THE DEGREE OF THE GAUSS MAP FOR
A GENERAL PRYM THETA-DIVISOR

Alessandro Verra
Dipartimento di Matematica, Università di Roma 3

1. Introduction
An interesting approach to the theta divisor $\Theta$ of a principally polarized abelian variety $A$ is certainly offered by its Gauss map $\gamma : \Theta \to \mathbf{P}^{d-1}$, $d = \dim A$. This is the natural map which associates to $x \in \Theta - \text{Sing}\Theta$ the tangent space $T_{\Theta,x} \subset H = T_{A,x}$. Since $H$ does not depend on $x$, $T_{\Theta,x}$ is a point of $\mathbf{P}(H^*) = \mathbf{P}^{d-1}$. As is well known $\gamma$ has finite degree unless $A$ is a product, moreover $\deg\gamma = d!$ if $\Theta$ is smooth. The degree of $\gamma$ varies according to the singularities of $\Theta$ and so it is interestingly related to the Andreotti-Mayer stratification of the moduli space of $(A, \Theta)$. On the other hand, if we consider pairs $(A, \Theta)$ such that $\dim\text{Sing}\Theta > 0$, we realize that, essentially, the map $\gamma$ is completely understood in the only case of Jacobians. Here $\gamma$ has degree $(2d^2 - d - 2)$ and its description is encoded in the beautiful geometry of the canonical curve.

After Jacobians of curves, the next usual step consists in considering Prym varieties. In this note we afford part of this step: we compute the degree of the Gauss map for the theta divisor $\Xi$ of a general Prym variety $P$.

Assume $P$ is defined by a pair $(C, \eta)$, where $C$ is a smooth, irreducible, canonically embedded curve of genus $g = d + 1$ and $\eta$ is a non trivial 2-torsion element of $\text{Pic}^0(C)$. Then $H = H^0(\omega_C \otimes \eta)^*$ so that $\mathbf{P}^{d-1} = |\omega_C \otimes \eta|$. Our result relies on a geometric description we give for the map $\gamma : \Xi \to |\omega_C \otimes \eta|$. This relates $\Xi$ to the family of rank three quadrics touching $C$ along a Prym-canonical divisor, that is to quadrics $q$ such that $q \cdot C = 2d$ with $d \in |\omega_C \otimes \eta|$. As we will see, there is a rational map $\alpha : \Xi \to Y^+$, where $Y^+$ is an irreducible component of this family of quadrics. $\alpha$ is the quotient map $\Xi \to \Xi/\langle -1 \rangle$ followed by a birational isomorphism. Moreover there is also the natural map

$$\lambda^+ : Y^+ \to |\omega_C \otimes \eta|$$

such that $\lambda(q) = d$. We will prove that $\gamma = \lambda \cdot \alpha$. Then we will use the construction to show the main result of the paper:

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THEOREM. Let $g \geq 3$ and let $\Xi$ be the Theta-Divisor of a general Prym variety of dimension $g - 1$. Then the degree of the Gauss map $\gamma : \Xi \to \mathbb{P}^{g-2}$ is

$$D(g) + 2^{g-3},$$

where $D(g)$ is the degree of the variety of all quadrics of rank $\leq 3$ in $\mathbb{P}^{g-1}$.

The number $D(g)$ can be found in the classical book of Room ([Ro] p.133, cfr. [HT]),

$$D(g) = \frac{(g-3)(g+1)}{(2g-7)(2g-9)} \cdots \frac{(3)}{(1)},$$

Since a general principally polarized abelian variety of dimension $\leq 5$ is a Prym, (cfr. [B1]), our formula yields $(g - 1)! = D(g) + 2^{g-3}$ if $3 \leq g \leq 6$. If $\dim P = 6$ we know from Debarre that the singular locus of $\Xi$ consists of 16 ordinary double points, ([D]). This implies $\deg \gamma = (g - 1)! - 32$ if $g = 7$ and gives $(g - 1)! - 32 = D(g) + 2^{g-3} = 688$. Assuming $P$ general, these seem to be the only cases where the degree of $\gamma$ was previously known.

Now we describe our strategy to compute the degree of the map $\lambda^+ : Y^+ \to |\omega_C \otimes \eta |$, hence to show the theorem. Assume $C$ has general moduli, we explain in section 2 that $Y^+$ is only one irreducible component of the variety $Y$ of all quadrics $q$ of rank $\leq 3$ satisfying $q \cdot C = 2d$, with $d \in |\omega_C \otimes \eta |$. There is another irreducible component $Y^-$ such that $Y = Y^+ \cup Y^-$. We can consider as above the map

$$\lambda : Y \to |\omega_C \otimes \eta |$$

sending $q$ to $\lambda(q) = d$, in particular $\lambda/Y^+ = \lambda^+$. Let $Q = PH^0(O_{\mathbb{P}^{g-1}}(2))$, it turns out that $\lambda = l/Y$ where

$$l : Q \to |\omega_C^{\otimes 2} |$$

is the restriction map. Note that $l$ is a linear projection, its center is the linear system $I_C$ of all quadrics through $C$, which is canonically embedded in $\mathbb{P}^{g-1}$. Let

$$Q^3 = \{ q \in Q/\text{rk}(q) \leq 3 \},$$

restricting $l$ to $Q^3$ we obtain a finite covering of degree $D(g) = \deg Q^3$. This happens because $\dim Q^3 = \dim |\omega_C^{\otimes 2} |$ and $I_C \cap Q^3$ is empty. On the other hand it holds

$$Y = l^{-1}(V) \cdot Q^3,$$

where $V$ is the embedding of $|\omega_C \otimes \eta |$ in $|\omega_C^{\otimes 2} |$ under the map $d \to 2d$. This implies

$$\deg \lambda^+ + \deg \lambda^- = D(g),$$

where $\lambda^- = \lambda/Y^-$. To distinguish $\deg \lambda^+$ from $\deg \lambda^-$ we specialize $C$ to a trigonal curve. Here some nice geometry appears, relating $Y^+$ and $Y^-$ (which are now reducible) to theta-characteristics on hyperelliptic curves of genus $g - 3$. 
If $C$ is trigonal the degree of $\lambda$ is no longer $D(g)$ because $S = I_C \cap \mathbb{Q}^3$ is not empty. Let $p = g - 3$, we prove that $\text{deg}\lambda = 2^{2p}$ in this case. Hence the contribution of the excess intersection $S$ to the degree of a general $\lambda$ is $D(g) - 2^{2p}$.

In addition we show that $\text{deg}\lambda^+ = 2^{p-1}(2p + 1)$ and $\text{deg}\lambda^- = 2^{p-1}(2p - 1)$, which are the numbers of even and odd theta’s on a curve of genus $p$. The reason for these numbers is that a trigonal $C$ is contained in a rational normal scroll $R$. Given $d \in |\omega_C \otimes \eta|$, there exists exactly one curve $E \in |\mathcal{O}_R(2)|$ such that $E \cdot C = 2d$. The fibre at $d$ of $\lambda^+$ (of $\lambda^-$) is then naturally bijective to the set of even (odd) theta’s on $E$.

Finally, using a standard parity lemma in the theory of Prym varieties and its geometric consequences, we deduce that the contribution of twice, hence the point $\gamma$|_$-t$|_ of fibre at $d$ of $\lambda^+$ (of $\lambda^-$) is then naturally bijective to the set of even (odd) theta’s on $E$.

The idea of using rank three quadrics is in some sense related to Tjurin’s paper [T], where $\gamma$ has a complementary description. Let $\pi : \tilde{C} \to C$ be the étale double covering defined by $\eta$. Then $P \subset \text{Pic}^{2g-2}(\tilde{C})$ and each point $x \in \Xi - \text{Sing}\Xi$ is a quadratic singularity for the theta divisor $\tilde{\Theta} = \{\tilde{L} \in \text{Pic}^{2g-2}(\tilde{C})/h^0(\tilde{L}) \geq 2\}$. As is well known the projectivized tangent cone of $\tilde{\Theta}$ at $x$ is a quadric of rank $\leq 4$

$$\tilde{Q} \subset \mathbf{P}(H^0(\omega_C \otimes \eta)^* \oplus H^0(\omega_C)^*).$$

As in [T] we can restrict $\tilde{Q}$ to $P(H^0(\omega_C \otimes \eta)^*)$: this is the hyperplane $\mathbf{P}T_{\Xi,x}$ counted twice, hence the point $\gamma(x)$. As in section 2 we can restrict $\tilde{Q}$ to $\mathbf{P}H^0(\omega_C)^*$: this is a quadric in the family $Y^+$.

To avoid any problem we work on the complex field. Nevertheless it seems quite possible that the proof extends to any algebraically closed field $k$, $\text{chark} \neq 2$. Finally, we wish to thank some friends for various interesting discussions on the subject, in particular V. Kanev, E. Sernesi, C. Ciliberto.

Some frequently used notations: - $C^{(d)}$: the $d$-symmetric product of a curve $C$. - $x + C^{(d-1)}$: the ample divisor $\{d \in C^{(d)}/d = x + d\}$. - $|L| (|V|)$: the linear system of the line bundle $L$ (of the space $V \subset H^0(L)$). - $<S>$: the linear span of a set $S$ in a vector or projective space. - $X \cdot Y$: the intersection scheme of $X$ and $Y$. $XY$: the intersection number. - $[Z]$ the class of $Z \subset X$ in the numerical equivalence ring of $X$. - $f^*(o)$: the scheme theoretic fibre at $o$ of a morphism $f : X \to Y$. - $S_t$: the fibre of a $T$-scheme $S$ at a point $t$ of $T$. - $E^*$: the dual of the vector (projective) space $E$. $\text{PE: Proj}E^*$. 

