Fully Dynamic Data Structures for Interval Coloring

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Abstract

We consider the dynamic graph coloring problem restricted to the class of interval graphs. At each update step the algorithm is presented with an interval to be colored, or a previously colored interval to delete. The goal of the algorithm is to efficiently maintain a proper coloring of the intervals with as few colors as possible by an online algorithm. In the incremental model, each update step presents the algorithm with an interval to be colored. The problem is closely connected to the online vertex coloring problem of interval graphs for which the Kierstead-Trotter (KT) algorithm achieves the best possible competitive ratio. We first show that a sub-quadratic time direct implementation of the KT-algorithm is unlikely to exist conditioned on the correctness of the Online Boolean Matrix Vector multiplication conjecture due to Henzinger et al. [10]. We then design an incremental algorithm that is subtly different from the KT-algorithm and uses at most \(3\omega - 2\) colors, where \(\omega\) is the maximum clique in the interval graph associated with the set of intervals. Our incremental data structure maintains a proper coloring in amortized \(O(\log n + \Delta)\) update time where \(n\) is the total number of intervals inserted and \(\Delta\) is the maximum degree of a vertex in the interval graph. We then consider the fully dynamic framework involving insertions and deletions. On each update, our aim is to maintain a \(3\omega - 2\) coloring of the remaining set of intervals, where \(\omega\) is the maximum clique in the interval graph associated with the remaining set of intervals. Our fully dynamic algorithm supports insertion of an interval in \(O(\log n + \Delta \log \omega)\) worst case update time and deletion of an interval in \(O(\Delta^2 \log n)\) worst case update time.

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Introduction

The graph coloring problem is an extensively studied problem. Similarly, maintenance of data structures for dynamic graphs has been extensively studied. The dynamic graph coloring problem is as follows: There is an online update sequence of insertion or deletion of edges or vertices and the goal is to maintain proper coloring after every update. Several works ([1], [3], [5]) propose heuristic and experimental results on the dynamic graph coloring problem. To the best of our knowledge, the only formal analysis of data structures for dynamic graph coloring are [10], [3], and [2]. Bhattacharya et.al. gives the current best fully dynamic randomized algorithm which maintains \(\Delta + 1\) vertex coloring in \(O(\log \Delta)\) expected amortized update time [1]. They also give the current best deterministic algorithm which maintains \(\Delta + o(\Delta)\) vertex coloring in \(O(polylog\Delta)\) amortized update time [1].

In this work we study dynamic data structures for coloring of interval graphs when the input is an online update sequence of intervals. The goal is to efficiently maintain a proper coloring with as few colors as possible by an online algorithm. In the incremental model intervals are inserted one after the other and we aim to efficiently maintain a proper coloring using as few
colors as possible after every update. Our approach is to consider efficient implementations of well-studied online algorithms for interval coloring. Note that an online algorithm is not allowed to re-color a vertex during the execution of the algorithm. On the other hand, an incremental algorithm is not restricted in any way during an update step except that we desire that the updates be done as efficiently as possible. Naturally, an online algorithm is a good candidate for an incremental algorithm as it only assigns a color to the current interval, and does not change the colour of any of the other intervals. Online algorithms for interval coloring and its variants is a rich area with many results. Epstein et.al. studied online graph coloring for interval graphs [8]. They studied four variants of the problem: online interval coloring with bandwidth, online interval coloring without bandwidth, lazy online interval coloring with bandwidth, and lazy online interval coloring without bandwidth. For online interval coloring with bandwidth, Narayanaswamy presented an algorithm with competitive ratio 10 [13] and Epstein et. al. showed a lower bound of 3.2609 [8]. For lazy online interval coloring with bandwidth and lazy online interval coloring without bandwidth, Epstein et.al. proved that competitive ratio can be arbitrarily bad for any online algorithm [8]. For the online interval coloring problem, Kierstead and Trotter presented a 3 competitive algorithm and they also proved that their result is tight [11]. In other words, The online algorithm (KT-algorithm) due to Kierstead and Trotter [11] is known to have the optimum competitive ratio. The tightness is by showing the existence of an adaptive adversary that forces an online algorithm to use $3\omega - 2$ colors where $\omega$ is the maximum clique size in the interval graph formed by the given set of $n$ intervals. On the other hand, the KT-algorithm uses at most $3\omega - 2$ colors. The KT-algorithm computes a proper coloring by assigning to each vertex a color which is 2-tuple denoted by $(\text{level},\text{offset})$. The value of level is in the range $[0, \omega - 1]$ and the value of offset is from the set $\{1, 2, 3\}$. Further, all the vertices whose level value is 0 form an independent set. Therefore, an efficient implementation of KT-algorithm gives us an incremental algorithm that maintains a proper coloring with at most $3\omega - 2$ colors.

Our Work: In Section 2.2, we show that we are unlikely to be able to design a sub-quadratic implementation of the KT-algorithm. The reason for this is that the level value assigned to an interval depends on the size of the maximum clique in the graph induced by the intervals intersecting with it. We show a reduction from the Online Boolean Matrix-Vector Multiplication(OMv) problem [10] to the problem of computing the induced subgraph of the neighbours of a vertex. By the conjecture on the OMv problem, it is unlikely to have a sub-quadratic algorithm for the Induced Subgraph Computation problem. Thus, we believe that any algorithm that depends on computing an induced subgraph is unlikely to have a sub-quadratic dynamic algorithm, even when the graph is an interval graph. We design an incremental algorithm which avoids this limitation by using a different approach to compute level value for an interval. Thus, we differ from KT-algorithm in computing the level value for an interval. However, our algorithm uses the same number of colors as the KT-algorithm. Our incremental algorithm (Theorem 3) supports insertion of a sequence of intervals in amortized $O((\log n + \Delta))$ update time. We have also considered the fully dynamic framework in which an interval that has already been colored can be deleted, apart from the insertions. At the end of each update, our aim is to maintain a $3\omega - 2$ coloring of the remaining set of intervals, where $\omega$ is the maximum clique in the interval graph associated with the remaining set of intervals. Our fully dynamic algorithm (Theorem 13) supports insertion of an interval in $O((\log n + \Delta \log \omega))$ worst case update time and deletion of an interval in $O(\Delta^2 \log n)$ worst case update time. Finally, the question of significant interest to us is whether the dependence on $\Delta$ can be sub-linear in the incremental case and whether it can be sub-quadratic in the
fully dynamic case. Another interesting direction is the nature of the trade-off between the number of colors used and the update time if we allow a change of color assigned to an interval.

2 Preliminaries

$I$ denotes a set of intervals and the associated interval graph $G(I)$ is denoted by $G(I)$. For an undirected graph $G$, $\omega(G)$ denotes the size of the maximum cardinality clique in $G$, $\Delta(G)$ denotes the maximum degree of a vertex in $G$ and $\chi(G)$ denotes the chromatic number of $G$. It is well-known that for interval graphs $\omega(G) = \chi(G)$. When the context is clear we denote $\omega(G)$ as $\omega$ and $\Delta(G)$ as $\Delta$.

2.1 Kierstead-Trotter Algorithm - Overview

Let $\sigma = v_1, v_2, v_3, \ldots, v_n$ be the ordering of vertices of an interval graph $G = G(I)$. Each vertex in $\sigma$ is presented to an online algorithm as the corresponding interval. Let $I$ denote the set of intervals corresponding to the vertices and vertex $v_j$ corresponds to the interval $I_j \in I$. We also refer to each vertex and interval along with a subscript whose range is from 1 to $n$. For example, when we use $v_j$ or $I_j$ we mean the $j$-th vertex in $\sigma$. Throughout the paper, $v_j$ will be the vertex corresponding to the interval $I_j$. For a given vertex $v_i$ from $\sigma$ the algorithm computes a color based on the color given to the vertices $v_1, \ldots, v_{i-1}$. The color assigned to a vertex $v$ is a tuple of two values and is denoted as $(p(v), o(v))$. In Step I, the first value called the level or position of the presented vertex denoted by $p(v)$, is computed, and in Step II the second value, called the offset denoted by $o(v)$, is computed from the set $\{1, 2, 3\}$. The key property is that for each edge $\{u, v\}$, the tuples $(p(u), o(u))$ is different from $(p(v), o(v))$.

