The problem of the “common inessential discriminant divisors” attracted the attention of Dedekind, Kronecker, and Hensel in the early days of algebraic number theory. Four sources are particularly important: Dedekind’s announcement, in 1871, of the second edition of Dirichlet’s lectures [2], Dedekind’s 1878 paper [4], the 25th section of Kronecker’s 1882 Grundzüge [17], and Hensel’s 1894 paper [15], which is our focus here. Both of the key papers of Dedekind were translated and annotated in [9]. A brief history of results related to this problem can be found in [21, 2.2.1].

We here present an annotated translation of Kurt Hensel’s “Arithmetische Untersuchungen über die gemeinsamen ausserwesentlichen Discriminantentheiler einer Gattung” (Journal für die Reine und Angewandte Mathematik, 113 (1894), 128–160) [15]. (All older volumes of this journal are available online, so the original paper is easy to access.) Our translation is based on a preliminary translation by Timothy Molnar and Jonathan Webster, completed by Fernando Q. Gouvêa, who also added explanatory footnotes.

Before giving the translation itself we provide a quick outline of the mathematical background of this paper. Some of what we say is informed speculation: it seems clear that both Dedekind and Kronecker started thinking about this subject early on, certainly by the 1860s, but neither one published anything for quite a while: Dedekind published his first version of the theory in 1871 [3] and Kronecker finally explained his theory in the Grundzüge of 1882 [17]. Few notes or unpublished manuscripts seem to have survived. Any account of their process is inferred from what they said later. For the evolution of Dedekind’s ideas, see also [7], [12], and [10].
1 The Mathematical Background

When Kronecker and Dedekind set out to generalize Kummer’s theory of cyclotomic integers, they quickly ran into obstacles. Finding a way around these difficulties led each of them to develop a far more complicated theory than Kummer’s. As a result, each had to justify the extra work by highlighting what made it necessary.

Suppose \( n > 0 \) is an integer and let \( \zeta \) be a primitive \( n \)-th root of unity. Kummer had found an explicit description in terms of congruences of how rational primes factor in the cyclotomic integers \( \mathbb{Z}[\zeta] \). It seems that both Dedekind and Kronecker\(^1\) saw that Kummer’s description could be interpreted in terms of congruences between polynomials (known as “higher congruences” at the time). In modern terms, it would go something like this.

**Theorem 1** Let \( n > 2 \) be an integer, let \( \zeta \) be a primitive \( n \)-th root of unity, and let \( \Phi_n(x) \) be the \( n \)-th cyclotomic polynomial. Fix a prime number \( p \in \mathbb{Z} \) and let

\[
\Phi_n(x) \equiv F_1(x)^{e_1} F_2(x)^{e_2} \ldots F_r(x)^{e_r} \pmod{p}
\]

be the factorization of \( \Phi(x) \) modulo \( p \), where the \( F_i(x) \) are distinct irreducible polynomials in \( \mathbb{F}_p[x] \). Then the factorization of \( \Phi(x) \) in \( \mathbb{Z}[\zeta] \) is

\[
\Phi(x) = p_1^{e_1} p_2^{e_2} \ldots p_r^{e_r},
\]

with distinct prime ideals \( p_i = (p, F_i(\zeta)) \).

Of course, Kummer did not speak of ideals; instead, he thought of \( p_i \) as the “ideal prime divisor” determined by \( p \) and \( F_i(x) \). He gave an explicit method for determining the exponent of \( p_i \) in a factorization. Thus, the “ideal prime divisor” is essentially the valuation corresponding to \( p_i \).

This beautiful result seemed to suggest the possibility of a very simple theory in the general case: for a general number field \( \mathbb{Q}(\alpha) \), let \( \Phi(x) \) be the minimal polynomial for \( \alpha \) and factor it modulo \( p \). One could then use this to define “ideal primes” à la Kummer.

The choice of \( \alpha \) is crucial, of course. At least one example would have been familiar to everyone: the field \( \mathbb{Q}(\sqrt{-3}) \) is the same as the cyclotomic field of order 3. Kummer’s approach worked if one took \( \alpha \) to be a cube root of 1 but would not work if we took \( \alpha = \sqrt{-3} \). Both Dedekind and Kronecker

\(^{1}\)And also Selling in [23].
figured out that one needed to work with all the algebraic integers in the field $\mathbb{Q}(\alpha)$.

That highlights the first difficulty: in the case of $\mathbb{Q}(\zeta)$ the ring of algebraic integers is exactly $\mathbb{Z}[\zeta]$, but this will not be true in general. If $K$ is a number field and $\mathcal{O} \subset K$ is its ring of algebraic integers there may not exist any $\alpha \in \mathcal{O}$ such that $K = \mathbb{Q}(\alpha)$ and $\mathcal{O} = \mathbb{Z}[\alpha]$. In such a situation, there is no obvious $\Phi(x)$ to work with.

Dedekind showed, however, that Under certain conditions we can still make it work. Given a prime number $p \in \mathbb{Z}$, suppose we can find an $\alpha$ such that $\mathbb{Z}[\alpha] \subset \mathcal{O}$ has index not divisible by $p$. Then factoring the minimal polynomial for $\alpha$ modulo $p$ gives the correct factorization of $(p)$ in $\mathcal{O}$. This theorem was announced by Dedekind in 1871 [2]; a proof appeared in 1878 [4]; see [9] for translations. It seems clear that Kronecker was also aware of this fact, since he says he too started by considering higher congruences.

This allowed one to hope, then, that an explicit factorization theory could be based on a local approach: for each prime $p$, find a generator $\alpha$ such that $p$ does not divide the index $(\mathcal{O} : \mathbb{Z}[\alpha])$. Then apply the theorem to find the factorization. Dedekind says in [4, §4] that he spent a long time trying to prove that such an $\alpha$ always exists (see [9, p. 39] for a translation).

Alas, this is not true: there exist number fields in which all of the indices have a common prime divisor. Dedekind pointed this out (and stated the factorization theorem) in [2], probably to explain why he had needed to take a different route. Kronecker says in his Grundzüge [17, §25, p. 384] of 1882 that he had found an example in 1858.

Both Dedekind and Kronecker pointed to this essential difficulty to justify introducing a new approach: ideals in Dedekind’s case, forms in many variables in Kronecker’s. Some years later, Zolotarev tried to extend Kummer’s theory directly in this style [26], but then realized that his approach would fail for finitely many primes. (Eventually, in a second paper [25], Zolotarev found still another way to work around the difficulty.) Dedekind’s paper [4] was, as is clear from the introduction, prompted by an announcement of Zolotarev’s work.

Kronecker also mentioned Zolotarev’s attempt in [17, §25], where he stated the problem in terms of discriminants. For each choice of $\alpha$, let $d(\alpha) = \text{disc}(\Phi(x))$ be the discriminant of its minimal polynomial. Let $d_K$ be the field discriminant. Then $d(\alpha) = m^2d_K$, where $m$ is exactly the index $(\mathcal{O} : \mathbb{Z}[\alpha])$. Kronecker, who always preferred specific elements to collections, thought about this as follows. The many element discriminants $d(\alpha)$ have
a common factor $d_K$ which is the essential part, attached to the “Gattungsbereich” $K$ rather than to a specific element. The other factors of $d(\alpha)$ (i.e., the factors of $m$) are therefore “inessential.” So in the “bad” examples what is happening is that some prime $p$ is an inessential divisor of every element discriminant. Such primes were the “common inessential discriminant divisors.”

The name is perhaps ill-chosen, because it is perfectly possible for a prime $p$ to divide the discriminant $d_K$ and also divide the index $(O : \mathbb{Z}[\alpha])$. Such a prime divisor is then both “essential” (it divides $d_K$) and “inessential”! Dedekind’s term “index divisor” seems more appropriate. In the later literature, the “index $i(K)$ of the field $K$” was defined to be the greatest common divisor of the indices of all the generators of $K$; then Kronecker’s common inessential discriminant divisors are just the divisors of $i(K)$. See [21, 2.2.1 item 3] for information on more recent work.

Kronecker’s example “in the thirteenth roots of 1” is probably the simplest one. He never gave the details, but they are probably as Hensel gave them in his Ph.D. thesis [13] (see also [22, 2.2]). Let $\zeta$ be a primitive 13-th root of unity. There is a unique subfield $K$ of degree 4 over $\mathbb{Q}$. Since the discriminant of $\mathbb{Q}(\zeta)$ is a power of 13, so is the discriminant of $K$ (in fact, $d_K = 13^3$). It follows from Kummer’s work that the prime number 3 is divisible by four ideal primes in $K$, each of which has norm 3; let $p$ be one of these. Since $N(p) = 3$, the field $\mathcal{O}/p$ has three elements. Consider some $\alpha \in K$. Since $K$ is a normal field, the discriminant of the minimal polynomial of an integer in $K$ is the square of the product of differences of the four roots, which are integers in $K$. Since there are only three congruence classes modulo $p$, at least two of the roots must be congruent modulo $p$, i.e., one of these differences must be divisible by $p$. Since $p$ lies above 3, the discriminant $d(\alpha) \in \mathbb{Z}$ must be divisible by 3. Since $d_K$ is a power of 13, the divisor 3 is inessential. This is true for any $\alpha$, so 3 is a common inessential discriminant divisor.

This set up the problem of determining exactly when this phenomenon happens. One of the things that interests us about this problem is that it was solved several times. Dedekind found a criterion in his paper [4]. It is a sign of how little Kronecker followed Dedekind’s work that he suggested the

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2See footnote 102 on page 30.

3This is global number field 4.0.2197.1 in [20].

4Hasse, in [11, p. 456], attributes this criterion to Hensel and says it was the first success of Hensel’s new methods, presumably meaning $p$-adic methods. In fact the criterion was first found by Dedekind and neither author used $p$-adic methods.
problem of common inessential discriminant divisors to Hensel for his Ph.D. in 1882. Hensel did not solve it completely in his thesis but he published a solution in 1894, in [15], which we translate here. (Petri argues in [22, 2.4] that the majority of the results were known to Hensel before 1886.) In the first paragraph gives the same criterion that had been found by Dedekind in 1878. While Hensel refers to Dedekind’s paper, it is not clear how carefully he had read it. In any case, he proceeds to find still another criterion in the second half of the paper.

As Kronecker’s student, Hensel does not work with ideals, but rather with forms in several variables as in Kronecker’s Grundzüge [17]. He probably learned this approach from Kronecker’s lectures, but those remained unpublished. Hensel refers to the Grundzüge as “Kummer’s Festschrift” because Kronecker originally published it to commemorate the 50th anniversary of Kummer’s doctorate.

We have tried, in our footnotes, to provide hints about how Kronecker’s approach works, without attempting a full account of Kronecker’s theory. For a modern attempt at explaining it, see [24] or [8]. The key thing to keep in mind for this paper is that one considers a kind of “generic algebraic integer”: given an integral basis $\xi_1, \xi_2, \ldots, \xi_n$ (Kronecker and Hensel call it a “fundamental system”) one considers the “fundamental form”

$$w_0 = u_1\xi_1 + \cdots + u_n\xi_n,$$

where the $u_i$ are indeterminates. This is a polynomial in the $n$ variables $u_1, u_2, \ldots, u_n$; choosing integer values for the $u_i$ produces an algebraic integer in $K$.

Replacing the $\xi_i$ by their conjugates gives a conjugate $w_j$ of the fundamental form; multiplying $x - w_j$ for all $j$ gives the “fundamental equation” for the domain. This is a polynomial in $\mathbb{Z}[u_1, u_2, \ldots, u_n, x]$. As before, specializing the $u_i$ to integer values gives the polynomial of degree $n$ that has the corresponding algebraic integer as a root.

Factoring the fundamental equation provides a method for finding the factorization of a rational prime $p$ in the ring of integers of $K$. This is just as in Dedekind’s theorem: reduce the fundamental equation modulo $p$ and

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5Hensel later generalized the numerical condition in Kronecker’s example to give a sufficient criterion for the existence of common inessential discriminant divisors, and even attempted to prove the condition was also necessary, which it is not. See the careful discussion in [22, 2.2].
factor it. The problem is that we are now trying to factor a polynomial in $n + 1$ variables.

The word “Gattung” means “kind” or “genus”; Kronecker used it to mean “type of algebraic number.” For example, $\sqrt{2}$ and $3 + \sqrt{2}$ are algebraic numbers of the same “Gattung” because each is a rational function of the other. Algebraic numbers of the same type belong to the same “Gattungsbereich,” which means something like “type domain.” What we would call the base field Kronecker called the “domain of rationality.” As usual, Kronecker did not think in set-theoretic terms and would have avoided thinking of a “Gattungsbereich” as a completed whole.

Writing after Kronecker’s death, Hensel seems a little more relaxed about completed wholes—but also a little fuzzier. He seems to use “Gattung” and “Gattungsbereich” almost interchangeably for both a field and its ring of integers. Since the word “genus” now means something completely different we have opted to translate both words as “domain” in most cases.

2 The 1894 paper

Hensel published two important papers in 1894. Both of them likely contain material he originally submitted for his Habilitation in 1886. None of these were published at the time, and we know of them only from Kronecker’s notes. See [22, 2.3] for a reconstruction. The two 1894 papers were published after Kronecker’s death, perhaps because Hensel expected to find proofs for many of these results among Kronecker’s papers; see his explicit comment on p. 37 below.

The first paper published in 1894 was [16] “Untersuchung der Fundamentalgleichung einer Gattung für eine reelle Primzahl als Modul und Bestimmung der Teiler ihrer Discriminante” (Journal für die Reine und Angewandte Mathematik 113 (1894), 61–83). In it Hensel proved something that had been stated by Kronecker in [17]: the discriminant of the fundamental equation, which is a polynomial in $\mathbb{Z}[u_1, u_2, \ldots, u_n]$, has the discriminant of the field $K$ as its largest integer factor. From this it follows that the factorization modulo $p$ of the fundamental equation corresponds exactly to the factorization of $p$.

The paper we translate builds on that to consider common inessential discriminant divisors. Hensel wants to characterize when such divisors occur. He finds several answers, the first of which is identical to the one presented
by Dedekind in 1878.

In our translation we have chosen to focus on getting the mathematical content right, preserving Hensel’s language, notations, and general point of view. We have not tried (and would not have succeeded) to preserve every nuance of meaning or to reproduce Hensel’s grammar precisely.

Our translation is based on the original publication in the *Journal für die Reine und Angewandte Mathematik* (113 (1894), 128–160); the original page numbers are indicated in the margin. Hensel’s own footnotes (which are few) are marked by asterisks, while our annotations are given in numbered footnotes.

Hensel indicates theorems by using indented text; we have used a modern simulacrum of the same device. Hensel does not signpost the beginning or end of a proof; we have usually added such signposts in the footnotes. We have retained Hensel’s notation as much as possible.

Every once in a while words have been inserted in square brackets when we felt it would clarify the meaning. Hensel often uses \( \geq \) to indicate inequality or incongruence; we have silently substituted \( \neq \) or \( \nequiv \). We have also rendered “ganze Functionen” and “ganze ganzzahlige Functionen” as “polynomials” and “integral polynomials” or “polynomials with integer coefficients”, respectively, without further comment. Hensel often says “order” when we would say “degree”; he sometimes also uses “dimension” in a similar sense. We have mostly translated “degree” when it was unambiguous what was meant; see the footnotes.

**Outline**

Hensel’s paper contains five sections which he labels §1 to §5. The main results in each section are as follows.

§1. The main theorem here is that a prime \( p \) is a common inessential discriminant divisor in a field \( K \) if and only if there are not enough irreducible polynomials modulo \( p \) to match the factorization of \( p \) in \( K \). This result was also in [4, 9].

§2. Using the criterion Hensel just found seems to require knowing the factorization of \( p \), but in fact all we need to know is how many prime divisors of \( p \) in \( K \) have a given degree. In this section Hensel shows that one can determine this number without knowing the factorization of \( p \). Petri argues that the material in this paragraph was not part of the Habilitation materials,
hence was new in 1894.

§3. The focus now changes to the index form \( \Delta(u_1, u_2, \ldots, u_n) \) (with respect to a fixed integral basis). Since \( p \) is a common inessential discriminant divisor exactly when plugging any \( n \)-tuple of integers into \( \Delta \) results in a number divisible by \( p \). Hensel derives a general criterion to recognize when a polynomial with integer coefficients has this property.

