Helmholtz-Hodge Theorems: Unification of Integration and Decomposition Perspectives

Jose G. Vargas †

May 8, 2014.

Abstract

We develop a Helmholtz-like theorem for differential forms in Euclidean space $E_n$ using a uniqueness theorem similar to the one for vector fields. We then apply it to Riemannian manifolds, $R_n$, which, by virtue of the Schlafli-Janet-Cartan theorem of embedding, are here considered as hypersurfaces in $E_N$ with $N \geq n(n + 1)/2$.

We obtain a Hodge decomposition theorem that includes and goes beyond the original one, since it specifies the terms of the decomposition.

We then view the same issue from a perspective of integrability of the system $(d\alpha = \mu, \delta\alpha = \nu)$, relating boundary conditions to solutions of $(d\alpha = 0, \delta\alpha = 0)$, $\delta$ is what goes by the names of divergence and co-derivative, inappropriate for the Kähler calculus with which we obtain the foregoing).

1 Introduction

We use the Kähler calculus (KC) [1], [2], [3] to extend Helmholtz theorem to differential $k-$forms in Euclidean spaces of arbitrary dimension. We then adapt it to Riemannian manifolds, which we view as embedded in Euclidean...
spaces. We thus achieve a Hodge theorem which, not only speaks of the decomposition of differential forms, but actually specifies the terms of such decomposition as integrands involving the exterior derivative and the co-differential of the differential form.

This paper has three major parts. In a first part we present the strategy to get into the deep results we have announced. We then recollect formulas from KC needed to get to those results. In a second part, we develop Helmholtz theorem for differential $k-$forms in Euclidean spaces of arbitrary dimension. In a third part, we adapt our most general Helmholtz theorem to Riemannian manifolds, where the condition of vanishing differential $k-$form at the boundary does not apply in general.

In the second part, the venue is a Euclidean space, $E_n$, and the boundary condition is at infinity, where the differential form in question is supposed to vanish. If it does not, we assume its vanishing and add to the result a constant differential needed for that vanishing.

In the third part, the venue is a region of a $E_n$, or a Riemannian manifold; in any of the last two cases, dealing with the boundary condition brings to the fore the emergence of a harmonic term which is additional to the two terms characteristic of Helmholtz theorems. From now on and for practical reasons, the appellative Helmholtz will be used whenever the specifics of a problem lead to expressing a differential $k-$form as sum of closed and co-closed terms exclusively. The appellative Hodge will be used when a harmonic term also enters the expression.

It is important to emphasize two main issues which, if not made explicit, may cause confusion. Helmholtz theorem is about integration. Hodge’s theorem is only about decomposition. In the first case, we speak of uniqueness of the solution (of a differential system) that satisfies certain conditions. In the second case, we speak of the uniqueness of the decomposition of a differential form into closed, co-closed and harmonic terms. For differential $k-$forms and only them, we go beyond Hodge by considering it from a perspective of integration and obtaining all three terms in the decomposition as integrals. The decomposition theorem is, for both differential $k-$forms and inhomogeneous ones, a by-product of the integration theorem for differential $k-$forms.

The second issue has to do with the nature of differential forms in KC, where components have three types of indices in the most general case. One type is constituted by valuedness superscripts, as usual. One type of subscripts is for components of multilinear functions of vector fields, whether antisymmetric or not. Covariant differentiation governs differentiation in-
volving indices of those types. A second type of subscript is for functions of \( r^{-}\)surfaces, their evaluation being given by their integration; exterior differentiation pertains to them. Thus, for example, \( dx \) and \( dx^i + dy^j + dz^k \) are functions of curves whose evaluation on a curve \( \gamma \) with end points \( A \) and \( B \) in \( E_3 \) is given by

\[
\int_{x_A}^{x_B} dx = x_B - x_A, \quad \int_{A}^{B} dr = (x_B - x_A)i + (y_B - y_A)j + (z_B - z_A)k.
\]

(1)

Differential \( r^{-}\)forms are here integrands (functions of \( r^{-}\)surfaces) like in Cartan [5], Kähler and Rudin [6].

The view of differential forms just expressed is particularly suited for a convenient and deep use of Laplacians for doing what Dirac delta functions do in \( ad hoc \) manner. Recall in this regard that the existence of a Stokes theorem allows for the definition of their exterior derivatives, as expressed by É. Cartan in 1922 ([7], section 74), thus years before the concept of Dirac’s delta. See also [8], sections 30 and 31.

Let \( z \) represent the unit differential \( n^{-}\)form in Euclidean space \( E_n \). Clearly \( z = dr \, r^{n-1} \Omega_{n-1} \), where the differential \( (n - 1)^{-}\)form \( \Omega_{n-1} \) is the unit element of “solid angle” and where \( r \) is the radial coordinate in \( n^{-}\)-dimensional space. Using a symbol \( \partial \) to be later explained, we write the Laplacian as \( \partial \partial \).

Given a region \( R \) (dimension \( n \)) of \( E_n \) and containing the origin, one applies Stokes theorem to the integral \( (\partial \partial r^{-\lambda})z \) with undetermined integer \( \lambda \), and where \( r \) is the length of the radius vector. One thus has

\[
\int_{R} \left( \partial \partial \frac{1}{r^{\lambda}} \right) z = - \int_{\partial R} \lambda r^{-(\lambda+1)} dr \, z = - \int_{\partial R} \lambda \Omega_{n-1} \frac{r^{n-1}}{r^{\lambda+1}} \]

(2)

With \( S_{n-1} \) as solid angle, the last term becomes \( -(n - 2)S_{n-1} \) if we choose \( \lambda \) to be equal to \( n - 2 \). We take \( -(n - 2)S_{n-1} \) to the opposite side in (2) to emphasize the independence of dimension:

\[
1 = - \frac{1}{(n - 2)S_{n-1}} \int_{R} \left( \partial \partial \frac{1}{r^{n-2}} \right) z.
\]

(3)

2 Strategy

Kähler based his calculus on Kähler algebra, i.e. the Clifford algebra defined by

\[
dx^i dx^j + dx^j dx^i = 2g^{ij}.
\]

(4)
When at least one of two factors in a Clifford product is of grade one, it can be decomposed as sum of exterior and interior products. Correspondingly, Kähler’s comprehensive derivative, here denoted as $\partial$, has two pieces. We shall refer to them as exterior and interior derivatives ($d$ and $\delta$ respectively), given that they emerge through that composition of the Clifford product (See equation (8)). Neither Kähler’s comprehensive derivative, nor its interior part, nor the exterior derivative of products other than exterior ones satisfies the standard Leibniz rule (See next section). Yet, as he did, we shall use the term derivative even in cases when that rule is not satisfied.

