THETA BLOCK CONJECTURE FOR PARAMODULAR FORMS OF WEIGHT 2

VALERY GRITSENKO AND HAOWU WANG

ABSTRACT. In this paper we construct an infinite family of paramodular forms of weight 2 which are simultaneously Borcherds products and additive Jacobi lifts. This proves an important part of the theta-block conjecture of Gritsenko–Poor–Yuen (2013) related to the only known infinite series of theta-blocks of weight 2 and \( q \)-order 1. We also consider some applications of this result.

1. Introduction

Paramodular forms are Siegel modular forms of degree two with respect to the symplectic group \( \Gamma_t \) of elementary divisors \((1,t)\), the paramodular group. There are two ways to construct paramodular forms from Jacobi forms. The first one is the Jacobi lifting due to Gritsenko (see [6] and [5]) which lifts a holomorphic Jacobi form to a paramodular form. The second method is the multiplicative lifting (Borcherds automorphic product, see [1], [2]) in a form, proposed by Gritsenko–Nikulin in [12], which lifts a weakly holomorphic Jacobi form of weight 0 to a meromorphic paramodular form. In [14], V. Gritsenko, C. Poor and D. Yuen investigated the paramodular forms which are simultaneously Borcherds products and Gritsenko lifts.

Let \( f : \mathbb{Z}_{\geq 0} \to \mathbb{Z}_{\geq 0} \) be a function with a finite support. We define pure theta block (see [15])

\[
\Theta_f(\tau, z) = \eta^{f(0)}(\tau) \prod_{a=1}^{\infty} (\vartheta_a(\tau, z)/\eta(\tau))^{f(a)}
\]

as a finite product of the Jacobi theta-series \( \vartheta_a(\tau, z) = \vartheta(\tau, az) \) divided by the Dedekind \( \eta \)-function \( \eta(\tau) = q^{\frac{1}{24}} \prod_{n \geq 1} (1 - q^n) \), where

\[
\vartheta(\tau, z) = q^{\frac{1}{2}} (\zeta - \zeta^{-1}) \prod_{n \geq 1} (1 - q^n \zeta)(1 - q^n \zeta^{-1})(1 - q^n)
\]

is the odd Jacobi theta-series which is a Jacobi form of weight \( \frac{1}{2} \) and index \( \frac{1}{2} \) with a multiplier system of order 8 (see [12]). In general, \( \Theta_f \) is a weak Jacobi form of weight \( f(0)/2 \) of index \( N = \frac{1}{2} \sum a^2 f(a) \) with a character. This is important that for some functions \( f \) the theta block is holomorphic at infinity as Jacobi modular form (see the general theory on theta blocks in [15]). In this way one gets the most important holomorphic Jacobi forms of small weights. The following conjecture was proposed in [14].

Date: December 21, 2018.

2010 Mathematics Subject Classification. 11F30, 11F46, 11F50, 11F55, 14K25.

Key words and phrases. Bocherds product, Gritsenko lift, Paramodular forms, Jacobi forms.
Theta block conjecture. Let the theta block $\Theta_f$ be a holomorphic Jacobi form of weight $k$ and index $N$ and vanishing order one in $q = e^{2\pi i T}$. We define a weak Jacobi form $\Psi_f = - (\Theta_f | T_-(2)) / \Theta_f$ of weight 0 and index $N$, where $T_-(2)$ is the index raising Hecke operator. Then

$$\text{Grit}(\Theta_f) = \text{Borch}(\Psi_f)$$

is a holomorphic symmetric paramodular form of weight $k$ with respect to $\Gamma_N^\perp$.

This conjecture gives a characterization of paramodular forms which are simultaneously Borcherds products and Gritsenko lifts. The conjecture was proved in [14, §8] for all known series of theta blocks of weights $3 \leq k \leq 11$. The last known infinite series is the theta blocks of weight 2 of type $10 \leq k \leq 11$. The sieve of paramodular cusp forms is denoted by $F_k$ with $(\Gamma_k, \chi)$ found in [15]:

$$\phi_{2,a} = \frac{\theta_{a_1} \theta_{a_2} \theta_{a_3} \theta_{a_4} \theta_{a_1+a_2} \theta_{a_2+a_3} \theta_{a_3+a_4} \theta_{a_1+a_2+a_3+a_4}}{\eta^6} \in J_{2,N}$$

where $a = (a_1, a_2, a_3, a_4) \in \mathbb{Z}^4$ and

$$N = 2a_1^2 + 3a_1a_2 + 2a_2a_3 + a_1a_4 + 3a_2^2 + 4a_2a_3 + 2a_3a_4 + 3a_2^2 + 3a_3a_4 + 2a_4^2.$$

**Theorem 1.1.** For any $a \in \mathbb{Z}^4$ such that $\phi_{2,a}$ is not identically zero, we have

$$\text{Grit}(\phi_{2,a}) = \text{Borch}(\frac{\phi_{2,a} | T_-(2)}{\phi_{2,a}}) \in M_2(\Gamma_N^\perp).$$

This theorem was announced in [16] with the main idea of the proof. In the paper we give its complete proof and present some applications.

## 2. Modular forms and lifts constructions

Let $N$ be a positive integer. The paramodular group of level (or polarisation) $N$ is a subgroup of $\text{Sp}_2(\mathbb{Q})$ defined by

$$\Gamma_N = \left( \begin{array}{cccc} * & N \ast & * & * \\ * & * & * & N \ast \\ * & N \ast & * & * \\ N \ast & N \ast & N \ast & * \end{array} \right) \cap \text{Sp}_2(\mathbb{Q}), \quad \text{all } \ast \in \mathbb{Z}. \quad (2.1)$$

For $N > 1$, we shall use the following double normal extension

$$\Gamma_N^\uparrow = \Gamma_N \cup \Gamma_NV_N, \quad V_N = \frac{1}{\sqrt{N}} \begin{pmatrix} 0 & N & 0 & 0 \\ -1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & -N & 0 \end{pmatrix}. \quad (2.2)$$

This group acts on the Siegel upper half plane of genus 2

$$\mathbb{H}_2 = \left\{ Z = \begin{pmatrix} \tau & z \\ z & \omega \end{pmatrix} \in M(2, \mathbb{C}) : \text{Im } Z > 0 \right\}$$

in the usual way $M(Z) = (AZ + B)(CZ + D)^{-1}$, $M = \begin{pmatrix} A & B \\ C & D \end{pmatrix} \in \text{Sp}_2(\mathbb{R})$. We put $(F|k)M(Z) = \det(CZ + D)^{-k}F(M(Z))$ for any integral $k$.

**Definition 2.1.** A holomorphic function $F : \mathbb{H}_2 \rightarrow \mathbb{C}$ is called a Siegel paramodular form of weight $k$ and level $N$ with character $\chi$ if $F|kM = \chi(M)F$ for any $M \in \Gamma_N$. We denote the space of such modular forms by $M_k(\Gamma_N, \chi)$. A paramodular form $F$ is called a cusp form if $\Phi(F|kM) = 0$ for all $g \in \text{Sp}_2(\mathbb{Q})$, where $\Phi$ is Siegel’s operator. The space of paramodular cusp forms is denoted by $S_k(\Gamma_N, \chi)$. 
Let $\chi_N : \Gamma_N^+ \to \{\pm 1\}$ be the nontrivial binary character such that $\chi_N(V_N) = -1$ and $\chi_N|_{G_N} = 1$. Then $M_k(\Gamma_N)$ is decomposed into the direct sum of plus and minus $V_N$-eigenspaces. For $F \in M_k(\Gamma_N^+, \chi_N)$ ($\epsilon = 0$ or 1), we consider its Fourier and Fourier–Jacobi expansions

\[ F(Z) = \sum_{m \geq 0} \sum_{n \in \mathbb{N}, r \in \mathbb{Z}} c(n, r, m) q^n \zeta^r \xi^{mN} = \sum_{m \geq 0} \phi_{mN}(\tau, z) \xi^{mN}, \]

where $q = \exp(2\pi i \tau)$, $\zeta = \exp(2\pi i z)$, $\xi = \exp(2\pi i \omega)$. Then we have the equality $(-1)^{k+\epsilon} F(\tau, z, \omega) = F(N\omega, z, \tau/N)$, which yields $c(n, r, m) = (-1)^{k+\epsilon} c(m, r, n)$. When $k + \epsilon$ is odd (even), $F$ is called antisymmetric (symmetric). We remark that $F$ is a cusp form if and only if $c(n, r, m) = 0$ unless $4mnN - r^2 > 0$.

