1 Introduction

In the 2013 paper [5] titled “MAP estimators and their consistency in Bayesian nonparametric inverse problems”, Dashti, Law, Stuart, and Voss showed for the first time how in a Bayesian inverse problem for Banach space valued parameters, maximizers of (infinitesimal) small ball probabilities can be identified with minimizers of the Onsager–Machlup functional usually interpreted as the logposterior density (although the latter is only strictly valid in finite dimensions). This seminal paper paved the way for further analysis of modes (in various senses) in [6, 8, 1, 7, 4] and is subject to ongoing research. Unfortunately, while the ideas of the paper are groundbreaking, the mathematical arguments are not valid in the general Banach space case and only constitute a rigorous proof in a Hilbert space setting. In addition, there were some other minor mistakes (for example, lemma 3.9. in the paper is slightly too weak for its use case) which were largely removed in [7] (but again in the Hilbert space setting).

The most prominent part of [5] consists of computing limits of ratios of small ball probabilities of type

$$\frac{\mu_0(B_\delta(z^\delta))}{\mu_0(B_\delta(\bar{z}))}$$

where $z^\delta$ is a sequence (or just a single point) and $\mu_0$ is a Gaussian measure on a Banach space $X$.

This manuscript demonstrates that the general claims of [5] are still true and that the technical difficulties can be overcome by digging a bit deeper in the necessary mathematical toolboxes. The main technical problem in the Banach space setting is the incompatibility between the Banach norm $\|\cdot\|_X$ and the Cameron–Martin norm $|\cdot|_E$.

1.1 Gaussian measures in Hilbert and Banach spaces

A Gaussian measure $\mu_0$ on a Hilbert space $X$ has a covariance operator $Q : X \to X$ which is a self-adjoint non-negative compact operator and this constitutes the existence of an orthonormal basis $(e_k)_k$ on $(X, \langle \cdot, \cdot \rangle)$. Because in the Hilbert space setting, the Cameron–Martin space is $E = Q^{1/2}X$ with the inner product $[h, k]_E = \langle Q^{-1/2}h, Q^{-1/2}k \rangle_X$, we immediately obtain an orthonormal basis in $E$ by scaling $h_n := Q^{1/2}e_n$. This works because $Q$ is a


“diagonal” operator on the basis vectors $e_n$. As a sideeffect, we can compute the following norms:

$$|x|^2_E = \sum_n \langle x, h_n\rangle^2$$

$$\|x\|^2_X = \sum_n \langle x, e_n\rangle^2 = \sum_n \langle x, Q^{-1/2} h_n\rangle^2 = \sum_n \frac{\langle x, h_n\rangle^2}{\sigma^2_n}$$

This means, $X$-norm and $E$-norm are just re-weighted variants of each other (because both are Hilbert norms and $E = Q^{1/2} X$).

If $X$ is only a Banach space, we cannot do this, because the covariance operator is merely a mapping $X^*_\mu_0 \times X^*_\mu_0 \to \mathbb{R}$ which does not give us a basis on $X$. While it is possible to construct a Karhunen-Loève decomposition using Rayleigh coefficients (see [2]), we cannot write the $X$-norm by rescaling the $E$-norm, because a general Banach-norm is not of the same form as a Hilbert-norm (just as there is no way to express the $l^1$-norm by rescaling an $l^2$-norm with some constant coefficients). For example, set $X = l^1$ and $\mu_0 = \varnothing \ N(0, \sigma^2_k)$ (with $\sum \sigma_k < \infty$ such that samples are indeed in $X$, as provable by the Kolmogorov two-series theorem). Then $X^*_\mu_0 = \{(y_k) : \sum y_k^2 \sigma^2_k < \infty\}$ and $E = \{(y_k) : \sum \frac{y_k^2}{\sigma^2_k} < \infty\}$. The canonical bases here are $h^*_k = (0, \ldots, 0, \frac{1}{\sigma_k}, 0, \ldots)$ (such that $h^*_k(x) = \frac{x_k}{\sigma_k}$) and $h_k = (0, \ldots, 0, \sigma_k, 0, \ldots)$. Then we can evaluate the Cameron–Martin norm of an element by computing

$$|x|^2_E = \sum_k |h^*_k(x)|^2$$

but the original space’s norm is given by $\|x\|^2_X = \left(\sum_{k=1}^{\infty} |\sigma_k \cdot h^*_k(x)|^2\right)^{1/2}$ which is not just a scaled version of $|x|^2_E$ (as compared to what we can do in the Hilbert space setting).

Note: The expression for $\|x\|_X$ above can be derived in this setting because we are in a sequence space and everything boils down to entries at some position. The situation is much more difficult if $X$ is not a sequence space.

In other words, in the Banach setting, there is no way of computing the $X$-norm with the use of the Cameron–Martin basis. The only connection is given by $E \subset X$, i.e. we can always bound $\|\cdot\|_X \leq C \|\cdot\|_E$.

The dominant technical issue here is the incompatibility of $X$-norm-balls with the $E$-norm. In the Hilbert space setting (thinking about level sets of the both norms) we can think of axis-aligned ellipsoids only differing in their principal axes so everything can be translated by scaling (and at most additional rotations if the ellipses’ axes are not aligned), but in the Banach space setting, we have to reconcile ellipsoids (from the $E$-norm) with more irregular shapes like an $l^1$-ball, which is not analytically tractable anymore.

The tools with which we overcome this problem, are the following:

First, we use a technique by Bay and Croix [2] (unknown at the time of [5]) which allows for explicit construction of a Karhunen-Loève decomposition in the Banach space setting.

Second, we drop explicit translation between $X$-norm and $E$-norm and instead prove an abstract norm bound result in lemmata 2 and 3 which allow lower bounds of the CM-norm in terms of the ambient space norm (and which get stricter on a sequence of projection subspaces). With that we can prove an explicit version of Anderson’s inequality which we can use.

All in all, the arguments in [5] work nicely in Hilbert space but need to be adapted in the general Banach space setting.

2
1.2 Construction of bases

A secondary issue is the construction of a basis of functions which works in both $X$ and $E$. In the Hilbert space setting it makes the most sense to construct a basis first on $X$ (using the spectral decomposition of the covariance operator) and then scale this basis to get a basis on $E$ and then subsequently on $X^*_\mu$ using the reproducing kernel/Cameron–Martin isometry $R_{\mu_0}$. The Riesz isomorphism allows for an easy identification of $X$ and $X^*$, essentially letting us handwave away all differences between the various spaces. Graphically, the basis is constructed as follows.

$$
X \xrightarrow{\text{scale}} E \xrightarrow{R_{\mu_0}^{-1}} X^*_\mu
$$

In particular, the “scale” arrow is reversible: We can translate the basis in $X$ and $E$ back and forth, compatible with the respective norms.

In the Banach space setting there are at least two different possibilities: By density of $X^*_\mu$ in $X^*_{\mu_0}$, we can construct (by Gram-Schmidt-Orthogonalization) an orthonormal basis in $(X^*_\mu, \langle \cdot, \cdot \rangle_{L^2(X, \mu_0)})$ consisting of elements in $X^*$. This can be transported to $E$ via the mapping $R_{\mu_0}$:

$$
X^*_\mu \xrightarrow{R_{\mu_0}} E
$$

The second way is to use Rayleigh coefficients of the operator $R_\mu$ to iteratively decompose the Cameron–Martin space and obtain a basis of $E$ (see [2]). One problem with that approach is that this can lead to a decomposition with “eigenvalues” (actually, Rayleigh coefficients) $\lambda_k$ that are not summable (see comment 4 of the mentioned paper), i.e. the operator $R_\mu$ is non-nuclear, but we will use this method nonetheless.

For both approaches, there is no way of “going back” to the original space $X$ and its norm. In the Banach space setting, this is a major issue that needs to be handled in some more depth.

