ON HAMILTONIAN CONTINUUM MECHANICS

MICHAL PAVELKA*1 & ILYA PESHKO2 & VÁCLAV KLIKA3

CONTENTS

1 Introduction 3
2 Hamiltonian mechanics 3
  2.1 Lagrangian frame 4
  2.2 Eulerian frame 6
  2.3 Non-Newtonian fluids 7
  2.4 SHTC equations 8
  2.5 Jacobi identity 10
  2.6 Gauge invariance, symmetries and conserved quantities 11
  2.7 Hyperbolicity 17
  2.8 Clebsch variables 20
  2.9 Semidirect product structure 22
  2.10 Onsager-Casimir reciprocal relations 23
3 Space-time formulation 24
  3.1 Lagrangian variational formulation 24
  3.2 Eulerian variational formulation 25
  3.3 F_A or A^A? 26
4 Conclusion 27
A Functional derivatives 32
  B From Lagrange to Euler 33
    B.1 Derivative of the Eulerian deformation gradient F(x) 34
    B.2 Derivative of the Eulerian mass density ρ(x) 34
    B.3 Derivative of the Eulerian entropy density s(x) 35
    B.4 Derivative of the Eulerian momentum density m(x) 35
    B.5 Derivative of an arbitrary Eulerian functional 36
    B.6 The Eulerian Poisson bracket 36
C From deformation gradient to the Left Cauchy-Green tensor 38
D From deformation gradient to distortion 38

LIST OF FIGURES

Figure 1 Demonstration of formula (118) (the dashed triangle). The left manifold represents the Lagrangian configuration while right represents the Eulerian configuration. 34
CONTINUUM MECHANICS CAN BE FORMULATED IN THE LAGRANGIAN FRAME (WHERE PROPERTIES OF CONTINUUM PARTICLES ARE ADDRESSED) OR IN THE EULERIAN FRAME (WHERE FIELDS LIVE IN AN INERTIAL FRAME). THERE IS A CANONICAL HAMILTONIAN STRUCTURE IN THE LAGRANGIAN FRAME. BY TRANSFORMATION TO THE EULERIAN FRAME WE FIND THE POISSON BRACKET FOR EULERIAN CONTINUUM MECHANICS WITH DEFORMATION GRADIENT (OR THE RELATED DISTORTION MATRIX). BOTH LAGRANGIAN AND EULERIAN HAMILTONIAN STRUCTURES ARE THEN DISCUSSED FROM THE PERSPECTIVE OF SPACE-TIME VARIATIONAL FORMULATION AND BY MEANS OF SEMI-DIRECT PRODUCTS OF LIE ALGEBRAS. FINALLY, WE DISCUSS THE IMPORTANCE OF THE JACOBI IDENTITY AND, IN PARTICULAR THE PROOF OF HYPERBOLICITY OF THE IMPLIED QUASILINEAR SYSTEMS OF FIRST-ORDER PARTIAL DIFFERENTIAL EVOLUTION EQUATIONS AND THEIR GAUGE INVARIANCE.

\*pavelka@karlin.mff.cuni.cz
1 Mathematical Institute, Faculty of Mathematics and Physics, Charles University, Sokolovská 83, 186 75 Prague, Czech Republic, pavelka@karlin.mff.cuni.cz
2 Institut de Mathématiques de Toulouse, Université Toulouse III, F-31062 Toulouse, France, peshkov@math.nsc.ru
3 Department of Mathematics, FNSPE, Czech Technical University in Prague, Trojanova 13, 120 00 Prague, Czech Republic, vaclav.klika@fjfi.cvut.cz
The usual way continuum mechanics is presented is based on a generalization of Newton’s laws to continuum particles [2, 3, 4]. As Newton’s laws can be seen as a consequence of the principle of least action or Hamiltonian mechanics (Hamilton canonical equations), see e.g. [5], so can be the continuum mechanics. Moreover, continuum mechanics can be formulated in the Lagrangian frame, where coordinates are attached to matter, or Eulerian frame, where coordinates are attached to an inertial frame of reference. We shall present the transformation of the Hamiltonian continuum mechanics from the Lagrangian frame to the Eulerian frame, which leads to the Poisson bracket generating reversible evolution equations for density, momentum density and entropy density (balance laws) coupled with evolution for the deformation gradient or its inverse, which is called the distortion. This is the Hamiltonian formulation of continuum mechanics in the Eulerian frame.

Subsequently, we recall the SHTC (Symmetric Hyperbolic Thermodynamically Compatible) equations for nonlinear Eulerian elasticity and fluid mechanics, we project the Poisson bracket to less detailed levels of description suitable for polymeric flows (based on the left Cauchy-Green or conformation tensor), and discuss the meaning of Jacobi identity and its relation to hyperbolicity of the governing evolution equations. In Sec. 2.6, we show how existence of a conserved quantity implies gauge invariance of the evolution equations (even in the non-canonical case). In Sec. 2.8, construction from the Clebsch variables (providing a variational principle) of fluid mechanics with the distortion field is demonstrated. In Sec. 2.9, the Poisson bracket for Eulerian continuum mechanics with distortion field is shown to have the structure of semidirect product of fluid mechanics and a cotangent bundle, which provides a geometric basis.

In Sec. 3, we identify the variational structure of Lagrangian continuum mechanics in the space-time settings and the complementary Eulerian variational principle. By invoking the gauge freedom of the Lagrangian (depends only on gradients of particle labels), the SHTC equations are identified as Euler-Lagrange equations in space-time Galilean settings and as integrability conditions.

Novelty of this paper lies (i) in the transformation from the Hamiltonian mechanics in the Lagrangian frame to the Eulerian frame, (ii) in the proof of automatic hyperbolicity and gauge invariance, (iii) in the semidirect-product structure of SHTC equations, (iv) and in the space-time variational formulation of the equations.

2 HAMILTONIAN MECHANICS

The principle of least action has long been the fundamental approach to modern mechanics [5]. Although initially formulated for mechanics of classical particles and optics, since the end of the 19th century it has been applied also to the mechanics of rigid bodies, field theories and continuum mechanics, e.g. [8]. In the 80s, Hamiltonian mechanics was connected with thermodynamics in several related for-
mulations [9, 10, 11, 12, 13] and culminated in the GENERIC (General Equation for Non-Equilibrium Reversible-Irreversible Coupling) framework [14, 15], which has been reviewed in monographies [13, 16, 17].

The usual obstacle, however, for understanding the GENERIC framework is the presence of Poisson brackets, which often is regarded as a mysterious mathematical concept. It is of course not mysterious at all, as it can be seen as a result of the principle of least action in its usual form or in the geometric Euler-Poincaré form [18]. Let us, however, demonstrate a simple-in-principle derivation of the Poisson bracket for Eulerian continuum mechanics elucidating the origin of the Hamiltonian structure (requiring only minimal geometric background).

2.1 Lagrangian frame

Let us consider a body, material points of which are described by a reference (Lagrangian) coordinates $X$. Position of the material point $X$ at time $t$ with respect to a chosen inertial laboratory frame is then given by the mapping $x(t,X)$ from the Lagrangian coordinates to the Eulerian coordinates. This mapping is usually assumed to be smooth enough and invertible. These properties will be violated later in this paper, but for the moment let us adopt those assumptions as well.

Mechanical state of a material point is characterized by its position $x(t,X)$ and velocity $\dot{x}(t,X)$ or the corresponding momentum density $M(t,X)$ (momentum per Lagrangian volume $dX$). In mathematical terms the couple $(x,\dot{x})$ forms a tangent bundle while the couple $(x,M)$ forms a cotangent bundle. Since we are seeking Hamiltonian evolution (generated by a Poisson bracket and energy), we choose the latter description. The Lagrangian state variables are thus the field of Eulerian positions $x(t,X)$ and the field of momentum density $M(t,X)$.

Since these state variables form a cotangent bundle, they are equipped with the canonical Poisson bracket in the Lagrangian frame, see e.g. [4],

$$\{F,G\}^{(L)} = \int dX \left( \frac{\delta F}{\delta x^i(X)} \frac{\delta G}{\delta M_i^j(X)} - \frac{\delta G}{\delta x^i(X)} \frac{\delta F}{\delta M_i^j(X)} \right),$$  \hspace{1cm} (1)

where $F$ and $G$ are two arbitrary functionals of the Lagrangian state variables. The explicit dependence on time is omitted from the notation as in the rest of the paper since now on. The derivatives stand for functional (or Volterra) derivatives, see Appendix A. This Poisson bracket clearly satisfies the Jacobi identity,

$$\{F,\{G,H\}\} + \{G,\{H,F\}\} + \{H,\{F,G\}\} = 0,$$  \hspace{1cm} (2a)

as can be seen by direct verification. The bracket is of course antisymmetric,

$$\{F,G\} = -\{G,F\},$$  \hspace{1cm} (2b)

and satisfies the Leibniz rule (assuming sufficient mathematical regularity),

$$\{F,GH\} = \{F,G\}H + G\{F,H\}.$$  \hspace{1cm} (2c)

Therefore, bracket (1) is indeed a Poisson bracket as it satisfies all the properties (2).

Denoting a general set of state variables by $q$, a Poisson bracket can be equivalently expressed by means of its Poisson bivector

$$L^{\alpha\beta} = \{q^\alpha, q^\beta\},$$  \hspace{1cm} (3)

Greek indeces denote state variables while latin space (or space-time) coordinates.
which is antisymmetric and can be used to reconstruct the bracket as follows,

$$\{F, G\} = \langle F_{q^\alpha} | L^{\alpha\beta} | G_{q^\beta}\rangle,$$

(4)

where $\langle \cdot | \cdot \rangle$ means a contraction (e.g. duality in distributions).

Once having state variables $q$, e.g. $q = (q(X), M(X))$ in the Lagrangian continuum mechanics, and the corresponding Poisson bracket, the reversible evolution of a functional $\mathcal{F}(q)$ of the state variables is given by

$$\dot{\mathcal{F}} = \{\mathcal{F}, E\},$$

(5)

where $E$ is the total energy of the (isolated) system. This is a sort of weak formulation of the problem. On the other hand, evolution of the functional $\mathcal{F}$ can be expressed using the chain rule as functional derivatives of the functional multiplied by evolution equations of the state variables,

$$\dot{q}^\alpha = \{q^\alpha, E\} = L^{\alpha\beta} \left| \frac{\delta E}{\delta q^\beta} \right|.$$

(6)

For instance, for the Lagrangian state variables we have

$$\dot{\mathcal{F}}(x(X), M(X)) = \{\mathcal{F}, E\}^{(L)},$$

as well as

$$\dot{\mathcal{F}}(x(X), M(X)) = \int dX \left( \frac{\delta \mathcal{F}}{\delta x^i} \partial_t x^i + \frac{\delta \mathcal{F}}{\delta M_i} \partial_t M_i \right).$$

(8)

By comparing these two equalities, we can conclude that the evolution equations for $x$ and $M$ are

$$\partial_t x^i(X) = \frac{\delta E}{\delta M_i(X)}$$

(9a)

$$\partial_t M_i(X) = -\frac{\delta E}{\delta x^i(X)}$$

(9b)

for any energy $E(x, M)$. This is a way to obtain evolution equations from a Poisson bracket.

Let us be more specific. Choosing the energy as

$$E = \int dX \left( \frac{M^2}{2\rho_0} + \rho_0 W(\nabla x x) \right),$$

(10)

where the first term denotes the kinetic energy and the second denotes elastic energy (dependent only on gradients of the field $x(X)$), equations (9) obtain the concrete form

$$\partial_t x^i(X) = \frac{\delta^{ij} M_j}{\rho_0}$$

(11a)

$$\partial_t M_i(X) = \frac{\partial}{\partial X^j} \left( \rho_0 \frac{\partial W}{\partial \delta x^i X} \right)$$

(11b)

Here, $\rho_0(X)$ is the reference mass density. The metric tensor $\delta^{ij}$ can be thought of as equal to the unit matrix in the Euclidean space endowed with Cartesian coordinates. Note that the Einstein summation convention is employed and that the capital index denotes coordinates in the Lagrangian frame. Also, apart from the field $\rho_0(X)$ the energy can depend on the field of entropy density $s_0(X)$ (per volume $dX$) to cope with non-isothermal bodies. Equations (11) are the reversible evolution equations.
for a continuous body with stored energy \( W(\nabla X x) \) in the Lagrangian frame, which are to be solved when initial and boundary conditions are supplied.

However, it is often preferable to formulate the evolution equations in the Eulerian frame because (i) the Lagrangian configuration may be inaccessible (as in the case of fluids), (ii) conservation laws are directly at hand in the Eulerian frame and so it is clearer how to add dissipative terms to the evolution equations, and (iii) also, the use of the Eulerian formulation can be advantageous in many practical situations \[19\]. The complementary equations in the Eulerian frame are shown in the next section.

2.2 Eulerian frame

First we have to declare what are the fields constituting the Eulerian state variables. We choose the fields

\[
\begin{align*}
\rho(x) &= \rho_0(X(x)) \cdot \det \frac{\partial X}{\partial x} \\
m(x) &= M(X(x)) \cdot \det \frac{\partial X}{\partial x} \\
s(x) &= s_0(X(x)) \cdot \det \frac{\partial X}{\partial x} \\
F_{1j}(x) &= \frac{\partial x^i}{\partial X^j | X(x)}
\end{align*}
\]

of local mass density (per volume \( dx \)), momentum density, entropy density and the deformation gradient. Note that the Eulerian deformation gradient \( F(x) \) is not a naturally Eulerian field and should be regarded rather as inverse of the distortion \( F^{-1} = A \). Indeed, the latter, which can be seen as Eulerian gradient of the fields of Lagrangian labels, is the natural Eulerian field in the Eulerian variational principle, see Sec. 3.

The goal is to project the Lagrangian Poisson bracket (1) to an Eulerian Poisson bracket by letting the functionals depend only on the Eulerian fields (12). After rather lengthy calculation (Appendix B), we obtain the Poisson bracket

\[
\{F, G\}^{(\text{Eulerian})} = \{F, G\}^{(\text{FM})} + \int d\delta x F_{1j} \left( \frac{\delta F}{\delta F_{1i}} \frac{\delta G}{\delta m_i} - \frac{\delta G}{\delta F_{1i}} \frac{\delta F}{\delta m_i} \right) - \int d\delta x \delta F_{1k} \left( \frac{\delta F}{\delta F_{1i}} \frac{\delta G}{\delta m_k} \frac{\delta G}{\delta m_i} - \frac{\delta G}{\delta F_{1i}} \frac{\delta G}{\delta m_k} \right),
\]

(13)

where \( \{F, G\}^{(\text{FM})} \) stands for the Poisson bracket of fluid mechanics,

\[
\{F, G\}^{(\text{FM})} = \int d\rho (\partial_1 F_\rho G_{m_1} - \partial_1 G_\rho F_{m_1}) + \int d\delta m_i (\partial_1 F_{m_i} G_{m_1} - \partial_1 G_{m_i} F_{m_1}) + \int d\delta s (\partial_1 F_s G_{m_1} - \partial_1 G_s F_{m_1}).
\]

(14)

For brevity, from now on, the functional derivatives in the Poisson brackets are denoted by subscript, e.g. \( \frac{\delta F}{\delta \rho} = F_\rho \), and if the functionals are assumed to be local.

