Geometric quantization of non-relativistic and relativistic Hamiltonian mechanics

Giovanni Giachetta†, Luigi Mangiarotti‡ and Gennadi Sardanashvily§

† Department of Mathematics and Physics, University of Camerino, 62032 Camerino (MC), Italy
‡ Department of Theoretical Physics, Physics Faculty, Moscow State University, 117234 Moscow, Russia

Abstract. We show that non-relativistic and relativistic mechanical systems on a configuration space $Q$ can be seen as the conservative Dirac constraint systems with zero Hamiltonians on different subbundles of the same cotangent bundle $T^*Q$. The geometric quantization of this cotangent bundle under the vertical polarization leads to compatible covariant quantizations of non-relativistic and relativistic Hamiltonian mechanics.

1 Introduction

We study covariant geometric quantization of Hamiltonian mechanics subject to time-dependent transformations, whose configuration space is an $(m+1)$-dimensional oriented smooth manifold $Q$ coordinated by $(q^\lambda)$. This is the case of non-relativistic mechanics on a configuration space and relativistic mechanics on a pseudo-Riemannian manifold.

In non-relativistic mechanics, $Q$ is a fibre bundle over the time axis $\mathbb{R}$ provided with the Cartesian coordinate $q^0 = t$ with affine transition functions $t' = t + \text{const}$. Its different trivializations $Q \cong \mathbb{R} \times M$ correspond to different non-relativistic reference frames. In relativistic mechanics, a configuration manifold $Q$ need no fibration over $\mathbb{R}$, but is a time-like oriented pseudo-Riemannian manifold with respect to a non-degenerate metric $g$ of signature $(+, -, \cdots)$. It admits a coordinate atlas $\{(U; q^0, q^k)\}$ where $g^{00} > 0$ on each coordinate chart. Such a coordinate chart provides the local fibration

$$ U \ni (q^0, q^k) \mapsto q^0 \in \mathbb{R}, $$

and can be seen as a local non-relativistic configuration space. In particular, if $Q = \mathbb{R}^4$ and $g$ is the Minkowski metric, one comes to Special Relativity.

Let $T^*Q$ be the cotangent bundle $T^*Q$ of $Q$. Coordinated by $(q^\lambda, p_\lambda = \dot{q}_\lambda)$, it is provided with the canonical Liouville form $\Xi = p_\lambda dq^\lambda$, the above mentioned symplectic form $\Omega = d\Xi$, and the corresponding Poisson bracket

$$ \{f, g\}_T = \partial_\lambda f \partial_\lambda g - \partial_\lambda f \partial_\lambda g' $$

1E-mail address: mangiaro@camserv.unicam.it
2E-mail address: mangiaro@camserv.unicam.it
3E-mail address: sard@grav.phys.msu.su
on the ring $\mathcal{C}^\infty(T^*Q)$ of smooth real functions on $T^*Q$. We will show that the cotangent bundle $T^*Q$ of $Q$ plays a role both of the homogeneous momentum phase space of non-relativistic mechanics and the momentum phase space of relativistic mechanics on $Q$, but non-relativistic and relativistic Hamiltonian systems occupy different one-codimensional subbundles $N$ and $N'$ of $T^*Q \to Q$. They are given by the constraints

\begin{align}
(a) \; \phi_N &= p_0 + \mathcal{H}(q^k, p_k) = 0, & (b) \; \phi_{N'} &= g_{\mu\nu} \partial^\mu \mathcal{H}' \partial^\nu \mathcal{H}' - 1 = 0,
\end{align}

where $\mathcal{H}$ and $\mathcal{H}'$ are non-relativistic and relativistic Hamiltonians, respectively. Solutions of non-relativistic and relativistic Hamiltonian systems are vector fields on $N$ and $N'$, which obey the constraint Hamilton equations

\begin{align}
(a) \; \gamma_j \Omega_N &= 0, & (b) \; \gamma_j' \Omega_{N'} &= 0,
\end{align}

where $\Omega_N$ and $\Omega_{N'}$ are the pull-back presymplectic forms on $N$ and $N'$. These are equations of the conservative Dirac constraint dynamics on the symplectic manifold $(T^*Q, \Omega)$, whose Hamiltonian is constant on a primary constraint space.

Therefore, in order to quantize non-relativistic and relativistic mechanics, one can provide the geometric quantization of the cotangent bundle $T^*Q$, where non-relativistic and relativistic Hamiltonian systems are characterized by quantum constraints

\begin{align}
(a) \; \hat{\phi}_N \psi &= 0, & (b) \; \hat{\phi}_{N'} \psi' &= 0
\end{align}

on elements $\psi$ of the quantum space.

Recall that the geometric quantization procedure falls into the following three steps: prequantization, polarization and metaplectic correction (e.g., [2, 15, 20]). Given a symplectic manifold $(Z, \Omega)$ and the corresponding Poisson bracket $\{,\}$, prequantization associates to each element $f$ of the Poisson algebra $\mathcal{C}^\infty(Z)$ on $Z$ a first order differential operator $\hat{f}$ in the space of sections of a complex line bundle $C$ over $Z$ such that the Dirac condition

\begin{align}
[\hat{f}, \hat{f}'] = -i \{\hat{f}, \hat{f}'\}
\end{align}

holds. Polarization of a symplectic manifold $(Z, \Omega)$ is defined as a maximal involutive distribution $T \subset TZ$ such that Orth$_\Omega T = T$, i.e.,

\begin{align}
\Omega(u, v) = 0, \quad \forall u, v \in T_z, \quad z \in Z.
\end{align}

