Stochastic hydrodynamic-type evolution equations driven by Lévy noise in 3D unbounded domains - abstract framework and applications.

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Abstract

The existence of martingale solutions of the hydrodynamic-type equations in 3D possibly unbounded domains is proved. The construction of the solution is based on the Faedo-Galerkin approximation. To overcome the difficulty related to the lack of the compactness of Sobolev embeddings in the case of unbounded domain we use certain Fréchet space. We use also compactness and tightness criteria in some nonmetrizable spaces and a version of the Skorokhod Theorem in non-metric spaces. The general framework is applied to the stochastic Navier-Stokes, magneto-hydrodynamic (MHD) and the Boussinesq equations.

Keywords: Lévy noise, martingale solution, compactness method

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1. Introduction.

Let $O \subset \mathbb{R}^d$, $d = 2, 3$, be an open connected possibly unbounded subset with smooth boundary $\partial O$. Let $\mathbb{H} \subset L^2(O; \mathbb{R}^d)$ and $\mathbb{V} \subset H^1(O; \mathbb{R}^d)$, where $d \in \mathbb{N}$, be two Hilbert spaces such that $\mathbb{V} \subset \mathbb{H}$, the embedding being continuous. Here $H^1(O; \mathbb{R}^d)$ stands for the Sobolev space. We consider the following stochastic equation

\[
\begin{align*}
&u(t) + \int_0^t [\mathcal{A}u(s) + \mathcal{B}(u(s)) + \mathcal{R}u(s)] \, ds = u_0 + \int_0^t f(s) \, ds + \int_0^t \int_{\mathbb{V}} F(s, u(s^-); y) \eta(dy, ds) \\
&\quad + \int_0^t \int_{\mathbb{V}} G(s, u(s)) \, dW(s), \quad t \in (0, T). \tag{1.1}
\end{align*}
\]

In this equation, $\mathcal{A}$, $\mathcal{B}$, $\mathcal{R}$ are maps defined in the spaces $\mathbb{H}$ or $\mathbb{V}$, satisfying appropriate conditions (A.1), (B.1)-(B.5) and (R.1), respectively, formulated in Section 2. Moreover, $W$ stands for a cylindrical Wiener process on a separable Hilbert space and $\eta$ is a time-homogeneous Poisson random measure on a measurable space $(Y, \mathcal{F})$ with a $\sigma$-finite intensity measure $\mu$ and $Y_0 \in \mathcal{F}$ such that $\mu(Y \setminus Y_0) < \infty$. The processes $W$ and $\eta$ are assumed to be independent. For example, if $L = (L(t))_{t \geq 0}$ is a Lévy process in a Hilbert space $E$ and $\eta$ is the Poisson random measure corresponding to the process of jumps $(\Delta L(t))_{t \geq 0}$, where

\[\Delta L(t) := L(t) - L(t^-), \quad t \geq 0,\]

then we can put $Y_0 := \{x \in E : \|x\|_E < 1\}$. In this case the noise terms considered in equation (1.1) correspond to the Lévy-Itô decomposition of the process $L$, see e.g. [3] and [40]. We impose rather general assumptions (F.1)-(F.3) and (G.1)-(G.3) on the noise terms, see Section 2. We prove the existence of a martingale solution of equation (1.1) understood as a system $(\Omega, \mathcal{F}, \mathbb{P}, \eta, W, u)$, where $(\Omega, \mathcal{F}, \mathbb{P}, \mathbb{P})$ is a filtered probability and $u = (u(t))_{t \in [0, T]}$ is a stochastic process satisfying appropriate regularity properties and integral identity. The trajectories of the process $u$ are, in particular,
H-valued weakly càdlàg functions such that

\[ \mathbb{E} \left[ \sup_{t \in [0,T]} |u(t)|_H^2 + \int_0^T \|u(t)\|_{\dot{H}^1}^2 \, ds \right] < \infty. \tag{1.2} \]

The construction of a solution is based on the Faedo-Galerkin method, i.e.

\[
\begin{align*}
    u_n(t) &= P_n u_0 - \int_0^t \left[ P_n \mathcal{A} u_n(s) + \mathcal{B}_n(u_n(s)) + P_n \mathcal{R} u_n(s) - P_n f(s) \right] \, ds \\
    &\quad + \int_0^t \int_{\mathbb{R}^d} P_n F(s, u_n(s), y) \eta(ds, dy) \\
    &\quad + \int_0^t P_n G(s, u_n(s)) \, dW(s), \quad t \in [0,T].
\end{align*}
\]

We prove that the processes \((u_n(t))_{t \in [0,T]}\), satisfy the following uniform estimates

\[ \sup_{n \in \mathbb{N}} \mathbb{E} \left[ \sup_{t \in [0,T]} |u_n(t)|_H^p \right] < \infty \quad \text{and} \quad \sup_{n \in \mathbb{N}} \mathbb{E} \left[ \int_0^T \|u_n(t)\|_{\dot{H}^1}^2 \, dt \right] < \infty, \tag{1.3} \]

where \(p \in [1,2+\gamma] \) and \(\gamma > 0\) is a given parameter. For each \(n \in \mathbb{N}\), the process \(u_n\) generates a probability measure \(\mathcal{L}(u_n)\) on appropriate functional space. We prove that the set of laws \(\{\mathcal{L}(u_n), n \in \mathbb{N}\}\) is tight in the space \(\mathcal{Z}\), where

\[ \mathcal{Z} := L^2(0,T;\mathbb{V}) \cap L^2(0,T;L^2_{\text{loc}}(O)) \cap \mathcal{D}(0,T;\mathbb{U}) \cap \mathcal{D}(0,T;\mathbb{H}_u), \]

defined in Section 4. To this end use the compactness and tightness criteria in the space \(\mathcal{Z}\), see Lemma 4.1 and Corollary 4.2 in Section 4. They are counterparts for the present abstract setting of the corresponding criteria proved in [39]. To prove the tightness of \(\{\mathcal{L}(u_n), n \in \mathbb{N}\}\) we use estimates (1.3) with \(p = 2\). Next, we apply a version of the Skorokhod Embedding Theorem for non-metric spaces, see Appendix C, following easily from the Jakubowski’s version of the Skorokhod Theorem [29] and from the version due to Brzeźniak and Hausenblas [10]. At this stage we need estimates (1.3) with \(p > 2\).

The abstract approach is applied to the stochastic

- Navier-Stokes equations,
- magneto-hydrodynamic equations (MHD),
- Boussinesq equations

in the domain \(O\). In applications, the present approach allows to consider the multiplicative Gaussian noise term, represented by \(\int_0^t G(s,u(s)) \, dW(s)\), dependent both on the state \(u\) and their spatial derivatives \(\frac{\partial u}{\partial n}\), \(1 \leq i \leq d, d = 2, 3\). Presence of the derivatives \(\frac{\partial u}{\partial n}\) in the noise term is important in modelling the turbulence, see [38] and [39]. Assumptions (G.1)-(G.3) formulated in Section 2 cover the following example

\[
G(t,u(t)) \, dW(t) = \sum_{i=1}^{\infty} \left[(b_i(x) \cdot \nabla) u(t,x) + c_i(x) u(x)\right] dB_i(t),
\]

where \((B_i)_{i \in \mathbb{N}}\) are independent real-valued standard Wiener processes, see Section 8 in [13].

The present paper is a straightforward generalization of the results of [39], where the stochastic Navier-Stokes equations are considered. Here, we construct an abstract framework which covers also other hydrodynamic-type equations, e.g. stochastic magneto-hydrodynamic and Boussinesq equations. In comparison to [39] we consider more general Lévy noise term and additionally we prove estimates (1.2) on the solution of equation (1.1). Moreover, to construct a process \(u\) it is sufficient to use estimates (1.3) with \(p > 2\) (instead of \(p > 4\)).
The theory of the stochastic Navier-Stokes equations driven by Gaussian noise were developed in many papers, see e.g. [7], [8], [15], [17], [25], [16], [38], [37], [41], [42] and [13]. The noise term of Poissonian type is considered in the papers [20], [19], [21] and [12], and more general Lévy noise in [39] and [45]. We consider these equations because of their importance in other hydrodynamic models, e.g. magneto-hydrodynamic equations and Boussinesq equations.

The stochastic magneto-hydrodynamic equations driven by Gaussian noise in 2D domains were considered by Barbu and Da Prato [5] for additive noise term and Chueshov and Millet [18] for multiplicative noise term. In the papers by Sritharan and Sundar [46], and by Sango [43] the analysis of the existence of solutions in 2D and 3D bounded domains is provided. In [43] the noise term depends both on the velocity $u$ and the magnetic field $b$ but does not depend on their spatial derivatives. This follows from assumptions (24) and (25) in [43]. Here we will generalize these results to the case of unbounded domain when the Gaussian noise term depends on $u$, $b$ and their derivatives $\frac{\partial u}{\partial x_i}$, $\frac{\partial b}{\partial x_i}$, $1 \leq i \leq d$, of the velocity and the magnetic field. Moreover, we add also Poissonian type noise term of the form given in equation (1.1).

The Boussinesq equations has been studied by Foiaş, Manley and Temam [26] and Ghidaglia [27] in the deterministic case. The stochastic Boussinesq equations driven by Gaussian noise is considered by Duan and Millet [22], Ferrario [24] in 2D domains of the form $\mathbb{R} \times [0, 1]$. Martingale solutions in 2D and 3D domains of the form $\mathbb{R}^{d-1} \times [0, 1]$ with periodic boundary conditions in the directions $x_i$, $1 \leq i \leq d - 1$ were considered in [14]. In the present paper, we generalize the results to the cases of unbounded domain $O$. Moreover, we consider a general Lévy noise.

The present paper is split into two main parts. The first one, consisting of Sections 2-5, concerns the abstract framework. In Section 2 we formulate the problem and the general assumptions. The compactness and tightness criterion are contained in Section 4. Section 5 contains the proof of the main theorem on the existence of a martingale solutions. The second part (Section 6) is devoted to applications. Some auxilliary results are given in Appendices.
2. Statement of the problem

Let \( O \subset \mathbb{R}^d \) be an open connected possibly unbounded subset with smooth boundary \( \partial O \), where \( d = 2, 3 \). Let \( (\mathcal{H}, (\cdot, \cdot)_\mathcal{H}) \) and \( (\mathcal{V}, (\cdot, \cdot)_\mathcal{V}) \) be two Hilbert spaces such that

\[
\mathcal{H} \subset L^2(O; \mathbb{R}^d) \quad \text{and} \quad \mathcal{V} \subset H^1(O; \mathbb{R}^d),
\]

where \( \hat{d} \) is a positive integer, and the norms in \( \mathcal{H} \) and \( \mathcal{V} \) induced by the inner products, denoted by \( (\cdot, \cdot)_\mathcal{H} \) and \( (\cdot, \cdot)_\mathcal{V} \), are equivalent to the norms inherited from the spaces \( L^2(O; \mathbb{R}^d) \) and \( H^1(O; \mathbb{R}^d) \), respectively. We assume that \( \mathcal{V} \hookrightarrow \mathcal{H} \) the embedding being dense and continuous. Moreover, we assume that the inner product in the space \( \mathcal{V} \) is of the following form

\[
(u, v)_\mathcal{V} = (u, v)_\mathcal{H} + \langle u, v \rangle, \quad u, v \in \mathcal{V} \tag{2.1}
\]

Then the norm in \( \mathcal{V} \) is of the form

\[
\|u\|_\mathcal{V}^2 = \|u\|_\mathcal{H}^2 + \|u\|_\mathcal{V}^2, \quad u \in \mathcal{V},
\]

where \( \|u\|_\mathcal{V}^2 = \langle u, u \rangle \). Identifying \( \mathcal{H} \) with its dual \( \mathcal{H}' \), we have the following continuous embeddings

\[
\mathcal{V} \hookrightarrow \mathcal{H} \cong \mathcal{H}' \hookrightarrow \mathcal{V}'.
\]

The dual pairing between a Hilbert space \( X \) and its dual space \( X' \) will be denoted by \( \langle \cdot, \cdot \rangle_{X', X} \). If no confusion seems likely we omit the subscripts \( X', X \) and write \( \langle \cdot, \cdot \rangle \).

Let \( (O_R)_{R \in \mathbb{N}} \) be a sequence of open and bounded subsets of \( O \) with regular boundaries \( \partial O_R \) such that

\[
O_R \subset O_{R+1} \quad \text{and} \quad \bigcup_{R=1}^{\infty} O_R = O.
\]

We will use the space \( L^2(0, T; L^2_{\text{loc}}(O)) \) of measurable functions \( u : [0, T] \times O \to \mathbb{R}^d \) such that for all \( R \in \mathbb{N} \)

\[
p_{T,R}(u) := \left( \int_0^T \int_{O_R} |u(t, x)|^2 \, dx \, dt \right)^{\frac{1}{2}} < \infty,
\]

with the Fréchet topology generated by the sequence of seminorms \( (p_{T,R})_{R \in \mathbb{N}} \).

**Assumptions.** We assume that \( \mathcal{A}, \mathcal{B} \) and \( \mathcal{R} \) are maps satisfying the following conditions.

(A.1) \( \mathcal{A} : \mathcal{V} \to \mathcal{V}' \) is a linear map such that

\[
\langle \mathcal{A}u, v \rangle = (u, v)_\mathcal{V}, \quad u, v \in \mathcal{V}. \tag{2.3}
\]

(B.1) \( \mathcal{B} : \mathcal{V} \times \mathcal{V} \to \mathcal{V}' \) is a bilinear map and there exists a constant \( c_1 > 0 \) such that

\[
|\mathcal{B}(u, v)|_{\mathcal{V}'} \leq c_1 \|u\|_\mathcal{V} \|v\|_\mathcal{V}, \quad u, v \in \mathcal{V}. \tag{2.4}
\]

(B.2) \( \mathcal{B} \) satisfies the following condition

\[
\langle \mathcal{B}(u, v), w \rangle = -\langle \mathcal{B}(u, w), v \rangle, \quad u, v, w \in \mathcal{V}. \tag{2.5}
\]

We will also use the following notation \( \mathcal{B}(u) := \mathcal{B}(u, u) \).

(B.3) \( \mathcal{B} : \mathcal{V} \to \mathcal{V}' \) is locally Lipschitz continuous, i.e. for every \( r > 0 \) there exists a constant \( L_r \) such that

\[
|\mathcal{B}(u) - \mathcal{B}(\bar{u})|_{\mathcal{V}'} \leq L_r \|u - \bar{u}\|_{\mathcal{V}}, \quad u, \bar{u} \in \mathcal{V}, \quad \|u\|_{\mathcal{V}}, \|\bar{u}\|_{\mathcal{V}} \leq r.
\]

(B.4) There exist a separable Hilbert space \( \mathcal{V}_0 \subset \mathcal{V} \), the embedding being dense and continuous, such that \( \mathcal{B} \) can be extended to a bilinear map from \( \mathcal{H} \times \mathcal{H} \) into \( \mathcal{V}'_0 \). Moreover, there exists a constant \( c_2 > 0 \) such that

\[
|\mathcal{B}(u, w)|_{\mathcal{V}_0'} \leq c_2 \|u\|_{\mathcal{H}} \|w\|_{\mathcal{H}}, \quad u, w \in \mathcal{H}. \tag{2.6}
\]

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Moreover, we impose the following conditions on the random forces, the deterministic force

\begin{equation}
\text{(B.5)} \quad \text{for all } \varphi \in \mathbb{V}, \text{ the map } \tilde{B}_\varphi \text{ defined by }
\end{equation}

\begin{equation}
(\tilde{B}_\varphi(u))(t) := \langle B(u(t), \varphi), \quad u \in L^2(0, T; \mathbb{H}), \quad t \in [0, T]
\end{equation}

restricted to bounded subsets of \(L^2(0, T; \mathbb{H})\) is a continuous map into \(L^1([0, T]; \mathbb{R})\) if in the space \(L^2(0, T; \mathbb{H})\) we consider the topology inherited from the space \(L^2(0, T; L^2_{\text{loc}}(\mathcal{O})).\)

\begin{equation}
\text{(R.1)} \quad R : \mathbb{H} \to \mathbb{V}' \text{ is linear and continuous and there exists a constant } c_3 > 0 \text{ such that }
\end{equation}

\[-\langle Ru, u \rangle \leq c_3|u|_{\mathbb{H}}^2, \quad u \in \mathbb{V}.\]

**Remark 2.1.** Condition (B.5) is equivalent to the following one

- if \((u_n)\) is a sequence bounded in \(L^2(0, T; \mathbb{H})\) and \(u_n \to u\) in \(L^2(0, T; L^2_{\text{loc}}(\mathcal{O})),\) then for all \(\varphi \in \mathbb{V}:
\end{equation}

\[\lim_{n \to \infty} \int_0^T \langle B(u_n(s)) - B(u(s)), \varphi \rangle \, ds = 0.\]

Moreover, we impose the following conditions on the random forces, the deterministic force \(f\) and the initial state \(u_0\). We assume that

- if \((u_n)\) is a sequence bounded in \(L^2(0, T; \mathbb{H})\) and \(u_n \to u\) in \(L^2(0, T; L^2_{\text{loc}}(\mathcal{O})),\) then for all \(\varphi \in \mathbb{V}:
\end{equation}

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\end{equation}

\[\lim_{n \to \infty} \int_0^T \langle B(u_n(s)) - B(u(s)), \varphi \rangle \, ds = 0.\]

Moreover, for all \(\varphi \in \mathbb{H}\) the mapping \(\tilde{B}_\varphi\) defined by

\begin{equation}
(\tilde{B}_\varphi(u))(t, y) := (F(t, u(t^{-}) ; y), \varphi)_{\mathbb{H}}, \quad u \in L^2(0, T; \mathbb{H}), \quad (t, y) \in [0, T] \times Y
\end{equation}

is a continuous from \(L^2(0, T; \mathbb{H})\) into \(L^2([0, T] \times Y, df \otimes \mu; \mathbb{R})\) if in the space \(L^2(0, T; \mathbb{H})\) we consider the topology inherited from the space \(L^2(0, T; L^2_{\text{loc}}(\mathcal{O})).\)

**G.1** \(W(t)\) is a cylindrical Wiener process in a separable Hilbert space \(\mathbb{V}\). The process \(W\) is independent of \(\eta\).

**G.2** \(G : [0, T] \times \mathbb{V} \to L_{HS}(\mathbb{V}_w, \mathbb{H})\) and there exists a constant \(L_G > 0\) such that

\[\|G(t, u_1) - G(t, u_2)\|_{HS(\mathbb{V}_w, \mathbb{H})}^2 \leq L_G\|u_1 - u_2\|_\mathbb{V}^2, \quad u_1, u_2 \in \mathbb{V}, \quad t \in [0, T].\]

Moreover there exist \(\lambda, \kappa \in \mathbb{R}\) and \(\alpha \in \{2 - \frac{2}{\gamma + \gamma}, 2\}\) such that

\[2\langle Ru, u \rangle - |G(t, u)|_{LS(\mathbb{V}_w, \mathbb{H})}^2 \geq \alpha |u|^2 - \lambda |u|_{\mathbb{H}}^2 - \kappa, \quad u \in \mathbb{V}, \quad t \in [0, T].\]

1Here \(l\) denotes the Lebesgue measure on the interval \([0, T]\).
Moreover, $G$ extends to a continuous mapping $G : [0, T] \times \mathbb{H} \to \mathcal{L}_{HS}(Y_W, V')$ such that
\begin{equation}
\|G(t, u)\|_{\mathcal{L}_{HS}(V', V')}^2 \leq C(1 + |u|_{L^2}^2), \quad u \in \mathbb{H}.
\end{equation}
for some $C > 0$. Moreover, for every $\varphi \in V$ the map $G_\varphi$ defined by
\begin{equation}
(G_\varphi(u))(t) := \langle G(t, u(t)), \varphi \rangle, \quad u \in L^2(0, T; \mathbb{H}), \quad t \in [0, T]
\end{equation}
is a continuous mapping from $L^2(0, T; \mathbb{H})$ into $L^2([0, T]; \mathcal{L}_{HS}(Y_W, \mathbb{R}))$ if the space $L^2(0, T; \mathbb{H})$ we consider the topology inherited from the space $L^2(0, T; \mathcal{L}_{HS}(O))$.

For any Hilbert space $E$ the symbol $\mathcal{L}_{HS}(Y_W; E)$ denotes the space of Hilbert-Schmidt operators from $Y_W$ into $E$.

Let us consider the following stochastic equation
\begin{equation}
\begin{aligned}
u(t) + \int_0^t \left( A\nu(s) + B(u(s)) + R\nu(s) \right) ds &= u_0 + \int_0^t f(s) ds + \int_0^t \int_{Y_0} F(s, u(s); y) \eta(ds, dy) \\
&+ \int_0^t \int_{Y_0} G(s, u(s)) dW(s), \quad t \in (0, T).
\end{aligned}
\end{equation}

**Definition 2.2.** A martingale solution of equation (2.16) is a system $(\tilde{\mathcal{A}}, \tilde{\eta}, \tilde{W}, \tilde{u})$, where
- $\tilde{\mathcal{A}} := (\tilde{\Omega}, \tilde{\mathcal{F}}, \tilde{\mathbb{P}}, \tilde{\mu})$ is a filtered probability space with a filtration $\tilde{\mathcal{F}} = \{\tilde{\mathcal{F}}_t\}_{t \geq 0}$.
- $\tilde{\eta}$ is a time homogeneous Poisson random measure on $(Y, \mathcal{F})$ over $\tilde{\mathcal{A}}$ with the intensity measure $\mu$.
- $\tilde{W}$ is a cylindrical Wiener process on the space $Y_W$ over $\tilde{\mathcal{A}}$.
- $\tilde{u} : [0, T] \times \tilde{\Omega} \to \mathbb{H}$ is a predictable process with $\tilde{\mathbb{P}}$-a.e. paths $\tilde{u}(\cdot, \omega) \in \mathbb{D}([0, T], \mathbb{H}_w) \cap L^2(0, T; V)$

such that for all $t \in [0, T]$ and all $\varphi \in V$ the following identity holds $\tilde{\mathbb{P}}$-almost surely
\begin{equation}
\begin{aligned}
\langle \tilde{u}(t), \varphi \rangle_{\mathbb{H}_w} + \int_0^t \langle A\tilde{u}(s), \varphi \rangle ds + \int_0^t \langle B(\tilde{u}(s)), \varphi \rangle ds + \int_0^t \langle R\tilde{u}(s), \varphi \rangle ds \\
= \langle u_0, \varphi \rangle_{\mathbb{H}_w} + \int_0^t \langle f(s), \varphi \rangle ds + \int_0^t \int_{Y_0} \langle F(s, \tilde{u}(s); y), \varphi \rangle_{\mathbb{H}_w} \tilde{\eta}(ds, dy) \\
+ \int_0^t \int_{Y_0} \langle G(s, \tilde{u}(s)) \rangle_{\mathbb{H}_w} \tilde{\eta}(ds, dy) + \int_0^t \langle G(s, \tilde{u}(s)) \rangle_{\mathbb{H}_w} \tilde{\eta}(ds, dy).
\end{aligned}
\end{equation}
The symbol $\mathbb{H}_w$ denotes the Hilbert space $\mathbb{H}$ endowed with the weak topology and $\mathbb{D}([0, T]; \mathbb{H}_w)$ is the space of weakly càdlàg functions $u : [0, T] \to \mathbb{H}$. Recall that $u : [0, T] \to \mathbb{H}$ is weakly càdlàg if for every $h \in \mathbb{H}$ the real-valued function $[0, T] \ni t \mapsto (u(t), h)_{\mathbb{H}}$ is càdlàg.

