Finite groups with permutable Hall subgroups

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Abstract

Let \( \sigma = \{ \sigma_i | i \in I \} \) be a partition of the set of all primes \( \mathbb{P} \) and \( G \) a finite group. A set \( \mathcal{H} \) of subgroups of \( G \) is said to be a complete Hall \( \sigma \)-set of \( G \) if every member \( \neq 1 \) of \( \mathcal{H} \) is a Hall \( \sigma_i \)-subgroup of \( G \) for some \( i \in I \) and \( \mathcal{H} \) contains exactly one Hall \( \sigma_i \)-subgroup of \( G \) for every \( i \) such that \( \sigma_i \cap \pi(G) \neq \emptyset \).

In this paper, we study the structure of \( G \) assuming that some subgroups of \( G \) permutes with all members of \( \mathcal{H} \).

1 Introduction

Throughout this paper, all groups are finite and \( G \) always denotes a finite group. We use \( \pi(G) \) to denote the set of all primes dividing \( |G| \). A subgroup \( A \) of \( G \) is said to permute with a subgroup \( B \) if \( AB = BA \). In this case they say also that the subgroups \( A \) and \( B \) are permutable.

Following [1], we use \( \sigma \) to denote some partition of \( \mathbb{P} \). Thus \( \sigma = \{ \sigma_i | i \in I \} \), where \( \mathbb{P} = \cup_{i \in I} \sigma_i \) and \( \sigma_i \cap \sigma_j = \emptyset \) for all \( i \neq j \).

A set \( \mathcal{H} \) of subgroups of \( G \) is a complete Hall \( \sigma \)-set of \( G \) [2, 3] if every member \( \neq 1 \) of \( \mathcal{H} \) is a Hall \( \sigma_i \)-subgroup of \( G \) for some \( \sigma_i \in \sigma \) and \( \mathcal{H} \) contains exactly one Hall \( \sigma_i \)-subgroup of \( G \) for every \( i \) such that \( \sigma_i \cap \pi(G) \neq \emptyset \). If every two members of \( \mathcal{H} \) are permutable, then \( \mathcal{H} \) is said to be a \( \sigma \)-basis [4] of \( G \). In the case when \( \mathcal{H} = \{ \{2\}, \{3\}, \ldots \} \) a complete Hall \( \sigma \)-set \( \mathcal{H} \) of \( G \) is also called a complete set of Sylow subgroups of \( G \).

We use \( \mathcal{H}_\sigma \) to denote the class of all soluble groups \( G \) such that every complete Hall \( \sigma \)-set of \( G \) forms a \( \sigma \)-basis of \( G \).

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A large number of publications are connected with studying the situation when some subgroups of $G$ permute with all members of some fixed complete set of Sylow subgroups of $G$. For example, the classical Hall’s result states: *$G$ is soluble if and only if it has a Sylow basis, that is, a complete set of pairwise permutable Sylow subgroups.* In [5] (see also Paragraph 3 in [6, VI]), Huppert proved that $G$ is a soluble group in which every complete set of Sylow subgroups forms a Sylow basis if and only if the automorphism group induced by $G$ on every its chief factor $H/K$ has the order divisible by at most one different from $p$ prime, where $p \in \pi(H/K)$. In the paper [7], Huppert proved that if $G$ is soluble and it has a complete set $S$ of Sylow subgroups such that every maximal subgroup of every subgroup in $S$ permutes with all other members of $S$, then $G$ is supersoluble.

The above-mentioned results in [5, 6, 7] and many other related results make natural to ask:

(I) Suppose that $G$ has a complete Hall $\sigma$-set $H$ such that every maximal subgroup of any subgroup in $H$ permutes with all other members of $H$. What we can say then about the structure of $G$? In particular, does it true then that $G$ is supersoluble in the case when every member of $H$ is supersoluble?

(II) Suppose that $G$ possesses a complete Hall $\sigma$-set. What we can say then about the structure of $G$ provided every complete Hall $\sigma$-set of $G$ forms a $\sigma$-basis in $G$?

Our first observation is the following result concerning Question (I).

**Theorem A.** Suppose that $G$ possesses a complete Hall $\sigma$-set $H$ all whose members are supersoluble. If every maximal subgroup of any non-cyclic subgroup in $H$ permutes with all other members of $H$, then $G$ is supersoluble.

