POINCARÉ SERIES OF MONOMIAL RINGS

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Abstract. Let $k$ be a field, let $I$ be an ideal generated by monomials in the polynomial ring $k[x_1, \ldots, x_t]$ and let $R = k[x_1, \ldots, x_t]/I$ be the associated monomial ring. The $k$-vector spaces $\text{Tor}^R_i(k, k)$ are $\mathbb{N}_t$-graded. We derive a formula for the multigraded Poincaré series of $R$,

$$P_R^k(x, z) = \sum_{i \geq 0, \alpha \in \mathbb{N}_t} \dim_k \text{Tor}_{i, \alpha}^R(k, k)x^\alpha z^i,$$

in terms of the homology of certain simplicial complexes associated to subsets of the minimal set of generators for $I$. The homology groups occurring in the formula can be interpreted as the homology groups of lower intervals in the lattice of saturated subsets of the generators for $I$.

1. Introduction

Let $I$ be an ideal generated by monomials in the polynomial ring $Q = k[x] = k[x_1, \ldots, x_t]$ over some field $k$ and let $R = Q/I$ be the associated monomial ring. The $\mathbb{N}_t$-grading of $Q$ assigning the degree $\alpha = (\alpha_1, \ldots, \alpha_t)$ to the monomial $x_1^{\alpha_1} \cdots x_t^{\alpha_t}$ is inherited by $R$ and by $k \cong Q/(x)$. Thus the $k$-vector space $\text{Tor}_i^R(k, k)$ can be equipped with an $\mathbb{N}_t$-grading for each $i$. The formal power series

$$P_R^k(x, z) = \sum_{i \geq 0, \alpha \in \mathbb{N}_t} \dim_k \text{Tor}_{i, \alpha}^R(k, k)x^\alpha z^i \in \mathbb{Z}[x_1, \ldots, x_t, z]$$

is called the multigraded Poincaré series of $R$. It is proved by Backelin in [3] that this series is rational of the form

$$P_R^k(x, z) = \prod_{i=1}^t (1 + x_i z) \frac{b_R(x, z)}{b_R(x, z)},$$

for a polynomial $b_R(x, z) \in \mathbb{Z}[x_1, \ldots, x_t, z]$.

In this paper we derive a formula for $b_R(x, z)$ in terms of the homology of certain simplicial complexes $\Delta_S$ associated to subsets $S$ of the minimal set of generators, $M_I$, for $I$.

Sets of monomials are considered as undirected graphs by letting edges go between monomials having non-trivial common factors. Denote by $c(S)$ the number of connected components of a monomial set $S$. If $S$ is a finite set of monomials let $m_S$ denote the least common multiple of all elements of $S$.

Given a pair $S \subseteq M$ of monomial sets, $S$ is called saturated in $M$ if for all $m \in M$ and all connected subsets $T$ of $S$, $m$ divides $m_T$ only if $m \in S$. Denote by $K(M)$ the set of non-empty saturated subsets of $M$.

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If $M$ is a monomial set and $M = M_1 \cup \ldots \cup M_r$ is its decomposition into connected components, then $\Delta'_M$ is the simplicial complex with vertex set $M$ and simplices
\[ \{ S \subseteq M \mid m_S \neq m_M \text{ or } M_i \cap S \text{ disconnected for some } i \} . \]

If $H = \bigoplus_{i \in \mathbb{Z}} H_i$ is a graded vector space with $H_i$ finite dimensional for each $i$, then denote $H(z) = \sum_{i \in \mathbb{Z}} \dim H_i z^i \in \mathbb{Z}[z^{-1}, z]$.

After these explanations we can state the main result.

**Theorem 1.** Let $k$ be any field. Let $I$ be an ideal in $Q = k[x_1, \ldots, x_t]$ generated by monomials of degree at least 2, and let $M$ be its minimal set of generators. The denominator of the Poincaré series of $R = Q/I$ is given by
\[ b_R(x, z) = 1 + \sum_{S \in K(M)} m_S(-z)^{c(S)+2} \bar{H}(\Delta'_S; k)(z), \]

In section 4 we will see that $\bar{H}(\Delta'_S; k)$ can be interpreted as the reduced homology of the open interval $(0, S)$ in the set $K(M)$ partially ordered by inclusion.

Theorem 1 has the following corollary, which gives a solution to a problem posed by Avramov in [2]. It improves the bound of the $z$-degree of $b_R(x, z)$ given in [2] considerably.

**Corollary 1.** With notations as in Theorem 1
\[ \deg b_R(z) \leq n + g, \]
where $b_R(z) = b_R(1, \ldots, 1, z)$, $n = |M|$ is the number of minimal generators of $I$ and $g$ is the largest size of a discrete subset of $M$. In particular
\[ \deg b_R(z) \leq 2n, \]
with equality if and only if $R$ is a complete intersection.

For a monomial set $M$, let $L_M$ denote the set $\{m_S \mid S \subseteq M \}$ partially ordered by divisibility. If $I$ is a monomial ideal, the set $L_I = L_{M_I}$ is called the lcm-lattice of $I$, cf. [10].

Let $I$ and $I'$ be monomial ideals in the polynomial rings $Q = k[x]$ and $Q' = k[x']$, respectively, where $x$ and $x'$ are finite sets of variables. One can introduce an equivalence relation on the class of monomial ideals by declaring that $I$ and $I'$ are equivalent if $L_I$ and $L_{I'}$ are isomorphic as partially ordered graphs, that is, if there is a bijection $f : L_I \rightarrow L_{I'}$ which respects both structures. Note that we do not require that the ideals live in the same polynomial ring. This equivalence relation is studied by Gasharov, Peeva and Welker in [10], where they prove that the ring $R = Q/I$ is Golod if and only if $R' = Q'/I'$ is.

Avramov proves in [2] that, with $R$ and $R'$ as above, there is an isomorphism of graded Lie algebras
\[ \pi^{\geq 2}(R) \cong \pi^{\geq 2}(R'). \]
Here $\pi(R)$ is the homotopy Lie algebra of $R$, cf. [1] chapter 10. A corollary of this result is the equality
\[ b_R(z) = b_{R'}(z), \]
where $b_R(z) = b_R(1, \ldots, 1, z)$. Thus after fixing the coefficient field, $b_R(z)$ depends only on the equivalence class of $I$. This generalizes the fact that whether $R$ is Golod or not depends only on the equivalence class of $I$.

As a consequence of our formula we obtain a slight strengthening of Avramov’s corollary, namely
Corollary 2. Let $I$ and $I'$ be ideals generated by monomials of degree at least 2 in the rings $k[x]$ and $k[x']$ respectively, where $x$ and $x'$ are finite sets of variables and $k$ is a field. Let $R = k[x]/I$, $R' = k[x']/I'$. If $f : L_I \to L_{I'}$ is an isomorphism of partially ordered graphs, then

$$b_{R'}(x', z) = f(b_R(x, z)),$$

where $f(b_R(x, z))$ denotes the result of applying $f$ to the coefficients of $b_R(x, z)$, regarding it as a polynomial in $z$.