2. Gauss map and rank three quadrics.

In the following we introduce the basic construction relating the Gauss map to rank three quadrics, moreover we fix our notations. The main object to be considered is
an étale double covering

\[ \pi : \tilde{C} \to C, \]

defined by the non trivial line bundle \( \eta \). The corresponding involution on \( \tilde{C} \) is denoted by \( i \). For simplicity, it will be assumed that both \( \tilde{C} \) and \( C \) are non hyperelliptic and that \( \omega_C \otimes \eta \) is very ample. The latter property is satisfied unless \( \eta \cong \mathcal{O}_C(x_1 + x_2 - y_1 - y_2) \) with \( x_1, x_2, y_1, y_2 \in C \), ([CD], 0.6). An element \( d \in |\omega_C \otimes \eta| \) is said to be a \textit{Prym-canonical divisor}. As usual \( \pi \) induces the Norm map

\[ Nm : \tilde{J} \to J, \]

where \( \tilde{J} = \text{Pic}^{2g-2}(\tilde{C}) \) and \( J = \text{Pic}^{2g-2}(C) \). By definition the image of \( \mathcal{O}_C(\Sigma x_i) \) by \( Nm \) is \( \mathcal{O}_C(\Sigma \pi(x_i)) \). The fibre of \( Nm \) is split in two isomorphic connected components. In particular \( Nm^{-1}(\omega_C) = P \cup P' \), where

\[ P = \{ \tilde{L} \in Nm^{-1}(\omega_C)/h^0(\tilde{L}) \text{ is even } \}, \quad P' = \{ \tilde{L} \in Nm^{-1}(\omega_C)/h^0(\tilde{L}) \text{ is odd } \}. \]

As usual, we will say that \( P \) is the \textit{Prym variety of} \( \pi \). \( P \) is biregular to a \((g-1)\)-dimensional abelian variety, namely to \( \text{Im}(1-i^*) \subset \text{Pic}^0(\tilde{C}) \). The restriction to \( P \) of the theta-divisor

\[ \tilde{\Theta} = \{ \tilde{L} \in \tilde{J}/h^0(\tilde{L}) > 0 \} \]

is twice a principal polarization \( \Xi \subset P \), in particular it holds \( \Xi = \text{Sing} \tilde{\Theta} \cdot P \), (cfr. [M] and [ACGH] p.295). By definition \( \Xi \) is the \textit{Prym Theta-divisor of} \( \pi \). To introduce some projective geometry related to \( \Xi \) we consider the vector spaces

\[ H = H^0(\omega_C)^*, \quad H^-, \quad H^+ \]

where \( H^- \), \( H^+ \) are the eigenspaces of the involution induced by \( i \) on \( H \). We notice some canonical identifications induced by \( \pi^* : H^+ = H^0(\omega_C)^*, H^- = H^0(\omega_C \otimes \eta)^* \). For the associated projective spaces the notations will be

(2.1) \[ P = PH^- \quad P^+ = PH^+ \quad P^- = PH^- \]

Let

\[ h^+ : P \to P^+ \quad \text{and} \quad h^- : P \to P^- \]

be the linear projections of centers \( P^- \) and \( P^+ \). We have the commutative diagram

(2.2) \[
\begin{array}{ccc}
\text{P}^- & \xleftarrow{\ h^- \ } & \text{P} & \xrightarrow{\ h^+ \ } & \text{P}^+ \\
\U & \U & \U \\
C^- & \xleftarrow{\ h^- / \tilde{C} \ } & \tilde{C} & \xrightarrow{\ h^+ / \tilde{C} \ } & C,
\end{array}
\]

where \( \tilde{C} \) is canonically embedded in \( P \) and

(2.3) \[ C^+ = h^+(\tilde{C}) \quad C^- = h^-(\tilde{C}). \]
It turns out that $\pi = h^+/\hat{C} = h^-/\hat{C}$. Moreover $\mathcal{O}_C(1) \cong \omega_C \otimes \eta$ so that $C^-$ is the Prym canonical embedding of $C$. On the other hand $C^+$ is the canonical model of $C$. For simplicity we will put

$$C = C^+,$$

when no confusion arises. To simplify notations we put

$$Q = \mathbb{P}H^0(\mathcal{O}_\mathbb{P}(2)) \quad , \quad B = \mathbb{P}H^0(\omega_C^\otimes 2).$$

The natural restriction map will be denoted by

$$\lambda : Q \rightarrow B.$$

$\lambda$ is induced from the standard exact sequence

$$0 \rightarrow H^0(\mathcal{I}(2)) \rightarrow H^0(\mathcal{O}_\mathbb{P}(2)) \rightarrow H^0(\omega_C^\otimes 2) \rightarrow 0,$$

where $\mathcal{I}$ is the Ideal of $C$, so $\lambda$ is the linear projection of center $| \mathcal{I}_C(2) |$. Let

$$Q^r = \{ q \in Q / \text{rank}(q) \leq r \},$$

we will be specially interested to the variety $Q^3$. *The degree of $Q^3$ is denoted as $D(g)$. $Q^3$ has dimension $3g - 4$, (cfr. [ACGH], p.100), as well as $B$. Therefore

$$\lambda/Q^3 : Q^3 \rightarrow B$$

is a finite morphism of degree $D(g)$ if $| \mathcal{I}_C(2) | \cap Q^3 = \emptyset$. *This condition is satisfied if $C$ is sufficiently general*, (see remark 2.9-1). Finally let

$$v : | \omega_C \otimes \eta | \rightarrow B.$$

be the 'squaring map' sending $d \in | \omega_C \otimes \eta |$ to $2d$. $v$ is obtained from the diagram

$$H^0(\omega_C \otimes \eta) \overset{\sigma}{\rightarrow} \text{Sym}^2 H^0(\omega_C \otimes \eta) \overset{\mu}{\rightarrow} H^0(\omega_C^\otimes 2)$$

as the projectivization of $\mu \cdot \sigma$, where $\mu$ is the multiplication map and $\sigma(x) = x^2$. We leave as an exercise to show that $v$ is an embedding. If $\mu$ is surjective then $v$ is the 2-Veronese embedding of $\mathbb{P}^{g-2}$ in a subspace of $B$ of appropriate dimension. It is known that $\mu$ is surjective if $C$ is general of genus $g \geq 6$, ([B2]). The image of $v$ in $B$ will be denoted as

$$V.$$

The inverse image of $V$ by $\lambda$ will be

$$W.$$
$W$ is a cone over $V$ in the projective space $Q_3$, in particular $\text{Sing}W = |I_C(2)|$.

2.9 REMARKS (1) A rank three quadric $q$ containing $C$ defines a line bundle $L$ of degree $\leq g-1$ with non injective Petri map. Hence $C$ is not general. To see this consider the moving part $M$ of the pencil which is cut on $C$ by the ruling of maximal linear subspaces of $q$. Observe that $M$ is defined by a vector space $U \subseteq H^0(L)$, where $L$ is a line bundle satisfying $L^\otimes 2 \cong \omega_C(-b)$ and $b \geq 0$. By the base-point-free-pencil-trick, ([ACGH] p.126), the Petri map $\mu : H^0(L) \otimes H^0(L(b)) \to H^0(\omega_C)$ is not injective on $U \otimes H^0(L(b))$.

(2) For each quadric $q \in W - I_C$, it holds $q \cdot C = 2d$, where $\lambda(q) = 2d$. Let $I_{2d}$ be the vector space of quadratic forms vanishing on $2d$. Then $|I_{2d}|$ is a space of maximal dimension in the cone $W$. For a general $2d$ the fibre of $\lambda/Q^3$ over $2d$ is the locus of rank three quadrics in $|I_{2d}| - I_C$.

2.10 DEFINITION. The intersection scheme $Y = W \cdot Q^3$ is the variety of Prym-canonical rank three quadrics.

2.11 LEMMA. (i) Every irreducible component of $Y$ has dimension $\geq g-2$.
(ii) If $C$ is sufficiently general $Y$ is reduced of pure dimension $g-2$.

Proof. (i) Dimension count, since $\text{codim}_Q W = 2g - 2$. (ii) If $C$ is general the map $\lambda/Q^3$ is finite, hence $Y = \lambda^*(V)$ is pure of dimension $g-2$. In particular each irreducible component $Z$ of $Y$ is horizontal i.e. $\lambda(Z) = V$. One can show that, moving $W$ in a smooth family, the limit of a horizontal irreducible component is still horizontal. Then, to prove the reducedness of a general $Y$, it suffices to produce one $Y$ such that all its irreducible horizontal components are reduced. This is the case if $C$ is a general trigonal curve: see corollary 4.8.

Throughout the paper the notions of general curve or of general cover $\pi : \tilde{C} \to C$ are used in different ways. The next definition will help to distinguish among them.

2.12 DEFINITION. (1) $\pi$ is Petri general if no element of $Y$ contains $C$.
(2) $\pi$ is standard if $X$ and $Y$ are reduced.
$R_{\text{Petri}}$ and $R_{\text{standard}}$ denote the corresponding subsets in the moduli space of $\pi$.

We will see very soon that $Y$ is strongly related to the Prym-Theta divisor $\Xi$ and to its Gauss map. To study this relation let

$$\pi^{(2g-2)} : \tilde{C}^{(2g-2)} \to C^{(2g-2)}$$

be the morphism defined by pushing-forward divisors via $\pi$ and let

$$G$$ and $$B$$

be the following objects:
- $B$ is the Hilbert scheme of conics in $C^{(2g-2)}$, that is of connected curves $B$ such that $B \cdot (x + C^{(2g-3)}) = 2$ and $p_a(B) = 0$.

- $\mathcal{G} = G^1_{2g-2}(\tilde{C})$. Here $G^1_{2g-2}(\tilde{C})$ denotes the Hilbert scheme of lines in $\tilde{C}^{(d)}$, that is of curves $P$ such that $P \cdot (x + C^{(d-1)}) = 1$ and $p_a(P) = 0$ (cfr. [ACGH] IV.3).

Each $P \in \mathcal{G}$ is a pencil of divisors of degree $2g-2$. If $P$ is general $\pi^{(2g-2)}$ embeds $P$ in $C^{(2g-2)}$ as a conic $B$. Hence $\pi^{(2g-2)}$ induces a rational map

$$h : \mathcal{G} \to B,$$

sending $P$ to $B$. It is easy to see that $h(P) = h(i^*P)$ and that $\pi^{(2g-2)}/P$ is an embedding iff $P' \neq i^*P'$, $P'$ being the moving part of $P$. Let $a : \mathcal{G} \to \tilde{J}$, $b : B \to J$ be the natural Abel maps, then

$$\xymatrix{
\mathcal{G} \ar[r]^{h} & B \\
\tilde{J} \ar[u]^{a} \ar[r]_{Nm} & J \ar[u]^{b}
}$$

is a commutative diagram. We are interested in the scheme

$$(2.14) \quad X = a^*Nm^*(o) = h^*b^*(o),$$

where $o$ is $\omega_C$ considered as a point of $J$. In other words $X$ is the fibre at $\omega_C$ of the natural norm map $G^1_{2g-2}(\tilde{C}) \to Pic^{2g-2}(C)$. Since $Nm^*(o) = P \cup P^-$, $X$ is not connected. The underlying set of $X$ is naturally identified with the family of pairs

$$(\tilde{L}, \Gamma),$$

where $\Gamma$ is a 2-dimensional subspace of $H^0(\tilde{L})$ and $Nm\tilde{L} = \omega_C$. Note that the image of $a$ is $Sing\tilde{\Theta}$. Let $\Xi^- = P^- \cdot Sing\tilde{\Theta}$, one has

$$X = X^+ \cup X^-,$$

where

$$X^+ = a^*\Xi, \quad X^- = a^*\Xi^-.$$

**2.15 THEOREM.** Every component of $X$ is at least $g-2$ dimensional. Moreover $X^+$ and $X^-$ are integral of dimension $g-2$ if $\pi$ is general and $g \geq 4$.

