$z$-Weyl gravity in higher dimensions

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Abstract. We consider higher dimensional gravity in which the four dimensional spacetime and extra dimensions are not treated on an equal footing. The anisotropy is implemented in the ADM decomposition of higher dimensional metric by requiring the foliation preserving diffeomorphism invariance adapted to the extra dimensions, thus keeping the general covariance only for the four dimensional spacetime. The conformally invariant gravity can be constructed with an extra (Weyl) scalar field and a real parameter $z$ which describes the degree of anisotropy of conformal transformation between the spacetime and extra dimensional metrics. In the zero mode effective 4D action, it reduces to four-dimensional scalar-tensor theory coupled with nonlinear sigma model described by extra dimensional metrics. There are no restrictions on the value of $z$ at the classical level and possible applications to the cosmological constant problem with a specific choice of $z$ are discussed.

Keywords: gravity, modified gravity

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1 Introduction

Conformal invariance is an important device to probe diverse areas of theoretical physics and its application to gravity theory was initiated with the idea that conformally invariant gravity in four dimensions \cite{1, 2} might result in an unified description of gravity and electromagnetism, thus extending the Einstein’s general relativity. It is well-known that the Einstein-Hilbert action of general relativity is not a conformally invariant theory and in order to realize the conformal invariance, one has to resort to curvature-squared gravity with Weyl tensor, or to introduce conformal scalar fields \cite{3, 4} which compensate the conformal transformation of the metric. In the latter Weyl gravity in which we are interested, a quartic potential for the scalar field can be allowed and its higher dimensional extensions are straightforward.\(^1\)

In yet alternative attempt to apply the conformal symmetry to gravity, an anisotropic extension in which the homogeneous conformal transformation is generalized to anisotropic scaling symmetry can be considered. The prototype gravity theory which exhibits such a feature is the Horava-Lifshitz (HL) gravity \cite{6} which was proposed as a quantum theory of gravity. Essential feature of HL gravity is the anisotropic scaling of space and time as a fundamental symmetry of quantum gravity, thus abandoning the Lorentz symmetry at short distance. This idea was extended \cite{7} to the five dimensional theory in which the higher dimensional theory breaks diffeomorphism invariance explicitly at the action level to its foliation-preserving subgroup. The foliation can be adapted to an extra space dimension, thus leaving the four-dimensional spacetime diffeomorphism intact.\(^2\)

The purpose of this work is to construct 4+D dimensional conformal gravity in which the action is invariant under anisotropic conformal transformations: the spacetime (4) and extra dimensional (D) metric enjoy conformal transformations with different conformal weights. This kind of anisotropic conformal transformation was considered before in an attempt to extend the original Weyl invariance of Horava-Lifshitz gravity by using an extra scalar field. An anisotropic Weyl invariant theory whose action is invariant under the anisotropic conformal transformations of the space and time metric components in the ADM formalism was constructed and its cosmological consequences were investigated \cite{8}.

\(^1\)See ref. \cite{5} and references therein.
\(^2\)The action considered in the paper \cite{7} (eq. (2.3) of the reference), is not the most general action with foliation preserving diffeomorphism symmetry containing up to quadratic derivatives. Compare with our eq. (4.1).
In extension to higher dimensional space-time and construction of the anisotropic invariant gravity, there are essentially two steps that can be taken. Since we are treating the space-time and extra dimensions on a different footing, we do not require the full diffeomorphism symmetry on 4 + D dimensions and in order to implement the anisotropy between them, we only demand the foliation preserving diffeomorphism (FPD) invariant where the foliation is adapted along the extra dimensions so that the spacetime 4D general covariance is intact. The next step is to contemplate the conformal transformations of the spacetime and extra dimensional metrics with a scalar field that compensates the conformal weight of the metrics. Here, a parameter which describes the degree of anisotropy of conformal transformation between the spacetime and extra dimensional metrics can be introduced. It will be denoted by \( z \) which becomes a dynamical exponent of the action. After these steps, one arrives at a conformally anisotropic higher dimensional gravity but which preserves spacetime diffeomorphism. Since the action breaks 4+D general covariance explicitly, there may arise instability due to ghost-like excitations in the quadratic perturbation around a vacuum. This is checked in detail in 5D case. Another point is that a priori an arbitrary real value of \( z \) can be assigned for the endowment of anisotropic conformal invariance and the resulting theory encompass various limit including Einstein and conformal Weyl gravity, but some specific value of \( z \) could be preferred.

The paper is organized as follows: in section 2, we explicitly construct 4+D dimensional gravity with FPD invariance. In section 3, it is extended to anisotropic conformal invariance and discussions on the dimensional reduction to effective four dimensional gravity are given. In section 4, a more detailed account is given for 5D case. Some exact solutions are given and discussions on the dimensional reduction to effective four dimensional gravity are given. Section 5 includes summary and possible applications to the cosmological constant problem with a specific choice of \( z \).

## 2 FPD invariant gravity

Let us consider \((d+D)\) dimensional spacetime where the foliation is adapted to \( D \) dimensions. The \((d+D)\) dimensional metric can be written as

\[
ds^2 = g_{AB}(x,y)(dx^A + N^A_M dy^M)(dx^B + N^B_N dy^N) + \gamma_{MN}(x,y)dy^M dy^N, \tag{2.1}
\]

where \( A, B, \cdots \) indices denote those for \( d \)-dimensional base spacetime with coordinate \( x \) and \( M, N \) are those for \( D \)-dimension with coordinate \( y \). Here, \( N^A_M \) is a shift vector along the base spacetime corresponding to the \( M \)-direction \( [9] \), \( g_{AB} \) and \( \gamma_{MN} \) correspond to \( d \) and \( D \)-dimensional metric, respectively.

The above equation (2.1) is a generalized ADM decomposition: (i) When \( A, B, \cdots = 1, 2, 3 \) and \( M, N, \cdots = 0, 1, 2, 3 \); extra dimensions are foliated along the spacetime and it is the usual Kaluza-Klein reduction \([10]\). (iii) The case we are interested in is when 4 dimensional hypersurface is foliated along the extra \( D \)-dimensions, which is given by \( A, B, \cdots = 0, 1, 2, 3 \) and \( M, N, \cdots = 4, 5, \cdots D + 3 \). From here on we consider \( d = 4 \). (Also, change the notation \( A, B, \cdots \rightarrow \mu, \nu, \cdots \) and \( M, N, \cdots \rightarrow m, n, \cdots \)). To proceed, we consider foliation preserving diffeomorphism:

\[
x^\mu \rightarrow x'^\mu \equiv x^\mu(x,y), \quad y^n \rightarrow y'^m \equiv y^m(y), \tag{2.2}
\]

whose infinitesimal transformations are given by

\[
x'^\mu = x^\mu + \xi^\mu(x,y), \quad y'^m = y^m + \eta^m(y); \quad \partial_\mu \eta^m = 0. \tag{2.3}
\]
Here $\eta^m(y)$ is a function of $y^m$ only. One can check that

\[
\delta g_{\mu\nu} = -\mathcal{L}_\xi(x)g_{\mu\nu} - \eta^m(y)\partial_m g_{\mu\nu},
\]
\[
\delta N^\mu_m = -\mathcal{L}_\xi(x)N^\mu_m - \mathcal{L}_\eta(y)N^\mu_m - \partial_m \xi^\mu,
\]
\[
\delta \gamma_{mn} = -\mathcal{L}_\eta(y)\gamma_{mn} - \xi^\mu \partial^\mu \gamma_{mn},
\]

where $\mathcal{L}_\xi(x)$ and $\mathcal{L}_\eta(y)$ with $\xi = \xi^\mu \partial_\mu$, $\eta = \eta^m \partial_m$ are the Lie-derivative acting only on the indices of $x$ and $y$, respectively.

