FRW Cosmology in Ghost Free Massive Gravity from Bigravity

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Abstract

We study FRW homogeneous cosmological solutions in the bigravity extension of the recently found ghost-free massive gravity. When the additional extra metric, needed to generate the mass term, is taken as nondynamical and flat, no homogeneous flat FRW cosmology exists. We show that, when the additional metric is a dynamical field a perfectly acceptable FRW solution exists. Solutions fall in two branches. In the first branch the massive deformation is equivalent to an effective cosmological constant determined by the graviton mass. The second branch is quite rich: we have FRW cosmology in the presence of a “gravitational” fluid. The control parameter $\xi$ is the ratio of the two conformal factors. When $\xi$ is small, the evolution is similar to GR and interestingly the universe flows at late time towards an attractor represented by a dS phase.

1 Introduction

Recently, there has been a renewed interest in the search of a modified theory of gravity at large distances through a massive deformation of GR (see [1] for a recent review). A great deal of effort was devoted to extend the seminal work of Fierz and Pauli (FP) [3] at nonlinear level [2]. The FP theory, defined at linearized level, is plagued by a number of diseases. In particular, the
modification of the Newtonian potential is not continuous when the mass $m^2$ vanishes, giving a large correction (25%) to the light deflection from the sun that is experimentally excluded [4]. A possible way to circumvent the physical consequences of the discontinuity was proposed in [5]; the idea is that the linearized approximation breaks down near a massive object like the sun and an improved perturbative expansion must be used that leads to a continuous zero mass limit. In addition, FP is problematic as an effective theory. Regarding FP as a gauge theory where the gauge symmetry is broken by an explicit mass term $m$, one would expect a cutoff $\Lambda_2 \sim mg^{-1} = (mM_{\text{pl}})^{1/2}$, however the real cutoff is $\Lambda_5 = (m^4M_{\text{pl}})^{1/5}$ or $\Lambda_3 = (m^2M_{\text{pl}})^{1/3}$, much lower than $\Lambda_2$ [6]. A would-be Goldstone mode is responsible for the extreme UV sensitivity of the FP theory, that becomes totally unreliable in the absence of proper UV completion. Recently it was shown that there exists a non linear completion of the FP theory that is free of ghosts up to the fourth order [7] and avoids the presence of the Boulware-Deser instability [8]. Then the propagation of only five degrees of freedom and the absence of instabilities was extended to the non perturbative level in [9]; this was shown also in the St"uckelberg language in [10]. The bigravity extension of this theory was shown to be ghost-free [11].

Quite naturally massive gravity leads to bigravity. Indeed, any massive deformation, obtained by adding to the Einstein-Hilbert action a non-derivative self-coupling of the metric $g$, requires the introduction of an additional metric $\tilde{g}$. This auxiliary metric may be a fixed external field, or be a dynamical one. When $\tilde{g}$ is non dynamical we are dealing with æther-like theories; on the other hand if it is dynamical we enter in the realm of bigravity [12], originally introduced by Isham, Salam and Strathdee [14]. The need for a second dynamical metric also follows from rather general grounds. Indeed, it was shown in [15] that in the case of non singular static spherically symmetric geometry with the additional property that the two metrics are diagonal in the same coordinate patch, a Killing horizon for $g$ must also be a Killing horizon for $\tilde{g}$. Thus, it seems that in order that the Vainshtein mechanism is effective and GR is recovered in the near horizon region of a black hole, $\tilde{g}$ has to be dynamical [16]. In this paper we also show that cosmology calls for the bigravity formulation of massive gravity. While in the St"uckelberg formulation there is no homogeneous flat FRW solution [17], (see also for a related work [18, 19, 20, 21]) in the present work we show that flat FRW homogeneous solutions do exist in the bigravity formulation.

In section 2 the formulation of massive gravity as bigravity is reviewed. The cosmological ansatz is introduced in section 3, where the structure of the modified Einstein equations and the consequences of Bianchi identities are studied. Cosmological evolution falls in two branches described in section 4 and section 5. In section 6 the results of the previous sections are extended.
to the case of spatially curved geometries. Section 7 contains our conclusions. The full set of Einstein equations can be found in the appendix.

