QUASI-REPRESENTATIONS OF FINSLER MODULES OVER $C^*$-ALGEBRAS

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Abstract. We show that every Finsler module over a $C^*$-algebra has a quasi-representation into the Banach space $\mathbb{B}(\mathcal{H}, \mathcal{K})$ of all bounded linear operators between some Hilbert spaces $\mathcal{H}$ and $\mathcal{K}$. We define the notion of completely positive $\varphi$-morphism and establish a Stinespring type theorem in the framework of Finsler modules over $C^*$-algebras. We also investigate the nondegeneracy and the irreducibility of quasi-representations.

1. INTRODUCTION

The notion of Finsler module is an interesting generalization of that of Hilbert $C^*$-module. It is a useful tool in the operator theory and the theory of operator algebras and may be served as a noncommutative version of the concept of Banach bundle, which is an essential concept in the Finsler geometry. In 1995 Phillips and Weaver [10] showed that if a $C^*$-algebra $\mathcal{A}$ has no nonzero commutative ideal, then any Finsler $\mathcal{A}$-module must be a Hilbert $C^*$-module. If $\mathcal{A}$ is the commutative $C^*$-algebra $C_0(X)$ of all continuous complex-valued functions vanishing at infinity on a locally compact Hausdorff space $X$, then any Finsler $\mathcal{A}$-module is isomorphic to the module of continuous sections of a bundle of Banach spaces over $X$. The concept of a $\varphi$-morphism between Finsler modules was introduced in [1].

The Gelfand–Naimark–Segal (GNS) representation theorem is one of the most useful theorems, which is applied in operator algebras and mathematical physics. That provides a procedure to construct representations of $C^*$-algebras. A generalization of GNS construction to a topological $*$-algebra established by Borchers, Uhlmann and Powers leading to unbounded $*$-representations of $*$-algebras; see [12]. Another is a generalization of a positive linear functional to a completely positive map studied by Stinespring [14], see also [6].

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Let $\mathcal{A}$ be a $C^*$-algebra and let $\mathcal{A}^+$ denote the positive cone of all positive elements of $\mathcal{A}$. We define a Finsler $\mathcal{A}$-module to be a right $\mathcal{A}$-module $E$ equipped with a map $\rho : E \to \mathcal{A}^+$ (denoted by $\rho_\mathcal{A}$ if there is an ambiguity) satisfying the following conditions:

(i) The map $\| \cdot \|_E : x \mapsto \| \rho(x) \|$ makes $E$ into a Banach space.
(ii) $\rho(xa)^2 = a^* \rho(x)^2 a$, for all $a \in \mathcal{A}$ and $x \in E$.

A Finsler module $E$ over a $C^*$-algebra $\mathcal{A}$ is said to be full if the linear span of $\{ \rho(x)^2 : x \in E \}$ is dense in $\mathcal{A}$. For example, if $E$ is a (full) Hilbert $C^*$-module over $\mathcal{A}$ (see [7]), then $E$ together with $\rho(x) = < x, x >^{\frac{1}{2}}$ is a (full) Finsler module over $\mathcal{A}$, since

$$\rho(xa)^2 = a^* < x, xa > a = a^* \rho(x)^2 a.$$ 

In particular, every $C^*$-algebra $\mathcal{A}$ is a full Finsler module over $\mathcal{A}$ under the mapping $\rho(x) = (x^*x)^{\frac{1}{2}}$.

Our goal is to extend the notion of a representation of a Hilbert $C^*$-module to the framework of Finsler $\mathcal{A}$-modules. We show that every Finsler $\mathcal{A}$-module has a quasi-representation into the Banach space $\mathbb{B}(\mathcal{H}, \mathcal{K})$ of all bounded linear operators between some Hilbert spaces $\mathcal{H}$ and $\mathcal{K}$. We define the notion of completely positive $\varphi$-morphism and establish a Stinespring type theorem in the framework of Finsler modules over $C^*$-algebras. We also introduce the notions of the nondegeneracy and the irreducibility of quasi-representations and study some interrelations between them.

2. QUASI-REPRESENTATIONS OF FINSLER MODULES

We start our work by giving the definition of a $\varphi$-morphism of a Finsler module.

