The Effective Action of $W_3$ Gravity to All Orders

Jan de Boer‡ and Jacob Goeree§

Institute for Theoretical Physics
University of Utrecht
Princetonplein 5
P.O. Box 80.006
3508 TA Utrecht

Abstract

The effective action for chiral $W_3$ gravity is studied. It is shown that the computation of the effective action can be reduced to that of a $SL(3,\mathbb{R})$ Wess-Zumino-Witten theory. If one assumes that the effective action for the Wess-Zumino-Witten model is identical to the WZW action up to multiplicative renormalizations, then the effective action for $W_3$ gravity is, to all orders, given by a constrained WZW model. The multiplicative renormalization constants of the WZW model are discussed and it is analyzed which particular values of these constants are consistent with previous one-loop calculations, and which reproduce the KPZ formulas for gravity and their generalizations for $W_3$ gravity.

‡e-mail: deboer@ruunts.fys.ruu.nl
§e-mail: goeree@ruunts.fys.ruu.nl
1. Introduction

Starting with a two-dimensional conformal field theory coupled to gravity one can construct an induced action for pure gravity by integrating out the matter degrees of freedom from the theory. One of the striking features of two dimensions is that the form of the induced action does not depend on the detailed form of the field theory, but only on its central charge. Therefore, this induced action is a good starting point to study the general properties of two-dimensional quantum gravity. The quantization of the induced action can proceed in several ways, depending on the gauge condition that one imposes on the remaining symmetries of the induced gravity action. In the conformal gauge this leads to Liouville theory, whereas in the chiral gauge the resulting theory is non-local. In this paper we deal with the quantization of the chiral induced action of \( W \) gravity. We do not discuss the quantization of the covariant induced action for \( W \) gravity in the conformal gauge which, as was demonstrated in [1, 2], would amount to the quantization of Toda theory. More on \( W \) gravity can be found in [3, 4, 5, 6] and references therein.

For ordinary gravity, the chiral induced action can be defined by

\[
e^{-\Gamma_c[\mu]} = \left\langle e^{-\frac{1}{\pi} \int d^2 z \mu T} \right\rangle_{\text{OPE}},
\]

where the right hand side is computed using the operator product expansion of \( T \) with itself. Here, the only remnant of the field theory one started with is the central charge \( c \) occurring in the operator product expansion; \( \mu \) refers to a component of the metric in the chiral gauge, \( ds^2 = dzd\bar{z} + \mu d\bar{z}d\bar{z} \), and for this reason \( \mu \) is sometimes denoted by \( h_{zz} \). To quantize this chiral induced action, we want to compute the generating functional for correlation functions of \( \mu \),

\[
e^{-\Gamma_{\text{eff}}[T]} = \int \mathcal{D} \mu e^{\frac{1}{\pi} \int d^2 z \mu T - \Gamma_c[\mu]},
\]

This functional \( \Gamma_{\text{eff}}[T] \) can be expanded in terms of \( 1/c \), \( \Gamma_{\text{eff}}[T] = \sum_{i \geq 0} c^{1-i} \Gamma^{(i)}_{\text{eff}}[T] \), and \( \Gamma^{(0)}_{\text{eff}} \) is the Legendre transform of \( \Gamma_c[\mu] \). As the precise form of \( \Gamma_c[\mu] \) is known [7],

\[
\Gamma_c[\mu] = -\frac{c}{24\pi} \int d^2 z \mu \partial^2 (1 - \frac{1}{\partial \mu \partial})^{-1} \partial \mu,
\]

one can in principle compute \( \Gamma_{\text{eff}}[T] \) order by order. This has been done up to one loop [8, 9, 10, 11], and the one loop result is \( -25\Gamma^{(0)}_{\text{eff}} + 13T \frac{\delta \Gamma^{(0)}_{\text{eff}}}{\delta T} \), so that the total
result up to one loop can be written as \((c - 25)\Gamma_{\text{eff}}^{(0)}[(1 + \frac{13}{c})T]\). This suggests [9] that the all order result for \(\Gamma_{\text{eff}}[T]\) is given by \(Z_k c \Gamma_{\text{eff}}^{(0)}[Z_T T]\), for certain constants \(Z_k\) and \(Z_c\) that are power series in \(1/c\). However, it is clear that it will become more and more cumbersome to go to higher orders, so we will choose a different strategy to compute \(\Gamma_{\text{eff}}[T]\).

Instead of using (1.1) as a starting point, we could also start with the following definition of \(\Gamma_c[\mu]\),

\[
e^{-\Gamma_c[\mu]} = \int \mathcal{D}\phi e^{-S(\phi)} \frac{1}{\pi} \int d^2 z \mu T(\phi), \tag{1.4}
\]

where \(T(\phi)\) is such that upon quantizing the \(\phi\) degrees of freedom, \(T(\phi)\) becomes an energy momentum tensor with central charge \(c\). One now immediately finds that

\[
e^{-\Gamma_{\text{eff}}[T]} = \int \mathcal{D}\phi e^{-S(\phi)} \delta(T - T(\phi)). \tag{1.5}
\]

In general, it is very difficult to perform the path integral over the \(\phi\) fields in the presence of this delta function, but it turns out that if we start with a constrained \(SL(2, \mathbb{R})\) WZW theory as an action, it is possible to perform this path integral and thus compute the effective action for gravity to all orders. This construction is closely related to the 'hidden \(SL(2, \mathbb{R})\) symmetry' in 2-d gravity [12, 7, 10]. In the constrained \(SL(2, \mathbb{R})\) WZW model one of the currents becomes, upon imposing the constraints, the energy momentum tensor of the theory, and therefore the delta function in (1.5) becomes a delta function for a current of the WZW theory and can be integrated out. For this last step one has to perform a change of variables in the WZW theory from the group variable \(g\) to the current \(g^{-1} \partial g\), and take the corresponding jacobian into account. The result of all this is that the effective action \(\Gamma_{\text{eff}}[T]\) is indeed of the form \(Z_k c \Gamma_{\text{eff}}^{(0)}[Z_T T]\), with \(Z_k\) and \(Z_T\) given by [5,7].

In the next section we perform this calculation for \(W_3\) gravity. Ordinary gravity can be treated in the same way, and we leave the detailed calculation for ordinary gravity to the reader. The induced and effective action for \(W_3\) action are defined similarly as for ordinary gravity. The induced action depends, besides on \(\mu\), on an extra field \(\nu\) (sometimes called \(B_{\bar{z} z \bar{z} z}\)), that couples to the \(W_3\) field. A difference with ordinary gravity is that the explicit form of \(\Gamma_c[\mu, \nu]\) is not known; for \(W_3\) gravity \(\Gamma_c[\mu, \nu]\) has an expansion in \(1/c\), and is known up to three loops [13]. For \(W_3\) gravity we start with a constrained \(SL(3, \mathbb{R})\) WZW model. It is known that imposing certain constraints on an \(SL(3, \mathbb{R})\) current algebra reduces the current algebra to a \(W_3\) algebra [12, 14]. The fields \(T(g)\) and \(W(g)\) that couple to the \(W_3\) gauge fields \(\mu, \nu\) are the generators of this \(W_3\) algebra. Furthermore, \(T(g)\) and \(W(g)\) can be chosen such as to preserve the
gauge invariance of the constrained WZW model. This enables us to perform a BRST quantization of the model. Because the BRST operator is nilpotent only on-shell, we need the Batalin-Vilkovisky quantization procedure to compute the quantum action.

To complete the computation, we need the jacobian for the change of variables from $g$ to $g^{-1} \partial g$. This is a rather difficult point, of which our understanding is incomplete. This point is discussed in section 4, where it is shown that knowing this jacobian is equivalent to knowing the effective action for ordinary WZW theory. Using the ansatz that this effective action is proportional, up to multiplicative renormalizations, to the WZW action, we then complete the calculation of the effective action of $W_3$ gravity. The result agrees with one-loop calculations \cite{11, 15} for $W_3$ gravity if the multiplicative renormalizations of the WZW model agree with the one-loop calculations for the WZW model \cite{25, 5}. The resulting effective action for $W_3$ gravity is proportional to a constrained WZW model, as conjectured in \cite{15}. Thus, $W_3$ gravity can be seen as an example of completely integrable nonlocal field theory. The crucial ingredient in establishing this integrability is demanding BRST invariance at the quantum level. The result also shows that the level of the $SL(3, \mathbb{R})$ current algebra in $W_3$ gravity is given by a KPZ-like formula as proposed in \cite{11, 12}. We would like to stress that none of these conclusions holds if the effective action of the WZW model is not simply proportional to a WZW action.