We shall first provide the main formulas of the KC to be used in the paper. We then proceed to do for differential 1–forms in 3-D Euclidean space, $E_3$, what Helmholtz did for vector fields. Here the main difficulty to be dealt with is that whereas the second exterior derivative is zero, the curl of the curl of a vector field is not zero in general. So, there is not total parallelism in the proofs pertaining to vector fields, on the one hand, and differential 1–forms, on the other. We have dealt with this issue in [4] and there is nothing to be added except for starting to use the notation that will later be valid for arbitrary grade in $E_n$.

A Helmholtz theorem for differential 2–forms in $E_3$ is achieved by expressing in the so obtained theorem the differential 1–form in terms of its dual. It is then simply a matter of solving for the differential 2–form. Some needed polishing of the proof in [4] is given. Comparison of those two theorems makes obvious their generalization to arbitrary grade in arbitrary dimension.

Retrospectively, Helmholtz theorem will be seen as a particular case of our better Hodge theorem, or the latter one as a generalization of the first. The proofs of the intermediate versions of theorems in the evolution from Helmholtz to Hodge are all cut by the same pattern. Because of the profuse intermingling of appearances of the $d$ and $\delta$ operators, we proceed to produce that general pattern.

We start, like Helmholtz did, with the hypothetical form of (in our case) a differential form $\alpha$ as a sum of terms where those operators appear in expressions of the form

$$\alpha = d \left( \ldots \int \ldots \delta' \alpha' \ldots \right) + \delta \left( \ldots \int \ldots d' \alpha' \ldots \right). \tag{5}$$

We are then led to compute only

$$\delta d \left( \ldots \int \ldots \delta' \alpha' \ldots \right), \quad \text{and} \quad d \delta \left( \ldots \int \ldots d' \alpha' \ldots \right). \tag{6}$$
since $dd$ and $\delta\delta$ vanish identically. The nature of the contents of the integrands allows one to, following due process, introduce $\delta d$ and $d\delta$ inside the integral as $\delta' d'$ and $d' \delta'$. One then replaces $\delta' d'$ with $\partial\partial' - d' \delta'$, and $d' \delta'$ with $\partial\partial' - \delta' d'$. The integration of the $\partial\partial'$ terms are the ones that will produce $\delta\alpha$ and $d\alpha$ when applying $\delta$ and $d$ to the right hand side of (5), respectively.

After doing that, we begin the transition to our richer Hodge theorem. It is based on embedding Riemannian manifolds $R^n$, on Euclidean spaces, $E_N$. Recall that the standard Hodge theorem has three terms, one of them closed, another one co-closed and a third one which is harmonic, the last one being zero in Helmholtz theorem. But it would not be zero if, instead of integrating to the whole of $E_n$, and of $E_3$ in particular, we integrated to regions thereof.

The next step in the argument consists in embedding a Riemannian manifold in a Euclidean space. Such embedding gives us a $n-$hypersurface in a $N-$Euclidean space. A hypersurface is not a region, which has the same dimension as the Euclidean space as a matter of definition of the term. By the time we shall have reached this step, it will be obvious that this is not a problem whatsoever, which would be if dealing with vector fields (we shall discuss this in a later section). In the case of a region as in the case of a hypersurface, it is the integration of the term containing $d' \delta'$ (arising from the replacement of $\delta' d'$ with $\partial\partial' - d' \delta'$), and of the term $\delta' d'$ (arising from the replacement of $d' \delta'$ with $\partial\partial' - \delta' d'$) that gives rise to the harmonic term in Hodge’s theorem. The harmonic term is an effect of the boundary condition. Said better, it is a reflection of what happens outside the boundary.

3 Basic Kähler calculus

3.1 Concepts

For dealing with Riemannian spaces, we shall embed them in Euclidean spaces. We may then use Cartesian coordinates to simplify the computations. We shall use Roman characters for differential forms in equations from Kähler’s papers and Greek letters otherwise.

Kähler defines covariant derivatives of differential forms. In terms of Cartesian coordinates, take the very simple form

$$d_h v = \frac{\partial v}{\partial x^h}.$$  \hspace{1cm} (7)
when they are scalar-valued. \( d_h \) satisfies the Leibniz rule,

\[
d_h(u \lor v) = d_h u \lor v + u \lor d_h v,
\]

(8)

As mentioned in the previous section,

\[
\partial v \equiv dx^h \lor d_h v = dv + \delta v.
\]

(9)

\[
dv \equiv dx^h \land d_h v, \quad \delta v \equiv dx^h \cdot d_h v.
\]

(10)

dv is the exterior derivative. For \( \delta v \), see below.

It follows from (9) that, in principle, \( \partial \partial \) consists of four terms. One of them is \( dd \), which, as we know, is zero. Similarly, \( \delta \delta \) equals 0 if \( d_h \) is computed with the Levi-Civita connection. This is automatically the case in Euclidean spaces. We thus have the well known equation

\[
\partial \partial = \delta d + d \delta.
\]

(11)

### 3.2 Differentiation of products

Let operators \( \eta \) and \( e^h \) be distributive operators defined on \( k \)-forms as

\[
\eta u_r = (-1)^r u_r, \quad e^h u = dx^h \cdot u,
\]

(12)

The following two sets of equations

\[
\partial(u \lor v) = \partial u \lor v + \eta u \lor \partial v + 2e^h u \lor d_h v,
\]

(13)

\[
d(u \lor v) = du \lor v + \eta u \lor dv + e^h u \lor d_h v - \eta d_h u \lor e_h v,
\]

(14)

\[
\delta(u \lor v) = \delta u \lor v + \eta u \lor \delta v + e^h u \lor d_h v + \eta d_h u \lor e_h v
\]

(15)

and

\[
\partial(u \land v) = \partial u \land v + \eta u \land \partial v + e^h u \land d_h v + \eta d_h u \land e_h v,
\]

(16)

\[
d(u \land v) = du \land v + \eta u \land dv,
\]

(17)

\[
\delta(u \land v) = \delta u \land v + \eta u \land \delta v + e^h u \land d_h v + \eta d_h u \land e_h v
\]

(18)

provide some of the flavor of differentiation in KC.

When using (8) to get Eq. (13), one has to pass \( dx^h \) to the right of \( u \) to multiply \( d_h v \). It is in this process that the last term arises. In Eqs. (14)-(16) and (18), there are other terms arising from the interplay of the products \( \lor \)
and ∧, one of them explicit and the other one implicit in each of $d(u \lor v)$ and $\partial(u \land v)$. Equations (15) and (8) are the respective differences between (13) and (14), on the one hand, and (16) and (17) on the other.

Of great importance is the concept of constant differential, $c$, defined by $d_h c = 0$. Then

$$\partial(u \lor c) = (\partial u) \lor c, \quad (19)$$

but $\partial(c \lor u) \neq c \lor (\partial u)$. For differential 0–forms, $f$, we have, using (18),

$$\delta(cf) = (\eta c)\delta f + (dx^h \cdot c) d_h f = 0 + df \cdot c = -\eta c \cdot df, \quad (20)$$

since $\delta f$ is zero and $d_h f$ is a 0–form. Polynomials in Cartesian $dx$'s with constant coefficients are constant differentials.

