1. Motivation

Often in practice jobs are partially ordered in such a way that the second job cannot begin before the first one is finished. Such situations are considered by Kyle Siegrist in his recent paper [S]. So we have a partially ordered network of jobs and are interested in, among others, the subset of jobs which are already completed at time $t$. Clearly this increases with $t$, and the mentioned subset can be viewed as partially ordered again, the order being inherited by the original order. So we have an increasing family of partially ordered sets. Such a situation asks itself for more algebraic structurisation. In fact there is the following new application of the [HR]–method.

2. The set–up of the Bauer simplex of continuous exchangeable probability measures on the set of order processes

We use here the notation and terminology of [HR2000]. In particular $V$ is the set of all reflexive, transitive (but not necessarily anti-symmetric) relations — called by us partial orders — on $M$, where $M$ is a countable infinite set, and we recall that $V$ carries a natural metrisable topology and that the diagonal relation $D$ is the neutral element of the semigroup $(V, \lor)$.)

**Definition 1.** We denote by $\mathcal{Y}$ the set of all left–continuous increasing maps $Y : [0, \infty) \mapsto V$ with $Y(0) = D$.

**Definition 2.** For $Y, Z \in \mathcal{Y}$ the join $Y \lor Z \in \mathcal{Y}$ is given by $(Y \lor Z)(t) := Y(t) \lor Z(t)$.

**Proposition 3.** $(\mathcal{Y}, \lor)$ is an idempotent abelian semigroup with neutral element the constant $D$ and absorbing element the map which switches from $D$ to $M \times M$ immediately at $t = 0$.

**Definition 4.** By the support of a $Y \in \mathcal{Y}$ we understand the set $\langle Y \rangle := \bigcup_{t \in [0, \infty]} \langle Y(t) \rangle \subseteq M$.

**Definition 5.** $Z := \{ Y \in \mathcal{Y} | \langle Y \rangle \text{ is finite.} \}$

**Lemma 6.** $\langle D \rangle = \emptyset$ and $\langle Y \lor Z \rangle \subseteq \langle Y \rangle \cup \langle Z \rangle$ for all $Y, Z \in \mathcal{Y}$, so $Z$ is a sub–semigroup of $\mathcal{Y}$.

**Definition 7.** For $Y, Z \in \mathcal{Y}$ we say $Y \leq Z$ if and only if $Y \lor Z = Z$.

**Lemma 8.** This is the case if and only if $Y(t) \leq Z(t)$ for all $t \in [0, \infty]$.

**Lemma 9.** “$\leq$” is a partial order on $\mathcal{Y}$.

**Definition 10.** For $Z \in Z$ we put $Q_Z := \{ Y \in \mathcal{Y} | Z \subseteq Y \}$.

**Lemma 11.** For all $Y, Z \in Z$ we have $Q_{Y \lor Z} = Q_Y \cap Q_Z$.

**Definition 12.** A sub–semigroup $I$ of $Z$ is called left–hereditary if for all $Z_1, Z_2 \in Z$ $Z_1 \leq Z_2 \in I$ implies $Z_1 \in I$.

**Proposition 13.** The join $\bigvee_{Z \in I} Z$ is left–continuous again.

**Proof.** For this let us remember why the semigroup operation $\lor : V \times V$ is continuous. We see $V$ as a closed subset of the compact metrisable space $\{0, 1\}^{M \times M}$. There is a natural mapping $p : \{0, 1\}^{M \times M} \mapsto V$ which assigns to any $\hat{V} \in \{0, 1\}^{M \times M}$ the smallest partial order containing $\hat{V}$. It is easily seen that $p$ is continuous, on the basis of the explicit construction with the chains and because the subsets $Q_U$ (with $U \in U$) generate the topology of $V$.
The semigroup operation \( \vee \) on \( V \) is now the obvious composition of continuous maps
\[
V \times V \rightarrow \{0,1\}^{M \times M} \times \{0,1\}^{M \times M} \rightarrow \{0,1\}^{M \times M} \rightarrow V ,
\]
so \( \vee : V \times V \rightarrow V \) is continuous. The problem with infinitely many factors instead of just two factors is the map in the middle in the above chain composition. I do not see that it is continuous. However, for our left–continuity we only need continuity from below. So, in terms of subsets of \( M \times M \), the question is, if \( A_{mn} \not\supset A_m \) as \( n \to \infty \) for each \( m \), whether or not \( \bigcup_m A_{mn} \) converges to \( \bigcup_m A_m \). It is definitively increasing in \( n \), so the question is whether or not
\[
\bigcup_n A_{mn} = \bigcup_m A_{mn}
\]
is true. Trivially, it is true, completing the proof of the proposition.

