Quantum Group Analysis of the Bound States in the Strong Coupling Regime of the Modified Sine-Gordon Model*

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Abstract

A quantum group analysis is applied to the Sine-Gordon model (or may be its version) in a strong-coupling regime. Infinitely many bound states are found together with the corresponding S-matrices. These new solutions of the Yang-Baxter equations are related to some reducible representations of the quantum \(sl(2)\) algebra resembling the Kac-Moody algebra representations in the Wess-Zumino-Witten-Novikov conformal field theory.

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0 Introduction.

A concept of the exact integrability played an outstanding role both in statistical mechanics and in the quantum field theory in two dimensions. Various aspects of it have become recently an object of a new close examination due to the observation by A. Zamolodchikov \cite{1} that certain deformations of conformal field theories may possess infinitely many non-trivial conserved charges giving rise to integrable field theories, whose $S$-matrices factorize and satisfy the Yang-Baxter equations (YBE).

Thus studying some deformations of the Virasoro algebra and the higher ones might lead to the essentially new understanding of the universal behavior in the vicinity of criticality. Knowledge of different near-critical regimes is essential for solving a problem of classification of the critical points being attacked since the advent of conformal invariance \cite{2}.

One of the first purely elastic scattering theory investigated in detail was a sine-Gordon model. It had received a lot of attention as a favorite toy of the theorists in the late seventies (see \cite{3} and references therein). Nevertheless, it is shown in the present paper that in a strong-coupling regime this model (or may be its version) exhibits a rich structure of the infinitely many bound states almost unexplored.\footnote{The existence of these bound states was mentioned in \cite{4}.}

The structure of these bound states resembles that of the Kac-Moody algebra representations in the Wess-Zumino-Novikov-Witten conformal field theory. Moreover, the similarity to the WZNW conformal field theory is by no means accidental since the non-local conserved currents of the sine-Gordon model derived in \cite{5, 6, 7} generate the $q$-deformed loop algebra which is isomorphic to the Kac-Moody one up to the central element.

However, the absence of central extension leads to the essential distinctions as far as the structure of the representations is concerned. The bound states discussed below are associated with the irreducible and some special reducible representations of the quantum $sl(2)$ algebra rather than with the integrable highest weight modules of the affine $A_1^{(1)}$ algebra. For some of these super-multiplets the new solutions of the YBE are found. It is shown that the number of these solutions is in fact infinite and finding the higher ones through the more elementary recursively one encounters more and more reducible representations.
Such reducible representations have been known in the purely elastic scattering theory already. For the rational $S$-matrices with the higher symmetries they were discussed by Karowsky [8] and in a series of works [9, 10, 11] dealing with the classification of the integrable vertex models in terms of Bethe ansatz. A general form of the Bethe ansatz entirely in terms of the representation theory of the simple Lie algebras has been conjectured in these papers. The limitations of the proposed formulas and the underlying algebraic structures remained somewhat obscure. I hope that the approach used in the present paper will help to elucidate the subject to some extent.

The technique used is a quantum group analysis introduced by Kulish and Reshetikhin just in the context of the higher spin representations of the sine-Gordon model [12] and developed in a series of works by Jimbo [13, 14, 15] who first realized the relation of the algebra considered to the affine algebra. The innovations I propose below make it possible in more or less regular way to produce new solutions of the Yang-Baxter equations recursively.

Despite the formal absence of the anomaly in the current algebra there is a strong evidence that it manifests itself indirectly since the requirements of unitarity and crossing permit only a finite number of the irreducible multiplets while the reducible ones may be regarded in a sense as their descendants. This conjecture of the dynamically generated anomaly is discussed in the end of the paper.

The paper is organized as follows. In the next section a brief review of the most important purely elastic scattering theory concepts is given in order to make the paper self-consistent and to fix the convenient notations for what it follows. The loop $sl(2)$ quantum algebra is derived from the YBE and discussed thoroughly in the sect. 3. Using these results the fusion procedure for the $S$-matrices and the massive fusion rules are considered in the sect. 4. The unsolved problems and the prospects for the future investigations are summarized in the last section. A cumbersome proof of the fusion theorem is presented in the Appendix.
1 Generalities of the purely elastic scattering theory.

Some general facts concerning the purely elastic scattering theory are collected in this section in order to fix notations mainly. First, a conventional parametrization for the 2D relativistic systems of the energy and momentum through the rapidity is used

\[ E = m \cosh \theta; \quad P = m \sinh \theta, \quad (1.1) \]

where \( m \) is the mass of the particle.

1.1 Bootstrap kinematics.

Consider a collision of two particles, say "a" and "b", with the masses \( m_a \) and \( m_b \) respectively, in the rest frame of their mass center. Suppose that a third particle "c" with the mass \( m_c \) is produced as an intermediate bound state in course of this scattering process. The requirement of the energy-momentum conservation leads to the following equations

\[ m_c = m_a \cosh \theta_a + m_b \cosh \theta_b; \quad 0 = m_a \sinh \theta_a + m_b \sinh \theta_b. \quad (1.2) \]

These constraints determine the rapidities of colliding particles \( \theta_a \) and \( \theta_b \) in terms of masses \( m_a, m_b \) and \( m_c \). It is more convenient, however, to parametrize the masses themselves in terms of rapidities.

\[ m_a = -m_0 \sinh \theta_b; \quad m_b = m_0 \sinh \theta_a; \quad m_c = m_0 \sinh (\theta_a - \theta_c). \quad (1.3) \]

Once the particle "c" is a bound state its mass should be less then the total mass of its parents. Hence,

\[ \frac{m_a + m_b}{m_c} = \frac{\cosh \frac{\theta_a - \theta_b}{2}}{\cosh \frac{\theta_a + \theta_b}{2}} > 1 \]

which becomes possible only when the rapidities \( \theta_a \) and \( \theta_b \) take the purely imaginary values. Put

\[ \theta_a = iU_{ac}; \quad \theta_b = -iU_{bc}. \quad (1.4) \]
Then
\[ m_a = m \sin U_{bc}^c; \quad m_b = m \sin U_{bc}^b; \quad m_c = m \sin U_{ab}^c, \] (1.5)

\[ U_{ab}^c = U_{bc}^b + U_{bc}^c. \] (1.6)

Here the bar over the indices means charge conjugation. This parametrization may be interpreted geometrically in a very transparent way. Three Euclidean two-vectors \( P_c = P_a + P_b \) form a triangle. Hence, for any of them the ratio of its length to the sine of the angle between two remaining ones is the same.

The intermediate virtual bound state reveals itself as a pole of a two-particle \( S \)-matrix

\[ S_{a,b}(\theta) = \frac{(f_{ab}^c)^2}{\theta - iU_{ab}^c}|c\rangle\langle c| + \cdots. \] (1.7)

Consider a three-particle amplitude

\[ S_{d,a,b}(\theta_d, \theta_a, \theta_b) = S_{a,b}(\theta_{da})S_{a,b}(\theta_{ab})S_{a,b}(\theta_{db}), \] (1.8)

where \( \theta_{da} = \theta_d - \theta_a \), etc. Examining its dependence on the argument \( \theta_{ab} \) one finds a pole at \( \theta_{ab} = iU_{ab}^c \). In the vicinity of this pole the amplitude is dominated by the scattering through the bound state \( c \)-channel. Therefore the residue of the above pole is considered as a scattering amplitude of \( d \) and \( c \) particles. However, the value of the rapidity \( \theta_c \) is not fixed by the above resonance condition and is determined via the kinematical relations (1.4)

\[ \theta_{ac} = -iU_{bc}^b; \quad \theta_{bc} = iU_{bc}^c. \] (1.9)

leading to the following celebrity bootstrap equation

\[ S_{d,c}(\theta) = P_{ab}^{c}S_{a,c}(\theta - iU_{ab}^b)S_{b,c}(\theta + iU_{bc}^c)P_{ab}^{c}. \] (1.10)

Here a projection operator \( P_{ab}^{c} \) on the state \( |c\rangle \) is introduced.

### 1.2 The integrals of motion and the mass spectrum.

The imaginary rapidities (or Euclidean angles) \( U_{bc}^a \) etc. extracted from the solution of the bootstrap equations (1.4) define not only the mass spectrum but also the spectra of the other integrals of motion [1, 3]. The existence of an infinite number of conservation laws is absolutely necessary
in order for the $S$-matrix to be factorizable. The commuting integrals of motion are usually supposed to be local, additive and Lorentz-covariant. Therefore the spin-$s$ integral of motion $I_s$ acts on some $N$-particle in-state $|A_1(\theta_1)A_2(\theta_2)\cdots A_N(\theta_N)\rangle$ just as if it acted on each of the one-particle states $|A_n(\theta_n)\rangle$ separately:

$$I_s|A_1(\theta_1)\cdots A_N(\theta_N)\rangle = \sum_{n=1}^{N} I_s^n \exp (s\theta_n)|A_1(\theta_1)\cdots A_N(\theta_N)\rangle \quad (1.11)$$

The exponential dependence of the one-particle eigenvalues on corresponding rapidities is dictated by the above mentioned Lorentz-covariance. The pre-exponential factors $I_s^n$ cannot be determined kinematically. However, they are related to each other in the same manner as the masses of the corresponding particles. Really, consider again the process of two-particle scattering through the bound state discussed in the beginning of the previous subsection. In conformity with the eq. (1.11) the value of the integral of motion $I_s$ in the initial state $|A_a(\theta_a)A_b(\theta_b)\rangle$ is given by

$$I_s^a \exp (s\theta_a) + I_s^b \exp (s\theta_b).$$

In the vicinity of the resonance (1.7) its value in the final state

$$S_{a,b}(\theta_{ab})|A_a(\theta_a)A_b(\theta_b)\rangle$$

is dominated by the pole contribution with the residue

$$I_s^a \exp (s\theta_a)$$

coming from the bound state channel. Hence, by virtue of the condition (1.9)

$$I_c^s = I_b^s \exp (-isU_{bc}^\gamma) + I_a^s \exp (isU_{bc}^\gamma) \quad (1.12)$$

This constraint can be resolved as follows:

$$I_a^s = I \sin (sU_{bc}^\gamma); \quad I_b^s = I \sin (sU_{ab}^\gamma); \quad I_c^s = I \sin (sU_{cb}^\gamma). \quad (1.13)$$

A parametrization of masses (1.5) is certainly a particular case of (1.13) for $s = \pm 1$. Since in general a given sort of particles, e.g. "$c$", may participate in several types of reactions either as a real particle or an intermediate bound
state its mass and any other integral of motion may be represented as a product over all such three-particle virtual processes:

\[ m_c = \prod_{a,b} m_a \sin U_{ab}^c, \quad I^s_c = \prod_{a,b} I_0 \sin(sU_{ab}^c) \]  \hfill (1.14)

The angles \( U_{ab}^c \) are usually rational multiples of \( \pi \) if a number of particles is finite in a theory. For this reason the integral of motion \( I_s \) with the spin \( s \) divisible by all denominators of these rational multipliers vanish identically for any multi-particle asymptotic state. This fact served as a foundation for a work [1] where a connection of a certain purely elastic scattering theory to the critical Ising model perturbed by a magnetic field was established.