### 3.3 About the interior derivative

When the connection is Levi-Civita's, $\delta$ is the co-derivative. In $E_n$:

$$\delta v = (-1)^{n(n-1)/2}zd(vz), \quad z\delta v = d(vz), \quad (21)$$

where $z$ is the unit $n$–form for given orientation. Its square is $(-1)^{n(n-1)/2}$. Define $u \equiv vz$. Then $v$ equals $(-1)^{n(n-1)/2}uz$. Multiply the second of (21) by $z$. We get:

$$z\delta[(-1)^{n(n-1)/2}uz] = \delta(uz) = zdu. \quad (22)$$

We have obtained the following useful formulas

$$zdu = \delta(uz), \quad z\delta u = d(uz), \quad (23)$$

the last one being the same as in (21), after a change of notation.

### 4 Helmholtz Uniqueness

We now show that, under the usual conditions for Helmholtz theorem, the solution is unique for differential $k$–forms, i.e. of defined grade (here named homogeneous). The proof is based on Kähler’s very comprehensive version for differential forms [1], [3] of Green’s symmetric theorem, also called second Green identity.
4.1 Kähler’s Green theorem

Any differential form can be written as a sum of monomials. Assume each monomial written as a product of differential 1-forms. The operator $\zeta$ will denote the reversion of all factors in such products.

Let the subscript zero denote the 0–form part. Kähler defines the scalar product of order zero as the differential $n$–form

$$(u, v) \equiv (\zeta u \vee v)_0 w z = (\zeta u \vee v) \wedge w. \quad (24)$$

If $u$ and $v$ are homogeneous, it is necessary but not sufficient condition that $r = s$ in $(u_r, v_s)$ to be different from zero. He defines the scalar product as the differential $(n - 1)$–form $(u, v)_1$ defined by

$$(u, v)_1 \equiv dx^i \cdot (dx^i \vee u, v). \quad (25)$$

He then proves the “Green-Kähler theorem”:

$$d(u, v)_1 = (u, \partial v) + (v, \partial u) \quad (26)$$

4.2 Helmholtz uniqueness for differential $k$–forms

Let $(u_1, u_2)$ be differential $k$–forms such that $du_1 = du_2$, $\delta u_1 = \delta u_2$ on a differential manifold, $R$, and such that, at the boundary, $\partial R$, $u_1$ equals $u_2$. Define $\beta = u_1 - u_2$. Hence

$$d\beta = 0 = \delta \beta \text{ on } R, \quad \beta = 0 \text{ on } \partial R, \quad (27)$$

and, locally,

$$(\beta = d\alpha, \delta d\alpha = 0) \text{ on } R, \quad d\alpha = 0 \text{ on } \partial R. \quad (28)$$

Equation (26) with $u = \alpha$ and $v = d\alpha$ reads

$$d(\alpha, d\alpha)_1 = (\alpha, \partial d\alpha) + (d\alpha, \partial \alpha). \quad (29)$$

By (9) and (28), we have

$$(\alpha, \partial d\alpha) = (\alpha, dd\alpha) + (\alpha, \delta d\alpha) = 0 + 0. \quad (30)$$

Consider next $(d\alpha, \partial \alpha)$ and use again (9). If $\alpha$ is of definite grade, so are $d\alpha$ and $\delta \alpha$, but their grades differ by two units. Their scalar product is,
therefore, zero. On the other hand, we have, with \( a_A \) defined by \( d\alpha = a_A dx^A \) (with summation over the algebra as a module),

\[
(d\alpha, \partial\alpha) = (d\alpha, \delta\alpha) + (d\alpha, d\alpha) = 0 + \sum |a_A|^2. \tag{31}
\]

Substituting (30)-(31) in (29), applying Stokes theorem and using \( d\alpha = 0 \), we get

\[
0 = \int_{\partial R} (\alpha, d\alpha) = \int_R d(\alpha, d\alpha) = \int_R \sum |a_A|^2. \tag{32}
\]

Thus \( d\alpha = 0 \), \( \beta = u_1 - u_2 \), and, therefore, \( u_1 = u_2 \). Uniqueness under conditions like those for Helmholtz theorem has been proved.

Let us not overlook that this theorem has been derived under the assumption that \( \alpha \) and, therefore, \( \beta \) and the \( u \)'s are of definite grades. It appears that there is no Helmholtz uniqueness theorem for inhomogeneous differential forms. The equation \( (d\alpha, \delta\alpha) = 0 \) constitutes a formidable system of equations, likely to have infinite solutions (even if the space were two dimensional!) This is nevertheless no impediment to prove Hodge’s standard decomposition theorem.

5 Helmholtz Theorems for \( k \)-forms

5.1 Helmholtz Theorem for \( k \)-forms in \( E_3 \)

In this section, we shall try to avoid potential confusion by replacing the symbol \( z \) with the symbol \( w \).

With \( r_{12} \equiv [(x-x')^2 + (y-y')^2 + (z-z')^2]^{1/2} \), the standard Helmholtz theorem states

\[
v = -\frac{1}{4\pi} \nabla \int_{E'_3} \frac{\nabla' \cdot v(r')}{r_{12}} dV' + \frac{1}{4\pi} \nabla \times \int_{E'_3} \frac{\nabla' \times v(r')}{r_{12}} dV'. \tag{33}
\]

By proceeding in parallel to the proof of (33), we showed in [4] that Helmholtz theorem for differential 1-forms in \( E_3 \) reads

\[
\alpha = -\frac{1}{4\pi} d \int_{E'_3} \frac{(\delta'\alpha')w'}{r_{12}} - \frac{\delta}{4\pi} \left( dx^j dx^k \int_{E'_3} \frac{d'\alpha' \wedge dx^h}{r_{12}} \right), \tag{34}
\]

This theorem is a particular case of the theorem in the next subsection. Needless to say that inputs \( \delta\alpha \) and \( d\alpha \) must be 0–form and 2–forms.
The equation $\alpha = w\beta$, uniquely defines $\beta$. We substitute it in (34) and solve for $\beta$:

$$\beta = \frac{1}{4\pi}wd\left(\int_{E_3'} \frac{\delta'(w'\beta')}{r_{12}} w'\right) + \frac{1}{4\pi}w\delta \left(\int_{E_3'} \frac{d'(w'\beta') \wedge dx'^i}{r_{12}}\right).$$  \hspace{1cm} (35)

We use (18) to move $w$ to the right of $d$ and $\delta$. For details, we refer to [4], except for the following simplifications.