In the classical case, when $\sigma = \{\{2\}, \{3\}, \ldots\}$, we get from Theorem A the following two known results.

**Corollary 1.1** (Asaad M., Heliel [8]). If $G$ has a complete set $S$ of Sylow subgroups such that every maximal subgroup of every subgroup in $S$ permutes with all other members of $S$, then $G$ is supersoluble.

Note that Corollary 1.1 is proved in [8] on the base of the classification of all simple non-abelian groups. The proof of Theorem A does not use such a classification.

**Corollary 1.2** (Huppert [6, VI, Theorem 10.3]). If every Sylow subgroup of $G$ is cyclic, then $G$ is supersoluble.

The class $1 \in \mathcal{F}$ of groups is said to be a *formation* provided every homomorphic image of $G/\Phi(G)$ belongs to $\mathcal{F}$. The formation $\mathcal{F}$ is said to be: *saturated* provided $G \in \mathcal{F}$ whenever $G/\Phi(G) \leq \Phi(G)$; *hereditary* provided $G \in \mathcal{F}$ whenever $G \leq A \in \mathcal{F}$.

Now let $p > q > r$ be primes such that $qr$ divides $p - 1$. Let $P$ be a group of order $p$ and $QR \leq \text{Aut}(P)$, where $Q$ and $R$ are groups with order $q$ and $r$, respectively. Let $G = P \rtimes (QR)$. Then, in view of the above-mentioned Hupper’s result in [5], $G$ is not a group such that every complete set of Sylow subgroups forms a Sylow basis of $G$. But it is easy to see that every complete
Hall $\sigma$-set of $G$, where $\sigma = \{\{2,3\}, \{7\}, \{2,3,7\}'\}$, is a $\sigma$-basis of $G$. This elementary example is a motivation for our next result, which gives the answer to Question (II) in the universe of all soluble groups.

**Theorem B.** The class $\mathfrak{H}_\sigma$ is a hereditary formation and it is saturated if and only if $|\sigma| \leq 2$. Moreover, $G \in \mathfrak{H}_\sigma$ if and only if $G$ is soluble and the automorphism group induced by $G$ on every its chief factor of order divisible by $p$ is either a $\sigma_i$-group, where $p \notin \sigma_i$, or a $(\sigma_i \cup \sigma_j)$-group for some different $\sigma_i$ and $\sigma_j$ such that $p \in \sigma_i$.

In the case when $\sigma = \{\{2\}, \{3\}, \ldots\}$ we get from Theorem B the following

**Corollary 1.3** (Huppert [5]). Every complete set of Sylow subgroups of a soluble group $G$ forms a Sylow basis of $G$ if and only if the automorphism group induced by $G$ on every its chief factor $H/K$ has order divisible by at most one different from $p$ prime, where $p \in \pi(H/K)$.

## 2 Proof of Theorem A

**Lemma 2.1** (See Knyagina and Monakhov [12]). Let $H$, $K$ and $N$ be pairwise permutable subgroups of $G$ and $H$ is a Hall subgroup of $G$. Then $N \cap HK = (N \cap H)(N \cap K)$.

**Proof of Theorem A.** Assume that this theorem is false and let $G$ be a counterexample of minimal order. Let $\mathfrak{H} = \{H_1, \ldots, H_t\}$. We can assume, without loss of generality, that the smallest prime divisor $p$ of $|G|$ belongs to $\pi(H_1)$. Let $P$ be a Sylow $p$-subgroup of $H_1$.

(1) If $R$ is a minimal normal subgroup of $G$, then $G/R$ is supersoluble. Hence $R$ is the unique minimal normal subgroup of $G$, $R$ is not cyclic and $R \notin \Phi(G)$.

We show that the hypothesis holds for $G/R$. First note that

$$\mathfrak{H}_0 = \{H_1R/R, \ldots, H_tR/R\}$$

is a complete Hall $\sigma$-set of $G/R$, where $H_iR/R \simeq H_i/H_i \cap R$ is supersoluble since $H_i$ is supersoluble by hypothesis for all $i = 1, \ldots, t$.

