Symmetry-protected self-correcting quantum memories

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A self-correcting quantum memory can store and protect quantum information for a time that increases without bound with the system size, without the need for active error correction. We demonstrate that symmetry can lead to self-correction in 3D spin lattice models. In particular, we investigate codes given by 2D symmetry-enriched topological (SET) phases that appear naturally on the boundary of 3D symmetry protected topological (SPT) phases. We find that while conventional onsite symmetries are not sufficient to allow for self-correction in commuting Hamiltonian models of this form, a generalized type of symmetry known as a 1-form symmetry is enough to guarantee self-correction. We illustrate this fact with the 3D ‘cluster state’ model from the theory of quantum computing. This model is a self-correcting memory, where information is encoded in a 2D SET ordered phase on the boundary that is protected by the thermally stable SPT ordering of the bulk. We also investigate the gauge color code in this context. Finally, noting that a 1-form symmetry is a very strong constraint, we argue that topologically ordered systems can possess emergent 1-form symmetries, i.e., models where the symmetry appears naturally, without needing to be enforced externally.

I. INTRODUCTION

Quantum error correcting codes can be used to protect information in a noisy quantum computer. While most quantum codes require complex active error correction procedures to be performed at regular intervals, it is theoretically possible for a code to be self-correcting [1–3]. That is, the energetics of a self-correcting quantum memory can suppress errors for a time that increases without bound in the system size, without the need for active control. Such a memory is typically envisioned as a many-body spin system with a degenerate ground space. Quantum information can then be stored in its degenerate ground space for an arbitrarily long time provided that the system is large enough.

In seeking candidate models for self-correction, inspiration has been drawn from recent advances in our understanding of topologically ordered spin lattice models. The simplest example of a two-dimensional topologically ordered model is Kitaev’s toric code [4], one of the most studied and pursued quantum error correcting codes. With active error correction, the toric code has a lifetime that grows exponentially with the number of qubits. However it is not self-correcting, as without active error correction the lifetime of encoded information is constant in the number of qubits. On the other hand, the four-dimensional generalization of the toric code [1] provides a canonical example of a self-correcting quantum memory.

Encouraged by the capabilities of the 4D toric code, there has been a substantial effort to find self-correcting quantum memories that meet more physically realistic constraints and, in particular, exist in three or fewer spatial dimensions. A number of no-go results make this search very challenging [5–11]. While there has been considerable progress with proposals that attempt to circumvent these constraints in various ways [6, 10, 12–18], none have yet provided a complete answer to the problem.

Symmetry can provide new directions in the search for self-correcting quantum memories, as the landscape of ordered spin lattice models becomes even richer when one considers the interplay of symmetry and topology. If a global symmetry is imposed on a model, a system can develop new quantum phases under the protection of this symmetry. The properties that distinguish such symmetry-protected phases from more conventional phases persist only when these symmetries are not broken. This has led to new types of phases protected by symmetry, including symmetry-protected topological (SPT) phases [19–23] (phases with no intrinsic topological order) and symmetry enriched topological (SET) [24–37] phases (those including both intrinsic topological order and symmetry). These phases have found many applications in quantum computing [38–59].

In this paper, we show that such phases can support self-correcting quantum memories in three-dimensions, provided an appropriate symmetry is enforced. We argue that the generic presence of point-like excitations in commuting Hamiltonian models protected by an onsite symmetry precludes thermal stability (mirroring the instability of the 2D toric code), and so we are naturally led to consider higher-form symmetries. Models with higher form symmetries have excitations that are higher-dimensional objects, such as strings or mem-
branes, rather than point-like excitations that are typical in models with onsite symmetries. With such exotic excitations, we can seek models with the type of energetics believed to be needed for self-correction. Focussing on models with symmetries that are not spontaneously broken, we consider models that have an SPT ordered bulk. We then give two examples of 3D models that are self-correcting when a 1-form symmetry is enforced. The first example is based on the 3D ‘cluster state’ model of Raussendorf, Bravyi and Harrington (RBH) \cite{RBH}; this model with a 1-form symmetry has a bulk that remains SPT-ordered at non-zero temperature \cite{Niu10}. We show that a self-correcting quantum memory can be encoded in a 2D SET boundary of this 3D model, and is protected by the thermally-stable SPT ordering of the bulk. The second example is based on the 3D gauge color code \cite{GCS}, which is conjectured to be self-correcting; we show that a commuting variant of this model is self-correcting when subject to a 1-form symmetry.

Finally, we consider whether 1-form symmetries that lead to self-correction can be emergent, rather than enforced. The analogy here is to the charge-parity symmetry that emerges in anyonic models such as the toric code; such symmetries need not be externally enforced, as they are intrinsic to the model and stable under perturbations. We give evidence that the 1-form symmetry used in the 3D gauge color code example may be emergent, arising as a result of emergent charge-parity symmetries on topologically-ordered codimension-1 submanifolds of the 3D bulk. In the gauge color code, this symmetry is the ‘color flux conservation’ identified by Bombin \cite{Bombin09}.

The paper is structured as follows. In Sec. \ref{sec:background} we review self-correction and the conditions required for it, as well as phases of matter protected by symmetry. We analyse the effect of coupling symmetry-protected models to a thermal bath in Sec. \ref{sec:him} and argue that onsite symmetries are insufficient to offer thermal stability of a symmetry-protected phase. In Sec. \ref{sec:ua} we present our first example of a self-correcting quantum memory protected by a higher (1-form) symmetry: a thermally-stable 3D SPT-ordered model with a protected 2D SET-ordered boundary. A second example, based on the 3D gauge color code, is analyzed in Sec. \ref{sec:ic}. We discuss the possibility of such 1-form symmetries being emergent in 3D topological models in Sec. \ref{sec:e} based around the gauge color code. We discuss some implications of these results and open questions in Sec. \ref{sec:ou}.

\section{Background} \label{sec:background}

In this section we briefly review self correcting quantum memories, as well topological phases with symmetry.
number of terms, costing a constant amount of energy. This immediately rules out all 2D stabilizer codes \[5\], and 3D stabilizer Hamiltonians that have translationally invariant terms and a ground space degeneracy that is independent of system size (the so-called STS models of Yoshida \[8\]). Quantum codes in 3D that are free of string-like logicals have been investigated by Haah \[6\] \[12\] and Michnicki \[13\] \[14\], however they do not achieve a memory time that is unbounded (with the size of the system) for a fixed temperature.

One class of proposals seeks to couple a 2D topologically ordered model, such as the toric code, to a 3D theory with long range interactions with the goal of confining the anyonic excitations. For example, excitations in the toric code can be coupled to the modes of a 3D bosonic bath \[10\] \[15\] \[16\] to introduce such long range interactions. The idea is that anyon pair production can be suppressed and pairs can be confined such that logical errors cannot be created in the aforementioned way. The 2D topologically ordered theory provides a natural robustness to perturbations and local errors, while the confinement of the anyons (and diverging chemical potential) provided by the 3D bulk suppresses logical failures from thermal errors. A complication with the above approach is that the confining mechanism seems to be in conflict with perturbative stability of the boundary theory, as confining the boundary anyons may lead to a loss in topological order (the presence of deconfined quasiparticles is the hallmark of 2D topological order).

Finally, while the search for self-correcting quantum memories has primarily focussed on stabilizer codes, subsystem codes \[65\] \[66\] are a promising direction because many of the no-go theorems described above do not directly apply. Briefly, a subsystem code is a stabilizer code where some of the logical qubits are chosen not to be used for encoding, and instead are left as redundant gauge degrees of freedom. For the purposes of quantum memories, the use of subsystem codes and gauge qubits offers much more flexibility in selecting a Hamiltonian for the code, and the spectral requirements of the model for self-correction are potentially more relaxed. The 3D gauge color code \[61\] is an example of a topological subsystem code with a variety of remarkable properties, including a fault-tolerant universal set of gates via a technique known as gauge fixing, and the ability to perform error-correction with only a single round of measurements. This later property is known as single-shot error correction \[62\] and arises from a special type of confinement of errors during the measurement step. It is conjectured in Ref. \[61\] that the 3D gauge color code is self-correcting.

### 1. Thermalization and memory time

The central question for a candidate self-correcting quantum memory is how long the encoded information can undergo thermal evolution while still being recoverable. For a self-correcting quantum memory, this time should grow with the system size provided the temperature is sufficiently low. In this section, we briefly review thermalization and motivate the energy barrier as useful tool to diagnose the memory time.

The standard approach to modelling thermalization of a many body system is to couple the system to a thermal bosonic bath. Let \( H_{\text{sys}} \) be the Hamiltonian describing the quantum memory of interest, and let \( H_{\text{bath}} \) be a Hamiltonian for the bosonic bath. Thermalization is modelled by evolution under the following Hamiltonian

\[
H_{\text{full}} = H_{\text{sys}} + H_{\text{bath}} + \lambda \sum_{\alpha} S_\alpha \otimes B_\alpha, \tag{1}
\]

where \( S_\alpha \otimes B_\alpha \) describe the system-bath interactions, \( S_\alpha \) is a local operator acting on the system side, \( B_\alpha \) is an operator acting on the bath side, and \( \alpha \) is an arbitrary index. It is assumed that the coupling parameter is small, \(|\lambda| \ll 1\).

Suppose that the state is initialized in a ground state \( \rho(0) \) of \( H_{\text{sys}} \). Then after some time \( t \) we are left with a noisy state \( \rho(t) \). Since the coupling operators \( S_\alpha \otimes B_\alpha \) are local, errors are introduced to the system in a local way. Errors result in the ground space due to couplings with the bath, and in general the system will have to pass through a highly excited space before a logical fault is introduced. One can give a precise description of this process using a perturbation theory analysis, such as a master equation approach like the well-known Davies formalism \[67\] \[68\] which we review in Appendix A.

For a self-correcting quantum memory, we wish to be able to recover the state \( \rho(0) \) from \( \rho(t) \) after some time \( t \) using error correction. Error correction consists of two steps, firstly a sequence of measurements is performed on the noisy state \( \rho(t) \) to obtain an error syndrome, then a recovery map is performed that depends on the syndrome (the measurement outcomes). The net action of the syndrome measurement and recovery map can be condensed into a map \( \Phi_{\text{ec}} : \mathcal{H} \rightarrow \mathcal{H} \), where \( \mathcal{H} \) is the Hilbert space of the memory system. For a fixed error rate \( \epsilon \), we can define the memory time \( \tau_{\text{mem}} \) as the maximum \( t \) for which the inequality

\[
\| \Phi_{\text{ec}}(\rho(t)) - \rho(0) \|_1 \leq \epsilon \tag{2}
\]

is satisfied.

An upper bound to the memory time, is the mixing time \( \tau_{\text{mix}} \), which is the time taken for \( \rho(t) \) to be \( \epsilon \) close to
to the Gibbs state (for some fixed $\epsilon$). This bound holds since once the system has thermalized to the Gibbs state, the system retains no information about the initial state. Note that in some cases, such as the 3D toric code, the memory time can be substantially less than the mixing time \cite{63}. A useful first probe into the feasibility of a SCQM is the energy barrier, since a growing energy barrier is necessary in many cases to achieve self-correction. In the following subsection we define this quantity.

2. Energy barrier

If we cannot recover the logical information after some time $t$ then we say a logical fault has occurred. One way of producing a logical fault is if a sequence of errors from the system-bath coupling results in a logical operator (or an operator near to a logical operator). Due to the locality of the coupling between the system and bath (in Eq. (1)), errors are introduced to the memory in a local way. The energy penalty incurred during any such process that results in a logical fault is is called the energy barrier.

We first define a local decomposition of a logical operator. In this paper we restrict to stabilizer Hamiltonians, however the energy barrier can similarly be defined for any commuting projector Hamiltonian. Let $H_S = -\sum_i h_i$ be a stabilizer Hamiltonian (i.e., each local term is a Pauli operator, and all terms mutually commute), and $\mathcal{I}$ a Pauli logical operator. A local decomposition of $\mathcal{I}$ is a sequence of Pauli operators $D(\mathcal{I}) = \{I^{(k)} \mid k = 1, \ldots, N\}$ such that $I^{(1)} = I$ and $I^{(N)} = \mathcal{I}$, and $I^{(k)}$ and $I^{(k+1)}$ differ only by a local (constant range) operator.

For any ground state $|\psi_0\rangle$ of $H_S$, the state $I^{(k)}|\psi_0\rangle$ is also an eigenstate of $H_S$ for each $k$ with energy $E^{(k)}$. We can use this to define the energy barrier $\Delta$ for a logical fault. Namely, the energy barrier for the local decomposition $D(\mathcal{I})$ is defined as

$$\Delta_{D(\mathcal{I})} = \max_k (E^{(k)} - E_0),$$

where $E_0$ is the ground space energy. The energy barrier for a logical fault in $H_S$ is defined as

$$\Delta = \min_{D(\mathcal{I})} \Delta_{D(\mathcal{I})}.$$ 

In other words, the energy barrier for a logical fault is the smallest energy barrier of any logical operator, minimized over all local decompositions. Intuitively, the energy barrier should be large in order to suppress logical faults from occurring. Indeed for stabilizer Hamiltonians, an energy barrier that grows with the size of the system is a necessary condition (although not sufficient) for self-correction \cite{63,64}.

B. Topological phases with symmetry

In the classification of phases of matter in the absence of symmetry, two gapped many body Hamiltonians are said to belong to the same phase if they can be interpolated between by a family of many body Hamiltonians, without closing the energy gap between the ground space and lowest excited space. When symmetry is at play, the classification becomes richer, as the interpolating family must also respect the symmetry. In particular, it is possible that two Hamiltonians that are equivalent in the absence of symmetry, become inequivalent when there are symmetry constraints. This leads to the notion of SPT and SET phases, which we now briefly define (see Ref. \cite{20} for a detailed discussion).

