Covariant representations for singular actions on $C^*$-algebras

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Abstract

Singular actions on $C^*$-algebras are automorphic group actions on $C^*$-algebras, where the group need not be locally compact, or the action need not be strongly continuous. We study the covariant representation theory of such actions. In the usual case of strongly continuous actions of locally compact groups on $C^*$-algebras, this is done via crossed products, but this approach is not available for singular $C^*$-actions (this was our path in a previous paper). The literature regarding covariant representations for singular actions is already large and scattered, and in need of some consolidation. We collect in this survey a range of results in this field, mostly known. We improve some proofs and elucidate some interconnections. These include existence theorems by Borchers and Halpern, Arveson spectra, the Borchers–Arveson theorem, standard representations and Stinespring dilations as well as ground states, KMS states and ergodic states and the spatial structure of their GNS representations.

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1 Introduction

Covariant representations of $C^*$- and $W^*$-dynamical systems $(A, G, \alpha)$ are fundamental objects in both $C^*$-algebra theory, as well as in mathematical quantum physics. Our interest here is in covariant representations for singular $C^*$-actions, i.e. automorphic group actions on $C^*$-algebras, where the group need not be locally compact, or the action need not be strongly continuous. Such actions are abundant in physics and arise naturally in mathematics. Many of the usual mathematical tools break down for these actions, e.g. crossed products, but a great deal of analysis has been done for their covariant representations. Though much of the theory is collected in monographs such as [BR02], [BR96], [Sa91], [Pe89] and [Ta03], unfortunately many important results are still widely scattered in the literature. We feel it necessary to collect here some of these scattered results, improve proofs where we can, and add some new examples and results which seem interesting. Our intention is to augment the material in the monographs, not to replace any of these sources. Whilst the usefulness of this is primarily for ourselves, we hope that this review will also be of use to practitioners in the area.

In a previous work, we studied crossed product constructions for singular actions cf. [GrN14], but in this review we will not include that. We will mainly concentrate on $W^*$-dynamical systems and singular actions, and for these, will focus on structural issues for covariant representations, leaving applications aside. Some of these issues include existence, spectrum conditions (cf. Borchers–Arveson Theorem), innerness, standard representation structures and Stinespring dilations. The most important types of states associated with a singular action are ground states, KMS states, and ergodic states, and we will briefly review these, as well as the properties of their GNS representations.

In more detail, what we will cover are the following. We start with the natural topologies of the automorphism groups of $C^*$-algebras and von Neumann algebras, and discuss the Borchers–Halpern Theorem characterizing existence of covariant representations in terms of their folia of normal states. We refine these conditions and consider covariance for cyclic representations where the generating vector is not necessarily $G$-invariant. We also consider conditions for covariant representations to
be inner. The universal covariant representation is a useful tool for analyzing a singular action as a $W^*$-dynamical system.

Next, we consider the standard form representations of a $W^*$-dynamical system, which is a special and heavily used covariant representation (Section 3.1). For a projection $P$ in a von Neumann algebra $\mathcal{M}$, we consider the reduced von Neumann algebra $P\mathcal{M}P$, and composing the reduction map with the standard representation of the image, we obtain a completely positive map $\varphi_P$ for which we can construct a Stinespring dilation representation $\pi_{\varphi_P}$ of $\mathcal{M}$. In particular, given a $W^*$-dynamical system and an invariant projection $P$, then $\pi_{\varphi_P}$ is covariant, and this generalizes the analogous theorem for the GNS representation of an invariant state for a $C^*$-dynamical system (Subsection 3.4).

We then consider covariant representations satisfying a spectral condition, study issues around the Borchers–Arveson Theorem (Section 4) and characterize the ground states whose GNS representations give rise to such covariant representations (Section 5). This is motivated by the fact that such states are of central importance in physics, in fact the existence of such an invariant state is an axiom for algebraic quantum field theory (cf. [Ar99, Axiom 4, p.104], [HK64]). We also consider the structure of these representations and clarify the role of ground states. We study the case where zero is isolated in the Arveson spectrum in detail.

We continue in Section 6 by recalling the basic structural facts of the GNS representation of a KMS state, since thermal quantum physics is based on such a setting. This is followed by a very short section on ergodic states.

Notation

For a $C^*$-algebra $\mathcal{A}$, we write $\mathcal{A}^*$ for the space of continuous linear functionals on $\mathcal{A}$ and $\mathcal{S}(\mathcal{A}) \subseteq \mathcal{A}^*$ for the set of states. For a $W^*$-algebra $\mathcal{M}$, we write $\mathcal{M}_* \subseteq \mathcal{M}^*$ for the predual of $\mathcal{M}$, i.e. the subspace of normal functionals and $\mathcal{S}_n(\mathcal{M}) \subseteq \mathcal{M}_*$ for the set of normal states. If $X$ and $Y$ are Banach spaces, we write $B(X,Y)$ for the space of bounded operators from $X$ to $Y$. For a topological group $G$, we write $G_d$ for the underlying discrete group.

2 Covariant representations

2.1 $C^*$-and $W^*$-dynamical systems

For a $C^*$-algebra $\mathcal{A}$ there are two natural topologies for its automorphism group Aut$(\mathcal{A})$ with respect to which it is a topological group. The norm topology of $B(\mathcal{A}) \supseteq$ Aut$(\mathcal{A})$, and the strong topology. In the strong topology, the open neighborhoods of a $\rho_0 \in$ Aut$(\mathcal{A})$ are

$$N_{\varepsilon}(\rho_0; A_1, \ldots, A_n) := \{ \rho \in$ Aut$(\mathcal{A}) \mid \| \rho(A_i) - \rho_0(A_i) \| < \varepsilon, \ i = 1, \ldots, n \}$$

for $\varepsilon > 0$, $A_i \in \mathcal{A}$ and $n \in \mathbb{N}$. Therefore if we want a topological group $G$ to act on $\mathcal{A}$, it is natural to look for homomorphisms $\alpha : G \rightarrow$ Aut$(\mathcal{A})$, $g \mapsto \alpha_g$, which are continuous with respect to one of these two topologies. The norm topology is too restrictive for most applications, hence one normally requires continuity with respect to the strong topology. We fix some terminology:

Definition 2.1. A $C^*$-action is a triple $(\mathcal{A}, G, \alpha)$, where $\mathcal{A}$ is a $C^*$-algebra, $G$ is a topological group and $\alpha : G \rightarrow$ Aut$(\mathcal{A})$ is a homomorphism (which is not assumed to have any continuity property). If $\alpha$ is strongly continuous, i.e. for every $A \in \mathcal{A}$, the orbit map $\alpha^A : G \rightarrow \mathcal{A}$, $g \mapsto \alpha_g(A)$ is continuous, we call $(\mathcal{A}, G, \alpha)$ a $C^*$-dynamical system (cf. [Pe89], [BR02 Def. 2.7.1]). The usual case will mean
that the action is strongly continuous and the group \(G\) is locally compact. A singular action is one which is not the usual case.

A \(C^*\)-action \((\mathcal{A}, G, \alpha)\) has a dual action \(\alpha^*: G \to \mathcal{B}(\mathcal{A}^*)\) by isometries on the topological dual \(\mathcal{A}^*\) given by

\[
(\alpha^*_g \omega)(A) := \omega(\alpha^{-1}_g(A)) \quad \text{for} \quad g \in G, A \in \mathcal{A}, \omega \in \mathcal{A}^*.
\]

(1)

The space of norm continuous elements of \(\alpha^*\) is denoted by

\[
(\mathcal{A}^*)_c := \{\omega \in \mathcal{A}^* | \lim_{g \to e} \|\alpha^*_g \omega - \omega\| = 0\}.
\]

(2)

Since \(G\) acts on \(\mathcal{A}^*\) by isometries, this subspace is norm closed and maximal with respect to the property that the \(G\)-action on \((\mathcal{A}^*)_c\) is continuous with respect to the norm topology on \((\mathcal{A}^*)_c\) (see [Bo96 Thm. II.2.2] for further properties). We write

\[
\mathcal{S}(\mathcal{A})_c := \mathcal{S}(\mathcal{A}) \cap (\mathcal{A}^*)_c
\]

for the set of states with continuous orbit maps. If \(\alpha: G \to \text{Aut}(\mathcal{A})\) is continuous with respect to the operator norm on \(\mathcal{B}(\mathcal{A})\), then \((\mathcal{A}^*)_c = \mathcal{A}^*\).

For the usual case, \(C^*\)-actions have been extensively analyzed, and there are many tools available, such as crossed products. However, this is frequently too restrictive, e.g. if we have a strongly continuous one-parameter automorphism group \(\alpha: \mathbb{R} \to \text{Aut}(\mathcal{A})\) where \(\mathcal{A}\) is a \(W^*\)-algebra, then the action must be inner (cf. [Ta03 Exercise XI.3.6]). In physics and some natural examples in mathematics, we have singular actions, and then the available theory is more limited. To analyze a singular action, one is often forced to choose some representation \(\pi\) with respect to which the \(\alpha_g\) are normal maps (i.e. each \(\alpha^*_g\) preserves the set of normal states of the von Neumann algebra \(\pi(\mathcal{A})''\)), and the orbit maps \(g \mapsto \pi(\alpha_g(A))\) are strong operator continuous and then analyze the action on the von Neumann algebra \(\pi(\mathcal{A})''\). The cost of this strategy is that the analysis is subject to the chosen representation \(\pi\). Not every automorphism of \(\pi(\mathcal{A})\) will extend to \(\pi(\mathcal{A})''\), only those which are normal maps with respect to \(\pi\). On the other hand, every automorphism of \(\pi(\mathcal{A})''\) is automatically normal, but not all will preserve \(\pi(\mathcal{A})\). We fix terminology for this context.

Let \(\mathcal{M}\) be a \(W^*\)-algebra, then every automorphism \(\rho\) of the \(W^*\)-algebra \(\mathcal{M}\) is already a normal map, i.e. a \(W^*\)-automorphism (cf. [Pe89 Thm. 2.5.2] or [Sa71 Cor. 4.1.23]), hence there is no need to restrict \(\text{Aut}(\mathcal{M})\). As any \(\rho \in \text{Aut}(\mathcal{M})\) is a normal map, the isometry \(\rho^*: \mathcal{M}^* \to \mathcal{M}^*\) (given by \(\rho^*(\omega) = \omega \circ \rho\)) preserves the predual \(\mathcal{M}_*\), hence by \(\mathcal{M} = (\mathcal{M}_*)^*\) the map \(\rho \to \rho^* | \mathcal{M}_*\) embeds \(\text{Aut}(\mathcal{M})\) as a group of isometries of the Banach space \(\mathcal{M}_*\).

The natural topology one would like to give \(\text{Aut}(\mathcal{M})\), is the coarsest topology which makes the orbit maps \(\text{Aut}(\mathcal{M}) \to \mathcal{M}_*, \rho \to \rho(A)\) continuous with respect to any of the strong operator, weak operator, ultraweak or ultrastrong topologies. Unfortunately \(\text{Aut}(\mathcal{M})\) is not a topological group with respect to such a topology, which leads us to the following. As \(\text{Aut}(\mathcal{M})\) is identified with a group of isometries of \(\mathcal{M}_*\), there are two natural group topologies on it (cf. [Haa75]):

**Definition 2.2.** Let \(\mathcal{M}\) be a \(W^*\)-algebra. Then the \(u\)-topology of \(\text{Aut}(\mathcal{M})\) is defined to be the coarsest topology which makes the orbit maps \(\text{Aut}(\mathcal{M}) \to \mathcal{M}_*, \rho \to \rho^*(\omega) \in \mathcal{M}_*\) norm continuous for each \(\omega \in \mathcal{M}_*\). This topology is also called the \(\sigma\)-weak topology (cf. [Sa91 p. 12]), and \(\text{Aut}(\mathcal{M})\) is a topological group with respect to this topology.

The \(p\)-topology of \(\text{Aut}(\mathcal{M})\) is the coarsest topology for which all maps \(\text{Aut}(\mathcal{M}) \to \mathbb{C}, \rho \mapsto \omega(\rho(M))\) for \(\omega \in \mathcal{M}_*\) and \(M \in \mathcal{M}\) are continuous, and this also makes \(\text{Aut}(\mathcal{M})\) into a topological group.

Clearly, the \(u\)-topology is finer than the \(p\)-topology, and we will derive the corresponding inequality in Example 2.6 below. However, the two topologies coincide for factors of type I and II \(_1\) ([Haa75 Cor. 3.8]). We define:
Definition 2.3. Let $G$ be a topological group and $\mathcal{M}$ be a $W^*$-algebra, and assume we have a homomorphism $\alpha : G \to \text{Aut}(\mathcal{M})$. We call $(\mathcal{M}, G, \alpha)$ a $W^*$-dynamical system if $\alpha$ is continuous with respect to the $u$-topology, i.e. $\mathcal{M}_u \subseteq (\mathcal{M}^*)_c$, i.e. the action of $G$ on the Banach space $\mathcal{M}_u$ is continuous.

For locally compact groups, this coincides with the naive notion by the following ([Ha72 Cor. 2.4], [Arv71], [Str81 §13.5], [Bla06 Thm. III.3.2.2]):

Theorem 2.4. Let $G$ be a locally compact group, $\mathcal{M}$ be a von Neumann algebra, and $\alpha : G \to \text{Aut}(\mathcal{M})$ a homomorphism. Then the following are equivalent:

(i) For each $M \in \mathcal{M}$, the map $\alpha^M : G \to \mathcal{M}$, $g \mapsto \alpha_g(M)$ is continuous with respect to the strong (or weak) operator topology.

(ii) For each $\omega \in \mathcal{M}_u$, the orbit map $\omega^\alpha : G \to \mathcal{M}_u$, $g \mapsto \alpha_g(\omega)$ is norm continuous.

(iii) For each $\omega \in \mathcal{M}_u$ and $M \in \mathcal{M}$, the map $\alpha^{\omega,M} : G \to \mathbb{C}$, $g \mapsto \omega(\alpha_g(M))$ is continuous.

Remark 2.5. That (ii) and (iii) need not be equivalent for a general topological group follows from the fact that the $u$-topology is strictly finer than the $p$-topology for some von Neumann algebras (cf. Example 2.3). For general topological groups it follows from properties of the standard representation that this extension of the definition of a $W^*$-dynamical system is the most useful one (cf. equation (11) below).

Example 2.6. (see [Ha72 Cor. 3.15] for a similar discussion of Aut($L^\infty([0,1])$)).

We consider $\mathcal{M} = L^\infty([0,1])$, $\mathcal{H} = L^2([0,1])$ and note that $\mathcal{M}_u \cong L^1([0,1])$. Let $G := \text{Homeo}(0,1)_b \subseteq \text{Homeo}(0,1)$ be the subgroup consisting of all homeomorphisms mapping Lebesgue zero sets to Lebesgue zero sets, i.e. $g$ and $g^{-1}$ are absolutely continuous. We topologize $G$ as a subgroup of $\text{Homeo}(0,1)$ which carries the group topology defined by the embedding

$$\text{Homeo}(0,1) \to C([0,1])^2, \quad g \mapsto (g,g^{-1})$$

([Stp06 Cor. 9.15]). Then $G$ acts by automorphisms on the von Neumann algebra $\mathcal{M}$ by $\alpha_g(f) := g_* f := f \circ g^{-1}$. We show that this action is continuous with respect to the $p$-topology but not with respect to the $u$-topology (Remark 2.5(b)). This implies in particular that on the group $\text{Aut}(L^\infty([0,1]))$, these two topologies do not coincide.

Continuity in $p$-topology: We consider the continuous bilinear map

$$\beta : L^\infty([0,1]) \times L^1([0,1]) \to \ell^\infty(G), \quad \beta(f,h)(g) := \int_0^1 (g_* f)(x)h(x) \, dx.$$ 

We have to show that all functions $\beta(f,h)$ are continuous on $G$. In view of $\beta(f,h)(g_1 g_2) = \beta((g_2)_* f,h)(g_1)$, it suffices to verify continuity in $e = \text{id}_{[0,1]} \in G$.

Since $\beta$ is continuous and bilinear and the subspace $C(G) \cap \ell^\infty(G)$ is closed in $\ell^\infty(G)$, it suffices to do that for the case where $h$ is bounded and $f = \chi_{[a,b]}$ is a characteristic function of an interval $[a,b] \subseteq [0,1]$. For $\|g-e\|_\infty < \varepsilon$, we observe that

$$E := g^{-1}([a,b]) \Delta [a,b] \subseteq [a - \varepsilon, a + \varepsilon] \cup [b - \varepsilon, b + \varepsilon],$$

which leads to

$$|\beta(\chi_{[a,b]}, h)(g) - \beta(\chi_{[a,b]}, h)(e)| = \left| \int_0^1 (g_* \chi_{[a,b]} - \chi_{[a,b]})(x)h(x) \, dx \right| \leq \int_0^1 \chi E(x) |h(x)| \, dx \leq 4\varepsilon \|h\|_\infty.$$

This proves that the function $\beta(\chi_{[a,b]}, h)$ is continuous at $e$, and hence that the homomorphism $\alpha : G \to \text{Aut}(\mathcal{M})$ is continuous with respect to the $p$-topology.
Discontinuity in the \( u \)-topology: Since the \( u \)-topology on \( \text{Aut}(\mathcal{M}) \) corresponds to the strong operator topology for the action on \( L^2([0,1]) \) (see Example 3.5(a) and Remark 3.7), we have to show that the representation \( U: G \to U(L^2([0,1])) \) defined by \( U_g f := \sqrt{|g'|} \cdot (f \circ g) \) is not continuous. This will be achieved by showing that the orbit map \( G \to L^2([0,1]), g \mapsto \sqrt{|g'|} \) for the constant function 1 is not continuous at \( e \).

For every \( n \in \mathbb{N} \), we consider the piecewise linear continuous function \( h_n: [0,1] \to \mathbb{R} \), determined by its values at the joining points to be:

\[
h_n(x) := \begin{cases} 
0 & \text{for } x = \frac{k}{2^n}, k = 0, \ldots, 2^n, \\
(1 - \frac{1}{2^n}) \frac{1}{2^{n+1}} & \text{for } x = \frac{2k+1}{2^{n+1}}, k = 0, \ldots, 2^n - 1.
\end{cases}
\]

Then

\[
g_n: [0,1] \to [0,1], \quad g_n(x) := x + h_n(x)
\]
defines a sequence in \( G \). Note that these homeomorphisms are piecewise linear with

\[
g_n'(x) := \begin{cases} 
2 - \frac{1}{2^n} & \text{for } \frac{k}{2^n} < x < \frac{2k+1}{2^{n+1}}, \\
\frac{1}{2^n} & \text{for } \frac{2k+1}{2^{n+1}} < x < \frac{k+1}{2^n}.
\end{cases}
\]

As \( g_n(x) = x \) for \( x = \frac{k}{2^n}, k = 0, \ldots, 2^n \), and \( g_n \) is strictly increasing, we have

\[
\|g_n - \text{id}\|_\infty \leq \frac{1}{2^n} \quad \text{and} \quad \|g_n^{-1} - \text{id}\|_\infty \leq \frac{1}{2^n}.
\]

This implies that \( \lim_{n \to \infty} g_n = e \) in \( G \). Next we observe that

\[
\| \sqrt{g_n} - \sqrt{g_{n+1}} \|_2^2 = 2 \left(1 - \int_0^1 \sqrt{g_n'} \sqrt{g_{n+1}'} \right).
\]

From

\[
\int_0^1 \sqrt{g_n'} \sqrt{g_{n+1}} = \frac{1}{4} \sqrt{2 - \frac{1}{2^n}} \sqrt{2 - \frac{1}{2^{n+1}}} + \frac{1}{4} \sqrt{2 - \frac{1}{2^n}} \sqrt{1 - \frac{1}{2^{n+1}}} + \frac{1}{4} \sqrt{\frac{1}{2^n}} \sqrt{2 - \frac{1}{2^{n+1}}} + \frac{1}{4} \sqrt{\frac{1}{2^n}} \sqrt{\frac{1}{2^{n+1}}} \to \frac{1}{2}
\]

it follows that \( \| \sqrt{g_n} - \sqrt{g_{n+1}} \|_2 \to 1 \). This shows that the sequence \( U_{g_n^{-1}} 1 = \sqrt{g_n} \) does not converge to 1 in \( L^2([0,1]) \).

Remark 2.7. Given any action \( (\mathcal{A}, G, \alpha) \), we can always define the strongly continuous part of it by

\[
\mathcal{A}_c := \{ A \in \mathcal{A} \mid \alpha^A: G \to \mathcal{A}, g \mapsto \alpha_g(A) \text{ is norm continuous} \} \quad \text{and} \quad \alpha^c_g := \alpha_g | \mathcal{A}_c.
\]

Unfortunately, as we will see in Example 2.3.2 below, it is possible that \( \mathcal{A}_c = \mathbb{C}1 \).

If we start from a \( W^* \)-dynamical system \( (\mathcal{M}, G, \beta) \) with \( G \) locally compact, then \( \mathcal{M}_c \) is weakly dense in \( \mathcal{M} \), and

\[
\mathcal{M}_c = C^* \{ \beta_f(A) \mid f \in L^1(G), A \in \mathcal{M} \},
\]

where the integrals \( \beta_f(A) := \int_G f(g) \beta_g(A) \, dg \) exist in the weak topology ([Pe89] Lemma 7.5.1)). In the case that \( \mathcal{M} = \mathcal{A}'' \) for some concrete \( C^* \)-algebra \( \mathcal{A} \) invariant with respect to \( G \), it is unfortunately possible that \( \mathcal{A} \cap \mathcal{M}_c = \mathbb{C}1 \). Moreover, in general only the representations of \( \mathcal{M}_c \) which are the restrictions of normal representations of \( \mathcal{M} \) will extend from \( \mathcal{M}_c \) to \( \mathcal{M} \) to produce representations on \( \mathcal{A} \). Thus the \( C^* \)-dynamical system \( (\mathcal{M}_c, G, \beta) \) is not a good vehicle to study the general covariant representations of \( (\mathcal{A}, G, \beta) \).
2.2 Covariant representations.

**Definition 2.8.** (a) A covariant representation for a $C^*$-action $(\mathcal{A}, G, \alpha)$ is a pair $(\pi, U)$, where $\pi: \mathcal{A} \to \mathcal{B}(\mathcal{H})$ is a non-degenerate representation of $\mathcal{A}$ on the Hilbert space $\mathcal{H}$ and $U: G \to \mathcal{U}(\mathcal{H})$ is a continuous unitary representation satisfying

$$U(g)\pi(a)U(g)^* = \pi(\alpha_g(a)) \quad \text{for} \quad g \in G, a \in \mathcal{A}. \quad (3)$$

For a fixed Hilbert space $\mathcal{H}$, we write $\text{Rep}(\alpha, \mathcal{H})$ for the set of covariant representations $(\pi, U)$ of $(\mathcal{A}, G, \alpha)$ on $\mathcal{H}$.

(b) A non-degenerate representation $(\pi, \mathcal{H})$ of $\mathcal{A}$ is called covariant if there exists a continuous representation $U$ of $G$ such that $(\pi, U)$ is a covariant representation of $(\mathcal{A}, G, \alpha)$. It is called quasi-covariant if $(\pi, \mathcal{H})$ is quasi-equivalent to a covariant representation (cf. [Bo69]). See Remark 2.14(c) below for more on quasi-equivalent representations.

(c) We write $\mathcal{S}_\alpha(\mathcal{A})$ for the set of those states of $\mathcal{A}$ arising as vector states in covariant representations of $(\mathcal{A}, G, \alpha)$. By the Lemma 2.9 below, we have in fact $\mathcal{S}_\alpha(\mathcal{A}) \subseteq (\mathcal{A}^*)_c$. A state $\omega \in \mathcal{S}_\alpha(\mathcal{A})$ is called covariant (resp. quasi-invariant) if the corresponding cyclic representation $\pi_\omega$ obtained by the GNS construction is covariant (resp. quasi-covariant) (cf. [GK70] Def. 6; see Theorem 2.20 for more on quasi-invariant states). Below we will characterize the covariant states.

**Lemma 2.9.** If $(\pi, U)$ is a covariant representation for $(\mathcal{A}, G, \alpha)$ and $S \in B_1(\mathcal{H})$ a trace class operator, then the continuous linear functional $\omega_S(\cdot) := \text{tr}(\pi(\cdot)S)$ on $\mathcal{A}$ is contained in $(\mathcal{A}^*)_c$.

**Proof.** (GK70 Prop. 3)) For $S \in B_1(\mathcal{H})$, we have

$$(\omega_S \circ \alpha_g^{-1} - \omega_S)(A) = \text{tr}(U_g^*\pi(A)U_gS - \pi(A)S) = \text{tr}(\pi(A)(U_gSU_g^* - S)),$$

and since the conjugation action of $G$ on $B_1(\mathcal{H})$ is continuous, $\omega_S \in (\mathcal{A}^*)_c$. \hfill $\square$

**Remark 2.10.** (1) For a covariant representation $(\pi, U) \in \text{Rep}(\alpha, \mathcal{H})$ of the $C^*$-action $(\mathcal{A}, G, \alpha)$, the map $U: G \to \mathcal{U}(\mathcal{H})$ is strong operator continuous. Therefore $\beta_g(A) := U_gAU_g^*$ defines a homomorphism $\beta: G \to \text{Aut}(\mathcal{B}(\mathcal{H}))$ and Lemma 2.9 shows that $(\mathcal{B}(\mathcal{H}), G, \beta)$ is a $W^*$-dynamical system. As the von Neumann algebra $\pi(\mathcal{A})'' \subseteq \mathcal{B}(\mathcal{H})$ is $\beta_G$-invariant, we also obtain a $W^*$-dynamical system $(\pi(\mathcal{A})'', G, \beta_{\pi(\mathcal{A}'')})$ (cf. [Pe89] 7.4.2)]. Conversely, given a $W^*$-dynamical system $(\mathcal{M}, G, \beta)$, it always has a faithful normal representation which is covariant (cf. equation (11) below).

(2) In the usual case, for vector states, Lemma 2.9 is [Bo69 Lemma II.2].

(3) Note that if $(\pi, \mathcal{H})$ is covariant, then $\ker(\pi)$ is preserved by $\alpha$, hence one can easily find non-covariant representations if $\mathcal{A}$ has ideals which are not preserved by $\alpha$.

Let $(\mathcal{A}, G, \alpha)$ be a $C^*$-action. In the usual case, where $G$ is locally compact and $\alpha$ is strongly continuous, the covariant representations are in bijective correspondence with the non-degenerate representations of the crossed product $C^*$-algebra $\mathcal{A} \rtimes_\alpha G$. For a singular action, it is not obvious in general that covariant representations exist. There always exist covariant representations of $(\mathcal{A}, G, \alpha)$, which is an instance of the usual case, and if covariant representations of $(\mathcal{A}, G, \alpha)$ exist, they will be amongst these. Here is an example of a singular action with no covariant representations.

**Example 2.11.** (A $C^*$-action $(\mathcal{A}, G, \alpha)$ with no non-zero covariant representations)

A topological group $G$ is called exotic if all its continuous unitary representations are trivial. In [Ba91] Ch. 2] one finds various constructions of such a group of the type $G = E/\Gamma$, where $E$ is a Banach space and $\Gamma \subseteq E$ is a discrete subgroup.

Let $G$ be an exotic topological group. Take the left regular representation on $\ell^2(G)$, i.e. $(V_g\psi)(h) := \psi(g^{-1}h)$ for $\psi \in \ell^2(G)$, $g, h \in G$. Then $V$ is a non-trivial unitary representation,
Definition 2.12. (a) For a representation \( (\pi, U) \) of its set of normal states, i.e., the corresponding folium \( F \) of a normal representation, then \( U \) must be trivial as \( G \) is exotic, hence \( \pi \circ \alpha_g = \pi \) for all \( g \in G \). However \( \mathcal{A} \) is simple and \( \pi \) is non-trivial hence \( \pi \) is injective, and then \( \alpha_g(A) = A \) for all \( g \in G \) and \( A \in \mathcal{A} \), which is a contradiction. Thus there are no non-trivial covariant representations, i.e. \( \mathcal{G}_\alpha(\mathcal{A}) = \emptyset \).

In the subsections below, we will consider the problem of the existence of covariant representations in some detail. In Corollary 2.23 we will obtain conditions characterizing the existence of a covariant representation. Regarding explicit constructions, it is well-known that one can obtain a covariant representation for singular actions either from

- standard form representations of \( W^* \)-dynamical systems,
- from the representations of \( W^* \)-crossed products of \( W^* \)-dynamical systems, or
- from invariant states with appropriate continuity conditions (cf. [DJP03]).

These will be considered below in Section 3.1 and Proposition 2.26 respectively. Below in Theorem 3.34 we will obtain a covariant representation via the Stinespring Dilation Theorem.

There are also natural uniqueness and structure questions, e.g. given a covariant representation \( (\pi, U) \) for a \( C^* \)-action \( (\mathcal{A}, G, \alpha) \), find and analyze all unitary representations \( U : G \to U(\mathcal{H}) \) for which \( (\pi, U) \in \text{Rep}(\alpha, \mathcal{H}) \) is a covariant representation. Below we will see that if a spectral condition is added, then we can find a natural “minimal” such \( U : G \to U(\mathcal{H}) \) which is unique.

### 2.3 Folia and the Borchers–Halpern Theorem

In this subsection we will characterize when a representation \( \pi \) is covariant in terms of properties of its set of normal states, i.e., the corresponding folium \( F(\pi) \).

**Definition 2.12.** (a) For a \( C^* \)-algebra \( \mathcal{A} \), we call a subset \( F \subseteq \mathcal{A}^* \) a folium if there exists a representation \( (\pi, \mathcal{H}) \) of \( \mathcal{A} \) with

\[
F = F(\pi) := \{ \omega_S \in \mathcal{G}(\mathcal{A}) \mid 0 \leq S \in \mathcal{B}(\mathcal{H}), \, \text{tr} S = 1 \}
\]

where \( \omega_S(A) := \text{tr}(\pi(A)S) \) as in Lemma 2.9.

(b) We likewise define the folium \( F(\pi) \subseteq \mathcal{M}_s \) of a normal representation \( (\pi, \mathcal{H}) \) of a \( W^* \)-algebra \( \mathcal{M} \).

As the normal states of \( B(\mathcal{H}) \) are identified with trace class operators by \( \omega_S(A) = \text{tr}(SA) \), we have

\[
F(\pi) = \{ \omega \circ \pi \mid \omega \text{ is a normal state of } \pi(\mathcal{A})'' \} \cong \mathcal{G}_n(\pi(\mathcal{A})'')
\]

(5)

because all normal states of \( \pi(\mathcal{A})'' \) extend to normal states of \( B(\mathcal{H}) \) (cf. [Bla06, Cor. III.2.1.10]). Clearly \( F(\pi) \) inherits from \( \pi(\mathcal{A})'' \) the convexity and invariance under conjugations. We verify that it also inherits norm closedness.

**Lemma 2.13.** Let \( (\pi, \mathcal{H}) \) be a representation of \( \mathcal{A} \).

(a) The folium \( F(\pi) \subset \mathcal{A}^* \) and its linear span are both norm closed.

(b) Moreover, \( F(\pi) \) coincides with the set of vector states of the representation \( (\rho, B_2(\mathcal{H})) \) of \( \mathcal{A} \) given by \( \rho(A)B := \pi(A)B, \ A \in \mathcal{A}, \ B \in B_2(\mathcal{H}). \)
Proof. (a) The restriction map $(\pi(A)''_* \to \pi(A)^*)$ is isometric (\cite[Prop. 2.12]{BN12}), and the subset $\mathcal{S}_n(\pi(A)'') \subseteq (\pi(A)'')_*$ of normal states is norm closed. This implies that

$$\text{span } F(\pi) = \{\omega_S | S \in B_1(\mathcal{H})\}$$

is norm-closed in $\mathcal{A}^*$, and this shows the norm closedness of $F(\pi)$.

(b) The vector states of $\rho$ are of the form

$$\omega_B(A) = \langle B, \pi(A)B \rangle = \text{tr}(B^* \pi(A)B) = \text{tr}(BB^* \pi(A)),$$

where $BB^*$ is a positive trace class operators with $\text{tr}(B^* B) = \|B\|_2^2 = 1$. Hence these are precisely the states of the form $\omega_S$, $0 \leq S \in B_1(\mathcal{H})$ with $\|S\| = 1$. Therefore $F(\pi)$ coincides with the set of vector states of $\rho$.

Remark 2.14. (a) In \cite{Ka62} it is shown that the set of vector states $V(\pi) \subseteq F(\pi) \subseteq \mathcal{A}^*$ of a representation $(\pi, \mathcal{H})$ of a C*-algebra $\mathcal{A}$ is a norm closed subset. This implies the closedness of $F(\pi)$ since $F(\pi) = V(\rho)$ by Lemma 2.13(b). However, the closedness of the larger set $F(\pi)$ is much easier to get.

(b) A folium $F \subseteq \mathcal{S}(\mathcal{A})$ can be abstractly characterized as a convex set of states which is norm closed, and contains with $\omega$ all states of the form

$$(B * \omega)(A) := \frac{\omega(B^* AB)}{\omega(B^* B)} , \quad \omega(B^* B) > 0$$

(\cite[p. 84]{HKK70}). This is a better intrinsic definition of a folium as it does not rely on the existence of a representation $\pi$.

(c) For a state $\omega \in \mathcal{S}(\mathcal{A})$, the folium $F(\pi, \omega)$ is the norm-closed convex subset generated by the set $\{B * \omega | \omega(B^* B) > 0\}$ (cf. \cite{E}). By polarization, $F(\pi, \omega)$ generates the same norm closed subspace of $\mathcal{A}^*$ as $A\omega A$, where we define

$$(A\omega)(B) := \omega(AB) \quad \text{and} \quad (\omega A)(B) := \omega(AB) \quad \text{for} \quad A, B \in \mathcal{A}.$$  

As $\text{span } F(\pi, \omega)$ is norm closed by Lemma 2.13, we see that

$$\text{span } F(\pi, \omega) = [A\omega A].$$

where $[\cdot]$ denotes the closed span of its argument.

(d) For two representations $\pi_1$ and $\pi_2$ of $\mathcal{A}$, their folia are equal $F(\pi_1) = F(\pi_2)$ if and only if they are quasi-equivalent, i.e. there is an isomorphism $\beta : \pi_1(\mathcal{A})'' \to \pi_2(\mathcal{A})''$ of $W^*$-algebras such that $\beta(\pi_1(A)) = \pi_2(A)$ for all $A \in \mathcal{A}$ (cf. \cite[Prop. 10.3.13]{KR86}). This means that each subrepresentation of $\pi_1$ has a subrepresentation which is unitarily equivalent to a subrepresentation of $\pi_2$, and vice versa (\cite[Cor. 10.3.4(ii)]{KR86}). This statement is also contained in \cite[Cor. 5.11]{AS01} as the corresponding “split faces” are the corresponding folia in our terminology.

(e) A subset $E \subseteq \mathcal{S}(\mathcal{A})$ is contained in the folium $F(\pi)$ of a representation $(\pi, \mathcal{H})$ if and only if the cyclic representations $(\pi, \mathcal{H}_\omega)$, $\omega \in E$, are contained in the corresponding left multiplication representation $(\rho, B_2(\mathcal{H}))$ with $\rho(A)(B) = \pi(A)B$, which satisfies $F(\rho) = F(\pi)$. Therefore every subset $E \subseteq \mathcal{S}(\mathcal{A})$ is contained in a minimal folium Fol(E) which can be obtained as $F(\bigoplus_{\omega \in E} \pi, \omega)$. This further implies that

$$\text{Fol}(E) = \left\{ \sum_{n=1}^{\infty} c_n \nu_n \bigg| 0 \leq c_n \leq 1, \sum_{n=1}^{\infty} c_n = 1, \nu_n \in F(\pi, \omega_n), \omega_n \in E \right\}.$$
Example 2.15. For $\mathcal{A} = C_0(X)$, $X$ locally compact, and a state $\omega \in \mathcal{S}(\mathcal{A})$ obtained from a probability measure by $\omega(A) = \int A\,d\mu$, the corresponding folium can be determined rather easily from \cite{[K]}{[H72]}. For $f \in \mathcal{A}$ with $\int_X |f|^2\,d\mu = 1$, we have $f \ast \omega = |f|^2\omega$. Since the embedding $L^1(X,\mu) \hookrightarrow \mathcal{A}^*$, $h \mapsto h \cdot \omega$ is isometric, it follows that

$$\text{Fol}(\omega) = \left\{ F\omega \mid F \in L^1(X,\mu), 0 \leq F, \int_X F\,d\mu = 1 \right\}$$

corresponds to the set of probability measures which are absolutely continuous with respect to $\mu$.

Theorem 2.16. (Borchers–Halpern Theorem) Let $(\mathcal{A},G,\alpha)$ be a $C^*$-action and $F \subseteq \mathcal{S}(\mathcal{A})$ be a folium. Then there exists a covariant representation $(\pi,U,\mathcal{H})$ of $(\mathcal{A},G,\alpha)$ with $F = F(\pi)$ if and only if $F$ is $\alpha^*_G$-invariant and contained in $(\mathcal{A}^*)_c$.

Proof. (cf. \cite{Hal72} p. 258, \cite{Bo83} Thm. III.2) Below we will also obtain a proof of this from standard forms in Remark 3.9.

Kadison’s old paper \cite{Ka65} already contains an interesting precursor of this theorem.

Corollary 2.17. Let $(\mathcal{A},G,\alpha)$ be a $C^*$-action and $(\pi,\mathcal{H})$ be a non-degenerate representation of $\mathcal{A}$. Then the following are equivalent:

(i) $\pi$ is quasi-covariant.

(ii) $F(\pi)$ is $\alpha^*_G$-invariant and contained in $(\mathcal{A}^*)_c$.

(iii) We have that $\ker\pi$ is $\alpha_G$-invariant, hence the induced action of $G$ on $\pi(\mathcal{A})$ is defined. Moreover, this induced action of $G$ on $\pi(\mathcal{A})$ extends to an action $\beta : G \to \text{Aut}(\pi(\mathcal{A})''_c)$, defining a $W^*$-dynamical system.

Proof. The equivalence between (i) and (ii) follows by applying the Borchers–Halpern Theorem to the folium $F = F(\pi)$.

Next we show the equivalence of (ii) and (iii). Note that if $\pi$ is quasi-covariant, then its kernel must coincide with the kernel of a covariant representation, and this is always invariant with respect to $\alpha_G$. Thus the induced action of $G$ on $\pi(\mathcal{A})$ is defined. As quasi-covariance of $\pi$ implies (iii), we only need to prove the converse. That $(\pi(\mathcal{A})''_c,G,\beta)$ is a $W^*$-dynamical system implies that $(\pi(\mathcal{A})''_c)$ is $\beta^*_G$-invariant, hence $\alpha^*_G$-invariant as a $\varphi \in (\pi(\mathcal{A})''_c)$ uniquely specified by its restriction to $\mathcal{A}$. Thus $F(\pi) = (\pi(\mathcal{A})''_c) \cap \mathcal{S}(\mathcal{A})$ is $\alpha^*_G$-invariant. By definition of a $W^*$-dynamical system, $G$ acts continuously on $(\pi(\mathcal{A})''_c)$, hence $F(\pi) \subseteq (\mathcal{A}^*)_c$. Thus we have obtained equivalence with (ii).

Remark 2.18. (a) The existence of non-zero covariant representations is equivalent to the existence of non-zero quasi-covariant representations of $\mathcal{A}$. Thus Corollary 2.17(iii) is a criterion for the existence of covariant representations.

(b) The question of when a quasi-covariant representation is actually covariant, was analyzed by Bulinskii in \cite{Bu73a} \cite{Bu73b}, but below in Subsection 3 we will see better conditions.

Corollary 2.17 has a specialization which can answer the following question. Given a $C^*$-algebra $\mathcal{A} \subseteq \mathcal{B}(\mathcal{H})$ and an automorphism $\gamma \in \text{Aut}\mathcal{A}$, when does $\gamma$ extend to an automorphism of $\mathcal{A}''$?

Corollary 2.19. Let $\mathcal{A}$ be a $C^*$-algebra, let $(\pi,\mathcal{H})$ be a non-degenerate representation of $\mathcal{A}$ and let $\gamma \in \text{Aut}\mathcal{A}$ be an automorphism such that $\ker\pi$ is $\gamma$-invariant. Then the following are equivalent:

(i) $F(\pi)$ is $\gamma$-invariant.

(ii) The induced automorphism of $\gamma$ on $\pi(\mathcal{A})$ extends to an automorphism on $\pi(\mathcal{A})''$. 

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Moreover, if this is the case, and if \( \pi \) is irreducible, then \( \gamma \) is unitarily implementable on \( \pi(A) \).

**Proof.** Let \( G \subset \text{Aut} A \) be the discrete group generated by \( \gamma \). This defines a \( C^* \)-action \( \langle A,G,\alpha \rangle \) for which \( F(\pi) \subset (A^*)_c \). Moreover, if \( \gamma \) extends to an automorphism on \( \pi(A)^\prime \), it automatically defines a \( W^* \)-dynamical system with respect to \( G \). Thus by Corollary 2.17, it follows that (i) and (ii) are equivalent.

If \( \pi \) is irreducible, then \( \pi(A)^\prime = B(H) \), so as all automorphisms of \( B(H) \) are inner, the last statement follows. \( \square \)

We now consider covariance conditions for states.

**Theorem 2.20.** For \( \langle A,G,\alpha \rangle \) and a state \( \omega \) of \( A \), the following are equivalent:

(i) \( \omega \in \mathcal{S}_\omega(A) \), i.e. \( \omega \) is a vector state of some covariant representation \( \langle \pi,U \rangle \) of \( \langle A,G,\alpha \rangle \).

(ii) \( A\omega A \subseteq (A^*)_c \).

(iii) \( \text{Fol}(\omega) := F(\pi_\omega) \subseteq (A^*)_c \).

(iv) \( \text{Fol}_G(\omega) := \text{Fol}(\alpha_G^*\omega) \subseteq (A^*)_c \).

Furthermore, the following are equivalent for \( \omega \in \mathcal{S}_\omega(A) \):

(a) \( \omega \) is quasi-invariant (cf. Def. 2.8(c)).

(b) \( \pi_\omega \) is quasi-covariant.

(c) \( F(\pi_\omega) = \text{Fol}(\omega) \) is \( \alpha_G^* \)-invariant.

(d) \( \pi_\omega \) is equivalent to a subrepresentation of a covariant representation \( \pi \) with \( F(\pi) = F(\pi_\omega) \).

**Proof.** Observe first, that for a subset \( E \subseteq \mathfrak{S}(A) \), the folium \( \text{Fol}(E) \) generated by \( E \) is equal to the norm closed convex hull of the union of the folia \( \text{Fol}(\nu) = F(\pi_\nu) \), \( \nu \in E \), and the span of each of these is equal to \( \llbracket \mathcal{A}_\nu \mathfrak{A} \rrbracket \) by [6]. Hence \( \mathcal{A}E \mathcal{A} \subseteq (A^*)_c \) is equivalent to \( \text{Fol}(E) \subseteq (A^*)_c \) as \( (A^*)_c \) is a norm-closed subspace of \( A^* \).

(ii) \( \Leftrightarrow \) (iii) follows directly from [6] and the norm closedness of \( (A^*)_c \).

(i) \( \Rightarrow \) (ii): If \( \langle \pi,U \rangle \) is a covariant representation with vector state \( \omega \), then \( \omega \in F(\pi) \). This implies that \( A\omega A \subseteq \text{span} F(\pi) \subseteq (A^*)_c \).

(ii) \( \Rightarrow \) (i): Since \( (A^*)_c \) is \( G \)-invariant, condition (ii) implies that

\[
A\alpha_G^*\omega B = \alpha_G^* (\alpha_g^{-1}(A) \cdot \omega \cdot \alpha_g^{-1}(B)) \in (A^*)_c,
\]

and hence that \( \mathcal{A}(\alpha_G^*\omega) \mathcal{A} \subseteq (A^*)_c \). We thus obtain by the first part of the proof that \( \text{Fol}(\alpha_G^*\omega) \subseteq (A^*)_c \). The \( G \)-invariance of \( \text{Fol}(\alpha_G^*\omega) \) follows from the fact that it is generated by a \( G \)-invariant subset of \( (A^*)_c \). Therefore the Borchers–Halmers Theorem 2.16 implies the existence of a covariant representation \( \langle \pi,U \rangle \) of \( \langle A,G,\alpha \rangle \) with \( F(\pi) = \text{Fol}(\alpha_G^*\omega) \supseteq \omega \).