1.3 Lemma 3.9 has the wrong form

For reference we recall:

Lemma 1 (Lemma 3.9, [5]). Let $(w^\delta)_\delta \subset X$ with $z^\delta \rightharpoonup 0$ weakly in $X$ but $z^\delta \not\to 0$ strongly in $X$. Then for all $\epsilon > 0$ there is a $\delta > 0$ such that

$$
\frac{\mu_0(B_\delta(w^\delta))}{\mu_0(B_\delta(0))} < \epsilon.
$$

In the proof of theorem 3.5. in [5], there is a sequence $z^\delta$ converging weakly, but not strongly, to some $\bar{z}$. The authors then continue to state that they apply lemma 3.9 now to the sequence $w^\delta := z^\delta - \bar{z}$. If we were to apply this lemma, this would yield a bound of the form

$$
\frac{\mu_0(B_\delta(w^\delta))}{\mu_0(B_\delta(0))} = \frac{\mu_0(B_\delta(z^\delta - \bar{z}))}{\mu_0(B_\delta(0))} < \epsilon,
$$

i.e. we bound balls centered at $z^\delta - \bar{z}$. On the other hand, the proof claims bounds on

$$
\frac{\mu_0(B_\delta(z^\delta))}{\mu_0(B_\delta(0))} < \epsilon
$$

(1)

which is not implied by an application of the lemma. This means that we need a slightly stronger version of the lemma here. Kretschmann [7] solved this problem in the Hilbert space by proving the lemma for ratios of the form $\frac{\mu_{0}(B_{\delta}(z^\delta))}{\mu_{0}(B_{\delta}(0))}$.

For the Banach space case we will complete this argument like this:

1. Prove a version of lemma 3.9. which says: If $z^\delta$ converges weakly in $X$, but not strongly in $X$, to some $\tilde{z} \in X$, then for all $\epsilon > 0$ there is a $\delta > 0$ such that

$$\frac{\mu_{0}(B_{\delta}(z^\delta))}{\mu_{0}(B_{\delta}(\tilde{z}))} < \epsilon.$$ 

In addition to the aforementioned problems (i.e. incompatibility of $X$- and $E$-norm and working with bases in both spaces), there is another technical problem which only arises in the true Banach space setting: The proof of lemma 3.9 in [5] makes heavy use of the following fact about sequences in Hilbert space: If a sequence $z^\delta \rightharpoonup \tilde{z}$ but $z^\delta \not\to \tilde{z}$, then $\liminf \|z^\delta\|_X > \|\tilde{z}\|_X$. This is also sometimes called the Radon-Riesz property and it is true for all uniformly convex complete vector spaces, including Hilbert spaces (but not Banach spaces in general): Consider $L^1(0,1)$ and $f_n(x) = 1 + \sin(nx)$. Then $f_n \rightharpoonup 1 = f$, and $\|f_n\|_1 \to 1 = \|f\|_1$ but $f_n$ does not converge strongly. The Radon-Riesz property leverages a strict margin between the norms $\|z^\delta\|_X$ of the sequence and its weak limit $\|\tilde{z}\|_X$ which is explicitly used in the computation of the proof of lemma 3.9.

This is another problem connected to making the proof work in the general Banach setting.

2. Then, when we need a bound like (1), we can use that

$$\frac{\mu_{0}(B_{\delta}(z^\delta))}{\mu_{0}(B_{\delta}(0))} = \frac{\mu_{0}(B_{\delta}(z^\delta))}{\mu_{0}(B_{\delta}(\tilde{z}))} \cdot \frac{\mu_{0}(B_{\delta}(\tilde{z}))}{\mu_{0}(B_{\delta}(0))}$$

and we can control the first term with our new lemma and the second term with Anderson’s inequality.

Note that this means that our version of lemma 3.9 is stronger than the versions both in [5] and in [7], because we can bound both $\frac{\mu_{0}(B_{\delta}(z^\delta))}{\mu_{0}(B_{\delta}(0))}$ and $\frac{\mu_{0}(B_{\delta}(z^\delta))}{\mu_{0}(B_{\delta}(\tilde{z}))}$ in the limit. The last quantity is of more canonical interest in its own regard.

1.4 Limit superior bounds on small ball probabilities

On page 12 of [5], it is claimed that for $z^\delta$ converging weakly in $E$ and strongly in $X$, but not strongly in $E$ to some $\tilde{z} \in E$, we have

$$\limsup_{\delta \to 0} \frac{\mu_{0}(B_{\delta}(z^\delta))}{\mu_{0}(B_{\delta}(\tilde{z}))} \leq 1.$$ 

It is not immediately clear that this is correct but was later shown by Dashti (cited as communicated personally in [7], lemma 4.13). In the Banach space setting we need a stronger version of this statement, replacing $\mu_{0}$ with a (not necessarily Gaussian) measure $\tilde{\mu}$ absolutely continuous with respect to a Gaussian $\mu_{0}$.
2 Small ball probabilities for Gaussian measures in Banach space

Let $\mu$ be a Gaussian measure\(^1\) on a Banach space $X$, with Cameron–Martin space $E$. We denote by $B_\delta(x)$ the ball of $X$-norm $\delta$ around $x \in X$. The missing fragments in [5] are the following three technical lemmata about small ball probabilities:

1. For $z \in X$ and $r \in [0, 1]$, there is a constant $C$ such that

$$\frac{\mu(B_\delta(z))}{\mu(B_\delta(r \cdot z))} \leq \exp \left[ -C^2 \cdot \frac{(\|z\|_X - \delta)^2 - (r \|z\|_X + \delta)^2}{2} \right].$$

This is a stronger version of lemma 3.6 in [5].

2. For $z^\delta \rightharpoonup \bar{z}$ weakly in $X$ for some $\bar{z} \in X \setminus E$,

$$\liminf_{\delta \to 0} \frac{\mu(B_\delta(z^\delta))}{\mu(B_\delta(0))} = 0.$$  

This is the statement of lemma 3.7 in [5].

3. For $z^\delta \rightharpoonup \bar{z}$ weakly in $X$, but $z^\delta \not \to \bar{z}$ in $X$ for some $\bar{z} \in E$,

$$\liminf_{\delta \to 0} \frac{\mu(B_\delta(z^\delta))}{\mu(B_\delta(\bar{z}))} = 0.$$  

This is a stronger version of lemma 3.9 in [5]. With these statements we will be able to prove theorem 1 in the Banach space setting, which is the main statement of [5], showing existence of a minimizing sequence of small ball probability centers converging to a point in $\bar{z} \in E$ (which then is the MAP estimator).

We start by deriving the proper way to finitely approximate probabilities of infinite-dimensional sets.

2.1 How to do finite approximation

The standing assumption here is that $X$ is a Banach space and we consider a centered (i.e. mean zero) Gaussian measure $\mu$ with its Cameron–Martin space $E$.

Gaussian measures in infinite dimensions do not have a density because there is no Lebesgue measure. But in a non-literal sense one could write

$$\mu(A) = \int_A \exp \left( -\frac{1}{2} \|x\|^2_E \right) dx.$$  

(2)

Note that the integrand is 0 almost surely as $\mu(E) = 0$. This counterbalances the “infinity” of what an equivalent object to Lebesgue measure in infinite dimensions (which of course does not exist) would do.

\(^1\)In [5], it is called $\mu_0$ because $\mu$ is a non-Gaussian posterior. But as we only work with the prior, we abbreviate here by dropping the 0 from now on.
Notation-wise we now follow \cite{3}, for example $R_{\mu} : X^*_\mu \to E$ is the isomorphism between $X^*_\mu = X^* \cdot L^2(\mu)$ and the Cameron–Martin space $E$.

Representation (2) is true at least asymptotically as we will see in the following. It is possible, as in \cite{2}, to find a sequence $\{h^*_k\}_k \subset X^*$ dense in $X^*_\mu$ such that $B_{\mu}(h^*_k, h^*_l) = \delta_{k,l}$. This corresponds to an orthonormal basis in Cameron–Martin space $(h^*_k)_k \subset E$ where $h^*_k = R_{\mu} h^*_k$.

Finite approximation in this context can be done by defining $Q^n x = \sum_{k=1}^n h^*_k \cdot h^*_k(x)$. The problem is that the $h^*_k$ have (interpreted as random variables) variance 1 and so the original covariance structure of $\mu$ is hidden in the product $h^*_k \cdot h^*_k(x)$. For this reason, it is useful to consider a different approximation operator, given by

\[ P^n : X \to \text{span}\{x_1, \ldots, x_k\}, \quad P^n(x) = \sum_{k=1}^n x^*_k(x) \cdot x_k \]

where $(x^*_k)_k$ are sequences in $X$ and $X^*$ respectively with $x_k = \sigma^{-1} h^*_k$ and $x^*_k = \sigma_k \cdot h^*_k$. Here $\sigma_k^2$ are the Rayleigh quotients obtained in the decomposition procedure and should be thought of as being the variance of $x^*_k$. The details for this construction can be retrieved from \cite{2}, with the most important properties being that $x^*_k = \sigma_k \cdot h^*_k$, i.e. the $x^*_k$ are i.i.d. $N(0, \sigma^2)$ r.v.s. Note also that $x^*_k(x_i) = h^*_k(h_i) = \delta_{k,l,i}$.

