A New Technique for Generating Distributions Based on a Combination of Two Techniques: Alpha Power Transformation and Exponentiated T-X Distributions Family

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Abstract: In the following article, a new five-parameter distribution, the alpha power exponentiated Weibull-exponential distribution is proposed, based on a newly developed technique. It is of particular interest because the density of this distribution can take various symmetric and asymmetric possible shapes. Moreover, its related hazard function is tractable and showing a great diversity of asymmetrical shaped, including increasing, decreasing, near symmetrical, increasing-decreasing-increasing, increasing-constant-increasing, J-shaped, and reversed J-shaped. Some properties relating to the proposed distribution are provided. The inferential method of maximum likelihood is employed, in order to estimate the model’s unknown parameters, and these estimates are evaluated based on various simulation studies. Moreover, the usefulness of the model is investigated through its application to three real data sets. The results show that the proposed distribution can, in fact, better fit the data, when compared to other competing distributions.

Keywords: alpha power transformation; exponentiated T-X family; Weibull distribution; exponential distribution; maximum likelihood estimation

1. Introduction

Various techniques for constructing families to generate new modified distributions that better fit applicable data sets have been suggested in the academic literature. For example, the development of beta family by [1] and Kumaraswamy family by [2], in which the baseline distributions are defined on the support [0, 1] as the generators. Ref. [3] developed a general technique that allows for the use of any baseline distribution as a generator. This innovative technique is described as the transformed transformer (T-X) family of distributions. Many families of distributions have been suggested and examined recently, based on this technique. This includes the gamma-G family of distributions by [4], the Weibull-G family of distributions by [5], the Lomax-G family of distributions by [6], the Lindley-G family of distributions by [7], the Gompertz-G family of distributions by [8], the power Lindley-G family of distributions by [9] and the odd Lomax-G family of distributions by [10], among others.

Following the T-X family of distributions, ref. [11] suggested a new generalization and named it the exponentiated T-X family of distributions. For any cumulative distribution function (CDF), \( G(x) \), the CDF of the exponentiated T-X family, can be expressed as:

\[
F_{ETX}(x) = \int_0^{-\log(1 - G^c(x))} R(t)dt = R[-\log(1 - G^c(x))] \quad (1)
\]
where \( r(t) \) is the probability density function (PDF) of a baseline random variable (RV) \( T \), in which \( T \) is defined over \([0, \infty)\). Let \( G(x) \) indicate the CDF of the RV \( X \). Thus, the PDF corresponding to (1) is

\[
f_{ETX}(x) = \frac{cg(x)G^{c-1}(x)}{1 - G(x)} r[- \log (1 - G(x))] \quad ; \quad c > 0. \tag{2}
\]

Additionally, ref. [11] defined the exponentiated Weibull-G (EW-G) family of distributions from (1) by taking \( r(t) \) to be the Weibull distribution with \( a \) and \( \gamma \) as the shape and scale parameters, respectively. The CDF of the EW-G family of distributions can be expressed as:

\[
F_{EWG}(x) = 1 - e^{- \left( -\log(1 - G(x)) \right)^\alpha}. \tag{3}
\]

Considering \( G \) to be the exponential CDF, we obtain the exponentiated Weibull-exponential (EW-E) as a member of the exponentiated Weibull-G (EW-G) family, with CDF defined as follows:

\[
F_{EWE}(x) = 1 - e^{- \left( -\log(1 - (1 - e^{-\lambda x})^\alpha) \right)^\gamma}. \tag{4}
\]

The corresponding PDF of the EW-E distribution can be expressed using (4), as

\[
f_{EWE}(x) = \frac{ac e^{-\lambda x} (1 - e^{-\lambda x})^{c-1}}{\gamma - 1 - (1 - e^{-\lambda x})^\epsilon} \left( -\log \left( 1 - (1 - e^{-\lambda x})^\alpha \right) \right)^{d-1} e^{- \left( -\log(1 - (1 - e^{-\lambda x})^\alpha) \right)^\gamma}. \tag{5}
\]

Further, ref. [12] have suggested an innovative technique for introducing an additional parameter to a family of distributions. This new technique has been referred as the alpha power transformation (APT) family of distributions. The CDF and PDF of the APT family, respectively, can be expressed as:

\[
F_{APT}(x) = \begin{cases} 
\frac{a^{G(x)-1}}{G(x)} & \text{if } \alpha > 0, \alpha \neq 1 \\
\frac{1}{G(x)} & \text{if } \alpha = 1, 
\end{cases} \tag{6}
\]

\[
f_{APT}(x) = \begin{cases} 
\frac{\log a}{\alpha} g(x) a^{G(x)} & \text{if } \alpha > 0, \alpha \neq 1 \\
g(x) & \text{if } \alpha = 1, 
\end{cases} \tag{7}
\]

where \( G(x) \) and \( g(x) \) represent the CDF and PDF of any continuous distribution.

In the literature, several studies have applied this technique to introduce some new distributions. These include the study of [12], in which the APT technique was applied to the exponential distribution; the study of [13], which introduced the alpha power Weibull distribution; the study of [14], which introduced the alpha power transformed Lindley distribution; the study of [15], which introduced the alpha power transformed inverse Lindley distribution; the study of [16], which presented the alpha power transformed inverse Lindley distribution; the study of [17], which presented the alpha power Gompertz distribution, and the study of [18], which presented the alpha power transformed log-logistic distribution.

Therefore, the main point of interest in this paper is the introduction of a new technique for constructing a family to generate some modified distributions with more flexibility in fitting data. To illustrate, this study combines two techniques-exponentiated T-X and APT-in order to develop a new five-parameter distribution, called the alpha power exponentiated Weibull-exponential distribution (APEWED). This rest of the article is organized as follows: Section 2 introduces the APEWED with some special cases along with its survival and hazard functions. Section 3 investigates some fundamental statistical properties of the APEWED. Section 4 discusses the estimation of its parameters using the maximum likelihood estimation. Section 5 provides a simulation study that evaluate these estimates.
Section 6 considers three applications, in order to show the efficiency of the introduced distribution. Finally, the article concluded in Section 7.

