Full-shell nanowires (semiconducting nanowires fully coated with a superconducting shell) have been recently presented as a novel means to create Majorana zero modes. In contrast to partially coated semiconducting nanowires, it has been argued that full-shell nanowires do not require high magnetic fields and low densities to reach a putative topological regime. We here present a theoretical study of these devices taking into account all the basic ingredients, such as inhomogeneous charge distributions across the section of the nanowire, required to qualitatively explain the first experimental results (S. Vaitiekunas et al. arXiv:1809.05513). We derive a criterion, dependent on the even-odd occupation of the radial subbands with zero angular momentum, for the appearance of Majorana zero modes. In the absence of subband mixing, these give rise to strong zero-bias anomalies in tunneling transport in roughly half of the system’s parameter space under an odd number of flux quanta. Due to their coexistence with gapless subbands, the zero modes do not enjoy generic topological protection. Depending on the details of any residual subband mixing, they can develop a minigap, delocalize or even be destroyed.
In this work we address this question by studying the spectral properties of more general full-shell nanowires, confirming previous findings where applicable, and generalizing them to the realistic case in which charge density is not uniform across the superconductor-semiconducting interface of the nanowire. We find that, while the gapless phase is not topological, it does not preclude the presence of unprotected but strong Majorana-like ZBAs. The sector with lowest angular momentum, when considered in isolation, becomes topological depending on the parity of the number of occupied normal-state radial subbands. We characterize this even-odd effect by computing the phase diagram of a full-shell, solid core nanowire model. We show that ZBAs arise throughout a substantial fraction of nanowire parameter space, and persist across the whole extension of odd lobes. The predicted transport spectroscopy shows a marked similarity to the experimental observations without the need of fine tuning. The resulting Majorana states are however unprotected against general subband-mixing perturbations (from e.g. interface disorder or a noncircular nanowire section or shell), since they coexist with a gapless background. We explore their fate in the presence of mode mixing. Depending on the mixing details, we find a variety of possible behaviors, including the development of a trivial or a non-trivial gap, a splitting or a broadening of the zero mode into a delocalized quasibound Majorana state. We conclude by commenting on possible alternative scenarios for the observations.

Model.—We first develop the simplest description of a solid semiconducting nanowire of radius \( R \), oriented along the \( z \) direction, and fully coated with a conventional superconductor of thickness \( d \). The Fermi energies of the two materials are denoted by \( \mu_N \) and \( \mu_S \) respectively, with \( \mu_S \gg \mu_N \). The associated Fermi wavelengths are denoted by \( \lambda_{N,S} = h/\sqrt{2m^*\mu_{N,S}} \), with \( m^* \) the effective mass (assumed uniform for simplicity). When contacted, the radial charge density profile becomes non-uniform, and in general \( \mu(r) \) acquires screening corrections. We assume the simple approximation \( \mu(r < R) = \mu_N, \mu(r > R) = \mu_S \). Similarly, \( |\Delta(|r| < R)| = 0, |\Delta(r > R)| = |\Delta| \). The dependence of \( |\Delta| \) with flux \( \Phi \) is incorporated from the LP Ginzburg-Landau theory results, see Appendix B whose high accuracy has been recently established [54]. The relevant spin-orbit Rashba coupling inside the nanowire is radial, \( \alpha(r) \parallel \hat{r}, \) and is much smaller in the superconductor than in the semiconductor. We approximate \( \alpha(r < R) = \alpha/2 \hat{r}, \alpha(r > R) = 0 \) [55,57]. The section of the nanowire is assumed circular for the moment, so that subbands have a well defined total angular momentum \( m_j \). The three dimensional Nambu Hamiltonian for this model can be written in cylindrical coordinates as

\[
H = \left[ \frac{(\mathbf{p} + eA)^2}{2m^*} - \mu(r) + \alpha(r) \cdot \sigma \times \mathbf{p} + eA(r) \right] \tau_z + \sigma_y \tau_y |\Delta(r)| e^{in\phi},
\]

(1)

where \( \sigma_i \) are Pauli matrices for spin, and \( \tau_i \) for the electron-hole sectors. The magnetic flux is incorporated through the \( n \)-fluxoid in the pairing term and through the axial gauge field \( A(r) = \frac{\pi R^2}{2\pi R} \phi \), where \( \phi \) is the axial unit vector in cylindrical coordinates, \( \Phi = \pi BR^2 \) is the flux and \( B \) is the magnetic field along the \( z \) direction.

