POWERS OF AMPLE DIVISORS AND SYZYGIES OF PROJECTIVE VARIETIES

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Abstract. Suppose $X$ is a projective variety (which needs not be smooth), and $L$ is an ample line bundle on $X$. We show that there exist integers $c$ and $b$ such that for any non-negative integer $p$ and any integer $d \geq cp + b$, $L^d$ is normally generated and embeds $X$ as a variety whose defining ideal is generated by quadratics and has linear syzygies up to the $p$-th step (i.e. $L^d$ has property $N_p$).

0. Introduction

In this paper we study syzygies of embeddings of projective varieties given by powers of an ample line bundle.

It is a well celebrated theorem of Mumford ([8, Theorem 5]) which stated that if $X$ is a projective variety and $L$ is an ample line bundle on $X$, then $L^d$ is normally generated and embeds $X$ as a variety whose defining ideal is generated by quadratics for all $d \gg 0$. Green ([4]) in 1984, developed a technique using Koszul cohomologies as the first step towards understanding higher syzygies of projective varieties. In particular, he proved that if $L = O_{\mathbb{P}^n}(1)$ then $L^d$ embeds $\mathbb{P}^n$ as a variety whose defining ideal is generated by quadratics and has linear syzygies up to the $p$-th step for all $d \geq p$. This property was later formalized by Green and Lazarsfeld ([5]) and called property $N_p$.

Definition. Let $Y$ be a projective variety and let $L$ be a very ample line bundle on $Y$ defining an embedding $\varphi_L : Y \hookrightarrow \mathbb{P} = \mathbb{P}(H^0(Y, L)^*)$.

1) The line bundle $L$ is said to have property $N_0$ if $\varphi_L(Y)$ is projectively normal, i.e. $L$ is normally generated.

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Let $S = \text{Sym}^* H^0(Y, \mathcal{L})$, the homogeneous coordinate ring of the projective space $\mathbb{P}$. Suppose $A$ is the homogeneous coordinate ring of $\varphi_{\mathcal{L}}(Y)$ in $\mathbb{P}(H^0(Y, \mathcal{L})^*)$, and

$$0 \rightarrow F_n \rightarrow F_{n-1} \rightarrow \ldots \rightarrow F_0 \rightarrow A \rightarrow 0$$

is a minimal free resolution of $A$. The line bundle $\mathcal{L}$ is said to have property $N_p$ (for $p \in \mathbb{N}$) if and only if it has property $N_0, F_0 = S$, and $F_i = S(-i - 1)^{\alpha_i}$ with $\alpha_i \in \mathbb{N}$ for all $1 \leq i \leq p$.

Mumford and Green’s results seem to suggest that if $X$ is a projective variety and $\mathcal{L}$ an ample line bundle on $X$, then for any non-negative integer $p$, $\mathcal{L}^d$ should have property $N_p$ for all $d$ large enough. Property $N_p$ of powers of ample line bundles have since then been studied extensively on various types of projective varieties, such as in [1], [2], [3], [6], [9] and [11]; and on abelian varieties in relation to a conjecture of Lazarsfeld ([7]), such as in [10] and [12]. When $X$ is a smooth projective variety and $\mathcal{L}$ is a very ample line bundle on $X$, Ein and Lazarsfeld ([3]) obtained an effective bound on the powers of $\mathcal{L}$ required to have property $N_p$. Without assuming the smoothness of $X$, Inamdar ([6]) was still able to show that $\mathcal{L}^d$ satisfies property $N_p$ when $d \gg 0$. However, Inamdar’s method could not give any bound for such values of $d$. In this paper, we give a different and simpler method to prove Inamdar’s result. Stronger, we will give an explicit bound for such values of $d$. Our bound, even though not as effective as that of Ein and Lazarsfeld (due to the lack of smoothness), is still linear on $p$. Our main theorem is the following.

**Theorem 0.1** (Theorem 2.2). Suppose $X$ is a projective variety of dimension $n$, $\mathcal{L}$ is a very ample divisor on $X$. Then, there exists an integer $b$ such that for any non-negative integer $p$, the line bundle $\mathcal{L}^d$ is normally generated and possesses property $N_p$ for all $d \geq pn + b$.

If $\mathcal{L}$ is only an ample line bundle (not necessarily very ample), we also obtain integers $c$ and $b$ such that for any non-negative integer $p$ and any $d \geq cp + b$, $\mathcal{L}^d$ has property $N_p$ (Corollary 2.2.1).