Actually, Knizhnik, Polyakov and Zamolodchikov derived their result for the level of the $SL(2, \mathbb{R})$ current algebra in gravity by an analysis of the gauge fixing of the covariant induced action for gravity \cite{10}. This procedure is closely related to the one used here, and can be generalized to $W_3$ gravity by gauge fixing the covariant action \cite{1, 2} for $W_3$ gravity. The advantage of this approach is that it makes the $SL(3, \mathbb{R})$ current algebra structure in $W_3$ gravity very clear. The disadvantage is that it is difficult to extract all order results from it, because the covariant action is only known to lowest order in $1/c$. This 'KPZ' approach to $W_3$ gravity will be discussed in a separate paper \cite{17}.

2. The Induced Action of $W_3$ Gravity

We start with the action for a constrained $SL(3, \mathbb{R})$ WZW model. The action is given by \cite{12}

$$ S_1 = k S_{WZW}(g) + \frac{1}{4\pi} \int d^2 z \ (\bar{A}^1 (J_1 - \xi) + \bar{A}^2 (J_2 - \xi) + \bar{A}^3 J_3), $$

(2.1)
where the current $\mathcal{J} = kg^{-1}\partial g$ is parametrized by

$$
\mathcal{J} = \begin{pmatrix} H_0 & J_1 & J_3 \\ K_1 & H_1 - H_0 & J_2 \\ K_3 & K_2 & -H_1 \end{pmatrix}.
$$

(2.2)

The action consists of a WZW model at level $k$, and three gauge fields $\bar{A}^i$ ($i = 1 \ldots 3$), that play the role of lagrange multipliers; $\xi$ is an arbitrary parameter different from zero, that is usually taken to be equal to one. It is well known that imposing the constraints $J_1 = J_2 = \xi, J_3 = 0$ on an $SL(3,\mathbb{R})$ current algebra reduces the current algebra to a $W_3$ algebra \cite{12, 14}. The WZW actions $S_{WZW}^\pm$ are given by

$$
S_{WZW}^\pm(g) = \frac{1}{2\pi} \int_\Sigma d^2z \Tr(g^{-1}\partial g g^{-1}\partial g) \pm \frac{1}{6\pi} \int_B \Tr(g^{-1}dg)^3,
$$

(2.3)

and satisfy the following Polyakov–Wiegmann identities \cite{18}:

$$
S_{WZW}^+(gh) = S_{WZW}^+(g) + S_{WZW}^+(h) + \frac{1}{\pi} \int d^2z \Tr(g^{-1}\partial g \partial hh^{-1}),
$$

$$
S_{WZW}^-(gh) = S_{WZW}^-(g) + S_{WZW}^-(h) + \frac{1}{\pi} \int d^2z \Tr(g^{-1}\partial g \bar{\partial hh}^{-1}).
$$

(2.4)

The action (2.1) has an invariance under the gauge transformations generated by the subgroup $N^-$ of lower triangular matrices. Explicitly, the action (2.1) is invariant under $\delta_\epsilon \mathcal{J} = k\partial \epsilon + [\mathcal{J}, \epsilon]$ (or $\delta_\epsilon g = g\epsilon$) and $\delta_\epsilon \bar{A}^1 = -\bar{\partial} \epsilon_1$, $\delta_\epsilon \bar{A}^2 = -\bar{\partial} \epsilon_2$, and $\delta_\epsilon \bar{A}^3 = -\bar{\partial} \epsilon_3 + A^2 \epsilon_1 - \bar{A}^1 \epsilon_2$, where

$$
\epsilon = \begin{pmatrix} 0 & 0 & 0 \\ \epsilon_1 & 0 & 0 \\ \epsilon_3 & \epsilon_2 & 0 \end{pmatrix}.
$$

(2.5)

As explained in the introduction we intend to couple this theory to the $W_3$ gauge fields $\mu, \nu$ by adding a term $\int \mu T(\mathcal{J}) + \int \nu W(\mathcal{J})$ to the action, while preserving the gauge invariance. To find $T(\mathcal{J})$ and $W(\mathcal{J})$ one uses the fact that there is a unique gauge transformation given by a lower triangular matrix $n$ with ones on the diagonal, such that

$$
\begin{pmatrix} 0 & \xi & 0 \\ T(\mathcal{J})/2\xi & 0 & \xi \\ W(\mathcal{J})/\xi^2 & T(\mathcal{J})/2\xi & 0 \end{pmatrix} = n^{-1} \begin{pmatrix} H_0 & \xi & 0 \\ K_1 & H_1 - H_0 & \xi \\ K_3 & K_2 & -H_1 \end{pmatrix} n + k\xi n^{-1} \partial n.
$$

(2.6)

The factors $1/2\xi$ and $1/\xi^2$ have been included for later convenience. The polynomials $T(\mathcal{J})$ and $W(\mathcal{J})$ are invariant under $N^-$ gauge transformations of the constrained
current $J_{\text{constr}} = J|_{j_1=\xi,j_2=\xi,j_3=0}$ that appears in (2.6): if we perform the $N^-$ gauge transformation $J'_{\text{constr}} = m^{-1}J_{\text{constr}}m + km^{-1}\partial m$, then the unique lower triangular matrix $n$ in (2.6) that brings $J'_{\text{constr}}$ in the right form is given by $n' = m^{-1}n$. Because $n^{-1}J_{\text{constr}}n + kn^{-1}\partial n = n^{-1}J'_{\text{constr}}n' + kn^{-1}\partial n'$, the left hand side of (2.6) does not change under this gauge transformation, and $T(J)$ and $W(J)$ are gauge invariant. Under a gauge transformation of the full current $J$, $T(J)$ and $W(J)$ are only invariant up to terms proportional to $J - J_{\text{constr}}$. Therefore, if we add $\int \mu T(J) + \int \nu W(J)$ to the action (2.1), the action is $N^-$ invariant up to terms proportional to the constraints. It is possible, by modifying the transformation rules for $\bar{A}^i$, to make the action exactly $N^-$ invariant. The expressions $T(J)$ and $W(J)$ are the so-called gauge invariant polynomials on the constrained phase space [19]. In terms of classical hamiltonian reduction, it are precisely these polynomials that survive the hamiltonian reduction of the WZW theory and it is known [19] that they generate an algebra that is isomorphic to the classical $W_3$ algebra. Of course, if $W$ and $T$ are gauge invariant polynomials on the constrained phase space, so are $W + n\partial T$ and $T$. The $T$ and $W$ we take, as defined by (2.6), correspond to a particular basis choice known as the ‘highest weight gauge’ [19], which guarantees that $W$ will transform as a spin three field.

