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TECHNOLOGY ADOPTION IN A DECLINING MARKET

By

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Technology adoption in a declining market

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Abstract

Rapid technological developments are inducing the shift in consumer demand from existing products towards new alternatives. When operating in a declining market, the profitability of incumbent firms is largely dependent on the ability to correctly time the introduction of product innovations. This paper contributes to the existing literature on technology adoption by considering the optimal innovation

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investment in the context of the declining market. We study the problem of a firm that has an option to undertake the innovation investment and thereby either to add a new product to its portfolio (add strategy) or to replace the established product by the new one (replace strategy). We are able to quantify the value of the option to adopt a new technology, as well as the optimal timing to exercise it. We find that it can be optimal for the firm to innovate not only because of the significant technological improvement, but also due to demand saturation. In the latter case profits of the established product may become so low that the firm will adopt a new technology even if the newest available innovation has not improved for some time. This way, our approach allows to explicitly account for the effect of a decline in the established market on technology adoption. Furthermore, we find that under certain conditions an inaction region exists, in which the firm does not innovate, while for lower technology levels it applies the add strategy and for higher technology levels the replace strategy.

\textit{JEL Classification:} C61, D81, O33

\textit{Keywords:} Technology adoption, Declining demand, Product innovation, Dynamic programming

1 Introduction

In 2007 Apple launched the iPhone. Since then Apple’s sales have been risen more than tenfold and its profits more than twentyfold. About ten years later the expectation is that, despite the upgrades that are expected to lead to new supercycles of sales, iPhone’s growth will slow. Therefore, Apple is anxiously looking for the next big thing. Apple TV, for instance, turned out not to be the disruptive offering that Apple executives promised. So, it will be difficult to find another product with the universal appeal and fat margins of the first iPhone\textsuperscript{1}.

In general, demand for existing products decreases over time due to the arrival of more exciting alternatives\textsuperscript{2}. This induces that firms need to change their product portfolio over time, and thus have to innovate

\textsuperscript{1}Source: \textit{The New Old Thing - Apple and the iPhone}, The Economist, July 1st, 2017.

\textsuperscript{2}An example, among many others is the introduction of solid state drives as an alternative for hard disk drives for data storage in computers. Before the current transition to solid state drives, the computer storage market has in the past decades gone through significant innovations from 14-inch, via 8-inch and 5.25-inch to 3.5-inch drives (see Kwon (2010)).
in order to keep on making profits. This paper has the aim to study optimal firm behavior in such a setting. To do so, we study a problem of an existing incumbent producing an established product of which demand declines over time. The firm has an option to innovate, where, due to technological progress, a newer technology can produce better products. The resulting higher demand of the better product leads to higher profits. As time passes the best available new technology that can be adopted by the firm improves. So, the longer the firm waits with investing, the better the technology is that the firm can acquire and the better the products are the firm can produce.

In such a scenario the firm has the necessity to innovate, because otherwise the declining demand of the existing product diminishes its revenue over time. In evaluating its innovation options the firm faces the following tradeoff. By adopting soon the firm is not affected too much by the reducing revenues from the existing technology, while it attracts a newer technology with higher profits. Adopting late means that, on the one hand, the firm suffers for a long time from declining profits due to the demand decrease of the established product. On the other hand, later adoption implies that, due to technological progress, the firm can attract a still better new technology with which the firm can obtain higher profits than when it adopted a new technology sooner.

The existing literature, like Balcer and Lippman (1984), Farzin et al. (1998), and Hagspiel et al. (2015), consider similar innovation problems (see Huisman (2001) and Hoppe (1999) for an extensive survey about decision theoretic models of technology adoption), but they do not consider the important characteristic of declining demand for the existing product. As a result we obtain that the time to innovate can be governed by two different causes. First, like in Farzin et al. (1998), a firm innovates right at the moment of arrival of a far better technology, the use of which enables the firm to produce products with much higher demand, leading to a considerable profit increase. Second, the fact that demand for the existing product declines over
time, implies that the firm’s revenue gets lower and lower as long as it does not innovate. For this reason it could be optimal for the firm to adopt a new technology a time lag after its introduction.

The latter result is as such not new in the literature, but what is new is that it is caused by declining demand for the existing product. To exemplify, first consider Balcer and Lippman (1984) that also shows that as time passes without new technological improvements, it may become profitable to purchase an existing technology that is superior to the one in place even though it was not profitable to do so in the past. However, in that paper this is caused by the fact that the discovery time was not memoryless. Hagspiel et al. (2015) show that changing arrival rates over time of new technologies can result in firms adopting a new technology at a later point in time than when it was available for the first time. McCardle (1985) argues that such a time lag can be explained by the uncertainty regarding the profit potential of a new technology. Doraszelski (2004), who distinguishes between innovations and improvements, concludes that the possibility of further improvements gives the firm an incentive to delay the adoption of a new innovative technology until it is sufficiently advanced.

Unlike the just mentioned contributions, Kwon (2010) has in common with our paper that it also considers a firm with a declining profit stream over time. However, Kwon (2010), and also Hagspiel et al. (2016), that extends Kwon (2010) by considering capacity optimization, does not consider a sequence of new technologies arriving over time. Instead, it analyzes whether to exercise a single innovation opportunity. In addition, the firm also has an option to exit the industry, which exists before and after the investment. Matomaki (2013) generalizes the work of Kwon (2010) by considering different stochastic processes representing profit uncertainty. Strategic interactions in a declining industry are studied by Fine and Li (1986) and Murto (2004).

The described product innovation problem is attacked as follows in this paper. As in Farzin et al. (1998), technological progress is modeled as a Poisson process, where the level of the frontier technology jumps up at unknown points in time. Demand for the existing product decreases over time, resulting in a reduction of the associated profit with a fixed rate. At the moment the firm adopts the new technology it either adds a new and technologically more advanced product to its product portfolio, meaning that it also keeps on producing the established product, or it replaces the old product by the new one. The revenue obtained
from selling the new product is deterministic and increasing in the level of the adopted technology.

We start out by considering only the option to replace. Here, we obtain a threshold level for the technology that needs to be reached in order for the firm to invest optimally. The threshold level is increasing in the profit level of the established product, i.e. the firm delays the product innovation if the established product market is more profitable. We carry out a comparative statics analysis assuming a specific functional form for the profit flow in the new market. Among others we find that the adoption threshold level is not affected by the rate of decline of the demand in the established market. As an extension, we allow the revenue of the new product to be stochastic. Particularly we impose that this revenue is governed by a geometric Brownian motion (GBM, for short) process. Remarkable is that the uncertainty parameter does not influence the investment threshold level if the revenue is linearly dependent on the stochastic variable. Otherwise, the standard result holds that larger uncertainty postpones investment.

We then proceed the analysis by also taking into account the option to add the innovative product to the product portfolio. The disadvantage of this strategy is that both products are competing in the sense that the new product cannibalizes demand of the old one and vice versa. Of course, the firm is still able to replace the old product by the new one, i.e. to stop production of the established product. Essentially, what we find is that the firm either innovates early and applies the add strategy or innovates late and applies the replace strategy. In the latter case, the firm waits for more technological improvements because its revenue solely depends on the new product upon adoption. Broadly speaking we found two different situations leading to qualitatively different solutions. In the first situation, it holds that the firm always innovates earlier if the current profit from selling the established product is lower, which is as expected. However, in the second situation an inaction region with respect to the technology level exists. In particular, in this inaction region the firm refrains from carrying out a product innovation, whereas for lower technology levels it would be optimal to innovate and add the new product to the product portfolio. If the technology level is sufficiently high the firm carries out the replace strategy. It turns out that such a situation occurs if the cannibalization effect is large enough, such that it dominates the increases revenue effect of a better technology.

The paper is organized as follows. Section 2 introduces the model. The replace strategy is analyzed in Section 3. Section 4 extends this analysis by also taking into account the add strategy. Section 5 concludes.
2 Model

We consider an incumbent firm currently producing an established product. As time passes, consumers get access to better alternatives in an evolving economy, shifting their demand away from the established product. Moreover, in case of durable goods the existing consumer base reduces as time passes, because more consumers already bought the product. For these reasons profits earned on the established product market decrease over time. The firm has been active in this market for some time, and we, therefore, assume that it has a perfect foresight about the future demand of the established product. Thus, the profit flow of the firm at time $t$ is deterministic and equals $\pi_0(X_t) = z_0 X_t$, with $z_0 > 0$. The declining nature of the established market is captured by process $X = \{X_t : t \geq 0\}$, where

$$dX_t = \alpha X_t dt,$$

with $X_0 = x_0$, where $x_0 > 0$ and $\alpha < 0$.

Facing a declining profit stream, the firm has an incentive to update its product portfolio. To do so it has to perform a product innovation by adopting a new, more advanced technology. Innovating requires an irreversible investment outlay of $I$. More significant technological improvements allow to produce products of higher quality. The adoption of the new technology, thus, boosts the firm’s revenue, as it is able to attract more consumers.

The development of technologies over time is governed by a stochastic process, which is exogenous to the firm. Similar to Huisman (2001) and Farzin et al. (1998), the state of the technological progress is given by a compound Poisson process, $\theta = \{\theta_t : t \geq 0\}$. We may express $\theta_t = \theta_0 + u N_t$, where $\theta_0 > 0$ denotes the state of technology at the initial point in time, $u > 0$ is the jump size and $\{N_t, t \geq 0\}$ follows a homogeneous Poisson process with rate $\lambda > 0$. This formulation implies that new technologies arrive at rate $\lambda$, and each arrival increases the technology level by $u$. Note that process $\theta$ is non-decreasing over time, which reflects the non-declining nature of the technological progress.

It follows that initially the firm is producing with a technology $\xi_0$, for which it holds that $\xi_0 \leq \theta_0$. Without loss of generality we impose that $\xi_0 < \theta_0$, implying that initially the firm is not producing with the best available technology. The reason for this could be that the firm exists for some time and adopted its current
technology at some time in the past.

Essentially, the firm has two reasons to innovate. The first reason is that the established product market profit has reduced too much so that to keep on producing this established product is not economically viable for the firm. The second reason is that over time alternative technologies have been invented with which the firm could enter markets that are more profitable than the established product market. Translated to our model, the first reason is equivalent to a low value of $X$, whereas the second reason implies a high value of $\theta$. We conclude that innovating is optimal for low values of $X$, and high values of $\theta$, while the firm should keep on being active on the established product market when $X$ is high and $\theta$ is low.

The objective of the firm is, thus, to determine the optimal time to adopt the new technology. At that time the firm has to decide whether to simply replace the old product by the new one, or to add the new product to the existing product portfolio, so that the firm will produce both products at the same time. The next section fully concentrates on the replace case. This is an interesting case by itself as, for instance, in the two-period model of Levinthal and Purohit (1989) it is established that replacing the existing version of the product with an upgrade gives higher profits than joint production. The option to add is taken into account in Section 4.

3 Option to replace

In case the firm replaces the old product by the new one, it has to decide on the timing. Therefore, the firm solves the following optimal stopping problem:

$$F(\theta, x) = \sup_{\tau} \mathbb{E} \left[ \int_0^{\tau} \pi_0(X_s) e^{-rs} \, ds + \left\{ \int_{\tau}^{+\infty} \pi_1(\theta_s) e^{-rs} \, ds - I e^{-r\tau} \right\} \chi_{\{\tau < +\infty\}} \bigg| \theta_0 = \theta, X_0 = x \right],$$  

(1)

in which $F$ is the value of the firm, $\tau$ denotes the investment timing, $r > 0$ represents the discount rate, and $\pi_1$ is the profit flow in the new market.

In this setting the firm faces a trade-off between early adoption and the significance of the technological improvement. In particular, waiting for a better technology comes at a cost of operating longer with lower profits.

If the firm decides to innovate at the current level of $\theta$, it earns a profit flow of $\pi_1(\theta)$. Adding it up and
discounting gives a total discounted profit stream $\frac{\pi_1(\theta)}{r}$. Since innovating requires an investment outlay of $I$, this results in the following value of instantaneous investment,

$$V(\theta) = \frac{\pi_1(\theta)}{r} - I.$$  \hspace{1cm} (2)

In this section we do not propose any particular instance of $\pi_1$; instead we simply assume that it is an increasing and concave function of $\theta$, with $\pi_1(0) = 0$ and $\lim_{\theta \to +\infty} \pi_1(\theta) = +\infty$. This entails that $V$ is also increasing and concave (Alvarez et al. (2003)), guaranteeing the existence of an unique solution of the optimal stopping problem.

The corresponding Hamilton–Jacobi–Bellman (HJB) equation for the optimization problem (1) is given by

$$\min \{rF(\theta, x) - [\pi_0(x) + LF(\theta, x)] , F(\theta, x) - V(\theta) \} = 0,$$  \hspace{1cm} (3)

where the infinitesimal generator is defined by

$$Lf(\theta, x) = \alpha x \frac{\partial f(\theta, x)}{\partial x} + \lambda [f(\theta + u, x) - f(\theta, x)],$$  \hspace{1cm} (4)

with $f$ being continuous in $\theta$ and continuous with derivative absolute continuous in $x$.

Let the set $\mathcal{C} := \{ (\theta, x) \in \mathbb{R}^+ \times \mathbb{R}^+ : F(\theta, x) > V(\theta) \}$ denote the continuation region, and $\mathcal{S} := \{ (\theta, x) \in \mathbb{R}^+ \times \mathbb{R}^+ : F(\theta, x) = V(\theta) \}$ denote the stopping region. The firm adopts the new technology at the moment that the boundary between stopping and continuation region is passed. This happens at the optimal investment timing, denoted by $\tau^*$, which is given by

$$\tau^* = \inf \{t > 0 : (\theta_t, X_t) \notin \mathcal{C} \}.$$  

The expression for the boundary, or threshold curve, is derived in the following proposition.

**Proposition 1** The boundary (threshold curve) that separates the continuation and stopping region is defined as follows:

$$\partial \mathcal{S} = \{(\theta, x) : \theta \geq \bar{\theta} \wedge x = b(\theta) \},$$  

where

$$b(\theta) = \frac{(r + \lambda)V(\theta) - \lambda V(\theta + u)}{z_0},$$  \hspace{1cm} (5)
and $\bar{\theta}$ is implicitly defined by $(r + \lambda)V(\bar{\theta}) - \lambda V(\bar{\theta} + u) = 0$.\(^5\) Moreover, $b$ is an increasing function of $\theta$.

**Proof of Proposition 1** See Appendix A.1 for the proof.

From Proposition 1 we conclude that $b$ is an upward sloping curve in the $(\theta, x)$–plane. This implies that adoption of the new technology does not happen only due to a technology arrival, which corresponds to a horizontal jump in the $(\theta, x)$–plane. It can also happen that the existing revenue for the established product becomes so low that innovating is optimal. This is reflected by the decrease in $x$ over time, which corresponds to a vertical movement in the $(\theta, x)$–plane, such that innovating takes place at the moment the $b$–curve is hit from above. These two possibilities are graphically illustrated in Figure 1.

**Figure 1:** Illustration of the two possible ways of adopting: at the arrival of a new technology (left-right horizontal crossing of threshold curve) or after a sufficient decrease of the profitability of the current market (downward vertical crossing of threshold curve).

In this figure the current level of $(\theta, x)$ is marked by a star ($\star$). The solid lines correspond to the profit decline in the established market, whereas the dashed lines illustrate the technology arrivals. As said before the threshold curve can be crossed in two ways. In one case, an additional decline in the market is necessary for the investment to be optimal after two technology arrivals, and $b$ is hit from above. In the other case, innovating is optimal immediately after two technology arrivals and $b$ is crossed from the left.

\(^5\)Here, $\bar{\theta}$ represents the level of the technology that triggers investment when there is no market left for the old product.
Passing the boundary in these different ways has to be taken into account in the derivation of the value function of the firm in the continuation region, which we present in Proposition 2.

**Proposition 2** Let the number of arrivals of new technologies until it is optimal to innovate be given by

\[ n(\theta, x) = \left\lceil \frac{b^{-1}(x) - \theta}{u} \right\rceil, \tag{6} \]

where, for \( k \geq 0 \), \( \left\lceil k \right\rceil = \min \{ n \in \mathbb{N} : n \geq k \} \). Then the value of the firm in the continuation region is equal to

\[ F(\theta, x) = \left( \frac{\lambda}{r + \lambda} \right)^{n(\theta, x)} V(\theta + n(\theta, x)u) + \frac{z_{0} x}{r - \alpha} \left[ 1 - \left( \frac{\lambda}{r + \lambda - \alpha} \right)^{n(\theta, x)} \right] \]

\[ + \sum_{k=0}^{n(\theta, x)-1} \left\{ \frac{x}{b(\theta + ku)} \right\} \frac{z_{0} b(\theta + ku) \lambda^{k}}{(r + \lambda - \alpha)^{k+1}} \sum_{m=0}^{k} \frac{1}{m!} (-\alpha)^{m} \left( \frac{1}{(r + \lambda)^{k-m+1}} - \frac{1}{(r + \lambda - \alpha)^{k-m+1}} \right) \ln \left[ \frac{x}{b(\theta + ku)} \right]^{m} \chi\{\theta > \bar{\theta} - ku\}, \tag{7} \]

where \( \chi_A \) represents the indicator function of set \( A \).

