A NOTE ON HIGHER-ORDER
DIFFERENTIAL OPERATIONS

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In this paper we consider successive iterations of the first-order differential
operations in space $\mathbb{R}^3$.

1. INTRODUCTION

Let $C^\infty(\mathbb{R}^3)$ be the set of scalar functions $f = f(x_1, x_2, x_3) : \mathbb{R}^3 \to \mathbb{R}$ which
have the continuous partial derivatives of the arbitrary order on coordinates $x_i$ ($i = 1, 2, 3$). Let $\tilde{C}^\infty(\mathbb{R}^3)$ be the set vector functions $\tilde{f} = (f_1(x_1, x_2, x_3), f_2(x_1, x_2, x_3), f_3(x_1, x_2, x_3)) : \mathbb{R}^3 \to \mathbb{R}^3$ which have the coordinately continuous partial derivatives of the arbitrary order on coordinates $x_i$ ($i = 1, 2, 3$). First-order differential operations of the vector analysis of the space $\mathbb{R}^3$ are defined on the following set of functions:

$F = \{ f : \mathbb{R}^3 \to \mathbb{R} | f \in C^\infty(\mathbb{R}^3) \}$ and $\tilde{F} = \{ \tilde{f} : \mathbb{R}^3 \to \mathbb{R}^3 | \tilde{f} \in \tilde{C}^\infty(\mathbb{R}^3) \}$.

First-order differential operations of the vector analysis of the space $\mathbb{R}^3$ are defined as the following three linear operations [1], denoted here by $\nabla_1, \nabla_2$ and $\nabla_3$ for convenience:

(1) $\text{grad } f = \nabla_1 f = \frac{\partial f}{\partial x_1} \tilde{e}_1 + \frac{\partial f}{\partial x_2} \tilde{e}_2 + \frac{\partial f}{\partial x_3} \tilde{e}_3 : F \to \tilde{F},$

(2) $\text{curl } \tilde{f} = \nabla_2 \tilde{f} = \left( \frac{\partial f_3}{\partial x_2} - \frac{\partial f_2}{\partial x_3} \right) \tilde{e}_1 + \left( \frac{\partial f_1}{\partial x_3} - \frac{\partial f_3}{\partial x_1} \right) \tilde{e}_2 + \left( \frac{\partial f_2}{\partial x_1} - \frac{\partial f_1}{\partial x_2} \right) \tilde{e}_3 : \tilde{F} \to \tilde{F},$

(3) $\text{div } \tilde{f} = \nabla_3 \tilde{f} = \frac{\partial f_1}{\partial x_1} + \frac{\partial f_2}{\partial x_2} + \frac{\partial f_3}{\partial x_3} : \tilde{F} \to F.$

Let $\Omega = \{ \nabla_1, \nabla_2, \nabla_3 \}$ be the set of above defined operations and let $\Sigma = F \cup \tilde{F}$. Then the first-order differential operations can be considered as partial operations $\Sigma \to \Sigma$, i.e. as operations whose domain (and codomain) are subsets $F$ or

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$\vec{F}$ of $\Sigma$. Second and higher-order differential operations are then defined as products of operations in $\Omega$ in the sense of composition of operations. Some of these products might be meaningful, like $\nabla_3 \circ \nabla_1$, while the others are meaningless, like $\nabla_1 \circ \nabla_1$.

To all meaningless products for any argument we associate the value of nowhere defined function $\vartheta$ ($\text{Dom}(\vartheta) = \emptyset$ and $\text{Ran}(\vartheta) = \emptyset$). Nowhere defined function $\vartheta(f)$ is a concept from the recursive function theory [2]. We do not consider the function $\vartheta$ as the starting argument for calculating the value of the higher-order differential operations. In that way we increase set $\Sigma$ into set $\Sigma = F \cup \vec{F} \cup \{\vartheta\}$.

