Phase Shift Sequences for an Adding Interferometer

Peter Hyland\textsuperscript{1}, Brent Follin\textsuperscript{2}, and Emory F. Bunn\textsuperscript{2}

\textsuperscript{1} Physics Department, University of Wisconsin - Madison, Madison, WI 53706
\textsuperscript{2} Physics Department, University of Richmond, Richmond, VA 23173

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\textbf{ABSTRACT}

Cosmic microwave background (CMB) polarimetry has the potential to provide revolutionary advances in cosmology. Future experiments to detect the very weak B mode signal in CMB polarization maps will require unprecedented sensitivity and control of systematic errors. Bolometric interferometry may provide a way to achieve these goals. In a bolometric interferometer (or other adding interferometer), phase shift sequences are applied to the inputs in order to recover the visibilities. Noise is minimized when the phase shift sequences corresponding to all visibilities are orthogonal. We present a systematic method for finding sequences that produce this orthogonality, approximately minimizing both the length of the time sequence and the number of discrete phase shift values required. When some baselines are geometrically equivalent, we can choose sequences that read out those baselines simultaneously, which has been shown to improve signal to noise ratio.

\textbf{Key words.} Techniques: interferometric - techniques: polarimetric - cosmic microwave background

1. Introduction

The field of observational cosmology has been advancing quickly in recent years. Observations of the cosmic microwave background (CMB) radiation have been leading the way, as evidenced by WMAP’s highly successful mapping of CMB anisotropy (Hinshaw et al. 2008), DASI’s detection of the polarized component of the CMB (Leitch et al. 2005), and the 2006 Nobel Prize in Physics awarded to John Mather and George Smoot. Momentum is building for experiments that characterize the CMB polarization in detail (Bock et al. 2006).

A linear polarization map can always be expressed as the sum of two component maps, denoted E and B (Seljak & Zaldarriaga 1997; Kamionkowski et al. 1997). CMB experiments to date have detected only the “curl-free” E component, which is produced primarily by (scalar) density perturbations. The “divergence-free” B component is not produced by scalar perturbations at linear order, and is therefore a clean probe of other, smaller effects. In particular, inflationary models predict a B-mode signal produced by gravitational wave (tensor) perturbations in the early universe. These B modes promise to hold key information about the process of inflation and particle physics above the Grand Unification scale. The challenge of finding the B modes is no small task, however: the B component is expected to be at least an order of magnitude weaker than the E component (which is itself small compared to the temperature anisotropy) over all angular scales. Experiments to search for B modes will require unprecedented sensitivity and control of systematic errors.

Bolometric interferometry is one proposed method for achieving these goals (Polenta et al. 2007; Charlassier 2008; Timbie et al. 2006; Korotkov et al. 2006; Tucker et al. 2008). A bolometric interferometer is a marriage between highly sensitive, incoherent bolometric detectors and the phase-sensitive, systematic-error-reducing observing technique of interferometry. Hamilton et al. (2008) have shown that a bolometric interferometer can achieve sensitivities comparable to traditional technologies. The question of whether bolometric interferometry is useful for CMB polarimetry will thus depend on the method’s ability to control systematic errors. Systematic errors in interferometers are certainly different from those in imaging experiments (Bunn 2007); it can be argued that interferometers are superior in this regard, although this question requires further research.

In a bolometric interferometer, the signals from a set of input feedhorns are combined with either a Butler combiner or a quasi-optical (Fizeau) combiner. In either case, bolometers measure the total power in the combined beam – that is, each bolometer is illuminated by signals from all of the input horns. Since the signal in each detector is the sum of all the inputs, a bolometric interferometer is an example of an “adding” interferometer (as opposed to traditional radio interferometers, which are “multiplying” interferometers). One of the keys to making this method work is to arrange for the phase information to be encoded in the bolometer signals, so that individual pairwise visibilities can be extracted. To achieve this goal, a sequence of phase shifts can be applied to each of the input horns, in such a way that each visibility is phase-shifted in an independent fashion. The resulting time series can be solved for the individual visibilities. The phase shift sequences should be chosen so that this inversion can be done with minimal noise.
We would like the length of the phase shift sequence to be as short as possible, to avoid error due to $1/f$ noise in the detectors. Clearly the number of phase shifts must be at least as large as the number of visibilities to be recovered. In the most general case, an $N$-horn interferometer has $N(N-1)/2$ distinct visibilities, requiring long phase shift sequences for interferometers with many inputs. On the other hand, if the input horns are arranged in a regular pattern, such as a square array, then many antenna pairs correspond to identical visibilities. These can be given identical phase shifts and read out together. This coherent treatment of equivalent (or redundant) baselines has two advantages. First, it allows for shorter phase shift sequences. Second, by coadding equivalent signals, the signal-to-noise ratio is improved (Charlassier et al. 2008, hereinafter C08).