**Proof of Proposition 2** See Appendix A.2 for the proof.

The value function in the continuation region consists of three parts. The first term in (7) can be interpreted as the expected discounted value of adopting the new technology upon its arrival. Here the fraction \( \frac{\lambda}{r + \lambda} \) accounts for the stochastic discount factor under a Poisson process (Huisman (2001, p.46)).

The second term in (7) represents what the firm earns on sales of the established product until it innovates. Here \( \frac{z_{0} x}{r - \alpha} \) stands for the discounted revenue stream if the firm were active on the established product market forever. However, after the firm innovates, it discontinues this activity. Therefore, we need to subtract the amount \( \frac{z_{0} x}{r - \alpha} \). The denominator \( r + \lambda - \alpha \) makes sure that the resulting expected revenue stream is discounted \( r \), it is corrected for the fact that the revenue stream lasts up until the innovation time \( \lambda \), and that the revenue decreases over time with rate \( -\alpha \) due to the declining demand of the established product.

The third term in (7) accounts for the fact that the innovation can occur not only due to the technology jump, but also by the decline in the established market. In order to illustrate this, consider the scenario when the current demand in the established market and the technology level are such that the innovation
will always be optimal after two jumps. Let \( C_n \) denote the subset of the continuation region where stopping is optimal after \( n \) jumps in \( \theta \), i.e., if \((\theta, x) \in C_n\) then \((\theta + nu, x) \in S\). Thus, in the region \( C_2 \) we can simplify the value function in (7) – considering \( n(\theta, x) = 2 \) – as follows

\[
\left(\frac{\lambda}{\gamma + \lambda}\right)^2 V(\theta + 2u) + \frac{z_0 x}{\gamma - \alpha} \left[ 1 - \left(\frac{\lambda}{\gamma + \lambda - \alpha}\right)^2 \right] + \\
\left\{ \frac{x}{b(\theta)} \right\}^{\gamma + \lambda} \frac{z_0 b(\theta)}{\gamma + \lambda} \left( 1 + \frac{1}{\gamma + \lambda - \alpha} \right) \chi_{\{\theta > \bar{\theta}\}} + \\
\left\{ \frac{x}{b(\theta + u)} \right\}^{\gamma + \lambda} \frac{z_0 b(\theta + u)}{\gamma + \lambda} \left( \frac{\lambda}{\gamma + \lambda} - \frac{1}{\gamma + \lambda - \alpha} \right) \ln \left( \frac{x}{b(\theta + u)} \right) \chi_{\{\theta > \bar{\theta} - u\}}.
\]

Figure 2 shows the four alternative ways the stopping region can be reached from an initial level of \((\theta, x) \in C_2\).

(a) Stopping region is reached by two jumps.
(b) Stopping region is reached by a decline in the established market after one jump.
(c) Stopping region is reached by one jump after a decline in the established market.
(d) Stopping region is reached by a decline in the established market.

Figure 2: Four different ways of reaching the stopping region from an initial level of \((\theta, x) \in C_2\).
The first two terms in (8) capture the case when the technology level $\theta + 2u$ is reached after two jumps, as depicted in Figure 2a. The last three terms in (8) correct for the fact that in certain scenarios the demand in the established market may decline enough for the firm to be willing to adopt a lower technology level than $\theta + 2u$. In particular, the firm might end up adopting a technology level, $\theta + u$, if the established market declines enough before the second jump takes place to trigger the investment. In this case the stopping region can be reached in two different ways. The first is illustrated in Figure 2b, where the first technology arrival happens relatively early. This brings the firm in the region one jump away from adopting, $C_1$, where a further decline in the established market triggers the investment. This situation is captured by the last correction term in (8). The second possibility, when the jump occurs relatively late, is shown in Figure 2c. In this case the decline in the established market brings the firm to the region $C_1$, after which the first technology arrival triggers the investment. This scenario is accounted for by the second correction term in (8). Finally, as shown in Figure 2d the firm may eventually adopt the current level of technology, $\theta$, if the market declines even further before any jump occurs. The first correction term in (8) corrects for that.

The following remark highlights important properties of function $F$ defined in (7).

**Remark 1** $F$ is continuous in both arguments, $\theta$ and $x$, and has derivative absolute continuous in $x$.

This result follows directly from the proof of Proposition 2 presented in Appendix A.2.

### 3.1 Example

In Section 3 we derived the analytical solution of the optimal stopping problem for a general expression of the profit flow in the new market. In this subsection we analyze a specific example for a functional form of $\pi_1$. In particular we consider a profit flow expressed by $\pi_1(\theta) = z_1 \theta^\beta$ with $0 < \beta \leq 1$. After plugging this expression into (2) and (5), we obtain that the threshold curve $b$ is given by the following equation

$$b(\theta) = \frac{1}{z_0} \left[ \frac{z_1 (r + \lambda) \theta^\beta - \lambda (\theta + u)^\beta}{r} - rI \right].$$

(9)

From the expression of the threshold curve (9) the following comparative statics results are derived.

**Proposition 3** The threshold curve $b$ is increasing in $z_1$, and decreasing in $\lambda$, $z_0$, $I$, and $u$, for a given value $\theta$. 

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**Proof of Proposition 3** See Appendix A.3 for the proof.

Proposition 3 implies that the firm will innovate later if profits on the established product market are higher. On the other hand, the firm will innovate sooner if the revenue from innovating is higher. Intuitively, waiting for the next technology arrival is more appealing if it is expected to occur sooner or when the technology arrival results in a higher increase of the technology level. If innovating is more expensive it will happen later. The location of the threshold curve is not affected by $\alpha$. However, a larger decline rate of the revenue in the established product market will result in reaching this threshold curve sooner.

Concerning the interest rate $r$ there are opposing effects. On the one hand, when $r$ increases the firm is less inclined to wait for future technological breakthroughs and therefore wants to innovate sooner. This effect dominates for small $r$. On the other hand, the firm innovates later, because the net present value of the investment decreases with $r$. Figure 3 illustrates the relationship between a point in the threshold curves for a fixed level of $\theta$ and different values of $r$.

![Figure 3: Example showing that the threshold curve $b(\theta)$ for a given $\theta = 5$ first increases in the discount rate $r$ and then decreases. Parameter values used: $\lambda = 0.05, u = 0.5, I = 50, z_0 = 50, z_1 = 10$ and $\beta = 1$.](image)

### 3.2 Extension

This section extends the analysis to the case where the profit flow in the new market is stochastic. Due to, e.g., consumer behavior risk, risk of product malfunction or unsatisfactory performance, the profitability of a new product is often exposed to uncertainty when entering a new market.

Similar to the previous example, we assume that at the moment the firm innovates, the instantaneous
profit obtained by selling the new product equals \( z_1 \theta^\beta \), with \( \tau \) being the moment of investment. Where in our benchmark model the instantaneous profit is fixed at this level, we now consider the situation that future profits are uncertain and follow a GBM. So, the profit flow of the new product now equals \( z_1 Y_t^\beta \), with

\[
dY_t = \mu Y_t dt + \sigma Y_t dW_t,
\]

where \( \mu \) denotes the drift, \( \sigma > 0 \) is the volatility, and \( Y_0 = \theta^\beta \).

Applying Itô’s lemma to determine the evolution of \( Y_t^\beta \), we get

\[
dY_t^\beta = \left( \mu \beta - \frac{1}{2} \beta(1 - \beta)\sigma^2 \right) Y_t^\beta dt + \sigma \beta Y_t^\beta dW_t.
\]

Thus, the process \( \{Y_t^\beta : t \geq 0\} \) also follows a GBM with drift \( \mu \beta - \frac{1}{2} \beta(1 - \beta)\sigma^2 \) and volatility \( \sigma \beta \). The value in the stopping region in this case is given by

\[
V(\theta) = \mathbb{E} \left[ \int_0^{+\infty} z_1 Y_t^\beta e^{-rt} dt - I \left| Y_0 = \theta \right. \right] = \frac{z_1 \theta^\beta}{r - \mu \beta + \frac{1}{2} \beta(1 - \beta)\sigma^2} - I.
\]

Note that the functional form of the value function is the same as in (2). The only difference is that the total discounted profit stream on the new market has been corrected for the trend parameter \( \mu \), and volatility parameter \( \sigma \). Moreover, the HJB equation for the optimization problem is similar to (3) and therefore, the resulting expression for the threshold curve \( b \) is given by

\[
b(\theta) = \frac{1}{z_0} \left[ \frac{(r + \lambda)\theta^\beta - \lambda(\theta + u)^\beta}{r - \mu \beta + \frac{1}{2} \beta(1 - \beta)\sigma^2} - rI \right].
\]

Proposition 4 shows how the threshold curve depends on the different parameters.

**Proposition 4** The threshold curve \( b \) is increasing in \( z_1 \) and \( \mu \), and decreasing in \( \sigma \), \( \lambda \), \( z_0 \), \( I \), and \( u \), for a given value \( \theta \).

**Proof of Proposition 4** See Appendix A.4 for the proof.

Compared to the deterministic model, the only change is that the drift \( \mu \) and the volatility \( \sigma \) enter the expression for the threshold curve. Thus, we can straightforwardly conclude that our earlier results still

\(^6\)Note that the technology parameter \( \theta \) can be interpreted as a value that is normalized to represent the effect of technology level on the profit.
hold for this model extension. In addition, we find that the firm invests earlier if the profit flow on the new
market is expected to increase more, i.e. when \( \mu \) is larger. The volatility parameter \( \sigma \) does not influence
the investment decision if the profit flow is linear in the technology level (\( \beta = 1 \)). For \( 0 < \beta < 1 \), a larger
volatility decreases the threshold curve, implying that the firm delays investment. This is due to the fact
that the new market is less attractive because an increased volatility decreases the trend of the revenue in
the new market. The latter is caused by the concavity of the profit function which implies a decreasing slope.
Therefore, a downward shock has a larger (negative) effect than the (positive) effect of the upward shock.

4 Option to add or replace

In this section we give the firm the option to keep producing the established product after investing in the
innovative product. The firm can still replace the established product right away, as we analyzed before. We
denote by \( \pi_1^A \) (respectively, \( \pi_1^R \)) the profit that results from adding the new product to the product portfolio
(respectively, replacing the old product by the new one).

In this problem the firm not only needs to decide on when to invest in the new product but also when to
stop producing the old product. This means that the firm solves the following optimal stopping problem

\[
F(\theta, x) = \sup_{\tau_1} \mathbb{E} \left[ \int_0^{\tau_1} \pi_0(X_s)e^{-rs}ds + \left\{ \sup_{\tau_2: \tau_2 \geq \tau_1} \mathbb{E} \left[ \int_{\tau_1}^{\tau_2} \pi_1^A(\theta_{\tau_1}, X_s)e^{-rs}ds - Ie^{-r\tau_1} \right] \right\} \chi_{\{\tau_1 < +\infty\}} \right] \chi_{\{\tau_1 < +\infty\}} \mid \theta_0 = \theta, X_0 = x \]

(10)

where \( \tau_1 \) denotes the time the firm adopts the new technology, i.e. adds the innovative product to its
product portfolio; and \( \tau_2 \) denotes the time the firm stops producing with the old technology, i.e. it replaces
the established product by the new one.

If the firm has an option to keep the old product alive after investing in the innovative product, it has to
take into account that some market share of the old product will be cannibalized by the upgrade. This is
because a fraction of the consumers will switch to the new version once it becomes available. To take this
effect into account, we specify the inverse demand functions for the old and new products, \( p_0^A \) and \( p_1^A \), as
follows\textsuperscript{7}

\[
p_0^A(\theta, x) = (1 - \gamma q_0 - \eta q_1 \theta) x, \\
p_1^A(\theta, x) = (1 - \gamma q_1 - \eta q_0 \theta),
\]

where \(\eta\) denotes the cannibalization coefficient, and \(\gamma\) represents the demand sensitivity of the product to its quantity. If the firm produces only one product, either solely the old one or the new one, the cannibalization effect is not present. Therefore, before the firm introduces the upgrade, the price for the existing product is given by \(p_0(x) = (1 - \gamma q_0) x\). If the firm decides to replace the old product by the new one, the price for the latter is \(p_1^R(\theta) = (1 - \gamma q_1)\theta\). In the notation of the model presented in Section 2, we have \(z_0 = (1 - \gamma q_0)q_0\).

We further introduce \(z_1 = (1 - \gamma q_1)q_1\) and \(\kappa = 2\eta q_0 q_1\), where the latter represents the strength of the cannibalization effect. This leads to the following profit functions

\[
\begin{align*}
\pi_0(x) &= p_0(x)q_0 = z_0 x, \\
\pi_1^R(\theta) &= p_1^R(\theta)q_1 = z_1 \theta, \\
\pi_1^A(\theta, x) &= p_0^A(\theta, x)q_0 + p_1^A(\theta, x)q_1 = \pi_0(x) + \pi_1^R(\theta) - \kappa \theta x,
\end{align*}
\]

where \(\pi_0\) denotes the profit before innovation, \(\pi_1^R\) is the profit of the firm producing only the innovative product, and \(\pi_1^A\) is the profit of the firm producing both products. Note that \(\pi_1^R\) corresponds to the linear profit function example in Section 3.

The following proposition states that in fact, given the chosen demand functions, the decision of the firm is either never replace the old product (and instead produce both products forever) or replace upon investing.

**Proposition 5** The firm will keep producing the old product upon adoption of a new technology with level \(\theta\) if \(0 < \theta < \hat{\theta}\), and will replace the old product if the new technology level is such that \(\theta \geq \hat{\theta}\), with \(\hat{\theta} = \frac{z_0}{\kappa}\).

**Proof of Proposition 5** See Appendix A.6 for the proof.

\textsuperscript{7}The demand system can be derived from the following utility function

\[
U = xq_0 - \frac{1}{2} \gamma q_0^2 x - \eta q_0 q_1 \theta x + \theta q_1 - \frac{1}{2} \gamma q_1^2 \theta + \lambda(y - p_0^A q_0 - p_1^A q_1),
\]

in which \(\lambda\) is a Lagrange parameter and \(y\) is the income of the representative consumer.
As a result, the space \((\theta, x)\) is split in two regions: for \(\theta < \hat{\theta}\), upon investment the firm produces both products, whereas for \(\theta \geq \hat{\theta}\) the firm produces just the innovative one upon investment.

Taking into account Proposition 5, simple manipulations allow us to rewrite the problem (10) as

\[
F(\theta, x) = \sup_{\tau_1} \mathbb{E} \left[ \int_0^{\tau_1} \pi_0(X_s)e^{-r_0s}ds + e^{-r_0\tau_1}V(\theta_{\tau_1}, X_{\tau_1}) \right] \bigg|_{\theta_0 = \theta, X_0 = x}, \tag{12}
\]

with

\[
V(\theta, x) = \frac{\pi_1^A(\theta, x) - \pi_1^R(\theta)}{r - \alpha} \chi_{\{0 < \theta < \hat{\theta}\}} + \frac{\pi_1^R(\theta)}{r} - I,
\]

which can be re-written as

\[
V(\theta, x) = \begin{cases} 
V^A(\theta, x) = \frac{(z_0 - \kappa \theta)x}{r - \alpha} + \frac{z_1 \theta}{r} - I & \text{if } 0 < \theta < \hat{\theta} \\
V^R(\theta, x) = \frac{z_1 \theta}{r} - I & \text{if } \theta \geq \hat{\theta}
\end{cases}
\]

The HJB equation corresponding to the optimization problem (12) is given by

\[
\min \{rF(\theta, x) - [\mathcal{L}F(\theta, x) + \pi_0(x)], F(\theta, x) - V(\theta, x)\} = 0,
\]

where the infinitesimal generator is the same as defined in (4). As in the previous section, let the set

\[
C := \{(\theta, x) \in \mathbb{R}^+ \times \mathbb{R}^+ : F(\theta, x) > V(\theta)\}
\]

denote the continuation region, and

\[
S := \{(\theta, x) \in \mathbb{R}^+ \times \mathbb{R}^+ : F(\theta, x) = V(\theta)\}
\]

denote the stopping region. Then the optimal investment timing, denoted by \(\tau^*\), is given by

\[
\tau^* = \inf \{t > 0 : (\theta_t, X_t) \notin C\}.
\]

Unlike in Section 3, we have to distinguish two different cases concerning the shape of the optimal exercise boundary, and, consequently of the continuation region \(C\). In the first case, the boundary is monotonically increasing in \(\theta\). In the second case, the boundary exhibits non-monotonic behavior. Proposition 6 states the condition that allows to distinguish between these two cases.

