ON EMBEDDINGS OF FULL AMALGAMATED FREE PRODUCT C*-ALGEBRAS

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Abstract. We examine the question of when the *-homomorphism \( \lambda : A \ast_D B \to \tilde{A} \ast_D \tilde{B} \) of full amalgamated free product C*-algebras, arising from compatible inclusions of C*-algebras \( A \subseteq \tilde{A}, B \subseteq \tilde{B} \) and \( D \subseteq \tilde{D} \), is an embedding. Results giving sufficient conditions for \( \lambda \) to be injective, as well as classes of examples where \( \lambda \) fails to be injective, are obtained. As an application, we give necessary and sufficient conditions for the full amalgamated free product of finite-dimensional C*-algebras to be residually finite dimensional.

1. Introduction

Given C*-algebras \( A, B \) and \( D \) with injective *-homomorphisms \( \phi_A : D \to A \) and \( \phi_B : D \to B \), the corresponding full amalgamated free product C*-algebra (see [1] or [9, Chapter 5]) is the C*-algebra \( \mathfrak{A} \), equipped with injective *-homomorphisms \( \sigma_A : A \to \mathfrak{A} \) and \( \sigma_B : B \to \mathfrak{A} \) such that \( \sigma_A \circ \phi_A = \sigma_B \circ \phi_B \), such that \( \mathfrak{A} \) is generated by \( \sigma_A(A) \cup \sigma_B(B) \) and satisfying the universal property that whenever \( \mathfrak{C} \) is a C*-algebra and \( \pi_A : A \to \mathfrak{C} \) and \( \pi_B : B \to \mathfrak{C} \) are *-homomorphisms satisfying \( \pi_A \circ \phi_A = \pi_B \circ \phi_B \), there is a *-homomorphism \( \pi : \mathfrak{A} \to \mathfrak{C} \) such that \( \pi \circ \sigma_A = \pi_A \) and \( \pi \circ \sigma_B = \pi_B \). This situation is illustrated by the following commutative diagram:

\[
\begin{array}{ccc}
A & \xrightarrow{\phi_A} & D \\
\downarrow{\sigma_A} & & \downarrow{\phi_B} \\
\mathfrak{A} & \xleftarrow{\sigma_B} & B.
\end{array}
\]

The full amalgamated free product C*-algebra \( \mathfrak{A} \) is commonly denoted by \( A \ast_D B \), although this notation hides the dependence of \( \mathfrak{A} \) on the embeddings \( \phi_A \) and \( \phi_B \).
Question 1.1. Let $D$, $A$, $B$, $\tilde{D}$, $\tilde{A}$ and $\tilde{B}$ be $C^*$-algebras and suppose there are injective $*$-homomorphisms making the following diagram commute:

$$
\begin{array}{ccc}
\tilde{A} & \xrightarrow{\phi_A} & \tilde{D} & \xrightarrow{\phi_B} & \tilde{B} \\
\lambda_A & & \lambda_D & & \lambda_B \\
A & \xleftarrow{\phi_A} & D & \xleftarrow{\phi_B} & B
\end{array}
$$

Let $A *_B \tilde{B}$ and $\tilde{A} *_{\tilde{B}} \tilde{B}$ be the corresponding full amalgamated free product $C^*$-algebras, and let $\lambda : A *_D B \to \tilde{A} *_{\tilde{B}} \tilde{B}$ be the $*$-homomorphism arising from $\lambda_A$ and $\lambda_B$ via the universal property. When is $\lambda$ injective?

We prove in [2] that $\lambda$ is injective when either (i) $D = \tilde{D}$ (or more precisely, when the $*$-homomorphism $\lambda_D$ is surjective), or (ii) there are conditional expectations $E_A : \tilde{A} \to A$ and $E_B : \tilde{B} \to B$ that send $\tilde{D}$ onto $D$ and agree on $D$. Injectivity in the case $D = \tilde{D}$ was previously proved by G. K. Pedersen [10]. (Moreover, earlier results of F. Boca [4] imply that the map $\lambda$ is injective when $D = \tilde{D}$ and when there are conditional expectations

$$
\begin{array}{ccc}
A & \xrightarrow{E_A} & \tilde{A} & \xrightarrow{E_{\tilde{A}}} & D & \xrightarrow{E_D} & B & \xrightarrow{E_B} & \tilde{B},
\end{array}
$$

an argument for the case $D = \tilde{D} = C$, which uses Boca’s results, is outlined in [5, 4.7].) However, we include our proof because it is different from that found in [10] and because it contains the main idea of our proof of injectivity in case (ii). In [4] we consider some general conditions and give some concrete examples when $\lambda$ fails to be injective. Finally, in [3] we apply this embedding result to extend a result from [5] about residual finite-dimensionality of full amalgamated free products of finite-dimensional $C^*$-algebras.

2. Embeddings of full free products

The following result is of course well known. We include a proof for completeness.

Lemma 2.1. Let $A$ be a $C^*$-subalgebra of a $C^*$-algebra $\tilde{A}$ and let $\pi : A \to B(\mathcal{H})$ be a $*$-representation. Then there is a Hilbert space $\mathcal{K}$ and a $*$-representation $\tilde{\pi} : \tilde{A} \to B(\mathcal{H} \oplus \mathcal{K})$ such that

$$
\tilde{\pi}(a)(h \oplus 0) = (\pi(a)h) \oplus 0 \quad (a \in A, h \in \mathcal{H}).
$$