A simpler version of the calculation leading to fluid mechanics was called a “small miracle” in \[20\], and similar procedure leading to fluid mechanics equipped with the left Cauchy-Green tensor was presented in \[13\].
(involving no spatial gradients), \( F_\rho \) stands for the partial derivative \( F_\rho = \partial F/\partial \rho \), \( f \) being volume density of \( F \). This slightly overloaded notation helps to keep the formulas clear and should not cause any confusion. Bracket (13) is certainly a Poisson bracket, i.e. it fulfills criteria (2), since it has been obtained by projection of Poisson brackets (see e.g. [21]). The bracket is compatible with Poisson bivector 5.12a of [22], and it expresses kinematics of the Eulerian state variables consisting of the state variables of fluid mechanics (\( \rho, m \) and \( s \)) and the deformation gradient \( F(x) \). The projection procedure can be summarized as the following theorem:

**Theorem 1.** Canonical Poisson bracket (1) of functionals \( F \) and \( G \) of the Eulerian fields (12) is equal to Poisson bracket (13). The latter bracket expresses kinematics of the Eulerian fields \( (\rho, m, s, F) \).

The reversible evolution equations implied by bracket (13) are

\[
\begin{align*}
\partial_t \rho &= -\partial_i (\rho E_{m_i}) \\
\partial_t m_i &= -\partial_j (m_i E_{m_j}) - \rho \partial_i E_\rho - m_j \partial_i E_{m_j} - s \partial_i E_s - F^j \partial_j E_{F^j} \\
&\quad + \partial_j (F^j \partial_i E_{F_i} + F^i \partial_j E_{F_i}) \\
\partial_t s &= -\partial_i (s E_{m_i}) \\
\partial_t F^i &= -E_{m_k} \partial_k F^i + F^j \partial_j E_{F^i}.
\end{align*}
\]

where the energy \( E = \int \text{d}x e(\rho, m, s, F) \) still remains to be specified. The functional derivatives of energy can be replaced by partial derivatives of total energy density \( e \) hereafter due to the algebraic dependence on the state variables. Note that the total momentum is of course conserved, since the first line (except the first term) in the evolution equation for \( m_i \) can be rewritten as gradient of generalized pressure \( \partial_1 p \) for

\[
p = -e + \rho \frac{\partial e}{\partial \rho} + m_j \frac{\partial e}{\partial m_j} + s \frac{\partial e}{\partial s} + F^i \frac{\partial e}{\partial F^i}.
\]

Total energy density \( e \) can be prescribed as

\[
e = \frac{m^2}{2\rho} + \varepsilon(\rho, s, F),
\]

where \( \varepsilon \) is the elastic and internal energy. In particular, \( E_m = m/\rho = v \) becomes the velocity. The evolution equation for the deformation gradient then gets the explicit form

\[
\partial_t F = -(v \cdot \nabla) F + \nabla v \cdot F,
\]

which is the usual evolution equation for \( F \) in the Eulerian frame, e.g. see [23, 19, 24]. Equations (15) represent evolution equations for density, momentum density, entropy density and deformation gradient in the Eulerian frame, and they attain an explicit form once total energy density is specified.

2.3 Non-Newtonian fluids

Since the Lagrangian configuration is usually irrelevant in the case of fluids (even non-Newtonian), the fluids are often described by state variables \( \rho, m, s \) and the left Cauchy-Green tensor

\[
B^{ij} (x) = F^i_1 (x) F^j_1 (x) \delta^{ij}.
\]
Note that $\delta^{ij}$ is actually an inverse body metric measuring lengths in the Lagrangian frame, see [4, 25]. By letting the functionals in bracket (13) depend on these state variables we arrive at Poisson bracket

$$\{F, G\}^{(\text{LCG})} = \{F, G\}^{(\text{FM})} + \int \text{d}x \left( F_{B\ell} (B^{jk} \partial_j G_{m_l} + B^{ij} \partial_j G_{m_k}) - G_{B\ell} (B^{ik} \partial_j G_{m_l} + B^{ij} \partial_j F_{m_k}) \right)$$

$$- \int \text{d}x \partial_i (B^{ik} F_{B\ell} G_{m_l} - G_{B\ell} F_{m_k}),$$

which expresses kinematics of fields $\rho, m, s$ and $B$. Details of the calculation can be found in Appendix C. The description of a fluid including the left Cauchy-Green tensor is suitable for non-Newtonian complex fluids, e.g. [26].

The evolution equations generated by bracket (19) are

$$\partial_t \rho = -\partial_i (\rho E_{m_i})$$

$$\partial_t m_i = -\partial_j (m_i E_{m_j}) - \rho \partial_i E_{\rho} - m_i \partial_i E_m - s \partial_i E_s - B^{jk} \partial_i E_{B\ell^k}$$

$$+ \partial_i (B^{k}\partial_j E_{B\ell^k}) + \partial_j (B^{\ell^k}(E_{B\ell^k} + E_{B\ell^i}))$$

$$\partial_t B^{lk} = -\nu \partial_j B^{lk} + B^{ij} \partial_j E_{m_k} + B^{ji} \partial_j E_{m_k}$$

$$\partial_t s = -\partial_i (s E_{m_i}).$$

The evolution equations are

$$\nabla = 0,$$

i.e. the upper-convected derivative of $B$ be equal to zero. Moreover, once the dependence of the generalized of $F$ (or of its inverse $A$) from being holonomic (i.e. being the gradient of the mapping $x^I(X^i)$) to non-holonomic triad is sufficient to describe fluid flows, either Newtonian [27] or non-Newtonian [28]. This, however, requires the introduction of the local reference configuration instead of the global Lagrangian configuration associated with the coordinates $X^i$, e.g. see [29].

### 2.4 SHTC equations

As it will be discussed later in Sec.3, the distortion matrix $A$ represents a rather natural Eulerian state variable instead of $F_m$ which is a natural Lagrangian state variable. Therefore, besides the projection from $F$ to $B$ one can also carry out transformation of variables from $F$ to $A = F^{-1}$,

$$A^T_i (x) = (F^{-1}(x))^T_i,$$

by letting the functionals depend only on $\rho, m, s$ and $A$, see appendix D for details.

The resulting Poisson bracket is

$$\{F, G\}^{(A)} = \{F, G\}^{(\text{FM})} - \int \text{d}x A^T_i (F_{A\ell^i} \partial_i G_{m_l} - G_{A\ell^i} \partial_i F_{m_l})$$

$$- \int \text{d}x \partial_i A^T_i (F_{A\ell^i} G_{m_l} - G_{A\ell^i} F_{m_l}).$$

Note that we assume that kinetic energy is in form $m^2/2\rho$ so that its conjugate is velocity, $E_m = v$. 

which is the Poisson bracket for the distortion. Thus we come to the following conclusion, which might be anticipated already from the results in [30].

Proposition 1 (On the origin of continuum mechanics with distortion). The Poisson bracket (22), which expresses kinematics of the Eulerian fields \((\rho, \mathbf{m}, s, \mathbf{A})\), is obtained by projection from the canonical Lagrangian Poisson bracket (1).

The reversible evolution equations generated by this Poisson bracket are

\[
\begin{align*}
\partial_t \rho &= -\partial_i (\rho \mathbf{m}_i) \\
\partial_t \mathbf{m}_i &= -\partial_j (\mathbf{m}_j \mathbf{E}_{m_i}) - \rho \partial_i \mathbf{E}_\rho - \mathbf{m}_j \partial_i \mathbf{E}_{m_j} - s \partial_i \mathbf{E}_s - \mathbf{A}^l \partial_i \mathbf{E}_{\mathbf{A}^l} \\
&\quad + \partial_i (\mathbf{A}^l \mathbf{E}_{\mathbf{A}^l}) - \partial_i (\mathbf{A}^l \mathbf{E}_{\mathbf{A}^l}) \\
\partial_t s &= -\partial_i (s \mathbf{E}_{m_i}), \\
\partial_t \mathbf{A}^l &= -\partial_i (\mathbf{A}^l \mathbf{E}_{m_i}) + (\partial_i \mathbf{A}^l - \partial_i \mathbf{A}^l \mathbf{E}_{m_i}).
\end{align*}
\]

Again, once the energy is specified, the equations acquire an explicit form. These evolution equations are part of the Symmetric Hyperbolic Thermodynamically Compatible (SHTC) equations, originally found in [17], see also [30, 31].

Although the distortion was defined as inverse of the deformation gradient, meaning that \(\partial_i \mathbf{A}^l = \partial_i \mathbf{A}^l \mathbf{e} \), we have actually never used this property. This is the crucial point making distortion advantageous, since by including dissipation this condition can be violated, i.e.

\[
\mathcal{T}_{ij} = \partial_i \mathbf{A}_j - \partial_j \mathbf{A}_i \neq 0 \quad \text{or} \quad \nabla \times \mathbf{A} \neq 0.
\]

Tensor \(\mathcal{T}_{ij}\) is called torsion tensor [29], and it expresses incompatibility in the deformation field. Its physical interpretation depends on the physical context. Usually it is interpreted as the dislocation density (or Burgers) tensor [33, 29] but also can be used to represent the spin of the distortion field \(\mathbf{A}\) which can be associated with the small-scale vortex dynamics in turbulent flows as discussed in [29]. The Lagrangian configuration is then no longer uniquely determined because integration of \(\mathbf{A}\) over a closed loop does not necessarily yield zero, which is how dislocations are naturally incorporated into the mechanics, e.g. see [34]. Hence, we have equipped Eulerian coordinates with natural state variables which do not dwell on the existence of a continuous mapping connecting the reference Lagrangian and actual configurations and allow to include formation and propagation of defects in continuous medium.

It is also important to note that Poisson bracket (22) satisfies Jacobi identity (2a) unconditionally, see [30], which determines the form of the bracket even for \(\nabla \times \mathbf{A} = 0\).

2.4.1 Fluid mechanics

The Poisson bracket expressing kinematics of fluid mechanics can be obtained for instance by projection from bracket (13) to the fluid fields, \(\rho, \mathbf{m}\) and \(s\), i.e. by letting the functionals \(F\) and \(G\) depend only on those fields. Bracket (14) is then obtained, and the evolution equations become

\[
\begin{align*}
\partial_t \rho &= -\partial_i (\rho \mathbf{m}_i) \\
\partial_t \mathbf{m}_i &= -\partial_j (\mathbf{m}_j \mathbf{E}_{m_i}) - \rho \partial_i \mathbf{E}_\rho - \mathbf{m}_j \partial_i \mathbf{E}_{m_j} - s \partial_i \mathbf{E}_s \\
\partial_t s &= -\partial_i (s \mathbf{E}_{m_i}).
\end{align*}
\]

With energy

\[
E(\rho, \mathbf{m}, s) = \int \mathrm{d}x \frac{\mathbf{m}^2}{2\rho} + \int \mathrm{d}x \epsilon(\rho, s)
\]

(26)
the usual form of compressible Euler equations are recovered. One can even get for instance the Korteweg fluid equations simply by letting the energy depend on \((\nabla \rho)^2\), see e.g. [17].

Fluid mechanics can be also seen as evolution on the coadjoint orbit of the infinite-dimensional Lie group of diffeomorphism of a domain to itself [35, 8]. The Hamiltonian formulation of fluid mechanics is especially useful for instance in stability analysis [36].

2.5 Jacobi identity

Jacobi identity (2a) is an inherent property of Poisson brackets, explicit verification of which is usually a formidable task. This difficulty was overcome by program [37] checking the identity in an automatized way. What is the reason for such interest in Jacobi identity? We address this question in the following sub-sections.

2.5.1 Self-consistency of Hamiltonian dynamics

Hamiltonian evolution of state variables \(q\) can be expressed by Eq. (6), and from the geometric point of view it can be seen as motion in the state space where \(q\) belongs. The curves \(q(t)\) are integral curves of the Hamiltonian vector field \(X_E\) the components of which represent the right hand side of Eq. (6),

\[
X_E \overset{\text{def}}{=} L^{\alpha\beta} E_{q^\beta} \partial_{q^\alpha}, \quad \partial_{q^\alpha} \overset{\text{def}}{=} \frac{\partial}{\partial q^\alpha},
\]

or

\[
\dot{q}^\alpha = \{q^\alpha, E\} = L^{\alpha\beta} E_{q^\beta} = (L \cdot dE)(q^\alpha) = X_E(q^\alpha) = X_E^\alpha.
\]

Hamiltonian evolution can be seen as motion along the Hamiltonian vector field generated by energy \(E\).

Having the Hamiltonian vector field, let us ask the question whether the structure of Eq. (28) is kept during the evolution. Taking arbitrary functionals \(F, G\) and \(E\), we have

\[
\{\{F, G\}, E\} = \mathcal{L}_{X_E} (dF \cdot L \cdot dG)
\]

\[= \mathcal{L}_{X_E} (dF) \cdot L \cdot dG + dF \cdot \mathcal{L}_{X_E} (L) \cdot dG + dF \cdot L \cdot dF \cdot \mathcal{L}_{X_E} (dG)
\]

\[= d\mathcal{L}_{X_E} F \cdot L \cdot dG + dF \cdot \mathcal{L}_{X_E} (L) \cdot dG + dF \cdot L \cdot dF \cdot \mathcal{L}_{X_E} G
\]

\[= \{\{F, E\}, G\} + dF \cdot \mathcal{L}_{X_E} (L) \cdot dG + \{F, \{G, E\}\}, \tag{29}
\]

where \(\mathcal{L}_{X_E}\) is the Lie derivative with respect to the Hamiltonian vector field, see e.g. [38], which commutes with differential \(d\). Using Jacobi identity, Eq. (2a), we obtain that

\[
dF \cdot \mathcal{L}_{X_E} (L) \cdot dG = 0 \quad \forall F, G, \text{which means that} \quad \mathcal{L}_{X_E} L = 0. \tag{30}
\]

We have thus proved the following proposition, c.f. [39],

**Proposition 2.** Lie derivative of Poisson bivector along the Hamiltonian vector field, given by Eq. (27), is zero,

\[
\mathcal{L}_{X_E} L = 0, \tag{31}
\]

see e.g. [39].

This tells that the Poisson bivector \(L\) does not change along the evolution of the system. Jacobi identity can be seen as a condition of self-consistency of the reversible Hamiltonian evolution.
2.5.2 Criterion when constructing Poisson brackets

Jacobi identity is also useful as a decisive criterion when choosing between several possible forms of a Poisson bracket. For instance, in [22], it led to the identification of coupling between the mechanics of the Eulerian deformation gradient $F(x)$ and fluid mechanics. Similarly, in [30] bracket (22) is derived by projection from a simpler bracket for fluid mechanics with labels (distortion being spatial gradient of labels). By adding terms to the bracket that are zero for compatible distortion matrices ($\nabla \times A = 0$), Jacobi identity becomes valid unconditionally (even with incompatible distortion), and bracket (22) is recovered.

