A Novel Sparse recovery based DOA estimation algorithm by relaxing the RIP constraint

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Abstract: Direction of Arrival (DOA) estimation of mixed uncorrelated and coherent sources is a long existing challenge in array signal processing. Application of compressive sensing to array signal processing has opened up an exciting class of algorithms. The authors investigated the application of orthogonal matching pursuit (OMP) for Direction of Arrival (DOA) estimation for different scenarios, especially to tackle the case of coherent sources and observed inconsistencies in the results. In this paper, a modified OMP algorithm is proposed to overcome these deficiencies by exploiting maximum variance based criterion using only one snapshot. This criterion relaxes the imposed restricted isometry property (RIP) on the measurement matrix to obtain the sources and hence, reduces the sparsity of the input vector to the local OMP algorithm. Moreover, it also tackles sources irrespective of their coherency. The condition for the weak-1 RIP on decreased sparsity is derived and it is shown that how the algorithm gives better result than the OMP algorithm. With an addition to this, a simple method is also presented to calculate source distance from the reference point in a uniform linear sensor array. Numerical analysis demonstrates the effectiveness of the proposed algorithm.

1. Introduction

The direction of arrival estimation or source localization problem refers to retrieving the direction information of multiple waves/sources from the data collected by sensor arrays which are arranged in different shapes. It has wide applications such as in mobile communications, sonar and radar technologies. Many proposals for DOA estimation algorithms have been given which can be broadly classified into following categories: data adaptive algorithms, subspace based algorithms and maximum likelihood algorithms. These methods depend on various statistical properties of received data, e.g. sample covariance matrix. They require a sufficient number of data snapshots to accurately estimate the data covariance matrix. Again, these can be sensitive to source correlations that tend to cause a rank deficiency in the covariance matrix. Many attempts have been made to modify the covariance matrix to solve the problem of coherency are available in the literature.

Compressive sensing (CS) is being widely utilised in different research areas for its various properties. It utilises the fact that the signals impinging on a sensor array are spatially sparse thus letting to exploit CS theory in DOA estimation problem. The novel contribution of CS-based DOA estimation methods is that very less samples are required and gives high-resolution performance even in highly noisy environment. A literature survey reveals some of the most interesting
applications of CS to DOA estimation problem in recent studies. Sparse recovery algorithms are equivalently effective with minimal snapshots, unlike as with multiple snapshots in case of conventional high-resolution DOA estimation algorithms. Hence, it is easier to work in scenarios like dynamic object tracking. The following work on application of CS to DOA estimation is a testimony to the advantages over traditional methods. In [18], Yang et al. have given an overview of these sparse methods for DOA estimation particularly recently developed gridless sparse methods. In [23, 21], a sparse covariance-based representation is exploited for source localization by applying a global matched filter. In [1], the \( l_1 \)-SVD sparse recovery algorithm is proposed for DOA estimation. In [2, 3] a iterative reweighted \( l_1 \) minimization is proposed for source estimation. In [6], a mixed \( l_{2,0} \)-norm based joint sparse approximation technique is proposed (JLZA-DOA) to solve the DOA estimation problem. Algorithms in [15, 7, 27, 26] address the DOA estimation problem by directly representing the array output in time domain with an over-complete basis from the array response vector.

As stated above, for subspace-based algorithms, the coherency of sources leads to serious performance degradation. Hence, the formulation of DOA estimation problem as a spectral estimation problem doesn’t give acceptable results. Our objective is to devise such a method for source localization such that the data or its covariance matrix doesn’t have to be manipulated in order to obtain the DOAs especially in case of source coherency. A major highlight of the existing sparse recovery approaches is that they are devised on a probabilistic guarantee contrast to array signal processing approaches which provides a deterministic guarantee. In this work, we extend the orthogonal array processing model matrix between covariance matrix and measured data to a deterministic setup and explore an intuitive link between array signal processing and compressive sensing theory for the complex DOA measurement similar to [8, 9]. Especially, we show that till a partial support of DOAs can be estimated deterministically using the proposed novel maximum variance criterion, the remaining unknown DOAs can be estimated with any greedy sparse recovery algorithm (in our case, the OMP algorithm) [24]. This is owing to the fact that we are forcing to relax the weak-1 RIP constant of the measurement matrix which makes this modified OMP algorithm independent of the columns of the measurement matrix. With introduction of such an hybridization, the proposed algorithm overcomes the drawbacks of the existing traditional approaches [31, 10, 11, 12] and yields a superior recovery performance. We also provide a a simple method to calculate the distance of a source from the sensor array. It uses the same sensor array unlike [30] and gives good results.

