Since January 2020 Elsevier has created a COVID-19 resource centre with free information in English and Mandarin on the novel coronavirus COVID-19. The COVID-19 resource centre is hosted on Elsevier Connect, the company's public news and information website.

Elsevier hereby grants permission to make all its COVID-19-related research that is available on the COVID-19 resource centre - including this research content - immediately available in PubMed Central and other publicly funded repositories, such as the WHO COVID database with rights for unrestricted research re-use and analyses in any form or by any means with acknowledgement of the original source. These permissions are granted for free by Elsevier for as long as the COVID-19 resource centre remains active.
Study of fractional order dynamics of nonlinear mathematical model

Kamal Shah\textsuperscript{a,b}, Amjad Ali\textsuperscript{b}, Salman Zeb\textsuperscript{b}, Aziz Khan\textsuperscript{a}, Manar A. Alqudah\textsuperscript{c}, Thabet Abdeljawad\textsuperscript{a,d,*}

\textsuperscript{a} Department of Mathematics and Sciences, Prince Sultan University, P.O. Box 66833, 11586 Riyadh, Saudi Arabia
\textsuperscript{b} Department of Mathematics, University of Malakand, Dir (L), Khyber Pakhtunkhwa, Pakistan
\textsuperscript{c} Department of Mathematical Sciences, College of Science, Princess Nourah bint Abdulrahman University, P.O. Box 84428, Riyadh 11671, Saudi Arabia
\textsuperscript{d} Department of Medical Research, China Medical University, Taichung 40402, Taiwan

Received 14 February 2022; revised 31 March 2022; accepted 21 April 2022
Available online 11 May 2022

KEYWORDS
COVID-19 Problem; Feasible region; Numerical solution; MEM and NSFD methods; Global and local stability; CPU time

Abstract This manuscript is devoted to establishing some theoretical and numerical results for a nonlinear dynamical system under Caputo fractional order derivative. Further, the said system addresses an infectious disease like COVID-19. The proposed system involves natural death rates of susceptible, infected and recovered classes respectively. By using nonlinear analysis feasible region and boundedness have been established first in this study. Global and Local stability analysis along with basic reproduction number have also addressed by using the next generation matrix method. Upon using the fixed point approach, existence and uniqueness of the approximate solution for the mentioned problem has also investigated. Some stability results of Hyers-Ulam (H-U) type have also discussed. Further for numerical treatment, we have exercised two numerical schemes including modified Euler method (MEM) and nonstandard finite difference (NSFD) method. Further the two numerical schemes have also compared with respect to CPU time. Graphical presentations have been displayed corresponding to different fractional order by using some real data.

© 2022 THE AUTHORS. Published by Elsevier BV on behalf of Faculty of Engineering, Alexandria University. This is an open access article under the CC BY license (http://creativecommons.org/licenses/by/4.0/).

1. Introduction

Fractional calculus has got much attention in recent times. The foundation of the said area had been provided by Newton and Lebznitz during seventeenth century. Later on Reimann, Liouville, Fourier, Abel and Euler developed this branch very well [1]. The first notable definition had been given by Reimann-Liouville in 1832. Also the mentioned definition has been mod-
ified by Caputo in 1967 to elaborate the concept more clearly in geometry [2,3]. Since fractional derivative infact is a definite integral which can be defined in numbers of ways. Geometrically the said operator gives accumulation of a function which includes its integer counterpart part as a special case. Therefore, various researchers have given different definitions for fractional differential operators [4]. The said operators have numerous application in modeling of various real world problems. As in most of the situations where memory terms or index laws are involved cannot be well explained via classical operators. For comprehensive descriptions and well explanations fractional calculus is powerful tool to use (see [5,6]). Researchers have used the concept of fractional calculus for detailed explanations of many real world problems including physical, biological, dynamical and chemical phenomena (we refer few article as [7–11]).

Recently various authors have used other kids of fractional differential operators like Hilfer, nonsingular kernel type, Caputo-Fabrizio operator to investigate various models. Here we state that each operator has own merits and de-merits. Here we remark that using various fractional differential operators numerous results relating controllability, existence theory, optimization and numerical analysis have been published in last few years. The concept of Hilfer derivative of fractional order has been applied to investigate some controllability results for delay type problems in [12,13]. In same line authors [14] have established existence and controllability results for nonlocal mixed Volterra–Fredholm type fractional delay integro-differential equations. Some advanced results in controllability using fractional order derivatives have been established in [15,16]. In same line authors have used neutral fractional order derivative to investigate a class of partial differential equations in [17]. Also authors have developed very applicable results for controllability and existence theory for neutral fractional integro-differential equations. For detail work, we refer [18–20]. Various results regarding analysis of various problems by using generalized proportional fractional integral operators, Atangana-Baleanu-Caputo operators have been established in last few years. Some recent contribution can be read as [21–23]. The analysis in aforesaid work has also applicable and can be further extended to COVID-19 or other infectious disease model. Here we remark that nonsingular kernel type and Caputo-Fabrizio operators suffer from initialization effect and extra assumptions need to be imposed on right hand side of a differential equations involving such like operators. This is not good effect and therefore now reducing the use of the said operators in applications. Although significant work has been done by using the aforesaid operators in epidemiology. But to the best of our knowledge in epidemiological models, the most powerful operators are those introduced by Reimann-Liouville and Caputo. Because they have clear geometrical interpretation like integer order differential equations and upon using integer value, the concerned equations reduce to their corresponding classical form.

The research area devoted to mathematical models for analysis of various physical phenomena in real life is very interesting and has received wide attention among researchers in last several years. The idea of mathematical model had been originated by Bernoulli in 1776 (see [26]). Later on the concept adopted by Mekendric and Kermark in 1927 to describe an infectious disease model which is known as SI SR. The said model then further modified and extended to form new models for various infectious disease like TB, Choleror, Typhoid, Dengue, HIV and AIDS, etc (see [27,28]). Recently the Coronavirus disease (COVID-19) which has been originated from China and transmitted all over the world. The aforesaid disease which reported in an outbreak in 2019 in Wuhan, Hubei province, China, is caused by the SARS-CoV-2 virus (for further etiological information see [29]). According to researchers of virology, the said virus belongs to the class of beta-Coronavirus and like Middle East Respiratory Syndrome Coronavirus and Severe Acute Respiratory Syndrome Coronavirus. The novel virus began to cause pneumonia, and was named as Coronavirus disease (2019) on 11 February 2020 by WHO (see detail in [30]).