If we compute $T(J)$ and $W(J)$ from (2.6) and add these to the action (2.1), the resulting action $S_2(\bar{A},g,\mu,\nu)$ reads

$$S_2 = kS_{\text{WZW}}(g) + \frac{1}{\pi} \int d^2z \left( \bar{A}^1(J_1 - \xi) + \bar{A}^2(J_2 - \xi) + \bar{A}^3J_3 \right)$$

$$+ \frac{N_T}{\pi} \int d^2z \mu(H_0^2 - H_0H_1 + H_1^2 + \xi(K_1 + K_2) - k\partial(H_0 + H_1))$$

$$+ \frac{N_W}{\pi} \int d^2z \nu(H_0^2H_1 - H_0H_1^2 + \xi(H_1K_1 - H_0K_2) + \xi^2K_3 + \frac{1}{2}k\xi\partial(K_2 - K_1)$$

$$+ \frac{1}{2}k^2\partial^2(H_0 - H_1) + k(-H_0\partial H_0 + H_1\partial H_1 + \frac{1}{2}H_0\partial H_1 - \frac{1}{2}H_1\partial H_0) \right),$$

(2.7)

where we have introduced two normalization factors $N_T$ and $N_W$. As explained above, the $N^-$ transformations that leave this action invariant are still given by $\delta_\epsilon J = k\partial \epsilon + [J, \epsilon]$ for the current, while for $\bar{A}^i$ they are extended to

$$\delta_\epsilon \bar{A}^1 = -\bar{\partial} \epsilon_3 + \bar{\partial}^2 \epsilon_1 - \bar{A}^1 \epsilon_2 + \epsilon_3(-N_T(\mu(H_0 + H_1) + 2k\partial \mu))$$

$$+ N_W(\nu(-H_0^2 + H_1^2 + \xi(K_2 - K_1) + k\partial(H_0 - H_1)) + \frac{1}{2}\partial \nu(H_1 - H_0)))$$

$$+ \frac{1}{2}k^2(\partial^2(H_0 - H_1) + k(-H_0\partial H_0 + H_1\partial H_1 + \frac{1}{2}H_0\partial H_1 - \frac{1}{2}H_1\partial H_0))$$

$$\delta_\epsilon \bar{A}^2 = -\bar{\partial} \epsilon_2 + N_T\mu(\epsilon_2(H_0 - 2H_1) - \xi\epsilon_3) - kN_T\epsilon_2\partial \mu$$

$$+ N_W(\epsilon_2\partial\nu(H_0 - 2H_1) - \xi\epsilon_3H_1 + \xi\epsilon_2K_1 + k\epsilon_2\partial H_0)$$

$$+ \frac{k^2}{2}N_W(\epsilon_2\partial\nu(H_0 + 2H_1) - \xi\epsilon_3\partial\nu) + \frac{k^2}{2}N_W\epsilon_2\partial^2 \nu$$

$$\delta_\epsilon \bar{A}^3 = -\bar{\partial} \epsilon_1 + N_T\mu(\epsilon_1(2H_0 - H_1) + \xi\epsilon_3) - kN_T\epsilon_1\partial \mu$$

$$- N_W(\epsilon_1\partial\nu(-\epsilon_1H_1 - 2H_0) + \xi\epsilon_3H_0 - \xi\epsilon_1K_2 + k\epsilon_1\partial H_1)$$
\[ -\frac{k}{2} N_W (\varepsilon_1 \partial \nu (2H_0 + H_1) + \xi \varepsilon_3 \partial \nu) - \frac{k^2}{2} N_W \varepsilon_1 \partial^2 \nu. \]  
(2.8)

As a generalization of (1.4) we consider the functional \( S_{\text{ind}}(\mu, \nu) \) defined by

\[ e^{-S_{\text{ind}}(\mu, \nu)} = \int \frac{D\bar{A}Dg}{\text{gauge volume}} e^{-S_2(\bar{A}, g, \mu, \nu)}. \]  
(2.9)

We shall prove shortly, that, with an appropriate choice of \( N_W \) and \( N_T \), the induced action \( S_{\text{ind}}(\mu, \nu) \) is equal to the induced action for \( W_3 \) gravity, to all orders in \( 1/c \). The induced action \( \Gamma_c[\mu, \nu] \) for \( W_3 \) gravity is defined by (cf. (1.1))

\[ e^{-\Gamma_c[\mu, \nu]} = \langle e^{-\frac{1}{\pi} \int d^2z (\mu T + \nu W)} \rangle_{\text{OPE}}, \]  
(2.10)

where the right hand side is computed using the operator product expansions of the \( W_3 \) algebra. Thus, in order to prove that \( S_{\text{ind}}(\mu, \nu) = \Gamma_c[\mu, \nu] \), we need to verify that upon quantizing \( g \) and \( \bar{A} \) the fields

\[ T_{\text{ind}} = \pi \frac{\delta S_2}{\delta \mu}, \quad W_{\text{ind}} = \pi \frac{\delta S_2}{\delta \nu}. \]  
(2.11)

generate a quantum \( W_3 \) algebra. This fixes the values of \( N_W \) and \( N_T \).

The quantization of \( S_2(g, \bar{A}, \mu, \nu) \) is most easily performed using BRST quantization (cf. [12]). The BRST transformation rules for \( g \) and \( \bar{A} \) are defined by replacing the parameters \( \varepsilon_i \) of the gauge transformations \( \delta \varepsilon = g \varepsilon \) and (2.8) by anti-commuting ghosts \( c_i \). We denote these transformation rules by \( \delta_B g \) and \( \delta_B \bar{A} \). The BRST transformation rules for the ghosts read \( \delta_B c_1 = \delta_B c_2 = 0 \) and \( \delta_B c_3 = c_1 c_2 \). However, due to the extra terms we added to the \( \bar{A} \) transformation rules in (2.8), the BRST operator \( \delta_B \) no longer satisfies \( \delta^2_B = 0 \). It only satisfies \( \delta^2_B = 0 \) when we use the \( \bar{A} \) equations of motion. In such a case a proper quantization and BRST gauge fixing of the theory requires that we use the Batalin-Vilkovisky formalism [20].

For all fields in the theory we introduce antifields \( (\bar{A}_i^*, g^* \text{ and } c^*) \) with opposite statistics. Because \( \delta_B^2 = 0 \) only on-shell, we typically need to include terms that are quadratic in the ghosts \( c_a \) and in the anti-fields to find a solution to the master equation. Because only \( \delta_B^2 \bar{A}_i \neq 0 \), the only terms quadratic in the antighosts that are needed are terms quadratic in \( \bar{A}_i^* \). Furthermore, if we compute \( \delta_B^2 \bar{A}_i \), we find that each term in the answer contains at most one derivative, and that the answer is proportional to the \( \bar{A}_i \) equations of motion. This leads us to write down the following ansatz for the
minimal solution to the master equation

\[ S_{\text{min}} = S_2 + \int \bar{A}_i^* \delta_B \bar{A}^i + \int g^* \delta_B g - \int \bar{c}_{\alpha} \int c_{\beta} + \int \bar{A}_i^* \bar{A}_j^* E^{ij,\alpha\beta} c_{\alpha} c_{\beta} \]
\[ + \int \bar{A}_i^* \bar{A}_j^* F^{ij,\alpha\beta} c_{\alpha} \partial c_{\beta} + \int \bar{A}_i^* \partial \bar{A}_j^* G^{ij,\alpha\beta} c_{\alpha} c_{\beta}. \]  

(2.12)

If we denote by \( \phi^I \) the set of fields \( (\bar{A}_i^*, g, c_i) \) and by \( \phi^*_I \) the corresponding set of anti-fields \( (\bar{A}_i^*, g^*, c_i^*) \), then the master equation reads \( (S_{\text{min}}, S_{\text{min}}) = 0 \), where

\[ (P, Q) = \frac{\partial P}{\partial \phi^I} \frac{\partial Q}{\partial \phi^*_I} - \frac{\partial P}{\partial \phi^*_I} \frac{\partial Q}{\partial \phi^I}. \]  

(2.13)

Working out the master equation for (2.12) yields, among others, the equation

\[ \delta_2^B(\bar{A}^k) = \frac{\delta S_2}{\delta A^k} \left( (2E^{jk,\alpha\beta} - \partial G^{jk,\alpha\beta}) c_{\alpha} c_{\beta} + (2F^{jk,\alpha\beta} - G^{jk,\alpha\beta} + G^{kj,\alpha\beta}) c_{\alpha} \partial c_{\beta} \right) 
\[ - \partial \left( \frac{\delta S_2}{\delta A^k} \right) (2\delta_{\alpha}^k + G^{jk,\alpha\beta}) c_{\alpha} c_{\beta}. \]  

(2.14)