From the transformations (2.4)–(2.6), one obtains finite coordinate transformations as follows

\[ g^\prime_{\mu\nu}(x', y') = \frac{\partial x^\rho}{\partial x^\prime_\mu} \frac{\partial y^\sigma}{\partial y^\prime_\nu} g_{\rho\sigma}(x, y), \]
\[ N^\prime_m = \frac{\partial y^\mu}{\partial y^\prime_m} [\frac{\partial x^\mu}{\partial x^\rho} N^\nu_m(x, y) - \frac{\partial x^\nu}{\partial y^m}], \]
\[ \gamma^\prime_{mn}(x', y') = \frac{\partial y^p}{\partial y^m} \frac{\partial y^q}{\partial y^n} \gamma_{pq}(x, y). \]

One can check that the above transformations (2.7)–(2.9) leave the line element (2.1) invariant. The next step is to construct Diff(4)-covariant derivative and field strength [11, 12]:

\[ D_m g_{\mu\nu} = \partial_m g_{\mu\nu} - \mathcal{L}_\xi(x)g_{\mu\nu} \]
\[ = \partial_m g_{\mu\nu} - N^\mu_m \partial_\mu g_{\mu\nu} - (\partial_\mu N^\nu_m) g_{\mu\nu} - (\partial_\nu N^\mu_m) g_{\mu\nu}, \]
\[ F_{mn} = [D_m, D_n], \quad \xi_m = N^\mu_m \frac{\partial}{\partial x^\mu}, \]
\[ F^\mu_{mn} = \partial_m N^\mu_n - \partial_n N^\mu_m - N^\nu_m \partial_\nu N^\mu_n + N^\nu_n \partial_\nu N^\mu_m. \]

One can check that under FPD (2.2), the above quantities (2.10), (2.11) transform covariantly as

\[ (D_m g_{\mu\nu})^\prime = \frac{\partial y^\mu}{\partial y^m} \frac{\partial x^\rho}{\partial x^\prime_\mu} \frac{\partial x^\sigma}{\partial x^\prime_\nu} D_n g_{\rho\sigma}, \]
\[ (F_{mn})^\prime = \frac{\partial y^\mu}{\partial y^m} \frac{\partial y^q}{\partial y^n} \frac{\partial x^\prime_\mu}{\partial x^\prime_\nu} F^\nu_{pq}. \]

One also has the derivative of the metric $\gamma_{mn}$ transforming as

\[ (\partial_\mu \gamma_{mn})^\prime = \frac{\partial x^\nu}{\partial x^\prime_\mu} \frac{\partial y^p}{\partial y^m} \frac{\partial y^q}{\partial y^n} (\partial_\nu \gamma_{pq}). \]

under FPD (2.2). The covariant quantities (2.12)–(2.13) can be contracted to generate FPD invariant quantities. Geometrically, $D_m g_{\mu\nu}$ is a generalized extrinsic curvature along $m$-th extra dimension and $F^\mu_{mn}$ is a Diff(4) vector-valued curvature associated with the holonomy of the shift vector $N^\mu_n$ along the extra dimensions.

Other invariants are curvatures of 4 dimensional spacetime and $D$ extra dimensions. In order to construct these, let us first define

\[ \hat{\partial}_m = \partial_m - N^\mu_m \partial_\mu. \]

3FPD in refs. [11, 12] is restricted to the case where $y^m = y^n, \eta^n = 0$. 

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\[ \hat{\partial}_m = \partial_m - N^\mu_m \partial_\mu. \]
Then, we note that under FPD (2.2),

$$\hat{\partial}_m = \left( \frac{\partial y^n}{\partial y^m} \right) \partial_n, \quad \hat{\partial}_\mu = \left( \frac{\partial x^\nu}{\partial x^\mu} \right) \partial_\nu.$$  \hspace{1cm} (2.16)

Therefore, from (2.7), the curvature scalar $R^{(4)} = g^{\mu\nu} R_{\mu\nu}$ is a FPD invariant. Likewise,

$$\hat{R}^{(D)} = \gamma^{mn} \hat{R}_{mn},$$  \hspace{1cm} (2.17)

with the connection $\hat{\Gamma}^{p}$ given by

$$\hat{\Gamma}^{p}_{mn} = \frac{1}{2} \gamma^{pq} \left( \hat{\partial}_m \gamma_{np} + \hat{\partial}_n \gamma_{mp} - \hat{\partial}_q \gamma_{mn} \right)$$  \hspace{1cm} (2.18)

is also a FPD invariant. Consequently, we can construct a general FPD invariant $(4 + D)$ dimensional action as follows:

$$S_{(FPD)} = \int d^{4+D} x \sqrt{g} M_*^{2+D} \sqrt{\gamma} \left[ (R^{(4)} - 2\Lambda) + \alpha_1 \hat{R}^{(D)} - \frac{\alpha_2}{4} \gamma^{mn} \gamma^{pq} g_{\mu\nu} F_{mp} F_{\nu q} \right. \right.$$

$$\left. - \frac{\alpha_3}{4} \gamma^{mn} g^{\mu\nu} g^{\alpha\beta} (\mathcal{D}_m g_{\mu\nu} \mathcal{D}_n g_{\alpha\beta} - \alpha_4 \mathcal{D}_m g_{\mu\nu} \mathcal{D}_n g_{\alpha\beta}) \right. \right.$$

$$\left. - \frac{\alpha_5}{4} g^{\mu\nu} \gamma^{mn} \gamma^{pq} (\partial_\mu \gamma_{mp} \partial_\nu \gamma_{nq} - \alpha_6 \partial_\mu \gamma_{mn} \partial_\nu \gamma_{pq}) \right] + S_m,  \hspace{1cm} (2.19)$$

where $\alpha_{1-6}$ are arbitrary constants and $M_*$ is the higher dimensional gravitational constant.

One can check that when $\alpha_1 = 1$, the action (2.19) combines into the $(4 + D)$ dimensional Einstein-Hilbert action with cosmological constant with the FPD being elevated to the $4 + D$ general covariance. $S_m$ is the matter action whose constituents will not be our concern, but we will compute general constraint on the energy-momentum tensor of $S_m$ imposed by FPD invariance in 5D case. For generic values of $\alpha$’s the theory exhibits pathological behaviors displaying ghost-like excitations in the perturbation theory [7]. This issue will be taken up later for the 5D conformally invariant case.

