THE ROLE OF HARNACK EXTENSION IN THE KURZWEIL-STIELTJES SENSE: INTEGRATING FUNCTIONS OVER ARBITRARY SUBSETS

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Abstract. The Harnack extension principle discussing a sufficient condition for the integrable functions on particular subsets of $(a, b)$ to be integrable on $[a, b]$ is already included in the Kurzweil-Henstock integral (see e.g. Theorem 1.1). The Kurzweil-Stieltjes integral reduces to the Kurzweil-Henstock integral whenever the integrator is an identity function. It is known that if the integrator $F$ is discontinuous on $[c, d] \subset [a, b]$, then the values of the Kurzweil-Stieltjes integrals

$$
\int_c^d (dF) g, \quad \int_{[c,d]} (dF) g, \quad \int_{[c,d)} (dF) g, \quad \text{and} \quad \int_{(c,d]} (dF) g
$$

need not coincide (see [37, Section 5]). This indicates that the Harnack extension principle in the Kurzweil-Henstock integral cannot be valid any longer for the Kurzweil type Stieltjes integrals with discontinuous integrators. The concepts of equiintegrability and equiregulatedness are pivotal to the notion of Harnack extension for the Kurzweil-Stieltjes integration.

Moreover, it is also known that, in general, the existence of the integral $\int_a^b (dF) g$ does not (even in the case of the identity integrator $F(x) = x$) always imply the existence of the integral $\int_T (dF) g$ for every subset $T$ of $[a, b]$. This follows from the well-known fact that, if e.g. $T \subset [a, b]$ is not measurable, then the existence of the Lebesgue integral $\int_a^b g \, dt$ (which is a particular case of the Kurzweil-Henstock one) does not imply, in general, that also the integral $\int_T g \, dt$ exists. Therefore, besides having an interest in constructing Harnack extension principle for the Kurzweil-Stieltjes integral, the aim of this paper is to investigate the existence of the integrals $\int_T (dF) g$ for arbitrary closed subsets $T$ of an elementary set $E$.

1. Introduction

One of the meaningful discussions in the topic of Kurzweil-Henstock integration concerns the Harnack extension principle and the Cauchy property (see e.g. [30, Corollaries 7.10 and 7.11] and [32, Theorems 1.4.6, 1.4.8 and 4.4.4]). The Cauchy property was first used for the Riemann integral in order to integrate functions unbounded in the neighborhood of a finite number of points (see e.g. [18, Theorems 2.12 to 2.16]). A similar idea has been also applied to integrate in a Lebesgue sense...
functions not summable in the neighborhood of some points. Based on the Cauchy property, C. G. A. Harnack suggested a way how to calculate integrals of functions defined on an open set. The Cauchy property in the integral theory gives a sufficient condition for the integrable functions on each \((c, d) \subset (a, b)\) to be integrable on \([a, b]\) (see e.g. [23], [30], [31], [32]). In the setting of the Kurzweil-Henstock integral for real-valued functions, the Harnack extension principle reads as follows (see e.g. [12, Theorem 9.22], [30, Corollary 7.11], [32, Theorem 4.4.4]):

**Theorem 1.1.** Let \(T \subset [a, b]\) be a closed set and let \(\{[a_i, b_i] : i \in \mathbb{N}\}\) be a collection of pairwise disjoint intervals such that \((a, b) \setminus T = \bigcup_{i=1}^{\infty} (a_i, b_i)\). Then, if \(g\) is a real-valued function and the Kurzweil-Henstock integrals \(\int_{a}^{b} g \chi_{T} \, dt\) and \(\int_{a_i}^{b_i} g \, dt\) exist for all \(i \in \mathbb{N}\) and the series

\[
\sum_{i=1}^{\infty} \sup \left\{ \left| \int_{a_i}^{b_i} g \, dt \right| : a_i \leq r \leq t \leq b_i \right\}
\]

converges, then the Kurzweil-Henstock integral \(\int_{a}^{b} g \, dt\) exists and

\[
\int_{a}^{b} g \, dt = \int_{a}^{b} g \chi_{T} \, dt + \sum_{i=1}^{\infty} \int_{a_i}^{b_i} g \, dt.
\]

The original definition of the gauge based integral, in general non-separated functions of two variables, was given by J. Kurzweil in the late 1950’s and published in his paper on generalized ordinary differential equations as an alternative definition of the Perron/Denjoy integral (see e.g. [25], [29]). In the early 1960’s, R. Henstock independently rediscovered the analogous definition of integral, and developed it into a systematic theory (see e.g. [15], [16], [17]). Nowadays, the integral is known as the Kurzweil-Henstock (or Henstock-Kurzweil) integral (see e.g. [9], [24], [28], [43]). Its definition is based on the Riemannian type sums and refinements controlled by gauges, for that known also as the generalized Riemann integral or the gauge integral, leads to a nonabsolutely convergent integral which is more powerful than the Lebesgue integral, and contains as a special case also the Stieltjes type integrals. Throughout this paper we work with the Kurzweil-Stieltjes integrals. The simplest integral of this type is the Riemann-Stieltjes integral of the form \(\int_{a}^{b} [d f] g\), in which a function \(g: [a, b] \to \mathbb{R}\) called the integrand is integrated with respect to another function \(f: [a, b] \to \mathbb{R}\) referred to as the integrator (see e.g. [12], [38], [41], [51]). This integral appeared for the first time in a famous treatise [51] by T. J. Stieltjes. Up to now, many authors considered various kinds of the Stieltjes integral using the gauge integration (see e.g. [14], [20], [35], [38], [41], [46], [52]) and becoming more popular in the field of differential equations and other applications (see e.g. [1], [5], [6], [9], [10], [21], [22], [33], [34], [42], [50], [54]). In the literature these integrals are known under several different names (Henstock-Stieltjes, Perron-Stieltjes, generalized Riemann-Stieltjes, etc.). To our opinion all of these integrals are special cases of the Kurzweil integral from [25] or [29]. Therefore, we prefer to call this integral \textit{the Kurzweil-Stieltjes integral}. 

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The Kurzweil-Henstock integral has been generalized in various ways; for instance, Sergio S. Cao ([7]) noticed that the Kurzweil’s definition can be easily extended to functions with values in Banach spaces and investigated some of the properties of the abstract Kurzweil-Henstock integral. This abstract Kurzweil-Henstock integral received further attention, for instance the monograph by Š. Schwabik and G. Ye [49] discussing these types of integrals, i.e. the McShane, Bochner, Dunford, and Pettis integrals for Banach space-valued functions, and proceeding to compare the relationship between these various integrals. Moreover, the fundamental results concerning the Kurzweil-Stieltjes integral for Banach space-valued functions integral were given by Š. Schwabik in [46] and [48], where he called it the abstract Perron-Stieltjes integral. These results obtained by Š. Schwabik have undergone expansion by G. A. Monteiro and M. Tvrdý completing them to theory such that it was applicable to prove some of the results on the continuous dependence of solutions to generalized linear differential equations in a Banach space (see [35] and [36]).

Convergence theorems for the integral concern the possibility of interchanging the limit and the integral (see e.g. [12], [13], [14], [31], [41]). Extension of the Kurzweil-Stieltjes integral to the integration over elementary sets, i.e. sets that are finite unions of bounded intervals, was given in [37, Section 5], where it was useful ingredient for proving the bounded convergence theorem for the abstract Kurzweil-Stieltjes integral. Unfortunately, by [37, Theorems 5.8, 5.9 and 5.10, Remark 5.12, Theorem 5.13], it seems that the Harnack extension principle for the Kurzweil-Henstock integral, see e.g. Theorem 1.1, cannot be easily extended to the integrals of the Stieltjes type as, whenever the integrator $F$ is not continuous on $[a, b]$, for a subinterval $J \subset [a, b]$ having an infimum and a supremum $c$ and $d$, respectively, the integrals

$$
\int_c^d [dF] g, \quad \int_{[c,d]} [dF] g, \quad \int_{(c,d]} [dF] g, \quad \int_{(c,d)} [dF] g, \quad \text{and} \quad \int_{[c,d)} [dF] g
$$

need not have the same values, even if they all exist (see [37, Remark 5.12]). The reason is the fact that the Stieltjes integrals over a one-point set need not be zero (see e.g. [37, Proposition 5.7]).

From the study of convergence theorems for gauge type integrals appeared the notion of equiintegrability whose idea behind was that there exists a single gauge $\delta$ which works for all the functions in the sequence (see e.g. [1], [13], [27], [38], [39], [47], [49]). Beside extending the definition of the Kurzweil-Stieltjes integral over arbitrary bounded sets (see Section 3), in order to deal with Harnack extension principle for the Kurzweil-Stieltjes integral based on [37], it is necessary to develop the notion of equiintegrability for Banach space-valued functions and investigate their fundamental properties, including some important results regarding equiregulatedness, proved in Section 4. Furthermore, we show that the theory from Sections 3 and 4 leads to a new Harnack extension principle for the Kurzweil-Stieltjes integral, which significantly improves the results from [37]. Therefore, the goal of this paper is to provide sufficient conditions vouching Harnack extension principle for
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the Kurzweil-Stieltjes integral and then to show that it is applicable to integrate functions over arbitrary closed subsets of an elementary set as shown in Section 5.

2. Preliminaries

In this section we will recall some terminology and notations commonly used in the literature.

Let $X$, $Y$, $Z$ be Banach spaces. As usual, the symbols $\| \cdot \|_X$, $\| \cdot \|_Y$, $\| \cdot \|_Z$ stand for the norm in $X$, $Y$, $Z$, respectively.

If there are bilinear mapping $B : X \times Y \to Z$ and $\beta \in [0, \infty)$ such that

$$\| B(x, y) \|_Z \leq \beta \| x \|_X \| y \|_Y$$

for $x \in X$, $y \in Y$, we say that the triple $(X, Y, Z)$ is a bilinear triple with respect to $B$. In such a case, we write $B = (X, Y, Z)$ and use the abbreviation $x \ y$ for $B(x, y)$.

Besides a classical situation with $X = Y = Z = \mathbb{R}$, a typical nontrivial example is e.g. $B = (\mathcal{L}(X, Z), X, Z)$ where $\mathcal{L}(X, Z)$ is the space of all linear bounded operators $L : X \to Z$, while $B(L, x) = L \ x \in Z$ for $x \in X$ and $L \in \mathcal{L}(X, Z)$. It is easy to see that, without any loss of generality we may assume that $\beta = 1$.

Two intervals in $\mathbb{R}$ are said to be disjoint if their intersection is empty, while they are said to be non-overlapping if their intersection contains at most one point. In this paper, by an elementary set we understand a finite union of mutually disjoint bounded intervals. Note that bounded intervals are themselves elementary sets.

A finite set $\alpha = \{\alpha_0, \alpha_1, \ldots, \alpha_m\} \subset [a, b]$ with $m \in \mathbb{N}$ is said to be a division of the interval $[a, b]$ if

$$a = \alpha_0 < \alpha_1 < \ldots < \alpha_m = b.$$

The set of all divisions of $[a, b]$ is denoted by $\mathcal{D}[a, b]$. The symbol $\nu(\alpha)$ will be kept for the number of subintervals $[\alpha_{j-1}, \alpha_j]$ generated by the division $\alpha$, i.e. $\nu(\alpha) = m$ in the above case.

