VIRASORO CONSTRAINTS OF CURVES AS RESIDUES

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Abstract. Inspired by Eynard-Orantin topological recursions, we reformulate the Virasoro constraints for curves as residues of multilinear differentials. As applications they can be used to compute the $n$-point functions of Gromov-Witten invariants of curves.

1. Introduction

This is a sequel to an earlier work [17] in which we reformulate the DVV Virasoro constraints [4] for the Witten-Kontsevich tau-function [16, 10] as Eynard-Orantin topological recursions on the Airy curve. The topological recursions in Eynard-Orantin formalism are expressed as residues of multilinear differentials on some spectral curve. In this formalism one starts with a spectral curve and some initial data and writes down the recursion relations in terms of residues, and so the spectral curve has to be specified beforehand in order to write down the recursion relations. It has been shown by Eynard and Orantin [8] that if one applies the Eynard-Orantin formalism to the Airy curve

$$x(z) = \frac{1}{2} z^2, \quad y(z) = z, \quad B(z_1, z_2) = \frac{dz_1 dz_2}{(z_1 - z_2)^2},$$

then the $n$-point functions are the $n$-point functions of the Witten-Kontsevich tau-function. They use the Kontsevich model [10] to achieve this. An alternative way is to explicitly write down the Eynard-Orantin topological recursions in this case in terms of the correlators, and then match with the well-known Virasoro constraints for intersection numbers on $\overline{M}_{g,n}$ derived by Dijkgraaf-Verlinde-Verlinde [4].

The approach in [17] is different. We start with the Virasoro constraints satisfied by the Witten-Kontsevich tau-function and rewrite them in terms of residues after multiplying with some kernel function. The kernel function is interpreted in terms of the genus zero two-point function of the Gromov-Witten theory of a point, while the Airy is given by the genus zero one-point function. Since $\overline{M}_{0,1}$ and $\overline{M}_{0,2}$ do not exists, one has to define the one-point function and the two-point...
function in genus zero using the formula for \( n \)-point correlators in genus zero for \( n \geq 3 \).

Later we interpret the above two different approaches as the reconstruction approach and the emergent approach respectively. In general, in a reconstruction approach, one starts with some building blocks and some fundamental laws and reconstruct the whole theory; in an emergent approach one starts with a given theory and uncover the building blocks and the fundamental laws.

By emergent geometry of a Gromov-Witten type theory, we mean the study of geometric structures that emerge from the general properties of the theory. Virasoro constraints and integrable hierarchies are the very general properties of Gromov-Witten theory that are pursued for any given symplectic or projective manifold. The emergent approach starts with one of these general properties and seek for suitable geometric reformulations.

In this paper we will focus on the Virasoro constraints of a curve proved by Okounkov and Pandharipande \cite{15}. We show that by exactly the same approach as in \cite{17} one can reformulate their strengthened Virasoro constraints in terms of residue computations for \( n \)-point functions.

As applications one can compute the \( n \)-point functions of Gromov-Witten invariants of curves. The Virasoro constraints reduce the computations of the Gromov-Witten invariants of a curve to its stationary sector. The disconnected stationary GW invariants of a curves has been shown by Okounkov and Pandharipande \cite{14} using GW/Hurwitz correspondence to be given by summations over partitions. As consequences, operator formulas for the disconnected \( n \)-point functions for \( \mathbb{P}^1 \) and elliptic curves are obtained. In particular, they give a method to compute the disconnected \( n \)-point function for \( \mathbb{P}^1 \) and they show that the disconnected \( n \)-point functions for elliptic curves are given by the Bloch-Okounkov character formula \cite{1}. The \( n \)-point functions of connected stationary GW invariants of \( \mathbb{P}^1 \) are conjectured by Norbury and Scott \cite{12} to satisfy the Eynard-Orantin topological recursion for

\[
\begin{align*}
x(z) &= z + \frac{1}{z}, \\
y(z) &= \ln z, \\
B(z_1, z_2) &= \frac{dz_1 dz_2}{(z_1 - z_2)^2}.
\end{align*}
\]

This conjecture has been proved by Dunin-Barkowski et al \cite{7} and extended to the case of equivariant GW invariants by Fang, Liu and Zong \cite{9}. More recently, Dubrovin and Yang \cite{5} conjecture a formula for the \( n \)-point functions of stationary GW invariants of the projective line. This formula has been proved by Marchal \cite{11} and Dubrovin-Yang-Zagier \cite{3} by different approaches. By combining with the method in
this paper, one can then get formulas for $n$-point functions of general GW invariants of the projective line. The case of the projective line is particularly interesting because all $n$-point functions can now be computed using residues. See also Borot-Norbury [2].

The rest of this note is arranged as follows. In §2 we recall the strengthened Virasoro constraints for algebraic curves. We reformulate the Virasoro constraints and its strengthening in §3 and §4 respectively. We recall in §5 some methods to compute the $n$-point functions of stationary GW invariants of $\mathbb{P}^1$. They can be combined with our results. In §6 we present some concluding remarks on some directions for further investigations.

2. STRENGTHENED VIRASORO CONSTRAINTS FOR ALGEBRAIC CURVES

We now recall the Virasoro constraints and their strengthening for algebraic curves proved by Okounkov and Pandharipande [15]. We will closely follow their notations with slight modifications.

2.1. Virasoro constraints for algebraic curves. Let $X$ be a non-singular algebraic curve of genus $g$. Let

$$
P \phantom{\text{;}}$$

$$\alpha_1, \ldots, \alpha_g, \beta_1, \ldots, \beta_g$$

$$Q \phantom{\text{;}}$$

be a basis of $H^*(X, \mathbb{C})$ with the following properties:

(i) the class $P = 1 \in H^0(X, \mathbb{C})$ is the identity,

(ii) the classes $\alpha_i \in H^{1,0}(X, \mathbb{C})$ and $\beta_j \in H^{0,1}(X, \mathbb{C})$ determine a symplectic basis of $H^1(X, \mathbb{C})$, i.e.,

$$\int_X \alpha_i \cup \beta_j = \delta_{ij},$$

(iii) The class $Q = \omega \in H^2(X, \mathbb{C})$ is the Poincaré dual of the point.

Introduce four sets of variables corresponding to the descendents of these classes respectively:

$$t^P_0, t^P_1, t^P_2, \ldots,$$

$$s^i_0, s^i_1, s^i_2, \ldots,$$

$$\tilde{s}^i_0, \tilde{s}^i_1, \tilde{s}^i_2, \ldots,$$

$$t^Q_0, t^Q_1, t^Q_2, \ldots,$$
where $s_k^i$ and $\bar{s}_k^j$ are odd variables that supercommute. Let $\xi$ denote the formal sum:

$$
\xi = \sum_{k \geq 0} t_k^P \tau_k(P) + \sum_{i=1}^g \sum_{k \geq 0} \left( s_k^i \tau_k(\alpha_i) + \bar{s}_k^j \tau_k(\beta_i) \right) + \sum_{k \geq 0} t_k^Q \tau_k(Q).
$$

The free energy $F_X$ is the generating series of Gromov-Witten invariants of $X$:

$$
F_X = \sum_{n \geq 0} \frac{1}{n!} (\xi^n)_X,
$$

the partition function $Z_X$ is

$$
Z_X = \exp F_X.
$$

The Virasoro operators for the GW theory of $X$ are defined by:

$$
L_{-1} = - \frac{\partial}{\partial t_0^P} + \sum_{n=0}^{\infty} \left( t_{n+1}^P \frac{\partial}{\partial t_n^P} + \sum_{i=1}^{g} \left( t_{n+1}^i \frac{\partial}{\partial s_n^i} + t_{n+1}^{\bar{i}} \frac{\partial}{\partial \bar{s}_n^i} \right) + (n+1) t_{n+1}^Q \frac{\partial}{\partial t_n^Q} \right)
$$

$$
+ t_0^P t_0^Q + \sum_{i=1}^{g} s_0^i \bar{s}_0^i,
$$

$$
L_0 = - \frac{\partial}{\partial t_1^P} - \chi(X) \frac{\partial}{\partial t_0^Q}
$$

$$
+ \sum_{n \geq 0} \left( n t_n^P \frac{\partial}{\partial t_n^P} + \sum_{i=1}^{g} \left( (n+1) s_n^i \frac{\partial}{\partial s_n^i} + n \bar{s}_n^i \frac{\partial}{\partial \bar{s}_n^i} \right) + (n+1) t_n^Q \frac{\partial}{\partial t_n^Q} \right)
$$

$$
+ \chi(X) \sum_{n \geq 0} t_{n+1}^P \frac{\partial}{\partial t_n^P} + \frac{\chi(X)}{2} t_0^P t_0^P,
$$

and for $k > 0$,

$$
L_k = -(k+1)! \frac{\partial}{\partial t_{k+1}^P} + \sum_{n=1}^{(k+n)!} \frac{(k+n)!}{(n-1)!} t_n^P \frac{\partial}{\partial t_{k+n}^P} + \sum_{n=0}^{\infty} \frac{(k+n+1)!}{n!} t_n^Q \frac{\partial}{\partial t_{k+n}^Q}
$$

$$
+ \sum_{n=0}^{\infty} \sum_{i=1}^{g} \left( \frac{(k+n+1)!}{n!} s_n^i \frac{\partial}{\partial s_{k+n}^i} + \frac{(n+k)!}{(n-1)!} \bar{s}_n^i \frac{\partial}{\partial \bar{s}_{k+n}^i} \right)
$$

$$
- \chi(X) (k+1)! \sum_{r=1}^{k+1} \frac{1}{r} \frac{\partial}{\partial t_r^Q} + \chi(X) \sum_{n=1}^{\infty} \sum_{r=n}^{k+n} \frac{1}{r} \frac{(k+n)!}{(n-1)!} t_n^P \frac{\partial}{\partial t_{k+n-1}^Q}
$$

$$
+ \frac{\chi(X)}{2} \sum_{n=0}^{k-2} (n+1)! (k-n-1)! \frac{\partial}{\partial t_n^Q} \frac{\partial}{\partial t_{k-n-2}^P}.
$$
2.2. Strengthening of the standard Virasoro constraints. Okounkov and Pandharipande also define two additional differential operators $D_k^i$ and $\bar{D}_k^i$ for $k \geq -1$ by:

\[
D_k^i = -(k + 1)! \frac{\partial}{\partial s_{k+1}^i} + \sum_{n \geq 0} \frac{(k + n + 1)!}{n!} t_n^p \frac{\partial}{\partial s_{k+n}^i},
\]

\[
\bar{D}_k^i = -(k + 1)! \frac{\partial}{\partial \bar{s}_{k+1}^i} + \sum_{n \geq 0} \frac{(k + n + 1)!}{n!} t_n^p \frac{\partial}{\partial \bar{s}_{k+n}^i}.
\]

The operators $L_k, D_k^i, \bar{D}_k^i$ annihilate the partition function $Z_X$, and they satisfy the following commutation and anti-commutation relations:

\[
[L_n, L_m] = (n - m)L_{n+m},
\]

\[
[L_n, D_m^i] = -(m + 1)D_{n+m}^i,
\]

\[
[L_n, \bar{D}_m^i] = (n - m)\bar{D}_{n+m}^i,
\]

\[
\{D_n^i, D_m^j\} = \{D_n^i, \bar{D}_m^j\} = \{\bar{D}_n^i, \bar{D}_m^j\} = 0.
\]

If the operators $\{L_k\}_{k \geq -1}$ are identified with the Lie algebra of holomorphic vector field

\[
\mathcal{V} = \{-z^{k+1} \frac{\partial}{\partial z}\}_{k \geq 1},
\]

then, the operators $\{D_k^i\}_{k \geq -1}$ define a $\mathcal{V}$-module isomorphic to $\{-z^{k+1}\}_{k \geq -1}$ with the action defined by differentiation, and the operators $\{\bar{D}_k^i\}_{k \geq -1}$ define a $\mathcal{V}$-module isomorphic to the adjoint representation.

