BOUNDEDNESS OF PSEUDODIFFERENTIAL OPERATORS OF $C^*$-ALGEBRA-VALUED SYMBOL.*

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Abstract

Let us consider the set $S^A(\mathbb{R}^n)$ of rapidly decreasing functions $G : \mathbb{R}^n \to A$, where $A$ is a separable $C^*$-algebra. We prove a version of the Calderón-Vaillancourt theorem for pseudodifferential operators acting on $S^A(\mathbb{R}^n)$ whose symbol is $A$-valued. Given a skew-symmetric matrix, $J$, we prove that a pseudodifferential operator that commutes with $G(x + JD)$, $G \in S^A(\mathbb{R}^n)$, is of the form $F(x - JD)$, for $F$ a $C^\infty$-function with bounded derivatives of all orders.

1 Introduction.

Throughout this work, $A$ denotes a separable $C^*$-algebra and $S^A(\mathbb{R}^n)$ denotes the $A$-valued Schwartz space of smooth and rapidly decreasing functions on $\mathbb{R}^n$. On $S^A(\mathbb{R}^n)$ we define the $A$-valued inner product

$$<f, g> = \int f(x)^*g(x)dx,$$

whose associated norm we denote by $\| \cdot \|_2$, $\|f\|_2 = \| <f, f> \|^\frac{1}{2}$.

The completion of $S^A(\mathbb{R}^n)$ with this norm is a Hilbert $A$-module that we denote by $E$. The set of all adjointable (and therefore bounded) operators on $E$ we denote by $B^*(E)$. Let $CB^\infty(\mathbb{R}^n, A)$ denote the set of $C^\infty$-functions with bounded derivatives of all orders.

In section 2, we see a generalization of the Calderón-Vaillancourt Theorem, \[\text{[1]},\] for a pseudodifferential operator, $a(x, D)$, whose symbol, $a$, is in $CB^\infty(\mathbb{R}^2n, A)$.

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Following the steps of Seiler’s proof in [8] (in fact going back to Hwang, [4]), we can see that $a(x,D)$ is bounded on $E$. Note that Seiler’s result needs that one works on a Hilbert space where we have that the Fourier transform is unitary in the usual sense. One of the advantages of working with this norm, $\| \cdot \|_2$, is that the Fourier transform becomes an "unitary" operator on the completion of $S^A(\mathbb{R}^n)$; i.e. it is a Hilbert-module adjointable operator on $E$, whose inverse is equal to its adjoint. This norm $\| \cdot \|_2$ allows for a proof of the Calderón-Vaillancourt Theorem for pseudodifferential operators whose symbols are $A$-valued functions.

We also prove that these operators are adjointable. So, we have $a(x,D) \in \mathcal{B}^*(E)$, $a \in CB^\infty(\mathbb{R}^{2n},A)$.

In [7], Rieffel defines a deformed product in $CB^\infty(\mathbb{R}^n,A)$, depending on an anti-symmetric matrix, $J$, by

$$F \times_J G(x) = \int (2\pi)^{-n} e^{iu \cdot v} F(x + Ju)G(x + v) dudv.$$  

It is not difficult to see that the left-regular representation of $CB^\infty(\mathbb{R}^n,A)$ defines pseudodifferential operators on $\mathcal{B}^*(E)$; in fact, for $F \in CB^\infty(\mathbb{R}^n,A)$, $L_F(\varphi) = F \times_J \varphi$, $\varphi \in S^A(\mathbb{R}^n)$, is the pseudodifferential operator of symbol $F(x - J\xi)$.

Rieffel proves that $L_F$, $F \in CB^\infty(\mathbb{R}^n,A)$ is a continuous operator on $E$, ([7], Corollary 4.7) and that $L_F$ is adjointable on $E$ ([7], Proposition 4.2).

The Heisenberg group acts on $\mathcal{B}^*(E)$ by conjugation in the following way.

Given $V \in \mathcal{B}^*(E)$,

$$(z, \zeta, t) \rightarrow E_{z,\zeta,t}^{-1}VE_{z,\zeta,t}, \quad (z, \zeta, t) \in \mathbb{R}^{2n} \times \mathbb{R},$$

where

$$E_{z,\zeta,t}f(x) = e^{it \zeta \cdot x} f(x - z), \quad f \in S^A(\mathbb{R}^n).$$

It is easy to see that $V_{z,\zeta} = E_{z,\zeta,0}^{-1}VE_{z,\zeta,0}$ does not depend on $t \in \mathbb{R}$.

