Rational solutions of dressing chains and higher order Painlevé equations.

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Abstract

We present a new approach to determine the rational solutions of the higher order Painlevé equations associated to periodic dressing chain systems ($A_n^{(1)}$-Painlevé systems). We obtain new sets of solutions, giving determinantal representations indexed by specific Maya diagrams in the odd case or universal characters in the even case.

1 Introduction

It is now well known that the six Painlevé equations PI-PVI, discovered more than one century ago by Painlevé and Gambier\textsuperscript{1,2} (see also Fuchs, Picard and Bonnet\textsuperscript{3}), define new transcendental objects which can be thought as nonlinear analogues of special functions. Except for the first one, the Painlevé equations all depend on some set of parameters and if for generic values of these parameters the solutions cannot be reduced to usual transcendental functions, it appears that for some specific values of these parameters we retrieve classical transcendental functions or even rational functions\textsuperscript{1,2,4,5}. In the last decades, the classification and the properties of these last solutions has been a subject of active research\textsuperscript{2,4}.

In the early nineties, Shabat, Veselov and Adler\textsuperscript{6,7,8} introduce the concept of dressing chains, showing that they possess the Painlevé property and that the Painlevé equations PII-PVI can be described in terms of dressing chains (scalar or matricial) of low orders. The higher order chains can then be considered as generalizing the classical Painlevé equations and the problem of finding their rational solutions arises naturally\textsuperscript{13,14}. For a Schrödinger operator, the scalar dressing chain of period $p$ constitutes an higher order generalization of PIV for $p$ odd (the symmetric form of PIV itself corresponds to the case $p = 3$) and an higher order generalization of PV for $p$ even (the symmetric form of PV itself corresponds to the case $p = 4$).

In the case of dressing chains of odd periodicity, previous results\textsuperscript{15} seem to indicate that these solutions are necessarily obtained from rational extensions of the harmonic oscillator (HO) potential\textsuperscript{9,10,11}. A remaining question is then to determine among all these rational dressings of the harmonic oscillator, the particular ones...
which allow to solve a dressing chain of given periodicity. An elegant, although indirect, way to answer to this problem pass through the approach developed principally by the Japanese school around Okamoto, Noumi, Yamada, Jimba, Tsuda and others (see \cite{13} and references therein, see also \cite{8}). It rests on the symmetry group analysis of the dressing chain in the parameters space. The parametric symmetries of the chain combined to B"acklund transformations constitute an "extended affine Weyl group" (for the scalar dressing chain of period \( p \), the associated extended affine Weyl group is \( A^{(1)}_{p-1} \)) which preserves the structure of the dressing chain. Starting from some a complete set of simple "fundamental solutions" possessing the rational character, successive applications of the transformations belonging to these Weyl groups generates step by step all the rational solutions of the dressing chain system.

In this article, we propose an alternative approach to build the rational solutions of the Schrödinger operators dressing chain systems (with non zero shift). It has the advantage to be direct and explicit, in the sense that it furnishes an immediate determinantal representation for the solutions of the differential system. Moreover, it links the existence of rational solutions for the dressing chain system to the analytical properties of the underlying quantum potential.

We consider the whole sets of rational extensions of the harmonic and isotonic potentials that we label by Maya diagrams and universal character \cite{16,17} respectively. Analyzing the combinatorial properties of these objects allows us to select the extended potentials which solve the odd and even periodic dressing chains respectively and gives closed form determinantal expressions for the solutions of dressing chain system.

The paper is organized as follows. We start by recalling some basic elements concerning the concept of dressing chains, cyclic potentials and their connection to the PIV and PV equations. In the second part, we introduce the notion of cyclic Maya diagram and associated \( \mathcal{F} \)-vector and give the general structure of a p-cyclic Maya diagram (Theorem 1). It allows in the next part, to show how the cyclic rational extensions of the HO give rational solutions of dressing chains system of period \( p \) for appropriate choices of the parameters (Theorem 2). As illustrating examples, we treat in details the cases \( p = 3 \) (ie the standard PIV equation) and \( p = 5 \) (also called the \( A_4 \)-PIV system).

In the fourth section, after recalling some properties of the isotonic and its rational extensions, which are expressed in terms of Laguerre pseudo-Wronskians and labelled by universal charaters, we show how these last ones allow to obtain rational solutions of the even periodic dressing chain systems for which we give new explicit representations (Theorem 4). We illustrate these general results for the particular cases \( p = 4 \) (ie the standard PV equation) and \( p = 6 \) (ie \( A_5 \)-PV system).

\section{Dressing chains and cyclic potentials}

Consider a potential \( U(x) \), the associated Schrödinger and Riccati-Schrödinger (RS) equation \cite{18} being respectively

\begin{equation}
\begin{aligned}
-\psi''(x) + U(x)\psi(x) &= E_\lambda \psi(x) \\
-w_\lambda'(x) + w_\lambda^2(x) &= U(x) - E_\lambda,
\end{aligned}
\end{equation}

where the RS function \( w_\lambda(x) \) is minus the logarithmic derivative of the eigenfunction \( \psi_\lambda(x) \): \( w_\lambda(x) = -\psi''_\lambda(x)/\psi_\lambda(x) \). The auxiliary spectral parameter \( \lambda \) allows us in what follows to define a sequence of eigenvalues and associated eigenfunctions.

Given a particular eigenfunction \( \psi_\nu(x) \) (or its associated RS function \( w_\nu(x) \)) of \( U(x) \) for the eigenvalue \( E_\nu \), we can build a new potential \( U^{(\nu)}(x) \) via the Darboux transformation (DT) of seed function \( \psi_\nu \) \cite{19}:

\begin{equation}
U^{(\nu)}(x) = U(x) + 2w_\nu'(x),
\end{equation}
called an **extension** of $U(x)$. Then $\psi^{(\nu)}_{\lambda}(x)$ and $w^{(\nu)}_{\lambda}(x)$ defined as

\[
\begin{cases}
\psi^{(\nu)}_{\lambda}(x) = W(\psi_{\nu}, \psi_{\lambda} | x)/\psi_{\nu}(x), \; \lambda \neq \nu \\
\psi^{(\nu)}_{\nu}(x) = 1/\psi_{\nu}(x),
\end{cases}
\]

and

\[
\begin{cases}
w^{(\nu)}_{\lambda}(x) = -w_{\nu}(x) + (E_{\lambda} - E_{\nu})/(w_{\lambda}(x) - w_{\nu}(x)), \; \lambda \neq \nu \\
w^{(\nu)}_{\nu}(x) = -w_{\nu}(x),
\end{cases}
\]

are solutions of

\[
\begin{align*}
-\psi^{(\nu)''}_{\lambda} + U^{(\nu)}\psi^{(\nu)}_{\lambda} &= E_{\lambda}\psi^{(\nu)}_{\lambda} \\
-\left(w^{(\nu)'}_{\lambda}\right)^2 + \left(w^{(\nu)}_{\lambda}\right)^2 &= U^{(\nu)} - E_{\lambda}.
\end{align*}
\]

By chaining such DT, we produce a sequence of extensions

\[
\begin{align*}
\psi^{(\nu_1)}_{\lambda} &\rightarrow \psi^{(\nu_1,\nu_2)}_{\lambda} \rightarrow \psi^{(\nu_1,\nu_2,\ldots,\nu_p)}_{\lambda} \\
U^{(\nu_1)} &\rightarrow U^{(\nu_1,\nu_2)} \rightarrow U^{(\nu_1,\nu_2,\ldots,\nu_p)},
\end{align*}
\]

where

\[
U^{(\nu_1,\ldots,\nu_p)}(x) = U(x) + 2 \left( \sum_{i=1}^{p} w^{(\nu_1,\ldots,\nu_{i-1})}_{\nu_i}(x) \right)'.
\]

The Crum formulas allow us to express the extended potentials as well as their eigenfunctions in terms of Wronskians, containing only eigenfunctions of the initial potential [20, 9]:

\[
\begin{align*}
U^{(\nu_1,\ldots,\nu_p)}(x) &= U(x) - 2 \left( \log W^{(\nu_1,\ldots,\nu_p)}(x) \right)'' \\
\psi^{(\nu_1,\ldots,\nu_p)}_{\lambda}(x) &= W^{(\nu_1,\ldots,\nu_p,\lambda)}(x)/W^{(\nu_1,\ldots,\nu_p)}(x),
\end{align*}
\]

where

\[
W^{(\nu_1,\ldots,\nu_p)}(x) = W(\psi_{\nu_1}, \ldots, \psi_{\nu_p} | x)
\]

is the Wronskian of the family $(\psi_{\nu_1}, \ldots, \psi_{\nu_p})$. Here we are using the convention that if a spectral index is repeated two times in the characteristic tuple $(\nu_1, \ldots, \nu_p)$, then we suppress the corresponding eigenfunction in the Wronskians of the right-hand members of Eq(5). In order to simplify the notation we temporarily use the simplified notation $\nu_i \rightarrow i$. We can now define the notion of **cyclicity**.

A potential $U(x)$ is said to be $p$-**cyclic** if there exists a chain of $p$ DT such that

\[
U^{(1,\ldots,p)}(x) = U(x) + \Delta,
\]

ie at the end of the chain we recover *exactly* the initial potential translated by an energy shift $\Delta$. This condition is then stronger than the usual $p^{th}$-order shape invariance [21, 9]. As shown by Veselov and Shabat [7], the successive RS seed functions then satisfy the following first order non linear system $(\varepsilon_{ij} = E_i - E_j)$.
called a dressing chain of period $p$. We will say that the potential $U(x)$ solves the dressing chain. $\Delta$ and the $\epsilon_{i,i+1}$ are called the parameters of the dressing chain.

The cyclicity condition Eq(10) gives also

$$2 \left( w_1(x) + \ldots + w_p^{(1\ldots p-1)}(x) \right)' = \Delta, \quad (12)$$

that is, with an appropriate choice of the integration constant

$$w_1(x) + \ldots + w_p^{(1\ldots p-1)}(x) = \frac{\Delta}{2} x. \quad (13)$$

In all what follows, we suppose that (non zero shift assumption)

$$\Delta \neq 0. \quad (14)$$

As proven by Veselov and Shabat [7], this system passes the Painlevé test and, as we will see below, for $p = 3$ and $p = 4$ the corresponding dressing chains can be seen as symmetrized forms of the PIV and PV equations respectively.

As illustrating examples we now consider the lowest order cases of cyclicity.

### 2.1 1-step cyclic potential

It is straightforward to show that the HO is in fact the unique potential possessing the 1-step cyclicity property. Indeed, if we want

$$U^{(\nu)}(x) = U(x) + \Delta, \quad (15)$$

Eq(13) gives immediately

$$w_\nu(x) = \Delta x/2 \quad (16)$$

and

$$U(x) = E_\nu - w_\nu'(x) + w_\nu^2(x) = \frac{\Delta^2}{4} x^2 + E_\nu - \frac{\Delta}{2}, \quad (17)$$

ie $U(x)$ is an HO potential with frequency $\omega = \Delta$, the 1-step cyclicity condition coinciding for this potential with the usual shape invariance property [21, 9].
2.2 2-step cyclic potentials

If the potential $U(x)$ is 2-step cyclic, the corresponding dressing chain of period 2 is (see Eq(11) and Eq(13))

$$
\begin{align*}
&\begin{cases}
-\left( w_2^{(1)}(x) + w_1(x) \right)' + \left( w_2^{(1)}(x) \right)^2 - (w_1(x))^2 = \varepsilon_{12} \\
-\left( w_1(x) + w_2^{(1)}(x) \right)' + (w_1(x))^2 - \left( w_2^{(1)}(x) \right)^2 = \varepsilon_{21} - \Delta,
\end{cases}
\end{align*}
$$

(18)

with

$$
\begin{align*}
w_1(x) + w_2^{(1)}(x) &= \frac{\Delta}{2} x.
\end{align*}
$$

(19)

Taking the difference of the two equations in Eq(18) and combining with Eq(19), we obtain in a straightforward way

$$
\begin{align*}
w_2^{(1)}(x) &= \frac{\Delta}{4} x + \frac{\varepsilon_{12}/\Delta + 1/2}{x}, \\
w_1(x) &= \frac{\Delta}{4} x - \frac{\varepsilon_{12}/\Delta + 1/2}{x}.
\end{align*}
$$

(20)

Consequently

$$
U(x) = E_1 - w_1'(x) + w_1^2(x) = \frac{\omega^2}{4} x^2 + \frac{(\alpha + 1/2)(\alpha - 1/2)}{x^2} - \omega(\alpha + 1) = V(x; \omega, \alpha),
$$

(21)

where $\omega = \Delta/2$ and $\alpha = \varepsilon_{12}/\Delta$. We deduce that the unique 2-step cyclic potential is the isotonic oscillator (IO) with frequency $\omega$ and "angular momentum" $a = \alpha - 1/2$ 26 [12].

2.3 3-step cyclic potentials and Painlevé IV

The dressing chain of period $p = 3$ has the form (see Eq(11))

$$
\begin{align*}
&\begin{cases}
-\left( w_2^{(1)}(x) + w_1(x) \right)' + \left( w_2^{(1)}(x) \right)^2 - (w_1(x))^2 = \varepsilon_{12} \\
-\left( w_1(x) + w_2^{(1)}(x) \right)' + \left( w_3^{(1,2)}(x) \right)^2 - (w_2^{(1)}(x))^2 = \varepsilon_{23} \\
-\left( w_1(x) + w_3^{(1,2)}(x) \right)' + (w_1(x))^2 - \left( w_3^{(1,2)}(x) \right)^2 = \varepsilon_{31} - \Delta,
\end{cases}
\end{align*}
$$

(22)

with (see Eq(13))

$$
\begin{align*}
w_1(x) + w_2^{(1)}(x) + w_3^{(1,2)}(x) &= \frac{\Delta}{2} x.
\end{align*}
$$

(23)

By defining

$$
\begin{align*}
\begin{cases}
y = \sqrt{\frac{2}{\Delta}} (w_1(x) - \Delta x/2) \\
t = \sqrt{\frac{2}{\Delta}} x,
\end{cases}
\end{align*}
$$

(24)
we can then easily show \([7, 8]\) that \(y\) satisfies the PIV equation (see \([1, 22, 2]\))

\[
y'' = \frac{1}{2y} (y')^2 + \frac{3}{2} y^3 + 4ty^2 + 2 \left(t^2 - a\right) y + \frac{b}{y},
\]

(here the prime denotes the derivative with respect to \(t\)) with parameters

\[
a = - (\Delta + \varepsilon_{23} + 2\varepsilon_{12}) / \Delta, \quad b = - \frac{2\varepsilon_{23}^2}{\Delta^2}.
\]

2.4 4-step cyclic potentials and Painlevé V

For \(p = 4\), the dressing chain of Eq(11) becomes

\[
\begin{align*}
&- (w_2^{(1)}(x) + w_1(x))' + \left(w_2^{(1)}(x)\right)^2 - (w_1(x))^2 = \varepsilon_{12} \\
&- (w_3^{(1,2)}(x) + w_2^{(1)}(x))' + \left(w_3^{(1,2)}(x)\right)^2 - (w_2^{(1)}(x))^2 = \varepsilon_{23} \\
&- (w_4^{(1,2,3)}(x) + w_3^{(1,2)}(x))' + \left(w_4^{(1,2,3)}(x)\right)^2 - (w_3^{(1,2)}(x))^2 = \varepsilon_{34} \\
&- (w_1(x) + w_4^{(1,2,3)}(x))' + (w_1(x))^2 - \left(w_4^{(1,2,3)}(x)\right)^2 = \varepsilon_{41} - \Delta.
\end{align*}
\]

with the cyclicity condition (see Eq(13))

\[
w_1(x) + w_2^{(1)}(x) + w_3^{(1,2)}(x) + w_4^{(1,2,3)}(x) = \frac{\Delta}{2} x.
\]

As shown by Adler \([8]\), the function

\[
y(t) = 1 - \frac{\Delta x}{2 (w_1(x) + w_2(x))}, \quad t = x^2,
\]

satisfies the PV equation (see \([1, 23, 2]\))

\[
y'' = \left(1 - \frac{1}{y - 1}\right) (y')^2 - \frac{y'}{t} + \frac{(y - 1)^2}{t^2} \left(ay + \frac{b}{y}\right) + \frac{c}{t} + \frac{d (y + 1)}{y - 1},
\]

with parameters

\[
a = \frac{\varepsilon_{12}^2}{2\Delta^2}, \quad b = - \frac{\varepsilon_{34}^2}{2\Delta^2}, \quad c = \frac{1}{4} (\Delta - \varepsilon_{41} + \varepsilon_{23}), \quad d = - \frac{\Delta^2}{32}.
\]

3 Cyclic Maya diagrams

3.1 First definitions

We define a Sato Maya diagram as an infinite row of boxes, called levels, labelled by relative integers and which can be empty or filled by at most one "particle" (graphically represented by a bold dot) \([24, 25, 11, 12]\). All the
levels sufficiently far away on the left are filled and all the levels sufficiently far away on the right are empty. The set of Sato Maya diagrams can be put in one to one correspondence with the set of tuple of relative integers of the form $N_m = (n_1, \ldots, n_m) \in \mathbb{Z}^m$. The tuple $N_m$ contains the indices of the filled levels above zero (included) and the indices of the empty levels strictly below zero, or in other words, the filled levels in corresponding Sato Maya diagram are indexed by the set

$$\{ j < 0 : j \notin N_m \} \cup \{ j \geq 0 : j \in N_m \}. \quad (32)$$

In the following we use the term **Maya diagram** to designate both the Sato Maya diagram in graphical form, and the associated tuple. If all the $n_i$ are positive then the Maya diagram is said to be **positive** (respectively **negative**). In the case of a positive Maya diagram, if all the $n_i$ are non-zero, then the Maya diagram is said to be **strictly positive**.

Two Maya diagrams $N_m$ and $N'_m$ are **equivalent** if they differ only by a global translation of all the particles in the levels and we note

$$N_m \approx N'_m. \quad (33)$$

The **canonical representative** of such an equivalence class is the unique strictly positive Maya diagram of the class for which the zero level is empty, i.e. the zero level is the first empty level.

A $k$-**translation** applied to the canonical Maya diagram $N_m = (n_1, \ldots, n_m) \in (\mathbb{N}^*)^m$ generates the Maya diagram, denoted $N_m \oplus k$, obtained from by shifting by $k$ all the particles in the levels of $N_m$. For $k > 0$, we have

$$N_m \oplus k = (0, \ldots, k-1) \cup (N_m + k), \quad (34)$$

where $N_m + k = (n_1 + k, \ldots, n_m + k)$.