1.3 The factorizability the unitarity and the crossing.

When the \( S \)-matrix cannot be diagonalized and thus cannot be reduced to some pure phase factors to a number of new restrictions arise.

Consider a scattering of two particles \( \alpha \) and \( \beta \). Let \( \alpha \) belongs to the multiplet \( a \) consisting of \( n_a \) particles with the equal masses \( m_a \) and \( \beta \) belongs to the multiplet \( b \) consisting of \( n_b \) particles having mass \( m_b \). The different particles \( \alpha \) and \( \beta \) are labeled by the isotopic quantum numbers including besides the common multiplet indices \( a \) and \( b \) also internal indices \( M_\alpha \) and \( M_\beta \) allowing to distinguish the particles inside each of the multiplets and referred below as the isospin projections. The latter may be changed after scattering. Therefore an \( S \)-matrix \( S_{[a,b]}(\theta_a - \theta_b) \) may be viewed as a mapping: \( V_a \otimes V_b \rightarrow V_a \otimes V_b \) where \( V_a \) and \( V_b \) are the \( n_a \)-dimensional and \( n_b \)-dimensional spaces of one-particle in-states.

A factorization hypothesis [3] for the multi-particle scattering in the completely integrable system implies that the net \( N \)-particle amplitudes do not depend on the order in which the intermediate two-particle processes take place. This assumption specified for a 3-particle scattering imposes stringent constraints on the admissible two-particle \( S \)-matrices known as the Yang-Baxter equations (YBE) [3, 16].

\[
S_{[a,b]}(\theta_a - \theta_b)S_{[a,c]}(\theta_a - \theta_c)S_{[b,c]}(\theta_b - \theta_c) =
S_{[b,c]}(\theta_b - \theta_c)S_{[a,c]}(\theta_a - \theta_c)S_{[a,b]}(\theta_a - \theta_b).
\]  \hfill (1.15)

The YBE combined with a standard initial condition

\[ S_{[a,b]}(\theta)|_{\theta=0} = P_{[a,b]}, \]  \hfill (1.16)
where $P_{[a,b]}$ is the permutation operator for two identical multiplets $a$ and $b$, lead to the following relations for their solutions

$$ S_{[b,a]}(-\theta)S_{[a,b]}(\theta) = f(\theta)\hat{I} \tag{1.17} $$

where $\hat{I}$ is an identity operator. This condition is almost equivalent to the unitarity provided the real analyticity

$$ S_{[b,a]}(-\theta^*) = S_{[a,b]}^\dagger(\theta) \tag{1.18} $$

is respected.

One more conventional dynamical principle in the purely elastic scattering theory is that of the crossing symmetry. We shall also use it in a slightly generalized form

$$ S_{[a,b]}(\theta) = U^{-1}S_{[\overline{b},a]}(i\pi - \theta)U \tag{1.19} $$

where a $b$-antiparticle is denoted by $\overline{b}$ and $U$ is the matrix of some unitary transformation.

## 2 Affine $U_q(sl(2))$ Symmetry of the YBE.

### 2.1 The Sine-Gordon S-matrix.

The simplest trigonometric solution of the YBE for the case when all three particles belong to the same doublet (i.e. all the spaces $V_a$, $V_b$ and $V_c$ are 2-dimensional) has been known since long ago [13].

$$ S_{[\pm\frac{1}{2},\pm\frac{1}{2}]}(\theta) = \left\{ Z(\theta) \left( \sinh \left( \nu \theta + i \frac{\gamma}{2} \left( 1 + \hat{\sigma}_a^3 \hat{\sigma}_b^3 \right) \right) + i \sin \gamma \left( e^{i\theta} \hat{\sigma}_a^- \hat{\sigma}_b^+ + e^{-i\theta} \hat{\sigma}_a^+ \hat{\sigma}_b^- \right) \right\}, \tag{2.1} $$

where both $\hat{\sigma}_a^\lambda, \hat{\sigma}_b^\lambda (\lambda = \pm, \mp)$ map $sl(2) \to \text{End}(V_a \otimes V_b)$:

$$ \hat{\sigma}_a = \hat{\sigma} \otimes \hat{I}; \quad \hat{\sigma}_b = \hat{I} \otimes \hat{\sigma} $$

and all the operators $\hat{\sigma}^3, \hat{\sigma}^\pm$ are just the Pauli matrices. The overall scalar factor $Z(\theta)$ doesn’t affect the YBE but is essential for the unitarity of the $S$-matrix and its appropriate behavior in the ultra-relativistic limit. In
particular it serves to eliminate the function \( f(\theta) \) in the eq. (1.17). Notice that the above \( S \)-matrix differs from the conventional one of the six-vertex model [10] but may be related to the latter via a simple transformation:

\[
S_{\left[\frac{i}{2}, \frac{3}{2}\right]}(\theta_{ab}) = \exp (-\nu(\theta_a \hat{\sigma}_a^3 + \theta_b \hat{\sigma}_b^3))S_{[a,b]}^{6V}(\theta_{ab}) \exp (\nu(\theta_a \hat{\sigma}_a^3 + \theta_b \hat{\sigma}_b^3)) \quad (2.2)
\]

which is compatible with the YBE. Both the original and the transformed \( S \)-matrices depend on the relative rapidity \( \theta_{ab} = \theta_a - \theta_b \) only, since they commute with the operator of the total isospin projection \( \hat{h}/2 = \hat{\sigma}_a^3 + \hat{\sigma}_b^3 \).

The parameter \( \nu \) may be chosen arbitrarily without violating both the YBE and the unitarity. However, it must be related to the parameter \( \gamma \) unambiguously once the crossing symmetry (1.19) is imposed. Namely,

\[
\nu = 1 - \gamma/\pi. \quad (2.3)
\]

The transformation matrix from the eq. (1.19) then reads

\[
U = \exp (i\nu\pi(\hat{\sigma}_a^3 - \hat{\sigma}_b^3)/2). \quad (2.4)
\]

Note that the factor \( Z(\theta) \) must satisfy two equations

\[
Z(\theta) = Z(i\pi - \theta); \quad Z(\theta)Z(-\theta) = (\sinh(i\gamma + \nu\theta)\sinh(i\gamma - \nu\theta))^{-1}. \quad (2.5)
\]

After the parameter \( \nu \) is chosen the \( S \)-matrix in question almost coincides with that of the sine-Gordon model [3]. There are two differences however. The first is due to the transformation (2.2) while the second is just the rescaling of the rapidities. One should replace the parameter \( \nu \) by \( 8\pi/\gamma_{SG} \) in order to relate the former to the latter exactly. The minimal solution of the eqs. (2.5) obtained in [3] reads

\[
Z(\theta) = \frac{i}{\pi}\Gamma(1 - \nu\alpha)\Gamma(1 - \nu(1 - \alpha)) \prod_{n=1}^{\infty} R_n(\alpha)R_n(1 - \alpha) \quad (2.6)
\]

where \( \alpha = \theta/i\pi \) and

\[
R_n(\alpha) = \frac{\Gamma(\nu(2n - \alpha))\Gamma(1 + \nu(2n - \alpha))}{\Gamma(\nu(2n + 1 - \alpha))\Gamma(1 + \nu(2n - 1 - \alpha))} \quad (2.7)
\]

\footnote{It should be real to preserve real analyticity.}
For $\nu > 1$ this function has the well known breather poles in the physical strip labeled by an integer $n$

$$\alpha = 1 - n/\nu; \quad \alpha = n/\nu \quad n \leq \nu. \quad (2.8)$$

The first series correspond to the zeros of $\sinh (i\gamma - \nu\theta) = i \sin \nu\pi (1 - \alpha)$ while the second one appears in the cross channel. When $\nu$ decreases and reaches the unity these poles leave the physical strip. However, a new pole corresponding to the zero of $\sinh (i\gamma + \nu\theta) = i \sin \nu\pi (1 + \alpha)$ emerges for $1/2 < \nu < 1$

$$\alpha = 1/\nu - 1, \quad \text{or} \quad \theta = i\gamma/\nu. \quad (2.9)$$

Due to the above inequality it is the only one in a physical strip. This important fact will be discussed in the sect. 3.11. Meanwhile we are going to describe the symmetry of the solution.

2.2 The symmetry.

The quantum group $U_q(sl(2))$ was discovered by Kulish and Reshetikhin [12] as the symmetry of the above solution and exploited to generalize it for the case of the higher spins. We recall briefly the derivation following [17].