Now let $V/R$ be a maximal subgroup of $H_iR/R$, so $|(H_iR/R) : (V/R)| = p$ is a prime. Then $V = R(V \cap H_i)$ and hence

$$p = |(H_iR/R) : (V/R)| = |(H_iR/R) : (R(V \cap H_i)/R)| = |H_iR : R(V \cap H_i)| =$$

$$= |H_i||R||R \cap (V \cap H_i)| : |V \cap H_i||R||H_i \cap R| = |H_i| : |V \cap H_i| = |H_i : (V \cap H_i)|,$$

so $V \cap H_i$ is a maximal subgroup of $H_i$. Assume that $H_iR/R$ is not cyclic. Then $H_i$ is not cyclic, so

$$(V \cap H_i)H_j = H_j(V \cap H_i)$$

for all $j \neq i$ by hypothesis and hence

$$(V/R)(H_jR/R) = (R(V \cap H_i)/R)(H_jR/R) = (H_jR/R)((V \cap H_i)R/R) = (H_jR/R)(V/R).$$
Consequently the hypothesis holds for \( G/R \), so \( G/R \) is supersoluble by the choice of \( G \). Moreover, it is well known that the class of all supersoluble groups is a saturated formation (see Ch. VI in [5] or ?? in [?]). Hence the choice of \( G \) implies that \( R \) is the unique minimal normal subgroup of \( G \), \( R \) is not cyclic and \( R \not\in \Phi(G) \).

(2) \( G \) is not soluble. Hence \( R \) is not abelian and \( 2 \in \pi(R) \).

Assume that this is false. Then \( R \) is an abelian \( q \)-group for some prime \( q \). Let \( q \in \pi_k \). Since \( R \) is non-cyclic by Claim (1) and \( R \leq H_k \), \( H_k \) is non-cyclic. Hence every member of \( \mathcal{H} \) permutes with each maximal subgroup of \( H_k \). Since \( R \not\in \Phi(G) \), \( R \not\in \Phi(H_k) \) and so there exists a maximal subgroup \( V \) of \( H_k \) such that \( R \not\in V \) and \( RV = H_k \). Hence \( E = R \cap V \neq 1 \) since \( |R| > q \) and \( H_k \) is supersoluble. Clearly, \( E \) is normal in \( H_k \). Now assume that \( i \neq k \). Then \( V \) permutes with \( H_i \) by hypothesis, so \( VH_i \) is a subgroup of \( G \) and

\[
R \cap VH_i = (R \cap V)(R \cap H_i) = R \cap V = E
\]

by Lemma 2.1 and so \( H_i \leq N_G(E) \). Therefore \( H_i \leq N_G(E) \) for all \( i = 1, \ldots, t \). This implies that \( E \) is normal in \( G \), which contradicts the minimality of \( R \). Hence we have (2).

(3) If \( R \) has a Hall \( \{2, q\} \)-subgroup for each \( q \) dividing \( |R| \), then a Sylow 2-subgroup \( R_2 \) of \( R \) is non-abelian.

Assume that this is false. Then by Claim (2) and Theorem 13.7 in [9, XI], the composition factors of \( R \) are isomorphic to one of the following groups: a) \( PSL(2, 2^f) \); b) \( PSL(2, q) \), where \( 8 \) divides \( q - 3 \) or \( q - 5 \); c) The Janko group \( J_1 \); d) A Ree group. But with respect to each of these groups it is well-known (see, for example [10, Theorem 1]) that the group has no a Hall \( \{2, q\} \)-subgroup for at least one odd prime \( q \) dividing its order. Hence we have (3)

(4) If at least one of the subgroups \( H_i \) or \( H_k \), say \( H_i \), is non-cyclic, then \( H_i H_k = H_k H_i \) (This follows from the fact that every maximal subgroup of \( H_i \) permutes with \( H_k \)).

(5) \( H = H_1 \) is not cyclic (This directly follows from Claim (2), [9, IV, 2.8] and the Feit-Thompson theorem).

In view of Claim (5), \( \mathcal{H} \) contains non-cyclic subgroups. Without loss of generality, we may assume that \( H_1, \ldots, H_r \) are non-cyclic groups and all groups \( H_{r+1}, \ldots, H_t \) are cyclic.