**Proposition 14.** If \( I \) is a left–hereditary sub–semigroup of \( Z \) and \( Z \in Z \) is such that \( Z \leq \bigvee_{Y \in I} Y \), then for each \( \varepsilon > 0 \) \( Z(t) \) is \( \leq \) the maximum of finitely many \( Y \in I \), taken at the time \( t + \varepsilon \), for all \( t \).

**Proof.** For all \( j,k \in M \) we put \( t_0 = t_0(j,k) : = \inf \{ t \in [0,\infty] \mid (j,k) \in Z(t) \} \) with the usual convention \( \inf(\emptyset) = \infty \). For each pair \( (j,k) \) with \( t_0(j,k) < \infty \) and each \( \varepsilon > 0 \) there is a \( Y_{jk} \in I \) such that we have \( (j,k) \in \bigvee_{Y \in I} (Y_{jk} + \varepsilon) \). We define \( Y_{jk}(t) \) to be \( D \) on \( [0,t_0 + \varepsilon] \) and to be \( Y_{jk}(t_0 + \varepsilon) \) on \( [t_0 + \varepsilon, \infty] \), and because of left-heredity we have still \( Y_{jk} \in I \). For \( j,k \in (Z) \) with \( t_0(j,k) = \infty \) we put \( Y_{jk} \) to be the constant \( D \), which is always \( \in I \). So we have \( Z(t) \leq \bigvee_{j,k \in (Z)} (Y_{jk}(t + \varepsilon)) \) with \( \bigvee_{j,k \in (Z)} Y_{jk} \in I \), and the proposition is proved.

Next, in order to endow \( Y \) with a suitable topology, so to introduce in a useful way the space \( M^1_+(Y) \) of (Radon) probability measures on \( Y \), we need to resort to a Skorokhod–like topology on \( Y \). For this we recall that \( V \) can be seen as a subset of \( \{0,1\}^{M \times M} \), and we start with the basic set \( \{0,1\} \) in the place of \( V \), denoting by \( Y' \) the set of all left–continuous increasing maps \( Y' : \{0,\infty\} \rightarrow \{0,1\} \) with \( Y'(0) = 0 \). Any such map can be identified with its jump point \( \in \{0,\infty\} \), and we let \( Y' \) inherit its (compact) topology from there. Our multidimensional set \( Y \) consists of certain maps from \( \{0,\infty\} \) to \( V \subset \{0,1\}^{M \times M} \), and viewed as maps from \( \{0,\infty\} \) to \( \{0,1\}^{M \times M} \) this splits into the component maps from \( \{0,\infty\} \) to \( \{0,1\} \), which are all left–continuous and increasing. In this way

**Definition 15.** \( Y \) inherits its topology from the isomorphic set
\[
\tilde{Y} := \{ f \in \{0,\infty\}^{(M \times M)\setminus D} \mid f(j,l) \leq \max( f(j,k) , f(k,l) ) \text{ for all pairwise different } j,k,l \in M \},
\]
[0,\infty]^{(M \times M)\setminus D} being endowed with the product topology, which is compact by Tychonov’s theorem and metrisable, and \( Y \) being compact as a closed subset of \( \{0,\infty\}^{(M \times M)\setminus D} \).

**Definition 16.** By \( M^1_+(Y) \) we denote the space of all Borel probability measures on \( Y \), all of which are automatically Radon, \( Y \) being endowed with the above compact metrisable topology.

**Proposition 17.** The Borel \( \sigma \)–algebra on \( Y \) is generated by the closed subsets \( Q_Z \).

**Proof.** In the framework of the isomorphic description of \( Y \) given by Definition 15, denoting the canonical projections from \( \tilde{Y} \) to \( \{0,\infty\} \) by \( pr_{jk} \), \( Q_Z \) corresponds to the intersection of finitely many sets of the form \( pr_{jk}^{-1}(\{0,1\}) \), where the \( t \)'s are the switching times of \( Z \) (since we require the switching times of the \( Y \)'s in \( Q_Z \) to be \( \leq \) those of \( Z \) ), so the sets \( Q_Z \) are closed.

