PERIODIC GEODESICS AND GEOMETRY OF COMPACT LORENTZIAN MANIFOLDS WITH A KILLING VECTOR FIELD

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ABSTRACT. We study the geometry and the periodic geodesics of a compact Lorentzian manifold that has a Killing vector field which is timelike somewhere. Using a compactness argument for subgroups of the isometry group, we prove the existence of one timelike non self-intersecting periodic geodesic. If the Killing vector field is never vanishing, then there are at least two distinct periodic geodesics; as a special case, compact stationary manifolds have at least two periodic timelike geodesics. We also discuss some properties of the topology of such manifolds. In particular, we show that a compact manifold $M$ admits a Lorentzian metric with a never vanishing Killing vector field which is timelike somewhere if and only if $M$ admits a smooth circle action without fixed points.

1. INTRODUCTION

A classical subject in Geometry is the question of existence and multiplicity of periodic geodesics. A well known result in differential geometry establishes the existence of a periodic geodesic in every compact Riemannian manifold; a Lorentzian analog of this result is still an open problem in its full generality. However, there are remarkable partial results in this direction. An earlier result by Tipler (see [34]) gives the existence of one periodic timelike geodesic in compact Lorentzian manifolds that admit a regular covering which has a compact Cauchy surface. Recently, this result has been extended by Guediri [14, 15, 18] and Sánchez [31] to the case that the Cauchy surface in the covering is not necessarily compact, but assuming certain hypotheses on the group of deck transformations. The existence of a periodic timelike geodesic has been established also by Galloway in [11], where he proves the existence of a longest periodic timelike curve, which is necessarily a geodesic, in each stable free timelike homotopy class. In [12], the same author proves the existence of a causal (i.e., nonspacelike) periodic geodesic in any compact two-dimensional Lorentzian manifold. More recently, Guediri has proved that compact flat spacetimes contain a causal periodic geodesic [14], and that such spacetimes contain a periodic timelike geodesic if and only if the fundamental group of the underlying manifold contains a nontrivial timelike translation [15]. Non existence results for periodic causal geodesics are also available, see [12,16,17].

In the particular case of static compact Lorentzian manifolds, using variational methods it has been established the existence of a periodic geodesic in each free homotopy class corresponding to an element of the fundamental group having finite conjugacy class, [7]. This result has been generalized to any free homotopy class containing a periodic timelike curve in [31], obtaining in particular that any compact static spacetime admits a periodic timelike geodesic. As to the stationary case, there exist in literature some previous results when the spacetime admits a standard stationary expression and, as a consequence, it is

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never compact (see [4, 24]). As suggested by the authors in [7, 18, 31], an interesting open question would be to determine if existence results for periodic geodesics also hold for any stationary compact Lorentzian manifold. Moreover, in general there does not exist a globally hyperbolic covering for this class of manifolds (see [31] pag. 23, observe also that, unlike the static case, compact stationary Lorentzian manifolds may be simply connected) and a different approach from that of [7, 14, 15, 18, 31, 34] is needed.

In this paper, we answer positively to the above question by using the strong relation between Killing vector fields and geodesics, and a compactness criterion for subgroups of the isometry group of a Lorentzian manifold. More precisely, we prove the following:

**Theorem.** Let $(M, g)$ be a compact Lorentzian manifold with $\dim(M) \geq 2$ that admits a Killing vector field $K$ that is timelike somewhere. Then there is some non trivial periodic non self-intersecting timelike geodesic in $(M, g)$. If either one of the following two conditions is satisfied, then there are at least two non trivial periodic non self-intersecting geodesics in $M$:

1. $\max_{q \in M} g(K_q, K_q) \neq 0$;
2. $K$ is never vanishing.

When either condition is satisfied, if in addition $K$ has at most one periodic integral line, then there are infinitely many geometrically distinct non trivial periodic non self-intersecting geodesics in $(M, g)$.

Recall that two periodic geodesics $c_1, c_2 : \mathbb{R} \to M$ are geometrically distinct if the sets $c_1(\mathbb{R})$ and $c_2(\mathbb{R})$ are distinct, i.e., if one cannot be obtained as an iteration of the other. The proof of our theorem employs Lie group techniques; the key result is a compactness criterion for subgroups of the isometry group of a compact Lorentzian manifold (Proposition 2.6). Using this criterion, one shows that a compact Lorentzian manifold with a Killing vector field that is timelike somewhere also has a Killing vector field $K$ that is timelike somewhere, and all of whose integral curves are periodic (this is called a closed Killing vector field in the paper). In this situation, the periodic geodesics are given by those integral curves of $K$ that pass through critical points for the function $f = g(K, K)$ on $M$ (Lemma 2.3). The existence of a never vanishing closed Killing vector field also gives us information on the topology of this manifold, that can be characterized as a compact manifold admitting a smooth action of the circle $S^1$ all of whose orbits have finite stabilizer or, equivalently, without fixed points (Section 3). Fixed point free actions of the circle on low dimensional compact (simply connected) manifolds are classified, see [32, 33] for actions in dimension 3, [9, 10] in dimension 4 and [22] in dimension 5.