3. Virasoro Constraints for Curves as Residues

3.1. Virasoro constraints of curves in terms of correlators. To illustrate the idea, let us first set all the odd variables to zero for the moment. Then the constraint $L_k Z_X = 0$ can be rewritten in the
following form:

\[
\begin{align*}
&k! \langle \tau_k(P) \prod_{i=1}^{n} \tau_{a_i}(P) \cdot \prod_{j=1}^{m} \tau_{b_j}(Q) \rangle \\
&= \sum_{i=1}^{n} \frac{(k + a_i - 1)!}{(a_i - 1)!} \langle \tau_{k+a_i-1}(P) \prod_{l \in [n]} \tau_{a_i}(P) \cdot \prod_{j=1}^{m} \tau_{b_j}(Q) \rangle \\
&+ \sum_{j=1}^{m} \frac{(k + b_j)!}{b_j!} \langle \prod_{i=1}^{n} \tau_{a_i}(P) \cdot \tau_{k+b_j-1}(Q) \prod_{l \in [m]} \tau_{b_j}(Q) \rangle \\
&- \chi(X) k! \sum_{r=1}^{k} \frac{1}{r} \langle \tau_{k-1}(Q) \prod_{i=1}^{n} \tau_{a_i}(P) \cdot \prod_{j=1}^{m} \tau_{b_j}(Q) \rangle \\
&+ \chi(X) \sum_{i=1}^{n} \sum_{r=a_i}^{k+a_i-1} \frac{1}{r} \frac{(k + a_i - 1)!}{(a_i - 1)!} \langle \tau_{k+a_i-2}(Q) \prod_{l \in [n]} \tau_{a_i}(P) \cdot \prod_{j=1}^{m} \tau_{b_j}(Q) \rangle \\
&+ \frac{\chi(X)}{2} \sum_{n=0}^{k-3} (n + 1)! (k - n - 2)! \langle \tau_{n}(Q) \tau_{k-n-3}(Q) \prod_{i=1}^{n} \tau_{a_i}(P) \cdot \prod_{j=1}^{m} \tau_{b_j}(Q) \rangle \\
&+ \frac{\chi(X)}{2} \sum_{n=0}^{k-3} \sum_{I_1 \cup I_2 = [n] \ J_1 \cup J_2 = [m]} (n + 1)! (k - n - 2)! \langle \tau_{n}(Q) \tau_{a_{I_1}}(P) \tau_{b_{J_1}}(Q) \\
&\cdot \tau_{k-n-3}(Q) \tau_{a_{I_2}}(P) \tau_{b_{J_2}}(Q) \rangle.
\end{align*}
\]

Here we have used the following notations: \([n]\) denotes the set of indices \(\{1, \ldots, n\}\), and for a subset \(I \subset [n]\), \(\tau_{a_I}(P) = \prod_{i \in I} \tau_{a_i}(P)\). Similar notations will be used below. The terms on the right-hand side can be interpreted as fusions and fissions of \(\tau_k(P)\) respectively.
Multiply both sides by $\prod_{i=1}^n a_i! \prod_{j=1}^m (b_j + 1)!$:

\[
\langle \sigma_k(P)\sigma_{a_1}(P) \cdots \sigma_{a_n}(P)\sigma_{b_1}(Q) \cdots \sigma_{b_m}(Q) \rangle \\
= \sum_{i=1}^n a_i \langle \sigma_{a_1}(P) \cdots \sigma_{k+a_i-1}(P) \cdots \sigma_{a_n}(P)\sigma_{b_1}(Q) \cdots \sigma_{b_m}(Q) \rangle \\
+ \sum_{j=1}^m (b_j + 1) \langle \sigma_{a_1}(P) \cdots \sigma_{a_n}(Q)\sigma_{b_1}(Q) \cdots \sigma_{k+b_j-1}(Q) \cdots \sigma_{b_m}(Q) \rangle \\
- \chi(X) \sum_{r=1}^k \frac{1}{r^2} \langle \sigma_{k-1}(Q)\sigma_{a_1}(P) \cdots \sigma_{a_n}(P)\sigma_{b_1}(Q) \cdots \sigma_{b_m}(Q) \rangle \\
+ \chi(X) \sum_{i=1}^n \sum_{r=a_i}^{k+a_i-1} \frac{1}{r} a_i \langle \sigma_{a_1}(P) \cdots \sigma_{k+a_i-2}(Q) \cdots \sigma_{a_n}(P)\sigma_{b_1}(Q) \cdots \sigma_{b_m}(Q) \rangle \\
+ \frac{\chi(X)}{2} \sum_{n=0}^{k-3} \langle \sigma_n(Q)\sigma_{k-n-3}(Q)\sigma_{a_1}(P) \cdots \sigma_{a_n}(P)\sigma_{b_1}(Q) \cdots \sigma_{b_m}(Q) \rangle \\
+ \frac{\chi(X)}{2} \sum_{n=0}^{k-3} \sum_{I_1 \sqcup I_2 = [n]} \prod_{J_1 \sqcup J_2 = [m]} \langle \sigma_n(Q)\sigma_{a_{I_1}}(P)\sigma_{b_{J_1}}(Q) \rangle \langle \sigma_{k-n-3}(Q)\sigma_{a_{I_2}}(P)\sigma_{b_{J_2}}(Q) \rangle,
\]

where

\[\sigma_a(P) = a! \tau_a(P), \quad \sigma_b(Q) = (b + 1)! \tau_b(Q).\]
3.2. Derivations of the recursion kernels. We now rewrite the above recursion relations in terms of generating series:

\[
\begin{align*}
&\sum_{k,a_i,b_j=0}^{\infty} \left( \frac{\sigma_k(P)}{z_0} \frac{\sigma_{a_1}(P)}{z_{1}^{a_1+1}} \cdots \frac{\sigma_{a_n}(P)}{z_{n}^{a_n+1}} \frac{\sigma_{b_1}(Q)}{w_1^{b_1+2}} \cdots \frac{\sigma_{b_m}(Q)}{w_{m+2}} \right) \\
&= \sum_{k,a_i,b_j=0}^{\infty} \sum_{i=1}^{n} a_i z_0^{k-1} \left( \frac{\sigma_{a_1}(P)}{z_{1}^{a_1+1}} \cdots \frac{\sigma_{k+a_i-1}(P)}{z_{i}^{k+a_i}} \cdots \frac{\sigma_{a_n}(P)}{z_{n}^{a_n+1}} \frac{\sigma_{b_1}(Q)}{w_1^{b_1+2}} \cdots \frac{\sigma_{b_m}(Q)}{w_{m+2}} \right) \\
&\quad + \sum_{k,a_i,b_j=0}^{\infty} \sum_{j=1}^{m} (b_j + 1) z_0^{k-1} \left( \frac{\sigma_{a_1}(P)}{z_{1}^{a_1+1}} \cdots \frac{\sigma_{k+b_j-1}(Q)}{z_{j}^{k+b_j+1}} \cdots \frac{\sigma_{b_m}(Q)}{w_{m+2}} \right) \\
&\quad - \sum_{k,a_i,b_j=0}^{\infty} \chi(X) \sum_{r=1}^{k} \frac{1}{r} \left( \frac{\sigma_{k-1}(Q)}{z_{1}^{k+1}} \frac{\sigma_{a_1}(P)}{z_{1}^{a_1+1}} \cdots \frac{\sigma_{k+a_i-2}(Q)}{z_{i}^{k+a_i}} \cdots \frac{\sigma_{a_n}(P)}{z_{n}^{a_n+1}} \frac{\sigma_{b_1}(Q)}{w_1^{b_1+2}} \cdots \frac{\sigma_{b_m}(Q)}{w_{m+2}} \right) \\
&\quad + \chi(X) \sum_{i=1}^{n} \sum_{r=a_i}^{k} \frac{1}{r} a_i \left( \frac{\sigma_{a_1}(P)}{z_{1}^{a_1+1}} \cdots \frac{\sigma_{k+a_i-2}(Q)}{z_{i}^{k+a_i}} \cdots \frac{\sigma_{a_n}(P)}{z_{n}^{a_n+1}} \frac{\sigma_{b_1}(Q)}{w_1^{b_1+2}} \cdots \frac{\sigma_{b_m}(Q)}{w_{m+2}} \right) \\
&\quad + \frac{\chi(X)}{2} \sum_{k=3}^{\infty} \sum_{n=0}^{k-3} \left( \frac{\sigma_{n}(Q)}{z_{0}^{n+2}} \frac{\sigma_{k-n-3}(Q)}{z_{0}^{k-n-1}} \prod_{i=1}^{n} \frac{\sigma_{a_i}(P)}{z_{i}^{a_i+1}} \prod_{j=1}^{m} \frac{\sigma_{b_j}(Q)}{w_{j}^{b_j+2}} \right) \\
&\quad + \frac{\chi(X)}{2} \sum_{n=0}^{k-3} \sum_{I_1 \mid I_2 = [n]} \prod_{I_1 \mid I_2 = [m]} \left( \frac{\sigma_{n}(Q)}{z_{0}^{n+2}} \frac{\sigma_{a_I}(P)}{z_{1}^{a_I+1}} \frac{\sigma_{b_J}(Q)}{w_{J}^{b_J+2}} \right) 
\end{align*}
\]

