Fermionic construction of tau functions and random processes

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Abstract

Tau functions expressed as fermionic expectation values [14] are shown to provide a natural and straightforward description of a number of random processes and statistical models involving hard core configurations of identical particles on the integer lattice, like a discrete version simple exclusion processes (ASEP), nonintersecting random walkers, lattice Coulomb gas models and others, as well as providing a powerful tool for combinatorial calculations involving paths between pairs of partitions. We study the decay of the initial step function within the discrete ASEP (d-ASEP) model as an example.

1 Introduction

Random processes and integrable systems are often regarded as opposites extremes in the study of dynamical systems, since the first, by its very definition involves indeterminacy and probabilistic considerations, while the second represents deterministic evolution with the highest degree of coherence in time and space. However if we consider the simplest example of motion of a free quantum particle we find that this simplest model combines integrability with randomness. The Brownian motion and the motion of quantum particles may be described by similar tools, say, in terms of correlation functions and integrals over paths.

If we consider quantum integrable systems, beginning with the earliest studies on spin systems solvable via the Bethe ansatz, it is clear that, like all quantum systems, these involve probabilistic considerations, involving averaging and statistical correlations, and therefore such systems combine integrability with randomness. In fact, many are solvable using the same methods as certain related statistical systems since the Hamiltonians may be embedded in the same family of commuting operators as the transfer matrices governing solvable statistical models. Later, starting with [21], [22], the success was achieved in applying methods of Bethe ansatz to various driven-diffusive and non-equilibrium systems like asymmetric simple exclusion process (ASEP).

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We shall focus on the relations of such systems to classical integrable systems rather than to quantum ones.

It was first shown in the work of McCoy, Tracy and Wu on the Ising model that certain correlation functions in solvable statistical models satisfy Painlevé equations, which naturally arise in integrable systems under scaling reductions. In the series of papers by the Kyoto school, a systematic approach to integrable systems based on fermionic constructions of tau functions was developed, in which remarkable links between the quantum and classical interpretations appeared. Classically, a tau function may be seen as a sort of universal “potential” for families of integrable equations like the KP and TL hierarchies. (In the Hamiltonian setting, it may be interpreted as Hamilton’s principle function evaluated along the integral manifolds mapped out by a complete set of commuting flows.) At the same time, it has an interpretation as a certain correlation function for models of free fermions. Although in the classical integrable systems context this formulation at first appeared to simply be an interesting reinterpretation of the previous approaches to classical integrable systems, based on isospectral flows of linear operators, inverse spectral theory and the Zakharov-Shabat’s dressing method [66], it subsequently turned out to be a powerful tool linking soliton theory with many other fields of physics and mathematics.

Tau functions have long been known to play a central role in the remarkable links found between random matrix models and integrable systems [19], [16], [3]. Subsequent applications of tau functions were also found in combinatorics, probability theory, statistical physics and quantum chaos. (See, e.g., the series of papers by C.Tracy, G. Widom, A.Okounkov, A. Borodin, K.Johansson, J. Baik, P. Forrester, P. Zinn-Justin, M. Adler, P. van Moerbeke. In particular see [63], [46], [48], [9], [10], [11], [12], [33], [6], [15], [16], [17], [18], [4], [5]). Some of these links are quite subtle, while others are still not completely understood. They appear however to be quite central to the seemingly endless new applications of ideas and methods originating in the modern theory of integrable systems to these other domains.

In this work, we present some direct links of the fermionic construction of tau functions as developed in [14], [32] with certain types of random processes and combinatorial problems that arise in relation to partitions. As in [14], [32], we use the language of free fermions, Maya diagrams and Young diagrams. This “quantum-like” formulation turns out to be very fruitful in providing natural links between many stochastic systems and classical integrability, as well as suggesting new methods for approaching the combinatorial aspects common to these problems. Suitably interpreted, it will be shown to provide a natural and straightforward description of a number of random processes and statistical models. This approach is based on relating a natural combinatorially defined basis for the fermionic Fock space to states of the random system. To each basis Fock vector is associated, both a $1D$ configuration of a lattice gas (via its “Maya diagram”) and a Young diagram, which may be viewed as a region of $2D$ hard core particles. This provides a fermionic setting for well-known models like asymmetric simple exclusion processes, 1D- lattice log gases subject non-intersecting lattice walkers and models of fluctuating interfaces. Other work along similar or related lines may be found, e.g. in the work of Okounkov, Olshansky, Borodin, Reshetihin, and others.

The specific random process or equilibrium statistical model is determined by specification of the tau function and the interpretation of time variables. A central tool consists of the well-known [60], [61, 62] Schur function expansions of tau functions,

\[ \tau(t, \bar{t}) = \sum_{\lambda, \mu} \tau_{\lambda, \mu} s_\lambda(t) s_\mu(\bar{t}), \]
where the sum is over all pairs of partitions $\lambda, \mu$, which may be deduced very simply from the fermionic representation. We show that the tau function $\tau_{\lambda, \mu}$ may be interpreted as a generating function for the (unnormalized) transition probabilities between states of certain models of stochastic systems. After a suitable normalization, the procedure also leads to discrete analogs of models of random orthogonal matrices.

In the theory of integrable systems the variables $t$ and $\bar{t}$ are known as “higher times” describing the simultaneous evolution under commuting flows, and providing solutions to integrable “hierarchies”, such as the Toda lattice TL and KP systems. In the present work, these variables are either chosen to have specific values, thereby defining the statistical weight of a given configuration, or just play the role of formal expansion parameters which, through the Schur function series allow the tau function to be interpreted as a generating function for transition probabilities.

We use the generating tau function for constructing models of random walks of non-intersecting ("hard core") particles along one dimensional lattice. We obtain two types of models introduced by M. Fisher [15]: (a) random turn models and (b) vicious walkers (whose particular case M. Fisher called lock step models). In models (a) at each tick of the clock a randomly chosen walker takes a random step, while in models (b) a certain number of walkers take random step in each time instant. In both types of models each site may be occupied by only one walker at the same time.

The alternative approach to the same models is to consider the related Young diagrams, which are 2D figures which are in one-to-one correspondence with 1D configurations of hard core particles. This yields models of oscillating domains.

Models (a) will be related to the repeating actions of elements of the so-called $\hat{gl}(\infty)$ Lie algebra to Fock vectors and tau functions. The number of these actions is identified with the discrete time $T$ parameter of the stochastic process. Models (b) will be related to the repeating action of elements of the related $\hat{GL}(\infty)$ group. The important role of group $\hat{GL}(\infty)$ and its algebra in theory of integrable systems was discovered in [14], [32].

The structure of the paper is as follows.

Second part of the introductory section is devoted to the general conception of our approach. Here we introduce a time-dependent normalization function $Z_\nu(T)$ which plays a central role in the approach. In section 2 we consider models of type (a). After necessary preliminaries we come to subsection 2.3 where we relate a graph to each Toda lattice tau function and consider a certain random turn walk along this graph. Here notions of path and of permutation factor of a path are introduced. In subsection 2.4 we show that each tau function is a generating function which counts differences of weights for two types of random motion along the related graph, which are notable by a value of a permutation factor of their paths. In subsection 2.5 we consider known models of Coulomb lattice gas in thermodynamical equilibrium, see [17] for a review. The Coulomb potential is modified in a way that the vacuum configuration (the step function presenting the Fermi surface) has minimal energy. We point out a two fold relation of these models to classical integrable systems. In subsection 2.6 we introduce a model of random turn walkers in external potential which may be viewed as a discrete-time version of ASEP (d-ASEP) with variable site-depending hopping rates and compare related normalization function with the partition function of a model of Coulomb gas in thermodynamical equilibrium. We show that $Z_0(T)$ may be viewed as a certain correlation function in a discrete model of a random ensemble of orthogonal $T \times T$ matrices. For constant hopping rate we find the asymptotic configuration in the large $T$ limit. We found that the limiting shape coincides with one obtained
by Vershik and Kerov in [38] in the study of a limiting shape of random partitions with fixed large area which are distributed according to the so-called Plancherel measure. In our case the size of asymptotic configuration depends on the hopping rate as it is given by formula (92). In subsection 2.7 we introduce additional dependence of the hopping rates which now depend on a mutual position of particles via the modified Coulomb potential with an arbitrary charge and show how it corrects the size of the asymptotic configuration. In section 3 we consider models (b). In subsection 3.1 we show how the so-called determinantal ensembles [13] appear as a result of actions of elements of \( \hat{GL}(\infty) \) group on tau functions. In subsection 3.2 we mark the convenience of usage of multi-component fermions which allows in a simple way to construct fermionic representations for models of vicious walkers in an external potential and via Wick’s theorem yields answers for correlation functions in form of determinants. In subsection 3.3 we consider models of ”random layering” and models of ”rocks” (figures which generalize diagrams of the so-called 3D partitions) which we believe were basically known in certain versions, for instance, as the so-called polynuclear growth model. Here again we introduce the external potential which results in a variable site-dependent rate (this is the hopping rate in 1D picture and a gluing rate in 2D one). The normalization function of some of these models coincides with the tau function of Pfaff lattice [4], [35].

Except the consideration in subsection 2.6 we restrict ourselves to the presentation of various models of random processes arising from integrable systems more than to the analysis of random models themselves which is mainly an analysis of the large time behavior of different correlation functions. As it was very successfully demonstrated in series of papers by A.Okounkov the usage of fermionic language may be quite helpful for such purposes. This part of work will be published separately.

1.1 Action on Fock vectors. Time-dependent normalization function \( Z_\nu(T) \)

We refer the reader to Appendices A.1- A.3 for a compendium useful facts and conventions regarding fermions, partitions and Schur functions that are used repeatedly in what follows. For present purposes, this involves only charged, one-dimensional free fermions satisfying the standard canonical anti-commutation relations

\[
f_i \bar{f}_j + \bar{f}_j f_i = \delta_{ij}, \quad f_i f_j + f_j f_i = 0, \quad \bar{f}_i \bar{f}_j + \bar{f}_j \bar{f}_i = 0, \quad i, j \in \mathbb{Z}
\]

(2)

These act on a fermionic Fock space whose “vacuum state” \( |0\rangle \) is, like the Dirac sea, annihilated by all negative component creation operators \( f_i \) and positive component annihilation operators \( \bar{f}_i \)

\[
f_{-i-1}|0\rangle = \bar{f}_i|0\rangle = 0, \quad i \geq 0
\]

(3)

More generally, we have the “charge \( n \)” vacuum states \( |n\rangle \),

\[
|n\rangle = f_{n-1} \ldots f_0|0\rangle \quad \text{for } n \geq 0, \quad |n\rangle = \bar{f}_n \ldots \bar{f}_{-1}|0\rangle \quad \text{for } n < 0,
\]

(4)

where the integer \( n \) denotes the Dirac “sea level”. We may visualize the level - \( n \) Dirac sea as a diagram consisting of a set of integer sites on a vertical axis, with all sites below \( n \) occupied, while the \( n \)-th site and all those above it are empty, as in Fig.1.

A complete set of basis vectors for the Fock space may be associated with Maya diagrams, which we shall denote by Greek letters \( \lambda, \nu, \ldots \), to be more precise, by two letters, Greek and Latin, the first denotes the configuration of particles, the second denotes the level of the Dirac
1. Dirac sea of level $n$

$\bullet$ $n$
$\bullet$ $n-1$
$\bullet$ $n-2$
$\bullet$ $n-3$
$\bullet$ $n+1$
$\bullet$ $n+2$

2. One particle one hole

$\circ$ $n+2$
$\circ$ $n+1$
$\circ$ $n$
$\circ$ $n-1$
$\circ$ $n-2$
$\circ$ $n-3$

1. Dirac sea of level $n$ 2. One particle one hole

Figure 1: Maya diagrams

sea, which we perturb by placing a finite number of particles of the Dirac sea to the upper positions. In this way all basis vectors of Fock space may be obtained. Since no two particles can occupy the same site, each Fock basis vector may be encoded by a partition, these are the ordered sets of weakly decreasing numbers, say, $\nu = (\nu_1, \nu_2, \ldots)$ where $\nu_1 \geq \nu_2, \ldots \geq 0$, and by a level, say $n$, related to the coordinates of the particles, $h_1 > h_2 > \cdots$, as follows:

$$\nu_i = h_i + i - n$$  \hspace{1cm} (5)

We number (identical) particles in downward direction, the highest one be the first. Provided we view each Fock vector as being created by a perturbation of vacuum ('initial') vector, one may say, that $\nu_i$ measures the deviation of a current ('Lagrangian') coordinate, $h_i$, of the $i$-th particle from its initial value, $n - i$, in the vacuum state.

The partitions are very useful and conventional notations, that is why we need them. There are two suitable way to figure partitions: via Maya diagram, which is just configuration of the particles $h = (h_1, h_2, \ldots)$ placed on the vertical line, and via Young diagram which is a set of $\nu = (\nu_1, \nu_2, \ldots)$ related to $h = (h_1, h_2, \ldots)$ by (5). They also yields links with the topic of growing interfaces, since the diagram of each partition (Young diagram) may be viewed as a discrete model of a two-dimensional region. A set of partitions placed one above another in three-dimensional space may be a discrete model of $3D$ region, and so on. A particular case of such $3D$ figure, called plane partition, was considered in the papers of Okounkov and Reshetikhin [48], [49] as a certain statistical model.

In the present paper we consider a sequence of Fock vectors obtained by application of operators $o^{i-1}$ to an initial basis Fock vector, say, $|\nu\rangle$, as follows

$$|\nu, n\rangle \rightarrow o^{1,0}|\nu, n\rangle \rightarrow o^{2,1}o^{1,0}|\nu, n\rangle \rightarrow \ldots \rightarrow o^{T,T-1}\cdots o^{1,0}|\nu, n\rangle \hspace{1cm} (6)$$

This sequence may be viewed as a $T$-step process on the space of configurations of hard core particles ($1D$ lattice gas) where each configuration is related to some basis Fock vector and may be visualized via the so-called Maya diagram, see the picture. The number $T$ will be related as discrete time of the process. Alternatively, (6) may be viewed as a $T$-step process on the Young diagrams (a type of $2D$ regions).

In what follows everywhere, where it will not be not confusing, we shall omit the label related to the level of configuration (or, the same, the level of Dirac sea), keeping only Greek
letters, say, \( \lambda, \mu, \nu \) for configurations of the hard core particles, having in mind the pairs \((\lambda, l), (\mu, m), (\nu, n)\).

The result of the application of

\[
o(T) := o^{T,T-1} \cdots o^{1,0}
\]

(7)

to the initial state \( \nu \) (corresponding to an initial configuration of hard core particles), is a Fock vector which is a linear combination of basis Fock vectors (=configurations of hard core particles). One of the objects of an interest is the relative weight of a given configuration, say, \( \lambda \),

\[
W_{\nu \rightarrow \lambda}(T) = \langle \lambda | o(T) | \nu \rangle
\]

(8)

with respect to the weight of all possible configurations,

\[
P_{\nu \rightarrow \lambda}(T) = \frac{W_{\nu \rightarrow \lambda}(T)}{Z_{\nu}(T)}
\]

(9)

where the normalization function in the denominator is

\[
Z_{\nu}(T) = \sum_{\lambda} \langle \lambda | o(T) | \nu \rangle
\]

(10)

the sum ranges over all possible configurations.

Matrix element

\[
o^{j,j-1}_{\lambda,\mu} = \langle \lambda, l | o^{j,j-1} | \mu, m \rangle
\]

(11)

will be referred as (one-step) transition weight between a configuration \( |\mu, m\rangle \) and a configuration \( |\lambda, l\rangle \) at moment \( j \).

If we want to keep in mind the probabilistic interpretation of the process we ask transition weights to be positive.

The completeness of the basis Fock vectors implies \( \sum_{\mu,m} |\mu, m\rangle \langle \mu, m| = 1 \). We have

\[
\langle \lambda, l | o^{T,T-1} \cdots o^{1,0} | \nu, n \rangle = \sum_{\{\mu^{(i)}, m^{(i)}\}} o^{T,T-1}_{\lambda,\mu^{(T-1)}} \cdots o^{1,0}_{\mu^{(1)},\nu}
\]

(12)

where one sums over all possible intermediate configurations \( \mu^{(i)}, m^{(i)}, i = 1, \ldots , T - 1 \).

Each monomial non-vanishing term in the sum is related to a certain set of configurations, \( \nu, n \rightarrow \mu^{(1)}, m^{(1)} \rightarrow \cdots \rightarrow \mu^{(T-1)}, m^{(T-1)} \rightarrow \lambda, l \),

(13)

which will be referred as a path from the configuration \( \nu, n \) to the configuration \( \lambda, l \), having length (or, the same, duration) \( T \). The value of the monomial is referred as the weight of the path, which is equal to a product of all one-step transition weights (11) along the path.

Let us notice that it may have sense to evaluate \( Z_{\nu}(T) \) also for the case where the positivity condition,

\[
o^{j,j-1}_{\lambda,\mu} \geq 0,
\]

(14)

is violated and therefore we have no a probabilistic interpretation of (12) as a random process on the space of configurations of the hard core particles.
In certain examples considered at the present paper transition weights (11) take the form

\[ \langle \lambda, l | o^{j,j-1} | \mu, m \rangle = e^{-U_{\lambda,j}(j) + U_{\mu,m}(j-1)} \]  

(15)

The case where the weight of each path depends only on the end points of the path, i.e. on initial and on final configurations shall be referred as potential weights. In what follows we shall consider only potential transition weights.

To compare with, let us assign to each configuration of hard-core particles, \( \lambda \), an energy \( U_{\lambda} \). What people are interested in statistical physics is the partition function of the system

\[ \sum_{\lambda} e^{-U_{\lambda}} \]  

(16)

(everywhere we shall put temperature times the Boltzmann constant to be one).

In case of random process the normalization function (10) - which yields the sum of weights of all pathes of durations \( T \) which start at the configuration \( \nu \) - plays the role similar to the role of partition function in thermodynamics. For potential transition weights the normalization function takes the form

\[ Z_{\nu}(T) = \sum_{\lambda} e^{-U_{\lambda}(T) + U_{\nu}(0)} N_{\lambda,\nu}(T) \]  

(17)

where \( N_{\lambda,\nu}(T) \) counts the number of pathes of duration \( T \). Given \( T \), this factor may be also included into the exponent to define a partition function of a \( T \)-dependent statistical model.

At the present paper we shall focus to figure out the links of random processes (6) to classical integrable systems, random matrices and growing surfaces. Benefits of these links will be presented in our forthcoming papers [29], [30].

Our main example is a discrete version of asymmetric simple exclusion process (d-ASEP), where we present the answer for the decay of the step function. In short we shall consider random walk of \( N \) non-intersecting particles and other examples.

The mentioned links of random processes with classical integrable systems are two-fold. The first one is the usage of series (1) as generating functions for transition weights between different configurations of the hard-core particles. Then higher times \( t \) and \( \bar{t} \) are just formal parameters. There are different tau functions, related to \( A, B, D \) and \( C \) root systems; the way of construction of these may involve different type of fermionic (or, bosonic) operators. At the present paper we deal with \( A \) type, which may be realized in terms of fermions (2). We show that \( A \)-type tau function may generate certain classes of transition weights (11). We consider two ways of generating weights, see below. This is one link between random processes and integrable systems.

