\( K^+ \rightarrow \pi^+\pi^0 \) decays at next-to-leading order in the chiral expansion on finite volumes

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We present the ingredients for determining \( K^+ \rightarrow \pi^+\pi^0 \) matrix elements via the combination of lattice QCD and chiral perturbation theory (\( \chiPT \)). By simulating these matrix elements at unphysical kinematics, it is possible to determine all the low-energy constants (LECs) for constructing the physical \( K^+ \rightarrow \pi^+\pi^0 \) amplitudes at next-to-leading order (NLO) in the chiral expansion. In this work, the one-loop chiral corrections are calculated for arbitrary meson four-momenta, in both \( \chiPT \) and quenched \( \chiPT \) (quenched and partially quenched)

1. INTRODUCTION

The need for a high-precision prediction for \( K \rightarrow \pi\pi \) amplitudes is underlined by the recent experimental measurement of Re(\( \epsilon'/\epsilon \)) and the long-standing puzzle, the \( \Delta I = 1/2 \) rule. Although the finite-volume (FV) techniques developed in Refs. \( ^2 \) can ultimately enable an accurate calculation of \( K \rightarrow \pi\pi \) decay rates, the most practical approach to the numerical calculation of these decay rates remains the combination of lattice QCD and (quenched and partially quenched) \( \chiPT \). From a calculation for the CP-conserving, \( \Delta I = 3/2 \), \( K \rightarrow \pi\pi \) decay in Ref. \( ^2 \), the large kaon mass and the presence of final state interactions, non-LO corrections in this expansion are significant. In a recent work \( ^3 \), we have proposed to perform lattice simulations at unphysical kinematics over a range of meson masses and momenta, from which we can determine all the necessary LECs for constructing the physical matrix element \( \langle \pi^+\pi^0 | O_{\Delta S=1} | K^+ \rangle \) at NLO in the chiral expansion. We have suggested a specific unphysical kinematics \(^4\) that enables such a procedure, and have studied the finite-volume effects as explained in detail in Refs. \( ^4 \), which arises from replacing sums by integrals in the one-loop calculation that involves the diagrams in Fig. 1. It can be shown

\[
\begin{align*}
Q_4 &= \langle \bar{s}_\alpha d_\alpha \rangle_L (\bar{u}_\beta u_\beta - \bar{d}_\beta d_\beta)_L \\
&\quad + \langle \bar{s}_\alpha u_\alpha \rangle_L (\bar{u}_\beta d_\beta)_L,

Q_7 &= \frac{3}{2} \langle \bar{s}_\alpha d_\alpha \rangle_L \sum_{q=u,d,s} e_q (\bar{q}_\beta q_\beta)_R,

Q_8 &= \frac{3}{2} \langle \bar{s}_\alpha d_\alpha \rangle_L \sum_{q=u,d,s} e_q (\bar{q}_\alpha q_\alpha)_R,
\end{align*}
\]

where \( e_q \) is the electric charge of \( q \) and \( (\bar{\psi}_1 \psi_2)_{L,R} \) means \( \psi_1 \gamma_\mu (1 \mp \gamma_5) \psi_2 \).

2. FINITE-VOLUME EFFECTS

In Ref. \( ^2 \), we investigate the FV corrections, power-like in \( 1/L \), which arise from replacing sums by integrals in the one-loop calculation that involves the diagrams in Fig. 1. It can be shown

\(^3\) Another choice is considered in Ref. \( ^3 \).
\(^4\) Such a calculation for \( \langle \pi^+\pi^0 | O_{\Delta S=1} | K^+ \rangle \) at two particular kinematics, \( M_K = M_\pi \) and \( M_K = 2M_\pi \) with all mesons
that a diagram which does not have an imaginary part in Minkowski space will only have FV corrections exponential in $L$, therefore only diagram (c) contributes to the $1/L^n$ corrections. Because the two-pion final state has $I = 2$, this diagram only contains four-quark intermediate states and there are no disconnected quark-loops in the quark-flow picture. For the same reason, it does not receive contributions from the $\eta'$ propagator. Hence the $1/L^n$ effects are identical in $\chi$PT and quenched $\chi$PT ($q\chi$PT) for $K^+ \rightarrow \pi^+\pi^0$ at this order, and only at this order. This is not true for $\Delta I = 1/2$ decay amplitudes [9,10].