If we introduce

\[
W(u_1, \ldots, u_n, x_1, \ldots, x_m) = \sum_{a_i, b_j=0}^{\infty} \left( \frac{\sigma_{a_1}(P)}{u_{1}^{a_1+1}} \cdots \frac{\sigma_{a_n}(P)}{u_{n}^{a_n+1}} \frac{\sigma_{b_1}(Q)}{x_{1}^{b_1+2}} \cdots \frac{\sigma_{b_m}(Q)}{x_{m}^{b_m+2}} \right) du_1 \cdots du_n dx_1 \cdots dx_m,
\]
then the above recursion relations can be rewritten in the following form:

\[
W(u_0, u_1, \ldots, u_n, x_1, \ldots, x_m) = \sum_{i=1}^{n} A(u; u_0, u_i)W(u_1, \ldots, u_{i-1}, u, u_{i+1}, \ldots, u_n, x_1, \ldots, x_m) \\
+ \sum_{j=1}^{n} B(x; u_0, x_i)W(u_1, \ldots, u_n, x_1, \ldots, x_{j-1}, x, x_{j+1}, \ldots, x_m) \\
- \chi(X)C(x; u_0)W(u_1, \ldots, u_n, x, x_1, \ldots, x_m) \\
+ \sum_{i=1}^{n} \chi(X)D(x; u_0, u_i)W(u_1, \ldots, u_{i-1}, u_{i+1}, \ldots, u_n, x, x_1, \ldots, x_m) \\
+ \frac{\chi(X)}{2}E(x; u_0)W(u_1, \ldots, u_n, x, x_1, \ldots, x_m)
\]

where \(A, B, C, D, E\) are operators to be specified below.

The operator \(A(u; u_0, u_i)\) has the following effect:

\[
\frac{1}{u^l}du \mapsto \sum_{k+a=l} \frac{a\cdot u_i^{a-1}}{u_0^{k+1}}du_idu_0.
\]

It describes a fusion of type \(P + P \to P\), and it can be realized by multiplying with the following differential form then take \(\text{res}_{u=0}\):

\[
\sum_{l=0}^{\infty} u_l \sum_{k+a=l} \frac{a\cdot u_i^{a-1}}{u_0^{k+1}}du_idu_0 = \sum_{k,a=0}^{\infty} \frac{a\cdot u^{k+a-1} \cdot u_i^{a-1}}{u_0^{k+1}}du_idu_0 \\
= \sum_{a=0}^{\infty} a\cdot u_i^{a-1} \cdot \sum_{k=0}^{\infty} \frac{u^k}{u_0^{k+1}}du_idu_0 = u_i^{-2} \left(1 - \frac{u}{u_i}\right)^{-2} \cdot \left(\frac{u}{u_0} - u\right)du_idu_0
\]

The operator \(B(x; u_0, x_i)\) has the following effect:

\[
\frac{1}{x^{l+1}}dx \mapsto \sum_{k+b=l} \frac{(b+1)x_j^{b-2}}{u_0^{k+1}}dx_jdu_0.
\]
It describes a fusion of type $P + Q \to Q$, and it can be realized by multiplying with the following differential form then take $\text{res}_{z=0}$:

\[
\sum_{l=0}^{\infty} x^l \sum_{k+b=l} \frac{(b+1)x_j^{-b-2} x^{k+b}}{u_0^{k+1}} dx_j du_0 = \sum_{k,b=0}^{\infty} x^b \frac{(b+1)x_j^{-b-2} x^{k+b}}{u_0^{k+1}} dx_j du_0 \\
= \sum_{b=0}^{\infty} x^b \left( \frac{b+1}{u_0} \right) dx_j \cdot \frac{dx_j}{(x_j-x)^2} \cdot \frac{du_0}{u_0-x}.
\]

The operator $C(x; u_0)$ has the following effect:

\[
\frac{1}{x^{k+1}} dx \mapsto \frac{1}{u_0^{k+1}} \sum_{r=1}^{k} \frac{1}{r} du_0.
\]

It describes a conversion of type $P \to Q$. It can be realized by multiplying with the following differential form then take $\text{res}_{z=0}$:

\[
\sum_{k=1}^{\infty} x^k \frac{1}{u_0^{k+1}} \sum_{r=1}^{k} \frac{1}{r} du_0 = \sum_{r=1}^{\infty} \frac{1}{r} \sum_{k=r}^{\infty} \frac{x^r}{u_0^{k+1}} = \sum_{r=1}^{\infty} \frac{1}{r} \sum_{k=r}^{\infty} \frac{1}{u_0^{k+1}} \frac{x^r}{1-x^{\frac{1}{u_0}}} \\
= - \log(1-x^{\frac{1}{u_0}}) \cdot \frac{du_0}{u_0-x}.
\]

The operator $D(x; u_0, u_i)$ has the following effect:

\[
\frac{1}{x^l} dx \mapsto \sum_{k+a=l, k,a \geq 1} \frac{a}{u_0^{k+1} u_i^{a+1}} \sum_{r=a}^{k+a-1} \frac{1}{r} du_0 du_i.
\]
It describes a fusion of type $P + P \rightarrow Q$. It can be realized by multiplying with the following differential form then take $\text{res}_{x=0}$:

$$
\sum_{l=2}^{\infty} x^{l-1} \sum_{k+a=l, a\geq 1} \frac{a}{u_0^{k+1} u_i^{a+1}} \sum_{r=a}^{k+a-1} \frac{1}{r} du_0 du_i
$$

$$
= \sum_{r=1}^{\infty} \frac{1}{r} \sum_{a=r}^{\infty} \frac{a}{u_0^{a+1}} du_i \sum_{k=0}^{\infty} \frac{x^k}{u_0^{k+1}} du_0
$$

$$
= \sum_{r=1}^{\infty} \frac{1}{r} \frac{(x)_r^{-1}}{u_i^r} \left( r - (r-1) \frac{x}{u_i} \right) \cdot \frac{du_i}{(u_i - x)^2} \frac{1}{u_0} \frac{1}{1 - \frac{x}{u_0}} du_0
$$

$$
= \left( 1 - \log(1 - \frac{x}{u_i}) \right) \frac{du_i}{(u_i - x)^2} \frac{du_0}{u_0 - x}.
$$

The operator $E(x; u_0)$ has the following effect:

$$
\frac{1}{x^{k+1}} (dx)^2 \mapsto \frac{du_0}{u_0^{k+1}}.
$$

It describes a fission of type $P \rightarrow Q + Q$. It can be realized by multiplying with the following expression then take $\text{res}_{u=0}$:

$$
\sum_{k=0}^{\infty} \frac{x^k}{dx} \cdot \frac{du_0}{u_0^{k+1}} = \frac{1}{dx} \cdot \frac{du_0}{u_0 - x}.
$$

To summarize, we have shown that:

$$
W(u_0, u_{[n]}, x_{[m]})
$$

$$
= \sum_{i=1}^{n} \text{res}_{u=0} \left( \frac{du_i du}{(u_i - u)^2} \cdot \frac{1}{du_0 - u} W(u, u_{[i]}, x_{[m]}) \right)
$$

$$
+ \sum_{j=1}^{n} \text{res}_{x=0} \left( \frac{dx_j dx}{(x_j - x)^2} \cdot \frac{1}{dx_0 - x} \cdot W(u_{[i]}, x, x_{[m]}) \right)
$$

$$
+ \text{res}_{x=0} \left( \chi(X) \log(1 - \frac{x}{u_i}) \cdot \frac{du_0}{u_0 - x} \cdot W(u_{[i]}, x, x_{[m]}) \right)
$$

$$
+ \text{res}_{x=0} \left( \left( 1 - \log(1 - \frac{x}{u_i}) \right) \frac{du_i dx}{(u_i - x)^2} \cdot \frac{\chi(X)}{dx} \frac{du_0}{u_0 - w} W(u_{[i]}, x, x_{[m]}) \right)
$$

$$
+ \frac{1}{2} \text{res}_{x=0} \left( \frac{\chi(X)}{dx} \cdot \frac{du_0}{u_0 - x} \cdot W(u_{[i]}, x, x_{[m]}) \right)
$$

$$
+ \frac{1}{2} \text{res}_{u=0} \left( \frac{\chi(X)}{dx} \cdot \frac{du_0}{u_0 - x} \cdot W(u_{[i]}, x, x_{[m]}) \right).
$$
Here we have used the following notations: $[n]$ denotes the set of indices $\{1, \ldots, n\}$, $u_{[n]} = u_1, \ldots, u_n$; $[n]_i = [n] - \{i\}$, $u_{[n]_i} = u_1, \ldots, \hat{u}_i, \ldots, u_n$ ($u_i$ is deleted).

Now we can formulate the following

**Theorem 3.1.** The following recursion relations hold for nonsingular algebraic curves:

\[
W(u_0, u_1, \ldots, u_n, x_1, \ldots, x_m)
= \sum_{i=1}^{n} \text{res}_{u=0} \left( K(u_0, u)W_0(u_i, u)W(u, u_{[n]_i}, x_{[m]}) \right)
+ \sum_{j=1}^{n} \text{res}_{x=0} \left( K(u_0, x)W_0(x_j, x)W(u_{[n]_j}, x, x_{[m]}) \right)
+ \text{res}_{x=0} \left( \chi(X)L(u_0, x) \cdot W(u_{[n]}, x, x_{[m]}) \right)
+ \sum_{i=1}^{n} \text{res}_{x=0} \left( \chi(X)K(u_0, x)W_0(u_i, x)W(u_{[n]}, x, x_{[m]}) \right)
+ \frac{1}{2} \text{res}_{x=0} \left( \chi(X)K(u_0, x) \cdot W(u_{[n]}, x, x_{[m]}) \right)
+ \frac{1}{2} \sum \prod_{I_1 \cup I_2 = [n]} \text{res}_{x=0} \left( \chi(X)K(u_0, x) \cdot W(u_{I_1}, x, x_{I_1})W(u_{I_2}, x, x_{I_2}) \right),
\]

where

\[
W_0(u_1, u_2) = \frac{du_1du_2}{(u_1 - u_2)^2},
W_0(x_1, x_2) = \frac{dx_1dx_2}{(x_1 - x_2)^2},
W_0(u, x) = \left( 1 - \log \left( \frac{1 - x}{u} \right) \right) du dx
\]

are genus zero two-point functions, and

\[
K(u_1, u_2) = \frac{1}{u_1 - u_2} \frac{du_1}{du_2},
K(u_1, x_2) = \frac{1}{u_1 - x_2} \frac{du_1}{dx_2},
L(u_1, x_2) = \log \left( 1 - \frac{x_2}{u_1} \right) \frac{du_1}{u_1 - x_2}.
\]
Remark 3.2. $L(u_1, x_2)$ is related by $W_0(u_1, x_2)$ by differentiation:

\begin{equation}
W_0(u_1, x_2) = -\frac{\partial L(u_1, x_2)}{\partial x_2}dx_2.
\end{equation}