The reminder of the paper is organized as follows. The preliminaries for the algorithm is given in Section 2. The proposed algorithm is provided in Section 3. Section 4 demonstrates the performance and effectiveness of the proposed algorithm using robust numerical analysis. Section 5 provides conclusion of the paper. Note that \( \cdot^H \) represents Hermitian transpose, \( \cdot^T \) represents Transpose, \( \cdot^{\dagger} \) represents Pseudo-inverse and supp \( \cdot \) represents the support of a matrix defined as subset of the matrix domain containing those elements which are not mapped to zero.

2. Preliminaries

2.1. CS Data Model for DOA Estimation

Consider an uniform linear array (ULA) having \( N \) omnidirectional sensors. Let \( M \) be number of narrow-band source signals that impinge upon the ULA from distinct directions \( \theta = (\theta_1, \theta_1, ..., \theta_M) \). Then, the output sample of the \( n^{th} \) sensor corrupted by additive white Gaussian noise, at the \( k^{th} \)
instant can be expressed as

\[ x_n(k) = \sum_{i=1}^{M} a_n(\theta_i)s_i(k) + w_n(k); \quad k = 1, 2, \ldots, K \] (1)

Here, \( K \) denotes number of snapshots, \( s_i(k) \) is the \( k^{th} \) sample of the \( i^{th} \) source signal, \( w_n(k) \) is the \( k^{th} \) noise sample at the \( n^{th} \) sensor, \( a_n(\theta_i) = e^{-j(\pi d_i n \gamma_i)} \) where \( \gamma_i = \frac{2\pi d_i \sin(\theta_i)}{\lambda} \), \( d \) and \( \lambda \) are inter-element spacing and wavelength, respectively. Writing (1) in matrix notation as in [29], we have

\[ \mathbf{x}(k) = \mathbf{A}(\theta)s(k) + \mathbf{w}(k); \quad k = 1, 2, \ldots, K \] (2)

Here, \( i = 1, 2, \ldots, M, \mathbf{x} = [x_1(k), x_2(k), \ldots, x_N(k)]^T \in \mathbb{C}^{N \times 1}, \mathbf{A} = [a(\theta_1), a(\theta_2), \ldots, a(\theta_M)] \in \mathbb{C}^{N \times M} \) with \( a(\theta_i) = [a_1(\theta_i), a_2(\theta_i), \ldots, a_N(\theta_i)]^T, \mathbf{s} = [s_1(k), s_2(k), \ldots, s_M(k)]^T \in \mathbb{C}^{M \times 1} \) and \( \mathbf{w} = [w_1(k), w_2(k), \ldots, w_N(k)]^T \in \mathbb{C}^{N \times 1} \). The DOA estimation problem is to find \( \theta_i \) by working on the received vector, \( \mathbf{x} \). As stated before, an adequately large number of snapshots is required to utilise the statistical properties. Apart from this, when sources are coherent or even partially coherent, \( \mathbf{R}_x \) becomes rank deficient where \( \mathbf{R}_x \) is the covariance matrix of \( \mathbf{x} \). In order to overcome these deficiencies, compressive sensing models the problem of estimating the DOAs of the sources based on the measurements of the form

\[ \mathbf{y} = \Phi \mathbf{x} \] (3)

where \( \Phi \in \mathbb{C}^{m \times N} \) is a measurement matrix and \( \mathbf{y} \in \mathbb{C}^{m \times 1} \) is the measured vector obtained from the received signal vector \( \mathbf{x} \in \mathbb{C}^{N \times 1} \). It is to be noted that \( m \) is taken equal to \( M \ln(N) \). So, now the DOA estimation problem can stated as a sparse recovery problem as following in noiseless case:

\[ \max_{\mathbf{s} \in \mathbb{C}^N} \|\mathbf{s}\|_0 \text{ s.t. } \Phi \mathbf{A}s = \mathbf{y} \] (4)

where, \( \|\mathbf{s}\|_0 \) is defined as the \( l_0 \) norm of a vector \( \mathbf{s} \) which contains the number of nonzero entries in \( \mathbf{s} \).