The YBE may be represented in the following form

$$S_{[a,b]}(\theta_{ab})T_{[0,ab]}(\theta_{0a}, \theta_{0b}) = T_{[0,ba]}(\theta_{0b}, \theta_{0a})S_{[ab]}(\theta_{ab}). \quad (2.10)$$

where the monodromy-matrix

$$T_{[0,ab]}(\theta_{0a}, \theta_{0b}) = S_{[0,a]}(\theta_{0a})S_{[0,b]}(\theta_{0b}). \quad (2.11)$$

is just the product of two $S$-matrices. Let us consider the YBE concentrating on the $\theta_0$-dependence. Using the eq. (2.1) an explicit expression for the monodromy-matrix easily derives of which three groups of terms (those proportional to $e^{\pm 2\nu\theta_0}$ and independent on $\theta_0$ respectively) can be selected. Thus eq. (2.10) splits into three equations. Examining each of them separately and extracting the terms proportional to $\hat{\sigma}_0^+ \lambda \Delta_{ab} \hat{\sigma}^{\lambda}$ and to $\hat{\sigma}_0^3 \delta_{ab} \hat{\sigma}^{\lambda}$ respectively one can get the following linear equations for the $S$-matrix

$$S_{[a,b]}(\theta_a - \theta_b)\Delta_{ab}(\hat{\sigma}^{\lambda}) = \Delta_{ba}(\hat{\sigma}_0^\pm)S_{[a,b]}(\theta_a - \theta_b); \quad (2.12)$$

$$S_{[a,b]}(\theta_a - \theta_b)\delta_{ab}(\hat{\sigma}^{\lambda}) = \delta_{ba}(\hat{\sigma}_0^3)S_{[a,b]}(\theta_a - \theta_b). \quad (2.13)$$
where the operators attending the l.h.s of the eq. (2.12) are:
\[
\Delta_{ab}(\hat{\sigma}^\pm) = \Delta(\hat{\sigma}^\pm) = (\hat{\sigma}_a^\pm e^{-i\frac{\gamma}{2} \hat{\sigma}_b^3} + e^{i\frac{\gamma}{2} \hat{\sigma}_b^3} \hat{\sigma}_b^\pm);
\]
\[
\Delta_{ab}(\hat{\sigma}^3) = \Delta(\hat{\sigma}^3) = (\hat{\sigma}_a^3 + \hat{\sigma}_b^3).
\] (2.14)

The operators in the r.h.s of the eq. (2.12) differ from those in the l.h.s by the permutation of \(a\) and \(b\):
\[
\Delta_{ba}(\hat{\sigma}^\pm) = \Delta(\hat{\sigma}^\pm) = (\hat{\sigma}_a^\pm e^{-i\frac{\gamma}{2} \hat{\sigma}_b^3} + e^{i\frac{\gamma}{2} \hat{\sigma}_b^3} \hat{\sigma}_b^\pm);
\]
\[
\Delta_{ba}(\hat{\sigma}^3) = \Delta(\hat{\sigma}^3) = (\hat{\sigma}_a^3 + \hat{\sigma}_b^3)
\] (2.15)

The operators involved in the eq. (2.13) are similar to the above ones but depend on the rapidities as follows
\[
\delta_{ab}(\hat{\sigma}^\pm) = \delta(\hat{\sigma}^\pm) = (e^{\mp 2\nu \theta_a} \hat{\sigma}_a^\pm e^{-i\frac{\gamma}{2} \hat{\sigma}_b^3} + e^{\mp 2\nu \theta_b} e^{i\frac{\gamma}{2} \hat{\sigma}_b^3} \hat{\sigma}_b^\pm)
\]
\[
\delta_{ab}(\hat{\sigma}^3) = \delta(\hat{\sigma}^3) = (\hat{\sigma}_a^3 + \hat{\sigma}_b^3).
\] (2.16)

The expressions in the l.h.s. (2.12) are related to the ones in the r.h.s. by the permutation as in the eq. (2.13)
\[
\delta_{ba}(\hat{\sigma}^\pm) = \delta(\hat{\sigma}^\pm) = (e^{\mp 2\nu \theta_a} \hat{\sigma}_a^\pm e^{-i\frac{\gamma}{2} \hat{\sigma}_b^3} + e^{\mp 2\nu \theta_b} e^{i\frac{\gamma}{2} \hat{\sigma}_b^3} \hat{\sigma}_b^\pm)
\]
\[
\delta_{ab}(\hat{\sigma}^3) = \delta(\hat{\sigma}^3) = (\hat{\sigma}_a^3 + \hat{\sigma}_b^3)
\] (2.17)

This system turns out to be equivalent to the initial YBE (1.15).

It can be easily checked that each of the four triples of the operators \(\Delta_{ab}(\hat{\sigma}^\lambda), \Delta_{ba}(\hat{\sigma}^\lambda)\) and \(\delta_{ab}(\hat{\sigma}^\lambda), \delta_{ba}(\hat{\sigma}^\lambda)\) provide four different (reducible) matrix representations of the three elements \(\hat{e}, \hat{f}, \hat{h}\) enjoying the following commutation relations
\[
[\hat{e}, \hat{f}] = \frac{\sin(\gamma \hat{h})}{\sin \gamma}; \quad [\hat{h}, \hat{e}] = 2\hat{e}; \quad [\hat{h}, \hat{f}] = -2\hat{f}
\] (2.18)

and generating an associative algebra called quantum universal enveloping algebra of sl(2) and denoted \(U_q(sl(2))\) \((q = e^{i\gamma})\). Below a shorthand notation \(U_q\) will be used. When \(\gamma \to 0\) the quantum universal enveloping algebra degenerates into an ordinary enveloping algebra of \(sl(2)\) denoted \(U(sl(2))\) or just \(U\) for brevity.

Note that the Pauli matrices with the usual commutation relations
\[
[\hat{\sigma}^+, \hat{\sigma}^-] = \hat{\sigma}^3; \quad [\hat{\sigma}^3, \hat{\sigma}^\pm] = \pm \hat{\sigma}^\pm
\] (2.19)
yield the simplest although trivial realization of \((2.18)\) since \(\sin(\gamma \hat{\sigma}^3) = \hat{\sigma}^3 \sin \gamma\).

Moreover, the six generators \(\Delta_{ab}(\hat{\sigma}^\lambda), \Delta_{ba}(\hat{\sigma}^\lambda)\) or \(\delta_{ab}(\hat{\sigma}^\lambda), \delta_{ba}(\hat{\sigma}^\lambda)\) combined provide a realization of the \(q\)-deformed \(A_1^{(1)}\) loop algebra referred as \(\mathcal{U}_q^L\) below. The latter is generated by six elements \(\hat{e}_1, \hat{f}_1, \hat{h}_1, \hat{e}_0, \hat{f}_0, \hat{h}_0\) obeying the following commutation relations

\[
[e_i, f_j] = \delta_{ij} \frac{\sin(\gamma \hat{h}_i)}{\sin \gamma}; \quad [h_i, e_j] = 2(-)^{i-j} e_j; \quad [h_i, f_j] = -2(-)^{i-j} f_j \quad (2.20)
\]

just stating that the loop algebra includes two \(q\)-deformed \(A_1\) subalgebras. These trivial relations are supplemented by Serre relations

\[
\hat{e}_i^3 \hat{e}_j - \hat{e}_j \hat{e}_i^3 - (1 + 2 \cos \gamma) \left( \hat{e}_i^2 \hat{e}_j \hat{e}_i - \hat{e}_i \hat{e}_j \hat{e}_i^2 \right) = 0; \quad
\hat{f}_i^3 \hat{f}_j - \hat{f}_j \hat{f}_i^3 - (1 + 2 \cos \gamma) \left( \hat{f}_i^2 \hat{f}_j \hat{f}_i - \hat{f}_i \hat{f}_j \hat{f}_i^2 \right) = 0; \quad (2.21)
\]

for \(i \neq j\)

The correspondence between the generators \(\Delta(\hat{\sigma}^\lambda), \delta(\hat{\sigma}^\lambda)\) is indicated in Table 1.

|    | \(\Delta(\hat{\sigma}^+)\) | \(\Delta(\hat{\sigma}^-)\) | \(\Delta(\hat{\sigma}^\lambda)\) | \(\delta(\hat{\sigma}^+)\) | \(\delta(\hat{\sigma}^-)\) | \(-\delta(\hat{\sigma}^\lambda)\) |
|----|--------------------------|--------------------------|--------------------------|--------------------------|--------------------------|--------------------------|
| \(\hat{e}_1\) | \(\hat{f}_1\) | \(\hat{h}_1\) | \(\hat{e}_0\) | \(\hat{f}_0\) | \(\hat{h}_0\) |

The above quantum algebra \(\mathcal{U}_q\) is an one-parametric deformation of a genuine \(sl(2)\) enveloping algebra. From this viewpoint the mapping \(\Delta : \mathcal{U}_q \to \text{End}(V_a \otimes V_b)\) called \textit{co-product} and defined via \((2.14)\) generalizes an action of the \(sl(2)\) for the case of two fundamental representations while eq. \((2.15)\) yields the analogous mapping \(\Delta\) for \(V_b \otimes V_a\). The same mappings may be defined for the direct product of any representations (even not necessarily irreducible) and are in fact the algebra homomorphisms \(\mathcal{U}_q \to \mathcal{U}_q \otimes \mathcal{U}_q\):

\[
\Delta_{12}(\hat{h}) \equiv \Delta(\hat{h}) = \hat{h} \otimes \hat{I} + \hat{I} \otimes \hat{h}; \quad (2.22)
\]

\[
\Delta_{12}(\hat{e}) \equiv \Delta(\hat{e}) = \hat{e} \otimes e^{-i \frac{\gamma}{2} \hat{h}} + e^{i \frac{\gamma}{2} \hat{h}} \otimes \hat{e}; \quad (2.23)
\]

\[
\Delta_{12}(\hat{f}) \equiv \Delta(\hat{f}) = \hat{f} \otimes e^{-i \frac{\gamma}{2} \hat{h}} + e^{i \frac{\gamma}{2} \hat{h}} \otimes \hat{f}; \quad (2.24)
\]
\[ \Delta_2(\hat{h}) \equiv \bar{\Delta}(\hat{h}) = \hat{h} \otimes \hat{I} + \hat{I} \otimes \hat{h}; \quad (2.25) \]
\[ \Delta_1(\hat{e}) \equiv \bar{\Delta}(\hat{e}) = \hat{e} \otimes e^{-i\frac{2\theta}{h}} + e^{i\frac{2\theta}{h}} \otimes \hat{e}; \quad (2.26) \]
\[ \Delta_1(\hat{f}) \equiv \bar{\Delta}(\hat{f}) = \hat{f} \otimes e^{-i\frac{2\theta}{h}} + e^{i\frac{2\theta}{h}} \otimes \hat{f}; \quad (2.27) \]