Theorem 1 is first proved in the case when $I$ is generated by square-free monomials and then a construction by Fröberg [9] is used to reduce the general case to this case. The starting point of the proof in the square-free case is the observation that the Poincaré series $P_{R}(x)$ is determined by a finite set of the multigraded deviations $\epsilon_{i,\alpha}(R)$ of the ring $R$. Then we use the fact that these multigraded deviations can be computed from a minimal model for $R$ over the polynomial ring.

Section 2 contains definitions and conventions concerning relevant combinatorial notions. The facts about minimal models needed are given in section 3. The proof of Theorem 1 along with some auxiliary results, is presented in section 4. Section 5 contains the corollaries of the main result and their proofs. In the concluding remark of section 5 we note how Theorem 1 gives a combinatorial criterion for a monomial ring to be Golod.

2. COMBINATORICS

2.1. Simplicial complexes. A simplicial complex on a set $V$ is a set $\Delta$ of subsets of $V$ such that $F \subseteq G \in \Delta$ implies $F \in \Delta$. $V$ is the vertex set of $\Delta$. The $i$-faces or $i$-simplices of $\Delta$ are the elements in $\Delta$ of cardinality $i + 1$. We do not require that \{v\} $\in \Delta$ for all $v \in V$, but if a simplicial complex $\Delta$ is given without reference to a vertex set $V$, then it is assumed that $V = \cup \Delta$.

If $\Delta$ is a simplicial complex then $\tilde{C}(\Delta)$ will denote the augmented chain complex associated to $\Delta$. Thus $\tilde{C}_i(\Delta)$ is the free abelian group on the $i$-faces of $\Delta$, $\emptyset$ being considered as the unique $(-1)$-face, and $\tilde{C}(\Delta)$ is equipped with the standard differential of degree $-1$. Therefore

$$H_i(\tilde{C}(\Delta)) = \tilde{H}_i(\Delta).$$

As usual, if $G$ is an abelian group, then $	ilde{C}(\Delta; G) = \tilde{C}(\Delta) \otimes G$ and $\tilde{H}_i(\Delta; G) = H_i(\tilde{C}(\Delta; G))$.

The Alexander dual of a simplicial complex $\Delta$ with vertices $V$ is the complex

$$\Delta^\vee = \{F \subseteq V \mid V - F \not\in \Delta\}.$$

The join of two complexes $\Delta_1, \Delta_2$ with disjoint vertex sets $V_1, V_2$ is the complex with vertex set $V_1 \cup V_2$ and faces

$$\Delta_1 \ast \Delta_2 = \{F_1 \cup F_2 \mid F_1 \in \Delta_1, F_2 \in \Delta_2\}.$$

With $\Delta_1$ and $\Delta_2$ as above, we define what could be called the dual join of them:

$$\Delta_1 \cdot \Delta_2 = (\Delta_1^\vee \ast \Delta_2^\vee)^\vee.$$

Thus $\Delta_1 \cdot \Delta_2$ is the simplicial complex with vertex set $V_1 \cup V_2$ and simplices

$$\{F \subseteq V_1 \cup V_2 \mid F \cap V_1 \in \Delta_1 \text{ or } F \cap V_2 \in \Delta_2\}.$$
We will now briefly review the effects of these operations on the homology groups when the coefficients come from a field $k$.

If $|V| = n$, then (\text{Lemma 5.5.3})

$$\widetilde{H}_i(\Delta; k) \cong \widetilde{H}_{n-i-3}(\Delta^\vee; k).$$

If $C$ is a chain complex, then $sC$ denotes the shift of $C$; $(sC)_i = C_{i-1}$. Because of the convention that a set with $d$ elements has dimension $d-1$ considered as a simplex there is a shift in the following formula:

$$\widetilde{H}(\Delta_1 \ast \Delta_2; k) \cong s(\widetilde{H}(\Delta_1; k) \otimes_k \widetilde{H}(\Delta_2; k)).$$

If $H = \bigoplus_{i \in \mathbb{Z}} H_i$ is a graded vector space such that each $H_i$ is of finite dimension, then let $H(z) = \sum_{i \in \mathbb{Z}} \dim H_i z^i$ be the generating function of $H$. The above isomorphisms of graded vector spaces can be interpreted in terms of generating functions. If $\Delta$ has $n$ vertices, then

$$z^n \widetilde{H}(\Delta^\vee; k)(z^{-1}) = z^n \widetilde{H}(\Delta; k)(z),$$

and if $\Delta = \Delta_1 \ast \Delta_2$, then

$$\widetilde{H}(\Delta; k)(z) = z \widetilde{H}(\Delta_1; k)(z) \cdot \widetilde{H}(\Delta_2; k)(z).$$

From these two identities and an induction one can work out the following formula. If $\Delta = \Delta_1 \ldots \Delta_r$, then

$$\widetilde{H}(\Delta; k)(z) = z^{2r-2} \widetilde{H}(\Delta_1; k)(z) \ldots \widetilde{H}(\Delta_r; k)(z).$$

### 2.2. Sets of monomials.

Let $x_1, \ldots, x_t$ be variables. If $\alpha \in \mathbb{N}^t$, then we write $x^\alpha$ for the monomial $x_1^{\alpha_1} \cdots x_t^{\alpha_t}$. The multidegree of $x^\alpha$ is $\deg(x^\alpha) = \alpha$. If $\alpha \in \{0, 1\}^t$, then both $\alpha$ and $x^\alpha$ are called square-free.

To a set $M$ of monomials we associate an undirected graph, with vertices $M$, whose edges go between monomials having a non-trivial common factor. This is the graph structure referred to when properties such as connectedness et c., are attributed to monomial sets. Thus, for instance, a monomial set is called discrete if the monomials therein are pairwise without common factors. By $D(M)$ we denote the set of non-empty discrete subsets of $M$. A connected component of $M$ is a maximal connected subset. Any monomial set $M$ has a decomposition into connected components $M = M_1 \cup \ldots \cup M_r$, and we let $c(M) = r$ denote the number of such.

If $I$ is an ideal in a polynomial ring generated by monomials there is a uniquely determined minimal set of monomials generating $I$. This minimal generating set, denoted $M_I$, is characterized by being an antichain, that is, for all $m, n \in M_I$, $m \parallel n$ implies $m = n$.

If $S$ is a finite set of monomials, then $m_S$ denotes the least common multiple of all elements of $S$. By convention $m_\emptyset = 1$. The set $L_M = \{m_S \mid S \subseteq M\}$ partially ordered by divisibility is a lattice with lcm as join, called the lcm-lattice of the set $M$. If $I$ is a monomial ideal, then $L_I := L_{M_I}$ is called the lcm-lattice of $I$.

Remark: The gcd-graph of $I$, studied in [22], is the complement of the graph $L_I$.

If $M, N$ are two sets of monomials then $MN$ denotes the set of those monomials in $M$ which divide some monomial in $N$. Write $M_m = M_{\{m\}}$, and $M_a = M_{P_a}$.