We are currently investigating the FV effects of these amplitudes in a moving frame. The Lellouch-Lüscher factor has not yet been derived for this, while the modification of Lüscher’s quantisation condition [13,14] relating the infinite-volume $\pi\pi$ scattering phase to the FV two-pion energy spectrum, due to the moving frame was obtained in Ref. [15]. As a by-product of our work, we verify that the energy shift obtained in one-loop perturbation theory in a moving frame agrees with the expansion of the quantization condition in Ref. [15] to the same order.

3. ONE-LOOP CHIRAL CORRECTIONS IN INFINITE VOLUME

We evaluate the one-loop correction by using dimensional regularisation and subtracting $\log(4\pi) - \gamma_E + 1 + 2/(4 - d)$. The lowest-order amplitudes are all proportional to $1/f^3$, where $f$ is the light pseudoscalar meson decay constant in the chiral limit. At NLO, we choose to express $1/f^3$ in terms of $1/(f_\pi^2 f_K)$. This factor fully absorbs the dependence upon the Gasser-Leutwyler LECs $L_4$ and $L_5$ introduced via wavefunction renormalisation.

In Ref. [6], the one-loop diagrams have been calculated for arbitrary external meson four-momenta in both $\chi$PT and $q\chi$PT. The results are lengthy and are presented on a web site [16]. In Fig. 2, we show an example of these results for $\langle \pi^+\pi^0|O_{7,8}\rangle [K^+]$. These plots are the ratios between the one-loop corrections and the lowest-order matrix elements with both final-state pions at rest. Fig. 2a is the result in $\chi$PT and Fig. 2b is that in $q\chi$PT. This figure suggests that in a quenched numerical calculation of these matrix elements, it is not appropriate in general to perform chiral extrapolation using unquenched $\chi$PT results [17]. This is confirmed by numerical data [18].

4. CONCLUSIONS

We have made theoretical progress towards the calculation of $K^+ \rightarrow \pi\pi$ decay amplitudes via the combination of lattice QCD and $\chi$PT. We find

Figure 1. One-loop diagrams for $K^+ \rightarrow \pi^+\pi^0$ amplitudes. The grey circles (squares) are weak (strong) vertices. The diagrams for wavefunction renormalisation are not shown here.
Figure 2. Ratio between the one-loop correction, at the renormalisation scale 0.7 GeV, and the lowest-order result for $\langle \pi\pi|O_{7,8}|K^+ \rangle$ in (a) $\chi$PT and (b) $q\chi$PT. The x axis is $M_\pi$ and the y axis is $M_K$. In (b), the coupling accompanying the kinetic term of the $\eta'$ propagator is set to zero, and the $\eta'$ mass is taken to be $M_0 = 0.5$ GeV. The one-loop results are not very sensitive to these parameters. The singular behaviour along the line $M_\pi = \sqrt{2}M_K$ in (b) is due to the fact that when performing the $q\chi$PT calculation, we use a basis in which the pseudo-Goldstone states are $\bar{q}q$ mesons, where $q$ and $q'$ are $u, d$ and $s$, and the $\bar{ss}$ meson becomes tachyonic when $M_\pi > \sqrt{2}M_K$.

it feasible to determine all the LECs for constructing $I_2^O(\pi\pi|O_{\Delta S=1}|K)$ at NLO in the chiral expansion. A quenched numerical study is in progress \cite{18}. As for the $\Delta I = 1/2$ channel, we find the situation to be considerably more complicated \cite{9}.

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