NEGATIVE VOLATILITY FOR A 2-DIMENSIONAL SQUARE ROOT SDE

PETER SPREIJ AND ENNO VEERMAN

ABSTRACT. In affine term structure models the short rate is modelled as an affine transformation of a multi-dimensional square root process. Sufficient conditions to avoid negative volatility factors are the multivariate Feller conditions. We will prove their necessity for a 2-dimensional square root SDE in canonical form by presenting a methodology based on measure transformations and the trivial fact that a random variable assumes negative values if it has negative expectation. We exploit the property that solutions to square root SDEs have expectations which solve a system of linear differential equations. As an aside we will present two proofs for the martingale property of the density processes used in completely affine models.

1. Introduction

1.1. Problem and motivation. In recent years, affine term structure models (ATSMs) have become a popular instrument for modelling the dynamics of a term structure, i.e. the dynamics of the short interest rate and the long interest rate. These models have been introduced by [5] and can be regarded as a multi-dimensional extension of the Cox-Ingersoll-Ross model [2]. The short rate is modelled as an affine transformation of a (possibly multi-dimensional) state factor $X$ which satisfies a multi-dimensional square root SDE. The diffusion part involves square roots of affine transformations of $X$, which are called volatility factors, or just volatilities. Conditions need to be imposed on the parameters to guarantee that the volatility factors do not become negative, in order to assure pathwise uniqueness and to justify the Feynman-Kac formula for the bond price, see [12] for a detailed discussion. As shown in [5] sufficient conditions for this are the so-called multivariate Feller conditions, but they are not known to be necessary (see [3] footnote 6).

Imposing the Feller conditions is not always desirable in practice, as they might contradict with certain economic principles. In [11] this is observed for a 2-factor model where the state factor consists of the interest rate and inflation. It turns out that estimating the model without the Feller conditions yields parameter values that are in agreement with economic theory. Therefore, the question is raised whether these conditions can be relaxed.
However, the model proposed in [11] is actually a discrete-time ATSM. Imposing the Feller conditions for excluding negative volatility factors is meaningless in discrete time, since negative volatilities always occur with positive probability, due to the normally distributed jumps of the process. Instead, [11] mainly investigates the mathematical correctness of the discrete-time ATSM, with or without the Feller conditions.

In the present paper we try to answer the initial question whether the Feller conditions are necessary for excluding negative volatility factors in continuous time. We focus on two 2-dimensional square root SDEs in canonical form, one with proportional volatilities and one with linear independent volatility factors, see further down for a precise formulation. The dimension is restricted to 2, since more or less only for this case explicit computations can be performed. The proportional case is interesting from a practical point of view, as it is the underlying SDE for one of the 2-factor models proposed in [11]. Simulations suggest that for some parameters, the volatility factor stays positive almost surely, even though the Feller conditions do not hold. We refer to Figure 1 in [11] for an illustration. However, in this paper we prove that this suggestion is false: without the Feller conditions there is always positive probability that the volatility attains a negative value.

1.2. Notation and definitions. With $S_m(p)$ we denote the class of $p$-dimensional square root SDEs with $m$ volatility factors. That is, an element in $S_m(p)$ is an SDE of the form

\begin{equation}
    dX_t = (aX_t + b)dt + \Sigma \sqrt{|v(X_t)|}dW_t, \quad X(0) = x_0 \in \mathbb{R}^p,
\end{equation}

with $X$ a $p$-dimensional stochastic process, $W$ a $p$-dimensional Brownian motion, $a \in \mathbb{R}^{p \times p}$, $b \in \mathbb{R}^p$, $\Sigma \in \mathbb{R}^{p \times p}$ non-singular. The $j$-th column of $\Sigma$ is denoted by $\Sigma^j$. Furthermore, $v(X_t)$ denotes the diagonal matrix with diagonal elements $V_i(t) = v_i(X_t) = \alpha_i + \beta_i X_t$, which we call volatility factors or volatilities. Here $\alpha_i \in \mathbb{R}$ and $\beta_i$ is a $p$-dimensional row-vector. We let $\alpha = (\alpha_1, \ldots, \alpha_p)$, $\beta$ the matrix with $i$-th row equal to $\beta_i$ and $m$ the rank of $\beta$. The initial value $x_0$ is taken such that $v_i(x_0) \geq 0$ for all $i$. We write $\sqrt{|v(X_t)|}$ for the diagonal matrix with diagonal elements $\sqrt{V_i(t)}$. When $V_{i,t} \geq 0$ a.s. for all $i$, we omit the absolute sign and write $\sqrt{V_t}$ instead. Later on we will use the notation $\text{sgn} (v(X_t))$ for the diagonal matrix with diagonal elements $\text{sgn} (V_i(t))$.

Existence of a weak solution to an SDE in $S_m(p)$ holds, since the drift and diffusion part are continuous functions which in addition fulfil a growth-condition, see Theorem IV.2.3 and IV.2.4 in [7]. To prove that existence and uniqueness of a strong solution holds (equivalent to pathwise uniqueness by Theorem IV.2.1 in [7]) appears very difficult for general square root SDEs. The diffusion part is not Lipschitz-continuous, so standard results, like Theorem IV.3.1 in [7], are not applicable. Instead, one can use Theorem 1 in [13], but this result only applies to square root SDEs which can be written in a certain canonical form (denoted by $A_m(p)$ and $C_p(p)$ below), for
example when the Feller conditions are satisfied. In general though, we do not know whether existence and uniqueness of a strong solution holds for a square root SDE in $\mathbb{S}_m(p)$.

Sufficient conditions for strictly positive volatility factors $V_{i,t}$ are the multivariate Feller conditions from [5], named after Feller’s test for explosions. We consider a weak version of these conditions which are sufficient for non-negative instead of strictly positive volatility factors. These weak Feller conditions are given by

\begin{align}
\forall i, \forall j : \beta_i \Sigma^j = 0 \text{ or } \partial D_i \subset \partial D_j, \tag{1.2}
\end{align}

\begin{align}
\forall i, \forall x \in \partial D_i : \beta_i(ax + b) \geq 0. \tag{1.3}
\end{align}

Here $\partial D_i$ denotes the boundary $\{x \in \mathbb{R}^p : v_i(x) = 0, v_j(x) \geq 0, \forall j\}$.

The subclass $\mathbb{C}_m(p) \subset \mathbb{S}_m(p)$ contains the square root SDEs which are in canonical form, i.e.

\begin{align}
\Sigma = I, \quad V_i = X_i, \text{ for } i \leq m, \tag{1.4}
\end{align}

with $I$ the identity matrix. We adopt the notation of [3] for the class $\mathbb{A}_m(p)$, the SDEs which are in canonical form and in addition satisfy the weak Feller conditions. In canonical form, the Feller conditions translate as

\begin{align}
i, j \leq m, k > m &\Rightarrow
a_{ij} \geq 0 \text{ for } i \neq j, \quad a_{ik} = 0, \quad b_i \geq 0, \quad \alpha_k \geq 0, \quad \beta_{ki} \geq 0. \tag{1.5}
\end{align}

If $X$ solves a square root SDE in $\mathbb{S}_m(p)$ which satisfies the weak Feller conditions, then there exists an affine transformation of $X$ which solves an SDE in canonical form, see the appendix in [3] and Chapter 4 in [12]. It is remarkable that when the volatilities are proportional, the Feller conditions are not needed for this. This will be proved in Proposition 4.1. We write $\mathbb{S}(p) \subset \mathbb{S}_1(p)$ for the $p$-dimensional square root SDEs with proportional volatilities (that is, $\alpha_i = \alpha_1$ and $\beta_i = \beta_1$ for all $i$), and similarly $\mathbb{C}(p)$ for those in canonical form.

1.3. **Set-up.** The rest of this paper is organized as follows. In Section 2 we present the methodology to prove the necessity of the Feller conditions for non-negative volatility factors for the general class $\mathbb{S}_m(p)$. Explicit computations are only possible for the special cases $\mathbb{C}_2(2)$ and $\mathbb{S}(2)$, which we provide in the remaining sections. The method is based on solving a system of linear ODEs satisfied by $E X_t$ and transforming the underlying probability measure via a certain exponential density process $L$. For the method to work, it is necessary that $L$ is a martingale. This is relatively straightforward for the class $\mathbb{C}_2(2)$, but much more difficult to show for $\mathbb{S}(2)$. Therefore, we first prove the necessity of the Feller conditions for the class $\mathbb{C}_2(2)$ in Section 3 before tackling the harder case $\mathbb{S}(2)$. We use two sections for working out the methodology for the latter. Section 4 is entirely devoted to verifying a local version of Novikov’s condition (as given in Corollary 3.5.14 in [10]), in order to prove the martingale property of $L$ for $\mathbb{S}(2)$. Section 5 deals with
solving the systems of linear ODEs and proving the necessity of the Feller conditions for \( S(2) \).

Though slightly off-topic, we have added an appendix with the proof that \( L \) is also a martingale for the class \( \mathbb{A}_m(p) \). We have two reasons for this. In completely affine models (see [4]) one uses this particular exponential process \( L \) to relate the physical with the risk-neutral measure for SDEs in the class \( \mathbb{A}_m(p) \). However, the fact that \( L \) is a legitimate density process (i.e. a martingale) is obscured in the literature. Therefore we clarify this once and for all. We give two proofs that \( L \) is a martingale. Both serve as underlying ideas for proving the martingale property of \( L \) for the other classes \( S(2) \) and \( C_2(2) \), which is the second reason.