Note that $P^n = Q^n$ is the same operator, but the normalization factor $\sigma_k$ is shifted in the product: $h^*_k \cdot h^*_k(x) = (\sigma_k \cdot x_k) \cdot (\sigma_k^2 \cdot x_k)$. Sometimes it's useful to work with a representation which is explicitly finite-dimensional, so we define

\[ S^n : X \to \mathbb{R}^n, \quad S^n(x) = (x^*_k(x))_{k=1}^n = (\sigma_k \cdot h^*_k(x))_{k=1}^n. \]

Note that this is an equivalent characterization, because $T : \mathbb{R}^n \to \text{ran}(P^n)$ with $T(\alpha) = \sum_{k=1}^n \alpha_k x_k$ is an isomorphism. For later convenience we write $a_k := \sigma_k^2 = \|h^*_k\|^2_x$.

Computation by finite approximation works like this: Note that $A \subset (S^n)^{-1} S^n A$. Define $N_n := (S^n)^{-1} S^n A \setminus A$, then $N_{n+1} \subset N_n$ with $\bigcap_{n} N_n = \emptyset$. The Gaussian measure $\mu$ is continuous from above, which means that $\lim_{n \to \infty} \mu(N_n) = 0$, i.e.

\[ \lim_{n \to \infty} \mu((S^n)^{-1} S^n A) = \mu(A). \]

In other words, we can approximate the measure of a set arbitrarily well by finite approximation.

Now we take a closer look at the form of this approximation. Note that $\mu(0) = 0$ and $\mu_{\sigma_0} = N(0, \sigma_0^2)$ due to the fact that the $h^*_k$ are i.i.d. $N(0, 1)$ random variables. Thus,

\[ \mu(A) = \lim_{n \to \infty} \int_{S^n A} \exp\left(-\frac{x_1^2}{2\sigma_1^2} - \cdots - \frac{x_n^2}{2\sigma_n^2}\right) dx = \lim_{n \to \infty} \int_{S^n A} \exp\left[-\frac{1}{2} \left(\sum_{k=1}^n a_k x_k^2 \right)\right] dx \]

Remark 1. In contexts where we can explicitly write down the $h^*_k$ and $h_k$ (for example if $X$ is a sequence space and $\mu$ a diagonal Gaussian measure with variance $\sigma^2$ on the entries), the sum $\sum_{k=1}^n a_k x_k^2$ is identical to the Cameron–Martin norm, which motivates the formal interpretation of the integral (2) above.

We will also need the following object, which is a “slice projection” on a subset of dimensions:

\[ S^n_m : X \to \mathbb{R}^{n-m}, \quad S^n_m(x) = (\sigma_k \cdot h^*_k(x))_{k=m+1}^n \]

Note that in particular $S^n = S^n$.

An important tool in the following will be various norm bounds between $\| \cdot \|_X$ and $| \cdot |_E$. 

6
2.2 Norm bounds between $X$ and $E$

We can always bound the Cameron–Martin norm from below by the ambient space norm.

$$\|x\|_E = \sup_{f \in X^*} \frac{|f(x)|}{\|f\|_{L^2}} = \sup_{f \in X^*} \frac{|f(x)|}{\|f\|_{L^2}} \cdot \frac{\|f\|_{L^2}}{\|f\|_{L^2}} \geq \frac{1}{C} \cdot \|x\|_X$$

where $C$ is the best constant.

Following Lemma 2, we get stronger bounds on the space $X$.

Note that for $x \in X$, we can compute

$$\left| \sum_{k=1}^n a_k x_k^* \right| \leq \left( \sum_{k=1}^n a_k^2 \right)^{\frac{1}{2}} \left( \sum_{k=1}^n \|x_k\|^2 \right)^{\frac{1}{2}} \leq \left( \sum_{k=1}^n \|x_k\|^2 \right)^{\frac{1}{2}} \sup_{f \in X} \frac{|f(x)|}{\|f\|_{L^2}} \cdot \frac{\|f\|_{L^2}}{\|f\|_{L^2}} \cdot \frac{\|f\|_{L^2}}{\|f\|_{L^2}}.$$

For brevity, denote the set of such functionals by $F_n = \{ f \in X^* : f(h_k) = 0, k = 1, \ldots, n \}$

see that using the reproducing kernel property $f(h) = \langle R_1 h, f \rangle_{L^2(X, \mu)}$ we can equivalently characterize $F_n = \{ f \in X^* : B_n(f, h_k^*) = 0, k = 1, \ldots, n \}$

Thus, for $x \in X^n$, we get

$$\left| \sum_{k=1}^n a_k x_k^* \right| \leq \left( \sum_{k=1}^n a_k^2 \right)^{\frac{1}{2}} \left( \sum_{k=1}^n \|x_k\|^2 \right)^{\frac{1}{2}} \sup_{f \in X} \frac{|f(x)|}{\|f\|_{L^2}} \cdot \frac{\|f\|_{L^2}}{\|f\|_{L^2}} \cdot \frac{\|f\|_{L^2}}{\|f\|_{L^2}}.$$

Note that for $n \to \infty$, this bound becomes stronger as $C_n := \frac{1}{K_n} \to \infty$ for $n \to \infty$. 

The next lemma shows that the norm bounds also hold for truncated (i.e. projected) elements in $E$:

**Lemma 3 (Norm bounds for projections).** For $x \in X$, we have $|P^i(x)|_E \geq C \|P^i(x)\|_X$. Also, similarly to lemma 2, we get stronger bounds on the space $X^\perp = (P^i)^\perp$, i.e. for all $x \in X^n$ and $n < 1$, we have $|P^i(x)|_E \geq C_n \|P^i(x)\|_X$ with $C_n \leq C_{n+1}^i \leq C_n^i$ with $\lim_{n \to \infty} C_n^i = C_n$ from lemma 2.

**Proof.** Define $\Sigma_i = \text{span}\{x_1^*, \ldots, x_i^*\}$, i.e. $f \in \Sigma_i$ has form $f = \sum_{k=1}^i a_k x_k^*$. Note that for such an $f$, we can compute $B_\mu(f, f) = f(R_\mu(f)) = \sum_{k=1}^i a_k x_k^* \left( \sum_{l=1}^i a_l R_\mu(x_l^*) \right) = \sum_{k=1}^i a_k^2 \cdot \sigma_k^2$ by
Consider $X$ on the Banach space case. Let’s illustrate the issue for a more concrete example.

As mentioned before, the proof in the Banach space $X$ we have

$$\|P^l(y)\|_X = \sup_{f \in X^*} \frac{f(P^l(y))}{\|f\|_{X^*}} = \sup_{f \in \Sigma_l} \frac{f(P^l(y))}{\|f\|_{X^*}} \leq \sup_{f \in \Sigma_l} \frac{f(P^l(y))}{\|f\|_{X^*}} \cdot \sup_{f \in X^*} \frac{B_\nu(f, f)}{\|f\|_{X^*}}$$

$$\leq \sup_{a \in \mathbb{R}^{l^2}} \sum_{k=1}^l a_k \cdot \sigma_k \cdot h_k^*(y) \cdot \sqrt{\sum_{k=1}^l a_k^2 \cdot \sigma_k^2} = C \cdot \sup_{\beta \in \mathbb{R}^{l^2}} \sum_{k=1}^l \beta_k \cdot h_k^*(y) \cdot \sqrt{\sum_{k=1}^l \beta_k^2} = C \cdot \sqrt{\sum_{k=1}^l (h_k^*(y))^2}$$

$$= C \cdot \|P^l(y)\|_{\tilde{F}}.$$

The proof for the stronger norm bounds follows along the same lines as in the proof of lemma 2.

Note that we do not need to have $x \in E$ (compare to lemma 2) because the range of the projection $P^l$ is a subset of $E$ already. \qed

The preceding lemma motivates ordering the functionals $h_k^*$ by requiring $\sigma_{k+1} \leq \sigma_k$ or equivalently $a_k \leq a_{k+1}$, which we will assume from now on.