Let $S$ be a subset of a monomial set $M$. The saturation of $S$ in $M$ is the set $\bar{S} = MN$, where $N = \{m_{S_1}, \ldots, m_{S_r}\}$ if $S_1, \ldots, S_r$ are the connected components of $S$. Clearly $S \subseteq \bar{S}$, and $S$ is called saturated in $M$ if equality holds. Equivalently, $S$ is saturated in $M$ if for all $m \in M$, $m \parallel m_T$ implies $m \in S$ if $T$ is a connected subset
of \( S \). Clearly \( S \) is saturated in \( M \) if and only if all the connected components of \( S \) are. The set of saturated subsets of \( M \) is denoted \( \bar{K}(M) \), and the set of non-empty such subsets is denoted \( K(M) \).

Two monomial sets \( M, N \) are called equivalent if there is an isomorphism of partially ordered sets \( f : L_M \rightarrow L_N \) which is also an isomorphism of graphs. Such a map \( f \) will be called an \textit{equivalence}.

To a monomial set \( M \) we associate a simplicial complex \( \Delta_M \), with vertex set \( M \) and faces

\[
\{ S \subseteq M \mid m_S \neq m_M \text{ or } S \text{ disconnected} \}.
\]

We state again the definition of the simplicial complex \( \Delta'_M \), given in the introduction. Let \( M = M_1 \cup \ldots \cup M_r \) be the decomposition of \( M \) into its connected components. The complex \( \Delta'_M \) has vertex set \( M \) and faces

\[
\{ S \subseteq M \mid m_S \neq m_M \text{ or } M_i \cap S \text{ disconnected for some } i \}.
\]

Note that

\[
\Delta'_M = \Delta_{M_1} \cdot \ldots \cdot \Delta_{M_r}.
\]

**Lemma 1.** Let \( M \) and \( N \) be antichains of monomials. An equivalence \( f : L_M \rightarrow L_N \) induces a bijection \( K(M) \rightarrow K(N) \), where \( S \in K(M) \) is mapped to \( f(S) \in K(N) \). Furthermore, for every \( S \subseteq M \), \( S \) and \( f(S) \) are isomorphic as graphs and the complexes \( \Delta'_S, \Delta'_{f(S)} \) are isomorphic.

**Proof.** If \( M \) is an antichain, then \( L_M \) is atomic with atoms \( M \). An isomorphism of lattices maps atoms to atoms, so \( f \) restricts to a graph isomorphism \( M \rightarrow N \) and hence gives rise to a bijection of subgraphs of \( M \) to isomorphic subgraphs of \( N \), which clearly maps saturated sets to saturated sets. Since the definition of \( \Delta'_S \) is phrased in terms of the graph structure of \( S \) and on the lattice structure of \( L_S \subseteq L_M \), it is clear that \( \Delta'_S \cong \Delta'_{f(S)} \) for any subset \( S \) of \( M \). \( \square \)

If \( k \) is a field and \( M \) is a monomial set in the variables \( x_1, \ldots, x_t \) then \( R = k[x_1, \ldots, x_t]/(M) \) is the monomial ring associated to \( M \). Monomial ideals are homogeneous with respect to the multigrading of \( k[x_1, \ldots, x_t] \), so monomial rings inherit this grading.

### 3. \( \mathbb{N}^t \)-graded Models and Deviations

In this section we will collect and adapt to the \( \mathbb{N}^t \)-graded situation some well known results on models of commutative rings. There is no claim of originality. Our main reference is \( \text{[1]} \) chapter 7.2. Inspiration comes also from the sources \( \text{[2]} \) and \( \text{[7]} \), where analogous but not directly applicable results can be found.

The notation \( |x| \) refers to the homological degree of an element \( x \). We use \( \deg(x) \) to denote the multidegree of \( x \).

#### 3.1. Deviations

Let \( R = Q/I \), where \( I \) is a monomial ideal in \( Q \). Recall that the \textit{ith deviation}, \( \epsilon_i = \epsilon_i(R) \), of the ring \( R \) can be defined as the number of variables adjoined in degree \( i \) in an acyclic closure, \( R(X) \), of \( k \) over \( R \), cf. \( \text{[1]} \) Theorem 7.1.3. See \( \text{[1]} \) section 6.3 for the construction of acyclic closures. The ring \( R \) is \( \mathbb{N}^t \)-graded and \( k \) is an \( \mathbb{N}^t \)-graded \( R \)-module, and one can show that there is a unique \( \mathbb{N}^t \)-grading on the acyclic closure \( R(X) \) which is respected by the differential. One may therefore introduce \( \mathbb{N}^t \)-graded deviations

\[
\epsilon_{i, \alpha} = | \{ x \in X \mid |x| = i, \deg(x) = \alpha \} |.
\]
It is clear that $\epsilon_{i,\alpha} = 0$ if $|\alpha| < i$.

By the general theory, $R(X)$ is a minimal resolution of $k$, and hence $R(X) \otimes_R k = H(R(X) \otimes_R k)$ is isomorphic as a multigraded vector space to $\text{Tor}^R(k, k)$. This yields a product representation of the multigraded Poincaré series

$$P_k^R(x, z) = \prod_{i \geq 1, \alpha \in \mathbb{N}} \frac{(1 + x^\alpha z^{2i-1})^{\epsilon_{2i-1, \alpha}}}{(1 - x^\alpha z^{2i})^{\epsilon_{2i, \alpha}}}.$$ 

It is a fundamental result that the deviations $\epsilon_i$ can be computed from a minimal model of $R$ over $Q$, cf. \cite{1} Theorem 7.2.6. Only trivial modifications are required in order to show that the $N^i$-graded deviations $\epsilon_{i, \alpha}$ can be computed from an $N^i$-graded minimal model of $R$ over $Q$. For this reason, we will need a few facts about minimal models.

### 3.2. Free dg-algebras

Let $V = \bigoplus_{i \geq 0} V_i$ be a graded vector space over $k$ such that $\dim_k V_i < \infty$ for each $i$. We denote by $\Lambda V$ the free graded commutative algebra on $V$, that is,

$$\Lambda V = \text{exterior algebra}(V_{\text{odd}}) \otimes_k \text{symmetric algebra}(V_{\text{even}}).$$

Denote by $(V)$ the ideal generated by $V$ in $\Lambda V$. A homomorphism $f : AV \to \Lambda V$ of graded algebras with $f(V) \subseteq (W)$ induces a linear map $Lf : V \to W$, called the linear part of $f$, which is defined by the requirement $f(v) - Lf(v) \in (W)^2$ for all $v \in V$.

If $x_1, \ldots, x_t$ is a basis for $V_0$, then $\Lambda V = Q \otimes_k \Lambda(V_+)$, where $Q = k[x_1, \ldots, x_t]$ and $V_+$ is the sum of all $V_i$ for positive $i$. Therefore $\Lambda V$ may be regarded as a $Q$-module and each $(\Lambda V)_n$ is a finitely generated free $Q$-module. Let $m \subseteq Q$ be the maximal ideal generated by $V_0$ in $Q$. Note that $(V) = (V_+) + m$ as vector spaces. The following basic lemma is a weak counterpart of Lemma 14.7 in \cite{2} and of Lemma 1.8.7 in \cite{3}.