Hence, if \( \mathbf{s} \) is sparse or compressible in some basis (here \( \mathbf{A} \)) and \( \Phi \) satisfies specified conditions, then compressive sensing algorithms can recover the signal vector, \( \mathbf{s} \), from the measured vector, \( \mathbf{y} \). If \( \Phi \) is a random matrix, it has to satisfy either Restricted Isometry Condition (RIP) or the incoherence condition. Here, we focus on the RIP condition, specifically on the weak-1 RIP condition [9].

2.2. Application of OMP Algorithm

Orthogonal Matching Pursuit (OMP) [13] is a greedy algorithm which computes the support of the sparse signal \( \mathbf{s} \) iteratively. After computing the support of the signal completely, the pseudo-inverse of the measurement matrix restricted to the corresponding columns, is used to reconstruct \( \mathbf{s} \). The algorithm has been presented in Algorithm 2.1. \textit{Note} that \( \Phi = [\phi_1, \phi_2, \ldots, \phi_N] \) where \( \phi_i \in \mathbb{C}^{m \times 1}, i = 1, 2, \ldots, N \) represents the \( i^{th} \) column of \( \Phi \). Also, in Algorithm 2.1, \( P_{R(\cdot)} \) denotes the projection of the matrix and \( P_R^\perp(\cdot) \) denotes the orthogonal projection of the matrix. Following is the sequence of steps for estimating the DOAs.:

\begin{enumerate}
  \item \textit{Step 1}: Obtain a single snapshot data vector \( \mathbf{x} \).
  \item \textit{Step 2}: Compute \( \mathbf{y} \) from \( \mathbf{x} \) using (3).
  \item \textit{Step 3}: Solve (4) using the OMP algorithm and obtain an estimate of \( \mathbf{s} \), say \( \hat{\mathbf{s}} \).
\end{enumerate}
Step 4: Plot the angle spectrum of $\hat{s}$ using (5). The $\theta$s corresponding to the peaks in the graph give an estimate of the DOAs.

$$P_\theta(\hat{s}) = ||\hat{s}_\theta||^2; \quad \theta \in \left[-\frac{\pi}{2}, \frac{\pi}{2}\right]$$

(5)

Algorithm 2.1 Orthogonal Matching Pursuit (OMP) algorithm

Input: $y \in \mathbb{C}^{m \times 1}$, $\Phi \in \mathbb{C}^{m \times N}$, $s \in \mathbb{N}$

Output: $\alpha \subset [N]$

1: $\alpha \leftarrow \emptyset$
2: while $|\alpha| < s$ do
3: $k \leftarrow \arg \max_{l \notin [N] \setminus \alpha} ||y \cdot P_{R(\Phi_\alpha)}^\perp \phi_l||$
4: $\alpha \cup k$
5: end while
6: return $\alpha$

In order to point out the discrepancies in the results, we did 10 Monte Carlo trials (Fig.1) with 17 sensors, 3 sources ($\theta_1 = 0^\circ$, $\theta_2 = 40^\circ$ and $\theta_3 = 60^\circ$).

![Graph showing consistency of OMP Algorithm](image)

Figure 1: Plot for consistency of OMP Algorithm (Monte Carlo trials = 5)

| Trial number | DOAs detected |
|--------------|---------------|
| 1            | $\theta_1$    |
| 2            | $\theta_1, \theta_2$ |
| 3            | $\theta_2, \theta_3$ |
| 4            | $\theta_2$    |
| 5            | $\theta_1, \theta_2, \theta_3$ |

Table 1: Trial Number 1-5

| Trial number | DOAs detected |
|--------------|---------------|
| 6            | $\theta_2$    |
| 7            | $\theta_1, \theta_3$ |
| 8            | $\theta_1, \theta_2, \theta_3$ |
| 9            | $\theta_1$    |
| 10           | $\theta_1, \theta_2, \theta_3$ |

Table 2: Trial Number 6-10

It can be observed that in trial 1, 4 and 6, only one source is detected. Similarly in trial 2, 3, 5 and 9, only two sources are detected whereas in trial 7, 8 and 10, all the three sources are detected.
Clearly the results didn’t detect all the DOAs in every simulation and they are highly inconsistent, thus unstable in performance.

