SLIDING SHILNIKOV CONNECTION AND CHAOS IN PREY SWITCHING MODEL

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Abstract. Recently, a piecewise smooth system was derived as a model of a 1 predator-2 prey interaction where the predator feeds adaptively on its preferred prey and an alternative prey. In such a model, strong evidence of chaotic behavior was numerically found. Here, we revisit this model and prove the existence of a Shilnikov sliding connection when the parameters are taken in a codimension one submanifold of the parameter space. As a consequence of this connection, we conclude that the model behaves chaotically for an open region of the parameter space.

1. Introduction

In ecology, prey switching refers to a predator’s adaptive change of habitat or diet in response to prey abundance and has been observed in many predator species [2,10,11,31]. Roughly speaking, a predator is said to be switching between prey species if the number of attacks upon a species is disproportionately large when the species is abundant relative to other prey, and disproportionately small when the species is relatively rare [20,31]. Switching in predators is often related to stabilizing mechanisms of prey populations and is a possible explanation for coexistence [11,16]. Intuitively, predators tend to feed most heavily upon the most abundant prey species. As this prey species declines the predator “switches” the major fraction of its attacks to another prey, which has become the most abundant. In this way, no prey population is drastically reduced nor becomes very abundant [20].

Using the principle of optimal foraging [27], Piltz et al. [23] have modeled a 1 predator-2 prey interaction as a piecewise dynamical system of kind

\[ \dot{x} = Z(x) = F(x) + \text{sign}(h(x))G(x), \]

where \( x \in \mathbb{R}^3 \geq 0 \), \( h : \mathbb{R}^3 \to \mathbb{R} \) is linear, and \( F, G \) are smooth functions. Here, \( \Sigma = h^{-1}(0) \) is called discontinuous manifold. It is assumed that the predator instantaneously switches its food preference according to the availability of preys in the environment. This sudden change in the food preference of the
The notion of trajectories for piecewise smooth systems of kind (1) was stated by Filippov in [8]. Nowadays, the differential equation (1) is called Filippov system. It is worthwhile to mention the existence of a vast literature on Filippov systems modeling real phenomena in many other areas of applied science. For instance, see [24] for applications in control theory, [3, 6, 19] in mechanical models, [4, 18] in electrical circuits, [5, 15] in relay systems, among others. In all those applications the discontinuity is due to an abrupt change in the system when some barrier is broken.

Piltz et al. [23] found evidence that for a given choice of parameters their model exhibits chaotic behavior. Roughly speaking, chaotic behavior can be understood as the existence of an invariant set for which the dynamics is transitive, sensitive to initial conditions, and have dense periodic points (see Definitions 2, 3, and 4 of Section 2). For the biological model in question, chaotic behavior means that for a given initial condition of population density of the species involved one cannot estimate (even vaguely) its long-term evolution. Hence, knowledge of chaotic behavior in a specific ecological model is of major importance, particularly, for experimentalists who need to be aware of potential implications of chaos for long-term predictions, and the fact that sustained “irregular” fluctuations may be due to chaos [13].

For smooth dynamical systems, chaotic behavior may be tracked by studying the existence of objects previously known as being chaotic. This is the case of a Shilnikov homoclinic orbit, which is a trajectory connecting a hyperbolic saddle–focus equilibrium to itself, bi–asymptotically (see, for instance, [25, 26, 30]).

In the Filippov context, pseudo–equilibria are special points on the discontinuous manifold that must be distinguished and treated as typical singularities (see, for instance, [8, 12]). These singularities give rise to the definition of the sliding Shilnikov orbit (see Definition 1), which is a trajectory in the Filippov sense connecting a hyperbolic pseudo saddle–focus to itself in an infinity time at least by one side, forward or backward. This object has been first considered in [22], where some of their properties were studied. In particular, it was proved the existence of infinitely many sliding periodic solutions near a sliding Shilnikov orbit.

In [21], using the well known theory of Bernoulli shifts, it was provided a full topological and ergodic description of the dynamics of Filippov systems near a sliding Shilnikov orbit \( \Gamma \). In particular, it was established the existence of a set \( \Lambda \subset \partial M^s \) such that the restriction to \( \Lambda \) of the first return map \( \pi \), defined near \( \Gamma \), is topologically conjugate to a Bernoulli shift with infinite topological entropy. This ensures \( \pi \), consequently the flow, to be as much chaotic as one wishes. In particular, given any natural number \( m \geq 1 \) one can find infinitely
Figure 1. The point $p_0 \in M^s$ is a hyperbolic pseudo saddle–focus. The trajectory $\Gamma$, called Shilnikov sliding orbit, connects $p_0$ to itself passing through the point $q_0 \in \partial M^s$. Notice that the flow leaving $q_0$ reaches the point $p_0$ in a finite positive time, and approaches backward to $p_0$, asymptotically.

many periodic points of the first return map with period $m$ and, consequently, infinitely many closed orbits near $\Gamma$.

Our approach consists in finding a set of parameters for which the considered model admits a sliding Shilnikov orbit. This ensures, analytically, that for parameters taken in a neighborhood of this set the model behaves chaotically.

2. Preliminary concepts and known results

This section is devoted to present the basic theory on Piecewise Smooth Vector Fields (PSVFs, for short). We shall also define the concept of Sliding Shilnikov orbits and state some known results regarding the chaotic behavior near a sliding Shilnikov orbit.

A PSVF on $\mathbb{R}^3$ is a pair of $C^r$-vector fields $X$ and $Y$, where $X$ and $Y$ are restricted to regions of $\mathbb{R}^3$ separated by a smooth codimension one manifold $\Sigma$. The discontinuous manifold $\Sigma$ is obtained considering $\Sigma = h^{-1}(0)$, where $h$ a differentiable function having 0 as a regular value. So, a PSVF is given by:

$$Z(x) = \begin{cases} X(x), & \text{if } h(x) \geq 0, \\ Y(x), & \text{if } h(x) \leq 0, \end{cases} \quad (2)$$

where $x \in \mathbb{R}^3$. As usual, system (2) is denoted by $Z = (X, Y)$.

The points on $\Sigma$ where both vectors fields $X$ and $Y$ simultaneously point outward or inward from $\Sigma$ define, respectively, the escaping $\Sigma^e$ or sliding $\Sigma^s$ regions, and the interior of its complement in $\Sigma$ defines the crossing region $\Sigma^c$. The complementary of the union of those regions is constituted by the tangency
points between $X$ or $Y$ with $\Sigma$. When the contact between the trajectories of $X$ with $\Sigma$ is quadratic (resp., cubic) on $x \in \Sigma$ we say that $x$ is a fold (resp., cusp) singularity.

