IF $B$ AND $f(B)$ ARE BROWNIAN MOTIONS, THEN $f$ IS AFFINE

MICHAEL R. TEHRANCHI

Abstract. It is shown that if the processes $B$ and $f(B)$ are both Brownian motions then $f$ must be an affine function. As a by-product of the proof, it is shown that the only functions which are solutions to both the Laplace equation and the eikonal equation are affine.

1. Statement of results

Suppose that the process $B$ is a Brownian motion and that the function $f$ is affine. Then the process $f(B)$ is again a Brownian motion. This short note proves the converse: if both $B$ and $f(B)$ are Brownian motions, then $f$ must be affine.

To be precise, we will use the following definition of Brownian motion:

Definition 1. The continuous process $B = (B_t)_{t \geq 0}$ is called an $n$-dimensional Brownian motion in a filtration $\mathcal{F} = (\mathcal{F}_t)_{t \geq 0}$ iff there exists an $n$-dimensional vector $b$ and an $n \times n$ non-negative definite matrix $A$ such that for all $0 \leq s \leq t$ the conditional distribution of the increment $X_t - X_s$ given $\mathcal{F}_s$ is Gaussian with mean $(t - s)b$ and covariance matrix $(t - s)A$.

The main result of this paper is this theorem:

Theorem 2. Suppose $B$ is an $n$-dimensional Brownian motion in the filtration $\mathcal{F}$ with non-singular diffusion matrix $A$. Suppose the process $f(B) = (f(B_t))_{t \geq 0}$ is an $m$-dimensional Brownian motion in the same filtration $\mathcal{F}$ for a measurable function $f : \mathbb{R}^n \to \mathbb{R}^m$. Then

$$f(x) = Px + q$$

for some $m \times n$ matrix $P$ and $q \in \mathbb{R}^m$.

The idea of the proof is simply an application of Jensen’s inequality. A similar argument yields a related theorem. We will use the notation $\| \cdot \|$ for the Euclidean norm and $\langle \cdot, \cdot \rangle$ the Euclidean inner product on $\mathbb{R}^n$.

Theorem 3. Let $D \subseteq \mathbb{R}^n$ be an open, connected set, and suppose $u : D \to \mathbb{R}$ is a classical solution to both the Laplace equation

$$\Delta u = 0$$

and the eikonal equation

$$\|\nabla u\| = 1.$$ 

Then $u(x) = \langle p, x \rangle + q$ for some constants $p \in \mathbb{R}^n$ and $q \in \mathbb{R}$, where $\|p\| = 1$.

Remark 1. Theorem 3 is contained in Lemma 4.1 of the recent pre-print of Garnica, Palmas and Ruiz-Hernandez [1]. Their proof appeals to methods of differential geometry, while the proof given below only uses Jensen’s inequality.

Remark 2. There is little loss in assuming that $u$ is a classical solution to the Laplace equation. Indeed, if $u$ is only assumed to be locally integrable and a solution to the Laplace equation in the sense of distributions, then $u$ is automatically infinitely differentiable, and in particular, a classical solution to the Laplace equation. See Chapter 9 of Lieb and Loss’s textbook [2].
2. Proofs

In this section, we prove the results presented above.

Proof of Theorem 2 Since every component of a vector-valued Brownian motion is a scalar Brownian motion, it is sufficient to consider the case $m = 1$.

First we show that $f$ is smooth. Now since the conditional distribution of $f(B_t)$ given $\mathcal{F}_0$ is Gaussian, we can conclude that

$$\mathbb{E}[|f(B_t)| | \mathcal{F}_0] < \infty$$

a.s. for all $t \geq 0$. In particular, we have the growth bound

$$x \mapsto f(x)e^{-\epsilon \|x\|^2}$$

is Lebesgue integrable on $\mathbb{R}^n$ for all $\epsilon > 0$. Now since $f(B)$ is a Brownian motion, there is a constant $\mu \in \mathbb{R}$ such that

$$\mathbb{E}[f(B_t)|\mathcal{F}_s] = (t-s)\mu + f(B_s),$$

and hence we have the representation

$$f(x) = -\tau \mu + \int f(y)\phi(\tau, x, y)dy$$

where

$$\phi(\tau, x, y) = (2\pi \tau)^{-n/2} \text{det}(A)^{-1/2} \exp \left(-\frac{1}{2\tau} (y - br - x, A^{-1}(y - br - x)) \right)$$

is the Brownian transition density. But by the boundedness property (1) and the smoothness of $x \mapsto \phi(\tau, x, y)$ combined with the dominated convergence theorem, the function $f$ is differentiable. Furthermore, its gradient $\nabla f$ has the representation

$$\nabla f(x) = \int \nabla f(y)\phi(\tau, x, y)dy$$

and also satisfies the boundedness property (1). By iterating this argument, we see that $f$ is infinitely differentiable.