The Borel \( \sigma \)–algebra on \( \{0,\infty\}^{(M \times M)\setminus D} \) is generated by the closed subsets, and the “finite–dimensional” closed subsets ( \( \times [0,\infty] \times [0,\infty] \times \ldots \) ) already suffice to generate this, since any other closed subset is the intersection of countably many “finite–dimensional” ones, the finite–dimensional projections being continuous. As we know from \( [0,\infty]^n \), the \( \sigma \)–algebra generated by the finite–dimensional closed intervals \( [0,\bar{t}] \) is equal to the one generated by all closed sets, and the intervals \( [0,\bar{t}] \) correspond to our sets \( Q_Z \). Finally, the transition from \( [0,\infty]^{(M \times M)\setminus D} \) to \( Y \) is achieved by taking the trace \( \sigma \)–algebra on the closed subset \( \tilde{Y} \), completing the proof of the Proposition.

**Theorem 18.** A function \( \varphi : Z \rightarrow \mathbb{R} \) is positive definite with respect to \( \vee \), normalised (i.e. \( \varphi(D) = 1 \)) and continuous from below if and only if
\[
\varphi(Z) = \mu(Q_Z) \text{ for all } Z \in Z
\]
for a (uniquely determined) \( \mu \in M_+^1(\mathcal{Y}) \).

Proof. For any \( \mu \in M_+^1(\mathcal{Y}) \) the function \( \varphi : \mathcal{Z} \mapsto [0,1] \) defined by \( \varphi(Z) = \mu(Q_Z) \) is positive definite and fulfills \( \varphi(D) = 1 \), as is easily seen in the same way as in previous work of Paul Ressell and myself. Moreover it is continuous from below, because this is totally analogous to the continuity from above for distribution functions of probability measures on (certain Borel–measurable subsets of) \([0,\infty]^n\), \( Q_Z = \{ Y \in \mathcal{Y} | Y \geq Z \} \) being the event that the switching times of \( Y \) are all \( \leq \) those of \( Z \), where only finitely many switching times of \( Z \) are allowed to be \( < \infty \).

Conversely, if \( \varphi \) is any positive definite function with \( \varphi(D) = 1 \), there is a unique Radon probability measure \( \nu \) on \( \mathcal{Z} \) representing \( \varphi \) via

\[
\varphi(Z) = \int 1_I(Z) \, d\nu(I) = \nu(\{ I \in \mathcal{I} | Z \in I \})
\]

where \( \mathcal{I} \) denotes the set of all left–hereditary sub–semigroups of \( \mathcal{Z} \), and where the identification of \( I \in \mathcal{I} \) with \( I \in \{ 0,1 \}^\mathcal{Z} \) is used to topologise \( \mathcal{I} \). We define \( h : \mathcal{I} \mapsto \mathcal{Y} \) by

\[
h(I) := \bigvee_{Z \in I} Z .
\]

In the following argument we mean by the sub–index “−ε” the shift operation \( f \mapsto f(\bullet - \varepsilon) \) ( where for times \( t \leq 0 \) \( f(t) \equiv D \) ) and by the sub–index “+ε” we mean \( f \mapsto f(\bullet + \varepsilon) \). Note that in \( \mathcal{Z} \) \( Z^{(1)} = Z^{(2)} \) is equivalent to \( Z^{(1)}_{-\varepsilon} = Z^{(2)}_{-\varepsilon} \), but not equivalent to \( Z^{(1)}_{+\varepsilon} = Z^{(2)}_{+\varepsilon} \). So in the argument below \( Z_{-\varepsilon} \in I \) clearly implies \( Z = (Z_{-\varepsilon})_{+\varepsilon} \in I_{+\varepsilon} \), and \( Z \in I_{+\varepsilon} \) implies \( Z_{-\varepsilon} \in (I_{+\varepsilon})_{-\varepsilon} \). Here, for \( I \), first taking “+ε” and afterwards “−ε” means replacing the elements of \( I \) on \([0,\varepsilon]\) by \( D \), and the result of this is still in \( I \) because of the left–hereditarity of \( I \), so we luckily arrive at \( Z_{-\varepsilon} \in I \) as desired. With the first of the following “≥” being by Proposition 14, we have