The orbit space $M/S^1$ may fail to be a manifold when the action is not free (see Remark 3.2), but in general it has the structure of a compact orbifold. Using equivariant Ljusternik–Schnirelman category theory, one obtains a slightly better estimate on the number of integral lines of $K$ that are geodesics (Corollary 3.3), such number is greater than or equal to the equivariant category $\text{cat}_{S^1}(M)$, which is greater than or equal to 2. If one assumes that $(M, g)$ has a non closed Killing vector field that is timelike somewhere, and that $\max g(K, K)$ is non zero, then we prove that $(M, g)$ has infinitely many closed Killing vector fields that are timelike somewhere, each of which gives rise to (at least) two periodic geodesics. Under the further assumption that the Killing vector field has at most one periodic integral curve, then we prove that the family of periodic geodesics produced in this way contains indeed infinitely many geometrically distinct periodic geodesics. Explicit examples of compact stationary Lorentzian manifolds with a timelike Killing vector field all of whose integral lines but one are non periodic are given in Subsection 3.2.

All the results of this paper apply in particular to compact stationary manifolds, i.e., Lorentzian manifolds $(M, g)$ that admit an everywhere timelike Killing vector field $K$. It should be observed that, in this case, there is an alternative approach to the problem which uses an auxiliary Riemannian metric $g_R$ naturally associated to $g$ and $K$ (see [24]).
A very interesting open question is whether a simply connected compact stationary Lorentzian manifold has compact isometry group. The answer is yes in the real analytic case (see [8]). In Subsection 3.3, we discuss briefly this issue, giving some partial results towards a positive answer to the compactness question. The question is studied in full generality in [27].

Finally, in Section 4 we present a few related results on the existence of periodic geodesics in compact semi-Riemannian manifolds, obtained using the techniques discussed in the paper.

2. Proof of the Theorem

2.1. Preliminaries on Killing vector fields. Semi-Riemannian manifolds are assumed to be connected with dimension \( n \geq 2 \). In the case of Lorentzian manifolds, we assume they are time-oriented (thus, time-orientable). Our notation and conventions follow the standard ones in Lorentzian Geometry, see [3, 26, 28].

Recall that a Killing vector field \( K \) in a semi-Riemannian manifold \((M, g)\) is a vector field whose flow preserves \( g \), or, equivalently, such that \( \nabla K \) is skew-symmetric, i.e., the bilinear map \( T_pM \times T_pM \ni (v, w) \mapsto g(\nabla_v K, w) \in \mathbb{R} \) is skew-symmetric for all \( p \in M \). Here \( \nabla \) is the Levi-Civita connection of \( g \). If \( K \) is a Killing vector field, then the function \( f = g(K, K) \) is constant along each integral curve of \( K \), namely, \( K(f) = 2g(\nabla_K K, K) = 0 \). The Lie bracket of Killing vector fields on \( M \) is a Killing vector field, so that the space \( \text{Kill}(M, g) \) of all Killing vector fields on \( M \) is a Lie algebra. Given a diffeomorphism \( \Phi : M \to M \) and a vector field \( K \) on \( M \), the push-forward of \( K \) by \( \Phi \) is the vector field \( \Phi_\ast(K) \) on \( M \) defined by \( \Phi_\ast(K)_p = d\Phi(\Phi^{-1}(p))K_{\Phi^{-1}(p)} \) for all \( p \in M \).

Let \((M, g)\) be a semi-Riemannian manifold, we will denote by \( \text{Iso}(M, g) \) the Lie group of all isometries of \((M, g)\), and by \( \mathfrak{iso}(M, g) \) its Lie algebra. It is well known that the isometry group of a compact Riemannian manifold is a compact Lie group. The natural action of \( \text{Iso}(M, g) \) on \( M \) is smooth, and the non trivial elements of \( \text{Iso}(M, g) \) cannot act trivially on a non empty open subset of \( M \) (see [26] Proposition 3.62).

A correspondence between elements of \( \mathfrak{iso}(M, g) \) and Killing vector fields on \( M \) is obtained as follows. For \( p \in M \) denote by \( \beta_p : \text{Iso}(M, g) \to M \) the smooth map \( \beta_p(\Phi) = \Phi(p) \). A Lie algebra anti-isomorphism \( \mathfrak{iso}(M, g) \ni \xi \mapsto K^\xi \in \text{Kill}(M, g) \) is obtained by setting:

\[
K^\xi_p = d\beta_p(1)\xi, \quad p \in M.
\]

For \( \Phi \in \text{Iso}(M, g) \) consider the Lie group isomorphism \( I_\Phi : \text{Iso}(M, g) \to \text{Iso}(M, g) \) given by \( I_\Phi(\Psi) = \Phi \circ \Psi \circ \Phi^{-1} \), and denote by \( \text{Ad}_\Phi : \mathfrak{iso}(M, g) \to \mathfrak{iso}(M, g) \) its differential at the identity. Using these notations, one has the following immediate equality:

\[
\Phi \circ \beta_{\Phi^{-1}(p)} = \beta_p \circ I_\Phi,
\]

for all \( \Phi \in \text{Iso}(M, g) \) and all \( p \in M \).