Starting from a given Maya diagram $N_m$ we can modify it by suppressing particles in the filled levels or by filling empty levels. We call such an action on a level, a **flip**. If $(\nu_1, \ldots, \nu_p)$ are the indices of the flipped levels then the characteristic tuple of the resulting Maya diagram can be denoted $(N_m, \nu_1, \ldots, \nu_p)$ with the convention that a twice repeated index has to be suppressed.

### 3.2 $\vec{s}$-vectors and cyclicity

A Maya diagram $N_m$ is said $p$-**cyclic with translation of** $k > 0$ if we can translate it by $k$, by acting on it with $p$ flips. In other words, there must exist a tuple of $p$ positive integers $(\nu_1, \ldots, \nu_p) \in \mathbb{N}^p$ such that

$$N_m \approx (N_m, \nu_1, \ldots, \nu_p). \quad (35)$$

In particular, if $N_m$ is canonical, there must exist a tuple of $p$ positive integers $(\nu_1, \ldots, \nu_p) \in \mathbb{N}^p$ such that

$$(N_m, \nu_1, \ldots, \nu_p) = N_m \oplus k. \quad (36)$$

The ordered tuple $(\nu_1, \ldots, \nu_p)$ is called a $p$-**cyclic chain** associated to $N_m$.

We can readily see that every extension is trivially $(2m + k)$-cyclic with translation of $k$ via the chain:

$$N_m \cup (N_m + k) \cup (0, \ldots, k-1) = (n_1, \ldots, n_m, n_1 + k, \ldots, n_m + k, 0, \ldots, k-1). \quad (37)$$
It is clear that every \( p \)-cyclic with translation of \( k \) Maya diagram is also \((p + 2l)\)-cyclic with translation of \( k \) for any integer \( l \), since Eq.(36) implies

\[
(N_m, \nu_1, \dots, \nu_p, \nu_{p+1}, \nu_{p+1}, \dots, \nu_l, \nu_l) = N_m \oplus k.
\] (38)

To a given Maya diagram \( N_m \), we can associate in a one to one way a \( \vec{s} \)-vector, which is an infinite sequence of “spin variables”, \( \vec{s} = (s_n)_{n \in \mathbb{Z}} \), where \( s_n = +1 \) (up spin) or \( s_n = -1 \) (down spin), in the following way

* If level \( n \) is filled then \( s_n = -1 \).
* If level \( n \) is empty then \( s_n = +1 \).

For a canonical Maya diagram, it means that \( N_m \) gives the positions of the down spins in the positively indexed part of the \( \vec{s} \)-vector:

* If \( n \in N_m \) or \( n < 0 \), then \( s_n = -1 \).
* If \( n \geq 0 \) and \( n \notin N_m \), then \( s_n = +1 \).

The \( \vec{s} \)-vector is subject to the topological constraint

\[
\lim_{n \to -\infty} s_n = -1, \quad \lim_{n \to +\infty} s_n = 1.
\] (39)

In the particular case of a canonical Maya diagram, we have more precisely

\[
\left\{ \begin{array}{c}
   s_{n<0} = -1 \\
   s_0 = +1 \\
   s_{n>n_m} = +1.
\end{array} \right.
\] (40)

A flip at the level \( \nu \) in \( N_m \) corresponds to change the sign of \( s_{\nu}(s_{\nu} \rightarrow -s_{\nu}) \): suppressing a particle in a level corresponds to a positive flip of \( s_{\nu} \) \((-1 \rightarrow +1)\), while filling a level corresponds to a negative flip of \( s_{\nu} \) \((+1 \rightarrow -1)\).

In the sequel, we let \( \mathcal{F}^{(i_1, \ldots, i_p)} \) denote the flip operator which, when acting on \( \vec{s} \), flips the spins \( s_{i_1}, \ldots, s_{i_p} \). We also let \( \mathcal{T}_k \) denote the translation operator of amplitude \( k \) on the \( \vec{s} \)-vector:

\[
\mathcal{T}_k \vec{s} = \vec{s}^{'}, \text{ with } s_n^{'} = s_{n-k}, \quad \forall n \in \mathbb{Z}.
\] (41)

A Maya diagram \( N_m \) is \( p \)-cyclic with translation of \( k \) iff \( \exists (i_1, \ldots, i_p) \in \mathbb{N}^p \) such that the corresponding \( \vec{s} \)-vector is in the kernel of \( \mathcal{F}^{(i_1, \ldots, i_p)} - \mathcal{T}_k \):

\[
\mathcal{F}^{(i_1, \ldots, i_p)} \vec{s} = \mathcal{T}_k \vec{s}.
\] (42)

Suppose that \( N_m \) is a canonical Maya diagram which is \( p \)-cyclic with translation of \( k > 0 \). We let \( p_- \) denote the number of negative flips and \( p_+ \) the number of positive flips in the \( p \)-cyclic chain. We then have the following lemma:
Lemma 1. \( k \) has the same parity as \( p \) and

\[
\begin{align*}
\{p = p_+ + p_- \in \mathbb{N}, \\
k = p_- - p_+ \in \{1, \ldots, p\}. \tag{43}
\end{align*}
\]

Proof. Initially, above \( n_m \) all the \( s_n \) are up and below 0 they are all down. Between 0 and \( n_m \), we have \( m \) spins down. After the \( p \)-cyclic chain, all the spins above \( n_m + k \) are up and below \( k \) they are all down. In the block of indices \( \{k, \ldots, n_m + k\} \) we still have \( m \) down spins and in the set of levels \( \{0, \ldots, n_m + k\} \) we now have \( m + k \) down spins before the \( p \)-cyclic chain the same set contained only \( m \) down spins. Out of the set \( \{0, \ldots, n_m + k\} \) the Maya \( \vec{s} \)-vector remains unchanged and at the end, the action of the \( p \)-cyclic chain with translation \( k \) is to have flipped negatively \( k \) spins. Consequently

\[
\begin{align*}
\{p = p_+ + p_- \in \mathbb{N}, \\
k = p_- - p_+ \in \{1, \ldots, p\}. \tag{44}
\end{align*}
\]

\( \square \)

For example, for a 3-cyclic chain we can have \( k = 1 \) \( (p_- = 2, p_+ = 1) \) or \( k = 1 \) \( (p_- = 3, p_+ = 0) \) and for a 4-cyclic chain, we can have \( k = 2 \) \( (p_- = 3, p_+ = 1) \) or \( k = 4 \) \( (p_- = 4, p_+ = 0) \).

### 3.3 Structure of the cyclic Maya diagrams

Let first give some supplementary definitions.

We let

\[
(r \mid s)_k = (r, r + k, \ldots, r + (s - 1)k), \ r, s \in \mathbb{N}, \tag{45}
\]

and call such a set of indices a block of length \( s \).

A block of the type \( \{r \mid s\}_1 \), containing \( s \) consecutive integers, is called a Generalized Hermite (GH) block of length \( s \) and in the particular case \( r = 0 \), \( \{0 \mid s\}_1 \) is called a removable block of length \( s \).

Be \( I = \{n_j\}_{j \in \mathbb{Z}} \subset \mathbb{Z} \) a subset of integers indices. We say that \( s_n \) is discontinuous at \( n_j \) on \( I \) if \( s_{n_{j+1}} = -s_{n_j} \). A subset of indices \( I_l = k\mathbb{Z} + l, \ l \in \{0, \ldots, k - 1\} \) is called a \( k \)-support. Note that there is only one \( 1 \)-support which identifies with \( \mathbb{Z} \).

We can now establish the main theorem concerning the cyclic Maya diagram.

**Theorem 1.** Any \( p \)-cyclic Maya diagram has the following form

\[
N_m = ((1 \mid \alpha_1)_k, \ldots, (k - 1 \mid \alpha_{k-1})_k; (\lambda_1 \mid \mu_1)_k, \ldots, (\lambda_j \mid \mu_j)_k), \tag{46}
\]

with \( j = (p - k) / 2 \in \mathbb{N} \) and where the \( \alpha_i, \lambda_i, \mu_i \) are arbitrary positive integers, constituting a set of \( p - 1 \) arbitrary integer parameters on which depends the \( p \)-cyclic Maya diagram \( N_m \). The \( k - 1 \) blocks of the type \( (l \mid \alpha_l)_k \), \( l \in \{1, \ldots, k - 1\} \), \( \alpha_l \) arbitrary, are called \( k \)-Okamoto blocks. The \( j \) blocks \( (\lambda_l \mid \mu_l)_k \), \( l \in \{1, \ldots, j\} \), are called blocks of the second type.

The corresponding \( p \)-cyclic chains are obtained by forming tuples from the set of indices

\[
\{0, 1 + \alpha_1 k, \ldots, (k - 1) + \alpha_{k-1} k, \ \lambda_1, \lambda_1 + \mu_1 k, \ldots, \lambda_j, \lambda_j + \mu_j k\}. \tag{47}
\]

Concretely, the above set contain 0, the first element of each block of the second type and the last element of each block (Okamoto and second type) after having increased its length by 1.
Proof. Eq (42) gives the following infinite linear system for the $s_n$

$$
\begin{cases}
  s_n = s_{n-k}, \text{ if } n \notin N_m \\
  s_n = -s_{n-k}, \text{ if } n \in N_m.
\end{cases}
$$

(48)

This implies that $s_n$, as a function of $n$, is piecewise constant on each $k$-support $I_l$ with a finite number of discontinuities on each of these supports and with the asymptotic behaviour (see Eq (40))

$$
\begin{cases}
  s_{l+jk} = +1, \text{ if } l + jk > n_m \\
  s_{l+jk} = -1, \text{ if } l + jk < 0.
\end{cases}
$$

(49)

The discontinuities of $s_n$ are located at the flip positions minus $k$, $(i_1 - k, ..., i_p - k)$, and the possible $p$-cyclic canonical Maya diagrams are obtained by sharing these $p$ discontinuities into the different $k$-supports $I_l$, $l \in \{0, ..., k-1\}$. Remark that the constraint Eq (49) implies that we have an odd number of discontinuities in each $k$-support $I_l$. Due to the canonical choice Eq (41) and the cyclicity condition Eq (36), we know that necessarily we have to do one flip at the level 0 (which is on the support $I_0$) and one flip at the level $n_m + k$ (which is on the $k$-support $I_{l_m}$ if $n_m = l_m \mod k$). This means also that we have necessarily at least one discontinuity in the $k$-support $I_0$ (in $-k$) and one in the $k$-support $I_{l_m}$ (in $n_m$).

Let $p_l \in 2\mathbb{N} + 1$ the number of discontinuities in the $k$-support $I_l$, $\sum_{l=0}^{k-1} p_l = p$ (with $p - k \in 2\mathbb{N}$), and denote $l + j^{(l)}_i k$, $i \in \{1, ..., p_l\}$, $j^{(l)}_i \in \mathbb{N}$, the positions of these discontinuities.

In the positively indexed part of the $k$-support $I_l$, the down spins are then located at

$$
\left(l, ..., l + \left(j^{(l)}_1 - 1\right)k\right), \left(l + j^{(l)}_2 k, ..., l + \left(j^{(l)}_3 - 1\right)k\right), ..., \left(l + j^{(l)}_{p_l-1} k, ..., l + \left(j^{(l)}_{p_l} - 1\right)k\right),
$$

(50)

which can be rewritten in terms of blocks as (see Eq (15)),

$$
\left(l \mid j^{(l)}_1 \right)_k, \left(l + j^{(l)}_2 k \mid j^{(l)}_3 - j^{(l)}_2 \right)_k, ..., \left(l + j^{(l)}_{p_l-1} k \mid j^{(l)}_p - j^{(l)}_{p_l-1}\right)_k.
$$

(51)

Note that due to the canonical choice, it gives in particular for the $k$-support $I_0$

$$
\left(j^{(0)}_2 k \mid j^{(0)}_3 - j^{(0)}_2 \right)_k, ..., \left(j^{(0)}_{p_0-1} k \mid j^{(0)}_p - j^{(0)}_{p_0-1}\right)_k.
$$

(52)

If we denote $j^{(l)}_1 = \alpha_l$ and

$$
\begin{cases}
  j^{(l)}_{2r+1} - j^{(l)}_2 = \beta^{(l)}_2, \\
  l + j^{(l)}_2 k = \beta^{(l)}_{2r}, \quad r \geq 1,
\end{cases}
$$

(53)

Eq (52) and Eq (51) become

$$
\left(\beta^{(0)}_1 \mid \beta^{(0)}_2 \right)_k, ..., \left(\beta^{(0)}_{p_0-2} \mid \beta^{(0)}_{p_0-1}\right)_k
$$

(54)

and

$$
\left(l \mid \alpha_l \right)_k, \left(\beta^{(l)}_1 \mid \beta^{(l)}_2 \right)_k, ..., \left(\beta^{(l)}_{p_l-2} \mid \beta^{(l)}_{p_l-2}\right)_k.
$$

(55)

By taking into account all the $k$-supports, globally we have $(k - 1) k$-Okamoto blocks $(l \mid \alpha_l)_k$, where the $\alpha_l$ are arbitrary, and $j = \sum_{l=0}^{k-1} (p_l - 1)/2 = (p - k)/2$ blocks of the second type $(\beta^{(l)}_{2r} \mid \beta^{(l)}_{2r})_k$, $r \geq 1$, where the $\beta^{(l)}_i$ are arbitrary. If we denote them as

$$
(\lambda_i \mid \mu_i)_k, \quad i \in \{1, ..., j\},
$$

(56)

where the $\lambda_i$ and $\mu_i$ are arbitrary, we arrive to Eq (40).

The positions of the flips, obtained by adding $k$ to the discontinuities positions, gives the $p$-cyclic chain associated to $N_m$. We have $p_+ = j$ positive flips located in $\lambda_1, ..., \lambda_j$ and $p_- = j+k$ negative flips located in $0, 1+\alpha_1 k, ..., (k-1)+\alpha_{k-1} k$, $\lambda_1 + \mu_1 k, ..., \lambda_j + \mu_j k$. 

\end{proof}
For example, consider the $p$-cyclic chain $(\lambda_1 + \mu_1 k, \ldots, \lambda_j + \mu_j k, \lambda_1, \ldots, \lambda_j, 1 + \alpha_1 k, \ldots, (k - 1) + \alpha_{k-1} k, 0)$. Its action on $N_m$ can be explicitly written as

$$N_m \overset{\lambda_1}{\to} ((1 \mid \alpha_1)_k, \ldots, (k - 1 \mid \alpha_{k-1})_k; (\lambda_1 + k \mid \mu_1)_k, \ldots, (\lambda_j \mid \mu_j)_k)$$

$$\vdots \overset{\lambda_j}{\to} ((1 \mid \alpha_1)_k, \ldots, (k - 1 \mid \alpha_{k-1})_k; (\lambda_1 + k \mid \mu_1)_k, \ldots, (\lambda_j + k \mid \mu_j - 1)_k)$$

$$\overset{\lambda_j + \mu_{j_1}}{\to} ((1 \mid \alpha_1)_k, \ldots, (k - 1 \mid \alpha_{k-1})_k; (\lambda_1 + k \mid \mu_1)_k, \ldots, (\lambda_j + k \mid \mu_j)_k)$$

$$\overset{1 + \alpha_1 k}{\to} ((1 \mid \alpha_1 + 1)_k, \ldots, (k - 1 \mid \alpha_{k-1})_k; (\lambda_1 + k \mid \mu_1)_k, \ldots, (\lambda_j + k \mid \mu_j)_k)$$

$$\overset{(k-1)+\alpha_{k-1} k}{\to} ((1 \mid \alpha_1 + 1)_k, \ldots, (k - 1 \mid \alpha_{k-1} + 1)_k; (\lambda_1 + k \mid \mu_1 - 1)_k, \ldots, (\lambda_j + k \mid \mu_j)_k)$$

$$\to N_m \oplus k,$$  \hspace{1cm} (57)

since

$$\begin{align*}
(0, (1 \mid \alpha_1)_k, \ldots, (k - 1 \mid \alpha_{k-1})_k; (\lambda_1 + k \mid \mu_1)_k, \ldots, (\lambda_j + k \mid \mu_j)_k) \\
= (0, 1, \ldots, k - 1) \cup ((1 + k \mid \alpha_1)_k, \ldots, (2k - 1 \mid \alpha_{k-1})_k; (\lambda_1 + k \mid \mu_1)_k, \ldots, (\lambda_j + k \mid \mu_j)_k) \\
= (0 \mid k)_1 \cup (N_m + k).
\end{align*}$$

(58)

Let us see now the two extremal cases $k = p$ and $k = 1$.

When in Eq(56) some blocks merge or overlap, we say that the block structure of $N_m$ is **degenerate**.

### 3.3.1 $k = p$

We have no second type block ($j = 0$) and ($p - 1$) $p$-Okamoto blocks:

$$N_m = \left( (1 \mid \alpha_1)_p, \ldots, (p - 1 \mid \alpha_{p-1})_p \right), \hspace{1cm} m = \alpha_1 + \ldots + \alpha_{p-1}. \hspace{1cm} (59)$$

Each $p$-support contains exactly one discontinuity ($p_l = 1$, $\forall l \in \{0, \ldots, p - 1\}$) and $N_m$ contains no multiple of $p$. We call $N_m$ a **$p$-Okamoto Maya diagram**.

The corresponding $p$-cyclic chains are built from

$$\{0, 1 + \alpha_1 p, \ldots, (p - 1) + \alpha_{p-1} p\}, \hspace{1cm} (60)$$

which contains $p$ negative flips.

### 3.3.2 $k = 1$

We have no Okamoto block and $(p - 1) / 2$ second type blocks:

$$N_m = \left( (\lambda_1 \mid \mu_1)_1, \ldots, (\lambda_{(p-1)/2} \mid \mu_{(p-1)/2})_1 \right), \hspace{1cm} m = \mu_1 + \ldots + \mu_{(p-1)/2}. \hspace{1cm} (61)$$
which are GH-blocks. There is only one 1-support which \( Z \) is itself and which contains all the \( p \) discontinuities \((k - 1 = 0 \text{ and } p_0 = p)\). \( N_m \) is called a \( p \)-GH Maya diagram.

