Single Squaring Verifiable Delay Function from Time-lock Puzzle in the Group of Known Order

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Abstract. A Verifiable Delay Function (VDF) is a function that takes a specified sequential time $T$ to be evaluated, but can be verified in $O(\log T)$-time. For meaningful security, $T$ can be at most subexponential in the security parameter $\lambda$ but has no lower bound. VDFs are useful in several applications ranging from randomness beacons to sustainable blockchains but are really rare in practice. The verification in all of these VDFs requires $\Omega(\lambda, \log T)$ sequential time. This paper derives a verifiable delay function that is verifiable in $O(1)$-sequential time. The key observation is that the prior works use subexponentially-hard algebraic assumptions for their sequentiality. On the contrary, we derive our VDF from a polynomially-hard sequential assumption namely the time-lock puzzle over the group of known order. In particular, we show that time-lock puzzle can be sequentially-hard even when the order of the group is known but the delay parameter is polynomially-bounded i.e., $T \leq \text{poly}(\lambda)$. As an add-on advantage, our VDF requires only one sequential squaring to verify. Thus, in our VDF, the sequential effort required for verification is fixed and independent of the delay parameter $T$.

Keywords: Verifiable delay function · Modulo exponentiation · Sequentiality · Time-lock puzzle · Random oracle model

1 Introduction

The notion of verifiable delay functions was introduced in [1]. A verifiable delay function is a function with the domain $\mathcal{X}$ and the range $\mathcal{Y}$, that takes a specified number of sequential steps $T$ to be evaluated (irrespective of the amount of parallelism) and can be verified efficiently (even without parallelism) and publicly. In order to avoid exponential (processors) adversary $T = 2^{o(\lambda)}$ at most.

Along with its introduction, a candidate VDF using injective rational maps has been proposed in [1]. However, its prover needs a certain amount of parallelism to evaluate. Wesolowski [8] and Pietrzak [6] come up with two VDFs separately, although based on the same hardness assumption of time-lock puzzle in the group of unknown order [7]. Feo et
al. [5] propose a VDF based on super-singular elliptic curves defined over finite fields. The verifiers in all of these VDFs incur $\Omega \lambda \log T$ sequential time in verification.

Therefore, an intriguing question would be is it possible to design a VDF that is verifiable in sequential time independent of $\lambda$ and $T$.

### 1.1 Technical Overview

We show that it is possible to have VDF from polynomially-hard sequential function i.e., the delay parameter $T \leq \text{poly}(\lambda)$. The prior works do not enjoy faster verification because of their hardness assumptions used in the sequentiality are essentially sub-exponentially hard.

First we find that time-lock puzzle over the group of known orders can also be sequentially but polynomially hard. Then, we propose a VDF that needs only a single sequential squaring for verification. Thus the verification effort (sequential time) is independent of the security parameter $\lambda$ and the delay parameter $T$.

Briefly, our scheme works as follows. Depending upon $\lambda$ and $T$, it chooses a random oracle $H : \mathcal{X} \rightarrow \mathbb{QR}_q^*$ such that $\mathbb{QR}_q^*$ is the semigroup of squares except 1 in a finite field $\mathbb{F}_q$ of order $q$. Then, the prover is asked to find the square root of an arbitrary element in $\mathbb{F}_q^*$. Given the input $x \in \mathcal{X}$, the prover needs to compute $H(x)^{\frac{q+1}{4}} \in \mathbb{F}_q^*$. During verification, the verifier accepts if and only if $y^2 = H(x)$.

We show that our construction is correct, sound and sequential. Further, our VDF needs no proof, thus is only a one round protocol. We summarize a bunch of other advantages and also a disadvantage of our VDF over the existing ones in Sect. 6.1.

### 1.2 Organization of the Paper

This paper is organized as follows. Section 2 discusses a few existing schemes known to be VDF. In Section 3, we present a succinct review of VDF. We propose our single squaring verifiable delay function in Section 4. In Section 5 we establish the essential properties correctness, sequentiality and soundness of the VDF. Finally, Section 7 concludes the paper.

### 2 Related Work

In this section, we mention some well-known schemes qualified as VDFs, and summarize their features in Table 1 in Sect. 6.1. We categorize them by the sequentiality assumptions.
Injective Rational Maps In 2018, Boneh et al. [1] propose a VDF based on injective rational maps of degree $T$, where the fastest possible inversion is to compute the polynomial GCD of degree-$T$ polynomials. They conjecture that it achieves $(T^2, o(T))$ sequentiality using permutation polynomials as the candidate map. However, it is a weak VDF as it needs $O(T)$ processors to evaluate the output in time $T$.

Time-lock Puzzle Rivest, Shamir and Wagner introduced the time-lock puzzle stating that it needs at least $T$ number of sequential squaring to compute $y = g^{2T} \mod \Delta$ when the factorization of $\Delta$ is unknown [7]. Therefore they proposed this encryption that can be decrypted only sequentially. Starting with $\Delta = pq$ such that $p, q$ are large primes, the key $y$ is enumerated as $y = g^{2T} \mod \Delta$. Then the verifier, uses the value of $\phi(\Delta)$ to reduce the exponent to $e = 2T \mod \phi(\Delta)$ and finds out $y = g^e \mod \Delta$. On the contrary, without the knowledge of $\phi(\Delta)$, the only option available to the prover is to raise $g$ to the power $2^T$ sequentially. As the verification stands upon a secret, the knowledge of $\phi(\Delta)$, it is not a VDF as verification should depend only on public parameters.

