Abstract. Almost hypercomplex pseudo-Hermitian manifolds are considered. Isotropic hyper-Kähler manifolds are introduced. A 4-parametric family of 4-dimensional manifolds of this type is constructed on a Lie group. This family is characterized geometrically. The condition a 4-manifold to be isotropic hyper-Kähler is given.

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Introduction

The general setting of this paper is inspired by the work of D. V. Alekseevsky and S. Marchiafava [1]. Our purpose is to develop a parallel direction including indefinite metrics. More precisely we combine the ordinary Hermitian metrics with the so-called by us skew-Hermitian metrics with respect to the almost hypercomplex structure.

In the first section we consider an appropriate decomposition of the space of all bilinear forms on a vector space equipped with a hypercomplex structure.

*Corresponding author
Here we emphasize on a notion of the skew-Hermitian metric. In fact, we construct three skew-Hermitian metrics and one Hermitian, i.e. a pseudo-Hermitian structure.

In the second we develop the notion of an almost hypercomplex manifold with a pseudo-Hermitian structure and particularly the so-called pseudo-hyper-Kählerian and isotropic Kähler structures.

Finally, in the third section we equip a 4-dimensional Lie group with an almost hypercomplex pseudo-Hermitian structure and we characterize it geometrically.

1 Hypercomplex pseudo-Hermitian structures on a vector space

Let \( V \) be a real 4\( n \)-dimensional vector space. A (local) basis on \( V \) is denoted by \( \{ \partial/\partial x^i, \partial/\partial y^i, \partial/\partial u^i, \partial/\partial v^i \}, i = 1, 2, \ldots, n \). Each vector \( \mathbf{x} \) of \( V \) is represented in the mentioned basis as follows

\[
\mathbf{x} = x^i \frac{\partial}{\partial x^i} + y^i \frac{\partial}{\partial y^i} + u^i \frac{\partial}{\partial u^i} + v^i \frac{\partial}{\partial v^i},
\]

(1)

A standard hypercomplex structure on \( V \) is defined as in [8]:

\[
\begin{align*}
J_1 \frac{\partial}{\partial x^i} &= \frac{\partial}{\partial y^i}, & J_1 \frac{\partial}{\partial y^i} &= -\frac{\partial}{\partial x^i}, & J_1 \frac{\partial}{\partial u^i} &= -\frac{\partial}{\partial v^i}, & J_1 \frac{\partial}{\partial v^i} &= \frac{\partial}{\partial u^i}; \\
J_2 \frac{\partial}{\partial x^i} &= \frac{\partial}{\partial u^i}, & J_2 \frac{\partial}{\partial u^i} &= -\frac{\partial}{\partial x^i}, & J_2 \frac{\partial}{\partial v^i} &= -\frac{\partial}{\partial y^i}, & J_2 \frac{\partial}{\partial y^i} &= \frac{\partial}{\partial v^i}; \\
J_3 \frac{\partial}{\partial x^i} &= -\frac{\partial}{\partial v^i}, & J_3 \frac{\partial}{\partial v^i} &= -\frac{\partial}{\partial x^i}, & J_3 \frac{\partial}{\partial u^i} &= \frac{\partial}{\partial y^i}, & J_3 \frac{\partial}{\partial y^i} &= -\frac{\partial}{\partial u^i}.
\end{align*}
\]

(2)

The following properties about \( J_i \) are direct consequences of (2)

\[
\begin{align*}
J_1^2 &= J_2^2 = J_3^2 = -\text{Id}, & J_1 J_2 &= -J_2 J_1 = J_3, & J_2 J_3 &= -J_3 J_2 = J_1, & J_3 J_1 &= -J_1 J_3 = J_2.
\end{align*}
\]

(3)

If \( x \in V \), i.e. \( x(x^1, \ldots, x^n, y^1, \ldots, y^n; u^1, \ldots, u^n; v^1, \ldots, v^n) \) then according to (2) and (3) we have

\[
\begin{align*}
J_1 x(-y^1, \ldots, -y^n; x^1, \ldots, x^n; v^1, \ldots, v^n; -u^1, \ldots, -u^n), \\
J_2 x(-u^1, \ldots, -u^n; -v^1, \ldots, -v^n; x^1, \ldots, x^n; y^1, \ldots, y^n), \\
J_3 x(v^1, \ldots, v^n; -u^1, \ldots, -u^n; y^1, \ldots, y^n; -x^1, \ldots, -x^n).
\end{align*}
\]

Definition 1.1 ([1])

1) A triple \( H = (J_1, J_2, J_3) \) of anticommuting complex structures on \( V \) with \( J_3 = J_1 J_2 \) is called a hypercomplex structure on \( V \);

2) The 3-dimensional subspace \( Q = \langle H \rangle = \mathbb{R} J_1 + \mathbb{R} J_2 + \mathbb{R} J_3 \) of the space of endomorphisms \( \text{End} V \) is called a quaternionic structure on \( V \). It is said that \( H = (J_3) \) is an admissible basis of \( Q \).
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Note that two admissible bases $H$ and $H'$ of $Q = \langle H \rangle = \langle H' \rangle$ are related by an orthogonal matrix in $\text{SO}(3)$.