Due to the axial symmetry of the model, the above \( H \) can be decomposed into decoupled sectors with differ-
ent total angular momentum $m_j$ [53], see Appendix A. By discretizing the resulting $H_m$ into a one-dimensional semi-infinite tight-binding Hamiltonian along the z direction, we can compute the total differential conductance $dI/dV$ from a tunneling probe coupled into the end of the nanowire as a sum of different $m_j$ contributions. At zero temperature and small bias voltage $V$, the $dI/dV$ is proportional to the local density of states (LDOS) at the edge of the nanowire for energy $eV$. We compute the zero-temperature tunneling $dI/dV$ using the Green’s function formalism for one-dimensional, semi-infinite conductors [55–60].

Results and discussion. — In Fig. 1 we show a comparison between the experimental $dI/dV$ as a function of bias $V$ and flux $\Phi$ (destructive LP regime) and our simulations (both in the weak and destructive LP regimes). The LP regime is mainly controlled by the ratio $R^2/(d\xi)$ between nanowire radius $R$ and superconductor coherence length $\xi$ and thickness $d$; a large ratio giving weak LP, see Appendix A. We also present the corresponding simulation for a hollow nanowire, wherein the superconductor is confined to a very thin shell $R - \delta < r < R$ with $\delta \ll R$, see sketch. The experiment shows a strong ZBA clearly visible throughout the odd $n = \pm 1$ LP lobes, including $\Phi/\Phi_0 \sim 1$ and its vicinity. This feature is reproduced by the solid nanowire simulation. Other qualitative features, such as the skewness in $V$ of the subgap $dI/dV$ peaks, are also in general agreement with the experiment, save for the smoothness of the latter, probably due to disorder and temperature. In contrast, the hollow nanowire never shows ZBAs around integer $\Phi/\Phi_0$, as already demonstrated in Ref. [53], and is found to be an inadequate model for these devices despite its prominent role in the textbook discussion of the LP effect [49].

Due to the non-gateable nature of the full-shell devices, $\mu_N$ and to some extent also $\mu_S$ are unknown. It is important to establish when ZBAs anomalies arise as a function of these two quantities. To this end we first analyse the solid nanowire $dI/dV$ for fixed $\mu_S$ and decreasing $\mu_N$. This corresponds to fixing $\lambda_S$ and increasing $\lambda_N$. Simultaneously we compute the $m_j$-resolved bandstructure at $\Phi/\Phi_0 = 1$, both in the superconducting and the normal phase. The combined results are shown in Fig. 2. We find a trivial phase with split ZBAs (panel a) that transitions to a non-trivial phase with an unsplit ZBA (panel c) corresponding to a Majorana bound state localized at the tunneling contact. This happens whenever an $m_j = 0$ Nambu subband, in red in panels (d-f), becomes inverted. We note that the amount of ZBA splitting in the trivial phase depends on various geometric and model parameters, and can be smaller than experimental temperatures, in which case a smoothed, unsplit ZBA should still be observed in the trivial phase.

The topological phases accurately correlate, in the limit of $\Delta \ll \mu_S, \mu_N$, with an odd occupation of the normal-state $m_j = 0$ radial subbands, panels (g-i). These normal subbands are spread throughout the inner core and the outer shell of the nanowire, so that the precise transition point depend on both $\lambda_N$ and $\lambda_S$. It also depends weakly on $\Phi$. This is demonstrated in Fig. 2(b), where $\lambda_N$ and $\lambda_S$ are tuned to the vicinity of an even-odd transition. There, $\Phi/\Phi_0 \lesssim 1$ has odd occupancy and a ZBA exactly at zero, while for $\Phi/\Phi_0 \gtrsim 1$ the occupancy is still even, and the ZBA exhibits a weak splitting. Such $\Phi$-dependence within odd lobes is however quite weak in practice.