Proof. Since in general $\pi$ is a direct sum of cyclic representations, we may without loss of generality assume that $\pi$ is a cyclic representation with cyclic vector $\xi$. Let $\phi$ be the vector state $\phi(\cdot) = \langle \pi(\cdot)\xi, \xi \rangle$ of $A$. Then $\mathcal{H}$ is identified with $L^2(\mathcal{A}, \phi)$ and $\pi$ is the associated GNS representation. Let $\tilde{\phi}$ be an extension of $\phi$ to a state of $\tilde{A}$, and let $\tilde{\mathcal{H}} = L^2(\tilde{\mathcal{A}}, \tilde{\phi})$. Then the inclusion $A \hookrightarrow \tilde{A}$ gives rise to an isometry $\mathcal{H} \to \tilde{\mathcal{H}}$, and we may thus write $\tilde{\mathcal{H}} = \mathcal{H} \oplus \mathcal{K}$ for a Hilbert space $\mathcal{K}$. If $\tilde{\pi} : \tilde{A} \to B(\mathcal{H} \oplus \mathcal{K})$ is the GNS representation associated to $\tilde{\phi}$, then (2) holds. \qed

The following result was first proved by G. K. Pedersen [10, Thm. 4.2]. We offer a new proof, which is perhaps more elementary. This proof contains essentially the same idea as our proof of Proposition 2.4 below.
Proposition 2.2. Let 
\[ \tilde{\mathcal{A}} \supseteq \mathcal{A} \supseteq \mathcal{D} \subseteq \mathcal{B} \subseteq \mathcal{\tilde{B}} \]
be inclusions of \( \mathcal{C}^*\)-algebras and let \( \mathcal{A} \ast_\mathcal{D} \mathcal{B} \) and \( \tilde{\mathcal{A}} \ast_\mathcal{D} \mathcal{\tilde{B}} \) be the corresponding full amalgamated free product \( \mathcal{C}^*\)-algebras. Let \( \lambda : \mathcal{A} \ast_\mathcal{D} \mathcal{B} \to \tilde{\mathcal{A}} \ast_\mathcal{D} \mathcal{\tilde{B}} \) be the \(*\)-homomorphism arising via the universal property from the inclusions \( \mathcal{A} \hookrightarrow \tilde{\mathcal{A}} \) and \( \mathcal{B} \hookrightarrow \mathcal{\tilde{B}} \). Then \( \lambda \) is injective.

**Proof.** Let \( \pi : \mathcal{A} \ast_\mathcal{D} \mathcal{B} \to \mathcal{B}(\mathcal{H}) \) be a faithful \(*\)-homomorphism. We will find a Hilbert space \( \mathcal{K} \) and a \(*\)-homomorphism \( \tilde{\pi} : \mathcal{A} \ast_\mathcal{D} \mathcal{B} \to \mathcal{B}(\mathcal{K} \oplus \mathcal{K}) \) such that
\[ \tilde{\pi}(\lambda(x))(h \oplus 0) = (\pi(x)h) \oplus 0 \quad (x \in \mathcal{A} \ast_\mathcal{D} \mathcal{B}, h \in \mathcal{K}). \]
This will imply that \( \lambda \) is injective.

Let \( \pi_A : \mathcal{A} \to \mathcal{B}(\mathcal{K}) \) and \( \pi_B : \mathcal{B} \to \mathcal{B}(\mathcal{K}) \) be the \(*\)-representations obtained by composing \( \pi \) with the inclusions \( \mathcal{A} \hookrightarrow \mathcal{A} \ast_\mathcal{D} \mathcal{B} \) and \( \mathcal{B} \hookrightarrow \mathcal{A} \ast_\mathcal{D} \mathcal{B} \). Let
\[ \sigma_{A,0} : \mathcal{A} \to \mathcal{B}(\mathcal{K} \oplus \mathcal{K}_{A,0}), \]
\[ \sigma_{B,0} : \mathcal{B} \to \mathcal{B}(\mathcal{K} \oplus \mathcal{K}_{B,0}) \]
be \(*\)-representations obtained from Lemma 2.1 such that
\[ \sigma_{A,0}(a)(h \oplus 0) = (\pi_A(a)h) \oplus 0 \quad (a \in \mathcal{A}, h \in \mathcal{K}), \]
and similarly with \( \mathcal{A} \) replaced by \( \mathcal{B} \). Note that \( 0 \oplus \mathcal{K}_{A,0} \) is reducing for \( \sigma_{A,0}(\mathcal{D}) \).

Using Lemma 2.1 we find Hilbert spaces \( \mathcal{K}_{B,1} \) and \( \mathcal{K}_{A,1} \) and \(*\)-representations
\[ \sigma_{B,1} : \mathcal{B} \to \mathcal{B}(\mathcal{K}_{A,0} \oplus \mathcal{K}_{B,1}), \]
\[ \sigma_{A,1} : \mathcal{A} \to \mathcal{B}(\mathcal{K}_{B,0} \oplus \mathcal{K}_{A,1}) \]
such that
\[ \sigma_{B,1}(d)(k \oplus 0) = \sigma_{A,0}(d)(0 \oplus k) \quad (d \in \mathcal{D}, k \in \mathcal{K}_{A,0}), \]
\[ \sigma_{A,1}(d)(k \oplus 0) = \sigma_{B,0}(d)(0 \oplus k) \quad (d \in \mathcal{D}, k \in \mathcal{K}_{B,0}). \]
Proceeding recursively, for every integer \( n \geq 2 \) we find \(*\)-representations
\[ \sigma_{B,n} : \mathcal{B} \to \mathcal{B}(\mathcal{K}_{A,n-1} \oplus \mathcal{K}_{B,n}), \]
\[ \sigma_{A,n} : \mathcal{A} \to \mathcal{B}(\mathcal{K}_{B,n-1} \oplus \mathcal{K}_{A,n}) \]
such that
\[ \sigma_{B,n}(d)(k \oplus 0) = \sigma_{A,n-1}(d)(0 \oplus k) \quad (d \in \mathcal{D}, k \in \mathcal{K}_{A,n-1}), \]
\[ \sigma_{A,n}(d)(k \oplus 0) = \sigma_{B,n-1}(d)(0 \oplus k) \quad (d \in \mathcal{D}, k \in \mathcal{K}_{B,n-1}). \]

