Existence and uniqueness of tronquée solutions of the third and fourth Painlevé equations

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Abstract
It is well known that the first and second Painlevé equations admit solutions characterized by divergent asymptotic expansions near infinity in specified sectors of the complex plane. Such solutions are pole-free in these sectors and called tronquée solutions by Boutroux. In this paper, we show that similar solutions exist for the third and fourth Painlevé equations as well.

Keywords: the third and fourth Painlevé equations, tronquée solutions, asymptotic analysis
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1. Introduction and statement of results

The third and fourth Painlevé equations are the following two second order nonlinear differential equations

\[ P_{III} : \frac{d^2 u}{dx^2} = \frac{1}{u} \left( \frac{du}{dx} \right)^2 - \frac{1}{x} \frac{du}{dx} + \frac{1}{x} \left( \alpha u^2 + \beta \right) + \gamma u^3 + \delta, \] (1.1)

\[ P_{IV} : \frac{d^2 u}{dx^2} = \frac{1}{2u} \left( \frac{du}{dx} \right)^2 + \frac{3}{2} u^3 + 4xu^2 + 2(x^2 - \alpha)u + \beta, \] (1.2)

where \( \alpha, \beta, \gamma \) and \( \delta \) are arbitrary constants. Their solutions, which are called the Painlevé transcendents, satisfy the famous Painlevé property, i.e. all of their movable singularities are

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poles. Although the Painlevé equations were originally studied from a purely mathematical point of view, it has been realized in recent years that they play an important role in many other mathematical and physical fields (see [6, 13, 16] and references therein) (see [5] and [29, chapter 32]), such as general relativity (see [32]), two-dimensional polymers (see [33]), nonlinear optics (see [12]) and quantum gravity (see [14]). Additionally, the third Painlevé transcendent has applications in the study of the Ising model (see [9, 26]) and the study of polyelectrolytes in excess salt solution (see [23]).

An important part of studying Painlevé transcendents is to investigate their asymptotic behaviour. More specifically, we discern two directions of research: the first direction is to study asymptotics of certain solutions on the real axis and find the relations between their asymptotic behaviours at two endpoints of an interval, usually $(-\infty, \infty)$ or $(0, \infty)$. These relations are called connection formulae (see, for example, [2, 3, 10, 17]). Connection formulae have also been studied on the imaginary axis (see [24, 25]). The other direction is to study the asymptotics in the complex plane. Usually, Painlevé transcendents have poles in the complex plane, whose locations depend on the initial conditions of the corresponding differential equation. But, in sectors containing one special ray in the complex $x$-plane, there also exist some solutions which are pole-free when $|x|$ is large. Such solutions were named tronquée solutions by Boutroux [4] when he studied Painlevé equations 100 years ago. These solutions usually have free parameters appearing in exponentially small terms for large $|x|$ (see Its and Kapaev [18]). Moreover, there are some solutions without any parameters, which are pole-free in larger sectors containing three special rays. Such solutions are called tritronquée solutions. Recently, Joshi and Kitaev [20] reconsidered these solutions and proved the existence and uniqueness of tritronquée solutions for the first Painlevé equation with more modern techniques. Later on, similar results were obtained for the first and second Painlevé hierarchies as well (see [8, 19, 21, 22]). Note that tronquée solutions for the fifth Painlevé equation are also studied by Andreev and Kitaev [1] and Shimomura [30]. In these two papers, the method used are different from that in [20].

In the literature, tronquée solutions to the first Painlevé equation

$$P_1 : \frac{d^2 u}{dx^2} = 6 u^2(x) - x$$

are well studied. For $P_1$, when studying the behaviour of their tritronquée solutions in the unbounded domain, people are also interested in their properties in the finite domain. Based on numerical results (see, for example, [15]), Dubrovin et al [11] conjecture that the tritronquée solutions for $P_1$ are analytic in a neighbourhood of the origin and in a sector of central angle $8\pi/5$ containing the origin. This conjecture was proved by Costin et al [7] very recently. It is not clear whether similar conjectures are true for other Painlevé equations.

In this paper, we will follow Joshi and Kitaev’s idea (see [20]) and study the tronquée solutions for the $P_{III}$ and $P_{IV}$ equations. Although some divergent power series solutions for these two equations are known, tronquée solutions, to the best of our knowledge, have not appeared in the literature. So we think it is worthwhile to prove the existence of these tronquée solutions, give the regions of validity of these solutions and their general asymptotic behaviour. Moreover, it is worth mentioning that compared to other methods like the isomonodromy method, Joshi and Kitaev’s approach used in this paper is more straightforward.

Before we state our main results, we note that (see [16, p 150] and [27]) using Bäcklund transformations it suffices to study only two cases for the $P_{III}$ equations: (i) $\gamma\delta \neq 0$ and (ii) $\gamma\delta = 0$. Moreover, these two cases can be reduced to the following canonical ones: (i) $\gamma = 1$ and $\delta = -1$ and (ii) $\alpha = 1, \gamma = 0$ and $\delta = -1$ without loss of generality (see [16, p 150]
and \([27]\)). Thus, for \(P_{III}\), we only need to study the following two equations:

\[
P_{III}^{(i)}: \frac{d^2 u}{dx^2} = \frac{1}{u} \left( \frac{du}{dx} \right)^2 - \frac{1}{x} \frac{du}{dx} + \frac{1}{x^2} (\alpha u^2 + \beta) + u^3 - \frac{1}{u},
\]

(1.4)

and

\[
P_{III}^{(ii)}: \frac{d^2 u}{dx^2} = \frac{1}{u} \left( \frac{du}{dx} \right)^2 - \frac{1}{x} \frac{du}{dx} + \frac{1}{x^2} (\alpha u^2 + \beta) - \frac{1}{u}.
\]

(1.5)

For the above two equations, we have the following results.

**Theorem 1.** For \(k = 0, 1\) and an \(x_0 \neq 0\) of sufficiently large modulus, the following two statements hold.

1. There exist one-parameter solutions of \(P_{III}^{(i)}\) in the sectors

\[
\Omega_k^{(m)} = \begin{cases} 
  \{ x \in \mathbb{C} \mid |x| > |x_0|, -\frac{\pi}{2} + k\pi < \arg x < \frac{\pi}{2} + k\pi \} & \text{for } m = 0, 2, \\
  \{ x \in \mathbb{C} \mid |x| > |x_0|, k\pi < \arg x < (k + 1)\pi \} & \text{for } m = 1, 3
\end{cases}
\]

with the following asymptotic expansion

\[
u(x) \sim \sum_{n=0}^{\infty} a_n^{(m)} x^{-n} \quad \text{as } |x| \to \infty.
\]

(1.7)

Here \(a_0^{(m)} = e^{\frac{\pi i m}{2}}\), \(m = 0, 1, 2, 3\), and the subsequent coefficients \(a_n^{(m)}, n \geq 0\), are determined by the recurrence relation

\[
2a_0^{(m)} A_0^{(m)} A_{n+1}^{(m)} = (\beta - 1 - n) a_n^{(m)} - A_0^{(m)} \sum_{i=1}^{n} a_i^{(m)} A_{n+1-i}^{(m)} - \sum_{k=1}^{n+1} \sum_{i=0}^{n+1-k} A_i^{(m)} A_{n+1-k-i}^{(m)}
\]

\[
- \sum_{k=1}^{n} a_k^{(m)} \sum_{i=0}^{n+1-k} A_i^{(m)} A_{n+1-k-i}^{(m)}
\]

(1.8)

with \(A_0^{(m)} = -(a_0^{(m)})^2\), \(\delta_{0,0} = 1\) and \(\delta_{n,0} = 0\) for \(n > 0\).