Denote the first integral in (35) as $I$ and the second one as $I_i$. We have $wdI = \delta(wI)$ and $w'\delta'(w'\beta') = w'(w'd\beta') = -d\beta' = -d\beta' \wedge 1$. The exterior product by 1 is superfluous, except for the purpose for making later Eq. (39) clear. Similarly, $w\delta(dx^{jk}I^i) = d(dx^{jk}I^i)$ and $d'(w'\beta') \wedge dx'^i = dx'^i \wedge d'(w'\beta') = 1$.

We use these results in (35), change the order of the terms and get

$$\beta = -\frac{1}{4\pi}d\left(dx^i \int_{E_3'} \frac{\delta' \beta' \wedge dx^{jk}}{r_{12}}\right) - \frac{1}{4\pi}\delta \left(w \int_{E_3'} \frac{\delta' \beta' \wedge 1}{r_{12}}\right).$$  \hspace{1cm} (37)

We use these results in (35), change the order of the terms and get

$$\beta = -\frac{1}{4\pi}d\left(\int_{E_3'} \delta' \beta' \wedge dx^{jk}\right) \wedge dx^i \wedge d'\left(w'\beta' \wedge dx'^i\right) = \frac{1}{2} \left[dx^i w'\delta' \beta' + w'\delta' \beta' dx^i\right] =$$

$$= \frac{1}{2} (dx^{jk} \delta' \beta' + \delta' \beta' dx^{jk}) = dx^{jk} \wedge \delta' \beta' = \delta' \beta' \wedge dx^{jk}. \hspace{1cm} (36)

We use these results in (35), change the order of the terms and get

$$\beta = -\frac{1}{4\pi}d\left(\int_{E_3'} \frac{\delta' \beta' \wedge dx^{jk}}{r_{12}}\right) - \frac{1}{4\pi}\delta \left(w \int_{E_3'} \frac{\delta' \beta' \wedge 1}{r_{12}}\right).$$  \hspace{1cm} (37)

Write the first term in (34) as

$$-\frac{1}{4\pi}d\left[1 \wedge \int_{E_3'} \frac{(\delta' \alpha') \wedge w'}{r_{12}}\right]. \hspace{1cm} (38)$$

Let the index $A$ label a Cartesian basis of the algebra as module. Let $dx^A$ be the unique element in the basis such that $dx^A \wedge dx^A = w$. Define $\int_{E_3} \gamma_r$ if the grade $r$ of $\gamma$ is different from 3. All four terms on the right of (34) and (37) are thus of the form

$$-\frac{1}{4\pi}d\left(dx^A \int_{E_3'} \frac{(\delta' \wedge dx^A)}{r_{12}}\right) \wedge dx^i \wedge d'\left(w' \beta' \wedge dx'^i\right) =$$

$$= -\frac{1}{4\pi}d\left(dx^A \int_{E_3'} \frac{(d' \wedge dx^A)}{r_{12}}\right).$$  \hspace{1cm} (39)
that may yield not zero integral since the sum of the respective grades must be 3. For each surviving value of the index $\bar{A}$, the value of the index $A$—thus the specific $dx^A$ at the front of the integral—is determined. We shall later show for ulterior generalization that we may replace the Cartesian basis with any other basis, which we shall choose to be orthonormal since they are the “canonical ones” of Riemannian spaces.

5.2 Helmholtz Theorem for Differential k-forms in $E_n$

Let $\omega^A(\equiv \omega^i \omega^j ... \omega^r)$ denote elements of a basis in the Kähler algebra of differential forms such that the $\omega^\mu$ are orthonormal. The purpose of using an orthonormal basis is that exterior products can be replaced with Clifford products. Let $\omega^A$ be the monomial (uniquely) defined by $\omega^A \omega^{\bar{A}} = z$, with no sum over repeated indices.

The generalized Helmholtz theorem in $E_n$ reads as follows

$$\alpha = -\frac{1}{(n-2)S_{n-1}}[d(\omega^A I_\delta^\bar{A}) + \delta(\omega^A I^d_\bar{A})],$$

with summation over a basis in the algebra and where

$$I_\delta^\bar{A} \equiv \int_{E'_n} \frac{(\delta' \alpha') \wedge \omega^{\bar{A}}}{r_{12}^{n-2}}, \quad I^d_\bar{A} \equiv \int_{E'_n} \frac{(d' \alpha') \wedge \omega^{\bar{A}}}{r_{12}^{n-2}}. \quad (41)$$

$r_{12}$ is defined by $r_{12}^2 = (x_1 - x'_1)^2 + ... + (x_n - x'_n)^2$ in terms of Cartesian coordinates.

It proves convenient for performing differentiations to replace $\omega^i$, $\omega^A$ and $\omega^{\bar{A}}$ with $dx^i$, $dx^A$ and $dx^{\bar{A}}$. If the results obtained are invariants, one can re-express the results in terms of arbitrary bases.

We proceed again via the uniqueness theorem, as in the vector calculus, with specification now of $d\alpha$, $\delta \alpha$ and that $\alpha$ goes sufficiently fast at $\infty$. vanishing of $\alpha$ at infinity. Because of the annulment of $d\delta$ and $\delta d$, the proof reduces to showing that $\delta d(dx^A I_\delta^\bar{A})$ and $d\delta(dx^A I^d_\bar{A})$ respectively yield $\delta \alpha$ and $d\alpha$, up to the factor at the front in (40). Since the treatment of both terms is the same, we shall carry them in parallel, as in

$$\left(\begin{array}{c} \delta \\ d \end{array}\right) \alpha \rightarrow \left(\begin{array}{c} \delta d \\ d\delta \end{array}\right) dx^A I_\delta^\bar{A} = \partial \delta dx^A I_\delta^\bar{A} \left(\begin{array}{c} \delta d \\ d\delta \end{array}\right) dx^A I^d_\bar{A}. \quad (42)$$

In the first term on the right hand side of (42), we move $\partial \delta$ to the right of $dx^A$, insert it inside the integral with primed variables, multiply by $-\frac{1}{(n-2)S_{n-1}}$ and
treat the integrand as a distribution. We easily obtain that the first term yields \( \delta \alpha \).

For the last term in (42), we have

\[
( \delta \alpha \cdot dx^A ) = \left( \delta \left[ (\eta dx^\alpha) \land dx^i \frac{\partial I_A}{\partial x^i} \right] \right).
\]

(43)

For the first line in (43), we have used (7) and the second equation (10). For the development of the second line, we have used the Leibniz rule.