Let $f : [a, b] \to X$ be a function with values in a Banach space $X$. As in the case of the real valued functions, the variation of $f$ on $[a, b]$ is defined by

$$\text{var}_a^b f = \sup_{\alpha \in \mathcal{D}[a, b]} \left\{ \sum_{j=1}^{\nu(\alpha)} \| f(\alpha_j) - f(\alpha_{j-1}) \|_X \right\}.$$

If $\text{var}_a^b f < \infty$, then we say that $f$ has a bounded variation on $[a, b]$. $BV([a, b], X)$ is the set of all functions $f : [a, b] \to X$ of bounded variation on $[a, b]$.

Let $B = (X, Y, Z)$ be a bilinear triple. For $f : [a, b] \to X$ and a division $\alpha = \{\alpha_0, \alpha_1, \ldots, \alpha_m\}$ of $[a, b]$ we define

$$(B) \ V_a^b(f, \alpha) := \sup \left\{ \left\| \sum_{j=1}^{\nu(\alpha)} [f(\alpha_j) - f(\alpha_{j-1})] y_j \right\|_Z : y_j \in Y, \| y_j \|_Y \leq 1, \ j = 1, 2, \ldots, \nu(\alpha) \right\}.$$
and 

\[(B) \var^b \alpha f = \sup \left\{ (B)V^b_\alpha(f, \alpha) : \alpha \in \mathcal{D}[a, b] \right\}.\]

A function \(f: [a, b] \rightarrow X\) with \((B) \var^b \alpha f < \infty\) is said to have a bounded \(B\)–variation on \([a, b]\) or also to have a bounded semi-variation. The set of all functions \(f: [a, b] \rightarrow X\) with bounded \(B\)–variation on \([a, b]\) is denoted by \((B)BV([a, b], X)\).

\(G([a, b], X)\) denotes the set of all \(X\)-valued functions which are regulated on \([a, b]\). Recall that \(f: [a, b] \rightarrow X\) is regulated on \([a, b]\) if for each \(t \in [a, b]\) there is a \(f(t+) \in X\) such that

\[
\lim_{s \rightarrow t^+} ||f(s) - f(t+)||_X = 0,
\]

and for each \(t \in (a, b)\) there is a \(f(t-) \in X\) such that

\[
\lim_{s \rightarrow t^-} ||f(s) - f(t^-)||_X = 0.
\]

For \(f \in G([a, b], X)\) and \(t \in [a, b]\) we denote \(\Delta^+ f(t) = f(t+) - f(t), \Delta^- f(t) = f(t) - f(t-)\) and \(\Delta f(t) = f(t+) - f(t-)\) (where by convention \(\Delta^- f(a) = \Delta^+ f(b) = 0\)).

Let \(B = (X, Y, Z)\) be a bilinear triple. A function \(f: [a, b] \rightarrow X\) is called \(B\)–regulated on \([a, b]\) (or simply-regulated on \([a, b]\)) if the function \(fy : t \in [a, b] \rightarrow f(t)y \in Z\) is regulated for all \(y \in Y\). The set of all simply-regulated functions \(f: [a, b] \rightarrow X\) is denoted by \((B)G([a, b], X)\).

Clearly, \(G([a, b], X) \subset (B)G([a, b], X)\). Moreover, it is known that

\[BV([a, b], X) \subset G([a, b], X) \quad \text{and} \quad BV([a, b], X) \subset (B)BV([a, b], X)\]

A finite set of points in \([a, b]\)

\[P = \{\alpha_0, \xi_1, \alpha_1, \xi_2, \ldots, \alpha_{m-1}, \xi_m, \alpha_m\}\]

where \(\{\alpha_0, \alpha_1, \ldots, \alpha_m\} \in \mathcal{D}[a, b]\) and \(\xi_j \in [\alpha_{j-1}, \alpha_j]\) for \(j = 1, 2, \ldots, \nu(P)\) is called a tagged partition of \([a, b]\). The point \(\xi_j\) is called tag of the subinterval \([\alpha_{j-1}, \alpha_j]\) for every \(j = 1, 2, \ldots, \nu(P)\). We then shall write

\[P = \{([\alpha_{j-1}, \alpha_j], \xi_j)\} \quad \text{or} \quad P = (\alpha, \xi)\]

with \(\alpha = \{\alpha_0, \alpha_1, \ldots, \alpha_m\}, \xi = \{\xi_1, \xi_2, \ldots, \xi_m\}\) and \(\nu(P) = \nu(\alpha)\).

Positive functions \(\delta: [a, b] \rightarrow (0, \infty)\) are called gauges on \([a, b]\). For a given gauge \(\delta\) on \([a, b]\), a tagged partition \(P = \{([\alpha_{j-1}, \alpha_j], \xi_j)\}\) of \([a, b]\) is called \(\delta\)–fine if

\[[\alpha_{j-1}, \alpha_j] \subset (\xi_j - \delta(\xi_j), \xi_j + \delta(\xi_j)) \quad \text{for} \quad j = 1, 2, \ldots, \nu(P)\].

The following lemma shows that the set of \(\delta\)–fine partitions is nonempty and this result is known as the Cousin lemma (see e.g. [3], [12, Lemma 9.2], [30, Theorem 2.3.1], [32, Theorem 1.1.5]).

**Lemma 2.1. [Cousin]** Given an arbitrary gauge \(\delta\) on \([a, b]\) there is a \(\delta\)–fine partition of \([a, b]\).
If $\mathcal{B} = (X, Y, Z)$ is a bilinear triple, then for functions $f: [a, b] \to X$, $g: [a, b] \to Y$ and a tagged partition $P = \{(\alpha_j, \xi_j)\}$ of $[a, b]$, we set
\[
S(\nu(P), f, g, P) = \sum_{j=1}^{\nu(P)} [f(\alpha_j) - f(\alpha_{j-1})] g(\xi_j)
\]
and
\[
S(f, \nu(P), d^g, P) = \sum_{j=1}^{\nu(P)} f(\xi_j)[g(\alpha_j) - g(\alpha_{j-1})].
\]

Now, we can present the definition of the abstract Kurzweil-Stieltjes integral as introduced by Š. Schwabik in [46, Definition 5].

**Definition 2.2.** Let $\mathcal{B} = (X, Y, Z)$ be a bilinear triple and let $f: [a, b] \to X$ and $g: [a, b] \to Y$ be given. We say that the Kurzweil-Stieltjes integral (shortly KS-integral) $\int_a^b [df] g$ exists if there is $I \in Z$ such that for every $\varepsilon > 0$ there is a gauge $\delta$ on $[a, b]$ such that
\[
\left\| S(\nu(P), f, g, P) - I \right\|_Z < \varepsilon
\]
holds for every $\delta$–fine partition $P$ of $[a, b]$. In such a case we put
\[
\int_a^b [df] g = I.
\]
Furthermore, we define
\[
\int_a^a [df] g = 0 \quad \text{and} \quad \int_a^b [df] g = -\int_b^a [df] g \quad \text{if} \quad b < a.
\]

Similarly, if $f: [a, b] \to X$ and $g: [a, b] \to Y$, then $\int_a^b f [dg] = I \in Z$ if and only if for every $\varepsilon > 0$ there is a gauge $\delta$ on $[a, b]$ such that
\[
\left\| S(f, \nu(P), d^g, P) - I \right\|_Z < \varepsilon
\]
holds for every $\delta$–fine partition $P$ of $[a, b]$.

The Kurzweil-Stieltjes integral is well defined by Definition 2.2 thanks to Lemma 2.1. The existence of the Kurzweil-Stieltjes integral is guaranteed, see e.g. [46, Proposition 15]. Obviously, it reduces to the Kurzweil-Henstock integral whenever the integrator $f$ in (2.1) (the integrator $g$ in (2.2)) is an identity function.

Throughout the paper, we assume that $\mathcal{B} = (X, Y, Z)$ is a bilinear triple. Furthermore, $[a, b]$ is a fixed bounded and closed interval in $\mathbb{R}$. All functions $f$ are supposed to be defined on the whole interval $[a, b]$ and extended outside the interval $[a, b]$ in such a way that $f(t) = f(a)$ and $f(s) = f(b)$ for $t < a$ and $s > b$. 
3. Integration over Arbitrary Bounded Sets

In [37, Section 5] the Kurzweil-Stieltjes integral of operator-valued functions over elementary subsets of \([a, b]\) was introduced and its basic properties were described. Of course, this definition can be easily extended to arbitrary subsets of \([a, b]\) and to setting in a general bilinear triple \(\mathcal{B} = (X, Y, Z)\).

**Definition 3.1.** Let \(f : [a, b] \rightarrow X\), \(g : [a, b] \rightarrow Y\) and let \(S\) be an arbitrary subset of \([a, b]\). Then the Kurzweil-Stieltjes integral (shortly KS-integral or integral) of \(g\) with respect to \(f\) over the set \(S\), denoted by \(\int_S [df] g\), is defined by
\[
\int_S [df] g := \int_a^b [df] (g\chi_S)
\]
whenever the integral on the right hand side exists.

Similarly, if \(f : [a, b] \rightarrow X\), \(g : [a, b] \rightarrow Y\), then the integral \(\int_S f [dg]\) is defined by
\[
\int_S f [dg] := \int_a^b (f\chi_S) [dg]
\]
whenever the integral on the right hand side exists.

**Remark 3.2.** By Definitions 2.2 and 3.1 the existence of the integral \(\int_S [df] g\) means that there exists \(I \in Z\) with the following property: for every \(\varepsilon > 0\) there exists a gauge \(\delta\) on \([a, b]\) such that
\[
\left\| S((df, g\chi_S), P) - I \right\|_Z < \varepsilon
\]
whenever \(P\) is a \(\delta\)-fine partition of \([a, b]\).

Definition 5.1 from [37] is a special case of Definition 3.1. However, it is easy to see that all the results presented in [37] for the special case \(\mathcal{B} = (\mathcal{L}(X, Z), X, Z)\) can be reformulated for the setting of this paper with a general bilinear triple \(\mathcal{B} = (X, Y, Z)\). In particular, the following Propositions 3.3 and 3.5 are valid.

**Proposition 3.3.** Let \(S\) be an arbitrary subset of \([a, b]\). Then the following assertions are true:

(i) Let \(f : [a, b] \rightarrow X\) and \(g_i : [a, b] \rightarrow Y\), \(i = 1, 2\), be such that the integrals \(\int_S [df] g_i\) for \(i = 1, 2\) exist. Then the integral \(\int_S [df] (c_1 g_1 + c_2 g_2)\) exists as well and
\[
\int_S [df] (c_1 g_1 + c_2 g_2) = c_1 \int_S [df] g_1 + c_2 \int_S [df] g_2
\]
for all \(c_1, c_2 \in \mathbb{R}\).

(ii) Let \(f_i : [a, b] \rightarrow X\), \(i = 1, 2\), and \(g : [a, b] \rightarrow Y\) be such that the integrals \(\int_S [df_i] g\) for \(i = 1, 2\) exist. Then the integral \(\int_S [d (c_1 f_1 + c_2 f_2)] g\) exists as well and
\[
\int_S [d (c_1 f_1 + c_2 f_2)] g = c_1 \int_S [df_1] g + c_2 \int_S [df_2] g
\]
for all $c_1, c_2 \in \mathbb{R}$.

