On the Average Complexity of the $k$-Level

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Abstract

Let $A$ be an arrangement of $n$ lines in the Euclidean plane. The $k$-level of $A$ consists of all intersection points $v$ of lines in $A$ which have exactly $k$ lines of $A$ passing below $v$. The complexity of the $k$-level in a line arrangement has been widely studied. In 1998 Dey proved an upper bound of $O(n \cdot (k+1)^{1/3})$. We investigate the complexity of $k$-levels in random line and hyperplane arrangements. When the arrangement is obtained from any fixed projective line arrangement of $n$ lines by choosing a random cell to contain the south-pole, we prove an upper bound of $O((k+1)^2)$ on the expected complexity of the $k$-level. As a byproduct we show that the complexity of any ($\leq j$)-zone in a $d$-dimensional simple arrangement of $n$ hyperplanes is of order $\Theta((j+1)^{d-1})$. The classical zone theorem is the case $j = 0$.

We also consider arrangements of great $(d-1)$-spheres on the sphere $S^d$ which are orthogonal to a set of random points on $S^d$. In this model we prove that the expected complexity of the $k$-level is of order $\Theta((k+1)^{d-1})$.

1 Introduction

Let $A$ be an arrangement of $n$ lines in the Euclidean plane. The vertices of $A$ are the intersection points of lines of $A$. Throughout this article we consider arrangements to be simple, i.e., no 3 lines intersect in a common vertex, we also assume that no two lines are parallel, and no line is vertical. The $k$-level of $A$ consists of all vertices $v$ which have exactly $k$ lines of $A$ below $v$. We denote the $k$-level by $V_k(A)$ and its size by $f_k(A)$. Moreover, by $f_k(n)$ we denote the maximum of $f_k(A)$ over all arrangements $A$ of $n$ lines, and by $f(n) = f_{\lceil (n-2)/2 \rceil}(n)$ the maximum size of the middle level.

A $k$-set of a finite point set $P$ in the Euclidean plane is a subset $K$ of $k$ elements of $P$ that can be separated from $P \setminus K$ by a line. Paraboloid duality is a bijection $P \leftrightarrow A_P$ between
point sets and line arrangements (for details on this duality see [O’R94, Chapter 6.5] or [Edc87, Chapter 1.4]). The number of $k$-sets of $P$ equals $|V_{k-1}(A_P)| \cup V_{n-1-k}(A_P)|$.

In discrete and computational geometry bounds on the number of $k$-sets of a planar point set, or equivalently on the size of $k$-levels of a planar line arrangement have important applications. The complexity of $k$-levels was first studied by Lovász [Lov71] and Erdős et al. [ELSS73], they bound the size of the $k$-level by $O(n \cdot (k + 1)^{1/2})$. Dey [Dev98] used the crossing lemma to improve the bound to $O(n \cdot (k + 1)^{1/3})$. In particular, the maximum size $f(n)$ of the middle level is $O(n^{4/3})$. Concerning the lower bound on the complexity, Erdős et al. [ELSS73] gave a construction showing that $f(2n) \geq 2f(n) + cn = \Omega(n \log n)$ and conjectured that $f(n) \geq \Omega(n^{1+\varepsilon})$. An alternative $\Omega(n \log n)$-construction was given by Edelsbrunner and Welzl [EW85]. The current best lower bound $f_k(n) \geq n \cdot e^{\Omega(\sqrt{\log n})}$ was obtained by Nivasch [Niv08] improving on a bound by Tóth [Tőt01]. For more background on the problem we refer to Chapter 11 of Matoušek’s book [Mat02].

1.1 Higher Dimensions

The problem of determining the complexity of the $k$-level admits a natural extension to higher dimensions: Consider a simple arrangement $\mathcal{A}$ of $n$ hyperplanes in $\mathbb{R}^d$, i.e., no $d+1$ hyperplanes intersect in a common point, we also assume that the intersection of any $d$ given hyperplanes is a single point, and no hyperplane is parallel to the $x_d$-axis. The $k$-level $V_k(\mathcal{A})$ of $\mathcal{A}$ consists of all vertices (i.e. intersection points of $d$ hyperplanes) which have exactly $k$ hyperplanes of $\mathcal{A}$ below them (with respect to the $d$-th coordinate). We denote the $k$-level by $V_k(\mathcal{A})$ and its size by $f_k(\mathcal{A})$. Moreover, by $f^{(d)}_k(n)$ we denote the maximum of $f_k(\mathcal{A})$ among all arrangements $\mathcal{A}$ of $n$ hyperplanes in $\mathbb{R}^d$.