The main result of the present paper is expressed in the following theorem.

**Theorem 2.3.** Let assumptions (A.1), (B.1)-(B.5), (R.1), (C.1), (F.1)-(F.3) and (G.1)-(G.3) be satisfied. Then there exists a martingale solution $(\tilde{\mathcal{A}}, \tilde{\eta}, \tilde{W}, \tilde{u})$ of problem (2.16) such that
\begin{equation}
\mathbb{E} \left[ \sup_{t \in [0, T]} \|\tilde{u}(t)\|^2_{\mathbb{H}_w} + \int_0^T \|\tilde{u}(t)\|^2_V ds \right] < \infty.
\end{equation}
Assumption (F.3) and second part of assumption (G.3) are important in the case of unbounded domain $O$. If $O$ is bounded, they can be omitted. Assumptions (G.2)-(G.3) allow to consider the Gaussian noise term dependent both on $u$ and $\partial_i u$, $i = 1, ..., d$. This corresponds to inequality (2.13) with $a < 2$. In the case when $a = 2$ the noise term $G$ depends on $u$ but not on its spatial derivatives.

The proof of Theorem 2.3 is based on the Faedo-Galerkin method. To this end we need appropriate orthonormal basis in the space $\mathbb{H}$. In the next section we recall a general approach used also in [13] and [39] in the case of Navier-Stokes equations.
3. Auxiliary results from functional analysis - space $\mathbb{U}$ and an orthonormal basis in $\mathbb{H}$

Let us recall that we have the following three separable Hilbert spaces such that

$$\mathbb{V}_* \subset \mathbb{V} \subset \mathbb{H},$$

the embedding being dense and continuous. Since $\mathbb{V}_*$ is a separable Hilbert space, there exists a Hilbert space $\mathbb{U}$ such that $\mathbb{U} \subset \mathbb{V}_*$, $\mathbb{U}$ is dense in $\mathbb{V}_*$ and the embedding

$$\mathbb{U} \hookrightarrow \mathbb{V}_*$$

is compact. In particular, $\mathbb{U}$ is compactly embedded into the space $\mathbb{H}$. Let us denote $\tilde{\mathbb{U}}$ and let $\iota^* : \mathbb{H} \to \mathbb{U}$ be its adjoint operator. Note that $\iota$ is compact and since the range of $\iota$ is dense in $\mathbb{H}$, $\iota^* : \mathbb{H} \to \mathbb{U}$ is one-to-one. Let us put $D(L) := \iota^*(\mathbb{H}) \subset \mathbb{U}$ and

$$Lu := (\iota^*)^{-1}u, \quad u \in D(L).$$

It is clear that $L : D(L) \to \mathbb{H}$ is onto. Let us also notice that

$$(Lu, w)_\mathbb{H} = (u, w)_\mathbb{U}, \quad u \in D(L), \quad w \in \mathbb{U}.$$  

By equality (3.4) and the densiness of $\mathbb{U}$ in $\mathbb{H}$, we infer that $D(L)$ is dense in $\mathbb{H}$.

Since $L$ is self-adjoint and $L^{-1}$ is compact, there exists an orthonormal basis $\{e_i\}_{i \in \mathbb{N}}$ of $\mathbb{H}$ composed of the eigenvectors of operator $L$. Let us fix $n \in \mathbb{N}$ and let $P_n$ be the operator from $\mathbb{U}'$ to $\text{span}[e_1, \ldots, e_n]$ defined by

$$P_n u^* := \sum_{i=1}^{n} (u^*|e_i)e_i, \quad u^* \in \mathbb{U}'$$

where $\langle \cdot, \cdot \rangle$ denotes the duality pairing between the space $\mathbb{U}$ and its dual $\mathbb{U}'$. Note that the restriction of $P_n$ to $\mathbb{H}$, denoted still by $P_n$, is given by

$$P_n u = \sum_{i=1}^{n} (u, e_i)e_i, \quad u \in \mathbb{H},$$

and thus it is the $(\cdot, \cdot)_\mathbb{H}$-orthogonal projection onto $\text{span}[e_1, \ldots, e_n]$. Restrictions of $P_n$ to other spaces considered in (3.1) will also be denoted by $P_n$. Moreover, it is easy to see that

$$(P_n u^*, v)_\mathbb{H} = \langle u^*, P_n v \rangle, \quad u^* \in \mathbb{U}' \quad v \in \mathbb{U}.$$  

Let us denote $\tilde{e}_i := \frac{e_i}{\|e_i\|_\mathbb{H}}$, $i \in \mathbb{N}$. The following lemma is a straightforward counterpart of Lemma 2.4 in [13] corresponding to our abstract setting.

**Lemma 3.1.**

(a) The system $\{\tilde{e}_i\}_{i \in \mathbb{N}}$ is the orthonormal basis in the space $(\mathbb{U}, \langle \cdot, \cdot \rangle_\mathbb{H})$.

(b) For every $n \in \mathbb{N}$ and $u \in \mathbb{U}$

$$P_n u = \sum_{i=1}^{n} (u, \tilde{e}_i)\tilde{e}_i,$$

i.e., the restriction of $P_n$ to $\mathbb{U}$ is the $(\cdot, \cdot)_\mathbb{H}$-projection onto the subspace $\text{span}[e_1, \ldots, e_n]$.

(c) For every $u \in \mathbb{U}$

(i) $\lim_{n \to \infty} \|P_n u - u\|_{\mathbb{U}} = 0$,

(ii) $\lim_{n \to \infty} \|P_n u - u\|_{\mathbb{V}_*} = 0$,

(iii) $\lim_{n \to \infty} \|P_n u - u\|_{\mathbb{V}} = 0$.

**Proof.** The proof is essentially the same as the proof of Lemma 2.4 in [13] and thus omitted. \hfill $\Box$
4. Compactness and tightness results

4.1. Deterministic compactness criterion

Let us recall that we have the following separable Hilbert spaces

\[ U \hookrightarrow V \hookrightarrow H \cong H' \hookrightarrow U', \]

where the embedding \( U \hookrightarrow V \) is dense and compact and the embedding \( V \hookrightarrow H \) is continuous. Let us consider the following functional spaces being the counterparts in our framework of the spaces used in [39], see also [35] and [36]:

- \( D([0, T], U') : = \) the space of càdlàg functions \( u : [0, T] \rightarrow U' \) with the topology \( T_1 \) induced by the Skorokhod metric,
- \( L^2_w(0, T; V) : = \) the space of measurable functions \( u : [0, T] \times O \rightarrow \mathbb{R}^d \) such that for all \( R \in \mathbb{N} \)
  \[ p_{T,R}(u) : = \left( \int_0^T \int_O |u(t, x)|^2 dxdt \right)^{\frac{1}{2}} < \infty, \]  
  \[ (4.1) \]
  with the topology \( T_3 \) generated by the seminorms \((p_{T,R})_{R \in \mathbb{N}}\).
- \( D([0, T]; U') : = \) the space of weakly càdlàg functions \( u : [0, T] \rightarrow H \) with the weakest topology \( T_4 \) such that for all \( h \in H \) the mappings \( D([0, T]; U') \ni u \mapsto (u(\cdot), h)_H \in D([0, T]; \mathbb{R}) \) (4.2) are continuous. In particular, \( u_n \rightarrow u \) in \( D([0, T]; U') \) iff for all \( h \in \mathbb{H} \) : \( (u_n(\cdot), h)_H \rightarrow (u(\cdot), h)_H \) in the space \( D([0, T]; \mathbb{R}) \).

We will use the following modification of [39, Theorem 2].

**Lemma 4.1.** Let

\[ Z := L^2_w(0, T; V) \cap L^2(0, T; L^2_{loc}(O)) \cap D([0, T]; U') \cap D([0, T]; U') \]  

and let \( T \) be the supremum of the corresponding topologies. Let

\[ K \subset L^\infty(0, T; \mathbb{H}) \cap L^2(0, T; V) \cap D([0, T]; \mathbb{H}) \]

satisfy the following three conditions

(a) for all \( u \in K \) and all \( t \in [0, T] \), \( u(t) \in \mathbb{H} \) and \( \sup_{t \in [0,T]} \sup_{x \in [0,T]} |u(x)|_H < \infty, \)

(b) \( \sup_{t \in [0,T]} \int_0^T \|u(s)|_V^2 ds < \infty \), i.e. \( K \) is bounded in \( L^2(0, T; V) \).

(c) \( \lim_{\delta \to 0} \sup_{t \in [0,T]} w_{[0,T],l^p}(u, \delta) = 0. \)

Then \( K \subset Z \) and \( K \) is \( T \)-relatively compact in \( Z \).

The proof of Lemma 4.1 is given in Appendix A.
4.2. Tightness criterion

Let \((\Omega, \mathcal{F}, \mathbb{P})\) be a probability space with filtration \(\mathcal{F} := (\mathcal{F}_t)_{t \in [0,T]}\) satisfying the usual hypotheses. Using Lemma 5.1 we get the corresponding tightness criterion in the measurable space \((\mathcal{Z}, \sigma(\mathcal{Z}))\), where \(\mathcal{Z}\) is defined by (4.3) and \(\sigma(\mathcal{Z})\) denotes the topological \(\sigma\)-field, see [39, Corollary 1].

**Corollary 4.2.** Let \((X_n)_{n \in \mathbb{N}}\) be a sequence of càdlàg \(\mathbb{F}\)-adapted \(\mathbb{U}'\)-valued processes such that

(a) there exists a positive constant \(C_1\) such that
\[
\sup_{n \in \mathbb{N}} \sup_{s \in [0,T]} |X_n(s)|_{\mathcal{H}} \leq C_1,
\]

(b) there exists a positive constant \(C_2\) such that
\[
\sup_{n \in \mathbb{N}} \mathbb{E} \left[ \int_0^T \|X_n(s)\|_{\mathcal{H}}^2 \, ds \right] \leq C_2.
\]

(c) \((X_n)_{n \in \mathbb{N}}\) satisfies the Aldous condition in \(\mathbb{U}'\).

Let \(\tilde{\mathbb{P}}_n\) be the law of \(X_n\) on \(\mathcal{Z}\). Then for every \(\varepsilon > 0\) there exists a compact subset \(K_\varepsilon\) of \(\mathcal{Z}\) such that
\[
\tilde{\mathbb{P}}_n(K_\varepsilon) \geq 1 - \varepsilon.
\]

Let us recall the Aldous condition in the form given by Métivier.

**Definition 4.3.** (M. Métivier) A sequence \((X_n)_{n \in \mathbb{N}}\) satisfies the **Aldous condition** in the space \(\mathbb{U}'\) iff
\[
\forall \varepsilon > 0 \ \forall \eta > 0 \ \exists \delta > 0 \text{ such that for every sequence } (\tau_n)_{n \in \mathbb{N}} \text{ of } \mathbb{F}\text{-stopping times with } \tau_n \leq T \text{ one has } \sup_{n \in \mathbb{N}} \sup_{0 \leq \theta \leq \delta} \mathbb{P}[|X_n(\tau_n + \theta) - X_n(\tau_n)|_{\mathbb{U}'} \geq \eta] \leq \varepsilon.
\]

5. Existence of solutions

5.1. Faedo-Galerkin approximation

Let \(\mathcal{U}\) be the space defined by (3.2). Let \(\{e_i\}_{i=1}^\infty\) be the orthonormal basis in \(\mathbb{H}\) composed of eigenvectors of the operator \(L\) defined by (3.3). In particular, \(\{e_i\}_{i=1}^\infty \subset \mathcal{U}\). Let \(\mathbb{H}_n := \text{span}\{e_1, \ldots, e_n\}\) be the subspace with the norm inherited from \(\mathbb{H}\) and let \(P_n\) be defined by (3.5). Consider the following map
\[
\mathcal{B}_n(u) := P_n \mathcal{B}(X_n(u), u), \quad u \in \mathbb{H}_n,
\]
where \(X_n : \mathbb{H} \to \mathbb{H}\) is defined by \(X_n(u) = \theta_n(|u|_{\mathcal{U}})u\), where \(\theta_n : \mathbb{R} \to [0, 1]\) of class \(C^\infty\) such that
\[
\theta_n(r) = 1 \quad \text{if} \quad r \leq n \quad \text{and} \quad \theta_n(r) = 0 \quad \text{if} \quad r \geq n + 1.
\]

Since \(\mathbb{H}_n \subset H, \mathcal{B}_n\) is well defined. Moreover, \(\mathcal{B}_n : \mathbb{H}_n \to \mathbb{H}_n\) is globally Lipschitz continuous. Let us consider the classical Faedo-Galerkin approximation in the space \(\mathbb{H}_n\)

\[
u_n(t) = P_n u_0 - \int_0^t \left[ P_n \mathcal{A} u_n(s) + \mathcal{B}_n(u_n(s)) + P_n \mathcal{R} u_n(s) - P_n f(s) \right] \, ds
+ \int_0^t \int_{\mathcal{Y}_0} P_n F(s, u_n(s^-), y) \eta(d\sigma, dy) + \int_0^t \int_{\mathcal{Y} \setminus \mathcal{Y}_0} P_n F(s, u_n(s^-), y) \eta(ds, dy)
+ \int_0^t P_n G(s, u_n(s)) \, dW(s), \quad t \in [0, T].
\]

**Lemma 5.1.** For each \(n \in \mathbb{N}\), there exists a unique \(\mathbb{F}\)-adapted, càdlàg \(\mathbb{H}_n\) valued process \(u_n\) satisfying the Galerkin equation (5.1).
Proof. The assertion follows from Theorem 9.1 in [28], see also [2].

In the following lemma we will prove uniform estimates of the solutions \( u_n \) of (5.1). Actually, these estimates hold provided the noise terms satisfy only condition (2.10) in assumption (F.2) and inequality (2.13) in assumption (G.2). The proof of the lemma is based on the Itô formula, see [28] or [34], and the Burkholder-Davis-Gundy inequality, see [40].

Lemma 5.2. The processes \((u_n)_{n \in \mathbb{N}}\) satisfy the following estimates.

(i) For every \( p \in [1, 2 + \gamma] \) there exists a positive constant \( C_1(p) \) such that

\[
\sup_{n \geq 1} \mathbb{E}\left( \sup_{0 \leq t \leq T} |u_n(s)|^p \right) \leq C_1(p). \tag{5.2}
\]

(ii) There exists a positive constant \( C_2 \) such that

\[
\sup_{n \geq 1} \mathbb{E}\left( \int_0^T |u_n(s)|^2 ds \right) \leq C_2. \tag{5.3}
\]

(Here \( \gamma > 0 \) is the constant defined in assumption (F.2).)

The proof of Lemma 5.2 is postponed to Appendix D.

The solutions \( u_n, n \in \mathbb{N} \), of the Galerkin equations define probability measures \( \mathcal{L}(u_n), n \in \mathbb{N} \), on the measurable space \((Z, \sigma(T))\), defined by (4.3) with the topological \( \sigma \)-field \( \sigma(T) \). Using Corollary 4.2 and Lemma 5.2 we will prove that the set of measures \( \{\mathcal{L}(u_n), n \in \mathbb{N}\} \) is tight on \((Z, \sigma(T))\). We use inequalities (5.3) and (5.2) with \( p = 2 \).

Lemma 5.3. The set of measures \( \{\mathcal{L}(u_n), n \in \mathbb{N}\} \) is tight on \((Z, \sigma(T))\).

The proof of Lemma 5.3 is given in Appendix D.

Further construction of a martingale solution of equation (4.16) is based on the Skorokhod Theorem for nonmetric spaces, see [10] and Appendix C of the present paper. This theorem guaranties, in particular, the existence of a sequence \((\bar{u}_n)\) of \( Z \)-valued stochastic processes such that \( \mathcal{L}(u_n) = \mathcal{L}(\bar{u}_n), n \in \mathbb{N} \), convergent almost surely to a limit process on a different probability space. The main difficulty occurs in passing to the limit in the nonlinear term, in the cases of unbounded domain. Here we need inequality (5.2) with \( p > 2 \), as well as Assumption (B.5). Actually, this is the only place, where we use (5.2) with \( p > 2 \). Let us mention that in the next section, devoted to applications, we will prove that the nonlinear terms appearing in the hydrodynamic-type equations satisfy Assumption (B.5). Similar problems occur in the noise terms, where Assumptions (F.3) and (G.3) are important. The method used in the following proof of Theorem 2.2 is closely related to the approach due to Brzeźniak and Hausenblas [10].

5.2. Proof of Theorem 2.2

We will apply the Skorokhod Theorem for the sequence of laws of \((u_n, \eta_n, W_n)\), where \( \eta_n := \eta \) and \( W_n := W, n \in \mathbb{N} \). Since by Lemma 5.3 the set of measures \( \{\mathcal{L}(u_n), n \in \mathbb{N}\} \) is tight on the space \( Z \), the set \( \{\mathcal{L}(u_n, \eta_n, W_n), n \in \mathbb{N}\} \) is tight on \( Z \times M_{\mathbb{S}}([0, T], Y) \times C([0, T]; Y) \). Here \( C([0, T]; Y) \) denotes the space of \( Y \)-valued continuous functions with the standard supremum-norm and \( M_{\mathbb{S}}([0, T], Y) \) is defined in Appendix B. By Corollary 7.1 and Remark 9.2, see Appendix C, there exists a subsequence \((n_k)_{k \in \mathbb{N}}\), a probability space \((\tilde{\Omega}, \tilde{\mathcal{F}}, \tilde{P})\) and, on this space, \( Z \times M_{\mathbb{S}}([0, T], Y) \times C([0, T]; Y) \)-valued random variables \((u_k, \eta_k, W_k), (\bar{u}_k, \bar{\eta}_k, \bar{W}_k), k \in \mathbb{N}\) such that

(i) \( \mathcal{L}((\bar{u}_k, \bar{\eta}_k, \bar{W}_k)) = \mathcal{L}(u_{n_k}, \eta_{n_k}, W_{n_k}) \) for all \( k \in \mathbb{N} \);
We will prove the following lemma.

Step 1.

Similarly, by inequality (5.5) with $\bar{\eta}_n(\tilde{\omega})$ and $(\bar{u}_n, \bar{\eta}_n, \bar{W}_n)_{n \in \mathbb{N}}$. Moreover, $\bar{\eta}_n, n \in \mathbb{N}$, and $\bar{\eta}_n$ are time homogeneous Poisson random measures on $(\mathbb{Y}, \mathcal{Y})$ with the intensity measure $\mu$ and $\bar{W}_n, n \in \mathbb{N}$, and $\bar{W}_n$ are cylindrical Wiener processes, see [10, Section 9]. By the definition of the space $\mathcal{Z}$, see (4.3), we have

$$\bar{u}_n \to u_\ast \text{ in } L^2_0(0, T; \mathcal{V}) \cap L^2(0, T; \mathcal{L}_2^{\infty}(O)) \cap D([0, T]; \mathbb{U}) \cap D([0, T]; \mathbb{Y}_n) \text{ } \tilde{\mathbb{P}}\text{-a.s.} \quad (5.4)$$

Since the random variables $\bar{u}_n$ and $u_\ast$ are identically distributed, we have the following inequalities. For every $p \in [1, 2 + \gamma]$

$$\sup_{n \geq 1} \mathbb{E}\left( \sup_{0 \leq s \leq T} |\bar{u}_n(s)|_p^p \right) \leq C_1(p). \quad (5.5)$$

and

$$\sup_{n \geq 1} \mathbb{E}\left( \int_0^T ||\bar{u}_n(s)||_2^2 \ ds \right) \leq C_2. \quad (5.6)$$

By inequality (5.6), there exists a subsequence of $(\bar{u}_n)$, still denoted by $(\bar{u}_n)$, convergent weakly in the space $L^2([0, T] \times \tilde{\Omega}; \mathcal{V})$. Since by (5.4) $\bar{u}_n \to u_\ast$ in $\mathcal{Z}$, we infer that $u_\ast \in L^2([0, T] \times \tilde{\Omega}; \mathcal{V})$, i.e.

$$\mathbb{E}\left[ \int_0^T ||u_\ast||_2^2 \ ds \right] < \infty. \quad (5.7)$$

Similarly, by inequality (5.5) with $p := 2$ we can choose a further subsequence of $(\bar{u}_n)$ convergent weak star in the space $L^2(\Omega; L^\infty(0, T; \mathbb{Y}))$, and using (5.4), deduce that

$$\mathbb{E}\left[ \sup_{t \in [0, T]} |u_\ast(t)|_{L_2}^2 \right] < \infty. \quad (5.8)$$

Step 1. Let us fix $\varphi \in \mathbb{U}$. Analogously to [10], let us denote

$$\Lambda_n(\bar{u}_n, \bar{\eta}_n, \bar{W}_n, \varphi)(t) := (\bar{u}_n(0), \varphi)_{\mathbb{Y}_n}$$

and

$$\Lambda(u_\ast, \eta_n, W_n, \varphi)(t) := (u_\ast(0), \varphi)_{\mathbb{Y}_n}$$

and

$$\Lambda(u_\ast, \eta_n, W_n, \varphi)(t) := (u_\ast(0), \varphi)_{\mathbb{Y}_n}$$

We will prove the following lemma.
Lemma 5.4. For all \( \varphi \in \mathbb{V} \)

(a) \( \lim_{n \to \infty} \mathbb{E}[\int_0^T |(\bar{u}_n(t) - u_*(t), \varphi)|^2 \, dt] = 0 \),

(b) \( \lim_{n \to \infty} \mathbb{E}[\|(\bar{u}_n(0) - u_*(0), \varphi)|^2] = 0 \),

(c) \( \lim_{n \to \infty} \mathbb{E}[\int_0^T \langle P_n \mathcal{A}\bar{u}_n(s) - \mathcal{A}u_*(s), \varphi \rangle \, ds \, dt] = 0 \),

(d) \( \lim_{n \to \infty} \mathbb{E}[\int_0^T \langle P_n B_n(\bar{u}_n(s)) - B(u_*(s)), \varphi \rangle \, ds \, dt] = 0 \),

(e) \( \lim_{n \to \infty} \mathbb{E}[\int_0^T \langle P_n \mathcal{A}\bar{u}_n(s) - \mathcal{A}u_*(s), \varphi \rangle \, ds \, dt = 0 \),

(f) \( \lim_{n \to \infty} \mathbb{E}[\int_0^T \int_0^t \langle P_n F(s, \bar{u}_n(s)) - F(s, u_*(s)), \varphi \rangle \, ds \, dt] = 0 \),

(g) \( \lim_{n \to \infty} \mathbb{E}[\int_0^T \int_0^t \langle P_n F(s, \bar{u}_n(s)) - F(s, u_*(s)), \varphi \rangle \, ds \, dy \, dt] = 0 \),

(h) \( \lim_{n \to \infty} \mathbb{E}[\int_0^T \langle \int_0^t \langle P_n G(s, \bar{u}_n(s)) - G(s, u_*(s)) \rangle \, dw_s(s, \varphi) \rangle \, dt] = 0 \).