We use the same rule to also transform the first line in (43),

\[
d \left( dx^i \cdot dx^A \frac{\partial I_A}{\partial x^i} \right) = \left[ \eta (dx^i, dx^A) \right] \land dx^i \frac{\partial^2 I_A}{\partial x^i \partial x^i} = (dx^A \cdot dx^i) \land dx^i \frac{\partial^2 I_A}{\partial x^i \partial x^i}.
\]

(44)

For the second line, we get

\[
\delta \left[ (\eta dx^A) \land dx^i \frac{\partial I_A}{\partial x^i} \right] = dx^i \cdot \left[ \frac{\partial^2 I_A}{\partial x^i \partial x^i} (\eta dx^A) \land dx^i \right].
\]

(45)

We shall use here that

\[
dx^i [ (\eta dx^A) \land dx^i ] = -\eta (dx^A \land dx^i) \cdot dx^i = (dx^A \land dx^i) \cdot dx^i,
\]

(46)

thus obtaining

\[
\delta \left[ (\eta dx^A) \land dx^i \frac{\partial I_A}{\partial x^i} \right] = (dx^A \land dx^i) \cdot dx^i \frac{\partial^2 I_A}{\partial x^i \partial x^i}.
\]

(47)

Getting (44) and (47) into (43), we obtain

\[
\left( \frac{d\delta}{\delta d} \right) dx^A I_A^{(\delta)} = \left[ dx^A (\cdot \cdot dx^i) \right] (\cdot) dx^i \int_{\partial E_n} \frac{\partial^2}{\partial x^i \partial x^i} \frac{1}{r_{12}^{n-2}} \left( \delta \alpha \right) \land dx^A.
\]

(48)

Integration by parts with respect to \( x^i \) yields two terms. The total differential term is

\[
\left[ dx^A (\cdot \cdot dx^i) \right] (\cdot) dx^i \int_{\partial E_n} \frac{\partial}{\partial x^i} \frac{1}{r_{12}^{n-2}} \left( \delta \alpha \right) \land dx^A.
\]

(49)

Application of Stokes theorem yields

\[
\left[ dx^A (\cdot \cdot dx^i) \right] (\cdot) dx^i \int_{\partial E_n} \frac{\partial}{\partial x^i} \frac{1}{r_{12}^{n-2}} \left\{ dx^i \cdot \left[ \left( \delta \alpha \right) \land dx^A \right] \right\}, \]

(50)
where we have indulged in the use of parentheses for greater clarity. It is null if the differentiations of $\alpha$ go sufficiently fast to zero at infinity.

The other term resulting from the integration by parts is

$$- \left[ dx^A(\cdot) dx^i \right] \int_{E_n^\prime} \left( \frac{\partial}{\partial x'^{\mu}} \frac{1}{r^{n-2}_{12}} \right) \frac{\partial}{\partial x'^{\mu}} \left( \left( \frac{\delta' \alpha'}{d' \alpha'} \right) \wedge dx'^A \right). \quad (51)$$

This is zero because of cancellations that take place in groups of three different indices. We shall devote the next subsection to dealing with the intricacies of such cancellations.

In terms of Cartesian bases, we have, on the top line of the left hand side of (42)

$$dx^A \int_{E_n^\prime} \frac{(\delta' \alpha') \wedge dx'^A}{r^{n-2}_{12}}. \quad (52)$$

It is preceded by invariant operators, which we may ignore for present purposes. We move $dx^A$ inside the integral, where we let $(\delta' \alpha')_A$ be the notation for the coefficients of $\delta' \alpha'$. We thus have, for that first term,

$$\int_{E_n^\prime} dx^A \wedge ([\delta' \alpha']_A dx'^A] \wedge dx'^A \quad (53)$$

The numerator can be further written as $(\delta' \alpha')_A dx^A z'$. It is clear that $z$ and $(\delta' \alpha')_A dx'^A$ are invariants, but not immediately clear that $(\delta' \alpha')_A dx^A$ also is so. Whether we have the basis $dx^A$ or $dx'^A$ as a factor is immaterial. since the invariance of $(\delta' \alpha')_A dx'^A$ can be seen as following from the matching of the transformations of $(\delta' \alpha')_A$ and $dx'^A$ each in accordance with its type of covariance. The same matching applies if we replace $\omega'^A$ with $dx^A$, since $dx'^A$ and $dx^A$ transform in unison.

We have shown that (40) constitutes the decomposition of $\alpha$ into closed and co-closed terms. Together with (41), it solves the problem of integrating the system $d\alpha = \mu, \delta \alpha = \nu$, for given $\mu$ and $\nu$, and with the stated boundary condition

**5.3 Identical vanishing of some integrals**

As we are about to show, expressions (51) cancel identically (Expression (50) cancels at infinity for fast vanishing, not identically).

Consider the first line in (51). Let $\alpha$ be of grade $h \geq 2$ (If $h$ were one, the dot product of $dx^A$ with $dx^i$ would be zero). Let $p$ and $q$ be a specific pair
of indices in a given term in $\alpha$, i.e. in its projection $a'_{pqC}dx^p \wedge dx^q \wedge dx^C$, upon some specific basis element $dx^A$. Such a projection can be written as
\[ (a'_{pqC}dx^p \wedge dx^q \wedge dx^C), \]
where $dx^A$ is a unit monomial differential 1–form (there is no sum over repeated indices. We could also have chosen to write the same term as
\[ (a'_{qpC}dx^q \wedge dx^p \wedge dx^C), \]
with $a'_{qpC} = -a'_{pqC}$. Clearly, $dx^C$ is uniquely determined if it is not to contain $dx^p$ and $dx^q$. We then have
\[ \delta'(a_{pqC}dx^p \wedge dx^q \wedge dx^C) = a'_{pqC}dx^q \wedge dx^C - a'_{pqC}dx^p \wedge dx^C. \quad (54) \]
The two terms on the right are two different differential 2–forms. They enter two different integrals, corresponding to $dx^q \wedge dx^C$ and $dx^p \wedge dx^C$ components of $\delta'\alpha'$. To avoid confusion, we shall refer to the basis elements in the integrals as $dx^B$ since they are $(h-1)$–forms, unlike the $dx^A$ of (54), which are differential $h$–forms.

When taking the first term of (54) with $i = p$ into the top line of (51), the factor at the front of the integral is
\[ -(dx^B \cdot dx^p) \wedge dx^l. \]
But this factor is zero since $dx^B$ is $dx^q \wedge dx^C$, which does not contain $dx^p$ as a factor. Hence, for the first term, we need only consider $i = q$. By the same argument, we need only consider $i = q$ for the second term in (54). Upon multiplying the $dx^B$’s by pertinent $dx^B$’s, we shall obtain the combination
\[ (a'_{pqC} \cdot pp' - a'_{pqC} \cdot qp') z', \]
as a factor inside the integral for the first line of (51). We could make this statement because the factor outside also is the same one for both terms: $(dx^q \wedge dx^C) \cdot dx^q$ and $(dx^p \wedge dx^C) \cdot dx^p$ are equal. The contributions arising from the two terms on the right hand side of (54) thus cancel each other out. We would proceed similarly with any other pair of indices, among them those containing either $p$ or $q$. The annulment of the top line of (51) has been proved.