In [40], the Poisson bracket for the infinite Grad hierarchy in kinetic theory was formulated. Projection for instance to the first ten moments (fluid mechanics and the matrix of second moments) does not end up in a closed form. Jacobi identity can be seen as a closure criterion so that the resulting evolution equations become frame invariant.

2.6 Gauge invariance, symmetries and conserved quantities

We shall now make few observations regarding conserved quantities, symmetries and transformation invariants in Hamiltonian systems. These links and properties follow again from the structure of Poisson bracket and allow stronger statements about symmetries and conservation laws than in Lagrangian systems as observed by Noether.

We approach this problem from two perspectives. The first one, Sec. 2.6.1, 2.6.2 and 2.6.3, is rather intuitive, easily understandable and invoking the properties of Hamiltonian systems but rather formal. Subsequently, in Sec. 2.6.4, we built upon rigorous results from [41].

2.6.1 Noether theorem

Assume now that there is a conserved quantity $G(q)$ of the Hamiltonian system (28), i.e. $\{G, E\} = 0$. This property can be rewritten as

$$0 = \{G, E\} = G_\alpha L^{\alpha \beta} E_{q^\beta} = X_E(G) = L_{X_E} G, \quad (32)$$

which means that the action of the vector field on functional $G$ (conserved quantity) is zero. From antisymmetry of the Poisson bracket we also have

$$0 = \{E, G\} = E_{q^\alpha} L^{\alpha \beta} G_{q^\beta} = X_G(E) = L_{X_G} E, \quad (33)$$

which means that field $X_G$ represents a symmetry of the Hamiltonian in the following sense.

**Definition 1** (Symmetry of Hamiltonian). A vector field $X$ is a symmetry of Hamiltonian $E$ when its action on the Hamiltonian is zero, $L_X E = X(E) = 0$.

Looking at Eqs. (32) and (33), we obtain the Hamiltonian version of famous Noether theorem.

**Theorem 2** (Emmy Noether and her inverse in Hamiltonian setting). *Any conserved quantity $G(q)$ of Hamiltonian system (28) generates a Hamiltonian vector field $X_G$, which is a symmetry of the Hamiltonian, $L_{X_G} E = 0$. Conversely, if a Hamiltonian vector field $X_G$ is a symmetry of the Hamiltonian, then generator $G$ of the field is conserved. See also [42].*
2.6.2 Symmetry of a Hamiltonian dynamical system

Let us now focus on infinitesimal symmetries of Hamiltonian dynamical systems (28). Note that the calculations are formal and well substantiated only for finite dimensional systems. However, we shall proceed in this formal treatment as it elucidates the geometric content.

A Hamiltonian vector field $X_G$ defines an infinitesimal transformation of state variables

$$q^\alpha = q^\alpha + \varepsilon X_G(q^\alpha) = q^\alpha + \varepsilon \pounds_{X_G} q^\alpha = q^\alpha + \varepsilon (q^\alpha, G).$$  \hspace{1cm} (34)

This formula actually represents infinitesimal transformations in a broader sense, and we shall comment on this later after introducing the rigorous definition.

For Lie derivatives it holds that, see e.g. [38],

$$\pounds_{[X,Y]} = \pounds_X \pounds_Y - \pounds_Y \pounds_X$$

for any vector fields $X$ and $Y$. Therefore, assuming that a vector field $X$ commutes with $X_E$, $[X,X_E] = 0$, it follows that

$$\pounds_X \pounds_{X_E} = \pounds_{X_E} \pounds_X.$$  \hspace{1cm} (36)

Considering Hamiltonian system (28), solution after an infinitesimal time $dt$ is equal to

$$q^\alpha(t + dt) = q^\alpha(t) + dt \pounds_{X_E} q^\alpha(t)$$

with correction terms of order $(dt)^2$. A vector field $X$ generates infinitesimal transformation (34). Assuming that it commutes with $X_E$, i.e. identity (36) holds, the transformed variables at time $t + dt$ are

$$q^\alpha(t + dt) = q^\alpha(t + dt) + \varepsilon \pounds_X q^\alpha(t + dt)$$

$$= q^\alpha(t) + dt \pounds_{X_E} q^\alpha(t) + \varepsilon \pounds_X q^\alpha(t) + \varepsilon \pounds_X (dt \pounds_{X_E} q^\alpha(t))$$

$$= q^\alpha(t) + \varepsilon \pounds_X q^\alpha(t) + dt \pounds_{X_E} (q^\alpha(t) + \varepsilon \pounds_X q^\alpha(t))$$

$$= q^\alpha(t) + \varepsilon \pounds_X q^\alpha(t).$$ \hspace{1cm} (38)

Therefore, the transformed quantities (34) obey the same evolution equations as the quantities before transformation, and thus they have the same set of solutions. Such a transformation is called a symmetry of the dynamical system. Hence we have shown the following proposition.

**Proposition 3** (Symmetries of Hamiltonian system). A symmetry of the dynamical system is any vector field $X$ that commutes with the Hamiltonian vector field,

$$0 = [X_E, X] = X_E X - XX_E = \pounds_{X_E} X - \pounds_X X_E.$$  \hspace{1cm} (39)

The symmetry induces an infinitesimal transformation (34), after which the Hamiltonian system has the same set of solutions (in short time) as the original system, see also [47].

2.6.3 Symmetries of Hamiltonian systems and conserved quantities

Finally, let us now turn to the question whether a conserved quantity also represents a symmetry of the Hamiltonian dynamical system. Using the definition of the symmetry, Eq. (39), we have

$$- \pounds_{X_G} X_E = \pounds_{X_E} X_G = \pounds_{X_E} (L \cdot dG) = \pounds_{X_E} L \cdot dG + L \cdot \pounds_{X_E} dG$$

$$= \pounds_{X_E} L \cdot dG + L \cdot d \pounds_{X_E} G = \pounds_{X_E} L \cdot dG = 0,$$ \hspace{1cm} (40)
where we used that the Lie derivative commutes with differential, $\mathcal{L}d = d\mathcal{L}$, see e.g. [38], and where Jacobi identity was used, see Proposition 2. From this observation it follows that if a functional $G$ is a conserved quantity, $\dot{G} = 0$, of a Hamiltonian dynamical system (28), the Hamiltonian vector field generated by $G$ is a symmetry of the Hamiltonian system. Using this identity in Eq. (40) we obtain the following theorem

**Theorem 3.** Assume that functional $G$ is a conserved quantity of a Hamiltonian dynamical system (28), see Eq. (32). Then the Hamiltonian vector field generated by $G$ is a symmetry of the Hamiltonian system in the sense of Eq. (39), cf. [41].

Hence we not only know the obvious fact that a conservation law generates a symmetry, but we know the relation explicitly, i.e. a direct relation between a conserved quantity and an invariance in the system. This result can be extended to a general case, but certain technical extensions of the concepts here have to be carried out [41], e.g. the relation (34) is no longer a transformation (strictly speaking), but can be shown to be a prolongation of a Hamiltonian vector field.

### 2.6.4 Rigorous generalization

Theorem 7.15 from [41] reveals an equivalence between a conservation law and a symmetry but on top of that it provides a connection between the conserved quantity $G$ and a particular symmetry of the system. This symmetry corresponds to a Hamiltonian vector field $\mathbf{v}_G$ with characteristic (components) $Q = \{q, G\}$ (in our notation) and where state variables are denoted as $q$. Now, a Hamiltonian vector field is a special type of evolutionary vector field (having non-zero components only those corresponding to the state variables) such that its prolongation is equal to the Poisson bracket in the following sense

$$\text{pr} \, \mathbf{v}_Q(F) \overset{\text{def}}{=} \{F, Q\} \quad \forall F(t, x, q).$$

(41)

Note that it can be shown that for any Poisson bracket and any functional $F$, such an evolutionary vector field exists. In particular, given $\mathbf{v} = \tau \partial_t + \xi^i \partial_{x^i} + \phi^\alpha \partial_{q^\alpha}$, $\tau$, $\xi^i$ and $\phi^\alpha$ being components of the vector field, then a corresponding evolutionary vector field (which form is particularly convenient for calculating prolongations) is

$$\mathbf{v}_Q = Q^\alpha \{t, x, q, \nabla q\} \partial_{q^\alpha}$$

(42)

where $Q^\alpha = \phi^\alpha - \xi^i \partial_{x^i} q^\alpha - \tau \partial_t q^\alpha$. A vector field $\mathbf{v}$ is a symmetry of a system if and only if its evolutionary representative vector field $\mathbf{v}_Q$ is a symmetry of the system.

First, we make few observations.

**Finite dimension** For a canonical Poisson bracket we have that

$$\mathbf{v} = \mathbf{v}_Q = \text{pr} \, \mathbf{v} = \text{pr} \, \mathbf{v}_Q,$$

as all the derivatives are only with respect to state variables and hence

$$\mathbf{v} = \text{pr} \, \mathbf{v}_Q = \{q, G\}$$

defines an infinitesimal transformation of dependent (state) variables

$$\bar{q} = q + \varepsilon \{q, G\}$$

which is actually a symmetry of the system. Note that this observation extends to any finite dimensional system due to Darboux’ theorem stating that any Poisson bracket in finite dimensions can be rewritten locally into a canonical form.
**ALGEBRAIC POISSON BRACKET** By repeating the same arguments as in the previous point we can similarly show that for an algebraic Poisson bracket (now independent of dimension) a transformation of dependent (state) variables

\[ \tilde{q} = q + \varepsilon(q, G) \]

is a symmetry of the system.

We shall now proceed to the general case where we already know that symmetry exists (the system is invariant to this transformation) once we know there is a conserved quantity \( G \). The aim is to find this transformation explicitly. From the theorem (7.15 in [41]) we know that \( \{q^\alpha, G\} = Q^\alpha \) defines a characteristic \( Q^\alpha \) of the evolutionary vector field and hence it has to be of the following form

\[ Q^\alpha = \phi^\alpha(t, x, q) - \xi^i(t, x, q) \partial_x^i q^\alpha - \tau(t, x, q) \partial_t q^\alpha, \]

which then determines a transformation (via the corresponding vector field \( \nu = \xi^i \partial_x^i + \tau \partial_t + \phi^\alpha \partial_{q^\alpha} \))

\[ \begin{align*}
\tilde{t} &= t + \varepsilon \tau(t, x, q), \\
\tilde{x}^i &= x^i + \varepsilon \xi^i(t, x, q), \\
\tilde{q}^\alpha &= q^\alpha + \varepsilon \phi^\alpha(t, x, q).
\end{align*} \tag{44} \]

The two above special cases can now be easily understood as well as they correspond to the special situation when \( \xi^i = 0, \tau = 0 \).

Hence we showed the following statement:

**Theorem 4.** Any Hamiltonian system with a conserved quantity \( G \) is invariant to transformation (44) with \( \{q^\alpha, G\} = \phi^\alpha(t, x, q) - \xi^i(t, x, q) \partial_x^i q^\alpha - \tau(t, x, q) \partial_t q^\alpha \), providing identification of \( \phi^\alpha, \xi^\alpha \) and \( \tau \) in a unique way, cf. [41].

Note that transformations (34) and (44) are compatible as infinitesimal transformations, since the former can be seen as Taylor series to the first order of the latter. Therefore, the former can be seen as transformations in a broader sense (allowing also for differential operators).

Let us illustrate the above observations on a few examples.

**HAMILTONIAN MECHANICS OF PARTICLES.** State variables are position and momentum, \( q = (r, p) \), while the only independent variable is time \( t \), and the Poisson bracket is in a canonical form. In this case we know that the relation between conserved quantity \( G \) and invariant transformation is particularly simple: \( \tilde{r} = r + \varepsilon(r, G), \tilde{p} = p + \varepsilon(p, G) \), as in Eq. (34).

From conservation of particle momentum, \( G = c^i p_i \) (for an arbitrary fixed vector \( c \)), we immediately get invariance to translations \( \tilde{r}^j = r^j + \varepsilon c^j, \tilde{p} = p \). Conservation of angular momentum \( G = c \cdot (r \times p) \) yields invariance to rotations \( \tilde{r} = r + \varepsilon c \times r \). Finally, conservation of Galilean boost \( G = m r - tp \) yields \( \tilde{r} = r + \varepsilon(r, G) = r - t \bar{I}, \tilde{p} = p + \varepsilon(p, G) = p - m \bar{I} \).

**KORTEWEG-DE VRIES.** equation reads

\[ u_t = u_{xxx} + uu_x = \{u, H\}, \]  

(45)

\begin{flushright}
not to be confused with continuum momentum density \( m \)
\end{flushright}
where 
\[ \mathcal{H}(u) = \int -\frac{1}{2}u_x^2 + \frac{1}{6}u^3\,dx, \quad \{A, B\} = \int \delta A \frac{d}{dx}\delta B\,dx. \]

Conservation of the following functionals
\[ P_1 = \int \frac{1}{2}u_x^2\,dx, \quad P_2 = \int \left( -\frac{1}{6}u^3 - \frac{1}{2}u_x^2 \right)\,dx, \quad P_3 = \int (ux + \frac{1}{2}tu^2)\,dx, \]
corresponds to characteristics
\[ Q_1 = \{u, P_1\} = u_x, \quad Q_2 = \{u, P_2\} = u_{xxx} + uu_x = u_4, \quad Q_3 = \{u, P_3\} = 1 + tu_x, \]
and hence, to invariants of the problem
\[ \tilde{t} = t + \varepsilon, \quad \tilde{x} = x + \varepsilon, \quad \tilde{u}(x, t) = u(x - \varepsilon t, t) + \varepsilon. \]

**LAGRANGIAN FRAME OF FLUID MECHANICS.** First, let us choose the Lagrangian frame, bracket (1), and \( G = \int \, dX M_i(X) \) equal to the \( j \)-th component of the total momentum. Using energy (10), the Poisson bracket of \( G \) and energy is
\[ \{G, E\}^{(L)} = -\int \, dX E \frac{\partial}{\partial X^i} \left( \rho_0(X) \frac{\partial W}{\partial X^i} \right) = 0, \quad (46) \]
which means that total the momentum is conserved. Infinitesimal transformations of the state variables \( x(X) \) and \( M(X) \) are (as above)
\[ \dot{x}^i(X) = x^i(X) + \varepsilon \{x^i(X), G\}^{(L)} = x^i(X) + \varepsilon \delta^i_k, \quad (47a) \]
\[ \dot{M}_i(X) = M_i(X) + \varepsilon \{M_i(X), G\}^{(L)} = M_i(X). \quad (47b) \]
The first equation expresses infinitesimal translation in the \( k \)-direction and hence, the evolution equations are invariant with respect to infinitesimal translations, as it follows directly from the conservation of the total momentum \( G \).

**EULERIAN FRAME OF FLUID MECHANICS.** In the Eulerian setting, let us choose the Poisson bracket for fluid mechanics (14) and state variables \( (\rho, m, s) \).