For practical purposes, in order to calculate the previously defined intrinsic objects, it is convenient to consider the Lie derivatives, given by $Xh(x) = \langle \nabla h(x), X(x) \rangle$ and, for $i \geq 2$, $X^i h(x) = \langle \nabla X^{i-1} h(x), X(x) \rangle$, where $\langle \cdot, \cdot \rangle$ is the usual inner product in $\mathbb{R}^3$. The points in $\Sigma^c$ satisfy $Xh(x) \cdot Yh(x) > 0$. The points in $\Sigma^s$ (resp. $\Sigma^u$) satisfy $Xh(x) < 0$ and $Yh(x) > 0$ (resp. $Xh(x) > 0$ and $Yh(x) < 0$). Finally, the tangency points of $X$ (resp. $Y$) satisfy $Xh(x) = 0$ (resp. $Yh(x) = 0$).

Observe that (2) is multivalued on $\Sigma$. In order to fix some ambiguity, we consider the Filippov’s convention (see [8]) and define a new vector field on $\Sigma^s$. This new vector field, called sliding vector field, is tangent to $\Sigma$ and is given by

$$Z^s(x) = \frac{Yh(x)X(x) - Xh(x)Y(x)}{Yh(x) - Xh(x)} \text{ for all } x \in \Sigma^s.$$

In this scenario, the trajectories $\Gamma_Z(t, q)$ of $Z$ are considered as a concatenation of trajectories of $X$, $Y$ and $Z^s$.

The points $x \in \Sigma^s$ such that $Z^s(x) = 0$ are called pseudo equilibrium of $Z$. A pseudo-equilibrium is called hyperbolic pseudo-equilibrium when it is a hyperbolic critical point of $Z^s$. Particularly, if $x^s \in \Sigma^s$ (resp. $x^s \in \Sigma^c$) is an unstable (resp. stable) hyperbolic focus of $Z^s$, then we call $x^s$ a hyperbolic saddle-focus pseudo-equilibrium or just hyperbolic pseudo saddle-focus.

**Definition 1.** Let $Z = (X, Y)$ be a piecewise continuous vector field having a hyperbolic pseudo saddle-focus $p \in \Sigma^s$ (resp. $p \in \Sigma^c$), and let $q \in \partial \Sigma^s$ (resp. $q \in \partial \Sigma^c$) be a visible fold point of the vector field $X$ such that

1. the orbit passing through $q$ following the sliding vector field $Z^s$ converges to $p$ backward in time (resp. forward in time);
2. the orbit starting at $q$ and following the vector field $X$ spends a time $t_0 > 0$ (resp. $t_0 < 0$) to reach $p$.

So, through $p$ and $q$ a sliding loop $\Gamma$ is easily characterized. We call $\Gamma$ a sliding Shilnikov orbit (see Figure 7).

**Definition 2.** System (2) is topologically transitive on an invariant set $W$ if for every pair of nonempty open sets $U$ and $V$ in $W$ there exist $x \in U$, a trajectory of $Z$, $\Gamma_Z(t, x)$, and $t_0 > 0$ such that $\Gamma_Z(t_0, x) \in V$.

**Definition 3.** System (2) exhibits sensitive dependence on a compact invariant set $W$ if there is a fixed $r > 0$ satisfying $r < \text{diam}(W)/2$ such that for each $x \in W$ and $\varepsilon > 0$ there exist a $y \in B_\varepsilon(x) \cap W$ and positive global
trajectories $\Gamma^+_x$ and $\Gamma^+_y$ passing through $x$ and $y$, respectively, satisfying
\[ d(\Gamma^+_x(t_0), \Gamma^+_y(t_0)) > r, \]
where $t_0 \in \mathbb{R}$ is positive, $d$ is the Euclidean distance and $\text{diam}(W)$ is the diameter of $W$, i.e., the biggest distance between two elements of $W$.

**Definition 4.** System (2) is chaotic on a compact invariant set $W$ if it is topologically transitive, exhibits sensitive dependence on $W$, and have dense periodic orbits in $W$.

Notice that the previous definitions coincide with those used for smooth flows.

The next theorem ensures that a PSVF presenting a sliding Shilnikov connection is, in fact, chaotic. The proof of Theorem 1 is performed in [21].

**Theorem 1** ([21]). Let $Z_0 = (X_0, Y_0)$ be given by (2). Assume that $Z_0$ admits a sliding Shilnikov orbit. Then, there exists a neighborhood $U \subset \chi^r$ of $Z_0$ such that each $Z \in U$ admits an invariant compact set $\Lambda_Z$ on which $Z$ is chaotic.

### 3. Main results

Ciliates are eukaryotic single cells that belong to the protist kingdom. They occur in aquatic environment and feed on small phytoplankton, constituting a relevant link between levels of marine and freshwater food webs (see [28]). Coexistence of species in a shared environment may arise from ecological trade-offs (see [17]), which appear in many situations in ecology. Lake Constance is a freshwater lake situated on the German-Swiss-Austrian border that has been under scientific investigation for decades and a substantial amount of data on the biomass of several phytoplankton and zooplankton species is available (see [1] [28] [29]). Based on the available data, Piltz et al. [23] derive the following piecewise smooth model for a 1 predator-2 prey interaction where the predator feeds adaptively on its preferred prey and an alternative prey:

\[
(\dot{p}_1, \dot{p}_2, \dot{P})^T = \begin{cases} 
(r_1 - \beta_1 P)p_1 \\
r_2p_2 \\
(eq_1\beta_1 p_1 - m)P 
\end{cases} & \text{if } H(p_1, p_2, P) > 0, \\
r_1p_1 \\
(r_2 - \beta_2 P)p_2 \\
(eq_2\beta_2 p_2 - m)P 
\end{cases} & \text{if } H(p_1, p_2, P) < 0,
\]

where $(p_1, p_2, P) \in \mathbb{R}^3_{\geq 0}$ and $H(p_1, p_2, P) = \beta_1 p_1 - q_1 \beta_2 p_2$. The plane $S = H^{-1}(0)$ is the discontinuous manifold for system (4). The variables of the
model (4), $P, p_1,$ and $p_2,$ represent the density of the predator population, preferred prey, and alternative prey, respectively. Regarding the parameters, $q_1 \geq 0$ represents the desire to consume the preferred prey, $a_q > 0$ is the slope of the preference trade-off, and $q_2 \geq 0$ is the desire to consume the alternative prey. It is assumed that the intercept of the preference trade-off $b_q = q_2 - a_q q_1$ satisfies $b_q \geq 0$. In addition, $e > 0$ is the proportion of predation that goes into predator growth, $\beta_1 > 0$ and $\beta_2 > 0$ are, respectively, the death rates of the preferred and alternative prey due to predation. Finally, $m > 0$ is the predator per capita death rate per day and $r_1 > r_2 > 0$ are the per capita growth rates of the preferred and alternative prey, respectively.

The above constraints imply that the parameters of the differential system (4) lie in a subset of the Euclidian space $\mathbb{R}^9$, namely $\eta = (r_1, r_2, a_q, q_1, q_2, \beta_1, \beta_2, e, m) \in M = R \times Q \times \mathbb{R}^4 \subset \mathbb{R}^9,$ where $R = \{(r_1, r_2) \in \mathbb{R}_0^2 : r_1 > r_2\}$ and $Q = \{(a_q, q_1, q_2) \in \mathbb{R}_{>0} \times \mathbb{R}_{>0}^2 : q_2 \geq a_q q_1\}.$ The set $M$ is called space of parameters which is a 9-dimensional submanifold of $\mathbb{R}^9$ with boundary and corner.

