String Theory and Quantum Spin Networks

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Abstract

We propose an approach to formulating string theory in a curved spacetime, which is based on the connection between the states of the WZW model for the isometry group of a background spacetime metric and the representations of the corresponding quantum group. In this approach the string states scattering amplitudes are defined by the evaluations of the theta spin networks for the associated quantum group. We examine the evaluations given by the spin network invariants defined by the spin foam state sum model associated to the two-dimensional BF theory for the background isometry group. We show that the corresponding string amplitudes are well-defined if the spacetime manifold is compact and admits a group metric. We compute the simplest scattering amplitudes in the case of the SU(2) background isometry group, and we provide arguments that these are the amplitudes of a topological string theory.

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1 Introduction

In the string theory approach to quantum gravity one is able to calculate perturbatively the graviton scattering amplitudes in a flat background metric spacetime, at least up to two loops [1], although there are strong indications that one can do this to an arbitrary number of loops [2]. From these amplitudes one can obtain the quantum effective action for general relativity [3, 4], and therefore in that sense the string theory can be considered as a quantum theory of gravity. However, beside proving the finiteness, another problem is that one is restricted only to the spacetimes with non-compact spatial sections, and hence addressing the problem of the phenomenologically relevant de-Sitter spacetime becomes a complicated task [5].

In the canonical loop quantum gravity (CLQG) approach, for a review and references see [6], the topology of the spatial slice is not a problem, although the spacetime topology is restricted to $\mathbb{R} \times \Sigma$, where $\Sigma$ is a three-manifold. In this approach one is dealing with the wavefunctions of the spatial spin connection, and therefore it is a nontrivial task to obtain an appropriate vacuum state, which has to be peaked around particular spacetime metric [7]. A related problem is to obtain an effective action which would correspond to general relativity, which is related to the problem of constructing the graviton scattering amplitudes.

The spin foam (SF) state sum models of quantum gravity, for a review and references see [8, 9], can be considered as the category theory generalizations of the Regge calculus approach to quantum gravity. As such, the SF models do not have the spacetime topology restriction as the CLQG models, although one can think of the SF models as the path-integral generalization of the CLQG models. All these features allow one to construct the SF scattering amplitudes in a straightforward manner [10, 11, 12]. However, because the spin foam models are the connection based formulations, it is difficult to extract the corresponding effective actions, and hence it is difficult to see whether they have a good semi-classical limit, which should be the general relativity with small quantum corrections.

Note that the string theory can be formulated as a two-dimensional (2d) state sum model, which is known as the random triangulations model (for a review and references see [13]). In this approach the string world-sheet is triangulated and embedded in a flat metric $d$-dimensional spacetime. One then assigns a unit length to all the edges and calculates the weights $e^{-\beta S(T)}$ for each triangulation $T$, where $S(T)$ is the string action and $\beta$ is a parameter.
The partition function is then obtained as a sum of the weights over all the triangulations. An analogous construction can be also made for the superstring [14, 15].

By using the dual triangulation of the world-sheet, one can represent the terms in the sum over the triangulations as the $d$-dimensional momentum space Feynman diagrams for a $\phi^3$ matrix field theory with the $\delta e^{-\nabla^2 \phi}$ kinetic term. Unfortunately, this approach is feasible only for strings propagating in $d \leq 1$ spacetimes. The $d = 0$ case can be interpreted as a partition function for the 2d general relativity, while the $d = 1$ case can be interpreted as a partition function for a 2d dilaton gravity with a single scalar field. Note that the Jackiw-Taitelboim 2d dilaton gravity model can be represented as the $SO(2,1)$ BF theory [10], so that the corresponding spin foam model in the Euclidian case can be also written as a $\phi^3$ matrix model [10]. However, while in the random triangulations approach one has a single matrix of a fixed dimension, in the spin foam case one has a multi-matrix model, with the matrices of all dimensions.

Therefore one can think of the string theory matrix models as special cases of the BF theory 2d spin foam models. These spin foam models can be embedded in $d > 1$ spacetimes via the choice of the BF theory Lie group $G$, so that the group manifold is the spacetime. The spin foam state sums can be defined rigorously via the representations of the quantum group for $G$. Since there is a close connection between the states of the Wess-Zumino-Witten (WZW) model for the Lie group $G$ and the representations of the corresponding quantum group [20, 21, 22], we propose in this paper to interpret certain spin network observables for the 2d BF spin foam model as the scattering amplitudes for a string propagating on the spacetime whose background metric is given by the Lie group manifold metric. In section two we briefly describe the construction of the string scattering amplitudes via the vertex operators and the relation to the spacetime fields effective action. In section three we apply the approach of the previous section to the case of the $SU(2)$ WZW model, and by using the connection of the WZW model to the quantum group theory, we propose a formula for the string states scattering amplitude as a group covariant linear combination of the $\theta_n$ spin network invariants. In section four we define these invariants via the spin foam state-sum model based on the 2d BF theory for the Lie group $G$, while in section five we show how this state sum can be evaluated via the quantum group spin networks. Because our amplitudes are based on a topological 2d theory, one can expect that these are the amplitudes of a topological string
theory, so that we present arguments for this in section five. We present our conclusions in section six, and in the appendix we prove the topological invariance of our state-sum model.

2 String theory approach

The string theory approach to quantizing matter and gravity can be described in the following way. Let $M$ be a $d$-dimensional manifold representing the spacetime ($d \geq 2$), and let $\Sigma$ be a surface (2d compact manifold) embedded in $M$, representing the string world-sheet. Given an embedding $X : \Sigma \to M$, we can associate to it a string action

$$S(X) = \frac{1}{l_s^2} \int_{\Sigma} d^2 \sigma \left[ \sqrt{|g|} g^{\alpha \beta} \partial_\alpha X^\mu \partial_\beta X^\nu G_{\mu \nu}(X) + \epsilon^{\alpha \beta} \partial_\alpha X^\mu \partial_\beta X^\nu B_{\mu \nu}(X) + \varphi(X) + l_s^2 \sqrt{|g| R \Phi(X)} \right], \quad (1)$$

where $l_s$ is the string length constant, $G_{\mu \nu}$ is a metric on $M$, $B_{\mu \nu}$ a two-form on $M$, $\varphi$ and $\Phi$ are scalar fields on $M$. The metric $g_{\alpha \beta}$ on $\Sigma$ is independent from the induced metric $\partial_\alpha X^\mu \partial_\beta X^\nu G_{\mu \nu}(X)$, but it should belong to the same signature class as the induced one. One can also introduce the open string ($\Sigma$ with boundaries) and the fermionic string coordinates ($M$ a super-manifold), but in order to keep the discussion simple, we will only consider the bosonic closed string.

