A Three-level Stochastic Linear-quadratic Stackelberg Differential Game with Asymmetric Information

1st Kaixin Kang  
School of Mathematics  
Shandong University  
Jinan, 250100, P. R. China  
202012062@mail.sdu.edu.cn

2nd Jingtao Shi  
School of Mathematics  
Shandong University  
Jinan, 250100, P. R. China  
shijingtao@sdu.edu.cn

Abstract—This paper is concerned with a three-level stochastic linear-quadratic Stackelberg differential game with asymmetric information, in which three players participate as Player 1, Player 2 and Player 3. Player 3 acts as the leader of Player 2 and Player 1, Player 2 acts as the leader of Player 1 and Player 1 acts as the follower. The asymmetric information considered is: the information available to Player 1 is based on the sub-$\sigma$-algebra of the information available to Player 2, and the information available to Player 2 is based on the sub-$\sigma$-algebra of the information available to Player 3. By maximum principle of forward-backward stochastic differential equations and optimal filtering, feedback Stackelberg equilibrium of the game is given with the help of a new system consisting of three Riccati equations.

Index Terms—Stackelberg differential game; linear-quadratic optimal control; maximum principle; forward-backward stochastic differential equation; Riccati equation; stochastic filtering

I. INTRODUCTION

In this paper, we will study a three-level stochastic linear-quadratic (LQ) Stackelberg differential game with asymmetric information. The Stackelberg game, also called leader-follower game, which was proposed by Stackelberg [13] in 1952, is a kind of game with hierarchical structure. In Stackelberg game, players play the role of the leader or the follower, and make decision sequentially. First of all, we give an example in the supply chain management to introduce the research motivation of this paper.

Example 1. (Cooperative advertising and pricing problem) He et al. [5] studied a cooperative advertising and pricing problem, in which there are two players, a manufacturer and a retailer. Chutani and Sethi [4] considered a cooperative advertising program under manufacturer and retailer level competition, with a finite number of independent manufacturers and retailers. Kennedy et al. [6] extended this problem to the one in a dynamic three-echelon supply chain, which is composed of a manufacturer, a distributor and a retailer. In their supply chain, the manufacturer sells his product to the retailer via the distributor.

We consider the following cooperative advertising and pricing model, which is an extension of that introduced in [6]:

\[
\begin{align*}
\dot{x}(t) &= \left[\psi_R(t)\alpha_R(t) + \psi_D(t)\alpha_D(t) + \psi_M(t)\alpha_M(t)\right] \sqrt{1-x(t)} - \Delta x(t) dt + \sigma(x(t)) dW(t), \quad t \in [0, T], \\
x(0) &= x_0,
\end{align*}
\]

where \(x(t)\) is the market awareness which determines the total sales, the constant \(\sigma(x) > 0\) reflects the rate which potential consumers are lost. \(\sigma(x)\) is a variance term, which is usually taken as \(\sigma(x) = C_\alpha \sqrt{x(1-x)}\), for some constant \(C_\alpha\). The retailer decides the retail price \(P_R(\cdot)\) and sets the local advertising effort \(\alpha_R(\cdot)\) with the advertising effectiveness \(\psi_R(\cdot)\). The distributor decides the distributor price \(P_D(\cdot)\) and sets the distributor’s advertising effort \(\alpha_D(\cdot)\) with the advertising effectiveness \(\psi_D(\cdot)\). The manufacturer decides a wholesale price \(P_M(\cdot)\), a national advertising effort \(\alpha_M(\cdot)\) with the advertising effectiveness \(\psi_M(\cdot)\) and a subsidy rate \(\phi(\cdot)\) to the retailer’s local advertising effort through a vertical cooperative advertising program.

Set \(V_R(\cdot) \triangleq (P_R(\cdot), \alpha_R(\cdot), \psi_R(\cdot))\), \(V_D(\cdot) \triangleq (P_D(\cdot), \alpha_D(\cdot))\) and \(V_M(\cdot) \triangleq (P_M(\cdot), \alpha_M(\cdot), \phi(\cdot))\), whose values are taken from some admissible control sets \(V_R, V_D\) and \(V_M\) respectively. Then we encounter a stochastic Stackelberg differential game with three players. In detail, first the retailer’s optimal strategy \(v_R^*(\cdot)\) is solved by:

\[
J_R(v_R^*(\cdot), v_D(\cdot), v_M(\cdot)) = \max_{v_R(\cdot) \in V_R} J_R(v_R(\cdot), v_D(\cdot), v_M(\cdot)),
\]

for all \(v_D(\cdot), v_M(\cdot)\), with

\[
J_R(v_R(\cdot), v_D(\cdot), v_M(\cdot)) = \mathbb{E}\left\{ \int_0^T e^{-rt} \left[ \left( P_R(t) D(P_R(t)) x(t) - (1 - \phi(t) \alpha_M^2(t)) \right) dt \right] \right\}.
\]

Then the distributor’s optimal strategy \(v_D^*(\cdot)\) is obtained by:

\[
J_D(v_R^*(\cdot), v_D^*(\cdot), v_M(\cdot)) = \max_{v_D(\cdot) \in V_D} J_D(v_R^*(\cdot), v_D(\cdot), v_M(\cdot)),
\]

and the manufacturer’s optimal strategy \(v_M^*(\cdot)\) is determined by:

\[
J_M(v_R^*(\cdot), v_D^*(\cdot), v_M^*(\cdot)) = \max_{v_M(\cdot) \in V_M} J_M(v_R^*(\cdot), v_D^*(\cdot), v_M(\cdot)).
\]
for all \( v_M(\cdot) \), where
\[
J_D(v_R(\cdot), v_D(\cdot), v_M(\cdot)) = \mathbb{E}\left\{ \int_0^T e^{-rt}\left[ (P_D(t) - P_M(t))D(P_R(t))x(t) - kD^2 D(P_R(t))x(t) - \alpha_2^2(t)\right]dt \right\}.
\]
Finally, the manufacturer’s optimal strategy \( v_M^* \) is given by
\[
J_M(v_R^*(\cdot), v_D^*(\cdot), v_M^*(\cdot)) = \max_{v_M(\cdot) \in V_M} J_M(v_R^*(\cdot), v_D^*(\cdot), v_M(\cdot)),
\]
with
\[
J_M(v_R(\cdot), v_D(\cdot), v_M(\cdot)) = \mathbb{E}\left\{ \int_0^T e^{-rt}\left[ (P_M(t) - c)D(P_R(t))x(t)
- kM D(P_R(t))x(t) - \alpha_3^2(t) - \phi(t)\alpha_4^2(t)\right]dt \right\}.
\]
Here, \( r > 0 \) is the discount rate, \( c > 0 \) is the manufacturing cost, \( k^M \) and \( k^B \) are the transport cost. \( 0 \leq D(p) \leq 1 \) is some demand function satisfying usual conditions.

This is a three-level stochastic Stackelberg differential game with three players. Each player hopes to maximize his/her target functional by selecting an appropriate control.