## 2 Massive Gravity and Bigravity

Any modification of GR that turns a massless graviton into a massive one calls for additional DoF (degree of freedom). An elegant way to provide them is to work with the extra tensor \( \tilde{g}_{\mu\nu} \). When coupled to the standard metric \( g_{\mu\nu} \), it allows to build non-trivial diff-invariant operators that lead to mass terms, when expanded around a background. Consider the action \[ S = \int d^4x \sqrt{\tilde{g}} \kappa M_{pl}^2 \tilde{R} + \sqrt{g} \left[ M_{pl}^2 \left( \mathcal{R} - 2m^2 V \right) + L_{\text{matt}} \right], \]

where \( \mathcal{R}(g_i) \) are the corresponding Ricci scalars and the interaction potential \( V \) is a scalar function of the tensor \( X^\mu_\nu = g^{\mu\alpha} \tilde{g}_{\alpha\nu} \). Matter is minimally coupled to \( g \) and it is described by \( L_{\text{matt}} \). The constant \( \kappa \) controls the relative size of the strength of gravitational interactions in the two sectors, while \( m \) sets the scale of the graviton mass. The action (1) brings us into the realm of bigravity theories, whose study started in the ‘60 [14]. An action of the form (1) can be also viewed as the effective theories for the low lying Kaluza-Klein modes in brane world models [12]. The massive deformation is encoded in the non-derivative coupling between \( g_{\mu\nu} \) and the extra tensor field \( \tilde{g}_{\mu\nu} \). Clearly the action is invariant under diffeomorphisms, which transform the two fields in the same way (diagonal diffs). Taking the limit \( \kappa \to \infty \), the second metric decouples, and gets effectively frozen to a fixed background value so that the “relative” diffeomorphisms are effectively broken, as far as the first metric is concerned. Depending on the background value of \( \tilde{g}_{\mu\nu} \) one can explore both the Lorentz-invariant (LI) and the Lorentz-breaking (LB) phases of massive gravity [13]. When the second metric is dynamical this is determined by its asymptotic properties, as discussed below. In this case notice that \( \tilde{g}_{\mu\nu} \) is determined by its equations of motion (for any finite \( \tilde{M}_{pl} \)) so that we will be working always with consistent and dynamically determined backgrounds. The role played by \( \tilde{g}_{\mu\nu} \) is very similar to the Higgs field, its dynamical part restores gauge invariance and its background value determines the realization of the residual symmetries.

The modified Einstein equations can be written as:

\[
E^\mu_\nu + Q^\mu_\nu = \frac{1}{2 M_{pl}^2} T^\mu_\nu \quad (2)
\]

\[
\kappa \tilde{E}^\mu_\nu + Q^\mu_\nu = 0; \quad (3)
\]

\(^1\)When not specified, indices of tensors related with \( g(\tilde{g}) \) are raised/lowered with \( g(\tilde{g}) \)
where we have defined \(Q_1\) and \(Q_2\) as effective energy-momentum tensors induced by the interaction term. The only invariant tensor that can be written without derivatives out of \(g\) and \(\tilde{g}\) is \(X^\mu_\nu = g^{\mu_\alpha}_1 \tilde{g}_{\alpha_\nu}\). The ghost free potential \(V\) is a particular 4-parameter scalar function of \(Y^\mu_\nu = (\sqrt{X})^\mu_\nu\) given by

\[
V = \sum_{n=0}^{4} a_n V_n, \quad n = 0 \ldots 4
\]

where The \(V_n\) are the symmetric polynomials of \(Y\)

\[
V_0 = 1 \quad V_1 = \tau_1, \quad V_2 = \tau_1^2 - \tau_2, \quad V_3 = \tau_1^3 - 3 \tau_1 \tau_2 + 2 \tau_3, \\
V_4 = \tau_1^4 - 6 \tau_1^2 \tau_2 + 8 \tau_1 \tau_3 + 3 \tau_2^2 - 6 \tau_4
\]

with \(\tau_n = \text{tr}(Y^n)\). In \([11]\) it was shown that in the bimetric formulation the potential \(V\) is ghost free as in the St"uckelberg formulation. We have that

\[
Q_1^\mu_\nu = m^2 \left[ V \delta^\mu_\nu - (V' Y)^\mu_\nu \right]
\]

\[
Q_2^\mu_\nu = m^2 q^{-1/2} (V' Y)^\mu_\nu
\]

where \((V')^\mu_\nu = \partial V / \partial Y^\nu_\nu\) and \(q = \det X = \det(\tilde{g})/\det(g)\).

### 3 Ansatz, Equations of Motions and Conservation Laws

For simplicity, here we consider the case of spatially flat geometries, the non-flat case is discussed in section 6.

