Coulomb potential in one dimension with minimal length: a path integral approach

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Abstract

We solve the path integral in momentum space for a particle in the field of the Coulomb potential in one dimension in the framework of quantum mechanics with the minimal length given by \((\Delta X)_0 = h\sqrt{\beta}\), where \(\beta\) is a small positive parameter. From the spectral decomposition of the fixed energy transition amplitude we obtain the exact energy eigenvalues and momentum space eigenfunctions.

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1 Introduction

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The Planck length defines a natural limit of the spacetime resolution and its extremely small value \( l_p \approx 10^{-35} \text{m} \) renders the physical effects at this scale beyond the actual accessible energies in particle accelerators. The main issue of recently proposed models, like models with extra dimensions and models with a minimal length scale, is to lower the Planck scale to experimentally accessible energies. Recently, in a series of papers, Kempf et al. [1, 2, 3, 4] and Hinrichsen and Kempf [5] developed a quantum mechanics based on a modified commutation relation
\[
\left[ \hat{x}_i, \hat{p}_j \right] = i\hbar \left( (1 + \beta \hat{p}^2) \delta_{ij} + \beta' \hat{p}_i \hat{p}_j \right),
\]
where \( \beta \) and \( \beta' \) are small parameters. This commutation relation leads to a generalized uncertainty principle (GUP) which defines a non-zero minimal length in position. A minimal length in position can be found in various contexts like string theory [6], loop quantum gravity [7], and noncommutative field theories [8] and the Holographic Principle [9, 10]. One major feature of the minimal length is that the physics below this scale becomes inaccessible and then defining a natural cut-off which prevents from the usual UV divergencies, and then put a limit to the possible resolution of the spacetime.

The other consequence of the presence of the minimal length is the appearance of an intriguing UV/IR mixing. This mixing between UV and IR divergences was also noticed in the ADS/CFT correspondence [11, 12] and in the canonical noncommutative quantum field theories [8]. On the other hand some scenarios have been proposed where the minimal length is related to large extra dimensions [13], to the running coupling constant [14] and to the physics of black holes production [15].

Recently, the effect of the minimal length in quantum mechanical problems has been studied in several works. Among them, we cite: the solution of the harmonic oscillator in arbitrary dimensions with minimal length in the momentum space representation [1, 2, 16]; the cosmological constant problem and the classical limit of the physics with minimal length [17, 18]; the solution of the Coulomb potential in one and three dimensions, has been studied respectively in [19] and [20, 21]; the Casimir
force for the electromagnetic field confined between parallel plates has been computed in [22], and the electron magnetism has been investigated in [23, 24]. In this paper, we use another approach, namely the path integral formalism in the momentum space representation, to obtain the energy eigenvalues and the momentum eigenfunctions of a particle in the field of Coulomb potential in one dimension in the presence of the minimal length. The physical interest in the 1D Coulomb potential comes from its potential applications in semiconductors or insulators [25].

The rest of the paper is organized as follows. In the next section, we introduce the main relations of quantum mechanics with minimal length, such as maximally localization states, modified orthogonality and completeness relations. In section 3, we extend the path integral in the momentum space representation and the fixed energy transition amplitude with regulating functions to time independent systems in the presence of the minimal length. In section 4 we determine, from the spectral decomposition of the fixed energy transition amplitude, the exact energy eigenvalues and momentum space eigenfunctions of a particle in the Coulomb potential in one dimension. The last section is left for concluding remarks.

2 Quantum mechanics with minimal length

Let us consider the following one dimensional momentum space realization of the position and momentum operators [2]

\[ X = i\hbar (1 + \beta p^2) \frac{\partial}{\partial p}, \quad P = p, \quad \beta \geq 0. \]  

(1)

These operators satisfy the following deformed commutator and GUP

\[ [X, P] = i\hbar (1 + \beta p^2), \quad (\Delta X)(\Delta P) \geq \frac{\hbar}{2} [1 + \beta (\Delta P)^2]. \]  

(2)
A minimization of the saturate GUP with respect to $(\Delta P)$ leads to a minimal length given by

$$(\Delta X)_0 = \hbar \sqrt{\beta}. \quad (3)$$

At this stage we introduce a first quantized Hilbert space equipped with the complete basis $\{ |p\> \}$ given by

$$\int Dp |p\> \langle p| = 1. \quad (4)$$

The hermiticity requirement of the position operator leads to the following squeezed momentum measure

$$Dp = \frac{dp}{(1 + \beta p^2)}, \quad (5)$$

and consequently we obtain

$$\langle p| p' \rangle = (1 + \beta p^2) \delta(p - p'). \quad (6)$$

The appearance of the minimal length given by Eq.(3) leads to a loss of the notion of localized states in the position space since we cannot probe the coordinates space with a resolution less than the minimal length. However, the information on position space is still accessible via the so called maximally localization states [2]. These states are squeezed states, which saturate the GUP and verify the constraint given by Eq.(3).

