Equivariant resolutions over Veronese rings

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**Abstract**
Working in a polynomial ring \(S = k[x_1, \ldots, x_n]\), where \(k\) is an arbitrary commutative ring with 1, we consider the \(d\)th Veronese subalgebras \(R = S^{(d)}\), as well as natural \(R\)-submodules \(M = S^{(\geq r,d)}\) inside \(S\). We develop and use characteristic-free theory of Schur functors associated to ribbon skew diagrams as a tool to construct simple \(GL_n(k)\)-equivariant minimal free \(R\)-resolutions for the quotient ring \(k = R / R_+\) and for these modules \(M\). These also lead to elegant descriptions of \(\text{Tor}^R_i(M, M')\) for all \(i\) and \(\text{Hom}_R(M, M')\) for any pair of these modules \(M, M'\).

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1 | INTRODUCTION

Let \(S = k[x_1, \ldots, x_n]\) be a polynomial ring where \(k\) is a commutative ring with unit. Consider \(S\) as a standard graded \(k\)-algebra \(S = \bigoplus_j S_j\) in which \(S_j\) are the homogeneous polynomials of degree \(j\). Then \(S\) contains for each \(d = 1, 2, 3, \ldots\) a Veronese subalgebra

\[
R = S^{(d)} := \bigoplus_{j \equiv 0 \mod d} S_j.
\]

Fixing \(d\), our goal is to study various \(R\)-modules \(M\) inside \(S\) of the form

\[
M = S^{(\geq r,d)} := \bigoplus_{j \geq r \mod d} S_j = S_r \oplus S_{r+d} \oplus S_{r+2d} \oplus S_{r+3d} \oplus \cdots,
\]
where \( r = 0, 1, 2, \ldots \). Our first main result, Theorem 1.1, describes a (generally infinite) minimal\(^\dagger\) \( R \)-free resolution for each such \( M \) as an \( R \)-module. When \( k \) is a field, it is known (see Subsection 4.3) that \( M \) always has a linear \( R \)-free resolution\(^\ddagger\)

\[
0 \leftarrow M \leftarrow R^{\beta_0} \leftarrow R(-1)^{\beta_1} \leftarrow R(-2)^{\beta_2} \leftarrow \cdots
\]

However, we will aim for more precise equivariant descriptions using the action of the general linear group \( GL(V) \) on the symmetric algebra

\[
S = k[x_1, \ldots, x_n] = S(V),
\]

in which \( V = k^n \) is a free \( k \)-module with \( k \)-basis \( x_1, \ldots, x_n \), so that \( S_j \cong S^j(V) \), the \( j \)th symmetric power of \( V \). We should note that, by contrast, there has been more extensive study of equivariant descriptions of the finite free resolutions for both \( R \) itself and the modules \( M \) over the polynomial ring \( S(S^d(V)) \); see [6, 15, 18, 25, 29, 30].

Our results use polynomial representations of \( GL(V) \), including the Schur functors \( V \mapsto S^\lambda(V) \) corresponding to number partitions \( \lambda \), and skew Schur functors \( V \mapsto S^\lambda/\mu(V) \) associated to a skew diagram \( \lambda/\mu \); see Akin, Buchsbaum, and Weyman [1], Macdonald [24, appendix I.A]. Certain skew diagrams play an important role here, namely the ribbon (skew/rim hook = border strip) diagrams that we denote \( \sigma(\alpha) \), whose row sizes from bottom to top are specified by an (ordered) composition \( \alpha = (\alpha_1, \alpha_2, \ldots, \alpha_\ell) \), with one column of overlap between consecutive rows. For example, \( \sigma((3, 1, 1, 2, 4)) \) is this diagram:

\[
\begin{bmatrix}
\cdot & \cdot & \cdot & \cdot & \cdot \\
\cdot & \cdot & \cdot & \cdot & \\
\cdot & \cdot & \cdot & \cdot & \\
\cdot & \cdot & \cdot & \cdot & \\
\cdot & \cdot & \cdot & \cdot & 
\end{bmatrix}
\]

Theorem 1.1. Fix \( d, r \geq 1 \) and let \( R = S^{(d)} \) and \( M = S^{(\geq r, d)} \) as above. Then one has an explicit \( GL(V) \)-equivariant minimal \( R \)-free resolution of \( M \) of the form

\[
0 \leftarrow M \leftarrow R \otimes_k S^{(r)}(V) \leftarrow R \otimes_k S^{\sigma(d, r)}(V) \leftarrow R \otimes_k S^{\sigma(d, d, d, r)}(V) \leftarrow \cdots
\]

whose \( i \)th resolvent \( R \otimes_k S^{\sigma(d, r)}(V) \) has \( R \)-basis elements in degree \( d_i + r \).

Corollary 1.2. In the setting of Theorem 1.1, \( \text{Tor}_i^R(M, k)_j \) vanishes for \( j \neq d_i + r \), and

\[
\text{Tor}_i^R(M, k)_{d_i + r} \cong S^{\sigma(d, r)}(V),
\]

as a polynomial \( GL(V) \)-representation.

Example 1.3. Taking \( d = 3 \) and \( r = 4 \), so that \( R = S^{(3)} \) and \( M = S^{(\geq 4, 3)} \), the \( R \)-free resolution of \( M \) has the form

\[
R \otimes_k S^{\sigma(3, 3)}(V) \leftarrow R \otimes_k S^{\sigma(3, 4)}(V) \leftarrow R \otimes_k S^{\sigma(3, 5)}(V) \leftarrow \cdots
\]

\(^\dagger\) We do not assume \( k \) is a field, so “minimal” here means the differentials have all entries in \( R_+ = S^{(d, d)} \).

\(^\ddagger\) That is, after rescaling the grading of \( R = S^{(d)} \) to make its generators of degree 1 (not degree \( d \)) and shifting the grading on \( M = S^{(\geq r, d)} \) to make its \( R \)-module generators of degree 0 (not degree \( r \)).
Example 1.4. We intentionally allow for the case where $d = 1$, so that $R = S^{(1)} = S$, and $M = S^{(2r,1)} = (x_1, x_2, ..., x_n)^r$. For example, when $d = 1$ and $r = 4$, so that $R = S^{(1)} = S$, then the $S$-free resolution of $M = S^{(2,4,1)} = (x_1, x_2, ..., x_n)^4$ looks like

$$S \otimes_k S^{d^{\bullet}}(V) \leftarrow S \otimes_k S^{d^{\bullet}}(V) \leftarrow S \otimes_k S^{d^{\bullet}}(V) \leftarrow ...$$

This recovers $GL(V)$-equivariant $S$-resolutions of powers $(x_1, ..., x_n)^r$ of the irrelevant ideal $S_+$ discussed by Buchsbaum and Rim [10], Buchsbaum and Eisenbud [9], and Srinivasan [33]; see also Kustin [20, sections 4 and 5]. For $r = 1$, it gives a Koszul complex resolving $(x_1, ..., x_n)$.

Example 1.5. Note that as Theorem 1.1 assumes $r \geq 1$, it deliberately excludes the case $r = 0$, where the $R$-free resolution of the module $M = S^{(0,d)} = S^{(d)} = R$ is trivial. On the other hand, taking $r = d$, one has

$$M = S^{(d,d)} = S_+^d = R_+,$$

the irrelevant ideal of $R = S^{(d)}$ whose quotient defines the module $k = R/R_+$. Thus, the $R$-free resolution of $M$ also gives the resolution of $k$, and $Tor^R_i(k, k) \cong Tor^R_{i-1}(R_+, k)$.

Building up to the proof of Theorem 1.1, Section 2 reviews equivariant resolutions, polynomial $GL(V)$-representations, and skew Schur functors. Section 3 then derives some special properties of skew Schur functors of ribbon shape that are crucial for our proof. In particular, given a sequence of ribbons, we construct in Definition 3.13 a right resolution of the Schur functor associated to the concatenation of the ribbons in terms of Schur functors associated to various near-concatenations of the ribbons. This complex, which may be of independent interest, turns out to be a categorification of a special case of the Hamel–Goulden determinantal identity (see [16]) for skew Schur functions. The modules and differentials of this complex are simple to describe, and analysis of a special case of this complex gives us an essential tool for computing the modules $Tor^R_i(M, M')$.

The maps in the resolution turn out to be part of an unexpected complex of $S$-modules $(\mathcal{R}_+, \partial)$, dubbed the complex of ribbons in Subsection 4.1. The differential in the complex of ribbons is the restriction of a simple tensor-degree-lowering map $\partial$ on $S \otimes_k T_k(S)$, where here $T_k(S)$ is a tensor algebra. However, this $\partial$ is not a differential ($\partial^2 \neq 0$) on all $S \otimes_k T_k(S)$; rather it only satisfies $\partial^2 = 0$ when restricted to the complex of ribbons $\mathcal{R}_+$. After proving Theorem 1.1 in Subsection 4.2, the next two subsections discuss a few of its corollaries, including a symmetric function identity (Subsection 4.4), and a poset homology calculation (Subsection 4.6).

A striking feature of the minimal resolution in Theorem 1.1 is its uniformity, both in the resolutions and the form of its differentials. This uniformity is utilized to great effect in Section 5, where we equivariantly describe $Tor^R(M, M')$ and also $Hom_R(M, M')$ for $R = S^{(d)}$ and all $M, M'$ of the form $S^{(2r,d)}$, $S^{(2r'd)}$. Even at the level of the standard un-derived functors $M \otimes_R M'$ and $Hom_R(M, M')$, one already sees evidence of the consistent structure for these modules in the next two results, proven in Subsections 5.1 and 5.3.

Theorem 1.6. Fix $d, r, r' \geq 1$ and let $R = S^{(d)}$ with the three $R$-modules

$$M = S^{(2r,d)},$$
$$M' = S^{(2r',d)},$$
$$M'' = S^{(2r'',d)},$$

where $r'' = r + r'$. 
(i) The multiplication map $M \otimes_R M' \xrightarrow{\varphi} M''$ gives rise to a $GL(V)$-equivariant short exact sequence of $R$-modules
\[
0 \to \mathcal{S}^{\sigma(r,r')}(V)(-r'') \to M \otimes_R M' \to M'' \to 0
\]
with the $R$-module $\mathcal{S}^{\sigma(r,r')}(V)(-r'')$ concentrated in degree $r''$, annihilated by $R_+$. 

(ii) The sequence splits as $R$-modules, giving an $R$-module isomorphism
\[
M \otimes_R M' \cong M'' \oplus \mathcal{S}^{\sigma(r,r')}(V)(-r'').
\]

(iii) When $\binom{r+r'}{r}$ lies in $k^\times$, the sequence also splits as $GL(V)$-representations.

**Theorem 1.7.** Assume $n \geq 2$, so that $S = k[x_1, \ldots, x_n]$ is not univariate. Fix an integer $d \geq 1$, defining $R = S^{(d)}$. For $r, r' \geq 0$, consider three $R$-modules
\[
M = S^{(\geq r, d)}, \\
M' = S^{(\geq r', d)}, \\
M'' = S^{(\geq r'', d)},
\]
defining $r'' := r' - r$ if $r \leq r'$, otherwise if $r > r'$, defining $r''$ to be the unique integer in $[0, d)$ congruent to $r' - r \mod d$. Then one has a $GL(V)$-equivariant $R$-module isomorphism
\[
M'' \longrightarrow \text{Hom}_R(M, M') \\
m'' \longmapsto (m \mapsto m'' \cdot m).
\]

Subsection 5.2 uses Theorem 1.1 to equivariantly describe $\text{Tor}_R^i(M, M')$ for $i \geq 1$, which is even simpler. Note that as $k = R/R_+$ where $R_+ = S^{(d,d)}$, dimension-shifting lets one rephrase Theorem 1.1 as saying that for $M, M'$ being $S^{(\geq r, d)}, S^{(\geq r', d)}$, one has $\text{Tor}_R^{i-1}(M, M') \cong \mathcal{S}^{\sigma(d,d^{-1},r)}(V)$ for $i \geq 1$. Our last main result elegantly generalizes this.

**Theorem 1.8.** Fix $d, r, r' \geq 1$, and let $R, M, M'$ denote $S^{(d)}, S^{(\geq r, d)}, S^{(\geq r', d)}$, as usual. Then for $i \geq 1$, the $R$-module $\text{Tor}_R^i(M, M')$ is annihilated by $R_+$, and as a module over $k = R/R_+$, has a $GL(V)$-isomorphism
\[
\text{Tor}_R^i(M, M') \cong \mathcal{S}^{\sigma(r,d,r')}(V).
\]

**Example 1.9.** Note that Theorem 1.8 is interesting even if $d = 1$, so $R = S = k[x_1, \ldots, x_n]$. It asserts, for instance, that there is a $GL(V)$-equivariant isomorphism
\[
\text{Tor}_3^S((S_+)^2, (S_+)^3) \cong \mathcal{S}^{\sigma(1,1,1)}(V).
\]

\(^1\) See Proposition 5.1 for the simple answer in the univariate case $n = 1$. 


2 | REPRESENTATION THEORY PRELIMINARIES

2.1 | Some generalities on equivariant resolutions

We mention a fact about equivariant resolutions that can be found in Broer, Reiner, Smith, and Webb [5, section 2].

Let $R$ be a commutative, graded Noetherian $k$-algebra, and $G$ a group of $k$-algebra automorphisms of $R$. For $M$ a Noetherian graded $R$-module with a $G$-action via $k$-module automorphisms, say $G$ acts \textit{compatibly} on $M$ if $g(r \cdot m) = g(r)g(m)$ for all $r$ in $R$, $m$ in $M$, $g$ in $G$.

\textbf{Proposition 2.1.} In the above setting, one has the following. If $M, N$ are two graded Noetherian $R$-modules with compatible $G$-actions as above, then $\text{Tor}^R_i(M, N)$ and $\text{Ext}^i_R(M, N)$ for all $i \geq 0$ are also graded Noetherian $R$-modules with compatible $G$-actions.

2.2 | Review of polynomial $GL(V)$ representations

We will need some of the basic facts on polynomial representations of $GL_n(k)$, and polynomial functors. References are Akin, Buchsbaum, and Weyman [1], Fulton [12, sections 8.2 and 8.3], Fulton and Harris [13, Lecture 6], Gandini [14], Macdonald [24, appendix I.A], Stanley [34, appendix 7.A.2].

Let $V = k^n$ be a free $k$-module of rank $n$, so that $GL(V) \cong GL_n(k)$. A representation $\varphi : GL(V) \rightarrow GL(U)$ (where $U$ is also a free $k$-module) is \textit{polynomial} if the matrix entries $[\varphi(A)_{k,\ell}]_{k,\ell=1,2,...}$ are all polynomial functions in the matrix entries of $A = [a_{ij}]_{i,j=1,2,...,n}$. If these entries $\varphi(A)_{k,\ell}$ are all homogeneous in $\{a_{ij}\}$ of degree $d$, then one says that $\varphi$ is a \textit{homogeneous} polynomial representation of degree $d$. When no confusion arises, we will again refer to $\varphi$ by $U$, in our standard abuse of notation. A polynomial representation $U$ has \textit{character}

$$ch_U = ch_U(x_1, ..., x_n) = \text{trace} \varphi(\text{diag}(x)),$$

where $\text{diag}(x)$ is an $n \times n$ diagonal matrix having eigenvalues $x_1, ..., x_n$. This character $ch_U$ is a symmetric polynomial in $x_1, x_2, ..., x_n$. However, all of the representations of $GL(V)$ that we consider here are restrictions to $GL(V)$ of polynomial functors from the category of finite-rank free $k$-modules and $k$-linear maps to itself. This means that they are defined for all $n$, and one can regard $ch_U$ as an element of the \textit{ring of symmetric functions} $\Lambda_k$ in the infinite variable set $x_1, x_2, ..., n$, with coefficients in $k$. When $k$ is a field of characteristic zero, the $GL(V)$-representation $U$ is determined up to isomorphism by its character $ch_U$.

\textbf{Remark 2.2.} For $k$ a field, the polynomial functors that we are considering are \textit{strict polynomial functors} in the sense of Friedlander and Suslin [11, Definition 2.1]. Although we will not make use of it, a more general theory of strict polynomial functors over arbitrary commutative rings $k$ is described by Krause [19, chapter 8].

Here are some examples of polynomial functors and their characters.

\footnote{Alternatively, $M$ is a module for the \textit{skew group algebra} $R\#G$; see Leuschke and Wiegand [22].}
Example 2.3. The $m$th tensor power $T^m(V) = V^\otimes m$, the $m$th symmetric power $S^m(V)$ and the $m$th exterior power $\wedge^m(V)$ are all polynomial functors, with
\[
\text{ch}_{T^m(V)} = h_m^m = e_m = (x_1 + x_2 + \cdots)^m,
\]
\[
\text{ch}_{S^m(V)} = h_m,
\]
\[
\text{ch}_{\wedge^m(V)} = e_m.
\]
Here $h_m$ is the complete homogeneous symmetric function, which is the sum of all monomials of degree $m$ in $x_1, x_2, \ldots$, and $e_m$ is the elementary symmetric function, which is the sum of all squarefree monomials of degree $m$ in $x_1, x_2, \ldots$.