The phase diagram of the model at fixed $\Phi = \Phi_0$ is shown in Fig. 3(a), where we compare the normal-phase odd-occupancy criterion (blue regions) to the emergence of Majorana zero modes (orange lines), as a function of $R/\lambda_S$ and $R/\lambda_N$. We find that, indeed, the two accurately match, except around complete depletion $R \lesssim \lambda_F, \lambda_S$. Panel (b) shows a wider parameter range, for which the orange boundaries are omitted, as their com-
FIG. 3. (Color online) (a,b) Phase diagram of a $\Phi = \Phi_0$ solid-core, full-shell nanowire of radius $R$, versus $R/\lambda_{N,S}$, where $\lambda_{N,S}$ are Fermi wavelengths in the semiconductor core and superconductor shell, respectively. Panel (a) focuses on low densities while (b) shows a wider range. The thick orange lines in (a) mark the boundaries of regions with a topologically non-trivial $m_j = 0$ subband with Majoranas. These are computed using exact tight-binding simulations with finite $\Delta$. Blue regions in (a,b) correspond to an odd occupancy of the normal-phase radial subbands, computed using wavematching at the core-shell boundary for $\Delta = \alpha = 0$, see Appendix A. Gray and black lines correspond to two analytical approximations in the phase diagram, colored squares in (a). (c-f) $dI/dV$ as a function of bias $V$ and subband-mixing strength $\eta$, starting at different points in the phase diagram, colored squares in (a). (d) Linecuts of the (c) that emphasize the opening and subsequent band inversion of a minigap induced by mixing, which eventually results in the destruction of the Majorana state. Parameters: (a) $R = 100 \text{ nm}, d = 100 \text{ nm}$ and $\Delta = 0.2 \text{ meV}, \alpha = 10 \text{ meV}$ nm (orange curve); (b) $R = 65 \text{ nm}, d = 25 \text{ nm}$.

computation at high density (using tight-binding) is considerably more costly than blue regions of odd-occupancy (based on wavematching, see Appendix A). We also plot, in gray and black lines, two analytical approximations for the even-odd boundaries, see Eqs. (A11) and (A10), respectively. (c-f) $dI/dV$ as a function of bias $V$ and subband-mixing strength $\eta$, starting at different points in the phase diagram, colored squares in (a). (d) Linecuts of the (c) that emphasize the opening and subsequent band inversion of a minigap induced by mixing, which eventually results in the destruction of the Majorana state. Parameters: (a) $R = 100 \text{ nm}, d = 100 \text{ nm}$ and $\Delta = 0.2 \text{ meV}, \alpha = 10 \text{ meV}$ nm (orange curve); (b) $R = 65 \text{ nm}, d = 25 \text{ nm}$.

are a common occurrence, occupying essentially half of the phase diagram. No fine tuning is thus necessary to achieve such a phase, which could explain why the gateable experimental nanowires are likely to show this phenomenology.

The background of gapless $m_j \neq 0$ modes provide a continuum of excitations for the $m_j = 0$ Majorana to couple to or decay into. As a result, the Majorana zero mode does not enjoy generic topological protection in full-shell nanowires, as mode mixing can potentially destroy it. To understand how, we have performed simulations using a minimal model for mode mixing, in line with our nanowire model, see Appendix C for implementation details. A single parameter $\eta$ controls the strength of mode mixing, with all preceding results corresponding to $\eta = 0$. In Fig. 3(c-d) we show the evolution of the $dI/dV$ in two topological points of the phase diagram as a uniform $\eta$ is increased throughout the semi-infinite nanowire. The $dI/dV$ first develops a small topological minigap (black background around the ZBA), which then closes and reopens at a critical value of $\eta$, destroying the ZBA (c). More complex behaviors can arise, such as intermediate phases with split near-zero modes (e), or mode broadening into a Majorana quasibound state for non-uniform $\eta$, see Appendix C. An initially trivial phase without a ZBA, panel (f), can even become topological at small $\eta$, although the eventual fate at large mode mixing is always the same: a destruction of the Majorana zero mode.

