DIMENSIONAL REDUCTION OVER THE QUANTUM SPHERE
AND NON-ABELIAN $q$-VORTICES

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Abstract. We extend equivariant dimensional reduction techniques to the case of quantum spaces which are the product of a Kähler manifold $M$ with the quantum two-sphere. We work out the reduction of bundles which are equivariant under the natural action of the quantum group $SU_q(2)$, and also of invariant gauge connections on these bundles. The reduction of Yang–Mills gauge theory on the product space leads to a $q$-deformation of the usual quiver gauge theories on $M$. We formulate generalized instanton equations on the quantum space and show that they correspond to $q$-deformations of the usual holomorphic quiver chain vortex equations on $M$. We study some topological stability conditions for the existence of solutions to these equations, and demonstrate that the corresponding vacuum moduli spaces are generally better behaved than their undeformed counterparts, but much more constrained by the $q$-deformation. We work out several explicit examples, including new examples of non-abelian vortices on Riemann surfaces, and $q$-deformations of instantons whose moduli spaces admit the standard hyper-Kähler quotient construction.

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INTRODUCTION

Let $M$ be a smooth manifold. In this paper we define and characterize vector bundles over the quantum space $\mathcal{M} := \mathbb{C}P^1_q \times M$ which are equivariant under an action of the quantum group $\text{SU}_q(2)$. Here $\mathbb{C}P^1_q$ is the quantum projective line which is defined in §1.1. The vector bundles will be given as (finitely-generated and projective) $\text{SU}_q(2)$-equivariant modules over the algebra of functions $\mathcal{A}(\mathcal{M}) = \mathcal{A}(\mathbb{C}P^1_q) \otimes \mathcal{A}(M)$. We will describe the dimensional reduction of invariant connections on the $\text{SU}_q(2)$-equivariant modules over the algebra $\mathcal{A}(\mathcal{M})$. In particular, we will reduce Yang–Mills gauge theory on $\mathcal{A}(\mathcal{M})$ to a type of Yang–Mills–Higgs theory on the manifold $M$. The vacuum equations of motion for this model give $q$-deformations of some known vortex equations, whose solutions possess, as we shall see, some remarkable properties.

In the $q = 1$ case, a general and systematic treatment of $\text{SU}(2)$-equivariant dimensional reduction over the product $\mathbb{C}P^1 \times M$ of the ordinary complex projective line $\mathbb{C}P^1$ with a Kähler manifold $M$ was first carried out in [1]. Here $\text{SU}(2)$ acts in the standard way by isometries of the homogeneous space $\mathbb{C}P^1$ and trivially on $M$. It was shown in [1] that there is a one-to-one correspondence between $\text{SU}(2)$-equivariant vector bundles over $\mathbb{C}P^1 \times M$ and $\text{U}(1)$-equivariant vector bundles over $M$, with $\text{U}(1)$ acting trivially on $M$. The reduced vector bundle has the structure of a quiver bundle, in this case a representation of the linear $A_{m+1}$ quiver chain in the category of complex vector bundles over $M$. Moreover, certain natural first order gauge theory equations on $\mathbb{C}P^1 \times M$ reduce to generalizations of vortex equations called holomorphic chain vortex equations, which contain a multitude of BPS-type integrable equations as special cases [20]. These include standard abelian and non-abelian vortex equations in two dimensions, and the self-duality and perturbed abelian Seiberg–Witten monopole equations in four dimensions. With suitable notions of stability for holomorphic bundles over $\mathbb{C}P^1 \times M$ and the corresponding quiver bundles over $M$, a variant of the Hitchin–Kobayashi correspondence identifies the necessary and sufficient conditions for the existence of solutions to these equations [1]. This particular reduction has been further developed in [31, 20, 12], where some physical applications are also considered. The reduction generalizes to the product of $M$ with any homogeneous space which is a flag manifold; see [2, 21] for the general theory.
In this paper we will study how equivariant dimensional reduction and the ensuing vortex equations are modified when the ‘internal’ sphere $\mathbb{C}P^1$ is replaced with a particular noncommutative deformation. Dimensional reduction over the fuzzy sphere $\mathbb{C}P^1_F$ was considered in [5, 4, 16], where it was shown that the deformation significantly alters the vacuum structure of the induced Yang–Mills–Higgs theory, which in some instances may not coincide with the standard vortex models in the commutative limit. In particular, solutions of abelian vortex equations are studied in [16] which correspond to instantons in the original Yang–Mills theory on $\mathbb{C}P^1_F \times M$ but are nevertheless non-BPS states of the dimensionally reduced field theory. In the following we will demonstrate that a similar vacuum structure emerges when the dimensional reduction is performed over a quantum sphere $\mathbb{C}P^1_q$. As discussed in [7], a basic problem with standard vortex equations is that it is not possible to reach the zeroes of the corresponding Yang–Mills–Higgs action functional by means of non-trivial vortex solutions, due to topological obstructions. In [3] it was shown that one can improve this functional by using the formalism of twisted quiver bundles, which yields zeroes of the action for bundles admitting flat connections. In the present paper we show that, in contrast to the usual quiver gauge theories that arise through dimensional reduction, the same is true for the Yang–Mills–Higgs models which are systematically obtained via SU$_q$(2)-equivariant dimensional reduction over $\mathbb{C}P^1_q$.

In order to rigorously carry out the dimensional reduction in parallel to the commutative case, it is necessary to extend the equivariant decompositions of [11, 31] within the algebraic framework of noncommutative geometry and in a Hopf algebraic framework appropriate to the action of the quantum group SU$_q$(2). This is the content of §1–§3 of the present paper. In §1 and §2 we extend the requisite geometry of the projective line $\mathbb{C}P^1$ to the $q$-deformed case $\mathbb{C}P^1_q$, using the fact that there are finitely-generated projective modules over the quantum sphere that correspond to the canonical line bundles on the Riemann sphere in the $q \to 1$ limit. In §3 we generalize the decompositions of [11, 31] to invariant gauge fields for the action of SU$_q$(2) on $M = \mathbb{C}P^1_q \times M$. In §4 we study the reduction of Yang–Mills theory on $M$. In particular, we formulate a suitable notion of generalized instanton on the quantum space $M$ which coincides with solutions of the vortex equations associated to minima of the induced $q$-deformed Yang–Mills–Higgs action functional on $M$; we call the (gauge equivalence classes of) solutions to these equations ‘$q$-vortices’. We also examine in detail the structure of the corresponding vacuum moduli spaces and the topological stability conditions for the existence of solutions to the $q$-vortex equations, finding in general that these moduli spaces are much more constrained than their classical $q \to 1$ limits. In §5 we study some explicit examples and compare with analogous results in the literature for the case $q = 1$, showing that the $q$-deformation generically improves the geometrical structure of the associated moduli spaces. In particular, we analyse moduli spaces of $q$-vortices on Riemann surfaces giving new examples of non-abelian vortices, and show that our $q$-deformations of instantons on Kähler surfaces are analogous to those of some previous noncommutative deformations of the self-duality equations.

Conventions. In the following we shall use the terminology covariance to mean both covariance for an action and ‘co-covariance’ for a coaction. The $q$-number

\begin{equation}
[s] = [s]_q := \frac{q^s - q^{-s}}{q - q^{-1}},
\end{equation}

(0.1)
is defined for $q \neq 1$ and any $s \in \mathbb{R}$. For a coproduct $\Delta$ we use the conventional Sweedler notation $\Delta(x) = x^{(1)} \otimes x^{(2)}$ (with implicit summation). This convention is iterated to give $(\text{id} \otimes \Delta) \circ \Delta(x) = (\Delta \otimes \text{id}) \circ \Delta(x) = x^{(1)} \otimes x^{(2)} \otimes x^{(3)}$, and so on.

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1. **SU$_q$(2)-equivariant bundles on the quantum projective line**

The quantum projective line $\mathbb{C}P^1_q$ is defined as a quotient of the sphere $S^3_q \simeq \text{SU}_q(2)$ with respect to an action of the group $U(1)$. It is the standard Podleś sphere $S^2_q$ of [28] with additional structure. The construction we need is the well-known quantum principal $U(1)$-bundle over $S^2_q$, whose total space is the manifold of the quantum group $\text{SU}_q(2)$.

1.1. **Quantum projective line $\mathbb{C}P^1_q$.**

We begin with the algebras of $S^3_q$ and $\mathbb{C}P^1_q$. The manifold of $S^3_q$ is identified with the manifold of the quantum group $\text{SU}_q(2)$. The deformation parameter $q \in \mathbb{R}$ can be restricted to the interval $0 < q < 1$ without loss of generality. The coordinate algebra $\mathcal{A}(\text{SU}_q(2))$ is the $\ast$-algebra generated by elements $a$ and $c$ with the relations

$$
ac = qca \quad \text{and} \quad c^* a^* = q a^* c^* , \quad a c^* = q c^* a \quad \text{and} \quad ca^* = q a^* c ,
$$

(1.1)

$$
c c^* = c^* c \quad \text{and} \quad a^* a + c^* c = a a^* + q^2 c c^* = 1 .
$$

These relations are equivalent to requiring that the ‘defining’ matrix

$$
U = \begin{pmatrix} a & -q c^* \\ c & a^* \end{pmatrix}
$$

is unitary, $UU^* = U^* U = 1$. The Hopf algebra structure for $\mathcal{A}(\text{SU}_q(2))$ is given by the coproduct

$$
\Delta \begin{pmatrix} a & -q c^* \\ c & a^* \end{pmatrix} = \begin{pmatrix} a & -q c^* \\ c & a^* \end{pmatrix} \otimes \begin{pmatrix} a & -q c^* \\ c & a^* \end{pmatrix} ,
$$

with a ‘tensor product’ of rows by columns, e.g. $\Delta(a) = a \otimes a - q c^* \otimes c$, etc., the antipode

$$
S \begin{pmatrix} a & -q c^* \\ c & a^* \end{pmatrix} = \begin{pmatrix} a^* & c^* \\ -q c & a \end{pmatrix} ,
$$

and the counit

$$
\epsilon \begin{pmatrix} a & -q c^* \\ c & a^* \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} .
$$

The quantum universal enveloping algebra $\mathcal{U}_q(\text{su}(2))$ is the Hopf $\ast$-algebra generated as an algebra by four elements $K, K^{-1}, E, F$ with $KK^{-1} = 1 = K^{-1} K$ and relations

$$
K^{\pm 1} E = q^{\pm 1} E K^{\pm 1} , \quad K^{\pm 1} F = q^{\mp 1} F K^{\mp 1} \quad \text{and} \quad [E, F] = \frac{K^2 - K^{-2}}{q - q^{-1}} .
$$

(1.2)
The $\ast$-structure is simply
\[ K^\ast = K, \quad E^\ast = F \quad \text{and} \quad F^\ast = E, \]
and the Hopf algebra structure is provided by the coproduct $\Delta$, the antipode $S$, and the counit $\epsilon$ defined by
\[ \Delta(K^{\pm 1}) = K^{\pm 1} \otimes K^{\pm 1}, \quad \Delta(E) = E \otimes K + K^{-1} \otimes E, \quad \Delta(F) = F \otimes K + K^{-1} \otimes F, \]
\[ S(K) = K^{-1}, \quad S(E) = -q E, \quad S(F) = -q^{-1} F, \]
\[ \epsilon(K) = 1, \quad \epsilon(E) = \epsilon(F) = 0. \]
There is a bilinear pairing between $U_q(\mathfrak{su}(2))$ and $A(SU_q(2))$ given on generators by
\[ \langle K, a \rangle = q^{-1/2}, \quad \langle K^{-1}, a \rangle = q^{1/2}, \quad \langle K, a^\ast \rangle = q^{1/2} \quad \text{and} \quad \langle K^{-1}, a^\ast \rangle = q^{-1/2}, \]
\[ \langle E, c \rangle = 1 \quad \text{and} \quad \langle F, c^\ast \rangle = -q^{-1}, \]
with all other couples of generators pairing to 0. One regards $U_q(\mathfrak{su}(2))$ as a subspace of the linear dual of $A(SU_q(2))$ via this pairing. There are canonical left and right $U_q(\mathfrak{su}(2))$-module algebra structures on $A(SU_q(2))$ such that
\[ \langle g, h \triangleright x \rangle := \langle gh, x \rangle \quad \text{and} \quad \langle g, x \triangleleft h \rangle := \langle hg, x \rangle \]
for all $g, h \in U_q(\mathfrak{su}(2))$, $x \in A(SU_q(2))$. They are given by $h \triangleright x := ((1 \otimes h), \Delta(x))$ and $x \triangleleft h := ((h \otimes \text{id}), \Delta(x))$, or equivalently
\[ h \triangleright x := x(1) \langle h, x(2) \rangle \quad \text{and} \quad x \triangleleft h := \langle h, x(1) \rangle x(2) \]
in the Sweedler notation. These right and left actions mutually commute,
\[ (h \triangleright x) \circ g = \langle x(1) \langle h, x(2) \rangle \rangle \circ g = \langle g, x(1) \rangle x(2) \langle h, x(3) \rangle = h \triangleright \langle (g, x(1)) x(2) \rangle = h \triangleright (x \circ g), \]
and since the pairing satisfies
\[ \langle S(h)^\ast, x \rangle = \overline{\langle h, x^\ast \rangle} \]
for all $h \in U_q(\mathfrak{su}(2))$, $x \in A(SU_q(2))$, the $\ast$-structure is compatible with both actions,
\[ h \triangleright x^\ast = (S(h)^\ast \triangleright x)^\ast \quad \text{and} \quad x^\ast \triangleleft h = (x \triangleleft S(h)^\ast)^\ast \]
for all $h \in U_q(\mathfrak{su}(2))$, $x \in A(SU_q(2))$. The left action for any $s \in \mathbb{N}_0$ is given explicitly by
\[ K^{\pm 1} \triangleright a^\ast = q^{\mp \frac{s}{2}} a^\ast, \quad \text{and} \quad K^{\pm 1} \triangleright a^s = q^{\mp \frac{s}{2}} a^s, \]
\[ K^{\pm 1} \triangleright c^s = q^{\mp \frac{s}{2}} c^s \quad \text{and} \quad K^{\pm 1} \triangleright c^\ast = q^{\pm \frac{s}{2}} c^\ast, \]
\[ F \triangleright a^\ast = 0 \quad \text{and} \quad F \triangleright a^s = q^{(1-s)/2} [s] c a^s a^{-1}, \]
\[ F \triangleright c^\ast = 0 \quad \text{and} \quad F \triangleright c^s = -q^{-(1+s)/2} [s] a c^s a^{-1}, \]
\[ E \triangleright a^\ast = -q^{3/2-s} [s] a^{-1} c^\ast \quad \text{and} \quad E \triangleright a^s = 0, \]
\[ E \triangleright c^\ast = q^{(1-s)/2} [s] c^{-1} a^s \quad \text{and} \quad E \triangleright c^s = 0. \]
The right action is given explicitly by
\[
\begin{align*}
\alpha^* \triangleleft K^{\pm 1} &= q^{\mp \frac{3}{2}} a^* \quad \text{and} \quad \alpha^* \triangleleft K^{\pm 1} = q^{\pm \frac{3}{2}} a^* \alpha^*, \\
\alpha^* \triangleleft F &= q^{(s-1)/2} [s] c a^{s-1} \quad \text{and} \quad \alpha^* \triangleleft F = 0, \\
\alpha^* \triangleleft E &= 0 \quad \text{and} \quad \alpha^* \triangleleft E = -q^{(3-s)/2} [s] c a^* s^{-1},
\end{align*}
\]
(1.4) \( \alpha^* \triangleleft E = q^{(s-1)/2} [s] c a^{s-1} \alpha^* \quad \text{and} \quad \alpha^* \triangleleft E = 0. \)

Now we describe the U(1)-principal bundle over \( S^2_q \), whose total space is the manifold of the quantum group \( SU_q(2) \). It is an example of a quantum homogeneous space \( [10] \) constructed as follows. If \( \mathcal{A}(U(1)) := \mathbb{C}[\zeta, \zeta^*] / (\zeta \zeta^* - 1) \) denotes the (commutative) algebra of coordinate functions on the group U(1), the map
\[
\pi : \mathcal{A}(SU_q(2)) \longrightarrow \mathcal{A}(U(1)), \quad \pi \left( \begin{array}{cc} a & -q c^* \\ c & a^* \end{array} \right) = \left( \begin{array}{cc} \zeta & 0 \\ 0 & \zeta^* \end{array} \right)
\]
is a surjective Hopf *-algebra homomorphism, so that \( \mathcal{A}(U(1)) \) becomes a quantum subgroup of \( SU_q(2) \) with a right coaction
\[
\Delta_R := (\text{id} \otimes \pi) \circ \Delta : \mathcal{A}(SU_q(2)) \longrightarrow \mathcal{A}(SU_q(2)) \otimes \mathcal{A}(U(1)).
\]
The coinvariant elements for this coaction, i.e. elements \( \{ x \in \mathcal{A}(SU_q(2)) \mid \Delta_R(x) = x \otimes 1 \} \), generate a subalgebra of \( \mathcal{A}(SU_q(2)) \) which is the coordinate algebra \( \mathcal{A}(S^2_q) \) of the standard Podleś sphere \( S^2_q \) first described in \( [28] \).

For the purposes of the present paper, it will be useful to also have an equivalent description of the bundle by taking an action (irrelevantly right or left) of the abelian group \( U(1) = \{ z \in \mathbb{C} \mid z z^* = 1 \} \) on the algebra \( \mathcal{A}(SU_q(2)) \), i.e. we consider the map
\[
\alpha : U(1) \longrightarrow \text{Aut} (\mathcal{A}(SU_q(2)))
\]
defined on generators by
\[
\alpha_z(a) = a z \quad \text{and} \quad \alpha_z(a^*) = a^* z^*,
\]
(1.7) \( \alpha_z(c) = c z \quad \text{and} \quad \alpha_z(c^*) = c^* z^* \),
and extended as an algebra map, \( \alpha_z(x y) = \alpha_z(x) \alpha_z(y) \) for \( x, y \in \mathcal{A}(SU_q(2)) \) and \( z \in U(1) \). Here the complex number \( z \) is the evaluation of the function \( \zeta \in \mathcal{A}(U(1)) \). The coordinate algebra \( \mathcal{A}(S^2_q) \) is then regarded as the subalgebra of invariant elements in \( \mathcal{A}(SU_q(2)) \),
\[
\mathcal{A}(S^2_q) := \mathcal{A}(SU_q(2))^{U(1)} := \{ x \in \mathcal{A}(SU_q(2)) \mid \alpha_z(x) = x \}.
\]
As a set of generators for \( \mathcal{A}(S^2_q) \) we may take
\[
B_- := a c^*, \quad B_+ := c a^* \quad \text{and} \quad B_0 := c c^*,
\]
for which one finds relations
\[
B_- B_0 = q^2 B_0 B_- \quad \text{and} \quad B_+ B_0 = q^{-2} B_0 B_+,
\]
\[
B_- B_+ = q^2 B_0 (1 - q^2 B_0) \quad \text{and} \quad B_+ B_- = B_0 (1 - B_0),
\]
and $*$-structure $(B_0)^* = B_0$ and $(B_+)^* = B_-$. The algebra inclusion $\mathcal{A}(S^2_q) \hookrightarrow \mathcal{A}(\text{SU}_q(2))$ is a quantum principal bundle and can be endowed with compatible calculi \cite{10}, a construction that we shall illustrate later on.

In \(2.2\) we will describe a natural complex structure on the quantum two-sphere $S^2_q$ for the unique two-dimensional covariant calculus on it. This will transform the sphere $S^2_q$ into a quantum riemannian sphere or quantum projective line $\mathbb{C}P^1_q$. Having this in mind, with a slight abuse of ‘language’ we will speak of $\mathbb{C}P^1_q$ rather than $S^2_q$ from now on.

The sphere $S^2_q$ (and hence the quantum projective line $\mathbb{C}P^1_q$) is a quantum homogeneous space of $\text{SU}_q(2)$ and the coproduct of $\mathcal{A}(\text{SU}_q(2))$ restricts to a left coaction of $\mathcal{A}(\text{SU}_q(2))$ on $\mathcal{A}(S^2_q)$ (or $\mathcal{A}(\mathbb{C}P^1_q)$):

$$\Delta_L : \mathcal{A}(\mathbb{C}P^1_q) \longrightarrow \mathcal{A}(\text{SU}_q(2)) \otimes \mathcal{A}(\mathbb{C}P^1_q).$$

In particular, the elements

$$Y_- := -ac^*, \quad Y_+ := qca^* \quad \text{and} \quad Y_0 := q^2 (1 + q^2)^{-1} - q^2 c c^*$$

transform according to the fundamental ‘vector corepresentation’ of $\text{SU}_q(2)$ given by

$$\Delta_L(Y_-) = a^2 \otimes Y_- - (1 + q^{-2}) Y_- \otimes Y_0 + c^2 \otimes Y_+,$$

$$\Delta_L(Y_0) = qac \otimes Y_- + (1 + q^{-2}) Y_0 \otimes Y_0 - c^* a^* \otimes Y_+,$$

and

$$\Delta_L(Y_+) = q^2 c^2 \otimes Y_- + (1 + q^{-2}) Y_+ \otimes Y_0 + a^* c^* \otimes Y_+.$$

The following result is evident.

**Proposition 1.12.** The element $1 \in \mathcal{A}(\mathbb{C}P^1_q)$ is the only coinvariant element for this coaction, i.e. the only $x \in \mathcal{A}(\mathbb{C}P^1_q)$ for which $\Delta_L(x) = 1 \otimes x$.

### 1.2. Equivariant line bundles on $\mathbb{C}P^1_q$.

Let $\rho : U(1) \rightarrow V$ be a representation of $U(1)$ on a finite-dimensional complex vector space $V$. The corresponding space of $\rho$-equivariant elements is given by

$$\mathcal{A}(\text{SU}_q(2)) \boxtimes_\rho V := \{ \varphi \in \mathcal{A}(\text{SU}_q(2)) \otimes V \mid (\alpha \otimes \text{id}) \varphi = ((\text{id} \otimes \rho^{-1})) \varphi \},$$

where $\alpha$ is the action \eqref{1.7} of $U(1)$ on $\mathcal{A}(\text{SU}_q(2))$. The space \eqref{1.13} is an $\mathcal{A}(\mathbb{C}P^1_q)$-bimodule. We shall think of it as the module of sections of the vector bundle associated with the quantum principal $U(1)$-bundle on $\mathbb{C}P^1_q$ via the representation $\rho$. There is a natural $\text{SU}_q(2)$-equivariance, in that the left coaction $\Delta$ of $\mathcal{A}(\text{SU}_q(2))$ on itself extends in a natural way to a left coaction on $\mathcal{A}(\text{SU}_q(2)) \boxtimes_\rho V$ given by

$$\Delta^\rho = \Delta \otimes \text{id} : \mathcal{A}(\text{SU}_q(2)) \boxtimes_\rho V \longrightarrow \mathcal{A}(\text{SU}_q(2)) \otimes (\mathcal{A}(\text{SU}_q(2)) \boxtimes_\rho V).$$

The irreducible representations of $U(1)$ are labelled by an integer $n \in \mathbb{Z}$. If $C_n \simeq \mathbb{C}$ is the irreducible one-dimensional left $U(1)$-module of weight $n$, they are given by

$$\rho_n : U(1) \longrightarrow \text{Aut}(C_n), \quad C_n \ni v \rightarrow z^n v \in C_n.$$

The corresponding spaces of equivariant elements are well-known and amount to a vector space decomposition \cite{24, eq. (1.10)]

$$\mathcal{A}(\text{SU}_q(2)) = \bigoplus_{n \in \mathbb{Z}} \mathcal{L}_n,$$

where
where
\begin{equation}
(1.17) \quad \mathcal{L}_n := \mathcal{A}(\text{SU}_q(2)) \boxtimes_{\rho_n} \mathbb{C} \simeq \{ x \in \mathcal{A}(\text{SU}_q(2)) \mid \alpha_z(x) = x(z^*)^n \} .
\end{equation}
In particular, \( \mathcal{L}_0 = \mathcal{A}(\mathbb{C}P^1_q) \). One has \( \mathcal{L}_+^* = \mathcal{L}_{-n} \) and \( \mathcal{L}_n \mathcal{L}_m = \mathcal{L}_{n+m} \). Each \( \mathcal{L}_n \) is clearly a bimodule over \( \mathcal{A}(\mathbb{C}P^1_q) \) and is naturally isomorphic to \( \mathcal{A}(\text{SU}_q(2)) \boxtimes_{\rho_n} \mathbb{C} \). It was shown in [33, Prop. 6.4] that each \( \mathcal{L}_n \) is a finitely-generated projective left (and right) \( \mathcal{A}(\mathbb{C}P^1_q) \)-module of rank one. They give the modules of \( \text{SU}_q(2) \)-equivariant elements or of sections of line bundles over the quantum projective line \( \mathbb{C}P^1_q \) with monopole charges \(-n\). One has the following results (cfr. [17, Prop. 3.1]).

**Lemma 1.18.**

1. Each \( \mathcal{L}_n \) is the bimodule of equivariant elements associated with the irreducible representation of \( \text{U}(1) \) with weight \( n \).