The maximally localization states denoted $| \psi_{ml}^{\xi} \rangle$ are defined as states localized around a position $\xi$ such that we have

$$\langle \psi_{\xi}^{\text{ml}} | X | \psi_{\xi}^{\text{ml}} \rangle = \xi, \quad (\Delta X)_{|\psi_{\xi}^{\text{ml}}\rangle} = (\Delta X)_0, \quad (7)$$

and are solutions of the following equation

$$\left( X - \langle X \rangle + \frac{\langle [X, P] \rangle}{2 (\Delta P)^2} (P - \langle P \rangle) \right) | \psi_{\xi}^{\text{ml}} \rangle = 0. \quad (8)$$
In the momentum space representation and with the expression of the position operator given by Eq.(1) we obtain
\[
\left( i\hbar (1 + \beta p^2) \frac{d}{dp} - \langle X \rangle + i\hbar \frac{(1 + \beta \langle P^2 \rangle)}{2 (\Delta P)^2} (p - \langle P \rangle) \right) \psi_{ml}^{\xi} (p) = 0. \tag{9}
\]
The solution to this equation is given by
\[
\psi_{ml}^{\xi} (p) = \frac{N}{\sqrt{2\pi \hbar}} (1 + \beta p^2)^{-\frac{1}{2}} e^{\left[ \frac{i\langle X \rangle}{\hbar \sqrt{\beta}} \frac{(1 + \beta (\Delta P)^2)^2}{2 (\Delta P)^2} \right] \tan^{-1} \left( \frac{p \sqrt{\beta}}{\Delta P} \right)} \tag{10}
\]
We set \( N = 1 \) by rescaling the states \( N^{-1} \psi_{ml}^{\xi} (p) \rightarrow \psi_{ml}^{\xi} (p) \). The states of absolutely maximal localization are those with \( \langle X \rangle = \xi, \langle P \rangle = 0 \) and, if we restrict these states to the ones for which \( \Delta P = 1/\sqrt{\beta} \), we obtain
\[
\psi_{ml}^{\xi} (p) = (1 + \beta p^2)^{-\frac{1}{2}} \exp \left[ - \frac{i\xi}{\hbar \sqrt{\beta}} \tan^{-1} \left( \frac{p \sqrt{\beta}}{\Delta P} \right) \right]. \tag{11}
\]
The states \( \psi_{ml}^{\xi} (p) \) are physically relevant ones with a finite energy
\[
\left\langle \frac{p^2}{2m} \right\rangle_{\psi_{ml}^{\xi}} = \frac{1}{4\pi \hbar m} \int \frac{p^2 dp}{(1 + \beta p^2)^3} = \frac{1}{8\beta^2 \hbar m}. \tag{12}
\]
On the other hand, the minimal length renders the maximally localized states no longer orthogonal like the coherent states. Indeed we show that
\[
< \psi_{\xi}^{ml} | \psi_{\xi'}^{ml} > = \frac{1}{2\pi \hbar} \int \frac{dp}{(1 + \beta p^2)^2} \exp \left( \frac{i (\xi - \xi')}{\hbar \sqrt{\beta}} \tan^{-1} \left( \frac{p \sqrt{\beta}}{\Delta P} \right) \right) = \frac{2}{\pi \hbar \sqrt{\beta}} \sin \left( \frac{u\pi}{2} \right) \tag{13}
\]
where we have set \( u = \frac{(\xi - \xi')}{\hbar \sqrt{\beta}} \).

Finally, it is easy to verify that the set \( \{ | \psi_{\xi}^{ml} > \} \) is complete in the sense that we have
\[
\int d\xi (1 + \beta p^2)^2 | \psi_{\xi}^{ml} > < \psi_{\xi}^{ml} | = 1. \tag{14}
\]
3 Path integral in momentum space

The path integral construction of the transition amplitude for quantum systems with minimal length in the momentum space representation follows the well known canonical steps. In momentum space representation the transition amplitude is given by

\[
(p_b T | p_a 0_a) = <p_b | U(T) | p_a >
\]

\[
= \lim_{N \to \infty} <p_b | \prod_{j=1}^{N+1} U(t_j, t_{j-1}) | p_a >, \tag{15}
\]

with the infinitesimal evolution operator \(U(t_j, t_{j-1}) = e^{-\frac{i}{\hbar} \hat{H}(t_j)}\) and \(\epsilon = (t_j - t_{j-1}) = \frac{T}{N+1}\). Inserting the completeness relation for the momentum states given by Eq.(4) between each pair of infinitesimal evolution operators gives

\[
(p_b T | p_a 0) = \lim_{N \to \infty} \prod_{j=1}^{N} \int dp_j \left(1 + \beta p_j^2\right) \prod_{j=1}^{N+1} (p_j t_j | p_{j-1} t_{j-1}), \tag{16}
\]

where the infinitesimal transition amplitude is given by

\[
(p_j t_j | p_{j-1} t_{j-1}) = <p_j | e^{-\frac{i}{\hbar} \hat{H}(t_j)} | p_{j-1} >. \tag{17}
\]