Pietrzak [6] and Wesolowski [8] circumvent this issue independently. We describe both the VDFs in the generic group $G$ as done in [2]. These protocols can be instantiated over two different groups – the RSA group $(\mathbb{Z}/N\mathbb{Z})^\times$ and the class group of imaginary quadratic number field $Cl(d)$.

Pietrzak’s VDF Pietrzak’s VDF exploits the identity $(g^{r \cdot 2^{T/2}} \times g^{2T}) = (g^r \times g^{2^{T/2}})^{2^{T/2}}$ for any random integer $r$. The prover $P$ needs to compute $y = g^{2T}$ and the verifier $V$ checks it by sampling a random $r$ in that identity. In the non-interactive setting, the integer $r \in \mathbb{Z}_{2\lambda}$ is sampled by another random oracle $H : \mathbb{Z} \times G \times G \times G \to \mathbb{Z}_{2\lambda}$. The above-mentioned identity relates all $v = g^{r \cdot 2^{T/2}} \times g^{2T}$ and $u = g^r \times g^{2^{T/2}}$ for any $r \in \mathbb{Z}^+$. So, $V$ engages $P$ in proving the identity for $T/2, T/4, \ldots, 1$ in $\log T$ number of rounds. In each round $i$, a new pair of $(u_i, v_i)$ is computed depending upon the integer $r_i$ as follows.

First, the prover $P$ initializes $u_1 = g$ and $v_1 = y$. Then $P$ is asked to compute the output $y = g^{2T}$ and the proof $\pi = \{z_1, z_2, \ldots, z_{\log T}\}$ such that $u_{i+1} = u_i^{r_i} \cdot z_i$ and $v_{i+1} = z_i^{r_i} \cdot v_i$ where $z_i = u_i^{2^{T/2^i}}$ and $r_i =$
H(u_i, T/2^{i-1}, v_i, z_i). The effort to generate the proof \( \pi \) is in \( O(\sqrt{T} \log T) \).

The verifier \( \mathcal{V} \) reconstructs \( u_{i+1} = u_i^{z_i} \cdot z_i \) and \( v_{i+1} = z_i^{v_i} \cdot v_i \) by fixing \( u_1, v_1 \) and \( r_i \) as before. Finally, \( \mathcal{V} \) accepts the proof if and only if \( v_{\log T} = u_{\log T}^{2^{\log T}} \).

As \( r_{i+1} \) depends on \( r_i \) they need to be computed sequentially. So, doing at most \( \log r_i \) squaring in each round, the verifier performs \( P \sum_{i=1}^{\log T} \log r_i = O(\lambda \log T) \) squaring in total. Sect. ?? presents a detailed inspection on the lower bound of verification.

The soundness of this protocol is based on the low order assumption in the group \( G \). It assumes that finding an element with the order < \( 2^\lambda \) is computationally hard. A crucial feature that comes from this assumption is that the statistical soundness. This VDF stands sound even against computationally unbounded adversary. The probability that this VDF accepts a false proof produced by even a computationally unbounded adversary is negligible in the security parameter \( \lambda \).

Wesolowski’s VDF It has been designed using the identity \( g^{2^T} = (g^q)^\ell \times g^r \) where \( 2^T = q\ell + r \). Here \( \ell \) is a prime chosen uniformly at random from the set of first \( 2^{2\lambda} \) primes \( \mathbb{P}_{2\lambda} \).

This protocol asks the prover \( \mathcal{P} \) to compute the output \( y = g^{2^T} \). Then, the verifier \( \mathcal{V} \) chooses the prime \( \ell \) from the set \( \mathbb{P}_{2\lambda} \). In the non-interactive version, the prime \( \ell = H_{\text{prime}}(\text{bin}(g)||\text{bin}(y)) \) is sampled by the random oracle \( H_{\text{prime}} : \mathcal{G} \times \mathcal{G} \rightarrow \mathbb{P}_{2\lambda} \). Now, \( \mathcal{P} \) needs to compute the proof \( \pi = g^q \) where the quotient \( q = \lfloor 2^T/\ell \rfloor \). The verifier \( \mathcal{V} \) finds the remainder \( r = 2^T \mod \ell \). Then \( \mathcal{V} \) checks if \( y = \pi^\ell \cdot g^r \). This holds true since \( 2^T = q\ell + r \) so \( g^{2^T} = (g^q)^\ell \times g^r \).

It has been shown in [8], that \( \mathcal{P} \) needs \( O(T/\log T) \) time to compute the proof \( \pi \). On the other hand, the verifier \( \mathcal{V} \) needs to compute \( \pi^\ell \) and \( g^r \). As \( r < \ell \), the verification needs at most \( 2\log \ell \) squaring. Since, \( \ell \) is a prime of size at most \( 2\lambda \log 2\lambda \) (by prime number theorem) the verification amounts to at most \( 2 \log \ell = O(\lambda \log \lambda) \) squaring. In Sect. ??, we review the lower bound of its verification.

