Curves of Finite Total Curvature

John M. Sullivan

Abstract. We consider the class of curves of finite total curvature, as introduced by Milnor. This is a natural class for variational problems and geometric knot theory, and since it includes both smooth and polygonal curves, its study shows us connections between discrete and differential geometry. To explore these ideas, we consider theorems of Fáry/Milnor, Schur, Chakerian and Wienholtz.

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Here we introduce the ideas of discrete differential geometry in the simplest possible setting: the geometry and curvature of curves, and the way these notions relate for polygonal and smooth curves. The viewpoint has been partly inspired by work in geometric knot theory, which studies geometric properties of space curves in relation to their knot type, and looks for optimal shapes for given knots.

After reviewing Jordan’s definition of the length of a curve, we consider Milnor’s analogous definition [Mil50] of total curvature. In this unified treatment, polygonal and smooth curves are both contained in the larger class of FTC (finite total curvature) curves. We explore the connection between FTC curves and BV functions. Then we examine the theorems of Fáry/Milnor, Schur and Chakerian in terms of FTC curves. We consider relations between total curvature and Gromov’s distortion, and then we sketch a proof of a result by Wienholtz in integral geometry. We end by looking at ways to define curvature density for polygonal curves.

A companion article [DS08] examines more carefully the topology of FTC curves, showing that any two sufficiently nearby FTC graphs are isotopic. The article [Sul08], also in this volume, looks at curvatures of smooth and discrete surfaces; the discretizations are chosen to preserve various integral curvature relations.

Our whole approach in this survey should be compared to that of Alexandrov and Reshetnyak [AR89], who develop much of their theory for curves having one-sided tangents everywhere, a class somewhat more general than FTC.
1. Length and total variation

We want to consider the geometry of curves. Of course curves—unlike higher-dimension-
al manifolds—have no local intrinsic geometry. So we mean the extrinsic geometry of
how the curve sits in some ambient space $M$. Usually $M$ will be in euclidean $d$-space $\mathbb{E}^d$,
but the study of space curves naturally leads also to the study of curves on spheres. Thus
we also allow $M$ to be a smooth Riemannian manifold; for convenience we embed $M$
isometrically into some $\mathbb{E}^d$. (Some of our initial results would still hold with $M$ being any
path-metric space; compare [AR89]. Here, however, our curves will be quite arbitrary but
not our ambient space.)

A curve is a one-dimensional object, so we start by recalling the topological clas-
sification of one-manifolds: A compact one-manifold (allowing boundary) is a finite dis-
joint union of components, each homeomorphic to an interval $I := [0, L]$ or to a circle
$S^1 := \mathbb{R}/\mathbb{LZ}$. Then a parametrized curve in $M$ is a continuous map from a compact
one-manifold to $M$. That is, each of its components is a (parametrized) arc $\gamma : I \to M$
or loop $\gamma : S^1 \to M$. A loop can be viewed as an arc whose endpoints are equal and
thus identified. A curve in $M$ is an equivalence class of parametrized curves, where the
equivalence relation is given by orientation-preserving reparametrization of the domain.

(An unoriented curve would allow arbitrary reparametrization. Although we will
not usually care about the orientation of our curves, keeping it around in the background
is convenient, fixing for instance a direction for the unit tangent vector of a rectifiable
curve.)

Sometimes we want to allow reparametrizations by arbitrary monotone functions
that are not necessarily homeomorphisms. Intuitively, we can collapse any time interval
on which the curve is constant, or conversely stop for some time at any point along the
curve. Since there might be infinitely many such intervals, the easiest formalization of
these ideas is in terms of Fréchet distance [Fré05].

The Fréchet distance between two curves is the infimum, over all strictly monotonic
reparametrizations, of the maximum pointwise distance. (This has also picturesquely been
termed, perhaps originally in [AG95], the “dog-leash distance”: the minimum length of
leash required for a dog who walks forwards along one curve while the owner follows
the other curve.) Two curves whose Fréchet distance is zero are equivalent in the sense
we intend: homeomorphic reparametrizations that approach the infimal value zero will
limit to the more general reparametrization that might eliminate or introduce intervals of
constancy. See also [Gra46 § X.7].

Given a connected parametrized curve $\gamma$, a choice of

$$0 \leq t_1 < t_2 < \cdots < t_n \leq L$$

gives us the vertices $v_i := \gamma(t_i)$ of an inscribed polygon $P$, whose edges are the mini-
mizing geodesics $e_i := v_iv_{i+1}$ in $M$ between consecutive vertices. (If $\gamma$ is a loop, then
indices $i$ are to be taken modulo $n$, that is, we consider an inscribed polygonal loop.) We
will write $P < \gamma$ to denote that $P$ is a polygon inscribed in $\gamma$.

The edges are uniquely determined by the vertices when $M = \mathbb{E}^d$ or more generally
whenever $M$ is simply connected with nonpositive sectional curvature (and thus a CAT(0)
space). When minimizing geodesics are not unique, however, as in the case when $M$ is a sphere and two consecutive vertices are antipodal, some edges may need to be separately specified. The mesh of $P$ (relative to the given parametrization of $\gamma$) is

$$\text{Mesh}(P) := \max_i (t_{i+1} - t_i).$$

(Here, of course, for a loop the $n^{th}$ value in this maximum is $(t_1 + L) - t_n$.)

The length of a polygon is simply the sum of the edge lengths:

$$\text{Len}(P) = \text{Len}_M(P) := \sum_i d_M(v_i, v_{i+1}).$$

This depends only on the vertices and not on which minimizing geodesics have been picked as the edges, since by definition all minimizing geodesics have the same length. If $M \subset N \subset \mathbb{E}^d$, then a given set of vertices defines (in general) different polygonal curves in $M$ and $N$, with perhaps greater length in $M$.

We are now ready to define the length of an arbitrary curve:

$$\text{Len}(\gamma) := \sup_{P < \gamma} \text{Len}(P).$$

When $\gamma$ itself is a polygonal curve, it is easy to check that this definition does agree with the earlier one for polygons. This fact is essentially the definition of what it means for $M$ to be a path metric space: the distance $d(v, w)$ between any two points is the minimum length of paths connecting them. By this definition, the length of a curve $\gamma \subset M \subset \mathbb{E}^d$ is independent of $M$; length can be measured in $\mathbb{E}^d$ since even though the inscribed polygons may be different in $M$, their supremal length is the same.

This definition of length for curves originates with Jordan [Jor93] and independently Scheeffer [Sch85], and is thus often called “Jordan length”. (See also [Ces56, §2].) For $C^1$-smooth curves it can easily be seen to agree with the standard integral formula.

**Lemma 1.1.** Given a polygon $P$, if $P'$ is obtained from $P$ by deleting one vertex $v_k$ then $\text{Len}(P') \leq \text{Len}(P)$. We have equality here if and only if $v_k$ lies on a minimizing geodesic from $v_{k-1}$ to $v_{k+1}$.

**Proof.** This is simply the triangle inequality applied to the triple $v_{k-1}, v_k, v_{k+1}$. \qed

A curve is called rectifiable if its length is finite. (From the beginning, we have considered only compact curves. Thus we do not need to distinguish rectifiable and locally rectifiable curves.)

**Proposition 1.2.** A curve is rectifiable if and only if it admits a Lipschitz parametrization.

**Proof.** If $\gamma$ is $K$-Lipschitz on $[0, L]$, then its length is at most $KL$, since the Lipschitz bound gives this directly for any inscribed polygon. Conversely, a rectifiable curve can be reparametrized by its arclength

$$s(t) := \text{Len}(\gamma|_{[0,t]})$$

and this arclength parametrization is 1-Lipschitz. \qed
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If the original curve was constant on some time interval, the reparametrization here will not be one-to-one. A nonrectifiable curve has no Lipschitz parametrization, but might have a Hölder-continuous one. (For a nice choice of parametrization for an arbitrary curve, see \[\text{Mor36}\].)