With the aid of the completeness relation for the maximally localization states and Eq.(11), and assuming the standard form of the Hamiltonian \(\hat{H} = \frac{\hat{p}^2}{2m} + \hat{V}(x)\), we arrive to the expression of the infinitesimal transition amplitude expressed as a phase space path integral

\[
(p_b t_b | p_a t_a) = \lim_{N \to \infty} \prod_{j=1}^{N} \int dp_j \left(1 + \beta p_j^2\right)
\]

\[
\times \prod_{j=1}^{N+1} \int \frac{dx_j}{2\pi\hbar} \exp \left\{ \frac{i\epsilon}{\hbar} \left\{ \frac{x_j}{\epsilon \sqrt{\beta}} \left[ \tan^{-1} \sqrt{\beta} p_j - \tan^{-1} \sqrt{\beta} p_{j-1} \right] - \frac{p_j^2}{2m} - V(x_j) \right\} \right\}. \tag{18}
\]

Now, following [26] we construct the path integral representation of the fixed energy transition amplitude. The later is defined by the following matrix element

\[
(p_b | p_a)_E^f = <p_b | \hat{R}(E) | p_a > \tag{19}
\]
of the resolvent operator

\[ \hat{R}(E) = \frac{i\hbar}{\hat{f}(E - \hat{H} + i\eta)} \hat{f}, \]  

(20)

where \( \hat{f} \) are regulating operators depending on \( X \) and \( P \). The resolvent operator is defined by the Fourier transform of the time evolution operator

\[ \hat{R}(E) = \int_0^T dT e^{\frac{i}{\hbar} ET} \hat{U}(T). \]  

(21)

Then, the path integral representation of the \( f \)-dependent fixed energy transition amplitude is given by

\[ (p_b \mid p_a)_E^f = f(0) \int_0^\infty dT(p_b T \mid p_a 0)_f, \]  

(22)

where \((p_b T \mid p_a 0)_f\) is the transition amplitude associated with the auxiliary Hamiltonian \( \hat{H} = \hat{f}(\hat{H} - E) \). The desired fixed energy transition amplitude is given by

\[ (p_b \mid p_a)_E = \int_0^\infty dT(p_b T \mid p_a 0) \]  

(23)

where \((p_b T \mid p_a 0)\) is obtained by integrating out the additional degree of freedom \( f \)

\[ (p_b T \mid p_a 0) = \int Df \Phi[f] (p_b T \mid p_a 0)_f, \]  

(24)

along with the following condition

\[ \int Df \Phi[f] = 1, \]  

(25)

with \( \Phi[f] \) a gauge fixing functional used to select a specific gauge.
4 Transition amplitude for the Coulomb potential in one dimension with minimal length

In this section we solve exactly the path integral of the Coulomb potential in one dimension in the presence of the minimal length. We start our calculation by substituting the potential $V(x) = -\alpha/x$ in the path integral representation of the transition amplitude given by Eq.(18),

$$\langle p_b T | p_a 0 \rangle_f = \lim_{N \to \infty} \prod_{j=1}^{N} \int \frac{dp_j}{(1 + \beta p_j^2)} \prod_{j=1}^{N+1} \int \frac{dx_j}{2\pi \hbar} \times \exp \left\{ \frac{i}{\hbar} \left\{ x_j \frac{\tan^{-1} \sqrt{\beta} p_j - \tan^{-1} \sqrt{\beta} p_{j-1}}{\epsilon} - f_j \left( \frac{p_j^2}{2m} - E \right) + f_j \frac{\alpha}{x_j} \right\} \right\}. \quad (26)$$

We shall work in the gauge given by

$$\Phi[f] = \prod_{t} \frac{1}{x} \exp \left\{ -\frac{i}{2\hbar x^2} \left[ f - x^2 \left( \frac{p^2}{2m} - E \right) \right]^2 \right\}. \quad (27)$$

Inserting $\Phi[f]$ in $\langle p_b T | p_a 0 \rangle$ given by Eq.(24) we obtain

$$\langle p_b T | p_a 0 \rangle = \lim_{N \to \infty} \prod_{j=1}^{N} \int \frac{dp_j}{(1 + \beta p_j^2)} \prod_{j=1}^{N+1} \int \frac{dx_j}{2\pi \hbar} \prod_{j=1}^{N+1} \int \frac{df_j}{\sqrt{2\pi \hbar x_j}} \times \exp \left\{ \frac{i}{\hbar} \left\{ x_j \frac{\tan^{-1} \sqrt{\beta} p_j - \tan^{-1} \sqrt{\beta} p_{j-1}}{\epsilon} - f_j \left( \frac{p_j^2}{2m} - E \right)^2 - f_j \frac{\alpha}{2x_j^2} \right\} \right\}. \quad (28)$$

