Entanglement of a qubit coupled to a resonator in the adiabatic regime

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We discuss the ground state entanglement of a bi-partite system, composed by a qubit strongly interacting with an oscillator mode, as a function of the coupling strength, the transition frequency and the level asymmetry of the qubit. This is done in the adiabatic regime in which the time evolution of the qubit is much faster than the oscillator one. Within the adiabatic approximation, we obtain a complete characterization of the ground state properties of the system and of its entanglement content.

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I. INTRODUCTION AND DESCRIPTION OF THE MODEL

The spin-boson model has been widely used to investigate the interaction between a two-level system, a qubit, and an harmonic oscillator environment, describing fluctuations of either electromagnetic or elastic origin. The coupling of the qubit with each environmental mode gives rise to a progressive entanglement, leading to the decoherence of the qubit itself.

This model has been largely employed in the weak coupling limit to explain noise effects in solid state devices which could be useful for quantum information processing. It has been also applied to describe the coupling of such devices to quantum detectors.

In the latter perspective, the strong coupling to a single bosonic mode has been analyzed in Ref. [2]. This kind of “restriction” to a single-mode environment appears to be useful for the decoherence problem too, as recent experimental and theoretical works have attributed a prominent role to the coupling of superconducting Josephson qubits with spurious micro-resonator resulting from the presence of switching charged impurities residing in the tunnel barrier [4, 5].

In this paper, we analyze the case of a qubit strongly coupled to a slow resonator, working in the adiabatic regime. Our model is a generalization of the one employed in Ref. [3], which turned out to also describe molecular Jahn-Teller effect. It can be also used to describe the coupling of a Josephson charge qubit to an electromagnetic resonator or to another (large) junction working in the harmonic regime, in the case of strong and off-resonant interaction. As it occurs for many solid state implementations, we assume, here, that the coupling can become so strong that the usual rotating wave approximation cannot be employed.

Our aim is to characterize the ground state of the system and, in particular, to evaluate the amount of quantum correlation present in (that is, the “entanglement content” of) the fundamental level. If the presence of the oscillator is spurious and un-wanted, this “residual” entanglement can produce errors in the information processing performed by the qubit. An investigation of the entanglement in the case of a two state system coupled to an ohmic environment has been performed by Costi and McKenzie, by exploiting the equivalence to the anisotropic Kondo model. They were able to show that the entanglement entropy, for level asymmetry different from zero, reaches a maximum at smaller values of dimensionless dissipation strength.

In fact, the calculation of ground state entanglement has been used to characterize complex quantum many body systems, with particular emphasis on its connection to quantum phase transitions [10, 11, 12, 13]. In our case, the system is bi-partite, and therefore there is no collective behavior to be examined; but nevertheless, a kind of criticality is present, as in the massive limit for the oscillator (and for qubit working at degeneracy), two regions in parameter space exist, exhibiting completely separable and entangled ground state, respectively. Furthermore, a sharp increase from zero is found at the onset of entanglement. This has been interpreted as a quantum reminiscence of the bifurcation of the fixed point of the oscillator in the corresponding classical model.

We show below that, within the adiabatic approximation scheme, this behavior can be obtained analytically together with the leading corrections for a finite tunnelling amplitude of the qubit. Our approach, however, is not limited to this region and we show that it can be used to systematically investigate ground state properties and entanglement in a broader parameter range, as we can account also for the effect of level asymmetry.

A qubit interacting with a single harmonic oscillator mode can be described by the Hamiltonian (in unit such that $\hbar = c = 1$)

$$H = \Delta \sigma_x + \left[ \epsilon + \frac{\lambda}{\sqrt{2m\omega}} (a + a^\dagger) \right] \sigma_z + \omega a^\dagger a$$

(1)

where $\Delta$ is the transition frequency of the qubit, $\epsilon$ is the level asymmetry, $\omega$ is the frequency of the oscillator and $\lambda$ is the coupling strength.
Hines et al. [6] start their description from the case of frozen qubit (i.e., $\Delta = \epsilon = 0$), which allows for an exact solution of the problem. Indeed, the Hamiltonian has doubly degenerate eigenstates which can be represented by displaced oscillator states $|\pm\rangle$. In this degeneracy limit, one obtains two displaced harmonic oscillator wells, with equilibrium positions $q_0 = \pm \lambda/m\omega^2$, so that the qubit can be localized in either the left or the right well.