In the corresponding \( p \)-cyclic chains, we have \((p + 1)/2\) negative flips at 0, \( \lambda_1 + \mu_1, \ldots, \lambda_j + \mu_j \) and \((p - 1)/2\) negative flips at \( \lambda_1, \ldots, \lambda_j \).

4 Rational extensions of the HO and rational solutions of the dressing chains of odd periodicity

4.1 Rational extensions of the HO

The HO potential is defined on the real line by

\[
V(x; \omega) = \frac{\omega^2}{4} x^2 - \frac{\omega}{2}, \quad \omega \in \mathbb{R}.
\]  

(62)

With Dirichlet boundary conditions at infinity and supposing \( \omega \in \mathbb{R}^+ \), \( V(x; \omega) \) has the following spectrum \((z = \sqrt{\omega/2})\)

\[
\left\{
\begin{array}{l}
E_n(\omega) = n\omega \\
\psi_n(x; \omega) = \psi_0(x; \omega) H_n(z), \quad n \geq 0,
\end{array}
\right.
\]  

(63)

with \( \psi_0(x; \omega) = \exp(-z^2/2) \). It is the most simple example of translationally shape invariant potential \[21, 18\]

with

\[
V^{(0)}(x; \omega) = V(x; \omega) + \omega.
\]  

(64)

It also possesses a unique parametric symmetry, \( \Gamma_3 \), which acts as \[11, 9\]

\[
\omega \xrightarrow{\Gamma_3} (-\omega) \quad \text{and} \quad V(x; \omega) \xrightarrow{\Gamma_3} V(x; -\omega) = V(x; \omega) + \omega,
\]  

(65)

and then generates the conjugate spectrum of \( V(x; \omega) \)

\[
\left\{
\begin{array}{l}
E_n(\omega) \\
\psi_n(x; \omega)
\end{array}\right\} \xrightarrow{\Gamma_3} \left\{
\begin{array}{l}
E_{-(n+1)}(\omega) = -n\omega < 0 \\
\psi_n(x; -\omega) = i H_n(iz) \exp(z^2/2) = \psi_{-(n+1)}(x; \omega).
\end{array}\right.
\]  

(66)

The union of the spectrum and the conjugate spectrum forms the extended spectrum of the HO which contains all the quasi-polynomial (ie polynomial up to a gauge factor) eigenfunctions of this potential. All the rational extensions of the HO are then obtained via chains of DT with seed functions chosen in the extended spectrum and they are then labelled by tuples of spectral indices which relative integers \( N_m = (n_1, \ldots, n_m) \in \mathbb{Z}^m \). This establishes a one to one correspondence between the set of rational extensions of the HO and the set of Maya diagrams. A rational extension associated to a canonical Maya diagram is said to be a canonical extension. The energy spectrum of \( V^{(N_m)}(x; \omega) \) is the set of \( \{E_j(\omega)\}_{j \in \mathbb{Z}} \) for the indices associated to the empty levels of the Maya diagram \( N_m \).

As shown in \[11\], this correspondence preserves the equivalence relation, in the sense that if

\[
N_m \approx N_m',
\]  

(67)
then the corresponding extended potentials are identical up to an additive constant:

\[ V^{(N_m)}(x; \omega) = V^{(N_m')} (x; \omega) + q\omega, \ q \in \mathbb{Z}. \]  

(68)

In particular, if \( N_m \) is canonical

\[ V^{(N_m \oplus k)}(x; \omega) = V^{(N_m)}(x; \omega) + k\omega. \]  

(69)

It follows that to describe all the rational extensions of the HO, we can restrict ourself to those associated to canonical Maya diagrams.

We have also equivalence relations for the Wronskians, in particular

\[ W^{(N_m \oplus k)}(x; \omega) = (\psi_0(x; \omega))^k W^{(N_m)}(x; \omega), \]  

(70)

or, if \( k \leq n_1 \leq \ldots \leq n_m \)

\[ W^{(0, \ldots, k-1) \cup N_m}(x; \omega) = (\psi_0(x; \omega))^k W^{(N_m - k)}(x; \omega). \]  

(71)

Using Eq(63) and the usual properties of the Wronskians [27]

\[
\begin{align*}
W (uy_1, \ldots, uy_m | x) &= u^n W (y_1, \ldots, y_m | x) \\
W (y_1, \ldots, y_m | x) &= (\frac{dz}{dx})^{m(m-1)/2} W (y_1, \ldots, y_m | z),
\end{align*}
\]

(72)

we can write

\[ W^{(N_m)}(x; \omega) \propto (\psi_0(x; \omega))^m H^{(N_m)}(z), \]  

(73)

where \( H^{(N_m)} \) is the following Wronskian determinant \( i = 0, \ldots, m - 1 \)

\[
H^{(N_m)}(z) = W \begin{pmatrix} H_{n_1}(z) & \ldots & H_{n_m}(z) \\
\vdots & \ddots & \vdots \\
(n_1)_i H_{n_1-i}(z) & \ldots & (n_m)_i H_{n_m-i}(z) \\
(n_1)_{m-1-i} H_{n_1-m+1}(z) & \ldots & (n_m)_{m-1} H_{n_m-m+1}(z) \end{pmatrix},
\]

(74)

\( i(x) \) and \( (x)_i \) being respectively the rising and falling factorials

\[ i(x) = x(x+1)\ldots(x+i-1), \ (x)_i = x(x-1)\ldots(x-i+1), \]  

(75)

with the convention that \( H_n(z) = 0 \) if \( n < 0 \).

Eq(70) and Eq(71) give then

\[
\begin{align*}
\mathcal{H}^{(N_m \oplus k)}(z) &\propto H^{(N_m)}(z) \\
\mathcal{H}^{(0, \ldots, k-1) \cup N_m}(z) &\propto H^{(N_m - k)}(z).
\end{align*}
\]

(76)
4.2 $p$-cyclic extensions of the HO and rational solutions of the periodic dressing chains

First we can prove the following lemma:

**Lemma 2.** A rational extension $V^{(N_m)}(x; \omega)$ of the HO is a $p$-cyclic potential iff its associated Maya diagram $N_m$ is $p$-cyclic.

**Proof.** If $N_m$ is a canonical Maya diagram which is $p$-cyclic with translation of $k > 0$, there exist a tuple of $p$ positive integers $(\nu_1, \ldots, \nu_p) \in \mathbb{N}^p$ such that (see Eq(63))

$$V^{(N_m,\nu_1,\ldots,\nu_p)}(x; \omega) = V^{(0,\ldots,k-1,\nu(N_m+k)}(x; \omega) = V^{(N_m)}(x; \omega) + k\omega$$

and consequently $V^{(N_m)}(x; \omega)$ is $p$-cyclic. More generally, using Eq(68), we deduce that if $N_m$ is an arbitrary $p$-cyclic Maya diagram, then $V^{(N_m)}(x; \omega)$ is a $p$-cyclic potential. Conversely, if $V^{(N_m)}(x; \omega)$ is $p$-cyclic, there exist a tuple of $p$ integers $(\nu_1, \ldots, \nu_p) \in \mathbb{Z}^p$ such that

$$V^{(N_m,\nu_1,\ldots,\nu_p)}(x; \omega) = V^{(N_m)}(x; \omega) + \Delta,$$

and necessarily the energy shift $\Delta$ must be an integer multiple of $\omega$. Due to the correspondence between rational extensions and Maya diagrams, this implies immediately that $(N_m,\nu_1,\ldots,\nu_p)$ and $N_m$ are equivalent. Consequently, $N_m$ is $p$-cyclic. \qed

We then deduce the theorem

**Theorem 2.** The rational extensions of the HO $V^{(N_m)}(x; \omega)$ where $N_m$ is given by Eq(10), solve the dressing chain of period $p$ Eq(II) for the set of parameters

$$\Delta = k\omega \text{ and } \varepsilon_{i,i+1} = (\nu_{P(i)} - \nu_{P(i+1)})\omega, \ i \in \{1, \ldots, p\},$$

where $P$ is any permutation of $S_p$ and $(\nu_{P+1} = \nu_1)$

$$(\nu_1, \ldots, \nu_p) = (0, 1 + \alpha_1 k, \ldots, (k-1) + \alpha_{k-1} k, \lambda_1, \lambda_2 + \mu_1 k, \ldots, \lambda_j, \lambda_j + \mu_j k).$$

The solutions of the dressing chain system are then given by $z = \sqrt{\omega/2x}$

$$\begin{cases} w^{(1,\ldots,i-1)}_i(x) = -\omega x/2 + \sqrt{\frac{\omega}{2x}} \log \left( \frac{\mathcal{H}^{(M_m,\nu_1,\ldots,\nu_{P(i)}-1)}(z)}{\mathcal{H}^{(M_m,\nu_1,\ldots,\nu_{P(i)})}(z)} \right), \text{ if the flip in } \nu_{P(i)} \text{ is positive}, \\ w^{(1,\ldots,i-1)}_i(x) = \omega x/2 + \sqrt{\frac{\omega}{2x}} \log \left( \frac{\mathcal{H}^{(M_m,\nu_1,\ldots,\nu_{P(i)}-1)}(z)}{\mathcal{H}^{(M_m,\nu_1,\ldots,\nu_{P(i)})}(z)} \right), \text{ if the flip in } \nu_{P(i)} \text{ is negative}, \end{cases}$$

with the convention that if a spectral index is repeated two times in the tuple $(\nu_1, \ldots, \nu_p)$ characterizing the chain, then we suppress the corresponding eigenfunction in the $\mathcal{H}$ determinants.

**Proof.** The first part is a direct consequence of the preceding lemma combined to Theorem 1. With the Crum formulas Eq(1), we can write

$$w^{(1,\ldots,i-1)}_i(x) = -\left( \log \left( \frac{W^{(N_m,\nu_1,\ldots,\nu_{P(i)}-1)}(x; \omega)}{W^{(N_m,\nu_1,\ldots,\nu_{P(i-1})}(x; \omega)} \right) \right)',$$

with the convention that if a spectral index is repeated two times in the tuple $(\nu_1, \ldots, \nu_p)$ characterizing the chain, then we suppress the corresponding eigenfunction in the Wronskians.

If $\nu_{P(i)} \in N_m$, the flip in $\nu_{P(i)}$ is positive and the tuple $(N_m,\nu_{P(1)},\ldots,\nu_{P(i)})$ contains one less index than $(N_m,\nu_{P(1)},\ldots,\nu_{P(i-1)})$. Using Eq(73) and Eq(63), we deduce the first equality of Eq(81).

If $\nu_{P(i)} \notin N_m$, the flip in $\nu_{P(i)}$ is negative and the tuple $(N_m,\nu_{P(1)},\ldots,\nu_{P(i)})$ contains one more index than $(N_m,\nu_{P(1)},\ldots,\nu_{P(i-1)})$. Using Eq(73) and Eq(63), we deduce the second equality of Eq(81). \qed
4.3 Examples

4.3.1 One-cyclic extensions of the HO and rational solutions of the dressing chain of period 1.

For \( p = 1 \), we have \( k = 1, \ j = 0 \) and then immediately Theorem 3 gives \( N_m = \emptyset \). It means that the unique rational potential solving the 1-cyclic chain is the HO itself and the corresponding cyclic chain reduces to \((0)\) ie to the SUSY partnership. This is perfectly coherent with our previous results on one step cyclicity (see Eq.(16)).

4.3.2 Two-cyclic extensions of the HO and rational solutions of the dressing chain of period 2.

For \( p = 2 \), we have \( k = 2, \ j = 0 \) and then we deduce from Theorem 2 that the Okamoto extension of the HO associated to the Maya diagram

\[
N_m = ((1 \ | \ m)_2) = (1, 3, \ldots, 2m - 1),
\]

(83)

solves the dressing chain of period 2.

This particular type of Okamoto Maya diagram is called an **Umemura staircase**. The corresponding 2-cyclic chain is \((0, 2m + 1)\).

Nevertheless, Theorem 3 is not applicable here since \( p \) is even and we cannot conclude that this is the most general rational solution of the 2-cyclic chain. In fact, we can write (see Eq.(5))

\[
V^{(N_m)}(x; \omega) = V(x; \omega) - 2 \left( \log W^{(1, 3, \ldots, 2m - 1)}(x; \omega) \right)'' ,
\]

(84)

where (see Eq(76) and Eq(63))

\[
W^{(1, 3, \ldots, 2m - 1)}(x; \omega) \propto e^{-mz^2/2}W(H_1(z), H_3(z), \ldots, H_{2m-1}(z) \mid z),
\]

(85)

with \( z = \sqrt{\omega/2x} \). But we also have \([29, 28]\)

\[
\begin{align*}
H_{2j+1}(z) &= (-1)^j 2^{2j+1} j! t^{1/2} L_j^{1/2}(t), \\
H_{2j}(z) &= (-1)^j 2^{2j} j! L_j^{-1/2}(t),
\end{align*}
\]

(86)

where \( L_j^\alpha \) is the usual Laguerre polynomial and \( t = z^2 \). It results (see Eq(72))

\[
W^{(1, 3, \ldots, 2m - 1)}(x; \omega) \propto e^{-mt/2}t^{m(m+1)/4}W(L_0^{1/2}(t), L_1^{1/2}(t), \ldots, L_m^{1/2}(t) \mid t).
\]

(87)

Using the well known derivation properties of the Laguerre polynomials \([29, 28]\)

\[
\frac{dL_j^\alpha(t)}{dt} = -L_j^{\alpha+1}(t),
\]

(88)

with \( L_0^\alpha(t) = 1 \) and \( L_n^\alpha(t) = 0 \), we obtain straightforwardly

\[
W(L_0^{1/2}(t), L_1^{1/2}(t), \ldots, L_m^{1/2}(t) \mid t) = (-1)^{m(m+1)/2}
\]

(89)
and

\[ W^{(1,3,\ldots,2m-1)}(x; \omega) \propto x^{m(m+1)/2} e^{-m\omega x^2/4}. \]  

(90)

Substituting in Eq(84), we arrive to

\[ V^{(N_m)}(x; \omega) = \omega^2 x^2/4 + \frac{m(m+1)}{x^2} - (m+1/2) \omega = V(x; \omega, m - 1/2), \]  

(91)

which is an isotonic oscillator with an integer angular momentum possessing, as expected the trivial monodromy property.

In this lowest even case, we notice that using Theorem 2 doesn’t allow us to recover the general IO potential for arbitrary values of the \( \alpha \) parameter, which is, as shown previously (see Eq(21)), the most general solution potential of the dressing chain of period 2.

4.3.3 Three-cyclic extensions of the HO and rational solutions of PIV.

We have \( p = 3 \) and consequently we can refer to Theorem 3 to determine all the rational solutions of the dressing chain of period 3, ie of the PIV equation. For the possible values of \( k \) (and \( j \)) we have only two possibilities: \( k = 1 \) (\( j = 1 \)) or \( k = 3 \) (\( j = 0 \)).

4.3.4 \( k = 1 \)

The 3-cyclic Maya diagram with \( k = 1 \) is a 3-GH Maya diagram of the form

\[ (\lambda | \mu)_1 = (\lambda, \ldots, \lambda + \mu - 1) = H_{\lambda,\mu}. \]  

(92)

Following Clarkson’s terminology [22, 2, 30], the 3-cyclic extensions \( V^{(H_{\lambda,\mu})}(x; \omega) \) are called the 3-generalized Hermite (3-GH) extensions.

From the Krein-Adler theorem [31, 32, 9], we deduce immediately that 3-GH extensions which are regular on the real line are those for which \( \mu \) is even.

The possible 3-cyclic chains corresponding to \( H_{\lambda,\mu} \) are built by permutation of \( \{0, \lambda, \lambda + \mu\} \). For the particular choice \( (0, \lambda, \lambda + \mu) \), the parameters in the dressing chain system of period 3 (see Eq(11)) are

\[
\begin{align*}
\varepsilon_{12} &= (-\lambda - \mu) \omega \\
\varepsilon_{23} &= \mu \omega \\
\varepsilon_{31} - \Delta &= (\lambda - 1) \omega.
\end{align*}
\]  

(93)

As for the associated parameters in the PIV equation associated to this chain they are given by

\[ a = -(1 - \mu - 2\lambda), \quad b = -2\mu^2, \]  

(94)

or

\[ a = m \in \mathbb{Z}, \quad b = -2(1 + m - 2n)^2. \]  

(95)
The same type of result (up to redefinition of \(a\) and \(b\)) can be obtained for all the possible choices of \(\varepsilon_1/\omega\) in the set \(\{0, \lambda, \lambda + \mu\}\).