The Fourier–Jacobi coefficient $\phi_{mN} \in J_{k,mN}$ is a holomorphic Jacobi form (see [4] and the more general Definition 3.1 below).

We introduce the additive lifting which is completely determined by its first Fourier–Jacobi coefficient. Let $\phi(\tau, z) \in J_{k,N}$. The index raising Hecke operator $T_-(m)$ is defined as follows

\[ \phi|_k T_-(m) = m^{-1} \sum_{a d = m \atop b \text{mod } d} a^k \phi \left( \frac{a \tau + b}{d}, az \right) \in J_{k,mN}. \]

We have

\[ c(n, r; \phi|_k T_-(m)) = \sum_{d \mid (n, r, m)} d^{k-1} e \left( \frac{nm}{d^2}, \frac{r}{d} \phi \right). \]

**Theorem 2.2 ([6]).** For $\phi \in J_{k,N}$, we obtain

\[ \text{Grit}(\phi)(Z) = c(0, 0; \phi) G_k(\tau) + \sum_{m \geq 1} (\phi|_k T_-(m)) (\tau, z) e^{2\pi imN\omega} \in M_k(\Gamma_N^+, \chi_N^k) \]

where $G_k(\tau) = (2\pi i)^{-k}(k-1)! \zeta(k) + \sum_{n \geq 1} \sigma_{k-1}(n) q^n$ is the Eisenstein series of weight $k$ on $\text{SL}_2(\mathbb{Z})$. Moreover, if $\phi$ is a Jacobi cusp form then $\text{Grit}(\phi)$ is a paramodular cusp form.

The paramodular forms Grit$(\phi)$ are always symmetric. We now describe the second lifting, the Borcherds automorphic product (see [1] and [2]) in a form proposed by Gritsenko and Nikulin (see [12]). The Borcherds automorphic product is defined by the first two Fourier–Jacobi coefficients.

**Theorem 2.3 ([12]).** Let $N$ be a positive integer. Assume that $\Psi \in J_{0,N}^1$ is a weakly holomorphic Jacobi form of weight 0 with index $N$ with Fourier expansion

\[ \Psi(\tau, z) = \sum_{n, r \in \mathbb{Z}} c(n, r) q^n \zeta^r \]

and $c(n, r) \in \mathbb{Z}$ for $4Nn - r^2 \leq 0$. We set $C = \frac{1}{2} \sum_{r \in \mathbb{Z}} r^2 c(0, r)$. Then the function

\[ Borch(\Psi) = \left( \eta^{-c(0,0)} \prod_{r > 0} \left( \frac{\vartheta_r}{\eta} \right) e^{c(0,r)} \xi^r \right) \cdot \exp \left( - \text{Grit}(\Psi) \right) \]

is a weakly holomorphic Jacobi form of weight $k$ with index $N$ and Fourier expansion

\[ \sum_{n, r \in \mathbb{Z}} c(n, r) q^n \zeta^r \]

where $c(n, r) \in \mathbb{Z}$ for $4Nn - r^2 \leq 0$. By Theorem 2.2, $Borch(\Psi)$ is an automorphic form of the weakly holomorphic Jacobi group $\Gamma_N^+ \times \mathbb{Z}$.
is a meromorphic paramodular form $\text{Borch}(\Psi) \in M_{k+D_0}^{\text{mero}}(\Gamma_N^+; \chi_\Psi)$ of weight $k = c(0, 0)/2$ whose divisor in $\Gamma_N^+ \backslash \mathbb{H}_2$ consists of Humbert modular surfaces

$$\text{Hum}(T_0) = \Gamma_N^+ \backslash \{Z \in \mathbb{H}_2 : n_0 \tau + r_0 z + Nm_0 \omega = 0\}, \quad T_0 = \left( \begin{array}{cc} n_0 & r_0/2 \\ r_0/2 & Nm_0 \end{array} \right)$$

with $\gcd(n_0, r_0, m_0) = 1$, $m_0 \geq 0$ and $\det(T_0) < 0$. The multiplicity of $\text{Borch}(\Psi)$ on $\text{Hum}(T_0)$ is $\sum_{n \geq 1} c(n^2 n_0 m_0, nr_0)$. For $\lambda \gg 0$, on \{ $Z \in \mathbb{H}_2 : \Im Z > \lambda E_2$ \} the following product expansion is valid

$$\text{Borch}(\Psi)(Z) = q^{A \zeta} B \zeta^C \prod_{n,r,m \in \mathbb{Z}; m \geq 0} (1 - q^n \zeta^r \xi^{N m})^{c(n, m, r)}$$

where

$$24 A = \sum_{r \in \mathbb{Z}} c(0, r), \quad B = \sum_{r \in \mathbb{N}} rc(0, r), \quad D_0 = \sum_{n < 0} \sigma_6(-n)c(n, 0).$$

The character $\chi_\Psi$ of the paramodular form is generated by the character (or the multiplier system) of the theta block in $\text{Borch}(\Psi)$ and the character $\chi_N^{k+D_0}$.

The automorphic product $\text{Borch}(\Psi)$ is uniquely defined by the first two Fourier–Jacobi coefficients. The first factor in (2.4) is a mixed (meromorphic) theta block $\Theta$ defined by the $q^A$-part of the Fourier expansion of $\Psi$. The second one is the product $-\Theta \Psi$. If $\text{Grit}(\phi)$ is a Borcherds automorphic product, then $\phi = \Theta$ and $\phi(T_-(2)) = -\Theta \Psi$. Thus, we conclude that $\phi$ is a mixed theta block and $\Psi = - (\phi|T_-(2))/\phi$. Furthermore, we can show that $\phi$ has exactly vanishing order one in $q = e^{2 \pi i r}$ otherwise the constant term $c(0, 0)$ in the Fourier expansion of $- (\phi|T_-(2))/\phi$ will be greater than $2k$ (see [19, Proposition 7.2] for a proof).

In [7] and [14] the conjecture was proved for the quasi-products of theta-functions

$$\eta^{3(8-1)} \theta_{d_1} \cdots \theta_{d_k} \in J_{2 - L, N}, \quad N = (d_1^2 + \cdots + d_k^2)/2 \in \mathbb{N},$$

and for the products of three theta-quarks

$$\theta_{a_1, b_1} \theta_{a_2, b_2} \theta_{a_3, b_3} \in J_{3, N}, \quad N = \sum_{i=1}^{3} (a_i^2 + a_i b_i + b_i^2),$$

where $\theta_{a, b} = \theta_a \eta b \theta_{a+b}/\eta$ is a holomorphic Jacobi form of weight 1 with a character of order 3 (see [3] and [15]).