The system’s wave function can be expanded in terms of a complete set of these orthonormal states and, for all sufficiently close to the degenerate limit, one obtains two displaced harmonic oscillator wells, with some necessary truncation of the Hilbert space [11], due to the lack of orthogonality between different displacements.

Here, we work in the opposite regime, and assume a fast qubit, $\Delta \gg \omega$, to perform the adiabatic approximation as described in the following section. Ground state entanglement is evaluated in section III, while section IV gives some concluding remarks.

II. ADIABATIC APPROACH

The Born-Oppenheimer approximation scheme can be followed more plainly by rewriting the Hamiltonian of Eq. (1) as follows

$$H = \frac{\omega}{2} \left[ Q^2 + P^2 + D\sigma_x + (W + LQ)\sigma_z \right],$$

where the oscillator coordinates representation has been introduced,

$$Q = \frac{1}{\sqrt{2}} (a^\dagger + a), \quad P = i\frac{1}{\sqrt{2}} (a^\dagger - a),$$

together with the dimensionless parameters $D = 2\Delta/\omega$, $W = 2\epsilon/\omega$ and $L = 2\lambda/\sqrt{m\omega^3}$. The basic assumption of the well-known adiabatic approximation is that the total wave function of a composite system with one fast (the qubit) and one slowly (the oscillator) changing part can be written as:

$$|\psi_{tot}\rangle = \int dQ \phi(Q) |Q\rangle \otimes |\chi(Q)\rangle$$

The states $|\chi(Q)\rangle$ are the eigenstates of the “adiabatic” equation of the qubit part for each fixed value of the slow variable $Q$,

$$[D\sigma_x + (W + LQ)\sigma_z] |\chi(Q)\rangle = E_{\pm}(Q) |\chi_{\pm}(Q)\rangle,$$

which gives the eigenvalues

$$E_{\pm}(Q) = \pm E(Q) = \pm \sqrt{D^2 + (W + LQ)^2}.$$  

The two eigenstates of Eq. (5) can be written as

$$|\chi_{\pm}(Q)\rangle = \frac{1}{\sqrt{2}} (A_-(Q)|+\rangle - A_+(Q)|-\rangle),$$

$$|\chi_{u}(Q)\rangle = \frac{1}{\sqrt{2}} (A_+(Q)|+\rangle + A_-(Q)|-\rangle),$$

where $|\pm\rangle$ are the eigenstates of $\sigma_z$ with eigenvalues $\pm 1$ and

$$A_{\pm}(Q) = \sqrt{1 \pm \frac{W + LQ}{E(Q)}}.$$  

The subscripts $l$ and $u$ refers to the lower and to the upper effective adiabatic potentials felt by the slow oscillator, respectively,

$$U_{u,l}(Q) = \frac{\omega}{2} [Q^2 \pm E(Q)].$$

As we are primarily interested on ground state properties, we will concentrate on $U_l$ from now on.

A special case of interest is the one with the qubit working at degeneracy, $W = 0$. In this case one obtains a symmetric Hamiltonian with conserved total parity (given by the joint transformation $Q \to -Q$ and $\sigma_z \to -\sigma_z$). Introducing the dimensionless parameter

$$\alpha = \frac{L^2}{2D} = \frac{\lambda^2}{m\omega^2\Delta},$$

one can show that for $\alpha \leq 1$, the potential $U_l^{W=0}(Q)$ can be viewed as a broadened harmonic potential well with its minimum at $Q = 0$ and $U_l^{W=0}(0) = -\Delta$. For $\alpha > 1$, on the other hand, the coupling of the oscillator with the qubit splits the oscillator potential producing a symmetric double well with the minima at

$$Q = \pm Q_0 = \pm \frac{D}{L} \sqrt{\alpha^2 - 1},$$

with

$$U_l^{W=0}(\pm Q_0) = -\frac{\Delta}{2} \left( \alpha + \frac{1}{\alpha} \right).$$

$Q_0$ is used as a kind of order parameter in Ref. [3], in the limit $D \to \infty$.