**Proposition 6** The investment threshold is monotonically increasing in \(\theta\) if the following condition is satisfied

\[
\kappa \left( \frac{rI}{z_1} + \frac{u\lambda}{r} \right) > \frac{\lambda}{r + \lambda - \alpha} z_0.
\]

**Proof of Proposition 6** See Appendix A.7 for the proof.

Before we give an economic interpretation to expression (13) in the last subsection, we first analyze the optimal solutions in the two different cases in the next two subsections.
4.1 Monotonic threshold boundary

Proposition 7 gives the expression for the optimal exercise boundary in case the boundary is monotonically increasing in \( \theta \).

**Proposition 7** Let us assume that the condition in Proposition 6 is satisfied. For \( \theta > \bar{\theta} \), where \( \bar{\theta} = \frac{rI}{z_1} + \frac{u\lambda}{r} \), the boundary between the stopping and the continuation region is given by

\[
b(\theta) = b^A(\theta)\chi_{\{\theta < \bar{\theta} - u\}} + b^{AR}(\theta)\chi_{\{\bar{\theta} - u \leq \theta < \bar{\theta}\}} + b^R(\theta)\chi_{\{\theta \geq \bar{\theta}\}}
\]

where

\[
b^A(\theta) = \frac{z_1 (\theta - \frac{u\lambda}{r}) - rI}{\kappa (\theta - \frac{u\lambda}{r - \alpha})} \]
\[
b^{AR}(\theta) = \frac{z_1 (\theta - \frac{u\lambda}{r}) - rI}{(r + \lambda - \alpha)\kappa - \lambda z_0} \]
\[
b^R(\theta) = \frac{z_1 (\theta - \frac{u\lambda}{r}) - rI}{z_0}.
\]

**Proof of Proposition 7** See Appendix A.8 for the proof.

Figure 4 provides an illustration of this exercise boundary. We also show the possible ways for the bivariate process \((\theta, x)\) to enter the stopping region. As before, this may happen either due to the decline in the profitability of the existing market, i.e. when \( x \) decreases (solid vertical arrow), or due to the arrival of a sufficiently better technology, i.e. a jump in \( \theta \) (dashed horizontal arrow). Figure 4 has in common with Figure 1 that both threshold curves are monotonically increasing in the \((\theta, x)\) – plane. However, whereas in Figure 1 the threshold curve has the same slope everywhere, in Figure 4 the threshold curve consists of three different pieces. To explain, we distinguish three subsets in the continuation region, which will be formally defined later in this section: \( \Omega^R \), \( \Omega^A \) and \( \Omega^{AR} \). In the first two the stopping region is either entered in the replace region or in the add region, respectively. In the latter both parts of the stopping region can still be reached, depending on the trajectory of the bivariate process \((\theta, x)\).

---

\( ^8 \)Depending on the jump size, \( u \), it can also be \( b(\theta) = b^{AR}(\theta)\chi_{\{\theta < \bar{\theta}\}} + b^R(\theta)\chi_{\{\theta \geq \bar{\theta}\}} \) or \( b(\theta) = b^R(\theta)\chi_{\{\theta \geq \bar{\theta}\}} \). In this section we are always considering the more comprehensive case.
On the part of the threshold curve, where $\theta \in [\hat{\theta}, +\infty)$, the firm is indifferent between waiting with investing and replacing production of the established product by producing the new one. For these values of $\theta$, the firm solves exactly the same optimal stopping problem as in Section 3. Therefore, in that part the position of the curves of Figures 1 and 4 coincide. The subset of the continuation region bounded to this part of the curve is $\Omega^R$. If the starting values of $(\theta, x)$ belong to $\Omega^R$, then upon investment the firm implements the strategy of replacing the old product by the new both in case of a decline in $x$ and in case of jumps in $\theta$.

On the part of the threshold curve, where $\theta \in [\bar{\theta}, \hat{\theta} - u)$, the firm is indifferent between waiting and investing in the new technology after which the firm will jointly produce the established and the new product. The subset of the continuation region, denoted by $\Omega^A$, contains the starting values of $(\theta, x)$ such that upon investment the firm implements the strategy of adding the new product to its product portfolio both in case of a decline in $x$ and in case of a jump in $\theta$.

In between, thus where $\theta \in [\hat{\theta} - u, \bar{\theta})$, on the threshold curve the firm is indifferent between waiting with investing and adding the innovative product to its product portfolio. In the case of waiting, the firm still applies the add strategy in case there is no jump during the next infinitesimal time period $dt$. If the jump occurs it will, however, replace. Indeed, $\Omega^{AR}$ denotes the subset of the continuation region, where it is not established beforehand, whether the firm will apply the “add” or “replace” strategy upon investment.

In what follows we present the formal definition of the subsets $\Omega^R$, $\Omega^A$ and $\Omega^{AR}$, as well as the optimal value functions in each subset.

- We can define $\Omega^R = \bigcup_{n=1}^{\infty} \Omega^R_n$, where

$$
\Omega^R_n = \{ (\theta, x) \in \mathbb{R}^+ \times \mathbb{R}^+ : \theta \geq \hat{\theta} \land b^R(\theta + (n-1)u) < x \leq b^R(\theta + nu) \}, \ n \in \mathbb{N}.
$$

In this region, the firm replaces the old product by the innovative one, once it decides to undertake an investment. The value function for the replace region is already derived in Section 3. Incorporating the new
notation, we let $F(\theta, x) \equiv F^R(\theta, x)$ for $(\theta, x) \in \Omega^R$, and get the following expression

$$
F^R(\theta, x) = \left( \frac{\lambda}{r+\lambda} \right)^{n(\theta,x)} V(\theta + n(\theta,x)u) + \frac{z_0 x}{r-\alpha} \left[ 1 - \left( \frac{\lambda}{r+\lambda-\alpha} \right)^{n(\theta,x)} \right] 
+ \sum_{k=0}^{n(\theta,x)-1} \left[ \frac{x}{b^R(\theta + ku)} \right]^\frac{\alpha}{\lambda} \left( z_1 \left( \theta + ku - \frac{u\lambda}{r} \right) - rI \right) \lambda^k \times 
\sum_{m=0}^{k} \frac{(-\alpha)^{-m}}{m!} \left[ \frac{1}{(r+\lambda)^{k-m+1}} - \frac{1}{(r+\lambda-\alpha)^{k-m+1}} \ln \left[ \frac{b^R(\theta + ku)}{b^R(\theta + ku)} \right] \right]^m,
$$

where

$$
n(\theta,x) = \left\lceil \frac{b^{-1}(x) - \theta}{u} \right\rceil. \tag{16}
$$

• We can define $\Omega^A = \bigcup_{n=1}^{\hat{n}} \Omega^A_n$, with $\hat{n} = \left[ \frac{\theta}{u} \right]$, which represents the number of jumps needed to exceed $\hat{\theta}$ starting from zero\(^9\), and, for $n \in \mathbb{N}$,

$$
\Omega^A_n = \{(\theta, x) \in \mathbb{R}^+ \times \mathbb{R}^+ : 0 < \theta < \hat{\theta} - nu \wedge b^A(\theta + (n-1)u) < x \leq b^A(\theta + nu) \wedge 0 < x \leq b^{AR}(\theta + nu)\}.
$$

In this region, the firm adds the innovative product to its product portfolio, i.e. keeps the old product alive, upon investment. Using a similar reasoning as the one for the replace case, we derive that the value function

\(^9\)The number of jumps in this case is limited, as $\theta$ and $x$ are bounded by 0.
in the this region, i.e. \( F(\theta, x) \equiv F^A(\theta, x) \) for \((\theta, x) \in \Omega^A\), is given by

\[
F^A(\theta, x) = \left( \frac{\lambda}{r + \lambda} \right)^{n(\theta, x)} V(\theta + n(\theta, x)u) + \frac{z_0 x}{r - \alpha} \left[ 1 - \frac{\kappa (\theta + n(\theta, x)u)}{z_0} \left( \frac{\lambda}{r + \lambda - \alpha} \right)^{n(\theta, x)} \right] + \sum_{k=0}^{n(\theta, x) - 1} \left\{ \frac{x}{b^A(\theta + ku)} \frac{\lambda^k}{r^{k+1}} \right\}^m \chi\{\theta > k \}
\]

\[ (17) \]

\[
\sum_{m=0}^{n(\theta, x) - 1} \left\{ \frac{x}{b^A(\theta + ku)} \frac{\lambda^k}{r^{k+1}} \right\}^m \chi\{\theta > k \}
\]

Finally, we can define \( \Omega^{AR} = \bigcup_{n=1}^{\infty} \bigcup_{p=1}^{\min\{n, \lambda\}} \Omega^R \), where, for \( n, p \in \mathbb{N} \), if \( n \neq p \) we have

\[
\Omega^R = \{ (\theta, x) \in \mathbb{R}^+ \times \mathbb{R}^+ : \theta < \hat{\theta} - (p - 1)u \land b^R(\theta + (n - 1)u) < x \leq b^R(\theta + nu) \}
\]

otherwise

\[
\Omega^R = \{ (\theta, x) \in \mathbb{R}^+ \times \mathbb{R}^+ : \theta < \hat{\theta} - (p - 1)u \land b^{AR}(\theta + (n - 1)u) < x \leq b^R(\theta + nu) \}
\]

In this region, upon investment the firm may either produce both products or replace the old one by the new one. In fact \( \Omega^R \) represents the sets of values for \( \theta \) and \( x \) such that either \( x \) decreases continuously and reaches region \( \Omega^R_{n-1} \), or a jump occurs, leading to the region \( \Omega^R_n \). These possible transitions for the case of \( \Omega^2 \) are illustrated in Figure 5, where the vertical solid arrow represents a continuous decrease in the value of \( x \), whereas the horizontal dashed arrow represents a jump in the technology level.

We note that for the values of \( \theta \) and \( x \) in the previous two regions, the decision regarding the type of investment is clear: when one is in \( \Omega^R \), we do not know for how many jumps in the technology we will need to wait and when the jumps will take place, but we do know for sure that once the firm invests, it will replace the old product by the new one. Similarly, in \( \Omega^A \) the same holds for the investment timing but the firm knows that upon investment, the two products will be produced forever. However, \( \Omega^{AR} \) is a region where not only there is uncertainty regarding the timing of the investment, but also regarding the strategy upon investment (either add or replace). In particular, for \((\theta, x) \in \Omega^{AR}\), the value function is given
Figure 5: Illustration of the increasing threshold curve $b(\theta)$.

by $F(\theta, x) \equiv F^{AR}(\theta, x)$, where

$$
F^{AR}(\theta, x) = \left( \frac{\lambda}{r + \lambda} \right)^{n(\theta, x)} V(\theta + n(\theta, x)u) + \frac{z_0x}{r - \alpha} \left[ 1 - \left( \frac{\lambda}{r + \lambda - \alpha} \right)^{n(\theta, x)} \right] 
$$

$$
+ \sum_{k=0}^{n(\theta, x)-1} \left\{ \frac{x}{b(\theta + ku)} \left[ i(\theta + ku) \lambda^k \times \sum_{m=0}^{k} (-\alpha)^{-m} \frac{1}{m!} \left[ \frac{1}{(r + \lambda)^{k-m+1}} - \frac{1}{(r + \lambda - \alpha)^{k-m+1}} \right] \ln \left[ \frac{x}{b(\theta + ku)} \right] \right]^m \right\} \chi_{\{\theta > \bar{\theta} - ku\}}.
$$

The function (18) reflects that, given the state of $(\theta, x)$, different parts of the boundary may be reached. In order to illustrate this, we consider a specific example of the value function, when $(\theta, x) \in \Omega_3^3$. In this case (18) becomes

$$
F^{AR}(\theta, x) = \left( \frac{\lambda}{r + \lambda} \right)^3 V(\theta + 3u) + \frac{z_0x}{r - \alpha} \left[ 1 - \left( \frac{\lambda}{r + \lambda - \alpha} \right)^3 \right] 
$$

$$
+ \left[ \frac{x}{b^A(\theta)} \right] \left( \frac{z_1}{r - \alpha} \right) \left[ \frac{1}{r + \lambda} - \frac{1}{r + \lambda - \alpha} \right] \right\} 
$$

$$
+ \left[ \frac{x}{b^{AR}(\theta + u)} \right] \left( \frac{z_1}{r - \alpha} \right) \left[ \sum_{m=0}^{2} \frac{\lambda^2(-\alpha)^{-m}}{m!} \left[ \frac{1}{(r + \lambda)^{3-m}} - \frac{1}{(r + \lambda - \alpha)^{3-m}} \right] \ln \left[ \frac{x}{b^{AR}(\theta + u)} \right] \right]^m,
$$

where $i(\theta) = z_1 \left( \theta - \frac{u\lambda}{r} \right) - r I$.  

22
The value function in (19) consists of five terms. The first two terms are similar to the value function in Section 3. They reflect the possibility to reach the stopping region after three jumps, and the revenues that the firm earns until that point. The last three terms account for the fact that different boundaries may be hit. If for \((\theta, x) \in \Omega^2_3\) no jump in technology occurs and \(x\) declines sufficiently then the Add boundary is hit, which is reflected by the third term. If before investment there occurs only one jump in technology, which brings it to the level \(\theta + u\), and \(x\) declines sufficiently, the firm will enter the stopping region through the Add/Replace boundary from \(\Omega^1_1\). This is captured by the fourth term. Finally, if after two jumps, i.e. when the firm will achieve the level of technology \(\theta + 2u\), it will reach the Replace boundary by decline in \(x\) and enter the stopping region from \(\Omega^1_R\). This is lastly reflected in the fifth term.

### 4.2 Non-monotonic threshold boundary

We start out with presenting the threshold boundary in Proposition 8 in case the boundary is non-monotonic in \(\theta\).

**Proposition 8** Let us assume that the condition of the Proposition 6 is not satisfied. For \(\theta > \bar{\theta}\), the optimal exercise boundary is given by

\[
b(\theta) = b^A(\theta) \chi_{\{\bar{\theta}_1 < \theta \leq \bar{\theta}_1\}} + b^{AC_A}(\theta) \chi_{\{\bar{\theta}_1 \leq \theta < \bar{\theta}_1 - u\}} + b^{AC_R}(\theta) \chi_{\{\bar{\theta}_2 - u < \theta < \bar{\theta}_2\}} + b^{AR}(\theta) \chi_{\{\bar{\theta}_2 - u < \theta < \bar{\theta}_2\}} + b^R(\theta) \chi_{\{\theta \geq \bar{\theta}\}},
\]

(20)

where \(b^{AC_A}\) and \(b^{AC_R}\) are implicitly defined in (49) and (50), respectively. Moreover, \(\bar{\theta}_1\) and \(\bar{\theta}_2\) are defined as \(b^A(\bar{\theta}_1) = b^{AR}(\bar{\theta}_1 + u)\) and \(b^{AR}(\bar{\theta}_2) = b^R(\bar{\theta}_2 + u)\), respectively, which are explicitly given by (47) and (48).

**Proof of Proposition 8** See Appendix A.9 for the proof.

In Figure 6 we provide an illustration for this case, and show one possible transition of the bivariate process \((\theta, x)\) from the continuation region to the stopping region. Where Figures 1 and 4 have in common that the threshold curve \(b(\theta)\) is monotonically increasing, it is evident from Figure 6 that the threshold curve has a decreasing part when the condition of Corollary 6 is not satisfied. The main difference with the boundary depicted in Figure 4, considering that we start in the continuation region, is the following: for the monotonic case, for all \(\theta > \bar{\theta} - u\), there always exist values of \(x\) for which a jump in the technology level
always leads to the investment. At the same, for all these points, it is also possible to achieve the threshold curve by a declining in $x$. For the non-monotonic case, however, for $\tilde{\theta}_1 < \theta < \tilde{\theta}_2$, there are no values of $x$ such that a jump in the technology level lead to investment, as it is exemplified by the horizontal arrow in Figure 6. However, starting from the point represented by a star in Figure 6, it is possible to reach the boundary through a decline in $x$.

In fact the optimal threshold curve is a piece-wise function consisting of five regions, as defined in Proposition 8 and illustrated in Figure 6. The derivation of the value function in this case can be done analogously to the previous case, i.e. we need to track the transitions of the bivariate process $(\theta, x)$ and take into account all possibilities for it to enter the stopping region. The difference is that now we need to take into account that instead of three potential boundary functions, in the non-monotonic case we have five. The value function, thus, has to reflect that either of the five different parts of the boundary can be reached from above.

