A numerical method for valuation of European option with regime-switching volatility and interest rate

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Abstract. This paper develops a numerical method for pricing European options with regime-switching volatility and interest rate governed by the Black-Scholes equation. The method is based on a finite volume method with a spatial discretization and implicit time stepping technique. We show that the system matrix of the discretized system is an M-matrix and present an algorithm for solving the problem. Numerical experiment are implemented for some European option problems to illustrate the usefulness of this method.

1. Introduction

Several studies of an option use the Black-Scholes model as a mathematical model—a second order partial differential equation with respect to the time horizon \( t \) and the underlying asset price \( S \). The model assumes the volatility of underlying asset is constant during the period of the contract. On the other hand, this assumption is not consistent with the market price. The value of an option depends on a stochastic volatility as well as the underlying stock price. Consequently, there is a need for more realistic models. One such formulation is a model with regime switching. Some research has shown that models based on stochastic volatility, jump diffusion and regime switching processes produce models that better fit market data.

The application of regime-switching for pricing option has been shown by Zhu et al [4] and Tauryawati et al [9]. In [4,9], the authors showed that the value of an option with regime-switching is governed by a pair of Black-Scholes model. This certainly makes the equation much more difficult to solve than those with constant volatilities. The numerical methods developed can easily be applied to these cases where the properties in each regime are more complex.
In this paper we present a finite volume method as a numerical method for pricing European call option with regime-switching volatility and interest rate. This method is based on a fitted finite volume method that uses a spatial discretization and an implicit time stepping technique. The formulation of finite volume method applied to the regime-switching volatility model, has been derived by Tauryawati [4]. The numerical results of some model problems will be presented in Sect. 3 to demonstrate the usefulness of the method.

2. Finite volume method

Let \( C \) denotes the value of a European option at time \( t \), \( \delta \) is a dividend and \( r \) is a risk-free interest rate and \( X \) is the price of the underlying asset. The Black-Scholes equation for regime-switching volatility and interest rate case referring to the work by Badran [2] and Tauryawati [4] is given by

\[
\frac{\partial C_1}{\partial t} + \frac{1}{2}\sigma_1^2 X^2 \frac{\partial^2 C_1}{\partial X^2} + (r_1 - \delta) X \frac{\partial C_1}{\partial X} - r_1 C_1 = \lambda_{12} (C_1 - C_2) \tag{1}
\]

\[
\frac{\partial C_2}{\partial t} + \frac{1}{2}\sigma_2^2 X^2 \frac{\partial^2 C_2}{\partial X^2} + (r_2 - \delta) X \frac{\partial C_2}{\partial X} - r_2 C_2 = \lambda_{21} (C_2 - C_1) \tag{2}
\]

with boundary conditions:

\[
C(0,t) = 0, \quad \lim_{X \to \infty} C(X,t) = X, \quad C(X,T) = \max \{X - E, 0\}.
\]

The PDE systems in Eq. (1) and Eq. (2) represent the price dynamic of European call options considering the economic changes by the regime-switching volatility and interest rate. The equation is solved using a finite volume method. The first step is to form a diffusion equation. For example in state 1:

\[
\frac{\partial C_1}{\partial \tau} = \frac{1}{2}\sigma_1^2 X^2 \frac{\partial^2 C_1}{\partial X^2} + (r_1 - \delta) X \frac{\partial C_1}{\partial X} - r_1 C_1 + \lambda_{12} (C_2 - C_1)
\]

\[
\frac{\partial C_1}{\partial \tau} = \frac{\partial}{\partial X} \left[ a X^2 \frac{\partial C_1}{\partial X} + b X C_1 \right] + p C_1 + \lambda_{12} C_2
\]

where \( a = \frac{1}{2}\sigma_1^2 \), \( b = r_1 - \delta - \sigma_1^2 \), \( p = -b - r_1 - \lambda_{12} \).

