Anomalous Decay and Decoherence in Atomic Gases

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Pair collisions in atomic gases lead to decoherence and decay. Assuming that all the atoms in the gas are equally likely to collide one is led to consider Lindbladian of mean field type where the evolution in the limit of many atoms reduces to a single qudit Lindbladian with quadratic non-linearity. We describe three smoking guns for non-linear evolutions: Power law decay and dephasing rates; Dephasing rates that take a continuous range of values depending on the initial data and finally, anomalous flow of the Bloch ball towards a hemisphere.

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Understanding and controlling decoherence is central to quantum sensing [11], time keeping [2] and quantum computing [3]. A basic mechanism of decoherence and decay in atomic gases is pair collisions [4][8]: The internal degrees of freedom of atomic vapor decohere by exchanging energy and angular momentum with the spatial degrees of freedom [9][10]. Such collisions can lead to non-linear evolutions [11] that deviate from the canonical Lindbladian evolution of linear open systems. Nonlinear Lindblad evolution have been treated by linearization [11], numerical calculation and perturbation theory [12]. Here we shall describe properties of collision-based nonlinear Lindblad evolution using analytically solvable examples.

The kinetic degrees of freedom of the atoms are viewed as (parts of) a thermal bath. We assume that all atoms are identical and any two atoms are equally likely to collide independent of how far they are. The time scale of the problem is determined by $\gamma > 0$, the (average) rate of collision of an atom with any one of the other $N$ atoms. Since $N$, the number of atoms in the gas, is large, it is unlikely that a given pair will collide twice. This is reflected in $\gamma/N$, the rate of collision of any fixed pair, being negligible as $N$ gets large.

Pair collisions are governed by a Poisson process, $N(t)$, which counts the number of collisions up to time $t$. In a collision, the state of the pair changes by a Kraus map $K_{jk}$, which may be viewed as the generalization of the scattering matrix to open systems. The stochastic evolution equation for the pair, (in the interaction picture,) is governed by

$$\dot{\rho}_2(t + dt) - \dot{\rho}_2(t) = (K - 1)\rho_2(t) \, dN$$

(1)

Since $E(dN) = (\gamma/N) \rho_2(t) \, dt$, the average evolution is

$$d\rho_2 = \frac{\gamma}{N} (K - 1)\rho_2 \, dt,$$

(2)

The factor $N^{-1}$ is interpreted as rare events rather than weak interactions. Since $K - 1$ has a Lindblad form [11] we shall henceforth denote it by $L$.

It follows that the density matrix of the internal degrees of freedom of the gas evolves by a Lindbladian of mean field type [14][15]:

$$\frac{d\rho^{(N)}}{dt} = \frac{\gamma}{N} \sum_{k>j=1}^{N} L_{jk} \rho^{(N)}.$$  

(3)

$L_{jk}$ are symmetric under interchange of atoms and we assumed $L_{jj} = 0$ (without loss of generality).

We assume initial data in which the atoms are uncorrelated

$$\rho^{(N)}(0) = \rho_0 \otimes^N$$

(4)

where $\rho_0$ may be a superposition of internal energy states.

The evolution takes place in a Hilbert space whose dimension is exponential in $N$.

It is a known fact about Eq. (3) with initial data Eq. (4) [13], that in the $N \to \infty$ limit, the evolution preserves the product structure of any finite subcluster and in particular for any pair:

$$\rho^{(j,k)}(t) = Tr_{N-2} \rho^{(N)}(t) = \rho(t) \otimes \rho(t)$$

(5)

(For additional details on this equation see appendix A). It follows that the linear $N$ body evolution in Eq. (3), reduces in the large $N$ limit, to a non-linear evolution of a single atom with quadratic non-linearity [14][15]:

$$\frac{d\rho}{dt} = \gamma Tr_2 (L_{12}\rho \otimes \rho).$$

(6)

The evolution is trace and positivity preserving [20].