The matrices of $J_1$ and $J_2$ are given in [8] by $(n \times n)$-sets of $(4 \times 4)$-matrices $J_\alpha = \text{diag}(I_\alpha, I_\alpha, \ldots, I_\alpha)$, where $I_\alpha$ ($\alpha = 1, 2, 3$) are respectively

\[
I_1 = \begin{pmatrix}
0 & 1 & 0 & 0 \\
-1 & 0 & 0 & 0 \\
0 & 0 & 0 & -1 \\
0 & 0 & 1 & 0
\end{pmatrix},
I_2 = \begin{pmatrix}
0 & 0 & 1 & 0 \\
0 & 0 & 0 & 1 \\
-1 & 0 & 0 & 0 \\
0 & -1 & 0 & 0
\end{pmatrix},
\]

and consequently

\[
I_3 = \begin{pmatrix}
0 & 0 & 0 & 1 \\
0 & 0 & -1 & 0 \\
0 & 1 & 0 & 0 \\
-1 & 0 & 0 & 0
\end{pmatrix}.
\]

The matrices $J_\alpha$ of the complex structures $J_\alpha$ ($\alpha = 1, 2, 3$) with respect to an admissible frame for $H = (J_\alpha)$ are called standard matrices.

A bilinear form $f$ on $V$ is defined as ordinary, $f : V \times V \to \mathbb{R}$. We denote by $\mathcal{B}(V)$ the set of all bilinear forms on $V$. Each $f$ is a tensor of type $(0, 2)$, and $\mathcal{B}(V)$ is a vector space of dimension $16n^2$.

Let $J$ be a given complex structure on $V$. A bilinear form $f$ on $V$ is called Hermitian (respectively, skew-Hermitian) with respect to $J$ if the identity $f(Jx, Jy) = f(x, y)$ (respectively, $f(Jx, Jy) = -f(x, y)$) holds true.

**Definition 1.2 ([1])** A bilinear form $f$ on $V$ is called an Hermitian bilinear form with respect to $H = (J_\alpha)$ if it is Hermitian with respect to any complex structure $J_\alpha$, $\alpha = 1, 2, 3$, i.e.

\[
f(J_\alpha x, J_\alpha y) = f(x, y), \quad \forall x, y \in V.
\]

We will denote by $\mathcal{B}_H(V)$ the set of all Hermitian bilinear forms on $V$.

In [6] is introduced the notion of pseudo-Hermitian bilinear forms, namely:

**Definition 1.3 ([6])** A bilinear form $f$ on $V$ is called a pseudo-Hermitian bilinear form with respect to $H = (J_1, J_2, J_3)$, if it is Hermitian with respect to $J_\alpha$ and skew-Hermitian with respect to $J_\beta$ and $J_\gamma$, i.e.

\[
f(J_\alpha x, J_\alpha y) = -f(J_\beta x, J_\beta y) = -f(J_\gamma x, J_\gamma y) = f(x, y), \quad \forall x, y \in V,
\]

where $(\alpha, \beta, \gamma)$ is a circular permutation of $(1, 2, 3)$.
Now, let us show the existence of the introduced bilinear forms on $V$.

We denote $f \in B_\alpha \subset B(V)$ ($\alpha = 1, 2, 3$) when $f$ satisfies the conditions (4). Let us remark that $B_H(V)$ is a subspace of the vector space $B(V)$. The projector $\Pi_H : B(V) \rightarrow B_H(V)$ is defined in [1] as follows

$$f \rightarrow (\Pi_H f)(x, y) := \frac{1}{4} \{ f(x, y) + f(J_1 x, J_1 y) + f(J_2 x, J_2 y) + f(J_3 x, J_3 y) \}. \quad (5)$$

For convenience we set $\Pi_0 := \Pi_H$ and $B_0 := B_H(V)$. Clearly, $\Pi_0$ is a projector, i.e. $\Pi_0^2 = \Pi_0$.

Analogously we define the operators: $\Pi_\alpha : B(V) \rightarrow B_\alpha$, $\alpha = 1, 2, 3$ as follows

$$f \rightarrow (\Pi_\alpha f)(x, y) := \frac{1}{4} \{ f(x, y) + f(J_\alpha x, J_\alpha y) - f(J_\beta x, J_\beta y) - f(J_\gamma x, J_\gamma y) \}, \quad (6)$$

where $(\alpha, \beta, \gamma)$ is a circular permutation of $(1, 2, 3)$. It is not difficult to see that $\Pi_\alpha f \in B_\alpha$, $\alpha = 1, 2, 3$.