For the soundness, Wesolowski’s VDF relies on the adaptive root assumption in group \( \mathcal{G} \). It states that given a prime \( \ell \) and an arbitrary element in the group \( \mathcal{G} \) it is computationally hard to find the \( \ell \)-th root of that element.

Continuous VDF In a beautiful adaption of Pietrzak’s construction, Ephraim et al. introduce a unique feature called continuous verifiability into the world of VDFs. Any intermediate state at time \( T' < T \) of their VDF can be continuously verified in \( O(\lambda) \) time [4].
Isogenies over Super-Singular Curves Feo et al. presents two VDFs based on isogenies over super-singular elliptic curves [5]. They start with five groups \(<G_1, G_2, G_3, G_4, G_5>\) of prime order \(T\) with two non-degenerate bilinear pairing maps \(e_{12} : G_1 \times G_2 \rightarrow G_5\) and \(e_{34} : G_3 \times G_4 \rightarrow G_5\). Also there are two group isomorphisms \(\phi : G_1 \rightarrow G_3\) and \(\varphi : G_4 \rightarrow G_2\). Given all the above descriptions as the public parameters along with a generator \(P \in G_1\), the prover needs to find \(\varphi(Q)\), where \(Q \in G_4\), using \(T\) sequential steps. The verifier checks if \(e_{12}(P, \varphi(Q)) = e_{34}(\phi(P), Q)\) in \(\text{poly}(\log T)\) time. It runs on super-singular curves over \(\mathbb{F}_p\) and \(\mathbb{F}_{p^2}\) as two candidate groups.

3 Preliminaries

Now we mention the notations and terminology used in this paper.

3.1 Notation

We denote the security parameter with \(\lambda \in \mathbb{Z}^+\). The term \(\text{poly}(\lambda)\) refers to some polynomial of \(\lambda\), and \(\text{negl}(\lambda)\) represents some function \(\lambda^{-\omega(1)}\). If any randomized algorithm \(A\) outputs \(y\) on an input \(x\), we write \(y \leftarrow A(x)\).

By \(x \leftarrow \mathcal{X}\), we mean that \(x\) is sampled uniformly at random from \(\mathcal{X}\). For any set \(\mathcal{X}\), \(|\mathcal{X}|\) denotes the cardinality of the set \(\mathcal{X}\), whereas for everything else encoded as strings, \(|x| = n\) denotes the bit-length of the string \(x \in \{0,1\}^n\).

We consider \(A\) as efficient if it runs in probabilistic polynomial time (PPT). We assume (or believe) a problem to be hard if it is yet to have an efficient algorithm for that problem. We say that an algorithm \(A\) runs in parallel time \(T\) with \(\Gamma\) processors if it can be implemented on a PRAM machine with \(\Gamma\) parallel processors running in time \(T\).

3.2 Square Roots in Finite Fields of Odd Characteristics

In this section, we discuss the issues of finding square roots of an element in a finite field. It will be required in the security proofs in Sect. 5.

Any field \(\mathbb{F}_q\) can be succinctly represented using the fact that \(\mathbb{F}_q\) is always isomorphic to the ring of polynomials \(\mathbb{F}_p[x]/(f(x))\) where \(p\) is a prime and \(f(x)\) is an irreducible polynomial over \(\mathbb{F}_p\) of degree \(n\) such that \(q = p^n\). Therefore, it suffices to specify \(p\) and \(f(x)\) to describe \(\mathbb{F}_q\). The multiplicative cyclic group of \(\mathbb{F}_q\) is denoted as \(\mathbb{F}_q^*\) of order \(q - 1\).
Let $\mathbb{Q}_q$ be the set of squares in $\mathbb{F}_q^*$ i.e., $\mathbb{Q}_q = \{s^2 \mid \forall s \in \mathbb{F}_q^*\}$ of order $(q - 1)/2$. We define the semigroup of non-trivial squares as,

$$\mathbb{Q}_q^* \overset{\text{def}}{=} \mathbb{Q}_q \setminus \{1\}.$$ 

**Lemma 1.** In a finite field of odd order $q$,

i. for all $a \in \mathbb{F}_q^*$, $a \in \mathbb{Q}_q$ if and only if $a^{\frac{q-1}{2}} = 1$,

ii. The order of $\mathbb{Q}_q$ is $|\mathbb{Q}_q| = \frac{q-1}{2}$.

**Proof.** By Lagrange’s theorem, all elements in $\mathbb{F}_q^*$ satisfies the equation $x^{q-1} = 1$. It means $x^{\frac{q-1}{2}} = \pm 1$ holds true for all elements in $\mathbb{F}_q^*$. Therefore, the polynomial $x^{q-1} - 1$ has exactly $(q - 1)$ roots over $\mathbb{F}_q^*$.

**Part i.** If $a \in \mathbb{Q}_q$ then for some $x \in \mathbb{F}_q^*$, $a = x^2$. Then, $a^{\frac{q-1}{2}} = x^{q-1} = 1$. Conversely, if $a^{\frac{q-1}{2}} = 1$ then $a^\frac{1}{2}$ must be a root of the equation $x^{q-1} = 1$. Then, by Lagrange’s theorem, $a^\frac{1}{2} \in \mathbb{F}_q^*$, so $a \in \mathbb{Q}_q$.