The second link is the fact that the normalization functions \( Z_0(T) \) may be related to a tau function of some other classical integrable systems. It occurs if we consider deformations of transition weights with a set of deformation parameters \( \tilde{t} = (\tilde{t}_1, \tilde{t}_2, \ldots) \)

\[ \langle \lambda, l | o^{j,j-1} | \mu, m \rangle \rightarrow \langle \lambda, l | o^{j,j-1}(\tilde{t}) | \mu, m \rangle = e^{-H_0(\tilde{t})} \langle \lambda, l | o^{j,j-1} | \mu, m \rangle \]  

(18)

specified later by (58) and caused by transforms of operators \( o^{j,j-1} \) of form

\[ o^{j,j-1} \rightarrow o^{j,j-1}(\tilde{t}) = e^{H_0(\tilde{t})} o^{j,j-1} e^{-H_0(\tilde{t})}, \quad j = 1, \ldots, T \]  

(19)
operators $H_0$ are defined in Appendix A.3). These deformation parameters may be recognized as higher times of different underlying integrable equations. These are dual integrable hierarchies in sense of papers [50] and [54] but not exactly\(^4\). In many cases, under flows (19), the normalization function $Z_0(t, \tilde{t})$ of (10) is a tau function of $B$-type, namely, the $B$ type in the realization of Kac-van de Leur, see [34] (also known as Pfaff lattice independently introduced as a tool for study of random matrices see [4], [40]). Thus, what we mean is a sort of duality between $A$ and $B$ type of equations.

The last remark is that we may consider a sort of coupling of random processes with statistical models on the space of configuration,

$$Z = \sum_{\lambda} e^{-q^2E_\lambda}, \quad (20)$$

where $q$ is some constant (charge) as follows. We use the energy $q^2E_\lambda$ of the model to modify each transition weight:

$$\langle \lambda | o^{j,j-1} | \mu \rangle \rightarrow e^{-q^2E_\lambda + q^2E_\mu} \langle \lambda | o^{j,j-1} | \mu \rangle, \quad j = 1, 2, \ldots, T \quad (21)$$

Then

$$Z_\nu(T) = \sum_{\lambda} \langle \lambda | o(T) | \nu \rangle \rightarrow \sum_{\lambda} e^{-q^2E_\lambda + q^2E_\nu} \langle \lambda | o(T) | \nu \rangle \quad (22)$$

As we shall show in subsection 2.7 this procedure may keep relations with integrable systems if $q^2E_\lambda$ is chosen in a special way.

\section{2 Action on Fock vectors via $\hat{gl}(\infty)$ algebra}

\subsection{Maya diagrams}

Let us consider an infinite set of vertexes labelled by integers. Each vertex will be sketched as either a white or black disk. Following Sato, we shall call it Maya diagram, (after a Maya game in which small stones are moved to occupy free sites according to a certain rule). In what follows we shall describe certain rules for motion of “stones” - hard core particles. The Maya diagrams for the vacuum state are presented in Figure 1.2.

We may consider each black ball as a particle and each white ball as the lack of a particle at this site (empty site).

The Maya diagram is a way of visualization of basis Fock vectors for fermions on the circle, sites correspond to Fourier components of the fermions, no two fermions can occupy the same site; therefore, there is one site for at most one particle (these are what is called hard core particles). Sites with positive large enough numbers are supposed to be free, while sites with large enough negative numbers are supposed to be occupied, that is the related Maya diagrams possesses this property: all sites to the up of a certain number are all white, while sites to the down of a certain number are all black. The Maya diagram where all sites below $n$-th one are occupied, and all sites above $n - 1$ are free, is called the vacuum Maya diagram of level $n$, see the Figure 1. It represent the so-called Dirac sea of level $n$.

\footnote{This duality might be related in a general notion of duality of integrable systems developed in [2], [24], [25], [7]}
Each Maya diagram may be obtained from the vacuum diagram of a certain level, say, \(n\), via placing of certain number, say, \(k\), of black balls (particles) to higher empty sites. We say that each Maya diagram has level \(n\).

**Basis Fock vectors, partitions and Maya diagrams**

By its origin Maya diagrams is a visualization of vectors in the fermionic Fock space, see Appendix A.1. Each black ball (a particle) located at a site \(i\) indicates that the site is occupied.

Let us introduce the following basis vectors in the fermionic Fock space

\[
|\lambda, n\rangle = (-1)^{\sum_{i=1}^{k} \beta_i} f_{n+\alpha_k} \tilde{f}_{n-1-\beta_k} \cdots f_{n+\alpha_1} \tilde{f}_{n-1-\beta_1} |n\rangle, \quad \alpha_1 > \cdots > \alpha_k \geq 0, \quad \beta_1 > \cdots > \beta_k \geq 0
\]

(23)

where \(\lambda = (\alpha_1, \ldots, \alpha_k|\beta_1, \ldots, \beta_k)\) is the Frobenius notation for the partition \(\lambda\), and \(n\) is the level of Dirac sea, see Introduction and Appendix A.1 for more details.

This vector is one-to-one related to the Maya diagram of level \(n\), where particles are situated as follows. A particle number \(i\) (black ball) counted from the top has a coordinate \(n+\alpha_i\), \(i = 1, \ldots, k\). Thus, the upmost particle has the coordinate \(\alpha_1\). Underneath of the sea level all sites are occupied except free sites (holes) which we count from the bottom. Hole number \(i\) has a coordinate \(n-1-\beta_i\), thus the down-most hole has the coordinate \(n-1-\beta_1\).

Notice that \(\beta_1 + 1\) is the length of partition lambda, \(\ell(\lambda)\).

One may give an alternative description of coordinates of particles on the Maya diagram related to the vector \(|\lambda, n\rangle\). Take any integer \(N\) which is not less than \(\ell(\lambda)\). Then all site with coordinates \(n-N-1, n-N-1, \ldots\) are occupied. On the sites \(n-N, n-N+1, \ldots\) there are \(N\) particles which occupy sites with coordinates

\[
h_i = \lambda_i - i + N, \quad i = 1, \ldots, N, \quad N \geq \ell(\lambda)
\]

(24)

All other sites including and above \(n-N\) are empty.

Dual vector is

\[
\langle \lambda, n | := (-1)^{\sum_{i=1}^{k} \beta_i} \langle n | f_{-1-\beta_1} \tilde{f}_{\alpha_1} \cdots f_{n-1-\beta_k} \tilde{f}_{n+\alpha_k} \cdots
\]

(25)

One can verify the orthonormality condition

\[
\langle \lambda, l | \nu, n \rangle = \delta_{l,n} \delta_{\lambda,\nu}
\]

(26)

Along these notations we shall identify partitions, Maya diagrams and basis Fock vectors. Being identical to basis Fock vectors, Maya diagrams may be multiplied by numbers and may form (formal) linear combinations.

**Action of \(gl(\infty)\) on Maya diagrams**

It was old observation that quadratic operators of type \(f_i \tilde{f}_k, i, k = 1, \ldots, N\), form \(gl(N)\) Lie algebra. In the papers of Kyoto school it was studied within symmetry analysis of integrable equations like Toda lattice equation. One of the observations made in the wonderful series devoted to hierarchies of integrable equations (in particular, see [14], [32]) is that \(N = \infty\) case needs a special consideration. When there are an infinite number of fermionic modes one needs normal ordering. Then, natural combination are

\[
E_{i,k} := f_i \tilde{f}_k - \langle 0 | f_i \tilde{f}_k | 0 \rangle \quad -\infty < i, k < +\infty
\]

(27)

which form an algebra called \(\hat{gl}(\infty)\) (the algebra of infinite matrices with central extension). We do not need properties of this algebra. Let us only note that such algebras play essential
role in the construction of hierarchies of integrable classical (not quantum) equations, where the algebra of infinite matrices appears as an underlying symmetry algebra of integrable equations.

This algebra naturally acts on the fermionic Fock space.

For our purposes we are interested in the action of $E_{i,k}$ on the basis Fock vectors, which may be viewed as an action on Maya diagrams. The point is that this action transforms one Maya diagram to the other one multiplied by a sign factor. Then any product of operators (27) do the same.

Keeping in mind definition of basis Fock vector and applying (27) one can verify that the action of $E_{i,k}$ on Maya diagram is as follows.

First, let us notice that the second term in the definition of operator $E_{i,k}$ is vanishing for $i \neq k$. Let us consider this case first. Each $E_{i,k}$ acts on a Maya diagram trivially (as multiplication by zero of the related basis Fock vector) in two cases. They occur either if $k$-th site is empty, or/and if $i$-th site is already occupied by a particle as on Figure 2.1. Otherwise $E_{i,k}$ action places black ball from its $k$-th site to the $i$ site, multiplying the new Maya diagram by $(-1)^{c_{ik}}$, where $c_{ik}$ is a number of black balls between $i$-th and $k$-th sites, in other words, the number of particles which the particle originally located at $k$-th site “jumps over” when moving to its new, $i$-th, site, see Figure 2.2. An arrow on this figure shows that the particle hope from the site $k$ to the site $i$.

$\begin{array}{c}
\bullet \quad \bullet \quad \bullet \quad \bullet \\
\cdot \quad \cdot \quad \cdot \quad \cdot \\
\cdot \quad \cdot \quad \cdot \quad \cdot \\
\cdot \quad \cdot \quad \cdot \quad \cdot \\
E_{i,k} = 0, \\
E_{i,k} = 0, \\
E_{i,k} = (-1)^{c_{ik}} \\
\end{array}$

1. Elimination of Maya diagrams  
2. Nontrivial action

**Figure 2**: Elementary $gl(\infty)$ actions on Maya diagrams

The action of operators $E_{i,i}$ on Maya diagrams may be called testifying action. They acts trivially on the vacuum Maya diagram of level 0. For $i < 0$ the operator $E_{i,i}$ eliminates any Maya diagram which (similar to the vacuum diagram) contains a particle at the site $i < 0$. If the site $i$ is empty then $E_{i,i}$ acts as the multiplication by unity. For $i \geq 0$ the operator $E_{i,i}$ eliminates any Maya diagram which (as the vacuum diagram of the level 0) contains the white ball at the site $i$. If there is a particle at the site $i$ then $E_{i,i}$ acts as multiplication by unity.

Because Maya diagrams are in one-to-one correspondence with basis Fock vectors we can act on them by sums and products of operators (27).

The element

$$Q = \sum_{i \in \mathbb{Z}} E_{i,i}$$  \hspace{1cm} (28)

plays a special role. One can easily check that operator $Q$ acts a Maya diagram of a level $n$ by the multiplication by $n$. It is called the charge operator.

**Graphs**
Let us take vertices of a Maya diagram as vertices of a graph, where vertices are labelled by integers and each pair of vertices may be connected by a pair of opposite directed arrows. Given
\[ A = \sum_{i \neq k} a_{i,k}E_{i,k} \in \hat{gl}(\infty), \]  
we assign the weight \( a_{i,k} \) to the arrow which starts at the vertex \( k \) and ends at the vertex \( i \). For the sake of simplicity, we shall not consider diagonal terms, \( E_{i,i} \). Also let us not draw arrows of zero weight. In this way we obtain some weighted graph, related to given \( A = \in \hat{gl}(\infty) \). For instance, for
\[ A = \sum_{i=n-2}^{n} (E_{i+1,i} + E_{i,i+1}) \]  
we obtain the graph on Figure 3.1.

![Graph](image)

Figure 3: Maya diagrams projected on graphs

In the present paper we are interested in motion of particles along arrows of the graph whose vertices coincide with vertices of Maya diagram. Having in mind this picture one may forget about all vertices of Maya diagrams which are not linked by vertices. For instance, the Maya diagrams depicted on Figures 3.2 and 3.3, projected on the graph depicted on Figure 3.1, respectively, are as on Figures 3.4 and 3.5. If the number of arrows is finite we shall ignore the rest part of a Maya diagram.

### 2.1 Young diagram versus Maya diagram. Interface between 2D area versus configuration of particles on the 1D lattice

For discrete systems the difference between 1D and 2D systems of hard core particles is rather conventional in the following sense: any 1D lattice gas configuration may be converted into a Young diagram viewed as an example of a condensate of hard core 2D particles.

Let us figure out an example:
These two figures show the relation between the motion of particles on Maya diagrams and a motion of the interface.

One may say, that each hop of a 1D particle upward is related to an increasing of the area of the corresponding Young diagram (a condensation of 2D particles to the Young diagram), while
1. Upward/downward hop of a particle on a Maya diagram

2. Adding/removing a box to a Young diagram

Figure 4: Equivalent representations of elementary steps

a hop downward results in the inverse process of 'evaporation of particles' and the decreasing of the area the Young diagram. (A figure similar to Fig. 4.2 may be found in [14] without relations to the topic of the present paper: to stochastic motion.)

2.2 Generating vector for Maya diagrams

There are vectors, coherent states, \( \langle n | e^{H(t)} \rangle \) and \( e^{H(t)} | n \rangle \), which depend on parameters \( t = (t_1, t_2, \ldots) \), see Appendix A.3, and which generate all Fock space with fixed given sea level \( n \),

\[
\langle n | e^{H(t)} \rangle = \sum_{\lambda \in P} \langle \lambda, n | s_\lambda(t) \rangle, \quad e^{H(t)} | n \rangle = \sum_{\lambda \in P} | \lambda, n \rangle s_\lambda(t) \tag{31}
\]

where \( s_\lambda(t) \) is the Schur polynomial [42]. The Schur polynomial is a polynomial in many variables \( t_1, t_2, \ldots \), which are labelled by partitions. The appearance of the Schur polynomial is typical for integrable systems.

2.3 Random turn motion on graphs. Permutation factor and weight of path

Consider the graph related to

\[
A = \sum_{i \neq k} a_{i,k} E_{i,k}, \quad E_{i,k} = f_i \bar{f}_k, \quad i \neq k \tag{32}
\]

which is a set of vertices, some of which may be linked by arrows: an arrow \( (i, k) \) starts on a vertex \( k \) and ends on a vertex \( i \). We assign a weight \( a_{i,k} \) to each arrow \( (i, k) \). In case \( a_{i,k} = 0 \) there is no arrow. The matrix \( a \) with entrances \( a_{i,k} \) is called connecting matrix for a graph. For simplicity of consideration we omit diagonal terms.

We shall consider a discrete dynamical motion along a Maya diagram with an initial diagram \( \nu \) given by

\[
|\nu\rangle \rightarrow A|\nu\rangle \rightarrow \ldots \rightarrow A^T|\nu\rangle \tag{33}
\]

Here \( T \) is viewed as discrete time variable.
As we have explained in the Introduction we have different paths connecting two different configurations, see (13); each path is related to a sequence of intermediate partitions.

Consider $A^T$ which is a sum of monomials, each one is a product of $T$ terms $a_{i_\alpha,k_\alpha} E_{i_\alpha,k_\alpha}^{\alpha}, \alpha = 1, \ldots, T$. Each path of process (33) is generated by a unique monomial applied to a initial Maya diagram. The number $T$ may be referred as duration of path. It may be described as set of consequent events, single hops. Each time instant, say, $\alpha$, a particle hops along an arrow $(i_\alpha,k_\alpha)$: one hop at each time instant. Say, monomial $a_{i_\alpha,k_\alpha} E_{i_\alpha,k_\alpha}^{\alpha}$, provided it acts nontrivially on $\nu$, at first time instant describes the hop of a particle along an arrow $(i_1,k_1)$ and at last time instant, $T$, describes the hop of particle along an arrow $(i_T,k_T)$. The product weights of $T$ arrows which were crossed by particles, $\prod_{\alpha=1}^{T} a_{i_\alpha,k_\alpha}$, is called the weight of path.

Each pair of configurations may be connected by a number of paths each of which has its own weight.

The fact that $A^T$ is a sum of weighted monomials means randomness of the motion described by (33).

Let us call it random turn motion along a graph, given by $A$.

The random motion where in each time instant only one chosen at random particle hops is called random turn motion.

To each path we shall assign permutation factor as follows.

Given acting nontrivially monomial, we can follow each particle involved into the motion along graph. Let us enumerate particles located on the initial Maya diagram from the top of diagram, and keep these (personal) numbers of particles on the target Maya diagram (where each particle 'remembers' it’s given number). Let us re-enumerate particles of the target Maya diagram from the top: this may be also obtained by a certain permutation of personal numbers which we assigned to the particles in the beginning of the motion. If the sign of the permutation is even then the permutational factor of the path is equal to $+1$. If the sign is odd the permutation factor of the path is equal to $-1$.

Thus, to a each path we assign a weight, a permutation factor, and a duration.

### 2.4 Graphs and tau functions. Tau function as counting function

Consider the random turn motion described in the previous subsection.

Consider all paths of duration $T$ starting with a configuration described by a partition $\lambda'$ and ending on a configuration $\lambda$. The sum of weights of all such paths with positive permutation factor we denote by

$$W_{\lambda' \to \lambda}^{(+)}(T)$$

The sum of weights of all such paths with negative permutation factor we denote by

$$W_{\lambda' \to \lambda}^{(-)}(T)$$

Let us note that permutation sign appears due to the fermionic approach to the problem. In problems of random motion of hard-core particles along graphs which know nothing about fermions, one may wonder about a total weight, which is

$$W_{\lambda' \to \lambda}(T) = W_{\lambda' \to \lambda}^{(+)}(T) + W_{\lambda' \to \lambda}^{(-)}(T)$$

From the previous consideration we have

$$\langle \lambda, n | A^T | \lambda', n \rangle = W_{\lambda' \to \lambda}^{(+)}(T) - W_{\lambda' \to \lambda}^{(-)}(T)$$
By (31) we relate these numbers to the so-called TL tau function \([14], [64]\) in the following way

\[
\tau_A(n, t, \bar{t}) := \langle n| e^{H(t)} e^{z A} e^{\bar{H}(\bar{t})} |n\rangle = \sum_{T=0}^{\infty} \sum_{\lambda, \lambda'} \frac{z^T}{T!} \left(W^{(+)}_{\lambda' \to \lambda}(T) - W^{(-)}_{\lambda' \to \lambda}(T)\right) s_\lambda(t) s_{\lambda'}(\bar{t})
\]

(38)

where \(A\) is

\[
A = \sum_{i \neq j} a_{i,j} f_i \bar{f}_j
\]

(39)

and \(z, t = (t_1, t_2, \ldots), \bar{t} = (\bar{t}_1, \bar{t}_2, \ldots)\) are formal parameters, and \(f_i, \bar{f}_i\) are free fermions.

Thus, we say, that the power of \(z\) counts the number of steps. \(T\)-step process is

\[
|\nu\rangle \to A |\nu\rangle \to \cdots \to A^T |\nu\rangle
\]

(41)

The variable \(T\) is treated as the (discrete) time of the random process described by \(A\).