We now define the Hilbert spaces
\[ \mathcal{\tilde{H}}_A = \mathcal{K} \oplus \mathcal{K}_{A,0} \oplus \mathcal{K}_{B,0} \oplus \mathcal{K}_{A,1} \oplus \mathcal{K}_{B,1} \oplus \mathcal{K}_{A,2} \oplus \cdots, \]
\[ \mathcal{\tilde{H}}_B = \mathcal{K} \oplus \mathcal{K}_{B,0} \oplus \mathcal{K}_{A,0} \oplus \mathcal{K}_{B,1} \oplus \mathcal{K}_{A,1} \oplus \mathcal{K}_{B,2} \oplus \cdots, \]
where the brackets indicate where the constructed representations act, and we let \( \sigma_{\tilde{A}} : \mathcal{A} \to \mathcal{B}(\mathcal{\tilde{H}}_A) \) and \( \sigma_{\tilde{B}} : \mathcal{B} \to \mathcal{B}(\mathcal{\tilde{H}}_B) \) be the \(*\)-representations
\[ \sigma_{\tilde{A}} = \sigma_{A,0} \oplus \sigma_{A,1} \oplus \sigma_{A,2} \oplus \cdots, \]
\[ \sigma_{\tilde{B}} = \sigma_{B,0} \oplus \sigma_{B,1} \oplus \sigma_{B,2} \oplus \cdots, \]
where the summands act as indicated by brackets in (3). Consider the unitary $U : \mathcal{H}_A \to \mathcal{H}_B$ mapping the summands in $\mathcal{H}_A$ identically to the corresponding summands in $\mathcal{H}_B$ as indicated by the arrows below:

$$\begin{align*}
\mathcal{H}_A &= \mathcal{H} \oplus \mathcal{K}_{A,0} \oplus \mathcal{K}_{B,0} \oplus \mathcal{K}_{A,1} \oplus \mathcal{K}_{B,1} \oplus \mathcal{K}_{A,2} \oplus \cdots \\
\mathcal{H}_B &= \mathcal{H} \oplus \mathcal{K}_{B,0} \oplus \mathcal{K}_{A,0} \oplus \mathcal{K}_{B,1} \oplus \mathcal{K}_{A,1} \oplus \mathcal{K}_{B,2} \oplus \cdots.
\end{align*}$$

Let $\mathcal{K} = \mathcal{K}_{A,0} \oplus \mathcal{K}_{B,0} \oplus \mathcal{K}_{A,1} \oplus \mathcal{K}_{B,1} \oplus \cdots$ and identify $\mathcal{H} \oplus \mathcal{K}$ with $\mathcal{H}_A$. Then we have the $*$-representations $\tilde{\pi}_A = \sigma_A : A \to B(\mathcal{H} \oplus \mathcal{K})$ and $\tilde{\pi}_B : B \to B(\mathcal{H} \oplus \mathcal{K})$, the latter defined by $\tilde{\pi}_B(\cdot) = U^* \sigma_B(\cdot) U$. By construction, the restrictions of $\tilde{\pi}_A$ and $\tilde{\pi}_B$ to $D$ agree, and we have

$$\begin{align*}
\tilde{\pi}_A(a)(h \oplus 0) &= (\pi_A(a)h) \oplus 0 \quad (a \in A, h \in \mathcal{H}), \\
\tilde{\pi}_B(b)(h \oplus 0) &= (\pi_B(b)h) \oplus 0 \quad (b \in B, h \in \mathcal{H}).
\end{align*}$$

Letting $\tilde{\pi} : \tilde{A} \ast_D \tilde{B} \to B(\mathcal{H} \oplus \mathcal{K})$ be the $*$-homomorphism obtained from $\tilde{\pi}_A$ and $\tilde{\pi}_B$ via the universal property, we have that (3) holds.

For a C*-algebra $A$, unital or not, let $A^u$ denote the unitization of $A$. Thus, as a vector space, $A^u = A \oplus \mathbb{C}$ with multiplication defined by $(a, \mu) \cdot (a', \mu') = (aa' + \mu a + \mu'a, \mu \mu')$. We identify $A$ with the ideal $A \oplus 0$ of $A^u$, which has codimension 1.

**Lemma 2.3.** Let $A \supseteq D \subseteq B$ be inclusions of C*-algebras. Consider the unitizations and corresponding inclusions

$$\begin{align*}
A^u &\hookrightarrow D^u \twoheadrightarrow B^u \\
A &\hookrightarrow D \twoheadrightarrow B.
\end{align*}$$

Let $\lambda : A \ast_D B \to A^u \ast_D B^u$ be the resulting $*$-homomorphism between full amalgamated free products. Then there is an isomorphism $\pi : A^u \ast_D B^u \to (A \ast_D B)^u$ such that $\pi \circ \lambda : A \ast_D B \to (A \ast_D B)^u$ is the canonical embedding arising in the definition of the unitization.

**Proof.** Since any $*$-representations of $A$ and $B$ that agree on $D$ extend to $*$-representations of $A^u$ and $B^u$ that agree on $D^u$, the $*$-homomorphism $\lambda$ is injective. Let $e \in A^u \ast_D B^u$ be the unit of $A^u$, which is of course identified with the units of $B^u$ and $D^u$. Clearly, $A^u \ast_D B^u$ is generated by the image of $\lambda$ together with $e$. One easily sees that

$$(\lambda(x) + \mu e)(\lambda(x') + \mu' e) = \lambda(xx') + \mu \lambda(x') + \mu' \lambda(x) + \mu \mu' e.$$  