**Proof.** The second statement follows from the first one and the theorems of Bertram and Welters about reducedness, irreducibility and codimension of Brill-Noether loci in a general Prym variety, ([B], [W]). The first statement too is a consequence of the same results.
We want to point out that the Hilbert scheme of conics of $|\omega_C|$ is the fibre
\[ B_{\omega_C} = b^{-1}(a) \]
of the map $b$. This scheme is related to $Q^3$ by the birational map
\[ \delta : B_{\omega_C} \to Q^3, \]
which associates to a smooth conic $B$ its dual hypersurface $B^* \subset \mathbb{P}^+$. This defines a map $\alpha = \delta \cdot h/X$ which is fundamental in this paper. In the beginning we want to define $\alpha$ in a slightly different way. To do this we consider on $X$ the universal line $p : \mathcal{P} \to X$ induced by the Hilbert scheme $\mathcal{G}$. $\mathcal{P}$ is the projectivization of a rank two vector bundle $\mathcal{U}$. Moreover the fibre product $\mathcal{P}^* \times_p \mathcal{P}^*$ is embedded in $\mathcal{P}(\mathcal{U} \otimes \mathcal{U})^*$ as the locus of indecomposable vectors. Over the point $P$ this embedding is precisely the Segre embedding of $\mathcal{P}(\mathcal{U}'_P) \times \mathcal{P}(\mathcal{U}'_P)$ as a quadric surface. This defines a section
\[ \sigma : X \to \mathbb{P}Sym^2(\mathcal{U} \otimes \mathcal{U}), \]
which associates to $P$ such a quadric. Then we consider the multiplication map
\[ \mu : \mathcal{U} \otimes \mathcal{U} \to H^0(\mathcal{O}_\mathcal{P}(1)) \otimes \mathcal{O}_C, \]
sending $s \otimes t$ to $s^*t$. The 2-symmetric product of $\mu$ induces a map
\[ m : \mathbb{P}Sym^2(\mathcal{U} \otimes \mathcal{U}) \to \mathbb{P}H^0(\mathcal{O}_\mathcal{P}(2)). \]
Taking the restriction of $m(\sigma(P))$ to $\mathbb{P}^+$ and $\mathbb{P}^-$ one defines two maps:
\[ m^+ : \mathbb{P}Sym^2(\mathcal{U} \otimes \mathcal{U}) \to \mathbb{P}H^0(\mathcal{O}^+(2)) \]
and
\[ m^- : \mathbb{P}Sym^2(\mathcal{U} \otimes \mathcal{U}) \to \mathbb{P}H^0(\mathcal{O}^-(2)). \]
Let $P$ be the point $(\tilde{L}, \Gamma)$ of $X$ and let
\[ \tilde{Q} = m(\sigma(P)). \]
We can describe the quadric $\tilde{Q}$ in a very concrete way. At first we have $\mathcal{U}_P = \Gamma$. Fix a basis $\{s_1, s_2\}$ of $\Gamma$ so that $z_{uv} = s_u \otimes s_v$ is a basis of $\Gamma \otimes \Gamma$. Then consider on $P$ the linear forms $x_{uv} = m(z_{uv}) = s_u^*s_v$ and the matrix
\[ A = (x_{uv}). \]
Since the image of $|\Gamma|^* \times |\Gamma|^*$ by Segre embedding is $\{det(z_{uv}) = 0\}$, it follows
\[ \tilde{Q} = \{detA = 0\}. \]

2.20 REMARK The construction of $\tilde{Q}$ is classical and well known: see e.g. [AM], [ACGH], [K]. If $\Gamma = H^0(\tilde{L})$ then, by Riemann-Kempf theorem, $\tilde{L}$ is a double point of $\tilde{\Theta}$ and $\tilde{Q}$ its projectivized tangent cone. $\tilde{Q}$ clearly contains $\tilde{C}$. 
2.21 LEMMA. Let $M$ and $\tilde{c}$ respectively be the moving and the fixed part of $| \Gamma |$. Assume $\Lambda \subset \tilde{Q}$ is a maximal linear subspace. Then $\Lambda \cdot \tilde{C} = \tilde{c} + i^*\tilde{c} + f$, where $f \in M \cup i^*M$. In particular $M$ is cut on $\tilde{C}$ by a ruling of $\tilde{Q}$.\[Proof.\] We have $s_u = \sigma s'_u$ ($u = 1, 2$), where $s'_1$, $s'_2$ generate $M$ and $\text{div}(\sigma) = \tilde{c}$. Moreover a system of equations for $\Lambda$ can be written as $(\ref{2.22})$.

Now we want to describe $m^+(\sigma(P))$ and $m^-(\sigma(P))$, that is the intersections

\[q = \tilde{Q} \cdot P^+ \quad \text{and} \quad t = P^- \cdot \tilde{Q}.\]

At first we recall how $i^*$ acts on a linear form $x$: $i^*x = x \iff x$ is zero on $P^-$, $i^*x = -x \iff x$ is zero on $P^+$. Secondly we observe that the terms in the matrices

\[(A + t A) \quad \text{and} \quad (A - t A)\]

are respectively invariant and anti-invariant by $i^*$. This follows from $i^*x_{uv} = x_{vu}$. Then one easily deduces that the restrictions of $\tilde{Q}$ to $P^+$ and $P^-$ are respectively

\[q = \{\text{det}(A + t A) = 0\} \quad \text{and} \quad t = \{\text{det}(A - t A) = 0\}.\]

Here we have identified, with some abuse, a linear form on $P$ and its restriction to $P^+$ or to $P^-$. We will do this when no confusion is possible. Note that

\[\text{det}(A + t A) = 4x_{11}x_{22} - (x_{12} + x_{21})^2 = 0 \quad \text{and} \quad \text{det}(A - t A) = (x_{12} - x_{21})^2,\]

so $q$ is a quadric of rank $\leq 3$ and $t$ is either $P^-$ or a double hyperplane in it. $m^+ \cdot \sigma$ is not defined at $P$ iff $t = P^-$. Since $x_{11}$, $x_{22}$ are linearly independent on $P^+$, $q$ has rank $\geq 2$. Hence $m^+ \cdot \sigma$ is a morphism.

2.24 PROPOSITION. The following conditions are equivalent:

(i) $t = P^-$,

(ii) the moving part of $| \Gamma |$ is the pull-back by $\pi$ of a pencil on $C$.

(iii) $q$ contains $C$.

Proof. Let $s_u = \sigma s'_u$ ($u = 1, 2$) as in the proof of (2.21). Then: (i) $\iff x_{12} - x_{21} = 0 \iff s_1i^*s_2 = s_2i^*s_1 \iff s'_1 = i^*s'_u$ ($u = 1, 2$) $\iff$ (ii). Moreover, it is easily seen that (i) $\iff \tilde{Q} \supset \tilde{C} \cup C^- \iff \tilde{Q} \supset \tilde{C} \cup C \iff$ (iii).

2.25 LEMMA. $t \cdot C^- = q \cdot C$. In particular $q$ belongs to the variety $Y$.

Proof. One immediately checks that $4x_{11}x_{22} - (x_{12} + x_{21})^2$ and $(x_{12} - x_{21})^2$ restrict to the same bicanonical section of $\tilde{C}$. Then the result follows from the equalities

\[\pi^*(t \cdot C^-) = \tilde{C} \cdot \{(x_{12} - x_{21})^2 = 0\} = \tilde{C} \cdot \{4x_{11}x_{22} - (x_{12} + x_{21})^2 = 0\} = \pi^*(q \cdot C).\]

By the previous lemma the image of $m^+ \cdot \sigma$ is in $Y$. On the other hand the image of $m^- \cdot \sigma$ is in $P^-\ast$, where $P^-\ast$ is embedded in the space of quadrics of $P^-$ by the squaring map. From now on we will adopt the following
2.26 DEFINITION. 
- \( \alpha : X \to Y \) is the morphism \( m^+ \cdot \sigma \).
- \( \beta : X \to P^- \) is the rational map \( m^- \cdot \sigma \).

2.27 THEOREM. Let \( v : P^- \to V \) be the restriction isomorphism sending \( t \) to \( t \cdot C^- \) and let \( \lambda \) be the restriction map considered in 2.6. Then the diagram

\[
\begin{array}{ccc}
X & \xrightarrow{\alpha} & Y \\
\downarrow{\beta} & & \downarrow{\lambda/Y} \\
P^- & \xrightarrow{v} & V
\end{array}
\]

is commutative.

Proof. It follows immediately from lemma 2.25 and the definition of \( m^+ \), \( m^- \).

Now we can relate the previous constructions to the Gauss map. First we recall that the target space of this map is \( P^- \), because the tangent sheaf of \( P \) is \( O_P \otimes H^- \). Therefore, the Gauss map

\[(2.28) \quad \gamma : \Xi \to P^- \]

associates to a point \( \tilde{L} \in \Xi - Sing\Xi \) a hyperplane \( \gamma(\tilde{L}) \) in \( P^- \). Since \( \tilde{L} \) is a smooth point, the space \( \Gamma = H^0(\tilde{L}) \) is two-dimensional, (cfr. [ACGH]p.298). Hence \( P = (\tilde{L}, \Gamma) \) is a point of \( X \). In particular \( \beta(P) \) is another hyperplane in \( P^- \). The main fact is now Tjurin’s geometric description of \( \gamma \) which says (cfr. [T],1.4):

\[(2.29) \quad \gamma(\tilde{L}) = \beta(P). \]

2.30 THEOREM. If \( X^+ \) is irreducible, then it holds \( \text{deg} \beta/X^+ = \text{deg} \gamma \).

Proof. The Abel map \( a/X^+ : X^+ \to \Xi \) sends \( P = (\tilde{L}, \Gamma) \) to \( \tilde{L} \). Moreover we have \( \beta/X^+ = \gamma \cdot a/X^+ \) because \( \gamma(\tilde{L}) = \beta(P) \). \( a/X^+ \) is clearly invertible on \( \Xi - Sing\Xi \). If \( X^+ \) is irreducible, then \( a/X^+ \) is birational and the result follows.

The statement is no longer true if \( X^+ \) is reducible. In this case the irreducible components of \( X^+ \) which are contracted by the Abel map \( a \) contribute to the degree of \( \beta/X^+ \). An example of this situation will appear in section 4, where \( C \) is trigonal. From now on we fix the following notations:

2.31 DEFINITION.
\( b_+(\pi) = \text{degree of} \ \beta \text{ on} \ X^+ \).
\( b_- (\pi) = \text{degree of} \ \beta \text{ on} \ X^- \).

It is clear that \( b_+ (\pi) \) is the degree of the Gauss map if \( \pi \) is general. The remaining part of this paper will be devoted to the solution of the following problem:

Compute \( b_+ (\pi) + b_- (\pi) \) and \( b_+ (\pi) - b_- (\pi) \) if \( \pi \) is general.
3. The maps $b_+ - b_-$ and $b_+ + b_-

Let $\mathcal{R}_g$ be the moduli space of $\pi: \tilde{C} \to C$. With some abuse, we will denote in the same way $\pi$ and its moduli point in $\mathcal{R}_g$. We have maps

$$b_+: \mathcal{R}_g \to \mathbb{Z} \quad \text{and} \quad b_-: \mathcal{R}_g \to \mathbb{Z},$$

sending $\pi$ respectively to $b_+(\pi)$ and to $b_-(\pi)$. It is clear that the values of $b_+ + b_-$ and of $b_+ - b_-$ are constant on some open dense subset. In this section we compute this constant value for $b_+ + b_-$. Then we introduce a method for computing the constant value of $b_+ - b_-$. The method will be applied in the next section to the case where $C$ is trigonal, this will make the computation effective.