2. The natural map \( \mathcal{L}_n \otimes \mathcal{L}_m \to \mathcal{L}_{n+m} \) defined by multiplication induces an isomorphism of \( \mathcal{A}(\mathbb{C}P^1_q) \)-bimodules
\[ \mathcal{L}_n \otimes \mathcal{A}(\mathbb{C}P^1_q) \mathcal{L}_m \simeq \mathcal{L}_{n+m} , \]
and in particular \( \text{Hom}_{\mathcal{A}(\mathbb{C}P^1_q)}(\mathcal{L}_m, \mathcal{L}_n) \simeq \mathcal{L}_{n-m} \).

**Proof.** These results follow by using the representation theory of \( \text{U}(1) \) as well as the relations
\[ a \otimes \mathcal{A}(\mathbb{C}P^1_q) c = q c \otimes \mathcal{A}(\mathbb{C}P^1_q) a , \quad a \otimes \mathcal{A}(\mathbb{C}P^1_q) c^* = q c^* \otimes \mathcal{A}(\mathbb{C}P^1_q) a , \quad c \otimes \mathcal{A}(\mathbb{C}P^1_q) c^* = c^* \otimes \mathcal{A}(\mathbb{C}P^1_q) c , \]
and so on, which are easily established. \( \square \)

From the transformations in (1.8), it follows that an \( \mathcal{A}(\mathbb{C}P^1_q) \)-module generating set for \( \mathcal{L}_n \) is given by elements
\begin{equation}
(1.19) \quad |\Psi^{(n)}\rangle = \begin{cases} 
\mu \geq 0, & \text{for } n \geq 0 , \\ e^{[n]-\mu} a^\mu & \text{for } n \leq 0 , \mu = 0,1,\ldots,|n| .
\end{cases}
\end{equation}
Then one writes equivariant elements as
\begin{equation}
(1.20) \quad \varphi_f = \begin{cases} 
\sum_{\mu=0}^{n} e^{\mu} a^{n-\mu} f_\mu = \sum_{\mu=0}^{n} \tilde{f}_\mu e^{\mu} a^{n-\mu} & \text{for } n \geq 0 , \\
\sum_{\mu=0}^{\lceil n \rceil} e^{[n]-\mu} a^\mu f_\mu = \sum_{\mu=0}^{\lceil n \rceil} \tilde{f}_\mu e^{[n]-\mu} a^\mu & \text{for } n \leq 0 ,
\end{cases}
\end{equation}
with \( f_\mu \) and \( \tilde{f}_\mu \) generic elements in \( \mathcal{A}(\mathbb{C}P^1_q) \). The elements in (1.19) are not independent over \( \mathcal{A}(\mathbb{C}P^1_q) \), since the bimodules \( \mathcal{L}_n \) are not free modules.

A generic finite-dimensional representation \((V, \rho)\) for \( \text{U}(1) \) is given by a weight decomposition
\begin{equation}
(1.21) \quad V = \bigoplus_{n \in \mathbb{W}(V)} C_n \otimes V_n , \quad \rho = \bigoplus_{n \in \mathbb{W}(V)} \rho_n \otimes \text{id} .
\end{equation}
Here \((C_n, \rho_n)\) is the one-dimensional irreducible representation of \( \text{U}(1) \) with weight \( n \in \mathbb{Z} \) given in (1.15), the spaces \( V_n = \text{Hom}_{\text{U}(1)}(C_n, V) \) are the multiplicity spaces, and the set
$W(V) = \{ n \in \mathbb{Z} \mid V_n \neq 0 \}$ is the set of weights of $V$. For the corresponding space of $\rho$-equivariant elements we have a corresponding decomposition

\begin{equation}
\mathcal{A}(SU_q(2)) \mathbb{C} \otimes V = \bigoplus_{n \in W(V)} \mathcal{L}_n \otimes V_n,
\end{equation}

with $\mathcal{L}_n$ the irreducible modules in (1.17) giving sections of line bundles over $\mathbb{C}P^1_q$.

The left action of the group-like element $K$ on $\mathcal{A}(SU_q(2))$ allows one to give a dual presentation of the line bundles $\mathcal{L}_n$ as

\begin{equation}
\mathcal{L}_n = \{ x \in \mathcal{A}(SU_q(2)) \mid K \triangleright x = q^{n/2} x \}.
\end{equation}

Indeed, if $H$ is the infinitesimal generator of the $U(1)$-action $\alpha$, the group-like element $K$ can be written as $K = q^{-H/2}$. Then from the relations (1.2) of $\mathcal{U}_q(\mathfrak{su}(2))$ one finds

\begin{equation}
E \triangleright \mathcal{L}_n \subset \mathcal{L}_{n+2} \quad \text{and} \quad F \triangleright \mathcal{L}_n \subset \mathcal{L}_{n-2}.
\end{equation}

On the other hand, commutativity of the left and right actions of $\mathcal{U}_q(\mathfrak{su}(2))$ yields

\begin{equation}
\mathcal{L}_n \triangleleft h \subset \mathcal{L}_n
\end{equation}

for all $h \in \mathcal{U}_q(\mathfrak{su}(2))$. It was shown in [33, Thm. 4.1] that there is also a decomposition

\begin{equation}
\mathcal{L}_n = \bigoplus_{J=\lfloor |n|/2 \rfloor + 1, \lfloor |n|/2 \rfloor + 2, \ldots} V_j^{(n)},
\end{equation}

with $V_j^{(n)}$ the spin $J$ representation space (for the right action) of $\mathcal{U}_q(\mathfrak{su}(2))$. Combined with (1.16), we get a Peter-Weyl decomposition for $\mathcal{A}(SU_q(2))$ [36]. A PBW-basis for $\mathcal{A}(SU_q(2))$ is given by monomials $a^m c^k c^l$ for $k, l = 0, 1, \ldots$ and $m \in \mathbb{Z}$, with the convention that $a^{-m}$ is short-hand notation for $a^{-m}$ when $m > 0$. Furthermore, a similar basis for $\mathcal{L}_n$ is given by the monomials $a^{-k} c^k c^l$, since from (1.3) it follows that $K \triangleright (a^m c^k c^l) = q^{(-m-k+l)/2} a^m c^k c^l$ and the requirement that $-m - k + l = n$ is met by redefining $l \rightarrow l + n$ forcing in turn $m = l - k$. In particular, the monomials $a^{-k} c^k c^l$ are the only $K$-invariant elements, thus providing a PBW-basis for $\mathcal{L}_0 = \mathcal{A}(\mathbb{C}P^1_q)$.

## 2. SU_q(2)-invariant gauge fields on the quantum projective line

We will now describe connections on the quantum projective line. For this, we will need an explicit description of the calculi on the quantum principal bundle over $\mathbb{C}P^1_q$. The principal bundle $(\mathcal{A}(SU_q(2)), \mathcal{A}(\mathbb{C}P^1_q), \mathcal{A}(U(1)))$ is endowed with compatible non-universal calculi [10, 11] obtained from the three-dimensional left-covariant calculus on $SU_q(2)$ [36], which we present first. We then describe the unique left-covariant two-dimensional calculus on the quantum projective line $\mathbb{C}P^1_q$ [29] obtained by restriction, and also the projected calculus on the structure group $U(1)$. The calculus on $\mathbb{C}P^1_q$ can be canonically decomposed into a holomorphic and an anti-holomorphic part. All the calculi are compatible in a natural sense. These constructions will produce a connection on the quantum principal bundle over $\mathbb{C}P^1_q$ with respect to the left-covariant calculus $\Omega^*(\mathbb{C}P^1_q)$, also with a natural holomorphic structure. This connection will determine a covariant derivative on the module of equivariant elements $\mathcal{L}_n$, which can be shown [19] to correspond to the canonical Grassmann connection on the associated projective modules over $\mathcal{A}(\mathbb{C}P^1_q)$. We also briefly recall how to compute the monopole number $n \in \mathbb{Z}$ by
Their coproducts and antipodes are easily found to be

$$X_z = \frac{1 - K^4}{1 - q^{-2}}, \quad X_- = q^{-1/2} FK \quad \text{and} \quad X_+ = q^{1/2} E K = X^*_+.$$ 

Their coproducts and antipodes are easily found to be

\begin{align}
\Delta(X_z) &= 1 \otimes X_z + X_z \otimes K^4 \quad \text{and} \quad \Delta(X_\pm) = 1 \otimes X_\pm + X_\pm \otimes K^2, \\
\Delta(X_z) &= 1 \otimes X_z + X_z \otimes K^4 \quad \text{and} \quad \Delta(X_-) = 1 \otimes X_- + X_- \otimes K^2.
\end{align}

The dual space of one-forms $\Omega^1(SU_q(2))$ has a basis

\begin{align}
\beta_z &= a^* da + c^* dc, \quad \beta_- = a^* da - q a^* dc^* \quad \text{and} \quad \beta_+ = a dc - q c da
\end{align}

of left-invariant forms. The differential $d : A(SU_q(2)) \rightarrow \Omega^1(SU_q(2))$ is given by

\begin{align}
df &= (X_\beta \triangleright f) \beta_- + (X_\beta \triangleright f) \beta_+ + (X_\beta \triangleright f) \beta_z
\end{align}

for all $f \in A(SU_q(2))$. If $\Delta^{(1)}$ is the (left) coaction of $A(SU_q(2))$ on itself extended to forms, the left-coariance of the basis forms is the statement that

\begin{align}
\Delta^{(1)}(\beta_s) &= 1 \otimes \beta_s,
\end{align}

while the left-covariance of the calculus is stated as

\begin{align}
(\Delta \otimes \text{id}) \circ \Delta^{(1)} = (\Delta^{(1)} \otimes \text{id}) \circ \Delta^{(1)} \quad \text{and} \quad (\epsilon \otimes \text{id}) \circ \Delta^{(1)} = 1.
\end{align}

The requirement that it is a $*$-calculus, i.e. $d(f^*) = (df)^*$, yields

\begin{align}
\beta_*^z = -\beta_+ \quad \text{and} \quad \beta_*^z = -\beta_+.
\end{align}

The bimodule structure is given by

\begin{align}
\beta_z a &= q^{-2} a \beta_z, \quad \beta_z a^* = q^2 a^* \beta_z, \quad \beta_\pm a = q^{-1} a \beta_\pm \quad \text{and} \quad \beta_\pm a^* = q a^* \beta_\pm,
\end{align}

\begin{align}
\beta_z c &= q^{-2} c \beta_z, \quad \beta_z c^* = q^2 c^* \beta_z, \quad \beta_\pm c = q^{-1} c \beta_\pm \quad \text{and} \quad \beta_\pm c^* = q c^* \beta_\pm.
\end{align}

Higher degree forms can be defined in a natural way by requiring compatibility with the commutation relations (the bimodule structure (2.6)) and that $d^2 = 0$. One has

\begin{align}
d\beta_z &= -\beta_- \wedge \beta_+, \quad d\beta_+ = q^2 (1 + q^2) \beta_z \wedge \beta_+ \quad \text{and} \quad d\beta_- = -q^{-2} (1 + q^2) \beta_z \wedge \beta_-
\end{align}

together with the commutation relations

\begin{align}
\beta_+ \wedge \beta_+ &= \beta_- \wedge \beta_- = \beta_z \wedge \beta_z = 0, \\
\beta_- \wedge \beta_+ + q^{-2} \beta_+ \wedge \beta_- &= 0, \\
\beta_+ \wedge \beta_- + q^4 \beta_- \wedge \beta_+ &= 0,
\end{align}

\begin{align}
\beta_z \wedge \beta_+ + q^{-4} \beta_+ \wedge \beta_z &= 0.
\end{align}
Finally, there is a unique top form $\beta_- \wedge \beta_+ \wedge \beta_2$. We may summarize the above results as follows.

**Proposition 2.9.** For the three-dimensional left-covariant differential calculus on $SU_q(2)$, the bimodules of forms are all trivial (left) $\mathcal{A}(SU_q(2))$-modules given explicitly as

$$
\begin{align*}
\Omega^0(SU_q(2)) &= \mathcal{A}(SU_q(2)) , \\
\Omega^1(SU_q(2)) &= \mathcal{A}(SU_q(2))\langle \beta_- , \beta_+ , \beta_2 \rangle , \\
\Omega^2(SU_q(2)) &= \mathcal{A}(SU_q(2))\langle \beta_- \wedge \beta_+ , \beta_- \wedge \beta_2 , \beta_+ \wedge \beta_2 \rangle , \\
\Omega^3(SU_q(2)) &= \mathcal{A}(SU_q(2))\beta_- \wedge \beta_+ \wedge \beta_2 .
\end{align*}
$$

The exterior differential and commutation relations are obtained from (2.7) and (2.8), whereas the bimodule structure is obtained from (2.6).

### 2.2. Holomorphic forms on $\mathbb{CP}^1_q$.

The restriction of the three-dimensional calculus of §2.1 to the quantum projective line $\mathbb{CP}^1_q$ yields the unique left-covariant two-dimensional calculus on $\mathbb{CP}^1_q$ [22]. Further development of this approach has led to a description of this calculus in terms of a Dirac operator [32]. The ‘cotangent bundle’ $\Omega^1(\mathbb{CP}^1_q)$ is shown to be isomorphic to the direct sum $\mathcal{L}_{-2} \oplus \mathcal{L}_{+2}$ of the line bundles with degree (monopole charge) $\pm 2$. Since the element $K$ acts as the identity on $\mathcal{A}(\mathbb{CP}^1_q)$, the differential (2.24) restricted to $\mathcal{A}(\mathbb{CP}^1_q)$ becomes

$$
\overline{\partial}f = (X_+ \triangleright f) \beta_- + (X_- \triangleright f) \beta_+ = q^{-1/2}(F \triangleright f) \beta_- + q^{1/2}(E \triangleright f) \beta_+
$$

for $f \in \mathcal{A}(\mathbb{CP}^1_q)$. This leads to a decomposition of the exterior differential into a holomorphic and an anti-holomorphic part, $d = \overline{\partial} + \partial$, with

$$
\overline{\partial}f = (X_+ \triangleright f) \beta_- \quad \text{and} \quad \partial f = (X_- \triangleright f) \beta_+.
$$

for $f \in \mathcal{A}(\mathbb{CP}^1_q)$. An explicit computation on the generators (1.10) of $\mathbb{CP}^1_q$ yields

$$
\begin{align*}
\overline{\partial}B_- &= -q^{-1} a^2 \beta_-, \\
\overline{\partial}B_0 &= -q^{-1} c a \beta_- \\
\overline{\partial}B_+ &= c^2 \beta_-
\end{align*}
$$

and

$$
\begin{align*}
\partial B_+ &= q a^2 \beta_+, \\
\partial B_0 &= c^* a^* \beta_+ \\
\partial B_- &= -q^2 c^* a^2 \beta_+.
\end{align*}
$$

It follows that

$$
\Omega^1(\mathbb{CP}^1_q) = \Omega^{0,1}(\mathbb{CP}^1_q) \oplus \Omega^{1,0}(\mathbb{CP}^1_q)
$$

where $\Omega^{0,1}(\mathbb{CP}^1_q) \simeq \mathcal{L}_{-2,\beta_-} \simeq \overline{\partial}(\mathcal{A}(\mathbb{CP}^1_q))$ is the $\mathcal{A}(\mathbb{CP}^1_q)$-bimodule generated by

$$
\{ \overline{\partial}B_- , \overline{\partial}B_0 , \overline{\partial}B_+ \} = \{ a^2 , c a , c^2 \} \beta_- = q^2 \beta_- \{ a^2 , c a , c^2 \}
$$

and $\Omega^{1,0}(\mathbb{CP}^1_q) \simeq \mathcal{L}_{+2,\beta_+} \simeq \partial(\mathcal{A}(\mathbb{CP}^1_q))$ is the $\mathcal{A}(\mathbb{CP}^1_q)$-bimodule generated by

$$
\{ \partial B_+ , \partial B_0 , \partial B_- \} = \{ a^* a^2 , c^* a^* , c^* a^2 \} \beta_+ = q^{-2} \beta_+ \{ a^* a^2 , c^* a^* , c^* a^2 \}.
$$

That these two modules of forms are not free is also expressed by the existence of relations among the differentials given by

$$
\begin{align*}
\partial B_0 - q^{-2} B_- \partial B_+ + q^2 B_+ \partial B_- = 0 & \quad \text{and} \quad \overline{\partial}B_0 - B_+ \overline{\partial}B_- + q^{-4} B_- \overline{\partial}B_+ = 0 .
\end{align*}
$$

The two-dimensional calculus on $\mathbb{CP}^1_q$ has then quantum tangent space generated by the two elements $X_+$ and $X_-$ (or, equivalently $F$ and $E$). It has a unique (up to scale) top
invariant form $\beta$, which is central, $\beta f = f \beta$ for all $f \in \mathcal{A}(\mathbb{C}P^1_q)$, and $\Omega^2(\mathbb{C}P^1_q)$ is the free $\mathcal{A}(\mathbb{C}P^1_q)$-bimodule generated by $\beta$, i.e. one has $\Omega^2(\mathbb{C}P^1_q) = \beta \mathcal{A}(\mathbb{C}P^1_q) = \mathcal{A}(\mathbb{C}P^1_q)\beta$. Both $\beta_\pm$ commute with elements of $\mathcal{A}(\mathbb{C}P^1_q)$ and so does $\beta_- \wedge \beta_+$, which may be taken as the natural generator $\beta = \beta_- \wedge \beta_+$ of $\Omega^2(\mathbb{C}P^1_q)$ (cfr. [22] or [32, App.]). Writing any one-form $\alpha = x \beta_- + y \beta_+ \in \mathcal{L}_{-2}\beta_- \oplus \mathcal{L}_{+2}\beta_+$, the product of one-forms is given by

$$(x \beta_- + y \beta_+) \wedge (t \beta_- + z \beta_+) = (xz - q^2 yt) \beta_- \wedge \beta_+ .$$

By (2.10) it is natural (and consistent) to demand $d\beta_- = d\beta_+ = 0$ when restricted to $\mathbb{C}P^1_q$. Then the exterior derivative of any one-form $\alpha = x \beta_- + y \beta_+ \in \mathcal{L}_{-2}\beta_- \oplus \mathcal{L}_{+2}\beta_+$ is

$$(2.10) \quad d\alpha = \partial x \wedge \beta_- + \partial y \wedge \beta_+ = \big( X_- \triangleright y - q^2 X_+ \triangleright x \big) \beta_- \wedge \beta_+ ,$$

since $K$ acts as $q^{\pm 1}$ on $\mathcal{L}_{\pm 2}$. Notice that in (2.10), both $X_+ \triangleright x$ and $X_- \triangleright y$ belong to $\mathcal{A}(\mathbb{C}P^1_q)$, as they should. We may summarize these results as follows.

**Proposition 2.11.** The two-dimensional differential calculus on the quantum projective line $\mathbb{C}P^1_q$ is given by

$$\Omega^\bullet(\mathbb{C}P^1_q) = \mathcal{A}(\mathbb{C}P^1_q) \oplus (\mathcal{L}_{-2}\beta_- \oplus \mathcal{L}_{+2}\beta_+) \oplus \mathcal{A}(\mathbb{C}P^1_q)\beta_- \wedge \beta_+ .$$

Moreover, the splitting $\Omega^1(\mathbb{C}P^1_q) = \Omega^{1,0}(\mathbb{C}P^1_q) \oplus \Omega^{0,1}(\mathbb{C}P^1_q)$, together with the two maps $\partial$ and $\bar{\partial}$ given above, constitute a complex structure for the differential calculus.

A Hodge operator at the level of one-forms is constructed in [22] via a left-covariant map $\hat{\star} : \Omega^1(\mathbb{C}P^1_q) \rightarrow \Omega^1(\mathbb{C}P^1_q)$ which squares to the identity id. In the description of the calculus as given in Proposition 2.11 it is defined by

$$(2.12) \quad \hat{\star}(\partial f) = \partial f \quad \text{and} \quad \hat{\star}(\bar{\partial} f) = -\bar{\partial} f$$

for all $f \in \mathcal{A}(\mathbb{C}P^1_q)$. One then demonstrates its compatibility with the bimodule structure, i.e. the map $\hat{\star}$ is a bimodule map. Thus $\hat{\star}$ has values $\pm 1$ on holomorphic or anti-holomorphic one-forms respectively, i.e. one has $\hat{\star} = \pm \text{id}$ on $\Omega^{1,0}(\mathbb{C}P^1_q)$ or $\Omega^{0,1}(\mathbb{C}P^1_q)$ respectively. In particular, $\hat{\star} \beta_\pm = \pm \beta_\pm$. The calculus has one central top two-form and the Hodge operator is naturally extended by requiring

$$(2.13) \quad \hat{\star}1 = \beta_- \wedge \beta_+ \quad \text{and} \quad \hat{\star}(\beta_- \wedge \beta_+) = 1 .$$

We conclude this section by mentioning the calculus on $U(1)$ which makes all three calculi compatible from the quantum principal bundle point of view. The strategy [10] consists in defining the calculus on the coordinate algebra $\mathcal{A}(U(1))$ via the Hopf projection $\pi$ in (1.3). One finds that the projected calculus is one-dimensional and bicovariant. Its quantum tangent space is generated by

$$(2.14) \quad X = X_z = \frac{1 - K^4}{1 - q^{-2}}$$

with dual one-form given by $\beta_z$. Explicitly, one finds

$$\beta_z = z^* \, dz , \quad dz = z \, \beta_z \quad \text{and} \quad dz^* = -q^2 \, z^* \, \beta_z$$

along with the noncommutative commutation relations

$$\beta_z \, z = q^{-2} \, z \, \beta_z , \quad \beta_z \, z^* = q^2 \, z^* \, \beta_z \quad \text{and} \quad z \, dz = q^2 \, dz \, z .$$
The data \((\mathcal{A}(SU_q(2)), \mathcal{A}(\mathbb{CP}^1_q), \mathcal{A}(U(1)))\) defines a ‘topological’ quantum principal bundle. There are differential calculi both on the total space \(\mathcal{A}(SU_q(2))\) (the three-dimensional left-covariant calculus) and on \(\mathcal{A}(U(1))\) (obtained from it via the same projection \(\pi\) in (1.5) giving the bundle structure). Moreover, from the calculus on \(\mathcal{A}(SU_q(2))\) one also obtains by restriction a calculus on the base space \(\mathcal{A}(\mathbb{CP}^1_q)\). The three calculi are compatible with the bundle structure [10] (see also [19]), thus constructing a quantum principal bundle with non-universal calculi. The vector field \(X_z\) is vertical for the fibration.

### 2.3. Connections on equivariant line bundles over \(\mathbb{CP}^1_q\).

The most efficient way to define a connection on a quantum principal bundle (with given calculi) is by decomposing the one-forms on the total space into horizontal and vertical forms [10] [11]. Since horizontal one-forms are given in the structure group of the principal bundle, one needs a projection onto forms whose range is the subspace of vertical one-forms. The projection is required to be covariant with respect to the right coaction of the structure Hopf algebra.

For the principal bundle over the quantum projective line \(\mathbb{CP}^1_q\) that we are considering, a principal connection is a covariant left module projection \(\Pi : \Omega^1_{\text{ver}}(SU_q(2)) \to \Omega^1_{\text{ver}}(SU_q(2))\), i.e. \(\Pi^2 = \Pi\) and \(\Pi(x \alpha) = x \Pi(\alpha)\) for \(\alpha \in \Omega^1(SU_q(2))\) and \(x \in \mathcal{A}(SU_q(2))\). Equivalently, it is a covariant splitting \(\Omega^1(SU_q(2)) = \Omega^1_{\text{ver}}(SU_q(2)) \oplus \Omega^1_{\text{hor}}(SU_q(2))\). The covariance of the connection is the requirement that

\[
\alpha_{R}^{(1)} \circ \Pi = \Pi \circ \alpha_{R}^{(1)},
\]

with \(\alpha_{R}^{(1)}\) the extension to one-forms of the action \(\alpha_R\) in (1.7)–(1.8) of the structure Hopf algebra \(U(1)\). It is not difficult to see that with the left-covariant three-dimensional calculus on \(\mathcal{A}(SU_q(2))\), a basis for \(\Omega^1_{\text{hor}}(SU_q(2))\) is given by \(\beta_{-}, \beta_{+}\). Furthermore, one has

\[
\alpha_{R}^{(1)}(\beta_{z}) = \beta_{z}, \quad \alpha_{R}^{(1)}(\beta_{-}) = \beta_{-} \cdot z^* \cdot z^2 \quad \text{and} \quad \alpha_{R}^{(1)}(\beta_{+}) = \beta_{+} \cdot z^2,
\]

and so a natural choice of connection \(\Pi = \Pi_z\) is to define \(\beta_{z}\) to be vertical [10] [22], whence

\[
\Pi_z(\beta_{z}) := \beta_{z} \quad \text{and} \quad \Pi_z(\beta_{\pm}) := 0.
\]

With a connection, one has a covariant derivative acting on right \(\mathcal{A}(\mathbb{CP}^1_q)\)-modules \(\mathcal{E}\) of equivariant elements,

\[
\nabla := (\text{id} - \Pi_z) \circ d : \mathcal{E} \to \mathcal{E} \otimes_{\mathcal{A}(\mathbb{CP}^1_q)} \Omega^1(\mathbb{CP}^1_q),
\]

and one readily proves the Leibniz rule \(\nabla(\varphi \cdot f) = (\nabla \varphi) \cdot f + \varphi \otimes df\) for all \(\varphi \in \mathcal{E}\) and \(f \in \mathcal{A}(\mathbb{CP}^1_q)\). We shall take for \(\mathcal{E}\) the line bundles \(\mathcal{L}_n\) of (1.17). Then with the left-covariant two-dimensional calculus on \(\mathcal{A}(\mathbb{CP}^1_q)\) (coming from the left-covariant three-dimensional calculus on \(\mathcal{A}(SU_q(2))\) as described in (2.2), we have

\[
(2.15) \quad \nabla \varphi = (X_+ \triangleright \varphi) \beta_+ + (X_- \triangleright \varphi) \beta_-
\]

with \(X_\pm \triangleright \varphi \in \mathcal{L}_{n \pm 2}\) for \(\varphi \in \mathcal{L}_n\). Using Lemma [1.18] we conclude that

\[
\nabla \varphi \in \mathcal{L}_{n-2} \beta_- \oplus \mathcal{L}_{n+2} \beta_+ \simeq \mathcal{L}_n \otimes_{\mathcal{A}(\mathbb{CP}^1_q)} \Omega^1(\mathbb{CP}^1_q)
\]

as required.
A generic covariant derivative on the module $\mathcal{L}_n$ is of the form $\nabla_\alpha = \nabla + \alpha$, with $\alpha$ an element in $\text{Hom}_{A(CP^1_q)}(\mathcal{L}_n, \mathcal{L}_n \otimes_A(CP^1_q) \Omega^1(CP^1_q))$. For later use it is helpful to characterize this space. More generally, from Lemma 1.18 we can infer the following results.