The most simple case is random processes where \(W^{(-)}_{\lambda' \to \lambda}(T) = 0\)

(42)

for all pairs \(\lambda' \lambda\). Then tau function may be interpreted as counting function for weights \(W^{(-)}_{\lambda' \to \lambda}(T)\). Then it follows that \(W^{(-)}_{\lambda' \to \lambda}(T)\) is a subject to certain discrete bilinear equations (Hirota equations), which we shall write down not here.

However we can weaken condition (42) keeping the interpretation of tau function as counting function. Consider two examples where particles may hop only to the nearest neighboring sties. One example is presented on the figures 5. Figures 5.1 and 5.3 depict graphs where initial step function, \(\nu = 0\), configuration will decay. In figure 5.1 at each time instant a particle chosen at random will hop upward along an arrow if the neighboring place is not occupied. In figure 5.2, motion is impossible: in this configuration particles are locked. Each finite perturbation of \(\nu = 0\) will come to \(\nu = 0\) in a final number of steps. In Figure 5.3 not only particles are chosen at random but also arrows are to be chosen at random: if the neighboring vertices are free, chosen at random particle may hop either upward or downward. It is called non-intersecting random turn walkers, (on this topic see [6], [17], [8] and [3]). These examples of random motion keep the condition (42) because it is ordered motion and the signs of each path is positive. For the choice of \(A\) as in Figures 5.1-5.3, namely, \(A = H_1, H_{-1}, H_1 + H_{-1}\) the weight of any path is either zero, or one. It means that tau function is a generating function for number of ways to get configuration \(\lambda\) starting with a configuration \(\lambda'\) in \(T\) steps.

Second example is a walk on a ring, where particles are allowed to hop to the nearest neighboring site. How to obtain such graphs is depicted on figure 6: figures 6.3 and 6.4 are obtained respectively from figure 6.1 and 6.2. On figures 6.2 (or, the same 6.4) condition (42) is violated because when a particle hops along the arrow linking vertices \(n\) and \(n+5\) permutation of particles is odd in case the total number of particles on the ring is even. Thus each path containing odd number of hops along arrows linking sites \(n\) and \(n+5\) has negative permutation factor. Nevertheless if we assign positive weight to all arrows except arrows linking \(n\) and \(n+5\)
1. $H_{-1}$
2. $H_1$
3. $H_{-1} + H_1$

Figure 5: Random turn motion along graphs

1. $n+5$
2. $n+4$
3. $n+3$
4. $n+2$
5. $n+1$
6. $n$

1. $n+5$
2. $n+4$
3. $n+3$
4. $n+2$
5. $n+1$
6. $n$

Figure 6: For the case depicted in 2 and 4, in order to keep the probabilistic interpretation, one has to assign negative weights to the arrows that link $n + 5$ and $n$ sites, tau function will count the transition weights between any pair of configurations. Say, if arrows linking $n$ and $n + 5$ have weight $-1$, while all other arrows have weight $1$ tau function counts number of ways linking any pair of configuration on the ring.

Processes like those depicted in Figures 5 and 6 are also known as discrete versions of simple exclusion processes (totally asymmetric (d-TASEP) for figures 5.1, 5.2, and symmetric for figure 5.3), which may be considered either on the line as on figures 1-3, or on a ring as in the Figure 6. This type of models was introduced by M.E.Fisher. It may be called discrete time asymmetric simple exclusion process (d-ASEP) in case the probability rate for hops in the opposite directions are different and does not depend upon site numbers. The solvability of ASEP was proven by H. Spohn in [21, 22] by method of Bethe anzats and a huge literature is devoted to the study of different aspects of this model which, in non-equilibrium physics, plays a role comparable to the role of Ising model in statistical physics.

The problem of evaluation of correlation functions, asymptotic behavior we hope to consider separately using fermionic approach.

The dependence of transition probabilities on parameters (weights of arrows) will be considered in subsection 2.6. Before this consideration of non-equilibrium system of particles we find that a consideration of fermions in thermal equilibrium will be helpful.
2.5 Fermions in thermal equilibrium

Consider the following statistical ensemble: there is an infinite set of levels labelled by integers. Each level \( i \) has energy \( U_i \) and may be occupied by a fermion. In thermodynamics the probability to occupy a site \( i \) is proportional to \( \exp(-U_i) \), where \( U_i \) is energy level numbered by \( i \) (we put the Boltzmann constant times temperature to be one). In case \( U_i \) is a monotonous function of the site number we can use Maya diagrams of some level, say \( n \). At zero temperature all lower site are occupied up to this level (in solid state physics called Fermi level). The bottom limit of Maya diagram is fully packed by particles, while top limit is free of particles. A fragment of the picture is sketched out in the following figure

\[
\begin{align*}
\vdots &\quad \bullet i+1 \quad \exp (-U_{i+1}) \\
\circ i &\quad \exp (-U_i) \\
\bullet i-1 &\quad \exp (-U_{i-1}) \\
\bullet i-2 &\quad \exp (-U_{i-2}) \\
\vdots &
\end{align*}
\]

Fig.1. Fermionic levels where some sites are occupied

Each configuration of fermions distributed among levels has its weight given by the product of the Gibbs factors related to the occupied sites. The contribution of the fragment depicted on the figure is \( e^{-U_{i-2}-U_{i-1}-U_{i+1}} \).

These are non-interacting fermions. The equilibrium ensemble is described by its partition function.

First, let us introduce the notation which we shall use throughout the paper

\[ U_\lambda(n) = \sum_{i=1}^{\infty} (U_{\lambda_i+i+n} - U_{-i+n}) \]  \hspace{1cm} (43)

Notice that for the vacuum configuration of any level \( n \) we have \( U_0(n) = 0 \).

The normalized partition function is then a sum over partitions

\[ e^{-F_0} = \sum_\lambda e^{-U_\lambda(n)} \]  \hspace{1cm} (44)

where \( F_0 \) is the free energy of the system of non-interacting fermions living on a Maya diagram of a level \( n \).

Remark. Let notice, that we can cut Maya diagram from the bottom via the following procedure. We send all energies of sites located below, say, site \( N \), to minus infinity. Without the loss of generality we take \( N = 0 \). Then, we have in particular,

\[ e^{U_{-1}-U_0} = 0 \]  \hspace{1cm} (45)

In this way we restrict our consideration by finite number of fermions, \( n \). It is available in case our system is related to one-dimensional solid state system where there are finite number
of particles. In such cases \( n \) will be the number of particles equal to Fermi level. If only \( n \) fermions are involved we have

\[
U_\lambda(n) = \sum_{i=1}^{n} (U_{\lambda, i-n} - U_{-i+n}),
\]

(46)

where \( \lambda_i - i \) is the coordinate of the particle numbered by \( i \) \((i = 1, \ldots, n)\) on a Maya diagram of a level \( n \) which is cut at zero site. We count particles from the top and their coordinates from the bottom.

We shall consider both cases, \( N = 0 \) and \( N \to -\infty \), the first will be described with the help of semi-infinite Maya diagram and the second by usual Maya diagram.

**Modified Coulomb potential** \( E_\lambda \). Now, suppose that fermions pair-wise interact, that is, now, the energy of the system is \( U_\lambda(n) \) plus the energy of pair-wise interaction. We choose this interaction as Coulomb interaction (if sites are interpreted as level under Fermi surface then, this is Coulomb interaction in the momentum space)

\[
q^2 \log(i-j),
\]

(47)

which describes the repulsion of fermions of a charge \( q \) at sites \( i \) and \( j \). The problem that Coulomb energy goes to \( +\infty \) for semi-infinite lattice. In order to avoid the divergency of the energy for semi-infinite lattice we shall modify it taking the energy of particles in a configuration \( \lambda \) in form

\[
E_\lambda = -\log s_\lambda(t_\infty) := \log \frac{\prod_{i<j}^{L} (h_i - h_j)}{\prod_{i=1}^{L} h_i!}, \quad h_i = \lambda_i - i + L, \ i = 1, \ldots, L, \ L \geq \ell(\lambda)
\]

(48)

Notation \( s_\lambda(t_\infty) \) is taken from the Appendix A.1, see (177).

**Remark 2.1.** This expression contains pair-wise Coulomb interaction (47) in the enumerator and an external electric potential in the denominator. As one may check this expression vanishes on the vacuum configuration for any value of \( L \geq 0 \). Let us mark that \( L \) is not a parameter: the energy does not depend on the choice of \( L \) provided it exceeds the length of a partition. It is reasonable to recall the meaning of numbers \( i, \lambda_i \) and \( h_i \). The number \( i \) numerate particles counted in the downward direction. The number \( \lambda_i \) measure the shift of the particle \( i \) in the upward direction from it’s homesite in the vacuum configuration. Given \( L \) the number \( h_i \) is the coordinate of the particle numbered by \( i \) which is counted from the origin located on the site \( n - L \) on the Maya diagram of level \( n \). The point that the interaction (48) does not depend on the choice of this origin which one can send to \( -\infty \). It is perfectly adopted for the lattice gas with fully packed “bottom” limit. Moreover, one can show that the vacuum configuration is the configuration with minimal energy equal to is zero.

The related free energy will be denoted by \( F_q \).

In this case the partition function of the system is

\[
e^{-F_q(n)} = \sum_\lambda e^{-U_\lambda - q^2 E_\lambda + q^2 |\lambda| \log(t_1 \bar{t}_1)} = \sum_\lambda e^{-U_\lambda (s_\lambda(t_\infty))^{q^2} (t_1 \bar{t}_1)^{|\lambda|}}
\]

(49)

where we introduce dependence on axillary parameters \( t_1 \bar{t}_1 \) for further convenience.

There are special cases, \( q^2 = 1 \) and \( q^2 = 2 \), where (49) may be identified with tau functions of quite different integrable systems. (The case \( q^2 = 4 \) is also related to integrable systems, however we will not need it at present paper). We shall consider these cases separately.
For $q^2 = 2$ the slightly modified series (49) is equal to
\[ e^{-F_2(n,t_1,t_1)} := \sum_\lambda e^{-U_\lambda q^2 \mathbb{E}_\lambda + |\lambda| \log(t_1 t_1)} = \sum_\lambda e^{-U_\lambda (t_1 t_1)} |\lambda| \left( s_\lambda(t_\infty) \right)^2 \]
\[ = c_n^{-1} \langle \lambda | e^{H_1} e^{\sum_{i \geq 0} U_i \tilde{t}_i - \sum_{i < 0} U_i \tilde{t}_i} e^{\tilde{t}_1 H_1} | \lambda, n \rangle \]
which is a Toda lattice (TL) tau function. (Here $c_n$ is a normalization constant chosen in a way that the right hand side is equal to unity for $t_1 = \tilde{t}_1 = 0$, see (211)). The equality directly follows from (31) and (26) if we take into account (210) and finely use (173) and (177). This formula is true for both cases $N = -\infty$ and $N = 0$. To obtain the last case we need keep in mind relation (45).

Let us refer to the paper [41] where the Coulomb gas on the lattice was related to the tau functions of different type.

In general, the following vacuum expectation value which is known to be an example of the Toda lattice tau function, namely
\[ \tau(n, t, \tilde{t}) = \langle \lambda | e^{H(t)} e^{\sum_{i \geq 0} U_i \tilde{t}_i - \sum_{i < 0} U_i \tilde{t}_i} e^{H(\tilde{t})} | n \rangle = c_n \sum_\lambda e^{-U_\lambda} s_\lambda(t) s_\lambda(\tilde{t}) \]
(51)
yields the partition function for the system of fermions in a configuration $|\lambda, n\rangle$ whose pair-wise interaction (47) is replaced by interaction of all particles via the potential given by
\[ \log s_\lambda(t) + \log s_\lambda(\tilde{t}) \]
(52)
which, in general, is not of a pair-wise type. The second equality of (51) is derived from relations (31),(26) and (210), exactly in the same way as previous formula (50) but without specification of Schur functions. The pair-wise type interactions one may obtain via specification of parameters $t$ and $\tilde{t}$ which enter the tau function (51), via formulas (179)-(176) of Appendix A.2. (Such specifications were considered in [52], [53], [54] and [50] in quite different contexts).

Remark 2.2. To complete the talk about links between the Coulomb lattice gas and tau functions we note, that if we choose a special parametrization of the set $\{U_i, i \in \mathbb{Z}\}$: via sets of variables $\tilde{t} := (\tilde{t}_1, \tilde{t}_2, \ldots)$ and $\tilde{t}^* := (\tilde{t}_{-1}, \tilde{t}_{-2}, \ldots)$ as follows:
\[ U_i = \sum_{m \neq 0}^{\infty} i^m \tilde{t}_m, \quad i \in \mathbb{Z} \]
(53)
we obtain (see [50]) that $e^{-F_2}$ is a TL tau function in variables of a certain dual hierarchy.

Similar parametrization of the parameters $U$ which enter tau function (50) was used in the context of completely different problems in [47] and in [1]. In [1] relation (53) has a meaning of the dispersion law for fermions in solid state physics, where $i$ plays the role of momentum and $U_i$ plays the role of energy of fermions. Authors thanks P. Wiegmann for explaining [1] before it was published.

The case $q^2 = 1$ is most important in view of its application in the next section, and is less trivial from the point of view of which it may be in integrable hierarchies. Given level of vacuum configuration $n$ consider
\[ \varrho_n(U) := \sum_\lambda \langle \lambda, n | e^{\sum_{i \geq 0} U_i \tilde{t}_i - \sum_{i < 0} U_i \tilde{t}_i} e^{H_1} | 0, n \rangle \]
(54)
\[ S(t) = c_n \sum_{\lambda} \sum_{\ell(\lambda) \leq n} e^{-U_{\lambda}(n)} \delta_{\lambda}(t) \]

where \( h_i = \lambda_i - i + n \). The second equality is obtained from formulas (210) and (206) and the last equality from (177) of Appendix A.2, and where \( c_n \) is given by (211) (We shall omit this constant \( c_n \) below). Then replacing summation over the cones \( h_1 > \cdots > h_n \geq 0 \) by summation over non-ordered non-negative \( h_1, \ldots, h_n \) which may be done due to the permutational symmetry of the terms (and getting the factor \((n!)^{-1}\)) we arrive at the partition function for \( n \) fermions with the electric charge 1:

\[ \varrho_n(U) = 1 \frac{1}{n!} \sum_{h_1, \ldots, h_n \geq 0} n \prod_{i=1}^{n} e^{-U_{h_i} + U_{i+n}} \prod_{i<j} |h_i - h_j| = e^{-F_1(n)} \] (55)

Notice that we do not need condition (45) because the reduction to the \( n \) particle partition function is achieved by imposing the condition \( \ell(\lambda) \leq n \).

**Remark 2.3.** The interesting fact is the following. Let us write the grand partition function

\[ \varrho(\mu, U) = \sum_{n=0}^{\infty} e^{n\mu} \varrho_n(U) \] (56)

where \( \mu \) is a chemical potential. Then one can show that this is an example of the infinite-soliton tau function of a version of the BKP hierarchy, suggested by V.Kac and J. van de Leur [35] (called them ”charged” BKP hierarchy). This BKP hierarchy is different from the BKP hierarchy presented in [14], [32].

The fermionic representation is a specification of the fermionic representation found in [40] in the context of the study of Pfaff lattice [4] and its generalizations. For the sake of simplicity we shall write down the fermionic representation only for \( \varrho_n(U) \), where \( n \) is an even number. It is

\[ \varrho_n(U) = n!(n) e^{H(\tilde{t})} \sum_{\tilde{t}} 1 + f(i) f(j) \text{sign}(j-i) e^{-H(\tilde{t}^*)} |0\rangle \] (57)

where

\[ f(i) = \sum_{m=0}^{+\infty} \tilde{t}^m f_m \]

and where \( H(\tilde{t}) \) and \( \tilde{H}(\tilde{t}^*) \) are given by (40) and sets of variables \( \tilde{t} := (\tilde{t}_1, \tilde{t}_2, \ldots) \) and \( \tilde{t}^* := (\tilde{t}_{-1}, \tilde{t}_{-2}, \ldots) \) are related to the set \( U := (U_0, U_1, \ldots) \) via

\[ U_i = 2 \sum_{m \neq 0} \tilde{t}_m \] (58)

where two semi-infinite sets of parameters \( \tilde{t}_m \) are called higher times of the coupled charged BKP hierarchy. Let us figure out the equidistant energy levels case:

\[ U_{i-1} - U_i = -2\tilde{t}_1 \] (59)

which is described by all \( \tilde{t}_m \) vanish except \( \tilde{t}_1 \).

Notice that (58) is not one-to-one correspondence, given sets \( \tilde{t} \) and \( \tilde{t}^* \) uniquely define the variables \( U \) but not vice versa.
One need to make some comments on (57). We obtain it by developing the exponential term in the middle of the vacuum expectation value into the Taylor series. Then, as one can see, only $1 \over 2$-th term contributes. After one uses certain tricks invented in [40] he gets (57).

In the same way, using results of [40] we obtain that $e^{-F_q}$ is also a tau function. We shall not write precise formulas for this case. Thus, for special values of electric charges, $q = 1, \sqrt{2}$ (and also for the case $q = 2$ which we omit in the present paper) we relate Coulomb particles to tau functions.

2.6 Random turn walk in an external potential. Decay of the step function

Let us describe a certain model. Consider the random turn walk related to the weighted graph given by

$$A(U) = A_+(U) + A_-(U),$$

and where

$$A_+(U) = \sum_{i \in \mathbb{Z}} e^{U_{i-1} - U_i} f_i \bar{f}_{i-1}, \quad A_-(U) = \sum_{i \in \mathbb{Z}} e^{U_i - U_{i-1}} f_{i-1} \bar{f}_i,$$

We use the same letters $U = \{U_i\}$ as in the previous subsection in order to pay attention to a certain similarity of the answers for probabilities to find particles in given configurations obtained in rather different problems: particles in the thermal equilibrium and particles subjected to the chosen random motion.

According to the previous consideration our model is the following. There is the infinite graph, which is the one-dimensional lattice $\mathbb{Z}$ (which we view as a set of integer points on a vertical line), with all neighboring sites linked by pairs of oppositely directed arrows. Each site may be either empty, or filled by at most one particle. The 'bottom' limit of this graph is fully packed by particles, while the 'top' limit is free of particles. At each time step, in the 'middle' of the graph we have some configuration of particles which are subject to random motion.

In each unit time interval a particle chosen at random hops to the neighboring position provided it is free. It may hop either upward, or downward with different probabilities. The rate of the hop of a particle from the site $i - 1$ to the site $i$ is given by

$$up : \quad r(i) = e^{-U_i + U_{i-1}}$$

while the rate of the hops from the site $i$ to the site $i - 1$ is given by

$$down : \quad r^*(i) = e^{-U_{i-1} + U_i}$$

which is the number inverse to (62).

One may say that we consider random turn walking particles in an external field, which affects the rates of the hops. Notice that this model should be considered as a generalization of the so-called random turn model which was introduced in [15]. (See e.g., [8], [18], for other approaches.)