Throughout this paper, we will work on projective varieties over an algebraically closed field of characteristic 0.
1. Preliminaries

In this section, we recall some notations and results that will be used in the proof of our main theorem. Suppose $X$ is a projective variety, $\mathcal{L}$ is an ample divisor on $X$ which is globally generated. Then, there is a canonical surjective evaluation map $e_{\mathcal{L}} : H^0(\mathcal{L}) \otimes \mathcal{O}_X \to \mathcal{L}$. We define $\mathcal{M}_\mathcal{L}$ to be the kernel of $e_{\mathcal{L}}$. By construction, we have an exact sequence

$$0 \to \mathcal{M}_\mathcal{L} \to H^0(\mathcal{L}) \otimes \mathcal{O}_X \to \mathcal{L} \to 0.$$  

(1.1)

The criterion to verify property $N_p$ we use in this paper is the following result of Ein and Lazarsfeld.

**Lemma 1.1** ([3], Lemma 1.6). Assume that $\mathcal{L}$ is very ample, and that $H^1(\mathcal{L}^k) = 0$ for all $k \geq 1$. Then $\mathcal{L}$ satisfies property $N_p$ if and only if

$$H^1(\wedge^a \mathcal{M}_\mathcal{L} \otimes \mathcal{L}^b) = 0, \ \forall \ a \leq p + 1 \text{ and } b \geq 1.$$  

Another criterion we will be using is the following generalized Castelnuovo’s Lemma of Mumford.

**Lemma 1.2** ([8], Theorem 2). Suppose $\mathcal{L}$ is a base-point-free ample line bundle on a projective variety $X$, and let $\mathcal{F}$ be a coherent sheaf on $X$. If $H^i(\mathcal{F} \otimes \mathcal{L}^{-i}) = 0$ for all $i \geq 1$, then the multiplication map

$$H^0(\mathcal{F} \otimes \mathcal{L}^i) \otimes H^0(\mathcal{L}) \to H^0(\mathcal{F} \otimes \mathcal{L}^{i+1})$$

is surjective for all $i \geq 0$.

2. Main Theorem

The core of the proof of our main theorem lies in the following proposition.

**Proposition 2.1.** Suppose $X$ is a projective variety of dimension $n$, and $\mathcal{L}$ is a very ample line bundle on $X$. Then, there exists an integer $b_0$ such that for any non-negative integer $p$, and for any integers $q \geq 1$ and $a \geq pn + b_0$, we have

$$H^i(\mathcal{M}_{\mathcal{L}}^p \otimes \mathcal{L}^a) = 0 \ \forall \ 1 \leq i \leq n.$$
Proof. Since \( L \) is a very ample divisor on \( X \), by Serre’s vanishing theorem, there exists an integer \( b_0 \) such that

\[
H^i(L^m) = 0 \quad \forall \ 1 \leq i \leq n \text{ and } m \geq b_0. \tag{2.1}
\]

We will now prove the proposition by using induction on \( p \).

For \( p = 0 \), the conclusion follows from (2.1). Suppose now that \( p \geq 1 \). Since \( L \) is very ample on \( X \), \( L^q \) is very ample on \( X \) for any \( q \geq 1 \), and so, \( L^q \) is globally generated for any \( q \geq 1 \). Let \( \mathcal{M}_{L^q} \) be the kernel of the evaluation map \( H^0(L^q) \otimes \mathcal{O}_X \to L^q \to 0 \). We have the following exact sequence

\[
0 \to \mathcal{M}_{L^q} \to H^0(L^q) \otimes \mathcal{O}_X \to L^q \to 0. \tag{2.2}
\]

By tensoring (2.2) with \( \mathcal{M}_{L^q}^{-1} \otimes L^a \), and taking the long exact sequence of cohomologies, we get

\[
H^0(L^q) \otimes H^0(\mathcal{M}_{L^q}^{-1} \otimes L^a) \to H^0(\mathcal{M}_{L^q}^{-1} \otimes L^{a+q}) \to H^1(\mathcal{M}_{L^q}^{-1} \otimes L^a) \to H^0(L^q) \otimes H^1(\mathcal{M}_{L^q}^{-1} \otimes L^a),
\]

and

\[
H^{i-1}(\mathcal{M}_{L^q}^{-1} \otimes L^{a+q}) \to H^i(\mathcal{M}_{L^q}^{-1} \otimes L^a) \to H^0(L^q) \otimes H^i(\mathcal{M}_{L^q}^{-1} \otimes L^a), \ \forall \ i \geq 2.
\]

Since \( a \geq pm + b_0 > (p-1)n + b_0 \), it follows from the inductive hypothesis that

\[
H^i(\mathcal{M}_{L^q}^{-1} \otimes L^a) = H^i(\mathcal{M}_{L^q}^{-1} \otimes L^{a+q}) = 0 \ \forall \ 1 \leq i \leq n.
\]