![Figure 6: Illustration of the non-monotonic threshold curve $b(\theta)$ and the hysteresis region (shaded area).](image)

As can be seen in Figure 6, due to the decreasing behavior of the threshold curve, a hysteresis region arises. This region is illustrated by the shaded area. In the hysteresis region the firm in fact refrains from investing, while at the same time for a smaller level of the technology parameter adopting the new technology is optimal applying the add strategy. This at first sight counter-intuitive innovation strategy makes sense, because in
the hysteresis region the firm wants to keep the option open to apply the replace strategy with a higher
technology level instead of jointly producing the established and the new product once the decrease of $x$ has
resulted in reaching the threshold boundary. This is beneficial here, because $\theta$ is relatively large, implying
that the cannibalization effect, given by $2nq_0q_1\theta x$, will make the add strategy unattractive compared to
replace.

4.3 Non-monotonicity condition

Surprisingly, Figure 6 shows that the threshold boundary $b(\theta)$ is decreasing for $\theta \in (\hat{\theta} - u, \tilde{\theta})$. The implica-
tion is that levels of $x$ exist for which it is not optimal to innovate for some level of the technology parameter
level $\theta$, while it is optimal to do so for smaller $\theta$–levels. This occurs when the condition of Proposition 6 is
violated, meaning that

$$\kappa \left( \frac{rI + u\lambda}{z_1} \right) < \frac{\lambda}{r + \lambda - \alpha} z_0. \quad (21)$$

A direct economic interpretation of this non-monotonicity condition is difficult to give. To understand better
what is going on, we now derive this condition in a more economically intuitive way. To do so, we start out
recognizing that right at the threshold boundary the firm is indifferent between investing now or waiting for
a period $dt$ and investing then. If the firm invests now, for a subset of the decreasing part of the boundary,
i.e. where $\theta \in (\tilde{\theta}, \hat{\theta})$, we know that upon investing the firm keeps on producing the old product after
investment. Therefore, the resulting payoff of investing immediately is given by

$$V^A(\theta, x) = z_0 - \kappa x - \alpha x + z_1 \theta x - I.$$  

To set up the payoff for the policy “waiting for a period $dt$ and investing then”, we have to notice that at
the part of the boundary that we focus on, thus $b(\theta)$ for $\theta \in (\tilde{\theta}, \hat{\theta})$, the firm invests such that it replaces
the old product by the new one after a technology jump, while it applies the add strategy if no technology
jump takes place. Consequently, the waiting policy leads to the following payoff

$$w(\theta, x) = z_0 x dt - \lambda dt \left[ \frac{z_1 (\theta + u)}{r} - I \right] + \alpha dt \left[ \frac{z_0 x - \kappa \theta x}{r - \alpha} \right] + (1 - \lambda dt - r dt) \left[ \frac{z_0 - \kappa \theta x}{r - \alpha} + \frac{z_1 \theta x}{r} - I \right].$$

Now, at the boundary it holds that

$$h(\theta, x) = w(\theta, x) - v(\theta, x) = 0. \quad (22)$$
In the neighbourhood of that part of the boundary the function \( h(\theta, x) \) can be interpreted as the relative value of waiting. It is straightforward to verify that this equality is equivalent to the expression for \( b(\theta) \) for \( \theta \in (\tilde{\theta}_2, \tilde{\theta}) \) in expression (14).

From totally differentiating \( h(\theta, x) \) it follows that the slope of the boundary satisfies

\[
\frac{dx}{d\theta} \bigg|_{h(\theta, x)=0} = -\frac{\partial h(\theta, x)}{\partial \theta} + \frac{\partial h(\theta, x)}{\partial x},
\]

which

\[
\frac{\partial h(\theta, x)}{\partial \theta} = \frac{r + \lambda - \alpha}{r - \alpha} \left( \kappa x - \frac{r - \alpha - z_1}{r + \lambda - \alpha} \right) dt,
\]

\[
\frac{\partial h(\theta, x)}{\partial x} = \frac{r + \lambda - \alpha}{r - \alpha} \left( \kappa \theta - \frac{\lambda}{r + \lambda - \alpha} z_0 \right) dt.
\]

From (23) we know that if the slope of the threshold boundary is negative, it has to hold that

\[
\text{sign} \left( \frac{\partial h(\theta, x)}{\partial \theta} \right) = \text{sign} \left( \frac{\partial h(\theta, x)}{\partial x} \right).
\]

From Proposition 7 we get that

\[
\kappa \theta - \frac{\lambda}{r + \lambda - \alpha} z_0 > 0
\]

is required for the boundary \( b^{AR}(\theta) \) to be positive, so that the slope of the threshold boundary being negative implies that

\[
\text{sign} \left( \frac{\partial h(\theta, x)}{\partial \theta} \right) = \text{sign} \left( \frac{\partial h(\theta, x)}{\partial x} \right) > 0.
\]

After substitution of \( x \) by \( b(\theta) \) for \( \theta \in (\tilde{\theta} - u, \tilde{\theta}) \) of (14), into (24), this condition becomes

\[
\frac{r + \lambda - \alpha}{r - \alpha} \left[ \kappa \left( \frac{r - \alpha}{r + \lambda - \alpha} \right) - \frac{\lambda}{r + \lambda - \alpha} z_0 \right] > 0,
\]

which can be simplified to condition (21).

We conclude from (27) that necessary conditions for the threshold boundary to decrease is that the relative value of waiting goes up in \( x \) as well as in \( \theta \). From (25) and (26) we obtain that the relative value of waiting increases in \( x \). This is because the lower is \( x \), the less incentive to wait with investment the firm has, as its revenues decline in the current market and the cannibalization effect it faces also declines with \( x \).

However, the effect of \( \theta \) on the value of waiting is ambiguous. From (26) we get that if per unit of \( \theta \) the cannibalization effect, \( \kappa x \), is small relative to the revenue \( z_1 \), the relative value of waiting is decreasing
in \( \theta \). This is because then the dominating effect is the forgone revenues of selling the innovative product, due to the fact that the investment is not yet undertaken. On the threshold boundary the relative value of waiting is zero (cf. Equation (22)), so that the decreasing value of waiting due to an increase of \( \theta \) needs to be compensated by an increasing value of waiting due to an increase of \( x \). Hence the threshold boundary is increasing, as usual.

However, if the cannibalization effect per unit of \( \theta \), \( \kappa x \), is larger than the forgone revenues from not investing, the value of postponed cannibalization dominates. This implies that the value of waiting increases with \( \theta \). That means that on the threshold boundary, for higher values of \( \theta \) the value of waiting at the same time should decrease due to a change of \( x \). From this we conclude that on the boundary an increase of \( \theta \) must be accompanied by a decrease of \( x \). For this reason the threshold boundary \( b(\theta) \) is decreasing.

From (24) we obtain that the scenario where the threshold boundary is decreasing on the interval \( \left( \hat{\theta}_2, \hat{\theta} \right) \), occurs when the level of cannibalization per unit \( \theta \), \( \kappa x \), is relatively large. In particular, from expression (24) it follows that at the threshold boundary \( b(\theta) \), we have that \( \kappa x = \kappa b(\theta) \) is larger than the revenue on the new market per unit \( \theta \), \( z_1 \).

The at first sight counter-intuitive feature is that \( \kappa b(\theta) \) is large when \( \kappa \) is small. To see this, it is important to notice that \( b(\theta) \) is negatively affected by \( \kappa \), because this means that increasing \( \kappa \) has two contradictory effects on the level of cannibalization on the threshold boundary per unit \( \theta \), \( \kappa b(\theta) \). On the one hand an increase of \( \kappa \) would mean an increase in the marginal effect of cannibalization with respect to \( \theta \), because of the direct effect of \( \kappa \). On the other hand, the indirect effect of an increase of \( \kappa \) is that the marginal effect of cannibalization with respect to \( \theta \), decreases, as \( b(\theta) \) decreases with \( \kappa \).

Mathematically, the marginal effect of cannibalization on the threshold boundary with respect to \( \theta \) is represented by

\[
\kappa b(\theta) = \frac{r - \alpha}{r + \lambda - \alpha} \left[ \frac{z_1 \left( \theta - \frac{\alpha \lambda}{\tau} \right) - r I}{\kappa \theta - \frac{\lambda}{\lambda + r - \alpha} z_0} \right] = \frac{r - \alpha}{r + \lambda - \alpha} \left[ \frac{z_1 \left( \theta - \frac{\alpha \lambda}{\tau} \right) - r I}{\theta - \frac{\lambda}{\lambda + r - \alpha} \frac{z_0}{\kappa}} \right],
\]

from which we conclude that overall \( \kappa b(\theta) \) decreases with \( \kappa \) so that the indirect effect described in the previous paragraph dominates. This explains why the scenario of a non-monotonic threshold boundary \( b(\theta) \) with decreasing slope for \( \theta \in \left( \hat{\theta}_2, \hat{\theta} \right) \) occurs when \( \kappa \) is small, as is confirmed in inequality (21).
5 Conclusion

This paper studies the product innovation option of an incumbent. Initially the firm is active in selling its established product. However, the firm’s profit associated with the established product decreases over time due to the facts that, in case of durable goods, over time the consumer base declines because more consumers have already bought the product, and other firms introduce products that compete with the established product of the focal firm. For this reason the firm wants to change its product portfolio by innovating. Due to technological progress the firm is able to introduce a better product if it innovates later. Therefore, the firm faces the following trade-off. If it innovates early, it stops the profit decline associated with its established product early, but the adopted new product only incrementally improves the established one. If the firm innovates late, it is able to launch a product of much better quality, but at the same time it has to deal with a long period of declining demand of its established product. Depending on the realizations of the technological breakthroughs, the paper determines the firm’s optimal product innovation timing. We show that such an innovation can occur either right at the moment of a technological breakthrough, or some time after such an event. In the latter case the firm adopts the new product, because demand of the established product has reduced too much. We further obtain an explicit expression for the value of the firm, reflecting a weighted average of all possible innovation patterns.

A product innovation implicitly creates another problem: what to do with the old product? To analyze this problem we explicitly distinguish between two strategies: introducing the new product while keeping the old product alive (add strategy), or abolishing the old product when introducing the new one. Producing both products at the same time generates a cannibalization effect. It turns out that in some scenario this cannibalization effect leads to a hysteresis effect. In particular, when a firm adopts a very advanced technology it abolishes the old product. If the adopted technology is less advanced the firm does not carry out a product innovation, while it applies the add strategy for lower technology levels.

This paper provides a solid basis for interesting extensions that could be topics for future research. Here we think about determining the optimal production capacity associated with launching the new product, including the innovation strategy of competitors and how to optimally react to that, and to incorporate learning effects regarding the production processes of the different products.
A Proofs of propositions

A.1 Proof of Proposition 1

Let us consider the set

\[ U = \{ (\theta, x) \in \mathbb{R}^+ \times \mathbb{R}^+ : rV(\theta) - LV(\theta) < 0 \}. \]

Note that \( rV(\theta) - LV(\theta) < 0 \Leftrightarrow (r + \lambda)V(\theta) - \lambda V(\theta + u) < \pi_0(x) \Leftrightarrow g(\theta) < \pi_0(x) \), where \( g(\theta) = (r + \lambda)V(\theta) - \lambda V(\theta + u) \). Given that \( \pi_1 \) is a concave function then \( V \) is also a concave function, implying that \( V' \) is a decreasing function. So, \( V'(\theta + u) > \frac{1}{\lambda + \tau} V'(\theta + u) \), which implies that \( g'(\theta) > 0 \), i.e. \( g \) is an increasing function. Thus, given that \( g(0) < 0 \), \( \lim_{\theta \to +\infty} g(\theta) = +\infty \) and \( \pi_0(x) > 0 \), for \( x > 0 \), \( U \) is of the form

\[ U = \{ (\theta, x) \in \mathbb{R}^+ \times \mathbb{R}^+ : \theta < h(x) \}, \]

where \( h \) is implicitly defined as \( g(\theta) < \pi_0(x) \Leftrightarrow \theta < h(x) \). Given that \( \pi_0 \) is an increasing function of \( x \), then \( h \) is also an increasing function of \( x \). In the limit case for \( x = 0 \), we have \( \theta < h(0) \Leftrightarrow g(\theta) < \pi_0(0) = 0 \). Let us define \( \bar{\theta} \) such that \( \bar{\theta} = h(0) \Leftrightarrow g(\bar{\theta}) = 0 \Leftrightarrow (r + \lambda)V(\bar{\theta}) - \lambda V(\bar{\theta} + u) = 0 \). Given that \( g \) is an increasing function with \( g(0) < 0 \), \( \lim_{\theta \to +\infty} g(\theta) = +\infty \), we can guarantee that \( \bar{\theta} \) exists and it is unique.

By Propositions 3.3 and 3.4 from Oksendal and Sulem (Oksendal and Sulem (2007)), we know that \( U \subseteq C \).

In particular, as \( h \) is an increasing function of \( x \), we notice that for \( 0 < \theta < \bar{\theta} \) we are always in the continuation region.

From the HJB Equation (3), we know that the value function in the stopping region is \( F(\theta, x) = V(\theta) \), whereas in the continuation region it satisfies the following differential equation

\[ (r + \lambda)F(\theta, x) = \pi_0(x) + \alpha x \frac{\partial F(\theta, x)}{\partial x} + \lambda F(\theta + u, x), \]

which is equivalent to

\[ \frac{\partial F(\theta, x)}{\partial x} - \frac{(r + \lambda)}{\alpha x} F(\theta, x) = -\frac{\lambda}{\alpha x} F(\theta + u, x) - \frac{z_0}{\alpha}. \]  \hspace{1cm} (28)

The equation (28) is of the form \( \frac{\partial y(x)}{\partial x} + P(x)y(x) = R(x) \), and it has a solution \( y(x) = \frac{1}{u(x)} \int u(x)R(x)dx \), where \( u(x) = e^{\int P(x)dx} \). Then the solution of (28) can be written as

\[ F(\theta, x) = x^{r + \lambda} \int x^{r + \lambda - \frac{\lambda}{\alpha}} \left( \frac{\lambda}{\alpha x} F(\theta + u, x) - \frac{z_0}{\alpha} \right) dx. \]  \hspace{1cm} (29)
The value of $F(\theta + u, x)$ depends on which set $(\theta + u, x)$ belongs to, which in its turn depends on which set $(\theta, x)$ belongs to. This implies that we need to split the continuation region in different subsets.

Let $C_n$ denote the subset of the continuation region where stopping is optimal after $n$ jumps in $\theta$, i.e. if $(\theta, x) \in C_n$ then $(\theta + nu, x) \in S$. Further, in region $C_n$ we denote the value function by $f_n(\theta, x)$. Thus

$$F(\theta, x) = f_n(\theta, x)\chi_{\{(\theta, x)\in C_n\}} + V(\theta)\chi_{\{(\theta, x)\in S\}}.$$  

In this problem the threshold is not only one point but a boundary $\theta^*(x)$, for $x > 0$. At the boundary between continuation and stopping regions, value matching and smooth pasting on $x$ should hold. In order to make the notation clearer, we will write the threshold in a different way. The boundary can also be expressed as a function of $\theta$, i.e. $\theta = \theta^*(x) \Leftrightarrow x = \theta^{*-1}(\theta) = b(\theta)$. Hence, from now on we consider that the boundary between the continuation and stopping regions is given by $x = b(\theta)$, for $\theta \geq \bar{\theta}$.

Using the new notation and taking into account the $U$ definition, we can define the stopping region and the subsets of the continuation region as follows:

$$S = \{ (\theta, x) \in \mathbb{R}^+ \times \mathbb{R}^+ : 0 < x \leq b(\theta) \}.$$  

and, for $n \in \mathbb{N}$,

$$C_n = \{ (\theta, x) \in \mathbb{R}^+ \times \mathbb{R}^+ : b(\theta + (n - 1)u) < x \leq b(\theta + nu) \}.$$  

We can also write the value matching and smooth pasting conditions as follows:

$$f_1(\theta, b(\theta)) = V(\theta) \quad \text{and} \quad \frac{\partial f_1(\theta, x)}{\partial x} \bigg|_{x=b(\theta)} = \frac{\partial V(\theta)}{\partial x} \bigg|_{x=b(\theta)} = 0 : (30)$$

Moreover, value matching between the different functions on the continuation region should also hold, i.e., for $n \in \mathbb{N}$ and $\theta \geq \bar{\theta} - nu$,

$$f_n(\theta, b(\theta, \theta + nu)) = f_{n+1}(\theta, b(\theta, \theta + nu)). \quad (31)$$

The first conditions in (30) and condition (31) imply that function $F$ is continuous everywhere.