Performing the Gaussian integrals over $f_j$ and $x_j$ we obtain

$$\langle p_b T | p_a 0 \rangle = (1 + p_b^2/p_E^2)^{-1/2} (1 + p_a^2/p_E^2)^{-1/2} \frac{(2m/p_E^2)}{\sqrt{2\pi i \hbar}} \lim_{N \to \infty} \prod_{j=1}^{N} \int \frac{dp_j}{\sqrt{2\pi i \hbar} \left( 1 + p_j^2/p_E^2 \right)} \left( 1 + \beta p_j^2 \right) \times \prod_{j=1}^{N+1} \exp \left\{ \frac{i}{\hbar} \left\{ \frac{2m^2}{\beta p_E^4 \epsilon^2} \left[ \tan^{-1} \sqrt{\beta} p_j - \tan^{-1} \sqrt{\beta} p_{j-1} \right]^2 \right\} \right\}. \quad (29)$$

At this stage we expand the first term in the exponent around $p_j$. This procedure leads to

$$\frac{2im^2}{\hbar \beta p_E^4 \epsilon} \left[ \arctan p_j \sqrt{\beta} - \arctan p_{j-1} \sqrt{\beta} \right]^2 = \frac{2im^2}{\hbar p_E^4 \epsilon} \frac{(\Delta p_j)^2}{(1 + \beta p_j^2)^2} \left\{ 1 - \frac{1}{6} \beta (\Delta p_j)^2 + \cdots \right\}, \quad (30)$$
where the omitted terms contain higher powers of $\Delta p_j$. Substituting in Eq.(29), we obtain

\[
(p_0 T | p_a 0) = (1 + p_b^2/p_E^2)^{-1/2} (1 + p_a^2/p_E^2)^{-1/2} \frac{1}{\sqrt{2\pi i\hbar M}} \lim_{N \to \infty} \prod_{j=1}^N \int \frac{dp_j}{\sqrt{2\pi i\hbar}} \frac{dp_j}{\sqrt{2\pi i\hbar}} \frac{1}{(1 + p_j^2/p_E^2)}
\]

\[
\times \prod_{j=1}^{N+1} \exp \left\{ \frac{i\varepsilon}{\hbar} \left( \frac{M}{2\varepsilon} \frac{(\Delta p_j)^2}{(1 + p_j^2/p_E^2)^2 (1 + \beta p_j^2)^2} - \frac{\beta m^2}{4p_E^4} \varepsilon \right) - \frac{\beta m^2}{4p_E^4} \varepsilon \right\} \right\}.
\]

The correction terms $(\Delta p_j)^n$ are calculated perturbatively and replaced by their expectation values $< (\Delta p_j)^n >$, using the following formula [26]

\[
< O(\Delta p) >= \int \frac{\sqrt{g(p)}}{\sqrt{2\pi i\hbar/M}} \frac{1}{\sqrt{2\pi i\hbar M}} \frac{1}{\sqrt{2\pi i\hbar M}} e^{\frac{i\hbar}{\hbar} g_{mn}(p)(\Delta p)^n} O(\Delta p) d(\Delta p).
\]

In our case, the metric $g(p)$ and the mass are given respectively by

\[
g_{mn}(p_j) = \frac{\delta_{mn}}{(1 + p_j^2/p_E^2)^2 (1 + \beta p_j^2)^2}, \quad M = \frac{4m^2}{p_E^4}.
\]

Then, for the first non vanishing expectation values we easily obtain

\[
< (\Delta p_j)^2 > = 3 \left( \frac{i\hbar}{4m^2/p_E^4} \right)^2 g_{mn}^2,
\]

\[
< (\Delta p_j)^4 > = 15 \left( \frac{i\hbar}{4m^2/p_E^4} \right)^3 g_{mn}^3.
\]

The remaining correction terms contains higher powers in $\varepsilon$. Substituting in Eq.(31), and considering only the contributions which are relevant to order $\varepsilon$ we obtain

\[
(p_0 T | p_a 0) = (1 + p_b^2/p_E^2)^{-1/2} (1 + p_a^2/p_E^2)^{-1/2} \frac{1}{\sqrt{2\pi i\hbar/M}} \lim_{N \to \infty} \prod_{j=1}^N \int \frac{dp_j}{\sqrt{2\pi i\hbar/M}} \frac{dp_j}{\sqrt{2\pi i\hbar/M}} \frac{1}{(1 + p_j^2/p_E^2)}
\]

\[
\times \prod_{j=1}^{N+1} \exp \left\{ \frac{i\varepsilon}{\hbar} \left( \frac{M}{2\varepsilon} \frac{(\Delta p_j)^2}{(1 + p_j^2/p_E^2)^2 (1 + \beta p_j^2)^2} + \frac{\beta m\hbar}{4p_E} \varepsilon \right) - \frac{\beta m}{4p_E^4} \varepsilon \right\} \right\}.
\]