We say that $V$ is Heisenberg-smooth if the map $(z, \zeta) \rightarrow V_{z,\zeta}$ is $C^\infty$, and, if $z \rightarrow V_z$ is $C^\infty$, where $V_z = V_{z,0}$, we say that $V$ is translation-smooth.

When we are dealing with the scalar case, $A = \mathbb{C}$, we have the remarkable characterization of Heisenberg-smooth operators in $\mathcal{B}^*(E)$ given by H. O. Cordes, [3]: these are the pseudodifferential operators whose symbols are in $CB^\infty(\mathbb{R}^{2n})$.

In section 3 we prove that if a skew-symmetric, $n \times n$, matrix is given and if the $C^*$-algebra $A$ is such that a suitably stated generalization of Cordes’ characterization can be proved, then any Heisenberg-smooth operator $T \in \mathcal{B}^*(E)$, which commutes with every pseudodifferential operator with symbol $G(x + J\xi)$, for some $G \in CB^\infty(\mathbb{R}^n,A)$, is also a pseudodifferential operator with symbol $F(x - J\xi)$, for
some $F \in CB^\infty(\mathbb{R}^n, A)$. This is a rephrasing of a conjecture stated by Rieffel for an arbitrary $A$ at the end of Chapter 4 of [7] (the operators $G(x+JD)$ are those obtained from the right regular representation for his deformed product on $CB^\infty(\mathbb{R}^n, A)$). That Cordes’ characterization implies Rieffel’s conjecture has already been proved for the scalar case, [5]. The Schwartz kernel argument used in [5] has to be avoided here, in the more general case.

2 $a(x, D) \in \mathcal{B}^s(E)$.

Let us consider a pseudodifferential operator on $E$ such that, if $\varphi \in S^A(\mathbb{R}^n)$,

$$a(x, D)\varphi(x) = \int e^{i(x-y)\xi}a(x, \xi)\varphi(y)d y d \xi,$$

for $a \in CB^\infty(\mathbb{R}^{2n}, A)$, where $d y = (2\pi)^{-\frac{n}{2}}dy$. As in the scalar case, we can see that $a(x, D)\varphi(x)$ is well defined for each $x \in \mathbb{R}^n$, if $\varphi \in S^A(\mathbb{R}^n)$.

An example of such an operator is given by Rieffel:

Given a function $F \in CB^\infty(\mathbb{R}^n, A)$,

$$L_F\varphi(x) = \int e^{i(x-y)\xi}F(x-J\xi)\varphi(y)d y d \xi.$$

The integrals considered here are oscillatory integrals ([7], Chapter 1).

Let us see next the fundamental ideas of a generalization of the Calderón-Vaillancourt Theorem for operators on $E$.

First, let us see that $a(x, D)(\varphi) \in E$, for $\varphi \in S^A(\mathbb{R}^n)$.

Considering $L^2(\mathbb{R}^n, A)$ as the set of all functions $f : \mathbb{R}^n \to A$ such that $\int \|f(x)\|^2dx < \infty$, with the “almost everywhere” equivalence relation, where we consider the norm $\| \cdot \|_{L^2}$ (defined in the usual way), we can prove that $a(x, D)(\varphi) \in L^2(\mathbb{R}^n, A)$, as follows:

Using integration by parts and the equation

$$(i + x)^\alpha e^{ixy} = (i + D_y)^\alpha e^{ixy}, \quad \alpha = (1, \ldots, 1), \quad \text{where} \quad D_y = -i\partial_y,$$

we obtain

$$a(x, D)(\varphi)(x) = \int e^{ix\xi}a(x, \xi)\hat{\varphi}(\xi)d \xi =$$

$$= (i + x)^{-\alpha} \int [(i + D_\xi)^{\alpha}e^{ix\xi}]a(x, \xi)\hat{\varphi}(\xi)d \xi =$$

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\[ (i + x)^{-\alpha} \int e^{i\xi} [(i - D\xi)^\alpha a(x, \xi)\hat{\varphi}(\xi)] d\xi, \]

where \( \hat{\varphi}(\xi) = \int e^{-i\xi y} \varphi(y)dy \). Since the last integral is bounded, \( a(x, D)(\varphi) \in L^2(\mathbb{R}^n, A) \).

On the other hand, since \( \| \cdot \|_2 \leq \| \cdot \|_{L^2} \), we can prove that \( L^2(\mathbb{R}^n, A) \subseteq E \), so that \( a(x, D)(\phi) \in E \).

