ON THE POSSIBLE IMAGES OF THE MOD $\ell$ REPRESENTATIONS ASSOCIATED TO ELLIPTIC CURVES OVER $\mathbb{Q}$

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ABSTRACT. Consider a non-CM elliptic curve $E$ defined over $\mathbb{Q}$. For each prime $\ell$, there is a representation $\rho_{E,\ell}: \text{Gal}(\overline{\mathbb{Q}}/\mathbb{Q}) \to \text{GL}_2(\mathbb{F}_\ell)$ that describes the Galois action on the $\ell$-torsion points of $E$. A famous theorem of Serre says that $\rho_{E,\ell}$ is surjective for all large enough $\ell$. We will describe all known, and conjecturally all, pairs $(E, \ell)$ such that $\rho_{E,\ell}$ is not surjective. Together with another paper, this produces an algorithm that given an elliptic curve $E/\mathbb{Q}$, outputs the set of such exceptional primes $\ell$ and describes all the groups $\rho_{E,\ell}(\text{Gal}(\overline{\mathbb{Q}}/\mathbb{Q}))$ up to conjugacy. Much of the paper is dedicated to computing various modular curves of genus 0 with their morphisms to the $j$-line.

1. Possible images

Consider an elliptic curve $E$ defined over $\mathbb{Q}$. For each prime $\ell$, let $E[\ell]$ be the $\ell$-torsion subgroup of $E(\overline{\mathbb{Q}})$, where $\overline{\mathbb{Q}}$ is a fixed algebraic closure of $\mathbb{Q}$. The group $E[\ell]$ is a free $\mathbb{F}_\ell$-module of rank 2 and there is a natural action of the absolute Galois group $\text{Gal}_Q := \text{Gal}(\overline{\mathbb{Q}}/\mathbb{Q})$ on $E[\ell]$ which respects the group structure. After choosing a basis for $E[\ell]$, this action can be expressed in terms of a Galois representation

$$\rho_{E,\ell}: \text{Gal}_Q \to \text{GL}_2(\mathbb{F}_\ell);$$

its image $\rho_{E,\ell}(\text{Gal}_Q)$ is uniquely determined up to conjugacy in $\text{GL}_2(\mathbb{F}_\ell)$. A renowned theorem of Serre [Ser72] says that $\rho_{E,\ell}$ is surjective for all but finitely many $\ell$ when $E$ is non-CM.

In this paper, we shall describe all known (and conjecturally all) proper subgroups of $\text{GL}_2(\mathbb{F}_\ell)$ that occur as the image of such a representation $\rho_{E,\ell}$. Applying our classification with earlier work, we will obtain an algorithm to determine the set $S$ of primes $\ell$ for which $\rho_{E,\ell}$ is not surjective and also compute $\rho_{E,\ell}(\text{Gal}_Q)$ for each $\ell \in S$.

Before stating our classification in §§1.1–1.7, let us make some comments. We will consider each prime $\ell$ separately. For simplicity, assume that the $j$-invariant $j_E \in \mathbb{Q}$ of $E/\mathbb{Q}$ is neither 0 nor 1728. Our first step in determining $\rho_{E,\ell}(\text{Gal}_Q)$ is to compute the group

$$G := \pm \rho_{E,\ell}(\text{Gal}_Q),$$

i.e., the group generated by $-I$ and $\rho_{E,\ell}(\text{Gal}_Q)$. The benefit of studying $G$, up to conjugacy in $\text{GL}_2(\mathbb{F}_\ell)$, is that it does not change if $E$ is replaced by a quadratic twist. Moreover, if $E'/\mathbb{Q}$ is a quadratic twist of $E/\mathbb{Q}$, then after choosing appropriate bases, we will have $\rho_{E',\ell} = \chi \cdot \rho_{E,\ell}$ for some quadratic character $\chi: \text{Gal}_Q \to \{\pm 1\}$. Since $j_E \notin \{0,1728\}$, all twists of $E$ are quadratic twists and hence $G$, up to conjugacy, depends only on the value $j_E$. The character $\det \circ \rho_{E,\ell}: \text{Gal}_Q \to \mathbb{F}_\ell^\times$ describes the Galois action on the $\ell$-th roots of unity, so $\det(G) = \mathbb{F}_\ell^\times$.

For a subgroup $G$ of $\text{GL}_2(\mathbb{F}_\ell)$ with $\det(G) = \mathbb{F}_\ell^\times$ and $-I \in G$, we can associate a modular curve $X_G$; it is a smooth, projective and geometrically irreducible curve defined over $\mathbb{Q}$. It comes with a natural morphism

$$\pi_G: X_G \to \text{Spec} \mathbb{Q}[j] \cup \{\infty\} =: \mathbb{P}^1_\mathbb{Q}$$

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such that for an elliptic curve $E/\mathbb{Q}$ with $j_E \notin \{0,1728\}$, the group $\rho_{E,\ell}(\text{Gal}_{\mathbb{Q}})$ is conjugate in $\text{GL}_2(\mathbb{F}_\ell)$ to subgroup of $G$ if and only if the $j_E = \pi_G(P)$ for some rational point $P \in X_G(\mathbb{Q})$.

Much of this paper is dedicated to describing those modular curves $X_h$ of genus 0 with $X_G(\mathbb{Q}) \neq \emptyset$. Such modular curves are isomorphic to the projective line and their function field is of form $\mathbb{Q}(h)$ for some modular function $h$ of level $\ell$. Giving the morphism $\pi_G$ is then equivalent to expressing the modular $j$-invariant in the form $J(h)$ for a unique rational function $J(t) \in \mathbb{Q}(t)$.

Once we have determined $G$, we know that $\rho_{E,\ell}(\text{Gal}_{\mathbb{Q}})$ will either be the full group $G$ or equal to an index 2 subgroup $H$ of $G$ for which $-I \notin H$. For each such $H$, it is then a matter of determining whether the quadratic character $\text{Gal}_{\mathbb{Q}} \xrightarrow{\rho_{E,\ell}} G \to G/H \cong \{\pm 1\}$ is trivial or not.

We will first focus on the general case of non-CM elliptic curves over $\mathbb{Q}$. In §1.9, we will give a complete description of the groups $\rho_{E,\ell}(\text{Gal}_{\mathbb{Q}})$ when $E/\mathbb{Q}$ has complex multiplication.

**Notation.** We now define some specific subgroups of $\text{GL}_2(\mathbb{F}_\ell)$ for an odd prime $\ell$. Let $C_s(\ell)$ be the subgroup of diagonal matrices. Let $\epsilon = -1$ if $\ell \equiv 3 \pmod{4}$ and otherwise let $\epsilon \geq 2$ be the smallest integer which is not a quadratic residue modulo $\ell$. Let $C_{ns}(\ell)$ be the subgroup consisting of matrices of the form $\left( \begin{smallmatrix} a & b \\ \epsilon b & a \end{smallmatrix} \right)$ with $(a,b) \in \mathbb{F}_\ell^2 - \{(0,0)\}$. Let $N_s(\ell)$ and $N_{ns}(\ell)$ be the normalizers of $C_s(\ell)$ and $C_{ns}(\ell)$, respectively, in $\text{GL}_2(\mathbb{F}_\ell)$. We have $[N_s(\ell) : C_s(\ell)] = 2$ and the non-identity coset of $C_s(\ell)$ in $N_s(\ell)$ is represented by $\left( \begin{smallmatrix} 0 & 1 \\ 1 & 0 \end{smallmatrix} \right)$. We have $[N_{ns}(\ell) : C_{ns}(\ell)] = 2$ and the non-identity coset of $C_{ns}(\ell)$ in $N_{ns}(\ell)$ is represented by $\left( \begin{smallmatrix} 1 & 0 \\ 0 & 1 \end{smallmatrix} \right)$. Let $B(\ell)$ be the subgroup of upper triangular matrices in $\text{GL}_2(\mathbb{F}_\ell)$.

1.1. $\ell = 2$. Up to conjugacy, there are three proper subgroups of $\text{GL}_2(\mathbb{F}_2)$:

$$G_1 = \{I\}, \quad G_2 = \{I, \left( \begin{smallmatrix} 1 & 1 \\ 0 & 1 \end{smallmatrix} \right)\}, \quad G_3 = \{I, \left( \begin{smallmatrix} 1 & 0 \\ 1 & 1 \end{smallmatrix} \right), \left( \begin{smallmatrix} 0 & 1 \\ 1 & 0 \end{smallmatrix} \right)\}.$$

For $i = 1, 2, 3$, the index $[\text{GL}_2(\mathbb{F}_2) : G_i]$ is 6, 3 and 2, respectively. Define the rational functions

$$J_1(t) = 256 \frac{(t^2 + t + 1)^3}{t^2(t + 1)^2}, \quad J_2(t) = 256 \frac{(t + 1)^3}{t}, \quad J_3(t) = t^2 + 1728.$$

**Theorem 1.1.** Let $E$ be a non-CM elliptic curve over $\mathbb{Q}$. Then $\rho_{E,2}(\text{Gal}_{\mathbb{Q}})$ is conjugate in $\text{GL}_2(\mathbb{F}_2)$ to a subgroup of $G_i$ if and only if $j_E$ is of the form $J_i(t)$ for some $t \in \mathbb{Q}$.

1.2. $\ell = 3$. Define the following subgroups of $\text{GL}_2(\mathbb{F}_3)$:

- Let $G_1$ be the group $C_3(3)$.
- Let $G_2$ be the group $N_3(3)$.
- Let $G_3$ be the group $B(3)$.
- Let $G_4$ be the group $N_{ns}(3)$.
- Let $H_{1,1}$ be the subgroup consisting of the matrices of the form $\left( \begin{smallmatrix} 1 & 0 \\ 0 & \ast \end{smallmatrix} \right)$.
- Let $H_{3,1}$ be the subgroup consisting of the matrices of the form $\left( \begin{smallmatrix} 1 & 0 \\ 0 & \ast \end{smallmatrix} \right)$.
- Let $H_{3,2}$ be the subgroup consisting of the matrices of the form $\left( \begin{smallmatrix} 1 & 0 \\ 0 & \ast \end{smallmatrix} \right)$.