2. The Alpha Power Exponentiated Weibull-Exponential Distribution (APEWED)

An RV X is said to have APEWED, with five parameters \( \alpha, \gamma, c, \lambda, \) and \( \lambda \), if its CDF is given as

\[
F(x) = \begin{cases} 
  a^{1-e^{-\left(-\log(1-(1-e^{-\lambda x}))\right)^a}} & \text{if } \alpha > 0, \alpha \neq 1, \\
  1 - e^{-\left(-\log(1-(1-e^{-\lambda x}))\right)^a} & \text{if } \alpha = 1 
\end{cases}
\]

for \( x \geq 0, a, \gamma, c, \lambda > 0 \).

The corresponding PDF is expressed as

\[
f(x) = \begin{cases} 
  \frac{\log \alpha}{e} a c \frac{\lambda e^{-\lambda x} (1-e^{-\lambda x})^{-1}}{1-(1-e^{-\lambda x})^c} \left(-\log(1-(1-e^{-\lambda x}))\right)^{a-1} \left(-\log(1-(1-e^{-\lambda x}))\right)^{a} & \text{if } \alpha > 0, \alpha \neq 1, \\
  ac \frac{\lambda e^{-\lambda x} (1-e^{-\lambda x})^{-1}}{1-(1-e^{-\lambda x})^c} \left(-\log(1-(1-e^{-\lambda x}))\right)^{a-1} e^{-\left(-\log(1-(1-e^{-\lambda x}))\right)^a} & \text{if } \alpha = 1. 
\end{cases}
\]

The survival function, \( S(x) \), for the APEWED is given by

\[
S(x) = \begin{cases} 
  \frac{\alpha}{a-1} \left(1 - \alpha^{-e^{-\left(-\log(1-(1-e^{-\lambda x}))\right)^a}}\right) & \text{if } \alpha > 0, \alpha \neq 1, \\
  e^{-\left(-\log(1-(1-e^{-\lambda x}))\right)^a} & \text{if } \alpha = 1, 
\end{cases}
\]

and the hazard rate function, \( H(x) \), for the APEWED is expressed as

\[
H(x) = \begin{cases} 
  \frac{\log \alpha}{e} a c \frac{\lambda e^{-\lambda x} (1-e^{-\lambda x})^{-1}}{1-(1-e^{-\lambda x})^c} \left(-\log(1-(1-e^{-\lambda x}))\right)^{a-1} \left(-\log(1-(1-e^{-\lambda x}))\right)^{a} -1 & \text{if } \alpha > 0, \alpha \neq 1, \\
  ac \frac{\lambda e^{-\lambda x} (1-e^{-\lambda x})^{-1}}{1-(1-e^{-\lambda x})^c} \left(-\log(1-(1-e^{-\lambda x}))\right)^{a-1} e^{-\left(-\log(1-(1-e^{-\lambda x}))\right)^a} & \text{if } \alpha = 1. 
\end{cases}
\]

2.1. Some Special Cases of the APEWED

- For \( \alpha = 1 \) in (9), we get the exponentiated Weibull-exponential distribution.
- For \( c = 1 \) in (9), we obtain the alpha power Weibull-exponential distribution.
- For \( a = 1, c = 1 \) in (9), we obtain the Weibull-exponential distribution.
- For \( a = \lambda = a = \gamma = 1 \) in (9), we get the Standard exponentiated exponential distribution.
- For \( a = \lambda = a = 1 \) in (9), we get the Kumaraswamy-Standard exponential distribution.
- For \( a = \lambda = c = 1 \) and \( a = -k \) in (9), we obtain the Type 2 extreme value distribution.
- For \( a = \lambda = c = 1 \) in (9), we obtain the Weibull distribution.

2.2. Graphical Presentations of the APEWED

The PDF and hazard rate function plots for the APEWED are given in Figures 1 and 2, respectively, for particular parameters.
Figure 1. Plots of the APEWED PDF for some certain values of \(a, \alpha, \gamma, c,\) and \(\lambda.\)

Figure 2. Plots of the hazard function of the APEWED for some certain values of \(a, \alpha, \gamma, c,\) and \(\lambda.\)

It can be seen from Figure 1, that the plots of the pdf of the APEWED enjoy various shapes including, symmetric, J-shaped, reversed J-shaped, right skewed, left-skewed. Additionally, Figure 2 reveals that, the hazard rate function of the APEWED possess a great diversity of symmetrical and asymmetrical shaped. Particularly, it can be increasing, decreasing, near symmetrical, increasing-decreasing-increasing, increasing-constant-increasing, J-shaped, and reversed J-shaped. These observations can be considered as strong signs indicating to the great flexibility of the APEWED in fitting data.

2.3. Expansion of the PDF of the APEWED

In this subsection, an expansion of the PDF of the APEWED is suggested. First, consider the following series representation

\[
\alpha^{-z} = \sum_{q=0}^{\infty} (-1)^q \frac{\log \alpha)^q}{q!} (z)^q, \tag{12}
\]
into (9), we obtain the PDF as follows:

\[
\begin{align*}
    f(x) &= \frac{\alpha}{e-1} \frac{\alpha c}{\alpha - 1} \lambda e^{-\lambda x} (1 - e^{-\lambda x})^{c-1} \left(1 - (1 - e^{-\lambda x})^c\right)^{-1}
    \sum_{q=0}^{\infty} (-1)^q \left(\frac{\log a}{q!}\right)^q e^{-xq + \gamma q}
    \left(\frac{1}{\gamma}\right)^q \left(-\log(1 - (1 - e^{-\lambda x})^c)\right)^{q+1}.
\end{align*}
\]

Then, by employing the following series of the exponential function,

\[
e^{-z} = \sum_{n=0}^{\infty} (-1)^n \frac{z^n}{n!},
\]

we obtain:

\[
\begin{align*}
    f(x) &= \frac{\alpha}{e-1} \frac{\alpha c}{\alpha - 1} \lambda e^{-\lambda x} (1 - e^{-\lambda x})^{c-1} \left(1 - (1 - e^{-\lambda x})^c\right)^{-1}
    \sum_{q=0}^{\infty} \sum_{n=0}^{\infty} (-1)^{q+n} \left(\frac{\log a}{q!}\right)^{q+1} (q+1)^n
    \left(\frac{1}{\gamma}\right)^q \left(-\log(1 - (1 - e^{-\lambda x})^c)\right)^{a(n+1)-1}. 
\end{align*}
\]