In view of (5)–(6) the following proposition holds:

**Proposition 1.1** The vector space $B(V)$ admits the following decomposition

$$B(V) = B_0 \oplus B_1 \oplus B_2 \oplus B_3, \quad B_\alpha = \text{Im} \Pi_\alpha, \quad \alpha = 0, 1, 2, 3,$$

where the operators $\Pi_0, \Pi_1, \Pi_2$ and $\Pi_3$ are projectors with values in $B(V)$ such that

$$\Pi_2^2 = \Pi_0, \quad \Pi_0 + \Pi_1 + \Pi_2 + \Pi_3 = \text{Id},$$

$$\Pi_\alpha \circ \Pi_\beta = \Pi_\beta \circ \Pi_\alpha = 0, \quad \alpha \neq \beta; \quad \alpha, \beta \in \{0, 1, 2, 3\}.$$

So, pseudo-Hermitian bilinear forms exist and moreover they are three types in any vector space $V$ equipped with a hypercomplex structure $H$, denoted by $(V, H)$.

Let $x$ determined by (1) and

$$y = a^i \frac{\partial}{\partial x^i} + b^i \frac{\partial}{\partial y^i} + c^i \frac{\partial}{\partial u^i} + d^i \frac{\partial}{\partial v^i}, \quad i = 1, 2, \ldots, n$$

be arbitrary vectors on $V$. Following [9], we define as in [6] a pseudo-Euclidean metric of signature $(2n, 2n)$ on $V$ by a symmetric bilinear form $g$ as follows

$$g(x, y) := \sum_{i=1}^{n} (-x^i a^i - y^i b^i + u^i c^i + v^i d^i).$$
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Hence for the local basis \( \{ \partial/\partial x^i, \partial/\partial y^i, \partial/\partial u^i, \partial/\partial v^i \} \), \( i = 1, 2, \ldots, n \) on \( V \) we have for \( i, j \in \{ 1, 2, \ldots, n \} \)

\[
-g \left( \frac{\partial}{\partial x^i}, \frac{\partial}{\partial x^j} \right) = -g \left( \frac{\partial}{\partial y^i}, \frac{\partial}{\partial y^j} \right) = g \left( \frac{\partial}{\partial u^i}, \frac{\partial}{\partial u^j} \right) = g \left( \frac{\partial}{\partial v^i}, \frac{\partial}{\partial v^j} \right) = \delta_{ij},
\]

\[
g \left( \frac{\partial}{\partial x^i}, \frac{\partial}{\partial y^j} \right) = g \left( \frac{\partial}{\partial x^j}, \frac{\partial}{\partial y^i} \right) = g \left( \frac{\partial}{\partial u^i}, \frac{\partial}{\partial v^j} \right) = g \left( \frac{\partial}{\partial v^i}, \frac{\partial}{\partial u^j} \right) = 0.
\]

Let us remark that if we denote \( e_i = \partial/\partial x^i \) \( (i = 1, 2, \ldots, n) \) then according to (2) the basis

\[
(e_1, e_2, \ldots, e_n; J_1 e_1, J_1 e_2, \ldots, J_1 e_n; \ldots; J_3 e_1, J_3 e_2, \ldots, J_3 e_n)
\]

is an an admissible basis of \( H \) and it is orthonormal with respect to \( g \).

Because of the properties

\[
g(J_1 x, J_1 y) = -g(J_2 x, J_2 y) = -g(J_3 x, J_3 y) = g(x, y), \tag{8}
\]

the pseudo-Euclidean metric \( g \) is a symmetric pseudo-Hermitian bilinear form and \( g \in \mathcal{B}_1 \). Moreover, \( g_1(x, y) := g(J_1 x, y) = -g(J_1 y, x) \) coincides with the known Kähler form with respect to \( J_1 \), i.e. \( \Phi(x, y) := g_1(x, y) \) [6].

The associated bilinear forms \( g_2(x, y) := g(J_2 x, y) \) and \( g_3(x, y) := g(J_3 x, y) \) of \( g \) are symmetric and \( \Phi \in \mathcal{B}_0, g \in \mathcal{B}_1, g_2 \in \mathcal{B}_3, g_3 \in \mathcal{B}_2 \), i.e. the Kähler form \( \Phi \) is Hermitian and \( g, g_2, g_3 \) are pseudo-Hermitian of different types, but they have the same signature \((2n, 2n)\). Then the structure \((H, G) := (H, g, \Phi, g_2, g_3)\) is called a hypercomplex pseudo-Hermitian structure on \( V \) [6].

According to [8], the matrices that commute with \( J_\alpha \) \( (\alpha = 1, 2, 3) \) are \( A = (A_{ij}) \), \( i, j \in \{ 1, 2, \ldots, n \} \), where every \( (A_{ij}) \) is a \((4 \times 4)\)-matrix of the form

\[
A_{ij} = \begin{pmatrix} P & Q \\ -Q & P \end{pmatrix}, \quad P = \begin{pmatrix} a & b \\ -b & a \end{pmatrix}, \quad Q = \begin{pmatrix} c & d \\ d & -c \end{pmatrix}, \quad a, b, c, d \in \mathbb{R}.
\]

The set of the \( J_\alpha \)-commuting matrices, that are invertible, is a group which is isomorphic to \( \text{GL}(n, \mathbb{H}) \).