Also many researchers have formulated various mathematical models for COVID-19. This disease has greatly affected the whole globe. Due to COVID-19, more than fifty millions people have been died. Nearly five hundred million people have got infectious throughout the world (see some detail in [31,32]). The said disease has greatly destroyed the life style and economical situation of many countries around the globe. Recently some countries have worked on to prepare vaccine for permanent cure of the disease in which they have got some success like UK, USA, China and Germany, etc (see [33,34]). Researchers from bioengineering side, virology, epidemiology and those working on mathematical biology have worked significantly in last two years to investigate various procedure how to eradicate or control the disease from further spreading in community. Those who are working on computational biology have developed several mathematical models to understand the transmission dynamics of the said disease. Therefore large numbers of mathematical models have been formulated for COVID-19. Authors [42] have formulated the following SIRD type model to demonstrate the diffusion of the infection in various communities

\[
\begin{align*}
\begin{cases}
\dot{S}(t) &= \gamma I S \\
\dot{I}(t) &= \gamma I S - (\alpha + d)I \\
\dot{R}(t) &= \alpha I \\
\dot{D}(t) &= d, \\
S(0) &= S_0 > 0, \quad I(0) = I_0 \geq 0, \\
R(0) &= R_0 \geq 0, \quad D(0) = D_0 \geq 0,
\end{cases}
\end{align*}
\]

where \(S\) stand for susceptible, \(I\) for infected, \(R\) for recovered and \(D\) for death class. Further, \(d\) denote death rate, \(\gamma\) is used for infection rates and \(\alpha\) represents rate of conversion to susceptible. Recently various analysis, investigations and procedure have been conducted to deal the disease from various aspects. The contribution done in this regards, we refer few article as [35–41]. Since the concept of fractional calculus has got significant attention during last few decades. Therefore mathematician as well as researchers in biomathematics have given much attention to study said models for the mentioned infection under various type derivatives of arbitrary orders (see [43–49]). The investigation of biological model of infection disease under fractional derivatives is an interesting study as compared to classical order derivative. Because fractional calculus provides dynamical interpretations of real world phenomena with more degree of freedom (see few as [50–53]).

Therefore inspired from the mentioned applications and importance of fractional calculus and biological models, we update model (1) by incorporating natural death rates of
For the last section seven is related to a brief conclusion. Section six is devoted to comparison of both schemes. Section five is related to qualitative analysis. Fourth section is about the model under consideration. Section three is devoted to some fundamental results. Section two is numerical analysis in this research work.

We will use Matlab for our comparison between both classical and fractional order model to consider the proposed problem is given. We will use Matlab for our comparison between both classical and fractional order model to consider the available real initial data. The said method (MEM) to describe the numerical results. The said method also has used in many papers for instance [50–52]. A comparison between both classical and fractional order model to conclude which one is more better for numerical interpretation of the proposed problem is given. We will use Matlab for our numerical analysis in this research work.

Here we organized our work as: In first part we give introduction. Then we give some elementary results in Section second. Third section is devoted to some fundamental results about the model under consideration. Fourth section is devoted to qualitative analysis. Fifth section is related to numerical analysis which is further divided into four subsections. Sixth section is devoted to comparison of both schemes. Further last section seven is related to a brief conclusion.

2. Fundamental result

We recall some basic definition of fractional calculus [1–3].

Definition 2.1. Arbitrary order integration of \( \mathcal{F} \in L[0, T] \) for \( \beta > 0 \) is define as

\[
\mathcal{F}_0^\beta[t] = \frac{1}{\Gamma(\beta)} \int_0^t (t - \rho)^{\beta - 1} \mathcal{F}(\rho) d\rho
\]

provided that integral on right exists pointwise on \((0, \infty)\).

Definition 2.2. If \( \mathcal{F} \in C[0, T] \), then Caputo derivative of arbitrary order with \( 0 < \alpha \leq 1 \) is recalled

\[
\mathcal{F}_0^\alpha(t) = \begin{cases} \frac{1}{\Gamma(\beta - \alpha)} \int_0^t (t - \rho)^{\beta - 1 - \alpha} \mathcal{F}(\rho) d\rho, & 0 < \beta < 1, \\ \frac{d^\beta \mathcal{F}}{dt^\beta}, & \beta = 1, \end{cases}
\]

provided that integral on right exists pointwise on \((0, \infty)\).

Lemma 2.3. The solution of FDE

\[
\mathcal{F}_0^\beta \mathcal{F}(t) = \mathcal{G}(t), \quad 0 < \beta \leq 1,
\]

is given by

\[
\mathcal{F}(t) = \mathcal{G}_0 + \mathcal{F}_0^\beta \mathcal{G}(t), \quad \mathcal{G}_0 \in \mathcal{R}.
\]

3. Feasible region and stability theory

Here we derive some conditions for feasible region as well as for stability theory using the next generation matrix method.

Lemma 3.1. The solution \( (\mathcal{S}, \mathcal{I}, \mathcal{R}, \mathcal{D}) \in \mathbb{R}_+^4 \) is bounded and attracted towards the feasible region defined by

\[
\mathcal{F} = \left\{ (\mathcal{S}, \mathcal{I}, \mathcal{R}, \mathcal{D}) \in \mathbb{R}_+^4 : 0 < \mathcal{N}(t) \leq \frac{\lambda}{(\delta_1 + \delta_2 + \delta_3)} \right\}.
\]

Proof. Summing all the equations of the Model (2) yields

\[
\begin{align*}
\frac{d}{dt} \mathcal{S} & = \lambda - \delta_1 \mathcal{S} - \delta_2 \mathcal{I} - \delta_3 \mathcal{R} - \alpha \mathcal{D}, \\
\frac{d}{dt} \mathcal{I} & \leq \lambda - \delta_1 \mathcal{S} - \delta_2 \mathcal{I} - \delta_3 \mathcal{R} - \alpha \mathcal{D}, \\
\frac{d}{dt} \mathcal{D} & \leq \lambda + (\delta_1 + \delta_2 + \delta_3) \mathcal{N}, \\
\frac{d}{dt} \mathcal{N} & = - (\delta_1 + \delta_2 + \delta_3) \mathcal{N} \leq \lambda.
\end{align*}
\]

By solving (3), we get

\[
\mathcal{N}(t) \leq \frac{\lambda}{(\delta_1 + \delta_2 + \delta_3)} + C \exp(- (\delta_1 + \delta_2 + \delta_3) t).
\]

when \( t \to \infty \), then one has \( \mathcal{N}(t) \leq \frac{\lambda}{(\delta_1 + \delta_2 + \delta_3)} \). Hence the required result. \( \square \)

Theorem 3.2. The disease free equilibrium and pandemic equilibrium points of the model (2) are given by \( \mathcal{E}_0 = (\frac{\lambda}{\alpha}, 0, 0, 0) \) and

\[
\mathcal{E}^* = (\mathcal{S}^*, \mathcal{I}^*, \mathcal{R}^*, \mathcal{D}^*),
\]

such that

![Flow chart of the proposed model (2).](image)
\[ \mathcal{R}_0 = \frac{\gamma \lambda}{\delta_1 (\delta_2 + \mu + d)}. \]