Example 2.4. For each partition $\lambda$ with $m = |\lambda| := \sum \lambda_i$, there is a degree $m$ homogeneous polynomial representation of $GL(V)$ called the Schur functor $S^\lambda(V)$, with
\[
\text{ch}_{S^\lambda(V)} = s_\lambda,
\]
where $s_\lambda$ is the Schur function in $x_1, x_2, \ldots$. When $k$ is a field of characteristic zero, Schur functors $S^\lambda(V)$ give all of the irreducible polynomial $GL(V)$-representations.

Example 2.5. The construction of $S^\lambda(V)$ is a special case of a Schur functor $S^D(V)$ defined for more general diagrams $D \subseteq \{1, 2, \ldots\} \times \{1, 2, \ldots\}$, such as when $D = \lambda / \mu$ is a skew Ferrers diagram for a pair of partitions $\mu \subseteq \lambda$. The skew Schur functor $S^{\lambda / \mu}(V)$ has $\text{ch}_{S^{\lambda / \mu}(V)} = s_{\lambda / \mu}$, the skew Schur function. We illustrate the construction of the Schur functor $S^D(V)$ here with an example.

Label the cells of $D$ bijectively with $1, 2, \ldots, m$, and define the sets of labels $C_1, \ldots, C_c$ in its columns (left-to-right) and rows $R_1, \ldots, R_r$ (bottom-to-top), as in this example:
\[
D = \begin{array}{ccc}
7 & 8 \\
3 & 5 & 6 \\
1 & 2 & 4 \\
\end{array}
\]
\[
R_1 = \{1, 2, 4\}, R_2 = \{3, 5, 6\}, R_3 = \{7, 8\},
\]
\[
C_1 = \{1\}, C_2 = \{2, 3\}, C_3 = \{4, 5\}, C_4 = \{6, 7\}, C_5 = \{8\}.
\]

Introduce the row and column compositions for $D$:
\[
\text{rows}(D) := (|R_1|, \ldots, |R_r|),
\]
\[
\text{cols}(D) := (|C_1|, \ldots, |C_c|).
\]

Given a composition $\alpha = (\alpha_1, \ldots, \alpha_c)$, abbreviate
\[
S^\alpha(V) := S^{\alpha_1}(V) \otimes_k \cdots \otimes_k S^{\alpha_c}(V),
\]
\[
\wedge^\alpha(V) := \wedge^{\alpha_1}(V) \otimes_k \cdots \otimes_k \wedge^{\alpha_c}(V).
\]
Then $S^D(V)$ will be a subfunctor of $S^{\text{rows}(D)}(V)$ defined as the image of the composite of two maps, the first an antisymmetrization map $\s_A$ along the columns of $D$, the second a symmetrization map.
\( \mathcal{R}_D \) along the rows of \( D \):

\[
\bigwedge^{\text{cols}(D)}(V) \xrightarrow{\mathcal{A}_D} T^m(V) \xrightarrow{\mathcal{R}_D} S^{\text{rows}(D)}(V).
\]

Here the maps \( \mathcal{A}_D, \mathcal{R}_D \) are described as follows. Assuming that the column indexing sets \( C_1, \ldots, C_c \) each consist of contiguous sequences of integers, let \( w_D \) be any permutation of the tensor positions that sends the row indexing sets \( R_1, \ldots, R_c \) to a contiguous sequence of integers. Then \( \mathcal{A}_D, \mathcal{R}_D \) can be described via tensor products as

\[
\mathcal{A}_D := \bigotimes_{j=1}^c \mathcal{A}_{|C_j|},
\]

\[
\mathcal{R}_D := \left( \bigotimes_{i=1}^r \mathcal{B}_{|R_i|} \right) w_D,
\]

where \( \mathcal{A}_\epsilon : \bigwedge^\epsilon(V) \to T^\epsilon(V) \) is the usual inclusion via antisymmetrization sending

\[
 v_1 \wedge v_2 \wedge \cdots \wedge v_\epsilon \mapsto \sum_{w \in \mathfrak{S}_\epsilon} \text{sgn}(w) \cdot v_{w(1)} \otimes v_{w(2)} \otimes \cdots \otimes v_{w(\epsilon)},
\]

and \( \mathcal{B}_\epsilon : T^\epsilon(V) \to S^\epsilon(V) \) is the usual symmetrization/multiplication map sending

\[
 v_1 \otimes v_2 \otimes \cdots \otimes v_\epsilon \mapsto v_1 \cdot v_2 \cdots v_\epsilon.
\]

In our example above, the map \( \mathcal{A}_D \) is

\[
\bigwedge^{\text{cols}(D)}(V) = \bigwedge^1(V) \otimes \bigwedge^2(V) \otimes \bigwedge^2(V) \otimes \bigwedge^2(V) \otimes \bigwedge^1(V) \to T^8(V)
\]

sending

\[
 v_1 \otimes (v_2 \wedge v_3) \otimes (v_4 \wedge v_5) \otimes (v_6 \wedge v_7) \otimes v_8 \\
 \mapsto v_1 \otimes (v_2 \otimes v_3 - v_3 \otimes v_2) \otimes (v_4 \otimes v_5 - v_5 \otimes v_4) \otimes (v_6 \otimes v_7 - v_7 \otimes v_6) \otimes v_8
\]

while the map \( \mathcal{R}_D \) is this permutation \( w_D \) and tensor product of symmetrizations:

\[
 V^\otimes^8 \mapsto S^3(V) \otimes S^3(V) \otimes S^2(V) = S^\text{rows}(D)(V)
\]

\[
 u_1 \otimes \cdots \otimes u_8 \mapsto u_1 u_2 u_4 \otimes u_3 u_5 u_6 \otimes u_7 u_8.
\]

As notation, given a filling \( T \) of the squares of \( D \) with vectors from \( V \), one can associate to \( T \) a tensor of decomposable wedges in \( \bigwedge^{\text{cols}(D)}(V) \), and then let \( [T] \) denote its image in \( S^{\text{rows}(D)}(V) \) under \( \mathcal{R}_D \circ \mathcal{A}_D \). For example, with \( D \) the diagram above,

\[
 \begin{bmatrix}
 v_1 & v_2 & v_3 & v_4 \\
 v_3 & v_5 & v_6 & v_7 \\
 v_2 & v_6 & v_8 & v_4
 \end{bmatrix} = \mathcal{R}_D \left( \mathcal{A}_D \left( v_1 \otimes (v_2 \wedge v_3) \otimes (v_4 \wedge v_5) \otimes (v_6 \wedge v_7) \otimes v_8 \right) \right).
\]
Akin, Buchsbaum, and Weyman prove several important properties of the Schur functors $\mathbb{S}^{\lambda/\mu}$ corresponding to skew shapes $\lambda/\mu$ in [1, Theorem II.2.16]. First, they show $\mathbb{S}^{\lambda/\mu}(V)$ has an explicit presentation as a quotient representation of $\bigwedge^{\text{cols}(\lambda/\mu)}(V)$, rather than as a subrepresentation of $S^{\text{rows}(\lambda/\mu)}(V)$. Specifically, they show that the kernel of the map

$$\mathcal{B}^{\lambda/\mu} \circ \mathcal{A}^{\lambda/\mu} : \bigwedge^{\text{cols}(\lambda/\mu)}(V) \to S^{\text{rows}(\lambda/\mu)}(V)$$

is generated by certain Garnir relations. Second, they show that $\mathbb{S}^{\lambda/\mu}(V)$ is not only a free $k$-module, but universally free, in the sense that it has a $k$-basis that commutes with any change of the base ring $k$, described as follows. Once one picks a $k$-basis $e_1, \ldots, e_n$ for $V$, one obtains an obvious monomial $k$-basis for $\bigwedge^{\text{cols}(D)}(V)$ consisting of tensors of wedges of the $\{e_i\}$, indexed by fillings $T : \lambda/\mu \to \{1, 2, \ldots, n\}$ that are column-increasing, that is, strictly increasing from top-to-bottom in each column of $\lambda/\mu$. Letting $[T]$ denote the image of such an element under $\mathcal{B}^{\lambda/\mu} \circ \mathcal{A}^{\lambda/\mu}$, they show $\mathbb{S}^{\lambda/\mu}(V)$ has a $k$-basis $\{[T]\}$ as $T$ runs through the subset of semistandard (column-strict) tableaux of shape $\lambda/\mu$: fillings $T$ that are strictly increasing top-to-bottom down each column, but also weakly increasing left-to-right in each row. Garnir relations let one rewrite any $[T]$ where $T$ is a column-increasing filling as a $k$-linear combination of $[T']$ indexed by semistandard tableaux $T'$.

We mention here a consequence of this universality that we will need.

**Proposition 2.6.** Fixing a skew shape $\lambda/\mu$, for any complex $C$ of $GL(V)$-modules, tensoring over $k$ with $\mathbb{S}^{\lambda/\mu}(V)$ is exact, so it commutes with taking homology: one has an isomorphism of $GL(V)$-modules

$$H_i(C \otimes_k \mathbb{S}^{\lambda/\mu}(V)) \cong H_i(C) \otimes_k \mathbb{S}^{\lambda/\mu}(V).$$

**Example 2.7.** When $k$ is a field, tensor products, subfunctors and quotient functors of polynomial functors remain polynomial. Hence, the homology groups of a complex of polynomial functors are all polynomial functors. In particular, in our setting where $R = S^{(d)}$ and $M = S^{(\geq r,d)}$ have each of their homogeneous components a polynomial functor of the form $S^i(V)$, one can conclude that homogeneous components of $\text{Tor}^R_i(M,k) = \text{Tor}^R_i(k,M)$ are polynomial functors, as they can be computed as the homology of a tensored Koszul complex $(R \otimes_k A^*) \otimes_R M$. Similar arguments show that the (N-graded) homogeneous components $\text{Tor}^R_i(M,M')_j$ of each $\text{Tor}^R_i(M,M')$ are polynomial functors, where $M = S^{(\geq r,d})$ and $M' = S^{(\geq r',d)}$. This will also follow from the polynomial form of the $R$-free resolution of $M$ in Theorem 1.1.

Note that in this setting, $\text{Hom}_R(M,N)$ (and hence $\text{Ext}^i_R(M,k)$) is not necessarily a polynomial functor. The action of an element $g$ in $GL(V)$ on $\varphi$ in $\text{Hom}_R(M,N)$ via $\varphi \mapsto g \varphi g^{-1}$ introduces denominators of $\det(g)$ into the action.

### 2.3 Schur–Weyl duality

Every homogeneous polynomial functor $V \mapsto \mathcal{P}(V)$ of degree $m$ not only gives rise to a family of polynomial representations of $GL(V) = GL_n(k)$ for all $n$, but also a representation of $\mathfrak{S}_m$, or $k \mathfrak{S}_m$-module, on its squarefree or multilinear or $x_1 x_2 \cdots x_m$-weight space: pick $V = k^m$, and consider the...
\( \mathbf{k} \)-vector space

\[
P(V)_{x_1 x_2 \cdots x_m} := \{ u \in P(V) : \text{diag}(x)u = x_1 x_2 \cdots x_m \cdot u \}.
\]

Conversely, when \( \mathbf{k} \) has characteristic zero, one can recover the functor \( V \mapsto P(V) \) from this \( \mathbf{k} \mathfrak{S}_m \)-module via the formula

\[
P(V) \cong \bigoplus_{m=0}^{\infty} \left( T^m(V) \otimes_{\mathbf{k} \mathfrak{S}_m} P(V)_{x_1 x_2 \cdots x_m} \right),
\]

where the \( m \)-fold tensor product \( T^m(V) = V \otimes^m \) is regarded as a left \( \mathbf{k} \text{GL}(V) \)-module under the diagonal action

\[
g(v_1 \otimes \cdots \otimes v_m) := g(v_1) \otimes \cdots \otimes g(v_m)
\]

and a right \( \mathbf{k} \mathfrak{S}_n \)-module using the tensor position action

\[
(v_1 \otimes \cdots \otimes v_m)\sigma := v_{\sigma(1)} \otimes \cdots \otimes v_{\sigma(m)}.
\]

The \( GL(V) \)-irreducible Schur functor \( S^\lambda(V) \) with \( |\lambda| = m \) has \( x_1 \cdots x_m \)-weight space \( S^\lambda(V)_{x_1 x_2 \cdots x_m} \) isomorphic to the irreducible Specht module \( S^\lambda \) for \( \mathfrak{S}_m \). More generally, each skew shape \( \lambda/\mu \) with \( m = |\lambda| - |\mu| \) has skew Schur functor \( S^{\lambda/\mu}(V) \) with \( x_1 \cdots x_m \)-weight space \( S^{\lambda/\mu}(V)_{x_1 x_2 \cdots x_m} \) isomorphic to the skew Specht module \( S^{\lambda/\mu} \) for \( \mathfrak{S}_m \).

**Notational warning.** To lighten notation, from here onward we often omit the \( V \) from the Schur functor notation \( S^{\lambda/\mu}(V) \), and denote this simply by \( S^{\lambda/\mu} \). In particular,

\[
\mathcal{S}^{(m)} = S^{(m)}(V) = S^m(V) = S^m = S_m,
\]

where \( S = \mathbf{k}[x_1, \ldots, x_n] = S(V) \). We hope context will resolve any confusion.

## 3 | RIBBON SCHUR FUNCTORS

Schur functors coming from ribbon diagrams play a central role in this work. These Schur functors are almost never irreducible as \( GL(V) \)-representations, but still give some of the simplest classes of Schur functors one can study. Our goal in the next few subsections is to present characteristic-free homological results valid for arbitrary commutative rings \( \mathbf{k} \), lifting symmetric function identities known for ribbon Schur functions. We will make heavy use of the universal freeness results for skew Schur functors of Akin, Buchsbaum and Weyman [1], for example, Proposition 2.6.

### 3.1 | Concatenation and near-concatenation

**Definition 3.1.** Given a composition \( \alpha = (\alpha_1, \ldots, \alpha_\ell) \), recall that the ribbon diagram \( \sigma(\alpha) \) has row sizes from bottom to top given by the entries of \( \alpha \), and each consecutive row overlaps in exactly one column. For two compositions \( \alpha = (\alpha_1, \ldots, \alpha_\ell), \beta = (\beta_1, \ldots, \beta_m) \), define their *concatenation* \( \alpha \cdot \beta \)
and near concatenation $\alpha \odot \beta$ by

$$\alpha \cdot \beta := (\alpha_1, \ldots, \alpha_{\ell}, \beta_1, \ldots, \beta_m),$$

$$\alpha \odot \beta := (\alpha_1, \ldots, \alpha_{\ell-1}, \alpha_{\ell} + \beta_1, \beta_2, \ldots, \beta_m).$$

**Example 3.2.** If $\alpha = (2, 1, 3)$ and $\beta = (2, 4, 2)$, then one has $\alpha \cdot \beta = (2, 1, 3, 2, 4, 2)$ and $\alpha \odot \beta = (2, 1, 5, 4, 2)$ with ribbon diagrams

There is a family of basic relations among ribbon Schur functions [24, chapter I.5, Example 21(a)], known to generate all other relations among them [4, Proposition 2.2]:

$$s_\alpha s_\beta = s_\alpha \cdot \beta + s_\alpha \odot \beta \tag{2}$$

We wish to lift this identity (2) to the following short exact sequence, which will form the base case in an inductive proof of the more general Theorem 3.14.

**Proposition 3.3.** For any compositions $\alpha, \beta$ one has maps of polynomial functors $\Delta_{\alpha, \beta}, m_{\alpha, \beta}$ giving rise to a $GL(V)$-equivariant short exact sequence of representations:

$$0 \rightarrow S^{\sigma(\alpha)} \Delta_{\alpha, \beta} S^{\sigma(\beta)} \rightarrow S^{\sigma(\alpha)} \otimes S^{\sigma(\beta)} m_{\alpha, \beta} S^{\sigma(\alpha \odot \beta)} \rightarrow 0. \tag{3}$$

This sequence is split over $k$, but not necessarily split over $GL(V)$.

When the compositions $\alpha$ and $\beta$ as in the statement of Proposition 3.3 are clear from context, we will use the more concise notation $m, \Delta$ to denote the maps $m_{\alpha, \beta}, \Delta_{\alpha, \beta}$.

In fact, it is not hard to generalize concatenation, near-concatenation, the identity (2) and sequence (3) to more general classes of diagrams.