The pseudo-Euclidean metric \( g \) has a matrix with respect to the basis (7) of the form \( \mathbf{g} = \text{diag}(g, g, \ldots, g) \), where

\[
g = \begin{pmatrix} -I_2 & O_2 \\ O_2 & I_2 \end{pmatrix}.
\]

The group preserving \( \mathbf{g} \) is defined by the condition \( A^T \mathbf{g} A = \mathbf{g} \) for arbitrary \( A \in \text{GL}(4n, \mathbb{R}) \). It is clear that the group which preserves \( \mathbf{g} \) is \( \text{O}(2n, 2n) \).
The structural group of \((V, H, G)\) has the property to preserve the structures \(J_\alpha\) and the metric \(g\) (consequently \(\Phi, g_2, g_3\), too). Then this structural group is the intersection of \(\text{GL}(n, \mathbb{H})\) and \(\text{O}(2n, 2n)\). We get immediately that

\[
A \in \text{GL}(n, \mathbb{H}) \cap \text{O}(2n, 2n) \iff a^2 + b^2 = 1, \ c = d = 0.
\]

Therefore \(\text{GL}(n, \mathbb{H}) \cap \text{O}(2n, 2n)\) is an 1-parametrical group, i.e. the elements \(A_{ij}\) of \(A\) depend on 1 real parameter.

## 2 Almost \((H, G)\)-structures on a manifold

Let \((M, H)\) be an almost hypercomplex manifold \([1]\). We suppose that \(g\) is a symmetric tensor field of type \((0, 2)\). If it induces a pseudo-Hermitian inner product in \(T_p M, p \in M\), then \(g\) is called a pseudo-Hermitian metric on \(M\). The structure \((H, G) := (J_1, J_2, J_3, g, \Phi, g_2, g_3)\) is called an almost hypercomplex pseudo-Hermitian structure on \(M\) or in short an almost \((H, G)\)-structure on \(M\). The manifold \(M\) equipped with \(H\) and \(G\), i.e. \((M, H, G)\), is called an almost hypercomplex pseudo-Hermitian manifold, or in short an almost \((H, G)\)-manifold.

The structural tensors of the almost \((H, G)\)-manifold are the three tensors of type \((0, 3)\) determined by

\[
F_\alpha(x, y, z) = g((\nabla_x J_\alpha) y, z) = (\nabla_x g_\alpha)(y, z), \quad \alpha = 1, 2, 3, \tag{9}
\]

where \(\nabla\) is the Levi-Civita connection generated by \(g\) \([6]\).

The properties of \(H\) and \(g\) imply the following properties of \(F_\alpha\):

\[
\begin{align*}
F_1(x, y, z) &= F_2(x, J_3 y, z) + F_3(x, y, J_2 z), \\
F_2(x, y, z) &= F_3(x, J_1 y, z) + F_1(x, y, J_3 z), \\
F_3(x, y, z) &= F_1(x, J_2 y, z) - F_2(x, y, J_1 z);
\end{align*} \tag{10}
\]

\[
\begin{align*}
F_1(x, y, z) &= -F_1(x, z, y) = -F_1(x, J_1 y, J_2 z), \\
F_2(x, y, z) &= F_2(x, z, y) = F_2(x, J_2 y, J_3 z), \\
F_3(x, y, z) &= F_3(x, z, y) = F_3(x, J_3 y, J_1 z).
\end{align*} \tag{11}
\]

Let us consider the Nijenhuis tensors \(N_\alpha\) for \(J_\alpha\) and \(X, Y \in \mathfrak{X}(M)\) given by

\[
N_\alpha(X, Y) = [J_\alpha X, J_\alpha Y] - J_\alpha [J_\alpha X, Y] - J_\alpha [X, J_\alpha Y] - [X, Y].
\]

It is well known that the almost hypercomplex structure \(H = (J_\alpha)\) is a hypercomplex structure if \(N_\alpha\) vanishes for each \(\alpha = 1, 2, 3\). Moreover, it is known that one almost hypercomplex structure \(H\) is hypercomplex if and only if two of the structures \(J_\alpha\) \((\alpha = 1, 2, 3)\) are integrable. This means that two of the tensors \(N_\alpha\) vanish \([1]\).

Let us note that according to \((8)\) the manifold \((M, J_1, g)\) is almost Hermitian and the manifolds \((M, J_\alpha, g), \alpha = 2, 3\), are almost complex manifolds with Norden
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metric (or B-metric) [2, 3]. The basic classes of the mentioned two types of manifolds for dimension $4n$ are:

1. $W_1(J_1) : F_1(x, y, z) = -F_1(y, x, z)$,
2. $W_2(J_1) : \mathcal{S}_{x,y,z}\{F_1(x,y,z)\} = 0$,
3. $W_3(J_1) : F_1(x, y, z) = F_1(J_1 x, J_1 y, z), \quad \theta_1 = 0$,
4. $W_4(J_1) : F_1(x, y, z) = \frac{1}{2^{n-1}} \left\{ g(x, y) \theta_1(z) - g(x, z) \theta_1(y) - g(x, J_1 y) \theta_1(J_1 z) + g(x, J_1 z) \theta_1(J_1 y) \right\}$,

where $\theta_1(\cdot) = g^{ij} F_1(e_i, e_j, \cdot)$ for an arbitrary basis $\{e_i\}_{i=1}^{4n}$ [5].