3. Proposed Algorithm

Now to resolve the above demonstrated problem, we move on to relax the RIP condition on the measurement matrix. This is done because intuitively we want to make this estimation of the support vector independent of the RIP condition. Also, it occurs in many sparse signal recovery problems, the condition of RIP is unnecessarily strict and that is not satisfied by many measurement matrices. Therefore, weaker versions of RIP are used and thus we use a weak - 1 RIP condition given below.

**Definition 3.1.** (Weak-1 Restricted Isometry Property) Matrix \( \Phi \in \mathbb{C}^{m \times N} \) satisfies the weak RIP of order \( s \) if there exists a constant \( \delta \in (0, 1) \) such that

\[
(1 - \delta)I_{s+1} \leq \Phi_{\beta}^H \Phi_{\beta} \leq (1 + \delta)I_{s+1} \quad \forall \beta \supset \alpha, |\beta| = s + 1
\]

Here, \( s \in \mathbb{N}, \alpha \subset [N] \) and \( |\alpha| = s \). \( \Phi_{\beta} \) is defined as all submatrices with \( |\beta| = s + 1 \) columns of \( \Phi \) which are uniformly well conditioned. This weak-1 RIP is satisfied by a less stringent condition on \( \Phi \). The measurement matrix we use here is Gaussian Matrix which satisfies the weak-1 RIP [9].

To bring in the effect of this weaker RIP condition, a maximum variance (MV) is formulated based searching criterion which exploits the orthogonality of covariance matrix \( R_y \) of measured vector \( y \) and measurement matrix \( \Phi \) and is independent of the RIP condition for detecting correct indices. This is implemented by using the following corollary obtained immediately from Conventional beamforming [31].

**Corollary 3.1.** (MV based Searching Criterion) [31]: Let \( M \) be number of sources, \( N \) be number of sensors and \( m \) be the measurements taken using measurement matrix \( \Phi \in \mathbb{C}^{m \times N} \) from received vector \( x \in \mathbb{C}^{N \times 1} \) to obtain vector \( y \in \mathbb{C}^{m \times 1} \). Define \( \phi_\Gamma \in \mathbb{C}^{m \times 1}, \Gamma \in 1, 2, ..., m \) in general position; that is, any collection of \( m \) columns \( \phi_\Gamma \) are linearly independent. Then, \( \Gamma \) will belong to \( \text{supp} [x] \), if and only if

\[
\phi_\Gamma^H R_y \phi_\Gamma = 0
\]

where \( R_y \) is the covariance matrix of \( y \).

**Proof.** By the assumption, \( y \) can be factored as a product \( y = \Phi_\xi x^\xi \), where \( \xi = \text{supp} [x] \). \( \Phi_\xi \) is the matrix which consists of columns \( \phi_i \) whose indices \( i \) are in \( \xi \) and \( x^\xi \) is the matrix that consists of rows whose indices are in \( \xi \). Note that \( \Phi_\xi \) has full column rank and \( x^\xi \) has full row rank.

From conventional beamforming method [3], we know that \( \phi_\Gamma^H R_y \phi_\Gamma = 0 \). This condition will satisfy if and only if \( \phi_\Gamma \in R(\Phi)^\perp R(\Phi_y) \) so that \( \phi_\Gamma \) can be expressed as a linear combination of \( \{\phi_k\}_{k \in \xi} \). Since the columns of \( \Phi \) are in general position, then above condition can only be satisfied if and only if \( k \in \text{supp}[x] \). Here \( R(\Phi) \) represents range space of the matrix \( \Phi \).