The quantization procedure amounts to defining the following path integral

$$Z[G, B, \varphi, \Phi] = \int \mathcal{D}g \mathcal{D}X e^{iS(X)}, \quad (2)$$

which will be a functional of the spacetime fields $G_{\mu \nu}, B_{\mu \nu}, \varphi$ and $\Phi$. In order to define $Z$ one sets $\Phi = \text{const.}$ on $\Sigma$ and splits the spacetime fields as

$$G_{\mu \nu} = G_{0\mu \nu} + h_{\mu \nu}, \quad B_{\mu \nu} = B_{0\mu \nu} + b_{\mu \nu}, \quad (3)$$

such that the part quadratic in derivatives of the action $S(G_0, B_0, X)$, which we denote as $S_0(X)$, is solvable. One can then write

$$Z[G, B, \varphi] = \int \mathcal{D}g \mathcal{D}X e^{iS_0(X) + iS(h, b, \varphi, X)} = \int \mathcal{D}g \mathcal{D}X e^{iS_0(X)} \sum_{n=0}^{\infty} \frac{i^n}{n!} S^n_{1}(h, b, \varphi, X)$$
\[
\sum_{n=0}^{\infty} \frac{i^n}{n!} \int \mathcal{D}g \mathcal{D}X e^{iS_0(X)} S^n(h, b, \varphi, X)
\]
\[
= \sum_{n=0}^{\infty} \frac{i^n}{n!} \langle S^n(h, b, \varphi, X) \rangle ,
\]
so that the problem of defining \( Z \) is reduced to the problem of defining simpler path-integrals.

The next step is to expand the fields \( h, b \) and \( \varphi \) over a basis of functions on \( M \). This basis is chosen from the particle-like unitary representations of the isometry group of the metric \( G_0 \). For example, when \( M = \mathbb{R}^d \) and \( G_0 = \text{diag}(-, +, \ldots, +) \) the isometry group is a \( d \)-dimensional Poincare group \( \text{ISO}(d-1, 1) \), whose particle unitary irreps are labelled by a pair \( (m, s) \), where \( m \geq 0 \) is the mass and \( s \) is an \( SO(d-1) \) or \( SO(d-2) \) irrep, corresponding to the spin \( (m > 0) \) or the helicity of the particle \( (m = 0) \). Since \( \mathbb{R}^d = \text{ISO}(d-1, 1)/\text{SO}(d-1, 1) \), one takes the plane waves \( u_p(X) = e^{ip_\mu X^\mu} \) as the basis functions. Hence given a function \( f \) on \( M \) one can write
\[
f(X) = \int dp \ u_p(X) f(p) .
\]
In general case \( \int dp \) could be also a sum, which depends on the choice of \( M \) and the background metric isometry group. In particular, when \( b = \varphi = 0 \), the Fourier expansion (5) gives
\[
Z[G] = \sum_{n=0}^{\infty} \frac{i^n}{n!} \int dp_1 \cdots \int dp_n h_{\mu\nu}(p_1) \cdots h_{\rho\sigma}(p_n) A^{\mu\nu\cdot\cdot\cdot\rho\sigma}(p_1, \ldots, p_n) ,
\]
where
\[
A^{\mu\nu\cdot\cdot\cdot\rho\sigma}(p_1, \ldots, p_n) = \int d^2\sigma_1 \cdots \int d^2\sigma_n \langle V^{\mu\nu}_{p_1}(X(\sigma_1)) \cdots V^{\rho\sigma}_{p_n}(X(\sigma_n)) \rangle ,
\]
and \( V^{\mu\nu}_{p}(X(\sigma)) \) is the trace-free symmetric part of
\[
\sqrt{|g|} g^{\alpha\beta} \partial_\alpha X^\mu \partial_\beta X^\nu u_p(X) .
\]
The tensor \( V^{\mu\nu}_{p} \) represents the graviton vertex operator, while the trace part of (8) gives the vertex operator for the dilaton \( \phi \). The antisymmetric part of (8) is the B-field vertex operator. The vertex operator \( V_p = u_p(X) \) corresponds to the ground state scalar field \( \varphi \), the tachyon.
The quantity \( \langle V_1 \cdots V_n \rangle \) can be identified as a correlation function of primary fields and their descendants in a conformal field theory defined by the action \( S_0 \) on the surface \( \Sigma \). The amplitude

\[
A(p_1, \ldots, p_n) = \int_{\Sigma} d^2\sigma_1 \cdots \int_{\Sigma} d^2\sigma_n \langle V_1 \cdots V_n \rangle,
\]

is then interpreted as the amplitude for scattering of \( n \) quanta of the massless string states via the world-sheet \( \Sigma \) in the spacetime \( M \) with the background metric \( G_{0\mu\nu} \) and the background two-form \( B_{0\mu\nu} \).

Since \( \Phi = \text{const.} \), then the quantity \( \lambda = e^{-i\Phi} \) can be interpreted as the string theory coupling constant, so that the contributions to the scattering amplitudes from the world-sheets of various genera can be written as

\[
A_{\text{tot}}(p_1, \ldots, p_n) = \sum_{g \geq 0} \lambda^{2g-2+n} A_g(p_1, \ldots, p_n),
\]

where \( g \) is the genus of the surface \( \Sigma \). In analogy to the Feynman diagrams of particle field theories, the functional

\[
Z[G, B, \phi, \varphi] = \sum_{g,n} \lambda^{2g-2+n} Z_{g,n}[G, B, \phi, \varphi],
\]

where \( n = n_G + n_B + n_\phi + n_{\varphi} \), can be interpreted as the quantum effective action for the massless string modes [3, 4].

One of the interesting features of string theory is that the genus zero functional \( Z_0 \) can be expanded in the powers of the string length \( \ell_s \) as

\[
Z_0[G, B, \phi] = S_0[G, B, \phi] + \ell_s^2 S_1[G, B, \phi] + \ell_s^4 S_2[G, B, \phi] + \cdots,
\]

where \( S_0 \) is the Einstein-Hilbert action coupled to the dilaton \( \phi \) and the \( B \) field\(^\dagger\), while the higher-order terms contain the higher powers of the Ricci curvature \( R_{\mu\nu}(G) \). The higher-genus terms in (11) can be then interpreted as the quantum corrections to this classical action. This is why the string theory can be considered as a quantum theory of general relativity. However, it is not obvious that the amplitudes \( A_g \) will be finite for every genus \( g \). There are strong indications that this can be achieved in the superstring theory [2], although an explicit proof has not been given yet. So far, only the \( g \leq 2 \) amplitudes have been proven finite [1].