Moreover, if $\rho : A^u \ast_D B^u \to \mathbb{C}$ is the $*$-homomorphism arising from the unital $*$-homomorphisms $A^u \to \mathbb{C}$ and $B^u \to \mathbb{C}$, then $\rho(e) = 1$ and $\lambda(A \ast_D B) \subseteq \ker \rho$. Hence $\lambda(A \ast_D B)$ has codimension 1 in $A^u \ast_D B^u$. Now $\pi$ can be defined by $\pi(\lambda(x) + \mu e) = (x, \mu)$. $\square$
Proposition 2.4. Suppose

\[ \begin{array}{ccc}
A & \xrightarrow{\pi} & B \\
\downarrow & & \downarrow \\
D & \xrightarrow{\pi_D} & E
\end{array} \]

is a commuting diagram of inclusions of C*-algebras. Let \( \lambda : A * D B \to \tilde{A} * D \tilde{B} \) be the resulting *-homomorphism of full free product C*-algebras. Suppose there are conditional expectations \( E_A : \tilde{A} \to A \), \( E_D : \tilde{D} \to D \) and \( E_B : \tilde{B} \to B \) onto \( A \), \( D \) and \( B \), respectively, such that the diagram

\[ \begin{array}{ccc}
\tilde{A} & \xleftarrow{\pi} & \tilde{D} \\
\downarrow & & \downarrow \\
A & \xleftarrow{\pi_A} & D \\
\downarrow & & \downarrow \\
E_A & \xrightarrow{E_D} & E_B \\
\end{array} \]

commutes. Then \( \lambda \) is injective.

Proof. By appealing to Lemma 2.3, we may without loss of generality assume that all the algebras and *-homomorphisms in \( (5) \) are unital. Let \( \tilde{\pi} : \tilde{A} * D \tilde{B} \to B(\mathcal{H}) \) be a faithful, unital *-representation. As in the proof of Proposition 2.2, we will find a Hilbert space \( \mathcal{K} \) and a *-homomorphism \( \tilde{\pi} : \tilde{A} * D \tilde{B} \to B(\mathcal{K} \oplus \mathcal{K}) \) such that

\[ \tilde{\pi}(\lambda(x))(h \oplus 0) = (\pi(x)h) \oplus 0 \quad (x \in A * D B, h \in \mathcal{H}). \]

Let \( \pi_A : A \to B(\mathcal{H}) \) and \( \pi_B : B \to B(\mathcal{K}) \) be the *-representations obtained by composing \( \pi \) with the embeddings \( A \hookrightarrow A * D B \) and \( B \hookrightarrow A * D B \), and let \( \pi_D : D \to B(\mathcal{K}) \) be their common restriction to \( D \). Consider the canonical left action of \( \tilde{D} \) on the right Hilbert \( D \)-module \( L^2(\tilde{D}, E_D) \), which is obtained from \( \tilde{D} \) by separation and completion with respect to the \( D \)-valued inner product \( \langle d_1, d_2 \rangle = E_D(d_1^* d_2) \). Consider the Hilbert space \( L^2(\tilde{D}, E_D) \otimes_D \mathcal{K} \), where the left action of \( D \) on \( \mathcal{K} \) is via \( \pi_D \). Since \( \pi_D \) is unital, \( \mathcal{K} \) embeds as a subspace, and we can write

\[ L^2(\tilde{D}, E_D) \otimes_D \mathcal{K} = \mathcal{K} \oplus \mathcal{K}_D. \]

Consider the left action of \( \tilde{D} \) on the Hilbert space \( \mathcal{K} \oplus \mathcal{K}_D \). The subspace \( \mathcal{K} \) is reducing for the restriction of \( \sigma_D \) to \( D \), and we have \( \sigma_D(d)(h \oplus 0) = (\pi_D(d)h) \oplus 0 \) for every \( d \in D \) and \( h \in \mathcal{K} \).

In a similar way, consider the Hilbert spaces

\[ L^2(\tilde{A}, E_A) \otimes_A \mathcal{K}, \quad L^2(\tilde{B}, E_B) \otimes_B \mathcal{K} \]

and the associated left actions \( \sigma_{A,0} \) of \( \tilde{A} \), respectively \( \sigma_{B,0} \) of \( \tilde{B} \). Since the diagram \( (5) \) commutes, the Hilbert space \( (6) \) embeds canonically as a subspace of both spaces \( (7) \). We may thus write

\[ L^2(\tilde{A}, E_A) \otimes_A \mathcal{K} = \mathcal{K} \oplus \mathcal{K}_A \oplus \mathcal{K}_{A,0}, \]

\[ L^2(\tilde{B}, E_B) \otimes_B \mathcal{K} = \mathcal{K} \oplus \mathcal{K}_D \oplus \mathcal{K}_{B,0}, \]

the subspace \( \mathcal{K} \oplus \mathcal{K}_D \oplus 0 \) is reducing for the restrictions of \( \sigma_{A,0} \) and \( \sigma_{B,0} \) to \( \tilde{D} \), and we have \( \sigma_{A,0}(d)(\eta \oplus 0) = (\sigma_D(d)\eta) \oplus 0 = \sigma_{B,0}(d)(\eta \oplus 0) \) for every \( d \in \tilde{D} \) and
Let \( \sigma_{A,0} \) denote the action of \( \tilde{D} \) on \( \mathcal{K}_{A,0} \) obtained by restricting \( \sigma_{A,0} \) to \( \tilde{D} \) and compressing, and similarly for \( \sigma_{B,0,\tilde{D}} \).