In the rest of this paper we shall consider examples of Kraus operators for which Eq. (6) can be solved exactly and where the solutions display unusual decoherence and decay properties. Each of the Kraus maps we consider is associated with only one transition in the system. This applies when the other transitions are forbidden, or if the power spectrum of the bath is appropriate. The maps were chosen to demonstrate the richness of non-linear Lindblad equation.

For the sake of simplicity and geometric visualization, we shall henceforth specialize to the case that $\rho$ describes.
The vector $\mathbf{u}$ represents the state in the Bloch ball. Under linear evolutions, the Bloch ball evolves into shrinking ellipsoids that eventually collapse on the stationary states \cite{16}. Quadratic evolutions allow for more complicated behavior, as we shall see.

A. Polynomial decay

Consider the situation where excited pairs of atoms decay together to their ground states through pairwise interaction. This scenario can be described by the Lindblad for a pair \cite{21}:

$$L_{12}\rho_2 = A\rho_2 A^\dagger - \frac{1}{2}\{A^\dagger A, \rho_2\}, \quad A = a_1 \otimes a_2$$ \hspace{1cm} (8)

with $a = |0\rangle\langle 1|$ the annihilation operator. Taking the partial trace in Eq. (6) gives a quadratic Lindbladian with an effective, state dependent, decay rate $\tilde{\gamma}(\rho)$

$$\frac{d\rho}{dt} = \frac{\tilde{\gamma}(\rho)}{2} (2a\rho a^\dagger - \{a^\dagger a, \rho\}), \quad \tilde{\gamma}(\rho) = \gamma \text{Tr} (a^\dagger a \rho)$$ \hspace{1cm} (9)

Plugging the expression for $\rho$ Eq. (7) in the Lindblad Eq. (5) results in the equation for the Bloch vector

$$\dot{\mathbf{u}} = -\frac{\tilde{\gamma}}{2}(u_x, u_y, 2(u_z - 1)), \quad \tilde{\gamma} = \gamma(1 - u_z)/2$$ \hspace{1cm} (10)

Since

$$2d\log u_x = 2d\log u_y = d\log(1 - u_z) = -\tilde{\gamma}dt$$ \hspace{1cm} (11)

the trajectories are parabolas, independent of $\tilde{\gamma}(\rho)$, Fig. 1. The atoms eventually relax to the ground state which is the (only) stationary point. The non-linearity only affects the schedule.

To find the schedule, consider the equation for $u_z$, which decouples from the rest

$$\dot{u}_z = \gamma(1 - u_z)^2/2$$ \hspace{1cm} (12)

The solution is a $1/t$ decay law

$$1 - u_z(t) = \left(\frac{1}{1 - u_z(0)} + \frac{\gamma t}{2}\right)^{-1}$$ \hspace{1cm} (13)

The polynomial decay is a smoking gun of non-linear evolution equations.

The slow decay has a simple interpretation: The effective rate $\tilde{\gamma}$ slows down as atoms relax to the ground state because excited atoms find it harder to mate with a partner that would allow both to decay.
Substituting Eqs. \[16\], \[17\] into the Lindblad Eq. \[3\] yields

$$\frac{d}{dt}p_N(t) = \frac{\gamma}{N} \sum_{n=1}^{N} p_n(t) \left( \frac{2n}{2} \right) (R_{n-1} - R_n) \quad (18)$$

Equating the coefficients of $R_n$ gives the Master equation for the vector of probabilities $p = (p_0, \ldots, p_{N/2})$

$$\dot{p}(t) = -\frac{1}{N} G p(t) \quad (19)$$

where $G$ is the upper triangular stochastic matrix \[22\] whose only non zero elements are

$$G_{nn} = -G_{n-1,n} = \gamma n(2n-1), \quad n = 0, 1, \ldots, N/2 \quad (20)$$

The solution of Eq. \[19\] is simply $p(t) = e^{-Gt/N} p(0)$. The rates are given by the eigenvalues of $G/N$. Being a triangular matrix, the eigenvalues are given by the diagonal, thus the eigenvalues of Eq. \[19\], i.e. the decay rates, are $\gamma n(2n-1)/N$. The nonzero rates span from $\gamma/N$ to $\gamma(N-1)/2$, with a decreasing density as they increase, see Fig. \[3\].