In general we cannot set both metrics in diagonal form and preserve homogeneity. For instance, take the following form for \(g\) and \(\tilde{g}\):

\[
ds^2 = a^2(t) \left( -dt^2 + dr^2 + r^2 d\Omega^2 \right)
\]

\[
\tilde{d}s^2 = \omega^2(t) \left[ -c^2(t) dt^2 + 2D(t) dt dr + dr^2 + r^2 d\Omega^2 \right]
\]

It is convenient to define

\[
\xi = \frac{\omega}{a}
\]

If \(D \neq 0\), the metric \(g\) is still homogeneous and the hypersurface of homogeneity is \(t = \text{const}\); on the other hand the metric \(\tilde{g}\) is homogeneous with respect to a different time slicing. In addition, \(Q_1, Q_2\) are not diagonal and \(Q_{1/2,r} \neq Q_{1/2,\theta} = Q_{1/2,\phi}\), as a result \(Q_1\) and \(Q_2\) are not energy momentum tensors (EMT) for a homogenous perfect fluid. In addition, using Bianchi identities and the equations of motion one can show that there is no physical

\[\text{A very similar potential having the same form but with } X \text{ instead of } X^{1/2} \text{ was considered in } [22].\]
solution when \( D(t) \neq 0 \). Indeed, from the fact that the Einstein tensor for \( g \) is diagonal, also \( Q_1 \) should be diagonal and with \( D \neq 0 \) one finds that \( \xi = \xi \) is constant with \( 6a_3 \xi^2 + 4a_2 \xi + a_1 = 0 \). Furthermore, Bianchi identities can be fulfilled only if \( a_2 + 3 \xi a_3 = 0 \). Finally, in order to solve the \( tt \) component of Einstein equations for the metric \( \tilde{g} \) for any value of \( r \), we are forced to set \( D = 0 \). Thus, we only need to consider the case \( D = 0 \).

The potential part of the action is separately invariant under diff and thus it gives two set of Bianchi identities \( \nabla_\nu Q_1^\nu = \tilde{\nabla}_\nu Q_2^\nu = 0 \) which are equivalent to the following conservation law

\[
\frac{d}{dt} (a^3 \rho_g) + p_g \frac{d}{dt} a^3 = 0, \tag{10}
\]

where

\[
\rho_g = \frac{m^2}{4\pi G} \left( 3a_3 \xi^3 + 3a_2 \xi^2 + \frac{3}{2}a_1 \xi + \frac{a_0}{2} \right),
\]

\[
p_g = -\frac{m^2}{4\pi G} \left[ 3a_3 c \xi^3 + a_2 (2c + 1) \xi^2 + \frac{a_1}{2} (c + 2) \xi + \frac{a_0}{2} \right].
\tag{11}
\]

Eq. (10) is equivalent to the following condition

\[
m^2 \left( 6a_3 \xi^2 + 4a_2 \xi + a_1 \right) (c \omega a' - a \omega') = 0. \tag{12}
\]

The conservation of the matter EMT gives

\[
\frac{d}{dt} (a^3 \rho_m) + p_m \frac{d}{dt} a^3 = 0. \tag{13}
\]

In order to determine the three functions of time \( a, c, \omega \) we need a set of three independent equations. As in GR, one can show that for both metrics the time-time plus the space-space component of the equations of motion leads to the Bianchi identities. As a result, also for massive gravity we can take the conservation equations (12-13), together with

\[
H^2 = \frac{8\pi G}{3} (\rho_m + \rho_g), \tag{14}
\]

\[
\frac{H \omega^2}{c^2} = \frac{m^2}{3\kappa} \left( a_1 \xi^{-3} + 6a_2 \xi^{-2} + 18a_3 \xi^{-1} + 24a_4 \right), \tag{15}
\]

as independent equations. We defined the Hubble parameters in conformal time \( H = a'/a^2 \) and \( H_\omega = \omega'/\omega^2 \). The full set of equations of motion is given in Appendix A.