**Definition 3.4.** Given two (not necessarily skew-shaped) diagrams $D, D'$, define their disjoint sum $D \oplus D'$, concatenation $D \cdot D'$ and near-concatenation $D \odot D'$ as follows, illustrated for two skew diagrams here, with two extreme cells labeled $x, x'$ for clarity:

$$D = \begin{array}{|c|c|}
\hline x &  \\
\hline \end{array} \quad D' = \begin{array}{|c|c|}
\hline &  \\
\hline \end{array}$$
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The disjoint sum \( D \oplus D' \) places \( D \) southwest of \( D' \), sharing no rows and columns.

The concatenation \( D \cdot D' \) is obtained from \( D \oplus D' \) by moving the cells of the rightmost column of \( D \) and leftmost column of \( D' \) to lie in the same column, with no other overlap of rows and columns.

The near-concatenation \( D \odot D' \) is obtained from \( D \oplus D' \) by moving the cells in the top row of \( D \) and bottom row of \( D' \) to lie in the same row, with no other overlap of rows and columns.

The following is an easy generalization of identity (2).

**Proposition 3.5.** For any two skew shapes \( D, D' \), one has the skew Schur function identity

\[
s_D s_{D'} = s_{D \cdot D'} + s_{D \odot D'}.
\]

**Proof.** Use the formula, the skew Schur function as

\[
s_D = \sum_T x_T \quad \text{where} \quad T \text{ runs over all column-strict tableaux of shape } D \text{ with entries in } \{1, 2, \ldots, n\}. \quad \text{Here } x_T := \prod_{i=1}^n x_{c_i(T)}^i \quad \text{where } c_i(T) \text{ is the number of occurrences of } i \text{ in the tableau } T.
\]

The left side in the proposition can be written as the sum \( \sum_{(T,T')} x_T x_{T'} \) over all pairs \( (T,T') \) of column-strict tableaux of shape \( D, D' \). Letting \( i, i' \), respectively, denote the entries filling the northeastmost cell \( x \) in \( T \), and the southwestmost cell \( x' \) in \( T' \), the two summands on the right side of the identity segregate these \( (T,T') \) summands on the left side according to whether

- \( i > i' \), so the tableaux \( T, T' \) glue to give one of shape \( D \cdot D' \), or
- \( i \leq i' \), so the tableaux \( T, T' \) glue to give one of shape \( D \odot D' \).

\( \square \)

The proof of Proposition 3.5 does not extend to characters \( \text{ch}_{\mathcal{D}(V)} \) associated with arbitrary (not necessarily skew) diagrams \( D \), as in general the tableaux with semistandard fillings need not form a basis for the Schur functor \( \mathcal{S}^D(V) \) (see, for instance, [36, Remark (ii), p. 489]). We next lift the identity in the Proposition 3.5 to an exact sequence.

**Proposition 3.6.** Let \( D, D' \) be any diagrams, not necessarily skew-shaped.

(i) There is an injective \( \text{GL}(V) \)-equivariant map \( \Delta: \mathcal{S}^{D \cdot D'} \hookrightarrow \mathcal{S}^{D \oplus D'} \).

(ii) There is a surjective \( \text{GL}(V) \)-equivariant map \( m: \mathcal{S}^{D \oplus D'} \twoheadrightarrow \mathcal{S}^{D \odot D'} \).

(iii) Assume \( D \) contains a northeastmost corner cell \( x \) and \( D' \) contains a southwestmost corner cell \( x' \), such as when \( D, D' \) are skew shapes. Then the maps \( m, \Delta \) in (i),(ii) satisfy \( m \circ \Delta = 0 \), giving a short complex, but not always exact at the middle term:

\[
0 \rightarrow \mathcal{S}^{D \cdot D'} \xrightarrow{\Delta} \mathcal{S}^{D \oplus D'} \xrightarrow{m} \mathcal{S}^{D \odot D'} \rightarrow 0.
\]
(iv) When $D, D'$ are both skew shapes, the complex in (iii) is short exact and split as $k$-modules (but not necessarily split as $GL(V)$-modules).

In particular, part (iv) applied to $D = \sigma(\alpha), D' = \sigma(\beta)$ gives Proposition 3.3, and the maps $\Delta, m$ are the maps $\Delta_{\alpha, \beta}, m_{\alpha, \beta}$.

Example 3.7. For diagrams and , the complex in (iii) takes the form

$$0 \rightarrow \mathbf{S} \xrightarrow{\Delta} \mathbf{S} \xrightarrow{m} \mathbf{S} \rightarrow 0.$$ (4)

Here one can check that the map $\Delta$ sends

$$\begin{bmatrix} a \\ b \\ c \\ d \end{bmatrix} \rightarrow \begin{bmatrix} a \\ b \\ c \\ d \end{bmatrix} - \begin{bmatrix} a \\ c \\ d \end{bmatrix} + \begin{bmatrix} b \\ c \\ d \end{bmatrix}.$$

Meanwhile, viewing the image of $m$ as lying in $S^{\text{rows}(D \otimes D')} = S^3(V) \otimes S^1(V)$, it sends

$$\begin{bmatrix} w \\ x \\ y \\ z \end{bmatrix} \rightarrow zxy \otimes w - zwy \otimes x.$$

Then their composite $m \circ \Delta$ is the zero map, as shown here:

$$\begin{bmatrix} a \\ b \\ c \\ d \end{bmatrix} \xrightarrow{m \circ \Delta} \begin{bmatrix} a \\ b \\ c \\ d \end{bmatrix} - m \begin{bmatrix} a \\ c \\ d \end{bmatrix} + m \begin{bmatrix} b \\ c \\ d \end{bmatrix} = c b d \otimes a - c a d \otimes b - (b c d \otimes a - b a d \otimes c) + a c d \otimes b - a b d \otimes c = 0.$$

However, one can check that in this case, the short complex (4) is not short exact, for example, by comparing their $GL(V)$-characters:

$$\text{ch}_{\mathbf{S}} = s_{(2, 1, 1)},$$

$$\text{ch}_{\mathbf{S}} = s_{(2, 1, 1)} + s_{(2, 2)} + s_{(3, 1)} + s_{(2, 2)},$$

$$\text{ch}_{\mathbf{S}} = s_{(3, 1)}.$$ 

The reader can also check that, replacing $D'$ above with the diagram in which one swaps the rows to make it a Ferrers diagram, the resulting complex would be short exact:

$$0 \rightarrow \mathbf{S} \xrightarrow{\Delta} \mathbf{S} \xrightarrow{m} \mathbf{S} \rightarrow 0.$$ 

Proof of Proposition 3.6. We prove each part separately.

Proof of (i). Let the last column of $D$ and first column of $D'$ contain $\ell, \ell'$ cells, respectively, so that they are merged to form a column with $\ell + \ell'$ cells in $D \cdot D'$. Then the factorization of the
antisymmetrization map
\[ \wedge^\ell (V) \to \wedge (V) \otimes T^\ell (V) = T^\ell (V) \]
leads to a factorization of the antisymmetrization map \( \mathcal{A}_{D \cdot D'} \) through \( \mathcal{A}_{D \oplus D'} \) as follows:
\[ \wedge^{\text{cols}(D \cdot D')}(V) \to \wedge^{\text{cols}(D \oplus D')}(V) \to T_{|D|+|D'|}(V). \]
As \( \mathcal{R}_{D \cdot D'} = \mathcal{R}_{D \oplus D'} \), this leads to an inclusion of the Schur functors
\[ S^{D \cdot D'} = \text{im}(\mathcal{R}_{D \cdot D'} \circ \mathcal{A}_{D \cdot D'}) \subset \text{im}(\mathcal{R}_{D \oplus D'} \circ \mathcal{A}_{D \oplus D'}) = S^{D \oplus D'}. \]

Proof of (ii). Similarly, let the top row of \( D \) and bottom row of \( D' \) contain \( r, r' \) cells, respectively, so that they are merged to form a row with \( r + r' \) cells in \( D \odot D' \). Then the factorization of the symmetrization map
\[ T^{r+r'}(V) = T^r(V) \otimes T^{r'}(V) \to S^r(V) \otimes S^{r'}(V) \to S^{r+r'}(V) \]
leads to a factorization of the symmetrization map \( \mathcal{R}_{D \odot D'} \) through \( \mathcal{R}_{D \oplus D'} \) as follows:
\[ T_{|D|+|D'|}(V) \to S^{\text{rows}(D \oplus D')}(V) \to S^{\text{rows}(D \odot D')}(V). \]
As \( \mathcal{A}_{D \odot D'} = \mathcal{A}_{D \oplus D'} \), this leads to a surjection of the Schur functors
\[ S^{D \oplus D'} = \text{im}(\mathcal{R}_{D \oplus D'} \circ \mathcal{A}_{D \oplus D'}) \to \text{im}(\mathcal{R}_{D \odot D'} \circ \mathcal{A}_{D \odot D'}) = S^{D \odot D'}. \]

Proof of (iii). We wish to show that when \( D, D' \) have northeastmost, southwestmost cells \( x, x' \) then the map \( m \circ \Delta: S^{D \cdot D'} \to S^{D \odot D'} \) is zero. Let \( N := |D| + |D'| \) denote the number of cells in any of the diagrams \( D \cdot D', D \oplus D', D \odot D' \). One can express \( m \circ \Delta \) as a composite
\[ \wedge^{\text{cols}(D \cdot D')}(V) \xrightarrow{\mathcal{A}_{D \cdot D'}} T^N(V) \xrightarrow{\mathcal{R}_{D \odot D'}} S^{\text{rows}(D \odot D')}(V). \]

as illustrated in the following example. Let
\[
\begin{align*}
D &= \begin{array}{cccc}
\cdot & \cdot & x & \cdot \\
\cdot & \cdot & \cdot & \cdot \\
\end{array} \\
D' &= \begin{array}{cccc}
\cdot & \cdot & \cdot & \cdot \\
\cdot & \cdot & \cdot & \cdot \\
\end{array}
\end{align*}
\]
so \( N = |D| + |D'| = 11 \), and use these consistent cell labelings in \( D \cdot D' \) and \( D \odot D' \), with \( x = 5, x' = 6 \) darkened, in bijection with the tensor positions in \( T^N(V) \):
\[
\begin{align*}
D \cdot D' &= \begin{array}{cccc}
9 & 7 & 11 & 10 \\
8 & 6 & 2 & 3 \\
1 & 4 & 5 & 6 \\
\end{array} \\
D \odot D' &= \begin{array}{cccc}
9 & 7 & 11 & 10 \\
8 & 6 & 2 & 3 \\
1 & 4 & 5 & 6 \\
\end{array}
\end{align*}
\]

Then the map \( \wedge^{\text{cols}(D \cdot D')}(V) \xrightarrow{\mathcal{A}_{D \cdot D'}} T^N(V) \) sends
\[
(u_1 \wedge u_2) \otimes u_3 \otimes (u_4 \wedge u_5 \wedge u_6 \wedge u_7) \otimes (u_8 \wedge u_9) \otimes (u_{10} \wedge u_{11}) \rightarrow (u_1 \otimes u_2 \otimes \cdots \otimes u_{10} \otimes u_{11}) \cdot \gamma_{D \cdot D'}^{-}
\]
where \( \gamma_{D,D'}^- \) is a particular column-antisymmetrizing element of the group algebra \( \mathbf{k}[\mathfrak{S}_n] \) acting (on the right) on the tensor positions in \( T^N(V) \). Specifically,

\[
\gamma_{D,D'}^- := \sum_w \text{sgn}(w) \cdot w
\]

summing over \( w \) in the group \( \mathfrak{S}_{\text{cols}(D \cdot D')} \) of column permutations of \( D \cdot D' \). In this example,

\[
\mathfrak{S}_{\text{cols}(D \cdot D')} = \mathfrak{S}_{\{1,2\}} \times \mathfrak{S}_{\{3\}} \times \mathfrak{S}_{\{4,5,6,7\}} \times \mathfrak{S}_{\{8,9\}} \times \mathfrak{S}_{\{10,11\}}.
\]

Meanwhile, the map \( T^N(V) \overset{\mathcal{B}_{D \cdot D'}}{\longrightarrow} S_{\text{rows}(D \cdot D')}(V) \) sends

\[
u_1 \otimes \nu_2 \otimes \cdots \otimes \nu_{10} \otimes \nu_{11} \longmapsto \nu_1 \nu_4 \otimes \nu_2 \nu_3 \nu_5 \nu_6 \nu_8 \nu_{10} \otimes \nu_7 \nu_9 \otimes \nu_{11}.
\]

Now note that \( x, x' \) lie in the same column of \( D \cdot D' \). Thus, in \( \mathbf{k}[\mathfrak{S}_n] \), one can factor

\[
\gamma_{D,D'}^- = \varepsilon \cdot (1 - t_{x,x'}),
\]

where \( t_{x,x'} \) is the transposition of the labels on the cells \( x, x' \) (so \( t_{5,6} = t_{x,x'} \) in the above example), and \( \varepsilon := \sum_w \text{sgn}(w) \cdot w \) where \( w \) ranges over any choice of coset representatives for the cosets \( \mathfrak{S}_{\text{cols}(D \cdot D')}/\langle t_{x,x'} \rangle \). Hence, the image of \( \mathcal{A}_{D \cdot D'} \) lies in \( T^N(V) \cdot (1 - t_{x,x'}) \).

On the other hand, as \( x, x' \) lie in the same row of \( D \odot D' \), one can check that \( \mathcal{B}_{D \odot D'} \) annihilates all of \( T^N(V) \cdot (1 - t_{x,x'}) \). Hence, the composite \( m \circ \Delta = \mathcal{B}_{D \odot D'} \circ \mathcal{A}_{D \cdot D'} = 0 \).

Proof of (iv). When \( D, D' \) are both skew shapes, so are all three of \( D \oplus D', D \cdot D', D \odot D' \), and hence all three Schur functors \( s_{D \oplus D'}, s_{D \cdot D'}, s_{D \odot D'} \) are universally free by [1, Theorem II.2.16]. Then Proposition 3.5 shows that for any coefficient ring \( \mathbf{k} \), the short complex in part (iii) is one of free \( \mathbf{k} \)-modules of the form \( 0 \rightarrow \mathbf{k}^a \rightarrow \mathbf{k}^{a+b} \rightarrow \mathbf{k}^b \rightarrow 0 \) for fixed integers \( a, b \) independent of \( \mathbf{k} \).

Consequently, whenever \( \mathbf{k} \) is a field, this complex is short exact by dimension-counting. We claim this implies it must also be exact when \( \mathbf{k} = \mathbb{Z} \): one has an inclusion of free \( \mathbb{Z} \)-modules \( \text{im}(\Delta) \subseteq \ker(m) \) inside \( \mathbb{Z}^{a+b} \), both of same rank \( a \), and if any prime \( p \) were to divide the index \( [\ker(m) : \text{im}(\Delta)] \), it would contradict exactness when \( \mathbf{k} = \mathbb{F}_p \).

Finally, once one knows that for \( \mathbf{k} = \mathbb{Z} \) the sequence is short exact, it must also split, as \( \mathbb{Z}^b \) is a projective \( \mathbb{Z} \)-module. Then applying \( (-) \otimes_{\mathbb{Z}} \mathbf{k} \) to this split short exact sequence of free \( \mathbb{Z} \)-modules shows that the sequence is split short exact over any coefficient ring \( \mathbf{k} \). \( \square \)

**Remark 3.8.** The short complexes and short exact sequences in parts (iii),(iv) of Proposition 3.6 are closely related to special cases of short complexes of Specht modules for \( \mathfrak{S}_n \) considered by James and Peel in [26]; see also Liu [23, section 2.2]. They are also dual to special cases of short complexes of Weyl modules for \( GL(V) \) considered by Shimozono in [31].

### 3.2 Resolving concatenations by near-concatenations

Note that both operations \( \alpha \cdot \beta \) and \( \alpha \odot \beta \) are associative, and that they associate with each other, meaning

\[
\alpha \cdot (\beta \odot \gamma) = (\alpha \cdot \beta) \odot \gamma,
\]

\[
\alpha \odot (\beta \cdot \gamma) = (\alpha \odot \beta) \cdot \gamma.
\]
Thus, one can write sequences of these operations unambiguously without parentheses. In particular, for any sequence \( \alpha = (\alpha^{(1)}, \ldots, \alpha^{(\ell)}) \) of compositions, one can define the compositions \( \alpha^{(1)} \cdot \alpha^{(2)} \cdots \alpha^{(\ell)} \) and \( \alpha^{(1)} \odot \alpha^{(2)} \odot \cdots \odot \alpha^{(\ell)} \).

We next note that one may recast the basic ribbon Schur function identity (2) in the form

\[
 s_{\alpha} \cdot s_{\beta} = s_{\alpha \odot \beta} = \det \begin{bmatrix} s_{\alpha} & s_{\alpha \odot \beta} \\ 1 & s_{\beta} \end{bmatrix}
\]

and regard it as the \( \ell' = 2 \) special case of the following family of identities.