In what follows, we state the main result of this paper.

**Theorem A.** There exists a codimension one submanifold $N$ of $M$ such that the differential system (4) possesses a sliding Shilnikov orbit whenever $\eta \in N.$ Moreover, there exists a neighborhood $U \subset M$ of $N$ such that the differential system (4) behaves chaotically whenever $\eta \in U.$

4. **Proof of the main result**

Consider the differential system (4). In order to eliminate the dependence of the switching manifold on the parameters, let us consider the change of variables $x = p_1/\beta_1,$ $y = p_2/(a_q \beta_2)$ and $z = P/\beta_1.$ In these new variables, system (4) reads

$$
\begin{align*}
X(x, y, z) &= \begin{pmatrix}
(r_1 - z)x \\
2y \\
(eq_1 x - m) z
\end{pmatrix} \\
Y(x, y, z) &= \begin{pmatrix}
(r_1 x) \\
(r_2 - \beta_2 \beta_1 z) y \\
(eq_2 \beta_1 a_q y - m) z
\end{pmatrix}
\end{align*}
$$

if $h(x, y, z) > 0;

if $h(x, y, z) < 0,

\begin{equation}
\left(\dot{x}, \dot{y}, \dot{z}\right)^T = \begin{cases}
(r_1 - z)x \\
2y \\
(eq_1 x - m) z
\end{cases} \\
\begin{cases}
(r_1 x) \\
(r_2 - \beta_2 \beta_1 z) y \\
(eq_2 \beta_1 a_q y - m) z
\end{cases}
\end{equation}

\text{ if } h(x, y, z) > 0;

\text{ if } h(x, y, z) < 0,
where \((x, y, z) \in \mathbb{R}_\geq 0^3\) and \(h(x, y, z) = x - y\). Now, the switching manifold is given by \(\Sigma = h^{-1}(0) = \{(x, x, z) : x \geq 0, z \geq 0\}\).

4.1. Dynamics of \(X\) and \(Y\) and their contacts with \(\Sigma\). Notice that the plane \(\Pi_y = \{y = 0\}\) is invariant through the flow of \(X\). The restriction of \(X\) onto the plane \(\Pi_y\) reads

\[
X(x, z) = \begin{pmatrix}
(r_1 - z)x \\
(eq_1x - m)z
\end{pmatrix}.
\]

Moreover, the projection of each orbit of \(X\) into the plane \(\Pi_y\) coincides with an orbit of \(\bar{X}\). Indeed, the subsystems \((\dot{x}, \dot{z})\) and \(\dot{y}\) are uncoupled.

The equilibria of \(X\) are \(E_1 = (0, 0)\) and \(E_2 = (m/(eq_1), r_1)\). The equilibrium \(E_1\) is a saddle with eigenvectors \((1, 0)\) and \((0, 1)\) associated to the eigenvalues \(r_1\) and \(-m\), respectively. The equilibrium \(E_2\) has pure imaginary eigenvalues, namely \(\pm i\sqrt{mr_1}\). Furthermore, \(\bar{X}\) is a Lotka-Volterra system which has the following first integral:

\[
F(x, z) = -m - r_1 + eq_1x + z - m\log\left(\frac{eq_1x}{m}\right) - r_1\log\left(\frac{z}{r_1}\right).
\]

It implies that the equilibrium \(E_2\) is a center (see Figure 2). Furthermore,

\[
y(t) = y_0\exp(r_2t),
\]

since

The dynamics on the \(y\)-direction is unbounded increasing and the \(X\)-trajectories spiral from \(\Pi_y\) toward \(\Sigma\), crossing \(\Sigma\). The trajectories of \(X\), on the domain

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**Figure 2.** Projection of \(X\) into the plane \(\Pi_y\).
$\mathbb{R}^3_{\geq 0}$, lie on cylinders around the straight line $\ell = \{(m/(eq_1), y, r_1) \mid y \geq 0\}$. See Figure 3.

The next step is to study the contacts of the vectors fields $X$ and $Y$ with the switching manifold $\Sigma$. Let $p = (x, x, z) \in \Sigma$, computing the Lie derivatives $Xh(p)$ and $Yh(p)$ we get:

$$Xh(p) = (r_1 - r_2 - z)x \quad \text{and} \quad Yh(p) = \left(r_1 - r_2 + \frac{\beta_2 z}{\beta_1}\right)x.$$  

Solving the equation $Xh(p) = 0$ we conclude that the contacts between the vector field $X$ and the switching manifold $\Sigma$ occur at $S_X = S_X^1 \cup S_X^2$, where $S_X^1 = \{(0, 0, z) : z > 0\}$ and $S_X^2 = \{(x, x, r_1 - r_2) : x > 0\}$. Analogously, solving the equation $Yh(p) = 0$, we conclude that the contacts between $Y$ and $\Sigma$ occurs at $S_Y = S_Y^1 \cup S_Y^2$ where $S_Y^1 = S_X^1$ and $S_Y^2 = \{(x, x, -r_1 + r_2)\}$. The switching manifold $\Sigma$ is then partitioned in two open regions, namely sliding region $\Sigma^s = \{(x, x, z) \in \Sigma \mid z > r_1 - r_2\}$ and crossing region $\Sigma^c = \{(x, x, z) \in \Sigma \mid 0 < z < r_1 - r_2\}$.

Notice that the tangency line $S_X^2$ is the boundary of the sliding region $\Sigma^s$. From Definition (1), $S_X^2$ will play an important role in finding a sliding Shilnikov orbit. In order to determine the kind of contact between $X$ and $\Sigma$ occurring on $S_X^2$, we compute the second Lie derivative $X^2h$. Accordingly, let $\overline{p} = (x, x, r_1 - r_2) \in S_X^2$, so

$$X^2h(\overline{p}) = (r_1 - r_2)(m - eq_1)x.$$
Proof. Take \((0, 0, r_1 - r_2)\) and \(c = \left(\frac{m}{eq_1}, \frac{m}{eq_1}, r_1 - r_2\right)\).

Moreover, we have that \(X^2h(x, x, r_1 - r_2) > 0\) for \(0 < y < m/(eq_1)\). So, \(S_X^e = \{(x, x, r_1 - r_2) \in S_X^c | 0 < x < m/(eq_1)\}\) is a curve of visible fold points of \(X\) and, therefore, the local trajectories of \(X\) remain at the region where \(X\) is defined (i.e. \(h(x, y, z) > 0\)), before and after the tangential contact with \(S_X^e\).

It is worthwhile to mention that \(c \in \mathbb{R}^3_0\) is a contact of cusp type.

