One-loop $f(R)$ gravity in de Sitter universe

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Received 14 January 2005
Accepted 14 February 2005
Published 24 February 2005

Abstract. Motivated by the dark energy issue, the one-loop quantization approach for a family of relativistic cosmological theories is discussed in some detail. Specifically, general $f(R)$ gravity at the one-loop level in a de Sitter universe is investigated, extending a similar program developed for the case of pure Einstein gravity. Using generalized zeta regularization, the one-loop effective action is explicitly obtained off shell, which allows us to study in detail the possibility of (de)stabilization of the de Sitter background by quantum effects. The one-loop effective action may also be useful for the study of the constant curvature black hole nucleation rate and it provides a plausible way of resolving the cosmological constant problem.

Keywords: dark energy theory, quantum field theory on curved space
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1. Introduction

Recent astrophysical data indicate that our universe is currently in a phase of accelerated expansion. One possible explanation for this fact is to postulate that gravity is nowadays being modified by some terms which grow when curvature decreases. This could be, for instance, an inverse curvature term [1] which might have its origin at a very fundamental level, such as string/M-theory [2] or the presence of higher dimensions [3], which seem able to explain such accelerated expansion. Gravity modified with inverse curvature terms is known to contain some instabilities [4] and cannot pass some solar system tests, but further modifications of the same which include higher derivative, curvature squared terms make it again viable [5]. (The Palatini formulation may also improve the situation; for recent discussion, see [6] and references therein.)

Having in mind possible applications of modified gravity for the late time universe, the following question appears. If it so happens that Einstein gravity is only an approximate theory, looking at the early universe should this (effective) quantum gravity be different from the Einsteinian one? A widely discussed possibility in this direction is quantum $R^2$ gravity (for a review, see [7]). However, other modifications are welcome as well, because they sometimes produce extra terms which may help to realize the early time inflation. This is supported by the possibility of accelerated expansion with simple modified gravity. Thus, we will study here general $f(R)$ gravity at the one-loop level in a de Sitter universe. A similar program for the case of pure Einstein gravity (recall that it is also multiplicatively non-renormalizable) has been initiated in [8]–[10] (see also [11]). Using generalized zeta-function regularization (see, for instance [12,13]), one can get the one-loop effective action and then study the possibility of stabilization of the de Sitter background by quantum effects. Moreover, such an approach hints also at a possible way of resolving the cosmological constant problem [10]. Hence, the study of one-loop $f(R)$ gravity is a natural step to be undertaken for the completion of such a programme, keeping always in...
mind, however, that a consistent quantum gravity theory is not available yet. But in any case, one should also not forget that, from our present knowledge, current gravity, which might indeed deviate from Einstein’s one, ought to have its origin in the Planck era and come from a more fundamental quantum gravity/string/M-theory approach.

Let us briefly review the classical modified gravity theory which depends only on scalar curvature:

\[ I = \frac{1}{\kappa^2} \int d^4x \sqrt{-g} f(R). \]  

(1.1)

Here \( \kappa^2 = 16\pi G \) and \( f(R) \) is, in principle, an arbitrary function. By introducing the auxiliary fields \( A \) and \( B \), one may rewrite the action (1.1) as

\[ I = \frac{1}{\kappa^2} \int d^4x \sqrt{-g} [B(R - A) + f(A)]. \]  

(1.2)

Using the equation of motion and deleting \( B \) one gets to the Jordan frame action. Using the conformal transformation \( g_{\mu\nu} \rightarrow e^\sigma g_{\mu\nu} \) with \( \sigma = -\ln f'(A) \), we obtain the Einstein frame action (scalar–tensor gravity), as follows:

\[ I_E = \frac{1}{\kappa^2} \int d^4x \sqrt{-g} \left\{ R - \frac{3}{2} \left[ \frac{f''(A)}{f'(A)} \right]^2 g^{\sigma\rho} \partial_\rho A \partial_\sigma A - \frac{A}{f'(A)} + \frac{f(A)}{f'(A)^2} \right\} \]

\[ = \frac{1}{\kappa^2} \int d^4x \sqrt{-g} \left[ R - \frac{3}{2} g^{\rho\sigma} \partial_\rho \sigma \partial_\sigma \sigma - V(\sigma) \right], \]  

(1.3)

\[ V(\sigma) = e^\sigma g(e^{-\sigma}) - e^{2\sigma} f(g(e^{-\sigma})) = \frac{A}{f'(A)} - \frac{f(A)}{f'(A)^2}. \]  

(1.4)

Note that two such classical theories, in these frames, are mathematically equivalent. It is known that they are not equivalent however at the quantum level (off shell), due to the use of different parametrizations. Even at the classical level, the physics they describe seems to be different. For instance, in the Einstein frame matter does not seem to freely fall along geodesics, which is a well established fact.

As an interesting and specific example of the general setting above, the following action corresponding to gravity modified at large distances may be considered [1,14]:

\[ I = \frac{1}{\kappa^2} \int d^4x \sqrt{-g} (R - \mu R^{-n}). \]  

(1.5)

Here \( \mu \) is an (extremely small) coupling constant and \( n \) is some number, assumed to be \( n > -1 \). The function \( f(A) \) and the scalar field \( \sigma \) are

\[ f(A) = A - \mu A^{-n}, \quad \sigma = -\ln \left( 1 + n\mu A^{-n-1} \right). \]  

(1.6)

The Friedman–Robertson–Walker (FRW) universe metric in the Einstein frame is chosen as

\[ ds^2_E = -dt_E^2 + a_E^2(t_E) \sum_{i=1}^3 (dx^i)^2. \]  

(1.7)
If the curvature is small, the solution of equation of motion is [14]
\[ a_E \sim t_E^{3(n+1)^2/(n+2)^2}, \quad \sigma = \frac{n+1}{n+2} \ln t_E, \quad t_{E_0}^2 = \frac{(n+1)^2}{(n+2)^2} \left( 1 - \frac{3(n+1)^2}{4(n+2)^2} \right). \]  
(1.8)

The FRW metric in the Jordan frame is
\[ ds^2 = -dt^2 + a^2(t) \sum_{i=1}^{3} (dx^i)^2, \]  
(1.9)

where the variables in the Einstein frame and in the physical Jordan frame are related to each other by
\[ t = \int e^{\sigma/2} dt_E, \quad a = e^{\sigma/2} a_E, \]  
(1.10)

which gives \( t \sim t_E^{1/(n+2)} \) and
\[ a \sim t^{(n+1)(2n+1)/(n+2)}, \quad w = -\frac{6n^2 + 7n - 1}{3(n+1)(2n+1)}. \]  
(1.11)

The first important consequence of the above equation (1.11) is that there is the possibility of accelerated expansion for some choices of \( n \) (a kind of effective quintessence). In fact, if \( n > (-1 + \sqrt{3})/2 \) or \(-1 < n < -1/2\), it follows that \( w < -1/3 \) and \( d^2a/dt^2 > 0 \). This is the reason why such a theory [1] and some modifications thereof [5], [15]–[18] have indeed been widely considered as candidates for gravitational dark energy models. The black holes and wormholes in such models have also been discussed [19, 20].

When \(-1 < n < -1/2\), one arrives at \( w < -1 \), i.e. the universe is shrinking. If we replace the direction of time by changing \( t \) to \(-t\), the universe is expanding but \( t \) should be considered to be negative so that the scale factor \( a \) ought to be real. Then there appears a singularity at \( t = 0 \), where the scale factor \( a \) diverges as \( a \sim (-t)^{2/3(n+1)} \). One may shift the origin of time by further changing \(-t\) to \( t_s - t \). Hence, in the present universe, \( t \) should be less than \( t_s \) and a singularity is seen to appear at \( t = t_s \) (for a discussion of this point, see [5]):
\[ a \sim (t_s - t)^{2/3(n+1)}. \]  
(1.12)

This is a sort of Big Rip singularity. We should note that in the Einstein frame (1.7) the solution (1.8) gives \( w \) in the Einstein frame as
\[ w_E = -1 + \frac{2(n+2)^2}{9(n+1)^2} > -1. \]  
(1.13)

Therefore, in the Einstein frame, there is no singularity of the Big Rip type. In general, for the scalar field \( \varphi \) with potential \( U(\varphi) \) and canonical kinetic term, the energy density \( \rho \) and the pressure \( p \) are given by
\[ \rho = \frac{1}{2} \dot{\varphi}^2 + U(\varphi), \quad p = \frac{1}{2} \dot{\varphi}^2 - U(\varphi). \]  
(1.14)

Therefore \( w \) is given by
\[ w = \frac{p}{\rho} = -1 + \frac{\dot{\varphi}^2}{(1/2)\dot{\varphi}^2 + U(\varphi)}. \]  
(1.15)
which is bigger than \(-1\) if \(U(\varphi) > -\frac{1}{2} \varphi^2\). Therefore, in order that the phantom with \(w < -1\) is realized, one needs a non-canonical kinetic term or a negative potential. For the action (1.3), the sign of the kinetic term for the scalar field \(\sigma\) is the same as that in the canonical action. Therefore, in order to obtain a phantom in the Einstein frame, at least, we need \(V(\sigma) < 0\) or \(A f'(A) < f(A)\). For the case (1.5), if \(\mu < 0\), \(V\) can be negative. From (1.6), however, if the curvature \(R = A\) is small, \(\sigma\) becomes imaginary if \(n > 0\), which indicates that the curvature cannot become so small. This is quite general. Let us assume that for a not very small curvature, \(f(R)\) could be given by the Einstein action, \(f(R) \sim R\); then, \(f'(R) \sim 1 > 0\). On the other hand, if we also assume that when the curvature is small, \(f(R)\) behaves as \(f(R) \sim \mu R^{-n}\), where \(\mu < 0\) and \(n > 0\), then \(f'(R) < 0\) for small \(R\). Hence, \(f'(R)\) should vanish for finite \(R\), which tells us that \(\sigma\) diverges since \(\sigma = -\ln f'(A)\). Therefore, \(R\) cannot become so small. Even in the case when \(0 > n > -1\), equation (1.8) tells us that \(t_{E_0}\) and therefore \(\sigma\) becomes imaginary. Then a negative \(\mu\) could be forbidden. At least the region where we have effective Einstein gravity does not seem to be continuously connected with the region where curvature is small for the case when a negative potential from modified gravity follows. Thus, classical modified gravity mainly supports the accelerated expansion.