Furthermore, we have the following two series expansions:

\[
(-\log(1-z))^r = \sum_{m=0}^{\infty} \sum_{k=0}^{m} \frac{(-1)^{m+k} (m-k) P_k, m (z)^{r+m}}{r!}.
\]

where

\[
P_k, m = m^{-1} \sum_{f=1}^{m} (m - f(k+1)) c_f P_{k, m-f},
\]

for \(m = 1, 2, ..., P_{k, 0} = 1\), and \(c_m = (-1)^{m+1}(m + 1)^{-1}\),

and

\[
(1 - z)^{-1} = \sum_{j=0}^{\infty} z^j.
\]

Consequently, by substituting in Equation (13), the PDF can be reduced to

\[
\begin{align*}
    f(x) &= \frac{\alpha}{e-1} \frac{\alpha c}{\alpha - 1} \lambda e^{-\lambda x} \sum_{q=0}^{\infty} \sum_{n=0}^{\infty} \sum_{m=0}^{\infty} \sum_{k=0}^{m} \sum_{j=0}^{\infty} (-1)^{m+k+q+n} a(n+1) \left(\frac{\log a}{q!}\right)^{q+1}
    \left(\frac{1}{\gamma}\right)^q \left(-\log(1 - (1 - e^{-\lambda x})^c)\right)^{a(n+1)-1}
    e^{-xq + \gamma q}
    \left(\frac{1}{\gamma}\right)^q \left(-\log(1 - (1 - e^{-\lambda x})^c)\right)^{q+1}.
\end{align*}
\]

In addition, we have the following binomial series expansion

\[
(1 - z)^{a-1} = \sum_{d=0}^{\infty} (-1)^d \binom{a-1}{d} z^d.
\]

Therefore, the PDF of APEWED can be rewritten as follows

\[
\begin{align*}
    f(x) &= \sum_{d=0}^{\infty} \psi_d e^{-\lambda x(d+1)}.
\end{align*}
\]

where
ψ_d = \frac{\alpha}{\alpha - 1} a \lambda \sum_{q=0}^{\infty} \sum_{n=0}^{\infty} \sum_{m=0}^{\infty} \sum_{k=0}^{\infty} (-1)^{m+k+q+n+d} (a(n+1) - 1) (m-(a(n+1)-1))^{m} (a(n+1)+m+j-1) \sum_{d=0}^{\infty} \psi_d \left( \frac{1}{\lambda(d+1)} \right)^{r+1} \Gamma(r+1),
\quad (15)

3. Some Statistical Properties of the APEWED

This section discusses some fundamental statistical properties of the APEWED.

3.1. Quantile Function and Median

The quantile function, say \( x_p = Q(p) \), of the APEWED can be derived as

\[
x_p = -\frac{1}{\lambda} \log \left[ 1 - \left\{ 1 - e^{-\gamma \left[ \frac{1}{\log \alpha} \right]} \right\}^{\frac{1}{2}} \right].
\quad (16)
\]

Then, we can obtain the median by putting \( p = 0.5 \) in Equation (16), as follows:

\[
x_{0.5} = -\frac{1}{\lambda} \log \left[ 1 - \left\{ 1 - e^{-\gamma \left[ \frac{1}{\log \alpha} \right]} \right\}^{\frac{1}{2}} \right].
\quad (17)
\]

3.2. Skewness and Kurtosis

The Bowley’s coefficient for skewness and Moors’ coefficient for kurtosis of the APEWED are, respectively, given as

\[
\text{Skewness} = \frac{Q(0.75) - 2Q(0.5) + Q(0.25)}{Q(0.75) - Q(0.25)},
\quad (18)
\]
and

\[
\text{Kurtosis} = \frac{Q(0.875) - Q(0.625) + Q(0.375) - Q(0.125)}{Q(0.75) - Q(0.25)},
\quad (19)
\]
where \( Q(\cdot) \) represents the quantile function of the APEWED in Equation (16).

3.3. Moments

If \( X \sim \text{APEWED} (\alpha, a, \gamma, c, \lambda) \), then the \( r \)th moment of \( X \) can be expressed as

\[
\mu_r = E(X^r) = \sum_{d=0}^{\infty} \psi_d \left( \frac{1}{\lambda(d+1)} \right)^{r+1} \Gamma(r+1),
\quad (20)
\]
where \( \psi_d \) is given by (15). The proof of (20) is given in Appendix A. Thus, the mean of the APEWED is given by setting \( r = 1 \) in Equation (20)

\[
\mu = \sum_{d=0}^{\infty} \psi_d \left( \frac{1}{\lambda(d+1)} \right)^2.
\quad (21)
\]

From (20) and (21), the variance for the APEWED can be easily expressed as

\[
\sigma^2 = E(X^2) - E(X)^2
\]
and

\[
\sigma^2 = 2 \sum_{d=0}^{\infty} \psi_d \left( \frac{1}{\lambda(d+1)} \right)^3 - \mu^2.
\quad (22)
\]
3.4. Moment Generating Function and Characteristic Function

If \( X \sim \text{APEWED} (a, a, \gamma, c, \lambda) \), then the moment generating function (MGF), \( M_x(t) \), takes the form of

\[
M_x(t) = \sum_{d=0}^{\infty} \sum_{r=0}^{\infty} \frac{t^r}{r!} \psi_d \left( \frac{1}{\lambda(d + 1)} \right)^{r+1} \Gamma(r+1). \tag{23}
\]

In the same way, the characteristic function of the APEWED with density (14), can be expressed as

\[
\phi_x(t) = E(e^{itx}) = \sum_{d=0}^{\infty} \sum_{r=0}^{\infty} \frac{(it)^r}{r!} \psi_d \left( \frac{1}{\lambda(d + 1)} \right)^{r+1} \Gamma(r+1) \tag{24}
\]

where \( \psi_d \) is given by (15). The proposed results of (23) is followed from some algebraic manipulation given in Appendix A.