In order to state the main result of this subsection (Lemma 1), we introduce the following new parameters

\[
\phi = r_1 - r_2 \quad \text{and} \quad \tau = \frac{m}{eq_1}.
\]

Solving the above relations for \(r_1\) and \(m\) we get \(S_X^2 = \{(x, x, \phi) \in S_X^c | y \geq 0\}\) and \(c = (\tau, \tau, \phi)\).

Lemma 1. For each \(x_0 \in (0, \tau)\), the forward trajectory of \(X\) passing through \((x_0, x_0, r_1 - r_2)\) intersects transversally the switching manifold \(\Sigma\) at \(\mu(x_0) = (u(x_0), u(x_0), v(x_0))\). In other words, the saturation of \(S_X^e\) through the forward flow of \(X\) intersects \(\Sigma\) transversally in the curve \(\{\mu(x_0) : 0 < x_0 < \tau\}\).

Moreover, the following statements hold:

i) for \(x_0 < \tau\) sufficiently close to \(\tau\) we have

\[
\begin{align*}
u(x_0) &= \tau - 2(x_0 - \tau) + O(x_0 - \tau)^2, \\
v(x_0) &= r_1 - r_2 + O(x_0 - \tau)^2;
\end{align*}
\]

ii) and given \(x_0 \in (0, \tau)\), for \(r_2 > 0\) sufficiently small we have

\[
\begin{align*}
u(x_0) &= x_0 + O(r_2), \\
v(x_0) &= r_1 + \sqrt{2r_1 T(x_0)(m - eq_1 x_0)} \sqrt{r_2} + O(r_2^{3/2}).
\end{align*}
\]

Proof. Take \((x_0, x_0, \phi) \in S_X^e\), such that \(0 < x_0 < \tau\). The parameter \(r_2\) will play an important role in this proof, so we shall make it explicit in what follows. Let us consider \(\psi(t, x_0; r_2) = (x(t, x_0; r_2), y(t, x_0; r_2), z(t, x_0; r_2))\) the solution of \(X\) such that \(\psi(0, x_0; r_2) = (x_0, x_0, \phi)\). Notice that

\[
\begin{align*}
\frac{\partial x}{\partial t}(0, x_0; r_2) &= r_2 x_0, \\
\frac{\partial^2 x}{\partial t^2}(0, x_0; r_2) &= r_2^2 x_0 + eq_1 x_0(\tau - x_0)\phi, \\
\frac{\partial y}{\partial t}(0, x_0; r_2) &= r_2 x_0, \quad \text{and} \quad \frac{\partial^2 y}{\partial t^2}(0, x_0; r_2) &= r_2^2 x_0.
\end{align*}
\]
In order to apply the implicit function theorem, we compute
\[
\frac{\partial x}{\partial t}(0, x_0; r_2) = \frac{\partial y}{\partial t}(0, x_0; r_2) = \frac{\partial^2 x}{\partial t^2}(0, x_0; r_2) > 0.
\]
we conclude that the curve \( y(t, x_0; r_2) \) starts below from the curve \( x(t, x_0; r_2) \), that is, \( y(t, x_0; r_2) < x(t, x_0; r_2) \) for \( t > 0 \) sufficiently small. However, \( x(t, x_0; r_2) \) is bounded and \( y(t, x_0; r_2) \) is unbounded increasing, therefore there exists a first positive time \( t_1(x_0; r_2) > 0 \) such that
\[
x(t_1(x_0; r_2), x_0; r_2) = y(t_1(x_0; r_2), x_0; r_2) = x_0 \exp(r_2 t_1(x_0; r_2)).
\]
It means that the trajectory of \( X \) passing tangentially by each \( (x_0, x_0, \phi) \in S^v_x \), for \( 0 < x_0 < \tau \), reaches transversally the switching manifold \( \Sigma \) at \( \psi(t x_0), x_0; r_2 \). Accordingly, for \( 0 < x_0 < \tau \), we define \( \mu(x_0) = \psi(t_1(x_0; r_2), x_0; r_2) \),
\[
\begin{align*}
    u(x_0) &= x(t_1(x_0; r_2), x_0; r_2) = x_0 \exp(r_2 t_1(x_0; r_2)), \\
v(x_0) &= z(t_1(x_0; r_2), x_0; r_2).
\end{align*}
\]
This concludes the proof of the first part of the lemma.

To see the “moreover” part we first prove that the Taylor series of \( u(x_0) \) around \( x_0 = \tau \) reads
\[
u(x_0) = \tau - 2(x_0 - \tau) + O_2(x_0 - \tau)^2.
\]
Notice that the difference \( x(t, x_0; r_2) - x_0 \exp(r_2 t) \) around \( t = 0 \) reads
\[
x(t, x_0; r_2) - x_0 \exp(r_2 t) = -\frac{eq_1 \phi x_0(x_0 - \tau)}{2} t^2 - \frac{eq_1 \phi x_0(r_2(4x_0 - 3\tau) + eq_1(x_0 - \tau)^2)}{6} t^3 + O_4(t).
\]
Therefore, the function
\[
\Delta(t, x_0) := \frac{x(t, x_0; r_2) - x_0 \exp(r_2 t)}{t^2}
\]
is well defined and, around \( t = 0 \), reads
\[
\Delta(t, x_0) = -\frac{eq_1 \phi x_0(x_0 - \tau)}{2} - \frac{eq_1 \phi x_0(r_2(4x_0 - 3\tau) + eq_1(x_0 - \tau)^2)}{6} t + O_2(t).
\]
In order to apply the implicit function theorem, we compute
\[
\Delta(0, \tau) = 0, \quad \frac{\partial \Delta}{\partial t}(0, \tau) = -\frac{eq_1 r_2 \tau^2 \phi}{6} \neq 0, \quad \text{and} \quad \frac{\partial \Delta}{\partial x_0}(0, \tau) = -\frac{eq_1 \tau \phi}{2}.
\]
Therefore, we find a unique function \( t_2(x_0) \) such that

\[
\begin{align*}
t_2(\tau) &= 0, \\
\frac{d^2}{dt^2}(0, \tau) &= -\frac{3}{r_2^\tau}.
\end{align*}
\] (15)

From the uniqueness of \( t_2 \) we conclude that, for \( x_0 \) sufficiently close to \( \tau \), \( t_1(x_0; r_2) = t_2(x_0) \). So, using (15), \( u(x_0) = x_0 \exp(r_2 t_1(x_0)) \) can be expanded around \( x_0 = \tau \) in order to get (14).

Finally, we shall prove that given \( \rho^* \in (0, \tau) \) there exists a neighborhood \( U \) of \( x_0^* \) and \( r_2^* > 0 \) such that \( u(x_0) < \tau \) and \( v(x_0) > r_1 \) for every \( (x_0, r_2) \in U \times (0, r_2^*) \).