As in the planar case, there remains a gap between lower and upper bounds;

$$\Omega(n^{d/2} k^{[d/2]-1}) \leq f^{(d)}_k(n) \leq O(n^{d/2} k^{[d/2]-1} c_d),$$

here $c_d > 0$ is a small positive constant only depending on $d$. Details and references can be found in Chapter 11 of Matoušek’s book [Mat02]. In dimensions 3 and 4 improved bounds have been established. For example, for $d = 3$, it is known that $f^{(3)}_k(n) \leq O(n(k + 1)^{3/2})$ (see [SST01]).

For the middle level in dimension $d \geq 2$ an improved lower bound $f^{(d)}_k(n) \geq n^{d-1} \cdot e^{\Omega(\sqrt{\log n})}$ is known (see [Tőt01] and [Niv08]).

2 Our Results

In the first part of this paper we consider arrangements of lines in the projective plane and investigate the average complexity of the $k$-level, when the arrangement is “randomly” projected to an Euclidean arrangement. This question was raised by Barba, Pilz, and Schnider while sharing a pizza [BPS19].

In the considered model, a cell $c$ of the arrangement is chosen uniformly at random and we consider a projected Euclidean arrangement where $c$ is mapped to the top-bottom-cell/southpole. In Section 3 we prove the following bound on the average complexity of the $k$-level in this model. Remarkably the bound is independent of the number $n$ of lines in the arrangement.

**Theorem 1.** Given a projective arrangement $\mathcal{A}$ of $n$ lines, the expected size of the $k$-level in the induced Euclidean arrangement is at most $8e \cdot (k + 1)^2$ when the southpole is chosen uniformly at random among the cells of $\mathcal{A}$.
Given a hyperplane arrangement $A$ in $\mathbb{R}^d$ and a hyperplane $H_0 \in A$, the zone of $H_0$, denoted by $Z_{\leq 0}(H_0, A)$, is the set of all faces (from 0-dimensional, i.e., vertices, to $d$-dimensional, i.e., cells) of $A$ which can be seen from $H_0$, i.e., can be connected to $H_0$ along a simple path which intersects hyperplanes of $A$ only at the endpoints. The classical zone theorem for hyperplane arrangements (cf. [ESS91] and [Mat02, Chapter 6.4]) bounds the size of any zone in an arrangement of $n$ hyperplanes by $O(n^{d-1})$.

Our proof of Theorem 1 uses the planar case of the following generalization of the zone theorem to higher orders. The $(\leq j)$-zone of a hyperplane $H_0$ in an arrangement $A$, denoted by $Z_{\leq j}(H_0, A)$, consists of all faces of $A$ which can be connected to $H_0$ with a simple path whose interior intersects at most $j$ hyperplanes of $A$.

**Theorem 2** (Generalized Zone Theorem). Let $A$ be a simple arrangement of $n$ hyperplanes in $\mathbb{R}^d$ and let $H_0 \in A$, then the complexity of the $(\leq j)$-zone of $H_0$ is of order $\Theta((j + 1)n^{d-1})$.

We prove this theorem in Section 4. For the planar case $d = 2$, we show that the number of vertices in the $(\leq j)$-zone of $H_0$ that also lie above or on $H_0$ is at most $2e(j + 2)n$. This bound is used in the proof of Theorem 1.

In Section 5 we consider “arrangements of randomly chosen lines”. Here we propose the following model of randomness. Think of a projective line arrangement as a great-circle arrangement on the unit sphere $S^2$ in $\mathbb{R}^3$. The correspondence between great-circles on $S^2$ and planes through the origin in $\mathbb{R}^3$ extends to a correspondence between arrangements of the respective objects, Figure 1 gives an illustration.

**Figure 1**: The correspondence between great-circles on the unit sphere $S^2$ and lines in a plane $\Pi$. Using the center of the sphere as the center of projection, the points $A, B, C, D$ on the sphere $S^2$ are projected to the points $A', B', C', D'$ in the plane $\Pi$.