3.1 LEMMA. The diagram

$$
\begin{array}{ccc}
X & \xrightarrow{h} & B_{\omega_C} \\
\downarrow{\sigma} & & \downarrow{\delta} \\
\text{PSym}^2(U \otimes U) & \xrightarrow{m^+} & \mathbb{Q}^3
\end{array}
$$

is commutative. In particular $\alpha = \delta \cdot h$.

Proof. Let $P = (\tilde{L}, \Gamma)$. Assume $q = (m^+ \cdot \sigma)(P)$ and $q' = (\delta \cdot h)(P)$. By definition of $\delta \cdot h$, the family $\{< \pi_* \tilde{f} >, \tilde{f} \in \Gamma \}$ is the family of tangent hyperplanes to $q'$. On the other hand we have $< \pi_* \tilde{f} > = \mathbb{P}^+ \cdot < \tilde{f} + i^* \tilde{f} >$ for each $\tilde{f}$. Keeping our notations as in 2.19, we can assume $\tilde{f} = \text{divs}_1$. Then $< \tilde{f} + i^* \tilde{f} > = \{x_{11} = 0\}$, which is tangent to $q$. Hence $< \pi_* \tilde{f} >$ is tangent to $q$ so that $q = q'$.

Now we want to describe with some detail the fibres of $\alpha$. Let

$$q \in Y,$$

to avoid unnecessary complications we will assume that $q$ has rank three. Let

$$c \in \text{Div}(C)$$

be the divisor associated to $C \cdot \text{Sing} q$, we consider the family

$$(3.2) \quad B_q = \{c + f \in | \omega_C | \mid < c + f > \text{ is a hyperplane tangent to } q \}.$$ 

We want to point out that $B_q$ is a smooth conic in $| \omega_C |$ and that $\delta(B_q) = q$. Moreover, if $q$ contains $C$, it holds $B_q = c + \pi_* M$, where $M = \pi^* N$ and $N$ is the base-point-free pencil which is induced by the ruling of $q$. 
3.3 LEMMA. On $\tilde{C}$ there exists a base-point-free pencil $M$ such that

$$B = c + \pi_* M = c + \pi_* i^* M.$$ 

Moreover $M$ is unique modulo the action of $i^*$.

Proof. Assume $M$ exists and is generated by $s_1, s_2$. Let $\tilde{Q} = \{det(x_{uv}) = 0\}$, $(u, v = 1, 2)$, where $x_{uv} = \sigma s_u i^* s_v$ and $div \sigma = \pi^* c$. $\tilde{Q}$ is a quadric in $P$ such that:

(i) $\tilde{Q} \cdot P^+ = q$, (ii) $\tilde{Q} \cdot P^-$ has rank $\leq 1$, (iii) the equation of $\tilde{Q}$ is $i^*$-invariant, (iv) $\tilde{Q} \supset \tilde{C}$. (ii),(iii),(iv) can be proved for $\tilde{Q}$ exactly as for the quadric considered in 2.19. In the same way it follows that $\{4x_{11}x_{22} - (x_{12} + x_{21})^2 = 0\}$ is a cone over $q'$, where $q' = \tilde{Q} \cdot P^+$. One can check that a hyperplane $h$ is tangent to this cone if $h = < \pi^* c + \tilde{m} + i^* \tilde{m} >$, for some $\tilde{m} \in M$. On the other hand, by the proof of 3.1, $< \pi^* c + \tilde{m} + i^* \tilde{m} >$ is tangent to $q$. Then $q' = \delta(B_q) = q$ and (i) holds. A quadric like $\tilde{Q}$ satisfies the following property too: the pencil $\pi^* c + M$ is cut on $\tilde{C}$ by a ruling of maximal linear subspaces of $\tilde{Q}$, the other ruling cuts $\pi^* c + i^* M$. Therefore $M$ is uniquely defined by $\tilde{Q}$ up to the action of $i^*$. The statement then follows if there exists a unique quadric $\tilde{Q}$ satisfying (i),(ii),(iii),(iv).

Uniqueness: by (iii) $\tilde{Q}$ has an equation of type $A + B$, with $A \in H^0(\mathcal{O}_P(2))$ and $B \in H^0(\mathcal{O}_P(-2))$. Let $A_1 + B_1$ and $A_2 + B_2$ be the equations of two quadrics satisfying (i),(ii),(iii),(iv). By (i) we can assume $A_1 = A_2$. By (ii) and (iv) $B_1 - B_2$ has rank $\leq 2$ and is zero on $\tilde{C}$. This implies $B_1 = B_2$.

Existence: $\tilde{Q}$ is the cone of vertex $P^-$ over $q$ if $q \supset C$. If not $q \cdot C = 2d$ and $d$ is cut on $C$ by a hyperplane. Let $B$ be a squared equation of it, $A$ an equation of $q$. Regarding $A, B$ as in $H^0(\mathcal{O}_P(2))$, we consider the pencil $S = \{z_0 A + z_1 B = 0\}$. Since $\{A = 0\} \cdot \tilde{C} = \pi^* 2d = \{B = 0\} \cdot \tilde{C}$, a quadric of $S$ contains $\tilde{C}$. This is $\tilde{Q}$.

3.4 COROLLARY. A point $P = (\tilde{L}, \Gamma)$ belongs to $\alpha^{-1}(q)$ if and only if $| \Gamma |$ is $\tilde{c} + M$ or $\tilde{c} + i^* M$, for some divisor $\tilde{c}$ on $\tilde{C}$ satisfying $\pi_* \tilde{c} = c$.

Proof. Immediate from the lemma.

3.5 PROPOSITION. Let $q \in Y$ and let $n = degc$. Then:

(1) $\alpha^* (q)$ is a zero-dimensional scheme of length $2^{n+1}$.

(2) Assume $c > 0$. Then $\alpha^* (q) \cdot X^+$ and $\alpha^* (q) \cdot X^-$ have the same length.

Proof. Let $B = \delta^* (q)$. $B$ is a point of the Hilbert scheme $\mathcal{B}_{\omega_C}$, that is a smooth conic in $\omega_C$. We consider $Z = \pi(2g-2)^* (o_B)$, where $o_B$ is the generic point of $B$ and $\pi(2g-2) : \tilde{C}(2g-2) \to C^{(2g-2)}$ is the push-down map. Since $h : \mathcal{G} \to B$ is the map of Hilbert schemes induced by $\pi(2g-2)$, it follows that $(\delta \cdot h)^* (q)$ is a subscheme of $Z$. The support of $(\delta \cdot h)^* (q)$ is the set of generic points $o_P$ of lines $P$ such that $\pi(2g-2)(P) = B$. The multiplicity of $o_P$ in $(\delta \cdot h)^* (q)$ is the ramification index $\nu_P$ of $\pi(2g-2)$ at $o_P$. Therefore $\nu_P$ is the index of $\pi(2g-2)$ at a general point of $P$ times
the degree of $\pi^{(2g-2)}/P$. To compute $\nu_P$ we consider the base extension

$$\begin{array}{ccc}
\tilde{C}^{(n)} \times \tilde{C}^{(m)} & \longrightarrow & \tilde{C}^{(2g-2)} \\
\pi^{(n)} \times \pi^{(m)} & \downarrow & \pi^{(2g-2)} \\
C^{(n)} \times C^{(m)} & \longrightarrow & C^{(2g-2)}
\end{array}$$

where the horizontal arrows are the sum maps and $m + n = 2g - 2$. By lemma 3.3 $o_P$ is in $(\delta \cdot h)^*(q)$ iff $P = \tilde{c} + M$ or $P = \tilde{c} + i^*M$, with $M$ base-point-free and $\pi_*\tilde{c} = c$. From the equality $P = \tilde{c} + M$ and the previous diagram it follows that $\nu_P$ is the ramification index of $\pi^{(n)} \times \pi^{(m)}$ at the generic point of $\{\tilde{c}\} \times M$.

Then $\nu_P = \nu_{\tilde{c}^*} \nu_M$, where $\nu_{\tilde{c}^*}$ is the index of $\pi^{(n)}$ at $\tilde{c}$ and $\nu_M$ is the index of $\pi^{(m)}$ at the generic point of $M$. Since $M$ is base-point-free, $\pi^{(m)}$ is not ramified along $M$. Hence $\nu_M$ coincides with the degree of $\pi^{(m)}/M$ onto its image. This is one if $M \neq i^*M$, two if $M = i^*M$. We can conclude that: length$(\delta \cdot h)^*(q) = \Sigma \nu_P = \Sigma \nu_{\tilde{c}^*} \nu_M = 2\Sigma \nu_{\tilde{c}} = 2\deg \pi^{(n)} = 2^{n+1}$. This shows (1), since $\alpha = \delta \cdot h$.

(2) $\pi^{(n)^*}(c)$ embeds in $\alpha^*(q)$ via the map sending $\tilde{c}$ to $\tilde{c} + M$. Let $U$ be the image of such an embedding. If $M \neq i^*M$, $\alpha^*(q)$ is the disjoint union of $U$ and $i^*U$. If $M = i^*M$, $\alpha^*(q)$ is $U$ with a double structure, because $\nu_M = 2$. On the other hand we have $\pi^{(n)^*}(c) \subset S$, where $S = \pi^{(n)^*} \{ c \}$. $S$ is the disjoint union of $S^+$ and $S^-$, where $S^+ = \{ \tilde{c} \in S/\tilde{c} + M \in X^+ \}$ and $S^- = S - S^+$. If $c$ is the zero divisor $S$ is trivially one point and $\{ S^+, S^- \} = \{ S, \emptyset \}$. If $c > 0$, $S^+$ and $S^-$ are well known special varieties. They have been studied by Beauville in [B2]. In particular they are numerically equivalent in $\tilde{C}^{(n)}$, ([B2] section 1). Hence $\pi^{(n)^*}(c) \cdot S^+$ and $\pi^{(n)^*}(c) \cdot S^-$ have the same length. This implies (2).

### 3.6 COROLLARY

$\alpha$ is a finite morphism onto $Y$.

Proof. It follows from statement 3.5(1).

### 3.7 PROPOSITION

Assume $\pi$ is standard and Petri general. Then the degree of $\alpha$ is two.