This homomorphism may be extended to the whole \( q \)-deformed loop algebra as follows

\[ \Delta_1(\hat{h}_i) \equiv \bar{\Delta}(\hat{h}_i) = \hat{h}_i \otimes \hat{I} + \hat{I} \otimes \hat{h}_i; \quad (2.28) \]
\[ \Delta_1(\hat{e}_i) \equiv \bar{\Delta}(\hat{e}_i) = \hat{e}_i \otimes e^{-i\frac{2\theta}{h_i}} + e^{i\frac{2\theta}{h_i}} \otimes \hat{e}_i; \quad (2.29) \]
\[ \Delta_1(\hat{f}_i) \equiv \bar{\Delta}(\hat{f}_i) = \hat{f}_i \otimes e^{-i\frac{2\theta}{h_i}} + e^{i\frac{2\theta}{h_i}} \otimes \hat{f}_i; \quad (2.30) \]
\[ \Delta_1(\hat{h}_i) \equiv \bar{\Delta}(\hat{h}_i) = \hat{h}_i \otimes \hat{I} + \hat{I} \otimes \hat{h}_i; \quad (2.31) \]
\[ \Delta_1(\hat{e}_i) \equiv \bar{\Delta}(\hat{e}_i) = \hat{e}_i \otimes e^{-i\frac{2\theta}{h_i}} + e^{i\frac{2\theta}{h_i}} \otimes \hat{e}_i; \quad (2.32) \]
\[ \Delta_1(\hat{f}_i) \equiv \bar{\Delta}(\hat{f}_i) = \hat{f}_i \otimes e^{-i\frac{2\theta}{h_i}} + e^{i\frac{2\theta}{h_i}} \otimes \hat{f}_i; \quad (2.33) \]

The above co-multiplication combined with the following realization of the \( U_q^L \) in terms of the Laurent polynomials in variable \( z = e^{2\nu\theta} \) over the underlying algebra \( U_q \)

| \( \hat{e} \) | \( \hat{f} \) | \( \hat{h} \) | \( z^{-1}\hat{e} \) | \( zf \) | \(-\hat{h}\) |
|---|---|---|---|---|---|
| \( \hat{e}_1 \) | \( \hat{f}_1 \) | \( \hat{h}_1 \) | \( \hat{e}_0 \) | \( \hat{f}_0 \) | \( \hat{h}_0 \) |

may be used to generalize the operators entering the eq. \( (2.13) \)

\[ \delta_{12}(\hat{e}) = \delta(\hat{e}) = e^{-2\nu\theta_1}\hat{e} \otimes e^{i\frac{2\theta}{h}} + e^{-2\nu\theta_2}e^{-i\frac{2\theta}{h}} \otimes \hat{e}; \quad (2.34) \]
\[ \delta_{12}(\hat{f}) = \delta(\hat{f}) = e^{2\nu\theta_1}\hat{f} \otimes e^{i\frac{2\theta}{h}} + e^{2\nu\theta_2}e^{-i\frac{2\theta}{h}} \otimes \hat{f}; \quad (2.35) \]
\[ \delta_{21}(\hat{e}) = \delta(\hat{e}) = e^{-2\nu\theta_1}\hat{e} \otimes e^{-i\frac{2\theta}{h}} + e^{-2\nu\theta_2}e^{i\frac{2\theta}{h}} \otimes \hat{e}; \quad (2.36) \]
\[ \delta_{21}(\hat{f}) = \delta(\hat{f}) = e^{2\nu\theta_1}\hat{f} \otimes e^{-i\frac{2\theta}{h}} + e^{2\nu\theta_2}e^{i\frac{2\theta}{h}} \otimes \hat{f}; \quad (2.37) \]

Kulish and Reshetikhin \cite{12} conjectured that the \( S \)-matrix \( S_{[a,b]}(\theta_a - \theta_b) \) solving the eqs. \( (2.12,2.13) \) in \( V_a \otimes V_b \) where \( V_a \) and \( V_b \) are two arbitrary irreducible modules of \( U_q \) and Pauli matrices should be substituted by the generators \( \hat{e}, \hat{f}, \hat{h} \) provide the solution of the YBE as well. This conjecture has been checked by Jimbo \cite{13} who proved the following
Theorem 1 (Jimbo 1986) Let \( \rho_a : \mathcal{U} \to \text{End}(V_a) \); \((a = 1, 2, 3)\) be three arbitrary finite-dimensional irreducible representations of \( \text{sl}(2) \). Assume that there exist three representations \( \hat{\rho}_a, \hat{\rho}_b \) and \( \hat{\rho}_c \) of \( \mathcal{U}_q \) such that \( \hat{\rho}_a \to \rho_a \) as \( \gamma \to 0 \).

Then the linear equations

\[
S_{[a,b]}(\theta_a - \theta_b)\Delta(\hat{e}) = \Delta(\hat{e})S_{[a,b]}(\theta_a - \theta_b); \quad (2.38)
\]

\[
S_{[a,b]}(\theta_a - \theta_b)\delta(\hat{f}) = \delta(\hat{f})S_{[a,b]}(\theta_a - \theta_b) \quad (2.39)
\]

in \( \text{End}(V_a \otimes V_b) \) has at most one solution for the general value of \( \gamma \).

If eqs. (2.38, 2.39) admit a nontrivial solution \( S_{[a,b]}(\theta_a - \theta_b) \) it also satisfy the equations

\[
S_{[a,b]}(\theta_a - \theta_b)\Delta(\hat{f}) = \Delta(\hat{f})S_{[a,b]}(\theta_a - \theta_b); \quad (2.40)
\]

\[
S_{[a,b]}(\theta_a - \theta_b)\delta(\hat{h}) = \delta(\hat{h})S_{[a,b]}(\theta_a - \theta_b); \quad (2.41)
\]

\[
S_{[a,b]}(\theta_a - \theta_b)\delta(\hat{e}) = \delta(\hat{e})S_{[a,b]}(\theta_a - \theta_b) \quad (2.42)
\]

Three solutions \( S_{[a,b]}(\theta_a - \theta_b) \) for \((a, b) = (1, 2), (1, 3), (2, 3)\) satisfy the YBE in \( \text{End}(V_1 \otimes V_2 \otimes V_3) \).

Remark 1 An explicit solution for the above linear equations has been found in [12] when one of the isospins either \( J_a \) or \( J_b \) is equal to \( \frac{1}{2} \) (see the next subsection for a precise definition of the isospin quantum numbers).

\[
S^{(J_a, J_b)}_{[a,b]}(\theta) = Z(\theta) \times \left\{ \sinh \left( \nu \theta + i\frac{\gamma}{2}(1 + \hat{h} \otimes \hat{\sigma}^3) \right) + i \sin \gamma \left( e^{i\theta} \hat{e} \otimes \hat{\sigma}^- + e^{-i\theta} \hat{f} \otimes \hat{\sigma}^+ \right) \right\} \quad (2.43)
\]

An examination of it shows that the above theorem is equivalent to the following statement.

Let the isospins \( J_a \) and \( J_b \) be arbitrary and \( J_0 = \frac{1}{2} \). Then there is the only \( S \)-matrix \( S^{(J_a, J_b)}_{[a,b]}(\theta) \) satisfying the YBE (1.12). Once three such solutions \( S^{(J_a, J_b)}_{[a,b]}(\theta), S^{(J_a, J_c)}_{[a,c]}(\theta) \) and \( S^{(J_b, J_c)}_{[b,c]}(\theta) \) for arbitrary \( J_a, J_b \) and \( J_c \) do exist they satisfy the YBE (1.12).
By means of this theorem an $S$-matrix $S_{[a,b]}(\theta)$ for two arbitrary irreducible representations $V_a$ and $V_b$ may be constructed. This is a well known fusion process [8, 18] when the $S$-matrices (2.1) are used as the elementary building blocks for the $S$-matrix of the higher representations. We shall analyse this procedure below considering it as a sort of bootstrap equations (1.10) and emphasizing the importance of the quantum group invariance (2.38, 2.39). A few basic facts concerning the finite-dimensional irreducible representations will be important.