From this one can compute the tensors \( E, F \) and \( G \). The components of these tensors either vanish, or can be determined from the following relations

\[ E^{jk,\alpha\beta} = -E^{kj,\alpha\beta} = -E^{jk,\beta\alpha}, \]
\[ E^{12,12} = \frac{1}{4}(N_T \mu - 2N_W H_0 \nu + 2N_W H_1 \nu), \]
\[ E^{13,13} = \frac{1}{4}(-N_T \mu - 2N_W H_0 \nu - kN_W \partial \nu), \]
\[ E^{23,23} = \frac{1}{4}(-N_T \mu + 2N_W H_1 \nu + kN_W \partial \nu), \]
\[ G^{jk,\alpha\beta} = G^{kj,\alpha\beta} = -G^{jk,\beta\alpha}, \]
\[ G^{12,12} = G^{13,13} = G^{23,23} = -\frac{k}{4}N_W \nu, \]
\[ F^{jk,\alpha\beta} = -F^{kj,\alpha\beta} = F^{jk,\beta\alpha}, \]
\[ F^{12,12} = F^{13,13} = F^{23,23} = -\frac{k}{4}N_W \nu. \]  

(2.15)

If we substitute this back into (2.12), we find that the master equation is satisfied. The full quantum action is given by

\[ S_q = S_{\text{min}} - \frac{1}{\pi} \int d^2z \left( b_1^* B_1 + b_2^* B_2 + b_3^* B_3 \right), \]  

(2.16)

where \( b_i^* \) are the anti-fields for the anti-ghosts \( b_i \), and the \( B_i \) are Lagrange multipliers, also known as the Nakanishi-Lautrup fields, that will impose the gauge condition. The
gauge fixing is done by replacing the antifields $\phi^*$ by $\partial\Psi/\partial\phi$ in the full quantum action (2.16), where $\Psi$, the gauge fermion, represents a particular gauge choice. We will choose

$$\Psi = \int d^2 z \left( b_1 \tilde{A}^1 + b_2 \tilde{A}^2 + b_3 \tilde{A}^3 \right),$$  

so that we put $c^i = g^i = 0$, $\tilde{A}^i = b_i$ and $b_i^* = \tilde{A}^i$ in (2.16). The resulting gauge fixed action is off-shell BRST invariant under the BRST transformations

$$\delta_B \phi^i = - \left. \frac{\partial S_q}{\partial \phi^i} \right|_{\phi^i = \partial\Psi/\partial\phi^i}.  \tag{2.18}$$

Note that the transformation rules for $\tilde{A}$ with respect to $\delta_B$ are different from those with respect to $\delta_B$, but we are going to integrate out the $\tilde{A}$, we do not give those (lengthy) transformation rules here. The gauge fixed action we have obtained can be written in a form that is remarkably similar to (2.7),

$$S_{gf} = kS_{WZW}(g) - \frac{1}{\pi} \int d^2 z \left( b_1 \partial c_1 + b_2 \partial c_2 + b_3 \partial c_3 \right)$$

$$+ \frac{1}{\pi} \int d^2 z \left( \tilde{A}^1 (\hat{J}_1 - \xi - B_1) + \tilde{A}^2 (\hat{J}_2 - \xi - B_2) + \tilde{A}^3 (\hat{J}_3 - B_3) \right)$$

$$+ \frac{N}{\pi} \int d^2 z \left( \hat{H}_0 \partial \hat{H}_0 - \hat{H}_0 \hat{H}_1 + \hat{H}_1^2 + \xi (\hat{K}_1 + \hat{K}_2) - k \partial (\hat{H}_0 + \hat{H}_1) \right)$$

$$+ \frac{Nw}{\pi} \int d^2 z \left( \hat{H}_0 \hat{H}_1 - \hat{H}_0 \hat{H}_1^2 + \xi (\hat{K}_1 \hat{K}_1 - \hat{H}_0 \hat{K}_2) + \xi^2 \hat{K}_3 + \frac{1}{2} k \xi \partial (\hat{K}_2 - \hat{K}_1) \right)$$

$$+ \frac{1}{2} k^2 \partial^2 (\hat{H}_0 - \hat{H}_1) + k (- \hat{H}_0 \partial \hat{H}_0 + \hat{H}_1 \partial \hat{H}_1 + \frac{1}{2} \hat{H}_0 \partial \hat{H}_1 + \frac{1}{2} \hat{H}_1 \partial \hat{H}_0)),  \tag{2.19}$$

where the hatted currents are the components of an $SL(3, \mathbb{R})$ valued object $\hat{J}$ and are defined by

$$\hat{J}_1 = J_1 + c_2 b_3, \quad \hat{J}_2 = J_2 - c_1 b_3, \quad \hat{J}_3 = J_3,$$

$$\hat{H}_0 = H_0 + c_1 b_1 + c_3 b_3, \quad \hat{H}_1 = H_1 + c_2 b_2 + c_3 b_3,$$

$$\hat{K}_1 = K_1 + c_3 b_2, \quad \hat{K}_2 = K_2 - c_3 b_1, \quad \hat{K}_3 = K_3.  \tag{2.20}$$

A simple way to define these hatted quantities is by means of the following expression

$$\hat{J} = J - \left[ \begin{array}{ccc} 0 & 0 & 0 \\ c_1 & 0 & 0 \\ c_3 & c_2 & 0 \end{array} \right] \left[ \begin{array}{ccc} 0 & b_1 & b_3 \\ 0 & 0 & b_2 \\ 0 & 0 & 0 \end{array} \right],  \tag{2.21}$$

where $[,]_+$ denotes an anticommutator. The meaning of these hatted currents becomes clear once we integrate out $B_i$ from the gauge fixed action $S_{gf}$, giving

$$S_{gf2} = kS_{WZW}(g) - \frac{1}{\pi} \int d^2 z \left( b_1 \partial c_1 + b_2 \partial c_2 + b_3 \partial c_3 \right)$$
\[ + \frac{N_T}{n} \int d^2z \mu T(\hat{J}) + \frac{N_W}{n} \int d^2z \nu W(\hat{J}). \]  

(2.22)

The BRST transformation rules for the anti-ghosts \( b_i \) now read

\[
\begin{align*}
\delta_B b_1 &= J_1 - \xi + c_2 b_3, \\
\delta_B b_2 &= J_2 - \xi - c_1 b_3, \\
\delta_B b_3 &= J_3.
\end{align*}
\]  

(2.23)

If we compare the BRST transformation rules of \( H_i \) and \( K_i \) with those for \( \hat{H}_i \) and \( \hat{K}_i \), we see that the transformation rules for \( \hat{H}_i \) and \( \hat{K}_i \) can be obtained from those for \( H_i \) and \( K_i \) by replacing \( J_1 \) and \( J_2 \) by \( \xi \) and \( J_3 \) by 0, and \( H_i \) and \( K_i \) by their hatted counterparts. The BRST transformation rules for \( \hat{H}_i \) and \( \hat{K}_i \) are therefore determined by the way the constrained current behaves under \( N^- \) gauge transformations, whereas the transformation rules for \( H_i \) and \( K_i \) were determined by the way in which the unconstrained current transformed under gauge transformations. Because \( T(\mathcal{J}) \) and \( W(\mathcal{J}) \) were constructed in such a way as to be exactly invariant under \( N^- \) gauge transformations of the constrained current, this automatically implies that \( T(\hat{\mathcal{J}}) \) and \( W(\hat{\mathcal{J}}) \) must be BRST invariant. The same procedure also works for constrained \( SL(N, \mathbb{R}) \) models for arbitrary \( N \). Instead of going through all the details of the Batalin-Vilkovisky procedure, one simply constructs the gauge invariant polynomials on the constrained phase space and then replaces currents by hatted currents, using the obvious generalizations of (2.6) and (2.21), to construct the BRST invariant gauge fixed action. Actually, this procedure is nothing but a classical version of the quantum hamiltonian reduction studied in [21]. In this approach, one computes the cohomology of the BRST operator generating the BRST transformations of the gauge fixed action (2.22) with \( \mu = \nu = 0 \) on the space of polynomials in the currents and the ghosts. The result is that the BRST cohomology is generated by quantum versions \( T_q \) and \( W_q \) of \( T(\hat{\mathcal{J}}) \) and \( W(\hat{\mathcal{J}}) \), which are given by (2.26) \[ \ast \]. Here we see that this quantum BRST cohomology is a quantization of the space of gauge invariant polynomials on the constrained phase space of the classical theory.