On $S^2$ we have the duality between points (each antipodal pair of points defines the normal vector of the plane containing a great-circle) and great-circles. Since we can choose points uniformly at random from $S^2$, we get random arrangements of great-circles. This duality clearly generalizes to higher dimensions, and we can therefore talk about random arrangements on $S^d$ for a fixed dimension $d \geq 2$. We call the intersection of $S^d$ with a central hyperplane in $\mathbb{R}^{d+1}$ a great-$(d-1)$-sphere of $S^d$. Using the duality between antipodal pairs of points on $S^d$ and great-$(d-1)$-spheres we prove the following bound on the expected size of the $k$-level in this random model:
Theorem 3. Let $d \geq 2$ be fixed. In an arrangement of $n$ great-$(d-1)$-spheres chosen uniformly at random on the unit sphere $S^d$ (embedded in $\mathbb{R}^{d+1}$), the expected size of the $k$-level is of order $\Theta((k+1)^{d-1})$ for all $k \leq n/2$.

Corollary 4. Let $d \geq 2$ be fixed. In an arrangement of $n$ hyperplanes, which arises as the projection of an arrangement of $n$ great-$(d-1)$-spheres chosen uniformly at random from the unit sphere $S^d$ (embedded in $\mathbb{R}^{d+1}$), the expected size of the $k$-level is of order $\Theta((k+1)^{d-1})$.

3 Proof of Theorem 1

As the preparation for the proof of Theorem 1 we introduce some terminology and prove a few preliminary results. Let each of $F$ and $F'$ be a vertex, edge, line, or cell of an arrangement $A$ of lines. We define their distance $d_A(F,F')$ as the minimum number of lines of $A$ intersected by the interior of a curve connecting a point of $F$ with a point of $F'$. Using this terminology the $(\leq j)$-zone $Z_{\leq j}(\ell,A)$ of a line $\ell$ in an arrangement $A$ is defined as the set of vertices, edges and cells from $A$ which have distance at most $j$ from $\ell$. See Figure 2 for an illustration. The classical zone theorem asserts that $Z_{\leq 0}(\ell,A)$ has linear complexity (see e.g. [Mat02, Chapter 6.4], [O'R94, Chapter 6.2], or [Ede87, Chapter 5.3]). By Theorem 2 the complexity of $Z_{\leq j}(\ell,A)$ is in $O((j+1)n)$.

![Figure 2: The higher order zones of a line \(\ell\).](image)

Fix a directed line $\ell \in A$ and assume without loss of generality that it is horizontal and directed from left to right. Our aim is to bound the size of the set $C_k(\ell)$ of pairs $(C,v)$ where $C$ is a cell of the zone below and touching $\ell$ and $v$ is a vertex above $\ell$ whose distance to $C$ is $k$. Clearly, $v$ has to belong to the $(\leq k-1)$-zone of $\ell$.

Consider a family $\mathcal{F}$ of half-intervals in $\mathbb{R}$. We have left-intervals of the form $(-\infty, a]$ and right-intervals $[b, \infty)$. A collection of $k$ half-intervals from $\mathcal{F}$ is a $k$-clique if there is a point $p \in \mathbb{R}$ that lies in all these $k$ half-intervals but not in any other half-interval of $\mathcal{F}$.

Lemma 5. Any family of half-intervals contains at most $k+1$ different $k$-cliques.

Proof. For $p \in \mathbb{R}$, let $l(p)$ be the number of left-intervals and $r(p)$ the number of right-intervals containing $p$. A point $p$ certifies a $k$-clique iff $l(p) + r(p) = k$. From the monotonicity of the functions $l$ and $r$ it follows that if $(l(p_1), r(p_1)) = (l(p_2), r(p_2))$ for two points $p_1$ and $p_2$, then they are contained in the same intervals. Thus the number of $k$-cliques is at most the number of pairs $(l, r)$ such that $l + r = k$ and $l, r \geq 0$, which is $k+1$. \qed
For a fixed vertex $v$ in the $(\leq k-1)$-zone above $\ell$, let $B_\ell(v)$ be the set of cells $C$ such that $(C, v) \in C_\ell(\ell)$.