Lemma 2.1. For all \( \Phi \in \text{Iso}(M, g) \) and all \( \xi \in \mathfrak{iso}(M, g) \), the following formula holds:

\[
\Phi_\ast K^\xi = K^{\text{Ad}_\Phi(\xi)}.
\]

Proof. A direct computation:

\[
(\Phi_\ast K^\xi)_p = d\Phi(\Phi^{-1}(p))K^\xi_{\Phi^{-1}(p)} = d\Phi(\Phi^{-1}(p)) d\beta_{\Phi^{-1}(p)}(1)\xi
\]

\[
= d(\Phi \circ \beta_{\Phi^{-1}(p)})(1)\xi \quad \text{by } \text{Eq.} \]

\[
= d\beta_p(1) dI_\Phi(1)\xi = d\beta_p(1)\xi = K^\xi_p.
\]

If \( K \) is a Killing vector field for \((M, g)\) corresponding to the element \( \xi \in \mathfrak{iso}(M, g) \), we will say that \( K \) is closed if the 1-parameter group of isometries \( \{ \exp(t\xi) : t \in \mathbb{R} \} \) of \( \text{Iso}(M, g) \) generated by \( \xi \) is closed. Here \( \exp \) is the exponential map of the Lie group \( \text{Iso}(M, g) \). The integral curves of \( K \) through some \( p \in M \) of \( K \) are given by \( t \mapsto \exp(t\xi) \cdot p \) (see [26] Lemma 34, p. 256)), thus, if \( K \) is a closed Killing vector field, then its integral
curves in $M$ are either circles or points. The converse of this statement is also true when $(M, g)$ is a compact Riemannian manifold.

**Lemma 2.2.** Let $(M, g)$ be a compact Riemannian manifold and let $K$ be a Killing vector field all of whose integral curves are periodic (i.e., circles or points). Then $K$ is closed.

**Proof.** Let $G$ be the 1-parameter group of $\mathfrak{g}$ and assume by absurd that $G$ is not closed. Denote by $\overline{G}$ its closure in $\text{Iso}(M, g)$, which is a compact abelian Lie group, thus $\overline{G}$ is a torus (see for instance [2, Theorem 10.4]) of dimension greater than or equal to 2 (it cannot be $S^1$, because there is no 1-dimensional proper subgroup of $S^1$). We claim that the orbits of the actions of $G$ and of $\overline{G}$ in $M$ coincide. Namely, if $p \in M$, then clearly $Gp \subset \overline{G}p$; on the other hand, $Gp$ is dense in $\overline{G}p$, because $G$ is dense in $\overline{G}$. But $Gp$ (and obviously $\overline{G}p$) is a closed subset of $M$, because it is an integral line of $K$, which is a circle or a point, and this proves that $Gp = \overline{G}p$. Since $\overline{G}p$ is a circle or a point for all $p$, it follows that the regular isotropy $H \subset \overline{G}$ of the action of the compact group $\overline{G}$ on $M$ is non trivial, for otherwise $\overline{G}$ will be diffeomorphic to a circle or a point. Since $\overline{G}$ is abelian, such regular isotropy subgroup is normal in $\overline{G}$, which implies that every point of the regular orbits is fixed by the elements of $H$. This is a contradiction, because the union of the regular orbits form a dense open subset of $M$ (see for instance the principal orbit theorem in [5]), and no nontrivial element of $\text{Iso}(M, g)$ acts trivially on a non empty open subset of $M$. □

### 2.2. Killing vector fields and geodesics

We will use the following result in [20, Chapter VI, Proposition 5.7, page 252], whose simple proof is reproduced here for the reader’s convenience.

**Lemma 2.3.** Let $(M, g)$ be a semi-Riemannian manifold, and let $K$ be a Killing vector field on $M$. Let $p_0 \in M$ be a critical point for the function $f(p) = g(K_p, K_p)$; then, the integral line of $K$ through $p_0$ is a geodesic.

**Proof.** Since $f$ is preserved by the flow of the Killing vector $K$, it follows that the integral line of $K$ through $p_0$ consists entirely of critical points of $f$. Thus, it suffices to show that $(\nabla_K K)|_{p_0} = 0$. Since $p_0$ is a critical point of $f$ and $K$ is Killing, then for all $v \in T_{p_0}M$ it is:

$$0 = v(f) = 2g(\nabla_v K, K) = -2g(\nabla_K K, v),$$

i.e., $(\nabla_K K)|_{p_0} = 0$, which concludes the proof. □

**Corollary 2.4.** Let $(M, g)$ be a compact semi-Riemannian manifold with $\dim(M) \geq 2$ that admits a non trivial Killing vector field $K$ all of whose integral lines are periodic. Then, there is some non trivial periodic non self-intersecting geodesic in $M$. If either one of the following two conditions is satisfied, then there are at least two non trivial periodic non self-intersecting geodesics in $M$:

- (a) $\min g(K, K)$ and $\max g(K, K)$ are both non zero;
- (b) $K$ is never vanishing.