**Lemma 2.** Let $f : \Lambda U \to \Lambda V$ be a homomorphism of graded algebras such that $f_0 : \Lambda(U_0) \to \Lambda(V_0)$ is an isomorphism and the linear part, $Lf : U \to V$, is an isomorphism of graded vector spaces. Then $f$ is an isomorphism.

**Proof.** Identify $Q = \Lambda(U_0) = \Lambda(V_0)$ via $f_0$. Since $Lf$ is an isomorphism, $\Lambda U$ and $\Lambda V$ are isomorphic. Thus to show that $f$ is an isomorphism it is enough to show that $f_n : (\Lambda U)_n \to (\Lambda V)_n$ is surjective in each degree $n$, because $f_n$ is a map between finitely generated isomorphic $Q$-modules. We do this by induction. The map $f_0$ is surjective by assumption. Let $n \geq 1$ and assume that $f_i$ is surjective for every $i < n$. Then since $Lf$ is surjective we have

$$(\Lambda V)_n \subseteq f((\Lambda U)_n) + ((V_+)^2)_n + m(\Lambda V)_n.$$ 

$((V_+)^2)_n$ is generated by products $vw$, where $|v|, |w| < n$, so by induction $((V_+)^2)_n \subseteq f((\Lambda U)_n)$. Hence

$$(\Lambda V)_n \subseteq f((\Lambda U)_n) + m(\Lambda V)_n.$$ 

$(\Lambda V)_n$ and $f((\Lambda U)_n)$ are graded $Q$-modules, so it follows from the graded version of Nakayama’s lemma that $(\Lambda V)_n = f((\Lambda U)_n)$. \hfill $\square$

By a free dg-algebra, we will mean a dg-algebra of the form $(\Lambda V, d)$, for some graded vector space $V$, where $d$ is a differential of degree $-1$ satisfying $dV \subseteq (V)$. The linear part $Ld$ of the differential $d$ on $\Lambda V$ is a differential on $V$, and will be
denoted \( d_0 \). A free dg-algebra \((\Lambda V, d)\) is called \textit{minimal} if \( dV \subseteq (V)^2 \). Thus \((\Lambda V, d)\) is minimal if and only if \( d_0 = 0 \).

If \((\Lambda V, d)\) is a free dg-algebra which is \( \mathbb{N}^-\)graded, that is, there is a decomposition

\[
(\Lambda V)_i = \bigoplus_{\alpha \in \mathbb{N}^-} (\Lambda V)_{i,\alpha}
\]

such that \( d(\Lambda V)_{i,\alpha} \subseteq (\Lambda V)_{i-1,\alpha} \), then denote

\[
H_{i,\alpha}(\Lambda V, d) = H_i((\Lambda V)_{\alpha}, d)
\]

The \( \mathbb{N}^-\)-grading is called non-trivial if \( \deg(v) \neq 0 \) for all \( v \in V \).

The following is a counterpart of Lemma 3.2.1 in [11], but taking the \( \mathbb{N}^-\)-grading into account. It tells us how to ‘minimize’ a given dg-algebra.

\textbf{Lemma 3.} Let \((\Lambda V, d)\) be a non-trivially \( \mathbb{N}^-\)-graded dg-algebra with \( dV_1 \subseteq m^2 \). Then there exists an \( \mathbb{N}^-\)-graded minimal dg-algebra \((\Lambda H, d')\) with \( H = H(V, d_0) \), together with a surjective morphism of dg-algebras

\[
(\Lambda V, d) \rightarrow (\Lambda H, d')
\]

which is a quasi-isomorphism if \( k \) has characteristic 0. For arbitrary \( k \) we have

\[
H_i(\Lambda V, d) \cong H_i(\Lambda H, d')
\]

for all square-free \( \alpha \in \mathbb{N}^- \) and all \( i \).

\textit{Proof.} Let \( W \) be a graded subspace of \( V \) such that \( V = \ker d_0 \oplus W \) and similarly split \( \ker d_0 \) as \( H \oplus \im d_0 \) (hence \( H \cong H(V, d_0) \)). Note that since \( dV_1 \subseteq m^2 \), \( W_0 = W_1 = 0 \). \( d_0 \) induces an isomorphism \( W \rightarrow \im d_0 \), so we may write

\[
V = H \oplus W \oplus d_0(W).
\]

Consider the graded subspace \( U = H \oplus W \oplus dW \) of \( \Lambda V \). The induced homomorphism of graded algebras

\[
f : \Lambda U \rightarrow \Lambda V
\]

is an isomorphism by Lemma 2, because \( f_0 \) is the identity on \( \Lambda H \) and the linear part of \( f \) is the map \( l_H \oplus l_W \oplus g \), where \( g : dW \rightarrow d_0(W) \) is the isomorphism taking an element to its linear part (isomorphism precisely because \( \ker d_0 \cap W = 0 \)). Thus we may identify \( \Lambda U \) and \( \Lambda V \) via \( f \). In particular \( f^{-1} df \) is a differential on \( \Lambda U \), which we also will denote by \( d \), and \((\Lambda U, d)\) is a dg-algebra in which \( \Lambda(W \oplus dW) \) is a dg-subalgebra.

The projection \( U \rightarrow H \) induces an epimorphism of graded algebras

\[
\phi : \Lambda U \rightarrow \Lambda H
\]

with kernel \((W \oplus dW)\Lambda U\), the ideal generated by \( W \oplus dW \) in \( \Lambda U \). Define a differential \( d' \) on \( \Lambda H \) by

\[
d'(h) = \phi d(h),
\]

where \( \iota \) is induced by the inclusion \( H \subseteq U \). With this definition \( \phi \) becomes a morphism of dg-algebras and it is evident that \((\Lambda H, d')\) is minimal. Furthermore \( H(H, d'_0) = H \cong H(V, d_0) \) by definition.