The index in $\text{GL}_2(\mathbb{F}_3)$ of the above subgroups are 12, 6, 4, 3, 24, 8 and 8, respectively. Each of the groups $G_i$ contain $-I$. The groups $H_{i,j}$ do not contain $-I$ and we have $G_i = \pm H_{i,j}$.

Define the rational functions:

$$J_1(t) = 27 \frac{(t + 1)^3(t + 3)^3(t^2 + 3)^3}{t^3(t^2 + 3t + 3)^3}, \quad J_2(t) = 27 \frac{(t + 1)^3(t - 3)^3}{t^3}, \quad J_3(t) = 27 \frac{(t + 1)(t + 9)^3}{t^3}, \quad J_4(t) = t^3.$$

For $t \in \mathbb{Q} - \{0\}$, let $E_{1,t}$ be the elliptic curve over $\mathbb{Q}$ defined by Weierstrass equation

$$y^2 = x^3 - 3(t + 1)(t + 3)(t^2 + 3)(2t^2 - 3)(t^4 + 6t^3 + 18t^2 + 18t + 9).$$
For $t \in \mathbb{Q} - \{0, -1\}$, let $\mathcal{E}_{s,t}$ be the elliptic curve over $\mathbb{Q}$ defined by Weierstrass equation
\[
y^2 = x^3 - 3(t + 1)^3(t + 9)x - 2(t + 1)^4(t^2 - 18t - 27).
\]
The $j$-invariant of $\mathcal{E}_{s,t}$ is $J_s(t)$.

**Theorem 1.2.** Let $E$ be a non-CM elliptic curve over $\mathbb{Q}$.

(i) If $\rho_{E,3}$ is not surjective, then $\rho_{E,3}(\text{Gal}_\mathbb{Q})$ is conjugate in $\text{GL}_2(\mathbb{F}_3)$ to one of the groups $G_i$ or $H_{i,j}$.

(ii) The group $\rho_{E,3}(\text{Gal}_\mathbb{Q})$ is conjugate to a subgroup of $G_i$ if and only if $J_s(t)$ is of the form $J_i(t)$ for some $t \in \mathbb{Q}$.

(iii) Suppose that $\pm \rho_{E,3}(\text{Gal}_\mathbb{Q})$ is conjugate to $G_1$. Fix an element $t \in \mathbb{Q}$ such that $J_1(t) = j_E$. The group $\rho_{E,3}(\text{Gal}_\mathbb{Q})$ is conjugate to $H_{1,1}$ if and only if $E$ is isomorphic to $\mathcal{E}_{1,t}$ or the quadratic twist of $\mathcal{E}_{1,t}$ by $-3$.

(iv) Suppose that $\pm \rho_{E,3}(\text{Gal}_\mathbb{Q})$ is conjugate to $G_3$. Fix an element $t \in \mathbb{Q}$ such that $J_3(t) = j_E$. The group $\rho_{E,3}(\text{Gal}_\mathbb{Q})$ is conjugate to $H_{3,1}$ if and only if $E$ is isomorphic to $\mathcal{E}_{3,1}$. The group $\rho_{E,3}(\text{Gal}_\mathbb{Q})$ is conjugate to $H_{3,2}$ if and only if $E$ is isomorphic to the quadratic twist of $\mathcal{E}_{3,1}$ by $-3$.

**Remark 1.3.**

(i) Let us briefly explain how Theorem 1.2 can be used to compute $\rho_{E,3}(\text{Gal}_\mathbb{Q})$; similar remarks will hold for the remaining primes (the case $\ell = 2$ is particularly simple since $-I = I$). If $J_E$ is not of the form $J_i(t)$ for any $i \in \{1, 2, 3, 4\}$ and $t \in \mathbb{Q}$, then $\rho_{E,3}(\text{Gal}_\mathbb{Q}) = \text{GL}_2(\mathbb{F}_3)$.

To check if $J_E$ is of the form $J(t)$, clear denominators in $J(t) - J_E$ to obtain a polynomial in $t$ which one can then determine whether it has rational roots or not.

So assume that $\rho_{E,3}$ is not surjective, and let $i$ be the smallest value in $\{1, 2, 3, 4\}$ for which $J_E = J_i(t)$ for some $t \in \mathbb{Q}$. By Theorem 1.2(i) and (ii), we deduce that $\pm \rho_{E,3}(\text{Gal}_\mathbb{Q})$ is conjugate to $G_i$; note that the groups $G_i$ are ordered by decreasing index in $\text{GL}_2(\mathbb{F}_3)$. After possibly conjugating $\rho_{E,3}$, we may assume that $\pm \rho_{E,3}(\text{Gal}_\mathbb{Q}) = G_i$. If $\rho_{E,3}(\text{Gal}_\mathbb{Q})$ does not equal $G_i$, then it is equal to one of the subgroups $H_{i,j}$, and parts (iii) and (iv) give necessary and sufficient conditions to check this.

(ii) Our rational functions $J_i(t)$ are certainly not unique. In particular, any function of the form $J_i((at + b)/(ct + d))$ will work with fixed $a, b, c, d \in \mathbb{Q}$ satisfying $ad - bc \neq 0$ (though in general, one needs to also consider the value of $J_i(t)$ at $\infty$). Given $J_i(t)$, our equations for $\mathcal{E}_{i,t}$ were produced by an algorithm that we will later describe; there are other possibly simpler choices.

### 1.3. $\ell = 5$

Define the following subgroups of $\text{GL}_2(\mathbb{F}_5)$:

- Let $G_1$ be the subgroup consisting of the matrices of the form $\pm \begin{pmatrix} 1 & 0 \\ 0 & * \end{pmatrix}$.
- Let $G_2$ be the group $C_4(5)$.
- Let $G_3$ be the unique subgroup of $N_{ns}(5)$ of index 3; it is generated by $\begin{pmatrix} 2 & 0 \\ 0 & 2 \end{pmatrix}$, $\begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$ and $\begin{pmatrix} 0 & 6 \\ 3 & 0 \end{pmatrix}$.
- Let $G_4$ be the group $N_4(5)$.
- Let $G_5$ be the subgroup consisting of the matrices of the form $\pm \begin{pmatrix} 5 & 1 \\ 0 & 1 \end{pmatrix}$.
- Let $G_6$ be the subgroup consisting of the matrices of the form $\pm \begin{pmatrix} 0 & 1 \\ 0 & * \end{pmatrix}$.
- Let $G_7$ be the group $N_{ns}(5)$.
- Let $G_8$ be the group $B(5)$.
- Let $G_9$ be the unique maximal subgroup of $\text{GL}_2(\mathbb{F}_5)$ which contains $N_4(5)$; it is generated by $\begin{pmatrix} 2 & 0 \\ 1 & 1 \end{pmatrix}$, $\begin{pmatrix} 0 & 2 \\ 1 & 0 \end{pmatrix}$, $\begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}$ and $\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$.
- Let $H_{1,1}$ be the subgroup consisting of the matrices of the form $\begin{pmatrix} 1 & 0 \\ 0 & * \end{pmatrix}$.
- Let $H_{1,2}$ be the subgroup consisting of the matrices of the form $\begin{pmatrix} 0^2 & 0 \\ 0 & a \end{pmatrix}$. 


Theorem 1.4. Let $H_{5,1}$ be the subgroup consisting of the matrices of the form $\begin{pmatrix} \ast & 0 \\ 0 & 1 \end{pmatrix}$.
- Let $H_{5,2}$ be the subgroup consisting of the matrices of the form $\begin{pmatrix} a & 0 \\ 0 & \ast \end{pmatrix}$.
- Let $H_{6,1}$ be the subgroup consisting of the matrices of the form $\begin{pmatrix} 1 & 0 \\ 0 & \ast \end{pmatrix}$.
- Let $H_{6,2}$ be the subgroup consisting of the matrices of the form $\begin{pmatrix} 0 & a \\ 1 & 0 \end{pmatrix}$.

The index in $GL_2(\mathbb{F}_5)$ of the above subgroups are 60, 30, 30, 15, 12, 12, 10, 6, 5, 120, 120, 24, 24, 24 and 24, respectively. Each of the groups $G_i$ contain $-I$. The groups $H_{i,j}$ do not contain $-I$ and we have $G_i = \pm H_{i,j}$.