C08 gave an excellent overview of how a bolometric interferometer works and considered the choice of phase shift sequences in detail. For the case of a square array of horns, the paper presented a method of phase modulating the inputs that gives equivalent baselines identical phase shift sequences. In this paper we present independently-developed work on phase modulation and coherent addition of baselines that complements the methods of C08. We consider general horn arrangements as well as a regular square lattice. In the general case, we present a method for finding the optimal phase shift sequence assuming no baselines are redundant. In the case of a square array, we present a refinement of the method of C08. Unlike the original method, which achieves optimal noise performance only in the limit as the number of time steps tends to infinity, our method is optimal for sequences of nearly or exactly the minimum possible length.

In Sect. 2 we present our formalism for denoting sequences of phase shifts and consider the criterion for an optimal phase shift sequence. Section 3 introduces a shorthand notation and applies this to a method for constructing bases for a square array of horns, accounting for redundant baselines. For simplicity, we consider only one linear polarization state. Here we assume that the phase shifts $\Delta \phi$ are fixed by the geometry of the system and do not vary in time. In this expression, the detector can correspond either to a single point on the focal plane of a quasi-optical combiner or to a single output of a Butler combiner.

We are assuming here that all inputs contribute to all detectors with equal amplitude. If this assumption is relaxed, then an additional real factor $A_{jm}$ would need to be included in each term of the sum. The presence of these factors would affect the overall sensitivity of the detector to the various visibilities, but we do not expect it to influence the optimal choice of phase shifts, so we omit it.

Let us assume for the moment that we wish to recover the visibility associated with each pair of horns separately; we will return below to the case in which redundant baselines are coherently added before detection. We wish to choose the phase shifts $\phi_j(t)$ to enable recovery of all of the cross terms $E_j^* E_k$ from each detector. In principle, we could aim for a weaker goal, namely to ensure that each cross term be recoverable from the full set of detector outputs $S_1(t), S_2(t), \ldots$; however, to avoid systematic errors resulting from subtracting signals in different detectors, it is preferable to insist that each visibility be recovered from each detector separately.

Since we are focusing on one detector at a time, we suppress the subscript $m$ in eq. (1). Furthermore, the time-independent phase shifts $\Delta \phi_i$ do not affect the problem of visibility recovery, so we suppress these as well. Finally, we assume that the phase shifts $\phi_j(t)$ are changed in discrete time steps, so we replace the functions $\phi_j(t)$ with sequences $\phi_{jt}$. Here $t = 1, 2, \ldots, M$, where $M$ is the number of steps in the phase shift sequence at each detector. With these changes, equation (1) becomes

$$S_t \propto \sum_{j=1}^{N} E_j e^{i(\Delta j + \phi_j)} = \sum_{j,k=1}^{N} E_j^* E_k e^{i(\Delta j - \Delta k + \phi_j - \phi_k)},$$

where the phase shifts $\Delta j$ are fixed by the geometry of the system and do not vary in time. In this expression, the detector can correspond either to a single point on the focal plane of a quasi-optical combiner or to a single output of a Butler combiner.

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$$S_t \propto \sum_{j,k=1}^{N} E_j^* E_k e^{i(\phi_j - \phi_k)} = V_{jk} e^{i(\phi_j - \phi_k)}.$$

Finally, we assume that the phase shifts can take on $P$ equally-spaced values from 0 to $2\pi$:

$$\phi_{jt} \in \{2\pi p/P \mid p = 0, 1, 2, \ldots, P - 1\}.$$  

Given the time sequence of measurements $S_1, S_2, \ldots, S_M$, we wish to recover all $N(N-1)$ complex visibilities $V_{jk} = E_j^* E_k$ with $j \neq k$ (or equivalently to recover both real and imaginary parts of all pairs with $j < k$). In addition,
we will always recover the total power \( \sum_j V_{jj} = \sum_j |E_j|^2 \), which enters each \( S_t \) equally. Solving for the cross terms is therefore simply inverting a linear system of \( M \) equations for \( N(N-1)+1 \) unknowns. Generically, we expect this to be possible as long as
\[
M \geq N(N-1) + 1 \equiv N_{\text{vis}}. \tag{4}
\]