In supply chain management problems, three-level supply chains are often encountered. For example, for a multinational company with multiple sales markets, it is difficult for suppliers to adjust their behavior in direct response to retailers, and the presence of distributors is necessary. This forms a three-level supply chain of suppliers, distributors, and retailers.

A schematic of the three-level supply chain is given in Figure 1.(a), and there can be multiple agents as suppliers, distributors, and retailers. Another example of the three-level supply chain is given in Figure 1.(b), in which one agent is the supplier, two agents are the distributors, and three agents are the retailers. The matrix can indicate whether there is a leadership relationship between agents, where the leadership relationship means that the information of the “follower” can be obtained by the “leader”. In Figure 1.(b), for example, \( S_1 \) has a leadership relationship with \( D_1 \), then the position of the matrix (1, 2) is 1, \( D_1 \) has no leadership relationship with \( R_1 \), then the position of the matrix (2, 6) is 0. Because the leadership of the supply chain is one-way, the shaded part of the matrix must be 0. If no distributors exist, it is a two-level stochastic Stackelberg differential game.

About stochastic Stackelberg differential games with one leader and one follower, refer to Yong [19] for indefinite LQ case and applications to insurer-reinsurer problem (Chen and Shen [3]), etc. Mukaidani and Xu [9] studied a stochastic Stackelberg differential game with one leader and multiple followers. Wang and Yan [15] researched a Pareto-based stochastic Stackelberg differential game with multi-followers.

However, practically in Stackelberg differential game, due to the emergence of various factors, players can not observe the complete information, but can only grasp part of the information. This kind of problem is called Stackelberg differential game with asymmetric information. Shi et al. [10], [11] studied the two-level stochastic Stackelberg differential games with asymmetric information, in which the information available to the follower is based on the sub-\( \sigma \)-algebra of that available to the leader. Shi et al. [12] studied a two-level stochastic LQ Stackelberg differential game with overlapping information, in which the information of the follower and the leader has some overlapping parts, but no mutual inclusion relationship. Wang and Zhang [14] studied an asymmetric information two-level stochastic LQ Stackelberg differential game of mean-field type with one leader and two followers. Li et al. [7] investigated a two-level stochastic LQ Stackelberg differential game under asymmetric information patterns, where the follower uses his observation information to design his strategy whereas the leader implements his strategy using complete information. Zheng and Shi [22], [23] investigated two-level stochastic Stackelberg differential games with partial observation, in which both the leader and the follower have their own observation equations, and the information filtration available to the leader is contained in that to the follower. Yuan et al. [16] discussed a robust reinsurance contract with asymmetric information in a stochastic Stackelberg differential game. Zhao et al. [21] discussed a stochastic LQ Stackelberg differential game with two leaders and two followers under an incomplete information structure.

Motivated by the above three-level supply chain and the related literatures about the stochastic Stackelberg differential game, in this paper we study a three-level stochastic LQ Stackelberg differential game with asymmetric information. We call the players in the game as Player 1, Player 2 and Player 3. Player 3 acts as the leader of Player 2 and Player 1, Player 2 acts as the leader of Player 1 and Player 1 acts as the follower. The asymmetric information considered is: the information available to Player 1 is based on the sub-\( \sigma \)-algebra of the information available to Player 2, and the information available to Player 2 is based on the sub-\( \sigma \)-algebra of the information available to Player 3. By maximum principle and optimal filtering, feedback Stackelberg equilibrium of the game is given with the help of a new system consisting of three Riccati equations.

The rest of this paper is organized as follows. In Section 2, we formulate our problem. Section 3 is devoted to find the feedback Stackelberg equilibrium of the game. Finally in Section 4, some concluding remarks are given.
II. Problem Formulation

Let $T > 0$ be a finite time horizon, $(\Omega, \mathcal{F}, \mathbb{P})$ be a complete probability space, and $\mathbf{W} = \{W_1(t), W_2(t), W_3(t)\}_{t \in [0,T]}$ be a three-dimensional standard Brownian motion defined on it. For $t \in [0, T]$, let $\mathcal{F}_t$ be the natural filtration generated by $\mathbf{W}$ and $\mathcal{F}_t^I \triangleq \sigma\{W_i(s); 0 \leq s \leq t\}$ for $i = 1, 2, 3$. In our Stackelberg game, at $t \in [0, T]$ we let $G^I_t \equiv \mathcal{F}_t^I$ be the information generated by Player 1, $G^2_t \triangleq \sigma\{W_2(s), W_3(s); 0 \leq s \leq t\}$ be the information generated by Player 2, and $G^3_t \equiv \mathcal{F}_t$ be the information generated by Player 3. Obviously, $G^I_t \subseteq G^2_t \subseteq G^3_t$.

Remark 1. Inspired by [21], we can represent the information owned by the $i$th agent as:

$$G^I_t \triangleq \bigcup_{j=1,2,3, a_i \neq 0} \mathcal{F}_j^I,$$

where $i, j = 1, 2, 3$.

And the adjacency matrix of the Stackelberg game is

$$A = \begin{bmatrix} 0 & 0 & 1 \\ 0 & 1 & 1 \\ 1 & 1 & 1 \end{bmatrix}.$$  

Let the state process $x^{\ast v_1,v_2,v_3}(\cdot)$ satisfy the following linear stochastic differential equation (SDE):

$$\begin{cases} dx^{\ast v_1,v_2,v_3}(t) = \left[ A(t)x^{\ast v_1,v_2,v_3}(t) + \sum_{i=1}^{3} B_i(t)v_i(t) + b(t) \right] dt \\ + \sum_{i=1}^{3} \left[ C_i(t)x^{\ast v_1,v_2,v_3}(t) + \sigma_i(t) \right] dW_i(t), \quad t \in [0,T], \\ x^{\ast v_1,v_2,v_3}(0) = x_0, \end{cases}$$

where $x_0 \in \mathbb{R}^n$, $A(\cdot), B_i(\cdot), C_i(\cdot) \in \mathbb{R}^{n \times n}$, $b(\cdot), \sigma_i(\cdot) \in \mathbb{R}^n$ are deterministic and uniformly bounded functions. $v_1(\cdot) \in \mathbb{R}^n$ is the control process of Player $i$, for $i = 1, 2, 3$. The admissible control set $V_i$ of Player $i$ is defined as follows:

$$V_i \triangleq \left\{ v_i : [0,T] \times \Omega \to \mathbb{R}^n | v_i(\cdot) \text{ is } G^I_t \text{-adapted, and} \right\} E \int_0^T |v_i(t)|^2 dt < \infty, \quad i = 1, 2, 3.$$

The quadratic cost functional of $i$th Player ($i = 1, 2, 3$) is defined as follows:

$$J_i(v_1(\cdot), v_2(\cdot), v_3(\cdot)) \triangleq \frac{1}{2} E \left[ \int_0^T \left[ Q_4(t)x^{\ast v_1,v_2,v_3}(t), x^{\ast v_1,v_2,v_3}(t) \right] dt + R_i(t)v_1(t), v_1(t) + 2m_i(t), x^{\ast v_1,v_2,v_3}(t) \right] dt + \left\{ G_i(T)x^{\ast v_1,v_2,v_3}(T), x^{\ast v_1,v_2,v_3}(T) \right\} \right]$$

(2)

where $Q_4(\cdot), R_i(\cdot), G_i \in \mathbb{R}^{n \times n}$, $m_i(\cdot), n_i(\cdot) \in \mathbb{R}^n$ ($i = 1, 2, 3$) are deterministic and uniformly bounded functions. In addition, $R_i(\cdot)$ is positive definite, $Q_4(\cdot)$ and $G_i$ are semi-positive definite, for $i = 1, 2, 3$.

