Abstract

We extend Choe's idea in [Ch] to nonpolyhedral calibrated surfaces and give some examples of polyhedral sets over right prisms and nonpolyhedral calibrated surfaces.

1 Introduction

In [Ch], Choe proved “Every stationary polyhedral set is area-minimizing under diffeomorphisms leaving the boundary fixed”. In his proof, a system of differential forms and orientations (of faces) was chosen at each singular edge. In fact, the differential forms are calibrations that calibrate the faces at each singular edge and have the vanishing sum. We observe that, the suitable orientations of faces at each singular edge determine the same orientation on it whenever it lies on the boundary of faces.

By the above observation, we extend Choe's idea by proving a sufficient condition for certain sets of calibrated surfaces (including polyhedral sets) to be area-minimizing under diffeomorphisms leaving the boundary fixed. This sufficient condition, when applies to polyhedral sets, is also necessary.

We give some more examples of polyhedral sets over right prisms and first examples of nonpolyhedral calibrated surfaces (2-dimensional ones with singular sets of dimension 1 in \( \mathbb{R}^4 \)).
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2 The theorem

We refer the readers to [Ch] for the definition of polyhedral sets.

Let \( \{ C_i \}_{i \in I} \) be a set of calibrated surfaces of dimension \( m \) in \( \mathbb{R}^n \) \((m < n)\) and \( \{ w_i \}_{i \in I} \) be the set of correspondent calibrations. That means for each \( i \in I \), \( w_i \) calibrates \( C_i \) with a suitable orientation. Note that if \( \omega_i \) calibrates \( C_i \), then \( -\omega_i \) calibrates \( C_i \) with opposite orientation. Depending on a chosen orientation on \( C_i \) we have the correspondent calibration to be \( \omega_i \) or \( -\omega_i \).

Let \( \Sigma \subset \mathbb{R}^n \) be a set satisfies the following conditions:

(i) \( \Sigma \subset \bigcup_{i \in I} C_i \),
(ii) the set \( E = \Sigma \cap (C_i \cap C_j) \) is of dimension \( m - 1 \) for every \( i, j \in I, i \neq j \).

We call each \( F_i = \Sigma \cap C_i \) a face, each \( E \) a singular edge, the union of all singular edges \( E \) the singular set \( S \), the closure of \( \partial F_i \sim S \) the boundary edge of \( \Sigma \) in \( F_i \), the union \( \bigcup_{i \in I} (\partial F_i \sim S) \) the boundary \( \partial \Sigma \) of \( \Sigma \).

\( \Sigma \) is said to be area-minimizing under diffeomorphisms leaving the boundary fixed if

\[ V o l(\Sigma) \leq V o l(\varphi(\Sigma)), \]

for any diffeomorphism \( \varphi \) of \( \mathbb{R}^n \) leaving the boundary of \( \Sigma \) fixed.

Suppose \( \{ E_j \}_{j \in J} \) is the set of all singular edges and \( \{ F_i \}_{i \in I} \) is the set of all faces of \( \Sigma \). Denote

\[ I_{E_j} = \{ i : F_i \supset E_j \} \subset I, \]
\[ J_{F_i} = \{ j : E_j \subset F_i \} \subset J. \]

**Theorem 2.1** Let \( \Sigma \) be a set defined as above. Suppose that every singular edge \( E_j \) lies on the boundary \( \partial F_i \), \( \forall i \in I_{E_j} \) and for each \( E_j \) we can choose suitable orientations on \( F_i, \forall i \in I_{E_j} \), such that:

(i) the orientations on \( F_i, \forall i \in I_{E_j} \) determine the same orientation on \( E_j \),

(ii) the correspondent calibrations have vanishing sum.

Then \( \Sigma \) is area-minimizing under diffeomorphisms leaving \( \partial \Sigma \) fixed.

**Proof.** The reasonings of the proof are very similar as that of the main theorem in [Ch] with some little changes.

Let \( \varphi \) be a diffeomorphism leaving \( \partial \Sigma \) fixed and \( \varphi_i \) be the homotopy from the identity to \( \varphi \). Suppose \( G_j \) is the \( m \)-dimensional smooth surface swept out by \( \varphi_i(E_j) \) and \( D_i \) is \((m + 1)\)-dimensional surface swept out by \( \varphi_i(F_i) \). We have

\[ \partial D_i = F_i \cup \varphi(F_i) \cup \bigcup_{j \in J_{F_i}} G_j, \]

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and hence
\[ \int_{\partial D_i} w_i = \int_{F_i} w_i + \int_{\varphi(F_i)} w_i + \sum_{j \in J_{F_i}} \int_{G_j} w_i. \]

Since \( w_i \) is a calibration that calibrates \( F_i \), we get the following inequality:
\[ Vol(F_i) \leq Vol(\varphi(F_i)) - \sum_{j \in J_{F_i}} \int_{G_j} w_i, \]
and finally
\[ Vol(\Sigma) \leq Vol(\varphi(\Sigma)) - \sum_{j \in J} \sum_{i \in I_{E_j}} \int_{G_j} w_i. \]

By virtue of the assumptions of the theorem, we can assume the orientations on \( F_i, \forall i \in I_{E_j} \), determine the same orientation on \( G_j \) and since \( \sum_{i \in I_{E_j}} w_i = 0 \), the last term equals zero. The theorem is proved.

