Examples of genuine false elliptic curves which are modular

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Abstract

Let \( K \) be an imaginary quadratic field. Modular forms for \( \text{GL}(2) \) over \( K \) are known as Bianchi modular forms. Standard modularity conjectures assert that every weight 2 rational Bianchi newform has either an associated elliptic curve over \( K \) or an associated abelian surface with quaternionic multiplication over \( K \). We give explicit evidence in the way of examples to support this conjecture in the latter case. Furthermore, the quaternionic surfaces given correspond to genuine Bianchi newforms, which answers a question posed by J. Cremona in 1992 as to whether this phenomenon can happen.

1 Introduction

Let \( K \) be an imaginary quadratic field. A simple abelian surface over \( K \) whose algebra of \( K \)-endomorphisms is an indefinite quaternion algebra over \( \mathbb{Q} \) is commonly known as a false elliptic curve. This name was coined by J.-P. Serre in the 1970s [1] based on the observation that such a surface is isogenous to the square of an elliptic curve modulo every prime of good reduction [25, Lemma 6].

It is well-known that one can obtain false elliptic curves over \( K \) by base changing suitable abelian surfaces over \( \mathbb{Q} \). Accordingly, let us call a false elliptic curve over \( K \) genuine if it is not the twist of base-change to \( K \) of an abelian surface over \( \mathbb{Q} \). Motivated by the conjectural connections with Bianchi modular forms, in 1992 J. Cremona asked whether genuine false elliptic curves over imaginary quadratic fields should exist (see Question 3). We answer this question in the positive by providing explicit genus 2 curves whose Jacobians are genuine false elliptic curves. To the best of our knowledge these are the first such examples in the literature. Furthermore, by carrying out a detailed analysis of the associated Galois representations and applying the Faltings-Serre-Livné criterion, we prove the modularity of these false elliptic curves.

Theorem 1 The Jacobians of the following genus 2 curves are false elliptic curves which are modular by a genuine Bianchi newform as in Conjecture 2. For more information about the curves see [2, 3]

1. \( C_1 : y^2 = x^6 + 4ix^5 + (-2i - 6)x^4 + (-i + 7)x^3 + (8i - 9)x^2 - 10ix + 4i + 3, \)  
   Bianchi newform: \( 2.0.4.1-34225.3-a; \)

2. \( C_2 : y^2 = x^6 + (-2\sqrt{-3} - 10)x^5 + (10\sqrt{-3} + 30)x^4 + (-8\sqrt{-3} - 32)x^3 + (-4\sqrt{-3} + 16)x^2 + (-16\sqrt{-3} - 12)x - 4\sqrt{-3} + 16, \)  
   Bianchi newform: \( 2.0.3.1-61009.1-a; \)

3. \( C_3 : y^2 = (104\sqrt{-3} - 75)x^6 + (528\sqrt{-3} + 456)x^5 + (500\sqrt{-3} + 1044)x^4 + (1038\sqrt{-3} + 2706)x^3 + (-1158\sqrt{-3} + 342)x - 612\sqrt{-3} - 1800, \)  
   Bianchi newform: \( 2.0.3.1-67081.3-a; \)
4. \( C_4 : y^2 = x^6 - 2\sqrt{-3}x^5 + (2\sqrt{-3} - 3)x^4 + 1/3(-2\sqrt{-3} + 54)x^3 + (-20\sqrt{-3} + 3)x^2 + (-8\sqrt{-3} - 30)x + 4\sqrt{-3} - 11 \).

Bianchi newform: 2.0.3.1-123201.1-b.

Proof. See §4. □

Over the rationals there is a celebrated result that establishes a connection between elliptic curves over \( \mathbb{Q} \) and classical newforms of weight 2. Extending this to number fields is an important aspect of the Langlands programme. However, in the case when the number field is totally complex the correspondence needs to be modified to include false elliptic curves. This was first observed by P. Deligne in a letter to J. Mennicke in 1979 [11] for imaginary quadratic fields. The construction is detailed in [14] and is illustrated with an explicit example. This makes for the following picture.

\[
\begin{array}{c}
\{ \text{weight 2 rational Bianchi newforms}/K \} \\
\downarrow 1:1 \\
\{ \text{non-CM by } K \text{ elliptic curves}/K \} \uparrow \\
\{ \text{false elliptic curves}/K \} \quad \text{up to isogeny}
\end{array}
\]

Thus the details of modularity for QM-surfaces are different from the case of \( \text{GL}_2 \)-type. Let us be completely explicit with the following conjecture [8, 23].

**Conjecture 2** Let \( K \) be an imaginary quadratic field.

1. Let \( f \) be a Bianchi newform over \( K \) of weight 2 and level \( \Gamma_0(\mathfrak{n}) \) with rational Hecke eigenvalues. Then there is either an ordinary elliptic curve \( E/K \) of conductor \( \mathfrak{n} \) such that \( L(E/K, s) = L(f, s) \) or there is a false elliptic curve \( A/K \) of conductor \( \mathfrak{n}^2 \) such that \( L(A/K, s) = L(f, s)^2 \).

2. Conversely, if \( E/K \) is an ordinary elliptic curve of conductor \( \mathfrak{n} \) then there is an \( f \) as above such that \( L(E, s) = L(f, s) \). Moreover, if \( A/K \) is a false elliptic curve of conductor \( \mathfrak{n}^2 \) then there is an \( f \) as above such that \( L(A, s) = L(f, s)^2 \).