(i) \( \Leftrightarrow \) (iv): The \( G \)-orbit \( \alpha_G^*\omega = \{ \omega \circ \alpha_g | g \in G \} \) generates a folium \( \text{Fol}_G(\omega) = \text{Fol}(\alpha_G^*\omega) \), and since

\[
\alpha_g^* \text{Fol}(E) = \text{Fol}(\alpha_g^*E) \quad \text{for} \quad E \subseteq \mathfrak{S}(A), g \in G,
\]

the folium \( \text{Fol}_G(\omega) \) is \( G \)-invariant. It is the minimal \( G \)-invariant folium containing \( \omega \). Hence the equivalence of (i) and (iv) follows from Theorem 2.16.

Now we prove the second part of the theorem. Assertions (a) and (b) are equivalent by definition. That (b) is equivalent to (c) follows from Theorem 2.16 and Corollary 2.17 since \( \omega \in \mathcal{S}_\omega(A) \) implies \( F(\pi_\omega) \subseteq (A^*)_c \) by (iii).
If (d) holds, i.e., $F(\pi) = F(\pi_\omega)$ for a covariant representation $\pi$, then (c) follows from Corollary 2.17. Suppose, conversely, that (c) holds. Then $F(\tau) = F(\pi_\omega)$ for a covariant representation $(\tau, U, H)$ by the Borchers–Halpern Theorem. Consider the representation $(\pi, V, B_2(H))$ with $\pi(A)B := \tau(\pi)B$ and $V_\omega B := U_\omega B$. This covariant representation satisfies $F(\pi) = F(\tau)$, but $\omega \in F(\pi_\omega) = F(\tau) = F(\pi)$ is a vector state of $\pi$ by Lemma 2.13. If $B \in B_2(H)$ is such that $\omega(A) = \text{tr}(B^*\tau(\pi)B)$ for $A \in A$, then the cyclic subrepresentation of $A$ on $\|\pi(A)B\|$ is equivalent to $\pi_\omega$. Then (d) follows.

**Remark 2.21.** (a) Theorem 2.20 improves [Bo83, Thm. III.2], in that we already obtain the existence of a covariant representation from the condition $\text{Fol}(\omega) = F(\pi_\omega) \subseteq (A^*)_c$, the $\alpha_G^*$-invariance of $\text{Fol}(\omega)$ is not required.

(b) Note that if $\alpha$ is uniformly continuous (but $G$ need not be locally compact) then by $(A^*)_c = A^*$ properties (ii) and (iii) are trivially satisfied, hence by (i) we have $\mathcal{S}_\alpha(A) = \mathcal{S}(A)$. So covariant representations always exist for this case.

For the usual case, the following corollary can already be found in [Bo69, Thm. III.1].

**Corollary 2.22.** Let $(A, G, \alpha)$ be a C*-dynamical system and $\omega \in \mathcal{S}(A)$. Then $\omega \in \mathcal{S}_\alpha(A)$ if and only if $\omega \in (A^*)_c$, i.e.,

$$\mathcal{S}_\alpha(A) = \mathcal{S}(A) \cap (A^*)_c.$$ 

**Proof.** In view of Theorem 2.20 we have to show that $\omega \in (A^*)_c$ implies $A\omega A \subseteq (A^*)_c$. The trilinear map

$$A \times A \times A^* \rightarrow (A, B, \omega) \mapsto A\omega B$$

is continuous because $\|A\omega B\| \leq \|A\|\|\omega\||B||$ (cf. (7)). This map is $G$-equivariant, and this implies that $A(A^*)_c \subseteq (A^*)_c$, using the strong continuity of $g \mapsto \alpha_g$.

Thus covariant representations always exist if $\alpha$ is strongly continuous and $\mathcal{S}(A) \cap (A^*)_c \neq \emptyset$.

Theorem 2.20 has the following corollary:

**Corollary 2.23.** Given $(A, G, \alpha)$, the following are equivalent:

(i) There exists a non-zero covariant representation $(\pi, U)$ of $(A, G, \alpha)$.

(ii) There is a state $\omega \in (A^*)_c$ such that $A\omega A \subseteq (A^*)_c$.

(iii) There exists an $\alpha^*_G$-invariant folium $F \subseteq (A^*)_c$.

(iv) There is a state $\omega \in (A^*)_c$ such that $B \ast \omega \in (A^*)_c$ whenever $\omega(B^*B) > 0$.

In the usual case the GNS representations of invariant states always produce covariant representations. In the next example we see that for singular actions invariant states need not even be in $\mathcal{S}_\alpha(A)$. By Corollary 2.22 this requires $\alpha$ to be discontinuous in the strong topology.

**Example 2.24.** We construct an example of an invariant state $\omega \notin \mathcal{S}_\alpha(A)$. Then $\text{Fol}(\omega)$ is $\alpha^*_G$-invariant but not contained in $(A^*)_c$, which implies that $\mathcal{S}(A)_c$ is not a folium.

Let $(X, \sigma)$ be the non-degenerate symplectic space over $\mathbb{R}$ given by $X = \mathbb{C}$, $\sigma(z, w) := \text{Im}(z\overline{w})$ and $\mathcal{A} := \Delta(X, \sigma)$ is the associated Weyl C*-algebra with generating unitaries $(\delta_z)_{z \in X}$ satisfying

$$\delta_z^* = \delta_{-z} \quad \text{and} \quad \delta_z \delta_w = e^{-i\sigma(z,w)/2} \delta_{z+w} \quad \text{for} \quad z, w \in X.$$ 

The tracial state $\omega_0$ defined by $\omega_0(\delta_z) = \delta_{z,0}$ is invariant with respect to the action of $G = \mathbb{R}$ on $\mathcal{A}$ by $\alpha_g(\delta_z) = \delta_{g \sigma z}$. For this action we clearly have that $\omega_0 \in (A^*)_c$ by its invariance. Now $\delta_z \omega_0 \in A\omega_0 A$ for $z \neq 0$. Thus

$$\alpha^*_g(\delta_z \omega_0)(\delta_{-z}) = \omega_0(\delta_{z} \alpha_g(\delta_{-z})) = \omega_0(\delta_{z} \delta_{-e^{i\sigma z}}) = e^{i\sigma(z,e^{i\sigma z})/2} \omega_0(\delta_{z})$$

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and this expression is nonzero only when \(e^{i\theta} = 1\) (when it has modulus 1) hence it is discontinuous with respect to \(\theta\). Thus \(\delta_2 \omega_0 \notin (A^*)_c\) and Theorem 2.20 implies that \(\omega_0 \notin \mathfrak{S}_\alpha(A)\).

**Remark 2.25.** (a) Example 2.24 shows that the inclusion \(\mathfrak{S}_\alpha(A) \hookrightarrow \mathfrak{S}(A)_c := \mathfrak{S}(A) \cap (A^*)_c\) may be proper.

(b) If \(\mathfrak{S}(A)_c\) is a folium, by its \(G\)-invariance, Borchers’ Theorem implies that \(\mathfrak{S}_\alpha(A) = \mathfrak{S}(A)_c\), which is not always the case by (a). Therefore \(\mathfrak{S}(A)_c\) is not always a folium.

(c) A similar situation arises for a \(W^*\)-dynamical system \((M, G, \alpha)\) because the weak-* continuity of orbit maps in \(M_\ast\) does not imply that \(M_\ast = M_{\ast,c}\) (cf. Example 2.0). Accordingly, Theorem 2.16 implies that the vector states of normal covariant representations can be characterized by

\[\mathfrak{S}_{n,\alpha}(M) = \{\omega \in \mathfrak{S}_n(M) \mid \text{Fol}_G(\omega) \subseteq M_{\ast,c}\} .\]

(d) Suppose now that \((\pi, M)\) is a normal representation of \(M\) whose folium \(F(\pi) \cong \mathfrak{S}_n(\pi(M))\) is \(G\)-invariant and contained in \(M_{\ast,c}\). Then \(\pi\) is \(G\)-invariant, so that we obtain a natural \(G\)-action \(\alpha^M\) on \(\mathcal{N} := \pi(M)\) for which the action on \(\pi(M)\) is continuous. We thus obtain a \(W^*\)-dynamical system \((\mathcal{N}, G, \alpha^M)\). By Corollary 2.17 we know that this has a covariant representation with folium \(F(\pi)\). Below, we will obtain such a representation from the standard form realization of \(\mathcal{N}\).

As invariant states are important to construct covariant representations (cf. ground states, Definition 5.5 below, as well as KMS states), we need to characterize when they do produce covariant representations for singular actions. Observe first that given a \(C^*\)-action \((A, G, \alpha)\) and an invariant state \(\omega \in \mathfrak{S}(A)\), then its GNS representation \((\pi_\omega, \Omega_\omega, \mathcal{H}_\omega)\) always gives a covariant representation \((\pi_\omega, \Omega_\omega^\omega)\) of \((A, G, \alpha)\), where \(U_\omega^\omega = \{\omega \in \mathfrak{S}_n(M) \mid \text{Fol}_G(\omega) \subseteq M_{\ast,c}\}\).

(cf. [Bo96, Lemma IV.4.4]). We then have:

**Proposition 2.26.** For a \(C^*\)-action \((A, G, \alpha)\) and an invariant state \(\omega \in \mathfrak{S}(A)\), the following are equivalent:

1. \(U^\omega : G \to U(\mathcal{H}_\omega)\) is continuous, i.e. \((\pi_\omega, U^\omega) \in \text{Rep}(\alpha, \mathcal{H})\).
2. \(\pi_\omega\) is covariant.
3. \(\mathcal{A} \omega \mathcal{A} \subseteq (A^*)_c\).
4. \(\mathcal{A} \omega \subseteq (A^*)_c\).

If \((A, G, \alpha)\) is a \(C^*\)-dynamical system, then (i)-(iv) are satisfied.

**Proof.** (i) \(\Rightarrow\) (ii) is trivial.

(ii) \(\Rightarrow\) (iii): If \((\pi_\omega, U) \in \text{Rep}(\alpha, \mathcal{H}_\omega)\) then \(\omega \in \mathfrak{S}_n(A)\), hence by Theorem 2.20 we obtain that \(\mathcal{A} \omega \mathcal{A} \subseteq (A^*)_c\).

(iii) \(\Rightarrow\) (iv): Assume that \(\mathcal{A} \omega \mathcal{A} \subseteq (A^*)_c\). Observe that as \(\pi_\omega(A)\Omega_\omega\) is dense in \(\mathcal{H}_\omega\), this condition just states that the bounded maps \(g \mapsto \pi_\omega(\alpha_g(A))\) are continuous in the weak operator topology for all \(A \in \mathcal{A}\). As \(\Omega_\omega \in \mathcal{H}_\omega\), this implies that \(\mathcal{A} \omega \cup \omega \mathcal{A} \cup \{\omega\} \subseteq (A^*)_c\).

(iv) \(\Rightarrow\) (i): Condition (iv) implies that for \(A, B \in \mathcal{A}\), the function

\[g \mapsto (\pi_\omega(A)\Omega_\omega, U_\omega^\omega \pi_\omega(B)\Omega_\omega) = \omega(A^* \alpha_g(B)) = (A^* \omega)(\alpha_g(B))\]

is continuous, hence that \(g \mapsto U_\omega^\omega\) is weak operator continuous, by density of \(\pi_\omega(A)\Omega_\omega\). As the weak operator topology coincides with the strong operator topology on the unitary group, we conclude that \(U_\omega\) is continuous.
Finally, we assume that the $G$-action on $\mathcal{A}$ is continuous, i.e. that $(\mathcal{A}, G, \alpha)$ is a $C^*$-dynamical system. Then (i) follows from the fact that the subspace $\mathcal{H}_{\omega,c}$ of $U^\omega$-continuous vectors in $\mathcal{H}_\omega$ is $\pi_\omega(\mathcal{A})$-invariant and contains $\Omega_\omega$. Hence it coincides with $\mathcal{H}_\omega$. \hfill $\Box$

If $\omega \in \mathcal{S}_\alpha(\mathcal{A})$ is not $G$-invariant, then the remaining condition $\mathcal{A}_\omega \mathcal{A} \subseteq (\mathcal{A}^*)_c$ is not enough to ensure that $\pi_\omega$ is covariant, as Example 2.27 shows. It does imply that $\pi_\omega$ is equivalent to a subrepresentation of a covariant representation by Theorem 2.21. Below in Theorem 3.34 we will obtain a generalization of this theorem to invariant projections, where the GNS representation $\pi_\omega$ has to be replaced with a Stinespring dilation.

**Example 2.27.** (A non-quasi-covariant representation) Let $\alpha : \mathbb{R} \to \text{Aut} \ C_0(\mathbb{R})$ be the action of translation on $\mathcal{A} = C_0(\mathbb{R})$. Consider the covariant representation $(\pi, U)$, where $C_0(\mathbb{R})$ acts by multiplication on $\mathcal{H} = L^2(\mathbb{R})$ and the implementing unitaries $U_t$ act by right translation on $L^2(\mathbb{R})$. Let $\xi = \chi_{[0,1]} \in L^2(\mathbb{R})$ and let $\omega_\xi$ be the associated vector state. Then for the positive vector functionals $\omega_A\xi = |A|^2\omega_\xi$, $A \in \mathcal{A}$ we have $\lim_{t \to 0} \|\alpha_t^*\omega_A\xi - \omega_A\xi\| = 0$ (cf. [Bo69] Lemma II.2]). By polarization we thus get $\mathcal{A}_\omega\xi \subset (\mathcal{A}^*)_c$. Now $\pi_{\omega_\xi}$ is unitarily equivalent to the restriction of $\pi(\mathcal{A})$ to $L^2[0,1] \subset \mathcal{H}$. As the kernel of $\pi_{\omega_\xi}$ is $\{ f \in C_0(\mathbb{R}) \mid f \upharpoonright [0,1] = 0 \}$ which is not translation invariant, the representation $\pi_{\omega_\xi}$ is not quasi-covariant. However, by construction there exists a covariant representation $(\pi, U)$ such that $\pi_{\omega_\xi}$ is a subrepresentation of $\pi$.

### 2.4 Covariance of cyclic representations

In Theorem 2.20 we saw that a state $\omega$ is a vector state of a covariant representation of $(\mathcal{A}, G, \alpha)$ if and only if $\mathcal{A}_\omega \mathcal{A} \subseteq (\mathcal{A}^*)_c$. In the case that $\omega$ is invariant, by Proposition 2.26 this condition is enough to ensure that $\pi_\omega$ is covariant. This raises the question of how one can characterize for the general case when a GNS representation $\pi_\omega$ is covariant. First, following the path of the Wigner Theorem, we can characterize for a single automorphism, whether it is implementable in $\pi_\omega$.

**Definition 2.28.** Let $\mathcal{M}$ be a $W^*$-algebra. For $\omega \in \mathcal{S}_n(\mathcal{M})$ we write $s(\omega) \in \mathcal{M}$ for the corresponding **carrier projection**, also called the **support of $\omega$**. It is the maximal projection $p \in \mathcal{M}$ with $\omega(p) = 1$ and

$$\{ M \in \mathcal{M} \mid \omega(M^*M) = 0 \} = \mathcal{M}(1 - p)$$

(c.f. [AS01] Def. 2.133 or [Pe89] 8.15.4]). The **central support of $\omega$**, denoted $z(\omega)$ is the infimum of all central projections $q \in Z(\mathcal{M})$ such that $s(\omega) \leq q$.

Let $\mathcal{A}$ be a $C^*$-algebra and $\mathcal{A}^{**}$ be its enveloping $W^*$-algebra. Realizing any state $\omega$ of $\mathcal{A}$ as a normal state of the $W^*$-algebra $\mathcal{A}^{**}$, we define $s(\omega)$, $z(\omega) \in \mathcal{A}^{**}$ as above. For a non-degenerate representation $(\pi, \mathcal{H})$ of $\mathcal{A}$, we write $\pi^{**} : \mathcal{A}^{**} \to B(\mathcal{H})$ for the corresponding weakly continuous representation of $\mathcal{A}^{**}$ extending $\pi$. Then $\ker \pi^{**}_\omega = \mathcal{A}^{**}(1 - z(\omega))$ and $\pi^{**}_\omega(\mathcal{A}^{**}) \cong z(\omega)\mathcal{A}^{**}$ contains $s(\omega)$.

**Remark 2.29.** If $\mathcal{A}$ is already a $W^*$-algebra, then for a normal state $\omega$ of $\mathcal{A}$ we have that $s(\omega) \in \mathcal{A} \subset \mathcal{A}^{**}$ by [AS01] Lemma 2.132), and thus the two definitions for $s(\omega)$ coincide. Note that if $\mathcal{M}$ is a $W^*$-algebra, and $\omega \in \mathcal{S}_n(\mathcal{M})$, then $\omega$ is faithful on $\pi_\omega(\mathcal{M}) \cong z(\omega)\mathcal{M}$ if and only if $s(\omega) = z(\omega)$.

For the next theorem we recall the Murray–von Neumann equivalence relation $\sim$ on the set $\text{Proj}(\mathcal{M})$ of projections in a $W^*$-algebra $\mathcal{M}$. It is defined by $P \sim Q$ if and only if there exists a $V \in \mathcal{M}$ with $V^*V = P$ and $VV^* = Q$. We write $[\text{Proj}(\mathcal{M})]$ for the set of equivalence classes of projections.
**Theorem 2.30.** (Equivalence Theorem for cyclic representations) For states \( \varphi, \psi \in \mathcal{S}(\mathcal{A}) \), the corresponding cyclic representations \((\pi_\varphi, \mathcal{H}_\varphi)\) and \((\pi_\psi, \mathcal{H}_\psi)\) are unitarily equivalent if and only if \( s(\varphi) \sim s(\psi) \), i.e. their support projections are equivalent in \( \mathcal{A}^{\ast\ast} \) in the sense of Murray-von Neumann.

**Proof.** Given \( \varphi, \psi \in \mathcal{S}(\mathcal{A}) \), recall from [Ta02 Cor. V.1.11] that \( \pi_\varphi \) is unitarily equivalent to a subrepresentation of \( \pi_\psi \), denoted \( \pi_\varphi \preceq \pi_\psi \), if and only if \( s(\varphi) \preceq s(\psi) \), i.e. \( s(\varphi) \) is equivalent to a subprojection of \( s(\psi) \). Thus

\[
\pi_\varphi \preceq \pi_\psi \iff s(\varphi) \preceq s(\psi)
\]

and thus

\[
\pi_\varphi \cong \pi_\psi \iff s(\varphi) \sim s(\psi).
\]

Here we use that \( s(\varphi) \preceq s(\psi) \preceq s(\varphi) \) is equivalent to \( s(\varphi) \sim s(\psi) \) ([Ta02 Prop. V.1.3]) and \( \pi_\varphi \preceq \pi_\psi \preceq \pi_\varphi \) is equivalent to \( \pi_\varphi \cong \pi_\psi \) ([DiX77 Cor. 5.1.5]).

**Remark 2.31.** (a) An analogous statement holds for the central support projections: for states \( \varphi, \psi \in \mathcal{S}(\mathcal{A}) \), we have that \( z(\varphi) = z(\psi) \) if and only if \( \pi_\varphi \) and \( \pi_\psi \) are quasi-equivalent (cf. [AS01 Prop. 5.10, Eq. (5.6)], or [Pe89 Thm. 3.8.2]). Thus, by Remark 2.14(d), their folia are equal \( F(\pi_\varphi) = F(\pi_\psi) \).

(b) Below we will present an alternative proof of Theorem 2.30 based on standard representations in Theorem 3.32.

(c) That \( s(\omega) = 1 \) means that \( \omega \) is a faithful state, i.e. \( \Omega_\omega \in \mathcal{H}_\omega \) is separating. So one particular case of the preceding theorem is the fact that if \( \varphi \) and \( \psi \) are faithful, then \( \pi_\varphi \cong \pi_\psi \) (cf. [Bl06 Thm. III.2.6.7]).

(d) If \( \mathcal{A} \) is unital, then for two pure states \( \varphi \) and \( \psi \), their GNS representations are equivalent if and only if \( \varphi(A) = \psi(UAU^{-1}) \) for some \( U \in U(\mathcal{A}) \) and all \( A \in \mathcal{A} \) (cf. [AS01 Thm. 5.19]).

(e) A set of states of which the support projections are equivalent, has a differential geometric structure. This is studied in [AY05, ACS00, and ACS01].

**Corollary 2.32.** For \( \omega \in \mathcal{S}(\mathcal{A}) \), an automorphism \( \alpha \in \text{Aut}(\mathcal{A}) \) can be implemented in \( \mathcal{H}_\omega \), i.e. there exists \( U \in U(\mathcal{H}_\omega) \) with

\[
U\pi_\omega(A)U^* = \pi_\omega(\alpha(A)) \quad \text{for} \quad A \in \mathcal{A},
\]

if and only if \( \alpha(s(\omega)) \sim s(\omega) \).

**Proof.** The implementability of \( \alpha \) is equivalent to \( \pi_\omega \cong \pi_\omega \circ \alpha \cong \pi_{\omega\circ\alpha} \) and hence to \( s(\omega) \sim s(\omega \circ \alpha) \) by Theorem 2.30. As \( s(\omega \circ \alpha) = \alpha^{-1}(s(\omega)) \), the claim follows.

The equivalence \( \alpha(s(\omega)) \sim s(\omega) \) is in \( \mathcal{A}^{\ast\ast} \) when \( \mathcal{A} \) is a \( C^* \)-algebra, but if \( \mathcal{A} \) is a \( W^* \)-algebra, then by Remark 2.29 \( s(\omega) \in \mathcal{A} \subset \mathcal{A}^{\ast\ast} \) and hence the equivalence \( \alpha(s(\omega)) \sim s(\omega) \) is in \( \mathcal{A} \). The next example applies these concepts concretely.

**Example 2.33.** Let \( G = \mathbb{R} \) and \( \mathcal{M} := L^\infty(\mathbb{R}, M_2(\mathbb{C})) = L^\infty(\mathbb{R}) \otimes M_2(\mathbb{C}) \) with the natural \( \mathbb{R} \)-action \( \alpha \) by translation. It has a representation \( \rho : \mathcal{M} \rightarrow \mathcal{B}(L^2(\mathbb{R}, \mathbb{C}^2)) \) by pointwise matrix multiplication, and \( Z(\mathcal{M}) = L^\infty(\mathbb{R}) \otimes 1 \). A projection \( P \in \mathcal{M} \) can be represented by a measurable function \( P : \mathbb{R} \rightarrow M_2(\mathbb{C}) \) whose range consists of projections in \( M_2(\mathbb{C}) \). For projections in \( \mathcal{M} \), the relation \( P \sim Q \) is equivalent to \( \text{tr} P = \text{tr} Q \) in \( L^\infty(\mathbb{R}) \). Let \( (E_{ij})_{1 \leq i,j \leq 2} \) in \( M_2(\mathbb{C}) \) denote the standard matrix basis.

(a) For \( f \in L^1(\mathbb{R}, \mathbb{R}) \) with \( 0 < f(x) \) for all \( x \in \mathbb{R} \) and \( \int_\mathbb{R} f(x) \, dx = 1 \), we consider the state

\[
\omega(B) := \int_\mathbb{R} f(x)B_{11}(x) \, dx,
\]

where 

\[
B(x) = \begin{pmatrix} B_{11}(x) & B_{12}(x) \\ B_{21}(x) & B_{22}(x) \end{pmatrix}.
\]
As the $\rho$-cyclic vector $v(x) := \sqrt{f(x)} \left( \begin{smallmatrix} 1 \\ 0 \end{smallmatrix} \right)$ in $L^2(\mathbb{R}, \mathbb{C}^2)$ produces the state $\omega(B) = \langle v, \rho(B)v \rangle$, there is a unitary $W : \mathcal{H}_\omega \to L^2(\mathbb{R}, \mathbb{C}^2)$ which intertwines $\rho$ and $\pi_\omega$. The support projection of $\omega$ is $s(\omega)(x) = E_{11}$ and its central support $z(\omega)$ is 1. Both are translation invariant, hence so are their equivalence classes. Thus all $\alpha_t$ are implementable in $\pi_\omega$, and, as $\rho$ is a product representation, it is easily seen to be covariant, using the implementers $U_t \otimes 1$ on $L^2(\mathbb{R}) \otimes \mathbb{C}^2$, where $U_t$ is translation.

(b) Now consider a state of the form

$$\omega(B) := \int_{-\infty}^0 f(x) B_{11}(x) \, dx + \int_0^\infty g(x) \text{tr}(B(x)) \, dx$$

$$= \int_\mathbb{R} (f(x) + g(x)) B_{11}(x) \, dx + \int_0^\infty g(x) B_{22}(x) \, dx$$

for $0 < f \in L^1((-\infty, 0))$, $0 < g \in L^1((0, \infty))$ with $\int_\mathbb{R} f = \frac{1}{2}$, $\int_\mathbb{R} g = \frac{1}{4}$.

Then $s(\omega) = E_{11} + \chi_{\mathbb{R}^+} E_{22}$ is not translation invariant but $z(\omega) = 1$ is. Then $\pi_\omega$ is not covariant because $\text{tr} s(\omega) = 1 + \chi_{\mathbb{R}^+}$ is not translation invariant. If $\gamma$ is the representation of $\mathcal{M}$ on $L^2((-\infty, 0), \mathbb{C}^2) \oplus L^2([0, \infty), M_2(\mathbb{C}))$ by matrix multiplication (equipping $M_2(\mathbb{C})$ with the inner product $\langle C, D \rangle := \text{tr}(CD^*)$), then $\omega$ is the vector state obtained from the $\gamma$-cyclic vector $w(x) := \sqrt{f(x)} \left( \begin{smallmatrix} 1 \\ 0 \end{smallmatrix} \right) \oplus \sqrt{g(x)} I$.

Thus there is a unitary $V : \mathcal{H}_\omega \to L^2((-\infty, 0), \mathbb{C}^2) \oplus L^2([0, \infty), M_2(\mathbb{C}))$ which intertwines $\gamma$ and $\pi_\omega$.

**Remark 2.34.** Let $(A, G, \alpha)$ be a $C^*$-action and $\omega \in \mathcal{S}_\alpha(A)$. We would like to characterize situations when $\pi_\omega$ is actually covariant, i.e. the $G$-action can be implemented on $\mathcal{H}_\omega$ by a continuous unitary representation. By Theorem 2.22 we need to assume at least that $\pi_\omega$ is quasi-covariant, i.e. that Fol($\omega$) = $F(\pi_\omega)$ is $\alpha_{\mathcal{G}}$-invariant. Then we obtain a $W^*$-dynamical system $(\pi_\omega(A)''', G, \beta)$ and $\omega$ extends naturally to a state on $\mathcal{M} := \pi_\omega(A)'''$. The implementability problem for $\mathcal{M}$ is equivalent to the corresponding problem for the von Neumann algebra $\mathcal{M}_t$, so that it suffices to deal with it on the $W^*$-level.

As a next condition, one should require implementability of $\alpha_G$ in $\pi_\omega$. By Corollary 2.32 it is necessary that the equivalence class

$$[s(\omega)] = \{ P \in \text{Proj}(\mathcal{M}) \mid P \sim s(\omega) \}$$

is invariant under $\beta_{\mathcal{G}}$. Suppose that this is the case. Then each $\beta_g \in \text{Aut}(\mathcal{M})$ can be implemented in $\mathcal{H}_\omega$. To characterize whether there are implementers which combine to give a group representation, hence a covariant representation, is a well-known problem in group cohomology. One chooses a set of unitary implementers, e.g. let $U_g$ implement $\beta_g$. Then the discrepancy $\sigma$ with the group law, i.e. $U_g U_h = \sigma(g, h) U_{gh}$, produces a (non-commutative) 2-cocycle with coefficients in the unitary group $U(\mathcal{M}')$. If $\mathcal{M}'$ is commutative (the representation of $\mathcal{M}$ is multiplicity free), then one needs to characterize when the cocycle $\sigma$ is a coboundary within a suitably continuous class of cochains.

If the appropriate second cohomology group is trivial, this would give a sufficient condition for obtaining a covariant representation. In the case that $G$ is locally compact, this leads to the study of Moore cohomology for the group (cf. [Ro86], [MOW16]).

More specifically, we consider the group

$$\hat{G}_\omega := \{ (g, U) \in G \times U(\mathcal{H}_\omega) \mid (\forall M \in \mathcal{M}) \ U \pi_\omega(M) U^{-1} = \pi_\omega(\beta_g(M)) \}.$$  

Then $\hat{G}_\omega$ is a closed subgroup of $G \times U(\mathcal{H}_\omega)$ and the projection onto the second factor provides a continuous unitary representation of $\hat{G}_\omega$ on $\mathcal{H}_\omega$. Since every $\beta_g$ is implementable on $\mathcal{H}_\omega$, the map

$$q : \hat{G}_\omega \to G, \quad (g, U) \mapsto g$$
is surjective and its kernel is isomorphic to the unitary group $U(\pi_\omega(\mathcal{M})) \cong U(\mathcal{M}^\omega_s)$. We thus have a short exact sequence

$$e \rightarrow U(\mathcal{M}^\omega_s) \rightarrow \hat{G}_\omega \rightarrow G \rightarrow e.$$  

The covariance of the representation $\pi_\omega$ is equivalent to the splitting of this extension of topological groups.

The question of covariance of $\pi_\omega$ for a a $C^*$-action $(\mathcal{A}, G, \alpha)$ with $\omega \in \mathcal{G}_G(\mathcal{A})$, given unitary implementability, can be answered in a more restricted context (cf. [Ka71]):

**Theorem 2.35.** (Kallman’s Theorem) Let $\mathcal{A} \subseteq \mathcal{B}(\mathcal{H})$ be a unital $C^*$-algebra where $\mathcal{H}$ is separable. Let $\alpha : \mathbb{R} \rightarrow \text{Aut} \mathcal{A}$ be a $C^*$-action such that

(i) $t \mapsto \alpha_t(A)$ is weak operator continuous for each $A \in \mathcal{A}$, and

(ii) for each $t \in \mathbb{R}$ there is a unitary $U_t \in \mathcal{A}''$ such that $\alpha_t = \text{Ad} U_t$ on $\mathcal{A}$.

Then there is a strongly operator continuous one parameter unitary group $W : \mathbb{R} \rightarrow \mathcal{A}''$ such that $\alpha_t = \text{Ad} W_t$ on $\mathcal{A}$.

As an application of this, consider a $C^*$-action $(\mathcal{A}, R, \alpha)$ where $\mathcal{A}$ is unital and separable, and let $\omega \in \mathcal{G}(\mathcal{A})$. Let $F(\omega)$ be $\alpha^*_R$-invariant and contained in $(\mathcal{A}^* )_c$ (cf. Corollary 2.17), so (i) is satisfied and $\mathcal{H}_\omega$ is separable. We can obtain (ii) by e.g. assuming $\omega$ is pure as all automorphisms of $\mathcal{B}(\mathcal{H}_\omega) = \pi_\omega(\mathcal{A})''$ are inner. Thus a pure state is covariant if and only if $F(\omega)$ is $\alpha^*_R$-invariant and contained in $(\mathcal{A}^* )_c$, as the converse follows from Corollary 2.17.

As a second application of Kallman’s Theorem, consider a $C^*$-action $(\mathcal{A}, R, \alpha)$, and define the discrete crossed product $\mathcal{A} \rtimes_\alpha \mathbb{R}_d =: \mathcal{B}$. Observe that $\alpha$ extends to an action on $\mathcal{B}$ by $\alpha_t(B) := (\text{Ad} \delta_t)(B)$ for $B \in \mathcal{B}$ where $\delta_t \in \ell^1(\mathbb{R}, \mathcal{A}) \subset \mathcal{B}$ is the function with value $1$ at $t$ and zero elsewhere (note that $(\text{Ad} \delta_t)(\delta_s) = \delta_{t+s}$). Let $(\pi, \mathcal{H})$ be a representation of $\mathcal{B}$ for which $\mathcal{H}$ is separable. This corresponds to a covariant representation of $(\mathcal{A}, \mathbb{R}_d, \alpha)$. Then, using the unitaries $\pi(\delta_t) \in \pi(\mathcal{B})$, we have satisfied (ii) of Kallman’s Theorem for $(\mathcal{B}, \mathbb{R}, \alpha)$. To satisfy (i), we need to also assume that on $\mathcal{B}$, $F(\pi)$ is $\alpha^*_R$-invariant and contained in $(\mathcal{B}^* )_c$. It then follows that $(\pi, \mathcal{H})$ is a covariant representation of $(\mathcal{B}, \mathbb{R}, \alpha)$, and restriction to $\mathcal{A}$ produces a covariant representation of $(\mathcal{A}, \mathbb{R}, \alpha)$.

### 2.5 Continuity properties of covariant representations

Henceforth we assume that non-zero covariant representations exist for $(\mathcal{A}, G, \alpha)$. In the usual case for $(\mathcal{A}, G, \alpha)$ (Subsection 2.1), the entire covariant representation theory is carried by the crossed product $\mathcal{A} \rtimes_\alpha G$. When we do not have the usual case, it may still be possible to find a $C^*$-algebra which can fulfill the role of the crossed product. This has already been analyzed in [GrNi14], and in a subsequent paper we will continue this analysis in the presence of spectral conditions. First, we consider natural structures associated with covariant representations.

There is a universal covariant representation, obtained as follows.

**Definition 2.36.** Given $(\mathcal{A}, G, \alpha)$, cyclic representations of $\mathcal{A} \rtimes_\alpha G_d$ are obtained from states through the GNS construction. Let $\mathcal{G}_{r\omega}$ denote the set of those states $\omega$ on $\mathcal{A} \rtimes_\alpha G_d$ which produce covariant representations $(\pi_\omega, U^\omega) \in \text{Rep}(\alpha, \mathcal{H}_\omega)$ (if $G$ is nondiscrete, then some states on $\mathcal{A} \rtimes_\alpha G_d$ need not be in $\mathcal{G}_{r\omega}$ due to the continuity requirement for $U^\omega$). This allows us to define the universal covariant representation $(\pi_{r\omega}, U_{r\omega}) \in \text{Rep}(\alpha, \mathcal{H}_{r\omega})$ by

$$\pi_{r\omega} := \bigoplus_{\omega \in \mathcal{G}_{r\omega}} \pi_\omega, \quad U_{r\omega} := \bigoplus_{\omega \in \mathcal{G}_{r\omega}} U^\omega \quad \text{on} \quad \mathcal{H}_{r\omega} = \bigoplus_{\omega \in \mathcal{G}_{r\omega}} \mathcal{H}_\omega.$$  

This is non-trivial as long as $\mathcal{G}_{r\omega} \neq \emptyset$. We obtain a canonical $W^*$-dynamical system $\alpha^{r\omega} : G \rightarrow \text{Aut}(\mathcal{M}_{r\omega})$, where $\mathcal{M}_{r\omega} := \pi_{r\omega}(\mathcal{A})''$ and $\alpha^{r\omega}(g) = \text{Ad} U_{r\omega}(g)$.
Proposition 2.37. Assume that $(A, G, \alpha)$ has non-zero covariant representations. Then $(\pi_{co}, U_{co}) \in \text{Rep}(\alpha, \mathcal{H}_{co})$ is non-zero, and the folium $F(\pi_{co})$ is the unique folium in $(A^*)_c$ which is maximal in the sense that it contains all other folia in $(A^*)_c$. Moreover $F(\pi_{co}) = \mathcal{G}_\alpha(A)$ and this folium is $G$-invariant.

Proof. Any covariant representation corresponds to a representation of $A \times_\alpha G_d$, and the cyclic subrepresentations for this $C^*$-algebra are still covariant, hence $\mathcal{G}_{co} \neq \emptyset$ and $(\pi_{co}, U_{co}) \in \text{Rep}(\alpha, \mathcal{H}_{co})$ is non-zero. Moreover every covariant representation $(\pi, U)$ of $(A, G, \alpha)$ is a direct sum of subrepresentations $(\pi_{co}, U_{co})$, hence $F(\pi) \subseteq F(\pi_{co})$. Since every folium $F \subseteq (A^*)_c$ is contained in a $G$-invariant folium $F_G := \text{Fol}(\alpha_G F) \subseteq (A^*)_c$ and $F_G = F(\pi)$ for some covariant representation $(\pi, U)$ (Borchers–Halpern Theorem 2.16), it follows that $F(\pi_{co})$ contains every folium in $(A^*)_c$. Clearly, there is only one folium in $(A^*)_c$ with this property.

Further, Theorem 2.20 implies that every $\omega \in \mathcal{G}_\alpha(A)$ is contained in the folium $\text{Fol}_G(\omega) \subseteq (A^*)_c$, so that we also obtain the inclusion $\mathcal{G}_\alpha(A) \subseteq F(\pi_{co})$. Conversely, let $\omega \in F(\pi_{co})$, then by $G$-invariance of $F(\pi_{co})$, we have that $\alpha_G^* \omega \subset F(\pi_{co})$ and hence $\text{Fol}_G(\omega) \subseteq F(\pi_{co}) \subset (A^*)_c$. Thus by the above characterization of $\mathcal{G}_\alpha(A)$ (Theorem 2.20), it follows that $\omega \in \mathcal{G}_\alpha(A)$ (Theorem 2.20). This proves the reverse inclusion, hence the equality $\mathcal{G}_\alpha(A) = F(\pi_{co})$. \qed

Remark 2.38. In [Bo69] Thm. II.3 Borchers states conditions which imply that $\mathcal{G}(A)$ is a folium, but there he assumes the usual case. This lead to a false statement in [GrN14 Prop. 8.9(ii)], where it is claimed that $(A^*)_c = \pi_{co}(A)_c^\prime$ In general this is false by Example 2.24.

Proposition 2.39. Given $(A, G, \alpha)$, let $\tau_1 \geq \tau_2$ be group topologies on $G$. If $(\pi, U)$ is a covariant representation with respect to $\tau_1$, then it contains a $\tau_2$-covariant subrepresentation $(\pi_{\tau_2}, U_{\tau_2})$ which is maximal, in the sense that it contains all other $\tau_2$-covariant subrepresentations of $(\pi, U)$.

Proof. Given a covariant representation $(\pi, U)$ with respect to $\tau_1$ on $\mathcal{H}$, we consider the closed subspace $\mathcal{H}_c$ of continuous vector for the representation of the topological group $(G, \tau_2)$. Then $\mathcal{H}_c$ is $G$-invariant and maximal with respect to the property that the action of $(G, \tau_2)$ on this subspace is continuous.

Now let $\mathcal{H}_2 := \{\xi \in \mathcal{H}_c \mid (\forall A \in A) \pi(A)\xi \in \mathcal{H}_c\}$ be the maximal $A$-invariant subspace of $\mathcal{H}_c$. Then $\mathcal{H}_2$ is also $G$-invariant because, for $g \in G$, $\xi \in \mathcal{H}_2$ and $A \in A$, we have $\pi(A)U_g\xi = U_g\pi(\alpha_g^{-1}(A))\xi \in U_g\mathcal{H}_c = \mathcal{H}_c$. It is also clear that $\mathcal{H}_2$ is maximal with respect to the property that it carries a covariant representation of $(A, (G, \tau_2), \alpha)$. \qed

If $\tau_1$ is the discrete topology and $\tau_2$ the given topology on $G$, then the preceding proposition implies that every covariant representation $(\pi, U)$ of $(A, G_d, \alpha)$ contains a maximal covariant subrepresentation for $(A, G, \alpha)$. If the covariant subrepresentation is zero, we will call $(\pi, U)$ a purely discontinuous covariant representation. An irreducible covariant representation of $(A, G_d, \alpha)$ is either covariant or purely discontinuous.

Given any $(A, G, \alpha)$, we can always define the strongly continuous part of it by

$$\mathcal{A}_c := \{A \in A \mid \alpha^A : G \to A, g \mapsto \alpha_g(A) \text{ is norm continuous}\} \quad \text{and} \quad \alpha_g^c := \alpha_g \mid \mathcal{A}_c.$$

Unfortunately, as we will see in Example 2.42 below, it is possible that $\mathcal{A}_c = \mathbb{C}1$. As we have seen in the Borchers–Halpern Theorem, it is much more the continuous portion $(A^*)_c$ of the $G$-action on $A^*$ than the continuous portion $\mathcal{A}_c$ of $A$ that is responsible for the covariant representations.

Remark 2.40. If we start from a $W^*$-dynamical system $(\mathcal{M}, G, \beta)$ with $G$ locally compact, then $\mathcal{M}_c$ is weakly dense in $\mathcal{M}$, and

$$\mathcal{M}_c = C^*\{\beta_f(A) \mid f \in L^1(G), A \in \mathcal{M}\},$$
where the integrals $\beta_f(A) := \int_G f(g)\beta_g(M)\,dg$ exist in the weak topology ([Pe89, Lemma 7.5.1]). Thus, associated with any $C^*$-action $(A, G, \alpha)$, there is a $C^*$-dynamical system $(M_{co,c}, G, \beta^{co,c})$, which in the locally compact case encodes the covariant representation theory of $(A, G, \alpha)$. As remarked, it is possible that $M_{co,c}$ intersects $\pi_{co}(A)$ only in $C1$, though in the usual case $M_{co,c} \geq \pi_{co}(A)$ where the inclusion may be proper.

**Remark 2.41.** In general, $M_{co} := \pi_{co}(A)^\prime$ produces the $W^*$-dynamical system $(M_{co}, G, \alpha^{co})$ whose covariant normal representations are in one-to-one correspondence with the covariant representations of $(A, G, \alpha)$ because the $\mathcal{G}_d(M_{co})$ can be identified with $\mathcal{G}_d(A)$ (Proposition 2.37). This $W^*$-dynamical system is therefore a suitable tool to analyze covariant representations of a given (possibly singular) $C^*$-action $(A, G, \alpha)$.

We list a few examples which will be useful for subsequent discussion. The reader in a hurry can proceed to the next subsection.

**Example 2.42.** (A case of $A_c = C1$ and $\omega \in \mathcal{G}(A)^G \setminus \mathcal{G}_d(A)$) We consider the rotation action of $T$ on the abelian group $(\mathbb{C}, +)$ by multiplication. This produces an action of $G = \mathbb{R}$ on the Weyl algebra $A := \Delta(\mathbb{C}, \sigma)$, where $\sigma(z, w) = \text{Im}(zw)$, by

$$\alpha_t(\delta_z) = \delta_{e^{it}z}, \quad t \in \mathbb{R}, z \in \mathbb{C}.$$  

We claim that $A_c = C1$. To verify this claim, we consider the covariant representation $(\pi_\omega, U_\omega, H_\omega)$ of $A$ on $\ell^2(\mathbb{C}) \cong H_\omega \subseteq A^*$ corresponding to the $\alpha$-invariant tracial state $\omega$ defined by $\omega(\delta_z) = \delta_{0,z}$, for which $U_\omega$ fixes the cyclic vector $\delta_0$.

Since $A$ is simple by [BR96, Thm. 5.2.8], the state $\omega$ is faithful by Lemma [A.2] below. Therefore

$$\eta: A \to \ell^2(\mathbb{C}), \quad \eta(A) = A\delta_0$$

is a faithful continuous injection mapping the generator $\delta_z$ of $A$ to the basis element $\delta_z = (\delta_{zw})_{w \in \mathbb{C}} \in \ell^2(\mathbb{C})$. Note that $\eta$ is equivariant with respect to the action $\alpha$ of $T$ on $A$ and the representation $U$ of $T$ on $\ell^2(\mathbb{C})$ defined by the permutation of the generators

$$U_t\delta_z = \delta_{e^{it}z}, \quad t \in \mathbb{R}, z \in \mathbb{C}.$$  

Lemma [A.1] implies that $\ell^2(\mathbb{C})_c = \mathbb{C}\delta_0$ for the unitary one-parameter group $U$ which in particular entails that $\omega \in \mathcal{G}(A)^T \setminus \mathcal{G}_d(A)$. The continuity of the inclusion $\eta: A \to \ell^2(\mathbb{C})$ now yields $A_c = C1$ for $\alpha$. Nevertheless, the Schrödinger representation is an example of a faithful covariant representation for $\alpha$.

With a similar argument as in the previous example, we even obtain an example where $A$ is commutative.

**Example 2.43.** (A case of $A_c = C1$, $\omega \in \mathcal{G}(A)^G \setminus \mathcal{G}_d(A)$ and $A$ commutative) We consider the rotation action of $T$ on the abelian group $(\mathbb{C}, +)$ by multiplication and the $C^*$-algebra $A := C^*(\mathbb{C}_d)$, where $\mathbb{C}_d$ is the discrete additive group of complex numbers. We thus obtain an action of $T$ on $A$ by

$$\alpha_t(\delta_z) = \delta_{t,z}, \quad t \in \mathbb{T}, z \in \mathbb{C}.$$  

(a) We claim that $A_c = C1$. To verify this claim, we consider the faithful covariant representation $(\pi_\omega, U_\omega, H_\omega)$ of $A$ on $\ell^2(\mathbb{C}) \cong H_\omega \subseteq A^*$ corresponding to the $\alpha$-invariant state $\omega$ defined by $\omega(\delta_z) = \delta_{0,z}$, for which $U_\omega$ fixes the cyclic vector $\delta_0 \in \ell^2(\mathbb{C})$.