Consider a lattice $\Lambda$ in $d$ dimensions with a $D$-dimensional spin placed at each site $i \in \Lambda$. We consider systems described by a gapped, local Hamiltonian $H = \sum_{X \subseteq \Lambda} H_X$. Here, ‘local’ means that each term $h_X$ is supported on a set of spins $X$ with bounded diameter. We also assume the system has a symmetry described by a group $G$ with a unitary representation $S$. We say two gapped Hamiltonians $H_0$ and $H_1$ with symmetry $S(g)$, $g \in G$ belong to the same phase if there exists a continuous path of gapped, local Hamiltonians $H(s)$ $s \in [0,1]$ with symmetry $S(g)$ such that $H(0) = H_0$ and $H(1) = H_1$.

For SPT and SET ordered systems, one commonly considers global symmetries $S(g)$ that act in an onsite fashion on the underlying degrees of freedom. The global action of these onsite symmetries $S(g)$ may be expressed as

$$S(g) = \bigotimes_{i \in \Lambda} u(g), \quad g \in G,$$

where $u(g)$ is a local, site-independent representation of $G$.

We will also consider a generalised class of global symmetries, known as higher-form symmetries, which have been recently of high interest in the condensed matter, high energy and quantum information communities \cite{52,56,69,72}. These higher-form symmetries form a family of increasingly stringent constraints that generalize the onsite case, and this will be central in the discussion of the interplay of symmetry and self-correction. We introduce these symmetries in Section II B 3 and for the present discussion and the definitions of SPT and SET phases, the symmetry $S(g)$ is left general.
1. Symmetry protected topological phases

An SPT phase with symmetry $S(g)$ is defined as an equivalence class of Hamiltonians in the same class as the trivial phase (a non-interacting spin model with a product ground state) in the absence of symmetry, but in the same class as a non-trivial phase in the presence of $S(g)$. Ground states of such models are short range entangled, meaning they can be mapped to a product state under a constant depth quantum circuit; however, such a circuit must break the symmetry. Key characteristics of such phases are the absence of anyonic excitations, and the absence of topology dependent ground space degeneracy. However, when defined on a lattice with boundary, these phases host protected modes localized on the boundary, meaning they can be mapped to a product state under a constant depth quantum circuit; however, such a circuit must break the symmetry. Key characteristics of such phases are the absence of anyonic excitations, and the absence of topology dependent ground space degeneracy. However, when defined on a lattice with boundary, these phases host protected modes localized on the boundary, meaning the boundary theory of an SPT phase must be either symmetry breaking, gapless, or topologically ordered (note that a topologically ordered boundary can only exist when the boundary has dimension $d \geq 2$). As such, these systems are typically regarded as having a trivial bulk, but exotic boundary theories. Some well known examples are the 1D cluster state and the spin-1 Haldane phase (with $\mathbb{Z}_2$ symmetry), both of which host degenerate boundary modes that transform as fractionalized versions of the symmetry. More generally the group cohomology models [19] provide a systematic way of constructing SPT ordered models.

2. Symmetry enriched topological phases

An SET phase with symmetry $S(g)$ is defined by a Hamiltonian that is distinct from the trivial phase, even without any symmetry constraint. These topological phases can form distinct equivalence classes under the symmetry $S(g)$, and are referred to as SET phases. The key characteristics of such phases are the presence of anyonic excitations, and topology-dependent ground space degeneracy. These anyons can carry fractional numbers of the symmetry group, or may even be permuted under the symmetry action. Such anyon permuting symmetries can be used to define symmetry defects on the lattice, which can be thought of as localized and immobile quasiparticles that transform anyonic excitations when they are mutually braided. Some well known examples of SETs are found in Refs. [24–28], and a general framework is given by the symmetry enriched string-nets of Refs. [29–30]. These SET phases fall into two categories. The first category consists of non-anomalous SET phases. These are standalone topological phases in $d$-dimensions with onsite symmetry $S(g)$ as in Eq. (5). Anyons may undergo transformations under the symmetry action $S(g)$. The second category consists of anomalous SET phases. These are $d$-dimensional topological phases with a symmetry action that cannot be realised in an onsite way on the degrees of freedom on the $d$-dimensional boundary. These anomalous phases appear only on the boundary of $(d+1)$-dimensional SPT phases.

It is conjectured that the topologically ordered boundary of an SPT phase with bulk onsite symmetry must always be anomalous. In particular, a wide class of 3-dimensional SPT phases can be classified by the group cohomology models [19], which are labelled by elements of the cohomology group $H^4(G, U(1))$. (See Refs. [73–76] for examples of models outside this classification.) Moreover, in 2 dimensions, anyonic systems with discrete unitary symmetry $G$ (that does not permute the anyons) also have a label in $H^4(G, U(1))$ that classifies the anomalies [77] (see also [26]). The case $\omega = 1$ (i.e., trivial) means that there is no anomaly, and $\omega \neq 1$ means the system is anomalous and cannot be realised in 2-dimensions in a standalone way with onsite symmetries [31–36]. A conjecture of Ref. [37] is that the gapped boundary topological theory of a group cohomology model must always have an anyon $\omega \in H^4(G, U(1))$ that agrees with the label specifying the bulk SPT order. This kind of bulk-boundary correspondence was proved in Ref. [78] in the case that the symmetry group $G$ is abelian and does not permute the boundary anyons. Moreover, in Ref. [78], a general procedure to extract a boundary anomaly label from a bulk SPT has been given, in agreement with the conjecture.

3. Higher form symmetries

We will make use of a family of symmetries called higher-form symmetries [52, 53, 69, 72], generalizing the onsite case. These symmetries have been of recent interest for several reasons, in particular, they provide a useful structure for error correction in quantum computation [56], have been used to construct new phases of matter [52], and to understand topological phases from the symmetry breaking paradigm [69, 72].

A $q$-form symmetry (for some $g \in \{0, 1, ..., D−1\}$) is given by a symmetry operator associated with every closed codimension-$q$ submanifold of the lattice; these operators are written as $S_M(g)$ where $M$ is a closed codimension-$q$ submanifold of $\Lambda$ and $g \in G$. On these codimension-$q$ submanifolds, the action of the symmetry operators takes an onsite form: for $g \in G$ and a codimension-$q$ submanifold $M$, the symmetry operator is

$$S_M(g) = \prod_{i \in M} u(g), \quad g \in G$$ (6)
where the product runs over all sites $i$ of the submanifold $\mathcal{M}$, and $u(g)$ is a local, site-independent representation of $G$. That is, higher form symmetries can be thought of as being onsite symmetries on lower dimensional submanifolds. For systems with boundary, the submanifolds that the higher form symmetries are supported need only be closed relative the boundary of the lattice. In other words, the manifold $\mathcal{M}$ on which the symmetry is supported may have a boundary on the boundary of the lattice $\Lambda$, i.e. $\partial\mathcal{M} \subset \partial\Lambda$.

Of particular interest will be when the symmetry operators $S_M(g)$ can be given by terms in the Hamiltonian

$$H = \sum_{X \subset \Lambda} h_X$$

where the product runs over a subset of terms contained within $\mathcal{M}$. As in this case, when the Hamiltonian $H$ is a frustration free and commuting, the symmetry cannot be spontaneously broken.

A key feature of these systems is that symmetric excitations in systems with $q$-form symmetries must form $q$-dimensional objects. Of particular interest in this paper will be 1-form symmetries in 3-dimensional systems, where the symmetry operators are supported on closed 2-dimensional surfaces and excitations form closed 1-dimensional loop-like objects.

4. Dimensionality of excitations and self-correction

We conclude our background section with a comment regarding the crucial role of the dimensionality of excitations in the feasibility of self-correction. The conventional wisdom is that deconfined point-like excitations are an obstruction to self-correction, as harmful errors can be introduced with a low energy cost due to excitations that are free to propagate. For models with higher dimensional excitations, the energy cost to growing and moving these excitations can be large, such that logical errors are suppressed.

The importance of the form of the symmetry in the consideration of self-correction then becomes apparent. As we will see in the next section, in the case of an onsite symmetry, excitations remain point-like objects that are free to propagate, and therefore there is no extra stability afforded by the onsite symmetry. This motivates the consideration of 1-form symmetries, the next weakest generalization of the onsite symmetry (in the family of higher-form symmetries) and its potential for self-correction. In Sections [ ] and [ ] we will look at two examples of self-correcting quantum memories protected by $\mathbb{Z}_2^2$ 1-form symmetries.

III. SYMMETRY CONSTRAINTS AND QUANTUM MEMORIES

In this section, we consider what types of symmetric models may be worth investigating as potential self-correcting quantum memories.

A. No spontaneous symmetry breaking

We first briefly note that we only consider enforcing symmetries $S(g)$ in models where this symmetry is not spontaneously broken. If a model is described by spontaneous symmetry breaking then the ground space has less symmetry than the Hamiltonian, and this can render the model trivial as a memory by disallowing logical operator actions at all. That is because different ground states will in general be in different eigenspaces of the symmetry operator, and thus enforcing the symmetry would be prohibit transitions between ground states. In the case that the spontaneously broken symmetry is higher-form, then enforcing it could potentially removing anyonic excitations from the model. (The 3D toric code provides an illustrative example, where one can trivially obtain a self-correcting quantum memory by enforcing a $\mathbb{Z}_2$ 1-form symmetry that prevents any of the “vertex terms” from flipping.) For this reason, we only consider models where the symmetry is not spontaneously broken, and SPT ordered systems provide a natural family of candidates.

B. Onsite symmetries are insufficient for stability

In this section we argue that onsite symmetries are insufficient to promote a 2D topological quantum memory to be self-correcting, even if such a phase lives on the boundary of a 3D SPT model. Our goal here is simply to motivate moving beyond onsite symmetries (to higher-form symmetries), not to rigorously rule out any role for onsite symmetries in the study of self-correction.

In particular, consider the case where the full system is given by a commuting Hamiltonian with boundary, and that the protecting symmetry is abelian and onsite (with possibly an anomalous boundary action). The excitations in such systems will be point-like, and their presence precludes the possibility of having thermally stable (symmetry-protected) topological order, as shown in Ref. [56]. This suggests that the boundary theory is also not thermally stable, and thus not self-correcting. Indeed, as we show in Appendix [ ] this is the case for the
class of models where the boundary is an abelian twisted quantum double with a potentially anomalous boundary symmetry. Specifically, we show that there is a constant (symmetric) energy barrier in this case. Therefore we see that in the case of onsite (0-form) symmetries, the SPT ordered bulk offers no additional stability to the boundary theory. This motivates us to consider the boundaries of SPTs protected by 1-form (or other higher-form) symmetries.

C. System-bath coupling with symmetry and the symmetric energy barrier

Consider the system bath coupling of Eq. (1) and a symmetry \( S(g) \) (with \( g \in G \) for some group \( G \)). If

\[
[H_{\text{full}}, S(g)] = 0,
\]

then all of the errors that are introduced due to interactions with the bath must be from processes that conserve \( S(g) \). In particular, only excitations that can be created by symmetric thermal errors will be allowed and the symmetry is preserved throughout the dynamics.

Under symmetric dynamics, we should only consider local decompositions of logical operators that commute with the symmetry when defining the energy barrier \( \Delta \). If a local decomposition \( D(\tilde{l}) = \{l^{(k)} | k = 1, \ldots, N\} \) of a logical operator \( \tilde{l} \) is such that \([l^{(k)}, S(g)] = 0\) for all \( k \) and all \( g \in G \), then we call \( D(\tilde{l}) \) a symmetric local decomposition of \( \tilde{l} \). We label such symmetric local decompositions with symmetry \( G \) by \( D_G(\tilde{l}) \). Then the symmetric energy barrier is defined as

\[
\Delta_G = \min_{\tilde{l}, D_G(\tilde{l})} \Delta_{D_G(\tilde{l})}.
\]

Namely, it consists of the smallest energy barrier for any logical operator, where the cost is minimized over all symmetric local decompositions. For notational simplicity, we often omit the subscript \( G \) as the symmetry is clear from context.

With the abundance of no-go results for self-correction in 2D and 3D stabilizer memories, the relevant question is whether one can achieve self-correction if the system bath coupling respects a symmetry. In particular, for a given model \( H_S \), what symmetry \( S(g) \) can be imposed such that \( H_{\text{sys}} \) has a macroscopic symmetric energy barrier.

IV. SELF CORRECTION WITH A 1-FORM SPT PHASE

Our simplest example of a 3D self-correcting model in the presence of a 1-form symmetry is the commuting Hamiltonian model due to Raussendorf, Bravyi, and Harrington (RBH) \[60\]. This model has been used in high-threshold schemes for fault tolerant quantum computation \[60\] \[79\] \[80\]. In particular, the RBH model underpins the topological formulation of measurement based quantum computation, where single qubit measurements are used to simulate the braiding of punctures in the 2D toric code.

The RBH model is an example of an SPT ordered system under 1-form symmetry, which is thermally stable \[50\]. It contains no anyonic excitations in the bulk, however when defined on a lattice with a boundary, the boundary theory possesses point-like anyonic excitations. In particular, on a lattice with boundary, each 2D boundary component supports a 2D surface code phase. Without any symmetry, the excitations of this 2D surface code phase are deconfined, and information encoded in this surface phase will thermalize in constant time in the absence of error-correction. However, in the presence of symmetry, a natural question is whether the boundary code inherits any protection from the bulk SPT order. We will show that in the presence of 1-form symmetry, the bulk SPT order gives rise to confinement of boundary excitations and ultimately a macroscopic lifetime of boundary information. As such, this model provides a simple example of a self-correcting SET phase on the boundary of a 3D higher-form SPT. We now define the model and its properties.

A. The RBH model

The model is defined on a 3D cubic lattice \( L \) with boundary \( \partial L \). For concreteness, we choose convenient boundary conditions such that \( L \) has the topology of the solid torus \( D^2 \times S^1 \), where \( D^2 \) is a disk and \( S^1 \) is a circle. The boundary of this lattice is a torus and we choose ‘smooth’ boundary conditions (see Fig. 1). Label the set of all vertices, edges, faces and volumes by \( V, E, F, Q \), respectively, and also label by \( V^0, E^0, F^0, Q^0 \) those vertices, edges, faces and cubes that are on the interior of \( L \) (i.e., away from the boundary). To every interior face \( f \in F^0 \) we place a qubit, and to every edge \( e \in E \) (including boundary edges) we place a qubit. We refer to qubits on faces as primal qubits, and qubits on edges as dual qubits.