For $W \neq 0$, the symmetry is broken, and for this reason we refer to $W$ as the asymmetry parameter. The form of lower potential for two different sets of parameters is shown in Fig. 11.

Having obtained the state of the qubit, the last step in the adiabatic procedure is now to evaluate the ground state wave function for the oscillator, $\phi_0(Q)$, to be inserted in Eq. (11) to obtain the fundamental level of the coupled system. This wave function satisfies the one-dimensional time independent Schrödinger equation

$$H_{ad} \phi_0(Q) = \left( -\frac{\omega}{2} \frac{d^2}{dQ^2} + U_l(Q) \right) \phi_0(Q) = E_0 \phi_0(Q),$$

(14)
can be written as

\[ \rho_0 = \int_{-\infty}^{\infty} dQ \ket{\psi_0(Q)} \bra{\psi_0(Q)} = \frac{1}{2} (I + b_x \sigma_x + b_z \sigma_z) \]  

(15)

where \( \vec{b} = \bra{\vec{\sigma}} \) is the Bloch vector, whose non-zero components are explicitly given by the following integrals

\[ b_x = -\int_{-\infty}^{\infty} \phi_0^2(Q) \frac{D}{E(Q)} dQ, \]  

(16)

and

\[ b_z = -\int_{-\infty}^{\infty} \phi_0^2(Q) \frac{W + LQ}{E(Q)} dQ. \]  

(17)

In Fig. (3) and (4), we show the dependence on the dimensionless quantity \( \alpha = L^2/2D \) of the ground state expectation values defined by Eqs. (10) and (11), respectively.

It is easily seen that \( b_z \) is different from zero only for \( W \neq 0 \), while in the symmetric case the population is equally distributed between the states \( \ket{\pm} \) of the qubit. This is due to the inversion symmetry of the adiabatic potential, which, for finite \( D \), implies that the system does not localize in any of the wells and, consequently, that the state of qubit does not have a well defined value of \( \sigma_z \).

In the case \( W = 0 \), the integrand of Eq. (10) becomes the product of the squared ground state wave function and the square root of a Lorentz function centered at \( Q = 0 \). In the limit \( \alpha \to 0 \), this integral reduces to the normalization condition for the ground state wave function and thus \( b_x \approx -1 \). In fact, it is possible to show that, for small \( \alpha \), the main effect of the qubit is to renormalize the value of the oscillator frequency by a factor \( k = \sqrt{1-\alpha} \). As a result, the adiabatic ground state wave function for the oscillator is approximately given by

\[ \phi_0(Q) \approx \left( \frac{k}{\pi} \right)^{\frac{1}{2}} \exp \left\{ -\frac{k}{2} Q^2 \right\}, \]  

(18)

so that

\[ b_x \approx -1 + \frac{\alpha}{2Dk}, \quad \text{for} \ \alpha \ll 1. \]  

(19)

For \( \alpha \gg 1 \) the ground state wave function is located in spatial regions far from \( Q = 0 \) and thus \( b_x \approx 0 \). To obtain an analytic estimation for large \( \alpha \), we can take as an approximate adiabatic ground state for the oscillator the symmetric superposition

\[ \phi_0(Q) \approx \frac{1}{\sqrt{2}} \left\{ \phi_+(Q) + \phi_-(Q) \right\}, \]  

(20)

with

\[ \phi_{\pm}(Q) = \left( \frac{k'}{\pi} \right)^{\frac{1}{2}} \exp \left\{ -\frac{k'}{2} (Q \mp Q_0)^2 \right\}, \]  

(21)

III. REDUCED QUBIT STATE AND ENTANGLEMENT

The reduced density operator describing the qubit alone, when the overall system is in the ground state
where \(k' = 1 - \frac{1}{D\alpha^2}\) is, again, a renormalization factor for the oscillator frequency.

Taking the dominant contribution in Eq. (20), one gets

\[
b_x \simeq -\frac{1}{\alpha} - \frac{2}{D\alpha^2} \quad \text{for } \alpha \gg 1,\]

which we checked to be in very good agreement with the numerical solution.