In the following we adopt the mid-point prescription [26, 27]. To mate the wild looking kinetic term we define a new path dependent time $\sigma$ by the following symmetrized expression

\[
\varepsilon = \frac{\sigma_j}{(1 + \beta p_j^2)(1 + \beta p_{j-1}^2)}.
\]
and expand around the mid-point $\bar{p}_j = (p_j + p_{j-1})/2$ to obtain

$$
\varepsilon = \frac{\sigma_j}{(1 + \beta \bar{p}_j^2)^2} \left[ 1 - \frac{1}{2} \beta (\Delta p_j)^2 + \mathcal{O}(\beta^2) \right].
$$

(39)

Then, the exponent in Eq.(37) reads as

$$
\text{Exp} = \frac{i}{\hbar} \left\{ \frac{M}{2\sigma_j} \frac{(\Delta p_j)^2}{(1 + p_j^2/p_E^2)^2} \left[ 1 + \frac{1}{2} \beta (\Delta p_j)^2 \right] + \beta \sigma_j \frac{h^2 p_j^4}{16 m^2} - \frac{\alpha^2}{2} \sigma_j (1 - 2\beta \bar{p}_j^2) + \mathcal{O}(\beta^2) \right\}.
$$

(40)

There is another correction term proportional to $(\Delta p_j)^2$ coming from the measure. Using Eq.(32) with the following metric

$$
g_{mn}(p_j) = \frac{\delta_{mn}}{(1 + p_j^2/p_E^2)^2},
$$

(41)

we replace $(\Delta p_j)^2$ by

$$< (\Delta p_j)^2 > = \left( \frac{i\hbar \sigma}{4m^2/p_E^4} \right) \left( 1 + p_j^2/p_E^2 \right)^2.
$$

(42)

Collecting all the three mentioned corrections terms we obtain

$$
\text{Exp} = \frac{i\sigma_j}{\hbar} \left\{ \frac{M}{2\sigma_j} \frac{(\Delta p_j)^2}{(1 + p_j^2/p_E^2)^2} + \left( \frac{\beta h^2 p_E^4}{16 m^2} \right)_1 - \left( \frac{3\beta h^2 p_E^4}{16 m^2} \right)_2 + \left( \frac{\beta h^2 p_E^4}{8 m^2} \right)_3 - \frac{\alpha^2}{2} (1 - 2\beta \bar{p}_j^2) \right\}
$$

$$= \frac{i\sigma_j}{\hbar} \left\{ \frac{M}{2\sigma_j} \frac{(\Delta p_j)^2}{(1 + p_j^2/p_E^2)^2} + \alpha^2 \beta \bar{p}_j^2 - \frac{1}{2} \alpha^2 \right\}
$$

(43)

Here, we observe a cancelation of the corrections arising from the time slicing. The same cancelation occurred in the path integral treatment of the Coulomb potential in two and three dimensions [26].

However, it is interesting to note that, even in one dimension, the presence of the minimal length generates quantum corrections similar to the quantum corrections generated by the motion of point particles on curved spaces. This fact clearly suggests some equivalence between the effects induced by space curvature and the ones induced by the minimal length [28].

Now, let us incorporate the path dependent new time $T = \int_0^{\sigma_b} \frac{d\sigma}{(1 + \beta p^2(\sigma))^2}$ by means of the following identity

$$
[(1 + \beta p^2) (1 + \beta p_n^2)]^{-1} \int_0^\infty d\sigma \delta \left( T - \int_0^{\sigma_b} \frac{d\sigma}{(1 + \beta p^2(\sigma))^2} \right) = 1.
$$