3.3. **Inclusion of the odd observables.** To understand the correlators with the insertions of odd observables, for simplicity of presentation let us first consider

\[ \langle \tau_k(P)\tau_a(\alpha_i)\tau_b(\beta_j) \rangle. \]

The Virasoro constraints $L_{k-1}Z_X = 0$ give us:

\begin{align*}
& k! \langle \tau_k(P)\tau_a(\alpha_i)\tau_b(\beta_j) \rangle \\
& = \frac{(k+a)!}{a!} \langle \tau_{k+a-1}(\alpha_i)\tau_b(\beta_j) \rangle + \frac{(k+b-1)!}{(b-1)!} \langle \tau_a(\alpha_i)\tau_{k+b-1}(\beta_j) \rangle
\end{align*}

Multiplying both sides by $(a+1)!b!$ we get:

\begin{align*}
& \langle \sigma_k(P)\sigma_a(\alpha_i)\sigma_b(\beta_j) \rangle \\
& = (a+1)\langle \sigma_{k+a-1}(\alpha_i)\sigma_b(\beta_j) \rangle + b\langle \sigma_a(\alpha_i)\sigma_{k+b-1}(\beta_j) \rangle,
\end{align*}

where

\[ \sigma_a(\alpha_i) = (a+1)!\tau_a(\alpha_i), \quad \sigma_b(\beta_j) = b!\tau_b(\beta_j). \]

In terms of the generating series:

\begin{align*}
& \sum_{k,a,b \geq 0} \frac{\langle \sigma_k(P)\sigma_a(\alpha_i)\sigma_b(\beta_j) \rangle}{u_0^{k+1}p_i^{a+2}\bar{p}_j^{b+1}} \\
& = \sum_{k,a,b \geq 0} (a+1)\frac{p_i^{k-1}}{u_0^{k+1}}\frac{\langle \sigma_{k+a-1}(\alpha_i)\sigma_b(\beta_j) \rangle}{p_i^{k+a+1}\bar{p}_j^{b+1}} \\
& + \sum_{k,a,b \geq 0} \frac{bp_j^{k-1}}{u_0^{k+1}}\frac{\langle \sigma_a(\alpha_i)\sigma_{k+b-1}(\beta_j) \rangle}{p_i^{a+2}\bar{p}_j^{k+b}}.
\end{align*}

We reformulate it as follows:

\[ W(u_0, p_i, \bar{p}_j) = F(q_i; u_0, p_i)W(q_i, \bar{p}_j) + G(\bar{q}_j; u_0, \bar{p}_j)W(p_i, \bar{q}_j), \]

where the operators $F$ and $G$ will be specified below.

The operator $F(q_i; u_0, p_i)$ has the following effect:

\[ \frac{dq_i}{q_i^{l+1}} \mapsto \sum_{k+a=l} (a+1)\frac{du_0}{u_0^{k+1}}\frac{dp_i}{p_i^{a+2}}. \]
It describes the fusion of type $P + \alpha_i \rightarrow \alpha_i$. It can be realized by multiplication with the following differential then take $\text{res}_{\alpha_i=0}$:

$$
\sum_{l=0}^{\infty} q_i^l \sum_{k+a=l} (a + 1) \frac{du_0}{u_0^{k+1}} \frac{dp_i}{p_i^{a+2}} = \\
= \sum_{k=0}^{\infty} q_i^k \frac{dz_0}{u_0^{k+1}} \sum_{a=0}^{\infty} (a + 1) \frac{q_i^a dp_i}{p_i^{a+2}} \\
= \frac{du_0}{u_0 - q_i (p_i - q_i)^2} = \\
K(u_0, q_i) \cdot W_0(p_i, q_i),
$$

where

$$
K(u_0, q_i) = \frac{du_0}{dq_i} \frac{1}{u_0 - q_i}, \\
W_0(p_i, q_i) = \frac{dp_i dq_i}{(p_i - q_i)^2}.
$$

The operator $G(\bar{q}_j; u_0, \bar{p}_j)$ has the following effect:

$$
\frac{d\bar{q}_j}{\bar{q}_j} \mapsto \sum_{k+b=l} \frac{bdu_0 dp_j}{u_0^{k+1} \bar{p}_j^{b+1}}.
$$

It describes the fusion of type $P + \beta_j \rightarrow \beta_j$. It can be realized by multiplication with the following differential then take $\text{res}_{\beta_j=0}$:

$$
\sum_{l=0}^{\infty} \bar{q}_j^{l-1} \sum_{k+b=l} bdu_0 dp_j = \\
= \sum_{k=0}^{\infty} \bar{q}_j^k \sum_{b=0}^{\infty} \frac{b\bar{q}_j^{b-1} du_0}{u_0^{k+1} \bar{p}_j^{b+1}} = \\
= \frac{du_0}{u_0 - \bar{q}_j (\bar{p}_j - \bar{q}_j)^2} = \\
K(u_0, \bar{q}_j) W_0(\bar{p}_j, \bar{q}_j),
$$

where

$$
K(u_0, \bar{q}_j) = \frac{du_0}{d\bar{q}_j} \frac{1}{u_0 - \bar{q}_j}, \\
W_0(\bar{p}_j, \bar{q}_j) = \frac{dp_i d\bar{q}_i}{(\bar{p}_i - \bar{q}_i)^2}.
$$
Now we can generalize the recursion formula in Theorem 3.1 to the generating series

\[ W(u, u_1, \ldots, u_n, p^i_1, \ldots, p^k_{i_k}, \bar{p}^j_{j_1}, \ldots, \bar{p}^l_{j_l}, x_1, \ldots, x_m) = \sum \left( \frac{\sigma_a(P) \sigma_1(P)}{u^{a+1}_1} \frac{\sigma_2(P) \sigma_{b_1}(\alpha_i)}{u^{b_1+1}_n} \frac{\sigma_{b_2}(\alpha_{i_k})}{(p^k_{i_k})^{b_{k+2}}} \right) \] \[ \cdot \frac{\sigma_{c_1}(\beta_{j_1})}{(\bar{p}^j_{j_1})^{c_1+1}} \ldots \frac{\sigma_{c_l}(\beta_{j_l})}{(\bar{p}^l_{j_l})^{c_l+1}} \cdot \frac{\sigma_{d_1}(Q)}{x^{d_1+2}_1} \ldots \frac{\sigma_{d_m}(Q)}{x^{d_m+2}_m} \right) \]

We omit the long formula which is awkward to present here.

4. Strenthened Virasoro Constraints of Curves as Residues

We now show how to reformulate the constraints given by operators \( D^k_i \) and \( \bar{D}^k_i \) as residues.

4.1. The case of \( D^k_i \). For simplicity of presentation, we again consider a special case. By the constraints \( D_{k-1} Z_X = 0 \) we get

\[ k!(\tau_k(\alpha_i)\tau_a(P)\tau_c(\beta_i)) = \frac{(k + a)!}{a!} \langle \tau_{k+a-1}(\alpha_i)\tau_c(\beta_i) \rangle + \frac{(k + c)!}{c!} \langle \tau_a(P)\tau_{k+c-1}(Q) \rangle. \]

We multiply both sides by \( (k + 1)a!c! \) to get:

\[ \langle \sigma_k(\alpha_i)\sigma_a(P)\sigma_c(\beta_i) \rangle = (k + 1)\langle \sigma_{k+a-1}(\alpha_i)\sigma_c(\beta_i) \rangle + (k + 1)\langle \sigma_a(P)\sigma_{k+c-1}(Q) \rangle. \]

In terms of generating series we get:

\[ \sum_{k,a,c \geq 0} \frac{\langle \sigma_k(\alpha_i) \sigma_a(P) \sigma_c(\beta_i) \rangle}{p^{k+2}_i u^{a+1}_1 \bar{p}^{c+1}_i} = \sum_{k,a,c \geq 0} \frac{(k + 1)p^{a-1}_i}{u^{a+1}_1} \frac{\langle \tau_{k+a-1}(\alpha_i) \sigma_c(\beta_i) \rangle}{p^{k+a+1}_i \bar{p}^{c+1}_i} + \sum_{k,a,c \geq 0} \frac{(k + 1)p^k_i}{\bar{p}^{k+2}_i} \frac{\langle \sigma_a(P) \sigma_{k+c-1}(Q) \rangle}{u^{a+1}_1 \bar{p}^{k+c+1}_i}. \]

We rewrite it as follows:

\[ W(p_i, u_1, \bar{p}_i) = G(v_i; u_1, p_i)W(v_i, \bar{p}_i) + H(x; p_i, \bar{p}_i)W(u_1, x). \]
Here the operator $G(q_i; u_i, p_i)$ has the following effect:

$$
\frac{1}{q_i + 1} dq_i \mapsto \sum_{k+a=l} (k+1) \frac{dp_i}{p_i^{k+2}} \frac{du_1}{u_1^{a+1}}.
$$

It describes the fusion of type $\alpha_i + P \rightarrow \alpha_i$. It can be realized by first multiplying with the following differential then take res$_{q_i=0}$:

$$
\sum_{l=0}^{\infty} q_i^l \sum_{k+a=l} (k+1) \frac{1}{p_i^{k+2}} \frac{1}{u_1^{a+1}} dp_i du_i = \sum_{k=0}^{\infty} (k+1) \frac{u_1^k}{p_i^{k+2}} \sum_{a=0}^{\infty} \frac{q_i^a}{u_1^{a+1}} dp_i du_i
$$

$$
= \frac{dp_i}{(p_i - q_i)^2} \cdot \frac{du_1}{u_1 - q_i} = K(u_1, q_i) \cdot W_0(p_i, q_i),
$$

where

$$
W_0(p_i, q_i) = \frac{dp_i dq_i}{(p_i - q_i)^2},
$$

$$
K(u_1, q_i) = \frac{1}{u_1 - q_i} \frac{du_1}{dq_i}.
$$

The operator $H(x; p_i, \bar{p}_i)$ has the following effect:

$$
\frac{1}{x + 1} dx \mapsto \sum_{k+c=l} (k+1) \frac{dp_i}{p_i^{k+1}} \frac{d\bar{p}_i}{\bar{p}_i^{c+1}}.
$$