**Definition 2.1.** Suppose that $(E, \rho_\mathcal{A})$ and $(F, \rho_\mathcal{B})$ are Finsler modules over $C^*$-algebras $\mathcal{A}$ and $\mathcal{B}$, respectively and $\varphi : \mathcal{A} \to \mathcal{B}$ is a $*$-homomorphism of $C^*$-algebras. A (not necessarily linear) map $\Phi : E \to F$ is said to be a $\varphi$-morphism of Finsler modules if the following conditions are satisfied:

(i) $\rho_\mathcal{B}(\Phi(x)) = \varphi(\rho_\mathcal{A}(x))$;
(ii) $\Phi(xa) = \Phi(x)\varphi(a)$.

for all $x \in E$ and $a \in \mathcal{A}$. In the case of Hilbert $C^*$-modules, $\Phi$ is assumed to be linear and then condition (ii) is deduced from (i).

Now we introduce the notion of a quasi-representation of a Finsler module. Due to $\mathbb{B}(\mathcal{H}, \mathcal{K})$ is a Hilbert $C^*$-module over $\mathbb{B}(\mathcal{H}, \mathcal{K})$ via $\langle T, S \rangle = T^*S$, we can endow $\mathbb{B}(\mathcal{H}, \mathcal{K})$ a Finsler
structure by
\[ \rho_0(T) = (T^*T)^{\frac{1}{2}}. \]  

**Definition 2.2.** Let \((\mathcal{E}, \rho)\) be a Finsler module over a \(C^*\)-algebra \(\mathcal{A}\). A map \(\Phi : \mathcal{E} \to \mathbb{B}(\mathcal{H}, \mathcal{K})\), where \(\varphi : \mathcal{A} \to \mathbb{B}(\mathcal{H})\) is a representation of \(\mathcal{A}\) is called a quasi-representation of \(\mathcal{E}\) if \(\rho_0(\Phi(x)) = \varphi(\rho(x))\) for all \(x \in \mathcal{E}\).

We are going to show that for every Finsler \(\mathcal{A}\)-module there is a quasi-representation to \(\mathbb{B}(\mathcal{H}, \mathcal{K})\) for some Hilbert spaces \(\mathcal{H}\) and \(\mathcal{K}\), see also [9].

**Theorem 2.3.** Suppose \(\mathcal{E}\) is a Finsler \(\mathcal{A}\)-module with the associated map \(\rho : \mathcal{E} \to \mathcal{A}^+\). Then there is a quasi-representation \(\Phi : \mathcal{E} \to \mathbb{B}(\mathcal{H}, \mathcal{K})\) for some Hilbert spaces \(\mathcal{H}\) and \(\mathcal{K}\).

**Proof.** By the Gelfand–Naimark theorem for \(C^*\)-algebras, there is a representation \(\varphi : \mathcal{A} \to \mathbb{B}(\mathcal{H})\) for some Hilbert space \(\mathcal{H}\). We want to construct a Hilbert space \(\mathcal{K}\). Put
\[ \mathcal{K}_0 := \text{span}\{\varphi(a)f : a \in \mathcal{A}, f : \mathcal{E} \to \mathcal{H}\} \] is a map with a finite support\}
and define on \(\mathcal{K}_0\) an inner product by
\[ \langle \varphi(a)f, \varphi(b)g \rangle = \sum_{x \in \mathcal{E}} \langle \varphi(a)f(x), \varphi(b)g(x) \rangle. \]

Note that if \(\sum_{i=1}^{n} \varphi(a_i)f_i, \sum_{i=1}^{n} \varphi(a_i)f_i = 0\), then
\[ \sum_{x \in \mathcal{E}} \langle \sum_{i=1}^{n} \varphi(a_i)f_i(x), \sum_{i=1}^{n} \varphi(a_i)f_i(x) \rangle = 0. \]

Thus \(\sum_{i=1}^{n} \varphi(a_i)f_i(x) = 0\) for each \(x \in \mathcal{E}\), whence \(\sum_{i=1}^{n} \varphi(a_i)f_i = 0\).