2. Methodology

This section presents the methodology to prove the necessity of the (weak) Feller conditions with respect to excluding negative volatility factors. The underlying idea applies to the general class \( S_m(p) \), though explicit computations are un-doable for higher dimensions. The general scheme for proving necessity of the Feller conditions consists of the following steps:

Step 1. Let \((X, W)\) be a weak solution to (1.1) on some filtered probability space \((\Omega, \mathcal{F}, (\mathcal{F}_t), \mathbb{P})\). Then \( \mathbb{E}_\mathbb{P} X_t \) solves a linear ODE by Lemma 2.1 below, so we can compute \( \mathbb{E}_\mathbb{P} V_{i,t} \).

Step 2. Let

\[
L_t^\lambda := \mathcal{E} \left( \int_0^t \lambda^\top \text{sgn}(v(X_s)) \sqrt{|v(X_s)|} dW_s \right),
\]

where \( \lambda \in \mathbb{R}^p \). Fix an arbitrary time interval \([0, T]\), with \( T > 0 \). If the process \( L_t^\lambda \) is a martingale on \([0, T]\), then we can transform the measure \( \mathbb{P} \) into an equivalent probability measure \( \mathbb{Q}^\lambda \) on \( \mathcal{F}_T \) by \( d\mathbb{Q}^\lambda = L_T^\lambda d\mathbb{P} \). By Girsanov’s Theorem (Theorem 3.51 in [10]), \( W^\lambda \) defined by \( dW_t^\lambda = dW_t - \text{sgn}(v(X_t)) \sqrt{|v(X_t)|} \lambda dt \) is a Brownian motion under \( \mathbb{Q}^\lambda \) on \([0, T]\). Moreover, the resulting SDE for \( X \) under \( \mathbb{Q}^\lambda \) is still a square root SDE:

\[
dX_t = (aX_t + b + \Sigma v(X_t)\lambda) dt + \Sigma \sqrt{|v(X_t)|} dW_t^\lambda,
\]

where we can view the integral with respect to \( W^\lambda \) as a stochastic integral under \( \mathbb{Q}^\lambda \) by Proposition 7.26 of [8]. As in Step 1, also \( \mathbb{E}_{\mathbb{Q}^\lambda} X_t \) solves a linear ODE.

Step 3. Under violation of the Feller conditions, for each \( t > 0 \) we find \( \lambda \in \mathbb{R}^p \) such that \( \mathbb{E}_{\mathbb{Q}^\lambda} V_{i,t} < 0 \). Then obviously \( \mathbb{Q}^\lambda(V_{i,t} < 0) > 0 \) and by equivalence of measures also \( \mathbb{P}(V_{i,t} < 0) > 0 \).

Lemma 2.1. Let \((X, W)\) be a weak solution on some filtered probability space \((\Omega, \mathcal{F}, (\mathcal{F}_t), \mathbb{P})\) to

\[
dX_t = (aX_t + b) dt + \sigma(t, X_t) dW_t,
\]
with $W$ a $p$-dimensional Brownian motion, $a \in \mathbb{R}^{p \times p}$, $b \in \mathbb{R}^p$, $\sigma : [0, \infty) \times \mathbb{R}^p \to \mathbb{R}^{p \times p}$ measurable and satisfying the growth condition
\[ \| \sigma(t,x) \|^2 \leq K(1 + \|x\|^2), \text{ for some positive constant } K. \]
If $E\|X_0\|^2 < \infty$, then $\bar{x}_t = EX_t$ solves the ODE
\[ d\bar{x}_t = (a\bar{x}_t + b)dt, \quad \bar{x}(0) = EX_0. \]

**Proof** Taking expectations gives
\[ EX_t = EX_0 + E \int_0^t (aX_s + b)ds + E \int_0^t \sigma(s,X_s)dW_s. \]
By application of Problem 5.3.15 in [10] it holds that
\[ E \max_{0 \leq s \leq t} \|X_s\|^2 < \infty. \]
In addition to the growth condition this implies that the stochastic integral is a martingale, whence its expectation equals zero. The result then follows by an application of Fubini. \qed

We apply this methodology to the classes $C_2(2)$ and $S(2)$. For SDEs in $C_2(2)$ pathwise uniqueness holds, which we can use in Step 2 to verify the martingale property of the exponential process $L^\lambda_t$. However, for SDEs in $S(2)$ it is not clear whether we have pathwise uniqueness. Therefore, we use an alternative method to prove that $L^\lambda_t$ is a martingale by verifying Novikov’s condition. We are able to do this only up to the stopping time $\tau = \inf \{ t > 0 : V_{1,t} < 0 \}$ though. Consequently, we have to take
\[ L^\lambda_t := \mathcal{E}(\int_0^t \Lambda^\top \sqrt{\nu(X_{s,t})}dW_s)_t, \]
for the density process instead of (2.1). The result $\mathbb{P}(V_{i,t} < 0) > 0$ obtained in Step 3 needs to be replaced by $\mathbb{P}(\tau < t) > 0$.

### 3. Negative volatility for $C_2(2)$ without Feller conditions

Consider the class $C_2(2)$ of square root SDEs with independent volatilities (meaning that $\beta$ has full rank). For notational convenience we write $V_t$ for the first coordinate of a solution to the square root SDE, and $Y_t$ for the second coordinate. So we consider SDEs of the form
\begin{align*}
V_t &= (a_{11}V_t + a_{12}Y_t + b_1)dt + \sqrt{V_t}dW_{1,t}, \quad V_0 = v_0 \geq 0, \\
Y_t &= (a_{21}V_t + a_{22}Y_t + b_2)dt + \sqrt{Y_t}dW_{2,t}, \quad Y_0 = y_0 \geq 0.
\end{align*}
The Feller conditions (1.5) in this case read $a_{12}, a_{21} \geq 0$ and $b_1, b_2 \geq 0$. We shall violate the condition $a_{12} \geq 0$ by assuming
\[ a_{12} < 0, \]
whereas we strengthen $a_{21} \geq 0$ and $b_2 \geq 0$ to
\[ a_{21} > 0, b_2 > 0. \]
In this section we apply the method as described in Section 2 to show that $V_t$ attains a negative value with positive probability, for all $t > 0$. The SDE (2.2), obtained after the measure transformation described in Step 2, assumes the form

\[(3.5) \quad dV_t = (a_{11} \lambda V_t + a_{12} Y_t + b_1)dt + \sqrt{|V_t|} dW_{1,t}^{\lambda}, \quad V_0 = v_0 \geq 0,\]

\[(3.6) \quad dY_t = (a_{21} V_t + a_{22} Y_t + b_2)dt + \sqrt{|Y_t|} dW_{2,t}^{\lambda}, \quad Y_0 = y_0 \geq 0,\]

with $a_{11}^{\lambda} = a_{11} + \lambda_1$ and $a_{22}^{\lambda} = a_{22} + \lambda_2$. So in the corresponding ODEs for the expectation, the parameters $a_{11}^{\lambda}$ and $a_{22}^{\lambda}$ depend on the chosen underlying probability measure and are thus free to choose. We show that these can be selected in such a way that the first coordinate $E V_t$ gets negative, from which it follows that $V_t$ gets negative with positive probability. Below we suppress the dependence of $a_{11}$ and $a_{22}$ on $\lambda$.

**Proposition 3.1.** Let $a_{12}, a_{21}, b_1, b_2, x_0 \geq 0, y_0 \geq 0$ be arbitrary but fixed parameters and let $a_{11}$ and $a_{22}$ be variable. Consider the family of systems of differential equations parameterized by $a_{11}, a_{22}$:

\[(3.7) \quad \dot{x} = a_{11} x + a_{12} y + b_1, \quad x(0) = x_0 \geq 0;\]

\[(3.8) \quad \dot{y} = a_{21} x + a_{22} y + b_2, \quad y(0) = y_0 \geq 0.\]

Write $x(t, a_{11}, a_{22})$ for the solution $x(t)$ depending on $a_{11}$ and $a_{22}$. Assume (3.3) and (3.4). Then for all $t_0 > 0$ there exist $a_{11}$ and $a_{22}$ such that $x(t_0, a_{11}, a_{22}) < 0$.

**Proof** We use the following notation: $\tau$ is the trace of $a$, $\Delta$ its determinant, $\rho = a_{12}b_2 - a_{22}b_1$, $\bar{x} = \rho/\Delta$, $D = \tau^2 - 4\Delta$. By eliminating $y$ we obtain a second order equation for $x$:

\[(3.9) \quad \ddot{x} - \tau \dot{x} + \Delta x - \rho = 0.\]

If $\Delta \neq 0$ this has the general solution

$$x(t) = B_1 e^{r_1 t} + B_2 e^{r_2 t} + \bar{x},$$

where $r_i = \frac{1}{2}(\tau \pm \sqrt{D})$, $i = 1, 2$. We take $a_{11} = 0$ and $a_{22} > 0$ such that $D > 0$. Notice that $a_{11} = 0$ implies $\Delta > 0$. Solving for $B_2$ gives

$$B_2 = \frac{r_1(\bar{x} - x_0) + a_{12}y_0 + b_1}{r_2 - r_1} = \frac{a_{12}y_0 + a_{12}(b_2 + a_{21}x_0)}{a_{22}^{\lambda}} + O(a_{22}^{-3}),$$

as $a_{22} \to \infty$. Hence $B_2 < 0$ for $a_{22}$ big enough by the assumptions (3.3) and (3.4). Furthermore it holds that

$$r_1 = O(a_{22}^{-1}), \quad r_2 = a_{22} - O(a_{22}^{-1}), \quad a_{22} \to \infty,$$

From this it easily follows that for arbitrary $t_0 > 0$ we have

$$x(t_0, 0, a_{22}) = B_1 e^{r_1 t_0} + B_2 e^{r_2 t_0} + \bar{x}$$

$$= x_0 e^{r_1 t_0} - \bar{x}(e^{r_1 t_0} - 1) + B_2(e^{r_2 t_0} - e^{r_1 t_0}) \to -\infty,$$
as $a_{22} \to \infty$, since $B_2 e^{r t_0}$ tends to $-\infty$ and dominates the other terms. Hence, the choice $a_{11} = 0$ and $a_{22} > 0$ big enough results in $x(t_0, a_{11}, a_{22}) < 0$. □

**Theorem 3.2.** Let $((V, Y), W)$ be a solution to (3.1), (3.2), on some filtered probability space $(\Omega, \mathcal{F}, (\mathcal{F}_t), \mathbb{P})$. Assume (3.4) and that the Feller conditions are violated by (3.3). Then for all $t > 0$ it holds that $\mathbb{P}(V_t < 0) > 0$.

