On classical solutions to the Cauchy problem of the 2D compressible non-resistive MHD equations with vacuum states

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Abstract
In this paper, we investigate the Cauchy problem of the compressible non-resistive MHD on $\mathbb{R}^2$ with a vacuum as the far field density. We prove that the two-dimensional (2D) Cauchy problem has a unique local strong solution provided that the initial density and magnetic field decay are not too slow at infinity. Furthermore, if the initial data satisfies some additional regularity and compatibility conditions, the strong solution becomes a classical one. Additionally, we establish a blowup criterion for the 2D compressible non-resistive MHD depending solely on the density and magnetic fields.

Keywords: 2D compressible non-resistive MHD equations, vacuum, classical solutions, blowup criterion

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1. Introduction

In this paper, we study the two-dimensional (2D) compressible non-resistive magnetohydrodynamics (MHD) equations which read as follows:

\[ \rho_t + \text{div}(\rho u) = 0, \]  
\[ (\rho u)_t + \text{div}(\rho u \otimes u) + \nabla P(\rho) = \mu \Delta u + (\mu + \lambda) \text{div}u + (\nabla \times H) \times H, \]  

(1.1) 

(1.2)
\[ H_t + u \cdot \nabla H + H \text{div} u = H \cdot \nabla u, \quad \text{div} H = 0, \tag{1.3} \]

with the initial condition
\[ (\varrho, u, H)(0, x) = (\varrho_0, u_0, H_0)(x), \quad x \in \mathbb{R}^2, \tag{1.4} \]

and far field behavior (in some weak sense)
\[ u(t, x) \to 0, \quad \varrho(t, x) \to 0, \quad H(t, x) \to 0 \quad \text{as} \ |x| \to \infty, \quad \text{for} \ t \geq 0. \tag{1.5} \]

Here \( \varrho = \varrho(t, x) \), \( u(t, x) = (u_1, u_2)(t, x) \) and \( H(t, x) = (H_1, H_2)(t, x) \) represent the unknown density, velocity and magnetic field of the fluid, respectively. The pressure \( P(\varrho) \) is given by
\[ P(\varrho) = A \varrho^\gamma, \tag{1.6} \]

where \( \gamma > 1 \) is the adiabatic exponent, and \( A > 0 \) is a constant. The viscosity coefficients \( \mu \) and \( \lambda \) satisfy the following physical restrictions
\[ \mu > 0, \quad \mu + \lambda \geq 0. \tag{1.7} \]

MHD studies the motion of electrically conducting media in the presence of a magnetic field. The dynamic motion of fluid and a magnetic field interact strongly with each other, so the hydrodynamic and electrodynamic effects are complicated either from a physical viewpoint, or from a mathematical consideration (see [1, 18] and references therein). In this paper, we restrict ourselves to equations (1.1)–(1.3) called the viscous and non-resistive MHD equations, which means that the conducting fluids have a very high conductivity, such as ideal conductors (see [2, 10]). In particular, the magnetic equation (1.3) implies that in a highly conducting fluid the magnetic field lines move along exactly with the fluid, rather than simply diffusing out. This type of behavior is physically expressed as the magnetic field lines frozen into the fluid. For more details regarding the physical background, we refer the readers to [1, 2, 10, 18, 32] and references therein.

Now, we briefly recall some results concerning the multi-dimensional compressible/incompressible MHD relating to the present paper. First, if there is no electromagnetic effect, that is \( H = 0 \), the MHD system reduces to the classical compressible/incompressible Navier–Stokes equations, which have been discussed by many mathematicians (see [4, 9, 11, 12, 14, 24, 25, 29, 31] and references therein). Next, if there exists the diffusion term \( \nu \Delta H \) in (1.3), which has also been discussed by many researchers, we refer the reader to [13, 21, 26, 27, 30] and references therein.

However, as far as we know, the non-resistive MHD equations have not been thoroughly studied. As \( \varrho = \) is constant, i.e. incompressible non-resistive MHD, Jiu et al [17] proved the local existence of solutions in 2D for initial data in \( H^s \) for integer \( s \geq 3 \), and soon after that, Ren et al [28] and Lin et al [23] established the existence of global-in-time solutions for initial data sufficiently close to certain equilibrium solutions in two spatial dimensions. Almost at the same time, Fefferman et al [7] established a local existence result for initial data in \( (u_0, H_0) \in H^s \) with \( s > d/2 \) for \( d = 2, 3 \); in order to assume less regularity on \( u_0 \) than that of \( H_0 \), due to the existence of the diffusive term in the momentum equations, Chemin et al [3] proved the local existence in Besov spaces when \( u_0 \in B^{d/2-1}_{2,1}(\mathbb{R}^d) \) and \( H_0 \in B^{d/2}_{2,1}(\mathbb{R}^d) \), and more recently, Fefferman et al [8] presented an inspiring local-in-time existence and uniqueness solution in nearly optimal Sobolev space in \( \mathbb{R}^d \) for \( H_0 \in H^s(\mathbb{R}^d) \) and \( u_0 \in H^{s+1+\varepsilon}(\mathbb{R}^d) \) with \( s > d/2 \) and \( 0 < \varepsilon < 1 \). Here we also want to mention that Fan et al [5] established the global-in-time existence of smooth solutions of a 2D generalized MHD system with fractional diffusion \( (-\Delta)^{\alpha/2} u \) \( (0 < \alpha < 1/2) \).
Let us come back to the compressible non-resistive MHD equations. Fan et al [6] and Li et al [22] independently proved the existence of a unique local strong solution in 3D. After that, Xu et al [32] established a blowup criterion to explain the mechanism of blow-up and the structure of possible singularities of strong solutions for compressible non-resistive MHD equations due to the lack of the global existence of a strong solution to (1.1)–(1.3) with large initial data. Very recently, Zhu [35] proved the existence of a unique local classical solution with regular initial data (almost the same compatibility conditions as that in [6]) and improved the blowup criterions obtained in [32].

When the far field density is a vacuum (in particular, the initial density may have compact support) in 2D, the methods successfully applied in 3D in [6, 22, 32, 35] are not valid here, since the $L^p$-norm of $u$ could not be bound in terms of $\|\sqrt{\mu} u\|_{L^q}$ and $\|\nabla u\|_{L^q}$. Therefore, it is still unknown whether the well-posedness of strong/classical solutions to the compressible non-resistive MHD equations in 2D exist or not, even the local ones. In this paper, we want to answer part of these questions.

In this section, for $1 \leq r \leq \infty$, we denote the standard Lebesgue and Sobolev spaces as follows:

$$L^r = L^r(\mathbb{R}^2), \quad W^{s,r} = W^{s,r}(\mathbb{R}^2), \quad H^s = W^{s,2}.$$  

Now, we wish to define precisely what we mean by strong solutions.

**Definition 1.1.** If all derivatives involved in (1.1)–(1.3) for $(\rho, u, H)$ are regular distributions, and equations (1.1)–(1.3) hold almost everywhere in $(0, T) \times \mathbb{R}^2$, then $(\rho, u, H)$ is called a strong solution to (1.1)–(1.3).

**Theorem 1.1.** Let $\eta_0$ be a positive constant and

$$\bar{\eta} \triangleq (\rho + |\vec{x}|^2)^{1/2} \ln^{1+\eta_0}(\rho + |\vec{x}|^2). \quad (1.8)$$

For constants $q > 2$ and $a > 1$, assume that the initial data $(\rho_0, u_0, H_0)$ satisfy

$$\rho_0 \geq 0, \quad \rho_0 \bar{\eta}^a \in L^1 \cap H^1 \cap W^{1,q}, \quad H_0 \bar{\eta}^a \in H^1 \cap W^{1,q}, \quad \nabla u_0 \in L^2, \quad \sqrt{\rho_0} u_0 \in L^2.$$  

Then there exists a positive time $T_0 > 0$ such that the problem (1.1)–(1.7) has a unique strong solution $(\rho, u, H)$ on $(0, T_0] \times \mathbb{R}^2$ satisfying that

$$\begin{align*}
\rho &\in C([0, T_0]; L^1 \cap H^1 \cap W^{1,q}), \quad g \bar{\eta}^a \in L^\infty(0, T_0; L^1 \cap H^1 \cap W^{1,q}), \\
H &\in C([0, T_0]; H^1 \cap W^{1,q}), \quad H \bar{\eta}^a \in L^\infty(0, T_0; H^1 \cap W^{1,q}), \\
\sqrt{\rho_0} u, \sqrt{\rho_0} u^{-1} u, \sqrt{\rho_0} \bar{\eta} u_t &\in L^\infty(0, T_0; L^2), \\
\nabla u &\in L^2(0, T_0; L^d) \cap L^{(q+1)/q}(0, T_0; W^{1,q}), \quad \sqrt{\nabla} u \in L^2(0, T_0; W^{1,q}), \\
\sqrt{\nabla} u_t, \sqrt{\nabla} u_t^{-1} u_t &\in L^2(\mathbb{R}^2 \times (0, T_0)).
\end{align*} \quad (1.9)$$

and that

$$\inf_{0 \leq t \leq T_0} \int_{B_2} \varrho(t, x) dx \geq \frac{1}{4} \int_{\mathbb{R}^2} \varrho_0(x) dx, \quad (1.11)$$

for some constant $N > 0$ and $B_N \triangleq \{ x \in \mathbb{R}^2 : |x| < N \}$.

Moreover, if the initial data $(\rho_0, u_0, H_0)$ satisfies some additional regularity and compatibility conditions, the local strong solution $(\rho, u, H)$ obtained in theorem 1.1 becomes a classical one—that is:
**Theorem 1.2.** In addition to (1.9), suppose that

\[
\begin{align*}
\nabla^2 \varrho, \nabla^2 P(\varrho), \nabla^2 \mathbf{H} & \in L^2 \cap L^q, \\
x^k \nabla^2 \varrho, x^k \nabla^2 P(\varrho), x^k \nabla^2 \mathbf{H} & \in L^\infty(0, T_0; L^2), \\
x^k \nabla u, \sqrt{\varrho} u, \sqrt{\varrho} \nabla u, \sqrt{\varrho} \nabla^{-1} u, r \nabla^2 u, r^2 \nabla u, r^2 \nabla^{-1} u & \in L^\infty(0, T_0; L^2), \\
x^k \nabla \varrho & \in L^\infty(0, T_0; L^q(\varrho^{q/2})),
\end{align*}
\]  

(1.12)

for some constant \( \delta_0 \in (0, 1) \). Moreover, assume that the following compatibility conditions hold for some \( g \in L^2 \):

\[
-\mu \Delta u_0 - (\mu + \lambda) \nabla \text{div} u_0 + \nabla P(\varrho_0) - (\nabla \times \mathbf{H}_0) \times \mathbf{H}_0 = g_0^{1/2} g.
\]  

(1.13)

Then, in addition to (1.10) and (1.11), the strong solution \((\varrho, u, \mathbf{H})\) obtained in theorem 1.1 satisfies

\[
\begin{align*}
\nabla^2 \varrho, \nabla^2 P(\varrho), \nabla^2 \mathbf{H} & \in C([0, T_0]; L^2 \cap L^q), \\
x^k \nabla^2 \varrho, x^k \nabla^2 P(\varrho), x^k \nabla^2 \mathbf{H} & \in L^\infty(0, T_0; L^2), \\
x^k \nabla u, \sqrt{\varrho} u, \sqrt{\varrho} \nabla u, \sqrt{\varrho} \nabla^{-1} u, r \nabla^2 u, r^2 \nabla u, r^2 \nabla^{-1} u & \in L^\infty(0, T_0; L^2), \\
x^k \nabla \varrho & \in L^\infty(0, T_0; L^q(\varrho^{q/2})).
\end{align*}
\]  

(1.14)

No global classical solution exists, even without the effect of magnetic fields, due to [31] (see [33]). So, one naturally wonders, in finite time, what kinds of singularities will form, or what is the main mechanism of the possible breakdown of smooth solutions for the 2D compressible non-resistive MHD equations? There are two main results [32, 35] concerning blowup criteria for strong/classical solutions to the 3D compressible non-resistive MHD equations, which is similar to that obtained by Huang et al [14] for strong solutions to compressible Navier–Stokes equations. As is well-known, partial differential equations (PDEs) entailing many independent variables are harder than PDEs entailing few independent variables. Therefore, an interesting question to ask is whether the blowup criteria [32, 35] could be improved for the 2D compressible non-resistive MHD or not. Based on subtle estimates, our next main result in this paper answered this question positively for classical solutions, which can be shown as follows.

**Theorem 1.3.** Assume that the initial data \((\varrho_0, u_0, \mathbf{H}_0)\) satisfies (1.9), (1.12) and the compatibility conditions (1.13). Let \((\varrho, u, \mathbf{H})\) be a classical solution to the Cauchy problem (1.1)–(1.7). If \( 0 < T^* < \infty \) is the maximal time of existence, then

\[
\limsup_{T \to T^*} \left( \| \varrho \|_{L^\infty(0, T; L^q)} + \| \mathbf{H} \|_{L^\infty(0, T; L^\infty)} + \| \nabla \mathbf{H} \|_{L^2(0, T; L^2(\mathbb{R}^2))} \right) = \infty.
\]  

(1.15)

A few remarks are in order:

**Remark 1.1.** To obtain the local existence and uniqueness of strong/classical solutions, in theorem 1.1 and 1.2, the compatibility conditions we need are much weaker than the ones used in [6], similar to that of [35] for 3D compressible non-resistive MHD equations.

**Remark 1.2.** When \( \mathbf{H} = 0 \), i.e. there is no magnetic field effect, (1.1) and (1.2) reduces to the compressible Navier–Stokes equations, and theorems 1.1 and 1.2 are similar to the results of Li et al [19]. Roughly speaking, we generalize the results of [19] to the 2D compressible non-resistive MHD equations. Furthermore, theorems 1.1 and 1.2 extend the corresponding 3D problem in [6] and [35] to a 2D problem.

**Remark 1.3.** Recently, Wang [30] established a blowup criterion for 2D compressible MHD equations which solely depends on the uniform (in time) upper bound of the density \( \varrho \), i.e.
\[ \lim_{T \to T^*} \| \varrho \|_{L^\infty(0,T;L^\infty)} = \infty. \]  
(1.16)

It is clear that the blowup criterion in (1.15) for compressible non-resistive MHD equations is stronger than the one in (1.16). This is mainly due to the lack of a resistivity term for \( \mathbf{H} \), which can improve the stability of the system. However, theorem 1.3 is indeed improved from the previous ones obtained in [32, 35].

**Remark 1.4.** Compared with the previous blowup criterion established in [34] by Zhou et al for a 2D incompressible MHD system with zero magnetic diffusivity, which depends only on the magnetic fields, specifically,

\[ \lim_{T \to T^*} \| \nabla \mathbf{H} \|_{L^1(0,T;\text{BMO}(\mathbb{R}^2))}, \]

our result is somewhat stronger due to the compressibility and the existence of a vacuum. We also want to refer the readers to [16], where Jiang et al built a series of blowup criterions for a 2D generalized incompressible MHD system.

**Remark 1.5.** Compared with the incompressible non-resistive MHD [3, 7, 8], we consider the compressible one. Moreover, we indeed assume less regularity on \( u_0 \) than that of \( H_0 \) due to the existence of the diffusive term in (1.2) (see (1.9) and (1.12)), although it may not be optimal.

We now make some comments on the analysis of this paper. The key difficulty in studying such MHD equations lies in the non-resistivity of the magnetic equations. However, for the 2D case, when the far field density is a vacuum, another main difficulty is to bound the \( L^p(\mathbb{R}^2) \)-norm of \( u \) compared with the 3D case, which means the methods that have been successfully used in [6, 22, 32, 35] cannot be directly applied to our cases. Fortunately, previous results [11, 19, 20, 24–27] have provided hope for solving this problem. Specifically, we use the weighted \( L^p \)-bounds for elements of Hilbert space (see (2.5) and (2.6) below). Furthermore, compared with [11, 19, 20, 25], for the 2D compressible non-resistive MHD equations, the strong coupling between the velocity vector field and the magnetic field, such as \( u \cdot \nabla \mathbf{H} \) and \( (\nabla \times \mathbf{H}) \times \mathbf{H} \) (which does not appear in compressible Navier–Stokes equations), will provide us with some new difficulties. We will borrow some ideas from [26], that is, in order to control the term \( u \cdot \nabla \mathbf{H} \) and related terms, we need a spatial weighted mean estimate of \( \mathbf{H} \) and \( \nabla \mathbf{H} \) (see (3.33) and (4.1) below). Compared with [26], we have to face other difficulties caused by the lack of a resistive term in magnetic equations. We manage to solve the problem because (1.1) and (1.3) have an analogous structure from a mathematical view point. Therefore, the magnetic field could be treated in a similar manner as that used for density, although (1.3) is more complicated than (1.1).

The rest of the paper is organized as follows: in section 2, we collect some elementary facts and inequalities which will be needed in later analysis. Sections 3 and 4 are devoted to the \textit{a priori} estimates which are needed to obtain the local existence and uniqueness of strong/classical solutions. The main results, theorems 1.1, 1.2 and 1.3, are proved in section 5.