Let us consider the closure \(\overline{\mathcal{K}_0}\) of \(\mathcal{K}_0\) to get a Hilbert space, which is denoted by \(\mathcal{K}\). For any \(y \in \mathcal{E}\) and \(h \in \mathcal{H}\), the map \(h_y : \mathcal{E} \to \mathcal{H}\) defined by
\[ h_y(x) = \begin{cases} h & x = y \\ 0 & x \neq y \end{cases} \]
has a finite support. For \(x \in \mathcal{E}\), define \(\Phi(x) : \mathcal{H} \to \mathcal{H}\) by \(\Phi(x)h = \varphi(\rho(x))h_x\). We show that \(\Phi(x) \in \mathbb{B}(\mathcal{H}, \mathcal{K})\). Clearly \(\Phi(x)\) is linear. Also \(\Phi(x)\) is bounded, since
\[ \|\Phi(x)h\|^2 = \langle \Phi(x)h, \Phi(x)h \rangle = \langle \varphi(\rho(x))h_x, \varphi(\rho(x))h_x \rangle \]
\[ = \sum_{y \in \mathcal{E}} \langle \varphi(\rho(x))h_x(y), \varphi(\rho(x))h_x(y) \rangle = \langle \varphi(\rho(x))h, \varphi(\rho(x))h \rangle \]
\[ \leq \|\varphi(\rho(x))\|^2\|h\|^2, \]
whence \( \| \Phi(x) \| \leq \| \varphi(\rho(x)) \| \).

Further,

\[
\langle \Phi(x)^* \Phi(x)h, h' \rangle = \langle \Phi(x)h, \Phi(x)h' \rangle = \langle \varphi(\rho(x))h_x, \varphi(\rho(x))h_x' \rangle \\
= \sum_{y \in \mathcal{E}} \langle \varphi(\rho(x))h_x(y), \varphi(\rho(x))h_x'(y) \rangle \\
= \langle \varphi(\rho(x))h, \varphi(\rho(x))h' \rangle = \langle \varphi(\rho(x)^2)h, h' \rangle,
\]

for all \( h, h' \in \mathcal{H} \) and \( x \in \mathcal{E} \). Hence \( \Phi(x)^* \Phi(x) = \varphi(\rho(x)^2) \). Hence

\[
(\Phi(x)^* \Phi(x))^{\frac{1}{2}} = \varphi(\rho(x)). \tag{2.2}
\]

It follows from (2.1) and equality (2.2) that \( \rho_0(\Phi(x)) = \varphi(\rho(x)) \).

\[\square\]

**Remark 2.4.** If \( \Phi \) is surjective and \( \mathcal{B}(\mathcal{H}, \mathcal{K}) \) is a full Finsler \( \mathcal{B}(\mathcal{H}) \)-module, then by [1, Theorem 3.4(iv)], \( \varphi \) is surjective.

In the next section the notion of completely positive \( \varphi \)-morphism is introduced and a construction of Stinespring’s theorem for Finsler modules is given.

### 3. A STINESPRING TYPE THEOREM FOR FINSLER MODULES

The Stinespring theorem was first introduced in the work of Stinespring in 1955 that described the structure of completely positive maps of a \( C^* \)-algebra into the \( C^* \)-algebra of all bounded linear operators on a Hilbert space; see [14]. Recently Asadi [3] proved this theorem for Hilbert \( C^* \)-modules. Further, Bhat et al. [4] improve the result of [3] with omitting a technical condition. In this section we intend to establish a Stinespring type theorem in the framework of Finsler modules over \( C^* \)-algebras.

A \( \varphi \)-morphism \( \Phi : \mathcal{E} \to \mathcal{B}(\mathcal{H}, \mathcal{K}) \) is called completely positive if the map \( \varphi : \mathcal{A} \to \mathcal{B}(\mathcal{H}) \) is completely positive.

**Theorem 3.1.** Let \( (\mathcal{E}, \rho) \) be a Finsler module over a unital \( C^* \)-algebra \( \mathcal{A} \), let \( \mathcal{H}, \mathcal{K} \) be Hilbert spaces and let \( \Phi : \mathcal{E} \to \mathcal{B}(\mathcal{H}, \mathcal{K}) \) be a completely positive map associated to a completely positive map \( \varphi : \mathcal{A} \to \mathcal{B}(\mathcal{H}) \). Then there exist Hilbert spaces \( \mathcal{H}', \mathcal{K}' \) and isometries \( V : \mathcal{H} \to \mathcal{H}', W : \mathcal{K} \to \mathcal{K}' \), a \( * \)-homomorphism \( \theta : \mathcal{A} \to \mathcal{B}(\mathcal{H}') \) and a \( \theta \)-morphism \( \Psi : \mathcal{E} \to \mathcal{B}(\mathcal{H}', \mathcal{K}') \) such that \( \varphi(a) = V^* \theta(a)V, \Phi(x) = W^* \Psi(x)V \) for all \( x \in \mathcal{E} \) and \( a \in \mathcal{A} \).