**Proof.** Let $K$ be a Killing vector field as in the assumption. By Lemma 2.3 the integral curves of $K$ through critical points of the function $f(p) = g(K_p, K_p)$ are geodesics. Evidently they are periodic; periodic integral curves of a vector field (considered with minimal period) are non self-intersecting. If $f$ is identically zero, then every point in $M$ is critical for $f$, and since $K$ is non trivial then there are infinitely many non trivial integral curves of $K$ that are geodesics. If $f$ is not identically zero, then either the minimum or the maximum of $f$ are non zero, and the corresponding integral curve of $K$ is a non trivial periodic geodesic.

If either (a) or (b) holds and $f$ is not identically zero, then the integral curves of $K$ through a minimum and a maximum of $f$ are non trivial distinct periodic geodesics. □
Proposition 2.5. Let \((M, g)\) be a compact semi-Riemannian manifold with \(\dim(M) \geq 2\) that admits a Killing vector field \(K\) that generates a precompact 1-parameter subgroup of \(\text{Iso}(M, g)\). Then \(K\) can be approximated by closed Killing vector fields that generate precompact 1-parameter subgroups. In particular, there is some non-trivial periodic non self-intersecting geodesic in \(M\). If \(K\) satisfies either (a) or (b) of Corollary 2.4 then there are at least two non trivial periodic non self-intersecting geodesics in \(M\).

Proof. Let \(\xi \in \text{Iso}(M, g)\) be the element corresponding to \(K\), and consider the precompact 1-parameter subgroup \(G = \{ \exp(t\xi) : t \in \mathbb{R} \}\) of isometries generated by \(K\). Its closure \(\overline{G}\) is a Lie subgroup of \(\text{Iso}(M, g)\), which is abelian and compact, so it is a torus; denote by \(g\) its (abelian) Lie algebra. Then, \(\xi\) can be approximated by a sequence \(\xi_n \in g\) of vectors generating a closed 1-parameter subgroup of \(\overline{G} \subset \text{Iso}(M, g)\). The Killing vector fields \(K^n = K^{\pm n}\) are closed; using the relation \((2.1)\) and the compactness of \(M\), one sees easily that \(\lim_{n \to \infty} K^n_p = K_p\) uniformly in \(p \in M\). In particular, if \(K\) satisfies either (a) or (b) of Corollary 2.4 for \(n\) large enough also \(K^n\) does, and so, the thesis directly follows by applying Corollary 2.4 to \(K^n\).

2.3. Proof of Theorem.

Proposition 2.6. Let \((M, g)\) be a compact Lorentzian manifold and let \(K\) be a Killing vector field on \(M\) which is timelike at some point. Given \(H \subset \text{Iso}(M, g)\), assume that for all \(\Phi \in H\) and all \(q \in M\) it is \(d\Phi_q(K_q) = \pm K_{\Phi(q)}\). Then \(H\) is precompact. In particular, the 1-parameter subgroup of isometries generated by \(K\) is precompact.

Proof. Let \(p \in M\) be such that \(g(K_p, K_p) < 0\). Consider the compact subsets of \(TM\) given by:

\[
V = \left\{ \pm K_q : q \in M, \text{ is such that } g(K_q, K_q) = g(K_p, K_p) \right\},
\]

and

\[
V^\perp = \left\{ v \in K_q^\perp : q \in M, \text{ such that } g(K_q, K_q) = g(K_p, K_p), g(v, v) = 1 \right\}.
\]

Consider an orthogonal basis \(b = (v_1, \ldots, v_n)\) of \(T_p M\) with \(v_1 = K_p\) and \(g(v_i, v_j) = \delta_{ij}\) for \(i, j \in \{2, \ldots, n\}\). Now recall that the subset \(H \subset \text{Iso}(M, g)\) can be identified with the \(H\)-orbit of the basis \(b\) by the action of \(\text{Iso}(M, g)\) on the frame bundle \(F(M)\) (see [21 Theorem 1.2, Theorem 1.3]). We claim that every vector of a basis of the \(H\)-orbit belongs to the compact subset \(V \cup V^\perp\), and this implies that the \(H\)-orbit of \(b\) is precompact in the frame bundle \(F(M)\). The claim follows easily from the assumption that \(d\Phi_p(K_p) = \pm K_{\Phi(p)}\) for all \(\Phi \in H\) and all \(p \in M\).

The conclusion applies in particular to the 1-parameter subgroup \(H \subset \text{Iso}(M, g)\) of isometries generated by \(K\); for \(\Phi \in H\), we have \(K = \Phi(K)\).

Proof of Theorem. From Proposition 2.6 the 1-parameter subgroup of isometries generated by \(K\) is precompact. Therefore, taking into account that \(K\) is timelike somewhere, the first and second assertions follow directly from Proposition 2.5. Note that when \(K\) is timelike somewhere, then automatically min \(g(K, K)\) is non-zero.