**Lemma 2.16.** For any $n \in \mathbb{Z}$ one has
\begin{equation}
\text{Hom}_{A(CP^1_q)}(\mathcal{L}_n, \mathcal{L}_n \otimes_A(CP^1_q) \Omega^1(CP^1_q)) \simeq \mathcal{L}_{-2} \beta_- \oplus \mathcal{L}_{+2} \beta_+ = \Omega^1(CP^1_q),
\end{equation}
while for any two distinct integers $n, m \in \mathbb{Z}$ one has
\begin{equation}
\text{Hom}_{A(CP^1_q)}(\mathcal{L}_n, \mathcal{L}_m \otimes_A(CP^1_q) \Omega^1(CP^1_q)) \simeq \mathcal{L}_{m-n-2} \beta_- \oplus \mathcal{L}_{m-n+2} \beta_+ \simeq \mathcal{L}_{m-n} \otimes_A(CP^1_q) \Omega^1(CP^1_q).
\end{equation}

Given the connection, we can work out an explicit expression for its curvature, defined to be the $A(CP^1_q)$-linear (by construction) map
\[ \nabla^2 : = \nabla \circ \nabla : \mathcal{L}_n \to \mathcal{L}_n \otimes_A(CP^1_q) \Omega^2(CP^1_q). \]

**Proposition 2.19.** Let $\hat{\nabla}_n$ be the connection on the line bundle $\mathcal{L}_n$ defined in (1.17), given in (2.15) for the canonical left-covariant two-dimensional calculus on $A(CP^1_q)$. Then, with $\varphi \in \mathcal{L}_n$, its curvature is given by
\begin{equation}
\hat{\nabla}_n^2 \varphi = \hat{(X_+ \triangleright \varphi)} \beta_- \wedge \beta_+.
\end{equation}

As an element in $\text{Hom}_{A(CP^1_q)}(\mathcal{L}_n, \mathcal{L}_n \otimes_A(CP^1_q) \Omega^2(CP^1_q))$, one has
\begin{equation}
\hat{\nabla}_n^2 = -q^{n+1} \llbracket n \rrbracket \beta_- \wedge \beta_+.
\end{equation}

**Proof.** Using (2.15), (2.8) and the fact that $d \beta_\pm = 0$ on $CP^1_q$, by the Leibniz rule we have
\begin{align*}
\hat{\nabla}_n(\hat{\nabla}_n \varphi) &= (X_- X_+ \triangleright \varphi) \beta_- \wedge \beta_+ + (X_- X_- \triangleright \varphi) \beta_- \wedge \beta_-
\ &= ((X_- X_+ - q^2 X_+ X_-) \triangleright \varphi) \beta_- \wedge \beta_+,
\end{align*}
and (2.20) follows from the relation $X_- X_+ - q^2 X_+ X_- = X_z$. Since $X_z \triangleright \varphi = -q^{n+1} \llbracket n \rrbracket \varphi$ for $\varphi \in \mathcal{L}_n$, one has (2.21). Since $X_z \triangleright A(CP^1_q) = 0$, the curvature is $A(CP^1_q)$-linear. \qed

We can also derive an explicit expression for the corresponding gauge potential $a_n$ defined by $\varphi a_n = \nabla \varphi - \nabla \varphi$ for $\varphi \in \mathcal{L}_n$. With $X_z$ the vertical vector field in (2.14), using (2.4) and (2.15) we find $\varphi a_n = -(X_z \triangleright \varphi) \beta_z = q^{n+1} \llbracket n \rrbracket \varphi \beta_z$, or
\begin{equation}
a_n = q^{n+1} \llbracket n \rrbracket \beta_z.
\end{equation}

As usual, $a_n$ is not defined on $CP^1_q$ but rather on the total space $SU_q(2)$ of the bundle, i.e. $a_n \in \text{Hom}_{A(CP^1_q)}(\mathcal{L}_n, \mathcal{L}_n \otimes_A(CP^1_q) \Omega^1(SU_q(2)))$. In terms of the gauge potential, the curvature is given by
\begin{equation}
f_n := \hat{\nabla}_n^2 = d a_n
\end{equation}
as a direct consequence of the first identity in (2.7).
2.4. Holomorphic structures.

The connection given in \(\S 2.3\) can be naturally decomposed into a holomorphic and an anti-holomorphic part, \(\nabla = \nabla^\partial + \nabla^\bar{\partial}\). They are given by

\[
(2.24) \quad \nabla^\partial \varphi = (X_+ \triangleright \varphi) \beta_+ \quad \text{and} \quad \nabla^\bar{\partial} \varphi = (X_- \triangleright \varphi) \beta_- \\
\]

with the corresponding Leibniz rules

\[
\nabla^\partial (\varphi \cdot f) = (\nabla^\partial \varphi) \cdot f + \varphi \otimes \partial f \\
\nabla^\bar{\partial} (\varphi \cdot f) = (\nabla^\bar{\partial} \varphi) \cdot f + \varphi \otimes \bar{\partial} f ,
\]

for all \(\varphi \in \mathcal{L}_n\) and \(f \in \mathcal{A}(\mathbb{C}P^1_q)\). They are both flat, i.e. \((\nabla^\partial)^2 = 0 = (\nabla^\bar{\partial})^2\), and so the connection \(\nabla\) is integrable.

Holomorphic ‘sections’ are elements \(\varphi \in \mathcal{L}_n\) which satisfy

\[
\nabla^\bar{\partial} \varphi = 0 .
\]

From the actions given in \((1.3)\) we see that \(F \triangleright a^s = 0\) and \(F \triangleright c^s = 0\) for any \(s \in \mathbb{N}_0\), while \(F \triangleright a^s \neq 0\) and \(F \triangleright c^s \neq 0\) for any \(s \in \mathbb{N}\). Then, from the expressions \((1.20)\) for generic equivariant elements, we see that there are no holomorphic elements in \(\mathcal{L}_n\) for \(n > 0\). On the other hand, for \(n \leq 0\) the elements \(c^{[n] - \nu} a^\mu\), \(\mu = 0, 1, \ldots, |n|\) are holomorphic,

\[
\nabla^\bar{\partial} (c^{[n] - \nu} a^\mu) = 0 .
\]

Since \(\ker \bar{\partial} = \mathbb{C}\) (as only the constant functions on \(\mathbb{C}P^1_q\) do not contain the generator \(a^s\) or \(c^s\)), so that the only holomorphic functions on \(\mathbb{C}P^1_q\) are the constants, these are the only invariants in degree \(n\). We may conclude that holomorphic equivariant elements are all polynomials in two variables \(a, c\) with the commutation relation \(a c = q c a\), which defines the coordinate algebra of the quantum plane. Further aspects of these holomorphic structures are reported in \([17]\).

2.5. Unitarity and gauge transformations.

On each line bundle \(\mathcal{L}_n\), \(n \geq 0\) there is an \(\mathcal{A}(\mathbb{C}P^1_q)\)-valued hermitian structure

\[
\hat{h}_n : \mathcal{L}_n \times \mathcal{L}_n \rightarrow \mathcal{A}(\mathbb{C}P^1_q)
\]

defined by

\[
(2.25) \quad \hat{h}_n \left( \sum_{\mu=0}^{n} c^\mu a^s \cdot f_\mu, \sum_{\nu=0}^{n} c^\nu a^s \cdot g_\nu \right) = \sum_{\mu=0}^{n} f_\mu a^{n-\mu} c^\mu c^\nu a^{s-n-\nu} g_\nu
\]

in the \(\mathcal{A}(\mathbb{C}P^1_q)\)-module basis \((1.19)-(1.20)\). Having taken the right \(\mathcal{A}(\mathbb{C}P^1_q)\)-module structure for \(\mathcal{L}_n\), the hermitian structure \((2.25)\) is right \(\mathcal{A}(\mathbb{C}P^1_q)\)-linear and left \(\mathcal{A}(\mathbb{C}P^1_q)\)-antilinear. It is covariant under the natural left coaction of \(\mathcal{A}(\text{SU}_q(2))\) on \(\mathcal{L}_n\) induced by the inclusion \(\mathcal{L}_n \subset \mathcal{A}(\text{SU}_q(2))\). There is an analogous formula for \(n \leq 0\). By composing \(\hat{h}_n\) with the Haar functional of \(\mathcal{A}(\text{SU}_q(2))\) restricted to \(\mathcal{A}(\mathbb{C}P^1_q)\), one obtains a \(\mathbb{C}\)-valued inner product on \(\mathcal{L}_n\). Since the Haar functional of \(\mathcal{A}(\text{SU}_q(2))\) is invariant under the coaction of \(\mathcal{A}(\text{SU}_q(2))\) on itself \([13] \S 4.2.6\), we get an \(\text{SU}_q(2)\)-invariant inner product on each \(\mathcal{L}_n\). If we write elements \(\varphi \in \mathcal{L}_n\) as vector-valued functions \(\varphi = (\varphi_\mu, \mu = 0, 1, \ldots, |n|)\), the hermitian structure is simply \(\hat{h}_n(\varphi; \psi) = \sum_\mu \varphi^*_\mu \psi_\mu\).
Lemma 2.26. The connection \( \hat{\nabla}_n \) is unitary, i.e. it is compatible with the hermitian structure \( \hat{h}_n \),

\[
\hat{h}_n(\hat{\nabla}_n \varphi, \psi) + \hat{h}_n(\varphi, \hat{\nabla}_n \psi) = d(\hat{h}_n(\varphi, \psi)) \quad \text{for any } \varphi, \psi \in L_n.
\]

Proof. On the one hand, \( d(\hat{h}_n(\varphi, \psi)) = (X_+ \triangleright \hat{h}_n(\varphi, \psi)) \beta_+ + (X_- \triangleright \hat{h}_n(\varphi, \psi)) \beta_- \). Using the coproducts (2.1) we have

\[
(X_\pm \triangleright \hat{h}_n(\varphi, \psi)) = \sum_{\mu=0}^{|n|} X_\pm \triangleright (\varphi^*_\mu \psi_\mu) = \sum_{\mu=0}^{|n|} \left( \varphi^*_\mu (X_\pm \triangleright \psi_\mu) + (X_\pm \triangleright \varphi^*_\mu) (K^2 \triangleright \psi_\mu) \right)
\]

\[
= \sum_{\mu=0}^{|n|} \left( \varphi^*_\mu (X_\pm \triangleright \psi_\mu) + q^n (X_\pm \triangleright \varphi^*_\mu) \psi_\mu \right),
\]

and in turn

\[
d(\hat{h}_n(\varphi, \psi)) = \sum_{\pm} \sum_{\mu=0}^{|n|} \left( \varphi^*_\mu (X_\pm \triangleright \psi_\mu) + q^n (X_\pm \triangleright \varphi^*_\mu) \psi_\mu \right) \beta_\pm .
\]

On the other hand, using the antipodes (2.2) and \( \beta_\pm^* = -\beta_\pm \) we have

\[
\hat{h}_n(\hat{\nabla}_n \varphi, \psi) = \sum_{\pm} \sum_{\mu=0}^{|n|} \beta^-_\mu (X_\pm \triangleright \varphi^*_\mu) \psi_\mu = q^n \sum_{\pm} \sum_{\mu=0}^{|n|} q^{\mp 2} \beta^-_\mu (X_\pm \triangleright \varphi^*_\mu) \psi_\mu
\]

\[
= q^n \sum_{\pm} \sum_{\mu=0}^{|n|} (X_\pm \triangleright \varphi^*_\mu) \psi_\mu \beta_\pm ,
\]

and in turn

\[
\hat{h}_n(\hat{\nabla}_n \varphi, \psi) + \hat{h}_n(\varphi, \hat{\nabla}_n \psi) = \sum_{\pm} \sum_{\mu=0}^{|n|} \left( q^n (X_\pm \triangleright \varphi^*_\mu) \psi_\mu + \varphi^*_\mu (X_\pm \triangleright \psi_\mu) \right) \beta_\pm .
\]

A direct comparison now gives the result. \( \Box \)

We already know that any other connection is written as \( \nabla_\alpha = \nabla + \alpha \) with \( \alpha \) a generic element in \( \text{Hom}_{A(CP^1_q)}(L_n, L_n \otimes A(CP^1_q)) \Omega^1(CP^1_q) \) which, for a unitary connection \( \nabla \), is necessarily anti-hermitian,

\[
\hat{h}_n(\alpha \varphi, \psi) + \hat{h}_n(\varphi, \alpha \psi) = 0 \quad \text{for } \varphi, \psi \in L_n.
\]

Lemma 2.27. Unitary elements \( \alpha \in \text{Hom}_{A(CP^1_q)}(L_n, L_n \otimes A(CP^1_q)) \Omega^1(CP^1_q) \) are of the form

\[
\alpha = x \beta_- + q^2 x^* \beta_+ = x \beta_- - (x \beta_-)^* ,
\]

with \( x \) a generic element in \( L_{-2} \).

Proof. From the identification (2.17), we seek elements in \( \Omega^1(CP^1_q) = L_{-2} \beta_- \oplus L_{+2} \beta_+ \) which are unitary. It is straightforward to verify that a generic one-form \( \alpha = x_- \beta_- + x_+ \beta_+ \) with \( x_\pm \in L_{\pm 2} \) is unitary with respect to the hermitian structure \( \hat{h}_n \) if and only if it is written as claimed. \( \Box \)
The group $\mathcal{U}(\mathcal{L}_n)$ of gauge transformations consists of unitary elements in $\text{End}_{\mathcal{A}(\mathbb{C}P^1_q)}(\mathcal{L}_n)$ (with respect to the hermitian structure $\hat{h}_n$). It acts on a connection $\nabla$ by

$$(u, \nabla) \mapsto \nabla^u = u \circ \nabla \circ u^*.$$ 

An arbitrary connection $\nabla_\alpha = \nabla + \alpha$ will then transform to $(\nabla_\alpha)^u = \nabla + \alpha^u$ with

$$\alpha^u = u(\nabla u^*) + u \alpha u^*.$$ 

We know from Lemma 1.18 that $\text{End}_{\mathcal{A}(\mathbb{C}P^1_q)}(\mathcal{L}_n) \simeq \mathcal{L}_0 = \mathcal{A}(\mathbb{C}P^1_q)$. Thus $\mathcal{U}(\mathcal{L}_n)$ consists of unitary elements in the coordinate algebra $\mathcal{A}(\mathbb{C}P^1_q)$. Of these there are none which are nontrivial. Indeed, in the coordinate algebra of $\mathcal{A}(SU_q(2))$ there are no nontrivial invertible elements [15 App.]. Since $\mathcal{A}(\mathbb{C}P^1_q)$ is a subalgebra of the latter, it cannot contain any nontrivial invertible (hence unitary) elements either.

### 2.6. $SU_q(2)$-invariant connections and gauge transformations.

Recall that there is a coaction (1.14) of $\mathcal{A}(SU_q(2))$ on modules of sections. Let us denote by $\Delta^{(n)}$ the coaction on $\mathcal{L}_n$,

$$(\Delta^{(n)} : \mathcal{L}_n \longrightarrow \mathcal{A}(SU_q(2)) \otimes \mathcal{L}_n), \quad \Delta^{(n)}(\varphi) = \varphi(-1) \otimes \varphi(0),$$

with implicit summation as usual. By combining it with the coaction $\Delta^{(1)}$ of $\mathcal{A}(SU_q(2))$ on the bimodule of one-forms $\Omega^1(\mathbb{C}P^1_q)$ we get an analogous coaction

$$\Delta^{(1)}_{(n)} : \mathcal{L}_n \otimes_{\mathcal{A}(\mathbb{C}P^1_q)} \Omega^1(\mathbb{C}P^1_q) \longrightarrow \mathcal{A}(SU_q(2)) \otimes (\mathcal{L}_n \otimes_{\mathcal{A}(\mathbb{C}P^1_q)} \Omega^1(\mathbb{C}P^1_q)), \quad \Delta^{(1)}_{(n)}(\omega) = \omega(-1) \otimes \omega(0).$$

Next we give an ‘adjoint’ coaction of $\mathcal{A}(SU_q(2))$ on the space $\mathcal{C}(\mathcal{L}_n)$ of unitary connections,

$$\Delta^C : \mathcal{C}(\mathcal{L}_n) \longrightarrow \mathcal{A}(SU_q(2)) \otimes \mathcal{C}(\mathcal{L}_n),$$

defined by

$$\Delta^C(-) = m_{12} \circ \left( \text{id} \otimes \Delta^{(1)}_{(n)} \right) \circ \left( \text{id} \otimes (-) \right) \circ (S \otimes \text{id}) \circ \Delta^{(n)}$$

with $m_{12}$ the multiplication in the first two factors of the tensor product and $S$ the antipode. Thinking of $\Delta^C(-)$ as acting on $1 \otimes \varphi$ with $\varphi \in \mathcal{L}_n$, and using (2.28), we get the ‘explicit’ expression

$$\Delta^C(\nabla_\alpha)(\varphi) = S(\varphi(-1)) \left( \nabla_\alpha(\varphi(0)) \right)(-1) \otimes \left( \nabla_\alpha(\varphi(0)) \right)(0).$$

**Lemma 2.30.** The canonical connection $\hat{\nabla}_n$ in (2.15) is the unique invariant connection for this coaction, i.e. the unique element $\nabla \in \mathcal{C}(\mathcal{L}_n)$ for which

$$\Delta^C(\nabla) = 1 \otimes \nabla.$$

In particular, there is no non-trivial element in $\text{Hom}_{\mathcal{A}(\mathbb{C}P^1_q)}(\mathcal{L}_n, \mathcal{L}_n \otimes_{\mathcal{A}(\mathbb{C}P^1_q)} \Omega^1(\mathbb{C}P^1_q))$ which is invariant.

**Proof.** The left-coinvariance of the canonical connection $\hat{\nabla}_n$ is most easily seen from the corresponding gauge potential in (2.22). This is clearly left-coinvariant from the properties (2.5) of the basis one-forms, and in particular of $\beta_z$. Since a unitary element $\alpha \in \text{Hom}_{\mathcal{A}(\mathbb{C}P^1_q)}(\mathcal{L}_n, \mathcal{L}_n \otimes_{\mathcal{A}(\mathbb{C}P^1_q)} \Omega^1(\mathbb{C}P^1_q))$ is of the form given in Lemma 2.27, it is evident that $\alpha = 0$ is the only such left-invariant element. □
An ‘adjoint’ coaction of \( \mathcal{A}(\text{SU}_q(2)) \) on the group \( \mathcal{U}(\mathcal{L}_n) \) of gauge transformations,

\[
\Delta^U : \mathcal{U}(\mathcal{L}_n) \rightarrow \mathcal{A}(\text{SU}_q(2)) \otimes \mathcal{U}(\mathcal{L}_n),
\]

would be defined analogously as above by

\[
\Delta^U(-) = m_{12} \circ (\text{id} \otimes \Delta(n)) \circ (\text{id} \otimes (-)) \circ (S \otimes \text{id}) \circ \Delta(n),
\]

and thinking of \( \Delta^U(-) \) as acting on \( 1 \otimes \varphi \), with \( \varphi \in \mathcal{L}_n \), one has

\[
\Delta^U(u)(\varphi) = S(\varphi(-1))\left[(u(\varphi(0)))_{(-1)} \otimes (u(\varphi(0)))_{(0)}\right].
\]

In fact, we already know that \( \mathcal{U}(\mathcal{L}_n) \) consists of unitary elements in \( \mathcal{A}(\mathbb{C}P^1_q) \). Then, the \( \mathcal{A}(\text{SU}_q(2))-\)coaction \( \Delta^U \) is just the restriction to \( \mathcal{U}(\mathcal{L}_n) \) of the canonical \( \mathcal{A}(\text{SU}_q(2))-\)coaction on \( \mathcal{A}(\mathbb{C}P^1_q) \) given in (1.11). Also, as \( \mathcal{U}(\mathcal{L}_n) \) is made only of complex numbers of modulus one, the following result is immediate.

**Lemma 2.31.** The element \( 1 \in \mathcal{U}(\mathcal{L}_n) \) is the unique invariant gauge transformation for this coaction, i.e. the unique element \( u \in \mathcal{U}(\mathcal{L}_n) \) for which

\[
\Delta^U(u) = 1 \otimes u.
\]

This also follow from Proposition 1.12 giving 1 as the only \( \text{SU}_q(2) \)-invariant element in the algebra \( \mathcal{A}(\mathbb{C}P^1_q) \).

### 2.7. K-theory charges.

The line bundles on the sphere \( \mathbb{C}P^1_q \) described in (1.12) are classified by their monopole number \( n \in \mathbb{Z} \). One writes \( \mathcal{L}_n = p^{(n)}(\mathbb{C}P^1_q) |_{n+1} \) with suitable projections \( p^{(n)} \) in \( \text{Mat}_{|n|+1}(\mathcal{A}(\mathbb{C}P^1_q)) \). They are given explicitly by

\[
p^{(n)}_{\mu \nu} = \begin{cases} 
\sqrt{\alpha_{n,\mu} \alpha_{n,\nu}} c^{n-\mu} a^\mu a^* c^{n-\nu}, & n \geq 0 \\
\sqrt{\beta_{n,\mu} \beta_{n,\nu}} c^\mu a^{n-\mu} a^* c^{n-\nu}, & n \leq 0
\end{cases}
\]

(2.32)

with \( \mu, \nu = 0, 1, \ldots, |n| \) and the numerical coefficients

\[
\alpha_{n,\mu} = \prod_{j=0}^{n-\mu-1} \frac{1 - q^{2(n-j)}}{1 - q^{2(j+1)}} \quad \text{and} \quad \beta_{n,\mu} = q^{2\mu} \prod_{j=0}^{\mu-1} \frac{1 - q^{-2(|n|-j)}}{1 - q^{-2(j+1)}}.
\]

We use the convention \( \prod_{j=0}^{-1} (-1) := 1 \).

The projections in (2.32) are representatives of classes in the K-theory of \( \mathbb{C}P^1_q \), i.e. \([p^{(n)}] \in K_0(\mathbb{C}P^1_q) \). One computes the corresponding monopole number by pairing them with a non-trivial element in the dual K-homology, i.e. with (the class of) a non-trivial Fredholm module \([\mu] \in K_0(\mathbb{C}P^1_q) \). For this, one first calculates the corresponding Chern characters in the cyclic homology \( ch^\bullet(p^{(n)}) \in HC_\bullet(\mathbb{C}P^1_q) \) and cyclic cohomology \( ch^\bullet(\mu) \in HC^\bullet(\mathbb{C}P^1_q) \) respectively, and then uses the pairing between cyclic homology and cohomology.
The Chern character of the projections $p^{(n)}$ has a non-trivial component in degree zero $\text{ch}_0(p^{(n)}) \in H_{C_0}(\mathbb{CP}^1_q)$ given simply by a (partial) matrix trace

$$\text{ch}_0(p^{(n)}) := \text{tr}(p^{(n)}) = \begin{cases} 
\sum_{\mu=0}^{n} \alpha_{n,\mu}(c^*c)^{n-\mu} \prod_{j=0}^{\mu-1} (1 - q^{2j} c^*c) , & n \geq 0 \\
\sum_{\mu=0}^{[n]} \beta_{n,\mu}(c^*c)^{\mu} \prod_{j=0}^{[n]-\mu-1} (1 - q^{-2j} c^*c) , & n \leq 0 ,
\end{cases}$$

and $\text{ch}_0(p^{(n)}) \in \mathcal{A}(\mathbb{CP}^1_q)$. Dually, one needs a cyclic zero-cocycle, i.e. a trace on $\mathcal{A}(\mathbb{CP}^1_q)$. This was obtained in [23] and it is a trace on $\mathcal{A}(\mathbb{CP}^1_q)/\mathbb{C}$, i.e. it vanishes on $\mathbb{C} \subset \mathcal{A}(\mathbb{CP}^1_q)$. On the other hand, its values on powers of the element $c^*c$ is given by

$$\mu((c^*c)^k) = (1 - q^{2k})^{-1} , \quad k > 0 .$$

The pairing was computed in [14] and results in

$$\langle [\mu] , [p^{(n)}] \rangle := \mu(\text{ch}_0(p^{(n)})) = -n . \tag{2.33}$$

This integer is a topological quantity that depends only on the bundle, both over the quantum sphere and over its classical limit which is an ordinary two-sphere. In this limit it could also be computed by integrating the curvature two-form of any connection. However, in order to integrate the gauge curvature on the quantum sphere $\mathbb{CP}^1_q$ one requires a ‘twisted integral’, and the result is no longer an integer but rather a $q$-integer. We recall here the main facts, refering to [19] for additional details.