**Part ii.** Since, the orders of all the elements divide the order of the group, $|\mathbb{Q}_q| = k \cdot \frac{(q-1)}{2}$ for some $k \in \mathbb{Z}^+$. Since, $x^{\frac{q-1}{2}} = -1$ for some $x \in \mathbb{F}_q^*$, we have $\mathbb{Q}_q \subset \mathbb{F}_q^*$, so $|\mathbb{Q}_q| < |\mathbb{F}_q^*|$. Both together imply that $|\mathbb{Q}_q| = \frac{(q-1)}{2}$.

$\square$

**Lemma 2.** In a finite field of odd order $q$, $-1 \not\in \mathbb{Q}_q$ if and only if $q = 3 \mod 4$.

**Proof.** If $q = 3 \mod 4$ then $\frac{q-1}{2} = 1 \mod 2$. Therefore, $-1^{\frac{q-1}{2}} = -1 \not\in \mathbb{Q}_q$. Conversely, if $-1 \not\in \mathbb{Q}_q$ then $2^{-1} = 1 \mod 2$. Hence, $q = 3 \mod 4$.

$\square$

The condition $q = 3 \mod 4$ can also be satisfied efficiently by the following lemma,

**Lemma 3.** For all $q, n \in \mathbb{Z}^+$ and primes $p$, the condition $q = p^n = 3 \mod 4$ holds true if and only if $p = 3 \mod 4$ and $n$ is odd.

**Proof.** The condition is,

$$q = p^n = 3 \mod 4$$

$$p^n \mod \phi(4) = 3 \mod 4$$

$$p^n \mod 2 = 3 \mod 4.$$

The left hand side yields $3 \mod 4$ if and only if $p = 3 \mod 4$ and $n$ is odd.

$\square$
Lemma 4. In a finite field of odd order \( q = 3 \mod 4 \), for all \( a \in \mathbb{QR}_q^* \), square root of \( a \) is computable in \( O(\log q) \) squaring as \( a^{\frac{q+1}{4}} = a^{\frac{1}{2}} \in F_q^* \).

Proof. The exponent \( \frac{q+1}{4} \in \mathbb{Z}^+ \) if and only if \( q = 3 \mod 4 \). By lemma 1, \( a^{\frac{q+1}{4}} = a \in \mathbb{QR}_q \). So, \( a^{\frac{q+1}{4}} = a^{\frac{1}{2}} \in F_q^* \). It takes \( (\lceil \log (q+1) \rceil - 2) \) squaring in \( F_q^* \) to compute \( a^{\frac{q+1}{4}} \). \( \square \)

Assumption 1 (Lower bound on finding square roots in \( F_q^* \)) In a finite field of odd order \( q \), for all \( a \in \mathbb{QR}_q \), computing the square root of \( a \) takes \( \Omega(\log q) \) time.

Proof. We consider the algorithm by Doliskani and Schost to be fastest algorithm to find the square roots in sufficiently large finite fields [3]. The proposition 3.2 in cf. [3] shows that their algorithm runs in time \( O(sM(n) \log p + C(n) \log n) \) for \( q = p^n \) where \( M(n) = n \log n \log \log n \) and \( C(n) = \mathcal{O}(n^{1.67}) \) denote the time for multiplication and composition of polynomials. For square roots \( s \) turns out to be 1. Therefore, it takes \( \Omega(n \log p) = \Omega(\log q) \)-time.

3.3 Verifiable Delay Function

We borrow this formalization from [1].

Definition 1. (Verifiable Delay Function). A verifiable delay function from domain \( X \) to range \( Y \) is a tuple of algorithms (Setup, Eval, Verify) defined as follows,

- **Setup**(\( 1^\lambda, T \)) \( \rightarrow pp \) is a randomized algorithm that takes as input a security parameter \( \lambda \) and a delay parameter \( T \), and produces the public parameters \( pp \). We require Setup to run in \( \text{poly}(\lambda, \log T) \)-time.
- **Eval**(\( pp, x \)) \( \rightarrow (y, \pi) \) takes an input \( x \in X \), and produces an output \( y \in Y \) and a (possibly empty) proof \( \pi \). Eval may use random bits to generate the proof \( \pi \). For all \( pp \) generated by Setup\((1^\lambda, T) \) and all \( x \in X \), the algorithm Eval\((pp, x) \) must run in time \( T \).
- **Verify**\((pp, x, y, \pi) \rightarrow \{0, 1\} \) is a deterministic algorithm that takes an input \( x \in X \), an output \( y \in Y \), and a proof \( \pi \) (if any), and either accepts (1) or rejects (0). The algorithm must run in \( \text{poly}(\lambda, \log T) \) time.

Before we proceed to the security of VDFs we need the precise model of parallel adversaries [1].
Definition 2. (Parallel Adversary) A parallel adversary \( A = (A_0, A_1) \) is a pair of non-uniform randomized algorithms \( A_0 \) with total running time \( \text{poly}(\lambda, T) \), and \( A_1 \) which runs in parallel time \( \sigma(T) \) on at most \( \text{poly}(\lambda, T) \) number of processors.

Here, \( A_0 \) is a preprocessing algorithm that precomputes some state based only on the public parameters, and \( A_1 \) exploits this additional knowledge to solve in parallel running time \( \sigma \) on \( \text{poly}(\lambda, T) \) processors.

Three essential security properties of VDFs are now described.