The definition of a three-level Stackelberg game’s equilibrium strategy is given in Başar and Olsder [2], Definition 3.37. In detail, the players’ optimal goals are as follows. First, for Player 1:

$$J_1(v_1(\cdot), v_2(\cdot), v_3(\cdot)) = \inf_{v_1(\cdot) \in V_1} J_1(v_1(\cdot), v_2(\cdot), v_3(\cdot))$$

for all $v_2(\cdot) \in V_2$, $v_3(\cdot) \in V_3$; then, for Player 2:

$$J_2(v_1^\ast(\cdot), v_2(\cdot), v_3(\cdot)) = \inf_{v_2(\cdot) \in V_2} J_2(v_1^\ast(\cdot), v_2(\cdot), v_3(\cdot))$$

for all $v_3(\cdot) \in V_3$; finally, for Player 3:

$$J_3(v_1^\ast(\cdot), v_2^\ast(\cdot), v_3^\ast(\cdot)) = \inf_{v_3(\cdot) \in V_3} J_3(v_1^\ast(\cdot), v_2^\ast(\cdot), v_3^\ast(\cdot))$$

If such $(v_1^\ast(\cdot), v_2^\ast(\cdot), v_3^\ast(\cdot))$ exists, it is called an open-loop Stackelberg equilibrium strategy of our stochastic LQ Stackelberg differential game with asymmetric information. In this paper, our ultimate goal is to find the optimal feedback strategies of Player 1, Player 2 and Player 3.

III. Stackelberg Equilibrium Strategies

In this section, the problem is resolved in three steps, to solve optimal strategies of three players in turn. The main tool is various stochastic versions of Pontryagin’s maximum principle. For any $\mathcal{F}_t$-adapted process $\xi(\cdot)$, we denote by

$$\xi(t) \triangleq E[\xi(t)|G^I_t], \quad (\xi(t) \triangleq E[\xi(t)|G^I_t])$$

(6)

its optimal filtering estimates, for $t \in [0,T]$.

A. Problem of Player 1

First, we define the Hamiltonian function of Player 1 $H_1 : [0,T] \times \mathbb{R}^n \times \mathbb{R}^n \times \mathbb{R}^n \times \mathbb{R}^n \times \mathbb{R}^n \times \mathbb{R}^n \rightarrow \mathbb{R}$ as

$$H_1(t,x,v_1,v_2,v_3,y_1,z_1,z_2,z_3) \triangleq A(t)x + \sum_{i=1}^{3} B_i(t)v_i + b(t) + \sum_{i=1}^{3} [C_i(t)x + \sigma_i(t)]^\top y_i + \sum_{i=1}^{3} [C_i(t)x + \sigma_i(t)]^\top z_i - \frac{1}{2} (Q_4(t)x,x) - \frac{1}{2} (R_i(t)v_i,v_i) - (m_i(t), x) - (n_i(t), v_i).$$

Thus, the optimal strategy of Player 1 can be obtained by the stochastic maximum principle with partial information (See, for example, Baghery and Øksendal [11], or Shi et al. [10]):

$$R_i(t)v_i^\ast(t) - R_i(t)^\top \hat{y}_i(t) + n_i(t) = 0, \quad a.e., \quad a.s.,$$

(8)

where the $\mathcal{F}_t$-adapted process quadruple $(y_1(\cdot), z_1(\cdot), z_2(\cdot), z_3(\cdot))$ is the unique solution to the backward SDE (BSDE):

$$\begin{cases} -dy_1(t) = \left[ A(t)^\top y_1(t) + \sum_{i=1}^{3} C_i(t)^\top z_i(t) - Q_4(t)x(t) \right] dt \\ -m_1(t) dx(t) - \sum_{i=1}^{3} z_i(t) dW_i(t), \quad t \in [0,T], \\ y_1(T) = -G_i x_1^\ast(T), \end{cases}$$

where for simplicity, we denote $x_1^\ast(\cdot) \equiv x^{\ast v_1^\ast,v_2,v_3}(\cdot)$.

We want to represent the optimal strategy in the form of state feedback. For this target, let

$$y_1(t) = -p(t)x_1^\ast(t) - \varphi(t).$$

(10)
where \( p(\cdot) \) is deterministic, differentiable with \( p(T) = G_1 \),
and \( \mathcal{F}_t \)-adapted processes pair \((\varphi(\cdot), \theta(\cdot))\) satisfies the BSDE:
\[
\begin{cases}
d\varphi(t) = \alpha(t)dt + \theta(t)dW_3(t), & t \in [0, T], \\
\varphi(T) = 0,
\end{cases}
\]
where \( \alpha(\cdot) \) is some \( \mathcal{F}_t \)-adapted process to be determined.

Let (10) be filtered on \( \mathcal{G}^1_t \) to obtain \( y_1(\cdot) \). Substituting it into (8), the optimal state feedback strategy of Player 1 is:
\[
v_1^1(t) = -R_1^{-1}\left[ B_1^\top \dot{p}x_1^1 + B_1^\top \varphi + n_1(t) \right], \quad \text{a.e., a.s.} \quad (12)
\]
Noting that time variables are usually omitted somewhere since now, for notational simplicity.

Similar as [11], we apply Itô's formula to (10). Then, making a comparison between that and (9), we obtain the following Riccati equation which admits a unique differentiable solution \( p(\cdot) \) (See, for example, Yong and Zhou [20], Chapter 6):
\[
\begin{aligned}
\dot{p} + pA + A^\top p + & \sum_{i=1}^{3} C_i^\top pC_i - pB_1 R_1^{-1} B_1^\top p + Q_1 = 0, \\
p(T) = G_1.
\end{aligned}
\]
Moreover, we can get that \((\varphi(\cdot), \theta(\cdot))\) satisfies the following BSDE:
\[
\begin{aligned}
d\varphi(t) = & \left[ pB_1 R_1^{-1} B_1^\top px_1^1 - pB_1 R_1^{-1} B_1^\top \dot{p}x_1^1 + A^\top \varphi \\
& - pB_1 R_1^{-1} B_1^\top \dot{\varphi} + C^\top \theta + \sum_{i=2}^{3} pB_i v_i + \sum_{i=1}^{3} C_i^\top \theta p_{1,i} \right] dt - \theta(t)dW_3(t), \\
\varphi(T) = 0.
\end{aligned}
\]

Remark 2. Due to the length of the article, details of the calculation in (13), (14) are omitted. Readers can refer to [11]. At this point, it is equivalent to decouple \( x^*(\cdot) \) and \( y_1(\cdot) \), which ensures that the filtering equation of (1), together with the filtering equation of (9), admits a unique solution \((x_1^*(\cdot), y_1(\cdot), z_1(\cdot), z_2(\cdot), z_3(\cdot))\).