**Proof.** For local and global stability analysis, the equilibrium points are important and can be computed from model (2) as

\[ \begin{align*}
\frac{\partial X}{\partial t} & = 0 \\
\frac{\partial Y}{\partial t} & = 0 \\
\frac{\partial Z}{\partial t} & = 0 \\
\frac{\partial D}{\partial t} & = 0.
\end{align*} \]  

(7)

After putting the value of \( \mathcal{E}_0 \) in (10), we have

\[
\mathcal{J} = \begin{bmatrix}
-\delta_1 - \gamma & -\delta & -\delta_2 - \mu - d & 0 \\
\gamma & -\delta & -\delta_2 - \mu - d & 0 \\
0 & -\frac{\mu}{\eta} & -\frac{\mu}{\eta} & -\frac{\mu}{\eta} \\
0 & 0 & -\frac{\mu}{\eta} & -\frac{\mu}{\eta}
\end{bmatrix}.
\]

Now the characteristics equation can be computed as

\[
\det(\mathcal{J} - \eta \mathcal{R}) = \begin{vmatrix}
-\delta_1 - \eta & -\delta & -\delta_2 - \mu - d & -\frac{\mu}{\eta} \\
\gamma & -\delta & -\delta_2 - \mu - d & 0 \\
0 & -\frac{\mu}{\eta} & -\frac{\mu}{\eta} & -\frac{\mu}{\eta} \\
0 & 0 & -\frac{\mu}{\eta} & -\frac{\mu}{\eta}
\end{vmatrix} = 0.
\]

Thus the eigen values are given by

\[
\begin{align*}
\eta_1 &= -\delta_1 \\
\eta_2 &= \frac{-\delta_2 - \mu + d}{\delta_1} \\
\eta_3 &= -(\lambda + \delta_3).
\end{align*}
\]

Further \( \eta_2 \) can be written as

\[
\begin{align*}
\eta_2 &= \frac{-\delta_2 - \mu + d}{\delta_1} \\
\eta_3 &= \frac{\lambda}{\delta_1} \left[ 1 - \frac{1}{\delta_1} \right].
\end{align*}
\]

Since we see that \( \eta_2 < 0 \) if \( \mathcal{R}_0 < 1 \). Hence the needed results received. \( \square \)

**4. Theoretical analysis**

Here in this part of our work, fixed point theory is used to establish sufficient conditions for existence and puniness of solution to the considered model. Therefore, the existence of approximate solution and its uniqueness are investigated by using Schauder and Banach fixed point results [59,60]. We write our proposed model (2) with \( 0 < \beta \leq 1 \) as

\[
\begin{align*}
\frac{\partial \mathcal{F}(t)}{\partial t} &= \mathcal{F}(t, \mathcal{F}, \mathcal{J}, \mathcal{R}, \mathcal{D}), \\
\frac{\partial \mathcal{J}(t)}{\partial t} &= \mathcal{J}(t, \mathcal{F}, \mathcal{J}, \mathcal{R}, \mathcal{D}), \\
\frac{\partial \mathcal{D}(t)}{\partial t} &= \mathcal{D}(t, \mathcal{F}, \mathcal{J}, \mathcal{R}, \mathcal{D}), \\
\frac{\partial \mathcal{R}(t)}{\partial t} &= \mathcal{R}(t, \mathcal{F}, \mathcal{J}, \mathcal{R}, \mathcal{D}),
\end{align*}
\]

(11)

**Theorem 1.** The proposed model (2) is locally asymptotically stable at \( \mathcal{E}_0 \) if \( \mathcal{R}_0 < 1 \), while the the model (2) is locally asymptotically stable at \( \mathcal{E}^* \) if \( \mathcal{R}_0 > 1 \).

**Proof.** Since the jacobian matrix is computed from all four equations of the model (2), but here we take the first three equations of the model as these are independent of \( \mathcal{D} \). Therefore the Jacobian matrix for the model (2) can be computed as
Also $t \in [0, T]$ with $T < \infty$, then $\mathcal{E}_1 = C([0, T] \times R^+_1, R_1)$ is the Banach space. Further $\mathcal{H} = \mathcal{E}_1 \times \mathcal{E}_2 \times \mathcal{E}_3 \times \mathcal{E}_4 \times$ is also complete norm space endowed with norm
\[
\|U\| = \sup_{t \in [0, T]} |U(t)| = \sup_{t \in [0, T]} \left[ |\mathcal{F}(t)| + |\mathcal{J}(t)| + |\mathcal{R}(t)| + |\mathcal{S}(t)| \right].
\]
Writing (12) as
\[
U(t) = U_0(t) + \frac{1}{\Gamma(\beta)} \int_0^t (t - \rho)^{\beta-1}\mathcal{G}(\rho, U(\rho))d\rho,
\]
where
\[
U(t) = \begin{pmatrix}
\mathcal{F}(t) \\
\mathcal{J}(t) \\
\mathcal{R}(t) \\
\mathcal{S}(t)
\end{pmatrix},
U_0(t) = \begin{pmatrix}
\mathcal{F}_0(t) \\
\mathcal{J}_0(t) \\
\mathcal{R}_0(t) \\
\mathcal{S}_0(t)
\end{pmatrix},
\]
\[
\mathcal{G}(t, U(t)) = \begin{pmatrix}
f_1(t, \mathcal{F}(t), \mathcal{J}(t), \mathcal{R}(t), \mathcal{S}(t)) \\
f_2(t, \mathcal{F}(t), \mathcal{J}(t), \mathcal{R}(t), \mathcal{S}(t)) \\
f_3(t, \mathcal{F}(t), \mathcal{J}(t), \mathcal{R}(t), \mathcal{S}(t)) \\
f_4(t, \mathcal{F}(t), \mathcal{J}(t), \mathcal{R}(t), \mathcal{S}(t))
\end{pmatrix},
\]
For establishing theoretical results, we need the given hypothesis to be hold:

(E1) There exists constant $L_\gamma > 0$, for each $\mathcal{H}(t), \mathcal{H}(t) \in R \times R$, such that
\[
|\mathcal{H}(t, U(t)) - \mathcal{H}(t, U(t))| \leq L_\gamma |U(t) - U(t)|,
\]
(E2) There exist constants $C_\gamma > 0$ and $M_\gamma > 0$, with $|\mathcal{H}(t, U(t))| \leq C_\gamma |U| + M_\gamma$.

**Theorem 4.1.** Under the hypothesis (E2) and continuity of $\mathcal{G}$, at least one solution will be exists corresponding to the model (2).

