Note on rainbow connection number of dense graphs*

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Abstract

An edge-colored graph $G$ is rainbow connected if any two vertices are connected by a path whose edges have distinct colors. The rainbow connection number of a connected graph $G$, denoted by $rc(G)$, is the smallest number of colors that are needed in order to make $G$ rainbow connected. Following an idea of Caro et al., in this paper we also investigate the rainbow connection number of dense graphs. We show that for $k \geq 2$, if $G$ is a non-complete graph of order $n$ with minimum degree $\delta(G) \geq \frac{n}{2} - 1 + \log_k n$, or minimum degree-sum $\sigma_2(G) \geq n - 2 + 2 \log_k n$, then $rc(G) \leq k$; if $G$ is a graph of order $n$ with diameter 2 and $\delta(G) \geq 2(1 + \log_{\frac{k}{2}} \log_k n)$, then $rc(G) \leq k$. We also show that if $G$ is a non-complete bipartite graph of order $n$ and any two vertices in the same vertex class have at least $2 \log_{\frac{k}{2}} k \log_k n$ common neighbors in the other class, then $rc(G) \leq k$.

Keywords: rainbow coloring, rainbow connection number, parameter $\sigma_2(G)$

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1 Introduction

All graphs under our consideration are finite, undirected and simple. For notation and terminology not defined here, we refer to [11]. Let $G$ be a graph. The length of a path in $G$ is the number of edges of the path. The distance between two vertices $u$ and $v$ in $G$, denoted by $d(u, v)$, is the length of a shortest path connecting them in $G$. If there

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is no path connecting \( u \) and \( v \), we set \( d(x, y) := \infty \). An edge-coloring of a graph is a function from its edges set to the set of natural numbers. A graph \( G \) is rainbow edge-connected if for every pair of distinct vertices \( u \) and \( v \) of \( G \), \( G \) has a \( u-v \) path whose edges are colored with distinct colors. This concept was introduced by Chartrand et al. \([4]\). The minimum number of colors required to rainbow color a connected graph is called its rainbow connection number, denoted by \( rc(G) \). Observe that if \( G \) has \( n \) vertices, then \( rc(G) \leq n - 1 \). Clearly, \( rc(G) \geq diam(G) \), the diameter of \( G \). In \([4]\), Chartrand et al. determined the rainbow connection numbers of wheels, complete graphs and all complete multipartite graphs. In \([3]\), Chakraborty et al. proved that given a graph \( G \), deciding if \( rc(G) = 2 \) is NP-Complete. In particular, computing \( rc(G) \) is NP-Hard.

If \( \delta(G) \geq \frac{n}{2} \), then \( diam(G) = 2 \), but we do not know if this guarantees \( rc(G) = 2 \). In \([2]\), Caro et al. investigated the rainbow connection number of dense graphs, and they got the following results.

**Theorem 1.1.** Any non-complete graph with \( \delta(G) \geq \frac{n}{2} + \log n \) has \( rc(G) = 2 \).

**Theorem 1.2.** Let \( c = \frac{1}{\log(9/7)} \). If \( G \) is a non-complete bipartite graph with \( n \) vertices and any two vertices in the same vertex class have at least \( 2\log n \) common neighbors in the other class, then \( rc(G) = 3 \).

We will follow their idea to investigate dense graphs again. And we get the following results.

**Theorem 1.3.** Let \( k \geq 2 \) be an integer. If \( G \) is a non-complete graph of order \( n \) with \( \delta(G) \geq \frac{n}{2} - 1 + \log n \), then \( rc(G) \leq k \).

**Theorem 1.4.** Let \( k \geq 2 \) be an integer. If \( G \) is a non-complete graph of order \( n \) with \( \sigma_2(G) \geq n - 2 + 2\log n \), then \( rc(G) \leq k \).

**Theorem 1.5.** Let \( k \geq 3 \) be an integer. If \( G \) is a non-complete bipartite graph of order \( n \) and any two vertices in the same vertex class have at least \( 2\log_{\frac{k^2}{k-2}} k\log n \) common neighbors in the other class, then \( rc(G) \leq k \).

In \([3]\), Chakraborty et al. showed the following result.

**Theorem 1.6.** If \( G \) is a graph of order \( n \) with diameter \( 2 \) and \( \delta(G) \geq 8\log n \), then \( rc(G) \leq 3 \). Furthermore, such a coloring is given with high probability by a uniformly random 3-edge-coloring of the graph \( G \), and can also be found by a polynomial time deterministic algorithm.

Now we get the following result.

**Theorem 1.7.** Let \( k \geq 3 \) be an integer. If \( G \) is a graph of order \( n \) with diameter \( 2 \) and \( \delta(G) \geq 2(1 + \log_{\frac{k^2}{k-2}} k)\log n \), then \( rc(G) \leq k \).
2 Proof of the theorems

Proof of Theorem 1.3: Let $G$ be a non-complete graph of order $n$ with $\delta(G) \geq \frac{n}{2} - 1 + \log kn$. We use $k$ different colors to randomly color every edge of $G$. In the following we will show that with positive probability, such a random coloring make $G$ rainbow connected. For any pair $u, v \in V(G), uv \notin E(G)$, since $d(u) \geq \frac{n}{2} - 1 + \log kn$, $d(v) \geq \frac{n}{2} - 1 + \log kn$, there are at least $2\log kn$ common neighbors between $u$ and $v$, that is $|N(u) \cap N(v)| \geq 2\log kn$. Hence there are at least $2\log kn$ edge-disjoint paths of length two from $u$ to $v$. For any $w \in N(u) \cap N(v)$, the probability that the path $uwv$ is not a rainbow path is $\frac{1}{k}$. Hence, the probability that all these edge-disjoint paths are not rainbow is at most $\left(\frac{1}{k}\right)^{2\log kn} = \frac{1}{n^2}$. Since there are less than $\binom{n}{2}$ pairs non-adjacent vertices, and $\left(\frac{n}{2}\right)\frac{1}{n^2} < 1$. We may get that with positive probability, each pair of non-adjacent vertices are connected by a rainbow path. This completes the proof of Theorem 1.3.