Let \( f \) be a classical newform of weight 2 with a real quadratic Hecke eigenvalue field \( K_f = \mathbb{Q}(\{a_i\}) \) and denote \( \sigma = \text{Gal}(K_f/\mathbb{Q}) \). We say that \( f \) has an inner twist if \( f^\sigma = f \otimes \chi_K \) where \( \chi_K \) is the quadratic Dirichlet character associated to some imaginary quadratic field \( K \). It follows that \( f \) and \( f^\sigma \) must base change to the same Bianchi newform \( F \) over \( K \).

From a geometric point of view, let \( A/\mathbb{Q} \) be the abelian surface of \( \text{GL}_2 \)-type corresponding to \( f, f^\sigma \) via \( L(A/\mathbb{Q}, s) = L(f, s)L(f^\sigma, s) \). If the base-change surface \( A \otimes \mathbb{Q} \) \( K \) remains simple then it is a false elliptic curve and is modular by \( F \), that is \( L(A/K, s) = L(F, s)^2 \). This motivates the following question (see [9 Question 1'] and also [12 Conjecture 1]).

**Question 3** If \( f \) is a rational weight 2 Bianchi newform over \( K \) which is genuine, does \( f \) have an associated elliptic curve over \( K \)?

The term genuine is used to mean that \( f \) is not (a twist of) base-change of a classical newform just as with abelian surfaces. Given the modular correspondence above, we could rephrase this question to ask whether all false elliptic curves arise from a \( \text{GL}_2 \)-type surface over \( \mathbb{Q} \). The genus 2 curves given in [Theorem 1] answer this question and say that such a newform \( f \) does not necessarily have to correspond to an elliptic curve. For more background on Bianchi newforms see [10].
The genuine false elliptic curves we present also have an interesting connection to the Paramodularity Conjecture. Recall that the Paramodularity Conjecture posits a correspondence between abelian surfaces $A/\mathbb{Q}$ with $\text{End}_{\mathbb{Q}}(A) \otimes \mathbb{Q} \simeq \mathbb{Q}$ and genus 2 paramodular rational Siegel newforms of weight 2 that are not Gritsenko lifts [5, Conjecture 1.1]. As discussed in [4, 6] the conjectural correspondence needs to include abelian 4-folds $B/\mathbb{Q}$ with $\text{End}_{\mathbb{Q}}(B) \otimes \mathbb{Q}$ an indefinite non-split quaternion algebra over $\mathbb{Q}$ (see the amended version in [5, §8]). This can be illustrated using our genuine false elliptic curves.

Let $C/K$ be any of the four curves given in Theorem 1. Define $A/K$ to be the genuine false elliptic curve given by taking the Jacobian of $C/K$ with $\text{End}_{K}(A) \otimes \mathbb{Q} \simeq D/\mathbb{Q}$ an indefinite non-split quaternion algebra. Then the Weil restriction $B = \text{Res}_{K/\mathbb{Q}}(A)$ of $A$ from $K$ to $\mathbb{Q}$ is a simple abelian 4-fold such that $\text{End}_{\mathbb{Q}}(B) \otimes \mathbb{Q} \simeq D/\mathbb{Q}$. It is proven that there is a genuine rational weight 2 Bianchi newform $f$ over $K$ such that $L(A/K, s) = L(f, s^2)$. Now let $F$ be the genus 2 paramodular rational Siegel newform of weight 2 that is the theta lift of $f$. It now follows from the properties of Weil restriction [20] and theta lifting [3] that $L(B/\mathbb{Q}, s) = L(A/K, s) = L(f, s^2) = L(F, s^2)$.

Similar to the case of false elliptic curves, at any prime $p$ unramified in $D$ the 8-dimensional $p$-adic Tate module of $B/\mathbb{Q}$ splits as the square of a 4-dimensional submodule [7]. Then the 4-dimensional $p$-adic Galois representation has similar arithmetic to the one that arises from a 'generic' abelian 4-fold such that $\text{End}_{\mathbb{Q}}(B) \otimes \mathbb{Q} \simeq D/\mathbb{Q}$. Indeed, our example above shows that via the representation afforded by the submodule, $B/\mathbb{Q}$ is modular to a Siegel newform of the type considered in the Paramodularity Conjecture.

The article will be laid out as following: in §2 we outline how these genus 2 curves were found and in §3 we discuss some arithmetic properties of the attached Galois representation in the case where $\ell$ divides the discriminant of the acting quaternion algebra. Then §4 will be dedicated to showing how the Faltings-Serre-Livné criterion can be applied in order to prove that the examples given are modular. The final section lists the examples and contains further details of interest about them.

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2 Rational points on Shimura curves

In this section we outline how the genus 2 curves in Theorem 1 were found. Let $A$ be an abelian surface defined over an imaginary quadratic field $K$. Define the endomorphism rings $\text{End}_{K}(A)$ and $\text{End}_{\overline{K}}(A)$ to be the endomorphisms of $A$ which are defined over $K$ and $\overline{K}$ respectively. We say that $A$ has quaternionic multiplication, or $QM$ for short, if $\text{End}_{\overline{K}}(A)$ is an order $\mathcal{O}$ in an indefinite non-split quaternion algebra $B_D$ over $\mathbb{Q}$. We say that $A$ is a false elliptic curve if the action of $\mathcal{O}$ is defined over $K$. $B_D$ is used to denote the unique quaternion algebra of discriminant $D$ up to isomorphism.
Families of QM-surfaces have been constructed by K. Hashimoto et al. (see [15]) for quaternion algebras of discriminant 6 and 10. Testing numerically it would seem these give rise to surfaces which are all (a twist of) base-change. So we instead utilise two families given by S. Baba and H. Granath that have been derived from the moduli space.