**Proof.** Thank to Schauder fixed point theorem, considering a closed set $\mathcal{A} \subset \mathcal{H}$ with
\[
\mathcal{B} = \{ U \in \mathcal{H} : \|U\| \leq r, \ r > 0 \}.
\]
If $B : \mathcal{A} \rightarrow \mathcal{A}$ be the operator, then in view of (13), one has
\[
B(U(t)) = U_0(t) + \frac{1}{\Gamma(\beta)} \int_0^t (t - \rho)^{\beta-1}\mathcal{G}(\rho, U(\rho))d\rho,
\]
For any $U \in \mathcal{A}$, one has
\[
|B(U(t))| \leq |U_0| + \frac{1}{\Gamma(\beta)} \int_0^t (t - \rho)^{\beta-1}|\mathcal{G}(\rho, U(\rho))|d\rho,
\]
\[
\leq |U_0| + \frac{1}{\Gamma(\beta)} \int_0^t (t - \rho)^{\beta-1} |C_\gamma| |U| + M_\gamma|d\rho|
\]
\[
\leq |U_0| + \frac{1}{\Gamma(\beta)} \int_0^t (t - \rho)^{\beta-1} |C_\gamma| r + M_\gamma|d\rho|
\]
which implies that
\[
|B(U)| \leq r.
\]
Thus $U \in \mathcal{A}$ which yields $B(\mathcal{A}) \subset \mathcal{A}$ and hence $B$ is bounded. Let $t_1 < t_2 \in [0, T]$, taking
\[
|B(U)(t_2) - B(U)(t_1)| = \left| \frac{1}{\Gamma(\beta)} \int_0^{t_2} (t_2 - \rho)^{\beta-1}\mathcal{G}(\rho, U(\rho))d\rho - \frac{1}{\Gamma(\beta)} \int_0^{t_1} (t_1 - \rho)^{\beta-1}\mathcal{G}(\rho, U(\rho))d\rho \right|
\]
\[
\leq \frac{1}{\Gamma(\beta)} \left[ \int_0^{t_1} ((t_2 - \rho)^{\beta-1} - (t_1 - \rho)^{\beta-1})|\mathcal{G}(\rho, U(\rho))|d\rho + \int_0^{t_2} (t_2 - \rho)^{\beta-1} |\mathcal{G}(\rho, U(\rho))|d\rho \right]
\]
\[
\leq \frac{1}{\Gamma(\beta)(t_2 - t_1)^\beta} \left[ \int_0^{t_1} (t_2 - t_1)^{\beta-1} + \int_0^{t_2} (t_2 - \rho)^{\beta-1}|\mathcal{G}(\rho, U(\rho))|d\rho \right]
\]
\[
\leq \frac{1}{\Gamma(\beta)(t_2 - t_1)^\beta} \left[ \int_0^{t_1} (t_2 - t_1)^{\beta-1} + \int_0^{t_2} (t_2 - \rho)^{\beta-1}|\mathcal{G}(\rho, U(\rho))|d\rho \right]
\]
Therefore
\[
\|B(U)(t_2) - B(U)(t_1)\| \to 0, \text{ as } t_1 \to t_2.
\]
Hence $B$ is equi-continuous operator. Hence model (2) has at least one solution.

For existence of unique solution, we have the following result.

**Theorem 4.2.** Under the hypothesis (E1) and if $\frac{\rho}{\Gamma(\beta + 1)} L_\gamma < 1$, then the model (2) has a unique solution.

**Proof.** Let $B : \mathcal{H} \rightarrow \mathcal{H}$ be the operator and taking, $U, \hat{U} \in \mathcal{H}$, then one has
\[
|B(U) - B(\hat{U})| = \frac{1}{\Gamma(\beta)} \int_0^t (t - \rho)^{\beta-1}\mathcal{G}(\rho, U(\rho))d\rho
\]
\[
- \frac{1}{\Gamma(\beta)} \int_0^t (t - \rho)^{\beta-1}\mathcal{G}(\rho, \hat{U}(\rho))d\rho,
\]
\[
\leq \frac{\rho}{\Gamma(\beta + 1)} L_\gamma \|U - \hat{U}\|.
\]

(18) yields
\[
\|B(U) - B(\hat{U})\| \leq \frac{\rho}{\Gamma(\beta + 1)} L_\gamma \|U - \hat{U}\|.
\]

Hence $B$ is a contraction mapping, so by the use of Banach theorem, the considered system has a unique solution. □

Let $I \in C[0, T]$ with $f(0) = 0$ independent of $U$ as.

- $|f(t)| \leq q$, for $q > 0$;
- $\frac{C_0}{\gamma}D_\gamma^\gamma U(t) = \mathcal{G}(t, U(t)) + f(t)$.

**Lemma 4.3.** The solution of nonlinear FDE
\[
\frac{C_0}{\gamma}D_\gamma^\gamma U(t) = \mathcal{G}(t, U(t)) + f(t),
\]
\[
U(0) = U_0,
\]
satisfies the given relation
\[
\|U(t) - \left(U_0 + \frac{1}{\gamma} \int_0^t (t - \rho)^{\beta-1}\mathcal{G}(\rho, U(\rho))d\rho \right)\| \leq \frac{\rho}{\Gamma(\beta + 1)} L_\gamma t^\beta
\]
\[
= \mathcal{O}_{\gamma \neq 0}.
\]

**Proof.** The proof is similar done in [44,48,50]. □

**Theorem 4.4.** Due to hypothesis (E2) and Lemma 4.3, the solution of model (2) is $H-U$ stable if $\Delta = \frac{\rho}{\Gamma(\beta + 1)} L_\gamma < 1$.

**Proof.** If $U \in \mathcal{E}$ is the unique solution of (13), then for any other solution $U \in \mathcal{E}$, we have
From (22), we have
\[ \|U - \bar{U}\| \leq \frac{\Omega r_{\rho}}{1 - \Delta} q. \]  
(23)

Thus (23) yields that solution of (2) is H-U stable. □

5. Numerical solution

This part is devoted to establish two numerical schemes. We apply these schemes to our model one by one and compared the results.