The Hamiltonian describing the model we are interested in is a sum of commuting stabilizer terms, and we
break them up into bulk and boundary terms

\[ H = H_{L^0} + H_{\partial L}. \]  

Here, \( H_{L^0} \) is the bulk Hamiltonian and \( H_{\partial L} \) is the boundary Hamiltonian. The bulk Hamiltonian is a sum of cluster terms

\[ H_{L^0} = \sum_{f \in F^0} K_f - \sum_{e \in E^0} K_e, \]  

where each cluster term is a 5-body operator

\[ K_f = X_f \prod_{e \in c \subset f} Z_e, \quad K_e = X_e \prod_{f \in c \subset f} Z_f, \]  

and \( X_e \) and \( Z_e \) are the usual Pauli-X and Pauli-Z operators acting on the qubit \( v \). The boundary Hamiltonian is a toric code Hamiltonian build out of the ‘cluster boundary degrees of freedom’ (this is formalized in Eq. (28) of the following section)

\[ H_{\partial L} = - \sum_{e \in \partial V} A_v - \sum_{f \in \partial F} B_f, \]  

where \( \partial V \) and \( \partial F \) are the set of all boundary vertices and faces (respectively), and

\[ A_v = \prod_{e \in \partial V : e \subset v} X_e \prod_{f : v \subset f} Z_f, \quad B_f = \prod_{e \in c \subset f} Z_e, \]

where \( \partial E \) is the set of all boundary edges. These terms are depicted in Fig. 2. Note that the boundary we choose here is slightly different to that chosen in [79, 80] for topological measurement-based quantum computation.

We note that all of these terms can be considered ‘dressed’ terms of a simpler model with a trivial bulk. Indeed let us define what we refer to as the “trivial model” \( H^{(0)} \) in a similar way

\[ H^{(0)} = H_{L^0}^{(0)} + H_{\partial L}^{(0)}. \]  

Here, \( H_{L^0}^{(0)} \) describes the trivial paramagnet in the bulk

\[ H_{L^0}^{(0)} = \sum_{i \in E \cup F} X_i, \]  

and \( H_{\partial L}^{(0)} \) is the usual 2D toric code on the boundary

\[ H_{\partial L}^{(0)} = - \sum_{e \in \partial L} A_v - \sum_{f \in \partial F} B_f, \]  

where

\[ A_v = \prod_{e \in \partial V : e \subset v} X_e \quad B_f = \prod_{e \in c \subset f} Z_e. \]

Now one can see that these two models are equivalent to each other up to a constant depth circuit

\[ H = U H^{(0)} U^\dagger, \]  

where \( U \) is a product of controlled-Z gates that act on all pairs of neighbouring qubits at sites \( i \) and \( j \) by

\[ CZ_{ij} = \exp(\pi i/4(1 - Z_i)(1 - Z_j)) \]

Indeed, let a face \( f \) and an edge \( e \) be referred to as neighbours if the edge is contained within the face \( e \subset f \). Then \( U \) is a product of controlled-Z gates over all neighbouring sites

\[ U = \prod_{f \in F^0} \prod_{e \subset f} CZ_{fe}. \]  

This subtle difference between models is crucial to demonstrate self-correction, as we will shortly describe. In order to do so, let us first consider the ground space
degeneracy and the 1-form symmetries of the model.

B. Ground space degeneracy

On the solid torus, the ground space of $H$ is 4-fold degenerate, which arises from the boundary dressed surface code. This is because the bulk is nondegenerate while the boundary toric code has a degeneracy $d_B = 2^{2g}$ where $g$ is the genus of the 2D manifold it is defined on ($g = 1$ for the torus), which for example, can be seen more easily in the trivial Hamiltonian $H^{(0)}$. Using the boundary as a code to encode quantum information, we can find Pauli logical operators that wrap around the cycles $a$ and $b$ depicted in Fig. 3. We say a collection of edges is a dual cycle if it corresponds to a cycle on the dual lattice. Similarly, for any string $l$, we write $t^l = \{ f \in F^o : \partial f \cap l \neq \emptyset \}$ to denote the set of faces sitting just inside the boundary incident to the string $l$. Then we can write the Pauli logical operators as

$$X_1 = \prod_{e \in a_d} X_e \prod_{f \in a_d^\perp} Z_f, \quad Z_1 = \prod_{e \in b_p} Z_e, \quad (22)$$

and

$$X_2 = \prod_{e \in b_d} X_e \prod_{f \in b_d^\perp} Z_f, \quad Z_2 = \prod_{e \in a_p} Z_e, \quad (23)$$

for cycles $a_p$, $b_p$ and dual-cycles $a_d$, $b_d$ wrapping around the $a$ and $b$ loops of the lattice. These operators form canonical anti-commuting pairs and are depicted in Fig. IVB.

![Fig. 3](image-url)

**FIG. 3.** (a) The logical operators $Z_1$ and $X_1$, the operators $\bar{X}$ refer to Pauli $X$ on the edge, with Pauli $Z$ on the adjacent face in the bulk. (b) The $e$ and $m$ type excitations on the boundary of string and dual-string operators.

C. 1-form symmetries

The model $H$ has a $\mathbb{Z}_2^2$ 1-form symmetry, consisting of operators supported on 2-dimensional closed surfaces on each of the primal and dual sublattices. In particular, a generating set are given by vertex and cube operators (for primal and dual qubits, respectively), for each $q \in Q$ and $v \in V$

$$S_q = \prod_{f : f \subset q} X_f, \quad S_v = \prod_{e : e \subset v} X_e. \quad (24)$$

Note that for interior cubes $q \in Q^i$ and vertices $v \in V^o$, the symmetry operators $S_q$ and $S_v$ are 6-body operators, while those on the boundary are 5-body. Taking products of these operators gives rise to the $\mathbb{Z}_2^2$ 1-form symmetry

$$G = \langle S_v, S_q \mid v \in V, q \in Q \rangle. \quad (25)$$

One can easily check that these operators commute with both $H$ and $H^{(0)}$. Indeed it has been shown that under these symmetries the bulk model $H_{BC}$ belongs to a nontrivial SPT phase while the trivial bulk $H_{E^0}$ belongs to the trivial phase. Moreover, this distinction persists to nonzero temperature, where $H$ remains SPT ordered. We can now also remark that the choice of boundary model $H_{BC}$ was not accidental, indeed, with cluster model in the bulk, any choice of symmetric boundary Hamiltonian must also commute with the dressed toric code Hamiltonian. In section IVB we show that boundary stability (i.e., self-correction) of $H$ is intimately related to bulk SPT order, that an SPT ordered bulk is necessary to have a self-correcting toric code boundary.

D. Excitations

1. Excitations without the symmetry

Let us now consider the excitations in the model in the absence of any symmetry considerations. In the bulk, we can create all excitations by products of Pauli-Z operators. Indeed, for any subset of edges $E' \subset E^o$ or subset of faces $F' \subset F^o$, the operator

$$Z(E', F') = \prod_{f \in F'} Z_f \prod_{e \in E'} Z_e \quad (26)$$

anti-commutes with precisely the cluster terms $K_e$ and $K_f$ for which $e \in E'$ and $f \in F'$, and commutes with all remaining bulk and boundary terms. Therefore, $Z(E', F')$ creates excitations at all sites in $E' \cup F'$, and the energy cost is given by $|E' \cup F'| \Delta_{\text{gap}}$, where $\Delta_{\text{gap}} = 2$ is the energy gap.

The boundary is more interesting, as there are anyonic excitations that are free to propagate. Indeed, for a string $l$ on the boundary, we can define the string operator $Z(l) = \prod_{e \in l} Z_e$ which creates $e$-excitations on the
boundary $\partial l$ of the string. These $e$-excitations occur on
the vertices in the boundary of $l$, as the string operator $Z(l)$ anti-commutes with vertex terms $\mathcal{A}_v$ with $v \in \partial l$, and commutes with all the remaining terms. Similarly, using a dual string operator $X(l') = \prod_{e \in l'} X_e \prod_{f \in V} Z_f$ for a string $l'$ we can create $m$-excitations on the faces $f$ at the ends of $l'$. The $m$-excitations occur on the ends of the string operator $X(l')$, as the plaquette operators $\mathcal{B}_f$ with $f$ on the ends of $l'$ anti-commute with the dual string operator, while all remaining terms commute. Examples of such operators are depicted in Fig. 3.

Note that since the boundary toric code itself has no boundary, we have the following constraint

$$\prod_{v \in \partial V} \mathcal{A}_v = \prod_{f \in \partial F} \mathcal{B}_f = I$$

(27)

means that $e$ and $m$ particles are always created in pairs, which is often referred to as a global $\mathbb{Z}_2$ conservation law for the parity of anyonic charge.

To create a logical error, we can simply create a pair of $e$ or $m$ particles, and propagate them along one of the cycles in Fig. 1, then annihilate them back to vacuum. Such an operation implements a logical operation, and can be implemented via a sequence of local operators at an energy cost of only $2\Delta_{\text{gap}}$. Therefore the model is not self-correcting, as any information encoded in the boundary toric code degree of freedom will thermalize in constant time, as has been rigorously shown in Refs. [63, 64]. This is perhaps not surprising as the model is equivalent (up to a constant depth circuit) to a 2D toric code with a non-interacting bulk.

2. Excitations with the symmetry

We now consider what stability the 1-form symmetry $G$ offers. If the system is coupled to a thermal bath in a symmetry respecting way, such that the environment can only implement symmetric and local operations, then we must consider excitations constrained to the symmetric sector. If we consider bulk excitations, then the excitation operator $Z(E', F')$ of Eq. (26) is symmetric if and only if $E'$ is a cycle (i.e., it has no boundary) and $F'$ is dual to a cycle on the dual lattice (where vertices are replaced with cubes, edges with faces, and so on). This means the only symmetric excitations in the bulk are formed by combinations of closed loop-like (i.e., 1-dimensional) objects.

On the boundary, we see that boundary excitations are symmetric only if they are accompanied by a bulk string excitation. In particular, a string operator $Z(l)$ creating $e$ particles at vertices $\mu$ and $\nu$ is made symmetric by attaching a bulk string operator $Z(E')$ whose boundary is at the location of the two particles $\partial E' = \{\mu, \nu\}$ (i.e., $l \cup E'$ is a cycle). Similarly, the dual string operator $X(l')$ that creates $m$ excitations at $\mu'$ and $\nu'$ can be made symmetric by attaching a bulk string operator $Z(F')$ such that the union $l'^{\perp} \cup F'$ is a dual cycle (i.e., has no boundary on the dual lattice). Such excitations will flip cluster stabilizers in the bulk, for all terms $K_l$ with $e \in E'$ and $K_f$ with $f \in F'$, but will only create a pair of $e$ or $m$ particles on the boundary at their endpoint.

3. Energy barrier in the 1-form symmetric model

When the dynamics are restricted to the 1-form symmetric sector, there are two key observations. Firstly we note that bulk excitations must form collections of closed loop-like objects. Secondly, we note that boundary excitations only appear at the boundary of a bulk string-like excitation. Therefore, in the presence of symmetry, the thermal properties of the boundary are no longer decoupled from the bulk. Indeed, this is enough to achieve a diverging energy barrier.

Consider now the symmetric energy barrier, defined in Eq. (9). Let \{l^{(k)} | k = 1, \ldots, N\} be any sequence of operators such that each $l^{(k)}$ is symmetric, $l^{(k)}$ and $l^{(k+1)}$ differ only locally, $l^{(1)} = I$ and $l^{(N)}$ is a logical operator supported on either the $a$ or $b$ loops. We decompose each $l^{(k)}$ into its $Z$ support and $X$ support by writing $l^{(k)} = l^{(k)} Z(c^{(k)})$, where $Z(c^{(k)})$ must be a collection of closed loops (relative boundary) of $Z$ operators, to satisfy the symmetry condition. Since each $l^{(k)}$ differs only by a local operation, we have that $Z(c^{(k)}) Z(c^{(k+1)})$ must be a trivial cycle, and the final loop $Z(c^{(N)})$ must have a length proportional to either the size of the $a$ cycle or the $b$ cycle.

For any pair of boundary anyons at points $a$ and $b$, let $d_{(a,b)}$ be the length of the shortest boundary path between $a$ and $b$. For a general configuration of boundary anyons, we label the positions of the anyons by $C$. Define $d_C = \min_P \sum_{(a,b) \in P} d_{(a,b)}$, where the minimisation is over all pairings $P$ that partition the elements of $C$ into pairs. (Recall, there are always an even number of anyons on the boundary.) Since at each timestep the separation between anyons can only increase by a constant amount, and that the total distance travelled by anyons is lower bounded by the smallest length of the $a$ or $b$ cycles, we can conclude that there is a timestep $k \in \{1, \ldots, N\}$ with a configuration of anyons given by $C_k$, such that $d_{C_k} \equiv \min\{\lfloor a/2\rfloor, \lfloor b/2\rfloor\} - r$, where $r$ is the largest range of the operator $l^{(j)} l^{(j+1)}$, which is assumed to be constant. Since at most at most half of the loop $Z(c^{(k)})$ can be supported on the boundary to give the configuration $C_k$, and that $Z(c^{(k)})$ loops in the bulk
flip cluster terms supported on them, we have the energy cost of the configuration of $C_k$ is lower bounded by $d_{C_k}\Delta_{\text{gap}} + |C_k|\Delta_{\text{gap}}$. Since $d_{C_k} \geq \min\{|a/2|, |b/2|\} - r$ at at least one point, we have the energy barrier is lower bounded by half of the smallest size of the $a$ or $b$ cycles, which grows with the system size.