\(^\dagger\)Since the tachyon \( \varphi \) is not present in the superstring case, we omit it here.
3 The SU(2) WZW model

Let us analyze the SU(2) WZW model in the framework of previous section. In this case $M = S^3$ and the background metric $G_0$ is given by

$$Tr \int_{\Sigma} d^2\sigma \eta^{\alpha\beta} g^{-1} \partial_\alpha g g^{-1} \partial_\beta g ,$$  

(13)

where $g : \Sigma \rightarrow SU(2), \eta_{\alpha\beta}$ is a conformal class metric on $\Sigma$, while the background field $B_0$ is given by

$$Tr \int_{N} g^{-1} dg \wedge g^{-1} dg \wedge g^{-1} dg ,$$  

(14)

where $N$ is a compact three-manifold such that $\partial N = \Sigma$. The functions on $S^3$ can be expanded via the Peter-Weyl formula for the functions on $SU(2)$

$$f(g) = \sum_{j,m,\bar{m}} D^{(j)}_{m,\bar{m}}(g) f^m_{\bar{m}} ,$$  

(15)

where $2j = 0, 1, 2, ...$ and $-j \leq m, \bar{m} \leq j$. Hence the analogs of the $u_p(X)$ functions are the $D^{(j)}_{m,\bar{m}}(g)$ functions where $g = e^X$. The tachyon vertex operators will be then given by the primary fields

$$V^j_{m,\bar{m}}(\sigma) = D^{(j)}_{m,\bar{m}}(e^{X(\sigma)}),$$  

(16)

which correspond to the “particle” states $|j, m\rangle \otimes |j, \bar{m}\rangle \in V_j \otimes V^*_j$ of the isometry group $SU(2)$.

Let $\Sigma = S^1 \times \mathbb{R}$, then the first-quantized Hilbert space of states is given by

$$\mathcal{H}_s = \bigoplus_j \hat{V}_j \otimes \hat{V}^*_j ,$$  

(17)

where $\hat{V}_j$, and its dual $\hat{V}^*_j$ are the highest-weight irreps of the left and the right Kac-Moody (KM) algebras $J_+$ and $J_-$. If we fix the level number $k$ of the KM algebras, then there is a restriction $j \leq k/2$, because only then the corresponding irreps are unitary, i.e. there are no negative norm states.

Note that the structure of the space $\mathcal{H}_s$ can be interpreted also as a Hilbert space of the WZW model for the coset

$$SO(4)/SO(3) = SU(2) \times SU(2)/SU(2) .$$  

(18)

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In the particle limit, the physical irreps are \((j, j)\), which satisfy \(J_+^2 = J_-^2\), and they are the class-one irreps with respect to the \(SU(2)\) subgroup (i.e. they contain an \(SU(2)\) invariant vector, see [29]). One can also define a chiral WZW model Hilbert space as

\[
\mathcal{H}_+ = \sum_j \hat{V}_j .
\]  

(19)

The correlation functions

\[
\langle V_{m_1, m_1}^{\sigma_1} \cdots V_{m_n, m_n}^{\sigma_n} \rangle ,
\]

(20)
can be calculated on the sphere (\(\Sigma = S^2\)) via the Knizhnik-Zamolodchikov equation [16, 17], or by its generalization when \(\Sigma\) is a higher-genus surface [18]. A less rigorous approach, but more familiar to physicists, is to reduce (20) to gaussian functional integrations via the BRST gauge fixing procedure [19].

As far as the graviton, dilaton and the antisymmetric field correlation functions are concerned, it is natural to conjecture that they would be given by the irreducible parts of the vertex operator

\[
V_{j, m, m}^{a b} (\sigma) = J_+^a (\sigma) J_+^b (\sigma) D_{m, m}^{(j)} (e^X (\sigma)) ,
\]

(21)

where \(J_\pm\) are the WZW chiral currents.

In order to obtain the scattering amplitude, one would have to integrate the correlation functions over the surface \(\Sigma\). One expects to obtain a finite (or a renormalisable) result, because of the topological nature of three-dimensional gravity. But when \(\text{dim } M > 3\), one expects to obtain non-renormalisable divergencies, and one would have to use a super group extension of the isometry group in order to tame these divergencies, as it is done in the case of flat spacetime superstring theory. However, this program is difficult to implement, and as in the superstring case, it is not obvious that the arbitrary genus amplitudes will be finite. The idea of this paper is to use the connection between the states of the WZW model and the representations of the corresponding quantum group in order to give a more direct and simpler definition of the scattering amplitudes \(A_g\).

The quantum group connection comes from the following fact. Let \(\Sigma = S^2\), then

\[
\langle V_{m_1, m_1}^{\sigma_1} \cdots V_{m_n, m_n}^{\sigma_n} \rangle = \sum_i \psi_{m_1, \ldots, m_n}^{j_1 \ldots j_n (i)} (z_1, \ldots, z_n) \tilde{\psi}_{m_1, \ldots, m_n}^{\bar{j}_1 \ldots \bar{j}_n (i)} (\bar{z}_1, \ldots, \bar{z}_n) ,
\]

(22)
where the vectors $\psi^{(i)}$ (as well as the $\bar{\psi}^{(i)}$) form a basis in a subspace $C_k(j_1, \ldots, j_n)$ of the Hom$(j_1, \ldots, j_n)$ space, where $k$ is the level. The vector space $C_k$ is known as the space of conformal blocks, and $C_k$ is isomorphic to the Hom$(j_1, \ldots, j_n)$ space for the irreps of the quantum group $U_q(su(2))$ for $q = \exp \left( \frac{i\pi k}{k+2} \right)$. More generally, if $g$ is a simple Lie algebra, then the category of the integrable irreducible representations (irreps) of the affine Lie algebra $\hat{g}$, based on the finite-dimensional irreps of $g$, at the level $k \in \mathbb{Z}_+$ is equiv- alent as a modular tensor category to the category of the finite-dimensional irreps of the quantum group $U_q(g)$ for $q = \exp \left( \frac{i\pi m}{m(k+h)} \right)$, where $m \in \mathbb{N}$ and $h$ is the dual Coxeter number [21]. In the case of Lie algebras $sl(n)$ or $su(n)$, $m = 1$ and $h = n$.