2.3 Description of the irreducible representations.

The structure of the finite-dimensional irreducible representations of $\mathcal{U}_q$ for the generic values of $\gamma$ mainly repeats that of $\mathcal{U}$. They are labeled by the highest weight $J$ (the maximal eigenvalue of $\hat{h}$) and will be denoted as $\{J\}$ below. The states inside each $\mathcal{U}_q$-module are labeled by the weight $M$ (the eigenvalue of $\hat{h}$). The action of the algebra $\mathcal{U}_q$ on the states is defined by the following formulas

\[ \hat{h} |J, M\rangle = M |J, M\rangle; \]
\[ \hat{e} |J, M\rangle = d_M^J |J, M + 1\rangle; \quad \hat{f} |J, M\rangle = d_{M-1}^J |J, M - 1\rangle, \]

where

\[ d_M^J = \sqrt{[J + M + 1]_q[J - M]_q}; \quad [n]_q = \frac{\sin \gamma n}{\sin \gamma}. \]

Note that $\hat{e} |J, J\rangle = 0$, $\hat{f} |J, -J\rangle = 0$. The normalization of the above basis is chosen in such a way that the operators $\hat{e}$ and $\hat{f}$ are transposed to each other. There is a Casimir operator

\[ \hat{K} = \hat{f} \hat{e} + \left[ \hat{h} \right]_q \left[ \frac{\hat{h}}{2} + 1 \right]_q \]

commuting with all generators $\hat{e}, \hat{f}, \hat{h}$ by virtue of the commutation relations (2.18) and hence proportional to the identity operator when restricted to an irreducible representation:

\[ \hat{K} |J, M\rangle = \left[ J \right]_q \left[ J + 1 \right]_q |J, M\rangle \]
Remark 2 For $J > \pi/\gamma$ the eigenvalues of Casimir may be negative indicating that the states with a negative norm appear. The multiplets with the higher isospin are thus forbidden in a physical sensible theory. However, for sufficiently large $J$ the Casimir eigenvalues may become positive again. The permitted $J$, though equidistant, may be separated by the intervals larger then unity and this new group contains a finite number of the spectral points. For generic $\gamma$ the permitted spectrum consists of the infinite number of such groups known as the Takahashi zones [4, 19, 20]. Moreover, this structure is supplemented by another one including the so called odd states. All these subtleties will ignored in the present paper and the consideration will be restricted by the first Takahashi zone.

The Clebsch-Gordan (C-G) decomposition may be performed in two different ways.

1. With respect to co-product $\Delta$:

$$|J_a, M_a\rangle \otimes |J_b, M_b\rangle = \sum_{J=|J_a - J_b|}^{J_a + J_b} C_{M_a + M_b, M_a, M_b}^J |J, M_a + M_b\rangle$$  \hspace{1cm} (2.49)

The highest weight vector $|J, J\rangle$ in any irreducible representation $\{J\}$ is annihilated by $\Delta(\hat{e})$ while all other states $|J, M\rangle$ are created by an operator $\left(\Delta(\hat{f})\right)^{J-M}$ acting on the highest weight vector.

2. With respect to co-product $\overline{\Delta}$:

$$|J_a, M_a\rangle \otimes |J_b, M_b\rangle = \sum_{J=|J_a - J_b|}^{J_a + J_b} \overline{C}_{M_a + M_b, M_a, M_b}^J |J, M_a + M_b\rangle$$  \hspace{1cm} (2.50)

The structure of the C-G basis formed by the states $|J, M\rangle$ in this case is basically the same.

We are going to discuss some representations of the whole $q$-deformed loop algebra in what it follows. Since there is no central extension the highest weight integrable representations do not exist. So, just the formal Laurent polynomials over the irreducible representations will be considered. They are certainly in one to one correspondence with the usual finite dimensional irreducible representations of the ordinary $q$-deformed $sl(2)$ algebra. However, as shown in the next section, their C-G decomposition give rise to various representations reducible with respect to the subalgebra $\mathcal{U}_q$. 

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3 Massive fusion rules.

3.1 The irreducible bound states.

Now suppose that two solutions of the YBE $S^{(J_0, J_1)}_{[0,1]}(\theta_{01})$ and $S^{(J_0, J_2)}_{[0,2]}(\theta_{02})$ are known (here the upper indices denote the isospins of the irreducible representations 0,1,2). Let us show how the $S$-matrix $S^{(J_0, J_1+J_2)}(\theta)$ may be obtained. Since both $S$-matrices must satisfy eqs. (2.38,2.39) the monodromy-matrix (2.11) also enjoys the following intertwining properties:

$$T_{[0,12]}(\theta_{01}, \theta_{02}) \Delta_{012}(\hat{e}) = \Delta_{120}(\hat{e}) T_{[0,12]}(\theta_{01}, \theta_{02})$$ (3.1)
$$T_{[0,12]}(\theta_{01}, \theta_{02}) \delta_{012}(\hat{f}) = \delta_{120}(\hat{f}) T_{[0,12]}(\theta_{01}, \theta_{02})$$ (3.2)

where

$$\Delta_{012}(\hat{e}) = \left( \hat{e} \otimes \exp \left( -\frac{i}{2} \Delta_{12}(\hat{h}) \right) + \exp \left( \frac{i}{2} \hat{h} \right) \otimes \Delta_{12}(\hat{e}) \right)$$ (3.3)
$$\Delta_{120}(\hat{e}) = \left( \hat{e} \otimes \exp \left( \frac{i}{2} \Delta_{12}(\hat{h}) \right) + \exp \left( -\frac{i}{2} \hat{h} \right) \otimes \Delta_{12}(\hat{e}) \right)$$ (3.4)

$$\delta_{012}(\hat{f}) = \left( e^{2i\nu \theta_{01}} \hat{f} \otimes \exp \left( \frac{i}{2} \Delta_{12}(\hat{h}) \right) + e^{-i\hat{h}} \otimes \delta_{12}(\hat{f}) \right)$$ (3.5)
$$\delta_{120}(\hat{f}) = \left( e^{2i\nu \theta_{02}} \hat{f} \otimes \exp \left( -\frac{i}{2} \Delta_{12}(\hat{h}) \right) + e^{i\hat{h}} \otimes \delta_{12}(\hat{f}) \right)$$ (3.6)

The monodromy-matrix (2.11) acting in $\{J_0\} \otimes \{J_1\} \otimes \{J_2\}$ performs a cyclic permutation in the co-product $\Delta_{012}(\hat{e})$ i.e. converts it into $\Delta_{120}(\hat{e})$. Since the co-product $\Delta_{12}(\hat{e})$ is left intact by this operation it may be transformed into the block-diagonal form by means of the C-G decomposition in $\{J_1\} \otimes \{J_2\}$. Multiplying both sides of the eq. (3.1) by the projectional operator $(1 \otimes P_{J_1,J_2})$ one can verify that the operator

$$(1 \otimes P_{J_1,J_2}) T_{[0,12]}(\theta_{01}, \theta_{02})(1 \otimes P_{J_1,J_2})$$

enjoys the intertwining property (2.38) in $\{J_0\} \otimes \{J\}$ because the co-product operator $\Delta_{12}(\hat{e})$ commutes with the projector. This is not the case for the operator $\delta_{12}(\hat{f})$ when the values of the rapidities $\theta_1, \theta_2$ take generic values. These values may be adjusted, however, in such a way that the restriction of the operator $\delta(\hat{f})$ to the $J_1 + J_2$ module will coincide with that of $\Delta(\hat{f})$. 

Proposition 1 Put $\nu \theta_1 = -i \gamma J_2$, $\nu \theta_2 = i \gamma J_1$. Then

$$P^{J_1 + J_2} \delta_{12}(\hat{f}) P^{J_1 + J_2} = P^{J_1 + J_2} \delta_{12}(\hat{f}) P^{J_1 + J_2}. \quad (3.7)$$

Proof: First note that

$$[\delta_{12}(\hat{f}), \Delta_{12}(\hat{f})] = 0. \quad (3.8)$$

Hence,

$$\delta_{12}(\hat{f}) |J_1 + J_2, M\rangle = \Delta_{12}(\hat{f}) |J_1 + J_2, M\rangle \quad (3.9)$$

if and only if

$$\delta_{12}(\hat{f}) |J_1 + J_2, J_1 + J_2\rangle = \Delta_{12}(\hat{f}) |J_1 + J_2, J_1 + J_2\rangle. \quad (3.10)$$

The latter equality may be easily checked with the help of the eqs. (2.24) and (2.35) accounting for the fact that the highest weight vector in the representation considered is just the direct product of the two highest weight vectors:

$$|J_1 + J_2, J_1 + J_2\rangle = |J_1, J_1\rangle \otimes |J_1, J_1\rangle.$$

Corollary 1 The following bootstrap relation holds:

$$P^{J_1 + J_2} S^{(J_0, J_1)}_{[0,1]} (\theta - \frac{i \gamma}{\nu} J_2) S^{(J_0, J_2)}_{[0,2]} (\theta + \frac{i \gamma}{\nu} J_1) P^{J_1 + J_2} = S^{(J_0, J_1 + J_2)}_{[0,12]} (\theta) \quad (3.11)$$

In terms of parametrization (1.3) the transition $\{J_1\} \otimes \{J_2\} \rightarrow \{J_1 + J_2\}$ is characterized by the angles:

$$\mathcal{U}^{J_1} = \frac{\gamma}{\nu} J_1; \quad \mathcal{U}^{J_2} = \frac{\gamma}{\nu} J_2; \quad \mathcal{U}^{J_1 + J_2} = \frac{\gamma}{\nu} (J_1 + J_2). \quad (3.12)$$

Now a derivation of the mass spectrum for the particles with the arbitrary isospin is straightforward:

$$m_J = m_* \sin \left(\frac{\gamma}{\nu} J\right) \quad (3.13)$$

where $m_*$ is a common mass scale.
3.2 The analytic structure of the $S$-matrix for two irreducible multiplets.

The multiplets above do not exhaust all the bound states in the theory. The most direct way to see this is to examine an analytic structure of the $S$-matrix $S_{1,2}^{(J_1,J_2)}(\theta)$. We shall use a modified version of analysis proposed by Jimbo [15] for two identical representations ($J_1 = J_2$) and going back to the work [18].