**Claim.** $|B_\ell(v)| \leq k$.

**Proof.** Consider a line $g$ in $A$ and let $a$ be its intersection with $\ell$. If $v$ is to the left of $g$, draw the half-interval $[a, \infty)$ on $\ell$. If $v$ is to the right of $g$, draw the half-interval $(-\infty, a]$ on $\ell$. Let $H$ be the set of these half-intervals. We claim that there is a bijection between $B_\ell(v)$ and the $(k-1)$-cliques in $H$. Indeed, if the intersection of the half-intervals of a clique $K$, viewed as a subset of $\ell$, is $I_K$, then $I_K$ is the subset of $\ell$ which is reachable from $v$ by crossing the lines corresponding to the half-intervals of $K$. If $C$ is a cell below $\ell$ at distance $k$ from $v$ then $\ell$ and a subset of $(k-1)$ additional lines have to be crossed to reach $v$ from $C$, i.e., there is a $(k-1)$-clique in $H$ whose intersection is $C \cap \ell$. The number of $(k-1)$-cliques in $H$ is at most $k$ by Lemma 5.

Let $C_k$ be the union of the $C_k(\ell)$ over all the $2^n$ choices of a directed line $\ell$ in $A$.

**Theorem 6.** Let $A$ be an arrangement of $n$ lines and let $1 \leq k \leq n$. Then $|C_k| \leq 4e \cdot k(k+1) \cdot n^2$.

**Proof.** For a fixed directed line $\ell$ the set $C_k(\ell)$ is the union of $B_\ell(v)$ over all vertices $v$ in $A$ in the $(\leq k-1)$-zone above $\ell$. From the proof of the Generalized Zone Theorem (see the end of Section 4), we get that the number of such vertices is at most $2e(k+1)n$. From the above claim we have $|B_\ell(v)| \leq k$ so that $|C_k(\ell)| \leq 2ek(k+1)n$. Since there are $2n$ directed lines we get $|C_k| \leq 4ek(k+1)n^2$.

We are ready to prove Theorem 1.

**Proof.** The $k$-level with the southpole chosen in cell $C$ consists of the vertices at distance $k$ from $C$. Thus, the expected complexity of the $k$-level when choosing $C$ uniformly at random equals $|C_k|$ divided by the number of cells. Since the number of cells in a projective arrangement of $n$ lines is $\binom{n}{2} + 1$ and $|C_k| \leq 4ek(k+1)n^2$ by Theorem 6, we can conclude the statement from

$$\frac{4e \cdot k(k+1) \cdot n^2}{\binom{n}{2} + 1} \leq 8e \cdot k(k+1) \cdot \frac{n}{n-1} \leq 8e \cdot (k+1)^2 \cdot \frac{k}{k+1} \cdot \frac{n}{n-1} \leq 1.$$ 

4 Proof of Theorem 2

Let $A$ be an arrangement of $n$ hyperplanes in $\mathbb{R}^d$ and let $H_0 \in A$ be a fixed hyperplane. For any $j = 0, 1, \ldots, n-1$ denote by $V_{\leq j}$ the set of vertices of $A$ contained in the $(\leq j)$-zone $Z_{\leq j}(H_0, A)$ of $H_0$ in $A$, i.e., $v \in V_{\leq j}$ if there is a simple path $P_v$ from $v$ to $H_0$ whose interior has at most $j$ intersections with hyperplanes from $A$. Note that $V_{\leq 0}$ is the set of vertices in the traditionally studied zone of $H_0$ in $A$. 

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**Lower Bound:** We claim that $|V_{\leq j}| \geq \frac{1}{d+1}(j+1)(n-1)^{d-1}$ for $n \geq j + d + 1$ (here we use the usual notation for falling factorials $x^k = x(x-1)\ldots(x-k+1)$). To prove this bound, we use induction on the dimension $d$. For the base case, let $d = 1$. Since $A$ is an arrangement of points on the line, it is clear that if $n \geq j + 2$, we have $|V_{\leq j}| \geq j + 1$, as claimed.

Now let $d \geq 2$ and assume that the bound holds for simple arrangements of $n \geq j + d$ hyperplanes in $\mathbb{R}^{d-1}$.