**Corollary 2.2** Let \( \Sigma \) be a polyhedral set. Then \( \Sigma \) is area-minimizing under diffeomorphisms leaving \( \partial \Sigma \) fixed if and only if \( \Sigma \) satisfies the assumptions in the Theorem 2.1.

**Proof.** The sufficiency follows from the above theorem and the necessity follows from the proof of the main theorem in [Ch].

### 3 Examples

1. At Ken Brakke’s homepage, [http://www.susqu.edu/facstaff/b/brakke/](http://www.susqu.edu/facstaff/b/brakke/) we can see eight nice polyhedral cones, that are made of flat sheets meeting along triple lines with an equal angle 120°. All of them are area-minimizing under diffeomorphisms leaving the boundary fixed by virtue of Theorem 2.1. Figures 1 provide more three polyhedral sets over right prisms.

2. Below are examples of nonpolyhedral calibrated surfaces that is area-minimizing under diffeomorphisms leaving the boundary fixed.

   Let \( \mathbb{C}^2 \equiv \mathbb{R}^4 \) be complex plane with the standard complex structure \( J_1 \), \( J_1 e_1 = e_3; J_1 e_2 = e_4 \).

   Let \( R_2, R_3, \ldots, R_n \) be the rotations of angles \( \alpha, 2\alpha, \ldots, (n-1)\alpha \) about the plane \( \{x_3 = x_4 = 0\} \), respectively, where \( \alpha \) satisfies the condition \( n\alpha = 2\pi, n \in \mathbb{N} \). And let \( J_2, J_3, \ldots, J_n \) be \( (n-1) \) complex structures on \( \mathbb{R}^4 \) induced by \( R_2, R_3, \ldots, R_n \):

   \[ J_i(e_1) = R_i(e_3), \quad J_i(e_2) = R_i(e_4); \quad i = 2, 3, \ldots, n. \]

   Denote \( w_1, w_2, \ldots, w_n \) the Kähler forms correspondent to \( J_1, J_2, \ldots, J_n \). We can easily to see that:

   \[ \sum_{i=1}^{n} w_i = 0. \]
Consider the complex curves:

\[ C = \{(z, w) \in \mathbb{C}^2 : z = w^2\} = \{(x_1, x_2, x_3, x_4) : x_2 = x_1^2 - x_3^2; \ x_4 = 2x_1x_3\}. \]

Let \( D \) be the intersection of \( C \) and \( \{\sum_{i=1}^{4} x_i^2 = 1; x_1 \geq 0; x_3 \geq 0\} \). Note that \( D \) contains two planar curves \( \{x_2 = x_1^2; \ x_3 = x_4 = 0; x_1^2 + x_3^2 \leq 1; x_1 \geq 0\} \), and \( \{x_2 = -x_3^2; \ x_1 = x_4 = 0; x_1^2 + x_3^2 \leq 1; x_3 \geq 0\} \). By using the rotations of angles \( k\alpha, \ k = 1, 2, \ldots, n-1 \) about the plane \( \{x_3 = x_4 = 0\} \) we get the images \( D_i \ (i = 2, 3, \ldots, n) \) of \( D \). Obviously, \( w_1 \) calibrates \( D \) and \( w_i \) calibrates \( D_i, \ i = 2, 3, \ldots, n \).

The set \( \Sigma = D \cup D_i \) contains one singular edge and is area-minimizing under diffeomorphisms leaving the boundary fixed by virtue of Theorem 2.1. Similarly, by using the rotations of angles \( k\beta, \ k = 1, 2, \ldots, m-1; m\beta = 2\pi \) about the plane \( \{x_1 = x_4 = 0\} \), we get the images \( D_j' \ (j = 2, 3, \ldots, m) \) of \( D \) and the images \( \Sigma_j \ (j = 2, 3, \ldots, m) \) of \( \Sigma \).

The set \( D \cup D_i \cup D_j' \) contains two singular edges. The set \( \Sigma' = \Sigma \cup \Sigma_j \) contains many singular edges. By the same reasoning as above, they are also area-minimizing under diffeomorphisms leaving the boundary fixed.

References

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[HL] R. Harvey and H. B. Lawson, *Calibrated geometries*, Acta Math., 104 (1982), 47-157.