Define the rational functions:

\begin{align*}
J_1(t) &= \frac{(t^{20} + 228t^{15} + 494t^{10} - 228t^5 + 1)^3}{t^5(t^{10} - 11t^5 - 1)^5} \\
J_2(t) &= \frac{(t^2 + 5t + 5)^3(t^4 + 5t^2 + 25)^3(t^4 + 5t^3 + 20t^2 + 25t + 25)^3}{t^5(t^4 + 5t^3 + 15t^2 + 25t + 25)^5} \\
J_3(t) &= \frac{5^4t^3(t^2 + 5t + 10)^3(2t^2 + 5t + 5)^3(4t^4 + 30t^3 + 95t^2 + 150t + 100)^3}{(t^2 + 5t + 5)^5(t^4 + 5t^3 + 15t^2 + 25t + 25)^5} \\
J_4(t) &= \frac{(t + 5)^3(t^2 - 5)^3(t^2 + 5t + 10)^3}{(t^2 + 5t + 5)^5} \\
J_5(t) &= \frac{(t^4 + 228t^3 + 494t^2 - 228t + 1)^3}{t^5(t^2 - 11t - 1)^5} \\
J_6(t) &= \frac{(t^4 - 12t^3 + 14t^2 + 12t + 1)^3}{t^5(t^2 - 11t - 1)} \\
J_7(t) &= \frac{5^3(t + 1)(2t + 1)^3(2t^2 - 3t + 3)^3}{(t^2 + t - 1)^5} \\
J_8(t) &= \frac{5^2(t^2 + 10t + 5)^3}{t^5} \\
J_9(t) &= \frac{t^3(t^2 + 5t + 40)}{}
\end{align*}

For $t \in \mathbb{Q} \setminus \{0\}$, let $E_{1,t}$ be the elliptic curve over $\mathbb{Q}$ defined by the Weierstrass equation

\begin{align*}
y^2 &= x^3 - 27(t^{20} + 228t^{15} + 494t^{10} - 228t^5 + 1)x \\
&\quad + 54(t^{30} - 522t^{25} - 10005t^{20} - 10005t^{10} + 522t^5 + 1).
\end{align*}

For $t \in \mathbb{Q} \setminus \{0\}$, let $E_{5,t}$ be the elliptic curve over $\mathbb{Q}$ defined by the Weierstrass equation

\begin{align*}
y^2 &= x^3 - 27(t^4 + 228t^3 + 494t^2 - 228t + 1)x + 54(t^6 - 522t^5 - 10005t^4 - 10005t^2 + 522t + 1).
\end{align*}

For $t \in \mathbb{Q} \setminus \{0\}$, let $E_{6,t}$ be the elliptic curve over $\mathbb{Q}$ defined by the Weierstrass equation

\begin{align*}
y^2 &= x^3 - 27(t^4 - 12t^3 + 14t^2 + 12t + 1)x + 54(t^6 - 18t^5 + 75t^4 + 75t^2 + 18t + 1)
\end{align*}

The $j$-invariant of $E_{i,t}$ is $J_i(t)$.

**Theorem 1.4.** Let $E$ be a non-CM elliptic curve over $\mathbb{Q}$.

(i) If $\rho_{E,5}$ is not surjective, then $\rho_{E,5}(\text{Gal}_\mathbb{Q})$ is conjugate in $GL_2(\mathbb{F}_5)$ to one of the groups $G_i$ or $H_{i,j}$.

(ii) The group $\rho_{E,5}(\text{Gal}_\mathbb{Q})$ is conjugate to a subgroup of $G_i$ if and only if $j_E$ is of the form $J_i(t)$ for some $t \in \mathbb{Q}$.

(iii) Suppose that $\pm \rho_{E,5}(\text{Gal}_\mathbb{Q})$ is conjugate to $G_i$ with $i \in \{1, 5, 6\}$. Fix an element $t \in \mathbb{Q}$ such that $J_i(t) = j_E$.

The group $\rho_{E,5}(\text{Gal}_\mathbb{Q})$ is conjugate to $H_{i,1}$ if and only if $E$ is isomorphic to $E_{i,t}$. 

The group $\rho_{E,5}(\text{Gal}_\mathbb{Q})$ is conjugate to $H_{1,2}$ if and only if $E$ is isomorphic to the quadratic twist of $\mathcal{E}_{1,1}$ by 5.

1.4. $\ell = 7$. Define the following subgroups of $\text{GL}_2(\mathbb{F}_7)$:

- Let $G_1$ be the subgroup of $N_s(7)$ consisting of elements of $C_s(7)$ with square determinant and elements of $N_s(7) - C_s(7)$ with non-square determinant; it is generated by $(\frac{2}{0} \ 0)$, $(\frac{0}{2} \ 0)$ and $(\frac{1}{0} \ 0)$.
- Let $G_2$ be the group $N_s(7)$.
- Let $G_3$ be the subgroup consisting of matrices of the form $\pm (\frac{1}{0} \ 0)$.
- Let $G_4$ be the subgroup consisting of matrices of the form $\pm (\frac{0}{1} \ 0)$.
- Let $G_5$ be the subgroup consisting of matrices of the form $(\frac{0}{a} \ 0)$.
- Let $G_6$ be the group $N_{as}(7)$.
- Let $G_7$ be the group $B(7)$.
- Let $H_{1,1}$ be the subgroup generated by $(\frac{2}{0} \ 0)$ and $(\frac{0}{2} \ 0)$.
- Let $H_{3,1}$ be the subgroup consisting of the matrices of the form $(\frac{1}{0} \ 0)$.
- Let $H_{3,2}$ be the subgroup consisting of the matrices of the form $(\frac{a^2}{0} \ a^2)$.
- Let $H_{4,1}$ be the subgroup consisting of the matrices of the form $(\frac{0}{1} \ 0)$.
- Let $H_{4,2}$ be the subgroup consisting of the matrices of the form $(\frac{a^2}{0} \ 0)$.
- Let $H_{5,1}$ be the subgroup consisting of the matrices of the form $(\frac{\pm a^2 \ a^2}{0} \ 0)$.
- Let $H_{5,2}$ be the subgroup consisting of the matrices of the form $(\frac{a^2}{0} \ \pm a^2)$.
- Let $H_{7,1}$ be the subgroup consisting of the matrices of the form $(\frac{0 \ a^2}{0} \ 0)$.
- Let $H_{7,2}$ be the subgroup consisting of the matrices of the form $(\frac{0 \ a^2}{0} \ a^2)$.

The index in $\text{GL}_2(\mathbb{F}_7)$ of the above subgroups are 56, 28, 24, 24, 24, 21, 8, 112, 48, 48, 48, 48, 16 and 16, respectively. Each of the groups $G_i$ contain $-I$. The groups $H_{i,j}$ do not contain $-I$ and we have $G_i = \pm H_{i,j}$.

Define the rational functions

$$J_1(t) = 3^3 \cdot 5 \cdot 7^3 / 2^7$$
$$J_2(t) = \frac{t(t + 1)^3(t^2 - 5t + 1)^3(t^2 - 5t + 8)^3(t^4 - 5t^3 + 8t^2 - 7t + 7)^3}{(t^3 - 4t^2 + 3t + 1)^7}$$
$$J_3(t) = \frac{(t^2 - t + 1)^3(t^6 - 11t^5 + 30t^4 - 15t^3 - 10t^2 + 5t + 1)^3}{(t - 1)t^7(t^3 - 8t^2 + 5t + 1)}$$
$$J_4(t) = \frac{(t^2 - t + 1)^3(t^6 + 229t^5 + 270t^4 - 1695t^3 + 1430t^2 - 235t + 1)^3}{(t - 1)t(t^3 - 8t^2 + 5t + 1)^7}$$
$$J_5(t) = \frac{(-t^2 - 3t - 3)^3(t^2 - t + 1)^3(3t^2 - 9t + 5)^3(5t^2 - t - 1)^3}{(t^3 - 2t^2 - t + 1)(t^3 - t^2 - 2t + 1)^7}$$
$$J_6(t) = \frac{64t^3(t^2 + 7)^3(t^2 - 7t + 14)^3(5t^2 - 14t - 7)^3}{(t^3 - 7t^2 + 7t + 7)^7}$$
$$J_7(t) = \frac{(t^2 + 245t + 2401)^3(t^2 + 13t + 49)}{t^7}$$

Let $\mathcal{E}_1$ be the elliptic curve over $\mathbb{Q}$ defined by the Weierstrass equation

$$y^2 = x^3 - 5^3 \cdot 7^3 x - 5^4 \cdot 7^2 106$$
For $t \in \mathbb{Q} - \{0, 1\}$, let $\mathcal{E}_{3,t}$ be the elliptic curve over $\mathbb{Q}$ defined by the Weierstrass equation
\[
y^2 = x^3 - 27(t^2 - t + 1)(t^6 - 11t^5 + 30t^4 - 15t^3 - 10t^2 + 5t + 1)x
+ 54(t^{12} - 18t^{11} + 117t^{10} - 354t^9 + 570t^8 - 486t^7
+ 273t^6 - 222t^5 + 174t^4 - 46t^3 - 15t^2 + 6t + 1).
\]

For $t \in \mathbb{Q} - \{0, 1\}$, let $\mathcal{E}_{4,t}$ be the elliptic curve over $\mathbb{Q}$ defined by the Weierstrass equation
\[
y^2 = x^3 - 27(t^2 - t + 1)(t^6 + 229t^5 + 270t^4 - 1695t^3 + 1430t^2 - 235t + 1)x
+ 54(t^{12} - 522t^{11} - 8955t^{10} + 37950t^9 - 70998t^8 + 131562t^7
- 253239t^6 + 316290t^5 - 218058t^4 + 80090t^3 - 14631t^2 + 510t + 1).
\]

For $t \in \mathbb{Q}$, let $\mathcal{E}_{5,t}$ be the elliptic curve over $\mathbb{Q}$ defined by the Weierstrass equation
\[
y^2 = x^3 - 27 \cdot 7(t^2 - 3t - 3)(t^2 - t + 1)(3t^2 - 9t + 5)(5t^2 - t - 1)x
- 54 \cdot 7^2(t^4 - 6t^3 + 17t^2 - 24t + 9)(3t^4 - 4t^3 - 5t^2 - 2t - 1)(9t^4 - 12t^3 - t^2 + 8t - 3).
\]

For $t \in \mathbb{Q} - \{0\}$, let $\mathcal{E}_{7,t}$ be the elliptic curve over $\mathbb{Q}$ defined by the Weierstrass equation
\[
y^2 = x^3 - 27(t^2 + 13t + 49)^3(t^2 + 245t + 2401)x
+ 54(t^2 + 13t + 49)^4(t^4 - 490t^3 - 21609t^2 - 235298t - 823543).
\]

The $j$-invariant of $\mathcal{E}_{i,t}$ is $J_i(t)$.