### 2.3 An explicit Anderson's inequality

In this section we will think about the statement of lemma 3.6 in [5], which we record here:

**Lemma 4** (Lemma 3.6 in [5]). Let $\delta > 0$. For any centred Gaussian measure $\mu$ on a separable Banach space $X$ we have

$$\frac{\mu(B_\delta(z))}{\mu(B_\delta(0))} \leq e^{-a_1/2(\|z\|_X - \delta^2 - \delta)}$$

where $a_1$ is a constant independent of $z$ and $\delta$.

**Remark 2.** A very cartoonish distillation of the proof idea in the Hilbert space setting proposed in [5] is the following:

$$\mu(B_\delta(z)) = \int_{B_\delta(z)} e^{-\frac{|x|^2}{2}} \, dx = \int_{B_\delta(0)} e^{-\frac{|x|^2}{2}} \, dx$$

$$\leq e^{-\frac{|w|^2}{2}} \cdot \int_{B_\delta(0)} e^{-\frac{|w|^2}{2}} \, dw$$

where the last inequality uses the relation $\|x\|_E^2 = \sum_k a_k x_k^2 \geq a_1 \cdot \sum_k x_k^2 = a_1 \cdot \|x\|_2^2 = \sum_k \frac{|x_k|^2}{\sigma_k^2}$. Now of course this is not a rigorous proof (especially the step from the first line to the second line, and also because it is trying to use an $\infty$-dimensional Lebesgue density). But the main idea remains: Extract the “modulus of continuity” between the Cameron–Martin norm and the space norm (here: $l^2$, because every Hilbert norm can be written as an $l^2$-norm) as an exponential decay rate.

As mentioned before, the proof in [5] is valid for a Hilbert space $X$ and we only need to work on the Banach space case. Let’s illustrate the issue for a more concrete example.

Consider $X = l^1$ with the $l^1$ norm, a Gaussian measure $\mu = \odot N(0, \sigma_k^2)$ such that $\sum_k \sigma_k^2 < \infty$ (then samples lie a.s. in $X$) with Cameron–Martin space $E = \{ y : \|y\|_{l^2}^2 := \sum_k \frac{|y_k|^2}{\sigma_k^2} < \infty \}$. For brevity we write $a_k := \sigma_k^{-2}$ such that we can write $|x|_{E}^2 = \sum_k a_k x_k^2$. 

8
If we want to apply the proof idea above to $l^1$ we can observe that

$$\|x\|_1 = \sum_k |x_k| = \sum_k \frac{|x_k|}{\sigma_k} \cdot \sigma_k \leq \sqrt{\sum_k \frac{x_k^2}{\sigma_k^2}} \cdot \sqrt{\sum_k \sigma_k}^2,$$

i.e. $\|x\|_E^2 \geq \frac{\|x\|_1^2}{\sum_k \sigma_k^2} = \frac{\|x\|_1^2}{\sum_k \sigma_k^2}$. We can hope that the result carries over by replacing the 2-balls with 1-balls and the rate by the correct modulus of continuity between the $l^1$-norm and the $E$-norm. This is indeed the case (but we obviously need to make the proof more rigorous than this).

We start with a sketch of the proof in the concrete setting $X = l^1$ in order to demonstrate the main ideas, after that we present a correct proof for the setting of a general Banach space $X$.

**Lemma 5** (Special case $l^1$). Let $\delta > 0$. Consider the $l^1$-balls $B_1^\delta(z)$ with radius $\delta$ around $z$. Then we have

$$\frac{\mu(B_1^\delta(z))}{\mu(B_1^\delta(0))} \leq c \cdot e^{-\frac{(\|z\|_1-\delta)^2}{2\sum_k \sigma_k^2}}.$$ 

**Sketch of proof.** The main idea is that we can bound $\sum_k \frac{|x_k|^2}{\sigma_k^2} \leq \|x\|_E^2$. Then (with finite-dimensional approximation), we approximate by

$$\frac{\mu(B_1^\delta(z))}{\mu(B_1^\delta(0))} \approx \frac{\int_{B_1^\delta(z)} e^{-\frac{a_1 x_1^2 + \ldots + a_n x_n^2}{2}} \, dx}{\int_{B_1^\delta(0)} e^{-\frac{a_1 x_1^2 + \ldots + a_n x_n^2}{2}} \, dx}$$

Now we write $a_1 x_1^2 + \cdots + a_n x_n^2 = (|x_1| + \cdots + |x_n|)^2 \cdot \left[ \frac{1}{\sum_k \sigma_k} - \varepsilon \right] + R(x)$. We know that $R(x)$ is positive because of the relation between the 1-norm and the $E$-norm. Furthermore, we can bound $|x_1| + \cdots + |x_n| > (\|z\|_1 - \delta)$ for $x \in B_1^\delta(z)$ and $|x_1| + \cdots + |x_n| \leq \delta$ for $x \in B_1^\delta(0)$. Then

$$\leq \frac{e^{-\frac{(\|z\|_1-\delta)^2}{2\sum_k \sigma_k^2}} \cdot \left[ \frac{1}{\sum_k \sigma_k} - \varepsilon \right]}{e^{-\frac{\varepsilon}{2} \cdot \left[ \frac{1}{\sum_k \sigma_k} - \varepsilon \right]}} \cdot \frac{\int_{B_1^\delta(z)} e^{-R(x)} \, dx}{\int_{B_1^\delta(0)} e^{-R(x)} \, dx}$$

Now we don’t need to know the specific form of $R$, just that it is positive and “coercive” such that $e^{-R(x)}$ is normalizable on $\mathbb{R}^n$ and thus constitutes a probability measure’s density for which Anderson’s inequality holds and we can drop the last fraction. Then as this holds for all $\varepsilon > 0$, we can take $\varepsilon \to 0$ (and make the transition $n \to \infty$) and obtain (rewriting $a_k^{-1} = \sigma_k^2$)

$$\frac{\mu(B_1^\delta(z))}{\mu(B_1^\delta(0))} \leq e^{-\frac{(\|z\|_1-\delta)^2}{2\sum_k \sigma_k^2}}.$$ 

Now we can rigorously prove the statement we are interested in.

---

2Although we are not interested in $R(x)$ outside of the ball $B_1^\|z\|_1$ due to the domains of the integrals we are considering.
On the other hand, for

Lemma 6 (An explicit Anderson’s inequality, similar to lemma 3.6 in [5]). Let \( \delta > 0 \) and \( \mu \) be a centred Gaussian measure on a Banach space \( X \). Then for \( z \in X \) and \( r \in [0, 1] \), there is a constant \( 0 < C = \inf_{x \in X} \| x \|_X \) such that

\[
\frac{\mu(B_{\delta}(z))}{\mu(B_{\delta}(r \cdot z))} \leq \exp \left[ -C^2 \cdot \frac{\| z \|_X^2 - (\| r \|_X + \delta)^2}{2} \right].
\]

Proof. Consider first approximation via \( S^n \), and set \( \mu_n := \mu \circ S^n^{-1} \). Note that \( S^n = T^{-1} \circ P^n \) (with \( S^n, T, P^n \) defined as in section 2.1).

Now use the definition of the push-forward measure \( \lambda \circ T^{-1} \) and we get

\[
\int_{p^nB_z} \exp \left( -\frac{1}{2}(a_1x_1^2 + \cdots + a_nx_n^2) \right) d\lambda \circ T^{-1}(y) = \exp \left( -\frac{1}{2}(a_1x_1^2 + \cdots + a_nx_n^2) \right) \int_{p^nB_z} \exp \left( -\frac{1}{2}(a_1x_1^2 + \cdots + a_nx_n^2) \right) d\lambda \circ T^{-1}(y)
\]

Note that by lemma 3, \( a_1x_1^2 + \cdots + a_nx_n^2 = \| P^n y \|_E^2 = \| P^n z \|_X^2 \cdot (C^2 - \varepsilon_0) + R_{n,\varepsilon_0}(y) \) with \( R_{n,\varepsilon_0}(y) \geq 0 \) after choosing an arbitrarily small \( \varepsilon_0 > 0 \). As \( P^n x \to x \mu \)-almost-surely, for any arbitrarily small \( \varepsilon_1 > 0 \) we can choose an \( N_1 \) such that for all \( n \geq N_1 \), \( \| P^n z \|_X \geq (1 - \varepsilon_1) \| z \|_X \). Then for any \( y \in P^n B_{\delta}(z) \), we know that \( \| P^n y \|_X \geq \| P^n z \|_X - \| P^n y - P^n z \|_X \geq (1 - \varepsilon_1) \| z \|_X - \delta \).