The paper is organized as follows. In the next section the classical dynamics of \(f(R)\) gravity in de Sitter space is considered. Section 3 is devoted to the calculation of the one-loop effective action in \(f(R)\) gravity in de Sitter space. The explicit off-shell and on-shell effective action is found. The study of quantum-corrected de Sitter geometry for specific model of modified gravity is done in section 4. It turns out that quantum gravity corrections shift the radius of de Sitter space, trying to destabilize it. The calculation of the entropy for de Sitter space and constant curvature black holes is done in section 5. Some remarks about black hole nucleation rate are made. The discussion section gives a summary and outlook. In appendix A the black hole solutions with constant curvature are explicitly given for \(f(R)\) gravity. Appendix B is devoted to the details of the calculation of functional determinants for scalars, vectors and tensors in de Sitter space.

2. Modified gravity models

Recall that the general relativistic theory we are interested in is described by the action

\[
I = \frac{1}{16\pi G} \int d^4x \sqrt{-g} f(R),
\]

(2.1)

\(f\) being a function of scalar curvature only.

We have many possible simple choices for the Lagrangian density \(f(R)\). Here we consider the following three examples:

\[
f(R) = R + pR^2 - 2\Lambda,
\]

(2.2)

that is Einstein’s gravity with quadratic corrections;

\[
f(R) = R - \frac{\mu_1}{R}, \quad \mu_1 > 0,
\]

(2.3)

the model proposed in [1], and its trivial generalization

\[
f(R) = R - \frac{\mu_1}{R} - \mu_2.
\]

(2.4)
Here we shall be interested in models which admit solutions with constant four-dimensional curvature $R = R_0$, an example being the one of de Sitter. The general equations of motion for the model described by equation (2.1) are (see, for example, [21])

$$f'(R)R_{\mu\nu} - \frac{1}{2}f(R)g_{\mu\nu} + (\nabla_\mu \nabla_\nu - g_{\mu\nu}\nabla^2) f'(R) = 0,$$

(2.5)

$f'(R)$ being the derivative of $f(R)$ with respect to $R$. If we now require the existence of solutions with constant scalar curvature $R = R_0$, we arrive at

$$f'(R_0)R_{\mu\nu} = \frac{f(R_0)}{2}g_{\mu\nu}.$$  

(2.6)

Taking the trace, we have the condition [21]

$$2f(R_0) = R_0 f'(R_0)$$

(2.7)

and this means that the solutions are Einstein’s spaces, namely they have to satisfy the equation

$$R_{\mu\nu} = \frac{f(R_0)}{2f'(R_0)}g_{\mu\nu} = \frac{R_0}{4}g_{\mu\nu},$$

(2.8)

$R_0$ being a solution of equation (2.7). This gives rise to the effective cosmological constant:

$$\Lambda_{\text{eff}} = \frac{f(R_0)}{2f'(R_0)} = \frac{R_0}{4}.$$  

(2.9)

For the model defined by equations (2.2) and (2.3) we have, respectively,

$$R_0 = 4\Lambda, \quad \Lambda_{\text{eff}} = \Lambda,$$

(2.10)

$$R_0^2 = 3\mu_1, \quad \Lambda_{\text{eff}} = \pm \sqrt{3\mu_1}.$$  

(2.11)

It is clear that this class of constant curvature solutions contains the four-dimensional black hole solutions in the presence of a non-vanishing cosmological constant, like the Schwarzschild–(anti-) de Sitter solutions and all the topological solutions associated with a negative $\Lambda_{\text{eff}}$. In particular, with $\Lambda_{\text{eff}} > 0$, there exist the de Sitter and Nariai solutions, while for $\Lambda_{\text{eff}} < 0$ there exists the anti-de Sitter solution. For the sake of completeness, in appendix A we shall review all of them.

### 3. Quantum field fluctuations around the maximally symmetric instantons

In this section we will discuss the one-loop quantization of the model on a maximally symmetric space. Of course this should be considered only an effective approach (see, for instance [7]). To start, we recall that the action describing a generalized Euclidean gravitational theory is

$$I_\text{E}[g] = -\frac{1}{16\pi G} \int d^4x \sqrt{g} f(R),$$

(3.1)
As usual, indices are lowered and raised by the metric $\eta^{ij}$. Another symmetric background is $H(4)$, associated with a negative cosmological constant. For the $S(2) \times S(2)$ instanton, the consideration we are going to extract is not valid, because it is not a maximally symmetric space and equation (3.3), which we shall use several times from now on, does not hold true. In that case, one should make use of the techniques described in [22,23]. To start with, let us consider small fluctuations around the maximally symmetric instanton

$$g_{ij} = g_{ij}^{(0)} + h_{ij}, \quad g^{ij} = g^{(0)ij} - h^{ij} + h^{ik}h_{kj} + \mathcal{O}(h^3), \quad h = g^{(0)ij}h_{ij}. \quad \text{(3.4)}$$

As usual, indices are lowered and raised by the metric $g^{ij}$. Up to second order in $h_{ij}$, one has

$$\frac{\sqrt{g}}{\sqrt{g^{ij}}} = 1 + \frac{1}{2}h + \frac{1}{8}h^2 - \frac{1}{4}h_{ij}h^{ij} + \mathcal{O}(h^3) \quad \text{(3.5)}$$

and

$$R \approx R_0 - \frac{R_0}{4}h + \nabla_i h_{ij} - \Delta h + \frac{R_0}{4}h^{jk}h_{jk}$$

$$- \frac{1}{2} \nabla_i h^{ij}h + \frac{1}{2} \nabla_i h_{ij} \nabla^{k}h^{ij} + \nabla_i h^k_{ij} \nabla^j h^{jk} - \frac{1}{2} \nabla_j h_{ik} \nabla^i h^{jk}, \quad \text{(3.6)}$$

where $\nabla_k$ represents the covariant derivative in the unperturbed metric $g^{ij}$. By performing a Taylor expansion of $f(R)$ around $R_0$, again up to second order in $h_{ij}$, we get

$$S_E[g] = -\frac{1}{16\pi G} \int d^4x \sqrt{-g^{(0)}} \left[ f_0 + \frac{1}{4}(2f_0 - R_0 f_0') h + \mathcal{L}_2 \right], \quad \text{(3.7)}$$

where, up to total derivatives,

$$\mathcal{L}_2 = \frac{1}{2} f''_0 h_{ij} \nabla_i h_{jk} \nabla_j h_{rs} + \frac{1}{12} h_{ij} [3f_0'\Delta - 3f_0 + R_0 f_0'] h_{ij} - \frac{1}{2} f_0 h_{ij} \nabla_i h_{jk} + \frac{1}{4} f_0 [4f''_0 \Delta - 2f''_0 + R_0 f''_0] \nabla_i h_{ij}$$

$$+ \frac{1}{36} h [48f''_0 \Delta^2 - 24(f_0 - R_0 f''_0) \Delta + 12f_0 - 8R_0 f_0' + 3R_0^2 f_0''] h. \quad \text{(3.8)}$$

For the sake of simplicity, we have used the notation $f_0 = f(R_0)$, $f_0' = f'(R_0)$ and $f_0'' = f''(R_0)$, and in what follows we will set $X \equiv \frac{1}{4}(2f_0 - R_0 f_0')$ (deviation from the on-shell condition).
It is convenient to carry out the standard expansion of the tensor field $h_{ij}$ in irreducible components [10], namely
\[
h_{ij} = \hat{h}_{ij} + \nabla_i \xi_j + \nabla_j \xi_i + \nabla_i \nabla_j \sigma + \frac{1}{4} g_{ij} (h - \Delta_0) \sigma, \tag{3.9}
\]
where $\sigma$ is the scalar component, while $\xi_i$ and $\hat{h}_{ij}$ are the vector and tensor components with the properties
\[
\nabla_i \xi_i = 0, \quad \nabla_i \hat{h}_{ij} = 0, \quad \hat{h}_{ii} = 0. \tag{3.10}
\]
In terms of the irreducible components of the $h_{ij}$ field, the Lagrangian density, again disregarding total derivatives, becomes
\[
\mathcal{L}_2 = \frac{1}{16} \hat{h}^{ij} (3 f_0^i \Delta_2 - 3 f_0^i + R_0 f_0^i) \hat{h}_{ij} + \frac{1}{16} (2 f_0 - R_0 f_0^i) \xi^i (4 \Delta_1 + R_0) \xi_i + \frac{1}{32} \sigma \left[ 9 f_0^i \Delta_1 - 3 (f_0^i - 2 R_0 f_0^i) \Delta + 2 f_0 - 2 R_0 f_0^i + R_0^2 f_0^i \right] h + \frac{1}{32} \sigma \left[ 9 f_0^i \Delta_2 - 3 (f_0^i - 2 R_0 f_0^i) \Delta_0 \right] \sigma - (6 f_0 - 2 R_0 f_0^i - R_0^2 f_0^i) \Delta_0 - R_0 (2 f_0 - R_0 f_0^i) \Delta_0 \] \tag{3.11}
\]
where $\Delta_0$, $\Delta_1$ and $\Delta_2$ are the Laplace–Beltrami operators acting on scalars, traceless-transverse vector and traceless-transverse tensor fields respectively. The latter expression is valid off shell. In obtaining such an expression, due to the huge number of terms appearing in the computation, we have used a program for symbolic tensor manipulations.