Thus,

\[
H^0(L^q) \otimes H^0(\mathcal{M}_{L^q}^{-1} \otimes L^a) \to H^0(\mathcal{M}_{L^q}^{-1} \otimes L^{a+q}) \to H^1(\mathcal{M}_{L^q}^{-1} \otimes L^a) \to 0,
\]

and

\[
H^i(\mathcal{M}_{L^q}^{-1} \otimes L^a) = 0, \ \forall \ i = 2, \ldots, n.
\]

Therefore, it now suffices to show that

\[
H^0(L^q) \otimes H^0(\mathcal{M}_{L^q}^{-1} \otimes L^a) \to H^0(\mathcal{M}_{L^q}^{-1} \otimes L^{a+q}). \tag{2.3}
\]

For any \( 1 \leq i \leq n \) and \( 0 \leq j \leq q - 1 \), we have

\[
a + j - i \geq np + b_0 + j - i \geq np + b_0 - i \geq np + b_0 - n = (p-1)n + b_0.
\]

Thus, by the inductive hypothesis, for any \( 0 \leq j \leq q - 1 \), we have

\[
H^i(\mathcal{M}_{L^q}^{-1} \otimes L^{a+j-i}) = 0, \ \forall \ 1 \leq i \leq n.
\]
This together with the generalized Castelnuovo's Lemma (Lemma 1.2) imply that
\[ H^0(M_{pq}^{-1} \otimes L^{a+j}) \otimes H^0(L) \rightarrow H^0(M_{pq}^{-1} \otimes L^{a+j+1}) \] for all \( 0 \leq j \leq q - 1 \).

For \( j = 0, \ldots, q - 1 \), let \( \alpha_j \) be the surjective map
\[ H^0(M_{pq}^{-1} \otimes L^{a+j}) \otimes \cdots \otimes H^0(L) \xrightarrow{\alpha_j} H^0(M_{pq}^{-1} \otimes L^{a+j+1}) \otimes \cdots \otimes H^0(L). \]

Let \( \alpha \) be the composition map \( \alpha_{q-1} \circ \alpha_{q-2} \circ \cdots \circ \alpha_0 \), i.e.
\[ \alpha : H^0(M_{pq}^{-1} \otimes L^a) \otimes \cdots \otimes H^0(L) \rightarrow H^0(M_{pq}^{-1} \otimes L^{a+q}). \]

Then, \( \alpha \) can also be factored as \( \alpha = \gamma \circ \beta \), where
\[ \beta : H^0(M_{pq}^{-1} \otimes L^a) \otimes \cdots \otimes H^0(L) \rightarrow H^0(M_{pq}^{-1} \otimes L^a) \otimes H^0(L^a), \]

and
\[ \gamma : H^0(M_{pq}^{-1} \otimes L^a) \otimes H^0(L^a) \rightarrow H^0(M_{pq}^{-1} \otimes L^{a+q}). \]

Since \( \alpha \) is surjective, \( \gamma \) is also surjective. Thus, (2.3) holds, and the proposition is proved. \( \square \)

Our main theorem is the following.

**Theorem 2.2.** Suppose \( X \) is a projective variety of dimension \( n \), \( L \) is a very ample line bundle on \( X \). Then, there exists an integer \( b \) such that for any non-negative integer \( p \), \( L^d \) possesses property \( N_p \) for all \( d \geq pn + b \).

**Proof.** We choose \( b_0 \) as in Proposition 2.1. Let \( b = n + b_0 \). By Lemma 1.1, we only need to verify that
\[ H^1(\wedge^{a+1}M_{pd} \otimes L^{ad}) = 0 \forall a \geq 1 \text{ and } d \geq pn + b. \]

Since we are in characteristic 0, it suffices to show that
\[ H^1(M_{pd}^{a+1} \otimes L^{ad}) = 0 \forall a \geq 1 \text{ and } d \geq pn + b. \]

This is indeed true by Proposition 2.1 and the fact that \( ad \geq d \geq pn + b = (p+1)n + b_0 \). \( \square \)

As a corollary, we obtain a stronger result than that of Inamdar as follows.
Corollary 2.2.1 (see [6], Theorem 1.6). Suppose $X$ is a projective variety of dimension $n$, and $L$ is an ample line bundle on $X$. Then, there exists integers $c$ and $b$ such that for every non-negative integer $p$, $L^d$ has property $N_p$ for all $d \geq cp + b$.

Proof. Since $L$ is ample, there exists an integer $m_0$ such that $L^{m_0}$ is very ample for all $m \geq m_0$. Choose $c = m_0 n$ and replace $L$ by $L^{m_0}$, the result now follows from Theorem 2.2. □

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