**Proof.** By [11, Theorem 4.1] there exist a Hilbert space \( \mathcal{H}' = \mathcal{A} \otimes \mathcal{H} \), a representation \( \theta : \mathcal{A} \to \mathcal{B}(\mathcal{H}') \) and an isometry \( V : \mathcal{H} \to \mathcal{H}' \) defined by \( V(h) = 1 \otimes h \) such that
\( \varphi(a) = V^* \theta(a) V \). We may consider a minimal Stinespring representation for \( \theta \), where \( \mathcal{H}' \) is the closed linear span of \( \{ \theta(a) V h : a \in \mathcal{A}, h \in \mathcal{H} \} \).

Now, we put \( \mathcal{K}' \) to be the closed linear span of \( \{ \Phi(x) h : x \in \mathcal{E}, h \in \mathcal{H} \} \) and define the mapping \( \Psi : \mathcal{E} \to \mathbb{B}(\mathcal{H}', \mathcal{K}'), x \mapsto \Psi(x) \), where \( \Psi(x) : \text{span}\{ \theta(a) V h, a \in \mathcal{A}, h \in \mathcal{H} \} \to \mathcal{K}' \) is defined by \( \Psi(x)(\sum_{i=1}^{n} \theta(a_i) V h_i) = \sum_{i=1}^{n} \Phi(xa_i) h_i \) for \( x \in \mathcal{E}, a_i \in \mathcal{A}, h_i \in \mathcal{H} \).

The map \( \Psi(x) \) is well-defined and bounded, since

\[
\left\| \Psi(x) \left( \sum_{i=1}^{n} \theta(a_i) V h_i \right) \right\| = \left\| \sum_{i=1}^{n} \phi(a_i) \right\|^2 \leq \sum_{i,j=1}^{n} \langle \phi(a_j) \phi(a_i) V h_i, h_j \rangle \leq \sum_{i,j=1}^{n} \langle \phi(a_j) \phi(a_i) V h_i, h_j \rangle.
\]
The mapping $\Psi$ is a $\theta$-morphism, since for all $a, b \in \mathcal{A}$ and $h, g \in \mathcal{H}$

\[
\langle \Psi(x)^*\Psi(x)(\theta(a)Vh), \theta(b)Vg \rangle = \langle \Psi(x)(\theta(a)Vh), \Psi(x)(\theta(b)Vg) \rangle = \langle \Phi(xa)h, \Phi(xb)g \rangle = \langle \Phi(x)\varphi(a)h, \Phi(x)\varphi(b)g \rangle = \langle \Phi(x)^*\Phi(x)\varphi(a)h, \varphi(b)g \rangle = \langle \varphi(\rho(x)^2)\varphi(a)h, \varphi(b)g \rangle = \langle \varphi(b^*\rho(x)^2a)h, g \rangle = \langle V^*\theta(b^*\rho(x)^2a)Vh, g \rangle = \langle \theta(\rho(x)^2)\theta(a)Vh, \theta(b)Vg \rangle,
\]

whence $\Psi(x)^*\Psi(x) = \theta(\rho(x)^2)$. Moreover

\[
\Psi(x)\theta(a)(\theta(b)Vh) = \Psi(x)(\theta(ab)Vh) = \Phi(x(ab))h = \Phi((xa)b)h = \Psi(xa)(\theta(b)Vh),
\]

so that $\Psi(x)\theta(a) = \Psi(xa)$.

Since $\mathcal{H}' \subseteq \mathcal{H}$ we can consider a map $W$ as the orthogonal projection of $\mathcal{H}$ onto $\mathcal{H}'$. Hence $W^* : \mathcal{H}' \to \mathcal{H}$ is the inclusion map, whence for any $k' \in \mathcal{H}'$ we have $WW^*(k') = W(k') = k'$, that is $WW^* = I_{\mathcal{H}'}$.

Finally we observe that $W^*\Psi(x)Vh = \Psi(x)Vh = \Psi(x)(\theta(1)Vh) = \Phi(x)h$, that is $W^*\Psi(x)V = \Phi(x)$. \[\square\]

4. NONDEGENERATE AND IRREDUCIBLE QUASI-REPRESENTATIONS

In this section we define the notions of nondegenerate and irreducible quasi-representations of Finsler modules and describe relations between the nondegeneracy and the irreducibility, see [2]. Throughout this section we assume that the quasi-representations satisfy condition (ii) of Definition 2.1.