4. Self-correction

We have shown that the 1-form symmetric RBH model inherits a macroscopic energy barrier to a logical fault, due to the string-like nature of excitations resulting from the 1-form symmetry together with its coupling of bulk and boundary excitations. The question is whether this is sufficient for an unbounded memory time. In Appendix C, we give an argument following the well-known Peierls argument (see also Ref. [1]) to show that this energy barrier implies self-correction of the 1-form symmetric RBH model. In brief, we estimate the probability that an excitation loop $l$ of size $w$ emerges within the Gibbs ensemble at inverse temperature $\beta$. We show that large loop errors are quite rare if the temperature is below a critical temperature $T_c$, and we give an lower bound on $T_c$ at $2/\log(5)$. As such, if the error rate is small enough (that is, the temperature is low enough), then the logical information in the code is stable against thermal logical errors and the encoded information on the boundary will be protected for a long time.

We additionally remark that the ground space of this system is perturbatively stable [81], and as a code it admits an efficient decoder [60]. Therefore this model meets the requirements for a self-correcting quantum memory when protected by the 1-form symmetry.

E. Bulk boundary correspondence at nonzero temperature

As shown above, the 1-form symmetries constrain the form of the excitations in the model and give rise to an energy barrier, and self-correction. These 1-form symmetries are a very strong constraint, and one may ask if a code is trivially guaranteed to be self-correcting whenever such symmetries are enforced. (As a example of a strong symmetry leading trivially to self-correction, consider the toric code where the symmetry of the full stabilizer group is strictly enforced.)

In this section we show that the 1-form symmetry, although strong, is itself not sufficient to lead to self-correction on its own. Specifically, we show that self-correction under 1-form symmetries depends on the bulk SPT order of the model, establishing a bulk-boundary correspondence for SPTs at nonzero temperature. Recall, at zero temperature, the correspondence is that a system with nontrivial SPT order in the bulk must have a protected boundary theory – meaning it is gapless or topologically ordered – whenever the symmetry is not broken [82, 83]. Here we show that the bulk boundary correspondence holds at nonzero temperature in the RBH model; that the stability of the boundary toric code phase (i.e., whether or not we have a SPQM) depends on the bulk SPT order at nonzero temperature.

In order to make this connection, we recall a formulation of phase equivalence due to Chen et al. [20]. Namely, two systems belong to the same phase if they can be related by a local unitary transformation (a constant depth quantum circuit), up to the addition or removal of ancillas. Importantly, with symmetries $S(g)$ present, the local unitary transformations must commute with the symmetry and the ancillas that are added or removed must be in a symmetric state.

We now remark on the earlier claim on the necessity of the SPT nontriviality of the bulk to achieve self-correction. To do so, we first note that the symmetric energy barrier is invariant under symmetric local unitaries (that is, it is a phase invariant). Indeed consider two Hamiltonians $H_A$ and $H_B$ (defining quantum memories) in the same phase. Then in particular, we have $H_A + H_B$ and $H_B$ are related by a symmetric local unitary $U$, where let $H_A$ consists of a sum of local projections on the ancillas $\mathcal{A}$ into a symmetric state. Since $H_A$ and $H_A + H_B$ differ only by a sum of non-interacting terms on the ancilla, they have the same energy barrier. Let $\mathcal{X}$ be a logical operator for $H_A$, and consider a local decomposition $\{I^{(k)}_X \mid k = 1, \ldots, N\}$ of $\mathcal{X}$ (recall $I^{(1)}_X = I$ and $I^{(N)}_X = \mathcal{X}$, and $I^{(k)}_X$ and $I^{(k+1)}_X$ differ only by a local operator). This is also a logical decomposition for $H_A + H_A$. Then $\{U I^{(k)}_X U^\dagger \mid k = 1, \ldots, N\}$ constitutes a local decomposition for a logical operator of $H_B$, with the same energy barrier. This works for all choices of logical operators $\mathcal{X}$ and the models have the same symmetric energy barrier.

The invariance of the energy barrier requires us to consider a SPT-nontrivial bulk to achieve self-correction in the presence of 1-form symmetries. Indeed, if we instead considered the SPT-trivial model $H^{(0)}$ in the presence of 1-form symmetries, we see that there is no energy barrier, in the following way. Consider one of the logical operators $\mathcal{X}$ that wraps around either the $a$ or $b$ loops of Fig. 1. Such an operator is a product of Pauli $X$ operators supported on a dual cycle on $\partial \mathcal{L}$ (it is not dressed like that of the RBH model $H$). Then the symmetric energy barrier for this error is a constant $2\Delta$, since the process of creating two $m$ particles and wrapping them around a boundary cycle is symmetric, and only flips two $B_f$ pla-
The boundary degrees of freedom. However, we will see that bulk 1-form symmetry induces a 1-form symmetry on these boundary degrees of freedom, we note that the degrees of freedom are given by the product of effective Pauli-operators at any given time. Therefore the trivial model is not self-correcting, even in the presence of 1-form symmetries. In particular, this also gives a simple argument for why $H$ belongs to a distinct SPT phase to $H^{(0)}$.

This bulk boundary correspondence (at nonzero temperature) holds for systems with onsite symmetries too; we have argued in Section III B that self-correction was not possible on the 2D boundary of a 3D SPT protected by onsite symmetry. This coincides with with the lack of bulk SPT order at $T>0$ when the protecting symmetry is onsite, as shown in Ref. [56].

1. Anomalies

Finally, we return to the connection between higher-form anomalies and stability of the boundary theory. Recall that anomalies arise when considering a system with a boundary and analysing the action of the symmetry on boundary degrees of freedom. Let us first clarify what we mean by a higher form anomaly, by examining the 1-form case in 3-dimensions. Consider the bulk RBH Hamiltonian of Eq. (26), which is symmetric under 1-form symmetries $S_q$, $S_v$, $q \in Q$, $v \in V$. In this case, there is an extensive degeneracy due to degrees of freedom localised on the boundary, with one effective qubit degree of freedom per boundary edges $e \in E \setminus E^o$. Effective Pauli-X and Pauli-Z operators for these boundary degrees of freedom are given by

$$\tilde{X}_e := UX_e U^\dagger = X_e Z_{f(e)}, \quad \tilde{Z}_e := U Z_e U^\dagger = Z_e$$

(28)

where $f(e)$ is the unique face $f \in F^o$ such that $e \subset f$ and $U$ is a product of $CZ$ gates as in Eq (20).

Now we analyse the action of the 1-form symmetry on these boundary degrees of freedom, we note that the bulk 1-form symmetry induces a 1-form symmetry on the boundary degrees of freedom. However, we will see that it cannot be strictly realised on the boundary $E \setminus E^o$. Indeed, from the commutation relations with 1-form symmetries, we have

$$S_v = \bar{A}_v, \quad S_q = \bar{B}_{f(q)}, \quad \forall \quad v \in V \cap \partial \mathcal{L}, \quad q \in Q \cap \partial \mathcal{L}$$

(29)

and act as the identity otherwise. Here $f(q)$ is the unique face $f(q) = \partial q \cap \partial \mathcal{L}$, and $\bar{A}_v$ and $\bar{B}_f$ are defined in Eq. (14). This boundary action is not strictly contained within $E \setminus E^o$. Indeed, there is no way to reduce the boundary action into a form that is contained entirely within the boundary, meaning that the boundary action is anomalous. We remark that the origin of the Hamiltonian of Eq. (10), is that the boundary is chosen as the minimal Hamiltonian that respects the symmetry.

Without the 1-form anomaly, there are no terms coupling the bulk and boundary, and one can choose the boundary theory to be a completely decoupled 2D theory, as in the example of $H^{(0)}$. In such a theory, one can find a logical operator that has a symmetric local decomposition with constant energy cost, meaning the anomaly is necessary to have a self-correcting boundary. Such anomalies should only occur when we have a SPT ordered bulk.

V. THE GAUGE COLOR CODE PROTECTED BY 1-FORM SYMMETRY

We now turn to a model based on the gauge color code in 3 dimensions as our second example of a symmetry-protected self-correcting quantum memory. The gauge color code [61] is an example of a topological subsystem code. Subsystem codes contain a gauge group in addition to the stabilizer group, which introduces an equivalence between encoded states. Namely, gauge transformations do not alter the encoded information; they may change the gauge degrees of freedom, but these redundant degrees of freedom do not store any logical information.

In this section we study a commuting Hamiltonian model with a 1-form symmetry based on the gauge color code. This model provides another example of a self-correcting quantum memory protected by a 1-form symmetry.

A. The lattice

The gauge color codes we consider are defined on a family of 3-dimensional lattices known as tetrahedral 3-colexes [84]. Such lattices are cellulations of the 3-ball that satisfy certain combinatorial properties. In particular, a tetrahedral 3-colex $C_3$ is a result of gluing together 3-cells (polyhedra) to form a tetrahedron such that each vertex is 4-valent (meaning each vertex belongs to 4 edges) and 4-colorable (meaning each polyhedral 3-cell can be given one of four colors such that neighbouring 3-cells are differently colored). Let these four colors be labelled $r$, $b$, $g$, and $y$ (for red, blue, green, and yellow). Each edge can be given a single color label, where the color is determined by that of the two 3-cells that it connects. Similarly, each face $f \in C_3$ can therefore be labelled by pairs of colors $uv \equiv vu$, inherited from the two neighbouring 3-cells that it belongs to. Namely, each face is colored by the complement of the two colors on the 3-cells the face is incident to (e.g., a face belonging to a $r$ and a $b$ 3-cell is colored $gy$).

We note that, similar to the RBH model, the gauge
color code must have boundaries in order to possess a nontrivial code space. For concreteness, we consider the tetrahedral boundary conditions of Ref. [5], but one could also consider more general boundary conditions, such as the solid torus of the previous section. The boundary of the tetrahedral 3-colex consists of four facets, with the requirement: for each boundary facet, only edges of one color can terminate on the boundary and this color is unique for each facet. Here, an edge terminating on the boundary means that precisely one of its vertices belongs to the boundary. We therefore color each boundary facet by the color of the edges that terminate on it. Equivalently, a boundary of color \( k \) consists of all plaquettes of color \( uv \) such that \( u, v \neq k \). We arbitrarily choose one of the boundary facets, the \( b \) facet, and call this the outer colex \( \mathcal{C}_{\text{out}} \), which consists of the vertices, edges and plaquettes strictly contained on the boundary. This outer colex is therefore a 2-colex (a trivalent and 3-colorable two-dimensional lattice), and can be used to define a 2-dimensional color code. The remainder of the lattice \( \mathcal{C}_3 \setminus \mathcal{C}_{\text{out}} \) is called the inner colex.

On the outer colex, each plaquette has one of three possible color pairs \{gy, ry, rg\}, which we relabel for simplicity according to \( \text{gy} \leftrightarrow \text{A} \), \( \text{ry} \leftrightarrow \text{B} \), \( \text{rg} \leftrightarrow \text{C} \) as in Fig. 4. Each edge of the outer colex neighbours two plaquettes of distinct colors, we color each edge the third remaining color. Moreover, each of the three boundaries of the outer colex can be given a single color according to what color edges can terminate on them, as depicted in Fig. 4.

**FIG. 4.** (a) The tetrahedral 3-colex. (b) The \( b \) boundary of the tetrahedral lattice consists of faces that are colored \( uv \) with \( u, v \neq b \), which are then relabelled according to \( \text{gy} \leftrightarrow \text{A} \), \( \text{ry} \leftrightarrow \text{B} \), and \( \text{rg} \leftrightarrow \text{C} \).

### B. The 3D gauge color code

To each vertex of the lattice \( \mathcal{C}_3 \) we place a qubit. The gauge color code is specified by the gauge group \( \mathcal{G} \), which is a subgroup of the Pauli group on \( n \) qubits (where \( n \) is the number of vertices). The stabilizer group \( \mathcal{S} \) is in the center of the gauge group \( \mathcal{S} \propto Z(\mathcal{G}) \), consisting of elements of the gauge group that commute with every other element and where the signs are chosen such that \(-1 \notin \mathcal{S} \). For the gauge color code, we have an \( X \) and \( Z \) gauge generator for each face of the lattice,

\[
\mathcal{G} = \{ G_f^X, G_f^Z \mid f \text{ a face of } \mathcal{C}_3 \},
\]

where \( G_f^X = \prod_{v \in f} X_v \) and \( G_f^Z = \prod_{v \in f} Z_v \) are Pauli operators supported on the face \( f \). The stabilizers of the code are given by \( X \) and \( Z \) on the 3-cells of the lattice

\[
\mathcal{S} = \{ S_q^X, S_q^Z \mid q \text{ a 3-cell of } \mathcal{C}_3 \},
\]

where \( S_q^X = \prod_{v \in q} X_v \) and \( S_q^Z = \prod_{v \in q} Z_v \) are Pauli operators supported on 3-cells. Codestates of the gauge color code are the states that are in the +1 eigenspace of all elements of the stabilizer group and that are invariant under operators in the gauge group \( \mathcal{G} \). With the aforementioned boundary conditions, the code encodes one logical qubit, and logical operators can be taken to be \( \mathcal{X} = \prod_{v \in \mathcal{C}_3} X_v \) and \( \mathcal{Z} = \prod_{v \in \mathcal{C}_3} Z_v \), where the products are over all vertices of the lattice. Importantly, note that equivalent logical operators (i.e., up to products of stabilizers) can be found on the outer colex, namely \( \mathcal{X} \sim \prod_{v \in \mathcal{C}_{\text{out}}} X_v \) and \( \mathcal{Z} \sim \prod_{v \in \mathcal{C}_{\text{out}}} Z_v \) are valid representatives. Similarly to the RBH model, we are therefore justified in viewing the logical information as being encoded on the boundary.

One possible choice of Hamiltonian that contains the codespace in its ground space is given by the sum of all local gauge terms,

\[
H_\mathcal{G} = -\sum_f G_f^X - \sum_f G_f^Z,
\]

which we refer to as the **full GCC Hamiltonian**. This Hamiltonian is frustrated, meaning one cannot exactly satisfy all of the constraints \( G_f^X \) and \( G_f^Z \) simultaneously, making it difficult to study. There are many different Hamiltonians whose ground spaces contain the codespace of the gauge color code, and in the next subsection we introduce a solvable model, consisting of mutually commuting terms.

