Computation of Domains of Analyticity for the dissipative standard map in the limit of small dissipation

Adrián P. Bustamante∗

School of Mathematics
Georgia Institute of Technology
686 Cherry St. Atlanta,
GA, 30332-0160, USA

Renato C. Calleja†

Department of Mathematics and Mechanics
IIMAS, National Autonomous University of Mexico (UNAM)
Apdo. Postal 20-126, C.P. 01000, Mexico, D.F. MEXICO

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Abstract
Conformally symplectic systems include mechanical systems with a friction proportional to the velocity. Geometrically, these systems transform a symplectic form into a multiple of itself making the systems dissipative or expanding. In the present work we consider the limit of small dissipation. The example we study is a family of conformally symplectic standard maps of the cylinder for which the conformal factor, \( b(\varepsilon) \), is a function of a small complex parameter, \( \varepsilon \).

We assume that for \( \varepsilon = 0 \) the map preserves the symplectic form and the dependence on \( \varepsilon \) is cubic, i.e., \( b(\varepsilon) = 1 - \varepsilon^3 \). We compute perturbative expansions formally in \( \varepsilon \) and use them to estimate the shape of the domains of analyticity of invariant circles as functions of \( \varepsilon \). We also give evidence that the functions might belong to a Gevrey class at \( \varepsilon = 0 \). We also perform numerical continuation of the solutions as they pass through the boundary of the domain to illustrate that the monodromy of the solutions is trivial. The numerical computations we perform support conjectures on the shape of the domains of analyticity.

∗ apb7@math.gatech.edu
† calleja@mym.iimas.unam.mx

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I. INTRODUCTION

We study the limit of small dissipation/expansion of a family of conformally symplectic standard maps. In particular, we approximate the shape of domains of analyticity of invariant circles of a family of conformally symplectic standard maps of the cylinder, $\mathcal{M} = S^1 \times \mathbb{R}$, depending on a small parameter, $\varepsilon$, that vanishes as the conformal factor tends to one.

It was noted in [CCdlL17] that the small divisors depend on the complex parameter $\varepsilon$ and give rise to regions where the functions parameterizing the circles cannot be analytic with respect to $\varepsilon$ but miss by very little. A conjecture in [CCdlL17] states that the tori are analytic in a domain in the complex $\varepsilon$ plane that is obtained by taking from a ball centered at zero, a sequence of small balls with centers along smooth curves passing through the origin. The radii of the excluded balls decreases faster that any power of the distance of the centers of the balls to the origin. In fact, it was rigorously proved in [CCdlL17] that this domain is a lower bound. The main objective of the present work is to illustrate the results in one example, provide numerical evidence and indications of new results. Our computations indicate that there are singularities which cluster around several points at which one does not expect the functions to be analytic. The singularities in the complex $\varepsilon$ plane, cluster inside balls whose radii decrease at the ratios predicted by the conjecture.

A common method to compute invariant circles of a map of the cylinder $f_\varepsilon : \mathcal{M} \rightarrow \mathcal{M}$, is by computing a parameterization $K_\varepsilon : S^1 \rightarrow \mathcal{M}$ of the invariant circle which satisfies an invariance equation. The invariance equation is

$$f_\varepsilon \circ K_\varepsilon = K_\varepsilon \circ T_\omega$$

with $T_\omega(\theta) = (\theta + \omega)$. The invariance equation states that the dynamics on the invariant circle are conjugated to a rigid rotation of the circle by an irrational number $\omega$. The parameterization function $K_\varepsilon$ can be written in terms of a periodic function, $u_\varepsilon : S^1 \rightarrow \mathbb{R}$, as in equation (2.7). The method we use to find the singularities is to approximate the conjugacy function $u_\varepsilon(\theta)$ by means of a Lindstedt series expansion in $\varepsilon$. The Lindstedt method produces polynomials in $\varepsilon$ of high order,

$$u_\varepsilon^{\leq N}(\theta) = \sum_{n=0}^{N} u_n(\theta) \varepsilon^n$$

with $N \approx 10^3$, whose coefficients $u_n : S^1 \rightarrow \mathbb{C}$ are periodic functions. We then use the Lindstedt series of the conjugacies to obtain Padé rational functions whose poles are expected
approximate the poles of the original function $u_\varepsilon$. Padé extrapolation methods of Lindstedt series have been widely used by several authors [BFG01, BG01, CF02, BG04, dlLT95] in the simplectic case. Since the Padé extrapolation method is based on approximating an analytic function with a rational function, the computation of poles is done by approximating the roots of the denominator of the Padé function. The denominator is a polynomial that can be of very high degree, and computing its roots depends heavily on numerical precision. Since the computations are very sensitive to precision, we perform them using $\approx 10^3$ digits which allows us to compute singularities for values of $\varepsilon$ that are at a distance $\approx 0.3$ from $\varepsilon = 0$ in the complex plane. We expect that higher precision together with higher order degree series, would allow us to compute poles that are closer to the origin. However, the method already allows us to have an approximation of the boundary of the domain in regions that are contained in the small balls that were predicted by the conjecture in [CCdlL17], even when the singularities are not very close to $\varepsilon = 0$. For this reason it is very hard to notice that the functions that we are computing are not analytic. We also find conjectures on the rate of growth of the terms of the Lindstedt series.

