On the “scattering law” for Kasner parameters in the model with one-component anisotropic fluid

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Abstract

A multidimensional cosmological type model with 1-component anisotropic fluid is considered. An exact solution is obtained. This solution is defined on a product manifold containing $n$ Ricci-flat factor spaces. We singled out a special solution governed by the function $\cosh$. It is shown that this special solution has Kasner-like asymptotics in the limits $\tau \to +0$ and $\tau \to +\infty$, where $\tau$ is a synchronous time variable. A relation between two sets of Kasner parameters $\alpha_\infty$ and $\alpha_0$ is found. This formula (of “scattering law”) is coinciding with that obtained earlier for the $S$-brane solution (when scalar fields are absent).

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1 Introduction

In this paper we continue our investigations (started in [1]) of multidimensional solutions defined on product of several Ricci-flat factor spaces which have two Kasner-like asymptotical regions.

Here we recall that Kasner-like solutions with a chain of $n$ Ricci-flat factor-spaces ($M_i, g^{(i)}$) have the following form [2]

$$g = w d\tau \otimes d\tau + \sum_{i=1}^{n} A_i^2 \tau^{2\alpha_i} g^{(i)}, \quad (1.1)$$

where $w = \pm 1$, $\tau > 0$,

$$\sum_{i=1}^{n} d_i \alpha_i = 1, \quad (1.2)$$

$$\sum_{i=1}^{n} d_i (\alpha_i)^2 = 1, \quad (1.3)$$

and for any $i = 1, \ldots, n$ ($n \geq 2$): $A_i > 0$ is constant, $g^{(i)}$ is a Ricci-flat metric defined on the manifold $M_i$ (for $w = -1$ see [2]).

These solutions with non-Milne-type sets of Kasner parameters are singular since the Riemann tensor squared is divergent as $\tau \to +0$ [3]. For Milne-type sets of parameters, i.e. when $d_i = 1$ and $\alpha_i = 1$ for some $i$ ($\alpha_j = 0$ for all $j \neq i$) the metric is regular as $\tau \to +0$, when either i) $g^{(i)} = -wdy^i \otimes dy^i$, $M_i = \mathbb{R} \ ( -\infty < y^i < +\infty )$, or ii) $g^{(i)} = wdy^i \otimes dy^i$, $M_i$ is circle of length $L_i$ ($0 < y^i < L_i$) and $A_i L_i = 2\pi$ (i.e. when the cone singularity is absent).

In this paper we consider an exact cosmological type solution with 1-component “perfect” fluid (Section 2). (For earlier publications on multidimensional cosmological models with perfect fluid see [4]-[14] and references therein.) This solution is defined on a product manifold containing $n$ Ricci-flat factor spaces. It is derived in the Appendix. For $w = -1$ it was found in [11, 12, 13] and generalized in [14] for the case when a scalar field was added. A special case of this solution with a $\Lambda$-term component was obtained in [15] (see also [16] for scalar field generalization).

We write the solution in a so-called “minisuperspace-covariant” form that significantly simplifies the forthcoming analysis. In Section 3 we single out a special solution governed by the $cosh$ function. We show that this solution
has a Kasner-like asymptotics in both limits $\tau \to +0$ and $\tau \to +\infty$, where $\tau$ is the synchronous time variable. We also find a relation between two sets of Kasner parameters $\alpha_\infty = (\alpha^i_\infty) \in \mathbb{R}^n$ and $\alpha_0 = (\alpha^i_0) \in \mathbb{R}^n$:

$$\alpha^i_\infty = \frac{\alpha^i_0 - 2U(\alpha_0)U^i(U,U)^{-1}}{1 - 2U(\alpha_0)(U,U^\Lambda)(U,U)^{-1}}, \quad (1.4)$$

$i = 1, \ldots, n$. Here $U = (U^i)$ is a co-vector corresponding to the fluid component, $\bar{U} = (U^i)$ is dual vector and $U^\Lambda$ is a co-vector, corresponding to the $\Lambda$-term. All these vectors and the scalar product $(.,.)$ are defined below (see Section 2). Here $U(\alpha_0) = U_i\alpha^i_0 > 0$ and $U(\alpha_\infty) = U_i\alpha^i_\infty < 0$.

A relation analogous to (1.4) (“scattering law” formula) was obtained earlier for S-brane solution with one brane in [1]. We note that in [1] the geometrical sense of the scattering law was clarified for $n > 2$. Namely, the scattering law transformation for a brane $U$-vector (obeying $(U,U^\Lambda) < 0$) was expressed in terms of a function mapping a “shadow” part of the Kasner sphere $S^{n-2}$ onto “illuminated” one. The shadow and illuminated parts of the Kasner sphere were defined w.r.t. a point-like source of light located outside the Kasner sphere $S^{n-2}$. (For details of this geometrical construction see [1]).

The relation (1.4) appears also when the billiard approach to multicomponent anisotropic fluid is considered [18, 19, 20, 21]. It may be shown (as it was done in [23, 24] for S-brane solutions) that after the collision with a billiard wall (corresponding to the fluid component) the set of Kasner parameters, is defined by the Kasner set before the collision through the formula analogous to (1.4), see [26]. For the billiard approach in models with scalar field and fields of forms see [22, 23, 25, 26] and refs. therein.