When the number of particles $N$ is very large, Eq. \[19\] reduces to a first order PDE. To see this, let us introduce the continuous variable $x$ for the fraction of excited atoms:

$$x = \frac{2n}{N}, \quad 0 \leq x \leq 1 \quad (21)$$

In the $N \to \infty$ limit, Eq. \[19\] reduces to the conservation law \[17\].

$$\partial_t p = \partial_x j, \quad j = x^2 p \quad (22)$$

The initial data is a probability distribution $p_0(x)$:

$$p(x, t = 0) = p_0(x) = \begin{cases} \geq 0 & x \in [0, 1] \\ 0 & \text{otherwise} \end{cases} \quad (23)$$

Eq. \[22\] may be solved by the method of characteristics \[18\]. One finds (see appendix \[B\] for more details)

$$p(x, t) = (1 - xt)^{-2} p_0 \left( \frac{x}{1 - xt} \right) \quad (24)$$

Taking $p_0$ which is sharply localized one sees that the long time decay of the initial data toward the ground state has $1/t$ behavior, see Fig. \[2\].

The power law decay in the limit $N \to \infty$, due to a continuum of rates with diverging density near zero, Eq. \[24\], coincides with the power law decay in Eq. \[13\] which is due to nonlinearity \[23\].

**B. Continuum of dephasing rates**

Consider a situation where atoms do not dephase spontaneously, but do so in pairs when atoms collide where the Lindbladian for a colliding pair is:

$$L_{12} p_2 = \gamma [K \otimes K, [K \otimes K, \rho \otimes \rho]], \quad K^\dagger = K \quad (25)$$

Taking the partial trace as in Eq. \[3\] gives a quadratic Lindbladian with an effective, state dependent, dephasing rate $\tilde{\gamma}(\rho)$ describing the gas of atoms

$$\frac{dp}{dt} = \tilde{\gamma}(\rho) [K, [K, \rho]], \quad \tilde{\gamma}(\rho) = \gamma Tr(K^2 \rho) \quad (26)$$

Choose the Pauli matrix $\sigma_z$ so that $\sqrt{2} K = 1 \cos \theta + \sigma_z \sin \theta \quad [24]$. The equations of motion for the Bloch vector that follow from Eq. \[26\] are

$$\dot{u} = -g(u_x, u_y, 0), \quad g = \gamma \sin^2 \theta (1 + u_z \sin 2\theta) \quad (27)$$

The $z$-axis is the stationary manifold and $u_z(t)$ is a constant of motion (and hence also $g$). The orbits are radial.
(in \(x, y\)) with constant \(u_z\) and the schedule is exponential

\[ u_x(t) = u_x(0)e^{-g(u_z)t} \tag{28} \]

(See Fig. 4). The rates \(g(u_z)\) depend on the initial condition \(u_z\) and take values in an interval

\[ g \in \gamma \sin^2 \theta [1 - \sin 2\theta, 1 + \sin 2\theta] \ tag{29} \]

The interval degenerates to a single point when \(\theta = \pi/2\), which corresponds to the special case of linear evolution with \(\tilde{\gamma} = \gamma/2\). An interval of decay rates is a smoking gun for the non-linear evolution of the qubits.

\[ |0\rangle \langle 0| \]
\[ |-\rangle \langle -| \]
\[ |+\rangle \langle +| \]
\[ |1\rangle \langle 1| \]

FIG. 4: Dephasing with a continuum of rates. The vectors represent the exponential dephasing rate to the fixed point, \(-\frac{d}{dt} \log |u(t) - u(\infty)|\).