Consider the increasing filtration

\[
F_p = (\Lambda H)_{\leq p} \cdot (\Lambda(W \oplus dW)).
\]
Obviously $\cup F_p = \Lambda U$, and $dF_p \subseteq F_p$ since $d$ preserves $\Lambda(W \oplus dW)$. The associated first quadrant spectral sequence is convergent, with

$$E^2_{p,q} = H_p(\Lambda H, d') \otimes_k H_q(\Lambda(W \oplus dW), d) \implies H_{p+q}(\Lambda U, d).$$

Since $W_0 = W_1 = 0$, we have $H_0(\Lambda(W \oplus dW), d) = k$, and therefore $H_0(\Lambda H, d') = E^2_{0,0} = E^3_{0,0} = \cdots = E_{0,0}^{\infty} = H_0(\Lambda U, d) = H_0(\Lambda V, d)$. If the field $k$ has characteristic zero, then $(\Lambda(W \oplus dW), d)$ is acyclic, so in this case the spectral sequence collapses, showing that $H(\Lambda H, d') = H(\Lambda V, d)$. However, $\Lambda(W \oplus dW)$ need not be acyclic in positive characteristic $p$ — if $x \in W$ is of even degree, then $x^{np}$ and $x^{np-1}dx$ represent non-trivial homology classes for all $n \geq 1$. Recall however that we are working with $\mathbb{N}^t$-graded objects and maps. Since we have a non-trivial $\mathbb{N}^t$-grading, $\Lambda(W \oplus dW)|_{\alpha}$ is acyclic for square-free $\alpha$, simply because no elements of the form $x^a$, for $x \in (W \oplus dW), a \in \Lambda(W \oplus dW), n > 1$, are there. In particular the dissidents $x^{np}$ and $x^{np-1}dx$ live in non-square-free degrees. Hence the spectral sequence collapses in square-free degrees, regardless of characteristic, and so

$$H_{i,\alpha}(\Lambda H, d') \cong H_{i,\alpha}(\Lambda V, d),$$

for all square-free $\alpha$ and all $i$. □

3.3. Models. Let $Q = k[x_1, \ldots, x_t]$ and let $R = Q/I$ for some ideal $I \subseteq \mathbb{m}^2$. A model for the ring $R$ over $Q$ is a free dg-algebra $(\Lambda V, d)$ with $(\Lambda V)_0 = Q$, such that $H_0(\Lambda V, d) = R$ and $H_i(\Lambda V, d) = 0$ for all $i > 0$. In particular a model for $R$ over $Q$ is a free resolution

$$\cdots \to (\Lambda V)_n \to (\Lambda V)_{n-1} \to \cdots \to (\Lambda V)_1 \to Q \to R \to 0$$

of $R$ as a $Q$-module. A model $(\Lambda V, d)$ is called minimal if it is a minimal dg-algebra. A minimal model for $R$ over $Q$ always exists, and is unique up to (non-canonical) isomorphism, cf. [1] Proposition 7.2.4.

If $R$ is $\mathbb{N}^t$-graded, one can ask for the minimal model of $R$ to be $\mathbb{N}^t$-graded.

Lemma 4. Let $(\Lambda V, d)$ be a minimal $\mathbb{N}^t$-graded dg-algebra with $H_0(\Lambda V, d) = R$, and assume that

$$H_{i,\alpha}(\Lambda V, d) = 0$$

for all $i > 0$ and all square-free $\alpha \in \mathbb{N}^t$. Then $(\Lambda V, d)$ can be completed to a minimal model $(\Lambda W, d)$ of $R$ such that $W_\alpha = V_\alpha$ for all square-free $\alpha$.

Proof. A minimal model can be constructed inductively, by successively adjoining basis elements to $V$ in order to kill homology, cf. [1] propositions 2.1.10 and 7.2.4 for details. Since $\Lambda V$ is $\mathbb{N}^t$-graded, we can do this one multidegree at a time. Adding a basis element of multidegree $\alpha$ will not affect the part of the algebra below $\alpha$. Since $H_i((\Lambda V)_\alpha) = 0$ for all $i > 0$ when $\alpha$ is square-free, we do not need to add variables of square-free multidegrees in order to kill homology. Thus, applying this technique, we get a minimal model $\Lambda W$ of $R$, where $W$ is a vector space obtained from $V$ by adjoining basis elements of non-square-free degrees. In particular $W_\alpha = V_\alpha$ for all square-free $\alpha$. □

Taking $\mathbb{N}^t$-degrees into account, it is not difficult to modify the proof of Theorem 7.2.6 in [1] to obtain the following result:
Lemma 5. Let \((AW, d)\) be an \(\mathbb{N}^t\)-graded minimal model for \(R\) over \(Q\). Then the \(\mathbb{N}^t\)-graded deviations \(\epsilon_{i, \alpha}\) of \(R\) are given by
\[
\epsilon_{i, \alpha} = \dim_k W_{i-1, \alpha},
\]
for \(i \geq 1, \alpha \in \mathbb{N}^t\).

\[\square\]

4. Poincaré series

This section is devoted to the deduction of Theorem 1.

Theorem 1. Let \(k\) be any field. Let \(I\) be an ideal in \(Q = k[x_1, \ldots, x_t]\) generated by square-free monomials of degree at least 2, and let \(M\) be its minimal set of generators. The denominator of the Poincaré series of \(R = Q/I\) is given by
\[
b_R(x, z) = 1 + \sum_{S \in K(M)} m_S(-z)^{\epsilon(S)+2} \tilde{H}(\Delta_S; k)(z).
\]

Some intermediate results will be needed before we can give the proof. Retain the notations of the theorem throughout this section. We will frequently suppress the variables and write \(b_R = b_R(x, z)\) and \(P_R^k = P_R^k(x, z)\).

4.1. An observation. Assume to begin with that the ideal \(I\) is minimally generated by square-free monomials \(M = \{m_1, \ldots, m_n\}\) of degree at least 2. If we are given a subset \(S = \{m_i_1, \ldots, m_i_r\}\) of \(M\), where \(i_1 < \ldots < i_r\), then set \(\text{sgn}(m_{i_j}, S) = (-1)^{j-1}\). By Backelin [3], the Poincaré series of \(R\) is rational of the form
\[
P_R^k(x, z) = \prod_{i=1}^r (1 + x_i z) b_R(x, z),
\]
where \(b_R(x, z)\) is a polynomial with integer coefficients and \(x_i\)-degree at most 1 for each \(i\). We start with the following observation made while scrutinizing Backelin’s proof.

Lemma 6. If \(I\) is generated by square-free monomials, then the polynomial \(b_R\) is square-free with respect to the \(x_i\)-variables. Moreover \(b_R\) depends only on the deviations \(\epsilon_{i, \alpha}\) for square-free \(\alpha\). In fact, there is a congruence modulo \((x_1^2, \ldots, x_t^2)\):
\[
b_R \equiv \prod_{\alpha \in \{0, 1\}^t} (1 - x^\alpha p_\alpha(z)),
\]
where \(p_\alpha(z)\) is the polynomial \(p_\alpha(z) = \sum_{i=1}^{\lvert \alpha \rvert} \epsilon_{i, \alpha} z^i\).