**Step I:** For $r \geq 0$, let $G_r(v_i)$ denote the induced subgraph of $G$ on the vertex set $\{v_i | v_j \in V(G), j < i, p(v_j) \leq r, (v_i, v_j) \in E(G)\}$. Define $p(v_i) = \min \{r | \omega(G_r(v_i)) \leq r\}$.

**Key Properties maintained by Step I:**

- For each vertex $v_i$, $p(v_i) \leq \omega - 1$.
- **Property P:** The set $\{v | \omega(v) = 0\}$ is an independent set. For each $i$, $1 \leq i \leq \omega - 1$, the subgraph of $G$ induced on $\{v_j | p(v_j) = i\}$ has maximum degree at most 2.

**Step II:** Since there are at most two neighbours of $v_i$ such that their level is $p(v_i)$, $o(v_i)$ is chosen to be the smallest value from the set $\{1, 2, 3\}$ different from the offset of these neighbours whose level is $p(v_i)$.

**Analysis:** All the vertices in level 0 form an independent set, and for all these vertices the offset value is 1. For the vertices in levels 1 to $\omega - 1$, maximum degree is 2. Therefore, the algorithm uses at most 3 colors to color all the vertices belonging to a particular level $l$ where $1 \leq l \leq \omega - 1$. Hence, total colors used by the algorithm is $1 + 3(\omega - 1) = 3\omega - 2$.

**Crucial step in the implementation of KT-algorithm:** Given a vertex $v_i$, the subgraph $G_r(v_i)$ is the induced subgraph among the neighbours of $v_i$ which have level value at most $r$. Computing $G_r(v_i)$ is a very crucial step in KT-algorithm for different values of $r$ starting from $r = 0$ until the that value of $r$ for which $\omega(G_r(v_i)) \leq r$ is true. In Section 2.2 we show that the search for such an $r$ can be done without computing the induced subgraphs $G_r(v_i), r \geq 0$. The motivation for this line of research is the result in Section 2.2.
2.2 KT-algorithm must avoid computing induced subgraphs

In this Section we show that the problem of computing the induced subgraph of the closed neighborhood of a set of vertices is unlikely to have a sub-quadratic time algorithm. The input to the Induced Neighbourhood Subgraph Computation problem consists of the adjacency matrix \( M \) of a directed graph and a set \( S \) of vertices. The aim is to compute the graph induced by \( N_{\text{out}}(S) \cup S \) and output the subgraph. Here \( N_{\text{out}}(S) \) is the set of those vertices which have a directed edge from some vertex in \( S \). In other words, there is a directed edge from \( v_j \) to \( v_k \) iff the entry \( M[k][j] \) is 1. Next we show that Induced Neighbourhood Subgraph Computation problem is at least as hard as the following problem.

**Online Boolean Matrix-Vector Multiplication** [10]: The input for this online problem consists of an \( n \times n \) matrix \( M \), and a sequence of \( n \) boolean column vectors \( v_1, \ldots, v_n \), presented one after another to the algorithm. For each \( 1 \leq i \leq n - 1 \), the online algorithm should output \( M \cdot v_i \) before \( v_{i+1} \) is presented to the algorithm. Note that in this product, a multiplication is an AND operation and the addition is an OR operation. According to the OMv conjecture, due to Henzinger et al. [10], the Online Boolean Matrix-Vector Multiplication problem does not have a \( O(n^{3-\epsilon}) \) algorithm for any \( \epsilon > 0 \). The current best algorithm for the Online Boolean Matrix-Vector Multiplication problems has an expected running time of \( O(n^{\frac{3}{2}} \log^n n) \) [12].

We now show that an algorithm to solve the Induced Neighbourhood Subgraph Computation problem can be used to solve the Online Boolean Matrix-Vector Multiplication problem. Let \( \mathcal{A} \) be an algorithm for the Induced Neighbourhood Subgraph Computation problem with a running time of \( \mathcal{A} \) is \( O(n^{2-\epsilon}) \), for some \( \epsilon > 0 \). Then, we use algorithm \( \mathcal{A} \) to solve the Online Boolean Matrix-Vector Multiplication problem in \( O(n^{3-\epsilon}) \) time as follows: Let \( M \) be the input matrix for the Online Boolean Matrix-Vector Multiplication problem and let \( V_1, \ldots, V_n \) be the column vectors presented to the algorithm one after another. For the column vector \( V_i \), let set \( S_i = \{ v_j | V_i[j] = 1, 0 \leq j \leq n - 1 \} \). To compute \( M \cdot V_i \), we invoke \( \mathcal{A} \) on input \( \{ M, S_i \} \). Let \( G_{S_i} \) denote the induced subgraph on \( N_{\text{out}}(S_i) \cup S_i \subseteq V \) computed by the algorithm \( \mathcal{A} \). Note that \( G_{S_i} \) is an induced subgraph of the directed graph whose adjacency matrix is \( M \). To output the column vector \( M \cdot V_i \), we observe that the \( j \)-th row in the output column vector is 1 if and only if \( v_j \in G_{S_i} \) and there is an edge \((u,v_j)\) in \( G_{S_i} \) such that \( u \in S_i \). Given that \( G_{S_i} \) has been computed in \( O(n^{2-\epsilon}) \) time, it follows that the number of edges in \( G_{S_i} \) is \( O(n^{2-\epsilon}) \) and consequently the column vector \( M \cdot V_i \) can be computed in \( O(n^{2-\epsilon}) \) time. Therefore, using the \( O(n^{2-\epsilon}) \) algorithm \( \mathcal{A} \) we can solve Boolean Matrix-Vector Multiplication problem in \( O(n^{3-\epsilon}) \) time. If we believe that the OMv conjecture is indeed true, then it follows that the Induced Neighbourhood Subgraph Computation problem cannot have a \( n^{2-\epsilon} \) algorithm for any \( \epsilon > 0 \). This conditional lower bound on the Induced Neighbourhood Subgraph Computation problem deters us from coming up with a direct implementation of the KT algorithm for the incremental setting of the online interval coloring problem. In the following section we design an online interval coloring algorithm that avoids an explicit computation of the induced subgraph on the neighbourhood of an input interval.

3 An incremental data structure for interval coloring

In this section we present an incremental algorithm which is essentially an implementation of the KT-algorithm [11]. The subtle difference is that our algorithm has a different definition of the level value of an interval. The level value that we assign is at most the level value assigned by the KT-algorithm, and thus our algorithm uses the same number of colors as the KT-algorithm. The difference in the level value is illustrated in Figure [1].
Let \( \mathcal{W} = \{0, 1, \ldots, \omega - 1\} \). We use \( L(v_i) \in \mathcal{W} \) to denote the level value computed by our algorithm and \( p(v_i) \) to denote the level value computed by the KT-algorithm. To compute the offset value \( o(v_i) \) we use the same approach as KT-algorithm as described in Section 2.1. We design appropriate data structures to compute the level \( L(v) \) and the offset \( o(v) \) associated with an vertex \( v \).

For a set of intervals \( \mathcal{J} \), define the set \( \text{levels}(\mathcal{J}) = \{ L(v_j) \in \mathcal{W} | I_j \in \mathcal{J} \} \) to be set of levels assigned to intervals in \( \mathcal{J} \). Recall from Section 2 that \( v_j \) is the \( j \)-th vertex in \( \sigma \). Let \( t \) be a non-negative real number and let \( \mathcal{I}_t \) be the set of all intervals in \( \mathcal{I} \) which contain the point \( t \). Define \( h_t = \min(\{y \in \mathcal{W} | y \notin \text{levels}(\mathcal{I}_t)\}) \) i.e. \( h_t \) is the smallest non-negative integer which is not the level value for any interval containing \( t \). If \( h_t \geq 1 \), then the Supporting Line Segment (SLS) at \( t \) is defined to be the set \( e_t = \{(t, 0), \ldots, (t, h_t - 1)\} \), and \( h_t \) is called the height of SLS \( e_t \).

### 3.1 Insertion

We show in Lemma 2 that the level \( L(v_i) \) of interval \( I_i \), which is given by the maximum height of the SLS at any point contained in \( I_i \), is at most \( p(v_i) \). We prove in Lemma 3 that level values computed by our algorithm satisfies Property \( P \) (described in Section 2.1). We also prove using Lemma 1 that the \( L(v_i) \) can be computed based on the height of the SLS at finite set of points in interval \( I_i \). This proof is one of our key contributions. In particular, we show that the finite set points that we need to consider in interval \( I_i \) is the set of endpoints of the intervals intersecting with \( I_i \). For \( n \) intervals we have at most \( 2n \) distinct endpoints and we denote this set of endpoints by \( \mathcal{E} \). Therefore, on insertion of interval \( I_i \), we first compute the set \( S \) of endpoints which are in \( I_i \). We then query the height of the SLS at each of the points in \( S \) and take the SLS with maximum height to compute \( L(v_i) \).