For \(\varepsilon_1/\omega = 0\), the corresponding solution of PIV is obtained as (see Eq. 24 and Eq. 73)

\[
y_0 = \sqrt{\frac{2}{\omega}} \left( w_0^{(H_{\lambda, \mu})} - w_0 \right) = -\frac{d}{dz} \left( \log \left( \frac{\psi_0^{(H_{\lambda, \mu})}}{\psi_0} \right) \right),
\]

with \((z = \sqrt{\omega/2x})\)

\[
\psi_0^{(H_{\lambda, \mu})} \propto \frac{W^{(0, H_{\lambda, \mu})}(x; \omega)}{W^{(H_{\lambda, \mu})}(x; \omega)} \propto \frac{H^{(H_{\lambda, \mu})}(z)}{H^{(H_{\lambda, \mu})}(z)} \psi_0 \propto W^{(\lambda, H_{\lambda, \mu})}(x; \omega) \psi_0 \propto W^{(H_{\lambda, \mu})}(x; \omega) \psi_0 \propto e^{z^2} \frac{H^{(H_{\lambda, \mu})}(z)}{H^{(H_{\lambda, \mu})}(z)}.
\]

where we have used Eq. 76 and the fact that

\[
H_{\lambda, \mu} - 1 = H_{\lambda-1, \mu}.
\]

Then

\[
y_0(z) = -2z + \frac{d}{dz} \left( \log \left( \frac{H^{(H_{\lambda, \mu})}(z)}{H^{(H_{\lambda, \mu-1})}(z)} \right) \right).
\]

The second possible choice is \(\varepsilon_1/\omega = \lambda\) in which case

\[
y_\lambda = \sqrt{\frac{2}{\omega}} \left( w_\lambda^{(H_{\lambda, \mu})} - w_0 \right) = -\frac{d}{dz} \left( \log \left( \frac{\psi_\lambda^{(H_{\lambda, \mu})}}{\psi_0} \right) \right),
\]

where, using Eq. 76 and the fact that \((H_{\lambda, \mu}, \lambda) = (\lambda, \lambda+1, \ldots, \lambda+\mu-1) = H_{\lambda+1, \mu-1}

\[
\psi_\lambda^{(H_{\lambda, \mu})} \propto \frac{W^{(\lambda+1, H_{\lambda, \mu})}(x; \omega)}{W^{(H_{\lambda, \mu})}(x; \omega)} \propto \frac{H^{(H_{\lambda+1, \mu-1})}(z)}{H^{(H_{\lambda, \mu})}(z)} \propto e^{z^2} \frac{H^{(H_{\lambda+1, \mu-1})}(z)}{H^{(H_{\lambda, \mu})}(z)}.
\]

Then

\[
y_\lambda(z) = -2z + \frac{d}{dz} \left( \log \left( \frac{H^{(H_{\lambda, \mu})}(z)}{H^{(H_{\lambda+1, \mu-1})}(z)} \right) \right).
\]

The last possible choice is \(\varepsilon_1/\omega = \lambda + \mu\) giving

\[
y_{\lambda+\mu} = \sqrt{\frac{2}{\omega}} \left( w_{\lambda+\mu}^{(H_{\lambda, \mu})} - w_0 \right) = -\frac{d}{dz} \left( \log \left( \frac{\psi_{\lambda+\mu}^{(H_{\lambda, \mu})}}{\psi_0} \right) \right),
\]

with \((H_{\lambda, \mu}, \lambda + \mu) = (\lambda, \ldots, \lambda + \mu) = H_{\lambda, \mu+1} \)

\[
\psi_{\lambda+\mu}^{(H_{\lambda, \mu})} \propto \frac{W^{(\lambda+\mu+1, H_{\lambda, \mu})}(x; \omega)}{W^{(H_{\lambda, \mu})}(x; \omega)} \propto \frac{H^{(H_{\lambda, \mu+1})}(z)}{H^{(H_{\lambda, \mu})}(z)}.
\]
Then

\[ \frac{d}{dz} \log \left( \frac{H_{H,\lambda,\mu}(z)}{H_{H,\lambda+1,\mu}(z)} \right). \]  

(105)

We retrieve then the three usual form for the rational solutions associated to the generalized Hermite polynomials, namely (with \( k = 1, z = t \)) \[22, 2, 30\]

\[
\begin{aligned}
    y_0(t) &= \frac{d}{dt} \log \left( \frac{H_{H,\lambda,\mu}(t)}{H_{H,\lambda-1,\mu}(t)} \right), \\
    y_\lambda(t) &= -2t + \frac{d}{dt} \log \left( \frac{H_{H,\lambda,\mu}(t)}{H_{H,\lambda+1,\mu-1}(t)} \right), \\
    y_{\lambda+\mu}(t) &= \frac{d}{dt} \log \left( \frac{H_{H,\lambda,\mu}(t)}{H_{H,\lambda+1,\mu}(t)} \right),
\end{aligned}
\]

(106)

obtained for the following integer values of the \( a \) and \( b \) parameters in Eq(25)

\[ a = -(1 - \mu - 2\lambda), \quad b = -2\mu^2. \]  

(107)

**4.3.5 \( k=3 \)**

Now consider the case \( k = 3 \). The 3-cyclic Maya diagram with \( k = 3 \) is a 3-Okamoto Maya diagram of the form

\[
((1 | \alpha_1)_3, (2 | \alpha_2)_3) = (1, ..., 1 + 3(\alpha_1 - 1); 2, ..., 2 + 3(\alpha_2 - 1)) = \Omega_{\alpha_1,\alpha_2}.
\]  

(108)

Note that for some small values of \( \alpha_1 \) and \( \alpha_2 \), the 3-Okamoto Maya diagrams are also 3-GH Maya diagram set

\[
\Omega_{1,1} = (1,2) = H_{1,2}; \quad \Omega_{1,0} = (1) = H_{1,1}; \quad \Omega_{0,1} = (2) = H_{2,1}.
\]  

(109)

The Krein-Adler theorem implies that the 3-Okamoto extensions which are regular on the real line are those for which \( \alpha_1 = \alpha_2 \), namely correspond to 3-Okamoto Maya diagrams of the form \( \Omega_{\alpha,\alpha} \).

The possible 3-cyclic chains associated to \( \Omega_{\alpha_1,\alpha_2} \) are built by permutation from \( \{0, 1 + 3\alpha_1, 2 + 3\alpha_2\} \). For the chain \( 0, 1 + 3\alpha_1, 2 + 3\alpha_2 \), the parameters of the dressing chain system of period 3 (see Eq(11)) are

\[
\begin{aligned}
    \varepsilon_{12} &= (-1 + 3(\alpha_1 - \alpha_2)) \omega, \\
    \varepsilon_{23} &= (2 + 3\alpha_2) \omega, \\
    \varepsilon_{31} - \Delta &= (-4 - 3\alpha_1) \omega.
\end{aligned}
\]  

(110)

The corresponding parameters in the PIV equation are then given by \( (\omega = 1, \Delta = k = 3) \)

\[ a = \alpha_1 + \alpha_2, \quad b = -\frac{2}{9} (-1 + 3(\alpha_1 - \alpha_2))^2, \]  

(111)

or

\[ a = j \in \mathbb{Z}, \quad b = -2 (1/3 - j + 2\alpha_2)^2. \]  

(112)
The same result (up to redefinition of the integers) can be obtained with all the possible choices of \( \varepsilon_1 \) in the set \( \{0, 1 + 3\alpha_1, 2 + 3\alpha_2\} \).

For \( \varepsilon_1/\omega = 0 \), the corresponding solution of PIV is obtained as (see Eq.(24))

\[ y_0 = \sqrt{\frac{2}{\Delta}} \left( w_0^{(\alpha_1, \alpha_2)} - 3w_0 \right) = -\sqrt{\frac{2}{3\omega}} \frac{d}{dx} \left( \log \left( \frac{\psi_0^{(\alpha_1, \alpha_2)}}{\psi_0^{3}} \right) \right), \] (113)

with \( z = \sqrt{\omega/2x} \). If we suppose \( \alpha_1, \alpha_2 \geq 2 \), by using Eq.(76), we obtain

\[ \psi_0^{(\alpha_1, \alpha_2)} \propto W^{(\alpha_1-1, \alpha_2-1, \alpha_1 \alpha_2)}(x; \omega) \propto \mathcal{H}^{(\alpha_1, \alpha_2)}(z) \psi_0^{3}(x; \omega) \propto e^z \mathcal{H}^{(\alpha_1-1, \alpha_2-1)}(z) \] (114)

Note that \((0, \Omega_{\alpha_1, \alpha_2}) = (0, 1, 2) \cup \{(4, \ldots, 1 + 3(\alpha_1 - 1); 5, \ldots, 2 + 3(\alpha_2 - 1)) = \Omega_{\alpha_1-1, \alpha_2-1} \oplus 3\).

Then

\[ y_0 = -\sqrt{\frac{2\omega}{3}} x + \sqrt{\frac{2}{3\omega}} \frac{d}{dx} \log \left( \frac{\mathcal{H}^{(\alpha_1, \alpha_2)}(z)}{\mathcal{H}^{(\alpha_1, \alpha_2)}(z)} \right), \] (115)

that is,

\[ y_0(t) = -\frac{2}{3} t + \frac{d}{dt} \log \left( \frac{\mathcal{H}^{(\alpha_1, \alpha_2)}(t/\sqrt{3})}{\mathcal{H}^{(\alpha_1, \alpha_2)}(t/\sqrt{3})} \right). \] (116)

The second possible choice is \( \varepsilon_1/\omega = 1 + 3\alpha_1 \) in which case

\[ y_{1+3\alpha_1}(x) = \sqrt{\frac{2}{\Delta}} \left( w_1^{(\alpha_1, \alpha_2)} - 3w_0 \right) = -\sqrt{\frac{2}{3\omega}} \frac{d}{dx} \left( \log \left( \frac{\psi_0^{(1+3\alpha_1)}}{\psi_0^{3}} \right) \right), \] (117)

with \((1 + 3\alpha_1, \Omega_{\alpha_1, \alpha_2}) = \Omega_{\alpha_1+1, \alpha_2}\)

\[ \psi_0^{(1+3\alpha_1)} \propto W^{(\alpha_1+1, \alpha_2)}(x; \omega) \propto \mathcal{H}^{(\alpha_1+1, \alpha_2)}(z) \psi_0^{3}(x; \omega) \propto e^z \mathcal{H}^{(\alpha_1+1, \alpha_2)}(z) \] (118)

Then

\[ y_{1+3\alpha_1} = -\sqrt{\frac{2\omega}{3}} x + \sqrt{\frac{2}{3\omega}} \frac{d}{dx} \log \left( \frac{\mathcal{H}^{(\alpha_1, \alpha_2)}}{\mathcal{H}^{(\alpha_1+1, \alpha_2)}} \right), \] (119)

that is

\[ y(t) = -\frac{2}{3} t + \frac{d}{dt} \log \left( \frac{\mathcal{H}^{(\alpha_1, \alpha_2)}(t/\sqrt{3})}{\mathcal{H}^{(\alpha_1+1, \alpha_2)}(t/\sqrt{3})} \right). \] (120)
The last possible choice is $\varepsilon_1/\omega = 2 + 3\alpha_2$ giving

$$y_{2 + 3\alpha_2} = \sqrt{\frac{2}{\Delta}} \left( \psi_{2 + 3\alpha_2}^{(\Omega_{\alpha_1, \omega_2})} - 3w_0 \right) = -\sqrt{\frac{2}{3\omega}} \frac{d}{dx} \left( \log \left( \frac{\psi_{2 + 3\alpha_2}^{(\Omega_{\alpha_1, \omega_2})}}{\psi_0} \right) \right), \quad (121)$$

with $((2 + 3\alpha_1, \Omega_{\alpha_1, \omega_2}) = \Omega_{\alpha_1, \omega_2 + 1})$

$$\frac{\psi_{2 + 3\alpha_2}^{(\Omega_{\alpha_1, \omega_2})}}{\psi_0} \propto \frac{W^{(\Omega_{\alpha_1, \omega_2 + 1})}(x; \omega)}{W^{(\Omega_{\alpha_1, \omega_2})}(x; \omega) \psi_0^3(x; \omega)} \times \frac{\mathcal{H}^{(\Omega_{\alpha_1, \omega_2 + 1})}(z)}{\mathcal{H}^{(\Omega_{\alpha_1, \omega_2})}(z)} \propto \varepsilon^2 \frac{\mathcal{H}^{(\Omega_{\alpha_1, \omega_2 + 1})}(z)}{\mathcal{H}^{(\Omega_{\alpha_1, \omega_2})}(z)}. \quad (122)$$

Then

$$y_{2 + 3\alpha_2} = -\sqrt{\frac{2\omega}{3}} x + \sqrt{\frac{2}{3\omega}} \frac{d}{dx} \log \left( \frac{\mathcal{H}^{(\Omega_{\alpha_1, \omega_2})}(z)}{\mathcal{H}^{(\Omega_{\alpha_1, \omega_2 + 1})}(z)} \right), \quad (123)$$

that is

$$y_{2 + 3\alpha_2}(t) = -\frac{2}{3} t + \frac{d}{dt} \log \left( \frac{\mathcal{H}^{(\Omega_{\alpha_1, \omega_2})}(t/\sqrt{3})}{\mathcal{H}^{(\Omega_{\alpha_1, \omega_2 + 1})}(t/\sqrt{3})} \right). \quad (124)$$

We retrieve then the three usual form for the rational solutions associated to the generalized Hermite polynomials, namely (with $k = 1, z = t$) [22, 2, 30]

$$\left\{ \begin{array}{l}
\frac{d}{dt} \log \left( \frac{\mathcal{H}^{(\Omega_{\alpha_1, \omega_2})}(t/\sqrt{3})}{\mathcal{H}^{(\Omega_{\alpha_1, \omega_2 - 1})}(t/\sqrt{3})} \right) \\
\frac{d}{dt} \log \left( \frac{\mathcal{H}^{(\Omega_{\alpha_1, \omega_2})}(t/\sqrt{3})}{\mathcal{H}^{(\Omega_{\alpha_1 + 1, \omega_2})}(t/\sqrt{3})} \right) \\
\frac{d}{dt} \log \left( \frac{\mathcal{H}^{(\Omega_{\alpha_1, \omega_2})}(t/\sqrt{3})}{\mathcal{H}^{(\Omega_{\alpha_1, \omega_2 + 1})}(t/\sqrt{3})} \right), \end{array} \right. \quad (125)$$

obtained for the following integer values of the $a$ and $b$ parameters in Eq(15)

$$a = \alpha_1 + \alpha_2 = j \in \mathbb{Z}, \quad b = -2(1/3 - j + 2\alpha_2)^2. \quad (126)$$

### 4.4 5-cyclic chains and new solutions of the $A_4$-PIV

We have $p = 5$ and consequently we can have $k = 1 \ (j = 2), \ k = 3 \ (j = 1)$ or $k = 5 \ (j = 0)$.

#### 4.4.1 k=1

The 5-cyclic extension with $k = 1$ is associated to a 5-GH Maya diagram of the form

$$N_m = ((\lambda_1 \mid \mu_1)_1, (\lambda_2 \mid \mu_2)_1), \ m = \mu_1 + \mu_2. \quad (127)$$
When $\lambda_1 + \mu_1 \geq \lambda_2$, the block structure of $N_m$ is degenerate and we recover a 3-GH Maya diagram $H_{\lambda, \mu}$ (see Eq(122)), which is 3 but also trivially 5-cyclic. For simplicity we suppose in the following $\lambda_1 + \mu_1 < \lambda_2$. Explicitely

\[ N_m = (\lambda_1, ..., \lambda_1 + \mu_1 - 1, \lambda_2, ..., \lambda_2 + \mu_2 - 1). \tag{128} \]

The corresponding 5-cyclic chain is built by permutation of

\[ \{\lambda_1, \lambda_1 + \mu_1, \lambda_2, \lambda_2 + \mu_2, 0\}. \tag{129} \]

For the chain $(\lambda_1, \lambda_1 + \mu_1, \lambda_2, \lambda_2 + \mu_2, 0)$, the parameters of the dressing chain system of period 5 (see Eq(11)) are

\[
\begin{cases}
\varepsilon_{12} = -\mu_1 \omega \\
\varepsilon_{23} = (\lambda_1 - \lambda_2 + \mu_1) \omega \\
\varepsilon_{34} = -\mu_2 \omega \\
\varepsilon_{45} = (\lambda_2 + \mu_2) \omega \\
\varepsilon_{51} = - (\lambda_1 + 1) \omega,
\end{cases}
\tag{130}
\]

and the solutions of this system are given by (see Eq(81))

\[
w_1(x) = w_{\lambda_1}^{(\lambda_1|\mu_1), (\lambda_2|\mu_2)}(x) = -\omega x/2 + \sqrt{\frac{\omega}{2}} \frac{d}{dz} \left( \log \left( \frac{H^{((\lambda_1|\mu_1), (\lambda_2|\mu_2))}(z)}{H^{((\lambda_1|\mu_1), (\lambda_2|\mu_2))}(z)} \right) \right),
\]

\[
w_2(x) = w_{\lambda_1+N_m}^{(\lambda_1, N_m)}(x) = \omega x/2 + \sqrt{\frac{\omega}{2}} \frac{d}{dz} \left( \log \left( \frac{H^{((\lambda_1+1|\mu_1-1), (\lambda_2|\mu_2))}(z)}{H^{((\lambda_1+1|\mu_1-1), (\lambda_2|\mu_2))}(z)} \right) \right),
\]

\[
w_3(x) = w_{\lambda_2}^{(\lambda_1+1|\mu_1), (\lambda_2|\mu_2)}(x) = -\omega x/2 + \sqrt{\frac{2}{\omega}} \frac{d}{dz} \left( \log \left( \frac{H^{((\lambda_1+1|\mu_1), (\lambda_2|\mu_2))}(z)}{H^{((\lambda_1+1|\mu_1), (\lambda_2|\mu_2))}(z)} \right) \right),
\]

\[
\tag{131}
\tag{132}
\tag{133}
\]

\[
21
\]
are given by

\[ w_4(x) = w_{(\lambda_1+1\mu_1), (\lambda_2+1\mu_2-1)}(x) \]

\[ = \omega x/2 + \sqrt{\frac{\omega}{2}} d \frac{d}{dz} \left( \log \left( \frac{\mathcal{H}(\lambda_1+1\mu_1), (\lambda_2+1\mu_2-1)}{\mathcal{H}(\lambda_2+\mu_2, (\lambda_1+1\mu_1), (\lambda_2+1\mu_2-1))} (z) \right) \right) \]

\[ = \omega x/2 + \sqrt{\frac{\omega}{2}} d \frac{d}{dz} \left( \log \left( \frac{\mathcal{H}(\lambda_1+1\mu_1), (\lambda_2+1\mu_2-1)}{\mathcal{H}(\lambda_1+1\mu_1), (\lambda_2+1\mu_2-1)} (z) \right) \right), \] (134)

and (see Eq[76])

\[ w_5(x) = w_{(\lambda_1+1\mu_1), (\lambda_2+1\mu_2)}(x) \]

\[ = \omega x/2 + \sqrt{\frac{\omega}{2}} d \frac{d}{dz} \left( \log \left( \frac{\mathcal{H}(\lambda_1+1\mu_1), (\lambda_2+1\mu_2)}{\mathcal{H}(\lambda_2+\mu_2, (\lambda_1+1\mu_1), (\lambda_2+1\mu_2))} (z) \right) \right) \]

\[ = \omega x/2 + \sqrt{\frac{\omega}{2}} d \frac{d}{dz} \left( \log \left( \frac{\mathcal{H}(\lambda_1+1\mu_1), (\lambda_2+1\mu_2)}{\mathcal{H}(\lambda_1+1\mu_1), (\lambda_2+1\mu_2)} (z) \right) \right), \] (135)

4.4.2 \( k=5 \)

The 5-cyclic extension with \( k = 5 \) is associated to a 5-Okamoto Maya diagram of the form

\[ N_m = ((1 | \alpha_1)_5, (2 | \alpha_2)_5, (3 | \alpha_3)_5, (4 | \alpha_4)_5) = \Omega_{\alpha_1, \alpha_2, \alpha_3, \alpha_4}, \ m = \alpha_1 + \alpha_2 + \alpha_3 + \alpha_4. \] (136)