Put the optimal strategy \( v_1^1(\cdot) \) of Player 1 into the state equation (1) of \( x_1^*(\cdot) \), and set
\[
\begin{aligned}
\bar{A} & = A - B_1 R_1^{-1} B_1^\top p, \\
\bar{F}_1 & = -B_1 R_1^{-1} B_1^\top \dot{p}, \\
\bar{F}_2 & = \bar{B}_2^\top p, \\
\bar{F}_3 & = \bar{B}_3^\top p, \\
\bar{b} & = b - B_1 R_1^{-1} n_1, \\
\bar{f}_1 & = \bar{p}b + \sum_{i=2}^{3} C_i^\top \theta p_{1,i} + m_1 - pB_1 R_1^{-1} n_1.
\end{aligned}
\]
Applying Lemma 5.4 in Xiong [17] to state equation of \( x_1^*(\cdot) \) and (14), we derive the optimal filtering equation:
\[
\begin{aligned}
d\dot{x}_1^*(t) = & \left[ \bar{A} \dot{x}_1^* + \bar{F}_1 \dot{\varphi} + \sum_{i=2}^{3} B_i \dot{v}_i + \bar{b} \right] dt + \left[ C_1 \dot{x}_1^* + \sigma_3 \right] dW_3(t), \\
-d\ddot{x}_1^*(t) = & \left[ \dot{A} \dot{x}_1^* + C_1 \dot{\varphi} + \sum_{i=2}^{3} \ddot{F}_i \dot{v}_i + \ddot{f}_1 \right] dt - \ddot{\theta}(t)dW_3(t), \\
\dot{x}_1^*(0) = x_0, & \quad \ddot{x}_1^*(T) = 0,
\end{aligned}
\]
which admits a unique \( \mathcal{G}_1^1 \)-adapted solution \((\dot{x}_1^*(\cdot), \ddot{x}_1^*(\cdot), \ddot{\theta}(\cdot))\).

For any \( v_2(\cdot) \in V_2 \) and \( v_3(\cdot) \in V_3 \), the problem of Player 1, first, is solved in the following theorem.

**Theorem 1.** Let \( p(\cdot) \) satisfy (13), Player 1’s optimal strategy \( v_1^1(\cdot) \) is given by (12), where \((\dot{x}_1^*(\cdot), \ddot{x}_1^*(\cdot), \ddot{\theta}(\cdot))\) is the unique \( \mathcal{G}_1^1 \)-adapted solution to (15).

**B. Problem of Player 2**

This section focuses on solving the optimal control problem of Player 2. After Player 1 exercises the optimal strategy \( v_1^1(\cdot) = v_1^1[\cdot; v_2(\cdot), v_3(\cdot)] \) (which means that \( v_1^1(\cdot) \) depends on \( v_2(\cdot) \) and \( v_3(\cdot) \)) that we got in the previous subsection, the state equation of Player 2’s problem becomes:
\[
\begin{aligned}
dx_2(t) = & \left[ A_{x_2} + (\bar{A} - A) x_2 + \bar{F}_1 \dot{\varphi} + \sum_{i=2}^{3} B_i \dot{v}_i + \bar{b} \right] dt \\
& + \sum_{i=1}^{3} C_{x_2} dW_i(t), \\
-d\ddot{x}_2(t) = & \left[ \dot{A} \dot{x}_2 + C_2 \dot{\varphi} + \sum_{i=2}^{3} \ddot{F}_i \dot{v}_i + \ddot{f}_1 \right] dt - \ddot{\theta}(t)dW_3(t), \\
x_2(0) = x_0, & \quad \ddot{x}_2(T) = 0,
\end{aligned}
\]
where \( x_2(\cdot) \triangleq x_1^*(\cdot); v_2(\cdot), v_3(\cdot) \=(x_2^*(\cdot), \dot{x}_2^*(\cdot), \ddot{x}_2^*(\cdot), \ddot{\theta}(\cdot)) \).

Noting that equation (16) is decoupled conditional mean-field forward-backward SDE (CMFFBSDE) (see [10]), which admits a unique adapted solution \((x_2(\cdot), \dot{x}_2(\cdot), \ddot{x}_2(\cdot))\).

Define the Hamiltonian function \( H_2 : [0, T] \times (\mathbb{R}^n)^{10} \rightarrow \mathbb{R} \) of Player 2 as
\[
H_2(t, x_2, \dot{x}_2, \ddot{x}_2, x_3, v_2, z_1, z_2, z_3, \psi_2) \triangleq \left[ A(t)x_2 + (\bar{A}(t) - A(t)) x_2 + \bar{F}_1(t) \dot{\varphi} + \bar{b}(t) \right]^\top \psi_2 + \sum_{i=2}^{3} C_{x_2} \dot{v}_i + \sum_{i=2}^{3} \ddot{F}_i \dot{v}_i + \ddot{f}_1 \right] dt - \ddot{\theta}(t)dW_3(t),
\]
\[
-x_2(0) = x_0, & \quad \ddot{x}_2(T) = 0.
\]

The optimal strategy \( v_2^2(\cdot) \) of Player 2 can be obtained by the maximum principle of FBSDE with asymmetric information (see for example, [10], [11]):
\[
\begin{aligned}
R_x^2 x_2^2(t) + B_2^\top \dot{y}_2^2 + \bar{F}_{2,2} \dot{\psi}_2 + n_2 = 0, & \quad \text{a.e., a.s.} \quad (18)
\end{aligned}
\]
where \((y_2(\cdot), z_1(\cdot), z_2(\cdot), z_3(\cdot), \psi_2(\cdot))\) is the unique \( \mathcal{F}_t \)-adapted solution to the forward FBSDE:
\[
\begin{aligned}
d\dot{y}_2(t) = & \left[ \bar{A} \dot{y}_2 + (\bar{A} - A) \dot{y}_2 + \sum_{i=1}^{3} C_i \dot{z}_i \\
& + Q_2 x_2^2 + n_2 \right] dt - \dot{\psi}_2(t)dW_3(t), \\
-d\ddot{y}_2(t) = & \left[ \ddot{A} \ddot{y}_2 + \bar{F}_2 \ddot{\psi}_2 + C_3 \ddot{\psi}_2 \right] dt - \ddot{\theta}(t)dW_3(t), \\
\dot{y}_2(T) = G_2 x_2^2(T), & \quad \psi_2(0) = 0,
\end{aligned}
\]
with \((x_2^2(\cdot), \dot{x}_2^2(\cdot), \ddot{x}_2^2(\cdot), \dot{\psi}(\cdot), \ddot{\psi}(\cdot))\) is the solution to (16) corresponding to \( v_2^2(\cdot) \).
Noting that (19) is a coupled CMFFBSDE, we need to decouple it to ensure its solvability. Inspired by Yong [19], let \((x^*_v(\cdot), \psi^*_v(\cdot))\) be the optimal “state”, and set

\[
X_2 = \frac{x_2}{v_2}, \quad X_3 = \frac{\eta}{\theta}, \quad \bar{Z}_1 = \frac{\bar{z}_1}{0}, \quad X_3 = \frac{\bar{z}_3}{0},
\]