Proof. Let us fix \( \varphi \in \mathbb{V} \). Ad. (a). Since by (5.4) \( \bar{u}_n \to u_* \) in \( \mathbb{D}([0, T]; \mathbb{H}_w) \), \( \mathbb{P} \)-a.s., \( (\bar{u}_n(\cdot), \varphi) \rightarrow (u_*(\cdot), \varphi) \in \mathbb{D}([0, T]; \mathbb{R}) \), \( \mathbb{P} \)-a.s. Hence, in particular for almost all \( t \in [0, T] \)

\[
\lim_{n \to \infty} (\bar{u}_n(t), \varphi)_{\mathbb{H}_1} = (u_*(t), \varphi)_{\mathbb{H}_1}, \quad \mathbb{P}\text{-a.s.}
\]

Since by (5.5), \( \sup_{t \in [0,T]} |\bar{u}_n(t)|^2 < \infty \), \( \mathbb{P} \)-a.s., using the Dominated Convergence Theorem we infer that

\[
\lim_{n \to \infty} \int_0^T |(\bar{u}_n(t) - u_*(t), \varphi)|^2 \, dt = 0 \quad \mathbb{P}\text{-a.s.}. \tag{5.11}
\]

Moreover, by the Hölder inequality and (5.5) for every \( n \in \mathbb{N} \) and every \( r \in (1, 1 + \frac{2}{T}] \)

\[
\mathbb{E}[\int_0^T |(\bar{u}_n(t) - u_*(t), \varphi)|^2 \, dt] \leq c \mathbb{E}\left[ \int_0^T (|\bar{u}_n(t)|^2 + |u_*(t)|^2) \, dt \right] \leq c C_1(2r), \tag{5.12}
\]

where \( c, \tilde{c} > 0 \) are some constants. The assertion (a) follows now from (5.11), (5.12) and the Vitali Theorem.

Ad. (b). Since by (5.4) \( \bar{u}_n \to u_* \) in \( \mathbb{D}([0, T]; \mathbb{H}_w) \), \( \mathbb{P} \)-a.s. and \( u_* \) is right-continuous at \( t = 0 \), we infer that \( (\bar{u}_n(0), \varphi) \rightarrow (u_*(0), \varphi)_{\mathbb{H}_1} \), \( \mathbb{P} \)-a.s. By (5.5) assertion (b) follows from the Vitali Theorem.

Ad. (c). By (5.4) \( \bar{u}_n \to u_* \) in \( L^2_{\omega}([0, T]; \mathbb{V}) \), \( \mathbb{P} \)-a.s. Moreover, since \( \varphi \in \mathbb{U} \), \( P_n \varphi \to \varphi \) in \( \mathbb{V} \), see Lemma 5.1(c) in Section 3. Thus by (2.3) we infer that \( \mathbb{P} \)-a.s.

\[
\lim_{n \to \infty} \int_0^T \langle P_n \mathcal{A}\bar{u}_n(s), \varphi \rangle \, ds = \lim_{n \to \infty} \int_0^T \langle \bar{u}_n(s), P_n \varphi \rangle \, ds = \int_0^T \langle u_*(s), \varphi \rangle \, ds = \int_0^T \langle u_*(s), \varphi \rangle \, ds \tag{5.13}
\]

By (2.3), the Hölder inequality and (5.6) we have the following inequality for all \( t \in [0, T] \) and \( n \in \mathbb{N} \)

\[
\mathbb{E}[\int_0^T (\langle P_n \mathcal{A}\bar{u}_n(s), \varphi \rangle \, ds)^2] \leq c \mathbb{E}\left[ \int_0^T |\bar{u}_n(s)|^2 \, ds \right] \leq c C_2 \tag{5.14}
\]

for some constants \( c, \tilde{c} > 0 \). By (5.13), (5.14) and the Vitali Theorem we infer that for all \( t \in [0, T] \)

\[
\lim_{n \to \infty} \mathbb{E}[\int_0^T \langle P_n \mathcal{A}\bar{u}_n(s) - \mathcal{A}u_*(s), \varphi \rangle \, ds] = 0.
\]

Hence assertion (c) follows from (5.6) and the Dominated Convergence Theorem.
Ad. (d). Let us move to the nonlinear term. Here assumption (B.5) will be very important. Since by (5.6) and (2.3) the sequence $\bar{u}_n$ is bounded in $L^2(0,T;\mathbb{H})$ and by (5.4) $\bar{u}_n \to u_*$ in $L^2(0,T;L^2_{loc}(O))$, $\tilde{\mathbb{P}}$-a.s., by assumption (B.5) (see Remark 2.1) we infer that $\tilde{\mathbb{P}}$-a.s. for all $t \in [0,T]$ and all $\varphi \in \mathcal{V}_*$

$$\lim_{n \to \infty} \int_0^t \langle \mathcal{B}(\bar{u}_n(s)) - \mathcal{B}(u_*(s)), \varphi \rangle \, ds = 0.$$ 

It is easy to see that for sufficiently large $n \in \mathbb{N}$, $\mathcal{B}_n(\bar{u}_n(s)) = P_n \mathcal{B}(\bar{u}_n(s)), s \in [0,T)$. Moreover, if $\varphi \in \mathcal{U}$ then by Lemma 5.1 (c), $P_n \varphi \to \varphi$ in $\mathcal{V}_*$. Since $\mathcal{U} \subset \mathcal{V}_*$, we infer that for all $\varphi \in \mathcal{U}$ and all $t \in [0,T]$

$$\lim_{n \to \infty} \int_0^t \langle \mathcal{B}_n(\bar{u}_n(s)) - \mathcal{B}(u_*(s)), \varphi \rangle \, ds = 0 \quad \tilde{\mathbb{P}}\text{-a.s.} \tag{5.15}$$

By the Hölder inequality, (2.6) and (5.5) we obtain for all $t \in [0,T], r \in (0,\frac{1}{2})$ and $n \in \mathbb{N}$

\begin{align*}
\mathbb{E} \left[ \left( \int_0^t |\mathcal{B}_n(\bar{u}_n(s)), \varphi | \, ds \right)^{2r} \right] & \leq \mathbb{E} \left[ \int_0^t \| \mathcal{B}_n(\bar{u}_n(s)) \|_{\mathcal{V}_*} \| \varphi \|_{\mathcal{V}_*} \, ds \right]^{2r} \\
& \leq (c_2 |\varphi|_{\mathcal{V}_*})^{2r} \mathbb{E} \left[ \int_0^t |\bar{u}_n(s)|_{\mathcal{H}_*}^{2+2r} \, ds \right] \\
& \leq c_2 \mathbb{E} \left[ \sup_{s \in [0,T]} |\bar{u}_n(s)|_{\mathcal{H}_*}^{2+2r} \right] \leq C_C(2 + 2r). \tag{5.16}
\end{align*}

In view of (5.15) and (5.16), by the Vitali Theorem we obtain for all $t \in [0,T]$

$$\lim_{n \to \infty} \mathbb{E} \left[ \left( \int_0^t |\mathcal{B}_n(\bar{u}_n(s)) - \mathcal{B}(u_*(s)), \varphi | \, ds \right)^{2r} \right] = 0 \tag{5.17}$$

Since by (5.5) for all $t \in [0,T]$ and all $n \in \mathbb{N}$

$$\mathbb{E} \left[ \left( \int_0^t |\mathcal{B}_n(\bar{u}_n(s)), \varphi | \, ds \right)^{2r} \right] \leq c \mathbb{E} \left[ \sup_{s \in [0,T]} |\bar{u}_n(s)|_{\mathcal{H}_*}^{2+2r} \right] \leq c C_C(2),$$

where $c > 0$ is a certain constant, by (5.17) and the Dominated Convergence Theorem, we infer that assertion (d) holds.

Ad. (e). Since by (5.4) $\bar{u}_n \to u_*$ in $L^2(0,T;\mathcal{V})$ and the embedding $\mathcal{V} \subset \mathcal{H}$ is continuous, $\bar{u}_n \to u_*$ in $L^2(0,T;\mathcal{H})$, $\tilde{\mathbb{P}}$-a.s. Furthermore, since $\mathcal{R} : \mathcal{H} \to \mathcal{V}$ is linear and continuous, $\mathcal{R}\bar{u}_n \to \mathcal{R}u_*$ in $L^2(0,T;\mathcal{V}_*)$, $\tilde{\mathbb{P}}$-a.s. Since moreover by Lemma 3.1 (c) $P_n \varphi \to \varphi$ in $\mathcal{V}_*$, we infer that

$$\lim_{n \to \infty} \int_0^t \langle P_n \mathcal{R}\bar{u}_n(s), \varphi \rangle \, ds = \lim_{n \to \infty} \int_0^t \langle \mathcal{R}\bar{u}_n(s), P_n \varphi \rangle \, ds = \int_0^t \langle \mathcal{R}u_*(s), \varphi \rangle \, ds \quad \tilde{\mathbb{P}}\text{-a.s.}$$

By assumption (R.1), Lemma 3.1 (c) and (5.5) we have the following inequalities for all $r \in (0,\gamma), t \in [0,T]$ and $n \in \mathbb{N}$

\begin{align*}
\mathbb{E} \left[ \left( \int_0^t |P_n \mathcal{R}\bar{u}_n(s), \varphi | \, ds \right)^{2+2r} \right] & = \mathbb{E} \left[ \int_0^t |\bar{u}_n(s), P_n \varphi | \, ds \right]^{2+2r} \\
& \leq c |\varphi|_{\mathcal{V}_*}^{2+2r} \mathbb{E} \left[ \sup_{s \in [0,T]} |\bar{u}_n(s)|_{\mathcal{H}_*}^{2+2r} \right] \leq \tilde{c} C_C(2 + r) \tag{5.18}
\end{align*}

for some constants $c, \tilde{c} > 0$. Therefore by (5.18), (5.18) and the Vitali Theorem we infer that for all $t \in [0,T]$

$$\lim_{n \to \infty} \mathbb{E} \left[ \left( \int_0^t |P_n \mathcal{R}\bar{u}_n(s) - \mathcal{R}u_*(s), \varphi | \, ds \right)^{2} \right] = 0.$$ 

Hence assertion (d) follows from (5.5) and the Dominated Convergence Theorem.

Ad. (f). Assume that $\varphi \in \mathcal{H}$. For all $t \in [0,T]$ we have

$$\int_0^t \int_Y |(F(s,\bar{u}_n(s));y) - F(s,u_*(s));y)|^2 \, d\mu(y) \, ds = \int_0^t \int_Y |\bar{F}_\varphi(\bar{u}_n)(s,y) - \bar{F}_\varphi(u_*)(s,y)|^2 \, d\mu(y) \, ds \leq \|\bar{F}_\varphi(\bar{u}_n) - \bar{F}_\varphi(u_*)\|_{L^2([0,T];\mathcal{Y};\mathcal{R})},$$
where $\tilde{F}_x$ is the mapping defined by (2.11). Since by (5.4) $\tilde{u}_n \to u_s$ in $L^2(0,T; L^2_{\text{loc}}(O))$, $\bar{P}$-a.s., by assumption (F.3) we infer that for all $t \in [0,T]$

$$\lim_{n \to \infty} \int_0^t \| (F(s, \tilde{u}_n(s^-); y) - F(s, u_s(s^-); y), \varphi) \|_{2Y}^2 \, d\mu(y) ds = 0. \quad (5.19)$$

Moreover, by inequality (5.10) in assumption (F.2) and by (5.5) for every $t \in [0,T]$ every $r \in (1, 1 + \frac{1}{2})$ and every $n \in \mathbb{N}$ the following inequalities hold

$$\mathbb{E} \left[ \| \int_0^t \int_Y (F(s, \tilde{u}_n(s^-); y) - F(s, u_s(s^-); y), \varphi) \|_{2Y}^2 \, d\mu(y) ds \right]$$

$$\leq 2' \| \varphi \|_{2Y}^2 \mathbb{E} \left[ \| \int_0^t \left( |F(s, \tilde{u}_n(s^-); y)|_{2Y}^2 + |F(s, u_s(s^-); y)|_{2Y}^2 \right) d\mu(y) ds \right]$$

$$\leq 2'C_2' \| \varphi \|_{2Y}^2 \mathbb{E} \left[ \| \int_0^t \{ 2 |\tilde{u}_n(s)\|_{2Y}^2 + |u_s(s)\|_{2Y}^2 \} d\mu(s) ds \right] \leq c(1 + \mathbb{E} \left[ \sup_{s \in [0,T]} |\tilde{u}_n(s)|_{2Y}^2 \right])$$

$$\leq c(1 + C_1(2r)) \quad (5.20)$$

for some constant $c > 0$. Thus by (5.19), (5.20) and the Vitali Theorem for all $t \in [0,T]$

$$\lim_{n \to \infty} \mathbb{E} \left[ \| \int_0^t \int_Y (F(s, \tilde{u}_n(s^-); y) - F(s, u_s(s^-); y), \varphi) \|_{2Y}^2 \, d\mu(y) ds \right] = 0, \quad \varphi \in \mathbb{H}. \quad (5.21)$$

Moreover, since the restriction of $P_n$ to the space $\mathbb{H}$ is the $(\cdot, \cdot)_{2Y}$-projection onto $\mathbb{H}_s$, see Section 3, we infer that also

$$\lim_{n \to \infty} \mathbb{E} \left[ \int_0^t \int_Y (P_n F(s, \tilde{u}_n(s^-); y) - F(s, u_s(s^-); y), \varphi) \tilde{\eta}_n(ds, dy) \right] = 0, \quad \varphi \in \mathbb{H}. \quad (5.22)$$

Since $\mathbb{U} \subset \mathbb{H}$, (5.22) holds for $\varphi \in \mathbb{U}$.

**Ad. (g).** By (5.22) and by the properties of the integral with respect to the compensated Poisson random measure and the fact that $\tilde{\eta}_n = \eta_s$, we have

$$\lim_{n \to \infty} \mathbb{E} \left[ \int_0^t \int_Y (P_n F(s, \tilde{u}_n(s^-); y) - F(s, u_s(s^-); y), \varphi) \tilde{\eta}_n(ds, dy) \right]^2 = 0. \quad (5.23)$$

Moreover, by inequality (5.20) with $r := 1$ we obtain the following inequality

$$\mathbb{E} \left[ \left| \int_0^t \int_Y (P_n F(s, \tilde{u}_n(s^-); y) - F(s, u_s(s^-); y), \varphi) \tilde{\eta}_n(ds, dy) \right|^2 \right]$$

$$= \mathbb{E} \left[ \left| \int_0^t \int_Y (P_n F(s, \tilde{u}_n(s^-); y) - F(s, u_s(s^-); y), \varphi) \tilde{\eta}(ds, dy) \right|^2 \right] \leq c(1 + C_1(2)). \quad (5.24)$$

Now assertion (g) follows from (5.23), (5.24) and the Dominated Convergence Theorem.

**Ad. (h).** Let us assume that $\varphi \in \mathbb{V}$. We have

$$\int_0^t \| (G(s, \tilde{u}_n(s)) - G(s, u_s(s)), \varphi) \|^2_{L^2_t(Y \times \mathbb{R})} ds$$

$$= \int_0^t \| \hat{G}_\varphi(\tilde{u}_n(s)) - \hat{G}_\varphi(u_s(s)) \|^2_{L^2_t(Y \times \mathbb{R})} ds \leq \| \hat{G}_\varphi(\tilde{u}_n) - \hat{G}_\varphi(u_s) \|^2_{L^2_t(Y \times \mathbb{R})},$$

where $\hat{G}_\varphi$ is the mapping defined by (2.15). Since by (5.4) $\tilde{u}_n \to u_s$ in $L^2(0,T; L^2_{\text{loc}}(O))$, $\bar{P}$-a.s., by the second part of assumption (G.3) we infer that for all $t \in [0,T]$ and all $\varphi \in \mathbb{V}$

$$\lim_{n \to \infty} \int_0^t \| (G(s, \tilde{u}_n(s)) - G(s, u_s(s)), \varphi) \|^2_{L^2_t(Y \times \mathbb{R})} ds = 0. \quad (5.25)$$
Moreover, by (2.14) and (5.5) we see that for every $t \in [0, T]$ every $r \in (1, 1 + \frac{1}{2})$ and every $n \in \mathbb{N}$

\[
\mathbb{E} \left[ \left\| \int_0^t \left( G(s, \tilde{u}_n(s)) - G(s, u_(s), \varphi) \right) dW_s \right\|^2 \right] \\
\leq c_1 \mathbb{E} \left[ \left\| \varphi \right\|^2_{L^2} \cdot \int_0^t \left\| \left( G(s, \tilde{u}_n(s)) - G(s, u(s), \varphi) \right) \right\|^2_{L^2} ds \right] \\
\leq c_1 \mathbb{E} \left[ \int_0^T \left( 1 + \left\| \tilde{u}_n(s) \right\|^2_{L^2} + \left\| u(s) \right\|^2_{L^2} \right) ds \right] \\
\leq c_1 \left[ 1 + \mathbb{E} \left[ \sup_{s \in [0, T]} \left\| \tilde{u}_n(s) \right\|^2_{L^2} + \sup_{s \in [0, T]} \left\| u(s) \right\|^2_{L^2} \right] \right] \leq c(1 + 2C_1(2r)) \tag{5.26}
\]

for some positive constants $c, c_1, \tilde{c}$. Thus by (5.25), (5.26) and the Vitali Theorem, we infer that

\[
\lim_{n \to \infty} \mathbb{E} \left[ \int_0^T \left\| \left( G(s, \tilde{u}_n(s)) - G(s, u(s), \varphi) \right) \right\|^2_{L^2} ds \right] = 0 \quad \text{for all } \varphi \in \mathbb{V}. \tag{5.27}
\]

For every $\varphi \in \mathbb{V}$ and every $s \in [0, T]$ we have

\[
\langle P_n G(s, \tilde{u}_n(s)) - G(s, u(s), \varphi), \varphi \rangle = \langle G(s, \tilde{u}_n(s)), P_n \varphi \rangle - \langle G(s, u(s), \varphi) \rangle \\
= \langle G(s, \tilde{u}_n(s)), P_n \varphi \rangle - \langle G(s, \tilde{u}_n(s)) \rangle + \langle G(s, u(s), \varphi) \rangle \\
\leq \left\| G(s, \tilde{u}_n(s)) \right\|_{L^2(Y_s, \mathcal{F}_s; \mathbb{R}^2)} \left\| P_n \varphi - \varphi \right\|_{L^2(Y_s, \mathcal{F}_s; \mathbb{R}^2)} + \langle G(s, u(s), \varphi) \rangle.
\]

Thus by inequality (2.14) in assumption (G.3) and by (5.5) we obtain

\[
\mathbb{E} \left[ \int_0^T \left\| \left( P_n G(s, \tilde{u}_n(s)) - G(s, u(s), \varphi) \right) \right\|^2_{L^2(Y_s, \mathcal{F}_s; \mathbb{R}^2)} ds \right] \\
\leq 2C \left\| P_n \varphi - \varphi \right\|_{L^2(Y_s, \mathcal{F}_s; \mathbb{R}^2)} \mathbb{E} \left[ \int_0^T \left( 1 + \left\| \tilde{u}_n(s) \right\|^2_{L^2} + \left\| u(s) \right\|^2_{L^2} \right) ds \right] \\
\leq 2CT(1 + C_1(2)) \left\| P_n \varphi - \varphi \right\|_{L^2(Y_s, \mathcal{F}_s; \mathbb{R}^2)} + 2 \mathbb{E} \left[ \int_0^T \left\| \left( G(s, \tilde{u}_n(s)) - G(s, u(s), \varphi) \right) \right\|^2_{L^2(Y_s, \mathcal{F}_s; \mathbb{R}^2)} ds \right].
\]

Since $\mathbb{U} \subset \mathbb{V}$ and by Lemma (5.4)(a), $\left\| P_n \varphi - \varphi \right\|_{Y_s} \to 0$ for all $\varphi \in \mathbb{U}$, by (5.27) we infer that

\[
\lim_{n \to \infty} \mathbb{E} \left[ \int_0^T \left\| \left( P_n G(s, \tilde{u}_n(s)) - G(s, u(s), \varphi) \right) \right\|^2_{L^2(Y_s, \mathcal{F}_s; \mathbb{R}^2)} ds \right] = 0 \quad \text{for all } \varphi \in \mathbb{U}.
\]

Hence by the properties of the Itô integral we infer that for all $t \in [0, T]$ and all $\varphi \in \mathbb{U}$

\[
\lim_{n \to \infty} \mathbb{E} \left[ \left\| \int_0^t \left( P_n G(s, \tilde{u}_n(s)) - G(s, u(s), \varphi) \right) dW_s \right\|^2 \right] = 0. \tag{5.28}
\]

Moreover, by the Itô isometry, inequality (2.14) in assumption (G.3), and (5.5) we have for all $t \in [0, T]$ and all $n \in \mathbb{N}$

\[
\mathbb{E} \left[ \left\| \left( \int_0^t \left( P_n G(s, \tilde{u}_n(s)) - G(s, u(s), \varphi) \right) dW_s \right) \right\|^2 \right] \\
= \mathbb{E} \left[ \left\| \int_0^t \left( P_n G(s, \tilde{u}_n(s)) - G(s, u(s), \varphi) \right) \right\|^2_{L^2(Y_s, \mathcal{F}_s; \mathbb{R}^2)} ds \right] \\
\leq c \left[ 1 + \mathbb{E} \left[ \sup_{t \in [0, T]} \left\| \tilde{u}_n(s) \right\|^2_{L^2} + \sup_{t \in [0, T]} \left\| u(s) \right\|^2_{L^2} \right] \right] \leq c(1 + 2C_1(2)) \tag{5.29}
\]

for some $c > 0$. By (5.28), (5.29) and the Dominated Convergence Theorem we infer that

\[
\lim_{n \to \infty} \int_0^T \mathbb{E} \left[ \left\| \left( \int_0^t \left( P_n G(s, \tilde{u}_n(s)) - G(s, u(s), \varphi) \right) dW_s \right) \right\|^2 \right] = 0. \tag{5.30}
\]

This completes the proof of Lemma 5.4 \( \square \)

As a direct consequence of Lemma 5.4 we get the following corollary

\[ \quad \]
Corollary 5.5. We have

\[ \lim_{n \to \infty} \| (\bar{u}_n(\cdot), \varphi)_{\mathbb{H}} - (u_\ast(\cdot), \varphi)_{\mathbb{H}} \|_{L^2([0, T] \times \Omega)} = 0 \]  

(5.31)

and

\[ \lim_{n \to \infty} \| \Lambda_n(\bar{u}_n, \bar{\eta}_n, W_n, \varphi) - \Lambda(u_\ast, \eta_\ast, W_\ast, \varphi) \|_{L^1([0, T] \times \Omega)} = 0. \]  

(5.32)

Proof. Assertion (5.31) follows from the equality

\[ \| (\bar{u}_n(\cdot), \varphi)_{\mathbb{H}} - (u_\ast(\cdot), \varphi)_{\mathbb{H}} \|_{L^2([0, T] \times \Omega)}^2 = \mathbb{E} \left[ \int_0^T |(\bar{u}_n(t) - u_\ast(t), \varphi)_{\mathbb{H}}|^2 \, dt \right] \]

and Lemma 5.4 (a). Let us move to (5.32). Note that by the Fubini Theorem, we have

\[ \| \Lambda_n(\bar{u}_n, \bar{\eta}_n, W_n, \varphi) - \Lambda(u_\ast, \eta_\ast, W_\ast, \varphi) \|_{L^1([0, T] \times \Omega)} \]

= \int_0^T \mathbb{E} \left[ |\Lambda_n(\bar{u}_n, \bar{\eta}_n, W_n, \varphi(t)) - \Lambda(u_\ast, \eta_\ast, W_\ast, \varphi(t))| \right] dt.