### C. A commuting model

Here we define an exactly solvable model for the gauge color code. The Hamiltonian is given by a sum of gauge terms that belong to 3-cells of a single color. Without loss of generality, fix this color to be \( b \) (blue), and take all faces \( X_f \) and \( Z_f \) belonging to the blue 3-cells or blue boundary facet. That is, all faces \( f \) that have color \( uv \)
with \( u, v \neq b \). Label the set of these faces by
\[
G_b = \{ G^X_b, G^Z_b | \mathcal{K}(f) \in \{ \text{gr}, \text{gy}, \text{ry} \} \},
\]
where \( \mathcal{K}(f) \) denotes the color of \( f \). Note that \( G_b \) consists of commuting terms, as all terms are supported on either a bulk 3-cell or the \( b \) boundary (which are both 3-colorable and 3-valent sublattices). Or equivalently, if two faces share a common color then the terms commute. We can define an exactly solvable Hamiltonian by
\[
H_{G_b} = - \sum_{G \in G_b} G.
\]
This Hamiltonian decomposes into a number of decoupled 2D color codes, one on the \( b \) boundary, and one for each bulk 3-cell of color \( b \). Additionally, every qubit is in the support of at least one \( G \in G_b \).

With the above choice of boundary conditions, the outer colex (the \( b \) boundary) encodes one logical qubit, while the bulk 2D color codes are non-degenerate (as they are each supported on closed 2-cells). The ground space of the model is the joint +1 eigenspace of all terms \( G \in G_b \), and the ground space degeneracy is two-fold. This choice of Hamiltonian explicitly represents the gauge color code codespace on the outer colex. This situation is reminiscent of the RBH model, where quantum information is encoded on the boundary of the 3D bulk. We remark that the ground state of \( H_{G_b} \) can be thought of as a gauge fixed version of the gauge color code \( G \).

Logical operators can be chosen to be string-like operators supported entirely on the outer colex (the \( b \) boundary facet). Recall that edges and plaquettes on the outer colex has one of three possible colors, \( A \), \( B \), or \( C \), as defined in Fig. 4 and the boundaries are given a single color according to what color edges can terminate on them, as depicted in Fig. 5. The logical operators take the form of strings that connect all three boundaries of the triangular facet as in Fig. 5. Logical Pauli operators are supported on at least \( d \) qubits, where \( d \) is the smallest side length of the boundary facet and referred to as the distance of the code.

On the outer colex, a string operator with color \( k \in \{ A, B, C \} \) will flip the two \( k \) colored plaquettes on the boundary of the string. In particular, a \( k \)-colored \( X \)-string will create \( m_k \) excitations on its boundary (corresponding to the flipped \( G^X_f \) plaquettes). Similarly, a \( k \)-colored \( Z \)-string will create \( \epsilon_k \) excitations on its boundary (corresponding to the flipped \( G^Z_f \) plaquettes). These are depicted in Fig. 5. On a \( k \) colored boundary, both \( \epsilon_k \) and \( m_k \) particles can condense, meaning they can be locally created or destroyed at the boundary as in Fig. 5.

As such, the action of logical \( \overline{X} (\overline{Z}) \) can then be interpreted as creating three \( m \)-type (\( \epsilon \)-type) quasiparticles of each color from the vacuum at a point, then moving each colored excitation to its like-colored boundary where it is destroyed.

1. Relation to the RBH model

To motivate how the model \( H_{G_b} \) was constructed, we draw a comparison to the RBH model of the previous section. In particular, the RBH also has the structure of a subsystem code, that on a certain lattice is dual to the gauge color code. For the RBH model, one can consider the gauge group \( G_C \) is given by
\[
G_C = \langle K_p, X_p | p \in E \cup F \rangle,
\]
where \( K_p \) are the cluster state stabilizers of Eq. 12 and \( X_p \) are local \( X \)-fields. The corresponding stabilizer group \( S_C \) is given by
\[
S_C = \langle S_p | p \in Q \cup V \rangle,
\]
where \( S_p \) are the 1-form symmetry generators of the RBH model, given by Eq. 24. (The choice of gauge generators \( X_p \) stems from the application of the RBH model to fault tolerant measurement-based quantum computing, where \( X \)-measurements are used to propagate information.)

The commuting model describing the RBH model was chosen by selecting a subset \( G' \) of local, commuting elements of \( G_C \) to define the Hamiltonian, and imposing symmetries given by the stabilizer \( S_C \). This choice is non-unique, as there are many other subsets of \( G \) that could be used to construct a commuting model \( G' \). However, the choice of commuting subgroup \( G' \) was such that \( S_C \) is a (higher-form) subgroup of \( G' \), as in this case the symmetries are of the form of Eq. 14 and we are without
spontaneous symmetry breaking. The same construction was also used to generate the commuting GCC model, and can be used more generally for subsystem codes with a stabilizer group that has the structure of a $\mathbb{Z}_k^3$ 1-form symmetry for some $k$. We note however there are many distinct ways generating such Hamiltonians, and not all of them will be self-correcting under the 1-form symmetry.

D. 1-form symmetry and color flux conservation

The commuting model $H_{G_b}$ without any symmetry constraints is easily shown to be disordered at any non-zero temperature. In this section, we identify a 1-form symmetry of this model that, when enforced, leads to a diverging energy barrier and therefore self-correction on the boundary code.

The Hamiltonian $H_{G_b}$ has a $\mathbb{Z}_2^3$ 1-form symmetry given by the stabilizer group $S$ of Eq. (31). Recall that $S$ is generated by the stabilizers $S_q^X$ and $S_q^Z$ on the 3-cells $q$ of the lattice, and consists of operators supported on closed codimension-1 (contractible) surfaces. The two copies of $\mathbb{Z}_2$ 1-form symmetry come from the independent $X$-type and $Z$-type operators. The symmetry $S$ give strong constraints (conservation laws) on the possible excitations in the model: this is the color flux conservation of Bombin [61]. To discuss the color flux conservation that arises from the $\mathbb{Z}_2^3$ 1-form symmetry, let us assume that the system $H_{G_b}$ is coupled to a thermal bath (as in Eq. (1)) such that the the whole system respects the symmetry $S$, and discuss what type of excitations are possible in the model.

The model $H_{G_b}$ is a stabilizer Hamiltonian, and so excitations are labelled in the standard way. Specifically, excited states can be labelled by the set of ‘flipped terms’ $G_{ex} \subseteq G_b$. Not all sets $G_{ex}$ can be reached from the ground space in the presence of the symmetry $S$. Since the ground space of $H_{G_b}$ consists of the states in the +1 eigenspace of all terms in $G_b$, it follows that the ground space is also the +1 eigenspace of all operators in $S$, and since they are conserved, only the excited states that satisfy color flux conservation on each cell (as we will describe) can be reached.

In particular, note that for any 3-cell $q$ of color $k \neq b$, there is precisely one way of obtaining the stabilizers $S_q^X$ and $S_q^Z$ from terms in $G_b$, while for a 3-cell of color $b$ there are three ways of obtaining the stabilizers. More precisely, for the $X$-type stabilizers we have

$$S_q^X = \prod_{f \subset q, \ K(f) = uv} G_f^X,$$

where

$$uv \in \begin{cases} \{gy\} & \text{if } K(q) = r \\ \{ry\} & \text{if } K(q) = g \\ \{rg\} & \text{if } K(q) = y \\ \{gy, ry, rg\} & \text{if } K(q) = b. \end{cases}$$

The above expression holds similarly for the stabilizer $S_q^Z$. This can be seen as any plaquette that neighbours a 3-cell of color $k$ must be of color $uv$ with $u, v \neq k$, for which there is only one choice within $G_b$ for $k \neq b$, and three choices when $k = b$. Note that the multiple ways of forming $S_q^X$ and $S_q^Z$ on blue 3-cells as per Eq. (38) leads to local product constraints on these blue 3-cells (further constraining the excitations) however this is not important for the present discussion.

To ensure that an excitation $G_{ex}$ is valid, we must remain in the +1-eigenspace of $S$. From Eq. (37) we see that every 3-cell $q$ must have an even number of flipped plaquettes belonging to its boundary. Indeed, a single flipped plaquette $G_f^X$ of color $uv$ would violate the two stabilizer operators $S_q^X$ and $S_q^Z$ on the neighbouring $u$ and $v$ colored 3-cells $q$ and $q'$. This constraint implies that symmetric excitation configurations consist of collections of closed loop-like sets of flipped plaquettes.

This can be more easily visualised on the dual lattice, where 3-cells are replaced by vertices, and faces by edges. On the dual lattice, vertices carry a single color, edges are labelled by pairs of colors, and excitations are therefore given by sets of edges. We call the edges on the dual lattice that define an excitation a flux string. The color flux conservation on these closed flux strings is as follows.

To satisfy the constraints of Eqs. (37), and (38), for each vertex $v$ of color $k \in \{b, r, g, y\}$ the number of edges in a flux string incident to $v$ must be even. Since the vertices of color $k \in \{r, g, y\}$ only support terms in $G_b$ on neighbouring edges of a single color type (e.g. a $r$ vertex only supports terms on its neighbouring $gy$-colored edges), then the color of the excitation is conserved at each one of these vertices. Similarly on a $b$ vertex, all pairs of colors are separately conserved. This means if a $uv$ colored edge excitation enters a vertex, there must be a $uv$ colored edge excitation leaving the vertex. In summary, bulk excitations must form closed loops, where the color is conserved at every vertex, and this is illustrated in Fig. 6.

Flux loops may terminate on the outer colex. Recall that for a boundary facet of color $k$, there are no faces of color $uk$ for any $u$. In particular, for $k \neq b$, there is a unique color $u$ such that there are terms $G_f^X$ and $G_f^Z$ of color $uk$ in $G_b$. Flux loops of color $uk$ can terminate on this $k$-colored boundary facet. For the $b$ colored boundary facet (the outer colex), all three color
pairs of flux loops can terminate on the outer colex. Flux loops terminating on the b-facet can be viewed as ending in a $k_b$ or $m_b$ anyonic excitation on the boundary for $k \in \{A, B, C\}$ as in Fig. 6 (recall the colors are relabelled on the outer colex according to gy $\leftrightarrow$ A, ry $\leftrightarrow$ B, rg $\leftrightarrow$ C). Moreover, in the same way, the only way anyons can exist on the outer colex is at the ends of a flux loop on the bulk, as stand-alone boundary anyonic excitations violate the symmetry.

E. Energy barrier

We are now equipped to calculate the symmetric energy barrier for $H_{G_b}$ in the presence of the symmetry $S$. Recall that a logical error occurs when a triple of excitations $\alpha_A, \alpha_B, \alpha_C$, where $\alpha = e$ or $m$, are created at a point, and each anyon travels to its like-colored boundary. Put another way, a logical error occurs if an anyonic excitations $\alpha_k$ is created at each boundary, and the three anyons move and fuse back to the vacuum in the bulk of the outer colex. In any case, the only way to achieve a logical Pauli error is to create a number of anyonic excitations, which must move a combined distance of at least $d$, the side length of the outer colex. In the symmetric sector, anyonic excitations can only be exist on the boundary if they are accompanied by a bulk flux loop, and so the above creation, movement and fusion process can only occur when accompanied by bulk flux loops.

Since boundary excitations $\alpha_k$ with $\alpha \in \{e, m\}$ and $k \in \{A, B, C\}$ appear on the end of flux loops (each of which can only terminate on its like-colored boundary) to calculate the energy barrier we need only track the smallest length flux loops required to move the boundary anyons to create a logical error. From any point $v$ on the outer colex, let $l_A(v)$, $l_B(v)$, $l_C(v)$ be the shortest flux loops from a face $f$ on the outer colex containing $v$, to a face on the $A$, $B$, and $C$ facets, respectively (these flux loops are dual to a closed path on the dual lattice). Let $|l_A(v)|, |l_B(v)|, |l_C(v)|$ be the lengths of these flux loops (i.e., the number of edges on the dual path) and define

$$d_\perp := \min_{v \in \text{out}} (|l_A(v)| + |l_B(v)| + |l_C(v)|) \quad (39)$$

to be the shortest combined distance from any point on the outer colex to all three other facets. Note that $d_\perp$ grows as all side lengths of the tetrahedral 3-colex are increased.

Then during anyon creation, movement and annihilation process resulting in a logical error, the bulk flux loops which accompany the boundary anyons must have a combined length of at least $d_\perp$. This will incur an energy penalty of $\Delta_E = 2d_\perp$ since each flux loop consists of a path of flipped terms $G^e_f \in G_b$. As such the energy is proportional to $d_\perp$, which scales linearly with the minimum side length of the tetrahedral 3-colex. In particular, the model $H'$ with symmetry $S$ has a macroscopic energy barrier, and the boundary information is protected in the presence of a 3D bulk and symmetry constraint.

We remark two things. Firstly, the energy barrier and conservation laws in this section were presented in terms of excitations rather than error operators (as opposed to the operator approach for the RBH model). For the purposes of calculating the energy barrier these two pictures are equivalent, since the sequence of local (symmetric) excitations corresponds to a sequence of local (symmetric) operators, and vice-versa. Secondly, we remark that a tri-string logical operator of the above form can be pushed onto a single boundary of the outer colex, giving rise to a string-like representative. As such, a logical error can arise from a pair of anyons of the same color being created and moved along the boundary of the outer colex. Such a process also has an energy lower bounded by $\Delta_E = 2d_\perp$ since a $k$-colored string on the boundary of the outer colex is never adjacent to a boundary where its $k$-flux loops can terminate.

The argument from the symmetric energy barrier to self-correction follows identically to that of the RBH model. That is, provided the temperature is sufficiently low, information can be stored for a time that grows exponentially with the system size. (Note that the critical temperature will depend on the specific choice of 3-colex.) As a result, our stabilizer model based on the 3D gauge color code protected by $Z_2^e$ 1-form symmetry provides another example of a self-correcting quantum memory.