(44)
Then we have
\[
(p_b T | p_a 0) = \frac{1}{2\pi\hbar} \int_{-\infty}^{+\infty} d\mathcal{E} e^{-\frac{\mathcal{E}}{2}} (p_b | p_a) \varepsilon, \tag{45}
\]
where \((p_b | p_a)\varepsilon\) is the fixed pseudo-energy to be evaluated at \(\mathcal{E} = 0\) and is given by
\[
(p_b | p_a)\varepsilon = \left[(1 + \beta p_b^2) (1 + \beta p_a^2)\right]^{-1} \int_0^\infty d\sigma (p_b\sigma_b | p_a 0), \tag{46}
\]
with
\[
(p_b\sigma_b | p_a 0) = (1 + p_b^2 / p_E^2)^{-1/2} (1 + p_a^2 / p_E^2)^{-1/2} \frac{(2m/p_E^2)}{\sqrt{2\pi i\sigma_j \hbar}} \lim_{N \to \infty} \prod_{j=1}^N \int \frac{(2m/p_E^2)}{\sqrt{2\pi i\sigma_j \hbar}} \frac{dp_j}{(1 + p_j^2 / p_E^2)} \times \prod_{j=1}^{N+1} \exp \left\{ \frac{i\sigma_j}{\hbar} \left\{ \frac{2m^2}{p_E^4\sigma_j^2} \left(1 + p_j^2 / p_E^2\right)^2 + \alpha^2 \beta p_j^2 - \frac{1}{2} \alpha^2 + \mathcal{E} \right\} \right\}. \tag{47}
\]
To reduce this path integral to a more tractable one we use again the mid-point prescription and calculate all the correction terms proportional to \((\Delta p)^4\) and \((\Delta p)^2\) using Eq.(32) with \(g\) given by Eq.(41). After a lengthly and straightforward calculation we obtain
\[
(p_b\sigma_b | p_a 0) = (1 + p_b^2 / p_E^2)^{-1/2} (1 + p_a^2 / p_E^2)^{-1/2} \frac{(2m/p_E^2)}{\sqrt{2\pi i\sigma_j \hbar}} \lim_{N \to \infty} \prod_{j=1}^N \int \frac{(2m/p_E^2)}{\sqrt{2\pi i\sigma_j \hbar}} \frac{dp_j}{(1 + p_j^2 / p_E^2)} \times \prod_{j=1}^{N+1} \exp \left\{ \frac{i\sigma_j}{\hbar} \left\{ \frac{2m^2}{p_E^4\sigma_j^2} \left(1 + p_j^2 / p_E^2\right)^2 - \frac{\hbar^2 p_E^4}{8m^2} + \alpha^2 \beta p_j^2 - \frac{1}{2} \alpha^2 + \mathcal{E} \right\} \right\}, \tag{48}
\]
where \(\frac{\hbar^2 p_E^4}{8m^2}\) is the resulting final correction.

In the continuum limit we write Eq.(48) as
\[
(p_b\sigma_b | p_a 0) = \frac{e^{-\frac{i}{\hbar}(\alpha^2 - 2\mathcal{E})\sigma}}{(1 + p_b^2 / p_E^2)^{1/2} (1 + p_a^2 / p_E^2)^{1/2}} \int \mathcal{D}p \exp \left\{ \frac{i}{\hbar} \int_0^\sigma d\sigma \left\{ \frac{2m^2}{p_E^4} \frac{\bar{p}^2}{(1 + p_j^2 / p_E^2)^2} + \alpha^2 \beta p^2 - \frac{\hbar^2 p_E^4}{8m^2} \right\} \right\}. \tag{49}
\]
with \(\mathcal{D}p = \frac{(2m/p_E^2)}{\sqrt{2\pi i\sigma_j \hbar}} \lim_{N \to \infty} \prod_{j=1}^N \int \frac{(2m/p_E^2)}{\sqrt{2\pi i\sigma_j \hbar}} \frac{dp_j}{(1 + p_j^2 / p_E^2)}\).

Now, let us bring the kinetic term to the standard form by using the following coordinate transformation \(p \in (-\infty, +\infty) \to \vartheta \in (-\pi p_E, +\pi p_E)\)
\[
\vartheta_j = 2p_E \arctan \frac{p}{p_E}, \tag{50}
\]
Then, the transition amplitude is written as

\[
(p_b \sigma_b \mid p_a 0) = \frac{e^{-\frac{i}{\hbar} (\alpha^2 - 2\mathcal{E}) \sigma}}{(1 + p_b^2/p_E^2)^{1/2} (1 + p_a^2/p_E^2)^{1/2}} (\vartheta_b \sigma_b \mid \vartheta_a 0).
\] (51)

The transformation given by Eq.(50) generates a correction term which cancels exactly the one in Eq.(49), and finally the transition amplitude can be obtained as

\[
(\vartheta_b \sigma_b \mid \vartheta_a 0) = \int D\theta \exp \left\{ \frac{i}{\hbar} \int_0^{\sigma_b} d\sigma \left\{ \frac{M_E}{2} \dot{\vartheta}^2 + \frac{\hbar^2}{8M_Ep_E^2} \lambda (\lambda - 1) \tan^2 \frac{\theta}{2p_E} \right\} \right\},
\] (52)

with

\[
D\theta = \frac{1}{\sqrt{2\pi i\sigma_j/M_E}} \lim_{N \to \infty} \prod_{j=1}^N \int \frac{d\sigma_j}{\sqrt{2\pi i\sigma_j/M_E}}, \quad M_E = \frac{m^2}{p_E^2}
\]

and the parameter \( \lambda \) given by

\[
\lambda = \frac{1}{2} \left( 1 + \sqrt{1 + 32\beta m^2 \alpha^2 / \hbar^2} \right).
\] (53)