### 2. Preliminaries

First, the following local existence theory on bounded balls, where the initial density is strictly away from a vacuum, can be shown by similar arguments as those described in [6, 22, 35].
Lemma 2.1. For \( R > 0 \) and \( B_R = \{ x \in \mathbb{R}^2 \mid |x| < R \} \), assume that \((\varrho_0, \mathbf{u}_0, \mathbf{H}_0)\) satisfies
\[
(\varrho_0, \mathbf{u}_0, \mathbf{H}_0) \in H^2(B_R), \quad \inf_{x \in B_R} \varrho_0(x) > 0.
\]
(2.1)

Then there exists a small time \( T_R > 0 \) and a unique classical solution \((\varrho, \mathbf{u}, \mathbf{H})\) to the following initial-boundary-value problem
\[
\begin{cases}
\varrho_t + \text{div}(\varrho \mathbf{u}) = 0, \\
\varrho \mathbf{u}_t + \varrho (\mathbf{u} \cdot \nabla) \mathbf{u} + \nabla P(\varrho) - \mu \Delta \mathbf{u} - (\mu + \lambda) \text{div} \mathbf{u} - (\nabla \times \mathbf{H}) \times \mathbf{H} = 0, \\
\mathbf{H}_t + (\mathbf{u} \cdot \nabla) \mathbf{H} + \text{Hdiv} \mathbf{u} = (\mathbf{H} \cdot \nabla) \mathbf{u}, \\
\mathbf{u} = 0, x \in \partial B_R, \quad t > 0,
\end{cases}
\]

on \( B_R \times (0, T_R) \) such that
\[
\begin{align*}
\varrho, \mathbf{H} &\in C([0, T_R]; H^2), \quad \mathbf{u} \in C([0, T_R]; H^3) \cap L^2(0, T_R; H^4), \\
\varrho \mathbf{u} &\in L^\infty(0, T_R; H^2) \cap L^2(0, T_R; H^3), \quad \sqrt{\varrho} \mathbf{u} \in L^2(0, T_R; L^2), \\
\sqrt{\varrho} \mathbf{u}_t &\in L^\infty(0, T_R; H^1), \quad \sqrt{\varrho} \mathbf{u}_x \in L^\infty(0, T_R; H^2), \quad \sqrt{\varrho} \mathbf{u}_{tx} \in L^2(0, T_R; H^1), \\
\sqrt{\varrho} \mathbf{u}_{tx} &\in L^\infty(0, T_R; H^2) \cap L^2(0, T_R; H^3), \\
\mathbf{H} &\in L^\infty(0, T_R; H^2) \cap L^2(0, T_R; H^3), \quad \mathbf{H}_{tx} \in L^2(0, T_R; L^2), \\
\mathbf{H}_{tx} &\in L^\infty(0, T_R; H^2) \cap L^2(0, T_R; H^3).
\end{align*}
\]

(2.2)

where we denote \( L^2 = L^2(B_R) \) and \( H^k = H^k(B_R) \) for some positive integer \( k \).

Then, for either \( \Omega = \mathbb{R}^2 \) or \( \Omega = B_R \) with \( R \geq 1 \), the following weighted \( L^p \)-bounds for elements of Hilbert space \( \tilde{D}^{1,2}(\Omega) \triangleq \{ v \in H^1_{\text{loc}}(\Omega) \mid \nabla v \in L^2(\Omega) \} \) will play a crucial role in our analysis, which can be found in [19, Lemma 2.4].

Lemma 2.2. Let \( \mathfrak{e} \) be as in (1.9) and \( \Omega = \mathbb{R}^2 \) or \( \Omega = B_R \) with \( R \geq 1 \). For \( \gamma > 1 \), suppose that \( \varrho \in L^1(\Omega) \cap L^\infty(\Omega) \) is a non-negative function such that
\[
M_1 \leq \int_{B_{N_1}} \varrho dx, \quad \int_\Omega P(\varrho) dx \leq M_2,
\]

(2.4)

for some positive constants \( M_1, M_2 \) and \( N_1 \geq 1 \) with \( B_{N_1} \subset \Omega \). Then, there exists a positive constant \( C \) depending only on \( M_1, M_2, N_1, \gamma \) and \( \eta_0 \) such that
\[
|v|^\gamma l^{-1}_2 \leq C |\sqrt{\varrho} v|_l^2 + C |\nabla v|_l^2,
\]

(2.5)

for any \( v \in \tilde{D}^{1,2}(\Omega) \).

Furthermore, for \( \varepsilon > 0 \) and \( \eta > 0 \), there is a positive constant \( C \) depending on \( \varepsilon, \eta, M_1, M_2, N_1, \gamma \) and \( \eta_0 \) such that any \( \mathbf{v} \in \tilde{D}^{1,2}(\Omega) \) satisfies
\[
|v|^\eta |l|(l+\alpha)/\theta \leq C |\sqrt{\varrho} v|_l^2 + C |\nabla v|_l^2,
\]

(2.6)

with \( \tilde{\eta} = \min\{1, \eta\} \).

Next, we consider the following Lamé system,
\[
\begin{cases}
-\mu \Delta \mathbf{u} - (\mu + \lambda) \nabla \text{div} \mathbf{u} = \mathbf{F}, \quad \text{in } B_R, \\
\mathbf{u} = 0, \quad \text{on } \partial B_R,
\end{cases}
\]

(2.7)
The proof of the following $L^p$-bound is similar to that of [4, lemma 12].

**Lemma 2.3.** Let $u \in W^{1,q}_0(B_R)$ be a weak solution of the system (2.7), where $q > 1$. If $F \in W^{k,q}(B_R)$ for $k \geq 0$, then $u \in W^{k+2,q}(B_R)$ and

$$
\|u\|_{W^{k+2,q}(B_R)} \leq C\|F\|_{W^{k+2,q}(B_R)},
$$

(2.8)

where $C$ is independent of $R$.

Then, for $\nabla^\perp \equiv (\nabla_{\perp} \cdot \partial_t, \partial_t)$, denoting the material derivative of $\dot{f} \equiv f_t + u_1 \cdot \nabla f$. We now state some elementary estimates which follow from the Gagliardo–Nirenberg inequality and the standard $L^p$-estimate for the following elliptic system derived from the momentum equations in (1.2):

$$
\nabla \cdot (\rho u - H \cdot \nabla H), \quad \mu \nabla \omega = \nabla^\perp \cdot (\rho u - H \cdot \nabla H),
$$

where

$$
F \equiv (2\mu + \lambda)\text{div}u - P(\rho) - \frac{1}{2}|H|^2, \quad \omega = \partial_t u^2 - \partial_t u^4.
$$

The proofs of the following results are similar to that of [27, lemma 2.5].

**Lemma 2.4.** Let $(\rho, u, H)$ be a classical solution of (1.1)–(1.7). Then for $p \geq 2$ there exists a positive constant $C$ depending only on $p, \mu$ and $\lambda$ such that

$$
\|\nabla F\|_{L^p(R^2)} + \|\nabla \omega\|_{L^p(R^2)} \leq C\left( \|\rho \dot{u}\|_{L^p(R^2)} + \|H\|_{L^p(R^2)} + \|\nabla H\|_{L^p(R^2)} \right),
$$

(2.9)

$$
\|F\|_{L^p(R^2)} + \|\omega\|_{L^p(R^2)} \leq C\left( \|\rho \dot{u}\|_{L^p(R^2)} + \|H\|_{L^p(R^2)} + \|\nabla H\|_{L^p(R^2)} \right)^{(p-2)/p} \left( \|\dot{\rho} u\|_{L^2(R^2)} + \|P(\rho)\|_{L^2(R^2)} + \|H\|_{L^p(R^2)} \right)^{2/p},
$$

(2.10)

$$
\|\nabla u\|_{L^p(R^2)} \leq C\left( \|\rho \dot{u}\|_{L^2(R^2)} + \|H\|_{L^2(R^2)} \right)^{(p-2)/p} + C\|H\|_{L^p(R^2)}^2
$$

$$
\cdot \left( \|\nabla u\|_{L^2(R^2)} + \|P(\rho)\|_{L^2(R^2)} + \|H\|_{L^p(R^2)} \right)^{2/p} + C\|P(\rho)\|_{L^p(R^2)}.
$$

(2.11)

Finally, the following Beale–Kato–Majda type inequality, which is similar to that of [15, lemma 2.3], will be used later to estimate $\|\nabla u\|_{L^\infty}$ and $\|\nabla \phi\|_{L^2 \cap L^q}$ ($q > 2$).

**Lemma 2.5.** For $2 < q < \infty$, there is a constant $C(q)$ such that the following estimate holds for all $\nabla u \in L^2(R^2) \cap D^{1,k}(R^2)$,

$$
\|\nabla u\|_{L^\infty(R^2)} \leq C\left( \|\text{div} u\|_{L^\infty(R^2)} + \|\omega\|_{L^\infty(R^2)} \right) \ln \left( e + \|\nabla^2 u\|_{L^2(R^2)} \right)
$$

$$
\|\nabla u\|_{L^2(R^2)} + C\|\nabla u\|_{L^2(R^2)} + C.
$$

(2.12)

### 3. A priori estimates (I)

In this section and the next, for $p \in [1, \infty]$ and $k \geq 0$, we denote

$$
\int_{B_R} f \, dx = \int_{B_R} f, \quad L^p = L^p(B_R), \quad W^{k,p} = W^{k,p}(B_R), \quad H^k = W^{k,2}.
$$

Moreover, for $R > 4N_0 \geq 4$, we assume that the smooth triplet $(\rho_0, u_0, H_0)$ satisfies, in addition to (2.1), that
\[
\frac{1}{2} \leq \int_{B_{\rho_0}} \varrho_0(x) \, dx < \int_{B_{\rho}} \varrho_0(x) \, dx \leq \frac{3}{2}. \tag{3.1}
\]

It follows from lemma 2.1 that there exists some \( T_R > 0 \) such that the initial-boundary-value problem (2.2) has a unique classical solution \((\varrho, u, H)\) on \([0, T_R] \times B_{\rho} \) satisfying (2.3). For \( x, \eta_0, a \) and \( q \) as in theorem 1.1, the main aim of this section is to derive the following key \textit{a priori} estimate on \( \phi(t) \), defined by

\[
\phi(t) \triangleq 1 + \| \sqrt{\varrho} u \|_{L^2} + \| \nabla u \|_{L^2} + \| H \|_{L^2} + \| H x^a \|_{H^{1-\gamma} W^{\gamma, q}} + \| \varrho x^a \|_{L^1} + \| \varrho t \|_{L^1} + H^{1-\gamma} W^{\gamma, q}.
\tag{3.2}
\]

**Proposition 3.1.** Assume that \((\varrho_0, u_0, H_0)\) satisfies (2.1) and (3.1). Let \((\varrho, u, H)\) be the solution to the initial-boundary-value problem (2.2) on \([0, T_R] \times B_{\rho}\) obtained by lemma 2.1. Then there exist positive constants \( T_0 \) and \( M \) both depending on \( \mu, \lambda, \gamma, q, a, \eta_0, N_0 \) and \( C_0 \) such that

\[
\sup_{0 \leq t \leq T_0} \phi(t) + \int_0^{T_0} \left( \| \nabla^2 u \|_{L^2}^{(q+1)/q} + t \| \nabla^2 u \|_{L^2} + \| \nabla^2 u \|_{L^2} \right) \, dt \leq M,
\tag{3.3}
\]

where

\[
C_0 = \| \sqrt{\varrho_0} u \|_{L^2} + \| \nabla u \|_{L^2} + \| H_0 \|_{L^2} + \| H_0 x^a \|_{H^{1-\gamma} W^{\gamma, q}} + \| \varrho_0 x^a \|_{L^1} + H^{1-\gamma} W^{\gamma, q}.
\]

The proof of proposition 3.1 will be postponed to the end of this section. First, we start with the following energy estimate for \((\varrho, u, H)\) and preliminary \(L^2\)-bounds for \(\nabla u\).

**Lemma 3.2.** Let \((\varrho, u, H)\) be a smooth solution to the initial-boundary-value problem (2.2). Then there exists a positive constant \( \alpha = \alpha(\gamma, q) > 1 \) and a \( T_1 = T_1(C_0, N_0) > 0 \) such that for all \( t \in [0, T_1] \),

\[
\sup_{0 \leq s \leq t} \left( \| \nabla u \|_{L^2}^2 + \| H \|_{L^2}^2 + \| \nabla \varrho \|_{L^2}^2 + \| P(\varrho) \|_{L^2} \right)
\]

\[
+ \int_0^t \left( \| \nabla u \|_{L^2}^2 + \| \nabla \varrho \|_{L^2}^2 \right) \, ds \leq C + C \int_0^t \phi^\alpha(s) \, ds.
\tag{3.4}
\]

**Proof.** First, multiplying (2.2)\( _2 \) and (2.2)\( _3 \) by \( u \) and \( H \), respectively, and integrating the resulting equalities over \( B_R \) and summing them together, then integration by parts shows that

\[
\sup_{0 \leq s \leq t} \left( \| \sqrt{\varrho} u \|_{L^2}^2 + \| H \|_{L^2}^2 + \| P(\varrho) \|_{L^2} \right) + \int_0^t \| \nabla u \|_{L^2}^2 \, ds \leq C.
\tag{3.5}
\]

Next, for \( N > 1 \) and \( \varphi_N \in C_0^\infty(B_R) \) such that

\[
0 \leq \varphi_N \leq 1, \varphi_N(x) = 1, \text{ if } |x| \leq N/2, \text{ |\nabla^k \varphi_N|} \leq CN^{-k} (k = 1, 2),
\tag{3.6}
\]

then it follows from (3.1) and (3.5) that

\[
\frac{d}{dt} \int \varrho \varphi_{2N} \, dx = \int \varrho u \cdot \nabla \varphi_{2N} \, dx
\]

\[
\geq -CN_0^{-1} \left( \int \varrho \, dx \right)^{1/2} \left( \int |u|^2 \right)^{1/2} \geq -\tilde{C}(C_0, N_0),
\tag{3.7}
\]

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where, in the last inequality we have used,
\[
\int \varrho \, dx = \int \varrho_0 \, dx,
\]
due to (2.2)_1 and (2.2)_0. Integrating (3.7) over \((0, T_1)\) shows
\[
\inf_{0 \leq t \leq T_1} \int_{B_{R_0}} \varrho \, dx \geq \inf_{0 \leq t \leq T_1} \int B_{R_0} \varrho \varphi_{2N_0} \, dx \geq \int \varrho_0 \varphi_{2N_0} \, dx - \tilde{C} T_1 \geq 1/4,
\]
where \(T_1 \triangleq \min\{1, (4 \tilde{C})^{-1}\}\). From now on, we will always suppose that \(t \leq T_1\). The combination of (2.6), (3.5) and (3.8) shows that for \(\varepsilon > 0\) and \(\eta > 0\), every \(v \in \tilde{D}^{1,2}(B_R)\) satisfies
\[
\|v\|_{L^2(\xi+\varepsilon)/\eta} \leq C(\varepsilon, \eta) \|\sqrt{\rho} v\|_{L^2}^2 + C(\varepsilon, \eta) \|\nabla v\|_{L^2}^2,
\]
with \(\tilde{\eta} = \min\{1, \eta\}\). In particular, we have
\[
\|\varrho^\alpha u\|_{L^2(\xi+\varepsilon)/\eta} + \|u\|_{L^2(\xi+\varepsilon)/\eta} \leq C(\varepsilon, \eta) \varrho^{1+\alpha}(t).
\]
Next, multiplying (2.2)_2 by \(u\), and integration by parts yields
\[
\frac{d}{dt} \left( \mu \|\nabla u\|_{L^2}^2 + (\mu + \lambda) \|\nabla u\|_{L^2}^2 \right) + \|\varrho u\|_{L^2}^2 \leq C \int \varrho |u|^2 \|\nabla u\|_{L^2}^2 \, dx + 2 \int (P(\varrho) \nabla u, \varphi) \, dx + \int (H \cdot \nabla) u \, dx + \int |H|^2 \, dv_{u} \, dx.
\]
Now we estimate each term on the right-hand side of (3.11). First, the Gagliardo–Nirenberg inequality implies that for all \(p \in (2, +\infty)\),
\[
\|\nabla u\|_{L^p} \leq C \|\nabla u\|_{L^2}^{2p/3} \|\nabla u\|_{L^2}^{-2/p} \leq C(\varrho(t)) + C(\varrho(t)) \|\nabla u\|_{L^2}^{1-2/p},
\]
which, together with (3.10), yields that for \(\eta > 0\) and \(\tilde{\eta} = \min\{1, \eta\}\),
\[
\int \varrho^\alpha |u|^2 \|\nabla u\|_{L^2}^2 \, dx \leq C \|\varrho^\alpha/2 u\|_{L^2(\xi+\varepsilon)/\eta}^2 \|\nabla u\|_{L^2(\xi+\varepsilon)/\eta}^2 \leq C(\eta) \varrho^{\alpha+2\eta}(t) \left( 1 + \|\nabla^2 u\|_{L^2}^{\alpha/2} \right) \leq C \varrho^{\alpha}(t) \varrho^{1+\alpha}(t) \|\nabla^2 u\|_{L^2}^2.
\]
Next, since \(P(\varrho)\) satisfies
\[
P(\varrho) + \text{div}(P(\varrho)u) + (\gamma - 1) P(\varrho) \, \text{div} u = 0,
\]
we deduce from (3.10), (3.13) and the Sobolev inequality that
\[
2 \int P(\varrho) \, dv_{u} \, dx \leq \frac{d}{dt} \int P(\varrho) \, dv_{u} \, dx - 2 \int P(\varrho) \cdot \varrho \, dv_{u} \, dx + 2(\gamma - 1) \int P(\varrho) \, dv_{u} \, dx \leq \frac{d}{dt} \int P(\varrho) \, dv_{u} \, dx + \varepsilon \varrho^{-1}(t) \|\nabla^2 u\|_{L^2}^2 + C(\varepsilon) \varrho^\alpha(t).
\]
Then, using integration by parts together with (2.2), one obtains
\[
2 \int (\mathbf{H} \cdot \nabla)\mathbf{H} \cdot \mathbf{u} \, dx = -2 \frac{d}{dt} \int (\mathbf{H} \cdot \nabla)\mathbf{u} \cdot \mathbf{H} 
+ 2 \int (\mathbf{H} \cdot \nabla)\mathbf{u} \cdot \mathbf{H} 
= -2 \frac{d}{dt} \int (\mathbf{H} \cdot \nabla)\mathbf{u} \cdot \mathbf{H} 
+ 2 \int ((\mathbf{u} \cdot \nabla)\mathbf{H} \cdot \nabla)\mathbf{u} \cdot \mathbf{H} 
- 2 \int (\text{div}\mathbf{u} \cdot \nabla)\mathbf{u} \cdot \mathbf{H} 
+ 2 \int (\mathbf{H} \cdot \nabla)\mathbf{u} \cdot (\mathbf{H} \cdot \nabla)\mathbf{u} 
- 2 \int (\mathbf{H} \cdot \nabla)\mathbf{u} \cdot \text{div}\mathbf{u} 
\]  
\begin{equation}
(3.16)
\end{equation}