If, in addition, \(K\) has at most one periodic integral curve, then the closed Killing vector fields \(K^n\) from Proposition 2.5 that approximate \(K\) must be \(K^n \neq K\) for all \(n\) (recall that \(\dim(M) \geq 2\)), and we can therefore assume that the \(K^n\)’s are pairwise distinct. For each \(n\), \(K^n\) determines at least two non-trivial periodic non self-intersecting geodesics. The family of all such periodic geodesics for all \(n\) cannot be finite. Namely, if it were, then we could find a subsequence of the \(K^n\) with the property that all the elements of the subsequence have two distinct fixed curves \(\gamma_1\) and \(\gamma_2\) as common periodic integral lines. But then, the limit \(K\) would also have \(\gamma_1\) and \(\gamma_2\) as periodic integral lines, which gives a contradiction. This concludes the proof.
Let \((M, g)\) be a stationary Lorentzian manifold and \(K\) the Killing timelike vector field. Consider the auxiliary Riemannian metric \(g_R\) defined using \(g\) and \(K\) by:

\[
g_R(v, w) = g(v, w) - 2g(v, K_p)g(w, K_p)g(K_p, K_p)^{-1},
\]

for all \(p \in M\) and \(v, w \in T_p M\). The Lorentzian metric \(g\) is given in terms of \(g_R\) and \(K\) by a similar formula:

\[
g(v, w) = g_R(v, w) - 2g_R(v, K_p)g_R(w, K_p)g_R(K_p, K_p)^{-1},
\]

for all \(p \in M\) and \(v, w \in T_p M\). Since the flow of \(K\) preserves the metric \(g\) and the field \(K\) itself, then using \(g_R\) one sees immediately that the flow of \(K\) preserves \(g_R\), i.e., \(K\) is a Killing vector field also for the metric \(g_R\). Using this fact, we get the following immediate corollary of Lemma 2.2.

**Corollary 2.7.** Let \((M, g)\) be a compact stationary Lorentzian manifold, and let \(K\) be a timelike Killing vector field on \(M\). Then, \(K\) is closed if and only if all its integral lines are periodic. \(\square\)

**Remark 2.8.** Even when the two geodesics \(\gamma_1\) and \(\gamma_2\) determined in our main theorem (as integral lines of a closed vector field) are timelike, it does not necessarily follow that they belong to the same free (timelike) homotopy class, see Remark 3.2, but rather that they have some iterate that belong to the same free (timelike) homotopy class.

### 3. On the topological structure of a compact stationary Lorentzian manifold

#### 3.1. Fibration associated to a closed Killing vector field

A compact manifold \(M\) will be called a generalized Seifert fibered space if it admits a smooth action of the circle \(S^1\) without fixed points or, equivalently, with finite isotropy. The orbits of a fixed point free action of \(S^1\), that are diffeomorphic to \(S^1\), are called the fibers of the fibered space. Low dimensional generalized Seifert fibered spaces are classified, see [9, 10, 22, 32, 33]. By standard results on group actions, probably going back to Seifert, the orbit space of a smooth action of a compact Lie group on a compact manifold having finite isotropy has the structure of a compact orbifold (see the Appendix of E. Salem in [25] for details on orbifolds; the book contains also a more general result on the orbifold structure of orbit spaces in the context of Riemannian foliations).

Using the results of Section 2, it is easy to prove the following:

**Proposition 3.1.** A compact manifold \(M\) admits a Lorentzian metric tensor with a never vanishing Killing vector field that is timelike somewhere if and only if it is diffeomorphic to a generalized Seifert fibered space. In this case, the metric can be chosen to have a timelike Killing vector field.

**Proof.** As shown in the proof of the Theorem, a Lorentzian manifold as in the hypotheses above has a never vanishing closed Killing vector field \(K\). The one-parameter group of isometries generated by such a Killing field gives a smooth action of \(S^1\) without fixed points; \(K\) is tangent to the fibers of this action. Conversely, given a smooth action of \(S^1\) on \(M\) without fixed points, by a standard averaging argument one can find a Riemannian metric tensor \(g_R\) which makes such action isometric, i.e., the infinitesimal generator \(K\) of this action is \(g_R\)-Killing (see for instance [19]). Consider the Lorentzian metric tensor \(g\) defined as in (2.4); then \(K\) is timelike and \(g\)-Killing. \(\square\)

**Remark 3.2.** Given a free action of \(S^1\) on a compact manifold \(M\), then the orbit space \(M/S^1\) is a smooth manifold (see for instance [6] Theorem 23.4) or [20] Theorem 4.3). We observe however that in general the quotient space \(M_0 = M/S^1\) is not a manifold. As an example, consider \(M\) to be the Lorentzian Klein bottle obtained as the quotient of \(\mathbb{R}^2\) endowed with the Minowski metric \(dx^2 - dt^2\) by the action of the group generated by the
isometries \((x, t) \mapsto (x + 1, t)\) and \((x, t) \mapsto (1 - x, t + 1)\). The vector field \(K = \frac{\partial}{\partial y}\) on \(M\) is timelike and Killing; all its integral lines are periodic. It is easily seen that in this case the \(S^1\)-action induced by the flow of \(K\) has exactly two exceptional orbits, and that the orbit space \(M/S^1\) is homeomorphic to the closed interval \([0, 1/2]\). Note that the periodic integral lines of \(K\) corresponding to the two exceptional orbits do not belong to the same free homotopy class of the other integral lines of \(K\), but rather their two-fold iteration is in the free homotopy class of the other integral lines of \(K\).