It describes a fusion of type $\alpha_i + \beta_i \rightarrow Q$. It can be realized by multiplying with the following differential and taking res$_{x=0}$:

$$
\sum_{l=0}^{\infty} x^l \sum_{k+c=l} (k+1) \frac{dp_i}{p_i^{k+1}} \frac{d\bar{p}_i}{\bar{p}_i^{c+1}} = \frac{dp_i}{(p_i - x)^2} \frac{d\bar{p}_i}{\bar{p}_i - x} = K(\bar{p}_i, x) W_0(p_i, x),
$$

where

$$
W_0(p_i, x) = \frac{dp_i dx}{(p_i - x)^2},
$$

$$
K(\bar{p}_i, x) = \frac{1}{\bar{p}_i - x} \frac{d\bar{p}_i}{dx}.
$$

4.2. The case of $\bar{D}^i_k$. We will again use an example to illustrate the idea. By the constraints $\bar{D}^i_{k-1} Z_X = 0$ we have:

$$
k! \langle \tau_k(\beta_i) \tau_a(P) \tau_c(\alpha_i) \rangle = \frac{(k + a - 1)!}{(a - 1)!} \langle \tau_{k+a-1}(\beta_i) \tau_c(\alpha_i) \rangle - \frac{(k + c)!}{c!} \langle \tau_a(P) \tau_{k+c-1}(Q) \rangle.
$$

We multiply both sides by $a!(c + 1)!$ to get:

$$
\langle \sigma_k(\beta_i) \sigma_a(P) \sigma_c(\alpha_i) \rangle = a \langle \sigma_{k+a-1}(\beta_i) \sigma_c(\alpha_i) \rangle - (c + 1) \langle \sigma_a(P) \sigma_{k+c-1}(Q) \rangle,
$$

where
\[\sum_{k,a,c \geq 0} \left( \sigma_k(\beta_i) \sigma_a(P) \sigma_c(\alpha_i) \right) \frac{z_{a+1}^{k+1}}{p_i^{c+2}} = \sum_{k,a,c \geq 0} a p_i^{k-1} \left( \sigma_k(\beta_i) \sigma_a(P) \sigma_c(\alpha_i) \right) \frac{z_{a+1}^{k+a}}{p_i^{c+2}} \]

\[\sum_{k,a,c \geq 0} (c+1) p_i^{k-1} \sigma_a(P) \sigma_{k+c-1}(Q) \frac{z_{a+1}^{k+a}}{p_i^{c+2}}.\]

So we need an operator such that

\[\frac{1}{q_i} d\bar{q}_i \mapsto \sum_{k+a=l} a \frac{d\bar{p}_i}{p_i^{a+1}} \frac{d\bar{u}_1}{u_1^{a+1}}\]

from this we get:

\[\sum_{l=0}^{\infty} \sum_{k+a=l} \frac{1}{p_i^{k+1}} \frac{a}{u_1^{a+1}} d\bar{p}_i d\bar{u}_1 = \sum_{k=0}^{\infty} \frac{\bar{q}_i^k}{p_i^{k+1}} \sum_{a=0}^{\infty} \frac{a \bar{q}_i^{a-1}}{u_1^{a+1}} d\bar{p}_i d\bar{u}_1\]

\[= \frac{d\bar{p}_i}{\bar{p}_i - \bar{q}_i} \cdot \frac{d\bar{u}_1}{(u_1 - \bar{q}_i)^2} = K(p_i, q_i) W_0(u_1, \bar{q}_i),\]

where

\[W_0(u_1, \bar{q}_i) = \frac{d\bar{u}_1 d\bar{q}_i}{(u_1 - \bar{q}_i)^2},\]

\[K(p_i, q_i) = \frac{1}{p_i - q_i} \frac{d\bar{u}_i}{d\bar{q}_i}.\]

We also need an operator such that:

\[\frac{1}{x^{l+1}} dx \mapsto \sum_{k+c=l} (c+1) \frac{d\bar{p}_i}{p_i^{k+1}} \frac{d\bar{p}_i}{p_i^{c+2}}\]

From this we get

\[\sum_{l=0}^{\infty} x^l \sum_{k+c=l} (c+1) \frac{d\bar{p}_i}{p_i^{k+1}} \frac{d\bar{p}_i}{p_i^{c+2}} = \frac{d\bar{p}_i}{\bar{p}_i - x} \frac{d\bar{p}_i}{(p_i - x)^2},\]

and so

\[W_0(p_i, x) = \frac{d\bar{p}_i dx}{(p_i - x)^2},\]

\[K(\bar{p}_i, x) = \frac{1}{d\bar{p}_i} \frac{1}{\bar{p}_i - x}.\]

From these examples, the reformulation for the general case can be easily deduced.
5. Applications to $\mathbb{P}^1$

In this Section we will present the applications to the case of $\mathbb{P}^1$. We first recall the three different methods to compute the $n$-point functions of stationary GW invariants of $\mathbb{P}^1$ mentioned earlier, then we note the explicit formulas obtained by these methods to carry out the computations for general $n$-point functions of GW invariants of $\mathbb{P}^1$ using our formula.

5.1. The formula of Okounkov-Pandharipande. We refer to the original paper by Okounkov-Pandharipande [14] for definitions and notations. They define the relative GW invariants of $\mathbb{P}^1$ parameterized by two partitions $\mu, \nu$ of the same size:

\begin{equation}
F_{\mu,\nu}^*(z_1, \cdots, z_n) = \sum_{k_1, \ldots, k_n = -2}^{\infty} \left\langle \mu, \prod_{i=1}^n \tau_{k_i}(Q), \nu \right\rangle \prod_{i=1}^n z_i^{k_i+1},
\end{equation}

then they show that they are given by expectation values of some operators on the fermionic Fock space:

\begin{equation}
F_{\mu,\nu}^*(z_1, \cdots, z_n) = \left\langle \prod_{i=1}^n \alpha_{\mu_i} \prod_{j=1}^n E_0(z_i) \prod \alpha_{-\nu_j} \right\rangle.
\end{equation}

Their formula for one-point function is

\begin{equation}
\sum_{g=0}^{\infty} z^{2g} \langle \mu, \tau_{2g-2+l(\mu)+l(\nu)}(Q), \nu \rangle^g = \frac{1}{|\text{Aut}(\mu)||\text{Aut}(\nu)|} \frac{\prod S(\mu_i z) \prod S(\nu_i z)}{S(z)},
\end{equation}

where, by definition,

\begin{equation}
S(z) = \frac{\sinh(z/2)}{z/2} = \sum_{k=0}^{\infty} \frac{z^{2k}}{2^{2k}(2k+1)!}.
\end{equation}

In particular, for $\mu = \nu = (1^d)$,

\begin{equation}
\sum_{g=0}^{\infty} z^{2g} \langle \tau_{2g-2+2d}(Q) \rangle^g_{g,1,d} = \frac{1}{(d!)^2} S(z)^{2d-1}.
\end{equation}

They also give an algorithm to inductively compute the $n$-point functions. An alternative way they give is to show that a certain partition function satisfies the Toda equations, and this is equivalent to the
following recurrence relation for n-point functions. For any partitions \( \mu \) and \( \nu \) of the same size, we have

\[
F_{\mu+1,\nu+1}^0(z_1, \ldots, z_n) = \frac{1}{(m_1(\mu) + 1)(m_1(\nu) + 1)} \sum_{\{S_i, \mu^i, \nu^i\}} \prod_i \zeta(\Sigma S_i)^2 F_{\mu^i, \nu^i}^0(z_{S_i}),
\]

(13)

where \( \zeta(z) = 2 \sinh(z/2) \), and the summation is over all sets of triples \( \{(S_i, \mu^i, \nu^i)\} \), such that \( \{S_i\} \) is a partition of the set \( [n] = \{1, \ldots, n\} \) into nonempty disjoint subsets:

\[ [n] = \bigsqcup_i S_i, \quad S_i \neq \emptyset, \]

similarly, \( \{\mu^i\} \) and \( \{\nu^i\} \) satisfy

\[ \mu = \bigsqcup \mu^i, \quad \nu = \bigsqcup \nu^i, \quad |\mu^i| = |\nu^i|, \]

and where, by definition, \( z_S = \{z_i\}_{i \in S} \) and \( \Sigma S = \sum_{i \in S} z_i \).

The following are some samples of computations using this formula:

\[
F_{(1),(1)}^0(z_1, z_2) = \zeta(z_1 + z_2)^2 F_{\emptyset,\emptyset}^0(z_1, z_2)
+ \zeta(z_1)^2 F_{\emptyset,\emptyset}^0(z_1) \cdot \zeta(z_2)^2 F_{\emptyset,\emptyset}^0(z_2)
= \zeta(z_1)^2 \cdot \frac{1}{\zeta(z_1)} \cdot \frac{1}{\zeta(z_2)}
= \zeta(z_1) \zeta(z_2),
\]

\[
F_{(1^2),(1^2)}^0(z_1, z_2) = \frac{1}{2 \cdot 2} \left[ \zeta(z_1 + z_2)^2 F_{(1),(1)}^0(z_1, z_2)
+ \zeta(z_1)^2 F_{\emptyset,\emptyset}^0(z_1) \cdot \zeta(z_2)^2 F_{(1),(1)}^0(z_2)
+ \zeta(z_1)^2 F_{(1),(1)}^0(z_1) \cdot \zeta(z_2)^2 F_{\emptyset,\emptyset}^0(z_2) \right]
= \frac{1}{4} \left[ \zeta(z_1 + z_2)^2 \zeta(z_1) \zeta(z_2)
+ \zeta(z_1)^2 \cdot \frac{1}{\zeta(z_1)} \cdot \zeta(z_2)^2 \cdot \zeta(z_2)
+ \zeta(z_1)^2 \cdot \zeta(z_1) \cdot \zeta(z_2)^2 \cdot \frac{1}{\zeta(z_2)} \right]
= \frac{\zeta(z_1) \zeta(z_2)}{2!^2} [\zeta(z_1 + z_2)^2 + (\zeta(z_1)^2 + \zeta(z_2)^2)].
\]
Here is another example:

\[ F_{(1^3), (1^3)}^\infty (z_1, z_2) = \frac{\zeta(z_1)\zeta(z_2)}{3!^2 \lambda^4} \left[ (z_1 + z_2)^4 + \zeta(z_1 + z_2)^2(\zeta(z_1)^2 + \zeta(z_2)^2) \right] \]

\[ + (\zeta(z_1)^4 + 2!^2 \zeta(z_1)^2 \zeta(z_2)^2 + \zeta(z_2)^4) \].