We want to insist not just that the visibilities be recovered, but that they be recovered with minimal possible noise. To be specific, the recovery problem we wish to solve is
\[
\mathbf{S} = \mathbf{AV}, \tag{5}
\]
where \( \mathbf{S} \) is the \( M \)-dimensional signal vector, \( \mathbf{V} \) is the \( N_{\text{vis}} \)-dimensional vector of visibilities to be recovered, and \( \mathbf{A} \) is a matrix whose elements are determined by the phase shifts:
\[
A_{tm} = e^{i(\phi_{jt}-\phi_{tk})}, \tag{6}
\]
where the \( m \)th visibility \( V_m \) corresponds to the horn pair \( jk \). In this situation, where all elements of the matrix \( \mathbf{A} \) have absolute value equal to one, the minimum possible noise contributions to the visibilities is achieved when all columns of \( \mathbf{A} \) are orthogonal:
\[
\sum_t A_{tm}^* A_{tm'} = 0 \quad \text{for } m \neq m' \tag{7}
\]
or equivalently for \( \mathbf{A}^\dagger \mathbf{A} \) proportional to the identity matrix.

A proof of this statement is provided in Appendix A. Intuitively, it says that the visibilities are recovered with minimum noise when they are maximally independent of each other, that is, when they contribute orthogonally to the time series of signals at the detector.

Let us summarize. For any pair of horns \( j, k \), define an \( M \)-dimensional vector
\[
\Phi_{jk} = (e^{i(\phi_{j1}-\phi_{k1})}, e^{i(\phi_{j2}-\phi_{k2})}, \ldots, e^{i(\phi_{jM}-\phi_{kM})}). \tag{8}
\]
Our goal is to choose the set of phase shifts \( \phi_j \) such that the vectors \( \Phi_{jk} \) and \( \Phi_{j'k'} \) are orthogonal whenever \( (jk) \neq (j'k') \). When this condition is satisfied, each visibility is recovered simply by taking the dot product of the detector signal with the corresponding vector \( \Phi \); the estimator of \( V_{jk} \) is
\[
\hat{V}_{jk} = \frac{1}{M} \Phi_{jk}^\dagger \mathbf{S} = \frac{1}{M} \sum_{t=1}^M e^{-i(\phi_{jt}-\phi_{kt})} S_t. \tag{9}
\]
We will call the vector \( \Phi_{jk} \) the “mask” for the baseline \( jk \).

Note that \( \Phi_{jk} \) and \( \Phi_{kj} \) are complex conjugates of each other. The requirement that these be orthogonal, which means roughly that the elements of \( \Phi_{jk} \) uniformly sample directions in the complex plane, is necessary for both the real and imaginary parts of \( V_{jk} \) to be recovered with minimum noise.

It may be instructive to compare the phase shift schemes for the bolometric interferometer with those applied in a traditional multiplying interferometer. In traditional interferometry, orthogonal patterns of square-wave phase shifts (e.g., Walsh functions) are applied to each of the input antennas in order to reduce the response of the instrument to spurious signals (e.g., Thompson et al. 2001). The phase shift patterns we require in the adding interferometer must obey a more stringent orthogonality requirement: rather than merely demanding orthogonality of all of the input phase shifts (i.e., demanding that the \( \phi_j \) be orthogonal), we require that the phase shifts associated with all visibilities (i.e., all \( \Phi_{jk} \)) be orthogonal.

### 3. Method for finding phase shifts

Let us suppose that the number \( N \) of horns is fixed, as is the number \( P \) of possible phase shift values. We wish to find the shortest sequence of time steps (that is, the minimum \( M \)) that satisfies our orthogonality criterion. Alternatively, given \( M, P \), we can ask for the maximum number of horns that can be accommodated.

We will introduce the following shorthand notation for the possible phase factors:
\[
[p] = e^{i2\pi p/P}, \quad p = 0, 1, 2, \ldots P-1. \tag{10}
\]
For purposes of illustration, we will consider the case \( P = 4 \) in this section, so that the four possible phase shift values are
\[
[0123] = (1, i, -1, -i). \tag{11}
\]
The method we outline generalizes to other values of \( P \).

