Efficient calculation of the mutual inductance of arbitrarily oriented circular filaments via a generalisation of the Kalantarov-Zeitlin method

Kirill V. Poletkin\textsuperscript{a,}\textsuperscript{*}, Jan G. Korvink\textsuperscript{a}

\textsuperscript{a}The Institute of Microstructure Technology, Karlsruhe Institute of Technology, Hermann-von-Helmholtz-Platz 1, 76344 Eggenstein-Leopoldshafen, Germany

Abstract

In this article, we present a new analytical formulation for calculation of the mutual inductance between two circular filaments arbitrarily oriented with respect to each other, as an alternative to Grover\textsuperscript{[1]} and Babič\textsuperscript{[2]} expressions reported in 1944 and 2010, respectively. The formula is derived via a generalisation of the Kalantarov-Zeitlin method, which showed that the calculation of mutual inductance between a circular primary filament and any other secondary filament having an arbitrary shape and any desired position with respect to the primary filament is reduced to a line integral. In particular, the obtained formula provides a solution for the singularity issue arising in the Grover and Babič formulas for the case when the planes of the primary and secondary circular filaments are mutually perpendicular. The efficiency and flexibility of the Kalantarov-Zeitlin method allow us to extend immediately the application of the obtained result to a case of the calculation of the mutual inductance between a primary circular filament and its projection on a tilted plane. Newly developed formulas have been successfully validated through a number of examples available in the literature, and by a direct comparison with the results of calculation performed by the FastHenry software.

Keywords: Inductance, circular filaments, coils, line integral, electromagnetic

*Corresponding author

Email address: kirill.poletkin@kit.edu (Kirill V. Poletkin)
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1. Introduction

Analytical and semi-analytical methods in the calculation of inductances, and in particular the mutual inductances of filament wires and their loops, play an important role in power transfer, wireless communication, and sensing and actuation, and is applied in different fields of science, including electrical and electronic engineering, medicine, physics, nuclear magnetic resonance, mechatronics and robotics, to name the most prominent. Collections of formulas for the calculation of mutual inductance between filaments of different geometrical shapes covering a wide spectrum of practical arrangements have variously been presented in classical handbooks by Rosa [3], Grover [4], Dwight [5], Snow [6], Zeitlin [7], Kalantarov [8], among others.

The availability of efficient numerical methods such as FastHenry [9] (based on the multipole expansion) currently provides an accurate and fast solution for the calculation of mutual and self-inductance for any circumstance, including the use of arbitrary materials, conductor cross-sections, loop shapes, and arrangements. However, analytical methods allow to obtain the result in the form of a final formula with a finite number of input parameters, which when applicable may significantly reduce computation effort. It will also facilitate mathematical analysis, for example when derivatives of the mutual inductance w.r.t. one or more parameters are required to evaluate electromagnetic forces via the stored magnetic energy, or when optimization is performed.

Analytical methods applied to the calculation of the mutual inductance between two circular filaments is a prime example, and has been successfully used in an increasing number of applications, including electromagnetic levitation [10], superconducting levitation [11, 12, 13], magnetic force interaction [14], wireless power transfer [15, 16, 17], electromagnetic actuation [18, 19, 20, 21], micro-machined contactless inductive suspensions [22, 23, 24, 25] and hybrid suspensions [26, 27, 28], biomedical applications [29, 30], topology optimiza-
tion [31], nuclear magnetic resonance [32, 33], indoor positioning systems [34], navigation sensors [35], and magneto-inductive wireless communications [36].

The original formula of the mutual inductance between two coaxial circular filaments was derived by Maxwell [37, page 340, Art. 701] and expressed in terms of elliptic integrals. Butterworth obtained a formula covering the case of circular filaments with parallel axes [38]. Then, a general formal expression made for cases where the axes of the circles are parallel, and where their axes intersect, was derived by Snow [39]. However, the Butterworth and Snow expressions suffer from a low rate of convergence. This issue was recognized and solved by Grover, who developed the most general method in the form of a single integral [1]. Using the vector potential method, as opposed to the Grover means, the general case for calculating the mutual inductance between inclined circular filaments arbitrarily positioned with respect to each other was subsequently obtained by Babić et al. [2].

Kalantarov and Zeitlin showed that the calculation of mutual inductance between a circular primary filament and any other secondary filament having an arbitrary shape and any desired position with respect to the primary filament can be reduced to a line integral [8, Sec. 1-12, page 49]. In the present paper, we report an adaptation of this method to the case of two circular filaments and then derive a new analytical formula for calculating the mutual inductance between two circular filaments having any desired position with respect to each other as an alternative to the Grover and Babić expressions.

In particular, the obtained formula provides a solution for the singularity issue arising in the Grover and Babić formulas for the case when the planes of the primary and secondary circular filaments are mutually perpendicular. The efficiency and flexibility of the Kalantarov-Zeitlin method allow us to extend immediately the application of the obtained result to a case of the calculation of the mutual inductance between a primary circular filament and its projection on a tilted plane. For instance, this particular case appears in micro-machined inductive suspensions and has a direct practical application in studying their stability [25] and pull-in dynamics [27, 28]. The new analytical formulae were
verified by comparison with series of reference examples covering all cases given by Grover\cite{4}, Kalantarov and Zeitlin \cite{8}, and using direct numerical calculations performed by the Babić Matlab function \cite{2} and the FastHenry software \cite{9}.

2. Preliminary discussion

Two circular filaments having radii of $R_p$ and $R_s$ for the primary circular filament (the primary circle) and the secondary circular filament (the secondary circle), respectively are considered to be arbitrarily positioned in space, namely, they have a linear and angular misalignment, as is shown in Figure 1. Let us assign a coordinate frame (CF) denoted as $XYZ$ to the primary circle in such a way that the $Z$ axis is coincident with the circle axis and the $XOY$ plane of the CF lies on the circle’s plane, where the origin $O$ corresponds to the centre of primary circle. In turn, the $xyz$ CF is assigned to the secondary circle in a similar way so that its origin $B$ is coincident with the centre of the secondary circle.