1. $W_1(J_\alpha) : F_\alpha(x, y, z) = \frac{1}{4n} \left\{ g(x, y) \theta_\alpha(z) + g(x, z) \theta_\alpha(y) + g(x, J_\alpha y) \theta_\alpha(J_\alpha z) + g(x, J_\alpha z) \theta_\alpha(J_\alpha y) \right\}$,
2. $W_2(J_\alpha) : \mathcal{S}_{x,y,z}\{F_\alpha(x,y,z)\} = 0, \quad \theta_\alpha = 0$,
3. $W_3(J_\alpha) : \mathcal{S}_{x,y,z}\{F_\alpha(x,y,z)\} = 0$,

where $\theta_\alpha(z) = g^{ij} F_\alpha(e_i, e_j, z)$, $\alpha = 2, 3$, for an arbitrary basis $\{e_i\}_{i=1}^{4n}$ and $\mathcal{S}$ is the cyclic sum by three arguments [2].

The special class $W_0(J_\alpha) : F_\alpha = 0 \ (\alpha = 1, 2, 3)$ of the Kähler-type manifolds belongs to any other class within the corresponding classification.

We say that an almost hypercomplex pseudo-Hermitian manifold is a pseudo-hyper-Kähler manifold if $\nabla J_\alpha = 0 \ (\alpha = 1, 2, 3)$ with respect to the Levi-Civita connection generated by $g$ [6].

Clearly, in this case we have $F_\alpha = 0 \ (\alpha = 1, 2, 3)$ or the manifold is Kählerian with respect to $J_\alpha$, i.e. $(M, H, G) \in W_0(J_\alpha)$.

Immediately we obtain

**Proposition 2.1** If $(M, H, G) \in W_0(J_\alpha) \cap W_0(J_\beta)$ then $(M, H, G) \in W_0(J_\gamma)$ for all cyclic permutations $(\alpha, \beta, \gamma)$ of $(1, 2, 3)$ and $(M, H, G)$ is pseudo-hyper-Kählerian.

A basic property of the pseudo-hyper-Kähler manifolds is given in [6] by the following

**Theorem 2.2** ([6]) Each pseudo-hyper-Kähler manifold is a flat pseudo-Riemannian manifold of signature $(2n, 2n)$.

As $g$ is an indefinite metric, there exist isotropic vector fields $X$ on $M$, i.e. $g(X, X) = 0$, $X \neq 0$, $X \in \mathfrak{X}(M)$. Following [4] we define the invariants

$$\|\nabla J_\alpha\|^2 = g^{ij} \ g^{kl} g((\nabla_i J_\alpha) e_k, (\nabla_j J_\alpha) e_l), \quad \alpha = 1, 2, 3, \quad (14)$$

where $\{e_i\}_{i=1}^{4n}$ is an arbitrary basis of $T_p M$, $p \in M$. Let us remark that the invariant $\|\nabla J_\alpha\|^2$ is the scalar square of the $(1, 2)$-tensor $\nabla J_\alpha$.  


Definition 2.1 We say that an \((H, G)\)-manifold is:

(i) isotropic Kählerian with respect to \(J_\alpha\) if \(\|\nabla J_\alpha\|^2 = 0\) for some \(\alpha \in \{1, 2, 3\}\);

(ii) isotropic hyper-Kählerian if it is isotropic Kählerian with respect to every \(J_\alpha\) of \(H\).

Clearly, if \((M, H, G)\) is pseudo-hyper-Kählerian, then it is an isotropic hyper-Kähler manifold. The inverse statement does not hold.

3 A Lie group as a 4-dimensional \((H, G)\)-manifold

In [7] is constructed an example of a 4-dimensional Lie group equipped with a quasi-Kähler structure and Norden metric \(g\), i.e. it is a \(\mathcal{W}_3\)-manifold according to (13). There it is characterized with respect to \(\nabla\) of \(g\).

