STRAINS AND JETS IN BLACK HOLE FIELDS

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Abstract. We study the behaviour of an initially spherical bunch of particles emitted along trajectories parallel to the symmetry axis of a Kerr black hole. We show that, under suitable conditions, curvature and inertial strains compete to generate jet-like structures.

1 Introduction

In recent papers, Bini, de Felice and Geralico (2006, 2007) considered how the relative deviations among the particles of a given bunch depend on the geometric properties of the frame adapted to a fiducial observer. Here we outline the main steps of their analysis which revealed how spacetime curvature and inertial strains compete to generate jet-like structures.

2 The relative deviation equation

Consider a bunch of test particles, i.e. a congruence $C_U$ of timelike world lines with unit tangent vector $U (U \cdot U = -1)$ parametrized by the proper time $\tau_U$. In general, the lines of the congruence $C_U$ are accelerated with acceleration $a(U) = \nabla_U U$. Let $C_\ast$ be a fixed world line of the congruence which we consider as that of the “fiducial observer.” The separation between the line $C_\ast$ and a line of $C_U$ is represented by a connecting vector $Y$, i.e. a vector undergoing Lie transport along $U$, namely $\mathcal{L}_U Y = 0$. The latter equation becomes:

$$\dot{Y} \equiv \frac{dY}{d\tau_U} = \nabla_Y U = -(Y \cdot U) a(U) - K(U) \mathcal{L}_U Y,$$

(2.1)

where the kinematical tensor $K(U)$ is defined as $K(U) = \omega(U) - \theta(U)$. Here the antisymmetric tensor $\omega(U)^\beta_{\alpha}$ represents the vorticity of the congruence $C_U$.

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and the symmetric tensor $\theta(U)^{\beta}_{\alpha}$ represents the expansion. The symbol $\sqcup$ stands for right-contraction operation among tensors. The covariant derivative along $U$ of both sides of the Lie transport equation gives rise to the “relative deviation equation” which, once projected on a given spatial triad $E_a$, reads:

$$\ddot{Y}^a + K_{(U,E)}^a b Y^b = 0,$$

(2.2)

where $K_{(U,E)}^a b = [T_{(tw,U,E)} - S(U) + E(U)]^a b$. Here $E(U)$ is the electric part of the Riemann tensor relative to $U$ namely $E(U)^{\alpha}_{\gamma} = R^{\alpha}_{\beta\gamma\delta}U^\beta U^\delta$. The strain tensor $S$ is defined as

$$S(U)_{ab} = \sqrt{\Delta z} \left[ \frac{d}{dz} + \frac{2Mz}{\Delta z} \right],$$

(2.3)

$S(U)$ depends only on the congruence $C_U$ and not on the chosen spatial triad $E_a$, while the tensor $T$, given by:

$$T_{(tw,U,E)}^a b = \delta^a_b \omega^{2}(tw,U,E) - \omega^a_{(tw,U,E)}\omega_{(tw,U,E)b} - e_{ab} \omega^{f}(tw,U,E) K(U)^{b}_{c},$$

(2.4)

as derived in detail in Bini et al. (2006), characterizes the given frame relative to a Fermi-Walker frame along the congruence. We like to stress here that the components of the deviation vector should be restricted to the reference observer’s world line; they may behave differently according to the reference frame set up for their measurements.

### 3 Axial observers in Kerr spacetime

Consider Kerr spacetime, whose line element in Kerr-Schild coordinates ($t, x^1 = x, x^2 = y, x^3 = z$) is given by

$$ds^2 = g_{\alpha\beta}dx^\alpha dx^\beta \equiv (\eta_{\alpha\beta} + 2Hk_\alpha k_\beta)dx^\alpha dx^\beta,$$

(3.1)

where $H = Mr^3/(r^4 + a^2 z^2)$, $\eta_{\alpha\beta}$ is the flat spacetime metric and

$$k_\alpha dx^\alpha = -dt - \frac{(rx + ay)dx + (ry - ax)dy}{r^2 + a^2} - \frac{z}{r}dz,$$

(3.2)

with $r$ implicitly defined by

$$\frac{x^2 + y^2}{r^2 + a^2} + \frac{z^2}{r^2} = 1.$$

(3.3)