To prove (5.32) it is sufficient to note that by Lemma 5.4 (b)-(g), each term on the right hand side of (5.32) tends at least in \( L^1([0, T] \times \Omega) \) to the corresponding term in (5.10). \( \square \)

Step 2. Since \( u_n \) is a solution of the Galerkin equation, for all \( t \in [0, T] \)

\[ (u_n(t), \varphi)_{\mathbb{H}} = \Lambda_n(u_n, \eta_n, W_n, \varphi)(t), \quad \mathbb{P}\text{-a.s.} \]

In particular,

\[ \int_0^T \mathbb{E} |(u_n(t), \varphi)_{\mathbb{H}} - \Lambda_n(u_n, \eta_n, W_n, \varphi)(t)| \, dt = 0. \]

Since \( \mathcal{L}(u_n, \eta_n, W_n) = \mathcal{L}(\bar{u}_n, \bar{\eta}_n, W_n) \),

\[ \int_0^T \mathbb{E} |(\bar{u}_n(t), \varphi)_{\mathbb{H}} - \Lambda_n(\bar{u}_n, \bar{\eta}_n, W_n, \varphi)(t)| \, dt = 0. \]

Moreover, by (5.31) and (5.32)

\[ \int_0^T \mathbb{E} |(u_\ast(t), \varphi)_{\mathbb{H}} - \Lambda(u_\ast, \eta_\ast, W_\ast, \varphi)(t)| \, dt = 0. \]

Hence for \( l \)-almost all \( t \in [0, T] \) and \( \hat{\mathbb{P}} \)-almost all \( \omega \in \hat{\Omega} \)

\[ (u_\ast(t), \varphi)_{\mathbb{H}} - \Lambda(u_\ast, \eta_\ast, W_\ast, \varphi)(t) = 0, \]

i.e. for \( l \)-almost all \( t \in [0, T] \) and \( \hat{\mathbb{P}} \)-almost all \( \omega \in \hat{\Omega} \)

\[ (u_\ast(t), \varphi)_{\mathbb{H}} + \int_0^t \langle A u_\ast(s), \varphi \rangle \, ds + \int_0^t \langle B(u_\ast(s)), \varphi \rangle \, ds + \int_0^t \langle R u_\ast(s), \varphi \rangle \, ds \]

= \( (u_\ast(0), \varphi)_{\mathbb{H}} + \int_0^t \langle f(s), \varphi \rangle \, ds + \int_0^t \int_{Y_0} (F(s, u(s), y), \varphi)_{\mathbb{H}} \, d\eta(s) \, dy \)

+ \int_0^t \int_{Y_0} (F(s, u_\ast(s), y), \varphi)_{\mathbb{H}} \, d\eta(s) \, dy + \int_0^t \int_{Y_0} \psi(s, u(s), y) \, dW(s, y). \]  

(5.33)

Since \( u_\ast \) is \( \mathcal{Z} \)-valued random variable, in particular \( u_\ast \in \mathcal{D}([0, T]; \mathbb{H}_\ast) \), i.e. \( u_\ast \) is weakly càdlàg, we infer that equality (5.33) holds for all \( t \in [0, T] \) and all \( \varphi \in \mathcal{U} \). Since \( \mathcal{U} \) is dense in \( \mathcal{V} \), equality (5.33) holds for all \( \varphi \in \mathcal{V} \), as well. Putting \( \hat{\mathbb{A}} := (\hat{\Omega}, \hat{\mathcal{F}}, \hat{\mathbb{P}}, \hat{\eta}) := (\eta_\ast, \hat{W} := W_\ast, \hat{u} := u_\ast \), we infer that the system \((\hat{\Omega}, \hat{\eta}, \hat{W}, \hat{u})\) is a martingale solution of the equation (2.16). By (5.7) and (5.8) the process \( \hat{u} \) satisfies inequality (2.17). The proof of Theorem 2.3 is thus complete. \( \square \)
6. Applications

In this section $O$ is an open connected possibly unbounded subset of $\mathbb{R}^d$, $d = 2, 3$, with smooth boundary $\partial O$.

6.1. Stochastic Navier-Stokes equations

Let us consider the stochastic Navier-Stokes equations

$$
\begin{align*}
\dot{u}(t) + (u \cdot \nabla)u - \frac{1}{Re} \Delta u + \nabla p &= f(t) + \int_{Y_0} F(t, u(t^+); y) \eta(dt, dy), \\
&+ \int_{Y \times Y_0} F(t, u(t^+); y) \eta(dt, dy) + G(t, u(t)) \, dW(t)
\end{align*}
$$

(6.1)

in $[0, T] \times O$, with the incompressibility condition

$$
\text{div} u = 0,
$$

(6.2)

the homogeneous boundary condition

$$
u_{\partial O} = 0,
$$

(6.3)

and the initial condition $u(0) = u_0$. In this problem $u = u(t, x) = (u_1(t, x), \ldots, u_d(t, x))$ and $p = p(t, x)$ represent the velocity and the pressure of the fluid. In equation (6.1) $Re > 0$ is the Reynolds number related to the kinematic viscosity (we may put $Re := 1$). Furthermore, $f$ stands for the deterministic external forces, $G(t, u) \, dW(t)$, where $W$ is a cylindrical Wiener process on a Hilbert space $Y_W$, $\int_{Y_0} F(t, u(t^+); y) \eta(dt, dy)$ and $\int_{Y \times Y_0} F(t, u(t^+); y) \eta(dt, dy)$, where $\eta$ is a time-homogeneous Poisson random measure on a measurable space $(Y, \mathcal{Y})$ and $Y_0 \in \mathcal{Y}$ is such that $\mu(Y \setminus Y_0) < \infty$, stands for the random forces. Processes $W$ and $\eta$ are assumed to be independent. Let us recall the functional setting of the problem (6.1)-(6.3), see e.g. Temam [47].

Function spaces. Let us recall basic spaces used in the theory of Navier-Stokes equations. We will denote them with subscript 0.

$$
\begin{align*}
V_0 &:= C_c^\infty(O, \mathbb{R}^d) \cap \{\text{div} = 0\}, \\
H_0 &:= \text{the closure of } V_0 \text{ in } L^2(O, \mathbb{R}^d), \\
V_0 &:= \text{the closure of } V_0 \text{ in } H^1(O, \mathbb{R}^d).
\end{align*}
$$

(6.4) - (6.6)

In the space $H_0$ we consider the inner product and the norm inherited from $L^2(O, \mathbb{R}^d)$ and denote them by $(\cdot, \cdot)_0$ and $|\cdot|_0$, respectively, i.e.

$$
(u, v)_0 := (u, v)_{L^2}, \quad |u|_0 := |u|_{L^2}, \quad u, v \in H_0.
$$

(6.7)

In the space $V_0$ we consider the inner product

$$
(u, v)_{V_0} := (u, v)_0 + \langle u, v \rangle_0,
$$

where

$$
\langle u, v \rangle_0 := \frac{1}{Re} (\nabla u, \nabla v)_{L^2} = \frac{1}{Re} \sum_{i=1}^d \int_O \frac{\partial u}{\partial x_i} \cdot \frac{\partial v}{\partial x_i} \, dx, \quad u, v \in V_0
$$

(6.8)

and the norm $|u|_{V_0} := (u, u)_{V_0}^{1/2} = |u|_{L^2}^{1/2} + \frac{1}{Re} |\nabla u|_{L^2}^{1/2}$.

The operator $A_0$. We define the operator $A_0 : V_0 \to V'_0$ by setting

$$
\langle A_0 u, v \rangle_0 = \langle u, v \rangle_0, \quad u, v \in V_0.
$$

(6.9)

The form $b$. Let us consider the following tri-linear form, see [47].

$$
b(u, w, v) = \int_O (u \cdot \nabla w) v \, dx = \sum_{i=1}^d \int_O u \, \frac{\partial w}{\partial x_i} v \, dx,
$$

(6.10)
where \( u: \mathcal{O} \to \mathbb{R}^d, \ w, \ v: \mathcal{O} \to \mathbb{R}^d \) and \( d_1 \in \mathbb{N} \). (We will consider the cases when \( d_1 = d \) or \( d_1 = 1 \).) We recall basic properties of the form \( b \) in some Sobolev spaces. We will use them also in the magneto-hydrodynamic equations and in the Boussinesq equations. Using the H"older inequality and the Sobolev embedding Theorem, see [1], we obtain the following inequalities

\[
|b(u, w, v)| \leq ||u||_{L^2} ||\nabla w||_{L^2} ||v||_{L^2},
\]

\[
\leq c||u||_{H^1} ||w||_{H^1} ||v||_{H^1}, \quad u \in H^1(\mathcal{O}, \mathbb{R}^d), \ v, w \in H^1(\mathcal{O}, \mathbb{R}^d),
\]

(6.11)

where \( c \) is a positive constant. Thus, the form \( b \) is continuous on \( H^1(\mathcal{O}, \mathbb{R}^d) \times H^1(\mathcal{O}, \mathbb{R}^d) \times H^1(\mathcal{O}, \mathbb{R}^d) \), see Temam [48] and [47]. If we define a bilinear map \( B \) by \( B(u, w) := b(u, w, \cdot) \), then by (6.11), we infer that \( B(u, w) \in H^{-1}(\mathcal{O}, \mathbb{R}^d) \) and

\[
|B(u, w)||_{H^{-1}} \leq c||u||_{H^1} ||w||_{H^1}, \quad u \in H^1(\mathcal{O}, \mathbb{R}^d), \ w \in H^1(\mathcal{O}, \mathbb{R}^d).
\]

(6.12)

(The symbol \( H^{-1}(\mathcal{O}, \mathbb{R}^d) \) stands for the dual space of \( H^1(\mathcal{O}, \mathbb{R}^d) \).) Furthermore, if \( \text{div} u = 0 \), then

\[
\sum_{i=1}^n \frac{\partial}{\partial x_i} (u_i w) = \left( \sum_{i=1}^n \frac{\partial u_i}{\partial x_i} \right) w + \sum_{i=1}^n u_i \frac{\partial w}{\partial x_i} = (\text{div} u) w + u \cdot \nabla w = u \cdot \nabla w.
\]

If moreover \((u \cdot n)_{\partial \mathcal{O}} = 0\), then by the integration by parts formula, see [47],

\[
b(u, w, v) = \int_\mathcal{O} (u \cdot \nabla w) v \, dx = \sum_{i=1}^n \int_\mathcal{O} \frac{\partial}{\partial x_i} (u_i w) v \, dx = - \sum_{i=1}^n \int_\mathcal{O} u_i \frac{\partial v}{\partial x_i} w \, dx = - \int_\mathcal{O} (u \cdot \nabla v) w \, dx = -b(u, v, w).
\]

Thus for \( u \in H^1(\mathcal{O}, \mathbb{R}^d) \) such that \( \text{div} u = 0 \) and \((u \cdot n)_{\partial \mathcal{O}} = 0\) we have

\[
b(u, w, v) = -b(u, v, w), \quad v, w \in H^1(\mathcal{O}, \mathbb{R}^d).
\]

(6.13)

In particular,

\[
b(u, v, v) = 0 \quad v \in H^1(\mathcal{O}, \mathbb{R}^d).
\]

(6.14)

Hence

\[
\langle B(u, w) \rangle = -(\langle B(u, v) \rangle, w), \quad v, w \in H^1(\mathcal{O}, \mathbb{R}^d)
\]

(6.15)

and, in particular,

\[
\langle B(u, v) \rangle = 0, \quad v \in H^1(\mathcal{O}, \mathbb{R}^d).
\]

(6.16)

Let \( m > \frac{d}{2} + 1 \). By the Sobolev embedding Theorem, see [1], we have \( H^{m-1}(\mathcal{O}, \mathbb{R}^d) \hookrightarrow L^m(\mathcal{O}, \mathbb{R}^d) \). Thus if \( u \in H^1(\mathcal{O}, \mathbb{R}^d) \), \( \text{div} u = 0 \) and \((u \cdot n)_{\partial \mathcal{O}} = 0\), \( w \in H^1(\mathcal{O}, \mathbb{R}^d) \) and \( v \in H^m(\mathcal{O}, \mathbb{R}^d) \), then

\[
|b(u, w, v)| = |b(u, v, w)| = \left| \sum_{i=1}^n \int_\mathcal{O} u_i w \frac{\partial v}{\partial x_i} \, dx \right| \leq ||u||_{L^2} ||w||_{L^2} ||\nabla v||_{L^\infty} \leq c||u||_{L^2} ||w||_{L^2} ||v||_{H^m},
\]

(6.17)

where \( c > 0 \) is a certain constant. Hence the operator \( B \) can be uniquely extended to the tri-linear form, denoted still by \( B \),

\[
B: H_0 \times L^2(\mathcal{O}, \mathbb{R}^d) \to H^{-m}(\mathcal{O}, \mathbb{R}^d),
\]

where \( H_0 \) is the space of \( L^2(\mathcal{O}, \mathbb{R}^d) \) defined by (6.3), and the following inequality holds

\[
|B(u, w)||_{H^{-m}} \leq c||u||_{L^2} ||w||_{L^2}, \quad u \in H_0, \ w \in L^2(\mathcal{O}, \mathbb{R}^d),
\]

(6.18)

see e.g. Vishik and Fursikov [49].

In the following lemma we prove the property of the map \( B \) related to assumption (B.5) in the abstract framework.

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Lemma 6.1. Let \( u \in L^2(0, T; H_0) \) and let \((u_n)_n\) be a bounded sequence in \( L^2(0, T; H_0) \) such that \( u_n \to u \) in \( L^2(0, T; L^2(\Omega; \mathbb{R}^d)) \). Let \( w \in L^2(0, T; L^2(\Omega; \mathbb{R}^d)) \) and let \((w_n)_n\) be a bounded sequence in \( L^2(0, T; L^2(\Omega; \mathbb{R}^d)) \) such that \( u_n \to u \) in \( L^2(0, T; L^2(\Omega; \mathbb{R}^d)) \). If \( m > \frac{d}{2} + 1 \), then for all \( t \in [0, T] \) and all \( \varphi \in H^m_0(\Omega; \mathbb{R}^d) \):

\[
\lim_{n \to \infty} \int_0^t \langle B(u_n(s), w_n(s)), \varphi \rangle \, ds = \int_0^t \langle B(u(s), w(s)), \varphi \rangle \, ds.
\]

Proof. Assume first that \( \varphi \in D(\Omega; \mathbb{R}^d) \). Then there exists \( R > 0 \) such that \( \text{supp } \varphi \) is a compact subset of \( \Omega_R \). Then, using the integration by parts formula, we infer that for every \( u \in H_0 \) and \( w \in L^2(\Omega; \mathbb{R}^d) \)

\[
|\langle B(u, w), \varphi \rangle| = \left| \int_{\Omega_R} (u \cdot \nabla \varphi) \, dx \right| 
\leq ||u||_{L^2(\Omega_R)} ||w||_{L^2(\Omega_R)} ||\nabla \varphi||_{L^2(\Omega_R)} \leq C ||u||_{L^2(\Omega_R)} ||w||_{L^2(\Omega_R)} ||\varphi||_{H^m}.
\]

(6.19)

We have \( B(u_n, w_n) - B(u, w) = B(u_n - u, w_n) + B(u, w_n - w) \). Using inequality (6.19) and the H"older inequality, we obtain

\[
\left| \int_0^t \langle B(u_n(s), w_n(s)), \varphi \rangle \, ds - \int_0^t \langle B(u(s), w(s)), \varphi \rangle \, ds \right| 
\leq \left| \int_0^t \langle B(u_n(s) - u(s), w_n(s)), \varphi \rangle \, ds \right| + \left| \int_0^t \langle B(u(s), w_n(s) - w(s)), \varphi \rangle \, ds \right| 
\leq C \cdot \left( ||u_n - u||_{L^2(0, T; L^2(\Omega_R))} ||w_n||_{L^2(0, T; L^2(\Omega_R))} + ||u||_{L^2(0, T; L^2(\Omega_R))} ||w_n - w||_{L^2(0, T; L^2(\Omega_R))} \right) ||\varphi||_{H^m} 
\leq C \cdot \left( ||u_n - u||_{L^2(0, T; L^2(\Omega))} + ||u||_{L^2(0, T; H_0)} ||w_n - w||_{L^2(0, T; L^2(\Omega))} \right) ||\varphi||_{H^m} 
\]

where \( pr_t \) is the seminorm defined by (4.1) and \( C \) stands for a positive constant. Since \( u_n \to u \) and \( w_n \to w \) in \( L^2(0, T; L^2(\Omega)) \), we infer that for all \( \varphi \in D(\Omega; \mathbb{R}^d) \)

\[
\lim_{n \to \infty} \int_0^t \langle B(u_n(s), w_n(s)), \varphi \rangle \, ds = \int_0^t \langle B(u(s), w(s)), \varphi \rangle \, ds.
\]

(6.20)

If \( \varphi \in H^m_0(\Omega; \mathbb{R}^d) \) then for every \( \varepsilon > 0 \) there exists \( \varphi_\varepsilon \in D(\Omega; \mathbb{R}^d) \) such that \( ||\varphi - \varphi_\varepsilon||_{H^m} \leq \varepsilon \). Then for all \( s \in [0, t] \)

\[
\left| \langle B(u_n(s), w_n(s)) - B(u(s), w(s)), \varphi \rangle \right| 
\leq \left| \langle B(u_n(s), w_n(s)) - B(u(s), w(s)), \varphi - \varphi_\varepsilon \rangle \right| 
+ \left| \langle B(u_n(s), w_n(s)) - B(u(s), w(s)), \varphi_\varepsilon \rangle \right| 
\leq \left| \langle B(u_n(s), w_n(s)) - B(u(s), w(s)), \varphi - \varphi_\varepsilon \rangle \right| 
+ \left| \langle B(u_n(s), w_n(s)) - B(u(s), w(s)), \varphi_\varepsilon \rangle \right| 
\leq \varepsilon ||u_n||_0 ||w_n||_{L^2(\Omega)} + ||u||_0 ||w||_{L^2(\Omega)} + ||B(u_n(s), w_n(s)) - B(u(s), w(s)), \varphi_\varepsilon \rangle||_{H^m} 
\]

Hence

\[
\left| \int_0^t \langle B(u_n(s), w_n(s)) - B(u(s), w(s)), \varphi \rangle \, ds \right| 
\leq \varepsilon \int_0^t \left( ||u_n||_0 ||w_n||_{L^2(\Omega)} + ||u||_0 ||w||_{L^2(\Omega)} \right) ds + \int_0^t \left| \langle B(u_n(s), w_n(s)) - B(u(s), w(s)), \varphi_\varepsilon \rangle \right| \, ds 
\leq \frac{\varepsilon}{2} \left( \sup_{n \geq 1} \left| \int_0^t ||u_n||_{L^2(\Omega)}^2 \, ds \right| + ||u||_2^2 \right) 
+ \frac{1}{2} \left( \sup_{n \geq 1} \left| \int_0^t ||w_n||_{L^2(\Omega)}^2 \, ds \right| + ||w||_2^2 \right) 
+ \int_0^t \left| \langle B(u_n(s), w_n(s)) - B(u(s), w(s)), \varphi_\varepsilon \rangle \right| \, ds 
\]

Passing to the upper limit as \( n \to \infty \), we obtain

\[
\limsup_{n \to \infty} \int_0^t \langle B(u_n(s), w_n(s)) - B(u(s), w(s)), \varphi \rangle \, ds \leq M\varepsilon.
\]
where \( M := \frac{1}{\varepsilon} \left( \sup_{t \in [0,1]} \left( ||u||_{L^2(0,T,H)}^2 + ||w||_{L^2(0,T,L^2(\Omega))}^2 + ||u||_{L^2(0,T,L^2(\Omega))}^2 + ||w||_{L^2(0,T,L^2(\Omega))}^2 \right) \right) < \infty \). Since \( \varepsilon > 0 \) is arbitrary, we infer that (6.20) holds for all \( \varphi \in H_0^m(\Omega; \mathbb{R}^d) \). The proof of the lemma is thus complete.

**The operator \( B_0 \).** We will now concentrate on the bilinear map \( B \) in the spaces \( H_0 \) and \( V_0 \) defined by (6.5) and (6.6), respectively. We will denote it by \( B_0 \). By (6.12) we infer that for \( u, w \in V_0, B_0(u, w) \in V_0^* \) and the following inequality holds

\[
|B_0(u, w)|_{V_0^*} \leq c ||u||_0 ||w||_0, \quad u, w \in V_0.
\]

(6.21)

In particular, the map \( B_0 : V_0 \times V_0 \to V_0^* \) is bilinear and continuous. Furthermore, by (6.13)

\[
\langle B_0(u, w), v \rangle = -\langle B_0(u, v), w \rangle, \quad u, w, v \in V_0
\]

(6.22)

and hence

\[
\langle B_0(u, v), v \rangle = 0, \quad u, v \in V_0.
\]

(6.23)

Let for any \( m > 0 \),

\[
U_m := \text{the closure of } V_0 \text{ in } H^m(\Omega, \mathbb{R}^d).
\]

(6.24)

In the space \( U_m \) we consider the inner product inherited from \( H^m(\Omega, \mathbb{R}^d) \). Let \( m > \frac{d}{2} + 1 \). By (6.18), \( B_0 \) is a bounded bilinear operator \( B_0 : H_0 \times H_0 \to U_m^* \) and the following inequality holds

\[
|B_0(u, w)|_{U_m^*} \leq c ||u||_0 ||w||_0, \quad u, w \in H_0.
\]

(6.25)

We will also use the following notation, \( B_0(u) := B_0(u, u) \).

Let us also recall that the mapping \( B_0 : V_0 \to V_0^* \) is locally Lipschitz continuous, i.e. for every \( r > 0 \) there exists a constant \( L_r \) such that

\[
|B_0(u) - B_0(\tilde{u})|_{V_0^*} \leq L_r ||u - \tilde{u}||_0, \quad u, \tilde{u} \in V_0, \quad ||u||_0, ||\tilde{u}||_0 \leq r.
\]

(6.26)

**Solution of the Navier-Stokes equations.** Let \( u_0 \in H_0, \ f \in L^2(0, T; V_0^*), \ G : [0, T] \times V_0 \to L_{HS}(Y_w, H_0), \) where \( Y_w \) is a separable Hilbert space, and \( F : [0, T] \times H_0 \times Y \to H_0, \) where \( (Y, \mathcal{F}) \) is a measurable space, be given.