It is known [18, Prop. 4.15] that the modular automorphism associated with the Haar state $H$ on the algebra $\mathcal{A}(\text{SU}_q(2))$ when restricted to the subalgebra $\mathcal{A}(\mathbb{CP}^1_q)$ yields a faithful, invariant state on $\mathcal{A}(\mathbb{CP}^1_q)$, i.e. $H(a \triangleleft X) = H(a) \epsilon(X)$ for $a \in \mathcal{A}(\mathbb{CP}^1_q)$ and $X \in \mathcal{U}_q(\mathfrak{su}(2))$, with modular automorphism

$$\vartheta(g) = g \triangleleft K^2 \quad \text{for} \quad g \in \mathcal{A}(\mathbb{CP}^1_q) , \tag{2.34}$$

such that

$$H(ab) = H(\vartheta(b) a) \tag{2.35}$$

for $a, b \in \mathcal{A}(\mathbb{CP}^1_q)$. With $\beta_- \land \beta_+$ the central generator of $\Omega^2(\mathbb{CP}^1_q)$, $H$ the Haar state on $\mathcal{A}(\mathbb{CP}^1_q)$, and $\vartheta$ its modular automorphism in (2.34), it was proven in [22] that the linear functional

$$\int_{\mathbb{CP}^1_q} : \Omega^2(\mathbb{CP}^1_q) \to \mathbb{C} , \quad \int_{\mathbb{CP}^1_q} a \beta_- \land \beta_+ := H(a) \tag{2.36}$$

defines a non-trivial $\vartheta$-twisted cyclic two-cocycle $\tau$ on $\mathcal{A}(\mathbb{CP}^1_q)$ given by

$$\tau(a_0, a_1, a_2) := \frac{1}{2} \int_{\mathbb{CP}^1_q} a_0 \, da_1 \land da_2 . \tag{2.37}$$

This means that $b_\vartheta \tau = 0 = \lambda_\vartheta \tau = \tau$, where $b_\vartheta$ is the $\vartheta$-twisted coboundary operator

$$(b_\vartheta \tau)(f_0, f_1, f_2, f_3) := \tau(f_0 f_1, f_2, f_3) - \tau(f_0, f_1 f_2, f_3) + \tau(f_0, f_1, f_2 f_3) - \tau(\vartheta(f_3) f_0, f_1, f_2) ,$$
and \( \lambda_\vartheta \) is the \( \vartheta \)-twisted cyclicity operator
\[
(\lambda_\vartheta \tau)(f_0, f_1, f_2) := \tau(\vartheta(f_2), f_0, f_1).
\]
The non-triviality means that there is no twisted cyclic one-cochain \( \alpha \) on \( \mathcal{A}(\mathbb{CP}_q^1) \) such that \( b_\vartheta \alpha = \tau \) and \( \lambda_\vartheta \alpha = \alpha \), where here the operators \( b_\vartheta \) and \( \lambda_\vartheta \) are defined by formulae like those above (and directly generalize in any degree). Thus \( \tau \) is a class in \( \text{HC}_q^2(\mathbb{CP}_q^1) \), the degree two twisted cyclic cohomology of the quantum space \( \mathbb{CP}_q^1 \).

In terms of the projections \( p^{(n)} \), the curvature \( (2.21) \) of the connection \( (2.15) \) is given by
\[
(2.38) \quad F_{\psi_n} := p^{(n)} \, dp^{(n)} \wedge dp^{(n)} = -q^{n+1} [n] \, p^{(n)} \beta_- \wedge \beta_+ .
\]
Using the normalization \( H(1) = 1 \) for the Haar state on \( \mathcal{A}(\mathbb{CP}_q^1) \), its integral \( (2.39) \) is computed to be
\[
(2.39) \quad q^{-1} \int_{\mathbb{CP}_q^1} \text{tr}_q(F_{\psi_n}) = -[n].
\]
Here \( \text{tr}_q \) stands for the twisted or ‘quantum’ trace defined as follows \([35]\). Given an element \( M \in \text{Mat}_{|n|+1}(\mathcal{A}(\mathbb{CP}_q^1)) \), its (partial) quantum trace is the element \( \text{tr}_q(M) \in \mathcal{A}(\mathbb{CP}_q^1) \) defined by
\[
\text{tr}_q(M) := \text{tr}(M \sigma_{|n|/2}(K^2)) = \sum_{j,l=0}^{|n|} M_{jl} \left( \sigma_{|n|/2}(K^2) \right)_{lj}
\]
where \( \sigma_{|n|/2}(K^2) \) is the matrix form of the spin \( J = |n|/2 \) representation of the modular element \( K^2 \). In particular, \( \text{tr}_q(p^{(n)}) = q^{-n} \). The \( q \)-trace is ‘twisted’ by the automorphism \( \vartheta \),
\[
\text{tr}_q(M_1 M_2) = \text{tr}_q \left( (M_2 \circ K^2) M_1 \right) = \text{tr}_q \left( \vartheta(M_2) M_1 \right). 
\]
From the definition \( (2.37) \) of the \( \vartheta \)-twisted cyclic two-cocycle \( \tau \) and the expression \( (2.38) \) of the curvature \( F_{\psi_n} \), the integral \( (2.39) \) is also found to coincide with the coupling of the cocycle \( \tau \) to the projection \( p^{(n)} \) as
\[
(2.40) \quad (2q^{-1} \tau) \circ \text{tr}_q(p^{(n)} , p^{(n)} , p^{(n)}) = -[n].
\]
The pairing in \( (2.33) \) is the index of the Dirac operator on \( \mathbb{CP}_q^1 \). In parallel, the pairing in \( (2.40) \) can be obtained \([27, 35]\) as the \( q \)-index of the same Dirac operator, i.e. the difference between the quantum dimensions of its kernel and cokernel computed using \( \text{tr}_q \). Thus the \( q \)-integer \( (2.39) \) may be naturally regarded as a quantum Fredholm index computed from the pairing between the \( \vartheta \)-twisted cyclic cohomology and the (Hopf algebraic) \( SU_q(2) \)-equivariant K-theory \( K^M_{0}(su(2))(\mathbb{CP}_q^1) \) \([27, 35]\).

3. Dimensional Reduction of Invariant Gauge Fields

For a smooth manifold \( M \), let \( M \) denote the quantum space \( \mathbb{CP}_q^1 \times M \). By this we mean the family of quantum projective lines \( \mathbb{CP}_q^1 \times \{ p \} \simeq \mathbb{CP}_q^1 \) parametrized by points \( p \in M \). Let \( \mathcal{A}(M) = C^\infty(M) \) be the commutative algebra of smooth functions on \( M \). Then the algebra of \( M \) is given by
\[
\mathcal{A}(M) := \mathcal{A}(\mathbb{CP}_q^1) \otimes \mathcal{A}(M).
\]
Using the connections on the quantum principal bundle over $\mathbb{C}P^1_q$ given in (2), we will now construct invariant connections on $SU_q(2)$-equivariant modules over the algebra $A(M)$ and describe their dimensional reduction over $\mathbb{C}P^1_q$.

### 3.1. Dimensional reduction of $SU_q(2)$-equivariant vector bundles

We start by giving a coaction of the quantum group $SU_q(2)$ on $A(M)$, by coacting trivially on $A(M)$ and with the canonical coaction $\Delta_L$ on $A(\mathbb{C}P^1_q)$ given in (1.11). This gives a map defined by

$$\Delta : A(M) \to A(SU_q(2)) \otimes A(M),$$

for $b \in A(\mathbb{C}P^1_q), f \in A(M)$, where we use the Sweedler-like notation $\Delta_L(b) = b_{(-1)} \otimes b_{(0)}$ (with implicit summation), and $m_{13}$ denotes multiplication in the first and third factors of the tensor product. In parallel with the description (1.9) of the two-sphere algebra $A(\mathbb{C}P^1_q)$ as the subalgebra of invariant elements in $A(SU_q(2))$, there is an analogous description of the algebra $A(M)$ in terms of invariant elements in $A(SU_q(2)) \otimes A(M)$. For this, we let $U(1)$ act trivially on $A(M)$ with corresponding map

$$\bar{\alpha}_z : U(1) \to \text{Aut}(A(SU_q(2)) \otimes A(M)), \quad \bar{\alpha}_z(x \otimes f) = \alpha_z(x) \otimes f,$$

with $\alpha_z$ the $U(1)$-action on $A(SU_q(2))$ given in (1.7)–(1.8). It is then evident that

$$A(M) = (A(SU_q(2)) \otimes A(M))^{U(1)} := \{ f \in A(SU_q(2)) \otimes A(M) \mid \bar{\alpha}_z(f) = f \}.$$

It is also useful to regard the algebra $A(M)$ itself as coming from $A(M)$ via a projection related to the map $\pi$ in (1.5) that establishes the ‘quantum group’ $A(U(1))$ as a quantum subgroup of $A(SU_q(2))$. Indeed, by restricting $\pi$ to the subalgebra $A(\mathbb{C}P^1_q) \subset A(SU_q(2))$ one gets a one-dimensional representation

$$\pi : A(\mathbb{C}P^1_q) \to \mathbb{C}, \quad \pi(B_-) = \pi(B_+) = \pi(B_0) = 0$$

on the generators and $\pi(1) = 1$, which is none other that the counit $\epsilon$ restricted to $A(\mathbb{C}P^1_q)$. We then have a surjective algebra homomorphism

$$\overline{\pi} = \pi \otimes \text{id} : A(M) \to A(M), \quad x \otimes f \mapsto \epsilon(x)f.$$

A right $A(M)$-module $\mathcal{E}$ is said to be $SU_q(2)$-equivariant if it carries a left coaction

$$\delta : \mathcal{E} \to A(SU_q(2)) \otimes \mathcal{E}$$

of the Hopf algebra $A(SU_q(2))$ which is compatible with the coaction $\Delta$ of $A(SU_q(2))$ on $A(M)$,

$$\delta(\varphi \cdot f) = \delta(\varphi) \cdot \Delta(\varphi) \text{ for all } \varphi \in \mathcal{E}, f \in A(M).$$

Similarly, one defines $SU_q(2)$-equivariant left $A(M)$-modules. The remainder of this section is devoted to relating $A(SU_q(2))$-equivariant bundles $\mathcal{E}$ on the quantum space $\mathbb{C}P^1_q$ to $U(1)$-equivariant bundles $E$ over the manifold $M$.

Let $E \to M$ be a smooth, $U(1)$-equivariant complex vector bundle, with $U(1)$ acting trivially on $M$. This induces an action $\rho$ of the group $U(1)$ on the (right) $A(M)$-module $\mathcal{E} = C^\infty(M, E)$ of smooth sections of the bundle $E$, making it $U(1)$-equivariant. By the classical Serre–Swan theorem, the module $\mathcal{E}$ is a finitely-generated (right) projective
module over \( \mathcal{A}(M) \). Consider now the space \( \mathcal{E} \) of equivariant elements, generalizing those in \((1.13)\), given by

\[
(3.6) \quad \mathcal{E} = \mathcal{A}(SU_q(2)) \boxtimes_p \mathcal{E} := \{ \varphi \in \mathcal{A}(SU_q(2)) \otimes \mathcal{E} \mid (\alpha \otimes \text{id}) \varphi = ((\text{id} \otimes \rho^{-1})) \varphi \}.
\]

There is a natural \( SU_q(2) \)-equivariance. Again the left coaction \( \Delta \) of \( \mathcal{A}(SU_q(2)) \) on itself extends naturally to a left coaction on \( \mathcal{A}(SU_q(2)) \boxtimes_p \mathcal{E} \) given by

\[
\Delta^p = \Delta \otimes \text{id} : \mathcal{A}(SU_q(2)) \boxtimes_p \mathcal{E} \rightarrow \mathcal{A}(SU_q(2)) \otimes \left( \mathcal{A}(SU_q(2)) \boxtimes_p \mathcal{E} \right).
\]

This coaction is naturally compatible with the corresponding \( SU_q(2) \)-coaction in \((3.1)\). The space \((3.6)\) is an \( \mathcal{A}(M) \)-bimodule. Any \( \varphi \in \mathcal{A}(SU_q(2)) \boxtimes_p \mathcal{E} \) can be written as \( \varphi = \varphi^{(1)} \otimes \varphi^{(2)} \) with \( \varphi^{(1)} \in \mathcal{A}(SU_q(2)) \) and \( \varphi^{(2)} \in \mathcal{E} \) (and an implicit sum understood). Then the bimodule structure is given as

\[
(b \otimes f) (\varphi^{(1)} \otimes \varphi^{(2)}) = (b \varphi^{(1)}) \otimes (f \varphi^{(2)}) \quad \text{and} \quad (\varphi^{(1)} \otimes \varphi^{(2)}) (b \otimes f) = (\varphi^{(1)} b) \otimes (\varphi^{(2)} f)
\]

for \( b \otimes f \in \mathcal{A}(CP^1_q) \otimes \mathcal{A}(M) \). As a right (or left) \( \mathcal{A}(M) \)-module, it is finitely-generated and projective when it is defined with the tensor product of modules \( \mathcal{E} \) which are finitely-generated and projective, respectively.

Conversely, let \( \mathcal{E}_p \) be a finitely-generated \( SU_q(2) \)-equivariant right (or left) projective \( \mathcal{A}(M) \)-module. The surjective algebra homomorphism \( \overline{\pi} : \mathcal{A}(M) \rightarrow \mathcal{A}(M) \) in \((3.5)\) (together with the quantum group surjection in \((1.5)\)) induces a map sending \( \mathcal{A}(M) \)-modules to \( \mathcal{A}(M) \)-modules, with a residual coaction of the ‘quantum group’ \( \mathcal{A}(U(1)) \) which is trivial on \( \mathcal{A}(M) \). From \( \mathcal{E}_p \) we obtain one such module \( \mathcal{E}_p \), such that the coaction of \( \mathcal{A}(U(1)) \) is an action of \( U(1) \) on \( \mathcal{E}_p \). Again by the Serre–Swan theorem, \( \mathcal{E}_p \) is the \( \mathcal{A}(M) \)-module of smooth sections \( \mathcal{E} = C^\infty(M, E) \) of a complex vector bundle \( E \rightarrow M \) which is equivariant with respect to the action of \( U(1) \) lifting the trivial action on \( M \).

An alternative way to understand this correspondence between \( SU_q(2) \)-equivariant modules over \( \mathcal{A}(M) \) and \( U(1) \)-equivariant bundles over \( M \) is as follows. Given \( p \in M \), consider the evaluation map \( e^p : \mathcal{A}(M) \rightarrow \mathbb{C} \) defined by \( e^p(f) = f(p) \) for \( f \in \mathcal{A}(M) \). By \( U(1) \)-equivariance, it induces a surjective algebra homomorphism \( e^p : \mathcal{A}(M) \rightarrow \mathcal{A}(CP^1_q) \). Let \( \mathcal{E}_p \) be a finitely-generated \( SU_q(2) \)-equivariant projective right (or left) \( \mathcal{A}(M) \)-module. Then the surjection \( e^p \) induces a finitely-generated \( SU_q(2) \)-equivariant projective right (or left) module \( \mathcal{E}_p \) over \( \mathcal{A}(CP^1_q) \). We may in this way regard \( \mathcal{E}_p \) also as a family of finitely-generated \( SU_q(2) \)-equivariant projective right (or left) \( \mathcal{A}(CP^1_q) \)-modules \( \mathcal{E}_p \) of the type described in \((1.2)\) parametrized by points \( p \in M \). The module \( \mathcal{E}_p \) is in correspondence with the representations of \( U(1) \) via the construction of \((1.2)\) and admits a decomposition \((1.22)\) into irreducible rank one modules \((1.17)\).

We are now ready to formulate the fundamental statement of dimensional reduction, which will enable us to think of \( \mathcal{E} = \mathcal{A}(SU_q(2)) \boxtimes_p \mathcal{E} \) as the module of sections of an \( SU_q(2) \)-equivariant vector bundle on \( CP^1_q \times M \). We begin with the following preliminary decomposition.

**Lemma 3.7.** Let \( M \) be a smooth manifold with trivial \( U(1) \)-action. Let \( C_n, n \in \mathbb{Z} \), denote the irreducible \( U(1) \)-module of weight \( n \) as given in \((1.15)\). Then every \( U(1) \)-equivariant
\( \mathcal{A}(M) \)-bimodule \( \mathcal{E} \) is isomorphic to a finite direct sum
\[
\mathcal{E} \simeq \bigoplus_{n \in W(\mathcal{E})} C_n \otimes \mathcal{E}_n ,
\]
where \( W(\mathcal{E}) \subset \mathbb{Z} \) is the set of eigenvalues for the \( U(1) \)-action on \( \mathcal{E} \), and \( \mathcal{E}_n \) are \( \mathcal{A}(M) \)-bimodules with trivial \( U(1) \)-coaction. If \( \mathcal{E} \) is finitely-generated (resp. projective) then the modules \( \mathcal{E}_n \) are also finitely-generated (resp. projective).

**Proof.** Denote by \( C_n \), with \( n \in \mathbb{Z} \), the \( \mathcal{A}(M) \)-bimodule of sections of the trivial bundle over \( M \) with typical fibre \( C_n \). It is naturally \( U(1) \)-equivariant. Using the decomposition (1.21) of a generic finite-dimensional representation \((V, \rho)\) for \( U(1) \), the dual formulation [1, Prop. 1.1] then gives a finite isotopical decomposition
\[
\mathcal{E} \simeq \bigoplus_{n \in W(\mathcal{E})} C_n \otimes \mathcal{E}_n ,
\]
where \( W(\mathcal{E}) \subset \mathbb{Z} \) is the set of eigenvalues for the \( U(1) \)-action on \( \mathcal{E} \), so that \( W(\mathcal{E}) = \{ n \in \mathbb{Z} \mid \mathcal{E}_n \neq 0 \} \) are the weights of \( \mathcal{E} \), and \( \mathcal{E}_n = \text{Hom}_{U(1)}(C_n, \mathcal{E}) \) are \( \mathcal{A}(M) \)-bimodules with trivial \( U(1) \)-action. Since \( C_n \) is associated to the trivial bundle, it is of the form \( C_n \simeq C_n \otimes \mathcal{A}(M) \) and the decomposition (3.8) follows.

**Proposition 3.9.** Every finitely-generated \( SU_q(2) \)-equivariant projective bimodule \( \mathcal{E} \) over \( \mathcal{A}(M) \) can be equivariantly decomposed, uniquely up to isomorphism, as
\[
\mathcal{E} = \bigoplus_{i=0}^{m} \mathcal{E}_i = \bigoplus_{i=0}^{m} \mathcal{L}_{m-2i} \otimes \mathcal{E}_i ,
\]
for some \( m \in \mathbb{N}_0 \), where \( \mathcal{E}_i \) are bimodules of sections of smooth vector bundles \( E_i \) over \( M \) with trivial \( SU_q(2) \) coactions and \( \mathcal{L}_n \) are bimodules (1.17) of sections of the \( SU_q(2) \)-equivariant line bundles over \( \mathbb{C}P^1_q \), together with morphisms
\[
\Phi_i \in \text{Hom}_{\mathcal{A}(M)}(\mathcal{E}_{i-1}, \mathcal{E}_i) , \quad i = 1, \ldots, m
\]
of \( \mathcal{A}(M) \)-bimodules.

**Proof.** Since the \( U(1) \)-action on \( \mathcal{A}(M) \) is trivial, by Lemma 3.7 we have that every \( U(1) \)-equivariant \( \mathcal{A}(M) \)-bimodule \( \mathcal{F} \) is isomorphic to a finite direct sum
\[
\mathcal{F} \simeq \bigoplus_{n \in W(\mathcal{F})} C_n \otimes \mathcal{F}_n ,
\]
where \( W(\mathcal{F}) \) are the weights of \( \mathcal{F} \) for the \( U(1) \)-action, and \( \mathcal{F}_n \) are \( \mathcal{A}(M) \)-bimodules with trivial \( U(1) \)-action. Putting this together with the decomposition (1.22) in terms of the line bundles \( \mathcal{L}_n \), we arrive at a decomposition for the corresponding induced bimodule over \( \mathcal{A}(M) \) given by
\[
\mathcal{F} = \mathcal{A}(SU_q(2)) \boxtimes \rho \mathcal{F} = \bigoplus_{n \in W(\mathcal{F})} \mathcal{L}_n \otimes \mathcal{F}_n .
\]
This decomposition describes the \( U(1) \)-action on \( \mathcal{F} \). The rest of the left \( SU_q(2) \)-coaction is incorporated by using the dual right \( U_q(\text{su}(2)) \)-action. From (1.25) the latter leaves each
line bundle $\mathcal{L}_n$ alone but this is not the case for the bimodules $\mathcal{F}_n$. From relations (1.2) the right action of $E$ sends $\mathcal{L}_n \otimes \mathcal{F}_n$ to $\mathcal{L}_n \otimes \mathcal{F}_{n-2}$ with corresponding $\varphi_n : \mathcal{F}_n \rightarrow \mathcal{F}_{n-2}$ that are $\mathcal{A}(M)$-bimodule morphisms. In particular, every indecomposable bimodule $\mathcal{F}$ has weight set of the form $W(\mathcal{F}) = \{ m_-, m_- + 2, \ldots, m_+ - 2, m_+ \}$ consisting of consecutive even or odd integers. By defining $m = \frac{1}{2} (m_+ - m_-)$, $\mathcal{E} = \mathcal{L}_{-m_- - m} \otimes \mathcal{F}$, $\mathcal{E}_i = \mathcal{F}_{m_+ - 2i}$, and $\mathcal{E}_i = \mathcal{L}_{m_+ - 2i} \otimes \mathcal{E}_i$, we find that the $K$-action is given by (3.10), while the $E$-action is determined by a chain of $\mathcal{A}(M)$-bimodule morphisms

$$E \xrightarrow{\phi_0} E_1 \xrightarrow{\phi_2} \cdots \xrightarrow{\phi_m} E_m$$

with $\phi_i := \varphi_{m_+ - 2i}$. By fixing $\mathcal{A}(M)$-valued hermitian structures $h_i : \mathcal{E}_i \otimes \mathcal{E}_i \rightarrow \mathcal{A}(M)$ on the modules $\mathcal{E}_i$, the action of $F = E^*$ is given by the adjoint morphisms $\phi_i^*$ in $\text{Hom}_{\mathcal{A}(M)}(\mathcal{E}_i, \mathcal{E}_{i-1})$. \hfill $\Box$

3.2. Covariant hermitian structures.

We will now give a gauge theory formulation of the equivalence between the $\text{SU}_q(2)$-equivariant bundles over $\mathcal{A}(\mathcal{M}) = \mathcal{A}(\mathbb{CP}^1_q) \otimes \mathcal{A}(M)$ and the module chains over $\mathcal{A}(M)$ described in Proposition 3.9. We first describe the reduction of $\text{SU}_q(2)$-covariant hermitian structures on the $\text{SU}_q(2)$-equivariant bimodules $\mathcal{E}$ of (3.11). On each line bundle $\mathcal{L}_n$, there is the $\mathcal{A}(\mathbb{CP}^1_q)$-valued hermitian structure defined in (2.23). Since we require an element in $\mathcal{A}(\mathbb{CP}^1_q)$, any two modules $\mathcal{L}_n$ and $\mathcal{L}_m$ with $m \neq n$ are taken to be orthogonal.

Let $\mathcal{E}$ be a finitely-generated $\text{SU}_q(2)$-equivariant projective right module over the algebra $\mathcal{A}(\mathcal{M})$, with corresponding equivariant decomposition (3.10). On each $\mathcal{A}(M)$-module $\mathcal{E}_i$ in this decomposition we fix an $\mathcal{A}(M)$-valued hermitian structure

$$h_i : \mathcal{E}_i \otimes \mathcal{E}_i \rightarrow \mathcal{A}(M).$$

Combined with (2.23) this gives an $\mathcal{A}(\mathcal{M})$-valued hermitian structure on $\mathcal{E}_i$, defined by

$$h_i = h_{m_+ - 2i} \otimes h_i : \mathcal{E}_i \otimes \mathcal{E}_i \rightarrow \mathcal{A}(\mathbb{CP}^1_q) \otimes \mathcal{A}(M),$$

and in turn a left $\text{SU}_q(2)$-covariant hermitian structure on $\mathcal{E}$ by

$$h = \bigoplus_{i=0}^m h_i : \mathcal{E} \otimes \mathcal{E} \rightarrow \mathcal{A}(\mathcal{M}).$$

By construction, the modules $\mathcal{E}_i$, $i = 0, 1, \ldots, m$ are $\text{SU}_q(2)$-covariantly mutually orthogonal, i.e. $h_i(\mathcal{E}_i, \mathcal{E}_j) = 0$ for $i \neq j$.