**Theorem 2.1.** Let \( a \in CB^\infty(\mathbb{R}^{2n}, A) \). Then, \( a(x, D) \) is a bounded operator on \( E \). In fact, \( \|a(x, D)\| \leq l\pi(a) \), for \( l \in \mathbb{R}^+ \) independent of \( a \), and \( \pi(a) = \sup\{\|\partial^\alpha_\xi \partial^\beta_y a\|_\infty \mid \beta, \gamma \leq \alpha = (1, 1, \cdots, 1)\} \).

**Proof.** To begin with, let us consider the case when \( a \) has compact support. Denoting \( a(x, D) \) by \( T \), for \( \varphi, \psi \in S^A(\mathbb{R}^n) \), we look at \( < \psi, T\varphi > \), which equals \( < \hat{\psi}, \hat{T}\varphi > \).

(Here we are dealing with the Fourier transform in \( L^2(\mathbb{R}^n, A) \), which is “unitary” on \( E \), in the sense that \( < f, g >= < \hat{f}, \hat{g} >, \quad f, g \in L^2(\mathbb{R}^n, A) \subseteq E \).)

Since

\[ \hat{T}\varphi(\eta) = \int e^{-i\eta x} T\varphi(x)dx = \int e^{-i\eta x} e^{i(x-y)\xi} a(x, \xi)\varphi(y)dyd\xi dx, \]

we have

\[ < \hat{\psi}, \hat{T}\varphi > = \int e^{-i\eta x} e^{i(x-y)\xi} \hat{\psi}^\ast(\eta) a(x, \xi)\varphi(y)dyd\xi dxd\eta. \]

Using integration by parts and the equation (2.1), we have

\begin{align*}
< \hat{\psi}, \hat{T}\varphi > & = \int e^{-i\eta x} (i + x - y)^{-\alpha} \hat{\psi}^\ast(\eta)(i + x - y)^\alpha e^{i(x-y)\xi} a(x, \xi)\varphi(y)dyd\xi dxd\eta = \\
& = \int e^{-i\eta x} (i + x - y)^{-\alpha} \hat{\psi}^\ast(\eta) e^{i\xi x} e^{-i\eta y} [(i - D\xi)^\alpha a(x, \xi)] \varphi(y)dyd\xi dxd\eta = \\
& = \int e^{-i\eta x} (i + x - y)^{-\alpha} \hat{\psi}^\ast(\eta) e^{i\xi x} (i + \xi - \eta)^{-\alpha} [(i - D\xi)^\alpha a(x, \xi)] \varphi(y)dyd\xi dxd\eta = \\
& = \int e^{i\xi x} e^{-i\eta x} (i + x - y)^{-\alpha} \hat{\psi}^\ast(\eta) e^{-i\eta y} (i + \xi - \eta)^{-\alpha} [(i - D\xi)^\alpha a(x, \xi)] \varphi(y)dy d\eta dy d\xi.
\end{align*}

Let us consider

\[ h(z) = (i - z)^{-\alpha} \quad \alpha = (1, \cdots, 1) \]

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and
\[ h_z(y) = h(y - z) \quad y, z \in \mathbb{R}^n. \]

Thus, we have
\[ \langle \hat{\psi}, \hat{T}_\varphi \rangle = \int e^{ix\xi} f(x, \xi) \left[ (i - D_x)^\alpha (i - D_\xi)^\alpha a(x, \xi) \right] g(x, \xi) dxd\xi, \]

with
\[ f(x, \xi) = \int e^{-ix\eta} h_\xi(\eta) \hat{\psi}(\eta)^* d\eta \]
and
\[ g(x, \xi) = \int e^{-iy\xi} h_x(y) \varphi(y) dy. \]

So, we can write (by abuse of notation),
\[ \| \langle \hat{\psi}, \hat{T}_\varphi \rangle \| = (2\pi)^{-n/2} \| e^{-ix\xi} f^*(x, \xi), [(i - D_x)^\alpha (i - D_\xi)^\alpha a(x, \xi)] g(x, \xi) \|. \]

If \( c(x, \xi) = (i - D_x)^\alpha (i - D_\xi)^\alpha a(x, \xi) \), there exists \( d_1 \in \mathbb{R}^+ \), not depending on \( a \), such that \( \sup_{(x, \xi) \in \mathbb{R}^{2n}} \| c(x, \xi) \| < d_1 \pi(a) \).