**Insertion Algorithm**

Let \( I_i = [l_i, r_i] \) be the interval that is inserted in the current update step. Our algorithm computes the color \( (L(v_i), o(v_i)) \) by implementing Step 1 and Step 2 as described below:

1. **Computing \( L(v_i) \):**
   - **Step 1**: Insert \( l_i \) and \( r_i \) into the set \( \mathcal{E} \).
   - **Step 2**: Compute \( S = \mathcal{E} \cap I_i \). For each \( t \in S \), compute \( h_t \), the height of the SLS \( e_t \) at \( t \). Let \( L(v_i) = \max_{t \in S} h_t \).
   - **Step 3**: Update the SLS \( e_t \) for each point \( t \in S \).

2. **Computing \( o(v_i) \):** We prove in Lemma 3 that level values computed by our algorithm in Step 1 satisfies Property \( P \). From Property \( P \), \( v_i \) has at most two neighbours whose levels are \( L(v_i) \). We compute \( o(v_i) \) to be that value from the set \( \{1, 2, 3\} \) that is different from the offset values of the neighbours of \( v_i \) which have the level \( L(v_i) \).

**Proof of Correctness**

As mentioned before the description of the algorithm, the correctness of the insertion algorithm follows from the Lemma 1, Lemma 2, and Lemma 3. In the following Lemma, we show that it is sufficient to use the endpoints of the intervals to compute the level of an interval.

**Lemma 1.** Let \( t \) be a non-negative real number and let \( \mathcal{I}_t \) be the set of intervals that contain \( t \). Then there exists at least one endpoint in the set \( \mathcal{E} \) which is contained in each interval in \( \mathcal{I}_t \). Further, the height of the SLS at this endpoint is at least the height of the SLS at \( t \).

**Proof.** If \( t \) is an endpoint of an interval, then \( t \in \mathcal{E} \) and hence the Lemma is proved. Suppose \( t \) is not an endpoint. Let \( l_t \) denote the largest left endpoint among all the intervals in \( \mathcal{I}_t \) and \( r_t \) denotes the smallest right endpoint among all the intervals in \( \mathcal{I}_t \). By definition, \( l_t \in \mathcal{E} \) and
Let us assume that \( v_i \) and \( v_j \) be adjacent and both have level values \( \sigma(v_i) = 0 \). Without loss of generality, let us assume that \( v_i \) appeared before \( v_j \) in \( \sigma \). Therefore, at the when \( v_j \) is presented to the algorithm, there is an endpoint of \( v_i \) contained in \( v_j \) where the height of the SLS is non-zero. Therefore, the insertion algorithm does not assign \( L(v_j) \) to be 0. Therefore, \( \{ v \mid L(v) = 0 \} \) is an independent set. Suppose not, and \( \{ v \mid L(v) = 0 \} \) has a clique of size \( h_t \). Therefore, it follows that \( G_{L(v_j)}(v_i) \) has a clique of size at least \( L(v_i) \). Therefore, it follows that \( p(v_i) \geq L(v_i) \). Hence the Lemma.

Lemma 2. For each \( i \in [n] \), \( p(v_i) \) is at least the maximum height of the SLS at any point in the interval \( I_i \).

**Proof.** By definition, \( L(v_i) \) is the maximum height of SLS at any point contained in \( v_i \). By the definition of the height of an SLS at a point \( t \), we know that for each \( 0 \leq r \leq h_t - 1 \) there is an interval \( v \in I_t \) such that \( L(v) = r \), and all these intervals form a clique of size \( h_t \). Therefore, it follows that \( G_{L(v_j)}(v_i) \) has a clique of size at least \( L(v_i) \). Therefore, it follows that \( p(v_i) \geq L(v_i) \). Hence the Lemma.

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**Figure 1** Example Showing \( L(v_i) < p(v_i) \)

We show an example in Figure 1 where level value computed by our algorithm is strictly less than the level value computed by the KT-algorithm. The intervals arrive in the following order: \( v_1 = [1,2] \), \( v_2 = [8,9] \), \( v_3 = [1,7] \), \( v_4 = [3,9] \), \( v_5 = [4,6] \), and \( v_6 = [4,6] \). Intervals \( v_1, v_2, v_3 \) and \( v_5 \) gets level value 0. KT-algorithm computes \( p(v_6) = 3 \) and our algorithm computes \( L(v_6) = 2 \). The portion colored as violet is the overlapping portion between interval \( v_3 \) and interval \( v_4 \).

Lemma 3. Our algorithm satisfies Property P and thus uses at most \( 3\omega - 2 \) colors.

**Proof.** We first prove that the set \( \{ v_j \mid L(v_j) = 0 \} \) is an independent set. Suppose not, and let two vertices \( v_i \) and \( v_j \) be adjacent and both have level values 0. Without loss of generality, let us assume that \( v_i \) appeared before \( v_j \) in \( \sigma \). Therefore, at the when \( v_j \) is presented to the algorithm, there is an endpoint of \( v_i \) contained in \( v_j \) where the height of the SLS is non-zero. Therefore, the insertion algorithm does not assign \( L(v_j) \) to be 0. Therefore, \( \{ v \mid L(v) = 0 \} \) is
an independent set. Further, for each $1 \leq l \leq \omega - 1$, the same argument is used to show that for each pair of intervals $I_i$ and $I_k$ corresponding to two distinct vertices in $\{v_j \mid L(v_j) = l\}$, $I_i \not\subseteq I_k$. We prove that for each $l$, $1 \leq l \leq \omega$, the subgraph of $G$ induced on $\{v_j \mid L(v_j) = l\}$ has maximum degree at most 2. Suppose not, and if some vertex is of degree 3 in level $l$. We know that for two vertices with the same level value, the corresponding intervals cannot have a containment relationship between them. Therefore, it follows that the vertex of degree 3 is in a clique of 3 vertices in level $l$. Let $v_i$, $v_j$, and $v_k$ be the clique of 3 vertices in level $l$. The intervals corresponding to the three vertices are such that one of the intervals is contained in the union of the other two. Therefore, one of the 3 intervals contains a point $t$ for which the SLS has height $l + 1$. Consequently, all the 3 intervals cannot be assigned the same level value. Therefore, for each level $l$, the maximum vertex degree in the graph induced on $\{v_j \mid L(v_j) = l\}$ is at most 2.

Since we have proved that the level value computed by our algorithm satisfies Property $P$, we use the same procedure as KT-algorithm (described in Section 2.1) to compute the offset value. From Lemma 2, we know that for any interval $v_i$, we have $L(v_i) \leq p(v_i)$. Therefore, maximum level value of any vertex is $\omega - 1$. For level 0 we use one color and for every other level we use 3 colors. Therefore, total colors used by our algorithm is $3(\omega - 1) + 1 = 3\omega - 2$. Hence the Lemma.

### 3.2 Implementation of the incremental algorithm

We use the following data structures in our incremental algorithm. The running times for different operations on these data structures are tabulated in Table 1.

| Interval Tree $I$ | Method | Description | Worst Case Running Time | Return Value |
|-------------------|--------|-------------|-------------------------|--------------|
| $I$.insert($v$)   | Inserts interval $v$ into $I$ | $O(\log(|I|))$ | - |
| $I$.delete($v$)   | Deletes interval $v$ from $I$ | $O(\log(|I|))$ | - |
| $I$.intersection($v$) | Returns a set of intervals $V$ in $I$ that intersect with $v$ | $O(\log(|I|) + |V|)$ | $V$ |

| Dynamic Array $V$ | Method | Description | Amortized Running Time | Return Value |
|------------------|--------|-------------|------------------------|--------------|
| $V$.insert($x$)  | Inserts element $x$ into $V$ | $O(1)$ | - |

| Red-Black Tree $R$ | Method | Description | Worst Case Running Time | Return Value |
|-------------------|--------|-------------|-------------------------|--------------|
| $R$.insert($x$)   | Inserts element $x$ into $R$. | $O(\log(|R|))$ | - |
| $R$.delete($x$)   | Deletes element $x$ from $R$. | $O(\log(|R|))$ | - |
| $R$.max()         | Returns the maximum element $x$ in $R$. | $O(\log(|R|))$ | $x$ |
| $R$.min()         | Returns the minimum element $x$ in $R$. | $O(\log(|R|))$ | $x$ |
| $R$.empty()       | Checks if the tree $R$ is empty. | $O(1)$ | True/False |

| Table 1 Data Structures Used in the Incremental Algorithm and the Fully Dynamic Algorithm |

1. $I$ of type Interval Tree: Every interval $I_i$ is of the form $[l_i, r_i]$ where $l_i$ denotes the left endpoint of the interval and $r_i$ denotes the right endpoint of the interval. We maintain all the vertices corresponding to the intervals in $I$ in an interval tree $I$. For
every $v_i$, along with maintaining the endpoints $\{l_i, r_i\}$, we also maintain the level $L(v_i)$ and offset $o(v_i)$.