Before we move forward, we analyse the limit case $x = 0$, this means that there is no declining market but only the option to invest in a new technology, which is a well known problem. Taking into account Proposition 4.2 of Pimentel (2018), given that $g$ is an increasing function with only one zero at $\bar{\theta}$, we

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10We already showed that for all $x$, if $0 < \theta < \bar{\theta}$ we are always in the continuation region.
conclude that the value function, for $0 < \theta < \bar{\theta}$, is 

$$V(\theta) = \left(\frac{\lambda}{r+\lambda}\right)^n V(\theta + nu),$$

where $n(\theta) = \lceil \bar{\theta} - \theta \rceil$ with, for $k \geq 0$, $[k] = \min \{n \in \mathbb{N} : n \geq k\}$. Therefore,

$$\lim_{x \to 0^+} f_n(\theta, x) = \left(\frac{\lambda}{r+\lambda}\right)^n V(\theta + nu). \quad (32)$$

Let us consider $(\theta, x) \in C_1$, then $(\theta + u, x) \in S$, meaning that $F(\theta + u, x) = V(\theta + u)$. From (29) we get

$$f_1(\theta, x) = \frac{\lambda}{r+\lambda} \int x - \frac{z_0}{r+\lambda} \left( -\frac{\lambda}{\alpha x} V(\theta + u) - \frac{z_0}{\alpha} \right) \, dx$$

$$= \frac{\lambda}{r+\lambda} V(\theta + u) + \frac{z_0 x}{r - \alpha} \left[ 1 - \frac{\lambda}{r+\lambda - \alpha} \right] + C_1(\theta) x \frac{r+\lambda}{r - \alpha}, \quad (33)$$

where $C_1(\theta)$ needs to be determined.

For $\theta > \bar{\theta}$, from the conditions presented in (30), we have

$$\frac{\lambda}{r+\lambda} V(\theta + u) + \frac{z_0 b(\theta)}{r - \alpha} \left[ 1 - \frac{\lambda}{r+\lambda - \alpha} \right] + C_1(\theta) b(\theta) \frac{r+\lambda}{r - \alpha} = V(\theta), \quad (34)$$

$$\frac{z_0}{r - \alpha} \left[ 1 - \frac{\lambda}{r+\lambda - \alpha} \right] + \frac{r+\lambda}{\alpha} C_1(\theta) b(\theta) \frac{r+\lambda}{r - \alpha} - 1 = 0. \quad (35)$$

Solving the system of equations (34) and (35), we can derive the threshold curve:

$$\{(\theta, x) \in \mathbb{R} \times \mathbb{R} : \theta > \bar{\theta} \land x = b(\theta)\}, \text{ i.e., for } \theta > \bar{\theta},$$

$$C_1(\theta) = \left[ \frac{1}{r+\lambda} - \frac{1}{r+\lambda - \alpha} \right] z_0 b(\theta) \frac{r+\lambda}{r - \alpha} + 1, \quad (36)$$

$$b(\theta) = \frac{1}{z_0} \left[ (r+\lambda)V(\theta) - \lambda V(\theta + u) \right].$$

Note that $b(\theta) = \frac{1}{z_0} y(\theta)$, implying that $b(\theta) = 0$ and $b$ increases in $\theta$, which finishes the proof.

\[\square\]

### A.2 Proof of Proposition 2

We start deriving $f_1$. Then, using similar arguments, we proceed to derive $f_2$ and $f_3$. We consider that it helps to understand better the proof. The reasoning used here is similar to the one developed in Chapter 3 of Pimentel (2018).

Regarding $f_1$, we notice that, for $\theta > \bar{\theta}$, $f_1$ is given by (33), with $C_1(\theta)$ satisfying (36). For $\theta - u < \theta \leq \bar{\theta}$, from the definition of $f_1$ and $f_2$ and condition (32), we obtain that

$$\lim_{x \to 0^+} f_1(\theta, x) = \left(\frac{\lambda}{r+\lambda}\right) V(\theta + u) \Leftrightarrow C_1(\theta) \lim_{x \to 0^+} x \frac{r+\lambda}{\alpha} = 0.$$
Given that $\alpha < 0$, we have $\lim_{x \to 0^+} x^{\frac{r+1}{\alpha}} = +\infty$, implying that, for $\bar{\theta} - u < \theta \leq \bar{\theta}$, $C_1(\theta) = 0$.

Summarizing, for $(\theta, x) \in \mathcal{C}_1$, $f_1$ is given by

$$
\lambda \frac{1}{r+\lambda} V(\theta + u) + \frac{z_0 x}{r - \alpha} \left[ 1 - \frac{\lambda}{r + \lambda - \alpha} \right] + \left\{ z_0 b(\theta) \left[ \frac{\lambda}{b(\theta)} \right]^{\frac{r+1}{\alpha}} \left[ \frac{1}{r + \lambda} - \frac{1}{r + \lambda - \alpha} \right] \right\} \chi_{\{\theta > \bar{\theta}\}}.
$$

Now we move on to the derivation of $f_2$, for which we use arguments similar to the ones presented for $f_1$ (and therefore we omit some remarks). Let us consider $(\theta, x) \in \mathcal{C}_2$, then $(\theta + u, x) \in \mathcal{C}_1$, meaning that $F(\theta + u, x) = f_1(\theta + u, x)$. From (29), we get

$$
f_2(\theta, x) = x^{\frac{r+1}{\alpha}} \int x^{-\frac{r+1}{\alpha}} \left( -\frac{1}{\alpha x} f_1(\theta + u, x) - \frac{z_0}{\alpha} \right) dx
$$

$$
= \left( \frac{\lambda}{\lambda + r} \right)^2 V(\theta + 2u) + \frac{z_0 x}{r - \alpha} \left[ 1 - \left( \frac{\lambda}{r + \lambda - \alpha} \right)^2 \right]
$$

$$
+ \left\{ z_0 b(\theta + u) \left[ \frac{x}{b(\theta + u)} \right]^{\frac{r+1}{\alpha}} \left( -\frac{\lambda}{\alpha} \right) \left[ \frac{1}{r + \lambda} - \frac{1}{r + \lambda - \alpha} \right] \ln x \right\} \chi_{\{\theta > \bar{\theta} - u\}} + C_2(\theta) x^{\frac{r+1}{\alpha}},
$$

where $C_2(\theta)$ needs to be determined.

For $\theta > \bar{\theta} - u$, we obtain from from condition (31) and $f_1(\theta, b(\theta + u)) = f_2(\theta, b(\theta + u))$, that

$$
\frac{\lambda}{\lambda + r} V(\theta + u) + \frac{z_0 b(\theta + u)}{r - \alpha} \left[ 1 - \frac{\lambda}{r + \lambda - \alpha} \right] + \left\{ z_0 b(\theta) \left[ \frac{b(\theta + u)}{b(\theta)} \right]^{\frac{r+1}{\alpha}} \left[ \frac{1}{r + \lambda} - \frac{1}{r + \lambda - \alpha} \right] \right\} \chi_{\{\theta > \bar{\theta}\}}
$$

$$
= \left( \frac{\lambda}{\lambda + r} \right)^2 V(\theta + 2u) + \frac{z_0 b(\theta + u)}{r - \alpha} \left[ 1 - \left( \frac{\lambda}{r + \lambda - \alpha} \right)^2 \right]
$$

$$
+ \left\{ z_0 b(\theta + u) \left( -\frac{\lambda}{\alpha} \right) \left[ \frac{1}{r + \lambda} - \frac{1}{r + \lambda - \alpha} \right] \ln b(\theta + u) \right\} \chi_{\{\theta > \bar{\theta} - u\}} + C_2(\theta) b(\theta + u)^{\frac{r+1}{\alpha}},
$$

which implies that

$$
C_2(\theta) = \left\{ z_0 b(\theta)^{-\frac{r+1}{\alpha}} \left[ \frac{1}{r + \lambda} - \frac{1}{r + \lambda - \alpha} \right] \right\} \chi_{\{\theta > \bar{\theta}\}} +
$$

$$
\left\{ z_0 b(\theta + u)^{-\frac{r+1}{\alpha}} \left[ \lambda \left[ \frac{1}{(r+\lambda)^2} - \frac{1}{(r+\lambda-\alpha)^2} \right] + \frac{\lambda}{\alpha} \left[ \frac{1}{r + \lambda} - \frac{1}{r + \lambda - \alpha} \right] \ln (b(\theta + u)) \right] \right\} \chi_{\{\theta > \bar{\theta} - u\}},
$$

where we use the fact that $(r + \lambda)V(\theta) - \lambda V(\theta + u) = z_0 b(\theta)$, for $\theta \geq \bar{\theta}$.

For $\bar{\theta} - 2u < \theta \leq \bar{\theta} - u$, from the definition of $f_2$ definition and condition (32) results into

$$
\lim_{x \to 0^+} f_2(\theta, x) = \left( \frac{\lambda}{r + \lambda} \right)^2 V(\theta + 2u) \leftrightarrow C_2(\theta) \lim_{x \to 0^+} x^{\frac{r+1}{\alpha}} = 0.
$$
Hence, we have that $C_2(\theta) = 0$. Thus, for $(\theta, x) \in C_2$, the expression for $f_2$ is given by

$$f_2(\theta, x) = \left(\frac{\lambda}{r + \lambda}\right)^2 V(\theta + 2u) + \frac{z_0 x}{r - \alpha} \left[1 - \left(\frac{\lambda}{r + \lambda - \alpha}\right)^2\right] + \left\{z_0 b(\theta) \left(\frac{x}{b(\theta)}\right)^{1+\lambda} \left[\frac{1}{r + \lambda} - \frac{1}{r + \lambda - \alpha}\right] \right\} \chi\{\theta > \bar{\theta}\} \chi\{\theta > 0\},$$

where $C_3(\theta) = z_0 (r + \lambda) + \frac{z_0 x}{r - \alpha} \frac{1}{r + \lambda - \alpha} \ln x$.

Let us consider $(\theta, x) \in C_3$, then $(\theta + u, x) \in C_2$, meaning that $F(\theta + u, x) = f_2(\theta + u, x)$. From (29) we get

$$f_3(\theta, x) = x^{1+\lambda} \int x^{-\frac{1}{2\alpha}} \left(-\frac{\lambda}{\alpha x} f_2(\theta + u, x) - \frac{z_0}{\alpha}\right) dx$$

$$= \left(\frac{\lambda}{r + \lambda}\right)^3 V(\theta + 3u) + \frac{z_0 x}{r - \alpha} \left[1 - \left(\frac{\lambda}{r + \lambda - \alpha}\right)^3\right]$$

$$+ \left\{z_0 b(\theta + u) \left[\frac{x}{b(\theta + u)}\right]^{1+\lambda} \left(-\frac{\lambda}{\alpha}\right) \left[\frac{1}{r + \lambda} - \frac{1}{r + \lambda - \alpha}\right] \ln x \right\} \chi\{\theta > \bar{\theta} - u\}$$

$$+ \left\{z_0 b(\theta + 2u) \left[\frac{x}{b(\theta + 2u)}\right]^{1+\lambda} \left(-\frac{\lambda}{\alpha}\right)^2 \left[\frac{1}{(r + \lambda)^2} - \frac{1}{(r + \lambda - \alpha)^2}\right] \ln x \right\} \chi\{\theta > 2\bar{\theta} - 2u\}$$

$$+ \frac{\lambda^2}{2\alpha^2} \left[\frac{1}{r + \lambda} - \frac{1}{r + \lambda - \alpha}\right] \left[(\ln x)^2 - 2 \ln b(\theta + 2u) \ln x \right] \chi\{\theta > 2\bar{\theta} - 2u\} + C_3(\theta)x^{1+\lambda},$$

where $C_3(\theta)$ need to be determined.
For $\theta > \bar{\theta} - 2u$, we obtain from condition (31) and $f_2(\theta, b(\theta + 2u)) = f_3(\theta, b(\theta + 2u))$ that

$$
\left(\frac{\lambda}{r + \lambda}\right)^2 V(\theta + 2u) + \frac{z_0 b(\theta + 2u)}{r - \alpha} \left[1 - \left(\frac{\lambda}{r + \lambda - \alpha}\right)^2\right]
+ \left\{z_0 b(\theta) \left[\frac{b(\theta + 2u)}{b(\theta)}\right]^{\frac{\lambda}{r + \lambda}} \left[\frac{1}{r + \lambda} - \frac{1}{r + \lambda - \alpha}\right]\right\} \chi_{\{\theta > \bar{\theta}\}}
+ \left\{z_0 b(\theta + u) \left[\frac{b(\theta + 2u)}{b(\theta + u)}\right]^{\frac{\lambda}{r + \lambda}} \left[\frac{1}{r + \lambda} - \frac{1}{r + \lambda - \alpha}\right] \ln b(\theta + 2u)\right\} \chi_{\{\theta > \bar{\theta} - u\}}
\left(\frac{\lambda}{r + \lambda}\right)^3 V(\theta + 3u) + \frac{z_0 b(\theta + 2u)}{r - \alpha} \left[1 - \left(\frac{\lambda}{r + \lambda - \alpha}\right)^3\right]
+ \left\{z_0 b(\theta + u) \left[\frac{b(\theta + 2u)}{b(\theta + u)}\right]^{\frac{\lambda}{r + \lambda}} \left[-\frac{\lambda}{\alpha} \left[\frac{1}{(r + \lambda)^2} - \frac{1}{(r + \lambda - \alpha)^2}\right] \ln b(\theta + 2u)\right\} \chi_{\{\theta > \bar{\theta} - u\}}
+ \left\{z_0 b(\theta + 2u) \left[-\frac{\lambda^2}{2\alpha^2} \left[\frac{1}{(r + \lambda)^2} - \frac{1}{(r + \lambda - \alpha)^2}\right] \ln b(\theta + 2u)\right\} \chi_{\{\theta > \bar{\theta} - 2u\}}
C_3(\theta) b(\theta + 2u)^{\frac{\lambda}{r + \lambda}}.
$$

so that

$$
C_3(\theta) = \left\{z_0 b(\theta)^{\frac{-\lambda}{r + \lambda} + 1} \left[\frac{1}{r + \lambda} - \frac{1}{r + \lambda - \alpha}\right] \chi_{\{\theta > \bar{\theta}\}} + \left\{z_0 b(\theta + u)^{\frac{-\lambda}{r + \lambda} + 1} \left[\frac{1}{(r + \lambda)^2} - \frac{1}{(r + \lambda - \alpha)^2}\right] \chi_{\{\theta > \bar{\theta} - u\}}
+ \left\{z_0 b(\theta + 2u)^{\frac{-\lambda}{r + \lambda} + 1} \left[\frac{1}{(r + \lambda)^3} - \frac{1}{(r + \lambda - \alpha)^3}\right] \chi_{\{\theta > \bar{\theta} - 2u\}}\right\} \chi_{\{\theta > \bar{\theta} - 2u\}} + \frac{\lambda}{\alpha} \left[\frac{1}{r + \lambda} - \frac{1}{r + \lambda - \alpha}\right] \ln b(\theta + 2u) + \frac{\lambda^2}{2\alpha^2} \left[\frac{1}{r + \lambda} - \frac{1}{r + \lambda - \alpha}\right] \ln b(\theta + 2u)\right\} \chi_{\{\theta > \bar{\theta} - 2u\}}
\left(\frac{\lambda}{r + \lambda}\right)^3 V(\theta + 3u) \equiv C_3(\theta) \lim_{x \to 0^+} x^{\frac{\lambda}{r + \alpha}} = 0.
$$

For $\bar{\theta} - 3u < \theta \leq \bar{\theta} - 2u$, we get from the definition of $f_3$ and condition (32) that

$$
\lim_{x \to 0^+} f_3(\theta, x) = \left(\frac{\lambda}{r + \lambda}\right)^3 V(\theta + 3u) \equiv C_3(\theta) \lim_{x \to 0^+} x^{\frac{\lambda}{r + \alpha}} = 0.
$$

This implies that $C_3(\theta) = 0$. 

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For \((\theta, x) \in C_3\), the expression for \(f_3\) is given by

\[
f_3(\theta, x) = \left(\frac{\lambda}{r + \lambda}\right)^3 V(\theta + 3u) + \frac{z_0 x}{r - \alpha} \left[1 - \left(\frac{\lambda}{r + \lambda - \alpha}\right)^3\right] + \left\{z_0 b(\theta) \left[\frac{\theta}{b(\theta)}\right] \left(\frac{1}{r + \lambda} - \frac{1}{r + \lambda - \alpha}\right)\right\} \chi_{\{\theta > \bar{\theta}\}}
\]

\[
+ \left\{z_0 b(\theta + u) \left[\frac{x}{b(\theta + u)}\right] \left[\frac{\lambda}{(r + \lambda)^2} - \frac{1}{(r + \lambda - \alpha)^2}\right]\right\} \chi_{\{\theta > \bar{\theta} - u\}}
\]

\[
- \frac{\lambda}{\alpha} \left[\frac{1}{r + \lambda} - \frac{1}{r + \lambda - \alpha}\right] \ln \left[\frac{x}{b(\theta + u)}\right]\right\} \chi_{\{\theta > \bar{\theta} - 2u\}}.
\]

Now we are able to understand that (7) is a general expression for \(f_1, f_2\) and \(f_3\).