Given this proposition, the standard theory of Lipschitz functions shows that a rectifiable curve \(\gamma\) has almost everywhere a well-defined unit tangent vector \(T = \gamma'\), its derivative with respect to its arclength parameter \(s\). Given a rectifiable curve, we will most often use this arclength parametrization. The domain is then \(s \in [0, L]\) or \(s \in \mathbb{R}/L\mathbb{Z}\), where \(L\) is the length.

Consider now an arbitrary function \(f\) from \(|\) or \(S^1\) to \(M\), not required to be continuous. We can apply the same definition of inscribed polygon \(P\), with vertices \(v_i = f(t_i)\), and thus the same definition of length \(\text{Len}(f) = \sup \text{Len}(P)\). This length of \(f\) is usually called the total variation of \(f\), and \(f\) is said to be BV (of bounded variation) when this is finite.

For a fixed ambient space \(M \subset \mathbb{E}^d\), the total variation of a discontinuous \(f\) as a function to \(M\) may be greater than its total variation in \(\mathbb{E}^d\). The supremal ratio here is

\[
\sup_{p, q \in M} \frac{d_M(p, q)}{d_{\mathbb{E}^d}(p, q)}
\]

what Gromov called the distortion of the embedding \(M \subset \mathbb{E}^d\). (See \[\text{Gro81}\] pp. 6–9, \[\text{Gro83}\] p. 114) and \[\text{Gro78}\], as well as \[\text{KS97}\] or \[\text{DS04}\].) When \(M\) is compact and smoothly embedded (like \(S^{d-1}\)), this distortion is finite; thus \(f\) is BV in \(M\) if and only if it is BV in \(\mathbb{E}^d\).

The class of BV functions (here, from \(|\) to \(M\)) is often useful for variational problems. Basic facts about BV functions can be found in the original book \[\text{Car18}\] by Carathéodory or in many analysis texts like \[\text{GP83}\] Sect. 2.19, \[\text{Boa96}\] Chap. 3 or \[\text{Ber98}\]. For more details and higher dimensions, see for instance \[\text{Zie89}\] or \[\text{AFP00}\].

Here, we recall one nice characterization: \(f\) is BV if and only if it has a weak (distributional) derivative. Here, a weak derivative means an \(\mathbb{E}^d\)-valued Radon measure \(\mu\) which plays the role of \(f' \, dt\) in integration by parts, meaning that

\[
\int_0^L f' \varphi' \, dt = -\int_0^L \varphi \, \mu
\]

for every smooth test function \(\varphi\) vanishing at the endpoints. (This characterization of BV functions is one form of the Riesz representation theorem.)

**Proposition 1.3.** If \(f\) is BV, then \(f\) has well-defined right and left limits

\[
f_{\pm}(t) := \lim_{\tau \to t^\pm} f(\tau)
\]

everywhere. Except at countably many jump points of \(f\), we have \(f_-(t) = f(t) = f_+(t)\).

**Sketch of proof.** We consider separately each of the \(d\) real-valued coordinate functions \(f^i\). We decompose the total variation of \(f^i\) into positive and negative parts, each of which is bounded. This lets us write \(f^i\) as the difference of two monotonically increasing functions.
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(This is its so-called Jordan decomposition.) An increasing function can only have countably many (jump) discontinuities. (Alternatively, one can start by noting that a real-valued function without, say, a left-limit at \( t \) has infinite total variation even locally.)

In functional analysis, BV functions are often viewed as equivalence classes of functions differing only on sets of measure zero. Then we replace total variation with essential total variation, the infimal total variation over the equivalence class. A minimizing representative will be necessarily continuous wherever \( f_- = f_+ \). A unique representative can be obtained by additionally requiring left (or right, or upper, or lower) semicontinuity at the remaining jump points.

Our definition of curve length is not very practical, being given in terms of a supremum over all possible inscribed polygons. But it is easy to find a sequence of polygons guaranteed to capture the supremal length:

**Proposition 1.4.** Suppose \( P_k \) is a sequence of polygons inscribed in a curve \( \gamma \) such that \( \text{Mesh}(P_k) \to 0 \). Then \( \text{Len}(\gamma) = \lim \text{Len}(P_k) \).

**Proof.** By definition, \( \text{Len}(\gamma) \geq \text{Len}(P_k) \), so \( \text{Len}(\gamma) \geq \lim \text{Len}(P_k) \). Suppose that \( \text{Len}(\gamma) > \lim \text{Len}(P_k) \). Passing to a subsequence, for some \( \varepsilon > 0 \) we have \( \text{Len}(\gamma) \geq \text{Len}(P_k) + 2\varepsilon \).

Then by the definition of length, there is an inscribed \( P_0 \) (with, say, \( n \) vertices) such that \( \text{Len}(P_0) \geq \text{Len}(P_k) \). The common refinement of \( P_0 \) and \( P_k \) is of course at least as long as \( P_0 \). But this refinement is \( P_k \) with a fixed number \( (n) \) of vertices inserted; for each \( k \), these \( n \) insertions together add length at least \( \varepsilon \) to \( P_k \). For large enough \( k \), these \( n \) insertions are at disjoint places along \( P_k \), so their effect on the length is independent of the order in which they are performed. Passing again to a subsequence, there is thus some vertex \( v_0 = \gamma(t_0) \) of \( P_0 \) such that if \( P_k^0 \) is \( P_k \) with \( v_0 \) inserted, we have \( \text{Len}(P_k^0) \geq \text{Len}(P_k) + \varepsilon/n \).

But \( \gamma \) is continuous, in particular at \( t_0 \). So there exists some \( \delta > 0 \) such that for \( t \in [t_0 - \delta, t_0 + \delta] \) we have \( d_M(\gamma(t), \gamma(t_0)) < \varepsilon/2n \). Choosing \( k \) large enough that \( \text{Mesh}(P_k) < \delta \), the vertices of \( P_k \) immediately before and after \( v_0 \) will be within this range, so \( \text{Len}(P_k^0) < \text{Len}(P_k) + \varepsilon/n \), a contradiction. \( \square \)

Although we have stated this proposition only for continuous curves \( \gamma \), the same holds for BV functions \( f \), as long as \( f \) is semicontinuous at each of its jump points.

An analogous statement does not hold for polyhedral approximations to surfaces. First, an inscribed polyhedron (whose vertices lie “in order” on the surface) can have greater area than the original surface, even if the mesh size (the diameter of the largest triangle) is small. Second, not even the limiting value is guaranteed to be correct. Although Serret had proposed [Ser68, p. 293] defining surface area as a limit of polyhedral areas, claiming this limit existed for smooth surfaces, Schwarz soon found a counterexample, now known as the “Schwarz lantern” [Sch90]: seemingly nice triangular meshes inscribed in a cylinder, with mesh size decreasing to zero, can have area approaching infinity.

Lebesgue [Leb02] thus defined surface area as the \( \lim \inf \) of such converging polyhedral areas. (See [AT72, Ces89] for an extensive discussion of related notions.) One can also rescue the situation with the additional requirement that the shapes of the triangles stay bounded (so that their normals approach that of the smooth surface), but we will not
explore this here. (See also [Ton21]. In this volume, the companion article [Sul08] treats curvatures of smooth and discrete surfaces, and [War08] considers convergence issues.) Historically, such difficulties led to new approaches to defining length and area, such as Hausdorff measure. These measure-theoretic approaches work well in all dimensions, and lead to generalizations of submanifolds like the currents and varifolds of geometric measure theory (see [Mor88]). We have chosen here to present the more “old-fashioned” notion of Jordan length for curves because it nicely parallels Milnor’s definition of total curvature, which we consider next.

2. Total curvature

Milnor [Mil50] defined a notion of total curvature for arbitrary curves in Euclidean space. Suppose \( P \) is a polygon in \( M \) with no two consecutive vertices equal. Its turning angle at an interior vertex \( v_n \) is the angle \( \varphi \in [0, \pi] \) between the oriented tangent vectors at \( v_n \) to the two edges \( v_{n-1}v_n \) and \( v_nv_{n+1} \). (Here, by saying interior vertices, we mean to exclude the endpoints of a polygonal arc, where there is no turning angle; every vertex of a polygonal loop is interior. The supplement of the turning angle, sometimes called an interior angle of \( P \), will not be of interest to us.)