3.5. Mean Residual Life and Mean Waiting Time

If \( X \) follows the APEWED with \( S(x) \) in (10), then the mean residual life function, \( \mu(t) \), can be obtained from

\[
\mu(t) = \frac{1}{S(t)} \left( E(t) - \int_0^t x f(x) dx \right) - t, \tag{25}
\]

Let the integral \( I = \int_0^t x f(x) dx \). Then, from (14), we obtain

\[
I = \sum_{d=0}^{\infty} \psi_d \left( \frac{1}{\lambda(d + 1)} \right)^2 \gamma(\lambda t(d + 1), 2), \tag{26}
\]

where \( \gamma(a, b) = \int_0^a x^{b-1}e^{-x} dx \) denotes the lower incomplete gamma function. Subsequently, using Equations (10), (21), and (26) in Equation (25), the \( \mu(t) \), according to the APEWED, is obtained by

\[
\mu(t) = \frac{\alpha - 1}{a} \left\{ \sum_{d=0}^{\infty} \psi_d \left( \frac{1}{\lambda(d + 1)} \right)^2 \left[ 1 - \gamma(\lambda t(d + 1), 2) \right] - t \right\} \tag{27}
\]

Similarly, the mean waiting time function, \( \bar{\mu}(t) \), can be derived as follows

\[
\bar{\mu}(t) = t - \frac{1}{F(t)} \int_0^t x f(x) dx, \tag{28}
\]

where \( F(t) \) and \( I = \int_0^t x f(x) dx \) are given in (8) and (26), respectively. Thus, we can obtain \( \bar{\mu}(t) \) as

\[
\bar{\mu}(t) = t - \frac{\alpha - 1}{a} \left\{ \sum_{d=0}^{\infty} \psi_d \left( \frac{1}{\lambda(d + 1)} \right)^2 \gamma(\lambda t(d + 1), 2) \right\} \tag{29}
\]

3.6. Shannon and Rényi Entropies

If an RV \( X \) follows APEWED with PDF (9), the Shannon entropy (\( SE_X \)) of \( X \) is described as

\[
SE_X = -E[\log f(x)] = - \int_0^\infty \log(f(x)) f(x) dx. \tag{29}
\]
We have
\[
\log(f(x)) = \log \left( \frac{ac\alpha \log \alpha}{\gamma(\alpha - 1)} \right) - \lambda x + (c - 1) \log(1 - e^{-\lambda x}) - \log(1 - (1 - e^{-\lambda x})^c)
\]
\[+ (a - 1) \log \left( -\frac{\log(1 - (1 - e^{-\lambda x})^c)}{\gamma} \right) - \left( -\frac{\log(1 - (1 - e^{-\lambda x})^c)}{\gamma} \right)^a
\]
\[+ e^{-\left( -\frac{\log(1 - (1 - e^{-\lambda x})^c)}{\gamma} \right)^a} \log \alpha.
\]

Then, after solving the integrals, the Rényi entropy of the APEWED can be written as
\[
SE_X = -\log \left( \frac{ac\alpha \log \alpha}{\gamma(\alpha - 1)} \right) + \lambda \mu
\]
\[+ \frac{\alpha}{\alpha - 1} \sum_{q=0}^{\infty} \frac{(-1)^q (\log \alpha)^q + 1}{q!} \left\{ \frac{(c - 1)}{c} \sum_{s=1}^{\infty} \sum_{n=0}^{\infty} \frac{(-1)^n}{n!} (s\gamma)^n \left( \frac{1}{q+1} \right)^{\frac{q+1}{a+1}} \Gamma \left( \frac{n+1}{a+1} \right) \right\}
\]
\[+ \frac{\gamma}{\alpha(q + 1)} \left( \frac{1}{\alpha + 1} \right) + \frac{(a - 1)}{a(q + 1)} \left( k + \log(q + 1) \right) + \frac{1}{(q + 1)^2} + \frac{\log \alpha}{q + 2},
\]
where \( k \) is the Euler constant.

Moreover, the Rényi entropy, \( REX(v) \), of \( X \) can be described as follows:
\[
RE_X(v) = \frac{1}{1 - v} \log \left( \int_0^\infty f(x)^v dx \right); \quad v > 0, v \neq 1.
\]

Then, applying the PDF (9) and solving the integral, the Rényi entropy of APEWED can be expressed as
\[
RE_X(v) = \frac{v}{1 - v} \log \left( \frac{ac\alpha \log \alpha}{\gamma(\alpha - 1)} \right) - \log(\lambda) + \frac{1}{1 - v} \log \left( \sum_{q=0}^{\infty} \sum_{n=0}^{\infty} \sum_{m=0}^{\infty} \sum_{k=0}^{\infty} \sum_{a=0}^{\infty} \sum_{b=0}^{\infty} \sum_{c=0}^{\infty} \sum_{d=0}^{\infty} \frac{(-1)^{m+k+q+n+d}(v(a - 1) + an)(m-v(a-1)-an)(m+w-1)_k(c[a(v+n)+m+w]-v)_k}_{v^n!(v(a - 1) + an) - k} \right)
\]
\[\times \left( \frac{\log \alpha}{\gamma} \right)^a \left( \frac{1}{v} \right) \frac{\log(v+q)^n}{v + d}.
\]

Table 1 presents some values of the mean, variance, skewness, and kurtosis for the APEWED, for some different values of \( \alpha, \gamma, c, \) and \( a \) while \( \lambda = 2 \) is kept fixed. For fixed \( \gamma \) and \( a \), the values of the mean and variance of the APEWED increase while increasing the values of \( \alpha \) and \( c \), while the skewness and kurtosis values decrease as the value of \( \alpha \) increases. In addition, when the values of \( a, c, \) and \( a \) do not change, the mean and the variance increase as the value of \( \gamma \) increases.
Table 1. Mean, variance, skewness and kurtosis of the APEWED for various values of \( \alpha, a, \gamma, c \) while \( \lambda = 2 \) is fixed.