\[
\bar{Z}_3 = \frac{\bar{z}_3}{\theta}, \quad X_0 = \frac{\eta_0}{0}, \quad \bar{A}_1 = \frac{A}{0}, \quad \bar{A}_2 = \frac{\bar{A} - \bar{A}}{0},
\]

\[
\bar{B}_2 = \frac{\bar{B}_2}{0}, \quad \bar{F}_1 = \frac{\bar{F}_1}{0}, \quad \bar{B}_3 = \frac{\bar{B}_3}{0}, \quad \bar{C}_1 = \frac{C_1}{0},
\]

\[
\bar{C}_2 = \frac{C_2}{0}, \quad \bar{C}_3 = \frac{C_3}{0}, \quad \bar{Q}_2 = \frac{Q_2}{0},
\]

\[
\bar{G}_2 = \frac{G_2}{0}, \quad \bar{F}_2 = \frac{(0 \bar{F}_2)}{0}, \quad \bar{F}_3 = \frac{(0 \bar{F}_3)}{0}, \quad \bar{b}_2 = \frac{b}{0},
\]

Then (16) and (19) can be rewritten as:

\[
dX_2(t) = [\bar{A}_2X_2 + \bar{A}_2X_2 + \bar{F}_1Y_2 + \bar{B}_2v_2 + \bar{B}_3v_3]
\]

\[
\quad + dX_3)dt + \sum_{i=1}^{3} \left[ \bar{C}_iX_2 + \bar{C}_iX_2 + dW_i(t),
\]

\[
-dY_2(t) = [\bar{Q}_2X_2 + \bar{A}_2Y_2 + \bar{A}_2Y_2 + \sum_{i=1}^{3} \bar{C}_i Y_i]
\]

\[
\quad + \bar{F}_2 v_2 + \bar{F}_2 v_2 + dW_3(t) - \sum_{i=1}^{3} \bar{Z}_i(t) dW_i(t),
\]

\[
X_2(0) = X_0, \quad Y_2(T) = \bar{G}_2X_2(T),
\]

and similarly, the optimal strategy \(v^*_2(\cdot)\) can be written as:

\[
v^*_2(t) = -R_{v_2}^{-1} [\bar{B}_2\bar{Y}_2 + \bar{F}_2\bar{X}_2 + n_2], \quad a.e., a.s.
\]

Next, in order to represent the optimal strategy \(v^*_2(\cdot)\) of Player 2 in the form of state feedback, we assume

\[
Y_2(t) = P_1(t)X_2(t) + P_2(t)\bar{X}_2(t) + \Phi(t),
\]

where \(P_1(\cdot), \ P_2(\cdot)\) are deterministic, differentiable with \(P_1(T) = G_2, \ P_2(T) = 0\), and \(F_T\)-adapted processes triple \((\Phi(\cdot), \Lambda_2(\cdot), \Lambda_3(\cdot))\) satisfies the BSDE:

\[
\begin{aligned}
\quad d\Phi(t) &= \Gamma(t)dt + \Lambda_2(t)dW_2(t) + \Lambda_3(t)dW_3(t),
\quad \Phi(T) = 0,
\end{aligned}
\]

where \(\Gamma(\cdot)\) is some \(F_T\)-adapted process to be determined later.

Let (22) be filtered on \(G^T_2\) to obtain \(Y_2(\cdot)\). Substituting it into (21), we can represent the optimal state feedback strategy \(v^*_2(\cdot)\) of Player 2 as:

\[
v^*_2(t) = -R_{v_2}^{-1} [\bar{B}_2\bar{Y}_2 + \bar{F}_2\bar{X}_2 + \bar{F}_2\bar{X}_2 + \bar{F}_2\bar{X}_2 + \bar{F}_2\bar{X}_2 + \bar{F}_2\bar{X}_2 + \bar{F}_2\bar{X}_2]dW_3(t),
\]

\[
\quad + dX_2(t) = \left( \bar{A}_2X_2 + \bar{A}_2X_2 + \bar{F}_1\bar{X}_2 + \bar{B}_2v_2 + \bar{B}_3v_3 \right) dt - \sum_{i=1}^{3} \bar{Z}_i(t) dW_i(t),
\]

\[
X_2(0) = X_0, \quad \Phi(T) = 0,
\]

using the same method as in Section 3.1, we introduce a system of two Riccati equations as follows:

\[
\begin{aligned}
\quad P_1 + P_1\bar{A}_1 + \bar{A}_1 P_1 + \sum_{i=1}^{3} \bar{C}_i P_1 \bar{C}_i + P_1 (\bar{F}_1 - \bar{B}_2R_2^{-1}\bar{B}_2) P_1
\quad + Q_2 = 0, \quad P_1(T) = G_2,
\end{aligned}
\]

(25)
respectively. The problem of Player 2 could be solved in the following theorem.

**Theorem 2.** Let \( P_1(\cdot), P_2(\cdot) \) satisfy (25) and (26), Player 2’s optimal strategy \( v_2^*(\cdot) \) is given by (24), where \((\hat{X}_2(\cdot), \hat{X}_2^*(\cdot), \hat{Y}_2(\cdot), \hat{A}_2(\cdot)) \) are the unique \( \mathcal{G}_2 \)-adapted solutions to (28), (29).

**C. Problem of Player 3**

This section focuses on solving the problem of Player 3. Putting Player 2’s optimal strategy \( v_2^*(\cdot) \) into (20), the “state” equation of Player 3 becomes:

\[
\begin{align*}
&dX_3(t) = \left\{ \hat{A}_1X_3 + \hat{A}_2X_3 + \hat{A}_3X_3 + \hat{F}_1\Phi + (\hat{F}_1 - \hat{F}_3)\Phi \\
&\quad + \hat{B}_3v_3 + \hat{b}_3 \right\} dt + \sum_{i=2}^3 \left[ C_i^T X_3 + \sigma_i X_3 \right] dW_i(t), \\
&-d\Phi(t) = \left\{ \hat{H}_X^T X_3 - \hat{H}_X^T X_3 + \hat{A}_1^T \Phi + \hat{A}_2^T \Phi + \hat{A}_3^T \Phi \\
&\quad + \sum_{i=2}^3 C_i^T \eta_i + \bar{F}_3X_3 + \bar{F}_3^T \eta_i + \bar{f}_3 \right\} dt \\
&\quad - \hat{A}_2^T dW_2(t) - \hat{A}_3^T dW_3(t), \\
&X_3(0) = \Phi(T) = 0,
\end{align*}
\]

where \( X_3(\cdot) \triangleq X_2^T v_2^*=X_2^T v_2^*\), \( \hat{X}_3(\cdot) \triangleq \hat{X}_2^T v_2^*, \), \( \hat{X}_3(\cdot) \triangleq \hat{X}_2^T v_2^*, \).

**Remark 4.** Applying Lemma 5.4 in [17] to the state equation (1) and (14), the equations about \((\hat{X}_3(\cdot), \hat{\Psi}_3(\cdot))\) and the equations about \((\hat{X}_3(\cdot), \hat{\Psi}_3(\cdot))\) are decoupled. After derivation, the solvability of (30) can be guaranteed though it is fully coupled. The derivation process is omitted here.