We note that the shapes of the domains that we present here are remarkably different from what one sees in the symplectic case, see [CF02, BG04, dlLT95, CdlL10]. This is partly due to the fact that in the symplectic case or in the dissipative case, the small divisors do not depend on the conformal factor $b(\varepsilon)$ which in our case is a function of $\varepsilon$.

Some explorations of the shape of the analyticity domains in the dissipative standard map have been performed using the parameterization method in [CC10], that is very similar to the one described in section III C. In [CC10], it is noticed that the breakdown of invariant tori in the conservative and the dissipative case are similar when the conformal factor $b$ is a constant, [CdlL10a]. A different behavior in the breakdown of invariant tori involving bundle collapse is observed in the dissipative standard map in [CF12]. Explorations of the shape of the domains of analyticity in $\varepsilon$ in the conservative case with the use of the parameterization method appear in [CdlL09, CdlL10b].
II. PRELIMINARIES

We consider the dissipative standard map defined on the cylinder $\mathcal{M} = S^1 \times \mathbb{R}$ given by

$$f_\varepsilon(x_n, y_n) = (x_{n+1}, y_{n+1})$$

and

$$y_{n+1} = b_\varepsilon y_n + c_\varepsilon + \varepsilon V'(x_n)$$

$$x_{n+1} = x_n + y_{n+1}, \quad (2.1)$$

where $y_n \in \mathbb{R}$, $x_n \in S^1$, $\varepsilon \in \mathbb{R}$, and $V'(x) = \frac{1}{2\pi} \sin(2\pi x)$ is an analytic, periodic function. Here we consider the case when the dissipative parameter, $b_\varepsilon$, is given by $b_\varepsilon = b(\varepsilon) = 1 - \varepsilon^3$, and the drift parameter $c = c(\varepsilon)$ is a function that depends on the small parameter $\varepsilon$. The dissipative parameter $b_\varepsilon$ coincides with the Jacobian of the function. We note that the Jacobian is the rate of dissipation/expansion of the map $(2.1)$, this rate will be dependent of the parameter $\varepsilon$. In particular, the case $\varepsilon = 0$ coincides with the zero dissipation limit.

In fact, it is discussed in [CCdlL17] that $(2.1)$ is conformally symplectic. If $\Omega = dy \wedge dx$ is the standard symplectic form of the cylinder, the map $f_\varepsilon$ satisfies that

$$f^*\Omega = b_\varepsilon \Omega. \quad (2.2)$$

For certain values of $c_\varepsilon$, we know that maps of the form $(2.2)$ have analytic invariant circles corresponding to quasi-periodic orbits with Diophantine rotation numbers, $\omega$. The Lindstedt series analysis in Section IIIA determines that one condition for the the mapping $(2.1)$ to admit an invariant circle is that $c_\varepsilon = \omega \varepsilon^3 + O(\varepsilon^4)$. In the following, we discuss the properties that the rotation number should satisfy so that one can have quasi-periodic orbits parameterized by a function.

A. Quasi-periodic orbits

We consider a frequency $\omega$ that satisfies the Diophantine condition,

$$|\omega q - p| \geq \nu |q|^{-\tau}, \quad p \in \mathbb{Z}, \quad q \in \mathbb{Z} \setminus \{0\} \quad (2.3)$$

where $\nu \in \mathbb{R}^+$ and $\tau \in \mathbb{R}$ with $\tau \geq 1$.

Quasi periodic orbits of the dissipative standard map $(2.1)$ are found using a parametric representation of the variable $x_n \in S^1$ as

$$x_n = \theta_n + u_\varepsilon(\theta_n), \quad \theta \in S^1, \quad (2.4)$$
where $u_\varepsilon : S^1 \to \mathbb{R}$ is a 1-periodic function. We assume that the variable $\theta_n$ varies linearly as $\theta_{n+1} = \theta_n + \omega$ where $\omega$ is the rotation frequency.

It follows from equation (2.1) that

$$x_{n+1} - (1 + b_\varepsilon)x_n + b_\varepsilon x_{n-1} + (1 - b_\varepsilon)\omega - c_\varepsilon + \varepsilon V'(x_n) = 0 \quad (2.5)$$

We look for quasi periodic solutions by finding $u_\varepsilon$ and $c_\varepsilon = c(\varepsilon)$ such that

$$E_{c_\varepsilon}[u_\varepsilon] \equiv u_\varepsilon(\theta + \omega) - (1 + b_\varepsilon)u_\varepsilon(\theta) + b_\varepsilon u_\varepsilon(\theta - \omega) - (1 - b_\varepsilon)\omega - c_\varepsilon + \varepsilon V'(\theta + u_\varepsilon(\theta)) = 0. \quad (2.6)$$

We remark that the nature of the two unknowns is different since $u_\varepsilon(\theta)$ is a smooth complex 1-periodic function of $\theta \in S^1$ depending on the complex parameter $\varepsilon$ and $c_\varepsilon$ is a complex number depending on $\varepsilon$. The conjecture in [CCdlL17], states that $\varepsilon$ is a complex parameter whose range lays in a complex domain that is obtained by taking out from a neighborhood of $\varepsilon = 0$, points inside balls with centers along smooth curves passing though the origin. In [CCdlL17] there is also a rigorous lower bound close to the domain described in the conjecture.