In general,

\[ F_{(1^m), (1^m)}^\infty (z_1, z_2) \]

\[ \frac{\zeta(z_1)\zeta(z_2)}{m!^2} \sum_{k=0}^{m-1} \zeta(z_1 + z_2)^{2m-2-2k} \sum_{j=0}^{k} \binom{k}{j}^2 \zeta(z_1)^{2j} \zeta(z_2)^{2k-2j}. \]

5.2. One-Point function by the formula of Okounkov-Pandharipande.

For the one-point function by \{(12)\} one has

\[ \sum_{g=0}^{\infty} \lambda^{2g} \sum_{d=0}^{\infty} q^d \langle \tau_{2g-2+2d}(Q) \rangle_{g,1;d} \cdot (2g - 1 + 2d)!x^{-2g-2d} \]

\[ = \sum_{g=0}^{\infty} \lambda^{2g} \sum_{d=0}^{\infty} q^d \cdot (2g - 1 + 2d)!x^{-2g-2d} \text{res}_{w=0} \frac{S(w)^{2d-1}}{w^{2g+1}d!^2} \]

Now one can proceed in two different ways. First one can take the summation over \( d \) to get:

\[ \sum_{g=0}^{\infty} \lambda^{2g} \text{res}_{w=0} \left( \frac{S(w)^{-1}}{z^{2g}w^{2g+1}} \sum_{d=0}^{\infty} q^d \cdot \frac{(2g - 1 + 2d)!}{d!^2} x^{-2d} S(w)^{2d} \right) \]

\[ = \text{res}_{w=0} \left( -\frac{S(w)^{-1}}{w} \ln \frac{1 + \sqrt{1 - 4qS(w)^2/x^2}}{2} \right) \]

\[ + \sum_{g=1}^{\infty} \lambda^{2g} \text{res}_{w=0} \left( \frac{S(w)^{-1}}{x^{2g}w^{2g+1}} \frac{(2g - 1)!}{(1 - 4qS(w)^2/x^2)^{(4g-1)/2}} \sum_{j=0}^{g-1} \frac{(2g - 1)!}{(2g - 1 - 2j)!(j!)^2} \right) \]

\[ = -\left( \ln \frac{x + \sqrt{x^2 - 4q}}{2} - \ln x \right) \]

\[ + \lambda^2 \frac{16q/x^2 - 1}{24z^2(1 - 4q/x^2)^{5/2}} \]

\[ + \lambda^4 \frac{18432(q/x^2)^3 + 8256(q/x^2)^2 - 94q/x^2 + 7}{960x^4(1 - 4q/x^2)^{11/2}} \]

\[ + \lambda^6 \frac{1}{8064x^6(1 - 4q/x^2)^{17/2}} \left[ -31 + 1180q/x^2 + 134886(q/x^2)^2 \right. \]

\[ + 5419360(q/x^2)^3 + 23229440(q/x^2)^4 + 13271040(q/x^2)^5 \] + \ldots \].
From this one can see that for $g = 0$,
\begin{equation}
\sum_{d=1}^{\infty} q^d (\tau_{2d-2}(Q))_{0,1;d} \cdot (2d - 1)! x^{-2d} = \ln x - \ln \frac{x + \sqrt{x^2 - 4g}}{2},
\end{equation}
and for $g \geq 1$,
\begin{equation}
\sum_{d=0}^{\infty} q^d (\tau_{2g-2+2d}(Q))_{g,1;d} \cdot (2g - 1 + 2d)! x^{-2g-2d}
\end{equation}
has the shape
\begin{equation}
\frac{1}{x^{2g+2}(1 - 4q/x^2)(6g-1)/2} \sum_{j=0}^{2g-1} a_{g,j} \left(\frac{q}{x^2}\right)^j.
\end{equation}

According to [13], the coefficients
\begin{equation}
\frac{n!}{(n - 2j)!j!j!}
\end{equation}
are the sequence A089627. They are the $\gamma$-vectors of n-dimensional type B associahedra. This suggests some possibility of connection with the theory of cluster algebras. They have the following generating series:
\begin{equation}
\sum_{n=0}^{\infty} \sum_{j=0}^{[n/2]} \frac{n!}{(n - 2j)!j!j!} x^ny^j = \sum_{m=0}^{\infty} \sum_{j=0}^{\infty} \frac{(m + 2j)!}{m!j!j!} (x^2y)^j
\end{equation}
\begin{equation}
= \sum_{j=0}^{\infty} \frac{(x^2y)^j}{j!2} \sum_{m=0}^{\infty} \frac{(m + 2j)!}{m!} x^m \sum_{j=0}^{\infty} \frac{(x^2y)^j}{j!2} \frac{(2j)!}{(1 - x)^{2j}}
\end{equation}
\begin{equation}
= \frac{1}{\sqrt{1 - \frac{4x^2y}{(1-x)^2}}},
\end{equation}
therefore,
\begin{equation}
\sum_{g=0}^{\infty} \sum_{j=0}^{g-1} \frac{(2g - 1)!}{(2g - 1 - 2j)!j!j!} x^{2g-1} y^j = \frac{1}{2} \frac{1}{\sqrt{1 - \frac{4x^2y}{(1-x)^2}}} - \frac{1}{2} \frac{1}{\sqrt{1 - \frac{4x^2y}{(1+x)^2}}}.
\end{equation}

Unfortunately, there is an extra factor $(2g - 1)!$ that prevents us to use this summation to simplify the above computations.
Secondly for \( d > 0 \) one can expand \( S(w)^{2d-1} \) as follows:

\[
S(w)^{2d-1} = \frac{1}{w^{2d-1}} (e^{w/2} - e^{-w/2})^{2d-1}
\]

\[
= \frac{1}{w^{2d-1}} \sum_{l=0}^{d-1} \binom{2d-1}{l} (-1)^l (e^{(2d-1-2l)w/2} - e^{-(2d-1-2l)w/2})
\]

\[
= \frac{1}{w^{2d-1}} \sum_{l=0}^{d-1} \binom{2d-1}{l} (-1)^l \sum_{j=0}^{\infty} \frac{(2d - 1 - 2l)^{2j+1}}{2^{2j}(2j + 1)!} w^{2j+1},
\]

and for \( d = 0 \),

\[
S(w)^{-1} = \frac{w}{e^{w/2} - e^{-w/2}} = \sum_{m=0}^{\infty} \frac{B_{2m}}{(2m)!} (2^{1-2m} - 1) w^{2m},
\]

\[
\sum_{g=0}^{\infty} \lambda^{2g} \sum_{d=0}^{\infty} q^d \langle \tau_{2g-2+2d}(Q) \rangle_{g,1,d} \cdot (2g - 1 + 2d)! x^{-2g-2d}
\]

\[
= \sum_{g=0}^{\infty} \lambda^{2g} \cdot (2g - 1)! x^{-2g} \frac{B_{2g}}{(2g)!} (2^{1-2g} - 1)
\]

\[
+ \sum_{j=0}^{\infty} \sum_{g+d=j+1; g \geq 0, d \geq 1} \lambda^{2g} q^d (2g - 1 + 2d)! x^{-2g-2d}
\]

\[
\cdot \frac{1}{d!^2} \sum_{l=0}^{d-1} \binom{2d-1}{l} (-1)^l \frac{(2d - 1 - 2l)^{2j+1}}{2^{2j}(2j + 1)!}
\]

\[
= \sum_{g=0}^{\infty} \lambda^{2g} \cdot x^{-2g} \frac{B_{2g}}{2g} (2^{1-2g} - 1)
\]

\[
+ \sum_{j=0}^{\infty} \frac{1}{4j^2 d!} \sum_{d=1}^{\infty} \lambda^{2(j+1-d)} q^d \frac{d!^2}{d!^2} \sum_{l=0}^{d-1} \binom{2d-1}{l} (-1)^l (2d - 1 - 2l)^{2j+1}.
\]

5.3. Two-point function by the formula of Okounkov-Pandharipande.

By (14) we get

\[
\sum_{g=0}^{\infty} \sum_{k_1, k_2 \geq 0} \langle \tau_{k_1}(Q) \tau_{k_2}(Q) \rangle_{g,2;d; \zeta z_1^{k_1+1}, \zeta z_2^{k_2+1}}
\]

\[
= \frac{\zeta(z_1)\zeta(z_2)}{d!^2} \sum_{k=0}^{d-1} \zeta(z_1 + z_2)^{2d-2-2k} \sum_{j=0}^{k} \binom{k}{j}^2 \zeta(z_1)^j \zeta(z_2)^{2k-j}.
\]
By the selection rule:

\[(k_1 + 1) + (k_2 + 1) = 2g - 2 + 2d + 2 = 2(g + d),\]

therefore,

\[
\sum_{d=0}^{\infty} q^d \sum_{g=0}^{\infty} \lambda^{2g} \sum_{k_1, k_2 \geq 0} \langle \tau_{k_1} (Q) \tau_{k_2} (Q) \rangle_{g, 2d, \frac{k_1 + 1}{2}, \frac{k_2 + 1}{2}} = \sum_{d=0}^{\infty} \sum_{g=0}^{\infty} \sum_{k_1, k_2 \geq 0} \langle \tau_{k_1} (Q) \tau_{k_2} (Q) \rangle_{g, 2d, (\lambda z_1)^{k_1 + 1}, (\lambda z_2)^{k_2 + 1}}
\]

\[
= \sum_{d=0}^{\infty} q^d \zeta (\lambda z_1) \zeta (\lambda z_2) \frac{d l^2 \lambda^{2d}}{d} \sum_{k=0}^{d-1} \zeta (\lambda z_1 + \lambda z_2)^{2d - 2 - 2k} \cdot \sum_{j=0}^{k} \left( \begin{array}{c} k \\ j \end{array} \right)^2 \zeta (\lambda z_1)^{2j} \zeta (\lambda z_2)^{2k - 2j}.
\]

If we consider the terms with \(\lambda^0\), we get:

\[
\sum_{d=0}^{\infty} q^d \sum_{k_1, k_2 \geq 0} \langle \tau_{k_1} (Q) \tau_{k_2} (Q) \rangle_{0, 2d, \frac{k_1 + 1}{2}, \frac{k_2 + 1}{2}} = \sum_{d=0}^{\infty} q^d \frac{z_1 z_2}{d l^2} \sum_{k=0}^{d-1} (z_1 + z_2)^{2d - 2 - 2k} \sum_{j=0}^{k} \left( \begin{array}{c} k \\ j \end{array} \right)^2 z_1^{2j} z_2^{2k - 2j}
\]

\[
= \sum_{d=0}^{\infty} q^d \frac{z_1 z_2}{d l^2} \sum_{k=0}^{d-1} \sum_{l=0}^{2d - 2 - 2k} \left( \begin{array}{c} 2d - 2 - 2k \\ l \end{array} \right) z_1^{l} z_2^{2d - 2 - 2k - l} \sum_{j=0}^{k} \left( \begin{array}{c} k \\ j \end{array} \right)^2 z_1^{2j+l+1} z_2^{2d - 1 - l - 2j}.
\]