**Proof** We follow the methodology as described in Section 2. Time is restricted to an arbitrary but finite interval $[0, T]$, with $T > 0$. In addition to the existence of a weak solution we have pathwise uniqueness by Theorem 1 in [13], which implies existence of a strong solution by Theorem IV.2.1 in [7]. Hence we can apply Proposition A.3 to obtain that $L_\lambda t$ as defined by (2.1) is a martingale for all $\lambda \in \mathbb{R}^2$. So we can change the measure $\mathbb{P}$ by $d\mathbb{Q}^\lambda = L_\lambda T d\mathbb{P}$ and obtain an SDE under $\mathbb{Q}^\lambda$ up to time $T$, as given by (3.5) and (3.6). By Lemma 2.1 and Proposition 3.1, for all $t \in (0, T]$ we can choose $\lambda$ such that $\mathbb{E}_{\mathbb{Q}^\lambda} V_t < 0$, which implies that $\mathbb{Q}^\lambda( V_t < 0) > 0$. By equivalence of measures it follows that $\mathbb{P}(V_t < 0) > 0$ for all $t \in (0, T]$. Since $T > 0$ was chosen arbitrarily, the result holds for all $t > 0$. □

**4. Measure transformation for $\mathbb{S}(2)$ without Feller conditions**

Now that we have applied the methodology of Section 2 to the class $\mathbb{C}_2(2)$, we would like to do the same for $\mathbb{S}(2)$. Since we do not know whether pathwise uniqueness holds for solutions to SDEs from this class, we need to do some extra work in Step 2 for verifying the martingale property of $L_\lambda t$. This is done in the current section by checking a local version of Novikov's condition. Then we work out the remaining steps for proving necessity of the Feller conditions in the next section. First, however, we show that every SDE in $\mathbb{S}(p)$ can be rewritten in canonical form.

**Proposition 4.1.** Let $(X, W)$ be a solution on some filtered probability space $(\Omega, \mathcal{F}, (\mathcal{F}_t), \mathbb{P})$ of a $p$-dimensional square root SDE from $\mathbb{S}(p)$ with one volatility factor $V_{1,t} = \alpha_1 + \beta_1 X_t$ ($\alpha_1 \in \mathbb{R}$, $\beta_1$ a $p$-dimensional row vector, not equal to zero):

$$dX_t = (aX_t + b)dt + \Sigma \sqrt{|V_{1,t}|}dW_t,$$

$a \in \mathbb{R}^{p \times p}$, $b \in \mathbb{R}^p$ and $\Sigma \in \mathbb{R}^{p \times p}$ non-singular. Then there exists an affine transformation $\widetilde{X}$ of $X$ such that $\widetilde{X}$ solves an SDE from $\mathbb{C}(p)$:

$$d\widetilde{X}_t = (\widetilde{a}\widetilde{X}_t + \widetilde{b})dt + \sqrt{|\widetilde{X}_{1,t}|}d\widetilde{W}_t,$$

where $\widetilde{W}$ is an orthogonal transformation of $W$, whence also a Brownian motion. Moreover, $\widetilde{X}_{1,t} = cV_{1,t}$ for some positive constant $c > 0$. In addition we can take $\widetilde{a}_{1j} \geq 0$ for $j \neq 1$. 
As in Section 3, we write \( \text{rem } 5.5.29 \) in [10]).

Remark 5.5.29. Then the resulting SDE is of the form \( \sqrt{\text{square root SDE}} \). Thus the SDE for \( \tilde{X} = KX + \ell \) is of the canonical form (4.3). If \( \tilde{a}_{1j} < 0 \) for \( j \neq 1 \), then we take \( -\tilde{X}_{j,t} \) instead of \( \tilde{X}_{j,t} \), which also gives a canonical SDE (replacing \( \tilde{W}_{j,t} \) by \( -\tilde{W}_{j,t} \) still gives a Brownian motion), but with \( \tilde{a}_{1j} > 0 \) instead.

Thus the SDE for \( \tilde{X} = KX + \ell \) is of the canonical form (4.1). If \( \tilde{a}_{1j} < 0 \) for \( j \neq 1 \), then we take \( -\tilde{X}_{j,t} \) instead of \( \tilde{X}_{j,t} \), which also gives a canonical SDE (replacing \( \tilde{W}_{j,t} \) by \( -\tilde{W}_{j,t} \) still gives a Brownian motion), but with \( \tilde{a}_{1j} > 0 \) instead.

As a consequence of the previous proposition, it is sufficient to consider necessity of the Feller conditions (1.3) for the SDEs in \( C(\mathbb{R}) \), which read \( a_{12} = 0 \) and \( b_1 \geq 0 \). We only consider the first condition. In view of Proposition 4.1 we may always assume \( a_{12} \geq 0 \), so we violate the first Feller condition by

\[
(4.2) \quad a_{12} > 0.
\]

Remark 4.2. When \( a_{12} = 0 \) we are essentially dealing with a 1-dimensional square root SDE. Then \( b_1 \geq 0 \) is the remaining Feller condition, for which necessity follows by the 1-dimensional Feller’s test for explosions (Theorem 5.5.29 in [10]).

As in Section 3, we write \( V_t \) for the volatility factor \( X_{1,t} \). Moreover, for \( a_{12} > 0 \) we may substitute \( Y_t = X_{2,t} + b_1/a_{12} \) for the second coordinate so that the resulting SDE is of the form

\[
(4.3) \quad dV_t = (a_{11}V_t + a_{12}Y_t)dt + \sqrt{|V_t|}dW_{1,t}, \quad V_0 = v_0 \geq 0,
\]

\[
(4.4) \quad dY_t = (a_{21}V_t + a_{22}Y_t + b_2)dt + \sqrt{|V_t|}dW_{2,t}, \quad Y_0 = y_0 \in \mathbb{R}.
\]

Hence, for \( a_{12} > 0 \) we can assume without loss of generality that \( b_1 = 0 \). Note that in the present notation we have \( \tau = \inf\{t > 0 : V_t < 0\} \).

We prove that \( L_t^\lambda \) defined by (2.3) is a martingale for all \( \lambda \in \mathbb{R}^2 \) by verifying Novikov’s condition. The proof is similar to the proof of Proposition A.1, but more complicated due to the violation of the Feller conditions. Notice that for \( A_m(p) \), Novikov’s condition is satisfied under the additional
requirement (A.2). Likewise, for $C(2)$ without Feller conditions we need an additional condition to justify Novikov’s condition, namely

\[(4.5) \quad a_{11} < 0, \quad a_{22} < 0 \quad \text{and} \quad \det a > 0.\]

Note that this implies negative real parts for the eigenvalues of the matrix $a$, whence a solution $X$ is mean-reverting. This latter property would suggest more “stability” for $X$, and hence more integrability properties (like Novikov’s condition).

We first prove some lemmas.

**Lemma 4.3.** Let $((V, Y), W)$ be a solution to (4.3), (4.4), on some filtered probability space $(\Omega, \mathcal{F}, (\mathcal{F}_t), \mathbb{P})$. Assume $a_{11} < 0$. For $0 \leq c \leq -a_{11}$ it holds that

\[
\mathbb{E}\exp(cV_{t\wedge \tau}) \leq \exp(cv_0) \left[ \mathbb{E}\exp \left( 2ca_{12} \int_0^{t\wedge \tau} Y_s ds \right) \right]^{1/2}.
\]

**Proof** From (4.3) one obtains

\[
\mathbb{E}\exp(cV_{t\wedge \tau}) = \exp(cv_0) \mathbb{E} \left[ \exp \left( c \int_0^{t\wedge \tau} \sqrt{|V_s|} dW_{1,s} + ca_{11} \int_0^{t\wedge \tau} V_s ds \right) \times \exp \left( ca_{12} \int_0^{t\wedge \tau} Y_s ds \right) \right]
\]

\[
\leq \exp(cv_0) \left[ \mathbb{E}\exp \left( 2c \int_0^{t\wedge \tau} \sqrt{|V_s|} dW_{1,s} + 2ca_{11} \int_0^{t\wedge \tau} V_s ds \right) \right]^{1/2}
\times \left[ \mathbb{E}\exp \left( 2ca_{12} \int_0^{t\wedge \tau} Y_s ds \right) \right]^{1/2},
\]

where the last inequality follows from Cauchy-Schwarz. For $0 \leq c \leq -a_{11}$ it holds that

\[
\exp \left( 2c \int_0^{t\wedge \tau} \sqrt{|V_s|} dW_{1,s} + 2ca_{11} \int_0^{t\wedge \tau} V_s ds \right) \leq \mathcal{E} \left( \int_0^{t\wedge \tau} 2c \sqrt{|V_s|} dW_{1,s} \right),
\]

since $2c(c + a_{11}) \leq 0$ and $V_s \geq 0$ holds for all $s \leq \tau$. By optional sampling (see for example Problem 1.3.23 in [10]), the stopped exponential process in the above display is also a supermartingale. So it has expectation less than or equal to 1 and the result follows. $\square$