The linear position of the secondary circle with respect to the primary one is defined by the coordinates of the centre $B (x_B, y_B, z_B)$. The angular position of the secondary circle can be defined in two ways. Firstly, the angular position is defined by the angle $\theta$ and $\eta$ corresponding to the angular rotation around an axis passing through the diameter of the secondary circle, and then the rotation
Figure 2: Two manners for determining the angular position of the secondary circle with respect to the primary one: \( x'y'z' \) is the auxiliary CF the axes of which are parallel to the axes of \( XYZ \), respectively; \( x''y''z'' \) is the auxiliary CF defined in such a way that the \( x' \) and \( x'' \) are coincide, but the \( z'' \) and \( y'' \) axis is rotated by the \( \alpha \) angle with respect to the \( z' \) and \( y' \) axis, respectively.

of this axis lying on the surface \( x'y'y' \) around the vertical \( z' \) axis, respectively, as it is shown in Figure 2(a). These angles for determination of angular position of the secondary circle was proposed by Grover and used in his formula numbered by (179) in [4, page 207] addressing the general case for calculation of the mutual inductance between two circular filaments.

The same angular position can be determined through the \( \alpha \) and \( \beta \) angle, which corresponds to the angular rotation around the \( x' \) axis and then around the \( y'' \) axis, respectively, as it is shown in Figure 2(b). This additional second manner is more convenient in a case of study dynamics and stability issues, for instance, applying to axially symmetric inductive levitation systems [23, 25] in compared with the Grover manner. These two pairs of angles have the following relationship with respect to each other such as:

\[
\begin{align*}
\sin \beta &= \sin \eta \sin \theta; \\
\cos \beta \sin \alpha &= \cos \eta \sin \theta.
\end{align*}
\]

The details of the derivation of this set presented above are shown in Appendix A.
3. The Kalantarov-Zeitlin method

Using the general scheme for two circular filaments shown in Figure 1 as an illustrative one, the Kalantarov-Zeitlin method is presented. The method reduces the calculation of mutual inductance between a circular primary filament and any other secondary filament having an arbitrary shape and any desired position with respect to the primary circular filament to a line integral [8, Sec. 1-12, page 49].

Indeed, let us choose an arbitrary point $P$ of the secondary filament (as it has been mentioned above the filament can have any shape), as shown in Figure 1. An element of length $dl''$ of the secondary filament at the point $P$ is considered. Also, the point $P$ is connected the point $Q$ lying on the $Z$ axis by a line, which is perpendicular to the $Z$ axis and has a length of $\rho$, as shown in Figure 3. Then the element $dl''$ can be decomposed on $dz$ along the $Z$ axis and on $d\rho$ along the $\rho$ line and $d\lambda$ along the $\lambda$-circle having radius of $\rho$ (see, Figure 3). It is obvious that the mutual inductance between $dz$ and the primary circular filament is equal to zero because $dz$ is perpendicular to a plane of primary circle. But the mutual inductance between $d\rho$ and the primary circular filament is also equal to zero because of the symmetry of the primary circle relative to the $d\rho$ direction.

Thus, the mutual inductance $dM$ between element $dl''$ and the primary circle...
Figure 4: The Kalantarov-Zeitlin method: projection of the secondary filament on the \( \rho \)-plane passed through the point \( P \) and parallel to the plane of the primary circular filament; \( d\ell \) is the projection of the element \( d\ell'' \) on the \( \rho \)-plane.

is equal to the mutual inductance \( dM_\lambda \) between element \( d\lambda \) and the primary circle. Moreover, due to the fact that the primary and the \( \lambda \)-circle are coaxial and, consequently, symmetric then we can write:

\[
\frac{dM_\lambda}{M_\lambda} = \frac{d\lambda}{\lambda} = \frac{d\lambda}{2\pi \rho},
\]

where \( M_\lambda \) is the mutual inductance of the primary coil and \( \lambda \)-circle.

From Figure 4 it is directly seen that

\[
d\lambda = dy \cos \varphi - dx \sin \varphi = (\cos \zeta \cos \varphi - \cos \varepsilon \sin \varphi) d\ell,
\]

where \( \cos \varepsilon \) and \( \cos \zeta \) are the direction cosines of element \( d\ell \) relative to the \( X \) and \( Y \) axis, respectively. Hence, accounting for (2) and (3), we can write:

\[
dM = dM_\lambda = M_\lambda \frac{\cos \zeta \cos \varphi - \cos \varepsilon \sin \varphi}{2\pi \rho} d\ell,
\]

and as a result, a line integral for calculation mutual inductance between the primary circle and a filament is

\[
M = \frac{1}{2\pi} \int \frac{M_\lambda \cos \zeta \cos \varphi - \cos \varepsilon \sin \varphi}{\rho} d\ell,
\]

where \( M_\lambda \) is defined by the Maxwell formula for mutual inductance between two coaxial circles [37] page 340, Art. 701]. Note that during integrating, the \( Z \)
coordinate of the element $d\ell$ is also changing and this dependency is taken into account by the $M_\lambda$ function directly.

4. Derivation of Formulas

Due to the particular geometry of secondary filament under consideration, its projection on the $\rho$-plane (the $\rho$-plane is parallel to the primary circle plane and passed through the point $P$) is an ellipse, which can be defined in a polar coordinate by a function $r = r(\varphi)$ with the origin at the point $B$ as it is shown in Figure 5. Hence, the distance $\rho$ can be expressed in terms of the parameter $s$, which is fixed, and the distance $r$ from the origin $B$, which is varied with
the angular variable $\varphi$. Introducing the angle $\gamma$ as shown in Figure 6, for the distance $\rho$ the following equations can be written:

$$\rho \cos \gamma = r + s \cos(\xi - \varphi), \quad (6)$$
$$\rho \sin \gamma = s \sin(\xi - \varphi).$$

Due to (6), we have:

$$\rho^2 = r^2 + r \cdot s \cos(\xi - \varphi) + s^2, \quad (7)$$

where the function $r = r(\varphi)$ can be defined as [40]:

$$r = \frac{R_s \cos \theta}{\sqrt{\sin^2(\varphi - \eta) + \cos^2 \theta \cos^2(\varphi - \eta)}}. \quad (8)$$

The angle $\theta$ and $\eta$ defines the angular position of the secondary circle with respect to the primary one according to manner I considered in Sec. 2. Note that the function $r$ can be also defined via the angles $\alpha$ and $\beta$ of manner II also considered in Sec. 2 as it is shown in Appendix B. However, for the further derivation, the angular position of the secondary circle is defined through manner I, since it is convenient for the direct comparison with Grover’s and Babić results.