Given the Lie algebra $\mathbf{T}(Z)$ of global sections of $\mathbf{T} \to Z$, let $\mathcal{A}_T \subset \mathcal{C}^\infty(Z)$ denote the subalgebra of functions $f$ whose Hamiltonian vector fields $u_f$ fulfill the condition

\begin{align}
[u_f, \mathbf{T}(Z)] \subset \mathbf{T}(Z).
\end{align}
Elements of this subalgebra are quantized only. Metaplectic correction provides the pre-Hilbert space $E_T$ where the quantum algebra $A_T$ acts by symmetric operators. This is a certain subspace of sections of the tensor product $C \otimes D$ of the prequantization line bundle $C \to Z$ and a bundle $D \to Z$ of half-densities on $Z$. The geometric quantization procedure has been extended to Poisson manifolds [17, 18] and to Jacobi manifolds [7].

Geometric quantization of the cotangent bundle $T^*Q$ is well-known (e.g., [2, 15, 20]). The problem is that geometric quantization of $T^*Q$ does not automatically imply quantization of non-relativistic mechanics.

The momentum phase space of non-relativistic mechanics is the vertical cotangent bundle $V^*Q$ of $Q \to \mathbb{R}$, endowed with the holonomic coordinates $(t = q^0, q^k, p_k)$. It is provided with the canonical Poisson structure

$$\{f, f'\}_V = \partial_k f \partial_k f' - \partial_k f \partial_k f', \quad f, f' \in C^\infty(V^*Q),$$

whose symplectic foliation coincides with the fibration $V^*Q \to \mathbb{R}$ [3, 4]. Of course, one can quantize directly the Poisson manifold $V^*Q$. The problem is that non-relativistic mechanics can not be described as a Poisson Hamiltonian system on the momentum phase space $V^*Q$. Its Hamiltonian $H$ is not an element of the Poisson algebra $C^\infty(V^*Q)$, but a global section of the one-dimensional affine bundle

$$\zeta : T^*Q \to V^*Q.$$  

Therefore, the non-relativistic Hamilton equation can be written as the constraint Hamilton equation [3] on the cotangent bundle $T^*Q$.

Thus, we need compatible geometric quantizations both of the cotangent bundle $T^*Q$ and the vertical cotangent bundle $V^*Q$ [3]. We use that the fibration [3] defines the symplectic realization of the Poisson manifold $(V^*Q, \{\cdot, \cdot\}_V)$, i.e.,

$$\zeta^* \{f, f'\}_V = \{\zeta^* f, \zeta^* f'\}_T$$

for all $f, f' \in C^\infty(V^*Q)$. As a consequence, there is the monomorphism

$$\zeta^* : (C^\infty(V^*Q), \{\cdot, \cdot\}_V) \to (C^\infty(T^*Q), \{\cdot, \cdot\}_T)$$

of the Poisson algebra on $V^*Q$ to that on $T^*Q$.

The problem is that, though prequantization of $T^*Q$ leads to prequantization of $V^*Q$, polarization of $T^*Q$ need not imply polarization of the Poisson manifold $V^*Q$. We will show that the Schrödinger representation of $T^*Q$ by operators on half-densities on $Q$ yields the geometric quantization of $V^*Q$ such that the monomorphism of Poisson algebras [11] can be prolonged to that of quantum algebras. This prolongation is also required in order that non-relativistic and relativistic geometric quantizations be compatible on a local chart [11] of the relativistic configuration space $Q$. Such a compatibility takes place in heuristic
quantum theory where space-time coordinates $q^\lambda$ and spatial momenta $p_k$ have the same Schrödinger representation in non-relativistic and relativistic quantum mechanics. Given the Schrödinger representation of $T^*Q$, the quantum constraint equations make a sense of the Schrödinger equation in non-relativistic quantum mechanics and the relativistic quantum wave equation in the presence of a background metric $g$.

A fault of the Schrödinger representation of $T^*Q$ is that the corresponding quantum algebra $A_T$ consists of functions which are at most affine in momenta. Therefore, it does not include the most of physically relevant Hamiltonians. If a Hamiltonian $H$ is a quadratic in momenta, one can represent it as an element of the universal algebra of the Lie algebra $A_T$, but this representation is not always globally defined. This problem becomes especially important if one considers the non-relativistic limit of a relativistic systems because a transitive Hamiltonian is not a polynomial. It means that there is no satisfactory transition between quantum non-relativistic and relativistic systems.

2 Non-relativistic Hamiltonian dynamics

The dynamic equation (3a) is obtained as follows. Every global section

$$h : V^*Q \to T^*Q, \quad p \circ h = -\mathcal{H}(t, q^j, p_j)$$

(12)

of the affine bundle $\zeta$ yields the pull-back Hamiltonian form

$$H = h^*\Xi = p_k dq^k - \mathcal{H}dt$$

(13)

on $V^*Q$. Given a trivialization

$$V^*Q \cong \mathbb{R} \times T^*M,$$

(14)

the form $H$ is the well-known integral invariant of Poincaré–Cartan, where $\mathcal{H}$ is a Hamiltonian $\mathcal{H}$. Given a Hamiltonian form $H$ (13), there exists a unique Hamiltonian connection

$$\gamma_H = \partial_t + \partial^k \mathcal{H} \partial_k - \partial_k \mathcal{H} \partial^k$$

(15)

on the fibre bundle $V^*Q \to \mathbb{R}$ such that

$$\gamma_H|dH = 0$$

(16)

It defines the Hamilton equations on $V^*Q$.