The main idea of the proof of the conjecture for the mentioned above theta blocks is the following. The odd theta-function $\vartheta(\tau, z)$ vanishes with order 1 for $\tau = \lambda z + \mu$ for any $\lambda, \mu \in \mathbb{Z}$. Therefore we know the divisor of the theta block $\Theta_f$ of $q$-order one. The Hecke operator $T_-(m)$ keeps this divisor of the Jacobi form. Therefore we know a part of the divisor of the lift of $\Theta_f$. According to Theorem 2.3, the divisor of $\text{Borch}(\Psi)$ is determined by the singular Fourier coefficients $c(n, r)$ with $4nN - r^2 < 0$ of $\Psi$ where $N$ is the index of the Jacobi forms $\Theta_f$. By the construction, $\text{Grit}(\Theta_f)$ vanishes on the divisors determined by $q^a$-term of $\Psi$. If $\text{Borch}(\Psi)$ does not have other divisor then $\text{Grit}(\phi)/\text{Borch}(\psi)$ is a holomorphic modular form of weight zero equal to constant by Kôcher’s principle. Unfortunately, $\text{Borch}(\Psi)$ usually has additional divisors determined by singular Fourier coefficients from the higher $q^n$-terms. In order to pass through this difficulty, we represent $\Theta_f$ as a pull-back of a Jacobi form $\Theta_L$ in many variables associated to a certain positive definite lattice
L. Since Jacobi forms in many variables have stronger symmetry, the function
\( \Psi_L = -\frac{\varphi_L(T_{(2)})}{\Theta} \) may have much simpler singular Fourier coefficients such that the
divisor of \( \text{Borch}(\Psi_L) \) are determined only by \( q^0 \)-term of \( \Psi_L \). In the next section we
give the corresponding definitions and results in the case of many variables.

3. ORTHOGONAL MODULAR FORMS AND JACOBI FORMS OF LATTICE INDEX

We start with the general setup. Let \( M \) be an even lattice of signature \((2, n)\)
with \( n \geq 3 \). Let
\[
D(M) = \{ [\omega] \in \mathbb{P}(M \otimes \mathbb{C}) : (\omega, \omega) = 0, (\omega, \bar{\omega}) > 0 \}^+ 
\]
be the corresponding Hermitian symmetric domain of type IV (here + denotes one
of its two connected components). By \( O^+(M) \) we denote the index 2 subgroup
of the integral orthogonal group \( O(M) \) preserving \( D(M) \). By \( \tilde{O}^+(M) \) we denote
the subgroup of \( O^+(M) \) acting trivially on the discriminant group of \( M \). For any
\( v \in M \otimes \mathbb{Q} \) satisfying \( (v, v) < 0 \), the rational quadratic divisor associated to \( v \) is
defined as
\[
D_v = \{ [Z] \in D(M) : (Z, v) = 0 \}.
\]
We assume that \( M \) contains two hyperbolic planes and write \( M = U \oplus U_1 \oplus L(-1) \),
where \( U = \mathbb{Z} e_1 \oplus \mathbb{Z} f \), \( (e, e) = (f, f) = 0, (e, f) = 1 \), \( U_1 = \mathbb{Z} e_1 \oplus \mathbb{Z} f_1 \) are two
hyperbolic planes and \( L \) is an even integral positive definite lattice. We choose a
basis of \( M \) of the form \((e, e_1, ..., f_1, f)\), where and ... denotes a basis of \( L(-1) \). We fix
a tube realisation of the homogeneous domain \( D(M) \) related to the 1-dimensional
boundary component determined by the isotropic plane \( F = \langle e, e_1 \rangle \):
\[
\mathcal{H}(L) = \{ Z = (\tau, \delta, \omega) \in \mathbb{H} \times (L \otimes \mathbb{C}) \times \mathbb{H} : (\text{Im} Z, \text{Im} Z) > 0 \},
\]
where \( (\text{Im} Z, \text{Im} Z) = 2 \text{Im} \tau \text{Im} \omega - (\text{Im} \delta, \text{Im} \delta)_L \). In this setting, a Jacobi form
is a modular form with respect to the Jacobi group \( \Gamma^J(L) \) which is the parabolic
subgroup preserving the isotropic plane \( F \) and acting trivially on \( L \). This group is
the semidirect product of \( \text{SL}_2(\mathbb{Z}) \) with the Heisenberg group \( H(L) \) of \( L \) (see [6] and
[3]). Let \( L^\vee \) denote the dual lattice of \( L \) and \( \text{rank}(L) \) denote the rank of \( L \). We
define Jacobi modular forms with respect to \( L \).

**Definition 3.1.** For \( k \in \mathbb{Z} \), \( t \in \mathbb{N} \), a holomorphic function \( \varphi : \mathbb{H} \times (L \otimes \mathbb{C}) \rightarrow \mathbb{C} \) is
called *weakly holomorphic* Jacobi form of weight \( k \) and index \( t \) associated to \( L \), if it
satisfies the functional equations
\[
\varphi \left( \frac{at + b}{ct + d}, \frac{\delta}{ct + d} \right) = (ct + d)^k \exp \left( i\pi t \frac{c(\delta, \delta)}{ct + d} \right) \varphi(\tau, \delta), \\
\varphi(\tau, \delta + x + y) = \exp \left( -i\pi t ((x, x) \tau + 2(x, \delta)) \right) \varphi(\tau, \delta),
\]
for \( \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \text{SL}_2(\mathbb{Z}) \), \( x, y \in L \) and it has a Fourier expansion
\[
\varphi(\tau, \delta) = \sum_{n \geq n_0} \sum_{\ell \in L^\vee} f(n, \ell) q^n \zeta^\ell,
\]
where \( n_0 \in \mathbb{Z} \), \( q = e^{2\pi i \tau} \) and \( \zeta^\ell = e^{2\pi i (\ell, \delta)} \). If the Fourier expansion of \( \varphi \) satisfies
the condition \( (f(n, \ell) \neq 0 \Longleftrightarrow n \geq 0) \) then \( \varphi \) is called *weak* Jacobi form. If
\( (f(n, \ell) \neq 0 \Longrightarrow 2n - (\ell, \ell) \geq 0) \) (respectively, \( > 0 \)) then \( \varphi \) is called *holomorphic*
(respectively, *cusp*) Jacobi form.
We denote by \( J^{\omega}_{k,L,t} \) (respectively, \( J^{\omega}_{k,L,t}, J^{\text{cusp}}_{k,L,t}, J^{\text{cusp}}_{k,L,t} \)) the vector space of weakly holomorphic Jacobi forms (respectively, weak, holomorphic or cusp Jacobi forms) of weight \( k \) and index \( t \). We note that the Jacobi forms in one variable \( J_{k,N} \) considered in the previous section are identical to the Jacobi forms \( J_{k,A_1,N} \) for the lattice \( A_1 = (2) \) of rank 1.

The Jacobi lifting and Borcherds product of Jacobi forms in many variables are very similar to the case of one variable. Let \( \varphi \in J^{\omega}_{k,L,t} \). For any positive integer \( m \), we have

\[
\varphi|_{k,t}T_{-}(m)(\tau,\zeta) = m^{-1} \sum_{a+b-d = \frac{m}{a} \in \mathbb{N}} a^{k} \varphi\left( \frac{a\tau + b}{d}, \frac{c}{d} \right) \in J^{\omega}_{k,L,m},
\]

and the Fourier coefficients of \( \varphi|_{k,t}T_{-}(m)(\tau,\zeta) \) are given by the formula

\[
f_{m}(n,\ell) = \sum_{\substack{n \in \mathbb{N} \ni a(n,\ell,m) \choose a \in (n,\ell,m)}} \quad a^{k-1} f\left( \frac{nm}{a}, \frac{\ell}{a} \right),
\]

where \( a \mid (n,\ell,m) \) means that \( a \mid (n,m) \) and \( a^{-1}\ell \in L^{\vee} \).