Here $M$ and $a$ are the total mass and specific angular momentum characterizing the spacetime. Consider now a set of particles moving along the $z$-direction, on and nearby the axis of symmetry, with four velocity

$$U = \gamma[m + \nu E(m)]_{\alpha}, \quad \gamma = (1 - \nu^2)^{-1/2},$$

(3.4)

where $m = \sqrt{(z^2 + a^2)/(\Delta z \partial_t, E(m))]_3 = \sqrt{\Delta z}[(z^2 + a^2)|\partial_z + (2Mz/\Delta z)\partial_t], \Delta z = z^2 - 2Mz + a^2$ and where the instantaneous linear velocity $\nu = \nu(z)$, relative to
the local static observers, is function of \( z \). These particles are accelerated except perhaps those which move strictly on the axis of symmetry. Their history forms a congruence \( \mathcal{C}_U \) of \( \infty^2 \) world lines and each of them can be parametrized by the pair \((x, y)\) of the spatial coordinates. A frame adapted to this kind of orbits was found explicitly in Bini et al. (2007) and termed \( \{ \hat{E}(U) \} \). The congruence is accelerated with non-vanishing components along the three spatial directions \( E(U)_a \). It is now possible to study the components of the vector \( Y \) connecting a specific curve of the congruence which we fix along the rotation axis with \( x = 0 = y \) with tangent vector \( \hat{U} \) and nearby world lines of the same congruence. The reference world line is accelerated along the direction \( E(\hat{U})_3 \), i.e. \( a(U)_3 = a(\hat{U})_3 \epsilon(U)_3 \), with

\[
a(U)_3 = \frac{d}{dz} \left( \frac{\gamma}{\sqrt{g_{33}}} \right), \tag{3.5}
\]

where \( \tilde{g}_{33} = (z^2 + a^2)/\Delta_z \) denotes the restriction on the axis of that metric coefficient.

4 Tidal forces, frame induced deformations and strains

In order to determine the deviations measured by the “fiducial observer” \( \hat{U} \) with respect to the chosen frame we need to evaluate the kinematical fields of the congruence, namely the acceleration \( a(U) \), the vorticity \( \omega(U) \) and the expansion \( \theta(U) \), the electric part of the Weyl tensor \( \mathcal{E}(U) \), the strain tensor \( S(U) \) and the characterization of the spatial triad \( \{ E(U)_a \} \) with respect to a Fermi-Walker frame given by the tensor \( T_{(\text{fw},U,E)} \). All these quantities are calculated on the axis of symmetry so we can drop the tilde (‘\( \sim \)’); a lengthy calculation leads to the following results.

i) The only nonvanishing components of the kinematical tensors \( \omega(U) \) and \( \theta(U) \) are given by

\[
\theta(U)_{\bar{3}\bar{3}} = \frac{d}{dz} \left( \frac{\gamma}{\sqrt{g_{33}}} \right), \quad \omega(U)_{\bar{1}\bar{2}} = \gamma(1 + \nu) \frac{2aMz}{\sqrt{\Delta_z(z^2 + a^2)^{3/2}}} . \tag{4.1}
\]

ii) The vorticity vector becomes \( \omega_{(\text{fw},U,E)} = \omega_{(\text{fw},U,E)} \epsilon(U)_{\bar{3}} \) with \( \omega_{(\text{fw},U,E)} E(U)_{\bar{3}} = \omega(U)_{\bar{1}\bar{2}} \).

iii) The electric part of the Weyl tensor is a diagonal matrix with respect to the adapted frame \( \{ E(U)_a \} \), namely

\[
\mathcal{E}(U) = \mathcal{M} \frac{z^2 - 3a^2}{(z^2 + a^2)^3} \text{diag} [1, 1, -2] . \tag{4.2}
\]

iv) The only nonvanishing components of the tensor \( T_{(\text{fw},U,E)} \) turn out to be

\[
T_{(\text{fw},U,E)\bar{1}\bar{1}} = T_{(\text{fw},U,E)\bar{2}\bar{2}} = -\omega_{(\text{fw},U,E)} \epsilon(\bar{3}), \\
T_{(\text{fw},U,E)\bar{1}\bar{2}} = -T_{(\text{fw},U,E)\bar{2}\bar{1}} = -\dot{\omega}_{(\text{fw},U,E)} \epsilon(\bar{3}) \\
= \gamma \nu \frac{d}{dz} \omega_{(\text{fw},U,E)} \epsilon(\bar{3}). \tag{4.3}
\]
v) The non-zero components of the strain tensor $S(U)_{ij}$ can be written as

\begin{align*}
S(U)_{11} &= S(U)_{22} = -\omega_{(w,U,E)}^2 + \mathcal{M} z \frac{z^2 - 3a^2}{(z^2 + a^2)^2}, \\
S(U)_{12} &= -S(U)_{21} = -\dot{\omega}_{(w,U,E)}, \\
S(U)_{33} &= \frac{\gamma}{\sqrt{g_{33}}} \frac{\partial}{\partial z} [a(U)^3] + [a(U)^3]^2.
\end{align*}