Since $A$ is commutative, the annihilator of the state $\omega$ coincides with $\ker \pi_\omega$ (Lemma [A.2]). Further, the amenability of the discrete abelian group $\mathbb{C}_d$ shows that the representation of $A =
implies that \( \mu \) directly sum we obtain an example of a \( \omega \)-representation \( (\pi,U) \) of \( \mathbb{C} \), but by the preceding, \( M_\beta \) satisfies \( \prod_{t > 0} C(rT) \) by restricting to circles of radius \( r > 0 \). By considering the \( L^2 \)-space of a measure \( \mu \) on \( \mathbb{C} \) concentrated on \( rT \) where it is the invariant measure, we obtain for any \( r > 0 \) a covariant representation \( (\pi_r,U_r) \) of \( (\mathbb{A},\mathbb{R},\alpha) \). These representations separate the points of \( \mathbb{A} \). By taking their direct sum we obtain an example of a \( \mathbb{C} \)-action \( (\mathbb{A},\mathbb{R},\alpha) \) with \( \mathbb{A}_c = \mathbb{C} \) and a faithful covariant representation \( (\pi,U) \).

Note that for the associated \( W^* \)-dynamical system \( \beta : G \to \text{Aut}(\mathcal{M}) \), where \( \mathcal{M} := \pi(\mathbb{A})'' \) and \( \beta(g) = \text{Ad}U(g) \), we do obtain a subalgebra \( \mathcal{M}_c \) which is strongly operator dense in \( \mathcal{M} \) ([Pe89 Lemma 7.5.1]), but by the preceding, \( \mathcal{M}_c \) intersects \( \pi(\mathbb{A}) \) only in \( \mathbb{C} \).

**Example 2.44.** Let \( \mathbb{A} := \ell^\infty(T_d) \) with pointwise operations and \( \| \cdot \|_\infty \)-topology. Here \( T_d \) denotes the discrete group underlying the circle group \( T \subseteq \mathbb{C}^\times \). Consider the action of the topological group \( G = T \) on \( \mathbb{A} \) by rotation, i.e. \( \alpha_t(\delta_z) = \delta_{tz} \) for \( t,z \in T \).

Let \( t_0 \in T \) be an element of infinite order, so that the subgroup \( T_0 \subseteq T_d \) generated by \( t_0 \) is infinite. Define a character

\[
    \xi_0 : T_0 \to T, \quad \xi_0(t_0^n) := (-1)^n
\]

and let \( \xi : T_d \to T \) be an extension of this character to all of \( T_d \) (cf. [HM13 Prop. A1.35]). Then

\[
    S := \xi^{-1}(\{e^i t : 0 \leq t < \pi \}) \subset T
\]

satisfies \( T = S \cup t_0 S \) because \( \xi(t_0) = -1 \).

Since the abelian discrete group \( T_d \) is amenable, there exists an invariant mean \( \omega \in \mathcal{A}^* = \ell^\infty(T_d)^* \) which is an \( \alpha \)-invariant state on \( \mathbb{A} \). It corresponds to a finitely additive measure \( \mu \) on \( T \) by \( \omega(\chi_E) = \mu(E) \). As \( 1 = \omega(1) = \omega(\chi_S + \chi_{t_0 S}) = \mu(S) + \mu(t_0 S) = 2\mu(S) \), invariance of \( \omega \) now implies that \( \mu(S) = \frac{1}{2} \), so that, for \( A = \chi_S \in \mathcal{A} \), we obtain

\[
    \omega(A\alpha_t^n(A)) = \mu(S \cap t_0^n S) = \frac{1}{4} (1 + (-1)^n).
\]

As there are elements from both \( S \) and \( t_0 S \) arbitrarily close to \( 1 \in T \), this implies that the function \( t \mapsto \omega(A\alpha_t(A)) \) is not continuous on \( T \). We conclude that

\[
    \omega \in \mathcal{G}(\mathcal{A})^T \setminus \mathcal{G}_\alpha(\mathcal{A}).
\]

In this example \( \mathbb{A}_c = C(T) \) is strictly larger than \( \mathbb{C} \).

### 2.6 Innerness for covariant representations

Given an action \( (\mathbb{A},G,\alpha) \), a desirable property for a covariant representation \( (\pi,U) \) is that it is inner, i.e. \( U_G \subseteq \pi(\mathbb{A})'' \). This is desirable from a physical point of view, as it implies that the generators of the one-parameter groups in \( G \) are affiliated with \( \pi(\mathbb{A})'' \), hence are observables. It
also is a peculiarly quantum requirement, as in the case that $\mathcal{A}$ is commutative and $\alpha$ is nontrivial, then there are no inner covariant representations which are faithful on $\mathcal{A}$. Below in Sect. 4.2 we will see the surprising fact that certain spectral conditions guarantee the existence of inner covariant representations (Borchers–Arveson Theorem 4.14). Even in the absence of spectral conditions, the innerness of covariant representations have been analyzed. Moreover, we saw above in Kallman’s Theorem 2.35 that in some circumstances, innerness of the action implies covariance.

A great deal is known about $W^*$-actions on factors, starting from the simple observation that all automorphisms of type I factors are inner (as they are isomorphic to some $\mathcal{B}(\mathcal{H})$). However, for the case where the von Neumann algebra is unrestricted, the best result seems to be the one in Kraus [Kr79, Theorem 3.2], which we state below. We first need to fix some notation.

Given a $W^*$-dynamical system $(\mathcal{M}, G, \beta)$ such that $G$ is locally compact and abelian, denote the fixed point algebra by $\mathcal{M}^\beta$ and the center of $\mathcal{M}$ by $Z(\mathcal{M})$. For any projection $P \in \mathcal{M}^\beta$, the reduced von Neumann algebra $\mathcal{M}_P = P\mathcal{M}P$ is left invariant by $\beta$, hence we can restrict the action to obtain a new action $(\mathcal{M}_P, G, \beta^P)$. Let $f \in L^1(G)$, which we recall has Fourier transform

$$\hat{f}(\gamma) := \int_G f(g)\gamma(g) \,dg \quad \text{for} \quad \gamma \in \hat{G}$$

where $\hat{G}$ is the dual group. If we define $\beta_f \in \mathcal{B}(\mathcal{M})$ by

$$\beta_f(M) := \int_G f(g)\beta_g(M) \,dg,$$

then $\text{Spec}(\beta)$ denotes the support of the map $\hat{f} \mapsto \beta_f$, i.e.

$$\text{Spec}(\beta) = \{ \gamma \in \hat{G} \mid (\forall f \in L^1(G)) \beta_f = 0 \Rightarrow \hat{f}(\gamma) = 0 \}. \quad (10)$$

Then for $P \in \mathcal{M}^\beta$ we have $\text{Spec}(\beta^P) \subset \text{Spec}(\beta)$.

**Theorem 2.45.** For a $W^*$-dynamical system $(\mathcal{M}, G, \beta)$ such that $G$ is connected, locally compact and abelian, the following are equivalent:

(i) $\beta$ is inner.

(ii) For every nonzero projection $P \in Z(\mathcal{M}^\beta)$ and every compact neighborhood $V$ of 0 in the dual group $\hat{G}$, there is a nonzero projection $Q \in Z(\mathcal{M}^\beta)$ such that $Q \leq P$ and $\text{Spec}(\beta^Q) \subset V$.

(iii) For every nonzero projection $P \in Z(\mathcal{M}^\beta)$, there is a nonzero projection $Q \in Z(\mathcal{M}^\beta)$ such that $Q \leq P$ and $\text{Spec}(\beta^Q)$ is compact.

This is proven in [Kr79 Thm. 3.2]. Thus innerness is characterized by the projections in the center of the fixed point algebra. Note that, by [BR02 Prop. 3.2.41], the condition that $\text{Spec}(\beta^Q)$ is compact is equivalent to the norm continuity of $\beta^Q : G \to \text{Aut}(\mathcal{M}_P)$.

### 3 Standard and $P$-standard representations of $W^*$-algebras

We saw above in Corollary 2.17 of the Borchers–Halpern Theorem, that a representation $\pi$ of $\mathcal{A}$ is quasi-covariant if and only if the $C^*$-action $(\mathcal{A}, G, \alpha)$ extends to a $W^*$-dynamical system $((\pi(\mathcal{A}))^\beta, G, \beta)$. However, no indication was given on how to construct the covariant representation needed for the quasi-covariance. In this section we want to address this question, i.e. if one is given such a $W^*$-dynamical system, how we can construct a faithful normal representation of it which is covariant. This will be done through standard forms of $W^*$-algebras (defined below). Standard forms also occur naturally as the GNS representations of KMS states. Below in Theorem 3.34 we will show that associated to every invariant nonzero projection, there is a normal Stinespring dilation representation which is covariant. This provides an interesting source of covariant representations.
3.1 Standard forms of $W^*$-algebras

Given a von Neumann algebra $\mathcal{M} \subseteq \mathcal{B}(\mathcal{H})$, we recall next what its standard form representation is. This can be defined either constructively, or by abstract characterization of its structure, and by uniqueness there is only one such standard form representation, up to unitary equivalence. We first state the structural definition.

**Definition 3.1.** ([Haa75]) (a) A von Neumann algebra $\mathcal{M} \subseteq \mathcal{B}(\mathcal{H})$ is said to be in standard form if there exist an anti-unitary involution $J$ on $\mathcal{H}$ and a cone $\mathcal{C} \subseteq \mathcal{H}$ which is self-dual in the sense that

$$\mathcal{C} = \{ \psi \in \mathcal{H} \mid \langle \forall \xi \in \mathcal{C} \rangle \langle \psi, \xi \rangle \geq 0 \}$$

and $\mathcal{M}$, $J$, $\mathcal{C}$ satisfy:

(S1) $JM = M'$.

(S2) $J\psi = \psi$ for every $\psi \in \mathcal{C}$.

(S3) $AJA \subseteq C$ for all $A \in \mathcal{M}$.

(S4) $JAJ = A^*$ for all $A \in \mathcal{M} \cap M'$.

A von Neumann algebra in standard form is denoted by $(\mathcal{M}, J, \mathcal{C})$.

(b) A normal representation $(\pi, \mathcal{H})$ of a $W^*$-algebra is called a standard (form) representation if there exist $J$ and $\mathcal{C}$ such that $(\pi(\mathcal{M}), \mathcal{H}, J, \mathcal{C})$ is a von Neumann algebra in standard form.

(c) From the definition it follows that $(\mathcal{M}, J, \mathcal{C})$ is in standard form if and only if $(\mathcal{M}', \mathcal{H}, J, \mathcal{C})$ is in standard form.

**Remark 3.2.** For a von Neumann algebra in standard form, the map

$$\mathcal{M} \to \mathcal{M}', \quad M \mapsto JM^*J$$

induces an isomorphism of $W^*$-algebras $\mathcal{M}^{\text{op}} \to \mathcal{M}'$. (The opposite algebra $\mathcal{M}^{\text{op}}$ is the space $\mathcal{M}$ equipped with the previous multiplication but where the order of terms are reversed and all other operations, including the scalar multiplication, are the same. It is isomorphic to the complex conjugate algebra, via the map $M \mapsto M^*$.)

We can also give a constructive definition of a standard form representation (cf. [Bla06, Def. III.2.6.1]). It states that $\mathcal{M}$ is in standard form, if it is unitarily equivalent to the GNS representation of a faithful normal semifinite weight on $\mathcal{M}$ (here normal means lower semicontinuous with respect to the ultraweak topology on $\mathcal{M}_+$). Recall that a weight $w$ on $\mathcal{M}$ is semifinite if the set

$$\{ M \in \mathcal{M}_+ \mid w(M) < \infty \}$$

generates a $\sigma$-algebra which is $\sigma(\mathcal{M}, \mathcal{M}_+)$–dense in $\mathcal{M}$. Every von Neumann algebra has a faithful normal semifinite weight (cf. [Bla06, III.2.2.26]), in fact all normal semifinite weights are obtained as sums of normal positive forms (cf. [Haa75b]). As a consequence, standard form representations exist. We now state three equivalent characterizations of a standard form representation.

**Theorem 3.3.** For a von Neumann algebra $\mathcal{M} \subseteq \mathcal{B}(\mathcal{H})$, the following are equivalent:

(i) $\mathcal{M}$ is in standard form,

(ii) There is a faithful normal semifinite weight on $\mathcal{M}$ such that its GNS representation is unitarily equivalent to the inclusion $\mathcal{M} \hookrightarrow \mathcal{B}(\mathcal{H})$. 

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(iii) There is an anti-unitary involution $J$ on $\mathcal{H}$ such that $JMJ = M'$ and $JZJ = Z^*$ for every $Z \in Z(M)$.

Moreover, a standard form representation exists for any von Neumann algebra, and it is unique up to unitary equivalence.

Proof. The equivalence of (i) and (ii) is in [103, Thm. IX.1.2], and the equivalence of (ii) and (iii) is in [09 Thm. III.4.5.7], which also states the existence and uniqueness. This is also in [75 Thm. 1.6] and [03 Thm. IX.1.14].

Remark 3.4. (a) It is of central importance that for a von Neumann algebra $M$, its faithful standard form representations are unitarily equivalent. Moreover, every cyclic normal representation of $M$ is contained in its standard form representation (this has a generalization below in Proposition 3.31). Note that, by condition (iii), the additional structure in Definition 3.1 is automatic, given $J$. Thus if $M \subseteq B(\mathcal{H})$ is standard and commutative, it is a maximal abelian subalgebra of $B(\mathcal{H})$.

(b) For any von Neumann algebra in standard form $(M, \mathcal{H}, J, \mathcal{C})$, the map
\[
\{ \xi \in \mathcal{C} \mid \|\xi\| = 1 \} \to \mathfrak{S}_n(M), \quad \xi \mapsto \omega_\xi, \quad \omega_\xi(A) := \langle \xi, \pi(A)\xi \rangle
\]
is a homeomorphism for the norm topology on $\mathfrak{S}_n(M)$ (Haa75 Lemma 2.10).

(c) By [96 Thm. 5.3.10], we see that a normal KMS state with respect to any $W^*$-dynamical system is faithful, hence it is a faithful normal semifinite weight on $M$, hence its GNS representation is in standard form. Below in Section 4 we will consider KMS states in greater detail.

Example 3.5. (a) Let $(X, \mathfrak{S}, \mu)$ be a semifinite measure space, i.e. for each $E \in \mathfrak{S}$ with $\mu(E) = \infty$, there exists a measurable subset $F \subseteq E$ satisfying $0 < \mu(F) < \infty$. Then the multiplication action of $M := L^\infty(X, \mathfrak{S}, \mu)$ on $\mathcal{H} := L^2(X, \mathfrak{S}, \mu)$ realizes $M$ as a von Neumann algebra in standard form $(M, \mathcal{H}, J, \mathcal{C})$. Here $Jf = \overline{f}$ and $\mathcal{C} = \{ f \in L^2(X, \mathfrak{S}, \mu) \mid 0 \leq f \}$. Any element of $\mathcal{H}$ vanishing only in a zero set is a cyclic separating vector, and such elements exist if and only if $\mu$ is $\sigma$-finite.

(b) If $\mathcal{K}$ is a complex Hilbert space, $M = B(\mathcal{K})$ and $\mathcal{H} := B_2(\mathcal{K})$ is the Hilbert space of Hilbert–Schmidt operators on $\mathcal{K}$, then the left multiplication representation of $M$ on $\mathcal{H}$ yields a standard form $(M, \mathcal{H}, J, \mathcal{C})$, where $J(A) = A^*$ and $\mathcal{C} = \{ A \in B_2(\mathcal{K}) \mid 0 \leq A \}$. A cyclic vector exists if and only if $K$ is separable.

The faithful normal semifinite weights on $M$ include the faithful normal states, if any exist. The existence of a faithful normal state, is equivalent to the property that $M \subseteq B(\mathcal{H})$ is countably decomposable, i.e. every mutually orthogonal family of projections in $M$ is at most countable (cf. [06 Prop. III.2.2.27]). This is the case if $\mathcal{H}$ is separable. Given a faithful normal state, then its GNS representation has a cyclic separating vector, hence we can apply the Tomita–Takesaki modular theory in the GNS representation. This is directly connected to the standard form structures by:

Proposition 3.6. ([75 Thm. 1.1, Rem. 1.2]) If $M \subseteq B(\mathcal{H})$ is a von Neumann algebra with a cyclic separating vector $\Omega \in M$, then the corresponding modular involution $J$ leads to a standard form realization $(M, \mathcal{H}, J, \mathcal{C})$, where $\mathcal{C} \subseteq \mathcal{H}$ is the closed convex cone generated by the elements $AJA^\ast$, $A \in M$.

Remark 3.7. (a) As we are concerned with $W^*$-dynamical systems $\beta : G \to \text{Aut}(M)$, the following property of the standard form representation $(M, \mathcal{H}, J, \mathcal{C})$ is of central importance to us. The group
\[
U(\mathcal{H},M) := \{ U \in U(\mathcal{H}) \mid UC \subseteq \mathcal{C}, UJ = JU, U,MU^* = M \}
\]
has a natural homomorphism to Aut$(M)$ by conjugation:
\[
\Gamma: U(\mathcal{H},M) \to \text{Aut}(M), \quad \Gamma(U)(M) := UMU^*, \tag{11}
\]
and this homomorphism is an isomorphism of topological groups with respect to the \( u \)-topology on \( \text{Aut}(\mathcal{M}) \) and the strong operator topology on \( U(\mathcal{H})_\mathcal{M} \) ([Haa75] Prop. 3.5, [Ta03] Thm. IX.1.15, [Str81] §2.23). In particular, the whole group \( \text{Aut}(\mathcal{M}) \) has a natural unitary representation on \( \mathcal{H} \).

(b) As \( U(\mathcal{H})_\mathcal{M} \) commutes with \( J \), we have \( U(\mathcal{H})_\mathcal{M} \cap M \subset U(M \cap M') \), so that \( U(\mathcal{H})_\mathcal{M} \cap M \subset \ker \Gamma = \{ e \} \). Therefore \( U(\mathcal{H})_\mathcal{M} \) intersects \( U(\mathcal{M}) \) and \( U(M') \) trivially.

As any \( W^* \)-dynamical system is given by a \( u \)-continuous homomorphism \( \beta : G \to \text{Aut}(\mathcal{M}) \), Remark 3.7(a) implies that:

**Proposition 3.8.** If \( (\mathcal{M}, G, \beta) \) is a \( W^* \)-dynamical system, then the standard form representation \((\mathcal{M}, \mathcal{H}, J, C)\) of \( \mathcal{M} \) is covariant for \( \beta \). Moreover, the unitary implementers for \( \beta \) can be taken to be in \( U(\mathcal{H})_\mathcal{M} \).

This proposition is what makes standard forms particularly useful for physics (cf. [DJP03, Pi06]). Note that from Remark 3.7(b), the implementers in \( U(\mathcal{H})_\mathcal{M} \) cannot be inner for nontrivial automorphisms. This proposition raises the question about how the Arveson spectrum of \((\mathcal{M}, R, \beta)\) is related to the covariant implementers in \( U(\mathcal{H})_\mathcal{M} \). This will be considered in Sect. 4.

**Remark 3.9.** As an application of Proposition 3.8, we give an alternative proof of the Borchers–Halpern Theorem using standard forms (cf. Theorem 2.16). If \((\pi, U, \mathcal{H})\) is a covariant representation with \( F = F(\pi) \), then the \( \alpha^*_G \)-invariance of \( F \) follows from

\[
\omega_S(\alpha_g(A)) = \text{tr}(\pi(\alpha_g(A))S) = \text{tr}(U_gAU_g^*S) = \text{tr}(AU_g^*SU_g) = \omega_{U_g^*SU_g}(A)
\]

for \( S \in B_1(\mathcal{H}) \), \( g \in G \) and \( A \in A \). Lemma 2.9 shows that \( F \subseteq (A^*)_c \).

Suppose, conversely, that \( F \) is \( \alpha^*_G \)-invariant and contained in \((A^*)_c\). We identify \( A^* \) with the predual of enveloping \( W^* \)-algebra \( A^* \) of \( A \) and write \( \alpha^{**} \) for the induced action of \( G \) on \( A^{**} \). Then \( F \) is a \( G \)-invariant folium of \( A^{**} \). Let \( Z \) be its central support. Then \( \mathcal{M} := ZA^{**} \) is a \( G \)-invariant weakly closed ideal of \( A^{**} \) with \( \mathcal{G}_n(\mathcal{M}) = F \) for which we have a natural morphism of \( C^* \)-algebras \( \eta : A \to \mathcal{M}, A \mapsto ZA \).

Let \( \pi : \mathcal{M} \to B(\mathcal{H}) \) be a standard form realization of \( \mathcal{M} \) on \( \mathcal{H} \) and observe that the action of \( G \) on \( \mathcal{M} \) leads to a unitary representation \( U : G \to U(\mathcal{H}) \) by Prop. 3.8. Since \( G \) acts continuously on \( \mathcal{G}_n(\mathcal{M}) \cong F \), this representation is continuous by [Haa75] Prop. 3.5. We thus have a faithful covariant representation \((\pi, U)\) of \((\mathcal{M}, G, \alpha^{**}, |_{\mathcal{M}})\), and by pullback via \( \eta \) we obtain a covariant representation of \((A, G, \alpha)\) whose folium is \( F(\pi) = \mathcal{G}_n(\mathcal{M}) = F \).

We will need the following lemma.

**Lemma 3.10.** ([Haa75] Lemma 2.6) Let \((\mathcal{M}, \mathcal{H}, J, C)\) be a von Neumann algebra in standard form and \( p \in \mathcal{M} \) a projection. Then \( q := pJpJ \) is a projection in \( B(\mathcal{H}) \) and \((q\mathcal{M}q, q(\mathcal{H}), qJq, q(C))\) is a von Neumann algebra in standard form.

**Remark 3.11.** If \( \pi \) is a normal representation of \( \mathcal{M} \), then \( \ker \pi = p\mathcal{M} \) for a central projection \( p \) (called the support projection of \( \pi \) - cf. Def. 3.15 below). Thus we obtain a direct sum of \( W^* \)-algebras

\[
\mathcal{M} \cong p\mathcal{M} \oplus (1 - p)\mathcal{M} \cong \ker \pi \oplus \mathcal{N},
\]

and Lemma 3.10 asserts that the standard form \((\mathcal{M}, \mathcal{H}, J, C)\) decomposes accordingly as a direct sum of standard form realizations of \( \mathcal{N} \) and \( \ker \pi \).

In the following subsections we will analyze those projections \( P \in \mathcal{M} \) for which \( P\mathcal{M}P \) is standard on \( PH \).

---

1In [Pi03] Thm. 14 it is asserted that the strong operator topology on \( U(\mathcal{H})_\mathcal{M} \) corresponds to the \( p \)-topology on \( \text{Aut}(\mathcal{M}) \), but this is inconsistent with [Haa75] Rem. 3.9 and contradicts Example 2.6.
3.2 Cyclic projections, reductions and dilations.

**Definition 3.12.** (i) Given a projection $P$ in a von Neumann algebra $\mathcal{M} \subseteq \mathcal{B}(\mathcal{H})$, we define the *reduced* von Neumann algebra by

$$\mathcal{M}_P := \mathcal{P}\mathcal{M} \upharpoonright \mathcal{P}\mathcal{H} \subseteq \mathcal{B}(\mathcal{P}\mathcal{H}), \quad \text{and the reduction map } \quad M \mapsto M_P := \mathcal{P}M \upharpoonright \mathcal{P}\mathcal{H}.$$ (ii) Given a projection in the commutant $P \in \mathcal{M}'$, then we will say that $(\mathcal{M}')_P$ is the von Neumann algebra *induced* by $P$.

Then $\mathcal{M}_P$ is isomorphic to $P\mathcal{M}P \subseteq \mathcal{B}(\mathcal{H})$ by restriction of the latter to $\mathcal{P}\mathcal{H}$. Henceforth we will not distinguish between $\mathcal{M}_P$ and $P\mathcal{M}P$. Note that every strongly closed hereditary $C^*$-subalgebra of $\mathcal{M}$ is of the form $P\mathcal{M}P$ for some $P \in \mathcal{M}$ (cf. [Mu90, Thm. 4.1.8]).

The next two lemmas recall some basic facts on reduction and induction for von Neumann algebras.

**Lemma 3.13.** Let $\mathcal{M} \subseteq \mathcal{B}(\mathcal{H})$ be a von Neumann algebra and $P \in \mathcal{M}$ be a projection. Then the following hold:

(i) $(\mathcal{M}_P)' = \mathcal{P}\mathcal{M}' \upharpoonright \mathcal{P}\mathcal{H} = (\mathcal{M}')_P \subseteq \mathcal{B}(\mathcal{P}\mathcal{H})$,

(ii) $\mathcal{Z}(\mathcal{M})P = \mathcal{Z}(\mathcal{M}_P)$, where $\mathcal{M}_P = P\mathcal{M}P$,

(iii) $\mathcal{M}_P$ is in standard form if and only if $(\mathcal{M}')_P = \mathcal{M}' \upharpoonright \mathcal{P}\mathcal{H}$ is in standard form.

Proofs of (i) and (ii) are in [SZ79, Thm. 3.13] and [Dix82, Part 1, Ch. 2, Prop. 1]. To see (iii), note that (i) implies that $\mathcal{M}_P$ is standard if and only if $(\mathcal{M}')_P = (\mathcal{M}')'$ is standard, as a von Neumann algebra is in standard form if and only if its commutant is in standard form (Theorem 3.3 (iii)).

**Lemma 3.14.** Let $\mathcal{M} \subseteq \mathcal{B}(\mathcal{H})$ be a von Neumann algebra and $P \in \mathcal{M}$ be a projection. Then the following are equivalent:

(i) The central support of $P$ i.e. $z(P) := \inf\{Z \in \mathcal{Z}(\mathcal{M}) \mid P \leq Z\}$ is 1.

(ii) $\mathcal{H}_P := \mathcal{P}\mathcal{H}$ is $\mathcal{M}$-generating, i.e. $[\mathcal{M}\mathcal{H}_P] = \mathcal{H}$.

(iii) The ideal $[P\mathcal{M}\mathcal{P}]$ is weakly dense in $\mathcal{M}$.

(iv) The restriction map $R : \mathcal{M}' \to \mathcal{M}_P'$, $R(M) := M|_{\mathcal{H}_P} = MP$ is an isomorphism of $\mathcal{M}'$ onto $(\mathcal{M}_P)'$.

If these conditions are satisfied, then we further have:

$$\mathcal{H}_P = \ker(P\mathcal{M}(1 - P)). \quad (12)$$

Proof. The projection $Z$ onto $[\mathcal{M}\mathcal{H}_P]$ is contained in $\mathcal{M}'$ because $Z\mathcal{H}$ is $\mathcal{M}$-invariant. Since $[\mathcal{M}\mathcal{H}_P]$ is also $\mathcal{M}'$-invariant, we likewise obtain $Z \in \mathcal{Z}(\mathcal{M}) = \mathcal{M}'$, so that $Z$ is central in $\mathcal{M}$. It coincides with the central support of $P$. Therefore (i) and (ii) are equivalent.

The equivalence of (i) and (iii) follows from the fact that the central support $Z$ of $P$ has the property that $Z\mathcal{M}$ is the weakly closed ideal of $\mathcal{M}$ generated by $P$, i.e. the weak closure of $P\mathcal{M}\mathcal{P}$. That (i) and (iv) are equivalent follows from [SZ79, Prop. 3.14] or [Dix82, Part 1, Ch. 2, Prop. 2] or [Pe89, Prop. 2.6.7].

Now we assume that (i)-(iv) are satisfied. Since $\mathcal{H}_P$ is $\mathcal{M}$-generating, $(1 - P)\mathcal{M}\mathcal{H}_P$ is dense in $\mathcal{H}_P^\perp = (1 - P)\mathcal{H}$. Therefore

$$\mathcal{H}_P = ((\mathcal{H}_P)^\perp)^\perp = [(1 - P)\mathcal{M}\mathcal{H}_P]^\perp = [(1 - P)\mathcal{M}\mathcal{H}P]^\perp$$

implies that $\mathcal{H}_P = \ker((1 - P)\mathcal{M}\mathcal{H})$. \qed
Below we will say that a projection $P \in \mathcal{M}$ is *generating* if the subspace $\mathcal{H}_P := P\mathcal{H}$ is $\mathcal{M}$-generating, i.e. $P$ has central support $1$. This property is also equivalent to the injectivity of the map $\mathcal{M}' \to P\mathcal{M}', M \mapsto MP$; in this sense $P$ is *separating* for $\mathcal{M}'$. These will be very important below, e.g. in Lemma 4.18.

**Remark 3.15.** Let $\mathcal{M}$ be a von Neumann algebra and $P \in \mathcal{M}$ be a projection with central support $1$. Then the preceding lemma shows that $\mathcal{M}' \cong \mathcal{M}'_P$. In general, the complementary projection $1 - P$ need not have central support $1$. In fact, there may be a non-zero central projection $Z \subseteq P$. Then $Z\mathcal{M} = Z\mathcal{M}_P$ is an ideal of $\mathcal{M}$ contained in $\mathcal{M}_P$. If $\mathcal{M}_P$ contains no proper ideals of $\mathcal{M}$, then $1 - P$ also has central support $1$, so that we obtain $\mathcal{M}'_P \cong \mathcal{M}' \cong \mathcal{M}'_{1-P}$. Therefore the von Neumann algebras $\mathcal{M}_P$ acting on $\mathcal{H}_P$ and the von Neumann algebra $\mathcal{M}_{1-P}$ acting on $\mathcal{H}_{1-P}$ have isomorphic commutants. This is in particular the case if $\mathcal{M}$ is a factor.

**Example 3.16.** If the projection $P$ is minimal with central support $1$, then $\mathcal{M}_P \cong \mathbb{C}$ implies that $Z(\mathcal{M}) \cong \mathbb{C}$, so that $\mathcal{M}$ is a factor. Further, the existence of minimal projections implies that $\mathcal{M}$ is of type I, hence isomorphic to some $B(K)$.

Let $\mathcal{M} \subseteq B(\mathcal{H})$ be a von Neumann algebra and $P \in \mathcal{M}$ be a projection. Then for any normal representation of the reduced algebra $\pi_0 : \mathcal{M}_P \to B(\mathcal{H}_0)$ there is a natural completely positive map $\varphi : \mathcal{M} \to B(\mathcal{H}_0)$ defined by

$$\varphi : \mathcal{M} \to B(\mathcal{H}_0), \quad \varphi(M) := \pi_0(PMP)$$

which is a normal map. Thus there exists a normal minimal Stinespring dilation $(\pi_\varphi, \mathcal{H}_\varphi, V_\varphi)$, which is unique up to unitary equivalence (cf. [Ta02, Thm. IV.3.6], [Bla06, Thm. III.2.2.4]). It consists of a normal representation $\pi_\varphi$ of $\mathcal{M}$ on $\mathcal{H}_\varphi$ and a continuous linear map $V_\varphi : \mathcal{H}_0 \to \mathcal{H}_\varphi$ with

$$\pi_0(PMP) = V_\varphi^* \pi_\varphi(M)V_\varphi \quad \text{for} \quad M \in \mathcal{M} \quad \text{and} \quad [\pi_\varphi(M)V_\varphi \mathcal{H}_0] = \mathcal{H}_\varphi. \quad (13)$$

We recall the construction for use below. There are several possible definitions, which coincide by the uniqueness theorem (cf. [Ta02 Thm. IV.3.6]).

**Definition 3.17.** Given a von Neumann algebra $\mathcal{M} \subseteq B(\mathcal{H})$, a projection $P \in \mathcal{M}$ and a normal representation of the reduced algebra $\pi_0 : \mathcal{M}_P \to B(\mathcal{H}_0)$, then the *minimal Stinespring dilation* $(\pi_\varphi, \mathcal{H}_\varphi, V_\varphi)$ with respect to $\varphi : \mathcal{M} \to B(\mathcal{H}_0)$, $\varphi(M) := \pi_0(PMP)$ is constructed as follows. Equip the algebraic tensor product $\mathcal{M} \otimes_{\mathcal{M}_P} \mathcal{H}_0 := (\mathcal{M} \otimes \mathcal{H}_0)/J$, where $J := \text{Span}\{MB \otimes \xi - M \otimes \pi_0(B) \xi | M \in \mathcal{M}, B \in \mathcal{M}_P, \xi \in \mathcal{H}_0\}$ with a sesquilinear inner product, given on the elementary tensors by

$$\langle M \otimes \xi, N \otimes \eta \rangle := \langle \varphi(N^* M) \xi, \eta \rangle, \quad M, N \in \mathcal{M}, \xi, \eta \in \mathcal{H}_0.$$

This is well defined because $\varphi(MB) = \varphi(M)\pi_0(B)$ for $M \in \mathcal{M}$ and $B \in \mathcal{M}_P$. Then factor out by the kernel $\mathcal{N} := \{ \psi \in \mathcal{M} \otimes_{\mathcal{M}_P} \mathcal{H}_0 | \langle \psi, \psi \rangle = 0\}$ and complete to obtain $\mathcal{H}_\varphi$. Denote the factoring map by $\gamma : \mathcal{M} \otimes_{\mathcal{M}_P} \mathcal{H}_0 \to \mathcal{H}_\varphi$, and define $\pi_\varphi : \mathcal{M} \to B(\mathcal{H}_\varphi)$ by

$$\pi_\varphi(A)\gamma(M \otimes \xi) := \gamma(AM \otimes \xi)$$

and then extending it to $\mathcal{H}_\varphi$. Define

$$V_\varphi : \mathcal{H}_0 \to \mathcal{H}_\varphi, \quad V_\varphi \xi := \gamma(1 \otimes \xi)$$

which is an isometry as $\varphi(1) = 1$, which allows us to identify $\mathcal{H}_0$ with the subspace $V_\varphi \mathcal{H}_0$ in $\mathcal{H}_\varphi$. 26
It is easy to verify the claimed properties of \((\pi_\varphi, \mathcal{H}_\varphi, V_\varphi)\) in \([13]\) from this construction. Note that \(M \otimes \xi = MP \otimes \xi\) in \(\mathcal{M} \otimes_{\mathcal{M}P} \mathcal{H}_0\).

For \(P = 1\) we have \(\varphi = \pi_0\), which implies that \(\pi_\varphi = \pi_0\) and that \(V_\varphi = 1\), where we use the canonical identification of \(\mathcal{M} \otimes \mathcal{M} \mathcal{H}_0\) with \(\mathcal{H}_0\).

The given definition is a restriction of a more general definition for any completely positive map \(\varphi\) (cf. \([13,02]\) proof of Thm. IV.3.6]). In this form, if \(\varphi\) is a state, then \(\pi_\varphi : \mathcal{M} \to \mathcal{B}(\mathcal{H}_\varphi)\) is just the GNS-representation of the state.

**Definition 3.18.** If \((\pi, \mathcal{H})\) is a normal representation of the \(W^*\)-algebra \(\mathcal{M}\), then we define the support of \(\pi\) as the unique central projection \(s(\pi)\) for which \(\ker \pi = (1 - s(\pi))\mathcal{M}\) (cf. \([\text{Sa}71]\) Def. 1.21.14)).

Then \(\pi(\mathcal{M}) \cong s(\pi)\mathcal{M}\).

**Lemma 3.19.** Let \(\mathcal{M} \subseteq \mathcal{B}(\mathcal{H})\) be a von Neumann algebra and \(P \in \mathcal{M}\) be a projection. Fix a normal representation of the reduced algebra \(\pi_0 : \mathcal{M}_P \to \mathcal{B}(\mathcal{H}_0)\) and define \(\varphi : \mathcal{M} \to \mathcal{B}(\mathcal{H}_0)\) by \(\varphi(M) := \pi_0(PMP)\). Then the representation \((\pi_\varphi, \mathcal{H}_\varphi)\) has the following properties:

(i) \(s(\pi_\varphi) = z(s(\pi_0))\), where \(z(M) \in Z(\mathcal{M})\) denotes the central support of \(M \in \mathcal{M}\) and \(s(\pi_0) \in Z(\mathcal{M}_P)\) is the central support of \(\pi_0\).

(ii) \(V_\varphi\) is \(\mathcal{M}_P\)-equivariant, i.e. \(\pi_\varphi(B)V_\varphi = V_\varphi\pi_0(B)\) for all \(B \in \mathcal{M}_P\). In particular, \(V_\varphi\mathcal{H}_0\) is \(\pi_\varphi(\mathcal{M}_P)\)-invariant.

(iii) \(V_\varphi\mathcal{H}_0 = \pi_\varphi(P)\mathcal{H}_\varphi\).

**Proof.** (i) We have \(A \in \ker(\pi_\varphi)\) if and only if for all \(M \in \mathcal{M}\) and \(\xi \in \mathcal{H}_0\) we have

\[
0 = \|\pi_\varphi(A)\gamma(M \otimes \xi)\|^2 = \|\gamma(AM \otimes \xi)\|^2 = (\varphi(M^*A^*AM)\xi, \xi).
\]

As this holds for all \(\xi\), it is equivalent to

\[
0 = \varphi(M^*A^*AM) = \pi_0(PM^*A^*AMP) \quad \forall M \in \mathcal{M}.
\]

Then the preceding is equivalent to

\[
0 = s(\pi_0)PM^*A^*AMPs(\pi_0) = (AMs(\pi_0))^*(AMs(\pi_0)), \quad \text{i.e.} \quad AMs(\pi_0) = 0.
\]

We conclude that \(A \in \ker(\pi_\varphi)\) is equivalent to \(AMs(\pi_0) = \{0\}\), and this is equivalent to \(Az(s(\pi_0)) = 0\).

This proves that \(s(\pi_\varphi) = z(s(\pi_0))\).

(ii) This follows from

\[
\pi_\varphi(A)V_\varphi\xi = \pi_\varphi(A)\gamma(1 \otimes \xi) = \gamma(A \otimes \xi) = \gamma(1 \otimes \pi_0(A)\xi) = V_\varphi\pi_0(A)\xi
\]

for \(A \in \mathcal{M}_P\) and \(\xi \in \mathcal{H}_0\).

(iii) For \(M \in \mathcal{M}\) and \(\xi \in \mathcal{H}_0\), we have

\[
\pi_\varphi(P)\gamma(M \otimes \xi) = \gamma(PM \otimes \xi) = \gamma(PMP \otimes \xi) = \gamma(1 \otimes \pi_0(PMP)\xi) = V_\varphi\pi_0(PMP)\xi.
\]

This shows that \(\pi_\varphi(P)\mathcal{H}_\varphi = V_\varphi\mathcal{H}_0\) because \(\pi_0(\mathcal{M}_P)\mathcal{H}_0 = \mathcal{H}_0\). \(\square\)

**Proposition 3.20.** Let \(\mathcal{M}\) be a \(W^*\)-algebra and let \(P \in \mathcal{M}\) be a projection. Given a normal representation \((\pi, \mathcal{H})\) of \(\mathcal{M}\) in which \(\mathcal{H}_0 := \pi(P)\mathcal{H}\) is \(\mathcal{M}\)-generating, construct the restricted representation \((\pi_0, \mathcal{H}_0)\) of the reduction \(\mathcal{M}_P = PMP \subset \mathcal{M}\) by \(\pi_0(PMP) := \pi(PMP) \mid \mathcal{H}_0\), \(M \in \mathcal{M}\).

Then the map \(\pi \to \pi_0\) defines a bijection between isomorphism classes of normal representations \((\pi, \mathcal{H})\) of \(\mathcal{M}\) generated by the spaces \(\pi(P)\mathcal{H}\), and isomorphism classes of normal representations \((\pi_0, \mathcal{H}_0)\) of the reduction \(\mathcal{M}_P\).
Proof. Since the assignment $\pi \mapsto \pi_0$ defines a functor from the category of normal $M$-representations in which the range of $P$ is generating to the category of normal $M_P$-representations, it induced a well-defined map on the level of isomorphism classes.

To see surjectivity, let $(\pi_0, H_0)$ be a normal representation of $M_P$ and define $\varphi$ as $\varphi(M) = \pi_0(PMP)$, $M \in M$, then the corresponding minimal dilation $(\pi_\varphi, H_\varphi)$ is a normal representation of $M$ for which $\pi_\varphi(P)H_\varphi$ is generating (cf. Lemma 3.19(iii)). The restriction $(\pi_\varphi)_0$ of $\pi_\varphi(M_P)$ to $\pi_\varphi(P)H_\varphi = V_\varphi H_0$ is then unitarily equivalent to $(\pi_0, H_0)$ by Lemma 3.19(ii).

To verify injectivity, we have to show that $\pi_0 \cong \pi_0'$ implies that $\pi \cong \pi'$. Since $H_0 \subseteq H$ is $\pi(M)$-generating, by defining $V : H_0 \to H$ to be the inclusion map, we can verify the conditions (13). Thus the representation $(\pi, H)$ is equivalent to the minimal Stinespring dilation $(\pi_\varphi, H_\varphi, V_\varphi)$ of the completely positive map

$$\varphi : M \to B(H), \quad \varphi(M) := \pi(P)(PMP)|_{H_0} = \pi_0(PMP).$$

As the Stinespring construction is functorial from normal $M_P$-representations to normal $M$-representations, it maps isomorphic $M_P$-representations to isomorphic $M$-representations. \qed

### 3.3 Standard projections

We now introduce the following key concept.

**Definition 3.21.** Let $M \subseteq B(H)$ be a von Neumann algebra.

(i) We call a projection $P \in M$ **standard** if it is generating (i.e. its central support is $1$, cf. Lemma 3.14, and $M_P$ on $H_P$ is standard (equivalently, the faithful representation of $M'$ on $H_P$ is standard (Lemma 3.13(iii))).

(ii) Let $\Omega \in H$, then the $\sigma(M, M_+)$-closed left ideal $\{M \in M : M\Omega = 0\}$ can be written as $M(1 - P)$ for a projection $P = s(\Omega) \in M$ (cf. [Sa71 Prop. 1.10.1]), which we will call the **carrier projection** of $\Omega$. This coincides with the carrier projection $s(\omega)$ for the vector state $\omega(M) := \langle \Omega, M\Omega \rangle$ as in Definition 2.28.

**Examples 3.22.** The notion of a standard projection depends on the realization of $M$ on some Hilbert space.

(a) For $M = B(H)$, we have $M' = C_1$ and therefore the rank-one projections are standard.

(b) For the representation of $M = B(K)$ by left multiplications on the Hilbert space $H := B_2(K)$, the commutant consists of $B(K)^{op}$ acting by right multiplications, and a projection $P \in M$ is standard if and only if $P = VV^*$ holds for an isometry $V : H \to H$, i.e., if $P \sim 1$ (see Lemma 3.26 below).

Below we shall see that the carrier projection of a cyclic vector is standard. Note that $1 \in M$ is standard if and only if $M$ is in standard form.

**Lemma 3.23.** A von Neumann algebra $M \subseteq B(H)$ contains a standard projection if and only if there exists an $M'$-invariant subspace $H_0 \subseteq H$ on which the representation of $M'$ is faithful and standard.

**Proof.** If $P \in M$ is standard, then $H_P := PH$ is generating for $M$, hence separating for $M'$. As the representation of $M'$ on $H_P$ thus leads to an isomorphism $M' \cong (M_P)'$ (Lemma 3.13), the representation of $M'$ on $H_P$ is standard.

Suppose, conversely, that $H_0$ is a closed subspace of $H$ on which the representation of $M'$ is faithful and standard and let $P \in M$ be the orthogonal projection onto $H_0$. Then $H_0$ is $M$-generating because it separates $M'$, and thus $z(P) = 1$. Further, the fact that $P(M_P)|_{H_0}$ is the commutant of $M_P$ (Lemma 3.13) implies that the representation of $M_P$ on $P$ is standard. \qed
It is instructive to observe that there are von Neumann algebras containing no standard projections. This happens if the representation is too large.

**Examples 3.24.** (a) We consider the von Neumann algebra \( M = C^* \subseteq B(H) \). Then \( M \) contains a standard projection if and only if \( 1 \) is standard, and this is equivalent to the representation of \( M_{\rho P} = M = C^* \) on \( H \) being standard. This is only the case for \( \dim H = 1 \).