If \( M \) is an oriented surface, for instance if \( M = \mathbb{E}^2 \) or \( S^2 \), then we can also define a signed turning angle \( \varphi \in [-\pi, \pi] \) at \( v_n \), except that where \( \varphi = \pm\pi \) its sign is ambiguous.

To find the total curvature \( TC(P) \) of a polygon \( P \), we first collapse any sequence of consecutive equal vertices to a single vertex. Then \( TC(P) \) is simply the sum of the turning angles at all interior vertices.

Here, we mainly care about the case when \( P \) is in \( M = \mathbb{E}^d \). Then the unit tangent vectors along the edges, in the directions \( v_{n+1} - v_n \), are the vertices of a polygon in \( S^{d-1} \) called the tantrix of \( P \). (The word is a shortening of “tangent indicatrix”.) The total curvature of \( P \) is the length of its tantrix in \( S^{d-1} \).

**Lemma 2.1.** (See [Mil50, Lemma 1.1] and [Bor47].) Suppose \( P \) is a polygon in \( \mathbb{E}^d \), If \( P' \) is obtained from \( P \) by deleting one vertex \( v_n \) then \( TC(P') \leq TC(P) \). We have equality here if \( v_{n-1}v_nv_{n+1} \) are collinear in that order, or if \( v_{n-2}v_{n-1}v_nv_{n+1}v_{n+2} \) lie convexly in some two-plane, but never otherwise.

**Proof.** Deleting \( v_n \) has the following effect on the tantrix: two consecutive vertices (the tangents to the edges \( v_{n-1}v_n \) and \( v_nv_{n+1} \)) are coalesced into a single one (the tangent to the edge \( v_{n-1}v_{n+1} \)). It lies on a great circle arc connecting the original two, as in Figure 1. Using the triangle inequality twice, the length of the tantrix decreases (strictly, unless the tantrix vertices \( v_{n-1}v_n \) and \( v_nv_{n+1} \) coincide, or the relevant part lies along a single great circle in \( S^{d-1} \)). \( \square \)

**Corollary 2.2.** If \( P \) is a polygon in \( \mathbb{E}^d \) and \( P' < P \) then \( TC(P') \leq TC(P) \).

**Proof.** Starting with \( P \), first insert the vertices of \( P' \); since each of these lies along an edge of \( P \), these insertions have no effect on the total curvature. Next delete the vertices not in \( P' \); this can only decrease the total curvature. \( \square \)
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**Figure 1.** Four consecutive edges of a polygon $P$ in space give four vertices and three connecting edges (shown here as solid lines) of its tantrix on the sphere. When the middle vertex $v_n$ of $P$ is deleted, two vertices of the tantrix get collapsed to a single new one (labeled $v_{n-1}v_{n+1}$); it lies somewhere along edge connecting the two original vertices. The two new edges of the tantrix are shown as dashed lines. Applying the triangle inequality twice, we see that the length of the new tantrix (the total curvature of the new polygon) is no greater.

**Definition.** For any curve $\gamma \subset \mathbb{E}^d$ we follow Milnor [Mil50] to define

$$TC(\gamma) := \sup_{P \subset \gamma} TC(P).$$

We say that $\gamma$ has *finite total curvature* (or that $\gamma$ is FTC) if $TC(\gamma) < \infty$.

When $\gamma$ is itself a polygon, this definition agrees with the first one by Corollary 2.2. Our curves are compact, and thus lie in bounded subsets of $\mathbb{E}^d$. It is intuitively clear then that a compact curve of infinite length must have infinite total curvature; that is, that all FTC curves are rectifiable. This follows rigorously by applying the quantitative estimate of Proposition 6.1 below to finely inscribed polygons, using Propositions 1.4 and 3.1.

Various properties follow very easily from this definition. For instance, if the total curvature of an arc is less than $\pi$ then the arc cannot stray too far from its endpoints. In particular, define the *spindle* of angle $\theta$ with endpoints $p$ and $q$ to be the body of revolution bounded by a circular arc of total curvature $\theta$ from $p$ to $q$ that has been revolved about $\overrightarrow{pq}$.

(The spindle is convex, looking like an American football, for $\theta \leq \pi$ and is a round ball for $\theta = \pi$.)

**Lemma 2.3.** Suppose $\gamma$ is an arc from $p$ to $q$ of total curvature $\varphi < \pi$. Then $\gamma$ is contained in the spindle of angle $2\varphi$ from $p$ to $q$.

**Proof.** Suppose $x \in \gamma$ is outside the spindle. Consider the planar polygonal arc $pxq$ inscribed in $\gamma$. In that plane, since $x$ is outside the circular arc of total curvature $2\varphi$, by
elementary geometry the turning angle of $pxq$ at $x$ is greater than $\varphi$, contradicting the definition of $\text{TC}(\gamma)$. \hfill \Box

We also immediately recover Fenchel’s theorem \cite{Fen29}:

**Theorem 2.4 (Fenchel).** Any closed curve in $E^d$ has total curvature at least $2\pi$.

**Proof.** Pick any two distinct points $p, q$ on the curve. (We didn’t intend the theorem to apply to the constant curve!) The inscribed polygonal loop from $p$ to $q$ and back has total curvature $2\pi$, so the original curve has at least this much curvature. \hfill \Box

For this approach to Fenchel’s theorem to be satisfactory, we do need to verify (as Milnor did \cite{Mil50}) that our definition of total curvature agrees with the usual one $\int \kappa \, ds$ for smooth curves. For us, this will follow from Proposition 3.1.

### 3. First variation of length

We can characterize FTC curves as those with BV tangent vectors. This relates to the variational characterization of curvature in terms of first variation of length. (The discussion in this section is based on \cite[Sect. 4]{CF+04}.)

We have noted that an FTC curve $\gamma$ is rectifiable, hence has a tangent vector $T$ defined almost everywhere. We now claim that the total curvature of $\gamma$ is exactly the length (or, more precisely, the essential total variation) of this tantrix $T$ as a curve in $S^{d-1}$. We have already noted this for polygons, so the general case seems almost obvious from the definitions. However, while the tantrix of a polygon inscribed in $\gamma$ is a spherical polygon, it is not inscribed in $T$; instead its vertices are averages of small pieces of $T$. Luckily, this is close enough to allow the argument of Proposition 1.4 to go through again: Just as for length, in order to compute total curvature it suffices to take any limit of finer and finer inscribed polygons.

**Proposition 3.1.** Suppose $\gamma$ is a curve in $E^d$. If $P_k$ is a sequence of polygons inscribed in $\gamma$ with $\text{Mesh}(P_k) \to 0$, then $\text{TC}(\gamma) = \lim \text{TC}(P_k)$. This equals the essential total variation of its tantrix $T \subset S^{d-1}$.

We leave the proof of this proposition as an exercise. The first statement essentially follows as in the proof of Proposition 1.4 if it failed there would be one vertex $v_0$ along $\gamma$ whose insertion would cause a uniform increase in total curvature for all polygons in a convergent subsequence, contradicting the fact that sufficiently small arcs before and after $v$ have arbitrarily small total curvature. The second statement follows by measuring both $\text{TC}(\gamma)$ and the total variation through limits of (different but nearby) fine polygons.

To summarize, a rectifiable curve $\gamma$ has finite total curvature if and only if its unit tangent vector $T = \gamma'(s)$ is a function of bounded variation. (Thus the space of FTC curves could be called $W^{1,BV}$ or $BV^1$.) If $\gamma$ is FTC, it follows that at every point of $\gamma$ there are well-defined left and right tangent vectors $T_\pm$; these are equal and opposite except at countably many points, the corners of $\gamma$.

Now, to investigate curvature from a variational point of view, suppose we consider a continuous deformation $\gamma_t$ of a curve $\gamma \subset E^d$: fixing any parametrization of $\gamma$, this means...
a continuous family $\gamma_t$ of parametrized curves with $\gamma_0 = \gamma$. (If we reparametrize $\gamma$, we must apply the same reparametrization to each $\gamma_t$.)