Definition 3. (Correctness) A VDF is correct if for all \( \lambda, T \in \mathbb{Z}^+ \), and \( x \in \mathcal{X} \), we have,

\[
\Pr \left[ \text{Verify}(pp, x, y, \pi) = 1 \right| pp \leftarrow \text{Setup}(1^\lambda, T), x \overset{\$}{\leftarrow} \mathcal{X}, (y, \pi) \leftarrow \text{Eval}(pp, x) \right] = 1.
\]

Definition 4. (Soundness) A VDF is computationally sound if for all non-uniform algorithms \( A \) that run in time \( \text{poly}(\lambda, T) \), we have,

\[
\Pr \left[ y \neq \text{Eval}(pp, x) \right| \text{Verify}(pp, x, y, \pi) = 1 \right| pp \leftarrow \text{Setup}(1^\lambda, T), (x, y, \pi) \leftarrow A(1^\lambda, 1^T, pp) \right] \leq \text{negl}(\lambda).
\]

Further, a VDF is called statistically sound when all adversaries (even computationally unbounded) have at most \( \text{negl}(\lambda) \) advantage. Even further, it is called perfectly sound if we want this probability to be 0 against all adversaries. Hence, perfect soundness implies statistical soundness which implies computational soundness but not the reverses.

Definition 5. (Sequentiality) A VDF is \((\Gamma, \sigma)\)-sequential if there exists no pair of randomized algorithms \( A_0 \) with total running time \( \text{poly}(\lambda, T) \) and \( A_1 \) which runs in parallel time \( \sigma(T) < T \) on at most \( \Gamma \) processors, such that

\[
\Pr \left[ y = \text{Eval}(pp, x) \right| pp \leftarrow \text{Setup}(1^\lambda, T), \text{state} \leftarrow A_0(1^\lambda, T, pp), x \overset{\$}{\leftarrow} \mathcal{X}, y \leftarrow A_1(\text{state}, x) \right] \leq \text{negl}(\lambda).
\]

\( \sigma(T) = T \) is impossible as Eval runs in time \( T \). In all practical applications it suffices to attain \( \sigma(T) = (1-\epsilon)T \) for sufficiently small \( \epsilon \) while an almost-perfect VDF would achieve \( \sigma(T) = T - o(T) \) [1].
4 Single Squaring Verifiable Delay Function

As before, $\lambda \in \mathbb{Z}^+$ denotes the security parameter, $T \in 2^{o(\lambda)}$ denotes the delay parameter. The three algorithms that specify our VDF are,

4.1 The Setup($1^\lambda, T$) Algorithm

This algorithm outputs the public parameters $\mathbf{pp} = \langle F_q, H \rangle$ having the following meanings.

1. $F_q$ is a finite field of order $q \geq 2^\lambda$ such that $q = 3 \mod 4$. We denote $F_q^*$ as the multiplicative group of $F_q$. It is a cyclic group of order $q - 1$.
2. We take $H : \mathcal{X} \rightarrow \mathbb{QR}_q^*$ to be a random oracle that maps an input statement $x \in \mathcal{X}$ to an element in $\mathbb{QR}_q^*$.

None of the public parameters needs to be computed. The cost of Setup has been analyzed in Sect. 6.

4.2 The Eval Algorithm

The prover $P$ needs to compute one of the square roots of the element $H(x)$ from the $\mathbb{QR}_q^*$. In particular, $P$ executes,

Algorithm 1 Eval($\mathbf{pp}, x, T$) → y
1: compute $g := H(x) \in \mathbb{QR}_q^*$.
2: compute $y := g^{q+1} \in F_q^*$. \hspace{1cm} $\triangleright$ By lemma 4, $y$ is a square root of $s$ over $F_q^*$.
3: return $y$

$P$ announces the triple $(x, T, y)$.

4.3 The Verify Algorithm

The verifier $V$ only checks if $y^2 = H(x)$. So, $V$ runs the following algorithm,

Verification needs no proof, so $\pi = \bot$.

5 Security Analysis

Here we show that the proposed VDF is correct, sound and sequential.
Algorithm 2 Verify$(\mathbf{pp}, x, T, y, \bot) \to \{0, 1\}$

1: compute $g := H(x) \in \mathbb{QR}_q^*$
2: if $g^2 = g$ then
3: return 1
4: else
5: return 0
6: end if

5.1 Correctness

The verifier should always accept a valid triple $(x, y, T)$.

**Theorem 1.** The single squaring VDF is correct.

**Proof.** Since $H$ is a random oracle, $g \in \mathbb{QR}_q^*$ is uniquely determined by the challenge $x \in \mathcal{X}$. If the prover $\mathcal{P}$ finds the square root of $H(x)$ then the verifier $\mathcal{V}$ accepts the tuple $(\mathbf{pp}, x, y, T)$ as $y^2 = H(x)$ by lemma 4. $\square$

5.2 Soundness

Using an invalid proof no adversary $\mathcal{A}$ having $\text{poly}(T)$ processors should convince the verifier with non-negligible probability.

**Definition 6.** (Square Root Game $\sqrt{\mathcal{G}_F}$) Let $\mathcal{A}$ be a party playing the game. The square root finding game $\sqrt{\mathcal{G}_F}$ goes as follows:

1. $\mathcal{A}$ is given a composite integer $\mathbb{F}_q$.
2. $\mathcal{A}$ is given an element $z \leftarrow \mathbb{QR}_q^*$ chosen uniformly at random.
3. Observing $z$, $\mathcal{A}$ outputs an element $w \in \mathbb{F}_q^*$.