**Definition 6.2. A martingale solution** of problem (6.1), (6.3) is a system \( (\bar{u}, \bar{\eta}, \bar{W}, \bar{\tilde{u}}) \), where \( (\bar{u}, \bar{\eta}, \bar{W}) \) is as in Definition 2.2 and \( \bar{\tilde{u}} : [0, T] \times \bar{\Omega} \to H_0 \) is a predictable process with \( \bar{\mathbb{P}} \)-a.e. paths

\[
\bar{\tilde{u}}(\cdot, \omega) \in D([0, T], H_0) \cap L^2(0, T; V_0)
\]

such that for all \( t \in [0, T] \) and all \( v \in V_0 \) the following identity holds \( \bar{\mathbb{P}} \)-a.s.

\[
(\tilde{u}(t), v)_0 + \int_0^t \langle \mathcal{A}_0 \tilde{u}(s), v \rangle \, ds + \int_0^t \langle B_0(\tilde{u}(s)), v \rangle \, ds
\]

\[
= (u_0, v)_0 + \int_0^t \langle f(s), v \rangle \, ds + \int_0^t \int_{\Omega_0} (F(s, \tilde{u}(s^-); y), v)_{H_0} \, \bar{\eta}(ds, dy)
\]

\[
+ \int_0^t \int_{\Omega_0} (F(s, \tilde{u}(s^-); y), v)_{H_0} \, \bar{\eta}(ds, dy) + \left( \int_0^t G(s, \bar{\tilde{u}}(s)) \, d\bar{W}(s, v) \right).
\]

We apply the abstract framework with \( \mathcal{H} := H_0, \ \mathcal{V} := V_0, \ \mathcal{V}_s := U_m \) with \( m > \frac{d}{2} + 1 \), defined by (6.5), (6.6) and (6.24), respectively. Furthermore,

\[
\mathcal{A} := \mathcal{A}_0, \quad \mathcal{R} := 0 \quad \text{and} \quad \mathcal{B} := B_0.
\]

By (6.9), it is evident that the operator \( \mathcal{A}_0 \) satisfies condition (A.1). By (6.21), (6.22), (6.25) and (6.26) the map \( B_0 \) satisfies conditions (B.1)-(B.4). By Lemma 6.1 the map \( B_0 \) satisfies assumption (B.5). Applying Theorem 2.4 we obtain the following result about the existence of the solution of the Navier-Stokes problem.
Corollary 6.3. For every $u_0 \in H_0$, $f \in L^2(0, T; V\nu_0)$, \( G : [0, T] \times V_0 \rightarrow \mathcal{L}_{HS}(Y, H_0) \) satisfying conditions \((G.1)-(G.3)\) and \( F : [0, T] \times H_0 \times Y \rightarrow H_0 \) satisfying conditions \((F.1)-(F.3)\) there exists a martingale solution \((\tilde{u}, \tilde{\eta}, W, \tilde{u})\) of problem \((6.1)-(6.3)\) such that

\[
\mathbb{E} \left[ \sup_{t \in [0, T]} ||\tilde{u}(t)||^2_{H_0} + \int_0^T ||\tilde{u}(t)||^2_{Y_0} \, dt \right] < \infty.
\]

6.2. Magneto-hydrodynamic equations (MHD)

The mathematical model of the motion of a resistive fluid is obtained by coupling the Navier-Stokes equations and the Maxwell equations (see Sermange and Temam [44], 1983). We will consider the following stochastic magneto-hydrodynamic (MHD) equations

\[
du(t) + (u \cdot \nabla) u - \frac{1}{Re} \Delta u - s(b \cdot \nabla) b + \nabla p + \nabla \left( \frac{1}{2} b^2 \right) = f_0(t) + G_0(t, u(t), b(t)) \, dw(t) + \int_{Y_0} F_0(t, u(\tau^-), b(\tau^-); y) \eta_0(dt, dy),
\]

\[
dv(t) + (u \cdot \nabla) b - (b \cdot \nabla) u - \frac{1}{Rm} \text{curl} (\text{curl} b) = f_1(t) + G_1(t, u(t), b(t)) \, dw(t) + \int_{Y_0} F_1(t, u(\tau^-), b(\tau^-); y) \eta_1(dt, dy),
\]

in \((0, T) \times \Omega\), with the conditions

\[
\text{div} u = 0, \quad \text{div} b = 0 \quad \text{in} \ (0, T) \times \Omega
\]

and the following boundary conditions

\[
u = 0 \quad \text{and} \quad b \cdot n = 0 \quad \text{on} \ \partial \Omega,
\]

where \( n = (n_1, ..., n_d) \) stands for the unit outward normal on \( \partial \Omega \). Moreover, we impose the initial conditions

\[
u(0) = u_0, \quad b(0) = b_0.
\]

Here \( u, p, b \) are interpreted as the velocity, the pressure and the magnetic field. The three positive constants \( \frac{1}{Re}, \frac{1}{Rm} \) and \( s \) correspond to the kinematic viscosity, the magnetic diffusivity and the Hartman number, respectively, see Duvaut and Lions [23] and Sermange and Temam [44]. These equations are used to describe the turbulent flows in magnetohydrodynamics. Moreover, \( f_0 \) and \( f_1 \) stand for deterministic external forces, \( W_0 \) and \( W_1 \) are cylindrical Wiener processes in Hilbert spaces \( Y_0^0 \) and \( Y_1^0 \), respectively, \( \eta_0 \) and \( \eta_1 \) are compensated time homogeneous Poisson random measures with intensities \( \mu_0 \) and \( \mu_1 \) on measurable spaces \( (\mathcal{Y}^0, \mathcal{F}^0) \) and \( (\mathcal{Y}^1, \mathcal{F}^1) \), respectively. The sets \( Y_0^0 \subset \mathcal{Y}^0 \) and \( Y_1^0 \subset \mathcal{Y}^1 \) are such that \( \mu_0(\mathcal{Y}^0 \setminus Y_0^0) < \infty \) and \( \mu_1(\mathcal{Y}^1 \setminus Y_1^0) < \infty \). The processes \( W_0, W_1, \eta_0, \eta_1 \) are assumed to be independent.

Function spaces. Let us recall that the spaces used in the theory of the magneto-hydrodynamic equations are products of the spaces used for the Navier-Stokes equations, i.e. \( V_0, H_0 \) and \( V_0 \) defined by \((6.4), (6.5), (6.6)\) and spaces used in the theory of the Maxwell equations (spaces denoted with the subscript 1). Namely, see [44],

\[
\mathcal{V}_1 = \{ \epsilon \in C^m(\overline{\Omega}, \mathbb{R}^d), \ \text{div} \epsilon = 0, \ \text{and} \ (\epsilon \cdot n)_{\partial \Omega} = 0 \},
\]

\[
V_1 = \text{the closure of } \mathcal{V}_1 \text{ in } H^1(\Omega, \mathbb{R}^d) = \{ \epsilon \in H^1(\Omega, \mathbb{R}^d), \ \text{div} \epsilon = 0 \ \text{and} \ (\epsilon \cdot n)_{\partial \Omega} = 0 \},
\]

\[
H_1 = \text{the closure of } V_1 \text{ in } L^2(\Omega, \mathbb{R}^d).
\]
In the space $H_1$ we consider the inner product and the norm defined by
\[ (b, c)_{H_1} := s b(c, b)_{L^2}, \quad ||b||^2 := (b, b)_1, \quad b, c \in H_1. \] (6.31)

In the space $V_1$, we consider the inner product $(b, c)_{V_1} := (b, c)_1 + \langle b, c \rangle_1$, where
\[ \langle b, c \rangle_1 := \frac{s}{Rm} (\text{curl } b, \text{curl } c)_{L^2}, \quad b, c \in V_1. \] (6.32)

and the norm $||b||^2 := (b, b)_{V_1}$. Finally, we define the spaces
\[ V := V_0 \times V_1, \quad \mathbb{H} := H_0 \times H_1, \quad V' := \text{the dual space of } V \] (6.33)

with the following inner products
\[ (\Phi, \Psi)_\mathbb{H} := (u, v)_0 + (b, c)_1 \quad \text{for all } \Phi = (u, b), \Psi = (v, c) \in \mathbb{H} \]
\[ (\Phi, \Psi)_V := (\Phi, \Psi)_\mathbb{H} + \langle \Phi, \Psi \rangle \quad \text{for all } \Phi, \Psi \in V, \]
where $\langle \Phi, \Psi \rangle := \langle u, v \rangle_0 + (b, c)_1$ and $(\cdot, \cdot)_0$ and $(\cdot, \cdot)_1$ are defined by (6.7) and (6.8), respectively. We have $V \subset \mathbb{H} \subset V'$, where the embeddings are dense and continuous.

**The operator $\mathcal{A}$**. We define the operators $\mathcal{A}_1$ and $\mathcal{A}$ by the following formulae
\[ \langle \mathcal{A}_1 b, c \rangle := (b, c)_1, \quad b, c \in V_1, \]
\[ \langle \mathcal{A} \Phi, \Psi \rangle := (\mathcal{A}_0 u, v) + \langle \mathcal{A}_1 b, c \rangle, \quad \Phi, \Psi \in V, \] (6.34)

where $\mathcal{A}_0$ is defined by (6.2). It is clear that $\mathcal{A}_1 \in \mathcal{L}(V_1, V'_1)$ and $\mathcal{A} \in \mathcal{L}(V, V')$. Let us also notice that
\[ \langle \mathcal{A} \Phi, \Psi \rangle = \langle \Phi, \Psi \rangle, \quad \Phi, \Psi \in V. \] (6.35)

**The form $\hat{b}$ and the operator $\hat{B}$**. Using the form $b$ defined by (6.10) we will consider the tri-linear form $\hat{b}$ on $V \times V \times V$, where $V$ is defined by (6.33), see Sango [43] and Sermange and Temam [44]. Namely,
\[ \hat{b}(\Phi^{(1)}, \Phi^{(2)}, \Phi^{(3)}) := b(u^{(1)}, u^{(2)}, u^{(3)}) - sb(b^{(1)}, b^{(2)}, b^{(3)}) + sb(u^{(1)}, b^{(2)}, b^{(3)}) - sb(b^{(1)}, u^{(2)}, b^{(3)}), \]
where $\Phi^{(i)} = (u^{(i)}, b^{(i)}) \in V$, $i = 1, 2, 3$. By (6.11) we see that the form $\hat{b}$ is continuous. Moreover, by (6.13) and (6.14) the form $\hat{b}$ has the following properties, see also [43],
\[ \hat{b}(\Phi^{(1)}, \Phi^{(2)}, \Phi^{(3)}) = -\hat{b}(\Phi^{(1)}, \Phi^{(3)}, \Phi^{(2)}), \quad \Phi^{(i)} \in V, \quad i = 1, 2, 3 \] (6.36)

and in particular
\[ \hat{b}(\Phi^{(1)}, \Phi^{(2)}, \Phi^{(3)}) = 0, \quad \Phi^{(1)}, \Phi^{(2)} \in V. \] (6.37)

Now, let us define a bilinear map $\hat{B}$ by
\[ \hat{B}(\Phi, \Psi) := \hat{b}(\Phi, \Psi, \cdot), \quad \Phi, \Psi \in V. \] (6.38)

We will also use the notation $\hat{B}(\Phi) := \hat{B}(\Phi, \Phi)$. For $m > 0$ we define the following space
\[ V_m := \text{the closure of } V_0 \times V_1 \text{ in the space } H^m(O, \mathbb{R}^d) \times H^m(O, \mathbb{R}^d). \] (6.39)

We will collect properties of the map $\hat{B}$ in the following lemma.

**Lemma 6.4.** (1) *There exists a constant $c_1 > 0$ such that*
\[ ||\hat{B}(\Phi, \Psi)||_{V'} \leq c_1 ||\Phi||_V ||\Psi||_V, \quad \Phi, \Psi \in V. \]

*In particular, the form $\hat{B} : V \times V \to V'$ is bilinear and continuous. Moreover,*
\[ \langle \hat{B}(\Phi, \Psi), \Theta \rangle = -\langle \hat{B}(\Phi, \Theta), \Psi \rangle, \quad \Phi, \Psi, \Theta \in V. \]
(2) The mapping $\hat{B}$ is locally Lipschitz continuous on the space $\mathcal{V}$, i.e. for every $r > 0$ there exists a constant $L_r > 0$ such that
\[
|\hat{B}(\Phi) - \hat{B}(\Phi')|_{\mathcal{V}} \leq L_r \|\Phi - \Phi'\|_{\mathcal{V}}, \quad \Phi, \Phi' \in \mathcal{V}, \quad \|\Phi\|_{\mathcal{V}}, \|\Phi'\|_{\mathcal{V}} \leq r.
\]

(3) If $m > \frac{d}{2} + 1$, then $\hat{B}$ can be extended to the bilinear mapping from $\mathbb{H} \times \mathbb{H}$ to $\mathcal{V}_m'$ (denoted still by $\hat{B}$) such that
\[
|\hat{B}(\Phi, \Psi)|_{\mathcal{V}_m'} \leq c_2 \|\Phi\|_{\mathbb{H}} \|\Psi\|_{\mathbb{H}}, \quad \Phi, \Psi \in \mathbb{H},
\]
where $c_2$ is a positive constant.

Proof. Using the definition (6.38) of the mapping $\hat{B}$, we infer that assertion (1) follows from (6.11), (6.36) and (6.37). Assertion (2) follows from the following inequalities
\[
|\hat{B}(\Phi) - \hat{B}(\Phi')|_{\mathcal{V}} \leq |\hat{B}(\Phi) - \hat{B}(\Phi')|_{\mathcal{V}} + |\hat{B}(\Phi') - \hat{B}(\Phi)|_{\mathcal{V}},
\]
\[
\leq \|\hat{B}\| \cdot \|\Phi - \Phi'\|_{\mathcal{V}} + \|\hat{B}\| \cdot \|\Phi' - \Phi\|_{\mathcal{V}} + \|\hat{B}\| \cdot \|\Phi - \Phi'\|_{\mathcal{V}}.
\]
Thus the Lipschitz condition holds with $L_r = 2r \|\hat{B}\|$, where $\|\hat{B}\|$ stands for the norm of the bilinear map $\hat{B} : \mathcal{V} \times \mathcal{V} \to \mathcal{V}'$. Assertion (3) follows from (6.17). The proof is thus complete. $\square$

Weak formulation of problem (6.27). Let $\mathbb{H}$ and $\mathcal{V}$ be the Hilbert spaces defined by (6.33).

- Let $f_0 \in L^2(0, T; V_0')$, $f_1 \in L^2(0, T; V_1')$, $u_0 \in H_0$ and $b_0 \in H_1$ be given and let
\[
f := (f_0, f_1), \quad \Phi_0 := (u_0, b_0).
\]
Then $f \in L^2(0, T; \mathcal{V}')$ and $\Phi_0 \in \mathbb{H}$.

- Let $Y_W := Y^0_W \times Y^1_W$ and let $W(t) = (W_0(t), W_1(t))$. Then $W$ is a cylindrical Wiener process on the space $Y_W$. Moreover, let $G_0 : [0, T] \times V_0 \times V_1 \to \mathcal{L}_{HS}(Y^0_W, H_0)$ and $G_1 : [0, T] \times V_0 \times V_1 \to \mathcal{L}_{HS}(Y^1_W, H_1)$ be given and let us define the map $G$ by the formula
\[
G(\Phi)h := (G_0(a)h_0, G_1(b)h_1),
\]
where $\Phi = (a, b) \in V_0 \times V_1$, $h := (h_1, h_2) \in Y_W$ and $t \in [0, T]$. Then $G : \mathcal{V} \to \mathcal{L}_{HS}(Y_W, \mathbb{H})$.

- Let $Y := Y^0 \times Y^1$. Then $(Y, Y')$, where $Y' := Y^0' \otimes Y^1$ is a measurable space and $\eta(dt, d\nu) := (\eta_0(dt, d\nu_0), \eta_1(dt, d\nu_2))$ is a time homogeneous Poisson random measure on $(Y, \mathcal{Y})$ with the intensity measure $\mu := \mu_0 \otimes \mu_1$. Let $Y_0 := Y^0_1 \times Y^1_1$. Let $F_0 : [0, T] \times H_0 \times H_1 \times Y_0 \to H_0$ and $F_1 : [0, T] \times H_0 \times H_1 \times Y_1 \to H_1$ be given and let us define the map $F$ by the formula
\[
F(t, \Phi; y) := (F_0(t, u, b; y_0), F_1(t, u, b; y_1)),
\]
where $\Phi = (a, b) \in H_0 \times H_1$, $y := (y_0, y_1) \in Y$ and $t \in [0, T]$. Then $F : [0, T] \times \mathbb{H} \times Y \to \mathbb{H}$.

We apply the abstract framework with the spaces $\mathbb{H}$ and $\mathcal{V}$ defined by (6.33), the space $\mathcal{V}_r := \mathcal{V}_m$ with $m > \frac{d}{2} + 1$ defined by (6.39), the operator $\mathcal{A}$ defined by (6.44),
\[
\mathcal{B}(\Phi, \Psi) := \hat{B}(\Phi, \Psi), \quad \Phi, \Psi \in \mathcal{V},
\]
where $\hat{B}$ is defined by (6.38), and $\mathcal{R} := 0$. By Lemma 6.4 the map $\mathcal{B}$ satisfies assumptions (B.1)-(B.4). By Lemma 6.1 the map $\mathcal{B}$ satisfies assumption (B.5).
Definition 6.5. A martingale solution of the problem (6.27)-(6.30) is a system $(\bar{\mathbf{S}}, \bar{\eta}, \bar{W}, \bar{\Phi})$, where $(\bar{\mathbf{S}}, \bar{\eta}, \bar{W})$ is as in Definition 2.2 and $\Phi : [0, T] \times \Omega \rightarrow \mathbb{H}$ is a predictable process with $\bar{\mathbb{P}}$-a.e. paths $\Phi(\cdot, \omega) \in \mathbb{D}([0, T], \mathbb{H}_u) \cap L^2(0, T; \mathcal{V})$ such that for all $t \in [0, T]$ and all $\Psi \in \mathcal{V}$ the following identity holds $\bar{\mathbb{P}}$-a.s.

\[
\begin{align*}
(\Phi(t), \psi) + \int_0^t \langle A\Phi(\sigma), \Psi \rangle \, d\sigma + \int_0^t \langle \bar{B}(\Phi(\sigma)), \Psi \rangle \, d\sigma = (\Phi_0, \Psi)_{\mathcal{H}} \\
+ \int_0^t \langle f(\sigma), \Psi \rangle \, d\sigma + \int_0^t \int_{Y_0} (F(x, \Phi(s^-); y), \Psi \eta(s, dy) \, ds, dy \\
+ \int_0^t \int_{Y_0} (F(x, \Phi(s^-); y), \Psi \eta(s, dy) \, ds, dy) + \left( \int_0^t G(\Phi(\sigma)) \, d\bar{W}(\sigma), \Psi \right). \end{align*}
\]

Applying Theorem 2.3 we obtain the following result about the existence of the martingale solution of the magneto-hydrodynamic equations.

Corollary 6.6. For every $\Phi_0 = (u_0, b_0) \in \mathbb{H}, f \in L^2(0, T; \mathcal{V}^\prime), G : [0, T] \times \mathcal{V} \rightarrow L_{HS}(Y, \mathcal{H})$ satisfying conditions (G.1)-(G.3) and $F : [0, T] \times \mathbb{H} \times Y \rightarrow \mathcal{H}$ satisfying conditions (F.1)-(F.3) there exists a martingale solution $(\bar{\mathbf{S}}, \bar{\eta}, \bar{W}, \bar{\Phi})$, where $\Phi = (\bar{u}, \bar{b})$, of problem (6.27)-(6.30) such that

\[ \mathbb{E} \sup_{t \in [0, T]} (\bar{u}^2(t)_{L^2(Y)}^2 + |\bar{b}(t)|^2_{H_1}) + \int_0^T (||\bar{u}(t)||_{L^2(Y)}^2 + ||\bar{b}(t)||_{H_1}^2) \, dt < \infty. \]

6.3. Boussinesq equations

We consider $\mathbb{R}^d$, where $d = 2, 3$, with the canonical basis $\{e_1, e_2\}$ or $\{e_1, e_2, e_3\}$ and the Boussinesq model for the Bénard problem with random influences in the domain $O$

\[
\begin{align*}
du(t) + (u \cdot \nabla)u - \frac{1}{Re} \Delta u + \theta \varepsilon_d + \nabla p &= f_0(t) + G_0(t, u(t), \theta(t)) \, dW_0(t) \\
+ \int_{Y_0^1} F_0(t, u(\Gamma), \theta(\Gamma); y) \eta_0(\sigma, dy) \, d\sigma, \\
\theta(t) + (u \cdot \nabla)\theta - \kappa \Delta \theta - u_d &= f_2(t) + G_2(t, u(t), \theta(t)) \, dW_2(t) \\
+ \int_{Y_0^2} F_2(t, u(\Gamma), \theta(\Gamma); y) \eta_1(\sigma, dy) \, d\sigma, \\
\end{align*}
\]

where $t \in [0, T]$, with the incompressibility condition

\[ \text{div} u = 0 \]

and with the homogeneous boundary conditions

\[ u_{|\partial O} = 0 \quad \text{and} \quad \theta_{|\partial O} = 0. \]

The functions $u = u(t, x) = (u_1(t, x), ..., u_d(t, x))$ and $p = p(t, x)$ are interpreted as the velocity and the pressure of the fluid. Function $\theta = \theta(t, x)$ represents the temperature of the fluid (see [26]) and a given parameter $\kappa > 0$ is the coefficient of the thermometric conductivity. Here $f_0, f_2$ stand for the deterministic external forces, $W_0$ and $W_2$ are cylindrical Wiener processes in Hilbert spaces $Y_0^1$ and $Y_0^2$, respectively, $\eta_0$ and $\eta_2$ are compensated time homogeneous Poisson random measures with intensities $\mu_0$ and $\mu_2$ on measurable spaces $(Y^0, \mathcal{Y}^0)$ and $(Y^2, \mathcal{Y}^2)$, respectively. Moreover, $Y_0^0 \in \mathcal{Y}^0$ and $Y_0^2 \in \mathcal{Y}^2$ are such that $\mu_0(Y^0 \setminus Y_0^0) < \infty$ and $\mu_2(Y^2 \setminus Y_0^2) < \infty$. The processes $W_0, W_2, \eta_0, \eta_2$ are assumed to be independent.

The functional setting of the problem (6.42)-(6.44) is analogous to that considered in [26] and [14].
Function spaces. The spaces used in the theory of the Boussinesq problem are products of spaces used for the Navier-Stokes equations, i.e. \( \mathcal{Y}_0, H_0 \) and \( V_0 \) defined by (6.2), (6.5), (6.6) and spaces used in the theory of the heat equation (spaces denoted with the subscript 2). They are

\[
\mathcal{Y}_2 = C_0^\infty(\Omega, \mathbb{R}), \\
V_2 = H_0^1(\Omega, \mathbb{R}) := \text{the closure of } \mathcal{Y}_2 \text{ in } H^1(\Omega, \mathbb{R}), \\
H_2 = L^2(\Omega, \mathbb{R}).
\]

In the space \( V_2 \), we consider the inner product

\[
\langle \theta, \vartheta \rangle_{V_2} := (\theta, \vartheta)_{V_2} + \kappa(\nabla \theta, \nabla \vartheta)_{V_2}
\]

and the norm \( \| \theta \|_{V_2}^2 := \| \theta \|_{V_2}^2 + \| \nabla \theta \|_{V_2}^2 \), where \( \theta \in V_2 \). Finally, we define

\[
\mathbb{V} := V_0 \times V_2, \quad \mathbb{H} := H_0 \times H_2, \quad \mathbb{V}' := \text{the dual space of } \mathbb{V}
\]

with the following inner products

\[
\langle \psi, \varphi \rangle_{\mathbb{V}} := \langle \psi, \varphi \rangle_{V_2} \quad \text{for all } \psi = (u, \theta), \varphi = (v, \vartheta) \in \mathbb{H}
\]

We have \( \mathbb{V} \subset \mathbb{H} \subset \mathbb{V}' \), where the embeddings are compact and each space is dense in the following one.