As a corollary of Proposition 3.1 we get a somewhat better estimate on the number of periodic geodesics given in terms of the Ljusternik–Schnirelman category. Recall that the Ljusternik–Schnirelman category (shortly, LS category) \(\text{cat}(\mathcal{X})\) of a topological space \(\mathcal{X}\) is the cardinality (possibly infinite) of a minimal family of closed contractible subsets of \(\mathcal{X}\) whose union covers \(\mathcal{X}\). If \(\mathcal{X}\) is \(G\)-space, i.e., a topological space on which a compact group \(G\) is acting continuously, then one can define the equivariant notion of Ljusternik–Schnirelman \(G\)-category \(\text{cat}_G(\mathcal{X})\) (see for instance [23]). A homotopy \(H : U \times [0, 1] \to \mathcal{X}\) of an open \(G\)-invariant set \(U \subset \mathcal{X}\) is called \(G\)-equivariant if \(gH(x, t) = H(gx, t)\) for any \(g \in G\), \(x \in U\) and \(t \in [0, 1]\). The set \(U\) is \(G\)-categorical if there is a \(G\)-homotopy \(H\) with \(H(\cdot, 0)\) the identity, and \(H(\cdot, 1)\) maps \(U\) to a single orbit. The equivariant category \(\text{cat}_G(\mathcal{X})\) is the cardinality of a minimal family of \(G\)-categorical open sets whose union covers \(\mathcal{X}\).

If \(G\) is a compact Lie group, \(\mathcal{X}\) is a smooth \(G\)-manifold, and \(h : \mathcal{X} \to \mathbb{R}\) is a smooth function which is \(G\)-invariant, then \(h\) has at least \(\text{cat}_G(\mathcal{X})\) distinct critical \(G\)-orbits (see [23] Th. 3.2).

Corollary 3.3. Let \((M, g)\) be a compact Lorentzian manifold with a never vanishing closed Killing vector field on \(M\). Consider the \(S^1\)-action on \(M\) determined by \(K\). Then, there are at least \(\text{cat}_{S^1}(M)\) distinct periodic non self-intersecting geodesics in \(M\).

Proof. The function \(f : M \to \mathbb{R}\) defined by \(f(p) = g(K_p, K_p)\) is constant on the orbits of \(G = S^1\), thus it has at least \(\text{cat}_{S^1}(M)\) critical orbits. Hence, the proof follows by observing that distinct critical \(G\)-orbits of \(f\) in \(M\) correspond to distinct non self-intersecting periodic geodesics.

In Corollary 3.3 note that \(\text{cat}_{S^1}(M) \geq 2\). Namely, if it were \(\text{cat}_{S^1}(M) = 1\), then \(M\) would be (equivariantly) homotopic to an orbit of \(S^1\), which is diffeomorphic to \(S^1\). But, no compact manifold of dimension greater than or equal to 2 is homotopic to \(S^1\). Observe also that in general the equivariant LS category \(\text{cat}_{S^1}(M)\) is greater than or equal to the LS category \(\text{cat}(M/S^1)\) of the quotient space \(M/S^1\). The Klein bottle in Remark 3.2 provides an example where such inequality is strict: here the quotient space \(M/S^1\) is contractible, and thus \(\text{cat}(M/S^1) = 1\), while it is easily computed \(\text{cat}_{S^1}(M) = 2\).

3.2. Some examples. It is easy to produce examples of compact stationary Lorentzian manifolds with a timelike Killing vector field having all of its integral curves periodic or having no periodic integral curve at all. In next examples we show two different constructions to produce compact stationary manifolds with a timelike Killing vector field having integral curves of mixed type.

Example 1. Consider the following smooth isometric action of \(T^2 = S^1 \times S^1\) on the round 3-sphere \(S^3\). Set \(S^3 = \{(z, w) \in \mathbb{C}^2 : |z|^2 + |w|^2 = 1\}\), and for \((\lambda_1, \lambda_2) \in S^1 \times S^1\), \((z, w) \in S^3\), define \((\lambda_1, \lambda_2) \cdot (z, w) = (\lambda_1 z, \lambda_2 w) \in S^3\). The circles \(c_1 = S^1 \times \{0\} \subset S^3\) and \(c_2 = \{0\} \times S^1 \subset S^3\) are orbits of this action, and they have non trivial isotropy. Every other orbit is regular, and it is diffeomorphic to \(T^2\). Now consider a 1-parameter subgroup \(G \subset T^2\) which is dense in \(T^2\) and the restriction of the action of \(T^2\) to \(G\). The singular orbits \(c_1\) and \(c_2\) are also orbits of \(G\), since the projections \(\pi_1, \pi_2 : G \to S^1\) given by the restrictions of the projections \(\pi_1, \pi_2 : S^1 \times S^1 \to S^1\) onto the first and the second factor respectively, are surjective. All the other orbits of \(G\) are clearly not closed. Consider the
Killing vector field $K$ for the Riemannian metric $g_R$ corresponding to the isometric action of $G$, and define a Lorentzian metric $g$ on $S^3$ using formula (2.4). Then, $K$ is a timelike Killing vector field for $g$, and we obtain an example of a compact stationary Lorentzian manifold admitting a timelike Killing vector field with exactly two periodic integral lines.

