UNION OF CHAINS OF PRIMES

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Abstract. The union of an ascending chain of prime ideals is not always prime. We show that this property is independent of the parallel property for semiprimes. We also show that the PI-class is a tight bound on the number of non-prime unions of subchains in a chain of primes in a PI-algebra.

1. Introduction

In a commutative ring, the union of a chain of prime ideals is prime, and the union of a chain of semiprime ideals is semiprime. This paper demonstrates and measures the failure of these chain conditions in general.

Definition 1.1. A ring has the (semi)prime chain property (denoted $P^\uparrow$ and $SP^\uparrow$, respectively) if the union of any countable chain of (semi)prime ideals is always (semi)prime.

The property $SP^\uparrow$ was recognized by Fisher and Snider [1] as the missing hypothesis for Kaplansky’s conjecture on regular rings, and they gave an example of a ring without $SP^\uparrow$.

Our focus is on $P^\uparrow$. The class of rings satisfying $P^\uparrow$ is quite large. An easy exercise shows that every commutative ring satisfies $P^\uparrow$, and the same argument yields that the union of strongly prime ideals is strongly prime. In fact, we have the following result:

Proposition 1.2. Every ring $R$ which is a finite module over a central subring, satisfies $P^\uparrow$.

Proof. Write $R = \sum_{i=1}^t Cr_i$, where $C \subseteq \text{Cent}(R)$. Suppose $P_1 \subseteq P_2 \subseteq \cdots$ is a chain of prime ideals, with $P = \cup P_i$. If $a, b \in R$ with

$$\sum Car_i b = \sum aCr_i b = aRb \subseteq P$$

then there is $n$ such that $ar_i b \in P_n$ for $1 \leq i \leq t$, implying $aRb = \sum aCr_i b \subseteq P_n$, and thus $a \in P_n$ or $b \in P_n$. $\square$

(For a recent treatment of the correspondence of infinite chains of primes between a ring $R$ and a central subring, see [2]).

The class of rings satisfying $P^\uparrow$ also contains every ring that satisfies ACC (ascending chain condition) on primes, and is closed under homomorphic images and central localizations.

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1For simplicity we deal only with countable chains throughout the paper, but the arguments are general.
This led some mathematicians to believe that it holds in general. On the other hand, Bergman produced an example lacking $\mathcal{P}^\uparrow$ (see Example 2.1 below), implying that the free algebra does not have $\mathcal{P}^\uparrow$.

Obviously, the property $\mathcal{P}^\uparrow$ follows from the maximum property on families of primes. On the other hand, $\mathcal{P}^\uparrow$ implies (by Zorn’s lemma) the following maximum property: for every prime $Q$ contained in any ideal $I$, there is a prime $P$ maximal with respect to $Q \subseteq P \subseteq I$.

In Section 3 we show that $\mathcal{P}^\uparrow$ and $\mathcal{S}\mathcal{P}^\uparrow$ are independent, by presenting an example (due to Kaplansky and Lanski) of a ring satisfying $\mathcal{P}^\uparrow$ and not $\mathcal{S}\mathcal{P}^\uparrow$, and an example of a ring satisfying $\mathcal{S}\mathcal{P}^\uparrow$ but not $\mathcal{P}^\uparrow$.

We say that an ideal is **union-prime** if it is a union of a chain of primes, but not prime. The maximal number of non-prime unions of subchains of a chain of prime ideals is called the $\mathcal{P}^\uparrow$-index of the ring (see Definition 4.2). Section 2 extends Bergman’s example by showing that the $\mathcal{P}^\uparrow$-index of the free (countable) algebra is infinity. In Section 4 we discuss PI-rings, showing that the $\mathcal{P}^\uparrow$-index is tightly bounded by the PI-class.

2. Monomial algebras

We show that $\mathcal{P}^\uparrow$ and $\mathcal{S}\mathcal{P}^\uparrow$ fail in the free algebra by constructing an (ascending) chain of primitive ideals whose union is not semiprime. Let us start with a simpler theme, whose variations have extra properties.

Example 2.1 (A chain of prime ideals with non-semiprime union). Let $R$ be the free algebra in the (noncommuting) variables $x, y$. For each $n$, let

$$P_n = \langle xx, xyx, xy^2x, \ldots, xy^{n-1}x \rangle.$$  

As a monomial ideal, it is enough to check primality on monomials. If $uRu' \subseteq P_n$ for some words $u, u'$, then in particular $uy^n u' \in P_n$, which forces a subword of the form $xy^i x$ (with $i < n$) in $u$ or $u'$; hence either $u \in P_n$ or $u' \in P_n$.