Conclusion.—We have established the minimal ingredients necessary to model and explain the subgap tunneling $dI/dV$ phenomenology of full-shell superconductor-semiconductor nanowires of recent experiments [46]. The hollow-core version never shows ZBAs throughout a full LP lobe. It is necessary to consider solid-core nanowires with a non-zero charge density throughout the full nanowire section to obtain ZBAs similar to the experiment. We showed that these emerge for odd normal-state occupation of the radial $m_j = 0$ subbands. We have mapped analytically and numerically this even-odd effect in the emergence of ZBAs at odd LP lobes throughout the full phase diagram of the system’s model, and established the connection between the ZBAs to topologically-protected $m_j = 0$ Majorana zero modes. We have found that, while they are not a signature of robust topologically protected zero modes, unsplit ZBAs should be a common occurrence in these devices, occupying roughly half of their microscopic parameter space at zero temperature. We also found that the effect of subband mixing on the Majoranas is quite complex, ranging from minigap opening to mode splitting or broadening, but always ends up by destroying the Majorana states as mixing is increased.

While the physical picture presented here is qualitatively consistent with the first batch of experimental results, it also implies that the emergence of Majoranas in these devices is hard to predict and control, owing to its dependence on the occupation and mixing of $m_j = 0$.
subbands. Another important prediction is the lack of a ZBA within even-\( n \) lobes. Given the current resolution, it is not clear from the available data in \([46]\) whether the \( n = 2 \) hosts a ZBA or a low-lying split resonance. Thus, other common ZBA-generating mechanisms, such as smooth-confinement at the tunnel contact and unintentional quantum dot formation, should not be discarded as alternative scenarios in future studies.

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**Appendix A: Derivation of the normal-phase odd-occupancy criterion in odd-flux lobes**

In this section we tackle the problem of computing the occupation of the \( m_j = 0 \) subbands in the normal-state, whose parity is an approximate criterion for the existence of Majorana zero modes. This approximation is shown to be valid in the main text throughout odd-flux Little-Parks lobes, in the Andreev limit (\( \Delta \ll \mu_S, \mu_N \)) and in the absence of subband mixing. We perform the calculation using wavematching techniques at the semiconductor-superconductor boundary, and derive analytical approximations for the odd-parity regions in parameter space.

At zero temperature, the occupation of the \( m_j = 0 \), normal-state, radial subbands changes when one of them crosses the Fermi level. This always happens at \( \bar{k}_z = 0 \) in odd-flux lobes. Therefore, we look for the solutions of

\[
H_{m_j=0}(r)\Psi(r) = 0, \quad (A1)
\]

where \( H_{m_j=0} \) is the \( m_j = 0 \) projection of the normal-state (i.e. \( \Delta = 0 \)) Hamiltonian of Eq. (1). We write it as a piecewise-constant combination of nanowire \( H_N(r) \) and shell \( H_S(r) \) Hamiltonians as \([53]\)

\[
H_{m_j=0}(r) = H_N(r)\theta(R-r) + H_S(r)\theta(r-R), \quad (A2)
\]

with

\[
H_N(r) = \left( -\frac{1}{2m^*} \partial_r r \partial_r - \mu_N \right) \sigma_0
- \frac{1}{8m^*r^2} \left( \sigma_z + \left( 1 - \frac{\Phi^2 r^2}{R^2} \right) \sigma_0 \right)^2 + \frac{\alpha}{2r} \sigma_z \left( \sigma_z + \left( 1 - \frac{\Phi^2 r^2}{R^2} \right) \sigma_0 \right) \quad (A3)
\]

and

\[
H_S(r) = \left( -\frac{1}{2m^*} \partial_r r \partial_r - \mu_S \right) \sigma_0
- \frac{1}{8m^*r^2} \left( \sigma_z + (1 - \Phi) \sigma_0 \right)^2. \quad (A4)
\]

