1(\xi, \varphi(\alpha))\partial_{\alpha_1} \varphi_2(\alpha)
\]

\[
= \left[\text{Im}\{d_0(\xi) \ln(\varphi(\alpha) - E_1)\} + O(1)\right]\left[\frac{1}{|\ln |\alpha - E_1||} + O\left(\frac{1}{|\ln |\alpha - E_1||^2 |\alpha - E_1|^2}\right)\right]
\]

\[
+ \left[\text{Im}\{id_0(\xi) \ln(\varphi(\alpha) - E_1)\} + O(1)\right]O\left(\frac{1}{|\ln |\alpha - E_1||^2 |\alpha - E_1|^2}\right)
\]

\[
= \frac{\text{Im}\{d_0(\xi) \ln(\alpha - E_1)\}}{|\ln |\alpha - E_1||} + O\left(\frac{\ln |\ln |\alpha - E_1||}{|\ln |\alpha - E_1||}ight)
\]

\[
= -\text{Im} d_0(\xi) + O\left(\frac{\ln |\ln |\alpha - E_1||}{|\ln |\alpha - E_1||}\right),
\]

and

\[
\partial_{\alpha_2} \tilde{F}_1(\xi, \alpha) = \partial_{\alpha_1} F_1(\xi, \varphi(\alpha))\partial_{\alpha_2} \varphi_1(\alpha) + \partial_{\alpha_2} F_1(\xi, \varphi(\alpha))\partial_{\alpha_2} \varphi_2(\alpha)
\]

\[
= \left[\text{Im}\{d_0(\xi) \ln(\varphi(\alpha) - E_1)\} + O(1)\right]O\left(\frac{1}{|\ln |\alpha - E_1||^2 |\alpha - E_1|^2}\right)
\]

\[
+ \left[\text{Im}\{id_0(\xi) \ln(\varphi(\alpha) - E_1)\} + O(1)\right]O\left(\frac{1}{|\ln |\alpha - E_1||^2 |\alpha - E_1|^2}\right)
\]

\[
= \frac{\text{Im}\{id_0(\xi) \ln(\alpha - E_1)\}}{|\ln |\alpha - E_1||} + O\left(\frac{\ln |\ln |\alpha - E_1||}{|\ln |\alpha - E_1||}ight)
\]

\[
= -\text{Re} d_0(\xi) + O\left(\frac{\ln |\ln |\alpha - E_1||}{|\ln |\alpha - E_1||}\right).
\]
Similarly, by (6.22),
\[ \partial_{\alpha_1} \tilde{F}_2(\xi, \alpha) = \partial_{\alpha_1} F_2(\xi, \varphi(\alpha)) \partial_{\alpha_2} \varphi_1(\alpha) \ln |\alpha - E_1| + \partial_{\alpha_2} F_2(\xi, \varphi(\alpha)) \frac{\partial_{\alpha_1} \varphi_1(\alpha)}{|\alpha - E_1|^2} \]
\[ = \left[ q_1(\xi) + O\left( \frac{|\alpha - E_1|}{|\ln |\alpha - E_1||} \right) \right] \left[ 1 + O\left( \frac{1}{|\ln |\alpha - E_1||} \right) \right] + O\left( \frac{|\alpha - E_1|}{|\ln |\alpha - E_1||} \right) \left[ \frac{1 - \alpha_1 - \text{Re} E_1}{|\alpha - E_1|^2} \right] \]
\[ = q_1(\xi) + O\left( \frac{1}{|\ln |\alpha - E_1||} \right) \]

and
\[ \partial_{\alpha_2} \tilde{F}_2(\xi, \alpha) = \partial_{\alpha_1} F_2(\xi, \varphi(\alpha)) \partial_{\alpha_2} \varphi_1(\alpha) \ln |\alpha - E_1| + \partial_{\alpha_2} F_2(\xi, \varphi(\alpha)) \frac{\partial_{\alpha_1} \varphi_1(\alpha)}{|\alpha - E_1|^2} \]
\[ = \left[ q_1(\xi) + O\left( \frac{|\alpha - E_1|}{|\ln |\alpha - E_1||} \right) \right] O\left( \frac{1}{|\ln |\alpha - E_1||} \right) \left[ 1 + O\left( \frac{1}{|\ln |\alpha - E_1||} \right) \right] + O\left( \frac{|\alpha - E_1|}{|\ln |\alpha - E_1||} \right) \left( - \frac{\alpha_2 - \text{Im} E_1}{|\alpha - E_1|^2} \right) \]
\[ = q_2(\xi) + O\left( \frac{1}{|\ln |\alpha - E_1||} \right) \].