2. $V_t$ of type dynamic array, $V_t'$ of type dynamic array, and $Z_t$ of type doubly linked list: We maintain the supporting line segment $e_t$ at point $t$ and the height of supporting line segment $h_t$ using dynamic arrays $V_t$, $V_t'$ and doubly linked list $Z_t$. If the supporting line segment $e_t$ intersects with an interval whose level value is $i$ then we set $V_t[i]$ to 1, otherwise we set $V_t[i]$ to 0. We define the height $h_t$ of the supporting line segment $e_i$ as the index of the first 0 in $V_t$. To maintain $h_t$, we define a doubly linked list $Z_t$ which stores every index $i$ in $V_t$ where $V_t[i]$ is 0 in the increasing order of the value of $i$. Note that the value stored at the head node of $Z_t$ is $h_t$. We augment $Z_t$ with another dynamic array $V_t'$. For an index $i$, if $V_t[i]$ is 0 then $V_t'[i]$ stores a pointer to the node in $Z_t$ which stores the index $i$. If $V_t[i]$ is 1 then $V_t'[i]$ stores NULL. Therefore, using the dynamic array $V_t'$, insert, delete, and search operation in $Z_t$ can be performed in constant time. Since our algorithm is incremental, the dynamic arrays only expand. Insertion into a dynamic array takes amortized constant time \([6]\). The size of the array $V_t$ is the length of $V_t$. During insertion, whenever size of $V_t$ is increased, size of $V_t'$ is also increased and appropriate nodes are inserted in $Z_t$. A Query for the value of $h_t$ can be answered in constant time by returning the value stored in head node of $Z_t$. Due to an insert if $h_t$ changes, then we need to change the head node of $Z_t$ to the next node in the list and delete the previous head node of $Z_t$. This operation also takes constant time.

3. $\mathcal{E}$ of type Interval Tree: We maintain the set of endpoints as an interval tree denoted by $\mathcal{E}$. For every interval $I_i = [l_i, r_i]$, we maintain the left endpoint and the right endpoint as interval $[l_i, l_i]$ and $[r_i, r_i]$ respectively in $\mathcal{E}$.

4. $T$ of type Map with domain as integers and range as Interval Tree: We use the interval tree $T[h]$ for maintaining the intervals at level $h$ to enable allocation of the offset values to the intervals whose level is $h$.

3.3 Analysis of the incremental algorithm

Let $I_i = [l_i, r_i]$ be the interval inserted in the current update step. We give pseudo code of our insertion algorithm (see Algorithm 1) in Section 5 and a summarized description of the procedures used in our insertion algorithm in Table 2. We analyse our algorithm by computing the time required in every step.

1. Computing $L(v_i)$:

   Step 1: $I_i = [l_i, r_i]$ is inserted into $\mathcal{I}$. Let $I'_i = [l_i, l_i]$. We check if $I'_i$ is present in $\mathcal{E}$ by an intersection query. This query takes $O(\log n)$ time in the worst case.
   = If $I'_i$ is in $\mathcal{E}$. Then proceed to Step 2.
   = If $I'_i$ is not in $\mathcal{E}$. We use the procedure $GET-\text{SLS}(\mathcal{I}, l_i)$ to create the SLS $e_l$ at endpoint $l_i$. The procedure $GET-\text{SLS}$ (see Algorithm 3) works as follows: It performs an intersection query on $\mathcal{I}$ with $I'_i$. The query returns all the intervals in $\mathcal{I}$ which contains $I'_i$. Let $\mathcal{I}'_i$ denote the set returned by the intersection query. The worst case time required for this query is $O(\log |\mathcal{I}| + |\mathcal{I}'_i|)$. We create dynamic arrays $V_i$ and $V'_i$, each of size $\max(\text{levels}(\mathcal{I}'_i))$. For every $i$ in the range $[0, \max(\text{levels}(\mathcal{I}'_i))]$, we set $V_i[i] = 1$ if $i \in \text{levels}(\mathcal{I}'_i)$ and $V_i[i] = 0$ otherwise. For every $V_i[i] = 0$, we insert a node to the doubly linked list $Z_i$, storing index $i$ and store the pointer to that node in $V'_i[i]$. For every $V_i[i] = 1$, we store a NULL in $V'_i[i]$. Time taken for creating the dynamic arrays and the associated linked list is $O(\max(\text{levels}(\mathcal{I}'_i)))$. Since $\max(\text{levels}(\mathcal{I}'_i)) \leq \omega$, time taken for the operations on dynamic arrays and associated linked list is $O(\omega)$. Total time taken by procedure $GET-\text{SLS}(\mathcal{I}, l_i)$ is $O(\log |\mathcal{I}| + |\mathcal{I}'_i|)$
| Procedure | Description | Worst-case Running Time | Return Value |
|-----------|-------------|-------------------------|--------------|
| GET_SLS($I, t$) | Computes the supporting line segment for endpoint $t$ and stores it in the dynamic array $V_t$. | $O(\log(n) + \omega)$ | $V_t$ |
| MAX_HEIGHT_OF_SLS_IN_INTERVAL($E, v_i$) | From the interval tree $E$, computes the set of endpoints $S$ contained in the interval $I_i$ and for each $t \in S$, from the dynamic array $V_t$ computes the height $h_t$ and returns $h = \max\{h_t | t \in S\}$. | $O(\log(n) + \Delta)$ | $S, h$ |
| UPDATE_END_POINTS($S, L(v_i)$) | Updates the endpoints in $S$ to reflect the addition of a new interval $I_i$ at level $L(v_i)$. This updates the dynamic array $V_t$ for each $t \in S$. | $O(\Delta)$ | - |
| OFFSET($v_i$) | Assigns an offset value to $v_i$ from $\{1, 2, 3\}$ by considering the offset of the intervals intersecting it in $T[L(v_i)]$ where $L(v_i)$ is the level value of $v_i$. | $O(\log(n))$ | - |

Table 2: Procedures used in the Insertion Algorithm

$$+ O(\omega) = O(\log |I| + |I_f| + \omega).$$ At any level, $e_i$ intersects with at most 2 intervals and we have $\omega$ many levels. Hence, $|I_f| = O(\omega)$. Again, $|I| \leq 2n$. Therefore, time taken by procedure GET-SLS in the worst case is $O(\log n + \omega)$.

Same processing is repeated for $I'_i = [r_i, r_i]$. Therefore, we have the following Lemma.

**Lemma 4.** Worst case time taken by Step 1 in computing $L(v_i)$ of interval $v_i$ is $O(\log(n) + \omega)$.

**Step 2:** We use procedure MAX-HEIGHT-OF-SLS-IN-INTERVAL($E, v_i$) (see Algorithm 4) for this step. The procedure works as follows: It performs an intersection query of $I_i = [l_i, r_i]$ on $E$. This query returns the set $S$ of all the endpoints that intersect with $I_i$. The worst case time taken by intersection query is $O(\log |E| + |S|)$. Further, finding the height of the SLS at one endpoint in $S$ takes constant time. Therefore, finding the maximum height of the SLS at any point in $S$ takes time $O(|S|)$. Let $h$ denote the maximum height for any endpoint $t \in S$. Therefore, the level value of $I_i$, $L(v_i)$ is set to $h$. Since $\Delta$ is the maximum degree in the graph, any interval $I_i$ can intersect with at most $\Delta$ intervals. Therefore, $|S| = O(\Delta)$. Again, $|E| \leq 2|I| \leq 2n$. Thus the worst case time taken by Step 2 is $O(\log n + \Delta)$. Therefore, we have the following Lemma.