By decomposing the problem in Eq.(1) into control volumes. The interval domain \( I \) is divided into two parts \( I_i \) and \( J_i \) (control volume) with subinterval \( i = 0, 1, 2, ..., N - 1 \). The first domain is \( I_i = (X_i, X_{i+1}) \) and the second is \( J_i = (X_{i-1/2}, X_{i+1/2}) \) with \( I_i = (X_i, X_{i+1}), i = 0, 1, 2, ..., N - 1 \) with \( X_{i-1/2} = (X_{i-1} + X_i)/2, X_{i+1/2} = (X_i + X_{i+1})/2, X_0 = X_{-1/2} \) and \( X_{N+1/2} = X_{N+1} \). Then, an integral balance equation for each control volume is formulated as follows

\[
\int_{J_i} \frac{\partial C_1}{\partial \tau} dX = \int_{J_i} \frac{\partial C_1}{\partial X} \left[ a X^2 \frac{\partial C_1}{\partial X} + b X C_1 \right] dX + \int_{J_i} p C_1 dS + \int_{J_i} \lambda_{12} C_2 dX \quad \tag{3}
\]
\[
\frac{\partial C_1}{\partial \tau} x_i = X_{i+1/2} \rho(C_1) \bigg|_{X_{i+1/2}} - X_{i-1/2} \rho(C_1) \bigg|_{X_{i-1/2}} + \rho C_1 x_i + \lambda_{i2} C_2 x_i
\]  
(4)

where \( x_i = X_{i+1/2} - X_{i-1/2} \) and

\[
\rho(C_1) = aX \frac{\partial C_1}{\partial X} + bC_1.
\]  
(5)

Eq.(5) is called a flux or surface integral which need to be approximated. The approximation is divided into two cases.

**Case I.** For \( i = 2, 3, \ldots, N - 1 \)

\( \rho(C_1) \) is approximated as a two-point boundary value problem at the point \( X_{i-1/2} \) and \( X_{i+1/2} \) of the interval \( I_i \). Two-point boundary value at the point \( X_{i+1/2} \) is as follows:

\[
(a_{i+1/2} X C_1' + b_{i+1/2} C_1)' = 0, S \in I
\]  
(6)

\[
C_1(X_i) = C_1_i, C_1(X_{i+1}) = C_1_{i+1},
\]  
(7)

and subsequently at the point \( X_{i-1/2} \) is as follows:

\[
(a_{i-1/2} X C_1' + b_{i-1/2} C_1)' = 0, S \in I
\]  
(8)

\[
C_1(X_i) = C_1_i, C_1(X_{i-1}) = C_1_{i-1}
\]  
(9)

Eq.(6)-(9) are solved analytically and resulted:

\[
\rho_i(C_1) = b_{i+1/2} \frac{C_1_i X_i^{k_i} - C_{1i+1} X_{i+1}^{k_i}}{X_i^{k_i} - X_{i+1}^{k_i}}
\]  
(10)

\[
\rho_i(C_1) = b_{i-1/2} \frac{C_1_i X_i^{k_i} - C_{1i-1} X_{i-1}^{k_i}}{X_i^{k_i} - X_{i-1}^{k_i}}
\]  
(11)

where

\[
k_i = \frac{b_{i+1/2}}{a_{i+1/2}}.
\]  
(12)

**Case II.** For \( i = 0, 1 \)

On the flux \( (0, X_1) \), the approximation of Case I, cannot be used in this case because the Eq.(6) is degenerated for \( S \to 0 \). Therefore we consider the formulation as follow

\[
(a_{1/2} X C_1' + b_{1/2} C_1)' = c, S \in I
\]  
(13)

\[
C_1(0) = C_1_0, C_1(X_1) = C_1_1
\]  
(14)

with the same procedure, then we get:

\[
\rho_i(C_1) = \frac{1}{2} ((a_{1/2} + b_{1/2})C_1_1 - (a_{1/2} - b_{1/2})C_1_0)
\]  
(15)
\begin{align} 
C_{11} &= C_{10} + \frac{X}{X_1} (C_{11} - C_{10}) 
\end{align}

by substituting Eq.(10)-(11) and Eq.(15) into Eq.(4), we obtain:

\begin{align} 
\frac{\partial C_{11}}{\partial \tau} x_i &= X_{3/2} b_{3/2} \left[ \frac{C_{11}X_{11}^k - C_{12}X_{12}^k}{X_{11}^k - X_{12}^k} \right] - X_{1/2} b_{1/2} \left[ \frac{C_{11}X_{11}^k - C_{10}X_{10}^k}{X_{11}^k - X_{10}^k} \right] + \nonumber 
&\quad p x_i + x_i \lambda_{12} C_2 
\end{align}