\footnote{Notice that the constraint coincides with the one found for branch one solutions below, see sect. 4.}

\footnote{One can easily show that diff invariance also implies that \( \nabla_\nu Q_1^\nu = 0 \iff \tilde{\nabla}_\nu Q_2^\nu = 0 \).}
Starting from eq. (12), the cosmological evolution can be classified in two branches: either there is an algebraic constraint for $\xi$

$$6a_3 \xi^2 + 4a_2 \xi + a_1 = 0,$$

(16)

or

$$\frac{\omega'}{\omega} = c \frac{a'}{a}.$$  

(17)

Notice that, by definition $c(t)$ is the positive root of $c^2$ and thus $\omega'$ and $a'$ must have the same sign. Since in our universe $a' > 0$, we will consider only the case $\omega' > 0$.

4 Cosmology: Branch one

Let us consider the case when the conservation equation (12) is solved as in (16), so that clearly $\xi(t) = \bar{\xi}$ is a constant satisfying the condition

$$6a_3 \bar{\xi}^2 + 4a_2 \bar{\xi} + a_1 = 0$$

(18)

and $H_\omega = H \bar{\xi}^{-1}$.

From (14) we have

$$H^2 = \frac{8\pi G}{3} \rho_m + \frac{m^2}{3} \left[ a_0 - 6\xi^2 (2a_3 \xi + a_2) \right].$$

(19)

As a result, in this branch the effect of the massive deformation is to induce an effective cosmological constant

$$\Lambda_{eff} = \frac{m^2}{8\pi G} \left[ a_0 - 6\xi^2 (2a_3 \xi + a_2) \right].$$

(20)

For what concerns the second metric, (15) turns into an algebraic equation for $c(t)$:

$$c^2 = \frac{3\kappa H^2}{2m^2 \left[ 6 \xi (2a_4 \xi + a_3) + a_2 \right]}.$$ 

(21)

Summarizing, branch one is equivalent to the usual FRW cosmology with a cosmological constant. This is not very surprising. Let us remind that in the bimetric formulation, when one expands around flat space at the quadratic level, there are two spin-2 modes: one is always massless by diff-invariance, the other has Pauli-Fierz mass, see for instance [12, 13]. The point here is that condition (18) leads to a zero quadratic Pauli-Fierz mass. The condition is the same found in [16] in the branch where no gravity modification are present. Notice that, even if at the quadratic level the potential has no effect, reappears in the theory at higher orders in perturbation theory.

Finally, let us comment on the $\kappa \rightarrow \infty$ limit which corresponds to freezing the second metric. From eq. (15) one gets that $H_\omega = 0$, but in the branch we are dealing with, $\xi$ is constant and so also $H = 0$. As result, for the branch one no flat FRW cosmology exists when the second metric is non dynamical.
5 Cosmology: Branch two

The Bianchi identities in such a branch give

\[ c = \frac{H_\omega}{H} \xi. \]  

(22)

Inserting the expression of \( c \) in (15) we have

\[ m^2 \left[ 6\xi \left( 4a_4 \xi^2 + 3a_3 \xi + a_2 \right) + a_1 \right] - 3\kappa \xi H^2 = 0. \]  

(23)

If we solve this equation for \( H^2 \) and compare with (14), we find

\[ m^2 \left[ \xi^2 \left( \frac{8a_4}{\kappa} - 2a_2 \right) + \xi \left( \frac{6a_3}{\kappa} - a_1 \right) + \frac{a_1}{3\kappa \xi} + \frac{2a_2}{\kappa} - 2a_3 \xi^3 - \frac{a_0}{3} \right] = \frac{8\pi G \rho_m}{3}. \]  

(24)

For matter with an equation of state \( p_m = w \rho_m \), we have that \( \rho_m = \rho_0 a^{-3(1+w)} \). Thus, eq. (24) can be solved for \( \xi \) in terms of the conformal scale factor \( a(t) \). The problem of the cosmological evolution is reduced to FRW cosmology in the presence of matter plus an additional form of “gravitational” matter. Though the solution of (24) can be written in a closed form, it is not particularly illuminating.

Alternatively, from the definition of \( \xi \) and from (22) we have

\[ \xi' = (c - 1) a H \xi, \]  

(25)

and from (24) we can express \( a \) in terms of \( \xi \): \( a = f_a(\xi, w) \). Finally, from (28) and (60) (see appendix A) \( c \) can be expressed in terms of \( \xi \)

\[ c \equiv 1 + (1 + w) f_c(\xi), \]  

(26)

\( f_c(\xi) \) is a function whose explicit form is not needed. At this point, the velocity field of \( \xi \) is function of \( \xi \) only

\[ \xi' = (1 + w) f_c(\xi) H(\xi, w) f_a(\xi, w) \xi, \]  