**Remark 3.4.** Since $g\chi_{[a,b]} = g$ on $[a, b]$, it is clear that $\int_a^b [df] g$ exists if and only if the integral $\int_{[a,b]} [df] g$ exists and in such a case these integrals have the same value, i.e.

$$\int_{[a,b]} [df] g = \int_a^b [df] g. \quad (3.1)$$

On the other hand,

$$g(t)\chi_{(a,b)}(t) - g(t) = \begin{cases} 
-g(a) & \text{if } t = a, \\
0 & \text{if } t \in (a, b), \\
-g(b) & \text{if } t = b.
\end{cases}$$

and hence, by [46, Lemma 12] we get for an arbitrary $d \in (a, b)$

$$\int_a^b [df] (g\chi_{(a,b)} - g) = \int_a^d [df] (g\chi_{(a,b)} - g) + \int_d^b [df] (g\chi_{(a,b)} - g)$$

$$= -\left( \lim_{r \to a^+} [f(r)g(a)] - f(a)g(a) \right) - (f(b)g(b) - \lim_{r \to b^-} [f(r)g(b)])$$

i.e. the integral $\int_a^b [df] g$ exists if and only if the integral $\int_{(a,b)} [df] g$ exists and in such case

$$\int_{(a,b)} [df] g = \int_a^b [df] g$$

$$+ f(a)g(a) - f(b)g(b) + \lim_{r \to a^+} [f(r)g(b)] - \lim_{r \to b^-} [f(r)g(a)]. \quad (3.2)$$

Next proposition summarizes the properties of the KS-integral over all possible kinds of subintervals of $[a, b]$. The proofs of its assertions are easy modifications of those of [37, Theorems 5.8, 5.10 and 5.11]. The above observations concerning the cases $c = a$ and/or $d = b$ will be included having in mind the convention that the functions $f$ and $g$ are to be considered as extended outside of the interval $[a, b]$ as constant functions on $(-\infty, a] \cup [b, \infty)$.

**Proposition 3.5.** Let $f \in (B)G([a,b]; X)$, $g: [a, b] \to Y$ and $a \leq c < d \leq b$. Then the following assertions are true:

(i) The integral $\int_{(c,d)} [df] g$ exists if and only if the integral $\int_c^d [df] g$ exists. In such a case,

$$\int_{(c,d)} [df] g = f(c)g(c) - \lim_{r \to c^+} [f(r)g(c)] + \int_c^d [df] g + f(c)g(c) - f(d)g(d) + \lim_{r \to d^-} [f(r)g(d)].$$
(ii) The integral \( \int_{[c,d]} [df] g \) exists if and only if the integral \( \int_c^d [df] g \) exists. In such a case,
\[
\int_{[c,d]} [df] g = f(c) g(c) - \lim_{r \to c^-} [f(r) g(c)] + \int_c^d [df] g \\
+ f(d) g(d) - \lim_{r \to d^+} [f(r) g(d)].
\]

(iii) The integral \( \int_{[c,d]} [df] g \) exists if and only if the integral \( \int_c^d [df] g \) exists. In such a case,
\[
\int_{[c,d]} [df] g = f(c) g(c) - \lim_{r \to c^-} [f(r) g(c)] + \int_c^d [df] g \\
+ \lim_{r \to d^+} [f(r) g(d)] - f(d) g(d).
\]

(iv) The integral \( \int_{[c,d]} [df] g \) exists if and only if the integral \( \int_c^d [df] g \) exists. In such a case,
\[
\int_{[c,d]} [df] g = f(c) g(c) - \lim_{r \to c^-} [f(r) g(c)] + \int_c^d [df] g \\
+ \lim_{r \to d^+} [f(r) g(d)] - f(d) g(d).
\]

**Remark 3.6.** If \( a \leq c < d \leq b \), \( f \in (\mathcal{B})G([a,b]; X) \) and \( g: [a,b] \to Y \), then Proposition 3.5 implies that if any one of the integrals
\[
\int_{[c,d]} [df] g, \int_{[c,d]} [df] g, \int_{[c,d]} [df] g, \int_{[c,d]} [df] g, \int_{[c,d]} [df] g (3.3)
\]
exists, then all the others exist as well. Of course, their values can differ, in general. If, in addition, \( f \) is continuous on \([a,b]\), then all the equalities
\[
\int_{[c,d]} [df] g = \int_{[c,d]} [df] g = \int_{[c,d]} [df] g = \int_{[c,d]} [df] g = \int_{[c,d]} [df] g (3.4)
\]
are true.

The existence of the integral \( \int_a^b [df] g \) need not (even in the case of the identity integrator \( f(x) = x \)) always imply the existence of the integral \( \int_T [df] g \) for every subset \( T \) of \([a,b]\). The examples for that can be constructed on the basis of the Lebesgue integral (the special case of the Kurzweil-Henstock integral) assuming e.g. that \( T \) is not measurable. As shown by the next assertion, this cannot happen when we restrict ourselves to elementary subsets of \([a,b]\).

**Theorem 3.7.** The following assertions are true for all \( f \in (\mathcal{B})G([a,b]; X) \) and \( g: [a,b] \to Y \).

(i) Let \( E \) be an elementary subset of \([a,b]\) such that the integral \( \int_E [df] g \) exists. Then the integral \( \int_T [df] g \) exists for every elementary subset \( T \) of \( E \).
(ii) Let \( E = \bigcup_{k=1}^{p} J_{k} \), where \( \{J_k : k = 1, 2, \ldots, p\} \) are mutually disjoint subintervals of \([a, b]\), and let the integral \( \int_{E} [df] g \) exist. Then all the integrals

\[
\int_{J_k} [df] g, \quad k = 1, 2, \ldots, p,
\]

exist as well and

\[
\int_{E} [df] g = \sum_{k=1}^{p} \int_{J_k} [df] g. \tag{3.5}
\]

**Proof.** (i) See \[37\] Corollary 5.15. 

(ii) By \[37\] Theorem 5.13, this assertion is true if the set \( \{J_k : k = 1, 2, \ldots, p\} \) is a minimal decomposition of \( E \), i.e. (cf. \[37\] Definition 4.9) the union \( J_k \cup J_\ell \) is not an interval whenever \( k \neq \ell \). Of course, we may assume that the intervals \( \{J_k\} \) are ordered in such a way that \( x \leq y \) holds whenever \( x \in J_k \), \( y \in J_\ell \) and \( k < \ell \). Then, if \( \{J_k : k = 1, 2, \ldots, p\} \) is not a minimal decomposition, there must exist \( k \in \{1, 2, \ldots, p-1\} \) such that \( J = J_k \cup J_{k+1} \) is an interval. Then, as \( J_k \cap J_{k+1} = \emptyset \), we get

\[
\int_{J_k} [df] g + \int_{J_{k+1}} [df] g = \int_{a}^{b} [df] g(\chi_{J_k} + \chi_{J_{k+1}}) = \int_{a}^{b} [df] (g \chi_{J}) = \int_{J} [df] g.
\]

This implies that when we replace in the sum on the right-hand side of (3.5) all such couples by their unions, this sum does not change, wherefrom the assertion (ii) follows. \( \square \)

**Remark 3.8.** Let \( E = \bigcup_{k=1}^{p} J_{k} \) be an elementary set of \([a, b]\).

(i) Let \( \{J_k^*: k = 1, 2, \ldots, p^*\} \) be the minimal decomposition of \( E \). Then, Theorem \[3.7\] (ii) implies that \( p^* \leq p \) and

\[
\int_{E} [df] g = \sum_{k=1}^{p} \int_{J_k} [df] g = \sum_{k=1}^{p^*} \int_{J_k^*} [df] g.
\]

(ii) Let \( c_k \) and \( d_k \) be an infimum and a supremum of \( J_k \), respectively, for every \( k = 1, 2, \ldots, p \). Then, from Remark \[3.6\] with Theorem \[3.7\] (ii), the equality

\[
\int_{E} [df] g = \sum_{k=1}^{p} \int_{J_k} [df] g = \sum_{k=1}^{p} \int_{c_k}^{d_k} [df] g \tag{3.6}
\]

holds for all continuous integrators \( f \), specially for the Kurzweil-Henstock integral. However, (3.6) is not valid any longer for the KS-integral.

**Remark 3.9.** If a function \( g : [a, b] \to Y \) and a subset \( S \) of \([a, b]\) are such that \( g = 0 \) on \( S \), then \( g \chi_{S} = 0 \) on \([a, b]\) and hence \( \int_{S} [df] g = \int_{a}^{b} [df] (g \chi_{S}) = 0 \) and \( \int_{T} [df] g = 0 \) as well for every subset \( T \) of \( S \) and every \( f : [a, b] \to X \). In particular, if the integral \( \int_{S} [df] g \) exists and \( h : [a, b] \to Y \) coincides with \( g \) on \( S \), then \( \int_{S} [df] h = \int_{S} [df] g \).

Next assertion disclose the additivity properties of the KS-integral over arbitrary subsets of \([a, b]\).
Proposition 3.10. Let \( f: [a, b] \rightarrow X \), \( g: [a, b] \rightarrow Y \) and subsets \( S_1, S_2 \) in \( [a, b] \) be given. Then, whenever three of the integrals
\[
\int_{S_1} [df] g, \quad \int_{S_2} [df] g, \quad \int_{S_1 \cup S_2} [df] g, \quad \int_{S_1 \cap S_2} [df] g
\]
exist, then there exists also the remaining one and the equality
\[
\int_{S_1} [df] g + \int_{S_2} [df] g = \int_{S_1 \cup S_2} [df] g + \int_{S_1 \cap S_2} [df] g
\]
holds.

Proof. follows directly from the identity \( \chi_{S_1} + \chi_{S_2} = \chi_{S_1 \cup S_2} + \chi_{S_1 \cap S_2} \).

Corollary 3.11. Let \( f: [a, b] \rightarrow X \), \( g: [a, b] \rightarrow Y \) and subsets \( S_1, S_2 \) in \( [a, b] \) be such that \( S_1 \cap S_2 = \emptyset \) and both the integrals \( \int_{S_1} [df] g \) and \( \int_{S_2} [df] g \) exist. Then there exists also the integral \( \int_{S_1 \cup S_2} [df] g \) and the equality
\[
\int_{S_1 \cup S_2} [df] g = \int_{S_1} [df] g + \int_{S_2} [df] g
\]
holds.

Proposition 3.12. Let \( f: [a, b] \rightarrow X \) and \( g: [a, b] \rightarrow Y \) be given and let \( S_1, S_2, \ldots, S_p \) be subsets of \( [a, b] \) and \( p \geq 2 \). Denote
\[
S = \bigcup_{j=1}^{p} S_j \quad \text{and} \quad T_i = \left( \bigcup_{j=1}^{i-1} S_j \right) \cap S_i \quad \text{for} \quad i = 2, 3, \ldots, p
\]
and assume that all the integrals
\[
\int_{S_i} [df] g, \quad \int_{T_i} [df] g, \quad i = 1, 2, \ldots, p,
\]
exist. Then the integral
\[
\int_{S} [df] g
\]
exist as well and
\[
\int_{S} [df] g = \sum_{i=1}^{p} \int_{S_i} [df] g - \sum_{i=2}^{p} \int_{T_i} [df] g. \quad (3.7)
\]

Proof. (i) First, notice that if \( p = 2 \), then \( T_2 = S_1 \cap S_2 \) and the assertion of the theorem follows by Proposition 3.10.