3 Conformal extension

In this section, we first consider anisotropic extension of the action (2.19), whose action is invariant with respect to the anisotropic conformal transformations:

$$g_{\mu\nu} \rightarrow e^{2\omega(x,y)} g_{\mu\nu}, \quad \gamma_{mn} \rightarrow e^{2\omega(x,y)} \gamma_{mn}, \quad \phi \rightarrow e^{-v(z)\omega(x,y)} \phi, \quad N^\mu_m \rightarrow N^\mu_m,  \hspace{1cm} (3.1)$$

Here, we have introduced a parameter $z$ which characterizes the anisotropy of the extra dimension and $v(z)$ is an arbitrary function of $z$. $z = 1$ and $v(z) = 1 + D/2$ corresponds to the isotropic conformal transformations [5]. Taking into account an additional scalar field $\phi$, ...
one can construct the anisotropic invariant action in $(4 + D)$ dimensions as

$$ S_{(C)} = \int d^{4+D}x \sqrt{-g^{(4)}(4)\gamma M_*^{2+D}} \left[ \phi^{2+D_\gamma} \left( R^{(4)} + A_1 \gamma \nabla^\mu \nabla_\mu \phi + A_2 \frac{\gamma \nabla^\mu \phi \nabla_\mu \phi}{\phi^2} \right) - V_0 \phi^{2N} ight] + \alpha_1 \phi^{4+D_\gamma} \left( R^{(D)} + B_1 \gamma \nabla^\mu \nabla_\mu \phi + B_2 \frac{\gamma \nabla^\mu \phi \nabla_\mu \phi}{\phi^2} \right) - \frac{\alpha_2}{4} \phi^{4+D_\gamma} \gamma \nabla^\mu \phi \nabla_\mu \phi,$$

$$ - \alpha_3 \phi^{4+D_\gamma} \left\{ \frac{1}{4} \gamma^{mn} g^{\mu \nu} \alpha^3 (D_m g_{\mu \alpha} D_n g_{\nu \beta} - \alpha_4 D_m g_{\mu \nu} D_n g_{\alpha \beta}) \right\} + C_1 \gamma^{mn} g^{\mu \nu} \nabla_n \phi \nabla_m \phi + C_2 \gamma^{mn} \frac{\nabla^2 \phi}{\phi^2} + C_3 \gamma^{mn} \nabla^2 \phi \right\},$$

$$ + D_1 \gamma^{mn} g^{\mu \nu} \gamma^{pq} \nabla_\mu \phi \nabla_\nu \phi + D_2 \gamma^{mn} \nabla_\mu \phi \nabla_\nu \phi \right\},$$

(3.2)

where $M_*$ sets up a higher dimensional gravity scale and the Weyl scalar $\phi$ is dimensionless field. The hat covariant derivative is given by $\hat{\nabla}_m = \hat{\delta}_m^j + \hat{F}^m_{NP} j^P$ and the coefficients $A_1, B_1, C_1, D_1, D_2$ are fixed as

$$ A_1 = \frac{-6}{v}, \quad A_2 = \frac{-6}{v} \left( \frac{1}{v} - 1 \right), \quad B_1 = \frac{2z(1-D)}{v}, \quad B_2 = \frac{2z(1-D)}{v} \left( \frac{z(D-2)}{2v} - 1 \right), \quad C_1 = \frac{-4(1-4\alpha_4)}{v^2}, \quad C_2 = \frac{-4(1-4\alpha_4)}{v^2}, \quad D_1 = \frac{z(1-\alpha_6 D)}{v}, \quad D_2 = \frac{z^2 D(1-\alpha_6 D)}{v^2}. \quad (3.3)$$

Also, the exponent $N$ of the scalar field can be found as

$$ N = \frac{4 + Dz}{2v}. \quad (3.4)$$

When $\alpha_1, \alpha_6 = 1$ and $z = 1$, the above action (3.2) combines into the $(4 + D)$ dimensional conformally invariant gravity with $4 + D$ general covariance [5]:

$$ S_{(CW)} = \int d^n x \sqrt{-g^{(4+D)}} \left\{ \phi^2 R^{(n)} + \frac{4(n-1)}{n-2} g^{ab} \nabla_a \phi \nabla_b \phi - V_0 \phi^{2N} \right\},$$

(3.5)

where $a, b = 0, \ldots, n-1$ and $n = 4 + D$.

In passing, we make the following remarks. The only dimensionful parameter inside the parenthesis of (3.2) is $V_0$ which has mass squared dimensions originating from the cosmological constant term in (2.19). Also, the action (3.2) has conformal invariance for any real values of $z$. That negative values of $z$ might lead to some interesting application regarding the cosmological constant will be discussed later. Finally, it is to be noticed that FPD (2.7)–(2.9) remains as symmetry along with the scalar transformation $\phi'(x', y') = \phi(x, y)$ in the action (3.2), and FPD invariance and conformal symmetry are compatible with each other.

\footnote{Note that the over-all scale $M_*$ cannot be absorbed into the scalar field all together in (3.2), unless $z = 1$ so that $v(z) = 1 + D/2$, i.e., the isotropic case. From here on, we choose $v = (2 + Dz)/2$, because it simply yields the conventional 4D non-minimal coupling term of $\phi^2 R^{(4)}$. The case $Dz = -2$ is given a separate treatment.}
Now we discuss 4-dimensional effective action and let us consider only zero modes [10]. It is convenient to go to a frame where $N_{\mu}^m = 0$ (“comoving” frame) and we impose y-independence (cylindrical condition) for $g_{\mu\nu} = g_{\mu\nu}(x), \phi = \phi(x)$ and $\gamma_{mn} = \gamma_{mn}(x)$. In this approximation, the action (3.2) can be expressed as

$$S_C = \frac{1}{16\pi G^D}\int d^4x \sqrt{-g^{(4)}}|\det \gamma(x)|^{\frac{1}{2}} \left\{ \phi^2 R^{(4)} + Ag^{\mu\nu}\nabla_\mu \phi \nabla_\nu \phi - V(\phi) \right\}$$

$$- \alpha_5 \phi^2 \left[ \frac{1}{4}g^{\mu\nu}\gamma_{mn}\gamma_{pq} \left( \nabla_\mu \gamma_{mp} \nabla_\nu \gamma_{nq} - \alpha_6 \nabla_\mu \gamma_{mn} \nabla_\nu \gamma_{pq} \right) \right] + \bar{D} \gamma_{mn}(\nabla_\mu \gamma_{mn})\phi \nabla^\mu \phi \right\}. \quad (3.6)$$

The constant $G'$ and the potential are given by

$$\frac{1}{16\pi G^D} = M_s^{2+D}V(D), \quad V(\phi) = V_0\phi^{2+\frac{1}{2}Dz}, \quad (3.7)$$

where $V(D)$ is the volume of the extra dimensions, $V(D) = \int d^D y$. And

$$\bar{A} = \frac{1}{v^2} \left[ 12v - 6 - \alpha_5 z^2 D(1 - \alpha_6 D) \right], \quad \bar{D} = \frac{1}{v} [3 - \alpha_5 z(1 - \alpha_6 D)]. \quad (3.8)$$

There are several interesting aspects worthy of discussions in the above action (3.6). The power of the potential is exotic and can become any, in principle. $z = 0$ gives the usual $\phi^4$ potential. To avoid possible discontinuities of the potential and to have the parity symmetry $\phi \to -\phi$, we can restrict to only even powers of potential which selects $z$ as

$$z = \frac{2}{D} \left( \frac{1}{N-1} - 1 \right) \quad (N = \text{integers}), \quad (3.9)$$

with potential of the form $\sim \phi^{2N}$ with positive as well as negative powers being allowed. The $\gamma_{mn}$ fields behave like 4D scalar fields under general covariance (i.e., eq. (2.9)) in the above action and it describes scalar-tensor theory of gravity coupled with a complex nonlinear sigma model type given by $\gamma_{mn}$’s. It will be demonstrated explicitly that the gravity sector of the action (3.6) has the Einstein-Hilbert action at one side and Weyl gravity at the other. The existence of ghost-like excitations of $\gamma$’s is dependent upon the generic values of $\alpha$’s. For example, when $\alpha_5 = \alpha_6 = 1$, this problem can be avoided (see below).