It is clear that once we find a pair $(u_\varepsilon, c_\varepsilon)$ satisfying (2.6), we can recover the embedding of the quasi-periodic orbit by the parameterization $K_\varepsilon : S^1 \to M$,

$$K_\varepsilon(\theta) = \begin{pmatrix} \theta + u_\varepsilon(\theta) \\ \omega + u_\varepsilon(\theta) - u_\varepsilon(\theta - \omega) \end{pmatrix} \quad (2.7)$$

III. METHODS FOR COMPUTING SOLUTIONS

We will use two different methods for finding the solution pair $(u_\varepsilon, c_\varepsilon)$ of (2.6). The first method is based on a Lindstedt series approximation of the solutions written as formal power series of the small parameter $\varepsilon$. In our case the small parameter $\varepsilon$ will account both for the size of the perturbation and the distance of the conformal factor to the symplectic case. This method produces approximate solutions in the sense that if

$$u_\varepsilon^{\leq N}(\theta) = \sum_{k=0}^{N} u_k(\theta)\varepsilon^k \quad \text{and} \quad c^{\leq N}(\varepsilon) = \sum_{k=0}^{N} c_k\varepsilon^k \quad (3.8)$$

are polynomials in $\varepsilon$, we say that (3.8) is an approximate solution of order $N$ whenever $\|E_{c^{\leq N}(\varepsilon)}[u_\varepsilon^{\leq N}]\| = O(\varepsilon^{N+1})$, where $E$ is the functional defined in (2.6) and $\| \cdot \|$ is the supremum norm over all $\theta \in S^1$. The Lindstedt series method that we describe in section III A provides a way to construct an approximate solution of any given order $N \in \mathbb{N}$.
In section III C, we include an algorithm to find the solution \((u_ε, c_ε)\) by means of a Newton method. The method starts from an approximate solution pair \((u_a, c_a)\) so that the norm of \(E_{c_a}[u_a]\) is small and provides a correction \((v, δ)\) by imposing that the new solution \((u_a + v, c_a + δ)\) satisfies the functional equation \(E_{c_a+δ}[u_a + v]\) up to first order in \((v, δ)\). This method can be shown to converge using scales of Banach spaces.

A. Lindstedt Series

The Lindstedt series method consists of performing a formal series expansion in a small parameter \(ε\). According to (2.6), and the fact that \(b(ε) = 1 − ε^3\), we look for a solution, \((u_ε, c_ε)\), of

\[
\begin{align*}
    u_ε(θ + ω) - (2 - ε^3)u_ε(θ) + (1 - ε^3)u_ε(θ - ω) + ε^3ω - c(ε) &= -εV'(θ + u_ε(θ)) \quad (3.9)
\end{align*}
\]

as a power series expansion. That is, we look for solutions

\[
\begin{align*}
    u_ε(θ) &= \sum_{k=0}^{∞} u_k(θ)ε^k \quad \text{and} \quad c(ε) = \sum_{k=0}^{∞} c_kε^k, \quad (3.10)
\end{align*}
\]

where each \(u_n\) is a function from \(S^1\) to \(C\) and each \(c_n \in C\). This solution can be computed by equating powers of \(ε\) in (3.9). Taking the Taylor expansion at \(ε = 0\)

\[
- εV'(θ + u_ε(θ)) = \sum_{k=1}^{∞} S_k(θ)ε^k
\]

and substituting (3.10) into (3.9), we have that

\[
\begin{align*}
    \sum_{k=0}^{∞} u_k(θ + ω)ε^k - (2 - ε^3) \sum_{k=0}^{∞} u_k(θ) + (1 - ε^3) \sum_{k=0}^{∞} u_k(θ - ω) + ε^3ω - \sum_{k=0}^{∞} c_kε^k &= \sum_{k=1}^{∞} S_k(θ)ε^k. \quad (3.12)
\end{align*}
\]

Remark III.1. When \(V'(θ) = \sin(2πθ)\), or a trigonometric polynomial, the \(S_k(θ)\)’s can be computed very efficiently in terms of the \(u_i(θ)\)’s. Following [eK69, FdLL92] and denoting \(S(θ, ε) = \sin(2π(θ + u_ε(θ)))\), \(C(θ, ε) = \cos(2π(θ + u_ε(θ)))\), the coefficients of the series expansions \(S(θ, ε) = \sum_{k=0}^{∞} S_k(θ)ε^k \) and \(C(θ, ε) = \sum_{k=0}^{∞} C_k(θ)ε^k\) are given by the following recurrence relations,

\[
\begin{align*}
    (N + 1)S_{N+1}(θ) &= 2π \sum_{m=0}^{N} C_{N-m}(m + 1)u_{m+1}(θ) \quad (3.13) \\
    (N + 1)C_{N+1}(θ) &= -2π \sum_{m=0}^{N} S_{N-m}(m + 1)u_{m+1}(θ).
\end{align*}
\]