Let $A$ be a simple arrangement of $n \geq j + d + 1$ hyperplanes in $\mathbb{R}^d$. For each hyperplane $H \in A \setminus \{H_0\}$, we denote the arrangement induced by the other hyperplanes of $A$ on $H$ by $A/H$, i.e., $A/H = \{H' \cap H \mid H' \in A \setminus \{H\}\}$. The arrangement $A/H$ is a simple arrangement of $n - 1$ hyperplanes in $(d-1)$-dimensional space, whose vertices are the vertices of $A$ contained in $H$. Now consider the $(\leq j)$-zone $Z_{\leq j}(H_0 \cap H, A/H)$ of the $(d-2)$-dimensional plane $H_0 \cap H$ within $A/H$. Every vertex in $Z_{\leq j}(H_0 \cap H, A/H)$ is a vertex of $Z_{\leq j}(H_0, A)$ and each vertex $v$ in $Z_{\leq j}(H_0, A)$ is a vertex of the induced arrangement $A/H$ for each of the $d$ hyperplanes $H$ incident to $v$. Using the induction hypothesis, we obtain $|V_{\leq j}| \geq \frac{1}{d+1} \left(\frac{1}{1-j/2}\right)^d (j+1)(n-2)^{d-2} = \frac{1}{d+1} (j+1)(n-1)^{d-1}$.

This proves the claim and shows that the $(\leq j)$-zone of $H_0$ is of size $\Omega((j+1)n^{d-1})$.

**Upper Bound:** Let $A$ be a simple arrangement of $n$ hyperplanes in $\mathbb{R}^d$. Let $R$ be a random sample of hyperplanes from $A$ where $H_0 \in R$ and each hyperplane $H \neq H_0$ independently belongs to $R$ with probability $p := \frac{1}{j+2}$. The probability that a vertex $v \in V_{\leq j}$ is present in the induced subarrangement $A(R)$ and appears at distance 0 from $H_0$ is at least $\left(\frac{1}{j+2}\right)^d \cdot (1 - \frac{1}{j+2})^r$, where $0 \leq r \leq j$ denotes the distance of $v$ from $H_0$ in $A$. The $d$ hyperplanes determining the vertex $v$ are present with probability $\left(\frac{1}{j+2}\right)^d$, and the $r$ hyperplanes intersecting a fixed witnessing path $P_v$ from $v$ to $H_0$ are not present with probability $\left(1 - \frac{1}{j+2}\right)^r$. Note that

$$\left(1 - \frac{1}{j+2}\right)^r \geq \left(1 - \frac{1}{j+2}\right)^{j+1} = \left(\frac{j+1}{j+2}\right)^{j+1} = \left(1 + \frac{1}{j+1}\right)^{(j+1)} \geq 1/e,$$

where $e = 2.718\ldots$ denotes Euler’s number. Figure 3 gives an illustration for the planar case.

**Figure 3:** A path $P_v$ witnessing that $v$ belongs to the $(\leq j)$-zone of $\ell$ for all $j \geq 2$.

Let $X$ be the number of vertices in the $0$-zone of $H_0$ in $A(R)$. For the expectation of this random variable we have $\mathbb{E}(X) \geq \frac{1}{d+2} \left(\frac{1}{j+2}\right)^d \cdot |V_{\leq j}|$.

On the other hand, by the classical zone theorem we have $X \leq c \cdot |R|^{d-1}$ for some constant $c = c(d)$ only depending on $d$. Therefore $\mathbb{E}(X) \leq c \cdot \mathbb{E}(Y^{d-1})$, where $Y$ is the number of hyperplanes in $R$. Note that $Y \sim B(n, p)$ is a binomially distributed random variable.

The above inequalities imply $|V_{\leq j}| \leq c \cdot e \cdot (j + 2)^d \cdot \mathbb{E}(Y^{d-1})$. From known bounds for the moments of the binomial distribution, we obtain the estimate $\mathbb{E}(Y^{d-1}) = \Theta((np)^{d-1})$ (see for
instance [PT10], Corollary 2.1). Hence
\[ |V_{\leq j}| \leq c \cdot e \cdot (j + 2)^d \cdot O((n/(j + 2))^{d-1}) = O((j + 1)n^{d-1}). \]

Every vertex \( v \) of \( A \) belongs to at most \( 3^d \) faces of \( A \). Every face \( F \) belonging to \( Z_{\leq j}(H_0, A) \) contains a vertex \( v \) which also belongs to \( Z_{\leq j}(H_0, A) \). Therefore, \( |Z_{\leq j}(H_0, A)| \leq 3^d |V_{\leq j}| \) whence \( |Z_{\leq j}(H_0, A)| = O((j + 1)n^{d-1}) \).