The operators \( \mathcal{A}_2 \) and \( \mathcal{A} \). We define the operators \( \mathcal{A}_2 \) and \( \mathcal{A} \) by

\[
\langle \mathcal{A}_2 \theta, \vartheta \rangle := \langle \theta, \vartheta \rangle_{V_2}, \quad \theta, \vartheta \in V_2 \\
\langle \mathcal{A}_0 \phi, \psi \rangle := \langle \mathcal{A}_0 u, v \rangle + \langle \mathcal{A}_0 \theta, \vartheta \rangle, \quad \phi, \psi \in \mathbb{V}
\]

where \( \mathcal{A}_0 \) is defined by (6.9). It is clear that \( \mathcal{A}_2 \in \mathcal{L}(V_2, V_2') \) and thus \( \mathcal{A} \in \mathcal{L}(\mathbb{V}, \mathbb{V}') \). Let us notice that

\[
\langle \mathcal{A}_2 \phi, \psi \rangle = \langle \psi, \phi \rangle_{V_2}, \quad \phi, \psi \in \mathbb{V}
\]

The operator \( B_2 \). Let us consider the following tri-linear form \( b \) defined by (6.10) with \( d_1 := 1 \).

If we define a bilinear map \( B_2 \) by \( B_2(u, \vartheta) := b(u, \vartheta, \cdot) \), then by (6.12), we infer that \( B_2(u, \vartheta) \in V_2' \) and that the following estimate holds

\[
|B_2(u, \vartheta)|_{V_2'} \leq c|u|_{L^2}||\vartheta||_{V_2}, \quad u \in V_0, \quad \vartheta \in V_2.
\]

In particular, the mapping \( B_2 : V_0 \times V_2 \rightarrow V_2' \) is bilinear and continuous. Furthermore, by (6.15)

\[
\langle B_2(u, \theta), \vartheta \rangle = -(B_2(u, \theta), \vartheta), \quad u \in V_0, \quad \theta, \vartheta \in V_2.
\]

and, in particular,

\[
\langle B_2(u, \theta), \vartheta \rangle = 0, \quad u \in V_0, \quad \vartheta \in V_2.
\]

If \( m > \frac{4}{3} + 1 \), the operator \( B_2 \) can be extended to \( B_2 : H_0 \times H_2 \rightarrow H_0^{-m}(\Omega, \mathbb{R}) \), and by (6.18) satisfies the following inequality

\[
|B_2(u, \vartheta)|_{H_0^{-m}(\Omega, \mathbb{R})} \leq c|u|_{H_0}||\vartheta||_{V_2}, \quad u \in H_0, \quad \vartheta \in H_2.
\]

Using the above notation, the Boussinesq problem can be written as a system of the following two equations

\[
du(t) + [\mathcal{A}_0 u + B_0(u, u) - \partial v] \, dt = f_0(t) \, dt + G_0(t, u(t), \theta(t)) \, dW_0(t) \\
+ \int_{\gamma_0}^{t} F_0(t, u(\tau), \theta(\tau); y) \, \tilde{\eta}_0(\tau, dt, dy) + \int_{\gamma_0}^{t} F_0(t, u(\tau), \theta(\tau); y) \, \eta_0(\tau, dt, dy) \\
d\theta(t) + [\mathcal{A}_2 \theta + B_2(u, \theta) - u \vartheta] \, dt = f_2(t) \, dt + G_2(t, u(t), \theta(t)) \, dW_2(t) \\
+ \int_{\gamma_2}^{t} F_2(t, u(\tau), \theta(\tau); y) \, \tilde{\eta}_2(\tau, dt, dy) + \int_{\gamma_2}^{t} F_2(t, u(\tau), \theta(\tau); y) \, \eta_2(\tau, dt, dy).
\]
with the initial conditions
\[ u(0) = u_0, \quad \theta(0) = \theta_0. \] (6.55)

**Weak formulation of problem (6.42).** Let \( H \) and \( V \) be the Hilbert spaces defined by (6.46).

- Let \( f_0 \in L^2(0, T; V'_0) \), \( f_2 \in L^2(0, T; V'_2) \), \( u_0 \in H_0 \) and \( \theta_0 \in H_2 \) be given and let
  \[ f := (f_0, f_2), \quad \phi_0 := (u_0, \theta_0). \]
  Then \( f \in L^2(0, T; V') \) and \( \phi_0 \in \mathbb{H} \).

- Let \( Y_W := Y^0_W \times Y^2_W \) and let \( W(t) = (W_0(t), W_2(t)) \). Then \( W \) is a cylindrical Wiener process on the space \( Y_W \). Moreover, let \( G_0 : [0, T] \times V_0 \times V_2 \rightarrow L_{HS}(Y^0_W, H_0) \) and \( G_2 : [0, T] \times V_0 \times V_2 \rightarrow L_{HS}(Y^2_W, H_2) \) be given. Let us define the map \( G \) by the formula
  \[ G(\phi)h := (G_0(\phi)h_0, G_1(\phi)h_1), \] (6.56)
  where \( \phi = (u, \theta) \in V, h := (h_1, h_2) \in Y_W \) and \( t \in [0, T] \). Then \( G : V \rightarrow L_{HS}(Y_W, \mathbb{H}) \).

- Let \( Y := Y^0 \times Y^2 \). Then \( (Y, \mathcal{Y}) \), where \( \mathcal{Y} := \mathcal{Y}^0 \otimes \mathcal{Y}^2 \) is a measurable space and \( \eta(dt, dy) := (\mu(dt, dy_0), \eta(dt, dy_2)) \) is a time homogeneous Poisson random measure on \( (Y, \mathcal{Y}) \) with the intensity measure \( \mu := \mu_0 \otimes \mu_2 \). Moreover, \( Y_0 := Y^0_0 \times Y^2_0 \). Let \( F_0 : [0, T] \times H_0 \times H_2 \rightarrow H_0 \) and \( F_2 : [0, T] \times H_2 \times H_2 \rightarrow H_2 \) be given and let us define the map \( F \) by the formula
  \[ F(t, \phi, y) := (F_0(t, u, \theta; y_0), F_1(t, u, \theta; y_1)), \] (6.57)
  where \( \phi = (u, \theta) \in \mathbb{H}, y := (y_0, y_2) \in Y \) and \( t \in [0, T] \). Then \( F : [0, T] \times \mathbb{H} \times Y \rightarrow \mathbb{H} \).

Using the operator \( \mathcal{A} \) defined by (6.47) and putting
\[
\mathcal{B}(\phi, \psi) := (B_0(u, v), B_2(u, \theta)), \quad \phi = (u, \theta), \psi = (v, \theta) \in V, \] (6.58)
\[
\mathcal{R}(\phi) := (-\theta e_d, -u_d), \quad \phi \in V, \] (6.59)
by (6.53), (6.54) and (6.55) we obtain the following equation for \( \phi = (u, \theta) \)
\[
d\phi(t) + [\mathcal{A}\phi + \mathcal{B}(\phi) + \mathcal{R}\phi] dt = f(t) dt + G(t, \phi(t)) dW(t) \]
\[
+ \int_{Y_0} F(t, \phi(t^-); y) \eta(dt, dy) + \int_{Y \setminus Y_0} F(t, \phi(t^-); y) \eta(dt, dy) \] (6.60)
with the initial condition
\[ \phi(0) = \phi_0. \] (6.61)

W will now be concerned with some properties of the maps \( \mathcal{B} \) and \( \mathcal{R} \) defined by (6.58) and (6.59), respectively. Let \( U_m \) be the space defined by (6.24) and let us define
\[ U_m := U_m \times H^m_2(O, \mathbb{R}). \] (6.62)

**Lemma 6.7.** (Properties of the map \( \mathcal{B} \))

1. There exists a constant \( c_2 > 0 \) such that
   \[ |\mathcal{B}(\phi, \psi)|_V \leq c_1||\phi||_V ||\psi||_V, \quad \phi, \psi \in V. \]
   Moreover,
   \[ \langle \mathcal{B}(\phi, \psi), \chi \rangle = -\langle B(\phi, \chi), \psi \rangle, \quad \phi, \psi, \chi \in V. \]

2. If \( m > \frac{d}{2} + 1 \), then \( \mathcal{B} \) can be extended to the bilinear mapping from \( \mathbb{H} \times H \) to \( U_m' \). Moreover, there exists a constant \( c_2 > 0 \) such that
   \[ |\mathcal{B}(\phi, \psi)|_{U_m'} \leq c_2||\phi||_{U_m'} ||\psi||_{U_m'}, \quad \phi, \psi \in \mathbb{H}. \]
(3) The mapping $\mathcal{B}$ is locally Lipschitz continuous, i.e. for every $r > 0$ there exists a constant $L_r$ such that
\[ |\mathcal{B}(\phi) - \mathcal{B}(\psi)|_V \leq L_r \| \phi - \psi \|_V, \quad \phi, \psi \in V, \quad \| \phi \|_V, \| \psi \|_V \leq r. \]

**Proof.** **Ad. (1)** Let $\phi = (u, \theta) \in V$ and $\psi = (v, \theta) \in V$. By inequalities (6.21) and (6.49) we obtain the following estimates
\[
|\mathcal{B}(\phi, \psi)|_V^2 = \left( |(B_0(u, v), B_2(u, \theta))|_V^2 + |B_0(u, v)|_V^2 + |B_2(u, \theta)|_V^2 \right)
\leq c^2 |u||v|_V^2 + c^2 |u||\theta|_V^2 = c^2 |u||v|_V^2 + |\theta|_V^2 \leq c_1 |\phi|^2 \| \phi \|_V^2,
\]
where $c_1 > 0$ is a certain constant. This completes the proof of inequality (2.4).

**Ad. (2)** Let $\phi = (u, \theta) \in H$ and $\psi = (v, \theta) \in H$. Then by inequalities (6.25) and (6.52) we have the following estimates
\[
|\mathcal{B}(\phi, \psi)|^2 = |B_0(u, v)|^2 + |B_2(u, \theta)|_{\|C\|}^2 \leq c^2 |u||v|_V^2 + c^2 |u||\theta|_V^2 \leq c_2 |\phi|^2 \| \phi \|_V^2
\]
for some constant $c_2 > 0$. The proof of inequality (2.5) is thus complete.

**Ad. (3)** Let us fix $r > 0$ and let $\phi = (u, \theta), \tilde{\phi} = (\tilde{u}, \tilde{\theta}) \in V$ be such that $\| \phi \|_V, \| \tilde{\phi} \|_V \leq r$. We have
\[
|\mathcal{B}(\phi) - \mathcal{B}(\tilde{\phi})|_V^2 = \left( |(B_0(u, u), B_2(u, \theta))|_V^2 + |B_0(u, \theta) - B_0(\tilde{u}, \tilde{\theta})|_V^2 \right)
\leq c^2 |u||\theta|_V^2 + c^2 |u||\theta|_V^2 \leq c_3 |\phi|^2 \| \phi \|_V^2.
\]
We will estimate each term of the right-hand side of the above equality. By inequality (6.21) we have the following estimates
\[
|B_0(u, \theta) - B_0(\tilde{u}, \tilde{\theta})|_V^2 \leq |B_0(u, \theta - \tilde{\theta})|_V^2 + |B_0(\tilde{u}, \tilde{\theta})|_V^2 \leq 2 |u||\theta - \tilde{\theta}|_V^2 + 2 |\theta||\tilde{\theta}|_V^2.
\]
By inequality (6.49) we obtain the following estimates
\[
|B_2(u, \theta) - B_2(\tilde{u}, \tilde{\theta})|_V^2 \leq |B_2(u, \theta - \tilde{\theta})|_V^2 + |B_2(\tilde{u}, \theta - \tilde{\theta})|_V^2 \leq 2 |u||\theta - \tilde{\theta}|_V^2 + 2 |\theta||\tilde{\theta}|_V^2.
\]
Hence
\[
|\mathcal{B}(\phi) - \mathcal{B}(\tilde{\phi})|_V^2 \leq 8r^2 c_3 |\phi|^2 \| \phi \|_V^2.
\]
Thus the Lipschitz condition holds with $L_r = 2 \sqrt{2} c_3 r$. The proof of Lemma is thus complete.

**Lemma 6.8.** Operator $\mathcal{R}$ defined by (6.59) has the following properties:

(1) For every $\phi \in H$, $\mathcal{R}\phi \in V'$ and there exists a constant $c > 0$ such that
\[ |\mathcal{R}\phi|_V \leq c |\phi|_H. \quad (6.63) \]

(2) For every $\phi \in V$:
\[ \langle \mathcal{R}\phi, \phi \rangle \geq -|\phi|^2. \quad (6.64) \]

**Proof.** To prove the first part of the statement let $\phi = (u, \theta) \in H$ and $\psi = (v, \theta) \in V$. Since
\[
|\mathcal{R}\phi|^2 = |(-\theta e_x, -u_d)|^2 = \theta^2 + u_d^2 \leq |\phi|^2,
\]
we have the following estimates
\[
\left| \int_0^1 (\mathcal{R}\phi) \cdot \psi \, dx \right| \leq \int_0^1 |\mathcal{R}\phi||\psi| \, dx \leq \int_0^1 |\phi| \, dx \leq \int_0^1 |\phi| \, dx \leq \left( \int_0^1 |\psi|^2 \, dx \right)^{1/2} \left( \int_0^1 |\phi|^2 \, dx \right)^{1/2} \leq c |\phi|_H \| \phi \|_V,
\]

\[ \int_0^1 |\mathcal{R}\phi| \, dx \leq \left( \int_0^1 |\phi|^2 \, dx \right)^{1/2} \left( \int_0^1 |\phi|^2 \, dx \right)^{1/2} \leq c |\phi|_H \| \phi \|_V,
\]

\[ \int_0^1 |\mathcal{R}\phi| \, dx \leq \left( \int_0^1 |\phi|^2 \, dx \right)^{1/2} \left( \int_0^1 |\phi|^2 \, dx \right)^{1/2} \leq c |\phi|_H \| \phi \|_V,
\]

where \( c > 0 \) is a constant. Thus \( \mathcal{R}\phi \in \mathbb{V}' \) and inequality (6.64) holds. Let us move to the second part of the statement. Let \( \phi = (u, \vartheta) \in \mathbb{V} \). Since \( \langle \mathcal{R}\phi \rangle \cdot \phi = (\vartheta \epsilon_d, -u_d) \cdot (u, \vartheta) = -2u_d \vartheta \) and \( 2u_d \vartheta \leq |\phi|^2 \), we infer that

\[
\langle \mathcal{R}\phi, \phi \rangle = \int_0^T (\mathcal{R}\phi) \cdot \phi \, dx \geq - \int_0^T |\phi|^2 \, dx = -|\phi|^2_{L_2}.
\]

This completes the proof of inequality (6.64) and the proof of the lemma.

Solution of the Boussinesq equations. We apply the abstract framework with the spaces \( \mathbb{H}, \mathbb{V} \) and \( \mathbb{V}_r := \mathbb{V}_m \) with \( m > \frac{4}{3} + 1 \), defined by (6.46) and (6.62) respectively, and the maps \( \mathcal{A}, \mathcal{B} \) and \( \mathcal{R} \) defined by (6.47), (6.58), and (6.59), respectively. By Lemma 6.1 the map \( \mathcal{B} \) satisfies conditions (B.1)-(B.4). By Lemma 6.7 the map \( \mathcal{B} \) satisfies assumption (B.5). By Lemma 6.9 the operator \( \mathcal{R} \) satisfies condition (R.1).

Definition 6.9. A martingale solution of the problem (6.42)-(6.44) is a system \((\bar{X}, \bar{Y}, \bar{W}, \bar{\varphi})\), where \((\bar{X}, \bar{Y}, \bar{W})\) is as in Definition 2.2 and \( \bar{\varphi} : [0, T] \times \bar{\Omega} \to \mathbb{H} \) is a predictable process with \( \bar{\mathbb{P}} \) - a.e. paths \( \bar{\varphi}(\cdot, \omega) \in \mathcal{D}([0, T], \mathbb{H}_m) \cap L^2(0, T; \mathbb{V}) \) such that for all \( t \in [0, T] \) and all \( \psi \in \mathbb{V} \) the following identity holds \( \bar{\mathbb{P}} \)-a.s.

\[
(\bar{\varphi}(t), \psi) + \int_0^t (\mathcal{R}\bar{\varphi}(s), \psi) \, ds + \int_0^t (\mathcal{B} \bar{\varphi}(s), \bar{\varphi}(s)), \psi) \, ds + \int_0^t (\mathcal{R}(\bar{\varphi}(s)), \psi) \, ds = (\phi_0, \psi) + \int_0^t (f(s), \psi) \, ds + \int_0^t G(\bar{\varphi}(s)) \, dW(s), \psi
\]

Applying Theorem 2.3 we obtain the following result about the existence of the martingale solution of the Boussinesq problem.

Corollary 6.10. For every \( \phi_0 \in \mathbb{H} \), \( f \in L^2(0, T; \mathbb{V}') \), \( G : \mathbb{V} \to \mathcal{L}_{HS}(Y, \mathbb{H}) \) satisfying conditions (G.1)-(G.3) and \( F : [0, T] \times \mathbb{H} \times \mathbb{Y} \to \mathbb{H} \) satisfying conditions (F.1)-(F.3) there exists a martingale solution \((\bar{X}, \bar{Y}, \bar{W}, \bar{\varphi})\), where \( \bar{\varphi} = (\bar{u}, \bar{\vartheta}) \), of problem (6.42)-(6.44) such that

\[
\bar{\mathbb{E}} \left[ \sup_{t \in [0, T]} (|\bar{u}(t)|^2_{\mathbb{H}_2} + |\bar{\vartheta}(t)|^2_{\mathbb{H}_2}) + \int_0^T (|\bar{u}(t)|^2_{\mathbb{H}_2} + |\bar{\vartheta}(t)|^2_{\mathbb{H}_2}) \, dt \right] < \infty.
\]

7. Appendix: Proof of Lemma 4.1

7.1. The space of càdlàg functions

Let \((\mathbb{X}, \mathcal{Q})\) be a separable and complete metric space. Let \( \mathbb{D}([0, T]; \mathbb{X}) \) denote the space of all \( \mathbb{X} \)-valued càdlàg functions defined on \([0, T] \), i.e. the functions which are right continuous and have left limits at every \( t \in [0, T] \). We consider the space \( \mathbb{D}([0, T]; \mathbb{X}) \) endowed with the Skorokhod topology. This topology is completely metrizable, see [30].

Let us recall the notion of a modulus of the function. It plays analogous role in the space \( \mathbb{D}([0, T]; \mathbb{X}) \) as the modulus of continuity in the space of continuous functions \( \mathbb{C}([0, T]; \mathbb{X}) \).

Definition 7.1. (see [33]) Let \( u \in \mathbb{D}([0, T]; \mathbb{X}) \) and let \( \delta > 0 \) be given. A modulus of \( u \) is defined by

\[
\omega_{[0,T]}(u, \delta) := \inf_{\Pi_\delta} \max_{t \in [0, T]} \sup_{t_i \leq s \leq t_{i+1} \leq T} \rho(u(t), u(s)),
\]

where \( \Pi_\delta \) is the set of all increasing sequences \( \bar{\omega} = \{0 = t_0 < t_1 < ... < t_n = T\} \) with the following property \( t_{i+1} - t_i \geq \delta \), \( i = 0, 1, ..., n - 1 \). If no confusion seems likely, we will denote the modulus by \( \omega_{[0,T]}(u, \delta) \).

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Let us also recall the basic criterion for relative compactness of a subset of the space \( D([0, T]; X) \), see [30], [35] Chapter II.

**Theorem 7.2.** A set \( A \subset D([0, T]; X) \) has compact closure iff it satisfies the following two conditions:

(a) there exists a dense subset \( J \subset [0, T] \) such that for every \( t \in J \) the set \( \{u(t), u \in A\} \) has compact closure in \( X \).

(b) \( \lim_{n \to \infty} \sup_{u \in A} \|w_{[0,T]}(u, \delta)\| = 0 \).

7.2. **Proof of Lemma 4.1.**

Let us note that since \( D([0, T]; X') \) is metrizable, see [39]. Let us denote by \( \mathcal{K} \subset D([0, T]; X) \), the space of weakly càdlàg functions \( u : [0, T] \to X \) and such that

\[
\sup_{t \in [0, T]} |u(t)|_X \leq r. \tag{7.2}
\]

The space \( D([0, T]; X) \) is completely metrizable as well.

The following lemma says that any sequence \( (u_n) \subset L^\infty([0, T]; \mathbb{H}) \) convergent in \( D([0, T]; \mathbb{U}') \) is also convergent in the space \( D([0, T]; \mathbb{B}_w) \).

**Lemma 7.3.** (see Lemma 2 in [30]) Let \( u_n : [0, T] \to \mathbb{H}, n \in \mathbb{N} \), be functions such that

(i) \( \sup_{n \in \mathbb{N}} \sup_{t \in [0, T]} |u_n(t)|_{\mathbb{H}} \leq r \),

(ii) \( u_n \to u \) in \( D([0, T]; \mathbb{U}') \).

Then \( u, u_0 \in D([0, T]; \mathbb{B}_w) \) and \( u_n \to u \) in \( D([0, T]; \mathbb{B}_w) \) as \( n \to \infty \).

**Proof of Lemma 4.1.** Let us note that since \( \mathcal{K} \subset L^\infty([0, T]; \mathbb{H}) \cap D([0, T]; \mathbb{U}') \), by Lemma 7.3 \( \mathcal{K} \subset D([0, T]; \mathbb{H}_w) \). Now, it is easy to see that \( \mathcal{K} \subset Z \). We can assume that \( \mathcal{K} \) is a closed subset of \( Z \). Because of the assumption (b), the weak topology in \( L^2_{\text{loc}}([0, T]; \mathbb{V}) \) induced on \( Z \) is metrizable. By assumption (a), it is sufficient to consider the metric subspace \( D([0, T]; \mathbb{B}_w) \subset D([0, T]; \mathbb{H}_w) \) with \( r := \sup_{u \in \mathcal{K}} \sup_{t \in [0, T]} |u(t)|_{\mathbb{H}} \). Thus compactness of a subset of \( Z \) is equivalent to its sequential compactness. Let \( (u_n) \) be a sequence in \( \mathcal{K} \). By the Banach-Alaoglu Theorem condition (b) yields that the set \( \mathcal{K} \) is relatively compact in \( L^2_{\text{loc}}([0, T]; \mathbb{V}) \).