The corresponding 5-cyclic chain are obtained by permutation from

\[ \{1 + 5\alpha_1, 2 + 5\alpha_2, 3 + 5\alpha_3, 4 + 5\alpha_4, 0\}. \] (137)

The parameters in the dressing chain system of period 5 (see Eq[11]) for the chain \((1 + 5\alpha_1, 2 + 5\alpha_2, 3 + 5\alpha_3, 4 + 5\alpha_4, 0)\) are given by

\[
\begin{align*}
\varepsilon_{12} &= (-1 - 5(\alpha_2 - \alpha_1))\omega \\
\varepsilon_{23} &= (-1 - 5(\alpha_3 - \alpha_2))\omega \\
\varepsilon_{34} &= (-1 - 5(\alpha_4 - \alpha_3))\omega \\
\varepsilon_{45} &= (5\alpha_4 + 4)\omega \\
\varepsilon_{51} - \Delta &= (-6 - 5\alpha_1)\omega.
\end{align*}
\] (138)

and the corresponding solutions of the dressing chain system are (see Eq[51])

\[ w_2(x) = w_{(1\alpha_1)_5, (2\alpha_2)_5, (3\alpha_3)_5, (4\alpha_4)_5}(x) \]

\[ = \omega x/2 + \sqrt{\frac{\omega}{2}} d \frac{d}{dz} \left( \log \left( \frac{\mathcal{H}(1\alpha_1)_5, (2\alpha_2)_5, (3\alpha_3)_5, (4\alpha_4)_5)}{\mathcal{H}(1+5\alpha_1, (1\alpha_2)_5, (2\alpha_3)_5, (3\alpha_4)_5)} (z) \right) \right) \]

\[ = \omega x/2 + \sqrt{\frac{\omega}{2}} d \frac{d}{dz} \left( \log \left( \frac{\mathcal{H}(1\alpha_1)_5, (2\alpha_2)_5, (3\alpha_3)_5, (4\alpha_4)_5)}{\mathcal{H}(1\alpha_1+1\alpha_2, (2\alpha_3)_5, (3\alpha_4)_5)} (z) \right) \right), \] (139)
\[
\begin{align*}
    w_2(x) &= \frac{(1|\alpha_1+1)_5,(2|\alpha_2)_5,(3|\alpha_3)_5,(4|\alpha_4)_5}{5}\times (x) \\
    &= \omega x/2 + \sqrt{\frac{\omega}{2}} \frac{d}{dz}\left( \log \left( \frac{\mathcal{H}_2(1|\alpha_1+1)_5,(2|\alpha_2)_5,(3|\alpha_3)_5,(4|\alpha_4)_5}{\mathcal{H}_3(2+5\alpha_2,1|\alpha_1+1)_5,(2|\alpha_2)_5,(3|\alpha_3)_5,(4|\alpha_4)_5)} (z) \right) \right) \\
    &= \omega x/2 + \sqrt{\frac{\omega}{2}} \frac{d}{dz}\left( \log \left( \frac{\mathcal{H}_2(1|\alpha_1+1)_5,(2|\alpha_2)_5,(3|\alpha_3)_5,(4|\alpha_4)_5}{\mathcal{H}_3(1|\alpha_1+1)_5,(2|\alpha_2+1)_5,(3|\alpha_3)_5,(4|\alpha_4)_5)} (z) \right) \right), \quad (140)
\end{align*}
\]

\[
\begin{align*}
    w_3(x) &= \frac{(1|\alpha_1+1)_5,(2|\alpha_2+1)_5,(3|\alpha_3)_5,(4|\alpha_4)_5}{5}\times (x) \\
    &= \omega x/2 + \sqrt{\frac{\omega}{2}} \frac{d}{dz}\left( \log \left( \frac{\mathcal{H}_2(1|\alpha_1+1)_5,(2|\alpha_2+1)_5,(3|\alpha_3)_5,(4|\alpha_4)_5}{\mathcal{H}_3(3+5\alpha_3,1|\alpha_1+1)_5,(2|\alpha_2+1)_5,(3|\alpha_3)_5,(4|\alpha_4)_5)} (z) \right) \right) \\
    &= \omega x/2 + \sqrt{\frac{\omega}{2}} \frac{d}{dz}\left( \log \left( \frac{\mathcal{H}_2(1|\alpha_1+1)_5,(2|\alpha_2+1)_5,(3|\alpha_3)_5,(4|\alpha_4)_5}{\mathcal{H}_3(1|\alpha_1+1)_5,(2|\alpha_2+1)_5,(3|\alpha_3+1)_5,(4|\alpha_4)_5)} (z) \right) \right), \quad (141)
\end{align*}
\]

\[
\begin{align*}
    w_4(x) &= \frac{(1|\alpha_1+1)_5,(2|\alpha_2+1)_5,(3|\alpha_3+1)_5,(4|\alpha_4)_5}{5}\times (x) \\
    &= \omega x/2 + \sqrt{\frac{\omega}{2}} \frac{d}{dz}\left( \log \left( \frac{\mathcal{H}_2(1|\alpha_1+1)_5,(2|\alpha_2+1)_5,(3|\alpha_3+1)_5,(4|\alpha_4)_5}{\mathcal{H}_3(4+5\alpha_4,1|\alpha_1+1)_5,(2|\alpha_2+1)_5,(3|\alpha_3+1)_5,(4|\alpha_4)_5)} (z) \right) \right) \\
    &= \omega x/2 + \sqrt{\frac{\omega}{2}} \frac{d}{dz}\left( \log \left( \frac{\mathcal{H}_2(1|\alpha_1+1)_5,(2|\alpha_2+1)_5,(3|\alpha_3+1)_5,(4|\alpha_4)_5}{\mathcal{H}_3(1|\alpha_1+1)_5,(2|\alpha_2+1)_5,(3|\alpha_3+1)_5,(4|\alpha_4+1)_5)} (z) \right) \right), \quad (142)
\end{align*}
\]

and (see Eq. (140))

\[
\begin{align*}
    w_5(x) &= \frac{(1|\alpha_1+1)_5,(2|\alpha_2+1)_5,(3|\alpha_3+1)_5,(4|\alpha_4+1)_5}{5}\times (x) \\
    &= \omega x/2 + \sqrt{\frac{\omega}{2}} \frac{d}{dz}\left( \log \left( \frac{\mathcal{H}_2(1|\alpha_1+1)_5,(2|\alpha_2+1)_5,(3|\alpha_3+1)_5,(4|\alpha_4+1)_5}{\mathcal{H}_3(0,1|\alpha_1+1)_5,(2|\alpha_2+1)_5,(3|\alpha_3+1)_5,(4|\alpha_4+1)_5)} (z) \right) \right) \\
    &= \omega x/2 + \sqrt{\frac{\omega}{2}} \frac{d}{dz}\left( \log \left( \frac{\mathcal{H}_2(1|\alpha_1+1)_5,(2|\alpha_2+1)_5,(3|\alpha_3+1)_5,(4|\alpha_4+1)_5}{\mathcal{H}_3(1|\alpha_1+1)_5,(2|\alpha_2)_5,(3|\alpha_3)_5,(4|\alpha_4)_5)} (z) \right) \right), \quad (143)
\end{align*}
\]

where we have used

\[
\begin{align*}
    (0,1|\alpha_1+1)_5 \cup (2|\alpha_2+1)_5 \cup (3|\alpha_3+1)_5 \cup (4|\alpha_4+1)_5 \\
    = (0,1,2,3,4) \cup ((6|\alpha_1+1)_5 \cup (7|\alpha_2+1)_5 \cup (8|\alpha_3+1)_5 \cup (9|\alpha_4+1)_5) = N_m \oplus 5. \quad (144)
\end{align*}
\]

#### 4.4.3 k=3

The 5-cyclic Maya diagram with k = 3 is of the form

\[
N_m = (1|\alpha_1)_3 \cup (2|\alpha_2)_3 \cup (\lambda_1|\mu_1)_3, \quad m = \alpha_1 + \alpha_2 + \mu_1. \quad (145)
\]

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with the convention that a two times repeated index is suppressed from the list.

The corresponding 5-cyclic chain is obtained by permutations from

\[
\{1 + 3\alpha_1, 2 + 3\alpha_2, \lambda_1, \lambda_1 + \mu_1, 0\}
\] (146)

and for the chain \((1 + 3\alpha_1, 2 + 3\alpha_2, \lambda_1, \lambda_1 + \mu_1, 0)\) the parameters in the 5-cyclic dressing chain system (see Eq(11)) are, with the order above

\[
\begin{align*}
\varepsilon_{12} &= (−1 − 3(\alpha_2 − \alpha_1))\omega \\
\varepsilon_{23} &= (2 + 3\alpha_2 − \lambda_1)\omega \\
\varepsilon_{34} &= −\mu_1\omega \\
\varepsilon_{45} &= (\lambda_1 + \mu_1 − 3)\omega \\
\varepsilon_{51} − \Delta &= (−4 − 3\alpha_1)\omega.
\end{align*}
\] (147)

The solutions of the dressing chain system are given by (see Eq(31))

\[
w_1(x) = w_{1+3\alpha_1}^{(1|\alpha_1)_3,(2|\alpha_2)_3,(\lambda_1|\mu_1)_3}(x) \\
= \omega x/2 + \sqrt{\frac{\omega}{2}} d \left( \log \left( \frac{\mathcal{H}((1|\alpha_1)_3,(2|\alpha_2)_3,(\lambda_1|\mu_1)_3)(z)}{\mathcal{H}((1+3\alpha_1,1|\alpha_1)_3,(2|\alpha_2)_3,(\lambda_1|\mu_1)_3)(z)} \right) \right),
\] (148)

\[
w_2(x) = w_{2+3\alpha_2}^{(1|\alpha_1+1)_3,(2|\alpha_2)_3,(\lambda_1|\mu_1)_3}(x) \\
= \omega x/2 + \sqrt{\frac{\omega}{2}} d \left( \log \left( \frac{\mathcal{H}((1|\alpha_1+1)_3,(2|\alpha_2)_3,(\lambda_1|\mu_1)_3)(z)}{\mathcal{H}((2+3\alpha_2,1|\alpha_1)_3,(2|\alpha_2)_3,(\lambda_1|\mu_1)_3)(z)} \right) \right),
\] (149)

\[
w_3(x) = w_{\lambda_1}^{(1|\alpha_1+1)_3,(2|\alpha_2+1)_3,(\lambda_1|\mu_1)_3}(x) \\
= −\omega x/2 + \sqrt{\frac{\omega}{2}} d \left( \log \left( \frac{\mathcal{H}((1|\alpha_1+1)_3,(2|\alpha_2+1)_3,(\lambda_1|\mu_1)_3)(z)}{\mathcal{H}(\lambda_1,(1|\alpha_1+1)_3,(2|\alpha_2+1)_3,(\lambda_1|\mu_1)_3)(z)} \right) \right),
\] (150)

\[
w_4(x) = w_{\lambda_1+\mu_1}^{(1|\alpha_1+1)_3,(2|\alpha_2+1)_3,(\lambda_1+3|\mu_1-1)_3}(x) \\
= \omega x/2 + \sqrt{\frac{\omega}{2}} d \left( \log \left( \frac{\mathcal{H}((1|\alpha_1+1)_3,(2|\alpha_2+1)_3,(\lambda_1+3|\mu_1-1)_3)(z)}{\mathcal{H}(\lambda_1+\mu_1,(1|\alpha_1+1)_3,(2|\alpha_2+1)_3,(\lambda_1+3|\mu_1-1)_3)(z)} \right) \right),
\] (151)
and (see Eq(76))

\[
\begin{align*}
w_5(x) &= w_0^{(1(\alpha_1+1)_3, 2(\alpha_2+1)_3, (\lambda_1+3|\mu_1)_3)}(x) \\
&= \sqrt{\omega} x + \sqrt{\omega} d \frac{d}{dz} \left( \log \left( \frac{H(1(\alpha_1+1)_3, 2(\alpha_2+1)_3, (\lambda_1+3|\mu_1)_3)}{H(0, 1(\alpha_1+1)_3, 2(\alpha_2+1)_3, (\lambda_1+3|\mu_1)_3)}(z) \right) \right),
\end{align*}
\]

where we have used:

\[
(0, (1 \mid \alpha_1 + 1)_3, (2 \mid \alpha_2 + 1)_3, (\lambda_1 + 3 \mid \mu_1)_3) = (0, 1, 2) \cup ((4 \mid \alpha_1 + 1)_3, (5 \mid \alpha_2 + 1)_3, (\lambda_1 + 3 \mid \mu_1)_3) = N_m \oplus 3.
\]

5 Rational extensions of the IO and rational solutions of the dressing chains of even periodicity

5.1 Rational extensions of the IO

The IO potential (with zero ground level \(E_0 = 0\)) is defined on the positive half line \([0, +\infty]\) by

\[
V(x; \omega, \alpha) = \frac{\omega^2}{4} x^2 + \frac{(\alpha + 1/2)(\alpha - 1/2)}{x^2} - \omega (\alpha + 1), \quad |\alpha| > 1/2.
\]

If we add Dirichlet boundary conditions at 0 and infinity and if we suppose \(\alpha > 1/2\), it has the following spectrum \((z = \omega x^2/2)\)

\[
\begin{align*}
E_n(\omega) &= 2n\omega \\
\psi_{n \otimes \varnothing}(x; \omega, \alpha) &= \psi_{0 \otimes \varnothing}(x; \omega, \alpha) E_n^\alpha(z), \quad n \geq 0,
\end{align*}
\]

with \(\psi_{0 \otimes \varnothing}(x; \omega, \alpha) = z^{(\alpha+1/2)/2} e^{-z/2}\). The interest of this notation for the spectral indices \((n \otimes \varnothing\) rather than \(n\)\) will become clear later.

\(V(x; \omega, \alpha)\) is translationally shape invariant with \((\alpha_n = \alpha + n)\)

\[
V^{(0 \otimes \varnothing)}(x; \omega, \alpha) = V(x; \omega, \alpha_1) + 2\omega.
\]

It possesses also three discrete parametric symmetries:

*The \(\Gamma_1\) symmetry, \((\omega, \alpha) \overset{\Gamma_1}{\rightarrow} (-\omega, \alpha)\), which acts as

\[
V(x; \omega, \alpha) \overset{\Gamma_1}{\rightarrow} V(x; -\omega, \alpha) = V(x; \omega, \alpha) + 2\omega (\alpha + 1),
\]

and generates the conjugate shadow spectrum of \(V(x; \omega, \alpha)\) :
are canonical Maya diagrams. If then be indexed by a couple of Maya diagrams, that we call a The rational extensions of the IO, which are obtained via chains of DT associated to seed functions of this type, extended shadow spectrum shadow and conjugate shadow spectra forms the The union of the spectrum and the conjugate spectrum forms the extended spectrum. All together they contain all the quasi-polynomial eigenfunctions of the IO. To avoid the specific case where the extended and extended shadow spectra merge, we restrict the values of \( \alpha \) to be non-integer:

\[
\alpha \notin \mathbb{N}. \tag{164}
\]

The union of the spectrum and the conjugate spectrum forms the extended spectrum and the union of the shadow and conjugate shadow spectra forms the extended shadow spectrum. All together they contain all the quasi-polynomial eigenfunctions of the IO. To avoid the specific case where the extended and extended shadow spectra merge, we restrict the values of \( \alpha \) to be non-integer:

\[
\alpha \notin \mathbb{N}. \tag{164}
\]

The rational extensions of the IO, which are obtained via chains of DT associated to seed functions of this type, can then be indexed by a couple of Maya diagrams, that we call a universal character (UC) \[16, 17, 33, 34\].

\[
N_m \otimes L_r = (n_1, ..., n_m) \otimes (l_1, ..., l_r), \tag{165}
\]

\( N_m = (n_1, ..., n_m) \) containing the spectral indices of the seed functions belonging to the extended spectrum and \( L_r = (l_1, ..., l_r) \) those belonging to the extended shadow spectrum. The UC is said to be canonical if \( N_m \) and \( L_r \) are canonical Maya diagrams. If \( N_m \) and \( L_r \) are respectively \( p_1 \)-cyclic with translation of \( k_1 > 0 \) and \( p_2 \)-cyclic with translation of \( k_2 > 0 \) then we say that is the UC \( N_m \otimes L_r \) is \( p \)-cyclic, \( p = p_1 + p_2 \). For reasons which will become clear later, we call the quantity \( k = k_1 - k_2 \), the balanced translation amplitude of \( N_m \otimes L_r \).