(b) If \( M \subseteq B(H) \) is a commutative von Neumann algebra, then \( P = 1 \) is the only projection with central support \( 1 \). Then \( H_P = H \) and \( P \) is standard if and only if the representation of \( M \) on \( H \) is. As \( M \) is commutative, we then have \( M' = JMJ = JZ(M)J = M \). In particular, the representation must be multiplicity free. For \( M = L^\infty(X, \mathcal{S}, \mu) \), where \( \mu \) is a finite measure, this means that the representation of \( M \) is equivalent to the multiplication representation on \( L^2(X, \mathcal{S}, \mu) \).

(c) If \( M \subseteq B(H) \) is a factor of type I, then \( H \cong K \otimes K' \) with \( M = B(K) \otimes 1 \cong B(K) \) and \( M' = 1 \otimes B(K') \cong B(K') \). Let \( P = Q \otimes 1 \in M \) be a projection. As \( M \) is a factor, \( z(P) = 1 \) whenever \( Q \neq 0 \). Further, \( M_P \cong B(K_Q) \) and \( H_P = K_Q \otimes K' \). The representation of \( M_P \cong B(K_Q) \) on this space is standard if and only if \( H_P \cong B_2(K_Q) \) (with the left multiplication representation), and this is equivalent to \( K' \cong K_Q \). Therefore \( M \) contains a standard projection if and only if \( \dim K' \leq \dim K \), i.e., if the multiplicity space \( K' \) is isomorphic to a subspace of \( K \).

The content of the following lemma can already be found in Størmer’s approach to modular invariants of von Neumann algebras in [St72].

**Lemma 3.25.** Let \( M \subseteq B(H) \) be a von Neumann algebra, let \( \Omega \in H \) be a unit vector and \( P = s(\Omega) \) be the corresponding carrier projection. Then

(i) \( [M'\Omega] = PH \),

(ii) If \( \Omega \) is \( M \)-cyclic, then \( \Omega \in H_P = PH \) is cyclic and separating for \( M_{\rho P} \). In particular, \( P = s(\Omega) \) is standard.

**Proof.** (i) Let \( Q \) be the projection onto the closed subspace \([M'\Omega]\). As \( QH \) is \( M' \)-invariant, the projection \( Q \) is contained in \( M'' = M \). For \( M \in M \), the condition \( M\Omega = 0 \) is equivalent to \( MM'\Omega = \{0\} \), resp., to \( MQ = 0 \). Therefore \( M(1 - P) = M(1 - Q) \), and this implies that \( P = Q \).

(ii) First we observe that \( \Omega \) is \( M \)-cyclic because \( H_P = PH = [PM\Omega] = [PM_{\rho P}\Omega] = [M_{\rho P}\Omega] \). To see that \( \Omega \) separates \( M_{\rho P} \), let \( M \in M_{\rho P} \) satisfies \( M\Omega = 0 \), then the definition of the carrier projection implies that \( M \in M_{\rho P} \cap M(1 - P) = \{0\} \). From Proposition 3.6 it now follows that the representation of \( M_{\rho P} \) on \( H_P \) is standard.

For the next lemma we use the Murray–von Neumann equivalence relation \( \sim \) recalled in the lines just above Theorem 3.30.

**Lemma 3.26.** If \( P \) is a standard projection in the von Neumann algebra \( M \), then a projection \( Q \in M \) is standard if and only if \( P \sim Q \).

**Proof.** That \( P \) is standard means that the representation of \( M' \) on \( H_P = PH \) is standard which implies in particular that \( M' \cong M'_{\rho P} \). Since two standard representations of \( M' \) are equivalent by Remark 3.4(a), it follows from [Sa71] Prop. 2.7.3, applied to \( A := M' \) and \( P, Q \in M'' = M \), that \( Q \) is standard if and only if \( P \sim Q \).

**Definition 3.27.** (\( P \)-standard representations) Let \( P \) be a projection in the \( W^* \)-algebra \( M \) and \( \rho_P : M_P = PMP \to B(H_0) \) be a faithful standard form representation of \( M_P \). Then

\[ \varphi_P : M \to B(H_0), \quad \varphi_P(M) := \rho_P(PMP) \]
is a normal completely positive function, so that there exists a normal minimal Stinespring dilation \((\pi_\varphi, \mathcal{H}_\varphi, V_\varphi)\), which is unique up to unitary equivalence (cf. \cite[Thm. IV.3.6]{Ta02}, \cite[Thm. III.2.2.4]{Bla06}, Proposition 3.20). It is called the \(P\text{-standard representation of } \mathcal{M}\). If there is no risk of confusion, we will omit the subscript \(P\) on \(\varphi_\varphi\) and just use \((\pi_\varphi, \mathcal{H}_\varphi, V_\varphi)\).

It consists of a normal representation \(\pi_\varphi\) of \(\mathcal{M}\) on \(\mathcal{H}_\varphi\), and a continuous linear map \(V_\varphi\colon \mathcal{H}_0 \to \mathcal{H}_\varphi\) with

\[
\rho_P(\mathcal{P}_M) = \varphi_\varphi(M) = V_\varphi^* \pi_\varphi(M) V_\varphi \quad \text{for } M \in \mathcal{M} \quad \text{and} \quad \mathbb{P}_\varphi(\mathcal{M}) V_\varphi \mathcal{H}_0 = \mathcal{H}_\varphi.
\]

The construction and properties of \((\pi_\varphi, \mathcal{H}_\varphi, V_\varphi)\) was given above in the previous subsection, but we list the properties again below.

**Lemma 3.28.** For a projection \(P\) in the \(W^*\)-algebra \(\mathcal{M}\), the Stinespring dilation \((\pi_\varphi, \mathcal{H}_\varphi, V_\varphi)\) for \(\varphi(M) := \varphi_\varphi(M) := \rho_P(\mathcal{P}_M)\) has the following properties:

(i) \(s(\pi_\varphi) = z(P)\) is the central support of \(P\).

(ii) \(\pi_\varphi\) is \(\mathcal{M}_P\)-equivariant, i.e. \(\pi_\varphi(B) V_\varphi = V_\varphi \rho_P(B)\) for all \(B \in \mathcal{M}_P\). Further, \(V_\varphi \mathcal{H}_0 = \pi_\varphi(\mathcal{M}_P)\)-invariant and the restriction of \(\pi_\varphi(\mathcal{M}_P)\) to this subspace is standard.

(iii) \(\mathcal{V}_\varphi(\mathcal{H}_0) = \pi_\varphi(P) \mathcal{H}_\varphi\).

(iv) If the central support of \(P\) is \(1\), then \(\pi_\varphi\) is a faithful normal representation for which the projection \(\pi_\varphi(P)\) onto \(\mathcal{V}_\varphi \mathcal{H}_0\) is standard.

(v) If \(\mathcal{M} \subseteq \mathcal{B}(\mathcal{H})\) is a von Neumann algebra and \(\mathcal{H}_P\) is \(\mathcal{M}\)-generating, then the identity representation of \(\mathcal{M}\) on \(\mathcal{H}\) is unitarily equivalent to \(\pi_\varphi\) if and only if \(P\) is standard.

**Proof.** In Lemma 3.19, replace \(\pi_0\) with \(\rho_P\) to obtain the \((\pi_\varphi, \mathcal{H}_\varphi, V_\varphi)\) here.

(i) As \(\rho_P\) is faithful, \(s(\rho_P) = P\), so that this follows from Lemma 3.19(i).

(ii) The equivariance was already proven in Lemma 3.19(ii). As the restriction of \(\pi_\varphi(\mathcal{M}_P)\) to \(\mathcal{V}_\varphi \mathcal{H}_0\) is unitarily equivalent to \(\rho_P\) it is clearly standard.

(iii) This is Lemma 3.19(iii).

(iv) If \(z(P) = 1\) then by (i) \(\pi_\varphi\) is faithful. The rest is clear.

(v) In view of Proposition 5.20, the identical representation of \(\mathcal{M}\) on \(\mathcal{H}\) is equivalent to \(\pi_\varphi\) if and only if the representation of \(\mathcal{M}_P\) on \(\mathcal{H}_P\) is equivalent to \((\pi_\varphi)_0 \cong \rho_P\), i.e. standard by (iii). This means that \(P\) is standard. \(\square\)

**Proposition 3.29.** For two projections \(P, Q\) in the \(W^*\)-algebra \(\mathcal{M}\), the representations \(\pi_\varphi\) and \(\pi_\varphi Q\) are unitarily equivalent if and only if \(P \sim Q\).

**Proof.** (a) Suppose first that \(P \sim Q\). Then both have the same central support. As \(P = z(P)P \in z(P)\mathcal{M} \cong \mathcal{N} := \pi_\varphi(P)\mathcal{M}\) has central support \(1\) in \(z(P)\mathcal{M}\), it follows by Lemma 3.28(iv) that \(\pi_\varphi(P)\) is a standard projection in \(\mathcal{N}\). Now Lemma 3.28 implies that the projection \(\pi_\varphi(Q)\) is also standard in \(\mathcal{N}\) and Lemma 3.28(v) implies that the representations \(\pi_\varphi\) and \(\pi_\varphi Q\) are unitarily equivalent.

(b) If, conversely, \(\pi_\varphi\) \(\cong \pi_\varphi Q\), then \(\pi_\varphi(Q)\) is a standard projection in \(\mathcal{N} = \pi_\varphi(P)\mathcal{M} \cong \mathcal{B}(\mathcal{H}_\varphi)\), hence equivalent to \(\pi_\varphi(P)\) by Lemma 3.28. As \(z(P) = s(\pi_\varphi) = s(\pi_\varphi Q) = z(Q)\), we have \(P, Q \in z(P)\mathcal{M} \cong \mathcal{N}\). As \(\pi_\varphi\) is a faithful representation of \(\mathcal{N}\), it follows that \(P \sim Q\) in \(z(P)\mathcal{M}\), and hence that \(P \sim Q\) in \(\mathcal{M}\). \(\square\)

**Proposition 3.30.** For a projection \(P\) with central support \(1\), the representation \((\pi_\varphi, \mathcal{H}_\varphi)\) is cyclic if and only if the \(W^*\)-algebra \(\mathcal{M}_P\) is countably decomposable.
Proof. If \( \mathcal{M}_P \) is countably decomposable, then its standard representation contains a cyclic vector \( \Omega \) by Remark 3.3.4(d) and therefore \( \Omega \) is \( \mathcal{M} \)-cyclic in \( \mathcal{H}_{\varphi_P} \).

Suppose, conversely, that \( \pi_{\varphi_P} \) has a cyclic vector \( \Omega \) and that \( Q \) is its carrier projection. Then \( \pi_{\varphi_P}(Q) \) is a standard projection by Lemma 3.27 and \( \mathcal{M}_Q \) is countably decomposable by [Bla00, Prop. III.2.2.27]. Since the projection \( \pi_{\varphi_P}(P) \) is also standard, \( \pi_{\varphi_P}(P) \sim \pi_{\varphi_P}(Q) \) by Lemma 3.26, which in turn leads to \( P \sim Q \). We conclude that \( \mathcal{M}_P \cong \mathcal{M}_Q \) is countably decomposable. \( \square \)

The following proposition generalizes the observation that a standard form realization contains all cyclic representations of \( \mathcal{M} \).

**Proposition 3.31.** Let \( (\mathcal{M}, \mathcal{H}, J, C) \) be a von Neumann algebra in standard form and \( P \in \mathcal{M} \) be a projection with central support \( z(P) = 1 \). Then the representation \( (\pi_{\varphi_P}, \mathcal{H}_{\varphi_P}) \) is unitarily equivalent to the representation of \( \mathcal{M} \) restricted to the range of the projection \( JPJ \in \mathcal{M}' \).

**Proof.** Consider the projection \( P' := JPJ \in \mathcal{M}' \). It has the same central support \( z(P') = 1 \). This implies that, for the projection \( Q := PP' = P'P \), the map

\[
\Phi: \mathcal{M}_P \rightarrow \mathcal{M}_Q := (\mathcal{M}_P)_{P'} = P'M_P, \quad M \mapsto P'M
\]

is an isomorphism of von Neumann algebras (cf. [Pe79, Prop. 2.6.7]). In fact, since \( P' \) is generating for \( \mathcal{M}' \) because \( z(P') = 1 \), it is separating for \( \mathcal{M} \). From Lemma 3.10 (cf. [Haa75, Lemma 2.6]), we know that \((\mathcal{M}_Q, Q\mathcal{H}, QJP\mathcal{Q}, Q\mathcal{C})\) is a von Neumann algebra in standard form. Consider the linear map

\[
\gamma: \mathcal{H}_P \rightarrow \mathcal{H}_Q = Q\mathcal{H} = P'PH = P'\mathcal{H}_P, \quad \xi \mapsto P'\xi = Q\xi.
\]

For \( M \in \mathcal{M}_P \) we then have \( \gamma(M\xi) = P'M\xi = MP'\xi = \Phi(M)\gamma(\xi) \), so that \( \gamma \) intertwines the representation \( \pi_{\mathcal{M}_P} \) on \( \mathcal{H}_P \) with the representation of \( \mathcal{M}_Q \) on \( \mathcal{H}_Q \). This implies that the representation \( \rho_P(M) := P'M \) of \( \mathcal{M}_P \) on the subspace \( \mathcal{H}_Q = \mathcal{H}_P \cap \mathcal{H}_{P'} \) of \( \mathcal{H}_P \) is a faithful standard form representation of \( \mathcal{M}_P \). As \( z(P) = 1 \), the subspace \( \mathcal{H}_P = P\mathcal{H} \) is \( \mathcal{M} \)-generating, so that

\[
\mathcal{H}_Q = P'\mathcal{H}_P \quad \text{and} \quad [\mathcal{M}\mathcal{H}_Q] = [P'\mathcal{M}\mathcal{H}_P] = P'\mathcal{H} = \mathcal{H}_{P'}.
\]

Therefore \( M \mapsto P'M \) defines a faithful representation of \( \mathcal{M} \) on \( \mathcal{H}_{P'} \) (by [Pe79, Prop. 2.6.7]) in which the subspace \( \mathcal{H}_Q = P\mathcal{H}_{P'} \) is \( \mathcal{M} \)-generating and carries a faithful standard representation of \( \mathcal{M}_P \). We conclude that this representation is \( P \)-standard, hence unitarily equivalent to \( (\pi_{\varphi_P}, \mathcal{H}_{\varphi_P}) \) (Lemma 3.28(v)). \( \square \)

### 3.4 Implementability for \( W^* \)-dynamical systems

We reconsider Theorem 2.30 above and we give another proof based on standard representations.

**Theorem 3.32.** (Equivalence Theorem for cyclic representations) For two normal states \( \omega, \eta \) of a \( W^* \)-algebra \( \mathcal{M} \), the corresponding cyclic representations are equivalent if and only if \( s(\omega) \sim s(\eta) \), i.e. their carrier projections are equivalent.

**Proof.** First we use Lemma 3.25 and Lemma 3.28(v) to see that, for the carrier projections \( P := s(\omega) \) and \( Q := s(\eta) \), we have \( \pi_\omega \cong \pi_{\varphi_P} \) and \( \pi_\eta \cong \pi_{\varphi_Q} \). Therefore Proposition 3.29 implies that \( \pi_\omega \cong \pi_\eta \) is equivalent to \( P \sim Q \). \( \square \)

**Remark 3.33.** For a \( W^* \)-dynamical system \( (\mathcal{M}, G, \beta) \), we obtain a similar picture than in Subsection 2.1 if we replace the state \( \omega \) by a projection \( P \) and consider the corresponding \( P \)-standard representation \( (\pi_{\varphi_P}, \mathcal{H}_{\varphi_P}) \). A necessary condition for \( (\pi_{\varphi_P}, \mathcal{H}_{\varphi_P}) \) to be covariant with respect to \( \beta \), is that \( \beta_G \) preserves the kernel of \( \pi_{\varphi_P} \), hence the central support \( z(P) \) of \( P \). If this is the case, then we may replace \( \mathcal{M} \) by \( \mathcal{M}z(P) \), so that we may assume that \( z(P) = 1 \) and that \( \pi_{\varphi_P} \) is faithful.
Another necessary condition is that $\beta_G$ preserves the equivalence class $[P]$ of projections (Proposition 3.29), hence fixes its central support $z(P)$. If this is the case, then $\pi_{\varphi_P} \circ \beta_g \cong \pi_{\varphi_{\beta^{-1}_g}(P)}$ implies that each automorphism $\beta_g$ can be implemented in $\mathcal{H}_{\varphi_P}$. This leads to a topological group extension

$$\hat{G}_P := \{ (g, U) \in G \times U(\mathcal{H}_{\varphi_P}) | (\forall M \in \mathcal{M}) U \pi_{\varphi_P}(M) U^{-1} = \pi_{\varphi_P}(\beta_g(M)) \}$$

of $G$ by $N := U(\pi_{\varphi_P}(\mathcal{M}))' \cong U(\mathcal{M}_P')$ and the covariance of the representation $\pi_{\varphi_P}$ is equivalent to the splitting of this extension of topological groups.

This is closely related to the Lie group extensions constructed in [Ne08] for smooth actions of a Lie group $G$ on a continuous inverse algebra $\mathcal{A}$. For a projective $\mathcal{A}$-right module of the form $P \mathcal{A}$, $\hat{G}_P$ is an extension of an open subgroup $G_{[P]} := \{ g \in G | \beta_g(P) \sim P \}$

of $G$ by the unit group $\mathcal{A}_P^\times = (P \mathcal{A})^\times$. In the unitary context, which corresponds to Hilbert-$C^*$-modules, where $\mathcal{A}$ is a $C^*$-algebra, one expects extensions by the unitary group $U(\mathcal{A}_P)$.

For the required smoothness it may be enough that the orbit of $P \in \mathcal{A}$ is smooth in $\mathcal{A}$; which is the case if $P$ is a smoothing operator for a unitary representation of $G$, i.e., $P \mathcal{H} \subseteq \mathcal{H}^\infty$ (cf. [NSZ17]).

**Theorem 3.34.** Given a $W^*$-dynamical system $(\mathcal{M}, G, \beta)$ and a projection $P \in \mathcal{M}$ such that $P$ is $\beta_G$-invariant, then $\beta$ can be continuously implemented in $(\pi_{\varphi_P}, \mathcal{H}_{\varphi_P})$, i.e. $\pi_{\varphi_P}$ is covariant. In particular, the extension $\hat{G}_P$ of $G$ splits.

**Proof.** If $P$ is $\beta_G$-invariant, then $\beta_G$ preserves the subalgebra $\mathcal{M}_P$ and can be continuously implemented in the standard representation $(\rho_P, \mathcal{H}_0)$ of $\mathcal{M}_P$ (cf. Proposition 3.32). Then the corresponding completely positive map

$$\varphi_P : \mathcal{M} \to B(\mathcal{M}_P), \ M \mapsto \rho_P(PMP)$$

is $\beta_G$-equivariant, and the naturality of the Stinespring dilation implies that $\beta_G$ can be continuously implemented in $(\pi_{\varphi_P}, \mathcal{H}_{\varphi_P})$. Explicitly, fix the unitary implementing group $V : G \to U(\mathcal{H}_0)$, $\rho_P(\beta_g(M)) = V_g \rho_P(M) V^*_g$ for $M, N \in \mathcal{M}_0$. Then

$$\left\langle \pi_{\varphi_P}(\beta_g(A)) \gamma(M \otimes \xi), \gamma(N \otimes \eta) \right\rangle = \left\langle \gamma(\beta_g(A) M \otimes \xi), \gamma(N \otimes \eta) \right\rangle$$

$$= \left\langle \varphi(N^* \beta_g(A) M) \xi, \eta \right\rangle = \left\langle \rho_P(PN^* \beta_g(A) MP) \xi, \eta \right\rangle$$

$$= \left\langle V_g \rho_P(\beta_{\gamma^{-1}}(N) A \beta_{\gamma^{-1}}(M) P) V^*_g \xi, \eta \right\rangle$$

$$= \left\langle \pi_{\varphi}(A) \gamma(\beta_{\gamma^{-1}}(M) \otimes V^*_g \xi), \gamma(\beta_{\gamma^{-1}}(N) \otimes V^*_g \eta) \right\rangle$$

$$= \left\langle U_g \pi_{\varphi}(A) U^*_g \gamma(M \otimes \xi), \gamma(N \otimes \eta) \right\rangle,$$

where

$$U_g \gamma(M \otimes \xi) := \gamma(\beta_g(M) \otimes V_g \xi) \quad \text{implies} \quad \pi_{\varphi}(\beta_g(A)) = U_g \pi_{\varphi}(A) U^*_g.$$

It is obvious that $U_g$ is a unitary group homomorphism, by letting $A = 1$ above, and weak operator continuity is also easy to see.

The following example shows that Theorem 3.34 does not extend directly to the case where only $[P]$ is $G$-invariant. This case requires the passage to possibly non-trivial central extensions.

**Example 3.35.** For $\mathcal{M} = B(\mathcal{H})$ and $\dim \mathcal{H} > 1$, we consider a one-dimensional projection $P \in \mathcal{M}$ and observe that it is standard by Lemma 3.25. Thus the representation $(\pi_{\varphi_P}, \mathcal{H}_{\varphi_P})$ is unitarily...
equivalent to the identical representation of $M$ on $H$ by Lemma 3.28(v). We consider the action of $G := PU(H)$ on $M$ induced by conjugation. This action leaves the class $[P]$ of the projection $P$ invariant, but to implement it on $H$, we have to pass to the non-trivial central extension $\hat{G} = U(H)$ of $G$ by $T \cong U(M_\rho)$. That this central extension is non-trivial follows for infinite dimensional Hilbert spaces from the fact that every unitary operator is a commutator ([Ha82, Prob. 239]), and for $H = C^n$, the subgroup $T1 \subseteq SU(n(C) \cong C_n1$ (cyclic group of order $n$) consists of commutators in $SU_n(C)$.

**Remark 3.36.** (i) If $P = 1$, then $\pi_{x_P}$ is the standard representation of $M$ and Theorem 3.34 implies that $Aut(M)$ can be implemented (which is already known from Proposition 3.38).

(ii) If $M$ is a von Neumann algebra, and the $G$-invariant projection $P$ is standard, then the covariant representation $\pi_{x_P}$ is faithful (cf. Lemma 3.28(v)) and unitarily equivalent to the identity representation of $M$. Hence the identity representation of $M$ is covariant.

(iii) [Hal72, Thm. 8] describes criteria for the implementability in terms of the $G$-action on $Z(M)$ and [Hal72, Cor. 10] concerns semi-finite von Neumann algebras.

(iv) [Bla06, III.2.6.15/16] has a criterion for a von Neumann algebra $M \subseteq B(H)$ to be in standard form: If $H$ is separable and $M'$ is properly infinite. In view of (i), this can be viewed as a sufficient condition for unitary implementability of the $G$-action.

**Remark 3.37.** (Equivalence classes of projections for factors)

(a) If $M = B(H)$ is a factor of type I, then two projections $P, Q \in M$ are equivalent if and only if $\dim PH = \dim QH$, i.e. the set of equivalence classes is parameterized by the Hilbert dimensions of closed subspaces of $H$, which is the set of all cardinals $\leq \dim_{Hilb} H$.

In this case $Aut(M) = PU(H)$ acts by conjugation, so that every class $[P]$ is invariant under $Aut(M)$.

(b) If $M$ is a factor of type II$_1$, then the set of equivalence classes of finite projections (this means that $P \sim Q \leq P$ implies $P = Q$) can be identified with the unit interval $[0, 1]$ because any normalized trace $\tau: M \to \mathbb{C}$ provides a complete invariant. Since $\tau$ is $Aut(M)$-invariant, the automorphism group also preserves all equivalence classes of projections.

(c) From [Bla06, Thm. III.1.7.9] we recall that, the set $[Proj(M)]$ for a countably decomposable factor can be described as:

- $\{0, 1, \ldots, n\}$ if $M$ is of type I$_n$, $n \in \mathbb{N}$,
- $\{0, 1, 2, \ldots, \infty\}$ if $M$ is of type I$_\infty$,
- $[0, 1]$ if $M$ is of Type II$_1$.
- $[0, \infty]$ if $M$ is of Type II$_\infty$.
- $\{0, \infty\}$ if $M$ is of Type III.

This shows that only for type II$_\infty$, there is no a priori reason for $Aut(M)$ to preserve all equivalence classes of projections. Let $M$ be a factor of type II$_\infty$. Let $M_+$ be its cone of positive elements and assume that $\tau_0: M_+ \to [0, \infty]$ is a semi-finite faithful normal trace. Then, for any $P, Q \in Proj(M)$ with $\min\{\tau_0(P), \tau_0(Q)\} < \infty$, we have $\tau_0(P) = \tau_0(Q)$ if and only if $P \sim Q$ (see [Dix82, Part III, Ch. 2, § 7, Prop. 13(iii)]). The trace $\tau_0$ is unique up to multiplication by a positive scalar by [Dix82, Part I, Ch. 6, §4, Cor.], hence there exists a group homomorphism $\mu: Aut(M) \to \mathbb{R}_+^\times$ depending on $\tau_0$, with $\tau_0 \circ \theta = \mu(\theta)\tau_0$ for every $\theta \in Aut(M)$. Thus, if $\theta_0 \in Aut(M)$ satisfies $\mu(\theta_0) \neq 1$, then for every $P \in Proj(M) \setminus \{0\}$ with $\tau_0(P) < \infty$ we have $\tau_0(\theta_0(P)) \neq \tau_0(P)$, hence $\theta_0(P) \not\sim P$. Specific examples of such automorphisms of factors of type II$_\infty$ occur in connection with the structure of factors of type III$_1$; see [Ta03, Ch. XII, Th. 1.1(ii) and Def. 1.5(iii)].
particular, the hyperfinite factor $\mathcal{R}_{0,1}$ of type $\text{II}_\infty$ admits automorphisms $\theta_0$ as above, because $\mathcal{R}_{0,1}$ is involved in the decomposition of the hyperfinite factor of type $\text{III}_1$ as the crossed product of a $W^*$-dynamical system $(\mathcal{R}_{0,1}, \mathbb{R}, \alpha)$.

It is easy to construct a concrete example of such automorphisms. We consider the hyperfinite type II$_1$-factor $\mathcal{N} = \bigotimes_{n \in \mathbb{N}} M_2(\mathbb{C})$. For the infinite dimensional Hilbert space $\mathcal{H}$, the tensor product $\mathcal{M} := \mathcal{B}(\mathcal{H}) \otimes \mathcal{N}$ is then a factor of type $\text{II}_\infty$. From any unitary operator $U : \mathcal{H} \to \mathcal{H} \otimes \mathcal{H}$, we obtain an isomorphism

$$\Phi_0 : \mathcal{B}(\mathcal{H}) \to \mathcal{B}(\mathcal{H} \oplus \mathcal{H}) \cong \mathcal{B}(\mathcal{H}) \otimes M_2(\mathbb{C}), \quad \Phi_0(A) = UAU^{-1}.$$ 

Now

$$\Phi : \mathcal{M} \to \mathcal{M}, \quad \Phi(A \otimes B) := \Phi_0(A) \otimes B, \quad B \in \mathcal{N}$$

is an automorphism of $\mathcal{M}$. On $\mathcal{M}$ we consider the tensor product trace $\tau = \text{tr} \otimes \tau_\mathcal{N}$, where $\tau_\mathcal{N}$ is the normalized trace on $\mathcal{N}$. For a minimal projection $P$ on $\mathcal{H}$, we have

$$\tau(\Phi(P \otimes 1)) = \tau(UPU^{-1} \otimes 1) = (\text{tr} \otimes \tau_{M_2(\mathbb{C})})(UPU^{-1}) = \frac{1}{2} \text{tr}(UPU^{-1}) = \frac{1}{2} \text{tr}(P) = \frac{1}{2} \tau(P \otimes 1).$$

This means that $\mu(\Phi) = \frac{1}{2}$.

## 4 Spectral theory for covariant representations

In this section we will assume that $G = \mathbb{R}$ for simplicity, i.e. we have the one-parameter case. The Arveson spectrum is defined for any locally compact abelian group.

### 4.1 Arveson spectrum and spectral conditions

**Definition 4.1.** For a covariant representation $(\pi, U)$ of $(A, \mathbb{R}, \alpha)$ on $\mathcal{H}$ we have $U_t = \exp(-itH)$, $t \in \mathbb{R}$, for some selfadjoint operator $H$ on $\mathcal{H}$. In this case, for a subset $C \subseteq \mathbb{R}$, a $C$-spectral condition will mean that the spectrum $\text{Spec}(H)$ is contained in $C$. We will mostly be interested in the case that $C = [0, \infty)$, i.e. $H \geq 0$, in which case we will call $U : \mathbb{R} \to U(\mathcal{H})$ positive. A covariant representation $(\pi, U) \in \text{Rep}(\alpha, \mathcal{H})$ will be called positive if $U : \mathbb{R} \to U(\mathcal{H})$ is positive.

[Bo84] seems to be the first paper where the spectrum condition is studied in a context where $\alpha$ is not strongly continuous. Note that by adding a real multiple of the identity to $H$ we can trivially convert a positive unitary one-parameter group to one satisfying a $[\lambda, \infty)$-spectral condition, for any $\lambda \in \mathbb{R}$. So the important property here is that $H$ is bounded below. However, by the next Proposition, this property need not hold for all implementing unitary groups.

**Proposition 4.2.** Let $(U_t)_{t \in \mathbb{R}}$ be a positive strongly continuous unitary one-parameter group in the von Neumann algebra $\mathcal{M} \subseteq \mathcal{B}(\mathcal{H})$. Then $\mathcal{M}'$ is finite dimensional if and only if for any strongly continuous unitary one-parameter group $(W_t)_{t \in \mathbb{R}} \subset \mathcal{M}'$ the spectrum of the one-parameter group $(U_t W_t)_{t \in \mathbb{R}}$ is also bounded from below.

**Proof.** It is clear that if $\mathcal{M}'$ is finite dimensional, then the right hand side follows. We prove the converse.

(a) We first deal with the special case where $(U_t)_{t \in \mathbb{R}} \subset \mathcal{M}$ is norm continuous. Thus $U_t = \exp(-itH)$ where $H \in \mathcal{M}$ and $H \geq 0$. Let $(W_t)_{t \in \mathbb{R}} \subset \mathcal{M}'$ be a strongly continuous unitary one-parameter group, hence $W_t = \exp(-itB)$ for $B$ a selfadjoint operator, possibly unbounded. Then $U_t W_t = \exp(-it(H + B))$, and the assumption is that $\text{Spec}(H + B)$ is bounded from below. If $E$ is the spectral measure of $B$, then the subspaces $E[n, n+1]\mathcal{H}$, $n \in \mathbb{Z}$ are all preserved by $H$ and $B$. 

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and \( H + B \) restricted to such a subspace has spectrum in \([n, n + 1 + \Vert H \Vert]\). Thus if \( \text{Spec}(H + B) \) is bounded from below, then there is a \( K \) such that \( E[n, n + 1] = 0 \) for \( n < K \). Hence \( \text{Spec}(B) \) is bounded from below. Thus the spectrum of every strongly continuous one-parameter group \((W_t)_{t \in \mathbb{R}} \subset \mathcal{M}' \) is bounded from below. Since this also applies to \((W_t)_{t \in \mathbb{R}} \), it follows that \((W_t)_{t \in \mathbb{R}} \) is norm continuous.

If all strongly continuous unitary one-parameter groups in \( \mathcal{M}' \) are norm continuous, then every orthogonal family of projections in \( \mathcal{M}' \) must be finite (or else from an infinite sequence of projections in \( \mathcal{M}' \) we can define an unbounded selfadjoint operator which generates a one-parameter unitary group in \( \mathcal{M}' \) which is not norm continuous). Thus \( \mathcal{M}' \) is finite dimensional by [Og54] (see also Lemma [A.4]).

(b) Now we turn to the general case. For \( a < b \), let \( P[a, b) \) denote the corresponding spectral projection of \( U \). Then the subspace \( \mathcal{H}[a, b) := P[a, b) \mathcal{H} \) is invariant under \( \mathcal{M}' \) and \( U \), and since the restriction of \( U \) to \( \mathcal{H}[a, b) \) is norm continuous, (a) implies that the subalgebra \( \mathcal{M}'[a, b) := P[a, b) \mathcal{M}' \) of \( \mathcal{M}' \) is finite dimensional.

Let \( Z_j \in \mathcal{M}' \) be the central support of \( \mathcal{M}'[0, j), j \in \mathbb{N}_0 \). If the set \( \{Z_j: j \in \mathbb{N}_0\} \) is infinite, then there exists a subsequence \((Z_{j_k})_{k \in \mathbb{N}}\) for which \( Q_k := Z_{j_{k+1}} - Z_{j_k} \neq 0 \). Then \( B := \sum_{k=1}^{\infty} j_k^2 Q_k \) has the property that \( H - B \) is not bounded from below. Hence there are only finitely many \( Z_j \). In particular, there is a maximal one \( Z_N \) which must be \( 1 \). Therefore the representation of \( \mathcal{M}' \) on \( \mathcal{H}[0, N] \) is faithful, and this implies that \( \mathcal{M}' \) is finite dimensional.

Thus in general, given one positive implementing unitary group \((U_t)_{t \in \mathbb{R}} \) of an action \( \alpha: \mathbb{R} \to \text{Aut}(\mathcal{M}) \), then other implementing unitary groups need not have generators bounded from below, except if \( \mathcal{M}' \) is finite dimensional.

We will follow the convention of [BR02] that a unitary one-parameter group \((U_t)_{t \in \mathbb{R}} \) is related to its spectral measure \( E \) by

\[
U_t = e^{-itH} = \int_\mathbb{R} e^{-itp} dE(p) \quad \text{where} \quad H = \int_\mathbb{R} p dE(p).
\]

In this picture, for \( f \in L^1(\mathbb{R}) \) we have

\[
U_f = \int_\mathbb{R} f(t)U_t dt = \int_\mathbb{R} \int_\mathbb{R} e^{-itp} f(t) dt dE(p) = \int_\mathbb{R} \hat{f}(p) dE(p) = \hat{f}(H) .
\]

(14)

Thus if \( H \geq 0 \) then \( U_f = 0 \) whenever \( \text{supp} \hat{f} \subset (-\infty, 0) \).

Given a covariant representation \((\pi, U)\), there are two spectral theories which we will use: that of \( U \) (i.e. of \( H \)), and the Arveson spectral theory for \( \alpha \) (cf. [Arv74]). The relation between them will be made explicit. Arveson’s spectral theory was motivated by the search for a constructive proof of Borchers’ theorem (cf. Theorem [A.1] below; see [Ta03, Ch. XI]). We first define the Arveson spectral subspaces \( \mathcal{M}^\alpha(S) \) (cf. [BR02, Def 3.2.37]- this can be done for any locally compact abelian group):

**Definition 4.3.** Let \((\mathcal{M}, \mathcal{R}, \alpha)\) be a \( W^*\)-dynamical system on a von Neumann algebra \( \mathcal{M} \subseteq \mathcal{B}(\mathcal{H}) \). For \( f \in L^1(\mathbb{R}) \), we write

\[
\alpha_f(A) := \int_\mathbb{R} f(t)\alpha_t(A) dt, \quad A \in \mathcal{M}
\]

for the corresponding integrated representation (cf. [Pe89, Lemma 7.5.1]), where \( \alpha_f(A) \) is a weak integral with respect to the weak operator topology. We define

1. the **spectrum** of an \( A \in \mathcal{M} \) with respect to \( \alpha \) as

\[
\text{Spec}_\alpha(A) := \{ p \in \mathbb{R} \mid (\forall f \in L^1(\mathbb{R})) \alpha_f(A) = 0 \Rightarrow \hat{f}(p) = 0 \},
\]

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where \( \hat{f}(p) = \int_{\mathbb{R}} e^{-i xp} f(x) \, dx \) is the Fourier transform. Then the Arveson spectrum of \( \alpha \), denoted \( \text{Spec}(\alpha) \), is the closure of the union of the sets \( \text{Spec}_a(A) \) for all \( A \in \mathcal{M} \). (This agrees with the generalization to arbitrary locally compact groups in [10] above. Useful equivalent definitions are listed in [BR02 Prop. 3.2.40]).

(2) For a subset \( S \subseteq \mathbb{R} \), the Arveson spectral subspace of \( \alpha \) is

\[
\mathcal{M}^a(S) := \{ A \in \mathcal{M} \mid \text{Spec}_a(A) \subseteq S \},
\]

where the closure is with respect to the \( \sigma(\mathcal{M}, \mathcal{M}_a) \)-topology. The subspace

\[
\mathcal{M}_0^a(S) := \text{span} \{ \alpha f(A) \mid A \in \mathcal{M}, f \in L^1(\mathbb{R}) \text{ such that } \text{supp}(\hat{f}) \subseteq S \}
\]

is contained in \( \mathcal{M}^a(S) \) and, if \( S \) is open, then \( \mathcal{M}^a(S) = \mathcal{M}_0^a(S) \) (cf. [BR02 Lemma 3.2.39(4)]).

By the definition of the Arveson spectrum \( \text{Spec}(\alpha) \), and the fact that \( \text{Spec}_a(A^*) = -\text{Spec}_a(A) \) [BR02 Prop. 3.2.42(1)], it follows that \( \text{Spec}(\alpha) \) is a symmetrical set.

The basic algebraic structure of the Arveson spectral spaces for \((\mathcal{M}, \mathbb{R}, \alpha)\) which we will need is:

1. \( \mathcal{M}^a(S)^* = \mathcal{M}^a(-S) \) for all subsets \( S \subseteq \mathbb{R} \) (cf. [BR02 Lemma 3.2.42(2)]),

2. \( \mathcal{M}^a(S_1) \mathcal{M}^a(S_2) \subseteq \mathcal{M}^a(S_1 \cup S_2) \) for all closed subsets \( S_1, S_2 \subseteq \mathbb{R} \) (cf. [BR02 Lemma 3.2.42(4)]).

3. The union of the spaces \( \mathcal{M}^a[t, \infty) \) for \( t \in \mathbb{R} \) is weak operator dense in \( \mathcal{M} \) (cf. Lemma 4.20(1) below).

The space \( \mathcal{M}^a(\{0\}) = \mathcal{M}^R \) is the von Neumann algebra of invariant elements, and if \( U : \mathbb{R} \to U(H) \) is a strong operator continuous unitary implementing group for \( \alpha \), then clearly \( \mathcal{M}^a(\{0\}) = U_a^\prime \cap \mathcal{M} \).

If \( U_R \subset \mathcal{M} \) then \( U_R^\prime \subset \mathcal{M}^a(\{0\}) \).

The Arveson spectral spaces determine uniquely the action \( \alpha : \mathbb{R} \to \text{Aut}(\mathcal{M}) \) by the following (cf. [BR02 Prop. 3.2.44]):

**Proposition 4.4.** Let \((\mathcal{M}, \mathbb{R}, \alpha)\) and \((\mathcal{M}, \mathbb{R}, \beta)\) be two \( W^* \)-dynamical systems on a von Neumann algebra \( \mathcal{M} \subseteq B(H) \) such that

\[ \mathcal{M}^a[t, \infty) \subseteq \mathcal{M}^\beta[t, \infty) \quad \text{for} \quad t \in \mathbb{R}. \]

Then \( \alpha_t = \beta_t \) for all \( t \in \mathbb{R} \).

One can obtain the Arveson spectral spaces from the spectral projections \( E[t, \infty) \) of a unitary group implementing \( \alpha \) by

\[
\mathcal{M}^a[t, \infty) = \{ A \in \mathcal{M} \mid (\forall s \in \mathbb{R}) \; A E[s, \infty) H \subseteq E[s + t, \infty) H \}
\]

(cf. [BR02 Lemma 3.2.39(3), Prop. 3.2.43]). Such an implementing unitary group will exist if we choose e.g. \( \mathcal{M} = \mathcal{M}_{co} \) as above for a given action \((\mathcal{A}, \mathbb{R}, \alpha)\). This suggests that \( \mathcal{M}^a[t, \infty) \) consists of “shift operators,” and indeed, we can write \( \mathcal{M} \) in terms of “matrix” expansions w.r.t \( E \) (or equivalently \( U(C^*(\mathbb{R})) = U_{L^1(\mathbb{R})} \)), and characterize the Arveson spectral subspaces \( \mathcal{M}^a(S) \) in these terms:

**Example 4.5.** In the case that the generator \( H \) of \( U \) has spectrum only in \( \mathbb{Z} \), (15) above shows that with respect to the matrix decomposition of \( A \) with respect to the eigenspaces of \( H \), an \( A \in \mathcal{M}^a[t, \infty) \) must consist of an upper triangular (infinite) matrix, cf. [GrN14 Rem. C.4].
Specifically, let $\alpha : \mathbb{R} \to \text{Aut} \mathcal{B}(\mathcal{H})$ be the conjugation $\alpha_t(A) = U_t A U_{-t}$, where $U_{2\pi} = 1$, so that it actually defines a representation of the circle group $T \cong \mathbb{R}/2\pi \mathbb{Z}$. Denote by

$$\mathcal{B}(\mathcal{H})_n := \{ A \in \mathcal{B}(\mathcal{H}) \mid (\forall t \in \mathbb{R}) \ \alpha_t(A) = e^{int} A \}$$

its eigenspaces in $\mathcal{B}(\mathcal{H})$ and similarly let $\mathcal{H}_n$ be the eigenspace of $U$ in $\mathcal{H}$ with the projection $P_n$ onto it. Note that $\mathcal{B}(\mathcal{H})_n = \mathcal{B}(\mathcal{H})^\alpha \{ n \}$, i.e. it coincides with the Arveson spectral subspace for $\{ n \}$. The Peter–Weyl Theorem generalizes to continuous Banach representations of $G$ (cf. [Sh55, Thm. 2] and [HM13, Thm. 3.51]), hence an application of it to $\alpha \mid \mathcal{B}(\mathcal{H})_c$ implies that

$$\mathcal{B}(\mathcal{H})_c = \overline{\text{span} \left( \bigcup_{n \in \mathbb{Z}} \mathcal{B}(\mathcal{H})_n \right)}.$$  \hfill (16)

Write $A = (A_{jk})_{j,k \in \mathbb{Z}}$ as a matrix with $A_{jk} \in \mathcal{B}(\mathcal{H}_k, \mathcal{H}_j)$, and keep in mind that the convergence $A = \sum_{j \in \mathbb{Z}} \sum_{k \in \mathbb{Z}} A_{jk} = \sum_{j \in \mathbb{Z}} \sum_{k \in \mathbb{Z}} P_j A P_k$ is in general with respect to the strong operator topology. We have

$$\alpha_t(A) = (e^{it(j-k)} A_{jk})_{j,k \in \mathbb{Z}},$$

so that

$$A \in \mathcal{B}(\mathcal{H})_n \iff (j - k \neq n \Rightarrow A_{jk} = 0).$$

For $A = (A_{jk})_{j,k \in \mathbb{Z}} \in \mathcal{B}(\mathcal{H})$, let $A_n := (A_{jk} \delta_{j-k,n})_{j,k \in \mathbb{Z}}$ and observe that $A_n$ defines a bounded operator on $\mathcal{H}_n$, hence an element of $\mathcal{B}(\mathcal{H})_n$. In fact, all elements of the Arveson spectral space $\mathcal{B}(\mathcal{H})^\alpha \{ n \}$ must be of this type, i.e. consist of a single diagonal in the $n$th position above the main diagonal. As $\mathcal{B}(\mathcal{H})^\alpha [t, \infty)$ is the strong operator closed span of all $\mathcal{B}(\mathcal{H})^\alpha \{ n \}$ with $n \geq t$, we see that the matrix decomposition of an $A \in \mathcal{B}(\mathcal{H})^\alpha [t, \infty)$ consists of upper triangular matrices for which the $n$th diagonal is zero if $n < t$.