**Lemma 4.4.** Let $f : [0, \infty) \to \mathbb{R}$ be integrable. For all $t \geq 0$, $\varepsilon > 0$ it holds that

\[
\exp(\int_t^{t+\varepsilon} f(s) ds) \leq \frac{1}{\varepsilon} \int_t^{t+\varepsilon} \exp(\varepsilon f(s)) ds.
\]

**Proof** Fix $t \geq 0$, $\varepsilon > 0$. Define a probability measure $\mu$ on the Borel sigma-algebra $\mathcal{B}(\mathbb{R})$ by

\[
d\mu = \frac{1}{\varepsilon} 1_{[t, t+\varepsilon]} d\lambda,
\]
with \( \lambda \) the Lebesgue-measure. The exponential function is convex, so we can apply Jensen’s inequality to obtain
\[
\exp\left(\int_t^{t+\varepsilon} f(s) ds\right) = \exp\left(\int f(s) \mu(ds)\right) \leq \int \exp(f(s)) \mu(ds) = \varepsilon^{-1} \int_t^{t+\varepsilon} \exp(f(s)) ds.
\]
\[\square\]

**Lemma 4.5.** Let \(((V, Y), W)\) be a solution to (4.3), (4.4), on some filtered probability space \((\Omega, \mathcal{F}, (\mathcal{F}_t), \mathbb{P})\). Assume \(a_{22} < 0\), \(\det a > 0\). Write
\[
c_1 = \frac{-2a_{22} \det a}{a_{12}^2 + a_{22}^2}, \quad c_2 = \frac{2a_{12} \det a}{a_{12}^2 + a_{22}^2}.
\]
Then for \(0 \leq c \leq c_2\) it holds that
\[
\mathbb{E}\exp(c Y^1_{t \wedge \tau}) \leq 1 + \exp(k(t)),
\]
with \(k(t) = c_1 v_0 + c_2 y_0 + c_2 b_2 t 1_{\{b_2 > 0\}}\).

**Proof** Since \(c_1 a_{12} + c_2 a_{22} = 0\) and \(c_1 a_{11} + c_2 a_{21} = -\frac{1}{2}(c_1^2 + c_2^2)\), (4.3) and (4.4) give
\[
\mathbb{E}\exp(c V^1_{t \wedge \tau} + c_2 Y^1_{t \wedge \tau}) =
= \mathbb{E}\exp\left(c_1 v_0 + c_2 y_0 + c_2 b_2 (t \wedge \tau) + \int_0^{t \wedge \tau} (c_1 a_{11} + c_2 a_{21}) V_s ds + \int_0^{t \wedge \tau} \sqrt{|V_s|} (c_1 c_2) dW_s\right)
\leq \exp(k(t)) \mathbb{E}\exp\left(\int_0^{t \wedge \tau} \sqrt{|V_s|} (c_1 c_2) dW_s - \frac{1}{2} \int_0^{t \wedge \tau} (c_1^2 + c_2^2) V_s ds\right)
= \exp(k(t)) \mathbb{E}\E\left(\int_0^{t \wedge \tau} \sqrt{|V_s|} (c_1 c_2) dW_s\right)
\leq \exp(k(t)).
\]
Note that \(c_1 \geq 0\) and \(c_2 \geq 0\), so for \(0 \leq c \leq c_2\) we have
\[
\mathbb{E}\exp(c Y^1_{t \wedge \tau}) \leq \mathbb{E}\exp(c Y^1_{t \wedge \tau}) 1_{\{Y^1 > 0\}} \leq \mathbb{E}\exp(c_2 Y^1_{t \wedge \tau}) 1_{\{Y^1 > 0\}} \leq 1 + \mathbb{E}\exp(c_2 Y^1_{t \wedge \tau}) \leq 1 + \mathbb{E}\exp(c V^1_{t \wedge \tau} + c_2 Y^1_{t \wedge \tau}),
\]
which gives the result. \[\square\]

**Proposition 4.6.** Let \(((V, Y), W)\) be a solution to (4.3), (4.4), on some filtered probability space \((\Omega, \mathcal{F}, (\mathcal{F}_t), \mathbb{P})\). Assume (4.3) and (4.4). Fix an arbitrary \(c > 0\) and define
\[
\varepsilon(t) = \min\left(-\frac{a_{11}}{c}, \frac{1}{2}\left(-t + \sqrt{t^2 + \frac{2c_2}{ca_{12}}}\right)\right),
\]
with $c_2$ as in Lemma 4.5. Then for all $t \geq 0$ it holds that
\[
E \exp \left( c \int_t^{t+\varepsilon(t)} V_{s\wedge \tau} ds \right) < \infty.
\]

Proof Fix $t \geq 0$ and $\varepsilon := \varepsilon(t)$. Applying respectively Lemmas 4.4, 4.3, 4.4 and 4.5, we obtain
\[
E \exp \left( c \int_t^{t+\varepsilon(t)} V_{s\wedge \tau} ds \right)
\leq \frac{1}{\varepsilon} \int_t^{t+\varepsilon(t)} E \exp (\varepsilon V_{s\wedge \tau}) ds
\leq \frac{1}{\varepsilon} \exp (\varepsilon c v_0) \int_t^{t+\varepsilon(t)} ds \left[ E \exp \left( 2\varepsilon c a_{12} \int_0^s Y_u 1_{\{u \leq \tau\}} du \right) \right]^{1/2}
\leq \frac{1}{\varepsilon} \exp (\varepsilon c v_0) \int_t^{t+\varepsilon(t)} ds \left[ \frac{1}{8} \int_0^s E \exp (s \varepsilon c a_{12} Y_u 1_{\{u \leq \tau\}}) du \right]^{1/2}
\leq \frac{1}{\varepsilon} \exp (\varepsilon c v_0) \int_t^{t+\varepsilon(t)} ds \left[ \frac{1}{8} \int_0^s (1 + \exp(k(u))) du \right]^{1/2}
< \infty,
\]
with $k(u) = c_1 v_0 + c_2 y_0 + c_2 b_2 u 1_{\{b_2 > 0\}}$. Note that to apply Lemma 4.3 it is necessary that
\[
0 \leq \varepsilon c \leq -a_{11},
\]
which holds true by definition of $\varepsilon(t)$. To apply Lemma 4.5 we need to check that
\[
0 \leq s 2 \varepsilon c a_{12} \leq c_2, \quad \text{for all } t \leq s \leq t + \varepsilon.
\]
Choosing $s = t + \varepsilon$ this comes down to
\[
\varepsilon^2 + t \varepsilon - \frac{c_2}{2c a_{12}} \leq 0.
\]
This is satisfied if and only if
\[
\frac{1}{2} \left( -t - \sqrt{t^2 + \frac{2c_2}{ca_{12}}} \right) \leq \varepsilon \leq \frac{1}{2} \left( -t + \sqrt{t^2 + \frac{2c_2}{ca_{12}}} \right),
\]
which indeed holds true.

Using this proposition we can now verify the local version of Novikov’s condition, as given in Corollary 3.5.14 in [10].

Proposition 4.7. Let $((V, Y), W)$ be a solution to (4.3), (4.4), on some filtered probability space $(\Omega, \mathcal{F}, (\mathcal{F}_t), \mathbb{P})$. Assume (4.2) and (4.5). Then for all $c > 0$ there exist $0 = t_0 < t_1 < t_2 < \ldots < t_n \uparrow \infty$ such that
\[
E \exp (c \int_{t_i}^{t_{i+1}} V_{s\wedge \tau} ds) < \infty, \quad \text{for all } i,
\]
and $L^\lambda_t = E(\int_0^t \lambda^\top \sqrt{V_{s\wedge \tau}} dW_s)_t$ is a martingale for all $\lambda \in \mathbb{R}^2$. 

Then the SDE (2.2), obtained after the measure transformation, assumes the form

\[ L = \text{the Feller conditions by completing Step 3.} \]

Recall that by Proposition 4.1 we can write an SDE from \( S \) other hand, \( \epsilon(t_n) \rightarrow 0 \), since the sum \( \sum_{i=0}^{n-1} \epsilon(t_i) \) converges. But on the other hand,

\[
\epsilon(t_n) = \min \left( -\frac{a_{11}}{c} \cdot \frac{1}{2} \left( -t_n + \sqrt{t_n^2 + \frac{2c_2}{ca_{12}}} \right) \right) \rightarrow \min \left( -\frac{a_{11}}{c} \cdot \frac{1}{2} \left( -M + \sqrt{M^2 + \frac{2c_2}{ca_{12}}} \right) \right) > 0,
\]

which is a contradiction. Hence \( t_n \uparrow \infty \). Since the local version of Novikov’s condition holds true, \( L_{\lambda}^a \) is a martingale for all \( \lambda \in \mathbb{R}^2 \).

\[ \square \]

5. Negative volatility for \( S(2) \) without Feller conditions

In the previous section we showed that the measure transformation given in Step 2 is legitimate for \( S(2) \). Now we complete the proof of the necessity of the Feller conditions by completing Step 3. Recall that by Proposition 4.1 we can write an SDE from \( S(2) \) in canonical form as given by (4.3) and (4.4). Then the SDE (2.2), obtained after the measure transformation, assumes the form

\[
(5.1) \quad dV_t = (a_{11}V_t + a_{12}Y_t)dt + \sqrt{V_t}dW_{1,t}^\lambda, \quad V_0 = v_0 \geq 0,
\]

\[
(5.2) \quad dY_t = (a_{21}V_t + a_{22}Y_t + b_2)dt + \sqrt{V_t}dW_{2,t}^\lambda, \quad Y_0 = y_0 \in \mathbb{R},
\]

with \( t < \tau \) and \( a_{11} = a_{11} + \lambda_1 \) and \( a_{21} = a_{21} + \lambda_2 \). So in the corresponding ODEs for the expectation, the parameters \( a_{11} \) and \( a_{21} \) depend on the chosen underlying probability measure and are thus free to choose. Analogously to Proposition 3.1 we prove:

**Proposition 5.1.** Let \( a_{12}, a_{22}, b_2, x_0 \geq 0, y_0 \) be arbitrary but fixed parameters and let \( a_{11} \) and \( a_{21} \) be variable. Consider the family of systems of differential equations parameterized by \( a_{11}, a_{21} \):

\[
(5.3) \quad \dot{x} = a_{11}x + a_{12}y, \quad x(0) = x_0 \geq 0;
\]

\[
(5.4) \quad \dot{y} = a_{21}x + a_{22}y + b_2, \quad y(0) = y_0.
\]

Write \( x(t, a_{11}, a_{21}) \) for the solution \( x(t) \) depending on \( a_{11} \) and \( a_{21} \). If \( a_{12} \neq 0 \) and \( (x_0, \dot{x}_0, y_0) \neq (0, 0, 0) \), then it holds that for all \( t_0 > 0 \) there exist \( a_{11} \) and \( a_{21} \) such that \( x(t_0, a_{11}, a_{21}) < 0 \).