Proof. Note that \(\epsilon_{1, e_i} = 1\) and \(\epsilon_{1, \alpha} = 0\) for \(\alpha \neq e_i (i = 1, \ldots, t)\). Hence using the product representation [3] and reducing modulo \((x_1^2, \ldots, x_t^2)\) we get (note that \((1 + mp(z))^{\alpha} \equiv 1 + nmp(z)\) for any integer \(n\) and any square-free monomial \(m\)):
\[
b_R = \prod_{i \geq 1} (1 - x_{\alpha} z^{2i})^{\epsilon_{2i, \alpha}}
\]
\[
= \prod_{i \geq 1} (1 - x_{\alpha} (\epsilon_{2i-1, \alpha} z^{2i-1} + \epsilon_{2i, \alpha} z^{2i}))
\]
\[
= \prod_{i \geq 1} (1 - x_{\alpha} p_\alpha(z)),
\]
product taken over all square-free \(\alpha\), where \(p_\alpha(z) \in \mathbb{Z}[z]\) is the polynomial \(p_\alpha(z) = \sum_{i=1}^{\lvert \alpha \rvert} \epsilon_{i, \alpha} z^i\). \[\square\]
This gives a formula for $b_R$ in terms of the square-free deviations $\epsilon_{i,\alpha}$. Therefore we are interested in the square-free part of an $\mathbb{N}^t$-graded minimal model of $R$ over $Q$.

### 4.2. Square-free deviations

In the square-free case, there is a nice interpretation of the square-free deviations in terms of simplicial homology. Recall the definition of $\Delta_M$ found in section 2.2. $M_\alpha$ denotes the set of monomials in $M$ which divide $x^\alpha$.

**Theorem 2.** Assume that $I$ is minimally generated by a set $M$ of square-free monomials of degree at least 2. Let $\alpha$ be square-free and let $i \geq 2$. If $x^\alpha \notin I_1$, then $\epsilon_{i,\alpha} = 0$, and if $x^\alpha \in I_1$ then

$$
\epsilon_{i,\alpha} = \dim_k \tilde{H}_{i-3}(\Delta_{M_\alpha}; k).
$$

The proof of this theorem depends on the construction of the square-free part of a minimal model for $R$, which we now will carry out.

Let $C$ be the set of connected non-empty subsets of $M$ and let $V$ be the $\mathbb{N} \times \mathbb{N}^t$-graded vector space with basis $\theta \cup \{x_1, \ldots, x_t; |x_i| = 0, \deg(x_i) = \epsilon_i\}$, where

$$
Y = \{y_S \mid S \in C, |y_S| = |S|, \deg(y_S) = \deg(m_S)\}.
$$

If $S$ is any subset of $M$ and $S = S_1 \cup \ldots \cup S_r$ is its decomposition into connected components, then define the symbol $y_S \in \Lambda V$ by

$$
y_S = y_{S_1} \cdots y_{S_r}.
$$

It follows at once that $|y_S| = |S|$ and that $\deg(y_S) = \deg(m_S)$.

The differential $d$ on $\Lambda V$ is defined on the basis $y_S, S \in C$, by

$$
dy_S = \sum_{s \in S} \sgn(s, S) \frac{m_S}{m_S - \{s\}} y_{S - \{s\}},
$$

and is extended to all of $\Lambda V$ by linearity and the Leibniz rule (and of course $dx_i = 0$). Note that it may happen that $y_{S - \{s\}}$ becomes decomposable as a product in the sum above. One verifies easily that the formula (5) remains valid for disconnected $S$. By definition $d$ is of degree $-1$ and respects the $\mathbb{N}^t$-grading.

Clearly, we have $H_0(\Lambda V) = R$. Let $\alpha \in \mathbb{N}^t$ be square-free. Then the complex $(\Lambda V)_\alpha$ is isomorphic to the degree $\alpha$-part of the Taylor complex on the monomials $M$ (cf. 5). It is well-known that the Taylor complex is a resolution of $R$ over $Q$, so in particular $H_i((\Lambda V)_\alpha) = 0$ for all $i > 0$. Since we assumed that the monomials $m_i$ are of degree at least 2, we have $dV_i \subseteq m^2$. Therefore, by Lemma 3 there is a minimal $\mathbb{N}^t$-graded dg-algebra $(\Lambda H, d')$ such that $H \cong H(V, d_0)$, $H_0(\Lambda H, d') = H_0(\Lambda V, d) = R$ and

$$
H_{i,\alpha}(\Lambda H, d') = H_{i,\alpha}(\Lambda V, d) = 0,
$$

for all $i > 0$ and all square-free $\alpha \in \mathbb{N}^t$. Now, by Lemma 4 we can construct an $\mathbb{N}^t$-graded minimal model $(\Lambda W, d)$ of $R$, such that $W_\alpha = H_\alpha$ for all square-free $\alpha$. This is all we need to know about the minimal model $(\Lambda W, d)$ in order to be able to prove Theorem 2.

**Proof of Theorem 2.** By Lemma 3 we get that

$$
\epsilon_{i,\alpha} = \dim_k W_{i-1,\alpha} = \dim_k H_{i-1,\alpha} = \dim_k H_{i-1,\alpha}(V, d_0),
$$

and this finishes the proof.
for square-free $\alpha$. We will now proceed to give a combinatorial description of the complex $V = (V, d_0)$. $V$ splits as a complex into its $\mathbb{N}$-graded components

$$V = \bigoplus_{\alpha \in \mathbb{N}} V_{\alpha}.$$ 

$V_{\alpha}$ is one-dimensional and concentrated in degree 0 for $i = 1, \ldots, t$. This accounts for the first deviations $c_{1,1} = 1$. If $|\alpha| > 1$, then $V_{\alpha}$ has basis $y_{\alpha}$ for $\alpha \in S$ in the set

$$C_{\alpha} = \{ S \subseteq M \mid m_S = x^{\alpha}, \text{ connected} \}.$$ 

In particular $V_{\alpha} = 0$ if $x^\alpha \not\in L_I$. The differential of $V_{\alpha}$ is given by

$$dy_{\alpha} = \sum_{s \in S} sgn(s, S)y_{s - \{s\}}.$$ 

Let $\Sigma_{\alpha}$ be the simplicial complex whose faces are all subsets of the set $M_{\alpha} = \{ m \in M \mid m \neq x^\alpha \}$, with orientation induced from the orientation $\{ m_1, \ldots, m_n \}$ of $M$. Define a map from the chain complex $\tilde{C}(\Sigma_{\alpha}; k)$ to the desuspended complex $s^{-1}V_{\alpha}$ by sending a face $S \subseteq M_{\alpha}$ to $s^{-1}y_{\alpha}$ if $S \subseteq C_{\alpha}$ and to 0 otherwise. In view of (7), this defines a morphism of complexes, which clearly is surjective. The kernel of this morphism is the chain complex associated to $\Delta_{M_{\alpha}}$, so we get a short exact sequence of complexes

$$0 \to \tilde{C}(\Delta_{M_{\alpha}}; k) \to \tilde{C}(\Sigma_{\alpha}; k) \to s^{-1}V_{\alpha} \to 0$$

Since $\Sigma_{\alpha}$ is acyclic, the long exact sequence in homology derived from the above sequence shows that $H_i(V_{\alpha}) \cong H_{i-2}(\Delta_{M_{\alpha}}; k)$. The theorem now follows from (8).

In terms of the polynomials $p_{\alpha}(z)$ the theorem may be stated as

$$p_{\alpha}(z) = z^3\tilde{H}(\Delta_{M_{\alpha}}; k)(z),$$

for $x^\alpha \in L_I$.