Consider the invariance subalgebra $\mathcal{B}(\mathcal{H})_0 = \mathcal{B}(\mathcal{H})^\alpha \{ 0 \}$, which we note from the matrix decomposition must consist of elements of the form $A = \sum_{k \in \mathbb{Z}} A_{kk} = \sum_{k \in \mathbb{Z}} P_k A P_k$ (strong operator convergence). We may therefore define a projection $p_0 : \mathcal{B}(\mathcal{H}) \to \mathcal{B}(\mathcal{H})_0$ onto the invariant algebra by

$$p_0(A) := \sum_{k \in \mathbb{Z}} P_k A P_k \in \mathcal{B}(\mathcal{H})_0 \quad \text{for} \quad A \in \mathcal{B}(\mathcal{H}).$$

As the maps $A \to P_k A P_k$ are completely positive, it is clear that $p_0$ is a strong operator limit of completely positive maps (the finite partial sums) hence it is completely positive. It coincides with the usual group-averaging projection onto $\mathcal{B}(\mathcal{H})_0$ by:

$$\int_T \alpha_z(M) \, dz = p_0 \left( \int_T \alpha_z(M) \, dz \right) = \sum_{k \in \mathbb{Z}} P_k \int_T \alpha_z(M) \, dz P_k$$

$$= \sum_{k \in \mathbb{Z}} \int_T \alpha_z(P_k M P_k) \, dz = \sum_{k \in \mathbb{Z}} P_k M P_k = p_0(M).$$

In this example, we obtained a completely positive projection $p_0 : \mathcal{B}(\mathcal{H}) \to \mathcal{B}(\mathcal{H})_0$. By applying the Stinespring Dilation Theorem (or more precisely the generalized GNS construction in its proof), any representation $(\rho, \mathcal{K})$ of $\mathcal{B}(\mathcal{H})_0$ leads to a new representation $(\rho, \mathcal{K})$ of $\mathcal{B}(\mathcal{H})$ with $\mathcal{K}_0 \subseteq \mathcal{K}$ for which $p_0(p_0(A)) = P^* \rho(A) P$ holds for the orthogonal projection $P : \mathcal{K} \to \mathcal{K}_0$ (cf. [Ta02, Thm IV.3.6]). The question now arises whether we have such a map $p_0$ in the general case. In fact we do by the following (cf. [EW74, Lemma 1.4]):

**Proposition 4.6.** Let $(\mathcal{M}, \mathbb{R}, \alpha)$ be $W^*$-dynamical system for a von Neumann algebra $\mathcal{M} \subseteq \mathcal{B}(\mathcal{H})$ and let $\eta$ be an invariant mean on $C_b(\mathbb{R})$. For each $M \in \mathcal{M}$ define $\hat{\eta}M \in \mathcal{M} = (\mathcal{M}_*)^*$ by

$$(\hat{\eta}M)(\varphi) := \eta(\varphi(\alpha^M)) \quad \text{for all} \quad \varphi \in \mathcal{M}_* \quad \text{and} \quad \alpha^M(t) := \alpha_t(M).$$

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Then the map $\hat{\eta} : \mathcal{M} \to \mathcal{M}$ is an $\alpha_\mathbb{R}$-invariant conditional expectation onto the fixed point algebra $\mathcal{M}^\alpha(\{0\}) = \mathcal{M}_\mathbb{R}$.

As conditional expectations are completely positive, it follows that the maps $\hat{\eta} : \mathcal{M} \to \mathcal{M}_\mathbb{R}$ are always completely positive (cf. [NTU60]). Under specific additional assumptions, the maps $\hat{\eta}$ can even be independent of the choice of $\eta$ (cf. [EW74]). Moreover, if the completely positive map is normal, then there is a normal version of the Stinespring Theorem which guarantees that the new representation must be normal (cf. [Blu06] Thm III.2.2.4). We note however that there may exist no invariant mean $\eta$ on $C_0(\mathbb{R})$ for which the map $\hat{\eta} : \mathcal{M} \to \mathcal{M}$ from Proposition 4.6 is normal, as the following example shows:

**Example 4.7.** Let $\mathcal{H} = L^2(\mathbb{R})$, $\mathcal{M} = B(\mathcal{H})$, and for every $f \in L^\infty(\mathbb{R})$ let $M_f \in \mathcal{M}$ be the operator defined by multiplication by $f$. Also, for every $t \in \mathbb{R}$, let $\chi_t \in L^\infty(\mathbb{R})$ be given by $\chi_t(x) := e^{itx}$ for all $x \in \mathbb{R}$. Defining $\alpha_t(A) := M_{\chi_t} AM_{\chi_t}^*$ for all $A \in \mathcal{M}$ and $t \in \mathbb{R}$, we claim that $(\mathcal{M}, \mathbb{R}, \alpha)$ is a $W^*$-dynamical system with the property that, for every invariant mean $\eta$ on $C_0(\mathbb{R})$, the conditional expectation $\hat{\eta}$ fails to be normal. In fact, as Proposition 4.6 shows that $\hat{\eta}$ is a conditional expectation onto $\mathcal{M}^\alpha(\{0\})$, it suffices to check that there exists no normal conditional expectation from $B(\mathcal{H})$ onto $\mathcal{M}^\alpha(\{0\})$. To this end, first note that $\mathcal{M}^\alpha(\{0\}) = \{ M_{\chi_t} | t \in \mathbb{R} \}'$. As the $\sigma$-algebra of Borel subsets of $\mathbb{R}$ is the smallest one for which all functions $\chi_t$ are measurable, they generate the von Neumann algebra $\mathcal{D} := L^\infty(\mathbb{R})$ by Corollary 4.2. As $L^\infty(\mathbb{R})$ is a maximal abelian self-adjoint subalgebra of $\mathcal{M}$ (see for instance [Dix82] Part I, Ch. 7, no. 3, Th. 2), it follows that $\mathcal{M}^\alpha(\{0\}) = \mathcal{D}' = \mathcal{D}$.

On the other hand, for every conditional expectation $E : B(\mathcal{H}) \to \mathcal{D}$ one has $K(\mathcal{H}) \subseteq \ker E$ by [KS59] Rem. 5], hence $E$ cannot be $\sigma$-weakly continuous, because $K(\mathcal{H})$ is $\sigma$-weakly dense in $B(\mathcal{H})$. This shows that our claim above holds true.

**Remark 4.8.** (i) This example can be easily generalized to $\mathcal{H} = L^2(G)$ for any non-discrete locally compact abelian group $G$ instead of $\mathbb{R}$, using the same averaging procedure (see also [BP07]). If $G$ is a discrete abelian group, its dual $\hat{G}$ is a compact abelian group and one has a normal conditional expectation from $B(\mathcal{H})$ onto its maximal abelian subalgebra consisting of the multiplications operators by functions in $L^\infty(G) = \ell^\infty(G)$, just as in the special case discussed in Example 4.5, where $G = \mathbb{Z}$ and $\hat{\mathbb{Z}} = \mathbb{T}$.

(ii) By the Kovacs & Szücs Theorem (cf. [BR02] Prop. 4.3.8, p. 383]), the statement of Proposition 4.6 can be strengthened to give a normal invariant conditional expectation. For this, we need to assume in addition, that the subspace of invariant vectors is $\mathcal{M}$-generating, and that the given representation $\mathcal{M} \subseteq B(\mathcal{H})$ is covariant for $\alpha$.

In the case that we have a representation in standard form, the connection between the Arveson spectrum of the $W^*$-dynamical system $(\mathcal{M}, \mathbb{R}, \beta)$ and the one-parameter group of unitary implementers (cf. Proposition 3.8) is more direct:

**Proposition 4.9.** For any $W^*$-dynamical system $(\mathcal{M}, \mathbb{R}, \beta)$ such that $\mathcal{M}$ has an invariant faithful normal weight, then in the standard form representation of $\mathcal{M}$, the Arveson spectrum $\text{Spec}(\beta)$ coincides with the spectrum of the one-parameter group $U : \mathbb{R} \to U(\mathcal{H})_{\mathcal{M}}$ which implements $\beta$.

Recall Proposition 3.8 which follows from the fact that $\text{Aut}(\mathcal{M}) \cong U(\mathcal{H})_{\mathcal{M}}$ in any standard form realization of $\mathcal{M}$. The proof of Proposition 4.9 is in [Ta03] Prop. XI.1.24]. Note that by uniqueness of the standard form, the existence of an invariant faithful weight (or an invariant faithful normal state) is enough.

**Remark 4.10.** As a selfadjoint operator $A$ on a Hilbert space $\mathcal{H}$ has a division of its spectrum

$$\text{Spec}(A) = \text{Spec}_{pp}(A) \cup \text{Spec}_{ac}(A) \cup \text{Spec}_{sing}(A)$$

with decomposition $\mathcal{H} = \mathcal{H}_{pp} \oplus \mathcal{H}_{ac} \oplus \mathcal{H}_{sing}$
one may look for a similar decomposition of the Arveson spectrum of a $C^*$-action, and to relate this to the decomposition of the spectrum of its implementing groups. This has indeed been done for the $C^*$-dynamical case with additional assumptions (cf. [Dy10]), but thus far not for our case.

We also have:

Lemma 4.11. Let $(\mathcal{M}, \mathcal{H}, J, C)$ be a standard form realization and $(\beta_t)_{t \in \mathbb{R}}$ be a strongly continuous one-parameter group of $U(\mathcal{H})_\mathcal{M} \cong \text{Aut}(\mathcal{M})$. Then the following assertions hold:

(i) If $\beta$ is implementable on $\mathcal{M}$ by a unitary one-parameter group $(U_t)_{t \in \mathbb{R}}$ in $U(\mathcal{M})$ and $V_t := JU_tJ$ is the corresponding one-parameter group of $U(\mathcal{M}')$, then $\beta_t = U_tV_t$ for all $t \in \mathbb{R}$.

(ii) If $\text{Spec}(U) \subseteq [0, \infty)$, then $\text{Spec}(V) \subseteq (-\infty, 0]$ and the factorization of $\beta$ corresponds to the factorization into the negative and positive spectral part.

(iii) If $\beta_t = e^{-itH}$, then $JHJ = -H$. In particular, the spectrum of $H$ is symmetric.

Proof. (i) Since $U_t$ implements the conjugation with $\beta_t$, both commute for every $t$. The same holds for $V_t$ because $\beta_t M \beta_t^{-1} = V_t M V_t^{-1}$ for $t \in \mathbb{R}, M \in \mathcal{M}'$ follows from $J\beta_t = \beta_t J$. Therefore $W_t := U_tV_t$ is a strongly continuous one-parameter group of $U(\mathcal{H})$. It satisfies $JW_tJ = V_tU_tU_tV_t = W_t$. Further $Z_t := \beta_t W_t^{-1}$ commutes with $\mathcal{M}$ and $\mathcal{M}'$, hence is contained in the center of $\mathcal{M}$. We conclude that $Z_t = JZ_tJ = Z_t^* = Z_t^{-1}$, and thus $Z_t^2 = 1$, which in turn implies that $Z_t = 1$.

(ii) is clear from the definitions.

(iii) follows immediately from $J\beta_tJ = \beta_t$ (Remark 3.7) because $J$ is antilinear. 

Proposition 4.12. For any $W^*$-dynamical system $(\mathcal{M}, \mathbb{R}, \alpha)$, the subspace $\mathcal{M}_c \subseteq \mathcal{M}$ is the closed subalgebra generated by the elements with bounded Arveson spectrum.

Proof. Every element $M$ with bounded spectrum lies in a closed subspace on which the action is norm continuous ([BR02 Prop. 3.2.41]), so that $M \in \mathcal{M}_c$. Conversely, hitting an element $M \in \mathcal{M}_c$ with an approximate identity $(u_n)_{n \in \mathbb{N}}$ of $L^1(\mathbb{R})$ for which the supports $\text{supp}(\widehat{u}_n)$ are compact leads to elements $\alpha_{u_n}(M)$ with bounded spectrum converging to $M$. 

4.2 The Borchers–Arveson Theorem and minimal implementing groups.

We first consider an easily proven result which shows a connection between spectral properties and innerness of covariant representations. For a locally compact abelian group $G$ and a continuous unitary representation $(U, \mathcal{H})$ of $G$, we write $\text{Spec}(U) \subseteq \hat{G}$ for its spectrum, i.e., the support of the corresponding spectral measure on $\hat{G}$.

Lemma 4.13. (Longo’s Lemma) Let $(U, \mathcal{H})$ be a continuous unitary representation of the abelian locally compact group $G$ and $\mathcal{M} \subseteq B(\mathcal{H})$ be a von Neumann algebra normalized by $U_G$. Suppose that

(i) there exists an $\mathcal{M}$-cyclic unit vector $\Omega$ fixed by $U_G$, and that

(ii) $\text{Spec}(U) \cap \text{Spec}(U)^{-1} \subseteq \{e\}$.

Then $U_G \subseteq \mathcal{M}$. 

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Proof. We consider the action of $G$ on the commutant $\mathcal{M}'$ defined by $\beta_g(M) := U_gMU_g^*$. We have to show that $\beta$ is trivial; then $U_G \subseteq \mathcal{M}' = \mathcal{M}$. As $\Omega$ is cyclic for $\mathcal{M}$, it is separating for $\mathcal{M}'$. Let $\mathcal{E} := [\mathcal{M}'\Omega]$, with projection $E : H \to \mathcal{E}$ onto it, and note that $U_G$ preserves $\mathcal{E}$. As $\mathcal{E} \ni \Omega$ is $\mathcal{M}$-generating, it follows from Lemma 3.14(iv) that the restriction map $\mathcal{M}' \to \mathcal{M}' \mid \mathcal{E}$ is an isomorphism. Thus it suffices to prove that the $W^*$-dynamical system $(\mathcal{M}'E, G, \beta^E)$ is trivial, where $\beta^E_g := \text{Ad} U^E_g$ and $U^E_g := U_g \mid \mathcal{E}$. As $\Omega$ is cyclic, separating and invariant for this $W^*$-dynamical system, it follows from [Ba03 Prop. XI.1.24] that $	ext{Spec}(U^E) = \text{Spec}(\beta^E)$. By $	ext{Spec}(U^E) \subseteq \text{Spec}(U)$, and condition (ii) we conclude that $\text{Spec}(\beta^E) \cap \text{Spec}(\beta^E)^{-1} \subseteq \{e\}$. However, the Arveson spectrum of an automorphic action is symmetric, i.e. $\text{Spec}(\beta^E) = \text{Spec}(\beta^E)^{-1}$ (Lemma 3.11(iii)), hence $\text{Spec}(\beta^E) = \{e\}$, i.e. $\beta^E$ is trivial.

Note that the preceding lemma applies in particular to positive covariant representations of actions of $G = \mathbb{R}$.

For positive covariant representations of von Neumann algebras, we have the stronger, and very important Borchers–Arveson Theorem (cf. [BR02 Thm. 3.2.46]), which conversely, gives us a way of constructing the spectral projections of an implementing unitary group from the Arveson spectral subspaces.

**Theorem 4.14.** (Borchers–Arveson) Let $(\mathcal{M}, \mathbb{R}, \alpha)$ be a $W^*$-dynamical system on a von Neumann algebra $\mathcal{M} \subseteq B(H)$. Then the following are equivalent.

(i) There is a positive strong operator continuous unitary one-parameter group $U : \mathbb{R} \to U(H)$ such that $\alpha_t = \text{Ad} U_t$ on $\mathcal{M}$.

(ii) There is a positive strong operator continuous unitary one-parameter group $U : \mathbb{R} \to \mathcal{M}$ such that $\alpha_t = \text{Ad} U_t$ on $\mathcal{M}$.

(iii) Let $\mathcal{M}^\alpha(S)$ denote the Arveson spectral subspace for $S \subseteq \mathbb{R}$. Then

$$\bigcap_{t \in \mathbb{R}} [\mathcal{M}^\alpha[t, \infty) H] = \{0\}.$$ 

If these conditions hold, then we may take $U : \mathbb{R} \to \mathcal{M}$ to be $U_t = \int_\mathbb{R} e^{-itx} dP(x)$, where $P$ is the projection-valued measure uniquely determined by

$$P[t, \infty) H = \bigcap_{s < t} [\mathcal{M}^\alpha[s, \infty) H].$$

**Proof.** (i)$\Rightarrow$(iii): Let $P$ denote the projection valued measure of $U$. As $U$ is positive, $P[0, \infty) = 1$, hence, using (i), we obtain

$$\mathcal{M}^\alpha[t, \infty) H = \mathcal{M}^\alpha[t, \infty) P[0, \infty) H \subseteq P[t, \infty) H.$$ 

Thus, as $P$ is a projection-valued measure,

$$\bigcap_{t \in \mathbb{R}} [\mathcal{M}^\alpha[t, \infty) H] \subseteq \bigcap_{t \in \mathbb{R}} P[t, \infty) H = \{0\}$$

which proves (iii).

(iii)$\Rightarrow$(ii): In this proof we will let $[S] \in B(H)$ denote the orthogonal projection onto the space $[S]$. For $t \in \mathbb{R}$ define

$$Q_t := \left[ \bigcap_{t \in \mathbb{R}} [\mathcal{M}^\alpha[t, \infty) H] \right] \in B(H).$$

Then the map $t \mapsto 1 - Q_t$ is a spectral family, as it is an increasing, strongly left continuous map such that $1 - Q_t = 0$ if $t \leq 0$ and it increases strongly to $1$ as $t \to \infty$ (cf. [Wei80 Def. 7.11]).
Thus there is a unique projection valued measure $P$ such that $P(t,\infty) = Q_t$ for all $t \in \mathbb{R}$. As the subspaces $[M^\alpha[t,\infty)H]$ are invariant with respect to $M'$ their projections are in $M'' = M$ and hence $P(t,\infty) \in M$ for all $t \in \mathbb{R}$. Define

$$U_t := \int e^{-ip\beta}dP(p) \in M$$

then by $P(0,\infty) = 1$ it is positive. Define $\beta_t := \text{Ad}(U_t) \in \text{Aut}M$. As

$$M^\alpha[s,\infty)P(t,\infty)H = \bigcap_{t \in \mathbb{R}} [M^\alpha[s,\infty)M^\alpha[t,\infty)H] \subseteq \bigcap_{t \in \mathbb{R}} [M^\alpha[s+t,\infty)H] = P[s+t,\infty)H$$

we obtain from [15] that $M^\alpha[s,\infty) \subseteq M^\beta(s,\infty)$ for all $s \in \mathbb{R}$. Thus by Proposition 4.4 we get that $\alpha_t = \beta_t$.

(ii) $\Rightarrow$ (i) is trivial. \hfill \qed

**Remark 4.15.** (a) The theorem gives a sharp criterion stating when we have a positive covariant representation. It states that amongst the implementing positive unitary one-parameter groups, we can find one which is inner, and it selects one by construction. Hence by (ii) in Theorem 4.14, every normal representation of $M$ is covariant.

Moreover, given a positive covariant representation $(\pi,U)$ of a $C^*$-action $(A,G,\alpha)$, we can always find a new positive covariant representation $(\pi,V)$ such that its generator is affiliated with $\pi(A)$.

(b) An important consequence of the Borchers–Arveson Theorem 4.14 is that for any covariant representation $(\pi,U)$ of $(M,\mathbb{R},\alpha)$ for which $\pi$ is faithful and $U$ has positive spectrum, the action $\alpha$ is trivial on the center of $M$. Hence, $M$ must be non-commutative in order to admit non-trivial actions and positive covariant representations. Moreover, as any commutative $C^*$-subalgebra of $M$ preserved by the action $\alpha$ must be in $M^\mathbb{R}$, it follows that $\alpha$ cannot have any normal eigenvectors except for the identity eigenvalue. It seems that a $[0,\infty)$-spectral condition is a quantum mechanical phenomenon, which cannot occur in classical systems. It is now easy to give examples of actions for which there are covariant representations, but no positive covariant representations, e.g. the translation action of $\mathbb{R}$ on $C_0(\mathbb{R})$.

(c) The Borchers–Arveson Theorem has been generalized by Kishimoto to $\mathbb{R}^n$ [Ki79 Thm. 2], and further to connected locally compact abelian groups in [Pe89 Cor. 8.4.12].

Apart from the observations in Remark 4.14(b), the existence of a positive covariant representation places strong algebraic restrictions on the $C^*$-action $(A,G,\alpha)$. This is explored in Section 4.3.

By the Borchers–Arveson Theorem 4.14 if we have an implementing positive unitary one-parameter group $U: \mathbb{R} \to U(H)$, we may take it to be inner, and then $U(C^*(\mathbb{R})) \subset U''_\mathbb{R} \subset M^\alpha\{0\} = M^\mathbb{R}$. Above we saw that the Arveson spectral subspaces can be written in terms of “matrix decompositions” with respect to $C^*(\mathbb{R})$ (cf. Example 4.3 and preceding discussion). Thus the subalgebra $M^\alpha\{0\}$ already contains the spectral information of $(M,\mathbb{R},\alpha)$ because it contains all the spectral projections of $U$.

**Definition 4.16.** Let $M \subseteq B(H)$ be a von Neumann algebra, and $(M,\mathbb{R},\alpha)$ be a $W^*$-dynamical system.

(a) Let $(U_t)_{t \in \mathbb{R}} \subset M$ be a weakly continuous unitary one-parameter group with non-negative spectrum implementing $\alpha$. We say that $(U_t)_{t \in \mathbb{R}}$ is minimal if, for all other one-parameter groups $(\tilde{U}_t)_{t \in \mathbb{R}} \subset B(H)$ with non-negative spectrum implementing the same automorphisms, i.e. $\text{Ad}(U_t) = \text{Ad}(\tilde{U}_t) = \alpha_t$ for $t \in \mathbb{R}$, the corresponding one-parameter group $Z_t := \tilde{U}_tU_t^*$ has non-negative spectrum. A minimal one-parameter group is clearly unique, if it exists.
(b) The set of projections \((Q_t)_{t \in \mathbb{R}} \subseteq \mathcal{B(H)}\) defined by

\[
Q_t \mathcal{H} := \bigcap_{s < t} \|\mathcal{M}^\alpha[s, \infty)\mathcal{H}\|
\]

are called Borchers–Arveson projections. We also put

\[
Q_\infty := \lim_{t \to -\infty} Q_t
\]

and observe that the limit exists because \(Q_s \leq Q_t\) for \(s \geq t\). (Note that \(Q_t \in \mathcal{M} \ni Q_\infty\) by the bicommutant theorem, as \(\mathcal{M}'\) preserves \(\|\mathcal{M}^\alpha[s, \infty)\mathcal{H}\|\).)

If \(\mathcal{H}_\infty := Q_\infty \mathcal{H} = \{0\}\), then the unitary one-parameter group \((U_t)_{t \in \mathbb{R}} \subseteq \mathcal{M}\) whose spectral measure \(P\) is determined by \(P[t, \infty) = Q_t\) for \(t \in \mathbb{R}\) is called the Borchers–Arveson group for the \(W^*\)-dynamical system \((\mathcal{M}, \mathbb{R}, \alpha)\) (cf. Theorem 4.14).

In terms of its generator, the unitary group \(U_t = \exp(-itH) \in \mathcal{M}, H \geq 0\), is minimal if for all other strongly continuous unitary one-parameter group implementing the same automorphisms, i.e. \(\text{Ad}(U_t) = \text{Ad}(\tilde{U}_t)\) for \(t \in \mathbb{R}\), we have \(\tilde{H} \geq H\).

The first part of the following lemma is \cite{Arv74} Prop. p. 235).

**Lemma 4.17.** Suppose that \(Q_{\infty} = 0\). Then the Borchers–Arveson subgroup \((U_t)_{t \in \mathbb{R}}\) in \(\mathcal{M}\) is minimal and, for every \(\varepsilon > 0\) the projection \(P[0, \varepsilon) = Q_0 - Q_\varepsilon \in \mathcal{M}\) has central support \(1\).

**Proof.** From the formula

\[
P(t, \infty) = \bigcup_{s < t} \|\mathcal{M}^\alpha[s, \infty)\mathcal{H}\|
\]

for the spectral projections of \(U\), one derives as follows that \(P[0, \varepsilon) \neq 0\) for any \(\varepsilon > 0\): If \(m := \inf \text{Spec}(U)\) and \(0 < s < \varepsilon\), then \(\mathcal{H} = \mathcal{H}^{U[m, \infty)}\), so that

\[
P[\varepsilon, \infty) \subseteq \mathcal{M}^\alpha[s, \infty)\mathcal{H} = \mathcal{M}^\alpha[s, \infty)\mathcal{H}^{U[m, \infty)} \subseteq \mathcal{H}^{U[m + s, \infty)},
\]

is a proper subspace of \(\mathcal{H}\). Since this remains valid for every subrepresentation, the central support of the projections \(P[0, \varepsilon), \varepsilon > 0\), is \(1\).

To see that \(U\) is minimal (\cite{Bo96} Thm. II.4.6] or \cite{Arv74} Prop. p. 235), let \((\tilde{U}_t)_{t \in \mathbb{R}}\) be another strongly continuous unitary one-parameter group implementing the same automorphisms, i.e. \(\text{Ad}(U_t) = \text{Ad}(\tilde{U}_t)\). As \((U_t)_{t \in \mathbb{R}} \subseteq \mathcal{M}\), we have that \((U_t)_{t \in \mathbb{R}}\) and \((\tilde{U}_t)_{t \in \mathbb{R}}\) commute, hence they can be diagonalized simultaneously. Then the spectral measure \(P\) of \(U\) satisfies

\[
\mathcal{M}^\alpha[s, \infty)\tilde{P}(t, \infty)\mathcal{H} \subseteq \tilde{P}(t + s, \infty)\mathcal{H} \quad \text{for} \quad t, s \in \mathbb{R},
\]

so that, for \(t \in \mathbb{R}\),

\[
P(t, \infty) = \bigcup_{s < t} \|\mathcal{M}^\alpha[s, \infty)\mathcal{H}\| = \bigcup_{s < t} \|\mathcal{M}^\alpha[s, \infty)\tilde{P}(0, \infty)\mathcal{H}\| = \bigcup_{s < t} \tilde{P}(s, \infty)\mathcal{H} = \tilde{P}(t, \infty)\mathcal{H}.
\]

We conclude that \(P(t, \infty) \leq \tilde{P}(t, \infty)\) for \(t \in \mathbb{R}\). We prove that this implies that \(H \leq \tilde{H}\) holds for the infinitesimal generators of \(U\) and \(\tilde{U}\), respectively (cf. \cite{PS12} Prop. 6.3), and hence that \(U\) is minimal. Let \(n \in \mathbb{N}\) then we approximate \(H\) from below by a step function in steps of \(1/n\) to get

\[
H \geq H_n := \frac{1}{n} \sum_{k=1}^{\infty} P[\frac{k}{n}, \infty) \quad \text{and} \quad \|H - H_n\| \leq \frac{1}{n}.\]

In particular, the operators \(H\) and \(H_n\) have the same domain. Likewise \(\tilde{H} \geq \tilde{H}_n := \frac{1}{n} \sum_{k=1}^{\infty} \tilde{P}[\frac{k}{n}, \infty) \quad \text{and} \quad \|\tilde{H} - \tilde{H}_n\| \leq \frac{1}{n}.\)

Let \(\xi \in \mathcal{D}(\tilde{H}) = \mathcal{D}(\tilde{H}_n)\). Then we also have \(\xi \in \mathcal{D}(H_n) = \mathcal{D}(H)\) and \(H \leq \tilde{H}\) follows from

\[
\langle \xi, H\xi \rangle = \lim_{n \to \infty} \langle \xi, H_n\xi \rangle \leq \lim_{n \to \infty} \langle \xi, \tilde{H}_n\xi \rangle = \langle \xi, \tilde{H}\xi \rangle.
\]
Thus for a positive covariant representation, a minimal positive implementing group exists.

**Lemma 4.18.** Let $\mathcal{M} \subseteq \mathcal{B}(\mathcal{H})$ be a von Neumann algebra and $(\mathcal{M}, \mathbb{R}, \alpha)$ a $W^*$-dynamical system. A unitary one-parameter subgroup $(U_t)_{t \in \mathbb{R}} \subset \mathcal{M}$ with non-negative spectrum implementing $\alpha$ on $\mathcal{M}$ is minimal if and only if, for every $\varepsilon > 0$, the central support of $P[0, \varepsilon)$ is $1$.

**Proof.** If $U$ is minimal, then it coincides with the Borchers–Arveson subgroup in a faithful normal representation of $\mathcal{M}$. Hence the central support of every $P[0, \varepsilon) , \varepsilon > 0$, is $1$ by Lemma 4.17.

Assume, conversely, that the central support of every $P[0, \varepsilon) , \varepsilon > 0$, is $1$. As $(U_t)_{t \in \mathbb{R}} \subset \mathcal{M}$ is positive, $\mathcal{M}$ also contains the minimal implementing group, and we only need to compare $(U_t)_{t \in \mathbb{R}}$ with that. Thus we have to show that, for every central subgroup $Z_t = e^{itW} \in Z(\mathcal{M})$ for which $(U_tZ_t)_{t \in \mathbb{R}}$ has non-negative spectrum, we have $W \geq 0$. We argue by contradiction. If $W$ is not positive, then the corresponding spectral projection $P^W((−\infty, −2\varepsilon])$ is non-zero for some $\varepsilon > 0$. Our assumption implies that $P^W((−\infty, −2\varepsilon])P[0, \varepsilon) \neq 0$ in any normal representation, hence $H + W$ is negative on the range of this projection, where $H$ is the infinitesimal implemenetter of $(U_t)_{t \in \mathbb{R}}$. Therefore $H + W$ is not positive, which contradicts the assumption that $(U_tZ_t)_{t \in \mathbb{R}}$ has non-negative spectrum.

From this we obtain that normal representations take minimal groups to minimal groups:

**Lemma 4.19.** Let $\mathcal{M} \subseteq \mathcal{B}(\mathcal{H})$ be a von Neumann algebra and $(\mathcal{M}, \mathbb{R}, \alpha)$ a $W^*$-dynamical system. Let $(U_t)_{t \in \mathbb{R}} \subset \mathcal{M}$ be a minimal implementing positive unitary group for $\alpha$. If $\pi : \mathcal{M} \rightarrow \mathcal{B}(\mathcal{H}_\pi)$ is a normal representation, then $(\pi(U_t))_{t \in \mathbb{R}} \subset \pi(\mathcal{M})$ is a minimal implementing positive unitary group for $(\pi(\mathcal{M}), \mathbb{R}, \alpha_\pi)$, where $\alpha_\pi(t)A = \pi(U_t)A\pi(U_t)^*$.

**Proof.** By the previous lemma, it suffices to prove that for every $\varepsilon > 0$, the central support of $\pi(P[0, \varepsilon))$ is $1$. If $Z := s(\pi)$ is the support of $\pi$, then $\pi(\mathcal{M}) \cong Z\mathcal{M}$ and $\pi(P[0, \varepsilon))$ corresponds to $ZP[0, \varepsilon) \in Z\mathcal{M}$. If $Z' \in Z\mathcal{M}$ is a central projection with $0 = Z'ZP[0, \varepsilon) = Z'P[0, \varepsilon)$, then $Z' = 0$ follows from the fact that $Z'$ is also central in $\mathcal{M}$ and the central support of $P[0, \varepsilon)$ is $1$.

We will use these lemmas in the next subsection when we study the structure of positive covariant representations. We next show that every covariant representation contains a maximal subrepresentation which satisfies the Borchers–Arveson criterion (Theorem 4.14), which we then apply to the universal covariant representation $(\pi_{\text{cov}}, U_{\text{cov}}) \in \text{Rep}(\alpha, \mathcal{H}_{\text{cov}})$. This is in fact already known through the “minimal positive representation” constructed in either [Bo96, Thm. II.4.6] or [Pe89, Thm. 8.4.3], but we will need to make some of its details explicit.

**Lemma 4.20.** Let $(\mathcal{M}, \mathbb{R}, \alpha)$ be a $W^*$-dynamical system on a von Neumann algebra $\mathcal{M} \subseteq \mathcal{B}(\mathcal{H})$. Then

(i) $\mathcal{M} = \bigcup \{ \mathcal{M}^\alpha[s, \infty) \mid s \in \mathbb{R} \}^{\text{w-op}}.$

(ii) The space $\mathcal{H}_\infty := Q_\infty \mathcal{H} = \bigcap_{s \in \mathbb{R}} [\mathcal{M}^\alpha[s, \infty)\mathcal{H}]$ is an invariant subspace for $\mathcal{M} \cup \mathcal{M}'$, i.e. $Q_\infty \in \mathcal{M} \cap \mathcal{M}'$ and in the case that $(\mathcal{M}, U)$ is covariant, $\mathcal{H}_\infty$ is also $U$-invariant.

**Proof.** (i) By [BR02, Lemma 3.2.38(3)], we know that, for $f \in L^1(\mathbb{R})$ such that $\sup \hat{f} \subseteq [s, \infty)$ and $A \in \mathcal{M}$, we have $\alpha_f(A) \in \mathcal{M}^\alpha[s, \infty)$. Let $f \in L^1(\mathbb{R})$ be such that $\hat{f}$ is a smooth function with support in $[-1, 1]$, and normalized such that $\int_{\mathbb{R}} |f|dt = 1$ (note that both $f$ and $\hat{f}$ are Schwartz functions). Let $f_n(x) := nf(nx)$. Then $\int_{\mathbb{R}} |f_n|dt = 1$ and $\hat{f}_n(p) = \hat{f}(p/n)$ which has support in $[-n, n]$. Moreover the $f_n$ are progressively narrower concentrated around 0, i.e. given any $a > 0$ and an $\varepsilon > 0$, then there is an $N \in \mathbb{N}$ such that $\int_{-N}^N |f_n|dt > 1 - \varepsilon$ for $n > N$. Note that all $\alpha_{f_n}(A) \in \bigcup \{ \mathcal{M}^\alpha[s, \infty) \mid s \in \mathbb{R} \}$ for every $n \in \mathbb{N}$. We want to show that $\alpha_{f_n}(A) \rightarrow A$ in the weak operator topology for all $A \in \mathcal{M}$.
Let $\omega$ be a vector state on $\mathcal{M}$, and fix $A \in \mathcal{M}$, so that $t \mapsto \omega(\alpha_t(A))$ is continuous. For $\varepsilon > 0$, there is a $\delta > 0$ such that $|t| < \delta$ implies that $|\omega(A - \alpha_t(A))| < \varepsilon$. Note that $|\omega(A - \alpha_t(A))| \leq 2\|A\|$ for all $t$. Then there exists an $N \in \mathbb{N}$ such that $\int_{-\delta}^{\delta} |f_n(t)| dt > 1 - \varepsilon$ for all $n > N$. Then

$$|\omega(A - \alpha_{f_n}(A))| \leq \int |f_n(t)| |\omega(A - \alpha_t(A))| dt$$

$$= \left( \int_{(-\delta,\delta)} + \int_{\mathbb{R}\setminus(-\delta,\delta)} \right) |f_n(t)| |\omega(A - \alpha_t(A))| dt < \varepsilon + 2\|A\|\varepsilon.$$

Thus $\alpha_{f_n}(A) \to A$ in the weak operator topology, which proves part (i).

(ii) According to \cite{BR02} Lemma 3.2.39(2), we have $U_t\mathcal{M}_\alpha[s, \infty) = \mathcal{M}_\alpha[s, \infty)U_t$ for all $s, t \in \mathbb{R}$, hence the last claim is clear. As

$$\mathcal{M}'\mathcal{M}_\alpha[s, \infty)\mathcal{H} = [\mathcal{M}_\alpha[s, \infty)\mathcal{M}'\mathcal{H}] = [\mathcal{M}_\alpha(s, \infty)\mathcal{H}]$$

it is also clear that $\mathcal{M}'\mathcal{H}_\infty \subseteq \mathcal{H}_\infty$. Finally, by \cite{BR02} Lemma 3.2.42(4), we have

$$\mathcal{M}_\alpha[s, \infty) \cdot \mathcal{M}_\alpha[t, \infty) \subseteq \mathcal{M}_\alpha[s + t, \infty)$$

and hence

$$\mathcal{M}_\alpha[s, \infty)\mathcal{H}_\infty \subseteq \bigcap_{t \in \mathbb{R}} [\mathcal{M}_\alpha[s + t, \infty)\mathcal{H}] = \mathcal{H}_\infty.$$

As $\mathcal{H}_\infty$ is closed, it follows from part (i) that $\mathcal{M}\mathcal{H}_\infty \subseteq \bigcup_{s \in \mathbb{R}} \mathcal{M}_\alpha[s, \infty)\mathcal{H}_\infty \subseteq \mathcal{H}_\infty$.

The Borchers–Arveson Theorem \cite{4,14} states that $(\mathcal{M}, \mathbb{R}, \alpha)$ has a positive strong operator continuous unitary one-parameter implementing group if and only if $\mathcal{H}_\infty = \{0\}$. This indicates how to select a state for which its GNS representation has a positive implementing unitary group for $\alpha$ (see below).

In the context of this lemma, let $Q_\infty$ be the orthogonal projection onto $\mathcal{H}_\infty$. It follows from Lemma \cite{120} ii) that $Q_\infty \in \mathcal{M}' \cap \mathcal{M}'' = Z(\mathcal{M})$, hence $\mathcal{M}$ is diagonal with respect to the decomposition $\mathcal{H} = \mathcal{H}_\infty \oplus \mathcal{H}_\infty^\perp =: \mathcal{H}_\infty \oplus \mathcal{H}^{(+)}$. Let $P^{(+)} := 1 - Q_\infty$. Then $\mathcal{M}$ is the direct sum of the two ideals $\mathcal{M}_\infty := \mathcal{M}Q_\infty$ and $\mathcal{M}^{(+)} := \mathcal{M}P^{(+)}$. Define the positive subrepresentation of $\mathcal{M}$ to be the representation $\pi^{(+)} : \mathcal{M} \to \mathcal{B}(\mathcal{H}^{(+)})$ by $\pi^{(+)}(A) := A \upharpoonright \mathcal{H}^{(+)}$, $A \in \mathcal{M}$, then clearly $\pi^{(+)}(\mathcal{M}) \cong \mathcal{M}^{(+)}$. Its name is justified by the following proposition:

**Proposition 4.21.** Let $U : \mathbb{R} \to U(\mathcal{H})$ be a strong operator continuous unitary one-parameter group such that $\alpha_t := \text{Ad} U_t$ defines an action $\alpha : \mathbb{R} \to \text{Aut}(\mathcal{M})$ on a given von Neumann algebra $\mathcal{M} \subseteq \mathcal{B}(\mathcal{H})$. Then its positive subrepresentation $\pi^{(+)} : \mathcal{M} \to \mathcal{B}(\mathcal{H}^{(+)})$ has the following properties:

(i) There is a positive strong operator continuous unitary one-parameter group $V : \mathbb{R} \to U(\mathcal{H}^{(+)})$ such that $\alpha_t = \text{Ad} V_t$ on $\pi^{(+)}(\mathcal{M})$. This unitary implementing group may be chosen to be inner.

(ii) $\pi^{(+)}$ is maximal, in the sense that any subrepresentation of $\mathcal{M}$ to which $U_t$ restricts, and which has a positive implementing unitary group, must be contained in the positive subrepresentation.

**Proof.** We first need to prove that if $\mathcal{H}_1 \subset \mathcal{H}$ is a subspace invariant with respect to $\mathcal{M}$ and $U_\mathbb{R}$, then the spectral subspaces restrict. That means, if we label the subrepresentation by $\pi_1 : \mathcal{M} \to \mathcal{B}(\mathcal{H}_1)$, $\pi_1(A) := A \upharpoonright \mathcal{H}_1$, $A \in \mathcal{M}$, then $\pi_1(\mathcal{M}_\alpha[s, \infty)) = \pi_1(\mathcal{M})^\beta[s, \infty))$ for all $s \in \mathbb{R}$, where $\beta_t := \text{Ad}(U_t \upharpoonright \mathcal{H}_1)$. But this follows from the characterization \cite{13} since the spectral projection of $U_t$ commutes with the projection onto $\mathcal{H}_1$. If we let $\mathcal{H}_1 = \mathcal{H}^{(+)}$, then the spectral subspaces of $\beta_t$ are the projections of the spectral subspaces of $\alpha_t$ by $P^{(+)}$, hence by construction $\beta_t$ satisfies

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the condition of the Borchers–Arveson Theorem [4.14], and this proves (i). Then it follows that the positive subrepresentation of its orthogonal subrepresentation is zero, which is equivalent to (ii) by the Borchers–Arveson Theorem [4.14].

Given a $C^*$-action $(\mathcal{A}, \mathbb{R}, \alpha)$, consider the universal covariant representation $(\pi_{co}, U_{co}) \in \text{Rep}(\alpha, \mathcal{H}_{co})$ with associated $W^*$-dynamical system $(\mathcal{M}_{co}, \mathbb{R}, \alpha^{co})$. Then the positive subrepresentation $\pi_{co}^{(+)} : \mathcal{A} \to \mathcal{B}(\mathcal{H}_{co}^{(+)})$ has the universal property that every cyclic positive covariant representation of $(\mathcal{A}, \mathbb{R}, \alpha)$ is unitarily equivalent to a subrepresentation of it. Moreover, it is also unitarily equivalent to the “minimal positive representation” constructed in [Bo96, Thm II.4.6] and [Pe89, Thm. 8.4.3].

Consider a state $\omega \in \mathcal{S}(\mathcal{A})$ which is quasi-invariant, i.e. $\pi_\omega$ is quasi-covariant (cf. Def. 5.3(c)). Then $\alpha$ induces a $W^*$-dynamical system $(\mathcal{M}, \mathbb{R}, \beta)$, where $\mathcal{M} := \pi_{\omega}(\mathcal{A})''$. Moreover $(\mathcal{M}, \mathbb{R}, \beta)$ has a positive strong operator continuous unitary one-parameter implementing group if and only if $(\pi_{\omega}(\mathcal{A}), \mathbb{R}, \alpha)$ has. In view of the Borchers–Arveson Theorem [4.14] this is equivalent to

$$\{0\} = \mathcal{H}_{\omega, \infty} := \bigcap_{s \in \mathbb{R}} [\mathcal{M}^\beta(s, \infty) \mathcal{H}_\omega]$$

Thus any equivalent condition to $\mathcal{H}_{\omega, \infty} = \{0\}$ would characterize the set of such states with positive implementing group:

**Proposition 4.22.** For a $C^*$-action $(\mathcal{A}, \mathbb{R}, \alpha)$, define

$$\mathcal{S}_{co}^{(\omega)}(\mathcal{A}) := \{ \omega \in \mathcal{S}(\mathcal{A}) \mid (\pi_{\omega}, V) \in \text{Rep}(\alpha, \mathcal{H}_{\omega}) \text{ for some positive } V : \mathbb{R} \to U(\mathcal{H}_\omega) \}.$$  

For a quasi-invariant state $\omega \in \mathcal{S}(\mathcal{A})$, let $Q^\omega_\infty \in \pi_{\omega}(\mathcal{A})''$ be the orthogonal projection onto $\mathcal{H}_{\omega, \infty}$. Then

$$\omega \in \mathcal{S}_{co}^{(\omega)}(\mathcal{A}) \iff \omega(Q^\omega_\infty) = 0 \iff \omega(Q^\omega_\infty A) = 0 \text{ for all } A \in \mathcal{A}.$$  

**Proof.** Let $\mathcal{M} := \pi_{\omega}(\mathcal{M})''$. By Lemma [4.20(ii)] we have that $Q^\omega_\infty \in \mathcal{M}' \cap \mathcal{M}'' = Z(\mathcal{M})$. From the Cauchy–Schwartz inequality

$$|\omega(Q^\omega_\infty A)|^2 \leq \omega(Q^\omega_\infty) \omega(A^* A) \text{ for } A \in \mathcal{A},$$

we get that $\omega(Q^\omega_\infty) = 0$ implies $\omega(Q^\omega_\infty A) = 0$ for all $A \in \mathcal{A}$. Conversely, as $\pi_{\omega}(\mathcal{A})$ acts non-degenerately on $\mathcal{H}_\omega$, $\pi_{\omega}(E_\lambda) \to 1$ in strong operator topology for any approximate identity $(E_\lambda)_{\lambda \in \Lambda}$ in $\mathcal{A}$, hence if $\omega(Q^\omega_\infty A) = 0$ for all $A \in \mathcal{A}$ then $\omega(Q^\omega_\infty) = \lim_{\lambda} \omega(Q^\omega_\infty E_\lambda) = 0$ which gives the converse implication, and hence the second equivalence is established. Moreover, if $\omega(Q^\omega_\infty) = 0$ then also all $\omega_B(Q^\omega_\infty) = 0$ where $\omega_B(A) := \omega(B^* A B)$ for $A, B \in \mathcal{A}, \|B\| = 1$, and hence all vector states of $\pi_{\omega}$ will also satisfy it. The vector state of any vector orthogonal to $\mathcal{H}_{\omega, \infty}$ clearly satisfies the condition, whereas any nonzero vector $\psi \in \mathcal{H}_{\omega, \infty}$ produces $\omega_\psi(Q^\omega_\infty) = \|\psi\|^2 \neq 0$. Thus the condition $\omega(Q^\omega_\infty) = 0$ is equivalent to $\mathcal{H}_{\omega, \infty} = \{0\}$, which by the Borchers–Arveson Theorem [4.14] characterizes $\mathcal{S}_{co}^{(\omega)}(\mathcal{A})$. 

This condition looks different in Borchers approach (cf. [Bo96, Def. II.4.3(i)]) as his selection condition is $\omega E(\infty) = E(\infty) \omega = \omega$ where $E(\infty)$ is the projection onto the subspace $\mathcal{H}_{co} \cap \mathcal{H}_{co, \infty}$ in the universal representation on $\mathcal{H}_{co}$. However, this condition clearly coincides with the condition above in the given context.