Given the order $\mathcal{O}$, the set of norm 1 elements is denoted by $\mathcal{O}^1$. These act as isometries on the upper half plane $\mathcal{H}_2$ via the embedding $\mathcal{O} \hookrightarrow M_2(\mathbb{R})$ and the resulting quotient $X_D = \mathcal{H}_2/\mathcal{O}^1$ is a moduli space for abelian surfaces with quaternionic multiplication by $\mathcal{O}$. It is well known that these are compact Riemann surfaces called Shimura curves and they admit a model defined over $\mathbb{Q}$. In particular, $X_6 : X^2 + 3Y^2 + Z^2 = 0$ and $X_{10} : X^2 + 2Y^2 + Z^2 = 0$.

Let us detail the family for $D = 6$, for more information see [2]. We take the point on the conic

$$P_j = (4 : 3\sqrt{j} : \sqrt{-27j - 16}) \in X_6$$

and define the genus 2 curve

$$C_j : y^2 = (-4 + 3s)x^6 + 6tx^5 + 3t(28 + 9s)x^4 - 4t^2x^3 + 3t^2(28 - 9s)x^2 + 6tx - t^3(4 + 3s),$$

where $t = -2(27j + 16)$ and $s = \sqrt{-6j}$. Then the Jacobian of $C_j$ is a QM-surface. The curve is defined over the field $\mathbb{Q}(\sqrt{7}, \sqrt{-6})$ and the field of moduli for $C_j$ is $\mathbb{Q}(j)$. In this way we can generate numerous QM-surfaces by, for example, taking any $j \in \mathbb{Q}$.

For the purposes of modularity we need to fix a field $K = \mathbb{Q}(\sqrt{-5})$ and define a QM-surface over $K$. To do this we first establish whether $X_6(K)$ is non-empty and if this is the case then find a $K$-rational point $(a : b : c)$ on $X_6$. Take the quantity $j = (\frac{4b}{a})^2 \in K$ and the corresponding genus 2 curve $C_j$ is defined over $K(\sqrt{-6})$. It has a model defined over $K$ if and only if $K$ splits the Mestre obstruction which is the quaternion algebra

$$\left( \frac{-6j, -2(27j + 16)}{\mathbb{Q}(j)} \right) \simeq \left( \frac{-6, 2}{K} \right) \simeq B_D \otimes_{\mathbb{Q}} K.$$

Hence $C_j$ has a model defined over $K$ if and only if $K$ splits $B_D$.

So let us suppose that $K \hookrightarrow B_D$. Once we have the curve $C_j$ defined over $K(\sqrt{-6})$ we wish to find an isomorphic curve defined over $K$. Using MAGMA it is possible to take Igusa-Clebsch invariants and then create a model defined over $K$ with the same Igusa-Clebsch invariants. This then allows us to test whether the curve is a twist of base-change. It can be easily shown that a curve is genuine if the Euler polynomials at a pair of conjugate primes are not the same (up to twists). If it is indeed genuine then we endeavour to find a smaller model for the curve.

The size of the level places a limitation on whether a Bianchi modular form can be computed. Hence we try to find surfaces with as small a conductor as possible. With false elliptic curves it can be a challenge to find examples with small conductor (see [5] Section 8).

It is necessary to know the conductor exactly since we wish to find the conjecturally associated Bianchi newform. The odd part of the conductor can be found using MAGMA. Computing the even part has recently been made possible using machinery developed in [13]. The support of the ideal generated by the discriminant of a genus 2 hyperelliptic curve contains the support of the conductor of its Jacobian and the inclusion can in fact be strict. This phenomenon arises especially when one works with curves that have very large coefficients.

The curves in Theorem 1 were then found by parameterising the conic $X_6(K)$ and conducting a large search with varying $j$-value. To control the support of the conductor and stay away from large primes we used the proposition below from [2] Proposition 20]. Once suitable curves were discovered, minimal models were found using as yet unpublished code by L. Dembélé.

**Proposition 4** Let $C_j$ be a genus 2 curve as above and $\mathbb{Q}(j)$ a number field. Then $C_j$ has good reduction at a prime $p \nmid 6$ if and only if $\nu_p(j) = 0$. 

4
3 Galois representations attached to QM-surfaces

In this section we describe the image of the Galois representation attached to a QM-surface when the prime $\ell$ divides the discriminant of the quaternion algebra. A similar construction for GL$_2$-type surfaces can be found in [24, Chapter 4]. Let $A/K$ be a false elliptic curve with $O \to \text{End}_K(A)$ an order in the quaternion algebra $B/Q$ and denote by $\sigma_\ell : G_K \to GL_4(\mathbb{Z}_\ell)$ the representation coming from the $\ell$-adic Tate module $T_\ell A = \varprojlim A[\ell^n]$.

For each prime $\ell$ there is an associated $\ell$-adic representation

$$\rho_\ell : G_K \to \text{Aut}_\mathbb{Q}(T_\ell A) \simeq O_\ell^\times \subseteq B_\ell^\times,$$

where $O_\ell = O \otimes \mathbb{Z}_\ell$ and $B_\ell = B \otimes Q_\ell$. Furthermore, the $\rho_\ell$ form a strictly compatible system of $\ell$-adic representations [16]. Let $N_{B_\ell}$ denote the reduced norm of $B_\ell/Q_\ell$. Then the characteristic polynomial of $\sigma \in \text{Gal}(\overline{K}/K)$ is

$$P_{\rho_\ell}(\sigma) = N_{B_\ell}(1 - \rho_\ell(\sigma)t).$$

If $\ell \mid \text{Disc}(B)$ this precisely means that $(O \otimes \mathbb{Z}_\ell)^\times \simeq GL_2(\mathbb{Z}_\ell)$ and in this case there is a decomposition [17]

$$\sigma_\ell \simeq \rho_\ell \oplus \rho_\ell.$$

For the remainder of the section let $\ell$ be a prime that divides $\text{Disc}(B)$. For an overview on the arithmetic of quaternion algebras see [19], we summarise some results here. That $\ell$ is ramified in $B$ is to say that $B_\ell$ is isomorphic to the unique division quaternion algebra over $Q_\ell$. This can be represented as

$$\left(\frac{\pi, u}{Q_\ell}\right) \simeq Q_\ell \cdot 1 + Q_\ell \cdot i + Q_\ell \cdot j + Q_\ell \cdot ij; \ i^2 = u, \ j^2 = \pi;$$

where $\pi$ is the uniformiser of $\mathbb{Z}_\ell$ and $Q_\ell(\sqrt{u})$ is the unique unramified quadratic extension of $Q_\ell$.