2. There exists a unique solution of \(P_{III}^{(ii)}\) in each of the following sectors

\[
\Omega_k^{(m)} = \begin{cases} 
  \{ x \in \mathbb{C} \mid |x| > |x_0|, -\frac{\pi}{2} + k\pi < \arg x < \frac{3\pi}{2} + k\pi \} & \text{for } m = 0, 2, \\
  \{ x \in \mathbb{C} \mid |x| > |x_0|, k\pi < \arg x < (k + 2)\pi \} & \text{for } m = 1, 3
\end{cases}
\]

with the asymptotic expansion given in (1.7).

**Remark 1.** It is known that there exist rational solutions for \(P_{III}^{(i)}\) if and only if \(\alpha \pm \beta = 4k, k \in \mathbb{Z}\) (see \([5, 27]\)). As these rational solutions satisfy the above asymptotic expansion (1.7), they are the unique tronquée solutions in the second part of theorem 1. They are pole-free in \(\Omega_k^{(m)}\) when \(|x|\) is large since all of their poles are located in a bounded region in the complex \(x\)-plane. Moreover, our theorem suggests that, even for arbitrary parameters \(\alpha\) and \(\beta\), there still exist tronquée solutions which are pole-free in a pretty big sector \(\Omega_k^{(m)}\) for large \(x\). We think this is an interesting observation.
Remark 2. Readers may wonder why there is another set of coefficients \( \{ A^{(m)}_{\rho} \} \) in the recurrence relations (1.8). The reason is that, instead of considering the \( P_{\text{III}} \) equation (1.4) directly, we consider an equivalent system of first-order differential equations instead. Similar situations occur in the \( P_{\text{III}} \) and \( P_{\text{IV}} \) cases as well. The detailed explanations will be provided later in this section.

Theorem 2. For \( k = 0, 1, 2, 3 \) and an \( x_0 \neq 0 \) of sufficiently large modulus, the following two statements hold true.

1. There exist one-parameter solutions of \( P_{\text{III}} \) in the sectors

\[
S_k^{(m)} = \begin{cases} 
 \{ x \in \mathbb{C} \mid |x| > |x_0|, -\frac{3\pi}{4} - \frac{k\pi}{2} < \arg x < \frac{3\pi}{4} - \frac{k\pi}{2} \}, & \text{for } m = 0, 2, \\
 \{ x \in \mathbb{C} \mid |x| > |x_0|, \frac{\pi}{4} - \frac{k\pi}{2} < \arg x < \frac{7\pi}{4} - \frac{k\pi}{2} \}, & \text{for } m = 1
\end{cases}
\]  

(1.10)

with the following asymptotic expansion

\[
u(x) \sim x^\lambda \sum_{n=0}^{\infty} a_{n+1}^{(m)} x^{-\frac{n}{2}} \quad \text{as } |x| \to \infty.
\]  

(1.11)

Here \( a_0^{(m)} = e^{\frac{\pi in}{\sqrt{3}}} \), \( m = 0, 1, 2 \), and the subsequent coefficients \( a_n^{(m)} \), \( n \geq 0 \), are determined by the recurrence relation

\[
\begin{aligned}
[a_0^{(m)}]^2 (6a_n^{(m)} - 3A_{n+1}^{(m)}) &= (2n + 2 - 3\beta)n^{(m)} - 3a_0^{(m)} \sum_{l=1}^{n} a_l^{(m)} a_{n+1-l}^{(m)} \\
+3 \sum_{k=1}^{n} A_k^{(m)} \sum_{l=0}^{n+1-k} a_{k+l}^{(m)} A_{n+1-l}^{(m)} \\
[a_0^{(m)}] (3a_n^{(m)} - 6A_{n+1}^{(m)}) &= (3\beta - 4 + 2n)A_n^{(m)} - 3a_0^{(m)} \sum_{l=1}^{n} A_l^{(m)} A_{n+1-l}^{(m)} \\
-3 \sum_{k=1}^{n} A_k^{(m)} \sum_{l=0}^{n+1-k} A_{k+l}^{(m)} A_{n+1-l}^{(m)}
\end{aligned}
\]  

(1.12)

with \( A_0^{(m)} = -a_0^{(m)} \).

2. For any branch of \( x^{1/3} \), there exists a unique solution of \( P_{\text{III}} \) in \( \mathbb{C} \setminus \Gamma \) with the asymptotic expansion given in (1.11). Here \( \Gamma \) is an arbitrary branch cut connecting 0 and \( \infty \).

Remark 3. In the literature, (1.5) is also called the \( P_{\text{III}} \) of type \( D_4 \) according to the algebro-geometric classification scheme in [28]. It is also called the degenerate \( P_{\text{III}} \) by Kitaev and Vartanian in [24, 25].

Remark 4. Kitaev and Vartanian studied the connection formulae for the \( P_{\text{III}} \) equation in [24, 25]. Using the isomonodromy deformation method, they obtained asymptotics for solutions \( u(x) \) when \( x \to \pm \infty \) and \( x \to \pm i \infty \). These solutions are similar to the one-parameter solutions in the first part of our theorem 2. Their asymptotic formulae involve a term \( x^{1/3} \) and a trigonometric term. See, for example, formulae (39), (41) and (42) in [24].

Remark 5. In [25], Kitaev and Vartanian obtained significant asymptotic formulae for \( u(x) \), which are valid in the neighbourhood of poles. In addition, they obtained asymptotics for these poles, see theorems 2.1–2.3 in [25]. Note that these poles are located on the real or imaginary axis depending on different monodromy data.
We obtain the following results for PIV.

**Theorem 3.** For \( k = 0, 1, 2, 3 \) and an \( x_0 \neq 0 \) of sufficiently large modulus, the following statements hold true.