The formulation of spatial discretization for \( i = 0, 1 \) is:

\begin{align} 
\frac{\partial C_{11}}{\partial \tau} &= \xi_1 C_{10} + \psi_1 C_{11} + \phi_1 C_{12} + \lambda_{12} C_{21} 
\end{align}

where

\begin{align*} 
\xi_1 &= \frac{X_1(a_{1/2} - b_{1/2})}{4 \xi_1} 
\psi_1 &= \frac{-X_1(a_{1/2} + b_{1/2})}{4 x_i} - \frac{X_{3/2} b_{3/2} X_{11}^k}{x_i (X_{21}^k - X_{11}^k)} - p 
\phi_1 &= \frac{X_{3/2} b_{3/2} X_{12}^k}{x_i (X_{21}^k - X_{11}^k)} 
\end{align*}

and for \( i = 2, 3, \ldots, N-1 \) is given as follows:

\begin{align} 
\frac{\partial C_{1i}}{\partial \tau} &= \xi_i C_{i-1} + \psi_i C_{1i} + \phi_i C_{1i+1} + \lambda_{12} C_{2i} 
\end{align}

where

\begin{align*} 
\xi_i &= \frac{X_{i-1/2} b_{i-1/2} X_{i-1}^k}{x_i (X_{i+1}^k - X_{i-1}^k)} 
\psi_i &= \frac{-X_{i-1/2} b_{i-1/2} X_{i}^k}{x_i (X_{i+1}^k - X_{i-1}^k)} - \frac{X_{i+1/2} b_{i+1/2} X_{i+1}^k}{x_i (X_{i+1}^k - X_{i-1}^k)} - p 
\phi_i &= \frac{X_{i+1/2} b_{i+1/2} X_{i+1}^k}{x_i (X_{i+1}^k - X_{i}^k)} 
\end{align*}

The next step is to apply an implicit time discretization. Let \( \tau \) denotes points from \([0, T]\) such that \( 0 = \tau_0 < \tau_1 < \ldots < \tau_M = T \) and \( \Delta \tau_n = \tau_n - \tau_{n-1} > 0 \), where \( M > 1 \) is a positive integer. Applying the fully implicit time discretization to Eq. (19) for simplicity, we get

\begin{align} 
\frac{C_{1i}^{n+1} - C_{1i}^n}{\Delta \tau_{n+1}} &= \xi_i^{n+1} C_{i-1}^{n+1} + \psi_i^{n+1} C_{1i}^{n+1} + \phi_i^{n+1} C_{1i+1}^{n+1} + \lambda_{12} C_{2i}^{n+1} 
\end{align}
so that we obtain $C_1^n$ i is the solution of $X_i$ with time $\tau_n$ as follows:

$$C_1^n = (1 - \Delta \tau_{n+1} M^{n+1}) C_1^{n+1} - \Delta \tau_{n+1} R^{n+1}$$

(21)

where

$$M^{n+1} = \begin{bmatrix}
\psi_1^{n+1} & \phi_1^{n+1} \\
\xi_2^{n+1} & \psi_2^{n+1} & \phi_2^{n+1} \\
& & \ddots & \ddots \\
& & & \psi_{N-2}^{n+1} & \phi_{N-2}^{n+1} \\
& & & & \xi_{N-1}^{n+1} & \psi_{N-1}^{n+1}
\end{bmatrix}^{(N-1)\times(N-1)}$$

$$C_1^n = \begin{bmatrix}
C_1^n \\
C_1^2 \\
C_1^3 \\
\vdots
\end{bmatrix}, R_{n+1} = \begin{bmatrix}
\xi_1^{n+1} C_1^n + \lambda_{12} C_2^{n+1} \\
\lambda_{12} C_2^n + \lambda_{12} C_2^{n+1} \\
\phi_{N-1}^{n+1} C_1^n + \lambda_{12} C_2^{n+1}
\end{bmatrix}.$$  