§4. Kronecker had observed in \cite{17, §25} an interesting property of the index form in Dedekind’s cubic field example. While every value obtained by plugging integers into \( \Delta(u_1, u_2, u_3) \) was divisible by 2, there are integers from the cyclotomic field \( L = \mathbb{Q}(\zeta_3) \) for which we get values that are not divisible by 3. In this section Hensel shows that for any polynomial with integer coefficients we can find an auxiliary field \( L \) with this property.

§5. Given the result in §4, it is natural to ask which field we need to use. Hensel shows that one can always choose a subfield of a cyclotomic field of prime order.

Translation

Arithmetical Investigations of the Common Inessential Discriminant Divisors of a Domain

(by Mr. K. Hensel)

§1

In a recently published work (this Journal, vol. 110, pp. 61–83) I considered the congruence of least degree modulo a prime \( p \) satisfied by the fundamental form

\[
(1.) \quad w_0 = u_1\xi_1 + \cdots + u_n\xi_n
\]

\[\text{This is a typo, as noted on page 160 of this issue; it should be volume 113. The paper is 116.}\]

\[\text{This alerts the reader that the } u_i \text{ in this equation are supposed to be indeterminates. Hensel, following Kronecker, uses “form” to mean a polynomial in several variables. The } \xi_i \text{ in this expression are what we would call an integral basis. One can think of } w_0 \text{ as a}\]
of a given domain\(^8\) of the \(n\)-th order.\(^9\) The main result, which will serve as the basis for this work, says that, for any prime number, \(w_0\) does not satisfy a congruence of degree smaller than the degree \(n\) of the domain.\(^10\)

The congruence of lowest degree which is satisfied by \(w_0\) modulo \(p\) is made up in a simple manner from the congruences satisfied by \(w_0\) modulo each of the prime divisors of \(p\). Let \(P\) be one of these factors in the domain\(^11\) \((\mathfrak{G})\), and let \(\kappa\) be its degree.\(^12\) Then \(w_0\), with indeterminates \(u_1, \ldots, u_n\), satisfies (modulo \(P\)) the congruence of degree \(\kappa\)

\[ \mathfrak{F}(w) = w^\kappa + U^{(1)}(u_1 \ldots u_n)w^{\kappa-1} + \cdots + U^{(\kappa)}(u_1 \ldots u_n) \equiv 0 \pmod{P}, \]

whose left side\(^13\) is irreducible modulo the prime \(p\), while the coefficients are integral polynomials in \(u_1 \ldots u_n\).

Let then

\[ p = P_1^{\delta_1}P_2^{\delta_2} \cdots P_h^{\delta_h} \]

be the decomposition of \(p\) into its prime factors in the domain \((\mathfrak{G})\) and let

\[ \mathfrak{F}_1(w), \mathfrak{F}_2(w), \ldots, \mathfrak{F}_k(w) \]

be the functions\(^14\) of lowest degree having the fundamental form \(w_0\) as a root modulo the \(h\) corresponding distinct prime divisors \(P_1, P_2, \ldots, P_h\).

The polynomials have (as polynomials in \(w\)) degrees

\[ \kappa_1, \kappa_2, \ldots, \kappa_h, \]

"generic algebraic integer." Multiplying \((w - w_0)\) with all its conjugates gives an equation of degree \(n\) whose coefficients are in \(\mathbb{Z}[u_1, u_2, \ldots, u_n]\). This is the “equation of smallest degree satisfied by \(w_0\).” Notice that \(w_0\) is the fundamental form and \(w\) is the variable in the polynomial it is a root of. Hensel will follow this notational pattern throughout.

\(^8\)"Gattungsbereiches"
\(^9\)We would say “degree” instead of “order.”
\(^10\)In modern terms, the element \(w_0\), considered modulo \(p\), is integral of degree \(n\) over \(\mathbb{F}_p[u_1, u_2, \ldots, u_n]\).
\(^11\)"Bereich." We do not know why Hensel chooses \((\mathfrak{G})\) as the notation.
\(^12\)"Ordnungszahl." We would call it the residual degree of \(P\).
\(^13\)Hensel thinks in terms of equations and congruences, not polynomials. The “left hand side” of the congruence \(\mathfrak{F}(w) \equiv 0\) is the minimal polynomial for \(w_0 \pmod{P}\).
\(^14\)In general, Hensel means “polynomial” when he says “function.”
which are equal to the degrees of the prime factors of \( p \) \(^{15}\). Then in the cited paper\(^{16}\) (p. 75) it is shown that the congruence of lowest degree satisfied by \( w_0 \) modulo the prime number \( p \) will be the following:

\[
\mathcal{F}_{\delta_1}^1(w) \mathcal{F}_{\delta_2}^2(w) \cdots \mathcal{F}_{\delta_h}^h(w) \equiv 0 \pmod{p},
\]

and its degree in \( w \) is

\[
(6^a.) \quad \kappa_1 \delta_1 + \kappa_2 \delta_2 + \cdots + \kappa_h \delta_h = n.
\]

If we plug in \( w_0 \) for \( w \), each of the \( \mathcal{F}_i(w_0) \) is divisible by the divisor \( P_i \), so the whole product is divisible by \( P_1^{\delta_1} \cdots P_h^{\delta_h} \) and so divisible by \( p \).

Instead of the fundamental form \( w_0 \) we now want to consider an algebraic integer of the domain (\( \mathcal{G} \))

\[
(7.) \quad \xi_0 = a_1 \xi_1 + a_2 \xi_2 + \cdots + a_n \xi_n,
\]

which is simply obtained\(^{17}\) from \( w_0 \) by giving the unknowns \((u_1, \ldots, u_n)\) the integer values \((a_1, \ldots, a_n)\). We would like to investigate which congruence with integer coefficients is satisfied by \( \xi_0 \) modulo \( p \).

Obviously \( \xi_0 \) satisfies the congruence of \( n \)-th degree given by (6.), since we are just replacing \((u_1, \ldots, u_n)\) by \((a_1, \ldots, a_n)\). In general\(^{18}\) if we have polynomials

\[
\mathcal{F}_1(w), \mathcal{F}_2(w), \ldots, \mathcal{F}_h(w)
\]

of degrees \( \kappa_1, \kappa_2, \ldots, \kappa_h \), we write the corresponding\(^{19}\) functions

\[
\mathcal{F}_1(\xi), \mathcal{F}_2(\xi), \ldots, \mathcal{F}_h(\xi)
\]

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\(^{15}\)So Hensel knows that the factorization of \( p \) is determined by the factorization mod \( p \) of the fundamental equation, which is the polynomial with coefficients in \( \mathbb{Z}[u_1, u_2, \ldots, u_n] \) having \( w_0 \) as a root.

\(^{16}\)This is \(^{10}\).

\(^{17}\)So \( \xi_0 \) is a particular algebraic integer, obtained as a linear combination of the integral basis \( \xi_1, \xi_2, \ldots, \xi_n \).

\(^{18}\)"In general" here seems to mean that this notation will always be used.

\(^{19}\)Hensel is introducing notation: these are the same functions as before, except that he has replaced the indeterminates \( u_i \) by the integers \( a_i \). The different variable \( \xi \) indicates that this has been done. Notice that the functions \( \mathcal{F}_i(\xi) \) therefore depend on our choice of \( \xi_0 \), i.e., depend on the choice of the integers \( a_i \). The main point of the notation is precisely not to have to show the dependence on the \( u_i \) or the \( a_i \).
where the variable \(w\) is now replaced by \(\xi\) to indicate the difference). Then \(\xi_0\) is a root of the congruence of degree \(n\) with integer\(^{20}\) coefficients

\[
\begin{align*}
\delta_1(\xi) \ldots \delta_h(\xi) & \equiv 0 \pmod{p}.
\end{align*}
\]

We know this to be the case because the individual numbers \(\delta_i(\xi_0)\) are obtained from the corresponding forms \(\delta_i(w_0)\) by replacing the unknowns \((u_1, \ldots, u_n)\) by the integers \((a_1, \ldots, a_n)\). Therefore each of these are divisible\(^{20}\) a fortiori by the each of the prime factors as before.

The fundamental form \(w_0\), as proved in the aforementioned work, does not satisfy any congruence whose degree is smaller than that of \((6.)\). But the congruence \((8.)\) need not be the congruence of lowest degree\(^{21}\) satisfied by \(\xi_0\); for that to be true we would need to choose \(\xi_0\) appropriately. We must then investigate the following question:

Under what conditions will the algebraic integer \(\xi_0\) satisfy no congruences modulo \(p\) of degree less than \(n\)?

It is very easy to give a system of necessary conditions; we will later prove that they are also sufficient. First\(^{22}\) the \(h\) integer functions of \(\xi\) in \((8.),\ \delta_1(\xi), \ldots \delta_h(\xi)\) must be irreducible modulo \(p\). In fact\(^{23}\) if for example

\[
\delta_1(\xi) \equiv F_1(\xi)G_1(\xi) \pmod{p},
\]

where \(F_1\) and \(G_1\) are functions of \(\xi\) of degree lower than \(\kappa_1\), then the same congruence also holds modulo the prime divisor \(P_1\) of \(p\). Substituting \(\xi\) by \(\xi_0\) and observing that \(\delta_1(\xi_0)\) is divisible by \(P_0\), we see the congruence

\[
F_1(\xi_0)G_1(\xi_0) \equiv 0 \pmod{P_1},
\]

from which it follows that one of these factors, say \(F_1(\xi_0)\), is divisible by the prime divisor \(P_1\). Then \(\xi_0\) clearly satisfies the congruence modulo \(p\)

\[
(8a.) \quad F_1^{\delta_1}(\xi)F_2^{\delta_2}(\xi) \ldots \delta_h(\xi) \equiv 0 \pmod{p},
\]

\(^{20}\)Whereas in \((6.)\) the coefficients were polynomials in \(n\) variables.

\(^{21}\)For example, if we choose all of the \(a_i\) divisible by \(p\), the congruence of lowest degree will be the degree one polynomial \(\xi\).

\(^{22}\)Hensel doesn’t do Lemmas, but this is one. Formally: if \(\xi_0\) does not satisfy any congruence of degree less than \(n\), then the polynomials appearing in \((8.)\) must be irreducible modulo \(p\). Notice that these are the polynomials after substituting the \(u_i\) by the \(a_i\), so they depend on the choice of \(\xi_0\).

\(^{23}\)Here begins the proof of the Lemma.
because the first factor on the left side is divisible by $P^\delta_1$, while the others are divisible by $P^\delta_2, \ldots, P^\delta_h$. But the degree of (8a.) is smaller than $n$, since it is smaller\footnote{The contradiction ends the proof of the Lemma.} than the degree of (8.).

Secondly, the $h$ irreducible polynomials $\mathfrak{F}_i(\xi)$ must be distinct\footnote{Lemma 2, still under the running assumption that $\xi_0$ satisfies no congruence of degree lower than $n$.} modulo $p$. If for example $\mathfrak{F}_1(\xi)$ and $\mathfrak{F}_2(\xi)$ were congruent for this modulus, then a fortiori

$$\mathfrak{F}_1(\xi) \equiv \mathfrak{F}_2(\xi) \pmod{P_1P_2},$$

since $P_1P_2$ is a divisor of $p$. Substituting again $\xi_0$ for $\xi$ and noticing that $\mathfrak{F}_1(\xi_0)$ is divisible by $P_1$ and $\mathfrak{F}_2(\xi_0)$ is divisible by $P_2$, it follows from the above congruence that $\mathfrak{F}_1(\xi_0)$ and $\mathfrak{F}_2(\xi_0)$ are both divisible by the product $(P_1P_2)$. Now if we take any two exponents $\delta_1, \delta_2$ with $\delta_1 \geq \delta_2$, the power $\mathfrak{F}_1^{\delta_1}(\xi)$ will, when $\xi = \xi_0$, be divisible by the product $(P_1P_2)^{\delta_1}$, so a fortiori by $P_1^{\delta_1}P_2^{\delta_2}$. Thus $\xi_0$ satisfies the congruence

$$(8^b.) \quad \mathfrak{F}_1^{\delta_1}(\xi), \mathfrak{F}_2^{\delta_2}(\xi) \ldots \mathfrak{F}_h^{\delta_h}(\xi) \equiv 0 \pmod{p},$$

whose degree is smaller than that of (8.), so smaller\footnote{End of the proof, again by contradiction.} than $n$. So we have the following result:

If the number $\xi_0$ does not satisfy any congruence of degree less than $n$ modulo $p$, then the $h$ polynomials $\mathfrak{F}_1(\xi), \ldots, \mathfrak{F}_h(\xi)$ in (8.) are all distinct and irreducible modulo $p$.

This theorem was given by Mr. Dedekind in his great work\footnote{The original is “grossen Arbeit.” This is \textbf{1}, which is really a short note rather than a full-length memoir, so “grossen” cannot mean “large.” For Dedekind it is an immediate consequence of his theorem about prime decomposition. See \textbf{9}.} “Ueber den Zusammenhang zwischen der Theorie der Ideale und der Theorie der höheren Congruenzen” (\textit{Abh. der Gött. Gesellschaft} Volume 23), although in slightly different form. It is noteworthy that he demonstrated that for it to be possible to find such $h$ functions $\mathfrak{F}_1(\xi), \ldots, \mathfrak{F}_h(\xi)$, it is necessary and sufficient that a number $\xi_0$ exists for which $p$ is not an inessential divisor of the discriminant.\footnote{This is Theorem (IV) in \textbf{4}; see \textbf{9}. So at this point Hensel seems to know Dedekind’s 1878 criterion for $p$ to be a common inessential discriminant divisor.}
This result enabled him to find a specific field of degree three for which the prime number 2 is a common inessential divisor of all equation discriminants. This alone shows that the theory of number fields cannot be founded upon higher congruences. This can be done, however, if, as in this and the previous work, we work with the linear form $w_0 = u_1 \xi_1 + \cdots + u_n \xi_n$ with indeterminates $u_1, \ldots, u_n$, rather than a specific number $\xi_0$ of the domain. This is because in the previous work it was indeed established that in the discriminant of the polynomial having $w_0$ as a root no prime $p$ is contained as other than an essential divisor. The results on common inessential divisors of the discriminant that follow in this paper have not, to my knowledge, been given before.

We now want to investigate when condition (A.) can actually hold. If the $h$ polynomials $\mathfrak{F}_1(\xi), \ldots, \mathfrak{F}_h(\xi)$ are irreducible modulo $p$, two of them, say $\mathfrak{F}_1(\xi)$ and $\mathfrak{F}_2(\xi)$, can only be congruent if they have the same degree, i.e., if the degrees $\kappa_1$ and $\kappa_2$ of the corresponding prime factors $P_1$ and $P_2$ are equal. So let us arrange the distinct prime divisors $P_1, P_2, \ldots, P_h$ by their degrees $\kappa_1, \kappa_2, \ldots, \kappa_h$, grouping together those that have equal degrees. Of the $h$ integers, suppose that there are $\lambda_1$ equal to $\kappa_1$, then $\lambda_2$ equal to $\kappa_2$, . . . then $\lambda_\gamma$ equal to $\kappa_\gamma$, where $\lambda_1 + \lambda_2 + \cdots + \lambda_\gamma = h$ and the degrees $\kappa_1, \kappa_2, \ldots, \kappa_\gamma$ are all different. For the moment, take $\kappa$ to be one of these $\gamma$ degrees $\kappa_1, \ldots, \kappa_\gamma$ and let

$$P^{(1)}, P^{(2)}, \ldots, P^{(\lambda)}$$

30. This result” must be the necessary criterion in Theorem A rather than the “if and only if” result just mentioned.
31. “Theorie der Gattungen.” The same observation is made by Dedekind in the 1878 paper [4] and by Kronecker in [17, §25]. This is exactly what Zolotarev tried to do but had to move beyond. Note the consensus here: the several different pioneers of algebraic number theory point to this issue to justify the complexity of their approaches.
32. I.e., basing the whole theory on polynomial congruences.
33. The reference is to [16].
34. The theorem is that if you compute the discriminant of the fundamental equation you obtain a polynomial in $n$ variables $u_i$ whose content is $d_K$.
35. In other words, Hensel is acknowledging that Theorem A was proved by Dedekind but claims that his remaining theorems are new. This is not quite correct, since Dedekind also knew Theorem B and knew that the criterion was sufficient, which Hensel will prove later; see Theorems C and D.
36. This section is about counting how many irreducible polynomials are available for Theorem A. The first step is to group them by degree.
37. Now we focus on all the irreducible factors of a given degree.
be the prime factors of $p$ whose degree is equal to $\kappa$. Likewise, let
\[ (9^a.) \quad \mathcal{F}^{(1)}(\xi), \mathcal{F}^{(2)}(\xi), \ldots, \mathcal{F}^{(\lambda)}(\xi) \]
be functions of degree $\kappa$ satisfied by $\xi_0$ modulo the prime factors $P^{(1)}, \ldots, P^{(\lambda)}$.