**Theorem 1.5.** Let $E$ be a non-CM elliptic curve over $\mathbb{Q}$.

(i) If $\rho_{E,7}$ is not surjective, then $\rho_{E,7}(\text{Gal}_\mathbb{Q})$ is conjugate in $\text{GL}_2(\mathbb{F}_7)$ to one of the groups $G_i$ or $H_{i,j}$.

(ii) The group $\rho_{E,7}(\text{Gal}_\mathbb{Q})$ is conjugate to a subgroup of $G_i$ if and only if $J_E$ is of the form $J_i(t)$ for some $t \in \mathbb{Q}$.

(iii) The group $\rho_{E,7}(\text{Gal}_\mathbb{Q})$ is conjugate to $H_{1,1}$ if and only if $E/\mathbb{Q}$ is isomorphic to $\mathcal{E}_1$ or to the quadratic twist of $\mathcal{E}_1$ by $-7$.

(iv) Suppose that $\pm \rho_{E,7}(\text{Gal}_\mathbb{Q})$ is conjugate to $G_i$ with $i \in \{3, 4, 5, 7\}$. Fix an element $t \in \mathbb{Q}$ such that $J_i(t) = J_E$.

The group $\rho_{E,7}(\text{Gal}_\mathbb{Q})$ is conjugate to $H_{i,1}$ if and only if $E$ is isomorphic to $\mathcal{E}_{i,t}$.

The group $\rho_{E,7}(\text{Gal}_\mathbb{Q})$ is conjugate to $H_{i,2}$ if and only if $E$ is isomorphic to the quadratic twist of $\mathcal{E}_{i,t}$ by $-7$.

1.5. $\ell = 11$.

- Let $G_1$ be the subgroup generated by $\pm (1 \ 1)$ and $(4 \ 0 \ 6)$.
- Let $G_2$ be the subgroup generated by $\pm (1 \ 1)$ and $(5 \ 0 \ 5)$.
- Let $G_3$ be the group $N_{ns}(11)$.
- Let $H_{1,1}$ be the subgroup generated by $(1 \ 1)$ and $(4 \ 0 \ 6)$.
- Let $H_{1,2}$ be the subgroup generated by $(0 \ 1)$ and $(7 \ 0 \ 2)$.
- Let $H_{2,1}$ be the subgroup generated by $(0 \ 1)$ and $(5 \ 0 \ 5)$.
- Let $H_{2,2}$ be the subgroup generated by $(1 \ 1)$ and $(6 \ 0 \ 4)$.

The index in $\text{GL}_2(\mathbb{F}_{11})$ of the above subgroups are 60, 60, 55, 110, 120, 120, 120 and 120, respectively. Each of the groups $G_i$ contain $-I$. The groups $H_{i,j}$ do not contain $-I$ and we have $G_i = \pm H_{i,j}$.

Let $\mathcal{E}$ be the elliptic curve over $\mathbb{Q}$ defined by the Weierstrass equation $y^2 + y = x^3 - x^2 - 7x + 10$ and let $O$ be the point at infinity. The Mordell-Weil group $\mathcal{E}(\mathbb{Q})$ is an infinite cyclic group generated
by the point \((4, 5)\). Define

\[ J(x, y) := \frac{(f_1 f_2 f_3 f_4)^3}{f_5 f_6^4}, \]

where

\[
\begin{align*}
  f_1 &= x^2 + 3x - 6, \\
  f_2 &= 11(x^2 - 5)y + (2x^4 + 23x^3 - 72x^2 - 28x + 127), \\
  f_3 &= 6y + 11x - 19, \\
  f_4 &= 22(x - 2)y + (5x^3 + 17x^2 - 112x + 120), \\
  f_5 &= 11y + (2x^2 + 17x - 34), \\
  f_6 &= (x - 4)y - (5x - 9).
\end{align*}
\]

We shall view \(J\) as a morphism \(\mathcal{E} \to \mathbb{A}^1_\mathbb{Q} \cup \{\infty\}\).

Let \(\mathcal{E}_1/\mathbb{Q}\) be the elliptic curve defined by the Weierstrass equation \(y^2 = x^3 - 27 \cdot 11^4 x + 54 \cdot 11^5 \cdot 43\)
Let \(\mathcal{E}_2/\mathbb{Q}\) be the elliptic curve defined by the Weierstrass equation \(y^2 = x^3 - 27 \cdot 11^3 \cdot 131 x + 54 \cdot 11^4 \cdot 4973\).

**Theorem 1.6.** Let \(E\) be a non-CM elliptic curve defined over \(\mathbb{Q}\).

1. If \(\rho_{E, 11}\) is not surjective, then \(\rho_{E, 11}(\text{Gal}_\mathbb{Q})\) is conjugate in \(\text{GL}_2(\mathbb{F}_{11})\) to one of the groups \(G_{i}\) or \(H_{i, j}\).
2. The group \(\pm \rho_{E, 11}(\text{Gal}_\mathbb{Q})\) is conjugate to \(G_1\) in \(\text{GL}_2(\mathbb{F}_{11})\) if and only if \(j_E = -11^2\).
3. The group \(\pm \rho_{E, 11}(\text{Gal}_\mathbb{Q})\) is conjugate to \(G_2\) in \(\text{GL}_2(\mathbb{F}_{11})\) if and only if \(j_E = -11 \cdot 131^3\).
4. The group \(\rho_{E, 11}(\text{Gal}_\mathbb{Q})\) is conjugate to \(G_3\) in \(\text{GL}_2(\mathbb{F}_{11})\) if and only if \(j_E = J(P)\) for some point \(P \in \mathcal{E}(\mathbb{Q}) - \{\mathcal{O}\}\).
5. For \(i \in \{1, 2\}\), the group \(\rho_{E, 11}(\text{Gal}_\mathbb{Q})\) is conjugate in \(\text{GL}_2(\mathbb{F}_{11})\) to \(H_{i, 1}\) if and only if \(E\) is isomorphic to \(\mathcal{E}_i\).
6. For \(i \in \{1, 2\}\), the group \(\rho_{E, 11}(\text{Gal}_\mathbb{Q})\) is conjugate in \(\text{GL}_2(\mathbb{F}_{11})\) to \(H_{i, 2}\) if and only if \(E\) is isomorphic to the quadratic twist of \(\mathcal{E}_i\) by \(-11\).

**Remark 1.7.** The modular curve \(X^+_{ns}(11) = X_{G_3}\) is the only one in our classification that has genus 1 with infinitely many rational points. Halberstadt [Hal98] showed that \(X^+_{ns}(11)\) is isomorphic to \(\mathcal{E}\) and that the morphism to the \(j\)-line corresponds to \(J(x, y)\).

In §4.5.5, we give explicit polynomials \(A, B, C \in \mathbb{Q}[X]\) of degree 55 such that for a non-CM elliptic curve \(E/\mathbb{Q}\), we have \(j_E = J(P)\) for some \(P \in \mathcal{E}(\mathbb{Q}) - \{\mathcal{O}\}\) if and only if the polynomial \(A(x)j_E^2 + B(x)j_E + C(x) \in \mathbb{Q}[x]\) has a rational root. This gives a straightforward way to check the criterion of Theorem 1.6(iv).

### 1.6. \(\ell = 13\)

Define the following subgroups of \(\text{GL}_2(\mathbb{F}_{13})\):

- Let \(G_1\) be the subgroup consisting of matrices of the form \(\begin{pmatrix} a & b \\ c & d \end{pmatrix}\).
- Let \(G_2\) be the subgroup consisting of matrices of the form \(\begin{pmatrix} 0 & a \\ b & 0 \end{pmatrix}\).
- Let \(G_3\) be the subgroup consisting of matrices \(\begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix}\) for which \((a/b)^4 = 1\).
- Let \(G_4\) be the subgroup consisting of matrices \(\begin{pmatrix} 0 & a^2 \\ 0 & 0 \end{pmatrix}\).
- Let \(G_5\) be the subgroup consisting of matrices \(\begin{pmatrix} a^2 & 0 \\ 0 & 0 \end{pmatrix}\).
- Let \(G_6\) be the group \(B(13)\).
- Let \(G_7\) be the subgroup generated by the matrices \(\begin{pmatrix} 2 & 0 \\ 0 & 2 \end{pmatrix}, \begin{pmatrix} 2 & 0 \\ 0 & 3 \end{pmatrix}, \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}\) and \(\begin{pmatrix} 1 & 1 \\ -1 & 1 \end{pmatrix}\); it contains the scalar matrices and its image in \(\text{PGL}_2(\mathbb{F}_{13})\) is isomorphic to \(\mathfrak{S}_4\).
- Let \(H_{4, 1}\) be the subgroup consisting of matrices of the form \(\begin{pmatrix} a \alpha & 0 \\ 0 & a \end{pmatrix}\).
- Let \(H_{4, 2}\) be the subgroup generated by matrices of the form \(\begin{pmatrix} a^2 & 0 \\ 0 & a \end{pmatrix}\) and \(\begin{pmatrix} 2 & 0 \\ 0 & 4 \end{pmatrix}\).
- Let \(H_{5, 1}\) be the subgroup consisting of matrices of the form \(\begin{pmatrix} a & 0 \\ 0 & a \end{pmatrix}\).
- Let \(H_{5, 2}\) be the subgroup generated by matrices of the form \(\begin{pmatrix} a^4 & 0 \\ 0 & a^2 \end{pmatrix}\) and \(\begin{pmatrix} 4 & 0 \\ 0 & 2 \end{pmatrix}\).
The index in $\text{GL}_2(\mathbb{F}_{13})$ of the above subgroups are 42, 42, 42, 28, 28, 14, 91, 56, 56, 56 and 56, respectively. Each of the groups $G_i$ contain $-I$. The groups $H_{i,j}$ do not contain $-I$ and we have $G_i = \pm H_{i,j}$.