In Prop. 2.1 below, we prove that there exists \( d_2 \in \mathbb{R}^+ \), not depending on \( \varphi \) or \( g \), such that \( \| g \|_2 \leq d_2 \| \varphi \|_2 \), so that we have
\[ \| cg \|_2 \leq d_1 \pi(a) d_2 \| \varphi \|_2. \]

In a similar way as in Proposition 2.1, we get that
\[ \| f^* \|_2 \leq d_2 \| \varphi \|_2, \]
then, for \( k = d_1 d_2^2 (2\pi)^{-n/2} \), we have, for all \( \varphi, \psi \in S^A(\mathbb{R}^n) \),
\[ \| \langle \psi, a(x, D) \varphi \rangle \| \leq k \pi(a) \| \varphi \|_2 \| \psi \|_2. \]

As for the general case, we consider the function \( a_\varepsilon \in CB^\infty(\mathbb{R}^{2n}, A) \) given by \( a_\varepsilon(x, \xi) = \phi(\varepsilon x, \varepsilon \xi) a(x, \xi) \), for \( 0 < \varepsilon \leq 1 \), and where \( \phi \in C_c^\infty(\mathbb{R}^{2n}) \) is such that \( \phi \equiv 1 \) close to zero. As we just have seen, we have that \( \| \langle \psi, a(\varepsilon, D) \varphi \rangle \| \leq k \pi(a_\varepsilon) \| \varphi \|_2 \| \psi \|_2 \). Just doing some computations, we get that there is \( m \in \mathbb{R}^+ \), not depending on \( a \) or \( \varepsilon \), such that \( \pi(a_\varepsilon) \leq m \pi(a) \). Besides, it is not difficult to see that we have \( \lim_{\varepsilon \to 0} \langle \psi, a_\varepsilon(x, D) \varphi \rangle = \langle \psi, a(x, D) \varphi \rangle \). Actually,
\[ \| \langle \psi, a(x, D) \varphi \rangle \| \xrightarrow{\varepsilon \to 0} \| \langle \psi, a(\varepsilon, D) \varphi \rangle \| \leq km \pi(a) \| \varphi \|_2 \| \psi \|_2. \]
Considering, now, \(a(x,D)\varphi\) in place of \(\psi\), since \(a(x,D)\varphi \in L_2(\mathbb{R}^n, A)\), we have that
\[
\| < a(x,D)\varphi, a(x,D)\varphi > \| \leq l\pi(\alpha)\|a(x,D)\varphi\|_2\|\varphi\|_2, \quad l = km, \forall \varphi \in S^A(\mathbb{R}^n)
\]
as before. So, there is \(l \in \mathbb{R}^+\), not depending on \(a\), such that \(\|a(x,D)\| \leq l\pi(\alpha)\).

**Proposition 2.1.** Given \(\varphi \in S^A(\mathbb{R}^n)\), let \(g(x,\xi) = \int e^{-idy}(i + x - y)^{-\alpha}\varphi(y)dy\), then there exists \(d \in \mathbb{R}^+,\) not depending neither on \(g\) nor on \(\varphi\), such that \(\|g\|_2 \leq c\|\varphi\|_2\).

**Proof.** Let \(h_x\) be as before, and put \(g(x,\xi) = \int e^{-idy}h_x(y)\varphi(y)dy = \widetilde{h_x\varphi}(\xi)\). Then
\[
\int g(x,\xi)^*g(x,\xi)dxd\xi = \int < \widetilde{h_x\varphi}, \widetilde{h_x\varphi} > dx = \int < h_x\varphi, h_x\varphi > dx
\]
\[
= \int \widetilde{h(x)h(x)}dx \int \varphi(\xi)^*\varphi(\xi)d\xi.
\]
If \(d = \left( \int |h(x)|^2 dx \right)^{\frac{1}{2}}\), we have \(\|g\|_2 \leq d\|\varphi\|_2\).

**Note 2.1.** If \(a \in CB^\infty(\mathbb{R}^{2n}, A)\), we denote by \(\mathcal{O}(a)\) the pseudodifferential operator whose symbol is \(a\).

**Proposition 2.2.** There exists \(p \in CB^\infty(\mathbb{R}^{2n}, A)\) such that
\[
< \mathcal{O}(a)\varphi, \psi > = < \varphi, \mathcal{O}(p)\psi > \quad \forall \varphi, \psi \in S^A(\mathbb{R}^n).
\]

**Proof.** First we prove that \(p(y,\xi) = \int e^{-idy}a(y - z,\xi - \eta)^*dzd\eta\) belongs to \(CB^\infty(\mathbb{R}^{2n}, A)\). As for this, we use strongly the definition of oscillatory integrals given in [7], where we consider for a while the corresponding Fréchet space \(CB^\infty(\mathbb{R}^{2n}, A)\).