Thus \(S_k(θ) = -S_{k-1}(θ)\) and \(S_0 \equiv 0\), by (3.11).
Defining the operator

\[ L_\omega \varphi(\theta) := \varphi(\theta + \omega) - 2\varphi(\theta) + \varphi(\theta - \omega) \]  

equation (3.12) can be rewritten as

\[
\sum_{k=1}^{\infty} S_k(\theta) \varepsilon^k = \sum_{k=0}^{2} (L_\omega u_k(\theta) - c_k) \varepsilon^k + (L_\omega u_3(\theta) - c_3 + u_0(\theta) - u_0(\theta - \omega) + \omega) \varepsilon^3 \\
+ \sum_{k=4}^{\infty} (L_\omega u_k(\theta) - c_k + u_3(\theta) - u_3(\theta - \omega)) \varepsilon^k,
\]  

(3.15)

Some properties of the operator \( L_\omega \) are summarized in the following Lemma. See \textit{[dlL01]} for details about the proof.

**Lemma III.2.** Let \( \eta : S^1 \to S^1 \) a continuous function such that \( \int_0^1 \eta(\theta)d\theta = 0 \). If \( \omega \) is Diophantine as in (2.3), then there exists a solution, \( \varphi(\theta) \), to the equation

\[ L_\omega \varphi(\theta) = \eta(\theta) \]  

(3.16)

such that \( \int_0^1 \varphi(\theta)d\theta = 0 \). In fact, the solution is given by

\[ \varphi(\theta) = \sum_{\ell \in \mathbb{Z} \setminus \{0\}} \frac{\hat{\eta}_\ell}{2(\cos(2\pi \ell \omega) - 1)} e^{2\pi i \ell \theta}, \]

where \( \hat{\eta}_\ell \) are the Fourier coefficients of \( \eta(\theta) \).

The Lindstedt process is as follows. Matching the coefficients of the same order in (3.15) we obtain the following relations to different orders of \( \varepsilon \). The zero-th order term tells us that the coefficients at order zero in \( \varepsilon \) have to be trivial. The equations are

\[ L_\omega u_0(\theta) - c_0 = 0. \]  

(3.17)

Choosing \( c_0 = 0 \), then \( u_0 \equiv 0 \) is the solution given by Lemma III.2. This construction is analogous to the zero-th order term in the symplectic case.

**Remark III.3.** This method has been used in \textit{[FdLL92, dlL01, BFG01, BG01, CF02, BG04, dlLT95]} for the symplectic case. That is, making the same process for the standard map, \( (x_{n+1}, y_{n+1}) = (x_n + y_{n+1}, y_n + \varepsilon V'(x_n)) \), gives the following equation to all orders \( \varepsilon^n \),

\[ L_\omega u_k(\theta) = S_k(\theta) \quad k \geq 0. \]  

(3.18)
Moreover, $\int_0^1 S_k(\theta)d\theta = 0$ for all $k \geq 0$. This is a consequence of the symplectic structure and the fact that $S_k(\theta)$ depends on the previously computed $u_0(\theta)$, $u_1(\theta)$, $u_{k-1}(\theta)$, $S_0(\theta)$, $S_1(\theta)$, $S_{k-1}(\theta)$ (see Remark III.1).

The first and second orders in $\varepsilon$ are also analogous to the symplectic case. For this reason the first two coefficients of $c_\varepsilon$ will be trivial.

$$L_\omega u_k(\theta) - c_k = S_k(\theta), \quad k = 1, 2.$$  \hfill (3.19)

Choosing $c_1 = 0 = c_2$ the equations are reduced to the non dissipative case and, by Remark III.3, the right hand side has zero average. Therefore, we can find solutions $u_1(\theta)$, $u_2(\theta)$.

The third order in $\varepsilon$ is the first one that is different from the conservative case.

$$L_\omega u_3(\theta) - c_3 + \omega = S_3(\theta).$$  \hfill (3.20)

Here we notice that the drift parameter starts playing a rôle in the existence of invariant tori. Taking $c_3 = \omega$, equation (3.20) becomes the same equation as in the symplectic case. Since $S_3(\theta)$ has zero average we find $u_3(\theta)$.

The equations for orders higher that 4 are remarkably different since we have a counter term coming from the previously computed orders. Namely,

$$L_\omega u_k(\theta) = S_k(\theta) - u_{k-3}(\theta) + u_{k-3}(\theta - \omega) + c_k, \quad k \geq 4.$$  \hfill (3.21)

Notice that, by construction, $\int_0^1 u_{k-3}(\theta - \omega)d\theta = \int_0^1 u_{k-3}(\theta)d\theta = 0$ (see Lemma III.2). Now, taking

$$c_k = -\int_0^1 S_k(\theta)d\theta,$$  \hfill (3.22)

we can find $u_k(\theta)$ solving (3.21) for all $k \geq 4$.