## 2 Model with anisotropic fluid and its exact solution

### 2.1 The set-up

Now, we consider a cosmological type solution to Einstein equations with an anisotropic (perfect) fluid matter source

$$R^M_N - \frac{1}{2}g^M_N R = k^2T^M_N \quad (2.1)$$
defined on $D$-dimensional manifold
\[ M = \mathbb{R} \times M_1 \times M_2 \times \ldots \times M_n, \] (2.2)

with block-diagonal metric
\[ g = we^{2\gamma(u)} du \otimes du + \sum_{i=1}^{n} e^{2\beta_i(u)} g^{(i)}. \] (2.3)

Here $\mathbb{R} = (u_-, u_+)$ is an interval, $w = \pm 1$ and $n \geq 2$. Manifold $M_i$ with the metric $g^{(i)}$ is a Ricci-flat space of dimension $d_i$: $R_{m,n}[g^{(i)}] = 0$, $i = 1, 2, \ldots, n$, and $\kappa^2$ is a multidimensional gravitational constant.

Energy-momentum tensor of anisotropic fluid is adopted in the following form:
\[ (T^M_{MN}) = \text{diag}(-\hat{\rho}, \hat{p}_1 \delta^m_{k_1}, \ldots, \hat{p}_n \delta^m_{k_n}), \] (2.4)

where $\hat{\rho}$ and $\hat{p}_i$ are “density” and “pressures”, respectively, depending upon radial variable $u$.

In the cosmological case when $w = -1$ and all metrics $g^{(i)}$ have Euclidean signatures, $\hat{\rho} = \rho$ is a density and $\hat{p}_i = p_i$ is a pressure in $i$-th space. For static configurations with $w = 1$, $g^{(1)} = -dt \otimes dt$ and all metrics $g^{(i)}$, $i > 1$, having Euclidean signatures, the physical density and pressures are related to the effective (“hat”) ones by formulas: $\rho = -\hat{p}_1$, $p_u = -\hat{\rho}$, $p_i = \hat{p}_i$, $(i \neq 1)$, where $p_u$ is the pressure in $u$-th direction.

We also impose the following equation of state
\[ \hat{p}_i = \left(1 - \frac{2U_i}{d_i}\right) \hat{\rho}, \] (2.5)

where $U_i$ are constants, $i = 1, 2, \ldots, n$.

In what follows we use a scalar product
\[ (U, U') = G^{ij} U_i U'_j = \sum_{i=1}^{n} \frac{U_i U'_i}{d_i} + \frac{1}{2 - D} \left(\sum_{i=1}^{n} U_i \left(\sum_{j=1}^{n} U'_j\right)\right), \] (2.6)

for $U = (U_i), U' = (U'_i) \in \mathbb{R}^n$, where
\[ G^{ij} = \delta^{ij} \frac{1}{d_i} + \frac{1}{2 - D} \] (2.7)
are components of dual minisuperspace metric. Recall that \((G^{ij}) = (G_{ij})^{-1}\), where
\[
G_{ij} = d_i \delta_{ij} - d_i d_j,
\]
are components of minisuperspace metric \([17]\).

We also define a co-vector
\[
U^\Lambda = (d_i),
\]
(2.9)
corresponding to the \(\Lambda\) -term and the vector \(\bar{U} = (U^i)\)
\[
U^i = G^{ij} U_j = \frac{U_i}{d_i} + \frac{1}{2-D} \sum_{j=1}^{n} U_j,
\]
(2.10)
which is dual to \(U\).

### 2.2 Exact solution

Here we consider an exact cosmological solution to Hilbert-Einstein equations (2.1) defined on the manifold (2.2). We impose the following restriction on the \(U\)-vector in (2.5)
\[
K = (U, U) = \sum_{i=1}^{n} \frac{U_i^2}{d_i} + \frac{1}{2-D} \left( \sum_{i=1}^{n} U_i \right)^2 \neq 0.
\]
(2.11)
(The case \(K = 0\) will be considered in a separate publication.)

The solution has the following form (see Appendix C)
\[
g = |f(u)|^{-2h(U^\Lambda)} \exp(2c^0 u + 2\bar{c}^0) w d u \otimes d u + \sum_{i=1}^{n} |f(u)|^{-2hU^i} \exp(2c^i u + 2\bar{c}^i) g^{(i)},
\]
\[
k^2 \dot{\rho} = -w A |f(u)|^{2h(U^\Lambda)^{-2}} \exp(-2c^0 u - 2\bar{c}^0),
\]
(2.13)
where \(w = \pm 1\), \(h = K^{-1}\), \(g^{(i)}\) is a Ricci-flat metric on \(M_i\), and
\[
(U, U^\Lambda) = \sum_{i=1}^{n} \frac{U_i}{2-D},
\]
(2.14)
i = 1, \ldots, n.
The moduli function $f$ reads

$$f(u) = R \sinh(\sqrt{C}(u - u_0)), \ C > 0, \ KA < 0; \tag{2.15}$$

$$R \sin(\sqrt{|C|}(u - u_0)), \ C < 0, \ KA < 0; \tag{2.16}$$

$$R \cosh(\sqrt{C}(u - u_0)), \ C > 0, \ KA > 0; \tag{2.17}$$

$$|2AK|^{1/2}(u - u_0), \ C = 0, \ KA < 0, \tag{2.18}$$

where $R = |2AK/C|^{1/2}$, and $C$, $u_0$ are constants. (In (2.12) and (2.13) $f(u) \neq 0$ is assumed for all $u \in (u_-, u_+).$)

Vectors $c = (c^i)$ and $\bar{c} = (\bar{c}^i)$ obey the following constraints:

$$U(c) = U_i c^i = 0, \quad U(\bar{c}) = U_i \bar{c}^i = 0 \tag{2.19}$$

$$CK^{-1} + G_{ij} c^i \bar{c}^j = 0, \tag{2.20}$$

where $G_{ij} c^i \bar{c}^j = \sum_{i=1}^n d_i(c^i)^2 - (\sum_{i=1}^n d_i(c^i)^2)^2.$

In (2.12) and (2.13) we also denote

$$c^0 = U^\Lambda(c) = \sum_{i=1}^n d_i c^i, \quad \bar{c}^0 = U^\Lambda(\bar{c}) = \sum_{i=1}^n d_i \bar{c}^i. \tag{2.21}$$

The special solution with $C = c_i = 0$ (for all $i$) and $w = -1$ was considered in detail in [28, 29]. For $U = U^\Lambda$ and $A > 0$ it contains a special solution with $d_i = 1$, $g^i = dy^i \otimes dy^i$ ($i = 1, \ldots, n$), describing either (a part of) de-Sitter space (for $w = -1$) or (a part of) anti-de-Sitter space (for $w = 1$).