Setting
\[
\begin{align*}
\hat{Q}_3 &\triangleq \begin{pmatrix} Q_3 & 0 \\ 0 & 0 \end{pmatrix}, & \hat{G}_3 &\triangleq \begin{pmatrix} G_3 & 0 \\ 0 & 0 \end{pmatrix}, & \hat{m}_3 &\triangleq \begin{pmatrix} m_3 \\ 0 \end{pmatrix},
\end{align*}
\]
we have
\[
\begin{align*}
J_3(v_1(\cdot)^*, v_2(\cdot)^*, v_3(\cdot)) &= \frac{1}{2} \mathbb{E}\left\{ \int_0^T \left[ \langle \hat{Q}_3(t)X_3(t), X_3(t) \rangle \right. \\
&\left. + \langle \hat{R}_3(t)v_3(t), v_3(t) \rangle + \langle \hat{m}_3(t), X_3(t) \rangle \right] dt + \langle \hat{m}_3(X_3(T), X_3(T)) \right\}.
\end{align*}
\]

**Remark 5.** Different from the existing literature, there are two different state filtering in (30). We need to use the following equalities in the following derivation process of this subsection. For any \( \mathcal{F}_T \)-adapted processes \( \xi(\cdot), \eta(\cdot), \)

\[
\begin{align*}
\mathbb{E}\int_0^T \langle \xi(t) \mid \mathcal{G}_1 \rangle dt &= \mathbb{E}\int_0^T \langle \xi(t) \mid \mathcal{G}_1 \rangle dt, \\
\mathbb{E}\mathbb{E}\langle \xi(t) \mid \mathcal{G}_2 \rangle &\mid \mathcal{G}_1 \rangle dt = \mathbb{E}\mathbb{E}\langle \xi(t) \mid \mathcal{G}_1 \rangle \mid \mathcal{G}_2 \rangle dt, \\
\mathbb{E}\mathbb{E}\langle \xi(t) \mid \mathcal{G}_2 \rangle \mid \mathcal{G}_0 \rangle &= \mathbb{E}\mathbb{E}\langle \xi(t) \mid \mathcal{G}_2 \rangle \mid \mathcal{G}_0 \rangle,
\end{align*}
\]

Define the Hamiltonian function \( H_3 : [0, T] \times (\mathbb{R}^{2n})^5 \times \mathbb{R}^n \times (\mathbb{R}^{2n})^5 \rightarrow \mathbb{R} \) of Player 3 as:

\[
H(t, X_3, \Phi, \Phi, \lambda, v_3, Y_3, Z_1, Z_2, Z_3, \Psi_3) = \left[ \hat{A}_3(t)X_3 + \hat{A}_3(t)X_3 + \hat{A}_3(t)X_3 + \hat{F}_3(t)\Phi + (\hat{F}_3 - \hat{F}_1(t))\Phi \\
+ \hat{B}_3(t)v_3 + \hat{b}_3(t) \right]^T Y_3 + \sum_{i=1}^3 \left[ C_i^T X_3 + \sigma_i(t) \right] Y_i \\
+ \left[ \hat{H}_X^T X_3 - \hat{H}_X^T X_3 + \hat{A}_1^T \Phi + \hat{A}_2^T \Phi + \hat{A}_3^T \Phi \\
+ \sum_{i=2}^3 C_i^T \eta_i + \bar{F}_3X_3 + \bar{F}_3^T \eta_i + \bar{f}_3 \right] dt \\
+ \frac{1}{2} \langle \hat{m}_3(t, X_3) \rangle + \frac{1}{2} \langle \hat{m}_3(t, X_3) \rangle + \langle \hat{m}_3(t, X_3) \rangle + \langle \hat{m}_3(t, X_3) \rangle \right).
\]
and similarly, the optimal strategy \( v_3^\ast (\cdot) \) of Player 3 can be written as:

\[
v_3^\ast (t) = -R_3^{-1} \left[ \tilde{B}_3^\top \dot{\tilde{Y}}_3 + \tilde{B}_4^\top \hat{X}_3 + \tilde{B}_5^\top \hat{X}_3 + \eta_3 \right], \quad a.e., \ a.s.
\]  

(37)

Next, in order to represent the optimal strategy \( v_2^\ast (\cdot) \) of Player 3 in the state feedback form, we assume that

\[
\mathcal{Y}_2(t) = P_1(t)X_2(t) + P_2(t)\hat{X}_2(t) + P_3(t)\hat{X}_3(t) + \Omega(t),
\]

(38)

where \( P_1(\cdot), P_2(\cdot), P_3(\cdot) \) are deterministic, differentiable with \( P_1(T) = \hat{\Theta}_1 \), \( P_2(T) = 0 \), \( P_3(T) = 0 \), and \( \mathcal{F}_T \)-adapted processes \( (\Omega(\cdot), \Pi_1(\cdot), \Pi_2(\cdot), \Pi_3(\cdot)) \) satisfies the BSDE:

\[
d\Omega(t) = \Delta(t)dt + \sum_{i=1}^3 \Pi_i(t)dW_i(t),
\]

\[
\Omega(T) = 0,
\]

where \( \Delta(\cdot) \) is some \( \mathcal{F}_T \)-adapted process to be determined.

Let (38) be filtered on \( \mathcal{G}_T^\ast \) to obtain \( \tilde{Y}_2(\cdot) \) and on \( \mathcal{G}_T^\ast \) to obtain \( \hat{Y}_3(\cdot) \). Substituting \( \hat{Y}_3(\cdot) \) into (37), we can get the optimal state feedback strategy \( v_2^\ast (\cdot) \) of Player 3 is:

\[
v_2^\ast (t) = -R_2^{-1} \left[ \tilde{B}_2^\top \dot{\tilde{Y}}_2 + \tilde{B}_3^\top \dot{\hat{X}}_2 + \tilde{B}_4^\top \dot{\hat{X}}_3 + \eta_2 \right], \quad a.e., \ a.s.
\]  

(40)

As in sections 3.1 and 3.2, we introduce

\[
\begin{align*}
\hat{P}_1 &= \hat{P}_1 \left[ \tilde{A}_1 - \tilde{B}_3 R_3^{-1} \tilde{B}_2 \right] + \left[ \tilde{A}_1^\top - \tilde{B}_3^\top R_3^{-1} \tilde{B}_2^\top \right] \hat{P}_1 \\
\hat{P}_2 &= \hat{P}_2 \left[ \tilde{A}_2 - \tilde{B}_3 R_3^{-1} \tilde{B}_2 \right] + \left[ \tilde{A}_2^\top - \tilde{B}_3^\top R_3^{-1} \tilde{B}_2^\top \right] \hat{P}_2 \\
\hat{P}_3 &= \hat{P}_3 \left[ \tilde{A}_3 - \tilde{B}_3 R_3^{-1} \tilde{B}_2 \right] + \left[ \tilde{A}_3^\top - \tilde{B}_3^\top R_3^{-1} \tilde{B}_2^\top \right] \hat{P}_3 \\
\end{align*}
\]