We assume that position of each point is (at least $C^1$) smooth in time; the initial velocity of $\gamma_t$ will then be given by some (continuous, $\mathbb{R}^d$-valued) vectorfield $\xi$ along $\gamma$.

Let $\gamma$ be a rectifiable curve parametrized by arclength $s$, with unit tangent vector $T = \gamma'(s)$ (defined almost everywhere). Suppose $\gamma_t$ is a variation of $\gamma = \gamma_0$ whose initial velocity $\xi(s)$ is a Lipschitz function of arclength. Then the arclength derivative $\xi' = \frac{\partial \xi}{\partial s}$ is defined almost everywhere along $\gamma$, and a standard first-variation calculation shows that

$$\delta \xi \operatorname{Len}(\gamma) := \left. \frac{d}{dt} \right|_{t=0} \operatorname{Len}(\gamma_t) = \int_{\gamma} \langle T, \xi' \rangle \, ds.$$

If $\gamma$ is smooth enough, we can integrate this by parts to get

$$\delta \xi \operatorname{Len}(\gamma) = - \int_{\gamma} \langle T', \xi \rangle \, ds - \sum_{x \in \partial \gamma} \langle \pm T, \xi \rangle,$$

where, in the boundary term, the sign is chosen to make $\pm T$ point inward at $x$. In fact, not much smoothness is required: as long as $\gamma$ is $\text{FTC}$, we know that its unit tangent vector $T$ is $\text{BV}$, so we can interpret $T'$ as a measure, and this first-variation formula holds in the following sense: the weak (distributional) derivative $K := T'$ is an $\mathbb{R}^d$-valued Radon measure along $\gamma$ which we call the 

**curvature force.**

On a $C^2$ arc of $\gamma$, the curvature force is $K = dT = \kappa N \, ds$ and is absolutely continuous with respect to the arclength or Hausdorff measure $ds = \mathcal{H}^1$. The curvature force has an atom (a point mass or Dirac delta) at each corner $x \in \gamma$, with $K\{x\} = T_+(x) + T_-(x)$. Note that at such a corner, the mass of $K$ is not the turning angle $\theta$ at $x$. Instead,

$$|K\{x\}| = |K\{x\}| = 2 \sin(\theta/2).$$

Therefore, the total mass (or total variation) $|K|(\gamma)$ of the curvature force $K$ is somewhat less than the total curvature of $\gamma$: at each corner it counts $2 \sin(\theta/2)$ instead of $\theta$. Whereas $\text{TC}(\gamma)$ was the length (or total variation) $\operatorname{Len}_{S^d-1}(T)$ of the tantrix $T$ viewed as a (discontinuous) curve on the sphere $S^d-1$, we recognize this total mass as its length $\operatorname{Len}_{\mathbb{R}^d}(T)$ in euclidean space. Thus we call it the euclidean total curvature of $\gamma$:

$$\text{TC}^*(\gamma) := \operatorname{Len}_{\mathbb{R}^d}(T) = |K|(\gamma).$$

Returning to the first variation of length, we say that a vectorfield $\xi$ along $\gamma$ is **smooth** if $\xi(s)$ is a smooth function of arclength. The first variation $\delta \operatorname{Len}(\gamma)$ can be viewed as a linear functional on the space of smooth vectorfields $\xi$ along $\gamma$. As such a distribution, it has degree zero, by definition, if $\delta \xi \operatorname{Len}(\gamma) = \int_{\gamma} \langle T, \xi' \rangle \, ds$ is bounded by $C \sup_{\gamma} |\xi|$ for some constant $C$. This happens exactly when we can perform the integration by parts above.
We collect the results of this section as:

**Proposition 3.2.** Given any rectifiable curve $\gamma$, the following conditions are equivalent:

(a) $\gamma$ is FTC.

(b) There exists a curvature force $K = dT$ along $\gamma$ such that

$$\delta_\xi \text{Len}(\gamma) = -\int_\gamma \langle \xi, K \rangle - \sum_{\partial \gamma} \langle \xi, \pm T \rangle.$$

(c) The first variation $\delta \text{Len}(\gamma)$ has distributional degree zero. \hfill \Box

Of course, just as not all continuous functions are BV, not all $C^1$ curves are FTC. However, given an FTC curve, it is piecewise $C^1$ exactly when it has finitely many corners, and is $C^1$ when it has no corners, that is, when $K$ has no atoms. The FTC curve is furthermore $C^{1,1}$ when $T$ is Lipschitz, or equivalently when $K$ is absolutely continuous (with respect to arclength $s$) and has bounded Radon/Nikodym derivative $dK/ds = \kappa N$.

### 4. Total curvature and projection

The Fáry/Milnor theorem says that a knotted curve in $\mathbb{E}^3$ has total curvature at least $4\pi$, twice that of an unknotted round circle. The different proofs given by Fáry [Fár49] and Milnor [Mil50] can both be interpreted in terms of a proposition about the average total curvature of different projections of a curve.

The Grassmannian $G_k \mathbb{E}^d$ of $k$-planes in $d$-space is compact, with a unique rotation-invariant probability measure $d\mu$. For $p \in G_k \mathbb{E}^d$, we denote by $\pi_p$ the orthogonal projection to $p$. When we speak about averaging over all projections, we mean using $d\mu$. This proposition is essentially due to Fáry [Fár49], though he only stated the case $d = 3, k = 2$.

**Proposition 4.1.** Given a curve $K$ in $\mathbb{E}^d$, and some fixed $k < d$, the total curvature of $K$ equals the average total curvature of its projections to $k$-planes. That is,

$$\text{TC}(K) = \int_{G_k \mathbb{E}^d} \text{TC}(\pi_p(K)) \, d\mu.$$

**Proof.** By definition of total curvature and Proposition 3.1, we may reduce to the case where $K$ is a polygon. (To interchange the limit of ever finer inscribed polygons with the average over the Grassmannian, we use the Lebesgue monotone convergence theorem.) Since the total curvature of a polygon is the sum of its turning angles, it suffices to consider a single angle. So let $P_\theta$ be a three-vertex polygonal arc with a single turning angle of $\theta \in [0, \pi]$. Defining

$$f_k^d(\theta) := \int_{p \in G_k \mathbb{E}^d} \text{TC}(\pi_p(P_\theta)) \, d\mu,$$

the rotation-invariance of $\mu$ shows this is independent of the position of $P_\theta$, and our goal is to show $f_k^d(\theta) = \theta$.

First note that $f_k^d$ is continuous. It is also additive:

$$f_k^d(\alpha + \beta) = f_k^d(\alpha) + f_k^d(\beta).$$
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when $0 \leq \alpha, \beta \leq \alpha + \beta \leq \pi$. This follows by cutting the single corner of $P_\theta$ into two corners of turning angles $\alpha$ and $\beta$. Any projection of the resulting convex planar arc is still convex and planar, so additivity holds in each projection, and thus also holds after averaging.

A continuous, additive function is linear: $f_k^d(\theta) = c_k^d \theta$ for some constant $c_k^d$. Thus we just need to show $c_k^d = 1$. But we can easily evaluate $f$ at $\theta = \pi$, where $P_\pi$ is a “cusp” in which the incoming and outgoing edges overlap. Clearly every projection of $P_\pi$ is again such a cusp with turning angle $\pi$ (except for a set of measure zero where the projection is a single point). Thus $f_k^d(\pi) = \pi$, so $c_k^d = 1$ and we are done. □

A curve in $\mathbb{E}^1$ has only cusps for corners, so its total curvature will be a nonnegative multiple of $\pi$. A loop in $\mathbb{E}^1$ has total curvature a positive multiple of $2\pi$. (In particular, the loop in $\mathbb{E}^1$ is a real-valued function on $\mathbb{S}^1$, and its total curvature is $2\pi$ times the number of local maxima.)