\( \Phi \) is the externally-applied magnetic flux normalized by the flux quantum, i.e. \( \Phi = \frac{\Phi}{\Phi_0} \). For the purpose of finding the occupation boundaries we take the \( \alpha = 0 \) limit in Eq. (A3). This is a good approximation, since occupation boundaries occur by band inversions at \( k = 0 \), where spin-orbit coupling is zero. Our steplike model for the chemical potential in the radial direction reads

\[
\mu(r) = \begin{cases} 
\mu_N, & \text{if } r \leq R \\
\mu_S, & \text{otherwise.}
\end{cases} \quad (A5)
\]

We proceed by wavematching the zero energy solutions for the core and shell regions at \( r = R \). The problem reduces to solving a system of four uncoupled equations corresponding to the semiconducting and superconducting bi-spinors: \( \phi^N = (\phi^N, \phi^N_S) \) and \( \phi^S = (\phi^S, \phi^S_S) \), respectively. These read

\[
0 = \phi^N_\uparrow(r)'' + \phi^N_{\downarrow}(r)'
+ \phi^N_\uparrow(r) \left( -\frac{1}{r^2} + \frac{2m^*\mu_N + \Phi}{R^2} - \frac{\Phi^2 r^2}{4R^4} \right),
\]

\[
0 = \phi^N_\downarrow(r)'' + \phi^N_\uparrow(r)'
+ \phi^N_\downarrow(r) \left( 2m^*\mu_N - \frac{\Phi^2 r^2}{4R^4} \right),
\]

\[
0 = \phi^S_\uparrow(r)'' + \phi^S_{\downarrow}(r)'
+ \phi^S_\uparrow(r) \left( \frac{2m^*\mu_S - 1}{4r^2} \left( 4 - 2\Phi - \Phi^2 \right) \right),
\]

\[
0 = \phi^S_\downarrow(r)'' + \phi^S_\uparrow(r)'
+ \phi^S_\downarrow(r) \left( 2m^*\mu_S - \frac{1}{4r^2} \right).
\]

The solutions to the above set of equations which satisfy regularity at the origin are given by

\[
\phi^N_\uparrow(r) = C_{N,1}e^{-\Phi r^2/(4R^2)}r^{-(1)} L_{-1/2} \left( \frac{\Phi r^2}{2R^2} \right), \quad (A6)
\]

\[
\phi^N_\downarrow(r) = C_{N,1}e^{-\Phi r^2/(4R^2)}L_{1/2} \left( \frac{\Phi r^2}{2R^2} \right), \quad (A7)
\]

\[
\phi^S_\uparrow(r) = C_{S,1}J_{1/2} \left( r/\lambda_S \right) + C_{S,2}Y_{1/2} \left( r/\lambda_S \right), \quad (A8)
\]

\[
\phi^S_\downarrow(r) = C_{S,1}J_{1/2} \left( r/\lambda_S \right) + C_{S,2}Y_{1/2} \left( r/\lambda_S \right), \quad (A9)
\]
with
\[ \nu_i = \frac{3}{2} + \frac{R^2}{2F\lambda_N^2} - i, \quad (i = 1, 2) \]
\[ \nu_\Phi = \sqrt{4 - 2\Phi + \Phi^2}/2. \]

\( C_\nu \) are arbitrary constants, \( \lambda_N^{1/2} = \sqrt{2m^*\mu_N/s} \) are the Fermi wavelengths in the semiconducting and superconducting region, respectively, \( L_{\nu_i}^{(\alpha)}(x) \) are the generalized Laguerre polynomials, and \( J_\nu(x) \) and \( Y_\nu(x) \) are the Bessel functions of first and second kind, respectively.