\[
\{ I \in \mathcal{I} | Z_{-\varepsilon} \in I \} = \{ I \in \mathcal{I} | Z \in I_{+\varepsilon} \} \supseteq \{ I \in \mathcal{I} | Z \leq h(I) \} = h^{-1}(Q_{\mathcal{Z}}) \supseteq \{ I \in \mathcal{I} | Z \in I \}
\]

where the \( \nu \)-measure of the leftmost resp. rightmost side is \( \varphi(Z_{-\varepsilon}) \) resp. \( \varphi(Z) \). For \( \varepsilon \searrow 0 \) the switching times of \( Z_{-\varepsilon} \) converge from above to those of \( Z \), which means \( Z_{-\varepsilon} \nearrow Z \) in \( \mathcal{Z} \). So with the additional assumption that \( \varphi \) is continuous from below, \( \varphi(Z_{-\varepsilon}) \) converges to \( \varphi(Z) \). This shows that \( \nu \) is measurable with respect to the \( \nu \)-completion of the \( \sigma \)-algebra on \( \mathcal{I} \) and that \( \varphi(Z) = \nu(h^{-1}(Q_{\mathcal{Z}})) =: \mu(Q_{\mathcal{Z}}) \) with \( \mu \) being the image measure \( \nu^b \in M_+^1(\mathcal{Y}) \) as desired, finishing the proof of the theorem.

Corollary 19. In \( M_+^1(\mathcal{Y}) \) a sequence \((\mu_n)_{n \in \mathbb{N}}\) converges to \( \mu \) if and only if the corresponding positive definite functions fulfill \( \varphi_n(Z) \to \varphi(Z) \) for all \( Z \in \mathcal{Z} \) with \( \mu(\partial Q_{\mathcal{Z}}) = 0 \).

Proof. By the portmanteau theorem, the implication “⇒” is obvious. Conversely, if \( \varphi_n(Z) \to \varphi(Z) \) for all \( Z \in \mathcal{Z} \) with \( \mu(\partial Q_{\mathcal{Z}}) = 0 \), then by the compactness of \( M_+^1(\mathcal{Y}) \) we have along some sub–sequence \( \mu_n \to \mu \) and \( \varphi_n(Z) \to \varphi(Z) \) for all \( Z \) with \( \mu(\partial Q_{\mathcal{Z}}) = 0 \), so \( \varphi(Z) = \varphi(Z) \) for all \( Z \) with \( \mu(\partial Q_{\mathcal{Z}}) = \mu(\partial Q_{\mathcal{Z}}) = 0 \). By an analogous argument as with distribution functions on \( \mathbb{R}^n \), this suffices to imply \( \mu = \mu \) and finally \( \mu_n \to \mu \) for the whole sequence.

Next we introduce our notion of exchangeability for measures in \( M_+^1(\mathcal{Y}) \). The permutations (finite or infinite ones, this does not matter here) of the countably infinite base set \( M \) act in the natural way on the set \( \mathcal{V} \) of partial orders, cf. [HR2000]. And on \( \mathcal{Y} \) such a permutation acts timepointwise in the same way as it acts on \( \mathcal{V} \), and so the action carries in the canonical way over to \( M_+^1(\mathcal{Y}) \).

Definition 20. If a \( \mu \in M_+^1(\mathcal{Y}) \) is invariant under all these permutations, then we call it exchangeable and denote this in signs by \( \mu \in M_+^{1,\mathcal{Y}}(\mathcal{Y}) \).

Next we express this property by means of a mapping

\[
g : \mathcal{Z} \to T ,
\]

where \((T,+)\) is another abelian (but not idempotent) semigroup. \( T \) is defined as the set of all isomorphy classes of \( \mathcal{Z} \), where \( Z_1, Z_2 \in \mathcal{Z} \) are called isomorphic if they are permutations of each other. \( g : \mathcal{Z} \to T \) is then the mapping which assigns to every \( Z \in \mathcal{Z} \) its isomorphy class \( g(Z) \in T \). The semigroup operation “+” on \( T \) is defined by

\[
g(Z_1) + g(Z_2) := g(Z_1 \lor Z_2) \text{ for representants } Z_1 \text{ and } Z_2 \text{ with disjoint supports}.
\]