Let us consider the pull-back $\zeta^*H$ of the Hamiltonian form $H = h^*\Xi$ onto the cotangent bundle $T^*Q$. It is readily observed that the difference $\Xi - \zeta^*H$ is a horizontal 1-form on $T^*Q \to \mathbb{R}$. Then the contraction

$$\mathcal{H}^* = \partial_t \vert (\Xi - \zeta^*H) = p + \mathcal{H}$$

(17)
is a function on $T^*Q$. It is exactly the constraint function $\phi_N$ of the image $N = \text{Im} h$ of the closed imbedding $h$ [12]. It is given by the constraint

$$\mathcal{H}^* = p + \mathcal{H}(t, q^k, p_k) = 0.$$  \hfill (18)

Let us consider the equation

$$\gamma_i^* i_N^* \Omega = 0, \quad i_N : N \to T^* Q,$$  \hfill (19)

for a vector field $\gamma$ on $N$. It defines a conservative Dirac constraint system with a zero Hamiltonian on the primary constraint subspace (2a) of the symplectic manifold $T^* Q$. Being a one-codimensional closed imbedded submanifold, the constraint space $N$ is coisotropic. Therefore, a solution of the equation (19) always exists [8, 12]. If $\gamma_i \frac{dt}{dt} = 1$, it is readily observed that $T\zeta \gamma = \gamma_H$, where $T\zeta$ denotes the tangent morphism to $\zeta$ (9).

A glance at the equation (16) shows that one can think of the Hamiltonian connection $\gamma_H$ as being the Hamiltonian vector field of a zero Hamiltonian with respect to the presymplectic form $dH$ on $V^* Q$. Therefore, one can consider geometric quantization of the presymplectic manifold $(V^* Q, dH)$, besides geometric quantization of the Poisson manifold $(V^* Q, \{, \})$. Given a trivialization (14), this quantization has been studied in [20].

Usually, geometric quantization is not applied directly to a presymplectic manifold $(Z, \omega)$, but to a symplectic manifold $(Z', \omega')$ such that the presymplectic form $\omega$ is a pullback of the symplectic form $\omega'$. Such a symplectic manifold always exists. The following two possibilities are customarily considered: (i) $(Z', \omega')$ is a reduction of $(Z, \omega)$ along the leaves of the characteristic distribution of the presymplectic form $\omega$ of constant rank [1, 10], and (ii) there is a coisotropic imbedding of $(Z, \omega)$ to $(Z', \omega')$ [1, 4].

In application to $(V^* Q, dH)$, the reduction procedure meets difficulties as follows. Since the kernel of $dH$ is generated by the vectors $(\partial_t, \partial_k \mathcal{H}^k - \partial^k \mathcal{H} \partial_k, k = 1, \ldots, m)$, the presymplectic form $dH$ in physical models is almost never of constant rank. Therefore, one has to provide an exclusive analysis of each physical model, and to cut out a certain subset of $V^* Q$ in order to use the reduction procedure.

The second variant of geometric quantization of the presymplectic manifold $(V^* Q, dH)$ seems more attractive because any section $h$ (12) is a coisotropic imbedding. Then the geometric quantization of the presymplectic manifold $(V^* Q, dH)$ consists in geometric quantization of the cotangent bundle $T^* Q$ and setting the quantum constraint condition (4a) on physically admissible quantum states.

Thus, presymplectic geometric quantization of $V^* Q$ agrees with the above manifested approach to quantization of non-relativistic mechanics which requires additionally that geometric quantization of $T^* Q$ also provides quantization of the Poisson algebra on $V^* Q$. 5
3 Relativistic Hamiltonian dynamics

We aim to show that classical Hamiltonian relativistic mechanics on the configuration space $Q$ can be seen as a conservative constraint Dirac system on the cotangent bundle $T^*Q$.

We start from describing the velocity and momentum phase spaces of relativistic mechanics \cite{8,14}.

The velocity phase space of relativistic mechanics is the first order jet manifold $J^1_1 Q$ of 1-dimensional submanifolds of the manifold $Q$. It consists of the equivalence classes $[S]^1_q$, $q \in Q$, of one-dimensional imbedded submanifolds of $Q$ which pass through $q \in Q$ and are tangent to each other at $q$. Given the coordinates $(q^0, q^k)$ on $Q$ with the transition functions

$$q^0 \to \tilde{q}^0(q^0, q^j), \quad q^k \to \tilde{q}^k(q^0, q^j),$$

(20)

the jet manifold $J^1_1 Q$ is endowed with the adapted coordinates $(q^0, q^k, \tilde{q}^0)$ whose transition functions are obtained as follows. Let $d_\tilde{q} = \partial_0 + q_0^k \partial_k$ be the total derivative. Given coordinate transformations (20), one can easily find that

$$d_{\tilde{q}^0} = d_{\tilde{q}^0}(q^0) d_\phi = \left( \frac{\partial q^0}{\partial \tilde{q}^0} + q_0^k \frac{\partial q^0}{\partial \tilde{q}^k} \right) d_\phi.$$

Then we obtain the equation

$$\tilde{q}_0^k = d_{\tilde{q}^0}(q^0) d_\phi(\tilde{q}^k) = \left( \frac{\partial q^0}{\partial \tilde{q}^0} + q_0^k \frac{\partial q^0}{\partial \tilde{q}^k} \right) \left( \frac{\partial q^k}{\partial q^0} + q_0^k \frac{\partial q^k}{\partial q^0} \right).$$

Its solution is

$$\tilde{q}_0^k = \left( \frac{\partial q^k}{\partial q^0} + q_0^k \frac{\partial q^k}{\partial q^0} \right) \left/ \left( \frac{\partial q^0}{\partial \tilde{q}^0} + q_0^k \frac{\partial q^0}{\partial \tilde{q}^k} \right) \right..$$

(21)

A glance at the transformation law (21) shows that the fibration $\pi : J^1_1 Q \to Q$ is a projective bundle.