As is well known, invariance under diffeormorphisms renders the operator in the $(h, \sigma)$ sector not invertible. One needs a gauge fixing term and a corresponding ghost compensating term. We consider the class of gauge condition, parametrized by the real parameter $\rho$,
\[
\chi_k = \nabla_j h_{jk} - \frac{1 + \rho}{4} \nabla_k h,
\]
the harmonic or de Donder one corresponding to the choice $\rho = 1$. As gauge fixing, we choose the quite general term [7]
\[
\mathcal{L}_{gf} = \frac{1}{2} \chi^i G_{ij} \chi^j, \quad G_{ij} = \alpha g_{ij} + \beta g_{ij} \Delta, \tag{3.12}
\]
where the term proportional to $\alpha$ is the one normally used in Einstein’s gravity. The corresponding ghost Lagrangian reads [7]
\[
\mathcal{L}_{gh} = B^i G_{ik} \frac{\delta \chi^k}{\delta \varepsilon^i} C^j, \tag{3.13}
\]
where $C_k$ and $B_k$ are the ghost and anti-ghost vector fields, respectively, while $\delta \chi^k$ is the variation of the gauge condition due to an infinitesimal gauge transformation of the field. It reads
\[
\delta h_{ij} = \nabla_i \varepsilon_j + \nabla_j \varepsilon_i \implies \frac{\delta \chi^i}{\delta \varepsilon^j} = g_{ij} \Delta + R_{ij} + \frac{1 - \rho}{2} \nabla_i \nabla_j. \tag{3.14}
\]
Neglecting total derivatives, one has
\[
\mathcal{L}_{gh} = B^i (\alpha H_{ij} + \beta \Delta H_{ij}) C^j, \tag{3.15}
\]
where we have set
\[ H_{ij} = g_{ij} \left( \Delta + \frac{R_0}{4} \right) + \frac{1 - \rho}{2} \nabla_i \nabla_j. \]  

(3.16)

In irreducible components, one obtains
\[
\mathcal{L}_{gf} = \frac{\alpha}{2} \left[ \xi^k \left( \Delta + \frac{R_0}{4} \right)^2 \xi_k + \frac{3\rho}{8} \hat{h} \left( \Delta_0 + \frac{R_0}{3} \right) \Delta_0 \sigma \right. \\
\left. - \frac{\rho^2}{16} h \Delta_0 h - \frac{9}{16} \sigma \left( \Delta_0 + \frac{R_0}{3} \right)^2 \Delta_0 \sigma \right] + \beta \left\{ \tilde{B}^i \left( \Delta_1 + \frac{R_0}{4} \right) \tilde{B}^j \right. \\
\left. \left[ \hat{h} \left( \Delta_0 + \frac{R_0}{4} \right) \Delta_0 \hat{h} - \frac{9}{16} \sigma \left( \Delta_0 + \frac{R_0}{4} \right) \left( \Delta_0 + \frac{R_0}{3} \right)^2 \Delta_0 \sigma \right] \right\},
\]

(3.17)

\[
\mathcal{L}_{gh} = \alpha \left\{ \hat{B}^i \left( \Delta_1 + \frac{R_0}{4} \right) \hat{B}^j + \frac{\rho - 3}{2} b \left( \Delta_0 - \frac{R_0}{\rho - 3} \right) \Delta_0 c \right\} + \beta \left\{ \hat{B}^i \left( \Delta_1 + \frac{R_0}{4} \right) \hat{B}^j \right. \\
\left. \left[ \hat{h} \left( \Delta_0 + \frac{R_0}{4} \right) \left( \Delta_0 - \frac{R_0}{\rho - 3} \right) \Delta_0 c \right] \right\},
\]

(3.18)

where ghost irreducible components are defined by
\[
C_k = \hat{C}_k + \nabla_k c, \quad \nabla_k \hat{C}^k = 0,
\]
\[
B_k = \hat{B}_k + \nabla_k b, \quad \nabla_k \hat{B}^k = 0.
\]

(3.19)

In order to compute the one-loop contributions to the effective action one has to consider the path integral for the bilinear part
\[
\mathcal{L} = \mathcal{L}_2 + \mathcal{L}_{gf} + \mathcal{L}_{gh}
\]

(3.20)

of the total Lagrangian and take into account the Jacobian due to the change of variables with respect to the original ones. In this way, one gets [10, 7]
\[
Z^{(1)} = (\det G_{ij})^{-1/2} \int D[h_{ij}] D[C_k] D[B^k] \exp \left( - \int d^4 x \sqrt{\hat{g}} L \right) \\
= (\det G_{ij})^{-1/2} \det J_1^{-1} \det J_2^{1/2} \\
\times \int D[h] D[\hat{h}_{ij}] D[\hat{C}^i] D[\sigma] D[\hat{C}^i] D[\hat{B}^k] D[c] D[b] \exp \left( - \int d^4 x \sqrt{\hat{g}} L \right),
\]

(3.21)

where \( J_1 \) and \( J_2 \) are the Jacobians due to the change of variables in the ghost and tensor sectors respectively [10]. They read
\[
J_1 = \Delta_0, \quad J_2 = \left( \Delta_1 + \frac{R_0}{4} \right) \left( \Delta_0 + \frac{R_0}{3} \right) \Delta_0.
\]

(3.22)
and the determinant of the operator $G_{ij}$, acting on vectors, can be written as

$$\det G_{ij} = \text{const} \times \det \left( \Delta_1 + \frac{\alpha}{\beta} \right) \det \left( \Delta_0 + \frac{R_0}{4} + \frac{\alpha}{\beta} \right),$$  \hspace{1cm} (3.23)

while it is trivial in the case $\beta = 0$.

Now, a straightforward computation leads to the following off-shell one-loop contribution to the ‘partition function’:

$$e^{-\Gamma^{(1)}} \equiv Z^{(1)} = \det \left[ \Delta_1 + \frac{R_0}{4} \right] \times \det \left[ \beta \Delta_1 + \alpha \right]^{1/2}$$

$$\times \det \left[ (\rho - 3)\Delta_0 - R_0 \right] \times \det \left[ \beta \left( \Delta_0 + \frac{R_0}{4} \right) + \alpha \right]^{1/2}$$

$$\times \det \left[ \left( \Delta_2 - \frac{R_0}{6} \right) - \frac{X}{2f''_0} \left( \Delta_2 + \frac{R_0}{3} \right) \right]^{-1/2}$$

$$\times \det \left[ 2(\alpha + \beta \Delta_1) \left( \Delta_1 + \frac{R_0}{4} \right) + X \right]^{-1/2}$$

$$\times \det \left\{ \left[ f''_0 \left( \Delta_0 + \frac{R_0}{3} \right) - \frac{2f_0}{3R_0} \right] \left[ \beta \left( \Delta_0 + \frac{R_0}{4} \right) + \alpha \right] \left[ (\rho - 3)\Delta_0 - R_0 \right]^2 \right\}^{-1/2}$$

$$+ XC_1 + X^2C_2 \right\}^{-1/2},$$  \hspace{1cm} (3.24)

where $\Gamma^{(1)}$ is the one-loop contribution to the partition function and $C_1$ and $C_2$ are operators, which read

$$C_1 = \frac{2f''_0}{3} + \frac{2\alpha R_0^2}{3} + \frac{\beta R_0^3}{6} + \frac{f''_0 R_0^3}{3}$$

$$+ \left( -\frac{4f''_0}{3} + 3\alpha R_0 + \frac{17\beta R_0^3}{12} + \frac{5f''_0 R_0^3}{3} \right) \Delta_0$$

$$- \left( \frac{2\alpha R_0 \rho}{3} - \frac{\beta R_0^2 \rho}{3} - \frac{\alpha R_0 \rho^2}{3} - \frac{\beta R_0 \rho^2}{3} \right) \Delta_0$$

$$+ \left( \frac{3\alpha + 15\beta R_0}{4} + 2f''_0 R_0 - 2\alpha \rho - \frac{7\beta R_0 \rho}{3} + \frac{\alpha \rho^2}{3} - \frac{\beta R_0 \rho^2}{4} \right) \Delta_0^2$$

$$+ \frac{\beta}{3} (\rho - 3)^2 \Delta_0^3,$$

$$C_2 = \frac{2}{3} (\Delta_0 + R_0).$$  \hspace{1cm} (3.25)

Equation (3.24) reduces to the corresponding one in [10], when $f'' = 0$ (Einstein’s gravity with a cosmological constant). For another approach to the same problem see [24].

In the derivation of (3.24), it is understood that the functional determinants have been regularized by means of zeta-function regularization (see, for example [12, 13]). However, we should recall that within the zeta-function regularization it is no longer true that

$$\det AB = \det A \det B,$$  \hspace{1cm} (3.26)
where $A$ and $B$ are two (elliptic) operators. In fact, in general, one has
\[
\det AB = e^{a(A,B)} \det A \det B,
\] (3.27)
where $a(A,B)$ is a local functional called the multiplicative anomaly (see, for example, [25,26]). As a consequence, in the above manipulations, we have assumed the multiplicative anomaly to be trivial, namely $a(A,B) = 0$. This is justified since here we are limiting ourselves to the one-loop approximation, and in such a case, a non-trivial multiplicative anomaly, which is a local functional of the fields, may be absorbed into the renormalization ambiguity [27].