Theorem 3.1 ([7]) Let \((L, J, g)\) be a 4-dimensional almost complex manifold with Norden metric, where \(L\) is a connected Lie group with a corresponding Lie algebra determined by the global basis of left invariant vector fields \(\{X_1, X_2, X_3, X_4\}\); \(J\) is an almost complex structure defined by

\[
JX_1 = X_3, \quad JX_2 = X_4, \quad JX_3 = -X_1, \quad JX_4 = -X_2; \quad (15)
\]

\(g\) is an invariant Norden metric determined by

\[
g(X_1, X_1) = g(X_2, X_2) = -g(X_3, X_3) = -g(X_4, X_4) = 1, \quad g(X_i, X_j) = 0, \quad i \neq j; \quad g([X_i, X_j], X_k) + g([X_i, X_k], X_j) = 0. \quad (16)
\]

Then \((L, J, g)\) is a quasi-Kähler manifold with Norden metric if and only if \(L\) belongs to the 4-parametric family of Lie groups determined by the conditions

\[
[X_1, X_3] = \lambda_2 X_2 + \lambda_4 X_4, \quad [X_2, X_4] = \lambda_1 X_1 + \lambda_3 X_3, \\
[X_2, X_3] = -\lambda_2 X_1 - \lambda_3 X_4, \quad [X_3, X_4] = -\lambda_4 X_1 + \lambda_3 X_2, \\
[X_4, X_1] = \lambda_1 X_2 + \lambda_4 X_3, \quad [X_2, X_1] = -\lambda_2 X_3 + \lambda_1 X_4, \quad (17)
\]

where \(\lambda_i \in \mathbb{R}\) \((i = 1, 2, 3, 4)\) and \((\lambda_1, \lambda_2, \lambda_3, \lambda_4) \neq (0, 0, 0, 0)\). □

The components of \(\nabla\) are determined ([7]) by (17) and

\[
\nabla_{X_i} X_j = \frac{1}{2}[X_i, X_j] \quad (i, j = 1, 2, 3, 4). \quad (18)
\]
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Hence the components $R_{i,j,k,s} = R(X_i, X_j, X_k, X_s)$ $(i,j,k,s = 1,2,3,4)$ of the curvature tensor $R$ on $(L,g)$ are: [7]

$$
\begin{align*}
R_{1221} &= -\frac{3}{4} (\lambda_1^2 + \lambda_2^2), \\
R_{1441} &= -\frac{3}{4} (\lambda_1^2 - \lambda_2^2), \\
R_{2442} &= \frac{1}{4} (\lambda_1^2 - \lambda_2^2), \\
R_{1341} &= R_{2342} = -\frac{3}{4} \lambda_1 \lambda_2, \\
R_{1231} &= -R_{1234} = \frac{3}{4} \lambda_1 \lambda_4, \\
R_{1241} &= -R_{3243} = \frac{3}{4} \lambda_2 \lambda_4, \\
R_{1331} &= \frac{1}{4} (\lambda_1^2 - \lambda_2^2), \\
R_{2332} &= \frac{1}{4} (\lambda_1^2 - \lambda_2^2), \\
R_{3443} &= \frac{1}{4} (\lambda_1^2 + \lambda_2^2), \\
R_{2132} &= -R_{4134} = \frac{3}{4} \lambda_1 \lambda_3, \\
R_{2142} &= -R_{3143} = \frac{3}{4} \lambda_2 \lambda_3, \\
R_{3123} &= R_{4124} = \frac{1}{4} \lambda_3 \lambda_4,
\end{align*}
$$

(19)

and the scalar curvature $\tau$ on $(L,g)$ is [7]

$$
\tau = -\frac{3}{2} (\lambda_1^2 + \lambda_2^2 - \lambda_3^2 - \lambda_4^2). 
$$

(20)

Now we introduce a hypercomplex structure $H = (J_1, J_2, J_3)$ by the following way. At first, let $J_2$ be the given almost complex structure $J$ by (15). Secondly, we define an almost complex structure $J_1$ as follows

$$
J_1 : \quad J_1 X_1 = X_2, \quad J_1 X_2 = -X_1, \quad J_1 X_3 = -X_4, \quad J_1 X_4 = X_3. 
$$

(21)

Finally, let the almost complex structure $J_3$ be the composition of $J_1$ after $J_2$, i.e. $J_3 = J_1 J_2$.

Then the introduced structure $(H,G)$ on L has the properties (3) and (8). Hence we have the following

**Theorem 3.2** The manifold $(L,H,G)$ is an almost hypercomplex pseudo-Hermitian manifold of dimension 4. \[\square\]

We continue by a characterization of the constructed manifold $(L,H,G)$.

Let $(F_\alpha)_{i,j,k} = F_\alpha(X_i, X_j, X_k)$ and $(\theta_\alpha)_i = \theta_\alpha(X_i)$ be the components of the structural tensor $F_\alpha$ and its Lee form $\theta_\alpha (\alpha = 1,2,3)$, respectively. The nonzero components of $F_2$ are: [7]

$$
\begin{align*}
-(F_2)_{122} &= -(F_2)_{144} = 2(F_2)_{212} = 2(F_2)_{221} = 2(F_2)_{234} \\
2(F_2)_{112} &= 2(F_2)_{121} = 2(F_2)_{134} = 2(F_2)_{143} = -(F_2)_{211} \\
-(F_2)_{233} &= -2(F_2)_{311} = 2(F_2)_{323} = 2(F_2)_{332} = -2(F_2)_{341} = \lambda_2, \\
2(F_2)_{214} &= -2(F_2)_{223} = -2(F_2)_{232} = 2(F_2)_{241} = (F_2)_{322} \\
(F_2)_{444} &= -2(F_2)_{412} = -2(F_2)_{421} = -2(F_2)_{434} = -2(F_2)_{443} = \lambda_3, \\
-2(F_2)_{114} &= 2(F_2)_{123} = 2(F_2)_{132} = -2(F_2)_{141} = -2(F_2)_{312} \\
-2(F_2)_{321} &= -2(F_2)_{334} = -2(F_2)_{343} = (F_2)_{411} = (F_2)_{433} = \lambda_4.
\end{align*}
$$