All meaningful second-order differential operations are:

(4) $\Delta f = \text{div grad } f = (\nabla_3 \circ \nabla_1)(f)$,
(5) $\text{curl curl } \vec{f} = (\nabla_2 \circ \nabla_2)(\vec{f})$,
(6) $\text{grad curl } \vec{f} = (\nabla_1 \circ \nabla_3)(\vec{f})$,
(7) $\text{div curl } \vec{f} = (\nabla_3 \circ \nabla_2)(\vec{f}) = 0,$ $\vec{f}, f \in \Sigma \setminus \{\vartheta\}$.

In this paper we consider higher-order differential operations, search for meaningful ones and present some applications.

2. HIGHER-ORDER DIFFERENTIAL OPERATIONS

**Theorem 1.** For arbitrary operations $\nabla_i, \nabla_j, \nabla_k \in \Omega$ ($i, j, k \in \{1, 2, 3\}$) and argument $\xi \in \Sigma \setminus \{\vartheta\}$ the associative law holds:

$$\nabla_i \circ (\nabla_j \circ \nabla_k)(\xi) = (\nabla_i \circ \nabla_j) \circ \nabla_k(\xi).$$

**Proof.** Choosing the $\nabla_i, \nabla_j, \nabla_k$ from $\Omega$ and argument $\xi$ from $\Sigma \setminus \{\vartheta\}$, (9) appears in 54 possible cases. It is directly verified that whenever the left side of the equality is meaningless, the right side is also meaningless. Than, all meaningless products have the same value of the nowhere defined function $\vartheta$, so that (9) is true in the following form: $\vartheta = \vartheta$. Also, whenever the left side of equality is meaningful, the right side is also meaningful. Then, according to the associative law of the meaningful functions, we conclude that (9) is true.

From Theorem 1 it follows (by induction) that the generalized associative law also holds, so we may write the product $\nabla_{i_1} \circ \nabla_{i_2} \circ \cdots \circ \nabla_{i_n}$ without brackets ($i_j \in \{1, 2, 3\}$ : $j = 1, 2, ..., n$).

For higher-order differential operations, given as meaningful products, we say that they are the trivial products if they are trivially annihilated, i.e. if they are identically the same as the annihilating functions $0, \vec{0}$ from $\Sigma$. Otherwise, we refer to the higher-order differential operations, given as meaningful products, as nontrivial products (if they are nontrivially annihilated).

Next, we prove the statement:
Theorem 2. Higher-order differential operations appear as nontrivial products in the following three forms:

\[
\text{(grad) div \ldots \text{grad div grad } f} = (\nabla_1 \circ \nabla_3 \circ \ldots \circ \nabla_1 \circ \nabla_3 \circ \nabla_1 f, \\
\text{curl curl \ldots \text{curl curl curl } \vec{f}} = \nabla_2 \circ \nabla_2 \circ \ldots \circ \nabla_2 \circ \nabla_2 \circ \nabla_2 \vec{f}, \\
\text{(div) grad \ldots \text{div grad div } \vec{f}} = (\nabla_3 \circ \nabla_1 \circ \nabla_3 \circ \nabla_1 \circ \nabla_3 \vec{f},
\]

for arbitrary functions \( f, \vec{f} \in \Sigma \setminus \{\vartheta\} \), where terms in brackets are included for odd number of terms and are left out otherwise. All other meaningful operations are identically zero in their domain.

Proof. Meaningful third-order differential operations appear in the form of eight compositions as follows:

\[
\begin{align*}
\text{(10) grad div grad } f &= \nabla_1 \circ \nabla_3 \circ \nabla_1 f, \\
\text{(11) curl curl curl } \vec{f} &= \nabla_2 \circ \nabla_2 \circ \nabla_2 \circ \nabla_2 \circ \nabla_2 \vec{f}, \\
\text{(12) div grad div } \vec{f} &= \nabla_3 \circ \nabla_1 \circ \nabla_3 \circ \nabla_1 \circ \nabla_3 \vec{f}, \\
\text{(13) div curl curl } \vec{f} &= \nabla_3 \circ \nabla_2 \circ \nabla_2 \circ \nabla_2 \circ \nabla_2 \vec{f} = 0, \\
\text{(14) div curl grad } \vec{f} &= \nabla_3 \circ \nabla_2 \circ \nabla_1 \circ \nabla_1 \circ \nabla_3 \vec{f} = 0, \\
\text{(15) curl curl grad } f &= \nabla_2 \circ \nabla_2 \circ \nabla_1 \circ \nabla_1 \circ \nabla_3 \vec{f} = 0, \\
\text{(16) curl grad div } \vec{f} &= \nabla_3 \circ \nabla_2 \circ \nabla_1 \circ \nabla_3 \circ \nabla_3 \vec{f} = 0, \\
\text{(17) grad curl curl } \vec{f} &= \nabla_1 \circ \nabla_3 \circ \nabla_3 \circ \nabla_3 \circ \nabla_3 \circ \nabla_3 \vec{f} = 0, \quad f, \vec{f} \in \Sigma \setminus \{\vartheta\}.
\end{align*}
\]

Annullations of the operations (13)–(17) follow directly from the annullations (4)–(5). The statement follows directly from the principle of mathematical induction by means of using the general associative law and formulas (10)–(17).

For a given sequence of operations \( \nabla_{i_1}, \nabla_{i_2}, \ldots, \nabla_{i_n} \) from the set \( \Omega \) of functions, let define the concept of the collection of functions as a subset of functions \( \Theta \subseteq \Sigma \setminus \{\vartheta\} \) such that all functions \( \xi \) from \( \Theta \) annullate the nontrivial product \( \nabla_{i_1} \circ \nabla_{i_2} \circ \ldots \circ \nabla_{i_n} (\xi) \).

Let us form some collections. Scalar functions \( f \) from \( \Sigma \), such that \( \Delta^n f = 0 \) is true, define harmonic collection \( H_n \) of order \( n \), as the form of the polyharmonic functions. Let us notice that in the case of two dimensions there is a general form of polyharmonic functions \( f \) as a solution of the equation \( \Delta^n f = 0 \), [3]. Vector functions \( \vec{f} \) from \( \Sigma \), such that \( \text{curl}^n \vec{f} = 0 \) is true, define curling collection \( C_n \) of order \( n \).

We can remark that besides the total scalar operation \( \Delta : F \mapsto F \) (partial scalar operation \( \Delta : \Sigma \mapsto \Sigma \)) we can also consider the total vector operation \( \vec{\Delta} : \vec{F} \mapsto \vec{F} \) (partial vector operation \( \vec{\Delta} : \Sigma \mapsto \Sigma \)) defined by:

\[
\vec{\Delta} \vec{f} = (\Delta f_1, \Delta f_2, \Delta f_3) = \Delta f_1 \cdot \vec{e}_1 + \Delta f_2 \cdot \vec{e}_2 + \Delta f_3 \cdot \vec{e}_3.
\]
Let set $\vec{H}_n$ be the sign for the vector functions $\vec{f}$ from $\Sigma$ such that $\vec{\Delta}^n(\vec{f}) = \vec{0}$, where $\vec{\Delta}^n$ is iteration of order $n$ of the vector operation $\vec{\Delta}$ given by (18). The set of vector harmonic functions $\vec{H}_n$ of order $n$, which is defined in such a way, is not in the list of collections which appear in the previous theorem because it is not obtained through the compositions of operations (1)–(3). For the set $\vec{H}_n$ we shall keep the term collection.

Let us notice that for scalar polyharmonic collections, vector polyharmonic collections and curling collections, related to the index-order, the following inclusions hold:

(19) $\mathcal{H} \subset \mathcal{H}_2 \subset \cdots \subset \mathcal{H}_{n-1} \subset \mathcal{H}_n \subset \cdots,$

(20) $\vec{\mathcal{H}} \subset \vec{\mathcal{H}}_2 \subset \cdots \subset \vec{\mathcal{H}}_{n-1} \subset \vec{\mathcal{H}}_n \subset \cdots,$

(21) $\mathcal{C} \subset \mathcal{C}_2 \subset \cdots \subset \mathcal{C}_{n-1} \subset \mathcal{C}_n \subset \cdots.$

Let emphasize that all previous considerations can be transformed in three-dimensional orthogonal curvilinear coordinate system by introducing of corresponding presumptions for functions from the sets $\mathcal{F}$, $\vec{\mathcal{F}}$ and Lamé’s coefficients.