**Proposition 3.9.** For any sequence of compositions \( \alpha = (\alpha^{(1)}, \ldots, \alpha^{(\ell)}) \), one has

\[
s_{\alpha^{(1)}, \alpha^{(2)}, \ldots, \alpha^{(\ell)}} = \det [H_{ij}]_{i,j=1}^{\ell} \quad \text{where} \quad H_{ij} := \begin{cases} 0 & \text{if } i \geq j + 2, \\ 1 & \text{if } i = j + 1, \\ s_{\alpha^{(i)} \odot (\alpha^{(i+1)} \odot \cdots \odot \alpha^{(j)})} & \text{if } i \leq j. \end{cases}
\]

**Example 3.10.** The determinantal identity in the proposition looks as follows for \( \ell = 3, 4 \):

\[
s_{\alpha \cdot \beta \cdot \gamma} = \det \begin{bmatrix} s_{\alpha} & s_{\alpha \odot \beta} & s_{\alpha \odot \beta \odot \gamma} \\ 1 & s_{\beta} & s_{\beta \odot \gamma} \\ 0 & 1 & s_{\gamma} \end{bmatrix}
\]

\[
= s_{\alpha} s_{\beta} s_{\gamma} - \left( \frac{s_{\alpha} s_{\beta} s_{\gamma}}{s_{\beta \odot \gamma}} + s_{\alpha \odot \beta \odot \gamma} \right) + s_{\alpha \odot \beta \odot \gamma}.
\]

\[
s_{\alpha \cdot \beta \cdot \gamma \cdot \delta} = \det \begin{bmatrix} s_{\alpha} & s_{\alpha \odot \beta} & s_{\alpha \odot \beta \odot \gamma} & s_{\alpha \odot \beta \odot \gamma \odot \delta} \\ 1 & s_{\beta} & s_{\beta \odot \gamma} & s_{\beta \odot \gamma \odot \delta} \\ 0 & 1 & s_{\gamma} & s_{\gamma \odot \delta} \\ 0 & 0 & 1 & s_{\delta} \end{bmatrix}
\]

\[
= s_{\alpha} s_{\beta} s_{\gamma} s_{\delta} - \left( \frac{s_{\alpha} s_{\beta} s_{\gamma} s_{\delta}}{s_{\beta \odot \gamma \delta}} + \frac{s_{\alpha \odot \beta \odot \gamma \delta}}{s_{\beta \odot \gamma \delta}} \right) + \frac{s_{\alpha \odot \beta \odot \gamma \delta}}{s_{\beta \odot \gamma \delta}} - s_{\alpha \odot \beta \odot \gamma \odot \delta}.
\]

**Proof.** Given \( \alpha = (\alpha^{(1)}, \ldots, \alpha^{(\ell)}) \), denote the matrix \( H \) in the proposition as \( H(\alpha) \), and define two sequences of compositions of length \( \ell - 1 \):

\[
\hat{\alpha} := (\alpha^{(2)}, \alpha^{(3)}, \ldots, \alpha^{(\ell)})
\]

\[
\alpha' := (\alpha^{(1)} \odot \alpha^{(2)}, \alpha^{(3)}, \ldots, \alpha^{(\ell)})
\]

The proposition follows by induction on \( \ell \), via Laplace expansion in the first column:

\[
\det H(\alpha) = s_{\alpha^{(1)}} \cdot \det H(\hat{\alpha}) - \det H(\alpha')
\]

\[
= s_{\alpha^{(1)}} \cdot \det [s_{\alpha^{(2)}, \alpha^{(3)}, \ldots, \alpha^{(\ell)}} - s_{\alpha^{(1)} \odot (\alpha^{(2)} \odot \cdots \odot \alpha^{(\ell)})}
\]

\[
= \left( s_{\alpha^{(1)}}, \alpha^{(2)}, \alpha^{(3)}, \ldots, \alpha^{(\ell)} \right) - s_{\alpha^{(1)} \odot (\alpha^{(2)} \odot \cdots \odot \alpha^{(\ell)})}
\]

\[
= s_{\alpha^{(1)}}, \alpha^{(2)}, \alpha^{(3)}, \ldots, \alpha^{(\ell)}.
\]

Here the equality marked (*) used identity (2).
Remark 3.11. Proposition 3.9 is a very special case of the Hamel–Goulden determinantal identity [16], which for any skew diagram $D$ expresses the skew Schur function $s_D$ as a determinant, based on any decomposition of $D$ into ribbon subdiagrams that they call a planar outside decomposition. Proposition 3.9 is the special case of the Hamel–Goulden identity in which $D = \alpha(1) \cdot \alpha(2) \cdots \alpha(\ell')$ and the planar outside decomposition of $D$ is into the individual ribbon subdiagrams $\alpha(i)$ for $i = 1, 2, \ldots, \ell'$ inside it.

We next seek to lift Proposition 3.9 to a homological result, Theorem 3.14, generalizing Proposition 3.3 in the case $\ell' = 2$, and using it as a base case for induction on $\ell'$. To this end, given any length $\ell'$ sequence $\alpha = (\alpha^{1}, \ldots, \alpha^{\ell'})$ of compositions, and for any choice of subset of indices $I \subseteq [\ell' - 1] := \{1, 2, \ldots, \ell' - 1\}$, let $\alpha(I)$ be the length $\ell' - |I|$ sequence of compositions that replaces the $i$th comma in $\alpha$ with the $\circ$ operation, for each $i$ in $I$.

Example 3.12. Let $\ell' = 5$ and $\alpha = (\alpha, \beta, \gamma, \delta, \epsilon)$. Then

$$
\alpha(\emptyset) = (\alpha, \beta, \gamma, \delta, \epsilon) = \alpha,
$$

$$
\alpha([3]) = (\alpha, \beta, \gamma \circ \delta, \epsilon)
$$

$$
\alpha([1, 3]) = (\alpha \circ \beta, \gamma \circ \delta, \epsilon)
$$

$$
\alpha([2, 3]) = (\alpha, \beta \circ \gamma \circ \delta, \epsilon)
$$

$$
\alpha([1, 2, 3, 4]) = (\alpha \circ \beta \circ \gamma \circ \delta \circ \epsilon)
$$

Using this notation, and also an abbreviation for $\alpha = (\alpha^{1}, \ldots, \alpha^{\ell'})$ of the product

$$
\mathbb{S}_{\alpha} := \mathbb{S}_{\alpha^{1}} \otimes_{k} \cdots \otimes_{k} \mathbb{S}_{\alpha^{\ell'}},
$$

one can rephrase the determinant expansion in Proposition 3.9 as follows:

$$
\det H = \sum_{I \subseteq [\ell' - 1]} (-1)^{|I|} s_{\alpha(I)} = \sum_{i=0}^{\ell'-1} (-1)^{i} \sum_{|I|=i} s_{\alpha(I)}.
$$

(8)

Definition 3.13. Given $\alpha = (\alpha^{1}, \ldots, \alpha^{\ell'})$, define

$$
\mathbb{S}_{\alpha} := \mathbb{S}_{\alpha^{(1)}} \otimes_{k} \cdots \otimes_{k} \mathbb{S}_{\alpha^{(\ell')}}.
$$

Create a cochain complex $(H(\alpha), \delta)$ whose $i$th term is

$$
H_i(\alpha) := \bigoplus_{|I|=i} \mathbb{S}_{\alpha(I)},
$$

where $I \subseteq [\ell' - 1]$. 
modeling the $i$th inner summand on the far right of (8). Define the differential

\[
\begin{array}{c}
H_i(\alpha) \\ \oplus \bigoplus_{I \subseteq \ell-1: \quad |I| = i} \mathbb{S}^\alpha(I)
\end{array} \xrightarrow{\delta_i} \begin{array}{c}
H_{i+1}(\alpha) \\ \oplus \bigoplus_{J \subseteq \ell-1: \quad |J| = i+1} \mathbb{S}^\alpha(J)
\end{array}
\]

in block matrix form as follows. The $(I, J)$-block of $\delta_i$ is zero unless $J \supset I$. If $J \supset I$, define $\text{sgn}(I, J) := (-1)^m$ where $J = \{j_0 < j_1 < \cdots < j_i\}$ and $J = I \cup \{j_m\}$. Then define the $(I, J)$-block of $\delta_i$ to be $\text{sgn}(I, J) \cdot \nabla^\alpha_J \otimes^{\alpha(I)}$, where

\[
\nabla^\alpha_J : \mathbb{S}^\alpha(I) \rightarrow \mathbb{S}^\alpha(J)
\]

is the identity in most tensor factors, except that it is the surjective map $m_{\alpha(j_m), \alpha(j_{m+1})}$ from Proposition 3.3 sending the two tensor factors of $\mathbb{S}^\alpha(I)$ that involve $\alpha(j_m), \alpha(j_{m+1})$ onto the unique tensor factor of $\mathbb{S}^\alpha(J)$ involving $\alpha(j_m) \odot \alpha(j_{m+1})$.

The following homological lift of Proposition 3.9 or (8) is the main result of this section.

**Theorem 3.14.** For any sequence $\alpha$, one has that $(H(\alpha), \delta)$ is a cochain complex, with

\[
H^0(H(\alpha)) \cong \mathbb{S}^{\alpha(1)} \otimes \mathbb{S}^{\alpha(2)} \otimes \cdots \otimes \mathbb{S}^{\alpha(\ell)}
\]

and which is acyclic in strictly positive homological degrees.

In other words, $H(\alpha)$ gives a (right or co-)resolution of the $GL(V)$-module $\mathbb{S}^{\alpha(1)} \otimes \mathbb{S}^{\alpha(2)} \otimes \cdots \otimes \mathbb{S}^{\alpha(\ell)}$:

\[
0 \rightarrow \mathbb{S}^{\alpha(1)} \otimes \mathbb{S}^{\alpha(2)} \otimes \cdots \otimes \mathbb{S}^{\alpha(\ell)} \rightarrow H(\alpha) \rightarrow 0.
\]

**Example 3.15.** With notation $\otimes_k = : \otimes$, the $n = 3$ lifting of (6) looks as follows:

\[
0 \rightarrow \mathbb{S}^{\sigma(\alpha, \beta, \gamma)} \rightarrow \mathbb{S}^{\sigma(\alpha)} \otimes \mathbb{S}^{\sigma(\beta)} \otimes \mathbb{S}^{\sigma(\gamma)} \xrightarrow{\bigoplus \begin{bmatrix} V^{[1]} \otimes V^{[2]} \otimes V^{[1,2]} \end{bmatrix}} \mathbb{S}^{\sigma(\alpha \odot \beta \odot \gamma)} \rightarrow 0
\]

(9)

**Proof of Theorem 3.14.** To see that $H(\alpha)$ is a complex, the definition of $\text{sgn}(I, J)$ reduces to checking that for any $j, j' \notin I$ one has commutativity of the appropriate $V$ maps in this diamond:
This commutativity is easy when $|j' - j| \geq 2$. For $|j' - j| = 1$, it follows from commutativity of the appropriate surjections $m_{\alpha, \beta}$ from Proposition 3.3 appearing here:

\[ S^\sigma(\alpha) \otimes_k S^\sigma(\beta) \otimes_k S^\sigma(\gamma) \]

\[ S^\sigma(\alpha \otimes \beta) \otimes_k S^\sigma(\gamma) \]

\[ S^\sigma(\alpha \otimes \beta \otimes \gamma) \]

(10)

For the remaining assertions, proceed by induction on $\ell$, with base case given by the case $\ell = 2$, which one can check is Proposition 3.3. In the inductive step, recall from the proof of Proposition 3.9 these shorter sequences of compositions

\[ \hat{\alpha} := (\alpha(2), \alpha(3), \ldots, \alpha(\ell)) \]

\[ \alpha' := (\alpha(1) \otimes \alpha(2), \alpha(3), \ldots, \alpha(\ell)) \]

whose associated matrices $H(\alpha)$ satisfied an identity, coming from Laplace expansion:

\[ \det H(\alpha) = s_{\alpha(1)} \cdot \det H(\hat{\alpha}) - \det H(\alpha'). \]

We next lift this relation homologically. Define a morphism of complexes

\[ \Phi : S^\sigma(\alpha(1)) \otimes_k H(\hat{\alpha} \to H(\alpha')) \]

by having it map between their terms indexed by $I \subseteq [\ell - 2]$ as follows:

\[ \nabla_{\alpha'(I)} : S^\sigma(\alpha(1)) \otimes_k S^{\hat{\alpha}(I)} \to S^{\alpha'(I)}. \]

Checking $\Phi$ is a morphism of complexes means checking for $j \notin I$ that this commutes:

\[ S^\sigma(\alpha(1)) \otimes_k S^{\hat{\alpha}(I)} \xrightarrow{\nabla} S^{\alpha'(I)} \]

\[ S^\sigma(\alpha(1)) \otimes_k S^{\hat{\alpha}(I,j)} \xrightarrow{\nabla} S^{\alpha'(I,j)}. \]

This is easy for $j \geq 2$, and for $j = 1$ is commutativity of (10) with $(\alpha, \beta, \gamma) = (\alpha(1), \alpha(2), \alpha(3))$. By construction one has an isomorphism between $H(\alpha)$ and the mapping cone of $\Phi$:

\[ \text{cone}(\Phi) \cong H(\alpha). \]

Hence, there is a short exact sequence of cochain complexes

\[ 0 \to H(\alpha')[-1] \to \text{cone}(\Phi) \to S^\sigma(\alpha(1))(V) \otimes_k H(\hat{\alpha}) \to 0 \]

\[ \| \quad \| \quad \| \]

\[ A \quad B \quad C \]
leading to a long exact sequence in cohomology:

\[ 0 \to H^0A \to H^0B \to H^0C \to H^1A \to H^1B \to H^1C \to \cdots \]  \hspace{1cm} (11)

Applying the inductive hypothesis for \( H(\alpha') \), \( H(\hat{\alpha}) \) and Proposition 2.6 to the subsequences

\[
\begin{array}{ccc}
H^iA & \to & H^iB \\
& \parallel & \parallel \\
H^iH(\alpha') & \to & H^iH(\alpha) \oplus_k H^iH(\hat{\alpha}) \\
& \parallel & \parallel \\
0 & \to & \mathbb{S}^{\alpha(1)} \oplus_k H^iH(\hat{\alpha}) \\
\end{array}
\]

shows that \( H^iH(\alpha) = 0 \) for \( i \geq 2 \). On the other hand, the outer terms in (11) also vanish:

\[
H^0A = H^1H(\alpha') = 0,
\]

\[
H^1C = H^1(\mathbb{S}^{\alpha(1)} \oplus_k H(\hat{\alpha})) = \mathbb{S}^{\alpha(1)} \oplus_k H^1H(\hat{\alpha}) = 0,
\]

where the second vanishing again used the inductive hypothesis for \( H(\hat{\alpha}) \) and Proposition 2.6. Thus, the long exact sequence starts

\[
\begin{array}{cccccc}
0 & \to & H^0B & \to & H^0C & \to & H^1A \\
& & \parallel & \parallel & \parallel & \parallel \\
0 & \to & H^0H(\alpha) & \to & H^0(\mathbb{S}^{\alpha(1)} \oplus_k H(\hat{\alpha})) & \to & H^1H(\alpha') \\
& & \parallel & \parallel & \parallel & \parallel \\
& & \mathbb{S}^{\alpha(1)} \oplus_k \mathbb{S}^{\alpha(2)} \cdots \mathbb{S}^{\alpha(\ell')} & \xrightarrow{f} & \mathbb{S}^{\alpha(1) \odot \alpha(2) \cdots \alpha(\ell')} & \to & 0
\end{array}
\]

where the bottommost equalities used the inductive hypothesis on \( H^0 \) applied to \( H(\alpha'), H(\hat{\alpha}) \).