On the other hand, for \( y \in P^n B_{\delta}(r z) \), we immediately see that \( \| P^n y \|_X \leq r \| z \|_X + \delta \). All in all,

\[
\left( C^2 - \varepsilon_0 \right) \frac{\| z \|_X - \delta}{1 - \varepsilon_1} \right) \int_{p^nB_z} \exp \left( -\frac{1}{2}(\| r \|_X + \delta)^2 \right) d\lambda \circ T^{-1}(y)
\]

Figure 1: Illustration of lemma 6 in two dimensions with \( \| \cdot \|_X = \| \cdot \|_1 \). Shaded ellipses represent level sets of the Cameron–Martin norm. Note how the two norms can be very non-conforming.
Now the ratio of integrals is bounded from above by 1 due to Anderson’s inequality. Thus, for all \( \varepsilon_0, \varepsilon_1 > 0 \) there is an \( N_1 \in \mathbb{N} \) such that for all \( n \geq N_1 \),
\[
\frac{\mu_n(S^n B_\delta(z))}{\mu_n(S^n B_\delta(r \cdot z))} \leq \exp\left[-\frac{(C^2 - \varepsilon_0)(1 - \varepsilon_1)^2}{2} (\|x - \frac{\delta}{1 - \varepsilon_1}\|^2)\right]
\]

Now note that \( \mu(B_\delta(z)) = \lim_{n \to \infty} \mu_n(S^n B_\delta(z)) \) (and equivalently for \( B_\delta(r \cdot z) \)). Thus for all \( \varepsilon_2 > 0 \) there is an \( N_2 \in \mathbb{N} \) such that for all \( n \geq N_2 \), we have \( \mu_n(S^n B_\delta(z)) \geq \mu(B_\delta(z))(1 - \varepsilon_2) \) and \( \mu_n(S^n B_\delta(r \cdot z)) \leq \mu(B_\delta(r \cdot z))(1 + \varepsilon_2) \). Then for all \( n \geq N_2 \),
\[
\frac{\mu(B_\delta(z))}{\mu(B_\delta(r \cdot z))} \leq \frac{\mu_n(S^n B_\delta(z))(1 + \varepsilon_2)}{\mu_n(S^n B_\delta(r \cdot z))(1 - \varepsilon_2)}
\]

Thus,
\[
\frac{\mu(B_\delta(z))}{\mu(B_\delta(r \cdot z))} \leq \frac{1 + \varepsilon_2}{1 - \varepsilon_2} \exp\left[-\frac{(C^2 - \varepsilon_0)(1 - \varepsilon_1)^2}{2} (\|x - \frac{\delta}{1 - \varepsilon_1}\|^2)\right]
\]

and the statement follows from choosing \( \varepsilon_0, \varepsilon_1, \varepsilon_2 > 0 \) arbitrarily small.

\[\Box\]

### 2.4 Small ball probabilities around weakly converging sequences

In this section we will study the behaviour of small ball probabilities if their centres are weakly converging subsequences themselves, in particular

1. For \( z^\delta \rightharpoonup \bar{z} \) weakly in \( X \) for some \( \bar{z} \in X \setminus E \),
   \[
   \liminf_{\delta \to 0} \frac{\mu(B_\delta(z^\delta))}{\mu(B_\delta(0))} = 0.
   \]

2. For \( z^\delta \rightharpoonup \bar{z} \) weakly in \( X \), but \( z^\delta \not\rightharpoonup \bar{z} \) in \( X \) for some \( \bar{z} \in E \),
   \[
   \liminf_{\delta \to 0} \frac{\mu(B_\delta(z^\delta))}{\mu(B_\delta(\bar{z}))} = 0.
   \]

Before we start, let’s interpret these statements first by looking at figure 2. These sketches are unavoidably an abridging explanation but will hopefully help transmit the right message. The first bound illustrates the situation that \( z^\delta \rightharpoonup \bar{z} \) for some \( \bar{z} \notin E \). As each coordinate \( x_m^*(z^\delta) \to x_m^*(\bar{z}) \) converges and the \( E \)-norm of \( \bar{z} \) can be written in terms of these coefficients, the \( E \)-norm of \( z^\delta \) diverges to \( \infty \). Even more, this pushes the sequence so much “out there”, that the probability of balls around \( z^\delta \) carry essentially no probability mass in the limit (as compared to balls around 0).

The second bound handles the setting where \( \bar{z} \in E \) and \( z^\delta \rightharpoonup \bar{z} \) weakly in \( X \) but \( z^\delta \not\rightharpoonup \bar{z} \) strongly in \( X \). This means that the difference between \( z^\delta \) and \( \bar{z} \) is a “travelling bump” which is evermore strongly penalized by the Cameron–Martin norm (which punishes “later” coefficients more strongly). This leads to the whole ball \( B_\delta(z^\delta) \) being attached with a diverging \( E \)-norm and thus a vanishing small ball probability. Note that we can not use
\[ \liminf \| z^\delta \|_X > \| \bar{z} \|_X, \text{ as the Radon-Riesz property does not necessarily hold for general Banach spaces.} \]

We start with the first statement.

**Lemma 7** (corresponds to Lemma 3.7. in [5].) Let \( \bar{z} \notin E \) and \( z^\delta \to \bar{z} \) in \( X \). Then for all \( \epsilon > 0 \) there is a \( \delta > 0 \) such that

\[ \mu (B_\delta (z^\delta)) \frac{\mu (B_\delta (0))}{\mu (B_\delta (0))} < \epsilon. \]

**Proof.** As before, we start by approximating, and we consider \( \mu_n (S^n B_\delta (z^\delta)) \frac{\mu_n (S^n B_\delta (0))}{\mu_n (S^n B_\delta (0))} \).

We claim the following: For any \( A > 0 \) there is an \( N_1 \in \mathbb{N} \) and a \( \delta_1 > 0 \) such that for all \( n \geq N_1 \) and \( \delta < \delta_1 \),

\[ \inf_{x \in S^n B_\delta (z)} \sum_{k=1}^n a_k x_k^2 \geq A^2. \]

The proof of this goes as follows: We can assume the contrapositive, i.e. there is a \( A > 0 \) such that for all \( N_1 \) and \( \delta_1 > 0 \), there are \( n \geq N_1 \) and \( \delta < \delta_1 \) such that \( \inf_{x \in S^n B_\delta (z)} \sum_{k=1}^n a_k x_k^2 = \inf_{x \in P^n B_\delta (z^n)} \| P^n x \|_E^2 < A^2 \). We fix \( \delta > 0 \) and \( n \) for the moment and define a minimizing sequence \( (x_{m,n}^\delta) \) in \( P^n B_\delta (z) \) with \( \| P^n x_{m,n}^\delta \|_E \leq 2 \cdot A \). By boundedness of this sequence in \( P^n E \), which is a (finite-dimensional) Hilbert space, we can extract a subsequence (which we denote by the same symbol) such that \( x_{m,n}^\delta \to x_{m}^{\delta,n} \) in \( P^n E \) for some \( x_{m}^{\delta,n} \). Clearly, \( \| P^n x_{m}^{\delta,n} \|_E \leq 2 \cdot A \), as well. Now we can do this for any \( \delta \), even for a sequence of \( (\delta_m)_{m \in \mathbb{N}} \) with \( \delta_m \to 0 \).

By this method, we obtain a sequence of sequences:

\[ x_j^{\delta_1,n} \to x^{\delta_1,n} \text{ in } P^n E \text{ and } \forall j : \| P^n x_j^{\delta_1,n} \|_E \leq 2 \cdot A, \| P^n x_{j^{\delta_1,n}} \|_E \leq A \]

\[ x_j^{\delta_2,n} \to x^{\delta_2,n} \text{ in } E \text{ and } \forall j : \| x_j^{\delta_2,n} \|_E \leq 2 \cdot A, \| x_{j^{\delta_2,n}} \|_E \leq A \]

\[ \vdots \]

Figure 2: Visualization of lemma 7 (top) and 9 (bottom).
As the $x^{\delta_i,n}$ are bounded in $P^n E$, there is a $\delta$-subsequence such that they converge weakly in $P^n E$. In addition, because $x^{\delta_i,n} \in P^n B_\delta(z)$, we necessarily have $x^{\delta_i,n} \xrightarrow{\delta} P^n z$. By uniqueness of the weak limit, the $x^{\delta_i,n}$ need to converge weakly (in $E$) to $P^n z$. But then

$$\|P^n z\|_E \leq \liminf_i \|P^n x^{\delta_i,n}\|_E \leq 2 \cdot A.$$ 

Now we can play the same game for larger values $n$ by choosing a much bigger $N_i$, such that we obtain the statement

$$\|P^{n'} z\|_E \leq \liminf_i \|P^n x^{\delta_i,n'}\|_E \leq 2 \cdot A.$$ 

for arbitrarily large $n'$, which is in contradiction to our assumption that $z \notin E$.