(27)

and it has two fixed points

\[ \begin{cases} 
\xi = \bar{\xi}, & c = 1 \text{ with } f_c(\bar{\xi}) = 0 \quad \rho_m \to 0 \\
\xi = \bar{\xi}, & \text{with } H(\bar{\xi}, w) = 0 \quad \text{generic } \rho_m.
\end{cases} \]  

(28)

While the fixed point with zero Hubble parameter is not cosmologically very significant, the \( c = 1 \) case instead corresponds to a De Sitter phase.
Far from the fixed points the main features can be deduced by a qualitative analysis of the solutions of (24). The variable $\xi$ is dimensionless and naturally is a function of $G \rho_m/m^2$. In the early time, when $G \rho_m/m^2 \gg 1$ we have two possible regimes: small $\xi \propto O(m^2/G \rho_m)$ and large $\xi \propto O(G \rho_m/m^2)$.

**Large $\xi$**

The regime of large $\xi$ is determined by the leading positive power of $\xi$ present in eq. (24). For simplicity, here we report only the leading results, the corrections to them are $O(m^2/G \rho_m)$. For $a_3 \neq 0$ the leading term is $\xi^3$ and a solution exists only if $a_3 < 0$, with

$$
\xi = \left[ \frac{8\pi G \rho_m}{m^2 6|a_3|m^2} \right]^{1/3} + \frac{4 a_4 - \kappa a_2}{3 a_3 \kappa};
$$

$$
\rho_m + \rho_g = 2 \frac{a_4}{\kappa} \left( \frac{6 m^2 \rho_m^2}{\pi G a_3^2} \right)^{1/3} \ll \rho_m;
$$

$$
w_{eff} \equiv \frac{p_g + \rho_m}{\rho_g + \rho_m} = \frac{2 w - 1}{3};
$$

$$
c = -w.
$$

When $a_3 = 0$, the leading term is $\xi^2$ and a solution exists if $4 a_4 - \kappa a_2 > 0$, with

$$
\xi = \left[ \frac{8\pi G \kappa \rho_m}{m^2 6(4 a_4 - a_2 \kappa)} \right]^{1/2};
$$

$$
\rho_m + \rho_g = \frac{4a_4}{(4 a_4 - \kappa a_2)} \rho_m;
$$

$$
w_{eff} = w;
$$

$$
c = -\frac{3w + 1}{2}.
$$

Finally, if $a_3 = 0$ and $\kappa a_2 - 4 a_4 = 0$, the leading term is $\xi$ and a large $\xi$ solutions exists when $a_1 < 0$, with

$$
\xi = -\frac{8 \pi G}{3 a_1 m^2} \rho_m;
$$

$$
\rho_m + \rho_g = \frac{16 \pi G a_2^2}{3 a_1 m^2} \rho_m^2 \gg \rho_m;
$$

$$
w_{eff} = 2w + 1;
$$

$$
c = -(3w + 2).
$$
The large $\xi$ regime is characterized by non standard cosmology and/or negative values of $c$. Except for the case where the quadratic term is dominating, the effective equation of state $w_{eff}$ largely deviates from $w$, but even in this case $c < 0$ when $w > 0$. As a result, we conclude that the large $\xi$ regime is not a physical one.

**Small $\xi$**

Solutions with small $\xi$ are present only when $a_1 > 0$ and in this case

\[ \xi = \frac{a_1 m^2}{8 \pi G \kappa \rho_m}; \quad (41) \]
\[ \rho_m + \rho_g = \rho_m; \quad (42) \]
\[ w_{eff} = w, \quad (43) \]
\[ c = 4 + 3w. \quad (44) \]

For simplicity, we give only the leading terms, the corrections are $O(m^2/G \rho_m)$ and can be systematically computed. In this regime cosmology is standard: once the matter is so diluted that $\rho_m$ is negligible in (24), the system falls in the fixed point region and $\xi$ is almost constant; the universe enters in a late time dS phase.