One can then write

$$\psi^{j_1 \ldots j_n}_{m_1 \ldots m_n}(z) = v^{(k)}(z_1, \ldots, z_n) C^{j_1 \ldots j_n}_{m_1 \ldots m_n}(\iota),$$

where $C^{(i)}$ is the intertwiner tensor from the Hom$(j_1, \ldots, j_n)$ space, and similarly for $\bar{\psi}$, so that the scattering amplitude will be given as

$$A^{j_1 \ldots j_n}_{m_1 \ldots m_n} = \sum_i N_i(j_1, \ldots, j_n|k) C^{j_1 \ldots j_n}_{m_1 \ldots m_n}(\iota) \left( C^{j_1 \ldots j_n}_{m_1 \ldots m_n}(\iota) \right)^*,$$

where

$$N_i(j_1, \ldots, j_n|k) = \int dz_1 d\bar{z}_1 \cdots \int dz_n d\bar{z}_n v^{(k)}(z) \bar{v}^{(k)}(\bar{z}).$$

The constant $N_i$ can be identified as a norm of the vector $v_i$ from the $C_k$ space, and the value of this norm can be related to an evaluation of the $\theta_n$ spin network [20, 22]. On the sphere, this evaluation is determined up to a numerical constant, i.e. one can have many different evaluations. We will take a particular evaluation, which is suitable for our purposes, given by

$$N_i(j_1, \ldots, j_n|k) = \langle \theta(j_1, \ldots, j_n|k) \rangle_q = \exp \left( \frac{i\pi k}{k+2} \right),$$

where $\langle \theta \rangle_q$ is the evaluation defined by a 2d spin foam state sum invariant for the theta spin network embedded in a triangulation of $\Sigma$. This is motivated by the results from [27, 22, 7] on the relation between the spin networks, quantum groups and the state-sum models.

Note that the two intertwiners for the $\theta_n$ spin network can be different in general, and hence one should specify

$$N_{\iota \iota'}(j_1, \ldots, j_n|k) = \langle \theta(j_1, \ldots, j_n|\iota, \iota') \rangle_q = \exp \left( \frac{i\pi k}{k+2} \right).$$
Therefore the string amplitude will be given by
\[
A_{m_1m_1...m_nm_n} = \sum_{\text{id}} N_{\text{id}'} (j_1, ..., j_n; \kappa) C_{m_1...m_n}^{j_1...j_n(\iota)} (C_{m_1...m_n}^{j_1...j_n(\iota)\text{'}*}). \tag{28}
\]

## 4 String spin-foam model

The considerations of the previous sections suggest that one should construct a theta spin network evaluation which would depend on the string world-sheet, i.e. the surface \(\Sigma\), and its embedding into a spacetime, i.e. the group manifold \(G\). This evaluation should be a 2d topological invariant, because of the invariance under the world-sheet reparametrizations (diffeomorphisms of \(\Sigma\)). A natural way to obtain this invariant is to take the 2d BF theory for the Lie group \(G\), and consider the spin network observable
\[
\langle \theta(j_1, ..., j_n; \iota, \iota') \rangle = \int D\text{B} D\text{A} e^{i Tr \int_{\Sigma} B \wedge F} \theta(j_1, ..., j_n; \iota, \iota'|A), \tag{29}
\]
where \(\theta(j_1, ..., j_n; \iota|A)\) is the spin network function associated to the \(\theta_n\) spin network, i.e. a product of the holonomies along the edges of the spin network contracted by the intertwiners \(\iota\) and \(\iota'\) [8].

Note that one can construct other 2d diffeomorphism invariant path-integral expressions based on the BF theory. One can also take a non-topological modification of the BF theory, and then sum over the triangulations in order to make a diffeomorphism invariant. We have taken a simple choice, based on a topological theory, so that one would not need to perform a sum over the triangulations. However, we will see that the amplitudes one obtains from (29) are those of a topological string theory.

The expression (29) is a generalization of the expectation value of the Wilson loop, and in order to get an invariant, we need to define this path-integral. This can be done by integrating the \(B\) field, which gives a \(\delta(F)\), and then triangulating the \(\Sigma\), so that the \(A\) integration is replaced by the integration over the group elements (holonomies) associated to the dual lattice links [23, 8, 24, 25]. The spin network function \(\theta(A)\) is then given by a product of the corresponding representation matrix group elements (associated to the edges of the spin network) contracted by the appropriate intertwiner tensors (associated to the vertices of the spin network). The group integrations can be performed by using the following group theory formulas
\[
\delta(g) = \sum_{\Lambda} \dim \Lambda \chi_{\Lambda}(g), \tag{30}
\]
and
\[ \int_G dg D^{(\Lambda_1)}_{\alpha_1\beta_1}(g) D^{(\Lambda_2)}_{\alpha_2\beta_2}(g) = \frac{1}{\dim \Lambda_1} \delta_{\Lambda_1, \Lambda_2} C_{\alpha_1\alpha_2} (C_{\beta_1\beta_2})^* \] (31)
\[ \int_G dg D^{(\Lambda_1)}_{\alpha_1\beta_1}(g) \cdots D^{(\Lambda_n)}_{\alpha_n\beta_n}(g) = \sum_i C_{\alpha_1 \cdots \alpha_n}(i) (C_{\beta_1 \cdots \beta_n}(i))^*, \quad n \geq 3, \] (32)

where \( D^{(\Lambda)} \) is the group representation matrix in the representation \( \Lambda \), and \( \chi_\Lambda \) is the corresponding trace. One then obtains
\[ \langle \theta_n(j, \iota, \iota') \rangle = \sum_{\Lambda_f, \iota_e} A(\Lambda_f, \iota_e, j_1, \ldots, j_n, \iota, \iota') \] (33)

where \( \Lambda_f \) are the irreps associated to the faces of the dual two-complex for \( \Sigma \), \( \iota_e \) are the edge intertwiners coming from (32) and \( A \) is the amplitude for such a colored two-complex. This sum is called the spin foam state sum, because the faces of the two-complex remind us of a soap foam, and coloring by the irreps, i.e. the spins, gives a spin foam.

The amplitude \( A \) turns out to be a product of the group theory spin network evaluations, and this form of the amplitude can be obtained in the following way. Let \( \Gamma \) be the one-complex for a dual triangulation of \( \Sigma \), i.e. \( \Gamma \) is a trivalent graph. We can thicken \( \Gamma \) by replacing the edges by the ribbons, and we denote the corresponding ribbon graph as \( \tilde{\Gamma} \), see Fig. 1. Let us color every face (closed loop of \( \tilde{\Gamma} \)) by the irreps \( l_1, \ldots, l_L \). On the thickened graph draw the theta graph, such that the vertices of the theta graph coincide with the vertices of \( \Gamma \), and the edges of the theta graph run along the edges of \( \tilde{\Gamma} \), see Fig. 2. Then label the edges of the theta graph by the irreps \( j_1, \ldots, j_n \), as well as the two vertices of theta graph by the intertwiners \( \iota \) and \( \iota' \). In this way we obtain a colored \( \tilde{\Gamma}_c \) graph, see Fig. 2. Now shrink the middle of each edge of \( \tilde{\Gamma}_c \) to a point. In this way we obtain a collection of spin networks, whose number is equal to the number of the vertices of \( \Gamma \), see Fig. 3. To this configuration of spin networks we associate the following amplitude
\[ A = \prod_f \dim(l_f) \prod_e A_e(l_f, j) \prod_v A_v(l_f, \iota_e, j, \iota, \iota') . \] (34)