On the other hand, $\bigcup P_n = (RxR)^2$ which is not semiprime. This example, due to G. Bergman, appears in [3, Exmpl. 4.2]. Interestingly, primeness is always maintained in the following sense ([4, Lem. 4.1], also due to Bergman): for every countable chain of primes $P_1 \subset P_2 \subset \cdots$ in a ring $R$, the union $\bigcup (P_n[[\zeta]])$ is a prime ideal of the power series ring $R[[\zeta]]$.

We can modify this example so that the chain has a unique prime lying over it.

Example 2.2 (A chain of prime ideals whose union is not semiprime, although its radical is a maximal ideal). Let $D$ be the quotient division ring of the free algebra $F(x, y)$. Let $R$ be the subalgebra generated by $x$ and the subfield $F(y)$. Extend $\text{deg}_y : F[y] \to \mathbb{N}$ to $\text{deg}_y : F(y)^\times \to \mathbb{Z}$ in the obvious manner. Similarly to the previous example, take the prime ideals

$$P_n = \langle xax : a \in F(y), |\text{deg}_y(a)| < n \rangle.$$  

Again $P = \bigcup P_n = (RxR)^2$, which is not semiprime. But now $R/\sqrt{P} = R/\langle x \rangle \cong F(y)$.

Since in Example 2.1 $\bigcup P_n = (RxR)^2$, if $Q < R$ is a prime containing the union then $x \in Q$ so $R/Q$ is commutative.
In particular a chain of prime ideals starting from the chain \( P_1 \subset P_2 \subset \cdots \) has only one union-prime. Let us exhibit a (countable) chain providing infinitely many union-primes.

**Example 2.3** (A prime chain with infinitely many union-primes). Let \( R \) be the free algebra generated by \( x, y, z \). For a monomial \( w \) we denote by \( \deg_y w \) the degree of \( w \) with respect to \( y \). Consider the monomial ideals

\[
I_{i, n} = RxxR + RxzR + \cdots + Rxz^{i-1}xR + \langle xz^i xwxyz^i x : \deg_y w < n \rangle
\]

which form an ascending chain with respect to the lexicographic order on the indices, since \( xz^i x \in I_{i, n} \) for every \( i' > i \). To show that \( I_{i, n} \) are prime, suppose \( u, u' \) are monomials such that \( u, u' \notin I_{i, n} \) but \( uRu' \subseteq I_{i, n} \). Then \( uz^iy^nz^i u' \in I_{i, n} \). Since none of the monomials \( xz^i x \) (\( i' < i \)) is a subword of \( u \) or \( u' \), they are not subwords of \( uz^iy^nz^i u' \), forcing \( uz^iy^nz^i u' \) to have a subword of the form \( xz^i xwxyz^i x \) where \( \deg_y w < n \). It follows that \( z^iy^nz^i \) is a subword of \( z^ixwxyz^i \), contrary to the degree assumption. Now, for every \( i \), \( \bigcup I_{i, n} = RxxR + RxzR + \cdots + Rxz^{i-1}xR + (Rxz^i xR)^2 \) which contains \( (Rxz^i xR)^2 \) but not \( Rxz^i xR \), so it is not semiprime.

The \( \mathcal{P}^\uparrow \)-index of \( R \) is thus infinity. In Section 4 we show that this phenomenon is impossible in PI algebras: there, the number of union-primes in a prime chain is bounded by the PI-class.

Meanwhile, we strengthen the properties of the ideals in the chain:

**Example 2.4** (A chain of primitive ideals with non-semiprime union). Let \( R \) be the free algebra in the variables \( e, y \), modulo the relation \( e^2 = e \). Every monomial has a unique shortest presentation as a word (replacing \( e \) by \( e^2 \) throughout). Ordering monomials first by length and then lexicographically, every element \( f \) has an upper monomial \( \hat{f} \). Notice that \( \hat{fy^n g} = \hat{f} y^n \hat{g} \).

For each \( n \), let

\[
P_n = \langle e y^2 e, \ldots, e y^{n-1} e \rangle.
\]

To show that \( P_n \) is a prime ideal, assume that \( f y^n g \in P_n \). Then \( \hat{f} y^n \hat{g} = \hat{f} y^n \hat{g} \in P_n \), forcing \( f \in P_n \) or \( g \in P_n \) as in Example 2.7. The claim follows by induction on the number of monomials.