Let the number of time steps \( M \) be a power of 4: \( M = 4^\mu \) for some positive integer \( \mu \). We can define a set of \( \mu \) mutually orthogonal \( M \)-dimensional vectors as follows: the vector \( \alpha_1 \) is obtained by stepping through the four possible
phase values as slowly as possible – that is, it consists of $M/4$ repetitions of $[0]$, followed by $M/4$ repetitions of $[1]$, etc. Each subsequent vector $\alpha_j$ cycles through the possible phases four times faster until the last one $\alpha_\mu$, which consists of $M/4$ repetitions of the sequence $[0123]$. To be explicit, here is the case $\mu = 3$:

$$\alpha_1 = [000000000000011111111111122222222222223333333333333]$$  \hspace{1cm} (12)  
$$\alpha_2 = [000011112222333300011112222233330000111122223333]$$  \hspace{1cm} (13)  
$$\alpha_3 = [0123012301230123012301230123012301230123012301230123012301]$$  \hspace{1cm} (14)  

Here is an alternative description of the construction of these vectors: the $k$th element of the vector $\alpha_j$ is the $j$th-most significant digit in the base-4 expression for $(k-1)$. In the general case with $M = P^\mu$, $\alpha_1$ steps through the $P$ values as slowly as possible and each subsequent $\alpha_j$ cycles $P$ times faster.

We now define

$$\langle j_\mu, \ldots, j_2, j_1 \rangle = \alpha_\mu^{j_\mu} \cdots \alpha_2^{j_2} \alpha_1^{j_1}$$  \hspace{1cm} (15)  

for integers $j_\mu, \ldots, j_2, j_1$ between 0 and 3. Here multiplication and exponentiation are performed elementwise in each vector. Since $[p]$ is shorthand for $e^{ip\pi/2}$, multiplication corresponds to addition modulo 4 on the values in square brackets. For instance, in the case $\mu = 2$,

$$(2,1) = \alpha_2^2 \alpha_1 = [01230123012301230123012301230123012301230123012301230123]$$

$$(17)$$

$$(18)$$

It is straightforward to check that the vectors $\langle j_\mu, \ldots, j_2, j_1 \rangle$ are all mutually orthogonal. Since there are $4^\mu$ distinct vectors, they are a maximal set of orthogonal vectors. We can therefore search among this set for the optimal set of $N$ phase shift patterns to apply to our input horns.

As an example, consider the case $\mu = 2$, that is, let the number of time steps be $M = 4^2 = 16$. We will determine the maximum number of $N$ that can be accommodated. We proceed by assigning phase shift sequences to the horns one at a time. Without loss of generality, we can assume that the first horn has no phase shift at all (since any phase shift sequence can be subtracted from all inputs without altering the solution):

$$\phi_0 = (0,0) = \alpha_0^0 \alpha_1^0 = \alpha_0 = [0000000000000000].$$  \hspace{1cm} (19)  

$$\phi_1 = (0,1) = \alpha_0^0 \alpha_1^1 = \alpha_1 = [0000111122223333].$$  \hspace{1cm} (20)  

Here $\phi_j$ refers to the $M$-dimensional vector $(e^{i\phi_j1}, e^{i\phi_j2}, \ldots, e^{i\phi_jM})$. We can accommodate two more inputs by choosing

$$\phi_2 = (1,0) = \alpha_0^1 \alpha_1^0 = \alpha_2 = [01230123012301230123012301230123012301230123012301230123]$$  \hspace{1cm} (21)  

For these three input horns, we have six distinct baselines, with masks [equation (5)]

$$\Phi_{01} = \langle 0,3 \rangle, \quad \Phi_{02} = \langle 3,0 \rangle, \quad \Phi_{12} = \langle 3,1 \rangle,$$

$$\Phi_{10} = \langle 0,1 \rangle, \quad \Phi_{20} = \langle 1,0 \rangle, \quad \Phi_{21} = \langle 1,3 \rangle.$$  \hspace{1cm} (22)  

These are obtained by subtracting the values in angle brackets for the two horns modulo 4. For instance, $\Phi_{12} = \langle 0,1 \rangle - (1,0) = \langle -1,1 \rangle = \langle 3,1 \rangle$. These masks are all distinct, and hence mutually orthogonal, and furthermore are all orthogonal to the vector $(0,0)$, which is sensitive to the total power.

This construction shows that we can accommodate three horns with a sequence of 16 time steps. We next ask whether it is possible to accommodate a fourth vector $\phi_3$ in such a way that the new masks $\Phi_{03}, \Phi_{13}, \Phi_{23}, \Phi_{33}, \Phi_{31}, \Phi_{32}$, etc. are independent of the ones we have already found. A search of the $16 - 7 = 9$ candidates reveals an affirmative answer: $\phi_3 = \langle 3,3 \rangle = \alpha_0^3 \alpha_1^3$ works.