**Theorem 3.2** (see [6]). Let \( \varphi \in J^{\omega}_{k,L,t} \). Then the function

\[\text{Grit}(\varphi)(Z) = f(0,0)G^{k}_{k}(\tau) + \sum_{m \geq 1} \varphi|_{k,t}T_{-}(m)(\tau,\zeta)e^{2\pi im\omega}\]

is a modular form of weight \( k \) for the stable orthogonal group \( \widetilde{O}^{+} (2U \oplus L(-t)) \). Moreover, this form is symmetric i.e. \( \text{Grit}(\varphi)(\tau,\zeta,\omega) = \text{Grit}(\varphi)(\omega,\zeta,\tau) \).

We fix an ordering \( \ell > 0 \) in the lattice \( L > 0 \) in a way similar to positive root systems.

**Theorem 3.3** (see [8] for details). Let

\[\varphi(\tau,\zeta) = \sum_{n \in \mathbb{Z},\ell \in L^{\vee}} f(n,\ell)q^{n}\zeta^{\ell} \in J^{\omega}_{0,L,1} \cdot\]

Assume that \( f(n,\ell) \in \mathbb{Z} \) for all \( 2n - (\ell,\ell) \leq 0 \). We obtain a meromorphic modular form of weight \( f(0,0)/2 \) with respect to \( \widetilde{O}^{+} (2U \oplus L(-1)) \) with a character \( \chi \)

\[\text{Borch}(\varphi) = \left( \Theta_{f(0,\ast)}(\tau,\zeta) \exp(2\pi i C\omega) \right) \exp(-\text{Grit}(\varphi))\]

where \( C = \frac{1}{2\text{rank}(L)} \sum_{\ell \in L^{\vee}} f(0,\ell)(\ell,\ell) \) and

\[\Theta_{f(0,\ast)}(\tau,\zeta) = \eta(\tau)^{f(0,0)} \prod_{\ell > 0} \left( \frac{\vartheta(\tau,\ell,\zeta)}{\eta(\tau)} \right)^{f(0,\ell)}\]

is a mixed theta block. The character \( \chi \) is induced by the character of the theta-product and by the relation \( \chi(V) = (-1)^{D} \), where \( V : (\tau,\zeta,\omega) \rightarrow (\omega,\zeta,\tau) \), and \( D = \sum_{n < 0} \sigma_{0}(-n)f(n,0) \).

The poles and zeros of \( \text{Borch}(\varphi) \) lie on the rational quadratic divisors \( D_{v} \), where \( v \in 2U \oplus L^{\vee}(-1) \) is a primitive vector with \( (v,v) < 0 \). The multiplicity of this divisor is given by

\[\text{mult} D_{v} = \sum_{d \in \mathbb{Z}, d > 0} f(d^{2}n,d\ell),\]
where $n \in \mathbb{Z}$, $\ell \in L^\vee$ such that $(v, v) = 2n - (\ell, \ell)$ and $v \equiv \ell \mod 2U \oplus L(-1)$.

The same function has the following infinite product expansion

$$Borch(\varphi)(Z) = q^A \prod_{n, m \in \mathbb{Z}, \ell \in L^\vee} (1 - q^n \xi^\ell \xi^m)^{f(n, \ell)},$$

where $Z = (\tau, \omega) \in \mathcal{H}(L)$, $q = \exp(2\pi i \tau)$, $\xi^\ell = \exp(2\pi i (\ell, \omega))$, $\xi = \exp(2\pi i \omega)$, the notation $(n, \ell, m) > 0$ means that either $m > 0$, or $m = 0$ and $n > 0$, or $m = n = 0$ and $\ell < 0$, and

$$A = \frac{1}{24} \sum_{\ell \in L^\vee} f(0, \ell), \quad B = \frac{1}{2} \sum_{\ell > 0} f(0, \ell) \ell.$$

**Remark.** We can write the divisors of the theorem in a way similar to Theorem 2.3. According to the Eichler criterion (see [6]), the $\tilde{O}^+(2U \oplus L(-1))$-orbit of any primitive vector $v \in 2U \oplus L^\vee(-1)$ is uniquely determined by its norm $(v, v)$ and by the image of $v$ in the discriminant group of $2U \oplus L(-1)$. Therefore, there exists $(0, n, \ell, 1, 0) \in 2U \oplus L^\vee(-1)$ such that $(v, v) = 2n - (\ell, \ell) < 0$ and $v \equiv \ell \mod 2U \oplus L(-1)$, which implies

$$\tilde{O}^+(2U \oplus L(-1)) \cdot \mathcal{D}_v = \tilde{O}^+(2U \oplus L(-1)) \{ (Z \in \mathcal{H}(L) : n\tau - (\ell, \omega) = 0) \}.$$

It is known that Fourier coefficient $f(n, \ell)$ of $\varphi \in J_{4, L, 1}$ depends only on the hyperbolic norm $2nt - (\ell, \ell)$ of its index and the class of $\ell \in L^\vee$ modulo $tL$ (see [12]). The divisor of the Borcherds product in Theorem 3.3 is determined by the so-called *singular* Fourier coefficients $f(n, \ell)$ with $2nt - (\ell, \ell) < 0$. There are only finite number of orbits of such coefficients. To find all of them, we have to consider $f(n, \ell)$ with $n \geq 0$.

We say that the lattice $L$ satisfies the condition $\text{Norm}_2$ (see [9] and [13]) if

$$\text{Norm}_2 : \forall \bar{e} \in L^\vee/L \exists h_c \in \bar{e} : (h_c, h_c) \leq 2.$$

**Lemma 3.4.** If $\varphi \in J_{0, L, 1}^w$ and $L$ satisfies the condition $\text{Norm}_2$, then the singular Fourier coefficients of $\varphi$ are determined entirely by its $q^0$-term.

**Proof.** We note above that $f(n, \ell)$ depends only on $2n - (\ell, \ell)$ and $\ell \mod L$. Suppose that $f(n, \ell)$ is singular, i.e. $2n - (\ell, \ell) < 0$. There exists a vector $\ell_1 \in L^\vee$ such that $(\ell_1, \ell_1) \leq 2$ and $\ell - \ell_1 \in L$ because $L$ satisfies $\text{Norm}_2$ condition. If $-2 \leq 2n - (\ell, \ell) < 0$, it follows that $2n - (\ell, \ell) = -(\ell_1, \ell_1)$ and $f(n, \ell) = f(0, \ell)$. If $2n - (\ell, \ell) < -2$, then there exists a negative integer $n_1$ satisfying $2n - (\ell, \ell) = 2n_1 - (\ell_1, \ell_1)$. Thus, there will be a Fourier coefficient $f(n_1, \ell_1)$ with negative $n_1$, which contradicts the definition of weak Jacobi forms. \hfill \Box

Many examples of the lattices of type $\text{Norm}_2$ were found in [9] and [13] by construction of reflective modular forms. We prove below that $A_4(5)$ is also a lattice from this class.

By definition, $A_n = \{(a_1, \ldots, a_n) \in \mathbb{Z}^n : a_1 + \cdots + a_n = 0\}$. We fix the set of simple roots in $A_4$

$$\alpha_1 = (1, -1, 0, 0, 0) \quad \alpha_2 = (0, 1, -1, 0, 0)$$

$$\alpha_3 = (0, 0, 1, -1, 0) \quad \alpha_4 = (0, 0, 0, 1, -1).$$
The Kac-Weyl denominator function $\Phi$ and the subgroup $C$ represent a system, ten of them are generated by two $D$-classes and the zero element form a representative system of vectors and each subgroup contains two $4$-vectors and two $5$-vectors.