Using the compactness criterion in the space of càdlàg functions contained in Theorem 7.2, we will prove that \( (u_n) \) is compact in \( D([0, T]; \mathbb{U}') \). Indeed, by (a) for every \( t \in [0, T] \) the set \( \{u_n(t), n \in \mathbb{N}\} \) is bounded in \( \mathbb{H} \). Since the embedding \( \mathbb{H} \subset \mathbb{U}' \) is compact, the set \( \{u_n(t), n \in \mathbb{N}\} \) is compact in \( \mathbb{U}' \). This together with condition (c) implies compactness of the sequence \( (u_n) \) in the space \( D([0, T]; \mathbb{U}') \).

Therefore there exists a subsequence of \( (u_{n_k}) \), still denoted by \( (u_n) \), such that

\( u_n \to u \) in \( L^2_{\text{loc}}([0, T]; \mathbb{V}) \cap D([0, T]; \mathbb{U}') \) as \( n \to \infty \).

Since \( u_n \to u \) in \( D([0, T]; \mathbb{U}') \), by assumption (a) and Lemma 7.3 we infer that \( u_n \to u \) in \( D([0, T], \mathbb{H}_w) \). We will prove that there exists another subsequence of \( (u_{n_k}) \) such that

\( u_{n_k} \to u \) in \( L^2([0, T]; L^2_{\text{loc}}(\mathcal{O})). \)

To this end let us fix \( R \in \mathbb{N} \). Let us consider the following spaces of restrictions to \( \mathcal{O}_R \) of functions defined on \( \mathcal{O} \)

\[
\mathbb{H}(\mathcal{O}_R) := \{ u|_{\mathcal{O}_R}, u \in \mathbb{H} \} \subset L^2(\mathcal{O}_R; \mathbb{R}^3) \quad \text{and} \quad \mathbb{V}(\mathcal{O}_R) := \{ u|_{\mathcal{O}_R}, u \in \mathbb{V} \} \subset H^1(\mathcal{O}_R; \mathbb{R}^3).
\]
Using again Theorem 7.2 we infer that the sequence \((u_n t O_k)\) is compact in \(D([0, T]; U')\). Thus there exists a subsequence \((u_{n_k}) \subset (u_n)\) such that \(u_{n_k} t O_k \rightarrow u t O_k\) in \(D([0, T]; U')\) as \(k \rightarrow \infty\). Since \(O_k\) is bounded and the norms in \(H\) and \(V\) are equivalent to the norms in \(L^2(O, \mathbb{R}^d)\) and \(H^1(O, \mathbb{R}^d)\), respectively, we infer that the embedding \(\forall t O_k \subset H(O_k)\) is compact. Moreover, the embeddings \(H(O_k) \hookrightarrow H(O_k) \rightarrow U'\) are continuous. Hence, by the Lions Lemma \([33]\), for every \(\varepsilon > 0\) there exists a constant \(C_{\varepsilon, R} > 0\) such that

\[
|u|^2_{L^2(O_k)} \leq \varepsilon |u|^2_{V} + C_{\varepsilon, R} |u_{t O_k}|^2_{U}, \quad u \in V.
\]

In particular, for almost all \(s \in [0, T]\)

\[
|u_n(s) - u(s)|^2_{L^2(O_k)} \leq \varepsilon |u_n(s) - u(s)|^2_{V} + C_{\varepsilon, R} |u_n t O_k - u t O_k|^2_{U}, \quad k \in \mathbb{N},
\]

and hence

\[
p^2_{T, R}(u_n - u) = \|u_n - u\|^2_{L^2([0, T]; L^2(O_k))} \leq \varepsilon \|u_n - u\|^2_{L^2([0, T]; V)} + C_{\varepsilon, R} \|u_n t O_k - u t O_k\|^2_{L^2([0, T]; U')}.
\]

Passing to the upper limit as \(k \rightarrow \infty\) in the above inequality and using the estimate

\[
\lim_{k \rightarrow \infty} p^2_{T, R}(u_n - u) = \lim_{k \rightarrow \infty} \|u_n - u\|^2_{L^2([0, T]; V)} + \|u\|^2_{L^2([0, T]; V)} \leq 4c_2,
\]

where \(c_2 = \sup_{\varepsilon \in \mathbb{R}} \|u\|^2_{L^2([0, T]; V)}\), we infer that \(\lim_{k \rightarrow \infty} p^2_{T, R}(u_n - u) \leq 4c_2\varepsilon\). By the arbitrariness of \(\varepsilon\),

\[
\lim_{k \rightarrow \infty} p^2_{T, R}(u_n - u) = 0.
\]

Using the diagonal method we can choose a subsequence of \((u_n)\) convergent in \(L^2(0, T; L^2_{loc}(O))\). The proof of Lemma 4 is thus complete. \(\square\)

8. Appendix B: Time homogeneous Poisson random measure

We follow the approach due to Brzeźniak and Hausenblas \([11], [10]\), see also \([28], [3]\) and \([40]\). Let us denote \(\mathbb{N} := \{0, 1, 2, \ldots\}\), \(\overline{\mathbb{N}} := \mathbb{N} \cup \{\infty\}\). Let \((S, \mathcal{S})\) be a measurable space and let \(\mathcal{M}(S)\) be the set of all \(\overline{\mathbb{N}}\) valued measures on \((S, \mathcal{S})\). On the set \(\mathcal{M}(S)\) we consider the \(\sigma\)-field \(\mathcal{M}(S)\) defined as the smallest \(\sigma\)-field such that for all \(B \in \mathcal{S}\): the map \(i_B : \mathcal{M}(S) \ni \mu \mapsto \mu(B) \in \overline{\mathbb{N}}\) is measurable.

Let \((\Omega, \mathcal{F}, \mathbb{P})\) be a complete probability space with filtration \(\mathbb{F} := (\mathcal{F}_t)_{t \geq 0}\) satisfying the usual hypotheses, see \([34]\).

**Definition 8.1.** (see \([3]\) and Appendix C in \([11]\)). Let \((Y, \mathcal{Y})\) be a measurable space. A time homogeneous Poisson random measure \(\eta\) on \((Y, \mathcal{Y})\) over \((\Omega, \mathcal{F}, \mathbb{P})\) is a measurable function

\[
\eta : (\Omega, \mathcal{F}) \rightarrow (\mathcal{M}(\overline{\mathbb{N}} \times \mathcal{Y}), \mathcal{M}(\overline{\mathbb{N}} \times \mathcal{Y}))
\]

such that

1. for all \(B \in \mathcal{B}(\mathbb{R}_+) \otimes \mathcal{Y}\), \(\eta(B) := i_B \circ \eta : \Omega \rightarrow \overline{\mathbb{N}}\) is a Poisson random variable with parameter \(\mathbb{E}[\eta(B)]\);
2. \(\eta\) is independently scattered, i.e. if the sets \(B_1 \in \mathcal{B}(\mathbb{R}_+) \otimes \mathcal{Y}, j = 1, \ldots, n\), are disjoint then the random variables \(\eta(B_j), j = 1, \ldots, n\), are independent;
3. for all \(U \in \mathcal{Y}\) the \(\overline{\mathbb{N}}\)-valued process \((N(t, U))_{t \geq 0}\) defined by

\[
N(t, U) := \eta(0, t) \times U, \quad t \geq 0
\]

is \(\mathbb{F}\)-adapted and its increments are independent of the past, i.e. if \(t > s \geq 0\), then \(N(t, U) - N(s, U) = \eta(s, t) \times U\) is independent of \(\mathcal{F}_s\).
If \( \eta \) is a time homogeneous Poisson random measure then the formula
\[
\mu(A) := \mathbb{E}[\eta((0, 1] \times A)], \quad A \in \mathcal{Y}
\]
defines a measure on \((Y, \mathcal{Y})\) called an intensity measure of \( \eta \). Moreover, for all \( T < \infty \) and all \( A \in \mathcal{Y} \) such that \( \mathbb{E}[\eta((0, T] \times A)] < \infty \), the \( \mathbb{R} \)-valued process \( \{\tilde{N}(t, A)\}_{t \in (0, T]} \) defined by
\[
\tilde{N}(t, A) := \eta((0, t] \times A) - t \mu(A), \quad t \in (0, T],
\]
is an integrable martingale on \((\Omega, \mathcal{F}, \mathbb{P})\). The random measure \( l \otimes \mu \) on \( \mathcal{B}(\mathbb{R}_+) \otimes \mathcal{Y} \), where \( l \) stands for the Lebesgue measure, is called an compensator of \( \eta \) and the difference between a time homogeneous Poisson random measure \( \eta \) and its compensator, i.e.
\[
\tilde{\eta} := \eta - l \otimes \mu,
\]
is called a compensated time homogeneous Poisson random measure.

Let us also recall basic properties of the stochastic integral with respect to \( \tilde{\eta} \), see \([11], [28]\) and \([40]\) for details. Let \( E \) be a separable Hilbert space and let \( \mathcal{P} \) be a predictable \( \sigma \)-field on \([0, T] \times \Omega \). Let \( \mathcal{L}^2_{\mu,T}(\mathcal{P} \otimes \mathcal{Y} \otimes \mathcal{P} \otimes \mu; E) \) be a space of all \( E \)-valued, \( \mathcal{P} \otimes \mathcal{Y} \)-measurable processes such that
\[
\mathbb{E}\left[ \int_0^T \int Y \|\xi(s, \cdot, y)\|^2_E d\mu(y) \right] < \infty.
\]
If \( \xi \in \mathcal{L}^2_{\mu,T}(\mathcal{P} \otimes \mathcal{Y} \otimes \mathcal{P} \otimes \mu; E) \) then the integral process \( \int_0^T \int Y \xi(s, \cdot, y) \tilde{\eta}(ds, dy), t \in [0, T], \) is a càdlàg \( L^2 \)-integrable martingale. Moreover, the following isometry formula holds
\[
\mathbb{E}\left[ \left\| \int_0^T \int Y \xi(s, \cdot, y) \tilde{\eta}(ds, dy) \right\|^2_E \right] = \mathbb{E}\left[ \int_0^T \int Y \|\xi(s, \cdot, y)\|^2_E d\mu(y) \right], \quad t \in [0, T]. \tag{8.1}
\]

9. Appendix C: A version of the Skorokhod Embedding Theorem

In the proof of Theorem \([23]\) we use the following version of the Skorokhod Embedding Theorem following from the version due to Jakubowski \([29]\) and the version due to Brzeźniak and Hausenblas \([10]\) Theorem E.1).

**Corollary 9.1.** (Corollary 2 in \([39]\)) Let \( X_1 \) be a separable complete metric space and let \( X_2 \) be a topological space such that there exists a sequence \( \{f_i\}_{i \in \mathbb{N}} \) of continuous functions \( f_i : X_2 \to \mathbb{R} \) separating points of \( X_2 \). Let \( X := X_1 \times X_2 \) with the Tychonoff topology induced by the projections
\[
\pi_i : X_1 \times X_2 \to X_i, \quad i = 1, 2.
\]
Let \((\Omega, \mathcal{F}, \mathbb{P})\) be a probability space and let \( \chi_n : \Omega \to X_1 \times X_2 \), \( n \in \mathbb{N} \), be a family of random variables such that the sequence \( \{\text{Law}(\chi_n), n \in \mathbb{N}\} \) is tight on \( X_1 \times X_2 \). Finally let us assume that there exists a random variable \( \rho : \Omega \to X_1 \) such that \( \text{Law}(\pi_1 \circ \chi_n) = \text{Law}(\rho) \) for all \( n \in \mathbb{N} \).

Then there exists a subsequence \( \{\chi_{n_k}\}_{k \in \mathbb{N}} \), a probability space \( (\tilde{\Omega}, \tilde{\mathcal{F}}, \tilde{\mathbb{P}}) \), a family of \( X_1 \times X_2 \)-valued random variables \( \{\tilde{\chi}_k, k \in \mathbb{N}\} \) on \((\tilde{\Omega}, \tilde{\mathcal{F}}, \tilde{\mathbb{P}})\) and a random variable \( \chi_* : \Omega \to X_1 \times X_2 \) such that
\begin{enumerate}
\item \( \text{Law}(\tilde{\chi}_k) = \text{Law}(\chi_{n_k}) \) for all \( k \in \mathbb{N} \);
\item \( \tilde{\chi}_k \to \chi_* \) in \( X_1 \times X_2 \) a.s. as \( k \to \infty \);
\item \( \pi_1 \circ \tilde{\chi}_k(\tilde{\omega}) = \pi_1 \circ \chi_*(\tilde{\omega}) \) for all \( \tilde{\omega} \in \tilde{\Omega} \).
\end{enumerate}
To prove Theorem 2.3 we use Corollary 9.1 for the space
\[ X_2 := \mathcal{Z} := L^2_{\text{loc}}(0, T; \mathcal{V}) \cap L^2(0, T; L^2_{\text{loc}}(\mathcal{O}_R)) \cap \mathbb{D}([0, T]; \mathcal{U'}) \cap \mathbb{D}([0, T]; \mathbb{H}_w). \]

We recall now the result about the existence of the countable family of real valued continuous mappings defined on \( \mathcal{Z} \) and separating points of this space.

**Remark 9.2.** (see Remark 2 in [39])

1. Since \( L^2(0, T; L^2_{\text{loc}}(\mathcal{O}_R)) \) and \( \mathbb{D}([0, T]; \mathcal{U'}) \) are separable and completely metrizable spaces, we infer that on each of these spaces there exists a countable family of continuous valued mappings separating points, see [4], exposé 8.

2. For the space \( L^2_{\text{loc}}(0, T; \mathcal{V}) \) it is sufficient to put
\[ f_m(u) := \int_0^T \langle u(t), v_n(t) \rangle \, dt \in \mathbb{R}, \quad u \in L^2(0, T; \mathcal{V}), \quad m \in \mathbb{N}, \]
where \( \{v_n, m \in \mathbb{N}\} \) is a dense subset of \( L^2(0, T; \mathcal{V}). \) Then \( \{f_m\}_{m \in \mathbb{N}} \) is a sequence of continuous real valued mappings separating points of the space \( L^2_{\text{loc}}(0, T; \mathcal{V}). \)

3. Let \( \mathbb{H}_0 \subset \mathbb{H} \) be a countable and dense subset of \( \mathbb{H}. \) Then by (4.2) for each \( h \in \mathbb{H}_0 \) the mapping
\[ \mathbb{D}([0, T]; \mathbb{H}_0) \ni u \mapsto (u(\cdot), h)_{\mathbb{H}_0} \in \mathbb{D}([0, T]; \mathbb{R}) \]
\( f_h(t) \) is continuous. Since \( \mathbb{D}([0, T]; \mathbb{R}) \) is a separable complete metric space, there exists a sequence \( \{g_l\}_{l \in \mathbb{N}} \) of real valued continuous functions defined on \( \mathbb{D}([0, T]; \mathbb{R}) \) separating points of this space. Then the mappings \( f_h, h \in \mathbb{H}_0, l \in \mathbb{N} \) defined by
\[ f_h(u) := g_l((u(\cdot), h)_{\mathbb{H}_0}), \quad u \in \mathbb{D}([0, T]; \mathbb{H}_0), \]
form a countable family of continuous mappings on \( \mathbb{D}([0, T]; \mathbb{H}_w) \) separating points of this space.

10. Appendix D: Proofs of Lemmas 5.2 and 5.3

**Proof of Lemma 5.2.** For every \( n \in \mathbb{N} \) and \( R > 0 \) let us define
\[ \tau_n(R) := \inf \{t \geq 0 : |u_n(t)|_{\mathbb{H}} \geq R \} \land T. \quad (10.1) \]
Since the process \( (u_n(t))_{t \in [0, T]} \) is \( \mathbb{P} \)-adapted and right-continuous, \( \tau_n(R) \) is a stopping time. Moreover, since the process \( (u_n) \) is càdlàg on \( [0, T], \) the trajectories \( t \mapsto u_n(t) \) are bounded on \( [0, T], \) \( \mathbb{P} \)-a.s. Thus \( \tau_n(R) \uparrow T, \mathbb{P} \)-a.s., as \( R \uparrow \infty. \)

Assume first that \( p = 2 \) or \( p = 2 + \gamma. \) Using the Itô formula to the function \( \phi(x) := |x|^p := |x|^p_{\mathbb{H}}, \) \( x \in \mathbb{H}, \) we obtain for all \( t \in [0, T] \)
\[ |u_n(t \wedge \tau_n(R))|^p_{\mathbb{H}} = |P_n u_n|_{\mathbb{H}}^p + \int_0^{t \wedge \tau_n(R)} \langle [p|u_n(s)|^p_{\mathbb{H}}^{-1} \Phi_n(u_n(s)), u_n(s)] \rangle \, ds \]
\[ + M_n(t \wedge \tau_n(R)) + L_n(t \wedge \tau_n(R)) + K_n(t \wedge \tau_n(R)) + N_n(t \wedge \tau_n(R)) + J_n(t \wedge \tau_n(R)), \quad (10.2) \]
where
\[ \Phi_n(v) := -P_n Av - B_n(v) - P_n Rv + P_n f, \quad v \in \mathbb{H}_w, \quad (10.3) \]
and for $t \in [0, T]$

$$
M_n(t) := \int_0^t \int_{Y_0} \left[ |u_n(s^-) + P_nF(s, u_n(s^-); y)|_{L^2}^2 - |u_n(s^-)|_{L^2}^2 \right] \eta(ds, dy) + \int_0^t \int_{Y \setminus Y_0} \left[ |u_n(s^-) + P_nF(s, u_n(s^-); y)|_{L^2}^2 - |u_n(s^-)|_{L^2}^2 \right] \tilde{\eta}(ds, dy),
$$

(10.4)

$$
I_n(t) := \int_0^t \int_{Y_0} \left[ |u_n(s) + P_nF(s, u_n(s); y)|_{L^2}^2 - |u_n(s)|_{L^2}^2 \right] d\mu(y)ds,
$$

(10.5)

$$
K_n(t) := \int_0^t \int_{Y \setminus Y_0} \left[ |u_n(s) + P_nF(s, u_n(s); y)|_{L^2}^2 - |u_n(s)|_{L^2}^2 \right] d\mu(y)ds,
$$

(10.6)

$$
N_n(t) := \int_0^t |u_n(s)|_{H^2}^{p-2} \langle u_n(s), G(s, u_n(s)) \rangle_{L^2} dW(s),
$$

(10.7)

$$
J_n(t) := \frac{1}{2} \int_0^t \Tr[P_nG(s, u_n(s)) \frac{\partial^2 \phi}{\partial x^2}(P_nG(s, u_n(s)))^T] ds.
$$

(10.8)

Since by (2.3) we have $\langle A u_n, u_n \rangle = \| u_n \|_V^2$ and by (2.5), $\langle B u_n, u_n \rangle = 0$, we infer that for all $s \in [0, T]$

$$
\langle \Phi_n(u_n(s)), u_n(s) \rangle = -\| u_n(s) \|_V^2 - \langle R u_n(s), u_n(s) \rangle + \langle f(s), u_n(s) \rangle.
$$

By assumptions (R.1) and (C.1), (2.2) and the Schwarz inequality, we obtain for every $\varepsilon > 0$ and for all $s \in [0, T]$

$$
\langle \Phi_n(u_n(s)), u_n(s) \rangle \leq (-1 + \varepsilon)\| u_n(s) \|_V^2 + \left( \frac{1}{2} |f(s)|_V + \varepsilon_1 \right) \| u_n(s) \|_{L^2}^2 + \frac{1}{8\varepsilon} |f(s)|_V^2.
$$

Hence

$$
\int_0^{\tau_n(R)} \{ p|u_n(s)|_{L^2}^{p-2} \langle \Phi_n(u_n(s)), u_n(s) \rangle \} ds 
\leq p \int_0^{\tau_n(R)} |u_n(s)|_{L^2}^{p-2} \left( (-1 + \varepsilon)\| u_n(s) \|_V^2 + \left( \frac{1}{2} |f(s)|_V + \varepsilon_1 \right) \| u_n(s) \|_{L^2}^2 + \frac{1}{8\varepsilon} |f(s)|_V^2 \right) ds.
$$

(10.9)

Again, by (2.3) inequality (2.13) in assumption (G.2) can be written equivalently in the following form

$$
\| G(s, u) \|_{L^2(Y_0, \tilde{\eta})}^2 \leq (2 - a)\| u \|_V^2 + \lambda \| u \|_{L^2}^2 + \kappa, \quad u \in \mathcal{V}.
$$

Hence

$$
J_n(t \land \tau_n(R)) \leq \frac{p(p - 1)}{2} \int_0^{\tau_n(R)} |u_n(s)|_{L^2}^{p-2} \left( (2 - a)\| u_n(s) \|_V^2 + \lambda \| u_n(s) \|_{L^2}^2 + \kappa \right) ds.
$$

(10.10)

From the Taylor formula, it follows that for every $p \geq 2$ there exists a positive constant $c_p > 0$ such that for all $x, h \in \mathbb{H}$ the following inequality holds

$$
|x + h|_{L^2}^p - |x|_{L^2}^p - p|x|_{L^2}^{p-2} \langle x, h \rangle_{L^2} \leq c_p(|x|_{L^2}^{p-2} + |h|_{L^2}^{p-2}) |h|_{L^2}^2.
$$

(10.11)

By (10.11) and (2.10) we obtain the following inequalities

$$
|I_n(t)| \leq c_p \int_0^t \int_{Y \setminus Y_0} |P_nF(s, u_n(s); y)|_{L^2}^2 \left( |u_n(s)|_{L^2}^{p-2} + |P_nF(s, u_n(s); y)|_{L^2}^{p-2} \right) d\mu(dy)ds
\leq c_p \int_0^t \left( C_1 |u_n(s)|_{L^2}^{p-2} + C_2 (1 + |u_n(s)|_{L^2}^2) \right) ds
\leq \hat{c}_p \int_0^t \left( 1 + |u_n(s)|_{L^2}^2 \right) ds = \tilde{c}_p t + \hat{c}_p \int_0^t |u_n(s)|_{L^2}^2 ds, \quad t \in [0, T],
$$

where $\hat{c}_p > 0$ is a constant. Thus by the Fubini Theorem, we obtain the following inequality

$$
\mathbb{E}[|I_n(t)|] \leq \hat{c}_p t + \hat{c}_p \int_0^t \mathbb{E}[|u_n(s)|_{L^2}^2] ds, \quad t \in [0, T].
$$

(10.12)
Let us move now to the term $K_n$ defined by (10.6). From (10.11) we obtain the following inequalities for all $x, h \in \mathbb{R}$

$$|x + h|_{p}^p - |x|_{p}^p \leq \frac{p}{2}|x|_{p}^{p-2}|h|_{p}^2 + c_p h^p.$$  \hfill (10.13)

By (10.13), (2.10) and the fact that $\mu(Y \setminus Y_0) < \infty$ we get

$$|K_n(t)| \leq \int_0^t \int_{Y \setminus Y_0} \left( \frac{p}{2}\mu_n(s)_{p}^p + (c_p + \frac{p}{2})|u_n(s)|_{p}^{p-2}F(s, u_n(s))_{p}^2 + c_p F(s, u_n(s))_{p}^2 \right) \mu(s) ds$$

$$\leq \int_0^t \left( \frac{p}{2}\mu(Y \setminus Y_0)|u_n(s)|_{p}^p ds + C(c_p + \frac{p}{2})|u_n(s)|_{p}^{p-2} \right. \left. (1 + |u_n(s)|_{p}^2) + c_p C_p (1 + |u_n(s)|_{p}^2) ds \right.$$ 

$$\leq \hat{c}_p \int_0^t [1 + |u_n(s)|_{p}^p] ds = \hat{c}_p t + \hat{c}_p \int_0^t |u_n(s)|_{p}^p ds, \quad t \in [0, T],$$

where $\hat{c}_p$ is a positive constant. Thus by the Fubini Theorem, we obtain the following inequality

$$\mathbb{E}[|K_n(t)|] \leq \hat{c}_p t + \hat{c}_p \int_0^t \mathbb{E}[|u_n(s)|_{p}^p] ds, \quad t \in [0, T].$$  \hfill (10.14)