The rates \(g(u_z)\) are constants of motion. The decay of \(u_z\) to the northern hemisphere is given by

\[ u_z(t) = g \tanh \left( \frac{gt}{4} + \tanh^{-1} \frac{u_z(0)}{g} \right) \tag{34} \]

C. Flow to a Bloch hemisphere

Consider a process where pairs of excited atoms in the singlet state \(\sqrt{2} |s\rangle = |01\rangle - |10\rangle\) decay to their ground state \(|g\rangle = |00\rangle\) by collisions. A simple Kraus operator that describes this process is

\[ K_\rho = K_1 \rho K_1^\dagger + K_2 \rho K_2^\dagger \tag{30} \]

with

\[ K_1 = |g\rangle \langle s|, \quad K_2 = |s\rangle \langle s| \tag{31} \]

Substituting in Eqs. (28) gives

\[ \frac{d\rho}{dt} = \frac{\sigma_z}{4} (1 - Tr(\rho^2)) = \frac{\sigma_z}{2} \det \rho \tag{32} \]

(See appendix C for more details). The corresponding equation for the Bloch vector is

\[ 4\dot{u} = (0, 0, 1 - u \cdot u) \tag{33} \]

\(u_x\) and \(u_y\) are constants of motion. The decay of \(u_z\) to the northern hemisphere is given by

\[ u_z(t) = g \tanh \left( \frac{gt}{4} + \tanh^{-1} \frac{u_z(0)}{g} \right) \tag{34} \]

FIG. 5: Trajectories of a purifying channel to the upper Bloch hemisphere. The vectors represent the exponential flow rate to the fixed point.

where \(g^2 = 1 - u_x^2 - u_y^2\). All the points of the Bloch ball float up to the upper hemisphere with a continuum of rates in the interval \([0, 1]\). This is a third smoking gun for non-linear evolution.

**Summary:** We describe models of collisions where the mean field equations are solved exactly and display anomalous behavior of decoherence and decay: Power law decay to the ground state; A continuous interval of dephasing rates that depend on the initial data; And finally, flow to the hemisphere of the Bloch ball. None of these features can occur in (time independent) linear Lindbladian of finite dimensional systems.

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**Appendix A: Finite clusters remain uncorrelated**

Our aim in this appendix is to explain why Eq. holds. For rigorous proofs see [14, 15].

Note that Eq. (6) immediately follows from Eq. (5), which states that the reduced 2-body density matrix \(\rho^{(2)} = \text{Tr}_{3, 4, \ldots, N} \rho\) preserves the product structure when \(N\) is large, i.e.

\[ \rho^{(12)}(t) = \rho^{(1)}(t) \otimes \rho^{(2)}(t) \]

It is instructive to see first what lies behind the proof. Consider

\[ \rho^{(12)} = \text{Tr}_{3, \ldots, N} L \rho = \frac{1}{N} \text{Tr} L_{12} \rho^{(12)} + \frac{1}{N} \sum_{j > 2} \left( \text{Tr} L_{1j} \rho^{(1j)} + \text{Tr} L_{2j} \rho^{(2j)} \right) \]

The only term that describes direct interaction that can lead to buildup of the correlations in the 1-2 cluster, is
the first term. This can be neglected when \( N \) is large. The remaining terms, of \( O(1) \), do not couple the pair 1-2. This is fundamentally why the product structure of the initial data in small clusters is preserved by the mean field evolution.

Eq. (3) gives rise to an infinite hierarchy of coupled equations for the partial traces

\[
\rho^{(1...k-1)} = \sum_{i=1}^{k-1} Tr_{k}L_{ik}\rho^{(1...k)} + O(k^2/N)
\]

The last term, describing interactions within the \( k \)-cluster, can be neglected when \( k^2 \ll N \). Assuming the limit \( N \to \infty \) exists, \([25]\), gives the simpler hierarchy

\[
\rho^{(1...k-1)} = \sum_{i=1}^{k-1} Tr_{k}L_{ik}\rho^{(1...k)}
\] (A1)

We shall now verify that Eq. (A1) is satisfied by the ansatz

\[
\rho = \sigma^{\otimes N}
\] (A2)

and obtain the equation for \( \sigma \).