First, it is easy to check that
\[
\left| \int ((\mathbf{H} \cdot \nabla)\mathbf{u} \cdot \nabla)\mathbf{u} \cdot \mathbf{H} \right| + \left| \int (\mathbf{H} \cdot \nabla)\mathbf{u} \cdot (\mathbf{H} \cdot \nabla)\mathbf{u} \right| 
+ \left| \int (\text{div}\mathbf{u} \cdot \nabla)\mathbf{u} \cdot \mathbf{H} \right| + \left| \int (\mathbf{H} \cdot \nabla)\mathbf{u} \cdot \text{div}\mathbf{u} \right| \leq C\|\mathbf{H}\|_{L^2} \|\nabla\mathbf{u}\|^2_{L^2}.
\]

Next, Hölder’s inequality and Young’s inequality, together with (3.12), yield
\[
\left| \int ((\mathbf{u} \cdot \nabla)\mathbf{H} \cdot \nabla)\mathbf{u} \cdot \mathbf{H} \right| + \left| \int (\mathbf{H} \cdot \nabla)\mathbf{u} \cdot (\mathbf{u} \cdot \nabla)\mathbf{H} \right| 
\leq C \int \|\mathbf{H}\|\|\nabla\mathbf{u}\|\|\mathbf{H}\|\|\nabla\mathbf{H}\| \, dx 
\leq C \|\mathbf{H}\|_{L^\infty} \|\mathbf{u}\|^{-1}_{L^\infty} \|\mathbf{H}\|_{L^{2\alpha(\alpha-1)}} \|\nabla\mathbf{u}\|_{L^{2\alpha/\alpha}} 
\leq C \phi^\alpha(t) + \varepsilon \phi^{-1}(t) \|\nabla^2\mathbf{u}\|^2_{L^2}.
\]

Substituting the above two estimates into (3.16) gives
\[
2 \int (\mathbf{H} \cdot \nabla)\mathbf{H} \cdot \mathbf{u} \, dx \leq -2 \frac{d}{dt} \int (\mathbf{H} \cdot \nabla)\mathbf{u} \cdot \mathbf{H} + C \phi^\alpha(t) + \varepsilon \phi^{-1}(t) \|\nabla^2\mathbf{u}\|^2_{L^2}.
\]
\begin{equation}
(3.17)
\end{equation}

Similarly,
\[
\int \|\mathbf{H}\|^2 \text{div}\mathbf{u} \, dx \leq \frac{d}{dt} \int \|\mathbf{H}\|^2 \text{div}\mathbf{u} + C \phi^\alpha(t) + \varepsilon \phi^{-1}(t) \|\nabla^2\mathbf{u}\|^2_{L^2}.
\]
\begin{equation}
(3.18)
\end{equation}

Inserting (3.13), (3.15), (3.17) and (3.18) into (3.11) shows
\[
\frac{d}{dt} \left( \mu \|\nabla\mathbf{u}\|^2_{L^2} + (\mu + \lambda)\|\text{div}\mathbf{u}\|^2_{L^2} \right) 
- 2 \frac{d}{dt} \int \left( P(\mathbf{u}) \text{div}\mathbf{u} + \frac{1}{2} \|\mathbf{H}\|^2 \text{div}\mathbf{u} - 2(\mathbf{H} \cdot \nabla)\mathbf{u} \cdot \mathbf{H} \right) 
\leq C \phi^\alpha(t) + 4 \varepsilon \phi^{-1}(t) \|\nabla^2\mathbf{u}\|^2_{L^2}.
\]
\begin{equation}
(3.19)
\end{equation}
To estimate the last term on the right-hand side of (3.19), it follows from (2.8) that for \( p \in [2, q] \),
\[
\| \nabla^2 u \|_{L^p} \leq C (\| \rho u \|_{L^p} + \| \rho u \cdot \nabla u \|_{L^p} + \| \nabla P(\rho) \|_{L^p} + \| |H| |\nabla H| \|_{L^p}),
\] (3.20)
which, together with (3.12) and (3.13), yields
\[
\| \nabla^2 u \|_{L^2} \leq C \phi^{1/2}(t) \| \rho u \|_{L^2} + C \| \rho u \cdot \nabla u \|_{L^2} + C \phi^{\alpha}(t)
\leq C \phi^{1/2}(t) \| \rho u \|_{L^2} + \frac{1}{2} \| \nabla^2 u \|_{L^2} + C \phi^{\alpha}(t).
\] (3.21)
Substituting (3.21) into (3.19), then integrating the resulting inequality over \((0, t)\), and choosing \( \varepsilon \) suitably small leads to
\[
\frac{\mu}{2} \| \nabla u \|_{L^2}^2 + (\mu + \lambda) \| \text{div} u \|_{L^2}^2 + \int_0^t \| \sqrt{\rho} u \|_{L^2}^2 \, ds
\leq C + C \| P(\rho) \|_{L^2}^2 + C \| P(\rho) \|_{L^2}^2 \int_0^t \phi^{\alpha}(s) \, ds
\leq C + C \| H \|_{L^4}^4 + C \int_0^t \phi^{\alpha}(s) \, ds,
\] (3.22)
where in the last inequality we have used
\[
\| P(\rho) \|_{L^2}^2 \leq C \| P(\rho_0) \|_{L^2}^2 + C \int_0^t \| P(\rho) \|_{L^6}^{6/2} \| P(\rho) \|_{L^4}^{1/2} \| \nabla u \|_{L^2} \, ds
\leq C + C \int_0^t \phi^{\alpha}(s) \, ds,
\]
due to (3.14).

To estimate the second term on the right-hand side of (3.22), multiplying (2.2) by \( 4|H|^2 H \) and integrating the resulting equality over \( B_R \), we have
\[
\frac{d}{dt} \| H \|_{L^4}^4 \leq C \int |\nabla u| |H|^2 \, dx \leq C \| \nabla u \|_{L^2} \| H \|_{L^2} \| H \|_{L^\infty}.
\]
Integrating the above inequality over \((0, t)\) yields
\[
\| H \|_{L^4}^4 \leq C + \int_0^t \phi^{\alpha}(s) \, ds.
\] (3.23)
Putting (3.23) into (3.22), together with (3.5), leads to (3.4). Therefore, we complete the proof of lemma 3.1. \( \square \)

**Lemma 3.3.** Let \((\rho, u, H)\) and \(T_1\) be as in lemma 3.2. Then for all \( t \in (0, T_1) \),
\[
\sup_{0 \leq s \leq t} s\| \sqrt{\rho} u \|_{L^2}^2 + \int_0^t s\| \nabla u \|_{L^2}^2 \, ds \leq C \exp \left( C \int_0^t \phi^{\alpha}(s) \, ds \right).
\] (3.24)
**Proof.** Differentiating (2.2) with respect to \( t \) yields

\[
\rho u_t + \rho u \cdot \nabla u_t = \mu \Delta u_t - (\mu + \lambda) \nabla \operatorname{div} u_t
\]

\[
= - \rho_t (u_t + u \cdot \nabla u) - \rho u_t \cdot \nabla u - \nabla P_t (\rho) - \frac{1}{2} \nabla \| H_t \|_L^2 + H_t \cdot \nabla H + H \cdot \nabla H_t.
\]  

(3.25)

Multiplying (3.25) by \( \rho u \), and integrating the resulting equation over \( B_R \), we obtain

\[
\frac{1}{2} \frac{d}{dt} \| \rho |u| \|_L^2 \| |u| \|_L^2 + \mu \|
\]

\[
\int \frac{d}{dt} \| \rho |u| \|_L^2 \| |u| \|_L^2 + (\mu + \lambda) \int \| \nabla \operatorname{div} |u| \|_L^2 dx
\]

\[
= - 2 \int \rho u \cdot \nabla u \cdot |u| \rho dx - \int \rho u \cdot \nabla (u \cdot \nabla u_t) dx
\]

\[
- \int \rho u_t \cdot \nabla u \cdot u_t dx + \int P_t (\rho) \nabla \operatorname{div} u dx + \frac{1}{2} \int \| \nabla H_t \|_L^2 \| F |u| \|_L^2 dx
\]

\[
- \int (H \otimes H_t) : \nabla u_t dx
\]

\[
\leq C \int \rho |u| \| u \|_1 \| |u| \|_L^2 + \| u \|_L^2 \| |u| \|_L^2 \rho dx + C \int \rho |u| \| \nabla u \|_L^2 \| |u| \|_L^2 \rho dx
\]

\[
+ C \int \rho |u| \| \nabla u \|_L^2 \| u \|_L^2 \| |u| \|_L^2 dx + C \int \| \nabla H_t \|_L^2 \| |u| \|_L^2 \rho dx.
\]  

(3.26)

We now estimate each term on the right-hand side of (3.26) as follows:

First, it follows from (3.2), (3.5), (3.9), (3.10) and (3.12) that for \( \varepsilon \in (0, 1) \),

\[
\int \rho |u| \| u \|_1 \| |u| \|_L^2 + \| u \|_L^2 \| |u| \|_L^2 \rho dx
\]

\[
\leq C \| \sqrt{\rho |u|} \|_L^2 \| \sqrt{\rho |u|} \|_L^{1/2} \| \sqrt{\rho |u|} \|_L^{1/2} (\| \nabla |u| \|_L^2 + \| \nabla u \|_L^2)
\]

\[
+ C \| \rho |u| \|_2 \| \sqrt{\rho |u|} \|_L^{1/2} \| \sqrt{\rho |u|} \|_L^{1/2} \| \nabla^2 u \|_L^2
\]

\[
\leq C \rho(t) \| |u| \|_L^2 \| \| \nabla |u| \|_L^{1/2} + \| \nabla |u| \|_L^{1/2} \| \nabla |u| \|_L^2 + \| \nabla^2 u \|_L^2 + \phi(t)
\]

\[
\leq \varepsilon \| \nabla |u| \|_L^2 + C \rho(t) (\| \nabla^2 u \|_L^2 + \| \nabla |u| \|_L^2 + 1).
\]  

(3.27)

Next, Hölder’s inequality together with (3.10) and (3.12) shows that

\[
\int \rho |u| \| \nabla |u| \|_L^2 \rho dx \leq C \| \sqrt{\rho |u|} \|_L^2 \| \nabla |u| \|_L^2 \| \nabla |u| \|_L^2
\]

\[
\leq \varepsilon \| \nabla |u| \|_L^2 + C (\rho(t) + \| \nabla^2 u \|_L^2).
\]  

(3.28)

Then, Hölder’s inequality and (3.9) yield that

\[
\int \rho |u| \| \nabla |u| \|_L^2 \leq \| \nabla |u| \|_L^2 \| \sqrt{\rho |u|} \|_L^{3/2} \| \sqrt{\rho |u|} \|_L^{1/2}
\]

\[
\leq \varepsilon \| \nabla |u| \|_L^2 + C \rho(t) \| \sqrt{\rho |u|} \|_L^2.
\]

(3.29)
Next, it follows from (3.10) and (3.14) that
\[
\int |P_t(\varrho)||\text{div} u| \, dx \\
\leq C \int (P(\varrho)|\text{div} u| + |\nabla P(\varrho)||u|) \, dx \\
\leq C \left( \|P(\varrho)\|_{L^\infty} \|\nabla u\|_{L^2} + \|\varrho\|_{L^{\infty}} \|\varphi\|_{L^{\infty}} \|u\|_{L^2(\bar{u} - u)} \right) \|\nabla u\|_{L^2} \\
\leq \varepsilon \|\nabla u\|_{L^2}^2 + C\phi^\alpha(t). \quad (3.30)
\]

Finally, Hölder’s inequality, (2.2)3 and (3.10) gives that
\[
\int |H| |\nabla u| \, dx \leq C \int (|H| |\nabla u| + |\nabla H| |u|) \, dx \\
\leq C \left( \|H\|_{L^\infty} \|\nabla u\|_{L^2} + \|H\|_{L^q} \|u\|_{W^{1,q}} \|u\|_{L^{2q/(q-1)}} \right) \|\nabla u\|_{L^2} \\
\leq \varepsilon \|\nabla u\|_{L^2}^2 + C\phi^\alpha(t). \quad (3.31)
\]

Inserting (3.27)-(3.31) into (3.26) and choosing \(\varepsilon\) suitably small yields that
\[
\frac{d}{dt} \|u\|_{L^2}^2 + \mu \int |\nabla u|^2 \, dx + (\mu + \lambda) \int |\text{div} u|^2 \, dx \\
\leq C\phi^\alpha(t) \left( 1 + \|\sqrt{\gamma} u\|_{L^2}^2 + \|\nabla^2 u\|_{L^2}^2 \right) \quad (3.32)
\]

where in the last inequality we have used (3.21). Then, multiplying (3.32) by \(t\), we finally obtain (3.24) after using Gronwall’s inequality and (3.4). Therefore, we complete the proof of lemma 3.3.

\[\square\]

**Lemma 3.4.** Let \((\varrho, u, H)\) and \(T_1\) be as in lemma 3.2. Then for all \(t \in (0, T_1]\),
\[
\sup_{0 \leq r \leq t} \left( \|\varphi x^a\|_{L^q(\gamma^{1+\eta} w^{1+\eta})} + \|H x^a\|_{L^q(\gamma^{1+\eta} w^{1+\eta})} \right) \leq C \exp \left( C \int_0^t \phi^\alpha(s) \, ds \right). \quad (3.33)
\]

**Proof.** First, multiplying (2.2) by \(x^a\) and integrating the resulting equality over \(B_R\) integration by parts and using (3.10), we have
\[
\frac{d}{dt} \|\varphi x^a\|_{L^1} \leq C \int \varrho |x^{a-1} \ln |x|^2 + |x|^2| \, dx \\
\leq C \|\varphi x^{a-1+8/(8+a)}\|_{L^{(8+a)/(7+a)}} \|u x^{-4/(8+a)}\|_{L^{8+a}} \\
\leq C\phi^\alpha(t),
\]

which implies
\[
\sup_{0 \leq r \leq t} \|\varphi x^a\|_{L^1} \leq C \exp \left( C \int_0^t \phi^\alpha(s) \, ds \right). \quad (3.34)
\]

Next, it follows from the Sobolev inequality and (3.10) that for \(0 < \delta < 1\),
\[
\|u x^{-\delta}\|_{L^\infty} \leq C \left( \|u x^{-\delta}\|_{L^{1/3}} + \|\nabla(u x^{-\delta})\|_{L^1} \right) \\
\leq C \left( \|u x^{-\delta}\|_{L^{1/3}} + \|\nabla u\|_{L^1} + \|u x^{-\delta}\|_{L^{1/3}} \right) u x^{-1/3} \|\nabla x\|_{L^{12/(4-\delta)}} \\
\leq C \left( \phi^\alpha(t) + \|\nabla^2 u\|_{L^2} \right). \tag{3.35}
\]

Then, one derives from (2.2) that \( \dot{H} \triangleq H x^a \) satisfies
\[
\dot{H} + u \cdot \nabla H - a u \cdot \nabla \ln x \cdot \dot{H} + \dot{H} \text{div} u = \dot{H} \cdot \nabla u, \tag{3.36}
\]
which, together with (3.35) and the Gagliardo–Nirenberg inequality, shows that
\[
\frac{d}{dt} \|\dot{H}\|_{L^2} \leq C \left( \|\nabla u\|_{L^\infty} + \|u \cdot \nabla \ln x\|_{L^\infty} \right) \|\dot{H}\|_{L^2} \\
\leq C \left( \phi^\alpha(t) + \|\nabla^2 u\|_{L^{27/10}} \right) \|\dot{H}\|_{L^2}. \tag{3.37}
\]
Moreover, (3.36) together with (3.35) and the Gagliardo–Nirenberg inequality gives that for \( p \in [2, q] \)
\[
\frac{d}{dt} \|\nabla \dot{H}\|_{L^p} \\
\leq C \left( 1 + \|\nabla u\|_{L^\infty} + \|u \cdot \nabla \ln x\|_{L^\infty} \right) \|\nabla \dot{H}\|_{L^p} \\
+ C \left( \|\nabla^2 u\|_{L^{27/10}} \right) \|\nabla \dot{H}\|_{L^p} \\
\leq C \left( \phi^\alpha(t) + \|\nabla^2 u\|_{L^{27/10}} \right) \left( 1 + \|\nabla \dot{H}\|_{L^p} + \|\nabla \dot{H}\|_{L^p} \right). \tag{3.38}
\]
Combining (3.37) and (3.38), one yields
\[
\frac{d}{dt} \left( \|\dot{H}\|_{L^2} + \|\nabla \dot{H}\|_{L^p} \right) \\
\leq C \left( \phi^\alpha(t) + \|\nabla^2 u\|_{L^{27/10}} \right) \left( 1 + \|\nabla \dot{H}\|_{L^p} + \|\nabla \dot{H}\|_{L^p} \right). \tag{3.39}
\]
Similarly, with the help of (3.34) and denoting \( w \triangleq g x^a \), one obtains for \( p \in [2, q] \) that
\[
\frac{d}{dt} \left( \|w\|_{L^2} + \|\nabla w\|_{L^p} \right) \leq C \left( \phi^\alpha(t) + \|\nabla^2 u\|_{L^{27/10}} \right) \left( 1 + \|\nabla w\|_{L^p} + \|w\|_{L^2} + \|\nabla w\|_{L^2} \right), \tag{3.40}
\]
where \( w \) satisfies the following equality
\[
w_t + u \cdot \nabla w - awu \cdot \nabla \ln x + w \text{div} u = 0. \tag{3.41}
\]
Next, we claim that
\[
\int_0^t \left( \|\nabla^2 u\|_{L^{27/10}}^{q+1/q} + s \|\nabla^2 u\|_{L^{27/10}}^2 \right) \, ds \leq C \exp \left( C \int_0^t \phi^\alpha(s) \, ds \right), \tag{3.42}
\]
which, together with (3.39), (3.40) and Gronwall’s inequality, gives (3.33).
Now, to finish the proof of lemma 3.4, it only has to prove (3.42). In fact, on the one hand, it follows from (3.4), (3.21) and (3.24) that
\[
\int_0^t \left( \|\nabla^2 u\|_{L^2}^{5/3} + s \|\nabla^2 u\|_{L^2}^2 \right) ds \\
\leq C \int_0^t \left( \|\nabla u\|_{L^2} + \phi^\alpha(s) \right) ds + C \exp \left( C \int_0^t \phi^\alpha(s) ds \right) \int_0^t \phi^\alpha(s) ds \\
\leq C \exp \left( C \int_0^t \phi^\alpha(s) ds \right).
\] (3.43)