**Definition 4.1.** Let $\Phi : \mathcal{E} \to \mathcal{B}(\mathcal{H}, \mathcal{H})$ be a quasi-representation of a Finsler module $\mathcal{E}$ over a $C^*$-algebra $\mathcal{A}$. The map $\Phi$ is said to be nondegenerate if $\Phi(\mathcal{E})\mathcal{H} = \mathcal{H}$ and $\Phi(\mathcal{E})^*\mathcal{H} = \mathcal{H}$ (or equivalently, if there exist $\xi \in \mathcal{H}, \eta \in \mathcal{H}$ such that $\Phi(\mathcal{E})\xi = 0$ and
\(\Phi(\mathcal{E})^*\eta = 0\), then \(\xi = \eta = 0\). Recall that a representation \(\varphi : \mathcal{A} \to \mathbb{B}(\mathcal{H})\) of a \(C^*\)-algebra \(\mathcal{A}\) is nondegenerate if \(\varphi(\mathcal{A})\mathcal{H} = \mathcal{H}\) (or equivalently, if there exists \(\xi \in \mathcal{H}\) such that \(\varphi(\mathcal{A})\xi = 0\), then \(\xi = 0\)), see [13, definition A.1.].

**Theorem 4.2.** If \(\Phi : \mathcal{E} \to \mathbb{B}(\mathcal{H}, \mathcal{K})\) is a nondegenerate quasi-representation, then \(\varphi : \mathcal{A} \to \mathbb{B}(\mathcal{H})\) is a nondegenerate representation. If \(\mathcal{E}\) is full and \(\varphi\) is nondegenerate, then \(\Phi\) is also nondegenerate.

**Proof.** Suppose that \(\Phi\) is nondegenerate and \(\varphi(\mathcal{A})\xi = 0\). It follows from the Hewitt–Cohen factorization theorem that \(\Phi(\mathcal{E})\xi = \Phi(\mathcal{E}\mathcal{A})\xi = \Phi(\mathcal{E})\varphi(\mathcal{A})\xi = 0\). We conclude that \(\xi = 0\). Thus \(\varphi\) is nondegenerate.

Suppose that \(\Phi(\mathcal{E})\xi = 0\) for some \(\xi \in \mathcal{H}\). Then for any \(x \in \mathcal{E}\) we have \(\|\Phi(x)\xi\|^2 = \langle \Phi(x)^*\Phi(x)\xi, \xi \rangle = \langle \varphi(\rho(x)^2)\xi, \xi \rangle = \|\varphi(\rho(x))\|^2 = 0\). Since \(\mathcal{E}\) is a full Finsler \(\mathcal{A}\)-module, \(a = \lim_{n \to \infty} \sum_{i=1}^{k_n} \lambda_{i,n} \rho(x_{i,n})^2\) for some \(k_n \in \mathbb{N}\), \(x_{i,n} \in \mathcal{E}\) and \(\lambda_{i,n} \in \mathbb{C}\). Hence

\[
\varphi(a)\xi = \lim_{n \to \infty} \sum_{i=1}^{k_n} \lambda_{i,n} \varphi(\rho(x_{i,n}))^2 \xi = \lim_{n \to \infty} \sum_{i=1}^{k_n} \lambda_{i,n} \varphi(\rho(x_{i,n})) \varphi(\rho(x_{i,n}))\xi = 0,
\]

whence \(\xi = 0\). \(\square\)

**Remark 4.3.** The second result of Theorem 4.2 may fail, if the condition of being full is dropped. To see this take \(\mathcal{A}\) to be a nondegenerate von Neumann algebra acting on a Hilbert space, which has a nontrivial central projection \(P\). Hence the identity map \(\varphi : \mathcal{A} \to \mathbb{B}(\mathcal{H})\) is assumed to be nondegenerate.