According to Figure 6, the relationship between the element $d\lambda$ of the $\lambda$-circle and an increment of the angle $\varphi$ is as follows:

$$d\lambda = r \cdot d\varphi \cos \gamma - dr \sin \gamma = \left( r \cos \gamma - \frac{dr}{d\varphi} \sin \gamma \right) d\varphi. \quad (9)$$

Then, accounting for (9), (7) and (6), line integral (5) can be replaced by a definite integral for the calculation of mutual inductance as follows:

$$M = \frac{1}{2\pi} \int_0^{2\pi} M_\lambda \frac{r^2 + r \cdot s \cos(\xi - \varphi) - \frac{dr}{d\varphi} s \sin(\xi - \varphi)}{\rho^2} d\varphi. \quad (10)$$

Now, let us introduce the following dimensionless parameters such as:

$$\bar{x}_B = \frac{x_B}{R_s}; \quad \bar{y}_B = \frac{y_B}{R_s}; \quad \bar{z}_B = \frac{z_B}{R_s}; \quad \bar{r} = \frac{r}{R_s};$$
$$\bar{\rho} = \frac{\rho}{R_s}; \quad \bar{s} = \sqrt{\bar{x}^2 + \bar{y}^2}. \quad (11)$$
Figure 7: The special case: the two filament circles are mutually perpendicular to each other.

The \( \varphi \)-derivative of \( \bar{r} \) is

\[
\frac{d\bar{r}}{d\varphi} = \frac{1}{2} \bar{r}^3 \tan^2 \theta \sin(2(\varphi - \eta)),
\]

(12)

The mutual inductance \( M_\lambda \) is

\[
M_\lambda = \mu_0 \frac{2}{k} \Psi(k) \sqrt{R_p R_s \bar{\rho}},
\]

(13)

where \( \mu_0 \) is the magnetic permeability of free space, and

\[
\Psi(k) = \left(1 - \frac{k^2}{2}\right) K(k) - E(k),
\]

(14)

where \( K(k) \) and \( E(k) \) are the complete elliptic functions of the first and second kind, respectively, and

\[
k^2 = \frac{4\nu \bar{\rho}}{(\nu \bar{\rho} + 1)^2 + \nu^2 \bar{z}_\lambda^2},
\]

(15)

where \( \nu = R_s/R_p \) and \( \bar{z}_\lambda = \bar{z}_B + \bar{r} \tan \theta \sin(\varphi - \eta) \). Accounting for dimensionless parameters \([11]\) and substituting \([12]\) and \([14]\) into integral \([10]\), the new formula to calculate the mutual inductance between two circular filaments having any desired position with respect to each other becomes

\[
M = \frac{\mu_0 \sqrt{R_p R_s}}{\pi} \int_0^{2\pi} \frac{\bar{r} + t_1 \cdot \cos \varphi + t_2 \cdot \sin \varphi}{k \bar{\rho}^{1.5}} \cdot \bar{r} \cdot \Psi(k) d\varphi,
\]

(16)
where terms $t_1$ and $t_2$ are defined as

\[
t_1 = \bar{x}_B + 0.5\bar{r}^2 \tan^2 \theta \sin(2(\varphi - \eta)) \cdot \bar{y}_B; \tag{17}
\]

\[
t_2 = \bar{y}_B - 0.5\bar{r}^2 \tan^2 \theta \sin(2(\varphi - \eta)) \cdot \bar{x}_B,
\]

and $\bar{\rho} = \sqrt{\bar{r}^2 + 2\bar{r} \cdot \bar{s} \cos(\xi - \varphi) + \bar{s}^2}$.

Formula (16) can be applied to any possible cases, but one is excluded when the two filament circles are mutually perpendicular to each other. In this case the projection of the secondary circle onto the $\rho$-plane becomes simply a line as it is shown in Fig. 7 and as a result to integrate with respect to $\varphi$ is no longer possible.

For the treatment of this case, the Kalantarov-Zeitlin formula (5) is directly used. Let us introduce the dimensionless variable $\bar{\ell} = \ell/R_s$ and then the integration of (5) is preformed with respect to this dimensionless variable $\bar{\ell}$ within interval $-1 \leq \bar{\ell} \leq 1$. The direction cosines $\cos \zeta$ and $\cos \varepsilon$ become as $\sin \eta$ and $\cos \eta$, respectively (see, Fig. 7). Accounting for

\[
\rho \cos \varphi = s \cos \xi + \ell \cos \eta,
\]

\[
\rho \sin \varphi = s \sin \xi + \ell \sin \eta, \tag{18}
\]

and the Maxwell formula (14) and (15), where the $Z$-coordinate of the element $d\bar{\ell}$ is defined as

\[
\bar{z}_\lambda = \bar{z}_B \pm \sqrt{1 - \bar{\ell}^2}, \tag{19}
\]

then the formula to calculate the mutual inductance between two filament circles, which are mutually perpendicular to each other, becomes as follows:

\[
M = \frac{\mu_0 \sqrt{R_p R_s}}{\pi} \left[ \int_{-1}^{1} \frac{t_1 - t_2}{k\bar{\rho}^{1.5}} \cdot \Psi(k) d\bar{\ell} \right.
\]

\[
+ \left. \int_{-1}^{1} \frac{t_1 - t_2}{k\bar{\rho}^{1.5}} \cdot \Psi(k) d\bar{\ell} \right], \tag{20}
\]

where terms $t_1$ and $t_2$ are defined as

\[
t_1 = \sin \eta(\bar{x}_B + \bar{\ell} \cos \eta);
\]

\[
t_2 = \cos \eta(\bar{y}_B + \bar{\ell} \sin \eta), \tag{21}
\]
and \( \bar{\rho} = \sqrt{s^2 + 2\ell \cdot \bar{s} \cos(\xi - \eta) + \ell^2} \). Note that integrating equation (20) between \(-1\) and \(1\) is calculated with the positive sign and for the other direction the negative sign is taken.