| \( \alpha \) | \( a \) | \( \gamma \) | \( c \) | Mean | Variance | Skewness | Kurtosis |
|---|---|---|---|---|---|---|---|
| 0.5 | 1 | 0.5 | 0.4 | 0.0535 | 0.0089 | 3.9306 | 23.9731 |
| | 2 | | | 0.0398 | 0.0013 | 1.6318 | 6.8016 |
| | 3 | | | 0.0397 | 0.0006 | 0.8843 | 3.8329 |
| | 5 | | | 0.0418 | 0.0003 | 0.2731 | 2.7411 |
| 0.9 | 1 | 0.5 | 0.6 | 0.1263 | 0.0225 | 2.3931 | 11.4318 |
| | 2 | | | 0.1046 | 0.0044 | 0.8636 | 3.7658 |
| | 3 | | | 0.1043 | 0.0021 | 0.3484 | 2.7980 |
| | 5 | | | 0.1072 | 0.0009 | -0.1022 | 2.6789 |
| 1.5 | 1 | 0.5 | 3 | 0.6479 | 0.0899 | 7.763 | 3.9419 |
| | 2 | | | 0.6316 | 0.0230 | -0.0514 | 2.7641 |
| | 3 | | | 0.6354 | 0.0110 | -0.3345 | 2.9290 |
| | 5 | | | 0.6422 | 0.0043 | -0.5839 | 3.2962 |
| 3 | 1 | 0.5 | 5 | 0.9373 | 0.1097 | 0.5414 | 3.5208 |
| | 2 | | | 0.8897 | 0.0258 | -0.2236 | 2.9108 |
| | 3 | | | 0.8834 | 0.0119 | -0.4861 | 3.1878 |
| | 5 | | | 0.8817 | 0.0045 | -0.716 | 3.6183 |
| 0.5 | 1 | 1 | 0.4 | 0.1749 | 0.0690 | 3.0658 | 16.2110 |
| | 2 | | | 0.1460 | 0.0141 | 1.2632 | 5.0540 |
| | 3 | | | 0.1491 | 0.0072 | 0.6530 | 3.2279 |
| | 5 | | | 0.1586 | 0.0032 | 0.1220 | 2.6491 |
| 0.9 | 1 | 1 | 0.6 | 0.3176 | 0.1222 | 2.0855 | 9.3817 |
| | 2 | | | 0.2743 | 0.0268 | 0.7252 | 3.4100 |
| | 3 | | | 0.2756 | 0.0132 | 0.2565 | 2.7059 |
| | 5 | | | 0.2840 | 0.0056 | -0.1659 | 2.7114 |
| 1.5 | 1 | 1 | 3 | 0.9761 | 0.2575 | 1.0088 | 4.6678 |
| | 2 | | | 0.9351 | 0.0625 | 0.0615 | 2.7700 |
| | 3 | | | 0.9387 | 0.0299 | -0.2523 | 2.8496 |
| | 5 | | | 0.9485 | 0.0118 | -0.5288 | 3.1966 |
| 3 | 1 | 1 | 5 | 1.3252 | 0.3019 | 0.8026 | 4.1435 |
| | 2 | | | 1.2302 | 0.0663 | -0.0912 | 2.8483 |
| | 3 | | | 1.2167 | 0.0304 | -0.3899 | 3.0526 |
| | 5 | | | 1.2123 | 0.0116 | -0.6522 | 3.4789 |

3.7. Order Statistics

Suppose \( x_1, x_2, x_3, \ldots, x_n \) are some observed values from the APEWED and \( x_{i:n} \) indicates the \( i \)th order statistic. Then, we can express the PDF of \( x_{i:n} \) as follows:

\[
f_{i:n}(x) = \frac{n!}{(i-1)!(n-i)!} f(x) [F(x)]^{i-1} [1 - F(x)]^{n-i}.
\]  

(33)

Then, applying Equations (8) and (9), and following the binomial series expansion,

\[ (x-z)^n = \sum_{y=0}^{n} (-1)^y \binom{n}{y} x^{n-y} z^y, \]

(34)

the PDF of \( x_{i:n} \) associated with the APEWED can be obtained as follows:
\[
  f_{i:n}(x) = \frac{\log a}{B(i, n-i+1)(a-1)^n} \prod_{y=0}^{i-1} \binom{n-i}{y} \binom{n-i}{y} (-1)^i \sum_{i=0}^{n-i} \left( \frac{n-i}{l} \right) \left( \frac{1}{\gamma} \right) \times e^{-\lambda x}(1 - e^{-\lambda x})^{i-1} \left( \frac{-\log(1 - (1 - e^{-\lambda x})^c)}{\gamma} \right)^{a-1} \times e^{-\left( -\frac{1}{\gamma} \log(1 - (1 - e^{-\lambda x})^c) \right)^{a}} \times a^{-(y+l+1)}e^{-\left( -\frac{1}{\gamma} \log(1 - (1 - e^{-\lambda x})^c) \right)^{a}}
\]

where \( B(a, b) = \int_0^1 x^{a-1}(1 - x)^{b-1}dx \) refers to the beta function.

4. Maximum Likelihood Estimation

Let \( x_1, x_2, x_3, \ldots, x_n \) be a random sample of size \( n \) extracted from the APEWED. The log-likelihood function, \( \ell(a, a, \gamma, c; \lambda; x) \), for the parameters of the distribution is obtained by

\[
  \ell(a, a, \gamma, c; \lambda; x) = n \log \left( \frac{a \log a}{\alpha - 1} \right) + n \log \left( \frac{c a \lambda}{\gamma} \right) - \lambda \sum_{i=1}^{n} x_i + (c - 1) \sum_{i=1}^{n} \log(1 - e^{-\lambda x_i}) - \sum_{i=1}^{n} \log(1 - (1 - e^{-\lambda x_i})^c) + (a - 1) \sum_{i=1}^{n} \log \left( \frac{-\log(1 - (1 - e^{-\lambda x_i})^c)}{\gamma} \right) - \sum_{i=1}^{n} \left( \frac{-\log(1 - (1 - e^{-\lambda x_i})^c)}{\gamma} \right)^a - \log a \sum_{i=1}^{n} e^{-\left( -\frac{1}{\gamma} \log(1 - (1 - e^{-\lambda x_i})^c) \right)^a}.
\]  