3.3. Decomposition of covariant connections.

Denote the left-covariant calculus on $\mathcal{A}(\mathbb{CP}^1_q)$ constructed in (2.22) by $(\Omega^1(\mathbb{CP}^1_q), \hat{d})$. Let $(\Omega^1(M), d)$ be the standard $*$-calculus on $\mathcal{A}(M)$, with $\Omega^1(M)$ the vector space of (complex) differential one-forms and $d$ the usual de Rham exterior derivative on the smooth manifold $M$. Then we define a calculus $(\Omega^1(\mathcal{M}), \hat{d})$ on $\mathcal{A}(\mathcal{M}) = \mathcal{A}(\mathbb{CP}^1_q) \otimes \mathcal{A}(M)$ by

$$\Omega^1(\mathcal{M}) = (\Omega^1(\mathbb{CP}^1_q) \otimes \mathcal{A}(M)) \oplus (\mathcal{A}(\mathbb{CP}^1_q) \otimes \Omega^1(M))$$

and

$$\hat{d} = \hat{d} \otimes \text{id} + \text{id} \otimes d.$$

Let $\mathcal{E}$ be a finitely-generated $\text{SU}_q(2)$-equivariant projective right $\mathcal{A}(\mathcal{M})$-module. Then we define

$$\Omega^1(\mathcal{E}) = \mathcal{E} \otimes_{\mathcal{A}(\mathcal{M})} \Omega^1(\mathcal{M}),$$

where $\mathcal{E} \otimes_{\mathcal{A}(\mathcal{M})} \Omega^1(\mathcal{M})$ is the $\mathcal{A}(\mathcal{M})$-bimodule on the left and the differential of $\mathcal{E}$ on the right.
and from the equivariant decomposition (3.10), \( \mathcal{E} = \bigoplus_i \mathcal{E}_i = \bigoplus_i \mathcal{L}_{m-2i} \otimes \mathcal{E}_i \), we get a corresponding decomposition

\[
\Omega^1(\mathcal{E}) = \bigoplus_{i=0}^m \Omega^1(\mathcal{E}_i)
\]

with

\[
\Omega^1(\mathcal{E}_i) = \mathcal{E}_i \otimes_{\mathcal{A}(\mathcal{M})} \Omega^1(\mathcal{M}) \cong (\Omega^1(\mathcal{L}_{m-2i}) \otimes \mathcal{E}_i) \oplus (\mathcal{L}_{m-2i} \otimes \Omega^1(\mathcal{E}_i)),
\]

and obvious notations \( \Omega^1(\mathcal{L}_{m-2i}) = \mathcal{L}_{m-2i} \otimes_{\mathcal{A}(\mathcal{CP}_q^1)} \Omega^1(\mathcal{CP}_q^1) \) and \( \Omega^1(\mathcal{E}_i) = \mathcal{E}_i \otimes_{\mathcal{A}(\mathcal{M})} \Omega^1(\mathcal{M}) \).

A connection on the right \( \mathcal{A}(\mathcal{M}) \)-module \( \mathcal{E} \) is given via a covariant derivative

\[
\nabla : \mathcal{E} \rightarrow \Omega^1(\mathcal{E})
\]

obeying the Leibniz rule

\[
\nabla (\varphi \cdot (b \otimes f)) = (\nabla \varphi) \cdot (b \otimes f) + \varphi \otimes_{\mathcal{A}(\mathcal{M})} d(b \otimes f),
\]

for \( \varphi \in \mathcal{E} \) and \( b \otimes f \in \mathcal{A}(\mathcal{CP}_q^1) \otimes \mathcal{A}(\mathcal{M}) \). The connection is unitary if in addition it is compatible with the hermitian structure \( \hbar \) of \( \mathcal{E} \) so that

\[
(3.12) \quad \hbar(\nabla \varphi, \psi) + \hbar(\varphi, \nabla \psi) = d(\hbar(\varphi, \psi)),
\]

for \( \varphi, \psi \in \mathcal{E} \). Here the metric \( \hbar \) is naturally extended to a map \( \Omega^1(\mathcal{E}) \times \Omega^1(\mathcal{E}) \rightarrow \Omega^2(\mathcal{M}) \) by the formulae

\[
\hbar(\varphi \otimes_{\mathcal{A}(\mathcal{M})} \eta, \psi) = \eta^* \hbar(\varphi, \psi) \quad \text{and} \quad \hbar(\varphi, \psi \otimes_{\mathcal{A}(\mathcal{M})} \xi) = \hbar(\varphi, \psi) \xi,
\]

for \( \varphi, \psi, \xi \in \mathcal{E} \) and \( \eta, \xi \in \Omega^1(\mathcal{M}) \), which respectively define metrics \( \Omega^1(\mathcal{E}) \times \mathcal{E} \rightarrow \Omega^1(\mathcal{M}) \) and \( \mathcal{E} \times \Omega^1(\mathcal{E}) \rightarrow \Omega^1(\mathcal{M}) \). For any \( p \geq 0 \), the connection \( \nabla \) is extended to a \( \mathbb{C} \)-linear map \( \nabla : \Omega^p(\mathcal{E}) \rightarrow \Omega^{p+1}(\mathcal{E}) \) by the graded Leibniz rule, where

\[
\Omega^p(\mathcal{E}) = \bigoplus_{i=0}^m \Omega^p(\mathcal{E}_i)
\]

with

\[
\Omega^p(\mathcal{E}_i) = (\mathcal{L}_{m-2i} \otimes \Omega^p(\mathcal{E}_i)) \oplus (\Omega^1(\mathcal{L}_{m-2i}) \otimes \Omega^{p-1}(\mathcal{E}_i)) \oplus (\Omega^2(\mathcal{L}_{m-2i}) \otimes \Omega^{p-2}(\mathcal{E}_i))
\]

and \( \Omega^0(\mathcal{E}_i) := \mathcal{E}_i \).

As usual, for any two connections \( \nabla, \nabla' \), their difference is an element

\[
\nabla' - \nabla = A \in \text{Hom}_{\mathcal{A}(\mathcal{M})}(\mathcal{E}, \Omega^1(\mathcal{E}))
\]

and if the connections are unitary then the ‘matrix of one-forms’ \( A \) is in addition anti-hermitian,

\[
\hbar(A \varphi, \psi) + \hbar(\varphi, A \psi) = 0 \quad \text{for} \quad \varphi, \psi \in \mathcal{E}.
\]

The collection of anti-hermitian elements in \( \text{Hom}_{\mathcal{A}(\mathcal{M})}(\mathcal{E}, \Omega^1(\mathcal{E})) \) will be denoted by \( \text{Hom}^a_{\mathcal{A}(\mathcal{M})}(\mathcal{E}, \Omega^1(\mathcal{E})) \). The group \( \mathcal{U}(\mathcal{E}) \) of gauge transformations consists of unitary elements in \( \text{End}_{\mathcal{A}(\mathcal{M})}(\mathcal{E}) \), with respect to the hermitian structure \( \hbar \). It acts on a connection \( \nabla \) by

\[
(u, \nabla) \mapsto \nabla^u = u \circ \nabla \circ u^*,
\]

where here \( u \) acts implicitly as \( u \otimes_{\mathcal{A}(\mathcal{M})} \text{id}_{\Omega^1(\mathcal{M})} \). A connection \( \nabla_A = \nabla + A \) will then transform to \( (\nabla_A)^u = \nabla + A^u \) with

\[
A^u = u(A \circ u^*) + u A u^*.
\]
That each $U(L_n) \simeq S^1$, the complex numbers of modulus one, means that the part of a gauge transformation in $U(L)$ acting on the bundles $L_{m-2i}$ is trivial. This fact will be used in §4.2 for the gauge invariance of the Yang–Mills action functional.

**Lemma 3.13.** Any unitary connection $\nabla$ on $(\mathcal{E}, h)$ decomposes as

$$\nabla = \sum_{i=0}^{m} \left( \nabla_i \right) \left( \mathcal{E}_i \right)$$

where:

1. Each $\nabla_i$ is a unitary connection on $(\mathcal{E}_i, h_i)$, i.e.
   $$h_i(\nabla_i \varphi, \psi) + h_i(\varphi, \nabla_i \psi) = d(h_i(\varphi, \psi)) \quad \text{for} \quad \varphi, \psi \in \mathcal{E}_i.$$

2. For $j \neq i$, $\beta_{ji} \in \text{Hom}_{A(M)}(\mathcal{E}_i, \Omega^1(\mathcal{E}_j))$ is the adjoint of $-\beta_{ij}$, i.e.
   $$h(\beta_{ji} \varphi, \psi) + h(\varphi, \beta_{ij} \psi) = 0 \quad \text{for} \quad \varphi \in \mathcal{E}_i, \psi \in \mathcal{E}_j.$$

**Proof.** Decompose the connection as

$$\nabla = \sum_{i=0}^{m} \nabla_i \quad \text{with} \quad \nabla_i : \mathcal{E}_i \rightarrow \Omega^1(\mathcal{E})$$

and

$$\nabla_i \mathcal{E}_i = \sum_{j=0}^{m} \beta_{ji} \quad \text{with} \quad \beta_{ji} : \mathcal{E}_i \rightarrow \Omega^1(\mathcal{E}_j).$$

Then, since $h(\mathcal{E}_i, \mathcal{E}_j) = 0$ when $i \neq j$, the unitarity condition (3.12) for $\nabla_i$ breaks into pieces giving the claimed decomposition with $\nabla_i = \beta_{ii}$. □

In a completely analogous way, one can decompose any given element $A$ of the space $\text{Hom}^a_{A(M)}(\mathcal{E}, \Omega^1(\mathcal{E}))$ as

$$A = \sum_{i=0}^{m} \left( A_i + \sum_{j<i} \left( A_{ji} - A_{ji}^* \right) \right),$$

with $A_i \in \text{Hom}^a_{A(M)}(\mathcal{E}_i, \Omega^1(\mathcal{E}_i))$ and $A_{ji} \in \text{Hom}_{A(M)}(\mathcal{E}_i, \Omega^1(\mathcal{E}_j))$, leading to a decomposition

$$\text{Hom}^a_{A(M)}(\mathcal{E}, \Omega^1(\mathcal{E})) \simeq \bigoplus_{i=0}^{m} \left( \text{Hom}^a_{A(M)}(\mathcal{E}_i, \Omega^1(\mathcal{E}_i)) \right) \oplus \bigoplus_{j<i} \text{Hom}_{A(M)}(\mathcal{E}_i, \Omega^1(\mathcal{E}_j)).$$
3.4. SU\(_q(2)\)-invariant connections and gauge transformations.

On \( \mathcal{E} = \bigoplus_i \mathcal{E}_i \), \( \mathcal{E}_i = \bigoplus_j \mathcal{L}_{m-2i} \otimes \mathcal{E}_i \), we denote by \( \Delta_{\mathcal{E}} \) the coaction of the Hopf algebra \( \mathcal{A}(\text{SU}_q(2)) \) which combines the natural coaction of \( \mathcal{A}(\text{SU}_q(2)) \) on the modules \( \mathcal{L}_{m-2i} \) given in \ref{subsec:2.28} and the trivial coaction on the modules \( \mathcal{E}_i \),

\[
\Delta_{\mathcal{E}} : \mathcal{E} \longrightarrow \mathcal{A}(\text{SU}_q(2)) \otimes \mathcal{E} ,
\]

and by \( \Delta_{\mathcal{E}}^{(1)} \) its lift to \( \Omega^1(\mathcal{E}) \), \( \Delta_{\mathcal{E}}^{(1)} : \Omega^1(\mathcal{E}) \rightarrow \mathcal{A}(\text{SU}_q(2)) \otimes \Omega^1(\mathcal{E}) \). In complete parallel with \ref{subsec:2.10} there are ‘adjoint’ coactions of \( \mathcal{A}(\text{SU}_q(2)) \) on the space \( \mathcal{C}(\mathcal{E}) \) of unitary connections on \( \mathcal{E}_i \), on the group \( \mathcal{U}(\mathcal{E}) \) of gauge transformations, as well as on the spaces \( \text{Hom}_{\mathcal{A}(\mathcal{M})}(\mathcal{E}_i, \mathcal{E}_j) \) and \( \text{Hom}_{\mathcal{A}(\mathcal{M})}(\mathcal{E}_i, \Omega^1(\mathcal{E}_j)) \). We shall denote by \( \mathcal{C}(\mathcal{E})^{\text{SU}_q(2)} \), etc. the corresponding space of coinvariant elements, i.e.,

\[
\mathcal{C}(\mathcal{E})^{\text{SU}_q(2)} = \{ \nabla \in \mathcal{C}(\mathcal{E}) \mid \Delta^\mathcal{C}(\nabla) = 1 \otimes \nabla \} ,
\]

and similarly for the other spaces and coactions. The spaces \( \mathcal{C}(\mathcal{E})^{\text{SU}_q(2)} \) and \( \mathcal{U}(\mathcal{E})^{\text{SU}_q(2)} \) of invariant connections and gauge transformations are described in terms of objects defined on \( M \) and of canonical (and unique) objects defined on \( \mathbb{C}P^1_q \). We begin with the space \( \mathcal{C}(\mathcal{E})^{\text{SU}_q(2)} \).

**Lemma 3.14.** One has

\[
(\text{Hom}_{\mathcal{A}(\mathcal{M})}^a(\mathcal{E}_i, \Omega^1(\mathcal{E}_i)))^{\text{SU}_q(2)} = \mathbb{C} \otimes \text{Hom}_{\mathcal{A}(\mathcal{M})}^a(\mathcal{E}_i, \Omega^1(\mathcal{E}_i)) ,
\]

while for \( i \neq j \) one has

\[
(\text{Hom}_{\mathcal{A}(\mathcal{M})}(\mathcal{E}_i, \Omega^1(\mathcal{E}_j)))^{\text{SU}_q(2)} = \begin{cases} 0 & \text{if } j \neq i \pm 1 , \\ \mathbb{C} \beta_- \otimes \text{Hom}_{\mathcal{A}(\mathcal{M})}(\mathcal{E}_i, \mathcal{E}_{i-1}) & \text{if } j = i - 1 , \\ \mathbb{C} \beta_+ \otimes \text{Hom}_{\mathcal{A}(\mathcal{M})}(\mathcal{E}_i, \mathcal{E}_{i+1}) & \text{if } j = i + 1 . \end{cases}
\]

**Proof.** For any \( i, j \) one has

\[
\text{Hom}_{\mathcal{A}(\mathcal{M})}(\mathcal{E}_i, \Omega^1(\mathcal{E}_j)) \simeq \left( \text{Hom}_{\mathcal{A}(\mathbb{C}P^1_q)}(\mathcal{L}_{m-2i}, \mathcal{L}_{m-2j}) \otimes \text{Hom}_{\mathcal{A}(\mathcal{M})}(\mathcal{E}_i, \Omega^1(\mathcal{E}_j)) \right) \\
\quad \oplus \left( \text{Hom}_{\mathcal{A}(\mathbb{C}P^1_q)}(\mathcal{L}_{m-2i}, \Omega^1(\mathcal{L}_{m-2j})) \otimes \text{Hom}_{\mathcal{A}(\mathcal{M})}(\mathcal{E}_i, \mathcal{E}_j) \right)
\]

and, since \( \text{SU}_q(2) \) coacts trivially on the bundles \( \mathcal{E}_i \), for the coinvariant elements one finds

\[
(\text{Hom}_{\mathcal{A}(\mathcal{M})}^a(\mathcal{E}_i, \Omega^1(\mathcal{E}_j)))^{\text{SU}_q(2)} \simeq \left( (\text{Hom}_{\mathcal{A}(\mathbb{C}P^1_q)}(\mathcal{L}_{m-2i}, \mathcal{L}_{m-2j}))^{\text{SU}_q(2)} \otimes \text{Hom}_{\mathcal{A}(\mathcal{M})}(\mathcal{E}_i, \Omega^1(\mathcal{E}_j)) \right) \\
\quad \oplus \left( (\text{Hom}_{\mathcal{A}(\mathbb{C}P^1_q)}(\mathcal{L}_{m-2i}, \Omega^1(\mathcal{L}_{m-2j})))^{\text{SU}_q(2)} \otimes \text{Hom}_{\mathcal{A}(\mathcal{M})}(\mathcal{E}_i, \mathcal{E}_j) \right) .
\]

In order to proceed we need only the fact that there are no non-trivial \( \text{SU}_q(2) \)-invariant elements in the modules \( \mathcal{L}_n \) for \( n \neq 0 \), while 1 is the only invariant element in the algebra \( \mathcal{L}_0 = \mathcal{A}(\mathbb{C}P^1_q) \). Then by Lemma \ref{lem:1.18} one has

\[
(\text{Hom}_{\mathcal{A}(\mathbb{C}P^1_q)}(\mathcal{L}_{m-2i}, \mathcal{L}_{m-2j}))^{\text{SU}_q(2)} \simeq (\mathcal{L}_{2i-2j})^{\text{SU}_q(2)} = \begin{cases} \mathbb{C} & \text{if } 2i = 2j , \\ 0 & \text{if } 2i \neq 2j . \end{cases}
\]
On the other hand, using (2.18) one finds
\[
\left( \text{Hom}_{A(\mathbb{C}P_1^i)}(\mathcal{L}_{m-2i}, \Omega^1(\mathcal{L}_{m-2j})) \right)^{SU_q(2)} \simeq \left( \mathcal{L}_{2i-2j-2\beta_-} \oplus \mathcal{L}_{2i-2j+2\beta_+} \right)^{SU_q(2)}
\]
\[
\simeq \begin{cases} 
0 & \text{if } 2i - 2j \neq \pm 2, \\
\mathbb{C}\beta_- & \text{if } 2i - 2j = 2, \\
\mathbb{C}\beta_+ & \text{if } 2i - 2j = -2, 
\end{cases}
\]
and the results now follow. \hfill \square

Using Lemma 3.14 and \( \beta_- = -\beta_+ \), an element \( A \in (\text{Hom}_{A(M)}(\mathcal{E}_i, \Omega^1(\mathcal{E}_i)))^{SU_q(2)} \) can be written as
\[
A = \sum_{i=0}^{m} \left( 1 \otimes A_i + \beta_+ \otimes \phi_{i+1} + \beta_- \otimes \phi^*_{i+1} \right),
\]
where the ‘gauge potentials’ \( A_i \in \text{Hom}_{A(M)}(\mathcal{E}_i, \Omega^1(\mathcal{E}_i)) \) and ‘Higgs fields’ \( \phi_{i+1} \) in the space \( \text{Hom}_{A(M)}(\mathcal{E}_i, \Omega^1(\mathcal{E}_{i+1})) \) for \( i = 0, 1, \ldots, m \) with \( \mathcal{E}_{m+1} := 0 \). Now let \( (\mathcal{E}_i, h) \) be an \( SU_q(2) \)-equivariant hermitian \( A(M) \)-module decomposed as in (3.10) with the metric \( h \) decomposed as in (3.11). Let \( (\mathcal{E}_i, h_i) \) for \( i = 0, 1, \ldots, m \) be the hermitian \( A(M) \)-modules composing \( (\mathcal{E}_i, h) \), and let \( \mathcal{C}(\mathcal{E}_i) \) be the corresponding spaces of unitary connections.

**Proposition 3.17.** There is a bijection between the spaces \( \mathcal{C}(\mathcal{E}_i)^{SU_q(2)} \) and
\[
\mathcal{C}(\mathcal{E}_i) := \prod_{i=0}^{m} \left( \mathcal{C}(\mathcal{E}_i) \times \text{Hom}_{A(M)}(\mathcal{E}_i, \mathcal{E}_{i+1}) \right)
\]
which associates to any element \( (\nabla, \phi) \) of \( \mathcal{C}(\mathcal{E}_i) \), given by connections \( \nabla_i \in \mathcal{C}(\mathcal{E}_i) \) and Higgs fields \( \phi_{i+1} \in \text{Hom}_{A(M)}(\mathcal{E}_i, \mathcal{E}_{i+1}) \) for \( i = 0, 1, \ldots, m \), the \( SU_q(2) \)-invariant unitary connection \( \nabla \in \mathcal{C}(\mathcal{E})^{SU_q(2)} \) given by
\[
\nabla = \sum_{i=0}^{m} \left( \nabla_i + \beta_+ \otimes \phi_{i+1} + \beta_- \otimes \phi^*_{i+1} \right).
\]
Here \( \nabla_i \) is the unitary connection on \( (\mathcal{E}_i, h_i) \) given by
\[
\nabla_i = \nabla_{m-2i} \otimes \text{id} + \text{id} \otimes \nabla_i,
\]
where \( \nabla_{m-2i} \) is the unique (by Lemma 2.13) \( SU_q(2) \)-invariant unitary connection on the hermitian line bundle \( (\mathcal{L}_{m-2i}, \hat{h}_{m-2i}) \) given in (2.15) and (2.23).

**Proof.** Fix a unitary connection \( \nabla_i^0 \in \mathcal{C}(\mathcal{E}_i) \) for each \( i = 0, 1, \ldots, m \). Define unitary connections on each \( (\mathcal{E}_i, h_i) \) by \( \nabla_i^0 = \nabla_{m-2i} \otimes \text{id} + \text{id} \otimes \nabla_i^0 \). They are clearly \( SU_q(2) \)-invariant and give rise to an \( SU_q(2) \)-invariant unitary connection \( \nabla^0 \) on \( (\mathcal{E}, h) \) defined by the sum \( \nabla^0 = \sum_i \nabla_i^0 \). All \( SU_q(2) \)-invariant unitary connections \( \nabla \) on \( (\mathcal{E}, h) \) take the form \( \nabla = \nabla^0 + A \) with \( A \in (\text{Hom}_{A(M)}(\mathcal{E}_i, \Omega^1(\mathcal{E}_i)))^{SU_q(2)} \). The general form of such an element is given in (3.16), from which the expression (3.18) follows. \hfill \square

Next we give a similar characterization of the space \( \mathcal{U}(\mathcal{E})^{SU_q(2)} \).
Lemma 3.19. One has
\[
\left(\text{Hom}_{\mathcal{A}(M)}(\mathcal{E}_i, \mathcal{E}_j)\right)^{\text{SU}_q(2)} = \begin{cases} 
\mathbb{C} \otimes \text{End}_{\mathcal{A}(M)}(\mathcal{E}_i) & \text{if } i = j, \\
0 & \text{if } i \neq j.
\end{cases}
\]

Proof. For any \(i, j\) one has
\[
\text{Hom}_{\mathcal{A}(M)}(\mathcal{E}_i, \mathcal{E}_j) \cong \text{Hom}_{\mathcal{A}(\mathbb{C}P_1)}(\mathcal{L}_{m-2i}, \mathcal{L}_{m-2j}) \otimes \text{Hom}_{\mathcal{A}(M)}(\mathcal{E}_i, \mathcal{E}_j)
\]
and, since SU_q(2) coacts trivially on the bundles \(\mathcal{E}_i\), for the invariant elements one finds
\[
\left(\text{Hom}_{\mathcal{A}(M)}(\mathcal{E}_i, \mathcal{E}_j)\right)^{\text{SU}_q(2)} \cong \left(\text{Hom}_{\mathcal{A}(\mathbb{C}P_1)}(\mathcal{L}_{m-2i}, \mathcal{L}_{m-2j})\right)^{\text{SU}_q(2)} \otimes \text{Hom}_{\mathcal{A}(M)}(\mathcal{E}_i, \mathcal{E}_j).
\]
The result now follows from (3.15).

Proposition 3.20. There is a bijection between the groups \(\mathcal{U}(\mathcal{E})^{\text{SU}_q(2)}\) and
\[
\mathcal{W}(\mathcal{E}) := \prod_{i=0}^{m} \mathcal{U}(\mathcal{E}_i),
\]
which associates to any element \(u = (u_0, u_1, \ldots, u_m) \in \mathcal{W}(\mathcal{E})\) the SU_q(2)-invariant gauge transformation of \((\mathcal{E}, \mathcal{L})\) given by
\[
u = \sum_{i=0}^{m} u_i,
\]
with \(u_i = 1 \otimes u_i \in \mathcal{U}(\mathcal{E}_i)^{\text{SU}_q(2)} \cong \mathbb{C} \otimes \mathcal{U}(\mathcal{E}_i).
\]

Proof. This follows from Lemma 2.31 and Lemma 3.19.

The group \(\mathcal{U}(\mathcal{E}_i)\) acts on both spaces Hom_{\mathcal{A}(M)}(\mathcal{E}_i, \mathcal{E}_{i+1}) and Hom_{\mathcal{A}(M)}(\mathcal{E}_{i+1}, \mathcal{E}_i)\) of Higgs fields by
\[
u_i(\phi_{i+1}) = \phi_{i+1} \circ u_i^{-1} \quad \text{and} \quad \nu_i(\phi^*_{i+1}) = u_i \circ \phi^*_{i+1},
\]
for \(u_i \in \mathcal{U}(\mathcal{E}_i)\) and \(\phi_{i+1} \in \text{Hom}_{\mathcal{A}(M)}(\mathcal{E}_i, \mathcal{E}_{i+1}), \phi^*_{i+1} \in \text{Hom}_{\mathcal{A}(M)}(\mathcal{E}_{i+1}, \mathcal{E}_i)\). There is also an induced action of the group \(\mathcal{W}(\mathcal{E})\) on the space \(\mathcal{C}(\mathcal{E})\) of connections. The following result is then immediate.