The edge amplitude \( A_e \) is given by \( \dim^{-1}(l_f(e)) \) if none of the theta graph edges pass through the ribbon edge \( e \), otherwise it is 1. The vertex amplitude \( A_v \) is given by the group theory spin network evaluation for the spin network at the vertex \( v \).
The group theory evaluation of a spin network is given by the product of the vertex intertwiner tensors $C^{\nu}(\iota)$, which are contracted by the $\delta^\beta_\alpha$, $C_{\alpha\beta}$ or $(C_{\alpha\beta})^* = C^{\alpha\beta}$ tensors associated to the edges. For example, the evaluation of the $\theta_3$ spin network is given by

$$\theta_3(\Lambda_1, \Lambda_2, \Lambda_3) = \sum_{\alpha, \beta, \gamma} C^{\Lambda_1\Lambda_2\Lambda_3}_{\alpha\beta\gamma} (C_{\alpha\beta\gamma})^*, \quad (35)$$

where $C$ are the normalized Clebsch-Gordan (CG) coefficients and

$$(C_{\alpha\beta\gamma})^* = C^{\alpha\mu} C^{\beta\nu} C^{\gamma\rho} = C^{\alpha\beta\gamma}. \quad (36)$$

In the formula (36) we have suppressed the $\Lambda$ indices on the CG tensors for simplicity. The normalization is such that when $\theta_3$ is non-zero, its value is one.

In the case of the $\theta_4$ spin network we have

$$\theta_4(\Lambda_1, \Lambda_2, \Lambda_3, \Lambda_4; \Lambda, \Lambda') = \sum_{\alpha, \beta, \gamma, \delta} C^{\Lambda_1\Lambda_2\Lambda_3\Lambda_4}_{\alpha\beta\gamma\delta} (C_{\alpha\beta\gamma\delta})^*, \quad (37)$$

where

$$C^{\Lambda_1\Lambda_2\Lambda_3\Lambda_4}_{\alpha\beta\gamma\delta} = \sum_{\rho\mu} C^{\alpha\mu\Lambda_2\Lambda_3\Lambda_4}_{\rho\beta\gamma\delta} C^{\rho\mu}_{\alpha\beta\gamma\delta}. \quad (38)$$

while the evaluation for the tetrahedral spin network is given by a normalized $6j$ symbol

$$Tetr(\Lambda_1, \ldots, \Lambda_6) = \sum_\alpha C^{\Lambda_1\Lambda_2\Lambda_3\Lambda_4\Lambda_5\Lambda_6}_{\alpha\alpha_1\alpha_2\alpha_3\alpha_4\alpha_5\alpha_6} C^{\alpha_1\alpha_2\alpha_3\alpha_4\alpha_5\alpha_6}_{\alpha_1'\alpha_2'\alpha_3'\alpha_4'\alpha_5'\alpha_6'}. \quad (39)$$

Let us now calculate the spin foam amplitude for a Wilson loop (a $\theta_2$ spin network) on the sphere. A triangulation of the sphere by four triangles can be represented by the Mercedes-Benz (MB) graph, whose ribbon version is shown in Fig. 1. The corresponding SF amplitude will be a product of

$$\prod_f A_f = \dim^3 l_1 \dim l_2, \prod_e A_e = \dim^{-3} l_1, \prod_v A_v = \dim l_1 (\theta_3(l_1, j, l_2))^3, \quad (40)$$

which can be seen from the Figs 2 and 3 by setting $j_1 = j_2 = 0, j_3 = j_4 = j$ and $l_1 = l_3 = l_4$. Therefore

$$\langle \theta_2(j) \rangle = \sum_{l_f} \prod_f A_f \prod_e A_e \prod_v A_v = \sum_{l_1, l_2} \dim l_1 \dim l_2 (\theta_3(l_1, j, l_2))^3. \quad (41)$$
Since \( \theta_2^3(a, b, c) = \theta_3(a, b, c) \) and \( d_a d_b = \sum_c N^c_{ab} d_c \) where \( \theta_3(a, b, c) = N^c_{ab} \) and \( d_a = \dim a \), we get

\[
\langle \theta_2(j) \rangle = d_j \sum_i d_i^2 .
\] (42)

The sum (42) is divergent and that will be a generic problem with the spin foam state sum (33). A topologically invariant way to regularize it is to use the fact that the spin network evaluations can be interpreted as traces of \( \mathbb{C} \)-linear morphisms in the tensor category \( \text{Cat}(G) \) of the irreps of the Lie group \( G \). Then the spin foam state sum \( \langle \theta_n \rangle \) can be understood as a trace of a functor defined by the spin network morphisms associated to the spin foam, which maps the \( \text{Cat}(G) \) to itself. This functor is topologically invariant because the spin network morphisms satisfy relations associated with the invariance under the Pachner moves (see the Appendix).

The irreps of the quantum group \( U_q(g) \) also form a tensor category \( \text{Cat}_q(G) \), and hence one can define the q-spin net evaluations as the traces of \( \mathbb{C} \)-linear morphisms defined by the same spin network graphs as in the \( \text{Cat}(G) \) case. As a result, the quantum spin network evaluations are given essentially by the same expressions as in the classical case, i.e. one replaces the intertwiner tensors \( C(\iota) \) by their quantum analogs, but one must take into account the over and the under-crossings, which give a non-trivial contribution in the quantum group case [27, 26, 28]. Therefore the quantum spin networks inherit the topological properties of the classical ones. When \( q \) is an appropriate root of unity, then one obtains a modular tensor category, which has a finite number of the irreducible objects. One can then define the analogous functor as in the classical case, which is now finite, and topologically invariant.

Hence let us define the invariant \( N_{\iota \iota'} \) as the state sum

\[
\langle \theta_n(j, t, t') \rangle = \sum_{l_f, l_{te}} \prod_f \Delta(l_f) \prod_e A^{(q)}(l_f, j) \prod_v A^{(q)}(l_f, l_{te}, j, t, t') ,
\] (43)

where \( \Delta(l) \) is the quantum dimension and \( A^{(q)} \) are the quantum evaluations of the spin networks appearing in the \( \text{Cat}(G) \) amplitude (34). In the \( SU(2) \) case, one can calculate the quantum spin networks evaluations via the representation theory of the \( U_q(sl(2)) \) for \( q \) a root of unity [26], and by using the fact that any spin network can be represented as the trace of a composition of the 3-morphisms (i.e. \( CG \) tensors) in the corresponding tensor category.