### 4.3 Positive covariant representations and obstruction results.

The Borchers–Arveson Theorem produces several obstruction results for positive covariant representations. By Remark [4.15(b)], for any covariant representation $(\pi, U)$ of $(\mathcal{M}, \mathbb{R}, \alpha)$ for which $\pi$ is faithful and $U$ has positive spectrum, if the action is nontrivial, the algebra $\mathcal{M}$ must be noncommutative. This obstruction result leads to further obstructions, which we now discuss.
Proposition 4.23. Let $\mathcal{B}$ be a $C^*$-algebra and $\mathcal{A} := C_0(\mathbb{R}, \mathcal{B})$, endowed with the automorphisms $(\alpha_t f)(x) := f(x - t)$. Then all positive covariant representations $(\pi, U)$ of $(\mathcal{A}, \mathbb{R}, \alpha)$ satisfy $\pi = 0$.

Proof. Writing $\mathcal{A} \cong C_0(\mathbb{R}) \otimes \mathcal{B}$, we see that every non-degenerate representation of $\mathcal{A}$ can be written as $\pi(A_1 \otimes A_2) = \pi_1(A_1)\pi_2(A_2)$, where $\pi_1 : C_0(\mathbb{R}) \to B(\mathcal{H})$ and $\pi_2 : \mathcal{B} \to B(\mathcal{H})$ are commuting representations (cf. [10a02 Prop. 4.7, Lemma 4.18]). Hence every positive covariant representation $(\pi, U)$ of $(\mathcal{A}, \mathbb{R}, \alpha)$ leads to a covariant representation of $(C_0(\mathbb{R}), \mathbb{R}, \alpha)$, so that the Borchers–Arveson Theorem implies that $U_R$ commutes with $\pi_1(C_0(\mathbb{R}))$ and $\pi_2(\mathcal{B})$, and this implies that $U_R$ commutes with $\pi(\mathcal{A})$.

A function $f \in C_c(\mathbb{R})$ is a derivative of a compactly supported function $F$ if and only if \( \int_\mathbb{R} f(x) \, dx = 0 \). Then
\[
 f(x) = \lim_{h \to 0} \frac{F(x + h) - F(x)}{h}
\]
shows that we must have $\pi_1(f) = 0$ for all these functions. Now the density of
\[
\{ f \in C_c(\mathbb{R}) : \int_\mathbb{R} f(x) \, dx = 0 \} = \{ f' : f \in C_c(\mathbb{R}) \}
\]
in $C_0(\mathbb{R})$ implies $\pi_1 = 0$. This in turn leads to $\pi = 0$. \qed

The translation action can be twisted by a cocycle without affecting the obstruction. To see this, modify the construction as follows. On $\mathcal{A} := C_0(\mathbb{R}, \mathcal{B})$, we consider the automorphisms
\[
(\alpha_t f)(x) = \beta_t(x)(f(x - t)),
\]
where $\beta : \mathbb{R} \to C_0(\mathbb{R}, \text{Aut}(\mathcal{B}))$ is a cocycle in the sense that the translation automorphism $(\alpha_t^0 f)(x) := f(x - t)$ satisfies $\alpha_t = \beta_t \cdot \alpha_t^0$. Then
\[
\beta_{t+s}(\alpha_t^0) = \alpha_{t+s} = \alpha_t \alpha_s = \beta_t \alpha_t^0 \beta_s \alpha_s^0 = \beta_t \beta_s \alpha_t^0 \alpha_s^0 = \beta_t \cdot (\alpha_t^0 \beta_s \alpha_s^0) \alpha_t^0
\]
leads to the cocycle relation
\[
\beta_{t+s} = \beta_t \cdot (\alpha_t^0 \beta_s \alpha_s^0) \alpha_t^0.
\]
This means that
\[
\beta_{t+s}(x) = \beta_t(x) \beta_s(x - t) \quad \text{for} \quad t, s, x \in \mathbb{R}.
\]

Corollary 4.24. For $\alpha$ as in (20), all positive covariant representations $(\pi, U)$ of $(\mathcal{A}, \mathbb{R}, \alpha)$ satisfy $\pi = 0$.

Proof. Let $(\pi, U)$ be a covariant representation of $(\mathcal{A}, \mathbb{R}, \alpha)$ and observe that it extends to a covariant representation of the multiplier algebra $(M(\mathcal{A}), \mathbb{R}, \alpha)$. In $M(\mathcal{A})$ we have the subalgebra $C_0(\mathbb{R}, \mathbb{C})$ obtained from the functions whose values are multiples of 1. On this subalgebra the $\mathbb{R}$-action takes the form $(\alpha_t f)(x) = f(x-t)$ because $f(\mathbb{R}) \subseteq \mathcal{B}$ is fixed by all automorphisms. Then Proposition 4.23 implies that $\pi(C_0(\mathbb{R}, \mathbb{C})) = \{0\}$. This in turn yields $\pi(\mathcal{A}) = \{0\}$. \qed

Remark 4.25. If $\widehat{G} \cong \mathbb{T} \rtimes \gamma$ is a central $\mathbb{T}$-extension of $G$, for a given 2-cocycle $\gamma : G \times G \to \mathbb{T}$, then we associate the corresponding twisted group $C^*$-algebra $\mathcal{A} := C^*_\gamma(G_d)$ defined by the unitary generators $(\delta_g)_{g \in G}$ satisfying the relations
\[
\delta_g \delta_h = \gamma(g, h) \delta_{gh}.
\]

Any $\mathbb{R}$-action by automorphisms on $\widehat{G}$ fixing the central subgroup $\mathbb{T}$ pointwise induces a homomorphism $\alpha : \mathbb{R} \to \text{Aut}(\mathcal{A})$. Now covariant projective unitary representations for the cocycle $\gamma$ correspond to covariant representations of $(\mathcal{A}, \mathbb{R}, \alpha)$. 46
Example 4.26. (The Weyl algebra) Let \( \text{Heis}(V, \sigma) = \mathbb{T} \times V \) be the Heisenberg group of the real symplectic topological vector space \((V, \sigma)\) with the multiplication
\[
(z, v)(z', v') := (zz'e^{-\frac{i}{2} \sigma(v, v')}, v + v'), \quad z \in \mathbb{T}, v \in V
\]
and let \( \mathcal{A} := \Delta(V, \sigma) \) be the corresponding Weyl algebra, which is the discrete twisted group algebra \( C^*_r(V_d) \), where \( \gamma(v, w) = e^{-\frac{i}{2} \sigma(v, w)} \). We consider a smooth one-parameter group \( (\tau_t)_{t \in \mathbb{R}} = e^{iY} \in \text{Sp}(V, \sigma), Y \in \mathfrak{sp}(V, \sigma) \). Here smoothness refers to the smoothness of the \( \mathbb{R} \)-action \( \mathbb{R} \times V \to V \). This defines an action \( \alpha_0 : \mathbb{R} \to \text{Aut}(\text{Heis}(V, \sigma)) \) by \( \alpha_{0, t}(z, v) := (z, \tau_t(v)) \), and as \( \alpha_{0, t} \) fixes all \((z, 0)\), it also defines an automorphic \( \mathbb{R} \)-action \( \alpha \) on \( \mathcal{A} \) which is singular, as it is not strongly continuous. Now \( \mathcal{A} \) has many representations which are not continuous with respect to the underlying group \( \text{Heis}(V, \sigma) \) (nonregular representations), so to avoid these, we consider the associated Lie groups.

As the action \( \alpha \) on \( \text{Heis}(V, \sigma) \) is smooth, we form the corresponding oscilator group
\[
G := \text{Heis}(V, \sigma) \rtimes_{\alpha} \mathbb{R}.
\]

It is a Lie group because the \( \mathbb{R} \)-action on \( \text{Heis}(V, \sigma) \) is smooth. Now any smooth unitary representation \( (\pi, \mathcal{H}) \) of \( G \) for which \( \pi(z, 0, 0) = z1 \) will define a covariant representation of \( (\mathcal{A}, \mathbb{R}, \alpha) \), where the unitary implementers of \( \alpha_t \) are \( U_t := \pi(0, 0, t) \). We analyze positivity for these covariant representations.

Proposition 4.27. If \( (\pi, \mathcal{H}) \) is a smooth unitary representation of \( G \) for which the one-parameter group \( U_t = \pi(0, 0, t) \) has positive spectrum and \( \pi(z, 0, 0) = z1 \), then the infinitesimal generator \( Y \in \mathfrak{sp}(V, \sigma) \) of \( \tau \) satisfies
\[
\sigma(Yv, v) \geq 0 \quad \text{for} \quad v \in V.
\]

Proof. Let \( d := (0, 0, 1) \in \mathfrak{g} \), then \( U_t = \pi(0, 0, t) = \exp(td\pi(d)) = \exp(-itH) \). Let \( \xi \in \mathcal{H}^\infty \) be a smooth vector of \( \pi \), let \( v \in V \), then by assumption we have for every \( t \in \mathbb{R} \) the inequality
\[
0 \leq \langle \pi(\exp tv) H \pi(\exp -tv) \xi, \xi \rangle = i \langle \pi(\exp tv) d\pi(d) \pi(\exp -tv) \xi, \xi \rangle = i \langle d\pi(e^{t adv}) d\xi, \xi \rangle.
\]
Now
\[
(ad v)d = (0, (ad d)v, 0) = -\frac{d}{dt}(0, \tau_t(v), 0) \bigg|_{t=0} = (0, -Yv, 0),
\]
\[
(ad v)^2d = -[v, Yv] = (\sigma(v, Yv), 0, 0),
\]
hence
\[
e^{t ad v}d = d + tvd + \frac{t^2}{2} [v, [v, d]] = \left( \frac{t^2}{2} \sigma(v, Yv), -tYv, 1 \right)
\]
and so
\[
0 \leq i \langle d\pi(e^{t adv}) d\xi, \xi \rangle = \frac{t^2}{2} \sigma(Yv, v) \langle \xi, \xi \rangle - it \langle d\pi(Yv) \xi, \xi \rangle + i \langle d\pi(d) \xi, \xi \rangle.
\]
Since this holds for all \( t \in \mathbb{R} \), we eventually obtain \( \sigma(Yv, v) \geq 0 \).

In the special case that \( V \) is a complex pre-Hilbert space \( \mathcal{D} \), \( \sigma(v, w) = \text{Im} \langle v, w \rangle \) and \( \tau_t \in U(\mathcal{D}) \), then \( \langle Yv, v \rangle \in i\mathbb{R} \), so that
\[
0 \leq \sigma(Yv, v) = \text{Im} \langle Yv, v \rangle = -i \langle Yv, v \rangle = \langle -iYv, v \rangle
\]
implies that the infinitesimal generator \(-iY\) of the unitary one-parameter group \((\tau_t)_{t \in \mathbb{R}} \) is non-negative if there exists a positive covariant representation for \((\mathcal{A}, G, \alpha)\). In this case, as Fock representations exist and the second quantization of a positive operator is positive, we also have the converse implication (cf. \[N\text{Z}13\], \[Z\text{c}13\]).
Example 4.28. Let $G = U_2(\mathcal{H}) := U(\mathcal{H}) \cap (1 + B_2(\mathcal{H}))$, where $B_2(\mathcal{H})$ is the ideal of Hilbert–Schmidt operators. Then $G$ is a Banach–Lie group with Lie algebra $\mathfrak{g} = u_2(\mathcal{H}) = \{ X \in B_2(\mathcal{H}) : X^* = -X \}$. It is an interesting problem to determine all projective unitary representations of $G$. That this problem is naturally linked to covariant representations is due to the fact that every continuous cocycle $\omega : \mathfrak{g} \times \mathfrak{g} \to \mathbb{R}$ is of the form

$$
\omega(X, Y) = \text{tr}([D, X]Y) = \text{tr}(D[X, Y])
$$

for some $D \in \mathfrak{u}(\mathcal{H})$ (see [Ne03] Prop. III.19 and its proof). Then $\alpha_t(g) := \exp(tD)g \exp(-tD)$ is a continuous one-parameter group of automorphisms of $G$ acting naturally on the central extension $\mathbb{R} \oplus \mathfrak{g}$ with the bracket $[(z, x), (z', x')] := (\omega(x, x'), [x, x'])$ by $\alpha_t(z, x) := (z, \alpha_t(x))$ and this action lifts to the corresponding simply connected group $\hat{G}$, which leads to a Lie group $G^\sharp := \hat{G} \ltimes_\alpha \mathbb{R}$. Its Lie algebra is the double extension

$$
\mathfrak{g}^\sharp = \mathbb{R} \oplus \mathfrak{g} \oplus \mathbb{R}, \quad [(z, X, t), (z', X', t')] = (\omega(X, X'), [X, X'] + t[D, X'] - t'[D, X], 0).
$$

Presently, the classification of all corresponding projective positive covariant representations is still open. However, the case where $D$ is diagonalizable and the representation is a highest weight representation has been treated fully in [MN16]; see also [Ne17] for more complete results.

Since projective positive covariant representations of $G$ lead to unitary representations $(U, \mathcal{H})$ of the corresponding doubly extended group $G^\sharp$ for which the convex cone

$$
W := \{ x \in \mathfrak{g}^\sharp : -\text{id}U(x) \geq 0 \}
$$

has interior points, the method developed in [NSZ17] provides a natural $C^*$-algebra whose representation corresponds to these representations of $G^\sharp$. From the perspective of Remark 4.25, these representations correspond as well to positive covariant representations of $(\mathcal{A}, \mathbb{R}, \alpha)$ for $\mathcal{A} = C^*_r(U_2(\mathcal{H})_d)$, where this denotes the twisted group algebra corresponding to a central extension $\hat{G}$ of $U_2(\mathcal{H})$ by $T$ corresponding to the Lie algebra extension defined by the cocycle $\omega$ (see also [Ne14]).

The Borchers–Arveson Theorem also produces obstructions for various actions of groups on $C^*$-algebras, as in the following framework:

- There is a unital $C^*$-algebra $\mathcal{A}$, and two actions $\alpha : \mathbb{R} \to \text{Aut}(\mathcal{A})$, $\beta : G \to \text{Aut}(\mathcal{A})$ for a topological group $G$ and a nontrivial group action $\gamma : \mathbb{R} \to \text{Aut}(G)$ which intertwines $\alpha$ and $\beta$, i.e. $\beta(\gamma_t(g)) = \alpha_t \circ \beta(g) \circ \alpha_{-t}$ for all $t \in \mathbb{R}$, $g \in G$.

- Given this setting, then a covariant representation is a triple $(\pi, U, V)$, where $\pi : \mathcal{A} \to \mathcal{B}(\mathcal{H})$ is a nondegenerate representation, $U : \mathbb{R} \to U(\mathcal{H})$ is a unitary one-parameter group, and $V : G \to U(\mathcal{H})$ is a continuous unitary representation such that

$$
U_t\pi(A)U_{-t} = \pi(\alpha_t(g)), \quad V_g\pi(A)V_{g^{-1}} = \pi(\beta_g(A)), \quad U_tV_gU_{-t} = V_{\gamma_t(g)}
$$

for all $A \in \mathcal{A}$, $g \in G$ and $t \in \mathbb{R}$. We will say it is positive if $U$ is positive.

This framework will occur for example if one tries to quantize Lagrangian classical gauge theory on Minkowski space (cf. [Ble81]). In such a quantum gauge theory, $\mathcal{A}$ will be the algebra of observables, $\alpha$ is time evolution, and $\beta$ gives the gauge transformations. As the base space of the gauge theory is Minkowski space, $G$ can be matrix-valued functions on the base space, and $\gamma$ will consist of translations along the time coordinate.

The important action in this setting which will prohibit covariant representations, is $\gamma : \mathbb{R} \to \text{Aut}(G)$. We give a class of relevant examples where no nontrivial positive covariant representations are possible.
Proposition 4.29. Let $X$ be a locally compact Hausdorff space, and let $F \subseteq U(\mathfrak{n})$ be a closed subgroup containing $\mathbb{T}^1$, and let $G \subset C_b(X, F)$ be a subgroup with respect to pointwise multiplication. Let $t \mapsto \varphi_t \in \text{Homeo}(X)$ be a one-parameter group of homeomorphisms, and assume that $g \circ \varphi_t \in G$ for all $t \in \mathbb{R}$ and $g \in G$. Consider the action

\[ \gamma : \mathbb{R} \to \text{Aut}(G), \quad \gamma_t(g)(x) := \lambda_t(x)(g(\varphi_{-t}(x))), \quad \text{where} \quad \lambda_t(x) \in \text{Aut}(F), \]

so that $\lambda_t(x)$ fixes $\mathbb{T}^1$ pointwise. If $(V, U)$ is a positive covariant representation of $\gamma$, then for any $g \in C_b(X, \mathbb{T}^1) \cap G$ we have $V_g = V_{\gamma_t(g)}$ for all $t \in \mathbb{R}$.

Assume as above, the three actions $\alpha, \beta$ and $\gamma$ and assume that $\alpha_t$ and $\beta_t$ do not commute in $\text{Aut } \mathcal{A}$ for some $g \in C_b(X, \mathbb{T}^1) \cap G$, and that $\mathcal{A}$ is simple. Then the only positive covariant representation is the zero representation.

Proof. Asssuming a positive covariant representation $(V, U)$ of $\gamma$, note that $(V_{\gamma_t(g)})_{t \in \mathbb{R}}$ is commutative if $g \in C_b(X, \mathbb{T}^1) \cap G$. Thus $N := (V_{\gamma_t(g)})'' \subset \mathcal{B}(\mathcal{H})$ is commutative, and by construction the action of $\text{Ad}(U_t) := \tilde{\alpha}_t$ will preserve $N$ (Remark 4.15(b)). Thus by the Borchers–Arveson theorem $N$ contains the minimal unitary implementers for $\tilde{\alpha}_t$ which therefore commutes with $V_g \in N$ and so by covariance $V_g = V_{\gamma_t(g)}$ for all $t \in \mathbb{R}$.

For the second part, let $(\pi, U, V)$ be a positive covariant representation. By the previous part we have that $V_g = V_{\gamma_t(g)}$ for all $t \in \mathbb{R}$ and for $g \in C_b(X, \mathbb{T}^1) \cap G$. Now

\[ \pi(\beta_t(A)) = \text{Ad}(V_g)\pi(A) = \text{Ad}(V_{\gamma_t(g)})\pi(A) = \pi(\beta_{\gamma_t(g)}(A)) = \pi(\alpha_t \circ \beta_t \circ \alpha_{-t}(A)), \]

hence $\alpha_t \circ \beta_t(A) - \beta_t \circ \alpha_t(A) \in \text{Ker } \pi$ for all $A \in \mathcal{A}$. By hypothesis there is an $A \in \mathcal{A}$ and $t \in \mathbb{R}$ for which this is nonzero, hence as $\mathcal{A}$ is simple, $\pi$ must be the zero representation. \qed

Remark 4.30. One way to circumvent the obstruction from Proposition 4.29 is to ask instead for a positive covariant representation $(\pi, U, V)$, where $V : G \to U(\mathcal{H})$ is a continuous projective unitary representation. It is interesting that even in the Hamiltonian approach to quantum gauge theory (where $\gamma$ is trivial), projective gauge transformations occur naturally. These are obtained e.g. by using a quasi-free Fock representation of the CAR-algebra to produce a positive implementing unitary group for the time evolutions (cf. [CRS7] [La94]).

In this context we also mention that the method to relate positive covariant representations to positivity of a Lie algebra cocycle that we have seen in Example 4.26 has been put to work extensively in the context of positive covariant representations for gauge groups corresponding to semi-simple structure groups in [JN17].

Given the obstruction in Proposition 4.29 one strategy is to weaken the requirements on the representation. Starting with the actions $\alpha : \mathbb{R} \to \text{Aut}(\mathcal{A}), \beta : G \to \text{Aut}(\mathcal{A})$ and $\gamma : \mathbb{R} \to \text{Aut}(G)$ such that $\beta(\gamma_t(g)) = \alpha_t \circ \beta(g) \circ \alpha_{-t}$ for all $t \in \mathbb{R}$, $g \in G$, one considers triples $(\pi, U, V)$, where $\pi : \mathcal{A} \to \mathcal{B}(\mathcal{H})$ is a nondegenerate representation, $U : \mathbb{R} \to U(\mathcal{H})$ is a positive unitary one-parameter group, and $V : G \to U(\mathcal{H})$ is a map (not necessarily a representation) such that

\[ U_t\pi(A)U_{-t} = \pi(\alpha_t(A)), \quad V_g\pi(A)V_{g^{-1}} = \pi(\beta_g(A)), \quad U_tV_gU_{-t} = V_{\gamma_t(g)} \]

Then it follows that $V : G \to U(\mathcal{H})$ must be a cocycle representation, i.e.

\[ V_gV_h = \mu(g, h)V_{gh} \quad \text{where} \quad \mu(g, h) \in \pi(\mathcal{A})' \cap (U_2)' \]

for $g, h \in G$. By Proposition 4.29 we know that the cocycle $\mu$ must be nontrivial.
5 Ground states and their covariant representations

Recall from Lemma 4.17 that if a positive unitary one-parameter subgroup \((U_t)_{t \in \mathbb{R}} \subset \mathcal{M}\) is minimal, then for every \(\varepsilon > 0\), the central support of \(P[0, \varepsilon)\) is \(1\). Below, in a suitable subrepresentation, we will find a similar property for \(P(\{0\})\), the projection onto the space of invariant vectors. In the next two theorems, we first investigate structures associated with projections of central support \(1\).

5.1 Ground states

**Definition 5.1.** Let \((\mathcal{M}, \mathbb{R}, \alpha)\) be a concrete \(W^*\)-dynamical system on \(\mathcal{H}\), i.e. \(\mathcal{M} \subseteq B(\mathcal{H})\) is a von Neumann algebra. The **ground state vectors** of a positive covariant representation are the \(U\)-invariant elements of \(\mathcal{H}\) with respect to the minimal one-parameter group from the Borchers–Arveson Theorem 4.14. (This should be distinguished from the ground states defined in Definition 5.5 below; but see Corollary 5.6).

In the physics literature, the ground state vectors are defined as the eigenvectors of the Hamiltonian corresponding to the lowest value of its spectrum. As is well-known, for e.g. the quantum oscillator in the Schrödinger representation, this lowest spectral value can be nonzero. However, this definition coincides with our definition, as we took the minimal one-parameter group, and for this, the lowest spectral value of its generator is zero. In the example of the quantum oscillator, the generator of the minimal group is the usual Hamiltonian plus the multiple of the identity needed to shift the lowest value of its spectrum to zero.

**Theorem 5.2.** Let \(\mathcal{M}\) be a \(W^*\)-algebra and \((U_t)_{t \in \mathbb{R}} \subset \mathcal{M}\) be a positive one-parameter group in \(\mathcal{M}\) which is minimal. We write \(P\) for the spectral measure of \(U\) for which \(U_t = \int_{\mathbb{R}} e^{-itp} dP(p)\) and put \(P_\varepsilon := P[0, \varepsilon]\) for \(\varepsilon \geq 0\). Let \(Z_0\) be the central support of \(P_0\). Then we obtain a direct sum decomposition

\[
\mathcal{M} = Z_0 \mathcal{M} \oplus (1 - Z_0) \mathcal{M}.
\]

Moreover, the following assertions hold:

(i) For all normal representations \((\pi, \mathcal{H})\) of the ideal \(Z_0 \mathcal{M}\), the subspace \(\mathcal{H}_0 := \pi(P_0) \mathcal{H}\) of ground state vectors is \(\pi(\mathcal{M})\)-generating.

(ii) All normal representations \((\pi, \mathcal{H})\) of \((1 - Z_0) \mathcal{M}\) are positive covariant representations with respect to \((\pi(U_t))_{t \in \mathbb{R}}\), but they contain no non-zero ground state vectors.

(iii) For all normal representations \((\pi, \mathcal{H})\) of \(\mathcal{M}\) and \(\varepsilon > 0\), the subspace \(\pi(P_\varepsilon) \mathcal{H}\) is \(\pi(\mathcal{M})\)-generating.

**Proof.** (i) Let \((\pi, \mathcal{H})\) be a normal representation of \(Z_0 \mathcal{M}\), which corresponds to a normal representation of \(\mathcal{M}\) with \(\pi(Z_0) = 1\). Then the central support of \(\pi(P_0)\) is \(1\), so that the assertion follows from Lemma 3.14.

(ii) The positivity of \(U\) implies that the one-parameter group \((\pi(U_t))_{t \in \mathbb{R}}\) has positive spectrum.

If \(\pi(Z_0) = 0\), then also \(\pi(P_0) = 0\), so that \(\pi(U_t)\) has no non-zero fixed vectors. The minimality of \(U\) implies that \(\pi \circ U\) is minimal in \(\pi(\mathcal{M})\) (Lemma 4.19), so that \(\inf \text{Spec}(\pi \circ U) = 0\). Hence there are no ground state vectors for \(\alpha\) in \(\mathcal{H}\).

(iii) follows immediately from \(\inf \text{Spec}(\pi \circ U) = 0\) in every normal representation \(\pi\) of \(\mathcal{M}\), Lemmas 3.14 and 4.17.

**Remark 5.3.** Suppose, in the context of Theorem 5.2 that \(0\) is isolated in the spectrum of the positive implementing unitary group \(U\). Then the central support of \(P_0\) is \(1\), hence \(\mathcal{M} = Z_0 \mathcal{M}\). This is clearly an important subcase, which we will analyze in detail in Subsect. 5.3 below.
**Proposition 5.4.** Let \((A, \mathbb{R}, \alpha)\) be a \(C^*\)-action and let \((\pi, U)\) be a positive covariant representation for which the subspace \(H_0\) of \(U\)-fixed vectors is generating. Then \((U_t)_{t \in \mathbb{R}}\) is the Borchers–Arveson minimal group, hence in particular \(U_\mathbb{R} \subseteq \pi(A)''\).

**Proof.** Let \(\mathcal{M} := \pi(A)''\). From the Borchers–Arveson Theorem \[1.14\] we obtain a uniquely determined minimal strongly continuous unitary one-parameter group \((V_t)_{t \in \mathbb{R}}\) in \(\mathcal{M}\) implementing the automorphisms \(\text{Ad}(U_t)\). Then \(W_t := U_t V_t^* \in \mathcal{M}'\) is a one-parameter group with positive spectrum (Lemma \[1.17\]). Let \(H, H_1\) and \(H_2\) denote the infinitesimal generators of \(U, V\) and \(W\), respectively. All these operators have non-negative spectrum, so that Lemma \[A.3\] implies that \(H = H_1 + H_2\). Therefore \(H_0 \subseteq D(H) = D(H_1) \cap D(H_2)\) and, for every \(\Omega \in H_0\), we have

\[
0 = \langle H\Omega, \Omega \rangle = \langle H_1\Omega, \Omega \rangle + \langle H_2\Omega, \Omega \rangle.
\]

This implies \(H_2\Omega = 0\), so that \(\Omega\) is fixed by \(W\). As \(H_0\) is \(\mathcal{M}\)-generating, it is separating for \(\mathcal{M}'\), which leads to \(W_t = 1\) for \(t \in \mathbb{R}\). This proves that \(U_t = V_t \in \pi(A)''\).

Recall for an invariant state \(\omega\), the GNS unitary group \(U^\omega\) from above (preceding Proposition \[2.26\]). We define:

**Definition 5.5.** Given a \(C^*\)-action \((A, \mathbb{R}, \alpha)\), then a **ground state** is an invariant state \(\omega \in \mathfrak{S}(A)\) for which its GNS unitary group \((U_t^\omega)_{t \in \mathbb{R}}\) is continuous and positive (cf. \[Bo96\] Def. IV.4.9)). Then \(\Omega_\omega\) is a ground state vector in the GNS representation by the next corollary.

**Corollary 5.6.** Assume a \(C^*\)-action \((A, \mathbb{R}, \alpha)\) and an invariant state \(\omega \in \mathfrak{S}(A)\).

(i) If \(\omega\) is a ground state, i.e. \((U_t^\omega)_{t \in \mathbb{R}}\) is continuous and positive, then \(U^\omega\) is the Borchers–Arveson minimal group, hence \(U^\omega_\mathbb{R} \subseteq \pi_\omega(A)''\), and the GNS cyclic vector \(\Omega_\omega\) is a ground state vector for \(U^\omega\).

(ii) If there is a Borchers–Arveson minimal group \((V_t)_{t \in \mathbb{R}}\) on \(H_\omega\) and \(\Omega_\omega\) is a ground state vector, then \(U^\omega\) is positive and coincides with the Borchers–Arveson minimal group. Hence \(\omega\) is a ground state.

**Proof.** (i) follows from \(\Omega_\omega \in H_0\) and Proposition \[5.4\].

For (ii), by assumption we have \(V_t\Omega_\omega = \Omega_\omega\) for all \(t \in \mathbb{R}\). Together with covariance, this implies that \(V_t = U_t^\omega\) for all \(t\), so that by the definition \(\omega\) is a ground state.

In Subsection \[5.2\] below we will study existence of ground states.

**Example 5.7.** A case of an invariant state \(\omega\) for which \((\pi_\omega, U^\omega)\) is not positive but \(\text{Spec}(U^\omega)\) is bounded from below, so that there exists a positive implementation, can be obtained as follows.

We consider \(A = M_2(\mathbb{C})\) with elements \(A = (a_{ij})_{1 \leq i,j \leq 2}\), and let \(\alpha_t(A) = \left(\begin{array}{cc} a_{11} & e^{-it}a_{12} \\ e^{it}a_{21} & a_{22} \end{array}\right)\).

Define the state \(\omega(A) = a_{11}\), which is a vector state invariant with respect to \(\alpha_t\). Then \(U_t^\omega = \text{diag}(1, e^{it})\), but \(U_t = \text{diag}(e^{-it}, 1)\) also implements \(\alpha_t\). Then \(\text{Spec}(U^\omega) = \{0, -1\}\) is not positive, and \(\text{Spec}(U) = \{1, 0\}\) is positive.

**Remark 5.8.** (a) For the case where \((A, \mathbb{R}, \alpha)\) is a \(C^*\)-dynamical system, i.e. \(\alpha\) is strongly continuous, then the analog of Corollary \[5.6\] follows from \[Pe89\] Thm. 8.12.5).

(b) The properties of ground states listed above in Corollary \[5.6\] are in the literature, though with more restrictive assumptions than ours. E.g. in the usual case, for a ground state \(\omega\), we know from Araki \[Ar64\] (cf. \[Sa91\] Cor. 2.4.7) that \(U^\omega_\mathbb{R} \subseteq \mathcal{M} = \pi_\omega(A)''\). If we do not have the usual case, but \(A\) is assumed to have a local net structure as in \[Bo96\] Sect. 1.1, then one obtains from \[Bo96\] Cor. IV.4.11(2) that the GNS unitary group \(U^\omega : \mathbb{R} \to U(H_\omega)\) of a ground state \(\omega\) coincides.
with the minimal positive representation \(V : \mathbb{R} \to \text{U}(\mathcal{M})\). The main assumption for a local net of observables is that \(\mathcal{A}\) is an inductive limit of “local” \(C^*\)-algebras \(\mathcal{A}(O)\) indexed by the bounded open sets \(O\) in \(\mathbb{R}^4\) such that \(O_1 \subset O_2\) implies \(\mathcal{A}(O_1) \subseteq \mathcal{A}(O_2)\), and \(\alpha\) is covariant with respect to time translations acting on the regions \(O \subset \mathbb{R}^4\).

(c) In general, the projection onto a generating subspace as in Proposition 5.4 need not be contained in \(\pi(\mathcal{A})''\). A typical example can be obtained for \(\mathcal{A} = \mathcal{B}(\mathcal{H}) \oplus \mathcal{B}(\mathcal{H})\) and the canonical representation on \(\mathcal{H} \oplus \mathcal{H}\). For any unit vector \(v \in \mathcal{H}\), the element \((v, v) \in \mathcal{H} \oplus \mathcal{H}\) is cyclic, but the projection onto \(\mathcal{C}(v)\) is not contained in the von Neumann algebra \(\mathcal{A}\).

**Proposition 5.9.** Let \((\mathcal{M}, \mathbb{R}, \alpha)\) be a \(W^*\)-dynamical system and \((U_t)_{t \in \mathbb{R}}\) be a weakly continuous unitary one-parameter group in \(\mathcal{M}\) with positive spectrum such that \(\alpha_t = \text{Ad}(U_t)\) for \(t \in \mathbb{R}\) and \(U\) is minimal. Given a normal representation \((\pi, \mathcal{H})\) of \(\mathcal{M}\) in which \(\mathcal{H}_0 := \pi(P_0)\mathcal{H}\) is generating, i.e. which is generated by the ground state vectors, construct the restricted representation \((\pi_0, \mathcal{H}_0)\) of the reduction \(\mathcal{M}_{P_0} = P_0\mathcal{M}P_0 \subset \mathcal{M}\), i.e. \(\pi_0(P_0\mathcal{M}P_0) := \pi(P_0\mathcal{M}P_0) |_{\mathcal{H}_0}, M \in \mathcal{M}\). Then the map \(\pi \to \pi_0\) is a bijection between isomorphism classes of normal representations of \(\mathcal{M}\) generated by ground state vectors and isomorphism classes of normal representations \((\pi_0, \mathcal{H}_0)\) of the reduction \(\mathcal{M}_{P_0}\).

**Proof.** This is an application of Proposition 3.20.

**Example 5.10.** Let \(P\) be a projection in the \(W^*\)-algebra \(\mathcal{M}\) and consider the corresponding unitary one-parameter group

\[U_t := P + e^{-it}(1 - P) = e^{-it\mathcal{H}}\quad \text{for}\quad \mathcal{H} = 1 - P.\]

We assume that the central support of \(P\) is \(1\), so that \((U_t)_{t \in \mathbb{R}}\) is minimal (Lemma 4.18). For any normal representation \((\pi, \mathcal{H})\) of \(\mathcal{M}\), the subspace \(\mathcal{H}_P := \pi(P)\mathcal{H}\) of ground states for \(U\) is generating. It carries a representation of the ideal \(\mathcal{M}_P = P\mathcal{M}P\) of \(\mathcal{M}_0\) which determines it uniquely.

### 5.2 Existence of ground states

Recall that above in Definition 5.5 we defined a ground state for a given \(C^*\)-action \((\mathcal{A}, G, \alpha)\), as an invariant state \(\omega \in \mathcal{S}(\mathcal{A})\) for which \((U_t^\omega)_{t \in \mathbb{R}}\) is continuous and positive. In this case \(\Omega_\omega\) is a ground state vector for \((\pi_\omega, U^\omega)\). Denote the set of ground states by \(\mathcal{S}_0^0(\mathcal{A})\).

**Remark 5.11.** In the usual case (i.e. \(\alpha\) is strongly continuous), the left ideal \(L = [\mathcal{A} \cdot \mathcal{A}_0^0(\mathcal{H})(-\infty, 0)]\) generated by the subspace \(\mathcal{A}_0^0(\mathcal{H})(-\infty, 0)\) (cf. Definition 4.3) selects the ground states by \(\omega(L) = \{0\}\), i.e. \(\omega\mathcal{A}_0(\mathcal{H})(-\infty, 0) = 0\). The left ideal \(L\) is the well-known *Doplicher ideal* used for algebraic characterization of a ground state (cf. [Dop65]), and leads to an alternative definition of a ground state (cf. [Ar59] Def. 4.3, p. 82 and [BR96] Prop. 5.3.19)). Then \(\mathcal{S}_0^0(\mathcal{A}) \neq \emptyset\) if and only if \(L\) is proper in \(\mathcal{A}\).

In our case, we need not have that \(\alpha\) is strongly continuous, hence we need to deal with \(\mathcal{M}_0(\mathcal{H})(-\infty, 0) \subset \mathcal{M} = \pi_\omega(\mathcal{A})'',\) hence the condition \(\mathcal{M}_0(\mathcal{H})(-\infty, 0)\Omega_\omega = 0\) is external to \(\pi_\omega(\mathcal{A})\). We first make our condition explicit in the next proposition.

**Lemma 5.12.** Given a \(C^*\)-action \((\mathcal{A}, \mathbb{R}, \alpha)\), consider the associated \(W^*\)-dynamical system \(\alpha^{co} : \mathbb{R} \to \text{Aut}(\mathcal{M}_{co})\), where \(\mathcal{M}_{co} := \pi_{co}(\mathcal{A})''\) and \(\alpha^{co}(t) = \text{Ad}U_{co}(t)\). Then \(\omega \in \mathcal{S}(\mathcal{A})\) is a ground state for \((\mathcal{A}, \mathbb{R}, \alpha)\) if and only if it has a normal extension to \(\mathcal{M}_{co}\) which is a ground state for \((\mathcal{M}_{co}, \mathbb{R}, \alpha^{co})\).

**Proof.** Let \(\omega \in \mathcal{S}(\mathcal{A})\) be a ground state of \((\mathcal{A}, \mathbb{R}, \alpha)\). Then the GNS covariant representation \((\pi_\omega, U^\omega, \Omega_\omega)\) extends to a cyclic representation of \(\mathcal{A} \rtimes_\alpha \mathbb{R} \supset \mathcal{A}\) for which \((\pi_\omega, U^\omega) \in \text{Rep}(\alpha, \mathcal{H}_\omega)\). Thus \((\pi_\omega, U^\omega)\) is a subrepresentation of \((\pi_{co}, U_{co})\) hence \(\omega\) is normal with respect to \(\pi_{co}\) hence it
has an extension $\tilde{\omega}$ to a normal state of $M_{co}$, and it is clear that it is invariant with respect to $\alpha^{co}$.

To see that $\tilde{\omega}$ is a ground state for $(M_{co}, \mathbb{R}, \alpha^{co})$, note first that the GNS representation $(\pi_\omega, \Omega_\omega)$ is just the restriction of $\pi_{co}$ to $\mathcal{H}_\omega$, where $\Omega_\omega = \Omega_{\omega}$ on which we have $\pi_{\omega}(M_{co}) = \pi_\omega(A)'$ and hence $(\pi_\omega, U^\omega)$ is a $\alpha^{co}$-covariant representation, hence $M_{co} \bar{\omega} M_{co} \subseteq (M_{co})_{co}'$. Moreover $U^\omega = U^\omega$ because $U_t^\omega M_{co} \omega = (U_t^\omega M_{co} \omega) \omega$ for all $M \in \pi_\omega(A)'$ and we know that $U^\omega$ is positive.

Conversely, let $\nu \in \mathcal{S}(M_{co})$ be a normal ground state for $(M_{co}, \mathbb{R}, \alpha^{co})$. Then $\nu \upharpoonright \pi_{co}(A)$ is an invariant state for $\alpha$. As $\nu$ is normal, it follows that $\pi_\nu(\pi_{co}(A))$ is strong operator dense in $\pi_\nu(\pi_{co}(M_{co}))$, hence $\Omega_\nu$ is cyclic with respect to both algebras, and $\pi_\nu \upharpoonright \pi_{co}$ $| A = \pi_{\nu \upharpoonright A}$. Furthermore

$$U_t^\nu(\pi_\nu \upharpoonright \pi_{co})(A) \Omega_\nu = (U_t^\nu(\pi_\nu \upharpoonright \pi_{co})(A) U_t^\nu) \Omega_\nu = \pi_\nu(\pi_{co}(\alpha_t(A))) \Omega_\nu \quad \text{for} \quad A \in A,$$

hence $U^\nu$ is the GNS implementing unitary group for both $\nu$ and $\nu \upharpoonright A$, and it is clear that it is positive and leaves $\Omega_\nu$ invariant. Thus $\nu \upharpoonright A$ is a ground state of $(A, G, \alpha)$. □

By the preceding lemma, the next proposition also covers $C^*$-actions $(A, G, \alpha)$.

**Proposition 5.13.** Let $M \subseteq B(H)$ be a von Neumann algebra and $(M, \mathbb{R}, \alpha)$ a $W^*$-dynamical system. Then the following conditions are equivalent for a normal state $\omega$ of $M$:

(i) $\omega$ is a ground state.

(ii) $\omega M^{\alpha}(-\infty, 0) = \{0\}$.

If these conditions are satisfied, then the corresponding GNS representation $(\pi_\omega, \mathcal{H}_\omega)$ is covariant.

**Proof.** Let $(\pi_\omega, \mathcal{H}_\omega, \Omega_\omega)$ be the GNS representation of a given normal state $\omega$ and $\mathcal{N} := \pi_\omega(M)$.

(i) $\Rightarrow$ (ii): Let $\omega$ be a normal ground state of $(M, \mathbb{R}, \alpha)$ and let $(U_t^\omega)_{t \in \mathbb{R}}$ be the minimal positive unitary one-parameter group in $M$ implementing $\beta_t := \pi_\omega \circ \alpha_t = \text{Ad}(U_t^\omega)$ for $t \in \mathbb{R}$. Then $\Omega_\omega \in \mathcal{H}_\omega^U(\{0\})$, so that

$$\mathcal{N}^\beta(-\infty, 0) \Omega_\omega \subseteq \mathcal{H}_\omega^U(-\infty, 0) = \{0\}$$

follows from $\text{Spec}(U^\omega) \subseteq [0, \infty)$. For $M \in M$ and $f \in L^1(\mathbb{R})$ with $\text{supp}(\hat{f}) \subseteq (-\infty, 0)$, we have

$$\beta_f(\pi_\omega(M)) := \int f(t) \beta_t(\pi_\omega(M)) dt = \pi_\omega(\alpha_f(M)) \in \mathcal{N}^\beta(-\infty, 0).$$

As $(-\infty, 0)$ is open, it follows by [BR02, Lemma 3.2.39(4)] that all elements in $M^{\alpha}(-\infty, 0)$ are $\sigma(M, M_{co})$-limits of such $\alpha_f(M)$. As $\pi_\omega$ is normal, we thus have

$$\pi_\omega(M^{\alpha}(-\infty, 0)) \subseteq \mathcal{N}^\beta(-\infty, 0)$$

from which it follows by the first part that $\omega M^{\alpha}(-\infty, 0) = \{0\}$.

(ii) $\Rightarrow$ (i): Assume that $\omega M^{\alpha}(-\infty, 0) = \{0\}$, then we first prove that $\omega$ is $\alpha$-invariant (using a short argument from [Pe79]). As $1 \in M$ we see that $\omega(M^{\alpha}(-\infty, 0)) = \{0\}$, and this yields

$$\omega(M^{\alpha}(0, \infty)) = \overline{\omega(M^{\alpha}(0, \infty)'')} = \overline{\omega(M^{\alpha}(-\infty, 0))} = \{0\}$$

i.e. $M^{\alpha}(-\infty, 0) \cup M^{\alpha}(0, \infty) \subseteq \ker \omega$. However $M$ is the $\sigma(M, M_{co})$-closure of the span of the $\alpha$-preserved spaces $M^{\alpha}(-\infty, 0)$, $M^{\alpha}(0, \infty)$ and $M^{\alpha}\{0\}$, and these spaces only intersect in zero. Thus $\omega$ is only nonzero on $M^{\alpha}\{0\}$ and as the action of $\alpha$ on this space is trivial, it follows that $\omega$ is $\alpha$-invariant. Thus the GNS unitary group $U^\omega$ implements $\alpha$ in $\omega$ (but at this point we do not know that it is continuous). Then $\beta_t := \text{Ad}(U_t^\omega)$ defines a $W^*$-dynamical system $(\pi_\omega(M), \mathbb{R}, \beta)$, because $\beta_t(\pi_\omega(M)) = \pi_\omega(\alpha_t(M))$ and the right hand side is a composition of the $\sigma(M, M_{co})$-continuous map $t \mapsto \alpha_t(M)$ with the normal map $\pi_\omega$. 53
Now by [BR02] Lemma 3.2.39(4) it also follows for the $W^*$-dynamical system $(\mathcal{N}, \mathbb{R}, \beta)$ where $\mathcal{N} = \pi_\alpha(M)$ and $\beta_t \circ \pi_\alpha = \pi_\alpha \circ \alpha_t$, that all elements in $\mathcal{N}^\beta(-\infty, 0)$ are $\sigma(\mathcal{N}, \mathcal{N}_r)$-limits of elements $\beta_{t_f}(\pi_\alpha(M)) = \pi_\alpha(\alpha_t(M))$ for $M \in M$ and $f \in L^1(\mathbb{R})$ such that $\text{supp}(\tilde{f}) \subseteq (-\infty, 0)$. Thus we get from our assumption that $\mathcal{N}^\beta(-\infty, 0)\pi_\omega = \{0\}$. For every $s > 0$ we have

$$\Omega_\omega \in \ker(\mathcal{N}^\beta(-\infty, -s]) = \ker(\mathcal{N}^\beta(s, \infty)^\perp) = [\mathcal{N}^\beta(s, \infty)\mathcal{H}_\omega]^{\perp}$$

(for the first equality, see [BR02] Lemma 3.2.42(ii)). Thus

$$\Omega_\omega \perp \bigcup_{t > 0}[\mathcal{N}^\beta[t, \infty)\mathcal{H}_\omega] \supset \bigcap_{t > 0}[\mathcal{N}^\beta[t, \infty)\mathcal{H}_\omega] \supset \bigcap_{t \in \mathbb{R}}[\mathcal{N}^\beta[t, \infty)\mathcal{H}_\omega].$$

Hence

$$\Omega_\omega \in \left( \bigcap_{t \in \mathbb{R}}[\mathcal{N}^\beta[t, \infty)\mathcal{H}_\omega] \right)^{\perp}.$$ 

The closed space on the right hand side is $\mathcal{N}$-invariant, so, as it contains a cyclic vector, it must be all of $\mathcal{H}_\omega$. Thus

$$\bigcap_{t \in \mathbb{R}}[\mathcal{N}^\beta[t, \infty)\mathcal{H}_\omega] = \{0\}$$

and so we may apply the Borchers–Arveson Theorem 4.1.13 to conclude that the minimal Borchers–Arveson subgroup $(U_t)_{t \in \mathbb{R}} \subset \mathcal{N}$ exists and that its spectral measure $P$ is given by

$$P[t, \infty)\mathcal{H}_\omega = \bigcap_{s < t}[\mathcal{N}^\beta[s, \infty)\mathcal{H}_\omega].$$

As $P(\mathbb{R}) = P(\{0\})$, it follows that $\Omega_\omega \in P(\{0\})$, so that $\Omega_\omega$ is $U$-invariant, hence $\omega$ is a ground state for $(M, \mathbb{R}, \alpha)$, and $U$ coincides with $U^\omega$ by Corollary 5.6(ii). Now the covariance of $\pi_\omega$ follows from Proposition 2.26.