Furthermore, another delicate point should be mentioned. The Euclidean gravitational action, due to the presence of $R$, is not bounded from below, since arbitrary negative contributions can be induced on $R$, by conformal rescaling of the metric. For this reason, we have also used the Hawking prescription of integrating over imaginary scalar fields. Furthermore, the problem of the presence of additional zero modes introduced by the decomposition (3.9) can be treated making use of the method presented in [10].

As one can easily verify, in the limit $X \to 0$ (on-shell condition), equation (3.24) does not depend on the gauge parameters and reduces to
\[
\Gamma_{\text{on-shell}} = I_E(g_0) + \Gamma_{\text{on-shell}}^{(1)} = \frac{24\pi f_0}{GR_0^2} + \frac{1}{2} \ln \det \left[ \ell^2 \left( -\Delta_2 + \frac{R_0}{6} \right) \right] - \frac{1}{2} \ln \det \left[ \ell^2 \left( -\Delta_1 - \frac{R_0}{4} \right) \right] + \frac{1}{2} \ln \det \left[ \ell^2 \left( -\Delta_0 - \frac{R_0}{3} + \frac{2f_0}{3R_0f_0''} \right) \right].
\] (3.28)

As usual, an arbitrary renormalization parameter $\ell^2$ has been introduced for dimensional reasons. When $f''_0 = 0$, namely in the case of Einstein’s gravity with a cosmological constant, $f(R) = R - 2\Lambda$, one obtains the well known result [8]–[10]
\[
\Gamma_{\text{on-shell}} = I_E(g_0) + \Gamma_{\text{on-shell}}^{(1)} = \frac{12\pi}{GR_0} + \frac{1}{2} \ln \det \left[ \ell^2 \left( -\Delta_2 + \frac{R_0}{6} \right) \right] - \frac{1}{2} \ln \det \left[ \ell^2 \left( -\Delta_1 - \frac{R_0}{4} \right) \right].
\] (3.29)

In order to simplify the off-shell computation, we choose the gauge parameters $\rho = 1, \beta = 0$ and $\alpha = \infty$ (Landau gauge). Thus, we obtain
\[
\Gamma = \frac{24\pi}{GR_0^2} f_0 + \frac{1}{2} \ln \det \left( -\Delta_2 - \frac{R_0}{6} \frac{X + 2f_0}{X - 2f_0} \right) - \frac{1}{2} \ln \det \left( -\Delta_1 - \frac{R_0}{4} \right) - \frac{1}{2} \ln \det \left( -\Delta_0 - \frac{R_0}{2} \right) + \frac{1}{2} \ln \det \left\{ \left( -\Delta_0 - \frac{5R_0}{12} - \frac{X - 2f_0}{6R_0f''_0} \right)^2 - \left( \frac{5R_0}{12} + \frac{X - 2f_0}{6R_0f''_0} \right)^2 - \frac{R_0^2}{6} - \frac{X - f_0}{3f''_0} \right\}.
\] (3.30)
One-loop $f(R)$ gravity in de Sitter universe

Recall now that the functional determinant of a differential operator $A$ can be defined in terms of its zeta function by means of (see for example [12,13])

$$\zeta(s|A) = \sum \lambda_n^{-s}, \quad \Re s > \frac{D}{2}, \quad (3.31)$$

$$\ln \det(\ell^2 A) = -\zeta'(0|\ell^2 A) = -\zeta'(0|A) + \ln \ell^2 \zeta(0|A), \quad (3.32)$$

where the prime indicates derivation with respect to $s$. Looking at equation (3.30), we see that the one-loop effective action can be written in terms of the derivative of zeta functions corresponding to Laplace-like operators acting on scalar, vector and tensor fields on a four-dimensional de Sitter space. In all such cases, the eigenvalues of the Laplace operator are explicitly known and the zeta functions can be computed directly using equation (3.31).

For the reader’s convenience, we have reported in appendix B all the details of the method used in the explicit computation for the example that will follow.

Finally, equations (3.30), (3.32) and (B.29), (B.35) and (B.38) in appendix B lead to the off-shell one-loop effective action

$$\Gamma = \frac{24\pi}{GR_0^2} f_0 - \frac{1}{2} Q_2(\alpha_2) + \frac{1}{2} Q_1(\alpha_1) + \frac{1}{2} Q_0(\alpha_0) - \frac{1}{2} Q_0(\alpha_+) - \frac{1}{2} Q_0(\alpha_-), \quad (3.33)$$

where (see appendix B)

$$\alpha_2 = \frac{17}{4} + q_2, \quad q_2 = 2\frac{X + 2f_0}{X - 2f_0}, \quad (3.34)$$

$$\alpha_1 = \frac{13}{4} + q_1 = \frac{25}{4}, \quad q_1 = 3, \quad (3.35)$$

$$\alpha_0 = \frac{9}{4} + q_0 = \frac{33}{4}, \quad q_0 = 6, \quad (3.36)$$

$$\alpha_\pm = \frac{9}{4} + q_\pm, \quad q_\pm = 5 + 2 \frac{X - 2f_0}{R_0^2 f_0^\mu} \pm \sqrt{\left(5 + 2 \frac{X - 2f_0}{R_0^2 f_0^\mu}\right)^2 - 24 \left(1 + 2 \frac{X - f_0}{R_0^2 f_0^\mu}\right)}. \quad (3.37)$$

Now, we would like to present the explicit example for the model described by equation (2.4). First of all we consider the simplest case in the class of models defined by equation (2.4), thus

$$f(R) = R - \frac{\mu_1}{R} - \mu_2, \quad X = R_0 - 3\frac{\mu_1}{R_0} - 2\mu_2. \quad (3.38)$$

We may eliminate $X$ and get

$$\alpha_2 = \frac{57\mu_1 + R_0(2\mu_2 - 7R_0)}{4(\mu_1 + R_0^2)}, \quad (3.39)$$

$$\alpha_\pm = \frac{33}{4} + \frac{R_0^2}{\mu_1} \pm \frac{1}{\mu_1} \sqrt{R_0^4 - 36\mu_1^2 + 12\mu_1R_0(R_0 - 2\mu_2)}. \quad (3.40)$$

Hence, the one-loop effective action in $f(R)$ gravity in de Sitter space is found.

In the next section the above effective action will be applied to study the back-reaction of $f(R)$ gravity to background geometry. However, several important remarks are in order. As usual, any perturbative calculation of the effective action in quantum gravity is gauge
dependent. The way to resolve such a problem is well known: to use the gauge-fixing
independent effective action (for a review, see [7]). A more serious problem is related
to the fact that quantum gravity under investigation is not renormalizable. Then, generally
speaking, higher order corrections are of the same order as one-loop ones (the same is
applied to all previous quantum considerations of Einstein gravity). As a result all one-
loop conclusions are highly questionable as they may be spoiled by higher loop effects. In
this respect, the results of our work are definitely useful in the following sense. One can
expect that perturbatively renormalizable gravity may be constructed for some version of
\( f(R) \) gravity. (So far only higher derivative gravity is known to be renormalizable.) In this
case, our work gives the necessary background for one-loop quantization of such a theory.
From another side, to get the meaningful results with non-renormalizable quantum gravity
one may apply the exact renormalization group scheme. In such a case, higher loop effects
are not important. Our work may also be considered as a necessary and important step
in this direction. Indeed, it is technically clear how to construct the exact RG equations
for \( f(R) \) gravity using results of this section in analogy with Einstein gravity [31].

4. Quantum-corrected de Sitter cosmology

Let us consider the role of quantum effects in the background cosmology. So far, such
study has only been done for Einstein or higher derivative gravity. In order to see the
difference with such models, we take the example of modified gravity with the action

\[ f(R) = R - \frac{\mu_1}{R}. \]  

(4.1)

It is interesting to investigate the region where curvature is not very big, as otherwise the
classical theory is effectively reduced to Einstein’s gravity. Moreover, if curvature is small
one can neglect the powers of curvature in the one-loop effective action, supposing that
logarithmic terms give the dominant contribution. The parameter \( \mu_1 \) is chosen to be very
small in order to avoid conflicts with Newton’s law. As a result, one obtains

\[ \Gamma(R_0) = \frac{24\pi}{G R_0^2} \left( R_0 - \frac{\mu_1}{R_0} \right) + \left( \alpha + \frac{\beta}{\mu_1 + R_0^2} \right) \ln \left( \frac{R_0^2}{12} \right). \]  

(4.2)

Here \( \alpha \) and \( \beta \) are constants. It is assumed that the curvature is constant, \( R = R_0 \). Let
us find the minimum of \( \Gamma \) with respect to \( R_0 \). One can write \( \Gamma' (R_0) \) as

\[ \Gamma' (R_0) = F (R_0) - G (R_0), \]

\[ F (R_0) \equiv \frac{24\pi}{G R_0^2} \left( -1 + \frac{3\mu_1}{R_0^2} \right), \]

\[ G (R_0) \equiv \frac{2\beta R_0}{(\mu_1 + R_0^2)^2} \ln \left( \frac{R_0^2}{12} \right) - \frac{1}{R_0} \left( \alpha + \frac{\beta}{\mu_1 + R_0^2} \right). \]  

(4.3)

When

\[ R_0 = R_c \equiv \sqrt{3\mu_1}, \]  

(4.4)
One-loop $f(R)$ gravity in de Sitter universe

$F(R_0) = 0$, which corresponds to the classical solution. When $R_0 \sim 0$, $F(R_0)$ behaves as $F(R_0) \sim 72\pi\mu_1/GR_0^2$, and when $R_0 \rightarrow +\infty$, $F(R_0) \rightarrow -24\pi/GR_0^2$. Since

$$F'(R_0) = \frac{48\pi}{GR_0^2} \left(1 - \frac{6\mu_1}{R_0^2}\right),$$

there is a minimum for $F(R_0)$ when $R_0 = \sqrt{6\mu_1} > R_c$. On the other hand, if $\alpha \neq 0$, $G(R_0)$ behaves as

$$G(R_0) \rightarrow \frac{1}{R_0} \left(\alpha + \beta \frac{\mu_1}{\mu_1}\right),$$

when $R_0 \rightarrow 0$ and

$$G(R_0) \rightarrow -\frac{\alpha}{R_0},$$

when $R_0 \rightarrow +\infty$. Hence for $R_0 > 0$, if $\alpha < 0$, $\Gamma'(R_0) > 0$ when $R_0 \rightarrow +0$ and $\Gamma'(R_0) < 0$ when $R_0 \rightarrow +\infty$. Therefore, there is a solution which satisfies $\Gamma'(R_0) = 0$ if $\alpha < 0$. When $\alpha > 0$, the existence of the solution depends on the details of the parameters.