1. There exist one-parameter solutions of PIV in the regions

\[
\Omega^{(m)}_k = \begin{cases} 
\{ x \in \mathbb{C} \mid |x| > |x_0|, \frac{1}{2} k \pi < \arg x < \frac{1}{2} (k + 2) \pi \}, & \text{for } m = 1, \\
\{ x \in \mathbb{C} \mid |x| > |x_0|, -\frac{3}{2} k \pi < \arg x < -\frac{3}{2} (k + 2) \pi \}, & \text{for } m = 2, 3, 4.
\end{cases}
\]

with the following asymptotic expansion

\[
u(x) \sim \begin{cases} 
\sum_{n=0}^{\infty} a^{(m)}_n x^{-2n}, & \text{for } m = 1, 2, \\
\frac{1}{x} \sum_{n=0}^{\infty} a^{(m)}_n x^{-2n}, & \text{for } m = 3, 4,
\end{cases}
\] as \( |x| \to \infty \). (1.13)

Here \( a^{(1)}_0 = -2 \), \( a^{(2)}_0 = -2 \), \( a^{(3)}_0 = \kappa_0 \), \( a^{(4)}_0 = -\kappa_0 \) with \( \alpha = -\kappa_0 + 2 \kappa_\infty + 1 \) and \( \beta = -2 \kappa_0^2 \), and the subsequent coefficients \( a^{(m)}_n \) can be determined by the following recurrence relations: if \( m = 1 \),

\[
\begin{align*}
2 a^{(1)}_{n+1} - 8 A^{(1)}_n &= (1 - 2n) a^{(1)}_n - \sum_{k=1}^{n} a^{(1)}_k (4A^{(1)}_{n+1-k} - a^{(1)}_{n+1-k}) , \\
2 A^{(1)}_n - 2 a^{(1)}_n &= (2n - 1) A^{(1)}_n + 2 \sum_{k=1}^{n} A^{(1)}_k (a^{(1)}_{n+1-k} - A^{(1)}_{n+1-k})
\end{align*}
\] (1.14)

with \( a^{(1)}_1 = \alpha \), \( A^{(1)}_0 = \frac{1}{2} \), \( A^{(1)}_1 = -\frac{1}{2} - \kappa_0 + \frac{1}{2} \kappa_\infty \); if \( m = 2 \),

\[
\begin{align*}
d^{(2)}_{n+1} &= (\frac{1}{2} - n) a^{(2)}_n + 4 A^{(2)}_n + \frac{n}{2} \sum_{k=1}^{n} A^{(2)}_k (a^{(2)}_{n+1-k} - 4A^{(2)}_{n+1-k}) , \\
A^{(2)}_{n+1} &= (\frac{1}{2} + n) A^{(2)}_n + \sum_{k=0}^{n} A^{(2)}_k (a^{(2)}_{n+1-k} - A^{(2)}_{n+1-k}), \quad n \geq 0
\end{align*}
\] (1.15)

with \( d^{(2)}_1 = -\alpha \), \( A^{(2)}_0 = -\frac{1}{2} \kappa_\infty \); if \( m = 3 \),

\[
\begin{align*}
d^{(3)}_{n+1} &= -(n + \frac{1}{2}) a^{(3)}_n + \frac{n}{2} \sum_{k=0}^{n} A^{(3)}_k (a^{(3)}_{n+1-k} - 4A^{(3)}_{n+1-k}) , \\
A^{(3)}_{n+1} &= -(n + \frac{1}{2}) A^{(3)}_n + a^{(3)}_n - \sum_{k=1}^{n} A^{(3)}_k (A^{(3)}_{n+1-k} - a^{(3)}_{n+1-k}), \quad n \geq 1
\end{align*}
\] (1.16)

with \( A^{(3)}_0 = 1 \), \( A^{(3)}_1 = -\frac{1}{2} (1 - 2 \kappa_0 + \kappa_\infty) \); and if \( m = 4 \),

\[
\begin{align*}
da^{(4)}_{n+1} &= (n + \frac{1}{2}) a^{(4)}_n - \frac{n}{2} \sum_{k=0}^{n} A^{(4)}_k (a^{(4)}_{n+1-k} - 4A^{(4)}_{n+1-k}) , \\
A^{(4)}_{n+1} &= -(n + \frac{1}{2}) A^{(4)}_n + \sum_{k=0}^{n} A^{(4)}_k (A^{(4)}_{n+1-k} - a^{(4)}_{n+1-k})
\end{align*}
\] (1.17)

with \( A^{(4)}_0 = \frac{1}{2} \kappa_\infty \).

2. There exist a unique solution of PIV whose asymptotic behaviour as \( |x| \to \infty \) is given by (1.13), respectively, in the regions

\[
\Omega^{(m)}_k = \begin{cases} 
\{ x \in \mathbb{C} \mid |x| > |x_0|, k \pi < \arg x < (k + 2) \pi \}, & \text{for } m = 1, \\
\{ x \in \mathbb{C} \mid |x| > |x_0|, -\frac{3}{2} k \pi < \arg x < -\frac{3}{2} (k + 2) \pi \}, & \text{for } m = 2, 3, 4.
\end{cases}
\]
Remark 6. Its and Kapaev applied isomonodromy and Riemann–Hilbert methods to obtain the asymptotics for the Clarkson–McLeod solutions \( u(x) \) as \( x \to -\infty \). This solution is also similar to the one-parameter solutions in the first part of our theorem 3. The asymptotic formula involves a term \(-\frac{2}{3}x\) and a cosine term, see [17, equation (1.10)].

To prove the above theorems, we need the following theorem by Wasow (see [31, theorem 12.1]).

**Theorem 4.** Let \( S \) be an open sector of the complex \( x \)-plane with vertex at the origin and a positive central angle not exceeding \( \pi/(q + 1) \) (\( q \) a nonnegative integer). Let \( f(x, w) \) be an \( N \)-dimensional vector function of \( x \) and an \( N \)-dimensional vector \( w \) with the following properties.

1. \( f(x, w) \) is a polynomial in the components \( w_j \) of \( w \), \( j = 1, \ldots, N \), with coefficients that are holomorphic in \( x \) in the region \( 0 < x_0 \leq |x| < \infty \), \( x \in S \), where \( x_0 \) is a constant.
2. The coefficients of the polynomial \( f(x, w) \) have asymptotic series in powers of \( x^{-1} \), as \( x \to \infty \), in \( S \).
3. If \( f_j(x, w) \) denotes the components of \( f(x, w) \) then all the eigenvalues \( \lambda_j \), \( j = 1, 2, \ldots, N \) of the Jacobian matrix

\[
\lim_{x \to \infty, x \in S} \left( \frac{\partial f_j}{\partial w_k} \bigg|_{w=0} \right)
\]

are different from zero.
4. The differential equation

\[
x^{-q}w' = f(x, w)
\]

is formally satisfied by a power series of the form \( \sum_{n=1}^{\infty} a_n x^{-n} \).

Then there exists, for sufficiently large \( x \) in \( S \), a solution \( w = \phi(x) \) of (1.19) such that, in every proper subsector of \( S \),

\[
\phi(x) \sim \sum_{n=1}^{\infty} a_n x^{-n}, \quad x \to \infty.
\]

Note that in Wasow’s theorem, the function \( f(x, w) \) in (1.19) needs to be polynomial in terms of the components of the unknown function \( w \). However, there are no terms in both (1.1) and (1.2). This means we cannot directly apply Wasow’s theorem as was done for the PI and PII cases in [20, 21]. Fortunately, there exist first-order differential systems for Painlevé equations where all unknown functions are given in polynomial terms. We will consequently use these first-order differential systems to derive our results. This also explains remark 2, where \( A_n^{(m)} \) are actually coefficients for the other solution in the systems.