Define the polynomials

$P_1(t) = t^{12} + 231t^{11} + 269t^{10} - 3160t^9 + 6022t^8 - 9616t^7 + 21880t^6$

$- 34102t^5 + 28297t^4 - 12455t^3 + 2876t^2 - 243t + 1$

$P_2(t) = t^{12} - 9t^{11} + 29t^{10} - 40t^9 + 22t^8 + 16t^7 + 40t^6 - 22t^5 + 23t^4 - 4t^3 - 3t + 1$

$P_3(t) = (t^4 - t^3 + 2t^2 - 9t + 3) (3t^4 - 3t^3 - 7t^2 + 12t - 4) (4t^4 - 4t^3 - 5t^2 + 3t - 1)$

$P_4(t) = t^8 + 235t^7 + 1207t^6 + 955t^5 + 3840t^4 - 955t^3 + 1207t^2 - 235t + 1$

$P_5(t) = t^8 - 5t^7 + 7t^6 - 5t^5 + 5t^3 + 7t^2 + t + 1$

$P_6(t) = t^4 + 7t^3 + 20t^2 + 19t + 1$

$Q_4(t) = t^{12} - 512t^{11} - 13079t^{10} - 323000t^9 - 104792t^8 - 111870t^7$

$- 419368t^6 + 111870t^5 - 104792t^4 + 323000t^3 - 13079t^2 + 512t + 1$

$Q_5(t) = t^{12} - 8t^{11} + 25t^{10} - 44t^9 + 40t^8 + 18t^7 - 40t^6 - 18t^5 + 40t^4 + 44t^3 + 25t^2 + 8t + 1$

and the rational functions

$J_1(t) = \frac{(t^2 - t + 1)^3 P_1(t)^3}{(t^3 - 4t^2 + t + 1)^{13}}$

$J_2(t) = \frac{(t^2 - t + 1)^3 P_2(t)^3}{(t^3 - 4t^2 + t + 1)^{13} (t^3 - 4t^2 + t + 1)^{13}}$

$J_3(t) = \frac{13^4 (t^2 - t + 1)^3 P_3(t)^3}{(t^3 - 4t^2 + t + 1)^{13} (5t^3 - 7t^2 - 8t + 5)}$

$J_4(t) = \frac{(t^4 - 3t^3 + 5t^2 + t + 1)^3 P_4(t)^3}{(t^2 - 3t - 1)^{13}}$

$J_5(t) = \frac{(t^4 - t^3 + 5t^2 + t + 1)^3 P_5(t)^3}{t^{13} (t^2 - 3t - 1)}$

$J_6(t) = \frac{(t^2 + 5t + 13) P_6(t)^3}{t}$

For $t \in \mathbb{Q} \setminus \{0\}$, let $E_{4,t}$ be the elliptic curve over $\mathbb{Q}$ defined by the Weierstrass equation

$y^2 = x^3 - 27(t^4 - t^3 + 5t^2 + t + 1)^3 P_4(t)x + 54(t^2 + 1)(t^4 - t^3 + 5t^2 + t + 1)^4 Q_4(t)$

For $t \in \mathbb{Q} \setminus \{0\}$, let $E_{5,t}$ be the elliptic curve over $\mathbb{Q}$ defined by the Weierstrass equation

$y^2 = x^3 - 27(t^4 - t^3 + 5t^2 + t + 1)^3 P_5(t)x + 54(t^2 + 1)(t^4 - t^3 + 5t^2 + t + 1)^4 Q_5(t)$

**Theorem 1.8.** Let $E$ be a non-CM elliptic curve over $\mathbb{Q}$.

(i) If $\rho_{E,13}(\text{Gal}_\mathbb{Q})$ is conjugate to a subgroup of $B(13)$, then $\rho_{E,13}(\text{Gal}_\mathbb{Q})$ is conjugate to one of the groups $G_i$ with $1 \leq i \leq 6$ or to a group $H_{i,j}$.

(ii) For $1 \leq i \leq 6$, $\rho_{E,13}(\text{Gal}_\mathbb{Q})$ is conjugate in $\text{GL}_2(\mathbb{F}_{13})$ to a subgroup of $G_i$ if and only if $j_E$ is of the form $J_i(t)$ for some $t \in \mathbb{Q}$.

(iii) For an $i \in \{4,5\}$, suppose that $J_i(t) = j_E$ for some $t \in \mathbb{Q}$.

The group $\rho_{E,13}(\text{Gal}_\mathbb{Q})$ is conjugate to $H_{i,1}$ if and only if $E$ is isomorphic to $E_{i,t}$.

The group $\rho_{E,13}(\text{Gal}_\mathbb{Q})$ is conjugate to $H_{i,2}$ if and only if $E$ is isomorphic to the quadratic twist of $E_{i,t}$ by 13.

(iv) If $j_E$ is

$\frac{2^4 \cdot 5 \cdot 3^4 \cdot 17^3}{3^{13}}$, \quad \frac{2^{12} \cdot 5^3 \cdot 11 \cdot 3^4}{3^{13}}$ \quad or \quad $\frac{2^{18} \cdot 3^3 \cdot 13^4 \cdot 127^3 \cdot 139^3 \cdot 57^3 \cdot 283^3 \cdot 929}{5^{13} \cdot 61^{13}}$

then $\rho_{E,13}(\text{Gal}_\mathbb{Q})$ is conjugate to $G_7$.  

Up to conjugacy, there are four maximal subgroups $G$ of $\text{GL}_2(\mathbb{F}_{13})$ that satisfy $\det(G) = \mathbb{F}_{13}^\times$; they are $G_6 = B(13)$, $N_s(13)$, $N_{ns}(13)$ and $G_7$. The cases concerning subgroups of $B(13)$ are completely handled in Theorem 1.8.

Baran [Bar14] has shown that the modular curves $X^+_s(13)$ and $X^+_{ns}(13)$ attached to $N_s(13)$ and $N_{ns}(13)$, respectively, are both isomorphic to the genus 3 curve $C$ defined in $\mathbb{P}^2_{\mathbb{Q}}$ by the equation

$$(-y - z)x^3 + (2y^2 + 3yz)x^2 + (-y^3 + 3y^2z - 2z^2y + z^3)x + (2z^2y^2 - 3z^3y) = 0.$$ 

In [Bar14], the morphism from the model of the modular curves to the $j$-line is given. The seven rational points $(0,0,1)$, $(0,1,0)$, $(0,3,2)$, $(1,0,-1)$, $(1,0,0)$, $(1,1,0)$ of $C$ all correspond to cusps and CM points on $X_s(13)$ and $X_{ns}(13)$. Conjecturally, there are no non-CM elliptic curves $E$ over $\mathbb{Q}$ with $\rho_{E,13}(\text{Gal}_\mathbb{Q})$ conjugate to a subgroup of $N_s(13)$ or $N_{ns}(13)$; equivalently, $C$ has no other rational points.

Denote by $X_\mathfrak{S}_4(13)$ the modular curve corresponding to $G_7$. Banwait and Cremona [BC14] have shown that $X_\mathfrak{S}_4(13)$ is isomorphic to the genus 3 curve $C'$ defined in $\mathbb{P}^2_{\mathbb{Q}}$ by the equation

$$4x^3y - 3x^2y^2 + 3xy^3 - x^3z + 16x^2yz - 11xy^2z + 5y^3z + 3x^2z^2 + 9xyz^2 + y^2z^2 + xz^3 + 2yz^3 = 0$$

and have found the morphism from the modular curve to the $j$-line. The four rational points $(0,1,0), (0,0,1), (1,0,0)$ and $(1,3,-2)$ of $C'$ correspond to a CM point and three non-CM points; the non-CM points give rise to the three $j$-invariants in Theorem 1.8(iv).

Suppose $E/\mathbb{Q}$ is an elliptic curve with one of the $j$-invariants from Theorem 1.8(iv). From [BC14], we find that the image of $\rho_{E,13}(\text{Gal}_\mathbb{Q})$ in $\text{PGL}_2(\mathbb{F}_{13})$ is isomorphic to $\mathfrak{S}_4$. Therefore, $\rho_{E,13}(\text{Gal}_\mathbb{Q})$ is conjugate to $G_7$ since $G_7$ has no proper subgroups $H$ whose image in $\text{PGL}_2(\mathbb{F}_{13})$ is isomorphic to $\mathfrak{S}_4$ and satisfies $\det(H) = \mathbb{F}_{13}^\times$. In particular, this proves Theorem 1.8(iv).

Conjecturally, if $E$ is a non-CM elliptic curve over $\mathbb{Q}$, then $\rho_{E,13}(\text{Gal}_\mathbb{Q})$ is conjugate to a subgroup of $G_7$ if and only if $j_E$ is one of three values from Theorem 1.8(iv); equivalently, $C'$ has no other rational points.

**Remark 1.9.** The case $\ell = 13$ is the first for which we do not have a complete description. As explained above, it remains to determine all the rational points of the genus 3 curves $C$ and $C'$.