Put \(\mathcal{E} = \mathcal{A}P = \{aP : a \in \mathcal{A}\}\) as a Finsler \(\mathcal{A}\)-module equipped with \(\rho(aP) = |aP|\). Clearly \(\mathcal{A}P\) is not full. The identity map \(\Phi : \mathcal{A}P \to \mathbb{B}(\mathcal{H})\) satisfies the following:

(i) \(\rho_0(\Phi(aP)) = \rho_0(\rho(aP)) = |aP| = \varphi(|aP|) = \varphi(\rho(aP))\), where \(\rho_0\) is defined as in (2.1).

(ii) \(\Phi(aPb) = \Phi(aP)\varphi(b)\) for all \(b \in \mathcal{A}\).

Hence \(\Phi\) is a quasi-representation of \(\mathcal{E}\), which is not clearly nondegenerate, since

\[
\overline{\Phi(\mathcal{E})\mathcal{H}} = \overline{\mathcal{A}P(\mathcal{H})} = \overline{P(\mathcal{A}\mathcal{H})} \subseteq \overline{P(\mathcal{H})} = P(\mathcal{H}) \neq \mathcal{H}.
\]

In the following corollary we investigate a condition under which the representation \(\varphi\) and the quasi-representation \(\Phi\) are nondegenerate.

**Corollary 4.4.** If \(\varphi(\rho(x)) = I_\mathcal{H}\), then both \(\Phi\) and \(\varphi\) are nondegenerate.
Proof. Suppose \( \Phi(\mathcal{E})\xi = 0 \) for some \( \xi \in \mathcal{H} \). Then for all \( x \in \mathcal{E} \) we have \( \|\Phi(x)\xi\|^2 = \langle \Phi(x)^*\Phi(x)\xi , \xi \rangle = \langle \varphi(\rho(x)^2)\xi , \xi \rangle = \|\xi\|^2 = 0 \), so that \( \xi = 0 \). The nondegeneracy of \( \varphi \) follows from Theorem 4.2. \( \Box \)

**Definition 4.5.** Let \( \Phi : \mathcal{E} \to B(\mathcal{H}, \mathcal{H}') \) be a quasi-representation of a Finsler module \( \mathcal{E} \) over a \( C^* \)-algebra \( \mathcal{A} \) and let \( \mathcal{H}, \mathcal{H}' \) be closed subspaces of \( \mathcal{H} \) and \( \mathcal{H}' \), respectively. A pair of subspaces \( (\mathcal{H}, \mathcal{H}') \) is said to be \( \Phi \)-invariant if \( \Phi(\mathcal{E})\mathcal{H} \subseteq \mathcal{H}' \) and \( \Phi(\mathcal{E})^*\mathcal{H}' \subseteq \mathcal{H} \). The quasi-representation \( \Phi \) is said to be irreducible if \( (0, 0) \) and \( (\mathcal{H}, \mathcal{H}') \) are the only \( \Phi \)-invariant pairs. Recall that a representation \( \varphi : \mathcal{A} \to B(\mathcal{H}) \) of a \( C^* \)-algebra \( \mathcal{A} \) is irreducible if 0 and \( \mathcal{H} \) are only closed subspaces of \( \mathcal{H} \) being \( \varphi \)-invariant, i.e. are invariant for \( \varphi(\mathcal{A}) \).

**Theorem 4.6.** Suppose that the quasi-representation \( \Phi : \mathcal{E} \to B(\mathcal{H}, \mathcal{H}') \) constructed in Theorem 2.3 is irreducible. Then so is \( \varphi : \mathcal{A} \to B(\mathcal{H}) \). If \( \mathcal{E} \) is full and \( \varphi \) is irreducible, then \( \Phi \) is irreducible.

Proof. Suppose that \( \Phi \) is irreducible and a closed subspace \( \mathcal{K} \) of \( \mathcal{H} \) is \( \varphi \)-invariant. Consider \( \mathcal{K}' = \overline{\Phi(\mathcal{E})\mathcal{K}} \). Clearly \( \Phi(\mathcal{E})\mathcal{K} \subseteq \mathcal{K}' \). Due to \( \overline{\varphi(\mathcal{A})\mathcal{K}} \subseteq \mathcal{K} \) we observe that \( \varphi(\rho(x)^2)\mathcal{K} \subseteq \mathcal{K} \), whence \( \Phi(\mathcal{E})\varphi(\rho(x))\mathcal{K} \subseteq \mathcal{K} \) for all \( x \in \mathcal{E} \). Now let \( x \neq y \). In the notation of Theorem 2.3 we have