(ii) For \( p = 3 \) we have \( S = S_1 \cup S_2 \cup S_3 \), \( T_2 = S_1 \cap S_2 \), \( R_3 = (S_1 \cup S_2) \cap S_3 \). Denote \( M = S_1 \cup S_2 \). Then \( S = M \cup S_3 \), \( T_3 = M \cap S_3 \) and by Proposition 3.10 we get
\[
\int_{M} [df] g = \int_{S_1} [df] g + \int_{S_2} [df] g - \int_{T_2} [df] g
\]
and
\[
\int_S [df] g = \int_M [df] g + \int_{S_3} [df] g - \int_{M \cap S_3} [df] g
\]
\[
= \sum_{i=1}^3 \int_{S_i} [df] g - \int_{T_2} [df] g - \int_{T_3} [df] g
\]
\[
= \sum_{i=1}^3 \int_{S_i} [df] g - \sum_{i=2}^3 \int_{T_i} [df] g.
\]

(iii) Let \( N > 3 \) and let (3.7) holds for \( p = N - 1 \). Denote
\[
M = \bigcup_{j=1}^{N-1} S_j, \quad S = \bigcup_{j=1}^N S_j, \quad \text{and} \quad T_i = \left( \bigcup_{j=1}^{i-1} S_j \right) \cap S_i \quad \text{for} \quad i = 2, 3, \ldots, N - 1.
\]
Then \( S = M \cup S_N, T_N = M \cap S_N \) and by (3.7) (with \( n = N - 1 \)) and Proposition 3.10 we have
\[
\int_M [df] g = \sum_{i=1}^{N-1} \int_{S_i} [df] g - \sum_{i=2}^{N} \int_{T_i} [df] g
\]
and
\[
\int_S [df] g = \int_M [df] g + \int_{S_N} [df] g - \int_{M \cap S_N} [df] g
\]
\[
= \sum_{i=1}^N \int_{S_i} [df] g - \sum_{i=2}^N \int_{T_i} [df] g - \int_{T_N} [df] g
\]
\[
= \sum_{i=1}^N \int_{S_i} [df] g - \sum_{i=2}^N \int_{T_i} [df] g.
\]
By the induction principle, this completes the proof of the theorem. \( \square \)

4. Equiintegrability

Convergence theorems belong to the most important topics discussed in the frames of integration theory. For the abstract KS-integral, the Uniform Convergence Theorem given by Š. Schwabik in [46, Theorem 11] is the most simple one. It says that if the sequence \( \{g_n\} \) tends uniformly to \( g \) on \([a, b]\) and if all the integrals \( \int_a^b [df] g_n, n \in \mathbb{N}, \) exist, then the integral \( \int_a^b [df] g \) exists as well and
\[
\int_a^b [df] g = \lim_{n \to \infty} \int_a^b [df] g_n.
\]
When the uniform convergence of \( \{g_n\} \) to \( g \) is replaced by a just pointwise convergence on \([a, b]\), the situation is more difficult. One possible way is indicated by the Bounded Convergence Theorem (for the abstract KS-integral see [37, Theorem 6.3]) which requires the uniform boundedness of the sequence \( \{g_n\} \) on \([a, b]\). Next
Theorem 4.1 (Equiintegrability Convergence Theorem). Let \( f_n: [a, b] \to X \) and \( g_n: [a, b] \to Y \), for \( n \in \mathbb{N} \), be such that the integral \( \int_{a}^{b} [df_n] g_n \) exists for each \( n \in \mathbb{N} \). Furthermore, let the functions \( f: [a, b] \to X \) and \( g: [a, b] \to Y \) be such that the sequences \( \{f_n\} \) and \( \{g_n\} \) converge pointwise on \([a, b]\) to \( f \) and \( g \), respectively. Finally, suppose that

for every \( \eta > 0 \) there is a gauge \( \delta \) on \([a, b]\) such that

\[
\left\| S(df_n, g_n, P) - \int_{a}^{b} [df_n] g_n \right\|_Z < \eta
\]

for every \( \delta \)-fine partition \( P \) of \([a, b]\) and every \( n \in \mathbb{N} \).

Then the integrals \( \int_{a}^{b} [df] g \) and \( \lim_{n \to \infty} \int_{a}^{b} [df_n] g_n \) exist and

\[
\int_{a}^{b} [df] g = \lim_{n \to \infty} \int_{a}^{b} [df_n] g_n.
\]

Proof. Step 1. Let \( \varepsilon > 0 \) be given and let \( \delta \) be the gauge corresponding to \( \eta = \frac{\varepsilon}{4} \) by (4.1). Then

\[
\left\| S(df_n, g_n, P) - \int_{a}^{b} [df_n] g_n \right\|_Z < \frac{\varepsilon}{4}
\]

for every \( n \in \mathbb{N} \) and every \( \delta \)-fine partition \( P \) of \([a, b]\).

By Cousin’s lemma we may fix an arbitrary \( \delta \)-fine partition \( \tilde{P} \) of \([a, b]\). Due to the pointwise convergence on \([a, b]\) of \( \{f_n\} \) to \( f \) and of \( \{g_n\} \) to \( g \) we have

\[
\lim_{n \to \infty} S(df_n, g_n, \tilde{P}) = S(df, g, \tilde{P}).
\]

Hence, we can choose \( n_0 \in \mathbb{N} \) such that the inequality

\[
\left\| S(df_n, g_n, \tilde{P}) - S(df, g, \tilde{P}) \right\|_Z < \frac{\varepsilon}{4}
\]

holds for all \( n \geq n_0 \). Let arbitrary \( n_1, n_2 \geq n_0 \) be given. Then, using (4.3) and (4.4) we deduce

\[
\left\| \int_{a}^{b} [df_n] g_n - \int_{a}^{b} [df_n] g_n \right\|_Z
\leq \left\| \int_{a}^{b} [df_n] g_n - S(df_n, g_n, \tilde{P}) \right\|_Z + \left\| S(df_n, g_n, \tilde{P}) - S(df, g, \tilde{P}) \right\|_Z
\]

\[
+ \left\| S(df, g, \tilde{P}) - S(df_n, g_n, \tilde{P}) \right\|_Z + \left\| S(df_n, g_n, \tilde{P}) - \int_{a}^{b} [df_n] g_n \right\|_Z < \varepsilon,
\]

which shows that \( \{\int_{a}^{b} [df_n] g_n\} \) is a Cauchy sequence in \( Z \). Let

\[
I = \lim_{n \to \infty} \int_{a}^{b} [df_n] g_n.
\]
Step 2. We shall prove that \( \int_a^b [df]g = I \). Let \( \varepsilon > 0 \) be given and let \( \delta \) be a corresponding gauge given by \((4.1)\). Furthermore, let \( n_\varepsilon \in \mathbb{N} \) be such that

\[
\left\| \int_a^b [df_n] g_n - I \right\|_Z < \varepsilon \quad \text{for all } n \geq n_\varepsilon.
\]

Now, let \( \tilde{P} \) be a fixed \( \delta \)-fine partition of \([a,b]\). Thanks to the pointwise on \([a,b]\) convergence of \( \{f_n\} \) to \( f \) and of \( \{g_n\} \) to \( g \), we can choose \( k_0 \geq n_\varepsilon \) such that

\[
\left\| S(df_{k_0}, g_{k_0}, \tilde{P}) - S(df, g, \tilde{P}) \right\|_Z < \varepsilon.
\]

Hence,

\[
\begin{align*}
\left\| S(df, g, \tilde{P}) - I \right\|_Z & \leq \left\| S(df, g, \tilde{P}) - S(df_{k_0}, g_{k_0}, \tilde{P}) \right\|_Z \\
& \quad + \left\| S(df_{k_0}, g_{k_0}, \tilde{P}) - \int_a^b [df_{k_0}] g_{k_0} \right\|_Z + \left\| \int_a^b [df_{k_0}] g_{k_0} - I \right\|_Z < 3\varepsilon.
\end{align*}
\]

It follows that \( \int_a^b [df] g = I \) and this completes the proof. \( \square \)

**Definition 4.2.** Let \( f, f_n : [a, b] \to X \) and \( g, g_n : [a, b] \to Y \) for \( n \in \mathbb{N} \). The sequence \( \{g_n\} \) is said to be **equiintegrable with respect to \( \{f_n\} \)** on \([a, b]\) if the integrals \( \int_a^b [df_n] g_n \) exist for all \( n \in \mathbb{N} \) and the condition \((4.1)\) in Theorem 4.1 is satisfied.

Similarly, for any \( E \subset [a, b] \), we say that \( \{g_n\} \) is **equiintegrable with respect to \( \{f_n\} \)** on \( E \) if \( \{g_n \chi_E\} \) is equiintegrable with respect to \( \{f_n\} \) on \([a, b]\).

In view of Definition 4.2, Theorem 4.1 may be reformulated as follows:

**Let** \( f, f_n : [a, b] \to X \) and \( g, g_n : [a, b] \to Y \), \( n \in \mathbb{N} \), **be such that** \( \lim_{n \to \infty} f_n(t) = f(t) \) **and** \( \lim_{n \to \infty} g_n(t) = g(t) \) **on** \([a, b]\) **and suppose that the sequence** \( \{g_n\} \) **is equiintegrable with respect to** \( \{f_n\} \) **on** \([a, b]\). **Then the integral** \( \int_a^b [df] g \) **and** \( \lim_{n \to \infty} \int_a^b [df_n] g_n \) **exist and** \((4.2)\) **holds.**

**Remark 4.3.**

(i) The equiintegrability is often met in the literature dealing with the theory of Kurzweil-Henstock integrals, see e.g. R. G. Bartle [3, Chapter 8], R. A. Gordon [12, Chapter 13] and [13], J. Kurzweil [26, Chapter 5], J. Kurzweil and J. Jarník [27], Š. Schwabik [15, Chapter 1], Š. Schwabik and I. Vrkoč [47], Š. Schwabik and G. Ye [49, Chapter 3]. Meanwhile, little is known about the conditions that ensure the equiintegrability for Stieltjes type integrals for real-valued functions, see [4], [38, Chapter 6], [39].

(ii) Referring to e.g. R. A. Gordon [12, Definition 13.15] or Š. Schwabik and I. Vrkoč [47, Remark 6], a sequence \( \{g_n\} \) equiintegrable with respect to \( \{f_n\} \) on \([a, b]\) can be called also **uniformly integrable** with respect to \( \{f_n\} \) on \([a, b]\).

(iii) If \( f_n = f \) for all \( n \in \mathbb{N} \), we will say that \( \{g_n\} \) is equiintegrable with respect to \( f \) on \([a, b]\).
In general, it is rather difficult to verify that the condition (4.1) is satisfied. The following statement at least enables us to decide whether a given sequence \( \{g_n\} \) is equiintegrable with respect to \( \{f_n\} \) on \([a, b]\) without evaluating the values of all the integrals \( \int_a^b [df_n] g_n, \ n \in \mathbb{N} \).