Example 1. Put $Q = \mathbb{R}^4$ whose Cartesian coordinates $(q^0, q^k)$ are subject to the Lorentz transformations

$$\tilde{q}^0 = q^0 \cosh \alpha - q^1 \sinh \alpha, \quad \tilde{q}^1 = -q^0 \sinh \alpha + q^1 \cosh \alpha, \quad \tilde{q}^{2,3} = q^{2,3}.$$

(22)

Then (21) is exactly the transformations

$$\tilde{q}_0^1 = -\sinh \alpha + q_0^1 \cosh \alpha \quad \c/ \c = \frac{q_0^{2,3}}{\c/ \c = q_0^{2,3}\c/ \c = q_0^{2,3}} \c/ \c = q_0^{2,3}\c/ \c = q_0^{2,3}\c/ \c = q_0^{2,3}.$$
of three-velocities in Special Relativity.

Thus, one can think of the velocity phase space $J^1 \mathcal{Q}$ as being the space of non-relativistic velocities of a relativistic system.

The space of relativistic velocities is the tangent bundle $T \mathcal{Q}$ of $\mathcal{Q}$ equipped with the holonomic coordinates $(q^\lambda, \dot{q}^\lambda)$. We have the map

$$\lambda : J^1 \mathcal{Q} \ni q^k_0 \mapsto (\dot{q}^0, \dot{q}^k = q^0 q^k_0) \subset T \mathcal{Q},$$

over $\mathcal{Q}$ which assigns to each point of $J^1 \mathcal{Q}$ a line in $T \mathcal{Q}$. There is the converse map

$$\varrho : T \mathcal{Q} \to J^1 \mathcal{Q}, \quad \varrho^k_0 \circ \varrho = \dot{q}^k/\dot{q}^0,$$

such that $\varrho \circ \lambda = \text{Id} J^1 \mathcal{Q}$. It should be emphasized that, though the expression (24) looks singular at $\dot{q}^0 = 0$, this point belongs to another coordinate chart, and the morphism $\varrho$ is well defined.

A pseudo-Riemannian metric $g$ on $\mathcal{Q}$ defines the subbundle of hyperboloids

$$W_g = \{ \dot{q}^\lambda \in T \mathcal{Q} : g_{\mu\nu}(q) \dot{q}^\mu \dot{q}^\nu = 1 \}$$

of $T \mathcal{Q}$. Then, restricting $\varrho$ (24) and the image of $\lambda$ (23) to $W_g$, we obtain the familiar relations between non-relativistic and relativistic velocities in relativistic mechanics. Since $\mathcal{Q}$ is assumed to be time-oriented, the subbundle of hyperboloids $W_g$ is a disjoint union of the subbundles $W^+_g (\dot{q}^0 > 0)$ and $W^-_g (\dot{q}^0 < 0)$. Hereafter, we will consider only its connected component $W^+_g$.

Remark 2. Note that, in non-relativistic mechanics on a configuration bundle $\mathcal{Q} \to \mathbb{R}$, relativistic velocities of a non-relativistic system live in the subbundle $\dot{q}^0 = 1$ of $T \mathcal{Q}$. In particular, any non-relativistic dynamic equation on $\mathcal{Q} \to \mathbb{R}$ is equivalent to a geodesic equation with respect to a connection on the tangent bundle $T \mathcal{Q} \to \mathcal{Q}$.

The standard Lagrangian formalism fails to be appropriate to relativistic mechanics in a straightforward manner because a Lagrangian

$$L = \mathcal{L}(q^\lambda, \dot{q}^k_0) dq^0,$$

is defined only locally on a coordinate chart $(U; q^0, q^k)$ of the velocity phase space $J^1 \mathcal{Q}$. Nevertheless, given a motion $q^k = c^k(q^0)$, the pull-back $c^*L$ of a Lagrangian (26) is well behaved under transformations (20) where $d\tilde{q}^0 = dq^0$.

Therefore, let us consider a local regular Lagrangian $L$ (23) on a coordinate chart $(U; q^0, q^i, \dot{q}^i_0)$ of $J^1 \mathcal{Q}$, treated as a local velocity phase space of a non-relativistic mechanical system. This system can be described as a local Dirac constraint system on the cotangent bundle $T^*\mathcal{Q}|_U$ in the framework of the well-known Hamilton–De Donder
formalism (e.g., [8]). Indeed, the Poincaré–Cartan form associated to the Lagrangian $L$ defines the Legendre morphism

$$
\tilde{H}_L : J^1Q|_U \to T^*Q|_U,
$$

where the cotangent bundle $T^*Q$ of $Q$ plays a role of the homogeneous momentum phase space of non-relativistic mechanics. If a Lagrangian $L$ is regular, the equations (27) are solvable uniquely for

$$
p_0 = \phi(q^\mu, p_i),
$$

Then a solution of the Lagrange equations is an integral curve of the vector field $\gamma$ on the constraint space $N$ (28) which fulfills the equation

$$
\gamma\lfloor\Omega_N = 0,
$$

where $\Omega_N$ is the pull-back onto $N$ of the symplectic form $\Omega$ on $T^*Q$. This is a Dirac constraint system on $T^*Q$ in the case of a constant Hamiltonian on the primary constraint space (28).