### 1.7. $\ell \geq 17$.

We first describe all the known cases of non-CM elliptic curves $E/\mathbb{Q}$ for which $\rho_{E,\ell}$ is not surjective for some prime $\ell \geq 17$. Define the following groups:

- Let $G_1$ be the subgroup of $\text{GL}_2(\mathbb{F}_{17})$ generated by $\left(\begin{array}{cc} 2 & 0 \\ 0 & 1 \end{array}\right)$, $\left(\begin{array}{cc} 4 & 0 \\ 0 & 4 \end{array}\right)$ and $\left(\begin{array}{cc} 1 & 1 \\ 0 & 1 \end{array}\right)$.
- Let $G_2$ be the subgroup of $\text{GL}_2(\mathbb{F}_{17})$ generated by $\left(\begin{array}{cc} 11 & 0 \\ 0 & 2 \end{array}\right)$, $\left(\begin{array}{cc} -4 & 0 \\ 0 & 4 \end{array}\right)$ and $\left(\begin{array}{cc} 1 & 1 \\ 0 & 1 \end{array}\right)$.
- Let $G_3$ be the subgroup of $\text{GL}_2(\mathbb{F}_{37})$ consisting of the matrices of the form $\left(\begin{array}{cc} a & * \\ 0 & a \end{array}\right)$.
- Let $G_4$ be the subgroup of $\text{GL}_2(\mathbb{F}_{37})$ consisting of the matrices of the form $\left(\begin{array}{cc} * & * \\ 0 & * \end{array}\right)$.

**Theorem 1.10.**

(i) If $E/\mathbb{Q}$ has $j$-invariant $-17 \cdot 373^3/2^{17}$ or $-17^2 \cdot 101^3/2$, then $\rho_{E,17}(\text{Gal}_\mathbb{Q})$ is conjugate in $\text{GL}_2(\mathbb{F}_{17})$ to $G_1$ or $G_2$, respectively.

(ii) If $E/\mathbb{Q}$ has $j$-invariant $-7 \cdot 11^3$ or $-7 \cdot 137^3 \cdot 2083^3$, then $\rho_{E,37}(\text{Gal}_\mathbb{Q})$ is conjugate in $\text{GL}_2(\mathbb{F}_{37})$ to $G_3$ or $G_4$, respectively.

**Theorem 1.11** (Mazur, Serre, Bilu-Parent-Rebolledo). **Fix a prime $\ell \geq 17$ and let $E$ be a non-CM elliptic curve defined over $\mathbb{Q}$. If $(\ell,j_E)$ does not belong to the set**

$$\{(17, -17 \cdot 373^3/2^{17}), (17, -17^2 \cdot 101^3/2), (37, -7 \cdot 11^3), (37, -7 \cdot 137^3 \cdot 2083^3)\},$$

**then either $\rho_{E,\ell}$ is surjective or $\rho_{E,\ell}(\text{Gal}_\mathbb{Q})$ is conjugate to a subgroup of $N_{ns}(\ell)$.
Proof. The group $GL_2(F_{\ell})$ has either three or four maximal subgroups with determinant $F_{\ell}^\times$. They are $B(\ell)$, $N_8(\ell)$, $N_{ns}(\ell)$ and when $\ell \equiv \pm 3 \pmod{8}$, we also have a maximal subgroup $H_{S_4}(\ell)$ whose image in $PGL_2(F_{\ell})$ is isomorphic to the symmetric group $S_4$.

Take any non-CM elliptic curve $E$ over $\mathbb{Q}$. Serre has shown that $\rho_{E,\ell}(\text{Gal}_{\mathbb{Q}})$ cannot be conjugate to a subgroup of $H_{S_4}(\ell)$, cf. [Ser81, §8.4]. Bilu and Parent have proved that $\rho_{E,\ell}(\text{Gal}_{\mathbb{Q}})$ cannot be conjugate to a subgroup of $N_8(\ell)$, cf. [BPR11] (they make effective the bounds in earlier works of Bilu and Parent using improved isogeny bounds of Gaudron and Rémond). The $B(\ell)$ case follows from a famous theorem of Mazur, cf. [Maz78]. The modular curves $X_0(17)$ and $X_0(37)$ each have two rational points which are not cusps or CM points and they are accounted for by the curves of Theorem 1.10. □

We conjecture that Theorem 1.11 describes all the reasons that $\rho_{E,\ell}$ can fail to be surjective for a non-CM $E/\mathbb{Q}$ and a prime $\ell \geq 17$; this is a problem raised by Serre, cf. [Ser81, p.399], who asked if $\rho_{E,\ell}$ is surjective whenever $\ell > 37$.

Conjecture 1.12. If $E$ is a non-CM elliptic curve over $\mathbb{Q}$ and $\ell \geq 17$ is a prime such that the pair $(\ell, j_E)$ does not belong to the set (1.1), then $\rho_{E,\ell}(\text{Gal}_{\mathbb{Q}}) = GL_2(F_{\ell})$.

Even if Conjecture 1.12 is false for some $E/\mathbb{Q}$ and $\ell \geq 17$, the following proposition gives at most two possibilities for the image of $\rho_{E,\ell}$ (they can be distinguished computationally by looking at the division polynomial of $E$ at $\ell$).

Proposition 1.13. Suppose that $\rho_{E,\ell}$ is not surjective for a non-CM elliptic curve $E/\mathbb{Q}$ and a prime $\ell \geq 17$ for which $(\ell, j_E)$ does not lie in the set (1.1).

(i) If $\ell \equiv 1 \pmod{3}$, then $\rho_{E,\ell}(\text{Gal}_{\mathbb{Q}})$ is conjugate in $GL_2(F_{\ell})$ to $N_{ns}(\ell)$.

(ii) If $\ell \equiv 2 \pmod{3}$, then $\rho_{E,\ell}(\text{Gal}_{\mathbb{Q}})$ is conjugate in $GL_2(F_{\ell})$ to $N_{ns}(\ell)$ or to the group

$$G := \{a^3 : a \in C_{ns}(\ell)\} \cup \{\begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \cdot a^3 : a \in C_{ns}(\ell)\}.$$ 

1.8. Algorithm. Let $E/\mathbb{Q}$ be a non-CM elliptic curve (when $E/\mathbb{Q}$ has complex multiplication, the groups $\rho_{E,\ell}(\text{Gal}_{\mathbb{Q}})$ are all described in §1.9 below). In [Zyw15], we give an algorithm to compute the set $S'$ of primes $\ell \geq 13$ for which $\rho_{E,\ell}$ is not surjective.

Combined with the theorems from §§1.1–1.5, we are now able to compute the (finite) set $S$ of primes $\ell$ for which $\rho_{E,\ell}$ is not surjective. Moreover, using the results from §§1.1–1.5, we can give the group $\rho_{E,\ell}(\text{Gal}_{\mathbb{Q}})$, up to conjugacy in $GL_2(F_{\ell})$, for each $\ell \in S$.

Sutherland has a probabilistic algorithm to determine the groups $\rho_{E,\ell}(\text{Gal}_{\mathbb{Q}})$ by consider Frobenius at many primes $p$, [Sut15]. His algorithm can in principle be made deterministic using effective versions of the Chebotarev density theorem. Sutherland’s algorithm has the advantage that it can be used for elliptic curves over a number field $K \neq \mathbb{Q}$ (for our approach, we would have more modular curves to consider and those modular curves not isomorphic to $\mathbb{P}^1_{\mathbb{Q}}$ would need to be reconsidered).

The next task that needs to be completed is to consider the images of $\rho_{E,\ell^n}$ for small primes $\ell$ and $n \geq 2$. Rouse and Zureick-Brown have already done this for $\ell = 2$, cf. [RZB14]; the case $\ell = 2$ is rather accessible since all the groups that occur are solvable.
1.9. Complex multiplication. Up to isomorphism over $\overline{\mathbb{Q}}$, there are thirteen elliptic curves with complex multiplication that are defined over $\mathbb{Q}$. In Table 1 below, we give an elliptic curve $E_{D,f}/\mathbb{Q}$ with each of these thirteen $j$-invariants (this comes from [Sil94, Appendix A §3] though with some different models). The curve $E_{D,f}$ has conductor $N$ and has complex multiplication by an order $R$ of conductor $f$ in the imaginary quadratic field with discriminant $-D$.

| $j$-invariant | $D$ | $f$ | Elliptic curve $E_{D,f}$ | $N$ |
|---------------|-----|-----|--------------------------|-----|
| 0             | 3   | 1   | $y^2 = x^3 + 16$         | $3^5$ |
| $2^{4}3^{2}5^{3}$ | 2   | 2   | $y^2 = x^3 - 15x + 22$   | $2^{3}2$ |
| $-2^{1}5^{3} \cdot 5^{3}$ | 3   | 3   | $y^2 = x^3 - 480x + 4048$ | $3^3$ |
| $2^{6}3^{2} = 1728$ | 4   | 1   | $y^2 = x^3 + x$          | $2^6$ |
| $3^{4}5^{3}$ | 7   | 1   | $y^2 = x^3 - 1715x + 33614$ | $7^2$ |
| $3^{5}5^{3}17^{3}$ | 7   | 2   | $y^2 = x^3 - 29155x + 1915998$ | $7^2$ |
| $-2^{5}5^{4}$ | 8   | 1   | $y^2 = x^3 - 4320x + 96768$ | $2^8$ |
| $-2^{15}$ | 11  | 1   | $y^2 = x^3 - 9504x + 365904$ | $11^2$ |
| $-2^{1}5^{3}3^{3}$ | 19  | 1   | $y^2 = x^3 - 608x + 5776$ | $19^2$ |
| $-2^{1}5^{3}3^{2}5^{4}$ | 43  | 1   | $y^2 = x^3 - 13760x + 621264$ | $43^2$ |
| $-2^{1}5^{3}3^{5}5^{3}11^{3}$ | 67  | 1   | $y^2 = x^3 - 117920x + 15585808$ | $67^2$ |
| $-2^{1}8^{3}3^{5}2^{3}2^{9}3^{3}$ | 163 | 1   | $y^2 = x^3 - 34790720x + 78984748304$ | $163^2$ |

Table 1. CM elliptic curves over $\mathbb{Q}$

We first describe the group $\rho_{E,\ell}(\text{Gal}_\mathbb{Q})$ up to conjugacy when $E$ is a CM elliptic curve with non-zero $j$-invariant and $\ell$ odd.