Because \(k_1 + k_2 = 2d - 2\) is even, we have two cases to consider. Case 1. \(k_1 = 2j + l = 2n_1, k_2 = 2d - 2 - l - 2j = 2n_2\),

\[
\langle \tau_{2n_1} (Q) \tau_{2n_2} (Q) \rangle_{0, 2n_1 + n_2 + 1} = \frac{q^{n_1 + n_2 + 1}}{(n_1 + n_2 + 1)!^2} \sum_{k=0}^{n_1 + n_2} \sum_{j=0}^{k} \left( \begin{array}{c} 2n_1 + 2n_2 - 2k \\ 2n_1 - 2j \end{array} \right) \left( \begin{array}{c} k \\ j \end{array} \right)^2.
\]

Case 2. \(k_1 = 2j + l = 2n_1 - 1, k_2 = 2d - 2 - l - 2j = 2n_2 - 1\),

\[
\langle \tau_{2n_1 - 1} (Q) \tau_{2n_2 - 1} (Q) \rangle_{0, 2n_1 + n_2} = \frac{q^{n_1 + n_2}}{(n_1 + n_2)!^2} \sum_{k=0}^{n_1 + n_2} \sum_{j=0}^{k} \left( \begin{array}{c} 2n_1 + 2n_2 - 2k \\ 2n_1 - 2j - 1 \end{array} \right) \left( \begin{array}{c} k \\ j \end{array} \right)^2.
\]
By using TRR in genus zero, string equation and the divisor equation one can show that \[3, 12\]:

\[
\langle \tau_{2n_1}(Q) \tau_{2n_2}(Q) \rangle_{0,2;n_1+n_2+1} = \frac{q^{n_1+n_2+1}}{n_1 + n_2 + 11!n_2!};
\]

\[
\langle \tau_{2n_1-1}(Q) \tau_{2n_2-1}(Q) \rangle_{0,2;n_1+n_2} = \frac{q^{n_1+n_2}}{n_1 + n_2 11!n_2!}.
\]

So one gets two combinatorial identities:

\[
\frac{1}{(n_1 + n_2 + 1)!^2} \sum_{k=0}^{n_1+n_2} \sum_{j=0}^{k} \left( \frac{2n_1 + 2n_2 - 2k}{2n_1 - 2j} \right) \binom{k}{j}^2 = \frac{1}{n_1 + n_2 + 11!n_2!},
\]

\[
\frac{1}{(n_1 + n_2)!^2} \sum_{k=0}^{n_1+n_2-1} \sum_{j=0}^{k} \left( \frac{2n_1 + 2n_2 - 2k}{2n_1 - 2j - 1} \right) \binom{k}{j}^2 = \frac{1}{n_1 + n_2 11!n_2!}.
\]

Reversely, if one proves these identities, then one can use them to get \([18]\).

If we consider the terms with $\lambda^2$, we get:

\[
\sum_{d=0}^{\infty} q^d \sum_{k_1, k_2 \geq 0} \langle \tau_{k_1}(Q) \tau_{k_2}(Q) \rangle_{1,2; d} z_1^{k_1+1} z_2^{k_2+1}
\]

\[
= -\frac{1}{24} \sum_{d=0}^{\infty} q^d \sum_{k=0}^{d-1} \frac{z_1^2 z_2^2 + 2(z_1 z_2)^{d-2} }{d!} \sum_{k=0}^{2d-2-2k} \sum_{j=0}^{k} \binom{k}{j}^2 z_1^{2j} z_2^{2k-2j}
\]

\[
- \frac{1}{24} \sum_{d=0}^{\infty} q^d \sum_{k=0}^{d-1} (2d-2)(z_1 + z_2)^{2d-2-2k} \sum_{j=0}^{k} \binom{k}{j}^2 z_1^{2j} z_2^{2k-2j}
\]

\[
- \frac{1}{24} \sum_{d=0}^{\infty} q^d \sum_{k=0}^{d-1} (z_1 + z_2)^{2d-2-2k} \sum_{j=0}^{k} \binom{k}{j}^2 z_1^{2j} z_2^{2k-2j}
\]

It becomes more involved to extract information about the two-point correlators in higher genera.
Another way to extract some information from (17) is to take the coefficients of $z^k$ on both sides. For $k = 1$ we get:

$$
\sum_{d=0}^{\infty} q^d \sum_{g=0}^{\infty} \lambda^{2g} \sum_{k_2 \geq 0} \langle \tau_0(Q) \tau_{k_2}(Q) \rangle_{g,2d} z_2^{k_2+1} = \sum_{d=1}^{\infty} q^d \frac{1}{d!(d-1)! \lambda^{2d-1}} \zeta(\lambda z_2)^{2d-1}.
$$

The right-hand side is essentially Bessel function of order one. One can expand the right-hand side as follows:

$$
\sum_{d=1}^{\infty} q^d \frac{1}{d!(d-1)! \lambda^{2d-1}} (e^{\lambda z_2/2} - e^{-\lambda z_2/2})^{2d-1}
$$

$$
= \sum_{d=1}^{\infty} q^d \frac{1}{d!(d-1)! \lambda^{2d-1}} \sum_{j=0}^{d-1} (-1)^j \binom{2d-1}{j} (e^{(2d-1-2j)\lambda z_2/2} - e^{-(2d-1-2j)\lambda z_2/2})
$$

$$
= \sum_{d=1}^{\infty} q^d \frac{1}{d!(d-1)! \lambda^{2d-1}} \sum_{j=0}^{d-1} (-1)^j \binom{2d-1}{j} \sum_{l=0}^{\infty} \frac{(2d - 2j)^{2l+1} \lambda^{2l+1} \zeta^{2l+1}}{4^l (2l+1)!}
$$

Similarly, the coefficients of $z^2$ on both sides of (17) give us

$$
\sum_{d=0}^{\infty} q^d \sum_{g=0}^{\infty} \lambda^{2g} \sum_{k_2 \geq 0} \langle \tau_1(Q) \tau_{k_2}(Q) \rangle_{g,2d} z_2^{k_2+1} = \sum_{d=0}^{\infty} q^d \frac{\lambda \zeta(\lambda z_2)}{d! \lambda^{2d}} \sum_{k=0}^{d-1} (2d - 2 - 2k) \lambda \zeta(\lambda z_2)^{2d-1-2k}.
$$

One can get similar formulas for other $k = k_1 + 1$ in the same fashion.

5.4. The formula of Dubrovin and Yang. The examples in the last two Subsections indicate that it involves some complicated combinatorial identities to simplify the formulas derived from the formula of Okounkov and Pandharipande. Dubrovin and Yang define

$$
C_n(x_1, \ldots, x_n; \lambda) := \lambda^n \sum_{k_1, \ldots, k_n \geq 0} \langle \tau_{k_1}(Q) \cdots \tau_{k_n}(Q) \rangle \prod_{i=1}^{n} \frac{(k_i + 1)!}{x_i^{k_i+2}}.
$$

They conjecture the following formulas. Define a $2 \times 2$ matrix-valued series by

$$
R(x; \lambda) := \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} + \begin{pmatrix} \alpha(x; \lambda) & \beta(x; \lambda) \\ \gamma(x; \lambda) & -\alpha(x; \lambda) \end{pmatrix}
$$
where

\[
\alpha(x; \lambda) = \sum_{j=0}^{\infty} \frac{1}{j!} \sum_{i=0}^{j} \frac{\epsilon^{2(j-i)}}{i!(i+1)!} \sum_{l=0}^{i} (-1)^l (2i + 1 - 2l)^{2j+1} \left( \begin{array}{c} 2i + 1 \\ l \end{array} \right),
\]

\[
\gamma(x; \lambda) = Q(x; \lambda) + P(x; \lambda),
\]

\[
\beta(x; \lambda) = Q(x; \lambda) - P(x; \lambda),
\]

\[
P(x; \lambda) := \sum_{j=0}^{\infty} \frac{1}{\lambda^{2j+1}} \sum_{i=0}^{j} \frac{\epsilon^{2(j-i)}}{i!^2} \sum_{l=0}^{i} (-1)^l (2i + 1 - 2l)^{2j} \left( \begin{array}{c} 2i \\ l \end{array} \right) - \left( \begin{array}{c} 2i \\ l - 1 \end{array} \right),
\]

\[
Q(x; \epsilon) := -\frac{1}{2} \sum_{j=0}^{\infty} \frac{1}{\lambda^{2j+2}} \sum_{i=0}^{j} \frac{\epsilon^{2(j-i)+1}}{i!^2} \sum_{l=0}^{i} (-1)^l (2i + 1 - 2l)^{2j} \left( \begin{array}{c} 2i \\ l \end{array} \right) - \left( \begin{array}{c} 2i \\ l - 1 \end{array} \right).
\]

Then

\[
C_2(x_1, x_2; \lambda) = \frac{\text{tr}[R(x_1；\lambda)R(x_2；\lambda)] - 1}{(x_1 - x_2)^2},
\]

and for \(n \geq 3\),

\[
C_n(x_1, \ldots, x_n; \lambda) = -\frac{1}{n} \sum_{\sigma \in S_n} \frac{\text{tr}[R(x_{\sigma_1；\lambda}) \cdots R(x_{\sigma_n；\lambda})]}{(x_{\sigma_1 - x_{\sigma_2}}) \cdots (x_{\sigma_{n-1} - x_{\sigma_n}})(x_{\sigma_n - x_{\sigma_1}})}.
\]

This conjecture has been proved by Marchal [11] using Eynard-Orantin topological recursions and by Dubrovin-Yang-Zagier [6] using Toda equations.