To show that the ideal \( P_n \) is primitive, it is enough by 3 to prove that \( e(R/P_n) e \) is a primitive ring. We construct an isomorphism between \( e(R/P_n) e \) and the countably generated free algebra \( F(z_1, z_2, \ldots) \) by sending \( ey^m e \) for \( m \geq n + 1 \) (which are clearly algebraically independent) to \( z_{m-n} \). But the free algebra is primitive (see 3 \( F < x, y > \)).

On the other hand \( \bigcup P_n = ReyR + ReyR \), which contains \( (Rey)^2 \) but not \( e y \), so is not semiprime.

**Remark 2.5.** We say that a ring is weakly-\( \mathcal{P}^\uparrow \) if it has a unique minimal prime over every countable chain of prime ideals.

Since the intersection of a descending chain of primes is prime, Zorn’s lemma shows that there are minimal primes over every ideal, in particular over any union-prime. In the topology of the spectrum, a series \( P_n \) of primes converges to a prime \( Q \) if and only if \( \bigcap_{n=1}^\infty \bigcup_{m=n}^\infty P_n \subseteq Q \); in particular when \( P_1 \subseteq P_2 \subseteq \cdots \) is a chain, \( \lim P_n = Q \) if and only if \( \bigcup P_n \subseteq Q \). Therefore, the spectrum cannot distinguish \( \mathcal{P}^\uparrow \) from weakly-\( \mathcal{P}^\uparrow \).
In the examples of this section, there is a unique minimal prime over every union-prime. In Example 3.5 the situation is different: the union-prime ideal constructed there is the intersection of two primes containing it.

3. Matrix constructions

This section shows that $P^\uparrow$ and $SP^\uparrow$ are independent: the algebra in Example 3.1 satisfies $P^\uparrow$ but not $SP^\uparrow$, and the algebra in Example 3.5 satisfies $SP^\uparrow$ but not $P^\uparrow$.

3.1. $P^\uparrow$ does not imply $SP^\uparrow$. As mentioned in the introduction, Kaplansky conjectured that a semiprime ring all of whose prime quotients are von Neumann regular, is regular. Fisher and Snider [1] proved that this is the case if the ring satisfies $SP^\uparrow$ (also see [2, Thm. 1.17]), and gave a counterexample which lacks this property, due to Kaplansky and Lanski [2, Example 1.19]. We repeat the example and exhibit, in this ring, an ascending chain of semiprime ideals whose union is not semiprime.

Example 3.1 (Kaplansky-Lanski). (A ring whose prime ideals are maximal, but without $SP^\uparrow$). Let $R$ be the ring of sequences of 2-by-2 matrices which eventually have the form \[
\begin{pmatrix}
\alpha & \beta
\
0 & \alpha
\end{pmatrix}
\]
in the $n$th place, clearly a semiprime ring.

Let $I_n$ be the set of sequences in $R$, which are zero from the $n$th place onward. Clearly $R/I_n \cong R$, so the ideals are semiprime. However $\bigcup I_n$ is composed of sequences of matrices which are eventually zero, and $aRa$ is eventually zero for $a = \left(\begin{smallmatrix} 0 & 1 \\ 0 & 0 \end{smallmatrix}\right), \left(\begin{smallmatrix} 0 & 1 \\ 0 & 0 \end{smallmatrix}\right), \ldots$; hence $R/\bigcup I_n$ is not semiprime. On the other hand by the argument in [1], every prime ideal of $R$ is maximal, so there are no infinite chains of primes and $P^\uparrow$ holds trivially.

3.2. $SP^\uparrow$ does not imply $P^\uparrow$. In the rest of this section we investigate $P^\uparrow$ and $SP^\uparrow$ for rings of the form $\hat{A} = \left(\begin{array}{cc} A & M \\
M & A \end{array}\right)$ where $A$ is an integral domain and $M \triangleleft A$ is a nonzero ideal. We show that they always satisfy $SP^\uparrow$, and give an example which does not have $P^\uparrow$. Clearly $\hat{A}$ is a prime ring. Let us describe the ideals of this ring.