For the planar case \( d = 2 \), we can provide reasonable bounds for the number of vertices in the \((\leq j)\)-zone: An inductive argument, as used to show the classical zone theorem (see e.g. [GHW13 page 136]), shows \( |V_{\leq j}^+| \leq 2n - 3 \). Using the constant 2 in the role of \( c(2) \), we obtain \( |V_{\leq j}^+| \leq 2e(j + 2)n \).

This concludes the proof of Theorem 2.

5 Proof of Theorem 3

Let \( C \) be a simple arrangement of \( n \) great-(\( d - 1 \))-spheres on the unit sphere \( S^d = \{ x \in \mathbb{R}^{d+1} : \|x\| = 1 \} \) with center \( o \) in \( \mathbb{R}^{d+1} \). For a vertex \( v \) of the arrangement, let \( \phi_C(v) \) denote the number of great-(\( d - 1 \))-spheres that are crossed by the geodesic arc from \( v \) to the south-pole \( s = (0, \ldots, 0, -1) \) of the sphere. The set of vertices \( v \) of \( C \) with \( \phi_C(v) = k \) is denoted \( V_k(C) \).

When \( C \) is projected to a \( d \)-dimensional plane \( H \) with the origin \( o = (0, \ldots, 0) \) as center of projection, we obtain an arrangement \( A \) of hyperplanes in \( \mathbb{R}^d \). Moreover, if the south pole \( s \) is projected to a point “at infinity” of \( H \), say to \((0, \ldots, 0, -\infty) \), then, for every point \( p \) in \( S^d \), the \( S^1 \) containing the geodesic arc from \( p \) to \( s \) is projected to the “vertical” line through \( p \), i.e., the line \( p + (0, \ldots, 0, \lambda) \). The geodesic is projected to one of the two rays starting from \( p \) on this line. In particular, all vertices \( v \) of \( C \) with \( \phi_C(v) = k \) are projected to vertices of \( A \) either at level \( k \) or \( n - k - d \).

Let \( C \) be an arrangement of randomly chosen great-(\( d - 1 \))-spheres and let \( B \) be a subset of size \( d \) in \( C \). Note that with probability 1, the random great-sphere-arrangement is in general position, and simple, i.e., no more than \( d \) great-spheres intersect in a common point. Let \( p' \) be one of the two intersection points of the great-(\( d - 1 \))-spheres in \( B \). Now consider the arrangement \( C' = C - B \) and note that \( (C', p') \) can be viewed as a random arrangement of great-(\( d - 1 \))-spheres together with a random point on \( S^d \). Hence, to estimate the expected size of \( V_k(C) \), we can estimate the probability that \( \phi_C(p') = k \). This is the purpose of the following Lemma.

**Lemma 7.** Let \( C \) be an arrangement of \( n \) great-(\( d - 1 \))-spheres chosen uniformly at random on the unit sphere \( S^d \) (embedded in \( \mathbb{R}^{d+1} \) and centered at the origin). Let \( p \) be an additional point chosen uniformly at random from \( S^d \), and let \( B \) be the geodesic arc from \( p \) to the south pole on \( S^d \). For all \( k \leq n/2 \), the probability \( q_k \) that exactly \( k \) great-(\( d - 1 \))-spheres from \( C \) intersect \( A \) is of order \( \Theta((k + 1)^{d-1}/n^d) \). More precisely, it satisfies
\[
\frac{2^{d-1} \rho \pi (k + 1)^{d-1} (n - k + 1)^{d-1}}{(n + 1)^{2d-1}} \leq q_k \leq \min \left\{ \frac{\rho \pi}{n + 1}, \frac{\rho n^{d}(k + 1)^{d-1}}{(n + 1)^{2d-1}} \right\},
\]
where \( a^\gamma = a(a+1) \cdots (a+b-1) \) denotes the rising factorial and \( \rho = \frac{\text{area}_{d-1}(S^d)}{\text{area}_{d-1}(S^d)} = \frac{\Gamma(\frac{d+1}{2})}{\pi^{\frac{d+1}{2}}} \) only depends on the dimension \( d \).