Then, applying proposition 1.6 of [7], we can begin working with \(a\) of compact support, for which we can work as Cordes in chapter 1 section 4 of [2].

To obtain the general case, we apply the Dominated Convergence Theorem. Please, see details at proposition 4.6 of [8].

**Remark 2.1.** The application \(\mathcal{O} : CB^\infty(\mathbb{R}^{2n}, A) \rightarrow B^*(E)\), given by \(a \mapsto a(x,D)\), is well defined and it is easy to see that it is injective.

**Remark 2.2.** As in the scalar case, [2], chapter 8, we see that a pseudodifferential operator is Heisenberg-smooth, because \(\|a(x,D)\|\) depends just on a finite number of seminorms of \(a \in CB^\infty(\mathbb{R}^{2n}, A)\). Besides, for \(T = a(x,D)\), we have \(\partial^\alpha_x \partial^\gamma_\xi T_{z,\zeta} = \mathcal{O}(\partial^\alpha_x \partial^\gamma_\xi a_{z,\zeta})\), where \(a_{z,\zeta}(x,\xi) = a(x + z,\xi + \zeta), \beta, \gamma \in \mathbb{N}^n\) (for proving these results, we just need to do some computations which we can check in proposition 4.7 of [6]).

**Note 2.2.** Let \(\mathcal{H}\) be the subset of \(B^*(E)\) formed by the Heisenberg-smooth operators. We have that \(\mathcal{O} : CB^\infty(\mathbb{R}^{2n}, A) \rightarrow \mathcal{H}\) is a well defined, injective application. For \(A = \mathbb{C}\), we have that \(\mathcal{O}\) is a bijection, [3].
3 Pseudodifferential operators that commute with \( R_G \).

Let us consider here the right regular representation of \( CB^\infty(\mathbb{R}^n, A) \) for the deformed product:

\[
R_G F = F \times_J G.
\]

**Lemma 3.1.** If an operator \( T \in B^*(E) \) commutes with \( R_\varphi \) for all \( \varphi \in S^A(\mathbb{R}^n) \), there exists a sequence \( F_k \) in \( E \) such that \( F_k \times_J \varphi \) converges to \( T(\varphi) \), for all \( \varphi \in S^A(\mathbb{R}^n) \).

**Proof.** Let us find, first, a sequence, \( e_k \), such that, for all \( \varphi \in S^A(\mathbb{R}^n) \), \( e_k \times_J \varphi \rightarrow \varphi \) (convergence in the \( \| \cdot \| \) norm). Since \( A \) is separable, it has an approximate unit \( (u_k)_{k \in \mathbb{N}} \). For each \( k \in \mathbb{N} \), let us consider a C\(^\infty\)-function \( \phi_k : \mathbb{R}^n \rightarrow A \), with support the set \( \{ x \in \mathbb{R}^n / \| x \| \leq \frac{1}{k} \} \) such that \( \int \phi_k(x)dx = u_k \). Then, let \( e_k = F^{-1}(\phi_k) \), where \( F \) is the Fourier transform on \( S^A(\mathbb{R}^n) \) (for details, see proposition 2.5 of [6]).

Then, since \( R_\varphi \) is a continuous operator on \( E \), letting \( F_k = Te_k \in E \), \( R_\varphi(F_k) \) is well defined and we have \( F_k \times_J \varphi = R_\varphi(Te_k) = TR_\varphi e_k = T(e_k \times_J \varphi) \). Hence, since \( e_k \times_J \varphi \rightarrow \varphi \), we have \( F_k \times_J \varphi \rightarrow T_\varphi \), for all \( \varphi \in S^A(\mathbb{R}^n) \). \( \square \)

**Proposition 3.1.** If \( T \) is an operator in \( B^*(E) \) which is such that \( [T, R_\varphi] = 0 \ \forall \varphi \in S^A(\mathbb{R}^n) \), then \( T_{z,\zeta} = T_{z-J\zeta,0} \).

**Proof.** Since \( R_\varphi \) is continuous, for any \( F \in E \) and \( \varphi \in S^A(\mathbb{R}^n) \), we may write \( L_F(\varphi) = R_\varphi(F) \) thus defining \( L_F \) as an operator from \( S^A(\mathbb{R}^n) \) to \( E \).