In analogy to the Doplicher existence criterion for ground states in the usual case, we then have:

**Corollary 5.14.**

(i) Let $(M, \mathbb{R}, \alpha)$ be a $W^*$-dynamical system. Then a normal ground state of $M$ exists if and only if the $\sigma(M, M_\alpha)$-closed left ideal generated in $M$ by $M^\alpha(-\infty, 0)$ is not all of $M$.

(ii) For a given $C^*$-action $(A, \mathbb{R}, \alpha)$, a ground state exists if and only if for the associated $W^*$-dynamical system $(M_{co}, \mathbb{R}, \alpha^{co})$, the $\sigma(M_{co}, (M_{co})_\alpha)$-closed left ideal generated in $M_{co}$ by $M_{co}^{\alpha\alpha}(-\infty, 0)$ is not all of $M_{co}$.

**Proof.** As (ii) is obvious, we only prove (i). By Proposition 5.13 the ground states are precisely the states in the annihilator in $M_\alpha$ of the $\sigma(M, M_\alpha)$-closed left ideal in $M$ generated by $M^\alpha(-\infty, 0)$. As the predual $M_\ast$ separates the $\sigma(M, M_\alpha)$-closed left ideals of $M$ by [Pe89] Thm 3.6.11, Prop. 2.5.4, we conclude that the annihilator in $M_\ast$ of a $\sigma(M, M_\alpha)$-closed left ideal in $M$ is nonzero if and only if this left ideal is not all of $M$.

If the $C^*$-action $(A, \mathbb{R}, \alpha)$, is not a $C^*$-dynamical system, it seems very difficult to obtain a similar internal criterion on $A$ alone for the existence of ground states.

**Remark 5.15.** (Weak clustering) If $\omega$ is a ground state, then the question arises whether its ground state vectors in its GNS representation are unique (up to multiples) or not. Let $P_0$ be the projection onto the fixed points of $U^\omega$, so $\Omega_\omega \in P_0\mathcal{H}_\omega$. If $\dim(P_0\mathcal{H}_\omega) = 1$ (the ground state vector is unique) then $\pi_\omega$ is irreducible (cf. [Sa91] Prop. 2.4.9). By Theorem 7.41 below, this will be the case if $(\pi_\omega(A)^\prime)^\mathbb{R} = \mathcal{C}1$.  

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Otherwise, if \( \dim(P_0 \mathcal{H}_\omega) > 1 \) and \( \mathcal{M}_{P_0} := P_0 \mathcal{M} P_0 \) is abelian, then \( \mathcal{M} \) is type I (cf. \cite[Prop. 2.4.11]{Sa91}). The condition that \( \mathcal{M}_{P_0} \) is abelian will be guaranteed in a local net of \( C^* \)-algebras as for the Haag–Kastler axioms (cf. \cite[Prop. 3]{Ar64}).

Recall from Proposition 4.22 the definition of \( \mathcal{S}_{\alpha,0}^+ (\mathcal{A}) \). If one assumes the Haag–Kastler axioms, then all states in \( \mathcal{S}_{\alpha,0}^+ (\mathcal{A}) \) are ground states (cf. \cite[Thm IV.10]{Bo96}) and for these the GNS unitary group \( U^\omega : \mathbb{R} \to U(\mathcal{H}_\omega) \) coincides with the minimal positive representation \( V : \mathbb{R} \to U(\mathcal{M}) \) (cf. \cite[Cor. IV.11]{Bo96}).

### 5.3 The case where 0 is isolated in \( \text{Spec}_\alpha (\mathcal{M}) \)

We now take a closer look at ground states under the assumption that 0 is isolated in \( \text{Spec}_\alpha (\mathcal{M}) \subseteq \mathbb{R} \) (this includes the case of \( T \)-actions). In the physics literature this is discussed as the “spectral gap,” and this is well-studied, e.g. in lattice systems \cite{HK00}, or the mass gap in quantum field theory \cite[Sec. 4.4]{Ar99}. We assume that there exists an \( \varepsilon > 0 \) such

\[
\text{Spec}_\alpha (\mathcal{M}) \cap [-\varepsilon, \varepsilon] = \{0\}. \tag{SG}
\]

Accordingly, we write

\[
\mathcal{M}_0 := \{ M \in \mathcal{M} \mid (\forall t \in \mathbb{R}) \alpha_t (M) = M \} = \mathcal{M}^\alpha (\{0\}),
\]

\[
\mathcal{M}_+ := \mathcal{M}^\alpha (0, \infty) = \mathcal{M}^\alpha [\varepsilon, \infty) \quad \text{and} \quad \mathcal{M}_- := \mathcal{M}^\alpha (-\infty, 0) = \mathcal{M}^\alpha (-\infty, -\varepsilon].
\]

These are weakly closed subalgebras with \( \{ M^* \mid M \in \mathcal{M}_\pm \} = \mathcal{M}_\mp \). For any \( f \in L^1 (\mathbb{R}) \) with \( \text{supp}(\hat{f}) \subseteq (\varepsilon, \varepsilon) \) and \( \hat{f}(0) = 1 \), we then have \( \alpha_f (\mathcal{M}_\pm) = \{0\} \) and \( \alpha_f (\mathcal{M}) = M \) for \( M \in \mathcal{M}_0 \), so that this element defines a weakly continuous projection

\[
p_0 = \alpha_f : \mathcal{M} \to \mathcal{M}_0 \quad \text{with} \quad \ker p_0 \supseteq \mathcal{M}_+ + \mathcal{M}_-.
\]

Further, any \( f \in \mathcal{S}(\mathbb{R}) \) can be written as a sum of three Schwartz functions \( f = f_- + f_0 + f_+ \) with

\[
\text{supp}(\hat{f}_0) \subseteq (-\varepsilon, \varepsilon), \quad \text{supp}(\hat{f}_-) \subseteq (-\infty, -\varepsilon/2) \quad \text{and} \quad \text{supp}(\hat{f}_+) \subseteq (\varepsilon/2, \infty).
\]

Then \( \alpha_f = \alpha_f_+ + \alpha_f_0 + \alpha_f_- \) with \( \alpha_f_\pm (\mathcal{M}) \subseteq \mathcal{M}_\pm \), so that \( \mathcal{M}_- + \mathcal{M}_0 + \mathcal{M}_+ \) is weakly dense in \( \mathcal{M} \), resp., \( \mathcal{M}_- + \mathcal{M}_+ \) is weakly dense in \( \ker p_0 \). In general we cannot expect that \( \mathcal{M} = \mathcal{M}_- + \mathcal{M}_0 + \mathcal{M}_+ \), as the example \( \mathcal{M} = B (\ell^2 (\mathbb{N})) \) and \( \alpha_t (M_{jk}) = (e^{it(j-k)} M_{jk}) \) shows (cf. Example 5.18 below). We also note that

\[
\mathcal{M}^\alpha [0, \infty) = \mathcal{M}_0 \oplus \mathcal{M}_+ \quad \text{and} \quad \mathcal{M}^\alpha (-\infty, 0] = \mathcal{M}_- \oplus \mathcal{M}_0. \tag{21}
\]

**Remark 5.16.** If \( (\mathcal{A}, \mathbb{R}, \alpha) \) is a \( C^* \)-dynamical system for which 0 is isolated in \( \text{Spec}_\alpha (\mathcal{A}) \), then we would like to have a direct decomposition into three closed subalgebras

\[
\mathcal{A} = \mathcal{A}_- \oplus \mathcal{A}_0 \oplus \mathcal{A}_+ , \tag{22}
\]

defined by the spectral projections corresponding to \( (-\infty, 0), \{0\} \) and \( (0, \infty) \). Such a decomposition always exists if \( \alpha \) is norm continuous (cf. \cite{Ne10}), but if the generator \( D := -i \alpha'(0) \) is unbounded, then the situation is more complicated. In any case we know from \cite[Thm. XI.1.23]{Ta03} that

\[
\text{Spec}_\alpha (\mathcal{A}) = -i \text{Spec}(\alpha'(0)).
\]

**Remark 5.17.** In \cite[Prop. 15.12]{Str81} it is shown that, for a \( W^* \)-dynamical system \( (\mathcal{M}, \mathbb{R}, \alpha) \), the existence of an \( \varepsilon > 0 \) with

\[
\text{Spec}_\alpha (\mathcal{M}) \cap (-2\varepsilon, -\varepsilon] \cup [\varepsilon, 2\varepsilon] = \emptyset \tag{23}
\]
implies the existence of a hermitian element \( A \in Z(\mathcal{M}^\alpha) \) with \( \|A\| \leq \varepsilon/2 \), such that the modified action \( \widetilde{\alpha}_t := \text{Ad}(e^{itA})\alpha_t \) satisfies

\[
\text{Spec}_{\alpha}(\mathcal{M}) \cap (-\varepsilon, \varepsilon) = \{0\}.
\]

Here the main point is that \( \mathcal{N} := \mathcal{M}^\alpha[-\varepsilon, \varepsilon] \) is a subalgebra of \( \mathcal{M} \) on which \( \alpha \) is uniformly continuous, hence of the form \( \text{Ad}(e^{-itA}) \).

**Example 5.18.** Define \( U_t \in \mathcal{U}(L^2(\mathbb{T})) \) by \( (U_t f)(z) = f(e^{it}z) \) and \( \alpha_t(A) := U_t A U_t^* \) for \( A \in \mathcal{A} := B(L^2(\mathbb{T})) \). Then \( \text{Spec}_{\alpha}(\mathcal{A}) \subseteq \mathbb{Z} \) but there is no splitting as in \( \text{Prop. 1.1} \). ([Be09, Prop. 1.1])

We now assume that \( \mathcal{M} \subseteq \mathcal{B}(\mathcal{H}) \) is a von Neumann algebra and that \( \alpha_t \) is implemented by a Borchers–Arveson one-parameter group \( (U_t)_{t \in \mathbb{R}} \) of \( \mathcal{M} \) with non-negative spectrum. Let \( P \) denote the \( \mathcal{M} \)-valued spectral measure of \( U \) with \( U_t = \int_{\mathbb{R}} e^{-itz} \, dP(z) \). For the spectral projections of \( U \), we then have

\[
P(t, \infty) \mathcal{H} = \bigcap_{s < t} [\mathcal{M}^\alpha[s, \infty) \mathcal{H}]
\]

(cf. Theorem \[1.14\]). For \( 0 < t < \varepsilon \), this leads with \( \text{Prop. 1.1} \) to

\[
P(t, \infty) \mathcal{H} = [\mathcal{M}^\alpha[\varepsilon, \infty) \mathcal{H}] = [\mathcal{M}_+ \mathcal{H}].
\]

We conclude that, for \( 0 < t < \varepsilon \),

\[
P[0, t] \mathcal{H} = (P[t, \infty) \mathcal{H})^\perp = \ker \mathcal{M}^\alpha(-\infty, -\varepsilon) = \ker \mathcal{M}_- = \ker \mathcal{M}^\alpha(-\infty, 0)
\]

consists of ground state vectors ([BR96, Prop. 5.3.19(4)]). By minimality of \( U \), ground states are contained in \( P_0 \mathcal{H} \) for \( P_0 := P(\{0\}) \) which leads to

\[
P_0 = P[0, \varepsilon).
\]

This leads to:

**Lemma 5.19.** If \( 0 \) is isolated in \( \text{Spec}_{\alpha}(\mathcal{M}) \), then \( 0 \) is isolated in \( \text{Spec}(U) \).

**Remark 5.20.** With \( P_+ := 1 - P_0 = P[\varepsilon, \infty) \), we now obtain

\[
\mathcal{M} = P_0 \mathcal{M} P_0 + P_+ \mathcal{M} P_0 + P_0 \mathcal{M} P_+ + P_+ \mathcal{M} P_+ \subseteq \mathcal{M}_+ + \mathcal{M}_-
\]

and as \( P_0 \) has central support 1 (since 0 is isolated in the spectrum of \( U \)), it follows from Lemma \[3.14\] that

\[
(1 - P_0) \mathcal{H} = [P_+ \mathcal{M} \mathcal{H}_0].
\]

**Remark 5.21.** Let \( \pi_0 \) be the representation of \( P_0 \mathcal{M} P_0 \) on \( \mathcal{H}_0 \). Then

\[
\varphi(M) := \pi_0(P_0 \mathcal{M} P_0)
\]

is a completely positive linear map vanishing on the subspace \( \mathcal{M}_- + \mathcal{M}_+ \) and its restriction to \( \mathcal{M}_0 \) is a representation. Further,

\[
\varphi(M^* M) = 0 \quad \text{for} \quad M \in \mathcal{M}_-,
\]

which is equivalent to

\[
\varphi(\mathcal{M}_- \mathcal{M}_-) = \{0\}.
\]

If, conversely, \( \pi_0(\mathcal{H}_0) \) is a normal representation of \( P_0 \mathcal{M} P_0 = \mathcal{M}_0 \), then \( \varphi(M) := \pi_0(P_0 \mathcal{M} P_0) \) is a completely positive function on \( \mathcal{M} \) with \( \varphi(1) = 1 \), so that dilation leads to a representation \( (\pi, \mathcal{M}) \) containing \( \pi_0(\mathcal{H}_0) \) as a subrepresentation with respect to \( \mathcal{M}_0 \). Clearly, \( \pi(U_t) \) defines a unitary one-parameter group with non-negative spectrum and \( \pi(\mathcal{M}) \)-generating space of ground states.
Proposition 5.22. If \( \mathcal{M} \) is a factor satisfying condition \((SG)\), then there exist at most countably many projections \((P_j)_{j\in J} \) and pairwise different \( \lambda_j \geq 0 \) with \( U_t = \sum_{j\in J} e^{-it\lambda_j} P_j \). For \( j \neq k \), we further have \( |\lambda_j - \lambda_k| \geq \varepsilon \).

Proof. Let \( 0 \leq a < b \) such that \( 2(b - a) \leq \varepsilon \) and \( M \in P[a,b]MP[a,b] \). Then Spec\(_{\alpha}(M) \subseteq [a,b] - [a,b] \subseteq [-\varepsilon, \varepsilon] \) implies that

\[
P[a,b]MP[a,b] \subseteq \mathcal{M}_0.
\]

For disjoint compact subsets \( S_1, S_2 \subseteq [a,b] \), this further leads to

\[
P(S_1)MP(S_2) \subseteq \mathcal{M}^\alpha(S_1 - S_2) = \{0\}
\]

because \( S_1 - S_2 \subseteq [-\varepsilon, \varepsilon] \) does not contain \( 0 \). In view of [Sa71 Prop. 1.10.7], the central supports of \( P(S_1) \) and \( P(S_2) \) are disjoint. As \( \mathcal{M} \) is a factor, we obtain \( P(S_1) = 0 \) or \( P(S_2) = 0 \). This implies that \( U_t \) has at most a single spectral value in the interval \([a,b]\), and from that we derive that Spec\((U)\) is discrete, so that \( U_t = \sum_{j \in J} e^{-it\lambda_j} P_j \) as asserted. Then the differences \( \lambda_j - \lambda_k \) are contained in Spec\(_{\alpha}(\mathcal{M})\), which implies that \( |\lambda_j - \lambda_k| \geq \varepsilon \) for \( j \neq k \). \(\square\)

Example 5.23. In general, if \( \mathcal{M} \) is not a factor, the assumption that \( 0 \) is isolated in Spec\(_{\alpha}(\mathcal{M})\) does not imply that Spec\((U)\) is discrete. In \( \mathcal{M} := \ell^\infty(\mathbb{N}, \mathcal{B}(\ell^2)) \), we consider the minimal unitary one-parameter group given by \( U_t = (U_t^{(1)}, U_t^{(2)}, \ldots) \) with \( U_t^{(n)} \in \mathcal{B}(\ell^2) \) defined by

\[
U_t^{(n)} := P_1 + \sum_{j=0}^{\infty} e^{it(j+1+f(n))} P_{j+2},
\]

where \( P_j, j \in \mathbb{N} \), is the orthogonal projection onto \( C_{e_j} \) and \( f : \mathbb{N} \to \mathbb{Q}_+ \) is surjective. Then

\[
\text{Spec}(U) = \{0\} \cup [1, \infty).
\]

and, for \( \alpha = \text{Ad}(U) \) the block diagonal structure leads to Spec\(_{\alpha}(\mathcal{M}) = \bigcup_n \text{Spec}_{\alpha(n)}(\mathcal{M}) \), which in turn leads to

\[
\text{Spec}_{\alpha}(\mathcal{M}) = (-\infty, -1] \cup \{0\} \cup [1, \infty).
\]

A specific instance where \( 0 \) is isolated in Spec\(_{\alpha}(\mathcal{M})\) is the periodic case. We continue analysis of the periodic case, started above in Example 4.5. Let \( (\mathcal{M}, T, \alpha) \) be a \( W^*\)-dynamical system and \( (U_t)_{t\in \mathbb{R}} \) be a weakly continuous unitary one-parameter group in \( \mathcal{M} \) with positive spectrum such that \( \alpha_{e^{it}} = \text{Ad}(U_t) \) for \( t \in \mathbb{R} \) and \( U \) is minimal (cf. Definition 4.10).

The \( 2\pi\)-periodicity of \( U \) implies the existence of projections \((P_n)_{n\in \mathbb{N}_0} \) in \( \mathcal{M} \) with

\[
U_t = \sum_{n=0}^{\infty} e^{-int} P_n \quad \text{for} \quad t \in \mathbb{R}.
\]

In this case \( P_0 = P[0,\varepsilon) \) for \( 0 \leq \varepsilon \leq 1 \), so that Lemma 4.14 implies that the central support of \( P_0 \) is \( 1 \). With Lemma 3.14 this leads to

\[
\mathcal{M} = MP_0 \mathcal{M}^w.
\]

Put \( \chi_n(t) := e^{-int} \) and

\[
\mathcal{M}_n := \{ M \in \mathcal{M} \mid (\forall t \in \mathbb{R}) \alpha_{e^{it}}(M) = e^{-int}M \}.
\]

Then the subspaces \( P_0 MP_m \) are \( \alpha \)-eigenspaces with respect to the character \( \chi_{n-m} \) and the direct vector space sum \( \sum_{k=-n}^{\infty} P_{k+n} MP_k \) is weakly dense in \( \mathcal{M}_n \) for \( n \in \mathbb{N} \). In particular, the fixed point algebra \( \mathcal{M}_0 \) is the weak closure of \( \sum_{k=0}^{\infty} P_k MP_k \), where the subalgebras \( P_k MP_k \) of \( \mathcal{M} \) are two-sided ideals of \( \mathcal{M}_0 \) (as \([\mathcal{M}_0, P_n] = 0\)).
Above in Example 4.5 we noted that the fixed point projection \( p_0 : \mathcal{M} \to \mathcal{M}_0 \) by
\[
p_0(M) := \sum_{k=0}^{\infty} P_k M P_k = \int_{\mathbb{T}} \alpha_z(M) \, dz \quad \text{for } M \in \mathcal{M}
\]
is completely positive. Hence we can use the Stinespring dilation to build representations on \( \mathcal{M} \) from representations on \( \mathcal{M}_0 \).

Consider the case of a \( \mathbb{T} \)-action, i.e. a \( \mathcal{W}^\ast \)-dynamical system. For \( A_n \in \mathcal{M}_n \) we have
\[
\alpha_f(A_n) = \int_{\mathbb{T}} f(t) \alpha_t(A_n) \, dt = \int_{\mathbb{T}} f(t) e^{-int} \, dt \cdot A_n = \hat{f}(n) A_n.
\]
Therefore
\[
\pi(\alpha_f(A_n)) \Omega = \hat{f}(n) \pi(A_n) \Omega
\]
vanishes if \( \Omega \) is a ground state vector and \( \text{supp}(\hat{f}) \subseteq -\mathbb{N} \).

6 KMS states and modular groups.

A major area where covariant representations of singular actions are studied is that of KMS states and their representations. This is a fundamental part of the study of thermal quantum systems, and the literature in this area is vast. This section is only a scratch on the surface, and we will concentrate on some of the main structural issues. The standard references include [BR96], [SZ79] and for the case of \( \mathcal{W}^\ast \)-actions, a useful review of results is in [DJP03]. For a particularly interesting application in QFT, see [CR94].

6.1 Modular groups

First, we need to define the modular group (proofs and constructions are in [SZ79, Ch. 10] and [Ta03, Sect. III.4]). Let \( \mathcal{M} \subset \mathcal{B}(\mathcal{H}) \) be a von Neumann algebra, and let \( \varphi \) be a faithful, normal semifinite weight on \( \mathcal{M} \). Recalling the GNS construction for it, consider the left ideal
\[
\mathcal{N}_\varphi := \{ A \in \mathcal{M} \mid \varphi(A^*A) < \infty \}.
\]
By faithfulness of \( \varphi \) the sesquilinear form \( \langle A, B \rangle := \varphi(A^*B) \), \( A, B \in \mathcal{N}_\varphi \) is positive definite, hence we may complete \( \mathcal{N}_\varphi \) to obtain the Hilbert space \( \mathcal{H}_\varphi \). Let \( \xi : \mathcal{N}_\varphi \to \mathcal{H}_\varphi \) denote the faithful linear imbedding. Then the GNS representation \( \pi_\varphi : \mathcal{M} \to \mathcal{B}(\mathcal{H}_\varphi) \) is given by
\[
\pi_\varphi(A) \xi(B) := \xi(AB) \quad \text{for } A \in \mathcal{M}, B \in \mathcal{N}_\varphi
\]
and it is faithful. There may be no cyclic vector in \( \mathcal{H}_\varphi \), unless \( \varphi \) is bounded. By Theorem 3.3, this GNS representation is unitarily equivalent to the standard form realization of \( \mathcal{M} \). On the subspace \( \mathcal{D}_\varphi := \xi(\mathcal{N}_\varphi \cap \mathcal{N}_\varphi^*) \subset \mathcal{H}_\varphi \) there is a closable conjugate linear operator \( S_0 \) defined by
\[
S_0 \xi(A) := \xi(A^*) \quad \text{for } A \in \mathcal{N}_\varphi \cap \mathcal{N}_\varphi^*.
\]
Denote its closure by \( S_\varphi \). Then the modular operator of \( \varphi \) is the invertible positive operator (in general, unbounded) given by
\[
\Delta_\varphi := S_\varphi^* S_\varphi.
\]
The modular conjugation of \( \varphi \) is the operator \( J_\varphi := \Delta_\varphi^{1/2} S_\varphi \). Then \( (\Delta_\varphi^i)_{t \in \mathbb{R}} \) defines a strong operator continuous one parameter unitary group, and as \( \Delta_\varphi^i \mathcal{M} \Delta_\varphi^{-i} = \mathcal{M} \), this defines the modular automorphism group \( (\sigma_t^\varphi)_{t \in \mathbb{R}} \) in \( \text{Aut} \mathcal{M} \) by
\[
\sigma_t^\varphi(A) := \Delta_\varphi^i A \Delta_\varphi^{-i} \quad \text{for } A \in \mathcal{M}, t \in \mathbb{R},
\]
which is obviously a $W^*$-dynamical system. It is covariant by construction, and the generator of its implementers is $L_\varphi := -\ln \Delta$ (called the standard Liouvillean), i.e.
\[
\Delta_{\varphi}^{it} = \exp(-itL_\varphi).
\]
The relation between different modular groups on the same von Neumann algebra is given by:

**Theorem 6.1.** (Connes) Let $\mathcal{M}$ be a von Neumann algebra, and let $\varphi$ be a faithful, normal semifinite weight on $\mathcal{M}$.

(i) If $\psi$ is another faithful, normal semifinite weight on $\mathcal{M}$, then there is a unique strongly continuous path of unitaries $(u_t)_{t \in \mathbb{R}} \subset \mathcal{M}$ such that
\[
\sigma_\psi^t(A) = u_t \sigma_\varphi^t(A) u_t^* \quad \text{and} \quad u_{t+s} = u_t \sigma_\varphi^s(u_s).
\]
We write $(D\psi : D\varphi)_t := u_t$.

(ii) Conversely, if a strongly continuous path of unitaries $(u_t) \subset \mathcal{M}$ satisfies $u_{t+s} = u_t \sigma_\varphi^s(u_s)$ for all $t, s \in \mathbb{R}$, then there is a unique faithful, normal semifinite weight $\psi$ on $\mathcal{M}$ with $(u_t) = (D\psi : D\varphi)_t$ for all $t$.

This is proved in [Bla06, Thm. III.4.7.5], and in [Ta03, Thms. VIII.3.3, VIII.3.8]. The modular group also affects the adjoint action of positive unitary one-parameter groups on $\mathcal{M}$ (cf. [F98] for a direct proof and [ArZs05, Th. 2.1] for a general version):

**Theorem 6.2.** (Borchers’ Theorem on modular inclusions; [Bo92]) Let $\mathcal{M} \subset \mathcal{B}(\mathcal{H})$ be a von Neumann algebra, and let $\Omega \in \mathcal{H}$ be a cyclic and separating vector with associated vector state $\omega(\mathcal{M}) := \langle \Omega, \mathcal{M}\Omega \rangle$. Let $(U_s)_{s \in \mathbb{R}}$ be a positive unitary one-parameter group on $\mathcal{H}$ such that $U_s\Omega = \Omega$ for $s \in \mathbb{R}$ and $U_s\mathcal{M}U_s^* \subset \mathcal{M}$ for $s \geq 0$.

Then

(i) $\sigma_\omega^t(U_s) = \Delta_{\omega}^{it}U_s\Delta_{\omega}^{-it} = U_{e^{-2\pi i s}}$ for $s, t \in \mathbb{R}$, and

(ii) $J_\omega U_s J_\omega = U_s^*$ for $s \in \mathbb{R}$.

It is quite remarkable that there exist homomorphisms $\alpha : \mathbb{R} \to \text{Inn}(\mathcal{M})$ defining $W^*$-dynamical systems which do not lift to $\text{U}(\mathcal{M})$, i.e. the corresponding central extension $\hat{\mathbb{R}} := \alpha^* \text{U}(\mathcal{M})$ of $\mathbb{R}$ by $\text{U}(\mathcal{Z}(\mathcal{M}))$ is non-trivial ([Str81, §15.16]). Here is the main result behind these examples:

**Theorem 6.3.** A $W^*$-algebra $\mathcal{M}$ is semifinite if and only if the modular automorphism group of one of its faithful normal semifinite weights is implemented by a unitary one-parameter group in $\text{U}(\mathcal{M})$. Then the modular automorphism groups of all faithful normal semifinite weights are implemented by a unitary one-parameter group in $\text{U}(\mathcal{M})$.

**Proof.** See [PT73, Th. 7.4], which goes back to [La70, Ch. 14].

It follows by Theorem 6.3 that for any faithful normal semifinite weight of a factor $\mathcal{M}$ of type III, its corresponding modular automorphism group cannot be implemented by a unitary one-parameter group of $\mathcal{M}$. Consequently, the factor $\mathcal{M}$ of type III given by [Co73, Cor. 1.5.8(c)] has the remarkable property that, for every faithful normal semifinite weight, its modular automorphism group consists of inner automorphisms and yet it is not implemented by any one-parameter unitary group in $\mathcal{M}$. As explained in [Ta83, p. 21], this property can be shared only by (possibly countably decomposable) $W^*$-algebras with nonseparable predual.
On the positive side, there are nice results of Kallman and Moore building on measurable sections and Polish group structures. This requires $\mathcal{M}_*$ to be separable. More concretely, in $[\text{Ka71}]$ one finds that, for $G = \mathbb{R}$ and $\mathcal{M}_*$ separable, all inner $W^*$-dynamical systems can be implemented by one-parameter groups $U: \mathbb{R} \to \mathcal{U}(\mathcal{M})$. Note that the separability of $\mathcal{M}_*$ implies that the standard representation of $M$ is separable because the cone $C \cong \mathcal{M}_{*,+}$ is separable.

### 6.2 KMS condition

A weight $\varphi$ and its modular group $\sigma^\varphi$ satisfy the modular condition:

**Definition 6.4.** Given a $C^*$-action $(\mathcal{A}, \mathbb{R}, \alpha)$, possibly singular, then a lower semicontinuous weight $\varphi$ on $\mathcal{A}$ is said to satisfy the *KMS condition for $\alpha$ at $\beta \neq 0$* if

1. $\varphi = \varphi \circ \alpha_t$ for all $t \in \mathbb{R}$,
2. for every pair $A, B \in \mathcal{N}_\varphi \cap \mathcal{N}_\varphi^*$, there exists a bounded continuous function $F$ on the closed horizontal strip $S_\beta \subset \mathbb{C}$ where

   $$S_\beta := \{ z \in \mathbb{C} \mid 0 \leq \pm \text{Im}(z) \leq \pm \beta \} \quad \text{if} \quad \pm \beta > 0 \quad (\text{matched signs}).$$

Moreover, $F$ is analytic on the interior of $S_\beta$ and satisfies for all $t \in \mathbb{R}$:

$$F(t) = \varphi(\alpha_t(A)B), \quad F(t + i\beta) = \varphi(B\alpha_t(A)).$$

(25)

For the case $\beta = 1$ we call the KMS condition the *modular condition*. By rescaling $\alpha$, we see that $\varphi$ satisfies the KMS condition for $\alpha$ at $\beta \neq 0$ if and only if it satisfies the modular condition for $\alpha \beta^t$.

If $\varphi$ is a state, it will be called a *KMS state for $\alpha$ at $\beta$* or just a KMS state for short.

**Remark 6.5.** (a) In physical models with KMS states, $\beta$ is identified with the (negative) inverse temperature. In the case that $\varphi$ is a state (which is the case if $\mathcal{A}$ is unital and $1 \in \mathcal{N}_\varphi$), the invariance condition (i) is redundant, as invariance then follows from (ii). To see this, note that condition (25) implies that for every $A \in \mathcal{N}_\varphi \cap \mathcal{N}_\varphi^*$, there exists a bounded continuous function $F$ on the closed horizontal strip $S_\beta \subset \mathbb{C}$ which is analytic on the interior, and such that

$$F(t) = \varphi(\alpha_t(A)) = F(t + i\beta).$$

This is obtained by either substituting $1$ for $B$ into (25), or by substituting an approximate identity for $B$ into (25), and taking the limit (which is uniform in $t$). This means that we can define a new function $\tilde{F}$ on the entire complex plane by tiling $\mathbb{C}$ with vertical translates of the strip $S_\beta$, carrying along the values of $F$ on $S_\beta$. Then $\tilde{F}$ is continuous, bounded and analytic everywhere except on the horizontal lines where the strips join. By Morera’s Theorem, it is in fact analytic also on these joining lines, i.e. it is entire, and as it is bounded, by Liouville’s theorem it is constant. Thus $F(t) = \varphi(\alpha_t(A))$ is constant, i.e. (i) holds (see $[\text{BR96}, \text{Prop. 5.3.3}]$ for more details).

(b) For $C^*$-dynamical systems, Pusz and Woronowicz showed that both ground states and KMS states are “passive” states ($[\text{PW78}, \text{Thm. 1.2}]$), i.e.

$$\omega\left( -iU^*\delta(U) \right) \geq 0 \quad \text{for all} \quad U \in U_0(\mathcal{A}) \cap D(\delta)$$

where $\delta$ is the generator of $\alpha$ with domain $D(\delta) \subseteq \mathcal{A}$ ($[\text{PW78}, \text{Thm. 2.1}]$). Conversely, if a passive state is weakly clustering, then it is either KMS or a ground state $[\text{PW78}, \text{Thm. 1.3}]$.

The modular condition in fact uniquely characterizes the modular group of a weight by:
Theorem 6.6. Let $\mathcal{M}$ be a von Neumann algebra, and let $\varphi$ be a faithful normal semifinite weight on $\mathcal{M}$. Then the modular automorphism group $(\sigma_t^\varphi)_{t \in \mathbb{R}}$ in Aut $\mathcal{M}$ satisfies the modular condition. Conversely, for any $W^*$-dynamical system $(\mathcal{M}, \mathbb{R}, \alpha)$ which satisfies the modular condition for $\varphi$, the modular group $\sigma^\varphi$ coincides with $\alpha$.

This is proven in [Ta03, Thm. VIII.1.2] and [SZ79, Thm. p. 289]. Thus, every faithful, normal semifinite weight on $\mathcal{M}$ is a KMS weight for a unique one-parameter automorphism group.

Theorem 6.7. Given a $C^*$-action $(\mathcal{A}, \mathcal{G}, \alpha)$, let $\omega$ be a faithful state on $\mathcal{A}$ which satisfies the modular condition for $\alpha$. Then the normal extension $\tilde{\omega}$ of $\omega$ to $\mathcal{M} := \pi_\omega(\mathcal{A})''$ is faithful, and satisfies

$$\pi_\omega \circ \alpha_t = \sigma_t^\omega \circ \pi_\omega \quad \text{for } t \in \mathbb{R}.$$ 

This is proven in [Ta03 Prop. VIII.1.5]. In fact, the requirement that $\omega$ is faithful is too strong, one only needs that $\tilde{\omega}$ is faithful on $\mathcal{M}$ and satisfies the KMS condition with respect to $\omega$. A state on $\mathcal{A}$ which satisfies the KMS condition for $\alpha$ can therefore be characterized by this condition, i.e. that its GNS representation $\pi_\omega$ intertwines $\alpha$ with a rescaled copy of its modular group.

Proposition 6.8. Given a $C^*$-action $(\mathcal{A}, \mathcal{G}, \alpha)$, possibly singular, let $\omega$ be a KMS state for $\alpha$ at $\beta$. Then the following hold:

(i) $(\pi_\omega, U^\omega)$ is covariant.

(ii) the normal extension $\tilde{\omega}$ of $\omega$ to $\mathcal{M} := \pi_\omega(\mathcal{A})''$ by $\tilde{\omega}(M) := \langle \Omega_\omega, M\Omega_\omega \rangle$ is faithful.

(iii) If $\tilde{\alpha}_t := \text{Ad} U^\omega_t$, then $(\mathcal{M}, \mathbb{R}, \tilde{\alpha})$ is a $W^*$-dynamical system for which $\tilde{\omega}$ is a KMS state for $\tilde{\alpha}$ at $\beta$.

(iv) $\mathcal{M}' \cap \mathcal{M} \subseteq \mathcal{M}^{\tilde{\alpha}}$, the set of invariant elements of $\mathcal{M}$ with respect to $\tilde{\alpha}$ (modular automorphisms act trivially on the center).

(v) Let $\mathcal{N} \subseteq \mathcal{M}$ be a commutative von Neumann subalgebra such that $\tilde{\alpha}_t(\mathcal{N}) \subseteq \mathcal{N}$ for all $t \in \mathbb{R}$. Then $\mathcal{N} \subseteq \mathcal{M}^{\tilde{\alpha}}$.

(vi) $\mathcal{M}^{\tilde{\alpha}} = \{ A \in \mathcal{M} \mid (\forall B \in \mathcal{M}) \omega([A, B]) = 0 \}$.

Proof. (i) By the KMS condition, for every $A, B \in \mathcal{A}$, the function $t \mapsto \omega(\alpha_t(A)B)$ is continuous. By invariance of $\omega$ this implies for the GNS unitaries that $(U^\omega_t)_{t \in \mathbb{R}}$ is strong operator continuous, and so $(\pi_\omega, U^\omega)$ is covariant.

(ii) By assumption $\tilde{\omega}$ satisfies the KMS condition with respect to $\tilde{\alpha}$ on the strong operator dense subalgebra $\pi_\omega(\mathcal{A}) \subseteq \mathcal{M}$. By substituting an approximate identity for $B$ in the KMS condition [25], taking the limit and using Liouville’s theorem, we conclude that $\omega$ is $\alpha$-invariant on $\pi_\omega(\mathcal{A})$, hence on all of $\mathcal{M}$. By Lemma 6.9 below, it then follows that $\tilde{\omega}$ satisfies the KMS condition with respect to $\tilde{\alpha}$ on all of $\mathcal{M}$.

(ii) By [BR96 Theorem 5.3.10] it follows from (iii) that $\tilde{\omega}$ is faithful on $\mathcal{M}$.

(iv) Let $C \in \mathcal{M}' \cap \mathcal{M}$ and $A, B \in \mathcal{M}$. Then, by (iii), we have for some continuous bounded function $F$ on the strip $S_\beta$ that it is holomorphic on the interior, and on the boundary

$$F(t) = \tilde{\omega}(\alpha_t(AB)C) = \tilde{\omega}(C\alpha_t(AB)) = F(t + i\beta).$$

Proceeding as above, we define a new function $\tilde{F}$ on the entire complex plane by tiling $\mathbb{C}$ with vertical translates of the strip $S_\beta$, carrying along the values of $F$ on $S_\beta$. Then $\tilde{F}$ is continuous, bounded and analytic everywhere except on the horizontal lines where the strips join. By Morera’s
Thus, \( F(t) = \tilde{\omega}(\alpha_t(A)C\alpha_t(B)) = \tilde{\omega}(A\alpha_{-t}(C)B) = \langle A^*\Omega_\omega, \alpha_{-t}(C)B\Omega_\omega \rangle \) for \( t \in \mathbb{R} \).

As \( \Omega_\omega \) is cyclic and \( F \) is constant, we get that \( C \in M^\alpha \).

(v) As the restriction \( \omega_0 \) of \( \tilde{\omega} \) to \( \mathcal{N} \) is still a KMS-state with respect to the restriction \( \alpha^{(0)} \) of \( \tilde{\alpha} \) to \( \mathcal{N} \), it follows that \( \alpha^{(0)} \) coincides with the modular automorphism with respect to \( \omega_0 \). Thus by (iv), as \( \mathcal{N} \) is commutative, we have that \( \pi_{\omega_0}(\mathcal{N}) \subseteq \pi_{\omega_0}(\mathcal{N})^{\alpha^{(0)}} \), i.e. \( \pi_{\omega_0}(\tilde{\alpha}_t(N) - N) = 0 \) for all \( N \in \mathcal{N} \) and \( t \in \mathbb{R} \). By (ii), \( \tilde{\omega} \) is faithful, hence its restriction \( \omega_0 \) is faithful, and so \( \pi_{\omega_0} \) is faithful.

Thus \( \tilde{\alpha}_t(N) = N \) for all \( N \), \( t \), i.e. \( \mathcal{N} \subseteq M^\tilde{\alpha} \).

(vi) is proven in [BR96] Prop. 5.3.28].

\[ \square \]

#### Lemma 6.9.
Let \((\mathcal{M}, \mathbb{R}, \alpha)\) be a \( W^*\)-dynamical system, and let \( \omega \) be a normal \( \alpha \)-invariant state satisfying the KMS condition \((25)\) for all \( A, B \) in some \( W^*\)-dense \( \alpha \)-invariant unital *-subalgebra \( \mathcal{D} \) of \( \mathcal{M} \). Then \( \omega \) satisfies \((25)\) on all of \( \mathcal{M} \), hence is a KMS state for \( \alpha \) at \( \beta \).

\[ \text{Proof.} \] (Adapted from that of [BR96 Prop. 5.3.7]) Let \( A, B \in \mathcal{M} \) be arbitrary, and let \((A_\nu)_{\nu \in \Gamma}\) and \((B_\nu)_{\nu \in \Gamma'}\) be nets in \( \mathcal{D} \) which \( W^*\)-converge to \( A \) and \( B \) respectively. We can choose the same directed set \( \Gamma = \Gamma' \) for both nets, and by Kaplansky’s density theorem (cf. [BR02 Thm. 2.4.16]) we may choose \( \|A_\nu\| \leq \|A\|, \|B_\nu\| \leq \|B\| \) for all \( \nu \in \Gamma \). By assumption, for each pair \( A_\nu, B_\nu \in \mathcal{D} \), there exists a bounded continuous function \( F_\nu \) on the closed horizontal strip \( S_\beta \subset \mathbb{C} \) which is holomorphic on the interior of \( S_\beta \), and satisfies

\[ F_\nu(t) = \omega(\alpha_t(A_\nu)B_\nu), \quad F_\nu(t + i\beta) = \omega(B_\nu\alpha_t(A_\nu)) \quad \text{for} \quad t \in \mathbb{R}. \]

Let \( \nu > \mu \in \Gamma \). Then, by [BR96 Prop. 5.3.5], the positive function \( z \mapsto |F_\nu(z) - F_\mu(z)| \) takes its maximum on the boundary of \( S_\beta \), and hence for any \( z \in S_\beta \) we obtain on \( \mathcal{M} \):

\[ |F_\nu(z) - F_\mu(z)| \leq \max \left\{ \sup_{t \in \mathbb{R}} |\omega(\alpha_t(A_\nu)B_\nu - \alpha_t(A_\mu)B_\mu)|, \sup_{t \in \mathbb{R}} |\omega(B_\nu\alpha_t(A_\nu) - B_\mu\alpha_t(A_\mu))| \right\}. \]

Now using the \( \alpha \)-invariance of \( \omega \) we have

\[ \|\omega(\alpha_t(A_\nu)B_\nu - \alpha_t(A_\mu)B_\mu)\| = \|\omega(\alpha_t(A_\nu - A)B_\nu - \alpha_t(A_\mu - A)B_\mu + \alpha_t(A)(B_\nu - B) - \alpha_t(A)(B_\mu - B))\| \]

\[ \leq \|B\| \left( \|\pi_\omega(A_\nu^* - A^*)\Omega_\omega\| + \|\pi_\omega(A_\mu^* - A^*)\Omega_\omega\| \right) = \|B\| \left( \sup_{t \in \mathbb{R}} |\omega(\alpha_t(A_\nu)B_\nu - \alpha_t(A_\mu)B_\mu)| \right). \]

This expression converges uniformly with respect to \( t \) to zero as \( \nu \) and \( \mu \searrow \infty \). Likewise the other term \( |\omega(B_\nu\alpha_t(A_\nu) - B_\mu\alpha_t(A_\mu))| \) converges uniformly with respect to \( t \) to zero as \( \nu \) and \( \mu \searrow \infty \), hence \( (F_\nu)_\nu \) converges uniformly with respect to \( z \) to zero as both \( \nu \) and \( \mu \searrow \infty \), hence \( (F_\nu)_\nu \in \varGamma \) is a Cauchy net which converges uniformly, hence the limit function \( F(z) \) is continuous and bounded on \( S_\beta \subset \mathbb{C} \) and analytic on its interior. As

\[ F(t) = \lim_{\nu} \omega(\alpha_t(A_\nu)B_\nu) = \omega(\alpha_t(A)B) \quad \text{and} \]

\[ F(t + i\beta) = \lim_{\nu} F_\nu(t + i\beta) = \lim_{\nu} \omega(B_\nu\alpha_t(A_\nu)) = \omega(B\alpha_t(A)), \]

it follows that \( \omega \) satisfies \((25)\) for all \( A, B \in \mathcal{M} \).

\[ \square \]

#### Remark 6.10.
Recall the context of Proposition 6.8.

(a) By Proposition 6.8(ii), \( \mathcal{M} = \pi_\omega(A)^\prime \) is in standard form.
(b) By Theorem 6.7, \( \tilde{\alpha} \) coincides with a rescaled copy of the modular group of \( \tilde{\omega} \), hence there are strong restrictions on the existence of a KMS state for a given \( C^* \)-action.