For the planar case \( d = 2 \), the two upper bounds from Lemma 7 coincide if \( k \approx n/\pi \), and we have \( \frac{\pi}{2} \cdot \frac{1}{n} \leq q_k \leq \frac{\pi}{2} \cdot \frac{1}{n} \) for \( k \ll n/2 \), and \( \frac{1}{\pi} \cdot \frac{1}{n} \leq q_k \leq \frac{1}{\pi} \cdot \frac{1}{n} \) for \( k \approx n/2 \).
Proof. Denote by $\phi$ the length of the geodesic arc $A$ on $\mathbb{S}^d$ from $p$ to $s$, i.e., $\phi$ is the angle between the two rays emanating from $o$ towards $s$ and $p$. Note that – independent from the dimension $d$ – the three points $o$, $s$, and $p$ lie in a 2-dimensional plane which also contains the geodesic arc $A$. Point $p$ lies on a $(d - 1)$-sphere $C$ of radius $\sin(\phi)$ in the $d$-dimensional hyperplane defined by the equation $x_d = -\cos(\phi)$. Figure 4 gives an illustration for the case $d = 2$, where $C$ is a circle.

The probability that the arc $A$ defined by the random point $p$ is intersected by exactly $k$ great-$(d-1)$-spheres from the random arrangement $C$ is

$$q_k = \int_0^{\pi} \frac{\text{Vol}_{d-1}(\mathbb{S}^{d-1}) \sin^{d-1}(\phi)}{\text{Vol}_d(\mathbb{S}^d)} \cdot \left(\frac{n}{k}\right) (\phi/\pi)^k (1 - \phi/\pi)^{n-k} d\phi.$$  

This can be rewritten as

$$q_k = \rho \cdot \left(\frac{n}{k}\right) \cdot \int_0^{\pi} \sin^{d-1}(\phi) \cdot (\phi/\pi)^k (1 - \phi/\pi)^{n-k} d\phi,$$

where $\rho = \rho(d) = \frac{\text{Vol}_{d-1}(\mathbb{S}^{d-1})}{\text{Vol}_d(\mathbb{S}^d)} = \frac{\Gamma\left(\frac{d+1}{2}\right)}{\pi^{\frac{d+1}{2}} \Gamma\left(\frac{d+1}{2}\right)}$ is a constant only depending on $d$. The latter equation follows from $\text{Vol}_d(\mathbb{S}^d) = 2\pi^{\frac{d+1}{2}} / \Gamma\left(\frac{d+1}{2}\right)$, where $\Gamma(x)$ is the Euler gamma function (see e.g. [Wikb]).

In the following we give upper and lower bounds for $q_k$. The Euler beta function $B$ turns out to be the tool to evaluate the integrals:

$$B(a + 1, b + 1) = \int_0^1 t^a (1 - t)^b dt = \frac{a! \cdot b!}{(a + b + 1)!}.$$

For this identity and more information see for example [Wika].

To show the first upper bound on $q_k$, we bound the integral above as follows: Since $\sin(\phi) \leq 1$ holds for every $\phi \in [0, \pi]$, we have

$$q_k \leq \rho \left(\frac{n}{k}\right) \int_0^{\pi} (\phi/\pi)^k (1 - \phi/\pi)^{n-k} d\phi = \rho \pi \left(\frac{n}{k}\right) \int_0^1 t^k (1 - t)^{n-k} dt$$

$$= \rho \pi \left(\frac{n}{k}\right) B(k + 1, n - k + 1) = \rho \pi \cdot \frac{n!}{k! (n - k)!} \cdot \frac{k! (n - k)!}{(n + 1)!} = \rho \pi \cdot \frac{1}{n + 1}.$$
Towards the second upper bound on \( q_k \), we use the fact that \( \sin(\phi) \leq \phi \) holds for every \( \phi \in [0, \pi] 
\\n\\n\\n\\n
\[
q_k \leq \rho n^{d-1} \binom{n}{k} \int_{\phi=0}^{\pi} (\phi/n)^{k+d-1} (1 - \phi/n)^{n-k} d\phi = \rho n^d \binom{n}{k} \int_{t=0}^{1} t^{k+d-1} (1 - t)^{n-k} dt
\\n= \rho n^d \cdot \frac{n!}{k!(n-k)!} \cdot \frac{(k + d - 1)!(n-k)!}{(n+d)!} = \rho n^d \cdot \frac{(k+1)^{d-1}}{(n+1)^{d-1}}.
\]