The consequence of energy conservation is that \( \{\rho, E\} \) has to be a characteristic of the evolutionary vector field. Indeed, we have
\[ \{\rho, E\} = Q_\rho = \partial_t \rho, \]
which yields \( (\phi_\rho = 0, \, \xi^i = 0, \, \tau = -1) \) invariance of the density field to the transformation \( \tilde{t} = t + \varepsilon \). In the same manner all other state variables are invariant translation in time.

Further let us inspect the implications of conservation of the total momentum \( G = \int \, dX m_i(X) \). Calculating the symmetry
\[ \{\rho, G\} = \{\rho, m_i\} = \int \, dx b \rho(x_b) \frac{\partial \delta(x_a - x_b)}{\partial x^i_b} = -\partial_i \rho \]
reveals that density is invariant to translations, \( \dot{x}^i = x^i + \varepsilon \). Similarly, from
\[ \{m_i, G\} = \{m_i, m_j\} = \int \, dx b m_i(x_b) \frac{\partial \delta(x_a - x_b)}{\partial x^i_b} - m_j(x_a) \frac{\partial \delta(x_a - x_b)}{\partial x^j_a} = -\partial_j m_i, \]
we have the same observation that the momentum is invariant to translations. Then, from
\[ \{s, G\} = \{s, m_j\} = \int dx_b s(x_b) \frac{\partial \delta(x_a - x_b)}{\partial x^j_b} = -\partial_j s, \]
we can conclude that fluid mechanics is invariant to translations as we know that the total momentum is conserved.

Finally, let us choose G as the overall Galilean booster, see e.g. [43],
\[ G_i = \int \left( \rho(x)x_i - tm_i(x) \right) g_i(x). \quad (48) \]
The booster is indeed conserved, since
\[ G_i = \int dx \delta_i g_i = \]
\[ = \int dx m_i + \int dx \left( \rho \delta_i E_i + \partial_i m_j + s \partial_i E_s \right) \]
\[ = \int dx (m_i - \rho \delta_i ) + t \int dx \partial_i p = 0 \quad (49) \]
for energy (26), p being the pressure, see e.g. [17]. The fluid fields transform to
\[ \bar{\rho} = \rho + \epsilon(\rho, G)^{FM} = \rho + \epsilon t \partial_j \rho \]
\[ \bar{m}_j = \rho + \epsilon(m_j, G)^{FM} = m_j + \epsilon t \partial_i m_j - \epsilon \rho \delta_{ij} \]
\[ \bar{s} = s + \epsilon(s, G)^{FM} = s + \epsilon t \partial_i s, \]
which corresponds to infinitesimal transformation
\[ \tau = 0, \quad (51) \]
\[ \xi^i = -\delta_{ij} t, \quad (52) \]
\[ \phi_\rho = 0, \quad (53) \]
\[ \phi_{m_i} = -\rho \delta_{ij}, \quad (54) \]
\[ \phi_s = 0, \quad (55) \]
representing Galilean transformation. Hence, we showed that Galilean boost is conserved and as a result the dynamics of the system is invariant to Galilean transformation.

Similarly, for the extended bracket (22), which expresses kinematics of fields \( \rho, m, s \) and \( A \), the Galilean booster (48) is also conserved as it is followed from a similar calculation as above. The infinitesimal Galilean transformation then reads
\[ \bar{\rho} = \rho + \epsilon(\rho, G_k)^{(A)} = \rho + \epsilon t \partial_k \rho \]
\[ \bar{m}_i = \rho + \epsilon(m_i, G_k)^{(A)} = m_i + \epsilon t \partial_k m_i - \epsilon \rho \delta_{ik} \]
\[ \bar{s} = s + \epsilon(s, G_k)^{(A)} = s + \epsilon t \partial_k s \]
\[ \bar{A}_j^i = A_j^i + \epsilon(A_j^i, G_k)^{(A)} = A_j^i + \epsilon t \partial_k A_j^i, \]
and evolution equations (23) are transformed in the same way: they are Galilean invariant.

For a Hamiltonian system, it is possible to prove invariance of the evolution equations with respect to an infinitesimal transformation by showing the conservation of the transformation generator. For instance, Galilean invariance can be shown relatively easily by proving conservation of Galilean booster, and the method is not restricted to evolution equations in the form of conservation laws.
2.7 Hyperbolicity

Hyperbolicity is an essential feature of many systems in continuum thermodynamics \cite{44, 45, 46, 47}, since it provides well-posedness of the Cauchy problem locally in time and causality. However, it is not easy to check hyperbolicity when the equations are not in the form of system of conservation laws admitting an extra conservation law (typically energy conservation), see e.g. \cite{30} and the Godunov-Boillat theorem, first proposed by Godunov \cite{6}, and generalized in \cite{48, 49, 50} and \cite{51}. Note that in this section no integration is meant when summing over repeated indices.

2.7.1 Riemannian approach

It is sometimes possible, however, to infer hyperbolicity just from the Hamiltonian character of the equations. Let us first recall some results by Dubrovin, Novikov \cite{52} and Tsarëv \cite{53}. A Poisson bracket in 1D is of hydrodynamic type if the corresponding Poisson bivector has the form

\[ L^{\alpha \beta} = \{ q^\alpha(x), q^\beta(y) \} = g^{\alpha \beta}(q(x)) \partial_x \delta(x-y) + b^{\alpha \beta}(q(x)) \partial_x q^\gamma \delta(x-y). \quad (57) \]

The restriction to 1D is not essential as shown in \cite{54}. Energy is of hydrodynamic type if it is an integral of a function of the state variables,

\[ E = \int dx e(q(x)). \quad (58) \]

The evolution equations generated by a hydrodynamic-type Poisson bracket and hydrodynamic-type energy are quasilinear partial differential equations of first order

\[ \partial_t q^\alpha = v^\alpha_\beta(q(x)) \partial_x q^\beta \quad (59) \]

with

\[ v^\delta_\gamma = g^{\delta \beta} \frac{\partial^2 e}{\partial q^\beta \partial q^\gamma} + b^{\delta \beta} \frac{\partial e}{\partial q^\beta}. \quad (60) \]

as follows by direct calculation. The Poisson brackets discussed in this paper are all of hydrodynamic type.

It was shown in \cite{52} that

1. Under local changes \( Q = Q(q) \) of the vector of state variables, the coefficient \( g^{\alpha \beta} \) in (57) is transformed as a tensor with upper indices; if \( \det g^{\alpha \beta} \neq 0 \), then the expression, cf. Eq. (57), \( b^{\alpha \beta} = -g^{\alpha \delta} \Gamma^\beta_\delta \) is transformed so that \( \Gamma^\beta_\delta \) is the Christoffel symbol of an affine connection.

2. For the Poisson bivector to be antisymmetric it is necessary and sufficient that the tensor \( g^{\alpha \beta} \) be symmetric (i.e. it defines a pseudo-Riemannian metric, if \( \det g^{\alpha \beta} \neq 0 \)) and the connection \( \Gamma^\alpha_\beta \) is metric compatible: \( \nabla_\gamma g^{\alpha \beta} = 0 \forall \alpha, \beta, \gamma \), where \( \nabla_\gamma \) is the associated covariant derivative in the space of state variables.

3. For the bracket to satisfy Jacobi identity it is necessary and sufficient (in the case \( \det g^{\alpha \beta} \neq 0 \)) that the connection \( \Gamma^\beta_\delta \) be torsion- and curvature-free.

Assuming the non-degenerate case, the covariant Hessian of energy can be rewritten as

\[ \nabla_\delta \nabla_\gamma e = \frac{\partial}{\partial q^\delta} \frac{\partial e}{\partial q^\gamma} - \Gamma^\beta_\delta \frac{\partial e}{\partial q^\beta}. \quad (61) \]

from which it follows that the matrix \( v^\delta_\gamma \) can be rewritten as

\[ v^\gamma_\delta = g^{\delta \beta} \nabla_\beta \nabla_\delta e. \quad (62) \]
Furthermore, let us assume that the energy \( e(q) \) is a proper scalar (i.e. \( e(q) = e(Q(q)) \)). Then, due to that there is neither torsion nor curvature, the covariant derivatives commute and the matrix \( e_{\delta \gamma} \) is symmetric. The evolution equations (59) then become

\[
\partial_t q^\alpha = g^{\alpha \beta} e_{\beta \gamma} \partial_x q^\gamma,
\]

which can be symmetrized by multiplying it by the covariant Hessian of energy \( e_{\delta \alpha} \). One obtains

\[
e_{\delta \alpha} \partial_t q^\alpha = e_{\delta \alpha} g^{\alpha \beta} e_{\beta \gamma} \partial_x q^\gamma.
\]

If the energy is convex, then it follows that when taking a curve in the space of state variables, second derivative of energy with respect to a parameter parametrizing the curve is positive. This holds true in particular for geodesic curves. Therefore, energy is also geodesic convex [55] and its covariant Hessian is positive definite, i.e. symmetric positive definite. Equations (63) are thus equivalent (at least regarding their strong solutions) to the original system (63), the matrix in front of the time-derivative is symmetric positive-definite, and the matrices in front of the spatial derivatives are symmetric. Equations (64) thus form a system of quasilinear first-order symmetric hyperbolic partial differential equations [56, 31]. These results can be summarized as the following theorem:

**Theorem 5.** Consider a Hamiltonian system of hydrodynamic type with non-degenerate metric (57). Assuming that the energy of the system is of hydrodynamic type, convex and a proper scalar, it follows that the evolution equations can be regarded as a first-order quasilinear symmetric hyperbolic PDE system.

Let us give a few examples. Isentropic fluids in one-dimension are described by the mass density and momentum density \((\rho, m)\) (so \(m\) is a one-dimensional covector field) and thus, the first two terms in Poisson bracket (14), have the metric

\[
g = \begin{pmatrix} 0 & -\rho \\ -\rho & -2m \end{pmatrix}.
\]

This metric is non-degenerate and symmetric, but also indefinite.

To include entropy, one has to add not only the entropy field, but also the field of conjugate entropy flux \( w \) (see [30]), since otherwise the metric would be degenerate [54]. The Poisson bracket, including \( s \) and \( w \) fields, is

\[
\{F, G\} = \{F, G\}^{(FM)} + \int dx (\nabla F_s \cdot G_w - \nabla G_s \cdot F_w),
\]

which leads to the metric (again in 1D, i.e. only first components \( w \) and \( m \) of \( w \) and \( m \) are considered)

\[
g = \begin{pmatrix} 0 & -\rho & 0 & 0 \\ -\rho & -2m & -s & -w \\ 0 & -s & 0 & -1 \\ 0 & -w & -1 & 0 \end{pmatrix},
\]

which is again symmetric, indefinite and non-degenerate.

Similarly, Poisson bracket (22) restricted to functionals dependent only on \((m, A)\) has non-degenerate metric. Entropy can be added by including also the \( w \) field as above. Density can be added either by its relation with \( \det(A) \) or by adding both density and a conjugate velocity-like field (similar to the \( w \) field) coupled to it, see [57].
2.7.2 Godunov-Boillat theorem

The usual way to show symmetric hyperbolicity of a system of quasilinear first order equations is the Godunov-Boillat theorem, see e.g. [30]. The theorem is based on the passage to a dual formulation by means of Legendre transformation

\[ q^\dagger_\alpha = \frac{\partial e}{\partial q^\alpha}, \quad q^\dagger_\alpha = \frac{\partial L}{\partial q^\dagger_\alpha} \quad \text{and} \quad L = -e + q^\alpha q^\dagger_\alpha, \quad (68) \]

and it works for systems of overdetermined conservation laws (automatically implying an extra conservation law, e.g. energy conservation). Note that \( L \) has the meaning of pressure, cf. Eq. (15e), and it is the complete Legendre transformation of energy density. For non-conservative systems, as for instance the SHTC equations in Section 2.4, the theorem can be applied either by restriction to the compatible systems (\( \text{curl} \, \mathbf{A} = 0 \)) or by extension promoting the incompatibility to an extra state variables (Burgers tensor), see [30]. The way based on the Hamiltonian structure of the equations, Eqs. (64), can be seen as an alternative (or extension) to the Godunov-Boillat theorem.

Let us now bring the two approaches to proving hyperbolicity closer to each other. If \( e_{\beta \gamma} \) were the usual derivatives (not covariant), i.e. \( e_{\beta \gamma} = \partial_\beta \partial_\gamma e \), the situation would be simple. Taking again \( q^\dagger_\alpha = e_{q^\alpha} \) and \( L = -e + q^\dagger_\alpha q^\alpha \), then \( q^\alpha = L q^\dagger_\alpha \) and the system of equations becomes

\[ \partial_t L q^\dagger_\alpha = g^{\alpha \beta} e_{\beta \gamma} \partial_x L q^\dagger_\gamma, \quad (69) \]

or

\[ L^{\alpha \delta} \partial_t q^\dagger_\delta = g^{\alpha \beta} e_{\beta \gamma} L^{\gamma \delta} \partial_x q^\dagger_\gamma, \quad \text{where} \quad L^{\alpha \delta} = L q^\dagger_\alpha q^\dagger_\delta. \quad (70) \]

Because \( e_{\beta \gamma} L^{\gamma \delta} = \delta^\delta_{\gamma} \), we would have a symmetric hyperbolic system

\[ L^{\alpha \delta} \partial_t p_\delta = g^{\alpha \beta} \partial_x p_\beta. \quad (71) \]

However, the Hessian \( e_{\beta \gamma} \) is made of covariant derivatives, not partial. By this remark, we would like to draw attention to this similarity between the two approaches (Godunov-Boillat and Riemannian). We would like to clarify this relation in more detail in future.

2.7.3 Euler-Poincaré equations

A yet another possibility to assess hyperbolicity can be based on the gauge invariance of the underlying Lagrangian. Consider a Lie group \( G \), e.g. the group of volume preserving diffeomorphisms within a domain, [8]. Having the Lie group, we can construct its Lie algebra \( \mathfrak{g} \) and the dual to the Lie algebra \( \mathfrak{g}^* \), which are naturally equipped with the adjoint (\( \text{ad} \)) and coadjoint (\( \text{ad}^* \)) actions, respectively. Dynamics on the Lie algebra dual is then given by the Euler-Poincaré equations, see e.g. [58, 59]

\[ \frac{d}{dt} \frac{\partial L}{\partial p_\alpha} = -\text{ad}^*_{p_\beta} \frac{\partial L}{\partial p_\alpha}, \quad (72) \]

where \( p_\alpha \) is an element of Lie algebra \( \mathfrak{g} \) and \( L \) is the Lagrangian. This resembles the Godunov-Boillat dualization, since \( q = L_p \) is the momentum \( m \in \mathfrak{g}^* \) for \( p \) being the velocity \( v \in \mathfrak{g} \).
The coadjoint action is defined through the corresponding adjoint action (noting that \( m \in \mathfrak{g}^* \)),

\[
\langle \text{ad}^*_m \mathbf{w} \rangle \overset{\text{def}}{=} -\langle m, \text{ad}_v \mathbf{w} \rangle = -\langle m, [\mathbf{v}, \mathbf{w}] \rangle = -\int \! \! dx m_i (v^j \partial_j w^i - w^j \partial_j v^i) \tag{73}
\]

\[
= \int \! \! dx \left( \partial_i m_i v^i + m_i \partial_i v^i + m_i \partial_i w^i \right) w^i = \int \! \! dx \left( \mathcal{L}_v m_i + \text{div}(v) m_i \right) w^i
\]

for all \( \mathbf{w} \in \mathfrak{g} \). The Euler-Poincaré equations \((72)\) then become

\[
\frac{\partial^2 L}{\partial v^i \partial v^j} \partial_t v^j = -v^j \frac{\partial^2 L}{\partial v^k \partial v^i} \partial_i v^k - \frac{\partial L}{\partial v^i} \partial_j v^j - \frac{\partial L}{\partial v^i} \partial_j v^j
\]

\[
= -\left( v^j \frac{\partial^2 L}{\partial v^k \partial v^i} + \frac{\partial L}{\partial v^i} \delta^j_k + \frac{\partial L}{\partial v^k} \delta^j_i \right) \partial_i v^k. \tag{74}
\]

Assuming that the Lagrangian is complete Legendre transformation of a convex Hamiltonian (energy), as in the Godunov-Boillat dualization, then it is also convex and the matrix on the left hand side of \((74)\) is symmetric and positive definite. Moreover, the matrix on the right hand side is symmetric (in \( i \leftrightarrow k \)), which leads to the following theorem.