For \(W \neq 0\), \(b_z\) becomes non-zero and decreases monotonically with increasing \(\alpha\) with \(-1\) as limiting value for \(\alpha \gg 1\). This is due to the fact that the \(\sigma_z\) contribution dominates in the qubit Hamiltonian in this regime, and therefore the qubit stays in the state \((-\).

Even in the presence of an asymmetry \(W\), the \(x\) component of the Bloch vector continues to grow monotonically from \(-1\) to \(0\) when \(\alpha\) increases from zero, with only quantitative deviation from the \(W = 0\) behavior.

We can, thus, summarize by saying that for small \(\alpha\), the state of the qubit is the lower eigenstate of \(\sigma_z\), while for large enough \(\alpha\) the qubit is found to be in the lower eigenstate of \(\sigma_z\). Between these two extreme cases, a cross-over occurs, which becomes a true, sharp transition for very large \(D\) and \(W = 0\) (see below).

The knowledge of the qubit reduced density matrix allows us to evaluate the entanglement in the ground state. A quantitative measure of the entanglement between the qubit and the oscillator is given by the tangle \[10\] which, for globally pure states, is defined as

\[
\tau = 2 \left| 1 - \text{Tr}(\rho_0^2) \right|. \tag{23}
\]

This is an entanglement monotone, giving \(\tau = 0\) for a separable state and reaching \(\tau = 1\) for maximally-entangled states. In our case, Eq. (23) becomes

\[
\tau = 1 - b_z^2 - b_z^2 \tag{24}
\]

The tangle is shown in Figs. (4), (6) and (7) for different values of the \(D\) and \(W\) parameters, as a function of the dimensionless quantity \(\alpha\). For any \(W \neq 0\), the entanglement increases with increasing \(\alpha\) before reaching a maximum value; after that, it decreases again to zero for \(\alpha \gg 1\). As stated above, this is due to the fact that the state of the system factorizes in this limit if \(W \neq 0\).

In the symmetric case, see Fig. (7), the entanglement becomes maximal as the coupling increases and, in the strict adiabatic limit \(D \to \infty\), the tangle becomes discontinuous at the critical value \(\alpha = 1\) and rapidly increases from zero to one when \(\alpha > 1\).

This result has a simple analytic derivation that is easily obtained from the thermal ground-state of the system. The reduced density operator describing the ground state of the system may be found by tracing out the oscillator variables from the thermal state

\[
\rho = e^{-\beta H}/Z(\beta), \tag{25}
\]

and by taking the limit \(\beta \to \infty\). Here \(Z(\beta) = \text{Tr} \{e^{-\beta H}\}\)

is the partition function. The thermal density \(\rho\) possesses the full symmetry as the Hamiltonian \(H\) and, if the ground state is non degenerate, the zero-temperature state coincides with the ground state of the system. It is important to stress that this is not generally true for degenerate ground states (as in the case \(\Delta = \epsilon = 0\)). When a degeneracy arises, each individual ground state may not possess the same symmetries of the Hamiltonian. Instead, they are always shared by the zero temperature state, which is just an equal mixture of all the possible ground states. This situation does not occur in our case.

By rewriting the Hamiltonian (2) as

\[
H = \frac{1}{2m}p^2 + \frac{m\omega^2}{2}q^2 + \Delta \sigma_x + (\epsilon + \lambda \sigma_z) \sigma_z, \tag{26}
\]

where

\[
q = \frac{1}{\sqrt{m\omega}}Q, \quad p = \sqrt{m\omega}P, \tag{27}
\]

we see that the limit \(m \to \infty\) and \(m\omega^2 \to \text{const.}\) is equivalent to neglect the kinetic energy of the oscillator. In this regime, thus, one gets

\[
Z(\beta) = \text{Tr} \int_{-\infty}^{\infty} dq q |e^{\beta mQ^2} e^{-\beta (\Delta \sigma_z + (\epsilon + \lambda \sigma_z) \sigma_z)}|q\rangle = 2 \int_{-\infty}^{\infty} dqe^{-\beta mQ^2} \cosh \left[\beta \sqrt{\Delta^2 + (\epsilon + \lambda \sigma_z)^2}\right] \tag{28}
\]
In the first row, the states $|q\rangle$, employed to perform the trace, are just the position eigenstate of the oscillator.