It is crucial to observe that the action (3.6) exhibits residual 4D conformal invariance

$$g_{\mu\nu} \to e^{2\omega(x)} g_{\mu\nu}, \quad \gamma_{mn} \to e^{2\omega(x)} \gamma_{mn}, \quad \phi \to e^{-(1 + \frac{D\omega}{2})\omega(x)} \phi, \quad (3.10)$$

where $\omega$ is now a function of $x$ only, $\omega \equiv \omega(x)$. The residual conformal invariance enables to fix further the conformal scalar to be $\phi = \phi_0$, and the resultant action can be expressed as

$$S_\gamma = \frac{1}{16\pi G_N} \int d^4x \sqrt{-g} |\det \gamma(x)|^{\frac{1}{2}} \left\{ R^{(4)} - 2\Lambda_\gamma \right\}$$

$$- \alpha_5 \left[ \frac{1}{4}g^{\mu\nu}\gamma_{mn}\gamma_{pq} \left( \nabla_\mu \gamma_{mp} \nabla_\nu \gamma_{nq} - \alpha_6 \nabla_\mu \gamma_{mn} \nabla_\nu \gamma_{pq} \right) \right], \quad (3.11)$$

where

$$\frac{1}{16\pi G_N} = M_s^{2+D} \phi_0^2 V(D), \quad \Lambda_\gamma = \frac{1}{2} V_0 \phi_0^{\frac{4}{2+Dz}}. \quad (3.12)$$
We remark that we would have arrived at the same action, had we taken the alternative route: we can first fix $\phi = \phi_0$ using the conformal invariance in the action (3.2). Then, restrictions to zero modes again with only $x$-dependence lead to an effective 4D theory which preserves 4D general covariance. In the comoving frame, the action assumes the same form as (3.11). Note that the residual symmetry is independent of the cylindrical condition: after fixing $\phi = \phi_0$, the action (3.2) still respects $4 + D$ FPD (2.7)–(2.9). Alternatively, in the comoving gauge with $N_\mu^\nu = 0$, the action (3.2) is still invariant under the conformal transformation (3.1). When $\alpha_5 = \alpha_6 = 1$, the action (3.11) coincides with the zero mode sector of the scalar-graviton action in ref. [13] with an extra cosmological constant given by eq. (3.12), and in this case, there is no instability problem. Finally, we remark that one can check that the above action (3.11) admits solution given by

$$R^{(4)}_{\mu\nu} = \Lambda g_{\mu\nu}, \quad \gamma_{mn} = \gamma_{mn}^0 (= \text{const}). \quad (3.13)$$

This solution describes 4D de Sitter space when $\Lambda > 0$.\footnote{The solution may not be the exact solutions of full $(4 + D)$-dimensional field equations. However, this point will not be pursued further in this work.}

We can go one step further to absorb the factor $|\det \gamma(x)|^{1/2}$ in (3.11) into the metric by using the conformal transformation via

$$\tilde{g}_{\mu\nu} = |\det \gamma(x)|^{1/2} g_{\mu\nu}. \quad (3.14)$$

Rewriting (3.11) in terms of $\tilde{g}_{\mu\nu}$, we obtain (with $\tilde{g}_{\mu\nu} \to g_{\mu\nu}$)

$$S'_{(\gamma)} = \frac{1}{16\pi G_N} \int d^4x \left\{ R^{(4)} - 2|\det \gamma(x)|^{-1/2} \Lambda \gamma 
- \alpha_5 \left[ \frac{1}{4} g^{\mu\nu} \gamma_{mn} \gamma_{pq} (\nabla_\mu \gamma_{mp} \nabla_\nu \gamma_{nq} - \alpha_6 \nabla_\mu \gamma_{mn} \nabla_\nu \gamma_{pq}) \right] \right\},$$

with $\alpha_6' = \alpha_6 - \frac{3}{2\alpha_5}$. The above action describes the Einstein-Hilbert action coupled to nonlinear sigma model with target space coordinates given by $\gamma_{mn}$’s with a potential term. In 5D, the nonlinear sigma model is simply a scalar field theory with an exponential potential ($\Phi \equiv \log \gamma_{55}$):

$$V(\Phi) = 2\Lambda \gamma \exp \left( -\frac{1}{2} \Phi \right). \quad (3.15)$$

The scalar field is canonical or ghost depending on $\alpha_5 + \frac{3}{2} - \alpha_5 \alpha_6 > 0$ or $< 0$ as can be readily checked.

There exists other route as far as gauge fixing is concerned. Instead of fixing $\phi$, we can fix one component of $\gamma_{mn}$ or a combination of $\gamma_{mn}$’s to be a constant. The most natural choice from the action (3.6) is put $|\det \gamma(x)| = 1$. Then, we end up with

$$S_{(\phi, \gamma)} = \frac{1}{16\pi G} \int d^4x \sqrt{-g} \left\{ \phi^2 R^{(4)} + \bar{A} g^{\mu\nu} \nabla_\mu \phi \nabla_\nu \phi - V(\phi) 
- \alpha_5 \phi^2 \left[ \frac{1}{4} g^{\mu\nu} \gamma_{mn} \gamma_{pq} (\nabla_\mu \gamma_{mp} \nabla_\nu \gamma_{nq}) \right] + \lambda (|\det \gamma(x)| - 1) \right\}, \quad (3.16)$$
where $\lambda$ is a Lagrange multiplier. Now, the gravity sector is given by the 4D Weyl gravity.\(^6\) There are, however, noticeable changes from the ordinary 4D Weyl gravity. The coefficient $\bar{A}$ is no longer the well-known $6$, but is given by the value in (3.8). Recall that in 4D, conformal invariance would inevitably render $\phi$ to be a ghost with a positive value of $6$. In our case, there arise additional contributions from the conformal invariance of $\gamma_{mn}$ sector which could add up to make the overall coefficient $\bar{A}$ to be negative, making $\phi$ carry positive kinetic energy for certain range of parameters. This renders the resulting 4D action to be the Brans-Dicke gravity. Another difference is that the power of the potential is drastically different from $4$. These properties are basically inherited from anisotropic conformal invariance of the higher dimensions. It is interesting to observe that both the Einstei-Hilbert action and Brans-Dicke gravity can originate as two different gauged fixed versions from the conformally invariant action (3.6).