**Minisuperspace-covariant form of solution.**

This solution is derived in Appendix C in terms of “minisuperspace-covariant” notations for functions $\gamma(u)$, $\beta^i(u)$ appearing in metric (2.3).

Solution for $\beta = (\beta^i(u))$ reads as follows:

$$\beta^i(u) = -\frac{U^i}{(U,U)} \ln |f(u)| + c^i u + \bar{c}^i, \tag{2.22}$$

where $f(u)$ was defined in (2.15)-(2.18) and

$$\gamma = \gamma_0 \equiv \sum_{i=1}^n d_i \beta^i = U^\Lambda \beta^i \tag{2.23}$$

and $u$ is the harmonic variable.
3 Scattering law for Kasner parameters

Now we restrict our consideration by a special solution with \( C > 0 \), \( K = (U, U) > 0 \) and \( A > 0 \). In this case the solution is governed by moduli function \( f(u) = R \cosh(\sqrt{C}(u-u_0)) \), \( u \in (-\infty, +\infty) \), and has two Kasner-like asymptotics in the limits \( \tau \to +0 \) and \( \tau \to +\infty \), where \( \tau \) is a synchronous time variable (see below).

Another case, when there are two Kasner-like asymptotical regions, takes place when \( C > 0 \), \( K = (U, U) < 0 \) and \( A < 0 \) (this will be a subject of a separate paper).

3.1 Kasner-like behaviour

Let us consider our solution in a synchronous time:

\[
\tau = \varepsilon \int_{u_0}^{u} d\bar{u} e^{\gamma_{0}(\bar{u})},
\]

where \( \varepsilon = \pm 1 \), and

\[
e^{\gamma_{0}(u)} = |f(u)|^{-h(U^\Lambda, U)} \exp(e^0 u + e^0) \tag{3.2}
\]

is a lapse function.

Due to

\[
f \sim \frac{R}{2} \exp(\pm \sqrt{C}(u-u_0)),
\]

for \( u \to \pm \infty \), we get asymptotical relations for the lapse function

\[
e^{\gamma_{0}} \sim \text{const} \exp(b_{\pm} \sqrt{C} u),
\]

as \( u \to \pm \infty \), with

\[
b_{\pm} = \mp h(U^\Lambda, U) + \frac{e^0}{\sqrt{C}}. \tag{3.5}
\]

Using relations (2.21) and \( h = (U, U)^{-1} \), we could rewrite parameters \( b_{\pm} \) in a minisuperspace-covariant form:

\[
b_{\pm} = \mp \frac{(U^\Lambda, U)}{(U, U)} + (s, U^\Lambda), \tag{3.6}
\]

where

\[
s = (s_i) = (G_{ij}e^j/\sqrt{C}) \tag{3.7}
\]
is a co-vector, obeying relations

\[(s, U) = 0,\]  
\[\frac{1}{(U, U)} + (s, s) = 0,\]  
(3.8)  
(3.9)

following just from (2.19) and (2.20). In derivation of (3.6) we used the relation

\[c^0 = (s, U^\Lambda)\sqrt{C},\]  
(3.10)

following from (2.21) and (3.7).

In what follows we will use the inequality

\[|(s, U^\Lambda)| > \frac{|(U^\Lambda, U)|}{(U, U)},\]  
(3.11)

proved in Appendix C. The proof used relations (3.8), (3.9) and \((U, U) > 0\).

The parameter \(c^0\) is a non-zero one (otherwise the relation (2.20) would be incompatible with the conditions \(C > 0, K > 0\)).

It follows from (3.11) that \(b_\pm\) are also non-zero and

\[\text{sign}(b_\pm) = \text{sign}((s, U^\Lambda)) = \text{sign}(c^0).\]  
(3.12)

It may be verified that due to (3.11) the lapse function \(e^{\gamma_0(u)}\) is monotonically increasing from \(+0\) to \(+\infty\) for \(c^0 > 0\) and monotonically decreasing from \(+\infty\) to \(+0\) for \(c^0 < 0\).

We define a synchronous-like variable to be

\[\tau = \int_{-\infty}^{u} d\bar{u}e^{\gamma_0(\bar{u})}\]  
(3.13)

for \(c^0 > 0\) and

\[\tau = \int_{u}^{+\infty} d\bar{u}e^{\gamma_0(\bar{u})}\]  
(3.14)

for \(c^0 < 0\). Then, \(\tau = \tau(u)\) is monotonically increasing from \(+0\) to \(+\infty\) for \(c^0 > 0\) and monotonically decreasing from \(+\infty\) to \(+0\) for \(c^0 < 0\).