The estimates can be derived by differentiating (36) for each parameter and then equating the result to zero; that is, we have:

\[
  \frac{\partial \ell}{\partial a} = \frac{1}{a} \left( \frac{n}{\log a} - \frac{n}{a - 1} - \frac{1}{\log a} \sum_{i=1}^{n} \xi_i(a, a, \gamma, c, \lambda) \right), \tag{37}
\]

\[
  \frac{\partial \ell}{\partial a} = \frac{n}{a} + \sum_{i=1}^{n} \log \left( \frac{-\log(1 - (1 - e^{-\lambda x_i})^c)}{\gamma} \right) \left\{ 1 - \left( \frac{-\log(1 - (1 - e^{-\lambda x_i})^c)}{\gamma} \right)^a \left[ 1 - \xi_i(a, a, \gamma, c, \lambda) \right] \right\}, \tag{38}
\]

\[
  \frac{\partial \ell}{\partial \gamma} = -\frac{a}{\gamma} \left\{ n - \frac{1}{\gamma^2} \sum_{i=1}^{n} \left( \frac{-\log(1 - (1 - e^{-\lambda x_i})^c)}{\gamma} \right)^a \left[ 1 - \xi_i(a, a, \gamma, c, \lambda) \right] \right\}, \tag{39}
\]

\[
  \frac{\partial \ell}{\partial c} = \frac{n}{c} + \sum_{i=1}^{n} \log(1 - e^{-\lambda x_i}) \left\{ 1 + \frac{(1 - e^{-\lambda x_i})^c}{1 - (1 - e^{-\lambda x_i})^c} \left[ 1 - \frac{a - 1}{\log(1 - (1 - e^{-\lambda x_i})^c)} \right] - \frac{a}{\gamma^a} \left( \frac{-\log(1 - (1 - e^{-\lambda x_i})^c)}{\gamma} \right)^a \left[ 1 - \xi_i(a, a, \gamma, c, \lambda) \right] \right\}, \tag{40}
\]

and
\[
\frac{\partial \ell}{\partial \lambda} = \frac{n}{\lambda} - \sum_{i=1}^{n} x_i \left\{ 1 - e^{-\lambda x_i} \left[ \frac{c - 1}{1 - e^{-\lambda x_i}} + \frac{c (1 - e^{-\lambda x_i})^{c-1}}{1 - (1 - e^{-\lambda x_i})^c} \left( \frac{a - 1}{\log(1 - (1 - e^{-\lambda x_i})^c)} \right) \right] \right\},
\]
(41)

where

\[
\zeta_i(a, a, \gamma, c, \lambda) = \log ae^{-\left(\frac{1-1-(1-e^{-\lambda x_i}c)}{\gamma}\right)^a}.
\]
(42)

In other words, one can obtain the MLEs for each parameter by solving the system of Equations (37)–(41), using any numerical techniques such as the Newton–Raphson iteration method. Additionally, the log-likelihood in Equation (36) can be alternatively maximized, using a non-linear optimization tool that available in any statistical/mathematical software such as R.

5. Simulation Study

We conducted some simulation studies, in order to examine the behavior of the MLEs for the parameters of the APEWED for various sample sizes and different values of the parameters. The random samples were obtained from Equation (16), with sizes \( n = 50, 100, 150, 200, 250, \) and \( 350 \). Two sets of the parameters for APEWED were selected-Set 1: \( \alpha = 5, a = 6.1, \gamma = 5, c = 6.1, \lambda = 0.1 \) and Set 2: \( \alpha = 5, a = 5.1, \gamma = 5, c = 6.1, \lambda = 0.1 \). For the different sample sizes, the average estimates and mean squared error (MSE) of the MLEs were calculated. The MLEs were obtained using the routine “optim” in R.

The results of the simulations, regarding the average estimates and MSEs, are displayed in Table 2. The results show that the MSE decreased as the sample size \( n \) increased. In addition, the average estimates increased, in order to be closer to the chosen true values of the parameters as the sample size increased.

Table 2. Simulation results for the APEWED.

| Sample Size | Par. | MLE | MSE | MLE | MSE |
|-------------|------|-----|-----|-----|-----|
| 50          | \( \alpha \) | 4.9945 | 1.5241 | 4.9771 | 1.9823 |
|             | \( a \) | 6.2423 | 0.5701 | 5.3094 | 0.5158 |
|             | \( \gamma \) | 5.1852 | 1.5270 | 5.3514 | 2.1370 |
|             | \( c \) | 6.2585 | 1.3997 | 6.1700 | 1.6173 |
|             | \( \lambda \) | 0.1027 | 0.0004 | 0.1047 | 0.0005 |
| 100         | \( \alpha \) | 4.9925 | 0.9658 | 4.9521 | 1.1873 |
|             | \( a \) | 6.2098 | 0.3477 | 5.2067 | 0.2582 |
|             | \( \gamma \) | 5.1686 | 1.1209 | 5.1371 | 1.1003 |
|             | \( c \) | 6.1576 | 0.9234 | 6.1745 | 1.1180 |
|             | \( \lambda \) | 0.1023 | 0.0003 | 0.1018 | 0.0002 |
| 150         | \( \alpha \) | 5.0304 | 0.8661 | 4.9990 | 1.0853 |
|             | \( a \) | 6.1964 | 0.2327 | 5.1969 | 0.1804 |
|             | \( \gamma \) | 5.1767 | 0.8036 | 5.1907 | 1.0389 |
|             | \( c \) | 6.1412 | 0.7601 | 6.1570 | 0.9395 |
|             | \( \lambda \) | 0.1025 | 0.0002 | 0.1026 | 0.0002 |
Table 2. Cont.

| Sample Size | Par. | Set 1 | | | Set 2 | | |
|--------------|------|------|------|------|------|------|
|              |      | MLE  | MSE  | MLE  | MSE  | |
| 200          |      |      |      |      |      | |
| α            | 4.9813 | 0.6709 | 4.9528 | 0.9304 | |
| a            | 6.1522 | 0.1631 | 5.1871 | 0.1551 | |
| γ            | 5.0843 | 0.6256 | 5.1880 | 0.9357 | |
| c            | 6.1311 | 0.6732 | 6.1213 | 0.8138 | |
| λ            | 0.1011 | 0.0001 | 0.1024 | 0.0002 | |
| 250          |      |      |      |      |      | |
| α            | 4.9740 | 0.5833 | 4.9381 | 0.7149 | |
| a            | 6.1530 | 0.1468 | 5.1366 | 0.1204 | |
| γ            | 5.1130 | 0.5719 | 5.0990 | 0.6879 | |
| c            | 6.1091 | 0.5527 | 6.1298 | 0.7156 | |
| λ            | 0.1015 | 0.0001 | 0.1013 | 0.0002 | |
| 350          |      |      |      |      |      | |
| α            | 4.9706 | 0.5692 | 4.9391 | 0.6277 | |
| a            | 6.1300 | 0.1174 | 5.1572 | 0.0917 | |
| γ            | 5.0767 | 0.4588 | 5.1098 | 0.5165 | |
| c            | 6.1270 | 0.5142 | 6.0922 | 0.6325 | |
| λ            | 0.1011 | 0.0001 | 0.1013 | 0.0001 | |