We have proved the following proposition which is a particular case of part A) of Theorem 12 in [CCdlL17].

**Proposition III.4.** For any $N \in \mathbb{N}$, the procedure presented above allows to find an approximate solution,

$$u^{\leq N}_\varepsilon(\theta) = \sum_{k=0}^{N} u_k(\theta)\varepsilon^k \quad \text{and} \quad c^{\leq N}(\varepsilon) = \sum_{k=0}^{N} c_k\varepsilon^k,$$  \hfill (3.23)

such that

$$\|E_{\varepsilon^{\leq N}(\varepsilon)}[u^{\leq N}_\varepsilon]\| = O(\varepsilon^{N+1})$$

where $E$ is the functional defined in (2.6).
B. Padé extrapolation

The domain of analyticity for the solution of (2.6) can be approximated by implementing a Padé method in which we use the approximate solutions obtained by the Lindstedt series constructed in Section III A.

The Padé method is quite standard and is presented in several places in the literature. Here, we follow the exposition in [BGM96]. A Padé approximant of order \([p/q]\) of a function \(g(\varepsilon) = \sum_{i=0}^{\infty} g_i \varepsilon^i\) is a rational function, \(P(\varepsilon)/Q(\varepsilon)\), which agrees with \(u\) to the highest possible order in \(\varepsilon\).

That is,
\[
g(\varepsilon) - \frac{P(\varepsilon)}{Q(\varepsilon)} = O(\varepsilon^{p+q+1}). \tag{3.24}
\]

Where \(P(\varepsilon)\) and \(Q(\varepsilon)\) are polynomials of degree \(p\) and \(q\) respectively, \(Q(0) = 1\).

The existence of the polynomials \(P\) and \(Q\) can be obtained by noticing that (3.24) is equivalent to
\[
g(\varepsilon)Q(\varepsilon) = P(\varepsilon) + O(\varepsilon^{p+q+1})
\]
and, then, considering \(P(\varepsilon) = \sum_{i=0}^{p} P_i \varepsilon^i\) and \(Q(\varepsilon) = \sum_{i=0}^{q} Q_i \varepsilon^i\) the coefficients of the polynomials can be found by solving the following systems of equations
\[
\begin{align*}
    u_i + \sum_{j=1}^{i} u_{i-j} Q_j &= P_i & 0 \leq i \leq p \\
    u_i + \sum_{j=1}^{q} u_{i-j} Q_j &= 0 & p < i \leq p + q. \tag{3.25}
\end{align*}
\]

The second equation of (3.25) gives the \(Q'_j\)s, and then we can find the \(P'_i\)s by substituting in the first equation. Then, the boundary of the domain of analyticity of a function can be approximated by the zeros of \(Q\) in the \([p/q]\) Padé approximant.

There are a number of implementations of the Padé methods that are used in a quite standard manner. In the present work we use the implementations included in Version 2.9.0 of GP/PARI, [BBB+].

C. Newton’s method

In this section we summarize an iterative scheme in scales of Banach spaces that can be very well adapted to perform numerical computations. The scheme is based in a Newton
iteration starting from approximate solutions to the equation (2.6). We briefly describe the
scheme here since details of schemes of these kind and numerical implementations have been
described already in the literature \cite{CC10,CCdlL17,CF12,CCdlL13}, and the reader can
refer to these works for more details.

We start from an approximate solution \((u_a, c_a)\) of equation (2.6). Namely, we have a
solution so that \(||E_c[u_a]||\) is small enough. The approximate solution could be obtained
by several means. One possibility is starting from the integrable case (for \(\varepsilon\) close to zero)
and performing continuation or from a Lindstedt series expansions like the ones obtained in
Section III A. We remark that in the dissipative standard map we are studying, \(\varepsilon = 0\) is the
point where the map becomes symplectic. Since we use methods for conformally symplectic
systems we actually start the continuation from values of \(\varepsilon\) that are not equal to zero but
small.

The Newton algorithm consists of adding a correction \((v, \delta)\) to the approximate solution so
that supremum norm of (2.6) evaluated in the function plus the corrections, \(||E_c[u_a + v]\||\),
is of the order of the square of the norm of (2.6) evaluated at the approximate solution,

\[||E_{c_a + \delta}[u_a + v]|| \leq C||E_{c_a}[u_a]||^2.\]

One obtains the correction by solving the linearized equation of \(E_{c_a + \delta}[u_a + v]\) for \((v, \delta)\) around
the approximate solution, \((u_a, c_a)\).