On the other hand, choosing \( p = q \) in (3.20) and using (3.9), (3.10) and (3.12), we have
\[
\|\nabla^2 u\|_{L^q} \leq C \left( \|\theta u\|_{L^p} + \|\theta u \cdot \nabla u\|_{L^p} + \|\nabla \rho\|_{L^p} + \|\nabla H\|_{L^p} \right) \\
\leq C \|\theta u\|_{L^q(\mathbb{R}^n)} \left( \|\theta u\|_{L^q(\mathbb{R}^n)} \|\nabla u\|_{L^q(\mathbb{R}^n)} + \|\phi\|_{L^q(\mathbb{R}^n)} \right) \\
\leq C \phi^\alpha(t) \left( \|\nabla u\|_{L^q(\mathbb{R}^n)} \|\nabla u\|_{L^q(\mathbb{R}^n)} + \|\nabla u\|_{L^q(\mathbb{R}^n)} \right) \\
+ C \phi^\alpha(t) \left( 1 + \|\nabla^2 u\|_{L^q(\mathbb{R}^n)}^{(q-1)/q} \right). \] (3.44)

Then, combining (3.44) with (3.5), (3.24) and (3.43) leads to
\[
\int_0^t \|\nabla^2 u\|_{L^q}^{(q+1)/q} ds \\
\leq C \int_0^t \phi^\alpha(s) s^{-q/2q} \left( s \|\sqrt{\rho} u\|_{L^2}^2 \right)^{(q-1)/(q+1)} \left( s \|\nabla u\|_{L^2}^2 \right)^{(q-1)/(q+1)} ds \\
+ C \int_0^t \|\sqrt{\rho} u\|_{L^2}^2 ds + C \exp \left( C \int_0^t \phi^\alpha(s) ds \right) \\
\leq C \exp \left( C \int_0^t \phi^\alpha(s) ds \right) \left( 1 + \int_0^t \left( \phi^\alpha(s) + s \frac{\|\nabla u\|_{L^2}^2}{\sqrt{s}} \right) ds \right) \\
\leq C \exp \left( C \int_0^t \phi^\alpha(s) ds \right). \] (3.45)

and that
\[
\int_0^t \|\nabla^2 u\|_{L^2}^2 ds \leq C \int_0^t \phi^\alpha(s) \left( s \|\sqrt{\rho} u\|_{L^2}^2 \right)^{(q-1)/(q+1)} \left( s \|\nabla u\|_{L^2}^2 \right)^{(q-1)/(q+1)} ds \\
+ C \int_0^t \phi^\alpha(s) \|\sqrt{\rho} u\|_{L^2}^2 ds + C \int_0^t \phi^\alpha(s) \left( 1 + \|\nabla^2 u\|_{L^2}^{(q-1)/q} \right) ds \\
\leq C \sup_{0 \leq s \leq t} \left( s \|\sqrt{\rho} u\|_{L^2}^2 \right)^{(q-1)/(q+1)} \left( \int_0^t \phi^\alpha(s) + s \|\nabla u\|_{L^2}^2 ds \right) \\
+ C \sup_{0 \leq s \leq t} \left( s \|\sqrt{\rho} u\|_{L^2}^2 \right) \int_0^t \phi^\alpha(s) ds + C \int_0^t \left( \phi^\alpha(s) + s \|\nabla^2 u\|_{L^2}^2 \right) ds \\
\leq C \exp \left( C \int_0^t \phi^\alpha(s) ds \right). \] (3.46)

Therefore, (3.43), (3.45) and (3.46) give (3.42) and we complete the proof of lemma 3.4. □
Now, we complete the proof of proposition 3.1, which is a direct consequence of lemmas 3.2–3.4.

**Proof of proposition 3.1.** It follows from (3.4) and (3.33) that
\[ \phi(t) \leq \exp \left( C \exp \left( C \int_0^t \phi(s) ds \right) \right). \]

Standard arguments thus show that for \( M \triangleq e^{Ct} \) and \( T_0 \triangleq \min\{ T_1, (CM^\alpha)^{-1} \} \),
\[ \sup_{0 \leq t \leq T_0} \phi(t) \leq M, \]
which, together with (3.4), (3.21) and (3.42), leads to (3.3). Then the proof of proposition 3.1 is finished. \( \square \)

4. **A priori estimates (II)**

In this section, in addition to \( \mu, \lambda, \gamma, q, a, \eta_0, N_0 \) and \( C_0 \), the generic positive constant \( C \) may also depend on \( \delta_n \), \( \| \nabla^2 u_0 \|_{L^2} \), \( \| \nabla^2 \rho_0 \|_{L^2} \), \( \| \nabla^2 P(\rho_0) \|_{L^s} \), \( \| \nabla^2 H_0 \|_{L^s} \), \( \| \nabla^2 \nabla^2 H_0 \|_{L^s} \) and \( \| \mathbf{g} \|_{L^2} \).

**Lemma 4.1.** It holds that
\[ \sup_{0 \leq t \leq T_0} \left( \| \nabla^2 \nabla^2 \rho \|_{L^2} + \| \nabla^2 \nabla^2 P(\rho) \|_{L^2} + \| \nabla^2 \nabla^2 H \|_{L^2} \right) \leq C. \]

**Proof.** First, due to (1.13), (2.1) and (2.2), defining
\[ \sqrt{\rho} u(t = 0, x) \triangleq -g - \sqrt{\rho_0} u_0 \cdot \nabla u_0, \]
integrating (3.32) over \((0, T_0)\) and using (3.3) and (3.4), we have
\[ \sup_{0 \leq t \leq T_0} \| \sqrt{\rho} u_0 \|_{L^2} + \int_0^{T_0} \| \nabla u_0 \|_{L^2} dt \leq C, \]
which, together with (3.3) and (3.21), leads to
\[ \sup_{0 \leq t \leq T_0} \| \nabla u \|_{H^1} \leq C. \]

This combined with (3.3) and (3.35) shows that for \( \delta \in (0, 1) \),
\[ \| \delta^\delta u \|_{L^\infty} + \| \nabla^{-\delta} u \|_{L^\infty} \leq C(\delta). \]

Directly calculate that for \( 2 \leq r \leq q \)
\[ \| \delta_1(x^{(1+s)/2} + |u|) \|_{L^r} + \| P_I(\rho)(1 + |u|) \|_{L^r} + \| H_I(x^{(1+s)/2} + |u|) \|_{L^r} \leq C, \]
due to (2.2), (2.3), (3.14), (4.3) and (4.4). It follows from (3.9), (4.2), (4.3) and (4.4) that for \( \delta \in (0, 1) \) and \( s > 2/\delta \),
\[ \| \nabla^{-\delta} u \|_{L^r} \leq C(\delta, s) \| \nabla u \|_{L^r}. \]
\[ \leq C(\delta, s) \| \nabla u \|_{L^r}. \]
Next, denoting $\tilde{H} \triangleq x^8 H$ and $v \triangleq \tilde{x}^8 f(\varrho)$ with $f(\varrho) = \varrho^p$ for $p \in [1, \gamma]$, we easily get from (3.3) that
\begin{equation}
\|v\|_{L^\infty} + \|\nabla v\|_{L^{2n/\gamma}} + \|\tilde{H}\|_{L^\infty} + \|\nabla \tilde{H}\|_{L^{2n/\gamma}} \leq C,
\end{equation}
where $\nu$ satisfies
\begin{equation}
\nu_{t} + u \cdot \nabla \nu - \delta_0 \nu \cdot \nabla \ln \varphi + \nu \varrho \text{div} u = 0.
\end{equation}
It follows from (2.2), similarly to (3.36), that we have
\begin{equation}
H_{t} = u \cdot \nabla H - \delta_0 \nu \cdot \nabla \ln \varphi + \text{Hdiv} u = H \cdot \nabla u.
\end{equation}
Therefore, direct calculations give that
\begin{equation}
\frac{d}{dt}\|\nabla^2 \tilde{H}\|_{L^2} \leq C (1 + \|\nabla u\|_{L^\infty} + \|u \cdot \nabla \ln \tilde{x}\|_{L^\infty}) \|\nabla^2 \tilde{H}\|_{L^2} + C \|\nabla^2 u \nabla \tilde{H}\|_{L^2}
\end{equation}
where in the second and third inequalities we have used (4.4) and (4.7). Similarly, we can also obtain from (4.8) after calculations
\begin{equation}
\frac{d}{dt} \|\nabla^2 v\|_{L^2} \leq C \|\nabla^2 v\|_{L^2} + C \|\nabla^2 u\|_{L^2}.
\end{equation}
Combining (4.10) with (4.11), we get
\begin{equation}
\frac{d}{dt} \left( \|\nabla^2 v\|_{L^2} + \|\nabla^2 \tilde{H}\|_{L^2} \right)
\leq C \|\nabla^2 v\|_{L^2} + \|\nabla^2 \tilde{H}\|_{L^2} + C \|\nabla^3 u\|_{L^2}.
\end{equation}
We use (2.8) and (3.3) to estimate the last term on the right-hand side of (4.12) as follows:
\begin{align}
\|\nabla^3 u\|_{L^2} & \leq C \|\nabla (\varrho u)\|_{L^2} + C \|\nabla (\varrho u \cdot \nabla u)\|_{L^2} + C \|\nabla^2 P(\varrho)\|_{L^2}
\end{align}
where in the last inequality we have used (3.3), (4.3), (4.4), (4.6) and the following fact:
\[
\|x^6 \nabla^2 \varphi\|_{L^2} + \|x^6 \nabla^2 P(\varphi)\|_{L^2} + \|x^6 \nabla^2 H\|_{L^2} \\
\leq C\|\nabla^2 (x^6 \varphi)\|_{L^2} + C\|\nabla^2 (x^6 P(\varphi))\|_{L^2} + \|\nabla^2 (x^6 H)\|_{L^2} + C.
\] (4.14)

Substituting (4.13) into (4.12) and noting the definition of \(\nu\), one has
\[
\frac{d}{dt} \left( \|\nabla^2 (x^6 \varphi)\|_{L^2} + \|\nabla^2 (x^6 P(\varphi))\|_{L^2} + \|\nabla^2 (x^6 H)\|_{L^2} \right)
\leq C(1 + \|\nabla^2 u\|_{L^2}) \left( \|\nabla^2 (x^6 \varphi)\|_{L^2} + \|\nabla^2 (x^6 P(\varphi))\|_{L^2} + \|\nabla^2 (x^6 H)\|_{L^2} \right)
+ C\|\nabla u\|_{L^2} + C,
\]
which, together with (3.3), (4.2), (4.14) and Gronwall’s inequality, gives (4.1) and completes the proof of lemma 4.1. \(\square\)

**Lemma 4.2.** It holds that
\[
\sup_{0 \leq t \leq T_0} \int \|\nabla u\|_{L^2}^2 + \int_0^{T_0} \left( \|\sqrt{\mu} u\|_{L^2}^2 + \|\nabla^2 u\|_{L^2}^2 \right) dt \leq C.
\] (4.15)

**Proof.** Multiplying (3.25) by \(u_t\) and integrating the resulting equality over \(B_R\), integration by parts leads to
\[
\frac{1}{2} \frac{d}{dt} (\mu \|\nabla u\|_{L^2}^2 + (\mu + \lambda) \|\text{div} u\|_{L^2}^2) + \|\sqrt{\mu} u\|_{L^2}^2
\]
\[
= - \int (2 \rho \mu \cdot \nabla u + \rho u \cdot \nabla u_t + \rho u_t \cdot \nabla u) \cdot u_t dx - \int \rho u \cdot \nabla (u \cdot \nabla u) \cdot u_t dx
\]
\[
- \int \rho u \cdot \nabla u_t \cdot u_t dx - \int \rho u \cdot \nabla u_t \cdot (u \cdot \nabla) u dx + \int P(\varphi) \text{div} u_t dx
\]
\[
+ \frac{1}{2} \int \|H_t\|_{L^2}^2 dx - \int H_t \cdot \nabla u_t \cdot H dx - \int H \cdot \nabla u_t \cdot H dx.
\] (4.16)

Now, we estimate each term on the right-hand side of (4.16). First, it follows from (3.3), (4.2)–(4.4) and (4.6) that
\[
\int (2 \rho u \cdot \nabla u_t \cdot u_t + \rho u_t \cdot \nabla u_t \cdot u_t) dx \leqslant e\|\sqrt{\mu} u\|_{L^2}^2 + C(\varepsilon) \left( \|\sqrt{\mu} u\|_{L^2}^2 \|\nabla u\|_{L^2}^2 + \|\sqrt{\mu} u\|_{L^2}^2 \|\nabla^2 u\|_{L^2}^2 \right)
\]
\[
+ C(\varepsilon) \left( \|\sqrt{\mu} u\|_{L^2}^2 \|\nabla^2 u\|_{L^2}^2 + (\mu + \lambda) \|\nabla u\|_{L^2}^2 \right)
\]
\[
\leq e\|\sqrt{\mu} u\|_{L^2}^2 + C(\varepsilon) \left( 1 + \|\nabla u\|_{L^2}^2 \right).
\] (4.17)

Next, direct calculations yield that
\[
\int \rho u \cdot \nabla u_t \cdot u_t dx - \int \rho u \cdot \nabla u_t \cdot (u \cdot \nabla) u dx
\]
\[
= - \frac{d}{dt} \int (\rho u \cdot \nabla u_t + \rho u_t \cdot \nabla u_t) dx + \int (\rho u_t) \cdot \nabla u_t \cdot u dx
\]
\[
+ \int (\rho u_t) \cdot \nabla u_t \cdot (u \cdot \nabla) u dx + \int \rho u \cdot \nabla u_t \cdot u_t dx
\]
\[
+ \int \rho u \cdot \nabla u_t \cdot (u_t \cdot \nabla) u dx + \int \rho u \cdot \nabla u_t \cdot (u \cdot \nabla) u dx.
\] (4.18)
First, Hölder’s inequality together with (3.3) and (4.4)—(4.6) gives
\[
\int (\rho u_t) \cdot \nabla u_t \cdot u_t \, dx \leq C \| \rho \|_{L^\infty} \| \nabla u_t \|_{L^2} \| \nabla u_t \|_{L^2}^2 \nonumber \\
+ C \| \nabla^{(1+\alpha)/2} \rho \|_{L^2} \| \nabla^{\alpha/2} u_t \|_{L^\infty} \| \nabla^{\alpha/2} u_t \|_{L^2} \| \nabla u_t \|_{L^2} \nonumber \\
\leq \delta \| \nabla^2 u_t \|_{L^2}^2 + C(\delta) \| \nabla u_t \|_{L^2}^4 + C(\delta). \tag{4.19}
\]
Similarly,
\[
\int (\rho u_t) \cdot \nabla u_t \cdot (u \cdot \nabla) \, dx 
\leq C \| \rho \|_{L^\infty} \| \nabla^{\alpha/2} u_t \|_{L^2} \| \nabla^{\alpha/2} u_t \|_{L^2} \| \nabla u_t \|_{L^2} \nonumber \\
+ C \| \nabla^{(1+\alpha)/2} \rho \|_{L^2} \| \nabla^{\alpha/2} u_t \|_{L^\infty} \| \nabla^{\alpha/2} u_t \|_{L^2} \| \nabla u_t \|_{L^2} \nonumber \\
\leq \delta \| \nabla^4 u_t \|_{L^2}^2 + C(\delta) \| \nabla u_t \|_{L^2}^4 + C(\delta). \tag{4.20}
\]
Then, Hölder’s inequality together with (4.4) leads to
\[
\int \rho u \cdot \nabla u_t \cdot u_t \, dx \leq C \| \sqrt{\rho u_t} \|_{L^2} \| \sqrt{\rho u} \|_{L^\infty} \| \nabla u_t \|_{L^2} 
\leq \varepsilon \| \sqrt{\rho u_t} \|_{L^2}^2 + C(\varepsilon) \| \nabla u_t \|_{L^2}^2. \tag{4.21}
\]
Next, it follows from (4.3), (4.4) and (4.6) that
\[
\int \rho u \cdot \nabla u_t \cdot (u_t \cdot \nabla) \, dx \leq C \| \rho \|_{L^\infty} \| \nabla^{\alpha/2} u_t \|_{L^2} \| \nabla^{\alpha/2} u_t \|_{L^2} \| \nabla u_t \|_{L^2} \| \nabla u_t \|_{L^2}^2 
\leq C + C(\varepsilon) \| \nabla u_t \|_{L^2}^2, \tag{4.22}
\]
and similarly,
\[
\int \rho u \cdot \nabla u_t \cdot (u \cdot \nabla \cdot u_t) \, dx \leq C \| \rho \|_{L^\infty} \| \nabla^{\alpha/2} u_t \|_{L^2}^2 \| \nabla u_t \|_{L^2}^2 
\leq C \| \nabla u_t \|_{L^2}^2. \tag{4.23}
\]
Inserting (4.19)–(4.23) into (4.18) shows
\[
-\int \rho u \cdot \nabla u_t \cdot \nabla u_t \, dx - \int \rho u \cdot \nabla u_t \cdot (u \cdot \nabla) \, dx 
= -\frac{d}{dt} \int (\rho u \cdot \nabla u_t \cdot u_t + \rho u \cdot \nabla u_t \cdot (u \cdot \nabla) \, dx + \varepsilon \| \sqrt{\rho u_t} \|_{L^2}^2 
+ C(\varepsilon, \delta) \| \nabla u_t \|_{L^2}^4 + \delta \| \nabla^2 u_t \|_{L^2}^2 + C(\varepsilon, \delta). \tag{4.24}
\]
Next, it follows from (3.14), (4.3) and (4.5) that
\[ \int P_t(\varrho) \div u_t \, dx = \frac{d}{dt} \int P_t(\varrho) \div u \, dx - \int (P_t(\varrho) u_t) \cdot \nabla \div u_t \, dx \]
\[ + (\gamma - 1) \int (P_t(\varrho) \div u_t) \, dx \]
\[ \leq \frac{d}{dt} \int P_t(\varrho) \div u_t \, dx + C (\|P_t(\varrho) u\|_{L^2} + \|P_t(\varrho) u_t\|_{L^2}) \|\nabla^2 u\|_{L^2} \]
\[ + C \|P_t(\varrho)\|_{L^2} \|\nabla u\|_{L^2} \|\nabla u_t\|_{L^2} + \|P_t(\varrho)\|_{L^\infty} \|\nabla u\|_{L^2}^2 \]
\[ \leq \frac{d}{dt} \int P_t(\varrho) \div u_t \, dx + \delta \|\nabla^2 u\|_{L^2}^2 + C(\|\nabla u\|_{L^2}^2 + 1) . \]  \tag{4.25}