6. Applications

The APEWED was applied to three real data sets with the aim being to examine its fit, compared to that of some other competitive models. The first data set was taken from [19], in which the data are the strengths of glass fibers measuring 1.5 cm. As for the second data set, it was obtained from [20]. This set provides the survival times of 121 patients with breast cancer. The third data was obtained from [21]. This data represents 40 losses due to wind-related catastrophes. The sorted values include claims of 2,000,000 and for convenience they have been recorded in millions.

We compared the fit of the APEWED with some other distributions; namely, Kumaraswamy Weibull (Ku-W) by [22], beta Weibull (BW) by [23], exponentiated generalized Weibull (EGW) by [24], alpha power inverted exponential (APIE) by [25], exponential (E) and exponentiated truncated inverse Weibull-inverse Weibull (ETIWIW) by [26].

\[
f(x)_{Ku-W} = a \beta \alpha \left(\frac{x}{\gamma}\right)^{\alpha-1} e^{-\left(\frac{x}{\gamma}\right)^{\alpha}} \left[1 - \left(1 - e^{-\left(\frac{x}{\gamma}\right)^{\alpha}}\right)^{a}\right]^{b-1},
\]

\[
f(x)_{BW} = \frac{\Gamma(\alpha + \beta) \gamma}{\Gamma(\alpha) \Gamma(\beta)} \alpha \left(\frac{x}{\gamma}\right)^{\alpha-1} e^{-\left(\frac{x}{\gamma}\right)^{\alpha}} \left[1 - e^{-\left(\frac{x}{\gamma}\right)^{\alpha}}\right]^{\beta-1},
\]

\[
f(x)_{EGW} = ab \left(\frac{x}{\beta}\right)^{a-1} e^{-\left(\frac{x}{\beta}\right)^{a}} \left[1 - e^{-\left(\frac{x}{\beta}\right)^{a}}\right]^{b-1},
\]

\[
f(x)_{APIE} = \frac{\log \alpha}{\alpha - 1} \left(\frac{\lambda}{x^\alpha}\right) e^{\left(\frac{\lambda}{x^\alpha}\right)},
\]

\[
f(x)_{E} = \lambda e^{-\lambda x},
\]

\[
f(x)_{ETIWIW} = ab \theta \theta x^{-\theta-1} e^{-\left(\frac{x}{\theta}\right)^{\theta}} \left[1 - e^{-\left(\frac{x}{\theta}\right)^{\theta}}\right]^{-b-1} \left[1 - e^{-\left(\frac{x}{\theta}\right)^{\theta}}\right]^{-b-1} \left[1 - e^{-\left(\frac{x}{\theta}\right)^{\theta}}\right]^{-b-1}.
\]
The MLEs of parameters, along with the standard errors (SEs), for all fitted distributions to the three real data sets can be found in Tables 3–5, respectively. The negative likelihood values $-L(\hat{\theta})$, the relevant Kolmogorov-Smirnov (K-S) test statistics, p-values, Akaike’s Information Criteria (AIC) and Corrected Akaike Information Criteria (CAIC) for different models with respect to the three real data sets are also recorded in Tables 6–8.

Table 3. The MLEs (SEs in parentheses) for data set 1.

| Distribution | Estimated Parameters |
|--------------|----------------------|
| APEWE        | 12.9282 (120.0167)   |
| (\hat{a}, \hat{\alpha}, \hat{\gamma}, \hat{\lambda}) | 4.6970 (0.1264) |
| Ku-W         | 0.6872 (2.2652)      |
| BW           | 0.6352 (7.6149)      |
| EGW          | 1.3687 (6.7155)      |
| APIE         | 29.1359 (0.4040)     |
| E            | 0.6636 (0.0000)      |
| ETIWIW       | 1.2850 (10.5030)     |

Table 4. The MLEs (SEs in parentheses) for data set 2.

| Distribution | Estimated Parameters |
|--------------|----------------------|
| APEWE        | 1.4234 (0.3805)      |
| (\hat{a}, \hat{\alpha}, \hat{\gamma}, \hat{\lambda}) | 1.3111 (0.0972) |
| Ku-W         | 2.3494 (0.0250)      |
| BW           | 34.6367 (0.0205)     |
| EGW          | 0.0716 (0.0079)      |
| APIE         | 38.8013 (12.4715)    |
| E            | 0.0216 (0.0020)      |
| ETIWIW       | 6.1168 (1.0285)      |
| (\hat{a}, \hat{\alpha}, \hat{\lambda}) | 1.0926 (5.9985) |


Table 5. The MLEs (SEs in parentheses) for data set 3.

| Distribution | Estimated Parameters | \( \hat{\alpha}, \hat{\delta}, \hat{\gamma}, \hat{\epsilon}, \hat{\lambda} \) | \( \hat{\alpha}, \hat{\beta}, \hat{\lambda} \) | \( \hat{\alpha}, \hat{\beta}, \hat{\lambda} \) | \( \hat{\alpha}, \hat{\beta}, \hat{\lambda} \) |
|--------------|---------------------|----------------------|----------------------|----------------------|----------------------|
| APEWE        | 0.4579              | 0.8456               | 5.0031               | 13.8172              | 0.5141               |
| Ku-W         | 0.6557              | 0.0894               | 0.7470               | 3.556                | -                    |
| BW           | 34.6367             | 0.1878               | 0.1775               | 11.6764              | -                    |
| EGW          | 0.0741              | 0.5814               | 0.7564               | 3.722                | -                    |
| APIE         | 39.0799             | 1.1482               | -                    | -                    | -                    |
| E            | 10.155              | -                    | -                    | -                    | -                    |
| ETIWIW       | 16.4351             | 0.0544               | 4.9506               | 0.0275               | -                    |

From Tables 6–8, it can be seen that the APEWED had the lowest values of \(- L(\hat{\theta})\), Kolmogorov-Smirnov (K-S) test values, AIC and CAIC values, as well as the best p-values, which means that the APEWED provided the most proper fit to the three sets of data, as compared to the other models.