In the remaining part of this paper, we will prove theorems 1–3 in sections 2–4, respectively. Because sections 2–4 are self-contained, we shall use the same notation for the functions and variables but it will mean different things in different sections. We hope that this will not cause any confusion.
2. The $P_{III}^{(i)}$ equations

The $P_{III}$ equations (1.1) can be written as the following system of first-order differential equations:

\[
\begin{align*}
\frac{dx}{du} &= (-\delta)^{1/2}x + hu + xu^2U, \\
\frac{dU}{dx} &= \alpha + \gamma xu - (1 + h)U - xu^2U.
\end{align*}
\]  

(2.1)

where \( h = 1 - \beta(-\delta)^{-1/2} \) (see [16, p 149]). That is, eliminating \( U \) from the above equations gives us (1.1). For case (i), \( \gamma = 1, \delta = -1 \) and the first-order equations are

\[
\begin{align*}
\frac{dx}{du} &= x + (1 - \beta)u + xu^2U, \\
\frac{dU}{dx} &= \alpha + xu - (2 - \beta)U - xu^2U.
\end{align*}
\]  

(2.2)

Then we get the formal solutions to the above system.

**Proposition 1.** We have the following formal solutions for (2.2):

\[
\begin{align*}
u^{(m)}_{fin}(x) &= \sum_{n=0}^{\infty} a^{(m)}_n x^{-n} \quad \text{and} \quad U^{(m)}_{fin}(x) = \sum_{n=0}^{\infty} A^{(m)}_n x^{-n},
\end{align*}
\]  

(2.3)

where

\[
a^{(m)}_0 = e^{m\pi i/2} \quad \text{and} \quad A^{(m)}_0 = -(a^{(m)}_0)^2, \quad m = 0, 1, 2, 3,
\]  

(2.4)

and the subsequent coefficients are determined by (1.8).

**Proof.** Inserting (2.3) into (2.2) gives us the results. \( \square \)

**Remark 7.** The systems of first-order differential equations for Painlevé equations are not unique. For $P_{III}^{(i)}$, in addition to (2.2), one can get the same results as in the above proposition by studying the following system obtained from the Hamiltonian $H_{III}$

\[
\begin{align*}
\frac{dx}{du} &= x \frac{\partial H_{III}}{\partial U} = 4u^2U - xu^2U + (1 - \beta)u + x, \\
\frac{dU}{dx} &= -x \frac{\partial H_{III}}{\partial u} = -4uU^2 + (2ux + \beta - 1)U + \frac{1}{4}(2 + \alpha - \beta)x,
\end{align*}
\]  

(2.5)

where

\[
x H_{III}(x, u, U) := 2u^2U^2 - xu^2U + (1 - \beta)uU + xU - \frac{1}{4}(2 + \alpha - \beta)xu.
\]  

(2.6)

Since (2.5) is only valid for case (i), where \( \gamma = 1 \) and \( \delta = -1 \), it is more convenient to use (2.1) to consider both cases in the current and next sections.

Although the original $P_{III}^{(i)}$ equations (1.4) have \( \frac{1}{x} \) terms, the right-hand side of the first-order system (2.2) is a polynomial in the functions \( u \) and \( U \). Thus Wasow’s theorem 4 is applicable and we have the following results.

**Proposition 2.** In any sector of angle less than \( \pi \), there exists a solution \( u(x) \) of $P_{III}^{(i)}$ whose asymptotic behaviour as \( |x| \to \infty \) is given by \( u^{(m)}_{fin}(x) \) in (2.3) with \( m = 0, 1, 2, 3 \).
To arrive at the standard form (1.19) in Wasow’s theorem, we introduce \( u(x) - a_0^{(m)} \) and \( V(x) = U(x) - A_0^{(m)} \). Then \((v, V)\) satisfies the following system:

\[
\frac{d}{dx} \begin{pmatrix} v \\ V \end{pmatrix} = \begin{pmatrix} 1 + \frac{1 - \beta}{x} (v + a_0^{(m)}) + (v + a_0^{(m)})^2 (V + A_0^{(m)}) \\ \alpha + v + a_0^{(m)} - \frac{2 - \beta}{x} (V + A_0^{(m)}) - (v + a_0^{(m)}) (V + A_0^{(m)})^2 \end{pmatrix}
\]

(2.7)

and has a formal expansion given by

\[
v_f^{(m)}(x) = \sum_{n=1}^{\infty} a_n^{(m)} x^{-n} \quad \text{and} \quad V_f^{(m)}(x) = \sum_{n=1}^{\infty} A_n^{(m)} x^{-n}.
\]

It is easy to see that conditions 1, 2 and 4 in Wasow’s theorem are satisfied with \( q = 0 \). To verify condition 3, let us denote the right-hand side of (2.7) by \((f_1, f_2)^T\). Then we have

\[
\lim_{x \to \infty} \begin{pmatrix} \frac{\partial f_1}{\partial v} \\ \frac{\partial f_2}{\partial v} \end{pmatrix} \bigg|_{v=V=0} = \begin{pmatrix} 2a_0^{(m)}A_0^{(m)} & (a_0^{(m)})^2 \\ 0 & -2a_0^{(m)}A_0^{(m)} \end{pmatrix} := J.
\]

(2.8)

The eigenvalues of \( J \) are

\[
\lambda_{1,2} = \pm 2a_0^{(m)}A_0^{(m)} = \mp 2e^{-\frac{m\pi i}{3}}, \quad m = 0, 1, 2, 3;
\]

(2.9)

(see (2.4) for the values of \( a_0^{(m)} \) and \( A_0^{(m)} \)). Thus our proposition follows from Wasow’s theorem.

The above proposition gives us the existence of solutions to \( P_{III}^{(m)} \) with specific asymptotic expansions given in (1.7). However, these solutions may not be unique because some exponentially small terms with arbitrary parameters could exist. To understand these solutions better, more detailed analysis about their properties is needed. This will prove our theorem 1.

**Proof of theorem 1.** Let \((u_0, U_0)\) be a solution of (2.2) with asymptotic behaviour as given in (2.3). Consider a perturbation

\[
u = u_0 + \tilde{u} \quad \text{and} \quad U = U_0 + \tilde{U}, \quad |\tilde{u}|, |\tilde{U}| \ll 1
\]

(2.10)

of this solution. Then \( \tilde{u} \) and \( \tilde{U} \) satisfy

\[
x \frac{d}{dx} \begin{pmatrix} \tilde{u} \\ \tilde{U} \end{pmatrix} = \begin{pmatrix} (1 - \beta + 2xu_0U_0)\tilde{u} + xu_0\tilde{U} + xu_0\tilde{U} + xU_0\tilde{U} + xu_0\tilde{U} + x\tilde{U} \frac{U_0}{x} \end{pmatrix} \begin{pmatrix} \tilde{u} \\ \tilde{U} \end{pmatrix} = B(x) \begin{pmatrix} \tilde{u} \\ \tilde{U} \end{pmatrix}.
\]

(2.11)

By (2.3), the coefficient matrix \( B(x) \) in the above equation has the following asymptotic behaviour:

\[
B(x) = J + O\left(\frac{1}{x}\right), \quad \text{as} \ |x| \to \infty,
\]

(2.12)

where \( J \) is the matrix in (2.8) with eigenvalues given in (2.9). So, by theorem 12.3 in [31],

\[
Y(x)x^D \exp\left(x \text{ diag}(\lambda_1, \lambda_2)\right),
\]

(2.13)
where $D$ is a constant diagonal matrix $D = \text{diag}(d_1, d_2)$ and $Y(x)$ has an asymptotic expansion

$$ Y(x) \sim \sum_{r=0}^{\infty} Y_r x^{-r}, \quad \det Y_0 \neq 0, \quad \text{as } |x| \to \infty. $$

Consequently, we have

$$ \hat{u}(x) \sim c_1 x^{d_1} e^{i \lambda_1 x} + c_2 x^{d_2} e^{i \lambda_2 x}, \quad \text{as } |x| \to \infty \quad (2.14) $$

with two arbitrary parameters $c_1$ and $c_2$. To ensure our assumption $|\hat{u}| \ll 1$, we need to choose either $c_1 = 0$, or $c_2 = 0$, or a correct sector such that $\text{Re} e^{i \lambda x} < 0$. According to the specific values of $\lambda_{1,2}$ in (2.9), we prove the first part of our theorem.