**Proposition 2**  
Let $\{J_1\}$ and $\{J_2\}$ be two irreducible representations. Then, the solution of the eqs. (2.38, 2.39) reads

$$S_{1,2}(\theta) = \sum_{J=|J_1-J_2|}^{J_1+J_2} \Lambda_J(\theta) |J_2,J_1\rangle P^J \langle J_1,J_2| \quad (3.14)$$

where the operator

$$|J_2,J_1\rangle P^J \langle J_1,J_2| = \sum_{M=-J}^{J} |J,M\rangle \langle J,M| \quad (3.15)$$

Bra-vectors and ket-vectors belong to the C-G decompositions (2.49) and (2.50) respectively. The "eigenvalues" $\Lambda_J(\theta)$ are related to each other recursively:

$$\frac{\Lambda_J(\theta)}{\Lambda_{J-1}(\theta)} = \frac{\sinh (\nu\theta + i\gamma J)}{\sinh (\nu\theta - i\gamma J)} \quad (3.16)$$

**Proof:** Due to eq. (2.38, 2.40, 2.41) the $S$-matrix enjoys the interwining property with respect to the Casimir operators

$$S_{1,2}(\theta) \Delta(\hat{K}) = \Delta(\hat{K}) S_{1,2}(\theta) \quad (3.17)$$

This property combined with the analogous one for the operator $\hat{h}$ (2.41) provides the selection rules for the $S$-matrix elements $\langle J,M|S_{1,2}(\theta)|J',M'\rangle$ which vanish if $J \neq J'$, $M \neq M'$.

The "eigenvalues" $\Lambda(\theta)$ may be extracted from the other interwining equation (2.39) which is convenient to rewrite as follows

$$\Lambda_J(\theta) \langle J, M - 1 | \delta(\hat{f}) | J', M \rangle = \langle J, M - 1 | \delta(\hat{f}) | J', M \rangle \Lambda_{J'}(\theta) \quad (3.18)$$
Though the number of the equations might seem to exceed the number of eigenvalues, this is not the case because the most of the matrix elements of $\delta(\hat{f})$ just vanish and most of the non-vanishing are related to each other. Since the operator $\delta(\hat{f})$ commutes with $\Delta(\hat{f})$ (see the eq. (3.8)) it always lowers the isospin projection quantum number by one and may change the isospin $J$ at most by one. Indeed, considering the matrix elements of the product $\Delta(\hat{f})$ and $\delta(\hat{f})$ one can derive the following relations

$$d_{M}^{J}(J, M-1|\delta(\hat{f})|J', M) = \langle J, M|\delta(\hat{f})|J', M + 1\rangle d_{M-1}^{J}$$

where the matrix elements $d_{M}^{J}$ are given by the eq. (2.46). Therefore any of the matrix elements may be expressed either through the highest one $\langle J, J' - 1|\delta(\hat{f})|J, J' \rangle$ or through the lowest one $\langle J, -J|\delta(\hat{f})|J', -J \rangle$. On the other hand the highest matrix element does vanish for all $J < J' - 1$ while the lowest one does vanish also for $J > J' - 1$. Note that the relations between the matrix elements of the operator $\delta(\hat{f})$ have exactly the same form as (3.19). Thus the only independent equations are

$$\Lambda_{J}(\theta) = \langle J + 1, \mp J - 1|\delta(\hat{f})|J, \mp J \rangle = \frac{\langle J + 1, \mp J - 1|\delta(\hat{f})|J, \mp J \rangle}{\langle J, J - 1|\delta(\hat{f})|J, J \rangle} \Lambda_{J}(\theta)$$

Examining the $\theta$-dependence of the above matrix elements one can write them as

$$\langle J + 1, \mp J - 1|\delta(\hat{f})|J, \mp J \rangle = f_{J}^{\mp}(\gamma) \exp(\nu(\theta_{1} + \theta_{2})) \sinh(\nu\theta_{12} - \phi_{J}^{\mp}(\gamma))$$

where $f_{J}^{\mp}(\gamma)$ and $\phi_{J}^{\mp}(\gamma)$ are some complex functions of $\gamma$ which may be expressed through certain matrix elements. Surprisingly enough $\phi_{J}^{\mp}(\gamma)$ can be found explicitly. It has been shown in [15] that

$$\langle J + 1, \mp J - 1|\delta(\hat{f})|J, \mp J \rangle|_{\nu\theta_{12} = \mp i\gamma J} = 0$$

and consequently $\phi_{J}^{\mp}(\gamma) = \mp i\gamma J$. Plugging these matrix elements into the eq. (3.20) and accounting for the fact that the substitution of the operator $\delta(\hat{f})$ by $\delta(\hat{f})$ results just in the reversal of sign of $\gamma$ one can observe that the relation (3.16) holds up to the $\theta$ independent factor. The latter is fixed by the unitarity (1.17) and the initial condition (1.16). Really, as it follows from the above arguments the eigenvalues may be represented in the following form

$$\Lambda_{J}(\theta) = Z(\theta, \gamma) \zeta_{J}(\gamma) \prod_{L = J}^{J_{1} + J_{2}} \sinh(\nu\theta + i\gamma L) \prod_{L = |J_{1} - J_{2}| + 1}^{J - 1} \sinh(\nu\theta - i\gamma L)$$

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where the common factor $Z(\theta, \gamma)$ and the coefficients $\zeta_J(\gamma)$ are subject to the following constraints

\[
Z(\theta, \gamma)Z(-\theta, \gamma) (\zeta_J(\gamma))^2 = \prod_{L=\lvert J_1-J_2\rvert+1}^{J_1+J_2} \text{cosech}(\nu\theta + i\gamma L)\text{cosech}(\nu\theta - i\gamma L);
\]

\[
Z(\theta = 0, \gamma)\zeta_J(\gamma) = (-1)^{J_1+J_2-J}.
\]

These constraints can be easily splitted giving

\[
Z(\theta, \gamma)Z(-\theta, \gamma) = \prod_{L=\lvert J_1-J_2\rvert}^{J_1+J_2} \text{cosech}(\nu\theta + i\gamma L)\text{cosech}(\nu\theta - i\gamma L); \quad \text{(3.23)}
\]

\[
Z(\theta = 0, \gamma) = 1; \quad \zeta_J(\gamma) = (-1)^{J_1+J_2-J} \quad \text{(3.24)}
\]

The validity of the relations (3.20) now follows from the Jimbo’s theorem and the explicit construction of the $S$-matrix (3.11).

Thus, the proof is completed.

The relation (3.24) is not sufficient to calculate a function $Z(\theta)$ and in this way to find out the poles of the $S$-matrix in the physical strip corresponding to some bound states. Unfortunately, I have not found any straightforward arguments in favor of the crossing-symmetry of of the $S$-matrix (3.11) for the generic values of the isospins $J_1$ and $J_2$. However, there are several solutions known explicitly. Those with $J_1 = \frac{1}{2}$ and $J_2$ arbitrary (or vice versa) were found in [12]. Another one with $J_1 = J_2 = 1$ was computed in [21]. The above examples are crossing-symmetric. So, the following conjecture seems to be reasonable.

**Conjecture 1** The $S$-matrix (3.17) for the generic values of the isospins $J_1$ and $J_2$ possess the crossing-symmetry (1.19) with the unitary matrix

\[
U = \exp \left(i\nu\pi (\hat{h} \otimes \hat{I} - \hat{I} \otimes \hat{h})/2 \right).
\]

This crossing-symmetry leads to the equation for the $Z$-factor

\[
Z(\theta) = Z(i\pi - \theta).
\]
The minimal solution of the eqs. (3.24, 3.26) having the poles at all the points
\[ \theta = \frac{i\gamma J}{\nu}; \quad (|J_1 - J_2| \leq J \leq J_1 + J_2) \]
reads
\[
Z(\theta) = \prod_{J = |J_1 - J_2|}^{J_1 + J_2} Z_J(\theta)
\]
(3.27)
where
\[
Z_J(\theta) = \frac{i}{\pi} \Gamma(J\beta - \nu\alpha) \Gamma(J\beta - \nu(1 - \alpha)) \prod_{n=1}^{\infty} R_n^J(\alpha) R_n^J(1 - \alpha).
\]
(3.28)
Here \( \beta = \frac{\gamma}{\pi} = 1 - \nu \) and
\[
R_n^J(\alpha) = \frac{\Gamma(1 - J\beta - \nu(2(n - 1) + \alpha)) \Gamma(1 - J\beta - \nu(2n + \alpha))}{\Gamma(1 - J\beta - \nu(2n - 1 + \alpha)) \Gamma(1 - J\beta - \nu(2n - 1 + \alpha))}.
\]
(3.29)
While the function \( R_n^J(\alpha) \) has no poles in the physical strip \( 0 < \alpha < 1 \) the pre-factor \( \Gamma \)-functions do have them at
\[
\alpha = \frac{\beta J}{\nu}; \quad \theta = \frac{i\gamma J}{\nu};
\]
\[
\alpha = 1 - \frac{\beta J}{\nu}; \quad \theta = \frac{i\pi}{\nu} - \frac{i\gamma J}{\nu};
\]
(3.30)
provided \( \gamma J/\nu < \pi \).

The value of \( \nu \theta = i\gamma(J_1 + J_2) \) at which the bound states \( |J_1 + J_2, M \rangle \) arise is just the value when the ratio (3.16) for \( J = J_1 + J_2 \) vanishes. The tensor representation \( \{J_1\} \otimes \{J_2\} \) becomes reducible with respect to the total algebra \( U_q^L \). This happens because, the usual mapping \( \delta(f) : \{J_1 + J_2\} \rightarrow \{J_1 + J_2\} \oplus \{J_1 + J_2 - 1\} \) due to the eq. (3.21) degenerates into \( \delta(f) : \{J_1 + J_2\} \rightarrow \{J_1 + J_2\} \) making the module \( \{J_1 + J_2\} \) irreducible.