A simplifying auxiliary picture may be the following. Each particle moves in the potential field given by $U$ where $U_i$ is referred as the potential of a site $i$. The rate of a hop depends on the difference of the potentials between final and target sites as given by (62) and (63).

The alternative picture is obtained via the Young diagram of a Maya diagram. The process under consideration is a random changing of the shape of a Young diagram, viewed as random gluing and evaporating of boxes to it in such a way that only one box is added (or removed)
at each time step. Each box has a 'gluing energy' $U_R - U_{R-1}$, given by the distance, $R$, of the box to the main diagonal of the Young diagram. (This distance is called a content of a box of a Young diagram [42]). An example is given below, where a Maya diagram of level 0 and the related Young diagram are drawn (the Young diagram is drawn by bold lines). The distances (the "contents") of boxes marked by x on the figure are $R = -1$ for the lower box and $R = 2$ for the upper box (these numbers coincide with the height coordinates of black balls on the first figure). For boxes marked by *, as going up and right we respectively obtain $R = -2, 0, 3$.

Figure 7: Two realizations of random turn walk

Let us note that the one directed process of only adding of boxes (obtained from the general case by the imposing of condition that all $U_i \ll U_{i-1}$) is well studied, then the problem is equivalent to the enumeration of the so-called standard Young tableau [42], see below.

Let us evaluate transition probabilities.

For the sake of simplicity we take Dirac level $n$ to be zero and denote $U_\lambda(n)$ by $U_\lambda$ where we use notation (43).

First of all let us note that the weight of the transition from an initial configuration $|\lambda'\rangle$ to the final configuration $|\lambda\rangle$ in $T$ steps is given by the formula

$$W_{\lambda'\to\lambda}(T) = e^{U_\lambda - U_{\lambda'}}N_{\lambda',\lambda}(T)$$

(64)

where $N_{\lambda',\lambda}(T)$ is the number of ways to come from the position $\lambda'$ to the position $\lambda$ in $T$ steps. This follows from the choice of the weight of a single hop in form (62)-(63) and from the definition of the weight.

For the fermions in the state of the thermal equilibrium we have the partition function

$$Z = \sum_\lambda e^{-U_\lambda}$$

(65)

Each $Z^{-1}e^{-U_\lambda}$ yields the probability to find fermions in a state $|\lambda\rangle$.

In our case of random turn walk, the hard core particles which start from a given initial configuration $|\lambda'\rangle$ are described via

$$Z_{\lambda'}(T) = \sum_\lambda e^{U_\lambda - U_{\lambda'}}N_{\lambda',\lambda}(T)$$

(66)
(One may call $N_{\lambda',\lambda}(T)$ kinematic entropy factor.) Then

$$P_{\lambda'\to\lambda}(T) = Z_{\lambda'}(T)^{-1}e^{U_{\lambda'}-U_{\lambda}}N_{\lambda',\lambda}(T)$$

yields the probability to find the hard core particles in a state $|\lambda\rangle$ as a result of $T$-step random turn walk of an initial configuration $|\lambda'\rangle$ in an external potential $\{U_i, i \in \mathbb{Z}\}$.

In particular, in case the random turn walk starts with the vacuum configuration, then

$$Z_0(T) = \sum_{\lambda} e^{-U_{\lambda}}N_{\lambda,0}(T)$$

where $N_{\lambda,0}(T)$ is the number of ways to gain the configuration $|\lambda\rangle$ in $T$ steps if the initial configuration is the vacuum one.

The probability to find the hard-core particles (subject to random turn walk in a potential $\{U_i, i \in \mathbb{Z}\}$) in a configuration $|\lambda\rangle$ is

$$P_{0\to\lambda}(T) = Z_0(T)^{-1}e^{-U_{\lambda}}N_{\lambda,0}(T)$$

yields the probability to find the hard core particles in a state $|\lambda\rangle$ as a result of $T$-step random turn walk of an initial configuration $|\lambda'\rangle$.

As we shall show below

$$N_{\lambda,0}(T) = 2^{\frac{|\lambda|-T}{2}} \frac{T!}{(T-|\lambda|)!} \cdot s_{\lambda}(t_{\infty})$$

where $s_{\lambda}(t_{\infty})$ is the Schur function evaluated in a special point, $t_{\infty}=(1,0,0,\ldots)$, see Appendix A.3 (177).

Let us obtain (68)-(70) via the fermionic representation. Notice that we take

$$e^{z(A_+ + A_-)} = e^{\frac{z^2}{2}}e^{zA_+}e^{zA_-}$$

At the present paper we shall study the simplest case: $\lambda' = 0$. Then, our model describes a decay of the step configuration as a function of the discrete time $T$. Let us take into account that $A_-(U)|0\rangle = 0$.

By (71) we have

$$e^{z(A_+ + A_-)} = e^{\frac{z^2}{2}}e^{zA_+}e^{zA_-}$$

which gives the weight of the random process where an initial configuration of hard core particles $|\lambda'\rangle$ comes to the final configuration $|\lambda\rangle$ in $T$ steps.

We are interested in the evaluation of

$$W_{\lambda'\to\lambda}(T) = \tau_{\lambda',\lambda}(T) = \langle \lambda | (A_+ + A_-)^T | \lambda' \rangle$$

where $s_{\lambda}(t_{\infty})$ is the Schur function evaluated in a special point, $t_{\infty}=(1,0,0,\ldots)$, see Appendix A.3 (177).

Now let us decompose both sides in Taylor series in $z$ to find the right hand side of (72)
For time duration $T = 2m + |\lambda|$ we obtain

$$W_{0\rightarrow\lambda}(T) = T! \left( \frac{1}{2^m m!} \right) e^{-U_\lambda s_\lambda(t_\infty)} = T! 2^{\frac{|\lambda| - T}{2}} \frac{1}{(\frac{|\lambda|}{2})!} \cdot e^{-U_\lambda s_\lambda(t_\infty)} \tag{75}$$

which, for $U = 0$, gives the number of ways to get a configuration $|\lambda\rangle$ starting from vacuum configuration $|0\rangle$ in $T$-steps (for $T - |\lambda| = 2m$ is even and non-negative). The index $m$ counts how many times particles, which are involved in the random process, move downward. The left hand side vanishes if $T - |\lambda|$ is odd.

Also, notice that the length of partition $\lambda$ can not exceed the duration of motion $T$. The equality $\ell(\lambda) = T$ is related to the motion (or, the same, to the "path" as was explained in the subsection 2.3) where a single (the upmost on the vacuum Maya diagram) particle moves, and each hop is upward one. Thus in this very case the distance it has pass (which is equal to the first part of the partition $\lambda$, or, the same, equal to the length of the first row of Young diagram) is equal to $T$.

Let us consider different cases described by relation (75).

(1) For $\lambda = 0$ the Boltzmann factor $e^{U_\lambda} = 1$ and the weight is equal to the number of ways to return back

$$W_{0\rightarrow0}(T) = N_{0,0}(T) = (T - 1)!! = e^{\frac{1}{2} T \log T + \ldots} \tag{76}$$

where the last relation describes the large $T$ limit.

(2) The case $T = |\lambda|$ corresponds to the non-stop forward motion: all jumps of the particles are in upward direction, and described by $W_{0\rightarrow\lambda}(|\lambda|) = \langle \lambda|A^{|\lambda|}_\infty|0\rangle$. If one turns to the description of the motion via Young diagrams, one may see that the number

$$N_{\lambda,0}(|\lambda|) = |\lambda|! s_\lambda(t_\infty) =: d(\lambda) \tag{77}$$

describes the number of ways to create a Young diagram of given shape $\lambda$ by gluing at random box by box in such a way that each time we have a Young diagram, see fig.7. (In other words it is the number of the so-called standard tableau of the shape $\lambda$, see Appendix A.1 and for details see [42]). Thus we have

$$W_{0\rightarrow\lambda}(|\lambda|) = e^{-U_\lambda}|\lambda|! s_\lambda(t_\infty) = e^{-U_\lambda} d(\lambda), \tag{78}$$

(3) The case $\ell(\lambda) = 1$ means that the final state is a single particle configuration (however, in the middle of the process more particles may be involved, thus, the problem is different from the random walk of a single particle restricted to the half-line, which in $T \rightarrow \infty$ limit yields Brownian motion on the half-line). Thus, in this case $\lambda = (\lambda_1)$ where $\lambda_1$ is equal to the shift of the upmost particle from its position in the initial vacuum configuration. Suppose the hopping rate

$$e^{-U_i + U_{i-1}} = r(i) = r \tag{79}$$

does not depend on $i$ (which was figured out in (59), $r = \exp(-2\tilde{t}_1)$). As a result of simple evaluation one shows that in $T \rightarrow \infty$ limit the dominant term in the subset of $\ell(\lambda) = 1$ configurations $\lambda$ is related to the partition $\tilde{\lambda} = (\tilde{\lambda}_1)$ given by

$$\tilde{\lambda}_1 = r\sqrt{T} + \frac{1 - r^2}{2} + \frac{1}{r\sqrt{T}} \left( r^2 + \frac{(1 - r^2)^2}{8} \right) + O\left(T^{-\frac{3}{2}}\right)$$
The weight for one-particles configurations in large $T$ limit near $\tilde{\lambda}$ is given by the formula which resembles formula for the Brownian motion

$$ W_{0,\lambda} \simeq T! \exp\left( -\frac{(\lambda - r\sqrt{T})^2}{2\sqrt{T}} \right) $$

As we see the variance is given by $T^{\frac{1}{2}}$. At last we note, that in case the rate (62) depends on site, then in a wide class of rates in large $T$ limit $\lambda_1$ may be evaluated as the solution of $\tilde{\lambda}_1 = r(\tilde{\lambda}_1) \sqrt{T}$. For instance, for Gauss potential, $U_i = \frac{1}{2} q^2$, one obtains $\tilde{\lambda}_1 \sim \log T$.

(4) The case where $T - |\lambda|$ is large enough. Via Stirling’s formula we obtain

$$ W_{0,\lambda}(T) \simeq \sqrt{2}(T)^{\frac{|\lambda|}{2}} e^{-\frac{T}{2}} e^{O(T)} e^{-U \lambda} s_\lambda(t_\infty), \quad |\lambda| \ll T \quad (80) $$

Remark 2.4. Let us notice that the sum in the right hand side of (68) may be considered as a certain correlation function of a discrete version of the orthogonal ensemble of random matrices. To see it we use the same trick as in [54]. First, according to (177) we write

$$ s_\lambda(t_\infty) = \frac{\prod_{i<j}(h_i - h_j)}{\prod_{i=1}^T h_i}, \quad h_i = \lambda_i - i + T \geq 0, \quad i = 1, \ldots, T \quad (81) $$

Then we use the fact that we summarize a symmetric in the variables $h_i$ function multiplied by the right hand side of (81). In this case we can use permutations of $h_1, \ldots, h_T$ to replace the sum over the cones $h_1 > \cdots > h_T \geq 0$ by sum over all $h_i \geq 0$ getting a factor 1 over $T!$. At last we obtain

$$ Z_0(T) = c \sum_{h_1, \ldots, h_T \geq 0} \Gamma \left( 1 + \frac{T^2 + T}{4} - \frac{1}{2} \sum_{i=1}^T h_i \right)^{-1} e^{\sum_{i=1}^T V(h_i)} \prod_{i,j=1}^T |h_i - h_j| \quad (82) $$

$$ = c \sum_{h_1, \ldots, h_T \geq 0} \Gamma \left( 1 + \frac{T^2 + T}{4} - \frac{1}{2} \sum_{i=1}^T h_i \right)^{-1} e^{E(h,1)} \quad (83) $$

$$ = \sum_\lambda \tau_{\lambda,0}(T) \quad (84) $$

where $c = 2^{-\frac{T}{2} + \frac{T}{2}}$ and where

$$ V(h_i) = -U h_i + U_{T-i} - \log \Gamma(h_i + 1) + \frac{\log 2}{2} h_i \quad (85) $$

and $E(h, q), h = (h_1, \ldots, h_T)$ is the electrostatic energy of $T$ charges $q$ located on the one-dimensional lattice and having coordinates $h_1, \ldots, h_T$:

$$ E(h, q) = \sum_{i=1}^T (U_{h_i} - U_{T-i}) - \frac{\log 2}{2} h_i - q^2 \log \prod_{i<j} |h_i - h_j| + q^2 \log \Gamma(h_i + 1) \quad (86) $$

The number of terms in sums (82) and (83) is finite thanks to the Gamma function insertion. However in $T \to \infty$ limit, the sum ranges over all positive integers $h_i$.

Notice that $E(h, q)$ vanishes on the vacuum configuration $\lambda = 0$.

Terms in the right hand side of (86) are to be interpreted as follows: the probability rates given by $U$ are related to the external electric potential (which may depend on a site coordinate), gives rise to
the first term. The third term is the Coulomb interaction of particles with the unit charge. The last term has a meaning of an external electric field which provides the vanishing of the Coulomb energy of the vacuum configuration of the particles. Two last terms appeared due to the Schur function in the right hand side of (75) and descended from the hard core interaction of the particles. The (discrete) time of the random turn walk is equated with the total number of Coulomb particles, or, the same, with the ‘size’ of orthogonal matrix whose eigenvalues are presented by the positive integers \( h_1, \ldots, h_T \).

The \( \tau_{\lambda,0}(T) \) gives the weight of the random process where the vacuum initial configuration of hard core particles decays to a final configuration given by coordinates \( h_1, \ldots, h_T \) in \( T \) steps.

Now let us turn to the problem of finding large \( T \) limit. In a usual way, we variate positions of particles in order to obtain the dominant term in sum (82).

First let us remind that the length, say \( R \), of the partition we are looking for does not exceed \( T \). Introducing the density of particles \( \sigma \leq 1 \) (which the number of particles for one site) as a function of the variable \( h_i = \lambda_i - i + R \) (the origin of Maya diagram is related to \( h = R \)) in a way

\[
\int_0^\infty \sigma(h) dh = R
\]  

we come to the equation for \( \sigma \) which defines the dominant configuration of particles at time \( T \gg 1 \) in the continues limit

\[
\log \frac{r(h - T)}{h} + P \int_0^\infty \frac{\sigma(x) dx}{h - x} + \frac{1}{2} \log \left( T + \frac{R^2}{2} - \frac{R}{2} - \int_0^\infty x \sigma(x) dx \right) = 0
\] (88)

where \( P \int \) stands for the principal value.

Let us note that this equation may have no solutions.

This equation one can solve in case the hopping rate (62) does not depend on site: \( r(i) = r \), see (79). With the help of the formula

\[
\int_0^{2R} \frac{1}{2 - \frac{1}{\pi}} \frac{\arcsin \left( \frac{h}{R} - 1 \right)}{x - h} dh = \log \frac{2x}{R}
\] (89)

(which may be extracted from [39], our proof the reader will find in the Appendix) we obtain the following solution to (88)

\[
\sigma(h) = \frac{1}{2} - \frac{1}{\pi} \arcsin \left( \frac{h}{R} - 1 \right), \quad h \in [0, 2R]
\] (90)

and

\[
\sigma(h) = 1, \quad h < 0; \quad \sigma(h) = 0, \quad h > 2R,
\] (91)

where the length \( R \) of the partition is found by substitution of (90)-(91) into (88) and is equal to

\[
R = 2 \sqrt{\frac{T}{1 + r^{-2}}},
\] (92)

see fig.8 which describes the decay of initial vacuum Maya diagram of level 0 in large \( T \) limit.

Let us consider the partition \( \lambda = \lambda(T) \) and it’s Young diagram related to the asymptotic density function \( \sigma(h) \). Then solution (90)-(92) correspond to a Young diagram symmetrical under the reflection with respect to the main diagonal (this follows from the symmetry between particles and holes which exists in case of constant rate). The length of the partition \( \ell (\lambda(T)) \), it’s
area \(|\lambda(T)|\) and the number of boxes on the main diagonal \(k(\lambda(T))\) are given by the following formulae

\[
\ell(\lambda(T)) = R = 2\sqrt{\frac{r}{1 + r^{-2}}}
\]

\[
|\lambda(T)| = \int_0^{2R} h\sigma(h)dh - \frac{R^2}{2} + \frac{R}{2} = \frac{R^2}{4} + \frac{R}{2} = \frac{T}{1 + r^{-2}} + \sqrt{\frac{T}{1 + r^{-2}}}
\]

\[
k(\lambda(T)) = \frac{2}{\pi}\sqrt{\frac{T}{1 + r^{-2}}}
\]

We obtain (95) using the fact that the last number is equal to the number of particles which passed the origin (the origin is related to \(h = R\)) after duration \(T\). This number is obtained by evaluating the following integral:

\[
k(\lambda(T)) = \int_R^{2R} \sigma(h)dh = \frac{R}{\pi}
\]

The derivative \(\frac{dk(\lambda(T))}{dT} = \frac{1}{2\pi}(1 + r^{-2})^{-\frac{1}{2}}\) yields current of particles through the origin related to asymptotic configuration.

The number of downward steps \(m(T)\) of particles (see (75)) related to the asymptotic configuration is

\[
m(T) = \frac{T - |\lambda|}{2} = \frac{T}{2(1 + r^{-2})} - \frac{1}{2}\sqrt{\frac{T}{1 + r^{-2}}}
\]

These formulae give simple answers for particular cases: (a) completely symmetric simple exclusion process, \(r = 1\), (b) locking potential \(r \to 0\) resulting to \(\lambda(T) = 0\) (c) totally asymmetric exclusion process (d-TASEP) where particles hop only upward, \(r \to \infty\). Two last cases were considered in the list of examples which follows (75).

Let us write down a formula for the Schur function evaluated for the asymptotic partition \(\lambda = \lambda(T)\)

\[
s_\lambda(t_\infty) = e^{2R\left(\int_0^1 \tilde{\sigma}(u)(u - \log uR)du + \frac{1}{2} \int \tilde{\sigma}(u)\tilde{\sigma}(u') \log(uR - u'R)du'\right) + \cdots} = e^{-\frac{1}{2}R^2 \log R + O(1)}
\]

where by dots we denote minor terms in \(T \to \infty\) limit and where \(\tilde{\sigma}(u) := \sigma(uR)\). Together with (70) and (97) it yields the number of ways to get the asymptotic configuration in \(T\) steps via random turn walk in large \(T\) limit as

\[
N_{\lambda(T),0}(T) = \frac{T!}{2^m(t)m(T)!} s_{\lambda(T)}(t_\infty) = T!e^{-\frac{T}{2(1+r^{-2})}\log T - \frac{r}{2(1+r^{-2})}\log T + \cdots} = e^{\frac{T}{2}\log T + \cdots}
\]

Notice that in the large \(T\) limit the leading term of \(\log N_{\lambda(T),0}(T)\) does not depend on the rate \(r\), though the asymptotic configuration \(\lambda(T)\) depends. In the large \(T\) limit the leading term of \(\log N_{\lambda(T),0}(T)\) is the same for the locking case \(r \to 0\) where answer is given by (76) and in the d-TASEP case \(r \to \infty\) where answer is given by (77).