**Lemma 5.** Worst case time taken by Step 2 in computing $L(v_i)$ of vertex $v_i$ is $O(\log(n) + \Delta)$.

**Step 3:** We use procedure UPDATE-END-POINTS($S, L(v_i)$) (see Algorithm 5) to perform this step. The procedure works as follows:
We use the set \( S \) and the level value \( L(v_i) \) of \( v_i \) computed in Step 2 to update the endpoints. Let \( l = L(v_i) \). For every endpoint \( t \in S \) we do the following: We check the length of \( V_t \) which is the size of the array \( V_t \) in constant time.

**a. Case A:** If \( l < \text{len}(V_t) \). In this case, we set \( V_t[l] \) to 1. We use the pointer in \( V_t'[l] \) to delete the node in \( Z_t \) storing the value \( l \) and set \( V_t'[l] \) to NULL. If deleted node in \( Z_t \) was the head node, then we update the head node to the next node in \( Z_t \) and thus the value of \( h_t \) also gets updated. All these operations take constant time in the worst case.

**b. Case B** If \( l \geq \text{len}(V_t) \). In this case, we use the standard doubling technique for expansion of dynamic arrays \(^6\) until \( \text{len}(V_t) \) becomes strictly greater than \( l \). We also expand \( V_t' \) along with \( V_t \) and insert appropriate nodes to \( Z_t \). Following the standard analysis for dynamic array expansion as shown in \(^6\), one can easily show that all these steps take amortized constant time. Once \( \text{len}(V_t) > l \), the remaining operations are same as in the case \( A \).

To analyse the time required in Step 3, we observe that every update must perform the operations as described in case \( A \). We refer to these operations as task \( M \) (M stands for mandatory). Some updates have to perform additional operations as described in case \( B \). We refer to these operations as task \( A \) (A stands for additional). The time taken by each update to perform task \( M \) is \( |S| \times O(1) = O(|S|) \). Since \( \Delta \) is the maximum degree, hence \( |S| \leq \Delta \). Therefore, every update takes \( O(\Delta) \) time to perform task \( M \) in the worst case. To analyse the time required to perform task \( A \), we crucially use the fact that our algorithm is incremental and hence only expansions of the dynamic arrays take place. Since \( \omega \) is the size of the maximum clique, it follows that the maximum size of a dynamic array throughout the entire execution of the algorithm is upper bounded by \( 2\omega \). Over a sequence of \( n \) insertions, the total number of endpoints is upper bounded by \( 2n \). Therefore, we maintain at most \( 4n \) dynamic arrays. For every such array, total number of inserts in the array and the associated doubly linked list is at most \( 2\omega \) in the entire run of the algorithm. An insertion into the dynamic array takes constant amortized time and insertion into doubly linked list takes constant worst case time. Therefore, during the entire run of the algorithm total time required to perform task \( A \) on one dynamic array and its associated doubly linked list is \( O(\omega) \). This implies that during the entire run of the algorithm total time spent on task \( A \) over all the updates is \( \leq 4n \times O(\omega) \). Let \( T \) be total time spent on Step 3 at the end of \( n \) insertions. This is the sum of the total time for task \( A \) and the total time in task \( M \). Therefore,\[
\begin{align*}
T &\leq 4n \times O(\omega) + n \times O(\Delta) \\
&= 4n \times O(\Delta) + n \times O(\Delta) \text{ [since } \omega \leq \Delta + 1]\end{align*}
\]
Therefore, \( T = O(n\Delta) \) and we have the following Lemma.

**Lemma 6.** The Amortized time taken by Step 3 in computing \( L(v_i) \) of vertex \( v_i \) is \( O(\Delta) \).

2. **Computing \( o(v_i) \):**
   We have assigned level \( L(v_i) \) to interval \( I_i \). We use the map \( T \) to assign color to interval \( I_i \). If \( T[L(v_i)] \) is NULL, then we create an interval tree \( T'[L(v_i)] \) with \( v_i \) as the first node. This takes constant time. Otherwise, \( T'[L(v_i)] \) gives us the interval tree which stores all the intervals at level \( L(v_i) \). We perform an intersection query on \( T'[L(v_i)] \) with \( I_i \) to obtain all the intervals that intersect with \( I_i \). From Property \( P \), the maximum intervals returned by the above query is \( 2 \). Therefore, worst case time taken by the intersection query is \( O(\log|I| + 2) = O(\log n + 2) = O(\log n) \). \( o(v_i) \) is the smallest color from \( \{1, 2, 3\} \) not assigned to any of the at most two neighbours of \( v_i \) in level \( L(v_i) \).
Thus we have the following Lemma.

**Lemma 7.** Compiling the offset value of the vertex \(v_i\) with level value \(L(v_i)\) takes \(O(\log n)\) time in the worst case.

The amortized update time of our incremental algorithm is given by the following theorem.

**Theorem 8.** There exist an incremental algorithm which supports insertion of a sequence of \(n\) intervals in amortized \(O(\log n + \Delta)\) time per update.

**Proof.** For interval graphs, it is well known that \(\omega = \chi(G) \leq \Delta + 1\). Therefore, using Lemma 3, Lemma 4, Lemma 5, and Lemma 6, we conclude that total time taken by our incremental algorithm for insertion of \(n\) intervals is:

\[
\mathcal{T} = \text{Total time for Step 1} + \text{Total Time for Step 2} + \text{Total Time for Step 3} + \text{Total time for clean-up}
\]

\[
\mathcal{T} = n \times O(\log n + \omega) + n \times O(\log n + \Delta) + n \times O(\Delta) + n \times O(\log n)
\]

\[
\mathcal{T} = O(n \log n + n\Delta)
\]

Therefore, the amortized update time over a sequence of \(n\) interval insertions is \(O(\log n + \Delta)\).

\(\blacksquare\)

## 4 Fully Dynamic Data Structures for interval coloring

In this Section we study the interval coloring problem in the fully dynamic framework. In this framework a new interval can be inserted and a previously colored interval can be deleted. Let \(I_j = [l_j, r_j]\) be an interval inserted in the current update step and let \(v_j\) be the vertex corresponding to \(I_j\). We use the insertion algorithm (see Algorithm 1) as described in Section 2 to compute the color \((L(v_j), o(v_j))\). Let \(I_i = [l_i, r_i]\) be an interval deleted and let its color be \((L(v_i), o(v_i))\). We ensure that for every other vertex \(v_k\), \(L(v_k) \leq h\) where \(h\) denotes the maximum height of the supporting line segments contained in the interval \(I_k\).

This step is crucial as it ensures that the number of levels used is at most \(\omega - 1\) and that the level assignment satisfies Property P. We now outline the challenges and our approach.

### 4.1 Challenges in implementing the Deletion and Insertion algorithms

Let \(I_i = [l_i, r_i]\) be an interval deleted and its color be \((L(v_i), o(v_i))\).

**Challenge 1:** Let \(E_{I_i} = E \cap I_i\) be the set of endpoints contained in \(I_i\). Let \(t \in E_{I_i}\). For every other interval \(I_k\) which contains \(t\), if \(L(v_k) = L(v_i)\) then the SLS \(e_t\) must be updated at \(L(v_i)\) to reflect the deletion of \(v_t\). Further, if \(h_t > L(v_i)\) and no other interval with level value \(L(v_i)\) contains \(t\), then we must modify \(h_t\) to \(L(v_i)\).

**Challenge 2:** Let \(I_j\) be an interval. Just after the deletion of the interval \(I_i\), let \(h\) denote the maximum height of supporting line segments contained in \(I_j\). The vertex which is not dirty is called *clean*. Let \(D\) denote the set of all vertices which are dirty after the deletion of \(v_i\). The efficiency of the delete step depends on the efficient computation of \(D\) and the clean-up of \(D\) so that at the end of the update, there are no dirty vertices. We also must ensure that during the clean-up of \(D\) no clean vertex becomes dirty.