This is not the only value of \( \theta \) for which the degeneracy occurs. Consider e.g. \( \nu \theta = i\pi - i\gamma(|J_1 - J_2| + 1) \). (Here \( i\pi \) is added in order to choose \( \theta \) within the physical strip \( 0 < \theta < \pi \).) This value corresponds to the decoupling of the lowest representation \( \{|J_1 - J_2|\} \) of the C-G series (2.49). Inserting \( \nu \) from (2.3) one easily finds the Euclidean angle
\[
U_{J_1, J_2}^{|J_1 - J_2|} = i\pi - \frac{\gamma}{\nu}|J_1 - J_2|.
\]
(3.31)
Substituting it into the eq. (1.6) and using the momentum conservation (1.2) one obtains two other angles characterizing the reaction

\[ U_{J_2,|J_1-J_2|} = \frac{\gamma}{\nu} J_2; \quad U_{J_2,|J_1-J_2|,J_1} = \pi - \frac{\gamma}{\nu} J_1; \quad \text{for } J_1 > J_2; \]
\[ U_{J_1,|J_1-J_2|} = \pi - \frac{\gamma}{\nu} J_2; \quad U_{J_1,|J_1-J_2|,J_1} = \frac{\gamma}{\nu} J_1; \quad \text{for } J_1 < J_2; \]
\[ U_{J_1,|J_1-J_2|} = \frac{1}{2} \pi; \quad U_{J_1,|J_1-J_2|,J_1} = \frac{1}{2} \pi \quad \text{for } J_1 = J_2; \]

Plugging the above values into the expressions (1.5) for the masses of the particles participating in the reaction yields

\[ m_{J_1} = m_\star \sin \frac{\gamma}{\nu} J_1; \quad m_{J_2} = m_\star \sin \frac{\gamma}{\nu} J_2; \quad m_{|J_1-J_2|} = m_\star \sin \frac{\gamma}{\nu} |J_1 - J_2| \]

in full agreement with the spectrum (3.13). \(^3\)

This is not an end of the story however. A similar considerations apply to all the values of \( \theta \) for which any ratio (3.16) either vanishes or turns into infinity.

In commodity with (3.21) when the rapidity takes the corresponding value \( \theta = i \gamma J/\nu \) the operator \( \delta(f) \) maps the \( U_q \)-module \( \{J\} \to \{J\} \oplus \{J+1\} \) while the usual mapping to \( \{J-1\} \) disappears. Hence, an irreducible module \( \{J_1\} \otimes \{J_2\} \) becomes reducible with respect to the whole algebra \( U_q^L \) and the following representation decouples:

\[ \{J_1, J_2||J_+\} = \{J\} \oplus \{J + 1\} \oplus \cdots \oplus \{J_1 + J_2\} \quad (3.34) \]

There exits another series of the rapidities \( \theta = i(\pi - \gamma J)/\nu \) for which the module \( \{J_1\} \otimes \{J_2\} \) splits since the mapping \( \{J\} \to \{J-1\} \oplus \{J\} \oplus \{J+1\} \) degenerates into \( \{J\} \to \{J - 1\} \oplus \{J\} \). The decoupling representation is now

\[ \{J_1, J_2||J_\_\} = \{J\} \oplus \{J - 1\} \oplus \cdots \oplus \{|J_1 - J_2|\} \quad (3.35) \]

The above super-multiplets labeled by three quantum numbers \( J_1, J_2 \) and \( J_\pm \), contain new bound states having equal masses:

\[ m_{\{J_1, J_2||J_\pm\} \}^2 = m_{J_1}^2 + m_{J_2}^2 \pm 2m_{J_1} m_{J_2} \cos \left( \frac{\gamma}{\nu} J \right) \quad (3.36) \]

\(^3\)Notice that the iso-scalar particles are massless.
The scattering angles $U_{J_2,J_\pm}^{J_1}$ and $U_{J_\pm,J_1}^{J_2}$ should be defined through the following transcendental equations:

$$ U_{J_2,J_\pm}^{J_1} + U_{J_\pm,J_1}^{J_2} = \frac{\gamma}{\nu} J; \quad U_{J_2,J_\mp}^{J_1} + U_{J_\mp,J_1}^{J_2} = \pi - \frac{\gamma}{\nu} J. \quad (3.37) $$

Now we are in position to prove that the multiplets under consideration really emerge from bootstrap equations. The following generalization of the fusion procedure proposed by Karowsky [8] for the $S$-matrices being rational functions of rapidities gives the unitary $S$-matrix satisfying the YBE for the representation considered.

**Proposition 3** Define the following operators projecting onto the subspaces $\{J_1,J_2\parallel J_\pm\}$

$$ Q_{\{J_1,J_2\parallel J_\pm\}} = \sum_{L=|J_1-J_2|}^{J} \mu_L^{-} P_{J_1,J_2}^{-} \quad Q_{\{-J_1,J_2\parallel J_\pm\}} = \sum_{L=|J_1-J_2|}^{J} \left(\mu_L^{+}\right)^{-1} P_{J_1,J_2}^{+}, $$

$$ Q_{\{J_1,J_2\parallel J_+\}} = \sum_{L=J}^{J_1+J_2} \mu_L^{+} P_{J_1,J_2}^{+} \quad Q_{\{-J_1,J_2\parallel J_+\}} = \sum_{L=J}^{J_1+J_2} \left(\mu_L^{+}\right)^{-1} P_{J_1,J_2}^{+}; \quad (3.38) $$

where

$$ \left(\mu_L^{+}\right)^2 = \frac{\Lambda_L(\theta)}{Z(\theta)} \bigg|_{\theta = \frac{\nu}{2\Delta}}; \quad \left(\mu_L^{-}\right)^2 = \frac{\Lambda_L(\theta)}{Z(\theta)} \bigg|_{\theta = \frac{\nu}{2\Delta}} \quad (3.39) $$

and the reciprocity may be understood literally once a restriction to the subspace $\{J_1,J_2\parallel J_\pm\}$ is imposed. Then the unitary $S$-matrix solving the YBE and respecting the condition of the real analyticity reads

$$ S^{(J_0,J_\pm)}_{[0,12]}(\theta) = Q_{\{-J_1,J_2\parallel J_\pm\}}^{-1} S^{(J_0,J_1)}_{[0,1]}(\theta - i\nu U_{J_2,J_\pm}^{J_1}) S_{[0,2]}^{(J_0,J_2)}(\theta + i\nu U_{J_\mp,J_1}^{J_2}) Q_{\{J_1,J_2\parallel J_\pm\}} \quad (3.40) $$

The proof is somewhat cumbersome and is presented in Appendix. Here I would like to emphasize that the choice of the eigenvalues for the operators (3.38) is determined by the condition of real analyticity and, hence, by unitarity while it is irrelevant for the YBE.
The spectral decomposition analogous to (3.14) for the $S$-matrix involving one of the representations $\{J_1, J_2\|J\pm\}$ and any irreducible one just as that involving two reducible representations hasn’t been yet computed. Nevertheless, there is no doubt that the new bound states corresponding to highly reducible representations do exist. The reduction of these super-multiplets should split them into a finite number of the irreducible ones with the isospins $J = J_{\text{min}}, J_{\text{min}} + 1, \ldots J_{\text{max}} - 1, J_{\text{max}}$. The multiplicities of these constituents should be equal to one either for the maximal or for the minimal isospins and ought to be greater then one for the other species. Their structure will be investigated elsewhere. However, I would like to finish this section with a following conjecture.

**Conjecture 2** The isotopic structure of multiplets together with the corresponding scattering angles may be found recursively. To do this one should consider all possible fusions of the multiplets already found. Any direct product of two multiplets would be irreducible with respect to $U_L^q$ for the generic values of the rapidities. There exist the imaginary value of the relative rapidity for which it becomes reducible. This value gives the scattering angles while extracting the irreducible part of the direct product one gets the isotopic contents of the new multiplet.

### 4 Concluding remarks

There are two main results obtained in the present work. First, a lot of new bound states yet unknown in the context of the sine-Gordon model have been found. Second, the particle contents of this system is shown to be in one to one correspondence with a set of very peculiar representations of the quantum loop algebra. An algorithm is proposed for generating these representations recursively.

In my opinion, it is tempting to match this purely elastic scattering theory to some integrable perturbation of the WZNW conformal field theory. The main reasons why this correspondence seems to be plausible is that the $S$-matrix is invariant under the action of the $q$-deformed loop algebra. Though this algebra does not have an anomalous central extention the latter might be produced dynamically. In fact the requirements of the unitarity and crossing permit only a finite number of the irreducible multiplets equal to the integer part of $\pi\nu/\gamma$ just as it happens in the WZNW model with...
the Kac-Moody central charge $k = [\pi \nu / \gamma]$. The reducible representations might be interpreted as some combinations of the descendent states. Since the central charge is known to serve as a measure for the number of degrees of freedom of the system \footnote{22} and the number of irreducible representations decreases with the increase of $\gamma$ it looks like that this parameter describes the Zamolodchikov’s \footnote{22} renormalization group flow along one of the trajectories. The rational values of $\gamma$ where some of the states become massless might correspond to some critical points.

On the other hand the integrable deformations of the minimal models are also known to have the symmetry algebra coinciding with the $q$-deformed $A_1^{(1)}$ algebra \footnote{3} \footnote{4} \footnote{5}. So, to establish the exact correspondence between the purely elastic scattering theory in question and some perturbed conformal field theory it is necessary to calculate at least the central charge of the system.

The most direct way to compute the universal characteristics of the model - its central charge and the primary conformal dimensions is the finite size corrections approach \footnote{23} \footnote{24} \footnote{25} modified specifically for the purely elastic scattering theory by Al. Zamolodchikov \footnote{26}. A thermodynamic version of the Bethe ansatz applies to a massive integrable quantum field theory at a temperature much lower than the smallest of the masses but sufficiently high to treat the system thermodynamically allowing to calculate a specific heat proportional to the central charge \footnote{24} \footnote{25}. Moreover, examining the collective temperature excitations with the complex rapidities one can in principle obtain the conformal dimensions \footnote{24}.