For instance, the Lagrangian

$$
L_m = -m(1 - \sum_i(q^i_0)^2)^{1/2}dq^0
$$

of a free relativistic point mass $m$ in Special Relativity is regular on the ball $\sum_i(q^i_0)^2 < 1$, and defines a Dirac constraint system with a constant Hamiltonian on the primary constraint space

$$
p_0^2 - \sum_i p_i^2 = m^2.
$$

Therefore, let us describe a relativistic mechanical system as a conservative Hamiltonian system on the symplectic manifold $T^*Q$ which is characterized by a relativistic Hamiltonian

$$
\mathbf{H} : T^*Q \to \mathbb{R}
$$

(31). One also considers $T^*Q$, but provided with another symplectic form, e.g., $\Omega + F$ where $F$ is the strength of an electromagnetic field [15].

Any relativistic Hamiltonian $\mathbf{H}$ (31) defines the Hamiltonian map

$$
\mathbf{H} : T^*Q \to TQ,
$$

$$
\dot{q}^\mu = \partial^\mu\mathbf{H},
$$

(32)
over \( Q \) from the relativistic momentum phase space \( T^*Q \) to the space \( TQ \) of relativistic velocities. Since the relativistic velocities of a relativistic system live in the velocity hyperboloid \( W^+_g \), we have the constraint subspace \( N' = \tilde{H}^{-1}(W^+_g) \) (2b) of the relativistic momentum phase space \( T^*Q \). Let us assume that \( N' \rightarrow Q \) is a closed imbedded one-codimensional subbundle of \( T^*Q \) and, consequently, is coisotropic. It takes place if the Hamiltonian map (22) is of constant rank. Then a relativistic mechanical system can be described as a conservative Dirac constraint system on the primary constraint space \( i_{N'} : N' \rightarrow T^*X \) (2b). Its solutions are integral curves of the Hamiltonian vector field \( \gamma' \) on \( N' \) which obeys the relativistic Hamilton equation

\[
\gamma' \rfloor i^*_N \Omega = -i^*_N dH. \tag{33}
\]

A simple calculation shows that, if

\[
\{ H, g_{\mu\nu} \partial^\mu H \partial^\nu H \} = 0, \tag{34}
\]

the equation (33) has a solution

\[
\gamma' = \partial^\lambda H, \quad \gamma'_\lambda = -\partial^\lambda H. \tag{35}
\]

Let us give a few basic examples of relativistic Hamiltonian systems.

**Example 3.** The relativistic Hamiltonian of a free relativistic point mass in Special Relativity is

\[
H = -\frac{1}{2m} \eta^{\mu\nu} p^\mu p^\nu, \tag{36}
\]

where \( \eta \) is the Minkowski metric, while the constraint space \( N' \) is given by the equation (30). It is readily that this Hamiltonian fulfills the condition (34). Moreover, its restriction to the constraint space is a constant function. Then the Hamilton equation (33) takes the form

\[
u \rfloor i^*_N \Omega = 0. \tag{37}
\]

Its solution (33) reads

\[
\gamma' = 0, \quad p_k = \text{const}, \quad p_0 = -(m^2 - \eta^{jk} p_j p_k)^{-1/2}, \quad \gamma' = -\frac{1}{m} \eta^{\lambda\nu} p_\nu. \tag{38}
\]

Then, we obtain the familiar expression for three-velocities

\[
q^i_0 = -\eta^{il} p_l (m^2 - \eta^{jk} p_j p_k)^{-1/2}. \tag{39}
\]
Example 4. Let us consider a point electric charge $e$ in the Minkowski space in the presence of an electromagnetic potential $A_\lambda$. Its relativistic Hamiltonian reads

$$H = -\frac{1}{2m} \eta^{\mu\nu}(p_\mu - eA_\mu)(p_\nu - eA_\nu),$$

while the constraint space $N'$ (\ref{eq:constraint_space}) is

$$\eta^{\mu\nu}(p_\mu - eA_\mu)(p_\nu - eA_\nu) = m^2.$$

As in the previous example, the Hamilton $H$ fulfills the condition (\ref{eq:condition}), and its restriction to the constraint space is a constant function. Therefore, the relativistic Hamilton equation (\ref{eq:hamilton_equation}) takes the form (\ref{eq:reduced_hamilton_equation}). Its solution (\ref{eq:solution}) is

$$\gamma'_\lambda = -\frac{e}{m} \eta^{\mu\nu} \partial_\lambda A_\mu (p_\nu - eA_\nu),$$

$$\gamma'^i = -\frac{1}{m} \eta^{ik}(p_k - eA_k), \quad \gamma'^0 = \frac{1}{m} \eta^{00}(m^2 - \eta^{ij}(p_i - eA_i)(p_j - eA_j))^{-1/2}. \tag{39}$$

The equality (\ref{eq:three velocities}) leads to the usual expression for the three-velocities

$$p_k = -m\eta_{ki}q^i_0(1 + \eta_{ij}q^i_0 q^j_0)^{-1/2} + A_k$$

Substituting this expression in the equality (\ref{eq:reduced_hamilton_equation}), we obtain the familiar equation of motion of a relativistic charge in an electromagnetic field.