We have the following asymptotical relations for \(\tau = \tau(u)\)

\[\tau \sim \text{const} \, b_\pm^{-1} \exp(b_\pm \sqrt{C}u),\]  
(3.15)

as \(u \to \pm\infty\).
For $\beta = (\beta^i)$ from (2.22) we get (see (3.3))

$$
\beta^i(u) \sim \mp \frac{U^i \sqrt{Cu}}{(U, U)} + c^i u + \hat{c}^i
$$

(3.16)
as $u \to \pm \infty$, where $\hat{c}^i$ are constants. Hence, due to (3.15), we are led to Kasner-like asymptotics

$$
\beta^i \sim \alpha^i_{\pm} \ln \tau + \beta^i_{\pm}
$$

(3.17)
for $u \to \pm \infty$, where $\beta^i_{\pm}$ are constants and

$$
\alpha^i_{\pm} = \left[ \mp \frac{U^i}{(U, U)} + s^i \right] / b^i_{\pm}
$$

(3.18)
are Kasner-like parameters corresponding to $u \to \pm \infty$.

Asymptotical relations (3.17) could be also rewritten in the form of proper time asymptotics, i.e.

$$
\beta^i \sim \alpha^i_0 \ln \tau + \beta^i_0, \quad \text{as } \tau \to +0,
$$

(3.19)

$$
\beta^i \sim \alpha^i_\infty \ln \tau + \beta^i_\infty, \quad \text{as } \tau \to +\infty.
$$

(3.20)
Here

$$
\alpha^i_0 = \alpha^i_-, \quad \alpha^i_\infty = \alpha^i_+ \quad (3.21)
$$

for $c^0 > 0$ and

$$
\alpha^i_0 = \alpha^i_+, \quad \alpha^i_\infty = \alpha^i_- \quad (3.22)
$$

for $c^0 < 0$ and $\beta^i_0$, $\beta^i_\infty$ are constants.

It follows from definitions of Kasner parameters (3.18) that

$$
G_{ij} \alpha^i_{\pm} \alpha^j_{\pm} = 0, \quad (3.23)
$$

$$
U(\alpha_{\pm}) = U_i \alpha^i_{\pm} = \mp \frac{1}{b^i_{\pm}}, \quad (3.24)
$$

$$
U^\Lambda(\alpha_{\pm}) = 1, \quad (3.25)
$$
see (3.6), (3.8) and (3.9).

In components relations (3.23) and (3.25) read as

$$
\sum_{i=1}^n d_i \alpha^i_{\pm} = \sum_{i=1}^n d_i (\alpha^i_{\pm})^2 = 1. \quad (3.26)
$$
Thus, we are led to Kasner-like relations (1.2) and (1.3) for $\alpha_\pm = (\alpha_i^\pm)$. Hence, $\alpha_0 = (\alpha_0^i)$ and $\alpha_\infty = (\alpha_\infty^i)$ also obey relations (1.2) and (1.3).

So, we obtained a Kasner-like asymptotical behaviour of our special solution (with $C > 0$, $K > 0$ and $A > 0$) for i) $\tau \rightarrow +0$ and for ii) $\tau \rightarrow +\infty$, as well. The Kasner-like behaviour in the case i) is in agreement with the general result of the billiard approach from [22]. The the case ii) was considered in [26].

Using (3.12) and (3.24) we get

$$U(\alpha_0) = U_i \alpha_0^i > 0, \quad (3.27)$$

$$U(\alpha_\infty) < 0. \quad (3.28)$$

### 3.2 Scattering law

Now, we derive a relation between Kasner sets $\alpha_0$ and $\alpha_\infty$.

We start with formulae:

$$b_+\alpha_+ - b_-\alpha_- = -\frac{2\bar{U}}{(U,U)} \quad (3.29)$$

and

$$b_+ - b_- = -\frac{2(U^\Lambda,U)}{(U,U)}, \quad (3.30)$$

following from (3.18) and (3.6), respectively. (Recall that $\bar{U} = (U^i)$. ) Using these relations and (3.24) we get

$$\alpha_\pm^i = \frac{\alpha_\mp^i - 2U^jU(\alpha_\mp^i)\frac{1}{(U,U)^{-1}}}{1 - 2U(\alpha_\mp^i)(U,U^\Lambda)(U,U)^{-1}}. \quad (3.31)$$

This formula gives a scattering law formula for Kasner parameters in our case (see definitions (2.10), (3.21) and (3.22)) or

$$\alpha_\infty = \frac{\alpha_0 - 2\bar{U}U(\alpha_0)(U,U)^{-1}}{1 - 2\bar{U}(\alpha_0)(U,U^\Lambda)(U,U)^{-1}} = S(\alpha_0). \quad (3.32)$$

coinciding with the scattering law formula (1.4) derived in [1] for another $S$-brane solution when scalar fields are absent and $U$ is coinciding with the brane $U$-vector.

Due to (3.31) the inverse function $S^{-1}$ is given by just the same relation

$$\alpha_0 = \frac{\alpha_\infty - 2\bar{U}U(\alpha_\infty)(U,U)^{-1}}{1 - 2\bar{U}(\alpha_\infty)(U,U^\Lambda)(U,U)^{-1}} = S^{-1}(\alpha_\infty). \quad (3.33)$$
3.3 Geometric meaning of the scattering law

Here we analyze the geometric meaning of the scattering for \( n > 2 \) as it was done in [1] for the S-brane solution.

The Kasner-like relations (1.2) and (1.3) describe an ellipsoid isomorphic to a unit \( (n-2) \)-dimensional sphere \( S^{n-2} \) belonging to \( \mathbb{R}^{n-1} \). The sets of Kasner parameters \( \alpha \) may be parametrized by vectors \( \vec{n} \in S^{n-2} \), i.e. \( \alpha = \alpha(\vec{n}) \).