By imposing continuity of the solutions and their derivatives at \( r = R \), and by requiring \( u_S(r) \) to vanish at the outermost radius \( r = R + d \), we obtain a set of two independent transcendental equations \( (\pi_1 = 0 \text{ and } \pi_2 = 0) \) for each spin component which relates parameters \( \mu_N, \mu_S, \) and \( \Phi \) at the occupation boundaries. For the sake of conciseness we omit the explicit and lengthy expressions, which corresponds to the boundaries of the blue regions in Fig. 3(a,b).

A more compact formula for the occupation boundaries can be derived by expanding to second order in \( \lambda_S/R \) in the asymptotic series of Eqs. [A8, A9] (since \( \lambda_S \ll R \) is the experimentally relevant situation) and to zero order in the expansion of \( \pi_{1,2} \) around \( \Phi = \Phi_0 \). Under such approximations, occupation boundaries satisfy
\[ \left( 2fL_{\nu_1}^{(-1)}(1/2) - \frac{\lambda_S}{R} \left( L_{\nu_2}^{(0)}(1/2) + 3L_{\nu_2}^{(-1)}(1/2) \right) \right) \times \left( 2fL_{\nu_2}^{(0)}(1/2) - \frac{\lambda_S}{R} \left( L_{\nu_2}^{(0)}(1/2) + 2L_{\nu_2}^{(-1)}(1/2) \right) \right) = 0, \tag{A10} \]
with
\[ f = -\cot \left( \frac{d}{\lambda_S} \right) + \frac{\lambda_S}{R} \left( 1 - \frac{\lambda_S}{R} \right) \cot \left( \frac{d}{\lambda_S} \right) \left( m + s \cot \left( \frac{d}{\lambda_S} \right)^2 \right), \]
\[ m = \frac{1}{8} \left( 4R + d \right), \]
\[ s = \frac{1}{8} \left( \frac{d}{R + d} \right), \]
\[ \rho_i = \frac{3}{2} + \frac{R^2}{2\lambda_N^2} - i. \quad (i = 1, 2, 3) \]

Note that replacing \( = 0 \) by \( > 0 \) in Eq. (A10) above actually selects the regions with odd normal occupation. Eq. (A10) is plotted in black in Fig. 3(a,b).

Alternatively, expanding \( \pi_1,2 \) up to leading order in both \( \lambda_S/\lambda_N \) and \( \lambda_S/R \), reduces to
\[ \cos \left( \frac{d}{\lambda_S} \right) L_{\nu_1}^{(-1)} \left( \frac{\Phi}{2} \right) L_{\nu_2}^{(0)} \left( \frac{\Phi}{2} \right) = 0. \tag{A11} \]
which corresponds to the square mesh plotted in gray in Fig. 3(a,b). Despite its simplicity, the above equation captures quite well the essence of the even-odd effect in the ZBA of our full-shell nanowire model.

**Appendix B: Destructive Little Parks Oscillations**

Superconducting cylinders with small radii and thickness, \( d \), compared to the zero-temperature coherence length, \( \xi \) (the relevant regime in the experiments of Refs. 16 and 54), and longitudinally threaded by a magnetic field, undergo a strong non-periodic modulation of their critical temperature, which deviates from the Little-Parks prescription (see [61, 62]). This is a consequence of the flux-induced superconducting fluctuations to the persistent current that may lead to a complete suppression of superconductivity in finite intervals centered at half-integer normalized fluxes, \( \Phi \). The critical temperature dependence on the flux \( (T_C(\Phi)) \) as shown in Ref. [62] can be obtained from
\[ \ln \left( \frac{T_C}{T^0} \right) = \psi(1/2) - \psi \left( \frac{\Lambda(\Phi)}{2\pi T_C(\Phi)} \right), \tag{B1} \]
where \( T^0 \) is the critical temperature of the bulk material at zero flux, \( \psi \) is the digamma function and \( \Lambda \) is a pair-breaking term, which depends on \( d \) and reads
\[ \Lambda(\Phi) = \frac{T^0}{\pi R^2} \left[ 4(n+\Phi)^2 + \frac{d^2}{R^2} \left( \Phi^2 + \left( \frac{1}{3} + \frac{d^2}{20R^2} \right) n^2 \right) \right], \]
where \( n \) is the fluxoid winding number.