The expression given by (52) is exactly the path integral representation of the transition amplitude of a point particle in the symmetric Poschl-Teller potential, whose spectral decomposition is given by [27]

\[
(\vartheta_b \sigma_b \mid \vartheta_a 0) = \sum_{n=0}^{\infty} A_n e^{-\frac{i p_E^2}{8M_E} \left[ n^2 + (2n+1)\lambda \right]} \frac{1}{2p_E} \left( \sin \frac{\theta_b}{2p_E} \sin \frac{\theta_a}{2p_E} \right)^\lambda C_\lambda^n \left( \cos \frac{\theta_b}{2p_E} \right) C_\lambda^n \left( \cos \frac{\theta_a}{2p_E} \right),
\] (54)

and where the normalization constant is

\[
A_n = (\Gamma (\lambda))^2 \frac{2^{2\lambda-1} n! (n + \lambda)}{\pi \Gamma (n + 2\lambda)},
\] (55)

and \( C_\lambda^n (x) \) are the Gegenbauer polynomials. Substituting Eqs.(54) and (51) in Eq. (46), and integrating over \( \sigma \) we obtain the following spectral decomposition of the fixed pseudo-energy transition amplitude

\[
(p_b \mid p_a) = \frac{i \hbar}{2p_E} \frac{[(1 + \beta p_b^2) (1 + \beta p_a^2)]^{-1}}{(1 + p_b^2/p_E^2)^{1/2} (1 + p_a^2/p_E^2)^{1/2}} \sum_{n=0}^{\infty} A_n \frac{\sin \theta_b \sin \theta_a}{2p_E} \frac{\lambda C_\lambda^n (\cos \theta_b) C_\lambda^n (\cos \theta_a)}{\frac{\hbar^2}{2M_E} \left[ n^2 + (2n+1)\lambda \right] - \frac{\alpha^2 p_E^2}{2} + 2\mathcal{E} p_E^2}.
\] (56)
Then, the fixed energy transition amplitude is simply obtained by setting $E = 0$ in the above expression

$$
(p_b \mid p_a)_E = f(0) \frac{i\hbar}{2p_E} \frac{[(1 + \beta p_b^2)(1 + \beta p_a^2)]^{-1}}{(1 + p_b^2/p_E^2)^{1/2}(1 + p_a^2/p_E^2)^{1/2}} \sum_{n=0}^{\infty} A_n \frac{(\sin \theta_b \sin \theta_a)^\lambda C_n^\lambda (\cos \theta_b) C_n^\lambda (\cos \theta_a)}{\frac{\hbar^2}{2M_E} [n^2 + (2n + 1)\lambda] - \frac{\alpha^2 p_E^2}{2}}.
$$

(57)

Therefore, using the expression of $M_E$, we derive the following spectral condition

$$
\frac{\hbar^2 p_E^4}{2m^2} [n^2 + (2n + 1)\lambda] - \frac{\alpha^2 p_E^2}{2} = 0,
$$

(58)

from which, using the expression of $\lambda$ and that $p_E^2 = -2mE_n$ for bound states, we obtain the expression for the energy spectrum

$$
E_n = -\frac{m\alpha^2}{2\hbar^2} \left[ n^2 + \left( n + \frac{1}{2} \right) \left( 1 + \sqrt{1 + \left( \frac{32m^2\alpha^2\beta}{\hbar^2} \right)} \right) \right], \quad n = 0, 1, 2, ...
$$

(59)

It is obvious that for $\beta = 0$ we recover the usual energy levels of the Coulomb potential in one dimension [30]. The corrections terms brought by the minimal length are obtained by exploiting the fact that the $\beta$-dependent term in Eq.(59) is a small quantity. Indeed, this term can be written as $\left( \frac{(\Delta X)_0}{a_0} \right)^2$, where $a_0 = \frac{\hbar^2}{ma}$ is the "Bohr radius" for the 1D Coulomb atom. However, since the Bohr radius is the natural distance scale for our system, and in order to be experimentally accessible it must be greater than the minimal length. Then expanding Eq.(59) in terms of $\frac{(\Delta X)_0}{a_0}$ we obtain

$$
E_{\tilde{n}} = -\frac{m\alpha^2}{2\hbar^2\tilde{n}^2} \left[ 1 - 8 \left( \frac{\Delta X_0}{a_0} \right)^2 \frac{(\tilde{n} + 3/2)}{\tilde{n}^2} \right], \quad \tilde{n} = 1, 2, ...
$$

(60)

We note that, besides numerical factors, we have reproduced the same corrections terms as [20]. In Ref. [21] the correction term proportional to $(1/\tilde{n}^3)$ is missed.