A couple of comments are in order. Conformal invariance alone does not put any restrictions on the value of $z$. When $Dz = -2$, one can not choose $v = 1 + \frac{Dz}{2}$ and this case has to be treated separately. Nevertheless, the anisotropically conformal invariant action is still given by eq. (3.2). If we impose $Dz = -2$ in the action (3.2), it assumes a similar form as (3.2) with the noticeable difference that the conformal factor $\phi^2$ is no longer multiplying $R(4)$. If we perform the dimensional reduction by imposing the cylindrical condition and fixing $\phi = \phi_0$ as before, we end up with the exactly same forms as (3.11) with $(v = 1)$

$$\frac{1}{16\pi G_N} = M^2(5) V(D), \quad \Lambda(2) = \frac{1}{2} V\phi_0^2.$$ (3.17)

Likewise, fixing $|\det(\gamma(x))| = 1$ yields action of the type (3.16) in which the Weyl $\phi^2 R(4)$ term is replaced by the Einstein-Hilbert $R(4)$ and the potential is quadratic, $V \sim \phi^2$. It can be readily checked that this $Dz = -2$ case corresponds to redefinition of $N \to \infty$ limit of the action (3.2) with the parity symmetry $\phi \to -\phi$ being imposed as in (3.9). Another comment is that the action (3.2) does not fit to describe 5D case, since scalar curvature $R(D=1)$ and $F_{\mu \nu}$ are always zero. Therefore, separate investigations of anisotropic invariant action in 5D are performed in the next section, in which we explore other ground states relaxing the cylindrical condition and considering ground states with $g$-dependence. We also perform the quadratic perturbations to check the ghost-like excitations.

4 \hspace{1em} z-Weyl gravity in 5D

In this section, we investigate 5D case in some detail and include $S_m$ later for generality. We search solutions of equations of motion and perform quadratic perturbations around a vacuum to check the instability. We intend to make this section be self-contained. For that purpose, let us first rewrite the anisotropic invariant action in 5D, whose form with slight adjustments of notations and rearrangements of terms (see below) can be expressed as

$$S_{(5D)} = \int dy dx N \sqrt{g(4)} M^3 \left[ \phi^2 \left( R(4) - \frac{12}{z+2} \nabla\mu \nabla^\mu \phi \phi + \frac{12z}{(z+2)^2} \frac{\nabla^\mu \phi \nabla^\mu \phi}{\phi^2} \right) + \beta_1 \phi^2 \frac{2(z-4)}{(z+2)^2} \left( B_{\mu \nu} B^{\mu \nu} - \lambda B^2 \right) + \beta_2 \phi^2 A_\mu A^\mu - V(\phi) \right].$$ (4.1)

\(^6\)In five dimensional case with $\gamma_{55} = N^2$, $\alpha_5$ term vanishes and only the gravity sector survives.
where the potential $V$ is
\[ V = V_0 \varphi^{2(z+4)/(z+2)}. \quad (4.2) \]
In the action (4.1), $B_{\mu\nu}$ and $A_\mu$ are given by
\[ B_{\mu\nu} = K_{\mu\nu} + \frac{2}{(z+2)N} \varphi \left( \partial_\nu \varphi - \nabla_\nu \varphi N^\nu \right), \]
\[ A_\mu = \frac{\partial_\mu N}{N} + \frac{2z}{z+2} \frac{\partial_\mu \varphi}{\varphi}, \quad (4.4) \]
where $K_{\mu\nu}$ is the extrinsic curvature tensor, $K_{\mu\nu} = (\partial_\mu g_{\nu\rho} - \nabla_\mu g_{\nu\rho} - \nabla_\nu g_{\mu\rho})/(2N)$. Note that the above action (4.1) can be obtained from the action (3.2), when replacing $\phi$ with $\varphi$ and choosing $D = 1$, $v = (2 + z)/2$, $\alpha_3 = -\beta_1$, $\alpha_4 = \lambda$, $\alpha_5(\alpha_6 - 1) = \beta_2$. When $\beta_1 = \lambda = z = 1$ and $\beta_2 = 0$ it is 5D Weyl gravity, eq. (3.5) with $D = 1$. Note also that the action (4.1) is simply the anisotropic invariant extension of the following action considered in ref. [7]:
\[ S_{(5F)} = \int d^2 x N \sqrt{|g(4)|} \left[ \left( R(4) - 2\Lambda_5 \right) + \beta_1 \{ K_{\mu\nu}K^{\mu\nu} - \lambda K^2 \} + \beta_2 N^{-2} g^{\mu\nu} \partial_\mu N \partial_\nu N \right], \]
obtained from the action (2.19) in 5D. One can check that the action (4.1) is anisotropically conformal invariant with respect to
\[ N \rightarrow e^{\omega} N, \quad \varphi \rightarrow e^{-\omega} \varphi, \quad \varphi \rightarrow e^{-\omega} \varphi, \quad (4.5) \]
where $\omega = \omega(x, y)$.

Now we compute the equations of motion for the action (4.1). Varying for $N$, $N^\nu$, $g^{\mu\nu}$, $\varphi$ in the action (4.1) leads to the equations of motion:
\[ \delta_N S_{(5D)}: \quad \varphi (R(4)) - 2(4-z) \nabla_\mu \nabla_\nu \varphi \varphi = \beta_1 \frac{2(z+4)}{z+2} \left( B_{\mu\nu} B^{\mu\nu} - \lambda B^2 \right) \]
\[ - \beta_2 \frac{2(z-2)}{z+2} \left( A_\mu A^\mu + 2 \nabla_\mu A^\mu + \frac{8}{z+2} \nabla_\mu \varphi \varphi \right) - V(\varphi) = 0, \quad (4.6) \]
\[ \delta_{N^\nu} S_{(5D)}: \quad \nabla_\mu B^{\mu \nu} + \frac{2(4-z) \nabla_\mu \nabla_\nu \varphi \varphi}{z+2} A^\mu + \frac{2(z^2 - 8z - 8)}{z+2} \nabla_\mu \varphi \varphi B - \lambda \nabla_\nu B = 0, \quad (4.7) \]
\[ \delta_{g^{\mu\nu}} S_{(5D)}: \quad \varphi (E^{(1)}_{\mu\nu}) + \beta_1 \varphi \varphi E^{(1)}_{\mu\nu} + \beta_1 \lambda \varphi \varphi E^{(3)}_{\mu\nu} + N \beta_2 \varphi^2 E^{(4)}_{\mu\nu} = 0, \quad (4.8) \]
where
\[ E^{(1)}_{\mu\nu} = \frac{R(4)_{\mu\nu} - 2(z+1) \nabla_\eta N \nabla_\gamma \varphi \varphi}{N} g_{\mu\nu} - \frac{2(z^2 - 2z - 2)}{z+2} \nabla_\gamma \varphi \varphi g_{\mu\nu} + \frac{4(z - 2) \nabla_\gamma \varphi \varphi}{(z+2)^2} \varphi \]
\[ + 2 \nabla_\gamma \nabla_\gamma \varphi \varphi - \frac{\nabla_\mu \nabla_\nu \varphi \varphi}{N} = \frac{2(z^2 - 8z - 8)}{z+2} \nabla_\mu \varphi \varphi + \frac{4(z - 4)}{z+2} N \nabla_\mu \nabla_\nu \varphi \varphi + \frac{4}{z+2} B_{\mu\nu} B^\rho_{\mu\rho} g_{\mu\nu} - N B B_{\mu\nu}, \]
\[ E^{(2)}_{\mu\nu} = -2N(\mu)N(\nu)B^\gamma_{\nu} + 2\nabla_\eta N^\rho B_{\rho\mu} + N g_{\mu\nu} - 2N B_{\mu\nu} + 2 \nabla_{\rho\nu} B_{\mu\nu} + \frac{N}{2} B_{\rho\nu} B_{\mu\nu} - N B B_{\mu\nu} - \partial_\gamma B_{\mu\nu} + (z - 2) \partial_\mu \varphi \frac{4(z - 4)}{z+2} N m B_{\nu} \nabla_\nu \varphi \varphi + \frac{4}{z+2} B_{\mu\nu} B_N \nabla_\nu \varphi \varphi, \]
\[ \beta_2 \text{ term is omitted in this reference.} \]
\[ E_{\mu\nu}^{(3)} = -N \frac{B^2}{2} g_{\mu\nu} + NB^2 g_{\mu\nu} + 2\nabla_\mu BN_\nu - \frac{4z}{z+2} B N(\mu \nabla_\nu) \varphi - z\thetaBg_{\mu\nu} - \nabla_\mu BN^\rho g_{\mu\nu} + \partial_\nu Bg_{\mu\nu}, \]
\[ E_{\mu\nu}^{(4)} = A_\mu A_\nu - \frac{1}{2} g_{\mu\nu} A_\gamma A^\gamma, \]