For \( (U, U^\Lambda) \neq 0 \) (or, equivalently, when \( \sum_{i=1}^{n} U_i \neq 0 \), see (2.14)) the scattering law formula (1.4) in terms of \( \vec{n} \)-vectors reads as in [1]

\[
\vec{n}_\infty = \frac{(\vec{v}^2 - 1)\vec{n}_0 + 2(1 - \vec{v}\vec{n}_0)\vec{v}}{\vec{v} - \vec{n}_0}^2
\]  

(3.34)

where \( \vec{v} \) is a vector belonging to \( \mathbb{R}^{n-1} \) with \( |\vec{v}| > 1 \).

Here

\[
\vec{v}\vec{n}_0 < 1 \quad \vec{v}\vec{n}_\infty > 1,
\]  

(3.35)

for \( (U, U^\Lambda) < 0 \) (or, equivalently, when \( \sum_{i=1}^{n} U_i > 0 \)) and

\[
\vec{v}\vec{n}_0 > 1 \quad \vec{v}\vec{n}_\infty < 1,
\]  

(3.36)

for \( (U, U^\Lambda) > 0 \) (or, equivalently, when \( \sum_{i=1}^{n} U_i < 0 \)).

The vector \( \vec{v} = (v_i) \in \mathbb{R}^{n-1} \) is defined by the formula

\[
v_i = -\dot{U}_i/\dot{U}_0,
\]  

(3.37)

i = 1, \ldots, n - 1, \text{ where}

\[
\dot{U}_a = e^a_i U_i,
\]  

(3.38)

and the invertible matrix \( (e^a_i) \) satisfies the relations

\[
\eta^{ab} = e^a_i G^{ij} e^b_j,
\]  

(3.39)

\( a, b = 0, \ldots, n - 1 \), with

\[
e^0_i = q^{-1}U^\Lambda_i,
\]  

(3.40)

and

\[
q = \left[ - (U^\Lambda, U^\Lambda) \right]^{1/2} = \left[ (D - 1)/(D - 2) \right]^{1/2}.
\]  

(3.41)

(Here \( (\eta_{ab}) = (\eta^{ab}) = diag(-1, +1, \ldots, +1) \).)

This implies

\[
\dot{U}_0 = -q^{-1}(U, U^\Lambda)
\]  

(3.42)
and hence $\hat{U}_0 \neq 0$ when $(U, U^\Lambda) \neq 0$.

Relations (3.34), (3.35) and (3.36) could be readily proved from (3.31), (3.27) and (3.28) if the following “frame” Kasner-like parameters

$$\hat{\alpha}^a = \epsilon^a_i \alpha^i, \hspace{1cm} (3.43)$$

with

$$\hat{\alpha}^0 = q^{-1}, \hspace{1cm} \hat{\alpha}^i = q^{-1} n^i, \hspace{1cm} (3.44)$$

$i = 1, \ldots, n-1$, are used (see [1]). An important relation here is the following one

$$U(\alpha) = U_A \alpha^A = \hat{U}_a \hat{\alpha}^a = q^{-1} \hat{U}_0 (1 - \vec{v} \vec{n}). \hspace{1cm} (3.45)$$

Thus, for $(U, U^\Lambda) \neq 0$ we get just a modified inversion with respect to a point $v$ located outside the Kasner sphere $S^{n-2}$ (see Fig. 1). For $(U, U^\Lambda) < 0$ the function (3.34) maps a shadow part of the Kasner sphere $S^{n-2}$ onto illuminated one, while for $(U, U^\Lambda) > 0$ this function maps an illuminated part of the Kasner sphere $S^{n-2}$ onto shadow one. Here the shadow and illuminated parts of the Kasner sphere are defined w.r.t. a point-like source of light located at $v$.

For $(U, U^\Lambda) = 0$ (or, equivalently, when $\sum_{i=1}^{n} U_i = 0$) the main formula (1.4) in terms of $\vec{n}$-vectors reads

$$\vec{n}_\infty = \vec{n}_0 - 2 (\vec{b} \vec{n}_0) \vec{b}, \hspace{1cm} (3.46)$$

where $\vec{b} = (b_i)$ is a unit vector belonging to $\mathbb{R}^{n-1}$ ($|\vec{b}| = 1$) with components

$$b_i = \hat{U}_i / (\sum_{j=1}^{n-1} \hat{U}_j)^{1/2}, \hspace{1cm} (3.47)$$

Figure 1: The graphical representation of the modified inversion $S$ w.r.t. a point $V$ for $n = 3$, and $(U, U^\Lambda) < 0$: $N^i = S(N)$.

The inequalities on Kasner-like parameters (3.27) and (3.28) in this case reads as follows

$$b\vec{m}_0 > 0, \hspace{1cm} b\vec{m}_\infty < 0. \hspace{1cm} (3.48)$$

Thus, for $(U, U^\Lambda) = 0$ the function (3.34) is just a reflection with respect to a hyperplane $\{\vec{y} : \vec{b} \vec{y} = 0\}$, which contains a center of the Kasner sphere.
Relations (3.47) and (3.48) may be obtained from (3.34), (3.35) and (3.36) by means of the limiting procedure: \( \hat{U}_0 \to \pm 0 \) \( (|\vec{v}| \to +\infty) \).

It should be noted that all formulas presented above are also valid for \( n = 2 \). In this case the zero-dimensional Kasner sphere \( S^0 = \{ -1, 1 \} \) should be considered.

4 Example: \( n = 2 \)

Here we consider the simplest case of the solution with \( C > 0 \), \( K > 0 \), when \( n = 2 \). We put \( U_1 \neq 0 \) and \( U_2 = 0 \), i.e. \( \hat{p}_1 = w_1 \hat{\rho} \) with \( w_1 \neq 1 \) and \( \hat{p}_2 = \hat{\rho} \).