To show the lower bound on \( q_k \), we split the integral in two parts: Since \( \sin(\phi) \geq 2 \cdot \phi/\pi \) holds for every \( \phi \in [0, \pi/2] \) and \( \sin(\phi) \geq 2 \cdot (1 - \phi/\pi) \) holds for every \( \phi \in [\pi/2, \pi] \), we have

\[
q_k \geq 2^{d-1} \rho \binom{n}{k} \left[ \int_{\phi=0}^{\pi/2} (\phi/n)^{k+d-1} (1 - \phi/n)^{n-k} d\phi + \int_{\phi=\pi/2}^{\pi} (\phi/n)^{k}(1 - \phi/n)^{n-k+d-1} d\phi \right]
\\\geq 2^{d-1} \rho \binom{n}{k} \int_{\phi=0}^{\pi} (\phi/n)^{k+d-1} (1 - \phi/n)^{n-k+d-1} d\phi
\\= 2^{d-1} \rho \pi \binom{n}{k} \int_{t=0}^{1} t^{k+d-1} (1 - t)^{n-k-d-1} dt
\\= 2^{d-1} \rho \pi \cdot \frac{n!}{k!(n-k)!} \cdot \frac{(k + d - 1)!(n-k + d - 1)!}{(n+2d-1)!}
\\= 2^{d-1} \rho (k+1)^{d-1} \cdot (n-k+1)^{d-1} \cdot (n+1)^{2d-1}.
\]

This completes the proof of Lemma 7.

---

**Proof of Theorem 3.** Consider an arrangement \( C \) of \( n + d \) great-(\( d - 1 \))-spheres \( C_1, \ldots, C_{n+d} \) chosen uniformly and independently at random from \( S^d \). Let \( p \) be a vertex of \( C \) chosen uniformly at random (i.e., one of the two points of intersection of \( d \) great-(\( d - 1 \))-spheres \( C_{i_1}, \ldots, C_{i_d} \) chosen u.a.r. from \( C \)). Note that \( p \) is a u.a.r. chosen point from \( S^d \).

We now apply Lemma 7 with \( p \) and \( C_p := C - \{C_{i_1}, \ldots, C_{i_d}\} \). Point \( p \) is separated from \( o \) by \( k \) great-(\( d - 1 \))-spheres from \( C_p \) with probability \( q_k = \Theta(k^{d-1}/n^d) \). Since \( p \) is chosen uniformly at random among the \( 2 \binom{n+d}{d} \) vertices of \( C \), we obtain the desired bound of \( \Theta(k^{d-1}) \).

---

**6 Discussion**

Due to the \( O(nk^{1/3}) \) upper bound for the complexity of the \( k \)-level, Theorem 3 is only interesting for small \( k \), i.e., \( k \ll n^{3/5} \). It would be interesting to have an improved upper bound for the expected size of the \( k \)-level when the south-cell is randomly chosen also in the range of values between \( \Omega(n^{3/5}) \) and \( n/2 \).

We have no non-trivial lower bound and would like to know the answer to the following question:

**Question 1.** Is there a family of line arrangements where the expected size of the middle level is superlinear when the southpole is chosen uniformly at random? What about other \( k \)-levels?
Recursive constructions from [ELSS73] and [EW85] show that the size of the $n/2 - s$ level can be in $\Omega(n \log n)$ for any fixed $s$. Nevertheless computer experiments suggest that if we choose a random southpole for these examples the expected size of the middle level drops to be linear.

In Section 5 we were concerned with a natural model of randomness, where great-$d - 1$-spheres are chosen independently and uniformly at random from the sphere. In the context of research on Erdős–Szekeres-type problems, several articles made use of point sets which are sampled uniformly at random from a convex shape (see e.g. [BF87] [Va95] [BGAS13] [BSV]). It would be interesting to obtain bounds on the number of $k$-sets also for random point sets in these models.

Also it is worth mentioning that the probabilistic method used in Section 4 was already used e.g. by Clarkson and Shor [CS89].

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