**Proof** We use the same notation as in the proof of Proposition 3.1, but for reasons of brevity we write \( x(t) \) instead of \( x(t, a_{11}, a_{21}) \). Again, by eliminating \( y \) we obtain a second order equation for \( x \):

\[
(5.5) \quad \ddot{x} - \tau \dot{x} + \Delta x - \rho = 0,
\]
If $D = r^2 - 4\Delta < 0$, then the characteristic equation $r^2 - \tau r + \Delta = 0$ has two different complex roots, which are $r_i = \frac{1}{2}(\tau \pm i\sqrt{|D|})$. In that case the differential equation for $x$ has the general solution
\[
x(t) = \exp(\frac{1}{2}\tau t)(c_1 \cos(\omega t) + c_2 \sin(\omega t)) + \bar{x},
\]
with $\omega = \frac{1}{2}\sqrt{|D|}$ and $c_1$, $c_2$ are determined by the initial conditions $x_0$ and $y_0$ of the original system. Solving for $c_1$ and $c_2$ yields
\[
c_1 = x_0 - \bar{x},
\]
\[
c_2 = \frac{1}{\omega}(\dot{x}_0 - \frac{1}{2}\tau(x_0 - \bar{x})).
\]
Without loss of generality we may assume $a_{12} > 0$ as we can substitute $-y$ for $y$ to change the sign of $a_{12}$. Note that
\[
(5.6) \quad a_{21} \to -\infty \implies \Delta \to \infty \implies D \to -\infty, \bar{x} \to 0 \text{ and } \omega \to \infty.
\]
Fix $t_0 > 0$ and suppose $x_0 > 0$. By (5.6) we can choose $a_{21}$ such that $D < 0$, $\omega = (\pi + 2\pi k)/t_0$ for some $k \in \mathbb{N}$, and $\bar{x} < (x_0 \exp(\frac{1}{2}\tau t_0))/(\exp(\frac{1}{2}\tau t_0) + 1)$. It follows that
\[
x(t_0) = -\exp(\frac{1}{2}\tau t_0)c_1 + \bar{x} = -x_0 \exp(\frac{1}{2}\tau t_0) + (\exp(\frac{1}{2}\tau t_0) + 1)\bar{x} < 0.
\]
If $x_0 = 0$ and $\rho \neq 0$ then we take $a_{11}$ such that $\text{sgn} (\tau) = \text{sgn} (\rho)$ and $a_{21}$ such that $D < 0$, $\omega = 2\pi k/t_0$ for some $k \in \mathbb{N}$, which is possible in view of (5.6). Then $\Delta > 0$ and $\text{sgn} (\bar{x}) = \text{sgn} (\tau)$, so
\[
x(t_0) = \exp(\frac{1}{2}\tau t_0)c_1 + \bar{x} = (1 - \exp(\frac{1}{2}\tau t_0))\bar{x} < 0.
\]
If $x_0 = \rho = 0$ and $x_0 \neq 0$, then we choose $a_{21}$ such that $D < 0$, $\omega = (\frac{1}{2}\pi + \pi k)/t_0$ with $k \in 2\mathbb{N}$ if $\dot{x}_0 < 0$ and $k \in 2\mathbb{N} + 1$ if $\dot{x}_0 > 0$. Then
\[
x(t_0) = \exp(\frac{1}{2}\tau t_0)c_2 \sin(\omega t_0)) = \exp(\frac{1}{2}\tau t_0)\frac{x_0}{\omega} \cdot (1_{\{x_0 < 0\}} - 1_{\{x_0 > 0\}}) < 0.
\]
If $x_0 = \rho = \dot{x}_0 = 0$ then $\dot{y}_0 = 0$, so this case is excluded by assumption. \( \Box \)

Since Novikov’s condition is only verified under (4.5), we do not know whether $L_t^\Delta$ defined by (2.3) is a martingale without this condition. Therefore, to apply the methodology of Section 2 to the more general case, we need to do some extra work. First we show in Theorem 5.2 the necessity of the Feller condition $a_{12} = 0$, when (4.5) does hold. Then in Theorem 5.3 we relax (4.5) to $a_{22} < 0$. We show that if $a_{22} < 0$ and the Feller condition $a_{12} = 0$ is violated, there exists a solution $((V,Y),W)$ to the SDE (5.1), (5.2), up to the stopping time $\tau$, such that $V$ gets negative with positive probability. This is done as follows.

We construct a solution to the SDE by first changing the other parameters $a_{11}$ and $a_{21}$ in such a way that (4.5) does hold, and obtaining a solution to the corresponding SDE under some measure $\mathbb{Q}$ (for which we know that $V$ gets negative with positive $\mathbb{Q}$-probability from Theorem 5.2). Then changing the measure $\mathbb{Q}$ into an equivalent measure $\mathbb{P}$, we retrieve the original SDE using
Girsanov’s Theorem. By equivalence of measures \( V \) will also get negative under \( \mathbb{P} \).

**Theorem 5.2.** Let \(((V, Y), W)\) be a solution to (4.3), (4.4), on some filtered probability space \((\Omega, \mathcal{F}, (\mathcal{F}_t), \mathbb{P})\). Assume the Feller conditions are violated by (4.2). In addition assume (4.5). Then for all \( T > 0 \) it holds that

\[ \mathbb{P}(\tau < T) > 0. \]

**Proof** We follow the methodology as described in Section 2 and give a proof by contradiction. Time is restricted to an arbitrary but finite interval \([0, T]\), with \( T > 0 \). Proposition 4.7 gives that \( L_t^\lambda \), as defined by (2.3), is a martingale for all \( \lambda \in \mathbb{R}^2 \). So we can change the measure on \( \mathcal{F}_T \) by \( d\mathbb{Q}^\lambda = L_T^\lambda d\mathbb{P} \) and obtain an SDE under \( \mathbb{Q}^\lambda \), as given by (5.1) and (5.2), up to \( \tau \land T \). Now assume \( \mathbb{P}(\tau < T) = 0 \). Then we can apply Lemma 2.1. By Proposition 5.1 for all \( t \in (0, T) \) we can choose \( \lambda \) such that \( E_{\mathbb{Q}^\lambda} V_t < 0 \), which implies that \( \mathbb{Q}^\lambda(V_t < 0) > 0 \) and by equivalence of measures also \( \mathbb{P}(V_t < 0) > 0 \), which contradicts the assumption that \( \mathbb{P}(\tau < T) = 0 \). \( \square \)

**Theorem 5.3.** Consider an SDE in \( \mathbb{C}(2) \) given by (4.3) and (4.4). Assume the Feller conditions are violated by (4.2). Let time be restricted to an arbitrary but finite interval \([0, T]\), with \( T > 0 \). Then there exists an adapted stochastic process \(((V, Y), W)\) on some filtered probability space \((\Omega, \mathcal{F}, (\mathcal{F}_t), \mathbb{P})\) which is a solution to (4.3), (4.4), up to \( \tau \land T \), such that \( \mathbb{P}(\tau < t) > 0 \) for all \( t \in (0, T) \).

**Proof** Take \( \lambda \in \mathbb{R}^2 \) such that (4.5) holds true with \( a_{i1} \) replaced by \( a_{i1} + \lambda_i \), for \( i = 1, 2 \). Let \(((V, Y), W^\lambda)\) be a weak solution to (5.1), (5.2), on some filtered probability space \((\Omega, \mathcal{F}, (\mathcal{F}_t), \mathbb{Q}^\lambda)\), with time unrestricted. For this the conditions of Theorem 5.2 and Proposition 4.7 hold true. Applying the theorem gives that \( \mathbb{Q}^\lambda(\tau < t) > 0 \) for all \( t \). Applying the proposition with \(-\lambda\) instead of \( \lambda \) gives that

\[ L_t^{-\lambda} := \mathcal{E}(\int_0^t -\lambda^T \sqrt{V_{s\wedge T}} dW_s^\lambda)_t \]

is a martingale. It follows that \( \mathbb{P} \) defined by \( d\mathbb{P} = L_T^{-\lambda} d\mathbb{Q}^\lambda \) is a probability measure on \( \mathcal{F}_T \) equivalent to \( \mathbb{Q}^\lambda \). Moreover, the process \( W \) defined by \( dW_t = dW_t^\lambda + \sqrt{V_{t\wedge T}} \lambda dt \) is a Brownian Motion on \([0, T]\) under \( \mathbb{P} \) by Girsanov’s Theorem and \(((V, Y), W)\) solves

\[
\begin{align*}
    dV_t &= (a_{11}^\lambda V_t + a_{12} Y_t - \lambda_1 V_{t\wedge T}) dt + \sqrt{|V_t|} dW_{1,t}, \\
    dY_t &= (a_{21}^\lambda V_t + a_{22} Y_t + b_2 - \lambda_2 V_{t\wedge T}) dt + \sqrt{|V_t|} dW_{2,t},
\end{align*}
\]

under \( \mathbb{P} \) with time restricted to \([0, T]\). Therefore, \(((V, Y), W)\) is a solution to (4.3), (4.4) under \( \mathbb{P} \), when time is restricted to \([0, T] \). By equivalence of \( \mathbb{P} \) and \( \mathbb{Q}^\lambda \), we have \( \mathbb{P}(\tau < t) > 0 \) for all \( t \in (0, T) \). \( \square \)
Appendix A. Measure transformation for $A_m(p)$

In this section we prove that the exponential process $L^\lambda_t$ defined by (2.1) is a martingale for all $\lambda \in \mathbb{R}^p$ for the class $A_m(p)$. We present two methods. The first is by verifying Novikov’s condition by making use of the explicit form of the square root SDE. The second method uses pathwise uniqueness and also applies to a more general situation.