This proposition could be used to immediately reduce the $d$-dimensional case of Fenchel’s theorem (Theorem 2.4) to the $k$-dimensional case. Historically, this could have been useful. For instance, the theorem is trivial in $\mathbb{E}^1$ by the previous paragraph. In $\mathbb{E}^2$, the idea that a simple closed curve has total signed curvature $\pm 2\pi$ essentially dates back to Riemann (compare [Che89, §1]). Fenchel’s proof [Fen29] (which was his 1928 doctoral dissertation in Berlin) was for $d = 3$, and the first proof for general dimensions seems to be that of Borsuk [Bor47].

For alternative proofs of Fenchel’s theorem—as well as comparisons among them—see [Hor71] and [Che89, §4], and the references therein, especially [Lie29] and [Fen51]. Voss [Vos55] related the total curvature of a space curve to the total Gauss curvature of a tube around it, and thereby gave new a new proof of the Fáry/Milnor theorem as well as Fenchel’s theorem.

Milnor’s proof [Mil50] of the Fáry/Milnor theorem can be rephrased as a combination of the case $k = 1$ of Proposition 4.1 with the following:

**Lemma 4.2.** If $K \subset \mathbb{E}^3$ is nontrivially knotted, then any projection of $K$ to $\mathbb{E}^1$ has total curvature at least $4\pi$.

**Sketch of proof.** If there were some projection direction in which the total curvature were only $2\pi$, then the corresponding (linear) height function on $\mathbb{E}^3$ would have only one local minimum and one local maximum along $K$. Then at each intermediate height, there are exactly two points of $K$. Connecting each such pair with a straight segment, we form a spanning disk showing $K$ is unknotted. □

(This proof isn’t quite complete as stated: at some intermediate heights, one or even both strands of $K$ might have a whole subarc at that constant height. One could still patch in a disk, but easier is to follow Milnor and at the very beginning replace $K$ by an isotopic inscribed polygon. Compare [DS08].)

Fáry [Fár49], on the other hand, used $k = 2$ in Proposition 4.1, having proved:

**Lemma 4.3.** Any nontrivial knot diagram (a projection to $\mathbb{E}^2$ of a knotted curve $K \subset \mathbb{E}^3$) has total curvature at least $4\pi$. 
Sketch of proof. Any knot diagram divides the plane into regions: one unbounded and several bounded. If every bounded region is adjacent to the unbounded region, the only possibility is a tree-like diagram as in Figure 2 (left); this is clearly unknotted no matter how we choose over- and under-crossings.

Thus every nontrivial knot diagram $D$ has some region $R$ which is “doubly enclosed” by the curve, not necessarily in the sense of oriented winding numbers, but in the sense that any ray outwards from a point $p \in R$ must cut the curve twice. Then $R$ is part of the second hull of the curve $[CK+03]$, and the result follows by Lemmas 5 and 6 there. To summarize the arguments there (which parallel those of $[F´ar49]$), note first that the cone over $D$ from $p$ has cone angle at least $4\pi$ at $p$; by Gauss/Bonnet, this cone angle equals the total signed geodesic curvature of $D$ in the cone, which is at most its total (unsigned) curvature.

\[ \square \]

Either of these lemmas, combined with the appropriate case of Proposition 4.1 immediately yields the Fáry/Milnor theorem.

**Corollary 4.4.** The total curvature of a nontrivial knot $K \subset \mathbb{E}^3$ is at least $4\pi$.  

While Fáry stated the appropriate case of Proposition 4.1 pretty much as such, we note that Milnor didn’t speak about total curvature of projections to lines, but instead only about extrema of height functions. (A reinterpretation more like ours can already be found, for instance, in $[Tan98]$.)

Denne $[Den04]$ has given a beautiful new proof of Fáry/Milnor analogous to the easy proof we gave for Fenchel’s Theorem $[2.4]$. Indeed, the hope that there could be such a proof had led us to conjecture in $[CK+03]$ that every knot $K$ has an alternating quadrisecant. A quadrisecant, by definition, is a line in space intersecting the knot in four points $p_i$. The $p_i$ form an inscribed polygonal loop, whose total curvature (since it lies in a line) is either $2\pi$ or $4\pi$, depending on the relative ordering of the $p_i$ along the line and along $K$. The quadrisecant is called alternating exactly when the curvature is $4\pi$. Denne proved this conjecture: every nontrivially knotted curve in space has an alternating quadrisecant. This gives as an immediate corollary not only the Fáry/Milnor theorem, but also a new proof that a knot has a second hull $[CK+03]$. 

![Figure 2. A diagram in which every region is adjacent to the outside is in fact unknotted (left), so a knot diagram has a doubly enclosed region (right).](image-url)
We should also note that a proof of Fáry/Milnor for knots in hyperbolic as well as euclidean space was given by Brickell and Hsiung [BH74]; essentially they construct a point on the knot that also lies in its second hull. The theorem is also now known for knots in an arbitrary Hadamard manifold, that is, a simply connected manifold of nonpositive curvature: Alexander and Bishop [AB98] find a finite sequence of inscriptions—first a polygon $P_1$ inscribed in $K$, then inductively $P_{i+1}$ inscribed in $P_i$—ending with a $pqpq$ quadrilateral, while Schmitz [Sch98] comes close to constructing a quadrisecant.

The results of this section would not be valid for $TC^*$ in place of $TC$: there is no analog to Proposition 4.1 and the Fáry/Milnor theorem would fail.

We conclude this section by recalling that another standard proof of Fenchel’s theorem uses the following integral-geometric lemma due to Crofton (see [Che89, §4] and also [San89, San04]) to conclude that a spherical curve of length less than $2\pi$ is contained in an open hemisphere:

**Lemma 4.5.** The length of a curve $\gamma \subset S^{d-1}$ equals $\pi$ times the average number of intersections of $\gamma$ with great hyperspheres $S^{d-2}$.

**Proof.** It suffices to prove this for polygons (appealing again to the monotone convergence theorem). Hence it suffices to consider a single great-circle arc. But clearly for such arcs, the average number of intersections is proportional to length. When the length is $\pi$, (almost) every great circle is intersected exactly once. \hfill $\Box$

This lemma, applied to the tantrix, is equivalent to the case $k = 1$ of our Proposition 4.1. Indeed, when projecting $\gamma$ to the line in direction $v$, the total curvature we see counts the number of times the tantrix intersects the great sphere normal to $v$. We note also that knowing this case $k = 1$ (for all $d$) immediately implies all other cases of Proposition 4.1 since a projection from $E^d$ to $E^1$ can be factored as projections $E^d \to E^k \to E^1$.

Finally, we recall an analogous statement of the famous Cauchy/Crofton formula. Its basic idea dates back to Buffon’s 1777 analysis [Buf77] of his needle problem (compare [KR97, Chap. 1]). Cauchy obtained the formula by 1841 [Cau41] and generalized it to find the surface area of a convex body. Crofton’s 1868 paper [Cro68] on geometric probability includes this among many integral-geometric formulas for plane curves. (See [AD97] for a treatment like ours for rectifiable curves, and also [dC76, §1.7C] and [San89].)

**Lemma 4.6 (Cauchy/Crofton).** The length of a plane curve equals $\pi/2$ times the average length of its projections to lines.

**Proof.** Again, we can reduce first to polygons, then to a single segment. So the result certainly holds for some constant; to check the constant is $\pi/2$, it is easiest to compute it for the unit circle, where every projection has length 4. \hfill $\Box$

In all three of our integral-geometric arguments (4.1, 4.5, 4.6) we proved a certain function was linear and then found the constant of proportionality by computing one (perhaps sometimes surprising) example. One finds also in the literature proofs where the integrals (over the circle or more generally the Grassmannian) are simply computed explicitly. Although the trigonometric integrals are not too difficult, that approach seems to obscure the geometric essence of the argument.
5. Schur’s comparison theorem

Schur’s comparison theorem \cite{Sch21} is a well-known result saying that straightening an arc will increase the distance between its endpoints.

Chern, in §5 of his beautiful essay \cite{Che89} in his MAA book, gives a proof for $C^2$ curves and remarks (without proof) that it also applies to piecewise smooth curves. In \cite{CKS02} we noted that Chern’s proof actually applies to $C^{1,1}$ curves, that is to curves with a Lipschitz tangent vector, or with bounded curvature density. In fact, the natural class of curves to which the proof applies is FTC curves.