The thermal reduced density matrix of the qubit can be formally written in the form of Eq. (15) and is obtained by tracing out the thermal reduced density matrix over the oscillator degree of freedom. The temperature-dependent expectation values of $\sigma_x$ and $\sigma_z$, respectively, are, then

$$b_x = -\frac{2}{Z(\beta)} \int_{-\infty}^{\infty} dq \frac{\Delta}{\Delta(q)} e^{-\frac{\beta m x^2}{2}} \sinh[\beta \Delta(q)], \quad (29)$$

$$b_z = -\frac{2}{Z(\beta)} \int_{-\infty}^{\infty} dq \frac{\epsilon + \lambda q}{\Delta(q)} e^{-\frac{\beta m x^2}{2}} \sinh[\beta \Delta(q)], \quad (30)$$

where $\Delta(q) = \sqrt{\Delta^2 + (\epsilon + \lambda q)^2}$.

We focus, again, our discussion on the ground state $(\beta \to \infty)$. In this limit, the partition function may be evaluated by the steepest descent method. For $\epsilon = 0$, the integrand of Eq. (29) is symmetric around $q = 0$. When $\beta \to \infty$ and $\alpha \leq 1$ this function has only a sharp maximum at the origin, while for $\alpha > 1$, the integrand has two sharp maxima at $q = \pm \Delta \sqrt{\alpha^2 - \frac{1}{\lambda} \epsilon},$ symmetrically displaced around zero, where the function has a shallow minimum. In this limit, one easily obtain $b_x = 0$ and

$$b_x = \begin{cases} -1, & \alpha \leq 1; \\ -1/\alpha, & \alpha > 1. \end{cases} \quad (31)$$

Then, for the tangle, one simply gets

$$\tau = \begin{cases} 0, & \alpha \leq 1; \\ 1 - 1/\alpha^2, & \alpha > 1. \end{cases} \quad (32)$$

The first $1/D$-correction to this result can be obtained, for large $\alpha$ by using the expansion given in Eq. (22) together with the definition of the tangle, Eq. (24). These results are shown in Fig. (7), where the solid line is a plot of the tangle in the asymptotic regime $(D \to \infty)$.

The same procedure used above can be carried out in the asymmetric case provided a value of $q$ is found, such that

$$q = \frac{\lambda}{m \omega^2} \frac{\epsilon + \lambda q}{\sqrt{\Delta^2 + (\epsilon + \lambda q)^2}}. \quad (33)$$

This equation has three nontrivial solutions. Within our saddle-point scheme, in the limit $\beta \to \infty$, we must retain only the solution $q_m$ that corresponds to the lowest minimum of the potential. Therefore, we can write

$$b_x = -\frac{\Delta}{\Delta(q_m)}, \quad b_z = -\frac{\epsilon + \lambda q_m}{\Delta(q_m)}, \quad (34)$$

and, thus, one gets $\tau = 0$ for any finite $\epsilon \neq 0$.

Indeed, it can be seen from Fig. (6) that the tangle (for any value of $\alpha$) decreases progressively with the increase of the asymmetry parameter $W$. Furthermore, as exemplified in Fig. (5), for any fixed non-zero value of $W$, the tangle approaches zero as $D$ increases; so that, asymptotically, the result implied by Eq. (34) is obtained.

**IV. CONCLUDING REMARKS**

In conclusion, we have discussed the adiabatic approximation for a qubit coupled to a single oscillator mode and we have derived the resulting entanglement in the

**FIG. 5:** The tangle $\tau$ as a function of $\alpha$ for $D = 10$. Different curves, corresponding to different values of $W$, indicate the entanglement progressively decreases with increasing the asymmetry.

**FIG. 6:** The tangle $\tau$ as a function of $\alpha$ for $W = 0.1$ and different values of $D$.

**FIG. 7:** The tangle as a function of $\alpha$ in the symmetric case $W = 0$ for different values of the qubit tunnelling amplitude $D$. One can appreciate that the result of Eq. (32) is indeed reached asymptotically.
ground state, by giving simple analytical results in the strict adiabatic limit. The advantage of our approach, that requires a very small computational effort and correctly describes the model system when $\Delta/\omega \gg 1$, is to give a physically more transparent description of the ground state.