It will be interesting to compare considered numbers (length, weight, current,...) evaluated on the asymptotic configuration with their average values.

Let us mark that the shape of the asymptotic Young diagram given by (90)-(91) coincides with the shape found by Kerov and Vershik [38] in the problem of study of limiting shape of
random partitions with fixed weight $|\lambda| = N$ distributed according to the so-called Plancherel measure. We obtain the identification if we put $N = R^2$. As soon as we arrive at (75) this fact is not so striking because the Plancherel measure is equal to $N! s_\lambda(t_\infty)^2$. Thus, it is similar to the variational problem for d-TASEP where (75) is replaced by (78). Then, the variational problem for d-ASEP may be split into two parts: given weight of partition and taking it into account via Lagrangian multiplier to variate the shape of the partition and obtain (90)-(91) (this is quite similar to the d-TASEP problem where the weight is fixed as $|\lambda| = t$), then to variate with respect to the weight. This might be the other way to get formulae (90)-(92).

The problem of evaluation of correlation functions for various versions of random turn walk is studied in the forthcoming paper [29].

2.7 Weights which depend on mutual configuration: coupling with Coulomb-type weight

As we have seen the case $U = 0$ may be interpreted as a free random turn walk of the non-intersecting particles on the lattice. Here ”free” means that each particle hops either upward or downward with the same probability (provided, the target site is empty). Then, as we have seen, in the answer for the weight of transition between initial vacuum configuration to a configuration given by (integer) coordinates $h$, hard-core interaction gives rise to the factor $\Delta(h)$. Then (83) can be interpreted as a partition function for the Coulomb gas of particles (each one has the unit charge) living on the lattice.

For non-vanishing $U$ the weights of hops also depend on the location of the hop. In Coulomb gas picture $U_i$ is interpreted as an electric potential at a point $i$.

Now let us consider a model of random motion where the weight of each step is additionally

Figure 8: Decay of step function for the case of a constant hopping rate in $T \to \infty$ limit
dependent on the particle configuration as follows. Consider the ‘coupling’ of the model studied in the previous subsection 2.6 to the modified Coulomb interaction (48) where particles possess an arbitrary charge $q$. Namely, suppose that the weight of the hop is additionally multiplied by a factor which depends on the configuration of the particles according to formula

\[
\left( \frac{h_i}{h_i + 1} \prod_{j=1, j \neq i}^{\infty} \frac{h_i + 1 - h_j}{h_i - h_j} \right)^q
\]  

for the hop of $i$-th particle from the site $h_i$ to the site $h_i + 1$ (provided this site is empty), and according to

\[
\left( \frac{h_i}{h_i - 1} \prod_{j=1, j \neq i}^{\infty} \frac{h_i - 1 - h_j}{h_i - h_j} \right)^q
\]  

for the hop of $i$-th particle from the site $h_i$ to the site $h_i - 1$ (provided this site is empty).

Then instead of (75) we obtain

\[
W_{0 \to \lambda}(T; q) = T! \left( \frac{1}{2^{m_m!}} \right) e^{-U_s} s_\lambda(t_\infty)^{1+q^2} = T! \frac{1}{(2^{1+q^2})!} \cdot e^{-U_s} s_\lambda(t_\infty)^{1+q^2}
\]  

Normalization function is a sum of the weights over all transitions of the vacuum configuration of particles which are described by the non-intersecting random turn walk in the Coulomb potential of time duration $T$. After the corresponding changing of the weights of steps according to the appearance of the Coulomb interaction with a charge $q$, we get the resulting normalization function as

\[
Z(T, q) := \sum_{\lambda \in P} (s_\lambda(t_\infty))^{q^2} \langle \lambda| (A_- + A_+)^T |0 \rangle
\]  

\[
= \sum_{h_1, \ldots, h_T = 1}^{\infty} \frac{1}{\Gamma\left(\frac{T-|\lambda(h)|}{2} + 1\right)} \left( \frac{1}{2} \right)^{\frac{T-|\lambda(h)|}{2}} \prod_{i=1}^{T} e^{-V(h_i)} \prod_{i,j}^{T} |h_i - h_j|^\beta
\]  

In (104) $h$ is (non-ordered) set of positive numbers, $h_1, \ldots, h_T$, $|\lambda(h)| := h_1 + \cdots + h_T + \frac{T}{2} (T - 1)$, and

\[
\beta = 1 + q^2
\]  

\[
V(h) = U_h + \beta \log h!
\]  

Thus, by choosing the rates of hops of the random turn walk we arrive to effective electric charges model in thermal equilibrium with particles forced by

(1) external field given by $U$

(2) hard-core interaction between particles

(3) Coulomb interaction between particles with coupling constant $q$

Note that if $q \neq 0$ we have no anymore need to mention that our particles are hard-core ones, since the Coulomb repulsion force does not allow to occupy the same place by more than one particle.
One can see that the special cases, \( q = 0, 1, \sqrt{3} \), result in links with discrete analogues of orthogonal \((\beta = 1)\), unitary \((\beta = 2)\) and symplectic \((\beta = 4)\) matrix ensembles [43] which as it is known may be related to the integrable hierarchies.

Then, for constant rate the asymptotic configuration will be the same (90)-(91), see fig.(8), where now
\[
R^{1+q^2} = 2\sqrt{\frac{T}{1 + q^2}} 
\]
As we see the size of the asymptotic domain shrinks because of the adding of the modified Coulomb interaction which tries to confine particles near the vacuum configuration, see Remark 2.1. For asymptotic configuration \( \lambda(q, T) \) we straightforwardly obtain
\[
\ell(\lambda(q, T)) = \left( \frac{4T}{1 + q^2} \right)^{\frac{1}{2(1+q^2)}}, \quad |\lambda(q, T)| = \frac{1}{4} \left( \frac{4T}{1 + q^2} \right)^{\frac{1}{2(1+q^2)}} + \frac{1}{2} \left( \frac{4T}{1 + q^2} \right)^{-\frac{1}{2(1+q^2)}} 
\]
\[
k(\lambda(q, T)) = \frac{1}{\pi} \left( \frac{4T}{1 + q^2} \right)^{\frac{1}{2(1+q^2)}}, \quad m(q, T) = \frac{T}{2} - \frac{1}{8} \left( \frac{4T}{1 + q^2} \right)^{\frac{1}{2(1+q^2)}} - \frac{1}{4} \left( \frac{4T}{1 + q^2} \right)^{-\frac{1}{2(1+q^2)}}
\]
which gives answers for the asymptotic partition length, weight, for the number of particles which passed the origin and for the number of backward steps. For \( q^2 > 0 \) in the large \( T \) limit and we also obtain
\[
\log s_\lambda(t_\infty) = - \frac{1}{2(1 + q^2)} \left( \frac{4T}{1 + q^2} \right)^{\frac{1}{2(1+q^2)}} \log T + \ldots
\]
\[
\log N_{\lambda(q, T), \ell}(T) = \frac{T}{2} \log T + \ldots
\]

At last let us mark that the introduction of the free parameter \( q \) is similar to the introduction of a parameter \( s \) in [37], where a notion of an entropy of stochastic dynamical systems was studied. We hope to consider this problem in future.

### 2.8 Determinantal formulae: Wick theorem and Gessel-Viennot formulae

Consider \( A_+ (U) \) of (61). Then
\[
\frac{1}{N!} \langle \lambda' | A_+ (U)^N | \lambda \rangle = \langle \lambda' | e^{A_+ (U)} | \lambda \rangle, \quad N = |\lambda'| - |\lambda| 
\]
\[
(108)
\]
The form of right hand side allows to apply the Wick theorem:
\[
\langle \lambda' | e^{A_+ (U)} | \lambda \rangle = \det \left( \langle m | \bar{f}_{m+h} e^{A_+ (U)} f_{m+h_0} | m \rangle \right)_{i,j=1,...,m} = \det \left( e^{U_{m+h_i} - U_{m+h'_i}} \right)_{i,j=1,...,m}
\]
where
\[
h_i = \lambda_i - i + m, \quad h'_i = \lambda'_i - i + m, \quad m \geq \ell(\lambda') \geq \ell(\lambda)
\]
\[
(110)
\]
where we use
\[
\langle 0 | \bar{f}_{i+N} e^{A_+ (U)} f_i | 0 \rangle = \frac{1}{N!} e^{U_{i-N} - U_{i+N}}
\]
\[
(111)
\]
The right hand side also allows to consider tau functions

\[ \tau(t, U, \tilde{t}) = \langle 0 | e^{H(t)} e^{A_+(U)} e^{\tilde{H}(\tilde{t})} | 0 \rangle \]  

as a generating function for the transition weights \( \langle \lambda | A_+(U)^N | \lambda \rangle \).

**Remark 2.5.** Let us mark the formula

\[ \langle \lambda' | e^{A_+(U)} | \lambda \rangle = s_{\lambda / \lambda}(t) \prod_{i=1}^{\ell(\lambda')} e^{-U_{h_i} + U'_{h'_i}} \]  

where \( t_{\infty} = (1, 0, 0, \ldots) \) and

\[ \lambda_i = h_i - i + m, \quad \lambda'_i = h'_i - i + m, \quad m = \max(\ell(\lambda), \ell(\lambda')) = \ell(\lambda') \]  

The left hand side of (108) is interpreted as a \( N \)-step random turn process, described above. Let us consider an example.

**Gessel-Viennot formula for binomial determinants**

Consider two sets of positive integers \( a_k > \cdots > a_1 \geq 0 \) and \( b_k > \cdots > b_1 \geq 0 \), where \( a_i \geq b_i, \; i = 1, \ldots, k \). Following [20] let us study the following determinant

\[ \det \begin{pmatrix} a_1 & \cdots & a_k \\ b_1 & \cdots & b_k \end{pmatrix} := \det \left( \begin{pmatrix} a_i \\ b_j \end{pmatrix} \right)_{i,j=1,\ldots,k} \]  

called binomial determinant. Let us identify sets of \( b_{k-i} \) and \( a_{k-i} \) with sets of \( h_i \) and \( h'_i \) related by (110) to partitions \( \lambda \) and \( \lambda' \) that is

\[ \lambda_i = b_{k-i} + i - k, \quad \lambda'_i = a_{k-i} + i - k, \quad i = 1, \ldots, k \]  

Take

\[ [x]_k := \left( \frac{1}{1} \sum_{i=1}^{k} x_i, \frac{1}{2} \sum_{i=1}^{k} x_i^2, \ldots \right), \quad [y]_k := \left( \frac{1}{1} \sum_{i=1}^{k} y_i, \frac{1}{2} \sum_{i=1}^{k} y_i^2, \ldots \right) \]  

It is known [42] that Schur functions \( s_{\lambda}(t) \), where \( t = [x]_k \), vanish on partitions whose length exceed \( k: \ell(\lambda) > k \).

Then, as a specification of (109) and by using formulas (31), we get that the following tau function

\[ \tau(k, [x]_k, [y]_k) = \langle k | e^{H([x]_k)} e^{A_+} e^{\tilde{H}([y]_k)} | k \rangle \]  

\[ = \sum_{\lambda, \lambda' \in P} s_{\lambda}([x]_k) s_{\lambda'}([y]_k) \det \begin{pmatrix} a_1 & \cdots & a_k \\ b_1 & \cdots & b_k \end{pmatrix}, \]  

where

\[ A = \sum_{i \in \mathbb{Z}} i f_i \tilde{f}_{i-1}, \]  

is a generating function for the binomial determinants. This \( A \) is related to the motion along the following graph.
Remark 2.6. A tau function similar to (118) was used in [27] for the study of the so-called two matrix model.

The graph related to (120) is sketched out above. The weight of an arrow may be considered as the number of identical arrows of unit weight. Thus, the product of weights of successive arrows connecting two vertices yields the number of ways connecting these vertices.

The number of ways connecting points $k$ and $i$ on the Maya diagram is counted by \( \binom{i}{j} \). Therefore,

\[
\langle 0 | \bar{f}_i A^{i-j} f_j | 0 \rangle = \binom{i}{j} \quad (121)
\]

From (113) we have the relation obtained in [20]:

\[
\det \left( \begin{array}{c}
\frac{h_i'}{h_j} \\
\frac{h_j'}{h_i}
\end{array} \right)_{i,j=1,\ldots,k} = \frac{(n)_\lambda}{(n')_{\lambda'}} s_{\lambda/\lambda'}(t_\infty) \quad (122)
\]

where is the so-called skew Schur function (170).

Gessel and Viennot suggested to interpret the binomial determinant as the number of non-intersecting ways connecting points $a_i$ with points $b_i$, where $i = 1, \ldots, k$ (in our approach connecting a pair of Maya diagrams) as it is depicted at the right hand side figure [20]. Each step is either down or left directed:

Notice that if we introduce the space-time axis \((x, t)\) as it is figured on the picture Fig.3 we observe non-intersecting one-dimensional random walk of $k$ vicious walkers, moving with speed $\pm 1$, changing at random the direction at each time instant $\Delta t = \frac{1}{2}$. The particles start at $t = 0$ their motion from positions $a_1, \ldots, a_k$. Later a particle by particle meets a ‘light’ signal spreading with the speed $-1$ which starts motion in the origin at $t = 0$. A particle which started at $a_i$ meets the signal at point $\frac{1}{2} b_i$.

Considering random path to be an interface one can relate the right hand figure to the randomly growing area below the path. By stars we mark the last portion of area (ice covering of a step scala). Then the parameter $k$ in the tau function (118) may be viewed as discrete time.

These problems will be considered in the forthcoming paper.
1. The transition weight is given by the number of paths to the next figure

2. Paths connecting a pair of Maya diagrams may be viewed as the interface of growing area

Figure 10: Counting connecting paths

If we replace $e^{A}$ by $e^{A} \cdots e^{A} = e^{TA}$ in the expression for tau function (118) we obtain $T$-ple mapping of partitions described in the next section. In certain sense the model considered in this section interpolates between random turn and vicious walkers models, which we consider in the next section.

3 Action of $\hat{GL}(\infty)$ group elements on Fock vectors and random processes

Below we shall construct certain versions of stochastic motion on the space of partitions, which may be also called random slices (“rocks”) and random layering. Namely, one may view stochastic dynamics on partitions as models of random slices, which may be visualized as a set of partitions placed one above another according to its time evolution. Each three-dimensional figure of height $T$ obtained in this way present a path (13) and may be compared with a rock inscribed into a corner between two walls. In this section we present only descriptions of the models in terms of fermions without study of these models.

3.1 Determinantal processes

Now consider the case where each $o^{j-1}$ in (6) has a special form

\[ o^{j-1} = g^{j-1} = e^{\sum_{i,k} a^{j-1}_{ik} f_i \bar{f}_k}, \quad o(T) = g(T) := g^{T,T-1} \cdots g^{1,0} \]  

These exponentials are known to be elements of $\hat{GL}(\infty)$ group, see [14], [32], where the importance of $\hat{GL}(\infty)$ for constructing of integrable equations together with their solutions was established.

Then

\[ \tau_T(x, y) = \langle 0 | \bar{f}(x_N^{-1}) \cdots \bar{f}(x_1^{-1}) g(T) f(y_1) \cdots f(y_N) | 0 \rangle \Delta(x)^{-1} \Delta(y)^{-1} \]

is known [14], [32] to be a TL tau function. In our case we want tau functions to be dependent on the axillary discrete variable $\tau$ via (123).
In what follows we shall assume the following simplifying condition
\[ a_{ik}^{(j-1)} = 0, \quad i, k < 0, \quad j = 1, 2, \ldots \]  
(124)

Let us remind definitions of the Schur function
\[ s_\nu(x) = \det(x^{h_b}) \Delta(x)^{-1}, \quad h_i = \nu_i - i + N \]
and basis Fock vectors
\[ |\nu, N\rangle = f_{h_N} \cdots f_{h_1} |0\rangle, \quad \langle \nu, N| = \langle 0| \bar{f}_{h_1} \cdots \bar{f}_{h_N} \]

Denoting the matrix elements of \( g(T) \):
\[ \tau_{\lambda,\nu}(T) = \langle \lambda, N| g(T) |\nu, N\rangle, \]  
(125)

we may rewrite tau function in form
\[ \tau_T(x, y) = \sum_{\lambda, \nu} \tau_{\lambda,\nu}(T) s_\lambda(x) s_\nu(y) \]  
(126)

If we have (124), then Wick’s theorem has especially simple form:
\[ \langle \mu^{(j)}| g^{(j-1)} |\mu^{(j-1)}\rangle = \det g^{(j-1)}(h_a^{(j)}, h_b^{(j-1)}), \quad g^{(j-1)}(a, b) = \langle 0| f_\alpha g^{(j-1)} \bar{f}_b |0\rangle \]  
(127)

Remark 3.1. Thanks to the Wick theorem, we write (12) as
\[ \tau_T(x, y) = \sum_{\lambda, \nu} p^T_M \left( \{ h_a^{(j)} \} \right) \Delta(x) \Delta(y) \]  
(128)

where the sum ranges over \( T \) sets of all admissible configurations \( h_a^{(j)}, \quad j = 0, \ldots, T \)

\[ p^T_M \left( \{ h_a^{(j)} \} \right) = \det(x_a^{(0)}) \det(y_a^{(0)}) \prod_{j=0}^{T-1} \det(g^{j+1}(h_a^{(j+1)}, h_b^{(j)})) \]  
(129)

Now,
\[ p^T_M \left( \{ h_a^{(j)} \} \right) = \frac{p^T_M \left( \{ h_a^{(j)} \} \right)}{\tau_T(x, y)} = \frac{p^T_M \left( \{ h_a^{(j)} \} \right)}{\sum_{\lambda, \nu} \tau_{\lambda,\nu}(T) s_\lambda(x) s_\nu(y)} \]  
(130)

almost coincides with the formula (1.10) of [23] (under some specification of functions \( \psi \) and \( \phi \) in this paper, and under certain notational replacements: say, \( h_a^{(j)} \) of (129) are \( x_a^{(j)} \) of [23] and \( \tau_T(x, y) \) of (129) is \( Z_{M,m} \) of [23]), where it describes the so-called reduced probability density. The only difference between formula (129) and formula (1.10) of [23] is that, in our case, variables \( h_a^{(j)} \) take discrete values as they are related to the parts of the intermediate partitions, while the main example of [23] is the multimatrix model where these variables coincide with eigenvalues of random matrices (then, \( a \) serves for the number of an eigenvalue, and \( j \) serves for a number of a matrix in the chain of coupled matrices).

More generally (129) describes a discrete analogue of the so-called multi-level determinantal ensembles where the "levels" are numbered by \( j \) in (128).