A sort of such calculation has been already done in \footnote{4} and the result is consistent with the WZNW interpretation. However, only the irreducible multiplets were taken into account in the cited work. The derivation of the Bethe ansatz for the infinite system of excitations looks as a tremendous task at present accounting for the fact especially that the above excitations have not been described yet and the corresponding $S$-matrix has not been calculated.

Investigating the structure of the bound states one encounters the following mathematical problem. The realization of the algebra $\mathcal{U}_q^L$ given in Table 2 is the only one for which an algebraic definition for the operators $\hat{e}_0$, $\hat{f}_0$ in terms of the $\hat{e}$, $\hat{f}$ is known. This realization is naturally associated with the irreducible representations of the $q$-deformed $sl(2)$ algebra. The action of the
operators $\hat{c}_0$ and $\hat{f}_0$ on the direct product of the irreducible representations is described by the eq. (2.35) quite satisfactory but the same action on the states of the module $\{J_1, J_2 \parallel J\}$ should be determined recursively.

Finally, the relation of the model in question to the genuine sine-Gordon model is far from being clear. Their $Z$-factors for the $S$-matrices of the fundamental representations are in fact different functions of $\theta$ though they do coincide at $\nu = 1$ or $\gamma_{SG} = 8\pi$. At this point the $S$-matrix degenerates into the identity operator since the reflection amplitude of the soliton-antisoliton scattering vanishes while the transition amplitude becomes equal to the amplitude of scattering of the two identical particles and both equal to unity. Thus, in my view this point might be the point of singularity for the purely elastic scattering theory considered i.e. the boundary point of two different regimes. Unfortunately, neither the semiclassical approximation nor perturbative approach can’t be applied for the values of coupling constants close to unity. So, a similarity of the $S$-matrices may occur just formal. However, if the physical significance of the relation would be established it could provide an interesting example of duality between the weak- and strong-coupling regimes of the sine-Gordon model.
In any case the system seems to me deserving further attention due to its highly non-trivial structure. An examination of the latter might provide new profound insights in the near-critical universality.

The same algebraic methods are applicable for the systems with the higher underlying symmetries. In particular the solutions of the YBE involving two arbitrary irreducible representations (not necessarily rectangular as in [9, 10]) can be found. But this is a subject of the future publication.

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Appendix

Proof of the Proposition 4.

It would be convenient to use the notion of the monodromy-matrix (2.11) introduced previously for the product of two S-matrices just as those entering the eq. (3.40). The monodromy-matrices themselves obey a sort of the YBE:

$$S_{ab}(\theta)T_{a12}(\theta a1, \theta a2)T_{b12}(\theta b1, \theta b2) = T_{b12}(\theta b1, \theta b2)T_{a12}(\theta a1, \theta a2)S_{ab}(\theta) \quad (A.1)$$

So, the S-matrices (3.40) could enjoy the YBE if the projectional operators (3.38) did not interfere. Indeed, the eq. (A.1) may be sandwiched between the operators $Q_{J_1, J_2 \| J_{\pm}}^{-1}$ and $Q_{J_1, J_2 \| J_{\pm}}$, so it’s sufficient to show that

$$Q_{J_1, J_2 \| J_{\pm}}^{-1}T_{a12}(\theta a1, \theta a2)T_{b12}(\theta b1, \theta b2)Q_{J_1, J_2 \| J_{\pm}} = \sum_L Q_{J_1, J_2 \| J_{\pm}}^{-1}T_{a12}(\theta a1, \theta a2)P_{J_1, J_2}^L T_{b12}(\theta b1, \theta b2)Q_{J_1, J_2 \| J_{\pm}} \quad (A.2)$$

provided $\theta a1 - \theta a2 = \theta b1 - \theta b2 = i(\gamma J)/\nu$ or $i(\pi - \gamma J)/\nu$. The summation runs over $J \leq L \leq J_1 + J_2$ or $|J_1 - J_2| \leq L \leq J$ respectively.

Substituting the decomposition (3.14) for $S_{12}(\theta)$ into the YBE (2.10) involving two irreducible representations $\{J_1\}$ and $\{J_2\}$ one may represent
them in the following form

\[ \Lambda_J(\theta) \langle J, M | T_{[a,12]}(\theta_{a1}, \theta_{a2}) | J', M' \rangle = \langle J, M | T_{[b,21]}(\theta_{b2}, \theta_{b1}) | J', M' \rangle \Lambda_{J'}(\theta) \] (A.3)

The subsequent application of the eqs. (2.10, A.3) to the product of two monodromy-matrices leads to the following identities

\[ \Lambda_J(\theta) \langle J, M | T_{[a,12]}(\theta_{a1}, \theta_{a2}) T_{[b,12]}(\theta_{b1}, \theta_{b2}) | J', M' \rangle = \sum_{J_1 + J_2} \sum_{J'' = |J_1 - J_2|}^{J_1 + J_2} \langle J, M | T_{[a,12]}(\theta_{a1}, \theta_{a2}) | J'', M'' \rangle \times \Lambda_{J''}(\theta) \langle J'', M'' | T_{[b,21]}(\theta_{b2}, \theta_{b1}) | J', M' \rangle \] (A.4)

Putting \( \theta_{12} = i\gamma J/\nu \) and using the YBE (A.3) one can easily derive the following relations

\[ \langle J', M' | T_{[a,12]}(\theta_{a1}, \theta_{a2}) T_{[b,12]}(\theta_{b1}, \theta_{b2}) | J'', M'' \rangle = \sum_{L=J}^{J_1 + J_2} \langle J', M' | T_{[a,12]}(\theta_{a1}, \theta_{a2}) | J', M' \rangle \times \langle J, M' | T_{[b,21]}(\theta_{b2}, \theta_{b1}) | J''', M''' \rangle \] (A.5)

provided both the quantum numbers \( J' \) and \( J'' \) labeling the matrix elements are restricted by an inequality \( J \leq J', J'' \leq J_1 + J_2 \). Note that the summation ranges over the irreducible modules \( \{L\}_{J_1, J_2 || J_+} \). Thus the above relations almost coincide with the eq. (A.2) we are going to proof. The latter may be reduced to the former by a similarity transformation of a monodromy-matrix with the matrices (3.38)

\[ T_{[a,12]}(\theta_{a1}, \theta_{a2}) \rightarrow Q_{\{J_1, J_2 || J_+\}}^{-1} T_{[a,12]}(\theta_{a1}, \theta_{a2}) Q_{\{J_1, J_2 || J_+\}} \] (A.6)

and the same for \( T_{[b,12]}(\theta_{b1}, \theta_{b2}) \). The similarity transformation does not certainly affect the validity of the YBE, but on the other hand it’s not clear yet.

\(^4\)This is a similarity transformation only in the irreducible module \( \{J_1, J_2 || J_+\} \).
what it is needed for. The explanation is rather simple - the transformation (A.6) is necessary due to the unitarity requirement.

Recall that the eq. (1.17) stems from the YBE and the initial conditions (1.16) only. However, this is not sufficient for the unitarity of the $S$-matrix. The real analyticity (1.18) must hold in addition

$$ (S_{[0,12]}(\theta))^\dagger = S_{[12,0]}(-\theta^*) $$  \hfill (A.7)

In terms of the matrix elements of the $S$-matrix (3.40) the above equation reads

$$ \mu_{j^\pm} (\mu_{j^\mp})^{-1} \hat{T}_0(J'', M'') |S_{[0,1]}(\theta - i\mathcal{U}_{J_{2},J_\pm}^{T_1}) S_{[0,2]}(\theta + i\mathcal{U}_{J_{2},J_\pm}^{T_2}) |J', M'^* = $$

$$ (\mu_{j^\mp})^{-1} \mu_{j^\pm} (J', M') |S_{[1,0]}(-\theta^* - i\mathcal{U}_{J_{2},J_\pm}^{T_1}) S_{[2,0]}(-\theta^* + i\mathcal{U}_{J_{2},J_\pm}^{T_2}) |J'', M''$$

where $\mathcal{T}_0$ is the operator of transposition in the 0-th space.

Since the complex conjugation converts the C-G decomposition (2.49) into (2.50) at real $\gamma$ the above equation may be rewritten as

$$ (\mu_{j^\pm})^2 \langle J'', M'' | \mathcal{T}_0 S_{[0,2]}^* (\theta + i\mathcal{U}_{J_{2},J_\pm}^{T_1}) \mathcal{T}_0 S_{[0,1]}^* (\theta - i\mathcal{U}_{J_{2},J_\pm}^{T_2}) |J', M'\rangle = $$

$$ (\mu_{j^\mp})^2 \langle J', M' | S_{[1,0]}(-\theta^* - i\mathcal{U}_{J_{2},J_\pm}^{T_1}) S_{[2,0]}(-\theta^* + i\mathcal{U}_{J_{2},J_\pm}^{T_2}) |J'', M'' \rangle $$

where each of the transposition operators in the l.h.s. of the last equality acts on the nearest $S$-matrix only. Finally, using the property of real analyticity for any of the $S$-matrices inside the matrix elements one comes to the equality

$$ (\mu_{j^\pm})^2 \langle J'', M'' | S_{[0,2]}(-\theta^* + i\mathcal{U}_{J_{2},J_\pm}^{T_1}) S_{[0,1]}(-\theta^* - i\mathcal{U}_{J_{2},J_\pm}^{T_2}) |J', M'\rangle = $$

$$ (\mu_{j^\mp})^2 \langle J', M' | S_{[1,0]}(-\theta^* + i\mathcal{U}_{J_{2},J_\pm}^{T_1}) S_{[2,0]}(-\theta^* - i\mathcal{U}_{J_{2},J_\pm}^{T_2}) |J'', M'' \rangle $$

which due to the eq. (3.39) is just the particular case of the eq. (A.3).

The proof is completed.

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