Remark 3.2. (1) The ideals of $\hat{A}$ have the form $I = \left(\begin{array}{cc} I_{11} & I_{12} \\
I_{21} & I_{22} \end{array}\right)$, where for every $i, j, I_{ij} \triangleleft A$ (not necessarily proper), $I_{ii} \subseteq M$, and $MI_{ij} \subseteq I_{ij} \cap I_{ij'}$, (where $1' = 2$ and $2' = 1$).

(2) The semiprime ideals of $\hat{A}$ are of the form $I = \left(\begin{array}{cc} I & M \cap I \\
M \cap I & I' \end{array}\right)$, where $I, I'$ are semiprime, and $M \cap I' = M \cap I$.

Proof. (1) This is easy.

(2) Write $A_{ij} = A$ if $i = j$ and $A_{ij} = M$ otherwise. Clearly $I$ is semiprime if for every $a_{11} \in A_{11}, \ldots, a_{22} \in A_{22}$, (for every $i, k, \sum_{j,k} A_{jk}a_{ij}a_{kl} \subseteq I_{kl}$) implies (for every $i, k, a_{ik} \in I_{kl}$). Assuming this is the case, fix $i, j$ and choose $a_{kl} = 0$ for every $(k, l) \neq (i, j)$; then
On the other hand if Condition (⋄) holds and for every $i, \ell$, $\sum_{j,k} A_{jk} a_{ij} a_{k\ell} \subseteq I_{i\ell}$, then in particular $A_{ij} a_{ij}^2 \subseteq I_{ij}$ so each $a_{ij} \in I_{ij}$. Therefore, $I$ is semiprime iff (⋄) holds for every $i, j$. Let us interpret Condition (⋄). For $i = j$ it requires that $I_{ii}$ are semiprime. Assuming this is the case, for $i \neq j$ the condition is “$Ma_{ij}^2 \subseteq I_{ij}$ implies $a_{ij} \in I_{ij}$”, which in light of the standing assumption that $a_{ij} \in A_{ij}$, is equivalent to $M \cap I_{11} \subseteq I_{ij}$, since for every $b \in M$, $Mb^2 \subseteq I_{ij}$ iff $b \in I_{11}$ (indeed, if $b^2 M \subseteq I_{ij}$ then $(bM)^2 \subseteq I_{ij} \subseteq I_{11}$ so $bM \subseteq I_{11}$ and $b \in M \cap (I_{11} : M) \subseteq I_{11}$. On the other hand if $b \in M \cap I_{11}$ then $b^2 \in IM \subseteq I_{ij}$ and $b^2 M \subseteq I_{ij}$).

Now assume that $I_{ii}$ are semiprime, and that $M \cap I_{11} \subseteq I_{12} \cap I_{21}$. Since $I_{12} M \subseteq I_{11}$, we have that $I_{12} \subseteq M \cap (I_{11} : M) = M \cap I_{11} \subseteq I_{12}$ so $I_{12} = M \cap I_{11}$ and likewise $I_{21} = M \cap I_{11}$. Finally $M \cap I_{11} = M \cap I_{22}$ since $b \in M \cap I_{11}$ iff $b \in M \cap (I_{11} : M)$ iff ($b \in M$ and $b^2 M \subseteq I_{ij}$) iff $b \in M \cap (I_{22} : M)$ iff $b \in M \cap I_{22}$.

\[
\text{Proposition 3.3. The ring } \hat{A} \text{ satisfies } SP^\dagger. 
\]

\textbf{Proof.} By Remark 3.2 every chain of semiprime ideals $T_1 \subseteq T_2 \subseteq \cdots$ in $\hat{A}$ has the form $T_n = \left( \begin{array}{c c} I_n & J_n \\ J_n & I_n' \end{array} \right)$, $I_n$ and $I_n'$ are ascending chains of semiprime ideals, and $J_n = M \cap I_n = M \cap I_n'$. The union of this chain is $\left( \bigcup L \cap I_n \bigcup L' \cap I_n' \right)$ where $L = M \cap \bigcup I_n = M \cap \bigcup I_n'$, which is semiprime.

Using the description of the semiprime ideals, it is not difficult to obtain the following.

\textbf{Proposition 3.4.} (1) The prime ideals of $\hat{A}$ are $\left( \begin{array}{c c} J & M \\ M & A \end{array} \right)$ and $\left( \begin{array}{c c} A & M \\ M & J \end{array} \right)$ for prime ideals $J \triangleleft A$ containing $M$, and $I^0 = \left( \begin{array}{c c} J \bigcap I & M \bigcap I \\ M \bigcap I & I \end{array} \right)$ for prime ideals $I \triangleleft A$ not containing $M$.