(22)
Then we have $\theta_2 = 0$. By this way we confirm the statement in Theorem (3.1) that the introduced manifold in [7] is of the basic class $\mathcal{W}_3$ with respect to $J_2$ within the classification (13), i.e.

$$(L, J_2, g) \in \mathcal{W}_3(J_2).$$

(23)

Having in mind (16)–(18), (21) and (9), we obtain the nonzero components of $F_1$ as follows

$$(F_1)_{114} = -(F_1)_{123} = (F_1)_{132} = -(F_1)_{141} = (F_1)_{213} = (F_1)_{224} = -(F_1)_{231} = -(F_1)_{242} = \frac{1}{2}\lambda_1;$$

$$-(F_1)_{113} = -(F_1)_{124} = (F_1)_{131} = (F_1)_{142} = (F_1)_{214} = -(F_1)_{223} = (F_1)_{232} = -(F_1)_{241} = \frac{1}{2}\lambda_2;$$

$$-(F_1)_{314} = (F_1)_{323} = -(F_1)_{332} = (F_1)_{341} = (F_1)_{413} = (F_1)_{424} = -(F_1)_{431} = -(F_1)_{442} = \frac{1}{2}\lambda_3;$$

$$-(F_1)_{313} = -(F_1)_{324} = (F_1)_{331} = (F_1)_{342} = -(F_1)_{414} = (F_1)_{423} = -(F_1)_{432} = (F_1)_{441} = \frac{1}{2}\lambda_4.$$  

(24)

Then we have

$$(\theta_1)_1 = -\lambda_4, \quad (\theta_1)_2 = \lambda_3, \quad (\theta_1)_3 = -\lambda_2, \quad (\theta_1)_4 = \lambda_1.$$  

Since $(\lambda_1, \lambda_2, \lambda_3, \lambda_4) \neq (0, 0, 0, 0)$ then the 4-dimensional almost Hermitian manifold $(L, J_1, g)$ is not Kählerian and $\theta_1 \neq 0$.

The validity of the property $F_1(X, Y, Z) = F_1(J_1X, J_1Y, Z)$ is verified by us in virtue of (21) and (24). It is equivalent to the vanishing of the Nijenhuis tensor of $J_1$, i.e. $N_1 = 0$. According to [5] for dimension 4 we get that the considered manifold belongs to the basic class $\mathcal{W}_4(J_1)$ within (12), i.e.

$$(L, J_1, g) \in \mathcal{W}_4(J_1).$$

(25)

As it is known [5], this class contains the conformally Kähler manifolds of Hermitian type. The necessary and sufficient condition a $\mathcal{W}_4(J_1)$-manifold to be locally or globally conformally Kählerian one is the Lee form $\theta_1$ to be closed or exact. The basic components of $d\theta_1$ are:

$$d\theta_1(X_1, X_2) = \lambda_1^2 + \lambda_2^2, \quad d\theta_1(X_2, X_4) = d\theta_1(X_3, X_1) = \lambda_1\lambda_4 + \lambda_2\lambda_3,$$

$$d\theta_1(X_3, X_4) = -\lambda_3^2 - \lambda_4^2, \quad d\theta_1(X_1, X_4) = d\theta_1(X_2, X_3) = \lambda_3\lambda_4 - \lambda_2\lambda_3.$$  

Hence, $\theta_1$ is not closed and therefore the constructed $\mathcal{W}_4(J_1)$-manifold is not conformally Kählerian.
Having in mind (10), (22), (24), we compute the following nonzero components of $F_3$:

$$(F_3)_{121} = (F_3)_{121} = -(F_3)_{134} = -(F_3)_{143} = -2(F_3)_{211} = -2(F_3)_{244}$$
$$= (F_3)_{413} = (F_3)_{431} = (F_3)_{424} = (F_3)_{442} = \frac{1}{2} \lambda_1,$$
$$2(F_3)_{122} = 2(F_3)_{313} = -(F_3)_{212} = -(F_3)_{221} = (F_3)_{234} = (F_3)_{243}$$
$$= -(F_3)_{313} = -(F_3)_{331} = -(F_3)_{324} = -(F_3)_{342} = \frac{1}{2} \lambda_2,$$
$$2(F_3)_{213} = (F_3)_{231} = (F_3)_{224} = (F_3)_{242} = -(F_3)_{312} = -(F_3)_{321}$$
$$= (F_3)_{343} = (F_3)_{334} = -2(F_3)_{222} = -2(F_3)_{343} = \frac{1}{2} \lambda_3,$$
$$-(F_3)_{113} = -(F_3)_{124} = -(F_3)_{131} = -(F_3)_{142} = 2(F_3)_{311} = 2(F_3)_{344}$$
$$= (F_3)_{412} = (F_3)_{421} = -(F_3)_{434} = -(F_3)_{443} = \frac{1}{2} \lambda_4.$$

Hence, we establish directly that $\theta_3 = 0$ and $\mathcal{S}_{i,j,k}(F_3)_{ijk} = 0$. Therefore we obtain that the considered manifold belongs to the basic class $\mathcal{W}_3(J_3)$, i.e.

$$(L, J_3, g) \in \mathcal{W}_3(J_3).$$

Let us summarize the conclusions (23), (25) and (27) in the following statement.