Example 5. The relativistic Hamiltonian for a point mass $m$ in a gravitational field $g$ on a 4-dimensional manifold $Q$ reads

$$H = -\frac{1}{2m} g^{\mu\nu}(q)p_\mu p_\nu, \tag{40}$$

while the constraint space $N'$ (\ref{eq:constraint_space}) is

$$g^{\mu\nu} p_\mu p_\nu = m^2.$$

As in previous Examples, the relativistic Hamilton equation (\ref{eq:hamilton_equation}) takes the form (\ref{eq:reduced_hamilton_equation}).
4 Prequantization

Basing on the standard prequantization of the cotangent bundle $T^*Q$, we will construct the compatible prequantizations of the Poisson bundle $V^*Q \to \mathbb{R}$.

Let us recall the prequantization of $T^*Q$ (e.g., [2, 15, 20]). Since its symplectic form $\Omega$ is exact and, consequently, is of zero de Rham cohomology class, the prequantization bundle is the trivial complex line bundle

$$C = T^*Q \times \mathbb{C} \to T^*Q,$$

(41)

whose Chern class $c_1$ is zero. Coordinated by $(q^\lambda, p_\lambda, c)$, it is provided with the admissible linear connection

$$A = dp_\lambda \otimes \partial^\lambda + dq^\lambda \otimes (\partial_\lambda + ip_\lambda c \partial c)$$

(42)

with the strength form $F = -i\Omega$ and the Chern form

$$c_1 = \frac{i}{2\pi} F = \frac{1}{2\pi} \Omega.$$

The $A$-invariant Hermitian fibre metric on $C$ is $g(c, c) = \bar{c}$. The covariant derivative of sections $s$ of the prequantization bundle $C$ (41) relative to the connection $A$ (42) along the vector field $u$ on $T^*Q$ takes the form

$$\nabla_u(s) = (u^\lambda \partial_\lambda - iu^\lambda p_\lambda)s.$$  

(43)

Given a function $f \in C^\infty(T^*Q)$, the covariant derivative (43) along the Hamiltonian vector field

$$u_f = \partial^\lambda f \partial_\lambda - \partial_\lambda f \partial^\lambda, \quad u_f \Omega = -df,$$

of $f$ reads

$$\nabla_{u_f} = \partial^\lambda f (\partial_\lambda - ip_\lambda) - \partial_\lambda f \partial^\lambda.$$

Then, in order to satisfy the Dirac condition (5), one assigns to each function $f \in C^\infty(T^*Q)$ the first order differential operator

$$\hat{f}(s) = -i(\nabla_{u_f} + if)s = [-iu_f + (f - p_\lambda \partial^\lambda f)]s$$

(44)

on sections $s$ of the prequantization bundle $C$ (41). For instance, the prequantum operators (44) of local functions $f = p_\lambda$, $f = q^k$ and global functions $f = t$, $f = 1$ read

$$\hat{p}_\lambda = -i\partial_\lambda, \quad \hat{q}^\lambda = i\partial^\lambda + q^\lambda, \quad \hat{1} = 1.$$
For elements \( f \) of the Poisson subalgebra \( \zeta^*C^\infty(V^*Q) \subset C^\infty(T^*Q) \), the Kostant–Souriau formula (44) takes the form

\[
\hat{f}(s) = [-i(\partial^k f \partial_k - \partial_k f \partial^k) + (f - p_k \partial^k f)]s.
\] (45)

Turn now to prequantization of the Poisson manifold \((V^*Q, \{,\}_V)\). The Poisson bivector \( w \) of the Poisson structure (8) on \( V^*Q \) reads

\[
w = \partial^k \wedge \partial_k = -[w, u]_{SN},
\] (46)

where \([,]_{SN}\) is the Schouten–Nijenhuis bracket and \( u = p_k \partial^k \) is the Liouville vector field on the vertical cotangent bundle \( V^*Q \to Q \). The relation (46) shows that the Poisson bivector \( w \) is exact and, consequently, has the zero Lichnerowicz–Poisson cohomology class \([8, 18]\). Therefore, let us consider the trivial complex line bundle

\[
C_V = V^*Q \times \mathbb{C} \to V^*Q
\] (47)
such that the zero Lichnerowicz–Poisson cohomology class of \( w \) is the image of the zero Chern class \( c_1 \) of \( C_V \) under the cohomology homomorphisms

\[
H^*(V^*Q, \mathbb{Z}) \to H^*_\text{derh}(V^*Q) \to H^*_\text{LP}(V^*Q).
\]

Since the line bundles \( C \) (41) and \( C_V \) (47) are trivial, \( C \) can be seen as the pull-back \( \zeta^*C \) of \( C \), while \( C_V \) is isomorphic to the pull-back \( h^*C \) of \( C \) with respect to a section \( h \) (42) of the affine bundle (8). Since \( C_V = h^*C \) and since the covariant derivative of the connection \( A \) (42) along the fibres of \( \zeta \) (8) is trivial, let us consider the pull-back

\[
h^*A = dp_k \otimes \partial^k + dq^k \otimes (\partial_k + ip_k c \partial_c) + dt \otimes (\partial_t - iH c \partial_c)
\] (48)
of the connection \( A \) (42) onto \( C_V \to V^*Q \) [18]. This connection defines the contravariant derivative

\[
\nabla_{\phi}s_V = \nabla_{w^s\phi}s_V
\] (49)
of sections \( s_V \) of \( C_V \to V^*Q \) along one-forms \( \phi \) on \( V^*Q \). This contravariant derivative is corresponded to a contravariant connection \( A_V \) on the line bundle \( C_V \to V^*Q \) [18]. It is readily observed that this contravariant connection does not depend on the choice of a section \( h \). By virtue of the relation (49), the curvature bivector of \( A_V \) is equal to \(-iw\) [19], i.e., \( A_V \) is an admissible connection for the canonical Poisson structure on \( V^*Q \). Then the Kostant–Souriau formula

\[
\hat{f}_V(s_V) = (-i\nabla_{aw,f} + f)s_V = [-i(\partial^k f \partial_k - \partial_k f \partial^k) + (f - p_k \partial^k f)]s_V
\] (50)
defines prequantization of the Poisson manifold $V^*Q$, where $s_V$ are sections of the line bundle $C_V$.