For Kasner set \( \alpha = (\alpha^1, \alpha^2) \) we get from (1.2) and (1.3) \( \alpha^\pm = (\alpha^1 \pm \alpha^2) = \frac{1}{d_1 + d_2} \left( 1 \pm \frac{r}{d_1}, 1 \mp \frac{r}{d_2} \right) \), (4.1)

where \( r = \sqrt{d_1 d_2 (d_1 + d_2 - 1)} \). (The number \( r > 0 \) is integer one when \( d_1 = 1 \) or \( d_2 = 1 \) and also for \( (d_1, d_2) = (3, 6), (5, 5), (2, 8), (13, 13) \) etc \( [30] \).

Let \( d_2 > 1 \). Then \( \alpha^+_1 > 0 \) and \( \alpha^+_2 < 0 \). Due to \( U_2 = 0 \): \( U(\alpha) = U_1 \alpha^1 \) and hence \( U(\alpha^+) > 0 \) and \( U(\alpha^-) < 0 \) for \( U_1 > 0 \) \( (w_1 < 1) \) and \( U(\alpha^+) < 0 \) and \( U(\alpha^-) > 0 \) for \( U_1 < 0 \) \( (w_1 < 1) \).

It follows from (3.27) and (3.28) that

\[
\alpha_0 = \alpha_+ , \quad \alpha_\infty = \alpha_- .
\]

(4.2)

for \( U_1 > 0 \) and

\[
\alpha_0 = \alpha_- , \quad \alpha_\infty = \alpha_+ .
\]

(4.3)

for \( U_1 < 0 \).

Relation \( U_i c^i = U_1 c^1 = 0 \) implies \( c^1 = 0 \). Here \( c^0 = d_2 c^2 \). Due to (3.21) and (3.22) we should put \( c^2 < 0 \) for \( U_1 > 0 \) and \( c^2 > 0 \) for \( U_1 < 0 \). In this case the sets \( \alpha_\pm \) given by (3.18) are coinciding with those given by (4.1). This may be also verified by straightforward calculations using the following relations

\[
U^1 = \frac{(d_2 - 1)U_1}{d_1(D - 2)} , \quad U^2 = \frac{U_1}{(2 - D)} = (U, U^\Lambda) , \quad K = U^1 U_1 , \quad C = K d_2 (d_2 - 1) c^2_2 , \quad (4.4)
\]
where $D = d_1 + d_2 + 1 > 3$.

**Accelerated expansion of 3-dimensional factor-space.** After replacing $\tau \to \tau_0 - 0$, where $\tau_0$ is constant, we get for $w = -1$ two asymptotical Kasner type metrics

$$g_{as} = -d\tau \otimes d\tau + \sum_{i=1}^{2} A_i^2 (\tau_0 - \tau)^{2\alpha^i} g^{(i)},$$

where either $\alpha^i = \alpha_0^i$ ($A_i = A_i,0 > 0$) as $\tau \to \tau_0 - 0$, or $\alpha^i = \alpha_\infty^i$ ($A_i = A_i,\infty > 0$) as $\tau \to -\infty$.

Let $M_1$ be a flat 3-dimensional factor space ($d_1 = 3$), with the metric $g^{(1)} = dy^1 \otimes dy^1 + dy^2 \otimes dy^2 + dy^3 \otimes dy^3$. Then, due to relations (4.2), (4.3) and $\alpha_\infty < 0$ for $d_2 > 1$, we get an asymptotical accelerated expansion of our 3-dimensional factor space $M_1$ either as $\tau \to \tau_0 - 0$ for $U_1 < 0$, $c_2 > 0$ or as $\tau \to -\infty$ for $U_1 > 0$ and $c_2 < 0$.

**Milne-type asymptotics.** Now we put $d_1 = 1$. We get

$$\alpha_+ = (1,0), \quad \alpha_- = \frac{1}{1 + d_2} (1 - d_2, 2).$$

For $M_1 = \mathbb{R}$, $g^{(1)} = -w dy^1 \otimes dy^1$, $-\infty < y^i < +\infty$, we get a Milne-type (flat) asymptotic:

i) as $\tau \to +0$ for $U_1 > 0$ and $c^2 < 0$;

ii) as $\tau \to +\infty$ for $U_1 < 0$ and $c^2 > 0$.

Both cases correspond to $u \to +\infty$.

For $M_1 = S^3$, $g^{(1)} = w dy^1 \otimes dy^1$, $0 < y^i < +2\pi$, we may get either non-singular (static) solution in the case i) ($\tau = \rho$) or asymptotically flat (static) solution in the case ii).

### 5 Conclusions and discussions

In this paper we have considered the exact cosmological type solution with 1-component anisotropic fluid. This solution is defined on the product manifold (2.2) containing $n$ Ricci-flat factor spaces $M_1,...,M_n$.

We have singled out a special solution governed by the $\cosh$ moduli function and shown that this solution has Kasner-like asymptotics in the limits $u \to \pm \infty$, where $u$ is the harmonic variable, or, equivalently, in the limits $\tau \to +0$ and $\tau \to +\infty$, where $\tau$ is the synchronous type variable.
We have found a relation between two sets of Kasner parameters $\alpha_\infty$ and $\alpha_0$. The relation between them $\alpha_\infty = S(\alpha_0)$ is coinciding with the “scattering law” formula obtained for the $S$-brane solution from [1] when scalar fields are absent and the fluid $U$-vector is equal to the brane one.