and
\[ \delta_\varphi S_{5D}; \quad 2\varphi R_{(4)} - \frac{48(z+1)}{(z+2)^2} \nabla_\gamma \nabla_\varphi - \frac{12}{(z+2)N} \varphi \nabla_\rho \nabla^\rho N - \frac{48(z+1)}{(z+2)^2} \nabla_\rho \varphi \nabla^\rho N \]
\[ + \frac{2\beta_1}{z+2} \varphi^\frac{3(z-2)}{z+2} \left[ (2-\lambda(z+4))B^2 - \frac{2}{N}(1-4\lambda)z\theta - \frac{2}{N} \nabla_\rho BN^\rho + (z-4)B_{\mu\nu}^2 \right. \]
\[ + 2(1-4\lambda) \frac{1}{N} \partial_\rho B \left] - 2\beta_2 \varphi [A_\mu A^\mu - \frac{4}{z+2} \nabla_\mu N N^\mu + \frac{2z}{z+2} \nabla_\mu A^\mu] - \frac{dV(\varphi)}{d\varphi} = 0, \]

where \( \theta = 2(\partial_\rho \varphi - \nabla_\rho \varphi N^\rho)/\{(z+2)\varphi \} \).

### 4.1 Exact solutions

In order to find ground state solutions for the eqs. (4.6)\(\sim\)(4.9) with \(y\)-dependence, we work in the comoving gauge with \(N_\mu = 0\), where the five dimensional metric tensor is given by
\[ ds^2 = g_{\mu\nu} dx^\mu dx^\nu + N^2 dy^2. \]

Let us consider the ansatz
\[ g_{\mu\nu} = a^2(y)\eta_{\mu\nu}, \quad N = N(y), \quad \varphi = \varphi(y), \]

which describes a warped spacetime\(^8\) with 4D Minkowski vacuum \((R_{\mu\nu}^{(4)} = 0)\). We include a matter term \(S_m\) in the action (4.1) at this stage and substitution of the above ansatz (4.11) into the eqs. (4.6)\(\sim\)(4.9) (plus matter) yields the following equations:
\[ (1-4\lambda)\beta_1 H^2 = -2(1-4\lambda)\beta_1 H\theta - (1-4\lambda)\beta_1 \theta^2 + \frac{N^2}{4} \varphi^{z-4} V + \frac{N^2}{4} \varphi^{\frac{2(z-4)}{z+2}} T_1, \]
\[ (1-4\lambda)\beta_1 \partial_\rho H = (1-4\lambda)\beta_1 \left[ z\theta^2 + zH\theta - \partial_\rho \theta + \frac{\partial_\rho N}{N}(H+\theta) \right] - \frac{N^2}{2} \varphi^{\frac{2(z-4)}{z+2}} (T_1 + T_2), \]
\[ 0 = (1-4\lambda)\beta_1 \left[ (z+4)H^2 - (z-4)\theta^2 + 8H\theta + 2(\partial_\rho H + \partial_\rho \theta) \right] - \frac{2\partial_\rho N}{N}(H+\theta) - \frac{N^2}{4} \varphi^{\frac{2(z-4)}{z+2}} \frac{dV}{d\varphi}, \]

where quantities \(H\) and \(T_{1,2}\) are given by
\[ H = \frac{\partial_\rho a}{a}, \quad T_1 = -\frac{1}{M_4^4 \sqrt{g_{(4)}}} \frac{\delta S_M}{\delta N}, \quad T_2 g_{\mu\nu} = -\frac{2}{N M_4^4 \sqrt{g_{(4)}}} \frac{\delta S_M}{\delta g_{\mu\nu}}. \]

After some manipulations of the eqs. (4.12)\(\sim\)(4.14), one can combine the above equations into a single equation:
\[ T = (z + 4) V - \frac{z + 2}{2} \varphi \frac{dV}{d\varphi}, \]

\(^8\)See ref. [14] and references therein.
where a quantity $T$ is defined by $T \equiv -zT_1 + 4T_2$. We note that $T$ is related to the trace of the energy momentum tensor for the matter $S_m$. In an isotropic case with $z = 1$, $T$ corresponds to the trace $T^{AA}$ for the energy momentum tensor $T_{AB}$ of matter in 5D. Note also that for the anisotropic invariant action (4.1) with the potential (4.2), the r.h.s. of the eq. (4.16) yields zero so in this case, the quantity $T$ should satisfy the following condition
\[ T = 0 \rightarrow T_2 = \frac{z}{4}T_1. \] (4.17)

Alternatively, the above condition (4.17) can be obtained, after some tedious manipulations with eqs. (4.6)–(4.9), by checking $\delta S_{5D} = 0$ for the infinitesimal $(\omega \ll 1)$ transformations (4.5):
\[ \delta N = z\omega N, \quad \delta N^\mu = 0, \quad \delta g^{\mu\nu} = -2\omega g^{\mu\nu}, \quad \delta \varphi = -\frac{z+2}{2}\omega\varphi, \]
\[ \delta S_{5D} = \frac{\delta S_{5D}}{\delta N}\delta N + \frac{\delta S_{5D}}{\delta g^{\mu\nu}}\delta g^{\mu\nu} + \frac{\delta S_{5D}}{\delta N^\mu}\delta N^\mu + \frac{\delta S_{5D}}{\delta \varphi}\delta \varphi = 0 \]
\[ \rightarrow T_2 = \frac{z}{4}T_1. \] (4.18)

The above equation says that only matter fields satisfying this condition can couple. In other words, eq. (4.17) is the anisotropic conformal condition that should be satisfied by the matter sector.