**Theorem 6** (Hyperbolicity of Euler-Poincaré equations). Assume that the Euler-Poincaré equations \((72)\) are equipped with a convex energy function and that the Lagrangian is the complete Legendre transformation of the Hamiltonian. Assume, moreover, that the adjoint action is generated by the usual Jacobi-Lie bracket (commutator) on vector fields. Then the resulting evolution equations represent a symmetric hyperbolic system of first-order quasilinear partial differential equations.

2.8 Clebsch variables

Variational principles for fluid mechanics have been of great importance in physics. Clebsch \([60]\) found the canonical variables providing Hamiltonian structure to fluid mechanics, Seliger, Whitham \([61]\) and Lin \([62]\) equipped fluid mechanics with labels to gain the variational structure, \([63]\). See \([64]\) and \([59]\) for a clearer and more geometric explanation of the results. For instance in \([17]\), the Clebsch variables were written as \( \rho(x), \rho^*(x), \lambda(x), \lambda^*(x), s(x) \) and \( s^*(x) \) equipped with the canonical Poisson bracket for fields,

\[
\{ \delta F, \delta G \}^{\text{Clebsch}} = \int \! \! dx \left( \frac{\delta F}{\delta \rho} \frac{\delta G}{\delta \rho^*} - \frac{\delta G}{\delta \rho} \frac{\delta F}{\delta \rho^*} \right)
\]

\[
+ \int \! \! dx \left( \frac{\delta F}{\delta \lambda} \frac{\delta G}{\delta \lambda^*} - \frac{\delta G}{\delta \lambda} \frac{\delta F}{\delta \lambda^*} \right)
\]

\[
+ \int \! \! dx \left( \frac{\delta F}{\delta s} \frac{\delta G}{\delta s^*} - \frac{\delta G}{\delta s} \frac{\delta F}{\delta s^*} \right) \tag{75}
\]

for all \( F \) and \( G \) smooth enough functionals of the Clebsch variables. The evolution equations implied by this canonical bracket are

\[
\partial_t \rho = \frac{\delta E}{\delta \rho^*}, \quad \partial_t \rho^* = -\frac{\delta E}{\delta \rho}, \quad \text{etc.,} \tag{76}
\]

and they can be seen as a consequence of the principle of least action (with variations vanishing at boundaries)

\[
\delta \int_{t_1}^{t_2} \! \! dx L(\rho, \rho, \lambda, \lambda, s, s) = 0, \tag{77}
\]
where $L$ is the Lagrangian density related to energy by the Legendre transformation.

Fluid mechanics is then obtained by the projection

$$
\rho = \rho_i, \quad (78a)
$$

$$
m_i = \rho \partial_i \rho^* + \lambda \partial_i \lambda^* + s \partial_i s^*, \quad (78b)
$$

$$
s = s, \quad (78c)
$$

under which the canonical Clebsch Poisson bracket turns to the Poisson bracket for fluid mechanics (14).

One can, however, project the Clebsch variables not only to the fluid mechanics, but also to keep the $\lambda$ field, which can be seen as volume density of labels. Indeed, the projection then leads to Poisson bracket

$$
\{F, G\}^{(FM)} + \int \rho \lambda \left( \nabla F_{\lambda} \cdot G_m - \nabla G_{\lambda} \cdot F_m \right), \quad (79)
$$

which implies the evolution equation for $\lambda$

$$
\partial_t \lambda = - \partial_i (\lambda \rho m_i). \quad (80)
$$

Subsequent change of variables to $X = \lambda/\rho$ then yields

$$
\partial_t X = - E_m \partial_i X, \quad (81)
$$

which is a simple advection of function $X$ (a marker or a label) by the fluid. Starting with the three fields $\lambda^i$, $i$ being a Lagrangian index, the resulting Poisson bracket (called Lin bracket) is

$$
\{F, G\}^{(Lin)} = \{F, G\}^{(FM)} + \int \rho \lambda \left( \partial_j X^i \left( F_{m_i} G_{X^j} - G_{m_i} F_{X^j} \right) \right), \quad (82)
$$

see [65, 17] for more details. The Lin Poisson bracket yields evolution equations for fluid mechanics equipped with labels

$$
\partial_t \rho = - \partial_i (\rho E_{m_i}), \quad (83a)
$$

$$
\partial_t m_i = - \rho \partial_i E_{\rho} - m_j \partial_i E_{m_j} - s \partial_i E_s - \partial_i X^i, \quad (83b)
$$

$$
\partial_t s = - \partial_i (s E_{m_i}), \quad (83c)
$$

$$
\partial_t X^i = - E_m \partial_i X^i. \quad (83d)
$$

Finally, the field of labels $X^i$ can be projected to the distortion matrix through

$$
A^i_{\lambda} = \frac{\partial X^i}{\partial \lambda}, \quad (84)
$$

and the consequent projection of the Lin Poisson bracket leads to the Poisson bracket for distortion matrix (13). Note however, that the terms proportional to $\nabla \times A$ do not appear in the result automatically. They are zero provided the construction (84) is smooth enough, which is why they do not appear. However, the Jacobi identity is then fulfilled provided that $\nabla \times A = 0$, and the terms in (13) proportional to $\nabla \times A$ have to be added to make the Jacobi identity fulfilled unconditionally, see [30].

In summary, Clebsch variables provide an alternative formulation of variational principle for fluid mechanics and fluid mechanics with distortion.
2.9 Semidirect product structure

It is well known that mechanics (i.e. reversible evolution) of fluids is Hamiltonian, see e.g. [8, 66] or [17]. Fluid mechanics, in particular, is a realization of Lie-Poisson dynamics, where the Poisson bracket is the Lie-Poisson bracket on a Lie algebra dual. Another examples of Lie-Poisson dynamics are rigid body rotation or kinetic theory. In [66] it is explained how to construct new Hamiltonian dynamics by letting one Hamiltonian dynamics be advected by another. Having a Lie algebra dual \( \mathfrak{l}^* \) (for instance fluid mechanics), an another Lie algebra dual or cotangent bundle is advected by \( \mathfrak{l}^* \) by the construction of semidirect product.

One can even think of mutual action of the two Hamiltonian dynamics, which leads to the structure of matched pairs [67], [68]. For the purpose of this paper, however, we restrict the discussion only to one-sided action of one Hamiltonian system to another, i.e. to the semidirect product. A general formula for the Poisson bracket of semidirect product of a Lie algebra dual \( \mathfrak{l}^* \) and cotangent bundle \( T^*M \) was presented for instance in [68, 69, 70],

\[
\{F, G\}^{(l^* \ltimes T^*M)} = \{F, G\}^{(l^*)} + \{F, G\}^{(T^*M)} + \{F_A G_m \triangleright A\} - \{A F_m \triangleright A\} + \{D | F_m \triangleright G_D\} - \{D | G_m \triangleright D\}
\]

(85)

where \( A \in V \) is a covector field, \( A = A_i dx^i \) and \( D \in V^* \) is a vector field \( D = D^i \partial_i \), \( \{F, G\}^{(l^*)} \) is the Lie-Poisson bracket on the Lie algebra dual, \( \{F, G\}^{(T^*M)} \) is the canonical Poisson bracket on the cotangent bundle, \( \langle \bullet, \bullet \rangle \) is a scalar product (usually \( L^2 \), i.e. integration over the domain), \( m \in l^* \) is the momentum density (element of the Lie algebra dual) and \( \triangleright \) is the action of \( l^* \) on \( T^*M \), minus the Lie derivative \( -\mathcal{L} \). Poisson bracket (85) can be thus rewritten as

\[
\{F, G\}^{(l^* \ltimes T^*M)} = \{F, G\}^{(l^*)} + \{F, G\}^{(T^*M)} - \{F_A \mathcal{L}_{G_m} A\} + \{A \mathcal{L}_{F_m} A\} - \{D | \mathcal{L}_{F_m} G_D\} + \{D | \mathcal{L}_{G_m} D\},
\]

(86)

with \( \{F, G\}^{(l^*)} = \{F, G\}^{(FM)} \) and

\[
\{F, G\}^{(T^*M)} = \{F_A, G_D\} - \{G_A, F_D\} = \int dx \left( \frac{\delta F}{\delta A_i} \frac{\delta G}{\delta D^i} - \frac{\delta G}{\delta A_i} \frac{\delta F}{\delta D^i} \right).
\]

(87)

Lie derivatives of vector and covector fields read (see e.g. [38])

\[
\mathcal{L}_\nu A = \left( \nu^j \frac{\partial A_i}{\partial x^j} + \frac{\partial \nu^j}{\partial x^i} A_j \right) dx^i
\]

(88a)

\[
\mathcal{L}_\nu D = \left( \nu^j \frac{\partial D^i}{\partial x^j} - \partial_j \nu^j \frac{\partial}{\partial x^i} \right)
\]

(88b)

and the Poisson bracket thus gains the explicit form (noting that \( F_D \) and \( G_D \) are covector fields)

\[
\{F, G\}^{(l^* \ltimes T^*M)} = \{F, G\}^{(FM)} + \int dx \left( \frac{\delta F}{\delta A_i} \frac{\delta G}{\delta D^i} - \frac{\delta G}{\delta A_i} \frac{\delta F}{\delta D^i} \right)
\]

(89)

\[
- \left( dxF_A (G_m \partial_i A_i + \partial_i G_m A_j) + dxA_A (F_m \partial_i A_i + \partial_i F_m A_j) \right)
\]

\[
- \left( dxD^i (F_m \partial_j G_{Dj} + \partial_i F_m G_{Di}) + dxD^i (G_m \partial_j F_{Dj} + \partial_i G_m F_{Di}) \right).
\]
This Poisson bracket expresses kinematics of a cotangent bundle (a vector field and a covector field) advected by a Lie algebra dual (for instance fluid mechanics).

Let us now equip the covector field $A$ with an additional (Lagrangian) index, which is equivalent to letting additional copies of the cotangent bundle be advected by the Lie algebra dual, $A_i \rightarrow A^i_1$. In analogy, the vector field $D$ becomes $D^i_1$, and the Poisson bracket becomes

$$\{F, G\}^{(\ast \times T^\ast M)} = \{F, G\}^{(FM)} + \int dx \left( \frac{\delta F}{\delta A^i_1} \frac{\delta G}{\delta D^i_1} - \frac{\delta G}{\delta A^i_1} \frac{\delta F}{\delta D^i_1} \right)$$

Evolution equations implied by this bracket are

$$\frac{\partial_t \rho}{\partial t} = -\partial_1 (\rho E_{m_1})$$

$$\frac{\partial_t m_i}{\partial t} = -\rho \partial_1 (m_i \rho E_{m_1} - m_j \partial_1 E_{m_j} - A^i_1 \partial_1 E_{A^i_1} - D^i_1 \partial_1 E_{D^i_1})$$

$$+ \partial_1 (A^j_1 \partial_1 E_{A^j_1}) - \partial_1 (A^i_1 \partial_1 E_{A^i_1}) + \partial_1 (D^j_1 \partial_1 E_{D^j_1})$$

$$\frac{\partial_t A^i_1}{\partial t} = E_{D^i_1} - \partial_1 A^i_1 \partial_1 E_{m_1} - A^j_1 \partial_1 E_{m_j}$$

$$= E_{D^i_1} - \partial_1 (A^j_1 \partial_1 E_{m_j}) + (\partial_1 A^j_1 - \partial_1 A^i_1) \partial_1 E_{m_j}$$

$$\frac{\partial_t D^i_1}{\partial t} = -E_{A^i_1} - \partial_1 (D^j_1 \partial_1 E_{m_j}) + D^j_1 \partial_1 E_{m_i}$$

$$\frac{\partial_t s}{\partial t} = -\partial_1 (s \partial_1 E_{m_1})$$

In order to have reversible dynamics, parity of $D$ with respect to time reversal must be odd (parity of $A$ is even). The vector field $D$ can thus be interpreted as the associated momentum of distortion (e.g. it may represent microinertia of the microstructure, e.g. see [29]). By letting the functional depend only on $(\rho, m, s, A)$ bracket (91) becomes bracket (22), and it can be thus seen as an extension of that bracket.

In summary, we have constructed the Hamiltonian dynamics of a cotangent bundle advected by fluid mechanics. Distortion $A$ is coupled to its associated momentum $D$ (similarly as in [29]), and the resulting equations are indeed Hamiltonian. In particular, by disregarding the associated momentum, the bracket for distortion matrix (22) can be seen as the Lie-Poisson bracket for semidirect product of fluid mechanics and a vector space, which is the geometric interpretation of the Poisson bracket.

