"Partial" quantum cloning and quantum cloning of the mixed states

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Abstract

We discuss the "partial" quantum cloning of the pure two-partite states, when the "part" of initial state related to the one qubit is copied only. The same approach gives the possibility to design the quantum copying machine for the mixed qubit states.

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1 Introduction

Main laws of quantum mechanics forbidden the perfect cloning of the quantum states, see corresponding discussion for the pure states in [1], [2], and for the mixed states in [3]. But it is possible to carry out an approximate copying of the quantum states [4]. Quantum cloning machines (QCM) depend on the conditions accepted at its designing. They can produce identical copies of the initial state (symmetric QCM), nonidentical copies (non-symmetric QCM), the quality of the copying can be either identical for all states (universal QCM) or depend on the state (state-dependent QCM). Detailed discussion of the different variants of QCM and theirs possible applications in quantum cryptography and quantum informatics can be found in [5], [6].

One possible application of the QCM is an eavesdropping of the quantum channel. The aim of such eavesdropping defines the main properties of the designing QCM. One can design QCM which copies only part of the quantum state, for instance. Such QCM can be useful if eavesdropper, usually called Eve, intends to catch part of the transmitted quantum information only. Some classical analogue of this situation can be classical eavesdropping of the key words in the transmitted classical information. At quantum cloning we can choose the different parts
of the quantum signal in which we are interested. In this paper we intend to discuss some "partial" QCM, which copies one constituent of the two-partite states.

Our approach gives the possibility to consider QCM for a mixed states too. It is well known fact, that any mixed state can be considered as a reduction of a pure state, which is called "purification" of the mixed state [7]. So, cloning of the mixed state can be considered as a "partial" cloning of the "purification" of the mixed state. Some difference between the "partial" cloning machine and the cloning machine for the mixed states is connected with the corresponding difference of the sets of the initial states, see details below. Note, that the main attention in the present literature was devoted to the cloning of the pure states [5], [6].

2 "Partial" quantum cloning machine

We consider two-partite qubit states, qubits are elements of two-dimensional Hilbert space $H$. In order to construct QCM we need in tensor product of three such spaces on the ancilla space: $H_1 \otimes H_2 \otimes H_3 \otimes H_4$, here different components are marked by indexes. The first and third qubit components constitute a quantum state which carries information in the quantum channel, and the state of first component is interesting for Eve. The second component is a blank state, where we will copy the first component, the last component is necessary for the realization of the QCM. Let quantum channel carries the quantum state $|\Psi>| \in H_1 \otimes H_3$,.

\[ |\Psi> = a_{00} |0_10_3> + a_{01} |0_11_3> + a_{10} |1_10_3> + a_{11} |1_11_3>, \]  

where normalization condition holds,

\[ |a_{00}|^2 + |a_{01}|^2 + |a_{10}|^2 + |a_{11}|^2 = 1. \]  

Here and below $|0>,$ $|1>$ are base vectors in $H$. We suppose, that Eve’s goal is a copying of the first component of this state. After tracing one can obtain:

\[ \rho_{\text{init}} = Tr_3 \ |\Psi><\Psi| = A |0><0| + B |0><1| + B^* |1><0| + C |1><1|, \]  

\[ A = |a_{00}|^2 + |a_{01}|^2, \quad B = a_{00}a_{10} + a_{01}a_{11}, \quad C = |a_{10}|^2 + |a_{11}|^2 = 1 - A. \]

So, Eve has to realize the cloning to produce the pair of states (in the first and second components respectively) closest to $\rho_{\text{init}}$. We consider here symmetric QCM, so, we suppose, that
states in the first and second components have to coincide. Then produced state must be symmetric with regard to permutation of the first and second components. Let us introduce the orthonormal basis in the subspace of $H_1 \otimes H_2$ symmetric regarding this permutation:

$$|\Phi_1> = |0_1 0_2>, |\Phi_2> = \frac{1}{\sqrt{2}} (|1_1 0_2> + |0_1 1_2>), |\Phi_3> = |1_1 1_2>.$$ 