**Theorem 4.4** (Cauchy Equiintegrability Criterion). Let \( f_n : [a, b] \to X \) and \( g_n : [a, b] \to Y, \) for \( n \in \mathbb{N} \). Then the sequence \( \{g_n\} \) is equiintegrable with respect to \( \{f_n\} \) on \([a, b]\) if and only if

\[
\text{for every } \varepsilon > 0 \text{ there is a gauge } \delta \text{ such that }
\left\| S(df_n, g_n, P) - S(df_n, g_n, Q) \right\|_Z < \varepsilon
\]

holds for all \( n \in \mathbb{N} \) and all \( \delta \)-fine partitions \( P \) and \( Q \) of \([a, b]\).

*Proof.* (i) Assume that the sequence \( \{g_n\} \) is equiintegrable with respect to \( \{f_n\} \) on \([a, b]\). Let \( \varepsilon > 0 \) be given and let \( \delta \) be an arbitrary gauge corresponding by (4.1) to \( \eta = \varepsilon/2 \). Then for any couple \( P, Q \) of \( \delta \)-fine partitions of \([a, b]\) and any \( n \in \mathbb{N} \), we obtain

\[
\left\| S(df_n, g_n, P) - S(df_n, g_n, Q) \right\|_Z \leq \left\| S(df, g, \tilde{P}) - S(df_{k_0}, g_{\delta_0}, \tilde{P}) \right\|_Z
\]

\[
\leq \left\| S(df_n, g_n, P) - \int_a^b [df_n] g_n \right\|_Z + \left\| S(df_n, g_n, Q) - \int_a^b [df_n] g_n \right\|_Z < \varepsilon.
\]

(ii) Assume that the condition (4.6) is satisfied. Then, by the Cauchy-Bolzano criterion [46, Proposition 7] for the existence of the KS-integral, the integral \( \int_a^b [df_n] g_n \) exists for every \( n \in \mathbb{N} \). For given \( n \in \mathbb{N} \), gauge \( \delta \) and \( \varepsilon > 0 \), denote

\[
\mathcal{I}_n(\varepsilon, \delta) = \{ S(df_n, g_n, P) : P \text{ is } \delta-\text{fine tagged partition of } [a, b] \}.
\]

Due to (4.6) we have

\[
diam(\mathcal{I}_n(\varepsilon, \delta)) = \sup \left\{ \|S(df_n, g_n, P) - S(df_n, g_n, Q)\|_Z : P, Q \text{ are } \delta-\text{fine partitions of } [a, b] \right\} < \varepsilon.
\]

By Cousin lemma, any \( \mathcal{I}_n(\varepsilon, \delta) \) is nonempty and, furthermore,

\[
0 < \varepsilon_1 < \varepsilon_2 \implies \mathcal{I}_n(\varepsilon_1, \delta) \subset \mathcal{I}_n(\varepsilon_2, \delta) \text{ for every } n \in \mathbb{N} \text{ and every gauge } \delta.
\]

Thus, using Cantor’s intersection theorem for complete metric spaces (see e.g. [53, Theorem 5.1.17]) we conclude that, for every \( n \in \mathbb{N} \), the intersection \( \bigcap_{\varepsilon > 0} \mathcal{I}_n(\varepsilon, \delta) \) is a one-point set \( \{I_n\} \) with

\[
I_n = \int_a^b [df_n] g_n \in Z.
\]

As a consequence, if an arbitrary \( \eta > 0 \) is given and a gauge \( \delta_\varepsilon \) is such that (4.6) is true with \( \varepsilon = \eta/2 \), then \( I_n \in \mathcal{I}_n(\varepsilon, \delta_\varepsilon) \) for every \( n \in \mathbb{N} \). In particular, due to (4.7),
Lemma 4.6. Let \( \{f_n\} \) be equiintegrable and let the sequence \( \{g_n\} \) be equiintegrable with respect to \( \{f_n\} \) on \([a, b]\). This completes the proof. \( \Box \)

If \( E = [c, d] \) is a closed subinterval of \([a, b]\), it seems to be natural to define the equiintegrability on \( E \) also in the following alternative way:

\( \{g_n\} \) is equiintegrable with respect to \( \{f_n\} \) on \([c, d]\) if the conditions

\[
\text{the integrals } \int_c^d [df_n] g_n \text{ exist for all } n \in \mathbb{N}
\]

and

\[
\text{for every } \eta > 0 \text{ there is a gauge } \delta \text{ on } [a, b] \text{ such that}
\|
\| S(df_n, g_n, P) - \int_c^d [df_n] g_n \|_Z < \eta
\]

holds for every \( \delta \)-fine partition \( P \) of \([c, d]\) and every \( n \in \mathbb{N} \)

are satisfied.

Thus, any comparison of these two possible definitions is urgently needed. To this aim we shall first recall the notion of equiregulatedness due to Fraňková [11].

**Definition 4.5.** A subset \( M \) of \( G([a, b]; X) \) is called *equiregulated* if the following conditions hold.

(i) For each \( \varepsilon > 0 \) and \( \tau \in (a, b) \) there is a \( \delta_1(\tau) \in (0, \tau - a) \) such that

\[
\| f(\tau^-) - f(t) \|_X < \varepsilon \quad \text{for all } t \in (\tau - \delta_1(\tau), \tau) \text{ and } f \in M.
\]

(ii) For each \( \varepsilon > 0 \) and \( \tau \in [a, b] \) there is a \( \delta_2(\tau) \in (0, b - \tau) \) such that

\[
\| f(\tau^+) - f(t) \|_X < \varepsilon \quad \text{for all } t \in (\tau, \tau + \delta_2(\tau)) \text{ and } f \in M.
\]

Definition 4.5 allows us to develop the concept of equiintegrability of sequence \( \{g_n\} \) of functions mapping \([a, b]\) into \( Y \) with respect to \( \{f_n\} \) of functions mapping \([a, b]\) into \( X \) over an arbitrary elementary set \( E \) of \([a, b]\).

**Lemma 4.6.** Let \( J = [c, d] \subset [a, b] \), let the sequence \( \{f_n\} \subset G([a, b]; X) \) be equiregulated and let the sequence \( \{g_n\} \) of functions mapping \([a, b]\) into \( Y \) be pointwise bounded on \([c, d]\), i.e. the sequence \( \{g_n(t)\} \) is bounded in \( Y \) for each \( t \in [c, d] \). Then \( \{g_n\} \) is equiintegrable with respect to \( \{f_n\} \) on \( J \) if and only if condition (4.8) is satisfied.

**Proof.** (i) Let the assumptions of the lemma be satisfied and let (4.8) be true. First, assume that \( a < c < d < b \).
Let an arbitrary \( \varepsilon > 0 \) be given and let \( \tilde{\delta} \) be a gauge on \([c, d]\) such that (4.8) holds for \( \eta = \varepsilon / 3 \), i.e.

\[
\left\| S(d f_n, g_n, \bar{P}) - \int_c^d [d f_n] g_n \right\|_Z < \frac{\varepsilon}{3}
\]

for all \( n \in \mathbb{N} \) and all \( \tilde{\delta} \) - fine partitions \( \bar{P} \) of \([c, d]\).

Put

\[
\delta(t) = \begin{cases} 
\min\left\{ \frac{1}{4}(c - t), 1 \right\} & \text{if } t \in [a, c), \\
\min\{\Delta, \tilde{\delta}(c)\} & \text{if } t = c, \\
\min\left\{ \frac{1}{4}(t - c), \frac{1}{4}(d - t), \tilde{\delta}(t) \right\} & \text{if } t \in (c, d), \\
\min\{\Delta, \tilde{\delta}(d)\} & \text{if } t = d, \\
\min\left\{ \frac{1}{4}(t - d), 1 \right\} & \text{if } t \in (d, b], 
\end{cases}
\]

where \( \Delta \in (0, \min\{c - a, b - d\}) \) is such that

\[
\|f_n(t) - f_n(c-\|_X < \frac{\varepsilon}{3(1 + \sup\{\|g_n(c)\|_Y : n \in \mathbb{N}\})}
\]

for \( n \in \mathbb{N} \) and \( t \in (c - \Delta, c) \),

and

\[
\|f_n(d+) - f_n(t)\|_X < \frac{\varepsilon}{3(1 + \sup\{\|g_n(d)\|_Y : n \in \mathbb{N}\})}
\]

for \( n \in \mathbb{N} \) and \( t \in (d, d + \Delta) \).

Such a \( \Delta \) may be chosen since we assume the equiregulatedness of \( \{f_n\} \) and the boundedness of \( \{g_n(c)\} \) and \( \{g_n(d)\} \) in \( Y \).

Now, let \( P =\{(\alpha_{j-1}, \alpha_j, \xi_j)\} \) be a \( \delta \) - fine partition of \([a, b]\) such that \( \nu(P) > 4 \). Then, cf. [38, Lemma 6.2.11], there are indices \( k, \ell \in \mathbb{N} \) such that \( 2 \leq k < \ell \leq \nu(P) - 2 \), \( \xi_k = c > \alpha_{k-1} > c - \Delta > a \) and \( \xi_\ell = d < \alpha_{\ell+1} < d + \Delta < b \). In addition, we may assume that \( \xi_{k-1} = c = \alpha_k \) and \( d = \alpha_\ell = \xi_{\ell+1} \). To summarize, we have

\[
a < c - \Delta < \alpha_{k-1} < \xi_k = c = \alpha_k = \xi_{k+1} < \alpha_{k+1} < \xi_\ell = d = \alpha_\ell = \xi_{\ell+1} < \alpha_{\ell+1} < d + \Delta < b.
\]

Hence, we can deduce successively

\[
S(d f_n, g_n \chi_{[c,d]}, P) = \sum_{j=k}^{\ell+1} [f_n(\alpha_j) - f_n(\alpha_{j-1})] g_n(\xi_j)
\]

\[
= \sum_{j=k+1}^{\ell+1} [f_n(\alpha_j) - f_n(\alpha_{j-1})] g_n(\xi_j)
\]

\[
+ [f_n(c) - f_n(\alpha_{k-1})] g_n(c) + [f_n(\alpha_{\ell+1}) - f_n(d)] g_n(d)
\]

\[
= \sum_{j=k+1}^{\ell+1} [f_n(\alpha_j) - f_n(\alpha_{j-1})] g_n(\xi_j) + \Delta^- f_n(c) g_n(c) + \Delta^+ f_n(d) g_n(d)
\]

\[
+ [f_n(c-) - f_n(\alpha_{k-1})] g_n(c) + [f_n(\alpha_{\ell+1}) - f_n(d+)] g_n(d),
\]

where

\[
\Delta^- f_n(c) g_n(c) = \Delta^- f_n(c) g_n(\alpha_{k-1}) + f_n(c) g_n(\alpha_{k-1}) - f_n(c) g_n(c)
\]

and

\[
\Delta^+ f_n(d) g_n(d) = \Delta^+ f_n(d) g_n(\alpha_{\ell+1}) + f_n(d) g_n(\alpha_{\ell+1}) - f_n(d) g_n(d).
\]
i.e.

\[
S(d f_n, g_n \chi_{[c,d]}, P) = \sum_{j=k+1}^{\ell} [f_n(\alpha_j) - f_n(\alpha_{j-1})] g_n(\xi_j)
\]

\[
+ \Delta^- f_n(c) g_n(c) + \Delta^+ f_n(d) g_n(d)
\]

\[
+ [f_n(c-) - f_n(\alpha_{k-1})] g_n(c) + [f_n(\alpha_{\ell+1}) - f_n(d+)] g_n(d).
\]