As far as the gravitons and the other excited states are concerned, to the best of our knowledge, we do not know of any result in the literature which
explores the connection between the corresponding affine Lie algebra representation states and the related quantum group representation states. A reasonable guess, based on the nature of our construction, is that the scattering of the excited states carrying the spins $s_i$ and the vacuum representations $j_i$, where $i = 1, 2, ..., n$, would be given by the expectation value of the $\theta_n(j)$ graph with the $s_i$ loops linked with the $j_i$ lines, see Fig. 9. This expectation value can be defined via the spin foam state sum for the corresponding graph drawn on the ribbon graph for the surface, see Fig. 9. The corresponding spin foam amplitude will have the same form as the amplitude in the state sum (43), but now the edge amplitude $A_e$ will be different when one or more $s$-loops sit on that edge. In that case the amplitude $A_e$ will be given by the corresponding evaluation of the theta spin network containing the $s_i$ links.

5 Calculation of the invariants

It is not difficult to show that in the case when $\Sigma = S^2$, the invariant $I$ is proportional to the usual $q$-spin network evaluation of the theta graph. The expressions for the $A$ amplitudes are the same as in the classical case, and the only difference is that the spin network evaluations are the quantum ones. In the $n = 2$ case (i.e. the Wilson loop), this can be seen from the formula (42), since in the quantum group case the sum $\sum_t d_t^2$ is finite. In the $SU_q(2)$ case for $q = e^{i\pi/k+2}$ one has $2l = 0, 1, 2, ..., k$ and the quantum Wilson loop evaluation (or the quantum dimension) is given by

$$\Delta_l = (-1)^{2l} \sin \frac{\pi (2l+1)}{k+2} \sin \frac{\pi}{k+2}, \quad (44)$$

so that

$$\sum_l \Delta_l^2 = \frac{k + 2}{2 \left( \sin \frac{\pi}{k+2} \right)^2} = c_k^2 \quad (45)$$

Therefore

$$\langle \theta_2(j) \rangle_{g=0} = c_k^2 \Delta_j \quad (46)$$

In the $n = 3$ case, we use the topological invariance of the sum, and instead of using the MB graph, which corresponds to triangulating a sphere with four triangles, we can triangulate the sphere with only two triangles.
The dual graph is the theta graph, see Fig. 4, and we obtain
\[ \prod_f A_f = \Delta_a \Delta_b \Delta_c \quad \text{and} \quad \prod_e A_e = 1 \quad , \]  
\[ \prod_v A_v = Tr(j_1, j_3, j_2, b, a, c) Tr(j_1, j_2, j_3, b, c, a) \quad , \]  
so that
\[ \langle \theta_3 \rangle_0 = \sum_{a,b,c} \Delta_a \Delta_b \Delta_c Tr(j_1, j_3, j_2, b, a, c) Tr(j_1, j_2, j_3, b, c, a) \quad . \]  
Due to ortogonality of the normalized 6j symbols [27]
\[ \sum_l \Delta_l Tr(i, j, l, a, b, c) Tr^*(i, j, l, a, b, c') = \Delta_c^{-1} \delta_{c,c'} N_{lc}^b N_{jc}^c \quad , \]  
and the symmetry under the permutations of the columns of the 6j symbol, the sum (49) becomes
\[ \langle \theta_3 \rangle_{g=0} = c_k^2 \theta_3(j_1, j_2, j_3) = c_k^2 N_{j_1j_2}^{j_3} \quad . \]  
By induction one can show that
\[ \langle \theta_n(j, l, l') \rangle_{g=0} = c_k^2 \theta_n(j, l, l') \quad , \]  
where \( \theta_n \) is a rescaled quantum evaluation of the \( \theta_n \) spin network, given by (A.3). Clearly the \( \theta_n \) spin network expectation value is independent of the way we have embed the \( \theta_n \) spin network in the ribbon graph, which is the consequence of the topological invariance of the state sum \( \langle \theta_n \rangle \) (see the Appendix).

The result (52) implies that the corresponding scattering amplitudes, given by the formula (28), are the amplitudes of a topological string theory. One can see this by taking \( G = U(1)^N \) and \( q = 1 \), so that the irrep labels become the discrete momenta, while the Clebsch-Gordon coefficients become the delta functions for the momenta meeting at a three-vertex. One can then calculate the four-point function \( A(p_1, ..., p_4) \) for the sphere, and one obtains
\[ A(p_1, ..., p_4) = \text{const.} \delta(p_1 + p_2 - p_3 - p_4) \quad , \]  
where the constant is infinite. It is plausible to assume that (53) will be the dominant term for \( q \) close to one, with the constant being a large finite

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number, which gives a topological theory amplitude. In the non-topological case, the constant in (53) will be replaced by a function of the momenta. For example, in the usual string theory, this function is given by the Koba-Nielsen amplitude $\Gamma(1 + s/2)\Gamma(1 + t/2)$, where $s = (p_1 + p_2)^2$ and $t = (p_2 + p_3)^2$.

In the case when $\Sigma$ is a torus, we can triangulate it by two triangles. The dual graph is then the theta graph with two edges over-crossed. The corresponding ribbon graph is the theta graph with each edge twisted, see Fig. 5. When we draw a Wilson loop on that ribbon graph, the corresponding spin-foam amplitude will be the product of

$$\prod_f A_f = \Delta_l, \quad \prod_e A_e = \Delta_l^{-1}, \quad \prod_v A_v = \theta_3(l, j, l)\theta_3(l, j, l),$$

which gives

$$\langle \theta_2(j) \rangle_{g=1} = \sum_l N_{j,l}.$$  \hspace{1cm} (55)

In the $\theta_3$ case we get

$$\langle \theta_3(j) \rangle_{g=1} = \sum_l \Delta_l (Tetr(j_1, j_2, j_3, l, l, l))^2,$$  \hspace{1cm} (56)

while in the $\theta_4$ case, we get

$$\langle \theta(j_1, j_2, j_3, j_4; i, j) \rangle_{g=1} = \sum_{l,p} \Delta_l Prism(j_1, j_4, i, j_2, j_3; l, p, l) Prism(j_4, j_1, j, j_3, j_2; l, p, l),$$

where $Prism$ is the evaluation of the spin network shown in Fig. 6.