Hence, in fact, for any $A > 0$ there is an $N_i \in \mathbb{N}$ and a $\delta_1 > 0$ such that for all $n \geq N_i$ and $\delta < \delta_1$,

$$\inf_{x \in S^n B_\delta(z)} \sum_{k=1}^n a_k x_k^2 \geq A^2.$$ 

Now in finite dimensions weak convergence is strong convergence, and then for fixed $n$ there exists $\delta_2 > 0$ such that for $\delta < \delta_2$, we have $S^n B_\delta(x^{\delta}) \subset S^n B_{\delta_2}(\bar{z})$. Therefore, we can choose $\delta < \min\{\delta_1, \delta_2\}$ and then

$$\inf_{x \in S^n B_\delta(x^{\delta})} \sum_{k=1}^n a_k x_k^2 \geq A^2.$$ 

Lastly, we can (again for fixed $n$) choose $\delta_3 > 0$ such that

$$\sup_{x \in S^n B_\delta(0)} \sum_{k=1}^n a_k x_k^2 \leq \frac{A^2}{2}.$$ 

Then in conclusion, for arbitrary $A > 0$ and $n \in \mathbb{N}$ we can find $\delta, i = 1, \ldots, 3$ such that for $\delta < \min\{\delta_1, \delta_2, \delta_3\}$,

$$\frac{\mu_n(S^n B_\delta(x^{\delta}))}{\mu_n(S^n B_\delta(0))} = \frac{\int_{S^n B_\delta(z^{\delta})} \exp\left(-\frac{1}{2}(a_1 x_1^2 + \cdots + a_n x_n^2)\right) dx}{\int_{S^n B_\delta(0)} \exp\left(-\frac{1}{2}(a_1 x_1^2 + \cdots + a_n x_n^2)\right) dx}$$

$$= \frac{\int_{S^n B_\delta(0)} \exp\left(-\frac{1}{4}(a_1 x_1^2 + \cdots + a_n x_n^2)\right) \exp\left(-\frac{1}{4}(a_1 x_1^2 + \cdots + a_n x_n^2)\right) dx}{\int_{S^n B_\delta(0)} \exp\left(-\frac{1}{4}(a_1 x_1^2 + \cdots + a_n x_n^2)\right) \exp\left(-\frac{1}{4}(a_1 x_1^2 + \cdots + a_n x_n^2)\right) dx}$$

$$\leq \frac{\int_{S^n B_\delta(0)} e^{-A^2/4} \exp\left(-\frac{1}{4}(a_1 x_1^2 + \cdots + a_n x_n^2)\right) dx}{\int_{S^n B_\delta(0)} e^{-A^2/8} \exp\left(-\frac{1}{4}(a_1 x_1^2 + \cdots + a_n x_n^2)\right) dx} \leq e^{-A^2/8}$$

The statement for the ratio $\frac{\mu_n(S^n B_\delta(x^{\delta}))}{\mu_n(S^n B_\delta(0))}$ then follows with an equivalent argument as in the proof of lemma ??

The second statement that we will need almost corresponds to lemma 3.9 in [5], which bounded $\frac{\mu_n(S^n B_\delta(x^{\delta}))}{\mu_n(S^n B_\delta(0))} = 0$, i.e. with balls centered around 0 instead of $\bar{z}$. A corrected proof for this lemma in the Hilbert case can be found in [7].
We will modify the original version of the lemma by bounding \( \frac{\mu(B_{\delta}(z^\delta))}{\mu(B_{\delta}(\bar{z}))} \). It is more natural to think about the quotient \( \frac{L^q(z^\delta)}{L^q(0)} \) for \( z^\delta \to \bar{z} \) and we can recover the former result by seeing that \( \frac{L^q(z^\delta)}{L^q(0)} \leq \frac{L^q(\bar{z})}{L^q(0)} \leq \frac{L^q(\bar{z})}{L^q(0)} \) due too Anderson's inequality.

In the Hilbert space case, all calculations of ratios of type \( \frac{L^q(z^\delta)}{L^q(0)} \) so far (i.e. in [5, 7]) heavily use some kind of thresholding argument which uses that elements in \( B_{\delta}(\bar{z}) \) can be separated from elements in \( B_{\delta}(\bar{z}) \) by a hard norm difference (as already remarked in 1.3). Then this can be leveraged as an exponential rate. In the Banach case this is not possible anymore: \( z^\delta \to \bar{z} \) but \( z^\delta \not\to \bar{z} \) does not imply \( \|z^\delta\| < \liminf \|z^\delta\| \)!! Hence we need to be more careful here.

We will need the following modification of a lemma by Masoumeh Dashti (which was already presented in a simpler form in [7]):

**Lemma 8** (Dashti's lemma, version for more general measures on a Banach space). Assume that \( \gamma \) is a centered Gaussian measure on a Banach space \( X \) with Cameron–Martin space \( E \) and \( \hat{\mu} \) is another measure with density \( C \cdot r(x) \) with respect to \( \gamma \), i.e. \( \frac{d\hat{\mu}}{d\gamma}(x) = C \cdot r(x) \). Assume that \( r \) is locally bounded from above and below (away from 0), i.e. for every \( R > 0 \) there are constants \( M_1, M_2 > 0 \) such that \( r(x) \in (M_1, M_2) \) for \( \|x\| \leq R \). Assume further that \( z^\delta \to \bar{z} \) weakly in \( X \) for \( \delta \to 0 \). Then

\[
\limsup_{\delta \to 0} \frac{\hat{\mu}(B_{\delta}(z^\delta))}{\hat{\mu}(B_{\delta}(\bar{z}))} \leq M
\]

for some \( M < \infty \).

**Proof.** We start by bounding the numerator and we choose some \( \hat{h} \in j(X^*) \subset X^* \), i.e. \( R_{\mu} \hat{h} \in E \) and write (using the Cameron–Martin theorem)

\[
\hat{\mu}(B_{\delta}(z^\delta)) = C \int_{B_{\delta}(z^\delta)} r(x) d\gamma(x) = C \int_{B_{\delta}(z^\delta - R_{\mu} \hat{h})} r(y + R_{\mu} \hat{h}) d\gamma(y)
\]

\[
= C \int_{B_{\delta}(z^\delta - R_{\mu} \hat{h})} r(y + R_{\mu} \hat{h}) \exp\left(-\frac{1}{2} |R_{\mu} \hat{h}|_{E}^{2} - \hat{h}(y) \right) d\gamma(y)
\]

\[
= C \cdot \exp\left(-\frac{1}{2} |R_{\mu} \hat{h}|_{E}^{2} \right) \cdot \mu(B_{\delta}(z^\delta - R_{\mu} \hat{h})) \cdot \sup_{y \in B_{\delta}(z^\delta - R_{\mu} \hat{h})} \left[ e^{-\hat{h}(y)} \cdot r(y + R_{\mu} \hat{h}) \right]
\]

\[
= C \cdot \exp\left(-\frac{1}{2} |R_{\mu} \hat{h}|_{E}^{2} \right) \cdot \mu(B_{\delta}(0)) \cdot \sup_{y \in B_{\delta}(z^\delta - R_{\mu} \hat{h})} \left[ e^{-\hat{h}(y)} \cdot r(y + R_{\mu} \hat{h}) \right]
\]

with the last step being valid after an application of Anderson's inequality. The denominator is bounded as follows (similarly shifting by \( \bar{z} \) using the CM theorem):

\[
\hat{\mu}(B_{\delta}(\bar{z})) = C \exp\left(-\frac{1}{2} |\bar{z}|_{E}^{2} \right) \int_{B_{\delta}(0)} r(y + \bar{z}) \exp(-R_{\mu}^{-1} \bar{z}(x)) d\mu(x)
\]