(4.12)

As for every \( n \in \mathbb{N} \) the integral \( \int_c^d [df_n] g_n \) exists, by Proposition 3.5 (iii) (see also Remark 3.6) also all the integrals \( \int_{[c,d]} [df_n] g_n \), \( n \in \mathbb{N} \), exist and, using (4.12), we obtain further

\[
\left\| S(d f_n, g_n \chi_{[c,d]}, P) - \int_{[c,d]} [df_n] g_n \right\|_Z
\]

\[
= \left\| S(d f_n, g_n \chi_{[c,d]}, P) - \int_a^b [df_n] (g_n \chi_{[c,d]}) \right\|_Z
\]

\[
\leq \left\| \sum_{j=k+1}^{\ell} [f_n(\alpha_j) - f_n(\alpha_{j-1})] g_n(\xi_j) - \int_c^d [df_n] g_n \right\|_Z
\]

\[
+ \left\| [f_n(\alpha_{k+1}) - f_n(c-)] g_n(c) \right\|_Z + \left\| [f_n(\alpha_{\ell+1}) - f_n(d+)] g_n(d) \right\|_Z.
\]

(4.13)

Thus, in view of (4.9), (4.10) and (4.13) and since \( \{([\alpha_{j-1}, \alpha_j], \xi_j) : j = k+1, \ldots, \ell\} \) is a \( \tilde{\delta} \)-fine partition of \([c, d]\), we have

\[
\left\| S(d f_n, g_n \chi_{[c,d]}, P) - \int_a^b [df_n] (g_n \chi_{[c,d]}) \right\|_Z < \left( 1 + \frac{\|g_n(c)\|_Y}{1+\|g_n(c)\|_Y} + \frac{\|g_n(d)\|_Y}{1+\|g_n(d)\|_Y} \right) \varepsilon < \varepsilon
\]

for every \( n \in \mathbb{N} \). This proves the equiintegrability of the sequence \( \{g_n\} \) with respect to \( \{f_n\} \) on \([c, d]\) according to Definition 4.2.

(ii) Let \( \{g_n \chi_{[c,d]}\} \) be equiintegrable with respect to \( \{f_n\} \) on \([a, b]\). In particular, all the integrals \( \int_a^b [df_n] (g_n \chi_{[c,d]}) \), \( n \in \mathbb{N} \), exist and for any \( \varepsilon > 0 \) there is a gauge \( \tilde{\delta} \) on \([a, b]\) such that

\[
\left\| S(d f_n, g_n \chi_{[c,d]}, \tilde{P}) - \int_a^b [df_n] (g_n \chi_{[c,d]}) \right\|_Z \leq \varepsilon
\]

for all \( \tilde{\delta} \)-fine partitions \( \tilde{P} \) of \([a, b]\) and all \( n \in \mathbb{N} \).

(4.14)

Note that, by Proposition 3.5 (iii), also all the integrals \( \int_c^d [df_n] g_n \), \( n \in \mathbb{N} \), exist.

Now, let \( \varepsilon > 0 \) be given, let the gauge \( \tilde{\delta} \) be such that (4.14) is true. Put \( \delta(t) = \tilde{\delta}(t) \) for \( t \in [c, d] \), and let \( P = \{([\alpha_{j-1}, \alpha_j], \xi_j)\} \) be an arbitrary \( \delta \)-fine partition of \([c, d]\). Further, let \( \tilde{P} = \{([\tilde{\alpha}_{j-1}, \tilde{\alpha}_j], \xi_j)\} \) be a \( \tilde{\delta} \)-fine partition of \([a, b]\) whose restriction to
\[ [c, d] \text{ coincides with } P. \text{ In other words, there are } k, \ell \in \mathbb{N} \text{ such that} \]
\[
2 \leq k < \ell = k + \nu(P) < \nu(\tilde{P}) - 2, \\
\tilde{\alpha}_{k+j} = \alpha_j \text{ for } j \in \{0, 1, \ldots, \ell - k\} \quad \text{and} \quad \tilde{\xi}_{k+j} = \xi_j \text{ for } j \in \{1, \ldots, \ell - k\}.
\]
Furthermore, we may assume that
\[
\tilde{\alpha}_k = \tilde{\xi}_k = \tilde{\xi}_{k+1} = \alpha_0 = \xi_1 = c, \quad \tilde{\alpha}_{\ell} = \tilde{\xi}_\ell = \tilde{\xi}_{\ell+1} = \alpha_\nu(P) = d, \\
c - \Delta < \tilde{\alpha}_{k-1} < c \quad \text{and} \quad d < \tilde{\alpha}_{\ell+1} < d + \Delta,
\]
where \( \Delta \in (0, \min\{b - d, c - a\}) \) is given by (4.10). Thus, having in mind Proposition 3.5 (iii), for an arbitrary \( n \in \mathbb{N} \) we can successively deduce
\[
\left\| S(df_n, g_n, P) - \int_c^d [df_n] g_n \right\|_Z \\
\leq \left\| \sum_{j=1}^{\nu(P)} [f_n(\alpha_j) - f_n(\alpha_{j-1})] g_n(\xi_j) \int_c^d [df_n] g_n \right\|_Z \\
\leq \left\| S(df_n, g_n\chi_{[c,d]}, \tilde{P}) - [f_n(\tilde{\alpha}_{\ell+1}) - f_n(d)] g_n(d) - [f_n(\tilde{\alpha}_{k-1}) - f_n(c)] g_n(c) \\
- \int_{[c,d]} [df_n] g_n + \Delta^- f_n(c) g_n(c) + \Delta^+ f_n(d) g_n(d) \right\|_Z,
\]
wherefrom, thanks to relations (4.10) and (4.14), we finally obtain
\[
\left\| S(df_n, g_n, P) - \int_c^d [df_n] g_n \right\|_Z \\
\leq \left\| S(df_n, g_n\chi_{[c,d]}, \tilde{P}) - \int_a^b [df_n] (g_n\chi_{[c,d]}) \right\|_Z \\
+ \|f_n(c- ) - f_n(\tilde{\alpha}_{k-1})\|_X \|g_n(c)\|_Y + \|f_n(\tilde{\alpha}_{\ell+1}) - f_n(d+ )\|_X \|g_n(d)\|_Y < \varepsilon.
\]
This verifies relations (4.8). \( \square \)

Rather surprisingly, the same equivalences like that provided by Lemma 4.7 holds also for other types on intervals.

**Lemma 4.7.** Let \( J \) be an arbitrary subinterval of \([a, b]\), let the sequence \( \{f_n\} \subset G([a, b]; X) \) be equiregulated and let the sequence \( \{g_n\} \) of functions mapping \([a, b]\) into \( Y \) be pointwise bounded on \([a, b]\). Then \( \{g_n\} \) is equiintegrable with respect to \( \{f_n\} \) on \( J \) if and only if condition (4.8) is satisfied.

**Proof.** Let \( c = \inf J \) and \( d = \sup J \). For the case \( J = [c, d] \), the proof was given by the previous lemma.
Let $J = (c, d)$. First, assume that (4.8) is true. Let $\varepsilon > 0$ be given and let $\delta$ be a gauge on $[c, d]$ such that

$$
\left\| S(df_n, g_n, P) - \int_c^d [df_n] g_n \right\|_Z < \frac{\varepsilon}{3}
$$

holds for any $\delta$–fine partition $P$ of $[c, d]$ and any $n \in \mathbb{N}$.

Since the sequence $\{f_n\}$ is equiregulated, we may choose $\Delta \in \left(0, \frac{d-c}{2}\right)$ in such a way that the inequalities

$$
\left\| f_n(t) - f_n(c+) \right\|_X < \frac{\varepsilon}{3} \left(1 + \sup \{\|g_n(c)\|_Y : n \in \mathbb{N}\}\right)
$$

for all $t \in (c, c+\Delta)$

and

$$
\left\| f_n(d-) - f_n(t) \right\|_X < \frac{\varepsilon}{3} \left(1 + \sup \{\|g_n(d)\|_Y : n \in \mathbb{N}\}\right)
$$

for all $t \in (d-\Delta, d)$

hold for each $n \in \mathbb{N}$. Let $P = \{([\alpha_{j-1}, \alpha_j], \xi_j)\}$ be an arbitrary $\delta$–fine partition of $[c, d]$. Denote $m = \nu(P)$. In view of [38 Lemma 6.2.11] we may assume the following relations

$$
c = \xi_1 < \alpha_1 < c + \Delta < d - \Delta < \alpha_{m-1} < d = \xi_m.
$$

Thus,

$$
S(df_n, g_n\chi(c,d), P) = \sum_{j=2}^{m-1} [f_n(\alpha_j) - f_n(\alpha_{j-1})] g_n(\xi_j)
$$

$$
= S(df_n, g_n, P) - [f_n(\alpha_1) - f_n(c)] g_n(c) - [f_n(d) - f_n(\alpha_{m-1})] g_n(c)
$$

$$
= S(df_n, g_n, P) - \Delta^+ f_n(c) g_n(c) - \Delta^- f_n(d) g_n(d)
$$

$$
- [f_n(\alpha_1) - f_n(c+)] g_n(c) - [f_n(d-) - f_n(\alpha_{m-1})] g_n(c),
$$

wherefrom, using (4.15), (4.16) and Proposition 3.5 (i) we get for any $n \in \mathbb{N}$

$$
\left\| S(df_n, g_n\chi(c,d), P) - \int_{(c,d)} [df_n] g_n \right\|_Z \leq \left\| S(df_n, g_n, P) - \int_c^d [df_n] g_n \right\|_Z
$$

$$
+ \left\| f_n(\alpha_1) - f_n(c+) \right\|_X \|g_n(c)\|_Y + \left\| f_n(d-) - f_n(\alpha_{m-1}) \right\|_X \|g_n(d)\|_Y < \varepsilon.
$$

This means that $\{g_n\}$ is equiintegrable with respect to $\{f_n\}$ on $(c, d)$.

Similarly, we would prove the reverse implication and also the corresponding equivalences for the remaining cases $J = [c, d)$ and $J = (c, d)$.

Next propositions summarize the properties of the equiintegrability on subintervals of $[a, b]$. 

20
Corollary 4.8. Let $J$ be an arbitrary subinterval of $[a, b]$, let the sequence $\{f_n\} \subset G([a, b]; X)$ be equiregulated and let the sequence $\{g_n\}$ of functions mapping $[a, b]$ into $Y$ be pointwise bounded on $[a, b]$. Then

(i) if the sequence $\{g_n\}$ is equiintegrable with respect to $\{f_n\}$ on $[a, b]$, then it is equiintegrable with respect to $\{f_n\}$ on each subinterval $J$ of $[a, b]$,

(ii) for every $c \in (a, b)$, the sequence $\{g_n\}$ is equiintegrable with respect to $\{f_n\}$ on $[a, b]$ if and only if it is equiintegrable with respect to $\{f_n\}$ on both intervals $[a, c]$ and $[c, b]$.

Proof. (i) By [46, Proposition 8], the integral $\int_a^b df_n g_n$ exists for all $n \in \mathbb{N}$. Moreover, by the Cauchy Equiintegrability Criterion [4.4] hypothesis (4.6) is satisfied. It can be shown in a rather routine way that then (4.6) holds also on $[c, d]$. Therefore, by Theorem [4.4] $\{g_n\}$ is equiintegrable with respect to $\{f_n\}$ on $[c, d]$.