Now, the reader may be interested in the difference between this proposed criterion and that of the equation of conventional beamforming, i.e. \( A^H R_x A = 0 \) where \( A \) is the steering matrix and \( R_x \) is the input covariance matrix. We give following two remarks two explain the differences:
Remark 1: As can be observed, the proposed equation involves the sensing matrix \( \Phi \) and covariance matrix of the measured vector \( y \) from compressive sensing theory whereas the conventional beamforming has the steering matrix \( A \) and the input covariance matrix \( R_x \) respectively.

Remark 2: Conventional beamforming is known to be erroneous in case of closely spaced coherent sources. This is because the orthogonality of \( A \) and \( R_x \) is being exploited here. In case of coherent sources, the covariance matrix is known to experience rank loss and hence this results in broad peaks in the power spectrum of the true DOAs, especially in case of closely spaced coherent sources. But in case of the proposed maximum variance criterion, this rank loss doesn’t come into picture as the orthogonality of the columns of \( \Phi \) and \( R_y \) is insensitive to it. Hence, the proposed criterion works well with the cases of closely spaced coherent sources.

This criterion will recover some \( r \) indices from the required support. Then the next step is to compute the rest of the \( k = s - r \) indices by using OMP algorithm locally on the reduced sparsity vector \( y \) as shown in the proposed algorithm (Algorithm 3.1). In the next section, it is shown how the proposed relaxation helps in exact recovery of complete support set and hence, making the algorithm stable in performance. Note that the Algorithm 3.1 mentions an update criterion in line 8. This criterion is taken from [9] as a modification to (5.5) and is given below in (8).

\[
k = \arg \max_{l \in [N] \setminus \Gamma_1} \frac{\|P_{R(R_P^{\perp}(y)\Phi_l)}\|_2}{\|P_{R(R_P^{\perp}(\Phi_l))}\|_2}
\]

**Algorithm 3.1 Proposed algorithm**

**Input:** \( y \in \mathbb{C}^{m \times 1}, \Phi \in \mathbb{C}^{m \times N}, s \in \mathbb{N}, \eta > 0 \)

**Output:** \( \Gamma \subset [N] \)

1. \( \Gamma \leftarrow \varnothing; \)
2. for \( l = 1, ..., N \) do
3. \( \zeta_l \leftarrow \frac{\|y_P\|_2}{\|\phi_l\|_2}; \)
4. end for
5. \( \Gamma \leftarrow \text{indices of the } r \text{ largest } \zeta_l \text{’s}; \)
6. \( \Gamma_1 \leftarrow \varnothing; \)
7. while \( \|y \cdot P_{R(\Phi)}\|_2 > \eta \) do
8. Select \( k \) from \( r \) indices using update criterion (8);
9. \( \Gamma_1 \leftarrow \Gamma_1 \cup k \)
10. \( \hat{y} \leftarrow y + P_{R(\Phi)}(y - \Phi_{\Gamma_1}) \)
11. for \( l \in [N] \setminus \Gamma_1 \) do
12. \( \zeta_k \leftarrow \frac{\|y_P\|_2}{\|\phi_l\|_2}; \)
13. end for
14. \( \Gamma \leftarrow \Gamma_1 \cup \text{(indices of the } k \text{ largest } \zeta_l \text{’s}); \)
15. end while
16. return \( \Gamma \)
3.1. On the bound of weak-1 RIP constant

It was shown in [20] that for perfect recovery, the sufficient condition for guaranteeing the perfect recovery of $s$-sparse signals via the OMP algorithm is

$$\delta_{s+1} < \frac{1}{\sqrt{s+1}}$$

(9)

But as observed in Fig.1 for checking the consistency of OMP’s performance, this bound needed to be relaxed more without making any changes to the measurement matrix. This is where our criterion comes into the picture. Our condition initially estimates some indices of the sparse vector and feeds the OMP algorithm with an reduced residual rather than earlier case. This leads to decrease in the sparsity from $s$ to $k$ and now OMP just needs to estimate a smaller support set with the same measurement matrix. So the relaxed bound now becomes

$$\delta_{k+1} < \frac{1}{\sqrt{k+1}}$$

(10)

Clearly $\delta_{k+1} > \delta_{s+1}$. In our case of DOA estimation, the number of sources represents the sparsity $s$. So, number of sources $M = s$. Now we give the following condition for OMP to detect a index belonging to the target sparse vector.