Then, let us turn to the spectral decomposition of the fixed energy transition amplitude. Indeed, we use the following relation

$$
\frac{ih f(0)}{F(E)} \approx \frac{f(0)}{F'(E_n) E - E_n + i\eta},
$$

(61)
which when applied to \( F(E) = 2\hbar^2 E^2 [n^2 + (2n + 1) \lambda] + m\alpha^2 E \) gives

\[
\frac{i\hbar f(0)}{F(E)} \approx -\frac{f(0) \, i\hbar}{m\alpha^2 E - E_n + i\eta}.
\]  

(62)

This result allows us to fix the unwanted arbitrary factor in Eq.(46) as \( f(0) = m\alpha^2 \). With the aid of the following relations

\[
\cos \frac{\theta}{2p_E} = \frac{1}{\sqrt{1 + p^2/p_E^2}},
\]

(63)

\[
\sin \frac{\theta}{2p_E} = \frac{p/p_E}{\sqrt{1 + p^2/p_E^2}},
\]

(64)

we finally obtain

\[
(p_b \mid p_a)_E = \sum_{n=0}^{\infty} \frac{i\hbar}{E - E_n + i\eta} \Psi_n(p_b) \Psi_n(p_a),
\]

(65)

where \( E_n \) is the energy spectrum given by (59), and \( \Psi_n(p) \) are the normalized momentum space eigenfunctions given by

\[
\Psi_n(p) = \frac{e^{i\pi/2} \sqrt{\Delta_n}}{(1 + \beta p^2) (1 + p^2/p_E^2)^{1/2}} \left( \frac{p/p_E}{\sqrt{1 + p^2/p_E^2}} \right)^\lambda C_n^\lambda \left( \frac{1}{\sqrt{1 + p^2/p_E^2}} \right). \]

(66)

Let us point, that although the deformed algebra in Eqs.(1-2) is only defined to first order in the minimal length, the energy eigenvalues and momentum eigenfunctions are exact expressions. Let us also remark, that the energy eigenvalues are independent from the parameter \( \gamma \) which serves just to fix the measure in the definition of the completeness relation. Therefore, it can be ignored in the calculation or replaced by a more general term like \( \gamma h(p) \) without affecting the physical quantities.

Finally, let us obtain the momentum space eigenfunctions of the 1D Coulomb potential without the minimal length. Indeed, taking the limit \( \beta \to 0 \) and using the following relation [29]

\[
C_n^1(\cos \theta) = \frac{\sin (n + 1) \theta}{\sin \theta},
\]

(67)
we obtain, from (66),

\[ \Psi_{\tilde{n}}(p) = \frac{\sqrt{\frac{1}{4\pi pE}}}{(1 + p^2/p_E^2)^{1/2}} \left[ e^{i\tilde{n}\arctan \frac{p}{pE}} - e^{-i\tilde{n}\arctan \frac{p}{pE}} \right]. \]  

(68)

This expression is different from the one obtained in [30], where the authors of this paper reject the first term on the basis that the coordinate space eigenfunction vanishes for \( x \leq 0 \), due to the singular nature of the Coulomb potential for \( x = 0 \). In our case the singular point \( x = 0 \) is now hidden by the minimal length, since we cannot probe distances below the minimal length by virtue of the GUP.

Finally, let us point that the problems arising from the noncommutativity of the position operators are absent in our one dimensional framework, and that the generalization to higher dimensions is not a simple task. In the case of higher even dimensional spacetimes in the canonical noncommutativity, the path integral construction of the generating functional has been done in close analogy with the commutative case [31].

**5 Conclusion**

In summary, we have shown that the path integral in the momentum space representation of 1D Coulomb potential in the presence of a minimal length remains exactly solvable. Using the mid-point expansion and the space-time transformations, we mapped the problem to the one of a point particle in the symmetric Poschl-Teller potential. Although the deformed algebra given by Eq.(1-2) is only defined to first order in the minimal length \((\Delta X)_0\), we have obtained from the spectral decomposition of the fixed energy amplitude exact expressions of the energy eigenvalues and momentum space eigenfunctions. A particular feature of our calculation is the generation of quantum corrections, although we considered a one dimensional model. These quantum corrections arise naturally from the particle motion on curved spaces. This property of the minimal length reveals the rich structure.
of the spacetime, even in one dimension. Finally, we have checked the correctness of our results by rederiving the results of the usual Coulomb potential in one dimension and by the same way, we have proved that the energy levels are proportional to $1/n^2$, exactly like Coulomb potential in two and three dimensions. The more important case of the Coulomb potential in three dimensions with minimal length is under investigation and will be published elsewhere.

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