It is immediately observed that the prequantum operator $\hat{f}_V$ (50) coincides with the prequantum operator $\hat{\zeta}^* f$ (45) restricted to the pull-back sections $s = \zeta^* s_V$ of the line bundle $C$. Thus, prequantization of the Poisson algebra $C^\infty(V^*Q)$ on the Poisson manifold $(V^*Q, \{,\})$ is equivalent to its prequantization as a subalgebra of the Poisson algebra $C^\infty(T^*Q)$ on the symplectic manifold $T^*Q$.

5 Polarization

Given compatible prequantizations of the cotangent bundle $T^*Q$ and the Poisson bundle $V^*Q \to \mathbb{R}$, let us now construct their compatible polarizations.

Recall that, given a polarization $T$ of a prequantum symplectic manifold $(Z, \Omega)$, the subalgebra $A_T \subset C^\infty(Z)$ of functions $f$ obeying the condition (7) is only quantized. Moreover, after further metaplectic correction, one considers a representation of this algebra in a quantum space $E_T$ such that

$$\nabla_u e = 0, \quad \forall u \in T(Z), \quad e \in E_T. \tag{51}$$

Recall that a polarization of a Poisson manifold $(Z, \{,\})$ is defined as a sheaf $T^*$ of germs of complex functions on $Z$ whose stalks $T^*_z, z \in Z$, are Abelian algebras with respect to the Poisson bracket $\{,\}$ [19]. One can also require that the algebras $T^*_z$ are maximal, but this condition need not hold under pull-back and push-forward operations. Let $T^*(Z)$ be the structure algebra of global sections of the sheaf $T^*$; it is also called a Poisson polarization [17, 18]. A quantum algebra $A_T$ associated to the Poisson polarization $T^*$ is defined as a subalgebra of the Poisson algebra $C^\infty(Z)$ which consists of functions $f$ such that

$$\{f, T^*(Z)\} \subset T^*(Z).$$

Polarization of a symplectic manifold yields its maximal Poisson polarization, and vice versa.

There are different polarizations of the cotangent bundle $T^*Q$. We will consider those of them whose direct images as Poisson polarizations onto $V^*Q$ with respect to the morphism $\zeta$ (11) are polarizations of the Poisson manifold $V^*Q$. This takes place if the germs of a polarization $T^*$ of the Poisson manifold $(T^*Q, \{,\}_T)$ are constant along the fibres of the fibration $\zeta$ (9) [19], i.e., are germs of functions independent of the momentum coordinate $p_0 = p$. It means that the corresponding symplectic polarization $T$ of $T^*Q$ is vertical with respect to the fibration $T^*Q \to \mathbb{R}$. 

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The vertical polarization $V T^* Q$ of $T^* Q$ obeys this condition. It is a strongly admissible polarization, and its integral manifolds are fibres of the cotangent bundle $T^* Q \to Q$. One can easily verify that the associated quantum algebra $\mathcal{A}_T$ consists of functions on $T^* Q$ which are at most affine in momenta $p_\lambda$. The quantum space $E_T$ associated to the vertical polarization obeys the condition

$$\nabla u^\lambda \partial_\lambda e = 0, \quad \forall e \in E_T. \quad (52)$$

Therefore, the operators of the quantum algebra $\mathcal{A}_T$ on this quantum space read

$$f = a^\lambda(q^\mu)p_\lambda + b(q^\mu), \quad \hat{f} = -i a^\lambda \partial_\lambda + b. \quad (53)$$

This is the Schrödinger representation of $T^* Q$.

The vertical polarization of $T^* Q$ defines the maximal polarization $T^*$ of the Poisson manifold $V^* Q$ which consists of germs of functions, constant on the fibres of $V^* Q \to Q$. The associated quantum space $E_V$ obeys the condition

$$\nabla u^k \partial_k e = 0, \quad \forall e \in E_V. \quad (54)$$

The quantum algebra $\mathcal{A}_V$ corresponding to this polarization of $V^* Q$ consists of functions on $V^* Q$ which are at most affine in momenta $p_k$. Their quantum operators read

$$f = a^k(q^\mu)p_k + b(q^\mu), \quad \hat{f} = -i a^k \partial_k + b. \quad (55)$$

This is the Schrödinger representation of $V^* Q$.

## 6 Metaplectic correction

To complete the geometric quantization procedure of $V^* Q$, let us consider the metaplectic correction of the Schrödinger representations of $T^* Q$ and $V^* Q$.

The representation of the quantum algebra $\mathcal{A}_T$ (53) can be defined in the subspace of sections of the line bundle $C \to T^* Q$ which fulfill the relation (52). This representation reads

$$f = a^\lambda(q^\mu)p_\lambda + b(q^\mu), \quad \hat{f} = -i a^\lambda \partial_\lambda + b, \quad (56)$$

and, therefore, can be restricted to sections of the pull-back line bundle $C_Q = \hat{0}^* C \to Q$, where $\hat{0}$ is the canonical zero section of the cotangent bundle $T^* Q \to Q$. However, this is not yet a representation in a Hilbert space.