We have proven in [11], that if we have two equivalent UC

\[
N_m \otimes L_r \approx N'_m \otimes L'_r, \tag{166}
\]
that is, if
\[ N_m \approx N'_m \text{ and } L_r \approx L'_r, \] (167)
then
\[ V^{N_m \otimes L_r}(x; \omega, \alpha) = V^{N'_m \otimes L'_r}(x; \omega, \alpha_s) + 2q\omega, \quad q, s \in \mathbb{Z}. \] (168)

In particular, if \( N_m \) and \( L_r \) are canonical and \( k_1, k_2 > 0 \)
\[ V^{(N_m \circ k_1) \otimes (L_r \circ k_2)}(x; \omega, \alpha) = V^{N_m \otimes L_r}(x; \omega, \alpha_{k_1-k_2}) + 2k_1\omega. \] (169)

For the Wronskians, we have the following equivalence relation [12]
\[ W^{(N_m \circ k_1) \otimes (L_r \circ k_2)}(x; \omega, \alpha) = \prod_{i=0}^{k_1-1} (\psi_{0 \otimes 0}(x; \omega, \alpha_i)) \prod_{j=0}^{k_2-1} (\psi_{0 \otimes 0}(x; \omega, \alpha_{k_1-j})) W^{N_m \otimes L_r}(x; \omega, \alpha_{k_1-k_2}) \] (170)
\[ = z^{\alpha(k_1-k_2)/2+(k_1-k_2)^2/4} e^{-(k_1+k_2)z/2} \times W^{N_m \otimes L_r}(x; \omega, \alpha_{k_1-k_2}), \]
or, if \( k_1 \leq n_1 \leq \ldots \leq n_m \) and \( k_2 \leq l_1 \leq \ldots \leq l_r \)
\[ W^{((0,\ldots,k_1-1) \cup N_m) \otimes ((0,\ldots,k_2-1) \cup L_r)}(x; \omega, \alpha) \] (171)
\[ = \prod_{i=0}^{k_1-1} (\psi_{0 \otimes 0}(x; \omega, \alpha_i)) \prod_{j=0}^{k_2-1} (\psi_{0 \otimes 0}(x; \omega, \alpha_{k_1-j})) W^{(N_m-k_1) \otimes (L_r-k_2)}(x; \omega, \alpha_{k_1-k_2}) \]
\[ = z^{\alpha(k_1-k_2)/2+(k_1-k_2)^2/4} e^{-(k_1+k_2)z/2} W^{(N_m-k_1) \otimes (L_r-k_2)}(x; \omega, \alpha_{k_1-k_2}). \]

Using Eq(155), Eq(160), Eq(72) and the derivation properties of the Laguerre [29, 28, 12] (see also Eq(75))
\[ \left\{ \begin{array}{c}
\frac{d^j}{dz^j} (L_n^\alpha (z)) = (-1)^j L_n^{\alpha+j} (z) \\
\frac{d^j}{dz^j} (z^{-\alpha} L_n^{-\alpha} (z)) = (n-\alpha)_j L_n^{-\alpha-j} (z),
\end{array} \right. \] (172)
we can write
\[ W^{N_m \otimes L_r}(x; \omega, \alpha) \propto z^{(m-r)^2/4-r(r-1)} \times e^{-(m+r)z/2} \times \mathcal{L}^{N_m \otimes L_r}(z; \alpha), \] (173)
where \( \mathcal{L}^{N_m \otimes L_r} \) is the following determinant \((i = 0, \ldots, m + r - 1)\)
\[ \mathcal{L}^{N_m \otimes L_r} (z; \alpha) = \left| \begin{array}{c}
\overrightarrow{L}_{n_1} (z; \alpha), \ldots, \overrightarrow{L}_{n_m} (z; \alpha), \overrightarrow{L}_{l_1} (z; \alpha), \ldots, \overrightarrow{L}_{l_r} (z; \alpha)
\end{array} \right|, \] (174)
with \( L_n^\alpha (z) = 0 \) if \( n < 0 \)
\[ \overrightarrow{L}_n (z; \alpha) = \begin{pmatrix}
L_n^\alpha (z) \\
(1)^j L_n^{\alpha+j} (z) \\
(1)^{m+r-1} L_n^{\alpha+m+r-1} (z)
\end{pmatrix}, \quad \overrightarrow{L}_l (z; \alpha) = \begin{pmatrix}
z^{m+r-1} L_l^{-\alpha} (z) \\
(l-\alpha)_1 z^{m+r-1} L_l^{-\alpha-j} (z) \\
(l-\alpha)_{m+r-1} L_l^{-\alpha-m-r+1} (z)
\end{pmatrix}. \] (175)
\( \mathcal{L}^{N_m \otimes L_r} \) is called a Laguerre pseudowronskian \([11][12]\).

Eq \([170] \) and Eq \([171] \) give then

\[
\begin{align*}
\mathcal{L}^{(N_m \otimes k_1) \otimes (L_r \otimes k_2)}(z; \alpha) & \propto z^{2rk_2+k_2(k_2-1)} \mathcal{L}^{(N_m) \otimes (L_r)}(z; \alpha k_1 - k_2) \\
\mathcal{L}^{((0, \ldots, k_1-1) \cup N_m) \otimes ((0, \ldots, k_2-1) \cup L_r)}(z; \alpha) & \propto z^{2rk_2+k_2(k_2-1)} \mathcal{L}^{(N_m-k_1) \otimes (L_r-k_2)}(z; \alpha k_1 - k_2).
\end{align*}
\]  

(176)

The equivalence property Eq \([168] \) can be viewed as the most general transcription of the shape invariance property Eq \([156] \) at the level of the rational extensions of the IO potential \([12]\).

Due to this equivalence property, to describe all the rational extensions of the IO, it is sufficient to consider those associated to canonical UC.

We can also note the useful symmetry relation

\[
W^{L_r \otimes N_m}(x; \omega, \alpha) = \Gamma_2 \left( W^{N_m \otimes L_r}(x; \omega, \alpha) \right),
\]

(177)

which leads immediately to

\[
V^{L_r \otimes N_m}(x; \omega, \alpha) = \Gamma_2 \left( V^{N_m \otimes L_r}(x; \omega, \alpha) \right) + 2\omega \alpha.
\]

(178)

5.2 \( p \)-cyclic extensions of the IO and rational solutions of the even periodic dressing chains

The cyclicity of the canonical UC \( N_m \otimes L_r \) alone is not sufficient to ensure the cyclicity of the associated rational extension of the IO \( V^{N_m \otimes L_r}(x; \omega, \alpha) \). Nevertheless we have the following lemma

**Lemma 3.** If the UC \( N_m \otimes L_r \) is \( p \)-cyclic with a zero balanced translation amplitude, then \( V^{N_m \otimes L_r}(x; \omega, \alpha) \) is a \( p \)-cyclic potential.

**Proof.** If \( N_m \otimes L_r \) is \( p \)-cyclic, there exists a chain of DT \((\nu_1, \ldots, \nu_{p_1}) \otimes (\lambda_1, \ldots, \lambda_{p_2})\) with \( p = p_1 + p_2 \), and \( k_1, k_2 \in \mathbb{N} \) such that

\[
V^{(N_m, \nu_1, \ldots, \nu_{p_1}) \otimes (L_r, \lambda_1, \ldots, \lambda_{p_2})}(x; \omega, \alpha) = V^{(N_m \otimes k_1) \otimes (L_r \otimes k_2)}(x; \omega, \alpha),
\]

(179)

which, combined to Eq \([169] \), leads to

\[
V^{(N_m, \nu_1, \ldots, \nu_{p_1}) \otimes (L_r, \lambda_1, \ldots, \lambda_{p_2})}(x; \omega, \alpha) = V^{N_m \otimes L_r}(x; \omega, \alpha k_1 - k_2) + 2k_1 \omega.
\]

(180)

In the case of a zero balanced translation amplitude \( k_1 = k_2 \) and \( V^{N_m \otimes L_r}(x; \omega, \alpha) \) is then \( p \)-cyclic with an energy shift \( \Delta = 2k_1 \omega \).

Combining this lemma with Theorem 1, we arrive directly to the theorem

**Theorem 3.** The rational extensions of the IO \( V^{N_m \otimes L_r}(x; \omega, \alpha) \) with

\[
\begin{align*}
N_m &= \{(1 | a_1)_k, \ldots, (k - 1 | a_{k-1})_k ; (\lambda_1 | \mu_1)_k, \ldots, (\lambda_{j_1} | \mu_{j_1})_k) \\
L_r &= \{(1 | b_1)_k, \ldots, (k - 1 | b_{k-1})_k ; (\rho_1 | \sigma_1)_k, \ldots, (\rho_{j_2} | \sigma_{j_2})_k) \}
\end{align*}
\]

(181)
where \( a_i, b_i, \lambda_i, \mu_i, \rho_i, \sigma_i \) are arbitrary positive integers, solve the dressing chain of even period \( p = p_1 + p_2 \) with \( p_1 = 2j_1 + k \), \( l = 1, 2 \), \( p_1, p_2 \) and \( k \) have the same parity with \( 0 < k \leq \min(p_1, p_2) \) for the following values of the parameters

\[
\Delta = 2k\omega \text{ and } \varepsilon_{i,j+1} = \begin{cases} 
2(\nu_{p(i)} - \nu_{p(i+1)}) \omega, & \text{if } \nu_{p(i)}, \nu_{p(i+1)} \in \{1, ..., p_1\} \text{ or } \nu_{p(i)}, \nu_{p(i+1)} \in \{p_1 + 1, ..., p\}, \\
2(\nu_{p(i)} - \nu_{p(i+1)} + \alpha) \omega, & \text{if } \nu_{p(i)} \in \{1, ..., p_1\} \text{ and } \nu_{p(i+1)} \in \{p_1 + 1, ..., p\}, \\
2(\nu_{p(i)} - \nu_{p(i+1)} - \alpha) \omega, & \text{if } \nu_{p(i+1)} \in \{1, ..., p_1\} \text{ or } \nu_{p(i)} \in \{p_1 + 1, ..., p\},
\end{cases}
\]

(182) 

\((\nu_{p+1} = \nu_1)\) where \( P \) is any permutation of \( S_p \) and

\[
(\nu_1, ..., \nu_{p_1}) \otimes (\nu_{p_1+1}, ..., \nu_p) = (0, 1 + a_1 k, ..., (k - 1) + a_{k-1} k, \lambda_1, \lambda_1 + \mu_1 k, ..., \lambda_{j_1}, \lambda_{j_1} + \mu_{j_1} k) \otimes (0, 1 + b_1 k, ..., (k - 1) + b_{k-1} k, \rho_1, \rho_1 + \sigma_1 k, ..., \rho_{j_1}, \rho_{j_1} + \sigma_{j_1} k).
\]

(183)

As for the \( w \) solutions of the dressing chain system, for the chain above, they are given by

\[
w_{\nu_i \otimes \psi}^{(N, \nu_1, ..., \nu_{i-1})}(x; \omega, \alpha) = -\omega x/2 + \frac{\alpha - 3/2 + m - r + i}{x} \frac{x}{\omega} \frac{1}{dz} \left( \log \left( \frac{\mathcal{L}^{(N, \nu_1, ..., \nu_{i-1}) \otimes L_r}(z; \alpha)}{\mathcal{L}^{(N, \nu_1, ..., \nu_i) \otimes L_r}(z; \alpha)} \right) \right), \\
\text{if } i \leq p_1 \text{ and the flip in } \nu_i \text{ is positive,}
\]

(184)

\[
w_{\nu_i \otimes \psi}^{(N, \nu_1, ..., \nu_{i-1})}(x; \omega, \alpha) = \omega x/2 - \frac{\alpha - 2/2 + m - r + i}{x} \frac{x}{\omega} \frac{1}{dz} \left( \log \left( \frac{\mathcal{L}^{(N, \nu_1, ..., \nu_{i-1}) \otimes L_r}(z; \alpha)}{\mathcal{L}^{(N, \nu_1, ..., \nu_i) \otimes L_r}(z; \alpha)} \right) \right), \\
\text{if } i \leq p_1 \text{ and the flip in } \nu_i \text{ is negative,}
\]

(185)

\[
w_{\psi \otimes \nu_i}^{(N, \nu_1, ..., \nu_{p_1})}(L_r, \nu_{p+1}, ..., \nu_{i-1})(x; \omega, \alpha) = -\omega x/2 + \frac{\alpha - 13/2 + m + p_1 + 3(r + i)}{x} \frac{x}{\omega} \frac{1}{dz} \left( \log \left( \frac{\mathcal{L}^{(N, \nu_1, ..., \nu_{p_1}) \otimes (L_r, \nu_{p_1+1}, ..., \nu_{i-1})}(z; \alpha)}{\mathcal{L}^{(N, \nu_1, ..., \nu_{p_1}) \otimes (L_r, \nu_{p+1}, ..., \nu_{i-1})}(z; \alpha)} \right) \right), \\
\text{if } i = p_1 + j, \; j > 0, \text{ and the flip in } \nu_i \text{ is positive,}
\]

(186)

\[
w_{\psi \otimes \nu_i}^{(N, \nu_1, ..., \nu_{p_1})}(L_r, \nu_{p+1}, ..., \nu_{i-1})(x; \omega, \alpha) = \omega x/2 + \frac{\alpha - 7/2 + m + p_1 + 3(r + i)}{x} \frac{x}{\omega} \frac{1}{dz} \left( \log \left( \frac{\mathcal{L}^{(N, \nu_1, ..., \nu_{p_1}) \otimes (L_r, \nu_{p_1+1}, ..., \nu_{i-1})}(z; \alpha)}{\mathcal{L}^{(N, \nu_1, ..., \nu_{p_1}) \otimes (L_r, \nu_{p+1}, ..., \nu_{i-1})}(z; \alpha)} \right) \right), \\
\text{if } i = p_1 + j, \; j > 0, \text{ and the flip in } \nu_i \text{ is negative,}
\]

(187)

with the convention that if a spectral index is repeated two times in the tuples constituting the UC, then we suppress the corresponding eigenfunction in the Laguerre pseudowronskians \( \mathcal{L} \).

Proof. Note that due to the relations Eq(177) and Eq(178), we can without loss of generality restrict the study of the solutions to the case \( p_1 \geq p_2 \).

If \( i \leq p_1 \) and the flip in \( \nu_i \) is negative, we have, using Eq(178)

\[
w_{\nu_i \otimes \psi}^{(N, \nu_1, ..., \nu_{i-1})}(x; \omega, \alpha) = -\omega x / 2 + \frac{1}{x} \frac{\omega x}{dz} \left( \log \left( \frac{\mathcal{L}^{(N, \nu_1, ..., \nu_{i-1}) \otimes L_r}(z; \alpha)}{\mathcal{L}^{(N, \nu_1, ..., \nu_i) \otimes L_r}(z; \alpha)} \right) \right) + \\
\omega x / 2 - \frac{\alpha - 2/2 + m - r + i}{x} \frac{x}{\omega} \frac{1}{dz} \left( \log \left( \frac{\mathcal{L}^{(N, \nu_1, ..., \nu_{i-1}) \otimes L_r}(z; \alpha)}{\mathcal{L}^{(N, \nu_1, ..., \nu_i) \otimes L_r}(z; \alpha)} \right) \right),
\]

(177)
In the same manner, if \( i \leq p_1 \) and the flip in \( \nu_i \) is positive, we have

\[
\begin{aligned}
    w_{\nu_1 \otimes \nu}^{N_m \otimes L_r}(x; \omega, \alpha) &= -\omega x/2 + \frac{\alpha - 1/2 + m - r}{x} + \omega x \frac{d}{dz} \left( \log \left( \frac{\mathcal{L}^{N_m \otimes L_r}(z; \alpha)}{\mathcal{E}^{(N_m, \nu) \otimes L_r}(z; \alpha)} \right) \right) .
\end{aligned}
\tag{189}
\]

If \( i > p_1 \) and the flip in \( \nu_i \) is negative, we have, using Eq\((173)\)

\[
\begin{aligned}
    w_{\nu_1 \otimes \nu}^{N_m \otimes L_r}(x; \omega, \alpha) &= - \left( \log \left( \frac{W_{N_m \otimes L_r}(x; \omega, \alpha)}{W_{N_m \otimes L_r}(x; \omega, \alpha)} \right) \right)' \\
    &= - \left( \log \left( x^{-\alpha+1/2-m-3r} \times e^{-\omega x^2/4} \times \frac{\mathcal{L}^{N_m \otimes L_r}(z; \alpha)}{\mathcal{E}^{(N_m, \nu) \otimes L_r}(z; \alpha)} \right) \right)' \\
    &= \omega x/2 + \frac{\alpha - 1/2 + m + 3r}{x} \\
    &+ \omega x \frac{d}{dz} \left( \log \left( \frac{\mathcal{L}^{N_m \otimes L_r}(z; \alpha)}{\mathcal{E}^{(N_m, \nu) \otimes L_r}(z; \alpha)} \right) \right) .
\end{aligned}
\tag{190}
\]

In the same manner, if \( i > p_1 \) and the flip in \( \nu_i \) is positive, we have

\[
\begin{aligned}
    w_{\nu_1 \otimes \nu}^{N_m \otimes L_r}(x; \omega, \alpha) &= -\omega x/2 - \frac{\alpha - 7/2 + m + 3r}{x} \\
    &+ \omega x \frac{d}{dz} \left( \log \left( \frac{\mathcal{L}^{(N_m, \nu_1, \ldots, \nu_p) \otimes (L_r, \nu_{p_1+1}, \ldots, \nu_{p_1+\nu_1-1})}(z; \alpha)}{\mathcal{E}^{(N_m, \nu_1, \ldots, \nu_p) \otimes (L_r, \nu_{p_1+1}, \ldots, \nu_{p_1+\nu_1-1})}(z; \alpha)} \right) \right) .
\end{aligned}
\tag{191}
\]

\[
\square
\]

5.3 Examples

5.3.1 Dressing chain of period 2.

We have \( p = 2 \) which implies \( p_1 = p_2 = k = 1 \) and \( j_1 = j_2 = 0 \). It leads to \( N_m \otimes L_r = \emptyset \otimes \emptyset \) and the potential which solves the dressing chain of period 2 is the IO itself. The corresponding 2-cyclic chain is \( (0) \otimes (0) \). These results are in agreement with section I.