We address challenge 1 in Section 4.3 by using a Red-Black tree to maintain supporting line segments. We address challenge 2 as follows: On deletion of \(I_i\), we initialize set \(D = \{v_j | L(v_j) > L(v_i), (v_i, v_j) \in E(G)\}\). That is, the set \(D\) consists of every vertex \(v_j\) such that \(L(v_j) > L(v_i)\) and \(I_j\) intersects with \(I_i\). In Lemma 9, we show that \(D\) is a superset of all dirty vertices created due to the deletion of \(I_i\). In other words, we show that only those vertices which are adjacent to \(v_i\) and those whose level number is more than that of \(v_i\) can be dirty during the update step after the deletion of \(I_i\).
4.2 Update Algorithms

The algorithm executed during each update is as below:

Deletion Algorithm (For the pseudo-code, see Algorithm [2]): Let \( I_i = [l_i, r_i] \) be the interval deleted during the current update step. Let \( (L(v_i), o(v_i)) \) be the color of \( v_i \). The steps in the algorithm are as follows:

Step 1: Remove interval \( I_i \).

Step 2: Compute \( E_{I_i} = E \cap I_i \).

Step 3: For each endpoint \( t \in E_{I_i} \), update SLS \( e_t \) to reflect the deletion of interval \( I_i \).

Step 4: Compute \( D = \{v_j | L(v_j) > L(v_i) \land (v_i, v_j) \in E(G)\} \).

Step 5: Consider \( D \) in the increasing order of level value. Ties are broken by considering first the intervals inserted earlier. Let \( v_j \) be the current vertex under consideration from the set \( D \). The following steps are executed for \( v_j \):

- Compute \( S = E \cap I_j \).
- For every endpoint \( t \in S \) compute \( h_t \) the height of the SLS \( e_t \) at \( t \).
- Compute \( h = \max(h_t | t \in S) \).
- If \( h \geq L(v_j) \). Then \( L(v_j) \) is unchanged. Prior to this update, we maintained that there is a \( t \in I_j \) such that \( h_t \geq L(v_j) \). This invariant persists for \( v_j \).
- If \( h < L(v_j) \). Then,
  - Change level value of \( v_j \) from \( L(v_j) \) to \( h \). This ensures that there is a \( t \in I_j \) such that \( h_t \geq L(v_j) + 1 \). This invariant is ensured after the change at \( v_j \).
  - Compute the offset value \( o(v_j) \) for \( v_j \) with level value \( h \).
  - Update SLS \( e_t \) for every point \( t \in S \) to reflect the change in level value of \( v_j \).
  - Continue to the next vertex in the set \( D \).

Insertion Algorithm (For the pseudo-code see Algorithm [1]): Let \( I_i = [l_i, r_i] \) be an interval inserted in the current update step. We use the insertion algorithm in Section 3 and compute the color \( (L(v_i), o(v_i)) \) for \( v_i \) (see Algorithm [6]).

Lemma 9. Let \( I_i \) be the deleted interval and let \( (L(v_i), o(v_i)) \) be the color of \( v_i \). Then all the dirty vertices during the update are contained in the set \( D \).

Proof. Let \( v_k \) be a vertex. By definition of the level of a vertex, we know that there is a point \( t \in I_k \) such that \( h_t \geq L(v_k) \). In other words, for the SLS \( e_t = \{(t, 0), \ldots , (t, h_t - 1)\} \), \( h_t \) is at least \( L(v_k) \). Further, the Deletion algorithm maintains the invariant that whenever \( L(v_j) \) is modified it is changed to be the maximum \( h_t \), over all \( t \in I_j \). Also, the vertices are processed in non-decreasing order of level number, and the level number is not increased.

We show that if \( L(v_k) < L(v_i) \), then \( v_k \) does not become dirty due to the deletion of \( I_i \) and during the update. Since \( L(v_i) > L(v_k) \), it follows that for each \( 0 \leq x \leq L(v_k) \) during the update the level is unchanged for the interval \( I_p \) for which \( t \in I_p \) and \( L(v_p) = x \). Therefore, the maximum height of a supporting line segment in \( I_k \) at the end of the update initiated by the deletion of \( I_i \) is at least \( L(v_k) \). Therefore, \( v_k \not\in D \).

Secondly, we show that if \( (v_k, v_i) \) is not an edge, then \( v_k \) does not become dirty due to the deletion of \( I_i \) and during the update. If \( (v_k, v_i) \) is not an edge, then the intervals \( I_k \) and \( I_i \) are disjoint. Therefore, \( t \not\in I_i \). Consequently, it follows that for each \( 0 \leq x \leq h_t - 1 \) during the update the level is unchanged for the interval \( I_p \) for which \( t \in I_p \) and \( L(v_p) = x \). Therefore, the maximum height of supporting line segments in \( I_k \) after the deletion of \( I_i \) and throughout the update is at least \( h_t \). Since \( h_t \geq L(v_k) \) after the deletion of \( I_i \) it follows that \( v_k \not\in D \). Hence the lemma.
Lemma 10. At the end of each update, the number of colors used is at most \(3\omega - 2\) colors, where \(\omega\) is the size of the maximum clique in the interval graph just after the update.

Proof. The algorithm maintains the crucial invariant after every update that for each vertex \(v_i\), there is a point \(t \in I_i\) such that \(h_t \geq L(v_i)\). Therefore, the largest level of any vertex is at most \(\omega - 1\). Further, the vertices whose level values are 0 form an independent set. Further, for each \(v_j \in D\) the algorithm maintains the following invariant that whenever \(L(v_j)\) is modified: \(L(v_j)\) is modified to be the maximum \(h_t\), over all \(t \in I_j\). Next, we claim that when \(L(v_j)\) changes it has at most two neighbors in the new level and none of the neighbors has a containment relationship with \(I_j\). The proof is by contradiction by assuming that \(v_j\) is the first vertex for which this claim is false. The contradiction is obtained by using the argument in Lemma 3. Therefore, it follows that after the update step, Property \(P\) is satisfied by the level assignment to the vertices. Consequently, the algorithm uses at most \(3\omega - 2\) colors after each update step. Hence the Lemma.

4.3 Implementation of the fully dynamic algorithm

In this section we describe the data structures that we use for the implementation of our fully dynamic algorithm. The following data structures are same as described in Section 3.2:

- Interval Tree \(I\) to maintain all the intervals. For every vertex \(v_i \in I\), we maintain \([l_i, r_i]\), the level value \(L(v_i)\), the offset value \(o(v_i)\) and the time of insertion (if \(k^{th}\) update inserts vertex \(v_i\) then we store \(k\) as the time of insertion for vertex \(v_i\)).

- Interval Tree \(E\) to maintain the set of endpoints.

- Map \(T\) with domain as integer and range as interval tree for computing the offset value.

The only data structure which is different from Section 3.2 is the data structure that we use to maintain the supporting line segments. For the fully dynamic algorithm, we maintain the supporting line segments as follows: For an SLS \(e_t\) at point \(t\), we use two Red-Black trees, \(Z_t\) and \(NZ_t\). A level value \(l\) is stored in \(Z_t\) if there is no interval in \(I\) whose level value is \(l\) and it contains \(t\). A level value \(l\) is stored in \(NZ_t\) if there is an interval in \(I\) whose level value is \(l\) and it contains \(t\). To compute the height \(h_t\) of an SLS \(e_t\) we do the following: If \(Z_t\) is non empty then minimum value in the tree \(Z_t\) is the height of SLS at \(t\). If \(Z_t\) is empty then height of SLS at \(t\) is one more than the maximum value in the tree \(NZ_t\). We know that \(|Z_t| \leq \omega\) and \(|NZ_t| \leq \omega\). Therefore, time required to find the height of an SLS \(e_t\) is \(O(\log \omega)\).

Running times for different operations on these data structures are tabulated in Table 1.

We address challenge 1 after deleting interval \(I_i\) as follows: We delete \(v_i\) from interval tree \(I\) and \(T[L(v_i)]\). To find the set of endpoints \(E_{I_i}\), we query the interval tree \(E\) with interval \(I_i\). For every endpoint \(t \in E_{I_i}\), we query \(T[L(v_i)]\). If this query returns NULL then we delete \(L(v_i)\) from the Red-Black tree \(NZ_t\) and insert \(L(v_i)\) to the Red-Black tree \(Z_t\). We analyse all these operations in details in Section 4.4.