In this case, the equation we have to solve is

\[D_u E_{c_a}[u_a] v - \delta = -E_{c_a}[u_a]\]

which involves unbounded operators in Banach spaces (namely \(D_u E_{c_0}[u_0] v\)) that are actually
bounded if one considers that the operators map into Banach spaces of less regularity. It is
a standard observation in Nash–Moser theory \cite{Zeh75,Zeh76}, that to set up a converging
iterative Newton scheme it is not necessary to find an exact inverse of the operator \(D_u E_{c_a}[u_a]\),
but an approximate inverse will suffice.

One obtains an approximate inverse by noticing that the modified Newton equation,

\[h' D_u E_{c_a}[u_a] v - v D_u E_{c_a}[u_a] h' = -h'(E_{c_0}[u_0] - \delta),\]

with \(h'(\theta) = 1 + \frac{\partial u(\theta)}{\partial \theta}\), factorizes in a sequence of operators that map Banach spaces of
regular functions to Banach spaces of functions with less regularity.
This method has been used in several works [CL09, dL08, dL01]. Here we only make a reference to the justification in [CC10], where the reader can refer to for details. Let the operators $D_{-1}, D_1^b$ by

$$
D_- f(\theta) = f(\theta - \omega) - f(\theta)
$$

$$
D_1^b f(\theta) = f(\theta + \omega) - bf(\theta)
$$

(3.28)

A small remark is that $3.28$ are operators that are diagonal in Fourier space. In the following lemma, we write the modified Newton as a sequence of operators that are either diagonal in Real or Fourier space.

**Lemma III.5.** The modified Newton equation in (3.27) with $E_{ca}[u]$ defined in (2.6) is equivalent to

$$
D_1^b[-h'(\theta)h'(\theta - \omega)D_-[(h')^{-1}(\theta)v(\theta)]] = -h'(E_{ca}[u_a](\theta) - \delta).
$$

(3.29)

**Remark III.6.** One notices that the operators involved in the l.h.s. of equation (3.29) only involve differentiation, multiplication, division, shifting the arguments of functions, and solving the difference equations with constant coefficients in (3.28). All this operations can be implemented very efficiently using the computer. For instance if we discretize the periodic functions using $n$ uniformly distributed points and we use a Fast Fourier Transform method, the modified Newton step equation can actually be solved in $O(n \log n)$ operations.

The factorization in equation (3.29) suggests an algorithm that is used to solve the modified Newton equation.

**Algorithm III.7.**

1) Find two functions $\varphi$ and $\nu$ solving the equations

$$
D_1^b \varphi(\theta) = -h'E_{ca}[u]
$$

(3.30)

and

$$
D_1^b \nu(\theta) = -h'(\theta)
$$

(3.31)

Notice that if $\varphi(\theta)$ and $\nu(\theta)$ are solutions of (3.30) and (3.31), respectively, then the equation $D_1^b(\varphi(\theta) - \delta\nu(\theta)) = -h'(\theta)(E_{ca}[u_0](\theta) - \delta)$ holds for any $\delta \in \mathbb{C}$. This will allow us to chose a complex number $\delta$ so that the average of $\varphi(\theta) - \delta\nu(\theta)$ vanishes.

2) Choose $\delta \in \mathbb{C}$ such that

$$
\int_{\mathbb{T}} \varphi(\theta) - \delta\nu(\theta) h'(\theta - \omega) d\theta = 0.
$$
iii) Obtain \( w \) from the solution of the constant coefficient difference equation

\[
D_w(\theta) = \frac{\varphi(\theta) - \delta \nu(\theta)}{-h'(\theta)h'(\theta - \omega)} .
\]  

(3.32)

Notice that after choosing a \( \delta \) in step ii) so that the right hand side has zero average we can always find a periodic function \( w \) solving (3.32) when the r.h.s. is smooth enough.

iv) Construct \( v(\theta) = h'(\theta)w(\theta) \) and obtain the improved solution \((\tilde{u}, \tilde{c})\) defined as

\[
\tilde{u}(\theta) = u_a(\theta) + v(\theta) , \quad \tilde{c} = c_a + \delta .
\]

The observation in remark \( \text{III.6} \) is that the operators in (3.29) are very efficiently implementable with the use of a computer either in Real or in Fourier space. This efficiency comes from the fact that all the operations involved in the four steps of Algorithm \( \text{III.7} \) are multiplications, additions and integrals of periodic functions that take only \( O(n) \) operations in Real space; and differentiation, shifts and solving cohomology equations with constant coefficients, that take only \( O(n) \) operations in Fourier space. Therefore, the most expensive operation in the Algorithm \( \text{III.7} \) is transforming from Real to Fourier space and back. This can be done in \( O(n \log n) \) operations by means of a Fast Fourier Transform.

Remark III.8. We note that the algorithm is guaranteed to converge inside the boundaries of the analyticity domain. Indeed, in \( \text{[CdlL10b]} \) was rigorously justified that the algorithm only fails to converge as the continuation reaches the boundary of analyticity. Therefore, the continuation method can also be used to assess the bounds on the domain of \( \varepsilon \).

IV. NUMERICAL RESULTS

In this section we present the results of implementing the methods described in Section \( \text{III} \). All the computations were done using the golden ratio, \( \omega = \frac{1 + \sqrt{5}}{2} \), which satisfies (2.3) \( \text{[dlL01]} \).