(6) Let \( E_{\{i,j\}} = H_i H_j \) where \( i \leq r \). If \( r \) is the smallest prime dividing \( |E_{\{i,j\}}| \), then \( E_{\{i,j\}} \) is \( p \)-nilpotent, so it is soluble. Therefore \( E_{\{i,j\}} \neq G \).

Clearly, the hypothesis holds for \( E_{\{i,j\}} \). Hence if \( E_{\{i,j\}} < G \), then this subgroup is supersoluble by the choice of \( G \), and so it is \( p \)-nilpotent. Now assume that \( E_{\{i,j\}} = G \). Then \( r = p = 2 \) and \( E_{\{i,j\}} = H_i H_j = H_j H_i \). Let \( V_1, \ldots, V_t \) be the set of all maximal subgroups of \( H_j \) of \( H \). Since \( H \) is supersoluble, it has a normal 2-complement \( S \). Then \( SV_i \) is a maximal subgroup of \( H_j \), so \( SV_i H_j = H_j SV_i \) is a subgroup of \( G \) by hypothesis. Moreover, this subgroup is normal in \( G = E_{\{i,j\}} \) since \( |G : H_j SV_i| = 2 \). Now let \( E = SV_i H_j \cap \cdots \cap SV_i H_j \). Then \( E \) is normal in \( G \) and clearly \( E \cap P \leq \Phi(P) \).
Now we show that for any prime $q$ dividing $|H_j|$, there are a Sylow $q$-subgroup $Q$ of $H_j$ and an element $h \in H$ such that $P \leq N_G(Q^h)$. Indeed, by the Frattini argument, $G = EN_G(Q)$. Hence by [6 VI, 4.7], there are Sylow 2-subgroups $G_2$, $E_2$ and $N_2$ of $G$, $E$ and $N_G(Q)$ respectively such that $G_2 = E_2 N_2$. Let $P = (G_2)^x$. Then $P = (E_2)^x (N_2)^x$, where $(E_2)^x$ is a Sylow 2-subgroup of $E$ and $(E_2)^x$ is a Sylow 2-subgroup of $(N_G(Q))^x = N_G(Q^x)$. Since $G = HH_j$, $x = hw$ for some $h \in H$ and $w \in H_j$. Hence

$$N_G(Q^x) = N_G(Q^{wh}) = N_G((Q^w)^h),$$

where $Q^w$ is a Sylow $q$-subgroup of $H_j$. Therefore $(E_2)^x = E \cap P \leq \Phi(P)$. Consequently, $P \leq N_G((Q^w)^h)$. This shows that for any prime $q$ dividing $|H_j|$, there is a Sylow $q$-subgroup $Q$ of $H_j$ and an element $h \in H$ such that $P \leq N_G(Q^h)$. Thus $G$ has a Hall $\{2,q\}$-subgroup $PQ^h$ for each $q$ dividing $|H_j|$. Moreover, since $H$ is supersoluble by hypothesis, $G$ has a Hall $\{2,s\}$-subgroup for each $s$ dividing $|H|$. Hence in view of Claim (3), $P$ is not abelian. Then $P \cap F(H) \neq 1$, so $P \cap F(H) \leq Z_\infty(H)$ since $H$ is supersoluble. Let $Z$ be a group of order 2 in $Z(H)$. Since $Z \leq P \leq N_G((Q^h))$, $Z = Z^{h^{-1}} \leq N_G(Q)$. It follows that $Z \leq N_G(H_j)$. Thus $Z^G = Z^{HH_j} = Z^{H_j} \leq ZH_j$. This shows that a Sylow 2-subgroup of $Z^G$ has order 2. Hence $Z^G$ is 2-nilpotent. Let $S$ be the 2-complement of $Z^G$. It is clear that $S \neq 1$. Since $S$ is characteristic in $Z^G$, it is normal in $G$. On the other hand, $S$ is soluble by the Feit-Thompson theorem. This induces that $G$ has an abelian minimal normal subgroup, which contradicts Claim (2). Thus (6) holds.

(7) $E_i = HH_i$ is supersoluble for all $i = 2, \ldots, t$ (Since the hypothesis holds for $E_i$ and $E_i < G$ by Claim (5), this follows from the choice of $G$).

(8) $E = H_1 \ldots H_r$ is soluble.