To prove the second part of our theorem and achieve uniqueness, we consider solutions in the following sectors:

$$ S(m) = \left\{ x \in \mathbb{C} \mid |x| > |x_0|, -\frac{\pi}{2} + k \pi \pm \varepsilon < \arg x < \frac{\pi}{2} + k \pi \pm \varepsilon \right\}, \quad \text{for } m = 0, 2, $$

$$ S_{k,\pi}(m) = \left\{ x \in \mathbb{C} \mid |x| > |x_0|, k \pi \pm \varepsilon < \arg x < (k + 1) \pi \pm \varepsilon \right\}, \quad \text{for } m = 1, 3 $$

with a small $\varepsilon > 0$. Let $(u_1, U_1)$ and $(u_2, U_2)$ be two solutions of (2.2) whose asymptotic expansions are given by (2.3) in sectors $S_{k,\pi}(m)$ and $S_{k+1,\pi}(m)$, respectively. Consider the difference of these solutions

$$ w := u_1 - u_2 \quad \text{and} \quad W := U_1 - U_2, \quad (2.15) $$

which we define in the overlap of the two sectors

$$ \mathcal{S}_{k,\pi}(m) = S_{k,\pi}(m) \cap S_{k+1,\pi}(m). \quad (2.16) $$

Since $(u_1, U_1)$ and $(u_2, U_2)$ satisfy the same asymptotic expansion, we have

$$ w(x) = O(x^{-\beta}) \quad \text{and} \quad W(x) = O(x^{-\beta}) \quad \text{as } |x| \to \infty \quad \text{and} \quad x \in \mathcal{S}_{k,\pi}(m) \quad (2.17) $$

for all $\beta \in \mathbb{N}$. Moreover, one can verify using (2.2) that $w(x)$ and $W(x)$ satisfy the following equations:

$$ \frac{d}{dx} \begin{pmatrix} w \\ W \end{pmatrix} = \begin{pmatrix} 1 - \beta \\ x \end{pmatrix} + (u_1 + u_2)U_1 - U_2 \begin{pmatrix} u_1^2 \\ \beta - 2x - u_1(U_1 + U_2) \end{pmatrix} \begin{pmatrix} w \\ W \end{pmatrix} := \tilde{B}(x) \begin{pmatrix} w \\ W \end{pmatrix}. \quad (2.18) $$

The coefficient matrix $\tilde{B}(x)$ satisfies the same asymptotic behaviour as $B(x)$ does in (2.12). Then by the same argument as in the first part, we get

$$ w(x) \sim c_1 x^{d_1} e^{i \lambda_1 x} + c_2 x^{d_2} e^{i \lambda_2 x}, \quad \text{as } |x| \to \infty. \quad (2.19) $$

To ensure $w(x)$ satisfies the asymptotic behaviour in (2.17) for $x \in \mathcal{S}_{k,\pi}(m)$, we must choose $c_1 = c_2 = 0$ in the above formula. This means $u_1 = u_2$ in $S_{k,\pi}(m)$ and the sector of validity can be extended to $\Omega_{k,\pi}(m)$ in (1.9). This completes the proof of our results. \qed

3. The $P_{III}^{(ii)}$ equations

For case (ii), where $\alpha = 1$, $\gamma = 0$ and $\delta = -1$, the first-order equations (2.1) reduce to the following form:

$$ \begin{cases} x \frac{du}{dx} = x + (1 - \beta)u + xu^2 U, \\ x \frac{dU}{dx} = 1 - (2 - \beta)U - xuU^2. \end{cases} \quad (3.1) $$

Then we obtain the formal solutions.
Proposition 3. We have the following formal solutions for (3.1)
\[ u^{(m)}_f (x) = x^{\frac{1}{3}} \sum_{n=0}^{\infty} a^{(m)}_n x^{-\frac{2n}{3}} \quad \text{and} \quad U^{(m)}_f (x) = x^{-\frac{1}{3}} \sum_{n=0}^{\infty} A^{(m)}_n x^{-\frac{2n}{3}}, \]
(3.2)
where
\[ a^{(m)}_0 = e^{\frac{2m\pi i}{3}} \quad \text{and} \quad A^{(m)}_0 = -a^{(m)}_0, \quad m = 0, 1, 2, \]
(3.3)
and the subsequent coefficients are determined by (1.12).

Proof. Inserting (3.2) into (3.1) gives us the results. \(\square\)

Again, applying Wasow’s theorem 4, we get the existence of the solutions to \(P_{III}\) with asymptotic expansions as given in (1.11).

Proposition 4. In any sector of angle less than \(\frac{3\pi}{2}\), there exists a solution \(u(x)\) of \(P_{III}\) whose asymptotic behaviour as \(|x| \to \infty\) is given by \(u^{(m)}_f (x)\) in (3.2) with \(m = 0, 1, 2\).

Proof. First we transform (3.1) into the standard form (1.19) in Wasow’s theorem. Let \(y = x^{\frac{1}{3}}\) and write
\[ u(x) = y(v(y) + a^{(m)}_0) \quad \text{and} \quad U(x) = \frac{1}{y^2} (V(y) + A^{(m)}_0). \]
Then \((v, V)\) satisfies the following equations:
\[ \begin{align*}
\frac{1}{y} \frac{d}{dy} \left( \frac{v}{V} \right) &= \left( 3 + \frac{2 - 3\beta}{y^2} (v + a^{(m)}_0) + 3(v + a^{(m)}_0)^2(V + A^{(m)}_0) \right) \\
&\quad \left( 3 + \frac{3\beta - 4}{y^2} (V + A^{(m)}_0) - 3(v + a^{(m)}_0)(V + A^{(m)}_0)^2 \right)
\end{align*} \]
(3.4)
and has a formal expansion given by
\[ u^{(m)}_f (y) = \sum_{n=1}^{\infty} a^{(m)}_n y^{-2n} \quad \text{and} \quad V^{(m)}_f (y) = \sum_{n=1}^{\infty} A^{(m)}_n y^{-2n}. \]
(3.5)
Again, let us denote the right-hand side of (3.4) by \((f_1, f_2)^T\). Then we have
\[ \lim_{y \to \infty} \begin{pmatrix} \frac{\partial f_1}{\partial v} & \frac{\partial f_1}{\partial V} \\ \frac{\partial f_2}{\partial v} & \frac{\partial f_2}{\partial V} \end{pmatrix} \bigg|_{v=V=0} = \begin{pmatrix} 6a^{(m)}_0 & 3(a^{(m)}_0)^2 \\ -3(A^{(m)}_0)^2 & -6a^{(m)}_0 A^{(m)}_0 \end{pmatrix} := J. \]
(3.6)
The eigenvalues of \(J\) are
\[ \lambda_{1,2} = \pm 3\sqrt{3} e^{-\frac{2m\pi i}{3} \neq 0}, \quad m = 0, 1, 2; \]
(3.7)
see (3.3) for the values of \(a^{(m)}_0\) and \(A^{(m)}_0\). Then, according to Wasow’s theorem, there exist true solutions \((v(y), V(y))\) to (3.4) in any sector of angle less than \(\frac{\pi}{2}\) with given asymptotic expansion in (3.5). Recalling \(y = x^{\frac{1}{3}}\) and the relations between \((u, U)\) and \((v, V)\), our proposition follows. \(\square\)

As in section 2, with the above results, we are ready to prove our theorem 2.