Let’s assume, that the second component be in state $|0>$. Description of the QCM is, in essence, the definition of the corresponding unitary operator $U$. Following to \cite{4, 8}, we set

$$U|0_1 0_2 0_4> = |\Phi_1>|Q_0> + |\Phi_2>|Y_0>.$$ 

(4)

$$U|1_1 0_2 0_4> = |\Phi_3>|Q_1> + |\Phi_2>|Y_1>,$$ 

(5)

where $|Q_0>, |Q_1>, |Y_0>, |Y_1>$ are some vectors, belonging to $H_4$. Symmetry of QCM is provided by the fact, that right-hand part of this relation contains linear combinations of vectors $|\Phi_k>$ only. Taking into account (1), we obtain:

$$|\Xi> = U|\Psi 0_2 0_4> = $$

$$|\Phi_1>(a_{00}|0_3> + a_{01}|1_3>)|Q_0> + |\Phi_2>(a_{00}|0_3> + a_{01}|1_3>)|Y_0> + $$

$$|\Phi_3>(a_{10}|0_3> + a_{11}|1_3>)|Q_1> + |\Phi_2>(a_{10}|0_3> + a_{11}|1_3>)|Y_1>.$$ 

Generally speaking, the choice of the unitary operator $U$ is very broad and corresponding analysis is quite complex even for the lowest dimensions, so usually one admits some additional restrictions. We suppose as in \cite{8}, that following conditions (which guarantee the unitarity of $U$) are fulfilled:

$$<Q_k|Q_k> + <Y_k|Y_k> = 1, k = 1, 2,$$ 

(6)

$$<Y_0|Y_1> = <Q_0|Q_1> = <Q_k|Y_k> = 0, k = 1, 2.$$ 

(7)

Let

$$<Q_0|Q_0> = <Q_1|Q_1> = \zeta, <Y_1|Q_0> = <Q_1|Y_0> = \nu \sqrt{(1 - \zeta)\zeta},$$ 

(8)
so as $< Y_0 | Y_0 > = < Y_1 | Y_1 > = 1 - \zeta$, $0 \leq | \nu | \leq 1$. In this case QCM produces the next state from $H_1 \otimes H_2$:

$$\rho^{(12)}_{out} = Tr_{34} | \Xi > < \Xi | =$$

$$A \zeta | \Phi_1 > < \Phi_1 | + (1 - A) \zeta | \Phi_3 > < \Phi_3 | + (1 - \zeta) | \Phi_2 > < \Phi_2 | +$$

$$+ B \nu \sqrt{\zeta(1 - \zeta)} (| \Phi_1 > < \Phi_2 | + | \Phi_2 > < \Phi_3 |) + B \nu \sqrt{\zeta(1 - \zeta)} (| \Phi_2 > < \Phi_1 | + | \Phi_3 > < \Phi_2 |).$$

Reducing this state on the first component, we obtain:

$$\rho^{(1)}_{out} = Tr_{234} | \Xi > < \Xi | = \tilde{A} | 0 > < 0 | + \tilde{B} | 0 > < 1 | + \tilde{B} | 1 > < 0 | + \tilde{C} | 1 > < 1 |,$$

$$\tilde{A} = 1/2 - \zeta(1/2 - A), \tilde{C} = 1/2 + \zeta(1/2 - A), \tilde{B} = B \nu \sqrt{2\zeta(1 - \zeta)}.$$

It is necessary to compare the initial state and state which is produced by the QCM, in other words, we have to choose the measure of the closeness of these states. There are different measures, specifically, fidelity. It is defined for the mixed states as

$$F = \left[ Tr \sqrt{\rho_{init} \rho_{out} \rho_{init}} \right]^2,$$

this value is not very suitable for the analytical considerations. We use here more convenient measure:

$$\| \rho^{(1)}_{init} \otimes \rho^{(2)}_{init} - \rho^{(12)}_{out} \|^2 = Tr \left[ \rho^{(1)}_{init} \otimes \rho^{(2)}_{init} - \rho^{(12)}_{out} \right]^2 = W(\zeta, \nu, \Psi),$$

where

$$W(\zeta, \nu, \Psi) = A^2(A - \zeta)^2 + (1 - A)^2(1 - \zeta)^2 + (1 - \zeta)^2 + 2 [A^2(1 - A)^2 + 2 | B |^4] +$$