The fundamental weights of $D_4$ are the vectors
\[
\begin{align*}
w_1 &= \left(\frac{4}{5}, \frac{-1}{5}, \frac{-1}{5}, \frac{-1}{5}, \frac{1}{5}\right), \\
w_2 &= \left(\frac{3}{5}, \frac{3}{5}, \frac{-2}{5}, \frac{-2}{5}, \frac{2}{5}\right), \\
w_3 &= \left(\frac{2}{5}, \frac{2}{5}, \frac{2}{5}, \frac{-3}{5}, \frac{-3}{5}\right), \\
w_4 &= \left(\frac{1}{5}, \frac{1}{5}, \frac{1}{5}, \frac{1}{5}, \frac{-4}{5}\right).
\end{align*}
\]

So $(\alpha_i, w_j) = \delta_{ij}$ and $A_4^\vee / A_4 = \{0, w_1, w_2, w_3, w_4\}$. Therefore the renormalisation $A_4^\vee(5) = Z w_1 + Z w_2 + Z w_3 + Z w_4, \quad (\cdot, \cdot)$, is an even integral lattice of determinant $125$ and its dual lattice is $(A_4^\vee(5))^\vee = \frac{1}{5} A_4$.

**Lemma 3.5.** The lattice $A_4^\vee(5)$ satisfies the Norm$_2$ condition.

**Proof.** Consider the discriminant group $D = (A_4^\vee(5))^\vee / A_4^\vee(5) = \frac{1}{5} A_4 / A_4^\vee(5)$. We see above that $A_4^\vee / A_4$ is the cyclic group of order 5. We conclude that $D \cong \mathbb{Z}/5\mathbb{Z} \times \mathbb{Z}/5\mathbb{Z} \times \mathbb{Z}/5\mathbb{Z}$ is the 5-group of order 125.

We see that any two vectors in $\frac{1}{5} A_4$ of norm $\frac{6}{5}$ (or of norm $\frac{8}{5}$) are not equivalent modulo $A_4^\vee(5)$. The vectors of norm $\frac{6}{5}$ in $\frac{1}{5} A_4$ have 30 equivalent classes in $D$ and each class contains 2 elements. The vectors of norm $\frac{8}{5}$ in $\frac{1}{5} A_4$ have 20 equivalent classes in $D$ and each class contains 3 elements. The vectors of norm 2 in $\frac{1}{5} A_4$ have 24 equivalent classes in $D$ and each class contains 5 elements. Moreover, the above 124 classes and the zero element form a representative system of $D$. We call this representative system the standard system of $D$.

As a 5-group, $D$ is the bouquet of 31 cyclic subgroups of order 5. In the standard representative system, ten of them are generated by $\frac{\alpha_4}{5}$-vectors and each subgroup contains two $\frac{\alpha_4}{5}$-vectors and two $\frac{\alpha_5}{5}$-vectors. Fifteen subgroups are generated by $\frac{\alpha_5}{5}$-vectors and each subgroup contains two $\frac{\alpha_4}{5}$-vectors and two $\frac{\alpha_5}{5}$-vectors. Six subgroups are generated by 2-vectors and each subgroup contains four 2-vectors.

For any vector $\beta$ of norm 2 in $\frac{1}{5} A_4$, the lattice $A_4^\vee(5) + \mathbb{Z} \beta$ is isomorphic to the root lattice $A_4$.

From the proof of the previous lemma we obtain

**Lemma 3.6.** Let $O(A_4^\vee(5))$ be the integral orthogonal group of $A_4^\vee(5)$ and $O(D)$ be the orthogonal group of the discriminant group $D$ of $A_4^\vee(5)$.

1. $O(A_4^\vee(5)) \cong O(A_4) = W(A_4) \ltimes C_2$, where $W(A_4)$ is the Weyl group of $A_4$ and the subgroup $C_2$ is of order 2 and generated by the operator $\frac{3}{2} \mapsto -\frac{1}{2}$.

2. In the standard system, $O(A_4^\vee(5))$ acts transitively on the set of classes of the same norm.

3. The natural homomorphism $O(A_4^\vee(5)) \to O(D)$ is surjective.

We construct a reflective modular form for the lattice $2U \oplus A_4^\vee(-5)$. For that we consider the Kac-Weyl denominator function $\Phi_{A_4}$ of the affine Lie algebra of type $A_4$ (see [17, 8])

\[
\Theta_{A_4}(\tau, j) = \eta(\tau)^4 \prod_{r \in R^+(A_4)} \frac{\vartheta(\tau, (r, j))}{\vartheta(\tau)}, \quad j \in A_4 \otimes \mathbb{C}
\]
where $R^+(A_4)$ is the set of positive roots of $A_4$ defined in (3.7).

It is easy to check that $\Theta_{A_4}$ is a weak Jacobi form of weight 2 and index 1 for the lattice $A_4^0(5)$ (see [8] for the case of any root system). Moreover, it is anti-invariant under the action of the Weyl group $W(A_4)$ and invariant under the action of $C_2$. From [17] it follows that $\Theta_{A_4}(\tau, 0)$ is holomorphic but we do not need this fact here.

In the dual basis $\mathfrak{h} = z_1w_1 + z_2w_2 + z_3w_3 + z_4w_4 \in A_4 \otimes \mathbb{C}$, we have

$$\Theta_{A_4}(\mathfrak{h}, 0) = \eta^{-6}\vartheta(z_1, 0)\vartheta(z_2, 0)\vartheta(z_3, 0)\vartheta(z_4, 0)\vartheta(z_1 + z_2, 0)\vartheta(z_2 + z_3, 0)\vartheta(z_3 + z_4, 0) \vartheta(z_1 + z_2 + z_3 + z_4),$$

where $\vartheta(\tau, z)$ is anti-invariant under $\chi_4$. The character $\chi_4$ is $\chi_4(z) = \chi(z_1, 0)\chi(z_2, 0)\chi(z_3, 0)\chi(z_4, 0)$.

Remark 3.3. The following identity is true for the modular form of weight 2

$$\Phi_{2, A_4^0(5)} = \text{Grit}(\Theta_{A_4}) = \text{Borch}(\Psi_{A_4}) \in M_2(O^+(2U \oplus A_4^0(5)), \chi_2),$$

and its divisor is equal to

$$\mathcal{D}_0 = O^+(2U \oplus A_4^0(5)) \cdot \mathcal{D}_{(0,0,0,0,0)}.$$

The function $\text{Grit}(\Theta_{A_4})$ is a modular form of weight 2 for $O^+(2U \oplus A_4^0(5))$ which is anti-invariant under $W(A_4)$ and invariant under $C_2$. The Hecke operators of the Jacobi lifting preserve the divisor of $\Theta_{A_4}$. Therefore we get $\text{Div}(\text{Grit}(\Theta_{A_4})) \supset \text{Div}(\text{Borch}(\Psi_{A_4}))$ and $\text{Grit}(\Theta_{A_4}) = \text{Borch}(\Psi_{A_4})$ according to the K"{o}cher principle. We have thus proved the following.

Theorem 3.7. The following identity is true for the modular form of weight 2

$$\Phi_{2, A_4^0(5)} = \text{Grit}(\Theta_{A_4}) = \text{Borch}(\Psi_{A_4}) \in M_2(O^+(2U \oplus A_4^0(5)), \chi_2),$$

and it has reflective divisor (3.11). The character $\chi_2$ is of order 2 and defined by the relations $\chi_2(\mathfrak{h}) = 1$, $\chi_2(\vartheta(z_1, 0)) = 1$, $\chi_2(\vartheta(z_2, 0)) = 1$ and $\chi_2(\vartheta(z_3, 0)) = 1$.