Proof of theorem 2. Instead of studying the original system (3.1), it is more convenient to consider (3.4). Let \((v_0, V_0)\) be a solution of (3.1) with asymptotic behaviour given in (3.5). Consider a perturbation of this solution
\[ v = v_0 + \tilde{v} \quad \text{and} \quad V = V_0 + \tilde{V}, \quad |\tilde{v}|, |\tilde{V}| \ll 1. \]
(3.8)
Then $\hat{v}$ and $\hat{V}$ satisfy
\[
\frac{1}{y}\frac{d}{dy} \hat{v} = \left[ \frac{2 - 3\beta}{y^2} + 6(v_0 + a_0^{(m)})(V_0 + A_0^{(m)}) - 3(v_0 + a_0^{(m)})^2 \right] \hat{v} + 3(v_0 + a_0^{(m)})^2 \hat{v} + 6(v_0 + a_0^{(m)}) \hat{V} + 3(V_0 + A_0^{(m)}) \hat{V}^2 + 3\hat{V}^2.
\]
and
\[
\frac{1}{y}\frac{d}{dy} \hat{V} = -3(V_0 + A_0^{(m)})^2 \hat{V} + \left[ \frac{3\beta - 4}{y^2} - 6(v_0 + a_0^{(m)})(V_0 + A_0^{(m)}) - 3(v_0 + a_0^{(m)})^2 \right] \hat{V}
- 6(V_0 + A_0^{(m)}) \hat{V} - 3(v_0 + a_0^{(m)}) \hat{V}^2 - 3\hat{V}^2.
\]

Since $|\hat{v}|, |\hat{V}| \ll 1$, it is sufficient to consider the following linear system to determine the asymptotic behaviour of $\hat{v}$ and $\hat{V}$
\[
\frac{1}{y}\frac{d}{dy} \left( \begin{pmatrix} \hat{v} \\ \hat{V} \end{pmatrix} \right) = \begin{pmatrix} \frac{2 - 3\beta}{y^2} + 6(v_0 + a_0^{(m)})(V_0 + A_0^{(m)}) - 3(v_0 + a_0^{(m)})^2 \\ -3(V_0 + A_0^{(m)})^2 + \frac{3\beta - 4}{y^2} - 6(v_0 + a_0^{(m)})(V_0 + A_0^{(m)}) - 3(v_0 + a_0^{(m)})^2 \end{pmatrix} \begin{pmatrix} \hat{v} \\ \hat{V} \end{pmatrix}.
\]

By (3.5), the coefficient matrix $B(y)$ in the above equation has the following asymptotic behaviour
\[
B(y) = J + \mathcal{O} \left( \frac{1}{y^2} \right), \quad \text{as } |y| \to \infty,
\]
where $J$ is the matrix in (3.6) with eigenvalues given in (3.7). By the similar analysis as in the proof of theorem 1, we have
\[
\hat{v}(y) \sim c_1 y^{d_1} e^{\lambda_1 y} + c_2 y^{d_2} e^{\lambda_2 y}, \quad \text{as } |y| \to \infty
\]
with two arbitrary parameters $c_1$ and $c_2$. To ensure our assumption $|\hat{v}| \ll 1$, we need to choose either $c_1 = 0$ or a correct sector such that $\text{Re} \hat{v}_y y^2 < 0$. According to the specific values of $\lambda_{1,2}$ in (3.7), we should choose the sectors in the $y$-plane to be
\[
\left\{ y \in \mathbb{C} \mid |y| > |y_0|, \left( \frac{m}{3} - \frac{1}{4} \right) \pi + \frac{k\pi}{2} < \arg y < \left( \frac{m}{3} + \frac{1}{4} \right) \pi + \frac{k\pi}{2} \right\}, \quad \text{for } m = 0, 1, 2,
\]
where $k = 0, 1, 2, 3$ and $y_0 \neq 0$. As $y = x^t$, then the first part of our theorem is proved.

To prove the second part of the theorem, we can use similar analysis as in the proof of theorem 1 to show that, there exists a unique solution for (3.4) in each of the following sectors with its asymptotic behaviour determined by the expansion in (3.5)
\[
\left\{ y \in \mathbb{C} \mid |y| > |y_0|, \left( \frac{m}{3} - \frac{1}{4} \right) \pi + \frac{k\pi}{2} < \arg y < \left( \frac{m}{3} + \frac{3}{4} \right) \pi + \frac{k\pi}{2} \right\}, \quad \text{for } m = 0, 1, 2.
\]

Note that the angles of the above $y$-sectors are $\pi$. Then, the angles of the corresponding $x$-sectors should be $3\pi$. So, for any branch of $x^{1/3}$ in the complex $x$-plane, we obtain a unique solution with the given asymptotic expansion in (1.11). This completes our proof. \hfill \Box
4. The PIV equations

The Hamiltonian $H_{IV}$ for PIV is well known in the literature

$$H_{IV} = 2uU^2 - (u^2 + 2xu + 2\kappa_0)U + \kappa_\infty U$$  \hspace{1cm} (4.1)

(see, for example, [16, p 220]). So, (1.2) can be written as the following system of first order differential equations:

$$\begin{align*}
d\frac{u}{dx} & = \frac{\partial H_{IV}}{\partial U} = 4uU - u^2 - 2xu - 2\kappa_0, \\
d\frac{U}{dx} & = -\frac{\partial H_{IV}}{\partial u} = -2U^2 + 2uU + 2xU - \kappa_\infty, \hspace{1cm} (4.2)
\end{align*}$$

where $\alpha = -\kappa_0 + 2\kappa_\infty + 1$ and $\beta = -2\kappa_0^2$. Then the formal solutions are obtained as follows.

**Proposition 5.** As $|x| \to \infty$, (4.2) has four different formal solutions.

Case 1: $u_{1,f}(x) = x \sum_{n=0}^{\infty} a_n^{(1)} x^{-2n}$ and $U_{1,f}(x) = x \sum_{n=0}^{\infty} A_n^{(1)} x^{-2n}$, \hspace{1cm} (4.3)

where $a_0^{(1)} = -\frac{1}{2}$, $a_1^{(1)} = \alpha$, $A_0^{(1)} = \frac{1}{2}$, $A_1^{(1)} = \frac{1}{2} - \kappa_0 + \frac{1}{2}\kappa_\infty$, and the subsequent coefficients are given by (1.14).