One can solve perturbatively (14) to find $a$. For a radiation dominated universe we find\(^5\)

\[ a(t) = \frac{t}{t_0} + a_0 m^2 \frac{t^5}{30 t_0^3} + \frac{m^4 t^9 (a_0^2 \kappa + 20 a_1^2)}{1080 \kappa t_0^6} + \cdots, \quad t_0 = \left( \frac{3}{8 \pi G \rho_0} \right)^{1/2}. \quad (45) \]

In the case of matter dominated universe

\[ a(t) = \frac{t^2}{t_0^2} + a_0 m^2 \frac{t^8}{84 t_0^6} + \frac{m^4 t^{14} (4 a_0^2 \kappa + 49 a_1^2)}{30576 \kappa t_0^{10}} + \cdots \quad t_0 = \left( \frac{3}{2 \pi G \rho_0} \right)^{1/2}. \quad (46) \]

For instance, taking $\rho_0$ of order of the critical density today and $m \sim 10^{-33} eV$ we have that during the radion era $\xi \sim 10 (1 + z)^{-4} << 1$.

Let us also discuss the $\kappa \rightarrow \infty$ limit in this branch. When the second metric is non dynamical, from eq.(23) we get that $H \rightarrow 0$. Thus, also in this branch no flat FRW cosmology exists with just a single dynamical metric.

\(^5\)As an example, in the following expressions we give the leading and the next to leading correction to $a$. 

8
6 FRW with Spatial Curvature

Let us discuss the case with spatial curvature (see [20] for the case with frozen second metric). The metrics take the form

\[ ds^2 = a^2(t) \left( -dt^2 + \frac{dr^2}{(1 - k_1 r^2)} + r^2 d\Omega^2 \right) \]

\[ \tilde{ds}^2 = \omega^2(t) \left[ -c^2(t) dt^2 + \frac{dr^2}{(1 - k_2 r^2)} + r^2 d\Omega^2 \right]. \]  

(47)

In the presence of curvature the conservation law for \( Q_1 \) and \( Q_2 \) take a different form

\[ (F - 1) \left[ 2c \xi \left( 3a_3 \xi + a_2 \right) + 2a_2 \xi + a_1 \right] = 0, \]  

(48)

\[ F(r) \xi \left[ \left\{ 2 \xi \left( 9a_3 \xi + 4a_2 \right) + a_1 \right\} H_\omega - 4cH \left( 3a_3 \xi + a_2 \right) \right] - 
\]

\[ cH \left( 6a_3 \xi^2 + 8a_2 \xi + 3a_1 \right) + 2 \xi \left( 2a_2 \xi + a_1 \right) H_\omega = 0, \]  

(49)

where

\[ F(r) = \left( \frac{k_1 r^2 - 1}{k_2 r^2 - 1} \right)^{1/2}. \]  

(50)

When \( k_1 \neq k_2 \), \( F \) is different from one and the Bianchi identities must hold for any value of \( r \); this gives three nontrivial relations. From (48), we can solve for \( c \)

\[ c = -\frac{2a_2 \xi + a_1}{6a_3 \xi^2 + 2a_2 \xi}; \]  

(51)

then inserting \( c \) in (49) we get that

\[ H_\omega = -\frac{2H \left( 2a_2 \xi + a_1 \right)}{\xi \left( 18a_3 \xi^2 + 8a_2 \xi + a_1 \right)}; \]  

(52)

\[ H \left( 2a_2 \xi + a_1 \right) \left( 6a_3 \xi^2 + 4a_2 \xi + a_1 \right)^2 = 0. \]  

(53)

The solution \( 2a_2 \bar{\xi} + a_1 = 0 \) would lead to \( c = 0 \) and to a degenerate second metric. Thus, either \( H = H_\omega = 0 \) or \( \xi = \bar{\xi} = const \), with

\[ 6a_3 \xi^2 + 4a_2 \xi + a_1 = 0. \]  

(54)

In the special case \( a_2 = a_3 = 0 \) we get the nonphysical solution \( H = H_\omega = 0 \). Thus, the only interesting solutions of Bianchi identities, when \( k_1 \neq k_2 \), are the ones with \( \xi = \bar{\xi} \) satisfying eq (54). Notice that for \( \xi = \bar{\xi} \) we have that \( c = 1 \). The further constraint coming from the equations of motion gives

\[ \frac{k_2 - k_1}{a^2} + \frac{8\pi G}{3} \rho_m + \Lambda = 0, \]  

(55)
where \( \Lambda \) is a function of \( \bar{\xi} \). The previous equation is algebraic in \( a \) and it gives \( a = \text{constant} \) and \( H = 0 \). Thus, in order to obtain a reasonable cosmology we are forced to set \( k_1 = k_2 = k_c \) and we are back to the Bianchi identity (12). The analysis of the previous sections is the same, except for the presence of the curvature term.