In order to prove the cancellation of the second line in (51), the following considerations will be needed. A given $dx^A$ determines its corresponding
and vice versa. It follows then that only the term proportional to \(dx^i\) in \(d'd\alpha'\) exterior multiplies \(dx^i\), which is of the same grade as \(d'a\), i.e. \(h+1\). Hence \(dx^A \wedge dx^i\) is of grade 3 or greater for \(h > 0\). If \(dx^A \wedge dx^i\) is not to be null, \(dx^i\) cannot be in \(dx^A\). Hence, \(dx^A\) contains \(dx^i\) as a factor.

Let \((p, q, r)\) be a triple of three different indices in \(dx^A \wedge dx^i\). When \(i\) is \(p\) or \(q\) or \(r\), the respective pairs \((q, r)\), \((r, p)\) and \((p, q)\) are in \(dx^A\). We may thus write

\[
dx'^A = dx'^C \wedge dx'^q \wedge dx'^r, \quad dx'^A = dx'^p \wedge dx'^B. \tag{55}
\]

The coefficient of \(dx'^A\) in \(d'd\alpha'\) will be the sum of three terms, one of which is

\[
(a'_C q r - a'_C q r) dx'^q \wedge dx'^C \wedge dx'^r, \tag{56}
\]

and the other two are cyclic permutations. We partial-differentiate (56) with respect to \(dx'^p\) and multiply by \(dx'^p \wedge dx'^B\) on the right. We proceed similarly with \(i = q\) and \(i = r\), and add all these contributions. We thus get

\[
(a'_C q r p - a'_C q r p + a'_C p r q - a'_C p r q + a'_C q p r - a'_C q p r) z'. \tag{57}
\]

By virtue of equality of second partial derivatives, terms first, second and third inside the parenthesis cancel with terms fourth, fifth and sixth. To complete the proof, we follow the same process with another \(dx'^C\) and the same triple \((p, q, r)\) until we exhaust all the options. We then proceed to choose another triple and repeat the same process until we are done with all the terms, which completes the proof of identical vanishing of the second term arising from one of the two integrations by parts of the previous subsection.

### 6 Hodge’s Theorems

The “beyond” in the title of this section responds to the fact that we shall be doing much more than reproducing Hodge’s theorem. As is the case with Helmholtz theorem, we are able to specify in terms of integrals what the different terms are.

We shall later embed Riemannian spaces \(R_n\) in Euclidean spaces \(E_N\), thus becoming \(n\)-surfaces. As an intermediate step, we shall apply the traditional Helmholtz approach to regions of Euclidean spaces, i.e. \(R_n\)'s ab initio embedded in \(E_n\). The harmonic form—which is of the essence in Hodge’s theorem—emerges from the Helmholtz process in the new venues.
6.1 Transition from Helmholtz to Hodge

Though visualization is not essential to follow the argument, it helps for staying focused. For that reason, we shall argue in 3-D Euclidean space. It does not interfere with the nature of the argument.

On a region $R$ of $E_3$, including the boundary, define a differential 1-form or 2-form $\alpha$. Let $A$ denote any continuously differentiable prolongation of $\alpha$ that vanish sufficiently fast at infinity. On $R$, we have $dA = d\alpha$ and $\delta A = \delta \alpha$. We can apply Helmholtz theorem to the differential forms $A$. In order to minimize clutter, we write it in the form

$$-4\pi A = d\ldots \int_{R'} \frac{\delta' A'}{r_{12}} + \delta \int_{R'} \frac{d' A'}{r_{12}} +$$

$$+d\ldots \int_{E_3-R'} \frac{\delta' A'}{r_{12}} + \delta \ldots \int_{E_3-R'} \frac{d' A'}{r_{12}},$$

(58)

where $r_{12} = [(x - x')^2 + (y - y')^2 + (z - z')^2]^{1/2}$. We shall keep track of the fact, at this point obvious, that in the first two integrals on the right, $r'$ is in $R'$. It is outside $R'$ in the other two integrals, which will depend on the prolongation. By representing those terms simply as $F$, we have

$$-4\pi A = d\ldots \int_{R'} \frac{\delta' \alpha'}{r} + \delta \ldots \int_{R'} \frac{d' \alpha'}{r} + F.$$  

(59)

Since these equations yield $A$ everywhere in $E_3$ (i.e. $r$ not limited to $R$), they yield in particular what $A$ and $F$ are in $R$. We can thus write

$$-4\pi \alpha = d\ldots \int_{R'} \frac{\delta' \alpha'}{r} + \delta \ldots \int_{R'} \frac{d' \alpha'}{r} + F,$$

(60)

$F$ not having changed except that $F$ in (60) refers only to what it is in $R$ but it remains a sum of integrals in $E_3-R$. The prolongations will be determined as different solutions of a differential system to be obtained as follows.

By following the same process as in Helmholtz theorem, we obtain, in particular,

$$-4\pi d\alpha = d\delta \ldots \int_{R'} \frac{d' \alpha'}{r} + dF,$$

(61)

and similarly for $-4\pi \delta \alpha$ (just exchange $d$ and $\delta$).

Now, the first term on the right hand side of (61) will not become simply $-4\pi d\alpha$ as was the case in the previous section. It will yield two terms.
One of them is $-4\pi d\alpha$, and the other one is made to cancel with $dF$, thus determining a differential equation to be satisfied by $F$. To this we have to add another differential equation arising from application of $\delta$ to (60). Together they determine the differential system to be determined by $F$. Thus $-4\pi \alpha$ will be given by the three term decomposition (60). Notice that, in the process, we avoid integrating over $E'_3 - R'$ and instead solving a differential system in $R$, since the left hand side and the first term on the right hand side of (61) pertain to $\alpha$.

From now one, we shall make part of the theorems that the prolongations are solutions of a certain differential systems, later to be made explicit.

### 6.2 Hodge theorem in regions of $E_n$

Let $\alpha$ be a differential $k$–form satisfying the equations $d\alpha = \mu$ and $\delta\alpha = \nu$, and given at the boundary of a region of $E_n$. We proceed to integrate this system. (60) now reads

$$-(n-2)S_{n-1}\alpha = d \left[ \int_{R'} \frac{\delta\alpha'}{r_{12}} \right] + \delta \left[ \int_{R'} \frac{d\alpha'}{r_{12}} \right] + F,$$

where $R$ is a region of Euclidean space that contains the origin and where $r_{12}$ is the magnitude of the Euclidean distance between hypothetical points of components $(x, y, ... u, v)$ and $(x', y', ... u', v')$, all the coordinates chosen as Cartesian to simplify visualization. We said hypothetical because the interpretation as distance only makes sense when we superimpose $E_n$ and $E'_n$.