We now want to know if it is possible for these $\lambda$ functions to be irreducible and incongruent modulo $p$.

Since $\mathcal{F}^{(1)}(\xi)$ is irreducible [and has $\xi_0$ as a root] modulo $P^{(1)}$, if $\xi_0$ satisfies another polynomial congruence
\[ \Phi(\xi) \equiv 0 \pmod{P^{(1)}}, \]
then $\Phi(\xi)$ must be divisible by $\mathcal{F}^{(1)}(\xi)$ modulo $P^{(1)}$, because otherwise $\Phi(\xi)$ and $\mathcal{F}^{(1)}(\xi)$ would have a greatest common divisor modulo $P^{(1)}$, which contradicts the irreducibility of $\mathcal{F}^{(1)}(\xi)$. Therefore for the modulus $P^{(1)}$ and therefore $p$ itself, we get a congruence of the form:
\[ \Phi(\xi) \equiv \mathcal{F}^{(1)}(\xi)\Phi^{(1)}(\xi) \pmod{p}. \]
The same is true for the functions $\mathcal{F}^{(2)}(\xi), \ldots, \mathcal{F}^{(\lambda)}(\xi)$ if they are also not decomposable for $p$.

Now all whole numbers from $(\mathfrak{g})$, and hence also $\xi_0$, satisfy the congruence
\[ \xi^{P^\kappa} - \xi \equiv 0 \pmod{P^{(i)}} \quad (i = 1, 2, \ldots, \lambda), \]
for each of the $\lambda$ divisors of degree $\kappa$ $P^{(1)}, \ldots, P^{(\lambda)}$. Hence the expression $(\xi^{P^\kappa} - \xi)$ is divisible modulo $p$ by each of the $\lambda$ functions $\mathcal{F}^{(1)}(\xi), \ldots, \mathcal{F}^{(\lambda)}(\xi)$, if they are assumed to be irreducible. If those functions are incongruent modulo $p$, then the function $(\xi^{P^\kappa} - \xi)$ must be divisible by their product, and so it must contain at least $\lambda$ irreducible factors modulo $p$ of degree $\kappa$.

\[ \text{If} \]

---

38 The observation is that any polynomial such that $F(\xi_0) \equiv 0 \pmod{P^{(i)}}$ must be divisible (modulo $p$) by the corresponding irreducible polynomial $\mathcal{F}^{(i)}$.

39 Since the polynomials all have integer coefficients.

40 The residue field of each of the primes $P^{(i)}$ has $p^\kappa$ elements, all of which are roots of $x^{P^\kappa} - x$.

41 Hensel has shown, then, that any irreducible polynomial of degree $\kappa$ is a divisor of $\xi^{P^\kappa} - \xi$ modulo $P^{(i)}$. In modern terms, adjoining a root of an irreducible polynomial of degree $\kappa$ to $\mathbb{F}_p$ always gives the same field, namely the splitting field of $\xi^{P^\kappa} - \xi$. In fact, Hensel also needs to know that any polynomial whose degree divides $\kappa$ is a factor modulo $p$ of $\xi^{P^\kappa} - \xi$. For that he quotes his older paper [14], which is one of many nineteenth century papers dealing with “higher congruences” that we would describe as being about the theory of finite fields.
we consider all the irreducible factors modulo $p$ one finds\footnote{The point is that we have a polynomial of degree $p^\kappa$ which is the product of all irreducible polynomials mod $p$ whose degree divides $\kappa$. Writing $p^\kappa$ as the sum of those degrees and using Möbius inversion gives formula (10.) for the total number of distinct irreducible polynomials of degree $\kappa$ in $\mathbb{F}_p[\xi]$. The same formula is also found in Dedekind’s Abriß \cite{1}, but Dedekind does not quote it in his 1878 paper.} that $(\xi^{p^\kappa} - \xi)$ is the product of all irreducible polynomials whose degree is equal either to $\kappa$ or to a divisor of $\kappa$ and that $(\xi^{p^\kappa} - x\bar{i})$ has exactly

\begin{equation}
\bar{g}(\kappa) = \frac{1}{\kappa}(p^\kappa - \sum p^{\bar{q}} + \sum p^{\bar{q}'\bar{q}''} - \sum p^{\bar{q}'\bar{q}''\bar{q}'''} + \ldots)
\end{equation}

distinct irreducible divisors of degree $\kappa$, where $q, q', q'', \ldots$ are the distinct prime factors of $\kappa$.\footnote{In the statement of this theorem Hensel uses “real prime” to refer to a prime in $\mathbb{Z}$. Similarly, he later uses “real integer” for an element of $\mathbb{Z}$. The modern usage is “rational integer” and “rational prime,” but we have preserved Hensel’s words.} So if $\lambda > \bar{g}(\kappa)$ then it is not possible for the $\lambda$ irreducible functions $\mathfrak{F}^{(1)}(\xi), \ldots, \mathfrak{F}^{\lambda}(\xi)$ to be distinct modulo $p$. If we now apply this result to all $\gamma$ of the distinct degrees $\kappa_1, \ldots, \kappa_\lambda$ of the prime factors of $p$, we obtain from theorem (A.).\footnote{Compare with my paper: Untersuchung der ganzen algebraischen Zahlen eines Gattungsbereiches für einen beliebigen algebraischen Primdivisor; this Journal, volume 101, pages 140 and 141.}
Suppose that \( p = P_1^{\delta_1} \cdots P_h^{\delta_h} \) is the decomposition of a real prime number \( p \) in a domain \((G)\), and that among the \( h \) nonequivalent prime factors \( P_1, P_2, \ldots, P_h \) there are

\[
\lambda_1 \text{ of degree } \kappa_1, \\
\lambda_2 \text{ of degree } \kappa_2, \\
\vdots \\
\lambda_\gamma \text{ of degree } \kappa_\gamma.
\]

So we can find a number \( \xi_0 \) in the domain \((G)\) that does not satisfy any congruence modulo \( p \) of degree less than \( n \) only if

\[
(11.) \quad \lambda_1 \leq g(\kappa_1), \quad \lambda_2 \leq g(\kappa_2), \quad \ldots, \quad \lambda_\gamma \leq g(\kappa_\gamma)
\]

holds (where the \( \gamma \) whole numbers \( g(\kappa) \) are as in (10.)). If however, even one of these conditions is not met, then every number \( \xi_0 \) from \((G)\) satisfies a polynomial congruence modulo \( p \) of degree less than \( n \).

It should now be proved that condition (11.) is also sufficient to guarantee that at least one number \( \xi_0 \) from \((G)\) satisfies no congruence modulo \( p \) of degree less than \( n \). Let \( P \) be one of the \( h \) prime divisors of \( p \) and let \( \kappa \) be its degree. Then we can choose \( \xi_0 \) to satisfy the irreducible polynomial congruence

\[
\mathfrak{f}(\xi) \equiv 0 \pmod{P},
\]

where \( \mathfrak{f}(\xi) \) is one of the \( g(\kappa) \) irreducible divisors of degree \( \kappa \) of \((\xi^{p^\kappa} - \xi)\).

\[\text{In the congruence}\]

\[
\xi^{p^\kappa} - \xi \equiv \mathfrak{f}(\xi)\Phi(\xi) \pmod{P},
\]

\[\text{This is just theorem A plus an explicit count of the number of irreducible polynomials in degree } \kappa \text{ in } \mathbb{F}_p[\xi].\]

\[\text{So another proof is beginning here: that the conditions (11.) imply the existence of at least one } \xi_0 \text{ with the desired property. This is Theorem (IV) in [4]. Hensel’s proof is identical to Dedekind’s. See [9].}\]

\[\text{This is the lemma to be proved next. Given an irreducible polynomial of degree } \kappa \text{ in } \mathbb{F}_p[x], \text{ we can choose } \xi_0 \text{ so that it is a root of that polynomial modulo } P.\]

\[\text{Here starts the proof of the lemma.}\]
both the left and ride side disappear for as many incongruent values of \( \xi \) as the degree (namely for the \( p^\kappa \) congruence classes modulo \( P \) of numbers in the domain (\( \mathcal{G} \))). There must therefore exist a number \( \xi_0 \) for which \( \mathfrak{F}(\xi_0) \) is divisible by \( P \), because if not the function \( \Phi(\xi) \) of degree \( (p^\kappa - \kappa) \) would vanish modulo \( P \) for \( p^\kappa \) incongruent values of \( \xi \), which is not possible. We can therefore \( ^{48} \) choose \( \xi_0 \) to be a root of the chosen irreducible congruence of degree \( \kappa \)

\[
\mathfrak{F}(\xi) \equiv 0 \pmod{P}.
\]

Now choose \( ^{49} \) \( h \) irreducible functions \( \mathfrak{F}_1(\xi), \mathfrak{F}_2(\xi), \ldots, \mathfrak{F}_h(\xi) \), all incongruent modulo \( p \), whose degrees are respectively equal to \( \kappa_1, \kappa_2, \ldots \kappa_h \), so that each of the \( \mathfrak{F}_i(\xi) \) is a divisor of \( \xi^{p^{\kappa_i}} - \xi \). Such a system of \( h \) functions can only exist when condition (11.) is satisfied. When it is, we can find such a system, since the quantity of irreducible functions incongruent modulo \( p \) of degrees \( \kappa_1, \ldots, \kappa_h \) is larger than the number of functions we need. Let then

(12.)

\[
\xi^{(1)}_0, \xi^{(2)}_0, \ldots, \xi^{(h)}_0
\]

be \( h \) integers from (\( \mathcal{G} \)) chosen so that for each \( i \) the integer \( \xi^{(i)}_0 \) satisfies modulo \( P_i \) the congruence

(12a.)

\[
\mathfrak{F}_i(\xi^{(i)}_0) \equiv 0 \pmod{P_i} \quad (i = 1, \ldots, h).
\]

As we proved above, we can find \( h \) such numbers \( \xi^{(i)}_0 \).

Moreover, we can also assume \( ^{50} \) from the outset that the left side in (12a.) is not divisible by \( P_i^2 \), so that

(12b.)

\[
\mathfrak{F}_i(\xi^{(i)}_0) \not\equiv 0 \pmod{P_i^2} \quad (i = 1, 2, \ldots h).
\]

This is possible because if for some \( i \)

\[
\mathfrak{F}_1(\xi^{(1)}_0) \equiv 0 \pmod{P_1^2},
\]

\( ^{48} \)Lemma has been proved.

\( ^{49} \)We have shown that for each prime \( P \) of degree \( \kappa \) and each irreducible polynomial of degree \( \kappa \) we can find an integer that is a root of that polynomial modulo \( P \). Now we apply this to each of the prime factors of \( p \).

\( ^{50} \)Another little lemma.
one can substitute $\xi_0^{(1)}$ by $\bar{\xi}_0^{(1)} = (\xi_0^{(1)} + \pi_1)$, where $\pi_1$ is a whole number divisible by $P_1$ but not $P_2^2$. Then using Taylor’s theorem, we have:

$$\bar{\mathfrak{F}}_1(\bar{\xi}_0^{(1)}) = \mathfrak{F}_1(\xi_0^{(1)} + \pi_1) = \mathfrak{F}_1(\xi_0^{(1)}) + \pi_1 \mathfrak{F}'_1(\xi_0^{(1)}) + \frac{1}{2} \pi_1^2 \mathfrak{F}''_1(\xi_0^{(1)}) + \ldots.$$ 

According to our assumptions, the first term as well as the third and all subsequent terms are divisible by $P_2^2$. But in the second term $\mathfrak{F}'_1(\xi_0^{(1)})$ is not divisible by $P_1$ (because $\mathfrak{F}'_1(\xi)$ cannot have a common divisor with the irreducible polynomial $\mathfrak{F}_1(\xi)$), we see the following congruence:

$$\mathfrak{F}_1(\xi_0^{(1)}) \equiv \pi_1 \mathfrak{F}'_1(\xi_0^{(1)}) \not\equiv 0 \pmod{P_1^2}.$$ 

Thus the numbers $\xi_0^{(1)}, \ldots, \xi_0^{(h)}$ can be chosen from the beginning in such a way that the terms

$$\mathfrak{F}_1(\xi_0^{(1)}), \mathfrak{F}_2(\xi_0^{(2)}), \ldots, \mathfrak{F}_h(\xi_0^{(h)})$$

are divisible once and only once by the corresponding divisors

$$P_1, P_2, \ldots, P_h.$$ 

If this occurs, then it is also possible to find an algebraic number $\xi_0$, so that

$$(13.) \quad \xi_0 \equiv \xi_0^{(i)} \pmod{P_i^2} \quad (i = 1, 2, \ldots, h).$$

There always exists a number $\Pi_1$ in the domain $(\mathfrak{G})$ that is not divisible by $P_1$, but is divisible by all other divisors of $p$. We can find another number $\varrho_1$ so that

$$\varepsilon_1 = \varrho_1 \Pi_1^2 \equiv 1 \pmod{P_1^2}.$$ 

Then the number $\varepsilon_1$ is divisible by each of the divisors $P_2^2, \ldots P_h^2$ while the

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$^{51}$Hensel doesn’t discuss the denominators in the Taylor expansion. What is actually needed is a version of Taylor’s theorem for polynomials: express the formal polynomial $f(X + Y)$ as a polynomial in Y with coefficients in $\mathbb{Z}[X]$.

$^{52}$If $p = 2$ it does not seem clear that the third term is divisible by $P_1^2$. See the previous footnote.

$^{53}$Hensel is assuming, perhaps without noticing it, that he doesn’t have to worry about the possibility that $\mathfrak{F}'_1(\xi) \equiv 0$. He is correct because finite fields are perfect.

$^{54}$The lemma has been proved.

$^{55}$In the parallel passage of [4], Dedekind simply invokes the Chinese Remainder Theorem from [3]; Hensel is going to give a proof.

$^{56}$Here begins the proof of the Chinese Remainder Theorem in this situation.

*Set $\varrho_1 = x_1 + \pi_1 y_1$, where $\pi_1$ is divisible by $P_1$ exactly once. Then the algebraic
remainder of division by $P_i^2$ is 1. So we take
\[ \varepsilon_1, \varepsilon_2, \ldots, \varepsilon_h \]
to be $h$ algebraic integers chosen so that
\begin{align*}
\varepsilon_i &\equiv 1 \pmod{P_i^2} \\
\varepsilon_i &\equiv 0 \pmod{P_i^2}, \quad (i \neq l)
\end{align*}
and set
\begin{equation}
(14.) \quad \xi_0 = \varepsilon_1 \xi^{(1)} + \varepsilon_2 \xi^{(2)} + \cdots + \varepsilon_h \xi^{(h)}.
\end{equation}
Then $\xi_0$ satisfies the $h$ conditions (13.).\(^{57}\) Now from (12\(a\)) and (12\(b\)), we have
\begin{equation}
(15.) \quad \begin{cases}
\mathfrak{f}_i(\xi_0) \equiv \mathfrak{f}_i(\xi^{(i)}_0) \equiv 0 \pmod{P_i} \\
\mathfrak{f}_i(\xi_0) \equiv \mathfrak{f}_i(\xi^{(i)}_0) \not\equiv 0 \pmod{P_i^2}
\end{cases}
\end{equation}
Moreover, the expression $\mathfrak{f}_i(\xi_0)$ is not divisible by any prime divisors $P_l$ distinct from $P_i$. (If so, since $\mathfrak{f}_l(\xi_0)$ is certainly divisible by $P_l$, the irreducible polynomials $\mathfrak{f}_i(\xi)$ and $\mathfrak{f}_l(\xi)$ would have a common divisor, which means they would be congruent to each other, which conflicts with the above assumption.)