Case 2: $u_{2,f}(x) = x \sum_{n=0}^{\infty} a_n^{(2)} x^{-2n}$ and $U_{2,f}(x) = \frac{1}{x} \sum_{n=0}^{\infty} A_n^{(2)} x^{-2n}$, \hspace{1cm} (4.4)

where $a_0^{(2)} = -2$, $a_1^{(2)} = -\alpha$, $A_0^{(2)} = -\frac{1}{2}\kappa_\infty$, and the subsequent coefficients are given by (1.15).

Case 3: $u_{3,f}(x) = \frac{1}{x} \sum_{n=0}^{\infty} a_n^{(3)} x^{-2n}$ and $U_{3,f}(x) = x \sum_{n=0}^{\infty} A_n^{(3)} x^{-2n}$, \hspace{1cm} (4.5)

where $a_0^{(3)} = \kappa_0$, $A_0^{(3)} = 1$, $A_1^{(3)} = -\frac{1}{2}(1 - 2\kappa_0 + \kappa_\infty)$, and the subsequent coefficients are given by (1.16).

Case 4: $u_{4,f}(x) = \frac{1}{x} \sum_{n=0}^{\infty} a_n^{(4)} x^{-2n}$ and $U_{4,f}(x) = \frac{1}{x} \sum_{n=0}^{\infty} A_n^{(4)} x^{-2n}$, \hspace{1cm} (4.6)

where $a_0^{(4)} = -\kappa_0$, $A_0^{(4)} = \frac{1}{2}\kappa_\infty$, and the subsequent coefficients are given by (1.17).

**Proof.** Substituting (4.3)–(4.6) into (4.2) gives us the results. \hspace{1cm} \Box

**Remark 8.** When $\kappa_0 = 0$ in cases 3 and 4, we obtain the trivial solutions $u_{3,f} = 0$ and $u_{4,f} = 0$. Similarly, when $\kappa_\infty = 0$ in cases 2 and 4, the trivial solutions $U_{2,f} = 0$ and $U_{4,f} = 0$ occur. We can exclude these trivial cases in our subsequent analysis.

Using Wasow’s theorem again, we get the following result.

**Proposition 6.** In any sector of angle less than $\pi/2$, there exists a solution $u(x)$ of PIV whose asymptotic behaviour as $|x| \to \infty$ is given by $u_{mf}(x)$ in proposition 5 with $m = 1, 2, 3, 4$. 

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Proof. To arrive at the standard form (1.19) in Wasow’s theorem, we consider the following change of variables.

Case 1: Let \( v(x) = x^{-1}u_{1,f}(x) - A_0^{(1)} \) and \( V(x) = x^{-1}U_{1,f}(x) - A_0^{(1)} \). Then \( (v, V) \) satisfies the following equations

\[
\frac{1}{x} \frac{d}{dx} \begin{pmatrix} v \\ V \end{pmatrix} = \begin{pmatrix} 4(v + A_0^{(1)})(V + A_0^{(1)}) - (v + A_0^{(1)})^2 - 2(v + A_0^{(1)}) - \frac{1}{x^2}(v + A_0^{(1)}) - \frac{2\kappa_0}{x^2} \\ -2(V + A_0^{(1)})^2 + 2(v + A_0^{(1)})(V + A_0^{(1)}) + 2(V + A_0^{(1)}) - \frac{1}{x^2}(V + A_0^{(1)}) - \frac{\kappa_\infty}{x^2} \end{pmatrix}
\]

and has a formal expansion given by

\[
v_f(x) = \sum_{n=1}^{\infty} a_n x^{-2n} \quad \text{and} \quad V_f(x) = \sum_{n=1}^{\infty} A_n x^{-2n}
\]

for constants \( a_n \) and \( A_n \). Let us denote the right-hand side of (4.7) by \( (f_1, f_2)^T \), then we have

\[
\lim_{x \to \infty} \begin{pmatrix} \frac{\partial f_1}{\partial v} & \frac{\partial f_1}{\partial V} \\ \frac{\partial f_2}{\partial v} & \frac{\partial f_2}{\partial V} \end{pmatrix}_{v = 0} = \begin{pmatrix} 2 & 8 \\ 3 & 3 \end{pmatrix} := J_1.
\]

The eigenvalues of \( J_1 \) are \( \lambda_{1,2} = \pm \frac{\sqrt{3}}{3} i \neq 0 \). So by Wasow’s theorem 4, we have, for any sector of angle smaller than \( \pi/2 \), there exist solutions \((u, U)\) of (4.2) with respective asymptotic behaviours \((u_{1,f}, U_{1,f})\) as described in (4.3). Next, we will prove the other cases.

Case 2: Let \( v(x) = x^{-1}u_{2,f}(x) - A_0^{(2)} \) and \( V(x) = xU_{2,f}(x) - A_0^{(2)} \). Inserting these identities into (4.2) then gives

\[
\frac{1}{x} \frac{d}{dx} \begin{pmatrix} v \\ V \end{pmatrix} = \begin{pmatrix} \frac{4}{x^2}(v + A_0^{(2)})(V + A_0^{(2)}) - (v + A_0^{(2)})^2 - 2(v + A_0^{(2)}) - \frac{1}{x^2}(v + A_0^{(2)}) - \frac{2\kappa_0}{x^2} \\ \frac{2}{x^2}(V + A_0^{(2)})^2 + 2(v + A_0^{(2)})(V + A_0^{(2)}) + 2(V + A_0^{(2)}) + \frac{1}{x^2}(V + A_0^{(2)}) - \kappa_\infty \end{pmatrix}
\]

and \((v, V)\) has a formal expansion given by (4.8).

Case 3: Let \( v(x) = xu_{3,f}(x) - A_0^{(3)} \) and \( V(x) = xU_{3,f}(x) - A_0^{(3)} \). Then \((v, V)\) satisfies

\[
\frac{1}{x} \frac{d}{dx} \begin{pmatrix} v \\ V \end{pmatrix} = \begin{pmatrix} \frac{4}{x^2}(v + A_0^{(3)})(V + A_0^{(3)}) - \frac{1}{x^2}(v + A_0^{(3)})^2 - 2(v + A_0^{(3)}) + \frac{1}{x^2}(v + A_0^{(3)}) - 2\kappa_0 \\ -2(V + A_0^{(3)})^2 + \frac{2}{x^2}(v + A_0^{(3)})(V + A_0^{(3)}) + 2(V + A_0^{(3)}) - \frac{1}{x^2}(V + A_0^{(3)}) - \frac{\kappa_\infty}{x^2} \end{pmatrix}
\]

and has a formal expansion given by (4.8).
Case 4: Let $v(x) = xu_{4, f}(x) - a_0^{(4)}$ and $V(x) = xU_{4, f}(x) - A_0^{(4)}$. Then $(v, V)$ satisfies
\[
\frac{1}{x} \frac{d}{dx} \begin{pmatrix} v \\ V \end{pmatrix} = \begin{pmatrix} \frac{4}{x^2} (v + a_0^{(4)})(V + A_0^{(4)}) - \frac{1}{x^2} (v + a_0^{(4)})^2 - 2(v + a_0^{(4)}) + \frac{1}{x^2} (v + a_0^{(4)}) - 2\kappa_0 \\ -\frac{2}{x^2} (V + A_0^{(4)})^2 + \frac{2}{x^2} (v + a_0^{(4)})(V + A_0^{(4)}) + 2(V + A_0^{(4)}) + \frac{1}{x^2} (V + A_0^{(4)}) - \kappa_\infty \end{pmatrix}
\]
and has a formal expansion given by (4.8).