**Theorem 3.1.** Given the reduced sparsity $k$-sparse vector $s$ with support denoted as $\Gamma$, the index $j$ chosen in an iteration of the local OMP algorithm belongs to the support i.e., $j \in \Gamma$ if the weak-1 RIP constant $\delta_{k+1}$ of a matrix $\Phi$ satisfies $\delta_{k+1} < \frac{1}{\sqrt{k+1}}$

This theorem is straight forward from the fact that as the sparsity is reduced the OMP algorithm will perform better as it will have lesser support set to estimate because as the MV criterion estimates a part of the support, the local OMP algorithm will receive a more sparse signal and hence will perform better in recovery. We also provided a proof of Theorem 3.1 in Appendix A.

3.2. Computational Complexity of the Proposed Algorithm

Now note that the above mentioned $s$-sparse signal in section 3.1 is the $M$-sparse signal $s$ in our case received from a $N$ - sensor ULA. Again $m$ is the dimension of $y$. We know that in OMP algorithm, the residual update to compute an estimate of $s$, the required flops is approximately $4im$ at any $i^{th}$ iteration of the algorithm. Moreover, $2im$ flops are required for residual update. In noiseless case, considering that the algorithm requires $M$ iterations, the total number of flops will be about $2MmN + 3M^2m$. Hence it can be inferred that more the signal being recovered is sparse, better will be the outcome of OMP algorithm. In the proposed MV criterion, we have $L(mN + N)$ multiplications and $L(mN -1)$ additions where $L$ is the number of angles to be scanned. So the overall complexity is $O[L(mN + 1) + L(mN + 1) + 2MmN + 3M^2m]$.

3.3. Distance of Sources

We now show a method to estimate the distance of an source using the same array as opposed to the set up in [30], in which the authors do this using two different arrays. The set up geometry to estimate the distance of source is shown in fig. 2.
The following table gives a list of all notations used in the fig. 2.

| Variable | Meaning                                      |
|----------|----------------------------------------------|
| d        | Inter-sensor distance                        |
| C        | Distance between origin of Reference 1 and Reference 2 |
| N        | Number of sensors                            |
| a        | x-coordinate of source                        |
| b        | y-coordinate of source                        |
| Z        | Distance of source from Reference 1           |
| $\theta_1$ | DOA of source w.r.t. Reference 1           |
| $\theta_2$ | DOA of source w.r.t. Reference 2           |

Table 3: Notations in fig. 2

Divide received vector $\mathbf{x} \in \mathbb{C}^{N \times 1}$ into two halves by following method using a matrix $\mathbf{D}$ such that

$$\mathbf{D} = [\mathbf{0}_{T \times (N - T)}, \mathbf{I}_{T \times T}]$$

Then obtain $\mathbf{x}_1$ as

$$\mathbf{x}_1 = \mathbf{Dx}$$

with

$$T = \begin{cases} 
\frac{N}{2}, & \text{if } N \text{ is even} \\
\frac{N+1}{2}, & \text{if } N \text{ is odd}
\end{cases}$$

The procedure for finding $Z$ are as follows.

**Step 1:** Find $\theta_1$ and $\theta_2$ using the proposed method by dividing the array frame as explained above.

**Step 2:** Generate two straight-line equations

$$x = m_1 y$$

8
\[ x = m_2y + C \]  
(14)

with \( m_1 = \frac{\pi}{2} - \tan \theta_1 \) (\( \theta_1 \) obtained from DOA estimation using Frame-1) and \( m_2 = \frac{\pi}{2} - \tan \theta_2 \) (\( \theta_2 \) obtained from DOA estimation using Frame-2). (13) is obtained from Frame-1 and (14) is obtained from Frame-2. Clearly \( C = Td \).

**Step 3:** Solve the two straight line equations (13) and (14). These intersection of these two lines gives the position of the source. Denote this point as \((a, b)\). Then distance \( Z \) is calculated as

\[ Z = \sqrt{a^2 + b^2}. \]
(15)

### 4. Numerical Analysis

The following simulations are being presented to validate the proposed work in various environment scenarios. Also, results are discussed subsequently with respective plots.