2.10 Onsager-Casimir reciprocal relations

We assume that the Hamiltonian evolution is time-reversible (at least short time, assuming the existence of strong solutions), which is usually the case. From the mathematical point of view time-reversibility follows directly from the construction of Poisson brackets on symplectic manifolds or from subsequent projections towards less detailed levels of description, see [71]. Alternatively, as follows from our discussion in Section 2.6, it is a consequence of energy conservation in Hamiltonian systems. From the physical point of view, Hamiltonian dynamics expresses mechanics, which constructed as reversible. Irreversibility appears once thermodynamic effects are taken into account.
Assuming, moreover, that all state variables have definite parities with respect to the time-reversal transformation (TRT),

\[
\text{TRT}(q^\alpha) = \mathcal{P}(q^\alpha)q^\alpha, \quad \mathcal{P}(q^\alpha) = \pm 1.
\]  

(92)

TRT inverts velocities of all particles and all velocity-like and momentum-like (general quantities odd w.r.t. TRT) fields. For instance, \( \mathcal{P}(x^i) = 1, \mathcal{P}(m_i) = -1, \mathcal{P}(s) = \mathcal{P}(e) = 1, \mathcal{P}(\rho) = 1 \) and \( \mathcal{P}(A) = 1 \). For the Hamiltonian evolution to generate reversible evolution, the Poisson bivector must satisfy

\[
\mathcal{P}(L^{\alpha\beta}) = -\mathcal{P}(q^\alpha)\mathcal{P}(q^\beta),
\]

see [71]. Therefore, \( \mathcal{P}(L^{\alpha\beta}) = 1 \) for \( \mathcal{P}(q^\alpha) = -\mathcal{P}(q^\beta) \) while \( \mathcal{P}(L^{\alpha\beta}) = -1 \) for \( \mathcal{P}(q^\alpha) = -\mathcal{P}(q^\beta) \). Regarding the Hamiltonian evolution equations (28), condition (93) together with antisymmetry of \( L^{\alpha\beta} = -L^{\beta\alpha} \), imply the following theorem:

**Theorem 7** (Onsager-Casimir reciprocal relations). *Assuming Hamiltonian evolution (28) with reversible Poisson bivector (condition (93)), then*

• variables with the same parity, \( \mathcal{P}(q^\alpha) = \mathcal{P}(q^\beta) \), are coupled by an operator symmetric with respect to simultaneous transposition and time reversal

• variables with opposite parities, \( \mathcal{P}(q^\alpha) = -\mathcal{P}(q^\beta) \), are coupled by an operator anti-symmetric with respect to simultaneous transposition and time reversal.

Onsager-Casimir reciprocal relations are thus automatically satisfied by reversible Hamiltonian evolution. Of course the irreversible part is also required to satisfy these relations, see [16, 17].

3 SPACE–TIME FORMULATION

In this section, we discuss a space-time formulation of Hamiltonian continuum mechanics discussed in Sec. 2. In particular, it is shown that the bracket formulation and the variational formulation of equations of motion (15), (23) perfectly agree. For instance, this might be useful for formulating of equations of relativistic Hamiltonian continuum mechanics.

The index convention used in this section is as follows. First letters of Latin alphabet \( a, b, \ldots \) and \( A, B, \ldots \) run from 0 to 3 while middle letters \( i, j, \ldots \) and \( I, J, \ldots \) run from 1 to 3. Also, small letters index quantities related to the Eulerian coordinates \( x^a \) while capital letters index quantities related to the Lagrangian coordinates \( X^A \).

3.1 Lagrangian variational formulation

Let us consider the mapping \( x^a(X^A) \), with the inverse mapping \( X^A(x^a) \), \( a = 0, 1, 2, 3 \), \( A = 0, 1, 2, 3 \) such that \( x^0 = t \) is the time of an Eulerian observer while \( X^0 = \tau \) is the time of a Lagrangian observer (co-moving with the medium). However, we shall assume non-relativistic settings and hence, \( t = \tau \).

Let us first consider a general Lagrangian density \( \bar{\Lambda}(X^A, x^a(X^A), \partial_A x^a) \) in the Lagrangian frame \( X^A \), where \( \delta_A = \delta x^a / \delta X^A \). We assume that \( \bar{\Lambda} \) depends only on the 4-potential \( x^a(X^A) \) and their first derivatives. Moreover, due to that the Lagrangian should transform as a proper scalar with respect to the Galilean group of transfor-
motions, \( \tilde{A} \) depends on \( X^\Lambda \) and \( x^a(X^\Lambda) \) only implicitly, i.e. \( \tilde{A}(X^\Lambda, x^a(X^\Lambda), \partial_a x^a) = \Lambda(\partial_a x^a) \). The action then reads

\[
S = \int \tilde{A}(X^\Lambda, x^a(X^\Lambda), \partial_a x^a) dX = \int \Lambda(\partial_a x^a) dX,
\]

(94)

variation of which with respect to \( \delta x^a \) gives the Euler-Lagrange equation

\[
\partial_a \Lambda \partial_a x^a = 0.
\]

(95)

This equation governs motion of the continuum, and is equivalent to (10), (11). Equation of motion (95) is a system of second-order PDEs on \( x^a \). It, however, can be seen as a first order system on 4-deformation gradient \( F^a_\Lambda = \partial_a x^a \) supplemented by the integrability condition \( \partial_a F^a_\Lambda - \partial_b F^a_\Lambda = 0 \), which is a trivial consequence of the definition \( F^a_\Lambda = \partial_a x^a \) and the continuity of \( x^a(X^\Lambda) \). Thus, the first-order system equivalent to (95) and (10), (11) is

\[
\partial_a \Lambda F^a_\Lambda = 0, \quad \partial_a F^a_\Lambda - \partial_b F^a_\Lambda = 0.
\]

(96)

Recall that the 4-velocity is conventionally defined as the tangent vector to the continuum particle trajectories

\[
u^a = \frac{\partial x^a}{\partial \tau} = \frac{\partial x^a}{\partial \xi^0} = F^a_\nu,
\]

(97)

which gives \( u^a = (1, v^1, v^2, v^3) \) with \( v^i = \frac{\partial x^i}{\partial \tau} \) being the ordinary 3-velocity. Therefore, the 4-velocity \( u^a \) is just the 0-th column of the 4-deformation gradient \( F^a_\Lambda \) [29]. Hence, equations (96) are essentially equations for \( u^a \).

3.2 Eulerian variational formulation

Alternatively, the action integral can be transformed into the Eulerian frame. For this purpose, it is sufficient to perform the change of variables \( X^\Lambda \rightarrow x^a(X^\Lambda) \) in the action (95):

\[
S = \int \tilde{A}(X^\Lambda, x^a(X^\Lambda), \partial_a x^a) dX = \int w (x^a(X^\Lambda), x^a, \partial_a X^\Lambda) dx = \int \tilde{\mathcal{L}}(X^\Lambda(x^a), x^a, \partial_a X^\Lambda) dx
\]

(98)

where \( w = \det(\partial_a X^\Lambda) \), \( \tilde{A}(\partial_a x^a) = \tilde{L}(\partial_a X^\Lambda) \), and \( \tilde{\mathcal{L}} = w \tilde{L} \). The Euler-Lagrange equation is

\[
\partial_a \mathcal{L}_{\partial_a X^\Lambda} = 0,
\]

(99)

where \( \mathcal{L}(\partial_a X^\Lambda) = \tilde{\mathcal{L}}(X^\Lambda(x^a), x^a, \partial_a X^\Lambda) \). Similarly as in (96), it can be viewed as an extended first-order system for \( A^\Lambda_a = \partial_a X^\Lambda \),

\[
\partial_a (A^\Lambda_b \partial_a X^\Lambda_a - \mathcal{L}_{\partial a}^\Lambda_b) = 0, \quad u^a (\partial_a A^\Lambda_b - \partial_b A^\Lambda_a) = 0,
\]

(100)

where the former equation is obtained from (99) using the fact that \( \partial_a A^\Lambda_b = \partial_b A^\Lambda_a \) while the latter is the result of transformation of (96) into the Eulerian frame. The tensor \( T^a_b = A_a^\Lambda_b \partial_a X^\Lambda_a - \mathcal{L}_{\partial a}^\Lambda_b \) can be naturally called the matter energy-momentum tensor.
3.3 $F^a_a$ or $A^A_a$?

Let us now show that the Eulerian equations (15d) and (23d) for $F^i_1$ and $A^j_k$ can be derived from the space-time formulation (96). We first note that the Lagrangian equation for $F^a_a$ is just an identity, so-called integrability condition,

$$\partial_a(\partial_b x^a) - \partial_b(\partial_a x^a) = \partial_a F^a_a - \partial_b F^b_a = 0. \quad (101)$$

Thus, if we consider the 0-th ($A^i = 0$) equation and use that $v^i = F^i_0$, we obtain the conventional time evolution equation for the spatial entries of $F^a_a$

$$\partial_t F^i_1 - \partial_1 v^i = 0. \quad (102)$$

One can easily get an Eulerian form of this PDE

$$\partial_t F^i_1 + v^k \partial_i F^j_k - F^j_k \partial_i v^j = 0 \quad (103)$$

by simply transforming the Lagrangian derivatives to Eulerian ones $\partial_A = F^a_A \partial_a$ (and in particular $\partial_A = F^0_A \partial_0 = \partial_1 + v^k \partial_k$). Similarly, (100) reduces to

$$\partial_t A^i_1 + v^k \partial_i A^j_k + A^i_1 \partial_i v^j = 0. \quad (104)$$

Equations (103) and (104) are exactly equations (15d) and (23d), correspondingly, generated by the corresponding Poisson brackets.

Having two Eulerian equations at hand, (103) and (104), it is naturally to question which one can be considered as a preferable equation to be used in the Eulerian frame.

First of all, we note that despite that from the mathematical standpoint these equations are equivalent, the variational formulation clearly suggests that $F^a_a$ is a natural state variable in the Lagrangian frame while $A^A_a$ is naturally to be used in the Eulerian frame, see action integrals (98).

Furthermore, in order to see more differences between these two equations, let us write them in equivalent semi-conservative forms. Thus, let us introduce $\hat{F}^i_1 = \rho F^i_1$, like in works [23, 22] with $\rho = \rho_0 / \det(F^1_1)$ being the mass density, and $\rho_0$ the reference mass density. Then, by multiplying (103) by $\rho$ and adding it with the continuity equation multiplied by $F^i_1$, one gets

$$\partial_t \hat{F}^i_1 + \partial_i (v^k \hat{F}^k_1 - v^j \hat{F}^j_1) + v^i \partial_k \hat{F}^k_1 = 0. \quad (105)$$

On the other hand, by adding $0 \equiv v^k \partial_i A^i_k - v^k \partial_i A^i_k$ to the equation for $A^i_1$, it can be written as

$$\partial_t A^i_1 + \partial_i (v^k A^i_k) + v^k (\partial_k A^i_k - \partial_i A^i_k) = 0. \quad (106)$$

A few remarks are in order, which discuss similarities and differences between formulations in terms of $F^a_a$ and $A^A_a$.

**Remark 1.** The last terms in (105) and (106) are connected as

$$\partial_k \hat{F}^k_1 = \rho F^i_1 \partial_i (\hat{F}^i_1 - \partial_1 A^i_1). \quad (107)$$

In particular, $\partial_k A^i_1 - \partial_i A^i_k = 0$ in elasticity and hence, $\partial_k \hat{F}^k_1 = 0$ as well. However, one should not remove these terms from the PDEs because the resulting equations would not be equivalent to the original ones. In particular, the removing of these terms changes the characteristic structure of the equations [30].

One should use the identity $u^a \partial_b A^A_a = -A^A_a \partial_b u^a$, and the so-called orthogonality condition $u^a A^A_a = F^S_\alpha A^A_\alpha = 0$ and hence $A^A_0 = -v^k A^A_k$.
Remark 2. In the space-time formulation, the equations for $F^a_A$ and for $A^A_a$ are nothing else but the Lie derivatives along the 4-velocity $u^a = F^a_0$:  

$$\mathcal{L}_u F^a_A = u^b \partial_b F^a_A - F^b_A \partial_b u^a = 0, \quad \mathcal{L}_u A^A_a = u^b \partial_b A^A_a + A^A_b \partial_a u^b = 0.$$  

(108)

Here, it is necessary to recall that $F^a_A$ and $A^A_a$ transform not as rank-2 space-time tensors but as tetrads of covariant and contravariant vectors correspondingly, see also Sec. 2.9. Using this fact, the ordinary partial derivatives $\partial_a$ in equations (108) can be replaced with the covariant derivatives $\nabla_a$ associated with the torsion-free Levi-Civita connection:  

$$u^b \nabla_b F^a_A - F^b_A \nabla_b u^a = 0, \quad u^b \nabla_b A^A_a + A^A_b \nabla_a u^b = 0.$$  

(109)

This means that both formulations are unconditionally covariant, that is they transform form-invariantly under general coordinate transformation (including transformations between non-inertial frames). Furthermore, as discussed in [72], because the time evolutions (109) are given by the Lie derivatives, they are intrinsically transformed as objective tensors, that is frame-independently, and moreover, they are defined unambiguously. This, in particular, makes such formulations attractive for using in rheology where the classical stress-based formulations (e.g. Maxwell’s visco-elastic model) are known to suffer from the non-uniqueness of the choice of objective time rates.

Remark 3. The 4D equations for $F^a_A$ and $A^A_a$ are related as  

$$\partial_A F^a_B - \partial_B F^a_A = -F^a_B F^c_B (\partial_a A^c_b - \partial_b A^c_a).$$  

(110)

Hence, in the elasticity settings, formulations in terms of $F^a_A$ and $A^A_a$ can be used interchangeably, and, we believe, there is no universal way to give a preference to one over another. However, the situation becomes different in the context of irreversible deformations of either solids or fluids, and $A^A_a$ formulation may give more insight into the inelasticity via the concepts of torsion $T^A_{ab} = \partial_a A^A_b - \partial_b A^A_a$ (24), non-holonomic tetrads, and non-Riemannian geometry, e.g see [29, 34].

We thus conclude by noting that, from the perspective of the space-time variational principle and modeling of inelastic deformations (which are not considered in this paper), $F^a_A$ can be considered as a natural measure of deformation in the Lagrangian frame, while it is natural to use the distortion field $A^A_a$ in the Eulerian frame.

4 CONCLUSION

Instead of Newton’s laws of motion, continuum mechanics can be constructed from the principle of least action or as Hamiltonian mechanics. Indeed, all approaches lead to the same results in the Lagrangian frame (attached to matter). In the Eulerian frame (attached to an inertial system), however, there are various forms of continuum mechanics. We advocate continuum mechanics described by state variables density, momentum density, entropy density and distortion (inverse deformation gradient), and demonstrate a step by step derivation by transformation from the Lagrangian frame. This form of continuum mechanics has the following features worth mentioning:

- Clear geometric origin, Sec. 2.2.
- Hamiltonian evolution: It is generated by a Poisson bracket and energy, which has the following implications:
  - Automatic energy conservation, Sec. 2.
– Automatic relation to hyperbolicity, see Sec. 2.7.
– Geometric self-consistency and gauge invariance, see Sec. 2.5.
– Variational principle in Clebsch variables, see Sec. 2.8.
– Structure of semidirect product, see Sec. 2.9.
– Automatic validity of Onsager-Casimir reciprocal relations, see Sec. 2.10.

• Unified description for both solids and fluids, Sec. 2.4.
• Robustness with respect to violation of deformation compatibility conditions (propagation of torsion tensor), sec. 2.4.
• Space-time variational formulation, see Sec. 3.1.

ACKNOWLEDGMENT

This research was originally initiated by Markus Hütter during the IWNET 2018 conference. M.P. is grateful to Miroslav Grmela, Petr Vágner, Oğul Esen and Vít Průša for many discussions on Hamiltonian formulation of solids and fluids and to Jan Zeman, Petr Pelech, Martin Šykora, Miroslav Hanzelka and Tomáš Los for patience and their opinions during the GENERIC course 2018/19. Especially Petr Pelech and Martin Šykora helped us checking a preliminary version of the manuscript.