Any quadratic extension of $Q_\ell$ splits the ramified quaternion algebra. So let $L = Q_\ell(\sqrt{u})$ and $R_L$ be its ring of integers. Then $B \otimes Q_\ell L \simeq M_2(L)$ and there is an explicit isomorphism of $Q_\ell$-algebras

$$B_\ell \simeq \left\{ \begin{pmatrix} \alpha & \beta \\ \alpha' & \beta' \end{pmatrix} \bigg| \alpha, \beta \in L, \ ' : L \to L \text{ is conjugation in } L/Q_\ell \right\} \subseteq \text{Mat}_2(L);$$

$$i \mapsto \begin{pmatrix} \sqrt{u} & \ 0 \\ 0 & -\sqrt{u} \end{pmatrix}, \ j \mapsto \begin{pmatrix} \pi & \ 1 \\ 0 & 0 \end{pmatrix}.$$  

If $\nu : Q_\ell \to \mathbb{Z}$ is the $\ell$-adic valuation then $w = \nu \circ N_{B_\ell/Q_\ell}$ defines a valuation on the quaternion algebra $B_\ell$. This gives us

$$O_\ell = \{ x \in B_\ell \mid w(x) \geq 0 \}$$

which is the unique maximal order of $B_\ell$ and

$$J = \{ x \in B_\ell \mid w(x) > 0 \},$$

a two-sided ideal. It is a principal ideal given by $J = O_\ell J$ and any two-sided ideal of $O_\ell$ is a power of $J$.

Define $\lambda_\ell \subseteq O$ to be the unique two-sided ideal of reduced norm $\ell$ such that $\lambda_\ell^2 = (\ell)$. We define the $\lambda_\ell$-adic Tate module as

$$T_{\lambda_\ell} A = \varprojlim \Lambda[A[\lambda_\ell^n]],$$

where

$$A[\lambda_\ell^n] = \{ P \in A \mid \gamma \cdot P = 0 \ \forall \gamma \in \lambda_\ell^n \}.$$
There is a commutative diagram

\[
\begin{array}{cccc}
A[\ell] & \xleftarrow{\epsilon} & A[\ell^2] & \xleftarrow{\epsilon} & A[\ell^3] & \xleftarrow{\epsilon} & \ldots \\
\downarrow{\lambda_\ell} & & \downarrow{\lambda_\ell^2} & & \downarrow{\lambda_\ell^3} & & \\
A[\lambda_\ell] & \xleftarrow{\rho_\ell} & A[\lambda_\ell^2] & \xleftarrow{\rho_\ell} & A[\lambda_\ell^3] & \xleftarrow{\rho_\ell} & \ldots
\end{array}
\]

Using the notation \(O_\ell = \lim_{n \to \infty} O/\ell^n\) and \(O_{\lambda_\ell} = \lim_{n \to \infty} O/\lambda_\ell^n\), the Tate modules \(T_\ell A\) and \(T_{\lambda_\ell} A\) are \(O_\ell\) and \(O_{\lambda_\ell}\) modules respectively and we know that \(T_\ell A\) is a free rank 1 \(O_\ell\)-module. So make the projection as above

\[\pi_\ell : T_\ell A \longrightarrow T_{\lambda_\ell} A\]

with \(\ker(\pi_\ell) \simeq T_{\lambda_\ell}\). In particular, \(4 = \text{rank}_{Z_p} T_\ell A = \text{rank}_{Z_p} \ker(\pi_\ell) + \text{rank}_{Z_p} \text{Im}(\pi_\ell) = 2\text{rank}_{Z_p} T_{\lambda_\ell} A\) and so \(T_{\lambda_\ell} A\) is a free rank 1 \(O_{\lambda_\ell}\)-module. Note that \(O_\ell \simeq R_L + R_{L,j}\) so it will often be useful to identify \(T_{\lambda_\ell} A\) with \(R_{L,j}\).

We now write \(\text{Aut}_O(T_\ell A) \simeq O_\ell^x\) and \(\text{Aut}_O(T_{\lambda_\ell} A) \simeq R_L^x\). The action of \(\text{Gal}(\overline{K}/K)\) commutes with \(\text{Aut}_O(T_\ell A)\) and preserves \(T_{\lambda_\ell} A\). This allows us to define the representations

\[
\rho_\ell : \text{Gal}(\overline{K}/K) \longrightarrow O_\ell^x \simeq \left\{ \left( \begin{array}{cc} \alpha & \beta \\ \pi \beta' & \alpha' \end{array} \right) \mid \alpha, \beta \in R_L, \ \alpha\beta - \pi\beta' \neq 0 \right\} \subseteq \text{GL}_2(R_L),
\]

\[
\rho_{\lambda_\ell} : \text{Gal}(\overline{K}/K) \longrightarrow R_L^x \simeq \left\{ \left( \begin{array}{cc} \alpha & 0 \\ 0 & \alpha' \end{array} \right) \mid \alpha \in R_L^x \right\} \subseteq \text{GL}_2(R_L).
\]

The images are related by the restriction homomorphism

\[r : \text{Aut}_O(T_\ell A) \longrightarrow \text{Aut}_O(T_{\lambda_\ell} A),\]

\[\sigma \longmapsto \sigma|_{T_{\lambda_\ell} A};\]

which results in the following.