One can then check that the map labeled \( f \) above, which is a connecting homomorphism in the long exact sequence, actually coincides with the surjective map \( m_{\alpha, \beta} \) in this instance of the short exact sequence of Proposition 3.3 with \( \alpha = \alpha^{(1)} \) and \( \beta = \alpha^{(2)} \cdot \alpha^{(3)} \cdots \alpha^{(\ell')} \):

\[
0 \to \mathbb{S}^{\sigma(\alpha^{(1)}, \cdots, \alpha^{(\ell')})} \xrightarrow{\Delta_{\alpha, \beta}} \mathbb{S}^{\alpha(1)} \oplus_k \mathbb{S}^{\sigma(\alpha^{(2)}, \cdots, \alpha^{(\ell')})} \xrightarrow{m_{\alpha, \beta}} \mathbb{S}^{\sigma(1) \odot \sigma(2) \cdots \sigma(\ell')} \to 0.
\]

This gives the last two desired conclusions of the theorem:

\[
\begin{align*}
H^1H(\alpha) &= \ker(f) \cong \ker(m) \cong 0, \\
H^0H(\alpha) &= \coker(f) \cong \coker(m) \cong \mathbb{S}^{\sigma(\alpha^{(1)}, \cdots, \alpha^{(\ell')})}.
\end{align*}
\]

The following result arises from studying the \( \ell' = 3 \) case of the complex introduced in Definition 3.13. The utility of the following proposition will become apparent in the Tor computations in Subsections 4.2 and 5.2.
Proposition 3.16. For any three compositions \((\alpha, \beta, \gamma)\) one has an equality
\[
\mathbb{S}^{\sigma(\alpha, \beta, \gamma)} = \left( \mathbb{S}^{\sigma(\alpha, \beta)} \otimes_k \mathbb{S}^{\sigma(\gamma)} \right) \cap \left( \mathbb{S}^{\sigma(\alpha)} \otimes_k \mathbb{S}^{\sigma(\beta, \gamma)} \right),
\]
where the intersection is taken inside
\[
\mathbb{S}^{\sigma(\alpha)} \otimes_k \mathbb{S}^{\sigma(\beta)} \otimes_k \mathbb{S}^{\sigma(\gamma)}.
\]

Proof. Exactness in the left and middle terms of (9) shows that
\[
\mathbb{S}^{\sigma(\alpha, \beta, \gamma)} = \ker(\nabla_1^\emptyset) \cap \ker(\nabla_2^\emptyset).
\]
Then Proposition 2.6 together with exactness on the left in Proposition 3.3 identifies the kernels of \(\nabla_1^\emptyset, \nabla_2^\emptyset\) with the terms intersected on the right in the proposition. \(\square\)

4 | THE COMPLEX OF RIBBONS AND PROOF OF THEOREM 1.1

The goal of this section is to produce an explicit \(GL(V)\)-equivariant minimal \(R\)-free resolution of \(M\), where \(R = S^{(d)}\) and \(M = S^{(r, d)}\), with form as predicted by Theorem 1.1.

4.1 | The complex of ribbons

The complex of ribbons was mentioned in the Introduction. It has a differential induced from a simple tensor-degree-lowering map (which is not itself a differential) on a tensor algebra.

Specifically, define a tensor-degree-lowering \(S\)-linear endomorphism \(\partial\) on
\[
S \otimes_k T_k(S) := \bigoplus_{\ell=0}^{\infty} S \otimes_k T_k^{\ell}(S)
\]
via this formula:
\[
S \otimes_k T_k(S) \xrightarrow{\partial} S \otimes_k T_k(S)
\]
\[
s \otimes_k (s_1 \otimes_k s_2 \otimes_k s_3 \otimes_k \cdots \otimes_k s_r) \mapsto s \cdot s_1 \otimes_k (s_2 \otimes_k s_3 \otimes_k \cdots \otimes_k s_r).
\]

The definition in Subsection 2.2 of \(S^{\sigma(\alpha)}\), gives an inclusion of (free) \(k\)-modules
\[
S^{\sigma(\alpha)} \subseteq S^{\sigma_1} \otimes \cdots \otimes S^{\sigma_r} \subseteq T_k^{\ell}(S)
\]
and hence also \(S\)-module inclusions
\[
S \otimes_k S^{\sigma(\alpha)} \subseteq S \otimes_k (S^{\sigma_1} \otimes \cdots \otimes S^{\sigma_r}) \subseteq S \otimes T_k^{\ell}(S).
\]

Definition 4.1. Define for each \(\ell \geq 1\) an \(S\)-submodule \(\mathcal{R}_\ell\) of \(S \otimes T_k^{\ell}(S)\) by
\[
\mathcal{R}_\ell := S \otimes_k \left( \bigoplus_{\alpha} S^{\sigma(\alpha)} \right).
\]
where the direct sum is over compositions $\alpha = (\alpha_1, \ldots, \alpha_\ell)$ of length $\ell$. Compile them as

$$\mathcal{R}_* : = \bigoplus_{\ell = 1}^{\infty} \mathcal{R}_\ell.$$  

**Proposition 4.2.** The $S$-module endomorphism $\partial$ on $S \otimes T_k(S)$ has the following properties.

(i) It restricts to an $S$-module endomorphism of $\mathcal{R}_*$.

(ii) Upon restriction to $\mathcal{R}_*$, it satisfies $\partial^2 = 0$.

For this reason, we will call $(\mathcal{R}_*, \partial)$ the complex of ribbons.

**Proof.** To prove (i), for $\alpha = (\alpha_1, \alpha_2, \ldots, \alpha_\ell)$, let $\hat{\alpha} := (\alpha_2, \ldots, \alpha_\ell)$, and note $\partial$ gives a map

$$S^{\sigma(\alpha)} \cong S^0 \otimes_k S^{\sigma(\alpha)} \xrightarrow{\partial} S^{\alpha_1} \otimes_k S^{\sigma(\hat{\alpha})}$$

which is the same as the map $\Delta$ from a special case of Proposition 3.3. Consequently, by $S$-linearity, $\partial$ maps $S \otimes_k S^{\sigma(\alpha)} \to S \otimes_k S^{\sigma(\hat{\alpha})}$, so that it restricts to $\mathcal{R}_*$.

To prove (ii), introduce $\hat{\hat{\alpha}} := (\alpha_3, \alpha_4, \ldots, \alpha_\ell)$, and note that, by $S$-linearity, it suffices to check that this restricted composite $\partial^2 = 0$:

$$S^{\sigma(\alpha)} \cong S^0 \otimes_k S^{\sigma(\alpha)} \xrightarrow{\partial} S^{\alpha_1} \otimes_k S^{\sigma(\hat{\alpha})} \xrightarrow{\partial} S^{\alpha_1 + \alpha_2} \otimes_k S^{\sigma(\hat{\hat{\alpha}})}.$$

We claim $\partial^2 = 0$ here, because one can check $\partial^2$ coincides with the composite of two maps $\nabla[1]_\emptyset \circ i$ inside a particular instance of the exact sequence (9) where $\alpha := ((\alpha_1), (\alpha_2), \hat{\alpha})$:

$$S^{\sigma(\alpha)} \xrightarrow{i} S^{\alpha_1} \otimes_k S^{\alpha_2} \otimes_k S^{\sigma(\hat{\alpha})} \xrightarrow{\nabla[1]_\emptyset \circ i} S^{\alpha_1 + \alpha_2} \otimes_k S^{\sigma(\hat{\hat{\alpha}})} \oplus S^{\alpha_1} \otimes_k S^{\sigma((\alpha_2) \circ \hat{\alpha})}$$

Exactness here implies vanishing of both composites $\nabla[1]_\emptyset \circ i = 0 = \nabla[2]_\emptyset \circ i$, so $\partial^2 = 0$. \qed

The next lemma analyzes some of the cycles and boundaries within certain summands of the complex of ribbons $(\mathcal{R}_*, \partial)$, and forms a key step in many of our later results. To state it, recall that for a composition $\alpha = (\alpha_1, \alpha_2, \ldots, \alpha_\ell)$, letting $\hat{\alpha} := (\alpha_2, \ldots, \alpha_\ell)$ as usual, one can restrict $\partial$ in the complex of ribbons $\mathcal{R}_*$ to a map

$$\partial^\alpha : S \otimes_k S^{\sigma(\alpha)} \to S \otimes_k S^{\sigma(\hat{\alpha})}.$$

The map $\partial$ and this restriction $\partial^\alpha$ are homogeneous with respect to the usual grading on $\mathcal{R}_*$ inherited from the grading on $S \otimes_k T_k(S)$, where $S^p \otimes_k S^{\sigma(\alpha)}$ lies within the homogeneous component of degree $p + |\alpha|$. The corresponding homogeneous component of the map $\partial^\alpha$ is therefore a map

$$(\partial^\alpha)_p|_{\alpha} : S^p \otimes_k S^{\sigma(\alpha)} \to S^{p+\alpha_1} \otimes_k S^{\sigma(\hat{\alpha})}.$$

**Lemma 4.3.** Let $p > 0$ be any positive integer.
(i) One has a $GL(V)$-isomorphism

$$\text{ker}(\partial^\alpha)_{p+|\alpha|} \cong S^{\sigma(p,\alpha)}.$$ 

(ii) Moreover, one has equalities

$$\text{ker}(\partial^\alpha)_{p+|\alpha|} = \text{im}(\partial^{(q,\alpha)})_{p+|\alpha|} \quad \text{for all } 0 < q \leq p.$$ 

Proof. To prove (i), start by factoring $(\partial^\alpha)_{p+|\alpha|} = \pi \circ i$ as the following composite

$$S^p \otimes_k S^{\sigma(\alpha)} \overset{i}{\rightarrow} S^p \otimes_k S^{\sigma(\alpha_1)} \otimes_k S^{\sigma(\hat{\alpha})} \overset{\pi}{\rightarrow} S^{p+\alpha_1} \otimes_k S^{\sigma(\hat{\alpha})},$$

where $i$ is injective and $\pi$ is surjective, using Proposition 2.6 together with these facts:

- $i = 1 \otimes_k \Delta(\alpha_1,\hat{\alpha})$ where $\Delta(\alpha_1,\hat{\alpha})$ is an instance of the injection in Proposition 3.3,
- $\pi = m \otimes_k 1$ where $m : S^p \otimes S^{\alpha_1} \rightarrow S^{p+\alpha_1}$ is multiplication in $S = S(V)$.

Therefore, one has that

$$\text{ker}(\partial^\alpha)_{p+|\alpha|} = \text{im}(i) \cap \text{ker}(\pi)$$

$$= i(S^p \otimes_k S^{\sigma(\alpha)}) \cap (S^{\sigma(p,\alpha_1)} \otimes_k S^{\sigma(\hat{\alpha})})$$

$$\cong S^{\sigma(p,\alpha_1,\hat{\alpha})}$$

$$= S^{\sigma(p,\alpha)},$$

where the identification $\text{ker}(\pi) = S^{\sigma(p,\alpha_1,\hat{\alpha})} \otimes_k S^{\sigma(\hat{\alpha})}$ uses injectivity of the map $\Delta(p,\alpha_1,\hat{\alpha})$ in Proposition 3.3 (along with Proposition 2.6), and the isomorphism of the intersection with $S^{\sigma(p,\alpha_1,\hat{\alpha})}$ applies Proposition 3.16 to the three compositions $(p), (\alpha_1), \hat{\alpha}$.

Assertion (ii) will then follow from assertion (i), together with the commutativity of this diagram for $0 < q \leq p$:

$$S^{p-q} \otimes_k S^{\sigma(q,\alpha)} \cong (\partial^{(q,\alpha)})_{p+|\alpha|} \rightarrow S^p \otimes_k S^{\sigma(\alpha)} \cong S^p \otimes_k S^{\sigma(\alpha_1)} \otimes_k S^{\sigma(\hat{\alpha})}$$

Surjectivity of the left vertical map follows because it is the surjection $m_{(p-q),(q,\alpha)}$ in Proposition 3.3, noting that the compositions $(p-q)$ and $(q,\alpha)$ have $(p-q) \odot (q,\alpha) = (p,\alpha).$ $\square$

4.2 Resolution of the Veronese modules

In this section, fix $d, r \geq 1$, and the Veronese ring $R = S^{(d)}$ with the $R$-module $M = S^{(2r.d)}$. We will show that a minimal $R$-free resolution of $M$ is obtained by restricting to a uniform class of ribbons in the ribbon complex $\mathcal{R}_r$. 

As $M$ is minimally generated as an $R$-module by the $r$th homogeneous component $S_r = S^r(V) = \mathbb{S}^\sigma(r)$ of $S$, one can start an $R$-free resolution of $M$ with the surjection

$$\partial_0 : R \otimes_k \mathbb{S}^\sigma(r) = R \otimes_k S_r \rightarrow M$$

given by multiplication within $S$. Note that to make $\partial_0$ homogeneous, one should shift the degree of the $R$-basis elements for its source into degree $r$, making it $R \otimes_k \mathbb{S}^\sigma(r)(-r)$. The following is then a precise version of Theorem 1.1 from the Introduction.

**Theorem 4.4.** The map $\partial_0$ starts a $GL(V)$-equivariant minimal $R$-free resolution of $M$

$$R \otimes_k \mathbb{S}^\sigma(r) \xrightarrow{\partial_1} R \otimes_k \mathbb{S}^\sigma(d,r) \xrightarrow{\partial_2} \cdots \xrightarrow{\partial_i} R \otimes_k \mathbb{S}^\sigma(d_i,r) \xleftarrow{\partial_{i+1}} \cdots$$

obtained by restricting the base ring in the complex of ribbons $(\mathcal{R}_*, d_\varphi)$ from $S$-modules to $R$-modules, and then taking only the summands indexed by compositions of the form $\alpha = (d^i, r)$ for $i = 0, 1, 2, \ldots$. Here the free $R$-module in homological degree $i$ has its basis elements in degree $d_i + r$, so it is actually $R \otimes_k \mathbb{S}^\sigma(d^i,r)(-(d_i + r))$.

**Proof.** As $\partial^2 = 0$, it remains to show $\ker(\partial_i) = \text{im}(\partial_{i+1})$. Using the $\mathbb{N}$-grading on $S$ inherited by $R$ and $M$, one need only check this equality on each homogeneous component, that is, $\ker(\partial_i)_{d(i+m)+r} = \text{im}(\partial_{i+1})_{d(i+m)+r}$ for all $m \geq 0$. Notice that if $m = 0$, then $\ker(\partial_i)_{d(i+m)+r} = 0$, as $(\partial_i)_{d(i+m)+r}$ is just the natural inclusion

$$\mathbb{S}^\sigma(d^i,r) \rightarrow S^d \otimes_k \mathbb{S}^\sigma(d^i-1,r).$$

When $m > 0$, applying Lemma 4.3 with $\alpha = (d^i, r)$, $p = dm$ and $q = d$ gives the equality:

$$\ker(\partial(d^i,r))_{d(i+m)+r} = \text{im}(\partial(d,d^i,r))_{d(i+m)+r} = \text{im}(\partial(d+1,r))_{d(i+m)+r} \quad \parallel \quad \ker(\partial(i+1))_{d(i+m)+r} = \text{im}(\partial_{i+1})_{d(i+m)+r}. \quad \square$$

**Example 4.5.** Take $d = 3$ and $r = 2$, so that $R = S^{(3)}$ and $M = S^{(\geq 2,3)}$. Then the $R$-free resolution of $M$ has the form

$$R \otimes_k \mathbb{S}^\sigma \xleftarrow{\partial_1} R \otimes_k \mathbb{S} \xleftarrow{\partial_2} R \otimes_k \mathbb{S} \xleftarrow{\partial_3} \cdots.$$
4.3 Tor and Koszulity

As mentioned in the Introduction, Theorem 1.1 has a corollary.

**Corollary 1.2.** In the setting of Theorem 1.1, $\text{Tor}_i^R(M, k)_j$ vanishes for $j \neq d_i + r$, and

$$\text{Tor}_i^R(M, k)_{d_i+r} \cong \mathcal{S}^{(d_i,r)},$$

as a polynomial $GL(V)$-representation.

In particular, the resolution of Theorem 1.1 is pure, in the sense that its $i$th resolvent is a free $R$-module whose basis elements lie in degree $d_i + r$.

This corollary has consequences for $R$ and $M$ as Koszul algebras and modules. We recall here the definitions of these concepts, and a few of their basic properties. Let $A$ be an associative graded connected $k$-algebra, so $A = \bigoplus_{m=0}^{\infty} A_m$ with $A_0 = k$ and $A_i A_j \subseteq A_{i+j}$. Say that $A$ is standard graded if it is generated as a $k$-algebra by $A_1$.

**Definition 4.6.** In the above setting, one calls $A$ a Koszul algebra if the quotient $k = A / A_+$ has a linear free $A$-resolution, that is, one of the form

$$0 \leftarrow k \leftarrow A \leftarrow A(-1)^{\beta_1} \leftarrow A(-2)^{\beta_2} \leftarrow \ldots.$$ 

Equivalently, $\text{Tor}_i^A(k, k)_j = 0$ unless $j = i$.

Given a graded $A$-module $M$, one says that $M$ is a Koszul module over $A$ if it is generated in degree 0 and has a linear free $A$-resolution, that is, of the form

$$0 \leftarrow M \leftarrow A^{\beta_0(M)} \leftarrow A(-1)^{\beta_1(M)} \leftarrow A(-2)^{\beta_2(M)} \leftarrow \ldots.$$ 

Equivalently, $\text{Tor}_i^A(M, k)_j = 0$ unless $j = i$.

The following basic facts appear, for example, in Positselski and Polishchuk\(^\dagger\) [28].

**Proposition 4.7.** Let $A$ be a Koszul algebra.

(a) [28, chapter 3, Proposition 2.2] Veronese subalgebras of $A$ are also Koszul.

(b) [28, chapter 2, Proposition 1.1] Let $M$ be a Koszul module over $A$. Then for every $q \geq 0$, the $q$th truncated module $M^{[q]}$ defined by

$$M^{[q]}_n = \begin{cases} M_{q+n} & \text{for } n \geq 0, \\ 0 & \text{for } n < 0, \end{cases}$$

is also a Koszul module over $A$.