Proposition 3.21. The bijections between invariant connections and between invariant gauge transformations of Proposition 3.17 and Proposition 3.20, respectively, are compatible with the actions of the groups of Proposition 3.20 on the connections of Proposition 3.17, and there is a bijection between gauge orbits
\[
\mathcal{C}(\mathcal{E})^{\text{SU}_q(2)} / \mathcal{U}(\mathcal{E})^{\text{SU}_q(2)} \equiv \mathcal{C}(\mathcal{E}) / \mathcal{W}(\mathcal{E}).
\]
3.5. Integrable connections.

In the sequel we will need to work with integrable connections as well. Let \( M \) be a complex manifold, with the standard complex structure for the complexified de Rham differential calculus. Combined with the complex structure for the differential calculus on \( A(\mathbb{C}P^1_q) \) described in Proposition 2.11, we get a natural complex structure for the calculus on the algebra \( A(\mathbb{C}P^1_q) = A(\mathbb{C}P^1_q) \otimes A(M) \). If \( \nabla \) is a connection on the \( A(M) \)-bimodule \( \mathcal{E} \) with equivariant decomposition (3.10), then the \((0,2)\)-component of its curvature \( F^{0,2}_{\nabla} \) is an element of \( \text{Hom}_{A(M)}(\mathcal{E}, \Omega^{0,2}(\mathcal{E})) \), where

\[
\Omega^{0,2}(\mathcal{E}) = \bigoplus_{i=0}^{m} \left( (\Omega^{0,2}(\mathcal{L}_{m-2i}) \otimes \mathcal{E}_i) \oplus (\Omega^{0,1}(\mathcal{L}_{m-2i}) \otimes \Omega^{0,1}(\mathcal{E}_i)) \oplus (\mathcal{L}_{m-2i} \otimes \Omega^{0,2}(\mathcal{E}_i)) \right).
\]

The connection \( \nabla \) is then integrable if \( F^{0,2}_{\nabla} = 0 \). In this case the pair \( (\mathcal{E}, \nabla) \) is a holomorphic vector bundle [17, §2].

By (3.18) an \( SU_q(2) \)-invariant unitary holomorphic connection on \( (\mathcal{E}, h) \) is of the form

\[
(3.22) \quad \nabla^\phi = \sum_{i=0}^{m} \left( \nabla^\phi_i + \beta_- \otimes \phi^*_i \right),
\]

where \( \nabla^\phi_i \) is the holomorphic connection on \( (\mathcal{E}_i, h_i) \) given by

\[
\nabla^\phi_i = \hat{\nabla}^\phi_i \otimes \text{id} + \text{id} \otimes \nabla^\phi_i
\]

with \( \hat{\nabla}^\phi_i \) the unique \( SU_q(2) \)-invariant unitary holomorphic connection on the hermitian line bundle \( (\mathcal{L}_{m-2i}, \hat{h}_{m-2i}) \) given in (2.24), and \( \nabla^\phi_i \) is a holomorphic unitary connection on \( (\mathcal{E}_i, h_i) \). As before the Higgs fields \( \phi^*_i \in \text{Hom}_{A(M)}(\mathcal{E}_{i+1}, \mathcal{E}_i) \). Its curvature is readily found to be

\[
(3.23) \quad F^{0,2}_{\nabla} := (\nabla^\phi)^2 = \sum_{i=0}^{m} \left( \text{id} \otimes (\nabla^\phi_i)^2 + \beta_- \otimes (\phi^*_i \circ \nabla^\phi_{i+1} - \nabla^\phi_i \circ \phi^*_i) \right),
\]

where we have used \( (\hat{\nabla}^\phi_m)^2 = 0 \) and \( \beta_- \wedge \beta_- = 0 \).

There is a natural coaction of the quantum group \( SU_q(2) \) on the collection \( \mathcal{C}(\mathcal{E})^{1,1} \) of integrable unitary connections on \( (\mathcal{E}, \lambda) \), obtained by restricting the ‘adjoint’ coaction \( \Delta^c \) of \( A(SU_q(2)) \). Let \( \left( \mathcal{C}(\mathcal{E})^{1,1} \right)^{SU_q(2)} \) be the \( SU_q(2) \)-invariant subspace. This is the space of holomorphic structures on \( \mathcal{E} \) [17] for which the coaction of the Hopf algebra \( A(SU_q(2)) \) is holomorphic. Let \( \mathcal{C}(\mathcal{E})^{1,1} \) be the collection of integrable unitary connections on \( (\mathcal{E}_i, h_i) \).

Proposition 3.24. Let \( \mathcal{C}(\mathcal{E})^{1,1} \) be the subspace of \( \mathcal{C}(\mathcal{E}) \) consisting of integrable connections \( \nabla^\phi_i \in \mathcal{C}(\mathcal{E}_i)^{1,1} \) and Higgs fields \( \phi^*_i \in \text{Hom}_{A(M)}(\mathcal{E}_{i+1}, \mathcal{E}_i) \) for \( i = 0, 1, \ldots, m \) on which the holomorphic connection \( \nabla^\phi_{i+1,i} \) on \( \text{Hom}_{A(M)}(\mathcal{E}_{i+1}, \mathcal{E}_i) \) induced by \( \nabla^\phi_{i+1} \) and \( \nabla^\phi_i \) vanishes,

\[
\nabla^\phi_{i+1,i}(\phi^*_i) := \phi^*_i \circ \nabla^\phi_{i+1} - \nabla^\phi_i \circ \phi^*_i = 0.
\]

Then the bijection of Proposition 3.17 defines a bijection between the spaces \( \left( \mathcal{C}(\mathcal{E})^{1,1} \right)^{SU_q(2)} \) and \( \mathcal{C}(\mathcal{E})^{1,1} \).

Proof. This is a direct consequence of the expression (3.23) for the curvature. \( \square \)
4. Quiver gauge theory and non-abelian coupled \( q \)-vortex equations

In the remainder of this paper we will assume that \( M \) is a connected Kähler manifold of complex dimension \( d \), with fixed Kähler form \( \omega \in \Omega^{1,1}(M) \). We then proceed to work out the equivariant dimensional reduction of Yang–Mills theory defined on the quantum space \( \overline{M} = \mathbb{C}P^1 \times M \). This will produce a \( q \)-deformation of the usual quiver gauge theories on \( M \) associated to the linear \( A_{m+1} \) quiver \([1, 2, 31, 20, 21, 12]\). The vacuum states of the resulting quiver gauge theory are described by \( q \)-deformations of chain vortex equations, which arise by dimensional reduction of BPS-type gauge theory equations on \( \overline{M} \) and whose solutions we call ‘\( q \)-vortices’. The \( q \)-vortices on the manifold \( M \) are in a bijective correspondence with generalized instantons on the quantum space \( \overline{M} \). The data in the space \( \mathcal{C}(\overline{E}) \) of Proposition 3.17 defining a \( q \)-vortex will be referred to as a ‘stable \( q \)-quiver bundle’ over \( M \). In contrast to the \( q = 1 \) case, the degree of a \( q \)-quiver bundle is generically non-zero, and there are \( q \)-vortices which are realized as zeroes of the quiver gauge theory action functional. In fact, we will find that the \( q \)-deformation of quiver bundles over the manifold \( M \) is analogous in some ways to the twistings of quiver bundles considered in [3]. In particular, we will find analogous constraints on the characteristic classes of stable \( q \)-quiver bundles over \( M \), which can be used to naturally construct flat connections on \( M \). Henceforth we fix a deformation parameter \( 0 < q < 1 \) and an integer \( m \geq 0 \) parametrizing an \( SU_q(2) \)-equivariant decomposition as in (3.10).

4.1. Metrics on \( SU_q(2) \)-equivariant vector bundles.

In the following we will make use of various \( SU_q(2) \)-invariant metrics defined on the equivariant modules over \( A(\overline{M}) \) considered in [3]. We start by defining a natural Hodge duality operator on the forms

\[
\Omega^p(\overline{M}) = (\mathcal{A}(\mathbb{C}P^1_q) \otimes \Omega^p(M)) \oplus (\Omega^1(\mathbb{C}P^1_q) \otimes \Omega^{p-1}(M)) \oplus (\Omega^2(\mathbb{C}P^1_q) \otimes \Omega^{p-2}(M))
\]

with \( \Omega^0(M) = A(M) \) and \( \Omega^{<0}(M) := 0 =: \Omega^{>2d}(M) \). Let \( \star : \Omega^p(M) \rightarrow \Omega^{2d-p}(M) \) be the Hodge operator corresponding to the Kähler metric of \( M \), with \( \star 1 = \frac{1}{d!} \) and \( \star 2 = \text{id} \). Using the left-covariant Hodge operator \( \hat{\star} \) on \( \Omega^*(\mathbb{C}P^1_q) \) defined in (2.22) we then define the bimodule map

\[
\star := \hat{\star} \otimes \star : \Omega^p(\overline{M}) \rightarrow \Omega^{2(d+1)-p}(\overline{M})
\]

with \( \star^2 = \text{id} \). Using the integration defined in (2.36), we define an integral

\[
\int_{\overline{M}} := \int_{\mathbb{C}P^1_q} \otimes \int_M : \Omega^2(\mathbb{C}P^1_q) \otimes \Omega^{2d}(M) \rightarrow \mathbb{C}
\]

when the integral over \( M \) exists. We set \( \int_{\overline{M}} \alpha := 0 \) whenever \( \alpha \notin \Omega^2(\mathbb{C}P^1_q) \otimes \Omega^{2d}(M) \). One then introduces a complex inner product on \( \Omega^p(\overline{M}) \) for each \( p \geq 0 \) by

\[
(\alpha, \alpha')_{\Omega^p(\overline{M})} := \int_{\overline{M}} \alpha^* \wedge \hat{\star} \alpha'
\]

for \( \alpha, \alpha' \in \Omega^p(\overline{M}) \). Forms of different degrees are defined to be orthogonal. This is a natural generalization of analogous inner products \( (\cdot, \cdot)_{\Omega^p(M)} \) for each \( p \geq 0 \) defined via the Hodge operator \( \star \) on \( M \).
Let $\mathcal{E}$ be a finitely-generated $\text{SU}(2)$-equivariant projective bimodule over the algebra $\mathcal{A}(\mathcal{M})$, with equivariant decomposition ($3.10$). Given hermitian structures on its components $h_i : \mathcal{E}_i \times \mathcal{E}_i \to \mathcal{A}(\mathcal{M})$, we define $L^2$-metrics and $L^2$-norms on the modules of sections $\mathcal{E}_i$ for each $i = 0, 1, \ldots, m$ by

$$(\varphi, \psi)_h = \int_M h_i(\varphi, \psi) \frac{\omega^d}{dt}$$

and

$$\|\varphi\|_h = (\varphi, \varphi)_h^{1/2}$$

for $\varphi_i, \psi_i \in \mathcal{E}_i$. The hermitian structures $h_i$ also induce a metric $h_{i,i+1}$ on each of the spaces of Higgs fields $\text{Hom}_{\mathcal{A}(\mathcal{M})}(\mathcal{E}_i, \mathcal{E}_{i+1})$ as follows. Since each bimodule $\mathcal{E}_i$ is finitely-generated and projective, it is of the form $\mathcal{E}_i = p_i(\mathcal{A}(\mathcal{M}))^{n_i}$ for some $n_i \in \mathbb{N}$ and a projection $p_i \in \text{Mat}_{n_i}(\mathcal{A}(\mathcal{M}))$; then $\text{End}_{\mathcal{A}(\mathcal{M})}(\mathcal{E}_i) \simeq p_i \text{Mat}_{n_i}(\mathcal{A}(\mathcal{M})) p_i$. We denote by $\text{tr}$ the partial matrix trace over ‘internal indices’ of the endomorphism algebra $\text{End}_{\mathcal{A}(\mathcal{M})}(\mathcal{E}_i)$ and define

$$h_{i,i+1}(\phi_{i+1}, \psi_{i+1}) = \text{tr} (\phi_{i+1}^* \circ \psi_{i+1})$$

for $\phi_{i+1}, \psi_{i+1} \in \text{Hom}_{\mathcal{A}(\mathcal{M})}(\mathcal{E}_i, \mathcal{E}_{i+1})$, where $\phi_{i+1}^* : \mathcal{E}_{i+1} \to \mathcal{E}_i$ is the adjoint morphism of the Higgs field $\phi_{i+1} : \mathcal{E}_i \to \mathcal{E}_{i+1}$ with respect to the hermitian metrics $h_i$ on $\mathcal{E}_i$ and $h_{i+1}$ on $\mathcal{E}_{i+1}$. The associated $L^2$-inner products and $L^2$-norms are obtained by integrating the hermitian metrics over $\mathcal{M}$ to get

$$(\phi_{i+1}, \psi_{i+1})_{h_{i,i+1}} = \int_M h_{i,i+1}(\phi_{i+1}, \psi_{i+1}) \frac{\omega^d}{dt}$$

and

$$\|\phi_{i+1}\|_{h_{i,i+1}} = (\phi_{i+1}, \phi_{i+1})_{h_{i,i+1}}^{1/2}.$$

Using the hermitian structure $\mathcal{h}$ in ($3.11$), we can further define a complex inner product on the bimodules $\mathcal{E}$ over $\mathcal{A}(\mathcal{M})$. Let $\varphi, \psi \in \mathcal{E}$ with decompositions $\varphi = \sum_i \varphi_i \otimes \varphi_i$ and $\psi = \sum_i \psi_i \otimes \psi_i$, where $\varphi_i, \psi_i \in \mathcal{E}_{m-2i}$ and $\varphi_i, \psi_i \in \mathcal{E}_i$. Using the orthogonality of the direct sum decomposition $\mathcal{E} = \bigoplus_i \mathcal{E}_i$, with respect to $\mathcal{h}$, we define an $L^2$-metric and $L^2$-norm on $\mathcal{E}$ by

$$(\varphi, \psi)_\mathcal{h} = \sum_{i=0}^m H(h_{m-2i}(\varphi_i, \psi_i)) (\varphi_i, \psi_i)_{h_i}$$

and

$$\|\varphi\|_\mathcal{h} = (\varphi, \varphi)_\mathcal{h}^{1/2},$$

where $H$ is the Haar functional on $\mathcal{A}(\mathbb{C}P^q)$.

Finally, we define $\text{SU}(2)$-invariant metrics on the spaces $\text{Hom}_{\mathcal{A}(\mathcal{M})}(\mathcal{E}, \Omega^p(\mathcal{E}))$ for each $p \geq 0$. Since $\mathcal{E}$ is finitely-generated and projective, any $\mathcal{A}(\mathcal{M})$-linear map $\mathcal{E} \to \Omega^p(\mathcal{E})$ can be regarded as an element in $\text{End}_{\mathcal{A}(\mathcal{M})}(\mathcal{E}) \otimes_{\mathcal{A}(\mathcal{M})} \Omega^p(\mathcal{M})$, i.e. as a matrix with entries in $\Omega^p(\mathcal{M})$. By composing the hermitian structure $\mathcal{h}$ on $\mathcal{E}$ with an ordinary (partial) matrix trace over ‘internal indices’, one constructs an inner product on $\text{End}_{\mathcal{A}(\mathcal{M})}(\mathcal{E})$. By combining this product with the inner product on $\Omega^p(\mathcal{M})$ given in ($4.1$), one then obtains a natural $L^2$-inner product $(-, -)_\mathcal{h}$ on the space $\text{Hom}_{\mathcal{A}(\mathcal{M})}(\mathcal{E}, \Omega^p(\mathcal{E}))$ with associated $L^2$-norm $\| - \|_\mathcal{h}$. In an analogous way, one defines $L^2$-inner products $(-, -)_{h_i}$ on the orthogonal components $\text{Hom}_{\mathcal{A}(\mathcal{M})}(\mathcal{E}_i, \Omega^p(\mathcal{E}_i))$ with associated $L^2$-norms $\| - \|_{h_i}$, and $L^2$-inner products and $L^2$-norms $(-, -)_{h_{i,i+1}}$ and $\| - \|_{h_{i,i+1}}$ on the spaces $\text{Hom}_{\mathcal{A}(\mathcal{M})}(\mathcal{E}_i, \Omega^p(\mathcal{E}_{i+1}))$.

4.2. Dimensional reduction of the Yang–Mills action functional.

Let $\mathcal{C}(\mathcal{E})$ be the space of unitary connections on an $\text{SU}(2)$-equivariant hermitian $\mathcal{A}(\mathcal{M})$-module $(\mathcal{E}, \mathcal{h})$. The Yang–Mills action functional $\text{YM} : \mathcal{C}(\mathcal{E}) \to [0, \infty)$ is defined by

$$(4.2) \quad \text{YM}(\nabla) = \| F_\mathcal{h} \|_\mathcal{h}^2,$$
where $F_{\nabla} = \nabla^2$ is the curvature of the connection $\nabla$, regarded as an element of $	ext{Hom}_{A(M)}(E, \Omega^2(E))$. Under a gauge transformation, i.e. the action of a unitary endomorphism $u \in \mathcal{U}(E)$ of the module $E$ on $C(E)$, one has

$$F_{\nabla^u} = uF_{\nabla}u^* ,$$

and by construction the functional (1.2) is gauge invariant, i.e. $YM(\nabla^u) = YM(\nabla)$ for all $\nabla \in \mathcal{C}(E)$ and $u \in \mathcal{U}(E)$. Consequently, the Yang–Mills action functional descends to a map on gauge orbits $YM : \mathcal{C}(E)/\mathcal{U}(E) \to [0, \infty)$. We have already remarked that the part of a gauge transformation that acts on the bundles $\mathcal{L}_{m-2i}$ is trivial. This entails that in proving the invariance of the Yang–Mills functional under gauge transformation, there is no problem coming from the integral over $\mathbb{CP}^1_q$ not being a trace but rather a twisted trace.

In this section we compute the restriction of the functional (1.2) to the corresponding $\text{SU}_q(2)$-invariant subspaces $\mathcal{C}(E)^{\text{SU}_q(2)}$ and $\mathcal{C}(E)/\mathcal{U}(E)^{\text{SU}_q(2)}$.

**Proposition 4.3.** Under the bijection of Proposition [3.17] the restriction of the Yang–Mills action functional $YM|_{\mathcal{C}(E)^{\text{SU}_q(2)}}$ on the quantum space $M$ to $\text{SU}_q(2)$-invariant unitary connections is equal to the Yang–Mills–Higgs functional $YM_{q,m} : \mathcal{E}(E) \to [0, \infty)$ on the manifold $M$ defined by

$$YM_{q,m}(\nabla, \phi) = \sum_{i=0}^{m} \left( \| F_{\nabla_i} \|_{h_i}^2 + (q^2 + 1) \| \nabla_{i-1,i}(\phi_i) \|_{h_{i-1,i}}^2 + \| \phi_{i+1}^* \circ \phi_i - q^2 \phi_i \circ \phi_{i+1}^* - q^{m-2i+1}[m - 2i] id_{\mathcal{E}} \|_{h_i}^2 \right),$$

with $\phi_0 := 0 =: \phi_i^*$ and $\phi_{m+1} := 0 =: \phi_{m+1}^*$. Here $F_{\nabla_i} = \nabla_i^2$ is the curvature of the connection $\nabla_i \in \mathcal{C}(E_i)$ on $M$, regarded as an element of $\text{Hom}_{A(M)}(E_i, \Omega^2(E_i))$, while $\nabla_{i-1,i}$ is the connection on $\text{Hom}_{A(M)}(E_{i-1}, E_i)$ induced by $\nabla_{i-1}$ on $E_{i-1}$ and $\nabla_i$ on $E_i$ with

$$\nabla_{i-1,i}(\phi_i) = \phi_i \circ \nabla_{i-1} - \nabla_i \circ \phi_i .$$

Under the bijection of Proposition [3.22] the functional (4.1) restricts to a map on gauge orbits $YM_{q,m} : \mathcal{C}(E)/\mathcal{U}(E) \to [0, \infty)$.

**Proof.** From (3.18), any $\text{SU}_q(2)$-invariant unitary connection $\nabla \in \mathcal{C}(E)^{\text{SU}_q(2)}$ is of the form $\nabla = \sum_i (\nabla_i + \beta_+ \otimes \phi_{i+1} + \beta_- \otimes \phi_i^*)$ with $\nabla_i = \nabla_{m-2i} \otimes id + id \otimes F_{\nabla_i}$. Thus its curvature $F_{\nabla} = \nabla \circ \nabla$ is given by

$$F_{\nabla} = \sum_{i=0}^{m} \left( \nabla_i^2 + \beta_+ \otimes \nabla_{i-1,i}(\phi) - \beta_- \otimes \nabla_{i+1,i}(\phi_i^*) + (\beta_+ \wedge \beta_-) \otimes (\phi_i \circ \phi_i^*) (\beta_- \wedge \beta_+) \otimes (\phi_{i+1}^* \circ \phi_{i+1}) \right) .$$

A straightforward computation gives

$$\nabla_i^2 = f_{m-2i} \otimes id + id \otimes F_{\nabla_i} ,$$
where $f_{m-2i} = \hat{\nabla}_{m-2i}^2$ is the curvature (2.23) of the canonical connection on the $\text{SU}_q(2)$-equivariant line bundle $L_{m-2i}$ over $\mathbb{C}P^1_q$. By (2.21) one has

$$f_{m-2i} = -q^{m-2i+1} [m - 2i] \beta_- \wedge \beta_+$$

as an element of $\text{Hom}_{\mathcal{A}(\mathbb{C}P^1_q)}(L_{m-2i}, \Omega^2(L_{m-2i}))$. By (2.8) one has $\beta_- \wedge \beta_- = 0 = \beta_+ \wedge \beta_+$ and $\beta_+ \wedge \beta_- = -q^{2} \beta_- \wedge \beta_+$. Substituting everything into (4.6), we arrive at

$$F_{\Sigma} = \sum_{i=0}^{m} \left( \text{id} \otimes F_{\nabla_i} + \beta_+ \otimes \nabla_{i-1,i}(\phi_i) - \beta_- \otimes \nabla_{i+1,i}(\phi_{i+1}^*) \right)$$

$$+ (\beta_- \wedge \beta_+) \otimes (\phi_{i+1}^* \circ \phi_{i+1} - q^2 \phi_i \circ \phi_i^* - q^{m-2i+1} [m - 2i] \text{id}) \right) \right).$$

(4.7)

We now use the definition of the Hodge operator and integral on $M$ from (4.4), together with $\hat{\beta}_\pm = \pm \beta_\pm$ and (2.13), and the $*$-structure $\beta_+^* = -\beta_-$, to compute the corresponding Yang–Mills action functional (4.2). One finds

$$\text{YM}(\Sigma) = \int_M \text{tr} \left( F_{\Sigma}^* \wedge * F_{\Sigma} \right)$$

$$= \sum_{i,j=0}^{m} \int_{\mathbb{C}P^1_q} \otimes \int_M \text{tr} \left[ \left( \text{id} \otimes F_{\nabla_i}^* - \beta_- \otimes \nabla_{i-1,i}(\phi_i)^* + \beta_+ \otimes \nabla_{i+1,i}(\phi_{i+1}^*) \right)^* \right.$$

$$+ (\beta_- \wedge \beta_+) \otimes (\phi_{i+1}^* \circ \phi_{i+1} - q^2 \phi_i \circ \phi_i^* - q^{m-2i+1} [m - 2i] \text{id})^* \right.$$

$$\wedge \left( (\beta_- \wedge \beta_+) \otimes (\ast F_{\nabla_j}) - \beta_+ \otimes (\ast \nabla_{j-1,j}(\phi_j)) - \beta_- \otimes (\ast \nabla_{j+1,j}(\phi_{j+1}^*)) \right)$$

$$+ \text{id} \otimes (\phi_{j+1}^* \circ \phi_{j+1} - q^2 \phi_j \circ \phi_j^* - q^{m-2j+1} [m - 2j] \text{id}) \frac{\omega^d}{dt} \right).$$

(4.8)

Now use orthogonality of the splitting $\Sigma = \bigoplus_i \Sigma_i$, together with the fact that only the top two-form $\beta_- \wedge \beta_+ \in \Omega^2(\mathbb{C}P^1_q)$ has a non-zero integral over $\mathbb{C}P^1_q$ in (1.8). Using the identities (2.3) once again and the normalization $H(1) = 1$ for the Haar state on $\mathcal{A}(\mathbb{C}P^1_q)$, one finds that (1.8) coincides with the Yang–Mills–Higgs action functional (4.4). By construction, the functional $\text{YMH}_{q,m}$ is invariant under the action of the gauge group $\mathbb{C}(\Sigma)$ of Proposition 3.20 and hence descends to a well-defined functional on the orbit space $\mathcal{C}(\Sigma)/\mathbb{C}(\Sigma) \to [0, \infty)$.