Remark 3.8. The function $\Phi_{2, A_4^0(5)}$ is a reflective modular form of singular weight and has been constructed by N. Scheithauer in another way (see [21]). Scheithauer constructed this function at the zero-dimensional cusp related to $O^+(2U \oplus A_4(5))$ using the lifting from scalar-valued modular forms on congruence subgroups to modular forms for the Weil representation of $SL_2(\mathbb{Z})$. By [18, Corollary 1.13.3], we conclude

$$U \oplus \mathcal{U}(5) \oplus A_4 \cong 2U \oplus A_4^0(5),$$

because the two lattices belong to the same genus. Our construction corresponds to the one-dimensional cusp related to the decomposition $2U \oplus A_4^0(5)$ and it gives the additive Jacobi lifting of this reflective modular form.
Remark 3.9. A hyperbolisation of the affine Lie algebra $\hat{\mathfrak{g}}(A_4)$. The result of Theorem 3.7 has important application to the theory of Lie algebras. It shows that there exists a hyperbolisation of the affine Lie algebra $\hat{\mathfrak{g}}(A_4)$, i.e., a Lorentzian Kac–Moody algebra with the following property: the first Fourier–Jacobi coefficient of the automorphic Kac–Weyl–Borcherds denominator function of such generalised hyperbolic Kac–Moody algebra is the Kac–Weyl denominator function of the affine Lie algebra $\hat{\mathfrak{g}}(A_4)$. The generators and relations of this new algebra are defined by the Fourier coefficients of the lift $\text{Grit}(\Theta_{\mathfrak{a}})$. This is a new example in a rather short series: $\hat{\mathfrak{g}}(A_1)$ (see 12), $\hat{\mathfrak{g}}(4A_1)$ and $\hat{\mathfrak{g}}(3A_2)$ (see 7), $\hat{\mathfrak{g}}(A_2)$ (see 11). Twenty three root systems of Niemeier lattices are also this type of examples (see 8).

4. Applications

4.1. The proof of Theorem 1.1. Now we can prove the main result of the paper that the theta-block conjecture is true for the theta blocks of type $\frac{10-q^6}{6-q^3}$.

We obtain Theorem 1.1 as a particular case of Theorem 3.7. We take the specialisation of the identity of Theorem 3.7 $\text{Grit}(\Theta_{\mathfrak{a}}) = \text{Borch}(\mathfrak{h}_{\mathfrak{A}})$ for $(z_1, z_2, z_3, z_4) = (a_1 z, a_2 z, a_3 z, a_4 z)$. It gives the identity of Theorem 1.1. □

4.2. Explicit divisors of the paramodular forms of weight 2 and linear relations between Fourier coefficients. Our construction gives explicit formulas for the divisors of the modular forms from Theorem 1.1.

The first modular form corresponds to $\mathfrak{a} = (1, 1, 1, 1)$. In this case we obtain the first complementary Jacobi–Eisenstein series $E_{2,25;1} = \eta^{-6} \vartheta_1^2 \vartheta_2^2 \vartheta_3 \vartheta_4 \in J_{2,25}$ (see 4, §2). The singular Fourier coefficients of $\psi_{0,25} = -(E_{2,25;1}(T_{-}(2))/E_{2,25;1}$ are represented by $\text{Sing}(\psi_{0,25}) = \zeta^5 + 2\zeta^3 + 3\zeta + 4 + q^6 \zeta^{30}$. Thus the divisor of $\text{Grit}(E_{2,25;1})$ is completely defined by divisors of the theta block, i.e., the corresponding Borcherds product has no additional divisor.

When the index is larger than 25, the corresponding Borcherds products have additional divisors in general and the additional divisors will yields certain relations between Fourier coefficients of theta blocks. We first recall one example in 14. Let $\mathfrak{a} = (1, 1, 1, 2)$, we get $\phi_{2,37} = \eta^{-6} \vartheta_1^2 \vartheta_2 \vartheta_3 \vartheta_4 \vartheta_5 \in J_{2,37}^{\text{cusp}}$. Note that $\dim J_{2,37}^{\text{cusp}} = 1$. Let $\psi_{0,37} = -(\phi_{2,37}(T_{-}(2))/\phi_{2,37}$. The singular Fourier coefficients of $\psi_{0,37}$ are represented by

$$\text{Sing}(\psi_{0,37}) = \zeta^5 + \zeta^4 + 2\zeta^3 + 3\zeta^2 + 3\zeta + 4 + q^6 \zeta^{30}.$$ 

The coefficient $q^6 \zeta^{30}$ determines the divisor which does not appear in the theta block $\phi_{2,37}$. We call it the additional divisor of $\text{Grit}(\phi_{2,37})$. This is the last term in the formula for the full divisor of $\text{Borch}(\psi_{0,37})$:

$$10 \text{Hum} \begin{pmatrix} 0 & 1/2 & 1/2 \hline 37 & 37 & 37 \end{pmatrix} + 4 \text{Hum} \begin{pmatrix} 0 & 1 \hline 1 & 37 \end{pmatrix} + 2 \text{Hum} \begin{pmatrix} 0 & 3/2 & 3/2 \hline 37 & 37 & 37 \end{pmatrix}$$
$$+ \text{Hum} \begin{pmatrix} 0 & 2 \hline 2 & 37 \end{pmatrix} + \text{Hum} \begin{pmatrix} 0 & 5/2 \hline 5/2 & 37 \end{pmatrix} + \text{Hum} \begin{pmatrix} 6 & 15 \hline 15 & 37 \end{pmatrix}.$$ 

The fact that $\text{Grit}(\phi_{2,37})$ vanishes on the additional divisor is equivalent (see 14, Page 170) that the Fourier coefficients of $\phi_{2,37}$ satisfy the linear relation

$$\forall n, r \in \mathbb{Z}, \sum_{a \in \mathbb{Z}} c(6a^2 + na, 30a + r; \phi_{2,37}) = 0.$$
Next, we establish similar relations for other Jacobi forms of weight 2 and small indices. We know from [4] that the dimensions of the spaces of Jacobi forms of weight 2 and index 43, 50, 53 are all 1. The generators can be constructed the theta blocks
\[a = (-1, 5, -1, -2): \quad \phi_{2,43} = \eta^{-6} \vartheta^3 \vartheta^2 \vartheta^3 \vartheta^4 \vartheta_5 \in J_{2,43} \]
\[a = (2, -1, -3, 6): \quad \phi_{2,50} = \eta^{-6} \vartheta^3 \vartheta^2 \vartheta^3 \vartheta^4 \vartheta_6 \in J_{2,50} \]
\[a = (1, -6, 3, 1): \quad \phi_{2,43} = \eta^{-6} \vartheta^3 \vartheta^2 \vartheta^3 \vartheta^4 \vartheta_5 \vartheta_6 \in J_{2,53} \cdot \]

We put \( \psi_{0,m} = -(\phi_{2,m}|T_{-}(2))/\phi_{2,m} \) for \( m = 43, 50, 53 \). Their singular Fourier coefficients are represented by
\[
\text{Sing}(\psi_{0,43}) = \zeta^5 + 2\zeta^4 + 2\zeta^3 + 2\zeta + 3\zeta + 4 + q^2 \zeta^{19} + q^3 \zeta^{23}, \\
\text{Sing}(\psi_{0,50}) = \zeta^6 + 2\zeta^4 + 2\zeta^3 + 2\zeta + 4 + q^5 \zeta^{32} + 2q^11 \zeta^{47} + 2q^{12} \zeta^{49}, \\
\text{Sing}(\psi_{0,53}) = \zeta^6 + \zeta^5 + \zeta^4 + 2\zeta^3 + 2\zeta + 3\zeta + 4 + q\zeta^{15} + q^2 \zeta^{21} + q^6 \zeta^{36}.
\]

Since \( \text{Grit}(\phi_{2,j}) \) vanish on the additional divisors, their Fourier coefficients satisfy the similar relations for all additional divisors. For example, in the case of \( \phi_{2,43} \), we obtain two relations
\[
\forall n, r \in \mathbb{Z}, \quad \sum_{a \in \mathbb{Z}} c(2a^2 + na, 19a + r; \phi_{2,43}) = 0 \\
\forall n, r \in \mathbb{Z}, \quad \sum_{a \in \mathbb{Z}} c(3a^2 + na, 23a + r; \phi_{2,43}) = 0.
\]

4.3. Reflective modular form of weight 12. We give one more property of the lattice \( A_1^+(5) \).

**Lemma 4.1.** There is a primitive embedding of \( A_1^+(5) \) into the Leech lattice \( \Lambda_{24} \).

**Proof.** In fact, the lattice \( A_1^+(5) < \Lambda_{24} \) is a fixpoint sublattice with respect to the automorphism of cycle shape 55/1. We refer to [20, §9] for details. □

Any sublattice of \( \Lambda_{24} \) with property \( \text{Norm}_2 \) produces a strongly 2-reflective modular form. This is the pull-back of the Borcherds reflective modular form \( \Phi_{12} \in M_{12}(O^+(112_{2,26}), \det) \) (see [13, Theorem 4.2]). According to Lemma 3.5 we obtain a new example of reflective modular form.