Similarly, \((2.2)_3, (4.1), (4.3)\) and \((4.5)\) give that
\[ \frac{1}{2} \int H_t^2 \div u_t \, dx = \frac{d}{dt} \int \left( \frac{1}{2} |H_t|^2 \div u_t - H_t \cdot \nabla u_t \cdot H_t - H \cdot \nabla u_t \cdot H_t \right) \, dx - \int |H_t|^2 \div u_t \, dx \]
\[ - \int H^2 \cdot \nabla u_t \cdot H_t \, dx + \int H_t^2 \div u_t \, dx + 2 \int H_t \cdot \nabla u_t \cdot H_t \, dx \]
\[ \leq \frac{d}{dt} \int \left( \frac{1}{2} |H_t|^2 \div u_t - H_t \cdot \nabla u_t \cdot H_t - H \cdot \nabla u_t \cdot H_t \right) \, dx + C \left( \|\nabla u_t\|_{L^2}^2 + 1 \right) . \]  \tag{4.26}

where in the last inequality we have used \((4.6)\) and the following simple fact:
\[ \|\nabla H_t\|_{L^2} \leq C \|\nabla u_t\|_{L^2} \|\nabla H\|_{L^2} + C \|u_t\|_{L^2} \|\nabla^2 H\|_{L^2} + C \|H\|_{L^\infty} \|\nabla^2 u\|_{L^2} \]
\[ \leq C \|\nabla u_t\|_{L^{2/(2-\delta)}} \|\nabla H\|_{L^2} + C \|u_t\|_{L^{2/(2-\delta)}} \|\nabla^2 H\|_{L^2} + C \|H\|_{L^\infty} \|\nabla^2 u\|_{L^2} \]
\[ \leq C , \]  \tag{4.27}

due to \((4.1), (4.3), (4.4)\) and \((4.7)\).

Inserting \((4.17)\) and \((4.24)-(4.26)\) into \((4.16)\) and choosing \(\varepsilon\) suitably small we follow that
\[ \Phi'(t) + \|\sqrt{\varepsilon} u_t\|_{L^2}^2 \leq C \delta \|\nabla^2 u\|_{L^2}^2 + C \|\nabla u_t\|_{L^2}^2 + C , \]  \tag{4.28}

where
\[ \Phi(t) \triangleq \mu \| \nabla u_t \|_{L^2}^2 + (\mu + \lambda) \| \text{div} u_t \|_{L^2}^2 + \int (\varrho u \cdot \nabla u_t + \varrho \dot{u} \cdot \nabla u_t - (u \cdot \nabla) u) \, dx \]
\[ - \int P_t(\varrho) \text{div} u_t \, dx - \left( \frac{1}{2} |H|^2 \text{div} u_t - H \cdot \nabla u_t \cdot H - H \cdot \nabla u_t \cdot H \right) \, dx \]

satisfies

\[ C(\mu) \| \nabla u_t \|_{L^2}^2 - C \leq \Phi(t) \leq C \| \nabla u_t \|_{L^2}^2 + C, \quad (4.29) \]

due to the following fact:

\[ \left| \int (\varrho u \cdot \nabla u_t + \varrho \dot{u} \cdot \nabla u_t - (u \cdot \nabla) u) \, dx \right| - \int P_t(\varrho) \text{div} u_t \, dx \]
\[ + \left| \left( \frac{1}{2} |H|^2 \text{div} u_t - H \cdot \nabla u_t \cdot H - H \cdot \nabla u_t \cdot H \right) \right| \]
\[ \leq C \sqrt{\varrho} \| u \|_{L^\infty} \| \nabla u_t \|_{L^2} \| \sqrt{\varrho} u_t \|_{L^2} + C \sqrt{\varrho} \| \nabla u_t \|_{L^2} \| \nabla u_t \|_{L^2} \]
\[ + C \| \nabla u_t \|_{L^2} \| \nabla u_t \|_{L^2} + C\varrho \| \nabla u_t \|_{L^2} \| \nabla u_t \|_{L^2} \]
\[ \leq \varepsilon \| \nabla u_t \|_{L^2}^2 + C(\varepsilon), \quad (4.30) \]

which is derived from (4.2)–(4.5).

Then, it remains to estimate the first term on the right-hand side of (4.27). In fact, it follows from (3.25) and lemma 2.3 that for \( s > 2 \),

\[ \| \nabla^2 u_t \|_{L^2}^2 \]
\[ \leq C \| u \|_{L^\infty} \| \sqrt{\varrho} u_t \|_{L^2}^2 + C \| u \|_{L^\infty} \| \sqrt{\varrho} u_t \|_{L^2}^2 \| \nabla u_t \|_{L^2}^2 + C \| \nabla^{(a+1)/2} \theta_t \|_{L^2}^2 \| \nabla^{-1} u_t \|_{L^{2n/(s-2)}}^2 \]
\[ + C \| \nabla(\varrho) \|_{L^2}^2 + C \| \nabla^{(a+1)/2} H_t \|_{L^2} \| \nabla H_t \|_{L^{2n/(s-2)}}^2 + C \| \nabla \varrho \|_{L^\infty} \| \nabla u_t \|_{L^2}^2 \]
\[ \leq C \sqrt{\varrho} \| u_t \|_{L^2}^2 + C \| \nabla u_t \|_{L^2}^2 + C, \quad (4.30) \]

due to (4.1)–(4.7), (4.27) and the following fact:

\[ \| \nabla P_t(\varrho) \|_{L^2} \leq C \| \nabla u_t \|_{L^{2n/(s-2)}} \| \nabla \varrho \|_{L^2} + C \| \nabla^{-\delta} u \|_{L^\infty} \| \nabla^{\delta} \nabla^2 P(\varrho) \|_{L^2} + C \| \nabla^2 u \|_{L^2} \]
\[ \leq C, \]

where we have used the Gagliardo–Nirenberg inequality, (4.1), (4.3), (4.4) and (4.7). Inserting (4.30) into (4.28) and choosing \( \delta \) suitably small leads to

\[ \Phi'(t) + \| \sqrt{\varrho} u_t \|_{L^2}^2 \leq C \| \nabla u_t \|_{L^2}^2 + C. \quad (4.31) \]

Multiplying (4.31) by \( t \) and integrating the resulting inequality over \( (0, T_0) \), we obtain from Gronwall’s inequality, (4.2) and (4.29)

\[ \sup_{0 \leq t \leq T_0} t \| \nabla u_t \|_{L^2}^2 + \int_0^{T_0} t \| \sqrt{\varrho} u_t \|_{L^2}^2 \, dt \leq C, \]

which, together with (4.30), yields (4.15) and completes the proof of lemma 4.2. \( \square \)
Lemma 4.3. It holds that
\[
\sup_{0 \leq t \leq T_0} \left( \| \nabla^2 u \|_{L^p} + \| \nabla^2 P(\theta) \|_{L^p} + \| \nabla^2 H \|_{L^p} \right) \leq C. 
\]  
(4.32)

Proof. Applying the differential operator \( \nabla^2 \) to (4.8) and (4.9), respectively, and multiplying each equality by \( q \| \nabla^2 v \|^{q-2} \nabla^2 v \) and \( q \| \nabla^2 H \|^{q-2} \nabla^2 H \), and integrating the resulting equalities over \( B_R \) leads to
\[
\frac{d}{dt} \left( \| \nabla^2 H \|_{L^p} + \| \nabla^2 v \|_{L^p} \right) 
\leq C \| \nabla u \|_{L^\infty} \left( \| \nabla^2 H \|_{L^p} + \| \nabla^2 v \|_{L^p} \right) + \left( \| \nabla H \|_{L^\infty} + \| \nabla v \|_{L^\infty} \right) \| \nabla^2 u \|_{L^p} 
\leq C \left( 1 + \| \nabla^2 u \|_{L^p} \right) \left( 1 + \| \nabla^2 H \|_{L^p} + \| \nabla^2 v \|_{L^p} \right) + C \| \nabla^3 u \|_{L^p}. 
\]  
(4.33)

Due to (2.8), the last term on the right-hand side of (4.33) can be estimated as follows:
\[
\| \nabla^3 u \|_{L^p} \leq C \| \nabla (\rho u_r) \|_{L^p} + C \| \nabla (\rho u : \nabla u) \|_{L^p} + C \| \nabla^2 P(\theta) \|_{L^p} 
+ C \| \nabla (\nabla (\nabla \times H) \times H) \|_{L^p} 
\leq C \| \nabla (\rho u_r) \|_{L^p} + C \| \nabla (\rho u : \nabla u) \|_{L^p} + C \| \nabla^2 P(\theta) \|_{L^p} 
+ C \| \nabla (\nabla (\nabla \times H) \times H) \|_{L^p} 
\leq C \| \nabla u_r \|_{L^p} + C \| \nabla (\rho u_r) \|_{L^p} + \frac{1}{2} \| \nabla^3 u \|_{L^p} + C \| \nabla^2 P(\theta) \|_{L^p} 
+ C \| \nabla^2 H \|_{L^p} + C \]  
(4.34)

where we have used (3.3), (4.3), (4.4), (4.6) and (4.7).

Next, it follows from (4.15) that
\[
\int_0^{T_0} \left( \| \nabla u_r \|_{L^2}^{2/q} \| \nabla^2 u_r \|_{L^2}^{(q-2)/q} \right)^{(q+1)/q} dt 
\leq C \sup_{0 \leq t \leq T_0} \left( \| \nabla u_r \|_{L^2}^{2/q} \| \nabla^2 u_r \|_{L^2}^{(q-2)/q} \right)^{(q+1)/q} \int_0^{T_0} \left( t \| \nabla^2 u_r \|_{L^2}^2 + t^{-1}\left( q^2/q + 2 \right) \| \nabla^2 u_r \|_{L^2}^2 \right) dt 
\leq C. 
\]  
(4.35)

Putting (4.34) into (4.33), we get (4.32) from Gronwall’s inequality, (3.3), (4.2) and (4.35).
Therefore, the proof of lemma 4.3 is finished. \( \square \)

Lemma 4.4. It holds that
\[
\sup_{0 \leq t \leq T_0} \left( \| \nabla^3 u \|_{L^\infty} + \| \nabla^2 u_r \|_{H^s} + \| \nabla^2 (\rho u_r) \|_{L^{(s+2)/s}} \right) 
\leq \int_0^{T_0} \left( \| \nabla u_r \|_{L^2}^2 + \| \nabla^{-1} u_r \|_{L^2}^2 \right) dt \leq C. 
\]  
(4.36)
**Proof.** First, we claim that
\[
\sup_{0 \leq t \leq T_0} \left\| \sqrt{\varrho} u_t \right\|^2_{L^2} + \int_0^{T_0} \left| \nabla^2 u \right|^2_{L^2} dt \leq C, \tag{4.37}
\]
which, together with (2.5), (4.15) and (4.30), yields that
\[
\sup_{0 \leq t \leq T_0} \left| \nabla u_t \right|_{H^1} + \int_0^{T_0} \left| \nabla^{-1} u_t \right|^2_{L^2} dt \leq C. \tag{4.38}
\]
This combined with (4.13), (4.32), (4.34) and (4.35) leads to
\[
\sup_{0 \leq t \leq T_0} \left| \nabla^3 u \right|_{L^\infty} \leq C, \tag{4.39}
\]
which, together with (3.3), (4.1) and (4.32), shows
\[
\left| \nabla^2 (\varrho u) \right|_{L^{(q+2)/2}} \leq C \left( \left| \nabla^2 \varrho \right|_{L^{(q+2)/2}} + C \left| \nabla \varrho \right|_{L^{(q+2)/2}} + C \left| \varrho \right|_{L^{(q+2)/2}} \right)
\leq C \left( \left| \nabla^2 \varrho \right|_{L^{(q+2)/2}} + C \left| \varrho \right|_{L^{(q+2)/2}} \right)
\leq C. \tag{4.40}
\]
Therefore, we complete the proof of (4.36) from (4.37)–(4.40).

Now, we focus on the estimates of (4.37). In fact, differentiating (3.25) with respect to \( t \) yields that
\[
\varrho u_t = \varrho u \cdot \nabla u_t - \mu \Delta u_t - (\mu + \lambda) \nabla \text{div} u_t
\]
\[
= 2 \text{div}(\varrho u) u_t + \text{div}(\varrho u) u_t - 2(\varrho u_t) \cdot \nabla u_t - \varrho u_t \cdot \nabla u - 2 \varrho u_{tt} \cdot \nabla u
\]
\[
- \varrho u_t \cdot \nabla u - \nabla P_{tt}(\varrho) - \frac{1}{2} \nabla |H|^2_t + H_t \cdot \nabla H + 2H_t \cdot \nabla H_t + H \cdot \nabla H_t,
\]
which, multiplied by \( u_t \) and integrated by parts over \( B_R \), shows that
\[
\frac{1}{2} \frac{d}{dt} \int \varrho |u_t|^2 dx + \int (\mu |\nabla u_t|^2 + (\mu + \lambda) |\text{div} u_t|^2) dx
\]
\[
= -4 \int \varrho u \cdot \nabla u_t \cdot u_t dx - \int (\varrho u_t) \cdot (\nabla (u_t \cdot u_t) + 2 \nabla u_t \cdot u_t) dx
\]
\[
- \int (\varrho u_t) \cdot \nabla (u \cdot u_t) dx - \int 2 \varrho u_{tt} \cdot \nabla u \cdot u_t dx
\]
\[
- \int \varrho u_{tt} \cdot \nabla u \cdot u_t dx + \int P_{tt}(\varrho) \text{div} u_t dx + \frac{1}{2} \int |H|^2_t \text{div} u_t dx
\]
\[
+ \int H_t \cdot \nabla H \cdot u_t dx + 2 \int H_t \cdot \nabla H_t \cdot u_t dx + \int H \cdot \nabla H_t \cdot u_t dx
\]
\[
\triangleq \sum_{i=1}^{10} f_i, \tag{4.41}
\]
Now, we will estimate each term on the right-hand side of (4.41) as follows:

First, it follows from (4.4) that

$$|I_1| \leq C \|\varrho u\|_{L^\infty} \|\sqrt{\varrho} u_{\varrho}\|_{L^2} \|\nabla u_{\varrho}\|_{L^2} \leq \varepsilon \|\nabla u_{\varrho}\|_{L^2} + C(\varepsilon) \|\sqrt{\varrho} u_{\varrho}\|_{L^2}^2. \quad (4.42)$$

Next, Hölder’s inequality leads to

$$|I_2| \leq C \|x \varrho u_x\|_{L^2} (\|x^{-1} u_x\|_{2^q/(q-2)} \|\nabla u_{\varrho}\|_{L^2} + \|x^{-1} u_x\|_{2^q/(q-2)} \|\nabla u_{\varrho}\|_{L^2})$$

$$\leq C (1 + \|\nabla u_{\varrho}\|_{L^2}^2) (\|\sqrt{\varrho} u_{\varrho}\|_{L^2} + \|\nabla u_{\varrho}\|_{L^2})$$

$$\leq \varepsilon (\|\sqrt{\varrho} u_{\varrho}\|_{L^2}^2 + \|\nabla u_{\varrho}\|_{L^2}^2) + C(\varepsilon) (1 + \|\nabla u_{\varrho}\|_{L^2}^2). \quad (4.43)$$

where we have used (2.6), (3.9) and (4.5) and the following facts:

$$\|x \varrho u_x\|_{L^2} \leq C \|x \varrho_{\varrho}\|_{L^2} + C \|x \varrho u_x\|_{L^2}$$

$$\leq C \|x \varrho(1 + a)/2 \|_{L^2} \|x^{-a-1/2} u\|_{L^\infty} + C \|\varrho x^2\|_{L^{2\alpha}/(\alpha-1)} \|u x^{-a}\|_{L^{2\alpha}/(\alpha-1)}$$

$$\leq C + C \|\nabla u_{\varrho}\|_{L^2} \quad (4.44)$$

by using (4.4)–(4.6), where $\tilde{a} = \min\{2, a\}$.