(ii) The necessity part of the assertion (ii) follows immediately from the first assertion of this corollary.

On the other side, let $\eta > 0$ and $c \in (a, b)$ be given. Let $\delta_a$ and $\delta_b$ be the gauges satisfying (4.11) for $[a, b]$ replaced by $[a, c]$ or $[c, b]$, respectively. Then for a given $\eta > 0$ the condition (4.11) will be satisfied if we define

$$
\delta(t) = \begin{cases} 
\delta_a(t) & \text{if } t \in [a, c), \\
\min\{\delta_a(c), \delta_b(c)\} & \text{if } t = c, \\
\delta_b(t) & \text{if } t \in (c, b].
\end{cases}
$$

The next assertion is a direct consequence of Lemma [4.7] and Corollary [4.8]

Corollary 4.9. Let the sequence $\{f_n\} \subset G([a, b]; X)$ be equiregulated and let the sequence $\{g_n\}$ of mappings of $[a, b]$ into $Y$ be pointwise bounded. Then, the sequence $\{g_n\}$ is equiintegrable with respect to $\{f_n\}$ on $[a, b]$ if and only if the sequence $\{g_n\chi_E\}$ is equiintegrable with respect to $\{f_n\}$ on $[a, b]$ for every elementary subset $E$ in $[a, b]$.

Well-known Saks-Henstock Lemma (see e.g. [46, Lemma 16]) states that the Riemannian sums not only approximate in the ”gauge topology” the integrals over the whole integral but also over suitably chosen systems of subintervals. Next we will show that the equiintegrability implies a uniform Saks-Henstock property. But, first, let us introduce the notion of $\delta-$fine system.

Definition 4.10. We will say (see e.g. [46, Lemma 16]) that the set $S = \{([\beta_j, \gamma_j], \xi_j) : j = 1, 2, \ldots, m\}$ is a $\delta-$fine system in $[a, b]$ if

$$a \leq \beta_1 \leq \xi_1 \leq \gamma_1 \leq \beta_2 \leq \xi_2 \leq \gamma_2 \leq \cdots \leq \beta_m \leq \xi_m \leq \gamma_m \leq b$$

and

$$[\beta_j, \gamma_j] \subset (\xi_j - \delta(\xi_j), \xi_j + \delta(\xi_j)) \quad \text{for } j = 1, 2, \ldots, m.$$
Similarly like for divisions and tagged partitions of the interval \([a, b]\), we will denote by \(\nu(S)\) the number of the intervals contained in \(S\), i.e. \(\nu(S) = m\) in the above situation.

Let \(T \subset [a, b]\) and let \(S = \{([\beta_j, \gamma_j], \xi_j) : j = 1, 2, \ldots, m\}\) be a \(\delta\)–fine system in \([a, b]\). Then \(P\) is called a \(T\)–tagged \(\delta\)–fine system in \([a, b]\) if \(\xi_j \in T\) for every \(j = 1, 2, \ldots, m\).

**Proposition 4.11.** Let \(\{f_n\} \subset G([a, b], X)\) be equiregulated and let the sequence \(\{g_n\}\) of functions mapping \([a, b]\) into \(Y\) be equiintegrable with respect to \(\{f_n\}\) on \([a, b]\). Further, let \(\varepsilon > 0\) be given arbitrarily and let the gauge \(\delta\) on \([a, b]\) be such that

\[
\left\| \int_a^b [d(f_n, g_n, P) - \int_a^b [d f_n] g_n] \right\| _Z < \varepsilon
\]

for every \(\delta\)–fine partition \(P\) of \([a, b]\) and every \(n \in \mathbb{N}\). Then

\[
\left\| \sum_{j=1}^{\nu(S)} \left( [f_n(\gamma_j) - f_n(\beta_j)] g_n(\xi_j) - \int_{\beta_j}^{\gamma_j} [d f_n] g_n \right) \right\| _Z \leq \varepsilon
\]

holds for every \(\delta\)–fine system \(S = \{([\gamma_j, \beta_j], \xi_j)\}\) in \([a, b]\) and every \(n \in \mathbb{N}\).

**Proof.** Assume that the system \(\{([\gamma_j, \beta_j], \xi_j) : j = 1, 2, \ldots, n\}\) satisfies conditions

\[
\begin{align*}
  a \leq \gamma_1 \leq \xi_1 \leq \beta_1 \leq \gamma_2 \leq \cdots \leq \gamma_n \leq \xi_n \leq \beta_n \leq b, \\
  [\gamma_j, \beta_j] \subset (\xi_j - \delta(\xi_j), \xi_j + \delta(\xi_j)) \quad \text{for } j = 1, \ldots, n,
\end{align*}
\]

and denote \(\gamma_0 = a\) and \(\beta_{n+1} = b\). Then, by Corollary 4.9 the sequence \(\{g_n\}\) is equiintegrable with respect to \(\{f_n\}\) on each subinterval \([\beta_j, \gamma_{j+1}]\). Hence, we can apply the method of the proof of the Saks-Henstock lemma in [38, Lemma 6.5.1] to complete the proof of this corollary.

□

5. **HARNACK EXTENSION PRINCIPLE AND ITS APPLICATIONS**

Assume that

\[
(a, b) \setminus T = \bigcup_{i \in \mathbb{N}} (a_i, b_i), \quad \text{where } (a_i, b_i), i \in \mathbb{N},
\]

are pairwise disjoint open subintervals of \((a, b)\).

For every \(i \in \mathbb{N}\), let \(J_i\) denote \([a_i, b_i]\), \([a_i, b_i)\), \((a_i, b_i]\), or \((a_i, b_i)\), let \(E\) be an ordered finite union of \(J_i\), i.e. \(E = \bigcup_{i=p}^{p+q} J_i\) for some \(p, q \in \mathbb{N}\), and let the (Kurzweil-Henstock) integrals \(\int_{J_i} g \, dt\) or \(\int_{a_i}^{b_i} g \, dt\) exist for each \(i \in \mathbb{N}\). The following equation

\[
\int_{[a,b]} g \, dt = \int_a^b g \, dt = \int_{(a,b)} g \, dt
\]

(provided one of the integrals \(\int_{[a,b]} g \, dt\), \(\int_a^b g \, dt\), or \(\int_{(a,b)} g \, dt\) exists) and the situations of (3.1) and (3.6) which are true for the Kurzweil-Henstock integral has verified the
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statements below

\[ \int_{[a,b] \setminus T} g \, dt = \int_{(a,b) \setminus T} g \, dt \quad \text{(in case \( \int_T g \, dt \) exists)}, \]  
(5.3)

\[ \int_{J_i} g \, dt = \int_{a_i}^{b_i} g \, dt, \quad \text{for each} \ i \in \mathbb{N}, \]  
(5.4)

and

\[ \int_E g \, dt = \int_{\bigcup_{i=p}^{p+q} J_i} g \, dt = \sum_{i=p}^{p+q} \int_{a_i}^{b_i} g \, dt, \]  
(5.5)

respectively. As a consequence, under the conditions in Theorem 1.1 for the Kurzweil-Henstock integral we have that the integrals

\[ \int_a^b g \, dt, \int_{(a,b) \setminus T} g \, dt, \text{ and } \int_{\bigcup_{i \in \mathbb{N}} J_i} g \, dt \]

exist and the following relation

\[ \int_a^b g \, dt = \int_a^b g \chi_T \, dt + \int_{\bigcup_{i \in \mathbb{N}} J_i} g \, dt = \int_a^b g \chi_T \, dt + \sum_{i=1}^{\infty} \int_{a_i}^{b_i} g \, dt \]

holds. The Harnack extension is also valid for the Kurzweil-Henstock integrable real-valued functions defined on measure spaces endowed with locally compact metric topologies, see e.g. [40, Theorem 5.1]. It applies two important concepts, namely the \( \delta \)-fine cover and the nonabsolute subset involving the integration over elementary sets and fulfilling (5.2), (5.3), (5.4), and (5.5).

On the other hand, in view due of (3.1), (3.2), (3.3), or Remark 3.8 (ii), the relations (5.2), (5.3), (5.4), and (5.5) need not be true for the KS-integral. Hence neither Theorem 1.1 nor [40, Theorem 5.1] could be simply extended to the KS-integration.

We need the following definition in order to construct Harnack extension principle for the KS-integral.

**Definition 5.1.** Let \( E \) be an elementary set in \( \mathbb{R} \) and let \( T \) be a closed subset of \( E \). The sequence \( \{E_i\} \) of elementary subsets of \( E \setminus T \) is called a *proper cover* of \( E \setminus T \) if \( E \setminus Y = \bigcup_{i \in \mathbb{N}} E_i \).

**Remark 5.2.** Let \( E \) be an elementary set in \( \mathbb{R} \) and let \( T \) be a closed subset of \( E \).

(i) Then by [40, Lemma 5.1] there must exist elementary sets \( E_1, E_2, \ldots \) such that \( E_i \subseteq E \setminus T \) for every \( i \in \mathbb{N} \) and

\[ \mu \left( (E \setminus Y) \setminus \bigcup_{i \in \mathbb{N}} E_i \right) = 0 \]  
(5.6)

(\( \mu \) is a signed measure). The sequence \( \{E_i\} \) satisfying (5.6) in such a way that \( (E \setminus Y) \setminus \bigcup_{i \in \mathbb{N}} E_i = \emptyset \), is a proper cover of \( E \setminus Y \).
(ii) Consider \( E = (a, b) \subset [a, b] \) and a closed subset \( T \subset [a, b] \). Then \( (a, b) \setminus T \) is an open set and thus can be written as the union of a countable of pairwise disjoint intervals \((a_i, b_i) \subset (a, b) \setminus T, i \in \mathbb{N}\). Hence, the sequence \( \{(a_i, b_i)\} \) is an proper cover of \( (a, b) \setminus T \) (see e.g. Theorem 1.1 or (5.1)).

Applying Theorem 4.1 and Definition 5.1 together, we are now ready to prove Harnack extension principle for the KS-integral by taking \( E_i, i \in \mathbb{N} \) to be mutually disjoint elementary sets in \([a, b]\) such that \([a, b] \setminus T = \bigcup_{i \in \mathbb{N}} E_i \) for a closed subset \( T \) of \([a, b]\).