4.3. **Proof of Theorem**

**Proof of Theorem**

**Square-free case.** By Theorem 2, $p_{\alpha}(z) = 0$ unless $x^\alpha \in L_I$, in which case $p_{\alpha}(z) = z^3\tilde{H}(\Delta_{M_{\alpha}}; k)(z)$. But $\Delta_{M_{\alpha}}$ is contractible if $M_{\alpha}$ is disconnected, so $p_{\alpha}(z) = 0$ unless $x^\alpha \in cL_I$, where $cL_I$ denotes the subset of $L_I$ consisting of elements $l \neq 1$ such that $M_l$ is connected.

Hence by Lemma 5

$$b_R \equiv \prod_{x^\alpha \in L_I} (1 - x^\alpha p_{\alpha}(z)) \mod (x_1^2, \ldots, x_t^2).$$

If we carry out the multiplication in the above formula and use that $b_R$ is square-free with respect to the $x_i$-variables (by Lemma 5), we get the equality

$$b_R = 1 + \sum_{N \in D(cL_I)} \prod_{x^\alpha \in N} (-x^\alpha p_{\alpha}(z)) = 1 + \sum_{N \in D(cL_I)} m_N(-1)^{|N|} \prod_{x^\alpha \in N} p_{\alpha}(z)$$

(the identity $\prod_{x^\alpha \in N} x^\alpha = m_N$ holds because $N$ is discrete). Using (8) the formula takes the form

$$b_R = 1 + \sum_{N \in D(cL_I)} m_N(-1)^{|N|} \prod_{x^\alpha \in N} z^3\tilde{H}(\Delta_{M_{\alpha}}; k)(z).$$
By (2) this may be written
\[ b_R = 1 + \sum_{N \in D(cL_I)} m_N(-1)^{|N|}z^{|N|} + 2\tilde{H}(\Gamma; k)(z), \]
where \( \Gamma = \Delta_{M_{\alpha_1}} \cdots \Delta_{M_{\alpha_r}} \), if \( N = \{x^{\alpha_1}, \ldots, x^{\alpha_r}\} \). The point here is that 
\( M_N = M_{\alpha_1} \cup \cdots \cup M_{\alpha_r} \) is the decomposition of \( M_N \) into its connected components: 
any saturated subset \( S \) connected components. Then \( N = \{x^{\alpha_1}, \ldots, x^{\alpha_r}\} \). The point here is that 
\( M_N = M_{\alpha_1} \cup \cdots \cup M_{\alpha_r} \) is the decomposition of \( M_N \) into its connected components: 
this sets up a one-to-one correspondence between 
to what we want:

By (2) this may be written
\[ F(M) = 1 + \sum_{S \in K(M)} m_S(-z)^{c(S)} + 2\tilde{H}(\Delta'_S; k)(z), \]
when \( M \) is a set of monomials of degree at least 2. If \( I \) is a monomial ideal in 
some polynomial ring \( Q \) over \( k \), then set \( F(I) = F(M_I) \). So far we have proved 
that \( b_{Q/I} = F(I) \) whenever \( I \) is generated by square-free monomials. The claim of 
Theorem 1 is that \( b_{Q/I} = F(I) \) for all monomial ideals \( I \).

**Lemma 7.** Let \( I \) and \( I' \) be equivalent monomial ideals, and let \( f: L_I \to L_{I'} \) be an 
equivalence. Then
\[ f(F(I)) = F(I'), \]
where \( f(F(I)) \) denotes the result of applying \( f \) to the coefficients \( m_S \) of \( F(I) \), 
regarding it as a polynomial in \( z \).

**Proof.** By Lemma 1, \( f \) induces a bijection of \( K(M_I) \) onto \( K(M_{I'}) \), mapping \( S \) to 
\( f(S) \), such that \( \Delta'_S \cong \Delta'_{f(S)} \) and \( c(S) = c(f(S)) \) for \( S \in K(M) \). Since \( f(m_S) = m_{f(S)} \) for all \( S \subseteq M_I \), the result follows. \( \square \)

**Proof of Theorem 1** General case. We invoke the construction of Fröberg, [9] pp. 
30, in order to reduce to the square-free case. Let \( I \) be any monomial ideal in 
\( Q = k[x_1, \ldots, x_t] \), and let \( M = M_I \). Let \( d_i = \max_{m \in M} \deg_{x_i}(m) \). To each \( m \in M \) 
we associate a square-free monomial \( m' \) in \( Q' = k[x_{i,j} \mid 1 \leq i \leq t, 1 \leq j \leq d_i] \) as follows: If \( m = x_1^{\alpha_1} \cdots x_t^{\alpha_t} \) then
\[ m' = \prod_{i=1}^{t} \prod_{j=1}^{a_i} x_{i,j}. \]
The set \( M' = \{m' \mid m \in M\} \) minimally generates an ideal in \( Q' \), which we denote by 
\( I' \). The map \( M' \to M, m' \mapsto m \), extends to a map \( f: L_{I'} \to L_I \) characterized by the
property that $x_{i,j}$ divides $m \in L_I$, if and only if $x_{i,j}^j$ divides $f(m)$. From this defining property it is easily seen that $f$ is an equivalence. Therefore $f(F(I')) = F(I)$, by Lemma 7.

Let $R$ and $R'$ be the monomial rings associated to $M$ and $M'$ respectively. Using the technique of Lemma 7 it is easy to see that

$$b_R(x_1, \ldots, x_t, z) = b_{R'}(x_1, \ldots, x_t, x_1, x_2, \ldots, x_t, z),$$

that is,

$$b_R = f(b_{R'}).$$

But $I'$ is generated by square-free monomials, so $b_{R'} = F(I')$, whence

$$b_R = f(b_{R'}) = f(F(I')) = F(I),$$

which proves Theorem 1 in general. □

5. Applications and remarks

We will here give the proofs of the corollaries to the main theorem and make some additional remarks.

**Corollary 1.** With notations as in Theorem 7

$$\deg b_R(z) \leq n + g,$$

where $b_R(z) = b_R(1, \ldots, 1, z)$, $n = |M_I|$ is the number of minimal generators of $I$ and $g$ is the independence number of $M_I$, i.e., the largest size of a discrete subset of $M_I$. In particular

$$\deg b_R(z) \leq 2n,$$

with equality if and only if $R$ is a complete intersection.

**Proof.** If $\Delta$ is a simplicial complex with $v$ vertices, then $\deg \bar{H}(\Delta; k)(z) \leq v - 2$, because either $\dim \Delta = v - 1$, in which case $\Delta$ is the $(v - 1)$-simplex and $\bar{H}(\Delta; k) = 0$, or else $\dim \Delta \leq v - 2$, in which case $\bar{H}_i(\Delta; k) = 0$ for $i > v - 2$. The simplicial complex $\Delta_S$ has $|S|$ vertices. Thus the $z$-degree of a general summand in the formula 11 for $b_R(x, z)$ is bounded above by $c(S) + 2 + |S| - 2 \leq g + n$, because the number of components of $S$ can not exceed the independence number of $M_I$.