**Theorem 5.1 (Schur’s Comparison Theorem).** Let $\bar{\gamma} \subset \mathbb{R}^2$ be a planar arc such that joining the endpoints of $\bar{\gamma}$ results in a convex (simple, closed) curve, and let $\gamma \subset \mathbb{R}^d$ be an arc of the same length $L$. Suppose that $\bar{\gamma}$ has nowhere less curvature than $\gamma$ with respect to the common arclength parameter $s \in [0, L]$, that is, that for any subinterval $I \subset [0, L]$ we have

$$ \text{TC}(\bar{\gamma}|_I) \geq \text{TC}(\gamma|_I). $$

(Equivalently, $|K| - |\bar{K}|$ is a nonnegative measure.) Then the distance between the endpoints is greater for $\gamma$:

$$ |\gamma(L) - \gamma(0)| \geq |\bar{\gamma}(L) - \bar{\gamma}(0)|. $$

**Proof.** By convexity, we can find an $s_0$ such that the (or, in the case of a corner, some supporting) tangent direction $T_0$ to $\gamma$ at $\gamma(s_0)$ is parallel to $\bar{\gamma}(L) - \bar{\gamma}(0)$. Note that the convexity assumption implies that the total curvature of either half of $\bar{\gamma}$ (before or after $s_0$) is at most $\pi$. Now move $\gamma$ by a rigid motion so that it shares this same tangent vector at $s_0$, as in Figure 3 (If $\gamma$ has a corner at $s_0$, so does $\bar{\gamma}$. In this case we want to arrange that the angle from $T_0$ to each one-sided tangent vector $T_{\pm}$ is at least as big for $\bar{\gamma}$ as for $\gamma$.)

By choice of $T_0$ we have

$$ |\bar{\gamma}(L) - \bar{\gamma}(0)| = \langle \bar{\gamma}(L) - \bar{\gamma}(0), T_0 \rangle = \int_0^L \langle \bar{T}(s), T_0 \rangle ds, $$
while for $\gamma$ we have

$$\left| \gamma(L) - \gamma(0) \right| \geq \left\langle \gamma(L) - \gamma(0), T_0 \right\rangle = \int_0^L \left\langle T(s), T_0 \right\rangle \, ds.$$ 

Thus it suffices to prove (for almost every $s$) that

$$\left\langle T(s), T_0 \right\rangle \geq \left\langle \tilde{T}(s), T_0 \right\rangle.$$ 

But starting from $s_0$ and moving outwards in either direction, $\tilde{T}$ moves straight along a great circle arc, at speed given by the pointwise curvature; in total it moves less than distance $\pi$. At the same time, $T$ moves at the same or lower speed, and perhaps not straight but on a curved path. Clearly then $T(s)$ is always closer to $T_0$ than $\tilde{T}(s)$ is, as desired. More precisely,

$$\left\langle \tilde{T}(s), T_0 \right\rangle = \cos TC(\tilde{\gamma}|_{[s_0,s]}) \leq \cos TC(\gamma|_{[s_0,s]}) \leq \left\langle T(s), T_0 \right\rangle. \quad \square$$

The special case of Schur’s theorem when $\gamma$ and $\tilde{\gamma}$ are polygons is usually called Cauchy’s arm lemma. It was used in Cauchy’s proof [Cau13] of the rigidity of convex polyhedra, although Cauchy’s own proof of the arm lemma was not quite correct, as discovered 120 years later by Steinitz. The standard modern proof of the arm lemma (due to Schoenberg; see [AZ98] or [Cro97, p. 235]) is quite different from the proof we have given here. For more discussion of the relation between Schur’s theorem and Cauchy’s lemma, see [Con82, O’R00].

The history of this result is somewhat complicated. Schur [Sch21] considered only the case where $\gamma$ and $\tilde{\gamma}$ have pointwise equal curvature: twisting a convex plane curve out of the plane by adding torsion will increase its chord lengths. He considers both polygonal and smooth curves. He attributes the original idea (only for the case where $\tilde{\gamma}$ is a circular arc) to unpublished work of H. A. Schwarz in 1884. The full result, allowing the space curve to have less curvature, is evidently due to Schmidt [Sch25]. See also the surveys by Blaschke in [Bla21] and [Bla24, §28–30].

In Schur’s theorem, it is irrelevant whether we use the spherical or euclidean version of total curvature. If we replace $TC$ by $TC^\ast$ throughout, the statement and proof remain unchanged, since the curvature comparison is pointwise.

### 6. Chakerian’s packing theorem

A less familiar result due to Chakerian (and cited for instance as [BS99, Lemma 1.1]) captures the intuition that a long rope packed into a small ball must have large curvature.

**Proposition 6.1.** A connected FTC curve $\gamma$ contained in the unit ball in $\mathbb{R}^d$ has length no more than $2 + TC^\ast(\gamma)$. (If $\gamma$ is closed, the 2 can be omitted.)
Proof. Use the arclength parametrization $\gamma(s)$. Then
\[
\text{Len}(\gamma) = \int 1 \, ds = \int \langle T, T \rangle \, ds \\
= \langle T, \gamma \rangle \bigg|_{\text{endpts}} - \int \langle \gamma, dK \rangle \\
\leq 2 + \int d|K| = 2 + TC^*(\gamma). \quad \Box
\]

Chakerian [Cha64] gave exactly this argument for $C^2$ curves and then used a limit argument (rounding the corners of inscribed polygons) to get a version for all curves. Note, however, that this limiting procedure gives the bound with $TC^*$ replaced by $TC$; this is of course equivalent for $C^1$ curves but weaker for curves with corners. For closed curves, Chakerian noted that equality holds in $\text{Len} \leq TC$ only for a great circle (perhaps traced multiple times). In our sharper bound $\text{Len} \leq TC^*$, we have equality also for a regular $n$-gon inscribed in a great circle.

(We recall that we appealed to this theorem in Section 2 to deduce that FTC curves are rectifiable. This is not circular reasoning: we first apply the proof above to polygons, then deduce that FTC curves are rectifiable and indeed have BV tangents, and finally apply the proof above in general.)

Chakerian had earlier [Cha62] given a quite different proof (following Fáry) that $\text{Len} \leq TC$. We close by interpreting that first argument in our framework. Start by observing that in $E^1$, where curvature is quantized, it is obvious that for a closed curve in the unit ball (which is just a segment of length 2)
\[
\text{Len} \leq TC^* = \frac{2}{\pi} TC.
\]
Combining this with Cauchy/Crofton (Lemma 4.6) and our Proposition 4.1 gives immediately $\text{Len} \leq TC$ for curves in the unit disk in $E^2$. With a little care, the same is true for curves that fail to close by some angular holonomy. (The two endpoints are at equal radius, and we do include in the total curvature the angle they make when they are rotated to meet.) Rephrased, the length of a curve $\gamma$ in a unit neighborhood of the cone point on a cone surface of arbitrary cone angle is at most the total (unsigned) geodesic curvature of $\gamma$ in the cone. Finally, given any curve $\gamma$ in the unit ball in $E^d$, Chakerian considers the cone over $\gamma$ from the origin. The length is at most the total curvature in the cone, which is at most the total curvature in space. Rather than trying to consider cones over arbitrary FTC curves, we can prove the theorem for polygons and then take a limit.

7. Distortion

We have already mentioned Gromov’s distortion for an embedded submanifold. For a curve $\gamma \subset E^d$, the distortion is
\[
\delta(\gamma) := \sup_{p \neq q \in \gamma} \delta(p, q), \quad \delta(p, q) := \frac{\text{Len}(p, q)}{|p - q|},
\]
where \( \text{Len}(p, q) \) is the (shorter) arclength distance along \( \gamma \). Here, we discuss some relations between distortion and total curvature; many of these appeared in the first version of [DS04], but later improvements to the main argument there made the discussion of FTC curves unnecessary.