Note that the potential (4.2) satisfies eq. (4.17) and we present solutions with (i) $V_0 = 0$ and (ii) $V_0 \neq 0$: explicitly, one finds the solutions as follows:

(i) $V_0 = 0$
\[ g_{\mu\nu} = e^{-2\zeta^2y^2}\eta_{\mu\nu}, \quad N = N_0 e^{-b\zeta^2y^2}, \quad \varphi = \varphi_0 e^{\frac{z+2}{2}\zeta^2y^2}; \]

(ii) $V_0 = 16(1 - 4\lambda)\frac{\zeta^4y_0^2}{N_0^2} \varphi_0 \frac{N_0^2}{\bar{\varphi}_0}$
\[ g_{\mu\nu} = e^{-2\zeta^2(y-y_0)^2}\eta_{\mu\nu}, \quad N = N_0 e^{-z\zeta^2y^2}, \quad \varphi = \varphi_0 e^{\frac{z+2}{2}\zeta^2y^2}, \]

where $\zeta$, $b$, $y_0$, $N_0$, $\varphi_0$ are some constants. Both of these solutions describe warped spacetime with Gaussian warp factor which has been discussed in a number of papers [15–18]. But we will not go further along this direction. Instead we proceed in the next section to investigate scalar perturbations and vacuum instability of case (i).

### 4.2 Scalar perturbations in the quadratic action

It is known that an explicit breaking of the five dimensional diffeomorphism invariance leads to a pathological behavior of the scalar graviton [7], which includes four time derivatives or ghost-like kinetic term of the graviton. Thus, it is a very important issue to check if such a scalar graviton problem does exist in the model (4.1), which explicitly breaks the 5D diffeomorphism invariance.

In order to investigate the behavior of the scalar mode, we consider the following scalar perturbations of the metric and the scalar field for the Minkowski background with $V_0 = 0$

\[ N = e^\alpha, \quad N_\mu = \partial_\mu \beta, \quad \varphi = \varphi_0 + \tilde{\varphi}, \] (4.19)
where $\alpha$, $\beta$, $\xi$, $\bar{\varphi}$ are function of $(x^\mu, y)$ and $\varphi_0$ is a constant. We notice that the above ones (4.19) correspond to the perturbations around the 5D Minkowski background solution (i), obtained when choosing $\zeta = 0$, $N_0 = 1$. Substituting the perturbations (4.19) into the action (4.1) and after a lengthy computation, we arrive at the following quadratic action:

\[
S^{(2)} = \int dyd^4x \left[ -6\varphi_0^2 \left\{ (\partial_\mu \xi)^2 + 2\xi \Box^{(4)} \xi + 2\varphi_0^{-1}\bar{\varphi} \Box^{(4)} \xi \right\} 
\right.
\]

\[
- \frac{12}{z + 2} \varphi_0 \left\{ 2\partial_\mu \xi \partial_\mu \bar{\varphi} + \alpha \Box^{(4)} \bar{\varphi} + 2\xi \Box^{(4)} \bar{\varphi} + \frac{2(z + 1)}{z + 2} \varphi_0^{-1}\bar{\varphi} \Box^{(4)} \bar{\varphi} \right\}
\]

\[
+ \beta_1 \varphi_0 \frac{2(z - 1)}{z + 2} \left\{ 4(1 - 4\lambda)(\partial_\beta \xi)^2 - 2(1 - 4\lambda)\partial_\beta \xi \Box^{(4)} \beta + (1 - \lambda) \Box^{(4)} \beta \right\}^2
\]

\[
+ \frac{16(1 - 4\lambda)}{z + 2} \varphi_0^{-1}\partial_\beta \xi \partial_\beta \bar{\varphi} - \frac{4(1 - 4\lambda)}{z + 2} \varphi_0^{-1}\Box^{(4)} \beta \partial_\beta \bar{\varphi}
\]

\[
+ \frac{16(1 - 4\lambda)}{(z + 2)^2} \varphi_0^{-2}(\partial_\beta \bar{\varphi})^2 \right\} + \beta_2 \varphi_0 \left\{ \partial_\mu \alpha + \frac{2z}{z + 2} \varphi_0^{-1}\partial_\mu \bar{\varphi} \right\}^2.
\]

Here, $\Box^{(4)}$ denotes the four dimensional d’Alembert operator, defined by $\Box^{(4)} \equiv -\partial_\beta^2 + \partial_\xi^2$. The action (4.20) includes a potentially dangerous term with the four time derivatives, given by $(\Box^{(4)} \beta)^2$ in the third line and this raises vacuum instability problem. We check whether this term can be removed as a consequences of equations of motion for the fluctuations.

Varying the above action (4.20) with respect to $\alpha$ and $\beta$, one obtains the equations of motion as

\[
3\Box^{(4)} \xi + \frac{2(3 + \beta_2 z)}{z + 2} \varphi_0^{-1}\Box^{(4)} \bar{\varphi} + \beta_2 \Box^{(4)} \alpha = 0,
\]

(4.21)

\[
(1 - \lambda) \Box^{(4)} \beta = (1 - 4\lambda) \left( \partial_\beta \xi + \frac{2}{z + 2} \varphi_0^{-1}\partial_\beta \bar{\varphi} \right).
\]

(4.22)

Note that for the case of $\beta_2 = 0$, $\alpha$ behaves like a Lagrange multiplier. In this case, the perturbations $\varphi$, $\beta$ can be completely eliminated due to the eqs. (4.21), (4.22) in favor of the scalar graviton mode $\xi$. It turns out that in this case, there is no dynamical scalar graviton in the quadratic action (4.20).

On the other hand, for $\beta_2 \neq 0$ one can replace $\alpha$, $\Box^{(4)} \beta$ with $\xi$, $\bar{\varphi}$ in the action (4.20) by using the eqs. (4.21), (4.22). It is found that after taking integration by parts, the quadratic action (4.20) can be written as

\[
S^{(2)} = \int dyd^4x \left\{ A_{1\xi}(\partial^{(4)} \xi)^2 + A_{1\varphi}(\partial^{(4)} \varphi)^2 + A_{2\xi}(\partial_\xi \xi)^2 + A_{2\varphi}(\partial_\xi \varphi)^2
\right.
\]

\[
+ A_{1\xi\varphi}(\partial^{(4)} \xi \partial^{(4)} \varphi + A_{2\xi\varphi}(\partial_\xi \xi \partial_\xi \varphi)\right\},
\]

(4.23)

where

\[
A_{1\xi} = \frac{6\varphi_0^2(\beta_2 - \frac{3}{2})}{\beta_2}, \quad A_{1\varphi} = \frac{24(\beta_2 - \frac{3}{2})}{(z + 2)^2\beta_2}, \quad A_{1\xi\varphi} = \frac{24\varphi_0(\beta_2 - \frac{3}{2})}{(z + 2)\beta_2},
\]

\[
A_{2\xi} = \frac{3\varphi_0 \frac{2(z - 1)}{z + 2} (4\lambda - 1)\beta_1}{\lambda - 1}, \quad A_{2\varphi} = \frac{12\varphi_0 (4(\lambda - 1)(z + 2)^2 - \beta_1)}{(\lambda - 1)(z + 2)^2}, \quad A_{2\xi\varphi} = \frac{12\varphi_0 \frac{2(z - 1)}{z + 2} (4\lambda - 1)}{(\lambda - 1)(z + 2)}.
\]