\[
= C \exp\left(-\frac{1}{2} |\bar{z}|_{E}^{2} \right) \int_{B_{\delta}(0)} r(y + \bar{z}) \frac{1}{2} \left( \exp(R_{\mu}^{-1} \bar{z}(x)) + \exp(-R_{\mu}^{-1} \bar{z}(x)) \right) d\mu(x)
\]

\[
\geq C \exp\left(-\frac{1}{2} |\bar{z}|_{E}^{2} \right) \cdot \mu(B_{\delta}(0)) \cdot \inf_{y \in B_{\delta}(0)} r(y + \bar{z})
\]
As \( z^\delta \to \bar{z} \) weakly in \( X \), there is an \( R_1 > 0 \) such that \( B_\delta(z^\delta) \subset B_R(0) \) for \( \delta < 1 \) (or for \( \delta < \delta^* \) for some suitably chosen \( \delta^* \)). Define \( R := \max\{R_1, \|\bar{z}\|_X + 1\} \). Then,

\[
\limsup_{\delta \to 0} \frac{\hat{\mu}(B_\delta(z^\delta))}{\hat{\mu}(B_\delta(z))} \leq \exp\left(\frac{1}{2} |\bar{z}|^2_E - \frac{1}{2} |R_\mu \hat{h}|^2_E \right) \limsup_{\delta \to 0} \frac{\sup_{x \in B_\delta(z^\delta - R_\mu \hat{h})} \exp(-\hat{h}(x)) \cdot r(x + R_\mu \hat{h})}{\inf_{y \in B_\delta(0)} r(y + \bar{z})}
\]

Now we first bound the \( \limsup \) and see that

\[
\limsup_{\delta \to 0} \frac{\sup_{x \in B_\delta(z^\delta - R_\mu \hat{h})} \exp(-\hat{h}(x)) \cdot r(x + R_\mu \hat{h})}{\inf_{y \in B_\delta(0)} r(y + \bar{z})} \leq \limsup_{\delta \to 0} \frac{\sup_{w \in B_\delta(0)} \exp(-\hat{h}(w - R_\mu \hat{h} + z^\delta)) \cdot \sup_{x \in B_\delta(z^\delta)} r(w)}{\inf_{y \in B_\delta(0)} r(y)}
\]

\[
\leq \frac{M_2}{M_1} \cdot \limsup_{\delta \to 0} \exp[\hat{h}(z^\delta - R_\mu \hat{h})] \cdot \sup_{w \in B_\delta(0)} \exp(\hat{h}(w))
\]

\[
\leq \frac{M_2}{M_1} \cdot \limsup_{\delta \to 0} \exp[\hat{h}(z^\delta - R_\mu \hat{h})] \cdot \exp[\|\hat{h}\|_X \cdot \delta]
\]

\[
\leq \frac{M_2}{M_1} \cdot \exp(\hat{h}(\bar{z} - R_\mu \hat{h}))
\]

due to \( \hat{h} \in X^* \) and \( z^\delta \to \bar{z} \). Then we can continue with

\[
\limsup_{\delta \to 0} \frac{\hat{\mu}(B_\delta(z^\delta))}{\hat{\mu}(B_\delta(z))} \leq \frac{M_2}{M_1} \cdot \exp\left(\frac{1}{2} |\bar{z}|^2_E - \frac{1}{2} |R_\mu \hat{h}|^2_E \right) \cdot \exp(\hat{h}(\bar{z} - R_\mu \hat{h})].
\]

Now we can choose (by density of \( j(X^*) \) in \( X^* \)) a sequence of \( \hat{h} \) such that \( R_\mu \hat{h} \to \bar{z} \) strongly in \( E \). This choice then proves

\[
\limsup_{\delta \to 0} \frac{\hat{\mu}(B_\delta(z^\delta))}{\hat{\mu}(B_\delta(z))} \leq \frac{M_1}{M_2}.
\]

Now we are ready to prove the remaining technical statement.

**Lemma 9** (Generalized version of lemma 3.9 in [5]). *Let \( z^\delta \to \bar{z} \) weakly in \( X \) for some \( \bar{z} \in E \), but \( z^\delta \not\to \bar{z} \) strongly in \( X \). Then for any \( \epsilon > 0 \) there exists a \( \delta > 0 \) such that

\[
\frac{\mu(B_\delta(z^\delta))}{\mu(B_\delta(z))} < \epsilon.
\]

**Proof.** As before, we do finite approximation first and consider \( \frac{\mu_n(S^n B_\delta(z^\delta))}{\mu_n(S^n B_\delta(z))} \).

First we try to bound the denominator:

\[
\mu_n(S^n B_\delta(z)) = Z_n \cdot \int_{S^n B_\delta(z)} \exp\left(-\frac{1}{2} (a_1 x_1^2 + \cdots + a_n x_n^2)\right) \, dx
\]

Now for \( m < n \) we can apply lemma 3 to see that we can write \( \|P^m_n(y)\|_E^2 = ((C_m^n)^2 - \epsilon_0) \cdot \|P^m_n(y)\|_X^2 + R_{m,n,\epsilon_0}(y) \) for arbitrarily small \( \epsilon_0 > 0 \) where the remainder \( R_{m,n,\epsilon_0} \) is positive.
Also, we will need the following: For any \( \epsilon_1 > 0 \) there is an \( m_0 \in \mathbb{N} \) such that for all \( m > m_0 \),
\[
\sup_{n>m} \|P_m^n \tilde{z}\|_X < \epsilon_1.
\]  
(3)

Note that \( P_m^n \tilde{z} \rightarrow \tilde{z} \) strongly in \( X \), i.e. \( \|(I - P_m^n)\tilde{z}\|_X \rightarrow 0 \). Now choose an \( m_0 \in \mathbb{N} \) such that \( \|(I - P_m^n)\tilde{z}\|_X < \epsilon_1/2 \) for all \( n > m_0 \). If we now set \( m > m_0 \) arbitrary, then
\[
\sup_{n>m} \|P_m^n \tilde{z}\|_X \leq \|(I - P_m^n)\tilde{z}\|_X + \sup_{n>m} \|(I - P_m^n)\tilde{z}\|_X < \epsilon_1/2 + \epsilon_1/2.
\]

Then,
\[
\mu_n(S^n B_{\delta}(\tilde{z})) = Z_n \cdot \int_{\mathcal{P}_n B_{\delta}(\tilde{z})} \exp\left(-\frac{1}{2}((C_m^n)^2 - \epsilon_0) \cdot \|P_m^n(y)\|_X^2 + R_{m,n,\epsilon_0}(y)\) \, d\lambda \circ T^{-1}(y)
\]
\[
= \frac{Z_n}{Z_n} \cdot \int_{\mathcal{P}_n B_{\delta}(\tilde{z})} \exp\left(-\frac{1}{2}((C_m^n)^2 - \epsilon_0) \cdot \|P_m^n(y)\|_X^2\) \, d\mu_{0,n}(y)
\]
\[
\geq \frac{Z_n}{Z_n} \cdot \exp\left(-\frac{1}{2}((C_m^n)^2 - \epsilon_0) \cdot (\epsilon_1 + \delta)^2\right) \, d\mu_{0,n}(P_n B_{\delta}(\tilde{z}))
\]

where \( d\mu_{0,n}(x) \propto \exp(-1/2R(x)) \, dx \) is a convex, centred measure.

Regarding the numerator: Note that, for \( y \in \mathcal{P}_n B_{\delta}(\tilde{z}) \), we can write
\[
\|P_m^n(y)\|_X \geq \|P_m^n(\tilde{z})\|_X - \|P_m^n(\tilde{z}) - P_m^n(y)\|_X.
\]

We can bound \( \|P_m^n(\tilde{z}) - P_m^n(\tilde{z})\|_X \) from below in the following sense: There exists an \( M > 0 \) such that for all \( m \in \mathbb{N} \) there is a \( \delta_0 > 0 \) and a \( n_0 \in \mathbb{N} \) such that for all \( \delta < \delta_0 \) and \( n > n_0 \),
\[
\|P_m^n(\tilde{z}) - P_m^n(\tilde{z})\|_X > M.
\]

In order to prove this, we assume the opposite, i.e. for all arbitrarily small \( M > 0 \) there is an \( m \in \mathbb{N} \) such that for all \( \delta_0 > 0 \) and \( n_0 \in \mathbb{N} \) there is a \( \delta < \delta_0 \) and a \( n > n_0 \) such that
\[
\|P_m^n(\tilde{z}) - P_m^n(\tilde{z})\|_X \leq M.
\]

Now we choose first \( \kappa > 0 \) arbitrarily small and \( \delta_1 > 0 \) such that for \( \delta < \delta_1 \), we have
\[
\|P^m \tilde{z} - P^n \tilde{z}\|_X < \kappa/3 \text{ (which is possible because } \tilde{z} \rightarrow \tilde{z}).
\]

Secondly, we can find \( n_1 \in \mathbb{N} \) such that for all \( n > n_1 \), we have \( \|(I - P_n)(\tilde{z} - \tilde{z})\|_X < \kappa/3 \).