When we apply either $d$ or $\delta$ to (62), we shall use, as before, $d\delta + \delta d = \partial \partial$, with one of the terms on the left moved to the right ($d\delta = ...$, $\delta d = ...$ respectively). By developing the $\partial \partial$ term, it becomes the same as term on the right (i.e $d\alpha$ or $\delta\alpha$). It will cancel with the term on the left. The terms that vanished identically also vanish now, precisely because this is an identical vanishing. We are thus left with the total differential terms. If apply Stokes theorem, as before, these terms no longer disappear at the boundary. Hence, we are left with the two equations

$$\left[dx^\Lambda(,^\Lambda)dx^i\right](^\Lambda)dx^\Lambda \int_{R'} \left( \frac{\partial}{\partial x^i} \right) \frac{1}{r_{12}} \left[ \left( \frac{\delta\alpha'}{d\alpha'} \right) \wedge dx^\Lambda \right] + \left( \frac{d}{\delta} \right) F = 0$$

(63)
(Refer to (50)). Hence, the solution to Helmholtz problem is given by the pair of equations (62)-(63).

We shall now show that $F$ is harmonic, i.e. $(d\delta + \delta d)F = 0$. We shall apply $\delta$ and $d$ to the first and second lines of (63). Start by rewriting the first terms in (63) in the form, (49), they took before applying Stoke’s theorem. Upon applying the $\delta$ operator to the first line, we have, for $d\delta F$,

$$dx^h \cdot [(dx^A \cdot dx^i) \wedge dx^j] \int_{R^l} \frac{\partial^2}{\partial x^h \partial x^n} \left[ \left( \frac{\partial}{\partial x^r} \frac{1}{r_{12}^{n-2}} \right) \left( \frac{\delta' \alpha'}{d' \alpha'} \right) \right] \wedge dx^t. \quad (64)$$

Since this term happens to vanish, the computation will take place up to the factor $-1$, provided it is common to all terms in a development into explicit terms. We do so because (64) will be shown to vanish identically.

For $dx^h \cdot [(dx^A \cdot dx^i) \wedge dx^j]$ to be different from zero, $h$ and $i$ must be different and contained in $A$. Since $dx^l$ is not in $dx^A$, the product $dx^h \cdot dx^l$ is zero. Hence

$$dx^h \cdot [(dx^A \cdot dx^i) \wedge dx^j] = [dx^h \cdot (dx^A \cdot dx^i)] \wedge dx^j. \quad (65)$$

We can always write $dx^A$ as

$$dx^h \wedge dx^j \wedge dx^C \wedge dx^i. \quad (66)$$

This is antisymmetric in the pair $(i, h)$, which combines with the symmetry inside the integral to annul this term. Notice that we did not have to assign specific values for $(i, h)$, but we had to “go inside” $dx^A$. We mention this for contrast with the contents for the next paragraph. We have proved so far that $\delta dF = 0$.

We proceed to prove that $d\delta F = 0$. We rewrite the left hand side of (63) as in (49) and proceed to apply $d$ to it. We shall now have

$$dx^h \wedge [(dx^A \wedge dx^i) \cdot dx^j] \int_{R^l} \frac{\partial^2}{\partial x^h \partial x^n} \left[ \left( \frac{\partial}{\partial x^r} \frac{1}{r_{12}^{n-2}} \right) \right] \delta' \alpha' \wedge dx^t. \quad (67)$$

It is clear that, when $l$ takes a value different from the value taken by $i$, we again have cancellation due to the same combination of antisymmetry-symmetry as before. But the terms $dx^i \cdot dx^l$ would seem to interfere with the argument, but it does not. We simply have to be more specific than before with the groups of terms that we put together. We put together only terms where we have $dx^r \wedge dx^s$ arising from $(h = r, i = s)$ and $(h = s, i = r)$.  

18
When the running index \( l \) takes the values \( r \) or \( s \), the resulting factor at the front of the integral will belong to a different group. We have thus shown that (67) cancels out and, therefore, \( d\delta F = 0 \). To be precise, we have not only proved that \( F \) is harmonic, but that it is “hyper-harmonic”, meaning precisely that: \( \delta dF = 0 \) and \( d\delta F = 0 \).

### 6.3 Hodge’s theorem in hypersurfaces of \( E_N \)

A manifold embedded in a Euclidean space of the same dimension will be called a region thereof. A hypersurface is a manifold of dimension \( n \) embedded in a Euclidean space \( E_N \) where \( N > n \). The treatment here is the same as in subsection 6.1, the hypersurface playing the role of the region. The only issue that we need to deal with is a practical one having to do with the experience of readers. Helmholtz magnificent theorem belongs to an epoch where vector (and tensor) fields often took the place of differential forms. This can prompt false ideas as we now explain.

Let \( v \) be a vector field \( v \equiv a^\lambda(u,v)\hat{a}_\lambda \) (\( \lambda = 1, 2 \)) on a surface \( x^i(u,v) \) \((i = 1, 2, 3)\) embedded in \( E_3 \), the frame field \( \hat{a}_\lambda \) being orthonormal. It can be tangent or not tangent. By default, the vector field is zero over the remainder of \( E_3 \). In its present form, Helmholtz theorem would not work for this field since the volume integrals over \( E_3 \) would be zero. This is a spurious implication because the theorem should be about algebras of differential forms, not tangent spaces.

Let \( \mu \) be the differential 1–form \( a_\lambda(u,v)\tilde{\omega}^\lambda \), the basis \( \tilde{\omega}^\lambda \) being dual to the constant orthonormal basis field \( a_\lambda \). This duality yields \( a_\lambda = a^\lambda \). No specific curve is involved in the definition of \( \mu \), which is a function of curves, function determined by its coefficients \( a_\lambda(u,v) \) The specification of a vector field on a surface, \( v \), on the other hand needs to make reference to a surface for its definition. And yet the components of \( d\mu \) and \( \delta\mu \) (which respectively are a 2–form and a 0–form) enter non-null volume integrals, which pertain to 3–forms. The fact that most components (in the algebra) of an \( k \)–form are zero is totally irrelevant. The Helmholtz theorem for, say, a differential 1–form \( \mu \) can be formulated in any sufficiently high dimensional Euclidean space regardless of whether the “associated” vector field \( v \) is zero outside some surface.

Similarly, Helmholtz theorem for a differential \( n \)–form in \( E_N \) involves the integration of differential \( N \)–forms, built upon the interior differential \( (n - 1) \)–form and the exterior differential \( (n + 1) \)–form. In considering
simple examples (say a plane in 3-space), one can be misled or confused if one does not take into account the role of $1/r$, or else we might be obtaining an indefinite integral. Assume finite $\int \lambda(x, y) dx \wedge dy$ when integrating over the $xy$ plane. The integral $\int \lambda(x, y) dx \wedge dy \wedge dz$ would be divergent, but need not be so if there is some factor that goes to zero sufficiently fast at infinity of $z$ and $-z$.

It is also worth mentioning that — in the case of a hypersurface like in the case of a surface in $E_3$, $r_{12}$ represents a chord, which is not in the hypersurface.