(41)

\[
\begin{align*}
P_1 &= P_1 \left[ \tilde{A}_1 - \tilde{B}_3 R_3^{-1} \tilde{B}_2 \right] + \left[ \tilde{A}_1^\top - \tilde{B}_3^\top R_3^{-1} \tilde{B}_2^\top \right] P_1 \\
P_2 &= P_2 \left[ \tilde{A}_2 - \tilde{B}_3 R_3^{-1} \tilde{B}_2 \right] + \left[ \tilde{A}_2^\top - \tilde{B}_3^\top R_3^{-1} \tilde{B}_2^\top \right] P_2 \\
P_3 &= P_3 \left[ \tilde{A}_3 - \tilde{B}_3 R_3^{-1} \tilde{B}_2 \right] + \left[ \tilde{A}_3^\top - \tilde{B}_3^\top R_3^{-1} \tilde{B}_2^\top \right] P_3 \\
\end{align*}
\]

(42)

Noting the Riccati equation system (41)-(43) is decoupled, under some assumptions (See, for example, Yong [18], or Lim and Zhou [8]) it admits a differentiable solution triple \( (P_1(\cdot), P_2(\cdot), P_3(\cdot)) \), since we can solve them sequentially.

Then, we can get \( (\Omega(\cdot), \Pi_1(\cdot), \Pi_2(\cdot), \Pi_3(\cdot)) \) satisfies the following BSDE:

\[
\begin{align*}
-d\Omega(t) &= \left[ \tilde{A}_1^\top - \tilde{B}_3^\top R_3^{-1} \tilde{B}_2^\top \right] \hat{\Pi}_1 + \left[ \tilde{A}_2^\top - \tilde{B}_3^\top R_3^{-1} \tilde{B}_2^\top \right] \hat{\Pi}_2 + \left[ \tilde{A}_3^\top - \tilde{B}_3^\top R_3^{-1} \tilde{B}_2^\top \right] \hat{\Pi}_3 + \hat{\Pi}_4(t) \\
\hat{\Pi}_1(t) &= \left[ \tilde{A}_1^\top - \tilde{B}_3^\top R_3^{-1} \tilde{B}_2^\top \right] \hat{\Pi}_1 + \left[ \tilde{A}_2^\top - \tilde{B}_3^\top R_3^{-1} \tilde{B}_2^\top \right] \hat{\Pi}_2 + \left[ \tilde{A}_3^\top - \tilde{B}_3^\top R_3^{-1} \tilde{B}_2^\top \right] \hat{\Pi}_3 \\
\hat{\Pi}_2(t) &= \left[ \tilde{A}_2^\top - \tilde{B}_3^\top R_3^{-1} \tilde{B}_2^\top \right] \hat{\Pi}_1 + \left[ \tilde{A}_3^\top - \tilde{B}_3^\top R_3^{-1} \tilde{B}_2^\top \right] \hat{\Pi}_2 + \left[ \tilde{A}_3^\top - \tilde{B}_3^\top R_3^{-1} \tilde{B}_2^\top \right] \hat{\Pi}_3 \\
\hat{\Pi}_3(t) &= \left[ \tilde{A}_3^\top - \tilde{B}_3^\top R_3^{-1} \tilde{B}_2^\top \right] \hat{\Pi}_1 + \left[ \tilde{A}_3^\top - \tilde{B}_3^\top R_3^{-1} \tilde{B}_2^\top \right] \hat{\Pi}_2 + \left[ \tilde{A}_3^\top - \tilde{B}_3^\top R_3^{-1} \tilde{B}_2^\top \right] \hat{\Pi}_3 \\
\end{align*}
\]

(44)

\[
X_3(0) = X_0, \quad \Omega(T) = 0.
\]

(45)

\[
\begin{align*}
-d\tilde{\Omega}(t) &= \left[ \tilde{A}_1^\top - \tilde{B}_3^\top R_3^{-1} \tilde{B}_2^\top \right] \tilde{\Pi}_1 + \left[ \tilde{A}_2^\top - \tilde{B}_3^\top R_3^{-1} \tilde{B}_2^\top \right] \tilde{\Pi}_2 + \left[ \tilde{A}_3^\top - \tilde{B}_3^\top R_3^{-1} \tilde{B}_2^\top \right] \tilde{\Pi}_3 + \tilde{\Pi}_4(t) \\
\tilde{\Pi}_1(t) &= \left[ \tilde{A}_1^\top - \tilde{B}_3^\top R_3^{-1} \tilde{B}_2^\top \right] \tilde{\Pi}_1 + \left[ \tilde{A}_2^\top - \tilde{B}_3^\top R_3^{-1} \tilde{B}_2^\top \right] \tilde{\Pi}_2 + \left[ \tilde{A}_3^\top - \tilde{B}_3^\top R_3^{-1} \tilde{B}_2^\top \right] \tilde{\Pi}_3 \\
\tilde{\Pi}_2(t) &= \left[ \tilde{A}_2^\top - \tilde{B}_3^\top R_3^{-1} \tilde{B}_2^\top \right] \tilde{\Pi}_1 + \left[ \tilde{A}_3^\top - \tilde{B}_3^\top R_3^{-1} \tilde{B}_2^\top \right] \tilde{\Pi}_2 + \left[ \tilde{A}_3^\top - \tilde{B}_3^\top R_3^{-1} \tilde{B}_2^\top \right] \tilde{\Pi}_3 \\
\tilde{\Pi}_3(t) &= \left[ \tilde{A}_3^\top - \tilde{B}_3^\top R_3^{-1} \tilde{B}_2^\top \right] \tilde{\Pi}_1 + \left[ \tilde{A}_3^\top - \tilde{B}_3^\top R_3^{-1} \tilde{B}_2^\top \right] \tilde{\Pi}_2 + \left[ \tilde{A}_3^\top - \tilde{B}_3^\top R_3^{-1} \tilde{B}_2^\top \right] \tilde{\Pi}_3 \\
\end{align*}
\]

(46)
and
\[
\frac{d\tilde{x}_3(t)}{dt} = \left\{ \sum_{i=1}^{3} \tilde{a}_i - \tilde{b}_3 R_3^{-1}(\tilde{a}_3 + \tilde{a}_4) + (\tilde{a}_1 - \tilde{b}_3 R_3^{-1}\tilde{b}_4) \times \sum_{i=1}^{3} p_i \tilde{x}_3 + \left[ \tilde{a}_3 - \tilde{b}_3 R_3^{-1}\tilde{b}_4 \right] \tilde{\Omega} - \tilde{b}_3 R_3^{-1}n_3 + \tilde{b}_4 \right\} dt \\
+ \left[ \tilde{e}_3 \tilde{x}_3 + \tilde{e}_4 \tilde{\Omega} \right] dW_3(t),
\]
\[
- d\tilde{\Omega}(t) = \left\{ \sum_{i=1}^{3} \tilde{a}_i - \tilde{b}_3 \tilde{b}_4 R_3^{-1}\tilde{b}_4 \right\} \tilde{\Omega} + \sum_{i=1}^{3} \tilde{e}_i \tilde{p}_i \tilde{\Sigma}_i + \tilde{e}_4 \tilde{p}_2 \tilde{\Sigma}_2 \\
+ \tilde{e}_4 \tilde{p}_3 \tilde{\Sigma}_3 + \tilde{p}_1 \tilde{b}_2 + \frac{\tilde{f}_3 - \sum_{i=1}^{3} \tilde{p}_i \tilde{b}_3 + \tilde{a}_3}{3} R_3^{-1}n_3 \right\} dt \\
- \tilde{\Pi}_3(t) dW_3(t),
\]
\[
\tilde{x}_3(0) = \tilde{x}_0_3, \quad \tilde{\Omega}(T) = 0.
\] (47)

The problem of Player 3 can be solved in the following.