Substituting ansatz (A2) in Eq. (A1) gives

\[
\sum_{i=1}^{k-1} \sigma^{\otimes i-1} \otimes \sigma^{\otimes k-i-1} = \sum_{i=1}^{k-1} Tr_{k}L_{ik}\sigma^{\otimes k}
\]

\[
= \sum_{i=1}^{k-1} \sigma^{\otimes i-1} \otimes Tr_{2}(L_{12}\sigma \otimes \sigma) \otimes \sigma^{\otimes k-i-1}
\] (A3)

It is evident that Eq. (A3) holds provided \( \sigma \) satisfies the non-linear Lindblad equation

\[
\frac{d\sigma}{dt} = Tr_{2}(L_{12}\sigma \otimes \sigma)
\]

which is Eq. (5).

### Appendix B: Solution to the PDE for the probability distribution

Consider the partial differential equation for the probability distribution for the fraction of excited atoms in the ensemble \( p(x,t) \),

\[
\frac{\partial}{\partial t} p = \frac{\partial}{\partial x} (x^2 p)
\] (B1)

After some rearranging,

\[
2xp = \frac{\partial}{\partial t} p - x^2 \frac{\partial}{\partial x} p
\] (B2)

Eq. (22) may be solved using the method of characteristics \([18]\), which reduces solving the partial differential equation to solving a set of ordinary differential equations. This is done by introducing a curve in the \( x,t \) plane along which the partial differential equation transforms into an ordinary differential equation,

\[
\frac{d}{ds} p(x(s),t(s)) = F(p,x(s),t(s)),
\] (B3)

where \( s \) is a variable associated with the curve, and \( F \) is a some function of \( p,x,t \).

The left hand side of the equation above may be rewritten as,

\[
\frac{d}{ds} p(x(s),t(s)) = \frac{\partial p}{\partial t} \frac{dt}{ds} + \frac{\partial p}{\partial x} \frac{dx}{ds}.
\] (B4)

Equating the coefficients between Eqs. (B4) and (22) yields a set of three ODEs,

\[
\frac{dt}{ds} = 1
\]

\[
\frac{dx}{ds} = -x^2
\]

\[
F(p,x,t) = 2xp
\]

The solution for \( t(s) \) is

\[
t(s) = s
\] (B6)

The solution for \( x(s) \) is

\[
x(s) = \frac{1}{s + \frac{1}{x_0}} = \frac{1}{t + \frac{1}{x_0}} \leftrightarrow x_0 = \left(\frac{1}{x} - t\right)^{-1}.
\] (B7)

Finally, the solution for \( p(s) \) is,

\[
\frac{d}{ds} p = 2 \frac{1}{s + \frac{1}{x_0}} p
\] (B8)

It follows that

\[
\ln p = 2 \ln \left(s + \frac{1}{x_0}\right) + \ln \left(f(x_0)\right) = \ln \left(x^{-2}f\left(x^{-1}\right)\right)
\]

\( f(x_0) \) is an integration constant. Therefore the solution to the partial differential equation for \( p(x,t) \) (Eq. (22)) is,

\[
p(x,t) = x^{-2}f\left(x^{-1}\right) \cdot
\] (B9)

The function \( f\left(x^{-1}\right) \) is determined by the initial condition,

\[
p(x,t=0) = x^{-2}f(x) \equiv p_0(x)
\] (B10)

substituting back in Eq. (B9) gives

\[
p(x,t) = (1-xt)^{-2} p_0\left(x\left(\frac{1}{1-xt}\right)\right).
\] (B11)
Appendix C: Derivation of Eq. (32) from the Kraus map