Taking $r = d$, one sees that Corollary 1.2 gives an alternate proof that $R = S^{(d)}$ is a Koszul algebra, after rescaling the grading so that its algebra generators lie in degree 1; this was originally proven by Barcanescu and Manolache [8].

\(^\dagger\) Actually, in [28] the authors assume $k$ is a field, although their methods do not require it. For Koszulity definitions and results over more general rings $k$, see Beilinson, Ginzburg, and Soergel [3].
On the other hand, taking $r$ arbitrary, they also prove that each module $M = S^{(\geq r, d)}$ is a Koszul $R$-module, after shifting its grading so that the $R$-generators of $M$ lie in degree 0. This Koszulity of $S^{(\geq r, d)}$ as an $R$-module was proven for $0 \leq r \leq d - 1$ by Aramova, Barcanescu, and Herzog [2]. As an alternative to using Corollary 1.2, one could deduce Koszulity of $S^{(\geq r, d)}$ for arbitrary $r \geq 1$ from the known $0 \leq r \leq d - 1$ case as follows. Write $r = q \cdot d + \hat{r}$ where $0 \leq \hat{r} \leq d - 1$, and then apply Proposition 4.7 with $A = R = S^{(d)}$, and $M = S^{(\geq \hat{r}, d)}$. This shows that the truncated module $M[q] = S^{(\geq r, d)}$ is Koszul.

### 4.4 A symmetric function identity

Exactness of the resolution in Theorem 1.1 implies an identity of $GL(V)$-characters

$$
\text{ch}(R) \cdot \sum_{i=0}^{\infty} (-1)^i \text{ch}(\text{Tor}_i^R(M, k)) = \text{ch}(M).
$$

Since as symmetric functions, one has

$$
\text{ch}(R) = 1 + h_d + h_{2d} + h_{3d} + \cdots,
$$

$$
\text{ch}(M) = h_r + h_{d+r} + h_{2d+r} + \cdots,
$$

$$
\text{ch}(\text{Tor}_i^R(M, k)) = s_{\sigma(i, r)},
$$

this becomes the following symmetric function identity:

$$
h_r + h_{d+r} + h_{2d+r} + \cdots = (1 + h_d + h_{2d} + h_{3d} + \cdots) \cdot \sum_{i=0}^{\infty} (-1)^i s_{\sigma(i, r)}. \tag{12}
$$

In fact, if one gives an independent proof of the symmetric function identity (12), this leads in the case where $k$ is a field of characteristic zero, to an alternate proof of Corollary 1.2:

- first appeal to the Koszulity result for $M$ proven as in Subsection 4.3, then
- use this to deduce that (12) determines each of the $GL(V)$-characters $\text{ch}(\text{Tor}_i^R(M, k))$,
- which then determine each $\text{Tor}_i^R(M, k)$ uniquely as a $GL(V)$-representation.

We therefore explain here briefly how (12) connects to known symmetric function identities. For each $m = 0, 1, 2, \ldots$, extracting the homogeneous component of degree $md + r$ shows that it is equivalent to the identity $h_{md+r} = \sum_{i=0}^{m} (-1)^i h_{d(m−i)} \cdot s_{\sigma(i, r)}$. This can be rewritten, isolating the last term $s_{\sigma(d^m, r)}$ in the sum, as

$$
s_{\sigma(d^m, r)} = +h_d \cdot s_{\sigma(d^{m−1}, r)} - h_{2d} \cdot s_{\sigma(d^{m−2}, r)} + h_{3d} \cdot s_{\sigma(d^{m−3}, r)} - \cdots
$$

$$
+ (-1)^{m−2} h_{(m−1)d} \cdot s_{\sigma(d^{1}, r)} + (-1)^{m−1} h_{md} \cdot s_{\sigma(r)} + (-1)^m h_{md+r} \tag{13}
$$

We claim that this identity (13) is a consequence of the Jacobi–Trudi formula [24, (5.4); 34, section 7.16] expressing $s_{\sigma(d^m, r)}$ as a determinant in $h_n$’s. For a general skew shape $\lambda/\mu$, one has $s_{\lambda/\mu} = \det[h_{\lambda_i−\mu_j−i+j}]_{i,j=1,2,\ldots,\ell(\lambda)}$. For the special case of the ribbon shape $\sigma(d^m, r)$, this takes the
following form:

\[
\begin{vmatrix}
    h_r & h_{d+r} & h_{2d+r} & h_{3d+r} & \cdots & h_{md+r} \\
    1 & h_d & h_{2d} & h_{3d} & \cdots & h_md \\
    0 & 1 & h_d & h_{2d} & \cdots & h_{(m-1)d} \\
    0 & 0 & 1 & h_d & \cdots & h_{(m-2)d} \\
    \vdots & \vdots & \ddots & \vdots & \ddots & \vdots \\
    0 & 0 & \cdots & 0 & 1 & h_d
\end{vmatrix}
\]

The identity (13) is simply the above determinant expanded along its last column.

### 4.5 A curious symmetry

When \( r = 1 \) and \( d = 2 \), then \( R = S(2) \) and \( M = S(\geq 1,2) \) are the polynomials of even degree and odd degree in \( S = k[x_1, \ldots, x_n] \), respectively. In this case, Theorem 1.1 shows that \( M \) has its minimal \( R \)-free resolution of the form

\[
0 \leftarrow M \leftarrow R \otimes S^d \leftarrow R \otimes S\mathbb{P} \leftarrow R \otimes S\mathbb{P} \leftarrow R \otimes S\mathbb{P} \leftarrow \ldots
\]

involving only ribbon skew Schur functors indexed by skew shapes that are invariant under transposition, that is, their associated skew Schur functions are stable under the fundamental involution \( \Lambda \mapsto \Lambda \) swapping \( h_n \leftrightarrow e_n \) and \( s_\lambda \leftrightarrow s_{\lambda'} \); see Stanley [34, sections 7.6, 7.15]. In this case, one could rewrite (12) as follows:

\[
\sum_{i=0}^{\infty} (-1)^i \text{ch}(\text{Tor}_i^{S(3)}(S(\geq 1,2), k)) = \frac{h_1 + h_3 + h_5 + \cdots}{1 + h_2 + h_4 + h_6 + \cdots} = s_\Box - s_{\mathbb{P}} + s_{\mathbb{P}} - s_{\mathbb{P} \mathbb{P}} + \cdots
\]

There is an a priori reason involving symmetric functions for why this transpose-invariance occurs, as follows. Recall that

\[
H(t) := 1 + h_1 t + h_2 t^2 + \cdots \\
E(t) := 1 + e_1 t + e_2 t^2 + \cdots
\]

satisfy \( H(t)E(-t) = 1 \). Also \( \omega \) swaps \( H(t) \leftrightarrow E(t) \) because it swaps \( h_n \leftrightarrow e_n \). Now note

\[
\frac{h_1 + h_3 + h_5 + \cdots}{1 + h_2 + h_4 + h_6 + \cdots} = \frac{1}{2}(H(1) - H(-1)) = \frac{1}{2}(H(-1) - H(1)) = \frac{1}{2}H(1) \cdot H(-1) = \frac{1}{2}H(-1) \cdot H(1) = 1 - H(-1) \cdot E(-1)
\]

which is stable under \( \omega \).

**Question 4.8.** Is there a more conceptual explanation for this \( \omega \)-stability of \( \text{Tor}_i^{S(2)}(S(\geq 1,2), k) \)?
4.6 | Poset homology

The rings $S = \mathbb{k}[x_1, \ldots, x_n], R = S(d)$ and $R$-modules $M = S^{(\geq r, d)}$ are examples of a known set-up involving affine semigroups, where one can re-interpret $\text{Tor}_i^R(M, \mathbb{k})$ in terms of homology of certain partially ordered sets (posets). We recall this set-up here, which in general has three players.

- An ambient affine semigroup ring $S$ over the scalar ring $\mathbb{k}$, with $\mathbb{k}$-basis the semigroup elements $\{\sigma\}$ (this applies to $S = \mathbb{k}[x_1, \ldots, x_n]$ whose semigroup is $\mathbb{N}^n$).
- A $\mathbb{k}$-subalgebra $R$ of $S$ which is the affine semigroup ring for a subsemigroup with $\mathbb{k}$-basis elements the semigroup elements $\{\rho\}$ (this applies to $R = S(d)$).
- An $R$-submodule $M$ of $S$ coming from a semigroup $R$-submodule, with $\mathbb{k}$-basis elements the semigroup elements $\{\mu\}$ (this applies to $M = S^{(\geq r, d)}$).

In this context, define a partial order $\leq_R$ on $M$ by $\mu_1 <_R \mu_2$ if $\mu_1 \rho = \mu_2$ for some $\rho$ in $R$. Let $M_{<\mu}$ be the subposet consisting of the elements strictly below $\mu$ in this order. Also recall that the order complex $\Delta P$ of a poset $P$ is the abstract simplicial complex having $P$ as vertex set and a simplex for each chain (=totally ordered subset) of $P$.

One then has the following result, whose idea goes back at least as far as Laudal and Slettersjøe [21]; see also Herzog, Reiner, and Welker [17], Peeva, Reiner, and Sturmfels [27]. As in those sources, the proof (whose details we omit) comes from computing $\text{Tor}_i^R(M, \mathbb{k})$, next applying $M \otimes R (-)$, then extracting the $\mu$-graded component, and finally taking homology.

**Proposition 4.9.** In the above setting, for each semigroup element $\mu$ in $M$ one has

$$\text{Tor}_i^R(M, \mathbb{k})_{\mu} \cong \tilde{H}_{i-1}(\Delta(M_{<\mu}), \mathbb{k}),$$

equivariant for any group $G$ acting on $S$ that stabilizes $R, M$ and the multidegree $\mu$.

This immediately gives the following corollary of Theorem 1.1.

**Corollary 4.10.** For any ring $\mathbb{k}$, fix $d, r \geq 1$, and let $R = S^{(d)}, M = S^{(\geq r, d)}$ as usual. Then for any multidegree $a = (a_1, \ldots, a_n)$ in $\mathbb{N}^n$ with $|a| := \sum_j a_j \geq r$ and $|a| \equiv r \mod d$, one has $\tilde{H}_{i-1}(\Delta(M_{<\mu}), \mathbb{k}) = 0$ unless $|a| = di + r$, in which case

$$\tilde{H}_{i-1}(\Delta(M_{<\mu}), \mathbb{k}) \cong (\mathbb{S}^\sigma_{(d^i, r)})_{x^a}.$$  

Here $(\mathbb{S}^D)_{x^a}$ denotes the $x^a$-weight space in the polynomial $GL(V)$-representation $\mathbb{S}^D$, and the isomorphism is equivariant with respect to the subgroup $\mathbb{S}_a$ within $\mathbb{S}_n$ stabilizing $a$.

An important special case occurs by specializing to $a = (1, 1, \ldots, 1) = (1^{di+r})$, so that $\mu = x^a = x_1x_2 \cdots x_{di+r}$, and both the left and right sides of Corollary 4.10 have simpler interpretations. On the right side, letting $m = di + r$, the $x_1x_2 \cdots x_m$-weight space in the $GL(V)$-representation $\mathbb{S}^\sigma_{(d^i, r)}$ is the skew Specht module $\mathbb{S}^\sigma_{(d^i, r)}$ for the symmetric group $\mathbb{S}_m$, as discussed at the end of Subsection 2.3.

On the left side, $M_{<\mu}$ is isomorphic to a well-studied poset from the literature on poset topology. The supports of the squarefree monomials appearing in $M_{<\mu}$ are exactly the proper subsets
A ⊆ \{1, 2, \ldots, id + r\} with cardinality |A| ≥ r and |A| ≡ r \text{ mod } d. This poset $M_{\leq A}$ is therefore obtained from the Boolean algebra $2^{[m]}$ of all subsets of $[m] := \{1, 2, \ldots, m\}$ where $m = id + r$, by selecting the elements whose ranks lie in $\{r, r + d, r + 2d, \ldots, r + (i - 1)d\}$. Thus, one can rephrase this special case of Corollary 4.10 as asserting that the following:

\textbf{Corollary 4.11.} The order complex $\Delta$ for the rank-selected subposet of $2^{[m]}$ with $m = id + r$ allowing only the subsets whose ranks lie in $\{r, r + d, r + 2d, \ldots, r + (i - 1)d\}$ has $\tilde{H}_j(\Delta, \k) = 0$ unless $j = i - 1$, in which case $\tilde{H}_{i-1}(\Delta, \k) \cong \mathbb{S}(d^i)$ as an $\mathfrak{S}_m$-representation.

Corollary 4.11 is also an instance of a result\(^\dagger\) of Solomon [32, section 6], calculating the homology of all rank-selected subposets of a Boolean algebra $2^{[m]}$, which has been re-examined many times; see Wachs [35, section 3.4]. Solomon's result says the following [35, Theorem 3.4.4].

\textbf{Theorem 4.12.} For any composition $\alpha = (\alpha_1, \ldots, \alpha_\ell)$ of $m$, the order complex $\Delta$ of the rank-selected subposet of $2^{[m]}$ allowing only the subsets whose ranks lie in the partial sums

$$\{\alpha_1, \alpha_1 + \alpha_2, \alpha_1 + \alpha_2 + \alpha_3, \ldots, \alpha_1 + \cdots + \alpha_{\ell-1}\}$$

will have $\tilde{H}_i(\Delta; \k) = 0$ unless $i = \ell - 2$, while $\tilde{H}_{\ell-2}(\Delta; \k) \cong S^\alpha(\alpha)$ as an $\mathfrak{S}_m$-representation.

\textbf{Remark 4.13.} On the other hand, as discussed in Subsection 2.3, when $\k$ is a field of characteristic zero, one can recover a polynomial $GL(V)$-representation such as $\text{Tor}_i^R(M, \k)$ for $R = S(d)$, $M = S(\geq r, d)$ from the $\mathfrak{S}_m$-representation on its $x_1x_2 \cdots x_m$-weight space, where $m = \dim(V)$. In this way, using Solomon’s Theorem 4.12 to deduce Corollary 4.11 gives yet another alternate proof of Corollary 1.2 over fields of characteristic zero.

\section{On Tensor Products, Tor and Hom Between the Modules}

Our next goal is to compute $\text{Tor}_i^R(M, M')$ and $\text{Hom}_R(M, M')$ where

$$R = S(d),$$

$$M = S(\geq r, d),$$

$$M' = S(\geq r', d).$$

We first dispense with the easy case where rank$_k V = 1$, that is, $S = k[x]$. This case is something of an outlier, but easily analyzed, as all of the modules over $R = S(d) = k[x^d]$ are free of rank 1; that is, $S(\geq r, d) = x^r k[x^d]$. This immediately implies the following:

\(^\dagger\) One also needs the $S_n$-representation isomorphism $S^{\sigma(d)} \cong S^{\sigma(d')}$, an instance of the more general isomorphism $S^{\sigma(\alpha)} \cong S^{\sigma(\text{rev}(\alpha))}$ where rev$(\alpha) = (\alpha_\ell, \alpha_{\ell-1}, \ldots, \alpha_2, \alpha_1)$. The latter follows, for example, from Theorem 4.12 using the poset anti-automorphism $A \mapsto [1, 2, \ldots, m] \setminus A$ to the Boolean algebra $2^{[m]}$.\]
**Proposition 5.1.** For \( n = 1 \), the higher derived functors vanish, that is,

\[
\text{Tor}_i^R(M, M') = 0 = \text{Ext}_i^R(M, M') \quad \text{for } i \geq 1
\]

and in the \( i = 0 \) case, one has that both are free \( R \)-modules of rank one:

\[
\begin{align*}
\text{Tor}_0^R(M, M') &= x^r k[x, x^{-1}] \otimes_{k[x, x^{-1}]} x^{r'} k[x, x^{-1}] \cong x^{r+r'} k[x, x^{-1}], \\
\text{Ext}_0^R(M, M') &= \text{Hom}_{k[x, x^{-1}]}(x^r k[x, x^{-1}], x^{r'} k[x, x^{-1}]) \cong x^{r'-r} k[x, x^{-1}] \quad (\subset k[x, x^{-1}]).
\end{align*}
\]

## 5.1 Tensor products and proof of Theorem 1.6

Recall the statement of the theorem.

**Theorem 1.6.** Fix \( d, r, r' \geq 1 \) and let \( R = S^{(d)} \) with the three \( R \)-modules

\[
M = S^{(\geq r, d)}, \quad M' = S^{(\geq r', d)}, \quad M'' = S^{(\geq r'', d)}, \quad \text{where } r'' = r + r'.
\]

(i) The multiplication map \( M \otimes_R M' \xrightarrow{\varphi} M'' \) gives rise to a \( GL(V) \)-equivariant short exact sequence of \( R \)-modules

\[
0 \rightarrow S^{(r, r')}(−r'') \rightarrow M \otimes_R M' \rightarrow M'' \rightarrow 0
\]

with the \( R \)-module \( S^{(r, r')}(−r'') \) concentrated in degree \( r'' \), annihilated by \( R_+ \).