The value $N = 4$ is the maximum that can be achieved for the case of $M = 4^2$ time steps, as is clear from a counting argument: $N = 5$ horns would require at least $M = N(N-1) + 1 = 21$ steps.

Table IV shows the maximum number of horns that can be accommodated for various values of $M$. These were found by recursively searching the space of possible phase shifts in the manner described above. The last column shows the maximum value that would be possible according to the simple counting argument that the number of time steps must exceed the number of baselines. We have repeated this analysis for the case $P = 2$, where the phase shifters are capable of only 0 and 180° shifts, and found very similar results for the relationship between $M$ and $N$.

As noted in the introduction, extremely large values of $M$ are impractical. This is one reason that a bolometric interferometer with a large number of horns should surely be designed with a high degree of symmetry, so that there are many equivalent baselines that can be read out coherently. (The other reason is the signal-to-noise advantage.)

In summary, this section has presented a procedure for selecting $\phi_j$ that yields fully orthogonal masks. This means that the result of applying the mask for a given baseline will only be sensitive to the signal from the desired baseline, or equivalently that the reconstruction of all visibilities is accomplished with minimal noise.
M & N_{\text{max}} \text{ (actual)} & N_{\text{max}} \text{ (counting)} \\
\hline
4 & 2 & 2 \\
4^2 = 16 & 4 & 4 \\
4^4 = 64 & 8 & 8 \\
4^8 = 256 & 15 & 16 \\
4^{16} = 1024 & 24 & 32 \\
4^{32} = 4096 & 40 & 64 \\
\hline
\end{tabular}

Table 1. The number of horns $N$ that can be accommodated with a given number of time steps $M$. We assume $P = 4$ distinct phase shift values. The second column shows the maximum number that can be accommodated, while the third column shows the number found by the simple counting argument $N(N - 1) + 1 \leq M$.

### 4. Square Array

We now consider the case where the input horns are arranged in a square array with $N_{\text{side}}$ horns on a side. In this case, many different baselines (i.e., pairs of horns) sample the same visibility. We wish to apply identical phase shifts to such equivalent baselines, so that a single mask reads out their sum. Naturally, we also require that inequivalent baselines have orthogonal masks. This is the case considered in detail by Charlassier et al. (C08). Our method parallels theirs in many respects but refines it in some ways.

Following the notation of C08, we parameterize the position of horns in the array in units of the minimum horn separation as a vector $d_j = (l_j, m_j)$. Here $l_j, m_j$ are integers running from 0 to $N_{\text{side}} - 1$, labeling the position of the horn in along the $x$ and $y$ directions. The index $j$ runs from 0 to $N_{\text{side}}^2 - 1$ according to $j = l_j + N_{\text{side}}m_j$. We can construct a set of phase shifts for all horns that satisfy the desired criteria using the basis described in the previous section with $\mu = 2$. We let the phase shift sequence for horn $j$ be

$$\phi_j = (l_j, m_j).$$

Below we have explicitly written out the modulations for each horn in a $6 \times 6$ array.

\begin{align*}
(0, 0) & \langle 0, 0 \rangle \langle 2, 0 \rangle \langle 3, 0 \rangle \langle 4, 0 \rangle \langle 5, 0 \rangle \\
(0, 1) & \langle 1, 1 \rangle \langle 2, 1 \rangle \langle 3, 1 \rangle \langle 4, 1 \rangle \langle 5, 1 \rangle \\
(0, 2) & \langle 1, 2 \rangle \langle 2, 2 \rangle \langle 3, 2 \rangle \langle 4, 2 \rangle \langle 5, 2 \rangle \\
(0, 3) & \langle 1, 3 \rangle \langle 2, 3 \rangle \langle 3, 3 \rangle \langle 4, 3 \rangle \langle 5, 3 \rangle \\
(0, 4) & \langle 1, 4 \rangle \langle 2, 4 \rangle \langle 3, 4 \rangle \langle 4, 4 \rangle \langle 5, 4 \rangle \\
(0, 5) & \langle 1, 5 \rangle \langle 2, 5 \rangle \langle 3, 5 \rangle \langle 4, 5 \rangle \langle 5, 5 \rangle \\
\end{align*}