Following a similar manner, we then construct the Jacobian of $(f_1, f_2)^T$ evaluated at $(v, V) = 0$ as $|x| \to \infty$.

\[
J_2 = \begin{pmatrix} 2 & 0 \\ -\kappa_\infty & -2 \end{pmatrix}, \quad J_3 = \begin{pmatrix} 2 & 4\kappa_0 \\ 0 & -2 \end{pmatrix}, \quad J_4 = \begin{pmatrix} -2 & 0 \\ 0 & 2 \end{pmatrix}
\]
for case 2, case 3 and case 4, respectively. The eigenvalues of $J$ are $\lambda_{1,2} = \pm 2$. Since each eigenvalue is different from zero, all the conditions of Wasow’s theorem are fulfilled. Then our proposition follows from Wasow’s theorem.

Now, we are ready to prove our last theorem.

Proof of theorem 3. For simplicity, we only consider case 1. Instead of studying (4.2), it is more convenient to consider (4.7). Let $(v_0, V_0)$ be a solution of (4.7) with asymptotic behaviour given in (4.8). Consider a perturbation of this solution
\[
\begin{aligned}
v &= v_0 + \hat{v} \\
V &= V_0 + \hat{V}, \quad |\hat{v}|, |\hat{V}| \ll 1.
\end{aligned}
\quad (4.10)
\]
Then $\hat{v}$ and $\hat{V}$ satisfy the equations
\[
\frac{1}{x} \frac{d}{dx} \hat{v} = 4(v_0 + \hat{v} + a_0^{(1)}) \hat{V} + 4\hat{v}(V_0 + A_0^{(1)}) - \hat{v}(2v_0 + \hat{v} + 2a_0^{(1)}) - 2\hat{v} - \frac{1}{x^2} \hat{v},
\]
and
\[
\frac{1}{x} \frac{d}{dx} \hat{V} = -2\hat{V}(2V_0 + \hat{V} + 2A_0^{(1)}) + 2(v_0 + \hat{v} + a_0^{(1)}) \hat{V} + 2\hat{v}(V_0 + A_0^{(1)}) + 2\hat{V} - \frac{1}{x^2} \hat{V}.
\]
Since $|\hat{v}|, |\hat{V}| \ll 1$, it is sufficient to consider the following linear system of differential equations to determine the asymptotic behaviour of $\hat{v}$ and $\hat{V}$
\[
\frac{1}{x} \frac{d}{dx} \begin{pmatrix} \hat{v} \\ \hat{V} \end{pmatrix} = \begin{pmatrix} 4(V_0 + A_0^{(1)}) - 2(v_0 + a_0^{(1)}) - 2 - \frac{1}{x^2} & 4(v_0 + a_0^{(1)}) \\ 2(V_0 + A_0^{(1)}) & -4(V_0 + A_0^{(1)}) + 2(v_0 + a_0^{(1)}) + 2 - \frac{1}{x^2} \end{pmatrix} \begin{pmatrix} \hat{v} \\ \hat{V} \end{pmatrix}.
\]
By (4.3), the coefficient matrix $B_1(x)$ in the above equation has the following asymptotic behaviour:
\[
B_1(x) = J_1 + O \left( \frac{1}{x} \right), \quad \text{as } |x| \to \infty,
\quad (4.11)
\]
where $J_1$ is the matrix in (4.9) with eigenvalues given by $\lambda_{1,2} = \pm \frac{2\sqrt{2}}{3} i$. By a similar analysis as the ones in the proofs of theorems 1 and 2, we have
\[
\hat{v}(x) \sim c_1 x^{\kappa_1} e^{\lambda_1 x} \hat{v} + c_2 x^{\kappa_2} e^{\lambda_2 x}, \quad \text{as } |x| \to \infty
\quad (4.12)
\]
with two arbitrary parameters $c_1$ and $c_2$. To ensure that $|\hat{v}| \ll 1$, we need to choose either $c_i = 0$ or a correct sector such that $\text{Re} e^{i\frac{\hat{v}}{2}} < 0$. According to the specific values of $\lambda_{1,2}$, we should choose the sectors to be

$$S_k^{(1)} = \{ x \in \mathbb{C} | |x| > |x_0|, \frac{1}{2} k \pi < \arg x < \frac{1}{2} k \pi + \frac{1}{2} k \pi \}.$$

To prove the second part of our theorem and achieve uniqueness, we consider solutions in the following sectors:

$$S_{k, \pm \epsilon} = \{ x \in \mathbb{C} | |x| > |x_0|, \frac{1}{2} k \pi \pm \epsilon < \arg x < \frac{1}{2} k \pi + 1 \}$$

with a small $\epsilon > 0$. Let $(u_1, U_1)$ and $(u_2, U_2)$ be two solutions to (4.2) whose asymptotic expansions are given by (4.3) in sectors $S_{k, \epsilon}$ and $S_{k+1, -\epsilon}$, respectively. Consider the difference of these solutions

$$w(x) := u_1 - u_2 \quad \text{and} \quad W(x) := U_1 - U_2,$$

which are defined in the overlap of the two sectors

$$\tilde{S}_{k, \epsilon} = S_{k, \epsilon} \cap S_{k+1, -\epsilon}.$$  

(4.14)

Since $(u_1, U_1)$ and $(u_2, U_2)$ satisfy the same asymptotic expansion, we have

$$w(x) = O(x^{-\frac{1}{2}}) \quad \text{and} \quad W(x) = O(x^{-\frac{1}{2}}) \quad \text{as} \quad |x| \to \infty \quad \text{and} \quad x \in \tilde{S}_{k, \epsilon}$$

(4.15)

for all $j \in \mathbb{N}$. Moreover, one can verify that $w(x)$ and $W(x)$ satisfy

$$\frac{1}{x} \frac{d}{dx} \begin{pmatrix} w \\ W \end{pmatrix} = \begin{pmatrix} 4U_1 - 2u_1 - 2 & 4u_1 \\ 2U_1 & -4U_1 + 2u_1 + 2 \end{pmatrix} \begin{pmatrix} w \\ W \end{pmatrix} := \tilde{B}_1(x) \begin{pmatrix} w \\ W \end{pmatrix}.$$  

By the same argument as in the first part, we have

$$w(x) \sim c_1 x^\frac{1}{2} e^{x^\frac{1}{2}} + c_2 x^\frac{1}{2} e^{-x^\frac{1}{2}}, \quad \text{as} \quad |x| \to \infty.$$  

(4.16)

To ensure that $w(x)$ satisfies the asymptotic behaviour described in (4.15) for $x \in \tilde{S}_{k, \epsilon}$, we must choose $c_1 = c_2 = 0$ in the above formula. This means $u_1 = u_2$ in $\tilde{S}_{k, \epsilon}$ and the sector of validity can be extended to $\Omega_k^{(1)}$ in theorem 3. The other cases can be handled in a similar manner. This completes the proof. □

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