5.5. **Eynard-Orantin topological recursions for stationary GW invariants of \(P^1\).** Let us now recall some details about the Eynard-Orantin topological recursions [8] satisfied by the stationary Gromov-Witten invariants of \(P^1\), conjectured by Norbury and Scott [12], proved in [7] and extended to the equivariant case by Fang-Liu-Zong [9]. We will interpret some aspects of these results from the point of view of emergent geometry. In genus zero,

\[
\langle \tau_{2n}(Q) \rangle_0 = \frac{1}{(n+1)!^2},
\]
we have seen in \[5.2\] that

\[
W_{0,1}(x) := \sum_{n=0}^{\infty} \langle \tau_{2n}(\omega) \rangle_0 \frac{(2n + 1)!}{x^{2n+2}} = \sum_{n=0}^{\infty} \frac{(2n + 1)!q^{2n+1}}{(n + 1)!2} \frac{1}{w^{2n+2}} = \ln x - \ln \frac{x + \sqrt{x^2 - 4q}}{2}.
\]

Define

\[
y(x) := W_{0,1}(x) - \ln x - \ln q.
\]

Then we have

(22) \[y(x) = W_{0,1}(x) - \ln x - \ln q.
\]

and so

(23) \[y = \ln \frac{x - \sqrt{x^2 - 4q}}{2},
\]

and

(24) \[e^y = \frac{x - \sqrt{x^2 - 4q}}{2},
\]

and

(25) \[e^{-y} = \frac{x + \sqrt{x^2 - 4q}}{2q}.
\]

Therefore,

(26) \[x = e^y + qe^{-y}.
\]

If we set \(z = e^y\), then

(27) \[x = z + \frac{q}{z}, \quad y = \ln z.
\]

This shows that the spectral curve is given by the one-point function in genus zero. Next we show that Bergman kernel is given by the two-point function in genus zero. By \[18\], we have

\[
W_{0,2}(x_1, x_2; q) = \sum_{n_1, n_2=0}^{\infty} \langle \tau_{2n_1}(Q)\tau_{2n_2}(Q) \rangle_0 \frac{(2n_1 + 1)!}{x_1^{2n_1+2}} \frac{(2n_2 + 1)!}{x_2^{2n_2+2}}
\]

\[
+ \sum_{n_1, n_2=1}^{\infty} \langle \tau_{2n_1-1}(Q)\tau_{2n_2-1}(Q) \rangle_0 \frac{(2n_1)!}{x_1^{2n_1+1}} \frac{(2n_2)!}{x_2^{2n_2+1}}
\]

\[
= \sum_{n_1, n_2=0}^{\infty} \frac{1}{n_1!^2n_2!^2} \frac{q^{n_1+n_2+1}}{n_1 + n_2 + 1} \frac{(2n_1 + 1)!}{x_1^{2n_1+2}} \frac{(2n_2 + 1)!}{x_2^{2n_2+2}}
\]

\[
+ \sum_{n_1, n_2=1}^{\infty} \frac{n_1 n_2}{n_1!^2n_2!^2} \frac{q^{n_1+n_2}}{n_1 + n_2} \frac{(2n_1)!}{x_1^{2n_1+1}} \frac{(2n_2)!}{x_2^{2n_2+1}}.
\]
To take the summations, we take derivatives with respect to $q$ to get:

\[
\frac{\partial}{\partial q} W_{0,2}(x_1, x_2) = \sum_{n_1, n_2=0}^{\infty} \frac{1}{n_1! n_2!} q^{n_1+n_2} \sum_{n_1, n_2=0}^{\infty} \frac{(2n_1+1)! (2n_2+1)!}{x_1^{2n_1+2} x_2^{2n_2+2}}
\]

\[
+ \sum_{n_1, n_2=1}^{\infty} \frac{n_1 n_2}{n_1! n_2!} q^{n_1+n_2-1} \frac{(2n_1)! (2n_2)!}{x_1^{2n_1+1} x_2^{2n_2+1}}
\]

\[
= \sum_{n_1=0}^{\infty} \frac{q^{n_1} (2n_1+1)!}{n_1!} \frac{1}{x_1^{2n_1+2}} \sum_{n_2=0}^{\infty} \frac{q^{n_2} (2n_2+1)!}{n_2!} \frac{1}{x_2^{2n_2+2}}
\]

\[
+ q \sum_{n_1=1}^{\infty} \frac{n_1}{n_1!} q^{n_1-1} \frac{(2n_1)!}{x_1^{2n_1+2}} \sum_{n_2=1}^{\infty} \frac{n_2}{n_2!} q^{n_2-1} \frac{(2n_2)!}{x_2^{2n_2+2}}
\]

\[
= \frac{x_1}{(x_1^2 - 4q)^{3/2}} \cdot \frac{x_2}{(x_2^2 - 4q)^{3/2}}
\]

After integration we get:

(28) \( W_{0,2}(x_1, x_2) = \frac{x_1 x_2 - 4q}{2\sqrt{x_1^2 - 4q} \sqrt{x_2^2 - 4q(x_1 - x_2)^2}} - \frac{1}{2(x_1 - x_2)^2} \),

where the right-hand side of this equality is understood as a power series in $q$:

\[
\frac{1}{2(x_1 - x_2)^2} \left( (1 - 4q/(x_1 x_2))(1 - 4q/x_1^2)^{-1/2}(1 - 4q/x_2^2)^{1/2} - 1 \right).
\]

The Bergman kernel is then given by:

(29) \( B(x_1, x_2) = \frac{dx_1 dx_2}{(x_1 - x_2)^2} + W_{0,2}(x_1, x_2) dx_1 dx_2. \)

By straightforward computations we then get the following:

**Proposition 5.1.** When $x_1 = z_1 + q/z_1$ and $x_2 = z_2 + q/z_2$, the Bergman kernel is given by

(30) \( B(x_1, x_2) = \frac{dz_1 dz_2}{(z_1 - z_2)^2}. \)

So now all the needed data

(31) \( x(z) = z + \frac{q}{z}, \quad y(z) = \ln z, \quad B(z_1, z_2) = \frac{dz_1 dz_2}{(z_1 - z_2)^2} \)

emerge from the computations of the one-point function and the two-point function of stationary GW invariants of $\mathbb{P}^1$. As conjectured in [12] and proved in [7], the Eynard-Orantin topological recursion for such
data produce a sequence of multilinear differentials $\omega_{g,n}(z_1, \ldots, z_n)$. After the change of coordinates $x_i = z_i + q/z_i$, one has

\begin{equation}
\omega_{g,n}(z_1, \ldots, z_n) = \sum_{k_1, \ldots, k_n \geq 0} \langle \tau_{k_1}(Q) \cdots \tau_{k_n}(Q) \rangle \prod_{i=1}^{n} \frac{(k_i + 1)! dx_i}{x_i^{k_i + 2}}.
\end{equation}

See also [9] and [2].

5.6. Computations of general $n$-point functions of $\mathbb{P}^1$. Using Theorem 3.1, one can compute the $n$-point function of GW invariants of $\mathbb{P}^1$: They can all be reduced to the $n$-point function of stationary GW invariants of $\mathbb{P}^1$. For example,

\begin{align*}
W(u_0) &= \sum_{n=0}^{\infty} \langle \tau_n(P) \rangle \cdot \frac{(n + 1)!}{u_0^{n+1}} \\
&= \text{res}_{x=0}(2L(u_0, x)W(x)) + \frac{1}{2} \text{res}_{x=0}(2K(u_0, x)W(x, x)) \\
&\quad + \frac{1}{2} \text{res}_{x=0}(2K(u_0, x)W(x)W(x)) \\
&= \text{res}_{x=0}(2 \log (1 - \frac{x}{u_0}) \frac{du_0}{u_0 - x} W(x)) \\
&\quad + \frac{1}{2} \text{res}_{x=0} \left( 2 \frac{1}{u_0 - x} \frac{du_0}{dx} W(x, x) \right) \\
&\quad + \frac{1}{2} \text{res}_{x=0} \left( 2 \frac{1}{u_0 - x} \frac{du_0}{dx} W(x) W(x) \right).
\end{align*}

For the genus zero part we have:

\begin{align*}
W_{0,1}(u_0) &= \sum_{n=0}^{\infty} \langle \tau_{2d-1}(P) \rangle_{0,1; d} \cdot \frac{(2d)!}{u_0^{2d}} \\
&= \text{res}_{x=0} \left( 2 \log \left( \frac{x}{u_0} \right) \frac{du_0}{u_0 - x} \cdot W_{0,1}(x) \right) \\
&\quad + \frac{1}{2} \text{res}_{x=0} \left( 2 \frac{1}{u_0 - x} \frac{du_0}{dx} W(x, x) \right) \\
&\quad + \frac{1}{2} \text{res}_{x=0} \left( 2 \frac{1}{u_0 - x} \frac{du_0}{dx} \cdot W_{0,1}(x) \cdot W_{0,1}(x) \right),
\end{align*}

where

\begin{equation}
W_{0,1}(x) = \ln \frac{x - \sqrt{x^2 - 4q}}{2} - \ln x = \sum_{d=1}^{\infty} \frac{(2d - 1)!}{d!^2} q^d x^{2d} dx,
\end{equation}

\begin{equation}
W_{0,1}(x) = \ln \frac{x - \sqrt{x^2 - 4q}}{2} - \ln x = \sum_{d=1}^{\infty} \frac{(2d - 1)!}{d!^2} q^d x^{2d} dx,
\end{equation}
and by (28), we have
\[ W_{0,2}(x, x) = \frac{q}{(x^2 - 4q)^2}. \]
A calculation gives us
\[ W_{0,1}(u_0) = -2 \sum_{d=1}^{\infty} \frac{(2d - 1)!}{d!^2} \sum_{j=1}^{2d-1} \frac{1}{j} u_0^{-2d} q^d + \frac{q}{(u_0^2 - 4q)^2} + \left( \ln u_0 - \sqrt{u_0^2 - 4q} - \ln u_0 \right)^2. \]

6. Concluding Remarks

In this note we have reformulated the strengthened Virasoro constraints proved by Okounkov and Pandharipande [15] as topological recursion relations similar to the Eynard-Orantin topological recursions [8]. In [15] it is also shown that there is another way to strengthen the standard Virasoro constraints to the case of relative Gromov-Witten invariants of the curves. The reformulation can be done in exactly the same way.

In the standard Eynard-Orantin formalism the insertions of odd observables do not seem to be have been considered in the literature. So this work suggests an extension in this direction.

Another direction of research suggested by this work is to consider the removal of the operators of the form \( \tau_n(Q) \) in the correlators, in other words, to find topological recursions for
\[ W(x, u_1, \ldots, u_n, p_{i_1}^1, \ldots, p_{i_k}^k, \bar{p}_{j_1}^1, \ldots, \bar{p}_{j_l}^l, x_1, \ldots, x_m) \]
of the EO type.

The earlier work [17] and this work have reformulated the Virasoro constraints of the Gromov-Witten invariants of a point and an algebraic curves respectively. In a work in preparation we will generalize to other spaces and Frobenius manifolds.

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