**Theorem 5** Let \(A\) be an abelian surface defined over an imaginary quadratic field \(K\) such that \(\text{End}_K(A)\) is isomorphic to an order \(O\) in an indefinite non-split quaternion algebra \(B/Q\). Suppose that the prime \(\ell\) divides \(\text{Disc}(B)\) and \(\rho_\ell, \rho_{\lambda_\ell}\) are the Galois representations on the Tate modules \(T_\ell A, T_{\lambda_\ell} A\) respectively. Then there is a short exact sequence

\[1 \longrightarrow \epsilon \longrightarrow \text{Im}(\rho_\ell) \longrightarrow \text{Im}(\rho_{\lambda_\ell}) \longrightarrow 1,\]

where \(\epsilon \simeq (R_L, +)\).

**Proof.** The restriction homomorphism \(r\) is applied to the image of Galois. To describe the kernel \(\epsilon\), first let \(\phi \in \ker(r)\) and note that \((\phi - 1) \in \text{End}_O(T_\ell A)\). Then \(P \in \ker(\phi - 1)\) if and only if \(P \in T_{\lambda_\ell} A\) by definition.

Denote by \(\kappa = \lim_{n \to \infty} \lambda_\ell^n / \lambda_\ell^{n+1} \subseteq O_\ell\) which is the kernel of the projection \(O_\ell \to O_{\lambda_\ell}\). Then \(\kappa \cdot P \in T_{\lambda_\ell} A\) for any \(P \in T_\ell A\). This means that \(\kappa \cdot ((\phi - 1)(P)) = (\phi - 1)(\kappa \cdot P) = 0\) and so \(\text{Im}(\phi - 1) \subseteq T_{\lambda_\ell} A\). Hence we can view \(\phi - 1\) as a \(O_{\lambda_\ell}\)-linear map in \(\text{Hom}(T_\ell A/T_{\lambda_\ell} A, T_{\lambda_\ell} A)\).

Furthermore, one sees that for \(\phi, \psi \in \ker(r)\) the composition \((\phi - 1)(\psi - 1) : T_\ell A = 0\) implies that \((\phi \psi - 1) = (\phi - 1) + (\psi - 1)\). Hence we have an injection

\[i : \epsilon \longhookrightarrow (R_L, +),\]

\[\phi \longmapsto \phi - 1.\]

\[\square\]
The torsion subgroups $A[\ell]$ and $A[\lambda \ell]$ are free of rank 1 over the $\mathbb{F}_\ell$-algebras $\mathcal{O}/\ell$ and $\mathcal{O}/\lambda \ell$ respectively [16]. Explicitly these have the structure

$$\mathcal{O}/\ell \cong \{ (\alpha, \beta) \in \mathbb{F}_{\ell^2} \} \subseteq M_2(\mathbb{F}_{\ell^2}),$$

$$\mathcal{O}/\lambda \ell \cong \mathbb{F}_{\ell^2}.$$

Denote the residual representations by

$$\tau_\ell : \text{Gal}(\overline{K}/K) \rightarrow \text{Aut}_\mathcal{O}(A[\ell]) \leq \text{GL}_2(\mathbb{F}_{\ell^2}),$$

$$\rho_\ell : \text{Gal}(\overline{K}/K) \rightarrow \text{Aut}_\mathcal{O}(A[\lambda \ell]) \cong \mathbb{F}_{\ell^2}^\times.$$

It will often be useful to consider the representation $\tau_\ell$ to be defined over $\text{GL}_2(\mathbb{F}_{\ell^2})$ and will naturally lie in

$$\{ (\alpha, 0) \in \mathbb{F}_{\ell^2}^\times \} \subseteq M_2(\mathbb{F}_{\ell^2}).$$

Furthermore, note that up to conjugation the image of $\tau_\ell$ can be seen to live in $\text{GL}_2(\mathbb{F}_{\ell^2})$. Specifically, it will be contained in the non-split Cartan subgroup of $\text{GL}_2(\mathbb{F}_{\ell^2})$, which is the unique cyclic subgroup of order $\ell^2 - 1$. This can be seen by the fact that $\mathbb{F}_{\ell^2}$ is a 2-dimensional $\mathbb{F}_\ell$ vector space and there is a natural map $\mathbb{F}_{\ell^2}^\times \rightarrow \text{Aut}_{\mathbb{F}_\ell}(\mathbb{F}_{\ell^2})$ given by left multiplication.

For a Frobenius element $F_v$ the characteristic polynomial is given by

$$P_{\rho_\ell}(F_v) = N_{B_t}(t)(1 - \rho_\ell(F_v)t) = 1 - a_v t + N_v t^2,$$

which residually is

$$P_{\rho_\ell}(F_v) \mod \ell = (1 - \alpha t)(1 - \alpha^\ell t)$$

with $\overline{\rho}_\ell(F_v) = \alpha \in \mathbb{F}_{\ell^2}^\times$. Furthermore, $N_{\overline{\rho}_\ell/\overline{F}_\ell} \circ \overline{\rho}_\ell$ is the cyclotomic character.

Just as with the representations on the Tate modules we can make an analogous statement relating the images of the residual representations.