We argue by induction on $r$. If $r = 2$, it is true by Claim (5). Now let $r > 2$ and assume that the assertion is true for $r - 1$. Then by Claim (4), $E$ has at least three soluble subgroups $E_1$, $E_2$, $E_3$ whose indices $E : E_1$, $|E : E_2|$, $|E : E_3|$ are pairwise coprime. But then $E$ is soluble by the Wielandt theorem [11 I, 3.4].

(9) $R$ has a Hall $\{2,q\}$-subgroup for each $q$ dividing $|R|$.

It is clear in the case when $q \in \pi(H)$. Now assume that $q \notin \pi(H)$ for some $i > 1$. Then Claim (6) implies that $B = HH_i$ is a Hall soluble subgroup of $G$. Hence $B$ has a Hall $\{2,q\}$-subgroup $V$ and so $V \cap R$ is a Hall $\{2,q\}$-subgroup of $R$.

(10) A Sylow 2-subgroup $R_2$ of $R$ is non-abelian (This follows from Claims (3) and (9)).

(11) If $q \in \pi(H_k)$ for some $k > r$, then $q$ does not divide $|R : N_R((R_2)'|)$.

By Claim (7), $B = HH_k$ is supersoluble. Hence there is a Sylow $q$-subgroup of $Q$ of $B$ such that $PQ$ is a Hall $\{2,q\}$-subgroup of $B$. Then $U = PQ \cap R = (P \cap R)(Q \cap R) = R_2(Q \cap R)$ is a Hall supersoluble subgroup of $R$ with cyclic Sylow $q$-subgroup $Q \cap R$. By [6 VI, 9.1], $Q \cap R$ is normal in $U$, and $U/C_U(Q \cap R)$ is an abelian group by [13 Ch. 5, 4.1]. Hence

$$R_2C_U(Q \cap R)/C_U(Q \cap R) \simeq R_2/R_2 \cap C_U(Q \cap R)$$
is abelian and so $(R_2)' \leq C_U(Q \cap R)$. Consequently, $Q \cap R \leq N_R((R_2)')$.

The final contradiction. In view of Claim (11), $R = (E \cap R)N_R((R_2)')$. Hence

$$( (R_2)')^R = ((R_2)')^{(E \cap R)N_R((R_2)'}) = ((R_2)')^{E \cap R} \leq E \cap R.$$ But by Claim (8), $E \cap R$ is soluble. On the other hand, Claim (10) implies that $(R_2)' \neq 1$ and so $R$ is soluble, contrary to Claim (2). The theorem is thus proved.

## 3 Proof of Theorem B

The following lemma can be proved by the direct calculations on the base of well-known properties of Hall subgroups of soluble subgroups.

**Lemma 3.1.** The class $\mathcal{H}_\sigma$ is closed under taking homomorphic images, subgroups and direct products.

**Proof of Theorem B.** Firstly, from Lemma 3.1, $\mathcal{H}_\sigma$ is a hereditary formation.

Now we prove that $G \in \mathcal{H}_\sigma$ if and only if $G$ is soluble and the automorphism group induced by $G$ on every its chief factor of order divisible by $p$ is either a $\sigma_i$-group, where $p \not\in \sigma_i$, or a $(\sigma_i \cup \sigma_j)$-group for some different $\sigma_i$ and $\sigma_j$ such that $p \in \sigma_i$.

**Necessity.** Assume that this is false and let $G$ be a counterexample of minimal order. Then $G$ has a chief factor $H/K$ of order divisible by $p$ such that $A = G/C_G(H/K)$ is neither a $\sigma_i$-group, where $p \not\in \sigma_i$, nor a $(\sigma_i \cup \sigma_j)$-group, where $\sigma_i \neq \sigma_j$ and $p \in \sigma_i$. Since

$$G/C_G(H/K) \simeq (G/K)/(C_G(H/K)/K) = (G/K)/C_{G/K}(H/K)$$

and the hypothesis holds for $G/K$ by Lemma 3.1, the choice of $G$ implies that $K = 1$.