The function $S$ (defined on the set of Kasner vectors obeying $U(\alpha) > 0$) is bijective. The inverse function $S^{-1}$ (defined on the set of Kasner vectors obeying $U(\alpha) < 0$) is given by the same formula as the function $S$. The function $S$ depends upon the co-vector $U = (U_i)$. It is invariant upon the replacement: $U \mapsto \lambda U$, where $\lambda > 0$ (see [26]). The transformation $U \mapsto -U$ implies the replacement $S \mapsto S^{-1}$.

We have also analyzed the geometric meaning of the scattering law formula in terms of transformation of the Kasner sphere $S^{n-2}$, $n \geq 2$. For $(U, U^\Lambda) \neq 0$ (or, equivalently, when $\sum_{i=1}^n U_i \neq 0$) we get just a modified inversion with respect to a point $v$ located outside the Kasner sphere $S^{n-2}$, while for $(U, U^\Lambda) = 0$ (or, equivalently, when $\sum_{i=1}^n U_i \neq 0$) we are led to a reflection with respect to a hyperplane which contains a center of the Kasner sphere.

The scattering law formula may be applied for the solutions with Kasner-like asymptotical behaviours (written in a slightly different form)

$$g_{as} = -d\tau \otimes d\tau + \sum_{i=1}^n A_i^2(\tau_0 - \tau)^{2\alpha_i} g^{(i)}, \quad (5.1)$$

where either $\tau \to \tau_0 - 0$, or $\tau \to -\infty$. In this case the metric (5.1) may describe an asymptotical accelerated expansion of flat 3-dimensional factor space $M_1$ if $d_1 = 3$, $g^{(i)} = dy^i \otimes dy^i + dy^2 \otimes dy^2 + dy^3 \otimes dy^3$ and $\alpha^1 < 0$.

Another application of the scattering law formula appears when $d_1 = 1$ and one of the asymptotical Kasner set of parameters in (1.1) is of Milne type: $\alpha = (1, 0, \ldots, 0)$, e.g. when static non-singular solutions ($w = +1$, $M_1 = S^1$) or cosmological solutions ($w = -1$, $M_1 = \mathbb{R}$) with a horizon (for $\tau \to +0$) are considered. (Compare with flux-brane and $S$-brane solutions [31, 32]). These topics (mentioned above) may be a subject of separate publications.
Appendix

A Solution for Liouville system

Let
\[ L = \frac{1}{2} \langle \dot{x}, \dot{x} \rangle - A \exp[2 \langle b, x \rangle] \]  
(A.1)
be a Lagrangian, defined on \( V \times V \), where \( V = \mathbb{R}^n \), \( A \neq 0 \), and \( \langle \cdot, \cdot \rangle \) is non-degenerate real-valued quadratic form on \( V \). (Here \( \dot{x} = \frac{dx}{dt} \) etc.)

Let \( \langle b, b \rangle \neq 0 \). Then, the Euler-Lagrange equations for the Lagrangian (A.1) have the following solution \(^{[13]}\)
\[ \ddot{x} + 2A \dot{b} \exp[2 \langle b, x \rangle] = 0 \]  
(A.2)

have the following solution \(^{[13]}\)
\[ x(t) = -\frac{b}{\langle b, b \rangle} \ln |f(t - t_0)| + t\alpha + \beta, \]  
(A.3)
where \( \alpha, \beta \in V \),
\[ \langle \alpha, b \rangle = \langle \beta, b \rangle = 0, \]  
(A.4)
and
\[ f(\tau) = \begin{cases} R \sinh(\sqrt{C}\tau), & C > 0, \ < b, b > A < 0, \\ R \sin(\sqrt{|C|}\tau), & C < 0, \ < b, b > A < 0, \\ R \cosh(\sqrt{C}\tau), & C > 0, \ < b, b > A > 0, \\ |2A < b, b > |^{1/2}\tau, & C = 0, \ < b, b > A < 0, \end{cases} \]  
(A.5)

where \( R = |\frac{2A\langle b, b \rangle}{C}|^{1/2} \) and \( C, t_0 \) are constants.

The energy
\[ E = \frac{1}{2} \langle \dot{x}, \dot{x} \rangle + A \exp[2 \langle b, x \rangle] \]  
(A.6)
calculated for the solution (A.2) reads
\[ E = \frac{C}{2 \langle b, b \rangle} + \frac{1}{2} \langle \alpha, \alpha \rangle. \]  
(A.7)
B Lagrange representation

The Einstein equations (2.1) imply the conservation law

\[ \nabla_M T^M_N = 0. \]  

(B.8)

that due to relations (2.3) and (2.4) may be written in the following form

\[ \dot{\rho} + \sum_{i=1}^{n} d_i \dot{\beta}^i (\dot{\rho} + \dot{\beta}^i) = 0. \]  

(B.9)

Using the equation of state (2.5) we get

\[ \kappa^2 \dot{\rho} = -w Ae^{2U_i \beta^i - 2\gamma_0}, \]  

(B.10)

where \( \gamma_0(\beta) = \sum_{i=1}^{n} d_i \beta^i \), and \( A \) is constant.

The Einstein equations (2.1) with the relations (2.5) and (B.10) imposed are equivalent to the Lagrange equations for the Lagrangian (for \( w = -1 \) see [14])

\[ L = \frac{1}{2} e^{-\gamma + \gamma_0(\beta)} G_{ij} \dot{\beta}^i \dot{\beta}^j - e^{\gamma - \gamma_0(\beta)} V, \]  

(B.11)

where

\[ V = Ae^{2U_i \beta^i}, \]  

(B.12)

is the potential and the components of the minisupermetric \( G_{ij} \) are defined in (2.8).