**Theorem 6** Let $A$ be an abelian surface defined over an imaginary quadratic field $K$ such that $\text{End}_K(A)$ is isomorphic to an order $\mathcal{O}$ in an indefinite non-split quaternion algebra $B/Q$. Suppose that the prime $\ell$ divides $\text{Disc}(B)$ and $\tau_\ell, \tau_\ell$ are the residual Galois representations on the torsion subgroups $A[\ell]$ and $A[\lambda \ell]$ respectively. Then there is a short exact sequence

$$1 \rightarrow \overline{\tau} \rightarrow \text{Im}(\tau_\ell) \rightarrow \text{Im}(\overline{\rho}_\ell) \rightarrow 1,$$

where $\overline{\tau} \leq \mathbb{F}_{\ell^2}^\times$.

**Proof.** The proof of this fact is very similar to that of Theorem 5, except applied to automorphisms of the torsion points $A[\ell]$ and $A[\lambda \ell]$ instead. \qed

**Remark 7** If $f$ is a Bianchi newform which corresponds to a false elliptic curve with QM by $B_D$, then the residual representation attached to $f$ has cyclic image at the primes dividing the discriminant $D$.

Given a Bianchi newform with rational coefficients, it would be desirable to have a criterion which determines whether $f$ should correspond to an elliptic curve or a false elliptic curve. The above gives a necessary condition for $f$ to correspond to a false elliptic curve. We wish to know whether a sufficient condition also exists and if so whether it can be determined from computing a finite set of trace of Frobenius.
4 Proof of modularity

Here we provide a proof that the second example in Theorem 1 is modular using the Faltings-Serre-Livné criterion, the other cases follow similarly. So let $C_2$ be the genus 2 curve

$$C_2 : y^2 = x^6 + (−2\sqrt{-3} - 10)x^5 + (10\sqrt{-3} + 30)x^4 + (-8\sqrt{-3} - 32)x^3$$

$$+ (-4\sqrt{-3} + 16)x^2 + (-16\sqrt{-3} - 12)x - 4\sqrt{-3} + 16.$$  

and $A$ the Jacobian of $C_2$. The surface $A = \text{Jac}(C_2)$ has conductor $p_1^{13,1}p_2^{19,1}$ with norm $61009^2$ and $O \hookrightarrow \text{End}_\mathbb{Q}(\sqrt{-3})(A)$ where $O$ is the maximal order of the rational quaternion algebra of discriminant $10$.

Let $f \in S_2(\Gamma_0(p_2^{13,1}p_2^{19,1}))$ be the genuine Bianchi newform which is listed on the LMFDB database with label $[2.0.3.1-61009.1-a]$. We will show that $f$ is modular to $A$. As in [21], we can associate an $\ell$-adic Galois representation $\rho_{f,\ell}$ to $f$ such that $L(f,s) = L(\rho_{f,\ell},s)$.

The Faltings-Serre-Livné method gives an effective way to prove that two Galois representations are isomorphic by showing that the trace of Frobenius agree on a finite computable set of primes. We follow the steps outlined in [12] which for practical reasons necessitates use of the prime $\ell = 2$. This prime is ramified in the acting quaternion algebra and as in [3] we can associate a representation to the 2-adic Tate module.

Lemma 8 The representations have image

$$\rho_{A,2}, \rho_{f,2} : \text{Gal}(\mathbb{Q}(\sqrt{-3})/\mathbb{Q}(\sqrt{3})) \rightarrow GL_2(E)$$

where $E$ is the unique unramified quadratic extension of $\mathbb{Z}_2$.

Proof. For $\rho_{A,2}$ this is a direct consequence of the way that the representation has been defined. Let us now consider $\rho_{f,2}$. The prime 31 is split in $\mathbb{Q}(\sqrt{-3})$ and the Hecke eigenvalues above these primes are distinct. Hence we can take the field by adjoining the roots of the Hecke polynomials which gives $\mathbb{Q}(\sqrt{-33}, \sqrt{-123})$. The completion at either of the primes above 2 in this field gives the unique unramified quadratic extension of $\mathbb{Z}_2$ and so we can take this as the coefficient field $E$ by [22, Corollary 1]. \hfill \Box

First we must show that the residual representations are isomorphic.

Lemma 9 The residual representations $\overline{\rho}_{A,2}, \overline{\rho}_{f,2}$ are isomorphic and have image $C_3 \subset GL_2(\mathbb{F}_2)$.

Proof. Denote by $F_A$ and $F_f$ the fields cut out by $\overline{\rho}_{A,2}$ and $\overline{\rho}_{f,2}$ respectively. The field given by the 2-torsion of $A$ is isomorphic to $A_4$ which has only one proper normal subgroup. This subgroup has order 4 and so applying the short exact sequence of Theorem 6 the image of $\overline{\rho}_{A,2}$ must be $C_3$.

To show that $\text{Im}(\overline{\rho}_{f,2}) = C_3$ let $m$ denote the modulus

$$m = p_2^3 \cdot p_{13,1} \cdot p_{19,1}.$$ 

Then if $\text{Im}(\overline{\rho}_{f,2})$ is not equal to $C_3$ there must be a quadratic extension of $\mathbb{Q}(\sqrt{-3})$ contained in $F_f$ which corresponds to a quadratic character of $\text{Cl}(\mathbb{Z}[^1_2\sqrt{-3}])$, $m$. We compute the ray class group to be

$$\text{Cl}(\mathbb{Z}[^1_2\sqrt{-3}], m) \cong (\mathbb{Z}/2\mathbb{Z})^2 \oplus (\mathbb{Z}/12\mathbb{Z}) \oplus (\mathbb{Z}/36\mathbb{Z}).$$

Let $\{\chi_1, \ldots, \chi_4\}$ be an $\mathbb{F}_2$-basis for the quadratic characters of $\text{Cl}(\mathbb{Z}[^1_2\sqrt{-3}], m)$. Then we find the set $S = \{p_{7,1}, p_{7,2}, p_{13,2}, p_{19,2}, p_5\}$ which is such that $\{\chi_1(p), \ldots, \chi_4(p)\}_{p \in S}$ spans $\mathbb{F}_2^4$. If $F_f$
contains a quadratic subfield then by [12, Proposition 5.4] the associated quadratic character must be non-zero for one of the primes in $S$. Hence there must be a prime $p \in S$ that is inert in this subfield and so $\overline{\rho}_{f,2}(\text{Frob}_p)$ must have order 2. However, we compute that the trace of Frobenius is odd for all primes in $S$ and therefore $F_f$ is a cubic extension of $\mathbb{Q}(\sqrt{-3})$.