\[
\langle \Phi(x)^*\Phi(y)h, h' \rangle = \langle \Phi(y)h, \Phi(x)h' \rangle = \langle \varphi(\rho(y))h_y, \varphi(\rho(x))h'_x \rangle = \sum_{z \in \mathcal{E}} \langle \varphi(\rho(y))h_y(z), \varphi(\rho(x))h'_x(z) \rangle = 0,
\]

for all \( h, h' \in \mathcal{H} \). Put \( h' = \Phi(x)^*\Phi(y)h \) to get \( \langle \Phi(x)^*\Phi(y)h, \Phi(x)^*\Phi(y)h \rangle = 0 \). So that \( \Phi(x)^*\Phi(y)h = 0 \). Therefore \( \Phi(x)^*\Phi(y)\mathcal{K} = 0 \mathcal{K} \subseteq \mathcal{K} \). It follows that \( \Phi(\mathcal{E})^*\Phi(\mathcal{E})\mathcal{K} \subseteq \overline{\Phi(\mathcal{E})^*\Phi(\mathcal{E})\mathcal{K}} \subseteq \mathcal{K}' \). Since \( \Phi \) is irreducible, we conclude that \( (\mathcal{K}, \mathcal{K}') = (0, 0) \) or \( (\mathcal{K}, \mathcal{K}') = (\mathcal{K}, \mathcal{K}') \), hence \( \mathcal{K} = 0 \) or \( \mathcal{K} = \mathcal{H} \). This implies that \( \varphi \) is irreducible.

Now assume that \( \varphi \) is irreducible. It follows from [8, Remark 4.1.4] that \( \varphi \) is nondegenerate. By Theorem 4.2, \( \Phi \) is nondegenerate.

Consider \( (\mathcal{K}, \mathcal{K}') \) as a \( \Phi \)-invariant pair of subspaces. Any \( a \in \mathcal{A} \) can be represented as \( a = \lim_{n \to \infty} \sum_{i=1}^{k_n} \lambda_{i,n}^2 \rho(x_{i,n})^2 \) for some \( k_n \in \mathbb{N} \), \( x_{i,n} \in \mathcal{E} \) and \( \lambda_{i,n} \in \mathbb{C} \). Hence

\[
\varphi(a)\mathcal{K} = \lim_{n \to \infty} \sum_{i=1}^{k_n} \lambda_{i,n}^2 \varphi(\rho(x_{i,n}))^2 \mathcal{K} = \lim_{n \to \infty} \sum_{i=1}^{k_n} \lambda_{i,n}^2 \Phi(x_{i,n})^*\Phi(x_{i,n})\mathcal{K} \subseteq \mathcal{K},
\]

Hence \( \mathcal{K} = 0 \) or \( \mathcal{K} = \mathcal{H} \).

If \( \mathcal{K} = 0 \) then \( \Phi(\mathcal{E})^*\mathcal{K}' \subseteq \mathcal{K} = 0 \), and for every \( \xi' \in \mathcal{K}' \) we have \( 0 = \langle \Phi(x)^*\xi', \xi \rangle = \langle \Phi(x)^*\Phi(y)h, \Phi(x)^*\Phi(y)h \rangle = \langle \varphi(\rho(y))h_y, \varphi(\rho(x))h'_x \rangle = \sum_{z \in \mathcal{E}} \langle \varphi(\rho(y))h_y(z), \varphi(\rho(x))h'_x(z) \rangle = 0 \).
\[ \langle \xi', \Phi(x) \xi \rangle \text{ for } x \in \mathcal{E} \text{ and } \xi \in \mathcal{H}, \] so that \( \mathcal{H}' \perp \Phi(\mathcal{E}) \mathcal{H} = \mathcal{H}' \). Since \( \mathcal{H}' \subseteq \mathcal{H}' \), we have \( \mathcal{H}' = 0 \).

If \( \mathcal{H} = \mathcal{H}' \), then \( \mathcal{H}' = \Phi(\mathcal{E}) \mathcal{H} = \Phi(\mathcal{E}) \mathcal{H} \subseteq \mathcal{H}' \). Hence \( \mathcal{H}' = \mathcal{H}' \). Therefore \( \Phi \) is irreducible.

\[ \text{Remark 4.7. The result may fail, if the condition of being full is dropped. The closed subspace } P(\mathcal{H}) \text{ in Remark 4.3 when } \varphi : \mathcal{A} \to \mathcal{B}(\mathcal{H}) \text{ is irreducible provides a counterexample.} \]

Next we present some conditions under which the quasi-representation \( \Phi \) is nondegenerate and irreducible.