The ULA is chosen with \( N = 15 \) sensors having an intersensor spacing of half a wavelength. The scanning direction grid contains 181 points being sampled from \(-90^\circ\) to \(90^\circ\) having \(1^\circ\) interval.

The coherent sources were selected as \(-10^\circ, 10^\circ\) and \(0^\circ\) with respective normalised frequencies as \(\pi/4\) (coherent sources). The proposed method is considered with time snapshot \( K = 1 \) where as the other algorithms are given \( K = 200 \) snapshots. The measurement matrix \( \Phi \) has entries drawn from a Gaussian random matrix of size \( m \times N \). Fig.3, shows the comparison plot of various array signal processing based DOA estimation methods for this case with SNR = 10 dB. It is observed that the proposed method gives sharp peaks only in the true DOAs with coherent signals resolved completely but the other algorithms [4, 22] are not able to do so.

![Figure 3: DOA estimation of coherent sources (Subspace based methods)](image1)

Similarly we perform an experiment with the same DOAs with normalised frequencies as \(\pi/2\), \(\pi/3\) and \(\pi/4\) (non-coherent sources). Rest of the parameters are kept same, except SNR which is changed to 5 dB. Fig.4, shows the comparison plot for this case. It is observed that the proposed method gives sharp peaks in the true DOAs but only with one snapshot.

![Figure 4: DOA estimation of non-coherent sources](image2)
We now observe the case with sparse recovery methods presented in [18, 13, 3]. The sources considered are $-35^\circ, 0^\circ, 42^\circ$ and $57^\circ$ with respective normalised frequencies as $\pi/3$, $\pi/2$, $\pi/4$ and $\pi/4$ (partial coherency). The measurement matrix $\Phi$ has entries drawn from a Gaussian random matrix of size $m \times N$ for all methods. Fig.5 shows the comparison plot for this experiment at SNR = 5 dB. It is observed that the proposed method gives peaks in the direction of sources with good resolution. Important to note that the plot in fig. 5 as well as fig. 6, for [18], requires more than one snapshot (we considered 200 snapshots) whereas our algorithm works well with one snapshot.

Similarly we perform an experiment with the same DOAs with normalised frequencies as $\pi/3$, $\pi/2$, $\pi/4$ and $\pi/5$ (non-coherent sources). The rest of the parameters are held same, except SNR which is changed to 0 dB. Fig.6 shows the comparison plot for this case. It is observed that the proposed method gives sharp peaks only in the true DOAs where as the rest of the methods show spurious peaks in false directions.
To compare the root mean square error (RMSE) results for various state-of-the-art subspace based methods for coherent sources [4, 22], we simulated a 2 source-7 sensor set up (fig.7). It is observed that proposed algorithm has much better accuracy in highly noisy environment than the compared methods.

The proposed algorithm works well with one snapshot whereas the MUSIC algorithm and Capon algorithm require at least 100 snapshots to show acceptable results. To verify this, we simulated a RMSE plot against snapshots varying from 1 - 250 and other parameters set as $N = 15$, $M = 3$ and SNR = 10 dB. It can be seen that the the proposed algorithm gives consistently accurate result where as the compared algorithms take at least 100 and above number of snapshots to have the same level of accuracy.
In the next simulation we show how the performance of the OMP algorithm is improved with our proposed MV criterion. For this, we simulated a 3 source-10 sensor in an environment of SNR = 10 db to show the consistency of both the algorithms (fig. 9) for 10 Monte Carlo trials. The sources were selected as $\theta_1 = 60^\circ$, $\theta_2 = 40^\circ$ and $\theta_3 = 0^\circ$ with respective normalised frequencies as $\pi/4$, $\pi/7$, $\pi/8$ and $\pi/6$ for all. It shows for all iterations, the proposed algorithm selects the true DOA correctly in all iterations. For example, in the trial number 1 in (a), only $\theta_1$ is detected, in trial number 5, $\theta_2$ and $\theta_3$ are detected and in trial number 10, only $\theta_3$ is detected. In case of (b) i.e. proposed algorithm, all sources $\theta_1$, $\theta_2$ and $\theta_3$ are detected in all trials from 1 - 10. This proves the claim that the proposed algorithm has overcome the deficiency of the OMP algorithm and is consistent in results.