Let $Q$ be an oriented manifold. Applying the general metaplectic technique [2, 20], we come to the vector bundle $\mathcal{D}_{1/2} \to Q$ of complex half-densities on $Q$ with the transition functions $\rho' = J^{-1/2} \rho$, where $J$ is the Jacobian of the coordinate transition functions on
Since $C_Q \to Q$ is a trivial bundle, the tensor product $C_Q \otimes \mathcal{D}_{1/2}$ is isomorphic to $\mathcal{D}_{1/2}$. Therefore, the quantization formula (56) can be extended to sections of the half-density bundle $\mathcal{D}_{1/2} \to Q$ as

$$f = a^\lambda(q^\mu)p_\lambda + b(q^\mu), \quad \hat{f}\rho = (-iL_{a^\lambda}\partial_\lambda + b)\rho = (-ia^\lambda\partial_\lambda - i\frac{1}{2}\partial_\lambda(a^\lambda) + b)\rho,$$

where $L$ denotes the Lie derivative. The second term in the right-hand side of this formula is a metaplectic correction. It makes the operator $\hat{f}$ (57) symmetric with respect to the Hermitian form

$$\langle \rho_1 | \rho_2 \rangle = \left(\frac{1}{2\pi}\right)^m \int_Q \rho_1 \overline{\rho_2}$$

on the pre-Hilbert space $E_T$ of sections of $\mathcal{D}_{1/2} \to Q$ of compact support. The completion $\overline{E_T}$ of $E_T$ provides a Hilbert space of the Schrödinger representation of the quantum algebra $\mathcal{A}_T$, where the operators (57) are essentially self-adjoint. Of course, functions of compact support on the time axis $\mathbb{R}$ have a limited physical application, but we can simply restrict our consideration to some bounded interval of $\mathbb{R}$.

Since, in the case of the vertical polarization, there is a monomorphism of the quantum algebra $\mathcal{A}_V$ to the quantum algebra $\mathcal{A}_T$, one can define the Schrödinger representation of $\mathcal{A}_V$ by the operators

$$f = a^k(q^\mu)p_k + b(q^\mu), \quad \hat{f}\rho = (-ia^k\partial_k - i\frac{1}{2}\partial_k(a^k) + b)\rho$$

in the same space of complex half-densities on $Q$ as that of $\mathcal{A}_T$.

### 7 Quantum relativistic Hamiltonians

The representation (58) can be extended locally to functions on $T^*Q$ which are polynomials of momenta $p_\lambda$ if they are represented by elements of the universal enveloping algebra $\overline{\mathcal{A}_T}$ of the Lie algebra $\mathcal{A}_T$. However, this representation is not necessarily globally defined. For instance, a generic relativistic quadratic Hamiltonian

$$\mathcal{H}' = a^{\alpha\beta}(q^\lambda)p_\alpha p_\beta + b^{\alpha}(q^\lambda)p_\alpha + c(q^\lambda)$$

is not an element of the enveloping algebra $\overline{\mathcal{A}_T}$ because of the quadratic term. Written locally as a Hermitian element $\hat{\rho}_a \overline{a^{\alpha\beta}\rho}_\beta$ of $\overline{\mathcal{A}_T}$, this term is quantized as

$$-\partial_\alpha(a^{\alpha\beta}\partial_\beta\rho)$$
where the derivative of half-density $\partial_\beta \rho$ is ill-behaved, unless the Jacobian of the coordinate transition functions on $Q$ is independent of fibre coordinates $q^k$ on $Q$. In particular, this is the case of Special Relativity. For instance, the quantum relativistic Hamiltonian of a free relativistic point mass in the Minkowski space reads

$$\hat{H} = \frac{1}{2m} \eta^{\mu\nu} \partial_\mu \partial_\nu.$$  

This is exactly the Laplace operator. Accordingly, the relativistic Hamiltonian of a relativistic point electric charge takes the form

$$\hat{H} = \frac{1}{2m} \eta^{\mu\nu} (\partial_\mu - ieA_\mu)(\partial_\nu - ieA_\nu).$$  

If the determinant of a pseudo-Riemannian metric $g$ on $Q$ is not constant, a rather sophisticated procedure of quantization of the Hamiltonian (40) has been suggested in [13]. However, the Laplace operator constructed in [13] does not fulfill the Dirac condition

$$[\hat{H}, \hat{f}] = -i \{\hat{H}, f\}_T, \quad f \in \mathcal{A}_T.$$  

Given a non-relativistic Hamiltonian $\mathcal{H}$ and its Schrödinger quantization $\hat{\mathcal{H}}$, the quantum constraint equation (4a) is exactly the Schrödinger equation

$$i\partial_t \psi = \hat{\mathcal{H}} \psi$$  

of non-relativistic quantum mechanics.

A short calculation shows that, in the case of a free point mass and a point relativistic charge, the quantum constraint equation (4b) leads to the well-known equations of a classical scalar field. Their interpretation as equations of relativistic quantum mechanics however is under discussion. One of the problems is that the procedure of the quantum non-relativistic limit fails to be well defined. For instance, the Schrödinger quantization (4b) of the classical constraint (30) for a free relativistic point mass in the Minkowski space reads

$$(\eta^{\mu\nu} \partial_\mu \partial_\nu + m^2) \psi = 0.$$  

The classical constraint can be rewritten in an equivalent form

$$p_0 \pm (m^2 - \eta^{kj} p_k p_j)^{-1/2} = 0,$$  

suitable for passing to the non-relativistic limit

$$p_0 \approx m - \frac{1}{2m} \eta^{kj} p_k p_j.$$  

However, in contrast with the operator (60), the operator (61) is not polynomial in momenta and its Schrödinger quantization is not defined even locally.
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