Examples like a steep logarithmic spiral show that arcs of infinite total curvature can have finite distortion, even distortion arbitrarily close to 1. However, there is an easy bound the other way:

**Proposition 7.1.** Any arc of total curvature \( \alpha < \pi \) has distortion at most \( \sec(\alpha/2) \).

**Proof.** First, note that it suffices to prove this for the endpoints of the arc. (If the distortion were realized by some other pair \((p, q)\), we would just replace the original arc by the subarc from \(p\) to \(q\).)

Second, note by that Schur’s Theorem 5.1 we may assume the arc is convex and planar: we replace any given arc by the locally convex planar arc with the same pointwise curvature. Because the total curvature is less than \( \pi \), the planar arc is globally convex in the sense of Theorem 5.1, and the theorem shows the endpoint separation has only decreased.

Now fix points \( p \) and \( q \) in the plane; for any given tangent direction at \( p \), there is a unique triangle \( pxq \) with exterior angle \( \alpha \) at \( x \). Any convex arc of total curvature \( \alpha \) from \( a \) (with the given tangent) to \( c \) lies within this triangle. By the Cauchy/Crofton formula of Lemma 4.6 its length is then at most that of the polygonal arc \( pxq \). Varying now the tangent at \( a \), the locus of points \( x \) is a circle, and it is easy to see that the length is maximized in the symmetric situation, with \( \delta = \sec(\alpha/2) \). \( \square \)

This result might be compared with the bound [KS97, Lemma 5.1] on distortion for a \( C^1 \) arc with bounded curvature density \( \kappa \leq 1 \). By Schur’s Theorem 5.1 such an arc of length \( 2a \leq 2\pi \) can be compared to a circle, and thus has distortion at most \( a/\sin a \).

For any curve \( \gamma \), the distortion is realized either by a pair of distinct points or in a limit as the points approach, simply because \( \gamma \times \gamma \) is compact. In general, the latter case might be quite complicated. On an FTC curve, however, we now show that the distortion between nearby pairs behaves very nicely. Define \( \alpha(r) \leq \pi \) to be the turning angle at the point \( r \in \gamma \), with \( \alpha = 0 \) when \( r \) is not a corner.

**Lemma 7.2.** On an FTC curve \( \gamma \), we have

\[
\lim_{p,q \to r} \delta(p, q) = \sec \left( \frac{\alpha(r)}{2} \right),
\]

with this limit realized by symmetric pairs \((p, q)\) approaching \( r \) from opposite sides.

**Proof.** The existence of one-sided tangent vectors \( T_\pm \) at \( r \) is exactly enough to make this work, since the quotient in the definition of \( \delta(p, q) \) is similar to the difference quotients defining \( T_\pm \). Indeed, near \( r \) the curve looks very much like a pair of rays with turning angle \( \alpha \). Thus the lim sup is the same as the distortion of these rays, which is \( \sec(\alpha/2) \), realized by any pair of points symmetrically spaced about the vertex. \( \square \)

This leads us to define \( \delta(r, r) := \sec(\alpha(r)/2) \), giving a function \( \delta : \gamma \times \gamma \to [1, \infty] \) that is upper semicontinuous. The compactness of \( \gamma \) then immediately gives:
Corollary 7.3. On an FTC curve $\gamma$, there is a pair $(p, q)$ of (not necessarily distinct) points on $\gamma$ which realize the distortion $\delta(\gamma) = \delta(p, q)$. □

Although distortion is not a continuous functional on the space of rectifiable curves, it is lower semicontinuous. A version of the next lemma appeared as [KS97 Lem. 2.2]:

Lemma 7.4. Suppose curves $\gamma_j$ approach a limit $\gamma$ in the sense of Fréchet distance. Then $\delta(\gamma) \leq \lim \delta(\gamma_j)$.

Proof. The distortion for any fixed pair of points is lower semicontinuous because the arclength between them is. (And length will indeed jump down in a limit unless the tangent vectors also converge in a certain sense. See [Ton21 Chap. 2, §29].) The supremum of a family of lower semicontinuous functions is again lower semicontinuous. □

8. A projection theorem of Wienholtz

In [KS97], we made the following conjecture:

Any closed curve $\gamma$ in $\mathbb{E}^d$ of length $L$ has some orthogonal projection to $\mathbb{E}^{d-1}$ of diameter at most $L/\pi$.

This yields an easy new proof of Gromov’s result (see [KS97 DS04]) that a closed curve has distortion at least $\pi/2$, that of a circle. Indeed, consider the height function along $\gamma$ in the direction on some projection of small diameter. For any point $p \in \gamma$, consider the antipodal point $p^*$ halfway around $\gamma$, at arclength $L/2$. Since the height difference between $p$ and $p^*$ is continuous and changes sign, it equals zero for some $(p, p^*)$. The distance between the projected images of these points is at most the diameter, at most $L/\pi$. But since the heights were equal, this distance is the same as their distance $|p - p^*|$ in $\mathbb{E}^d$.

For $d = 2$, we noted that our conjecture follows immediately from Cauchy/Crofton: a closed plane curve of length $L$ has average width $L/\pi$ and thus has width at most $L/\pi$ in some direction. But for higher $d$, the analogs of Cauchy/Crofton give a weaker result. (A curve of length $L$ in $\mathbb{E}^3$, for instance, has projections to $\mathbb{E}^2$ of average length $\pi L/4$, and thus has some planar projection of diameter at most $\pi L/8$.)

In a series of Bonn preprints from 1999, Daniel Wienholtz proved our conjecture and in fact somewhat more: a closed curve in $\mathbb{E}^d$ of length $L$ has some orthogonal projection to $\mathbb{E}^{d-1}$ which lies in a ball of diameter $L/\pi$. Because Wienholtz’s work has unfortunately remained unpublished, we outline his arguments here.

Proposition 8.1. Given any closed curve $\gamma$ in $\mathbb{E}^d$ for $d \geq 3$, there is some slab containing $\gamma$, bounded by parallel hyperplanes $h_1$ and $h_2$, with points $a_i, b_i \in \gamma \cap h_i$ occurring along $\gamma$ in the order $a_1 a_2 b_1 b_2$. (We call the $h_i$ a pair of parallel interleaved bitangent support planes for $\gamma$.)

Proof. Suppose not. Then for any unit vector $v \in \mathbb{S}^{d-1}$, we can divide the circle parameterizing $\gamma$ into two complementary arcs $\alpha(v), \beta(v)$, such that the (global) maximum of the height function in direction $v$ is achieved only (strictly) within $\alpha$, and the minimum
Lemma 8.2. If $\gamma$ is a curve in $\mathbb{R}^{m+n}$ of length $L$, and its projections to $\mathbb{R}^m$ and $\mathbb{R}^n$ have lengths $a$ and $b$, then $a^2 + b^2 \leq L^2$.

Proof. By Proposition 8.4 it suffices to prove this for polygons. Let $a_i, b_i \geq 0$ be the lengths of the two projections of the $i$th edge, and consider the polygonal arc in $\mathbb{R}^2$ with successive edge vectors $(a_i, b_i)$. Its total length is $\sum \sqrt{a_i^2 + b_i^2} = L$, but the distance between its endpoints is $\sqrt{a^2 + b^2}$. □

Theorem 8.3. Any closed curve $\gamma$ in $\mathbb{R}^d$ of length $L$ lies in a cylinder of diameter $L/\pi$.