Since $\xi_0$ is chosen according to the condition (14.), for each $i$ this number satisfies the irreducible congruence of degree $\kappa_i$
\[ \mathfrak{f}_i(\xi_0) \equiv 0 \pmod{P_i} \]
and this expression is not divisible by any other prime factor, which means\(^{60}\)
\[ p + u \mathfrak{f}_i(\xi_0) \sim P_i. \]

---

\(^{57}\)So we have proved the Chinese Remainder Theorem.

\(^{58}\)Because they are irreducible.

\(^{59}\)Hensel means “by any other prime factor of $p$.”

\(^{60}\)Here $\sim$ denotes equivalence of divisors in Kronecker’s sense. We would say that $P_i$ is the greatest common divisor of $p$ and $\mathfrak{f}_i(\xi_0)$.

numbers $x_1$ and $y_1$ are defined by the system of linear congruences for the Modulus $P_1$
\begin{align*}
x_1P_1^2 &\equiv 1 \pmod{P_1}, \quad \frac{(x_1 + \pi_1 y_1)P_1^2 - 1}{\pi_1} \equiv 0 \pmod{P_1},
\end{align*}
which always has a solution.
If this is the case, then we can prove exactly the same way as was done for
the fundamental form of \( w_0 \) in §3 of the previous work\(^{61}\) that the congruence
of degree \( n \)
\[
\mathfrak{f}_1(\xi) \mathfrak{f}_2(\xi) \cdots \mathfrak{f}_h(\xi) \equiv 0 \pmod{p},
\]
is the smallest\(^{62}\) that \( \xi_0 \) satisfies, which means it is the element we need for
the converse\(^{63}\) of Theorem (B.).

Recalling that the degree \( \kappa \) of a prime divisor \( P \) of \( p \) coincides with the
degree (as a polynomial in \( w \)) of the corresponding irreducible modulo \( p \)
factor \( \mathfrak{f}(w) \) from (2.), we can restate\(^{64}\) the result without using \(^{65}\) the prime
factorization of \( p \), in the following manner:

\(^{61}\)Again, this is \([10]\).
\(^{62}\)I.e., is of the smallest degree.
\(^{63}\)So this concludes the proof of the converse: if the inequalities (11.) hold, then an
element \( \xi_0 \) as in the theorem can be found. See \([9]\) for a numerical example.
\(^{64}\)So here we are back to the factorization of the fundamental equation, which is a
polynomial in \( w \) with coefficients in \( \mathbb{Z}[u_1, u_2, \ldots, u_n] \).
\(^{65}\)The idea, of course, is that in general it is hard to find the factorization of \( p \), and
especially so when the condition in the theorem holds. Alas, factoring the fundamental
equation is hard as well. Hensel will address this in the next section.
If
\[ \mathfrak{F}(w) \equiv \mathfrak{F}_1^{\delta_1}(w) \ldots \mathfrak{F}_h^{\delta_h}(w) \pmod{p} \]
is the decomposition of the fundamental equation of a field \((\mathfrak{G})\) into its irreducible factors modulo \(p\), and if the numbers
\[ \lambda_1, \lambda_2, \ldots, \lambda_\gamma, \]
indicate how many of the \(h\) factors \(\mathfrak{F}_1(w), \ldots, \mathfrak{F}_h(w)\) have corresponding degree
\[ \kappa_1, \kappa_2, \ldots, \kappa_\gamma, \]
then all values \(\xi_0\) of domain modulo satisfy a congruence of degree lower than \(n\) if and only one of the \(\gamma\) conditions
\[ \lambda_i > \bar{g} (\kappa_i) (i = 1, 2, \ldots, \gamma) \]
is satisfied. Here the term \(\bar{g}(\kappa)\) is
\[ \bar{g}(\kappa) = \frac{1}{\kappa} (p^\kappa - \sum p^{\delta} + \sum p^{\kappa q^i} - \ldots), \]
and \(q, q', \ldots\) are the distinct prime factors of the number \(\kappa\).

Now take \(\xi_0\) to be some number in the domain \((\mathfrak{G})\); listing the first \(n\) powers of \(\xi_0\) in terms of the fundamental system \(\xi_1, \ldots, \xi_n\), we get \(n\) equations with rational integer coefficients
\[
\begin{align*}
1 &= a_{10} \xi_1 + \cdots + a_{n0} \xi_n, \\
\xi_0 &= a_{11} \xi_1 + \cdots + a_{n1} \xi_n, \\
\vdots \\
\xi_0^{n-1} &= a_{1,n-1} \xi_1 + \cdots + a_{n,n-1} \xi_n.
\end{align*}
\]
Now \(\xi_0\) satisfies a polynomial congruence modulo \(p\) of degree less than \(n\) if

\[66\text{The next few paragraphs relate the field discriminant and the discriminant of an element, introducing the notions of “index” and of “inessential divisor.” Hensel is still tracking Dedekind quite closely, but he would have seen some of this in Kronecker as well. The first step is to relate the fact that }\xi_0\text{ satisfies a congruence of degree less than }n\text{ to its index.}

\[67\text{“Fundamental system” is Kronecker’s name for an integral basis.} \]
and only if the determinant

\[ |a_{ik}| \]

\[(i = 1, \ldots, n)\]

\[(k = i, 1, \ldots, n - 1)\]

of the \(n\) linear equations (16.) is divisible by \(p\). This is because only in this case can we find \(n\) numbers \(A_0, A_1, \ldots, A_{n-1}\) (not all divisible by \(p\)), such that the sum of the first equation multiplied with \(A_0\), the second with \(A_1\), \ldots, and the last with \(A_{n-1}\), gives

\[ A_0 + A_1 \xi_0 + \cdots + A_{n-1} \xi_0^{n-1} \equiv 0 \pmod{p} \]

by making the coefficients of \(\xi_1, \ldots, \xi_n\) on the right side of the equation all divisible by \(p\). When we form the \(n\) systems of equations from (16.) and consider the \(n\) conjugate domains to \(G\), then we see the validity of the equation

\[ \mathcal{D}(\xi_0) = |a_{ik}|^2 D, \]

where \(\mathcal{D}(\xi_0)\) is the discriminant of the equation for \(\xi_0\) and \(D\) is the discriminant of the domain \((G)\).

Each equation discriminant consists, then, of two essentially different parts: on the one hand, the domain discriminant \(D\), which is the same in all discriminants, and on the other the squared determinant \(|a_{ik}|^2\), which is dependent on the choice of \(\xi_0\). For this reason, Kronecker called the first the essential and the second the inessential divisor of the discriminant \(\mathcal{D}(\xi_0)\). A prime \(p\) is contained in \(|a_{ik}|^2\) (and so is an inessential divisor of the discriminant \(\mathcal{D}(\xi_0)\)) if and only if \(\xi_0\) satisfies a congruence modulo \(p\) of degree less than \(n\). From our Theorem (C.), it follows now that the first part \(|a_{ik}|^2\) of the discriminant (although it depends on \(\xi_0\)) can contain factors which remain the same whatever \(\xi_0\) is chosen, and so cannot be removed by an

---

\[ ^{68}\text{A bit of linear algebra modulo } p: \text{ we want the system } A[\xi] = [0] \text{ to have a nontrivial solution mod } p, \text{ which requires the determinant to be zero mod } p. \]

\[ ^{69}\text{Since } \xi_1, \ldots, \xi_n \text{ is an integral basis, the only way an algebraic integer will be divisible by } p \text{ is by having all the coefficients divisible by } p. \]

\[ ^{70}\text{This is basically matrix multiplication: the matrix whose columns are the powers of } \xi_0 \text{ and its conjugates is equal to } |a_{ik}| \text{ times the matrix whose columns are the integral basis and its conjugates. Hensel thinks of } n \text{ conjugate domains rather than doing the computation in a normal closure.} \]

\[ ^{71}\text{Dedekind} \ [4] \text{ called the (absolute value of the) determinant } |a_{ik}| \text{ the index of the algebraic integer } \xi_0. \text{ Hensel seems to be content not to have a name for it.} \]
appropriate choice of $\xi_0$. These “common inessential divisors” of all equation discriminants of a domain are the primes $p$ (and only these) for which every number $\xi_0$ of the domain satisfies a congruence of lower than $n$-th degree. By applying Theorem (C.) we get the following:

If

$$F(w) \equiv S_{1}^{d_1}(w) \cdots S_{h}^{d_h}(w) \pmod{p}$$

is the decomposition of the fundamental equation of a field $(\mathfrak{G})$ into its irreducible factors modulo $p$, and if the numbers $\lambda_1, \ldots, \lambda_\gamma$ indicate how many factors have degree $\kappa_1, \ldots, \kappa_\gamma$ as polynomials in $w$, then $p$ is a common inessential divisor of all equation discriminants $D(\xi_0)$ from $(\mathfrak{G})$ if and only if at least one of the $\gamma$ conditions

$$\lambda_1 > \bar{g}(\kappa_1), \ldots, \lambda_\gamma > \bar{g}(\kappa_\gamma)$$

is satisfied.

§2

The result from the previous section can also be expressed in another form that is remarkable in that to apply it we do not need to know the decomposition of $p$ within the domain $(\mathfrak{G})$ or the factorization of the fundamental equation modulo $p$.

Let $P$ be an arbitrary prime factor of $p$ and let $\kappa$ be its [residual] degree. Now if

$$w_0 = u_1 \xi_1 + \cdots + u_n \xi_n$$

...
is a fundamental form\textsuperscript{73} for the domain \((\mathfrak{G})\), we set

\[ w_h = u_1\xi_1^h + \cdots + u_n\xi_n^h \quad (h = 0, 1, \ldots). \]

We know from the previous work\textsuperscript{74} (page 65), that only the first \(\kappa\) of these infinitely many forms

\[ w_0, w_1, \ldots, w_{\kappa-1} \]

are distinct modulo \(P\) (for indeterminate \(u_1, \ldots, u_n\)). In fact, \(w_\kappa \equiv w_0\), \(w_{\kappa+1} \equiv w_1, \ldots\), and in general

\[ w_{h+\kappa} \equiv w_h \pmod{P}. \]

From this it follows that the linear form\textsuperscript{76}

\[ w_\nu - w_0 = u_1(\xi_1^{\nu} - \xi_1) + u_2(\xi_2^{\nu} - \xi_2) + \cdots + u_n(\xi_n^{\nu} - \xi_n) \]

is divisible by the prime divisor \(P\) if and only if the number \(\nu\) is a multiple of the degree \(\kappa\). If this is the case, then the following simple considerations show that \(P\) is only contained once in that linear form\textsuperscript{77}

\textsuperscript{73}That is, \(\{\xi_1, \xi_2, \ldots, \xi_n\}\) is an integral basis and the \(u_i\) are indeterminates.

\textsuperscript{74}Namely \(\text{[16]}\).

\textsuperscript{75}Most readers will not be used to Kronecker’s form-based version of algebraic number theory. The thing to note is that the form \(w_h\) is a “generic element” of the ideal generated by \(\xi_h^i\). Whenever Hensel talks of such a form, one can translate to Dedekindian terms by considering the ideal generated by the coefficients.

The forms \(w_h\) are lifts of the images of \(w_0\) under the Frobenius automorphism modulo \(P\), which is of order \(\kappa\). This gives the congruence claims that follow immediately. Hensel does not have any of this language at his disposal, of course.

\textsuperscript{76}This is the key object for this section. In Dedekindian terms, we are considering the ideal \(I_\nu\) generated by the elements \(\xi_1^{\nu} - \xi_1\). Notice that these ideals depend on the choice of integral basis.

\textsuperscript{77}The claim is that when \(\kappa|\nu\) the ideal \(I_\nu\) is divisible exactly once by each prime of degree \(\kappa\) appearing in the factorization of \(p\). It is clear that \(P\) divides \(w_\nu - w_0\), so the key thing to prove is that \(P^2\) does not. The next two paragraphs appear to provide a proof of this claim. In fact, however, what they show is that the forms \(w_\nu - w_0\) may need to be modified so that this is true. Hensel phrases this as modifying the integral basis, but after the modification he suggests the list \(\xi_1, \xi_2, \ldots, \xi_n\) is no longer an integral basis.

The “proof” is divided into two cases: when \(P^2\) divides \(p\) and when it does not. In the first case, we have an actual proof. In the second a change to the integral basis is needed. Hensel is assuming we do not know the factorization of \(p\), however, so he will make the modification in any case.
If, first, \( P \) is a multiple divisor of \( p \), then \( P^2 \) is contained in \( p \) and we know from Fermat’s [little] theorem, that for all integer values of \( u_1, \ldots, u_n \) the congruence
\[
w_\nu \equiv w_0^\nu \pmod{p}
\]
holds. The same is also fulfilled a fortiori modulo the divisor \( P^2 \) of \( p \). If it were true that even for indeterminate \( u_1, \ldots, u_n \) we had \( P^2 \) dividing the linear form \( w_\nu - w_0 \), then all numbers \( \xi_0 = a_1\xi_1 + \cdots + a_n\xi_n \) in (\( \mathfrak{S} \)) would satisfy the congruence
\[
\xi_0^{p^\nu} - \xi_0 \equiv 0 \pmod{P^2}.
\]
That this is not the case\(^{78}\) can be easily seen if we consider, for example, \( \xi_0 = \pi \), where \( \pi \) is divisible by \( P \), but not by \( P^2 \); in this case, the left side of the congruence reduces modulo \( P^2 \) to \((-\pi)\), since \( \pi^{p^\nu} \) is clearly divisible by \( P^2 \).

Next, if \( P \) divides \( p \) only once and if the linear form \( w_\nu - w_0 \) for indeterminate \((u_1, \ldots, u_n)\) is divisible by \( P^2 \), then we must have
\[
\xi_i^{p^\nu} - \xi_i \equiv 0 \pmod{P^2} \quad (i = 1, 2, \ldots, n)
\]
for the \( n \) elements of the fundamental system, which as can easily be seen is generally\(^{79}\) not the case. For if the system \((\xi_1, \ldots, \xi_n)\) did have this property, we could modify it, without changing its character modulo \( p \), so that this  

\(^{78}\)So when \( P^2 \) divides \( p \) the form \( w_\nu - w \) is never divisible by \( P^2 \). The argument involves choosing a uniformizer at \( p \).

\(^{79}\)In the unramified case it is indeed possible for \( \xi_i^{p^\nu} - \xi_i \) to be divisible by \( P^2 \). A simple example is to take \( K = \mathbb{Q}(\sqrt{3}) \) with \( \xi_1 = 1 \) and \( \xi_2 = \sqrt{3} \). Let \( p = 11 \). Then \( \xi_1^{11} - \xi_1 = 0 \) and
\[
\xi_2^{11} - \xi_2 = (\sqrt{3})^{11} - \sqrt{3} = 242\sqrt{3} = 2 \cdot 11^2\sqrt{3}.
\]
If, as Hensel suggests, we instead use \( \xi_1 = 12 \), then it works, since \( 12^{11} - 12 \) is divisible by \( 11 \) only once.

The most dramatic example is the cyclotomic field generated by an \( \ell \)-th root of unity when \( p \equiv 1 \pmod{\ell} \). If we take the standard integral basis, then \( w_\nu = w_0 \) for all \( \nu \), and \( w_\nu - w_0 = 0 \).

For a cubic example, let \( K = \mathbb{Q}(\alpha) \) with \( \alpha^3 - 6\alpha^2 - 9\alpha - 1 = 0 \) (number field 3.3.3969.2 in \([20]\)). The integral basis is \((1, \alpha, \alpha^2)\) and when \( p = 5 \) both \( \alpha^5 - \alpha \) and \( \alpha^{10} - \alpha^2 \) turn out to be divisible by the square of one of the primes dividing \( 5 \).

\(^{80}\)Hensel is correct that this does not change the reduction mod \( p \) of the \( \xi_i \), but of course they may no longer be an integral basis.
exception does not occur. It suffices to replace one of the \( n \) elements \( \xi_i \) by \( \xi_i + p \). We know from the binomial theorem that

\[
(\xi_i + p)^{p^\nu} - (\xi_i + p) \equiv (\xi_i^{p^\nu} - \xi_i) - p \equiv -p \not\equiv 0 \pmod{P^2},
\]

because all other terms are divisible by \( p^2 \) and so by \( P^2 \). The easiest way of avoiding the occurrence of this case, without knowing the prime factors of \( p \), is, as is always possible, to assume that the first element of the fundamental system a priori equals one, and then instead introduce \( \xi_1 = 1 + p \), which does not change the character of the fundamental system modulo \( p \) at all. Then this exception can not occur for any prime factor of \( p \) because

\[
\xi_1^{p^\nu} - \xi_1 = (1 + p)^{p^\nu} - (1 + p) \equiv -p \pmod{p^2}.
\]

In this case \( w_\nu - w_0 \) will not contain any prime divisors from \( p \) more than once.