5.1. Numerical simulation by MEM

For the model (2), the numerical approximations are performed in this section by using MEM. Let \([0, T]\) be the set of points, on which we must have to evaluate the series solution of the model (2). Upon further subdivision of \([0, T]\) into \(m\) sub-intervals \([t_p, t_{p+1}]\) of equal difference \(j = \frac{T}{m}\) between consecutive points using \(t_p = pj\) with \(p = 0, 1, \ldots, m\), then obviously

\[ \mathcal{S}(t), \mathcal{I}(t), \mathcal{R}(t), \mathcal{D}(t), \mathcal{S}^{(0)} \frac{D^j [\mathcal{S}(t)]}{D^j t}, \mathcal{S}^{(1)} \frac{D^j [\mathcal{I}(t)]}{D^j t}, \mathcal{S}^{(2)} \frac{D^j [\mathcal{R}(t)]}{D^j t}, \mathcal{S}^{(3)} \frac{D^j [\mathcal{D}(t)]}{D^j t} \]

are continuous on \([0, T]\). Applying the modified Euler’s or Taylor’s method about \(t = t_0\) to the considered model expressed in (?) and for each value of \(t\) taking \(a \in (0, T)\), the expressions for \(t_1\) is given as

\[ \mathcal{S}(t_1) = \mathcal{S}(t_0) + f_1(t_0, \mathcal{S}(t_0), \mathcal{I}(t_0), \mathcal{R}(t_0), \mathcal{D}(t_0)) \frac{D^j [\mathcal{S}(t)]}{D^j t_{p+1}} + \mathcal{S}^{(1)} \frac{D^j [\mathcal{I}(t)]}{D^j t_{p+1}} + \mathcal{S}^{(2)} \frac{D^j [\mathcal{R}(t)]}{D^j t_{p+1}} + \mathcal{S}^{(3)} \frac{D^j [\mathcal{D}(t)]}{D^j t_{p+1}}, \]
\[ \mathcal{I}(t_1) = \mathcal{I}(t_0) + f_1(t_0, \mathcal{S}(t_0), \mathcal{I}(t_0), \mathcal{R}(t_0), \mathcal{D}(t_0)) \frac{D^j [\mathcal{S}(t)]}{D^j t_{p+1}} + \mathcal{I}^{(1)} \frac{D^j [\mathcal{I}(t)]}{D^j t_{p+1}} + \mathcal{I}^{(2)} \frac{D^j [\mathcal{R}(t)]}{D^j t_{p+1}} + \mathcal{I}^{(3)} \frac{D^j [\mathcal{D}(t)]}{D^j t_{p+1}}, \]
\[ \mathcal{R}(t_1) = \mathcal{R}(t_0) + f_1(t_0, \mathcal{S}(t_0), \mathcal{I}(t_0), \mathcal{R}(t_0), \mathcal{D}(t_0)) \frac{D^j [\mathcal{S}(t)]}{D^j t_{p+1}} + \mathcal{R}^{(1)} \frac{D^j [\mathcal{I}(t)]}{D^j t_{p+1}} + \mathcal{R}^{(2)} \frac{D^j [\mathcal{R}(t)]}{D^j t_{p+1}} + \mathcal{R}^{(3)} \frac{D^j [\mathcal{D}(t)]}{D^j t_{p+1}}, \]
\[ \mathcal{D}(t_1) = \mathcal{D}(t_0) + f_1(t_0, \mathcal{S}(t_0), \mathcal{I}(t_0), \mathcal{R}(t_0), \mathcal{D}(t_0)) \frac{D^j [\mathcal{S}(t)]}{D^j t_{p+1}} + \mathcal{D}^{(1)} \frac{D^j [\mathcal{I}(t)]}{D^j t_{p+1}} + \mathcal{D}^{(2)} \frac{D^j [\mathcal{R}(t)]}{D^j t_{p+1}} + \mathcal{D}^{(3)} \frac{D^j [\mathcal{D}(t)]}{D^j t_{p+1}}. \]

(24)

By taking the difference between consecutive points as \(j\) very very small, such that we may ignore terms containing higher-order derivatives to get

\[ \mathcal{S}(t_1) = \mathcal{S}(t_0) + f_1(t_0, \mathcal{S}(t_0), \mathcal{I}(t_0), \mathcal{R}(t_0), \mathcal{D}(t_0)) \mathcal{S}^{(1)} \frac{D^j [\mathcal{I}(t)]}{D^j t_{p+1}}, \]
\[ \mathcal{I}(t_1) = \mathcal{I}(t_0) + f_1(t_0, \mathcal{S}(t_0), \mathcal{I}(t_0), \mathcal{R}(t_0), \mathcal{D}(t_0)) \mathcal{I}^{(1)} \frac{D^j [\mathcal{I}(t)]}{D^j t_{p+1}}, \]
\[ \mathcal{R}(t_1) = \mathcal{R}(t_0) + f_1(t_0, \mathcal{S}(t_0), \mathcal{I}(t_0), \mathcal{R}(t_0), \mathcal{D}(t_0)) \mathcal{R}^{(1)} \frac{D^j [\mathcal{I}(t)]}{D^j t_{p+1}}, \]
\[ \mathcal{D}(t_1) = \mathcal{D}(t_0) + f_1(t_0, \mathcal{S}(t_0), \mathcal{I}(t_0), \mathcal{R}(t_0), \mathcal{D}(t_0)) \mathcal{D}^{(1)} \frac{D^j [\mathcal{I}(t)]}{D^j t_{p+1}}. \]

(25)

On repeating the procedure, we get sequence of point to approximate \((\mathcal{S}(t), \mathcal{I}(t), \mathcal{R}(t), \mathcal{D}(t))\) is formed. A generalized formula in this regard at \(t_{p+1} = t_p + h\) is given by

\[ \mathcal{S}(t_{p+1}) = \mathcal{S}(t_p) + f_1(t_p, \mathcal{S}(t_p), \mathcal{I}(t_p), \mathcal{R}(t_p), \mathcal{D}(t_p)) \mathcal{S}^{(1)} \frac{D^j [\mathcal{I}(t)]}{D^j t_{p+1}}, \]
\[ \mathcal{I}(t_{p+1}) = \mathcal{I}(t_p) + f_1(t_p, \mathcal{S}(t_p), \mathcal{I}(t_p), \mathcal{R}(t_p), \mathcal{D}(t_p)) \mathcal{I}^{(1)} \frac{D^j [\mathcal{I}(t)]}{D^j t_{p+1}}, \]
\[ \mathcal{R}(t_{p+1}) = \mathcal{R}(t_p) + f_1(t_p, \mathcal{S}(t_p), \mathcal{I}(t_p), \mathcal{R}(t_p), \mathcal{D}(t_p)) \mathcal{R}^{(1)} \frac{D^j [\mathcal{I}(t)]}{D^j t_{p+1}}, \]
\[ \mathcal{D}(t_{p+1}) = \mathcal{D}(t_p) + f_1(t_p, \mathcal{S}(t_p), \mathcal{I}(t_p), \mathcal{R}(t_p), \mathcal{D}(t_p)) \mathcal{D}^{(1)} \frac{D^j [\mathcal{I}(t)]}{D^j t_{p+1}}. \]

(26)

where \(r = 0, 1, 2, \ldots, n - 1\).