(4.24)
It should be pointed out that the resulting quadratic action (4.23) does not include the fourth order derivative terms and also, the problems of the ghost/instability can be cured for

\[ 0 < \beta_2 < \frac{3}{2}, \quad \beta_1 \begin{cases} > 0 & (1/4 < \lambda < 1) \\ < 0 & (\lambda < 1/4 \text{ or } 1 < \lambda) \end{cases}, \]  

(4.25)

which corresponds to \( A_{1\xi,1\varphi} < 0, \ A_{2\xi,2\varphi} < 0 \), respectively. Also, it can be checked that for \( \lambda = 1 \) or \( \lambda = 1/4 \), substitution of (4.21) and (4.22) into (4.20) leads to

\[
S^{(2)} = \int dy d^4x \left\{ A_{1\xi}(\partial^{(4)}\xi)^2 + A_{1\varphi}(\partial^{(4)}\varphi)^2 + A_{1\xi\varphi}\partial^{(4)}\xi\partial^{(4)}\varphi \right\},
\]

(4.26)

which implies that there are no the ghost/instability problems for \( 0 < \beta_2 < 3/2 \).

Consequently, we remark that the above result (4.25) is in contrast to that in [7] without the Weyl scalar mode, where there remains ghost scalar mode in the spectrum: either from the quadratic kinetic term, or from the higher time derivative terms. It is to be noticed that the Weyl scalar mode \( \tilde{\varphi} \) in the quadratic action (4.20) plays an essential role of taming the scalar graviton \( \xi \) and fourth order derivative terms, as in eqs. (4.20)∼(4.22), thus curing the pathologies of other modes at least in the quadratic perturbations (4.23). It remains to be checked whether such nice behavior persists at the cubic order.\(^9\)

5 Conclusion and discussions

In this paper, we considered the higher dimensional anisotropic invariant gravity \((z-Weyl \ gravity)\) in which the four dimensional spacetime and extra dimensions are not treated on an equal footing. In order to implement the anisotropy, we considered higher dimensional ADM decomposition and impose the foliation preserving diffeomorphism where the foliation is adapted along the extra dimensions, thus keeping the general covariance for the four dimensional spacetime. The resulting gravity is extended to the conformally invariant case with an extra (Weyl) scalar field as usual, but with a new ingredient of a parameter \( z \) which describes the degree of anisotropy of conformal transformation between the spacetime and extra dimensional metrics. Effective four-dimensional gravity reduces to a scalar-tensor theory coupled with nonlinear sigma model. We have performed scalar perturbations in the quadratic action in 4+1 dimensions, and found a range of parameter space which does not lead to four time derivatives or ghost-like kinetic term of the scalar graviton. Extension of the analysis to the general 4 + \( D \) dimensional case remains to be done.

The anisotropic conformal invariance puts no restriction a priori on the value of the conformal weight \( z \), at least at the classical level and this may lead to some new aspects hidden in the anisotropic theory. For example, in \( 5D \), the isotropic potential yields a fractional power \( V \sim \phi^{5/3} \) (18). Such a potential renders a perturbative approach inaccessible, and could be plagued with possible discontinuities of the potential. On the other hand, in the anisotropic case, any even power is allowed. This has the potential to provide a single framework for the descriptions of both positive and negative powers of potentials.

In fact, the conformal weight is a dynamical exponent: one can easily check the global scaling symmetry of the action (3.2) under the following transformations

\[
x_\mu \rightarrow bx_\mu, \quad y_m \rightarrow b^z y_m, \quad N^\mu_m \rightarrow b^{1-z} N^\mu_m, \quad \phi \rightarrow b^{-v(z)} \phi.
\]

\(^{9}\)It was shown in [19, 20] that the 4D anisotropic action with scalar field does not show any pathological behaviors at the quadratic as well as cubic order in the perturbative action.
Horava-Lifshitz gravity exhibits dynamical scaling with $z = 3$ in the UV, flowing to the relativistic value $z = 1$ in the IR where Lorentz symmetry is restored. Since there is no symmetry that protects $z$, quantum corrections would modify the scaling laws, producing non-integer $z$'s in general. This leads to consider some cases where an exotic values of $z$ can be considered. In ref. [21], dynamical critical behavior of gravity in the extreme ultra-infrared sector and a mechanism to relax the cosmological constant was speculated and it is argued that $z \sim 20 - 30$ would produce the correct factor $\sim 10^{-120}$ to address the cosmological constant problem. In our case, perhaps the most interesting aspect arises when $z$ assumes a negative value close to $-2$. If we consider the cosmological constant given by (3.12) with $D = 1$, we have (with $V_0 \sim \mathcal{O}(1)$)

$$\Lambda \sim M_*^2 \phi_0^{-\frac{4}{1+z}}.$$  

(5.2)

Note that the detailed value of $\Lambda$ does not depend on reduction scheme (or brane world scenario) like the size of the extra dimension, but only on the scale $M_*$ and $z$. Then the essential point is that when $z$ is very close to $-2$, the $\phi_0$ part can produce a very small number. For example, with $\phi_0 \sim 10$ and $z = -2.04$ ($N = -49$ in Eq. (3.9)), this factor can produce a number like $\sim 10^{-100}$. More fine-tuning would be necessary in order to produce the correct factor for the cosmological constant problem, but such tuning can be compared with the conventional one which requires a fine-tuning of the order $10^{-120}$ for the cosmological constant.

The negative critical exponent is very rare in physical situations [22], but that does not necessarily mean that it is forbidden at a fundamental level. Such a negative value would enforce a bizarre scaling properties of extra dimensions, scaling like positive powers of mass dimension.\(^\text{10}\) It also gives unconventional uncertainty relation for a negative value of $z$. Interpreting the scaling law (5.1) in energy-momentum space, we obtain scaling relations like $E \sim p_x \sim (p_y)^{\frac{1}{z}}$. As the energy approaches UV region, the spatial $x$-momentum also increases, but the $y$-momentum along the extra dimensions decreases, and vice versa for a negative value of $z$. Another way of saying this is that if we probe a short distance behavior in $x$-space with high energy, we are at the same time looking at the long-distance properties in the extra dimensions. This type of behavior seems to be counter intuitive from the ordinary quantum mechanical point of view, but is reminiscent in spirit of the T-duality [27] in string theory which does not distinguish the large scale from the smaller one.

It is to be remarked that the idea of negative dynamical exponent for the extra dimension and approaching the cosmological constant problem based upon it remains an open possibility. One could perform a cosmological analysis and compare with observational data in order to actually test the value of $z$. This work is in progress and will be reported elsewhere.

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\(^{10}\)In the so-called space-time-matter theory (see ref. [23] and references therein), the extra dimension is treated as an explicit mass dimension with $y = \frac{G_N m}{c^2}$. It is also worth mentioning discrete extra dimension which is dynamically created at large distances in UV completing higher-dimensional field theories [24, 25] and can be utilized to construct multigravity theories free of ghosts [26].
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