I.P. acknowledges a financial support by Agence Nationale de la Recherche (FR) (Grant No. ANR-11-LABX-0040-CIMI) within the program ANR-11-IDEX-0002-02. M.P. and V.K. were supported by Czech Science Foundation, Project No. 17-15498Y, and by Charles University Research Program No. UNCE/SCI/023.

REFERENCES

[1] H. Poincaré. Science et méthode. Bibliothèque de philosophie scientifique. Flammarion, 1918.
[2] M.E. Gurtin. An Introduction to Continuum Mechanics. Mathematics in Science and Engineering. Elsevier Science, 1982.
[3] J.E. Marsden and T.J.R. Hughes. Mathematical Foundations of Elasticity. Dover Civil and Mechanical Engineering. Dover Publications, 2012.
[4] Juan C. Simo, Jerrold E. Marsden, and P. S. Krishnaprasad. The hamiltonian structure of nonlinear elasticity: The material and convective representations of solids, rods, and plates. Archive for Rational Mechanics and Analysis, 104(2):125–183, Jun 1988.
[5] L.D. Landau and E.M. Lifshitz. Mechanics. Butterworth-Heinemann. Butterworth-Heinemann, 1976.
[6] S. K. Godunov. An interesting class of quasi-linear systems. Soy. Math., 2:947, 1961.
[7] S. Godunov, T. Mikhailova, and E. Romenskii. Systems of thermodynamically coordinated laws of conservation invariant under rotations. Siberian Mathematical Journal, 37(4):660–705, 1996.
[8] V.I. Arnold. Sur la géométrie différentielle des groupes de Lie de dimension infini et ses applications dans l’hydrodynamique des fluides parfaits. *Annales de l’institut Fourier*, 16(1):319–361, 1966.

[9] I. E. Dzyaloshinskii and G. E. Volovick. Poisson brackets in condense matter physics. *Annals of Physics*, 125(1):67–97, 1980.

[10] P.J. Morrison. Bracket formulation for irreversible classical fields. *Phys. Lett. A*, 100:423, 1984.

[11] M. Grmela. Particle and bracket formulations of kinetic equations. *Contemporary Mathematics*, 28:125–132, 1984.

[12] B.J Edwards and AN Beris. Non-canonical poisson bracket for nonlinear elasticity with extensions to viscoelasticity. *Journal of Physics A: Mathematical and General*, 24(11):2461, 1991.

[13] A.N. Beris and B.J. Edwards. *Thermodynamics of Flowing Systems*. Oxford Univ. Press, Oxford, UK, 1994.

[14] Miroslav Grmela and Hans Christian Öttinger. Dynamics and thermodynamics of complex fluids. i. development of a general formalism. *Phys. Rev. E*, 56:6620–6632, Dec 1997.

[15] Hans Christian Öttinger and Miroslav Grmela. Dynamics and thermodynamics of complex fluids. ii. illustrations of a general formalism. *Phys. Rev. E*, 56:6633–6655, Dec 1997.

[16] H.C. Öttinger. *Beyond Equilibrium Thermodynamics*. Wiley, 2005.

[17] Michal Pavelka, Václav Klika, and Miroslav Grmela. *Multiscale Thermodynamics*. De Gruyter, Berlin, Boston, aug 2018.

[18] Darryl D. Holm. *Euler-Poincaré Dynamics of Perfect Complex Fluids*, pages 169–180. Springer New York, New York, NY, 2002.

[19] Ilya Peshkov, Walter Boscheri, Raphaël Loubère, Evgeniy Romenski, and Michael Dumbser. Theoretical and numerical comparison of hyperelastic and hypoelastic formulations for Eulerian non-linear elastoplasticity. *Journal of Computational Physics*, 387:481–521, jun 2019.

[20] Henry D. I. Abarbanel, Reggie Brown, and Yumin M. Yang. Hamiltonian formulation of inviscid flows with free boundaries. *The Physics of Fluids*, 31(10):2802–2809, 1988.

[21] Michal Pavelka, Václav Klika, Ögul Esen, and Miroslav Grmela. A hierarchy of Poisson brackets in non-equilibrium thermodynamics. *Physica D: Nonlinear Phenomena*, 335:54–69, nov 2016.

[22] Markus Hütter and Theo A. Tervoort. Coarse graining in elasto-viscoplasticity: Bridging the gap from microscopic fluctuations to dissipation. volume 42 of *Advances in Applied Mechanics*, pages 253 – 317. Elsevier, 2009.

[23] S K Godunov and E I Romenskii. *Elements of continuum mechanics and conservation laws*. Kluwer Academic/Plenum Publishers, 2003.

[24] M.B. Rubin. An elastic-viscoplastic model exhibiting continuity of solid and fluid states. *International Journal of Engineering Science*, 25(9):1175–1191, jan 1987.

[25] Tamás Fülöp and Péter Ván. Kinematic quantities of finite elastic and plastic deformation. *Mathematical Methods in the Applied Sciences*, 35(15):1825–1841, 2012.
[26] J. Málek, K.R. Rajagopal, and K. Tůma. On a variant of the maxwell and oldroyd-b models within the context of a thermodynamic basis. *International Journal of Non-Linear Mechanics*, 76:42 – 47, 2015.

[27] Michael Dumbser, Ilya Peshkov, Evgeniy Romenski, and Olindo Zanotti. High order ADER schemes for a unified first order hyperbolic formulation of continuum mechanics: Viscous heat-conducting fluids and elastic solids. *Journal of Computational Physics*, 314:824–862, Jun 2016.

[28] Haran Jackson and Nikos Nikiforakis. A numerical scheme for non-Newtonian fluids and plastic solids under the GPR model. *Journal of Computational Physics*, 387:410–429, Jun 2019.

[29] Ilya Peshkov, Evgeniy Romenski, and Michael Dumbser. Continuum mechanics with torsion. *Continuum Mechanics and Thermodynamics*, Apr 2019.

[30] Ilya Peshkov, Michal Pavelka, Evgeniy Romenski, and Miroslav Grmela. Continuum mechanics and thermodynamics in the Hamilton and the Godunov-type formulations. *Continuum Mechanics and Thermodynamics*, 30(6):1343–1378, Nov 2018.

[31] S K Godunov. An interesting class of quasilinear systems. *Dokl. Akad. Nauk SSSR*, 139(3):521–523, 1961.

[32] S. K. Godunov, T. Yu. Mikhailova, and E. I. Romenskii. Systems of thermodynamically coordinated laws of conservation invariant under rotations. *Siberian Mathematical Journal*, 37(4):690–705, Jul 1996.

[33] L.D. Landau, E.M. Lifshitz, A.M. Kosevich, and L.P. Pitaevskii. *Theory of Elasticity*. Course of theoretical physics. Butterworth-Heinemann, 1986.

[34] Arash Yavari and Alain Goriely. Riemann–Cartan Geometry of Nonlinear Dislocation Mechanics. *Archive for Rational Mechanics and Analysis*, 205(1):59–118, Jul 2012.

[35] Jerrold E Marsden, Tudor Raţiu, and Alan Weinstein. Semidirect products and reduction in mechanics. *Transactions of the American Mathematical Society*, 281(1):147–177, 1984.

[36] H. D. I. Abarbanel, D. D. Holm, J. E. Marsden, and T. S. Ratiu. Nonlinear stability analysis of stratified fluid equilibria. *Philos. Trans. Roy. Soc. London Ser. A*, 318:349–409, 1986.

[37] M. Kroeger and M. Huetter. Automated symbolic calculations in nonequilibrium thermodynamics. *Comput. Phys. Commun.*, 181:2149–2157, 2010.

[38] M. Fecko. *Differential Geometry and Lie Groups for Physicists*. Cambridge University Press, 2006.

[39] Charles-Michel Marle. Symmetries of hamiltonian dynamical systems, momentum maps and reductions. In *Proceedings of the Fifteenth International Conference on Geometry, Integrability and Quantization*, pages 11–52, Sofia, Bulgaria, 2014. Avangard Prima.

[40] Miroslav Grmela, Liu Hong, David Jou, Georgy Lebon, and Michal Pavelka. Hamiltonian and godunov structures of the grad hierarchy. *Physical Review E*, 95(033121), 2017.

[41] Peter J Olver. *Applications of Lie groups to differential equations*, volume 107. Springer Science & Business Media, 2000.

[42] J. Butterfield. On symplectic reduction in classical mechanics. In Jeremy Butterfield and John Earman, editors, *Philosophy of Physics*, Handbook of the Phi-
P. Ván, M. Pavelka, and M. Grmela. Extra mass flux in fluid mechanics. *Journal of Non-Equilibrium Thermodynamics*, 42(2), 2017.

I. Müller and T. Ruggeri. *Rational extended thermodynamics*. Springer tracts in natural philosophy. Springer, 1998.

Arthur E. Fischer and Jerrold E. Marsden. The einstein evolution equations as a first-order quasi-linear symmetric hyperbolic system, i. *Communications in Mathematical Physics*, 28(1):1–38, Oct 1972.

G. M. Kremer. Extended thermodynamics of ideal gases with 14 fields. *Ann. Inst. H. Poincaré*, 45:401, 1986.

Jan Sbierski. On the existence of a maximal cauchy development for the einstein equations: a dezornification. *Annales Henri Poincaré*, 17(2):301–329, Feb 2016.

T. Ruggeri. Galilean invariance and entropy principle for systems of balance laws. *Continuum Mechanics and Thermodynamics*, 1(1):3–20, Feb 1989.

Guy Boillat. Sur l’existence et la recherche d’équations de conservation supplémentaires pour les systèmes hyperboliques. *C. R. Acad. Sc. Paris, Sér A*, 278, 1974.

T. Ruggeri and A. Strumia. Main field and convex covariant density for quasi-linear hyperbolic systems. *Ann. Inst. H. Poincaré*, 34:65, 1981.

K.O. Friedrichs and P.D. Lax. Systems of conservation equations with a convex extension. *Proc. Natl. Acad. Sci. USA*, 68:1686–1688, 1971.

B. A. Dubrovin and S. P. Novikov. Hamiltonian formalism of one-dimensional systems of hydrodynamic type, and the Bogolyubov-Whitham averaging method. *Dokl. Akad. Nauk SSSR*, 270:781–785, 1983. English transl. in Soviet Math. Dokl. 27 (1983).

S.P. Tsarëv. The geometry of Hamiltonian systems of hydrodynamic type. The generalized hodograph method. *Math. USSR Izv.*, 37(2):397, 1991.

B. A. Dubrovin and S. P. Novikov. Hydrodynamics of weakly deformed soliton lattices. Differential geometry and Hamiltonian theory. *Russ. Math. Survo.*, 44(35), 1989.

Nisheeth K. Vishnoi. Geodesic convex optimization: Differentiation on manifolds, geodesics, and convexity. *arXiv:1806.06373*, 2018.

K O Friedrichs. Symmetric positive linear differential equations. *Communications on Pure and Applied Mathematics*, 11(3):333–418, aug 1958.

Ilya Peshkov, Miroslav Grmela, and Evgeniy Romenski. Irreversible mechanics and thermodynamics of two-phase continua experiencing stress-induced solid-fluid transitions. *Continuum Mechanics and Thermodynamics*, 27(6):905–940, 2015.

Darryl D Holm, Jerrold E Marsden, and Tudor S Ratiu. The Euler–Poincaré equations and semidirect products with applications to continuum theories. *Advances in Mathematics*, 137(1):1 – 81, 1998.

C. J. Cotter and D. D. Holm. Continuous and discrete clebsch variational principles. *Foundations of Computational Mathematics*, 9(2):221–242, Apr 2009.

A. Clebsch. Über die Integration der Hydrodynamische Gleichungen. *Journal für die reine und angewandte Mathematik*, 56:1–10, 1895.
A FUNCTIONAL DERIVATIVES

The purpose of this section is to recall the concept of functional derivative. Consider a functional $F$ of field $f(X)$ that is Fréchet differentiable, i.e.

$$F(f + \delta f) = F(f) + DF_f(\delta f) + O(\delta f)^2, \quad (111)$$

where $\delta f \in C^\infty_0$ is a smooth variation with compact support (zero at the boundaries). Topology can be specified for instance as that of $\mathcal{D}$ space [73]. The Fréchet differential is then equal to the Gateaux derivative

$$\left. \frac{d}{d\lambda} \right|_{\lambda=0} F(f + \lambda \delta f) = DF_f(\delta f). \quad (112)$$

Since the Fréchet differential is linear in its argument ($\bullet$), it can be seen as an element of the dual space to $\delta f$, which is the space of distributions $\mathcal{D}'$. Therefore, it can be represented by

$$DF_f(\delta f) = \int dX \frac{\delta F}{\delta f}, \quad (113)$$
where the integral is understood as a notational shorthand for duality in the distribu-
tional sense $\langle \bullet, \bullet \rangle$. By combining the last two equalities, we obtain
\[
\frac{d}{d\lambda} \bigg|_{\lambda=0} F(f + \lambda \delta f) = \int dX \frac{\delta F}{\delta f},
\]
which is the usual way for calculation of functional derivatives $\frac{\delta F}{\delta f}$.

For instance if $F$ is integral of a smooth real-valued function $g(f)$ of field $f(X)$, the functional derivative becomes
\[
\frac{\delta}{\delta f} \int dX g(f(X)) = g'(f(X)),
\]
which is just the ordinary derivative of $g$.

If the functional depends on gradient of $f$, we have to carry out integration by parts (recalling that $\delta f$ vanishes at the boundaries), e.g.
\[
\frac{\delta}{\delta f} \int dX \frac{1}{2} \nabla f(X) \cdot \nabla f(X) = -\nabla f(X) \cdot (\nabla f(X)).
\]

Finally, if the functional simply picks a value of the field at given position, its derivative is the Dirac $\delta$–distribution,
\[
F(f) = f(X_0) = \int dX \delta(X - X_0) f(X) \Rightarrow \frac{\delta F}{\delta f(X)} = \delta(X - X_0).
\]

Equipped with these instruments, we can approach the transformation of Poisson brackets from the Lagrangian frame to the Eulerian frame.

B FROM LAGRANGE TO EULER

The purpose of this rather technical Appendix is to show in detail how the Eulerian bracket (13) is obtained from the Lagrangian canonical bracket (1). The latter bracket expresses kinematics of fields $x(X)$ and $M(X)$, while the former bracket has only Eulerian state variables $\rho(x)$, $m(x)$, $s(x)$ and $F(x)$. Note that the calculations are purely formal as we do not discuss the analytical details of the studied objects, which are thus assumed to be regular enough.

Before carrying out the actual transformation, we make a few observations about the mapping $x(X)$ and its inverse $X(x)$ and their behavior with respect to perturbations $\delta x(X)$. Firstly, we see that
\[
(x(X) + \delta x(X)) \circ X(x - \delta x(X(x))) = x - \delta x(X(x)) + \delta x(X(x)) + O(\delta x)^2 = x + O(\delta x)^2,
\]
which helps when calculating functional derivatives with respect to $x(X)$. This identity is demonstrated on Fig. 1.