Any \( a^{(j-1)} \) in form (123) gives rise to a multi-level determinantal ensemble [13].
In our approach to random processes the tau function $\tau_T(x, y)$ plays the role of the generating function for $\tau_{\lambda, \nu}(T)$. The normalization function is

$$Z_\nu(T) = \sum_\lambda \tau_{\lambda, \nu}(T)$$

In the present paper we are interested not in the ratio $P_T^M \left( \{ h^{(j)}_a \} \right)$ mentioned in the remark (see (129)) but in:

$$P_{\nu \to \lambda}(T) = \frac{\tau_{\lambda, \nu}(T)}{Z_\nu(T)}$$

(130)
yielding the normalized weight (which may be interpreted as the probability in case all weights are positive) to arrive to a configuration $\lambda$ in $T$ steps if an initial configuration is $\nu$.

Let us note that the most natural way to consider (128) is to use the so-called multi-component fermions and multi-component integrable hierarchies. This will be done below.

### 3.2 Multi-component fermions

For certain problems it is suitable to re-write (128) with the help of multi-component fermions,

$$[f^{(j)}_n, f^{(i)}_m]_+ = [\bar{f}^{(j)}_n, \bar{f}^{(i)}_m]_+ = 0, \quad [f^{(j)}_n, \bar{f}^{(i)}_m]_+ = \delta_{j,i}\delta_{n,m}$$

(see Appendix A.2 for details). The multicomponent fermions were used in [14], [32] to construct hierarchies of multicomponent integrable equations.

We shall use the following notations. Given set of partitions, $\nu^{(j)}$, $j = 0, \ldots, T$, we introduce a basis Fock vector

$$| \nu^{(0)}, n^{(0)}; \ldots; \nu^{(T)}, n^{(T)} \rangle := \Pi^{(T)} \cdots \Pi^{(1)} | 0, \ldots, 0 \rangle$$

where

$$\Pi^{(j)} = f^{(j)}_{h^{(j)}} \cdots f^{(j)}_{h^{(1)}}, \quad h^{(j)}_i = \nu^{(j)}_i - i + n^{(j)}, \quad n^{(j)} = \ell(\nu^{(j)})$$

The dual Fock vector we denote by $\langle \nu^{(0)}, n^{(0)}; \ldots; \nu^{(T)}, n^{(T)} |$;

$$\langle \lambda^{(0)}, l^{(0)}; \ldots; \lambda^{(T)}, l^{(T)} | \nu^{(0)}, n^{(0)}; \ldots; \nu^{(T)}, n^{(T)} \rangle = \delta_{\lambda^{(0)}, \nu^{(0)}} \cdots \delta_{\lambda^{(T)}, \nu^{(T)}}$$

Now, we replace (6) by

$$| \nu, N; 0, 0; \ldots; 0, 0 \rangle \rightarrow o^{1,0} | \nu, N; 0, 0; \ldots; 0, 0 \rangle \rightarrow \cdots \rightarrow o^{T, T-1} \cdots o^{1,0} | \nu, N; 0, 0; \ldots; 0, 0 \rangle$$

(131)

where

$$N = \ell(\nu^{(0)})$$

(132)

and where each operator $o^{j+1, j}$ is an intertwining operator between different Fock spaces $F^{(j+1)}$ and $F^{(j)}$:

$$o^{j+1, j} = G^{j+1, j} := e^{\sum_{k, k \ge 0} g^{j+1, j}_{k,i} f^{(j+1)}_k f^{(j)}_i}$$

(133)

Then (131) describes the same process on partitions as (6) conditioned by (123) and by (124) provided that $g^{j+1, j}_{k,i}$ coincides with $g^{j+1, j}(k, i)$ of (126).
Denoting

\[ G(T) = G^{T,T-1} \cdots G^{1,0} \]  

we re-write tau function (127) as

\[
\tau_T(x,y) = (N!)^T \langle 0, \ldots, 0 | f^{(T)}(x_1) \cdots f^{(T)}(x_N) G(T) \tilde{f}^{(1)}(y_N) \cdots \tilde{f}^{(1)}(y_1) | 0, \ldots, 0 \rangle \Delta(x)^{-1} \Delta(y)^{-1}
\]

\[ = \sum_{\lambda, \nu} \tau_{\lambda, \nu}(T) s_{\lambda}(x)s_{\nu}(y) \]  

where \( \tau_{\lambda, \nu}(T) \) of (125) is reexpressed as

\[
\tau_{\lambda, \nu}(T) = \langle 0, 0; \ldots; 0, 0 | G(T) | \nu, N; 0, 0; \ldots; 0, 0 \rangle
\]

Tau function (135) is a specification of the multi-component tau function introduced in [14] (see also [34]), and almost coincides with the fermionic representation of the multi-matrix model presented in [31].

Regardless to the choice of representation for \( \tau_{\lambda, \nu}(T) \) which is obtained either from the one-component fermionic expectation value (125), or from the multi-component one, (136), the formulae (130) yields the transition probability to come to a configuration \( \lambda \) if the initial configuration is \( \nu \).

**Example:** \( N \) vicious walkers (discrete Brownian motion of \( N \) hard core particles)

Let us apply it to the model of \( N \) vicious walkers introduced by M.Fisher in [15] (actually he mainly considered the so-called lock step model where all walkers start on odd (or even) numbered coordinates on the lattice). At each tick of the clock each walker moves either up or downward subject to no two walkers occupying the same site at the same time. In various contexts this model was studied in [6], [17], [18], [63].

In this case for (133) we take

\[
G^{j,j-1} = \exp \sum_{i=0}^{\infty} \left( e^{-U_{i+1}^{(j)} + U_{i}^{(j-1)}} f_{i+1}^{(j)} \tilde{f}_{i}^{(j-1)} + e^{-U_{i-1}^{(j)} + U_{i}^{(j-1)}} f_{i-1}^{(j)} \tilde{f}_{i}^{(j-1)} \right)
\]  

(137)

The number \( j \) plays the role of physical time, the parameter \( T \) coincides with the duration of the process. Application of operator \( G^{j,j-1} \) to a Fock vector (=Maya diagram) describes the unit-time transition of \( N \) particles, each of which hops to the nearest site conditioned that in both initial and final configurations there are no sites occupied by more than one particle. In (137) \( U \) plays the role of an external field: a particle located at a site \( i \) hops upward with the rate \( e^{-U_{i+1}^{(j)} + U_{i}^{(j-1)}} \) and downward with the rate \( e^{-U_{i-1}^{(j)} + U_{i}^{(j-1)}} \).

Then (131) describes the motion of hard-core particles which start with a configuration \( \nu \). As it follows from (184) the particles are constrained to move only along the positive half-line. Particles are not influenced by this restriction in case their initial configuration (given by \( \nu \)) is far enough from the origin.

The exponent in (137) should be compared with expression (61) which was used earlier to get the random turn motion.

**Vicious walkers on the circle**

We need to replace (137) by

\[
G^{j,j-1} = \exp g^{j,j-1}, \quad g^{j,j-1} = e^{-U_{0}^{(j)} + U_{n}^{(j-1)}} f_{0}^{(j)} \tilde{f}_{n}^{(j-1)} + e^{-U_{n}^{(j)} + U_{0}^{(j-1)}} f_{n}^{(j)} \tilde{f}_{0}^{(j-1)}
\]  

(138)
where $h_i = \nu_i - i + N$. Introduce
\[
ochi(T) = o^{\tau - 1, \nu (\tau - 1)} o^{\tau - 1, \nu - 2} \ldots o^{2,1, \nu (1)} o^{1,0}
\]
(140)

Then, the weight we are looking for is the following correlation function
\[
K_{\nu \to \lambda}(\rho^{(1)}, \ldots, \rho^{(\tau - 1)}) = \frac{\langle 0, 0; \ldots, 0; \lambda, N | \nochi(T) | \nu, N; 0, 0; \ldots, 0, 0 \rangle}{Z_\nu(T)} = \frac{\det \left( \langle 0 | f_{h_i}^{(\nu)} o_{\chi}(T) f_{h_i}^{(0)} | 0 \rangle \right)}{Z_\nu(T)}
\]
(141)

where $h_i = \lambda_i - i + N$ and $h_i' = \nu_i - i + N$ ($i = 1, \ldots, N$), and where we use the notation $|0\rangle := |0, 0; \ldots, 0, 0\rangle$. The last equality is due to the Wick theorem.

(b) what is the weight of the process where all paths (13) constrained by the condition that no one part of an $j$-th intermediate partition $\mu^{(j)}$ belonging to the path coincides with any of parts of the $\rho^{(j)}$, where $j = 1, \ldots, \tau - 1$.

Let us consider ‘characteristic operators’
\[
\chi^{(j)} := \chi(\rho^{(j)}) = \bar{f}^{(j)}_{h_N^{(j)}} \ldots \bar{f}^{(j)}_{h_i^{(j)}} \bar{f}^{(j)}_{h_i^{(j)}} f^{(j)}_{h_i^{(j)}} f^{(j)}_{h_i^{(j)}}
\]
(142)

where $h_i^{(j)} = \rho_i^{(j)} - i + N$. Introduce
\[
o_\chi(T) = o^{\tau, \nu (\tau - 1)} o^{\nu - 1, \nu - 2} \ldots o^{2,1, \nu (1)} o^{1,0}
\]
(143)

The weight of this process is the following correlation function
\[
K_{\nu \to \lambda}(\bar{\rho}^{(1)}, \ldots, \bar{\rho}^{(\tau - 1)}) = \frac{\langle 0, \ldots, 0; \lambda | o_\chi(T) | \nu, 0, \ldots, 0 \rangle}{Z_\nu(T)} = \frac{\det \left( \langle 0 | \bar{f}_{h_i}^{(\nu)} o_\chi(T) f_{h_i}^{(0)} | 0 \rangle \right)}{Z_\nu(T)}
\]
(144)

where $h_i = \lambda_i - i + N$ and $h_i' = \nu_i - i + N$ ($i = 1, \ldots, N$). The last equality is due to the Wick theorem.

Equivalent representation is given by
\[
K_{\nu \to \lambda}(\bar{\rho}^{(1)}, \ldots, \bar{\rho}^{(\tau - 1)}) = \frac{\langle 0, 0; \rho^{(\tau - 1)}, \ldots; \rho^{(1)}, \rho^{(1)}; \lambda, N | o(T) | \nu, N; \rho^{(\tau - 1)}, \ldots; \rho^{(1)}, 0, 0 \rangle}{Z_\nu(T)}
\]
(145)
where \( n^{(r-1)} = \ell(\rho^{(r-1)}) \)

This form allow to write it via the Wick theorem as

\[
K_{\nu \rightarrow \lambda}(\rho^{(1)}, \ldots, \rho^{(r-1)}) = \frac{\det (\langle 0| f_{h_{0}}^{\nu} O(T) f_{h_{0}}^{\nu} | 0 \rangle)}{Z_{\nu}(T)}
\]

By an analogy with the so-called spectral correlation functions in random matrices, we call the weights \( K_{\nu \rightarrow \lambda}(\rho^{(1)}, \ldots, \rho^{(r-1)}) \) and \( K_{\nu \rightarrow \lambda}(\bar{\rho}^{(1)}, \ldots, \bar{\rho}^{(r-1)}) \) correlation functions.

### 3.3 Random layering. Chains of Darboux transformations

Below we consider simple application of certain \( GL_{\infty} \) transformations, sometimes called Darboux transformations. For \( U = 0 \) below and for the case of growing partitions these examples are mainly reformulation of different known random models considered in literature.

(1) **Random lay**

Let us consider four basic examples of \( \widehat{GL}(\infty) \) operators. They are basic in the sense that their non-vanishing matrix elements \( o_{\lambda,\lambda'} \) are related to pairs of Young diagrams different by adding/removing layers (strips) to Young diagrams: vertical or horizontal ones. Thus, we pick up four different types of \( o \), say, \( o^{(i)} = \exp A^{(i)} \), \( i = 1, 2, 3, 4 \). Let us consider these four cases separately.

1. The exponential of

\[
A^{(1)}(x, U) := -A^{(1)}(-x, U) = - \sum_{k=1,2,\ldots} \sum_{i \in \mathbb{Z}} e^{-U_{i,+}U_{i,-}(-x)^k} f_i \bar{f}_{i+k} \tag{146}
\]

yields laying of a given Young diagram by a vertical strip, see fig. 11, as it is given by

\[
o^{(1)}(x, U)|\lambda'\rangle = \sum_{\lambda} |\lambda\rangle o^{(1)}_{\lambda,\lambda'}(x, U)
\]

where

\[
o^{(1)}_{\lambda,\lambda'}(x, U) := \langle \lambda|o^{(1)}|\lambda'\rangle = e^{U_{\lambda',-}U_{\lambda,+}x} |\lambda'\rangle - |\lambda\rangle
\]

is non-vanishing only if the difference between \( \lambda \) and \( \lambda' \) is the so-called vertical strip [42].

Example:

Now turn to the related Maya diagrams which describes 1D configuration of the related lattice gas. Let us compare it with the random turn walk, see Fig.7 in the subsection “Decay of the step function”. In the random turn walk model in each time instant only one particle hops either one step upward or one step downward to the nearest neighboring site. The Fig.11 above describes the one step of a group of randomly chosen particles upward to the nearest neighboring sites which occurs at one time instant, this hop being conditioned that after the hop each site is occupied by no more than one particle. One may say that Fig.11b describes ‘trains’ of particles which hop one step upward. The potential \( U \) gives rise to the gluing rate in 2D picture and the hopping rate in 1D picture which site-depending. The weight to absorb \( n \) boxes is proportional to \( x^n \).

2. The exponential of

\[
A^{(2)}(x, U) := - \sum_{k=1,2,\ldots} \sum_{i \in \mathbb{Z}} e^{-U_{i,+}U_{i,-}(-x)^k} f_i \bar{f}_{i+k} \tag{147}
\]
Fig. 11a. Condensation of a lay onto a Young diagram $\lambda'$: Random 'rain' of boxes from rightward. These boxes may be fitted only to admissible places.

Fig. 11b. Examples of diagrams of $\lambda$ where $o^{(1)}(\lambda)$ is nontrivial. One layer (one strip) is added at random to the Young diagram $\lambda'$. Boxes marked by x depict the difference between $\lambda'$ and $\lambda$, called the vertical strip

Figure 11: A layering by a vertical strip

yields getting away of a given Young diagram a vertical strip, that is

$$o^{(2)}(x, U)|\lambda'\rangle = \sum_{\lambda} |\lambda\rangle o^{(2)}_{\lambda, \lambda'}(x, U)$$

where

$$o^{(2)}_{\lambda, \lambda'}(x, U) := \langle \lambda | o^{(2)} | \lambda' \rangle = e^{U_{\lambda'} - U_{\lambda} x |\lambda'\rangle - |\lambda\rangle}$$

is non-vanishing only if the difference between $\lambda'$ and $\lambda$ is the vertical strip [42].

Presenting this process as a random motion of $1D$ particles on Maya diagram we see that it describes a hop downward to the nearest site of randomly chosen group of particles (they may form 'trains') at each time instant.

(3) The exponential of

$$A^{(3)}(x, U) = \sum_{k=1,2,\ldots} \sum_{i \in \mathbb{Z}} e^{-U_i + U_{i-k} x^k_f i - f_{i-k}}$$

yields laying of a given Young diagram by a horizontal strip, as it is given by

$$o^{(3)}(x, U)|\lambda'\rangle = \sum_{\lambda} |\lambda\rangle o^{(3)}_{\lambda, \lambda'}(x, U)$$
where the transition weight

\[ o_{\lambda,\lambda'}^{(3)}(x, U) := \langle \lambda | o^{(3)} | \lambda' \rangle = e^{U_{\lambda'} - U_{\lambda} x} | \lambda' | - | \lambda | \]

is non-vanishing only if the diagram \( \lambda \) includes the diagram \( \lambda' \), and the difference between these diagrams is the so-called horizontal strip [42].

Example is given by fig. 12:

Fig.12a. Condensation of a lay onto a Young diagram \( \lambda' \). Random 'rain' of boxes from downward. These boxes may be fitted only to admissible places, which means that new figure should be a Young diagram again

Fig.12b. Examples of diagrams of \( \lambda \) where \( o_{\lambda,\lambda'}^{(1)} \) is nontrivial. One layer (one strip) is added at random to the Young diagram \( \lambda' \). Boxes marked by x depict the difference between \( \lambda' \) and \( \lambda \), called the horizontal strip

Figure 12: A layering by a horizontal strip

Turning to a 1D configuration of the related lattice gas we see that it is suitable to interchange roles of particle (pictured as black balls) and free sites (white balls). In this dual picture trains of holes hop one step downward at each time instant. The rate of this process depends on the difference of potential in initial and final positions and is proportional to \( x^n \), where \( n \) is the number of absorbed boxes.

(4) The exponential of

\[ A^{(4)}(x, U) := \sum_{k=1,2,...} \sum_{i \in \mathbb{Z}} e^{-U_i + U_{i+k} x} f_i \bar{f}_{i+k} \]

yields the random process of getting away of a given Young diagram a horizontal strip, that is

\[ o^{(4)}(x, U) | \lambda' \rangle = \sum_{\lambda} | \lambda \rangle o_{\lambda,\lambda'}^{(4)}(x, U) \]
where

\[ o^{(4)}_{\lambda, \lambda'}(x, U) := \langle \lambda | o^{(4)} | \lambda' \rangle = e^{U_{\lambda'} - U_{\lambda} x} | \lambda' | - | \lambda | \]

is non-vanishing only if the difference between \( \lambda' \) and \( \lambda \) is the horizontal strip [42].

These are four cases. Now one can consider a chain of transformations, each transformation is given by a set of corresponding \( U \) and \( x \) which define gluing rates.

Examples of these chains were considered in the reviewing paper [65] from the point of view of discrete Hirota equations.

(II) Chains of Darboux transformations. Random layering and "rocks"

The chain of transformations of initial Fock vector is as follows:

\[ | \lambda' \rangle \rightarrow e^{A_{\sigma_1}(x_1)} | \lambda' \rangle \rightarrow e^{A_{\sigma_2}(x_2)} \cdots e^{A_{\sigma_T}(x_T)} | \lambda' \rangle \tag{150} \]

where \( \sigma_j = 1, 2, 3, 4 \), and where \( T \) is a discrete time. Given \( j = 1, 2, 3, \ldots, T \), each \( A_{\sigma_j}(x_j) \) is characterized by a given set of \( \{ U^{(j)}_i, i \in \mathbb{Z} \} \) which enters definitions (146),(147),(148) and (149).

Given set \( \{ \sigma_j, j = 1, 2, 3, \ldots, T \} \) and \( \{ U^{(j)}_i, j = 1, 2, 3, \ldots, T, i \in \mathbb{Z} \} \) gives rise to a chain of Darboux transformations. Such chains may be viewed as words of lengths \( T \) formed by four types of characters (a 'DNA' coding a set of random processes of adding or eliminating of layers). Tau function is constructed as

\[ \tau(T, t, \bar{t}) = \langle 0 | e^{H(t)} o(T) e^{H(\bar{t})} | 0 \rangle = \sum_{\lambda, \lambda'} \tau_{\lambda, \lambda'}(T) s_{\lambda}(t) s_{\lambda'}(\bar{t}) \]

then, each word is related to a certain generalized Darboux transformation of the tau function. This tau function may be evaluated because we know all matrix elements of each 'letter' \( e^{A_{\sigma_j}(x_j)} \). Let us note that tau functions of this type were considered in [52], [51] as series in skew Schur functions which in special cases provided examples of Gelfand-Graev hypergeometric series.