**Corollary 4.8.** Let \( \mathcal{E} \) be a full Finsler \( \mathcal{A} \)-module and let \( \varphi : \mathcal{A} \to \mathcal{B}(\mathcal{H}) \) is irreducible. Then the quasi-representation \( \Phi : \mathcal{E} \to \mathcal{B}(\mathcal{H}, \mathcal{H}) \) is nondegenerate and irreducible.

**Proof.** Since \( \varphi \) is irreducible, it is nondegenerate. Since \( \mathcal{E} \) is full, by Theorem 4.2, \( \Phi \) is nondegenerate and by Theorem 4.6, \( \Phi \) is irreducible. \( \square \)

**Theorem 4.9.** Let \( \mathcal{E} \) be a full Finsler \( \mathcal{A} \)-module. Then \( \Phi(\mathcal{E}) \) is a subset of the space \( \mathbb{K}(\mathcal{H}, \mathcal{H}') \) of all compact operators from \( \mathcal{H} \) into \( \mathcal{H}' \) if and only if \( \varphi(\mathcal{A}) \subseteq \mathbb{K}(\mathcal{H}) \).

**Proof.** Suppose \( \varphi(\mathcal{A}) \subseteq \mathbb{K}(\mathcal{H}) \). Applying the Hewitt–Cohen factorization theorem we have \( \Phi(\mathcal{E}) = \Phi(\mathcal{E} \mathcal{A}) = \Phi(\mathcal{E}) \varphi(\mathcal{A}) \subseteq \mathbb{K}(\mathcal{H}, \mathcal{H}') \).

Conversely, suppose that \( \Phi(\mathcal{E}) \subseteq \mathbb{K}(\mathcal{H}, \mathcal{H}') \). Since \( \mathcal{E} \) is full we have

\[ \varphi(a) = \lim_{n \to \infty} \sum_{i=1}^{k_n} \lambda_{i,n} \varphi(\rho(x_{i,n}))^2 = \lim_{n \to \infty} \sum_{i=1}^{k_n} \lambda_{i,n} \Phi(x_{i,n})^* \Phi(x_{i,n}) \in \mathbb{K}(\mathcal{H}), \]

where \( a = \lim_{n \to \infty} \sum_{i=1}^{k_n} \lambda_{i,n} \rho(x_{i,n})^2 \) for some \( k_n \in \mathbb{N}, x_{i,n} \in \mathcal{E} \) and \( \lambda_{i,n} \in \mathbb{C} \). \( \square \)

Now in the next two examples we illustrate the considered situations in the notation of Theorem 2.3.

**Example 4.10.** By [5, Theorem 1.10.2] the identity map \( \varphi : \mathbb{K}(\mathcal{H}) \to \mathcal{B}(\mathcal{H}) \) is irreducible. It is known that the \( \mathcal{C}^* \)-algebra \( \mathbb{K}(\mathcal{H}) \) is a full Finsler module over \( \mathbb{K}(\mathcal{H}) \) with \( \rho(T) = |T| \). Hence the quasi-representation \( \Phi : \mathbb{K}(\mathcal{H}) \to \mathcal{B}(\mathcal{H}, \mathcal{H}) \) is nondegenerate and irreducible.

**Example 4.11.** Consider \( \varphi = I : \mathcal{B}(\mathcal{H}) \to \mathcal{B}(\mathcal{H}) \). Then \( \varphi(\mathcal{B}(\mathcal{H}))^c = \{ T \in \mathcal{B}(\mathcal{H}) ; \varphi(S)T = T\varphi(S) \}, \) for all \( S \in \mathcal{B}(\mathcal{H}) \} = \{ T \in \mathcal{B}(\mathcal{H}) ; ST = TS, \) for all \( S \in \mathcal{B}(\mathcal{H}) \} = \mathbb{C}I \). Hence \( \varphi \) is irreducible. Also \( \mathcal{B}(\mathcal{H}) \) is a full Finsler \( \mathcal{B}(\mathcal{H}) \)-module, so that the quasi-representation \( \Phi : \mathcal{B}(\mathcal{H}) \to \mathcal{B}(\mathcal{H}, \mathcal{H}) \) is nondegenerate and irreducible.
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