If $\alpha = 0$, when $R_0 \rightarrow 0$, $G(R_0)$ behaves as

$$G(R_0) \rightarrow -\frac{\beta}{\mu_1 R_0},$$

and when $R_0 \rightarrow +\infty$, we find

$$G(R_0) \rightarrow \frac{2\beta}{R_0} \ln \left(\frac{t^2 R_0}{12}\right).$$

When $R_0 > 0$, if $\beta > 0$, $\Gamma'(R_0) > 0$ when $R_0 \rightarrow +0$ and $\Gamma'(R_0) < 0$ when $R_0 \rightarrow +\infty$. Hence even if $\alpha = 0$, when $\beta > 0$, there is a solution for equation $\Gamma'(R_0) = 0$. When $\beta < 0$, the existence of the solution depends on the details of the parameters again. Thus, there could be a positive non-trivial solution for $R_0$, which describes the quantum-corrected de Sitter space. One may play with the parameters of the theory under consideration in such a way that the quantum-corrected de Sitter space can provide a solution to the cosmological constant problem. The above results indicate that the classical de Sitter solution (4.4) can survive when one takes into account the quantum corrections. A similar consideration can be done for any specific $f(R)$ gravity.

Let us demonstrate that indeed with some fine-tuning the obtained effective action may be used to resolve the cosmological constant problem. One can present (3.33) corresponding to (4.1) as

$$\Gamma = \frac{24\pi}{GR_0^2} \left(R_0 - \frac{\mu_1}{R_0}\right) + Q \left(t^2; R_0, \mu_1\right).$$

In general $Q(t^2; R_0, \mu_1)$ has a structure

$$Q \left(t^2; R_0, \mu_1\right) = Q_0 \left(\frac{R_0^2}{\mu_1}\right) \ln \frac{t^2 R_0}{12} + Q_1 \left(\frac{R_0^2}{\mu_1}\right).$$

By the condition that $\Gamma$ takes a minimum value with the variation over $R_0$, we obtain

$$0 = \frac{\partial \Gamma}{\partial R_0} = \frac{24\pi}{G} \left(-\frac{1}{R_0^2} + \frac{3\mu_1}{R_0^4}\right) + \frac{\partial Q \left(t^2; R_0, \mu_1\right)}{\partial R_0}.$$
The convenient choice between the parameters is
\[
\left(\frac{12}{L^2}\right)^2 = c_0^2 \mu_1. \tag{4.13}
\]
Here $c_0$ is a constant which could be determined later. Then $Q$ has the following form:
\[
Q = Q \left(\frac{R_0^2}{\mu_1}\right) = Q_0 \left(\frac{R_0^2}{\mu_1}\right) \ln \left(\frac{1}{c_0} \sqrt{\frac{R_0^2}{\mu_1}}\right) + Q_1 \left(\frac{R_0^2}{\mu_1}\right). \tag{4.14}
\]

We now consider the possibility that the vanishing cosmological constant could be obtained by (fine-) tuning the parameters. The corresponding condition that the vacuum energy, or cosmological constant, vanishes requires
\[
\Gamma = 0, \tag{4.15}
\]
which may be solved with respect to $\mu_1$ as $\mu_1 = \mu_1 (R_0)$, which gives
\[
0 = \frac{\partial \Gamma}{\partial R_0} + \frac{d \mu_1}{dR_0} \frac{\partial \Gamma}{\partial \mu_0}. \tag{4.16}
\]

By combining (4.12) and (4.16) with (4.14), one gets
\[
0 = \frac{\partial \Gamma}{\partial \mu_0} = \frac{24\pi}{GR_0^3} - \frac{R_0^2}{\mu_1^2} Q' \left(\frac{R_0^2}{\mu_1}\right), \tag{4.17}
\]
which gives
\[
Q' \left(\frac{R_0^2}{\mu_1}\right) = - \frac{24\pi \mu_1^2}{GR_0^5}. \tag{4.18}
\]

Then by using (4.12), (4.14), and (4.18), it follows that
\[
0 = \frac{24\pi}{G} \left(- \frac{1}{R_0^2} + \frac{3\mu_1}{R_0^3} + \frac{2R_0}{\mu_1} Q' \left(\frac{R_0^2}{\mu_1}\right)\right) = \frac{24\pi}{G} \left(- \frac{1}{R_0^2} + \frac{\mu_1}{R_0^4}\right). \tag{4.19}
\]

Hence,
\[
R_0^2 = \mu_1. \tag{4.20}
\]

By using (4.10), (4.14), and (4.20), we find
\[
Q(1) = 0. \tag{4.21}
\]

Then equation (4.14) shows that
\[
c_0 = \exp \left(- \frac{Q_1(1)}{Q_0(1)}\right). \tag{4.22}
\]

Therefore, including the quantum corrections and (fine-) tuning the theory parameters, we may obtain the solution expressing the vanishing (effective) cosmological constant. Of course, such a solution of the cosmological constant problem is one loop, and in higher orders better fine-tuning may be required.
The effective action (4.2) has been evaluated in the Euclidean signature, in which case we should recall that the 4D de Sitter space with positive constant curvature $R_0$ becomes a sphere of radius

$$a = \sqrt{\frac{12}{R_0}}. \quad (4.23)$$

The volume (area) of the sphere $V$ is

$$V = \frac{8\pi^2 a^4}{3} = \frac{384\pi^2}{R_0^2}. \quad (4.24)$$

Identifying $\int d^4x \sqrt{g} \sim V = 384\pi^2 R_0^2$, one may reasonably assume the local effective Lagrangian corresponding to (4.2) to be

$$\Gamma = \frac{1}{384\pi^2} \int \sqrt{g} L_{\text{eff}}(R),$$

$$L_{\text{eff}}(R) = \frac{24\pi}{G} \left( R - \frac{\mu_1}{R} \right) + R^2 \left( \alpha + \frac{\beta}{\mu_1 + R^2} \right) \ln \left( \frac{l^2 R}{12} \right). \quad (4.25)$$

The effective equation of motion is

$$0 = \frac{1}{2} g_{\mu\nu} L_{\text{eff}}(R) - R_{\mu\nu} L'_{\text{eff}}(R) + \nabla_\mu \nabla_\nu L'_{\text{eff}}(R) - g_{\mu\nu} \nabla^2 L'_{\text{eff}}(R), \quad (4.26)$$

with the curvature being covariantly constant, $\nabla_{\rho} R_{\mu\nu} = 0$, equation (4.26) reduces to $\Gamma'(R_0) = 0$ in (4.3). Supposing the FRW metric with flat three-dimensional part,

$$ds^2 = -dt^2 + e^{2a(t)} \sum_{i=1,2,3} \left( dx^i \right)^2, \quad (4.27)$$

the $(t, t)$-component of (4.26) has the following form:

$$0 = -\frac{1}{2} L_{\text{eff}}(6\dot{H} + 12H^2) + 3(\dot{H} + H^2)L'_{\text{eff}}(6\dot{H} + 12H^2) - 3H \frac{d}{dt}(L'_{\text{eff}}(6\dot{H} + 12H^2)). \quad (4.28)$$

The Hubble parameter $H$ is defined by $H \equiv \dot{a}/a$, as usual. We now split $L_{\text{eff}}(R) = L_c(R) + L_q(R)$, with

$$L_c(R) \equiv \frac{24\pi}{G} \left( \frac{R^3}{4\mu_1} - \frac{R}{2} + \frac{5\mu_1}{4R} \right),$$

$$L_q(R) \equiv R^2 \left( \alpha + \frac{\beta}{\mu_1 + R^2} \right) \ln \left( \frac{l^2 R}{12} \right). \quad (4.29)$$

Let us assume that $L_q(R)$ are much smaller than $L_c(R)$ and consider the perturbation from the classical solution in (4.4), by putting

$$H = h_c + \delta h, \quad h_c \equiv \sqrt{\frac{R_c}{12}} = \sqrt{\frac{3\mu_1}{12}}, \quad (4.30)$$

or

$$R = R_c + \delta R, \quad \delta R \equiv 6\delta h + 24h_c\delta h. \quad (4.31)$$
Note that $L_c(R)$ contains the quantum correction. From (4.28) it follows that

$$0 = -18h_c L''_0 \delta \dot{h} - 54h_c^2 L'_0 \delta h + \left( -6h_c L'_0 + 72h_c^2 L''_0 \right) \delta h - \frac{1}{2} L_{q0} + 3h_c^2 L'_0$$

$$= 24\pi G h_c \left[ \frac{\delta \ddot{h}}{G} + \frac{72\pi h_c}{G} \delta \dot{h} - \frac{288\pi h_c^2}{G} \delta h \right.$$

$$+ 24h_c^2 \left( \alpha - \frac{\beta}{768\mu_1} \right) \ln \left( l^3 h_c^2 \right) + 12h_c^2 \left( \alpha + \frac{\beta}{192\mu_1} \right) \left( \alpha - \frac{\beta}{768\mu_1} \right) \ln \left( l^3 h_c^2 \right) + \left( \alpha + \frac{\beta}{192\mu_1} \right) \right] .$$

(4.32)

Here $L''_0 = L'_c(R_0)$ and $L''_0 = L''_c(R_0)$. Then the solution is

$$\delta h = h_1 + A_+ e^{\alpha t} + A_- e^{\beta t},$$

(4.33)

with

$$h_1 = \frac{G h_c}{2\pi} \left\{ 2 \left( \alpha - \frac{\beta}{768\mu_1} \right) \ln \left( l^3 h_c^2 \right) + \left( \alpha + \frac{\beta}{192\mu_1} \right) \right\} \right].$$

(4.34)

Since $A_+ > 0$, the de Sitter solution becomes unstable under the perturbation.