5.2. Numerical results and Discussion

Here we take real data of Pakistan as the total pollution of the country \(N = 220 \text{ millions} [61]\) and the other values are given in Table 1: Here we now present numerical interpretation of the proposed model (2) using MEM in Figs. 2–5. From Fig. 2, we see that susceptibility is decreasing at various rate due to different fractional orders. As fractional order tends to integer order the solution converges to the integer order solution. In same line the infected class is increasing in initial 100 days with various rate of transmission due to various fractional order then it turns to decrease with same behaviors as in Fig. 3. Consequently the increase in death class cause increase in recovered class. The recovery class dynamics raises at various fractional order as shown in Fig. 4. In Fig. 5, the dynamics of death class is also increasing until become stable due to faster infection rate.

| Table 1 | Interpretation and approximate values of parameters involve in the model (2). |
|------------------------|---------------------------------------------------------------|
| Compartment/Parameters | Description of parameter | Approximate value |
| \(\mathcal{S}_0\) | Initial density of susceptible class | 217.388901 millions |
| \(\mathcal{I}_0\) | Initial density of infected class | 1.29 million |
| \(\mathcal{R}_0\) | Initial density of recovered class | 1.256337 millions |
| \(\mathcal{D}_0\) | Initial density of death class | 0.028921 million |
| \(i\) | Recruitment rate | 0.0009 assumed |
| \(\pi\) | transmission rate from infection | 0.009978 |
| \(\mu\) | Recovery rate | 0.0025 assumed |
| \(d\) | Death rate due to infection | 0.019 |
| \(\gamma\) | rate of infection | 0.0028 assumed |
| \(\delta_1\) | Natural death rate of Susceptible class | 0.0009 assumed |
| \(\delta_2\) | Natural death rate of infected class | 0.00056 assumed |
| \(\delta_3\) | Natural death rate of recovered class | 0.000013 assumed |
Using NSFD scheme under the concept of fractional order derivative, we approximate the proposed model (2). We use Grünwald-Letnikov approximation for Caputo derivative. About some detail for this scheme, we refer [54–58]. Since for nonlinear systems the investigation of exact solution in most cases is impossible, so we focus on best approximation of the solution. In this regards, some NSFD schemes have been introduced. The said scheme has the ability to avoid the full implicit scheme and preserve positivity, monotonicity and convergence which make this method popular. For detail applications of the method see [62].

To construct the scheme, consider Grünwald-Letnikov approximation for the fractional order derivative given by

$$
\rho_0 D_0^\rho [V(t)] = \lim_{\rho \to 0^+} \rho^\rho \sum_{k=0}^\infty (-1)^k \binom{\rho}{k} V(t - k\rho),
$$

with $\rho = \eta h$, such that $h$ is the step size. Consider the following FDE as

$$
\rho_0 D_0^\rho [V(t)] = \mathcal{F}(t, V(t)), \quad t \in [0, T], T < \infty,
\begin{align*}
V(t_0) &= V_0.
\end{align*}
$$

Fig. 2  Transmission dynamics of susceptible class at various fractional order of the proposed model (2).

Fig. 3  Transmission dynamics of infected class at various fractional order of the proposed model (2).
Using (27), from (12) one has
\[ X_{q}^{\alpha} = 0 \]
for \( n = 1, 2, 3, \ldots \), 
where \( t_n = nh \) and \( K^\alpha_i \) are the Grünwald-Letnikov coefficients calculated as
\[ K^\alpha_i = \left( 1 - \frac{1 + \beta}{t} \right) K^\alpha_{i-1}, \quad i = 1, 2, 3, \ldots \]
and
\[ K^\alpha_0 = h^\beta. \]

Based on the above definition, our considered model (2) can be discretized as
\[
\begin{align*}
S(t_{n+1}) &= \frac{1}{K^\alpha_0} \left[ \sum_{j=1}^{n+1} K^\alpha_j S(t_{n+1-j}) + \lambda - \gamma S(t_n) - \delta_1 S(t_n) + \alpha R(t_n) \right] \\
I(t_{n+1}) &= \frac{1}{K^\alpha_0} \left[ \sum_{j=1}^{n+1} K^\alpha_j I(t_{n+1-j}) + \lambda I(t_n) - \delta_1 I(t_n) + 2 I(t_n) \right] \\
D(t_{n+1}) &= \frac{1}{K^\alpha_0} \left[ \sum_{j=1}^{n+1} K^\alpha_j D(t_{n+1-j}) \right].
\end{align*}
\]
5.4. Numerical interpretation and explanation

Here, we now present the transmission dynamics of the proposed model (2) using NSFD scheme in Figs. 6–9. The dynamical behaviors presented in Figs. 6–9 is also same as given in Fig. 2–5 respectively.

6. Comparison of both methods

Here we compare graphs of both methods for fixed time $t = 300$ in Figs. 10–13 as Here in Figs. 10–13, we compare the numerical solutions of various compartments of the proposed model by using MEM and NSFD schemes respectively. Both have similar results for same range of time. As MEM slightly simpler than NSFD scheme. Here we compare the numerical simulations of both methods with some real data of Pakistan for 200 days [63] in Fig. 14. We see that our simulated results have close agreement with real data in both cases for the given fractional orders. This shows that both schemes can be use as a powerful tool to investigate fractional order dynamics. Further we compare CPU time of both method by using Matlab 13 and Machine Cori-7 of HP with 8th generation in Table 2.
7. Conclusion

In this research work, we have established a modified type COVID-19 model by incorporating natural death rates of susceptible, infected and recovered classes respectively. The model under investigation has been studied under fractional order derivative of Caputo type. Further by using fixed point theory, we have established existence theory for numerical solutions. Also we have developed various results for global and local stability by using the next generation matrix method. The basic reproduction number has been computed. Moreover, the feasible region has also established for the proposed model along with its boundedness. The numerical interpretations have been performed by using two different numerical approaches based on MEM and NSFD methods. Both methods provide nearly same results for our model. Therefore we have compared both procedures in CPU time to see which one is most expensive with respect to time. In this regards NSFD method which is slightly general than MEM. Further MEM is slightly expensive in time than MEM for same number of iteration corresponding to different range of time. Graphical presentations have been provided for taking some real data about COVID –19 transmission in Pakistan. Both scheme have been compared
Fig. 10  Transmission dynamics of susceptible class at various fractional order using: (a). Modified Euler method. (b). Nonstandard Finite Difference Method.

Fig. 11  Transmission dynamics of infected class at various fractional order using: (a). Modified Euler method. (b). Nonstandard Finite Difference Method.

Fig. 12  Transmission dynamics of recovered class at various fractional order using: (a). Modified Euler method. (b). Nonstandard Finite Difference Method.
with real data. The concerned graphs have been plotted against different fractional order. Hence we concluded that fractional calculus provides more best explanations to real world problems for understanding their dynamics. In future, we will investigate some mathematical models under fractional order stochastic differential equations. Also piecewise concept will be applied for investigating epidemiological models.

Fig. 13 Transmission dynamics of death class at various fractional order using: (a). Modified Euler method. (b). Nonstandard Finite Difference Method.