In this case, the mask for the visibility corresponding to any pair of horns is simply $\langle \Delta l, \Delta m \rangle$. This means that all pairs with the same relative spacing get the same mask. Furthermore, as long as the number of phase shift steps $P$ is large enough, all inequivalent visibilities correspond to orthogonal masks as desired. The minimum value of $P$ is set by the fact that phases are only defined modulo $P$. Since $\Delta l, \Delta m$ can range from $-(N_{\text{side}} - 1)$ to $N_{\text{side}} - 1$, we need at least $P = 2(N_{\text{side}}^2 - 1) + 1$ distinct phase shifts. (As we will see in the next section, it may be desirable for $P$ to be a multiple of 3, in which case we simply round up to the nearest such value.) If $P$ is smaller than this, then distinct visibilities will be mapped onto the same phase shift sequence. For the above case, for example, we require $P \geq 11$. If we tried a smaller value, say $P = 10$, then the visibility corresponding to horns $(0, 0)$ and $(5, 1)$, for example, would get the same phase shift sequence as $(5, 0)$ and $(0, 1)$, namely $(5, 1) = (5, 1)$.

It is instructive to compare this scheme with the very similar one of C08. In both methods, the phase shift sequence for horn $(l, m)$ is expressed in the form $\langle h + mv \rangle$ for two basis shift patterns $h, v$. In order to achieve the desired orthogonality properties, Charlassier et al. choose $h, v$ to be independent random vectors of phase shifts. The randomness ensures approximate orthogonality, up to errors of order $M^{-1/2}$, where $M$ is the length of the phase shift sequence. In contrast, we choose $h = (1, 0)$ and $v = (0, 1)$. This results in strict orthogonality, as opposed to approximate orthogonality.

The number $M$ of distinct phase shift values required is essentially the same in the two methods. As in our method, C08 found that $P \approx 2N_{\text{side}}$ was required in order to produce orthogonal phase shifts using random basis vectors.

Using either method, the length $M$ of the modulation sequences is greatly reduced compared to the case of inequivalent baselines. The number of required phase shifts is $M = P^2 = (2N_{\text{side}}^2 - 1)^2$ when redundant baselines are tagged equivalently. If we instead used the methods of the previous section, we would require $M = N_{\text{side}}^2(N_{\text{side}}^2 - 1) + 1 \approx N_{\text{side}}^4$. For the $6 \times 6$ array denoted above, this is the difference between a 121-step sequence and a 1261-step sequence. Even more important is the signal-to-noise benefit of coadding equivalent baselines.

Although we have described this procedure as applying to a square array, it is in fact more general. It applies whenever the horn positions can be expressed as integer multiples of any two basis vectors, even if the two are not orthogonal, or in other words, to any parallelogram-shaped array. Furthermore, it can be applied to any subset of such a parallelogram-shaped array, since we can simply ignore the parts of the array with no horns in them. In particular, this means that the method can be applied to a hexagonal close-packed array of horns, as shown in Fig. 11.
For the other polarization state, we apply a slow three-phase modulation to this sequence: the first sequence of 3 for a particular baseline, then the next two measured visibilities, each of which contains a contribution proportional to the much larger Stokes parameter $I$. As eq. (24) indicates, the visibility for Stokes $Q$ in this case, we can define phase shift sequences similarly in terms of the triple $\langle l, m, 0 \rangle$ whose phase is $\frac{\pi}{3}l$. Note that this scheme is most natural to apply when $p$ is a multiple of 3 so that for every phase state $p$ there is another whose phase is $p + 2\pi/3$. Otherwise, the set of phase shifts involved in the sequences $\langle l, m, 1 \rangle$ will be larger than that involved in $\langle l, m, 0 \rangle$. In implementing this scheme, one would surely round $P$ up to the nearest multiple of 3.

However, as C08 have pointed out, it is impossible to recover all visibilities while taking full advantage of the noise reduction resulting from coadding redundant baselines. C08 describe two schemes for recovering some of the Stokes parameters with full accuracy, one of which (mode 2 of C08) involves measuring the visibilities for Stokes $I, U, V$ but not $Q$. Stokes $Q$ can then be measured by rotating the instrument 45°. In this section, we show how to implement this mode of operation using our phase shifting scheme.

Aside from the phase shifting scheme, there is another reason for adopting an observing scheme in which Stokes $Q$ is measured only by rotating the instrument. As eq. (24) indicates, the visibility for Stokes $Q$ is obtained by subtracting two measured visibilities, each of which contains a contribution proportional to the much larger Stokes parameter $I$. As a result, this visibility is likely to be subject to much larger errors than the other linear polarization (Stokes $U$).