Then, it follows from (3.9), (4.3), (4.4) and (4.44) that

$$|I_3| \leq C \int \left| (\varrho u_x) | (u) | \nabla^2 u | u_x | + |u| |\nabla u| |\nabla u_x | + |\nabla u|^3 | u_x | \right| \ dx$$

$$\leq C \|x \varrho u_x\|_{L^2} \|x^{-1/2} u\|_{L^\infty} \|\nabla u\|_{L^2} \|x^{-1/2} u_{\varrho}\|_{L^2}$$

$$+ C \|x \varrho u_x\|_{L^2} \|x^{-1/2} u\|_{L^\infty} \|\nabla u\|_{L^2} \|\nabla u_{\varrho}\|_{L^2}$$

$$\leq C (1 + \|\nabla u_{\varrho}\|_{L^2}) (\|\sqrt{\varrho} u_{\varrho}\|_{L^2} + \|\nabla u_{\varrho}\|_{L^2})$$

$$\leq \varepsilon (\|\sqrt{\varrho} u_{\varrho}\|_{L^2}^2 + \|\nabla u_{\varrho}\|_{L^2}^2) + C(\varepsilon) (1 + \|\nabla u_{\varrho}\|_{L^2}^2). \quad (4.45)$$

Next, it follows from Cauchy’s inequality, together with (3.9), (4.5) and (4.6), that

$$|I_4| \leq C \int \varrho |u_x| \|\nabla u| u_x| \ dx$$

$$\leq C \|x \varrho u_x\|_{L^2} \|x^{-1/2} u\|_{L^2/(\alpha-2)} \|\nabla u\|_{L^2} \|x^{-1/2} u_{\varrho}\|_{L^2/(\alpha-2)}$$

$$\leq C (1 + \|\nabla u_{\varrho}\|_{L^2}) (\|\sqrt{\varrho} u_{\varrho}\|_{L^2} + \|\nabla u_{\varrho}\|_{L^2})$$

$$\leq \varepsilon (\|\sqrt{\varrho} u_{\varrho}\|_{L^2}^2 + \|\nabla u_{\varrho}\|_{L^2}^2) + C(\varepsilon) (1 + \|\nabla u_{\varrho}\|_{L^2}^2). \quad (4.46)$$

Then, the Gagliardo–Nirenberg inequality together with (4.3) gives

$$|I_5| \leq C \|\nabla u\|_{L^\infty} \|\sqrt{\varrho} u_{\varrho}\|_{L^2} \leq C (1 + \|\nabla^2 u\|_{L^2}) \|\sqrt{\varrho} u_{\varrho}\|_{L^2}^2. \quad (4.47)$$

Next, it follows from (3.3), (3.14) and (4.3)–(4.6) that
\[ \|P_u(\varrho)\|_{L^2} \leq C\|u\|\|\nabla P(\varrho)\|_{L^2} + C\|u\|\|\nabla P(\varrho)\|_{L^2} + C\|u\|\|\nabla P(\varrho)\|_{L^2} + C\|u\|\|\nabla P(\varrho)\|_{L^2} \]

where in the last inequality we have used the following simple fact that

\[ \|x^{\delta/2}\nabla P(\varrho)\|_{L^2} \leq C\|x^{\delta/2}|u|^2\nabla P(\varrho)\|_{L^2} + C\|x^{\delta/2}|\nabla P(\varrho)\|_{L^2} + C\|x^{\delta/2}\nabla P(\varrho)\|_{L^2} + C\|x^{\delta/2}\nabla P(\varrho)\|_{L^2} \]

due to (3.3), (4.1), (4.3) and (4.4). Then, (4.48), together with Cauchy’s inequality, leads to

\[ |I_6| \leq C\|P_u(\varrho)\|_{L^2}\|\text{div}u\|_{L^2} \leq \varepsilon\|\nabla u\|_{L^2} + C(\varepsilon)\left(1 + \|\nabla u\|_{L^2}^2\right). \]

Finally, it follows from (2.2)_3, (3.3) and (4.3)–(4.6) that

\[ \|H_0\|_{L^2} \leq C\|x^{\delta/2}u\|_{L^2}/(q-2)\|x^{\delta/2}\nabla H\|_{L^2}/(q-2)\|x^{\delta/2}\nabla H\|_{L^2} + C\|\nabla u\|_{L^2} \]

where, in the last inequality, we have used the following fact:

\[ \|x^{\delta/2}\nabla H\|_{L^2} \leq C\|x^{\delta/2}|\nabla H\|_{L^2} + C\|x^{\delta/2}|\nabla H\|_{L^2} + C\|x^{\delta/2}H\|_{L^2} + C\|x^{\delta/2}H\|_{L^2} \]

due to (4.1), (4.3), (4.4) and (4.7). Then (4.50) and integration by parts leads to

\[ |I_7| + |I_8| + |I_9| + |I_{10}| \leq C\|\|H_l\|_{L^2}\|\nabla u\|_{L^2} + C\|\|H_l\|_{L^2}\|\nabla u\|_{L^2} \]

in terms of (4.5), (4.7), (4.27) and (4.50).

Substituting (4.42), (4.43), (4.45)–(4.49) and (4.51) into (4.41), choosing \( \varepsilon \) suitably small, and multiplying the resulting inequality by \( r^2 \), we get (4.37) after using Gronwall’s inequality and (4.15). Therefore, the proof of lemma 4.4 is completed.
5. Proof of theorems 1.1–1.3

With all the a priori estimates obtained in sections 3 and 4 at hand, we are now ready to prove the main results of this paper in this section.

**Proof of theorem 1.1.** Let \((g_0, u_0, H_0)\) be as in theorem 1.1. Without loss of generality, we assume that the initial density \(g_0\) satisfies

\[
\int_{\mathbb{R}^2} g_0 \, dx = 1,
\]

which means that there is a positive constant \(N_0\) such that

\[
\int_{B_{R_0}} g_0 \, dx \geq \frac{3}{4} \int_{\mathbb{R}^2} g_0 \, dx = \frac{3}{4}. \tag{5.1}
\]

We construct that \(g_0^k = g_0 + R^{-1} e^{-|x|^2}\) where \(0 \leq g_0^k \in C_0^\infty(\mathbb{R}^2)\) satisfies

\[
\begin{cases}
\int_{B_R} g_0^k \, dx \geq 1/2, \\
\tilde{x}^a g_0^k \to \tilde{x}^a g_0 \quad \text{in } L^1(\mathbb{R}^2) \cap H^1(\mathbb{R}^2) \cap W^{1,q}(\mathbb{R}^2), & \text{as } R \to \infty.
\end{cases} \tag{5.2}
\]

Then, we choose \(H_0^k \in \{ w \in C_0^\infty(B_R) | \text{div} w = 0 \}\) satisfying

\[
H_0^k \tilde{x}^a \to H_0 \tilde{x}^a \quad \text{in } H^1(\mathbb{R}^2) \cap W^{1,q}(\mathbb{R}^2), & \text{as } R \to \infty. \tag{5.3}
\]

Next, since \(\nabla u_0 \in L^2(\mathbb{R}^2)\), choosing \(v_i^k \in C_0^\infty(B_R) (i = 1, 2)\) such that

\[
\lim_{R \to \infty} \| v_i^k - \partial_i u_0 \|_{L^2(\mathbb{R}^2)} = 0, \quad i = 1, 2, \tag{5.4}
\]

we consider the unique smooth solution \(u_0^k\) of the following elliptic problem:

\[
\begin{cases}
-\Delta u_0^k + g_0^k u_0^k = \sqrt{g_0^k} h^k \partial_1 v_i^k, & \text{in } B_R, \\
u_0^k = 0, & \text{on } \partial B_R, \tag{5.5}
\end{cases}
\]

where \(h^k = (\sqrt{g_0} u_0) * j_1/\delta\) with \(j_\delta\) is the standard mollifying kernel of width \(\delta\). Extending \(u_0^k\) to \(\mathbb{R}^2\) by defining 0 outside of \(B_R\) and denoting it by \(\tilde{u}_0^k\), we claim that

\[
\lim_{R \to \infty} \left( \| \nabla (\tilde{u}_0^k - u_0) \|_{L^2(\mathbb{R}^2)} + \sqrt{g_0^k} \| \tilde{u}_0^k \|_{L^2(\mathbb{R}^2)} \right) = 0. \tag{5.6}
\]

In fact, it is easy to find that \(\tilde{u}_0^k\) is also a solution of (5.5) in \(\mathbb{R}^2\). Multiplying (5.5) by \(\tilde{u}_0^k\) and integrating the resulting equality over \(B_R\) leads to

\[
\int_{B_R} g_0^k |\tilde{u}_0^k|^2 \, dx + \int_{B_R} |\nabla \tilde{u}_0^k|^2 \, dx
\]

\[
\leq C \| \sqrt{g_0^k} \tilde{u}_0^k \|_{L^2(B_R)} \| h^k \|_{L^2(B_R)} + C \| v_i^k \|_{L^2(B_R)} \| \partial_i \tilde{u}_0^k \|_{L^2(B_R)}
\]

\[
\leq \varepsilon \| \nabla \tilde{u}_0^k \|_{L^2(B_R)}^2 + \varepsilon \| \sqrt{g_0^k} \tilde{u}_0^k \|_{L^2(B_R)}^2 + C(\varepsilon),
\]

which yields
\[ \| \sqrt{\rho R} \tilde{u}_0 \|_{L^2(B_{R})} + \| \nabla \tilde{u}_0 \|_{L^2(B_{R})} \lesssim C, \]  
(5.7)

for some constant \( C \) independent of \( R \).

Therefore, we conclude from (5.2) and (5.7) that there exists a subsequence \( R_j \to \infty \) and a function \( \tilde{u}_0 \in H^1_{\text{loc}}(\mathbb{R}^2) \) such that

\[
\begin{cases}
\sqrt{\rho R} u_{0j} \rightharpoonup \sqrt{\rho_0} \tilde{u}_0 \quad \text{weakly in } L^2(\mathbb{R}^2), \\
\nabla u_{0j} \rightharpoonup \nabla \tilde{u}_0 \quad \text{weakly in } L^2(\mathbb{R}^2).
\end{cases}
\]  
(5.8)

Next, it follows from (5.4), (5.5) and (5.8) that, for any \( \psi \in C^\infty_0(\mathbb{R}^2) \),

\[
\int_{\mathbb{R}^2} \partial_i (\tilde{u}_0 - u_0) \cdot \partial_i \psi \, dx + \int_{\mathbb{R}^2} \rho_0 (\tilde{u}_0 - u_0) \cdot \psi \, dx = 0,
\]

which gives that

\[ \tilde{u}_0 = u_0. \]  
(5.9)

Furthermore, we get from (5.5) that

\[
\limsup_{R_j \to \infty} \int_{\mathbb{R}^2} (|\nabla u_{0j}|^2 + \rho_0 |u_{0j}|^2) \, dx \leq \int_{\mathbb{R}^2} (|\nabla u_0|^2 + \rho_0 |u_0|^2) \, dx,
\]

which, combined with (5.8), shows

\[
\lim_{R_j \to \infty} \int_{\mathbb{R}^2} |\nabla u_{0j}|^2 \, dx = \int_{\mathbb{R}^2} |\nabla u_0|^2 \, dx, \quad \lim_{R_j \to \infty} \int_{\mathbb{R}^2} \rho_0 |u_{0j}|^2 \, dx = \int_{\mathbb{R}^2} \rho_0 |u_0|^2 \, dx.
\]

This, along with (5.8) and (5.9), yields (5.6).

Then, due to lemma 2.1, the initial-boundary value problem (2.2) with the initial data \((\rho_0^R, u_0^R, H_0^R)\) has a classical solution \((\rho^R, u^R, H^R)\) on \( B_R \times [0, T_R] \). Moreover, proposition 3.1 gives that there has been a \( T_0 \) independent of \( R \) such that (3.3) holds for \((\rho^R, u^R, H^R)\). Extending \((\rho^R, u^R, H^R)\) by zero on \( \mathbb{R}^2 \setminus B_R \) and denoting it by

\[
\rho_R \triangleq \varphi_R \rho^R, \quad u_R \triangleq \varphi_R u^R, \quad H_R \triangleq \varphi_R H^R,
\]

with \( \varphi_R \) as in (3.6), we first deduce from (3.3) that

\[
\sup_{0 \leq t \leq T_0} \left( \| \sqrt{\rho_R} \bar{u}_0 \|_{L^2} + \| \nabla \bar{u}_0 \|_{L^2} \right) \leq C + C \sup_{0 \leq t \leq T_0} \| \nabla u_0 \|_{L^2} \lesssim C, \]  
(5.10)

and

\[
\sup_{0 \leq t \leq T_0} \| \rho_R \bar{u}_0 \|_{L^1 \cap L^\infty} \leq C. \]  
(5.11)

Next, for \( p \in [2, q] \), it follows from (3.3) and (3.33) that
\[
\sup_{0 \leq t \leq T_0} \left( \| \nabla (x^a \partial_t \bar{u}^R) \|_{L^p(\mathbb{R}^2)} + \| x^a \nabla \bar{u}^R \|_{L^p(\mathbb{R}^2)} + \| \nabla (x^a \bar{H}^R) \|_{L^p(\mathbb{R}^2)} + \| x^a \nabla \bar{H}^R \|_{L^p(\mathbb{R}^2)} \right)
\leq C \sup_{0 \leq t \leq T_0} \left( \| x^a \nabla \bar{e}^R \|_{L^p(B_a)} + \| x^a \partial_t \nabla \bar{e}^R \|_{L^p(B_a)} + \| \partial_t \nabla x^a \|_{L^p(B_a)} \right)
+ C \sup_{0 \leq t \leq T_0} \left( \| x^a \nabla \bar{H}^R \|_{L^p(B_a)} + \| x^a \partial_t \nabla \bar{H}^R \|_{L^p(B_a)} + \| \partial_t \nabla x^a \|_{L^p(B_a)} \right)
\leq C + C \| x^a \partial_t \bar{u}^R \|_{L^p(B_a)} + C \| x^a \bar{H}^R \|_{L^p(B_a)}
\leq C. \tag{5.12}
\]

Then, it follows from (3.3) and (3.35) that
\[
\int_0^{T_0} \left( \| \nabla^2 \bar{u}^R \|_{L^p(\mathbb{R}^2)} + t \| \nabla^2 \bar{u}^R \|_{L^p(\mathbb{R}^2)} + \| \nabla^2 \bar{u}^R \|_{L^p(\mathbb{R}^2)} \right) \, dt \leq C, \tag{5.13}
\]
and that for \( p \in [2, q] \),
\[
\int_0^{T_0} \left( \| x^a \partial_t \bar{u}^R \|_{L^p(\mathbb{R}^2)} + \| x^a \bar{H}^R \|_{L^p(\mathbb{R}^2)} \right) \, dt
\leq C \int_0^{T_0} \left( \| x^a \bar{u} \|_{L^p(B_a)} \right) \, dt
+ C \int_0^{T_0} \left( \| x^a \bar{u} \|_{L^p(B_a)} \right) \, dt
\leq C \int_0^{T_0} \| x^a \bar{u} \|_{L^p(B_a)} \, dt \leq C. \tag{5.14}
\]
Next, one derives from (3.3) and (2.24) that
\[
\sup_{0 \leq t \leq T_0} t \| \nabla^2 \bar{u}^R \|_{L^2(\mathbb{R}^2)}^2 + \int_0^{T_0} t \| \nabla \bar{u}^R \|_{L^2(\mathbb{R}^2)}^2 \, dt
\leq C + C \int_0^{T_0} t \| \nabla \bar{u}^R \|_{L^2(\mathbb{R}^2)}^2 \, dt \leq C. \tag{5.15}
\]

With all these estimates (5.10)–(5.15) at hand, we find that the sequence \((\bar{u}^R, x^a \bar{H}^R, \bar{H}^R)\) converges, up to the extraction of subsequences, to some limit \((\bar{u}, x^a \bar{H}, \bar{H})\) in the obvious weak sense, that is, as \( R \to \infty \), we have
\[
x^a \bar{e}^R \to x^a \bar{e}, \quad x^a \bar{H}^R \to x^a \bar{H}, \quad \text{in } C(\bar{B}_N \times [0, T_0]), \text{ for any } N > 0, \tag{5.16}
\]
\[
x^a \partial_t \bar{u}^R \to x^a \partial_t \bar{u}, \quad x^a \partial_t \bar{H}^R \to x^a \partial_t \bar{H}, \quad \text{weakly * in } L^\infty(0, T_0; H^1(\mathbb{R}^2) \cap W^{1,q}(\mathbb{R}^2)), \tag{5.17}
\]
\[
\sqrt{\partial_t^2} \bar{u}^R \to \sqrt{\partial_t^2} \bar{u}, \quad \nabla \bar{u}^R \to \nabla \bar{u}, \quad \text{weakly * in } L^\infty(0, T_0; L^2(\mathbb{R}^2)), \tag{5.18}
\]
\[
\nabla^2 \bar{u}^R \to \nabla^2 \bar{u}, \quad \text{weakly in } L^\infty(0, T_0; L^2(\mathbb{R}^2) \times L^2((0, T_0) \times \mathbb{R}^2)), \tag{5.19}
\]
\[
l^{1/2} \nabla^2 \bar{u}^R \to l^{1/2} \nabla^2 \bar{u}, \quad \text{weakly in } L^2(0, T_0; L^2(\mathbb{R}^2)), \tag{5.20}
\]
\[ t^{1/2} \sqrt{\sigma^2} \tilde{u}_t^k \to t^{1/2} \sqrt{\sigma} u_t, \quad \nabla \tilde{u}_t^k \to \nabla u_t, \quad \text{weakly * in } L^\infty(0, T_0; L^2(\mathbb{R}^2)), \] (5.21)

\[ t^{1/2} \nabla \tilde{u}_t^k \to t^{1/2} \nabla u_t, \quad \text{weakly in } L^\infty((0, T_0) \times \mathbb{R}^2), \] (5.22)

and

\[ x^a \rho \in L^\infty(0, T_0; L^1(\mathbb{R}^2)), \quad \inf_{0 \leq t \leq T_0} \int_{B_{2r_0}} g(t, x) \text{d}x \geq \frac{1}{4}. \] (5.23)

Then, letting \( R \to \infty \), it follows from (5.16)–(5.23) that \((\varrho, u, H)\) is a strong solution of (1.1)–(1.7) on \((0, T_0) \times \mathbb{R}^2\) satisfying (1.10) and (1.11). Therefore, the proof of the existence part of theorem 1.1 is completed.