And in position state 2 we obtain,

$$C_2^n = (1 - \Delta \tau_{n+1} M^{n+1}) C_2^{n+1} - \Delta \tau_{n+1} R^{n+1}$$

(22)

where:

$$M^{n+1} = \begin{bmatrix}
\psi_1^{n+1} & \phi_1^{n+1} \\
\xi_2^{n+1} & \psi_2^{n+1} & \phi_2^{n+1} \\
& & \ddots & \ddots \\
& & & \psi_{N-2}^{n+1} & \phi_{N-2}^{n+1} \\
& & & & \xi_{N-1}^{n+1} & \psi_{N-1}^{n+1}
\end{bmatrix}^{(N-1)\times(N-1)}$$

$$C_2^n = \begin{bmatrix}
C_2^n \\
C_2^2 \\
C_2^3 \\
\vdots
\end{bmatrix}, R_{n+1} = \begin{bmatrix}
\xi_1^{n+1} C_2^n + \lambda_{21} C_1^{n+1} \\
\lambda_{21} C_2^n + \lambda_{21} C_2^{n+1} \\
\phi_{N-1}^{n+1} C_2^n + \lambda_{21} C_1^{n+1}
\end{bmatrix}.$$  

3. Simulation and discussion

In this section, we present several numerical experiments to illustrate the performance and convergence of the method. We solve numerically various problems governing with different conditions and different choices of parameters.
Figure 1. $E = 100$, $r_1 = 0.1$, $r_2 = 0.1$
$s_1 = 0.2$, $s_2 = 0.2$, $X = 200$, $T = 1$,
$\delta = 0.05$, $\lambda_{12} = 1$, $\lambda_{21} = 1$

Figure 2. $E = 100$, $r_1 = 0.1$, $r_2 = 0.1$
$s_1 = 0.2$, $s_2 = 0.3$, $X = 200$, $T = 1$,
$\delta = 0.05$, $\lambda_{12} = 1$, $\lambda_{21} = 1$

Figure 3. $E = 100$, $r_1 = 0.1$, $r_2 = 0.2$
$s_1 = 0.2$, $s_2 = 0.3$, $X = 200$, $T = 1$,
$\delta = 0.05$, $\lambda_{12} = 1$, $\lambda_{21} = 1$

Figure 4. $E = 100$, $r_1 = 0.1$, $r_2 = 0.3$
$s_1 = 0.2$, $s_2 = 0.2$, $X = 200$, $T = 1$,
$\delta = 0.05$, $\lambda_{12} = 1$, $\lambda_{21} = 1$
Figure 5. $E = 100, r_1 = 0.1, r_2 = 0.8$
$\sigma_1 = 0.2, \sigma_2 = 0.7, X = 200, T = 1,$
$\delta = 0.05, \lambda_{12} = 1, \lambda_{21} = 1$

Figure 6. $E = 100, r_1 = 0.1, r_2 = 0.6$
$\sigma_1 = 0.2, \sigma_2 = 0.7, X = 200, T = 1,$
$\delta = 0.05, \lambda_{12} = 0.6, \lambda_{21} = 0.7$

Figure 1. shown the condition when $\sigma_1 = \sigma_2$ and $r_1 = r_2$ and the results are the same as conditions $C_1 = C_2$ and without regime switching. While the numerical solution of European call option with regime-switching volatility and interest rate for $C_1 \neq C_2$ or $\sigma_1 \neq \sigma_2$ and $r_1 \neq r_2$ is given by Figure 2-4. In the scheme it can be seen that there is a choice of price positions that can be used by holder and writer to make a profit. And in Figure 5-6 the parameter $\sigma$ and $r$ is given with a considerable value to show a significant economic difference condition.

4. Conclusion

In this paper, we presented a finite volume method as a numerical method based on fitted finite volume method. The method used a spatial discretization and implicit time stepping technique for solving the pair of Black-Scholes equations governing European call option with regime-switching volatility and interest rate. We have shown that the numerical scheme results in a linear system with an M-system matrix. Numerical results were presented to demonstrate the usefulness of the method. It also shows the benefit for investor by giving a choice of prices in the scheme for transaction.

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