$$2 | B |^2 \left( | \sqrt{2}A - \nu \sqrt{(1 - \zeta)\zeta} |^2 + | \sqrt{2}(1 - A) - \nu \sqrt{(1 - \zeta)\zeta} |^2 \right) -$$

$$2(1 - \zeta) [A(1 - A) + | B |^2].$$

This value estimates the difference between initial and final states with fixed parameters $a_{00}, a_{01}, a_{10}, a_{11}$. For the determination of the QCM parameters we average this value respect to the set of all initial states. We use here the next parametrization of the initial state $| \Psi >$:

$$a_{00} = \cos \theta_1, a_{01} = \exp(i\gamma_1) \sin \theta_1 \cos \theta_2,$$

$$a_{10} = \exp(i\gamma_2) \sin \theta_1 \sin \theta_2 \cos \theta_3, a_{11} = \exp(i\gamma_3) \sin \theta_1 \sin \theta_2 \sin \theta_3,$$
where

\[ 0 \leq \theta_k \leq \pi/2, 0 \leq \gamma_m \leq 2\pi. \]

Here the first component has zero phase due to the corresponding freedom of the choice. For the averaging we need in corresponding measure. Supposing that all states \( |\Psi> \) are equiprobable, we choose as such a measure

\[
G(\zeta, \nu) = \| \rho_{\text{init}}^{(1)} \otimes \rho_{\text{init}}^{(2)} - \rho_{\text{out}}^{(12)} \|_\text{aver}^2 = \frac{1}{\pi^5} \int_0^{\pi/2} d\theta_1 \int_0^{\pi/2} d\theta_2 \int_0^{2\pi} d\gamma_1 \int_0^{2\pi} d\gamma_2 \int_0^{2\pi} d\gamma_3 \sin^2 \theta_1 \sin \theta_2 W(\zeta, \nu, \Psi)
\]

Simple calculations lead to the conclusion, that \( G(\zeta, \nu) \) takes its minimal value at \( \nu = 1, \zeta \approx 0.725 \). This value \( \nu \) implies, that vectors \( Q_0, Y_1 \) and \( Q_1, Y_0 \) are parallel. The values of the fidelity \( F = \left[ Tr \sqrt{\rho_{\text{init}}^{(1)} \rho_{\text{out}} \rho_{\text{init}}^{(1)} \rho_{\text{out}}} \right]^2 \), calculated at \( \zeta = 0.725 \), for the states on the "real" part of the Bloch sphere,

\[
\rho = \frac{1}{2} \begin{pmatrix}
1 + r \cos \theta & r \sin \theta \\
r \sin \theta & 1 - r \cos \theta
\end{pmatrix},
\]

\( 0 \leq r \leq 1, 0 \leq \theta \leq \pi/2, \) are plotted in fig.1. Evidently, that this QCM is state-dependent one, because the quality of the cloning depends on the quantum state.

3 QCM for the mixed states.

The construction described above can be used for the cloning of the mixed states. Note, that final state produced by QCM depend on the reduction of the initial state \( \rho_{\text{init}} \) only. It means, that one can reverse our considerations and take the mixed state \( \rho_{\text{init}} \) as initial one. Then pure state \( |\Psi> \) defined by relation (1) belongs to the space of the larger dimension and it is a "purification" of the state \( \rho_{\text{init}} \). "Purification" of the given mixed state \( \rho_{\text{init}} \) can be realized by different methods, as it follows from (11), but this nonuniqueness has not influence in the results. QCM constructed in accordance with relations (4), (5) produces states (9), (10), which depend on the parameters of the initial state \( \rho_{\text{init}} \) only. But the set of the initial states is changing, one has to use another parametrization for this set. As such parametrization one can take a Bloch sphere, see [7]. Namely, density matrix \( \rho_{\text{init}} \) can be described as

\[
\rho_{\text{init}} = \frac{1}{2} \begin{pmatrix}
A & B \\
B & C
\end{pmatrix} = \frac{1}{2} \begin{pmatrix}
1 + P_3 & P_1 - iP_2 \\
P_1 + iP_2 & 1 - P_3
\end{pmatrix}
\]
where
\[ P_1^2 + P_2^2 + P_3^2 \leq 1. \]