It only remains to prove the uniqueness of the strong solution satisfying (1.10) and (1.11). Let \((\varrho_1, u_1, H_1)\) and \((\varrho_2, u_2, H_2)\) be two strong solutions satisfying (1.10) and (1.11) with the same initial data, and denote

\[ \Psi \triangleq \varrho_1 - \varrho_2, \quad U \triangleq u_1 - u_2, \quad \Phi \triangleq H_1 - H_2. \]

First, subtracting the mass equation satisfied by \((\varrho_1, u_1, H_1)\) and \((\varrho_2, u_2, H_2)\) yields that

\[ \Psi_t + u_2 \cdot \nabla \Psi + \Psi \text{div} u_2 + \varrho_1 \text{div} U + U \cdot \nabla \varrho_1 = 0. \] (5.24)

Multiplying (5.24) by \(2 \Psi x^a\) for \(r \in (1, \bar{a})\) with \(\bar{a} = \min\{2, a\}\), and integrating by parts gives

\[
\begin{align*}
\frac{d}{dt} \|\Psi x^a\|^2_{L^2} &\leq C \left( \|\nabla u_2\|_{L^\infty} + \|u_2 x^{-1/2}\|_{L^\infty} \right) \|\Psi x^a\|^2_{L^2} + C \|\varrho_1 x^a\|_{L^\infty} \|\nabla U\|_{L^2} \|\Psi x^a\|_{L^2} \\
&\quad + C \|\Psi x^a\|_{L^2} \|U x^{-(a-\gamma)}\|_{L^2/(\tilde{\gamma} - 2)(\tilde{\gamma} - 2)} \|\nabla \varrho_1\|_{L^2/(\tilde{\gamma} - 2)(\tilde{\gamma} - 2)} \|\nabla U\|_{L^2} \\
&\leq C \left( 1 + \|\nabla u_2\|_{W^{1,\alpha}} + \|\nabla u_1\|_{W^{1,\alpha}} \right) \|\Psi x^a\|^2_{L^2} + C \|\Psi x^a\|_{L^2} \left( \|\sqrt{\varrho_1} U\|_{L^2} + \|\nabla U\|_{L^2} \right),
\end{align*}
\]

which, together with Gronwall’s inequality, yields that for all \(0 \leq t \leq T_0\)

\[ \|\Psi x^a\|_{L^2} \leq C \int_0^t \left( \|\sqrt{\varrho_1} U\|_{L^2} + \|\nabla U\|_{L^2} \right) \text{d}s. \] (5.25)

Then, subtracting the magnetic equation satisfied by \((\varrho_1, u_1, H_1)\) and \((\varrho_2, u_2, H_2)\) leads to

\[ \Phi_t + u_1 \cdot \nabla \Phi + U \cdot \nabla H_2 + H_1 \text{div} U + \Phi \text{div} u_2 = H_1 \cdot \nabla U + \Phi \cdot \nabla u_2. \] (5.26)

Multiplying (5.26) by \(2 \Phi x^a\), and integrating by parts shows that

\[
\begin{align*}
\frac{d}{dt} \|\Phi x^a\|^2_{L^2} &\leq C \left( \|\nabla u_2\|_{L^\infty} + \|\nabla u_1\|_{L^\infty} + \|u_1 x^{-1/2}\|_{L^\infty} \right) \|\Phi x^a\|^2_{L^2} + C \|H_1 x^a\|_{L^\infty} \|\nabla U\|_{L^2} \|\Psi x^a\|_{L^2} \\
&\quad + C \|\Phi x^a\|_{L^2} \|U x^{-(a-\gamma)}\|_{L^2/(\tilde{\gamma} - 2)(\tilde{\gamma} - 2)} \|\nabla \varrho_1\|_{L^2/(\tilde{\gamma} - 2)(\tilde{\gamma} - 2)} \|\Phi x^a\|_{L^2} \\
&\leq C \left( 1 + \|\nabla u_1\|_{W^{1,\alpha}} + \|\nabla u_2\|_{W^{1,\alpha}} \right) \|\Phi x^a\|^2_{L^2} + C \|\Phi x^a\|_{L^2} \left( \|\sqrt{\varrho_1} U\|_{L^2} + \|\nabla U\|_{L^2} \right),
\end{align*}
\]

which, together with Gronwall’s inequality, gives that
\[ \| \Phi \mathbf{x}' \|_{L^2} \leq C \int_0^t (\| \sqrt{\rho_1} U \|_{L^2} + \| \nabla U \|_{L^2}) \, ds, \quad (5.27) \]

for all \( t \in [0, T_0] \).

Next, subtracting the momentum equation satisfied by \((\varrho_1, \mathbf{u}_1, \mathbf{H}_1)\) and \((\varrho_2, \mathbf{u}_2, \mathbf{H}_2)\) shows that

\[
\varrho_1 U_t + \varrho_1 \mathbf{u}_1 \cdot \nabla U - \mu \Delta U - (\mu + \lambda) \nabla \text{div} U = - \varrho_1 \mathbf{u}_1 \cdot \nabla \mathbf{u}_2 - \Psi (\mathbf{u}_2 \cdot \mathbf{u}_2 - \mathbf{H}_2) - \nabla (P(\varrho_1) - P(\varrho_2)) - \frac{1}{2} \nabla (|\mathbf{H}_1|^2 - |\mathbf{H}_2|^2) + \mathbf{H}_1 \cdot \nabla \Phi + \Phi \cdot \nabla \mathbf{H}_2. \quad (5.28)
\]

Multiplying (5.28) by \( U \) and integrating by parts yields that

\[
\frac{d}{dt} \| \sqrt{\varrho_1} U \|_{L^2}^2 + 2 \mu \| \nabla U \|_{L^2}^2 + 2(\mu + \lambda) \| \text{div} U \|_{L^2}^2 \\
\leq C(\| \nabla u_2 \|_{L^\infty} + \| \sqrt{\varrho_1} U \|_{L^2}^2 + C(\| \mathbf{H}_1 \mathbf{x}'' \|_{L^\infty} + \| \mathbf{H}_2 \mathbf{x}'' \|_{L^\infty}) \| \Phi \mathbf{x}' \|_{L^2} \| \nabla U \|_{L^2} \\
+ C \int |\Phi(\mathbf{u}_2) + |\mathbf{u}_2| \nabla \mathbf{u}_2)| dx \\
+ C(\| P(\varrho_1) - P(\varrho_2) \|_{L^2} + \| \mathbf{H}_1 \|^2 - \| \mathbf{H}_2 \|^2 \|_{L^2}) \| \text{div} U \|_{L^2} \\
\leq \varepsilon \| \nabla U \|_{L^2}^2 + C(1 + \| \nabla u_2 \|_{L^\infty} + \| \sqrt{\varrho_1} U \|_{L^2}^2 + \| \Phi \mathbf{x}' \|_{L^2}^2) + J_1 + J_2. \quad (5.29)
\]

Then, Hölder’s inequality yields that

\[
J_1 \leq C \varepsilon \| \mathbf{x}' \|_{L^2} \| \mathbf{x}' \|_{L^\infty} (\| \mathbf{u}_2 \mathbf{x}' \|_{L^2} + \| \nabla \mathbf{u}_2 \|_{L^\infty} \| \mathbf{u}_2 \mathbf{x}' \|_{L^2}) \\
\leq C \varepsilon \left( \| \sqrt{\varrho_1} U \|_{L^2}^2 + \| \nabla \mathbf{u}_2 \|_{L^2}^2 + \| \nabla \mathbf{u}_2 \|_{L^\infty} \right) \| \Phi \mathbf{x}' \|_{L^2}^2 + \varepsilon \left( \| \sqrt{\varrho_1} U \|_{L^2}^2 + \| \nabla U \|_{L^2}^2 \right). \quad (5.30)
\]

Next, Lagrange’s mean value theorem together with (5.25) and (5.27) gives that

\[
J_2 \leq C(\| \varrho_1 \mathbf{x}' \|_{L^\infty} + \| \varrho_2 \mathbf{x}' \|_{L^\infty} + \| \mathbf{H}_1 \mathbf{x}'' \|_{L^\infty} + \| \mathbf{H}_2 \mathbf{x}'' \|_{L^\infty}) \left( \| \Phi \mathbf{x}' \|_{L^2} + \| \Phi \mathbf{x}' \|_{L^2} \right) \| \nabla U \|_{L^2} \\
\leq \varepsilon \| \nabla U \|_{L^2}^2 + C \varepsilon \left( \| \Phi \mathbf{x}' \|_{L^2}^2 + \| \Phi \mathbf{x}' \|_{L^2}^2 \right) \\
\leq \varepsilon \| \nabla U \|_{L^2}^2 + C \int_0^t (\| \sqrt{\varrho_1} U \|_{L^2}^2 + \| \nabla U \|_{L^2}^2) \, ds. \quad (5.31)
\]

Denoting

\[ G(t) \triangleq \| \sqrt{\varrho_1} U \|_{L^2}^2 + \int_0^t (\| \sqrt{\varrho_1} U \|_{L^2}^2 + \| \nabla U \|_{L^2}^2) \, ds, \]

and putting (5.30) and (5.31) into (5.29) and choosing \( \varepsilon \) small enough, one gives that

\[ G'(t) \leq C (1 + \| \nabla \mathbf{u}_2 \|_{L^\infty} + t \| \nabla^2 \mathbf{u}_2 \|_{L^2}^2 + t \| \nabla \mathbf{u}_2 \|_{L^2}^2) G(t), \]

which, together with Gronwall’s inequality and (1.10), yields that \( G(t) = 0 \). Therefore, \( U(t, x) = 0 \) for almost everywhere \((t, x) \in (0, T_0) \times \mathbb{R}^2\). Then, (5.25) and (5.27) imply that \( \Psi(t, x) = \Phi(t, x) = 0 \) for almost everywhere \((t, x) \in (0, T_0) \times \mathbb{R}^2\). The uniqueness of the strong solution is finished and we complete the proof of theorem 1.1. \( \square \)
Proof of theorem 1.2. Let \((\varrho_0, \mathbf{u}_0, \mathbf{H}_0)\) be as in theorem 1.2. Without loss of generality, assume that
\[
\int_{\mathbb{R}^2} \varrho_0 \, dx = 1,
\]
which implies that there exists a positive constant \(N_0\) such that (5.1) holds. We construct that \(\varrho_0^R = \varrho_0^R + R^{-1}e^{-|x|^2}\) where \(0 \leq \varrho_0^R \in C_0^\infty(\mathbb{R}^2)\) satisfies (5.2) and
\[
\begin{cases}
\nabla^2 \varrho_0^R \to \nabla^2 \varrho_0, & \text{in } L^2(\mathbb{R}^2),
\nabla \cdot \nabla^2 \varrho_0^R \to \nabla \cdot \nabla^2 \varrho_0, & \text{in } L^2(\mathbb{R}^2),
\end{cases}
\]
as \(R \to \infty\). Then, we also choose \(\mathbf{H}_0^R \in \{ w \in C_0^\infty(B_R) | \nabla w = 0 \}\) satisfying (5.3) and
\[
\begin{cases}
\nabla^2 \mathbf{H}_0^R \to \nabla^2 \mathbf{H}_0, & \text{in } L^2(\mathbb{R}^2),
\nabla \cdot \nabla^2 \mathbf{H}_0^R \to \nabla \cdot \nabla^2 \mathbf{H}_0, & \text{in } L^2(\mathbb{R}^2),
\end{cases}
\]
as \(R \to \infty\).

Then, we consider the unique smooth solution \(\mathbf{u}_0^R\) of the following elliptic problem
\[
\begin{cases}
-\mu \Delta \mathbf{u}_0^R - (\mu + \lambda) \nabla \mathbf{H}_0^R \cdot \nabla \mathbf{u}_0^R \cdot \nabla \mathbf{P}(\varrho_0^R) + \nabla P(\varrho_0^R)
= (\nabla \times \mathbf{H}_0^R) \times \mathbf{H}_0^R - \varrho_0^R \mathbf{u}_0^R + \sqrt{\varrho_0^R} \mathbf{h}_R, & \text{in } B_R,
\mathbf{u}_0^R = 0, & \text{on } \partial B_R,
\end{cases}
\]
where \(\mathbf{h}_R = (\sqrt{\varrho_0^R} \mathbf{u}_0^R + \mathbf{g}) * j_{1/R}\) with \(j_{1/R}\) is the standard mollifying kernel of width \(\delta\). Multiplying (5.34) by \(\mathbf{u}_0^R\) and integrating the resulting equation over \(B_R\), it is easy to show that
\[
\| \sqrt{\varrho_0^R} \mathbf{u}_0^R \|^2_{L^2(B_R)} + \mu \| \nabla \mathbf{u}_0^R \|^2_{L^2(B_R)} + (\mu + \lambda) \| \nabla \mathbf{u}_0^R \|^2_{L^2(B_R)}
\leq \int_{B_R} P(\varrho_0^R) |\nabla \mathbf{u}_0^R|^2 \, dx + \frac{1}{2} \int_{B_R} |\mathbf{H}_0^R|^2 |\nabla \mathbf{u}_0^R|^2 \, dx + \int_{B_R} |\mathbf{H}_0^R| |\nabla \mathbf{H}_0^R| |\mathbf{u}_0^R| \, dx
+ \| \sqrt{\varrho_0^R} \mathbf{u}_0^R \|_{L^2(B_R)} \| \mathbf{h}_R \|_{L^2(B_R)}
\leq\varepsilon \left( \| \sqrt{\varrho_0^R} \mathbf{u}_0^R \|^2_{L^2(B_R)} + \| \nabla \mathbf{u}_0^R \|^2_{L^2(B_R)} \right) + C(\varepsilon),
\]
which implies that
\[
\| \sqrt{\varrho_0^R} \mathbf{u}_0^R \|^2_{L^2(B_R)} + \| \nabla \mathbf{u}_0^R \|^2_{L^2(B_R)} \leq C,
\]
for some constant \(C\) independent of \(R\). Due to (2.8), we have
\[
\| \nabla \mathbf{u}_0^R \|_{L^2(B_R)} \leq C \| \nabla P(\varrho_0^R) \|_{L^2(B_R)} + C \| \mathbf{H}_0^R \| |\nabla \mathbf{H}_0^R| \|_{L^2(B_R)}
+ C \| \varrho_0^R \mathbf{u}_0^R \|_{L^2(B_R)} + C \| \sqrt{\varrho_0^R} \mathbf{h}_R \|_{L^2(B_R)}
\leq C.
\]

Next, extending \(\mathbf{u}_0^R\) to \(\mathbb{R}^2\) by defining 0 outside \(B_R\) and denoting it by \(\tilde{\mathbf{u}}_0^R\), we deduce from (5.35) and (5.36) that
\[
\| \nabla \tilde{\mathbf{u}}_0^R \|_{H^1(\mathbb{R}^2)} \leq C.
\]
which, together with (5.32) and (5.35), gives that there exists a subsequence $R_j \to \infty$ and a function $	ilde{u}_0 \in \{ u_0 \in H^s_0(\mathbb{R}^2) \cap \sqrt{\theta_0} u_0 \in L^2(\mathbb{R}^2), \nabla \tilde{u}_0 \in H^1(\mathbb{R}^2) \}$ such that

$$
\begin{align*}
\sqrt{\theta_0} \tilde{u}_0^R \rightharpoonup \sqrt{\theta_0} u_0, & \quad \text{weakly in } L^2(\mathbb{R}^2), \\
\nabla \tilde{u}_0^R \rightharpoonup \nabla u_0, & \quad \text{weakly in } H^1(\mathbb{R}^2).
\end{align*}
$$

(5.37)

It is easy to check that $\tilde{u}_0^R$ satisfies (5.34), and then one can deduce from (5.32)–(5.34) and (5.37) that $\tilde{u}_0$ satisfies

$$
-\mu \Delta \tilde{u}_0 - (\mu + \lambda) \nabla \text{div} \tilde{u}_0 + \nabla P(\theta_0) + \theta_0 \tilde{u}_0 = (\nabla \times H_0) \times H_0 + \theta_0 u_0 + \sqrt{\theta_0} g,
$$

which, combined with (1.13), yields that

$$
\tilde{u}_0 = u_0.
$$

(5.38)

Next, we get from (5.34) that

$$
\limsup_{R_j \to \infty} \int_{\mathbb{R}^2} \left( |\nabla \tilde{u}_0^R|^2 + \theta_0^R |\tilde{u}_0^R|^2 \right) \, dx \leq \int_{\mathbb{R}^2} \left( |\nabla u_0|^2 + \theta_0 |u_0|^2 \right) \, dx,
$$

which, combined with (5.37), shows

$$
\lim_{R_j \to \infty} \int_{\mathbb{R}^2} |\nabla \tilde{u}_0^R|^2 \, dx = \int_{\mathbb{R}^2} |\nabla u_0|^2 \, dx, \quad \lim_{R_j \to \infty} \int_{\mathbb{R}^2} \theta_0^R |\tilde{u}_0^R|^2 \, dx = \int_{\mathbb{R}^2} \theta_0 |u_0|^2 \, dx.
$$

This, along with (5.37) and (5.38), shows that

$$
\lim_{R \to \infty} \left( \| \nabla \tilde{u}_0^R - u_0 \|_{L^2(\mathbb{R}^2)} + \| \sqrt{\theta_0^R} \tilde{u}_0^R - \sqrt{\theta_0} u_0 \|_{L^2(\mathbb{R}^2)} \right) = 0.
$$

(5.39)

Similar to (5.39), we can also obtain that

$$
\lim_{R \to \infty} \| \nabla^2 (\tilde{u}_0^R - u_0) \|_{L^2(\mathbb{R}^2)} = 0.
$$

Finally, in terms of lemma 2.1, the initial-boundary value problem (2.2) with the initial data $(\theta_0^R, u_0^R, H_0^R)$ has a classical solution $(\theta^R, u^R, H^R)$ on $[0, T_R] \times B_R$. Hence, there is a generic positive constant $C$ independent of $R$ such that all those estimates stated in proposition 3.1 and lemmas 4.1–4.4 hold for $(\theta^R, u^R, H^R)$. Extending $(\theta^R, u^R, H^R)$ by zero on $\mathbb{R}^2 \setminus B_R$ and denoting

$$
\tilde{\theta}^R \triangleq \varphi_R \theta^R, \quad \tilde{u}^R, \quad \tilde{H}^R \triangleq \varphi_R H^R,
$$

with $\varphi_R$ as in (3.6). We deduce from (3.3) and lemmas 4.1–4.4 that the sequence $(\tilde{\theta}^R, \tilde{u}^R, \tilde{H}^R)$ converges weakly, up to the extraction of subsequences, to some limit $(\theta, u, H)$ satisfying (1.10), (1.11) and (1.14). Moreover, standard arguments show that $(\theta, u, H)$ is in fact a classical solution to the problem (1.1)–(1.7). The proof of theorem 1.2 is finished.