To show that the representations are isomorphic let $\psi_A$ denote the cubic character associated to $F_A$. Extend this to an $\mathbb{F}_3$-basis $\{\psi_A, \chi_1\}$ of the cubic characters of $\text{Cl} (\mathbb{Z}[\frac{1+\sqrt{-3}}{2}], m)$. We find that the prime $p_{37,1}$ is such that $\psi_A(p_{37,1}) = 0$ and $\chi_1(p_{37,1}) \neq 0$. So if $\chi_f$ is the cubic character associated to $F_f$ and $\chi_f$ is not in the span of $\chi_A$ then $\psi_f(p_{37,1})$ must be non-zero. In particular, $\overline{\rho}_{f,2}(\text{Frob}_p)$ must have order 3 but we find that $\text{Tr}(\overline{\rho}_{f,2}(\text{Frob}_{p_{37,1}})) = \text{Tr}(\overline{\rho}_{A,2}(\text{Frob}_{p_{37,1}}))$ and so we can conclude that the residual representations are isomorphic. 

Now that we have shown that the residual representations are isomorphic it remains to show that the full representations are isomorphic. The residual images are cyclic and note that this will always be the case when the prime $\ell$ divides the discriminant of the acting quaternion algebra. Since the images are cyclic we can use a result due to Livné, which we recall here.

**Theorem 10** Let $K$ be a number field, $E$ a finite extension of $\mathbb{Q}_2$ and $\mathcal{O}_E$ its ring of integers with maximal ideal $\mathfrak{P}$. Let

$$\rho_1, \rho_2 : \text{Gal}(\overline{K}/K) \to GL_2(E)$$

be two continuous representations unramified outside of a finite set of primes $S$ and $K_{2,S}$ the compositum of all quadratic extensions of $K$ unramified outside of $S$.

Suppose that

1. $\text{Tr}(\rho_1) \equiv \text{Tr}(\rho_2) \equiv 0 \pmod{\mathfrak{P}}$ and $\text{Det}(\rho_1) \equiv \text{Det}(\rho_2) \equiv 1 \pmod{\mathfrak{P}}$;
2. There is a finite set of primes $T$ such that the characteristic polynomials of $\rho_1$ and $\rho_2$ are equal on the set $\{\text{Frob}_p \mid p \in T\}$.

Then $\rho_1$ and $\rho_2$ have isomorphic semisimplifications.

**Proof.** See [17].

It is now possible to show that the representations attached to $A$ and $f$ are isomorphic.

**Proof (Theorem 1).** Restricting the representations to the cubic extension cut out by the residual representation, the mod $\mathfrak{P}$ image becomes trivial. We are then in a position to apply Livné’s criterion.

Let $\{\chi_1, \ldots, \chi_6\}$ be a basis of quadratic characters. Any set of primes $\{p_i\}$ for which the vectors $\{\chi_1(p_i), \ldots, \chi_6(p_i)\}$ cover $\mathbb{F}_2^6 \setminus \{0\}$ will satisfy the criterion. Following [12, §2.3 step (7)] we compute the set $T = \{3, 37, 43, 61, 67, 73, 97, 103, 127, 151, 157, 193, 211, 307, 313, 343, 373, 433, 463, 499, 523, 631, 661, 823, 1321, 2197, 2557, 2917\}$. The trace of Frobenii agree on this set and hence we can conclude that the two representations are isomorphic. 

**5 Examples**

At the time of writing there are 161343 rational Bianchi newforms of weight 2 in the LMFDB [18] and these are for the quadratic fields $\mathbb{Q}(\sqrt{-d})$ with $d = 1, 2, 3, 7, 11$. Up to conjugation and twist there are only four with square level for which no corresponding elliptic curve has been found. These are all accounted for by Theorem 1.
Curve 1 Let $C_1$ be the genus 2 curve as in Theorem 1:

\[ C_1 : y^2 = x^6 + 4ix^5 + (-2i - 6)x^4 + (-i + 7)x^3 + (8i - 9)x^2 - 10ix + 4i + 3. \]

- The surface $A = \text{Jac}(C_1)$ has conductor $p_{5,1}^3 \cdot p_{37,2}^2$ with norm $34225^2$.
- $O \hookrightarrow \text{End}_{\mathcal{O}(i)}(A)$ where $O$ is the maximal order of the rational quaternion algebra of discriminant 6.
- There is a genuine Bianchi newform $f \in S_2(\Gamma_0(p_{5,1}^2 \cdot p_{37,2}^2))$ which is modular to $A$ and is listed on the LMFDB database with label $2.0.4.1-34225^3-a$.