As in the previous section, we assume an $N_{\text{side}} \times N_{\text{side}}$ array of horns, but now we introduce an orthomode transducer for each horn, doubling the number of signals to be interfered. We can represent each of these $2N_{\text{side}}^2$ signals with a triple of labels $(l_j, m_j, n_j)$ where $(l_j, m_j)$ label the position of the horn as in the previous section, and $n_j = 0, 1$ labels the polarization state. In the previous section, we identified the horn $(l_j, m_j)$ with a phase shift sequence $\langle l_j, m_j, 0 \rangle$. In the present case, we can define phase shift sequences similarly in terms of the triple $\langle l_j, m_j, n_j \rangle$, each of which represents a sequence of $3P^2$ time steps, where $P \geq 2N_{\text{side}} - 1$ as in the previous section. For one of the two polarization states, we divide the phase shift sequences from each polarization state into three groups, each containing $3P^2$ elements:

$$\langle l, m, 0 \rangle = \langle (l, m), (l, m), (l, m) \rangle.$$  

(27)

For the other polarization state, we apply a slow three-phase modulation to this sequence: the first $P^2$ steps are unchanged, the next $P^2$ steps are multiplied by $e^{2\pi i/3}$, and the final block is multiplied by $e^{4\pi i/3}$:

$$\langle l, m, 1 \rangle = \left( \langle l, m \rangle, e^{2\pi i/3} \langle l, m \rangle, e^{4\pi i/3} \langle l, m \rangle \right).$$  

(28)

Note that this scheme is most natural to apply when $P$ is a multiple of 3 so that for every phase state $p$ there is another whose phase is $p + 2\pi/3$. Otherwise, the set of phase shifts involved in the sequences $\langle l, m, 1 \rangle$ will be larger than that involved in $\langle l, m, 0 \rangle$. In implementing this scheme, one would surely round $P$ up to the nearest multiple of 3.

It is straightforward to check that all equivalent baselines have identical phase shift sequences as desired. All pairs that interfere $x$ and $y$ polarization have independent, orthogonal phase shift sequences, allowing optimal reconstruction of Stokes $U, V$ visibilities [eqs. (25), (26)]. Those that interfere $x$ and $x$ have identical sequences to those that interfere $y$ and $y$. Applying these phase shift masks therefore allows recovery of the sum of these visibilities, which is $V_I$.

As in the previous section, this method is similar to that of C08, except that our method imposes strict orthogonality on distinct baselines, as opposed to relying on the approximate statistical orthogonality that results from choosing random phase shift sequences.

As an example, consider a square array with $N_{\text{side}} = 8$. The number of different phase shift values must satisfy $P \geq 2N_{\text{side}} - 1 = 15$. The length of the phase shift sequences is $M = 3P^2 = 675$. The shortest sequence of phase shifts we could possibly hope for would have $M$ equal to the number of unknowns we are trying to solve for. In this arrangement,
there are 112 inequivalent baselines, each of which has three complex visibilities that are measured, and in addition the total power in $I, Q, U$ are measured, resulting in a total of $6 \times 112 + 3 = 675$ unknowns. Our phase shift sequence is therefore as short as possible. For comparison, according to Fig. 4 of C08, the optimal noise levels in the C08 scheme are obtained only when the phase shift sequence is therefore as short as possible. For example, hexagonal arrays.

The ability to shorten the sequence of phase shifts is likely to be important in instrument design, because it reduces the degree to which $1/f$ noise must be controlled. For example, suppose that we can shift phase states at a rate of one state per 10 ms (either because of the design of the phase shifters or the bolometer time constants). As we saw in the previous section, an $8 \times 8$ array requires $\sim 1000$ phase shifts, which would take 10 seconds. We therefore require the $1/f$ noise knee to be below $\sim 0.1$ Hz. An alternative scheme involving a longer phase shift sequence would require correspondingly tighter control of the $1/f$ knee.