Thus, for a specific modified gravity model we have demonstrated that the quantum gravity correction shifts the radius of the de Sitter space trying to destabilize the de Sitter phase. This may find interesting applications in the study of the issue of the exit from inflation or in the study of the decay of the dark energy phase.

5. Black hole nucleation rate

We have remarked (see appendix A) that within the modified gravitational models we are dealing with there is room for black hole solutions, formally equivalent to black hole solutions of the Einstein theory with a non-vanishing cosmological constant. As in the Einstein case, one is confronted with the black hole nucleation problem [22]. We review here the discussion reported in [22,23].

To begin with, we recall that we shall deal with a tunnelling process in quantum gravity. On general backgrounds, this process is mediated by the associated gravitational instantons, namely stationary solutions of Euclidean gravitational action, which dominate the path integral of Euclidean quantum gravity. It is a well known fact that as soon as an imaginary part appears in the one-loop partition function, one has a metastable thermal state and thus a non-vanishing decay rate. Typically, this imaginary part comes from the existence of a negative mode in the one-loop functional determinant. Here, the semiclassical and one-loop approximations are the only techniques at our disposal, even though one should bear in mind their limitations as well as their merits.

Let us consider a general model described by $f(R)$ with $\Lambda_{\text{eff}} > 0$. Thus, we may have de Sitter and Nariai Euclidean instantons. Making use of the instanton approach, we have
for the Euclidean partition function
\[ Z \simeq Z(S_4) + Z(S_2 \times S_2) = Z^{(1)}(S_4)e^{-I(S_4)} + Z^{(1)}(S_2 \times S_2)e^{-I(S_2 \times S_2)}, \] (5.1)
where \( I \) is the classical action and \( Z^{(1)} \) the quantum correction, typically a ratio of functional determinants. The classical action can be easily evaluated and reads
\[ I(S_4) = -\frac{24f_0}{GR_0}, \quad I(S_2 \times S_2) = -\frac{16f_0}{GR_0}. \] (5.2)

At this point, we make a brief digression regarding the entropy of the above black hole solutions. To this end, we follow the arguments reported in [19]. If one makes use of the Noether charge method for evaluating the entropy associated with black hole solutions with constant curvature in modified gravity models, one has
\[ S = \frac{A_H}{4G}f'(R_0). \] (5.3)
As a consequence, in general, one obtains a modification of the ‘area law’. For stable models like (2.2) with \( p > 0 \) (see below), one has \( f'_0 = 1 + 8p\Lambda \). For the model (2.3), \( f'_0 = 4/3 \), and thus [19]
\[ S = \frac{A_H}{3G}. \] (5.4)
In the above equations, \( A_H = 4\pi r_H^2 \), \( r_H \) being the radius of the event horizon or cosmological horizon related to a black hole solution. This turns out to be model dependent. It is interesting to note that for unstable modified gravity (with negative first derivative of \( f \)) the entropy may be negative!

We are interested in the case of de Sitter space, and we have
\[ A_H = \frac{12\pi}{\Lambda_{\text{eff}}} = \frac{48\pi}{R_0}. \] (5.5)
Thus, for the de Sitter solution,
\[ S(S_4) = \frac{12\pi}{GR_0}f'(R_0) \] (5.6)
and if we take into account the equation (2.7), one gets
\[ S(S_4) = -I(S_4). \] (5.7)
With the help of the one-loop effective action one can calculate the quantum correction for classical entropy.

We may introduce the free energy \( F = -S/\beta \), where \( \beta = 2\pi\sqrt{12/R_0} \) is the inverse of the Hawking temperature for the de Sitter space. As a consequence, we have [23]
\[ Z \simeq Z^{(1)}(S_4)e^{-\beta\tilde{F}}, \] (5.8)
where
\[ \tilde{F} = F(S_4) - \frac{Z((S_2 \times S_2))}{\beta Z(S_4)}. \] (5.9)
The rate of quasiclassical decay in the de Sitter space is present as soon as $\hat{F}$ has a non-vanishing imaginary part and it is given by $N = 2 \text{Im} \hat{F}$. When $f(R) = R - 2\Lambda$, the Einstein case, it turns out that $Z((S_2 \times S_2))$ has an imaginary part but $Z(S_4)$ is real. As a result, in the Einstein case, the nucleation rate is \cite{22, 23}

$$N = -2 \frac{\text{Im} Z((S_2 \times S_2))}{\beta Z(S_4)}. \quad (5.10)$$

Within our generalized models, the dynamics of the gravitational field is different. In fact, also in the de Sitter case, due to the presence of an additional term in the on-shell one-loop effective action related to the operator $L_0$ (see equation (3.28)), there exists the possibility of negative modes. In fact, from appendix B, one has

$$\lambda_n(L_0) = \frac{R_0}{12} \left( n^2 + 3n - 4 + \frac{4f'_0}{R_0 f''_0} \right), \quad (5.11)$$

where $n = 0, 1, 2, 3, \ldots$. It is clear that we have negative modes as soon as

$$\frac{4f'_0}{R_0 f''_0} < 0. \quad (5.12)$$

For example, for the model

$$f(R) = R - \frac{\mu}{R^n}, \quad (5.13)$$

the quantity (5.12) is always negative and one obtains, at least, two negative modes.

For the model

$$f(R) = R + pR^2 - 2\Lambda, \quad (5.14)$$

there are no negative modes as soon as $p > 0$, in agreement with the classical stability observed in \cite{21}. For $p < 0$, one has only a negative mode when

$$p < -\frac{1}{8\Lambda}. \quad (5.15)$$

Thus, in general, within these specific modified gravitational models, de Sitter space is unstable due to quantum corrections.

### 6. Discussion

In summary, we have here calculated the one-loop effective action for general $f(R)$ gravity in the de Sitter space. Generalized zeta regularization has been used to obtain a finite answer for the functional determinants in the effective action, which has proven to be a very convenient procedure. The important lesson to be drawn from this calculation, generalizing the previous programme for one-loop Einstein gravity in the de Sitter background, is that quantum corrections tend to destabilize the classical de Sitter universe. The constant curvature black hole nucleation rate can be discussed within our scheme. Note that, typically, $f(R)$ models may contain (depending of course on the explicit form of $f$) more negative modes, aside from the Einstein one.

Another lesson we have learned is that the inverse powers of the curvature (which are important at the current epoch) do not arise from the perturbative quantum corrections,
One-loop $f(R)$ gravity in de Sitter universe

as has been explicitly demonstrated here. Some remarks about black holes and associated entropy in modified gravity have been also made.

There are several directions in which our results can be extended and applied. First of all, one can repeat the whole calculation of the functional determinants for the case of anti-de Sitter space. Then, the one-loop effective action found in the third section can also be given for the anti-de Sitter universe. In relation to the AdS/CFT correspondence, our approach can be very interesting for the study of the (de)stabilization of such a universe against quantum corrections. On the other hand, it can be also important in the search for the solution of the cosmological constant problem within hyperbolic backgrounds in the large distance limit [30], which will be discussed elsewhere. Second, having in mind the current interest in gauged supergravities in de Sitter space, one can try to formulate the theory of $f(R)$ gauged supergravity and to study the relevance of quantum effects for such theories.

Acknowledgments

We would like to thank L Vanzo for useful discussions. Support from the programme INFN(Italy)–CICYT(Spain), from DGICYT (Spain), project BFM2003-00620 and SEEU grant PR2004-0126 (EE), is gratefully acknowledged.