The gauge fixed action (2.22) consists of a WZW model and of three \( b, c \) systems. Because we know how to quantize these [22, 23], we can now try to find out whether \( S_{\text{ind}}(\mu, \nu) \) defined in (2.9) really is the induced action for \( W_3 \) gravity to all orders. As we explained previously, for this we need that \( N_T T(\hat{\mathcal{J}}) \) and \( N_W W(\hat{\mathcal{J}}) \) generate, at the

\footnote{In [21], it is shown that the BRST cohomology is isomorphic to the algebra generated by \( T_q \) and \( W_q \) with \( \hat{K}_i = 0 \). The actual BRST representatives of the cohomology were not constructed in [21], but one can show that they are given by (2.26). Indeed, the algebra generated by \( T_q \) and \( W_q \) does not change if one puts \( \hat{K}_i = 0 \) in (2.26).}
term with the largest number of currents is the same, both for the classical expressions
of a product of a certain number of currents always differ by terms that contain fewer
currents than the original product. This indicates that the coefficient in front of the
term with the largest number of currents is the same, both for the classical expressions
$T(\hat{\mathcal{J}})$ and $W(\hat{\mathcal{J}})$ and their normal ordered versions $T_q(\mathcal{J}, \text{ghosts}), W_q(\mathcal{J}, \text{ghosts})$. To obtain
the full expressions for $T_q$ and $W_q$, we need some extra principle that tells us how
to do this. The extra principle we choose is that of BRST invariance. As $T(\hat{\mathcal{J}}), W(\hat{\mathcal{J}})$
were classically BRST invariant, we require that $T_q, W_q$ are quantum BRST invariant.
Together with the requirement that the coefficients for the terms with the largest
number of currents do not change, this will completely fix the form of $T_q$ and $W_q$.
Classically, the BRST charge is given by

$$Q = \oint \frac{dz}{2\pi i} (c_1(\xi - J_1) + c_2(\xi - J_2) - c_3 J_3 - c_1 c_2 b_3).$$ \tag{2.24}$$

The quantum BRST operator is given by the same expression, with products of fields
replaced by normal ordered products. Notice that there is no normal ordering ambiguity in the definition of $Q$. The OPE’s of the ghosts and the currents are given by

$$c_i(z) b_j(w) = \frac{-\delta_{ij}}{(z-w)},$$
$$\mathcal{J}_a(z) \mathcal{J}_b(w) = \frac{-k \eta_{ab}}{(z-w)^2} + \frac{-f^{ac} \mathcal{J}_c(w)}{(z-w)},$$ \tag{2.25}$$

where we decomposed the current $\mathcal{J} = T^a, \eta^{ab} = \text{Tr}(T^a T^b)$, $\eta_{ab}$ is the inverse of $\eta^{ab}$, $f^{abc} T^c = [T^a, T^b]$, and indices are raised and lowered using $\eta^{ab}$. It is now a straightforward computation to show that the fields

$$T_q = N_T ((\hat{H}_0 \hat{H}_0) - (\hat{H}_0 \hat{H}_1) + (\hat{H}_1 \hat{H}_1) + \xi (\hat{K}_1 + \hat{K}_2) - (k-2)\partial (\hat{H}_0 + \hat{H}_1)), $$

$$W_q = N_W ((\hat{H}_0 \hat{H}_0 \hat{H}_1)) - (\hat{H}_0 \hat{H}_1 \hat{H}_1) + \xi ((\hat{H}_1 \hat{K}_1) - (\hat{H}_0 \hat{K}_2)) + \xi^2 \hat{K}_3
+ \frac{1}{2}(k-2)\xi \partial (\hat{K}_2 - \hat{K}_1) + \frac{1}{2}(k-2)^2 \partial^2 (\hat{H}_0 - \hat{H}_1)
+ (k-2)(-\hat{H}_0 \partial \hat{H}_0) + (\hat{H}_1 \partial \hat{H}_1) + \frac{1}{2}(\hat{H}_0 \partial \hat{H}_1) - \frac{1}{2}(\hat{H}_1 \partial \hat{H}_0)), \tag{2.26}$$

form a quantum $W_3$ algebra, with central charge

$$c = 50 + 24 \left( (k-3) + \frac{1}{(k-3)} \right). \tag{2.27}$$
Here, the hatted fields are still given by (2.20), and have the OPE’s
\[
\hat{J}_a(z)\hat{J}_b(w) = \frac{-(k-3)\eta_{ab}}{(z-w)^2} + \frac{-f_{ab}^c \hat{J}_c(w)}{(z-w)^2},
\]
for \(\hat{J}_a, \hat{J}_b \in \{\hat{H}_i, \hat{K}_i\}\). The normalization constants \(N_T\) and \(N_W\) must be equal to
\[
N_T = \frac{-1}{k-3},
\]
\[
N_W = \left(\frac{-6}{15k^4 - 146k^3 + 519k^2 - 792k + 432}\right)^{\frac{1}{2}} = \left(\frac{-48}{(k-3)^3(5c+22)}\right)^{\frac{1}{2}}\]
(2.29)
This shows that \(S_{ind}(\mu, \nu)\) is indeed the all order induced action for \(W_3\) gravity, where \(c\) is related to \(k\) via (2.27) and \(N_T\) and \(N_W\) must be chosen according to (2.29). The constant \(\xi\) can be chosen arbitrarily.

To summarize, we have shown that the constrained WZW model can be coupled to the \(W_3\) gauge fields in such a way that the resulting induced action for the \(W_3\) gauge fields is precisely the all order (chiral) induced action for \(W_3\) gravity.

3. The Effective Action of \(W_3\) Gravity

The effective action for \(W_3\) gravity is obtained by quantizing the induced action, and is defined by the following path integral (cf. (1.2))
\[
e^{-\Gamma_{eff}[T,W]} = \int \frac{DgD\bar{A}}{\text{gauge volume}} D\mu D\nu e^{\frac{1}{2} \int d^2z (\mu T + \nu W) - S_2(g, \bar{A}, \mu, \nu)},
\]
(3.1)
where \(S_2\) is the action (2.7). In the previous section we performed a BRST quantization of \(S_2(g, \bar{A}, \mu, \nu)\), by gauge fixing \(\bar{A}^i = 0\). This is a convenient gauge condition for proving that the induced action for \(\mu\) and \(\nu\) is the same as the induced action for \(W_3\) gravity, but not for the computation of the effective action. Therefore, we will use a different gauge here, namely \(H_0 = H_1 = K_1 - K_2 = 0\). Because the BRST operator \(\delta_B\) satisfies \(\delta_B^2 H_0 = \delta_B^2 H_1 = \delta_B^2 (K_1 - K_2) = 0\), there is no need to use Batalin-Vilkovisky quantization here. Under gauge transformations \(H_0, H_1\) and \(K_1 - K_2\) transform as
\[
\delta_\epsilon H_0 = J_1 \epsilon_1 + J_3 \epsilon_3, \\
\delta_\epsilon H_1 = J_2 \epsilon_2 + J_3 \epsilon_3, \\
\delta_\epsilon (K_1 - K_2) = (H_1 - 2H_0) \epsilon_1 + (H_0 - 2H_1) \epsilon_2 + (J_2 - J_1) \epsilon_3 + k \partial (\epsilon_1 - \epsilon_2). \]
(3.2)
This shows that gauge fixing \( H_0 = H_1 = K_1 - K_2 = 0 \) produces a Faddeev-Popov contribution to the path integral which is equal to

\[
\int \mathcal{D} \beta_1 \mathcal{D} \gamma_1 \mathcal{D} \beta_2 \mathcal{D} \gamma_2 \mathcal{D} \beta_3 \mathcal{D} \gamma_3 \exp \left( -\frac{1}{\pi} \int d^2 z \left( \xi \beta_1 \gamma_1 + \xi \beta_2 \gamma_2 + 2 \xi \beta_3 \gamma_3 + k \beta_3 \partial(\gamma_1 - \gamma_2) \right) \right),
\]