In the case of a higher-genus surface we can use a simple triangulation associated to the decomposition of the genus $g$ surface into $g$ tori connected by $g - 1$ tubes. A triangulation of a torus with a hole can be represented by a twisted theta ribbon graph with a hole cut around one of the vertices and three ribbons attached to the border of that cut, see Fig. 7. By joining these ribbon graphs along the free ribbon edges, one obtains a trivalent ribbon graph whose Euler characteristic $V - E + L$ is $2 - 2g$, where $V$ is number of vertices, $E$ is the number of edges and $L$ is the number of loops. For example, in the $g = 2$ case one obtains a trivalent ribbon graph with 14 vertices, 21 edges and 5 loops, see Fig. 8.

As far as the amplitudes for the excited states are concerned, let us analyze the case of a $\theta_3$ spin network linked with three loops carrying the irreps
\[ s_1, s_2 \text{ and } s_3. \] On the sphere, see Fig. 9, we will have
\[ \langle \tilde{\theta}_3(j, s) \rangle = \sum_{a,b,c} \Delta_a \Delta_b \Delta_c Tetr(1, 3, 2, b, a, c) Tetr(1, 2, 3, b, c, a) \]
\[ \tilde{\theta}_3(c, (j_1, s_1), a) \tilde{\theta}_3(a, (j_2, s_2), b) \tilde{\theta}_3(b, (j_3, s_3), c) \cdot (58) \]

Since
\[ \tilde{\theta}_3(i, (j, s), l) = \theta_3(i, j, l) \frac{Hopf(j, s)}{\Delta_j} , \]
where \( Hopf \) is the evaluation of two linked Wilson loops, we obtain
\[ \langle \tilde{\theta}_3(j, s) \rangle = c_2^2 N_{j_1, j_2}^j \prod_{i=1}^{3} \frac{Hopf(j_i, s_i)}{\Delta_i} . \]

In terms of the modular S-matrix elements, which can be defined as
\[ S_{j l} = \frac{Hopf(j, l)}{c_k} , \]
we obtain
\[ \langle \tilde{\theta}_3(j, s) \rangle = c_2^5 N_{j_1, j_2}^j \prod_{i=1}^{3} \frac{S_{j_i, s_i}}{\Delta_i} . \]

In the \( SU(2) \) case these expressions can be calculated by using the formula
\[ S_{j l} = \sqrt{\frac{2}{k + 2}} \sin \left( \frac{(2j + 1)(2l + 1)\pi}{k + 2} \right) . \]

6 Conclusions

We have proposed to define the \( n \)-point scattering amplitude for a string theory in a curved spacetime which is determined by the group manifold of a Lie group \( G \), as a linear combination of expectation values of the \( \theta_n \) spin network (28). We have defined the \( \theta_n \) expectation values by the state sum (43) for the 2d spin foam model based on the quantum group \( G_q \), where \( q \) is an appropriate root of unity. This definition is very natural when one considers the scattering of string ground states, due to the known connection between the space of conformal blocks for the WZW model for the group \( G \) at the level \( k \) and the corresponding Hom spaces for the \( G_q \) irreps for \( q = \exp \frac{2\pi i}{k + h} \), where \( h \) is the dual Coexter number. However, one can argue
that this definition gives a topological string theory scattering amplitudes, see the eq. (53). Therefore a further work is necessary to modify the path-integral expression (29) in order to obtain a non-topological string theory amplitudes.

The ground states scattering amplitudes based on the expectation value of the $\theta_n$ spin networks can be naturally extended to the case of scattering of the excited states with spins (irreps) $s_i$, $1 \leq i \leq n$, by considering the $\theta_n$ spin network linked with the loops $s_i$. However, in this case the precise connection between the states of the $g$ integrable irrep and the corresponding quantum group $U_q(g)$ irrep is not understood, so that this issue would require more study in order to find a correct formulation for the excited states.

Note that $G$ can be also a non-compact group, as long as we use the category of finite-dimensional irreps. However, the particle-like representations used in string theory are unitary, and hence infinite-dimensional. In this case the complication is that the spin net evaluations like (35), (37) and (39) become infinite sums or improper integrals, and therefore there is no guarantee that they will be convergent. However, the work on the Lorentzian spin foam models has demonstrated that a large class of such spin network evaluations can be defined, i.e. the class of simple spin networks [31, 32]. Furthermore, one can pass to the quantum group representations, in which case the convergence properties are enhanced [28]. Note that in the $SO(d-1,2)$ case, one can construct unitary finite-dimensional irreps at roots of unity [34], so that all the spin network evaluations will be defined in that case. In the Lorentzian case, the state sum integral over the continuous parameter of the unitary irreps becomes an integral over a compact interval when $q$ is real, while it is expected that the root of unity irreps will have the discrete parameter truncated [31], just as in the compact group case. Hence there are strong indications that the spin-foam string amplitudes could be constructed in the case of unitary irreps for non-compact groups.

As far as the flat backgrounds are concerned, the relevant groups are $G = U(1)^N$ or $G = \mathbb{R}^N$ in the non-compact case, and one can explore these examples before going to the more difficult non-abelian cases, especially when the properties of the corresponding quantum torii and the quantum planes have been analyzed in the literature, see for example [35].

We have formulated our string model only for the spacetimes which are group manifolds. Although this set of manifolds contains a lot of physically
relevant ones, for example

\[ SO(2,1) = AdS_3 = S^1 \times \mathbb{R}^2, \quad SL(2, \mathbb{R}) = \mathbb{R}^3, \quad SL(2, \mathbb{C}) = \mathbb{R}^3 \times S^3, \]

e tc., one would like to find an analogous construction for the case of coset manifolds \(G/H\) since the Minkowski, de Sitter and anti-de Sitter spaces, given respectively by

\[ Mink_d = \frac{ISO(d-1,1)}{SO(d-1,1)}, \quad AdS_d = \frac{SO(d-1,2)}{SO(d-1,1)}, \quad dS_d = \frac{SO(d,1)}{SO(d-1,1)}, \]

are even more important examples of such spaces. The work on the higher-dimensional spin foam models revealed that in the case of \(G/H\) homogeneous spaces one can use the simple spin network evaluations [30, 29, 31], but the problem is that the corresponding state sums are not topologically invariant. In the 2d case the topological invariance means invariance under the world-sheet reparametrizations, and the only way to restore it would be to sum over the different triangulations of the surface \(\Sigma\), which is in general a difficult thing to do. Perhaps using the quantum simple spin networks [33, 28] could improve the convergence of that sum.

As far as the supersymmetry is concerned, it can be implemented straightforwardly by replacing the spacetime isometry group by a supergroup extension, for example \(SO(N) \to OSp(N, 2M)\).

Inclusion of boundaries in our formalism will be an important task, since this will be a way to treat the open strings and the D-branes. This could provide an interpretation in terms of quantum spin networks for the amplitudes of the topological strings propagating on Calabi-Yau manifolds [36].