In the spherical coordinates we have:
\[ P_3 = r \cos \theta, P_1 = r \sin \theta \cos \varphi, P_2 = r \sin \theta \sin \varphi, \]
\[ 0 < r \leq 1, 0 \leq \theta \leq \pi, 0 \leq \varphi \leq 2\pi. \]

and
\[ A = \frac{1}{2}(1 + r \cos \theta), B = \frac{1}{2}r \sin \theta \exp(-i\varphi), C = \frac{1}{2}(1 - r \cos \theta). \]

In order to obtain the parameters of the QCM one has to average the value \( W(\zeta, \nu, \rho_{\text{init}}) \) on the Bloch sphere. We suppose that all states in the Bloch sphere are equiprobable, so averaging is reduced to the integral
\[ G(\zeta, \nu) = \frac{3}{4\pi} \int_0^1 r^2 dr \int_0^\pi \sin \theta d\theta \int_0^{2\pi} d\varphi W(\zeta, \nu, \rho_{\text{init}}). \]

Note, that if Eve has a priori information about transmitted quantum information she has to choose corresponding weight multiplier. We are searching in such values \( \zeta, \nu \) which correspond to the minimal value \( G(\zeta, \nu) \). As a result we obtain \( \nu = 1, \zeta \approx 0.715 \). The plot of fidelity for these parameters differs in a small way from the preceding one, so we omit it.

4 Conclusion

We have discussed here a "partial" quantum cloning, when only one component of the two-partite pure state is cloning. Such cloning can be considered as a variant of the eavesdropping of the quantum channel. The choice of the parameters of the QCM was realized with help of the some natural criterion. Namely, we seek in parameters corresponding to the minimum of the integral average of the "distance" between the initial state and output state. Note, that fidelity of the initial and output states for the most part of the Bloch sphere exceeds value \( 5/6 \) which corresponds to the universal QCM for the pure states \( \mathbb{I} \). This fact has two reasons. Firstly, described QCM copies only part of the two-partite state. Second, this QCM is state-dependent one, and this non-universality raises the quality of the cloning.
Moreover, here was discussed the cloning of the mixed qubit states. In order to consider such device we use the "purification" of the mixed state and then apply "partial" cloning machine. Let’s emphasize, that "purification" of the mixed state is not unique, but the output state produced by our QCM does not depend on this nonuniqueness. Parameters of the QCM was sought by minimization of the integral average of the distance between the initial and output states. Note, that this averaging differs from the used above because the sets of the states are different. In this case the fidelity of the initial and output states on the real part of the Bloch sphere exceeds the value 5/6 too. Evidently, that choice of the parameters $\nu, \zeta$ for the QCM is defined by the strategy of the eavesdropping.

We have discussed here only one possible QCM for the mixed states. Evidently, that there are many other variants for the QCM, may be, without restrictions like (4), (5), (8), asymmetric QCM etc.

References

[1] W.K. Wooters, W.H. Zurek. Nature, 299, 802 (1982).

[2] D. Dieks. Phys. Lett. A, 54, 1844 (1982).

[3] H. Barnum, C. M. Caves, C. A. Fuchs, R. Jozsa, B. Schumacher. Phys. Rev. Lett., 76, 2818 (1996) (arXiv quant-ph/9511010).

[4] V. Buzek, M. Hillery. Phys. Rev. A, 54, 1844 (1996).

[5] V. Scarani, S. Ibilsdur, N. Gisin. Rev. Mod. Phys., v.77, 1225-1256 (2005).

[6] N.J. Cerf, J. Fiurasek. Optical quantum cloning. In: Progress in Optics, v.49, ed. E. Wolf, Elsevier, 2006.

[7] J. Preskill. Lecture Notes for Physics: Quantum Information and Computation, 1998.

[8] S. Adhikari, A. K. Pati, I. Chakrabarty, B. S. Choudhury. arXiv:quant-ph/0705.0631, 2007.
Figure 1: Fidelity on real part of Bloch sphere