In order to demonstrate the efficiency and flexibility of the Kalantarov-Zeitlin method, a formula for the calculation of the mutual inductance between the primary circular filament and its projection on a tilted plane is obtained as follows. In this case, the function of \( r = r(\varphi) \) is constant and defined through the radius of primary coil as \( r = R_p \). Since the centre of the projection is coincide with the Z-axis, thus \( s = 0 \). Then, the formula is derived from (16) as its particular case (\( \bar{s} = 0 \) and \( \bar{r} = \bar{\rho} = 1 \)) and becomes, simply,

\[
M = \frac{\mu_0 R_p}{\pi} \int_0^{2\pi} \frac{1}{k} \Psi(k) d\varphi.
\]

The obtained formulas can be easily programmed, they are intuitively understandable for application. Also, the singularity arises in Grover’s and Babič’s formula for the calculate of the mutual inductance between two filament circles, which are mutually perpendicular to each other, is solved in developed formula (20). The Matlab files with the implemented formulas (16), (20) and (22) are available from the authors as supplementary materials to this article. Also, in Appendix B the developed formulas can be rewritten through the pair of the angle \( \alpha \) and \( \beta \).

5. Examples of Calculation. Numerical Verification

In this section, developed new formulas (16), (20) and (22) are verified by the examples taken from Grover [4] and Kalantarov [8] books and Babič article [2]. The special attention was addressed to the singularity case arisen when the two filament circles are mutually perpendicular to each other. Then, formula (22) was validated with the FastHenry software [9]. All calculations for considered cases proved the robustness and efficiency of developed formulas.

Note that the notation proposed in Grover’s and Kalantarov’s books in order to define the linear misalignment of the secondary coil is different from the
notation used in the Babič article and in our article as well. Also, the angular misalignment in the Babič formula must be defined through the parameters of the secondary coil plane. These particularities of the notation will be discussed specifically for each case. For all calculation, the primary coil is located on the plane \( XOY \) and its centre at the origin \( O(0,0,0) \).

5.1. Mutual inductance of coaxial circular filaments

Let us consider the circular filaments, which are coaxial and have a distance between their centres, as shown in Fig. 8. Then, this case in the notation proposed in this article is defined as \( R_p = A \), \( R_s = a \), the linear misalignment is \( z_B = d \), \( x_B = y_B = 0 \), the angular misalignment (manner I, Sec. 2) is \( \theta = 0 \) and \( \eta = 0 \). For the Babič formula the linear misalignment is defined in the same way, but for angular one the parameters of the secondary circle plane must be calculated and becomes \( a = 0 \), \( b = 0 \) and \( c = 1 \) (the Babič notation). These parameters have the following relationship with the angle \( \theta \) and \( \eta \): \( a = \sin \eta \sin \theta \); \( b = -\cos \eta \sin \theta \) and \( c = \cos \theta \) [2, Eq. (27), page 3597].
Example 1 (Example 24, page 78 in Grover’s book [4])

Let us suppose that two circles of radii \( a=20 \text{ cm} \) and \( A=25 \text{ cm} \) with their planes \( d=10 \text{ cm} \) apart are given. The results of calculation are

| Grover’s book | The Babič formula | This work, Eq. (16) |
|---------------|-------------------|-------------------|
| \( M, \text{nH} \) | 248.79            | 248.7874          | 248.7874          |

Example 2 (Example 25, page 78 in Grover’s book [4])

Two circles of radii \( a=2 \text{ in}=5.08 \text{ cm} \) and \( A=5 \text{ in}=12.7 \text{ cm} \) with their planes \( d=4 \text{ in}=10.16 \text{ cm} \) apart, the results become

| Grover’s book | The Babič formula | This work, Eq. (16) |
|---------------|-------------------|-------------------|
| \( M, \text{nH} \) | 18.38             | 18.3811           | 18.3811           |

Example 3 (Example 5-4, page 215 in Kalantarov’s book [8])

For two circles having the same radii of 10.00 cm with their planes \( d=4 \text{ cm} \) apart, the calculation shows the following

| Kalantarov’s book | The Babič formula | This work, Eq. (16) |
|-------------------|-------------------|-------------------|
| \( M, \text{nH} \) | 135.1             | 135.0739          | 135.0739          |

Example 4 (Example 5-5, page 215 in Kalantarov’s book [8])

Circles having the same radii as in Example 3, but their planes \( d=50 \text{ cm} \) apart are given. The results are

| Kalantarov’s book | The Babič formula | This work, Eq. (16) |
|-------------------|-------------------|-------------------|
| \( M, \text{nH} \) | 1.41              | 1.4106            | 1.4106            |
Example 5 (Example 5-6, page 224 in Kalantarov’s book \[8\])

Two circular coaxial filaments, radii of which are \( A = 25 \) cm and \( a = 20 \) cm, with their planes \( d = 8 \) cm apart are given. The results are as follows

|                  | Kalantarov’s book | The Babić formula | This work, Eq. \[16\] |
|------------------|-------------------|-------------------|------------------------|
| \( M, \text{nH} \) | 289.11            | 289.0404          | 289.0404               |

5.2. Mutual inductance of circular filaments with parallel axes

The scheme for calculation of the mutual inductance between circular filaments with parallel axes is shown in Fig. 9. The linear misalignment in the Grover notation can be defined by \( d \) is the distance between the planes of circles (the same parameter as in Sec. 5.1) and \( \rho \) is the distance between axes or via \( r \) is the distance between the centres and \( \varphi \) is the angle between the \( Z \)-axis and the radius vector \( r \). These parameters have the following relationship to the notation defined in this article, namely, \( z_B = d = r \cos \varphi \) and \( \rho = \sqrt{x_B^2 + y_B^2} = r \sin \varphi \). The angular misalignment is defined in the same way as described in Sec. 5.1.

Example 6 (Example 62, page 178 in Grover’s book \[4\])

Two circles of radii \( a = A = 15 \) cm have a distance between their centres \( r = 20 \) cm and an angle \( \varphi = \cos^{-1} 0.8 \) between the \( Z \)-axis and the radius vector.
r (please, see Fig. 9). Assuming that $y_B = \rho = r \sin \varphi = 12$ cm and $z_B = r \cos \varphi = 16$ cm, the results of calculation are as follows

\[
\begin{array}{|c|c|c|c|}
\hline
\text{Grover’s book} & \text{The Babić formula} & \text{This work, Eq. (16)} \\
\hline
M, \text{nH} & 45.31 & 45.3342 & 45.3342 \\
\hline
\end{array}
\]

**Example 7** (Example 63, page 178 in Grover’s book [4])

Two circles of the same diameter of $2a = 2A = 48$ in $= 121.92$ cm are arranged so that the distance between their planes $d = 15$ in $= 38.1$ cm and the distance between their axes is $\rho = 47.7$ in $= 121.158$ cm (please, see Fig. 9). Thus, we have $R_p = R_s = 60.96$ cm, $y_B = \rho$ and $z_B = d$, the results of calculation are as follows

\[
\begin{array}{|c|c|c|c|}
\hline
\text{Grover’s book} & \text{The Babić formula} & \text{This work, Eq. (16)} \\
\hline
M, \text{nH} & -24.56 & -24.5728 & -24.5728 \\
\hline
\end{array}
\]