Consider the Kraus map

$$K\rho = K_1\rho K_1^\dagger + K_2\rho K_2^\dagger$$  \hspace{1cm} (C1)

with

$$K_1 = |g\rangle \langle s|, \quad K_2 = 1 - |s\rangle \langle s|$$  \hspace{1cm} (C2)

where

$$|g\rangle = |00\rangle, \quad |s\rangle = \frac{|01\rangle - |10\rangle}{\sqrt{2}}.$$  \hspace{1cm} (C3)

The mean field Lindbladian that arises from this Kraus map takes the form

$$L\rho = Tr_2 [(K - 1) (\rho \otimes \rho)]$$  \hspace{1cm} (C4)

$$= (|s\rangle \otimes \rho) (Tr_2(|g\rangle \langle g| + |s\rangle \langle s|))$$  \hspace{1cm} (C5)

$$- Tr_2 \{|s\rangle \langle s|, \rho \otimes \rho\}.$$  \hspace{1cm} (C6)

It is convenient to use the notation $$|a\rangle = \sum_j a_{jk} |jk\rangle$$, $$|b\rangle = \sum_j b_{jk} |jk\rangle$$. This gives

$$Tr_2 \{|a\rangle \langle b| \rho \otimes \sigma\} = a\sigma^t b^t \rho$$  \hspace{1cm} (C7)

$$Tr_2 \{\rho \otimes \sigma |a\rangle \langle b|\} = \rho \sigma^t b^t.$$  \hspace{1cm} (C8)

In particular since $$\sqrt{2} |s\rangle = \sum_j \varepsilon_{jk} |jk\rangle$$, expression (C6) gives

$$Tr_2 \{|s\rangle \langle s|, \rho \otimes \rho\} = \frac{1}{2} (\varepsilon^t \varepsilon^t \rho + \rho \varepsilon \varepsilon^t \rho).$$  \hspace{1cm} (C9)

One can validate that

$$\varepsilon^t \varepsilon^t \rho = \rho \varepsilon \varepsilon^t \rho = 1 \det \rho,$$  \hspace{1cm} (C10)

thus expression (C6) simplifies to

$$Tr_2 \{|s\rangle \langle s|, \rho \otimes \rho\} = 1 \det \rho.$$  \hspace{1cm} (C11)

Expression (C5) is similarly simplified,

$$Tr \{|s\rangle \langle s|, \rho \otimes \rho\} (Tr_2(|g\rangle \langle g|) + Tr_2(|s\rangle \langle s|))$$  \hspace{1cm} (C12)

$$= Tr \left(\frac{\det \rho}{2} 1\right) \left(|0\rangle \langle 0| + \frac{1}{2} 1\right)$$

$$= \det \rho \left(|0\rangle \langle 0| + \frac{1}{2} 1\right)$$

where we used the identity: $$Tr_2(|s\rangle \langle s|) = \frac{1}{2} 1$$. The resulting Lindbladian is then

$$L\rho = (\det \rho) \left(|0\rangle \langle 0| + \frac{1}{2} 1 - 1\right) = \frac{\det \rho}{2} \sigma_z.$$  \hspace{1cm} (C13)

A density matrix of a qubit satisfies

$$\det \rho = \frac{1 - Tr(\rho^2)}{2},$$  \hspace{1cm} (C14)

so the Lindbladian can be expressed as

$$L\rho = \frac{1}{4} \left(1 - Tr(\rho^2)\right) \sigma_z.$$  \hspace{1cm} (C15)

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Linear Lindbladians are also complete positivity preserving. However, the notion of complete positivity is not defined for non-linear systems.

This corresponds to weak collision in the sense that the Kraus operators for the collisions are $K_1 = \sqrt{2}\epsilon A$ and $K_2 = 1 - \epsilon A\dagger A$. We have suppressed $\epsilon$ in the Lindbladian.

Columns sum to zero.

Under the identification $2x = 1 - u_z$.

With $\sin \theta \neq 0, \pi$ to avoid the trivial case.

The rigorous proofs, [14, 15], establish the existence of the limit.