This is a certain random process on partitions (and their Young diagrams) describing random adding and elimination of layers to their Young diagrams, each time step gives rise to an act of either creation, or elimination of layers, as it is shown in figures 11-14.

For the process of time duration \( T \), the relative weight of a configuration \( \lambda \) is given by the ratio

\[ P_{\lambda' \rightarrow \lambda}(T) = \frac{\tau_{\lambda, \lambda'}(T)}{Z_{\lambda'}(T)} \]

where

\[ Z_{\lambda'}(T) = \sum_{\lambda} \tau_{\lambda, \lambda'}(T) \]

is the sum of weights of all processes of duration \( T \) started with an initial configuration \( \lambda' \).

One may call the random process of creating and eliminating of layers as model of random slices, or, a "rock" inscribed into a corner between two walls, which may be viewed as a set of Young diagrams placed one above another as sections of 3D figure according to its time evolution. Each three-dimensional figure of height \( T \) obtained in this way presents a path (13).
Examples. Choose $\lambda' = 0$, and let $\sigma_j = 1$ (which means that we shall consider growing Young diagrams) and $U_i^{(j)} = U_i$, for all $j = 1, \ldots, T$.

Let us note that for $U = 0$ models considered below may be connected to various models related to the so-called semi-standard tableau (see Appendix A.1 and for details [42]), like a model of polynuclear growth (for instance, see [15] for a review).

A step number is again a Young diagram). This is a Markov process. The transition weight for each step, say,

$$A_j^{(1)}(x_j, U) = A^{(1)}(x_j, U) = -\sum_{k=1,2,\ldots} \sum_{i \in \mathbb{Z}} e^{-U_i + U_i^{(k)}} (-x_j)^k f_i \bar{f}_{i-k}$$  \quad (151)

In this model, $x_j$, $j = 1, \ldots, T$ and $U_i$, $i \geq 0$ are sets of parameters which define transition probabilities. Then the process may be called a version of random process of growing Young diagram which is as follows. The initial configuration is the zero diagram. Each time step one lay (a set of strips) is added at random to the previous Young diagram (in a way that new figure and the next configuration (related to the moment $j - 1$) is, say, $\mu^{(j-1)}$, and the next configuration (related to the moment $j$) is $\mu^{(j)}$, is equal to

$$W_{\mu^{(j-1)} \rightarrow \mu^{(j)}}(1) = e^{-U_{\mu^{(j)}} + U_{\mu^{(j-1)}} x_j^{\mu^{(j)} - \mu^{(j-1)}}}$$  \quad (152)

This yields a un-normalized probability to add a lay (a strip) of a weight (i.e. number of boxes) equal to $|\mu^{(j)} - \mu^{(j-1)}|$ to a Young diagram.

One can see that for this random process

$$W_{0 \rightarrow \lambda}(T) = e^{-U_{\lambda} s_{\lambda}(x_1, \ldots, x_T)} \quad \text{and} \quad Z_0(T) = \sum_{\lambda} e^{-U_{\lambda} s_{\lambda}(x_1, \ldots, x_T)}$$  \quad (153)

Since $U_0 = 0$ and $s_0 = 1$ one can notice notice that $1/Z_0(T)$ yields the probability to create zero Young diagram, $\lambda = 0$, i.e. the probability to create no any nonvanishing diagram).

Let us consider these formulae and find links to a few known random topics.

(1) First, consider the Gauss potential $U_i = e^{\frac{x_i^2}{2}}$ and chose $x_i = e^{c h_i'}$, where $c$ is a constant and $h_1', \ldots, h_T'$ are ordered positive integers $h_1' > \cdots > h_T'$. Then the weight

$$W_{0 \rightarrow \lambda}(T) = e^{c(h_1'^2 + \cdots + h_T'^2)} \prod_{i<j} (e^{c h_i'} - e^{c h_j'}) \det \left( e^{-c(h_i - h_j')^2} \right)_{i,j=1,\ldots,T}, \quad h_i = \lambda_i - i + T,$$

unexpectedly coincides with the (un-normalized) transition probability for $T$ non-intersecting ("ordered") Brownian particles on the line whose initial and final coordinates are positive integers $h_1', \ldots, h_T'$ and $h_1, \ldots, h_T$. In large $T$ limit it may be also related to the Itsykson-Zuber integral (relation between this integral and the Brownian motion is quite known topic, for instance see [36]).

Then there are few limiting cases for (153), which are simple.

(2) The first is $x_1 = \cdots = x_T = x < 1$. Then using (174) we obtain

$$Z_0(T) = e^{-U_T x^T \frac{1}{2} x^2 - \frac{1}{2} \prod_{i=0}^{T} e^{U_{T-i}} \prod_{i=0}^{T} \sum_{h_{i+1} \geq 0} \prod_{i,j=1}^{T} e^{-V_{h_i}} |h_i - h_j|^\beta}, \quad \beta = 1$$

where

$$V_{h_i} = U_{h_i} - h_i \log x$$

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Thus $Z_0(T)$ is the partition function of a discrete version of ensemble of random orthogonal matrices, which is also a certain tau function: the tau function of the Pfaff lattice introduced in [3], [34]. Let us note that the ensemble of random orthogonal matrices is well studied object, see [63], [6], [33]. For $U_i = 0$ it is a version of the well-known Laguerre ensemble.

(3) A similar result we obtain in case we take $x_j = e^{\sqrt{-1}(\phi + (j-1)\vartheta)}$, where $\phi, \vartheta$ are parameters. Then with the help of (176) we obtain unitary orthogonal ensemble

$$Z_0(T) = \frac{e^{\beta(\frac{1}{2}T^2 - \frac{1}{2}T)}}{T!} \prod_{i=1}^{T} (q; q)_{T-i} \sum_{h_1, \ldots, h_T \geq 0} \prod_{i=1}^{T} V_{h_i} \prod_{i,j=1}^{T} |e^{\phi h_i} - e^{\phi h_j}|^\beta, \quad \beta = 1 \quad (155)$$

where

$$V_{h_i} = U_{h_i} - h_i \vartheta \sqrt{-1}$$

where q-factorials $(q; q)_n$ are defined in (172), and $U_i$ may be chosen as series analogues to (53)

$$U_i = \sum_{m \neq 0} e^{im\sqrt{-1} t_m}, \quad i \in \mathbb{Z} \quad (156)$$

(Such parametrization was used in [50] for different purposes. These three cases can be also compared with [54]).

At last let us write down a group of examples (found in [42]) where $Z_0(T)$ (and, therefore, $P_{0-0}(T) = Z_0(T)^{-1}$) may be explicitly evaluated:

(a) Take all $U_i = 0$. Then, there is a simple formula

$$Z_0(T) = \sum_{\lambda} s_{\lambda}(x_1, \ldots, x_T) = \prod_{j=1}^{T} (1 - x_j)^{-1} \prod_{i<j}^{T} (1 - x_i x_j)^{-1}$$

(b) Take all $e^{U_i} = 1$, $i \leq m$ and $e^{U_i} = 0$, $i > m$. Then

$$Z_0(T) = \sum_{\lambda} s_{\lambda}(x_1, \ldots, x_T) = D_m / D_0, \quad D_m = \det \left(x_j^{2T+m-i} - x_j^{-1}\right)$$

where sum is going over all partitions whose Young diagrams are contained in the rectangle $T \times m$.

(c) Take all $e^{U_i} = 1$, $i \leq m$ and $e^{U_i} = 0$, $i > m$, as before, and put $x_j = e^{(2T-2j+1)\phi}$. Then

$$Z_0(T) = \sum_{\lambda} s_{\lambda}(e^{(2T-1)\phi}, \ldots, e^{\phi}) = \prod_{j=1}^{T} e^{\phi (m+2j-1)} - 1 \prod_{i<j}^{T} e^{2\phi (m+i+j-1)} - 1 = \sum_{\Theta} N(\Theta) e^{\Theta|\phi}$$

where first sum is going over all partitions whose Young diagrams are contained in the rectangle $T \times m$.

The second sum ranges over all symmetric plane partitions $\Theta$ whose diagram are contained in the box $T \times T \times m$. (Diagram of a plane partition is a 3D figure constructed from identical cubes, whose sections by horizontal planes are Young diagrams, see [42]). Here $N(\Theta)$ is the number of symmetric plane partitions of weight $|\Theta|$. Detailed explanations of the right hand sides of formulae in the last three examples (together with some other different examples of particular cases of summation formulae for $\sum_{\lambda} s_{\lambda}$) may be found in [42] from where they were borrowed by the authors.

Let us also note that the model of growing partition via layering (151) is directly related to the problem of enumerating of the so-called semi-standard tableau.
4 Discussion

In the present paper we want to figure out links between classical integrable systems and random system. These links in a natural way suggest to introduce site-dependent weights for hops of the particles and provide certain analogies between hard-core particles in equilibrium and non-equilibrium states. Along this line we introduced a version of discrete ASEP with site depending hopping rates. We show that the normalization function for probabilities of this model is related to classical integrable hierarchy of type $B$ rather than to the hierarchy of the type $A$ (the Toda lattice hierarchy) we started with. The normalization function plays the role similar to the role of partition function for statistical ensembles. We present a compact formula which describes the decay of the step function which also converts this problem to the evaluation of a certain correlation function for a discrete version of orthogonal matrix ensemble, where potential is defined by the site-dependent hopping rates of our model.

The further project aims the following problems:

(1) to study a model which is a modification of the model considered in the subsection 2.6. Namely, we replace the operator $A_+ + A_-$ by a linear combination of the so-called Virasoro generators $L_1, L_0$ and $L_{-1}$ which form $SL(2)$ algebra. This model is also quite solvable, however we will show that it presents different behavior

(2) to present a description of models of stochastic motion which may be obtained from various integrable hierarchies (various realizations of hierarchies of type A,B,C and D and their multi-component versions) [30]

(3) to study the asymptotic behavior of transition probabilities between different configurations and of certain correlation functions [29]

(4) In particular for random turn decay of step function to evaluate the time dependence of:
   (a) the mean number of involved particles, $\langle \ell(\lambda) \rangle$
   (b) the mean height $\langle |\lambda| \rangle$
   (c) the mean flow through the origin $\langle k(\lambda) \rangle$ ($k(\lambda)$ is the number of hooks, or, the same, the number of Frobenius coordinates

(5) to consider d-ASEP with various boundary conditions. To study the phenomenon of shock waves and phase transition of the first order known for the problem with open boundaries [55], [58]

(6) to study the phenomenon of thermalization which should occur for finite graphs

(7) to understand links with other approaches to the $1D$ non-equilibrium phenomena, in particular with these of papers [21], [22], [59], [57].

(8) to understand relations with the Bethe anzats method for the ASEP model

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A Appendixes

A.1 Partitions and Schur functions

Partitions. Polynomial functions in many variables are parameterized by partitions. A partition is any (finite or infinite) sequence of non-negative integers in decreasing order:

\[ \lambda = (\lambda_1, \lambda_2, \ldots, \lambda_r, \ldots) , \quad \lambda_1 \geq \lambda_2 \geq \cdots \geq \lambda_r \geq \cdots > 0 \] (157)

The numbers \( \lambda_i \) are called the parts of the \( \lambda \). The number of the parts is the length of the \( \lambda \), denoted \( \ell(\lambda) \). The sum of the parts denoted \( |\lambda| \), is called the weight of \( \lambda \). If \( |\lambda| = n \), we say that \( \lambda \) is a partition of \( n \). It is often convenient to extend the \( \lambda_i \)'s to an infinite sequence \( \{\lambda_i\}_{i \in \mathbb{N}} \) where

\[ \lambda_i := 0 \quad \text{if} \quad i > \ell(\lambda) \] (158)

The zero partition, with \( \ell(\lambda) = 0 \) is denoted by 0. The set of all partitions, including 0, is denoted by \( P \).

The Young diagram of a partition is defined as the set of points (or nodes) \((i, j) \in \mathbb{Z}^2 \) such that \( 1 \leq j \leq \lambda_i \). The Young diagram is viewed as a subset of entries in a matrix with \( \ell(\lambda) \) rows and the \( \lambda_1 \) columns, with the nodes denoted by squares aligned adjacently to form rows of length \( \lambda_1, \lambda_2, \ldots \), stacked so that successive rows downwards have equal or diminishing lengths, as in the example below

which is the diagram of the partition \( \lambda = (4, 3, 1) \), with weight \( |\lambda| = 8 \) and length \( \ell(|\lambda|) = 3 \).

The partition whose diagram is obtained by transposition of the diagram \( \lambda \) with respect to the main diagonal is called the conjugate partition and denoted by \( \nu \).

Another notation is due to Frobenius. Suppose that the main diagonal of the diagram of \( \lambda \) consists of \( r \) nodes \((i, i) \) \( (1 \leq i \leq r) \). For \( 1 \leq i \leq r \), let \( \alpha_i = \lambda_i - i \) be the number of nodes in the \( i \)th row to the right of \((i, i)\), and \( \beta_i = \nu_i - i \) the number of nodes in the \( i \)th column below \((i, i)\). We then have

\[ \alpha_1 > \alpha_2 > \cdots > \alpha_r \geq 0 \]
\[ \beta_1 > \beta_2 > \cdots > \beta_r \geq 0 \] (160)

The Frobenius notation for the partition \( \lambda \) is then

\[ \lambda = (\alpha_1, \ldots, \alpha_r | \beta_1, \ldots, \beta_r) = \alpha | \beta). \] (161)
The Frobenius notation may be viewed as a decomposition of a diagram \( \lambda \) into hooks with corners situated along the main diagonal, the largest hook being \((\alpha_1|\beta_1)\), the next one \((\alpha_2|\beta_2)\), and so on down to the smallest \((\alpha_r|\beta_r)\).

For example, the partition \((4, 3, 1)\) consists of two hooks \((3, 2)\) and \((1, 0)\):

![Diagram of a Young diagram with two hooks](image)

and in Frobenius notation is written \((3, 1|2, 0)\). If \( \lambda = (\alpha|\beta) \), then \( \lambda^t = (\beta|\alpha) \).

If we insert the increasing sequence of integers \(1, 2, \ldots|\lambda|\) into a Young diagram such that \(1\) is in the first box, and the numbers are increasing to the right within each row and downward within each column, the result is called a (standard) Young Tableau. If we insert a sequence \(1, 2, \ldots n\) with \(n \leq |\lambda|\) such that the numbers are nondecreasing to the right in each row, and decreasing downward in each column, this is called a semi-standard Young Tableau.

**Schur functions.**

The Schur functions \( S_\lambda([x]) \) associated with the partition \( \lambda \) may be viewed either as a symmetric homogeneous polynomial in \( N \) variables \((x_1, \ldots, x_N)\), where \( N \) may be any integer \( \geq |\ell(\lambda)| \) or, equivalently a weighted homogeneous polynomial in the infinite sequence of variables \((t_1, t_2, \ldots)\) defined by:

\[
s_\lambda([x]) := \left( \sum_{a=1}^{N} x_a , \frac{1}{2} \sum_{a=1}^{N} x_a^2 , \ldots , \frac{1}{i} \sum_{a=1}^{N} x_a^i , \ldots \right) := (t_1, t_2, \ldots, t_i, \ldots) \tag{163}
\]

where each \( t_i \) has weight \( i \), having total weight \( |\lambda| \). Because of the homogeneity condition, \( S_\lambda \) can only depend on the finite set of variables \((t_1, \ldots, t_{|\lambda|})\). Viewed as functions of the \( x_a \)'s, they may be defined as the following ratio of determinants (Jacobi-Trudi formula)

\[
S_\lambda([x]) = \frac{\det(x_i^{\lambda_j-j+N})}{\Delta(x_1, \ldots, x_N)} \quad 1 \leq i, j \leq N \tag{164}
\]

where

\[
\Delta(x_1, \ldots, x_N) = \prod_{i<j}^{N} (x_i - x_j) = \det(x_i^{\lambda_{N-j}}) \tag{165}
\]

is the Vandermonde determinant. For \( N \leq |\lambda| \), this is equivalent to the following combinatorial definition:

\[
S_\lambda([x]) = \sum_{x_1^{\mu_1} \ldots x_N^{\mu_N}} \quad \tag{166}
\]

where the sum is over all semi-standard Young Tableau of shape \( \lambda \) and \( \mu_i \) is the number of times \( i \) appears in the Tableau.

Representation theoretically, the significance of this is that \( S_\lambda([x]) \) is the character of the irreducible, rank \( |\lambda| \) tensor representation of \( U(N) \) or \( GL(N) \) whose symmetry properties are given by the Young diagram of \( \lambda \). (i.e. the irreducible representation consisting of tensors obtained by first symmetrizing all components labelled by the rows of the Young diagram, and then antisymmetrizing those labelled by the columns. The \( x_a \)'s are viewed as eigenvalues of the \( U(N) \) or \( GL(N) \) group element or one simply chooses diagonal elements \( X = \text{diag}(x_1, \ldots, x_N) \) and \( S_\lambda([x]) \) is the trace of the representation evaluated at \( X \).)
Polynomial functions in many variables, like the Schur functions, are parameterized by partitions.

Consider a semi-infinite set of variables \( t = (t_1, t_2, t_3, \ldots) \). Given partition \( \lambda \), the Schur function \( s_\lambda(t) \) is defined by

\[
s_\lambda(t) = \det(h_{\lambda_i-i+j}(t))_{1 \leq i,j \leq \ell(\lambda)}, \quad \text{where} \quad \sum_{k=0}^{\infty} z^k h_k(t) = \exp \sum_{m=1}^{\infty} z^m t_m, \tag{167}
\]

and, for \( k < 0 \), we put \( h_k(t) = 0 \). The \( h_k(t) \) is called the elementary Schur function.

If \( t = t(x^{(n)}) = (t_1(x^{(n)}), t_2(x^{(n)}), \ldots) \), \( t_m(x^{(n)}) = \frac{1}{m} \sum_{i=1}^{\infty} x_i^m \), then definitions (167) and (164) are equivalent [42]:

\[
s_\lambda(t(x^{(n)})) = \underline{s}_\lambda(x^{(n)}). \tag{169}
\]

From definition (167) it follows that \( s_\lambda(t(x^{(n)})) = 0 \) if \( \ell(\lambda) > n \).

We use the underline in \( \underline{s}_\lambda \) only to distinguish the two definitions. If an \( n \times n \) matrix \( X \) has eigenvalues \( x_1, \ldots, x_n \), we may denote \( \underline{s}_\lambda(x_1, \ldots, x_n) \) by \( s_\lambda(X) \), without underline, since in this paper the Schur function with uppercase argument is used only in this sense.

**Skew Schur function** is defined as follows

\[
s_{\lambda/\mu} = \det (h_{\lambda_i-\mu_j-i+j})_{1 \leq i,j \leq n}, \tag{170}
\]

Properties of these polynomials are described in details in [42].