Finally we set \( M := \kappa/3 \), thus there is (from the assumption above) an \( m = m(\kappa/3) \) and for \( \delta_0 := \delta_1 \) and \( n_0 := n_1 \) there is a \( \delta < \delta_0 \) and a \( n > n_0 \) such that \( \|P_m^n(\tilde{z}) - P_m^n(\tilde{z})\|_X < \kappa/3 \).

Thus, all in all, for the parameters chosen, we can bound
\[
\|\tilde{z} - \tilde{z}\|_X = \|P^n \tilde{z} + P^n \tilde{z} + (I - P^n)(\tilde{z} - \tilde{z})\|_X
\]
\[
\leq \|P^m \tilde{z} - P^n \tilde{z}\|_X + \|P^n \tilde{z} - P^m \tilde{z}\|_X + \|(I - P^n)(\tilde{z} - \tilde{z})\|_X
\]
\[
< \kappa.
\]
for $\kappa > 0$ arbitrarily small, which is in contradiction to $z^\delta \not\rightarrow \bar{z}$ in $X$. Thus, indeed, there exists an $M > 0$ such that for all $m\in\mathbb{N}$ there is a $\delta_0 > 0$ and a $n_0 \in \mathbb{N}$ such that for all $\delta < \delta_0$ and $n > n_0$,
\[ \|P^n_m(z^\delta) - P^n_m(\bar{z})\|_X > M. \]
This means that
\[
\|P^n_m(y)\|_X \geq \|P^n_m(z^\delta) - P^n_m(\bar{z})\|_X - \|P^n_m(\bar{z})\|_X - \|P^n_m(z^\delta) - P^n_m(y)\|_X
\]
\[ \geq M - \varepsilon_1 - \delta \]
for $n > n_0$ and $\delta < \delta_0$. Note that we bounded again $\|P^n_m(\bar{z})\|_X \leq \sup_n \|P^n_m\|_X < \varepsilon_1$ for $m > m_0$.
Thus,
\[
\mu_n(S^n_B(\delta)) = Z_n \cdot \int_{P^n_B(z)} \exp \left( \frac{-1}{2} (C^n_m z^2 - \varepsilon_0) \cdot \|P^n_m(y)\|_X^2 + R_{m,n,\varepsilon_0}(y) \right) \, d\lambda \circ T^{-1}(y)
\]
\[ = \frac{Z_n}{Z_n} \int_{P^n_B(z)} \exp \left( \frac{-1}{2} (C^n_m z^2 - \varepsilon_0) \cdot \|P^n_m(y)\|_X^2 \right) \, d\mu_{0,n}(y)
\]
\[ \leq \frac{Z_n}{Z_n} \exp \left( \frac{-1}{2} (C^n_m z^2 - \varepsilon_0) \cdot (M - \varepsilon_1 - \delta)^2 \right) \cdot \hat{\mu}_{0,n}(P^n_B(\delta)). \]

Now we can apply lemma 8 and obtain
\[
\limsup_{\delta \to 0} \frac{\mu_{0,n}(P^n_B(\delta))}{\mu_{0,n}(P^n_B(z))} \leq \exp \left( \frac{-1}{2} (C^n_m z^2 - \varepsilon_0) \cdot (M - \varepsilon_1 - \delta)^2 - (\varepsilon_1 + \delta)^2 \right) \cdot \hat{\mu}_{0,n}(P^n_B(\delta)). \]
Note that $(M - \varepsilon_1 - \delta)^2 - (\varepsilon_1 + \delta)^2 = M^2 - 2M\varepsilon_1 + \delta^2 \geq \frac{1}{2}M^2 - 2\varepsilon_1^2 + \delta^2$. Now choose $\delta$ and $\varepsilon_1$ (small enough) such that this number is positive, say, $(M - \varepsilon_1 - \delta)^2 - (\varepsilon_1 + \delta)^2 \geq \frac{1}{3}M^2$ and we obtain
\[
\limsup_{\delta \to 0} \frac{\mu_{0,n}(P^n_B(\delta))}{\mu_{0,n}(P^n_B(z))} \leq \exp \left( \frac{-1}{2} (C^n_m z^2 - \varepsilon_0) \cdot M^2}{6} \right) \cdot \hat{M}. \]
By choosing $m > m_0$ large enough and $n > \max\{m, n_0\}$, the constant $C^n_m$ from lemma 3 is larger than any positive number and we have proven the statement.

\[ \square \]

Note that the following theorem states a subset of theorem 3.5 in [5], but now it can be proven for Banach spaces $X$. Additionally, as in [7], we can drop all regularity assumptions on $\Phi$ but locally Lipschitz continuity. With the supporting lemmata 6, 7, and 9 complete, the proof is almost identical to the one in [7] so we refrain from copying it here.

**Theorem 1** (Part of original theorem 3.5). *Let $X$ be a Banach space and $\mu_0$ be a Gaussian measure on $X$ with Cameron–Martin space $E$. Consider a measure $\mu \ll \mu_0$ with $\frac{d\mu}{d\mu_0}(u) = Z^{-1} \cdot \exp(-\Phi(u))$. Suppose that $\Phi$ is locally Lipschitz continuous, i.e. for every $r > 0$ there exists $L(r) \in (0, \infty)$ such that for all $u_1, u_2 \in X$ with $\|u_1\|_X, \|u_2\|_X < r$, we have
\[ |\Phi(u_1) - \Phi(u_2)| \leq L(r) \cdot \|u_1 - u_2\|_X. \]

Then,
Let $z^\delta = \arg\max_{z \in X} \mu(B_\delta(z))$. There is a $\bar{z} \in E$ and a subsequence of $\{z^\delta\}_\delta$ which converges to $\bar{z}$ strongly in $X$.***
Remark 3. Note that this is only part of what is labelled theorem 3.5 in [5], but the other part concerning identification of the small ball probability minimizer $\bar{z}$ with minimizers of the Onsager-Machlup functional is unchanged.

Acknowledgments

The author would like to express his gratitude to Birzhan Ayanbayev, Martin Burger, Remo Kretschmann, Claudia Schillings, Björn Sprungk, and Tim Sullivan for helpful discussions and support.

References

[1] S. Agapiou, M. Burger, M. Dashti, and T. Helin. Sparsity-promoting and edge-preserving maximum a posteriori estimators in non-parametric bayesian inverse problems. *Inverse Problems*, 34(4):045002, 2018.

[2] X. Bay and J.-C. Croix. Karhunen-loève decomposition of gaussian measures on banach spaces. *arXiv preprint arXiv:1704.01448*, 2017.

[3] V. I. Bogachev. *Gaussian measures*. American Mathematical Soc., 1998.

[4] C. Clason, T. Helin, R. Kretschmann, and P. Piironen. Generalized modes in bayesian inverse problems. *SIAM/ASA journal on uncertainty quantification*, 7(2):652–684, 2019.

[5] M. Dashti, K. J. Law, A. M. Stuart, and J. Voss. Map estimators and their consistency in bayesian nonparametric inverse problems. *Inverse Problems*, 29(9):095017, 2013.

[6] T. Helin and M. Burger. Maximum a posteriori probability estimates in infinite-dimensional bayesian inverse problems. *Inverse Problems*, 31(8):085009, 2015.

[7] R. Kretschmann. *Nonparametric Bayesian Inverse Problems with Laplacian Noise*. PhD thesis, Universität Duisburg-Essen, 2019.

[8] H. C. Lie and T. Sullivan. Equivalence of weak and strong modes of measures on topological vector spaces. *Inverse Problems*, 34(11):115013, 2018.