By (10.11), (2.10) and (10.11), the process $(M_n(t \wedge \tau_n(R)))_{t \in [0, T]}$ is an integrable martingale. Hence $\mathbb{E}[M_n(t \wedge \tau_n(R))] = 0$ for all $t \in [0, T]$. Similarly, by (2.13) and (10.11), the process $(\mathbb{N}_n(t \wedge \tau_n(R)))_{t \in [0, T]}$ is an integrable martingale and thus $\mathbb{E}[\mathbb{N}_n(t \wedge \tau_n(R))] = 0$ for all $t \in [0, T]$. By (10.2), (10.9), (10.10), (10.12) and (10.14), we have for all $t \in [0, T]$

$$\mathbb{E}[|u_n(t \wedge \tau_n(R))|_{p}^p] + p[1 - \varepsilon - \frac{1}{2}(p - 1)(2 - a)]\mathbb{E}\left[ \int_0^{T \wedge \tau_n(R)} |u_n(s)|_{p}^{p-2}||u_n(s)||^2 ds \right]$$

$$\leq c(p) + \tilde{c}(p) \int_0^{T \wedge \tau_n(R)} \mathbb{E}[|u_n(s)|_{p}^p] ds,$$  \hfill (10.15)

where $c(p)$ and $\tilde{c}(p)$ are some positive constants. Let us choose $\varepsilon > 0$ such that $1 - \varepsilon - \frac{(p-1)(2-a)}{2} > 0$. Note that since by assumption (G.2) $a \in (2 - \frac{2}{p}, 2]$, such an $\varepsilon$ exists. By (10.15) we have, in particular, the following inequality

$$\mathbb{E}[|u_n(t \wedge \tau_n(R))|_{p}^p] \leq c(p) + \tilde{c}(p) \int_0^{T \wedge \tau_n(R)} \mathbb{E}[|u_n(s)|_{p}^p] ds.$$

By the Gronwall Lemma we infer that for all $t \in [0, T]$: $\mathbb{E}[|u_n(t \wedge \tau_n(R))|_{p}^p] \leq \hat{C}_p$ for some constant $\hat{C}_p$ independent of $t \in [0, T], R > 0$ and $n \in \mathbb{N}$, i.e.

$$\sup_{n \geq 1} \sup_{t \in [0, T]} \mathbb{E}[|u_n(t \wedge \tau_n(R))|_{p}^p] \leq \hat{C}_p.$$

Hence, in particular, $\sup_{n \geq 1} \mathbb{E}\left[ \int_0^{T \wedge \tau_n(R)} |u_n(s)|_{p}^p ds \right] \leq \hat{C}_p$ for some constant $\hat{C}_p > 0$. Passing to the limit as $R \uparrow \infty$, by the Fatou Lemma we infer that

$$\sup_{n \geq 1} \mathbb{E}\left[ \int_0^{T} |u_n(s)|_{p}^p ds \right] \leq \hat{C}_p.$$  \hfill (10.16)

By (10.15) and (10.16), we infer that $\sup_{n \geq 1} \mathbb{E}\left[ \int_0^{T \wedge \tau_n(R)} |u_n(s)|_{p}^{p-2}||u_n(s)||^2 ds \right] \leq C_p$ for some positive constant $C_p$. Passing to the limit as $R \uparrow \infty$ and using again the Fatou Lemma we infer that

$$\sup_{n \geq 1} \mathbb{E}\left[ \int_0^{T} |u_n(s)|_{p}^{p-2}||u_n(s)||^2 ds \right] \leq C_p.$$  \hfill (10.17)

In particular, putting $p := 2$ by (2.2), (10.17) and (10.16) we obtain inequality (5.3).
Let us move to the proof of inequality (5.2). By the Burkholder-Davis-Gundy inequality we obtain
\[
\mathbb{E}\left[ \sup_{t \in [0, T]} |M_n(t \wedge \tau_n(R))| \right] \leq \tilde{K}_p \mathbb{E}\left[ \left( \int_0^{\tau_n(R)} |u_n(s) + P_n F(s, u_n(s); y)|^p_{L^p} - |u_n(s)|^p_{L^p} \right)^2 \mu(dy)ds \right]^{\frac{1}{2}}
\] (10.18)
for some constant \( \tilde{K}_p > 0 \). Let us recall that \( M_n \) is defined by (10.14). By (10.11) and the Schwarz inequality we obtain the following inequalities for all \( x, h \in \mathbb{H} \)
\[
(\|x + h\|^p_{L^p} - \|x\|^p_{L^p})^2 \leq 2\|x\|^{2p-2}\|h\|^2_{L^p} + c_p^2(\|x\|^{2p-2} + \|h\|^{2p-2})^2
\]
\[
\leq 2\|x\|^{2p-2}\|h\|^2_{L^p} + 4c_p^2\|x\|^{2p-4}\|h\|^4_{L^p} + 4c_p^2\|h\|^2_{L^p}.
\]
Hence by inequality (2.10) in assumption (F:2) we obtain for all \( s \in [0, T] \)
\[
\int_0^{\tau_n(R)} \left( |u_n(s) + P_n F(s, u_n(s); y)|^p_{L^p} - |u_n(s)|^p_{L^p} \right)^2 \mu(dy)ds \leq 2\tilde{c}_1\int_0^{\tau_n(R)} |F(s, u_n(s); y)|^{2p}_{L^p} \mu(dy)
\]
\[
+ 4c_p^2\int_0^{\tau_n(R)} |F(s, u_n(s); y)|^{2p}_{L^p} \mu(dy) + 4c_p^2\int_0^{\tau_n(R)} |F(s, u_n(s); y)|^{2p}_{L^p} \mu(dy)
\]
\[
\leq C_1 + C_2|u_n(s)|^{2p-4}_{L^p} + C_3|u_n(s)|^{2p-2}_{L^p} + C_4|u_n(s)|^{2p}_{L^p}
\] (10.19)
for some positive constants \( C_i, i = 1, \ldots, 4 \). By (10.19) and the Young inequality we infer that
\[
\left( \int_0^{\tau_n(R)} \left( |u_n(s) + P_n F(s, u_n(s); y)|^p_{L^p} - |u_n(s)|^p_{L^p} \right)^2 \mu(dy)ds \right)^{\frac{1}{2}} \leq \tilde{c}_1 + \tilde{c}_2\left( \int_0^{\tau_n(R)} |u_n(s)|^{2p}_{L^p} \mu(dy)ds \right)^{\frac{1}{2}}
\] (10.20)
where \( \tilde{c}_1 \) and \( \tilde{c}_2 \) are some positive constants. By (10.18), (10.20) and (10.16) we obtain the following inequalities
\[
\mathbb{E}\left[ \sup_{t \in [0, T]} |M_n(t \wedge \tau_n(R))| \right] \leq \tilde{K}_p \tilde{c}_1 + \tilde{K}_p \tilde{c}_2\mathbb{E}\left[ \left( \int_0^{\tau_n(R)} |u_n(s)|^{2p}_{L^p} \mu(dy)ds \right)^{\frac{1}{2}} \right]
\]
\[
\leq \tilde{K}_p \tilde{c}_1 + \tilde{K}_p \tilde{c}_2\mathbb{E}\left[ \left( \sup_{s \in [0, T]} |u_n(s \wedge \tau_n(R))|^{p}_{L^p} \right)^{\frac{1}{2}} \left( \int_0^{\tau_n(R)} |u_n(s)|^{2p}_{L^p} \mu(dy)ds \right)^{\frac{1}{2}} \right]
\]
\[
\leq \tilde{K}_p \tilde{c}_1 + \frac{1}{4}\mathbb{E}\left[ \sup_{t \in [0, T]} |u_n(t \wedge \tau_n(R))|^{p}_{L^p} \right] + \tilde{K}_p \tilde{c}_2^2\mathbb{E}\left[ \int_0^\tau |u_n(s)|^{2p}_{L^p} \mu(dy)ds \right]
\]
\[
\leq \frac{1}{4}\mathbb{E}\left[ \sup_{t \in [0, T]} |u_n(t \wedge \tau_n(R))|^{p}_{L^p} \right] + \tilde{c},
\] (10.21)
where \( \tilde{c} = \tilde{K}_p \tilde{c}_1 + \tilde{K}_p^2 \tilde{c}_2^2 \tilde{c}_p \). (The constant \( \tilde{C}_p \) is the same as in (10.16)).

Similarly, by the Burkholder-Davis-Gundy inequality we obtain
\[
\mathbb{E}\left[ \sup_{t \in [0, T]} |N_n(t \wedge \tau_n(R))| \right] \leq C_p \cdot \mathbb{E}\left( \left( \int_0^{\tau_n(R)} |u_n(s)|^{2p-2}_{L^p} \cdot \|G(s, u_n(s))\|^2_{L^p} \mu(dy)ds \right)^{\frac{1}{2}} \right]
\]
\[
\leq C_p \mathbb{E}\left( \left( \sup_{t \in [0, T]} |u_n(t \wedge \tau_n(R))|^{p}_{L^p} \right)\left( \int_0^{\tau_n(R)} |u_n(s)|^{2p-2}_{L^p} \cdot \|G(s, u_n(s))\|^2_{L^p} \mu(dy)ds \right)^{\frac{1}{2}} \right],
\]
where \( N_n \) is defined by (10.7). By inequality (2.13) in assumption (G:2) and estimates (10.17).
Using inequalities (10.16), (10.21) and (10.23) in (10.25) we infer that

\begin{equation}
E \left[ \sup_{r \in [0,T]} |N_n(r \cap \tau_n(R))| \right] 
\leq C_p \left( \sup_{r \in [0,T]} |u_n(s \cap \tau_n(R))| \right)^p \left( \int_0^{\sup \tau_n(R)} |u_n(s)|^p \left[ A |u_n(s)|^{\alpha-2} + k \right] ds \right)^{\frac{1}{p}} 
\leq \frac{1}{4} E \left[ \sup_{r \in [0,T]} |u_n(r \cap \tau_n(R))| \right] + \tilde{c},
\end{equation}

(10.22)

where \( \tilde{c} = C^2 p^2 [ \tilde{C}_p + k \tilde{C}_{p-2} + (2 - \alpha) C_2 ] \). (The constants \( \tilde{C}_p, \tilde{C}_{p-2} \) are the same as in (10.16) and \( C_2 \) is the same as in (10.17).) Therefore by (10.24) for all \( t \in [0, T] \)

\begin{equation}
|u_n(t \cap \tau_n(R))| \leq c(p) + \tilde{c}(p) \left( \int_0^T |u_n(s)|^p \right)^{\frac{1}{p}} 
+ \sup_{r \in [0,T]} |M_n(r \cap \tau_n(R))| + \sup_{r \in [0,T]} |N_n(r \cap \tau_n(R))|.
\end{equation}

(10.24)

Since inequality (10.24) holds for all \( t \in [0, T] \) and the right-hand side of (10.24) in independent of \( t \), we infer that

\begin{equation}
E \left[ \sup_{r \in [0,T]} |u_n(t \cap \tau_n(R))| \right] \leq c(p) + \tilde{c}(p) \left( \int_0^T |u_n(s)|^p \right)^{\frac{1}{p}} 
+ E \left[ \sup_{r \in [0,T]} |M_n(r \cap \tau_n(R))| \right] + E \left[ \sup_{r \in [0,T]} |N_n(r \cap \tau_n(R))| \right].
\end{equation}

(10.25)

Using inequalities (10.16), (10.24) and (10.25) in (10.25) we infer that

\begin{equation}
E \left[ \sup_{r \in [0,T]} |u_n(t \cap \tau_n(R))| \right] \leq C_1(p),
\end{equation}

where \( C_1(p) > 0 \) is a constant independent of \( n \in \mathbb{N} \) and \( R > 0 \). Passing to the limit as \( R \to \infty \), we obtain inequality (5.1). Thus the lemma holds for \( p \in [2, 2 + \gamma] \).

Let now \( p \in [1, 2 + \gamma) \setminus \{2\} \). Let us fix \( n \in \mathbb{N} \). Then

\begin{equation}
|u_n(t)|^p \leq \left( \sup_{r \in [0,T]} |u_n(r)| \right)^p \leq \left( \sup_{r \in [0,T]} |u_n(r)|^{2+\gamma} \right)^{\frac{p}{2+\gamma}}, \quad t \in [0, T].
\end{equation}

Thus

\begin{equation}
\sup_{r \in [0,T]} |u_n(t)|^p \leq \left( \sup_{r \in [0,T]} |u_n(r)|^{2+\gamma} \right)^{\frac{p}{2+\gamma}}
\end{equation}

and by the Hölder inequality and inequality (5.1) with \( p := 2 + \gamma \)

\begin{equation}
E \left[ \sup_{r \in [0,T]} |u_n(t)|^p \right] \leq E \left[ \left( \sup_{r \in [0,T]} |u_n(r)|^{2+\gamma} \right)^{\frac{p}{2+\gamma}} \right] \leq \left( E \left[ \sup_{r \in [0,T]} |u_n(r)|^{2+\gamma} \right] \right)^{\frac{p}{2+\gamma}} \leq \left( |C_1(2 + \gamma)| \right)^{\frac{p}{2+\gamma}}.
\end{equation}

Since \( n \in \mathbb{N} \) was chosen in an arbitrary way, we infer that

\begin{equation}
\sup_{n \in \mathbb{N}} E \left[ \sup_{r \in [0,T]} |u_n(t)|^p \right] \leq C_1(p),
\end{equation}

where \( C_1(p) = |C_1(2 + \gamma)|^{\frac{p}{2+\gamma}} \). The proof of Lemma 5.3 is thus complete.

To prove Lemma 5.3 we will use Corollary 5.2. To check condition (c) in Corollary 4.2 we will use the following lemma.
Lemma 10.1. (Lemma 9 in [39]) Let \((X_n)_{n \in \mathbb{N}}\) be a sequence of \(\mathcal{U}^r\)-valued random variables. Assume that there exist constants \(\alpha, \beta > 0\) and \(C > 0\) such that for every sequence \((\tau_n)_{n \in \mathbb{N}}\) of \(\mathbb{F}\)-stopping times with \(\tau_n \leq T\) and for every \(n \in \mathbb{N}\) and \(\theta \geq 0\) the following condition holds
\[
\mathbb{E}[(|X_n(\tau_n + \theta) - X_n(\tau_n)|^p_{\mathcal{U}^r})] \leq C\theta^\beta.
\] (10.26)

Then the sequence \((X_n)_{n \in \mathbb{N}}\) satisfies the Aldous condition in the space \(\mathcal{U}'\).

Proof of Lemma [5.3] The proof is essentially the same as the proof of Lemma 5 in [39]. We provide the details because of its importance in the proof of Theorem 2.3. We will apply Corollary 4.2. By estimates (5.2) and (5.3), conditions (a), (b) are satisfied. Using Lemma 10.1, we will prove that the sequence \((u_n)_{n \in \mathbb{N}}\) satisfies the Aldous condition in the space \(\mathcal{U}'\). Let \((\tau_n)_{n \in \mathbb{N}}\) be a sequence of stopping times such that \(0 \leq \tau_n \leq T\). By (5.1), we have

\[
 u_n(t) = P_n u_0 - \int_0^{\tau_n} P_n A u_n(s) ds - \int_0^{\tau_n} P_n R u_n(s) ds - \int_0^{\tau_n} B_n(u_n(s)) ds + \int_0^{\tau_n} P_n f(s) ds + \int_0^{\tau_n} P_n^* G(s, u_n(s)) ds W(s) =: \sum_{i=1}^8 J_i(t), \quad t \in [0, T].
\]

Let \(\theta > 0\). We will check that each term \(J_i, i = 1, \ldots, 8\), satisfies condition (10.26) in Lemma 10.1.

Since by assumption (A.1) \(A : \mathcal{V} \to \mathcal{V}'\) and \(|A(u)|_{\mathcal{V}'} \leq |u|\) and the embedding \(\mathcal{V}' \hookrightarrow \mathcal{U}'\) is continuous, using inequality (5.3) we obtain the following estimates
\[
\mathbb{E}[|J_2(\tau_n + \theta) - J_2(\tau_n)_{|\mathcal{U}'})|] = \mathbb{E} \left| \int_{\tau_n}^{\tau_n+\theta} P_n A u_n(s) ds \right|_{\mathcal{U}'} \leq c \mathbb{E} \int_{\tau_n}^{\tau_n+\theta} |A u_n(s)_{|\mathcal{U}'}| ds
\]
\[
\leq c \mathbb{E} \int_{\tau_n}^{\tau_n+\theta} |u_n(s)|_{\mathcal{U}'} ds \leq c \mathbb{E} \left[ \theta^2 \left( \int_0^T |u_n(s)|^2_{\mathcal{U}'} ds \right)^{1/2} \right] \leq c \sqrt{C_2} \cdot \theta^2 \cdot c_2 \cdot \theta^d.
\]

By Assumption (R.1) and estimate (5.2) we have
\[
\mathbb{E}[|J_3(\tau_n + \theta) - J_3(\tau_n)|_{\mathcal{U}'})] = \mathbb{E} \left| \int_{\tau_n}^{\tau_n+\theta} P_n R u_n(s) ds \right|_{\mathcal{U}'} \leq \mathbb{E} \left| \int_{\tau_n}^{\tau_n+\theta} R u_n(s)_{|\mathcal{U}'}| ds \right|
\]
\[
\leq c \mathbb{E} \int_{\tau_n}^{\tau_n+\theta} |u_n(s)|_{\mathcal{U}'} ds \leq c \theta \mathbb{E} \sup_{s \in [0, T]} |u_n(s)|_{\mathcal{U}'} \leq c C_1(1) \theta =: c_3 \theta.
\]

Since \(\mathcal{U} \hookrightarrow \mathcal{V}\), by (2.6) and (5.2) we have the following inequalities
\[
\mathbb{E}[|J_4(\tau_n + \theta) - J_4(\tau_n)|_{\mathcal{U}'})] = \mathbb{E} \left| \int_{\tau_n}^{\tau_n+\theta} B u_n(s) ds \right|_{\mathcal{U}'} \leq \mathbb{E} \left| \int_{\tau_n}^{\tau_n+\theta} B u_n(s)_{|\mathcal{U}'}| ds \right|
\]
\[
\leq c \mathbb{E} \int_{\tau_n}^{\tau_n+\theta} |B| \cdot |u_n(s)|_{\mathcal{V}'}^2 ds \leq c ||B|| \cdot \mathbb{E} \sup_{s \in [0, T]} |u_n(s)|_{\mathcal{V}'}^2 \cdot \theta \leq c ||B|| C_1(2) \cdot \theta =: c_4 \cdot \theta,
\]

where \(||B||\) stands for the norm of \(B : \mathbb{H} \times \mathbb{H} \to \mathcal{V}'\).

Let us move to the term \(J_5\). By the Hölder inequality, we have
\[
\mathbb{E}[|J_5(\tau_n + \theta) - J_5(\tau_n)|_{\mathcal{U}'})] \leq c \mathbb{E} \left| \int_{\tau_n}^{\tau_n+\theta} P_n f(s) ds \right|_{\mathcal{U}'} \leq c \cdot \theta^d \cdot ||f||_{L^{2}(0, T; \mathcal{V}')} =: c_5 \cdot \theta^d.
\]

Let us consider the noise term \(J_6\). Since \(\mathbb{H} \hookrightarrow \mathcal{U}'\), by (2.10) with \(p = 2\) in Assumption (F.2) and by (5.2), we obtain the following inequalities
\[
\mathbb{E}[|J_6(\tau_n + \theta) - J_6(\tau_n)|_{\mathcal{U}'})] = \mathbb{E} \left| \int_{\tau_n}^{\tau_n+\theta} P_n F(s, u_n(s); y) \tilde{\eta}(ds, dy) \right|_{\mathcal{U}'}^2
\]
\[
\leq c \mathbb{E} \int_{\tau_n}^{\tau_n+\theta} \int_{\mathcal{H}} \left| P_n F(s, u_n(s); y) \tilde{\eta}(ds, dy) \right|^2_{\mathcal{U}'} ds \leq c \mathbb{E} \int_{\tau_n}^{\tau_n+\theta} \int_{\mathcal{H}} \left| P_n F(s, u_n(s); y) \right|^2_{\mathcal{U}'} \mu(dy) ds
\]
\[
\leq C \mathbb{E} \int_{\tau_n}^{\tau_n+\theta} \left( 1 + |u_n(s)|_{\mathcal{U}'}^2 \right) ds \leq C \cdot \theta \cdot (1 + \mathbb{E} \sup_{s \in [0, T]} |u_n(s)|_{\mathcal{U}'}^2) \leq C \cdot (1 + C_1(2)) \cdot \theta =: c_6 \cdot \theta.
\]
By condition (2.10) with $p = 1$ in Assumption (F.2) and estimate (5.2), we have

$$
\mathbb{E}[|J_\gamma(\tau_n + \theta) - J_\gamma(\tau_n)|_{U'}^2] = \mathbb{E}\left[\int_{\tau_n}^{\tau_n + \theta} \int_Y P_n F(s, u_n(s)) \, d\mu(y)\, ds \right]
$$

$$
\leq \mathbb{E}\left[\int_{\tau_n}^{\tau_n + \theta} \int_Y |F(s, u_n(s))|_{U'} \, ds \right] \leq C_1 \mathbb{E}\left[\int_{\tau_n}^{\tau_n + \theta} (1 + |u_n(s)|_{E_7}) \, ds \right]
$$

$$
\leq C_1 \sqrt{\theta} \mathbb{E}[1 + \sup_{s \in [0, T]} |u_n(s)|_{E_7}] \leq C_1 (1 + C_1(1)) \theta =: c_\theta \theta.
$$

Let us consider the term $J_\gamma$. By the Itô isometry, condition (2.14) in assumption (G.3), continuity of the embedding $U' \hookrightarrow U$ and inequality (5.2), we have

$$
\mathbb{E}[|J_\gamma(\tau_n + \theta) - J_\gamma(\tau_n)|_{U'}^2] = \mathbb{E}\left[\int_{\tau_n}^{\tau_n + \theta} P_n G(s, u_n(s)) \, dW(s) \right]^2
$$

$$
\leq c \mathbb{E}\left[\int_{\tau_n}^{\tau_n + \theta} (1 + |u_n(s)|_{E_7}) \, ds \right] \leq c(1 + \mathbb{E}[1 + \sup_{s \in [0, T]} |u_n(s)|_{E_7}]) \leq c(1 + C_1(1)) \theta =: c_\theta \theta.
$$

By Lemma 10.1, the sequence $(u_n)_{n \in \mathbb{N}}$ satisfies the Aldous condition in the space $U'$. This completes the proof of Lemma 5.3. \qed

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