Fig. 14 Comparison between real and simulated results by using NSFD schemes and MEM.

| Table 2 | Comparison of both methods. |
|---------|-----------------------------|
| Range of $t$ | CPU time of MEM in seconds | CPU time of NSFD method in seconds |
| 50 | 4.99 | 4.66 |
| 100 | 4.75 | 4.71 |
| 150 | 4.83 | 4.73 |
| 200 | 4.85 | 4.76 |
| 250 | 4.90 | 4.80 |
| 300 | 4.95 | 4.83 |

Funding

There does not exist any funding source.

Availability of data

All data used in this paper is included within the article.

Authors contributions

All authors played their role equally. First and second authors designed the models and did theoretical analysis. Third author did numerical. Last two authors edited and drafted the article.

Declaration of Competing Interest

The authors declare that they have no known competing financial interests or personal relationships that could have appeared to influence the work reported in this paper.
Study of fractional order dynamics

Acknowledgments

Authors Kamal Shah, Aziz Khan and Thabet Abdeljawad would like to thank Prince Sultan University for the support through the TAS research lab.

Manar A. Alqudah: Princess Nourah bint Abdulrahman University Researchers Supporting Project number (PNURSP2022R14), Princess Nourah bint Abdulrahman University, Riyadh, Saudi Arabia.