(3.3)

where we put \( J_1 = J_2 = \xi \) and \( J_3 = 0 \), which can be done safely after performing the \( \tilde{A} \) integration. It is clear that (3.3) is just some numerical factor, and we will ignore this factor. Then we can remove the volume of the gauge group in (3.1) by inserting the combination \( \delta(H_0)\delta(H_1)\delta(K_1 - K_2) \) into the path integral. The \( \tilde{A} \) and \( \mu, \nu \) integrations yield five more delta function insertions in the path integral. Altogether this shows that

\[
e^{-\Gamma_{\text{eff}}[T,W] = \int \mathcal{D} g \delta(J_1 - \xi)\delta(J_2 - \xi)\delta(J_3)\delta(H_0)\delta(H_1)\delta(K_1 - K_2)
\delta(T - N_T\xi(K_1 + K_2))\delta(W - N_W\xi^2 K_3) e^{-kS_{WZW}(g)}},
\]

(3.4)

It seems that we are already done, as the delta functions absorb all the degrees of freedom, and that we are left with a constrained WZW model. However, before we can integrate out the delta functions, we must first change variables from \( g \) to \( g^{-1}\partial g \), and compute the corresponding jacobian. This change of variables is a rather tricky point, which we now discuss in some detail.

4. The Effective Action of the WZW Model

It is generally believed \([18, 25]\), that the jacobian corresponding to the change of variables from \( A_z = g^{-1}\partial g \) to \( g \) leads to

\[
\mathcal{D} A_z = \exp(-2h_G S_{WZW}(g)) \mathcal{D} g,
\]

(4.1)

where \( h_G \) is the dual Coxeter number of the group under consideration. The computation of this jacobian proceeds by noticing that \( \delta A_z = \partial A_z(g^{-1}\delta g) \), so that the jacobian is equal to \( \det(\partial A_z) \), and then by writing this determinant as the path integral \( \int \mathcal{D} \psi \mathcal{D} \bar{\psi} \exp(-\int \bar{\psi} \partial A_z \psi) \), where \( \psi, \bar{\psi} \) are fermions transforming in the adjoint

*The symbols \( A_z \) and \( J_z \) used in this section should not be confused with \( \tilde{A}^i \) and \( J_i \) used in the previous sections.
representation of the group. Finally, one can derive a Ward identity for this fermionic path integral and show that the solution to this Ward identity is indeed given by (4.1).

Actually, (4.1) is in disagreement with one-loop calculations for the WZW model [23, 5]. If (4.1) were true, then one could easily compute the effective action for the WZW model to all orders: first, we compute the generating functional of connected diagrams $G[J_z]$, given by

$$\exp -G[J_z] = \int \mathcal{D}A_z \exp(-kS_{WZW}(A_z) + \frac{1}{\pi} \int d^2z \text{Tr}(A_zJ_z)). \quad (4.2)$$

If we change variables from $A_z$ to $g$ with $A_z = g^{-1} \partial g$, and parametrize $J_z$ by $J_z = -(k+2h_G)\partial hh^{-1}$, we can use the Polyakov-Wiegmann identity (2.4) to write the right-hand side of (4.2) as

$$\int \mathcal{D}g \exp(-(k+2h_G)S_{WZW}(gh) + (k+2h_G)S_{WZW}(h)). \quad (4.3)$$

We can safely replace the variable $g$ by $g' = gh^{-1}$, because this does not change the measure $\mathcal{D}g$, and we see that if we ignore an infinite factor, the generating functional $G[J_z] = -(k+2h_G)S_{WZW}(h)$. The effective action $S_{\text{eff}}(A_z)$ is the Legendre transform of $G[J_z]$,

$$S_{\text{eff}}(A_z) = \min_{J_z} \left(-G[J_z] - \frac{1}{\pi} \int d^2z \text{Tr}(A_zJ_z)\right)$$

$$= \min_h \left((k+2h_G)S_{WZW}(h) + \frac{(k+2h_G)}{\pi} \int d^2z \text{Tr}(A_z\partial hh^{-1})\right)$$

$$= \min_h \left((k+2h_G)S_{WZW}(h^{-1}) - \frac{(k+2h_G)}{\pi} \int d^2z \text{Tr}(A_zh^{-1}\partial h)\right). \quad (4.4)$$

The extremum is attained for $A_z = h^{-1}\partial h$, and we find that the effective action is simply

$$S_{\text{eff}}(A_z) = -(k+2h_G)S_{WZW}(A_z). \quad (4.5)$$

On the other hand, one can also perform a one-loop computation of the effective action [23, 5], and check the above result. In (4.2), the saddle point of the action $-kS_{WZW}(A_z) + \frac{1}{\pi} \int d^2z \text{Tr}(A_zJ_z)$ is at $A_z^{(0)}(J_z)$, where $A_z^{(0)}$ is defined by the equation $F(A_z^{(0)}, \frac{1}{k}J_z) = 0$. If we write $A_z = A_z^{(0)} + \tilde{A}_z$, and $J_z = -k\partial hh^{-1}$, so that $A_z^{(0)} = -\partial hh^{-1}$, then we can expand (4.2)

$$\exp -G[J_z] = \int \mathcal{D}\tilde{A}_z \exp(-kS_{WZW}(h) - \frac{k}{\pi} \int d^2z \text{Tr}(\tilde{A}_z\partial^{-1}_z A_z^{(0)} \tilde{A}_z + \ldots)), \quad (4.6)$$

$^1F(A_z, A_z)$ denotes the curvature of the connection $\partial + A_z + \bar{\partial} + \bar{A}_z$, and is given by $F = \partial A_z - \bar{\partial}A_z + [A_z, A_z]$.
where \( \partial A_z^{(0)} = \partial + \text{ad}(A_z^{(0)}) \) and \( \bar{\partial} A_z^{(0)} = \bar{\partial} + \text{ad}(A_z^{(0)}) \), with \( A_z^{(0)} = \frac{1}{k} J_z \). This shows that the one-loop contribution to \( G[J_z] \) is given by \( \frac{1}{2} \log \det(\delta^{-1} \bar{\partial} A_z^{(0)}) \). If we assume that this determinant is equal to \( \frac{1}{2} \log \det(\bar{\partial} A_z^{(0)}) - \frac{1}{2} \log \det(\partial A_z^{(0)}) \), then we can compute these determinants as explained below (4.1), to obtain

\[
G_{\text{one-loop}}[J_z] = \frac{1}{2} \log \det(\delta^{-1} \bar{\partial} A_z^{(0)}) + \frac{1}{2} \log \det(\partial A_z^{(0)}) = \frac{1}{2} \log \det(\bar{\partial} A_z^{(0)}) - \frac{1}{2} \log \det(\partial A_z^{(0)}). \tag{4.7}
\]

The effective action up to one loop can be computed in the same way as in (4.4)

\[
S_{\text{eff}}(A_z) = \min_h \left( (k + 2h_G)S_W^{-} (h^{-1}) - \frac{h_G}{\pi} \int d^2z \text{Tr}(h^{-1} \partial h h^{-1} \bar{\partial} h) \right), \tag{4.8}
\]

where in the last line we changed variables from \( h \) to \( h' \), with \( h'^{-1} \bar{\partial} h' = (1 - \frac{h_G}{k}) h^{-1} \bar{\partial} h \). The extremum is at \( A_z = (1 + \frac{h_G}{k}) h'^{-1} \partial h' \), and we find that up to one loop the effective action is given by

\[
S_{\text{eff}}(A_z) = -(k + 2h_G)S_W^-((1 - \frac{h_G}{k}) A_z). \tag{4.9}
\]