Proof. As we have noted, the case $d = 2$ follows directly from the Cauchy/Crofton formula (Lemma 8.6), since $\gamma$ has width at most $L/\pi$ in some direction. We prove the general case by induction. So given $\gamma$ in $\mathbb{R}^d$, find two parallel interleaved bitangent support planes with normal $v$, as in Proposition 8.4. Let $\tau$ be the distance between these planes, the thickness of the slab in which $\gamma$ lies. Project $\gamma$ to a curve $\gamma_1$ in the plane orthogonal to $v$, and call its length $L$. By induction, $\gamma_1$ lies in a cylinder of radius $L/2\pi$. Clearly, $\gamma$ lies in a parallel cylinder of radius $r$, centered in the middle of the slab, where $r^2 = (L/2\pi)^2 + (\tau/2)^2$. So we need to show that $(L/2\pi)^2 + (\tau/2)^2 \leq (L/2\pi)^2$, i.e., $L^2 + \pi^2\tau^2 \leq L^2$. In fact, since the length of the projection of $\gamma$ to the one-dimensional space in direction $v$ is at least $4\tau$, Lemma 8.2 gives us $L^2 + 4\tau^2 \leq L^2$, which is better than we needed. □

If we are willing to settle for a slightly worse bound in the original conjecture, Wienholtz also shows that we can project in a known direction:

Proposition 8.4. Suppose a closed curve $\gamma \subset \mathbb{R}^d$ has length $L$, and $p_1, p_2 \in \gamma$ are points realizing its diameter. Then its projection $\gamma_1$ to the plane orthogonal to $p_1 - p_2$ has diameter at most $L/2\sqrt{2}$. This estimate is sharp for a square.

Proof. Let $a_1, a_2$ be the preimages of a pair of points realizing the diameter $D$ of the projected curve $\gamma_1$. We may assume $\gamma$ is a quadrilateral with vertices $a_1, p_1, a_2, p_2$, since any other curve would be longer. (This reduces the problem to some affine $\mathbb{R}^3$ containing these four points.)

Along $\gamma$, first suppose the $a_i$ are interleaved with the $p_i$, so that the quadrilateral is $a_1p_1a_2p_2$, as in Figure 4 (left). We can now reduce to $\mathbb{R}^2$: rotating $a_1, a_2$ independently about the line $\overline{p_1p_2}$ fixes the length, but maximizes the diameter $D$ when the points are coplanar, with $a_1$ on opposite sides of the line. Now let $R$ be the reflection across this line $\overline{p_1p_2}$, and consider the vector $R(p_1 - a_1) + (a_1 - p_2) + (p_1 - a_2) + R(a_2 - p_2)$.
Figure 4. These quadrilaterals show the sharp bounds for the two cases in Proposition 8.4. The original diameter is \( p_1 p_2 \) in both cases, and the projected diameter is \( D \) (shown at the bottom). The square \( a_1 p_1 a_2 p_2 \) (left) has length \( 2\sqrt{2}D \). The bowtie \( a_1 a_2 p_1 p_2 \) (right) has length \( 2D/\sin \theta + D/\cos \theta \), minimized at about 3.33 for \( \theta \approx 76^\circ \) as shown, but in any case clearly greater than \( 3D \).

Its length is at most \( L \), but its component in the direction \( p_1 - p_2 \) is at least \( 2D \), as is its component in the perpendicular direction. Therefore \( L \geq 2\sqrt{2}D \).

Otherwise, write the quadrilateral as \( a_1 a_2 p_1 p_2 \) as in Figure 4 (right), and let \( \theta \) be the angle between the vectors \( a_2 - a_1 \) and \( p_1 - p_2 \). Suppose (after rescaling) that \( 1 = |a_2 - a_1| \leq |p_1 - p_2| \). Then the projected diameter is \( D = \sin \theta \). By the triangle inequality, the two remaining sides have lengths summing to at least

\[
|p_1 - a_2 + a_1 - p_2| \geq 2 \sin(\theta/2).
\]

(Equality here holds for instance when the quadrilateral is a symmetric bowtie.) Thus

\[
L/D \geq 2 + 2 \sin(\theta/2)/\sin \theta = 2 \frac{\sin \theta}{\sin \theta} + \frac{1}{\cos(\theta/2)} \geq 2 + 1 > 2\sqrt{2}. \quad \square
\]

9. Curvature density

We have found a setting which treats polygons and smooth curves in a unified way as two special cases of the more general class of finite total curvature curves. Many standard results on curvature, like Schur’s comparison theorem, work nicely in this class.

However, there is some ambiguity in how to measure curvature at a corner, reflected in our quantities \( TC \) and \( TC^* \). A corner of turning angle \( \theta \) is counted either as \( \theta \) or as \( 2 \sin(\theta/2) \), respectively.

At first, \( TC \) seems more natural: if we round off a corner into any convex planar arc, its curvature is \( \theta \). And the nice behavior of \( TC \) under projection (Proposition 4.1) explains why it is the right quantity for results like the Fáry/Milnor theorem.
But from a variational point of view, $TC^*$, which measures the mass of the curvature force $K$ at a corner, is sometimes most natural. Proposition 6.1 is an example of a result whose sharp form involves $TC^*$. An arbitrary rounding of a corner, whether or not it is convex or planar, will have the same value of

$$K = T_+ - T_- = 2 \sin(\theta/2) N.$$  

When we do choose a smooth, convex, planar rounding, we note that

$$\int |\kappa N| \, ds = \int \kappa \, ds = \theta, \quad \text{while} \quad \left| \int \kappa N \, ds \right| = 2 \sin(\theta/2).$$

For us, the curvature of an FTC curve has been given by the measure $K$. For a polygon, of course, this vanishes along the edges and has an atom at each vertex. But sometimes we wish to view a polygon as an approximation to a smooth curve and thus spread this curvature out into a smooth density. For instance the elastic energy, measured as $\int \kappa^2 \, ds$ for smooth curves, blows up if measured directly on a polygon; instead of squaring $K$, we should find a smoothed curvature density $\kappa$ and square that.

For simplicity, we will consider here only the case of equilateral polygons in $\mathbb{R}^d$, where each edge has length 1. To each vertex $v$, we allocate the neighborhood $N_v$ consisting of the nearer halves of the two edges incident to $v$, with total length 1. Depending on whether we are thinking of $TC$ or $TC^*$, we see total curvature either $\theta$ or $2 \sin(\theta/2)$ at $v$, and so it would be natural to use either $\theta$ or $2 \sin(\theta/2)$ as the curvature density $\kappa$ along $N_v$. The latter has also a geometric interpretation: if $uvw$ are consecutive vertices of the equilateral polygon, then the circle through these three points has curvature density $\kappa = 2 \sin(\theta/2)$.

However, from a number of points of view, there is another even better measure of curvature density. Essentially, what we have ignored above is that when we round off the corners of a polygon to make a smooth curve, we tend to make the curve shorter. Thus, while $\theta$ or $2 \sin(\theta/2)$ might be the correct total curvature for a neighborhood of $v$, perhaps it should get averaged over a length less than 1.

Let us consider a particularly simple smoothing, which gives a $C^{1,1}$ and piecewise circular curve. Given a polygon $P$, replace the neighborhood $N_v$ of each vertex $v$ by an “inscribed” circular arc, tangent at each endpoint. This arc turns a total angle $\theta$, but since it is shorter than $N_v$, its curvature density is $\kappa = 2 \tan(\theta/2)$.

As a simple example, suppose $P$ is a regular $n$-gon in the plane with edges of length 1 and turning angles $2\pi/n$. Its inscribed circle has curvature density $2 \tan(\theta/2)$, while its circumscribed circle has (smaller) curvature density $2 \sin(\theta/2)$). (Of course, the value $\theta$ lies between these two.)

Using the formula $2 \tan(\theta/2)$ for the curvature density along a polygon $P$ has certain advantages. For instance if $\kappa(P) \leq C$ then we know there is a nearby $C^{1,1}$ curve (the smoothing by inscribed circular arcs we used above) with this same curvature bound. The fact that $2 \tan(\theta/2)$ blows up for $\theta = \pi$ reflects the fact that a polygonal corner of turning angle $\pi$ is really a cusp. For instance, when the turning angle of a polygon in the plane passes through $\pi$, the total signed curvature (or turning number) jumps. For a smooth curve, such a jump cannot happen, unless the bending energy blows up because...
of a cusp. A bending energy for polygons based on \( \kappa = 2 \tan(\theta/2) \) will similarly blow up if we try to change the turning number.

Of course, the bending energy for curves is one conserved quantity for the for the integrable system related to the Hasimoto or smoke-ring flow. In the theory of discrete integrable systems, it seems clear due to work of Hoffmann and others that \( 2 \tan(\theta/2) \) is the right notion of curvature density for equilateral polygons. See [HK04, Hof08].

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