Now we show that $f$ must satisfy an eikonal equation. Note that Itô’s formula says

$$df(B_t) = (\nabla f(B_t), dB_t) + \frac{1}{2} \Delta f(B_t)dt.$$ 

Since $f(B)$ is a Brownian motion, the quadratic variation is

$$[f(B)]_t = \int_0^t \|\nabla f(B_s)\|^2 ds = \sigma^2 t$$

for some constant $\sigma \geq 0$. Hence $\nabla f$ is a solution of the eikonal equation

$$\|\nabla f\| = \sigma,$$

almost everywhere. But since $f$ is smooth, it solves the eikonal equation everywhere. Hence by Jensen’s inequality

$$\sigma = \|\nabla f(x)\|$$

$$\leq \int \|\nabla f(y)\|\phi(\tau, x, y)dy$$

$$= \sigma$$

Since the Euclidean norm is strictly convex, Jensen’s inequality says that for every $x$ there exists a vector $p_x \in \mathbb{R}^n$, possibly depending on $x$, such that $\nabla f(y) = p_x$, a.e. $y \in \mathbb{R}^n$. Since $\nabla f$ is continuous, we must have $\nabla f(y) = p$ for all $y$ and for some constant vector $p$. Hence $f(y) = \langle p, y \rangle + q$ as claimed.

We now proceed to the proof of the Theorem 3. It follows the same pattern, but it differs in a few details which we spell out for completeness.
Proof of Theorem 3. We will show that there is a unit vector $p$ such that $\nabla u(x) = p$ everywhere in $D$. Below we will use the notation $B = \{x \in \mathbb{R}^n : \|x\| < 1\}$ to denote the open unit ball in $\mathbb{R}^n$, and hence $x + rB$ denotes the ball of radius $r \geq 0$ centred at the point $x \in \mathbb{R}^n$.

Since $u$ is harmonic, it is well known (again, see Chapter 9 of [2]) that $u$ has the mean-value property: for every constant $r > 0$ such that $x + rB \subseteq D$ we have
\[
  u(x) = \frac{1}{r^n V_n} \int_{rB} u(x + y)dy
\]
where
\[
  V_n = \frac{\pi^{n/2}}{\Gamma(n/2)}
\]
denotes the Lebesgue measure of the unit ball $B$. Since $u$ is continuously differentiable in $D$, the gradient $\nabla u$ is bounded on compact sets, so the dominated convergence theorem allows us to differentiate both sides of the above equation, yielding
\[
  \nabla u(x) = \frac{1}{r^n V_n} \int_{rB} \nabla u(x + y)dy
\]
Now for each $x \in \mathbb{R}^n$, Jensen’s inequality yields
\[
  1 = \|\nabla u(x)\| = \frac{1}{r^n V_n} \left\| \int_{rB} \nabla u(x + y)dy \right\| \\
  \leq \frac{1}{r^n V_n} \int_{rB} \|\nabla u(x + y)\| dy \\
  = 1.
\]
Again, since the Euclidean norm is strictly convex, Jensen’s inequality says that there is a vector $p_x$, possibly depending on $x$, such that $\nabla u(z) = p_x$ a.e $z \in x + rB$ and $\|p_x\| = 1$. Since $\nabla u$ is continuous, we must have $\nabla u(z) = p_x$ for all $z$ such that $\|x - z\| \leq r$.

Now fix two points $x$ and $x'$ in $D$. Since $D$ is open and connected, there exists a path $C \subseteq D$ connecting them. Hence there exists a finite number of points $x = x_1, \ldots, x_N = x' \in D$ and radii $r_1, \ldots, r_N > 0$ such that $\{x_i + r_i B\}_{i=1}^N$ is a cover of the compact set $C \subseteq D$. In particular, the $p_x = p_{x'}$ and hence $\nabla u$ is constant on $D$ as claimed. □

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References

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