4.4 Analysis of the fully dynamic algorithm

We give a summarized description of the procedures used in our fully dynamic algorithm in Table 3. We present pseudo codes for our algorithm and procedures in Section 5. In this Section we present a worst case analysis of every step in our algorithm. We first analyse worst case performance of every step in our deletion algorithm and present a worst update time for deletion. Then we analyse worst case performance of every step in our insertion algorithm and present a worst update time for insertion.
### Table 3: Procedures used in the Fully Dynamic Algorithm

| Procedure                  | Description                                                                 | Worst-case Running Time | Return Value |
|----------------------------|-----------------------------------------------------------------------------|--------------------------|--------------|
| GET_SLS(\(\mathcal{I}, e\)) | Computes the supporting line segment for endpoint \(t\) and stores it in the Red Black Tree \(Z_t\) and \(NZ_t\). | \(O(\log(n) + \omega \log \omega)\) | \(Z_t, NZ_t\) |
| MAX_HEIGHT_OF_SLS_IN_INTERVAL(\(\mathcal{E}, v_i\)) | From the interval tree \(\mathcal{E}\), computes the set of endpoints \(S\) contained in the interval \(I_t\) and for each \(t \in S\), using the Red Black trees \(Z_t\) and \(NZ_t\), computes the height \(h_t\) and returns the maximum height \(h = \max \{h_t | t \in S\}\). | \(O(\log(n) + \Delta \log \omega)\) | \(S, h\) |
| UPDATE_END_POINTS(\(S, L(v_i)\)) | Updates the endpoints in \(S\) to reflect the addition of a new interval \(I_t\) at level \(L(v_i)\). This updates the Red Black trees \(Z_t\) and \(NZ_t\) for each \(t \in S\). | \(O(\Delta \log \omega)\) | - |
| OFFSET(v)                  | Assigns an offset value to \(v\) from \(\{1, 2, 3\}\) by considering the offset of the intervals intersecting it in \(T[L(v)]\) where \(L(v)\) is the level value of \(v\). | \(O(\log(n))\) | - |

#### 4.4.1 Analysis of Deletion Algorithm:

Let \(I_t = [l_t, r_t]\) be the interval that is deleted in the current update step. Let \(n\) be the total number of intervals inserted during the course of execution of the algorithm. We analyse the worst case time required in every step.

**Step 1:** We remove \(I_t\) from interval tree \(\mathcal{I}\) and interval tree \(T[L(v_i)]\). We know that \(|\mathcal{I}| \leq n\) and \(|T[L(v_i)]| \leq n\). Therefore, this step takes \(O(\log n)\) time in the worst case.

**Step 2:** To compute \(E_{I_t} = \mathcal{E} \cap I_t\), we perform an intersection query on \(\mathcal{E}\) with \(v_i\). The time required by the query to return the set \(E_{I_t}\) is \(O(\log |\mathcal{E}| + |E_{I_t}|)\). We know that \(|\mathcal{E}| \leq 2n\) and \(|E_{I_t}| \leq \Delta\). Therefore, worst case time required by this step is \(O(\log(n + \Delta))\).

**Step 3:** We need to update SLS \(e_t\) for every endpoint \(t \in E_{I_t}\). To do so, we check if there is another interval with level value equal to \(L(v_i)\) that contains \(t\). From property \(P\), after removing \(v_i\), there can be at most two intervals with level value \(L(v_i)\) that contains \(t\). We perform this check using an intersection query on \(T[L(v_i)]\) with \(t\). Worst case time required by this query is \(O(\log n + 2) = O(\log n)\). If this query returns a NULL, then we delete \(L(v_i)\) from the tree \(NZ_t\) and insert \(L(v_i)\) into the tree \(Z_t\). Since \(|Z_t| \leq \omega\) and \(|NZ_t| \leq \omega\), time required for deletion from \(NZ_t\) and insertion to \(Z_t\) is \(O(\log \omega)\). Therefore, total time required for one \(t \in E_{I_t}\) is \(O(\log n + \log \omega)\). We know that \(|E_{I_t}| \leq \Delta\). Therefore, worst case time required by this step is \(O(\Delta \log n + \Delta \log \omega) = O(\Delta \log n)\).

**Step 4:** We need to compute the set \(\mathcal{D} = \{v_j | L(v_j) > L(v_i), (v_i, v_j) \in E(G)\}\). To compute \(\mathcal{D}\), we perform an intersection query on \(\mathcal{I}\) with \(v_i\). Let \(H\) be the set returned by this query.
We follow the same analysis as described in Section 3.3. Let $T = T[\text{Step 1} + \text{Step 2} + \text{Step 3} + \text{Step 4}]$.

\[ T = O(\log n) + O(\log n + \log \omega) + O(\Delta \log n) + O(\Delta^2 \log n) \]

\[ T = O(\Delta^2 \log n) \]

4.4.2 Analysis of Insertion Algorithm

We follow the same analysis as described in Section 3.3. Let $I_i = [l_i, r_i]$ be the interval inserted in the current update step.

**Step 1**: Except procedure GET-SLS (see Algorithm 3) all other operations take same time as described in Section 3.3. For the SLS $e_i = [l_i, r_i]$, procedure GET-SLS will make at most $\omega$ insertion in the tree $Z_{l_i}$ and $NZ_{L_i}$. Time taken for all the operation on $Z_{l_i}$

Clearly $D \subseteq H$, $|H| \leq \Delta$ and $|Z| \leq \omega$. Therefore, worst case time required by this query is $O(\log n + \Delta)$. We scan through the set $H$ and discard the intervals whose level value is $\leq L(v_i)$ in $O(\Delta)$ time. Then we sort all the remaining intervals in the set in increasing order of their level value in $O(\Delta \log n)$ time. We use increasing order of insertion time to break the ties. This gives us the set $D$. Again $|D| \leq |H| \leq \Delta$. We work on every element $v_j$ in the set $D$ as follows:

- We need to compute the set $S$ of supporting line segments contained in $v_j$. To compute $S$, we perform an intersection query on $E$ with $v_j$. We know that $|E| \leq 2n$ and $|S| \leq 2\Delta$. Therefore, the query takes $O(\log n + \Delta)$ time in the worst case. Finding the height $h_t$ of SLS $e_t$ for $t \in S$ takes $O(\log \omega)$ time. Therefore, finding the height of all the supporting line segments in $S$ takes $O(\Delta \log \omega)$ time and finding the maximum, $h = \max\{h_t | t \in S\}$, takes another $O(\Delta)$ time. Therefore, this step takes $O(\log n + \Delta \log \omega)$ time in the worst case.

- If $h \geq L(v_j)$ then no further processing is required for $v_j$ and we move to the next vertex in $D$.

- If $h < L(v_j)$. Then we need to change level value of $v_j$ from $L(v_j)$ to $h$. We do the following:
  - We delete $I_j$ from $T[L(v_j)]$ and insert $I_j$ into $T[h]$. We set $h' = L(v_j)$ and $L(v_j) = h$. This step takes $O(\log n)$ time in the worst case.
  - We perform an intersection query on $T[h]$ with $I_j$. This takes $O(\log n)$ time in the worst case and return at most two intervals. We assign an unused offset value to $v_j$ from the set $\{1, 2, 3\}$ in constant time.

- For every endpoint $t \in S$, we check if there is another interval at level $h'$ that contains $t$. From property $P$, after changing level value of $v_j$ there can be at most two more intervals with level value $h'$ that contains $t$. We check this by performing an intersection query on $T[h']$ with $t$. This query takes $O(\log n) = O(\log \omega)$ time in the worst case. If the query returns a NULL then we delete $h'$ from $NZ_t$ and insert $h'$ to $Z_t$. Time taken to perform all these operation for one SLS $e_t$ is $O(\log n + \log \omega)$. Therefore, worst case time taken for the entire set $S$ is $O(\log \omega + \Delta \log \omega) = O(\Delta \log n)$. Therefore, the worst case time taken for one vertex $v_j \in D$ is $O(\Delta \log n)$. Hence, the worst case time taken for all the vertices in $D$ is $O(\Delta^2 \log n)$. This implies that the worst case time taken by Step 4 is $O(\Delta^2 \log n)$.

**Lemma 11.** Deletion of an interval takes $O(\Delta^2 \log n)$ time in the worst case.

**Proof.** Our deletion algorithm involves four steps. Therefore, worst case time $T$ taken by the deletion algorithm is:

\[ T = \text{Worst Case Time for Step 1} + \text{Worst Case Time for Step 2} + \text{Worst Case Time for Step 3} + \text{Worst Case Time for Step 4} \]

\[ T = O(\log n) + O(\log n + \Delta) + O(\Delta \log n) + O(\Delta^2 \log n) \]

\[ T = O(\Delta^2 \log n) \]

4.4.2 Analysis of Insertion Algorithm

We follow the same analysis as described in Section 3.3. Let $I_i = [l_i, r_i]$ be the interval inserted in the current update step.