**Example 8** (Example 65, page 183 in Grover’s book [4])

Two circles with radii of $A = 10$ cm and $a = 8$ cm have the distance between their centres $r = 50$ cm and an angle of $\cos \varphi = 0.4$ (please, see Fig. 9). Hence, we have $R_p = A$ and $R_s = a$, assuming that $y_B = r \sin \varphi = 45.83$ cm and $z_B = r \cos \varphi = 20.0$ cm, the results of calculation are as follows

\[
\begin{array}{|c|c|c|c|}
\hline
\text{Grover’s book} & \text{The Babić formula} & \text{This work, Eq. (16)} \\
\hline
M, \text{nH} & -0.2480 & -0.24828 & -0.24828 \\
\hline
\end{array}
\]

**Example 9** (Example 66, page 184 in Grover’s book [4])

Two circles with radii of $A = 10$ cm and $a = 8$ cm have the distance between their centres $r = 20$ cm and an angle of $\cos \varphi = 0.6$ (please, see Fig. 9), to find the mutual inductance between these circles. Hence, we have $R_p = A$ and
\( R_s = a \), assuming that \( y_B = r \sin \varphi = 16.0 \text{ cm} \) and \( z_B = r \cos \varphi = 12.0 \text{ cm} \), the results of calculation are as follows

|                      | Grover’s book | The Babić formula | This work, Eq. (16) |
|----------------------|---------------|--------------------|---------------------|
| \( M, \text{nH} \)   | 4.405         | 4.465              | 4.465               |

**Example 10 (Example 5-8, page 231 in Kalantarov’s book 8)**

Two circular filaments of the same radius of \( A = a = 5 \text{ cm} \) are arranged that the distance between their centres is \( r = 40 \text{ cm} \) and an angle of \( \cos \varphi = 0.4 \). Hence, we have \( R_p = R_s = 5 \text{ cm} \), assuming that \( y_B = r \sin \varphi = 36.66 \text{ cm} \) and \( z_B = r \cos \varphi = 16.0 \text{ cm} \), the results are as follows

|                      | Kalantarov’s book | The Babić formula | This work, Eq. (16) |
|----------------------|--------------------|--------------------|---------------------|
| \( M, \text{nH} \)   | -0.049             | -0.048963          | -0.048963           |

**Example 11 (Example 5-9, page 233 in Kalantarov’s book 8)**

Two circular filaments of radii of \( A = 10 \text{ cm} \) and \( a = 5 \text{ cm} \) are arranged that the distance between their centres is \( r = 20 \text{ cm} \) and an angle of \( \cos \varphi = 0.8 \). Hence, we have \( R_p = 10 \text{ cm} \) and \( R_s = 5 \text{ cm} \), assuming that \( y_B = r \sin \varphi = 36.66 \text{ cm} \) and \( z_B = r \cos \varphi = 16.0 \text{ cm} \), the results are as follows

|                      | Kalantarov’s book | The Babić formula | This work, Eq. (16) |
|----------------------|--------------------|--------------------|---------------------|
| \( M, \text{nH} \)   | 2.95               | 3.0672             | 3.0672              |

**Example 12 (Example 5-10, page 234 in Kalantarov’s book 8)**

Two circular filaments of radii of \( A = 20 \text{ cm} \) and \( a = 4 \text{ cm} \) are arranged that the distance between their centres is \( r = 2 \text{ cm} \) and an angle of \( \cos \varphi = 0.66 \). Hence, we have \( R_p = 20 \text{ cm} \) and \( R_s = 4 \text{ cm} \), assuming that \( y_B = r \sin \varphi = 1.5 \text{ cm} \)
Figure 10: Geometrical scheme of inclined circular filaments with intersect axes denoted via Grover’s notation: $x_1$ and $x_2$ are the distances from $S$, $\theta$ is the angle of inclination of the axes.

and $z_B = r \cos \varphi = 1.32$ cm, the results become as follows

|                  | Kalantarov’s book | The Babič formula | This work, Eq. [16] |
|------------------|-------------------|--------------------|---------------------|
| $M$, nH          | 15.99             | 15.9936            | 15.9936             |

5.3. Mutual inductance of inclined circular filaments with intersect axes

The general scheme of the arrangement of two inclined circular filaments whose axes intersect for calculation of mutual inductance is shown in Fig. 10. In Grover’s notation, we have $A$ and $a$ are radii of circular filaments and $S$ is the point of intersection of the circles axes. The centres of circles are at the distances $x_1$ and $x_2$ from $S$ of the primary and secondary circle, respectively. $\theta$ is the angle of inclination of the axes. Also, the following relationships are true, namely, $d = x_1 - x_2 \cos \theta$ and $\rho = x_2 \sin \theta$. Thus, the linear misalignment in the notation of this paper is defined again through $z_B = d$, $\rho = \sqrt{x_B^2 + y_B^2}$ and the angular misalignment is defined by the angle $\theta$, but $\eta$ is equal to zero.

For the Babič formula, the angular misalignment is defined by the parameters of the secondary circle plane as follows: $a = 0$; $b = -\sin \theta$ and $c = \cos \theta$.

From the general scheme shown in Fig. 10 two particular cases can be recognized. Namely, the first case is corresponded to concentric circles, when $x_1 = x_2 = 0$ and the second case is corresponded to circular filaments whose
axes intersect at the centre of one of the circle, when \( x_2 = 0 \). For the first case, to calculate the mutual inductance, the angle \( \theta \) and radii of circles must be known. For the second particular case, in addition to the distance \( d \) between the centre \( B \) of the secondary circle and the plane of the primary circle must be given.