We now proceed recursively as in the proof of Proposition 2.2. If Hilbert spaces \( \mathcal{K}_{A,-1} \) and \( \mathcal{K}_{B,-1} \) have been constructed with actions \( \sigma_{A,-1,\tilde{D}} \) and \( \sigma_{B,-1,\tilde{D}} \), respectively, of \( \tilde{D} \), then use Lemma 2.1 to construct Hilbert spaces \( \mathcal{K}_{B,n} \) and \( \mathcal{K}_{A,n} \) and *-homomorphisms

\[
\sigma_{B,n} : \tilde{B} \to B(\mathcal{K}_{A,n-1} \oplus \mathcal{K}_{B,n}),
\]

\[
\sigma_{A,n} : \tilde{A} \to B(\mathcal{K}_{B,n-1} \oplus \mathcal{K}_{A,n}),
\]

such that

\[
\sigma_{B,n}(\tilde{d})(k \oplus 0) = (\sigma_{A,n-1,\tilde{D}})(\tilde{d})k \oplus 0 \quad (\tilde{d} \in \tilde{D}, k \in \mathcal{K}_{A,n-1}),
\]

\[
\sigma_{A,n}(\tilde{d})(k \oplus 0) = (\sigma_{B,n-1,\tilde{D}})(\tilde{d})k \oplus 0 \quad (\tilde{d} \in \tilde{D}, k \in \mathcal{K}_{B,n-1}).
\]

Then let \( \sigma_{B,n,\tilde{D}} \) be the action of \( \tilde{D} \) on \( \mathcal{K}_{B,n} \) obtained from the restriction of \( \sigma_{B,n} \) to \( \tilde{D} \) by compressing, and similarly define the action \( \sigma_{A,n,\tilde{D}} \) of \( \tilde{D} \) on \( \mathcal{K}_{A,n} \).

We may now define the Hilbert spaces

\[
\tilde{\mathcal{H}}_A = \mathcal{H} \oplus \mathcal{K}_D \oplus \mathcal{K}_{A,0} \oplus \mathcal{K}_{B,0} \oplus \mathcal{K}_{A,1} \oplus \mathcal{K}_{B,1} \oplus \mathcal{K}_{A,2} \oplus \cdots,
\]

\[
\tilde{\mathcal{H}}_B = \mathcal{H} \oplus \mathcal{K}_D \oplus \mathcal{K}_{B,0} \oplus \mathcal{K}_{A,0} \oplus \mathcal{K}_{A,1} \oplus \mathcal{K}_{B,1} \oplus \mathcal{K}_{B,2} \oplus \cdots,
\]

where the brackets indicate where the constructed representations act. We let \( \sigma_A : \tilde{A} \to B(\tilde{\mathcal{H}}_A) \) and \( \sigma_B : \tilde{B} \to B(\tilde{\mathcal{H}}_B) \) be the *-representations

\[
\sigma_A = \sigma_{A,0} \oplus \sigma_{A,1} \oplus \sigma_{A,2} \oplus \cdots,
\]

\[
\sigma_B = \sigma_{B,0} \oplus \sigma_{B,1} \oplus \sigma_{B,2} \oplus \cdots,
\]

where the summands act as indicated by brackets in (10). Consider the unitary \( U : \tilde{\mathcal{H}}_A \to \tilde{\mathcal{H}}_B \) mapping the summands in \( \tilde{\mathcal{H}}_A \) identically to the corresponding summands in \( \tilde{\mathcal{H}}_B \) as indicated by the arrows below:

\[
U
\]

\[
\tilde{\mathcal{H}}_A = \mathcal{H} \oplus \mathcal{K}_D \oplus \mathcal{K}_{A,0} \oplus \mathcal{K}_{B,0} \oplus \mathcal{K}_{A,1} \oplus \mathcal{K}_{B,1} \oplus \mathcal{K}_{A,2} \oplus \cdots
\]

\[
\tilde{\mathcal{H}}_B = \mathcal{H} \oplus \mathcal{K}_D \oplus \mathcal{K}_{B,0} \oplus \mathcal{K}_{A,0} \oplus \mathcal{K}_{B,1} \oplus \mathcal{K}_{A,1} \oplus \mathcal{K}_{B,2} \oplus \cdots
\]

Let \( \mathcal{H} = \mathcal{K}_D \oplus \mathcal{K}_{A,0} \oplus \mathcal{K}_{B,0} \oplus \mathcal{K}_{A,1} \oplus \mathcal{K}_{B,1} \oplus \cdots \) and identify \( \mathcal{H} \oplus \mathcal{K} \) with \( \tilde{\mathcal{H}}_A \). Then we have the *-representations \( \tilde{\pi}_A = \sigma_A : \tilde{A} \to B(\mathcal{H} \oplus \mathcal{K}) \) and \( \tilde{\pi}_B : \tilde{B} \to B(\mathcal{H} \oplus \mathcal{K}) \),
of Question 1.1 fails to hold in general. Notice that in these examples, $B \cap D = B \supsetneq D$.

**Proposition 3.2.** Suppose

\[
\begin{array}{ccc}
A & \rightarrow & B \\
\downarrow & & \downarrow \\
\tilde{A} & \rightarrow & \tilde{B}
\end{array}
\]

is a commutative diagram of inclusions of $C^*$-algebras, and let $\lambda : A*D \rightarrow \tilde{A}*D\tilde{B}$ be the resulting $*$-homomorphism of full free product $C^*$-algebras. Suppose there are conditional expectations $E^A_D : A \rightarrow D$ and $E^B_D : B \rightarrow D$ with $E^B_D$ faithful. Suppose there are $d \in D$, $a \in A$ and $b \in B$ satisfying $ad \in A$, $db \in B$,

\begin{align}
(11) & \quad D(db) \cap Db = \{0\}, \\
(12) & \quad E^A_D(d^*a^*ad)b \neq 0.
\end{align}

Then $\lambda$ is not injective.