Indeed, consider the function

\[
\delta(t, r_2) = x(t, x_0; r_2) - y(t, x_0; r_2) = x(t, x_0; r_2) - x_0e^{r_2 t}.
\]

We know that \( (x(t, x_0; r_2), z(t, x_0; r_2)) \) is periodic in the variable \( t \). In fact, this is the solution of the Lotka-Volterra system (6) with initial condition \( (x_0, 1 - r_2) \) and, therefore, satisfies (7)

\[
F(x(t, x_0; r_2), z(t, x_0; r_2)) = F(x_0, 1 - r_2),
\] (16)

for every \( r_1 > r_2 > 0 \), \( 0 < x_0 < \tau \), and \( t \) on its interval of definition. Thus, for \( r_2 = 0 \), denote by \( T(x_0) > 0 \) the period of the solution \( (x(t, x_0; 0), z(t, x_0; 0)) \), that is, \( (x(T(x_0), x_0; 0), z(T(x_0), x_0; 0)) = (x_0, 1) \). Consequently, \( \delta(T(x_0), 0) = 0 \). We shall see that there is a saddle–node bifurcation occurring at \( t = T(x_0) \) for the critical value of the parameter \( r_2 = 0 \). Computing the derivative in the variable \( r_2 \) of (16) at \( t = T(x_0) \) and \( r_2 = 0 \) we get

\[
\frac{\partial x}{\partial r_2}(t(x_0), x_0, 0) = 0.
\]

So, we get

\[
\frac{\partial \delta}{\partial t}(t(x_0), 0) = 0, \quad \frac{\partial^2 \delta}{\partial t^2}(t(x_0), 0) = r_1(m - eq_1 x_0)x_0 > 0,
\]

and

\[
\frac{\partial \delta}{\partial r_2}(t(x_0), 0) = -x_0 T(x_0) < 0.
\] (17)

This implies the existence of a saddle–node bifurcation. In order to conclude this proof, we shall explicitly compute the solutions bifurcating from \( t = T(x_0) \). From (17), applying the Implicit Function Theorem, we get the existence of neighborhoods \( I_1 \) and \( V_1 \) of \( T(x_0) \) and \( 0 \), respectively, and a unique differentiable function \( \rho : I_1 \to V_1 \) such that \( \delta(t, \rho(t)) = 0 \) for every \( t \in I_1 \). Moreover,

\[
\rho(T(x_0)) = \rho'(T(x_0)) = 0 \quad \text{and} \quad \rho''(T(x_0)) = \frac{r_1(m - eq_1 x_0)}{2T(x_0)}.
\]
Notice that we are taking
\begin{equation}
(18) \quad r_2 = \rho(t) = \frac{r_1(m - eq_1x_0)}{2T(x_0)}(t - T(x_0))^2 + \mathcal{O}(t - T(x_0))^3.
\end{equation}

Proceeding with the change \( s = (t - T(x_0))^2 \), equation (18) is equivalent to
\[ r_2 = \frac{r_1(m - eq_1x_0)}{2T(x_0)}s + \mathcal{O}(s^{3/2}). \]

It is easy to see that the above equation can be inverted using the Inverse Function Theorem. So, we get the existence of neighborhoods \( U_2 \) and \( I_2 \) of 0, and a unique differentiable function \( \sigma : U_2 \rightarrow I_2 \) such that
\[ s = \sigma(r_2), \quad \sigma(0) = 0, \quad \sigma'(0) = \frac{2T(x_0)}{r_1(m - eq_1x_0)} > 0. \]

Going back through the change \( s = (t - T(x_0))^2 \) we get two distinct positive times \( t = T(x_0) \pm \sqrt{\sigma(r_2)} \) bifurcating from \( t = T(x_0) \). Since \( t_1(x_0; r_2) \) is the first return time we conclude that
\[ t_1(x_0; r_2) = T(x_0) - \sqrt{\sigma(r_2)} = T(x_0) + \sqrt{\frac{2T(x_0)}{r_1(m - eq_1x_0)}} \sqrt{r_2} + \mathcal{O}(r_2^{3/2}). \]

Finally, from (13) we compute
\begin{align}
(19) \quad u(x_0) &= x_0 + \mathcal{O}(r_2), \\
v(x_0) &= r_1 + \sqrt{2r_1T(x_0)(m - eq_1x_0)} \sqrt{r_2} + \mathcal{O}(r_2^{3/2}).
\end{align}

This concludes the proof. \( \square \)

**Lemma 2.** There exist \( a, b, c, \) and \( d \), with \( 0 < a < b < \tau \) and \( 0 < c < d \), such that \( 0 < u(x_0) < \tau \) and \( v(x_0) > r_1 \) for every \( (x_0, r_2) \in [a, b] \times [c, d] \). Moreover, for \( r_2 \in [c, d] \), \( \mu(x_0) \) is differentiable on \( [a, b] \) and \( u'(x_0)^2 + v'(x_0)^2 \neq 0 \) for every \( (x_0, r_2) \in [a, b] \times [c, d] \).

**Proof.** From (14) we have that \( u(x_0) > \tau \) for \( x_0 \) sufficiently close to \( \tau \), and from (19) we have that for a fixed \( x_0 \in (0, \tau) \) there exists \( r_2^* > 0 \) such that \( u(x_0) > \tau \) for every \( r_2 \in (0, r_2^*) \). Therefore, for \( r_2 = r_2^* < r_2^* \) there exists \( x_1^* \in (0, \tau) \) such that \( u(x_1^*) = \tau \) and \( u(x_0) < \tau \) for \( x_0 < x_1^* \) sufficiently close to \( x_1^* \). Moreover, \( v(x_1) > r_1 \) and, consequently, \( v(x_0) > r_1 \) for \( x_0 < x_1^* \) sufficiently close to \( x_1^* \) because \( (\tau, r_1) \) is a critical point for the first integral (7). Hence, take \( x_{1*} < x_1^* \) such that \( v(x_{1*}) > r_1 \) and \( u(x_{1*}) < \tau \). So, from the continuous dependence of the solutions on the initial conditions and parameters, we first get the existence of \( a, b, c, \) and \( d \), with \( 0 < a < x_{1*} < b < \tau \) and \( 0 < c < r_{2*} < d \) such that \( 0 < u(x_0) < \tau \) and \( v(x_0) > r_1 \) for every \( (x_0, r_2) \in [a, b] \times [c, d] \).
Now, in order to get the differentiability of $\mu$, define
\[ \Gamma(t, x_0, r_2) = x(t, x_0; r_2) - x_0 \exp(r_2 t). \]
From the proof of Lemma 1, for each $(x_0, r_2) \in [a, b] \times [c, d]$ we got the existence of $t_1(x_0; r_2) > 0$ such that $\Gamma(t_1(x_0; r_2), x_0, r_2) = 0$. Since
\[ \frac{\partial \Gamma}{\partial t}(t_1(x_0; r_2), x_0, r_2) = (r_1 - v(x_0))u(x_0) \neq 0, \]
we obtain, from the implicit function theorem, the existence of a unique differentiable function $t_2(x_0, r_0)$, defined in a neighborhood of $(x_0, r_2)$, such that $t_2(x_0, r_2) = t_1(x_0; r_2)$ and $\Gamma(t_2(x_0, r_2), x_0, r_2) = 0$ for every $(x_0, r_2)$ in this neighborhood. From the uniqueness property it follows that $t_1 = t_2$, which implies the differentiability of $\mu$ at $x_0 = x_0$ for $r_2 = r_2$. Since $(x_0, r_2)$ was taken arbitrary in the compact set $[a, b] \times [c, d]$, we conclude the differentiability of $\mu$ for every $(x_0, r_2) \in [a, b] \times [c, d]$.