(ii) The sequence splits as \( R \)-modules, giving an \( R \)-module isomorphism

\[
M \otimes_R M' \cong M'' \oplus S^{(r, r')}(−r'').
\]

(iii) When \( \frac{r + r'}{r} \) lies in \( k^\times \), the sequence also splits as \( GL(V) \)-representations.

**Remark 5.2.** Note that Theorem 1.6 is consistent with the description of \( \text{Tor}_0^R(M, M') \) in Proposition 5.1, because \( S^{(r, r')}(V) = 0 \) when \( \dim(V) = 1 \) and \( r, r' \geq 1 \).

**Remark 5.3.** Theorem 1.6 holds for \( r = 0 \) or \( r' = 0 \) assuming \( S^{(0, r')}(V) = S^{(r, 0)}(V) = 0 \).

The essence of Theorem 1.6 is the next lemma. For multidegrees \( \alpha \) in \( \mathbb{N}^n \) and monomials \( x^\alpha \), let \( |\alpha| := \sum_{i=1}^n \alpha_i \), and similarly denote the \( \mathbb{N} \)-degree of \( x^\alpha \) by \( |x^\alpha| := |\alpha| \).

**Lemma 5.4.** In the above setting, consider any multidegree \( \gamma \) in \( \mathbb{N}^n \) occurring in \( M'' \) with \( |\gamma| > r'' \), so that \( |\gamma| = r + r' + kd \) with \( k \geq 1 \). Then the \( \gamma \)-homogeneous component of the multiplication map \( (M \otimes_R M')_\gamma \rightarrow M''_\gamma \) is a \( k \)-module isomorphism. In other words, one has \( x^\alpha \otimes_R x^{\alpha'} = x^\beta \otimes_R x^{\beta'} \) whenever \( \alpha + \alpha' = \gamma = \beta + \beta' \).
Proof. Let \( \mathbf{a}, \mathbf{a}', \mathbf{b}, \mathbf{b}' \) be monomials with \( \mathbf{a} \otimes_R \mathbf{a}' \) and \( \mathbf{b} \otimes_R \mathbf{b}' \) in \((M \otimes_R M')_\gamma\), so that \( \mathbf{a} \cdot \mathbf{a}' = x = \mathbf{b} \cdot \mathbf{b}' \), and assume \( |\gamma| = (r + r') + kd \) with \( k \geq 1 \). We wish to show that \( \mathbf{a} \otimes_R \mathbf{a}' = \mathbf{b} \otimes_R \mathbf{b}' \). By moving elements of \( R = S(d) \) across the tensor symbol, one may assume without loss of generality that \( |\mathbf{a}| = |\mathbf{b}| = r \), so \( |\mathbf{a}'| = |\mathbf{b}'| = r' + kd \). Also without loss of generality, assume \( r \leq r' \).

Let \( \mathbf{c} = \gcd(\mathbf{a}, \mathbf{b}) \), and write

\[
\begin{align*}
\mathbf{a} &= \mathbf{c} \cdot \mathbf{c}', \\
\mathbf{b} &= \mathbf{c} \cdot \mathbf{c}''.
\end{align*}
\]

Then \( \mathbf{a} \cdot \mathbf{a}' = \mathbf{b} \cdot \mathbf{b}' \) implies \( \mathbf{c}' \) divides \( \mathbf{b}' \). As \( |\mathbf{c}'| \leq |\mathbf{a}| = r \leq r' \), while \( |\mathbf{b}'| = r' + kd \), one can express \( \mathbf{b}' = \mathbf{c}' \cdot \mathbf{p} \cdot \mathbf{q} \) with \( |\mathbf{c}' \cdot \mathbf{p}| \equiv 0 \mod d \), that is, with \( \mathbf{c}' \cdot \mathbf{p} \) lying in \( R \). Therefore,

\[
\begin{align*}
\mathbf{b} \otimes_R \mathbf{b}' &= \mathbf{c} \cdot \mathbf{c}'' \otimes_R \mathbf{c}' \cdot \mathbf{p} \cdot \mathbf{q} \\
&= \mathbf{c} \cdot \mathbf{c}'' \cdot \mathbf{c}' \cdot \mathbf{p} \otimes_R \mathbf{q} \\
&= \mathbf{a} \cdot \mathbf{c}'' \cdot \mathbf{p} \otimes_R \mathbf{q} \\
&= \mathbf{a} \otimes_R \mathbf{c}'' \cdot \mathbf{p} \cdot \mathbf{q} \\\n&= \mathbf{a} \otimes_R \mathbf{a}'.
\end{align*}
\]

**Proof of Theorem 5.6 (i).** Note that both \( R \)-modules \( M \otimes_R M' \) and \( M'' \) vanish in \( \mathbb{N} \)-degrees strictly below \( r'' = r + r' \). Lemma 5.4 already shows that the \( GL(V) \)-equivariant short exact sequence of \( R \)-modules

\[
0 \to \ker(\varphi) \to M \otimes_R M' \to M'' \to 0
\]

has \( \ker(\varphi) \) vanishing in \( \mathbb{N} \)-degrees strictly above \( r'' \). On the other hand, restricting the sequence to degree exactly \( r'' \) gives this short exact sequence of \( GL(V) \)-modules

\[
0 \to \ker(\varphi)_{r''} \to S^r \otimes_k S^{r'} \to S^{r+r'} \to 0.
\]

This shows that \( \ker(\varphi)_{r''} \cong \mathcal{O}(r,r')(−r'') \) as \( GL(V) \)-representations, by comparing it to the special case of Proposition 3.3 with \( \alpha = (r), \beta = (r') \).

Finally, as \( \varphi \) is an \( R \)-module map, \( \ker(\varphi) \) must be an \( R \)-submodule, and as it is concentrated in \( \mathbb{N} \)-degree \( r'' \), it must be annihilated by \( R_+ \).

Lemma 5.4 also leads to a somewhat surprising family of \( R \)-module splittings for the multiplication map \( M \otimes_R M' \overset{\varphi}{\to} M'' \). Order multidegrees \( \alpha, \beta \) in \( \mathbb{N}^n \) componentwise by \( \beta \leq \alpha \) if \( \beta_i \leq \alpha_i \) for \( i = 1, 2, \ldots, n \), and choose for each \( \alpha \) in \( \mathbb{N}^n \) a family of scalars \( \{c_{\alpha, \beta} : \beta \leq \alpha \text{ and } |\beta| = r\} \) satisfying this requirement:

\[
\sum_{\beta : \beta \leq \alpha, \quad |\beta| = r} c_{\alpha, \beta} = 1.
\]
For example, a choice of such scalars arises by fixing any total ordering $<$ on $\mathbb{N}^n$ and letting
\[
c_{\alpha, \beta} = \begin{cases} 
1 & \text{if $\beta$ is the $<$ -minimum of $\{ \beta' : \beta' \leq \alpha$ and $|\beta'| = r \}$,} \\
0 & \text{otherwise.}
\end{cases}
\]

The next lemma then immediately applies assertion (ii) of Theorem 1.6.

**Lemma 5.5.** For any choice of $\{c_{\alpha, \beta}\}$ as above, the $k$-module map $\psi : M' \to M \otimes_R M'$ defined for $\alpha$ in $\mathbb{N}^n$ with $|\alpha| = r + r'$ by
\[
\psi(x^\alpha) := \sum_{\beta \leq \alpha \atop |\beta| = r} c_{\alpha, \beta} x^\beta \otimes_R x^{\alpha - \beta}
\]
gives an $R$-module splitting of the multiplication map $M \otimes_R M' \xrightarrow{\varphi} M''$.

**Proof.** One has $\varphi \circ \psi = 1$ because
\[
\varphi(\psi(x^\alpha)) = \sum_{\beta \leq \alpha \atop |\beta| = r} c_{\alpha, \beta} x^\beta \cdot x^{\alpha - \beta} = x^\alpha \sum_{\beta \leq \alpha \atop |\beta| = r} c_{\alpha, \beta} = x^\alpha.
\]

What is perhaps surprising is that $\psi$ is $R$-linear, which one checks as follows. Using the $k$-linearity of $\psi$, it suffices to show for monomials $x^\alpha, x^\gamma$ in $M'', R$, that one has an equality
\[
\psi(x^\gamma x^\alpha) = x^\gamma \psi(x^\alpha).
\]
If $x^\gamma = 1$, this is vacuously true. If $x^\gamma \neq 1$, then $|\gamma + \alpha| > r''$, and the desired equality is
\[
\sum_{\beta' \leq \gamma + \alpha \atop |\beta'| = r} c_{\gamma + \alpha, \beta'} x^{\beta'} \otimes_R x^{\gamma + \alpha - \beta'} = \sum_{\beta \leq \alpha \atop |\beta| = r} c_{\alpha, \beta} x^{\gamma + \beta} \otimes_R x^{\alpha - \beta}
\]
which follows from Lemma 5.4 together with (14) for $\gamma + \alpha$ and $\alpha$. \qed

Assertion (iii) of Theorem 1.6 follows from our next lemma, the last of this section.

**Lemma 5.6.** When $\binom{r + r'}{r}$ lies in $k^\times$, the $R$-module splitting $\psi : M'' \to M \otimes_R M'$ can be chosen to be $GL(V)$-equivariant.

**Proof.** We will define $\psi$ on monomials $x^\alpha$ in $M'' = S^{(\geq r'', d)}$, and extend $k$-linearly.

Case 1. $|\alpha| > r''$. Define $\psi(x^\alpha) := x^\beta \otimes_R x^{\alpha - \beta}$, where $\beta \leq \alpha$ has $|\beta| = r$ and is otherwise arbitrary, for example, $\beta$ is the $<$-minimum of $\{ \beta' : \beta' \leq \alpha$ and $|\beta'| = r \}$ for a total order $<$ on $\mathbb{N}^n$.

Case 2. $|\alpha| = r''$. Define
\[
\psi(x^\alpha) = \frac{1}{\binom{r + r'}{r}} \sum_{\beta \leq \alpha \atop |\beta| = r} \binom{\alpha_1}{\beta_1} \binom{\alpha_2}{\beta_2} \cdots \binom{\alpha_n}{\beta_n} x^\beta \otimes_R x^{\alpha - \beta}.
\]
(15)
One readily checks that condition (14) holds in Case 1. For Case 2, use the identity
\[
\binom{r + r'}{r} = \sum_{\substack{\beta \in \alpha : \\
|\beta| = r}} \binom{\alpha_1}{\beta_1} \binom{\alpha_2}{\beta_2} \ldots \binom{\alpha_n}{\beta_n}
\]
coming from expanding both sides of \((1 + t)^{r + r'} = (1 + t)^{\alpha_1} (1 + t)^{\alpha_2} \ldots (1 + t)^{\alpha_n}\) binomially, and comparing coefficients of \(t^r\). Thus, by Lemma 5.5, this \(\psi\) defines an \(R\)-module splitting.

To check that this splitting \(\psi\) is also \(GL(V)\)-equivariant, one can check it separately in each \(\mathbb{N}\)-degree \(m \geq r''\). We distinguish the same two cases as before.

Case 1. \(m > r''\). It suffices to check for monomials \(x^\alpha\) with \(|\alpha| > r''\) and \(g\) in \(GL(V)\), that
\[
g(\psi(x^\alpha)) = \psi(g(x^\alpha)).
\]
Here \(\psi(x^\alpha) = x^\beta \otimes_R x^{\alpha - \beta}\) for some \(\beta\) with \(|\beta| = r\). Naming the coefficients \(c_\gamma, d_\delta\) in \(k\) appearing in these unique expansions
\[
g(x^\beta) = \sum_{|\gamma| = r} c_\gamma x^\gamma,
\]
\[
g(x^{\alpha - \beta}) = \sum_{|\delta| = r'} d_\delta x^{\delta},
\]
one finds that
\[
g(\psi(x^\alpha)) = g(x^\beta \otimes_R x^{\alpha - \beta}) = g(x^\beta) \otimes_R g(x^{\alpha - \beta}) = \sum_{|\epsilon| = r''} \sum_{\begin{array}{c} \gamma + \delta = \epsilon \\
|\gamma| = r \\
|\delta| = r'
\end{array}} c_\gamma d_\delta x^\gamma \otimes_R x^{\delta}.
\]

(16)

On the other hand, one also finds that
\[
g(x^\alpha) = g(x^\beta \cdot x^{\alpha - \beta}) = g(x^\beta) \cdot g(x^{\alpha - \beta}) = \sum_{|\epsilon| = r''} \left( \sum_{\begin{array}{c} \gamma + \delta = \epsilon \\
|\gamma| = r \\
|\delta| = r'
\end{array}} c_\gamma d_\delta \right) x^\epsilon
\]
and therefore
\[
\psi(g(x^\alpha)) = \sum_{|\epsilon| = r''} \left( \sum_{\begin{array}{c} \gamma + \delta = \epsilon \\
|\gamma| = r \\
|\delta| = r'
\end{array}} c_\gamma d_\delta \right) \psi(x^\epsilon).
\]

(17)

Lemma 5.4 shows that (16) and (17) are equal.

Case 2. \(m = r''\). One can check that the formula (15) was chosen so as to make the splitting \(\psi\) have its homogeneous component \(\psi_{r''} : M''_{r''} \rightarrow (M \otimes_R M')_{r''}\) equal to \(\frac{1}{(r+r')\cdot r'}\) times the following
composite of $GL(V)$-equivariant maps:

$$S^{r+r'} \hookrightarrow S \xrightarrow{\Delta} S \otimes_k S \xrightarrow{} S^r \otimes_k S^{r'}$$

Here $\Delta$ in the middle is the comultiplication in the usual bialgebra structure on $S = S(V)$, defined on the algebra generators $x_i$ in $V$ via $\Delta(x_i) := 1 \otimes_k x_i + x_i \otimes_k 1$.

**Example 5.7.** When $d = 2$ and $r = r' = 1$, one has $R = S^{(2)}$ and $M = M' = S^{(\geq 1,2)}$, $M'' = S^{(\geq 2,2)} = S^{(2)}_+$, with $S^{\sigma(1,1)} = \wedge^2$. Here the exact sequence of Theorem 1.6 is

$$0 \to \wedge^2(-2) \to S^{(\geq 1,2)} \otimes S^{(2)} \to S^{(2)}_+ \to 0,$$

which for any $k$ is an exact sequence of $GL(V)$-representations, and a split exact sequence of $S^{(2)}$-modules. However, if $k$ has characteristic 2 and $\dim(V) \geq 2$, it does not split as $GL(V)$-representations in its $\mathbb{N}$-degree 2, where it gives this well-known nonsplit sequence:

$$0 \to \wedge^2(V) \to V \otimes_k V \to S^2(V) \to 0.$$

### 5.2 Higher Tor and proof of Theorem 1.8

We recall the statement of the theorem.

**Theorem 1.8.** Fix $d, r, r' \geq 1$, and let $R, M, M'$ denote $S^{(d)}, S^{(\geq r, d)}, S^{(\geq r', d)}$, as usual. Then for $i \geq 1$, the $R$-module $\text{Tor}_i^R(M, M')$ is annihilated by $R_+$, and as a module over $k = R / R_+$, has a $GL(V)$-isomorphism

$$\text{Tor}_i^R(M, M') \cong S^{\sigma(r,d'_i,r')}.$$

**Proof.** Resolve $M'$ over $R$ as in Theorem 1.1, and apply $M \otimes_R (\_)$ to give a complex

$$0 \leftarrow M \otimes_k S^{(r)} \leftarrow M \otimes_k S^{(d,r')} \leftarrow M \otimes_k S^{(d,d',r')} \leftarrow \ldots$$

whose homology computes $\text{Tor}_i^R(M, M')$. The $i$th term $M \otimes_k S^{\sigma(d,i,r')}$ vanishes in degrees below $d_i + r + r'$. For $m \geq 0$, the degree $d(i+m)+r+r'$ component of the boundary map

$$(M \otimes_k S^{\sigma(d_i',r')}_{d(i+m)+r+r'}) \xrightarrow{(\delta)_i_{d(i+m)+r+r'}} (M \otimes_k S^{\sigma(d_i-1',r')}_{d(i+m)+r+r'})$$

may be identified with the component $\delta_i^{(d,r')}_{d(i+m)+r+r'}$ within the complex of ribbons.

In particular, if $m = 0$, then Lemma 4.3(i) gives a $GL(V)$-isomorphism

$$\ker \delta_i^{(d,r')}_{d(i+m)+r+r'} \cong S^{\sigma(r,d',r')}.$$
and for \( m \geq 1 \), Lemma 4.3(ii) shows that
\[
\ker \delta^{(d,r')}_{d(i+m)+r+r'} = \operatorname{im} \delta^{(d+1,r')}_{d(i+m)+r+r'}.
\]
Thus, \( \text{Tor}_i^R(M, M') \cong S_{(r,d,r')} \), which is pure of degree \( di + r + r' \) and annihilated by \( R_+ \). □

### 5.3 Hom and proof of Theorem 1.7

Recall the statement of the theorem.