First we show that $H \neq C_G(H)$. Indeed, assume that $H = C_G(H)$. By hypothesis, every complete Hall $\sigma$-set $W = \{W_1, \ldots, W_t\}$ of $G$ forms a $\sigma$-basis of $G$. Without loss of generality, we can assume that $p \in \pi(W_1)$. It is clear that $t \geq 2$. Since $H = C_G(H)$, $H$ is the unique minimal normal subgroup of $G$ and $H \not\leq \Phi(G)$ by [11] Ch.A, 9.3(c)] since $G$ is soluble. Hence $H = O_p(G) = F(G)$ by [11] Ch.A, 15.6]. Then for some maximal subgroup $M$ of $G$ we have $G = H \ltimes M$. Let $V = W_3$. We now show that $V^x \leq C_G(W_2)$ for all $x \in G$. First note that $W_2V^x = V^xW_2$ is a Hall $(\sigma_2 \cup \sigma_3)$-subgroup of $G$. Since $|G : M|$ is a power of $p$, any Hall $\sigma_0$-subgroup of $M$, where $p \not\in \pi_0$, is a Hall $\pi_0$-subgroup of $G$. Hence we can assume without loss of generality that $W_2V^x \leq M$ since $G$ is soluble. By hypothesis, $W_2(V^x)y = (V^x)^yW_2$ for all $y \in G$, so

$$D = \langle (W_2)^{V^x} \rangle \cap \langle (V^x)^{W_2} \rangle$$

is subnormal in $G$ by [11], 1.1.9(2)]. But $D \leq \langle W_2, V^x \rangle \leq M$, so

$$D^G = D^{HM} = D^M \leq M_G = 1$$

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by [11, Ch. A, 14.3], which implies that $[W_2, V^x] = 1$. Thus $V^x \leq C_G(W_2)$ for all $x \in G$. It follows that $H \leq (W_3)^G \leq N_G(W_2)$ and therefore $W_2 \leq C_G(H) = H$, a contradiction. Hence $H \neq C_G(H)$.

Finally, let $D = G \times G$, $A^* = \{(g, g) | g \in G\}$, $C = \{(c, c) | c \in C_G(H)\}$ and $R = \{(h, 1) | h \in H\}$. Then $C \leq C_D(R)$, $R$ is a minimal normal subgroup of $A^*R$ and the factors $R/1$ and $RC/C$ are $(A^*R)$-isomorphic. Moreover,

$$C_{A^*R}(R) = R(C_{A^*R}(R) \cap A^*) = RC,$$

so

$$A^*R/C = (RC/C) \rtimes (A^*/C),$$

where $A^*/C \simeq A$ and $RC/C$ a minimal normal subgroup of $A^*R/C$ such that $C_{A^*R/C}(RC/C) = RC/C$. As $H < C_G(H)$, we see that $|A^*R/C| < |G|$. On the other hand, by Lemma 3.1, the hypothesis holds for $A^*R/C$, so the choice of $G$ implies that $A \simeq A^*/C$ is either a $\sigma_i$-group, where $p \not\in \sigma_i$, or a $(\sigma_i \cup \sigma_j)$-group for some different $\sigma_i$ and $\sigma_j$ such that $p \in \sigma_i$. This contradiction completes the proof of the necessity.

**Sufficiency.** Assume that this is false and let $G$ be a counterexample of minimal order. Then $G$ has a complete Hall set $W = \{W_1, \ldots, W_t\}$ of type $\sigma$ such that for some $i$ and $j$ we have $W_iW_j \neq W_jW_i$. Let $R$ be a minimal normal subgroup of $G$. Then:

1. $G/R \in \mathcal{H}_\sigma$, so $R$ is a unique minimal normal subgroup of $G$.

   It is clear that the hypothesis holds for $G/R$, so $G/R \in \mathcal{H}_\sigma$ by the choice of $G$. If $G$ has a minimal normal subgroup $L \neq R$, then we also have $G/L \in \mathcal{H}_\sigma$. Hence $G$ is isomorphic to some subgroup of $(G/R) \rtimes (G/L)$ by [6, I, 9.7]. It follows from Lemma 3.1 that $G \in \mathcal{H}_\sigma$. This contradiction shows that we have Claim (1).