**Appendix C: Fate of the \( m_j = 0 \) Majorana under interband mixing**

The analysis of the full-shell nanowire based on decoupled \( m_j \)'s is valid in the idealized limit of nanowires with perfect cylindrical symmetry. Any deviation, such as a non-circular section, disorder in the semiconductor, in the superconductor shell or contact, or produced by the presence of a substrate, should be expected to break the assumption of decoupled \( m_j \)'s to some degree. To assess the likelihood of observing the \( m_j = 0 \) ZBA phenomenon connected to Majorana states, we compute and analyze the local density of states (LDOS) under an increasing coupling between a small set of angular momenta \( m_j = 0, \pm 1 \) at \( \Phi = \Phi_0 \) (adding higher bands does not change the qualitative results). As we discussed in the main text, this simplified model is enough to produce a very rich set of possible evolutions of the \( m_j = 0 \) Majoranas, eventually leading to its destruction at strong enough mixing.

The interband mixing is introduced as a uniform coupling \( \eta \) between \( m_j = 0 \) and \( m_j = \pm 1 \). We first assume \( \eta \) is independent of position. With a finite \( \eta \), the LDOS is no longer decomposable into different \( m_j \) contributions. In Fig. 3(c-f) we present the total LDOS at \( \Phi = \Phi_0 \) for increasing \( \eta \), starting from different points in the phase...
diagram Fig. 3(a). In (c) we see the simplest possibility. Starting in a non-trivial configuration, a small \( \eta \) creates a minigap in the \( m_j \neq 0 \) subbands by making them susceptible to superconducting pairing at zero energy, which otherwise only affects the \( m_j = 0 \) sector. This minigap acts as a proper topological gap, and protects the Majorana much as in conventional Oreg-Lutchyn nanowires. As \( \eta \) is increase further, however, the minigap eventually closes and reopens as a trivial gap, destroying the Majorana.

Starting from a different topological point in the phase diagram can produce a more complicated behavior, whereby the Majorana is not destroyed after the minigap is reopened, but instead splits into two near-zero modes, see Fig. 3(e). This split resonance is also eventually destroyed at higher mixing. A more exotic possibility may occur whereby a nanowire in the trivial regime (even occupation, no ZBA), may enter the non-trivial regime when adding a finite mixing, see Fig. 3(f). The resulting Majorana bound state survives once more until its minigap closes and reopens, either becoming split or disappearing altogether.

Finally, a quite different scenario can take place. If \( \eta \) is zero within the bulk of the nanowire, or due to some symmetry some of the \( m_j \neq 0 \) modes remain ungapped, the Majorana may become coupled to such gapless states by a local mixing \( \eta \) confined to the tip of the nanowire, where the Majorana wavefunction is concentrated. Such a local mode mixing is a likely occurrence in experimental devices, since the tip of the nanowire is not covered by a superconducting shell, and is therefore more susceptible to mode-mixing perturbations from the substrate or tunnel probes. The result of such a local \( \eta \) is shown in Fig. 4. The background LDOS does not develop a minigap. Instead, the zero mode becomes broadened into a quasibound Majorana state, with a width that grows with \( \eta \), and which represents its decay rate into the gapless nanowire bulk. All these results assume a semi-infinite nanowire, without any longitudinal quantization of the different \( m_j \) subbands. For finite nanowires the phenomenology becomes even more complicated, although in such case one can no longer rigorously speak about topological non-triviality (at least in closed systems \[63\]). The general conclusion, however, is that while a small amount of mode mixing can be beneficial to stabilize the \( m_j = 0 \) Majorana, it eventually leads to its destruction, either by broadening, splitting or a minigap closing and reopening.

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