The result of these quick observations is summarized in the following theorem:

When \( u_1, \ldots, u_n \) are indeterminates, the linear form

\[
w_\nu - w_0 = u_1(\xi_1^{p^\nu} - \xi_1) + \cdots + u_n(\xi_n^{p^\nu} - \xi_n),
\]

which is of degree \( p^\nu \) with respect to the elements of the fundamental system \( \xi_1, \ldots, \xi_n \), is divisible by the product of all distinct prime divisors of \( p \) whose degree \( \kappa \) is an exact divisor of \( \nu \), and contains each of these exactly once.

From this theorem we can draw an interesting conclusion, which is of [141]
importance for a subsequent investigation:

If the prime number $p$ contains prime factors which are pairwise distinct, so that

$$p \sim P_1 P_2 \ldots P_h,$$

then we can always find a linear form $w_\nu - w_0$ divisible by the prime $p$, or, equivalently, such that the $n$ congruences:

$$\xi_i^{\nu} \equiv \xi_i \pmod{p} \quad (i = 1, 2, \ldots, n)$$

are all satisfied. Given the theorem, we only need to choose $\nu$ to be the least common multiple of the $h$ degrees $\kappa_1, \ldots, \kappa_h$ of the prime divisors $P_1, \ldots, P_h$.

On the other hand, if $p$ contains a prime divisor more than once, then none of the linear factors $w_\nu - w_0$ is divisible by $p$, because none of those differences can contain a multiple factor of $p$ more than once. Thus we have the following theorem:

The prime $p$ decomposes in the domain $(\mathfrak{G})$ into a product of distinct prime divisors if and only if at least one of the linear forms $w_\nu - w_0$ is divisible by $p$.

Since the number $p$ is a divisor of the domain discriminant when and only when it contains at least one multiple prime factor, we can state as a corollary of the previous result the following theorem:

The prime $p$ is contained in the discriminant of the domain $(\mathfrak{G})$ if and only if one of the linear forms $(w_\nu - w_0)$ is divisible by a fractional power of $p$, but not divisible by $p$ itself.

---

83Hensel will use the theorem to derive a criterion for $p$ to be ramified in $(\mathfrak{G})$. The point is that $p$ is unramified if and only if it divides one of the ideals $I_\nu$. Note that he knows that this is equivalent to $p$ not dividing the field discriminant.

84As Petri notes in [22, 2.4], it is unclear which “subsequent investigation” Hensel has in mind.

85Sic, but he wants to say “linear forms.”

86“Gattungsdiscriminante.”

87We have translated the theorem as Hensel states it, but the statement seems incorrect. What he had proved is that $p$ is unramified if and only if it divides one of the forms $w_\nu - w$ (equivalently, one of the ideals $L_\nu$). The negation would then say that $p$ is ramified if and only if it divides none of them. It is also unclear why he brings in “a fractional power of $p$.”
Now let \( \kappa \) be an arbitrary whole number. We will form the product:

\[
F_\kappa(w_0) = \frac{(w_\kappa - w_0) \prod (w_{\frac{\kappa}{q'}} - w_0) \prod (w_{\frac{\kappa}{qq''}} - w_0) \ldots}{\prod (w_{\frac{\kappa}{q}} - w_0) \prod (w_{\frac{\kappa}{qq'}} - w_0) \ldots},
\]

where \( q, q', q'', \ldots \) are the distinct prime factors of \( \kappa \). More simply,

\[
F_\kappa(w_0) = \prod_{d|\kappa} (w_d - w_0)^{\varepsilon_d},
\]

where \( \varepsilon_d = \pm 1 \), according to whether the divisor of \( \kappa \) complementary to \( d \) is a product of an even or odd number of distinct prime factors \( q, q', q'' \ldots \) of \( \kappa \), and where \( \varepsilon_d = 0 \) when the ratio \( \frac{\kappa}{d} \) contains repeated prime factors.\(^{88}\)

This quotient is a rational function\(^{89}\) of the elements \((\xi_1, \ldots, \xi_\ell)\) of the fundamental system of \((G)\). Its dimension\(^{90}\) with respect to these elements is

\[
g(\kappa) = p^\kappa - \sum_q p_{\frac{\kappa}{q}} - \sum_q p_{\frac{\kappa}{qq'}} - \cdots = \sum_{d|\kappa} \varepsilon_d p^d.
\]

We immediately recognize that it [namely, \( F_\kappa(w_0) \)] is equivalent to\(^{91}\) the product of all distinct prime divisors of \( p \) whose degree is exactly equal to \( \kappa \).

If \( \bar{p} \) is a prime divisor of \( p \) whose degree \( \bar{\kappa} \) is not a divisor of \( \kappa \), then \( \bar{p} \) is contained in neither the numerator or denominator of \( F_\kappa(w_0) \).\(^{92}\) However, \( \bar{\kappa} \) is a divisor of \( \kappa \) then one shows exactly as in the corresponding question\(^{94}\) in the theory of cyclotomic equations, that \( \bar{p} \) occurs in the denominator just as often as it occurs in the numerator of \( F_\kappa(w_0) \).\(^{95}\)

\(^{88}\)In modern terms \( \varepsilon_d = \mu(\kappa/d) \), where \( \mu \) is the Möbius function. It apparently was introduced by Möbius in 1832, but was clearly not part of the standard toolkit.

\(^{89}\)In terms of ideals, the \( F_\kappa(w_0) \) correspond to fractional ideals \( F_\kappa = \prod_{d|\kappa} I_d' \), where the \( I_d \) are as above.

\(^{90}\)Hensel means the degree of the rational function in the symbols \( \xi_i \).

\(^{91}\)The claim is that this rational function is equivalent, in the sense of Kronecker, to a product of prime divisors. In Dedekind’s terms, Hensel is saying that the ideal \( F_\kappa \) is the product of these primes. This is incorrect, since Hensel ignores completely the primes that do not divide \( p \). In other words, the argument continues to be local.

\(^{92}\)First, divisors of degree not dividing \( \kappa \) do not divide any of the forms that make up \( F_\kappa(w_0) \).

\(^{93}\)Next, divisors of degree dividing \( \kappa \) but unequal to \( \kappa \) cancel out. This is the point of the complicated quotient.

\(^{94}\)The reference is to the formula for the \( n \)-th cyclotomic polynomial \( \Phi_n(x) \) in terms of the polynomials \( x^d - 1 \) for \( d \) dividing \( n \). See, for example, \(^{19}\) p. 285, where the notation \( \varepsilon_d \) is also used.

\(^{95}\)Divisors of degree \( \kappa \) occur exactly once.
contained once and only once in the numerator of the $F_\kappa(w_0)$, in the linear form $(w_\kappa - w_0)$, and thus our claim is proved.

If then $P^{(1)}, P^{(2)}, \ldots P^{(\lambda_\kappa)}$ are all the distinct prime factors of $p$ whose degree is equal to $\kappa$, then $F_\kappa(w_0)$ is equal to their product. This means, we have an equivalence:

$$F_\kappa(w_0) = \prod_{d|\kappa} (w_d - w_0)^{\epsilon_d} \sim P^{(1)} P^{(2)} \ldots P^{(\lambda_\kappa)}.$$

Taking norms it follows that

$$N(F_\kappa(w)) = p^{\kappa\lambda_\kappa} = p^{L_\kappa}$$

[So we get]

The form $F_\kappa(w)$, which has dimension $g(\kappa)$ with respect to $\xi_1, \ldots, \xi_n$, has degree $\kappa\lambda_\kappa$, where $\lambda_\kappa$ is the number of distinct prime factors of $p$ of degree $\kappa$. If no prime factor of degree $\kappa$ exists, then $\lambda_\kappa = 0$.

If we construct the $n$ forms

$$F_\kappa(w_0) \quad (\kappa = 1, 2, \ldots, n),$$

we know that their degree $L_\kappa$ equals $\kappa\lambda_\kappa$; from the previously proven theorem (D.), the prime $p$ is an inessential divisor of all equation discriminants \[143\] $D(\xi_0)$ from (G) when at least one of the inequalities

$$\lambda_\kappa > \bar{g}(\kappa) = \frac{1}{\kappa} g(\kappa) \quad \text{or} \quad \kappa\lambda_\kappa > g(\kappa)$$

96 Paragraph break inserted here to improve readability. Note that what is missing here is any attempt to deal with primes that are not divisors of $p$. See the numerical example below.

97 As a divisor in Kronecker’s sense.

98 The norm of a prime of degree $\kappa$ is of course $p^\kappa$. The equation is not actually true, since Hensel is silently ignoring primes that are not divisors of $p$.

99 The equation effectively defines the number $L_\kappa$. Numerical examples show (see below) that the norm need not be a power of $p$, so we should take $L_\kappa$ as the $p$-adic valuation of the norm instead.

100 I have left the statement of both theorems in Hensel’s terms, “dimension” and “degree” unchanged. The first means the degree of the rational function, while the second means the residual degree of the corresponding divisor. The “dimension” is just $g(\kappa)$, which we can easily compute in any case.

101 “Ordnungszahlen.” That is not quite the right word, since only the $p$-part of the norm has been computed. $L_\kappa$ is actually the $p$-adic valuation of the norm.
holds. Now since $\kappa \lambda_\kappa$ is the degree of the form $F_\kappa(w_0)$ and $g(\kappa)$ is the dimension with respect to $\xi_1, \ldots, \xi_n$, we can state the previously found result in a more elegant and simple form:

The prime $p$ is a common inessential divisor of the equation discriminants $D(\xi_0)$ of the ring of integers ($\mathfrak{G}$) if among the forms

$$F_\kappa(w_0) = \prod_{d|\kappa} (w_d - w_0)^{\kappa_d} \quad (\kappa = 1, \ldots, n)$$

at least one exists whose dimension with respect to the elements $\xi_1, \ldots, \xi_n$ of the fundamental system is smaller than the degree, that is than the exponent $L_\kappa$ of $p$ in the equation

$$N(F_\kappa(w)) = p^{L_\kappa}.$$

§3

The question of common inessential discriminant divisors can now be handled in an entirely different fashion, leading to an entirely different criterion for them to occur.

Let

$$(\xi_1^{(0)}, \ldots, \xi_n^{(0)})$$

A numerical example is clarifying. Suppose $K$ is the number field obtained by adjoining a root $\alpha$ of the polynomial $x^4 + x^3 + 6x^2 + 2x + 12$. (Global field 4.0.13564.1 in [20].) An integral basis is then $(1, \xi_2 = \alpha, \xi_3 = \frac{1}{2}(\alpha^2 + \alpha^3), \xi_4 = \alpha^3)$. Let $p = 2$ and set $\xi_1 = 1 + p = 3$. We want to consider the ideals $I_k$ generated by $\xi_1^{k_d} - \xi_i$. Then $N(F_1) = N(I_1) = 24 = 2^3 \cdot 3$, so $L_1 = 1 \lambda_1 = 3$. We have $N(F_2) = N(I_2 I_1^{-1}) = 1$ and $N(F_3) = N(I_3 I_1^{-1}) = 1$, so $L_2 = L_3 = 0$. Finally, $N(F_4) = N(I_4 I_2^{-1}) = 11$ is not divisible by 2, so $L_4 = 0$. This tells us 2 is divisible by three prime ideals of degree 1 and by none of degrees 2, 3, or 4. Since there are only two irreducible polynomials of degree 1 in $F_2[x]$, it follows that 2 is a common inessential discriminant divisor in this field. (In fact, the factorization is $(2) = p_1^2 p_2 p_3$ with all prime factors of degree 1.)

Notice that in this case 2 is ramified. Hensel claimed above that this happens if and only if 2 does not divide any of the ideals $I_k$, which is easy to check is the case. So for this field 2 it is both an essential and an inessential divisor of the discriminant.
be a fundamental system for the field \((G)\) and let

\[
\begin{align*}
  w^{(0)} &= u_1\xi_1^{(0)} + \cdots + u_n\xi_n^{(0)}, \\
  w^{(1)} &= u_1\xi_1^{(1)} + \cdots + u_n\xi_n^{(1)}, \\
  \vdots \\
  w^{(n-1)} &= u_1\xi_1^{(n-1)} + \cdots + u_n\xi_n^{(n-1)}
\end{align*}
\]

be the fundamental forms for the field \((G)\) and its conjugates. Then the discriminant of the fundamental equation is

\[
D = \prod_{a \neq \beta} (w^{(a)} - w^{(\beta)}) \quad (a, \beta = 0, 1, \ldots, n - 1).
\]

This is a homogeneous function of \(u_1, \ldots, u_n\) with integer coefficients. The greatest common divisor of all of these coefficients is a whole number, which I have shown in the previous work (page 78) agrees with the field discriminant, which is to say, with the square of the determinant

\[
|\xi_1^{(\kappa)}|^2 \quad (i = 1, 2, \ldots, n) \\
|\xi_i^{(\kappa)}|^2 \quad (\kappa = 0, 1, \ldots, n - 1).
\]

So we have

\[
D(u_1, \ldots, u_n) = \Delta^2(u_1, \ldots, u_n)|\xi_1^{(\kappa)}|^2,
\]

where \(\Delta(u_1, \ldots, u_n)\) is a homogeneous function of \(u_1, \ldots, u_n\), whose dimension is clearly equal to \(\frac{n(n-1)}{2}\), and whose coefficients no longer have any common divisors, which is to say \(\Delta(u_1, \ldots, u_n)\) is a primitive polynomial function in \(u_1, \ldots, u_n\).

Now suppose that instead of \(w^{(0)}\) we choose a number from the domain

\[
\xi^{(0)} = a_1\xi_1^{(0)} + \cdots + a_n\xi_n^{(0)}
\]

\[\text{I.e., an integral basis. The subscripts (0) have been added because Hensel is about to consider conjugates.}\]

\[\text{104 As usual, this is [16].}\]

\[\text{105 There are no equations in this section marked (1.) or (2.).}\]

\[\text{106 In fact } \Delta \text{ is the “index form,” i.e., in computes the index of } \mathbb{Z}[\xi^{(0)}] \text{ in the ring of integers. For his example of a cubic field in which 2 is a common inessential discriminant divisor in } [4], \text{ Dedekind computed it explicitly.}\]
together with its conjugates. Then we obtain its equation discriminant if in (3.) we substitute the indeterminates $u_1, \ldots, u_n$ by the whole numbers $a_1, \ldots, a_n$. The prime $p$ is an inessential divisor of the discriminant if and only if it is contained in $\Delta(a_1, \ldots, a_n)$. So we have the following theorem:

The prime $p$ is a common inessential equation discriminant divisor for the field $(\mathfrak{G})$ if and only if the primitive form

$$\Delta(u_1, \ldots, u_n)$$

is divisible by $p$ for all integer values of the indeterminates $u_1, \ldots, u_n$.

The question of when a polynomial form has values divisible by $p$ for all integer values of the indeterminates is fully answered by the following theorem:

A form $U(u_1, \ldots, u_n)$ has value divisible by a prime $p$ for all integer values of the indeterminates if it contains the module system:

$$(4.) \quad (p; u_1^p - u_1, \ldots, u_n^p - u_n);$$

that is, when $U$ can be written in the form

$$(4^a.) \quad U(u_1, \ldots, u_n) = pU_0 + (u_1^p - u_1)U_1 + \cdots + (u_n^p - u_n)U_n,$$

where $U_0, U_1, \ldots, U_n$ are integral polynomials in $u_1, \ldots, u_n$.