**Proposition 1.14.** Let $E$ be a CM elliptic curve defined over $\mathbb{Q}$ with $j_E \neq 0$. The ring of endomorphisms of $E_{\overline{\mathbb{Q}}}$ is an order of conductor $f$ in the ring of integers of an imaginary quadratic field of discriminant $-D$. Take any odd prime $\ell$.

(i) If $\left( \frac{D}{\ell} \right) = 1$, then $\rho_{E,\ell}(\text{Gal}_\mathbb{Q})$ is conjugate in $\text{GL}_2(\mathbb{F}_\ell)$ to $N_s(\ell)$.

(ii) If $\left( \frac{D}{\ell} \right) = -1$, then $\rho_{E,\ell}(\text{Gal}_\mathbb{Q})$ is conjugate in $\text{GL}_2(\mathbb{F}_\ell)$ to $N_{ns}(\ell)$.

(iii) Suppose that $\ell$ divides $D$ and hence $D = \ell$. Define the groups

$$G = \{ (a \ b \ c) : a \in \mathbb{F}_\ell^\times, b \in \mathbb{F}_\ell \},$$

$$H_1 = \{ (a \ b) : a \in (\mathbb{F}_\ell^\times)^2, b \in \mathbb{F}_\ell \}, \quad \text{and} \quad H_2 = \{ (\pm a \ b) : a \in (\mathbb{F}_\ell^\times)^2, b \in \mathbb{F}_\ell \}$$

If $E$ is isomorphic to $E_{D,f}$, then $\rho_{E,\ell}(\text{Gal}_\mathbb{Q})$ is conjugate in $\text{GL}_2(\mathbb{F}_\ell)$ to $H_1$.

If $E$ is isomorphic to the quadratic twist of $E_{D,f}$ by $-\ell$, then $\rho_{E,\ell}(\text{Gal}_\mathbb{Q})$ is conjugate in $\text{GL}_2(\mathbb{F}_\ell)$ to $H_2$.

If $E$ is not isomorphic to $E_{D,f}$ or its quadratic twist by $-\ell$, then $\rho_{E,\ell}(\text{Gal}_\mathbb{Q})$ is conjugate in $\text{GL}_2(\mathbb{F}_\ell)$ to $G$.

The following deals with the excluded prime $\ell = 2$.

**Proposition 1.15.** Let $E/\mathbb{Q}$ be a CM elliptic curve. Define the subgroup $G_2 = \{ I, (\frac{1}{1} \frac{1}{1}) \}$ of $\text{GL}_2(\mathbb{F}_2)$.

(i) If $j_E \in \{ 2^{4}3^{3}5^{3}, 2^{3}3^{11}3^{3}, -3^{3}5^{3}, 3^{3}5^{3}17^{3}, 2^{6}5^{3} \}$, then $\rho_{E,2}(\text{Gal}_\mathbb{Q})$ is conjugate to $G_2$.

(ii) If $j_E \in \{ -2^{15}3^{5}5^{3}, -2^{15}, -2^{15}3^{3}, -2^{18}3^{3}5^{3}, -2^{15}3^{5}5^{3}11^{3}, -2^{18}3^{5}5^{3}23^{3}29^{3} \}$, then $\rho_{E,2}(\text{Gal}_\mathbb{Q}) = \text{GL}_2(\mathbb{F}_2)$. 

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(iii) Suppose that \( j_E = 1728 \). The curve can be given by a Weierstrass equation \( y^2 = x^3 - dx \) for some \( d \in \mathbb{Q}^\times \).
   If \( d \) is a square, then \( \rho_{E,2}(\text{Gal}_\mathbb{Q}) = \{I\} \).
   If \( d \) is not a square, then the group \( \rho_{E,2}(\text{Gal}_\mathbb{Q}) \) is conjugate to \( G_2 \).
(iv) Suppose that \( j_E = 0 \). The curve \( E \) can be given by a Weierstrass equation \( y^2 = x^3 + d \) for some \( d \in \mathbb{Q}^\times \).
   If \( d \) is a cube, then \( \rho_{E,2}(\text{Gal}_\mathbb{Q}) \) is conjugate in \( \text{GL}_2(\mathbb{F}_2) \) to the group \( G_2 \).
   If \( d \) is not a cube, then \( \rho_{E,2}(\text{Gal}_\mathbb{Q}) = \text{GL}_2(\mathbb{F}_2) \).

It remains to consider the situation where \( \ell \) is an odd prime and \( E/\mathbb{Q} \) is an elliptic curve with \( j_E = 0 \). That such curves have cubic twists make the classification more involved.

**Proposition 1.16.** Let \( E \) be an elliptic curve over \( \mathbb{Q} \) with \( j_E = 0 \). Take any odd prime \( \ell \).

(i) If \( \ell \equiv 1 \pmod{9} \), then \( \rho_{E,\ell}(\text{Gal}_\mathbb{Q}) \) is conjugate to \( N_s(\ell) \) in \( \text{GL}_2(\mathbb{F}_\ell) \).
(ii) If \( \ell \equiv 8 \pmod{9} \), then \( \rho_{E,\ell}(\text{Gal}_\mathbb{Q}) \) is conjugate to \( N_{ns}(\ell) \) in \( \text{GL}_2(\mathbb{F}_\ell) \).
(iii) Suppose that \( \ell \) is congruent to 4 or 7 modulo 9. Let \( E'/\mathbb{Q} \) be the elliptic curve over \( \mathbb{Q} \) defined by \( y^2 = x^3 + 16\ell^e \), where \( e \in \{1,2\} \) satisfies \( \frac{\ell-1}{3} \equiv e \pmod{3} \).
   If \( E \) is not isomorphic to a quadratic twist of \( E' \), then \( \rho_{E,\ell}(\text{Gal}_\mathbb{Q}) \) is conjugate to \( N_s(\ell) \) in \( \text{GL}_2(\mathbb{F}_\ell) \).
   If \( E \) is isomorphic to a quadratic twist of \( E' \), then \( \rho_{E,\ell}(\text{Gal}_\mathbb{Q}) \) is conjugate in \( \text{GL}_2(\mathbb{F}_\ell) \) to the subgroup \( G \) of \( N_s(\ell) \) consisting of the matrices of the form \( \begin{pmatrix} a & 0 \\ 0 & b \end{pmatrix} \) or \( \begin{pmatrix} b & a \\ 0 & 0 \end{pmatrix} \) with \( a/b \in (\mathbb{F}_\ell^\times)^3 \).
(iv) Suppose that \( \ell \) is congruent to 2 or 5 modulo 9. Let \( E'/\mathbb{Q} \) be the elliptic curve over \( \mathbb{Q} \) defined by \( y^2 = x^3 + 16\ell^e \), where \( e \in \{1,2\} \) satisfies \( \frac{\ell+1}{3} \equiv -e \pmod{3} \).
   If \( E \) is not isomorphic to a quadratic twist of \( E' \), then \( \rho_{E,\ell}(\text{Gal}_\mathbb{Q}) \) is conjugate to \( N_{ns}(\ell) \) in \( \text{GL}_2(\mathbb{F}_\ell) \).
   If \( E \) is isomorphic to a quadratic twist of \( E' \), then \( \rho_{E,\ell}(\text{Gal}_\mathbb{Q}) \) is conjugate in \( \text{GL}_2(\mathbb{F}_\ell) \) to the subgroup \( G \) of \( N_{ns}(\ell) \) generated by the unique index 3 subgroup of \( C_{ns}(\ell) \) and by \( \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \).
(v) Suppose that \( \ell = 3 \). The curve \( E \) can be given by a Weierstrass equation \( y^2 = x^3 + d \) for some \( d \in \mathbb{Q}^\times \). Fix notation as in §1.2.
   If \( d \) or \(-3d \) is a square and \(-4d \) is a cube, then \( \rho_{E,3}(\text{Gal}_\mathbb{Q}) \) is conjugate to \( H_{1,1} \).
   If \( d \) and \(-3d \) are not squares and \(-4d \) is a cube, then \( \rho_{E,3}(\text{Gal}_\mathbb{Q}) \) is conjugate to \( G_1 \).
   If \( d \) is a square and \(-4d \) is not a cube, then \( \rho_{E,3}(\text{Gal}_\mathbb{Q}) \) is conjugate to \( H_{3,1} \).
   If \(-3d \) is a square and \(-4d \) is not a cube, then \( \rho_{E,3}(\text{Gal}_\mathbb{Q}) \) is conjugate to \( H_{3,2} \).
   If \( d \) and \(-3d \) are not squares and \(-4d \) is not a cube, then \( \rho_{E,3}(\text{Gal}_\mathbb{Q}) \) is conjugate to \( G_3 \).

1.10. **Overview.** We now give a very brief overview of the paper. In §2, we describe applicable subgroups \( G \) of \( \text{GL}_2(\mathbb{F}_\ell) \); these groups have many of the properties that the groups \( \pm \rho_{E,\ell}(\text{Gal}_\mathbb{Q}) \) do.

In §3, we recall what we need concerning the modular curve \( X_G/\mathbb{Q} \); we will identify its function field with a subfield of the field of modular function for the congruence subgroup \( \Gamma(\ell) \).

In §4, we prove the parts of our main theorems that determine \( \pm \rho_{E,\ell}(\text{Gal}_\mathbb{Q}) \). We describe the rational points of \( X_G \) when \( \ell \) is small. When \( X_G \) has genus 0 and \( X_G(\mathbb{Q}) \neq \emptyset \), then the function field of \( X_G \) is of the form \( \mathbb{Q}(h) \) for some modular function \( h \). Much of this section is dedicated to describing such \( h \) and determining the rational function \( J(t) \in \mathbb{Q}(t) \) such that \( J(h) \) is the modular \( j \)-invariant.