A. Lindstedt expansions

The construction of Lindstedt series in Section \( \text{III.A} \) was implemented as a numerical algorithm. The statement of Proposition \( \text{III.4} \) tells us that given any \( N \in \mathbb{N} \), the outcome
of the method is the pair of polynomials of degree $N$ in $\mathbf{III.4}$. The observation of Lemma $\mathbf{III.2}$ is that the operator $L_\omega$ defined in equation (3.14) is diagonal in Fourier Space and equation $3.16$ can be solved for $\phi$ if we allow to obtain functions with less regularity than the right hand side, $\eta$. We find the solution numerically by transforming to Fourier space and solving for the $u_k$'s from expressions (3.19) to (3.21). At every order of the process we obtain the $c_k$'s as a byproduct of imposing the condition that every order should have zero average.

The Lindstedt series expansions are used to obtain an approximate solution to the functional equation in (2.6) at some high order. Indeed, we discovered that with our implementations it is very hard to notice that the functions are not analytic. Namely, the singularities that exist close to the point $\varepsilon = 0$ are very hard to detect so we have no evidence that the radius of convergence of the series is exactly zero in the complex plane. Thus, if the solution belongs to a Gevrey class then the Gevrey exponent would be very close to one.

We approximated several norms of the coefficients, $u_k(\theta)$, to have an indication of how far the functions are from being analytic. First, we use the norm on the complex strip of size $\rho > 0$, i.e., $\theta \in S^1_\rho$ if $|\text{Im} (\theta)| < \rho$. Let $f : S^1_\rho \rightarrow S^1_\rho$ be a function of $S^1_\rho$ then the norm we use is

$$\|f\|_\rho = \sum_{\ell \in \mathbb{Z}} |\hat{f}_\ell|^2 e^{2\pi |\ell| \rho}$$

where $\hat{f}_\ell$ are the Fourier coefficients of $f$.

We say that the function $f(\varepsilon)$ belongs to the Gevrey class $G^\sigma$ with respect to the norm $\| \cdot \|_B$ at $\varepsilon = 0$ whenever

$$\|\partial_\varepsilon^k f(\varepsilon)\|_B \leq CR^k k^{\sigma k},$$

for $\varepsilon = 0$, [Gev18].

Since we want to check if the function $u_\varepsilon(\theta)$ belongs to a Gevrey class at $\varepsilon = 0$ with the analytic norms it is convenient to compute the following expressions as functions of $k$,

$$A_\rho(k) \equiv \frac{1}{k} \log \|u_k(\theta)\|_\rho,$$

(4.33)

and then approximate the constant $\sigma$.

The expressions (4.33) as functions of $k$ for the coefficients of the approximate solution are shown in Figure [I]
FIG. 1: Analytic norms of the coefficients of the Lindstedt expansion plotted as in expression (4.33).

We also used Sobolev norms defined for a real number \( r > 0 \) by the \( L^2 \)-norm of the \( r^{th} \) derivative with respect to \( \theta \),

\[
\|f\|_r = \|\partial_\theta^r f\|_{L^2}.
\]

Notice that when \( r = 0 \), \( \|f\|_0 \) corresponds to the \( L^2 \) norm of \( u \). The Sobolev norms can also be written in terms of Fourier coefficients as follows,

\[
\|f\|_r = \left( \sum_{k \in \mathbb{Z}} (2\pi k)^{2r} |\hat{f}_k|^2 \right)^{1/2}.
\]

As in the case for analytic norms we define the following expressions for the Sobolev norms,

\[
H^r(k) \equiv \frac{1}{k} \log \|u_k(\theta)\|_r.
\]  \hspace{1cm} (4.34)

We include the values of \( H^r(k) \) for the coefficients of the approximate solution and several values of \( r \) in Figure 2.
FIG. 2: Sobolev norms of the coefficients of the Lindstedt expansion plotted as functions
of the order of $\varepsilon$.

\[
A_\rho(k) = \log(a) + c \log(k + b)
\]

| $\rho$ | $a$         | $b$                 | $c$          |
|--------|-------------|---------------------|--------------|
| 0.5    | 0.719467892 | 27.3937734029272    | 1.00001766652951 |
| 0.1    | 0.719927523 | -6.67612303717059   | 0.999991308925563 |
| 0.01   | 0.719826182 | -12.4920865977342   | 0.999998383397402 |
| 0.001  | 0.719813322 | -13.0568139120788   | 0.999999331056825 |

TABLE I: Numerical fit of analytic norms in expression (4.33) for different values of $\rho$.

In both cases, the behavior of the norms coefficients $\|u_k(\theta)\|_B$ with respect to $k$ seem to belong to Gevrey classes. In Tables I and II we include the fit of the plots in Figures 1 and 2.