The disagreement between (4.5) and (4.9) is quite puzzling. The one-loop calculation involves the computation of a determinant, requiring a choice of regularization procedure. For 2-d quantum gravity and \( W_3 \) gravity, a computation similar to the one above agrees with independent one-loop calculations performed in momentum space [11]. In these momentum space calculations one has to deal with momentum routing ambiguities, and the agreement is only obtained after fixing these in a rather ad hoc way. Nevertheless, this provides an independent indication that (4.9) is a correct result. If one assumes that (4.9) is correct, then there is something wrong with the derivation of (4.5). The only non-classical step in this derivation is the replacement \( DA_z \to Dg \), and the only source of trouble can be that the jacobian for the change of variables \( A_z \to g \) is not the same as the jacobian for the change of measures \( DA_z \to Dg \). Although our understanding of how this could come about is incomplete, the problem (if any) seems to be related to giving a proper definition of \( DA_z \). The basic property that fixes this measure is demanding that for arbitrary functions \( f(A_z) \),

\[
\int DA_z \delta(A_z - X) f(A_z) = f(X). \tag{4.10}
\]
However, the measure $\mathcal{D}g$ stems from the inner product
\[
\langle \delta g, \delta' g \rangle = \int d^2 z \sqrt{\det h_{ab}} \text{Tr}(g^{-1}\delta gg^{-1}\delta' g),
\]
where $h_{ab}$ is the two-dimensional metric. The change of variables $g^{-1}\delta g \rightarrow \delta A_z$ produces a measure $\mathcal{D}A_z$ coming from the inner product
\[
\langle \delta A_z, \delta' A_z \rangle = \int d^2 z \sqrt{\det h_{ab}} \text{Tr}(\delta A_z \delta' A_z),
\]
and this inner product is ill-defined, because the integrand is not a density, but something of conformal weight $(\Delta, \bar{\Delta}) = (3, 1)$. A much more natural measure would for instance be the one coming from the inner product
\[
\langle \delta A_z, \delta' A_z \rangle = \int d^2 z \text{Tr}(\delta A_z \delta' A_z(A_z) + \delta' A_z \delta A_z(A_z)),
\]
where $A_z(A_z)$ is such that the connection $A$ has vanishing curvature. This measure is not consistent with (4.10) and not with the one-loop calculations, because the measure used there is determined by treating $A_z$ as a free field, i.e. one decomposes $A_z$ in Fourier modes and defines the inner product such that these are orthogonal. How one should compute the jacobian for the change of measure from $\mathcal{D}g$ to such a measure is not clear.

To proceed, we will assume that the effective action for the WZW model acquires only multiplicative renormalizations, keeping the above mentioned subtleties in mind. If we do not assume this then the computation of the effective action for $W_3$ gravity stops at (3.4). In any case both (4.5) and (4.9) are in agreement with this assumption. Thus, the rest of the computations are based on

**conjecture 1**

The effective action for the WZW model is given by $-k Z_k S_{WZW}(Z_A A_z)$, where $Z_k = 1 + \mathcal{O}(h_G/k)$ and $Z_A = 1 + \mathcal{O}(h_G/k)$.

As one can easily check with a calculation similar to the one that led to (4.5), this conjecture follows if one assumes that the following identity between path integrals is valid, relating $\mathcal{D}A_z$ and $\mathcal{D}g$:

**conjecture 1’**

For an arbitrary local functional $f$,
\[ \int \mathcal{D}A_z \ f(A_z) \exp(-kS_{WZW}(A_z)) = \int \mathcal{D}g \ f(Z^{-1}_Ag^{-1}\partial g) \exp(-kZ_kS_{WZW}(g)). \]

It is possible to prove the converse as well: conjecture 1 implies conjecture 1’. For the proof of this fact one first decomposes the function \( f \) into Fourier modes, and then parametrizes an arbitrary mode with a group valued variable \( h \) via

\[ f_h(A_z) = \exp(-\frac{1}{2} \int d^2z \ Tr(Z_AZ_kk\bar{\partial}hh^{-1}A_z)). \] (4.14)

Some manipulations, using the Polyakov-Wiegmann identity and the definition of the effective action, are then sufficient to derive conjecture 1’ for an arbitrary Fourier mode, and thus for arbitrary functions \( f \).

As far as the values of \( Z_k \) and \( Z_A \) are concerned, both the calculation using the naive jacobian and the one-loop calculation seem to suggest that \( Z_k = 1 + \frac{2h_G}{k} + \mathcal{O}(h_G/k)^2 \). For compact groups, the level of the WZW action must be an integer for the action to be well defined, suggesting that the level does not renormalize beyond one-loop, and that \( Z_k = (1 + \frac{2h_G}{k}) \) is indeed the full answer. This is the value of \( Z_k \) that we will use in the rest of the paper. The same value for \( Z_k \) was proposed in [23, 5]. As (4.5) and (4.9) predict different values for \( Z_A \) we will for \( Z_A \) take some arbitrary function \( Z_A(k) \).

5. The Effective Action of \( W_3 \) Gravity, Continued

Using conjecture 1’ and \( Z_k = (1 + \frac{2h_G}{k}) \) it is straightforward to work out the effective action for \( W_3 \) gravity. Starting with (3.4), and using that \( h_G = 3 \) for \( SL(3, \mathbb{R}) \), one finds:

\[ e^{-\Gamma_{eff}[T,W]} = \int \mathcal{D}g \delta(J_1 - \xi)\delta(J_2 - \xi)\delta(J_3)\delta(H_0)\delta(H_1)\delta(K_1 - K_2) \]
\[ \delta(T - N_T\xi(K_1 + K_2))\delta(W - N_W\xi^2K_3)e^{-kS_{WZW}(g)} \]
\[ = \int \mathcal{D}A_z \delta(J'_1 - \xi)\delta(J'_2 - \xi)\delta(J'_3)\delta(H'_0)\delta(H'_1)\delta(K'_1 - K'_2) \]
\[ \delta(T - N_T\xi(K'_1 + K'_2))\delta(W - N_W\xi^2K'_3)e^{-(k-6)S_{WZW}(A_z)}, \] (5.1)

where \( J' = kZ_A(k-6)A_z \). We can substitute the delta functions into the WZW action, and obtain the effective action for \( W_3 \) gravity to all orders. The final result reads, in
terms of the renormalized level \( k_c = k - 6 \):

\[
\Gamma_{\text{eff}}[T, W] = k_c S_{WZW}^{-} \begin{pmatrix}
0 & T & 0 \\
2N_T \xi_{(k_c+6)Z_A(k_c)} & 0 & T \\
N_W \xi^2{(k_c+6)Z_A(k_c)} & 2N_T(k_c+6)Z_A(k_c) & 0 \\
\end{pmatrix} \ .
\] (5.2)

The induced action for classical \( W_3 \) gravity, \( \Gamma_L[T, W] \), can also be obtained from a constrained WZW model; one can take the large \( k_c \) limit of (5.2), but one can also directly compute the Ward identities for a constrained WZW model and compare those with the Ward identities for the classical \( W_3 \) algebra. In any case the result for \( \Gamma_L[T, W] \) reads

\[
\Gamma_L[T, W] = k S_{WZW}^{-} \begin{pmatrix}
0 & \alpha & 0 \\
\beta T & 0 & \alpha \\
\gamma W & \beta T & 0 \\
\end{pmatrix} \ ,
\] (5.3)

where \( c = 24k \), \( 2\alpha \beta k = -1 \), and \( \gamma^2 = -10\beta^2/\alpha^2 \). Both (5.2) and (5.3) contain one free parameter, and we can choose \( \xi/(k_c+6)Z_A(k_c) = \alpha = 1 \). This proves that

\[
\Gamma_{\text{eff}}[T, W] = Z_k \Gamma_L[Z_T T, Z_W W] ,
\] (5.4)

and using (2.27) and (2.29) we find that \( k_c \) and the central charge \( c \) are related through

\[
c = 50 + 24 \left( (k_c + 3) + \frac{1}{(k_c + 3)} \right)
\] (5.5)

and that the renormalizations \( Z_k \), \( Z_T \) and \( Z_W \) are given by

\[
Z_k = \frac{24}{c} k_c = 1 - \frac{122}{c} + \ldots ,
\]

\[
Z_T = \frac{c(k_c + 3)}{24(k_c + 6)^2Z_A(k_c)^2} ,
\]

\[
Z_W = \frac{c \sqrt{(5c + 22)(k_c + 3)^3/2}}{48 \sqrt{30}(k_c + 6)^3Z_A(k_c)^3} .
\] (5.6)