Finally, we are ready to prove that expression (7) is indeed the value function. In order to prove so, we need to prove that it is the solution of the optimal stopping problem (1), i.e., we need to show that

\[
rV(\theta) - \mathcal{L}V(\theta) \geq 0 \quad \forall \ (\theta, x) \in S.
\]

and

\[
rF(\theta, x) - \mathcal{L}F(\theta, x) = 0 \quad \forall \ (\theta, x) \in C.
\]

Let us now consider \((\theta, x) \in S\), then \(F(\theta, x) = V(\theta)\) and \(F(\theta + u, x) = V(\theta + u)\). So,

\[
rV(\theta) - \mathcal{L}V(\theta) = (r + \lambda)V(\theta) - \lambda V(\theta + u) - z_0 x = z_0 (b(\theta) - x),
\]

which is non-negative, given that in the stopping region we always have \(0 < x \leq b(\theta)\). Thus, condition (37) is already verified.

Let us consider \((\theta, x) \in C\) such that \(n(\theta, x) = n\) and \(n(\theta + u, x) = n - 1\), with \(n \in \mathbb{N}\). We start realizing that

\[
rF(\theta, x) - \mathcal{L}F(\theta, x) = 0 \iff \alpha x \frac{\partial F(\theta, x)}{\partial x} - (r + \lambda)F(\theta, x) + z_0 x = -\lambda F(\theta + u, x).
\]

\(^{11}\)We are considering that \(n = 0\) represents the stopping region.
After some calculations we obtain that $\alpha x \frac{\partial F(\theta,x)}{\partial x}$ is equivalent to

$$z_0 x \frac{\alpha}{r - \alpha} \left[ 1 - \left( \frac{\lambda}{r + \lambda - \alpha} \right)^n \right] + (r + \lambda) z_0 x \frac{\alpha}{r - \alpha} \sum_{k=0}^{n-1} \left\{ b(\theta + ku) - \frac{\alpha}{\lambda} k + 1 \lambda x \right\} =$$

$$\sum_{m=0}^{k} \frac{1}{m! (-\alpha)^m} \left[ \frac{1}{(r + \lambda)^{k-m+1}} - \frac{1}{(r + \lambda - \alpha)^{k-m+1}} \right] \left[ \ln \left[ \frac{x}{b(\theta + ku)} \right] \right]^{m-1} \chi\{\theta > \theta - ku\}$$

Based on previous calculations, we note that

$$\alpha x \frac{\partial F(\theta,x)}{\partial x} - (r + \lambda) F(\theta,x) + z_0 x =$$

$$- (r + \lambda) \left( \frac{\lambda}{r + \lambda} \right)^n V(\theta + nu) - \frac{z_0 x}{r - \alpha} \left\{ (r + \lambda - \alpha) \left[ 1 - \left( \frac{\lambda}{r + \lambda - \alpha} \right)^n \right] - (r - \alpha) \right\}$$

$$- z_0 x \frac{\alpha}{r - \alpha} \sum_{k=1}^{n-1} \left\{ b(\theta + ku) - \frac{\alpha}{\lambda} k + 1 \lambda x \right\} \sum_{m=1}^{k} \frac{1}{(m-1)! (-\alpha)^{m-1} \left[ (r + \lambda)^{k-m+1} - (r + \lambda - \alpha)^{k-m+1} \right] \left[ \ln \left[ \frac{x}{b(\theta + ku)} \right] \right]^{m-1} \chi\{\theta > \theta - ku\}.}$$

Changing the variable in the first sum and simplifying the second term of the previous expression, we get

$$- \lambda \left( \frac{\lambda}{r + \lambda} \right)^{n-1} V(\theta + nu) - \lambda \frac{z_0 x}{r - \alpha} \left[ 1 - \left( \frac{\lambda}{r + \lambda - \alpha} \right)^{n-1} \right]$$

$$- z_0 x \frac{\alpha}{r - \alpha} \sum_{k=0}^{n-2} \left\{ b(\theta + (k + 1)u) - \frac{\alpha}{\lambda} k + 1 \lambda x \right\} \sum_{m=1}^{k+1} \frac{1}{(m-1)! (-\alpha)^{m-1} \left[ (r + \lambda)^{k-m+2} - (r + \lambda - \alpha)^{k-m+2} \right] \left[ \ln \left[ \frac{x}{b(\theta + (k + 1)u)} \right] \right]^{m-1} \chi\{\theta > \theta - (k+1)u\}.}$$

Then, continuing with the simplification of the first two terms of the previous expression and changing the variable in the second sum, we obtain

$$- \lambda \left\{ \left( \frac{\lambda}{r + \lambda} \right)^{n-1} V((\theta + u) + (n - 1)u) + \frac{z_0 x}{r - \alpha} \left[ 1 - \left( \frac{\lambda}{r + \lambda - \alpha} \right)^{n-1} \right] \right\}$$

$$+ z_0 x \frac{\alpha}{r - \alpha} \sum_{k=0}^{n-2} \left\{ b((\theta + u) + ku) - \frac{\alpha}{\lambda} k + 1 \lambda x \right\} \sum_{m=0}^{k} \frac{1}{m! (-\alpha)^m \left[ (r + \lambda)^{k-m+1} - (r + \lambda - \alpha)^{k-m+1} \right] \left[ \ln \left[ \frac{x}{b((\theta + u) + ku)} \right] \right]^{m} \chi\{\theta + u > \theta - ku\}.}$$

which is exactly $- \lambda F(\theta + u, x)$.

Finally, it only remains to prove that $F(\theta, x) \geq V(\theta), \forall (\theta, x) \in C$. 36
Let us consider a fixed $\theta > 0$. From Proposition 4.2 of Pimentel (2018), we obtain that, for $0 < \theta \leq \bar{\theta}$,

$$
\lim_{x \to 0^+} F(\theta, x) = \left( \frac{\lambda}{r + \lambda} \right)^{n(\theta, x)} V(\theta + n(\theta, x)u) \geq V(\theta).
$$

Moreover, for $\theta > \bar{\theta}$, we have $F(\theta, b(\theta)) = V(\theta)$. If we prove that $F$ is an increasing function on $x$, we conclude that $F(\theta, x) > V(\theta)$ for all $(\theta, x) \in \mathcal{C}$.

We start proving that, for a fixed $\theta > 0$, when $(\theta, x) \in \mathcal{C}$, $f_n(\theta, x)$ increases as a function of $x$, for all $n \in \mathbb{N}$. For that, we use an induction argument. Thus, consider initially $f_1$, which is given by expression

$$
\frac{\lambda}{r + \lambda} V(\theta + u) + \frac{z_0 x}{r - \alpha} \left[ 1 - \frac{\lambda}{r + \lambda - \alpha} \right] + \left\{ z_0 b(\theta) \left[ \frac{x}{b(\theta)} \right]^{\frac{r + \lambda}{\alpha}} - \frac{1}{r + \lambda - \alpha} \right\} \lambda \{ \theta > \theta \}.
$$

If $0 < \theta \leq \bar{\theta}$, since $V$ is an increasing function of $\theta$, it is straightforward that $f_1$ increases with $x$. For $\theta > \bar{\theta}$, to prove the monotony in $x$ we need to use the first derivative, i.e.

$$
\frac{\partial f_1(\theta, x)}{\partial x} = \frac{z_0}{r - \alpha} \left[ 1 - \frac{\lambda}{r + \lambda - \alpha} \right] + \left\{ z_0 b(\theta) \left[ \frac{x}{b(\theta)} \right]^{\frac{r + \lambda}{\alpha}} - \frac{1}{r + \lambda - \alpha} \right\} \lambda \{ \theta > \theta \}.
$$

We notice that, for $\theta > \bar{\theta}$, if $(\theta, x) \in \mathcal{C}$ this implies that $x > b(\theta)$, thus $\frac{\partial f_1(\theta, x)}{\partial x} > 0$. Then, for $(\theta, x) \in \mathcal{C}$, the function $f_1$ is always increasing in $x$.

Now, let us consider the function $f_{n+1}$ which, by (7), is given by

$$
f_{n+1}(\theta, x) = \left( \frac{\lambda}{r + \lambda} \right)^{n+1} V(\theta + (n+1)u) - \frac{z_0 x}{r - \alpha} \left[ 1 - \left( \frac{\lambda}{r + \lambda - \alpha} \right)^{n+1} \right] + z_0 x \frac{\frac{r + \lambda}{\alpha}}{\lambda} \sum_{k=0}^{n} b(\theta + ku) - \frac{\frac{r + \lambda}{\alpha}}{\lambda} + \sum_{m=0}^{k} \frac{1}{m! (\alpha)^m} \left[ \frac{x}{(r + \lambda)^{k-m+1}} \ln \left[ \frac{x}{b(\theta + ku)} \right] \right] m \chi \{ \theta > \theta - ku \}.
$$

Similar to the case $n = 1$, for $0 < \theta \leq \bar{\theta} - u$, the function $f_n$ increases with $x$. For $\theta > \bar{\theta} - u$ let us define

$$
j_n(\theta, x) = f_{n+1}(\theta, x) - f_n(\theta, x).$$

Doing some comprehensive calculations, we can write the formula as an equality, as follows:

$$
j_n(\theta, x) = z_0 \lambda^n \left\{ \left[ \frac{x}{(r + \lambda - \alpha)^{n+1}} - \frac{(r + \lambda) V(\theta + nu) - \lambda V(\theta + (n+1)u)}{(r + \lambda)^{n+1}} \right] + x \frac{\frac{r + \lambda}{\alpha}}{\lambda} \right\} b(\theta + nu) - \frac{\frac{r + \lambda}{\alpha}}{\lambda} + \sum_{m=0}^{n} \frac{1}{m! (\alpha)^m} \left[ \frac{1}{(r + \lambda)^{n-m+1}} - \frac{1}{(r + \lambda - \alpha)^{n-m+1}} \right] \ln \left[ \frac{x}{b(\theta + nu)} \right] m \chi \{ \theta > \theta - nu \}.
$$

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Given that $\theta > \bar{\theta} - nu$ we have

$$j_n(\theta, x) = z_0 \lambda^n \left\{ \left[ \frac{x}{(r + \lambda - \alpha)^{n+1}} - \frac{b(\theta + nu)}{(r + \lambda)^{n+1}} \right] + x^\frac{r+\lambda}{\alpha} b(\theta + nu)^{-\frac{r+\lambda}{\alpha}+1} \times \right.$$ 

$$\sum_{m=0}^{n} \frac{1}{m! (-\alpha)^m} \left[ \frac{1}{(r + \lambda)^{n-m+1}} - \frac{1}{(r + \lambda - \alpha)^{n-m+1}} \right] \left[ \ln \left[ \frac{x}{b(\theta + nu)} \right] \right]^m \left. \right\}.$$ 

Further, it holds that

$$\frac{\partial j_n(\theta, x)}{\partial x} = z_0 \lambda^n \left\{ \left[ \frac{x}{(r + \lambda - \alpha)^{n+1}} - \frac{b(\theta + nu)}{(r + \lambda)^{n+1}} \right] \right.$$ 

$$- \left( \frac{x}{b(\theta + nu)} \right)^\frac{r+\lambda}{\alpha} \cdot \sum_{m=1}^{n} \frac{1}{m! (-\alpha)^m} \left[ \frac{1}{(r + \lambda - \alpha)^{n-m+1}} \right] \left[ \ln \left[ \frac{x}{b(\theta + nu)} \right] \right]^m \left. \right\}$$

and

$$\frac{\partial^2 j_n(\theta, x)}{\partial x^2} = z_0 \lambda^n b(\theta + nu) \left( \frac{x}{b(\theta + nu)} \right)^\frac{r+\lambda}{\alpha} - 2 \cdot \frac{1}{n! (-\alpha)^{n+1}} \left[ \ln \left[ \frac{x}{b(\theta + nu)} \right] \right]^n.$$  

It is easy to check that $\left. \frac{\partial j_n(\theta, x)}{\partial x} \right|_{x=b(\theta+nu)} = 0$, which, together with the second condition in (30), means that the function $F(\theta, x)$ is smooth in $x$.

For a fixed $\theta > \bar{\theta} - nu$, we only want to study $j_n$ for $x > b(\theta + nu)$. Since $x > b(\theta + nu)$ and $\alpha < 0$, we have $\frac{\partial^2 j_n(\theta, x)}{\partial x^2} > 0$. Thus, $\frac{\partial j_n(\theta, x)}{\partial x}$ increases as a function of $x$, and, given that $\left. \frac{\partial j_n(\theta, x)}{\partial x} \right|_{x=b(\theta+nu)} = 0$, we conclude that $\frac{\partial j_n(\theta, x)}{\partial x}$ is always positive for $x > b(\theta + nu)$. Then $j_n$ also increases as a function of $x$.

Recall that $f_{n+1}(\theta, x) = f_n(\theta, x) + j_n(\theta, x)$ for $n \in \mathbb{N}$. We proved that $f_1$ increases as a function of $x$, for $x > b(\theta)$. Assuming that $f_n$ increases as a function of $x$, for $x > b(\theta + nu)$ (induction assumption), we just proved that $j_n$ also increases as a function of $x$. Then, by mathematical induction, $f_{n+1}$ also increases as a function of $x$, for $x > b(\theta + nu)$.

At this point we know that each function $f_n$ increases with $x$ for a fixed $\theta > 0$. In fact, given that $f_{n+1}(\theta, b(\theta + nu)) = f_n(\theta, b(\theta + nu))$, this also means that the function $F$ increases with $x$ when $(\theta, x) \in \mathcal{C}$, which concludes the proof.

\[\blacksquare\]
A.3 Proof of Proposition 3

For $\theta > \bar{\theta}$, straightforward differentiation gives

\[
\frac{\partial b(\theta)}{\partial z_1} = \frac{(r + \lambda)\theta^\beta - \lambda(\theta + u)^\beta}{z_0r} > 0,
\]

\[
\frac{\partial b(\theta)}{\partial \lambda} = \frac{z_1 \left[ \theta^\beta - (\theta + u)^\beta \right]}{z_0r} < 0,
\]

\[
\frac{\partial b(\theta)}{\partial z_0} = -\frac{1}{z_0^2} \left[ \frac{z_1 (r + \lambda)\theta^\beta - \lambda(\theta + u)^3}{r} - rI \right] < 0,
\]

\[
\frac{\partial b(\theta)}{\partial I} = -\frac{r}{z_0} < 0,
\]

\[
\frac{\partial b(\theta)}{\partial u} = -\frac{\lambda z_1(\theta + u)^{3-1}}{z_0r} < 0.
\]

A.4 Proof of Proposition 4

For $\theta > \bar{\theta}$, straightforward differentiation gives

\[
\frac{\partial b(\theta)}{\partial \mu} = \frac{\beta z_1 \left[ (r + \lambda)\theta^\beta - \lambda(\theta + u)^\beta \right]}{[z_0r - \mu \beta + \frac{1}{2}\beta(1 - \beta)\sigma^2]^3} > 0,
\]

\[
\frac{\partial b(\theta)}{\partial \sigma} = \frac{-\beta(1 - \beta)z_1 \left[ (r + \lambda)\theta^\beta - \lambda(\theta + u)^\beta \right]}{z_0[r - \mu \beta + \frac{1}{2}\beta(1 - \beta)\sigma^2]^3} < 0.
\]

The result for the other parameters is analogous to the proof of Proposition 3.

A.5 Restrictions in the stochastic processes for add strategy

In case the optimal strategy for the firm is to produce both products simultaneously, there are some restrictions on price and technology level in order to assure that both prices, $p_{0}^{A}$ and $p_{1}^{A}$, are positive. In particular, we need to assure that\(^{12}\)

$$\theta < \frac{2z_0}{\kappa} \equiv \theta^{up} \quad \text{and} \quad x < \frac{2z_1}{\kappa} \equiv x^{up}.$$

Moreover, for large values of $x$ the firm would never invest and keep both products alive. This is because the profit of the original product is larger than the profit the firm would receive from both products, due to

\(^{12}\)We consider $\gamma, \eta, q_0, q_1 \in \mathbb{R}^+$ and, in order to ensure that $z_0 > 0$ and $z_1 > 0$, we assume that $\gamma < \min \left\{ \frac{1}{q_0}, \frac{1}{q_1} \right\}$.  