Proof. By assumption no element of $Y$ contains $C$, moreover $X$ and $Y$ are reduced of dimension $g - 2$. Let $S = \lambda^*(d)$ be a general fibre of $\lambda/Y$, then $S$ is smooth and $\dim S = 0$. $S$ is the subset of rank three quadrics in the linear system $| I_{2d}(2) |$ of all quadrics cutting $2d$ on $C$. By proposition 3.5(1) it suffices to show that $Singq \cap C = \emptyset$, for some $q \in S$. Let $T$ be the embedded tangent space to $S$ at $q$ and let $I_{Singq}$ be the Ideal of $Singq$, then $T = | I_{2d}(2) | \cap | I_{Singq}(2) |$. (cfr.[ACGH]A-5 p.101). In other words $T = PKer\rho$, where $\rho : H^0(I_{2d}(2)) \to H^0(O_{Singq}(2))$ is the restriction map. Note that $h^0(I_{2d}(2)) - h^0(O_{Singq}(2)) = 1$. Assume $x \in Singq \cap C$, then $\rho$ is not surjective and hence $\dim T \geq 1$: against the smoothness of $S$. 
3.8 THEOREM. \( b_+(\pi) + b_-(\pi) = 2D(g) \) if \( \pi \) is standard and Petri general.

Proof. By assumption \( X, Y \) are reduced. Moreover \( \deg \alpha = 2 \) and \( \deg \lambda/Y = D(g) \) so that \( \deg(\lambda \cdot \alpha) = 2D(g) \). From \( \beta = \lambda \cdot \alpha \) one has \( b_+(\pi) + b_-(\pi) = \deg \beta = 2D(g) \).

Now we study the difference map \( b_+ - b_- \). We will work under the following

3.9 ASSUMPTION.

(1) Let \( C \subset q \), where \( q \) is a rank three quadric. Then \( \text{Sing}q \cap C \neq \emptyset \).

(2) No irreducible component of \( Y \) is in the center of \( \lambda \).

(3) The general fibre of \( \lambda/Y : Y \to V \) is finite.

Condition (1) guarantees that \( \alpha^*(q) \cdot X^+ \) and \( \alpha^*(q) \cdot X^- \) have the same length as soon as \( q \) contains \( C \). (1) is not satisfied iff there exists a globally generated theta characteristic \( \theta \) on \( C \). (2) and (3) seem to be always true.

3.10 DEFINITION. \( R \subset R_g \) is the subset where the assumption holds.

Note that (1),(2),(3) are satisfied if no rank three quadric contains \( C \). Hence \( R \) contains \( R_{P\text{etri}} \). In particular \( R \) contains a dense open subset. A good reason for considering \( R \) instead of \( R_{P\text{etri}} \) is that \( R \) has non empty intersection with the locus of points \( \pi \) such that \( C \) is trigonal: see section 4. Now we fix a smooth family

\[
(3.11) \quad \pi : \tilde{\mathcal{C}} \to \mathcal{C}
\]

of étale double coverings of genus \( g \) curves. This means that \( \pi \) is a connected étale double covering of a smooth family

\[
(3.12) \quad p : \mathcal{C} \to \mathcal{T}
\]

of irreducible genus \( g \) curves, we will assume that each fibre of \( p \) is not hyperelliptic. \( \pi \) is defined by \( \eta \in \text{Pic} \mathcal{C} \), \( \mathcal{T} \) is a smooth integral affine curve. For each \( t \in \mathcal{T} \), \( \pi \) induces an étale double covering

\[
\pi_t : \tilde{\mathcal{C}}_t \to \mathcal{C}_t.
\]

Using the previous data we construct the families of symmetric products

\[
p(n) : \tilde{\mathcal{S}}(n) \to \mathcal{T} \quad \text{and} \quad p(n) : \mathcal{S}(n) \to \mathcal{T},
\]

with fibres \( \tilde{\mathcal{C}}_t(n) \) and \( \mathcal{C}_t(n) \). Then we consider the Hilbert scheme \( \mathcal{B} \) of conics in \( \mathcal{S}(2g-2) \) and the Hilbert scheme \( \mathcal{B}_H \) of conics in \( \mathcal{H} \), where \( \mathcal{H} = \mathbb{P}p_*\omega_{\mathcal{C}/\mathcal{T}} \). Due to the equality \( \mathcal{H}_t = |\omega_{\mathcal{C}_t}| \) there is a natural inclusion \( \mathcal{H} \subset \mathcal{S}(2g-2) \), so that \( \mathcal{B}_H \subset \mathcal{B} \). Finally we also consider the Hilbert scheme \( \mathcal{G} \) of lines in \( \tilde{\mathcal{S}}(2g-2) \). All these are schemes over \( T \), they admit standard definitions we have omitted for brevity. Let

\[
\phi : \tilde{\mathcal{S}}(2g-2) \to \mathcal{S}(2g-2)
\]
be the morphism sending \( d \in \tilde{C}_t^{(n)} \) to \( \pi_t^*d \in C_t^{(n)} \). Then \( \phi_* \) induces a map

\[ h : \mathcal{G} \to \mathcal{B} \]

sending a general line \( P \in \mathcal{G} \) to the conic \( h(P) = \phi_*P \). By definition we put

(3.13) \[ \mathcal{X} = h^*\mathcal{B}_H. \]

For each \( t \) \( \mathcal{X}_t \) is our usual variety \( X \), defined from \( \pi_t \) as in 2.14. We have

(3.14) \[ \mathcal{X} = \mathcal{X}^+ \cup \mathcal{X}^-, \]

\( \mathcal{X}^+ \) and \( \mathcal{X}^- \) being disjoint components. Each component of \( \mathcal{X} \) is at least \( g - 1 \) dimensional: the same proof used in 2.15 works. By definition \( P \) is a point of \( \mathcal{X}^+ \) iff \( P \) is a line in a complete linear system of odd dimension. Let

\[ Q = \mathbb{P}Sym^2 p_*\omega_{\mathcal{C}/T} \]

and let \( \mathcal{Q}^3 \subset Q \) be the locus of quadrics of rank \( \leq 3 \). As in 2.16 we define a map

\[ \delta : \mathcal{B}_H \to Q, \]

sending a smooth conic to its dual hypersurface. Then we define

(3.15) \[ \alpha = \delta \cdot h/\mathcal{X} : \mathcal{X} \to Q. \]

\( \mathcal{Q}^3 \) is the birational image of \( \delta \). Let us fix a general very ample section

\[ s : T \to \mathbb{P}p_*(\omega_{\mathcal{C}/T} \otimes \eta). \]

Squaring \( s \) we obtain the family \( \{2d_t \in | \omega_{\mathcal{C}_t}^{\otimes 2} |, d_t = s(t)\} \). Then we construct, in the usual standard way, a vector bundle

(3.16) \[ \mathcal{F} \subset Sym^2 p_*\omega_{\mathcal{C}/T} \]

such that \( \mathcal{F}_t \subset Sym^2 H^0(\omega_{\mathcal{C}_t}) \) is the space of quadratic forms vanishing on \( 2d_t \). Here, and in what follows, each \( C_t \) is canonically embedded. Let

\[ F = \mathbb{P}\mathcal{F}, \]

we remark that each \( F_t \) contains a hyperplane \( I_t \) which is the linear system of quadrics through \( C_t \). \( I_t \) is the fibre at \( t \) of a projective bundle \( I \subset Q \) and \( I \) is the indeterminacy locus of the natural restriction map

(3.17) \[ \lambda : Q \to \mathbb{P}p_*\omega_{\mathcal{C}/T}^{\otimes 2}. \]
The resolution of the indeterminacy is \( \sigma \cdot \lambda, \sigma : Q' \to Q \) being the blowing up of \( I \).

Let \( \mathcal{Y}' \) be the strict transform of \( \mathcal{Y} = \alpha(\mathcal{X}) \), we consider the commutative diagram

\[
\begin{array}{ccc}
\mathcal{X}'^+ \cup \mathcal{X}'^- & \xrightarrow{\alpha'} & \mathcal{Y}' \\
\downarrow & & \downarrow \sigma/\mathcal{Y}' \quad \downarrow id \\
\mathcal{X}^+ \cup \mathcal{X}^- & \xrightarrow{\alpha} & \mathcal{Y} \\
\end{array}
\]

Here \( \lambda' : Q' \to \mathbb{P}p_\bullet \omega_{\mathcal{C}/T}^{\otimes 2} \) is the projective bundle structure induced by \( \lambda \) and the left square is a base extension. Recall that each component of \( \mathcal{X} \) has dimension \( \geq g - 1 \), the same property holds for \( \mathcal{X}'^+ \cup \mathcal{X}'^- \) because \( \alpha \) is finite and \( \sigma \) is birational. Let

\[ F' \subset Q' \]

be the strict transform of \( F \) by \( \sigma \). Since \( I \) is a divisor in \( F, \sigma/F' : F' \to F \) is a biregular map. In particular \( F' \) is locally a complete intersection in \( Q' \) and its codimension is \( g - 2 \). This implies that each irreducible component of

\[ S = \alpha'^* F' \]

has dimension \( \geq 1 \). Finally we fix a point \( o \in T \) and make the following assumption:

(i) \( \pi_t \) is Petri general if \( t \neq 0 \), so that no rank 3 quadric contains \( \mathcal{C}_t \),

(ii) \( d_t \) is general in its Prym-canonical linear system, in particular \( \lambda^*(d_o) \) is finite,

(iii) \( \pi_o \) is in the subset \( R \subset R_g \), defined in 3.10.

3.18 LEMMA. Each irreducible component of \( S \) is a curve which maps onto \( T \) via the rational map \( \lambda' \cdot \alpha' \).

Proof. It suffices to show that \( S_t = \alpha'^*F'_t \) is finite. Let \( t \neq o \), one can check in the previous diagram that \( \alpha^*F_t \) is biregular to \( S_t \). By (i) \( I_t \cap Q^3 = \emptyset \), hence \( F_t \cdot Q^3 \) is finite as well as \( \alpha^*F_t \). Let \( t = o \), consider the strict transform \( \mathcal{Y}'' \) of \( \mathcal{Y}_o \) and the Zariski closure \( E \) of \( \mathcal{Y}_o'' - \mathcal{Y}'' \). Then it holds \( S_o = \alpha'^*(F'_o \cdot \mathcal{Y}'' \) \( \cup \alpha'^*(F'_o \cdot E) \). The scheme \( F'_o \cdot \mathcal{Y}'' \) is the fibre over the point 2\( d_o \) of the map \( \lambda/\mathcal{Y}_o : \mathcal{Y}_o \to | \omega_{\mathcal{C}_o}^{\otimes 2} | \). Therefore it is finite because, by (ii) and (iii), \( d_o \) is general and \( \pi_o \) is in \( R \). The same holds for \( \alpha'^*(F'_o \cdot \mathcal{Y}_o'' \) because \( \alpha' \) is a finite map. Assume \( E \) is not empty and consider the Zariski closure \( \mathcal{Z}_t \) of \( \cup \mathcal{Y}_t \), \( t \neq o \). Then \( E \) is a component of the fibre \( \mathcal{Z}_o \). By (i) \( \mathcal{Z}_t \) is split in two distinguished irreducible components of dimension \( g - 2 \), for each \( t \neq o \). This implies \( dim E = g - 2 \). Now observe that \( F'_o \) is a general fibre of a projective bundle over the \( g - 2 \)-dimensional projective space \( | \omega_{\mathcal{C}_o} \otimes e | \).