**Schur functions evaluated at special points.**

We need notations:

\[
(a)_\lambda := (a)_{\lambda_1}(a - 1)_{\lambda_2} \cdots (a - k + 1)_{\lambda_k}, \quad (a)_m := \frac{\Gamma(a + m)}{\Gamma(a)}, \tag{171}
\]

\[
(q^a; q)_\lambda := (q^a; q)_{\lambda_1}(q^{a-1}; q)_{\lambda_2} \cdots (q^{a-k+1}; q)_{\lambda_k}, \quad (q^a; q)_m := (1 - q^a) \cdots (1 - q^{a+m-1}), \tag{172}
\]

where \( k = \ell(\lambda) \). We set \((a)_0 = (q^a; q)_0 = 1\) and \((a)_{-k} = (q^a; q)_{-k} = 0\) for \( k > 0 \).

We introduce the following notations [54]:

\[
t_\infty = (1, 0, 0, 0, \ldots), \tag{173}
\]

\[
t(a, 1) = \left( \frac{a}{1}, \frac{a}{2}, \frac{a}{3}, \ldots \right), \tag{174}
\]

\[
t(\infty, q) = (t_1(\infty, q), t_2(\infty, q), \ldots), \quad t_m(\infty, q) = \frac{1}{m(1 - q^m)}, \quad m = 1, 2, \ldots, \tag{175}
\]

\[
t(a, q) = (t_1(a, q), t_2(a, q), \ldots), \quad t_m(a, q) = \frac{1 - (q^a)^m}{m(1 - q^m)}, \quad m = 1, 2, \ldots \tag{176}
\]

Note that \( t(a, q) \) tends to \( t(\infty, q) \) (resp. \( t(a, 1) \)) as \( a \to \infty \) (resp. \( q \to 1 \)). As for \( t_\infty \), if \( f \) satisfies \( f(c_1 t_1, c_2 t_2, c^3 t_3, \ldots) = c^d f(t_1, t_2, t_3, \ldots) \) for some \( d \in \mathbb{Z} \), we have \( h^d f(t(\infty, q)) \to f(t_\infty) \) as \( h := \ln q \to 0 \). Below \( \Delta(h) := \prod_{i<j} (h_i - h_j) \).
Lemma 1. For a partition \( \lambda = (\lambda_1, \lambda_2, \ldots) \), let \( h_i := n + \lambda_i - i \) (\( 1 \leq i \leq n \)), where \( n \geq \ell(\lambda) \). Then

\[
\begin{align*}
{s}_\lambda(t, \infty) &= \Delta(h) \prod_{i=1}^{n} h_i! 
\quad \text{for } \lambda = (\lambda_1, \lambda_2, \ldots),
\end{align*}
\]

(177)

\[
{s}_\lambda(t(a, 1)) = \frac{\Delta(h)}{\prod_{i=1}^{n} h_i!} \prod_{i=1}^{n} \frac{\Gamma(a - n + h_i + 1)}{\Gamma(a - i + 1)},
\quad \text{for } \lambda = (\lambda_1, \lambda_2, \ldots),
\]

(178)

\[
{s}_\lambda(t(\infty, q)) = \frac{\Delta(q^h)}{\prod_{i=1}^{n} (q; q)_{h_i}}
\quad \text{for } \lambda = (\lambda_1, \lambda_2, \ldots),
\]

(179)

\[
{s}_\lambda(t(a, q)) = \frac{\Delta(q^h)}{\prod_{i=1}^{n} (q^a-q^i+1; q)_{h_i-n+i}}
\quad \text{for } \lambda = (\lambda_1, \lambda_2, \ldots),
\]

(180)

Note that those quantities (177)–(180) are independent of the choice of \( n \geq \ell(\lambda) \). We also mark that for integer positive \( a \) the Schur functions (178) and (180) vanish if \( \ell(\lambda) > a \).

A.2 Fermionic Fock space, \( gl(\infty) \) and \( GL(\infty) \)

The following is a summary regarding the one and two-component free fermion algebra based on the introductory section of [28]. The reader may refer to [32], [14] for further details.

In the following, \( A \) denotes the complex Clifford algebra over \( \mathbb{C} \) generated by charged free fermions \( \{f_i, \bar{f}_i\}_{i \in \mathbb{Z}} \), satisfying the anticommutation relations

\[
[f_i, f_j]_+ = [\bar{f}_i, \bar{f}_j]_+ = 0, \quad [f_i, \bar{f}_j]_+ = \delta_{ij}.
\]

(181)

where \([,]_+\) denotes the anticommutator.

Elements of the linear part

\[
W := (\oplus_{m \in \mathbb{Z}} \mathbb{C} f_m) \oplus (\oplus_{m \in \mathbb{Z}} \mathbb{C} \bar{f}_m)
\]

(182)

will be referred to as a free fermions. The fermionic free fields

\[
f(x) := \sum_{k \in \mathbb{Z}} f_k x^k, \quad \bar{f}(y) := \sum_{k \in \mathbb{Z}} \bar{f}_k y^{k-1}
\]

(183)

may be viewed as generating functions for the \( f_j, \bar{f}_j \)'s.

This Clifford algebra has a standard Fock space representation \( F \) and dual space \( \bar{F} \) (see e.g. [26, 28]) which contain unique vacuum states \( |0\rangle \) and \( \langle 0| \) respectively satisfying the properties

\[
\begin{align*}
|0\rangle f_m &= 0 & (m < 0), & \quad \langle 0| \bar{f}_m &= 0 & (m \geq 0), \\
\langle 0| f_m &= 0 & (m \geq 0), & \quad |0\rangle \bar{f}_m &= 0 & (m < 0).
\end{align*}
\]

(184)

The Fock spaces \( F \) and \( \bar{F} \) are mutually dual, with the hermitian pairing defined via the linear form \( \langle 0|0\rangle \) on \( A \) called the vacuum expectation value. This satisfies

\[
\begin{align*}
\langle 0|1|0\rangle &= 1; & \langle 0| f_m \bar{f}_m |0\rangle &= 1, & m < 0; & \langle 0| \bar{f}_m f_m |0\rangle &= 1, & m \geq 0, \\
\langle 0| f_n |0\rangle &= \langle 0| \bar{f}_n |0\rangle = \langle 0| f_m f_n |0\rangle = \langle 0| \bar{f}_m \bar{f}_n |0\rangle = 0; & \langle 0| f_m \bar{f}_n |0\rangle &= 0, & m \neq n.
\end{align*}
\]

(185)

(186)
Note that the expression on the right hand side is actually defined, by (185), as the infinite series
\[ \langle 0| w_1 \cdots w_{2n} |0 \rangle = \sum_{\sigma \in S_{2n}} \text{sgn}\sigma \langle 0| w_{\sigma(1)} w_{\sigma(2)} |0 \rangle \cdots \langle 0| w_{\sigma(2n-1)} w_{\sigma(2n)} |0 \rangle. \] (187)

Here \( \sigma \) runs over permutations for which \( \sigma(1) < \sigma(2), \ldots, \sigma(2n-1) < \sigma(2n) \) and \( \sigma(1) < \sigma(3) < \cdots < \sigma(2n-1) \).

If \( \{ w_i \}_{i=1,\ldots,N} \) are linear combinations of the \( f_j \)'s only, \( j \in \mathbb{Z} \), and \( \{ \bar{w}_i \}_{i=1,\ldots,N} \) linear combinations of the \( \bar{f}_j \)'s, \( j \in \mathbb{Z} \), then (187) implies
\[ \langle 0| w_1 \cdots w_n \bar{w}_n \cdots \bar{w}_1 |0 \rangle = \text{det} \left( \langle 0| w_i \bar{w}_j |0 \rangle \right) |_{i,j=1,\ldots,N} \] (188)

Following [14], [32], for all \( N \in \mathbb{Z} \), we also introduce the states
\[ \langle N | := \langle 0| C_N \] (189)
where
\[ C_N := f_0 \cdots f_{N-1} \quad \text{if} \quad N > 0 \] (190)
\[ C_N := f_{-1} \cdots f_N \quad \text{if} \quad N < 0 \] (191)
\[ C_N := 1 \quad \text{if} \quad N = 0 \] (192)
and
\[ |N \rangle := \bar{C}_N |0 \rangle \] (193)
where
\[ \bar{C}_N := f_{N-1} \cdots f_0 \quad \text{if} \quad N > 0 \] (194)
\[ \bar{C}_N := \bar{f}_N \cdots \bar{f}_{-1} \quad \text{if} \quad N < 0 \] (195)
\[ \bar{C}_N := 1 \quad \text{if} \quad N = 0 \] (196)

The states (189) and (193) are referred to as the left and right charged vacuum vectors, respectively, with charge \( N \). From the relations
\[ \langle 0| f_{N-k} f(x_i) |0 \rangle = x_i^{N-k}, \quad \langle 0| f_{-N+k-1} \bar{f}(y_i) |0 \rangle = y_i^{N-k}, \quad k = 1, 2, \ldots, N, \] (197)
and (188), it follows that
\[ \langle N| f(x_1) \cdots f(x_n) |0 \rangle = \delta_{n,N} \Delta_N(x), \quad N \in \mathbb{Z}, \] (198)
\[ \langle -N| \bar{f}(y_1) \cdots \bar{f}(y_n) |0 \rangle = \delta_{n,N} \Delta_N(y), \quad N \in \mathbb{Z}. \] (199)

For free fermion generators with \( |x| \neq |y| \),
\[ \langle 0| f(x) \bar{f}(y) |0 \rangle = \frac{1}{x-y} \] (200)

Note that the expression on the right hand side is actually defined, by (185), as the infinite series
\[ \sum_{n=0}^{\infty} y^n x^{-n-1} \] which converges only inside \( |x| < |y| \). However one can consider expression (200) for the whole region of \( x \) and \( y \) (when \( |x| \neq |y| \)) in the sense of analytical continuation.

From Wick’s theorem it follows that
\[ \langle n-m| f(x_1) \cdots f(x_n) \bar{f}(y_1) \cdots \bar{f}(y_m) |0 \rangle = \frac{\Delta_n(x) \Delta_m(y)}{\prod_{j=1, \ldots, m} (x_i - y_j)} \] (201)
A.3 Commuting flows, τ functions and Schur functions

In the theory of integrable systems the following $\hat{gl}(\infty)$ operators are important:

$$H_m = \sum_{i=-\infty}^{+\infty} f_i \bar{f}_{i+m}, \quad m = \pm 1, \pm 2, \ldots$$

(202)

These operators form Heisenberg algebra relations

$$[H_m, H_n] = m\delta_{m+n,0}$$

(203)

These oscillator algebra properties together with

$$H_m|n\rangle = 0; \quad \langle n|H_{-m} = 0, \quad m > 0, n \in \mathbb{Z}$$

(204)

allow to refer vectors $\langle 0|e^{H(t)}$ and $\bar{H}(\bar{t})$ as generalized coherent states which depend on parameters $t = (t_1, t_2, \ldots)$ and $\bar{t} = (\bar{t}_1, \bar{t}_2, \ldots)$. Here

$$H(t) = \sum_{m=1}^{\infty} H mt_m, \quad \bar{H}(\bar{t}) = \sum_{m=1}^{\infty} H_{-m} \bar{t}_m$$

(205)

We have the following fermionic representation of Schur functions (which follows, as usual from Wick’s theorem)

**Lemma 1** [14] For $\alpha_1 > \cdots > \alpha_k \geq 0, \beta_1 > \cdots > \beta_k \geq 0$ the next formula is valid:

$$\langle 0|e^{H(t)} \bar{f}_{-\beta_1-1} \cdots \bar{f}_{-\beta_k-1} f_{\alpha_1} \cdots f_{\alpha_k}|0\rangle = (-1)^{\beta_1+\cdots+\beta_k+k} s_{\lambda}(t),$$

(206)

where in the Frobenius notation $\lambda = (\alpha_1, \ldots, \alpha_k|\beta_1, \ldots, \beta_k)$.

Defining, as in eq. (61)

$$A^+_+(U) := \sum_{i \in \mathbb{Z}} e^{U_{i-1}-U_i} f_i \bar{f}_{i-1},$$

(207)

we have the following expression for the matrix elements of its exponential in terms of the Schur function evaluate at the special value $t_\infty := (1, 0 \ldots)$

**Lemma 2** [52]

$$\langle \lambda, n|e^{A^+_+(U)}|n\rangle = e^{-U_\lambda(n)} s_{\lambda}(t_\infty)$$

(208)

This relation follows from

$$A^+_+(U) = e^{H_0(U)} H_{-1} e^{-H_0(U)}, \quad H_0(U) := \sum_{i<0} U_i \bar{f}_i f_i - \sum_{i\geq0} U_i f_i \bar{f}_i$$

(209)

Indeed, by definitions of $H_0(U)$ and $\langle \lambda, n|$ one evaluates:

$$\langle \lambda, n|e^{H_0(U)} = c_n e^{-U_\lambda(n)} \langle \lambda, n|$$

(210)

where $U_\lambda(n)$ is given by (43), where $c_n$ is defined by

$$\langle n|e^{H_0(U)}|n\rangle = c_n, \quad c_n = e^{-U_0 \cdots -U_{n-1}}, n > 0; \quad c_n = e^{U_{n-1} \cdots + U_n}, n < 0$$

(211)
Then, (208) follows (210) and (206). Different detailed proof was written down in [26], where the notation \( r \) of (62) was used. In papers [50], [26] instead of \( e^{-U_\lambda(n)} \) of the present paper the notation \( r_\lambda(n) \) was used.

**Skew Schur function** is defined as follows

\[
s_{\lambda/\mu} = \det (h_{\lambda_i-\mu_j-i+j})_{1 \leq i,j \leq n},
\]

The following is a generalization of (208) in terms of skew Schur functions

**Lemma 3** [52]

\[
\langle \lambda, n \mid e^{A_+}\mid \mu, n \rangle = e^{U_\mu(n)} - e^{U_\lambda(n)} s_{\lambda/\mu}(t_\infty)
\]

### A.4 Derivation of (90) and some other formulae of subsection 2.6

Formula (90) results from

\[
\int_0^2 \frac{1}{2} - \frac{1}{\pi} \arcsin \left( \frac{y - 1}{u - y} \right) dy = \log 2u, \quad u \in [0, 2]
\]

where we put \( u = \frac{h}{R} \). We have obtained formula (214) as follows. First, it is widely used in random matrix theory (the proof may be found, say, in [43] or in [17]) the relation called the Wigner semi-circle law, which is the first relation of

\[
P \int_{-1}^{1} \frac{\sqrt{1-v^2}}{x-v} dv - x = 0, \quad x \in [-1, 1],
\]

while the last relation is obtained from the first one via taking the derivative with respect to \( x \) and via integrating by parts. Now, keeping in mind \((\arcsin v)' = (1 - v^2)^{-\frac{1}{2}}\) and taking the derivative of the l. h. s. of (214) with respect to \( u \), using \(((u - y)^{-1})_u = -(u - y)^{-1}_y\) and integrating by parts after a shift of the integration variable \((y - 1 = v)\) we obtain

\[
\frac{1}{u} + \frac{1}{\pi} P \int_{-1}^{1} \frac{1}{\sqrt{1-v^2}} \frac{dv}{u - 1 - v}
\]

where as \( u - 1 \in [-1, 1] \) the second term vanishes due to (215), while the first term (which originates from the lower boundary term; upper boundary term vanishes) coincides with the derivative of the r.h.s. of (214) with respect to \( u \). Thus we have proven (214) up to a term, say \( C \), independent of \( u \). Let us find it. Consider

\[
P \int_{0}^{2} \frac{1}{2} - \frac{1}{\pi} \arcsin \left( \frac{y - 1}{u - y} \right) dy = P \int_{0}^{2} \left( \frac{1}{2} - \frac{1}{\pi} \arcsin \left( \frac{y - 1}{u - y} \right) \right) \left( -\frac{d}{dy} \right) \log |u - y| dy
\]

\[
= \log u - \frac{1}{\pi} P \int_{0}^{2} \frac{\log |u - y| dy}{\sqrt{1 - (y - 1)^2}} =: \log u - C, \quad u \in [0, 2],
\]

where we integrated by parts. We should find the last term, \( C \), which (we know) is the constant we are looking for. We have

\[
C = \frac{1}{\pi} P \int_{0}^{2} \frac{\log |u - y| dy}{\sqrt{1 - (y - 1)^2}} = \frac{1}{\pi} \int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} \log(x - \sin \phi) d\phi, \quad x \in [-1, 1],
\]
which we evaluate at point \( x = 1 \) (since it does not depends on \( x \) we are free to choose any point of the interval \([-1, 1]\)):

\[
C = \frac{1}{\pi} \int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} \log(1 - \sin \phi) d\phi = \frac{1}{\pi} \int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} \sum_{n=1}^{\infty} \frac{d\phi \sin^{2n} \phi}{2n} = -\sum_{n=1}^{\infty} \frac{2^{-2n}}{2n} \binom{2n}{n} = \]

\[
= -\int_{0}^{1} \frac{dz}{z} \left( (1 - z^2)^{-\frac{1}{2}} - 1 \right) = -\int_{0}^{\frac{\pi}{2}} \frac{1 - \cos \phi}{\sin \phi} d\phi = 2 \log \frac{\phi}{2} \bigg|_{\frac{\pi}{2}}^{\frac{\pi}{2}} = -\log 2
\]

which completes the proof of (214).

Now let us obtain (94).

\[
|\lambda| = \sum_{i=1}^{R} (h_i + i - R) = \sum_{i=1}^{R} h_i - \frac{R^2}{2} + \frac{R}{2} \to \int_{0}^{2R} h\sigma(h) dh - \frac{R^2}{2} + \frac{R}{2}
\]

Then

\[
\int_{0}^{2R} h\sigma(h) dh = \frac{1}{2} \int_{0}^{2R} \left( \frac{1}{2} - \frac{1}{\pi} \arcsin \left( \frac{h}{R} - 1 \right) \right) \left( \frac{dh^2}{dh} \right) dh
\]

\[
= -\frac{1}{2\pi R} \int_{0}^{2R} \frac{h^2 dh}{\sqrt{1 - \left( \frac{h}{R} - 1 \right)^2}} = \frac{R^2}{2\pi} \int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} (1 + \sin^2 \phi) d\phi = \frac{3}{4} R^2
\]

Now let us derive (92). We have

\[
\frac{1}{2} \log(T - |\lambda|) = T - \frac{R^2}{4} - \frac{R}{2}
\]

which we equate to \( \log \frac{T}{R} \) and obtain (92).

At last let us obtain (96):

\[
\int_{R}^{2R} \sigma(h) dh = \int_{R}^{2R} \sigma(h) \left( \frac{dh}{dh} \right) dh = -\frac{R}{2} - \frac{1}{\pi R} \int_{R}^{2R} \frac{h dh}{\sqrt{1 - \left( \frac{h}{R} - 1 \right)^2}} = \frac{R}{\pi}
\]

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