**Example 13 (Example 5-7, page 227 in Kalantarov’s book [8])**

Two circular filaments of radii of \( A = 10 \text{ cm} \) and \( a = 2.5 \text{ cm} \) are concentric and an angle of inclination of the plane of the secondary circle is \( \theta = 60^\circ \). Hence, assuming that \( x_B = y_B = z_B = 0 \), the calculation of mutual inductance shows

| Kalantarov’s book | The Babič formula | This work, Eq. (16) |
|-------------------|-------------------|-------------------|
| \( M, \text{nH} \) | 6.044             | 6.0431            |
| 6.044             | 6.0431            | 6.0431            |

**Example 14 (Example 69, page 194 in Grover’s book [4])**

Two concentric circles with radii of \( A = 20 \text{ cm} \) and \( a = 14 \text{ cm} \) are arranged that an angle of inclination of the plane of the secondary circle is \( \cos \theta = 0.3 \). Hence, assuming that \( x_B = y_B = z_B = 0 \) and \( \theta = 72.5424^\circ \), the results of calculation are as follows

| Grover’s book | The Babič formula | This work, Eq. (16) |
|---------------|-------------------|-------------------|
| \( M, \text{nH} \) | 47.44             | 47.4431           |
| 47.44          | 47.4431           | 47.4431           |

**Example 15 (Example 70, page 194 in Grover’s book [4])**

Two circles with radii of \( A = 10 \text{ in}=25.4 \text{ cm} \) and \( a = 3 \text{ in}=7.62 \text{ cm} \) are arranged that an angle of inclination of the plane of the secondary circle is \( \cos \theta = 0.4 \) and a distance \( d = 3 \text{ in}=7.62 \text{ cm} \). Hence, assuming that \( x_B = y_B = 0 \),
\(z_B = d\) and \(\theta = 66.4218^\circ\), the results of calculation are as follows

| Grover’s book | The Babić formula | This work, Eq. (16) |
|---------------|--------------------|----------------------|
| \(M, \text{nH}\) | 15.543 | 15.5435 | 15.5435 |

*Example 16 (Example 71, page 201 in Grover’s book [4]*)

Two circles of radii \(A = 20.0\) cm and \(a = 10.0\) cm are considered with the centre of one on the axis of the other and a distance, \(d\), of 20.0 cm between the centres. The axes are to be inclined at an angle, \(\theta\), of 30°. The results of calculation show

| Grover’s book | The Babić formula | This work, Eq. (16) |
|---------------|--------------------|----------------------|
| \(M, \text{nH}\) | 29.436 | 29.4365 | 29.4365 |

*Example 17 (Example 5-11, page 235 in Kalantarov’s book [8]*)

Two circular filaments have radii of \(A = 10\) cm and \(a = 8\) cm. The axis the primary circle is crossed through the centre of the secondary circle at a distance \(d\) of 8 cm between their centres. The axes are to be inclined at an angle, cos \(\theta = 0.7\). Hence, assuming that \(x_B = y_B = 0, z_B = d\) and an angle of 45.5730°, the results of calculation are

| Kalantarov’s book | The Babić formula | This work, Eq. (16) |
|------------------|--------------------|----------------------|
| \(M, \text{nH}\) | 23.2 | 24.3794 | 24.3794 |

*Example 18 (Example 73, page 204 in Grover’s book [4]*)

Two circles of radii \(A = 16.0\) cm and \(a = 10.0\) cm are considered to be intersect the axes at point \(S\) in such a way that distances \(x_1\) and \(x_2\) are to be 20.0 cm and 5.0 cm, respectively. An angle of inclination between axes is \(\cos \theta = 0.5\). Hence,
assuming that \( x_B = 0, \ y_B = 4.3301 \text{ cm}, \ z_B = 17.5 \text{ cm} \) and an angle of 60.0°, we have

| \( M, \text{nH} \) |
|-----------------|
| Grover’s book   |
| The Babič formula |
| This work, Eq. (16) |
|-----------------|
| 13.612          |
| 13.6113         |
| 13.6113         |

Table 1: Calculation of mutual inductance for Example 19

| \( \eta \) | The Grover formula, Eq. (179) | The Babič formula, Eq. (24) | This work, Eq. (16) |
|-----------|-----------------|-----------------|-----------------|
| 0         | 13.6113         | 13.6113         | 13.6113         |
| \( \pi/6 \) | 14.4688         | 14.4688         | 14.4688         |
| \( \pi/4 \) | 15.4877         | 15.4877         | 15.4877         |
| \( \pi/4 \) | 16.8189         | 16.8189         | 16.819          |
| \( \pi/2 \) | 20.0534         | 20.0534         | 20.0534         |
| \( 2\pi/3 \) | 23.3252         | 23.3252         | 23.3252         |
| \( 3\pi/4 \) | 24.6936         | 24.6936         | 24.6936         |
| \( 5\pi/6 \) | 25.7493         | 25.7493         | 25.7493         |
| \( \pi \) | 26.6433         | 26.6433         | 26.6433         |
| \( 7\pi/6 \) | 25.7493         | 25.7493         | 25.7493         |
| \( 5\pi/4 \) | 24.6936         | 24.6936         | 24.6936         |
| \( 4\pi/3 \) | 23.3253         | 23.3253         | 23.3252         |
| \( 3\pi/2 \) | 20.0534         | 20.0534         | 20.0534         |
| \( 5\pi/3 \) | 16.8189         | 16.8189         | 16.819          |
| \( 7\pi/4 \) | 15.4877         | 15.4877         | 15.4877         |
| \( 11\pi/6 \) | 14.4688         | 14.4688         | 14.4688         |
| \( 2\pi \) | 13.6113         | 13.6113         | 13.6113         |
5.4. Mutual inductance of circular filaments arbitrarily positioned in the space

The validation of the developed formulas (16) and (20) for the general case, when the angular misalignment is defined through the angle $\theta$ and $\eta$ as shown in Fig. 2(a) in a range from 0 to $360^\circ$, the examples from the Babič article [2] were used. Also, we utilized the Matlab functions with the Grover formula [4, page 207, Eq. (179)] and the Babič formula [2, page 3593, Eq. (24)] implemented by F. Sirois and S. Babič.