A.1. Using Novikov’s condition. As mentioned in [1] page 129, a local version of Novikov’s condition holds for square root SDEs which satisfy the Feller conditions. A good reference is lacking though. The 1-dimensional case, equivalent to the Cox-Ingersoll-Ross model, is treated in [15] and the proof uses an application of the Feynman-Kac formula. For the general case $A_m(p)$ we will present a different method to verify Novikov’s condition. The underlying idea has also been used for verifying Novikov’s condition for the class $C(2)$ without the Feller conditions, see Section 4. Note that under the Feller conditions $V_{i,t} \geq 0$ almost surely, so $L^\lambda_t$ defined by (2.1) can be written as

$$L^\lambda_t = \mathcal{E}\left(\int_0^t \lambda^\top \sqrt{v(X_s)}dW_s\right).$$

We prove that a local version of Novikov’s condition holds for $A_m(p)$ under the additional requirement

$$\exists c_i > 0 \text{ for } i \leq m \text{ such that } \sum_j c_j a_{ji} \leq -\frac{1}{2}mc_i^2 \text{ for all } i \leq m. \tag{A.2}$$

For $m = p = 2$, elementary but tedious computations show that (A.2) is satisfied in the following four cases:

(i) $a_{12}, a_{21} \geq 0, a_{11}, a_{22} < 0, \det a > 0$
(ii) $a_{12}, a_{21} < 0, a_{11}, a_{22} \geq 0, \det a < 0$
(iii) $a_{11}, a_{12} < 0$
(iv) $a_{22}, a_{21} < 0$.

Notice that the first case involves a sharpening of the weak Feller conditions for $C_2(2)$ on the elements of $a$. This illustrates that Condition (A.2) is not vacuous and not in contradiction with the Feller conditions.

**Proposition A.1.** Consider a solution $(X, W)$ on some filtered probability space $(\Omega, \mathcal{F}, (\mathcal{F}_t), \mathbb{P})$ to a square root SDE from $A_m(p)$ with parameters $a, b, \alpha, \beta$. Assume in addition that (A.2) holds. Then for all $\lambda \in \mathbb{R}^p$ Novikov’s condition is satisfied for $L^\lambda_t$ defined by (2.1), whence $L^\lambda_t$ is a martingale for all $\lambda \in \mathbb{R}^p$.

**Proof** Let $\lambda \in \mathbb{R}^p$ be arbitrary. It is sufficient to find $\varepsilon > 0$ such that for all $t \geq 0$ we have $\mathbb{E}\exp\left(\frac{1}{2} \int_t^{t+\varepsilon} \lambda^\top v(X_s)ds\right) < \infty$. Note that by the canonical representation and by the Feller conditions [15], this expectation reduces to the form

$$\mathbb{E}\exp\left(\int_t^{t+\varepsilon} (q_0 + \sum_{i=1}^m q_i X_{i,s})ds\right), \text{ for some } q_j \geq 0, j = 0, 1, \ldots, m.$$
Take \( \varepsilon > 0 \) such that \( \varepsilon q_i \leq c_i \) for all \( i \leq m \). Since \( X \) solves \([1.1]\) in canonical form (see \([1.4]\)), one gets

\[
E \exp(\int_t^{t+\varepsilon} \sum_{i=1}^m q_i X_{i,s} \, ds) \leq \frac{1}{\varepsilon} \int_t^{t+\varepsilon} \sum_{i=1}^m c_i X_{i,s} \, ds
\]

Interchanging the summation indices, applying a general form of Hölder’s inequality and using the assumptions on \( c_i \), we obtain that

\[
E \exp(\int_0^s \sum_{j=1}^m c_i a_{ij} X_{j,u} \, du + \int_0^s c_i \sqrt{X_{i,u}} \, dW_{i,u}) \, ds
\]

where the last inequality holds by the supermartingale property of an exponential process. The result follows. \( \square \)

When the additional requirement \([A.2]\) does not hold, we cannot verify Novikov’s condition for proving that \( L^\lambda_t \) is a martingale. However, applying the above proposition twice solves this problem. We first transform the SDE such that \([A.2]\) does hold and then transform it back to the desired SDE. This is possible since the above proposition is valid for all \( \lambda \in \mathbb{R}^p \).

**Proposition A.2.** Consider a solution \((X, W)\) on some filtered probability space \((\Omega, \mathcal{F}, (\mathcal{F}_t), \mathbb{P})\) to a square root SDE from \( \mathcal{A}_m(p) \) with parameters \( a, b, \alpha, \beta \). Then \( L^\lambda_t \) defined by \([A.7]\) is a martingale for all \( \lambda \in \mathbb{R}^p \).

**Proof** Let \( c_i > 0 \) be arbitrary, \( i \leq m \). It is possible to choose \( \mu \in \mathbb{R}^p \) such that

\[
c_i(a_{ii} + \mu_i) + \sum_{j \neq i} c_j a_{ji} \leq -\frac{1}{2} mc_i^2, \text{ for } i \leq m, \text{ and } \mu_i = 0 \text{ for } i > m.
\]

We first show that \( L^\mu_t = \mathcal{E}(\mu^T \sqrt{v(X)} \cdot W)_t \) is a martingale. Therefore, we consider the SDE in \( \mathcal{A}_m(p) \) with parameters \( a^\mu, b, \alpha, \beta \), with \( a^\mu_{ii} = a_{ii} + \mu_i \), \( a^\mu_{ij} = a_{ij} \) otherwise. This SDE satisfies the conditions of Proposition \([A.1]\). By existence of a (strong or weak) solution, there exists a
filtered probability space \((\tilde{\Omega}, \tilde{\mathcal{F}}, (\tilde{\mathcal{F}}_t), \tilde{\mathbb{Q}}^\mu)\) with an adapted process \(\tilde{X}\) and a \(\tilde{\mathbb{Q}}^\mu\)-Brownian motion \(\tilde{W}^\mu\) such that
\[
d\tilde{X}_t = (a^\mu \tilde{X}_t + b)dt + \sqrt{v(\tilde{X}_t)}d\tilde{W}^\mu_t.
\]
By Proposition A.1, the exponential process
\[
\hat{L}^-_t := \mathcal{E}(-\mu^\top \sqrt{v(\hat{X})} \cdot \hat{W}^\mu)_t
\]
is a \(\hat{\mathbb{Q}}^\mu\)-martingale. Moreover, for a fixed arbitrary \(T > 0\), the stopped process \(\hat{L}^-_T\) is uniformly integrable. Hence we can change the measure \(\hat{\mathbb{Q}}^\mu\) into an equivalent measure \(\hat{\mathbb{P}}\) on \(\mathcal{F}_\infty\) by \(d\hat{\mathbb{P}} = \hat{L}^-_T d\hat{\mathbb{Q}}^\mu\). Then \(\hat{W}\) defined by
\[
d\hat{W}_t = d\hat{W}_t^\mu + \mu^\top \sqrt{v(\hat{X}_t)} 1_{\{t \leq T\}} dt,
\]
is a \(\hat{\mathbb{P}}\)-Brownian motion. Furthermore, \((\tilde{X}, \hat{W})\) is a second solution to the initial square root SDE
\[
d\tilde{X}_t = (a\tilde{X}_t + b)dt + \sqrt{v(\tilde{X}_t)}d\hat{W}_t, \ \hat{\mathbb{P}}\text{-a.s.},
\]
with time restricted to \([0, T]\). Hence applying Proposition B.2 (with \(A = \emptyset\)) gives that \(X^T\) and \(\hat{X}^T\) have the same law. Moreover, for \(t \leq T\) we have
\[
(\hat{L}^-_t)^{-1} = \mathcal{E}((\mu^\top \sqrt{v(\hat{X})} \cdot \hat{W})_t = \mathcal{E}((\mu^\top \sqrt{v(\hat{X})} \cdot \hat{W})_t = \mathcal{E}((\mu^\top (\hat{X} - \tilde{X}_0 - \int_0^t (a\tilde{X}_s + b)ds))_t, \ \hat{\mathbb{P}}\text{-a.s.},
\]
and
\[
L_t^\mu = \mathcal{E}(\mu^\top \sqrt{v(X)} \cdot W)_t = \mathcal{E}(\mu^\top (X - X_0 - \int_0^t (aX_s + b)ds))_t, \ \mathbb{P}\text{-a.s.},
\]
so \((\hat{L}^-_t)^{-1}\) and \(L_t^\mu\) are equal in law for \(t \leq T\). By equivalence of \(\hat{\mathbb{P}}\) and \(\mathbb{Q}^\mu\), it holds that \(E_{\hat{\mathbb{P}}}(\hat{L}^-_t)^{-1} = 1\) for all \(t \geq 0\). Therefore, \(E_{\hat{\mathbb{P}}}L_t^\mu = 1\) for \(t \leq T\).