**Theorem 3.3** The constructed 4-dimensional almost hypercomplex pseudo-Hermitian manifold $(L, H, G)$ on the Lie group $L$ belongs to basic classes with respect to the three almost complex structures of different types as follows

$$(L, H, G) \in \mathcal{W}_4(J_1) \cap \mathcal{W}_3(J_2) \cap \mathcal{W}_3(J_3).$$

The square norm $||\nabla J_\alpha||^2$ of $\nabla J_\alpha$ for an almost complex structure $J_\alpha$ is defined in [4] by (14). Having in mind the definition $F_\alpha(X, Y, Z) = g((\nabla X) J_\alpha, Y, Z)$ of the tensor $F_\alpha$, we obtain the following equation for the square norm of $\nabla J_\alpha$

$$||\nabla J_\alpha||^2 = g^{ij} g^{kl} g^{pq} (F_\alpha)_{ikp} (F_\alpha)_{jlp},$$

therefore

$$||\nabla J_\alpha||^2 = ||F_\alpha||^2, \quad \alpha = 1, 2, 3.$$

By virtue of (24), (22), (26) we receive immediately that

$$-2 ||\nabla J_1||^2 = ||\nabla J_2||^2 = ||\nabla J_3||^2 = 4 \left(\lambda_1^2 + \lambda_2^2 - \lambda_3^2 - \lambda_4^2\right).$$

The last equations and Equation (20) imply

**Proposition 3.4** (i) If the manifold $(L, H, G)$ is isotropic Kählerian with respect to some $J_\alpha$ ($\alpha = 1, 2, 3$) then it is isotropic hyper-Kählerian;

(ii) The manifold $(L, H, G)$ is isotropic hyper-Kählerian if and only if the condition $\lambda_1^2 + \lambda_2^2 - \lambda_3^2 - \lambda_4^2 = 0$ holds;
The manifold \((L, H, G)\) is isotropic hyper-Kählerian if and only if it has zero scalar curvature \(\tau\).

The space of unitary invariants of order 2 for a 4-dimensional Hermitian manifold is determined by the three quantities:

\[
\tau, \tau^*_1, \|\nabla \Phi\|_2 = 2 \|\delta \Phi\|_2,
\]

where \(\tau^*_1 = \frac{1}{2} g^{ij} g^{kl} R(X_i, J_1 X_j, X_k, J_1 X_l)\) [5].

In other words, as \(\|F_1\|^2 = \|\nabla \Phi\|^2\) and \(\|\theta_1\|^2 = \|\delta \Phi\|^2\) we get

\[
2\tau^*_1 = -\|\theta_1\|^2 = \lambda_1^2 + \lambda_2^2 - \lambda_3^2 - \lambda_4^2.
\]

Let us compute the associated scalar curvatures \(\tau^*_\alpha\) on \((L, J_\alpha, g)\) for \(\alpha = 2, 3\) by \(\tau^*_\alpha := g^{ij} g^{kl} R(X_i, X_k, J_\alpha X_l, X_j)\) [3]. Then, using (19), we obtain

\[
\tau^*_2 = \lambda_1 \lambda_3 + \lambda_2 \lambda_4, \quad \tau^*_3 = \lambda_1 \lambda_4 - \lambda_2 \lambda_3.
\]

Having in mind the definitions of the Nijenhuis tensors \(N_\alpha\) of \(J_\alpha\) \((\alpha = 2, 3)\) and the commutators (17), we get the components \(N_\alpha(X_i, X_j)\) and after that the square norm of \(N_\alpha\) as follows

\[
\|N_\alpha\|^2 = 32 \left(\lambda_1^2 + \lambda_2^2 - \lambda_3^2 - \lambda_4^2\right), \quad \alpha = 2, 3.
\]

It is clear, according to Proposition (3.4) that the manifold \((L, H, G)\) is isotropic hyper-Kählerian and scalar flat if and only if it has isotropic Nijenhuis tensors of \(J_2\) and \(J_3\).

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Kostadin Gribachev, Mancho Manev
University of Plovdiv
Faculty of Mathematics and Informatics
Department of Geometry
236 Bulgaria blvd.
Plovdiv 4003
Bulgaria

e-mail: costas@uni-plovdiv.bg, mmanev@yahoo.com
http://www.fmi-plovdiv.org/manev