**Example 19 (Example 12, page 3597 in the Babič article [2])**

Using the geometrical arrangement as in Example 18 (two circles with radii $A = 16.0$ cm and $a = 10.0$ cm and the centre of the secondary circle is located at $x_B = 0$, $y_B = 4.3301$ cm, $z_B = 17.5$ cm and the angle, $\theta$ of $60.0^\circ$), but the angle $\eta$ is varied in a range from 0 to $360^\circ$. The results of calculation are summed up in Table 1. Analysis of Table 1 shows that the developed formula (16) works identically to the Grover and Babič formula.

|                      | The Grover formula | The Babič formula | This work, Eq. (20) |
|----------------------|--------------------|-------------------|--------------------|
| $M$, nH              | -10.73             | -10.73            | -10.7272           |

**Example 20 (Example 11, page 3596 in the Babič article [2])**

Let us consider two circular filaments having radii of $R_p = 40$ cm and $R_s = 10$ cm, which are mutually perpendicular to each other that angles of $\eta = 0$ and $\theta = 90.0^\circ$. The centre of the secondary circle has the following coordinates: $x_B = 0$, $y_B = 20$ cm, and $z_B = 10$ cm. The problem illustrates the application of new formula (20). The results are

**Example 21**

Now we again apply formula (20) to the problem considered in Example 20, but in this case the centre of the secondary coil is located at origin, thus
Figure 11: Distribution of the error of the Babić formula in dependent on changing the \( \eta \)-angle within a range \( 0 < \eta \leq 360^\circ \) for Example 21 \((x_B = y_B = z_B = 0)\).

\[ x_B = y_B = z_B = 0. \] Hence, we have

| The Grover formula | The Babić formula | This work, Eq. (20) |
|--------------------|------------------|--------------------|
| \( M, \text{nH} \) | \( \text{NaN} \) | \( \text{NaN} \) |

Thus, the calculation shows that the Babić and Grover formula gives an indeterminate results, but developed formula (20) equals explicitly zero as expected for this case. Then, rotating the angle \( \eta \) in a range \( 0 < \eta \leq 360^\circ \), we reveal that the calculation of mutual inductance performed by developed formula (20) shows zero within this range of the \( \eta \)-angle, but the Babić formula demonstrates a small error, which is not exceeded \( M = 2.9205 \times 10^{-15} \text{nH} \) and distributed with the \( \eta \)-angle as shown in Figure 11.

**Example 22**

Let us consider mutually perpendicular circles (angles of \( \theta = 90.0^\circ \) and \( \eta = 0 \)) having the same radii as in Example 20, but the centre of the secondary coil occupies a position on the \( XOY \)-surface with the following coordinates \( x_B = \)

\[^1\text{Not-a-Number}\]
Figure 12: Chaotic distribution of the error of the Babič formula in dependent on changing the \( \eta \)-angle within a range \( 0 < \eta \leq 360^\circ \) for Example 22 (\( x_B = y_B = 10 \text{ cm} \) and \( z_B = 0 \)): the scaled-up images show the interruption of continuity of the curve at \( \eta = 90^\circ \) and \( 270^\circ \).

Figure 13: Geometrical scheme for calculation of mutual inductance between a primary circular filament and its projection. The angular misalignment is given by an angle of \( \theta \), while the linear misalignment by the coordinate \( z_B \).

\( y_B = 10 \text{ cm} \) and \( z_B = 0 \). Results of calculation are

|                      | The Grover formula | The Babič formula | This work, Eq. (20) |
|----------------------|--------------------|-------------------|--------------------|
| \( M, \text{nH} \)    | \( 4.013 \times 10^{-15} \) | \( 2.416 \times 10^{-15} \) | 0 |

Thus, the calculation expresses that the Babič and Grover formula gives the small errors, but developed formula (20) shows, explicitly, zero. Then, again let us rotate the angle \( \eta \) in a range \( 0 < \eta \leq 360^\circ \), the calculation of mutual
inductance performed by developed formula (20) reveals zero within this range of the η-angle, but the Babić formula demonstrates the chaotic distribution of the small error of calculation as shown in Fig. 12 which is in a range from $-0.7 \times 10^{-14}$ to $2.921 \times 10^{-14}$ nH and at the η-angle of $90^\circ$ and $270^\circ$ the calculation of mutual inductance is indeterminate (see, the scaled-up images of Fig. 12 which present the interruption of continuity at $\eta = 90^\circ$ and $270^\circ$).

5.5. Mutual inductance between a primary circular filament and its projection on a tilted plane

In this section new formula (22) for calculation of mutual inductance between a primary circular filament and its projection on a tilted plane is validated by comparison with the calculation performed via the FastHenry software [9]. The angular misalignment is given by the angle $\theta$ and $\eta$, while the linear misalignment is defined by the coordinate $z_B$ of the point $B$ crossing the tilted plane and the $Z$-axis. Fig. 13 shows a geometrical scheme for the calculation. The shown arrangement of the primary circle and its projection on the tilted plane corresponds to a particular case, when $\eta = 0$. Worth noting that the $\eta$-angle has no effect on the result of the calculation of the mutual inductance.

**Example 23**

Let us consider primary circle having a radius of 10.00 cm and a tilting plane crosses the $Z$-axis at the point $z_B=4$ cm. When a tilting angle of zero, then the geometry and arrangement corresponds to Example 3 (Example 5-4, page 215...
in Kalantarov’s book) for the case of two coaxial circles with the same radii. We calculate the mutual inductance for three values of a tilted angle at 0°, 10° and 15°. The results of calculation are shown in Table 2.

Although, there is the small deviation between results obtained with the FastHenry software and analytical formula (22), but this deviation is not significant and can be explained by the fact that the circles in the FastHenry software are divided into straight segments with a finite cross section in comparing with the analytical formula where the circles have no segments and a cross section. We are concluding the validity of developed formula (22).

| θ   | This work, Eq. (22) | FastHenry [9] |
|-----|--------------------|--------------|
| M, nH | M, nH              |              |
| 0°   | 1.4106             | 1.3761       |
| 5°   | 1.4117             | 1.3844       |
| 10°  | 1.4151             | 1.3933       |
| 15°  | 1.421              | 1.4031       |
| 20°  | 1.4298             | 1.4120       |
| 25°  | 1.4422             | 1.4202       |
| 30°  | 1.4594             | 1.4268       |
| 35°  | 1.4831             | 1.4336       |
| 40°  | 1.5161             | 1.4574       |
| 45°  | 1.5631             | 1.5230       |
| 50°  | 1.6329             | 1.6180       |
| 55°  | 1.7425             | 1.7273       |
| 60°  | 1.9299             | 1.8765       |
| 65°  | 2.2971             | 2.2416       |
| 70°  | 3.2127             | 3.1806       |
| 75°  | 7.1274             | 7.1679       |
Example 24

In this last example, we increase a distance between the centre of the primary circle with a radius of 10.00 cm and a tilting plane to \( z_B = 50 \text{ cm} \). Hence, a range of the tilted angle becomes larger then in example 23. Note that for zero tilting angle the geometry of the considered problem corresponds to example 4 (Example 5-5, page 215 in Kalantarov’s book). The results of calculation are shown in Table 3. Analysis of Table 3 shows a good agreement between the calculations, which confirms the validity of developed formula (22).