These results are in agreement with the one-loop results obtained in [11, 15], if \( Z_A(k_c) = 1 - \frac{2}{k_c} + O(1/k_c)^2 \), as predicted by (4.9). Note that the ‘KPZ’ relation between the level \( k_c \) and \( c \) given in (5.5) is independent of \( Z_A \), and always comes out of this analysis as

18
long as $Z_k = 1 + \frac{2h_G}{k}$. Clearly, the techniques used here can be applied to $W_N$ gravity for arbitrary $N$, and in particular to 2-d quantum gravity, yielding

$$\Gamma_{\text{eff}}[T] = Z_k \Gamma_L[Z_T T],$$

$$c = 13 + 6 \left( \frac{k_c + 2}{k_c + 2} \right),$$

$$Z_k = \frac{6}{c} k_c,$$

$$Z_T = \frac{c(k_c + 2)}{6(k_c + 4)^2 Z_A(k_c)^2}. \quad (5.7)$$

These results agree with those obtained in [10, 9, 8, 11], if $Z_A(k_c) = 1 - \frac{2}{k_c} + O(1/k_c)^2$.

6. Conclusions

We have shown how one can obtain the effective action for $W_3$ gravity if the effective action for the WZW theory is known. It is still an open problem to give a proof of conjecture 1, or to show that it is false.[*] A clue towards the validity of conjecture 1 can be obtained by performing a two-loop calculation for the WZW theory, which is under current investigation [27]. Assuming the validity of conjecture 1, what could be the exact values of $Z_k$ and $Z_A$? As we explained at the end of section 4, it is reasonable to expect that $Z_k = 1 + \frac{2h_G}{k}$, from which one can derive the ‘KPZ’ relation (5.5) between the central charge and the renormalized level $k_c$ for $W_3$ gravity. As for $Z_A$, one might ‘argue’ as follows (cf. [5]): the effective action of the WZW model has a current algebra of level $-k - 2h_G$, and the fact that the classical equation $(g J) = (-k - 2h_G) \partial g$ renormalizes on the quantum level to $(g J) = (-k - h_G) \partial g$ suggests that

$$Z_A(k) = \frac{k + h_G}{k + 2h_G} = 1 - \frac{h_G}{k} + \ldots \quad (6.1)$$

This value of $Z_A(k)$ has certain nice features: if we substitute it in (5.6) and (5.7), the expressions given there simplify considerably and agree with one-loop calculations. On the other hand, if conjecture 1’ were also valid for nonlocal functionals $f$, one could

[*] It may also turn out that the computation of the effective action of the WZW model is so ambiguous, that one can impose conjecture 1 as a regularization prescription.
take $f(A_z) = g(A_z) \exp(-l S_{WZW}(A_z))$ with an arbitrary functional $g$, and evaluate the left hand side of conjecture 1’ in two different ways. It turns out that the two answers agree for generic $g$ only if $Z_A = 1$ and $Z_k = 1 + \frac{a}{k}$ for some constant $a$. Thus, this suggests

$$Z_A(k) = 1,$$  \hspace{1cm} (6.2)

leading to renormalization constants for gravity and $W_3$ gravity that disagree with the one-loop calculations. However, these one-loop calculations were performed along the same lines as the computation of (4.9), and as soon as one claims that there is something wrong with (4.9), the one-loop results for gravity become doubtful as well. Clearly, a more precise analysis is needed to settle these issues.

The relation between the constrained WZW model presented here and Toda theory becomes clear if one picks in (3.1) the gauge choice $K_1 = K_2 = K_3 = 0$. Ignoring the non-trivial contribution of the Faddeev-Popov ghosts in this case, the action (3.1) reduces to a Toda action, and $T$ and $W$ can be identified with the conserved currents of the Toda theory.

All the computations in this paper have been done on the complex plane. Working on a non-trivial Riemann surface will probably introduce many extra subtleties, in particular one has to work with generalized WZW actions [6]. We leave this and other issues to future study.

Acknowledgement

We would like to thank B. de Wit for carefully reading the manuscript. This work was financially supported by the Stichting voor Fundamenteel Onderzoek der Materie (FOM).
References

[1] J. de Boer and J. Goeree, Phys. Lett. B274 (1992) 289.

[2] J. de Boer and J. Goeree, Nucl. Phys. B381 (1992) 329.

[3] C. M. Hull, 'Classical and Quantum W-Gravity’, QMW/PH/92/1.

[4] C. N. Pope, 'Lectures on W Algebras and W Gravity’, CTP TAMU-103/91, Lectures given at the Trieste Summer School in High-Energy Physics, August 1991.

[5] K. Schoutens, A. Sevrin and P. van Nieuwenhuizen, 'Induced Gauge Theories and W gravity’, ITP-SB-91-54, CERN-TH.6330/91, LBL-31381, UCB-PTH-91/51, to appear in 'Strings and Symmetries 1991', Stony Brook, May 1991.

[6] J. de Boer and J. Goeree, 'The Covariant Action and Its Moduli Space from Gauge Theory’, THU-92/14.

[7] A. M. Polyakov, Mod. Phys. Lett. A2 (1987), 893.

[8] K.A. Meissner, J. Pawelczyk, Mod. Phys. Lett. A10 (1990) 763.

[9] Al. B. Zamolodchikov, preprint ITEP 84-89 (1989).

[10] V. G. Knizhnik, A. M. Polyakov and A. B. Zamolodchikov, Mod. Phys. Lett. A3 (1988) 819.

[11] M.T. Grisaru, P. van Nieuwenhuizen, ‘Loop Calculations in Two-Dimensional Non-local Field Theories’, CERN-TH.6388/92.

[12] M. Bershadsky and H. Ooguri, Comm. Math. Phys. 126 (1989) 49.

[13] K. Schoutens, A. Sevrin and P. van Nieuwenhuizen, Nucl. Phys. B364 (1991) 584.

[14] V. G. Drinfel’d and V. V. Sokolov, J. Sov. Math. 30 (1985) 1975, Sov. Math. Doklady 3 (1981) 457.

[15] K. Schoutens, A. Sevrin and P. van Nieuwenhuizen, Nucl. Phys. B371 (1992) 315.

[16] Y. Matsuo, Phys. Lett. B227 (1989) 209.

[17] J. de Boer and J. Goeree, in preparation.

[18] A. M. Polyakov and P. B. Wiegmann, Phys. Lett. B131 (1983) 121; Phys. Lett. B141 (1984) 233.
[19] J. Balog, L. Fehér, L. O’Raifeartaigh, P. Forgács and A. Wipf, Ann. Phys. 203 (1990) 76.

[20] I.A. Batalin, G.A. Vilkovisky, Phys. Lett. B102 (1981) 27; Phys. Rev. D28 (1983) 2567.

[21] B. Feigin and E. Frenkel Phys. Lett. B246 (1990) 75; E. Frenkel, ’W-Algebras and Langlands-Drinfel’d Correspondence’, lectures given at Cargèse Summer School on ’New Symmetry Principles in QFT’, July 1991.

[22] D. Friedan, E. Martinec and S. Shenker, Nucl. Phys. B271 (1986) 93.

[23] V. Knizhnik and A. B. Zamolodchikov, Nucl. Phys. B247 (1984) 83.

[24] F. Bais, P. Bouwknegt, M. Surridge and K. Schoutens, Nucl. Phys. B304 (1988) 348.

[25] A. M. Polyakov, in Les Houches 1988, Fields, Strings and Critical Phenomena, eds. E. Brézin and J. Zinn-Justin, North-Holland, 1990.

[26] H. Ooguri, K. Schoutens, A. Sevrin and P. van Nieuwenhuizen, ’The Induced Action of $W_3$ Gravity’, ITP-SB-91/16, RIMS-764 (June 1991).

[27] B. de Wit, M. Grisaru, P. van Nieuwenhuizen, work in progress.