Clearly this is well–defined. In the same way as in our previous papers it is straightforward that \( g \) is strongly almost additive and hence strongly positivity forcing. The mapping \( g \) now expresses exchangeability as follows: let
\( \mu \in M^+_{s}(\mathcal{Y}) \); then we have the corresponding positive definite function \( \varphi : \mathcal{Z} \rightarrow \mathbb{R} \) with \( \varphi(\mathcal{Z}) = \mu(\mathcal{Q}_\mathcal{Z}) \), and \( \mu \) is exchangeable if and only if \( \varphi(\mathcal{Z}) \) depends only on the isomorphy class of \( \mathcal{Z} \), factorising over \( g \), i.e. \( \varphi = f \circ g \) for some mapping \( f : T \rightarrow \mathbb{R} \). This mapping \( f \) is necessarily positive definite again, since \( g \) is positivity forcing.

Because \( g \) is even strongly positivity forcing, the set \( \mathcal{P}^{1,g}(\mathcal{Z}) \) of positive definite functions on \( \mathcal{Z} \) factorising over \( g \) is a Bauer simplex whose extreme points are precisely the functions of the form \( \rho \circ g \) with \( \rho \) being a bounded character on the semigroup \( (T,+) \).

**Definition 21.** Let \( \mathcal{P}^{c,g}(\mathcal{Z}) \) denote the set of those functions in \( \mathcal{P}^{1,g}(\mathcal{Z}) \) which are continuous from below.

**Lemma 22.** \( \mathcal{P}^{c,g}(\mathcal{Z}) \) is an extreme subset of \( \mathcal{P}^{1,g}(\mathcal{Z}) \), i.e. there is no \( \varphi \in \mathcal{P}^{c,g}(\mathcal{Z}) \) which is equal to a non-trivial convex linear combination of functions in \( \mathcal{P}^{1,g}(\mathcal{Z}) \).

**Proof.** For decreasing functions, the property of being continuous from below is obviously an extreme one.

**Lemma 23.** \( \mathcal{P}^{c,g}(\mathcal{Z}) \) is a Bauer simplex with respect to the topology of full pointwise convergence, whose extreme points are precisely the below-continuous functions of the form \( \rho \circ g \) with bounded characters \( \rho \) on \( T \).

**Proof.** Immediate by the above and Lemma 22.

In Lemma 23, the below-continuous functions of the form \( \rho \circ g \) with bounded characters \( \rho \) on \( T \) correspond to \( \mu \in M^1_{s,c}(\mathcal{Y}) \) with \( \mathcal{Q}_1, \ldots, \mathcal{Q}_m \) \( \mu \)-independent for any \( \mathcal{Z}_1, \ldots, \mathcal{Z}_m \in \mathcal{Z} \) with pairwise disjoint supports. The question is whether this extreme boundary is not only closed with respect to full pointwise convergence, but also with respect to weak convergence in \( M^1_{s,c}(\mathcal{Y}) \). So let \( \mu_1, \mu_2, \ldots \) have the above property and \( \mu_n \rightarrow \mu \). Then, as \( \text{pars pro toto} \), \( \mu_n(\mathcal{Q}_1 \cap \mathcal{Q}_2) = \mu_n(\mathcal{Q}_1) \cdot \mu_n(\mathcal{Q}_2) \), and in case that \( \mu(\mathcal{Q}_1 \cap \mathcal{Q}_2) = \mu(\mathcal{Q}_1) = \mu(\mathcal{Q}_2) = 0 \) we get what we want. By the same countability and approximation argument as in \( M^1_{s}(\mathbb{R}^d) \) this is enough to get the desired closedness property.

But for the uniqueness of mixtures of extreme points, there is still a problem with the two different topologies involved, which give rise to different sets of mixing Radon measures.

**Acknowledgements**

*I am most grateful to my PhD supervisor Prof Dr Paul Ressel (Eichstätt) for having introduced me in a stimulating way into the powerful set of methods which has led to [HR1999], [HR2000], [H2003], the present work and a DFG application on invariant probability measures on combinatorial structures. I am also grateful to my present patron Prof Dr Joachim Gwinner (UniBw München) for the work environment and the freedom which allowed me to write [H2003], the present paper and the mentioned DFG application.*

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