We can rewrite the Yang–Mills–Higgs functional (4.4) in a more suggestive way. For this, consider the direct sum of $\mathcal{A}(M)$-modules with induced connection $(\Sigma, \nabla)$,

$$\Sigma = \bigoplus_{i=0}^{m} \Sigma_i, \quad \nabla = \bigoplus_{i=0}^{m} \nabla_i.$$

The induced hermitian structure $h = \bigoplus_i h_i$ on $\Sigma$ defines an inner product given by $(\varphi, \psi)_h = \sum_i (\varphi_i, \psi_i)_h$ for $\varphi, \psi \in \Sigma$, with corresponding norm $\|\varphi\|_h = (\varphi, \varphi)_h^{1/2}$ and extension to $\text{Hom}_{\mathcal{A}(M)}(\Sigma, \Omega^p(\Sigma))$ as described in (4.4). The Higgs fields $\phi_i \in \text{Hom}_{\mathcal{A}(M)}(\Sigma_{i-1}, \Sigma_i)$, with $i = 1, \ldots, m$, induce an element

$$\phi = \bigoplus_{i=1}^{m} \phi_i$$
of the quiver representation module \( \mathcal{R}(\mathcal{E}) \subset \text{End}_{A(M)}(\mathcal{E}) \) given by
\[
\mathcal{R}(\mathcal{E}) = \bigoplus_{i=1}^{m} \text{Hom}_{A(M)}(\mathcal{E}_{i-1}, \mathcal{E}_{i}) ,
\]
with induced connection \( \nabla = \bigoplus_{i} \nabla_{i,i-1} \). The induced hermitian structure \( h = \bigoplus_{i} h_{i-1,i} \) defines an inner product given by \( (\phi, \phi')_h = \sum_{i} (\phi_i, \phi'_i)_{h_{i-1,i}} \), for \( \phi, \phi' \in \mathcal{R}(\mathcal{E}) \), with associated \( L^2 \)-norm \( \| \phi \|_h = (\phi, \phi')_h^{1/2} \). Given \( \phi, \phi' \in \mathcal{R}(\mathcal{E}) \), we introduce the endomorphisms \( \phi \circ \phi'^* = \bigoplus_{i} (\phi \circ \phi'^*)_i \) and \( \phi^* \circ \phi' = \bigoplus_{i} (\phi^* \circ \phi')_i \), in \( \bigoplus_{i} \text{End}_{A(M)}(\mathcal{E}_i) \subset \text{End}_{A(M)}(\mathcal{E}) \) by
\[
(\phi \circ \phi'^*)_i = \phi_i \circ \phi'^*_i \quad \text{and} \quad (\phi^* \circ \phi')_i = \phi^*_i \circ \phi'_i .
\]
Using these maps we define the \( q \)-commutator \( [\phi^*, \phi']_q \in \text{End}_{A(M)}(\mathcal{E}) \) by
\[
[\phi^*, \phi']_q = \phi^* \circ \phi' - q^2 \phi \circ \phi^* .
\]
Finally, define an endomorphism \( \Sigma_{q,m} \) of \( \mathcal{E} \) by
\[
\Sigma_{q,m} = \bigoplus_{i=0}^{m} q^{m-2i+1} [m-2i] \text{id}_{\mathcal{E}_i} .
\]
Then the Yang–Mills–Higgs action functional \( (4.3) \) can be succinctly rewritten in compact form as a functional on \( \text{YM}_{q,m} : \mathcal{C}(\mathcal{E}) \times \mathcal{R}(\mathcal{E}) \to [0, \infty) \) given by
\[
\text{YM}_{q,m}(\nabla, \phi) = \text{YM}(\nabla) + (q^2 + 1) \| \nabla(\phi) \|_h^2 + \| [\phi^*, \phi]_q - \Sigma_{q,m} \|_h^2 .
\]

4.3. Holomorphic chain \( q \)-vortex equations.

Given a unitary connection \( \nabla \in \mathcal{C}(\mathcal{E}) \), there is a well-defined map \( [\nabla, -] \) from the space \( \text{Hom}_{A_{(\mathcal{M})}}(\mathcal{E}, \Omega^*(\mathcal{E})) \) to itself, where \( \Omega^*(\mathcal{E}) = \bigoplus_{p \geq 0} \Omega^p(\mathcal{E}) \). On homogeneous morphisms \( T \in \text{Hom}_{A_{(\mathcal{M})}}(\mathcal{E}, \Omega^p(\mathcal{E})) \) it is defined by
\[
[\nabla, T] := \nabla \circ T - (-1)^p T \circ \nabla .
\]
For the curvature \( F_{\nabla} = \nabla^2 \in \text{Hom}_{A_{(\mathcal{M})}}(\mathcal{E}, \Omega^2(\mathcal{E})) \), one then has the Bianchi identity
\[
[\nabla, F_{\nabla}] = 0 .
\]
As the space of connections \( \mathcal{C}(\mathcal{E}) \) is an affine space modeled on \( \text{Hom}_{A_{(\mathcal{M})}}(\mathcal{E}, \Omega^1(\mathcal{E})) \), as usual, for the critical points of the Yang–Mills action functional \( (4.2) \) one needs to compute it on a one-parameter family \( \nabla + t \eta \), and equate to zero the linear term in \( t \) for the corresponding expansion. By using properties of the complex inner product \( (-, -)_h \) on \( \text{Hom}_{A_{(\mathcal{M})}}(\mathcal{E}, \Omega^*(\mathcal{E})) \), this variational problem for the Yang–Mills action functional \( (4.2) \) shows that its critical points \( \nabla \) obey the Euler–Lagrange equation
\[
[\nabla^*, F_{\nabla}] = 0 ,
\]
where the adjoint operator of \( [\nabla, -] \) is defined with respect to the inner product as
\[
([\nabla^*, T], T')_h = (T, [\nabla, T'])_h
\]
for \( T, T' \in \text{Hom}_{A_{(\mathcal{M})}}(\mathcal{E}, \Omega^*(\mathcal{E})) \). Using the definition of the inner product given in \( (4.1) \) one easily shows that \( [\nabla^*, T] = -\frac{1}{2} [\nabla, \nabla T] \).

The purpose of this section is to characterize stable critical points of the Yang–Mills action functional \( (4.2) \) on \( \mathcal{M} \), and to study their dimensional reduction to configurations
on $M$. For this, following [34] §1.2 we introduce the notion of generalized instanton on
the quantum space $\underline{M}$.

Lemma 4.11. Let $\nabla \in \mathcal{C}(\underline{E})$ be a unitary connection and $\Xi \in \Omega^{2d-2}(\underline{M})$ a closed form
of degree $2d - 2$, regarded as the element $\text{id}_{\underline{E}} \otimes A(\underline{M}) \Xi$, such that

$$ (4.12) \quad \ast F_{\nabla} = - F_{\nabla} \wedge \Xi. $$

Then $\nabla$ solves the Yang–Mills equation (4.9) and $\text{YM}(\underline{\nabla}) = \text{Top}_2(\underline{E}, \Xi)$, where

$$ (4.13) \quad \text{Top}_2(\underline{E}, \Xi) = - (F_{\underline{\nabla}}, \ast (F_{\underline{\nabla}} \wedge \Xi)). $$

Proof. Using (4.12) and the graded right Leibniz rule we compute

$$ [\nabla^*, F_{\nabla}] = - \ast [\nabla, \ast F_{\nabla}] = \ast (\ast [\nabla, F_{\nabla} \wedge \Xi] + F_{\nabla} \wedge d \Xi) = 0, $$

where in the last line we used the Bianchi identity and $d \Xi = 0$. The second statement (4.13) follows from direct substitution of the equation (4.12) into the Yang–Mills action functional $\text{YM}(\underline{\nabla}) = (F_{\underline{\nabla}}, F_{\underline{\nabla}})_h$. □

Using $d \Xi = 0$, the definition of the inner product $(-, -)_h$, and the Bianchi identity $[\nabla, F_{\nabla}] = 0$, one shows in the standard way that the functional (4.13) does not depend on
the choice of connection $\nabla$ on $\underline{E}$. It thus defines a ‘topological action’ which depends only
on the $\mathcal{A}(\underline{M})$-module $\underline{E}$ and the closed form $\Xi$, and hence provides an a priori lower bound on the Yang–Mills action functional. We refer to the gauge invariant equation (4.12) as the $\Xi$-anti-selfduality equation. Gauge equivalence classes in $\mathcal{C}(\underline{E})/\mathcal{U}(\underline{E})$ of solutions to this first order equation are called generalized instantons or $\Xi$-instantons.

In the sequel we will use the natural closed $(1,1)$-form

$$ \omega = (\beta_- \wedge \beta_+) \otimes 1 + 1 \otimes \omega $$
on $\mathcal{A}(\underline{M})$, and set

$$ (4.14) \quad \Xi = \frac{\omega^{d-1}}{(d-1)!} = 1 \otimes \frac{\omega^{d-1}}{(d-1)!} + (\beta_- \wedge \beta_+) \otimes \frac{\omega^{d-2}}{(d-2)!}, $$

where the second term is absent when $d = 1$. We write $\text{Top}_2(\underline{E}, \omega)$ for the corresponding topological action functional (4.13). For simplicity, we also assume that $\text{tr}(F_{\underline{\nabla}}) = 0$ for any connection $\underline{\nabla} \in \mathcal{C}(\underline{E})$ without loss of generality, for otherwise one can consider $\tilde{F}_{\underline{\nabla}} = F_{\underline{\nabla}} - \frac{1}{r} \text{tr}(F_{\underline{\nabla}}) p$ where $r$ is the rank of the projection $p$ defining the bimodule $\underline{E}$.

We recall that on the algebra $\mathcal{A}(\underline{M}) = \mathcal{A}(\mathbb{C}P^1_q) \otimes \mathcal{A}(M)$ there is a natural complex structure which combines the complex structure of $M$ with the complex structure for the differential calculus on $\mathcal{A}(\mathbb{C}P^1_q)$ given in Proposition 2.11. Using it we write

$$ \Omega^p(\underline{E}) = \bigoplus_{i+j=p} \Omega^{i,j}(\underline{E}) $$
for the corresponding splitting of the space of $\underline{E}$-valued $p$-forms into $(i,j)$-forms. The $*$-involution maps $\Omega^{i,j}(\underline{E})$ to $\Omega^{j,i}(\underline{E})$, while the Hodge operator $\ast$ maps $\Omega^{i,j}(\underline{E})$ to
Consider then the linear operator

\[ L_\omega : \Omega_{i,j}(E) \rightarrow \Omega_{i+1,j+1}(E), \quad L_\omega(\alpha) := \alpha \wedge \omega. \]

Let \( L^*_\omega : \Omega_{i,j}(E) \rightarrow \Omega_{i-1,j-1}(E) \) be its adjoint with respect to the \( L^2 \)-inner product defined in \( \text{(4.11)} \). If \( \alpha \in \Omega^{1,1}(E) \), then

\[ \alpha^\omega := L^*_\omega(\alpha) = (\alpha^* \wedge * \omega) \]

is the component of \( \alpha \) (in \( \Omega^{0,0}(E) = E \)) along the closed \((1,1)\)-form \( \omega \). As in \( \text{(4.11)} \) this definition is extended to the modules \( \text{End}_A(M)(E) \) over \( A(M) \) and the corresponding modules over \( A(M) \) in the obvious ways.

For a connection \( \nabla \in \mathcal{C}(E) \), we decompose the curvature

\[ F_\nabla = F_{\nabla}^{2,0} + F_{\nabla}^{1,1} + F_{\nabla}^{0,2} \]

into its \((2,0)\), \((1,1)\), and \((0,2)\) components. Since \( \nabla \) is unitary, one has \( F_{\nabla}^{2,0} = -(F_{\nabla}^{0,2})^* \) and \( F_{\nabla}^{1,1} = -(F_{\nabla}^{1,1})^* \). We recall that the connection \( \nabla \) is called integrable if \( F_{\nabla}^{0,2} = 0 \).

**Lemma 4.16.** A connection \( \nabla \) solves the generalized instanton equation \( \text{(4.12)} \) if and only if it is an integrable unitary connection in \( \mathcal{C}(E)^{1,1} \) which satisfies the hermitian Yang–Mills equation

\[ F_{\nabla}^\omega = 0. \]

**Proof.** The Hodge operator \( * \) acts with a grading \((-1)^j\) on \( \Omega^{i,j}(E) \). Substituting the decomposition \( \text{(4.15)} \) into \( \text{(4.12)} \) with the choice \( \text{(4.14)} \) we thus find

\[ F_{\nabla}^{0,2} = 0, \]

whence \( \nabla \in \mathcal{C}(E)^{1,1} \). For the remaining \((1,1)\)-component, we note first the identity

\[ F_{\nabla} \wedge \frac{\omega^d}{d!} = F_{\nabla} \wedge * \omega = -F_{\nabla}^\omega \frac{\omega^{d+1}}{(d+1)!}. \]

But by the \( \Xi \)-anti-selfduality equation \( \text{(4.12)} \), the left-hand side is also equal to

\[ F_{\nabla} \wedge \frac{\omega^d}{d!} = -* F_{\nabla} \wedge \omega = +F_{\nabla}^\omega \frac{\omega^{d+1}}{(d+1)!}, \]

and comparing the two expressions gives \( \text{(4.17)} \). \( \square \)

We next describe the \( SU_q(2) \)-equivariant dimensional reduction of the above generalized instanton equations.

**Proposition 4.18.** Under the bijection of Proposition \( \text{[3.24]} \), the subspace of \( SU_q(2) \) invariant connections \( \nabla^\pi \in (\mathcal{C}(E)^{1,1})^{SU_q(2)} \) solving the generalized instanton equation on the quantum space \( M \) corresponds bijectively to the subspace of \( \mathcal{C}(E)^{1,1} \) consisting of elements \( (\nabla^\pi, \phi^*) \) which satisfy the holomorphic chain \( q \)-vortex equations on the manifold \( M \) given by

\[ F_{\nabla}^{\omega_i} = q^2 \phi_i \circ \phi^*_i - \phi^*_{i+1} \circ \phi_{i+1} + q^{m-2i+1} [m-2i] \text{id}_{E_i} \]

for each \( i = 0, 1, \ldots, m \). Here \( F_{\nabla}^{\omega_i} = *((F_{\nabla^{\omega_i}}^{1,1})^* \wedge * \omega) \) is the component (in \( \text{End}_A(M)(E_i) \)) of the curvature of \( \nabla_i \) along the Kähler form of \( M \).
Proof. By Proposition 3.24, the SU$_q$(2)-invariant subspace of generalized instanton connections on $\mathcal{M}$ described by Lemma 4.16 consists of connections and Higgs fields on $\mathcal{M}$ satisfying

$$F_{\nabla_i}^0 = 0 \quad \text{and} \quad \nabla_{\nabla_{i+1}}^\mathbf{\bar{\tau}} (\phi_i^*) = 0$$

for $i = 0, 1, \ldots, m$. Whence the $(1, 1)$-component of the curvature (4.7) for a connection $\nabla_{\nabla} \in \mathcal{O}(\mathcal{E})^{1,1}$ is given by

$$F_{\omega_{\nabla}} = \sum_{i=0}^{m} \left( \text{id} \otimes F_{\nabla_i}^1 + (\beta_- \wedge \beta_+) \otimes (\phi_i^* \circ \phi_{i+1} - q^2 \phi_i \circ \phi_i^* - q^{m-2i+1}[m - 2i] \text{id} ) \right).$$

We now use (2.13) and multiply this form with $\star \omega = 1 \otimes \omega \text{d} \text{d}! + (\beta_- \wedge \beta_+) \otimes (\star \omega)$ to get

$$F_{\omega_{\nabla}} = \sum_{i=0}^{m} \text{id} \otimes (F_{\nabla_i}^1 \circ \phi_i^* \circ \phi_{i+1} - q^2 \phi_i \circ \phi_i^* - q^{m-2i+1}[m - 2i] \text{id} ).$$

Using orthogonality of the direct sum $\mathcal{E} = \bigoplus_i \mathcal{E}_i$, the hermitian Yang–Mills equation (4.17) then corresponds to the set of equations (4.19).

The equations (4.19) are naturally invariant under the action of the group $\mathcal{U}(\mathcal{E})$. Under the bijection of Proposition 3.21, an SU$_q$(2)-invariant generalized instanton reduces to a gauge equivalence class of solutions to (4.19) in $\mathcal{C}(\mathcal{E})^1$/$\mathcal{U}(\mathcal{E})$. We call such a class a $q$-vortex on the manifold $\mathcal{M}$.

This equation illustrates the crucial feature of our deformation of standard quiver vortex equations, in that the commutator of Higgs fields is replaced with the $q$-commutator. We shall find various interesting consequences of this deformation below.

4.4. Vacuum structure.

In order to establish a correspondence between the generalized instanton equations on $\mathcal{M}$ and the non-abelian $q$-vortex equations on $\mathcal{M}$, we will now show directly how the latter equations describe stable critical points of the Yang–Mills–Higgs functional (4.4). For this, we will assume in the following that, for each $i = 1, \ldots, m$, the hermitian endomorphism $F_{\nabla_i}^\mathbf{\bar{\tau}}$ of $\mathcal{E}_i$ is non-negative, i.e. it defines a non-negative sesquilinear form on $\text{Hom}_{\mathcal{A}(\mathcal{M})}(\mathcal{E}_{i-1}, \mathcal{E}_i)$ given by $(F_{\nabla_i}^\mathbf{\bar{\tau}} \circ \phi_i, \phi'_i)_{\text{h}_{i-1,i}}$, for $\phi_i, \phi'_i \in \text{Hom}_{\mathcal{A}(\mathcal{M})}(\mathcal{E}_{i-1}, \mathcal{E}_i)$. Summing these forms we then get a non-negative sesquilinear form on $\mathcal{R}(\mathcal{E})$ defined by

$$(\phi, \phi')_{\mathcal{R}(\mathcal{E})} := \sum_{i=1}^{m} (F_{\nabla_i}^\mathbf{\bar{\tau}} \circ \phi_i, \phi'_i)_{\text{h}_{i-1,i}},$$

for $\phi, \phi' \in \mathcal{R}(\mathcal{E})$, with corresponding norm $\|\phi\|_{\mathcal{R}(\mathcal{E})}^2 := (\phi, \phi)_{\mathcal{R}(\mathcal{E})} \geq 0$ for each $\phi \in \mathcal{R}(\mathcal{E})$. 

Theorem 4.20. The restriction of the Yang–Mills–Higgs functional $\text{YMH}_{q,m}$ to elements $(\nabla^\partial, \phi^*) \in \mathcal{C}(\mathcal{E})^{1,1}$, for which $F^\omega_{i,i} \in \text{End}_{A(M)}(\mathcal{E}_i)$ is non-negative for $i = 1, \ldots, m$, is given by

$$\text{YMH}_{q,m} \left( \nabla^\partial, \phi^* \right) = \sum_{i=0}^{m} \left( 2q^{m-2i+1} [m-2i] \text{Top}_1(\mathcal{E}_i, \omega) - \text{Top}_2(\mathcal{E}_i, \omega) \right) - 2(1-q^2) \left\| \phi \right\|_{\mathcal{A}(\mathcal{E})}^2$$

(4.21)

where

$$\text{Top}_1(\mathcal{E}_i, \omega) := (F^\omega_{i,i}, \text{id}_{\mathcal{E}_i})_{h_i} \quad \text{and} \quad \text{Top}_2(\mathcal{E}_i, \omega) := - \left( F^\omega_{i,i} \ast (F^\omega_{i,i} \wedge \omega^{d-2} \left( \frac{d-2}{2} \right) )_{h_i} \right)_{h_i}.$$

Proof. We will apply a Bogomol’nyi-type transformation to the action functional (4.4). Firstly, we have

$$\left\| F^\omega_{i,i} \right\|_{h_i}^2 + (q^2 + 1) \left\| \nabla_{i-1,i}(\phi_i) \right\|_{h_{i-1,i}}^2$$

(4.22)

$$= 4 \left\| F^\theta_{i,i} \right\|_{h_i}^2 + \left\| F^\omega_{i,i} \right\|_{h_i}^2 + 2(q^2 + 1) \left\| \nabla_{i-1,i}^\partial (\phi_i) \right\|_{h_{i-1,i}}^2 - \text{Top}_2(\mathcal{E}_i, \omega)$$

for each $i = 0, 1, \ldots, m$ (see e.g. [3 §4]). For $(\nabla^\partial, \phi^*) \in \mathcal{C}(\mathcal{E})^{1,1}$, this is equal to

$$\left\| F^\omega_{i,i} \right\|_{h_i}^2 - \text{Top}_2(\mathcal{E}_i, \omega).$$

We now combine $\left\| F^\omega_{i,i} \right\|_{h_i}^2$ with the last set of norms in (4.4). For this, we expand out the last set of inner products in (4.21) to get

$$\left\| F^\omega_{i,i} \right\|_{h_i}^2 + \phi_{i+1}^* \circ \phi_{i+1} - q^2 \phi_i \circ \phi_i^* - q^{m-2i+1} [m-2i] \text{id} \right\|_{h_i}^2$$

$$= \left\| F^\omega_{i,i} \right\|_{h_i}^2 + \phi_{i+1}^* \circ \phi_{i+1} - q^2 \phi_i \circ \phi_i^* - q^{m-2i+1} [m-2i] \text{id} \right\|_{h_i}^2$$

$$+ 2 (F^\omega_{i,i}, \phi_{i+1}^* \circ \phi_{i+1} - q^2 \phi_i \circ \phi_i^* - q^{m-2i+1} [m-2i] \text{id} \right)_{h_i}.$$ 

The last term in the inner product here gives $-2q^{m-2i+1} [m-2i] \text{Top}_1(\mathcal{E}_i, \omega)$. The first two terms in this inner product can be evaluated by using (1.5) and the graded commutator to deduce that the curvature of the induced connection $\nabla_{i-1,i}$ on $\text{Hom}_{A(M)}(\mathcal{E}_{i-1}, \mathcal{E}_i)$ is

$$F^\omega_{i-1,i}(\phi_i) := \nabla_{i-1,i} \circ \nabla_{i-1,i}(\phi_i) = \phi_i \circ F^\omega_{i-1} - F^\omega_{i,i} \circ \phi_i.$$

Since $F^\theta_{i,i} = 0$ for each $i = 0, 1, \ldots, m$, we can use standard Kähler identities [3 eq. (4.10)] for the induced holomorphic structures on the $A(M)$-bimodules $\text{Hom}_{A(M)}(\mathcal{E}_{i-1}, \mathcal{E}_i)$ to get

$$\left( F^\omega_{i-1,i}(\phi_i), \phi_i \right)_{h_{i-1,i}} = \left\| \nabla_{i-1,i}^\partial (\phi_i) \right\|_{h_{i-1,i}}^2 - \left\| \nabla_{i-1,i}^\partial (\phi_i) \right\|_{h_{i-1,i}}^2,$$
which vanishes for \((\nabla^\theta, \phi^*) \in \mathcal{C}(\mathcal{E})^{1,1}\) by Proposition 3.24. It follows that

\[
\sum_{i=0}^{m} (F^\omega_{\nabla_i}, \phi_{i+1}^* \circ \phi_{i+1} - q^2 \phi_i \circ \phi_i^*)_{h_i} = \sum_{i=1}^{m} (\phi_i \circ F^\omega_{\nabla_{i-1}} - q^2 F^\omega_{\nabla_i}, \phi_i)_{h_{i-1,i}}
\]

\[
= \sum_{i=1}^{m} \left[ (F^\omega_{\nabla_{i-1}}, (\phi_i), (\phi_i))_{h_{i-1,i}} + (1 - q^2) (F^\omega_{\nabla_i}, \phi_i, \phi_i)_{h_{i-1,i}} \right]
\]

\[
= (1 - q^2) \|\phi\|^2_{\mathcal{A}(\mathcal{E})}.
\]

Putting everything together yields (4.21). \(\square\)

**Corollary 4.23.** The minima of the Yang–Mills–Higgs action functional \(\text{YMH}_{q,m}\) on \(\mathcal{C}(\mathcal{E})\), having values in \([0, \infty)\), are given by elements \((\nabla^\theta, \phi^*) \in \mathcal{C}(\mathcal{E})^{1,1}\) which satisfy the holomorphic chain \(q\)-vortex equations (4.19), and for which the curvature projections \(F^\omega_{\nabla_i}\) in \(\text{End}_{A(M)}(\mathcal{E}_i)\) for each \(i = 1, \ldots, m\) are non-negative with

\[
\|\phi\|^2_{\mathcal{A}(\mathcal{E})} \leq \frac{1}{2(1 - q^2)} \sum_{i=0}^{m} (2q^{m-2i+1} [m - 2i] \text{Top}_1(\mathcal{E}_i, \omega) - \text{Top}_2(\mathcal{E}_i, \omega)) \cdot
\]

When equality holds in (4.24), the minima achieve the infimum \(\text{YMH}_{q,m}(\nabla^\theta, \phi^*) = 0\).

**Proof.** From (4.22) it follows that the action functional (4.4) is minimized by taking \((\nabla^\theta, \phi^*) \in \mathcal{C}(\mathcal{E})^{1,1}\). From (4.21) one then gets that there is a Bogomol'nyi-type inequality

\[
\text{YMH}_{q,m}(\nabla, \phi) \geq \sum_{i=0}^{m} (2q^{m-2i+1} [m - 2i] \text{Top}_1(\mathcal{E}_i, \omega) - \text{Top}_2(\mathcal{E}_i, \omega)) - 2(1 - q^2) \|\phi\|^2_{\mathcal{A}(\mathcal{E})},
\]

with \(\text{Top}_1(\mathcal{E}_i, \omega)\) and \(\text{Top}_2(\mathcal{E}_i, \omega)\) not dependent on the choice of connection \(\nabla_i\) on the \(A(M)\)-module \(\mathcal{E}_i\). This inequality is saturated by solutions to the holomorphic \(q\)-vortex equations, with the bound (4.22) since \(\text{YMH}_{q,m}(\nabla, \phi) \geq 0\) and \(0 < q < 1\). \(\square\)

**Corollary 4.25.** A stable \(q\)-quiver bundle on \(M\) has characteristic classes constrained by the Bogomolov–Gieseker-type inequality

\[
2 \sum_{i=0}^{m} q^{m-2i} [m - 2i] \text{Top}_1(\mathcal{E}_i, \omega) \geq \sum_{i=0}^{m} \text{Top}_2(\mathcal{E}_i, \omega).
\]

If equality holds, then the connections \(\nabla_i\) are flat and

\[
\phi_{i+1}^* \circ \phi_{i+1} - q^2 \phi_i \circ \phi_i^* = q^{m-2i+1} [m - 2i] \text{id}_{\mathcal{E}_i}
\]

for each \(i = 0, 1, \ldots, m\).