Appendix A. Black hole solutions in modified gravity

Here we revisit the class of general static neutral black hole solutions in four dimensions and non-vanishing cosmological constant. Let us start with the usual ansatz for the metric

$$ds^2 = A(r) dt^2 - \frac{dr^2}{A(r)} - r^2 d\Sigma_k^2,$$  \hspace{1cm} (A.1)

where $k = 0, \pm 1$ and the horizon manifolds are $\Sigma_1 = S^2$, the two-dimensional sphere, $\Sigma_0 = T^2$, the two-dimensional torus, and $\Sigma_{-1} = H^2/\Gamma$, the two-dimensional compact Riemann surface. The scalar curvature and the non-vanishing components of the Ricci tensor read

$$R = -\frac{1}{r^2} \left[ r^2 A'' + 4r A' + 2A - 2k \right],$$  \hspace{1cm} (A.2)

$$R_{tt} = -\frac{1}{2r} \left[ r A'' + 2A' \right],$$  \hspace{1cm} (A.3)

$$R_{rr} = \frac{1}{2rA} \left[ r A'' + 2A' \right],$$  \hspace{1cm} (A.4)

$$R_{ab} = g_{ab} \left[ k - r A' - A \right].$$  \hspace{1cm} (A.5)

If we look for a constant curvature solution, we should have

$$r^2 A'' + 4r A' + 2A - 2k = -r^2 R_0.$$  \hspace{1cm} (A.6)

The general exterior solution depends on two integration constants, $c$ and $b$, and reads

$$A(r) = \frac{b}{r^2} + k - \frac{c}{r} - \frac{R_0}{12} r^2.$$  \hspace{1cm} (A.7)
However, this solution has to satisfy the equation of motion

\[ R_{\mu\nu} = \frac{R_0}{4} g_{\mu\nu}. \]  

(A.8)

As a result, it is easy to show that these equations are satisfied with \( b = 0 \) and \( c \) arbitrary, and we have (see for instance [28, 19])

\[ A(r) = k - \frac{c}{r} - \frac{R_0}{12} r^2. \]  

(A.9)

Since \( A(r) > 0 \), we can have \( k = 1 \) only for positive \( R_0 \), and this is the Schwarzschild–de Sitter solution. For \( R_0 < 0 \), we may have \( k = 1 \), namely the Schwarzschild–AdS solution. We may also have \( k = 0 \), with a torus topology for the horizon manifold, and \( k = -1 \), with an hyperbolic topology for the horizon topology, the so called topological black holes [28]. The constant \( c \) is related to the mass of the black hole. The de Sitter solution is obtained when \( R_0 > 0 \) and with \( c = 0 \) and \( k = 1 \); the AdS solution is obtained when \( R_0 < 0 \) and with \( c = 0 \) and \( k = 1 \). For \( c \) non-vanishing, one has black hole solutions. These black hole solutions may have extremal cases and extremal limits. The extremal case exists for \( k = -1 \). For \( k = 1 \), \( R_0 > 0 \), one has only the extremal limit of the Schwarzschild–de Sitter solution (see, for example [29] and references therein), and the metric reduces to

\[ ds^2 = 4 \frac{R_0}{R_0} (dS_2^2 + dS_2^2), \]  

(A.10)

This is a space with constant curvature \( R_0 \) solution of equations (A.8).

**Appendix B. Evaluation of functional determinants**

Here we shall make use of zeta-function regularization for the evaluation of the functional determinants appearing in the one-loop effective action, equation (3.30) computed in the previous section.

First, we outline the standard technique, based on binomial expansion, which relates the zeta functions corresponding to the operators \( \hat{A} \), with eigenvalues \( \hat{\lambda}_n > 0 \), and \( A = (R_0/12)(\hat{A} - \alpha) \), with eigenvalues \( \lambda_n = (R_0/12)(\hat{\lambda}_n - \alpha) \), \( \alpha \) being a real constant. With this choice, \( \hat{\lambda}_n \) and \( \alpha \) are dimensionless. We assume we are dealing with a second-order differential operator on a \( D \)-dimensional compact manifold. Then, by definition, for \( \text{Re} \ s > D/2 \) one has

\[
\zeta(s) \equiv \zeta(s|\hat{A}) = \sum_n \hat{\lambda}_n^{-s},
\]

(B.1)

\[
\zeta_\alpha(s) \equiv \zeta(s|A) = \sum_n \lambda_n^{-s} = \left( \frac{R_0}{12} \right)^{-s} \sum_n (\hat{\lambda}_n - \alpha)^{-s},
\]

(B.2)

where, as usual, zero eigenvalues have to be excluded from the sum. In order to use the binomial expansion in (B.2), we have to treat separately the several terms satisfying the condition \( |\hat{\lambda}_n| \geq |\alpha| \). So, we write

\[
\zeta_\alpha(s) = \left( \frac{R_0}{12} \right)^{-s} \left[ F_\alpha(s) + \sum_{k=0}^{\infty} \frac{\alpha^k \Gamma(s + k) \hat{G}(s + k)}{k! \Gamma(s)} \right],
\]

(B.3)
where we have set

\[ F_\alpha(s) = \sum_{\hat{\lambda}_n \leq |\alpha|; \hat{\lambda}_n \neq \alpha} (\hat{\lambda}_n - \alpha)^{-s}, \quad \hat{F}(s) = \sum_{\hat{\lambda}_n \leq |\alpha|} \hat{\lambda}_n^{-s}, \quad (B.4) \]

\[ \hat{G}(s) = \sum_{\hat{\lambda}_n > |\alpha|} \hat{\lambda}_n^{-s} = \zeta(s) - \hat{F}(s), \quad F_\alpha(0) - \hat{F}(0) = N_0, \quad (B.5) \]

\( N_0 \) being the number of zero modes. It has to be noted that (B.3) is also valid in the presence of zero modes or negative eigenvalues for the operator \( A \). In many interesting cases, \( F_\alpha(s) \) and \( \hat{F}(s) \) are vanishing and thus \( \hat{G}(s) = \zeta(s) \).

As is well known, the zeta function has simple poles on the real axis for \( s \leq D/2 \) but it is regular at the origin. Of course, the same analytic structure is also valid for the function \( \hat{G}(s) \). One has

\[ \Gamma(s)\hat{\zeta}(s) = \sum_{n=0}^{\infty} \frac{\hat{K}_n}{s + (n-D)/2} + \hat{J}(s), \quad (B.6) \]

\( \hat{J}(s) \) being an analytic function and \( \hat{K}_n \) the heat-kernel coefficients depending on geometrical invariants. In the physical applications we have to consider, we have to deal with the zeta function and its derivative at zero, thus it is convenient to consider the Laurent expansion around \( s = 0 \) of the functions

\[ \Gamma(s+k)\hat{\zeta}(s+k) = \frac{\hat{b}_k}{s} + \hat{a}_k + O(s), \quad (B.7) \]

\[ \Gamma(s+k)\hat{G}(s+k) = \frac{\hat{b}_k}{s} + a_k + O(s), \quad (B.8) \]

\[ b_0 = \hat{b}_k - \hat{F}(0), \quad a_0 = \hat{a}_0 + \gamma \hat{F}(0), \quad (B.9) \]

\[ b_k = \hat{b}_k - \hat{K}_{D-2k}, \quad a_k = \hat{a}_k - \Gamma(k)\hat{F}(k), \quad 1 \leq k \leq \frac{D}{2}, \quad (B.10) \]

\[ b_k = \hat{b}_k = 0, \quad \hat{G}(k) = \zeta(k) - \hat{F}(k), \quad k > \frac{D}{2}. \quad (B.11) \]

Now, from previous considerations, one obtains

\[ \zeta_\alpha(s) = \left( \frac{R_0}{12} \right)^{-s} \left[ \sum_{0 \leq k \leq D/2} \left( \frac{b_k \alpha^k}{k!} + s \frac{(a_k + \gamma b_k)\alpha^k}{k!} \right) \ight. \]

\[ + \left. F_\alpha(s) + s \sum_{k > D/2} \frac{\alpha^k \hat{G}(k)}{k} + O(s^2) \right], \quad (B.12) \]
and finally

\[ \zeta_\alpha(0) = F_\alpha(0) + \sum_{0 \leq k \leq D/2} \frac{b_k \alpha^k}{k!}, \]  

\[ \zeta'_\alpha(0) = -\zeta_\alpha(0) \ln \frac{R_0}{D} + \sum_{0 \leq k \leq D/2} \frac{(a_k + \gamma b_k) \alpha^k}{k!} + F'_\alpha(0) + \sum_{k > D/2} \frac{\alpha^k \hat{G}(k)}{k}, \]  

\( \gamma \) being the Euler–Mascheroni constant. If there are negative eigenvalues then \( F'_\alpha(0) \) has an imaginary part, which reflects instability of the model.

In the paper we have to deal with Laplace-like operators acting on scalar and constrained vector and tensor fields in a four-dimensional de Sitter space \( SO(4) \). In all such cases, the eigenvalues \( \lambda_n \) and relative degeneracies \( g_n \) can be written in the form

\[ \lambda_n = \frac{R_0}{12} \left( \hat{\lambda}_n - \alpha \right), \quad g_n = c_1 (n + \nu) + c_3 (n + \nu)^3, \quad \hat{\lambda}_n = (n + \nu)^2, \]  

where \( n = 0, 1, 2 \ldots \) and \( c_1, c_2, \nu, \alpha \) depend on the operator one is dealing with. In our cases we have

\[ L_0 = -\Delta_0 - \frac{R_0}{12} q \implies \begin{cases} \nu = \frac{3}{2}, & \alpha = \frac{9}{4} + q, \\ c_1 = \frac{1}{12}, & c_3 = \frac{1}{3}, \end{cases} \]  

\[ L_1 = -\Delta_1 - \frac{R_0}{12} q \implies \begin{cases} \nu = \frac{5}{2}, & \alpha = \frac{43}{12} + q, \\ c_1 = \frac{9}{4}, & c_3 = 1, \end{cases} \]  

\[ L_2 = -\Delta_2 - \frac{R_0}{12} q \implies \begin{cases} \nu = \frac{7}{2}, & \alpha = \frac{17}{4} + q, \\ c_1 = \frac{125}{12}, & c_3 = \frac{5}{3}, \end{cases} \]  

where \( q \) are dimensionless parameters depending on the specific choice of \( f(R) \).