**Theorem 3.** Let \( P_1(\cdot), P_2(\cdot), P_3(\cdot) \) satisfy the system of Riccati equations (41)-(43). Player 3’s optimal strategy is given by (40), where \( (X_3(\cdot), \tilde{x}_3(\cdot), \tilde{\Omega}(\cdot), \Pi_1(\cdot), \Pi_2(\cdot), \Pi_3(\cdot) ) \) are the unique adapted solutions to (45)-(47).

In addition, we can rewrite Player 1 and Player 2’s optimal feedback strategies (12) and (24) as follows:

\[
v_2(t) = -R_2^{-1} \left\{ \left[ \begin{array}{cc} B_2 & 0 \end{array} \right] P_1(t) \left[ \begin{array}{c} 0 \\ P_2(t) \end{array} \right] + \left[ \begin{array}{cc} 0 & B_2 \end{array} \right] \tilde{p}_1(t) \tilde{p}_2(t) \right\} \tilde{x}_3(t) + \left[ \begin{array}{cc} 0 & \tilde{b}_2 \end{array} \right] \tilde{\Omega}(t) + n_2(t), \quad a.e., \ a.s.,
\]
\[
v_1(t) = -R_1^{-1} \left\{ \left[ \begin{array}{cc} B_1' & 0 \end{array} \right] P_1(t) \left[ \begin{array}{c} 0 \\ P_2(t) \end{array} \right] + \left[ \begin{array}{cc} 0 & B_1' \end{array} \right] \tilde{p}_1(t) \tilde{p}_2(t) \right\} \tilde{x}_3(t) + \left[ \begin{array}{cc} 0 & \tilde{b}_1 \end{array} \right] \tilde{\Omega}(t) + n_1(t), \quad a.e., \ a.s.
\] (48)

Up to now, we have obtained the optimal feedback equilibrium strategies of our stochastic LQ Stackelberg differential game with asymmetric information.

**IV. CONCLUSIONS AND PROSPECTS**

In this paper, we have studied a three-level stochastic LQ Stackelberg differential game with asymmetric information, and the state feedback representation of the players’ equilibrium strategies is given. The solution is performed to solve the optimal control problems of Player 1, Player 2, and Player 3 in turn. Maximum principle with partial information and optimal filtering are used to seek the equilibrium strategies of Player 1 and Player 2. When Player 1 and Player 2 exercise their optimal strategies, the state equation for the optimal control problem faced by Player 3 is a fully coupled FBSDE with two different types of stochastic filtering terms, which is different from existing literatures. A new system of high-dimensional Riccati equations is introduced to get Player 3’s optimal state feedback strategy.

In the future, it is worthy to study the three-level Stackelberg game in general, not just linear quadratic problem, and it can be used to solve cooperative advertising and pricing problem in the dynamic three-echelon supply chain with asymmetric information, such as Example 1.

**REFERENCES**

[1] Baghery F., Øksendal B. A maximum principle for stochastic control with partial information. *Stochastic Analysis and Applications*, 2007, 25: 705-717.

[2] Basar T., Olsder G.J. *Dynamic Noncooperative Game Theory, Second Edition*. SIAM, Philadelphia, 1999.

[3] Chen L., Shen Y. On a new paradigm of optimal reinsurance: A stochastic Stackelberg differential game between an insurer and a reinsurer. *Astin Bulletin*, 2018, 48(2): 905-960.

[4] Chutani A., Sethi S.P. Dynamic cooperative advertising under manufacturer and retailer level competition. *European Journal of Operational Research*, 2018, 268: 635-652.

[5] He X., Prasad A. and Sethi S.P. Cooperative advertising and pricing in a dynamic stochastic supply chain: Feedback Stackelberg strategies. *Production and Operations Management*, 2009, 17(1): 1-44.

[6] Kennedy A. P., Sethi S.P., Siu C.C. and Yam S.C.P. Cooperative advertising in a dynamic three echelon supply chain. *Production and Operations Management*, 2021, 30(11): 3881-3905.

[7] Li Z., Marelli D., Fu M., Cai Q. and Meng W. Linear quadratic Gaussian Stackelberg game under asymmetric information patterns. *Automatica*, 2021, 125: 109406.

[8] Lim A.E.B., Zhou X. Linear-quadratic control of backward stochastic differential equations. *SIAM Journal on Control and Optimization*, 2001, 40(2): 450-474.

[9] Mukaidani H., Xu H. Stackelberg strategies for stochastic systems with multiple followers. *Automatica*, 2015, 53: 53-59.

[10] Shi J., Wang G. and Xiong J. Leader-follower stochastic differential game with asymmetric information and applications. *Automatica*, 2016, 63: 60-73.

[11] Shi J., Wang G. and Xiong J. Linear-quadratic stochastic Stackelberg differential game with asymmetric information. *Science China Information Sciences*, 2017, 60(9): 1-15.

[12] Shi J., Wang G. and Xiong J. Stochastic linear quadratic Stackelberg differential game with overlapping information. *ESAIM: Control, Optimisation and Calculus of Variations*, 2020, 26: 83.

[13] Stackelberg H. *The Theory of the Market Economy*. Oxford University Press, London, 1952.

[14] Wang G., Zhang S. An asymmetric information mean-field type linear-quadratic stochastic Stackelberg differential game with one leader and two followers. *Optimal Control Applications and Methods*, 2020, 41(4): 1034-1051.

[15] Wang Y., Yan Z. Pareto-based Stackelberg differential game for stochastic systems with multi-followers. *Applied Mathematics and Computation*, 2023, 436: 127512.

[16] Yuan Y., Liang Z. and Han X. Robust reinsurance contract with asymmetric information in a stochastic Stackelberg differential game. *Scandinavian Actuarial Journal*, 2022, 2022(4): 328-355.

[17] Xiong J. *An Introduction to Stochastic Filter Theory*. Oxford University Press, London, 2008.

[18] Yong J. *Linear Forward-Backward Stochastic Differential Equations*. Springer-Verlag, New York, 1999.

[19] Zheng Y., Shi J. Stackelberg stochastic differential game with asymmetric information and applications. *Applied Mathematics and Computation*, 2020, 41(4): 1034-1051.

[20] Zheng Y., Shi J. Stackelberg stochastic differential game with asymmetric observations. *International Journal of Control*, 2022, 95(5): 2510-2530.