5.3.2 4-cyclic extensions of the IO and rational solutions of PV

For \( p = 4 \), we can have \( p_1 = 3 \) and \( p_2 = 1 \) with \( k = 1 \) \( (j_1 = 1, j_2 = 0) \) or \( p_1 = p_2 = k = 2 \) \( (j_1 = j_2 = 0) \). As shown before, the dressing chain system is equivalent to the PV equation (see Eq\((30)\)) whose rational solutions are associated to the Umemura polynomials \([23]\).

\[
(p_1, p_2) = (3, 1) \quad \text{Theorem 4} \quad \text{gives then} \quad N_m \otimes L_r = (\lambda \mid \mu)_1 \otimes \emptyset \quad \text{and a corresponding 4-cyclic chain is given by} \quad (\lambda, \lambda + \mu, 0) \otimes (0). \quad \text{The solutions of the dressing chain system of period 4 with parameters}
\]

\[
\begin{aligned}
    \varepsilon_{12} &= -2\mu \omega \\
    \varepsilon_{23} &= 2(\lambda + \mu) \omega \\
    \varepsilon_{34} &= 2\alpha \omega \\
    \varepsilon_{41} - \Delta &= 2(-1 - \lambda - \alpha) \omega,
\end{aligned}
\tag{192}
\]

are (see Eq\((184,187)\))
\[ w_1(x) = w^{(\lambda | \mu)}_{\lambda \otimes \mathcal{O}}(x; \omega, \alpha) = -\omega x/2 + \frac{\alpha + \mu - 1/2}{x} + \omega x \frac{d}{dz} \left( \log \left( \frac{\mathcal{L}^{(\lambda | \mu)}_{\lambda \otimes \mathcal{O}}(z; \alpha)}{\mathcal{L}^{(\lambda + 1 | \mu - 1)}_{\lambda \otimes \mathcal{O}}(z; \alpha)} \right) \right) \] (193)

(the flip in \( \lambda \otimes \mathcal{O} \) is positive, \( m = \mu, \ r = 0 \)),

\[ w_2(x) = w^{(\lambda, (\lambda | \mu))}_{\lambda + \mu \otimes \mathcal{O}}(x; \omega, \alpha) = w^{(\lambda + 1 | \mu - 1)}_{\lambda + \mu \otimes \mathcal{O}}(x; \omega, \alpha) = \omega x/2 - \frac{\alpha + \mu + 1/2}{x} + \omega x \frac{d}{dz} \left( \log \left( \frac{\mathcal{L}^{(\lambda + 1 | \mu - 1)}_{\lambda + \mu \otimes \mathcal{O}}(z; \alpha)}{\mathcal{L}^{(\lambda + 1 | \mu - 1)}_{\lambda + \mu \otimes \mathcal{O}}(z; \alpha)} \right) \right) \] (194)

(the flip in \((\lambda + \mu) \otimes \mathcal{O}\) is negative, \(m = \mu - 1, \ r = 0\)),

\[ w_3(x) = w^{(\lambda, \lambda, (\lambda | \mu))}_{\lambda \otimes \mathcal{O}}(x; \omega, \alpha) = w^{(\lambda + 1 | \mu, 0)}_{\lambda \otimes \mathcal{O}}(x; \omega, \alpha) = \omega x/2 - \frac{\alpha + \mu + 1/2}{x} + \omega x \frac{d}{dz} \left( \log \left( \frac{\mathcal{L}^{(\lambda + 1 | \mu, 0)}_{\lambda \otimes \mathcal{O}}(z; \alpha)}{\mathcal{L}^{(\lambda + 1 | \mu, 0)}_{\lambda \otimes \mathcal{O}}(z; \alpha)} \right) \right) \] (195)

(the flip in \(0 \otimes \mathcal{O}\) is negative, \(m = \mu, \ r = 0\) and (see Eq(176))

\[ w_4(x) = w^{(0, \lambda + \mu, \lambda, (\lambda | \mu))}_{\mathcal{O} \otimes \mathcal{O}}(x; \omega, \alpha) = w^{(0, \lambda + 1 | \mu)}_{\mathcal{O} \otimes \mathcal{O}}(x; \omega, \alpha) = \omega x/2 + \frac{\alpha + \mu + 1/2}{x} + \omega x \frac{d}{dz} \left( \log \left( \frac{\mathcal{L}^{(0, \lambda + 1 | \mu)}_{\mathcal{O} \otimes \mathcal{O}}(z; \alpha)}{\mathcal{L}^{(0, \lambda + 1 | \mu)}_{\mathcal{O} \otimes \mathcal{O}}(z; \alpha)} \right) \right) \] (196)

(the flip in \(\mathcal{O} \otimes 0\) is negative, \(m = \mu + 1, \ r = 0\), where we have used

\[ \{ \begin{align*} (\lambda, (\lambda | \mu))_1 &= (\lambda + 1 | \mu - 1)_1 \\ (\mu, (\lambda + 1 | \mu - 1)_1) &= (\lambda + 1 | \mu)_1 \end{align*} \] (197)

Taking \(\omega = 2 (t = z)\), the corresponding solution of \(PV\) (see Eq(30)) with parameters (see Eq(31))

\[ \begin{align*} a &= 2\mu^2 \\ b &= -2\alpha^2 \\ c &= 4(\alpha + 2\lambda + \mu + 1) \\ d &= -1/2, \end{align*} \] (198)

is then (see Eq(29))

\[ y(t) = 1 - \frac{1}{dt} \left( \log \left( \frac{\mathcal{L}^{(\lambda | \mu)}_{\mathcal{O} \otimes \mathcal{O}}(t; \alpha)}{\mathcal{L}^{(\lambda + 1 | \mu)}_{\mathcal{O} \otimes \mathcal{O}}(t; \alpha)} \right) \right). \] (199)
\((p_1, p_2) = (2, 2)\) \ Theorem 4 \ gives \ \(N_{m} \otimes L_r = (1 \mid a_1)_2 \otimes (1 \mid b_1)_2\) \ and the corresponding 4-cyclic chain is \ \((1 + 2a_1, 0) \otimes (1 + 2b_1, 0)\). \ The solutions of the dressing chain system of period 4 with parameters

\[
\begin{align*}
\varepsilon_{12} &= 2(1 + 2a_1) \omega \\
\varepsilon_{23} &= 2(\alpha - 1 - 2b_1) \omega \\
\varepsilon_{34} &= 2(1 + 2b_1) \omega \\
\varepsilon_{41} - \Delta &= 2(-3 - 2a_1 - \alpha) \omega,
\end{align*}
\]

are (see Eq\((184-187)\) and Eq\((176)\)).

\[
w_1(x) = w_{(1|a_1)_2 \otimes (1|b_1)_2}(x; \omega, \alpha) = 2 \alpha + 1/2 + a_1 - b_1 + x \frac{d}{dz} \left( \log \left( \frac{L((1|a_1)_2 \otimes (1|b_1)_2)(z; \alpha)}{L((1|a_1+1)_2 \otimes (1|b_1)_2)(z; \alpha)} \right) \right)
\]

(the flip in \((1 + 2a_1) \otimes \emptyset\) is negative, \(m = a_1, \ r = b_1\),

\[
w_2(x) = w_{(0, (1|a_1+1)_2 \otimes (1|b_1)_2)(x; \omega, \alpha) = 2 \alpha + 3/2 + a_1 - b_1 + x \frac{d}{dz} \left( \log \left( \frac{L((1|a_1+1)_2 \otimes (1|b_1)_2)(z; \alpha)}{L((1|a_1+1)_2 \otimes (1|b_1)_2)(z; \alpha)} \right) \right)
\]

(the flip in \(0 \otimes \emptyset\) is negative, \(m = a_1 + 1, \ r = b_1\),

\[
w_3(x) = w_{(0, (1|a_1+1)_2 \otimes (1|b_1+1)_2)(x; \omega, \alpha) = 2 \alpha + 3/2 + a_1 + 3b_1 + x \frac{d}{dz} \left( \log \left( \frac{L((1|a_1+1)_2 \otimes (1|b_1+1)_2)(z; \alpha)}{L((1|a_1+1)_2 \otimes (1|b_1+1)_2)(z; \alpha)} \right) \right)
\]

(the flip in \(\emptyset \otimes (1 + 2b_1)\) is negative, \(m = a_1 + 2, \ r = b_1\) and

\[
w_4(x) = w_{(a_1)_2 \otimes (1|b_1+1)_2}(x; \omega, \alpha_2) = 2 \alpha + 5/2 + a_1 + 3b_1 + x \frac{d}{dz} \left( \log \left( \frac{L((1|a_1)_2 \otimes (1|b_1+1)_2)(z; \alpha_2)}{L((1|a_1)_2 \otimes (1|b_1+1)_2)(z; \alpha_2)} \right) \right)
\]

(the flip in \(\emptyset \otimes 0\) is negative, \(m = a_1, \ r = b_1 + 1\), where we have used

\[
\begin{align*}
(1 + 2a_1, (1 \mid a_1)_2) &= (1 \mid a_1 + 1)_2 \\
(0, (1 \mid a_1 + 1)_2) &= (0, 1) \cup (3 \mid a_1 + 1)_2 = (1 \mid a_1)_2 \oplus 2.
\end{align*}
\]
Taking $\omega = 2 \ (t = z)$, the corresponding solution of PV Eq(30) with parameters (see Eq(31))

\[
\begin{align*}
  a &= (1 + 2a_1)^2 / 8 \\
  b &= -(1 + 2b_1)^2 / 8 \\
  c &= 8(\alpha + 1 + a_1 - b_1) \\
  d &= -2,
\end{align*}
\]

is then (see Eq(29))

\[
y(t) = 1 - 2 / \left(1 - \frac{\alpha + 1 + a_1 - b_1}{t} + \frac{d}{dt} \left(\log \left(\frac{\mathcal{L}^{(1|a_1|b_1)}(z;\alpha)}{\mathcal{L}^{(1|a_1|b_1)}(z;\alpha_2)}\right)\right)\right),
\]

(207)

5.3.3 Rational solutions of the Painlevé chain of period 6 ($A_5$-PV)

For $p = 6$, we can have:

* $p_1 = 5$ and $p_2 = 1$ with $k = 1 \ (j_1 = 2, j_2 = 0)$
* $p_1 = 4$ and $p_2 = 2$ with $k = 2 \ (j_1 = 1, j_2 = 0)$
* $p_1 = p_2 = k = 3 \ (j_1 = j_2 = 0)$.

$(p_1, p_2) = (5, 1)$ Theorem 4 gives then $N_m \otimes L_r = ((\lambda_1 | \mu_1)_1, (\lambda_2 | \mu_2)_1) \otimes \emptyset$ and the corresponding 6-cyclic chain is given by $(\lambda_1, \lambda_1 + \mu_1, \lambda_2, \lambda_2 + \mu_2, 0) \otimes (0)$. The solutions of the dressing chain system of period 6 with parameters

\[
\begin{align*}
  \varepsilon_{12} &= -2\mu_1 \omega \\
  \varepsilon_{23} &= (\lambda_1 + \mu_1 - \lambda_2) \omega \\
  \varepsilon_{34} &= -2\mu_2 \omega \\
  \varepsilon_{45} &= (\lambda_2 + \mu_2) \omega \\
  \varepsilon_{56} &= 2\alpha \omega \\
  \varepsilon_{61} - \Delta &= 2(-\lambda_1 - \alpha) \omega,
\end{align*}
\]

(208)

are (see Eq(184, 187))

\[
\begin{align*}
  w_1(x) &= w_{\lambda_1 \otimes \emptyset}^{(\lambda_1 | \mu_1)_1, (\lambda_2 | \mu_2)_1} \otimes \emptyset (x;\omega, \alpha) \ 	ext{(the flip in $\lambda_1 \otimes \emptyset$ is positive, $m = \mu_1 + \mu_2$, $r = 0$)} \\
  &= -\omega x / 2 + \frac{\alpha + \mu_1 + \mu_2 - 1/2}{x} + \omega x \frac{d}{dz} \left(\log \left(\frac{\mathcal{L}^{(\lambda_1 | \mu_1)_1, (\lambda_2 | \mu_2)_1} \otimes \emptyset (z;\alpha)}{\mathcal{L}^{(\lambda_1 | \mu_1)_1, (\lambda_2 | \mu_2)_1} \otimes \emptyset (z;\alpha)}\right)\right),
\end{align*}
\]

(209)

\[
\begin{align*}
  w_2(x) &= w_{\lambda_1 + \mu_1 \otimes \emptyset}^{(\lambda_1+1 | \mu_1-1)_1} \otimes \emptyset (x;\omega, \alpha) \\
  &= \omega x / 2 - \frac{\alpha + \mu_1 + \mu_2 - 1/2}{x} + \omega x \frac{d}{dz} \left(\log \left(\frac{\mathcal{L}^{(\lambda_1+1 | \mu_1-1)_1, (\lambda_2 | \mu_2)_1} \otimes \emptyset (z;\alpha)}{\mathcal{L}^{(\lambda_1+1 | \mu_1-1)_1, (\lambda_2 | \mu_2)_1} \otimes \emptyset (z;\alpha)}\right)\right),
\end{align*}
\]

(210)

(the flip in $(\lambda_1 + \mu_1) \otimes \emptyset$ is negative, $m = \mu_1 + \mu_2 - 1$, $r = 0$),

\[
\begin{align*}
  w_3(x) &= w_{\lambda_2 \otimes \emptyset}^{(\lambda_1+1 | \mu_1)_1, (\lambda_2 | \mu_2)_1} \otimes \emptyset (x;\omega, \alpha) \\
  &= -\omega x / 2 + \frac{\alpha + \mu_1 + \mu_2 - 1/2}{x} + \omega x \frac{d}{dz} \left(\log \left(\frac{\mathcal{L}^{(\lambda_1+1 | \mu_1)_1, (\lambda_2 | \mu_2)_1} \otimes \emptyset (z;\alpha)}{\mathcal{L}^{(\lambda_1+1 | \mu_1)_1, (\lambda_2 | \mu_2)_1} \otimes \emptyset (z;\alpha)}\right)\right),
\end{align*}
\]

(211)
(the flip in $\lambda_2 \otimes \emptyset$ is positive, $m = \mu_1 + \mu_2$, $r = 0$),

\[
\begin{align*}
    w_4(x) &= w^{((\lambda_1+1)\mu_1),(\lambda_2+1)\mu_2-1)}_{\emptyset \otimes \emptyset}(x;\omega,\alpha) \\
    &= \omega x/2 - \frac{\alpha + \mu_1 + \mu_2 - 1/2}{x} + \omega x \frac{d}{dz} \left( \log \left( \frac{\mathcal{L}^{((\lambda_1+1)\mu_1),(\lambda_2+1)\mu_2-1)}{\emptyset \otimes \emptyset}(z;\alpha)}{\mathcal{L}^{((\lambda_1+1)\mu_1),(\lambda_2+1)\mu_2-1)}{\emptyset \otimes \emptyset}(z;\alpha)} \right) \\
    \end{align*}
\]

(212)

(the flip in $(\lambda_2 + \mu_2) \otimes \emptyset$ is negative, $m = \mu_1 + \mu_2 - 1$, $r = 0$),

\[
\begin{align*}
    w_5(x) &= w^{((\lambda_1+1)\mu_1),(\lambda_2+1)\mu_2+1)}_{0 \otimes 0}(x;\omega,\alpha) \\
    &= \omega x/2 - \frac{\alpha + \mu_1 + \mu_2 - 1/2}{x} + \omega x \frac{d}{dz} \left( \log \left( \frac{\mathcal{L}^{((\lambda_1+1)\mu_1),(\lambda_2+1)\mu_2+1)}{0 \otimes 0}(z;\omega,\alpha)}{\mathcal{L}^{((\lambda_1+1)\mu_1),(\lambda_2+1)\mu_2+1)}{0 \otimes 0}(z;\omega,\alpha)} \right) \\
    \end{align*}
\]

(213)

(the flip in $0 \otimes \emptyset$ is negative, $m = \mu_1 + \mu_2$, $r = 0$) and

\[
\begin{align*}
    w_6(x) &= w^{((\lambda_1+1)\mu_1),(\lambda_2+1)\mu_2+1)}_{0 \otimes \emptyset}(x;\omega,\alpha) \\
    &= \omega x/2 + \frac{\alpha + \mu_1 + \mu_2 - 1/2}{x} + \omega x \frac{d}{dz} \left( \log \left( \frac{\mathcal{L}^{((\lambda_1+1)\mu_1),(\lambda_2+1)\mu_2+1)}{0 \otimes \emptyset}(z;\alpha,\omega)}{\mathcal{L}^{((\lambda_1+1)\mu_1),(\lambda_2+1)\mu_2+1)}{0 \otimes \emptyset}(z;\alpha,\omega)} \right) \\
    \end{align*}
\]

(214)

(The flip in $\emptyset \otimes 0$ is negative, $m = \mu_1 + \mu_2$, $r = 0$).

$$(p_1, p_2) = (4, 2)$$  Theorem 4 gives $N_m \otimes L_r = ((1 \mid a_1)_{b_1}, (\lambda_1 \mid \mu_1)_{b_2}) \otimes (1 \mid b_1)_{2}$ and the corresponding 6-cyclic chain is $(1 + 2a_1, \lambda_1, \lambda_1 + \mu_1, 0) \otimes (1 + 2b_1, 0)$. The solutions of the dressing chain system of period 6 with parameters

$$(\varepsilon_{12} = 2(1 + 2a_1 - \lambda_1) \omega, \varepsilon_{23} = -2\mu_1 \omega, \varepsilon_{34} = 2(\lambda_1 + \mu_1) \omega, \varepsilon_{45} = 2(\alpha - 1 - 2b_1) \omega, \varepsilon_{56} = 2(1 + 2b_1) \omega, \varepsilon_{61} - \Delta = 2(-3 - 2a_1 - \alpha) \omega),$$

are (see Eq (184) and Eq (176))

\[
\begin{align*}
    w_1(x) &= w^{((1\mid a_1)_{b_1},(\lambda_1\mid \mu_1)_{b_2}) \otimes (1\mid b_1)_{2}}_{(1 + 2a_1) \otimes \emptyset}(x;\omega,\alpha) \\
    &= \omega x/2 - \frac{\alpha + 1/2 + a_1 + \mu_1 - b_1}{x} + \omega x \frac{d}{dz} \left( \log \left( \frac{\mathcal{L}^{((1\mid a_1)_{b_1},(\lambda_1\mid \mu_1)_{b_2}) \otimes (1\mid b_1)_{2}}{(1\mid a_1)_{b_1},(\lambda_1\mid \mu_1)_{b_2}) \otimes (1\mid b_1)_{2}}(z;\alpha)}{\mathcal{L}^{((1\mid a_1)_{b_1},(\lambda_1\mid \mu_1)_{b_2}) \otimes (1\mid b_1)_{2}}{(1\mid a_1)_{b_1},(\lambda_1\mid \mu_1)_{b_2}) \otimes (1\mid b_1)_{2}}(z;\alpha)} \right) \\
    \end{align*}
\]

(216)

\[
\begin{align*}
    w_2(x) &= w^{((1\mid a_1)_{b_1},(\lambda_1\mid \mu_1)_{b_2}) \otimes (1\mid b_1)_{2}}_{(1 + 2a_1) \otimes \emptyset}(x;\omega,\alpha) \\
    &= -\omega x/2 + \frac{\alpha + 1/2 + a_1 + \mu_1 - b_1}{x} + \omega x \frac{d}{dz} \left( \log \left( \frac{\mathcal{L}^{((1\mid a_1)_{b_1},(\lambda_1\mid \mu_1)_{b_2}) \otimes (1\mid b_1)_{2}}{(1\mid a_1)_{b_1},(\lambda_1\mid \mu_1)_{b_2}) \otimes (1\mid b_1)_{2}}(z;\alpha)}{\mathcal{L}^{((1\mid a_1)_{b_1},(\lambda_1\mid \mu_1)_{b_2}) \otimes (1\mid b_1)_{2}}{(1\mid a_1)_{b_1},(\lambda_1\mid \mu_1)_{b_2}) \otimes (1\mid b_1)_{2}}(z;\alpha)} \right) \\
    \end{align*}
\]

(217)