6. Conclusion

We derived and validated new formulas (16) and (20) for calculation of the mutual inductance between two circular filaments arbitrarily oriented with respect to each other. These analytic formulas have been developed based on the Kalantarov-Zeitlin method, which showed that the calculation of mutual inductance between a circular primary filament and any other secondary filament having an arbitrary shape and any desired position with respect to the primary filament is reduced to a line integral. In particular, the developed formula (20) provides a solution for the singularity issue arising in Grover’s and Babič’ formulas for the case when the planes of the primary and secondary circular filaments are mutually perpendicular.

Moreover, a curious reader can already recognize that formula (20) can be applied for calculation of the mutual inductance between the circular filament and a line, position of which with respect to the circle is defined through the linear and angular misalignment. For this reason, in Eq. (19) we assume that \( \bar{z}_\lambda = \bar{z}_B \) and formula (20) is integrated only from \(-1\) to \(1\). This fact proves again the efficiency and flexibility of the Kalantarov-Zeitlin method.

The advantages of the Kalantarov-Zeitlin method allow us to extend immediately the application of the obtained result to a case of the calculation of the mutual inductance between a primary circular filament and its projection on a tilted plane and to furnish this case via formula (22). For instance, this par-
ticular case appears in micro-machined inductive suspensions and has a direct practical application in studying their stability and pull-in dynamics.

New developed formulas have been successfully validated through a number of examples available in the literature. Also, the direct comparison the results of calculation with the numerical results obtained by utilizing the FastHenry software shows a good agreement. Besides, the obtained formulas can be easily programmed, they are intuitively understandable for application.

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Figure .14: The relationship between the angles of two manners for determining angular misalignment of the secondary circle: I and II are denoted for two spherical triangles highlighted by arcs in red color.
Appendix A. Determination of angular position of the secondary circular filament

The angular position of the secondary circle can be defined through the pair of angle \( \theta \) and \( \eta \) corresponding to manner I and the angle \( \alpha \) and \( \beta \) manner II. The relationship between two pairs of angles can be determined via two spherical triangles denoted in Roman number I and II as shown in Fig. 14. According to the law of sines, for spherical triangle I we can write the following relationship:

\[
\frac{\sin \eta}{\sin \frac{\pi}{2}} = \frac{\sin \beta}{\sin \theta}.
\]

(A.1)

For spherical triangle II, we have

\[
\frac{\sin(\frac{\pi}{2} - \eta)}{\sin(\frac{\pi}{2} - \beta)} = \frac{\sin \alpha}{\sin \theta}.
\]

(A.2)

Accounting for (A.1) and (A.2), the final set determining the relationship between two pairs of angles becomes as

\[
\begin{align*}
\sin \beta &= \sin \eta \sin \theta; \\
\cos \beta \sin \alpha &= \cos \eta \sin \theta.
\end{align*}
\]

(A.3)

Appendix B. Presentation of developed formulas via the pair of angles \( \alpha \) and \( \beta \)

Using set (A.3), we can write the following equations:

\[
\begin{align*}
\cos^2 \theta &= \cos^2 \beta (1 + \sin^2 \alpha); \\
\sin^2 \theta &= \sin^2 \beta + \cos^2 \beta \sin^2 \alpha; \\
\tan^2 \theta &= \frac{\sin^2 \beta + \cos^2 \beta \sin^2 \alpha}{\cos^2 \beta (1 + \sin^2 \alpha)}; \\
\cos^2 \eta &= \cos^2 \beta \sin^2 \alpha / \sin^2 \theta; \\
\sin^2 \eta &= \sin^2 \beta / \sin^2 \theta.
\end{align*}
\]

(B.1)

Now, applying set (B.1) to (8), the square of the dimensionless function \( \hat{r} \) becomes as

\[
\hat{r}^2 = \frac{\cos^2 \beta (1 + \sin^2 \alpha)(\sin^2 \beta + \cos^2 \beta \sin^2 \alpha)}{(\sin \varphi \cos \beta \sin \alpha - \cos \varphi \sin \beta)^2 + \cos^2 \beta (1 + \sin^2 \alpha)(\cos \varphi \cos \beta \sin \alpha + \sin \varphi \sin \beta)^2}.
\]

(B.2)
Then, for the dimensionless parameter $\tilde{z}_\lambda$, we have
\[
\tilde{z}_\lambda = \tilde{z}_B + \tilde{r} \left( \sin \varphi \cos \beta \sin \alpha - \cos \varphi \sin \beta \right) \sqrt{\cos^2 \beta (1 + \sin^2 \alpha)}.
\] (B.3)

Substituting (B.2), (B.3) and
\[
t_1 = \bar{x}_B + \bar{y}_B \cdot \tilde{r}^2 \frac{\sin^2 \beta + \cos^2 \beta \sin^2 \alpha}{\cos^2 \beta (1 + \sin^2 \alpha)} \times (\sin \varphi \cos \beta \sin \alpha - \cos \varphi \sin \beta).
\] (B.4)
\[
t_2 = \bar{y}_B - \bar{x}_B \cdot \tilde{r}^2 \frac{\sin^2 \beta + \cos^2 \beta \sin^2 \alpha}{\cos^2 \beta (1 + \sin^2 \alpha)} \times (\cos \varphi \cos \beta \sin \alpha + \sin \varphi \sin \beta).
\]
into Eq. (16), the angular misalignment of the secondary circle are defined through the pair of angle $\alpha$ and $\beta$ corresponding to the manner II.

For the case when the two circles are mutually perpendicular to each other, assuming that $\alpha = \pi/2$ then just replacing the angle $\eta$ by $\beta$ in formula (20), it can be used for calculation with new pair of angle $\alpha$ and $\beta$.

Since formula (22) is a particular case of (16), when $\bar{s} = 0$ and $\bar{r} = \bar{\rho} = 1$. Hence, substituting $\bar{r} = 1$ into (B.3) and using this modified equation for formula (22), it can be used for calculation of mutual inductance between the primary circle and its projection on a tilted plane, an angular position of which is defined by the pair of angle $\alpha$ and $\beta$.

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