Finally, notice that $F(u(x_0), v(x_0)) = F(x_0, r_1 - r_2)$. Assuming that $u'(x_0) = v'(x_0) = 0$ and computing the derivative of the last identity in the variable $x_0$ we get that $x_0 = \tau$. Hence, we conclude that $u'(x_0)^2 + v'(x_0)^2 \neq 0$ for every $(x_0, r_2) \in [a, b] \times [c, d]$. 

\[ \square \]

\textbf{Figure 4.} Saturation of the curve $\mu$. 
4.2. **Dynamics of the sliding vector field.** In this subsection we are going to look more closely at the sliding vector field. Firstly, consider system \([5]\) in the variables \((x, w, z) = (x, x-y, z), (x, w, z) \in \mathbb{R}_{\geq 0} \times \mathbb{R} \times \mathbb{R}_{\geq 0}\). So, \((\dot{x}, \dot{w}, \dot{z})^T = Z(x, w, z)\), where

\[
Z(x, w, z) = \begin{cases} 
(r_1 - z)x \\
r_2w + x(r_1 - r_2 - z) \\
(eq_1x - m)z
\end{cases}
\]

if \( w > 0 \),

\[
Z(x, w, z) = \begin{cases} 
  r_1x \\
  r_1x - (x - w) \left( r_2 - \frac{\beta_2}{\beta_1}z \right) \\
  \left( \frac{eq_2}{a_q}(x-w-m) \right) z
\end{cases}
\]

if \( w < 0 \).

The switching manifold is now given by \( \{(x, 0, z) : x \geq 0, z \geq 0\} \) and the associated sliding vector field reads:

\[
Z^*(x, z) = \begin{pmatrix} \frac{\beta_1 r_2 + \beta_2 r_1}{\beta_1 + \beta_2} x - \frac{\beta_2}{\beta_1 + \beta_2} x z \\
\frac{e(a_q q_1 - q_2)(r_1 - r_2)\beta_1}{a_q(\beta_1 + \beta_2)} x - m z + \frac{e(\beta_1 q_2 + a_q \beta_2 q_1)}{a_q(\beta_1 + \beta_2)} x z
\end{pmatrix}.
\]

The vector field \([21]\) admits two equilibria, namely \((0, 0)\) and

\[
(x_c, z_c) = \left( \frac{a_q m(\beta_1 r_2 + \beta_2 r_1)}{e(\beta_1 q_2 r_2 + a_q \beta_2 q_1 r_1)}, \frac{r_1 \beta_1}{\beta_2} \right).
\]

Notice that, from the original condition \( \eta \in \mathcal{M} \),

\[
0 < x_c < \tau \quad \text{and} \quad z_c > r_1 = \phi - r_2.
\]

The next lemma is the main result of this subsection.

**Lemma 3.** Let \( \eta \in \mathcal{M} \) and assume that

\[
m < \frac{4(\beta_1 + \beta_2)(r_2 \beta_1 + r_1 \beta_2)(q_2 r_2 \beta_1 + a_q q_1 r_1 \beta_2)^2}{(q_2 - a_q q_1)^2(r_1 - r_2)^2 \beta_1^2 \beta_2^2}.
\]

Then, the following statements hold:

(i) the equilibrium \((x_c, z_c)\) is a repulsive focus;

(ii) there exists \( x^* \in [0, \tau) \) such that the backward orbit of \( Z^* \) of any point of the straight segment \( L = \{(x, \phi) : x^* < x \leq \tau \} \subset S^w_X \) is contained in \( \Sigma^s \) and converges asymptotically to the equilibrium \((x_c, z_c)\).
Remark 1. If we consider the following change on the parameters

\[
e = \frac{a_q m z_c}{(a_q q_1 r_1 + q_2 (z_c - r_1)) x_c}
\]

and \( \beta_2 = \frac{r_2 \beta_1}{z_c - r_1} \),

then the inequality (23) becomes

\[
m < \frac{4 r_2 z_c (z_c - \phi)(a_q q_1 r_1 + q_2 (z_c - r_1))^2}{\phi^2 (q_2 - a_q q_1)^2 (z_c - r_1)^2}.
\]

Proof. Denote by \( \alpha \pm ib \) the eigenvalues of \( dZ^*(x_c, z_c) \). It is straightforward to see that (23) implies that \( b \neq 0 \). In this case

\[
\alpha = \frac{m (q_2 - a_q q_1) (r_1 - r_2) \beta_1 \beta_2}{2 (\beta_1 + \beta_2) (a_q q_1 r_1 \beta_2 + q_2 r_2 \beta_1)} > 0.
\]

Hence, \((x_c, z_c)\) is a repulsive focus. This concludes the proof of statement (i).

In order to prove statement (ii), we first claim that the sliding vector field \( Z^*(x, z) = (Z_1^*(x, z), Z_2^*(x, z)) \) given by equation (21) does not admit limit cycles contained in the region \( \{(x, z) \in \mathbb{R}^2 : x > 0, z > 0\} \). Indeed,

\[
\frac{\partial}{\partial x} \left( \frac{Z_1^*(x, z)}{x z} \right) + \frac{\partial}{\partial z} \left( \frac{Z_2^*(x, z)}{x z} \right) = \frac{e (q_2 - a_q q_1) (r_1 - r_2) \beta_1}{a_q (\beta_1 + \beta_2) z^2} > 0,
\]

for \( x, z > 0 \). Since the function \((x, y) \mapsto \frac{1}{x z}\) is \( C^1 \) in the region \( \{(x, z) \in \mathbb{R}^2 : x > 0, z > 0\} \), the claim follows by the Bendixson-Dulac criterion (see [7]).