It is easy to see that

\[
E_{z,\zeta}^{-1}L_FE_{z,\zeta} = E_{z-J\zeta,0}^{-1}L_FE_{z-J\zeta,0},
\]

for \( f \in S^A(\mathbb{R}^n) \). Then, using that \( E_{z,\zeta} \) leaves \( S^A(\mathbb{R}^n) \) invariant (here we are writing \( E_{z,\zeta} \) for \( E_{z,\zeta,0} \), earlier defined), we get

\[
E_{z,\zeta}^{-1}L_FE_{z,\zeta} = E_{z-J\zeta,0}L_FE_{z-J\zeta,0}, \quad (3.1)
\]

for \( F \in E \), so we have that \( (L_F)_{z,\zeta} = (L_F)_{z-J\zeta,0} \). By lemma 3.1 there is a sequence \( F_k \) in \( E \) such that, for all \( \varphi \in S^A(\mathbb{R}^n) \), \( \lim_{k \rightarrow \infty} (L_{F_k})_{z,\zeta}(\varphi) = T_{z,\zeta}(\varphi) \) (by equation (3.1)), so that \( T_{z,\zeta} = T_{z-J\zeta,0} \). \( \square \)

**Corollary 3.1.** If \( T \in B^*(E) \) is such as in proposition 3.1 and is translation-smooth, then \( T \) is Heisenberg-smooth.

**Lemma 3.2.** Given \( a \in CB^\infty(\mathbb{R}^{2n}, A) \), let \( b = \Pi_{j=1}^n(1 + \partial_{y_j})^2(1 + \partial_{\xi_j})^2a \), and \( \gamma(x) = \Pi_{j=1}^n f(x_j) \), with
Then we have \( a(x,\xi) = \int \gamma(-z)\gamma(-\zeta)b(x + z,\xi + \zeta)dzd\zeta. \)

In the scalar case, \( A = \mathbb{C} \), we can see the proof in [2], chapter 8, corollary 2.4. The same argument is valid for the general case.

**Theorem 3.1.** Let \( A \) be a \( C^* \)-algebra for which the above defined application \( O : CB^\infty(\mathbb{R}^{2n}, A) \rightarrow \mathcal{H} \) is a bijection. Then, given an operator \( T \in B^*(E) \), translation-smooth, that commutes with \( R_\varphi \) for all \( \varphi \in S^A(\mathbb{R}^n) \), there exists a function \( F \) in \( CB^\infty(\mathbb{R}^n, A) \) such that \( T = L_F \).

**Proof.** Since \( O \) is a bijection, and by corollary 3.1, there exists \( a \in CB^\infty(\mathbb{R}^{2n}, A) \) such that \( T = O(a) \). As in lemma 3.2 let \( b = \prod_{j=1}^{n}(1 + \partial x_j)^2(1 + \partial \xi_j)^2a \) and \( B = O(b) \).

Note that \( B_{z,\zeta} = \prod_{j=1}^{n}(1 + \partial z_j)^2(1 + \partial \zeta_j)^2T_{z,\zeta} \), see remark 2.2.

Since \( TR_\varphi = R_\varphi T \), it is not difficult to see that \( T_{z,\zeta}R_\varphi = R_\varphi T_{z,\zeta} \), for all \( \varphi \in S^A(\mathbb{R}^n) \). Then, we have \( [B, R_\varphi] = 0 \) for all \( \varphi \in S^A(\mathbb{R}^n) \). So, by proposition 3.1 \( B_{z,\zeta} = B_{z - J\zeta, 0} \), so that \( b(x + z, \xi + \zeta) = b(x + z - J\zeta, 0) \).

By lemma 3.2 we get \( a(z, \zeta) = a(z - J\zeta, 0) \). Choosing \( F(z) = a(z, 0) \), we have \( T = L_F \), with \( F \in CB^\infty(\mathbb{R}^n, A) \), as was to be proved. \( \blacksquare \)

**Remark 3.1.** We have just proved that a pseudodifferential operator that commutes with all operators \( G(x + JD) \), \( G \in S^A(\mathbb{R}^n) \), where \( J \) is a fixed skew-symmetric matrix, is of the form \( F(x - JD) \), \( F \in S^A(\mathbb{R}^n) \).

**Remark 3.2.** If \( A = \mathbb{C} \), since we have the Cordes’ characterization [3], we can see that theorem 3.1 gives us another proof of the main result of [5], without needing to apply the Schwartz kernel.

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