Validating the Distance estimation method: We simulated a scenario to check the validity of estimating the distance using the proposed distance estimation algorithm with synthetic data.
In Table 1, the true value of distance along with the DOA and the estimated values of distance is shown percentage errors in their estimation for various SNR levels. The percentage error is calculated as

\[
\text{Error(\%)} = \frac{|\text{True Distance} - \text{Est. Distance}|}{\text{True Distance}} \times 100
\]  \hspace{1cm} (16)

It is clear from the table that, with the increase in SNR level the percentage error in estimation changes very little.

| S. No. | SNR (dB) | True Distance (true DOA) (m) | Est. Distance (m) | Error (%) |
|--------|----------|-----------------------------|------------------|-----------|
| 1      | -5       | 500 (45°)                   | 500.083          | 0.0166    |
| 2      | 0        | 4500 (45°)                  | 4499.821         | 0.00397   |
| 3      | 10       | 2000 (60°)                  | 1999.991         | 0.00045   |

Table 4: Results on Source Distance Estimation

5. Conclusion

In this paper, a robust CS based sparse recovery algorithm for DOA estimation problem is introduced. The key idea lies in the concept of relaxation of the RIP constraint on the measurement matrix. The application of OMP algorithm to tackle DOA of sources gave inconsistent results. Thus, the proposed algorithm was developed which exploits a maximum variance criterion for the partial support recovery. Then it is shown how a local OMP algorithm will detect all sources with reduced sparsity. Also, a simple method is presented to find the distance of a source location from the reference point of the sensor array. CS has no strict requirement for the position of sensors (flexible for actual sites), hence this algorithm can be very much useful for practical applications.

A. Appendix A

Following is the proof of the Theorem 3.1:

\textbf{Proof.} If } t_k \text{ is maximally correlated index of the } i_{th} \text{ column of } \Phi, \varsigma_i \text{ with the residual } r_{k-1}. \text{ With } r_{k-1} = y, \text{ clearly}

\[ t_k = \arg \max_i |\langle \varsigma_i, y \rangle| \]  \hspace{1cm} (17)

From (17), for all } t_k \text{ we have

\[ |\langle \varsigma_i, y \rangle| = \|\Phi_i^T y\|_\infty \]  \hspace{1cm} (18)

\[ \geq \frac{1}{\sqrt{k}} \|\Phi_i^T y\|_2 \quad \text{(Using norm inequality)} \]  \hspace{1cm} (19)
But $y = \Phi_{\Gamma}x_{\Gamma}$, this gives

\[
\therefore |\langle \varsigma_i, y \rangle| \geq \frac{1}{\sqrt{k}} \|\Phi_{\Gamma}^T\Phi_{\Gamma}x_{\Gamma}\|_2 \quad (20)
\]

\[
\geq \frac{1}{\sqrt{k}}(1 - \delta_k)\|x_{\Gamma}\|_2 \quad (21)
\]

Now suppose $t_k \notin \Gamma$. Then

\[
|\langle \varsigma_i, y \rangle| = \|\varsigma_i^T\Phi_{\Gamma}^T\Phi_{\Gamma}x_{\Gamma}\|_2 \quad (22)
\]

\[
\leq \delta_{k+1}\|x_{\Gamma}\|_2 \quad \text{(from Lemma 3 of [20])} \quad (23)
\]

So, to avoid a case where $t_k \notin \Gamma$, we should have

\[
\frac{1}{\sqrt{k}}(1 - \delta_k)\|x_{\Gamma}\|_2 > \delta_{k+1}\|x_{\Gamma}\|_2 \quad (24)
\]

\[
\implies \sqrt{k}\delta_{k+1} + \delta_k < 1 \quad (25)
\]

\[
\implies \sqrt{k}\delta_{k+1} + \delta_{k+1} < 1 \quad (\because \delta_k \geq \delta_{k+1}) \quad (26)
\]

\[
\implies \delta_{k+1} < \frac{1}{\sqrt{k} + 1} \quad (27)
\]

Hence, it is proved that with the relaxed bound, the local OMP algorithm will detect the complete support set.

\[\square\]

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**B.1. Journal articles**

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