**Proof.** The first statement follows from the inequality (4.22) together with \(\|\phi\|^2_{\mathcal{A}(\mathcal{E})} \geq 0\) and \(0 < q < 1\). For the second statement, we use (4.21) to get

\[
\text{YMH}_{q,m}(\nabla, \phi) = -2(1 - q^2) \|\phi\|^2_{\mathcal{A}(\mathcal{E})} \leq 0.
\]

But from its definition (1.4) the Yang–Mills–Higgs functional is a sum of non-negative terms, whence \(\text{YMH}_{q,m}(\nabla, \phi) = 0\) and thus \(F^\omega_{\nabla_i} = 0\) for each \(i = 0, 1, \ldots, m\). \(\square\)

Let us demonstrate explicitly that the vacuum moduli space of the Yang–Mills–Higgs functional is generically non-empty.
Proposition 4.27. Suppose that the finitely-generated projective $\mathcal{A}(M)$-bimodules $\mathcal{E}_i$ have the same rank for all $i = 0, 1, \ldots, m$. Then a class of explicit solutions to the vacuum equations (4.26) is given by

$$\phi_i = \phi_i^0 := \sqrt{\lambda_i} u_i, \quad i = 1, \ldots, m,$$

where $u_i \in \text{Hom}_{\mathcal{A}(M)}(\mathcal{E}_{i-1}, \mathcal{E}_i)$ are arbitrary holomorphic morphisms unitary with respect to the hermitian structures $h_{i-1}$ on $\mathcal{E}_{i-1}$ and $h_i$ on $\mathcal{E}_i$, and the induced connection $\nabla_{\partial_{i-1},i}$, and $\lambda_i$ are the $q$-numbers

$$\lambda_i = \frac{q^{m-2i+3}}{(1-q^2)(1-q^6)} \left( [m - 2i + 2] - q^4 [m - 2i] - q^{4i} ([m + 2] - q^4 [m]) \right).$$

Proof. Substituting (4.28) into (4.26) gives the linear recursion relation

$$\lambda_{i+1} - q^2 \lambda_i = q^{m-2i+1} [m - 2i].$$

With $\lambda_0 = 0 = \lambda_{m+1}$, the solution of (4.30) is given by

$$\lambda_i = \sum_{k=0}^{i-1} q^{m-2(i-2k)+3} [m - 2(i - k - 1)].$$

We now use the definition of the $q$-integers $[m - 2(i - k - 1)]$ given in (0.1) and perform the sums over $k$ using

$$\sum_{k=0}^{i-1} x^k = \frac{1 - x^i}{1 - x},$$

with $x = q^6$ and $x = q^2$. This gives

$$\lambda_i = \frac{q^{m-2i+3}}{(q - q^{-1})(1-q^6)(1-q^2)} \left( (1 - q^{6i}) (q^{m-2i+2} - q^{m-2i+4}) - (1 - q^{2i}) (q^{-m+2i-2} - q^{-m+2i+4}) \right),$$

which is easily manipulated into the form (4.29). \hfill \Box

4.5. Stability conditions.

We can also derive topological obstructions to the existence of solutions to the $q$-vortex equations (4.19). For this, we suppose that $M$ is compact, and define the degree of a hermitian finitely-generated projective $\mathcal{A}(M)$-module $(\mathcal{E}, h)$ by

$$\deg(\mathcal{E}) = \frac{\text{Top}_1(\mathcal{E}, \omega)}{\text{vol}_\omega(M)},$$

where $\text{vol}_\omega(M) = \int_M \frac{\omega^m}{m!}$ is the Kähler volume of $M$. The degree depends on the cohomology class of $\omega$. The slope of $\mathcal{E}$ is the number

$$\mu(\mathcal{E}) = \frac{\deg(\mathcal{E})}{\text{rank}(\mathcal{E})},$$

with the rank, $\text{rank}(\mathcal{E})$, defined as the trace of the identity endomorphism acting on $\mathcal{E}$. 
Proposition 4.32. A stable \( q \)-quiver bundle on \( M \) has slopes constrained by the inequalities:

(a) \( \mu(\mathcal{E}_0) \leq q^{m+1} \lfloor m \rfloor \), with equality if and only if \( \mathcal{E}_0 \) admits a holomorphic connection \( \nabla_0 \) solving the hermitian Yang–Mills equation \( F_{\nabla_0}^g = q^{m+1} \lfloor m \rfloor \) \( \text{id}_{\mathcal{E}_0} \).

(b) \( \mu(\mathcal{E}_m) \geq -q^{-m+1} \lfloor m \rfloor \), with equality if and only if \( \mathcal{E}_m \) admits a holomorphic connection \( \nabla_m \) solving the hermitian Yang–Mills equation \( F_{\nabla_m}^g = -q^{-m+1} \lfloor m \rfloor \) \( \text{id}_{\mathcal{E}_m} \).

(c) \( \mu_{q,m}(\mathcal{E}_i) \leq 0 \), with equality if and only if \( \mathcal{E}_i \) admits a holomorphic connection \( \nabla_i \) solving the hermitian Yang–Mills equation \( F_{\nabla_i}^g = q^{m-2i+1} \lfloor m - 2i \rfloor \) \( \text{id}_{\mathcal{E}_i} \) for each \( i = 0, 1, \ldots, m \).

Proof. Point (a) follows from the \( q \)-vortex equation (4.19) for \( i = 0 \) after taking inner products on both sides with \( \text{id}_{\mathcal{E}_0} \), and using \( (\text{id}_{\mathcal{E}_0}, \text{id}_{\mathcal{E}_0}) = \text{rank}(\mathcal{E}_0) \text{vol}_g(M) \) together with \( \phi_0 := 0 \) and \( \|\phi_i\|_{H_0,1}^2 \geq 0 \). Point (b) follows similarly from (4.19) with \( i = m \) and \( \phi_{m+1} := 0 \). For point (c), we take inner products on both sides of (4.19) with \( \text{id}_{\mathcal{E}_i} \) and sum over \( i = 0, 1, \ldots, m \) to get the constraint

\[
\sum_{i=0}^m \deg(\mathcal{E}_i) = \sum_{i=0}^m q^{m-2i+1} \lfloor m - 2i \rfloor \text{rank}(\mathcal{E}_i) + \frac{q^2 - 1}{\text{vol}_g(M)} \sum_{i=1}^m \|\phi_i\|_{H^{-1},i}^2 ,
\]

and the result follows since \( 0 < q < 1 \) and \( \|\phi_i\|_{H^{-1},i}^2 \geq 0 \) (with \( \|\phi_i\|_{H^{-1},i}^2 = 0 \) if and only if \( \phi_i = 0 \)).

5. Examples

In this final section we will briefly consider some explicit examples of the \( q \)-vortex equations (4.19). In particular, we will describe how the \( q \)-deformations affect stability.
conditions for the existence of solutions to these equations and the structure of the corresponding moduli spaces. These considerations provide the first step to formulating a general algebro-geometric notion of stability for $SU_q(2)$-equivariant modules over $\mathcal{A}(M)$ and the corresponding $q$-quiver bundles over $M$.

5.1. Deformations of holomorphic triples and stable pairs.

A holomorphic triple $(E_0, E_1, \phi)$ on a compact Kähler manifold $(M, \omega)$ consists of a pair of holomorphic vector bundles $E_0, E_1$ over $M$ and a holomorphic morphism $E_0 \xrightarrow{\phi} E_1$ which together obey coupled vortex equations [13, 8]. For $m = 1$, our $q$-vortex equations (4.19) provide a $q$-deformation of such triples. In this case the equations (4.19) read

$$\begin{align*}
F_\omega \nabla_0 &= q^2 \left( \text{id}_{E_0} - q^{-2} \phi \circ \phi^* \right), \\
F_\omega \nabla_1 &= -\left( \text{id}_{E_1} - q^2 \phi^* \circ \phi \right),
\end{align*}$$

(5.1)

where $\phi := \phi_1$. The topological stability conditions on these triples are governed by Proposition 4.32 for $m = 1$. Additionally, by taking inner products on both sides of the equations (5.1) with $\text{id}_{E_0}$ and $\text{id}_{E_1}$ respectively, and summing shows that the degrees of the bundles are related by

$$\deg(E_0) + q^{-2} \deg(E_1) = q^2 \text{rank}(E_0) - q^{-2} \text{rank}(E_1).$$

Substituting this into the formula for the $(q, 1)$-degree of the quiver bundle then yields

$$\deg_{q, 1}(E) = \left( 1 - q^{-2} \right) \left( \deg(E_1) + \text{rank}(E_1) \right).$$

These criteria are much more stringent than the stability condition of [8].

Let us now denote $E := E_0, \nabla := \nabla_0$, and take $E_1$ to be the $\mathcal{A}(M)$-module of sections of the trivial holomorphic line bundle over $M$, i.e. $E_1 = \mathbb{C} \otimes \mathcal{A}(M) \simeq \mathcal{A}(M)$. Then $F_{E_1} = 0$, and $\text{Hom}_{\mathcal{A}(M)}(E_0, E_1) \simeq E_0$ so that the Higgs field $\phi$ can be regarded as an element of $\mathcal{E}$, i.e. as a holomorphic section. We set $\phi := q^{-1} \varphi$ with $\varphi \in \mathcal{E}$, so that $\varphi^* \otimes_{\mathcal{A}(M)} \varphi = \text{id}_{\mathcal{A}(M)}$ by the second equation of (5.1). The first equation of (5.1) can then written as

$$F_\varphi = q^{-2} \varphi \otimes_{\mathcal{A}(M)} \varphi^* = q^2 \text{id}_E.$$

Thus in this case the triple describes a $q$-deformation of the stable pairs $(E, \varphi)$ considered in [7]. The stability condition reads

$$\deg(E) = q^2 \text{rank}(E) - q^{-2}$$

which is much more restrictive than the undeformed one of [7, 13].

5.2. $q$-vortices on Riemann surfaces.

Let $M$ be a compact oriented Riemann surface of genus $g$. Then the equations (5.1) describe $q$-deformations of non-abelian vortices on $M$. For $g \neq 1$, the area of $M$ is

$$\text{vol}_\omega(M) = \frac{8\pi}{\kappa} (1 - g)$$

by the Gauss–Bonnet theorem, where $\kappa$ is the scalar curvature of $M$ with respect to the Kähler metric corresponding to $\omega$. Let us again consider a particular case. If $E := E_0, \nabla := \nabla_0$ and $E_1 \simeq \mathbb{C}^r \otimes \mathcal{A}(M)$ with $r = \text{rank}(E)$, then the Higgs field $\phi = q^{-1} \varphi$ can
be regarded as an element of $\mathbb{C}^r \otimes \mathcal{E}$. The characteristic class \( \frac{1}{2} \) \( \text{Top}_1(M, \omega) \) is the first Chern class \( c_1(\mathcal{E}) \) of the bundle \( \mathcal{E} \). A non-empty moduli space of solutions to the \( q \)-vortex equations, formally the same as in \([5.2]\), is ensured in this case by the stability condition

\[
c_1(\mathcal{E}) = \frac{4r}{\kappa} (q^2 - q^{-2})(1 - g)
\]

for \( g \neq 1 \). Since \( 0 < q < 1 \), this degree satisfies the bound \( c_1(\mathcal{E}) < \frac{4\kappa^2}{\kappa} (1 - g) \). Hence the pair \((\mathcal{E}, \varphi)\) is \( \tau \)-stable in the sense of \([2]\), and by the Hitchin–Kobayashi correspondence it is gauge equivalent to a solution of the non-abelian \( q \)-vortex equations. The corresponding moduli space of solutions is described explicitly in \([6]\). For abelian vortices, \( r = 1 \), this moduli space coincides with the \( |n| \)-th symmetric product orbifold of \( M \), i.e. the space of effective divisors on \( M \) of degree \( n = c_1(\mathcal{E}) \).

By taking \( \mathcal{E}_1 \) to be a (generically non-trivial) holomorphic line bundle, one also obtains from \((5.1)\) a \( q \)-deformation of the non-abelian vortex equations studied in \([30]\). However, contrary to the \( q = 1 \) case, wherein the reduction \( r = 1 \), \( \mathcal{E}_0 \simeq \mathcal{E}_1 \) and \( \nabla_0 = -\nabla_1 \) would lead to the standard abelian BPS vortex equations on \( M \), such an abelian reduction in \((5.1)\) is not consistent for \( q \neq 1 \). Indeed, as we first witnessed in point (c) of Proposition \( 4.32 \), the \( q \)-deformation generically imposes very stringent constraints on the allowed stable quiver bundles. Moreover, the \( q \)-vortices do not exist on the complex plane \( M = \mathbb{C} \), wherein the formal limit \( \text{vol}_\omega(M) = \infty \) would necessitate infinite vortex number and action. The features spelled out in this section are generic properties of the \( q \)-vortex equations \((5.1)\).

### 5.3. \( q \)-instantons.

Let \((M, \omega)\) be a Kähler surface. Set \( \mathcal{E}_0 \simeq \mathcal{E}_1 =: \mathcal{E} \), with \( r = \text{rank}(\mathcal{E}) \), and \( \phi = \text{id}_\mathcal{E} \). Then since \( \phi \) is a holomorphic section, \( \nabla_{0,1}(\phi) = 0 \), from \((4.5)\) we have \( \nabla_0 = \nabla_1 =: \nabla \) and both equations in \((5.1)\) simplify to

\[
F_\nabla^\mathcal{E} = (q^2 - 1) \text{id}_\mathcal{E}.
\]

For bundles \( \mathcal{E} \) of vanishing degree and \( q^2 \neq 1 \), this equation gives a deformation of the hermitian Yang–Mills equation on \( M \), and hence of the standard anti-selfduality equations

\[
*F_\nabla = -F_\nabla.
\]

Gauge equivalence classes of solutions to \((5.3)\) are thus called \( q \)-instantons, and their moduli spaces can be described explicitly in the following way.

The fixed points on the space \( \mathcal{C}(\mathcal{E}) \) of unitary connections on \( \mathcal{E} \) under the action of the group of gauge transformations \( \mathcal{U}(\mathcal{E}) \) are given by integrable connections \( \nabla \in \mathcal{C}(\mathcal{E})^{1,1} \). The natural \( \mathcal{U}(\mathcal{E}) \)-invariant symplectic form \( \omega_\mathcal{C} \) on \( \mathcal{C}(\mathcal{E}) \) is thus given by

\[
\omega_\mathcal{C}(\alpha, \alpha') = \frac{1}{2} \int_M \text{tr} \left( \alpha^* \wedge \alpha' \right)^\omega \omega^2
\]

for \( \alpha, \alpha' \in \text{Hom}^*_{\mathcal{A}(M)}(\mathcal{E}, \Omega^1(\mathcal{E})) \). We implicitly use the inclusion of the Lie algebra of \( \mathcal{U}(\mathcal{E}) \) in its dual space by means of the hermitian structure \( h \) on \( \mathcal{E} \). Then the corresponding moment map \( \mu_\mathcal{C} : \mathcal{C}(\mathcal{E}) \to (\text{Lie } \mathcal{U}(\mathcal{E}))^* \) is given by

\[
\mu_\mathcal{C}(\nabla) = F_\nabla^\mathcal{E}.
\]

The moduli space of \( q \)-instantons on \( M \) is thus realized as the symplectic quotient

\[
\mu_\mathcal{C}^{-1}((q^2 - 1) \text{id}_\mathcal{E}) / \mathcal{U}(\mathcal{E}),
\]
and hence the \( q \)-vortices in this case correspond to points of \( \mu^{-1}_C((q^2 - 1) \text{id}_\mathcal{E}) \) which lie inside the Kähler submanifold \( \mathcal{C}(\mathcal{E})^{1,1} \) (outside the singularities).

When \( M = \mathbb{C}^2 \), the constant shift in the moment map condition from \( \mu_C = 0 \) to \( \mu_C = (q^2 - 1) \text{id}_\mathcal{E} \) induces a shift in the corresponding real ADHM equation. The effect of this shift is to augment \([25]\) the moduli space of holomorphic instanton bundles to the moduli space of torsion free sheaves on the projective plane \( \mathbb{C}P^2 \) with a trivialization on a fixed projective line \( \mathbb{C}P^1 \subset \mathbb{C}P^2 \). This resolves the small instanton singularities and turns the instanton moduli space into a hyper-Kähler manifold of complex dimension \( 4r_k \), where \( k = \frac{1}{2\pi} \text{Top}_q(M, \omega) = c_2(\mathcal{E}) \). It is well-known \([26]\) that this modification arises explicitly in the equations which determine instantons on a certain noncommutative deformation of \( \mathbb{R}^4 \). Here we have shown that the same sort of resolution of instanton moduli space is achieved via our \( q \)-deformed dimensional reduction procedure over the quantum projective line \( \mathbb{C}P^1_q \). The essential feature behind such resolutions, provided here by the deformation in \((5.3)\), lies in the content of point (c) of Proposition \([4,32]\).

**Final remarks**

In this paper we have shown that the formalism of \( \text{SU}(2) \)-equivariant dimensional reduction over the sphere has a natural \( \text{SU}_q(2) \) Hopf algebraic generalization to reductions over the quantum sphere. This was achieved by recasting the standard dimensional reduction procedure into a purely algebraic framework and using the fact that much of the geometry of the projective line survives \( q \)-deformation to the quantum projective line (the quantum sphere with additional structure). We obtained a \( q \)-deformed Yang–Mills–Higgs theory from the reduction of Yang–Mills theory, and also \( q \)-deformations of quiver chain vortex equations from the reduction of natural first order gauge theory equations. We demonstrated that the moduli spaces of solutions to these \( q \)-vortex equations are more constrained but generically better behaved than their \( q \to 1 \) limits. In some instances, the vacuum moduli space can be described as a symplectic or even hyper-Kähler quotient. It would be interesting to explore whether the generic \( q \)-vortex equations admit such a moment map interpretation, as they do in the \( q = 1 \) case from the action of a unitary group on a representation space of quiver modules (see \([3, \S 2.2]\)). This presumably involves interpreting the \( q \)-commutator terms as moment map equations for a sort of quantum group action on the space of quiver gauge connections. This may also help fill an important gap in our construction, namely the proper formulation of stability conditions and the ensuing Hitchin–Kobayashi-type correspondence which relates the existence of solutions to the gauge equations with a stability criterion. This problem appears to lie in the general realm of extending noncommutative geometry into the algebro-geometric setting, which is not yet fully developed. It would be interesting to see if the \( q \)-deformations of the Yang–Mills–Higgs models derived in this paper improve the phenomenological viability of the models constructed in \([12]\). The somewhat intricate \( q \)-dependence of the vacuum Higgs field configurations described by Proposition \([1,27]\) may drastically alter the dynamical mass generation in these models. It would also be interesting to extend our constructions to Hopf algebraic equivariant dimensional reduction over other quantum homogeneous spaces.
References

[1] L. Álvarez-Cónsul, O. García-Prada, Dimensional reduction, SL(2, C)-equivariant bundles and stable holomorphic chains, Int. J. Math. 12 (2001) 159–201.
[2] L. Álvarez-Cónsul, O. García-Prada, Dimensional reduction and quiver bundles, J. Reine Angew. Math. 556 (2003) 1–46.
[3] L. Álvarez-Cónsul, O. García-Prada, Hitchin–Kobayashi correspondence, quivers and vortices, Comm. Math. Phys. 238 (2003) 1–33.
[4] P. Aschieri, T. Grammatikopoulos, H. Steinacker, G. Zoupanos, Dynamical generation of fuzzy extra dimensions, dimensional reduction and symmetry breaking, J. High Energy Phys. 0609 (2006) 026.
[5] P. Aschieri, J. Madore, P. Manousselis, G. Zoupanos, Dimensional reduction over fuzzy coset spaces, J. High Energy Phys. 0404 (2004) 034.
[6] J.M. Baptista, Non-abelian vortices on compact Riemann surfaces, Comm. Math. Phys. 291 (2009) 799–812.
[7] S.B. Bradlow, Special metrics and stability for holomorphic bundles with global sections, J. Diff. Geom. 33 (1991) 169–213.
[8] S.B. Bradlow, O. García-Prada, Stable triples, equivariant bundles and dimensional reduction, Math. Ann. 304 (1996) 225–252.
[9] S.B. Bradlow, G. Daskalopoulos, O. García-Prada, R. Wentworth, Stable augmented bundles over Riemann surfaces, London Math. Soc. Lect. Notes Ser. 208 (1995) 15–67.
[10] T. Brzeziński, S. Majid, Quantum group gauge theory on quantum spaces, Comm. Math. Phys. 157 (1993) 591–638 [Erratum ibid. 167 (1995) 235].
[11] T. Brzeziński, S. Majid, Quantum differential and the q-monopole revisited, Acta Appl. Math. 54 (1998) 185–233.
[12] B.P. Dolan, R.J. Szabo, Dimensional reduction, monopoles and dynamical symmetry breaking, J. High Energy Phys. 0903 (2009) 059.
[13] O. García-Prada, Dimensional reduction of stable bundles, vortices and stable pairs, Int. J. Math. 5 (1994) 1–52.
[14] P.M. Hajac, Bundles over quantum sphere and noncommutative index theorem, K-Theory 21 (2000) 141–150.
[15] P.M. Hajac, S. Majid, Projective module description of the q-monopole, Commun. Math. Phys. 206 (1999) 247–264.
[16] D. Harland, S. Kürkçüoğlu, Equivariant reduction of Yang–Mills theory over the fuzzy sphere and the emergent vortices, Nucl. Phys. B 821 (2009) 380–398.
[17] M. Khalkhali, G. Landi, W.D. van Suijlekom, Holomorphic structures on the quantum projective line, Int. Math. Res. Not. 4 (2010) 851–884.
[18] A. Klinyk and K. Schmüdgen, Quantum Groups and their Representations (Springer, 1997).
[19] G. Landi, C. Reina, A. Zampini, Gauged laplacians on quantum Hopf bundles, Comm. Math. Phys. 287 (2009) 179–209.
[20] O. Lechtenfeld, A.D. Popov, R.J. Szabo, Rank two quiver gauge theory, graded connections and noncommutative vortices, J. High Energy Phys. 0609 (2006) 054.
[21] O. Lechtenfeld, A.D. Popov, R.J. Szabo, Quiver gauge theory and noncommutative vortices, Prog. Theor. Phys. Suppl. 171 (2007) 258–268.
[22] S. Majid, Noncommutative riemannian and spin geometry of the standard q-sphere, Comm. Math. Phys. 256 (2005) 255–285.
[23] T. Masuda, Y. Nakagami, J. Watanabe, Noncommutative differential geometry on the quantum two-sphere of P. Podleś. I: An algebraic viewpoint, K-Theory 5 (1991) 151–175.
[24] T. Masuda, K. Mimachi, Y. Nakagami, M. Noumi, K. Ueno, Representations of the quantum group SU_q(2) and the little q-Jacobi polynomials, J. Funct. Anal. 99 (1991) 357–387.
[25] H. Nakajima, Heisenberg algebra and Hilbert schemes of points on projective surfaces, Ann. Math. 145 (1997) 379–388.
[26] N.A. Nekrasov, A.S. Schwarz, *Instantons on noncommutative $\mathbb{R}^4$ and (2,0) superconformal six-dimensional theory*, Comm. Math. Phys. 198 (1998) 689–703.

[27] S. Neshveyev, L. Tuset, *A local index formula for the quantum sphere*, Comm. Math. Phys. 254 (2005) 323–341.

[28] P. Podleś, *Quantum spheres*, Lett. Math. Phys. 14 (1987) 193–202.

[29] P. Podleś, *Differential calculus on quantum spheres*, Lett. Math. Phys. 18 (1989) 107–119.

[30] A.D. Popov, *Non-abelian vortices on Riemann surfaces: An integrable case*, Lett. Math. Phys. 84 (2008) 139–148.

[31] A.D. Popov, R.J. Szabo, *Quiver gauge theory of non-abelian vortices and noncommutative instantons in higher dimensions*, J. Math. Phys. 47 (2006) 012306.

[32] K. Schmüdgen, E. Wagner, *Dirac operator and a twisted cyclic cocycle on the standard Podleś quantum sphere*, J. Reine Angew. Math. 574 (2004) 219–235.

[33] K. Schmüdgen, E. Wagner, *Representations of cross product algebras of Podleś quantum spheres*, J. Lie Theory 17 (2007) 751–790.

[34] G. Tian, *Gauge theory and calibrated geometry. I*, Ann. Math. 151 (2000) 193–268.

[35] E. Wagner, *On the noncommutative spin geometry of the standard Podleś sphere and index computations*, J. Geom. Phys. 59 (2009) 998–1016.

[36] S.L. Woronowicz, *Twisted SU_q(2) group. An example of a noncommutative differential calculus*, Publ. Res. Inst. Math. Sci., Kyoto Univ. 23 (1987) 117–181.

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