The statements of the lemma follow from the above expansions. \( \square \)

Lemma 6.9 implies that \( \tilde{F} : \tilde{S}_r \to \mathbb{R}^2 \) is a \( C^1 \) map such that \( \tilde{F}(\mathbf{x}_0) = 0 \) and
\[ \det \begin{pmatrix} \partial_{\xi} \tilde{F}_1 & \partial_{\omega_2} \tilde{F}_1 \\ \partial_{\xi} \tilde{F}_2 & \partial_{\omega_2} \tilde{F}_2 \end{pmatrix} = \det \begin{pmatrix} f_0(\xi_{E_1}) & -\text{Re} \, f_{00}(\xi_{E_1}) \\ 0 & q_2(\xi_{E_1}) \end{pmatrix} = -8\sqrt{B} (\text{Im} \sqrt{E_2}) q_2(\xi_{E_1}) \neq 0, \]
where we have used the fact that \( q_2(\xi_{E_1}) \neq 0 \) (see Lemma 6.5) in the last step. Hence we can apply the implicit function theorem to conclude that there exists a \( \delta > 0 \) and a \( C^1 \)-curve
\[ \gamma : [\text{Re} \, E_1, \text{Re} \, E_1 + \delta] \to \tilde{S}_r \]
\[ \alpha_1 \mapsto (\alpha_1 + \delta) \gamma \] such that \( \gamma(\text{Re} \, E_1) = \mathbf{x}_0 \), the function \( \tilde{F} \) vanishes identically on the image of \( \gamma \), and
\[ \begin{pmatrix} \xi(\alpha_1) \\ \alpha_2(\alpha_1) \end{pmatrix} = - \begin{pmatrix} \partial_{\xi} \tilde{F}_1 & \partial_{\omega_2} \tilde{F}_1 \\ \partial_{\xi} \tilde{F}_2 & \partial_{\omega_2} \tilde{F}_2 \end{pmatrix}^{-1} \begin{pmatrix} \partial_{\alpha_1} \tilde{F}_1 \\ \partial_{\alpha_2} \tilde{F}_2 \end{pmatrix} \].

The technical complication that \( \mathbf{x}_0 \) lies on the boundary of \( \tilde{S}_r \) can be overcome either by appealing to a boundary version of the implicit function theorem (see [8, Theorem 5]) or by first constructing a \( C^1 \) extension of \( \tilde{F} \) to an open neighborhood of \( \mathbf{x}_0 \) (the existence of such an extension follows, for example, from the Whitney extension theorem) and then applying the standard implicit function theorem.

It follows from the definition (6.33) of \( \tilde{F} \) that \( F \) vanishes on the image of the curve \( \Phi \circ \gamma \), where \( \Phi \) denotes the map \( (\xi, \alpha) \mapsto (\xi, \varphi(\alpha)) \) which is a bijection from \( \tilde{S}_r \) to a subset of \( \tilde{S}_R \). At the
endpoint $x_0$, a computation gives
\[
\left( \begin{array}{c}
\xi' (\text{Re } E_1) \\
\alpha'_2 (\text{Re } E_1) 
\end{array} \right) = \frac{f_0 (\xi E_1) - \text{Re } d_0 (\xi E_1)}{q_2 (\xi E_1)} \left( \begin{array}{c}
- \text{Im } d_0 (\xi E_1) \\
q_1 (\xi E_1) 
\end{array} \right) = \left( \begin{array}{c}
-3A^4 + 7AB^2 - 5B - 8B^3 (E_2 + 3B) \\
\frac{A (E_2 + B)}{B (B + 3 |E_2|)} 
\end{array} \right).
\]

In particular, $\xi' (\text{Re } E_1) < 0$ and $\alpha'_2 (\text{Re } E_1) > 0$.

We finally show (6.10). Let $t \mapsto \gamma(t)$ be a parametrization of $\gamma$ such that $\gamma(0) = x_0$. Since $\gamma$ is $C^1$, we have
\[
\gamma(t) = x_0 + (at, bt, ct) + o(t) \quad t \downarrow 0,
\]
where $\gamma'(0) = (a, b, c)$ with $b > 0$ is proportional to $(\xi' (\text{Re } E_1), 1, \alpha'_2 (\text{Re } E_1))$; in particular, $a < 0$ and $c > 0$. Letting $w := b + ic$, we find
\[
\Phi (\gamma(t)) = x_0 + \left( at + o(t), \frac{wt + o(t)}{|\ln wt + o(t)|} \right) = x_0 + \left( at + o(t), \frac{wt}{|\ln t|} + o \left( \frac{t}{|\ln t|} \right) \right).
\]
Introducing a new parameter $s$ by $s = -at + o(t)$, this becomes
\[
\Phi (\gamma(t)) = x_0 + \left( -s, \frac{c_1 s}{|\ln s|} + o \left( \frac{s}{|\ln s|} \right) \right), \quad s \downarrow 0,
\]
where
\[
c_1 := -\frac{w}{a} = -\frac{1 + i\alpha'_2 (\text{Re } E_1)}{\xi' (\text{Re } E_1)} = \frac{2BE_1}{A^2 + 4AB - 3B^2 - B |E_2|}
\]
satisfies $\text{Re } c_1 > 0$ and $\text{Im } c_1 > 0$. In terms of the curve $\alpha(\xi)$ in (6.9), this can be expressed as (let $s = \xi E_1 - \xi$)
\[
\alpha(\xi) = E_1 + c_1 \frac{\xi E_2 - \xi}{|\ln (\xi E_1 - \xi)|} + o \left( \frac{\xi E_2 - \xi}{|\ln (\xi E_1 - \xi)|} \right), \quad \xi \uparrow \xi E_1,
\]
which proves (6.10). This completes the proof of Theorem 6.1.

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