The player $\mathcal{A}$ wins the game $\sqrt{\mathcal{G}_F}$ if $w^2 = z \in \mathbb{QR}_q^*$.

By lemma 1, there exists exactly two square roots of each $z \in \mathbb{QR}_q^*$. However, we need another version of the game $\sqrt{\mathcal{G}_F}$ in order to reduce the soundness-breaking game. Instead of sampling $z$ uniformly at random, in this version we compute $H : \mathcal{X} \to z \in \mathbb{QR}_q^*$.

**Definition 7.** (Square Root Oracle Game $\sqrt{\mathcal{G}_F^H}$). Let $\mathcal{A}$ be a party playing the game. Suppose, $H : \{0, 1\}^* \to \mathbb{QR}_q^*$ is random oracle that always maps any element from its domain to a fixed element chosen uniformly at random from its range, with the probability $\frac{1}{|\mathbb{QR}_q^*|}$. The Square Root Function Game $\sqrt{\mathcal{G}_F^H}$ goes as follows:
1. A is given the output of the Setup\((1^\lambda, T) \rightarrow \langle H, F_q \rangle\).
2. An element \(x \overset{\$}{\leftarrow} \mathcal{X}\) is chosen uniformly at random.
3. A is given the element \(H(x) \rightarrow z \in \mathbb{QR}_q^*\).
4. Observing \(z\), A outputs an element \(w \in \mathbb{F}_q^*\).

The player A wins the game \(\sqrt{\mathcal{G}_H}\) if \(w^2 = z \in \mathbb{QR}_q^*\).

Under the assumption that \(H\) samples elements from its range uniformly at random the distributions of \(z\) in both the games \(\sqrt{\mathcal{G}_F}\) and \(\sqrt{\mathcal{G}_H}\) should be indistinguishable for A. Therefore, A has exactly two solutions to win the game \(\sqrt{\mathcal{G}_H}\).

**Theorem 2. (Soundness).** Suppose A be a player who breaks the soundness of the single squaring VDF with probability \(p_{\text{win}}\). Then there is a player B who wins the square root oracle game \(\sqrt{\mathcal{G}_H}\) with the probability \(p_{\text{win}}\).

**Proof.** Run the Setup\((1^\lambda, T)\) to obtain \(\langle H, F_q \rangle\). The player B calls A on the input statement \(x \overset{\$}{\leftarrow} \mathcal{X}\). Suppose A outputs \(y \neq \text{Eval}(pp, x, y, T) \in \mathbb{F}_q^*\). As A breaks the soundness of this VDF, \(\text{Verify}(pp, x, y, T) = 1\), so \(y^2 = H(x) \in \mathbb{F}_q^*\) with the probability \(p_{\text{win}}\). So B computes \(y^2\) from \(y\) and outputs it to win the game \(\sqrt{\mathcal{G}_H}\) with probability \(p_{\text{win}}\). \(\square\)

### 5.3 Sequentiality

The sequentiality analysis of our VDF scheme is based on the sequentiality of time-lock puzzle [7].

**Definition 8. (Generalized Time-lock Game \(\mathcal{G}_G^T\))** Let \(A = (A_0, A_1)\) be a party playing the game. For the security parameter \(\lambda \in \mathbb{Z}^+\) and the delay parameter \(T = T(\lambda) \in \mathbb{Z}^+\), the time-lock game \(\mathcal{G}_G^T\) goes as follows:

1. \(A_0\) is given a group \(G\) of order \(|G| \geq 2^\lambda\).
2. \(A_0\) computes some information \(\text{state} \in \{0, 1\}^*\) on the input \(G\).
3. \(A_1\) is given the information \(\text{state} \) and an element \(g \overset{\$}{\leftarrow} G\) chosen uniformly at random.
4. Observing \(g\) and \(\text{state}\), \(A_1\) outputs an element \(h \in G\) in time \(< T\).

The player \(A = (A_0, A_1)\) wins the game \(\mathcal{G}_G^T\) if \(h = g^{2T} \in G\).
Hardness of Time-lock puzzle Here we describe two cases where time-lock puzzles are believed to be sequentially hard. The first one is celebrated as the time-lock puzzle in the group of unknown order \[7\]. The same idea has been conjectured for the class group of imaginary quadratic number fields \(Cl(d)\) in \[8\]. We bring into a similar sequentially hard assumption but in the group of known order.

Assumption. 2 (Time-lock puzzle in the group of unknown order). For all algorithms \(A = (A_0, A_1)\) running in time \(T\) on \(\text{poly}(T)\) number of processors, the probability that \(A\) wins the time-lock game \(G_T^G\) is at most \(\text{negl}(\lambda)\) when the order of the group \(G\) is unknown \[7\]. Formally,

\[
\Pr [A \text{ wins } G_T^G \mid |G| \text{ is unknown}] \leq \text{negl}(\lambda).
\]

However, knowing the order does not necessarily ease the computation sequentiality. It is because, till date, the only faster algorithm that we know to compute \(g^e\) for any \(e \in \mathbb{Z}^+\) is to reduce it to \(g^e \mod |G|\) assuming that the group \(G\) is not efficiently solvable into smaller groups. Therefore, when \(e < |G|\) there are only two options we are left with. If \(e \leq |G|\) then compute \(g^e\), otherwise, compute \((g^{-1})^{|G|-e}\). Clearly, the knowledge of \(|G|\) reduces the number of required squaring only if \(e > |G|/2\) but keeps the same when \(e \leq |G|/2\). Therefore, time-lock puzzle holds true even when the order \(|G|\) is known and the exponent is \(e \leq |G|/2\).