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APPENDIX A

Let us demonstrate the topological invariance of the $\theta_n$ spin network state sum (43). First, we consider the case when there is no spin network. Then the state sum (43) becomes the partition function for the 2d BF theory. It is given by

$$Z_g = \sum_{\Lambda} \Delta_{\Lambda}^{L-E+V} = \sum_{\Lambda} \Delta_{\Lambda}^{2-2g} ,$$

(A.1)

which is obviously a topological invariant. However, it will be instructive to prove this by proving the invariance of the state sum under the 2d Pachner moves [37] by examining how the corresponding amplitude changes.

The invariance under the (2, 2) move can be represented graphically by Fig. 10, which is equivalent to

$$\Delta_a \frac{1}{\Delta_a} \Delta_a = \Delta_a \frac{1}{\Delta_a} \Delta_a .$$

The invariance under the (1, 3) move can be represented graphically by Fig. 11, which is equivalent to

$$\Delta_a = \Delta_a \Delta_a^{-3} \Delta_a^3 .$$

Note that the same equations are obtained when a $\theta_n$ spin network is embedded in $\Sigma$, if the Pachner moves act on the part of the surface graph where no $\theta_n$ edge passes.

Now let us analyze the action of the Pachner moves on a region of the surface graph where only one edge of the $\theta_n$ graph passes. The invariance under the (2, 2) move can be represented by Fig. 12, which is equivalent to

$$\theta_3(a, j, b) \theta_3(a, j, b) = \Delta_a^{-1} \Delta_a \theta_3(a, j, b) .$$

This is an identity because of $\theta_3^2 = \theta_3$. The invariance under the (1, 3) move can be represented graphically in two inequivalent ways, see Fig.13 and Fig. 14. The Fig. 13 implies

$$\theta_3(a, j, b) = \Delta_a \Delta_a^{-2} \Delta_a \theta_3(a, j, b) \theta_3(a, j, b) ,$$

while the Fig. 14 gives

$$\theta_3(a, j, b) = \Delta_b \Delta_b^{-1} \theta_3(a, j, b) \theta_3(a, j, b) \theta_3(a, j, b) .$$
These are again identities due to $\theta^2_3 = \theta_3$.

Now let us assume that the topological invariance holds for a region containing $m \geq 1$ edges of a $\theta_n$ spin network. We will prove that this implies the topological invariance for a region containing $m + 1$ edges. In order to prove this, let us first prove the following lemma:

**L1** Given a region with $m$ edges one can remove one edge from that region without changing the state sum.

**Proof:** It is sufficient to prove the invariance of the state sum under the elementary move represented in Fig. 15, since any larger move of a $\theta_n$ edge can be represented as a composition of the elementary moves and their mirror images. The Fig. 15 implies the identity represented in Fig. 16. Note that the quantum evaluation of the $\theta_n$ spin network is given by [26]

$$\theta(j_1, \ldots, j_n; I, L) = \frac{\theta(j_1, j_2, i_1)\theta(i_1, j_3, i_2)\cdots\theta(i_{n-3}, j_{n-1}, j_n)}{\Delta_{i_1}\Delta_{i_2}\cdots\Delta_{i_{n-3}}} \delta_{I, L}, \quad (A.2)$$

where $I = (i_1, \ldots, i_{n-3})$, $L = (l_1, \ldots, l_{n-3})$ and $n \geq 4$. The identity in Fig. 16 is satisfied if

$$\theta(j_1, \ldots, j_n; I, L) = \theta(j_1, j_2, i_1)\theta(i_1, j_3, i_2)\cdots\theta(i_{n-3}, j_{n-1}, j_n) \delta_{I, L} \quad (A.3)$$

Since the evaluation (A.3) is a rescaled quantum evaluation (A.2), this means that the vertex amplitude for the $\theta_n$ spin network must be rescaled as

$$\theta(j_1, \ldots, j_n; I, L) \rightarrow \Delta_{i_1}\cdots\Delta_{i_{n-3}} \theta(j_1, \ldots, j_n; I, L) \quad (A.4)$$

in order to have an invariant state sum.

Now, given a region with $m + 1$ $\theta_n$ edges, by the lemma L1 one can remove one $\theta_n$ edge from that region without changing the state sum. By the assumption, any region containing $m \theta_n$ edges is invariant under the Pachner moves, and hence the larger region which contains the $m$ edges plus the one which was moved will be invariant under the Pachner moves.

In order to complete the proof, one should also demonstrate the invariance under the movement of the vertices of the $\theta_n$ spin network. This is easy to establish by examining the state-sum invariance under the movement of one vertex of the $\theta_n$ spin network from a vertex $v_1$ of the ribbon graph to a vertex $v_2$ connected by a ribbon edge. This is equivalent to an equation that the evaluation of the vertex spin network at $v_1$ contracted by the $\theta_m(v_1)$ evaluation is the same as the evaluation of the vertex spin network at $v_2$ contracted by
the $\theta_{m(n)}$ evaluation. Since we use the rescaled evaluation for $\theta_m$, given by the eq. (A.3), which is essentially the delta function of the intertwiners, it is easy to verify the equation for the $\theta_n$ vertex movement. QED

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Figure I: Ribbon graph for a sphere triangulated by four triangles.
Figure II: Colored ribbon graph for a sphere with an embedded $\theta_4$ spin network.

Figure III: Vertex spin networks for the ribbon graph from Fig. 2.
Figure IV: Ribbon graph for a sphere triangulated by two triangles with an embedded $\theta_3$ spin network.

Figure V: Ribbon graph for a torus triangulated by two triangles.
Figure VI: Vertex spin network for a $\theta_4$ spin network embedded in a torus.

Figure VII: Ribbon graph for a triangulated torus with a hole.
Figure VIII: Dual one-complex for a triangulation of a genus two surface.

Figure IX: Ribbon graph for a sphere with an embedded theta spin network linked with three Wilson loops.
Figure X: Invariance of the state sum under the $(2, 2)$ Pachner move which acts on a region where none of the edges of the $\theta_n$ spin network pass.

Figure XI: Invariance of the state sum under the $(1, 3)$ Pachner move which acts on a region where none of the edges of the $\theta_n$ spin network pass.
Figure XII: Invariance of the state sum under the $(2, 2)$ Pachner move which acts on a region where one of the $\theta_n$ spin network edges pass.

Figure XIII: Invariance of the state sum under the $(1, 3)$ Pachner move which acts on a region where one of the $\theta_n$ spin network edges pass.
Figure XIV: Another form of the invariance under the (1, 3) Pachner move from Fig. 13.

Figure XV: Invariance of the state sum under the elementary move of one of the $\theta_n$ spin network edges.
Figure XVI: Spin network identity following from the elementary move invariance from Fig. 15.