Since \( dim E = g - 2 \) it follows that \( \alpha'^*(F'_o \cdot E) \) is finite. This completes the proof.

Let

\[ S^+ = S \cdot \mathcal{X}'^+ \quad \text{and} \quad S^- = S \cdot \mathcal{X}'^- \]
we consider on $F'$ the family of zero-cycles
\[(3.19) \quad \delta_t = [\alpha'_+ S^+][F'_t] - [\alpha'_- S^-][F'_t], \quad t \in T.\]
The degree of $\delta_t$ is constant, moreover $\deg \delta_t = b_+(\pi_t) - b_-(\pi_t)$ if $t \neq o$. On the other hand we can write
\[\delta_o = e + \delta'_o,\]
where $e$ is supported on points $q'$ such that $\sigma(q') \in I_o$ and $\text{Supp}(e) \cap \text{Supp}(\delta'_o) = \emptyset$.

3.20 LEMMA. $\text{Supp}(e)$ is empty, so that $e = 0$

Proof. Let $q = \sigma(q')$, where $q \in I_o$. Then $\alpha'_*(q')$ and $\alpha^*(q)$ are biregular: this just follows from the definition of base-extension. Moreover $S^+ \cdot \alpha'_*(q')$ and $S^- \cdot \alpha^*(q')$ have the same length. This follows because, by proposition 3.5(1), $\lambda^+ \cdot \alpha^*(q)$ and $\lambda^- \cdot \alpha^*(q)$ have the same length. Then $q'$ is not in $\text{Supp}(e)$, which must be empty.

We have shown that degree of $\delta'_o$ is exactly $b_+(\pi_o) - b_-(\pi_o)$, therefore
\[\deg \delta_t = b_+(\pi_t) - b_-(\pi_t), \quad \forall t \in T.\]
In particular $b_+ - b_-$ is constant on the curve $T' = \{\pi_t, t \in T\} \subset R$. Let $\rho$ be any point of $R$, it is clear that there are families $\pi : \tilde{C} \to C$ satisfying the previous assumption and moreover such that $\rho = \pi_o$. Then any two points are connected by a chain of curves as $T'$ and the next theorem follows.

3.21 THEOREM. The map $b^+ - b^-$ is constant on $R$.

The constant value of $b^+ - b^-$ on $R$ is $2g-2$: this is the argument of the next section.

4. Trigonal Curves and Hyperelliptic Thetas.

In this section we study the case where $\pi : \tilde{C} \to C$ is an étale double covering of a general trigonal curve. We will see that in this special case the fibre of the map $\lambda : Y \to V$ is naturally related to the set of theta-characteristics on a hyperelliptic curve of genus $p = g - 3$. Using this relation we will be able to compute that $b^+(\pi) - b^-(\pi) = 2g-2$ and to show that this is the value of $b^+ - b^-$ at a general point of $R_g$. To simplify the exposition we will assume $g \geq 4$, leaving the case $g = 3$ as an exercise. Since $C$ is trigonal one has
\[(4.1) \quad C \subset R.\]

$R$ is a general rational normal scroll in the canonical space of $C$, that is the Hirzebruch surface $F_0$ ($g$ even) or $F_1$ ($g$ odd). The Picard group of $R$ is $\mathbb{Z}[h] \oplus \mathbb{Z}[f]$, where $h$ is a hyperplane section and $f$ is a fibre of $R$. One computes
\[C \sim 3h - (g - 4)f, \quad K_R \equiv -2h + (g - 4)f.\]

Let $I_R$, ($I_C$), be the linear system of quadrics through $R$, (through $C$), then
\[(4.2) \quad I_R = I_C.\]

In particular $I_R$ is the center of the linear projection $\lambda : Q \to B$ defined in (2.5). As we will see $I_R$ intersects $Y$, so that the degree of $\lambda/Y$ is different from $D(g)$.
4.3 Lemma.  (i) Let \( q \in Y \cap I_C \), then \( \text{Sing} q \cap C \) is not empty.
(ii) The dimension of \( Y \cap I_C \) is \( g - 5 \).

Proof. (i) Since each line in \( q \) intersects \( \text{Sing} q \), the same holds for each fibre \( f \) of \( R \). This implies that \( \text{Sing} q \cap R \) contains a curve \( b \). On the other hand we have \( C \sim h - K_R \), which is easily seen to be ample on \( R \). Hence \( C \cap b \neq \emptyset \).

(ii) Restricting to \( R \) the hyperplanes through \( \text{Sing} q \) we obtain a net of divisors \( N \subset | h | \). Let \( b \) be the fixed curve of \( N \), it is easy to see that \( bf = 1 \). Then \( h - b \sim mf \) and the moving part of \( N \) is a base-point-free net \( M \) in \( | m f | \). Let \( \phi_M : R \to \mathbb{P}^2 \) be the morphism defined by \( M \). We observe that \( \phi_M \) is simply the restriction to \( R \) of the linear projection of center \( \text{Sing} q \). This implies that \( \phi_M(R) \) is a smooth conic. But then \( \phi_M = v \cdot \phi_P \), where \( v \) is the 2-Veronese embedding of \( \mathbb{P}^1 \) and \( \phi_P \) is defined by a pencil \( P \subset | k f | \), \( m = 2k \). \( (P,b) \) is a point of \( G_k \times | h - 2kf | \), where \( G_k \) is the Grassmannian of pencils in \( | k f | \). Inverting the construction, one can easily see that any pair \( (P,b) \in \cup_k G_k \times | h - 2kf | \) defines a rank three quadric containing \( R \) and hence \( C \). This yields a surjective

\[
f : \cup_k (G_k \times | h - 2kf |) \to (I_R \cap Y)
\]

sending \( (P,b) \) to \( q \). Counting dimensions it follows \( \dim G_k \times | h - 2kf | \leq g - 5 \).

Now we consider the restriction map \( r : | 2h | \to | \omega_C^\otimes 2 | \). Since \( 2h - C \sim h + K_R \), we have \( h^0(\mathcal{O}_R(2h - C)) = h^1(\mathcal{O}_R(2h - C)) = 0 \). Then, from the long exact sequence of

\[
0 \to \mathcal{O}_R(2h - C) \to \mathcal{O}_R(2h) \to \mathcal{O}_C(2h) \to 0,
\]

it follows that \( r \) is an isomorphism. We are interested in the family of curves

\[
V_R = r^{-1}(V),
\]

where \( V \) is the image of \( | \omega_C \otimes \eta | \) under the ‘squaring map’. An element \( E \in V_R \) is a hyperelliptic curve of arithmetic genus \( p = g - 3 \). \( E \) satisfies the condition \( E \cdot C = 2d \), for a given \( 2d \in V \). The hyperelliptic map of \( E \) is induced by the natural projection of \( R \) onto \( \mathbb{P}^1 \).

4.5 Lemma. The general \( E \in V_R \) is a smooth irreducible curve.

Proof. It suffices to produce a smooth \( E_0 \in | 2h | \) and a stable canonical curve \( C_0 \sim C \) such that: (i) \( E_0 \cdot C_0 = 2d_0 \), (ii) \( \text{Supp} d_0 \cap \text{Sing} C_0 = \emptyset \), (iii) \( \dim | d_0 | = g - 2 \).

Indeed \( (C_0, d_0) \) deforms in a family of pairs \( \{(C_t, d_t), t \in T\} \), such that \( C_t \) is smooth and \( d_t \) is a Prym-canonical divisor. Moreover for each \( d_t \) there exists exactly one \( E_t \sim E_0 \) such that \( E_t \cdot C_t = 2d_t \). The general \( E_t \) is smooth because \( E_0 \) is smooth: this implies the lemma. For brevity we only sketch the construction of a pair \( (C_0, d_0) \). Fix \( E_0 = q \cdot R \), where \( q \) is a rank three quadric transversal to \( R \). A general tangent hyperplane to \( q \) cuts on \( R \) a smooth curve \( A_0 \) such that \( A_0 \cdot E_0 = 2a_0 \). In the very ample linear system \( | C - A_0 | \) choose a smooth \( B_0 \) which is transversal
to $A_0$ and moreover satisfies $B_0 \cdot E_0 = 2b_0$, $B_0 \cap A_0 \cap E_0 = \emptyset$. The required pair is $(C_0, d_0)$, with $C_0 = A_0 + B_0$ and $d_0 = a_0 + b_0$. Note that $C_0$ is the union of a rational and an elliptic curve intersecting transversally in $g$ points.

Let $I_{2d}$ be the linear system of quadrics through $2d = E \cdot C$ and let $I_E$ be the linear system of quadrics through $E$. We want to point out that

\begin{equation}
I_{2d} = I_E.
\end{equation}

This follows from $I_E \subseteq I_{2d}$ and the fact that $\dim I_E = \dim I_{2d} = \dim I_R + 1$, because $I_R = I_C$. We observe that the fibre at a general $2d$ of the map $\lambda/Y : Y \rightarrow V$ is

$$\hat{Q}_E = (I_E - I_R) \cdot \mathbb{Q}^3.$$  

4.7 LEMMA. Let $T_q$ be the projective tangent space to $q \in \hat{Q}_E$. If $\dim T_q > 0$ then $\text{Sing}q \cap R \neq \emptyset$.

Proof. The statement is obvious if $q$ has rank $\leq 2$, therefore we assume $rk(q) = 3$. The embedded tangent space to $\hat{Q}_E$ at $q$ is the linear system $T_q$ of all quadrics through $E \cup \text{Sing}q$. If $\dim T_q > 0$ it follows $T_q \cap I_R \neq \emptyset$. Hence there exists a quadric $q'$ of rank $\leq 6$ which contains both $R$ and $\text{Sing}q$. We claim that this implies $\text{Sing}q \cap R \neq \emptyset$. The proof is immediate if $q'$ has rank $\leq 4$. Assume $q'$ has rank 6 and consider the projection $\sigma : q' \rightarrow \mathbb{P}^5$ of center $\text{Sing}q'$ and the smooth quadric $G = \sigma(q')$. Note that $\text{Sing}q' \subset \text{Sing}q$ and that $\sigma(\text{Sing}q)$ is a plane. Moreover it can be assumed that $\sigma(R)$ is a surface too: if not $R \cap \text{Sing}q$ is already not empty. $\sigma(R)$ is not a plane because $R$ is not degenerate, hence $\sigma(R)$ has positive intersection index with $\sigma(\text{Sing}q)$ and this implies $\text{Sing}q \cap R \neq \emptyset$. If $q'$ has rank 5 the proof is analogous.

4.8 COROLLARY. $\hat{Q}_E$ is smooth and finite if $E$ is smooth. In particular, every irreducible component $Z \subset Y$ such that $\lambda(Z) = V$ is reduced.