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References

Bock, J. et al. 2006, arXiv:astro-ph/0604101
Bunn, E. F. 2007, Phys. Rev. D, 75, 083517, arXiv:astro-ph/0607312
Charllassier, R., 2008, arXiv:astro-ph/0805.4527
Charllassier, R., Hamilton, J., Bréelle, É., Ghribi, A., Giraud-Héraud, Y., Kaplan, J., Piat, M., & Prêle, D. 2008, arXiv:astro-ph/0806.0380, (C08)
Hamilton, J., Charllassier, R., Cressiot, C., Kaplan, J., Piat, M., & Rosset, C. 2008, arXiv:astro-ph/0807.0438
Hinshaw, G. et al. 2008, arXiv:astro-ph/0803.0732
Kamionkowski, M., Kosowsky, A., & Stebbins, A. 1997, Physical Review Letters, 78, 2058, arXiv:astro-ph/9609132
Korotkov, A. L. et al. 2006, in Presented at the Society of Photo-Optical Instrumentation Engineers (SPIE) Conference, Vol. 6275, Millimeter and Submillimeter Detectors and Instrumentation for Astronomy III. Edited by Zmuidzinas, Jonas; Holland, Wayne S.; Withington, Stafford; Duncan, William D. Proceedings of the SPIE, Volume 6275, pp. 62750X (2006).
Leitch, E. M., Kovac, J. M., Halverson, N. W., Carlstrom, J. E., Pryke, C., & Smith, M. W. E. 2005, ApJ, 624, 10, arXiv:astro-ph/0409357
Polenta, G. et al. 2007, New Astronomy Review, 51, 256
Seljak, U., & Zaldarriaga, M. 1997, Physical Review Letters, 78, 2054, arXiv:astro-ph/9609169
Thompson, A. R., Moran, J. M., & Swenson, Jr., G. W. 2001, Interferometry and Synthesis in Radio Astronomy, 2nd Edition (Wiley)
Timbie, P. T. et al. 2006, New Astronomy Review, 50, 999
Tucker, G. S. et al. 2008 in (SPIE), 70201M

Appendix A: Proof of minimum-noise condition

In this section we provide a proof of the assertion that orthogonal phase shift patterns minimize the noise in the recovered visibilities.

Assume that the visibilities are arranged in an $N_{\text{vis}}$-dimensional vector $\mathbf{V}$, and the observed signals are arranged in an $M$-dimensional vector $\mathbf{S}$. The two are related by an $M \times N_{\text{vis}}$ matrix $\mathcal{A}$:

$$\mathbf{S} = \mathcal{A} \mathbf{V}$$  \hspace{1cm} (A.1)

All entries of $\mathcal{A}$ are complex numbers with absolute value 1. We assume that $M \geq N_{\text{vis}}$ and that the matrix $\mathcal{A}$ has maximal rank, so that it is possible to solve for the unknown visibilities.

Assuming that the signals are contaminated with white noise with variance $\sigma^2$, the optimal reconstruction of the visibilities is the least-squares vector

$$\hat{\mathbf{V}} = (\mathcal{A}^\dagger \mathcal{A})^{-1} \mathcal{A}^\dagger \mathbf{S}.$$  \hspace{1cm} (A.2)
The noise covariance matrix for $\hat{V}$ is

$$\mathcal{N} = \sigma^2 (A^\dagger A)^{-1}. \quad (A.3)$$

The noise in the $j$th recovered visibility has variance $\mathcal{N}_{jj}$. We wish to show that this noise is minimized when the matrix $A$ has orthogonal columns.

The diagonal elements of the inverse noise matrix are

$$\langle \mathcal{N}^{-1} \rangle_{jj} = \sigma^{-2} (A^\dagger A)_{jj} = \sigma^{-2} \sum_{m=1}^{M} A^*_{mj} A_{mj} = \frac{M}{\sigma^2}.$$ \quad (A.4)

We can therefore write

$$\mathcal{N}^{-1} = \frac{M}{\sigma^2} (I + D), \quad (A.5)$$

where $I$ is the identity matrix and $D$ is a hermitian matrix with zeroes along the diagonal.

The noise covariance matrix is

$$\mathcal{N} = \frac{\sigma^2}{M} (I - D + D^2 - D^3 + \ldots) = \frac{\sigma^2}{M} [I - D + D(1 - D + D^2 - \ldots)D] = \frac{\sigma^2}{M} (I - D + DND). \quad (A.6)$$

Since $D$ has no diagonal elements, an arbitrary diagonal element of the noise covariance matrix is

$$\mathcal{N}_{jj} = \frac{\sigma^2}{M} [1 + (DND)_{jj}] = \frac{\sigma^2}{M} [1 + v^\dagger \mathcal{N} v], \quad (A.7)$$

where $v_k = D_{kj}$. Since $\mathcal{N}$ is a positive definite matrix, we conclude that

$$\mathcal{N}_{jj} \geq \frac{\sigma^2}{M}. \quad (A.8)$$

That is, the minimum noise variance achievable on any one visibility is $\sigma^2/M$. This value is achieved when the matrix $A$ is column orthogonal, since in this case $A^\dagger A = (M/\sigma^2) I$ and $\mathcal{N} = (\sigma^2/M) I$. 