Table 6. The goodness-of-fit measures for data set 1.

| Distribution | Statistics | \(- L(\hat{\theta})\) | K-S | \(p\)-Value | AIC | CAIC |
|--------------|------------|------------------------|-----|-------------|-----|------|
| APEWE        |            | 13.4256                | 0.1045 | 0.4977 | 36.8512 | 37.9038 |
| Ku-W         |            | 17.6139                | 0.2206 | 0.0043 | 43.2277 | 43.9173 |
| BW           |            | 14.6198                | 0.1298 | 0.2392 | 37.2396 | 37.9293 |
| EGW          |            | 14.6755                | 0.1461 | 0.1356 | 37.3511 | 38.0407 |
| APIE         |            | 96.7291                | 0.4593 | 5.8 \times 10^{-12} | 197.4581 | 197.6581 |
| E            |            | 88.8303                | 0.4180 | 5.5 \times 10^{-10} | 179.6606 | 179.7262 |
| ETIWIW       |            | 18.6649                | 0.2110 | 0.0073 | 45.3298 | 46.0195 |

Table 7. The goodness-of-fit measures for data set 2.

| Distribution | Statistics | \(- L(\hat{\theta})\) | K-S | \(p\)-Value | AIC | CAIC |
|--------------|------------|------------------------|-----|-------------|-----|------|
| APEWE        |            | 578.9279               | 0.0471 | 0.951 | 1167.856 | 1168.378 |
| Ku-W         |            | 583.7524               | 0.1141 | 0.0854 | 1175.505 | 1175.85 |
| BW           |            | 590.968                | 0.1108 | 0.1026 | 1189.936 | 1190.281 |
| EGW          |            | 603.9844               | 0.2032 | 9.1 \times 10^{-5} | 1215.969 | 1216.314 |
| APIE         |            | 625.8144               | 0.1881 | 0.0004 | 1255.629 | 1255.730 |
| E            |            | 585.1277               | 0.1203 | 0.0602 | 1172.255 | 1172.289 |
| ETIWIW       |            | 615.6412               | 0.1830 | 0.0006 | 1239.282 | 1239.627 |
Table 8. The goodness-of-fit measures for data set 3.

| Distribution | $-L(\hat{\theta})$ | K-S | $p$-Value | AIC  | CAIC  |
|--------------|---------------------|-----|-----------|------|-------|
| APEWE        | 115.2202            | 0.1783 | 0.1784 | 240.4404 | 242.3154 |
| Ku-W         | 127.1697            | 0.3038 | 0.0018 | 262.3394 | 263.5516 |
| BW           | 117.3411            | 0.1996 | 0.0968 | 242.6821 | 243.8942 |
| EGW          | 132.9958            | 0.4281 | $1.8 \times 10^{-6}$ | 273.9917 | 275.2038 |
| APIE         | 121.0846            | 0.1807 | 0.1671 | 246.1692 | 246.5121 |
| E            | 123.5491            | 0.1898 | 0.1292 | 249.0982 | 249.2093 |
| ETIWIW       | 118.1282            | 0.17996 | 0.1705 | 244.2563 | 245.4684 |

Figures 3–5 show the histograms and fits of the models for the three data sets. It can be seen, from Figures 3–5, that the APEWED provided the closest fit to the observed distribution (i.e., to the histogram) for the three data sets.
Figure 5. Observed and expected frequencies for each model for data set 3.

7. Conclusions

In this study, the five-parameter APEWED distribution was introduced, based on a new technique for generating distributions. This method combines two well-known families of distributions—namely, exponentiated T-X and APT—in order to allow for more flexibility and adaptability in fitting practical data sets. Both the density and hazard rate functions of the proposed distribution have attractive shapes for adopting different data behaviors. To illustrate, the APEWED can be adapted to the analysis of symmetrical and asymmetrical shaped data. Several general statistical properties of this new distribution were obtained and discussed. The parameters were estimated by employing the maximum likelihood technique. The performance of these MLEs were tested through various simulation studies. Three real data sets were considered, in order to assess the applicability of the proposed distribution, as compared to some other existing distributions. The results indicates that, the propose distribution, APEWED, can offer the best fit, compared to the competing distributions, considering its promising results.

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Appendix A

Proof. Proof of (20), by definition, the rth moment of APEWED is obtained by

\[ E(X^r) = \int_0^\infty x^r f(x) dx. \]

Then, from (14), we have

\[ E(X^r) = \int_0^\infty x^r \sum_{d=0}^\infty \psi_d e^{-\lambda x(d+1)} dx. \]
Integrating by substitution provides the rth moment of APEWED as

$$E(X^r) = \sum_{d=0}^{\infty} \psi_d \left( \frac{1}{\Lambda(d + 1)} \right)^{r+1} \Gamma(r + 1).$$

\[\square\]

**Proof.** Proof of (23), let X be an RV with PDF (14), where \(\psi_d\) is given by (15). Then, the MGF of APEWED can be derived as

$$M_x(t) = \int_0^\infty e^{tx} \sum_{d=0}^{\infty} \psi_d e^{-\lambda x(d+1)} dx.$$

Applying the exponential expansion of the following form

$$e^{tx} = \sum_{r=0}^{\infty} \frac{t^r x^r}{r!},$$

we have

$$M_x(t) = \sum_{d=0}^{\infty} \sum_{r=0}^{\infty} \frac{t^r}{r!} \psi_d \int_0^\infty x^r e^{-\lambda x(d+1)} dx$$

where \(\mu_r\) is calculated from (20). Thus, the MGF of APEWED can be easily obtained as

$$M_x(t) = \sum_{r=0}^{\infty} \frac{t^r \mu_r}{r!}$$

$$M_x(t) = \sum_{d=0}^{\infty} \sum_{r=0}^{\infty} \frac{t^r}{r!} \psi_d \left( \frac{1}{\lambda(d + 1)} \right)^{r+1} \Gamma(r + 1).$$

\[\square\]

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