This theorem can most easily be proved through induction. It is obviously true when no variable is present; we now assume that it is proved for the case of $n - 1$ variables $(u_2, \ldots, u_n)$ and prove it for $n$ variables. If the form $U(u_1, \ldots, u_n)$ has degree higher than $p - 1$ in $u_1$, then it can be reduced modulo $u_1^p - u_1$ to another form $\bar{U}(u_1, \ldots, u_n)$, whose degree in $u_1$ is at most

---

107 This is known as Hensel’s criterion for common inessential discriminant divisors. The idea is to compute the index form and then check that its values are always divisible by $p$. Hensel will state it first, then give an explicit way to test a form to see if all its values are indeed divisible by $p$, then summarize the whole thing into a theorem.

108 The proof is in this and the next paragraph.

109 In fact the proof is constructive: take the form, divide it by $u_1^p - u_1$, look at the coefficients of the resulting polynomial in $u_1$, rinse, repeat.
equal to \( p - 1 \), since clearly
\[
   u_1^p \equiv u_1, u_1^{p+1} \equiv u_1^2, \ldots, u_1^{p+i} \equiv u_1^{i+1} \pmod{(u_1^p - u_1)}.
\]
If we write the functions according to the powers of \( u_1 \), we see get congruence:
\[
(5.) U(u_1, \ldots, u_n) \equiv \bar{U} = \bar{U}_0 u_1^{p-1} + \bar{U}_1 u_1^{p-2} + \cdots + \bar{U}_{p-1} \pmod{(u_1^p - u_1)},
\]
and the function \( \bar{U}(u_1, \ldots, u_n) \) will be divisible by the prime \( p \) for every whole integer value of \( (u_1, \ldots, u_n) \) when the same is also the case for \( U(u_1, \ldots, u_n) \) and conversely, since they differ by a multiple of \( u_1^p - u_1 \), which for every whole value of \( u_1 \) is itself a multiple of \( p \) according to Fermat’s theorem.

Now if we give \( u_2, \ldots, u_n \) integer values \( a_2, \ldots, a_n \), \( \bar{U}_0, \ldots, \bar{U}_{p-1} \) become equal to integers \( A_0, \ldots, A_{p-1} \). The resulting expression for \( \bar{U} \)
\[
   A_0 u_1^{p-1} + A_1 u_1^{p-2} + \cdots + A_{p-1},
\]
must be divisible by \( p \) when we let \( u_1 \) be equal to each of the \( p \) incongruent numbers \( 0, 1, \ldots, p - 1 \). But an expression of degree \( p - 1 \) can only vanish modulo \( p \) for \( p \) incongruent values of \( u_1 \) if all its coefficients are divisible by \( p \). So it follows that \( \bar{U}(u_1, \ldots, u_n) \) is only divisible by \( p \) for all integer value systems if the same is true for the \( p \) coefficients \( \bar{U}_0(u_2, \ldots, u_n), \ldots, \bar{U}_{p-1}(u_2, \ldots, u_n) \), which are functions only of \( u_2, \ldots, u_n \). If this is the case, then according to inductive assumption all of these coefficients contain the divisor system \( p; u_2^p - u_2, \ldots, u_n^p - u_n \). The same now follows for the whole form \( \bar{U}(u_1, u_2, \ldots, u_n) \), and from equation (5) it follows that the function being investigated contains the divisor system
\[
   (p; u_1^p - u_1, \ldots, u_n^p - u_n),
\]
since it differs from the previous by only a multiple of \( (u_1^p - u_1) \); and thus the theorem is proved.

With the help of this theorem we now have the following criterion for the occurrence of a common inessential discriminant divisor:

\[\text{[146]}\]
\[\text{I.e., a polynomial of degree } p - 1 \text{ cannot have } p \text{ roots.}\]
If
\[ \Delta^2(u_1, \ldots, u_n) \]
is the discriminant of the fundamental equation of the domain \((\mathfrak{G})\) with its numerical factor removed, then the prime \(p\) is a common inessential divisor of the equation discriminants for \((\mathfrak{G})\) if and only if the primitive polynomial form \(\Delta(u_1, \ldots, u_n)\) of degree \(\frac{1}{2}n(n-1)\) contains the divisor system
\[ (p; u_1^p - u_1, \ldots, u_n^p - u_n), \]
(in the sense of Kronecker’s Festschrift), that is, if there is an equation
\[ \Delta(u_1, \ldots, u_n) = U_0p + U_1(u_1^p - u_1) + \cdots + U_n(u_n^p - u_n), \]
where \(U_0, U_1, \ldots, U_n\) are polynomials in \(u_1, \ldots, u_n\).

In his Festschrift for E. E. Kummer’s doctorate anniversary, L. Kronecker mentioned\(^{111}\) the possibility of such common inessential discriminant divisors, adding that this occurs, for example, when the primitive form \(\Delta(u_1, \ldots, u_n)\) can be expressed as a homogeneous polynomial function of \((u_i^p - u_i)\). The preceding simple observations show us\(^ {112}\) that this is the essential issue, as long as we add a term of the form \(pU_0(u_1, \ldots, u_n)\) to every expression. But to complete the proof we required the result from the previous paper (Page 78) that the form \(\Delta(u_1, \ldots, u_n)\) is primitive, which means that the discriminant \(D(u_1, \ldots, u_n)\) of the fundamental equation contains no numerical divisor beyond the field discriminant.

If we know the primitive form \(\Delta(u_1, \ldots, u_n)\), then it will be very easy to determine whether or not \(p\) is a common inessential discriminant divisor in \((\mathfrak{G})\). We first reduce the coefficients of this form to their smallest remainder modulo \(p\) and all exponents of \(u_1, \ldots, u_n\) larger than \(p - 1\) to their smallest remainder modulo \(p - 1\); \(p\) is a common inessential divisor of the discriminants\(^{147}\) of \((\mathfrak{G})\) if and only if the resulting form is identical to 0.

\(^{111}\)In \([17] \S 25\). Kronecker’s Gundzüge was first published as to celebrate the fiftieth anniversary of Kummer’s doctorate.

\(^{112}\)This paragraph is difficult to understand, but we think we caught the basic meaning. Hensel is saying that Kronecker observed that this condition was sufficient but that what he has added is that it is also necessary, since the content of the discriminant form is exactly the field discriminant.
To illustrate this way of handling the problem, I will now give the following simple example\(^\text{113}\) which was considered from a different point of view in my doctoral dissertation.

Let \(\nu\) be an arbitrary real prime of the form \(3\mu + 1\) and \(\mathfrak{G}_3(\epsilon)\) be the field generated by the three \(\mu\)-fold periods \(\epsilon_1, \epsilon_2, \epsilon_3\) of the \(\nu\)th roots of unity. We want to find the common inessential discriminant divisors of this field.

The three periods \(\epsilon_1, \epsilon_2, \epsilon_3\) form a fundamental system for the field \(\mathfrak{G}_3(\epsilon)\). Let

\[
\begin{align*}
w_1 &= u_1\epsilon_1 + u_2\epsilon_2 + u_3\epsilon_3, \\
w_2 &= u_1\epsilon_2 + u_2\epsilon_3 + u_3\epsilon_1, \\
w_3 &= u_1\epsilon_3 + u_2\epsilon_1 + u_3\epsilon_2,
\end{align*}
\]

so that the product \((w_1 - w_2)(w_2 - w_3)(w_3 - w_1)\) will be the square root of the discriminant of the fundamental equation. By using the known expressions for the resolvent of the cubic period equation, we get without difficulty the expression

\[
\prod (w_i - w_{i+1}) = -\nu(\alpha\Delta_1 + \beta\Delta_2).
\]

Here \(\alpha\) and \(\beta\) are the integers which occur in the decomposition of \(\nu\) into its prime factors in the field of the third roots of unity, so that

\[
(6.) \quad \nu = (\alpha + 3\beta\rho)(\alpha + 3\beta\rho^2) = \alpha^2 - 3\alpha\beta + 9\beta^2 \quad (\rho^2 + \rho + 1 = 0),
\]

and \(\Delta_1\) and \(\Delta_2\) are the primitive forms:

\[
\begin{align*}
\Delta_1 &= (u_1 - u_2)(u_2 - u_3)(u_3 - u_1) = \sum_{i=1}^{3} u_i^2u_{i+1} - \sum_{i=1}^{3} u_i^2u_{i+1}, \\
\Delta_2 &= \sum (u_i^3 - 3u_i^2u_{i+1}) + 6u_1u_2u_3.
\end{align*}
\]

\(^{113}\)In modern language, Hensel looks at the cyclotomic field corresponding to a prime number \(\nu = 3\mu + 1\). This has a unique cyclic cubic subfield \(\mathfrak{G}_3(\epsilon)\) generated by Gaussian periods \(\epsilon_1, \epsilon_2, \epsilon_3\). In his thesis, Hensel found a sufficient condition for such fields (and more general versions of them) to have common inessential discriminant divisors.

\(^{114}\)Hensel doesn’t say, but he knows and uses, that the three periods are cyclically permuted by the Galois action. The three forms \(w_1, w_2, w_3\) below are, of course, Galois conjugates.
Since the form $\Delta_1$ is only of degree 2 with respect to the quantities $u_1, u_2, u_3$, only the number 2 can occur as a common discriminant divisor. If we reduce the forms $\Delta_1$ and $\Delta_2$ modulo the divisor system

$$M = (2, u_1^2 - u_1, u_2^2 - u_2, u_3^2 - u_3),$$

we quickly see that $\Delta_1$ contains the same and we get the following congruence for the primitive form $(\alpha \Delta_1 + \beta \Delta_2)$:

$$\alpha \Delta_1 + \beta \Delta_2 \equiv \beta[(u_1 + u_2 + u_3) + (u_1 u_2 + u_2 u_3 + u_3 u_1)] \pmod{M},$$

which means the number 2 is only a common discriminant divisor when

$$\beta \equiv 0 \pmod{2}.$$

We can now give this result a more elegant form. The decomposition of $\nu$ in (6.) allows us to represent $4\nu$ as:

$$4\nu = (2\alpha - 3\beta)^2 + 27\beta^2 = A^2 + 27B^2,$$

so that the number $4\nu$ can always be expressed as $A^2 + 27B^2$, and the uniqueness of the decomposition in (6.) shows us that our depiction is also unique. From the two equations

$$A = 2\alpha - 3\beta, B = \beta$$

it follows that $A$ and $B$ are only divisible by 2 if the same is true for $\beta$, which means that in this case not only $4\nu$ but also $\nu$ itself can be expressed in the form $A^2 + 27B^2$. So we have the following theorem:

If $\nu = 3\mu + 1$ is an arbitrary real prime, then the number 2 is a common inessential discriminant divisor in the field $\mathbb{Q}_3(\epsilon)$, if and only if the number $\nu$ can be written in the form

$$\nu = A^2 + 27B^2.$$

For primes less than two hundred whose fields of roots of unity contain cubic subfields, this case occurs for

31, 43, 109, 127, 157, 189.

Note that this is stated incorrectly in [21, 2.2.1, item 3].

Hensel’s actual sentence is something like: “For primes $\nu$ in the first and second hundred this case happens for cubic period equations which are formed by roots of unity of order…”
In his discussion mentioned above of the common inessential discriminant divisors of a domain, Kronecker calls attention to the remarkable circumstance that this can be eliminated if the coefficients \( u_1, u_2, \ldots, u_n \) of the linear form

\[
    w_0 = u_1 \xi_1 + \cdots + u_n \xi_n
\]

are no longer in the domain of the real integers, but rather in the bigger realm of the algebraic numbers from another domain \( \Gamma(\zeta) \). Kronecker does not prove this interesting theorem, however, nor does he specify how the ring of integers should be chosen such that we can avoid the occurrence of a common discriminant divisor. Kronecker did not return to this subject later, and so far I cannot find any hint of a proof in his papers.

I would like to briefly touch on this point to show how to choose the field \( \Gamma(\zeta) \) for the coefficients \( u_1, \ldots, u_n \) so that the prime \( p \) is not a common discriminant divisor, and to determine the field of smallest degree for the adjunction. Finally, we wish to show that for this goal we only need the simplest algebraic numbers, which stem from the roots of unity of prime degree.

---

117Hensel is unclear on what exactly can be “eliminated” here. Suppose \( p \) is a common inessential discriminant divisor for a number field. By the previous theorem, the index form is a primitive form in the variables \( u_i \) which becomes divisible by \( p \) whenever we replace each \( u_i \) by an integer \( a_i \in \mathbb{Z} \). Kronecker’s observation is that we can obtain a value that is not divisible by \( p \) if we allow the values \( a_i \) to be algebraic integers in a larger field (which Hensel calls \( \Gamma(\zeta) \)). (In terms of theorem A, we are replacing \( \mathbb{F}_p \) by a finite field extension to get more irreducible polynomials.)

Hensel proposes to give a proof of Kronecker’s remark and to explain how to find a field \( \Gamma(\zeta) \). In fact, he will claim that he can take \( \Gamma(\zeta) \) to be a subfield of a cyclotomic field \( \mathbb{Q}(\mu_\ell) \) with \( \ell \) a prime.

Denote the original number field by \( K \). In modern terms, Hensel wants to construct an auxiliary number field \( L \), contained in a prime-order cyclotomic field. If we assume that \( K \) and \( L \) are linearly disjoint, then the integral basis \( \{\xi_1, \xi_2, \ldots, \xi_n\} \) of \( K \) over \( \mathbb{Q} \) will also be an integral basis for \( KL \) over \( L \) and the relative discriminant \( d(KL/L) \) is the ideal in \( \mathcal{O}_L \) generated by \( d_K \). Hensel’s result then means that \( p \) is no longer a common index divisor for \( KL/L \). That is, there exists an element \( \theta \in KL \) such that the index of \( \mathcal{O}_L[\theta] \) in \( \mathcal{O}_{KL} \) is not divisible by \( p \).

118This observation is in [17] §25 (p. 384 in volume II of [18]).

119Did Hensel wait until after Kronecker’s death to publish these results because he expected to find such a proof?
The following theorem, which is a simple extension of the one established in the previous sections, leads to these results:

Let

\[(1.) \quad F_1(u_1), F_2(u_2), \ldots, F_n(u_n)\]

be \(n\) integral polynomials, each in one variable \(u_1, \ldots, u_n\), such that each \(F_i(u_i)\) modulo \(p\) has as many incongruent integer roots \(z_i\) as its degree. Further, let \(F(u_1, \ldots, u_n)\) be an integral polynomial in all \(n\) indeterminates \(u_1, \ldots, u_n\). Then the congruence

\[(2.) \quad F(z_1, z_2, \ldots, z_n) \equiv 0 \pmod{p}\]

holds for all congruence roots \(z_1, \ldots, z_n\) of the \(n\) functions (1.) if and only if \(F(u_1, \ldots, u_n)\) contains the divisor system

\[(p; F_1(u_1), \ldots, F_n(u_n))\]

in the sense of Kronecker’s theory.

The proof of this theorem can be carried out in the same way as before, since the argument was based solely on the fact that the congruences of degree \(p\)

\[F_i(u_i) = u_i^p - u_i \equiv 0 \pmod{p}\]

have exactly \(p\) incongruent roots modulo \(p\), so as many as their degree, together with the fact that \(p\) is a prime number.

This theorem can finally also be extended in the following way: we can assume that the coefficients of the \(n + 1\) functions \(F_1(u_1), \ldots, F_n(u_n), F(u_1, \ldots, u_n)\) are no longer real integers, but rather elements of the ring of

\[\text{Working over } \mathbb{F}_p, \text{ the argument boils down to the observation that modulo } F_1(u_1) \text{ the polynomial } F \text{ becomes a polynomial of degree } \deg(F_1) - 1 \text{ in } u_1 \text{ whose coefficients are polynomials in the other variables. Replace } u_2, \ldots, u_n \text{ with arbitrarily chosen roots } z_2, \ldots, z_n. \text{ We get a polynomial in } u_1 \text{ which is zero for all possible choices of } z_1. \text{ Since the number of choices is } \deg(F_1), \text{ so higher than the degree of the specialized polynomial, this polynomial must be identically } 0. \text{ Hence each of the coefficient polynomials has the property that it is zero for all choices of } z_2, \ldots, z_n. \text{ Now use induction.}\]

\[\text{Now we allow the polynomials to have coefficients in some ring of integers and replace } p \text{ by one of its prime divisors; this amounts to working over a finite extension of } \mathbb{F}_p, \text{ and the argument goes through as before.}\]
integers of a field \(\Gamma(\zeta)\) determined by an arbitrary algebraic integer \(\zeta\); only now we must work modulo a prime divisor \(p(\zeta)\) of \(p\) instead of the element \(p\), which inside \(\Gamma(\zeta)\) loses the property of being indecomposable. Now if the functions

\[ F_i(u_i, \zeta) \]

modulo \(p(\zeta)\) contain the same number of incongruent roots \(\zeta_i\) inside the domain \(\Gamma(\zeta)\) as their degree indicates, then we can show as before that for a polynomial \(F(u_1, \ldots, u_n, \zeta)\), the congruences

\[ F(\zeta_1, \zeta_2, \ldots, \zeta_n; \zeta) \equiv 0 \pmod{p(\zeta)} \]

will hold for all value systems \((\zeta_1, \ldots, \zeta_n)\) if and only if the polynomial \(F(u_1, \ldots, u_n, \zeta)\) can be represented by the elements of the divisor system

\[ (p(\zeta); F_1(u_1, \zeta), \ldots, F_n(u_n, \zeta)) \]

in a homogeneous and linear way with integer\(^{123}\) coefficients.