4.9 PROPOSITION. Assume $C$ is a general trigonal curve then $\pi$ belongs to the set $R$ defined in 3.10.

Proof. By the previous lemmas 4.8 and 4.5 the general fibre of $\lambda/Y$ is finite so that condition 3.9 (3) holds for $\pi$. By lemma 2.11 and 4.4 conditions 3.9 (1) and 3.9(2) are satisfied too. Hence $\pi$ belongs to $R$.

Now we want to compute the degree of $\lambda$ on the two components $Y^+ = \alpha(X^+)$ and $Y^- = \alpha(X^-)$ of $Y$. Let $E$ be a smooth element of $V_R$, $E$ is a hyperelliptic curve. We will show that there is a natural identification between the set of even (odd) theta’s on $E$ and the fibre of $\lambda/Y^+$ (of $\lambda/Y^-$). We will also see that both $Y^+$ and $Y^-$ are reducible. Indeed $Y$ contains the components $S_k$, where $S_k$ is defined by the following condition: the fibre of $\lambda/S_k$ over any smooth $E$ is the subset of $\theta$’s satisfying $h^0(\theta) = k$.

We associate to $E$ the following sets:

\begin{equation}
\hat{Q}_E = (I_E - I_R) \cdot \mathbb{Q}^3.
\end{equation}
The set $T_E$ of theta characteristics on $E$.

The set $Q_E$ of line bundles $A \in \text{Pic}^{p+1}(E)$ such that $A^{\otimes 2} \cong O_E(h)$.

The fibre $\hat{Q}_E$ of $\lambda/Y$ at $E$.

The condition $E \cdot C = 2d$ defines a bijection

$$\epsilon_1 : Q_E \rightarrow T_E,$$

such that $\epsilon_1(A) = A^{\otimes 3}(-d)$. Indeed it holds $(\epsilon_1(A))^{\otimes 2} \cong O_E(3h - C) \cong \omega_E$.

**4.14 Lemma.** Let $C$ be general and let $E$ be general in $V_R$. Then $E$ satisfies the following property: (*) each $A \in Q_E$ is globally generated and has $h^0(A) = 2$.

**Proof.** It suffices to prove the statement for a general smooth $E \in |2h|$, because such an $E$ is in some $V_R$ as above. In the Hilbert scheme of $E$ we have the subscheme $E'$ parametrizing non degenerate curves which are biregular to $E$. One can easily show that $E'$ contains a dense open set $U_1$ of elements $E' \in |2h|$ satisfying (*). On the other hand let $R$ be the Hilbert scheme of $R$. We consider the morphism $\psi : E \rightarrow R$ which is so defined: $\psi(E') = R' = \bigcup_{x \in E'} \langle x, j(x) \rangle$, where $j$ is the hyperelliptic involution of $E'$. $R'$ is a Hirzebruch surface $F_n$. We recall that the locus of points $R' \in R$ for which $n$ is minimal, (i.e. $n = 0$ or $1$), is an open set $U_2$. An element of $U_1 \cap \psi^{-1}(U_2)$ is projective-isomorphic to some $E' \in |2h|$ satisfying (*).

In the following we will assume that $E$ is smooth and general as above. Let $q \in \hat{Q}_E$. Restricting the ruling of $q$ to $E$ we obtain a base-point-free pencil of divisors of degree $p + 1$. This yields a line bundle $A \in Q_E$. By definition

$$\epsilon_2 : \hat{Q}_E \rightarrow Q_E$$

is the injective map sending $q$ to $A$. Let $A \in Q_E$, consider the rank 3 quadric

$$q = \bigcup \langle a \rangle, \quad a \in |A|.$$

Since $|A|$ is base-point-free $\text{Sing} q \cap E$ is empty. Hence $q$ does not contain $R$ and moreover belongs to $\hat{Q}_E$. Then $\epsilon_2(q) = A$ and $\epsilon_2$ is bijective.

**4.16 Proposition.** The degree of $\lambda/Y : Y \rightarrow V$ is $2^{2p}$.

**Proof.** $\hat{Q}_E$ is the fibre of $\lambda/Y$. $\epsilon_1 \cdot \epsilon_2 : \hat{Q}_E \rightarrow T_E$ is bijective. $\#T_E = 2^{2p}$.

Now we want to study the theta characteristic

$$\theta_q = (\epsilon_1 \cdot \epsilon_2)(q)$$

for any $q \in \hat{Q}_E$. Let

$$(\tilde{L}_q, \Gamma), (i^* \tilde{L}_q, i^* \Gamma) = \alpha^*(q),$$

in particular we want to understand the relation between $\theta_q$ and the line bundle $\tilde{L}_q$. 

(4.10) the set $T_E$ of theta characteristics on $E$.

(4.11) the set $Q_E$ of line bundles $A \in \text{Pic}^{p+1}(E)$ such that $A^{\otimes 2} \cong O_E(h)$.

(4.12) the fibre $\hat{Q}_E$ of $\lambda/Y$ at $E$.

The condition $E \cdot C = 2d$ defines a bijection

$$\epsilon_1 : Q_E \rightarrow T_E,$$

such that $\epsilon_1(A) = A^{\otimes 3}(-d)$. Indeed it holds $(\epsilon_1(A))^{\otimes 2} \cong O_E(3h - C) \cong \omega_E$.
4.19 THEOREM. For each \( q \in \hat{Q}_E \) one has
\[
h^0(E, \theta_q) = 2 + h^0(\tilde{C}, \tilde{L}_q) = 2 + h^0(\tilde{C}, i^* \tilde{L}_q).
\]
Before giving the proof let us deduce the main consequences of the theorem. At first it follows that \( q \) is in the fibre of \( \lambda/Y^+ \) if and only if \( \theta_q \) is even. Therefore, recalling the number of even and odd thetas, we obtain

4.20 COROLLARY. \( \deg \lambda/Y^+ = 2p-1(2^p+1) \), \( \deg \lambda/Y^- = 2p-1(2^p-1) \).

Secondly, since \( E \) is hyperelliptic, we have a partition of the set of its thetas according to the dimension of the space of global sections. This defines a splitting of \( Y \) in more than two components. Let

\[
S_k = \{ q \in Y/ q \in \hat{Q}_E \text{ for a smooth } E, h^0(\theta_q) = k \}
\]

where the overline denotes Zariski closure in \( Y \), from the description of theta characteristics on a hyperelliptic curve it follows
\[
\#(S_k \cap \hat{Q}_E) = \binom{2p+2}{p+1-2k}, \quad k = 0, \ldots, \lfloor \frac{p+1}{2} \rfloor.
\]

Each \( S_k \) is a component of \( Y \) and they are reduced by lemma 4.8.

4.22 REMARK The degree of \( \lambda/S_k \) is \( \binom{2p+2}{p+1-2k} \). Applying the theorem, we can also remark that
\[
\alpha^*(S_k) = \{(\tilde{L}, \Gamma) \in X/ h^0(\tilde{L}) = 2+k \}.
\]

This implies that \( \alpha^*(S_0) \) is birational to the Prym-Theta divisor and that the degree of its Gauss map is \( \binom{2p+2}{p+1} \). It is not surprising that this is also the degree of the Gauss map for the theta divisor of a \( g-1 \)-dimensional Jacobian. Since \( C \) is trigonal, we know that the Prym of \( \pi : \tilde{C} \to C \) is indeed a Jacobian, ([R]).

PROOF OF THE THEOREM 4.19. Let us fix some preliminary constructions.

(1) The Castelnuovo surface of \( E \).

Let
\[
\rho : \overline{R} \to R
\]
be the double covering of \( R \) branched on \( E \). \( \overline{R} \) is a smooth rational surface endowed with the pencil of rational curves \( |\overline{f}| = |\rho^* f| \). One can show that
\[
|\overline{h}| = |\rho^* h|
\]
defines a natural embedding

\[ \phi : \overline{R} \rightarrow \mathbb{P}^g, \]  

therefore we will identify from now on \( \overline{R} \) to \( \phi(\overline{R}) \). Since \( f \overline{h} = 2 \), it follows that \( \overline{R} \) is a conic bundle in \( \mathbb{P}^g \). In particular \( \overline{R} \) is a surface of degree \( 2g - 4 \) with hyperelliptic hyperplane sections. It will be called here Castelnuovo surface, (cfr. [C]). Let

\[ \rho^* : \mathbb{P}^g \rightarrow \mathbb{P}^g \]

be the involution induced by \( \rho \), then \( \rho \) acts on \( H^0(\mathcal{O}_\overline{R}(\overline{h})) \). The +1 eigenspace is \( \rho^* H^0(\mathcal{O}_R(h)) \). The -1 eigenspace is generated by a section vanishing on \( \rho^{-1}(E) \). Then \( \rho^* H^0(\mathcal{O}_R(h)) \) has codimension 1 in \( H^0(\mathcal{O}_\overline{R}(\overline{h})) \), moreover the map \( \rho \) is induced by the projection from a point \( o \in \mathbb{P}^g \). One can also show that \( \overline{R} \) is the intersection of the cone over \( R \) of vertex \( o \) with a quadric hypersurface.

(2) **Pencils on the Castelnuovo surface.**

Let \( A \in Q_E \). For each \( a \in |A| \), the linear span \(<2a>\) is a tangent hyperplane to the quadric cone \( q \) such that \( \varepsilon_2(q) = A \). This follows from the definition of \( \varepsilon_2 \).

Claim. The curve \( H_a = <2a> \cdot R \) is irreducible for a general \( a \).

**Proof** Assume \( H_a \) is reducible for each \( a \). Then \( H_a = b_a + \Sigma f_{ia} \), where \( b_a \) is an irreducible section of \( R \) and \( f_{ia} \sim f \). Note that \( |a| \) is base-point-free because \( E \) is general as in lemma (4.14). Then, applying to \( |a| \) the uniform position lemma, it turns out that \( a = \Sigma x_i \), with \( \{x_i\} = b_a \cap f_{ia} \). But then \( f_{ia} \cdot E = 2x_i \) and \( x_i \) is a Weierstrass point of \( E \). This is impossible for a general \( a \).

By the claim \( H_a \) is a smooth rational curve. Since \( H_a \cdot E = 2a \), we have a splitting

\[ \rho^* H_a = A_a + j^* A_a, \]

where \( \rho/A_a : A_a \rightarrow H_a \) is birational. By adjunction formula we have \( H_a K_R = -g \). Since \( A_a \rho^{-1}(E) = H_a E = g - 2 \), it follows \( A_a K_R = -2 \) and \( A_a^2 = 0 \). Moreover \( A_a \) is not linearly equivalent to \( j^* A_a \). Since \( R \) is regular, the curves of the family \( F = \{A_a, j^* A_a / a \in |A|\} \) belong to finitely many linear equivalence classes. One can show easily from this that \( F \) is the disjoint union of two pencils: \( |A| \) and \( |j^* A| \). From the exact sequence

\[ 0 \rightarrow H^0(\mathcal{O}_R) \rightarrow H^0(\mathcal{O}_R(A)) \rightarrow H^0(\mathcal{O}_A(A)) \rightarrow 0 \]

it follows that \( |A| \) is an irreducible base-point-free pencil on \( \overline{R} \). We define as

\[ P_E \]
the set of all unordered pairs of line bundles \((O_{\tilde{R}}(A), O_{\tilde{R}}(j^*A))\) which are constructed as above from some \(A \in Q_E\). Let
\[
(4.30) \quad \epsilon_3 : Q_E \to P_E
\]
be the map sending \(A\) to \((O_{\tilde{R}}(A), O_{\tilde{R}}(j^*A))\). \(\epsilon_3\) is surjective by definition of \(P_E\).