Assumption. 3 (Time-lock puzzle in the group of known order). For all algorithms \(A = (A_0, A_1)\) running in time \(T\) on \(\text{poly}(T)\) number of processors, the probability that \(A\) wins the time-lock game \(G_T^G\) is at most \(\text{negl}(\lambda)\) when the order of the group \(G\) is known to be at least \(2^{T+1}\). Formally,

\[
\Pr [A \text{ wins } G_T^G \mid |G| \geq 2^{T+1}] \leq \text{negl}(\lambda).
\]

Definition 9. (Time-lock Oracle Game \(G_T^{HT}\)) Let \(A = (A_0, A_1)\) be a party playing the game. For the security parameter \(\lambda \in \mathbb{Z}^+\) and the delay parameter \(T = T(\lambda) \in \mathbb{Z}^+\), the time-lock oracle game \(G_T^{HT}\) goes as follows:

1. \(A_0\) is given the output of the \(\text{Setup}(1^\lambda, T) \rightarrow \langle H, G \rangle\).
2. \(A_0\) computes some information \(\text{state} \in \{0, 1\}^*\) on the input \(G\) and \(H\).
3. An element \(x \xleftarrow{\$} X\) is chosen uniformly at random.
4. \(A_1\) is given the information \(\text{state}\) and an element \(g \leftarrow H(x)\).
5. Observing \(g\) and \(\text{state}\), \(A_1\) outputs an element \(h \in G\) in time \(< T\).

The player \(A = (A_0, A_1)\) wins the game \(G_T^G\) if \(h = g^{2^T} \in G\).
Under the assumption that the random oracle $H$ samples strings from its range uniformly at random the view of the distributions of $g$ in both the games $G_T^H$ and $G_H^T$ are identical to $\mathcal{A}$. Thus we infer that,

**Assumption. 4 (Time-lock Oracle Assumption).** For all algorithms $\mathcal{A} = (\mathcal{A}_0, \mathcal{A}_1)$ running in time $< T$ on $\text{poly}(T)$ number of processors wins the time-lock oracle game $G_H^T$ with the probability at most negligible in the security parameter $\lambda$. Mathematically,

$$\Pr \left[ g^{2^T} = h \mid (H, G) \leftarrow \text{Setup}(1^\lambda, T), \text{state} \leftarrow \mathcal{A}_0(1^\lambda, T, H, G), x \leftarrow \mathcal{X}, g \leftarrow H(x), h \leftarrow \mathcal{A}_1(g, \text{state}) \right] \leq \text{negl}(\lambda).$$

The game $\sqrt{G_F^H}$ (Def. 7) is an example of time-lock puzzle in the group of known order as $|\mathbb{F}_q^*| = q - 1$ and $w = z^{\frac{q+1}{4}} \in \mathbb{F}_q^*$. By assumption 1, computing $w \in \mathbb{G}$ needs $\Omega(\log q)$ squaring for any $z \in \mathbb{QR}_q^*$. Therefore, the game $\sqrt{G_F^H}$ (Def. 7) satisfies assumption 4 for $\mathbb{G} = \mathbb{F}_q^*$ and $T = \log q$.

**Theorem 3. (Sequentiality).** Suppose $\mathcal{A}$ be a player who breaks the sequentiality of the single squaring VDF with probability $p_{\text{win}}$. Then there is a player $\mathcal{B}$ who wins the square root oracle game $\sqrt{G_F^H}$ (Def. 7) with probability $p_{\text{win}}$.

**Proof.** Run the $\text{Setup}(1^\lambda, T = \log q)$ to obtain $(H, F_q)$. $\mathcal{B}$ calls $\mathcal{A}$ on the input statement $x \leftarrow \mathcal{X}$. Against $x$, $\mathcal{A}$ outputs $y \in \mathbb{F}_q^*$ in time $< \log q$. As $\mathcal{A}$ breaks the sequentiality of this VDF, $y^2 = H(x) \in \mathbb{F}_q^*$ with the probability $p_{\text{win}}$. So $\mathcal{B}$ outputs $y$ and wins the game $\sqrt{G_F^H}$ with probability $p_{\text{win}}$. \(\square\)

In the next section, we show that our VDF is $(\text{poly}(\lambda), (1 - \epsilon)T)$-sequential.

**6 Efficiency Analysis**

Here we discuss the efficiencies of both the prover $\mathcal{P}$ and the verifier $\mathcal{V}$ in terms of number of modulo squaring and the memory requirement.
Cost of Setup The cost of constructing $H$ is equal to that of constructing another random oracle $H': \mathcal{X} \to \{F_q^* \setminus 1\}$ plus a single squaring over $F_q^*$ because $H(x) \overset{\text{def}}{=} H'(x)^2$. The random oracles of type $H'$ are standard in this domain of research [8,6].