In fact, \(E_{\hat{\mathbb{P}}}L_t^\mu = 1\) for all \(t \geq 0\), as \(T\) can be chosen arbitrarily, whence \(L_t^\mu\) is a \(\mathbb{P}\)-martingale.

Now we show that \(L_t^\mu\) is a \(\mathbb{P}\)-martingale. Again fix an arbitrary \(T > 0\). We change the measure \(\mathbb{P}\) into an equivalent measure \(\mathbb{Q}^\mu\) on \(\mathcal{F}_T\) by \(d\mathbb{Q}^\mu = L_T^\mu d\mathbb{P}\) and see that
\[
dX_t = (a^\mu X_t + b)dt + \sqrt{v(X_t)}dW^\mu_t,
\]
for \(t \leq T\), with \(W^\mu\) a \(\mathbb{Q}^\mu\)-Brownian motion on \([0, T]\) given by \(dW^\mu_t = dW_t - \mu^\top \sqrt{v(X_t)}dt\). Applying Proposition A.1 again gives that \(\mathcal{E}(\nu^\top \sqrt{v(X)} \cdot W^\mu)_t\) is a \(\mathbb{Q}^\mu\)-martingale for all \(\nu \in \mathbb{R}^p\) on \([0, T]\), whence on \([0, \infty)\) since \(T > 0\) is arbitrary. Choosing \(\nu = \lambda - \mu\) gives that
\[
E_{\mathbb{P}}L_t^\lambda = E_{\mathbb{Q}^\mu}L_t^\lambda(L_t^\mu)^{-1} = E_{\mathbb{Q}^\mu}\mathcal{E}(\lambda^\top \sqrt{v(X)} \cdot W)_t \mathcal{E}(\mu^\top \sqrt{v(X)} \cdot W^\mu)_t
\]
\[
= E_{\mathbb{Q}^\mu}\mathcal{E}((\lambda - \mu)^\top \sqrt{v(X)} \cdot W^\mu)_t = 1,
\]
for all \(t \geq 0\), which completes the proof. \(\square\)
A.2. Using pathwise uniqueness. For verifying Novikov’s condition one needs the explicit form of the underlying SDE. There are more general results in the literature for proving that an exponential process is a martingale without using the parameters of the SDE explicitly, for example those contained in [9], which treats the problem for Doléans exponentials of affine semimartingales under Feller conditions. These results cannot directly be used to show that the process defined in (A.1) is a martingale. Other results are the theorems in [14], Theorem 1 in [1] and Theorem A.1 in [6]. The latter theorem generalizes Theorem 1 in [1], but is not applicable for the square root SDEs in $\mathbb{A}_m(p)$, since strictly positiveness of the diffusion part $\sigma$ is required. Moreover, it only treats the one-dimensional case. Therefore, we give another generalization of Theorem 1 in [1] which also applies to $\mathbb{A}_m(p)$. The proof goes along the same line of thought, but for clarity we give it again, also to emphasize the need for existence and uniqueness of strong solutions, an aspect not mentioned in [1]. Note that the latter implies that we cannot apply the result to the general class $\mathcal{S}_m(p)$, as it is not clear whether pathwise uniqueness holds for general square root SDEs.

In the proof of the next proposition we use a uniqueness in law result for two weak solutions to an SDE up to a stopping time. This result is stated and proved in Proposition B.2 in the next section. Its proof uses the results in the literature for proving that an exponential process is a martingale.

**Proposition A.3.** Let $\mu : \mathbb{R}^p \to \mathbb{R}^p$, $\sigma : \mathbb{R}^p \to \mathbb{R}^{p \times p}$, $\gamma : \mathbb{R}^p \to \mathbb{R}^p$. Suppose we have weak existence for the $p$-dimensional SDE

\begin{equation}
  dX_t = \mu(X_t)dt + \sigma(X_t)dW_t, \tag{A.3}
\end{equation}

as well as existence and uniqueness of a strong solution for the SDE

\begin{equation}
  dX_t = (\mu(X_t) + \sigma(X_t)\gamma(X_t))dt + \sigma(X_t)dW_t. \tag{A.4}
\end{equation}

Let $(X, W)$ be a weak solution to (A.3) on some filtered probability space $(\Omega, \mathcal{F}, (\mathcal{F}_t), \mathbb{P})$ and $(\hat{X}, \hat{W})$ a solution to (A.4) on a (possibly different) probability space $(\hat{\Omega}, \hat{\mathcal{F}}, (\hat{\mathcal{F}}_t), \hat{\mathbb{P}})$. Suppose $\gamma(X_t)$ and $\gamma(\hat{X}_t)$ have continuous sample paths under $\mathbb{P}$ respectively $\hat{\mathbb{P}}$. Then $Y_t = \mathcal{E}(\gamma(X_t)\cdot W)_t$ is a $\mathbb{P}$-martingale.

**Proof** Fix $T > 0$ arbitrarily and define for $n \in \mathbb{N}$

$$\tau_n = \inf\{t > 0 : \|\gamma(X_t)\| \geq n\} \land T, \quad \hat{\tau}_n = \inf\{t > 0 : \|\gamma(\hat{X}_t)\| \geq n\} \land T.$$ 

Then $Y^n_t = \mathcal{E}(\gamma(X))1_{[0, \tau_n]} \cdot W)_t$ is a $\mathbb{P}$-martingale, since Novikov’s condition holds. Furthermore, $Y^n_{t \land T} = Y^n_t$, so $Y^n_t$ is uniformly integrable, whence $Y^n_T$ exists and equals $Y^T_T$. We can change the measure $\mathbb{P}$ into an equivalent measure $\mathbb{Q}^n$ on $\mathcal{F}_\infty$ by $d\mathbb{Q}^n = Y^n_T d\mathbb{P}$. Then $W^n_t$ defined by $dW^n_t = dW_t - Y^n_t dt$ is a $\mathbb{Q}^n$-Brownian motion and $(X, W^n)$ satisfies

$$dX_t = (\mu(X_t) + \sigma(X_t)\gamma(X_t)1_{[t \leq \tau_n]}dt + \sigma(X_t)dW^n_t,$$

under $\mathbb{Q}^n$. Therefore, $(X, W^n)$ is a solution to (A.4) on $(\Omega, \mathcal{F}_\infty, (\mathcal{F}_t), \mathbb{Q}^n)$ with time restricted to $[0, \tau_n]$. 


By continuity of $\gamma(X_t)$ under $\mathbb{P}$, we have $\mathbb{P}(\|\gamma(X_{\tau_n})\| \geq n)$ or $\tau_n = T = 1$, whence $\mathbb{Q}^{\gamma}(\|\gamma(X_{\tau_n})\| \geq n)$ or $\tau_n = T = 1$, for all $n$, by equivalence of $\mathbb{P}$ and $\mathbb{Q}$. So we can apply Proposition B.2 and obtain that $\tau_n$ and $\hat{\tau}_n$ have the same distribution under $\mathbb{Q}^{\gamma}$ respectively $\hat{\mathbb{P}}$.

By continuity of $\gamma(X_t)$ under $\mathbb{P}$, we have $\tau_n \uparrow T$, $\mathbb{P}$-a.s., which implies that $Y_t^{\tau_n 1(t \leq \tau_n)} = Y_{\tau_n 1(t \leq \tau_n)} \uparrow Y_{1(t \leq T)}$, $\mathbb{P}$-a.s. Hence we can apply the Monotone Convergence Theorem and obtain for $t \leq T$ that

$$\mathbb{E}_\mathbb{P} Y_t = \lim_{n \to \infty} \mathbb{E}_\mathbb{P} Y_t^{\tau_n 1(t \leq \tau_n)} = \lim_{n \to \infty} \mathbb{Q}^{\gamma}(t < \tau_n) = \lim_{n \to \infty} \hat{\mathbb{P}}(t < \hat{\tau}_n) = 1,$$

where the last equality holds since $\hat{\mathbb{P}}(\hat{\tau}_n \uparrow T) = 1$, by continuity of $\gamma(\hat{X}_t)$ under $\hat{\mathbb{P}}$. Because $T > 0$ was chosen arbitrarily, $\mathbb{E}_\mathbb{P} Y_t = 1$ holds for all $t \geq 0$, whence $Y_t$ is a $\mathbb{P}$-martingale.

**Remark A.4.** Note that the above proposition implies existence and uniqueness of a strong solution for (A.3). Indeed, suppose $(X^1, W)$ and $(X^2, W)$ are solutions to (A.3) on some filtered probability space $(\Omega, \mathcal{F}_\infty, (\mathcal{F}_t), \mathbb{P})$. Fix $T > 0$. Then $\hat{\mathbb{P}}$ defined by $d\hat{\mathbb{P}} = Y_T d\mathbb{P}$ is a probability measure on $\mathcal{F}_T$, equivalent to $\mathbb{P}$. Furthermore, $\hat{W}$ defined by $d\hat{W}_t = dW_t - Y_t dt$ is a $\hat{\mathbb{P}}$-Brownian motion on $[0, T]$ and $(X^1, \hat{W})$ and $(X^2, \hat{W})$ are solutions to (A.4) up to $T$. By pathwise uniqueness for (A.4) it holds that

$$\hat{\mathbb{P}}(X^1_t = X^2_t, t \in [0, T]) = 1,$$

which implies $\mathbb{P}(X^1_t = X^2_t, t \in [0, T]) = 1$, as $\mathbb{P}$ and $\hat{\mathbb{P}}$ are equivalent. Since $T$ was chosen arbitrarily, it follows that $\mathbb{P}(X^1_t = X^2_t, \forall t \geq 0) = 1$. Hence pathwise uniqueness holds for (A.3), which implies existence and uniqueness of a strong solution, by Theorem IV.2.1 in [7].