Proof of Theorem 1.4: Let $G$ be a non-complete graph of order $n$ with $\sigma_2(G) \geq n - 2 + 2\log kn$. We use $k$ different colors to randomly color every edge of $G$. In the following we will show that with positive probability, such a random coloring make $G$ rainbow connected. For any pair $u, v \in V(G), uv \notin E(G)$, as $\sigma_2(G) \geq n - 2 + 2\log kn$, it follows that $|N(u) \cap N(v)| \geq 2\log kn$. Similar to the proof of Theorem 1.3, we may get that with positive probability, each pair of non-adjacent vertices are connected by a rainbow path. This completes the proof of Theorem 1.4.

Proof of Theorem 1.5: Let $G$ be a non-complete bipartite graph of order $n$ and any two vertices in the same vertex class have at least $2\log \frac{k^2}{k-2} \log kn$ common neighbors in the other class. We use $k$ different colors to randomly color every edge of $G$. In the following we will show that with positive probability, such a random coloring make $G$ rainbow connected. For every pair $u, v \in V(G)$ and $u, v$ are in the same class of $V(G)$, then the distance of $d(u, v) = 2$, as $|N(u) \cap N(v)| \geq 2\log \frac{k^2}{k-2} \log kn$, there are at least $2\log \frac{k^2}{k-2} \log kn$ edge-disjoint paths of length two from $u$ to $v$. The probability that all these edge-disjoint paths are not rainbow is at most $\left(\frac{1}{k}\right)^{2\log \frac{k^2}{k-2} \log kn} < \left(\frac{1}{k}\right)^{2\log kn} = \frac{1}{n^2}$. For every pair $u, v \in V(G)$ from different classes of $G$ and $uv \notin E(G)$, then the distance of $d(u, v)$ is 3. Fix a neighbor $w_u$ of $u$, for any $u_i \in N(w_u) \cap N(v)$, the probability that $uw_uu_iv$ is not a rainbow path is $\frac{3k-2}{k^2}$. We know $|N(w_u) \cap N(v)| \geq 2\log \frac{k^2}{k-2} \log kn$. Hence, the probability that all these edge-disjoint paths are not rainbow is at most $\left(\frac{3k-2}{k^2}\right)^{2\log \frac{k^2}{k-2} \log kn} = \frac{1}{n^2}$. Thus, we may get that with positive probability, each pair of non-adjacent vertices are connected by a rainbow path. This completes the proof of Theorem 1.5.

Proof of Theorem 1.7:
Let $G$ be a graph of order $n$ with diameter 2. We use $k$ different colors to randomly color every edge of $G$. In the following we will show that with positive probability, such a random coloring make $G$ rainbow connected. For any two non-adjacent vertices $u, v$, if $|N(u) \cap N(v)| \geq 2\log_k n$, then there are at least $2\log_k n$ edge-disjoint paths of length two from $u$ to $v$. The probability that all these edge-disjoint paths are not rainbow is at most $\left(\frac{1}{k} \right)^{2\log_k n} = \frac{1}{n^2}$. Otherwise, $|N(u) \cap N(v)| < 2\log_k n$. Let $A = N(u) \setminus N(v)$, $B = N(v) \setminus N(u)$, then $|A| \geq 2\log_{\frac{k^2}{3}} k\log_k n$, $|B| \geq 2\log_{\frac{k^2}{3}} k\log_k n$. As the diameter of $G$ is two, for any $x \in A$, $\exists y_x \in N(v)$ such that $xy_x \in E(G)$, that is $xy_x v$ is a path of length 2. Now, we will consider the set of at least $2\log_{\frac{k^2}{3}} k\log_k n$ edge-disjoint paths $P = \{uxy_x v: x \in A\}$. For every $x \in A$, the probability that $uxy_x v$ is not a rainbow path is $\frac{3k-2}{k^2}$. Moreover, this event is independent of the corresponding events for all other members of $A$, because this probability does not change even with full knowledge of the colors of all edges incident with $v$. Therefore, the probability that all these edge-disjoint paths are not rainbow is at most $\left(\frac{3k-2}{k^2} \right)^{2\log_{\frac{k^2}{3}} k\log_k n} = \frac{1}{n^2}$. Since there are less than $\binom{n}{2} \frac{1}{n^2} < 1$. We may get that with positive probability, each pair of non-adjacent vertices are connected by a rainbow path. This completes the proof of Theorem 1.7.

References

[1] J. A. Bondy, U. S. R. Murty, Graph Theory, GTM 244, Springer, 2008.

[2] Y. Caro, A. Lev, Y. Roditty, Z. Tuza, R. Yuster, On rainbow connection, Electron. J. Combin. 15(2008), R57.

[3] S. Chakraborty, E. Fischer, A. Matsliah, R. Yuster, Hardness and algorithms for rainbow connection, 26th International Symposium on Theoretical Aspects of Computer Science STACS 2009(2009), 243C254. Also, see J. Combin. Optim. 21 (2011) 330C347.

[4] G. Chartrand, G. L. Johns, K. A. McKeon, P. Zhang, Rainbow connection in graphs, Math. Bohem.133(1)(2008)85-98.