The numerical results in Tables I and II lead us to think that the solutions that we
$H^r(k) = \log(a) + c \log(k + b)$

|   | a          | b            | c                      |
|---|------------|--------------|------------------------|
| $r = 0$ | 0.71981186150290 | -13.11936925724 | 0.999999439039333    |
| $r = 1$ | 0.71990246696872 | -8.455098498275  | 0.9999929620721     |
| $r = 2$ | 0.71995774061454 | -3.699094566292  | 0.99998926987818    |
| $r = 3$ | 0.71997659277593 | 1.15141934192   | 0.99998826853277    |
| $r = 4$ | 0.71995790042650 | 6.099307051805  | 0.9999897277892     |
| $r = 5$ | 0.71990050891201 | 11.14758050573  | 0.99999346098676     |
| $r = 6$ | 0.71980323117743 | 16.29938193670  | 0.99999928901585     |

TABLE II: Numerical fit of analytic norms in expression (4.34) for different exponents, $r$.

approximate are functions that are very hard to distinguish from analytic functions by just examining the a truncated expansion series. One of the rigorous results in [CCdlL17] states that the functions that satisfy equation (2.6) fail to be analytic since there is no ball around $\varepsilon = 0$ where the formal power series converges. Therefore, we conjecture that the solutions belong to a Gevrey class with an exponent that is very close to the analytic class.

**Conjecture IV.1.** The parametrization $u_\varepsilon$ belongs to a Gevrey class, $G^\sigma$, as a function of $\varepsilon$. The index, $\sigma$, is close to 1.

**B. Approximation of poles of the Lindstedt series**

Here we include the poles of the Lindstedt polynomial found with the Padé method. In Figure 3, we show the poles of the series approximated by means of the Padé method. It is well known that the Padé method computations are very sensitive to precision, see [BGM96], so we have implemented the computations with extended precision using the software gp/Pari [BBB]. We show the values of the poles in the $\varepsilon$ complex plane, and the complex values of the function $b(\varepsilon) = 1 - \varepsilon^3$. Figure 4 contains the comparison of the values of the function $b(\varepsilon)$ with the unit circle. We also include zoomed in versions of the values of $b(\varepsilon)$ in Figure 4.
FIG. 3: Points which are simultaneously poles of Padé approximants of degree $[475,475]$ and $[500,500]$. The implementation was done with 1000 digits. Left panel: Poles in the complex plane $\epsilon \in \mathbb{C}$. Right panel: Poles evaluated in the function $b(\epsilon) = 1 - \epsilon^3$, with $\epsilon \in \mathbb{C}$.

C. Newton method

We used Newton’s method and continuation to explore the monodromy of the solutions in the domains. A rigorous result in [CCdlL17] states that the solutions defined in the domain of analyticity in $\epsilon$ have trivial monodromy. We verified this fact numerically by performing continuation of the solutions $(u_\epsilon, c_\epsilon)$ around the poles that were previously approximated using the Padé series method described in Section IV B.

We used the approximated poles as centers of circular paths in $\epsilon$ over which we performed continuation while solving the invariance equation (2.6) using Algorithm III.7. Once the continuation completes a complete turn around a chosen pole, one verifies that the solution always arrives to the same starting point. This is an effect of the monodromy of the functions being trivial.

We present several instances of the functions for different parameter values along a circle winding around a pole in Figure 6. The path we used to surround the pole is presented in Figure 5. The continuation was performed using FFTW3, [FJ05], with the libquadmath library, [HLB00]. The radii of the continuation paths were chosen so that the path did
FIG. 4: The poles compared to the unit circle. Upper panel: Evaluation of the poles of the series by the function $b(\varepsilon) = 1 - \varepsilon^3$. Lower panels: Two zoomed in versions of the set.

not come very close to the poles. Indeed, when the continuation comes close to a pole our implementation of the Newton method becomes degenerate in the sense that one needs to compute quotients of very small quantities. The reason is that when solving equations (3.30) and (3.31), the divisors depending on $\varepsilon$, are below machine precision close to the pole and dividing over those quantities leads to large numerical errors.
FIG. 5: Poles of the series and two different continuations done with the Newton algorithm. The continuation is done around the pole in order to illustrate that the monodromy is trivial.

| Instance | $\epsilon$ | $c(\epsilon)$ |
|----------|------------|---------------|
| 1        | $0.3202966 + i0.1460915$ | $0.01994937 - i0.06774761$ |
| 2        | $0.3008391 + i0.1527000$ | $0.009976542 - i0.06136120$ |
| 3        | $0.2830167 + i0.1871540$ | $-0.01146081 - i0.06221038$ |
| 4        | $0.3122423 + i0.2245263$ | $-0.02718298 - i0.08804174$ |
| 5        | $0.3613448 + i0.1973876$ | $0.007928831 - i0.1127768$ |
| 6        | $0.3242691 + i0.1460201$ | $0.02157160 - i0.06953580$ |

TABLE III: Values of $\epsilon$ and $c(\epsilon)$ for different instances taken from the small circle in FIG.5

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FIG. 6: Real and imaginary part of different instances of a continuation by the Newton algorithm including the initial and final functions. One observes that there is no monodromy after a full turn around the pole.

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