We will now make a special assumption, that the coefficients of each of the \(n+1\) function \(F_i(u_i, \zeta)\) and \(F(u_1, \ldots, u_n, \zeta)\) are, as before, real integers.\(^{124}\) To call attention to this assumption, we will denote them as before by \(F_i(u_i)\) and \(F(u_1, \ldots, u_n)\). The congruence roots \(\zeta_i\) of the function \(F_i(u_i)\) modulo \(p(\zeta)\), however, are still assumed to belong to the ring of integers \(\Gamma(\zeta)\). Now if we reduce the integral function \(F(u_1, \ldots, u_n)\) to the smallest remainder modulo the integral module system

\[ (F_1(u_1), \ldots, F_n(u_n)), \]

we get an integral function \(\bar{F}(u_1, \ldots, u_n)\) of \(u_1, \ldots, u_n\) with real integer coefficients, whose degree in \(u_i\) will always be smaller than the degree of the function \(F_i(u_i)\). This reduced function can only be represented in a homogeneous way by the elements of the system

\[ (p(\zeta), F_1(u_1), \ldots, F_n(u_n)), \]

\(^{122}\)Hensel does not just assume the coefficients are now in \(\mathbb{Z}[\zeta]\), but rather indicates this explicitly in his notation.

\(^{123}\)Hensel probably means that the coefficients are polynomials in the \(u_i\) with integer coefficients.

\(^{124}\)Hensel will assume the polynomials in question have rational integer coefficients, but still wants to allow the variables to take values in \(\Gamma(\zeta)\). He claims that in this case the divisor \(p(\zeta)\) above can in fact be replaced by \(p\).
if all of their coefficients are divisible by $p(\zeta)$, that is to say by $p$ itself. Thus $F(u_1, \ldots u_n)$ contains the divisor system

$$(p(\zeta), F_1(u_1), \ldots, F_n(u_n))$$

if and only if it can be represented in a homogeneous and linear way by the rational system

$$(p, F_1(u_1), \ldots, F_n(u_n))$$

The function $F(u_1, \ldots, u_n)$ vanishes modulo $p(\zeta)$ for all congruence roots of the $n$ functions $F_i(u_i)$ if and only if the congruence

$$(3.) \quad F(u_1, \ldots, u_n) \equiv 0 \modd(p, F_1(u_1), \ldots, F_n(u_n))$$

is satisfied.

We can use this theorem to solve easily the question posed at the beginning of this paper. Take, as in the first paragraph of this paper,

$$w_0 = u_1 \xi_1 + u_2 \xi_2 + \cdots + u_n \xi_n$$

the fundamental form of the ring of integers $(\mathfrak{G})$, and let $w_1, w_2, \ldots, w_{n-1}$ denote the $n-1$ fundamental forms conjugate to $w_0$. Finally, let

$$\Delta^2(u_1, \ldots, u_n)$$

be the discriminant of the fundamental equation freed from its numerical divisors (the field discriminant). Now if $\Gamma(\zeta)$ is another arbitrary field domain and $p(\zeta)$ is a prime divisor of the real prime $p$ in $\Gamma(\zeta)$, we can investigate under which conditions $p(\zeta)$ is a common inessential divisor of all discriminants $\prod (w_i - w_n)$, by now letting $u_1, \ldots, u_n$ be arbitrary algebraic integers of the domain $\Gamma(\zeta)$ instead of arbitrary real integers. Equivalently, we can investigate the conditions under which the primitive form $\Delta(u_1, \ldots, u_n)$ is always divisible by $p(\zeta)$, if we replace $u_1, \ldots, u_n$ by arbitrary integers belonging to the domain $\Gamma(\zeta)$.

Now if $k$ is the degree of the prime divisor $p(\zeta)$ for the domain $\Gamma$, then the number of integers in $\Gamma$ that are incongruent modulo $p(\zeta)$ is equal to

---

125 Recall that Hensel uses “real” to mean “rational.”

126 As before $\Delta$ is the index form.

127 Hensel always thinks in terms of representatives rather than congruence classes; so it’s “the number of incongruent integers” rather than “the number of congruence classes.”
So every number $\zeta$ of this domain satisfies the congruence:

$$u^{p^k} - u \equiv 0 \pmod{p(\zeta)},$$

and so this congruence contains the same number of incongruent roots inside of $\Gamma$ as its degree displays. The above question can now be stated as follows: Under what conditions is the primitive form $\Delta(u_1, \ldots, u_n)$ divisible by $p(\zeta)$ for the congruence roots modulo $p(\zeta)$ of the $n$ functions

$$u_1^{p^k} - u_1, u_2^{p^k} - u_2, \ldots, u_n^{p^k} - u_n.$$

This question is directly answered by the last theorem, if we replace the $n$ functions $F_i(u_i)$ by $u_i^{p^k} - u_i$. So we get the theorem:\[128\]

The prime divisor $p(\zeta)$ in the field of rationality $\Gamma(\zeta)$ is a common inessential divisor of all equation discriminants of $(\mathfrak{G})$ if and only if $\Delta(u_1, \ldots, u_n)$, the discriminant of $(\mathfrak{G})$ freed of its numerical factor, contains the divisor system

$$(4.) \quad P_k = (p; u_1^{p^k} - u_1, \ldots, u_n^{p^k} - u_n)$$

in Kronecker’s sense, where $k$ is the degree of $p(\zeta)$ for the domain $\Gamma$.

Since the above criterion is solely dependent on the degree $k$ of $p(\zeta)$, it applies equally to all divisors of $p$ in $\Gamma$ with the given degree.

It follows from this that in the rationality domain $\Gamma(\zeta)$ the prime divisor $p(\zeta)$ is not a common inessential discriminant divisor of $(\mathfrak{G})$ if and only if the following condition is satisfied:

$$\Delta(u_1, \ldots, u_n) \not\equiv 0 \pmod{(p; u_1^{p^k} - u_1, \ldots, u_n^{p^k})}.$$

This result can now be used to decide what assumptions need to be made on the values taken by coefficients $u_1, \ldots, u_n$ of the fundamental form of $(\mathfrak{G})$, $w_0 = u_1\xi_1 + \cdots + u_n\xi_n$, inside a domain $\Gamma(\zeta)$, so that the discriminant $D(w_0)$ of the $n$ conjugate values $w_0, w_1, \ldots, w_{n-1}$ does not contain any prime factor of $p$ other than those in the discriminant, or equivalently, whether it is possible to choose values for the unknowns $u_1, \ldots, u_n$ in $\Gamma(\zeta)$ so that the form $\Delta(u_1, \ldots, u_n)$ is relatively prime to $p$.

\[128\] The statement is confusing because it refers to the “equation discriminants of $(\mathfrak{G})$” being divisible by a divisor in $\Gamma(\zeta)$. See footnote[117] for our interpretation.
Clearly we will first need to choose \( u_1, \ldots, u_n \), so that the primitive form \( \Delta(u_1, \ldots u_n) \) is coprime to every prime divisor of \( p \) in \( \Gamma(\zeta) \), i.e., so that none of the prime divisors [of \( p \) in \( \Gamma(\zeta) \)] are common inessential divisors of the discriminants \( D(w_0) \) from (\( \mathfrak{S} \)). If

\[
p = p_1^{\epsilon_1}(\zeta) \cdots p_l^{\epsilon_l}(\zeta)
\]

is the decomposition of \( p \) into its prime factors in \( \Gamma(\zeta) \), and if

\[
k_1, \ldots, k_l
\]

are the degrees of the individual distinct prime divisors, then \( p \) can only have the required properties if the primitive form \( \Delta(u_1, \ldots, u_n) \) does not contain any of the \( l \) divisor systems

\[
P_{k_i} = (p; u_1^{p_{ki}} - u_1, \ldots, u_n^{p_{ki}} - u_n) \quad (i = 1, 2, \ldots, l),
\]

where of course we only need to investigate those systems for which the numbers \( k_i \) are distinct. If these conditions are satisfied, then it is easy to see that for the unknowns \( u_1, \ldots, u_n \), such integers \( \zeta_1, \ldots, \zeta_n \) of the domain \( \Gamma(\zeta) \) can be chosen so that the number \( \Delta(\zeta_1, \ldots, \zeta_n) \) is co-prime to \( p \). If each of the \( p_i(\zeta) \) is not a common inessential divisor of the equation discriminant \( D(w_0) \), then for each \( i \) we can find \( n \) numbers \( \zeta^{(i)}_1, \ldots, \zeta^{(i)}_n \) such that

\[
\Delta(\zeta^{(i)}_1, \ldots, \zeta^{(i)}_n) \not\equiv 0 \pmod{p_i(\zeta)}.
\]

If we now consider these numbers for each of \( l \) prime divisors of \( p \), we can choose other numbers \( \zeta_1, \ldots, \zeta_n \) so that for each \( i \) we have

\[
\zeta_1 \equiv \zeta^{(i)}_1, \zeta_2 \equiv \zeta^{(i)}_2, \ldots, \zeta_n \equiv \zeta^{(i)}_n \pmod{p_i(\zeta)} (i = 1, 2, \ldots, l),
\]

and so for each \( i \):

\[
\Delta(\zeta_1, \ldots, \zeta_n) \equiv \Delta(\zeta^{(i)}_1, \ldots, \zeta^{(i)}_n) \not\equiv 0 \pmod{p_i(\zeta)}.
\]

This means the number \( \Delta(\zeta_1, \ldots, \zeta_n) \) is in fact coprime to \( p \).\(^{130}\)

\(^{129}\)Hensel claims here that if we know we can make each of a set of conditions hold separately, we can make them hold simultaneously. The “Chinese remainder theorem” argument is given in the rest of this paragraph.

\(^{130}\)Added a paragraph break here.
We will call the domain $\Gamma(\zeta)$ a *supplementary domain* for the domain $\mathcal{G}(\xi)$ with respect to the prime $p$ if we can choose values in $\Gamma(\zeta)$ for the unknowns $u_1, \ldots, u_n$ in the $n$ conjugate fundamental forms $w_0, w_1, \ldots, w_{n-1}$ so that the discriminant $\prod (w_i - w_k)$ contains the prime $p$ no more than the domain discriminant of $(\mathcal{G})$. Then we can state the necessary and sufficient conditions for $\Gamma(\zeta)$ to be a supplementary domain for $\mathcal{G}(\xi)$ with respect to $p$:

Let $p = p_1^{e_1} p_2^{e_2} \cdots p_l^{e_l}$ be the decomposition of the real prime $p$ into prime factors inside the domain $\Gamma$, and let $k_1, k_2, \ldots, k_\lambda$ be the distinct [residual] degrees [of the $p_i$]. Then $\Gamma(\zeta)$ is a supplementary domain for $(\mathcal{G})$ with respect to the prime $p$ if and only if the discriminant of the fundamental equation of $(\mathcal{G})$ freed from its numerical factor does not contain any of the $\lambda$ divisor systems:

$$P_{k_1}, P_{k_2}, \ldots, P_{k_\lambda}.$$

If the adjoined domain $\Gamma(\zeta)$ is *Galois*, then the degrees of all prime divisors of $p$ are equal. Setting $k$ to be the common value of all the degrees, we can replace the more complicated condition above by the simpler condition that the form $\Delta(u_1, \ldots, u_n)$ does not contain a divisor system

$$P_k = (p; u_1^{p^k} - u_1, \ldots, u_n^{p^k} - u_n).$$

Further, I would also like to remark that in the above theorem all of the divisor systems $P_k$ can be omitted if the index $k$ is a multiple of one of the other numbers $k_1, \ldots, k_\lambda$. If

$$\Delta(u_1, \ldots, u_n) \not\equiv 0 \pmod{P_k},$$

then a fortiori

$$\Delta(u_1, \ldots, u_n) \not\equiv 0 \pmod{P_{ak}};$$

because from the congruence

$$u^{p^k} - u \equiv (u^{p^k} - u)^{p(a-1)k} \equiv 0 \pmod{(p; u^{p^k} - u)}$$

it follows that every divisor system $P_{ak}$ is a multiple of $P_k$: so $\Delta(u_1, \ldots, u_n)$ cannot be divisible by $P_{ak}$ if it does not contain the system $P_k$.  

[154]
§5

We should now discuss which is the supplementary domain \( \Gamma(\zeta) \) of lowest degree for a given domain \((\mathfrak{G})\) with relation to an arbitrary prime \( p \).

To this end, we investigate discriminant of the fundamental equation freed from its numerical factor
\[
\Delta(u_1, \ldots, u_n)
\]
as to its divisibility by the divisor systems
\[
P_1, P_2, P_3, \ldots,
\]
where in each case
\[
P_k = (p; u_1^{p^k} - u_1, \ldots, u_n^{p^k} - u_n).
\]
If the primitive form does not already contain the first system
\[
P_1 = (p; u_1^p - u_1, \ldots, u_n^p - u_n),
\]
then \( p \) is not at all an inessential divisor for \( \mathfrak{G} \), which means the supplementary domain of lowest degree is that of natural integers. If however, \( \Delta(u_1, \ldots, u_n) \) is divisible by \( P_1 \), in the sequence \( P_1, P_2, \ldots \) we must come at last to a divisor system
\[
P_k = (p; u_1^{p^k} - u_1, \ldots, u_n^{p^k} - u_n),
\]
which is no longer contained\(^{131}\) in \( \Delta \). Now choose \( \mu \) large enough so that the power \( p^\mu \) is bigger than \( \frac{n(n-1)}{2} \), which is to say larger than the dimension\(^{132}\) of the primitive form \( \Delta(u_1, \ldots, u_n) \). Then this form cannot be reduced to one of lower degree modulo the system \( (u_1^{p^k} - u_1, \ldots, u_n^{p^k} - u_n) \), because all exponents of \( u_1, \ldots, u_n \) will be smaller than \( p^\mu \). Therefore, the form \( \Delta(u_1, \ldots, u_n) \) could only contain the module system
\[
(p; u_1^{p^\mu} - u_1, \ldots, u_n^{p^\mu} - u_n),
\]
\(^{131}\)Hensel is stating a lemma: it is not possible that for all \( k \) the form \( \Delta \) is a linear combination of the elements in \( P_k \). The proof, given in the rest of this paragraph, is easy: no form can be a linear combination of polynomials of degree bigger than its own degree unless it is zero, so if \( \Delta \in P_k \) for large enough \( k \) it would have to be divisible by \( p \). But \( \Delta \) is primitive. That puts an upper bound on \( k \).
\(^{132}\)As before, Hensel seems to use “dimension” for the degree of a homogeneous form.
if all its coefficients were divisible by $p$, which contradicts the assumption that $\Delta(u_1, \ldots, u_n)$ is primitive. Thus, the form can only contain a finite number of divisor systems $P_1, P_2, \ldots$, and the number of systems it contains will be smaller than $\mu$, where $p^\mu$ is the lowest power of $p$, which is bigger than $\frac{n(n-1)}{2}$.

Now take $P_k$ to be the first module system of the series $P_1, P_2, \ldots, P_{k-1}, P_k$, which is not contained in $\Delta$, so that each of the previous contain the form. Now if

$$\varphi(\zeta) = \zeta^k + a_1\zeta^{k-1} + \cdots + a_k = 0$$

is an equation of degree $k$, whose left side is also irreducible modulo $p$, then the domain $\Gamma(\zeta)$ that it defines will be a supplementary domain for $\mathfrak{G}(\zeta)$ with respect to $p$, and indeed it will be one of the lowest possible degree.