In the RBH model, the fact that the boundary was self-correcting in the presence of 1-form symmetries could be interpreted as directly resulting from the thermally stable bulk SPT order. In this stabilizer model of the gauge color code, the boundary stability and bulk SPT
VI. EMERGENT 1-FORM SYMMETRIES

As we have shown, SET models protected by a 1-form symmetry can be self-correcting. However, enforcing such 1-form symmetries is a very strong constraint, and in addition these symmetries are unusual in physics compared with the more prevalent onsite (0-form) symmetries. Here we explore the idea that 1-form symmetries may actually appear naturally in 3D topological models, and not require any sort of external enforcement. We refer to such a symmetry as emergent. It sounds too good to be true, but note that emergent symmetries in 2D topological models are ubiquitous (while perhaps poorly understood). In this section, we review emergent (0-form) symmetries in 2D topological models, as first highlighted by Kitaev [4]; here we will focus on the 2D color code. We then show that 3D models may possess emergent 1-form symmetries associated with such emergent 0-form symmetries on closed 2D submanifolds of the 3D model. We revisit the 3D gauge color code in light of emergent symmetries. Here we explore the idea that 1-form symmetries are a generic property of 2D topologically ordered models. We begin this section by reviewing an instructive first example: the 2D color code. We demonstrate the stability of emergent 1-form symmetries in topologically ordered models, and discuss the implications for self-correction.

A. Emergent 0-form symmetries in 2D

Kitaev observed the emergence of symmetry in 2D topological models such as the toric code and referred to this as a ‘miracle’ [4]. As we now know, emergent symmetries are a generic property of 2D topologically ordered models. We begin this section by reviewing an instructive first example: the 2D color code. We demonstrate the emergence of a \( \mathbb{Z}_2^4 \)-type 0-form symmetry in this 2D code, and how this gives rise to the well known anyonic color conservation (see for example Ref. [51]).

We first consider a 2D color code defined on the surface of a sphere (one can equivalently consider any closed surface for the discussion that follows). Recall, a 2D color code is defined on a lattice known as a 2-colex, which is a 3-colorable, 3-valent cellulation \( \Lambda \) of a 2-dimensional surface, which in this case is a sphere. We place a qubit on each edge of \( \Lambda \), and define the familiar \( X \)-type and \( Z \)-type face operators \( G_f^X = \prod_{e \in f} X_e \), and \( G_f^Z = \prod_{v \in f} Z_v \) for each face \( f \subset \Lambda \). In particular, since the lattice is 3-colorable and 3-velant, these face operators \( G_f^X \) and \( G_f^Z \) all commute. These operators generate the 2D color code stabilizer group \( S_{cc} = \langle G_f^X, G_f^Z | f \text{ a face of } \Lambda \rangle \), and define a corresponding Hamiltonian \( H_{2D-cc} \) by

\[
H_{2D-cc} = - \sum_{\text{faces } f} \left( G_f^X + G_f^Z \right).
\]

This 2D color code differs only from that defined on the outer colex (considered in Section [V] Fig. [5]) by a choice of boundary conditions.

Recall, a generating set for the anyonic excitations of this model can be labelled by \( m_k \) and \( e_k \), where \( k \in \{ \mathbf{A}, \mathbf{B} \} \) labels a color, \( e \)-type anyons corresponds to flipped \( X \)-type plaquettes, and \( m \)-type anyons correspond to flipped \( Z \)-plaquettes. One can obtain \( \mathbf{C} \) colored anyons as the fusion of an \( \mathbf{A} \) and \( \mathbf{B} \) colored anyon of the same type. This set of anyons forms a group under fusion \( A_{2D-cc} \cong \mathbb{Z}_2^4 \), with the above choice of generators.

However, not all anyonic excitation configurations are possible as there are global constraints that need to be satisfied in this model. In particular, since our model is defined on a closed surface, we have the following identities for each \( \alpha \in \{ X, Z \} \)

\[
\prod_{f \subset A} G_f^\alpha = \prod_{f \subset B} G_f^\alpha = \prod_{f \subset C} G_f^\alpha = \prod_{v \in A} \alpha_v, \tag{41}
\]

Letting \( N^e_B \) and \( N^m_B \) be the number of \( e_k \) and \( m_k \) anyonic excitations respectively, then the above equation implies the following relation

\[
N^e_B = N^e_C \mod 2, \tag{42}
\]

and similarly for \( N^m_B \). In particular this means that the number of \( e_{\mathbf{A}} \), \( e_{\mathbf{B}} \) and \( e_{\mathbf{C}} \) anyons is conserved mod 2 (and similarly for \( m_{\mathbf{A}}, m_{\mathbf{B}}, \) and \( m_{\mathbf{C}} \)).

If we regard anyons of color \( \mathbf{C} \) as being comprised of an \( \mathbf{A} \) color and a \( \mathbf{B} \) color anyon, we can obtain further constraints. Namely, for any two colors, \( \mathbf{u}, \mathbf{v} \in \{ \mathbf{A}, \mathbf{B}, \mathbf{C} \} \), we have a product constraint

\[
\prod_{f \subset A} G_f^\alpha \prod_{f \subset \mathbf{u}} G_f^\alpha = I. \tag{43}
\]

This implies a constraint on the parity of anyons

\[
N^e_\mathbf{u} + N^e_\mathbf{v} = 0 \mod 2, \tag{44}
\]

which along with the fact that we are regarding \( N^e_C = N^e_A + N^e_B \), means that \( N^e_A = N^e_B = 0 \mod 2 \) (and similarly for \( m \)-type anyons). The product constraint of Eq. (43) exists on the whole 2-dimensional lattice (that is, a codimension-0 surface), and gives rise to 4 independent anyonic constraints: that the number of \( e_{\mathbf{A}} \) anyons must be created or destroyed in pairs, and similarly for
Here we demonstrate how emergent 1-form symmetries can arise in a 3D model, in a sense by bootstrapping from the 2D case.

### B. Emergent 1-form symmetries in 3D

For illustrative purposes, we first consider a single charge sector of the 3D gauge color code $H_3$. This single-sector model is not topologically ordered, and so does not possess emergent symmetries; nonetheless it will be useful to illustrate the connection between 1-form symmetries in a 3D model and 0-form symmetries in associated 2D models existing across all codimension-1 submanifolds of the 3D model. The 1-form symmetries fix excitations to be 1-dimensional objects that conserve color flux.

Recall, the gauge color code is defined on a 3-colex $C_3$ (a 4-colorable, 4-valent cellulation) with a qubit on each vertex. For concreteness, we restrict our discussion to the $X$-sector of the gauge color code (the $Z$-sector follows similarly). That is, we consider the Hamiltonian

$$H_X = - \sum_f G_f^X,$$

consisting of the sum of all face terms over a 3-colex.

The ground space of $H_X$ is the mutual $+1$ eigenspace of all terms $G_f^X$, and excitations are eigenstates of the Hamiltonian in the $-1$ eigenspace of some terms (we say these terms are $G_f^X = -1$). We can label excited states uniquely by specifying which terms are $G_f^X = -1$, but importantly not all configurations are allowed, as there are algebraic constraints amongst terms.

Consider any closed codimension-1 submanifold $M$ of the 3-colex. Such a submanifold is a 2-colex with the color-pairs $A_M, B_M, C_M$ selected from the 6 possible color-pairs of faces in $C_3$. On this sub-2-colex, we have the familiar constraints from the 2-colex. Namely, for any 2-color-pairs $u, v \in \{A_M, B_M, C_M\}$, we have

$$\prod_{f \subset M, K(f) = u} G_f^X \prod_{f \subset M, K(f) = v} G_f^X = I,$$

mirroring the constraints of Eq. (43). In particular, this relation holds in the smallest instance when $M$ is the boundary of a 3-cell.

The product relations of Eq. (48) lead to constraints on excitations. Namely, for each codimension-1 submanifold (that is a 2-colex), the number of faces $f \subset q$ with $G_f^X = -1$ carrying a color $k$ must sum to (0 mod 2), and this holds for each (single) color $k$. This in turn requires excitations (which carry pairs of colors) to form closed loop-like objects that conserve color. The dual lattice again provides the visualization, where excitations correspond to sets of edges and edges carry a pair of colors. At each vertex $v$ of the dual lattice, let $N_k^v$ be the number of flux loops carrying the (single) color $k$ that...
contain \( v \). Then the constraints of Eq. (48) mean that
\[
N^v_k = 0, \quad \forall \, k, v,
\]
which is precisely the color flux conservation discussed in Sec. VD. In particular, this implies that excitations must form closed loop-like objects.

Not all flux loops are independent. A string excitation of a color \( \mathbf{xz} \) may branch into a pair of strings with colors \( \mathbf{xk} \) and \( \mathbf{kz} \) for \( k \neq x, z \). This then means there are three independent color pairs, such that all flux loops can be regarded as the fusion of these loops. The flux conservation can be regarded as three independent constraints on loop-like excitations.

Similar to the 0-form case, 1-form symmetries also imply a constraint (conservation law) for the loop-like excitations. We can infer a generalization of the law for detecting topological charge, which in this case applies to color flux, by considering codimension-1 submanifolds that are not closed. In particular, let \( M' \) be a codimension-1 submanifold with a boundary. Then it holds that
\[
\prod_{f \in M'} G_f^X \prod_{f \in M'} G_f^X = h_{\partial M'},
\]
where \( h_{\partial M'} \) is an operator supported on the (1-dimensional) boundary of \( M \). This means that the number (mod 2) of \( u \) colored and \( v \) colored flux loops that thread the region \( M' \) is detected by an operator \( h_{\partial M'} \) on the boundary of that region. Again, we can use the constraints to determine this number on each independent color pair.

This 3D example, then, gives the appearance of an emergent \( \mathbb{Z}_2^3 \) 1-form symmetry arising from a 0-form symmetry on codimension-1 submanifolds. We note, however, that by restricting to the \( X \)-sector, we do not have a topologically ordered model; the codimension-1 submanifolds do not have an emergent 0-form symmetry without both sectors, and so an emergent 1-form symmetry does not appear in the 3D model. Both electric and magnetic sectors are required simultaneously in order to have the emergent symmetry associated with either \( \mathbb{Z}_2 \). Regardless, our purpose here was simply illustrative—we are not fundamentally interested in this single-sector model, but rather a topologically-ordered 3D model with both sectors such as the gauge color code. We turn to that model now.

### 2. The gauge color code and color flux conservation

Does the topologically-ordered 3D gauge color code have an emergent 1-form symmetry associated with color flux conservation? Each sector of the gauge color code on its own, \( H_X \) and \( H_Z \), has loop-like, color-flux-conserving excitations. Proliferation of such excitations is therefore suppressed, as they are energetically confined. For the full gauge color code Hamiltonian,
\[
H_G = -\sum_f G_f^X - \sum_f G_f^Z,
\]
it is tempting to conclude that a \( \mathbb{Z}_2^6 \) 1-form symmetry will emerge, and lead to confined errors and suppression of logical faults. However, the terms of \( H_G \) are not mutually commuting (and indeed frustrated), and therefore we cannot immediately label excited states by specifying terms \( G_f^X, G_f^Z = \pm 1 \). In other words, this frustrated model’s excitations are not guaranteed to be well-defined extended objects with well-defined color flux as appear in each sector separately. If they were, then this would be strong evidence that the model was self-correcting.

Unfortunately, there are few tools available to understand the spectrum of a frustrated Hamiltonian such as \( H_G \), and without such information it is a very difficult task to analyse the thermal stability and memory time of the code. In this sense, one can view the exactly solvable model \( H_{\partial h} \) as the result of removing terms from the Hamiltonian until it is commuting, in the process losing its emergent 1-form symmetries and supplementing them with enforced 1-form symmetries. Understanding the excitations in \( H_G \) remains an important problem, to determine if it is self-correcting.

### 3. Higher-dimensional generalizations and emergent q-form symmetries

We briefly generalize the discussion to emergent \( q \)-form symmetries in \( d \)-dimensional systems that arise from (product) constraints residing on codimension-\( q \) submanifolds. In particular, a commuting Hamiltonian \( H = \sum_{X \subset \Lambda} h_X \) in \( d \)-dimensions has an emergent \( \mathbb{Z}_2 \) \( q \)-form symmetry if for all closed codimension-\( q \) submanifolds \( \mathcal{M} \), there exists an constraint
\[
\prod_{X \subset \Lambda} h_X = I.
\]
If there are multiple independent such constraints on the submanifolds, then there are multiple copies of emergent \( \mathbb{Z}_2 \) \( q \)-form symmetries. Importantly, we note that these constraints all look like emergent \( \mathbb{Z}_2 \) 0-form symmetries.
on codimension-\(q\) submanifolds. The generalized conservation law states that the number (mod 2) of excitations (which must be \(q\)-dimensional objects) threading the codimension-\(q\) region \(\mathcal{M}'\) can be measured by the operator \(\mathcal{H}_{\partial M'}\) on the codimension-(\(q+1\)) boundary of the region. In particular, if \(\mathcal{H}\) has a \(q\)-form emergent symmetry, let \(\mathcal{M}'\) be a codimension-\(q\) submanifold with a boundary, then it holds that
\[
\prod_{i \in \mathcal{M}'} h_i = \mathcal{H}_{\partial M'},
\]
where \(h_{\partial M'}\) is an operator supported on a small neighbourhood of the boundary of \(\mathcal{M}\). (This is because if we chose a complementary codimension-\(q\) submanifold \(\mathcal{M}''\) such that \(\partial M' = \partial M''\), then if \(\mathcal{M}\) is the result of gluing \(\mathcal{M}'\) and \(\mathcal{M}'\) along their boundary, we would have the usual constraint of Eq. \((52)\). Thus \(\prod_{i \in \mathcal{M}'} h_i\) can only differ from the identity by an operator supported on a small neighbourhood of \(\partial M'\).)

Examples of models with emergent higher-form symmetries include toric codes in various dimensions. For dimensions \(d \geq 2\), there are \(d-1\) distinct ways of defining a toric code. Namely, for each \(k \in \{1, \ldots, d-1\}\), we define the \((k, d-k)\) toric code that has \(k\)-dimensional logical \(X\) operators, and \((d-k)\)-dimensional logical \(Z\) operators. One can confirm that these models have emergent \(\mathbb{Z}_2\) \((k-1)\)-form and \(\mathbb{Z}_2\) \((d-k-1)\)-form symmetries. The smallest dimension that allows for a toric code with emergent \(\mathbb{Z}_2\) 1-form symmetries is \(d = 4\), with the \((2, 2)\) toric code, which is a self-correcting quantum memory.