Next, we show a construction of an isometric action of $\mathbb{R}$ on a compact manifold with only one periodic orbit.

**Example 2.** Let $(M_0, h)$ be a compact Riemannian manifold (without boundary) that admits an isometry $\psi : M_0 \to M_0$ which has exactly one fixed point $p_0$ and having no other periodic point, i.e., all the powers $\psi^N, N \in \mathbb{N} \setminus \{0\}$, only have $p_0$ as fixed point.

An example of this situation can be obtained as follows. Consider the rotation $R_\theta : S^2 \to S^2$ around the north-south axis by an angle $\theta \in [0, 2\pi]$ which is not a rational multiple of $\pi$. Here the two-sphere $S^2$ is endowed with the round metric. Then, $R_\theta$ induces a map $\tilde{R}_\theta : \mathbb{RP}^2 \to \mathbb{RP}^2$ on the projective plane $\mathbb{RP}^2$ which is an isometry with only one fixed point and no other periodic point.

Let $M$ be the manifold obtained as a quotient of the product $M_0 \times \mathbb{R}$, by identifying $(p, n)$ with $(\psi(p), n + 1)$, $n \in \mathbb{Z}$. The Lorentzian metric $h \oplus (-dt^2)$ on $M_0 \times \mathbb{R}$ induces a Lorentzian metric $g$ on $M$, for which the maps $T_\theta$, induced by the translations $M_0 \times \mathbb{R} \ni (p, t) \to (p, s + t) \in M_0 \times \mathbb{R}$ are isometries. The 1-parameter group of isometries of $(M, g)$ given by $\mathbb{R} \ni s \to T_s \in \text{Isom}(M, g)$ has a timelike Killing vector field $K = \frac{\partial}{\partial t}$ as infinitesimal generator. It is easy to see that $K$ has exactly one closed integral curve, which is the one passing through $p_0$.

3.3. **On the isometry group of a compact stationary Lorentzian manifold.** As to the isometry group of a compact stationary Lorentzian manifold, it is known that it may fail to be compact (see for instance [29, Remark 4.3] or [8]). In fact, there exists a complete classification of Lie groups that appear as connected components of the identity of the isometry group of compact Lorentzian manifolds, which is due independently to Adams/Stuck [1] and to Zeghib [35].

We have the following partial results concerning the compactness of the isometry group.

**Proposition 3.4.** Let $(M, g)$ be a compact Lorentzian manifold with a never vanishing closed Killing vector field $K$ that is timelike somewhere. Let $M_0 = M/\mathbb{S}^1$ be the orbit space of the corresponding $\mathbb{S}^1$-action and let $\pi : M \to M_0$ be the canonical projection. Then, the group $\text{Iso}(M, g; \pi)$ consisting of all isometries of $(M, g)$ that preserve the fibration is compact.

**Proof.** A diffeomorphism $\Phi$ of $M$ preserves the fibration if and only if the push-forward $\Phi_\ast(K)$ is a pointwise multiple of $K$, that is, $\Phi_\ast(K) = \lambda K$ for some function $\lambda$ on $M$. Moreover, as $\Phi$ is an isometry, $\Phi_\ast K$ must be Killing and it can be easily proved that $\lambda$ has to be constant (if $K$ is Killing, $\lambda K$ is Killing iff $\lambda$ is constant). Moreover, we have that

$$\min_{p \in M} g(K_p, K_p) = \min_{p \in M} g(K_{\Phi^{-1}(p)}p, K_{\Phi^{-1}(p)}p) = \min_{p \in M} g((\Phi_\ast K)_p, (\Phi_\ast K)_p)$$

$$= \lambda^2 \min_{p \in M} g(K_p, K_p),$$

and hence $\lambda^2 = 1$. Applying Proposition 2.6 to $H = \text{Iso}(M, g; \pi)$ and $K$, we obtain that $\text{Iso}(M, g; \pi)$ is precompact. But it is clearly closed. Whence, $\text{Iso}(M, g; \pi)$ is compact. $\square$

It is proven in [29, Lemma 4.4] that a compact Lorentzian manifold admitting a Killing vector field which is timelike at one point and whose isometry group is one-dimensional, then the isometry group must be compact. We have the following similar result:

**Corollary 3.5.** Let $(M, g)$ be a compact Lorentzian manifold with a Killing vector field that is timelike somewhere and assume that $\text{Iso}(M, g)$ is abelian. Then, $\text{Iso}(M, g)$ is compact.
**Proof.** If Iso($M, g$) is abelian then $Ad_{\Phi} = I$ for all $\Phi$. Therefore, the thesis directly follows from Lemma 2.1 and Proposition 2.6 applied to $H = Iso(M, g)$. □

4. Final results and remarks

The technique used in this paper for proving the existence of periodic geodesics is not limited to the hypotheses of our main theorem. It can essentially be applied to any compact semi-Riemannian manifold $(M, g)$ under the more general hypotheses of Proposition 2.5. To illustrate this, we give below some simple examples:

**Example 3.** Any compact two-dimensional Lorentzian manifold $(M, g)$ admits some timelike periodic geodesic [12]. If, in addition, we assume that $M \cong T^2$ is diffeomorphic to a torus and that it admits a (non-trivial) Killing vector field $K$, then $K$ does not vanish at any point [30] Th. 4.2]. Moreover, we have two possibilities: either $(T^2, g)$ is flat, and so, it contains infinitely many periodic timelike geodesics; or $(T^2, g)$ is non flat, which from [30] Th. 4.2] implies that Iso$(T^2, g)$ is compact. Whence, applying Proposition 2.5.

**Proposition 4.1.** Any Lorentzian torus $(T^2, g)$ with a Killing vector field $K \neq 0$ contains at least two geometrically distinct non trivial periodic non self-intersecting geodesics. □

**Example 4.** From [3], any simply connected compact real-analytic Lorentzian manifold $(M, g)$ has compact isometry group. Therefore:

**Proposition 4.2.** Any simply connected compact real-analytic Lorentzian manifold $(M, g)$ with $\dim(Iso(M, g)) > 0$ contains some non trivial periodic geodesic.

**Example 5.** Consider a compact semi-Riemannian manifold $(M, g)$ of index $m \geq 1$ that admits $m$ Killing vector fields $K^1, \ldots, K^m$ which generate a negative definite subspace for $g$ of dimension $m$ at some point $p$. If $m = 1$, then we are in the Lorentzian case studied in this paper. The geodesic connectedness in the particular case of generalized stationary semi-Riemannian manifolds (i.e., semi-Riemannian manifolds that admit a timelike distribution $D \subset TM$ of rank $m$ which is generated by $m$ pointwise linearly independent commuting timelike Killing vector fields $K^1, \ldots, K^m$) is studied in [13]. Denote by $\mathcal{A} : M \to \text{Mat}_m(\mathbb{R})$ the map taking values in the space of $m \times m$ real symmetric matrices defined by $\mathcal{A}(q) = (g(K^i_q, K^j_q))_{ij}$, and consider the compact subsets of $TM$ given by:

$$V = \left\{ K^i_q : i = 1, \ldots, m, \, q \in M \text{ is such that } \mathcal{A}(q) = \mathcal{A}(p) \right\}$$

and

$$V^\perp = \left\{ v \in \bigcap_{i=1}^m (K^i_q)^\perp : q \in M \text{ is as above and } g(v, v) = 1 \right\}.$$

Consider a basis $b = (v_1, \ldots, v_n)$ of $T_p M$ with $v_i = K^i_q$ for all $1 \leq i \leq m$ and $g(v_i, v_j) = \delta_{ij}$ for $i \in \{m + 1, \ldots, n\}$ and all $j$. The Killing vector fields $K^1, \ldots, K^m$ generate a precompact subgroup $H$ of Iso$(M, g)$ (as in Proposition 2.6). Under the assumptions that the $K^i$ commute, i.e., $[K^i, K^j] = 0$ for all $i, j = 1, \ldots, m$, then $H$ is abelian, and thus the closure $\overline{H}$ is a torus. Consider a non zero vector $T$ in the Lie algebra of $\overline{H}$ such that the corresponding Killing vector field $K^T$ is closed. Then, as in the proof of our main result, there is some integral curve of $K^T$ that is a geodesic, say $\gamma : \mathbb{R} \to M$. If the $K^i$’s generate a distribution of rank $k > 1$, we claim that there are infinitely many periodic integral curves of $K^T$ that are geodesics. Namely, if $k > 1$, there is at least one $\ell \in \{1, \ldots, m\}$ such that $K^\ell_q(0)$ and $K^{\ell\ell}_q(0)$ are linearly independent. Denote by $(\varphi_t)_{t \in \mathbb{R}}$ the flow of $K^\ell$; since $K^\ell$ and $K^T$ commute then, for all $t \in \mathbb{R}$, $\varphi_t \circ \gamma$ is an integral line of $K^\ell$. On the other hand, since $K^T$ is Killing, $\varphi_T \circ \gamma$ is a periodic geodesic in $(M, g)$. Since $K^\ell_q(0)$ and $K^{\ell\ell}_q(0)$ are linearly independent, then for $t \in \mathbb{R}$ sufficiently small the curves $\varphi_t \circ \gamma$ and $\gamma$ are distinct. We have proven the following:
Proposition 4.3. Compact semi-Riemannian manifolds as above of index $k > 1$ have infinitely many distinct periodic geodesics.

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