34
(the flip in $\lambda_1 \otimes \emptyset$ is positive, $m = a_1 + 1 + \mu_1$, $r = b_1$),

$$w_3(x) = w_0^{((1|a_1+1)_2, (\lambda_1, a_1-1)_2) \otimes (1|b_1)_2}(x; \omega, \alpha)$$

$$= \frac{\omega x}{2} - \frac{\alpha + 1/2 + a_1 + \mu_1 - b_1}{x} + \frac{\omega x}{x} \left( \log \frac{\mathcal{L}^{((1|a_1+1)_2, (\lambda_1, 2|\mu_1)_2) \otimes (1|b_1)_2}(z; \omega)}{\mathcal{L}^{(1|a_1+1)_2, (\lambda_1, 2|\mu_1)_2 \otimes (1|b_1)_2}(z; \alpha)} \right)$$  \hspace{1cm} (218)

(the flip in $(\lambda_1 + \mu_1) \otimes \emptyset$ is negative, $m = a_1 + \mu_1$, $r = b_1$),

$$w_4(x) = w_0^{((1|a_1+1)_2, (\lambda_1, \mu_1)_2) \otimes (1|b_1)_2}(x; \omega, \alpha)$$

$$= \frac{\omega x}{2} - \frac{\alpha + 3/2 + a_1 + \mu_1 + 3b_1}{x} + \frac{\omega x}{x} \left( \log \frac{\mathcal{L}^{((1|a_1+1)_2, (\lambda_1 + 2|\mu_1)_2) \otimes (1|b_1)_2}(z; \omega)}{\mathcal{L}^{(1|a_1+1)_2, (\lambda_1 + 2|\mu_1)_2 \otimes (1|b_1)_2}(z; \alpha)} \right)$$  \hspace{1cm} (219)

(the flip in $0 \otimes \emptyset$ is negative, $m = \mu_1 + a_1 + 1$, $r = b_1$, and we have $(0, (1 | a_1 + 1)_2, (\lambda_1 + 2 | \mu_1)_2) = (0, 1) \cup (3 | a_1 + 1)_2, (\lambda_1 + 2 | \mu_1)_2) = ((1 | a_1)_2, (\lambda_1 \mid \mu_1)_2 \oplus 2),$

$$w_5(x) = w_0^{((1|a_1+1)_2, (\lambda_1, a_1+2|\mu_1)_2) \otimes (1|b_1)_2}(x; \omega, \alpha)$$

$$= \frac{\omega x}{2} - \frac{\alpha + 3/2 + a_1 + a_1 + 3b_1}{x} + \frac{\omega x}{x} \left( \log \frac{\mathcal{L}^{(0, (1|a_1+1)_2, (\lambda_1, a_1+2|\mu_1)_2) \otimes (1|b_1)_2}(z; \omega)}{\mathcal{L}^{(1|a_1+1)_2, (\lambda_1, a_1+2|\mu_1)_2 \otimes (1|b_1)_2}(z; \alpha)} \right)$$  \hspace{1cm} (220)

and $((0, (1 | b_1 + 1)_2) = (0, 1) \cup (1 \mid b_1 + 1)_2 = (1 \mid b_1)_2 \oplus 2$, see also Eq(176))

$$w_6(x) = w_0^{((1|a_1)_2, (\lambda_1, \mu_1)_2) \otimes (1|b_1+1)_2}(x; \omega, \alpha_2)$$

$$= \frac{\omega x}{2} - \frac{\alpha + 9/2 + a_1 + 3b_1}{x} + \frac{\omega x}{x} \left( \log \frac{\mathcal{L}^{((1|a_1)_2, (\lambda_1, \mu_1)_2) \otimes (1|b_1+1)_2}(z; \omega_2)}{\mathcal{L}^{(1|a_1)_2, (\lambda_1, \mu_1)_2 \otimes (1|b_1+1)_2}(z; \alpha_2)} \right)$$  \hspace{1cm} (221)

(the flip in $\emptyset \otimes 0$ is negative, $m = \mu_1 + a_1$, $r = b_1 + 1$).

$(p_1, p_2) = (3, 3)$ and $k = 3$ Theorem 4 gives $N_m \otimes L_r = ((1 \mid a_1)_3, (2 \mid a_2)_3) \otimes ((1 \mid b_1)_2, (2 \mid b_2)_2)$ and the corresponding 6-cyclic chain is $(1 + 3a_1, 2 + 3a_2, 0) \otimes (1 + 3b_1, 2 + 3b_2, 0)$. The solutions of the dressing chain system of period 6 with parameters
are (see Eq (184) and Eq (176))

\[
\begin{align*}
\varepsilon_{12} &= 2 (-1 + 3 (a_1 - a_2)) \omega \\
\varepsilon_{21} &= 2 (2 + 3a_2) \omega \\
\varepsilon_{34} &= 2 (\alpha - 1 - 3b_1) \omega \\
\varepsilon_{31} &= 2 (-1 + 3 (b_1 - b_2)) \omega \\
\varepsilon_{41} &= 2 (2 + 3b_2) \omega \\
\end{align*}
\] (222)

\[
w_1(x) = w^{((1|a_1)_3, (2|a_2)_3)\otimes((1|b_1)_3, (2|b_2)_3)}_{(1+3a_1)\otimes\emptyset} (x; \omega, \alpha)
\]

\[
= \omega x/2 - \frac{\alpha + 1/2 + a_1 + a_2 - b_1 - b_2}{x} + \omega x \frac{d}{dz} \left( \log \left( \frac{L^{((1|a_1)_3, (2|a_2)_3)\otimes((1|b_1)_3, (2|b_2)_3)}(z; \alpha)}{L^{((1|a_1+1)_3, (2|a_2)_3)\otimes((1|b_1)_3, (2|b_2)_3)}(z; \alpha)} \right) \right)
\] (223)

(the flip in $1 + 3a_1 \otimes \emptyset$ is negative, $m = a_1 + a_2$, $r = b_1 + b_2$),

\[
w_2(x) = w^{((1|a_1+1)_3, (2|a_2)_3)\otimes((1|b_1)_3, (2|b_2)_3)}_{(2+3a_2)\otimes\emptyset} (x; \omega, \alpha)
\]

\[
= \omega x/2 - \frac{\alpha + 3/2 + a_1 + a_2 - b_1 - b_2}{x} + \omega x \frac{d}{dz} \left( \log \left( \frac{L^{((1|a_1+1)_3, (2|a_2)_3)\otimes((1|b_1)_3, (2|b_2)_3)}(z; \alpha)}{L^{((1|a_1+1)_3, (2|a_2+1)_3)\otimes((1|b_1)_3, (2|b_2)_3)}(z; \alpha)} \right) \right)
\] (224)

(the flip in $2 + 3a_2 \otimes \emptyset$ is negative, $m = a_1 + a_2 + 1$, $r = b_1 + b_2$),

\[
w_3(x) = w^{((1|a_1)_3, (2|a_2+1)_3)\otimes((1|b_1)_3, (2|b_2)_3)}_{\emptyset\otimes\emptyset} (x; \omega, \alpha)
\]

\[
= \omega x/2 - \frac{\alpha + 5/2 + a_1 + a_2 - b_1 - b_2}{x} + \omega x \frac{d}{dz} \left( \log \left( \frac{L^{((1|a_1)_3, (2|a_2+1)_3)\otimes((1|b_1)_3, (2|b_2)_3)}(z; \alpha)}{L^{((1|a_1)_3, (2|a_2)_3)\otimes((1|b_1+1)_3, (2|b_2)_3)}(z; \alpha)} \right) \right)
\] (225)

(the flip in $0 \otimes \emptyset$ is negative, $m = a_1 + a_2 + 2$, $r = b_1 + b_2$ and we have used $(0, 1 | a_1 + 1)_3, (2 | a_2 + 1)_3) = (0, 1, 2) \cup ((4 | a_1)_3, (5 | a_2)_3) = ((1 | a_1)_3, (2 | a_2)_3) \oplus 3$),

\[
w_4(x) = w^{((1|a_1)_3, (2|a_2)_3)\otimes((1|b_1)_3, (2|b_2)_3)}_{\emptyset\otimes(1+2b_1)} (x; \omega, \alpha)
\]

\[
= \omega x/2 + \frac{\alpha + 5/2 + a_1 + a_2 + 3b_1 + 3b_2}{x} + \omega x \frac{d}{dz} \left( \log \left( \frac{L^{((1|a_1)_3, (2|a_2)_3)\otimes((1|b_1)_3, (2|b_2)_3)}(z; \alpha)}{L^{((1|a_1+1)_3, (2|a_2)_3)\otimes((1|b_1+1)_3, (2|b_2)_3)}(z; \alpha)} \right) \right)
\] (226)

(the flip in $\emptyset \otimes (1+2b_1)$ is negative, $m = a_1 + a_2 + 3$, $r = b_1 + b_2$),

\[
w_5(x) = w^{((1|a_1)_3, (2|a_2)_3)\otimes((1|b_1+1)_3, (2|b_2)_3)}_{\emptyset\otimes(2+3b_1)} (x; \omega, \alpha_3)
\]

\[
= \omega x/2 + \frac{\alpha + 11/2 + a_1 + a_2 + 3b_1 + 3b_2}{x} + \omega x \frac{d}{dz} \left( \log \left( \frac{L^{((1|a_1)_3, (2|a_2)_3)\otimes((1|b_1+1)_3, (2|b_2)_3)}(z; \alpha_3)}{L^{((1|a_1)_3, (2|a_2)_3)\otimes((1|b_1+1)_3, (2|b_2+1)_3)}(z; \alpha_3)} \right) \right)
\] (227)

(the flip in $\emptyset \otimes (2+3b_1)$ is negative, $m = a_1 + a_2$, $r = b_1 + b_2 + 1$)
\[ w_0(x) = w_{\emptyset \otimes \emptyset}^{(1|a_1)_3(2|a_2)_3} \otimes (1|b_1+1)_3(2|b_2+1)_3}(x; \omega, \alpha_3) \]
\[ = \omega x/2 + \frac{\alpha + 17/2 + a_1 + a_2 + 3b_1 + 3b_2}{x} + \omega x \frac{d}{dz} \left( \log \frac{L^{(1|a_1)_3(2|a_2)_3} \otimes (1|b_1+1)_3(2|b_2+1)_3}(z; \alpha_3)}{L^{((1|a_1)_3(2|a_2)_3) \otimes (1|b_1)_3(2|b_2)_3}}(z; \alpha) \right) \]
\[ = \omega x/2 + \frac{\alpha + 7/2 + a_1 + a_2 - 9b_1 - 9b_2}{x} + \omega x \frac{d}{dz} \left( \log \frac{L^{((1|a_1)_3(2|a_2)_3) \otimes (1|b_1+1)_3(2|b_2+1)_3}(z; \alpha_3)}{L^{((1|a_1)_3(2|a_2)_3) \otimes (1|b_1)_3(2|b_2)_3}}(z; \alpha) \right) \]

(the flip in \( \emptyset \otimes 0 \) is negative, \( m = a_1 + a_2, \ r = b_1 + b_2 + 2 \), see also Eq (176)).

\( (p_1, p_2) = (3, 3) \) and \( k = 1 \) Theorem 4 gives \( N_m \otimes L_r = (\lambda_1 | \mu_1)_1 \otimes (\rho_1 | \sigma_1)_1 \) and the corresponding 6-cyclic chain is \( (\lambda_1, \lambda_1 + \mu_1, 0) \otimes (\rho_1, \rho_1 + \sigma_1, 0) \). The solutions of the dressing chain system of period 6 with parameters

\[ \begin{align*}
\varepsilon_{12} &= -2\mu_1 \omega \\
\varepsilon_{23} &= 2(\lambda_1 + \mu_1) \omega \\
\varepsilon_{34} &= 2(\alpha - \rho_1) \omega \\
\varepsilon_{34} &= -2\sigma_1 \omega \\
\varepsilon_{41} - \Delta &= 2(-1 - \lambda_1 - \alpha) \omega,
\end{align*} \]

(see Eq (184, 187) and Eq (176)):

\[ w_1(x) = w_{(\lambda_1|\mu_1)_1 \otimes (\rho_1|\sigma_1)_1}(x; \omega, \alpha) \]
\[ = -\omega x/2 + \frac{\alpha - 1/2 + \mu_1 - \sigma_1}{x} + \omega x \frac{d}{dz} \left( \log \frac{L^{(\lambda_1|\mu_1)_1 \otimes (\rho_1|\sigma_1)_1}(z; \alpha)}{L^{(\lambda_1+1|\mu_1-1)_1 \otimes (\rho_1|\sigma_1)_1}}(z; \alpha) \right) \]

(the flip in \( \lambda_1 \otimes \emptyset \) is positive, \( m = \mu_1, \ r = \sigma_1 \)),

\[ w_2(x) = w_{(\lambda_1+1|\mu_1-1)_1 \otimes (\rho_1|\sigma_1)_1}(x; \omega, \alpha) \]
\[ = \omega x/2 - \frac{\alpha - 1/2 + \mu_1 - \sigma_1}{x} + \omega x \frac{d}{dz} \left( \log \frac{L^{(\lambda_1+1|\mu_1-1)_1 \otimes (\rho_1|\sigma_1)_1}(z; \alpha)}{L^{(\lambda_1+1|\mu_1)_1 \otimes (\rho_1|\sigma_1)_1}}(z; \alpha) \right) \]

(the flip in \( (\lambda_1 + \mu_1) \otimes \emptyset \) is negative, \( m = \mu_1 - 1, \ r = \sigma_1 \)),

\[ w_3(x) = w_{0 \otimes \emptyset}^{(\lambda_1+1|\mu_1)_1 \otimes (\rho_1|\sigma_1)_1}(x; \omega, \alpha) \]
\[ = \omega x/2 - \frac{\alpha + 1/2 + \mu_1 - \sigma_1}{x} + \omega x \frac{d}{dz} \left( \log \frac{L^{(\lambda_1+1|\mu_1)_1 \otimes (\rho_1|\sigma_1)_1}(z; \alpha)}{L^{((\lambda_1)|\mu_1)_1 \otimes (\rho_1|\sigma_1)_1}}(z; \alpha) \right) \]
\[ = \omega x/2 - \frac{\alpha + 1/2 + \mu_1 - \sigma_1}{x} + \omega x \frac{d}{dz} \left( \log \frac{L^{((\lambda_1)|\mu_1)_1 \otimes (\rho_1|\sigma_1)_1}(z; \alpha)}{L^{((\lambda_1)|\mu_1)_1 \otimes (\rho_1|\sigma_1)_1}}(z; \alpha) \right) \]
Consider a rational extension of the HO associated to the canonical Maya diagram \( N_m \). By splitting \( N_m \) into two subsets of even and odd integers respectively, we can write

\[
W^{(N_m)}(x;\omega) \propto e^{-mx^2/2} \times W(H_{2a_1+1}(\sqrt{\frac{\omega}{2}}), ..., H_{2a_{m_1}+1}(\sqrt{\frac{\omega}{2}}), H_{2b_1}(\sqrt{\frac{\omega}{2}}), ..., H_{2b_{m_2}}(\sqrt{\frac{\omega}{2}}) | x)
\]  

where \( z = \sqrt{\omega x^2} \). Using Eq(72) and Eq(80), we obtain immediately \((z = \omega x^2/2)\)

\[
W^{(N_m)}(x;\omega) \propto e^{-mx^2}W \left( z^{1/2} L_{a_1}^{1/2}(z), ..., z^{1/2} L_{a_{m_1}}^{1/2}(z), L_{b_1}^{1/2}(z), ..., L_{b_{m_2}}^{1/2}(z) | x \right)
\]  

(176)
ie (see Eq\textup{(155)} and Eq\textup{(160)})

\[ W^{(N_m)}(x; \omega) \propto W^{A_{m_1} \otimes B_{m_2}}(x; \omega, 1/2), \]  

where

\[ A_{m_1} \otimes B_{m_2} = (a_1, ..., a_{m_1}) \otimes (b_1, ..., b_{m_2}). \]  

Since (see Eq\textup{(62)} and Eq\textup{(154)})

\[ V(x; \omega, 1/2) = V(x; \omega) - \omega, \]  

we deduce

\[ V^{(N_m)}(x; \omega) = V^{A_{m_1} \otimes B_{m_2}}(x; \omega, 1/2) + \omega. \]  

It means that all the rational extensions of the HO can be considered as rational extensions of the IO in the limit case where the \( \alpha \) parameter tends to \( 1/2 \). In this limit, the impenetrable barrier term in the IO potential disappears and the IO potential degenerates into the harmonic potential which is regular on the whole real line, the spectrum of the IO potential giving to the odd indexed eigenstates of the HO potential and the shadow spectrum of the IO the even indexed eigenstates of the HO. The result above shows that this degeneracy is still valid at the level of the rational extensions.

We can note that if we start from an extension of the IO with \( \alpha \) half integer \( \alpha = k + 1/2 \), associated to the UC \( A_{m_1} \otimes B_{m_2} \), Eq\textup{(169)} allows us to write

\[ V^{A_{m_1} \otimes B_{m_2}}(x; \omega, k + 1/2) = V^{(A_{m_1} \oplus k) \otimes B_{m_2}}(x; \omega, 1/2) + 2k\omega, \]  

and, with the correspondence Eq\textup{(242)} above

\[ V^{A_{m_1} \otimes B_{m_2}}(x; \omega, k + 1/2) = V^{(\tilde{N}_m)}(x; \omega) + (2k - 1)\omega, \]  

where

\[ \tilde{N}_m = (1, ..., 2k - 1, 2a_1 + 2k + 1, ..., 2a_{m_1} + 2k + 1) \cup (2b_1, ..., 2b_{m_2}). \]  

Consequently, the rational extensions of the IO with half integer \( \alpha \) parameter can be identified to rational extensions of the HO and are monodromy free. Conversely, the rational extensions of the which solves the even periodic dressing chains cover the subset of monodromy free solutions.

6 Conclusion

We propose a new method for building the rational solutions of the dressing chains for a Schrödinger operator. The results are directly obtained in closed determinantal form and we give some explicit examples which had never been previously presented. For the chains with odd periodicity (\( A_{2n-1} \)-PIV system) as for those with even periodicity (\( A_{2n+1} \)-PV system), we describe in a new systematic way some sets of rational solutions of the dressing
chain system. In the odd periodicity case we conjecture, based on the theory of potentials with trivial monodromy\cite{35, 36, 15, 10}, that the construction described in this work covers all rational solutions to higher order Painlevé systems. The even periodicity case is more involved, but it seems plausible to surmise that this is also the case: the trivial monodromy property having to be replaced by a constraint of fixed monodromy at one point for all the eigenfunctions of the potentials. These conjectures are the object of further investigations. The question of the optimal pseudo-Wronskian representation of these solutions\cite{11} will also be addressed in a forthcoming work.

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