For \( \gamma = \gamma_0(\beta) \), i.e. when the harmonic time gauge is considered, we get the set of Lagrange equations for the Lagrangian

\[ L = \frac{1}{2} G_{ij} \dot{\beta}^i \dot{\beta}^j - V, \]  

(B.13)

with the zero-energy constraint imposed

\[ E = \frac{1}{2} G_{ij} \dot{\beta}^i \dot{\beta}^i + V = 0. \]  

(B.14)
The exact solutions for the Lagrangian (B.13) with the potential (B.12) could be readily obtained using the relations from Appendices A and B.

The solutions read:

\[ \beta^i(u) = -\frac{U^i}{(U, U)} \ln |f(u)| + c^i u + \bar{c}^i, \]  

(C.15)

where \( u_0 \) is constant. Function \( f(u) \) in (C.15) is the following:

\[ f(u) = R \sinh(\sqrt{C}(u - u_0)), \quad C > 0, \quad KA < 0; \]  

(C.16)

\[ R \sin(\sqrt{|C|}(u - u_0)), \quad C < 0, \quad KA < 0; \]  

(C.17)

\[ R \cosh(\sqrt{C}(u - u_0)), \quad C > 0, \quad KA > 0; \]  

(C.18)

\[ |2AK|^{1/2}(u - u_0), \quad C = 0, \quad KA < 0, \]  

(C.19)

where \( K = (U, U), \ R = |2AK/C|^{1/2} \) and \( C, \ u_0 \) are constants.

Vectors \( c = (c^i) \) and \( \bar{c} = (\bar{c}^i) \) satisfy the linear constraint relations (see (A.4) in Appendix A)

\[ U(c) = U_i c^i = 0, \]  

(C.20)

\[ U(\bar{c}) = U_i \bar{c}^i = 0. \]  

(C.21)

The zero-energy constraint reads (see (A.6) in Appendix A)

\[ E = \frac{C}{2(U, U)} + \frac{1}{2} G_{ij} c^i c^j = 0. \]  

(C.22)

D Proof of the inequality (3.11)

Let us prove the inequality (3.11)

\[ |(s, U^\Lambda)| > \frac{|(U^\Lambda, U)|}{(U, U)} > 0, \]

for a vector \( s = (s^A) \in \mathbb{R}^n \) obeying relations \( (s, U) = 0, \ (s, s) = -1/(U, U) \). Here the scalar-product \( (U, U') = G^{ij} U_i U'_j \), where \( G^{ij} = \delta^{ij} d_i^{-1} + (2 - D)^{-1} \). We also use here the following relations \( (U, U) > 0, \ U^\Lambda = (d_i) \) and \( (U^\Lambda, U^\Lambda) < 0 \).
Proof. Let us define the vector
\[ U_1 = U - \frac{(U, U^\Lambda)}{(U^\Lambda, U^\Lambda)} U^\Lambda. \]  
(D.23)
It is clear that \((U_1, U^\Lambda) = 0\) and
\[ (U_1, U_1) = (U, U) - \frac{(U, U^\Lambda)^2}{(U^\Lambda, U^\Lambda)} > 0. \]  
(D.24)
since \((U, U) > 0\) and \((U^\Lambda, U^\Lambda) < 0\). Let us define vectors:
\[ s_0 = \frac{(s, U^\Lambda)}{(U^\Lambda, U^\Lambda)} U^\Lambda, \]  
(D.25)
\[ s_1 = \frac{(s, U_1)}{(U_1, U_1)} U_1, \]  
(D.26)
\[ s = s - s_0 - s_1. \]  
(D.27)
s\_0, s\_1 and s\_2 are mutually orthogonal and hence
\[ (s, s) = (s_0, s_0) + (s_1, s_1) + (s_2, s_2). \]  
(D.28)
For the first two terms in r.h.s. of (D.28) we get
\[ (s_0, s_0) = \frac{(s, U^\Lambda)^2}{(U^\Lambda, U^\Lambda)}, \]  
(D.29)
\[ (s_1, s_1) = \frac{(s, U_1)^2}{(U_1, U_1)} = \frac{(s, U^\Lambda)^2}{(U^\Lambda, U^\Lambda)} \frac{(U, U^\Lambda)^2}{[(U, U)(U^\Lambda, U^\Lambda) - (U, U^\Lambda)^2]} \]  
(D.30)
that implies
\[ (s, s) = \frac{(s, U^\Lambda)^2(U, U)}{(U, U)(U^\Lambda, U^\Lambda) - (U, U^\Lambda)^2} + (s_2, s_2). \]  
(D.31)
For the third term in r.h.s. of (D.28) the following inequality is valid
\[ (s_2, s_2) \geq 0, \]  
(D.32)
Indeed, due to \((s_2, U^\Lambda) = 0\), or, equivalently, \(\sum_{i=1}^n s_i^2 d_i = 0\), we obtain
\[ (s_2, s_2) = G_{ij} s_i^2 s_j^2 = \sum_{i=1}^n (s_i^2)^2 d_i \geq 0. \]  
(D.33)
Using this inequality, \((D.31)\), \((U^\Lambda, U^\Lambda) < 0\) and \((s, s) = -1/(U, U)\) we get

\[
(s, U^\Lambda)^2 = \left[ \frac{(U, U^\Lambda)^2}{(U, U)} - (U^\Lambda, U^\Lambda) \right] [(U, U)^{-1} + (s_2, s_2)] > \frac{(U, U^\Lambda)^2}{(U, U)^2} > 0, \quad (D.34)
\]

that is equivalent to the inequality \((3.11)\). Thus, \((3.11)\) is proved.

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