We note that \( \hat{\zeta}(s) \) is related to well known Hurwitz functions \( \zeta_H(s, \nu) \) by

\[ \hat{\zeta}(s) = \sum_{n=0}^{\infty} g_n \hat{\lambda}_n^{-s} = \sum_{n=0}^{\infty} \left[ c_1 (n + \nu)^{2s-1} + c_3 (n + \nu)^{2s-3} \right] \]

\[ = c_1 \zeta_H(2s - 1, \nu) + c_3 \zeta_H(2s - 3, \nu) \]  

and

\[ \hat{G}(s) = c_1 \zeta_H(2s - 1, \nu) + c_3 \zeta_H(2s - 3, \nu) - \hat{F}(s) \]

\[ = c_1 \zeta_H(2s - 1, \nu + \hat{n}) + c_3 \zeta_H(2s - 3, \nu + \hat{n}), \]  

\( \hat{n} \) being the number of terms not satisfying the condition \( \hat{\lambda}_n > |\alpha| \). In order to proceed, we have to compute the quantities \( \hat{b}_k \) and \( \hat{a}_k \) for \( k = 0, 1, 2 \). To this aim, we note that Hurwitz functions have only a simple pole at 1 and, more precisely,

\[ \zeta_H(s + 1, \nu) = \frac{1}{s} - \psi(\nu) + O(s), \]  

where \( \psi(\nu) = \sum_{n=1}^{\infty} \frac{\nu}{n^2} \) is the digamma function.
\( \psi(s) \) being the logarithmic derivative of Euler’s gamma function. After a straightforward computation, we get

\[ \hat{b}_0 = \hat{\zeta}(0) = c_1 \zeta_H(-1, \nu) + c_3 \zeta_H(-3, \nu), \quad \hat{b}_1 = \frac{c_1}{2}, \quad \hat{b}_2 = \frac{c_3}{2}, \quad (B.22) \]

\[ \hat{a}_0 = \hat{\zeta}'(0) - \gamma \hat{\zeta}(0) = c_1 [2\zeta_H'(-1, \nu) - \gamma \zeta_H(-1, \nu)] + c_3 [2\zeta_H'(-3, \nu) - \gamma \zeta_H(-3, \nu)], \quad (B.23) \]

\[ \hat{a}_1 = -c_1 \left[ \psi(\nu) + \frac{3}{2} \right] + c_3 \zeta_H(-1, \nu), \quad (B.24) \]

\[ \hat{a}_2 = c_1 \zeta_H(3, \nu) - c_3 \left[ \psi(\nu) + \frac{\gamma - 1}{2} \right]. \quad (B.25) \]

Using (B.14) we obtain

\[ \zeta'_a(0) \ell^2 L = \left( F_a(0) + \sum_{k=0}^{2} \frac{b_k \alpha^k}{k!} \right) \ln \frac{\ell^2 R_0}{12} + \sum_{k=0}^{2} \left( \frac{a_k + \gamma b_k}{k!} \right) + F_a'(0) + \sum_{k=3}^{\infty} \frac{\alpha^k \hat{G}(k)}{k}. \quad (B.26) \]

Now we have to consider separately the operators \( L_0, L_1, L_2 \) we are dealing with and explicitly compute \( b_k, a_k, G(k) \) using (B.15), (B.22)–(B.25), and (B.16)–(B.18).

**The scalar case**

The eigenvalues of \( L_0 \) are of the form

\[ \lambda_n = \frac{R_0}{12} \left( n + \frac{3}{2} \right)^2 - \alpha, \quad \alpha = \frac{9}{4} + q, \quad n = 0, 1, 2, \ldots \quad (B.27) \]

This case is model dependent since the parameter \( q \) explicitly depends on the choice of the Lagrangian \( f(R) \). Then one could have zero modes and also negative eigenvalues, but we take them into account by the functions \( F_a \) and \( \hat{F} \), both of which will in general appear in the final result. For \( k \geq 3 \), we have

\[ \hat{G}(k) = -\frac{1}{12} \zeta_H \left( 2k - 1, \frac{3}{2} + \hat{n} \right) + \frac{1}{3} \zeta_H \left( 2k - 3, \frac{3}{2} + \hat{n} \right), \quad (B.28) \]

where \( \hat{\lambda}_n > |\alpha| \) per \( n > \hat{n} \). Then

\[ Q_0(\alpha) \equiv \zeta'_a(0) \ell^2 L_0 = \left[ N_0 - \frac{17}{2880} - \frac{\alpha^2}{24} - \frac{\alpha^2}{12} \right] \ln \frac{\ell^2 R_0}{12} \]

\[ + \frac{1}{3} \left[ 3 F_a'(0) + 4 \zeta_H'(-3, 3/2) - \zeta_H'(-1, 3/2) \right] - \left[ 72 \hat{F}(1) + 11 - 6 \psi(3/2) \right] \frac{\alpha}{72} \]

\[ - \left[ 12 \hat{F}(2) + 4 \psi(3/2) + 7 \zeta_H(3) - 10 \right] \frac{\alpha^2}{24} + \sum_{k=3}^{\infty} \frac{\alpha^k \hat{G}(k)}{k}, \quad (B.29) \]

\( \zeta_R(s) \) being the Riemann zeta function. One of the three scalar Laplacian-like operators appearing in the one-loop effective action (3.33) does not depend on the model since for
such a case $\alpha = \alpha_0 = 33/4$. Then $\hat{\lambda}_0$ and $\hat{\lambda}_1$ are smaller than $\alpha$ ($\hat{n} = 2$) and so
\[
F_\alpha(s) = (-6)^{-s} + 5 (-2)^{-s}, \quad \hat{F}(s) = \left(\frac{4}{7}\right)^{-s} + 5 \left(\frac{25}{7}\right)^{-s},
\]
\[
\hat{G}(k) = -\frac{1}{12} \zeta_H \left(2k - 1, \frac{7}{2}\right) + \frac{1}{3} \zeta_H \left(2k - 3, \frac{7}{2}\right).
\]
From these equations it follows that
\[
Q_0(33/4) \sim -18.32 - 6\pi i + \frac{479}{90} \ln \frac{\ell^2 R_0}{12}.
\]
We see that there is an imaginary part since there are negative eigenvalues.

**The vector case**

The eigenvalues of $L_1$ are of the form
\[
\lambda_n = \frac{R_0^2}{12} \left( n + \frac{5}{2} \right)^2 - \alpha, \quad \alpha = \frac{13}{4} + q, \quad n = 0, 1, 2 \ldots
\]
For the vector case, $q = 3$ is a pure number and so $\alpha = 25/4 = \hat{\lambda}_0$. Thus there is a zero mode with multiplicity equal to 10 ($N_0 = 10$) and this has to be excluded in the evaluation of the zeta function. As a consequence, we have $F_\alpha(s) = 0$, $\hat{F}(s) = 10 \alpha^{-s}$, $b_0 = \hat{b}_0 - 10$, $a_0 = \hat{a}_0 + 10\gamma$, $a_1 = \hat{a}_1 - 10\alpha^{-1}$, $a_2 = \hat{a}_2 - 10 \alpha^{-2}$, and for $k \geq 3$
\[
\hat{G}(k) = -\frac{1}{4} \zeta_H \left(2k - 1, \frac{7}{2}\right) + \zeta_H \left(2k - 3, \frac{7}{2}\right).
\]
Finally,
\[
Q_1(25/4) \equiv \zeta'_0(0|\ell^2 L_1) = -\frac{191}{30} \ln \frac{\ell^2 R_0}{12} + \frac{22215}{64} + 4\zeta'_H(-3, 5/2)
- 9\zeta_H(-1, 5/2) - \frac{39375}{128} \zeta_R(3) - \frac{175}{32} \psi(5/2)
+ \sum_{k=3}^{\infty} \frac{\alpha^k \hat{G}(k)}{k} \sim -18.91 - \frac{191}{30} \ln \frac{\ell^2 R_0^2}{12}.
\]

**The tensor case**

The eigenvalues of $L_2$ are of the form
\[
\lambda_n = \frac{R_0^2}{12} \left( n + \frac{7}{2} \right)^2 - \alpha, \quad \alpha = \frac{17}{4} + q, \quad n = 0, 1, 2 \ldots
\]
As for the scalar case, here also zero modes could appear and/or negative eigenvalues, depending on the parameter $q$. Then, in general, we have to introduce the functions $F_\alpha(s)$ and $\hat{F}(s)$. For $k \geq 3$, we have
\[
\hat{G}(k) = \hat{\zeta}(k) = -\frac{125}{72} \zeta_H \left(2k - 1, \frac{7}{2}\right) + \frac{5}{3} \zeta_H \left(2k - 3, \frac{7}{2}\right) - \hat{F}(k),
\]
and

\[ Q_2(\alpha) \equiv \zeta'_0(0)\ell^2 L_2 = \left[ \frac{N_0 + 5833}{576} - \frac{125\alpha}{24} + \frac{5\alpha^2}{12} \right] \ln \frac{\ell^2 R_0}{12} + \frac{3}{8} \left[ 3F'_{\alpha}(0) + 20\zeta_H(-3, 7/2) - 125\zeta_H(-1, 7/2) \right] - \left[ 72\ell^2(1) + 535 - 750\psi(7/2) \right] \frac{\alpha}{72} - \left[ 324F(2) + 540\psi(7/2) + 23625\zeta_R(3) - 28386 \right] \frac{\alpha^2}{648} + \sum_{k=3}^{\infty} \frac{\alpha^k \hat{G}(k)}{k}. \]

(B.38)

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Journal of Cosmology and Astroparticle Physics 02 (2005) 010 (stacks.iop.org/JCAP/2005/i=02/a=010)
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