We list some of the Hecke eigenvalues here:

| Prime ideal $\mathfrak{p}$ | $P_2$ | $P_{5,1}$ | $P_{5,2}$ | $P_3$ | $P_{13,1}$ | $P_{13,2}$ | $P_{17,2}$ | $P_{17,1}$ | $P_{29,1}$ | $P_{29,2}$ | $P_{37,1}$ | $P_{37,2}$ | $P_{41,1}$ | $P_{41,2}$ | $P_7$ |
|--------------------------|------|----------|----------|------|----------|----------|----------|----------|----------|----------|----------|----------|----------|----------|------|
| $a_2(f)$                 | -2   | 0        | -1       | -3   | -4       | -5       | 5         | -1       | -9       | 0         | -5       | -3       | -3       | -9       | -9   |

Curve 2 Let $C_2$ be the genus 2 curve as in Theorem 2:

\[ C_2 : y^2 = x^6 + (2\sqrt{-3} - 10)x^5 + (10\sqrt{-3} + 30)x^4 + (-8\sqrt{-3} - 32)x^3 + (-4\sqrt{-3} + 16)x^2 + (-16\sqrt{-3} - 12)x - 4\sqrt{-3} + 16. \]

- The surface $A = \text{Jac}(C_2)$ has conductor $p_{13,1}^4 \cdot p_{19,1}^4$ with norm $61009^2$.
- $O \hookrightarrow \text{End}_{\mathcal{O}(\sqrt{-3})}(A)$ where $O$ is the maximal order of the rational quaternion algebra of discriminant 10.
- There is a genuine Bianchi newform $f \in S_2(\Gamma_0(p_{13,1}^2 \cdot p_{19,1}^2))$ which is modular to $A$ and is listed on the LMFDB database with label $2.0.3.1-61009.1-a$.

We list some of the Hecke eigenvalues here:

| Prime ideal $\mathfrak{p}$ | $P_3$ | $P_2$ | $P_{7,1}$ | $P_{7,2}$ | $P_{13,1}$ | $P_{13,2}$ | $P_{19,1}$ | $P_{19,2}$ | $P_{31,1}$ | $P_{31,2}$ | $P_{37,1}$ | $P_{37,2}$ | $P_{43,1}$ | $P_{43,2}$ |
|--------------------------|------|------|----------|----------|----------|----------|----------|----------|----------|----------|----------|----------|----------|----------|
| $a_2(f)$                 | -2   | -2   | -1       | -1       | 0        | -5       | 0         | -3       | -5       | -1       | 9         | -4       | -6       | -2       | 3     |

Curve 3 Let $C_3$ be the genus 2 curve as in Theorem 3:

\[ C_3 : y^2 = (104\sqrt{-3} - 75)x^6 + (528\sqrt{-3} + 456)x^4 + (500\sqrt{-3} + 1044)x^3 + (-1038\sqrt{-3} + 2706)x^2 + (-1158\sqrt{-3} + 342)x - 612\sqrt{-3} - 1800. \]

- The surface $A = \text{Jac}(C_3)$ has conductor $p_{7,1}^4 \cdot p_{37,2}^2$ with norm $67081^2$.
- $O \hookrightarrow \text{End}_{\mathcal{O}(\sqrt{-3})}(A)$ where $O$ is the maximal order of the rational quaternion algebra of discriminant 10.
- There is a genuine Bianchi newform $f \in S_2(\Gamma_0(p_{7,1}^2 \cdot p_{37,2}^2))$ which is modular to $A$ and is listed on the LMFDB database with label $2.0.3.1-67081.3-a$.

We list some of the Hecke eigenvalues here:

| Prime ideal $\mathfrak{p}$ | $P_3$ | $P_2$ | $P_{7,1}$ | $P_{7,2}$ | $P_{13,1}$ | $P_{13,2}$ | $P_{19,1}$ | $P_{19,2}$ | $P_{31,1}$ | $P_{31,2}$ | $P_{37,1}$ | $P_{37,2}$ | $P_{43,1}$ | $P_{43,2}$ |
|--------------------------|------|------|----------|----------|----------|----------|----------|----------|----------|----------|----------|----------|----------|----------|
| $a_2(f)$                 | -2   | -2   | 0        | 1        | -3       | 0        | -2       | 4         | 5         | 9         | -2       | -9       | 0         | -2       | 8     |

Curve 4 Let $C_4$ be the genus 2 curve as in Theorem 4:

\[ C_4 : y^2 = x^6 - 2\sqrt{-3}x^5 + (2\sqrt{-3} - 3)x^4 + 1/3(-2\sqrt{-3} + 54)x^3 + (20\sqrt{-3} + 3)x^2 + (-8\sqrt{-3} - 30)x + 4\sqrt{-3} - 11. \]
• The surface $A = \text{Jac}(C_4)$ has conductor $p_3^{12} \cdot p_{13,1}^1$ with norm $123201^2$.

• $O \to \text{End}_{\mathbb{Q}(\sqrt{-3})}(A)$ where $O$ is the maximal order of the rational quaternion algebra of discriminant 6.

• There is a genuine Bianchi newform $f \in S_2(\Gamma_0(p_3^{12} \cdot p_{13,1}^1))$ which is modular to $A$ and is listed on the LMFDB database with label $[2,0,3,1:123201:1-b]$. We list some of the Hecke eigenvalues here:

| Prime ideal $p$ | $p_3$ | $p_2$ | $p_{17,1}$ | $p_{19,1}$ | $p_{19,2}$ | $p_5$ | $p_{31,1}$ | $p_{31,2}$ | $p_{41,1}$ | $p_{41,2}$ | $p_{47,1}$ | $p_{47,2}$ | $p_{43,1}$ | $p_{43,2}$ |
|----------------|-------|-------|-------------|-------------|-------------|-------|-------------|-------------|-------------|-------------|-------------|-------------|-------------|-------------|
| $a_p(f)$       | 0     | 0     | -2          | -3          | 0           | -3    | -3          | -5          | -9          | -3          | -1          | 5           | 3           | 7           | -1          |

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