(4.4)

From Eqs. (4.2)–(4.4) it follows that the deviation matrix $K(U,E)$ has only the non-zero component

\[ K(U,E)_{33} = -\frac{\gamma}{\sqrt{g_{33}}} \frac{\partial[a(U)^3]}{\partial z} - [a(U)^3]^2 + \mathcal{E}(U)_{33}; \]

hence, as expected, the particles emitted nearby the axis in the $z$-direction will be relatively accelerated in the $z$-direction only. The spatial components of the connecting vector $Y$ are then obtained by integrating the deviation equation (2.2) which now reads

\[ \ddot{Y}^1 = 0, \quad \ddot{Y}^2 = 0, \quad \ddot{Y}^3 = -K(U,E)_{33} Y^3. \]

(4.6)

Equations (4.6) can be analytically integrated to give

\[ Y^1 = Y_0^1, \quad Y^2 = Y_0^2, \quad Y^3 = C \frac{\gamma \nu}{\sqrt{g_{33}}}, \]

(4.7)

where $C$ is a constant. This result shows that the conditions imposed on the particles of the bunch to move parallel to the axis of rotation is assured by a suitable balancing among the gravitoelectric (curvature) tensor (4.2), the inertial tensor (4.3) and the strain tensor (4.4).

5 Reference world line uniformly accelerated

Let us now consider the case of the reference world line with tangent vector $U$ constantly accelerated, namely with $a(U)^3 = A = \text{const.}$ The instantaneous linear velocity relative to a local static observer is given by

\[ \nu_A = \left[ 1 - \frac{1}{g_{33} \tilde{\kappa} + A(z - z_0)^2} \right]^{1/2}, \]

(5.1)

where the positive value has been selected for $\nu_A$ in order to consider outflows. $\tilde{\kappa}$ is a constant which acquires the meaning of the conserved (Killing) energy of the particle if the latter is not accelerated. The solution of equation (4.7) is given by

\[ Y^1 = Y_0^1, \quad Y^2 = Y_0^2, \quad Y^3 = Y_0^3 \frac{\gamma A \nu_A}{\sqrt{g_{33} \tilde{\kappa} \nu_A}}, \]

(5.2)
where $\nu_A^0 = \nu_A(z)$. Figure 1 shows the behaviour of an initially spherical bunch of particles in the $Y^1 - Y^3$ plane for increasing values of the coordinate $z$. We clearly see a stretching along the $z$ axis leading to a collimated axial outflow of matter, clearly suggestive of an “astrophysical jet.” Figure 2 shows qualitatively the behaviour of an initially spherical bunch of particles. Note that in this case the acceleration acts contrarily to the curvature tidal effect; indeed, $S(U), T_{(\omega U, U, E)}$ and $E(U)$ act in competition leading to a quite unexpected result. It is easy to show that the stretching shown in Figs. 1 and 2 for uniformly accelerated outgoing particles persists also with a general acceleration. Moreover this behaviour does not depend of the acceleration mechanism itself. Observed jets (see Figs. 3 and 4 as an example) appear to contain spherical bunches of particles emerging from the central black hole.

References

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Fig. 1. An initially circular bunch of particles on the $y^i-y^j$ plane emitted along the $z$-axis at $z_0/M = 2$ is shown to spread for the choice of parameters $a/M = 0.5$, $\kappa = 1.5$ and $M_A = 0.3$ as an example. The curves correspond to increasing values of the coordinate $z/M = [2, 4, 6, 8, 10, 12, 14, 16, 18, 20]$. 
Fig. 2. The qualitative behaviour of an initially spherical bunch of particles moving out along the axis of a rotating black hole.
Fig. 3. The jet from the Galaxy M87 (Credit: J.A. Biretta et al. Hubble Heritage Team (STScI/AURA), NASA) We can see almost spherical bunches of particles along the jet.
Fig. 4. Details of the jet from M87 close to the central black hole (Credit: J.A. Biretta et al. Hubble Heritage Team (STScI/AURA), NASA). Here we see a spherical bunch of particles apparently emerging from nearby the hole.