**Computing $L(v_i)$**:

**Step 1**: Except procedure GET-SLS (see Algorithm 3) all other operations take same time as described in Section 3.3. For the SLS $e_i = [l_i, r_i]$, procedure GET-SLS will make at most $\omega$ insertion in the tree $Z_{l_i}$ and $NZ_{L_i}$. Time taken for all the operation on $Z_{l_i}$.
and \(NZ_t\) is \(O(\omega \log \omega)\). Therefore, following the same argument as in Section 3.3 worst case time taken by Step 1 is \(O(\log n + \omega \log \omega)\).

**Step 2:** We use procedure \(\text{MAX-HEIGHT-OF-SLS-IN-INTERVAL}\) (see Algorithm 4) for this step. As described in Section 3.3 time taken to obtain the set \(S\) is \(O(\log n + \Delta)\). For SLS \(e_t\) at every \(t \in S\), it takes \(O(\log \omega)\) time to obtain the value of height \(h_t\). Therefore, to obtain the height of supporting line segments at all the points in \(S\), total time required is \(O(\Delta \log \omega)\). Finding the maximum height takes another \(O(\Delta)\) time. Therefore, worst case time taken by Step 2 is \(O(\log n + \Delta \log \omega)\).

**Step 3:** We use procedure \(\text{UPDATE-END-POINTS}\) (see Algorithm 5) for this step. This procedure does the following: For every \(t \in S\), it deletes \(L(v_i)\) from \(Z_t\) and inserts \(L(v_i)\) to \(NZ_t\). For every SLS \(e_t\) this takes \(O(\omega \log \omega)\) time. Therefore, worst case time taken by Step 3 is \(O(\Delta \log \omega)\).

**Lemma 12.** Insertion of an interval takes \(O(\log n + \Delta \log \omega)\) time in the worst case.

**Proof.** Worst case time \(T\) taken by the insertion algorithm is:

\[
T = \text{Worst case time taken to compute the level value} + \text{Worst case time taken to compute the offset value}
\]

\[
T = \text{Worst case time taken by Step 1} + \text{Worst case time taken by Step 2} + \text{Worst case time taken to compute the offset value}
\]

\[
T = O(\log n + \omega \log \omega) + O(\log n + \Delta \log \omega) + O(\Delta \log \omega) + O(\log n)
\]

\[
T = O(\log n + \Delta \log \omega)
\]

The proof of the Theorem 13 follows from Lemma 11 and Lemma 12.

**Theorem 13.** There exist a fully dynamic algorithm which supports insertion of an interval in \(O(\log n + \Delta \log \omega)\) worst case time and deletion of an interval in \(O(\Delta^2 \log n)\) worst case time.

## 5 Pseudo-Code of the algorithms

We present the pseudo-code for the functions used in both the incremental algorithm and the fully dynamic algorithm. The notation described in Section 2 is followed here.
Algorithm 1 Insertion($I_i = [l_i, r_i]$)

1: Computing $L(v_i)$:
2: $I_i$.insert($v_i$)
3: $I'_i \leftarrow [l_i, l_i]$  
4: $I'_i \leftarrow [r_i, r_i]$  
5: if len($E_i$.intersection($I'_i$)) = 0 then  
6:  $e_{l_i} =\text{GET}_\text{SLS}(I_i, l_i)$  
7:  $E_i$.insert($I'_i$)  
8: end if  
9: if len($E_i$.intersection($I'_i$)) = 0 then  
10:  $e_{r_i} =\text{GET}_\text{SLS}(I_i, r_i)$  
11:  $E_i$.insert($I'_i$)  
12: end if  
13: $S, h = \text{MAX\_HEIGHT\_OF\_SLS\_IN\_INTERVAL}(E_i, v_i)$  
14: $L(v_i) = h$  
15: UPDATE\_END\_POINTS($S, L(v_i)$)  
16: Computing $o(v_i)$:  
17: offset($v_i$)

Algorithm 2 Deletion($I_i = [l_i, r_i]$)

1: $T[L(v_i)].delete(v_i)$  
2: $I_i.delete(v_i)$  
3: $E_{I_i} = E_i$.intersection($v_i$)  
4: for $t$ in $E_{I_i}$ do  
5:  if len($T[L(v_j)].intersection(t)) == 0 then  
6:   $NZ_t.insert(L(v_i))$  
7:  end if  
8: end for  
9: $H = I_i$.intersection($v_j$)  
10: Do a linear search on the set $H$ and discard all the intervals whose level value is smaller than $L(v_j)$. Sort the intervals in the increasing order of level value. If two intervals belong to the same level then sort them in the increasing order of time of insertion. This gives us the set $D$  
11: for $v_j$ in $D$ do  
12:  if $h \geq L(v_j)$ then  
13:   continue  
14:  end if  
15: $T[L(v_j)].delete(v_j)$  
16: $T[h].insert(v_j)$  
17: for $t$ in $S$ do  
18:  if len($T[L(v_j)].intersection(t)) == 0 then  
19:   $NZ_t.insert(L(v_j))$  
20:  end if  
21: $NZ_t.delete(h)$  
22: $Z_t.delete(h)$  
23: end if  
24: $NZ_t.insert(h)$  
25: $Z_t.delete(h)$  
26: end for  
27: $L(v_j) = h$  
28: OFFSET($v_j$)  
29: end for
Algorithm 3 This code maintains SLS as Red-Black Trees. Same code can be easily modified to maintain SLS as dynamic arrays.

```plaintext
1: function GET_SLS(I, t)
2:   \( I_t = I \_\text{intersection}([t, t]) \)
3:   \( h_t, e_t = 0, [ ] \)
4:   for \( v \) in \( I_t \) do
5:       \( e_t.\text{append}(L(v)) \)
6:       \( h_t = \text{max}(h_t, L(v)) \)
7:   end for
8:   \( Z_t = \text{Red-Black-Tree()} \)
9:   \( NZ_t = \text{Red-Black-Tree()} \)
10:  for \( i \) in \{0, 1, 2, ...h_t\} do
11:      \( Z_t.\text{insert}(i) \)
12:  end for
13:  for \( i \) in \( P \) do
14:      \( NZ_t.\text{insert}(i) \)
15:      \( Z_t.\text{delete}(i) \)
16:  end for
17:  return \( e_t \)
18: end function
```

Algorithm 4 This code maintains SLS as Red-Black Trees. Same code can be easily modified to maintain SLS as dynamic arrays.

```plaintext
1: function MAX_HEIGHT_OF_SLS_IN_INTERVAL(E, v)
2:   \( S = E \_\text{intersection}(v) \)
3:   \( h = 0 \)
4:   for \( t \) in \( S \) do
5:       if \( Z_t.\text{empty()} \) \& \( NZ_t.\text{empty()} \) then
6:           \( h_t = 0 \)
7:       else if \( Z_t.\text{empty()} \) then
8:           \( h_t = NZ_t.\text{max()} + 1 \)
9:       else
10:          \( h_t = Z_t.\text{min()} \)
11:     end if
12:     \( h = \text{max}(h, h_t) \)
13:   end for
14:  return \( S, h \)
15: end function
```
Algorithm 5. This code maintains SLS as Red-Black Trees. Same code can be easily modified to maintain SLS as dynamic arrays.

```
1: function update_end_points(S, L(v))
2:     for t in S do
3:         if ¬Z_t.empty() then
4:             h_t = Z_t.min()
5:         end if
6:         if Z_t.empty() then
7:             h_t = NZ_t.max() + 1
8:         end if
9:         q = h_t
10:     while q < L(v) do
11:         Z_t.insert(q)
12:         q = q + 1
13:     end while
14:     if L(v) in Z_t then
15:         Z_t.delete(L(v))
16:     end if
17:     NZ_t.insert(L(v))
18: end for
19: end function
```

Algorithm 6

```
1: function offset(v)
2:     if T[L(v)] is None then
3:         T[L(v)] = INTERVAL_TREE()
4:     end if
5:     S = T[L(v)].intersection(v)
6:     o(v) = GREEDY_COLORING(S)
7:     T[L(v)].insert(v)
8: end function
```

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