By lemma 3 we need a prime $p = 3 \mod 4$ and an odd $n$ for $q = 3 \mod 4$. Any prime selected uniformly at random satisfies the condition $p = 3 \mod 4$ with probability at least $1/2$. The dimension $n$ is chosen to satisfy $p^n \geq 2^\lambda$. The enumeration of the irreducible polynomial $f(x)$ need not be the part of Setup as it may be sampled from a precomputed list. However, we recommend to use computer algebra systems like SAGEMATH in order to reliably sample $f(x)$.

Proof Size The proof size is essentially zero as the proof $\pi = \bot$ is empty. The output $y$ is an element in the group $F_q^*$. So the size of the output is $\log q$-bits.

Prover’s Efficiency As already mentioned in Theorem 3, the prover $P$ needs at least $T = \log q$ sequential time in order to compute $H(x)^{\frac{q+1}{4}}$. The prover needs no proof. Thus the prover’s effort is not only at least $T = \Theta(\log q)$. We use the $\Theta$ notation to mean $(1 - \epsilon)T$ for some $\epsilon \to 0$. We keep this provision because $P$ may speed up the field multiplications using $\text{poly}(T)$ processors but not in time $< T - o(T)$.

Verifier’s Efficiency In this VDF, verification needs only a single squaring over $F_q^*$.

Upper bound on $T$ As the output by the prover needs $\log q$-bits, we need to restrict $\log q \in \text{poly}(\lambda)$ at most. Cryptographic protocols that deal with objects larger than the polynomial-size in their security parameter are impractical. Hence, for the group $F_q^*$ the delay parameter $T = \log q$ must be bounded by $\text{poly}(\lambda)$.

We note that handling an element in the group $F_q^*$ does not imply the sequential effort by the verifier is proportional to $T = \log q$. As a comparison, we never consider the complexity of operation over the group $(\mathbb{Z}/N\mathbb{Z})^\times$ for the VDFs in [8,6] in table 1. In fact, the sequentiality of squaring does not depend on the complexity of the elementary group operation in the underlying group. The sequentiality should hold even if these elementary operations are of in $O(1)$-time.
6.1 Performance Comparison

In this section, we demonstrate a few advantages of our VDF.

**Lower bound on prover’s parallelism** The VDF based on injective rational maps demands $\text{poly}(\lambda, T)$-parallelism to compute the $\text{Eval}$ in time $T$ [1]. In our case, no parallelism is required to compute the VDF in time $T$.

**Proof size** The proofs in the Wesolowski’s and Pietrzak’s VDFs consume one and $\log T$ group elements in their underlying group $\mathbb{G}$ (i.e., $(\mathbb{Z}/N\mathbb{Z})^\times$ and $C(l(d))$) [8,6]. In order to communicate the proof and to verify efficiently $\log |\mathbb{G}| \in \text{poly}(\lambda)$. Our VDF requires no proof.

**Efficiency of Setup** The $\text{Setup}$ in the isogeny-based VDF may turn out to be as slow as the $\text{Eval}$ itself [5]. On the other hand, the $\text{Setup}$ in our VDF is as efficient as that in all other time-lock based VDFs [8,6].

**$\Omega(\lambda)$-Verifiability** All the existing VDFs except [1] at least $\lambda$ sequential effort for verification. Sect. ?? describes the concrete bound of the time-lock puzzle based VDFs. On the contrary, the most important advantage of our VDF is that the verification requires only one squaring over $\mathbb{F}_q$.

**Largeness of $T$** A limitation of our VDF is that it works only when $T = \log q \in \text{poly}(\lambda)$ as the output size is dominated by $T$, while all the above-mentioned VDFs allow $T \in 2^{\omega(\lambda)}$.

In Table 1, we summarize the above description.

| VDFs (by authors) | Eval size | Eval Parallel | Verify size | Setup size | Proof size |
|-------------------|-----------|---------------|-------------|------------|------------|
| Boneh et al. [1]  | $T^2$     | $T^2$         | $\log T$   | $\log T$  | --         |
| Wesolowski [8]    | $(1 + \frac{T}{\log T})T$ | $(1 + \frac{T}{\log T})T$ | $\lambda \log \lambda$ | $\lambda^3$ | $\lambda^3$ |
| Pietrzak [6]      | $(1 + \frac{T}{\sqrt{T}})T$ | $(1 + \frac{T}{\sqrt{T}})T$ | $\lambda \log T$ | $\lambda^3$ | $\log T$ |
| Ephraim et al. [4]| $(1 + \frac{T}{\sqrt{T}})T$ | $(1 + \frac{T}{\sqrt{T}})T$ | $\lambda$ | $\lambda^3$ | $\log T$ |
| Feo et al. [5]    | $T$       | $T$           | $T \log \lambda$ | --         | --         |
| **Our work**      | $T$       | $(1 - \epsilon)T$ | $1$         | $\lambda^3$ | --         |
7 Conclusions

This chapter derives the first VDF that verifies its computation with a single squaring. It has been derived from a new sequential assumption namely the time-lock puzzle in the group of known order. At the same time, this is a VDF from the complexity class of all deterministic computation as we have shown the delay parameter $T$ is in $\text{poly}(\lambda)$.

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