To show the injectivity let \(A, A'\) be distinct elements of \(Q_E\) and let \(\epsilon_3(A) = (O_{\tilde{R}}(A), O_{\tilde{R}}(j^*A)), \epsilon_3(A') = (O_{\tilde{R}}(A'), O_{\tilde{R}}(j^*A'))\). The long exact sequence of
\[
0 \to O_{\tilde{R}}(-E + A - A') \to O_{\tilde{R}}(A - A') \to A \otimes A^* \to 0
\]
gives \(h^0(O_{\tilde{R}}(A - A')) = 0\), so that the two pairs are distinct. \(\epsilon_3\) is bijective.

(3) The surface \(\tilde{R}\).
At first we consider the curve \(\tilde{C} = \rho^*(C)\). This is a birational model of \(\tilde{C}\) with singular locus \(\rho^*(d)\), where \(2d = C \cdot E\). We can assume that \(\text{Sing} \tilde{C}\) is a set of \(2g - 2\) ordinary nodes
\[
\{x_1, \ldots, x_{2g-2}\}
\]
such that \(2\Sigma \rho(x_i) = C \cdot E = 2d\). Let
\[
(3.31) \quad \sigma : \tilde{R} \to \tilde{R}
\]
be the blowing up along \(\{x_1, \ldots, x_{2g-2}\}\). Then the strict transform of \(\tilde{C}\) is \(\tilde{C}\), moreover the involution \(j/\tilde{C} : \tilde{C} \to \tilde{C}\) lifts to the fixed-point-free involution \(i\) of \(\tilde{C}\). Up to the action of \(i^*\), each \((O_{\tilde{R}}(A), O_{\tilde{R}}(j^*A)) \in P_E\) uniquely defines a pair
\[
(\tilde{L}_q, \Gamma_q),
\]
where \(\Gamma_q\) is the image of the restriction map \(H^0(O_{\tilde{R}}(\sigma^*A)) \to H^0(O_{\tilde{C}}(\sigma^*A))\) and \(\tilde{L}_q \cong O_{\tilde{C}}(\sigma^*A)\). Here the index \(q\) denotes the quadric cone corresponding to the element \((O_{\tilde{R}}(A), O_{\tilde{R}}(j^*A))\) under the bijection \(\epsilon_3 \cdot \epsilon_2\).

From \(\rho_*\sigma_*(\sigma^*A) = \rho_*A \in |h|\) it follows \(Nm\tilde{L}_q \cong O_{\tilde{C}}(\rho_*A) \cong \omega_{\tilde{C}}\). Let \(\tilde{l} = \sigma^*A \cdot \tilde{C}\); by projection formula one has \(\rho_*\sigma_*(\sigma^*A \cdot \tilde{C}) = \rho_*(A \cdot \sigma_*\tilde{C}) = \rho_*A \cdot C = \pi_s\tilde{l}\). On the other hand \(\rho_*A \cdot E = 2a\) for some \(a \in |A|\). Since
\[
<\pi_s\tilde{l} > = < 2a > = < \rho_*A >,
\]
the family \(\{< \pi_s\tilde{l} >, \tilde{l} \in |\Gamma_q|\}\) is exactly the family of tangent hyperplanes to \(q\).

Hence
\[
\alpha(\tilde{L}_q, \Gamma_q) = q.
\]

Let \(K\) be a canonical divisor on \(\tilde{R}\) and let
\[
(3.33) \quad u : \tilde{R} \to \mathbb{P}^{2g-2}
\]
be the map defined by \(| K + \tilde{C} |\). Then \(u/\tilde{C}\) is the canonical embedding and we can put \(P = \mathbb{P}^{2g-2}\). We consider a pair \((\tilde{L}_q, \Gamma_q)\) as above: for a divisor \(\tilde{l} \in |\Gamma_q|\) we have
\[
(3.34) \quad \tilde{l} = \tilde{C} \cdot \sigma^*A
\]
for some \(A_\tilde{l}\) in the pencil \(| A |\).
\textbf{4.35 Lemma.} \( h^0(\tilde{L}_q) = h^0(\mathcal{O}_{\tilde{R}}(K + \tilde{C} - \sigma^* A)) \).

\textit{Proof.} Recall that \( h^1(\tilde{L}_q) = h^0(\tilde{L}_q) \) and consider the standard exact sequence

\[
0 \to \mathcal{O}_{\tilde{R}}(-\tilde{C} + \sigma^* A) \to \mathcal{O}_{\tilde{R}}(\sigma^* A) \to \tilde{L}_q \to 0.
\]

Since \( h^0(-\tilde{C} + \sigma^* A) = h^0(\sigma^* A) = 0 \), it follows \( h^1(\tilde{L}_q) = 2 + h^1(\mathcal{O}_{\tilde{R}}(K + \tilde{C} - \sigma^* A)) \).

Now consider the exact sequence

\[
0 \to \mathcal{O}_{\tilde{R}}(K + \tilde{C} - \sigma^* A) \to \mathcal{O}_{\tilde{R}}(K + \tilde{C}) \to \mathcal{O}_{\sigma^* A}(K + \tilde{C}) \to 0.
\]

We have \( \mathcal{O}_{\sigma^* A}(K + \tilde{C}) \cong \mathcal{O}_{\mathbb{P}^1}(2g - 4) \) and \( \chi(\mathcal{O}_{\tilde{R}}(K + \tilde{C})) = 2g - 1 \), so that

\[
\chi(\mathcal{O}_{\tilde{R}}(K + \tilde{C} - \sigma^* A)) = \chi(\mathcal{O}_{\tilde{R}}(K + \tilde{C})) - \chi(\mathcal{O}_{\sigma^* A}(K + \tilde{C})) = 2.
\]

This implies \( h^0(\mathcal{O}_{\tilde{R}}(K + \tilde{C} - \sigma^* A)) = 2 + h^1(\mathcal{O}_{\tilde{R}}(K + \tilde{C} - \sigma^* A)) \) and the statement.

Let \( E' \subset \tilde{R} \) be the strict transform of \( E \) by \( \sigma \), we want to point the equality

\[
(4.36) \quad | K + \tilde{C} | = | E' + \sigma^* \theta |.
\]

Indeed it holds \( K \sim -\sigma^* \theta + (g - 4)\sigma^* \mathcal{F} + G \), \( G \) being the exceptional divisor of \( \sigma \).

Moreover \( E' \sim \sigma^* h - G \) and \( \tilde{C} \sim 3\sigma^* \theta - (g - 4)\sigma^* \mathcal{F} - 2G \). In particular we have \( K + \tilde{C} \sim 2\sigma^* \theta - G \sim E' + \sigma^* \theta \). Notice also that

\[
(4.37) \quad \mathcal{O}_{E'}(K + \tilde{C}) \cong \mathcal{O}_E(2h - d)
\]

with \( 2d = C \cdot E = 2\Sigma \rho(x_i) \).

\textbf{4.38 Lemma.} \( h^0(\tilde{L}_q) = 2 + h^0(\mathcal{O}_{E'}(K + \tilde{C} - \sigma^* A)) \).

\textit{Proof.} The previous lemma implies \( h^0(\tilde{L}_q) = h^0(\mathcal{O}_{\tilde{R}}(K + \tilde{C} - \sigma^* A)) \). From (4.36) and \( A + j^* A \sim \theta \) we have \( K + \tilde{C} - \sigma^* A - E' \sim \sigma^* j^* A \). Since \( | j^* A | \) is a base-point-free pencil of irreducible rational curves, the same holds for \( | \sigma^* j^* A | \). In particular we have \( h^0(\mathcal{O}_{\tilde{R}}(\sigma^* j^* A)) = 2 \) and \( h^1(\mathcal{O}_{\tilde{R}}(\sigma^* j^* A)) = 0 \). Then the statement follows from the long exact sequence of

\[
0 \to \mathcal{O}_{\tilde{R}}(\sigma^* j^* A) \to \mathcal{O}_{\tilde{R}}(K + \tilde{C} - \sigma^* A) \to \mathcal{O}_{E'}(K + \tilde{C} - \sigma^* A) \to 0.
\]

(4) \textit{Completion of the proof of theorem 4.19.}

By (4.37) \( \mathcal{O}_{E'}(\tilde{C} + K - \sigma^* A_\theta) \cong \mathcal{O}_E(2h - d) \otimes \mathcal{A}^* \). Since \( \mathcal{A}^{\otimes 2} \cong \mathcal{O}_E(h) \), it follows \( \mathcal{O}_{E'}(\tilde{C} + K - \sigma^* A) \cong \mathcal{A}^{\otimes 3}(-d) \cong \theta_q \), where \( \theta_q \) is the theta characteristic considered in (4.7) and \( \epsilon_2(q) = \mathcal{A} \). By lemma (4.38), \( h^0(\tilde{L}_q) = 2 + h^0(\theta_q) \).

The motivation for studying the trigonal case was the proof of the next theorem.
4.39 THEOREM. Generically the function $b_+ - b_- : \mathcal{R}_g \to \mathbb{Z}$ has value $2^{g-2}$.

Proof. By 3.21 it suffices to compute $b_+ - b_-$ at a point of the set $R$ defined in 3.10. By 4.9 an étale double covering of a general trigonal curve $C$ defines a point in $R$. We have seen that in this special case $b_+(\pi) - b_-(\pi)$ is twice the difference between the number of even and odd thetas on a curve of genus $g - 3$, that is $2^{g-2}$.

5. Conclusion

5.1 THEOREM. The degree of the Gauss map for the Theta divisor of a general Prym variety $P$ is $D(g) + 2^{g-3}$.

Proof. Let $\pi$ be the étale double covering defining $P$. Then $\pi$ is general and $b_+(\pi)$ is the degree of the Gauss map. By theorems 3.8 and 4.39 we have $b_+(\pi) + b_-(\pi) = 2D(g)$ and $b_+(\pi) - b_-(\pi) = 2^{g-2}$. Hence $b_+(\pi) = D(g) + 2^{g-3}$.

6. References

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AUTHOR’s ADDRESS:
Dipartimento di Matematica, Università di Roma Tre
L.go S.Leonardo Murialdo 1
00146 ROME (Italy)
e-mail: verra@matrm3.mat.uniroma3.it