Since $g \leq n$ we get in particular that

$$\deg b_R(z) \leq 2n,$$

with equality if and only if $M_I$ is discrete itself, which happens if and only if $R$ is a complete intersection. □

Now that we know that $R$ and $R'$ below satisfy $b_R = F(I)$ and $b_{R'} = F(I')$, the next corollary is merely a restatement of Lemma 4.

**Corollary 2.** Let $I$ and $I'$ be ideals generated by monomials of degree at least 2 in the rings $k[\mathbf{x}]$ and $k[\mathbf{x}']$ respectively, where $\mathbf{x}$ and $\mathbf{x}'$ are finite sets of variables. Let $R = k[\mathbf{x}]/I$, $R' = k[\mathbf{x}']/I'$. If $f : L_I \to L_{I'}$ is an equivalence, then

$$b_{R'}(\mathbf{x}', z) = f(b_R(\mathbf{x}, z)),$$

where $f(b_R(\mathbf{x}, z))$ denotes the result of applying $f$ to the coefficients of $b_R(\mathbf{x}, z)$, regarding it as a polynomial in $z$. □
Remark 1. Given formula (11), it is easy to reproduce the result, implicit in [8] and explicit in [5], that
\[ b_R(x, z) = \sum_{S \subseteq M_I} (-1)^{|c(S)|} z^{|S| + c(S)} m_S, \]
when the Taylor complex on \( M_I \) is minimal. The Taylor complex is minimal precisely when \( m_T = m_S \) implies \( S = T \), for \( S, T \subseteq M_I \), i.e., when \( L_I \) is isomorphic to the boolean lattice of subsets of \( M_I \). In this case every non-empty subset \( S \) of \( M_I \) is saturated, because \( m | m_T \) implies \( m \in T \) for any \( T \subseteq M_I \), and \( \Delta'_S \) is a triangulation of the \((|S| - 2)\)-sphere, because \( m_S = m_M \) only if \( S = M \).

Remark 2. The set of saturated subsets of \( M \) constitutes a lattice with intersection as meet and the saturation of the union as join. For each \( m \in M \), the singleton \( \{m\} \) is saturated, and the set \( M' = \{\{m\} \mid m \in M\} \) is a cross-cut of the lattice \( \hat{K}(M) \), in the sense of [7], that is, it is a maximal antichain. A subset \( S \subseteq M' \) is said to span if its supremum in \( \hat{K}(M) \) is \( M \) and if its infimum is \( \emptyset \). The simplicial complex of subsets of \( M' \) that do not span can be identified with the complex
\[ \Gamma_M = \{S \subseteq M \mid \tilde{S} \neq M\}. \]
Therefore, by [7] Theorem 3.1, there is an isomorphism
\[ \text{H} (\Gamma_M) \cong \text{H} (\hat{K}(M)), \]
where \( \hat{K}(M) \) is the partially ordered set \( \hat{K}(M) - \{\emptyset, M\} \). As usual, the homology of a partially ordered set \( P \) is defined to be the homology of the simplicial complex of chains in \( P \).

Actually, one can check that \( \tilde{S} \neq M \) is equivalent to \( m_S \neq m_M \) or \( S \cap M_i \) disconnected for some connected component \( M_i \) of \( M \), so in fact we have an equality \( \Gamma_M = \Delta'_M \). Thus \( \Delta'_M \) computes the homology of \( \hat{K}(M) \). Moreover, if \( S \in \hat{K}(M) \) then \( \hat{K}(S) \) is equal to the sublattice \( \hat{K}(M)_{\leq S} = \{T \in \hat{K}(M) \mid T \subseteq S\} \) of \( \hat{K}(M) \). Therefore the homology groups occurring in (11) can be interpreted as the homology groups of the lower open intervals \( \hat{K}(S) = (\emptyset, S)_{\hat{K}(M)} \) of the lattice \( \hat{K}(M) \) and we may rewrite (11) as
\[ b_R(x, z) = 1 + \sum_{S \in \hat{K}(M)} m_S (-z)^{c(S)} + 2^{|S|} \text{H} ((\emptyset, S)_{\hat{K}(M)}; k)(z). \]

This could be compared with the result of [10] that the Betti numbers of a monomial ring \( R = k[x_1, \ldots, x_t]/I \) can be computed from the homology of the lower intervals of the lcm-lattice, \( L_I \), of \( I \). Specifically, Theorem 2.1 of [10] can be stated as
\[ P^Q_R(x, z) = 1 + \sum_{1 \neq m \in L_I} m z^2 \text{H} ((1, m)_{L_I}; k)(z). \]

Here \( P^Q_R(x, z) \) is the polynomial
\[ P^Q_R(x, z) = \sum_{i \geq 0, \alpha \in \mathbb{N}^t} \dim_k \text{Tor}^{Q}_i (R, k)x^\alpha z^i. \]

The lattices \( \hat{K}(M) \) and \( L_M \) are not unrelated. There is a surjective morphism of join-semilattices from \( \hat{K}(M) \) to \( L_M \) sending \( S \) to \( m_S \), so \( L_M \) is always a quotient semilattice of \( \hat{K}(M) \). If the graph of \( M \) is complete, then the morphism is an isomorphism.
As a conclusion, we note how our formula gives a combinatorial criterion for when a monomial ring is Golod. Interesting sufficient combinatorial conditions have been found earlier, see for instance [12], but the author is not aware of any necessary condition which is formulated in terms of the combinatorics of the monomial generators.

Recall that \( R \) is called a \textit{Golod ring} if there is an equality of formal power series

\[
P_k^R(x, z) = \frac{\prod_{i=1}^{t}(1 + x_iz)}{1 - z(P^Q_R(x, z) - 1)}.
\]

In terms of the denominator polynomial the condition reads

\[b_R(x, z) = 1 - z(P^Q_R(x, z) - 1).\]

It is easily seen that \( S \) is saturated in \( M \) if and only if \( S \) is saturated in \( M_{ms} \).

Note also that \((1, m)_{LM} = L_{M_m} \setminus \{1, m\} =: L_{M_m}\). Therefore, after plugging the formulas (9) and (10) into (11) and equating the coefficients of each \( m \in LI \), we get a criterion for \( R \) to be a Golod ring as follows:

Call a monomial set \( N \) \textit{pre-Golod over} \( k \) if

\[
\tilde{H}(\bar{L}N, k)(z) = \sum_{S \in K(N) \atop m_S = m_N}^{} (-z)^{\epsilon(S)}H((\emptyset, S)_{K(N)}; k)(z).
\]

\textbf{Theorem 3.} Let \( k \) be a field and let \( I \) be a monomial ideal in \( k[x_1, \ldots, x_t] \) with minimal set of generators \( M \). Then the monomial ring \( R = k[x_1, \ldots, x_t]/I \) is Golod if and only if every non-empty subset of \( M \) of the form \( M_{m} \), with \( m \in LI \), is pre-Golod over \( k \).

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