(2) The union-prime ideals of $\hat{A}$ are of the form $\left( \begin{array}{c c} M' & M \\ M & M' \end{array} \right)$ where $M' \triangleleft A$ is a prime containing $M$, which can be presented as a union over an ascending chain of primes not containing $M$.

If the chain of primes includes an ideal with $A$ in one of the corners, then every higher term has the same form, and the union is determined by the union of entries in the other corner, which is prime since $\hat{A}$ is commutative. We thus assume the chain has the form $I_1^0 \subseteq I_2^0 \subseteq \cdots$ where $I_1 \subseteq I_2 \subseteq \cdots$ are primes in $\hat{A}$, not containing $M$. The union is clearly $M^0$ where $M' = \bigcup I_n$ is prime, and $M^0$ is not a prime iff $M \subseteq M'$.

\textbf{Example 3.5} (A prime PI-ring, integral over its center, satisfying $SP^\dagger$ but not (weakly-)P$^\dagger$). Let $k$ be a field. Let $\hat{A} = \left( \begin{array}{c c} A & M \\ M & A \end{array} \right)$, where $A = k[\lambda_1, \ldots]$ is the ring of polynomials in countably many variables $\lambda_1, \lambda_2, \ldots$, and $M = (\lambda_1, \ldots)$. Clearly $\hat{A} \subset M_2(A)$ is integral over $A$. Choose $I_n = (\lambda_1, \ldots, \lambda_n)$. Then $T_n =$
In Proposition 3.4.1, but their union \( \bigcup T_n = \begin{pmatrix} M & M \\ M & M \end{pmatrix} \) is obviously not prime. Therefore \( \hat{A} \) satisfies \( SP^\uparrow \) (Proposition 3.3) but not \( P^\uparrow \). Furthermore \( \hat{A}/\bigcup T_n \cong A/M \times A/M \) which is equal to its radical, so weakly-\( P^\uparrow \) also fails.

4. The property \( P^\uparrow \) in PI-rings

We define a \( P^\uparrow \)-index, and show that for PI-rings, it is bounded by the PI-class.

**Proposition 4.1.** Any Azumaya algebra satisfies \( P^\uparrow \) (and \( SP^\uparrow \)).

**Proof.** Let \( A \) be an Azumaya algebra over a commutative ring \( C \). There is a 1:1 correspondence between ideals of \( A \) and the ideals of \( C \), preserving inclusion, primality and semiprimality. The claim follows since the center satisfies \( P^\uparrow \) (and \( SP^\uparrow \)). \( \square \)

Recall that by Posner’s theorem, a prime PI-ring \( R \) is representable, namely embeddable in a matrix algebra \( M_n(C) \) over a commutative ring \( C \). The minimal such \( n \) is the PI-class of \( R \), denoted \( PI(R) \).

Although PI-rings do not have to satisfy the property \( P^\uparrow \), we show that the PI-class bounds the extent in which \( P^\uparrow \) may fail. To be more precise, we define the notion of the \( P^\uparrow \)-index. Recall that an ideal is union-prime if it is the union over an ascending chain of prime ideals, but not prime.

**Definition 4.2.** Define the \( P^\uparrow \)-index of \( R \) as the maximal number of non-prime unions of subchains in an ascending chain of prime ideals of \( R \).

In other words:
\[
\mathcal{P}^\uparrow (R) = \begin{cases} 
0 & \text{if } R \text{ has } P^\uparrow \\
\sup_I P^\uparrow (R/I) + 1 & \text{otherwise}
\end{cases}
\]

where the supremum is taken over the union-prime ideals of \( R \) (when they exist). For example, \( \mathcal{P}^\uparrow (R) = 0 \) if and only if \( R \) satisfies \( P^\uparrow \), and \( \mathcal{P}^\uparrow (R) = 1 \) if the union of an ascending chain of primes starting from a union-prime ideal, is prime.

We are now ready for our main positive result about PI-rings.