## C. Stability of emergent symmetries

Our discussion of emergent symmetries has focussed on Hamiltonians with commuting terms. This property allowed for the simple identification of product constraints. One can ask if the resulting emergent symmetries are a property of a finely tuned system alone, or if they hold more generally. In this section, we show that these symmetries are robust features of phases of matter, that they cannot be broken by local perturbations, irrespective of any symmetry considerations, provided they are sufficiently small. The argument uses the idea of quasi adiabatic continuation, following Ref. \[86\].

Consider a family of local Hamiltonians \(H_s\), labelled by a continuous parameter \(s \in [0, 1]\), such that \(H_0 = H\) is the original Hamiltonian, and \(H_s\) remains gapped for all \(s \in [0, 1]\). This family of Hamiltonians can be used to describe the situation where a perturbation is added to \(H\). We label ground states of \(H\) by \(|\psi_i\rangle\), and groundstates of \(H_s\) by \(|\psi_i^s\rangle\). Note that the ground states can be unitarily related by an adiabatic continuation. Then, following Ref. \[89\], there exists a unitary \(U(s)\) corresponding to a quasi-adiabatic change of the Hamiltonian with the following properties. For any operator \(O\), one can find a dressed operator \(O_s = U(s)O\mathcal{U}(s)^\dagger\), such that \(O_s\) has approximately the same expectation value in \(|\psi_i^s\rangle\) as \(O\) does in \(|\psi_i\rangle\). Moreover, if \(O\) is local, then \(O_s\) is local too. (The support of the dressed operators increases by a size determined by the choice of quasi-adiabatic continuation unitary \(U(s)\). The approximate ground state expectation values improve exponentially in the range of increased support of dressed operators.)

Importantly, one can use quasiadiabatic continuation to find dressed versions \(h_{\mathcal{X}}(s) = U(s)h_{\mathcal{X}}U(s)^\dagger\) of the Hamiltonian terms that have approximately the same ground space expectation values as those in the unperturbed Hamiltonian. These Hamiltonian terms will also have the same constraints. In particular, if \(H\) had an emergent \(q\)-form symmetry, then the dressed Hamiltonian also has an emergent \(q\)-form symmetry. Note that the dressed terms will in general be supported in a larger region, meaning one may need to rescale the lattice to resolve excitations and faithfully capture the generalized conservation law in the perturbed Hamiltonian. For example, consider the color code in the presence of perturbations, then one can renormalize the lattice such that individual excitations are well defined. Then in the renormalized lattice, these excitations still conserve anyon parity, and they still obey a conservation law for topological charge.

We remark that we required the gap to remain open in the presence of the perturbations. This can be guaranteed for any local perturbation (provided it is sufficiently weak), if \(H\) satisfies the conditions of TQO-1 and TQO-2 of Ref. \[81\]. In particular, the example models we have considered in Sections \[IV\] and \[V\] satisfy the conditions.

## D. Duality between emergent and enforceable symmetries

For emergent symmetries, we are faced with the puzzle that we have a conservation law without any symmetry operator. What is the origin of this symmetry? As pointed out by Kitaev in the case of the 2D toric code \[4\], we can always recover symmetry operators by introducing redundant “unphysical” degrees of freedom, viewed as gauge degrees of freedom. Here we briefly consider how Kitaev’s approach can be applied to higher-form symmetries. In particular, for systems with emergent symmetries, we will construct symmetry operators on an enlarged Hilbert space. This construction provides a duality between systems where the \(q\)-form symmetry is emergent and systems where it is enforced.
We will begin with the color code in 2D, and then show how to lift the construction to the 1-form case in 3D. We start by introducing new ancillary degrees of freedom—one ancilla for each term in the Hamiltonian. Label these ancilla by \( a_X(f) \) and \( a_Z(f) \) corresponding to the terms \( G_X^f \) and \( G_Z^f \) and fixed them in the \(+1\) eigenspace of Pauli operators \( X \) and \( Z \), respectively. We can now regard the new Hilbert space as \( \mathcal{H} \otimes \mathcal{A} \), and states in \( \mathcal{H} \) are embedded according to the isometry \( |\psi\rangle \mapsto |\psi\rangle \otimes |a\rangle \), where \( |a\rangle = (\otimes_{a_X(f)} (+)) (\otimes_{a_Z(f)} |0\rangle) \).

We refer to the (original) degrees of freedom in \( \mathcal{H} \) as matter, and those in \( \mathcal{A} \) as gauge. Importantly, not all states \( |\varphi\rangle \in \mathcal{H} \otimes \mathcal{A} \) are physical, only the subspace of states satisfying \( X_{a_X(f)} |\varphi\rangle = |\varphi\rangle \) and \( Z_{a_Z(f)} |\varphi\rangle = |\varphi\rangle \) are physical. At this point, it is clear from the embedding that the physical state space is the same as the original state space.

We now couple the matter and gauge degrees of freedom with an entangling unitary. Consider the mapping of gauge terms and matter Hamiltonian terms

\[
\begin{align*}
X_{a_X(f)} &\mapsto S_f^X, & G_X^f &\mapsto G_X^f, \\
Z_{a_Z(f)} &\mapsto S_f^Z, & G_Z^f &\mapsto G_Z^f.
\end{align*}
\]

Such a mapping can be achieved with a unitary \( U \) as we show below. In this new Hilbert space, which we label \( U(\mathcal{H} \otimes \mathcal{A})U^\dagger \), the physical state space is the subspace satisfying

\[
S_f^X |\varphi\rangle = S_f^Z |\varphi\rangle = |\varphi\rangle,
\]

where \( S_f^X = X_{a_X(f)}G_X^f \) and \( S_f^Z = Z_{a_Z(f)}G_Z^f \). The symmetry operators \( S_f^X \) and \( S_f^Z \) are known as gauge transformations, and states and operators that are related by them are thought of as equivalent.

The entangling unitary \( U \) that will result in the above mapping can be constructed out of 2-qubit CNOT gates, \( A_{i,j} \), which act by conjugation on Pauli operators as follows

\[
\begin{align*}
X_i &\mapsto X_iX_j, & Z_i &\mapsto Z_i, \\
X_j &\mapsto X_j, & Z_j &\mapsto Z_iZ_j.
\end{align*}
\]

Then for each face \( f \), we define the following unitaries

\[
U_f^X = \prod_{v \in f} A_{a_X(f),v}, \quad U_f^Z = \prod_{v \in f} A_{v, a_Z(f)}.
\]

Note that \( U_f^X \) has the following action:

\[
U_f^X Z_{a_X(f)} U_f^X \dagger = \begin{cases} S_f^X & \text{if } f = f' \\ X_{a_X(f)} & \text{otherwise.} \end{cases}
\]

Moreover, \( U_f^X \) commutes with all Hamiltonian terms \( G_X^f \) and \( G_Z^f \) \( \forall f \) (this statement only needs to be verified for terms \( G_f^f \) where \( f' \) and \( f \) are neighbours, where it holds because neighbouring terms intersect an even number of times—as is always the case for commuting CSS stabilizer Hamiltonians). A similar calculation gives the action of \( U_f^Z \)

\[
U_f^Z Z_{a_X(f)} U_f^Z \dagger = \begin{cases} S_f^Z & \text{if } f = f' \\ Z_{a_X(f)} & \text{otherwise.} \end{cases}
\]

where again \( U_f^Z \) commutes with all Hamiltonian terms \( G_X^f \) and \( G_Z^f \) \( \forall f \). Then the desired unitary \( U \) is given by \( U = \prod_f U_f^X U_f^Z \).

Since the Hamiltonian is unchanged by \( U \), one can ask what the excitations in the physical space of \( U(\mathcal{H} \otimes \mathcal{A})U^\dagger \) look like. Namely, for each flipped term \( G_X^f \) (\( G_Z^f \)) we must also flip the ancilla \( a_X(f) \) (\( a_Z(f) \)). Thus one can equally label excitations by the terms \( G_X^f \) and \( G_Z^f \), or the terms \( X_{a_X(f)} \) and \( Z_{a_Z(f)} \), as the two sets are gauge equivalent. The emergent 0-form symmetry manifests itself as product constraints amongst Hamiltonian terms (following Eq. 43). Specifically, it is equivalent to the following constraints, for any color \( u \neq v \)

\[
\prod_{f|K(f)=u} S_f^X \prod_{f|K(f)=v} S_f^X = \prod_{a_X(f)} X_{a_X(f)},
\]

and similarly for the \( Z \)-terms. Here, we see that the operator \( \prod_{a_X(f)} X_{a_X(f)} \) (which is gauge equivalent to a product of color code terms \( G_X^f \)) counts the number of excitations mod 2. As it is a product of symmetry operators, any physical state must lie in its \(+1\) eigenspace. That is, we have found a symmetry operator that determines the parity conservation of anyons, by introducing gauge degrees of freedom.

In the same way, we can perform an analogous procedure for each sector in the 3D gauge color code. Again, we associate ancilla to each term in the Hamiltonian, and then apply the unitary \( U \) that entangles gauge and matter degrees of freedom. Much like the 2D case, this leads to symmetry operators constructed on all codimension-1 submanifolds (out of products of \( S_f^X \) and \( S_f^Z \) on these surfaces) and a requirement that the physical states must live in their common \(+1\) eigenspace (the enforced 1-form symmetry). These symmetry operators mirror the 1-form operators that we have seen in sections \( \ref{section:1-forms} \) and \( \ref{section:2-forms} \). In fact, this construction works for any CSS stabilizer code (in any dimension), where the product over \( v \in f \) in Eq. \ref{eq:3d-unitary} is replaced by product over the qubits in the support of the stabilizer term.

By introducing redundant degrees of freedom, we have related a model with an emergent symmetry to one with
an enforced symmetry. The duality mapping known as *gauging* \[52, 87–91\] formalizes this relationship. Gauging a model with an onsite (0-form) symmetry produces a model with an emergent 0-form symmetry. Gauging also provides a potential direction for identifying models with emergent 1-form symmetries. We note that formalisms for gauging/ungauging more general types of symmetries have been explored by Vijay, Haah, and Fu \[92\], Williamson \[89\], as well as Kubica and Yoshida \[94\]; these approaches provide potentially powerful tools to identify self-correction and error correction. In fact, the coupling of gauge degrees of freedom is similar to many schemes of syndrome extraction, where measurement of ancillas is used to infer the eigenvalues of stabilizer terms. Measurement errors can break this correspondence, however, and result in a misidentification of errors. This is typically accommodated by requiring many rounds of measurements. For single shot error correction (such as in the GCC \[62\]), only a single round of measurements is needed, owing to the extensive number of symmetry constraints present, whose violation indicates a measurement error. In the case of emergent 0-form symmetries, the global constraint alone cannot provide sufficient information to correct for measurement errors. In a similar vein to self-correction in 3D, it would be interesting find 2D topological codes (if they exist) with emergent \(\mathbb{Z}_2\) 1-form symmetries, as such codes could in principle admit single-shot error correction.

**VII. DISCUSSION**

We have shown that spin lattice models corresponding to 2D SET ordered phases protected by a suitable 1-form symmetry can be self-correcting quantum memories. The key features of these 1-form symmetric models are that the bulk excitations are string-like and confined, and that the symmetry naturally couples bulk and boundary excitations to confine the later as well.

We have presented two explicit examples of 3D self-correcting quantum memories protected by 1-form symmetries. The understanding and classification of such 3D models remains largely unexplored. A natural class of candidates are the (modular) Walker Wang models \[36, 95–98\], which possess many of the desirable properties we seek. In particular, if the input anyon theory to the Walker Wang construction is modular, then all bulk excitations are confined, while the 2D boundary contains a copy of the input anyon theory. One can consider building 1-form symmetries into these types of models, as has been done by Williamson and Wang \[99\] for a class of models based on the state sum TQFTs of Ref. \[100\]. (We note this is similar to the way that Ref. \[36\] ‘decorates’ a Walker Wang model with a (0-form symmetry.) The 2-group construction of Ref. \[71\] presents another interesting family of models that warrants further investigation. In the stabilizer case, another possible approach to construct 3D models with 1-form symmetries is to “foliate” \[101\] a topological stabilizer code with emergent 0-form symmetries. As an example, foliation of a \(d\)-dimensional topological CSS code with emergent \(q\)-form symmetry generates a \((d+1)\)-dimensional generalized RBH-type model with a \((q+1)\)-form symmetry.

In the examples we have explored, we have seen the necessity of the bulk SPT-ordering in order to have a self-correcting boundary, and for the bulk SPT-ordering of these models to be thermally stable. A common viewpoint is that a self-correcting quantum memory should be topologically ordered at nonzero temperature. While this has not been proved in general, it has been observed to be true for many examples under Hastings’ definition for topological order at \(T \geq 0\) \[102\]. (For example, 2D commuting projector Hamiltonian models and the 3D toric code all lack topological order at \(T>0\), corresponding to the absence of self-correction.) Our examples provide further support to this perspective.

We briefly consider what our results imply for self-correction in the 3D gauge color code. As we have shown in Sec. \[7\] the 3D gauge color code realized as commuting Hamiltonians protected by an (enforced) 1-form symmetry is self-correcting. If we consider the full Hamiltonian of Eq. \((51)\), the model is frustrated and it is difficult to prove that it possesses the string-like excitations with well-defined topological charge required for our arguments. We have also argued that the full model possesses an emergent 1-form symmetry: the color flux conservation as previously identified by Bombin \[61\]. This emergent symmetry gives strong supporting evidence that proving self-correction for the full Hamiltonian of Eq. \((51)\) (without enforcing any symmetry requirement) may be possible provided that its spectral properties can be better understood.