We may observe that the sliding vector field writes

\[
Z^*(x, z) = Z_{LV}(x, z) + \left( 0, \frac{e (a_q q_1 - q_2) (r_1 - r_2) \beta_1}{a_q (\beta_1 + \beta_2)} x \right),
\]

where \( Z_{LV} \) is a Lotka-Volterra vector field. We know that \( Z_{LV} \) admits the following first integral

\[
H(x, z) = -m - \frac{r_2 \beta_1 + r_1 \beta_2}{\beta_1 + \beta_2} + \frac{e (a_q q_1 \beta_2 + q_2 \beta_1)}{a_q (\beta_1 + \beta_2)} x + \frac{\beta_2}{\beta_1 + \beta_2} z
\]

\[
- m \log \left( \frac{e (a_q q_1 \beta_2 + q_2 \beta_1)}{a_q m (\beta_1 + \beta_2)} \right) - \frac{r_2 \beta_1 + r_1 \beta_2}{r_2 \beta_1 + r_1 \beta_2} \log \left( \frac{\beta_2}{r_2 \beta_1 + r_1 \beta_2} z \right),
\]

that is, \( \langle \nabla H(x, z), Z_{LV}(x, z) \rangle = 0 \). Now, let

\[
a = \frac{(\beta_1 + \beta_2) (q_2 r_2 \beta_1 + a_q q_1 r_1 \beta_2)}{(q_2 \beta_1 + a_q q_1 \beta_2) (r_2 \beta_1 + r_1 \beta_2)} > 0.
\]

Since

\[
\langle \nabla H(ax, z), Z^*(x, y) \rangle = \frac{(q_2 - a_q q_1) (r_1 - r_2) \beta_1 (r_2 \beta_1 + (r_1 - z) \beta_2)^2}{r_2 \beta_1 + r_1 \beta_2} > 0,
\]
From Lemma 3 and Remark 1 we have that \((x, z)\) is a repulsive focus of \(Z\). Let \(L\) be the straight segment through the fold-regular point \((\phi, \tau)\). If there exists \(t^* > 0\) such that \(\varphi(t_z) \in S^u_X\), then the orbit of \(Z\) between \((\phi, \tau)\) and \((\phi, 0)\) is negatively invariant. Since \(Z\) has no limit cycles, the focus \((x, z)\) must attract, backward in time, the orbits of every point in the positive quadrant.

Finally, consider \(\varphi^*(t)\) the trajectory of \(Z\) passing through \((\phi, \tau)\). If there exists \(t^* > 0\) such that \(\varphi(t_z) \in S^u_X\), then take \(x^* = \varphi(t_z)\). Otherwise take \(x^* = 0\). It is easy to see that, in this case, \(x^* < \tau\). Indeed, \(x^* \neq \tau\), otherwise there would exist a periodic solution passing through \((\tau, r_1 - r_2)\), and \(\pi_2 Z^*(x, r_1 - r_2) = (r_1 - r_2)(eq_2x - m) > 0\) for every \(x > \tau\). Hence, the proof of statement \((ii)\) follows.

4.3. The Shilnikov sliding connection. Let us guarantee the existence of a Sliding Shilnikov Connection according to Definition 1. Lemma 1 ensures that the saturation of \(S^u_X\) through the forward flow of \(X\) intersects transversally the switching manifold \(\Sigma\) in a curve \(\mu(x_0) = (u(x_0), u(x_0), v(x_0))\), \(0 < x_0 < \tau\). Moreover, from Lemma 2 there exist \(a, b, c, d\), with \(0 < a < b < \tau\) and \(0 < c < d\), such that the curve \(0 < u(x_0) < \tau\) and \(v(x_0) > r_1\) for every \((x_0, r_2) \in [a, b] 	imes [c, d]\). Accordingly, for some \(x_0 \in [a, b]\), take \((x, z) = (u(x_0), v(x_0))\) and assume

\[
c < r_2 < d \quad \text{and} \quad m < \frac{4r_2v(x_0)(v(x_0) - \phi)(a_2q_1r_1 + q_2(v(x_0) - r_1))^2}{\phi^2(q_2 - a_2q_1)^2(v(x_0) - r_1)^2}.
\]

From Lemma 3 and Remark 1 we have that \((x, z) \in \Sigma^s \subset \Sigma\) is a repulsive focus of \(Z\) and there exists \(x^* \in [0, \tau)\) such that the backward orbit of any point in the straight segment \(L = \{(x, \phi) : x^* < x \leq \tau\} \subset S^u_X\) is contained in \(\Sigma^s\) and converges asymptotically to \(p\).

If \(x^* = 0\), then from Lemma 3 we have characterized a sliding Shilnikov connection through the fold-regular point \((x_0, x_0, \phi)\) and the pseudo-equilibrium \((u(x_0), u(x_0), v(x_0))\).

Now, assume that \(x^* \neq 0\). It remains to prove \((x, z) = (u(x_0), v(x_0))\) implies that \(x^* < x_0\). Notice that, in this case, the points \((x, z)\) and \((x_0, \phi)\) lie in the same level set of \(F\). Recall that \(F\) is the first integral of the Lotka-Volterra system. Denote \(C = F^{-1}(F(x_c, z_c))\). Firstly, we shall study the behavior of \(Z\) on \(C\), which is equivalent to analyze the sign of the product

\[
\langle \nabla F(x, z), Z^*(x, z) \rangle = \frac{(r_1 - r_2 - z)(eq_2x(r_1 - z) + a_q(-eq_1r_1x + mz)\beta_1)}{a_qz(\beta_1 + \beta_2)}.
\]

for \((x, z) \in C\).

Since \(a_qz(\beta_1 + \beta_2) > 0\) and \(r_1 - r_2 - z \leq 0\) for \(z \geq r_1 - r_2\), it is sufficient to analyze the sign of \(eq_2x(r_1 - z) + a_q(-eq_1r_1x + mz)\) on \(C\). Notice that the equation

\[
eq 0
\]
describes a hyperbola containing the points \((0,0)\) and \((x_c,z_c)\). Indeed, \([24]\) is a quadratic equation of the form \(Ax^2 + Bxz + Cz^2 + Dx + Ez + F = 0\), where \(B = -eq_2\), \(D = r_1e(q_2 - a_qq_1)\), \(E = a_qm\), \(A = C = F = 0\), and so \(B^2 - 4AC = B^2 > 0\). From convexity, each connected component of the hyperbola \([24]\) intersects \(C\) at most in two points. Solving \([24]\), we get

\[
z = z_h(x) = \frac{e(q_2 - a_qq_1)r_1x}{eq_2x - a_qm}.
\]

Define

\[
F_c(x) = F(x, z_h(x)) - F(x, z_c).
\]

Notice that \(F_c(x) > 0\), \(F_c(x) = 0\), and \(F_c(x) < 0\) imply \((x, z_h(x)) \in \text{ext}(C)\), \((x, z_h(x)) \in C\), and \((x, z_h(x)) \in \text{int}(C)\), respectively. Clearly, \(F(x_c) = 0\). Moreover,

\[
F'(x_c) = -\frac{eq_1r_1(\tau - x_c)^2 + \tau(r_1 - x_c)^2}{r_1(\tau - x_c)x_c} < 0.
\]

This implies that \(F_c(x) > 0\) for every \(x \in (0, x_c)\). Consequently, \(eq_2x(r_1 - z) + a_q(-eq_1r_1x + mz) < 0\) and \(\langle \nabla F(x, z), Z^s(x, z) \rangle > 0\) for every \((x, z) \in C\) such that \(x \in (0, x_c)\). It means that the vector field \(Z_s\) points outwards \(C\) provided that \((x, z) \in C\) and \(x \in (0, x_c)\).