**Theorem 4.3.** Let \( R \) be a (prime) PI-ring. Then \( \mathcal{P}^\uparrow (R) < PI(R) \).

**Proof.** Let \( R \) be a prime PI-ring of PI-class \( n \). If the PI-class is 1 then \( R \) is commutative, and has \( P^\uparrow (R) = 0 \). We continue by induction on \( n \). Let \( 0 = P_0 \subset P_1 \subset \cdots \) be an ascending chain of primes, and assume that \( \bigcup P_n \) is not a prime. Let \( Q \supset \bigcup P_n \) be a prime ideal. We want to prove that the PI-class of \( R/Q \) is smaller than that of \( R \).

Assume otherwise. Let \( g_n \) be a central polynomial for \( n \times n \) matrices (see [5], p. 26). Since \( PI(R/Q) = n \), there is a value \( \gamma \neq 0 \) of \( g_n \) in the center of \( R \), which is not in \( Q \). Since the center is a domain we can consider the localization \( A[\gamma^{-1}] \), which is Azumaya by Artin-Procesi [5, Theorem 1.8.48], since 1 is a value of \( g_n \) on this algebra. But then the union of \( 0 \subset P_1[\gamma^{-1}] \subset P_2[\gamma^{-1}] \subset \cdots \) is prime by Proposition 3.4.1 so \( \bigcup P_n \) is prime as well, contrary to assumption. \( \square \)

We now show the bound is tight. Notice that the ring constructed in Example 3.5 has PI-class 2 and is not \( P^\uparrow \) (and thus has \( \mathcal{P}^\uparrow (R) = 1 \)). Let us generalize this.

\[
\begin{pmatrix} I_n & I_n \\
I_n & I_n \end{pmatrix} \text{ form an ascending chain of primes by Proposition 3.4.1},
\]
Example 4.4 (An algebra of PI-class $n$ which has $\mathcal{P}^\uparrow$-index $n - 1$). Let $A_{(n)} = k \left[ \lambda_{i}^{(j)} : 1 \leq j < n, i = 1, 2, \ldots \right]$. Let $M_n = 0$ and for $j = n - 1, n - 2, \ldots, 1$, take $M_j = M_{j+1} + \langle \lambda_{1}^{(j)}, \lambda_{2}^{(j)}, \ldots \rangle$, so that $M_n \subset M_{n-1} \subset \cdots \subset M_1$. Now let

$$ R_{(n)} = \begin{pmatrix} A & M_1 & M_2 & \cdots & M_{n-1} \\ M_1 & A & M_2 & \cdots & M_{n-1} \\ M_2 & M_2 & A & \cdots & M_{n-1} \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ M_{n-1} & M_{n-1} & M_{n-1} & \cdots & A \end{pmatrix}. $$

This ring is prime, of PI-class $n$.

When $n = 2$ we obtain the ring of Example 3.5, so that $\mathcal{P}^\uparrow(R_{(2)}) = 1$. Consider the chain $I_1 \subset I_2 \subset \cdots$ of ideals of $A$ defined by $I_i = \langle \lambda_{1}^{(n-1)}, \ldots, \lambda_{i}^{(n-1)} \rangle$; thus $\bigcup I_i = M_{n-1}$. Let $\tilde{I}_n = I_n R_{(n)}$. Each ideal $\tilde{I}_n$ is prime, and the union is the set of matrices over $M_{n-1}$. The quotient ring is therefore $R_{(n)} / \bigcup \tilde{I}_n \cong R_{(n-1)} \oplus A_{(n-1)}$, which is not prime, and has a quotient $R_{(n-1)}$ with $\mathcal{P}^\uparrow(R_{(n-1)}) = n - 2$ by induction. This proves $\mathcal{P}^\uparrow(R_{(n)}) = n - 1$.

Remark 4.5. (1) Although PI-rings do not satisfy $\mathcal{P}^\uparrow$ (Example 3.5), affine PI-rings over Noetherian commutative rings satisfy ACC on semiprime ideals, and in particular are SP$^\uparrow$ and $\mathcal{P}^\uparrow$. (Schelter’s theorem, [5, Thm 4.4.16]).

(2) PI-rings with finite Gelfand-Kirillov dimension satisfy ACC on primes, since a prime PI-ring is Goldie, and then every prime ideal contains a regular element which reduces the dimension.

(3) On the other hand, we have examples of (non-affine) locally finite nil algebras (in particular $\text{GKdim}=0$), which do not satisfy $\mathcal{P}^\uparrow$, and of affine algebras of finite GK-dimension which do not satisfy $\mathcal{P}^\uparrow$. (Details will appear elsewhere.)

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