**Theorem 5.3.** (Harnack Extension Principle) Let \( f \in (\mathcal{B})G([a, b], X) \) and \( g : [a, b] \to Y \). Let \( T \) be a closed subset of \([a, b]\) and

\[
S := [a, b] \setminus T = \bigcup_{i \in \mathbb{N}} E_i,
\]

where \( E_i, i \in \mathbb{N} \), are mutually disjoint elementary sets in \([a, b]\). Further, put \( S_n = \bigcup_{i=1}^{n} E_i \) for \( n \in \mathbb{N} \) and assume that the integral \( \int_{T}[df] \) exists and the sequence \( \{g\chi_{S_n}\} \) is equiintegrable with respect to \( f \) on \([a, b]\). Then the integrals \( \int_{S}[df] \), \( \int_{[a,b]}[df] \), and \( \int_{E_i}[df] \) exist for all \( i \in \mathbb{N} \). Moreover,

\[
\int_{[a,b]} [df] g = \int_{[a,b]} [df] g + \int_{S}[df] g, \tag{5.7}
\]

where

\[
\int_{S}[df] g = \sum_{i=1}^{\infty} \int_{E_i}[df] g. \tag{5.8}
\]

**Proof.** It is obvious that \( S_n \) is an elementary set in \([a, b]\) for every \( n \in \mathbb{N} \) and \( E_i \subset S_n \) for every \( i = 1, 2, \ldots, n \). Since the integral \( \int_{S_n}[df] g = \int_{[a,b]}[df] (g\chi_{S_n}) \) exists for every \( n \in \mathbb{N}, \ E_i \) for \( i = 1, 2, \ldots, n \) are mutually disjoint, and \( f \in (\mathcal{B})G([a, b], X) \), by Theorem 3.7 (i), Corollary 3.11 and Proposition 3.12 we have that the integrals \( \int_{E_i}[df] g \), for \( i = 1, 2, \ldots, n \), exist and

\[
\int_{S_n}[df] g = \sum_{i=1}^{n} \int_{E_i}[df] g \quad \text{for any} \quad n \in \mathbb{N}. \tag{5.9}
\]

Furthermore, as

\[
(g\chi_{S})(t) = \lim_{n \to \infty} (g\chi_{S_n})(t) \quad \text{for all} \quad t \in [a, b], \tag{5.10}
\]

making use of Theorem 4.1 and (5.9), we obtain that

\[
\int_{S}[df] g = \lim_{n \to \infty} \int_{S_n}[df] g = \sum_{i=1}^{\infty} \int_{E_i}[df] g,
\]

i.e. (5.8) is true. Finally, (5.8) together with Proposition 3.10 and Corollary 3.11 imply (5.7). This completes the proof. \( \square \)
Remark 5.4. Recall, cf. Remark 3.4, that under the assumptions of Theorem 5.3 the integrals \( \int_{[a,b]} [df] g \) and \( \int_a^b [df] g \) coincide.

If \( T \subset [a, b] \), then, in general, the existence of the integral \( \int_{[a,b]} [df] g \) does not imply the existence of the integral \( \int_T [df] g \) (except when \( T \) is an elementary subset of \([a, b]\), see Proposition 3.7 (i)). This is demonstrated e.g. by the following example. Let \( g : [0, 1] \to \mathbb{R} \) be defined by
\[
g(t) = \begin{cases} 
0 & \text{if } t = 0, \\
2t \cos \frac{\pi}{t} + 2\pi \sin \frac{\pi}{t} & \text{if } 0 < t \leq 1.
\end{cases}
\]
Then \( g \) is Kurzweil-Henstock integrable on \([0, 1]\). However, the integral \( \int_T g \, dt = \infty \) if \( T := \{ t \in [0, 1] : g(t) \geq 0 \} \) (see e.g. [26], [30]). This means that the opposite implication to that given by the previous theorem does not hold, in general. Next theorem provides a certain affirmative result for the case that the integrator \( f \) is simply-regulated.

Theorem 5.5. Let \( f \in (B)G([a,b], X) \) and \( g : [a, b] \to Y \). Let \( T \) be a closed subset of \([a, b]\) and
\[
S := [a, b] \setminus T = \bigcup_{i \in \mathbb{N}} E_i,
\]
where \( E_i, i \in \mathbb{N}, \) are mutually disjoint elementary sets in \([a, b]\). Further, put \( S_n = \bigcup_{i=1}^n E_i \) for \( n \in \mathbb{N} \) and assume that the integral \( \int_{[a,b]} [df] g \) exists and the sequence \( \{g \chi_{S_n}\} \) is equiintegrable with respect to \( f \) on \([a, b]\). Then

(i) all the integrals \( \int_{E_i} [df] g \) for \( i \in \mathbb{N} \), \( \int_{S} [df] g \), and \( \int_{T} [df] g \) exist and the equalities
\[
\int_{S} [df] g = \sum_{i=1}^{\infty} \int_{E_i} [df] g \quad \text{and} \quad \int_{[a,b]} [df] g = \int_{T} [df] g + \int_{S} [df] g
\]
are true,

(ii) for every \( \varepsilon > 0 \) there is a gauge \( \delta \) on \( T \) such that
\[
\left\| \sum_{j=1}^{\nu(Q)} \left( \int_{\alpha_j}^{\beta_j} [df] g \chi_{T} - \int_{\alpha_j}^{\beta_j} [df] g \right) \right\|_Z < \varepsilon
\]
for every \( T \)-tagged \( \delta \)-fine system \( Q = \{([\alpha_j, \beta_j], \xi_j)\} \) in \([a, b]\).

Proof. (i) As a result of the equiintegrability property of the sequence \( \{g \chi_{S_n}\} \) with respect to \( f \in (B)G([a,b], X) \), as in the proof of Theorem 5.3 we can show that all the integrals \( \int_{E_i} [df] g \), \( i \in \mathbb{N} \), exist and
\[
\int_{S} [df] g = \sum_{i=1}^{\infty} \int_{E_i} [df] g.
\]
Moreover, notice that $S \cup T = [a, b]$ and $S \cap T = \emptyset$. Therefore, by Proposition 3.10 and Corollary 3.11, where we put $S_1 = S$ and $S_2 = T$, we obtain that also the integral $\int_T [df] g$ exists and the equality
\[
\int_T [df] g = \int_{[a, b]} [df] g - \int_S [df] g
\]
holds.

(ii) Let $\varepsilon > 0$ be given and let $\delta$ be a gauge on $[a, b]$ such that
\[
\left\| S(\,d\!f, g, P) - \int_a^b \,d\!f \right\|_Z < \frac{\varepsilon}{6}
\]
and
\[
\left\| S(\,d\!f, g\chi_T, P) - \int_a^b \,d\!f \right\|_Z < \frac{\varepsilon}{6}
\]
whenever $P$ is a $\delta$--fine partition of $[a, b]$.

Suppose now that $Q = \{(\alpha_j, \beta_j, \xi_j)\}$ is a $T$--tagged $\delta$--fine system in $[a, b]$. Then, using the Saks-Henstock lemma (see [46, Lemma 16]), we deduce
\[
\left\| \sum_{j=1}^{\nu(Q)} \left( \int_{\alpha_j}^{\beta_j} \,d\!f \right) (g\chi_T) - \int_{\alpha_j}^{\beta_j} \,d\!f \right\|_Z
\leq \left\| \sum_{j=1}^{\nu(Q)} \left( \int_{\alpha_j}^{\beta_j} \,d\!f \right) (g\chi_T) - \left[ f(\beta_j) - f(\alpha_j) \right] (g\chi_T)(\xi_j) \right\|_Z
+ \left\| \sum_{j=1}^{\nu(Q)} \left[ f(\beta_j) - f(\alpha_j) \right] g(\xi_j) - \int_{\alpha_j}^{\beta_j} \,d\!f \right\|_Z < \varepsilon.
\]
This completes the proof. \hfill \Box

The next result is essentially a corollary of Theorem 5.5.

**Theorem 5.6.** Let $f \in G([a, b], X)$ and $g : [a, b] \to Y$. Let $E$ be an elementary set in $[a, b]$, $T$ be a closed subset of $E$, and
\[
[a, b] \setminus T = \bigcup_{i \in \mathbb{N}} E_i,
\]
where $E_i$, $i \in \mathbb{N}$, are mutually disjoint elementary sets in $[a, b]$. Further, put $S_n = \bigcup_{i=1}^n E_i$ for $n \in \mathbb{N}$ and assume that the integral $\int_E [df] g$ exists and the sequence $\{g\chi_{S_n}\}$ is equiintegrable with respect to $f$ on $[a, b]$. Then the integral $\int_T [df] g$ exists.

**Proof.** Without loss of generality, by Remark 3.8, throughout the proof we may assume that $E = \bigcup_{k=1}^m J_k$ where $\{J_1, J_2, \ldots, J_m\}$ is a minimal decomposition of $E$, and hence by the hypothesis the integrals $\int_{J_k} [df] g$, for $k = 1, 2, \ldots, m$, exist. For every $k = 1, 2, \ldots, m$, we denote $c_k = \inf(J_k)$ and $d_k = \sup(J_k)$ and set
\[
T_k = T \cap [c_k, d_k].
\]
It is obvious that $\bigcup_{k=1}^{m} T_k = T$ and $\bigcap_{k=1}^{m} T_k = \emptyset$ due to the minimal decomposition of $\{J_1, J_2, \ldots, J_m\}$. And then, from Remark 3.6 we obtain that for every $k = 1, 2, \ldots, m$, the integrals

$$\int_{(c_k,d_k)} [df] g, \int_{[c_k,d_k]} [df] g, \int_{(c_k,d_k]} [df] g, \int_{[c_k,d_k]} [df] g, \text{ and } \int_{c_k} [df] g$$

exist.

Furthermore, for every subset $[c_k,d_k] \subset [a, b]$, $k = 1, 2, \ldots, m$, we may construct a sequence

$$\left\{ F_i^{(k)} = E_i \cap [c_k,d_k]; \ i \in \mathbb{N} \right\}$$

of mutually disjoint elementary sets in $[c_k,d_k]$ such that

$$\bigcup_{i \in \mathbb{N}} F_i^{(k)} = [c_k,d_k] \setminus T_k = [c_k,d_k] \setminus T, \text{ for } k = 1, 2, \ldots, m.$$ 

Moreover, if we put $H_n^{(k)} = \bigcup_{i=1}^{n} F_i^{(k)}$ for $n \in \mathbb{N}$, then we will have $H_n^{(k)} \subseteq S_n$ for every $n \in \mathbb{N}$.

Since, from (5.10) in the proof of Theorem 5.3 the sequence $\{g \chi_{S_n}\}$ converges pointwise on $[a, b]$ which further implies the boundedness of the sequence $\{(g \chi_{S_n})(t)\}$ in $Y$ for each $t \in [a, b]$, we obtain that the sequence $\{(g \chi_{H_n^{(k)}})(t)\}$ is bounded in $Y$ for each $t \in [a, b]$ and $k = 1, 2, \ldots, m$. As a consequence, by Proposition 4.8 (i) and Corollary 4.9 the sequence $\{g \chi_{H_n^{(k)}}\}$ is equiintegrable with respect to $f$ on $[c_k,d_k]$, for every $k = 1, 2, \ldots, m$. Therefore, by Theorem 5.5 the integrals $\int_{\bigcup_{i \in \mathbb{N}} F_i^{(k)}} [df] g$ and $\int_{T_k} [df] g$ exist and

$$\int_{T_k} [df] g = \int_{[c_k,d_k]} [df] g - \int_{\bigcup_{i \in \mathbb{N}} F_i^{(k)}} [df] g$$

holds for every $k = 1, 2, \ldots, m$. Finally, the existence of the integral $\int_{T} [df] g$ is obtained by making use of Remark 3.9 and Proposition 3.12

6. Conclusion

In the present paper, using the known results for the Kurzweil-Stieltjes integrals, we were able to prove theorems concerning the equiintegrability, to construct the Harnack extension principle in the setting of Kurzweil-Stieltjes integration, and to show its essential role in solving a problem on how to get the existence of integral $\int_{T_k} [df] g$ for an arbitrary closed subset $T$ of the elementary set $E$ if the integral $\int_{E} [df] g$ exists.

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