2. The hypothesis holds for any subgroup $E$ of $G$.

   Let $H/K$ be any chief factor of $G$ of order divisible by $p$ such that $H \cap E \neq K \cap E$. Then $G/C_G(H/K)$ is either a $\sigma_i$-group, where $p \not\in \sigma_i$, or a $(\sigma_i \cup \sigma_j)$-group for some different $\sigma_i$ and $\sigma_j$ such that $p \in \sigma_i$. Let $H_1/K_1$ be a chief factor of $E$ such that $K \cap E \leq K_1 < H_1 \leq H \cap E$. Then $H_1/K_1$ is a $p$-group and

$$EC_G(H/K)/C_G(H/K) \simeq E/(E \cap C_G(H/K))$$

is either a $\sigma_i$-group or a $(\sigma_i \cup \sigma_j)$-group. Since

$$C_G(H/K) \cap E \leq C_E(H \cap E/K \cap E) \leq C_E(H_1/K_1),$$

$E/C_E(H_1/K_1)$ is also either a $\sigma_i$-group or a $(\sigma_i \cup \sigma_j)$-group. Therefore the hypothesis holds for every factor $H_1/K_1$ of some chief series of $E$. Now applying the Jordan-Hölder Theorem for chief series we get Claim (2).

3. $R$ is a Sylow $p$-subgroup of $G$. 

Since $G/R \in \mathcal{F}_\sigma$ by Claim (1),

$$(W_i R/R)(W_j R/R) = (W_j R/R)(W_i R/R),$$

so $W_i W_j R$ is a subgroup of $G$. Assume that $R$ is not a Sylow $p$-subgroup of $G$ and let $B = W_i W_j R$. Then $B \neq G$. On the other hand, the hypothesis holds for $B$ by Claim (2). The choice of $G$ implies that $B \in \mathcal{F}_\sigma$, so $W_i W_j = W_j W_i$, a contradiction. Hence Claim (3) holds.

**Final contradiction for sufficiency.** In view of Claims (1) and (3), there is a maximal subgroup $M$ of $G$ such that $G = R \times M$ and $M_G = 1$. Hence $R = C_G(R) = O_p(G)$ by [11] Ch.A, 15.6]. Since $p$ does not divide $|G : R| = |G : C_G(R)|$ by Claim (3), the hypothesis implies that $M \cong G/R$ is a Hall $\sigma_k$-group for some $\sigma_k \in \sigma$, so one of the subgroups $W_i$ or $W_j$ coincides with $R$. Thus $G = W_i W_j = W_j W_i$. This contradiction completes the proof of the sufficiency.

Finally we prove that $\mathcal{F}_\sigma$ is saturated if and only if $|\sigma| \leq 2$. It is clear that $\mathcal{F}_\sigma$ is a saturated formation for any $\sigma$ with $|\sigma| \leq 2$. Now we show that for any $\sigma$ such that $|\sigma| > 2$, the formation $\mathcal{F}_\sigma$ is not saturated.

Indeed, since $|\sigma| > 2$, there are primes $p < q < r$ such that for some distinct $\sigma_i$, $\sigma_j$ and $\sigma_k$ in $\sigma$ we have $p \in \sigma_i$, $q \in \sigma_j$ and $r \in \sigma_k$. Let $C_q$ and $C_r$ be groups of order $q$ and $r$, respectively. Let $P_1$ be a simple $\mathbb{F}_p C_q$-module which is faithful for $C_q$, $P_2$ be a simple $\mathbb{F}_p C_r$-module which is faithful for $C_r$. Let $H = P_1 \times C_q$ and $Q$ be a simple $\mathbb{F}_q H$-module which is faithful for $H$. Let $E = (Q \times H) \times (P_2 \times C_r)$.

Let $A = A_p(E)$ be the $p$-Frattini module of $E$ ([11] p.853)), and let $G$ be a non-splitting extension of $A$ by $E$. In this case, $A \subseteq \Phi(G)$ and $G/A \cong E$. Then $G/\Phi(G) \in \mathcal{F}_\sigma$, where $\sigma = \{\sigma_i, \sigma_j, \sigma_k\}$. By Corollary 1 in [15], $QP_1 P_2 = O_{p'}(E) = C_E(A/\text{Rad}(A))$. Hence for some normal subgroup $N$ of $G$ we have $A/N \not\cong \Phi(G/N)$ and $G/C_G(A/N) \cong C_q \times C_r$ is a $(\sigma_i \cup \sigma_j)$-group. But neither $p \not\in \sigma_i$ nor $p \in \sigma_j$. Hence $G \not\in \mathcal{F}_\sigma$ by the necessity. The theorem is proved.

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