As we have shown, the procedure is easily extended to the asymmetric case and this is important since the entanglement changes dramatically for any finite (however small) value of the asymmetry in the qubit Hamiltonian. As mentioned in section 11 above, this is due to the fact the this term modifies the symmetry properties of the Hamiltonian, so that the form of the ground state changes radically and the same occurs to the reduced qubit state. For example, for a large enough interaction strength, the qubit state is a complete mixture if $W = 0$, while it becomes the lower eigenstate of $\sigma_z$ if $W \neq 0$. As a result, for large $\alpha$, there is much entanglement if $W = 0$, while the state of the system is factorized and thus $\tau = 0$ if $W \neq 0$. This is seen explicitly in Fig. 5. Furthermore, from the comparison of Figs. 3, 4, and 6, one can see that, with increasing $\alpha$, the tangle increases monotonically in the symmetric case, while it reaches a maximum before going down to zero if $W \neq 0$.

This is due to the fact that, in the first case, the ground state of the system becomes a Schrödinger cat-like entangled superposition, approximately given by

$$|\psi\rangle \approx \frac{1}{\sqrt{2}} \left( |\phi_+\rangle |- |\phi_-\rangle + \right), \quad \text{for } \alpha \gg 1, \quad (35)$$

where $|\phi_{\pm}\rangle$ are the two coherent states for the oscillator defined in Eq. 21, centered in $Q = \pm Q_0$, respectively, and almost orthogonal if $\alpha \gg 1$.

In the presence of asymmetry, on the other hand, the oscillator localizes in one of the wells of its effective potential and this implies that, for large $\alpha$, the ground state is given by just one of the two components superposed in Eq. 35. This is, clearly, a factorized state and therefore one gets $\tau = 0$.

Since $\tau$ is zero for uncoupled sub-systems (i.e., for very small values of $\alpha$), weather $W = 0$ or not, and since, for $W \neq 0$, it has to decay to zero for large $\alpha$, it follows that a maximum is present in between.

In fact, for intermediate values of the coupling, there is a competition between the $\alpha$-dependences of the two non-zero components of the Bloch vector. In particular, the length $|\vec{b}|$ is approximately equal to one for both small and large $\alpha$’s, see Figs. 4, 5, but the vector points in the $x$ direction for $\alpha \ll 1$ and in the $z$ direction for $\alpha \gg 1$. The maximum of the tangle in the asymmetric case occurs near the point in which $b_x \approx b_z$.

For the symmetric case, we were also able to derive analytically the sharp increase of the entanglement at $\alpha = 1$. This behavior appears to be reminiscent of the super-radiant transition in the many qubit Dicke model, which, in the adiabatic limit, shows exactly the same features described here, and which can be described along similar lines.

Finally, we would like to comment on the relationship of this work with those of Refs. 3 and 4. The approach proposed by Levine and Muthukumar, Ref. 3, employs an instanton description for the effective action. This has been applied to obtain the entropy of entanglement in the symmetric case, in the same critical limit described above. It turns out that this description is equivalent to a fourth order expansion of the lower adiabatic potential $U_i$. This approximation, although retaining all the distinctive qualitative features discussed above, gives slight quantitative changes in the results.

Concerning the asymmetric case, our results for the ground state entanglement appear similar to those found by Costi and McKenzie in Ref. 4, where the interaction of a qubit with an ohmic environment was numerically analyzed. It turns out that, for a bath with finite bandwidth, the entanglement displays a behavior analogous to that reported in Figs. 5-8, when plotted with respect to the value of the impedance of the bath. Here, instead, we concentrated on the dependence of the tangle on the coupling strength between the qubit and the environmental oscillator. Unfortunately, the coupling strength is not easily related to the coefficient of the spectral density used in Ref. 3, and therefore one cannot make a precise comparison between the two results. At least qualitatively, however, we can say that the ground state quantum correlations induced by the coupling with an ohmic environment are already present when the qubit is coupled to a single oscillator mode.

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