**Corollary A.5.** Consider a solution $(X, W)$ on some filtered probability space $(\Omega, \mathcal{F}, (\mathcal{F}_t), \mathbb{P})$ to a square root SDE from $A_m(p)$ with parameters $a, b, \alpha, \beta$. Then $L^\lambda_t$ defined by (A.7) is a martingale for all $\lambda \in \mathbb{R}^p$.

**Proof** This follows from Proposition A.3 with $\gamma(X_t) = \sqrt{v(X_t)} \lambda$. Both SDEs from the proposition belong to $A_m(p)$, for which weak existence holds by continuity of the parameters and satisfaction of a growth condition. Pathwise uniqueness holds by Theorem 1 in [13], which implies existence and uniqueness of a strong solution, by Theorem IV.2.1 in [7].

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**Appendix B. Uniqueness for a stopped SDE**

As mentioned in the remark preceding Proposition A.3 in this section we state and prove a uniqueness in law result for two weak solutions to an SDE, possibly defined on different probability spaces, up to a stopping time. This result is stated in Proposition B.2. In the proof we use a measurability lemma, which we prove first in Lemma B.1.

**Lemma B.1.** Let $(\Omega, \mathcal{F}, (\mathcal{F}_t), \mathbb{P})$ be a filtered probability space, $\tau$ a finite stopping time and $X$ a stochastic process with continuous sample paths. If
$X_t : \Omega \to \mathbb{R}$ is $\mathcal{F}_{\tau+t}$-measurable for all $t$, then $X_{t-\tau}1_{\tau<t}$ is $\mathcal{F}_t$-measurable for all $t$.

**Proof** It is possible to choose a sequence of stopping times $\tau_n \downarrow \tau$ a.s. such that $\tau_n$ only assumes countably many values. Since $X_{t-\tau}1_{\tau_n<t}$ converges to $X_{t-\tau}1_{\tau<t}$ a.s., it is enough to prove the statement for $\tau_n$ instead of $\tau$. For arbitrary Borel set $B$ it holds that

\[
\{X_{t-\tau_n}1_{\tau_n<t} \in B\} = \{X_{t-\tau_n} \in B, \tau_n < t\} \cup \{0 \in B, \tau_n \geq t\}
\]

\[
= \bigcup_{k<t} \{X_{t-k} \in B, \tau_n = k\} \cup \{0 \in B, \tau_n \geq t\}
\]

\[
= \bigcup_{k<t} \{(X_{t-k} \in B, \tau \leq k) \cap \{\tau_n = k\}\} \cup \{0 \in B, \tau_n \geq t\}.
\]

Since $\tau_n$ is a stopping time, we have $\{0 \in B, \tau_n \geq t\} \in \mathcal{F}_t$ as well as $\{\tau_n = k\} \in \mathcal{F}_t$ for $k < t$. Moreover, $X_t$ is $\mathcal{F}_{\tau+t}$-measurable for all $t$, which means that

\[
\{X_t \in B\} \cap \{\tau + t \leq s\} \in \mathcal{F}_s, \quad \text{for all } t \text{ and } s.
\]

Choosing $s = t$ and substituting $t - k$ for $t$ in the above display gives

\[
\{X_{t-k} \in B\} \cap \{\tau \leq k\} \in \mathcal{F}_t,
\]

which completes the proof. \(\square\)

**Proposition B.2.** Consider an SDE

(B.1) \[dX_t = \mu(X_t)dt + \sigma(X_t)dW_t,\]

which has a unique strong solution $(X, W)$ on $(\Omega, \mathcal{F}, (\mathcal{F}_t), \mathbb{P})$. Let $\tau$ be a stopping time of the form

\[\tau = \inf\{t > 0 : X_t \in A\} \land T,\]

with $A$ a measurable set and $T > 0$. Let $(\hat{X}, \hat{W})$ be an adapted stochastic process on a filtered probability space $(\hat{\Omega}, \hat{\mathcal{F}}, (\hat{\mathcal{F}}_t), \hat{\mathbb{P}})$, with $\hat{W}$ a $\hat{\mathbb{P}}$-Brownian motion. Suppose $(\hat{X}, \hat{W})$ is also a solution to (B.1) under $\hat{\mathbb{P}}$, but on the stopped interval $[0, \hat{\tau}]$, where we write

\[\hat{\tau} = \inf\{t > 0 : \hat{X}_t \in A\} \land T.\]

If $\hat{\mathbb{P}}(\hat{X}_{\hat{\tau}} \in A \text{ or } \hat{\tau} = T) = 1$, then the stopping times $\tau$ and $\hat{\tau}$ as well as the stopped processes $X^\tau$ and $\hat{X}^\hat{\tau}$ have the same distribution under $\mathbb{P}$ respectively $\hat{\mathbb{P}}$.

**Proof** We extend the solution $\hat{X}$ to (B.1) on $[0, \hat{\tau}]$ to a solution $Y$ to (B.1) on the whole interval $[0, \infty)$, for which we use existence of a strong solution. Define a filtration $(\mathcal{G}_t)$ by $\mathcal{G}_t := \hat{\mathcal{F}}_{\hat{\tau}+t}$. Then $\hat{W}_t := \hat{W}_{\hat{\tau}+t} - \hat{W}_{\hat{\tau}}$ is a $\hat{\mathbb{P}}$-Brownian motion with respect to $(\mathcal{G}_t)$. By existence of a strong solution,
there exists a process $Z$ adapted to $\mathcal{G}$ with initial value $\hat{X}_\tau$ such that

$$Z_t = \hat{X}_\tau + \int_0^t \mu(Z_s)ds + \int_0^t \sigma(Z_s)d\hat{W}_s.$$  

Define

$$Y_t = \hat{X}_t \mathbf{1}_{t<\hat{\tau}} + Z_{t-}\mathbf{1}_{t>\hat{\tau}}.$$  

By Lemma [B.1], $Y_t$ is $\hat{\mathcal{F}}_t$-measurable. It holds that $Y_{\hat{\tau}+t} = Z_t$ for $t \geq 0$, so that

$$Y_{\hat{\tau}+t} = \hat{X}_t + \int_0^t \mu(Y_{\hat{\tau}+s})ds + \int_0^t \sigma(Y_{\hat{\tau}+s})d(\hat{W}_{\hat{\tau}+s} - \hat{W}_\tau)$$

$$= \hat{X}_\tau + \int_\tau^{\hat{\tau}+t} \mu(Y_s)ds + \int_\tau^{\hat{\tau}+t} \sigma(Y_s)d\hat{W}_s.$$  

Note that

$$\hat{X}_\tau = \hat{X}_0 + \int_0^{\hat{\tau}} \mu(\hat{X}_s)ds + \int_0^{\hat{\tau}} \sigma(\hat{X}_s)d\hat{W}_s$$

$$= \hat{X}_0 + \int_0^{\hat{\tau}} \mu(Y_s)ds + \int_0^{\hat{\tau}} \sigma(Y_s)d\hat{W}_s,$$

whence

$$Y_t \mathbf{1}_{t>\hat{\tau}} = Y_{\hat{\tau}+t-\hat{\tau}} \mathbf{1}_{t>\hat{\tau}} = (\hat{X}_\tau + \int_\tau^{\hat{\tau}} \mu(Y_s)ds + \int_\tau^{\hat{\tau}} \sigma(Y_s)d\hat{W}_s) \mathbf{1}_{t>\hat{\tau}}$$

$$= (Y_0 + \int_0^t \mu(Y_s)ds + \int_0^t \sigma(Y_s)d\hat{W}_s) \mathbf{1}_{t>\hat{\tau}}.$$  

On the other hand it holds that

$$Y_t \mathbf{1}_{t\leq\hat{\tau}} = \hat{X}_t \mathbf{1}_{t\leq\hat{\tau}} = (\hat{X}_0 + \int_0^t \mu(\hat{X}_s)ds + \int_0^t \sigma(\hat{X}_s)d\hat{W}_s) \mathbf{1}_{t\leq\hat{\tau}}$$

$$= (Y_0 + \int_0^t \mu(Y_s)ds + \int_0^t \sigma(Y_s)d\hat{W}_s) \mathbf{1}_{t\leq\hat{\tau}}.$$  

Hence $(Y, \hat{W})$ is a solution to (B.1) on $(\hat{\Omega}, \hat{\mathcal{F}}, (\hat{\mathcal{F}}_t), \hat{\mathbb{P}})$. By uniqueness in distribution, $Y$ and $X$ have the same distribution. Since the paths of $\hat{X}$ and $Y$ coincide for $t \leq \hat{\tau}$ and since $\hat{\mathbb{P}}(\hat{X}_\hat{\tau} \in A \text{ or } \hat{\tau} = T) = 1$, it holds $\hat{\mathbb{P}}$-almost surely that

$$\hat{\tau} = \inf\{t > 0 : Y_t \in A\} \wedge T.$$  

Comparing this with the expression for $\tau$, we see that $\hat{\tau}$ and $\tau$ as well as $Y^\hat{\tau}$ and $X^\tau$ have the same distribution. From $Y^\hat{\tau} = \hat{X}^\hat{\tau}$ it follows that $\hat{X}^\hat{\tau}$ and $X^\tau$ have the same distribution. $\square$
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Korteweg-de Vries Institute of Mathematics, University of Amsterdam, Plantage Muidergracht 24, Amsterdam, The Netherlands

E-mail address: spreij@uva.nl

Korteweg-de Vries Institute of Mathematics, University of Amsterdam, Plantage Muidergracht 24, Amsterdam, The Netherlands

E-mail address: e.veerman@uva.nl