**Proof of theorem 1.3.** Now, we prove (1.15). Let $(\theta, u, H)$ be the unique classical solution to (1.1)–(1.7) obtained in theorem 1.2. We assume that the opposite holds, i.e.,

$$
\limsup_{T \to T^*} \left( \| \theta \|_{L^\infty(0,T;L^\infty(\mathbb{R}^2))} + \| H \|_{L^\infty(0,T;L^\infty(\mathbb{R}^2))} + \| \nabla H \|_{L^2(0,T;L^2(\mathbb{R}^2))} \right) = M_0 < \infty.
$$

(5.40)
First, a standard energy estimate gives
\[
\sup_{0 \leq t \leq T} \left( \| \sqrt{\varrho} u \|_{L^2}^2 + \| \mathbf{H} \|_{L^2}^2 + \| P(\varrho) \|_{L^1} \right) + \int_0^T \| \nabla u \|_{L^2}^2 \, dt \leq C.  \tag{5.41}
\]

Based on the assumption (5.40) and the energy estimates (5.41), we derive the following estimates (5.42), (5.51), (5.59) and (5.65) by lemma 5.1-5.4, respectively, which will be used to complete the proof of theorem 1.3. In the rest of this section, the generic positive constant \( C \) may depend on \( M_0, \mu, \lambda, \gamma, \varrho, a, \eta_0, N_0, \| g \|_{L^2}, C_0, \delta_0 \) and initial data assumed in theorem 1.2.

**Lemma 5.1.** Let \((\varrho, u, H)\) be a classical solution obtained in theorem 1.2. Under the condition (5.40), it holds that for \( 0 \leq t \leq T^* \)
\[
\sup_{0 \leq t \leq T} \| \nabla u \|_{L^2}^2 + \int_0^T \| \sqrt{\varrho} u \|_{L^2}^2 \, dt \leq C.  \tag{5.42}
\]

**Proof.** Multiplying (1.2) by \( \dot{u} \) and integrating by parts over \( \mathbb{R}^2 \), direct calculations yield that
\[
\int \varrho |\dot{u}|^2 \, dx = - \int \dot{u} \cdot \nabla P(\varrho) \, dx + \mu \int \dot{u} \cdot \Delta u \, dx + (\mu + \lambda) \int \dot{u} \cdot \nabla \div u \, dx
\]
\[
- \frac{1}{2} \int \dot{u} \cdot \nabla |H|^2 \, dx + \int H \cdot \nabla H \cdot \dot{u} \, dx. \tag{5.43}
\]

Now we estimate each term on the right-hand side of (5.43). First, it follows from (3.14) and integration by parts that
\[
- \int \dot{u} \cdot \nabla P(\varrho) \, dx = \frac{d}{dt} \int P(\varrho) \div u \, dx + \int \left[ (\gamma - 1) P(\varrho) |\div u|^2 + P(\varrho) \partial_i u_j x_i u_j \right] \, dx
\]
\[
\leq \frac{d}{dt} \int P(\varrho) \div u \, dx + C \| \nabla u \|_{L^2}^2. \tag{5.44}
\]

Then, integration by parts leads to
\[
\mu \int \Delta u \cdot \dot{u} \, dx = - \frac{\mu}{2} \frac{d}{dt} \int |\nabla u|^2 \, dx + \mu \int \Delta u (u \cdot \nabla) u \, dx
\]
\[
\leq - \frac{\mu}{2} \frac{d}{dt} \int |\nabla u|^2 \, dx + C \| \nabla u \|_{L^2}^2, \tag{5.45}
\]
and similarly,
\[
(\mu + \lambda) \int \nabla \div u \cdot \dot{u} \, dx \leq - \frac{\mu + \lambda}{2} \frac{d}{dt} \int (\div u)^2 \, dx + C \| \nabla u \|_{L^2}^2. \tag{5.46}
\]

Next, (1.3) and integration by parts yields that
\[
- \frac{1}{2} \int \dot{u} \cdot \nabla |H|^2 \, dx = \frac{1}{2} \frac{d}{dt} \int |H|^2 \div u \, dx - \frac{1}{2} \int |H|^2 \nabla \div u \, dx
\]
\[
- \frac{1}{2} \int |H|^2 u \cdot \nabla u \, dx + \frac{1}{2} \int |H|^2 (\div u)^2 \, dx
\]
\[
\leq \frac{1}{2} \frac{d}{dt} \int |H|^2 \div u \, dx + C \| \nabla u \|_{L^2}^2. \tag{5.47}
\]
and similarly, that
\[
\int H \cdot \nabla u \cdot \dot{u} \, dx \leq \frac{d}{dt} \int H \cdot \nabla u \cdot H \, dx + C \| \nabla u \|_{L^2}^2. \tag{5.48}
\]

Substituting (5.44)–(5.48) into (5.43), it follows from (2.11) and (5.40) that
\[
A'(t) + \| \sqrt{\rho} \dot{u} \|_{L^2}^2 \leq C \| \nabla u \|_{L^2}^2 + C \| \nabla u \|_{L^2}^2 + C \| \nabla H \|_{L^2}^2 + C \tag{5.49}
\]
where
\[
A(t) \triangleq \frac{\mu}{2} \| \nabla u \|_{L^2}^2 + \frac{\mu + \lambda}{2} \| \text{div} u \|_{L^2}^2 - \int \text{div} \rho P \, dx - \frac{1}{2} \int |H|^2 \text{div} dx + \int H \cdot \nabla u \cdot H \, dx \geq \frac{\mu}{4} \| \nabla u \|_{L^2}^2 + \mu + \frac{\lambda}{2} \| \text{div} u \|_{L^2}^2 - C, \tag{5.50}
\]
due to (5.40) and (5.41). Then integrating (5.49) over \((0, T)\), together with (5.40), (5.41) and (5.50), Gronwall’s inequality leads to (5.42). Therefore, we complete the proof of lemma 5.1. \(\square\)

**Lemma 5.2.** Let \((\rho, u, H)\) be a classical solution obtained in theorem 1.2. Under the condition (5.40), it holds that for \(0 \leq t \leq T^\ast\)
\[
\sup_{0 \leq t \leq T^\ast} \| \sqrt{\rho} \dot{u} \|_{L^2}^2 + \int_0^T \| \nabla u \|_{L^2}^2 \, dt \leq C. \tag{5.51}
\]

**Proof.** Operating \(\dot{u}^j \left[ \frac{\partial}{\partial x} + \text{div}(u \cdot ) \right] \) on (1.2)\(^j\), summing with respect to \(j\), and integrating the resulting equation by parts over \(\mathbb{R}^2\), one obtains
\[
\frac{1}{2} \frac{d}{dt} \left( \int \rho |\dot{u}|^2 \, dx \right) = - \int \dot{u}^j \left[ \frac{\partial}{\partial x} P \rho \right] \, dx + \int \dot{u}^j \left[ \nabla u \right] \, dx + \mu \int \dot{u}^j \left[ \nabla u \right] \, dx + \mu + \lambda \int \dot{u}^j \left[ \nabla u \right] \, dx + \mu + \lambda \int \dot{u}^j \left[ \nabla u \right] \, dx + \frac{1}{2} \int \dot{u}^j \left[ \frac{\partial}{\partial x} |H|^2 \right] \, dx + \frac{1}{2} \int \dot{u}^j \left[ \frac{\partial}{\partial x} (H \cdot \nabla H) \right] \, dx + \frac{1}{2} \int \dot{u}^j \left[ \frac{\partial}{\partial x} (H \cdot \nabla H) \right] \, dx. \tag{5.52}
\]

Now we estimate each term on the right-hand side of (5.52). First, it follows from (3.14), integration by parts and careful calculations, that
\[ -\int \dot{u}^i [\partial_j P_i (\varrho) + \text{div} (\partial_j P (\varrho) u)] \, dx \]
\[ = \int \left[ -P (\varrho) \varrho \text{div} u \dot{u}^i + \partial_\kappa (\partial_j \dot{u}^i u^j) P (\varrho) - P (\varrho) \partial_j \dot{u}^i u^j \right] \, dx \]
\[ \leq \varepsilon \| \nabla \dot{u} \|^2_{L^2} + C \| \nabla u \|^2_{L^2}. \quad (5.53) \]

Then, due to integration by parts and after subtle calculations, we have
\[ -\mu \int \dot{u}^i [\nabla \dot{u}^i + \text{div} (u \nabla u^i)] \, dx \]
\[ = -\mu \int \left[ |\nabla \dot{u}|^2 + \partial_\kappa \dot{u}^i \partial_\kappa u^j - \partial_i \dot{u}^i \partial_\kappa u^j - \partial_i u^i \partial_\kappa \partial_\kappa \dot{u}^j \right] \, dx \]
\[ \leq -\mu \| \nabla \dot{u} \|^2_{L^2} + C \| \nabla u \|^4_{L^2}, \quad (5.54) \]

and similarly, that
\[ (\mu + \lambda) \int \dot{u}^i [\partial_j \partial_\kappa \text{div} u + \text{div} (u \partial_j \text{div} u)] \, dx \leq -(\mu + \lambda) \| \nabla \dot{u} \|^2_{L^2} + C \| \nabla u \|^4_{L^2}. \quad (5.55) \]

Next, it follows from (1.3) and integration by parts, that
\[ -\frac{1}{2} \int \dot{u}^i (\partial_j \partial_\kappa |H|^2 + \text{div} (u \partial_j |H|^2)) \, dx \]
\[ = \frac{1}{2} \int \partial_\kappa \dot{u}^i \text{div} u |H|^2 \, dx - \frac{1}{2} \int \partial_\kappa u \cdot \nabla \dot{u}^i |H|^2 \, dx + \int \partial_\kappa \dot{u}^i H \cdot (H \cdot \nabla u - H \text{div} u) \, dx \]
\[ \leq \varepsilon \| \nabla \dot{u} \|^2_{L^2} + C \| \nabla u \|^2_{L^2}. \quad (5.56) \]

and similar calculations lead to
\[ \int \dot{u}^i (\partial_j (H \cdot \nabla H^j) + \text{div} (u (H \cdot \nabla H^j))) \, dx \leq \varepsilon \| \nabla \dot{u} \|^2_{L^2} + C \| \nabla u \|^2_{L^2}. \quad (5.57) \]

Then, inserting (5.53)–(5.57) into (5.52) and choosing \( \varepsilon \) suitably small leads to
\[ \frac{d}{dt} \left( \| \nabla \dot{u} \|^2_{L^2} + \| \nabla u \|^2_{L^2} \right) \leq C \| \dot{u} \|^2_{L^2} + C \| \nabla \dot{u} \|^2_{L^2}, \quad (5.58) \]

where we have used (2.11), (5.40), (5.42). Then, it follows from (5.58) and the compatibility conditions (1.13) that we obtain (5.51) after using Gronwall’s inequality and (5.40). Thus, we complete the proof of Lemma 5.2. \( \square \)

**Lemma 5.3.** Let \((\varrho, u, H)\) be a classical solution obtained in theorem 1.2. Under the condition (5.40), it holds that for \(0 < T \leq T^*\)
\[ \sup_{0 \leq \tau \leq T} \left( \| \varrho \mathcal{A} \|^2_{L^2(I; W^{1,q}_{2,1} \cap W^{1,q}_{2,2})} + \| H \mathcal{A} \|^2_{H^1(I; W^{1,q}_{2,1})} \right) \leq C. \quad (5.59) \]
Proof. First, multiplying (1.1) by \( \overline{\omega} \), and integrating the resulting equality over \( \mathbb{R}^2 \), leads to

\[
\frac{d}{dt} \int \overline{\omega} \partial_t \mathbf{u} \, dx \leq C \int \overline{\omega} \left| \mathbf{u} \right|^{q + \eta} (e + |\mathbf{x}|^2) \, dx \\
\leq C \left\| \overline{\omega} \right\|_{L^{q+\eta}} \left\| \mathbf{u} \right\|_{L^2} \left\| \nabla \mathbf{u} \right\|_{L^2} \\
= C,
\]

where in the last inequality we have used (2.6), (5.41) and (5.42).

Next, it follows from (3.36) and (3.41) that for \( p \in [2, q] \), we have

\[
\frac{d}{dt} \left( \left\| \mathbf{H} \right\|_{L^2} + \left\| \nabla \mathbf{H} \right\|_{L^p} + \left\| w \right\|_{L^2} + \left\| \nabla w \right\|_{L^p} \right) \\
\leq C \left( 1 + \left\| \mathbf{u} \right\|_{L^\infty} + \left\| \mathbf{u} \cdot \nabla \mathbf{x} \right\|_{L^\infty} \right) \left( \left\| \mathbf{H} \right\|_{L^2} + \left\| \nabla \mathbf{H} \right\|_{L^p} + \left\| w \right\|_{L^2} + \left\| \nabla w \right\|_{L^p} \right) \\
+ C \left( \left\| \nabla \mathbf{u} \right\|_{L^p} + \left\| \nabla \mathbf{x} \right\|_{L^\infty} \right) \left( \left\| \mathbf{H} \right\|_{L^2} + \left\| \nabla \mathbf{H} \right\|_{L^p} + \left\| w \right\|_{L^2} + \left\| \nabla w \right\|_{L^p} \right) \\
\leq C \left( 1 + \left\| \mathbf{u} \right\|_{L^\infty} + \left\| \nabla \mathbf{u} \right\|_{L^2} \right) \left( 1 + \left\| \mathbf{H} \right\|_{L^2} + \left\| \nabla \mathbf{H} \right\|_{L^p} + \left\| w \right\|_{L^2} + \left\| \nabla w \right\|_{L^p} \right) \\
\leq C \left( 1 + \left\| \mathbf{u} \right\|_{L^\infty} + \left\| \nabla \mathbf{u} \right\|_{L^2} + \left\| \nabla \mathbf{u} \right\|_{L^2} \right) \left( 1 + \left\| \mathbf{H} \right\|_{L^2} + \left\| \nabla \mathbf{H} \right\|_{L^p} + \left\| w \right\|_{L^2} + \left\| \nabla w \right\|_{L^p} \right)
\]

(5.61)

where in the second inequality we have used (5.41), (5.42) and similar discussions to those in (3.37) and (3.38); in the third inequality, we have used the following facts:

\[
\left\| \nabla \mathbf{u} \right\|_{L^p} \leq C \left( \left\| \mathbf{H} \right\|_{L^2} + \left\| \nabla \mathbf{H} \right\|_{L^p} \right)
\]

(5.62)

which come from \( L^p \)-estimates on the Lamé system. It follows from (2.12), that

\[
\left\| \mathbf{u} \right\|_{L^\infty} \leq C \left( \left\| \nabla \mathbf{u} \right\|_{L^2} + \left\| \omega \right\|_{L^2} \right) \ln (e + \left\| \mathbf{u} \right\|_{L^2}) \leq C \left\| \mathbf{u} \right\|_{L^2} + C
\]

(5.63)

which, together with (5.42), (5.51), (5.60) and (5.61), after using Gronwall’s inequality, gives

\[
\left\| \mathbf{H} \right\|_{L^2} + \left\| \nabla \mathbf{H} \right\|_{L^p} + \left\| w \right\|_{L^2} + \left\| \nabla w \right\|_{L^p} \leq C.
\]

Then, it follows from (5.51), (5.62) and (5.63) that

\[
\int_0^T \left\| \nabla \mathbf{u} \right\|_{L^\infty} \, dt \leq C.
\]
which, together with (5.40), (5.51) and (5.61), leads to

\[
\sup_{0 \leq t \leq T} \left( \|\nabla H\|_{L^2} + \|\nabla w\|_{L^2} \right) \leq C.
\]  

(5.64)

Therefore, combining (5.60), (5.63) and (5.64), one obtains (5.59) and the proof of lemma 5.3 is finished.

With the a priori estimates obtained in lemmas 5.1–5.3 at hand, the following higher order estimates of the solutions which are needed to guarantee that the local strong solutions are classical are similar to those obtained in lemmas 4.1–4.4, so we omit their detailed proofs here.

**Lemma 5.4.** Let \((\rho, u, H)\) be a classical solution obtained in theorem 1.2. Under the condition (5.40), it holds that for \(0 \leq t \leq T^*\)

\[
\sup_{0 \leq t \leq T} \left( \|\nabla^2 \rho\|_{L^2} + \|\nabla^2 \rho \nabla^2 P(\rho)\|_{L^2} + \|\nabla^2 \nabla u\|_{L^2} + \|\nabla^2 \nabla u^t\|_{L^2} + \|\nabla^2 H\|_{L^q} + \|\nabla^2 P(\rho)\|_{L^q} + \|\nabla^2 H\|_{L^q} + \|\nabla^2 u\|_{L^2} + \|\nabla^2 u^t\|_{L^2} + \|\nabla^2 \rho \nabla^2 u\|_{L^2} + \|\nabla^2 \rho \nabla^2 \rho^t\|_{L^2} \right) \\
+ \int_0^{T^*} \left( t \|\nabla u\|_{L^2}^2 + t \|\nabla^2 u\|_{L^2}^2 + t \|\nabla^2 u^t\|_{L^2}^2 + t \|\nabla^2 \rho^t\|_{L^2}^2 \right) \, dt \leq C.
\]  

(5.65)

Now we are ready to finish the proof of theorem 1.3. In fact, in view of the estimates obtained in lemmas 5.1–5.4, one can easily show that the functions

\((-\rho, u, H)(T^*, x) = \lim_{T \to T^*} (-\rho, u, H)(T, x)\)

satisfy the conditions imposed on the initial data (1.9) and (1.12). Therefore, we can take \((-\rho, u, H)(T^*, x)\) as the initial data and apply theorem 1.2 to extend our local classical solution beyond \(t > T^*\). This contradicts the assumption on \(T^*\). Therefore, we complete the proof of theorem 1.3.

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