**Proof.** Letting

\begin{align}
\sigma_A : A & \rightarrow A*D, \\
\sigma_B : B & \rightarrow A*D, \\
\sigma_{\tilde{A}} : \tilde{A} & \rightarrow \tilde{A}*D\tilde{B}, \\
\sigma_{\tilde{B}} : \tilde{B} & \rightarrow \tilde{A}*D\tilde{B}
\end{align}

be the embeddings as in (11), we have

\[
\lambda(\sigma_A(\tilde{a})\sigma_B(b)) = \sigma_{\tilde{A}}(\tilde{a})\sigma_B(b) = \sigma_{\tilde{A}}(a)\sigma_{\tilde{B}}(db) = \lambda(\sigma_A(a)\sigma_B(db)).
\]

Thus we need only show that

\[
\sigma_A(ad)\sigma_B(b) \neq \sigma_A(a)\sigma_B(db).
\]

We consider the reduced amalgamated free product of $C^*$-algebras (see [11] or [12]),

\[
(A*D^{\text{red}} B, E_D) = (A, E^A_D) * (B, E^B_D)
\]
Corollary 3.4. Suppose we can reduce to the case in which (ii) holds by application of Lemma 2.3. We may without loss of generality assume 

and the natural quotient ∗–homomorphism $A *_{D} B \rightarrow A *_{D}^{\text{red}} B$. Let $L^2(A *_{D}^{\text{red}} B, E_D)$ be the right Hilbert $D$–module obtained by separation and completion from $A *_{D}^{\text{red}} B$ with respect to the $D$–valued inner product $(x, y) = E_D(x^* y)$, and given $x \in A *_{D}^{\text{red}} B$, let $\hat{x}$ denote the corresponding element in $L^2(A *_{D}^{\text{red}} B, E_D)$. Let $\mathcal{H}_A = L^2(A, E_D^A)$ and $\mathcal{H}_B = L^2(B, E_D^B)$ be similarly defined. Then in $L^2(A *_{D}^{\text{red}} B, E_D)$, the closure of the subspace spanned by elements of the form $(ab)^*$ for $a \in A$ and $b \in B$ is isomorphic to the tensor product $\mathcal{H}_A \otimes_D \mathcal{H}_B$ of Hilbert $D$–modules. In order to show \[ (14) \], it will suffice to show

\[(ad^*) \otimes \hat{b} \neq \hat{a} \otimes (\hat{db})^* \]

in $\mathcal{H}_A \otimes_D \mathcal{H}_B$. Let $\zeta_B \in \mathcal{H}_B$. Then

\[
\langle (ad^*) \otimes \zeta_B, (ad^*) \otimes \hat{b} \rangle = \langle \zeta_B, (E_D^A(\hat{d^*} a^* ad)b^*) \rangle,
\]

\[
\langle (ad^*) \otimes \zeta_B, \hat{a} \otimes (\hat{db})^* \rangle = \langle \zeta_B, (E_D^A(\hat{d^*} a^* a)\hat{d})^* \rangle.
\]

From assumptions \[ (11) \] and \[ (12) \], we obtain $E_D^B(\hat{d} a^* a)\hat{d} b \neq E_D^B(\hat{d} a^* ad)\hat{d} b$. Since $E_D^B$ is faithful, there is $\zeta_B \in \mathcal{H}_B$ such that the right-hand sides of \[ (15) \] and \[ (16) \] are not equal.

Remark 3.3. From the above proof, one sees that the hypotheses of Proposition 3.2 can be weakened as follows: Assumptions \[ (11) \] and \[ (12) \] can be dropped, and $E_D^B$ need not be assumed faithful, but instead one must assume

\[
E_D^B(b^* (E_D^A(\hat{d}^* a^* ad) - E_D^A(\hat{d} a^* a)\hat{d} - \hat{d}^* E_D^A(a^* ad)) + \hat{d}^* E_D^A(a^* a)\hat{d}) b^*) \neq 0.
\]

Note that the LHS of \[ (17) \] is nothing other than

\[
\langle (ad^*) \otimes \hat{b} - \hat{a} \otimes (\hat{db})^* \rangle, (ad^*) \otimes \hat{b} - \hat{a} \otimes (\hat{db})^* \rangle.
\]

Corollary 3.4. Suppose

\[
(18) \quad \begin{array}{ccc}
\tilde{A} & \leftrightarrow & \tilde{D} \\
\downarrow & & \downarrow \\
A & \leftrightarrow & B
\end{array}
\]

is a commutative diagram of inclusions of $C^*$–algebras and let $\lambda : A *_{D} B \rightarrow \tilde{A} *_{D} \tilde{B}$ be the resulting ∗–homomorphism of full free product $C^*$–algebras. Suppose one of the following holds:

\[(1) \quad D = 0,
\]

\[(2) \quad D = \mathbb{C}, \text{ A and B are unital and the inclusions } D \hookrightarrow A \text{ and } D \hookrightarrow B \text{ are unital}.
\]

Suppose there are $\tilde{a} \in \tilde{D}$, $a \in A$ and $b \in B$ such that $a\tilde{a} \in A \setminus \{0\}$, $\tilde{d}b \in B$ and $\tilde{d}b \notin \mathbb{C}b$. Then $\lambda$ is not injective.

Proof. We can reduce to the case in which (ii) holds by application of Lemma 2.26. We may without loss of generality assume $A$ and $B$ are separable. Letting $E_D^A : A \rightarrow \mathbb{C}$ and $E_D^B : B \rightarrow \mathbb{C}$ be faithful states, we find that the hypotheses of Proposition 3.2 are satisfied.

From this corollary, we have the following class of concrete examples, which shows that $\lambda$ may be non-injective even if

\[
(19) \quad B \cap D = D = A \cap \tilde{D}.
\]
Example 3.5. Let $\mathcal{H}$ be an infinite-dimensional, separable Hilbert space. Inside $B(\mathcal{H})$, let $D = C_1$ and let $A = B = D + K(\mathcal{H})$, where $K(\mathcal{H})$ is the set of compact operators. Let $u \in B(\mathcal{H})$ be a unitary operator that does not belong to $D$, and let $\tilde{D} = C^*(u)$, $\tilde{A} = \tilde{B} = \tilde{D} + K(\mathcal{H})$. Let $\lambda : A * D B \to \tilde{A} * D \tilde{B}$ be the $*$-homomorphism arising from the inclusions. Then $\lambda$ is not injective.