References

[1] K. Diethelm, N.J. Ford. Analysis of fractional differential equations, J. Math. Anal. Appl. 265 (2) (2002) 229–248.
[2] I. Podlubny. Fractional Differential Equations, Elsevier, Amester Dam, North Holland, 1998.
[3] A.A. Kilbas, H.M. Srivastava, J.J. Trujillo. Theory and Applications of Fractional Differential Equations, Elsevier, Gordon and Breach, Holland, 2006.
[4] V. Lakshmikantham, A.S. Vatsala. Basic theory of fractional differential equations, Nonl. Anal.: Theory, Methods Appl. 69 (8) (2008) 2677–2682.
[5] S. Abbas, M. Benchohra, G.M.N. Guerekata. Topics in Fractional Differential Equations, Science & Business Media, Springer, Berlin, 2012.
[6] M. Rahimy. Applications of fractional differential equations, Appl. Math. Sci. 4 (50) (2010) 2453–2461.
[7] R. Almeida, N.R. Bastos, M.T.T. Monteiro. Modeling some real phenomena by fractional differential equations, Math. Method. Appl. Sci. 39 (16) (2016) 4846–4855.
[8] Y. Zhou, J. Wang, L. Zhang. Basic Theory of Fractional Differential Equations, World Scientific, Singapore, 2016.
[9] R. Almeida, N.R. Bastos, M.T.T. Monteiro. Modeling some real phenomena by fractional differential equations, Math. Method. Appl. Sci. 39 (16) (2016) 4846–4855.
[10] K. Diethelm, N.J. Ford, A.D. Freed. A predictor-corrector approach for the numerical solution of fractional differential equations, Nonlinear Dyn. 29 (1) (2002) 3–22.
[11] K.B. Oldham. Fractional differential equations in electrochemistry, Adv. Eng. Soft. 41 (1) (2010) 9–12.
[12] K. Kavitha, V. Vijayakumar, R. Udhayakumar, C. Ravichandran. Results on controllability of Hilfer fractional differential equations with infinite delay via measures of noncompactness, Asian J. Control (2021), https://doi.org/10.1002/asjc.2549.
[13] K.S. Nisar, V. Vijayakumar, Results concerning to approximate controllability of non-densely defined Sobolev-type Hilfer fractional neutral delay differential system, Math. Methods Appl. Sci. 44 (17) (2021) 13615–13632.
[14] W. Kavitha Williams, V. Vijayakumar, R. Udhayakumar, S.K. Panda, K.S. Nisar. Existence and controllability of nonlocal mixed Volterra-Fredholm type fractional delay integro-differential equations of order $1 < r < 2$, Num. Methods Partial Diff. Eqs. (2020), https://doi.org/10.1002/num.22697.
[15] V. Vijayakumar, C. Ravichandran, K.S. Nisar, K.D. Kučche. New discussion on approximate controllability results for fractional Sobolev type Volterra-Fredholm integro-differential systems of order $1 < r < 2$, Num. Methods Partial Diff. Eqs. (2021), https://doi.org/10.1002/num.22772.
[16] Y.K. Ma, M.M. Raja, K.S. Nisar, A. Shukla, V. Vijayakumar. Results on controllability for Sobolev type fractional differential equations of order $1 < r < 2$ with finite delay,AIMS Math. 7 (6) (2022) 10215–10233.
[17] C. Ravichandran, K. Munusamy, K.S. Nisar, N. Valliammal. Results on neutral partial integro-differential equations using Monch-Krasnoselskii fixed point theorem with nonlocal conditions, Fractal Fract. 6 (2) (2022) 75.
[18] K. Jothimani, N. Valliammal, C. Ravichandran. Existence result for a neutral fractional integro-differential equation with state dependent delay, J. Appl. Nonlinear Dyn. 7 (4) (2018) 371–381.
[19] E.F.D. Goufo, C. Ravichandran, G.A. Birjdar. Self-similarity techniques for chaotic attractors with many scrolls using step series switching. Math. Modell. Anal. 26 (4) (2021) 591–611.
[20] V. Valliammal, C. Ravichandran. Results on fractional neutral integro-differential systems with state-dependent delay in Banach spaces, Nonlinear Stud. 25 (1) (2018) 159–171.
[21] C. Ravichandran, K. Logeswari, F. Jarad, New results on existence in the framework of Atangana-Baleanu derivative for fractional integro-differential equations, Chaos, Solitons Fractals 125 (2019) 194–200.
[22] S. Rashid, F. Jarad, M.A. Noor, H. Kalsoom, Y.M. Chu. Inequalities by means of generalized proportional fractional integral operators with respect to another function, Mathematics 7 (12) (2019) 1225.
[23] Y.Y. Gambo, F. Jarad, D. Baleanu, T. Abdeljawad, On Caputo modification of the Hadamard fractional derivatives, Adv. Diff. Eqs. 2014 (1) (2014) 1–12.
[24] A. Khan, H. Khan, J.F. Gómez-Aguilar, T. Abdeljawad. Existence and Hyers-Ulam stability for a nonlinear singular fractional differential equations with Mittag-Leffler kernel, Chaos, Solitons Fractals 127 (2019) 422–427.
[25] H. Khan, F. Jarad, T. Abdeljawad, A. Khan. A singular ABC-fractional differential equation with p-Laplacian operator, Chaos, Solitons Fractals 129 (2019) 56–61.
[26] A.C. Fowler. Mathematical Models in the Applied Sciences, Cambridge University Press, Cambridge, 1997.
[27] L.O. Tedeschi. Assessment of the adequacy of mathematical models, Agri. Sys. 89 (2–3) (2006) 225–247.
[28] F.B. Castillo-Chavez, Z. Feng. Mathematical Models in Epidemiology, Springer, New York, 2019.
[29] A. Rauf et al. COVID-19 pandemic: epidemiology, etiology, conventional and non-conventional therapies, Int. J. Environ. Res. Public Health 17 (21) (2020) 8155.
[30] M.Y. Shehelnakov, L.V. Kolobukhina, O.A. Burgasova, I.S. Kruzhkova, V.V. Maleev. COVID-19: etiology, clinical picture, treatment, Russian J. Infect. Immunity 10 (3) (2020) 421–445.
[31] J. Wang. Mathematical models for COVID-19: applications, limitations, and potentials, J. Pub. Health Emer. 4 (2020) 1–12.
[32] Z. Ceylan. Estimation of COVID-19 prevalence in Italy, Spain, and France, Sci. Total Environ. 729 (2020) 138817.
[33] D.M. Thomas, R. Sturdivant, N.V. Dhurandhar, S. Debroy, N. Clark, A Primer on COVID-19 Mathematical Models, Obesity (2020).
[34] S.K. Mustafa, M.A. Ahmad, S. Sotnik, O. Zelenyi, V. Lyashenko, O. Alzahrani. Brief review of the mathematical models for analyzing and forecasting transmission of COVID-19, J. Crit. Rev. 7 (2020) 4206–4210.
[35] C.E. Wagner, C.M. Saad-Dow, B.T. Grenfell. Modelling vaccination strategies for COVID-19, Nat. Rev. Immunol. 2022 (2022) 1–3.
[36] M. Zhang et al. Human mobility and COVID-19 transmission: A systematic review and future directions, Annals GIS 2022 (2022) 1–14.
[37] F.W.C. Jasper et al. Genomic characterization of the 2019 novel human-pathogenic coronavirus isolated from patients with acute respiratory disease in Wuhan, Hubei, China, Emerging Microbes Infect. 9 (1) (2020) 221–236.
[38] A. Mourah, F. Mroue, Z. Taha. Stochastic mathematical models for the spread of COVID-19: a novel epidemiological approach, Math. Med. Biol.: J. IMA 39 (1) (2022) 49–76.
[39] J. Riuo, C.L. Althaus. Pattern of early human-to-human transmission of Wuhan 2019 novel coronavirus (2019-nCoV), Eurosurveillance 25 (4) (2020) 2000058.
[40] R.J. Pais, N. Taveira, Predicting the evolution and control of the COVID-19 pandemic in Portugal, F1000Research 9 (283) (2020) 283.
[41] Q. Lin et al, A conceptual model for the coronavirus disease (COVID-19) outbreak in Wuhan, China with individual reaction and governmental action, Int. J. Infec. Dis. 93 (2021) 211–216.
[42] D. Fanelli, F. Piazza, Analysis and forecast of COVID-19 spreading in China, Italy and France, Cha. Solit. Fract. 134 (2020) 109761.
[43] E. Atangana, A. Atangana, Face masks simple but powerful weapons to protect against COVID-19 spread: Can they have sides effects?, Results Phys 19 (2020) 103425.
[44] K. Shah, T. Abdeljawad, I. Mahariq, F. Jarad, Qualitative analysis of a mathematical model in the time of COVID-19, Bio. Res. Int. 2020 (2020) 16.
[45] K. Shah, Z.A. Khan, A. Ali, R. Amin, H. Khan, A. Khan, Haar wavelet collocation approach for the solution of fractional order COVID-19 model using Caputo derivative, Alex. Eng. J. 59 (5) (2020) 3221–3231.
[46] A. Atangana, A novel Covid-19 model with fractional differential operators with singular and non-singular kernels: Analysis and numerical scheme based on Newton polynomial, Alex. Eng. J. 60 (4) (2021) 3781–3806.
[47] A. Atangana, Modelling the spread of COVID-19 with new fractal-fractional operators Can the lockdown save mankind before vaccination, Chaos Solitons Fractals 136 (2020) 109860.
[48] A. Atangana, S.I. Araz, Mathematical model of COVID-19 spread in Turkey and South Africa: theory, methods, and applications, Adv. Diff. Equus. 2020 (1) (2020) 1–89.
[49] A. Boudaoui, M.Y. El hadj, Z. Hammouch, S. Allah, A fractional-order model describing the dynamics of the novel coronavirus (COVID-19) with nonsingular kernel, Chaos, Solitons Fractals 146 (2021) 110859.
[50] M. Sher, K. Shah, Z.A. Khan, H. Khan, A. Khan, Computational and theoretical modeling of the transmission dynamics of novel COVID-19 under Mittag-Leffler power law, Alex. Eng. J. 59 (5) (2020) 3133–3147.
[51] M. Mehran, A.V. Kamyad, Modified fractional Euler method for solving fuzzy fractional initial value problem, Commun. Nonl. Sci. Numer. Simul. 18 (1) (2013) 12–21.
[52] H. Ahmed, Fractional Euler method; an effective tool for solving fractional differential equations, J. Egyptian Mathe. Soc. 26 (1) (2018) 38–43.
[53] S. Treibert, H. Brunner, M. Ehrhardt, A nonstandard finite difference scheme for the SVICDR model to predict COVID-19 dynamics, Mathe. Biosci. Eng. 19 (2) (2022) 1213–1238.
[54] M. Khalsaraii, M.M. Rashidi, A. Shokri, H. Ramos, P. Khazad, A nonstandard finite difference method for a generalized Black-Scholes equation, Symmetry 14 (1) (2022) 141.
[55] S. Treibert, H. Brunner, M. Ehrhardt, A nonstandard finite difference scheme for the SVICDR model to predict COVID-19 dynamics, Math. Bio. Eng. 19 (2) (2022) 1213–1238.
[56] R.E. Mickens, Applications of nonstandard finite difference schemes, World Scientific, Singapore, 2000.
[57] R.E. Mickens, Nonstandard finite difference models of differential equations, World Scientific, Singapore, 1994.
[58] S. Treibert, H. Brunner, M. Ehrhardt, A nonstandard finite difference scheme for the SVICDR model to predict COVID-19 dynamics, Math. Bio. Eng. 19 (2) (2022) 1213–1238.
[59] A. Granas, J. Dugundji, Elementary fixed point theorems, Springer, New York, 2003.
[60] A.B. Amar, A. Jeribi, M. Mnif, Some fixed point theorems and application to biological model, Numer. Func. Anal. Opt. 29 (1–2) (2008) 1–23.
[61] https://www.worldometers.info, Pakistan COVID - Coronavirus Statistics, 23 December 2021.
[62] M.Y. Ongun, D. Arslan, R. Garrappa, Nonstandard finite difference schemes for a fractional-order Brusselator system, Adv. Diff. Equ. 2013 (1) (2013) 1–13.
[63] https://www.worldometers.info/coronavirus/country/pakistan/2021.