Indeed, $\Gamma(\zeta)$ is a supplementary domain of $(\mathfrak{G})$. The function $\varphi(\zeta)$ is irreducible modulo $p$, so $p$ is itself a prime inside the domain $\Gamma(\zeta)$ whose [residual] degree is $p^k$. The prime $p$ will be a common inessential divisor in the rationality domain $\Gamma(\zeta)$ of the equation discriminants of $(\mathfrak{G}$ if and only if the form $\Delta(u_1, \ldots, u_n)$ contains the divisor system $P_k = (p; u_i^{p^k} - u_i)$, which contradicts the previous assumption.

Furthermore, if $\Gamma_1(\eta)$ is a different domain whose degree is smaller than $k$, and $p(\eta)$ is a prime divisor of $p$, then its degree $k_1$ is at most equal to the degree of $\Gamma_1(\eta)$, and so is smaller than $k$. Therefore $p(\eta)$ is a common inessential divisor for $(\mathfrak{G})$ in the domain $\Gamma_1(\eta)$, because the form $\Delta(u_1, \ldots, u_n)$ contains the divisor system $P_{k_1} = (p; u_i^{p^{k_1}} - u_i)$, whose index is smaller than $k$.

133 This claim is proved in the next two paragraphs. Notice that Hensel doesn’t care if the supplementary field is linearly disjoint from his original field; that only matters if we want to interpret his result in terms of a relative extension $LK/L$ as above. So he can use any equation of degree $k$ which is irreducible mod $p$.

134 Sic, but Hensel means $k$. He says “ihre Ordnung für denselben ist $p^k$,” literally “the order for the same is $p^k$,” so perhaps he means the number of congruence classes?

135 Since all he really needs is for the residual degree of at least one of the factors of $p$ to be equal to $k$, the smallest possible degree for the supplementary field is realized when $p$ is inert and $f = k$.

*A polynomial of degree $k$ that is irreducible modulo $p$ always exists because the number

$$\bar{g}(k) = \frac{1}{k} \sum_{d|k} \epsilon_d p^d$$

of such functions is never equal to zero.
we have the following theorem:

If

\[ P_k = (p; u_1^p - u_1, \ldots, u_n^p - u_n) \]

is the divisor system of lowest degree that is not contained in the primitive form \( \Delta(u_1, \ldots, u_n) \), then the smallest supplementary domain \( \Gamma \) of \( \mathfrak{S}(\xi) \) for the prime \( p \) has degree \( k \). Such a domain will be defined by every polynomial equation of degree \( k \), whose left side is irreducible modulo \( p \).

From the proof it follows also that we can only obtain a supplementary domain defined by an equation of degree \( k \) if the left side is irreducible modulo \( p \). Further [136], the domain \( \Gamma(\zeta) \) defined by the above equation, that is, the totality of rational functions of \( \zeta \) contain all functions, which are irreducible equations modulo \( p \) of \( k \)th degree, and so we see that there is only one domain \( \Gamma(\zeta) \), which is a supplementary domain of lowest degree with respect to \( \mathfrak{S}(\xi) \).

This last statement should be understood as follows. The algebraic integers in two fields of degree \( k \) [defined by equations] which are irreducible modulo \( p \), are pairwise congruent modulo this prime, so that for the question we are considering one domain can be substituted for the other. The algebraic character of the supplementary domains \( \Gamma \) can, however, be very different, which raises the question of which is the algebraically simplest supplementary domain for a given domain \( (\mathfrak{S}) \) with respect to the prime \( p \) [157].

In the previously discussed place [138], in his Festschrift, Kronecker considered the field defined by the cubic equation

\[ \alpha^3 - \alpha^2 - 2\alpha - 8 = 0, \]

for which the number 2 is a common inessential discriminant divisor (as was first noted by Mr. Dedekind in the cited paper). [Kronecker suggests] that

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136We don’t understand this paragraph. Hensel seems to be claiming that there is only one field \( \Gamma(\zeta) \) of this type, but that is not true. The residue fields will all be \( \mathbb{F}_p^k \), of course, and that may be what he is referring to. See the next paragraph, where he admits he doesn’t really mean what he has said.

137Hensel doesn’t say what he means by “algebraically simplest,” of course. It will turn out that he can always choose a cyclotomic field.

138Kronecker’s *Grunzüge* [17] §25.

139Hensel says something like “Kronecker suggests that for this field... for which 2... (as was first noted...), that... That was too much, so we broke it up.

140This is [4], but Dedekind had actually given this example earlier, in [2].
this prime stops being an inessential divisor if the domain $\Gamma$ of the third roots of unity is adjoined to the rational domain. For this domain $(\mathfrak{G})$, then, the very simple cyclotomic field of the third roots of unity is a supplementary domain with respect to the inessential divisor 2.

This suggests an interesting theorem, that for every domain $(\mathfrak{G})$ and an arbitrary prime $p$ we can find a supplementary domain of the greatest algebraic simplicity, namely one constructed from roots of unity of prime degree. In what follows we prove this theorem and develop a method to find the smallest such domain.

Let $(\mathfrak{G})$ be a domain of $n$th degree and let $\Delta(u_1, \ldots, u_n)$ be the discriminant of the fundamental equation of $(\mathfrak{G})$ freed of its numerical factor. Further let

$$P_{k_1}, P_{k_2}, \ldots, P_{k_l}$$

be the divisor systems

$$P_k = (p; u_1^{p^k} - u_1, \ldots, u_n^{p^k} - u_n)$$

in which the primitive form $\Delta(u_1, \ldots, u_n)$ contains. We need only consider those whose index $k$ is not contained in one of the other indexes $k_1, \ldots, k_i$ as a divisor, because according to the observation above, the form $\Delta(u_1, \ldots, u_n)$ (if contained in the system $P_k$) is also divisible by every system $P_d$, whose index is a divisor of $k_i$. So we form the whole number:

$$F(p) = (p^{k_1} - 1)(p^{k_2} - 1) \ldots (p^{k_l} - 1)$$

With the integral basis $\{1, \alpha, 4/\alpha = \frac{1}{2}(\alpha^2 + \alpha - 1)\}$ given by Dedekind, the index form in this case is $\Delta(u_1, u_2, u_3) = 2u_2^3 - u_3^2u_1 - u_2u_3^2 - 2u_3^3$, which is clearly always even for integer values of the $u_i$. If, however, $\zeta$ is a cube root of unity, then $f(0, \zeta, \zeta^2) = 1$, as Kronecker says.

It seems likely that when Hensel says “constructed from roots of unity” he means a subfield of a cyclotomic field. But one could also ask for a cyclotomic field rather than a subfield. In fact, in what follows Hensel first finds the smallest prime-order cyclotomic field that has the desired property, then finds the smallest subfield of that field that still has the property.

To clarify the argument, we exemplify using the cubic field above. We use the integral basis $\{1, \alpha, \beta\}$ where $\beta = 4/\alpha$.

The discriminant of the fundamental equation is $-503(2u_2^3 - u_3^2u_1 - u_2u_3^2 - 2u_3^3)^2$. While Hensel never says it explicitly, to find his $\Delta$ we first divide by the field discriminant (here, $-503$) and then take the square root, so in this case $\Delta(u_1, u_2, u_3) = 2u_2^3 - u_3^2u_1 - u_2u_3^2 - 2u_3^3$, as mentioned above. That 2 is a common inessential discriminant divisor follows from $\Delta(u_1, u_2, u_3) = -u_3(u_2^2 - u_2) - u_2(u_3^2 - u_3) + 2(u_3^3 + u_3^3 - u_2u_3)$.

In our example there is only one, with $k = 1$. 

47
and look for the smallest prime \( \nu \) different from \( p \) which is not contained in \( F(p) \). Then the domain \( \Gamma_\nu \) of the \( \nu \)-th roots of unity is the smallest which is a supplementary domain for \((\mathfrak{S})\) with respect to the prime \( p \).

Now we easily prove that \( \Gamma_\nu \) is in fact a supplementary domain for \((\mathfrak{S})\). If \( p \) modulo \( \nu \) belongs to the exponent \( k \), then \( p \) decomposes in \( \Gamma_\nu \) into distinct prime factors of degree \( k \). So \( \Gamma_\nu \) is a supplementary domain of \((\mathfrak{S})\) if and only if \( \Delta(u_1, \ldots, u_n) \) does not contain the divisor system \( P_k \). This claim is only satisfied if the index \( k \) is not a divisor of any of the \( l \) numbers \( k_1, k_2, \ldots, k_l \). If this were however the case, then at least one of the \( l \) factors of the product \( F(p) \), and hence also \( F(p) \) itself, would be divisible by \( \nu \). Since \( \nu \) is the smallest prime not occurring in the product, the first part of the claim is proved.

Further, if \( \nu_1 \) is a prime smaller than \( \nu \) and if \( \Gamma_{\nu_1} \) is the field constituted by the \( \nu_1 \)-th roots of unity, then \( \Gamma_{\nu_1} \) can not be a supplementary domain of \((\mathfrak{S})\). In fact, according to the previous assumption, \( \nu_1 \) is a divisor of the product \( F(p) \), which means \( \nu_1 \) is contained in at least one of the factors \((p^{k_i} - 1)\). Therefore, the exponent \( k' \) of \( p \) modulo \( \nu_1 \) is a divisor of one of the \( l \) numbers \( k_1, \ldots, k_l \), and so the divisor system \( P_k \) is a divisor of the form \( \Delta(u_1, \ldots, u_n) \), or equivalently, \( p \) is an inessential divisor for \((\mathfrak{S})\) in the domain \( \Gamma_{\nu_1} \).

Now take \( \Gamma_\nu \) to be the supplementary domain of roots of unity we have just determined, of lowest degree. Then it will contain as many subdomains \( \Gamma_\nu(\lambda) \) as the number of divisors of \( \nu - 1 \). Namely, if

\[ \nu - 1 = \lambda \mu \]

is a decomposition of \( \nu - 1 \) in two factors, then under \( \nu \) there is contained a domain of degree \( \lambda \), namely the one containing the \( \lambda \) periods of \( \mu \) terms formed from the \( \nu \)-th roots of unity. We should now investigate which of these period domains \( \Gamma_\nu(\lambda) \) is of lowest degree \( \lambda \) and is still a supplementary domain.

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\(^{146}\)In our example \( F(2) = (2^1 - 1) = 1 \), so \( \nu = 3 \).

\(^{147}\)This result is to be proved in the next few paragraphs. Note that in our example it is, as Kronecker pointed out, the field of cube roots of unity.

\(^{148}\)So \( \Gamma_\nu \) is a supplementary domain.

\(^{149}\)So no cyclotomic field corresponding to a smaller prime \( \nu_1 \) will do the job.

\(^{150}\)We have found a cyclotomic field that will serve as a supplementary domain. Now Hensel wants to show that we can take a specific subfield.

\(^{151}\)So \( \Gamma_\nu(\lambda) \) is the unique cyclic subfield of degree \( \lambda \) in the cyclotomic field of \( \nu \)-th roots of unity. The “periods” are those defined by Gauss in the last chapter of his *Disquisitiones Arithmeticae*; they give an explicit basis of the field \( \Gamma_\nu(\lambda) \).
To answer this question, we think of all the divisors of $\nu - 1$ arranged according to their value in descending order and denote them by

$$\mu_1, \mu_2, \mu_3, \ldots, \mu_\varrho,$$

so that $\mu_1 = \nu - 1$, $\mu_\varrho = 1$, and

$$\mu_1 > \mu_2 > \mu_3 > \cdots > \mu_\varrho.$$

Substituting $p$ in $F(p)$ by each element of the sequence $p^{\mu_1}, p^{\mu_2}, \ldots, p^{\mu_\varrho}$, we see first that $F(p^{\mu_1})$ is divisible by $\nu$, because every one of the factors

$$p^{\mu_1 k_i} - 1 = (p^{r - 1})^{k_i} - 1$$

contains this prime. But $F(p^{\mu_\varrho}) = F(p)$ does not contain this prime, according to the above assumption on $\nu$. Let then $\mu$ be the first, and therefore largest, of these numbers such that

$$F(p^{\mu}) = (p^{\mu k_1} - 1) \cdots (p^{\mu k_l} - 1)$$

is no longer divisible by $\nu$. Let $\lambda$ be the complementary divisor of $\nu - 1$ to $\mu$, that is,

$$\lambda \mu = \nu - 1.$$

Then the domain $\Gamma_\nu(\lambda)$ containing the $\lambda$ periods with $\mu$ terms of the $\nu$th roots of unity, is the smallest [subfield] which is still a supplementary domain for $\mathfrak{G}$.

That this is a supplementary domain, one sees as follows: If $\kappa$ is the exponent of $p^\mu$ modulo $\nu$, that is, the smallest whole number for which the difference $(p^{\mu \kappa} - 1)$ is divisible by $\nu$, then $p$ decomposes in $\Gamma_\nu(\lambda)$ into distinct prime factors of degree $\kappa$. The domain $\Gamma_\nu(\lambda)$ is then a supplementary domain of $\mathfrak{G}$ if the primitive form $\Delta(u_1, \ldots, u_n)$ is not contained the module system $P_\kappa$, so if $\kappa$ is not among the numbers $k_1, \ldots, k_l$. But if this were the case, then one of the factors $p^{\mu k_i} - 1$, and so the number $F(p^{\mu})$, would be divisible by $\nu$, which is not the case.

\[^{152}\text{Hensel first states the result: we choose the largest divisor of $\nu$ that satisfies a divisibility condition.}\]
\[^{153}\text{This is the claim. The proof follows. In our example, of course, the only divisors of $3 - 1 = 2$ are $\mu_1 = 2$ and $\mu_2 = 1$ and the only subfield that works is $\Gamma_3$ itself.}\]
\[^{154}\text{So we have shown that the chosen $\Gamma_\nu(\lambda)$ is a supplementary domain. It remains to show that it is the smallest.}\]
Now taking a different period domain $\Gamma_{\nu}(\lambda')$ of smaller degree, so that $\lambda' < \lambda$, and since $\lambda'\mu' = \nu - 1$, so that $\mu' > \mu$, then this cannot be a supplementary domain for $(\mathfrak{G})$. Namely, if $\kappa'$ is the exponent of $p^{\mu'}$ modulo $\nu$, then $p^{\mu'\kappa'} - 1$ is divisible by $\nu$, and $\kappa'$ must be a divisor of one of the numbers $k_1, \ldots, k_l$. Since $F(p^{\mu'})$ is divisible by $\nu$ for every $\mu' > \mu$, one of the factors $(p^{\mu'k_i} - 1)$ is a multiple of $\nu$, which means the exponent $\kappa'$ belonging to $p^{\mu'}$ modulo $\nu$ is contained in $k_i$, and from this it follows that $p$ is still an inessential divisor of the discriminants of $(\mathfrak{G})$ in the field of rationality $\Gamma_{\nu}(\lambda')$. With this, we have proved the above conjecture, and we can summarize the end result of all the last investigations in the following elegant theorem:

Let $(\mathfrak{G})$ be a given field of degree $n$ and let $\Delta(u_1, \ldots, u_n)$ be the discriminant of the fundamental equation freed from its numerical factor. Further let $p$ be any real prime and denote by

$$P_{k_1}, P_{k_2}, \ldots, P_{k_l}$$

the divisor systems of the form

$$P_k = (p; u_1^{p^{k_1}} - u_1, \ldots, u_n^{p^{k_l}} - u_n)$$

which $\Delta$ contains, chosen so that among their indexes $k_1, \ldots, k_l$ none is a multiple of the others.

Let $\nu$ be the smallest prime which does not divide the integer

$$F(p) = (p^{k_1} - 1) \ldots (p^{k_l} - 1).$$

Then the field $\Gamma_{\nu}$ of the $\nu$-th roots of unity is the smallest [cyclotomic field] which is a supplementary domain of the domain $(\mathfrak{G})$. Further, if $\mu$ is the largest divisor of $\nu - 1$, for which the number

$$F(p^{\mu})$$

does not contain the prime $\nu$, the period field $\Gamma_{\nu}(\lambda)$ contained in it, defined by the $\lambda$ periods of $\mu$ terms in the $\nu$th roots of unity, is the smallest [subfield] which still has this property.

Berlin, November 17, 1893
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