Let us now discuss the differences with the St"uckelberg approach where \( \tilde{g} \) is non dynamical. Formally, the non-dynamical limit would correspond to \( \kappa \to \infty \). To make contact with the existing literature, we take \( \tilde{g} \) equivalent to the Minkowski flat metric implying clearly that \( c \) and \( \omega \) cannot be arbitrary. Imposing that the Riemann curvature tensor of \( \tilde{g} \) vanishes we find the conditions

\[
 k_c < 0, \quad c = \frac{\omega H \omega}{\sqrt{-k_c}}. \tag{56}
\]

Therefore, FRW cosmology with frozen second metric exists only with a negative non-vanishing spatial curvature \([17]\). When (56) holds, Bianchi identities can be realized only within branch one, leading to eq. (18).

7 Conclusions

In this paper we studied FRW cosmological solutions of the bigravity extension of the recently found ghost free massive gravity theory. It was shown in ref. [17, 20] that when the auxiliary metric needed for the formulation of the theory is taken to be non dynamical, no flat FRW cosmology exists. On the other hand, as we showed here, flat FRW solutions are allowed when the second metric is promoted to a dynamical field. This result, together with the analysis of spherically symmetric solutions \([16]\), indicate that bigravity is more than just a tool; it is an important ingredient in the formulations of a physically acceptable theory of massive gravity. The cosmological evolution is governed by Bianchi identities (BI). Depending on the way they are realized, cosmology falls in two separate branches. When the BI are implemented by an algebraic constraint on the ratio \( \xi \) of the two scale factors, we find standard FRW cosmology, in the presence of a cosmological constant proportional to the graviton mass scale. When BI are implemented by a differential relation between the Hubble parameters of the two metrics, cosmology is instead richer. Cosmological evolution at the early times, where \( G \rho_m/m^2 \gg 1 \), is controlled by the parameter \( \xi \). In the presence of matter with an equation of state \( w \), the large \( \xi \) regime leads to unphysical solutions with \( c < 0 \). On the contrary, the small \( \xi \) regime is safe and reproduces the standard early FRW cosmology; at late time the universe sets in a dS attractor region. The model is interesting and though there are a number of free parameters in the potential, it is surprisingly predictive. The next step is to study perturbations to check whether massive gravity cosmology is viable.
We will report these results in a future work.

**Note added**

During the completion of this work, a study on the same topic appeared \[23\]. Our results seem to agree with \[23\], except for some exotic case. Almost at the same time of our paper, another paper \[24\] on the same subject was announced in the arXiv.

### A Modified Einstein Equations

For completeness we collect the full set of equations and we restore the presence of the spatial curvature $k$. The non vanishing equations for the metric $g$ are the $tt$ component and the space-space components and read

$$
m^2 \left[ 6a_3 \xi^3 + 6a_2 \xi^2 + 3a_1 \xi + a_0 \right] + 8\pi G \rho_m = \frac{3k}{a^2} + 3H^2; \quad (57)$$

$$
m^2 \left[ 6a_3 c \xi^3 + 2a_2 (2c+1) \xi^2 + a_1 (c+2) \xi + a_0 \right] - 8\pi G p_m
\frac{k}{a^2} + \frac{2H'}{a} + 3H^2. \quad (58)$$

For the metric $\tilde{g}$ the structure is the same and we have

$$
m^2 \left( \frac{a_1}{\xi^3} + \frac{6a_2}{\xi^2} + \frac{18a_3}{\xi} + 24a_4 \right) = \frac{3\kappa k}{a^2 \xi^2} + \frac{3\kappa H^2}{c^2 \xi}; \quad (59)$$

$$
m^2 \left( \frac{a_1}{c \xi^3} + \frac{4a_2}{c \xi^2} + \frac{6a_3}{c \xi} + \frac{2a_2}{\xi^2} + \frac{12a_3}{\xi} + 24a_4 \right)
\frac{\kappa k}{a^2 \xi^2} + \frac{2\kappa H' \omega}{ac^2 \xi} + \frac{2\kappa H^2 \omega}{ac^2 \xi} - \frac{2\kappa c' \omega H}{ac^3 \xi}. \quad (60)$$

Except for section \[6\] in order to de-clutter formulae we have not included $k$; as we can see from the previous equations the curvature can be easily restored without effecting the analysis. A linear combination of eq. (57) and its time derivative together with (58) gives (12) and the conservation law for matter. The same is true for (59) and (60).

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