A final issue worth addressing is the following. If the Laplacian of the appropriate power of $1/r$ is zero and it multiplies the differential form outside the region or outside the hypersurface, why does the prolongation make a difference. This is a pseudo-problem easy to understand already at the point of equations (2) and (3), i.e. before we deal with prolongations. Those Laplacians are not generalizations of functions and cannot, therefore, be treated as such \[9\].

We conclude this subsection with the observation that, in Helmholtz theorems for differential forms, the variety of disconnected concepts that enter Helmholtz theorem for vector fields on surfaces in $E_3$ (a vector field, a surface, a gradient, a divergence, a curl, integrands and $E_3$) merge into or directly connect with the concept of differential form as an integrand in the Helmholtz theorem of the Kähler calculus, where we have

(a) a differential 1-form, in lieu of a vector field,
(b) from which we construct through Kähler differentiation an inhomogeneous differential form, in lieu of divergence, gradient and curl,
(c) from which in turn we build and evaluate (read integrate) a differential 3-form, in lieu of three volume integrals, one each for the components of the vector field,
(d) and we restrict the coefficients of the differential forms in (a) and (b) to surfaces, in lieu of vector fields defined on surfaces.

6.4 Hodge theorems in Riemannian spaces

We shall consider a Helmholtz-Hodge extension of Hodge’s theorem (i.e. a theorem of integration) and the standard Hodge theorem, which is a consequence of the former.

Consider now a differentiable manifold $R_n$ endowed with a Euclidean metric. By the Schlafli-Janet-Cartan theorem [10],[11],[12], it can be embedded
in a Euclidean space of dimension $N = n(n+1)/2$. Hence, a Helmholtz-Hodge theorem follows for orientable Riemannian manifolds that satisfy the conditions for application of Stokes theorem by viewing them as hypersurfaces in Euclidean spaces. At this point in our argument, the positive definiteness of the metric is required, or else we would have to find a replacement for the Laplacians considered in previous sections. The result is local, meaning non global, remark made in case the term local might send some physicists in a different direction. For clarity, the evaluation of the Laplacian now satisfies

$$1 = \frac{1}{(N - 2)S_{N-1}} \int_{E_N} \partial \partial \frac{1}{r^{N-2}} z,$$

where $r$ is the radial coordinate in $N$-dimensional space. Needless to say that it also applies to regions and hypersurfaces of $E_N$ that contain the origin. As a consequence of the results in the previous subsections, we have the following.

**Helmholtz-Hodge theorem:**

Hodge’s theorem is constituted by Eqs. (69)-(71): For differential $k-$forms in Riemannian spaces $R_n$

$$-(N-2)S_{N-1} \alpha = d \left[ \omega^A \int_{R_n} \frac{(\delta' \alpha') \wedge \omega^{A'}}{r_{12}^{N-2}} \right] + \delta \left[ \omega^A \int_{R_n} \frac{(d' \alpha') \wedge \omega^{A'}}{r_{12}^{N-2}} \right] + \mathcal{F},$$

with $r_{12}$ being defined in any Euclidean space of dimension $N \geq n(n + 1)/2$ where we consider $R_n$ to be embedded.

As previously discussed, $r_{12}$ represents a chord. We insist once more that $\omega^A$ is determined by the specific term in $\delta' \alpha'$ and $d' \alpha'$ that it multiplies. $\mathcal{F}$ is undetermined by solutions of the system $\delta \alpha = 0$, $d \alpha = 0$. So is, therefore $\alpha$.

Hodge’s theorem, as opposed to Helmholtz-Hodge theorem, is about decomposition. Hence, once again, uniqueness refers to something different from the uniqueness in the theorem of subsection (3.2), which refers to a differential system.
One might be momentarily tempted to now apply (69) to (70). We would get an identity, \( \mathcal{F} = \mathcal{F} \), by virtue of the orthogonality of the subspace of the harmonic differential forms to the subspaces of closed and co-closed differential forms.

**Hodge’s theorem:**

Any differential \( k \)-form, whether of homogeneous grade or not, can be uniquely decomposed into closed, co-closed and hyper-harmonic terms. For differential \( k \)-forms, the theorem is an immediate consequence of (69). For differential forms which are not of homogeneous grade, the theorem also applies because one only needs to add the decompositions of the theorem for the different homogeneous \( k \)-forms that constitute the inhomogeneous differential form.

### 7 Concluding Remarks.

We have obtained by computation results that, to our knowledge, had not been addressed before. We have gone as far as obtaining a Helmholtz theorem for differential \( k \)-forms in Riemannian manifolds. By virtue of the acquisition in general of a third, harmonic term, we have come to call it Helmholtz-Hodge, which goes far beyond Hodge’s theorem.

The more general results may still keep us far away from practical applications of these theorems, except in isolated cases in low dimensions. The reason is that the solution of problems of embedding are not trivial. It is a symptom of their difficulty that the embedding of a 3-D manifold in a 6-D Euclidean space takes the last pages of Cartan’s treatise of integration of exterior systems, a book dedicated to the Cartan-Kähler theory [8].

Of more practical interest is the fact that these results show the tremendous potential of the Kähler calculus, both in physics and mathematics. It would be too self-serving to mention here specific results. Interested readers can go to the authors web site www.physical-unification.com for references.

### 8 Acknowledgements

Conversations with Prof. Z. Oziewicz are acknowledged. Funding from PST Associates is deeply appreciated.
References

[1] Kähler, E.: Innerer und äußerer Differentialkalkül. Abh. Dtsch. Akad. Wiss. Berlin, Kl. Math. Phys. Tech. 4, 1-32 (1960).

[2] Kähler, E.: Die Dirac Gleichung Abh. Dtsch. Akad. Wiss. Berlin, Kl. Math. Phys. Tech

[3] Kähler, E.: Der innere Differentialkalkül. Rendiconti di Matematica, 21, 425-523 (1962).

[4] Vargas, J. G.: Helmholtz Theorem for Differential Forms in 3-D Euclidean Space, arXiv:143679v2[math.GM] 20 April 2014.

[5] Cartan, É.: Sur les variétés à connexion affine et la théorie de la relativité généralisée, Ann. Ec. Norm. 40, 325-412 (1923).

[6] Rudin, W.: Principles of Mathematical Analysis. Mc-Graw-Hill, New York (1976).

[7] Cartan, J. É.: Leçons sur les invariants intégraux, Hermann, Paris (1922).

[8] Cartan, J. É.: Les systèmes différentiels extérieurs et leurs applications géométriques, Hermann, Paris (1934).

[9] Schwarz, L.: Théorie des distributions, Hermann, Paris (1966).

[10] Schläfli, L.: Ann. di mat., 2nd series 5, 170-193 (1871-1873).

[11] Janet, M.: Ann. Soc. Pol. Math. 5, 38-73 (1926).

[12] Cartan, J. É.: Sur la possibilité de plonger un space riemannien donné dans un space euclidien. Ann. Soc. Pol. Math. 6, 1-7 (1927). Accessible through his Complete Works.