**Theorem 4.2.** Consider the primitive sublattice \( 2U \oplus A_1^+(-5) \hookrightarrow 2U \oplus \Lambda_{24} \) and the embedding of the homogenous domains \( D(2U \oplus A_1^+(-5)) \hookrightarrow D(2U \oplus \Lambda_{24}) \). Then
\[
\Phi_{12,A_1^+(-5)} = \Phi_{12}|D(2U \oplus A_1^+(-5)) \in M_{12}(O^+(2U \oplus A_1^+(-5)), \det)
\]
is strongly reflective modular form with the complete \((-2)\)-divisor
\[
\text{Div}(\Phi_{12,A_1^+(-5)}) = \sum_{r \in 2U \oplus A_1^+(-5)} D_r.
\]

**Corollary 4.3.** The modular form \( \Phi_{12,A_1^+(-5)} \) determines a Lorentzian Kac–Moody algebra. For this algebra, the 2-reflective Weyl group of the lattice \( U \oplus A_1(-5) \) has a Weyl vector of norm 0, i.e. it has parabolic type.

**Proof.** See a general construction of Lorentzian Kac–Moody algebras by 2-reflective modular forms in [13]. □
Corollary 4.4. The modular variety \( \tilde{O}^+(2U \oplus A_4^2(-5)) \setminus \mathcal{D}(2U \oplus A_4^2(-5)) \) is at least uniruled.

Proof. One can use the automorphic criterion proved in [9]. We note that this modular variety can be considered as a moduli space of lattice polarized K3 surfaces. \( \Box \)

4.4 Quasi pull-backs. We proved the main theorem using a pull-back of the reflective modular form \( \Phi_{2,A_3^1(5)} \). In this subsection we construct more pull-backs and quasi pull-backs of \( \Phi_{2,A_3^1(5)} \) in order to obtain interesting relations between liftings of multidimensional theta blocks. Using the same arguments about quasi pull-backs as in [10]–[13], we obtain the next proposition.

Proposition 4.5. Assume that \( T = 2U \oplus T_0(-1) \to 2U \oplus A_4^2(-5) \) is a primitive sublattice of signature \((2,n)\) with \( n = 3, 4, 5. \) We consider the corresponding embedding of the homogenous domains \( \mathcal{D}(T) \leftrightarrow \mathcal{D}(2U \oplus A_4^2(-5)) \) and a finite set

\[
\mathcal{R}_{\frac{3}{2}}(T_0^+) = \left\{ v \in \frac{1}{5} A_4(-1) : (v,v) = -\frac{2}{5}, (v,T_0) = 0 \right\}.
\]

Then the function

\[
\Phi_{2,A_3^1(5)}|T = \Phi_{2,A_3^1(5)}(Z) \left. \prod_{v \in \mathcal{R}_{\frac{3}{2}}(T_0^+)} (Z,v) \right|_{\mathcal{D}(T)}
\]

is a nontrivial modular form of weight 2 + \( \frac{1}{2} |\mathcal{R}_{\frac{3}{2}}(T_0^+)| \) with respect to \( \tilde{O}^+(T) \) with a character of order 2. The modular form \( \Phi_{2,A_3^1(5)}|T \) vanishes only on rational quadratic divisors of type \( \mathcal{P}_{u}(T) \), where \( u \) is the orthogonal projection of one vector \( v \in 2U \oplus \frac{1}{5} A_4(-1) \) with \( (v,v) = -\frac{2}{5} \) to \( T' \) satisfying \( -\frac{2}{5} \leq (u,u) < 0 \). If the set \( \mathcal{R}_{\frac{3}{2}}(T_0^+) \) is non-empty then \( F|T \) is a cusp form.

Example 4.6. Let \( T_0 = Z\alpha_1 + Z\alpha_2 + Z\alpha_3. \) Then \( T_0 \) is isomorphic to the lattice \( A_3(5). \) In this case \( T_0^+ = T_{\alpha}(A_4,5) \) and the set \( \mathcal{R}_{\frac{3}{2}}(T_0^+) \) is empty. Let \( z = z_1 + z_2 + z_3 = (2z_1 - z_2)w_1 + (2z_2 - z_1 - z_3)w_2 + (2z_3 - z_2)w_3 - z_3w_4. \)

The pull-back on any sublattice of this type commutes with the Jacobi lifting. In the coordinates fixed above we have

\[
\Theta_{A_4|A_3^1(5)}(\tau, z_3) = \eta^{-6} \theta(2z_1 - z_2) \theta(z_1 + z_2 - z_3) \theta(z_1 + z_3) \theta(z_1) \\
\theta(2z_2 - z_1 - z_3) \theta(z_2 + z_3 - z_1) \theta(z_2 - z_1) \\
\theta(2z_3 - z_2) \theta(z_3 - z_2) \theta(z_3) \in J_{2,A_3,5}.
\]

Therefore, \( \text{Grit}(\Theta_{A_4|A_3^1(5)}) \) is a modular form of weight 2 for \( \tilde{O}^+(2U \oplus A_3^1(-5)) \) and also a Borcherds product.

Example 4.7. The sublattice \( Z\alpha_1 + Z\alpha_3 \) is isomorphic to \( 2A_1(5). \) By taking \( z_2 = 0 \) in (4.2), we get

\[
\Theta_{2A_1(5)} = \frac{\theta^2(z_1 + z_3) \theta^2(z_1 - z_3) \theta^2(z_1) \theta(2z_1) \theta(2z_3)}{\eta^6} \in J_{2,2A_1,5}.
\]

Note that \( \text{dim} J_{2,2A_1,5} = 1 \) and 5 is the smallest index such that there exists a nontrivial Jacobi form of weight 2 for \( 2A_1. \) Therefore, \( \text{Grit}(\Theta_{2A_1(5)}) \) is a modular form of weight 2 for \( \tilde{O}^+(2U \oplus 2A_1(-5)) \) and a Borcherds product.
We note that the Jacobi form $\Theta_{A_4}$ of type $10-\eta/6-\eta$ generates a tower of the lifts of quasi pull-backs of type $9-\eta/3-\eta$ of weight 3 and $8-\eta$ of weight 4. The theta-block conjecture for these theta block was proved in [14] based on the reflective modular forms constructed in [7]. The function considered below have less parameters than the modular forms in [7] but they are cusp forms.