Proof. Take $\tilde{a} = u \in A$ and $a \in K(\mathcal{H}) \setminus \{0\}$. Since $u \notin C_1$, there is $b \in K(\mathcal{H})$ such that $ub \notin C_b$. Now apply Corollary [13, 4] One can choose $u$ so that $C^*(u) \cap (C_1 + K(\mathcal{H})) = C_1$, in order to get [19].

Proposition 3.6. Suppose

$$\begin{array}{cccc}
A & \xrightarrow{\sigma_A} & \tilde{D} & \xrightarrow{\sigma_{\tilde{D}}} & \tilde{B} \\
\uparrow & & \uparrow & & \uparrow \\
A & \xleftarrow{\sigma_{\tilde{A}}} & \tilde{D} & \xleftarrow{\sigma_D} & B
\end{array}$$

is a commutative diagram of inclusions of $C^*$-algebras, and let $\lambda : A * D B \to \tilde{A} * D \tilde{B}$ be the resulting $*$-homomorphism of full free product $C^*$-algebras. Suppose one of the following holds:

1. $D = 0$,
2. $D = C$, $A$ and $B$ are unital and the inclusions $D \hookrightarrow A$ and $D \hookrightarrow B$ are unital.

Suppose there are $\tilde{a} \in \tilde{D}$, $a_1, a_2 \in A$ and $b \in B \setminus D$ such that $a_1 \tilde{d}, a_2 \in A$, $a_1 \tilde{d} \notin C$ and $\tilde{d}b = bd\tilde{d}$. Then $\lambda$ is not injective.

Proof. We can reduce to the case in which (ii) holds by application of Lemma [23]. We use the same notation as in [13]. We have

$$\lambda(\sigma_A(a_1 \tilde{d})\sigma_B(b)\sigma_A(a_2)) = \sigma_{\tilde{A}}(a_1 \tilde{d})\sigma_{\tilde{B}}(b)\sigma_{\tilde{A}}(a_2)$$

$$= \sigma_{\tilde{A}}(a_1)\sigma_{\tilde{B}}(b)\sigma_{\tilde{A}}(\tilde{d}a_2) = \lambda(\sigma_A(a_1)\sigma_B(b)\sigma_A(\tilde{d}a_2)),$$

and we must only show

$$\sigma_A(a_1 \tilde{d})\sigma_B(b)\sigma_A(a_2) \neq \sigma_A(a_1)\sigma_B(b)\sigma_A(\tilde{d}a_2).$$

Without loss of generality, assume $A$ and $B$ are separable. Let $\phi_A : A \to C$ and $\phi_B : B \to C$ be faithful states. By adding a scalar multiple of the identity, if necessary, we may without loss of generality assume $\phi_B(b) = 0$. Let

$$(A *_{C} B, \phi) = (A, \phi_A) *_{C} (B, \phi_B)$$

be the reduced free product of $C^*$-algebras. Using arguments and notation as in the proof of Proposition [22] the closure of the subspace of $L^2(A *_{C} B, \phi)$ spanned by elements of the form $(a_1 a')^\circ$ for $a, a' \in A$ is isomorphic to $\mathcal{H}_A \otimes (C_\tilde{b}) \otimes \mathcal{H}_A$. To show [21], it will suffice to show

$$(a_1 \tilde{d})^\circ \otimes \tilde{b} \otimes \tilde{a}_2 \neq \tilde{a}_1 \otimes \tilde{b} \otimes (\tilde{d}a_2)^\circ$$

in $\mathcal{H}_A \otimes (C_\tilde{b}) \otimes \mathcal{H}_A$. However, this follows from the assumptions.

From the above proposition, we get the following example, which requires only “bad” relations between $A$ and $\tilde{D}$, not between $B$ and $\tilde{D}$.
Example 3.7. Let \( D, \tilde{D}, A \) and \( \tilde{A} \) be as in Example 3.5. Let \( B \) be any unital C*-algebra of dimension greater than 1, and let \( \tilde{B} = B \otimes \tilde{D} \) (for the unique C*-tensor norm). Then the \(*\)-homomorphism \( \lambda : A *_D B \rightarrow \tilde{A} *_{\tilde{D}} \tilde{B} \) arising from the inclusions (20) is not injective.

Remark 3.8. The problem with injectivity of \( \lambda \) in Examples 3.5 and 3.7 arises already at the algebraic level

\[
A *_{\text{alg}}^D B \rightarrow \tilde{A} *_{\text{alg}}^{\tilde{D}} \tilde{B}.
\]

On the other hand, in Examples 3.1, we can arrange that the map between algebras (22) is injective, while \( \lambda \) fails to be injective, e.g. by taking \( E \) to be a reduced free product. However, we do not know of an example where \( \lambda \) fails to be injective and where the algebraic map (22) is injective, but where \( A \cap \tilde{D} = D = B \cap \tilde{D} \).

4. An application to residual finite-dimensionality

A C*-algebra is said to be residually finite dimensional (r.f.d.) if it has a separating family of finite-dimensional \(*\)-representations. The first result linking full free products and residual finite dimensionality was M.-D. Choi’s proof [6] that the full group C*-algebras of nonabelian free groups are r.f.d. In [7], Exel and Loring proved that the full free product of any two r.f.d. C*-algebras \( A \) and \( B \) with amalgamation over either the zero C*-algebra or over the scalar multiples of the identity (if \( A \) and \( B \) are unital) is r.f.d. In [5], N. Brown and Dykema proved that a full amalgamated free product of matrix algebras \( M_k(C) *_D M_l(C) \) over a unital subalgebra \( D \) is r.f.d. provided that the normalized traces on \( M_k(C) \) and \( M_l(C) \) restrict to the same trace on \( D \). In this section, we observe that by applying Proposition 2.2, one obtains (as a corollary of the result from [5]) the analogous result for full amalgamated free products of finite-dimensional algebras.