The idea that 1-form symmetries may be emergent in 3D topological models is extremely intriguing, both from the perspective of self-correction and more generally. We have argued that 1-form symmetries may emerge in 3D models that possess emergent 0-form symmetries on all codimension-1 submanifolds, which in turn can be guaranteed by topological ordering of these submanifolds. We can ask whether the 1-form symmetries of the RBH model or commuting GCC model can be realised in an emergent fashion in a 3D commuting, frustration-free...
Hamiltonian. It is not clear if this is possible. The key goal here is to identify models that possess well-defined bulk excitations together with sufficient emergent 1-form symmetries to guarantee confinement for all of such excitations (not just, say, a single sector as in the 3D toric code). Topological subsystem codes, such as the gauge color code, are natural candidates. Along with obviating the need to enforce symmetries, another advantage of emergent symmetries is that the conservation laws are manifestly true, without putting any restrictions on the system-bath coupling.

A key open question is how to construct more general families of models with emergent higher-form symmetries. We have discussed a simple duality between emergent and enforceable symmetries, that symmetries can be introduced by adding gauge degrees of freedom in systems with emergent symmetries. In the case of 0-form symmetries, a simple well-known gauging map [52, 87–91] can be used to obtain a model with emergent 0-form symmetry from a model with enforced $\mathbb{Z}_2$ 0-form symmetry. Investigating this more generally in the presence of both enforced and emergent higher form symmetries may lead to interesting new models, and here we point the interested reader to new results by Kubica and Yoshida on generalized gauging and ungauging maps [94].

We have not considered the issue of efficient decoding for these self-correcting quantum memories. We note that our two examples, the RBH model and the gauge color code, have efficient decoders with the additional feature of being single-shot [50, 60, 103]. In general, we note that the string-like nature of the excitations (errors) in these 1-form symmetric self-correcting quantum memories ensure that efficient decoders exist in general [104].

Finally, there are many avenues for further investigation into the role of symmetry in self-correcting quantum memories. In particular, one can consider the stability and feasibility of self-correction in defect-based encodings, for example in twist defects [42, 54] or the “Cheshire charge” loops of Refs. [105, 106]. Such defects have a rich connection with SPT order, as well as with both enforced and emergent symmetries. Namely, as shown in Ref. [51], one can view topological phases with nontrivial domain walls as having SPT ground states protected by 0-form symmetries, where the protecting symmetry comes from the emergent 0-form symmetries of the topological model. It would be interesting to see if SPTs protected by higher-form symmetries also arise in this way, that is, from domain walls of topological models with emergent higher-form symmetries, and whether these associated domain walls (and symmetry defects that live on their boundaries) can be thermally stable. For example, the SPT order (at temperature $T \geq 0$) in the RBH model manifests as a thermally stable domain wall in the 4D toric code [56]. Whether one can construct similarly stable domain walls in 3D or less is an open problem. Another direction is to consider more general subsystem symmetries, where the dimension need not be an integer. For example, fracton topological orders (which can be partially self-correcting [12]) have been of great interest recently [32, 83, 107].

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Appendix A: Davies Formalism

In this appendix we briefly review the Davies formalism. Recall the system-bath coupling

\[ H_{\text{full}} = H_{\text{sys}} + H_{\text{bath}} + \lambda \sum_{\alpha} S_\alpha \otimes B_\alpha, \]  

(A1)

where \( S_\alpha \otimes B_\alpha \) describe the system-bath interaction for \( S_\alpha \) a local operator acting on the system side, \( B_\alpha \) is an operator acting on the bath side, and \( \alpha \) is an arbitrary index. It is assumed that the coupling parameter is small, \( |\lambda| \ll 1 \). Suppose that the state is initialized in a ground state \( \rho(0) \) of \( H_{\text{sys}} \), then the state evolves under a Markovian master equation

\[ \dot{\rho}(t) = -i[H_{\text{sys}}, \rho(t)] + \mathcal{L}(\rho(t)), \]  

(A2)

where \( \mathcal{L} \) is the Lindblad generator. Then the initial ground state \( \rho(0) \) evolves under this master equation according to

\[ \rho(t) = e^{t\mathcal{L}}(\rho(0)). \]  

(A3)

Here, the Lindblad generator is given by

\[ \mathcal{L}(\rho) = \sum_{\alpha, \omega} h(\alpha, \omega) \left( A_{\alpha, \omega} \rho A_{\alpha, \omega}^\dagger - \frac{1}{2} \{ \rho, A_{\alpha, \omega}^\dagger A_{\alpha, \omega} \} \right). \]  

(A4)

In the above, \( A_{\alpha, \omega} \) are the Fourier components of \( A_\alpha(t) = e^{iH_{\text{sys}}t} A_\alpha e^{-iH_{\text{sys}}t} \), meaning they satisfy

\[ \sum_\omega e^{-i\omega t} A_{\alpha, \omega} = e^{iH_{\text{sys}}t} A_\alpha e^{-iH_{\text{sys}}t}. \]  

(A5)

One can think of \( A_{\alpha, \omega} \) as the component of \( A_\alpha \) that transfers energy \( \omega \) from the system to the bath. Note that when the Hamiltonian \( H_{\text{sys}} \) is comprised of commuting terms, the terms \( A_\alpha(t) \) and therefore also \( A_{\alpha, \omega} \) are local operators. The function \( h(\alpha, \omega) \) can be thought of as determining the rate of quantum jumps induced by \( A_\alpha \) that transfer energy \( \omega \) from the system to the bath, and is the only part that depends on the bath temperature. It must satisfy detailed balance condition \( h(\alpha, -\omega) = e^{-\beta \omega} h(\alpha, \omega) \), which ensures that the Gibbs state

\[ \rho_\beta = e^{-\beta H_{\text{sys}}}/\text{Tr}(e^{-\beta H_{\text{sys}}}), \]  

(A6)

at inverse temperature \( \beta \) is a fixed point of the dynamics of Eq. (A1). That is, \( \rho_\beta = \lim_{t \to \infty} \rho(t) \). Moreover, under natural ergodicity conditions (see [108, 109] for more details), it is the unique fixed point.
In the case that we have a symmetry, 

$$[H_{\text{full}}, S(g)] = 0, \quad (A7)$$

then all of the errors that are introduced due to interactions with the bath must be from processes that conserve $S(g)$. In particular, only excitations that can be created by symmetric thermal errors will be allowed. Indeed, in the case that Eq. (A7) holds, we will have that 

$$e^{\mathcal{L}_t}(S(g)\rho S(g)) = S(g)\rho S(g), \quad (A8)$$

which justifies the consideration of the symmetric energy barrier in Eq. (9).

We note that the assumptions of this formalism are satisfied for systems where the terms are comprised of commuting Paulis, as in this case the system Hamiltonian has a discrete spectrum with well separated eigenvalues. However the formalism will not necessarily work beyond this exact case, for instance, when perturbations are added and small energy splittings are introduced between previously degenerate eigenvalues. The study of thermalization times for many body stabilizer Hamiltonians in the presence of perturbations is an interesting problem.

Appendix B: Thermal instability of 0-form SPT ordered memories

In this appendix we argue that onsite symmetries are insufficient to promote a 2D topological quantum memory to be self-correcting, even if such a phase lives on the boundary of a 3D SPT model. We restrict our discussion to the case where the boundary Hamiltonian is an abelian twisted quantum double. The interesting case is where the boundary symmetry action is anomalous. (However we don’t allow this boundary symmetry action to permute the anyon types.)

We will argue that the boundary theory of a 3D SPT ordered bulk phase, if topologically ordered, will necessarily possess deconfined anyons. That is, the boundary string operators corresponding to error chains can be deformed while still respecting the symmetry, even with anomaly. We focus on (twisted) quantum doubles on the boundary of 3D group cohomology SPTs, and rather than going into the details of their construction, we focus on the key features. In particular, local degrees of freedom (of both bulk and boundary) for these models are labelled by group elements, as $|g\rangle$, $g \in G$. The symmetry action of these 2D (boundary) systems takes the form $S(g) = R(g)N(g)$, where $R(g) = \otimes_i u(g)$, with $u(g) = \sum_{h \in G} |gh\rangle \langle h|$ and $N(g)$ is diagonal in the $|g\rangle$ basis and can be represented as a constant depth quantum circuit. One can think of $R(g)$ as the onsite action, and $N(g)$ as an anomaly. This anomaly must be trivial in a strictly 2D system, or equivalently if the system is at the boundary of a trivial SPT phase.

There are two types of excitation operators in the (twisted) quantum doubles. One type of excitation string operator for the boundary system is diagonal in the $|g\rangle$ basis (i.e., it is the same as in the untwisted theory), so it commutes with $N(g)$. This excitation string operator commutes with $u(g)$, up to a phase (that is a $k$th root of unity for some $k \in \mathbb{N}$), so to commute with $R(g)$ we need to consider excitation string operators of certain lengths. In particular, the process of creating an anyonic excitation at one boundary and dragging it to another boundary (or creating an anyon pair and dragging one around a nontrivial cycle before annihilating them again) can be done in a symmetric way. Since such an operation results in a logical error and only costs a constant amount of energy, we see that the boundary theory is unstable.

Thus we see that the anomaly affords no extra stability, and the model has the same stability as a topological model with an extra onsite symmetry on top. That is, like genuine 2D topological models of this type, the model has a constant symmetric energy barrier. Note that this argument can break down in 4D, where the boundary is a 3D twisted quantum double.

Therefore we see that in the case of onsite (0-form) symmetries, the SPT ordered bulk offers no additional stability to the boundary theory. Indeed, the symmetric energy barrier for the abelian twisted quantum double remains the same as the energy barrier without symmetry: constant in the size of the system. This motivates us to consider the boundaries of SPTs protected by 1-form (or other higher-form) symmetries.

Appendix C: Energy barrier is sufficient

In this appendix, we consider the timescale for logical faults in the 1-form symmetric RBH model. We estimate the probability that an excitation loop $l$ of size $w$ emerges within the Gibbs ensemble at inverse temperature $\beta$. We show that large loop errors are quite rare if the temperature is below a critical temperature $T_c$, which we lower bound by $2/\log(5)$.

Recall the symmetric excitations are given by applying operators $Z(E', F') = \prod_{f \in F'} Z_f \prod_{e \in E'} Z_e$, where $E'$ is a cycle (i.e., has no boundary) and $F'$ is dual to a cycle on the dual lattice. We will refer to both such subsets $E'$ and $F'$ as cycles, $l = E' \cup F'$, and the resulting excitation $|\psi(l)\rangle$ as an excitation loop configuration. Moreover, we will refer to each connected component of $l$ as a loop (intuitively loops are minimal in
that no proper subset of a loop can be a cycle). The energy $E(\gamma)$ of such an excitation configuration is given by $2|E' \cup F'| \cap L| + 2|\partial (E' \cup F') \cap \partial L|$, i.e., it is proportional to the length of the bulk cycle plus the number of times a bulk cycle touches the boundary. Then the Gibbs state $\rho_\beta$ is given by the weighted mixture of all symmetric excitations, where the weights are given by

$$P_\beta(\gamma) = \frac{1}{Z} e^{-\beta E(\gamma)}, \quad Z = \sum_\gamma P_\beta(\gamma),$$

(C1)

and $\gamma = (E', F')$ represents a valid (i.e., symmetric) excitation.

Define $d = \min\{|a/2|, |b/2|\}$. From section IV D 3, for a logical error to have occurred during the system-bath interaction, we must pass through an excited state $|\psi(c)\rangle$ such that $c$ contains a bulk loop with length $w \geq d - r$, for some constant $r$ independent of system size. (Here a bulk loop is one where at least half of its support is away from the boundary). Let us bound the probability that configurations containing such a loop occurs. Define $B_w$ to be the set of cycles containing a bulk loop with size at least $w$. Then

$$\sum_{c \in B_w} P_\beta(c) \leq \sum_{\text{loops } l} \sum_{|l| \geq w, c \cap l \neq \emptyset} P_\beta(c) \leq \sum_{\text{loops } l} e^{-\beta E(l)} \sum_{c \cap l \neq \emptyset} P_\beta(c) \leq \sum_{\text{loops } l} e^{-\beta E(l)},$$

(C3)

(C4)

where from the first to the second line we have used that a configuration $c$ containing a loop $l$ differs in energy from the configuration $c \setminus l$ by $E(c) = e^{-\beta E(l)} E(c \setminus l)$. Now the last line can be rewritten to give

$$\sum_{c \in B_w} P_\beta(c) \leq \sum_{k \geq w} N(k) e^{-2\beta k},$$

(C5)

where we have ignored contributions to $E(l)$ due to the boundary (these will only decrease the right hand side of Eq. (C4)) and $N(k)$ counts the number of loops of size $k$. Since a loop $l$ resides on either the primal or dual sublattice, each of which has the structure of a cubic lattice, we can obtain a crude upper bound on $N(k)$ by considering a loop as a non-backtracking walk, where at each step one can move in $5$ independent directions. This gives the bound $N(k) \leq p(d) 5^m = k$, where $p(d)$ is a polynomial in $d$, and is in particular proportional to the number of qubits.

Then, provided $T \leq 2 / \log(5)$, we have

$$\sum_{c \in B_w} P_\beta(c) \leq p(d) \sum_{k \geq w} e^{k \log(5) - 2\beta} = p(d) \frac{1}{1 - e^{(\log(5) - 2\beta)}} e^{k \log(5) - 2\beta}$$

(C6)

(C7)

which is exponentially decaying in $k$ (again provided $T \leq 2 / \log(5)$). Since errors can be achieved only if we pass through a configuration with a bulk loop of length $d - r$, we have the contribution of configurations that can cause a logical error is bounded by

$$\text{poly}(d) \frac{1}{1 - e^{-\sigma}} e^{-\alpha d}$$

(C8)

where $\alpha = 2\beta - \log(5) > 0$ is satisfied when the temperature is small enough. One can show that the decay rate of the logical operators is exponentially long, and therefore the fidelity of the logical information is exponentially long in the system size (see Proposition 1 of Ref. [2]). One could perform a more detailed calculation to show that, with a suitable decoder, error correction succeeds after an evolution time that grows exponentially in the system size (i.e., that logical faults are also not introduced during the decoding).

We also note that a similar argument can be made for the commuting gauge color code model of Section V. A different critical temperature will be observed that depends on the choice of 3-colex.