**Example 4.8.** Let $T_0 = Z\omega_1 + Z\omega_2 + Z\omega_3$. Its Gram matrix is

$$A_0 = \begin{pmatrix} 4 & 3 & 2 \\ 3 & 6 & 4 \\ 2 & 4 & 6 \end{pmatrix}, \quad \det(A_0) = 50, \quad A_0^{-1} = \frac{1}{5} \begin{pmatrix} 2 & -1 & 0 \\ -1 & 2 & -1 \\ 0 & -1 & 3/2 \end{pmatrix}.$$

In this case $T_0^\perp = Z\alpha_4$ and $R_\mathbb{Z}(T^\perp) = \{\pm \alpha_4/5\}$. We write $\mathbf{z} = \mathbf{z}_1 + \mathbf{z}_2$ with $\mathbf{z}_1 \in T_0 \otimes \mathbb{C}$, and $\mathbf{z}_2 \in T_0^\perp \otimes \mathbb{C}$. The quasi pull-back on $T_0 \subset A_4^\vee(5)$ can be written in the affine coordinate as the derivative at $\mathbf{z}_2 = 0$ (see [10] and [11, §8.4]). In this particular case we have the differential operator with respect to $z_4$ which commutes with the Hecke operators in the Jacobi lifting. As a result we obtain the following Jacobi form of weight 3 of type $9-\eta/3-\eta$

$$\Theta_{T_0} = \frac{\partial(z_1)\partial(z_2)\partial(z_1 + z_2)\partial^2(z_3)\partial^2(z_2 + z_3)\partial^2(z_1 + z_2 + z_3)}{\eta^3} \in J_{3,0}^{\text{cusp}}.$$

Therefore, $\text{Grit}(\Theta_{T_0}) \in S_3(\tilde{O}^+(2U \oplus A_0(-1)), \chi_2)$ is a cusp form of weight 3 with the Borcherds automorphic product constructed by $\Psi_{A_4}(\tau, \mathbf{z})|_{z_4=0}$.

**Example 4.9.** We can continue the construction of quasi pull-back setting $z_2 = 0$. The we get a Jacobi lifting of canonical weight with Borcherds product in four variables

$$\text{Grit}(\theta^2(z_1)\theta^4(z_3)\theta^2(z_1 + z_3)) \in S_4(\tilde{O}^+(2U \oplus B_0(-1)), \chi_2), \quad B_0 = \begin{pmatrix} 4 & 2 \\ 2 & 6 \end{pmatrix}.$$

The considered above examples support the following generalisation of the theta block conjecture formulated in the introduction.

**Conjecture 4.10.** Let $\phi \in J_{k,L,1}$. Then $\text{Grit}(\phi)$ has Borcherds product expansion, i.e.

$$\text{Grit}(\phi) = \text{Borch} \left( -\frac{\phi(T_0(2))}{\phi} \right),$$

if and only if $\phi$ is a pure theta block of type $(3.5)$ with $f(0, \ell) \geq 0$ for all $\ell$ and $\phi$ has vanishing order one in $q$.

**Remark 4.11.** The “only if” part of the above conjecture has an immediate corollary. If there exists a non-constant modular form of weight $k$ associated to $\tilde{O}^+(2U \oplus L(-1))$ which is simultaneously a Gritsenko lift and a Borcherds product, then $\text{rank}(L) \leq 8$ and $\text{rank}(L)/2 \leq k \leq 12 - \text{rank}(L)$. In fact, since $\phi$ has vanishing order one in $q$, the number $A$ in Theorem 3.3 is equal to 1. Equation (3.5) defines a Jacobi form for $L$. The Fourier coefficients of any Jacobi form of weight 0 define a generalised 2-design in the dual lattice $L^\vee$

$$\sum_{\ell \in L^\vee} f(0, \ell)(\ell, \mathbf{z})^2 = 2C(\mathbf{z}, \mathbf{z}) \quad \forall \mathbf{z} \in L \otimes \mathbb{C}$$

(see [7, Proposition 2.2]). Therefore, the number of $\ell > 0$ with non-zero $f(0, \ell)$ is at least $\text{rank}(L)$. In view of the singular weight i.e. $k \geq 1/2 \text{rank}(L)$, we prove the above claim.
Acknowledgements. The first author is supported by the Laboratory of Mirror Symmetry NRU HSE (RF government grant, ag. N 14.641.31.0001). The second author is supported by the Labex CEMPI (ANR-11-LABX-0007-01) of the University of Lille.

References

1. R.E. Borcherds, Automorphic forms on $O_{s+2}(R)$ and infinite products. Invent. Math. 120 (1995), 161–213.
2. R.E. Borcherds, Automorphic forms with singularities on Grassmannians. Invent. Math., 123 (1998), no. 3, 491–562.
3. F. Cléry, V. Gritsenko, Modular forms of orthogonal type and Jacobi theta-series. Abh. Math. Semin. Univ. Hambg 83 (2013), 187–217.
4. M. Eichler, D. Zagier, The theory of Jacobi forms. Progress in Mathematics 55, Birkhäuser, Boston, Mass., 1985.
5. V. Gritsenko, Modular forms and moduli spaces of Abelian and K3 surfaces. Algebra i Analiz 6 (1994), 65–102; English translation in St. Petersburg Math. J. 6 (1995), 1179–1208.
6. V. Gritsenko, Irrationality of the moduli spaces of polarized abelian surfaces. Int. Math. Res. Not. IMRN 6 (1994), 235–243.
7. V. Gritsenko, Reflective modular forms in algebraic geometry. arXiv:1005.3753.
8. V. Gritsenko, 24 faces of the Borcherds modular form $\Phi_{24}$. arXiv: 1203.6503.
9. V. Gritsenko, K. Hulek, Uniruledness of orthogonal modular varieties. J. Algebraic Geom. 23 (2014), 711–725.
10. V. Gritsenko, K. Hulek, G.K. Sankaran, The Kodaira dimension of the moduli of K3 surfaces. Invent. Math. 169 (2007), 519–567.
11. V. Gritsenko, K. Hulek, G.K. Sankaran, Moduli of K3 surfaces and irreducible symplectic manifolds, from: “Handbook of moduli, I” (editors G Farkas, I Morrison), Adv. Lect. Math. 24, International Press (2013) 459–526.
12. V. Gritsenko, V.V. Nikulin, Automorphic forms and Lorentzian Kac–Moody algebras. Part II. Internat. J. Math. 9 (1998), 201–275.
13. V. Gritsenko, V.V. Nikulin, Lorentzian Kac-Moody algebras with Weyl groups of 2-reflections. Proc. London Math. Soc. 116 (2018), no.3, 485–533.
14. V. Gritsenko, C. Poor, D. Yuen, Borcherds products everywhere. J. Number Theory 148 (2015), 164–195.
15. V. Gritsenko, N.-P. Skoruppa, D. Zagier, Theta blocks, preprint 2018, 56 pp. https://math.univ-lille1.fr/~d7/sites/default/files/THETA%20BLOCKS22.09.18_1.pdf
16. V. Gritsenko, H. Wang, Conjecture on theta-blocks of order 1. Uspekhi Matematicheskikh Nauk 72:5(437) (2017), 191–192. English version: Russian Mathematical Surveys, Volume 72, no. 5 (2017), 968–970.
17. V. Kac, D.H. Peterson, Infinite-dimensional Lie algebras, theta functions and modular forms. Adv. in Math. 53 (1984), 125–264.
18. V.V. Nikulin, Integer symmetric bilinear forms and some of their geometric applications. Math. USSR Izv. 14, 103–167 (1980).
19. C. Poor, J. Shurman, D. Yuen, Theta block Fourier expansions, Borcherds products, and a sequence of Newman and Shanks. Bull. Aust. Math. Soc. 98 (2018), 48–59.
20. N.R. Scheithauer, Generalized Kac-Moody algebras, automorphic forms and Conways group $I$. Adv. Math. 183 (2004), 240–270.
21. N.R. Scheithauer, On the classification of automorphic products and generalized Kac-Moody algebras. Invent. Math. 164 (2006), 641–678.