The functional derivatives of the Eulerian fields with respect to the Lagrangian fields are necessary to perform the projection. Let us start with the Lagrangian density in the Eulerian frame, $\rho_0(x) \overset{def}{=} \rho_0(X(x))$. This slightly overloaded notation is used throughout this appendix. To avoid confusion, the spatial variables on which the fields depend will be always indicated. To find functional derivative of $\rho_0(x)$ with respect to $x(X)$ we study
\[
\rho_0(x; x + \delta x) \overset{def}{=} \rho_0(X(x) + \delta X(x)),
\]
where \( \delta X(x) \) is the perturbation of mapping \( X(x) \) induced by perturbation \( \delta x(X) \). Using relation \((118)\), we obtain

\[
\rho_0(x; x + \delta x) = \rho_0(X(x - \delta x(X(x)))) = \rho_0(x - \delta x(X(x)))
\]

\[
= \rho_0(x) - \partial_k \rho_0(x) \delta x_k(X(x)) + O(\delta x)^2
\]

\[
= \rho_0(x) + \int dX \left[ - \partial_k \rho_0(x) \delta (X - X(x)) \right] \delta x_k(X) + O(\delta x)^2, \quad (120)
\]

which means that

\[
\frac{\delta \rho_0(x)}{\delta x_k(X)} = - \partial_k \rho_0(x) \delta (X - X(x)). \quad (121)
\]

### 8.1 Derivative of the Eulerian deformation gradient \( F(x) \)

The Eulerian mass density \( \rho(x) \) is equal to \( \rho_0(x) \) multiplied by Jacobian of the mapping \( X(x) \), and so to acquire the functional derivative of \( \rho(x) \) we have to first deal with functional derivative of the Eulerian deformation gradient

\[
F(x) \overset{\text{def}}{=} \frac{\partial x}{\partial X}\bigg|_{x(x)}, \quad (122)
\]

Using again relation \((118)\) we have

\[
F_i^i(x; x + \delta x) \overset{\text{def}}{=} \frac{\partial x^i(X) + \delta x^i(X)}{\partial X} \bigg|_{x(x - \delta x(x(x)))}
\]

\[
= \frac{\partial x^i}{\partial X} \bigg|_{x(x - \delta x(x(x)))} + \frac{\partial \delta x^i}{\partial X} \bigg|_{x(x)} + O(\delta x)^2
\]

\[
= \int dX \delta (X - X(x)) \frac{\partial \delta x_i}{\partial X} \bigg|_{x(x)} - \partial_k F_i^i(x) \delta x^k(X(x)) + O(\delta x)^2
\]

\[
= F_i^i(x) - \int dX \frac{\partial \delta (X - X(x))}{\partial X} \delta x_i(X) - \int dX \partial_k F_i^i(x) \delta (X - X(x)) \delta x^k(X),
\]

which means that the sought functional derivative reads

\[
\frac{\delta F_i^i(x)}{\delta x^k(X)} = - \frac{\delta (X - X(x))}{\partial X} \delta^i_k - \partial_k F_i^i(x) \delta (X - X(x)). \quad (124)
\]

### 8.2 Derivative of the Eulerian mass density \( \rho(x) \)

Now we can finish the calculation of the functional derivative of the Eulerian density \( \rho(x) \) with respect to \( x(X) \). The first part, derivative of \( \rho_0(x) \), has already been
obtained before in Eq. (121). What remains is to calculate derivative of determinant \( \text{det}(F(x)) \) with respect to \( x(X) \).

Considering determinant of a general matrix \( C \), its variation when the matrix is perturbed by \( \delta C \) reads

\[
\text{det}(C + \delta C) = \text{det}(C) \cdot \text{det}(I + C^{-1} \cdot \delta C) = \text{det} C \cdot \text{Tr}(C^{-1} \cdot \delta C) + O(\delta C)^2. \tag{125}
\]

Therefore, derivative of \( \text{det}(F) \) is

\[
\frac{\delta \text{det}(F)}{\delta x^k(X)} = -\frac{1}{(\text{det} F(x))^2} \text{det} F(x) \frac{\partial X^l}{\partial x^i} \frac{\delta F^i_l(x)}{\delta x^k(X)}. \tag{126}
\]

Derivative of the Eulerian density finally reads

\[
\frac{\delta \rho(x)}{\delta x^k(X)} = \frac{\delta \rho_0(x)}{\delta x^k(X)} \text{det} \frac{\partial X}{\partial x} \left( \frac{\rho_0(x)}{(\text{det} F(x))^2} \text{det} F(x) \frac{\partial X^l}{\partial x^i} \frac{\delta F^i_l(x)}{\delta x^k(X)} \right)
\]

\[
- \partial_k \rho_0(x) \delta(x - X(x)) \frac{1}{\text{det} F(x)}
\]

\[
+ \frac{\rho_0(x)}{\text{det} F(x)} \frac{\partial X^l}{\partial x^i} \frac{\delta(x - X(x))}{\partial X^l} \delta_k^i
\]

\[
+ \frac{\rho_0(x)}{\text{det} F(x)} \frac{\partial X^l}{\partial x^i} \partial_k F^i_l(x) \delta(x - X(x)). \tag{127}
\]

### 3.3 Derivative of the Eulerian entropy density \( s(x) \)

Having calculated derivative of mass density, the result for entropy density \( s(x) = s_0(x)/\text{det} F(x) \) is analogous,

\[
\frac{\delta s(x)}{\delta x^k(X)} = - \partial_k s_0(x) \delta(x - X(x)) \frac{1}{\text{det} F(x)}
\]

\[
+ \frac{s_0(x)}{\text{det} F(x)} \frac{\partial X^l}{\partial x^i} \frac{\delta(x - X(x))}{\partial X^l} \delta_k^i
\]

\[
+ \frac{s_0(x)}{\text{det} F(x)} \frac{\partial X^l}{\partial x^i} \partial_k F^i_l(x) \delta(x - X(x)). \tag{128}
\]

### 3.4 Derivative of the Eulerian momentum density \( m(x) \)

The functional derivative of \( m(x) = M(x(x))/\text{det} F(x) \) with respect to \( x(X) \) has the same form as derivatives of \( \rho(x) \) and \( s(x) \),

\[
\frac{\delta m_l(x)}{\delta x^k(X)} = - \partial_k M_l(x) \delta(x - X(x)) \frac{1}{\text{det} F(x)}
\]

\[
+ \frac{M_l(x)}{\text{det} F(x)} \frac{\partial X^l}{\partial x^i} \frac{\delta(x - X(x))}{\partial X^l} \delta_k^i
\]

\[
+ \frac{M_l(x)}{\text{det} F(x)} \frac{\partial X^l}{\partial x^i} \partial_k F^i_l(x) \delta(x - X(x)). \tag{129}
\]

But the field \( m(x) \) also depends on the yet unused Lagrangian field \( M(X) \). Derivative with respect to this fields is

\[
\frac{\delta m_l(x)}{\delta M_k(X)} = \delta_l^i \frac{\delta(x - X(x))}{\text{det} F(x)} \]
as follows from the formulas
\[ M(X(x)) = \int dX \delta(X - X(x)) M(X) \] (131)
and
\[ \frac{\delta M_k(X(x))}{\delta M_l(X)} = \delta_{k,l} \delta(X - X(x)). \] (132)

### 3.5 Derivative of an arbitrary Eulerian functional

Derivative of an arbitrary smooth enough functional of the Eulerian fields \( C(\rho(x), m(x), s(x), F(x)) \) with respect to the Lagrangian field \( x(X) \) can be calculated by chain rule as
\[ \frac{\delta C}{\delta x^k(X)} = \int dx \left( \frac{\delta C}{\delta \rho(x)} \frac{\delta \rho(x)}{\delta x^k(X)} + \frac{\delta C}{\delta m_l(x)} \frac{\delta m_l(x)}{\delta x^k(X)} + \frac{\delta C}{\delta s(x)} \frac{\delta s(x)}{\delta x^k(X)} + \frac{\delta C}{\delta F^l(x)} \frac{\delta F^l(x)}{\delta x^k(X)} \right). \] (133)

Similarly derivative of an arbitrary functional \( D(\rho(x), m(x), s(x), F(x)) \) with respect to the Lagrangian \( M(X) \) field is
\[ \frac{\delta D}{\delta M_k(X)} = \int dx \frac{\delta D}{\delta m_l(X)} \frac{\delta m_l(X)}{\delta M_k(X)}. \] (134)
which, using Eq. (130), can be rewritten more explicitly as
\[ \frac{\delta D}{\delta M_k(X)} = \int dx \frac{\delta D}{\delta m_l(X)} \frac{\delta m_l(X)}{\delta M_k(X)} \frac{\delta(X - X(x))}{\det \frac{\partial X}{\partial x}} \]
\[ = \int dx \frac{\delta D}{\delta m_l(X)} \delta(x - X(x)) \det \frac{\partial X}{\partial x}. \] (135)

The \( \delta \)-distribution can be seen as the limit of a sequence of smooth functions (e.g. Gaussians), \( f_n(x) \overset{D}{\to} \delta(x) \). Therefore, the last integral can be rewritten as
\[ \frac{\delta D}{\delta M_k(X)} = \lim_{n \to \infty} \int dx \frac{\delta D}{\delta m_k(X)} f_n(X - X(x)) \det \frac{\partial X}{\partial x} \]
\[ = \lim_{n \to \infty} \int dx' \delta \left( \frac{\delta D}{\delta m_k(X)} \right)_{x(x')} f_n(X - X(x(x'))) \]
\[ = \int dx' \frac{\delta D}{\delta m_k(X)} \left. \delta(X - X') \right|_{x(x')} = \int dx' \frac{\delta D}{\delta m_k(X)} \left. \delta(X - X') \right|_{x(x')} \] (136)

Now we are finally in position to calculate the Lagrangian Poisson bracket Eq. (1) for the Eulerian functionals \( C \) and \( D \).

### 3.6 The Eulerian Poisson bracket

Bracket (1) is the sum of terms like
\[ \int dX \delta C \frac{\delta D}{\delta x^k(X)} \frac{\delta M_k(X)}{\delta M_l(X)}. \] (137)

Functionals \( F \) and \( G \) are not used here to avoid confusing with tensor \( F \).
where the former functional derivative consists of all terms in Eq. (133). Let us first take only the term with derivative $C_{\rho(x)}$: 

$$
\int dX \int d\delta C \frac{\delta C}{\delta \rho(x)} \left[ -\partial_k \rho_0(x) \delta [X - X(x)] \right] \frac{1}{\det F(x)} \\
+ \frac{\rho_0(x)}{\det F(x)} \frac{\partial X^i}{\partial x^i} \frac{\partial \delta [X - X(x)]}{\partial x^i} \frac{\delta x^i}{\partial \delta C} \\
+ \frac{\rho_0(x)}{\det F(x)} \frac{\partial X^i}{\partial x^i} \frac{\partial \delta k}{\partial \delta C} \\
\int dX \delta [X - X(x)] \frac{\partial}{\partial x^i} \frac{\delta D}{\delta m_k(x)} \bigg|_{x(x)}
$$

which is obviously a part of the final Eulerian Poisson bracket (13). In the same fashion we obtain 

$$
\int dX \int d\delta C \frac{\delta C}{\delta m_1(x)} \frac{\delta m_1(x)}{\delta \delta C} \frac{\delta D}{\delta \delta C} = \int dX \int d\delta C \frac{\delta C}{\delta \delta C} \frac{\delta m_1(x)}{\delta m_1(x)} \frac{\delta D}{\delta m_1(x)} \bigg|_{x(x)}
$$

and 

$$
\int dX \int d\delta C \frac{\delta C}{\delta s(x)} \frac{\delta s(x)}{\delta \delta C} \frac{\delta D}{\delta \delta C} = \int dX \int d\delta C \frac{\delta C}{\delta \delta C} \frac{\delta s(x)}{\delta s(x)} \frac{\delta D}{\delta m_1(x)} \bigg|_{x(x)}
$$

So far we have recovered the $[C, D] \big|_{FM}$ part of the bracket (the antisymmetric part is obtained as negative of the same terms with C and D swapped).

The part dependent on the Eulerian deformation gradient $F(x)$ is calculated similarly as follows.

$$
\int dX \int d\delta C \frac{\delta C}{\delta F_{i_1}^l(x)} \frac{\delta F_{i_1}^l(x)}{\delta \delta C} \frac{\delta D}{\delta \delta C} \\
= \int dX \int d\delta C \frac{\delta C}{\delta F_{i_1}^l(x)} \left[ \frac{\partial \delta (X - X(x))}{\partial X^i} \frac{\partial \delta (X - X(x))}{\partial x^i} \right] \frac{\partial \delta D}{\partial \delta m_k(x)} \bigg|_{x(x)}
$$

which is the remaining part of bracket (13).

In summary, Eulerian Poisson bracket (13), which expresses kinematics of fields $\rho(x), m(x), s(x)$ and $F(x)$, has been derived by projection from the Lagrangian canonical Poisson bracket (i), expressing kinematics of $x(X)$ and $M(X)$. 

from Lagrange to Euler | 37
Derivative of the left Cauchy-Green tensor $B(x)$ with respect to the Eulerian deformation gradient is

$$\frac{\partial B_{ij}^k(x)}{\partial F_{k}^k} = \delta^{ij}(\delta_{i}^{k}\delta_{k}^j F_{j}^j + F_{i}^i \delta_{k}^k) = \delta^{kj} \delta_{i}^{k} F_{j}^j + \delta^{ik} F_{i}^i \delta_{j}^j.$$ (142)

Derivative of a functional $C(F)$ then becomes

$$\frac{\delta C}{\delta F_{k}^k(x)} = \frac{\delta C}{\delta B_{k}^k(x)} \delta^{kj} F_{j}^j + \frac{\delta C}{\delta B_{k}^k(x)} \delta^{ik} F_{i}^i,$$ (143)

and after plugging this relation into bracket (13) we obtain bracket (19) easily.

D FROM DEFORMATION GRADIENT TO DISTORTION

The purpose of this section is to show more details on the projection from bracket (13) to bracket (22), expressing kinematics of distortion. The distortion is the inverse of the Eulerian deformation gradeint $F(x)$,

$$A_{i}^j(x)F_{j}^j(x) = \delta_{i}^j.$$ (144)

Taking derivative of this equality with respect to $F_{k}^k(x)$ leads to

$$\frac{\partial A_{i}^j}{\partial F_{k}^k} = -A_{i}^j \delta_{k}^j.$$ (145)

After multiplication by $A_{j}^j$ we obtain

$$\frac{\partial A_{i}^j}{\partial F_{j}^j} = -A_{i}^j A_{j}^j,$$ (146)

from which it follows that

$$\frac{\delta C}{\delta F_{j}^j} = \frac{\delta C}{\delta A_{i}^j(x)} \frac{\partial A_{i}^j}{\partial F_{j}^j} = -\frac{\delta C}{\delta A_{i}^j} A_{i}^j A_{j}^j$$ (147)

for arbitrary functional $C(F)$.

Plugging this last relation into bracket (13) immediatly leads to bracket (22).