Finally, let state a few examples where scalar and vector polyharmonic collections appear.

Example 1. All meaningful products of third-and-higher-order differential operations for vector functions $\vec{f} \in \vec{\mathcal{H}}$ and scalar functions $f \in \mathcal{H}$ are anullated.

For vector functions $\vec{f} \in \vec{\mathcal{H}}$ the following equation holds:

(22) $\text{curl} \text{curl} \vec{f} = \text{grad} \text{div} \vec{f}.$

Hence, for $f \in \mathcal{H}$ and $\vec{f} \in \vec{\mathcal{H}}$, on the basis of formulas (22) and (10)–(17) the following is true:

$$\text{grad} \text{div} \text{grad} f = \text{grad} (\Delta f) = \vec{0},$$
$$\text{curl} \text{curl} \text{curl} \vec{f} = \text{curl} (\text{grad} \text{div}) \vec{f} = \vec{0},$$
$$\text{div} \text{grad} \text{div} \vec{f} = \text{div} (\text{curl} \text{curl}) \vec{f} = 0.$$  

Thus, all eight meaningful products of third-order differential operations are anulled, so that the statement is true.

Example 2. If $f \in \mathcal{H}_{n-1}$, then $x \cdot f \in \mathcal{H}_n$, $n \geq 2$.

Let us notice that if $f \in \mathcal{F}$, then $x \cdot f \in \mathcal{F}$. For an arbitrary scalar function $f \in \mathcal{F}$ the following equation is directly verified:

$$\Delta (x \cdot f) = 2 \partial f / \partial x + x \cdot \Delta (f).$$

Inductive generalization is the following equation:

$$\Delta^n (x \cdot f) = 2n \cdot \partial (\Delta^{n-1} (f)) / \partial x + x \cdot \Delta^n (f).$$

Thus, for $(n - 1)$-harmonic function $f \in \mathcal{H}_{n-1}$ the conclusion $x \cdot f \in \mathcal{H}_n$ is true.
Example 3. If \( f \in H_{n-1} \), then \((x^2 + y^2 + z^2) \cdot f \in H_n\), \( n \geq 2 \).

Let us notice that if \( f \in F \), then \((x^2 + y^2 + z^2) \cdot f \in F\).

For the arbitrary scalar function \( f \in F \) the following equations are directly verified:

\[
\Delta(x^2 \cdot f) = 2 \cdot f + 4x \cdot \partial f / \partial x + x^2 \cdot \Delta(f), \\
\Delta^2(x^2 \cdot f) = 8 \cdot \partial^2 f / \partial x^2 + 8x \cdot \partial(\Delta(f)) / \partial x + 4 \cdot \Delta(f) + x^2 \cdot \Delta^2(f). 
\]

Inductive generalization is the equation as follows:

\[
\Delta^n(x^2 \cdot f) = 4n(n - 1) \cdot \partial^2(\Delta^{n-2}(f)) / \partial x^2 + 4nx \cdot \partial(\Delta^{n-1}(f)) / \partial x + 2n \cdot \Delta^{n-1}(f) + x^2 \cdot \Delta^n(f). 
\]

Thus, if \( f \in H_{n-1} \), then \((x^2 + y^2 + z^2) \cdot f \in H_n\).

Two previous examples are the generalizations of the corresponding problems contained in [4].

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REFERENCES

1. M. L. Krasnov, A. I. Kiselev, G. I. Makarenko: Vector Analysis. Moscow 1981.
2. N. Cutland: Computability. Cambridge University Press, London 1980.
3. D. S. Mitrinović, J. D. Kečkić: Jednačine matematičke fizike. Beograd 1985.
4. D. S. Mitrinović, in association with P. M. Vasić: Diferencijalne jednačine, Novi zbornik problema 4. Beograd 1986.
5. M. J. Crowe: A History of Vector Analysis. University of Notre Dame Press, London 1967.