Lemma 4.1. Let \( S = \{ x \in \mathbb{R}^n \mid Ax = 0 \} \), where \( A \) is an \( m \times n \) matrix having only rational entries. Then vectors having only rational entries are dense in \( S \).

Proof. By considering the reduced row-echelon form of \( A \), we see that there is a basis for \( S \) consisting of rational vectors.

Theorem 4.2. Consider unital inclusions of C*-algebras \( A \supseteq D \subseteq B \) with \( A \) and \( B \) finite dimensional. Let \( A *_D B \) be the corresponding full amalgamated free product. Then \( A *_D B \) is residually finite dimensional if and only if there are faithful tracial states \( \tau_A \) on \( A \) and \( \tau_B \) on \( B \) whose restrictions to \( D \) agree.

Proof. Since every separable r.f.d. C*-algebra has a faithful tracial state, the necessity of the existence of \( \tau_A \) and \( \tau_B \) is clear.

Let us recall some well-known facts about a unital inclusion \( D \subseteq A \) of finite-dimensional C*-algebras (see e.g. Chapter 2 of [8]). Let \( p_1, \ldots, p_m \) be the minimal central projections of \( A \) and \( q_1, \ldots, q_n \) the minimal central projections of \( D \). Then the inclusion matrix \( A_D^A \) is an \( m \times n \) integer matrix whose \((i,j)\)th entry is \( \text{rank}(q_i p_j A q_j) / \text{rank}(q_i D) \), where the rank of a matrix algebra \( M_k(C) \) is \( k \). To a trace \( \tau \) on \( A \), we associate the column vector \( s \) of length \( m \) whose \( i \)th entry is the
trace of a minimal projection in $p_i A$. Then the restriction of $\tau$ to $D$ has associated column vector $(A^*_D)^{}s$, where the superscript $t$ indicates transpose.

Thus, given $A \supseteq D \subseteq B$ as in the statement of the theorem, the existence of faithful tracial states $\tau_A$ and $\tau_B$ agreeing on $D$ is equivalent to the existence of column vectors $s_A$ and $s_B$, none of whose components are zero, such that $(A_D^*)^{}s_A = (B_D^*)^{}s_B$, i.e.,

$$\begin{bmatrix} (A_D^*)^{} \\ -(B_D^*)^{} & \end{bmatrix}^{} \begin{bmatrix} s_A \\ s_B \end{bmatrix} = 0. \quad \tag{23}$$

Supposing now that such traces $\tau_A$ and $\tau_B$ exist, by Lemma 4.1 there is a solution $[s_A^*]$ to (23) whose entries are all strictly positive and rational. Therefore, the traces $\tau_A$ and $\tau_B$ agreeing on $D$ can be chosen to take only rational values on minimal projections of $A$ and, respectively, $B$. Hence there are unital inclusions into matrix algebras,

$$M_k(C) \supseteq A \supseteq D \subseteq B \subseteq M_\ell(C),$$

so that $\tau_A$ is the restriction of the tracial state on $M_k(C)$ to $A$ and $\tau_B$ is the restriction of the tracial state on $M_\ell(C)$ to $B$. By Proposition 2.2, $A *_D B$ is a subalgebra of $M_k(C) *_D M_\ell(C)$. By Theorem 2.3 of [5], $M_k(C) *_D M_\ell(C)$ is r.f.d. Therefore, $A *_D B$ is r.f.d. \hfill $\Box$

**References**

[1] B. Blackadar, *Weak expectations and nuclear C*-algebras*, Indiana Univ. Math. J. 27 (1978), 1021-1026. MR 80d:46110

[2] E. Blanchard and K. Dykema, *Embeddings of reduced free products of operator algebras*, Pacific J. Math. 199 (2001), 1-19. MR 2002f:46115

[3] D. Blecher and V. Paulsen, *Explicit construction of universal operator algebras and applications to polynomial factorization*, Proc. Amer. Math. Soc. 112 (1991), 839-850. MR 91j:46093

[4] F. Boca, *Free products of completely positive maps and spectral sets*, J. Funct. Anal. 97 (1991), 251-263. MR 92f:46064

[5] N. P. Brown and K. Dykema, *Popa algebras in free group factors*, J. reine angew. Math., to appear.

[6] M.-D. Choi, *The full C*-algebra of the free group on two generators*, Pacific J. Math. 87 (1980), 41-48. MR 82b:46069

[7] R. Exel and T. Loring, *Finite-dimensional representations of free product C*-algebras*, Internat. J. Math. 3 (1992), 469-476. MR 93f:46091

[8] F. M. Goodman, P. de la Harpe and V.F.R. Jones, *Coxeter graphs and towers of algebras*, Mathematical Sciences Research Institute Publications, vol. 14, Springer-Verlag, New York, 1989. MR 91c:46082

[9] T. Loring, *Lifting solutions to perturbing problems in C*-algebras*, Fields Institute Monographs, vol. 8, American Mathematical Society, Providence, RI, 1997. MR 98a:46090

[10] G. K. Pedersen, *Pullback and pushout constructions in C*-algebra theory*, J. Funct. Anal. 167 (1999), 243-344. MR 2000j:46105

[11] D. Voiculescu, *Symmetries of some reduced free product C*-algebras*, Operator Algebras and Their Connections with Topology and Ergodic Theory, Lecture Notes in Mathematics, Volume 1132, Springer-Verlag, Berlin, 1985, pp. 556-588. MR 87d:46075
D. V. Voiculescu, K. J. Dykema, and A. Nica, *Free Random Variables, A noncommutative probability approach to free products with applications to random matrices, operator algebras and harmonic analysis on free groups*, CRM Monograph Series *1*, American Mathematical Society, Providence, RI, 1992. MR 94c:46133

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