(c) By Proposition 6.8(iv), the modular automorphism group of a KMS state acts trivially on the center of the corresponding von Neumann algebra. This means that it adapts to the central disintegration of this algebra into factors. Therefore the main point in understanding modular automorphism groups concerns factors.

(d) By the fact that the modular automorphism group of a KMS state acts trivially on the center of the corresponding von Neumann algebra, it is easy to give an example of a \( W^* \)-dynamical system which has no normal faithful KMS states. Just take any one with an automorphism group which is not trivial on the center.

(e) By Proposition 6.8(iv), if \( M \) is commutative, it can only have KMS states for the trivial action. Compare this with the analogous property for ground states (cf. Remark 4.15(b)). Moreover, by Proposition 6.8(v), the group \( \tilde{\alpha} \) cannot have normal eigenvectors, unless they are invariant.

(f) The spectrum of the implementing group \( U^\omega \) has been examined, and under some conditions one can even prove that \( \text{Sp}(U^\omega) \) is independent of \( \omega \) and \( \beta \) (cf. [BW76 Thm. A]). However, as \( M \) is in standard form and \( U^\omega \) are the standard form implementers given by Proposition 3.8 (using [Ta03 Prop. IX.1.17]), and the spectrum of \( U^\omega \) equals the Arveson spectrum of \( \tilde{\alpha} \) by Proposition 6.9, the reason for this is clear.

We list a few equivalent conditions, where the extension of the \( \mathbb{R} \)-action to \( \pi_\omega(A)'' \) is assumed; criteria for this are given in Corollary 2.17(iii).

**Theorem 6.11.** Given a \( C^* \)-action \( (A, \mathbb{R}, \alpha) \), possibly singular, let \( \omega \) be a state on \( A \) such that the induced action of \( \mathbb{R} \) on \( \pi_\omega(A) \) extends to an action \( \tilde{\alpha} : \mathbb{R} \to \text{Aut}(\pi_\omega(A)'' \alpha) \), and defines a \( W^* \)-dynamical system. Denote the normal extension of \( \omega \) to \( M := \pi_\omega(A)'' \) by \( \tilde{\omega} \).

Then the following are equivalent for \( \beta > 0 \):

(i) \( \omega \) is a KMS state on \( A \) for \( \alpha \) at \( \beta \)

(ii) \( \tilde{\omega}(A\tilde{\alpha}_t(B)) = \tilde{\omega}(BA) \) for all \( A, B \) in some \( W^* \)-dense \( \tilde{\alpha} \)-invariant *-subalgebra of the entire elements in \( M \) of \( \tilde{\alpha} \).

(iii) \( \tilde{\omega} \) is \( \tilde{\alpha} \)-invariant, and satisfies the spectral condition:

\[
\tilde{\omega}(A^*A) \leq e^{\beta \lambda} \tilde{\omega}(AA^*) \quad \text{for all } A \in \mathcal{M}\tilde{\alpha}(-\infty, \lambda) \text{ and } \lambda \in \mathbb{R},
\]

where \( \mathcal{M}\tilde{\alpha}(-\infty, \lambda) \) denotes the Arveson spectral subspaces.

(iv) For all \( A, B \in A \) and \( f \) with \( \tilde{f} \in C_c^\infty(\mathbb{R}) \), we have:

\[
\int_{\mathbb{R}} f(t) \omega(A\alpha_t(B)) \, dt = \int_{\mathbb{R}} f(t + i\beta) \omega(\alpha_t(B)A) \, dt.
\]

**Proof.** (i)\( \Leftrightarrow \) (ii): (i) gives via Proposition 6.8 the \( W^* \)-dynamical system \( \tilde{\alpha} : \mathbb{R} \to \text{Aut}(\pi_\omega(A)'' \alpha) \), satisfying the KMS condition for \( \tilde{\omega} \). By [BR96 Prop. 5.3.7, Def. 5.3.1], this is equivalent to the condition given in (ii).

(ii)\( \Leftrightarrow \) (iii): The restriction of the \( W^* \)-dynamical system \( \tilde{\alpha} : \mathbb{R} \to \text{Aut}(\mathcal{M}) \) to its \( W^* \)-dense continuous subalgebra \( \mathcal{M}_c \) is a \( C^* \)-dynamical system. Assume (ii). Then the restriction of \( \tilde{\omega} \) to \( \mathcal{M}_c \) is still KMS, hence by [IC82 Thm. 1.1], this is equivalent for the \( \tilde{\alpha} \)-invariant \( \tilde{\omega} \) to satisfy

\[
\tilde{\omega}(A^*A) \leq e^{\beta \lambda} \tilde{\omega}(AA^*) \quad \text{for all } A \in (\mathcal{M}_c)\tilde{\alpha}(-\infty, \lambda) \text{ and } \lambda \in \mathbb{R}.
\]
Recall from [BR02 Lemma 3.2.39(4)] that

\[ \mathcal{M}^\alpha(\mathbb{R}^n, \lambda) = \mathcal{M}^\alpha_{\mathbb{R}}(\mathbb{R}^n, \lambda) = \text{Span}\{\tilde{\alpha}_f(M) \mid M \in \mathcal{M}, \ f \in L^1(\mathbb{R}), \ \text{supp}\tilde{f} \subset (-\infty, \lambda)\}, \]

where the closure is \( W^\ast \)-closure, but for the \( C^\ast \)-dynamical system on \( \mathcal{M}_c \), the corresponding expression has a norm closure. However, as the maps \( M \mapsto \tilde{\alpha}_f(M) \) are \( \sigma(\mathcal{M}, \mathcal{M}_c) \)-continuous (cf. [BR02 Prop. 3.1.4]) it follows that \( (\mathcal{M}_c)^\alpha(\mathbb{R}^n, \lambda) \) is \( W^\ast \)-dense in \( \mathcal{M}^\alpha(\mathbb{R}^n, \lambda) \). As \( \tilde{\omega} \) is normal, by substituting for \( A \) in condition (25) a net in \( (\mathcal{M}_c)^\alpha(\mathbb{R}^n, \lambda) \) which it holds, may have zero intersection with \( \mathcal{M}_c \). Hence, \( \mathcal{M}_c \) is the smallest closed left ideal in \( \mathcal{M} \). However, as the maps \( A \mapsto M \) are \( \sigma(\mathcal{M}, \mathcal{M}_c) \)-continuous (cf. [BR02 Prop. 3.1.4]) it follows that (ii) is satisfied.

For the converse, assume (iii). Then the condition restricts to the \( C^\ast \)-dynamical system on the \( W^\ast \)-dense continuous subalgebra \( \mathcal{M}_c \), using \( (\mathcal{M}_c)^\alpha(\mathbb{R}^n, \lambda) \subseteq \mathcal{M}^\alpha(\mathbb{R}^n, \lambda) \). Thus, by [dC82 Thm. 1.1], \( \tilde{\omega} \) is KMS on \( \mathcal{M}_c \), hence it satisfies (ii) for the norm-dense subalgebra of \( \mathcal{M}_c \) consisting of the entire elements of \( \tilde{\alpha} \) (cf. [BR96 Prop. 5.3.7, Def. 5.3.1]). As this subalgebra is \( W^\ast \)-dense in \( \mathcal{M} \), (ii) is satisfied.

(ii) \( \Leftrightarrow \) (iv): First write condition (iv) as

\[ \omega(A\alpha_f(B)) = \omega(\alpha_{f_\beta}(B)A) \quad \text{for} \quad f_\beta(t) := f(t + i\beta) \]

where \( A, B \in \mathcal{A} \) and \( f \in D \). As the maps \( M \mapsto \tilde{\alpha}_f(M) \) are \( \sigma(\mathcal{M}, \mathcal{M}_c) \)-continuous (cf. [BR02 Prop. 3.1.4]), we can extend this condition to all \( \mathcal{M} \). Then the equivalence of (iv) with (ii) on the \( C^\ast \)-dynamical subsystem of \( \tilde{\alpha} \) restricted to \( \mathcal{M}_c \) is given in [BR96 Prop. 5.3.12]. As the dense \( \tilde{\alpha} \)-invariant *-subalgebra of the entire elements in \( \mathcal{M}_c \) of \( \tilde{\alpha} \) are \( W^\ast \)-dense in \( \mathcal{M} \), the equivalence with (ii) follows.

Note that condition (ii), whilst commonly used for \( C^\ast \)-dynamical systems, is not that useful for singular actions, as the subalgebra of analytic elements on which it holds, may have zero intersection with \( \pi_\omega(A) \) by Example 2.42. There is a range of other equivalent conditions for the KMS condition, e.g. in terms of correlation functions (cf. [FvB77]), Green’s functions (cf. [GuO79]), spectral passivity (cf. [dC82]), and in terms of stability with respect to local perturbations of the dynamics (cf. [HKTP77]).

Regarding the question of the existence of KMS states for a given \( C^\ast \)-action, there are very few general results, and most analyses are done in particular contexts. In the \( C^\ast \)-dynamical case, existence of KMS states is proven for approximately inner dynamics if there is a trace state (cf. [FSS77]), time evolutions of Haag-Kastler quantum field theories, satisfying a nuclearity condition (cf. [BJ80]), for the Cuntz algebra (cf. [CP78]), for the CAR-algebra (cf. [RST69]) and many others. For a singular action on the Weyl algebra there is an existence condition in [RST70].

For a general condition for existence of KMS states, the only one we know of is by Woronowicz (cf. [WW85]).

**Theorem 6.12.** (Woronowicz) Let \( (\mathcal{A}, \mathbb{R}, \alpha) \) be a unital \( C^\ast \)-dynamical system. Then there is a KMS state \( \omega \) on \( \mathcal{A} \) for \( \alpha \) at \( \beta = 1 \) if and only if \( \mathcal{L} \neq \mathcal{A}^\text{op} \otimes \mathcal{A} \) (maximal tensor product), where \( \mathcal{A}^\text{op} \) is the opposite algebra of \( \mathcal{A} \) and \( \mathcal{L} \) is the smallest closed left ideal in \( \mathcal{A}^\text{op} \otimes \mathcal{A} \) containing the set

\[ \{A \otimes 1 - 1 \otimes \alpha_{i/2}(A^\ast) \mid A \in \mathcal{A} \text{ an entire element}\}. \]

This is [WW85 Thm. 3]. The set \( \mathcal{G}_\beta \) of KMS states for \( \alpha \) at \( \beta \) has an interesting structure.

**Theorem 6.13.** Let \( (\mathcal{M}, \mathbb{R}, \alpha) \) be a \( W^\ast \)-dynamical system. Then

(i) \( \mathcal{G}_\beta \subset \mathcal{M}_c \) is convex and weakly closed, but need not be compact nor have extreme points.

(ii) \( \omega \in \mathcal{G}_\beta \) is extremal in \( \mathcal{G}_\beta \) if and only if it is a factor state.
(iii) Two extremal points of $\mathcal{S}_A$ are either equal or disjoint (i.e. have disjoint GNS representations).

See the paragraph below [BR96, Prop. 5.3.30]. Note that the proofs of (ii) and (iii) carry over directly from the corresponding proofs in [BR96, Prop. 5.3.30]. If $(\mathcal{A},\mathbb{R},\alpha)$ is a $C^*$-dynamical system, then far stronger properties listed in [BR96, Prop. 5.3.30] hold.

There is a great deal more structure for KMS states, e.g. much is known about the behavior of KMS states with respect to perturbation of $\alpha$ (cf. [BR96, Ch. 5.4], and [DJP03]). We leave this large topic for the monographs.

## 7 Ergodic states

**Definition 7.1.** (a) Let $(\mathcal{M},G,\alpha)$ be a $W^*$-dynamical system. We say that it is **ergodic** if $\mathcal{M}^G = \mathbb{C}1$ (cf. [Ta03, Def. X.3.13]).

(b) For a $C^*$-action $(\mathcal{A},G,\alpha)$ a $G$-fixed state $\omega$ is called **ergodic** if it is an extreme point of the convex set $\mathcal{S}(\mathcal{A})^G$ of all $G$-fixed states. The state $\omega$ is called **weakly ergodic** if $\mathcal{H}_\omega^G = \mathbb{C}\Omega_\omega$ holds in the corresponding covariant GNS representation $(\pi_\omega,U_\omega,\Omega_\omega)$.

**Remark 7.2.** (a) For a $C^*$-action $(\mathcal{A},G,\alpha)$, if $\mathcal{S}(\mathcal{A})^G \neq \emptyset$, then extreme points, i.e. ergodic states, exist: First, if $\mathcal{A}$ is unital, then the state space $\mathcal{S}(\mathcal{A})$ is weak-$*$-compact, and it is easy to see that the subspace of invariant states $\mathcal{S}(\mathcal{A})^G \subset \mathcal{S}(\mathcal{A})$ is weak-$*$-closed, hence is also weak-$*$-compact and convex, and nonempty. It follows from the Krein–Milman theorem that $\mathcal{S}(\mathcal{A})^G$ has extreme points, and $\mathcal{S}(\mathcal{A})^G$ is equal to the closed convex set they generate. So ergodic states exist.

If $\mathcal{A}$ is nonunital, then augment $\mathcal{A}$ with the identity to obtain $\tilde{\mathcal{A}}$. It contains the maximal ideal $\mathcal{A}$ and $\tilde{\mathcal{A}}/\mathcal{A} \cong \mathbb{C}$. Extend the action $\alpha$ to $\tilde{\mathcal{A}}$ by setting $\tilde{\alpha}_g(1) = 1$ for all $g \in G$. We identify the set $\mathcal{S}(\mathcal{A})$ of states of $\mathcal{A}$ with those states $\omega$ of $\tilde{\mathcal{A}}$ for which $\omega \upharpoonright \mathcal{A}$ is a state of $\mathcal{A}$, so that $\omega$ is uniquely determined by this restriction. Each state $\omega$ on $\tilde{\mathcal{A}}$ has a unique decomposition

$$\omega = \lambda \omega_0 + (1 - \lambda)\varphi \quad \text{with } \lambda = 1 - \|\omega \upharpoonright \mathcal{A}\| \in [0,1],$$

where $\omega_0$ is the unique state satisfying $\omega(\mathcal{A}) = 0$ and $\varphi \in \mathcal{S}(\mathcal{A})$. Then $\omega$ is invariant if and only if $\varphi \in \mathcal{S}(\mathcal{A})^G$, so that

$$\mathcal{S}(\tilde{\mathcal{A}})^G = \text{conv} \left( \{\omega_0 \} \cup \mathcal{S}(\mathcal{A})^G \right)$$

and therefore we have

$$\text{Ext}(\mathcal{S}(\tilde{\mathcal{A}})^G) = \{\omega_0\} \cup \text{Ext}(\mathcal{S}(\mathcal{A})^G),$$

where Ext($C$) denotes the set of extreme points of the convex set $C$. Here the inclusion $\subseteq$ is immediate and for the converse we use that $\mathcal{S}(\mathcal{A})$ is a face of the convex set $\mathcal{S}(\tilde{\mathcal{A}})$ which follows from the convexity of the functional $\omega \mapsto \|\omega \upharpoonright \mathcal{A}\|$. This describes the ergodic states of $(\tilde{\mathcal{A}},G,\tilde{\alpha})$ in terms of those of $(\mathcal{A},G,\alpha)$.

(b) If $(\mathcal{M},G,\alpha)$ is a $W^*$-dynamical system and a normal state $\omega$ is an extreme point in $\mathcal{S}_n(\mathcal{M})^G$, then it also is an extreme point in the larger set $\mathcal{S}(\mathcal{M})^G$ of all $G$-invariant states of the $C^*$-algebra $\mathcal{M}$. This is due to the fact that $\mathcal{S}_n(\mathcal{M}) \subseteq \mathcal{S}(\mathcal{M})$ is a face, which in turn follows from the continuity characterization in [BR02, Thm. 2.4.21]. Conversely, if it is an extreme point of $\mathcal{S}(\mathcal{M})^G$ it is an extreme point in $\mathcal{S}_n(\mathcal{M})^G$. Hence a normal state is ergodic if and only if it is extreme in the set of invariant normal states $\mathcal{S}_n(\mathcal{M})^G$.

(c) If $\omega$ is an ergodic state of a $C^*$-action $(\mathcal{A},G,\alpha)$, then the associated $W^*$-dynamical system $(\pi_\omega(\mathcal{A})^\prime\prime,G,\tilde{\alpha})$ need not be ergodic, though the converse is true. For instance, if $G = \{1\}$ or, more generally, if $G$ is arbitrary and its action on $\mathcal{A}$ is trivial, then the ergodic states of $(\mathcal{A},G,\alpha)$ are exactly the pure states of $\mathcal{A}$, and for every pure state $\omega$ of $\mathcal{A}$ one has $\pi_\omega(\mathcal{A})^\prime\prime = B(H_\omega)$. Hence the $W^*$-dynamical system $(\pi_\omega(\mathcal{A})^\prime\prime,G,\tilde{\alpha})$ is not ergodic unless $\dim H_\omega = 1$. Examples of this
type can also be constructed for nontrivial group actions, cf. Example 7.6 below. This discrepancy between ergodicity of the state $\omega$ and ergodicity of the $W^*$-dynamical system $(\pi_\omega(A)^{\prime\prime}, G, \alpha)$ is discussed in Theorem 7.4 below.

(d) Ergodic states for singular actions need not have covariant GNS representations, unlike ground states and KMS states, so are less useful. To get a covariant GNS representation, one needs also a condition in Proposition 7.2. It seems for singular actions this must be added to obtain useful ergodic states. We now give an example of an ergodic state where the GNS-representation is not covariant.

**Example 7.3.** We continue the context of Example 2.11. Let $G$ be an abelian exotic topological group. Take the left regular representation on $\ell^2(G)$, i.e. $(V_g \psi)(h) := \psi(g^{-1}h)$ for $\psi \in \ell^2(G)$, $g, h \in G$. Let $A = K(\ell^2(G))$ which is a simple ideal of $B(\ell^2(G))$. Define $\alpha : G \to \text{Aut}(A)$ by $\alpha_g(A) := V_g A V^*_g$. We showed above that the $C^*$-action $(A, G, \alpha)$ has no covariant representations, so it suffices to show that it has ergodic states. As $G$ is abelian, it is amenable (with respect to any topology), hence $(A, G, \alpha)$ has an invariant state, i.e. $\mathcal{S}(A)^G \neq \emptyset$. By (a) above, it has ergodic states.

**Theorem 7.4.** Let $(A, G, \alpha)$ be a $C^*$-action, $\omega \in \mathcal{S}(\mathcal{A})^G$ and $(\pi_\omega, U_\omega, \mathcal{H}_\omega, \Omega_\omega)$ be the corresponding covariant GNS representation. Consider the following properties:

(a) $(\pi_\omega(A)^{\prime\prime})^G = \mathbb{C}1$, i.e., the action of $G$ on $\pi_\omega(A)^{\prime\prime}$ is ergodic.

(b) $\mathcal{H}_\omega^G = \mathbb{C}\Omega_\omega$, i.e., $\omega$ is weakly ergodic.

(c) $\omega$ is $G$-ergodic, i.e., an extreme point of $\mathcal{S}(\mathcal{A})^G$.

(d) $\pi_\omega(A) \cup U_\omega(G)$ acts irreducibly on $\mathcal{H}_\omega$.

(e) $\pi_\omega(A)^{\prime\prime}$ is of type III or $\Omega_\omega$ is a trace vector for $\pi_\omega(A)$.''

Then the implications (a) $\Rightarrow$ (b) $\Rightarrow$ (c) $\Leftrightarrow$ (d) and (b) $\Rightarrow$ (e) hold. Moreover, (a) implies that $\Omega_\omega$ is separating for $\pi_\omega(A)^{\prime\prime}$. On the other hand, if $\Omega_\omega$ is a separating vector for $\pi_\omega(A)^{\prime\prime}$, then the four conditions (a)–(d) are equivalent.

The relations between (a) to (d) are in [BR02 Thm. 4.3.20], whereas the implication (b) $\Rightarrow$ (e) is in [Lo79 Thm. 1], which is a Theorem by Hugenholtz and Størmer (cf. [Hu67, St67]).

**Lemma 7.5.** Let $\mathcal{M} \subseteq B(\mathcal{H})$ be a von Neumann algebra and $G \subseteq U(\mathcal{H})$ be a subgroup normalizing $\mathcal{M}$. Suppose further that $\Omega \in \mathcal{H}_G^G$ is a $G$-fixed cyclic separating unit vector for $\mathcal{M}$. Then $G$ commutes with the corresponding modular objects $J$ and $\Delta$.

**Proof.** Denote the action of $G$ on $\mathcal{M}$ by $\alpha_g(M) := g M g^*$. As $G$ fixed $\Omega$, we have $g(\mathcal{M} \Omega) = \alpha_g(M) \Omega$, and this implies that the unbounded antilinear involution defined by $S(M \Omega) := M^* \Omega$ for $M \in \mathcal{M}$ commutes with $G$. Now $J$ and $\Delta$ are uniquely determined by the polar decomposition $S = J \Delta^{1/2}$, hence also commute with $G$.

The following theorem is a refinement of the preceding one for von Neumann algebras with a cyclic vector $\Omega$. It clarifies in particular to which extent (c) implies (a), resp., (b). Note that $\Omega$ is separating if and only if $p = s(\omega) = 1$.

**Theorem 7.6.** Let $\mathcal{M} \subseteq B(\mathcal{H})$ be a von Neumann algebra, $G \subseteq U(\mathcal{H})$ be a subgroup normalizing $\mathcal{M}$, $\Omega \in \mathcal{H}_G^G$ be an $\mathcal{M}$-cyclic vector and $\omega \in \mathcal{S}(\mathcal{M})^G$ be the corresponding state. We write $p = s(\omega) \in \mathcal{M}$ for its carrier projection. Then the following are equivalent:

(i) $\mathcal{M} \cup U_\omega(G)$ acts irreducibly on $\mathcal{H}_\omega$.

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Lemma 3.13). Since $\Omega$ is cyclic for $M$, the equivalence of (iii) and (iv) follows by applying the equivalence of (i) and (iii) to $J$-conjugation follows from the fact that, on $H$, (v) are equivalent.

The cyclic representation of $M$ of $M$ is a simplex if and only if every invariant ergodic state $G$ isomorphic to $\mathcal{G}_0$. Hence (iii) and (iv) follows from Theorem 7.4 because $\Omega$ is a separating cyclic vector for $M$. Therefore the equivalence of (iii) and (iv) follows by applying the equivalence of (i) and (iii) to $M_p$ instead of $M$.

(iii) $\Leftrightarrow$ (v): As the representation of $M_p$ on $H_p := pH$ is standard by Lemma 7.6, the corresponding conjugation $J'$ yields an antilinear $G$-equivariant bijection $M_p \to M'$. Here the $G$-equivariance follows from the fact that, on $H_p$, the $G$-action commutes with $J$ by Lemma 7.5. Hence (iii) and (v) are equivalent.

(v) $\Leftrightarrow$ (vi) follows from Theorem 7.4 because $\Omega$ is a separating cyclic vector for $M_p$. 

Remark 7.7. We have seen above that a weakly ergodic state is in particular ergodic. So it is natural to look for sufficient conditions for the converse to be true. Suppose that $A$ is a separable $C^*$-algebra, $G$ locally compact separable and $(A, G, \alpha)$ a $C^*$-dynamical system. Then $A$ is $G$-abelian (i.e. $\mathcal{E}(A)^G$ is a simplex) if and only if every invariant ergodic state $\omega \in \mathcal{E}(A)$ is weakly ergodic (cf. [DNN75 Thm. 2]).

Proposition 7.8. Let $(M, H, J, \mathcal{C})$ be a von Neumann algebra in standard form, identify $\text{Aut}(M)$ with $U(H)_M$ and consider a subgroup $G \subseteq U(H)_M$. The following are equivalent

(i) $M \cup G$ acts irreducibly on $H$.

(ii) $(M')^G = \mathbb{C}1$.

(iii) $M' \cup G$ acts irreducibly on $H$.

(iv) $M^G = \mathbb{C}1$.

Proof. Conjugating with $J$ implies the equivalence of (i)/(iii), (ii)/(iv). The equivalence between (i) and (ii) and of (iii) and (iv) follows from Schur’s Lemma.

Remark 7.9. Suppose that $(M, G, \alpha)$ is a $W^*$-dynamical system where $M$ is commutative and $\omega \in \mathcal{E}_n(M)$ is a faithful separating normal state. Then $M$ is countably decomposable, hence isomorphic to $L^\infty(X, \mathcal{E}, \mu)$ for a finite measure space. Then $M_* := L^1(X, \mathcal{E}, \mu)$ and the standard representations can be realized on $H := L^2(X, \mathcal{E}, \mu)$. The group $G$ acts on this space by

$$U_g f = \delta(g)^{1/2}(g_* f),$$

where $\delta(g) \in L^1(X, \mathcal{E}, \mu)$ is the Radon–Nikodym derivative defined by $g_* \mu = \delta(g) \mu$. Note that the implementability of $G$ on the measurable space $(X, \mathcal{E})$ may be problematic if $G$ is not locally compact second countable, but in any case the unitary representation on $H$ exists and so does the action of $G$ on the Boolean $\sigma$-algebra $\mathcal{E}_\mu = \mathcal{E} / \sim$, where $E \sim F$ with $\mu(E \Delta F) = 0$. This Boolean $\sigma$-algebra is the space of projections in $M$. 

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That $\mu$ is ergodic means that $(\mathcal{M}')^G = \mathcal{M}^G = \mathbb{C}1$. Now $\mathcal{H}^G \neq \{0\}$ holds if $[\mu]$ contains a $G$-invariant finite measure. In fact, $f \in \mathcal{H}^G$ implies that $|f|^2 \mu$ is $G$-invariant. For the translation action of $\mathbb{R}$ on itself we have $\mathcal{H}^G = \{0\}$.

A Auxiliary results

Lemma A.1. Let $G$ be a connected topological group acting on a nonempty set $X$. We consider the corresponding unitary representation $(\pi, \ell^2(X))$. Then

(i) every $G$-continuous vector $\xi \in \ell^2(X)$ is fixed, and

(ii) $\ell^2(X)^G$ is generated by the characteristic functions of the finite $G$-orbits in $X$.

Proof. (i) Let $\xi \in \ell^2(X)$ be non-zero $G$-continuous vector and $c \in \mathbb{C}^X$ be such that $\xi_x = c$ for some $x \in X$. Then $F_c := \{x \in X : \xi_x = c\}$ is a finite subset of $X$. We write $P_c : \ell^2(X) \rightarrow \ell^2(F_c)$ for the corresponding orthogonal projection. Let $\varepsilon > 0$ be such that $|\xi_y - c| > \varepsilon$ for $y \not\in F_c$. If $g \in G$ satisfies $\|P_c(g\xi - \xi)\| < \varepsilon$, then, for every $x \in F_c$, we have $|\xi_x - g^{-1}x| < \varepsilon$, hence $g^{-1}x \in F_c$. Now the finiteness of $F_c$ implies that $g.F_c = F_c$ and hence $P_c(g\xi - \xi) = 0$. We conclude that

$$U := \{g \in G : \|P_c(g\xi - \xi)\| < \varepsilon\}$$

is an open closed identity neighborhood of $G$. Since $G$ is connected, it follows that $G = U$. This shows that all the subsets $F_c$ are $G$-invariant, and this in turn entails that $\xi$ is fixed under $G$.

(ii) is trivial. \qed

Lemma A.2. If $\omega \in \mathcal{A}^*$ is a tracial state, then

$$\ker \pi_\omega = \{A \in \mathcal{A} : \omega A = 0\}.$$ 

Proof. This follows from the fact that the cyclic element $\Omega \in \mathcal{H}_\omega$ is also separating: If $\omega A = 0$ and $B \in \mathcal{A}$, then [BR02] Rem. 3.2.66] yields

$$\langle \pi_\omega(AB)\Omega, \pi_\omega(AB)\Omega \rangle = \omega(B^*A^*AB) = \omega(BB^*A^*A) = 0. \quad \Box$$

Lemma A.3. Let $(U_t)_{t \in \mathbb{R}}$ and $(V_t)_{t \in \mathbb{R}}$ be two commuting continuous unitary one-parameter groups on $\mathcal{H}$ with non-negative spectrum, and put $W_t := U_tV_t$. If $A$ and $B$ are the infinitesimal generators of $U$ and $V$, respectively, then $A + B$ is closed and the infinitesimal generator of $W$.

Proof. Decomposing $\mathcal{H}$ into cyclic subspaces with respect to the representation of $\mathbb{R}^2$, defined by $(t, s) \rightarrow U_tV_s$, we may w.l.o.g. assume that $\mathcal{H} = L^2(\mathbb{R}^2, \mu)$ for a finite measure $\mu$ and that

$$(U_tF)(x, y) = e^{-itx}F(x, y) \quad \text{and} \quad (V_sF)(x, y) = e^{-isy}F(x, y).$$

Our assumption now implies that $\text{supp}(\mu) \subseteq [0, \infty)^2$. We further have $(AF)(x, y) = xF(x, y)$ and $(BF)(x, y) = yF(x, y)$. We define $(CF)(x, y) := (x + y)F(x, y)$ on its maximal domain

$$\mathcal{D}(C) := \left\{F \in L^2(\mathbb{R}^2, \mu) : \int_{\mathbb{R}^2} (x + y)^2 |F(x, y)|^2 d\mu(x, y) < \infty \right\}$$

and note that this is the infinitesimal generator of $W$. Then $\mathcal{D}(C) = \mathcal{D}(A) \cap \mathcal{D}(B)$ follows from the positivity of the functions $x$ and $y$ $\mu$-almost everywhere. \qed

Lemma A.4. Let $\mathcal{A}$ be a unital $C^*$-algebra for which the spectrum of every hermitian element is finite. Then $\mathcal{A}$ is finite dimensional.
**Proof.** Let $C \subseteq A$ be maximal abelian. Then $C$ inherits the finite spectrum property from $A$, and this implies that $C \cong C(X)$, where $X$ is a compact Hausdorff space on which every continuous function has finitely many values. This implies that $X$ is finite.

If $|X| = n$, then $C$ has a basis $(p_1, \ldots, p_n)$ consisting of minimal mutually orthogonal projections. Now

$$1 = p_1 + \cdots + p_n \quad \text{and} \quad p_ip_j = \delta_{ij}p_i.$$  

This leads to the decomposition $A = \sum_{i,j=1}^n p_iA_j$. Put $A_{ij} := p_iA_j$. The minimality of each $p_i$ implies that $A_{ii} = \mathbb{C}p_i$ is one-dimensional. Now let $i \neq j$ and $0 \neq z \in A_{ij}$. Then $0 \neq zz^* \in A_{ii} = \mathbb{C}p_i$. Hence

$$zw^* := h(z, w)p_i$$

defines a positive definite hermitian form $h$ on $A_{ij}$. If $w \in A_{ij}$ is orthogonal to $z$, then $zw^* = 0$ leads to $zw^*w = 0$. As $w^*w \in A_{jj} = \mathbb{C}p_j$ is non-zero if $w \neq 0$, it follows that $w^*w = 0$. Therefore $\dim A_{ij} = 1$ and thus $\dim A \leq n^2$.

With the preceding lemma one easily verifies the following (see the proof of [CM80 Thm. 1]):

**Proposition A.5.** Let $A$ be a unital $C^*$-algebra and let $\Gamma \subseteq \text{Aut}(A)$ be a subgroup which is compact in the norm topology. If $\Gamma$ acts ergodically on $A$, i.e., $A^\Gamma = \mathbb{C}1$, then $A$ is finite dimensional.

**Proof.** We consider the conditional expectation

$$f: A \to \mathbb{C}, \quad f(A)1 = \int_\Gamma \alpha_\gamma(A)\ d\gamma,$$

where $d\gamma$ refers to the normalized Haar measure $\mu_\Gamma$ on $\Gamma$, using the assumption that $A^\Gamma = \mathbb{C}1$.

For $\varepsilon \in (0, 1)$ we pick an open 1-neighborhood $U \subseteq \Gamma$ such that $\|\alpha_\gamma - \text{id}_A\| < \varepsilon$ for $\gamma \in U$. For $0 \leq A \in A$ we then have

$$f(A)1 \geq \int_U \alpha_\gamma(A)\ d\gamma = \int_U (\alpha_\gamma(A) - A)\ d\gamma + \mu_\Gamma(U)A \geq 0.$$  

As $\|\int_U (\alpha_\gamma(A) - A)\ d\gamma\| \leq \varepsilon\mu_\Gamma(U)\|A\|$, this leads to

$$f(A) = \|f(A)1\| \geq \mu_\Gamma(U)\|A\| - \mu_\Gamma(U)\varepsilon\|A\| = c\|A\| \quad (28)$$

where $c := \mu_\Gamma(U)(1 - \varepsilon)$. If $p_1, \ldots, p_n \in A$ satisfy $0 \leq p_i \leq 1$, $\|p_i\| = 1$, and $\sum_{i=1}^n p_j = 1$, then $1 = f(1) = \sum_{i=1}^n f(p_i) \geq cn$, and hence $n \leq c^{-1}$. Thus, if $C \cong C(X)$ is a commutative subalgebra of $A$, then all partitions of unity of $X$ are finite, and hence $X$ is finite. Now the proof of Lemma A.4 shows that $A$ is finite dimensional with $\dim A \leq c^{-2}$.

**Example A.6.** Examples of an ergodic state $\omega$ of a $C^*$-action $(A, G, \alpha)$, where the associated $W^*$-dynamical system $(\pi_\omega(A)^G, G, \tilde{\alpha})$ need not be ergodic, for nontrivial group actions.

Let $(A, G, \alpha)$ be a $C^*$-dynamical system where $G$ is a compact group, and consider the faithful conditional expectation

$$E: A \to A^G, \quad E(A) = \int_G \alpha_\gamma(A)\ d\gamma,$$

obtained by averaging with respect to the probability Haar measure on $G$. Then it is easily checked that $\mathcal{G}(A)^G = \{\omega \in \mathcal{G}(A) \mid \omega \circ E = \omega\}$ and the map $\mathcal{G}(A)^G \to \mathcal{G}(A)^G, \omega_0 \mapsto \omega_0 \circ E$, is an affine isomorphism. Hence the ergodic states of $A$ are exactly the states $\omega = \omega_0 \circ E$ where $\omega_0 = \omega|_{A^G} \in \mathcal{G}(A)^G$ is a pure state of $A^G$.

For any $\omega = \omega_0 \circ E \in \mathcal{G}(A)^G$ with $\omega_0 \in \mathcal{G}(A)$, the inclusion map $A^G \hookrightarrow A$ leads to an isometric embedding of Hilbert spaces $\mathcal{H}_{\omega_0} \hookrightarrow \mathcal{H}_{\omega}$ and the corresponding orthogonal projection $P: \mathcal{H}_{\omega} \to \mathcal{H}_{\omega_0}$ is the extension by continuity of the conditional expectation $E: A \to A^G$. Moreover,
for every \( A \in \mathcal{A}^\sigma \) one has \( E(AB) = AB \) for all \( B \in \mathcal{A}^\sigma \), hence \( P\pi_\omega(A)|_{\mathcal{H}_0} = \pi_\omega(A) \). This shows that one has the well-defined surjective linear map \( \pi_\omega(A) \to \pi_\omega(A), T \mapsto PT|_{\mathcal{H}_0} \), which implies \( \dim \pi_\omega(A) \leq \dim \pi_\omega(A) \).

If, moreover, the group \( G \) is finite and \( \omega \in \mathfrak{S}(\mathcal{A}) \) is a state whose corresponding \( W^* \)-dynamical system \( (\pi_\omega(A)^\sigma, G, \alpha) \) is ergodic, then \( \dim \pi_\omega(A)^\sigma < \infty \) by Proposition \ref{prop:ergodicity} hence \( \dim \pi_\omega(A) < \infty \) by the preceding paragraph. But at least for the permutation group \( G = S_n \), there are many dynamical systems \((\mathcal{A}, G, \alpha)\) and pure states \( \omega_\alpha \in \mathfrak{S}(\mathcal{A}^\sigma) \) with \( \dim \pi_\omega(A) = \infty \), with \( S_n \) acting by permutations on \( \mathcal{A} = B^{\otimes n} \) for various \( C^* \)-algebras \( B \). See for instance \cite[Ex. 2.3]{BN16}.

## B Commutative von Neumann algebras

Let \((X, \mathcal{S}, \mu)\) be a \( \sigma \)-finite measure space. Then we may identify \( L^\infty(X, \mathcal{S}, \mu) \) with the algebra \( \mathcal{M} \) of multiplication operators on \( L^2(X, \mathcal{S}, \mu) \) and any function \( f \in L^2(X, \mathcal{S}, \mu) \) for which \( f^{-1}(0) \) is a zero-set is a cyclic separating vector, from which one easily derives that \( \mathcal{M} = \mathcal{M}' \) is maximal abelian in \( B(\mathcal{H}) \); in particular \( \mathcal{M} \) is a commutative von Neumann algebra.

The following theorem provides an effective tool to determine when a \( * \)-invariant subset \( S \subseteq \mathcal{M} \) generates \( \mathcal{M} \) as a von Neumann algebra, i.e., \( S'' = \mathcal{M} \). This is achieved by a description of all von Neumann subalgebras of the von Neumann subalgebra \( \mathcal{M} = L^\infty(X, \mathcal{S}, \mu) \subseteq B(L^2(X, \mathcal{S}, \mu)) \).

**Theorem B.1.** (The \( L^\infty \)-Subalgebra Theorem) Let \((X, \mathcal{S}, \mu)\) be a \( \sigma \)-finite measure space and \( \mathcal{A} \subseteq L^\infty(X, \mathcal{S}, \mu) \subseteq B(L^2(X, \mathcal{S}, \mu)) \) be a von Neumann algebra. Then

\[
\mathfrak{A} := \{ E \in \mathcal{S} \colon \chi_E \in \mathcal{A} \}
\]

is a \( \sigma \)-subalgebra of \( \mathfrak{S} \) and

\[
\mathcal{A} \cong L^\infty(X, \mathfrak{A}, \mu|_{\mathfrak{A}}).
\]

Conversely, for every \( \sigma \)-subalgebra \( \mathfrak{A} \subseteq \mathfrak{S} \), \((X, \mathfrak{A}, \mu|_{\mathfrak{A}})\) is a von Neumann subalgebra of \( L^\infty(X, \mathcal{S}, \mu) \).

**Proof.**

**Step 1:** First we show that \( \mathfrak{A} \) is a \( \sigma \)-algebra. Clearly \( 0 \in \mathcal{A} \) implies \( 0 \in \mathfrak{A} \), and since \( 1 \in \mathcal{A}' = \mathcal{A} \), we also have \( \chi_{E^c} = 1 - \chi_E \in \mathcal{A} \) for each \( E \in \mathfrak{A} \). From \( \chi_{E^c} \cdot \chi_F = \chi_{E \cap F} \) we derive that \( \mathfrak{A} \) is closed under finite intersections. Now let \( (E_n)_{n \in \mathbb{N}} \) be a sequence of elements in \( \mathfrak{A} \). It remains to show that \( E := \bigcap_{n \in \mathbb{N}} E_n \in \mathfrak{A} \). Let \( F_n := E_1 \cap \cdots \cap E_n \). Then \( F_n \in \mathfrak{A} \) implies \( \chi_{F_n} \in \mathcal{A} \). Moreover, \( \chi_{F_n} \to \chi_F \) holds pointwise, so that \( \chi_{F_n} \to \chi_F \) in the weak operator topology, so that \( \chi_{F_n} \to \chi_F \) in \( \mathcal{A} \) and thus \( E \in \mathfrak{A} \). This proves that \( \mathfrak{A} \) is a \( \sigma \)-algebra.

**Step 2:** That \( \mathcal{A} \supseteq L^\infty(X, \mathfrak{A}, \mu|_{\mathfrak{A}}) \) follows directly from the fact that \( \mathcal{A} \) contains all finite linear combinations \( \sum_j c_j \chi_{E_j} \), \( E_j \in \mathfrak{A} \), the norm-closedness of \( \mathcal{A} \) and the fact that every element \( f \in L^\infty(X, \mathfrak{A}, \mu|_{\mathfrak{A}}) \) is a norm-limit of a sequence of step functions \( f_n \).

**Step 3:** Finally we show that \( \mathcal{A} \subseteq L^\infty(X, \mathfrak{A}, \mu|_{\mathfrak{A}}) \), i.e., that all elements of \( \mathcal{A} \) are \( \mathfrak{A} \)-measurable (if possibly modified on sets of measure zero).

Note that \( \mathcal{A} \) is closed under bounded pointwise limits. Let \( (p_n)_{n \in \mathbb{N}} \) be the sequence of polynomials converging on \([0, 1]\) uniformly to the square root function. For \( 0 \neq f \in \mathcal{A} \), we consider the functions \( p_n(\frac{\sqrt{|f|}}{\|f\|_\infty}) \), which also belong to \( \mathcal{A} \). Since they converge pointwise to \( \frac{|f|}{\|f\|_\infty} \), we see that \( |f| \in \mathcal{A} \). For real-valued elements \( f, g \in \mathcal{A} \), this further implies that

\[
\max(f, g) = \frac{1}{2}(f + g + |f - g|) \in \mathcal{A}.
\]

For any \( c \in \mathbb{R} \), it now follows that \( \max(f, c) \in \mathcal{A} \). The sequence \( e^{-n(\max(f, c) - c)} \in \mathcal{A} \) is bounded and converges pointwise to the characteristic function \( \chi_{\{f \leq c\}} \) of the set \( \{f \leq c\} := \{ x \in X : f(x) \leq c \} \).
We thus obtain that \(\chi_{\{f \leq c\}} \in \mathcal{A}\). We conclude that the set \(\{f \leq c\}\) is contained in the \(\mu\)-completion \(\mathfrak{A}_\mu\) of \(\mathfrak{A}\), and this finally shows that \(f \in L^\infty(X, \mathfrak{A}_\mu, \mu) = L^\infty(X, \mathfrak{A}, \mu)\).

\[\square\]

**Corollary B.2.** If \((X, \mathcal{G}, \mu)\) is a \(\sigma\)-finite measure space and \(\mathcal{F} \subseteq L^\infty(X, \mathcal{G}, \mu)\) is a subset with the property that \(\mathcal{G}\) is the smallest \(\sigma\)-algebra for which all elements of \(\mathcal{F}\) are measurable, then \(\mathcal{F}'' = L^\infty(X, \mathfrak{A}, \mu|_\mathfrak{A})\) is contained in the \(\mu\)-completion \(\mathcal{A}_\mu\) of \(\mathcal{A}\), and this finally shows that \(f \in L^\infty(X, \mathfrak{A}_\mu, \mu)\).

**Proof.** We have seen in Theorem [B.1] that \(\mathcal{F}'' = L^\infty(X, \mathfrak{A}, \mu|_\mathfrak{A})\) holds for a \(\sigma\)-subalgebra \(\mathfrak{A} \subseteq \mathcal{G}\). Then all elements of \(\mathcal{F}\) are measurable with respect to the \(\mu\)-completion \(\mathfrak{A}_\mu\) of \(\mathfrak{A}\), so that \(\mathcal{G} \subseteq \mathfrak{A}_\mu\). This implies that

\[\mathcal{F}'' = L^\infty(X, \mathfrak{A}, \mu|_\mathfrak{A}) = L^\infty(X, \mathfrak{A}_\mu, \mu|_\mathfrak{A}) \supseteq L^\infty(X, \mathcal{G}, \mu)\]

\[\square\]

**C A corrigendum to \([Ne14]\)**

In this short section we provide a corrigendum for a few wrong statements in \([Ne14]\) which have no consequences in that paper.

We consider a \(C^*\)-action \((A, G, \alpha)\). In the introduction of \([Ne14]\) and in \([Ne14]\) p. 314] we say that in \([Bo83]\) a state \(\omega \in \mathcal{S}(A)\) occurs in a covariant representation if and only if \(\omega \in (A^*)_c\). This is not correct in general and rectified by Theorem [2.20] but it is ok for \(C^*\)-dynamical systems (Corollary [2.22]). We need, in addition, that \(A_\omega A \subseteq (A^*)_c\).

Note also that \([Ne14]\) Cor. 6.3(ii)] is correct because there it is assumed that the action of \(T\) on \(G\) is continuous.

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