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the competition effect.

Straightforward calculus shows us that the optimal stopping time problem defined in (12) can be written as

$$F(\theta, x) = \frac{\pi_0(x)}{r - \alpha} + \sup_{\tau_1} \mathbb{E} \left[ e^{-r\tau_1} \right. V(\theta_{\tau_1}, X_{\tau_1}) \left. - \frac{\pi_0(X_{\tau_1})}{r - \alpha} \right] \chi_{(\tau_1 < +\infty)} \bigg| \theta_0 = \theta, X_0 = x \bigg].$$

Evidently, the firm will never invest for values $\theta, x$ such that $V(\theta, x) - \frac{\pi_0(x)}{r - \alpha} \leq 0$. If upon investment the firm decides to keep both products alive, then $V(\theta, x) - \frac{\pi_0(x)}{r - \alpha} = \frac{\pi^R_1(\theta)}{r} - \frac{\theta x}{r - \alpha} - I$. Thus, the firm will never decide to add the new product for $V(\theta, x) - \frac{\pi_0(x)}{r - \alpha} \leq 0$ if $x \geq \nu(\theta)$, with $\nu(\theta) = \frac{r - \alpha}{\kappa^R} \left[ \frac{\pi^R_1(\theta)}{r} - I \right]$. In particular, given that $\nu$ is an increasing function of $\theta$, with $\lim_{\theta \to +\infty} \nu(\theta) = \frac{r - \alpha}{\kappa^R}$, then, for all values of $\theta$, the firm will never invest and keep both products alive if the following condition holds:

$$x \geq \frac{r - \alpha z_1}{r - k}.$$  \hspace{1cm} (38)

### A.6 Proof of Proposition 5

Recalling the optimal stopping problem defined in (10) and taking into account (11) we have

$$\int_{\tau_1}^{\tau_2} \pi^A_1(\theta_{\tau_1}, X_s) e^{-rs} ds = \int_{\tau_1}^{\tau_2} [z_0 - \kappa \theta_{\tau_1}] X_s e^{-rs} ds + \int_{\tau_1}^{\tau_2} \pi^R_1(\theta_{\tau_1}) e^{-rs} ds. \hspace{1cm} (39)$$

which allows us to rewrite (10) as

$$F(\theta, x) = \sup_{\tau_1} \mathbb{E} \left[ \int_0^{\tau_1} \pi_0(X_s) e^{-rs} ds + \left\{ \int_{\tau_1}^{+\infty} \pi^R_1(\theta_{\tau_1}) e^{-rs} ds - 1 e^{-r\tau_1} + \right. \right. \sup_{\tau_1 \succ \tau_2 \geq \tau_1} \left. \left. \left\{ \int_{\tau_1}^{\tau_2} [z_0 - \kappa \theta_{\tau_1}] X_s e^{-rs} ds \right\} \chi_{(\tau_1, \tau_2)} \bigg| \theta_0 = \theta, X_0 = x \bigg] \right].$$

Given that we want to investigate when it is optimal to replace the old product, we need to analyze first the optimal stopping problem w.r.t. $\tau_2$, i.e., we need to solve

$$G(\theta_{\tau_1}, X_{\tau_1}) = \sup_{\tau_2 \succ \tau_2 \geq \tau_1} \mathbb{E} \left[ \int_{\tau_1}^{\tau_2} [z_0 - \kappa \theta_{\tau_1}] X_s e^{-rs} ds \bigg| \theta_{\tau_1}, X_{\tau_1} \right].$$

As $X$ is a deterministic process, it follows that

$$\mathbb{E} \left[ \int_{\tau_1}^{\tau_2} [z_0 - \kappa \theta_{\tau_1}] X_s e^{-rs} ds \bigg| \theta_{\tau_1}, X_{\tau_1} = x \right] = \frac{[z_0 - \kappa \theta_{\tau_1}] x}{r - \alpha} \left[ e^{-(r - \alpha) \tau_2} - e^{-(r - \alpha) \tau_1} \right], \quad \tau_2 \geq \tau_1.$$

Let us consider $g(t) = \frac{[z_0 - \kappa \theta_{\tau_1}] x}{r - \alpha} [e^{-(r - \alpha) \tau_2} - e^{-(r - \alpha) \tau_1}]$, with $t \geq \tau_1$, and $\hat{\theta} = \frac{z_0}{\kappa}$. We have $g(\tau_1) = 0$, and, moreover, $g$ increases when $0 < \theta_{\tau_1} < \hat{\theta}$ and decreases for $\theta_{\tau_1} > \hat{\theta}$. Then either
• for $0 < \theta_1 < \hat{\theta}$, the function $\rho$ is maximized when $\tau_2 = +\infty$, meaning that the firm, upon investing in the innovative product, keeps both products alive\textsuperscript{13}. In this case, $G(\theta_{\tau_1}, X_{\tau_1}) = \left[ z_0 - \kappa \theta_1 \right] X_{\tau_1} e^{-r\tau_1}$;

• for $\theta_1 \geq \hat{\theta}$, the function $\rho$ is maximized for $\tau_2 = \tau_1$, meaning that when the firm invests in the new product, it replaces the old one by the innovative one. Thus $G(\theta_{\tau_1}, X) = 0$.

Therefore, the decision between adding or replacing is purely deterministic, and it depends only on the relationship between the level of the new technology, $\theta_{\tau_1}$, and $\hat{\theta}$.

\textbf{A.7 Proof of Proposition 6}

We define the following set $U = \{ (\theta, x) \in \mathbb{R}^+ \times \mathbb{R}^+ : rV(\theta, x) - [\pi_0(x) + LV(\theta, x)] < 0 \}$, with $U \subseteq \mathcal{C}$ (by Propositions 3.3 and 3.4 from Oksendal and Sulem (2007)). Simple calculations show that the boundary of $U$ is the set of values for $\theta$ and $x$ such that

$$rV(\theta, x) - [\pi_0(x) + LV(\theta, x)] = i(\theta) - j(\theta)x,$$

where $i(\theta) = z_1 \left( \theta - \frac{u\lambda}{r} \right) - rI$ and $j(\theta) = \kappa \left( \theta - \frac{u\lambda}{r - \alpha} \right) + \left[ \frac{(r + \lambda - \alpha)\kappa \theta - \lambda z_0}{r - \alpha} \right] \chi_{\{ \theta < \bar{\theta} \}} + z_0 \chi_{\{ \theta \geq \bar{\theta} \}}$\textsuperscript{14}.

To ease the presentation, we define $d(\theta) = \frac{i(\theta)}{j(\theta)}$, with $d(0) = \frac{r - \alpha}{\kappa} \left[ \frac{\bar{\theta} - u}{u\lambda} \right]$.

Next we prove the monotonicity result for the boundary of $U$. Later, we will establish the result for the actual boundary of $\mathcal{C}$.

• For $0 < \theta < \hat{\theta} - u$, $d$ is an increasing function, with a vertical asymptote at $z = \frac{u\lambda}{r - \alpha} < \hat{\theta}$, and an horizontal asymptote at $\frac{\bar{\theta} - u}{\kappa} > 0$. Then it follows that

$$U = \left\{ (\theta, x) : (0 < \theta < z \land x < d(\theta)) \lor (z < \theta < \hat{\theta} - u, x > \max(0, d(\theta))) \right\}.$$

Furthermore, as for $0 < \theta < \hat{\theta} - u$, the investment decision leads to an add strategy (see Proposition 5), when $x$ is such that $x > \frac{r - \alpha}{\kappa} \frac{\bar{\theta} - u}{\lambda}$, the firm will not invest (according to Condition (38) in Appendix A.5).

\textsuperscript{13}Note that $\hat{\theta} = \frac{\theta_{\text{up}}}{2} < \theta_{\text{opt}}$, meaning that the firm always decides to add for admissible values of $\theta$.

\textsuperscript{14}To be precise we should write $\max\left\{ 0, \hat{\theta} - u \right\}$ instead of only $\hat{\theta} - u$. In the sequel we assume that $\hat{\theta} > u$, solving then the more complete case.
Then \( \{(\theta, x) : 0 < \theta < \bar{\theta} - u \land x > \frac{r - \alpha}{r} \} \) must be contained in the continuation region (although it does not belong to \( U \)).

Thus we conclude that \( C \) contains the following set:

\[
\left\{ (\theta, x) \in \mathbb{R}^+ \times \mathbb{R}^+ : 0 < \theta \leq \bar{\theta} \lor [\bar{\theta} < \theta < \bar{\theta} - u \land x > d(\theta)] \right\}
\]

and the boundary is monotonically increasing with \( \theta \). In case \( \bar{\theta} - u < \theta \), then \( \{(\theta, x) : 0 < \theta < \bar{\theta} - u \} \) is contained in \( C \), with boundary that does not change with \( \theta \).

- if \( \bar{\theta} - u \leq \theta < \bar{\theta} \) then \( d \) has a vertical asymptote at \( \frac{\lambda}{r + \lambda - \alpha} \bar{\theta} \equiv \upsilon \) and an horizontal asymptote at \( \frac{r - \alpha}{r + \lambda - \alpha} \frac{2z}{\kappa} > 0 \). Similar to the previous case, investment leads to add the second product to the first one, and investment occurs only for levels of inverse demand lower than \( \frac{r - \alpha}{r} \frac{2z}{\kappa} \).

But contrary to the last case, the sign of the first order derivative of \( d \) is not the same: if \( \upsilon < \bar{\theta} \) then \( d \) is increasing; otherwise it decreases with \( \theta \). The condition \( \upsilon < \bar{\theta} \) is equivalent to \( u > r \left[ \frac{\bar{\theta}}{r + \lambda - \alpha} - \frac{rI_u}{\lambda} \right] \equiv \phi \). Therefore, \( d \) increases with \( \theta \) for large values of \( u \) and decreases otherwise. Moreover, \( \upsilon < \bar{\theta} \) but the relation between \( \upsilon \) and \( \bar{\theta} - u \) depends on the magnitude of the jumps, as \( \bar{\theta} - u < \upsilon \iff u > \frac{r - \alpha}{r + \lambda - \alpha} \bar{\theta} \equiv \psi_1 \), with \( 0 < \phi_1 < \psi_1 < \bar{\theta} \). Hence, it is essential to study the behavior of \( d \) as a function of the jump magnitude, which we address next.

- If \( 0 < u < \phi_1 < \psi_1 \), it holds that \( \bar{\theta} < \upsilon < \bar{\theta} - u < \bar{\theta} \). Then it means that \( C \) contains the following set:

\[
\left\{ (\theta, x) : \bar{\theta} - u \leq \theta < \bar{\theta} \land x > d(\theta) \right\}
\]

and \( d \) decreases with \( \theta \). Additionally, note that for all \( \theta \in U \), \( d(\theta) < d(\bar{\theta} - u) = \frac{2z}{\kappa} \left[ \frac{\phi}{r + \lambda - \alpha} \left( \psi_1 - u \right) \right] < \frac{2z}{\kappa} < x_{up} \), as \( \psi_1 < \bar{\theta} \), and therefore \( x > x_{up} \).

- If \( \phi_1 < u < \psi_1 \), then \( \upsilon < \bar{\theta} - u < \bar{\theta} \) and therefore \( d \) increases. But there are several possible relations between \( \bar{\theta} \) relative to \( \bar{\theta} - u \) and \( \bar{\theta} \), which lead to different sets that are contained in the continuation region. In fact, there are three possibilities outcomes:

(i) \( \upsilon < \bar{\theta} < \bar{\theta} - u < \bar{\theta} \), and in that case \( \left\{ (\theta, x) : \bar{\theta} - u \leq \theta < \bar{\theta} \land x > d(\theta) \right\} \subseteq C \).

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15 The following conditions are equivalent: \( u > \phi_1 \iff \theta > \frac{\lambda}{r + \lambda - \alpha} \bar{\theta} \equiv \phi \left( \frac{rI_u}{\lambda} + \frac{u\lambda}{r} \right) > \frac{\lambda}{r + \lambda - \alpha} z_0 \).
(ii) $v < \hat{\theta} - u < \hat{\theta} < \hat{\theta}$, and therefore \( \{(\theta, x) : \hat{\theta} - u \leq \theta \leq \hat{\theta} \land x > d(\theta)\} \subseteq C. \)

(iii) $v < \hat{\theta} - u < \hat{\theta} < \hat{\theta}$ then \( \{(\theta, x) : \hat{\theta} - u < \theta < \hat{\theta} \} \subseteq C. \)

Moreover, note that for these cases, as $\lim_{\theta \to +\infty} d(\theta) = r_{r, x}^\alpha$. It follows that $x < d(\hat{\theta}) < \frac{r - \alpha r}{r + \lambda - \alpha} \kappa < z_1^\alpha < x$ up, which shows that these sets are non-empty sets of the state space of the process, with all having increasing boundary.

- Finally, we consider $u > \psi_1 > \phi_1$. In this case $\hat{\theta} - u < v < \hat{\theta}$ and $d$ increases. Also it holds that $\hat{\theta} > v$. Moreover, either $\psi_1 < u < \hat{\theta}$ or $u > \hat{\theta}$, such that

(i) For the case $\psi_1 < u < \hat{\theta}$, using simple arguments we prove that $d(\hat{\theta} - u) > \frac{r - \alpha \kappa}{r}.$

(ii) If $u > \hat{\theta}$, then $d(0) > \frac{r - \alpha \kappa}{r}.$

Then in both cases it means that the lowest value of $d$ for the relevant interval $(\hat{\theta} - u, \hat{\theta})$ (for case (i)) or $(0, \hat{\theta})$ for case (ii) is larger than $\frac{r - \alpha \kappa}{r}.$

Also we may have $\bar{\theta} < \hat{\theta}$ or the other way around. In case $\theta < \hat{\theta}$, it follows that

\[ \{(\theta, x) \in \mathbb{R}^+ \times \mathbb{R}^+ : \max \{0, \hat{\theta} - u\} \leq \theta \leq \hat{\theta} \lor \hat{\theta} < \theta < \hat{\theta} \land x > d(\theta)\} \subseteq C, \]

whereas in case $\hat{\theta} < \theta$

\[ \{(\theta, x) \in \mathbb{R}^+ \times \mathbb{R}^+ : \max \{0, \hat{\theta} - u\} \leq \theta \leq \hat{\theta}\} \subseteq C. \]

- if $\theta \geq \hat{\theta}$ then $j(\theta) = z_0$, and therefore it is trivial to conclude that for the set $U$ it holds that:

(i) If $\hat{\theta} \leq \bar{\theta}$,

\[ U = \{(\theta, x) \in \mathbb{R}^+ \times \mathbb{R}^+ : \hat{\theta} \leq \theta \leq \hat{\theta} \lor \hat{\theta} > \theta \land x > d(\theta)\}. \]

(ii) If $\hat{\theta} > \bar{\theta}$,

\[ U = \{(\theta, x) \in \mathbb{R}^+ \times \mathbb{R}^+ : \theta \geq \hat{\theta} \land x > d(\theta)\}. \]

Joining all the previous cases, we realize that it is possible to write a $W$ set, with $U \subseteq W \subseteq C$, as follows:

\[ W = \{(\theta, x) \in \mathbb{R}^+ \times \mathbb{R}^+ : 0 < \theta \leq \hat{\theta} \lor \hat{\theta} > \theta \land x > d(\theta)\}. \]

Looking at the $W$ set we realize that for $\theta \leq \hat{\theta}$, the pairs $(\theta, x)$ are all contained in the continuation region. Furthermore, for $\theta > \hat{\theta}$, the number of branches that the function $d$ has, depends on the position of
relative to $\hat{\theta} - u$ and $\tilde{\theta}$. After some calculations, we conclude that $
abla \theta < \hat{\theta} - u \Leftrightarrow u < \frac{\hat{\theta} - \tilde{\theta}}{r + \lambda} \equiv \phi_2$ and $
abla \theta < \hat{\theta} \Leftrightarrow u < \frac{\hat{\theta} - \tilde{\theta}}{r + \lambda} \equiv \phi_3$, with $\phi_1 < \phi_2 < \phi_3$. In a nutshell, we have:

- If $0 < u < \phi_2$ the function $d$ has three branches, the first two leading to a add region and the last one leading to a replace region. The first and the last branches increase with $\theta$, but the behavior of the second one depends on $u$:
  - If $0 < u < \phi_1$, the second $d$ decreases.
  - If $\phi_1 < u < \phi_2$, the function $d$ increases.

- If $\phi_2 < u < \phi_3$ the function $d$ increases for all values of $\theta$, and it has two branches, the first leading to an add region and the second leading to a replace region.

- If $u > \phi_3$ the function $d$ increases for all values of $\theta$, and it has only one branch, which leads to a replace region. This means that in this case adding is never an optimal decision.