**Theorem 1.7.** Assume \( n \geq 2 \), so that \( S = k[x_1, \ldots, x_n] \) is not univariate. Fix an integer \( d \geq 1 \), defining \( R = S(d) \). For \( r, r' \geq 0 \), consider three \( R \)-modules
\[
M = S_{(\geq r,d)},
M' = S_{(\geq r',d)},
M'' = S_{(\geq r'',d)},
\]
defining \( r'' := r' - r \) if \( r \leq r' \), otherwise if \( r > r' \), defining \( r'' \) to be the unique integer in \([0, d)\) congruent to \( r' - r \mod d \). Then one has a \( GL(V) \)-equivariant \( R \)-module isomorphism
\[
M'' \longrightarrow \operatorname{Hom}_R(M, M')
\]
\[
m'' \longmapsto (m \mapsto m'' \cdot m).
\]

Note that this map sending \( m'' \) to multiplication by \( m'' \) is \( GL(V) \)-equivariant and an \( R \)-module map, and it is always injective. What is not obvious is its surjectivity. We build up the proof of the theorem with a few lemmas.

**Lemma 5.8.** For any integers \( r, r', r'' \geq 0 \), there is an isomorphism
\[
\operatorname{Hom}_R(S_{(\geq r+ r',d)}, S_{(\geq r'',d)}) \cong \operatorname{Hom}_R(S_{(\geq r,d)} \otimes_R S_{(\geq r',d)}, S_{(\geq r'',d)}).
\]

**Proof.** Apply \( \operatorname{Hom}_R(-, S_{(\geq r'',d)}) \) to the split exact sequence from Theorem 1.6(i)
\[
0 \to S_{(r,r')} \to S_{(\geq r,d)} \otimes_R S_{(\geq r',d)} \to S_{(\geq r+ r',d)} \to 0
\]
and note \( \operatorname{Hom}_R(S_{(r,r')}, S_{(\geq r'',d)}) = 0 \) because \( S_{(r,r')} \) is all \( R \)-torsion. □

**Lemma 5.9.** For \( n := \dim V \geq 2 \), with \( 0 \leq r < d \) and \( i \geq 1 \), one has
\[
\operatorname{Ext}_R^i(S(d)/S_{(\geq di,d)}, S_{(\geq r,d)}) = 0.
\]

**Proof.** Note \( R = S(d) \) has \( I := S_{(\geq di,d)} = (R_+)^i \), the \( i \)th power of the graded maximal ideal \( R_+ \). Denoting \( M := S_{(\geq r,d)} \), it suffices to show that whenever \( 0 \leq r < d \) we have that \( \operatorname{Ext}_R^j(R/I, M) = 0 \) for \( 0 \leq j \leq n - 1 \). This vanishing is controlled by the grade of \( I \) on \( M \), denoted grade\((I, M)\), defined as the maximal length \( \ell \) of an \( M \)-regular sequence \( (\vartheta_1, \ldots, \vartheta_\ell) \) contained in \( I \): one has (see, e.g.,
Bruns and Herzog [7, Theorem 1.2.5]),

\[
\text{grade}(I, M) := \min \{ \ell : \text{Ext}^\ell_R(R/(I, M) \neq 0 \}.
\]

Now \( S \) is a Cohen–Macaulay \( R \)-module, and an \( R \)-module direct sum \( S = \bigoplus_{r=0}^{d-1} S^{(r,d)} \). Each \( R \)-summand \( M = S^{(r,d)} \) is therefore also a Cohen–Macaulay \( R \)-module. So \( x_1^{di}, \ldots, x_n^{di} \) in \( I = (R_+)^i \) give an \( M \)-regular sequence of length \( n \), showing \( \text{grade}(I, M) = n \).

**Remark 5.10.** The hypothesis \( 0 \leq r < d \) in Lemma 5.9 is important. When \( r \geq d \), the \( R \)-modules \( S^{(r,d)} \) are not \( R \)-summands of \( S \), and are not Cohen–Macaulay for \( n \geq 2 \). They have \( R \)-module depth 1 for \( r \geq d \), as observed by Greco and Martino in [15, Theorem 3.5].

The next lemma is the case \( r = r' \) in Theorem 1.7.

**Lemma 5.11.** The inclusion \( R \hookrightarrow \text{Hom}_R(S^{(r,d)}, S^{(r,d)}) \) is an isomorphism for all \( r \geq 0 \).

**Proof.** We prove this first for \( r \equiv 0 \mod d \), and then induct downward for \( r \not\equiv 0 \mod d \).

Fix any \( i \geq 1 \) with \( r = di \), and apply \( \text{Hom}_R(\cdot, S^{(di,d)}) \) to the short exact sequence

\[
0 \to S^{(di,d)} \to R \to Q \to 0,
\]

where \( R := S^{(d)} \) and \( Q := S^{(d)}/S^{(di,d)} \), giving a long exact sequence in \( \text{Ext} \). Its first term vanishes \( \text{Hom}_R(Q, S^{(di,d)}) = 0 \) because \( Q \) is all \( R \)-torsion. As \( \text{Ext}_1^1(R, S^{(di,d)}) = 0 \), one has a short exact sequence

\[
0 \to \text{Hom}_R(R, S^{(di,d)}) \to \text{Hom}_R(S^{(di,d)}, S^{(di,d)}) \to \text{Ext}_1^1(R, S^{(di,d)}) \to 0.
\]

On the other hand, applying \( \text{Hom}_R(Q, \cdot) \) to the sequence (18) yields

\[
0 \to \text{Hom}_R(Q, S^{(di,d)}) \to \text{Hom}_R(Q, R) \to \text{Hom}_R(Q, Q)
\]

\[
\to \text{Ext}_1^1(R, S^{(di,d)}) \to \text{Ext}_1^1(R, Q) \to \ldots
\]

but again \( \text{Hom}_R(Q, S^{(di,d)}) = \text{Hom}_R(Q, R) = 0 \), as \( Q \) is \( R \)-torsion. Also \( \text{Hom}_R(Q, Q) \cong Q \) because \( Q \) is a cyclic \( R \)-module. Finally, \( \text{Ext}_1^1(Q, R) = 0 \) by the \( r = 0 \) case of Lemma 5.9. Thus, (19) yields an isomorphism \( Q \cong \text{Ext}_1^1(R, S^{(di,d)}) \). One can check that this lets one assemble a morphism of short exact sequences given by the following commutative diagram:

\[
\begin{array}{cccccc}
0 & \to & \text{Hom}_R(R, S^{(di,d)}) & \to & \text{Hom}_R(S^{(di,d)}, S^{(di,d)}) & \to & \text{Ext}_1^1(Q, S^{(di,d)}) & \to & 0 \\
& & \uparrow & & \uparrow & & \uparrow & & \\
0 & \to & S^{(di,d)} & \to & R & \to & Q & \to & 0
\end{array}
\]

The outer two vertical maps are isomorphisms, so the snake lemma implies the middle map is an isomorphism, proving the \( r \equiv 0 \mod d \) case.

In the case \( d(i-1) < r < di \), we proceed by downward induction on \( r \). Apply the (left-exact) functor \( \text{Hom}_R(S^{(r,d)}, \cdot) \) to the inclusion \( S^{(r,d)} \hookrightarrow \text{Hom}_R(S^{(1,d)}, S^{(r+1,d)}) \). This induces an inclusion:

\[
\text{Hom}_R(S^{(r,d)}, S^{(r,d)}) \hookrightarrow \text{Hom}_R(S^{(r,d)}, \text{Hom}_R(S^{(1,d)}, S^{(r+1,d)}))
\]

\[
\cong \text{Hom}_R(S^{(r,d)} \otimes_R S^{(1,d)}, S^{(r+1,d)}),
\]

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where the latter isomorphism is Hom-Tensor adjunction. Lemma 5.8 gives an isomorphism

\[
\text{Hom}_R(S^{(\geq r, d)} \otimes_R S^{(\geq r+1, d)}, S^{(\geq r+1, d)}) \cong \text{Hom}_R(S^{(\geq r+1, d)}, S^{(\geq r+1, d)}),
\]

and by induction, \(\text{Hom}_R(S^{(\geq r+1, d)}, S^{(\geq r+1, d)}) \cong S^{(d)}\). It follows that there are inclusions

\[
S^{(d)} \hookrightarrow \text{Hom}_R(S^{(\geq r, d)}, S^{(\geq r, d)}) \hookrightarrow S^{(d)},
\]

and one can check that composing these two inclusions is the identity. Thus, the inclusion

\[
\text{Hom}_R(S^{(\geq r, d)}, S^{(\geq r, d)}) \hookrightarrow S^{(d)}
\]

has a right inverse and is hence also a surjection, so the isomorphism follows. \(\square\)

The next lemma is the case \(r' = 0\) in Theorem 1.7.

**Lemma 5.12.** For \(\dim V \geq 2\) and any \(r \geq 0\), one has that

\[
\text{Hom}_R(S^{(\geq r, d)}, S^{(d)}) \cong S^{(\geq d_i - r, d)},
\]

where \(i\) is the smallest integer such that \(r \leq d_i\).

**Proof.** Observe first that

\[
\text{Hom}_R(S^{(\geq r, d)}, S^{(d)}) \cong \text{Hom}_R(S^{(\geq r, d)}, \text{Hom}_R(S^{(\geq d_i - r', d)}, S^{(d)})) \quad \text{(Lemma 5.11)}
\]

\[
\cong \text{Hom}_R(S^{(\geq r, d)} \otimes_R S^{(\geq d_i - r', d)}, S^{(\geq d_i - r', d)}) \quad \text{(adjunction)}
\]

\[
\cong \text{Hom}_R(S^{(\geq d_i, d)}, S^{(\geq d_i - r', d)}) \quad \text{(Lemma 5.8)}.\]

Letting \(Q := S^{(d)} / S^{(\geq d_i, d)}\) again, and applying \(\text{Hom}_R(-, S^{(\geq d_i - r', d)})\) to the exact sequence

\[
0 \rightarrow S^{(\geq d_i, d)} \rightarrow R \rightarrow Q \rightarrow 0
\]

yields, as in the proof of Lemma 5.11, the short exact sequence

\[
0 \rightarrow S^{(\geq d_i - r', d)} \rightarrow \text{Hom}_R(S^{(\geq d_i, d)}, S^{(\geq d_i - r', d)}) \rightarrow \text{Ext}_R^1(Q, S^{(\geq d_i - r', d)}) \rightarrow 0.
\]

One can apply Lemma 5.9 to conclude \(\text{Ext}_R^1(Q, S^{(\geq d_i - r', d)}) = 0\), as the assumption on \(i\) implies that \(0 \leq d_i - r' < d\). Thus, \(\text{Hom}_R(S^{(\geq d_i, d)}, S^{(\geq d_i - r', d)}) \cong S^{(d_i - r', d)}\). \(\square\)

**Proof of Theorem 1.7.** We induct on \(r\), with base cases \(r = 0, 1\) handled separately.

The base case \(r = 0\). There is not much to prove, as \(r = 0\) implies \(M = R\) and \(r'' = r'\) so \(M'' = M'\). Here the theorem says the inclusion \(M' \hookrightarrow \text{Hom}_R(R, M')\) is an isomorphism.

The base case \(r = 1\). When \(r = 1\) and \(r' = 0\), the theorem is an instance of Lemma 5.12. When \(r = 1\) and \(r' = 1\), the theorem is an instance of Lemma 5.11. Consequently, one may assume without loss of generality that \(r' \geq 2\). Choose \(i\) to be the smallest integer such that \(r' \leq d_i\). The preceding lemmas yield a string of isomorphisms:

\[
\text{Hom}_R(S^{(\geq 1, d)}, S^{(\geq r', d)}) = \text{Hom}_R(S^{(\geq 1, d)}, \text{Hom}_R(S^{(\geq d_i - r', d)}, S^{(d)})) \quad \text{(Lemma 5.12)}
\]
\[ \cong \text{Hom}_R(S^{(\geq 1,d)} \otimes_R S^{(\geq d_i-r',d')}, S^{(d)}) \]  
(adjunction)

\[ \cong \text{Hom}_R(S^{(\geq d_i-r'+1,d')}, S^{(d)}) \]  
(Lemma 5.8)

\[ \cong S^{(\geq r'-1,d)} \]  
(Lemma 5.12).

The inductive step where \( r \geq 2 \). When \( r' = 0 \), the theorem is an instance of Lemma 5.12, so assume without loss of generality that \( r' \geq 1 \). There is then a string of isomorphisms:

\[ \text{Hom}_R(S^{(\geq r,d)}, S^{(\geq r',d)}) = \text{Hom}_R(S^{(\geq r-1,d)} \otimes_R S^{(\geq 1,d)}, S^{(\geq r',d)}) \]  
(Lemma 5.8)

\[ \cong \text{Hom}_R(S^{(\geq r-1,d)}, \text{Hom}_R(S^{(\geq 1,d)}, S^{(\geq r',d)})) \]  
(Hom-tensor adjunction)

\[ \cong \text{Hom}_R(S^{(\geq r-1,d)}, S^{(\geq r'-1,d)}) \]  
(above \( r = 1 \) base case)

\[ \cong S^{(\geq r''',d)} \]  
(inductive hypothesis). \( \square \)

**Remark 5.13.** Under some extra hypotheses on the scalars \( k \), one can regard the special case where \( 0 \leq r, r' \leq d - 1 \) in Theorem 1.7 as an instance of a more general statement, Proposition 5.14. Let \( k \) be a field, and \( G \) a finite abelian subgroup of \( GL_n(k) \) acting on \( S = k[x_1, \ldots, x_n] \). For a linear character \( \chi : G \to k^\times \) define the \( S^G \)-module of relative invariants in \( S \) as follows:

\[ S^{G,\chi} := \{ f \in S : g(f) = \chi(g)f \}. \]

**Proposition 5.14.** In the above setting with \( k \) a field, assume furthermore that

- \( \#G \) lies in \( k^\times \),
- \( k \) contains the \( e \)th roots of unity, for \( e := \text{lcm}\{\text{multiplicative orders of } g \text{ in } G\} \), and
- \( G \) contains no pseudo-reflections, that is, no \( g \) with \( \ker(g-I_n) \) a hyperplane in \( k^n \).

Then for any two linear characters \( \chi, \chi' : G \to k^\times \), defining a third character \( \chi'' := \chi^{-1}\chi' \), the following map is an \( S^G \)-module isomorphism:

\[ S^{G,\chi''} \longrightarrow \text{Hom}_{S^G}(S^{G,\chi}, S^{G,\chi'}) \]

\[ s'' \longrightarrow (s \mapsto s'' \cdot s). \]

The proof, which we omit here, uses the following result of Auslander [22, Theorem 5.15] about the skew group ring \( S \# G \), which is defined to be a free \( S \)-module on \( S \)-basis \( \{g\}_{g \in G} \), with multiplication defined for \( s, t \in S \) and \( g, h \in G \) by

\[ (t \cdot h) \cdot (s \cdot g) := t \cdot h(s) \cdot hg. \]

**Theorem 5.15** (Auslander’s theorem). Let \( k \) be a domain, and \( G \) a finite subgroup of \( GL_n(k) \) that contains no pseudo-reflections and has \( \#G \) in \( k^\times \). Then the following map is is an isomorphism of \( S^G \)-algebras, and hence also of \( S^G \)-modules:

\[ S \# G \longrightarrow \text{Hom}_{S^G}(S, S) \]

\[ s \cdot g \longrightarrow (t \mapsto s \cdot g(t)). \]
To recover the case \(0 \leq r, r' \leq d - 1\) in Theorem 1.7 from Proposition 5.14, take \(G\) to be the cyclic subgroup of order \(d\) given by all of the scalar matrices in \(GL_n(\mathbb{k})\) having a \(d\)th-root-of-unity on the diagonal. When \(n \geq 2\), this group \(G\) contains no pseudoreflections.

**Remark 5.16.** A natural next subject of study after computing \(\text{Hom}_R(S(\geq r, d), S(\geq r', d))\) is on the behavior of the higher derived functors \(\text{Ext}_R^i(S(\geq r, d), S(\geq r', d))\) for \(i > 0\). It turns out that this case is rather subtle in contrast to the case of Tor, and there is no similarly “clean” statement as in Theorem 1.7.

There are commonalities with the case of Tor worth mentioning: it turns out that the higher Ext modules are also annihilated by \(R^+\), and the Ext-modules satisfy a more delicate persistence of nonvanishing than the Tor-modules. The tools and ideas used to prove these statements, along with some surprising connections between the Yoneda algebra and categorification of certain noncommutative Hopf algebras, will be the subject of a future paper.

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