Finally, let \(\varphi^s(t)\) be the trajectory of \(Z^s\) passing through \((\phi, \tau)\). Since \(x^* = \pi_x\varphi(t_s)\) for some \(t_s > 0\) and \(\varphi(t_s) \in S_X^0\), there exists \(t'_s \in (0, t_s)\) such that \(\pi_x\varphi(t'_s) = x_c\). Moreover, \(\varphi(t'_s) \in \text{ext}(C)\) because \((x_c, z_c)\) is a repulsive focus lying on \(C\). From the previous comments, the trajectory \(\varphi^s(t)\) remains in the exterior of \(C\) for every \(t \in [t'_s, t_s]\). Hence, \(\varphi(t)(s) \in \text{ext}(C)\) implying that \(x^* < x_0\). Then, applying Lemma \([3]\) we have characterized a sliding Shilnikov connection through the fold-regular point \((x_0, x_0, \phi)\) and the pseudo-equilibrium \((u(x_0), u(x_0), v(x_0))\).

Now, define \(\tilde{N}\) as the set of parameter vectors \(\eta = (r_1, r_2, a_q, q_1, q_2, \beta_1, \beta_2, e, m) \in \mathcal{M}\) satisfying the inequalities

\[
a \leq x_0 \leq b, \quad c \leq r_2 \leq d, \quad \text{and} \quad m < M(x_0, r_2) := \frac{4r_2v(x_0)(v(x_0) - \phi)(a_qq_1r_1 + q_2(v(x_0) - r_1))^2}{\phi^2(q_2 - a_qq_1)^2(v(x_0) - r_1)^2},
\]

and the identities

\[
e = E(x_0) := \frac{a_qmv(x_0)}{(a_qq_1r_1 + q_2(v(x_0) - r_1))u(x_0)} \quad \text{and} \quad \beta_2 = B(x_0) := \frac{r_2\beta_1}{v(x_0) - r_1},
\]
The identities (26) come from assuming \((x_c, z_c) = (u(x_0), v(x_0))\) (see Remark 1). From the construction above, the differential system (4) possesses a sliding Shilnikov orbit whenever \(\eta \in \tilde{N}\).

In what follows, we shall identify a codimension one submanifold \(\mathcal{N} \subset \tilde{\mathcal{N}}\) of \(\mathcal{M}\). Firstly, notice that \(M\) is a positive continuous function and, therefore, assumes a minimum \(M_0 > 0\) on the compact set \([a, b] \times [c, d]\). Moreover, \(E'(x_0)^2 + B'(x_0)^2 \neq 0\) for every \(\tau \in (0, \tau)\). Indeed, it is easy to see that \(E'(x_0)^2 + B'(x_0)^2 = 0\) if, and only if, \(u'(x_0)^2 + v'(x_0)^2 = 0\), which would contradict Lemma 2. Without loss of generality, assume that for some \(x_0 \in (a, b)\), \(B'(x_0) \neq 0\). From the inverse function theorem, the function \(B\) can be locally inverted, that is, there exists a neighborhood \(B\) of \(B(x_0)\) and a unique function \(B^{-1} : \mathcal{B} \rightarrow (a, b)\) such that \(B \circ B^{-1}(\beta_2) = \beta_2\) whenever \(\beta_2 \in \mathcal{B}\). Hence, taking

\[
c \leq r_2 \leq d, \quad m \leq M_0, \quad \beta_2 \in \mathcal{B} \quad \text{and} \quad e = E \circ B^{-1}(\beta_2),
\]

the inequalities (25) and the identities (26) are fulfilled. So, for \(\eta = (r_1, r_2, a_q, q_1, q_2, \beta_1, \beta_2, c, m) \in \mathcal{M}\), define

\[
\mathcal{N} = \{\eta \in \mathcal{M} : c < r_2 < d, \ m < M_0, \ \beta_2 \in \mathcal{B} \quad \text{and} \quad e = E \circ B^{-1}(\beta_2)\} \subset \tilde{\mathcal{N}}.
\]

Notice that \(\mathcal{N}\) is a graph defined in an open domain. Therefore, \(\mathcal{N}\) is a codimension one submanifold of \(\mathcal{M}\).

Finally, the existence of the neighborhood \(U \subset \mathcal{M}\) of \(\mathcal{N}\), satisfying that the differential system (4) behaves chaotically whenever \(\eta \in \mathcal{U}\), follows directly from Theorem 1.

5. Numerical Simulations

In order to corroborate our results, we perform a numerical simulation that puts in evidence the existence of the Shilnikov sliding connection obtained in Section 4.3. We were able to find parameter values for which the repulsive focus \((x_c, z_c)\) of the sliding vector field \(Z_\mu\) changes its position crossing the curve \(\mu\). Recall that the curve \(\mu\), given by Lemma 1, is the saturation of \(S_X^\epsilon\) through the flow of \(X\) intersected with \(\Sigma\). The simulation rely on computer algebra and numerical evaluations carried out with the software MATHEMATICA (see [14]).

Notice that the vector field \(X\) given by (5) does not depend on the parameter \(\beta_1\). So, the repulsive focus \((x_c, z_c)\) can be moved by varying the parameter \(\beta_1\) keeping the trajectories of \(X\) unchanged. Accordingly, we shall fix all the parameter values but \(\beta_1\) (see Table 1).

Taking either \(\beta_1 = 0.994\) or \(\beta_1 = 10\) and considering the parameter values given by Table 1 we see that the conditions of Lemma 3 are satisfied, that is, \((x_c, z_c)\) is a repulsive focus. For \(\beta_1 = 0.994\), the numerical simulation indicates that \((x_c, z_c)\) is below the curve \(\mu\) (see Figure 5(a)). For \(\beta_1 = 10\), the numerical simulation indicates that \((x_c, z_c)\) is above the curve \(\mu\) (see Figure 5(c)).
Parameter & Value \\
--- & --- \\
m & 0.790 \\
r_1 & 0.836 \\
e & 0.948 \\
q_1 & 0.772 \\
a_q & 0.660 \\
q_2 & 1.084 \\
\beta_2 & 0.896 \\
r_2 & 0.126 \\

Table 1. Parameters for the numerical analysis.

Finally, from the continuous dependence on the parameter \( \beta_1 \), there exists \( \beta_1^* \), with \( 0.994 < \beta_1^* < 10 \), such that for \( \beta_1 = \beta_1^* \) the repulsive focus \((x_c, z_c)\) belongs to the curve \( \mu \). This gives rise to a Shilnikov sliding connection (see Figure 5(b)). We mention that the return \( x^* \) of the sliding vector field through the point \((\tau, \phi)\) on \( S^X_{\tau} \) satisfies \( x^* < a \) where \( 0 < a < \tau \) is the x-coordinate of the fold point which is connected to the repulsive focus through an orbit of \( X \). In other words, \( a \) belongs to the segment \( L \) given by Lemma 3.

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Figure 5. Relative position between the curve $\mu$ and the repulsive pseudo-focus $(x_c, z_c)$. In (a) and (c) we are taking $\beta_1 = 0.994$ and $\beta_1 = 10$, respectively. In (b) there exists $\beta_1^*$ such that $(x_c, z_c) \in \mu$.

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