We conclude that, if $u > \phi_1$, which is equivalent to $\kappa \left( \frac{\tilde{\theta}}{z_1} + \frac{\alpha \lambda}{r} \right) > \frac{\lambda}{r + \lambda - \alpha} z_0$, the function $d$ is defined in three different branches, and increases for all values of $\theta$.

We end the proof of this proposition with the following remark: at this point, formally, we cannot yet conclude that $W = C$ and that the threshold curve increases. But this will be finally proved in the proof of Proposition 7. Furthermore, in the proof of Proposition 8 we show that, when the previous condition does not hold, $W \subset C$ and the threshold curve has a decreasing branch.

A.8 Proof of Proposition 7

In this proposition we assume that $u > \phi_1$. Without loss of generality, we also assume that $\phi_1 < u < \phi_2$. The objective is only to guarantee that the function $d$, defined in the proof of the Proposition 6, has three different branches.

Using the same reasoning as we have used in the proof of Proposition 1 (as the HJB equation is the same), we end up with the same equation (29), i.e.,

$$F(\theta, x) = x^{-\frac{\alpha \lambda}{\alpha}} \int x^{-\frac{\alpha \lambda}{\alpha}} \left( -\frac{\lambda}{\alpha x} F(\theta + u, x) - \frac{z_0}{\alpha} \right) dx.$$  

(40)
As before, the value of \( F(\theta + u, x) \) depends on where the set \((\theta + u, x)\) belongs to, which in turn depends on where the set \((\theta, x)\) belongs to. Following the same notation used for the replace case, let \( C_1 \) denote the subset of the continuation region where stopping is optimal after one jump in \( \theta \), i.e. if \((\theta, x) \in C_1 \) then \((\theta + u, x) \in S \). Contrary to the case where the firm only has an option to replace, it holds when the firm invests, it can either add the new product to the portfolio (and in this case we denote its value function by \( V^A \)), or replace the old product by the new one (and we denote its value function by \( V^R \)). Furthermore, the decision depends on whether there is a decline in \( x \), hitting the boundary, if there is a jump in the technology level, passing the boundary. Thus, we split \( C_1 \) in three different sets: \( C_1 = \Omega_A^1 \cup \Omega_{AR}^1 \cup \Omega_R^1 \). Here \( \Omega_A^1 \) and \( \Omega_R^1 \) are the sets of values of the state space such that any movement (in \( x \) or \( \theta \)) will always lead to the add and replace region, respectively, whereas \( \Omega_{AR}^1 \) is the set of values where either the add region can be reached by a decline in \( x \), or the replace region can be reached by a jump in \( \theta \). The value functions for states \((\theta, x)\) in \( \Omega_A^1, \Omega_{AR}^1 \) and \( \Omega_R^1 \) will be denoted by \( f_A^1 \), \( f_{AR}^1 \) and \( f_R^1 \), respectively.

Following the same reasoning notation-wise, we propose that the threshold curve is given by 16

\[
b(\theta) = b^A(\theta)\chi_{\{\theta \leq \theta < \theta^* - u\}} + b^{AR}(\theta)\chi_{\{\theta^* - u \leq \theta < \theta^*\}} + b^R(\theta)\chi_{\{\theta \geq \theta^*\}},
\]

where \( b^A \), \( b^{AR} \) and \( b^R \) need to be derived yet.

We can write the \( C_1 \) as follows:

\[
C_1 = \{(\theta, x) \in \mathbb{R}^+ \times \mathbb{R}^+: b(\theta) < x \leq b(\theta + u)\}. \tag{41}
\]

So next we consider the different scenarios for the state \((\theta, x) \in C_1 \) in order to derive the corresponding value functions.

(i) For \((\theta, x) \in \Omega_1^R \) or \((\theta, x) \in \Omega_{AR}^1 \), the process will hit the stopping region after the next jump, and the decision will be to replace the old product by the new one. Therefore, \( f_R^1 \) and \( f_A^1 \) satisfy the following equation:

\[
f(\theta, x) = x^{-\frac{\lambda}{\alpha}} \int x^{-\frac{\lambda}{\alpha}} \left( -\frac{\lambda}{\alpha} \frac{V^R(\theta + u)}{x} - \frac{z_0}{\alpha} \right) dx. \tag{42}
\]

However the value matching and smooth pasting conditions are different:

\[
f_R^1(\theta, b_R^1(\theta)) = V^R(\theta) \quad \text{and} \quad \frac{\partial f_R^1(\theta, x)}{\partial x} \bigg|_{x=b_R^1(\theta)} = \frac{\partial V^R(\theta)}{\partial x} \bigg|_{x=b_R^1(\theta)} = 0. \tag{43}
\]

16If \( \phi_2 < u < \phi_3 \), it will be \( b(\theta) = b^{AR}(\theta)\chi_{\{\theta \leq \theta^*\}} + b^R(\theta)\chi_{\{\theta \geq \theta^*\}} \) and if \( u > \phi_3 \), it will be \( b(\theta) = b^R(\theta)\chi_{\{\theta \geq \theta^*\}} \).
and
\[ f_1^{AR}(\theta, b^{AR}(\theta)) = V^A(\theta, b^{AR}(\theta)) \quad \text{and} \quad \frac{\partial f_1^{AR}(\theta, x)}{\partial x} \bigg|_{x=b^{AR}(\theta)} = \frac{\partial V^A(\theta, x)}{\partial x} \bigg|_{x=b^{AR}(\theta)}. \] (44)

Given that Equation (42) is the same we have already solved in (33), it is easy to conclude that
\[ f_1^R(\theta, x) = \frac{\lambda}{r+\lambda} V^R(\theta + u) + \frac{z_0 x}{r+\lambda-\alpha} + C_1^R(\theta)x^{-\frac{\alpha}{\alpha-1}}, \]
with
\[ C_1^R(\theta) = -\frac{\alpha z_0}{(r+\lambda)(r+\lambda-\alpha)} b^R(\theta)^{-\frac{\alpha}{\alpha-1}} \quad \text{and} \quad b^R(\theta) = \frac{1}{z_0} \left[ \left( \theta - \frac{u \lambda}{r} \right) z_1 - rI \right]; \]
and
\[ f_1^{AR}(\theta, x) = \frac{\lambda}{r+\lambda} V^R(\theta + u) + \frac{z_0 x}{r+\lambda-\alpha} + C_1^{AR}(\theta)x^{-\frac{\alpha}{\alpha-1}}. \]
with
\[ C_1^{AR}(\theta) = -\frac{\alpha z_0}{(r+\lambda)(r-\alpha)} \left[ \kappa \theta - \frac{\lambda}{r+\lambda-\alpha} z_0 \right] b^{AR}(\theta)^{-\frac{\alpha}{\alpha-1}} \quad \text{and} \quad b_1^{AR}(\theta) = \frac{\left( \theta - \frac{u \lambda}{r} \right) z_1 - rI}{(r+\lambda-\alpha) \kappa \theta - r \alpha z_0}. \]

(ii) It remains to consider the states \((\theta, x) \in \Omega_1^A\), where after one jump the process will be in the stopping region, and the optimal strategy is to add. Then
\[ f_1^A(\theta, x) = x^{-\frac{\alpha}{\alpha-1}} \int x^{-\frac{\alpha}{\alpha-1}} \left( -\frac{\lambda}{\alpha} V^A(\theta + u, x) - \frac{z_0}{\alpha} \right) dx, \] (45)
where the value matching and smooth pasting conditions are the same as \( f_1^{AR} \), assuming that \( \theta > \bar{\theta} \)
\[ f_1^A(\theta, b^A(\theta)) = V^A(\theta, b^A(\theta)) \quad \text{and} \quad \frac{\partial f_1^A(\theta, x)}{\partial x} \bigg|_{x=b^A(\theta)} = \frac{\partial V^A(\theta, x)}{\partial x} \bigg|_{x=b^A(\theta)}. \]

It can be derived that
\[ f_1^A(\theta, x) = \frac{\lambda}{r+\lambda} V^R(\theta + u) + \frac{1}{r-\alpha} \left[ z_0 - \frac{\lambda}{r+\lambda-\alpha} \kappa(\theta + u) \right] x + C_1^A(\theta)x^{-\frac{\alpha}{\alpha-1}}, \] (46)
with
\[ C_1^A(\theta) = -\frac{\alpha}{(r+\lambda)(r+\lambda-\alpha)} \left[ \kappa \theta + \frac{u \lambda \kappa}{r-\alpha} \right] b^A(\theta)^{-\frac{\alpha}{\alpha-1}} \quad \text{and} \quad b_1^A(\theta) = \frac{\left( \theta - \frac{u \lambda}{r} \right) z_1 - rI}{\kappa \left( \theta - \frac{u \lambda}{r-\alpha} \right)}. \]

For \( u > \phi_1 \), we have proved that the threshold curve is given by (14). Moreover, the continuation region, \( C \), coincides with the \( W \) set defined before. Thus, the threshold curve is always increasing in \( \theta \).

\[ \text{[17] We proved previously that, for all } x, \text{ if } 0 < \theta < \bar{\theta} \text{ we are always in the continuation region.} \]
A.9 Proof of Proposition 8

In this proposition we assume that $0 < u < \phi_1$. This implies that the function $d$, defined in the proof of the Proposition 6, has a decreasing branch.

As before, we need to solve the following equation:

$$F'(\theta, x) = x^{\frac{\alpha}{\alpha - 1}} \int x^{-\frac{\alpha}{\alpha - 1}} \left(-\frac{\lambda}{\alpha x} F(\theta + u, x) - \frac{z_0}{\alpha}\right) dx.$$  

In the previous case (where $d$ increases everywhere), for $(\theta, x) \in C_1$, we can guarantee that: if $\bar{\theta} \leq \theta < \hat{\theta} - u$ the threshold can be reached through a decline in $x$ or a jump in $\theta$ (both to the Add region); if $\hat{\theta} - u \leq \theta < \hat{\theta}$ the threshold can be reached through a decline in $x$ (to the Replace region) or a jump in $\theta$ (to the Replace region); and if $\theta \geq \hat{\theta}$ the threshold can be reached through a decline in $x$ or a jump in $\theta$ (both to the Replace region). Consequently, the threshold curve is defined by three parts, as can be seen in (14).

In this case, we also have that the threshold curve also consists of the parts $b^A, b^{AR}$ and $b^R$, but they do not always cover the same sets. Namely, $b^A$ is only defined for $\bar{\theta} \leq \theta < \bar{\theta}_1$, where $\bar{\theta}_1$ is defined by $b^A(\bar{\theta}_1) = b^{AR}(\bar{\theta}_1 + u)$; and $b^{AR}$ is only defined for $\bar{\theta}_2 \leq \theta < \hat{\theta}$, where $\bar{\theta}_2$ is defined by $b^{AR}(\bar{\theta}_2) = b^R(\bar{\theta}_2 + u)$. Indeed, $b^R$ is the only one defined in the same set, i.e. for $\theta \geq \hat{\theta}$. Furthermore, in this case, $C_1$ does not have the same definition as in (41). In fact, there exist no points $(\theta, x) \in C$ with $\bar{\theta}_1 < \theta < \bar{\theta}_2$ such that $(\theta + u, x) \in \mathcal{S}$.

We can then write

$$C_1 = \left\{ (\theta, x) \in \mathbb{R}^+ \times \mathbb{R}^+ : \theta \notin \left(\bar{\theta}_1, \bar{\theta}_2\right) \land b(\theta) < x \leq b(\theta + u) \right\},$$

but it remains to define the threshold curve for $\bar{\theta}_1 < \theta < \bar{\theta}_2$. We propose the following definition

$$b(\theta) = b^A(\theta) \chi_{\{\bar{\theta}_1 \leq \theta < \bar{\theta}_1\}} + b^{AC}_A(\theta) \chi_{\{\bar{\theta}_1 \leq \theta < \bar{\theta}_2 - u\}} + b^{AC}_n(\theta) \chi_{\{\bar{\theta}_2 - u \leq \theta < \bar{\theta}_2\}} + b^{AR}(\theta) \chi_{\{\bar{\theta}_2 \leq \theta < \hat{\theta}\}} + b^R(\theta) \chi_{\{\theta \geq \hat{\theta}\}},$$

where $b^{AC}_A$ and $b^{AC}_n$ need to be derived. Note that $\bar{\theta}_1$ and $\bar{\theta}_2$ can be explicitly obtained as follows:

$$\bar{\theta}_1 = \frac{b_1 + \sqrt{b_1^2 - 4a_1c_1}}{2a_1}, \quad (47)$$

with $a_1 = \lambda \kappa z_1$, $b_1 = \lambda (\kappa r I + z_1(z_0 - 2u\kappa)) + \frac{u\lambda z_1}{r}$, $c_1 = r I (\lambda z_0 - u\kappa(r + 2\lambda - \alpha)) + \frac{u\lambda z_1(u\kappa(\alpha - 2\lambda) + \lambda z_0)}{r}$ and

$$\bar{\theta}_2 = \frac{b_2 + \sqrt{b_2^2 - 4a_2c_2}}{2a_2}, \quad (48)$$

\[18\] Note that in this case $C_1$ is an union of two disjoint sets.
where \( a_2 = \kappa z_1 \), \( b_2 = \kappa r + z_1(z_0 - u\kappa) + \frac{u\lambda z_1}{r} \) and \( c_2 = r z_0 I + \frac{u\lambda z_1(\lambda - \alpha)}{(r + \lambda - \alpha)} \).

For \( \tilde{\theta}_1 \leq \theta < \tilde{\theta}_2 \), the threshold curve can be reached only through a decline in \( x \) (to the Add region), as a jump in \( \theta \) will lead still to the continuation region. If \( \tilde{\theta}_1 \leq \theta < \tilde{\theta} - u \) it will jump to the region in which the value function is defined by \( f_1^{AR} \), whereas if \( \tilde{\theta} - u \leq \theta < \tilde{\theta}_2 \) it will go to the region in which the value function is defined by \( f_1^R \). Denoting the value functions in these two regions as \( f_1^{ACA} \) and \( f_1^{ACR} \), respectively, it is clear that they are defined by the following equations:

\[
f_1^{ACA}(\theta, x) = x^{r+\lambda} \int x^{-\frac{r+\lambda}{\alpha}} \left( -\frac{\lambda}{\alpha x} f_1^{AR}(\theta + u, x) - \frac{z_0}{\alpha} \right) dx,
\]

and

\[
f_1^{ACR}(\theta, x) = x^{r+\lambda} \int x^{-\frac{r+\lambda}{\alpha}} \left( -\frac{\lambda}{\alpha x} f_1^R(\theta + u, x) - \frac{z_0}{\alpha} \right) dx.
\]

The value matching and smooth pasting conditions are the same for both cases, that is,

\[
f_1^{ACA}(\theta, b^{ACA}(\theta)) = V^A(\theta, b^{ACA}(\theta)) \quad \text{and} \quad \frac{\partial f_1^{ACA}(\theta, x)}{\partial x} \bigg|_{x=b^{ACA}(\theta)} = \frac{\partial V^A(\theta, x)}{\partial x} \bigg|_{x=b^{ACA}(\theta)}
\]

and

\[
f_1^{ACR}(\theta, b^{ACR}(\theta)) = V^A(\theta, b^{ACR}(\theta)) \quad \text{and} \quad \frac{\partial f_1^{ACR}(\theta, x)}{\partial x} \bigg|_{x=b^{ACR}(\theta)} = \frac{\partial V^A(\theta, x)}{\partial x} \bigg|_{x=b^{ACR}(\theta)}.
\]

For \( b^{ACA} \) and \( b^{ACR} \) we cannot obtained explicitly expressions, but they are implicitly defined as follows:

\[
\frac{r + \lambda - \alpha}{(r + \lambda)(r - \alpha)} \left[ \kappa \theta - \left( \frac{\lambda}{r + \lambda - \alpha} \right)^2 z_0 \right] x + \frac{\lambda}{r + \lambda} C_1^{AR}(\theta + u) x^{\frac{r+\lambda}{\alpha}} = V^R(\theta) - \left( \frac{\lambda}{r + \lambda} \right)^2 V^R(\theta + 2u) \quad (49)
\]

and

\[
\frac{r + \lambda - \alpha}{(r + \lambda)(r - \alpha)} \left[ \kappa \theta - \left( \frac{\lambda}{r + \lambda - \alpha} \right)^2 z_0 \right] x + \frac{\lambda}{r + \lambda} C_1^{R}(\theta + u) x^{\frac{r+\lambda}{\alpha}} = V^R(\theta) - \left( \frac{\lambda}{r + \lambda} \right)^2 V^R(\theta + 2u). \quad (50)
\]

This concludes the proof.

\[\blacksquare\]

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