Assuming that \( G := \pm \rho_{E,\ell}(\text{Gal}_\mathbb{Q}) \) is known, with \( E/\mathbb{Q} \) non-CM, we describe in §5 how to determine the (finite number of) quadratic twists of \( E' \) of \( E \) for which \( \rho_{E',\ell}(\text{Gal}_\mathbb{Q}) \) is not conjugate to \( G \). In §6, we prove the parts of our main theorems that determine \( \rho_{E,\ell}(\text{Gal}_\mathbb{Q}) \) given \( \pm \rho_{E,\ell}(\text{Gal}_\mathbb{Q}) \).

In §7, we prove the propositions from §1.9 concerning CM elliptic curves defined over \( \mathbb{Q} \). The \( j \)-invariant 0 case requires special attention since one has to worry about cubic twists. Finally, in §8, we prove Proposition 1.13.
The equations in §1.1–1.7 and Magma code verifying some claims in §4 and §6 can be found at: http://www.math.cornell.edu/~zywina/papers/PossibleImages/

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2. Applicable Subgroups

Fix an integer $N \geq 2$. For an elliptic curve $E/\mathbb{Q}$, let $E[N]$ be the $N$-torsion subgroup of $E(\mathbb{Q})$. After choosing a basis for $E[N]$ as a $\mathbb{Z}/N\mathbb{Z}$-module, the natural Galois action on $E[N]$ can be expressed in terms of a Galois representation

$$\rho_{E,N}: \text{Gal}_\mathbb{Q} \to \text{GL}_2(\mathbb{Z}/N\mathbb{Z}).$$

When $N$ is a prime, these agree with the representations of §1. We now describe some restrictions on the possible images of $\rho_{E,N}$.

Definition 2.1. We say that a subgroup $G$ of $\text{GL}_2(\mathbb{Z}/N\mathbb{Z})$ is applicable if it satisfies the following conditions:

- $G \neq \text{GL}_2(\mathbb{Z}/N\mathbb{Z})$,
- $-I \in G$ and $\det(G) = (\mathbb{Z}/N\mathbb{Z})^\times$,
- $G$ contains an element with trace 0 and determinant $-1$ that fixes a point in $(\mathbb{Z}/N\mathbb{Z})^2$ of order $N$.

This definition is justified by the following.

Proposition 2.2. Let $E$ be an elliptic curve over $\mathbb{Q}$ for which $\rho_{E,N}$ is not surjective. Then $\pm \rho_{E,N}(\text{Gal}_\mathbb{Q})$ is an applicable subgroup of $\text{GL}_2(\mathbb{Z}/N\mathbb{Z})$.

Proof. The group $G := \pm \rho_{E,N}(\text{Gal}_\mathbb{Q})$ clearly contains $-I$. The character $\det \circ \rho_{E,N}: \text{Gal}_\mathbb{Q} \to (\mathbb{Z}/N\mathbb{Z})^\times$ is the surjective homomorphism describing the Galois action on the group of $N$-th roots of unity in $\mathbb{Q}$, i.e., for a $N$-th root of unity $\zeta \in \mathbb{Q}$, we have $\sigma(\zeta) = \zeta^{\det(\rho_{E,N}(\sigma))}$ for all $\sigma \in \text{Gal}_\mathbb{Q}$. Therefore, $\det \circ \rho_{E,N}$ is surjective and hence $\det(G) = (\mathbb{Z}/N\mathbb{Z})^\times$.

Let $c \in \text{Gal}(\overline{\mathbb{Q}}/\mathbb{Q})$ be an automorphism corresponding to complex conjugation under some embedding $\overline{\mathbb{Q}} \hookrightarrow \mathbb{C}$. Set $g := \rho_{E,N}(c)$. As a topological group, the connected component of $E(\mathbb{R})$ containing the identity is isomorphic to $\mathbb{R}/\mathbb{Z}$. Therefore, $E(\mathbb{R})$ contains a point $P_1$ of order $N$. We may assume that $\rho_{E,N}$ is chosen with respect to a basis whose first term is $P_1$, and hence $g$ is upper triangular whose first diagonal term is 1. We have $\det(g) = -1$ since $c$ acts by inversion on $N$-th roots of unity. Therefore, $g$ is upper triangular with diagonal entries 1 and $-1$, and hence $\text{tr}(g) = 0$.

Now suppose that $G = \text{GL}_2(\mathbb{Z}/N\mathbb{Z})$. Define $S = \rho_{E,N}(\text{Gal}_\mathbb{Q}) \cap \text{SL}_2(\mathbb{Z}/N\mathbb{Z})$. Since $G = \text{GL}_2(\mathbb{Z}/N\mathbb{Z})$, $\rho_{E,N}(\text{Gal}_\mathbb{Q}) \neq \text{GL}_2(\mathbb{Z}/N\mathbb{Z})$ and $\det(\rho_{E,N}(\text{Gal}_\mathbb{Q})) = (\mathbb{Z}/N\mathbb{Z})^\times$, we deduce that $S \neq \text{SL}_2(\mathbb{Z}/N\mathbb{Z})$ and $\pm S = \text{SL}_2(\mathbb{Z}/N\mathbb{Z})$. However, this is impossible by Lemma 2.3 below, so we must have $G \neq \text{GL}_2(\mathbb{Z}/N\mathbb{Z})$. \hfill \Box

Lemma 2.3. There is no proper subgroup $S$ of $\text{SL}_2(\mathbb{Z}/N\mathbb{Z})$ such that $\pm S = \text{SL}_2(\mathbb{Z}/N\mathbb{Z})$.

Proof. Suppose that $S$ is a subgroup of $\text{SL}_2(\mathbb{Z}/N\mathbb{Z})$ for which $\pm S = \text{SL}_2(\mathbb{Z}/N\mathbb{Z})$. By [Zyw10, Lemma A.6], we deduce that there is a prime power $\ell^e$ dividing $N$ such that the image $S$ of $S$ in $\text{SL}_2(\mathbb{Z}/\ell^e\mathbb{Z})$ is a proper subgroup satisfying $\pm S = \text{SL}_2(\mathbb{Z}/\ell^e\mathbb{Z})$. So without loss of generality, we may assume that $N = \ell^e$.

The group $S$ has index 2 in $\text{SL}_2(\mathbb{Z}/\ell^e\mathbb{Z})$. Therefore, $S$ is normal in $\text{SL}_2(\mathbb{Z}/\ell^e\mathbb{Z})$ and the quotient is cyclic of order 2. However, the abelianization of $\text{SL}_2(\mathbb{Z}/\ell^e\mathbb{Z})$ is a cyclic group of order $\text{gcd}(\ell^e, 12)$, cf. [Zyw10, Lemma A.1]. Therefore, we must have $\ell = 2$. Since the abelianization of $\text{SL}_2(\mathbb{Z}/2^e\mathbb{Z})$ is
cyclic of order 2 or 4, we find that $S$ is the unique subgroup of $\text{SL}_2(\mathbb{Z}/2^r\mathbb{Z})$ of index 2. The group $S$ is now easy to describe; it is the group of elements in $\text{SL}_2(\mathbb{Z}/2^r\mathbb{Z})$ whose image in $\text{SL}_2(\mathbb{Z}/2\mathbb{Z})$ lies in the unique cyclic group of order 3. However, this implies that $\pm S \neq \text{SL}_2(\mathbb{Z}/2^r\mathbb{Z})$ since $-I \equiv I \pmod{2}$. This contradiction ensures that no such $S$ exists.

\textbf{Remark} 2.4. When $N$ is a prime $\ell$, which is the setting of this paper, the last condition in the definition of applicable subgroup can be simplified to say simply that $G$ contains an element with trace 0 and determinant $-1$.

3. Modular curves

Fix an integer $N \geq 1$; in our later application, we will take $N$ to be a prime $\ell$. In §3.1, we recall the Galois theory of the field of modular functions of level $N$. In §3.2, we define modular curves in terms of their functions fields. We take an unsophisticated approach to modular curves and develop what we need from Shimura’s book [Shi94]; it will be useful for reference in future work. Alternatively, one could develop modular curves as in [DR73, IV-3].

3.1. Modular functions of level $N$. The group $\text{SL}_2(\mathbb{Z})$ acts on the complex upper half plane $\mathbb{H}$ via linear fractional transformations, i.e., $\gamma* = (a\tau + b)/(c\tau + d)$ for $\gamma = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \text{SL}_2(\mathbb{Z})$ and $\tau \in \mathbb{H}$. Let $\Gamma(N)$ be the congruence subgroup consisting of matrices in $\text{SL}_2(\mathbb{Z})$ that are congruent to $I$ modulo $N$. The quotient $\Gamma(N)\backslash \mathbb{H}$ is a Riemann surface and can be completed to a compact and smooth Riemann surface $X_N$. Let $\tau$ be a variable of the complex upper half plane.

Every meromorphic function $f$ on $X_N$ has a $q$-expansion $\sum_{n \in \mathbb{Z}} c_n q^{n/N}$; here the $c_n$ are complex numbers which are 0 for all but finitely many negative $n$ and $q^{1/N} := e^{2\pi i \tau/N}$. We define $\mathcal{F}_N$ to be the field of meromorphic functions on $X_N$ whose $q$-expansion has coefficients in $\mathbb{Q}(\zeta_N)$, where $\zeta_N$ is the $N$-th root of unity $e^{2\pi i /N}$. For example, $\mathcal{F}_1 = \mathbb{Q}(j)$ where $j = j(\tau)$ is the modular $j$-invariant with the familiar expansion

$$j = q^{-1} + 744 + 196884q + 21493760q^2 + 864299970q^3 + \ldots.$$

For each $d \in (\mathbb{Z}/N\mathbb{Z})^\times$, let $\sigma_d$ be the automorphism of the field $\mathbb{Q}(\zeta_N)$ for which $\sigma_d(\zeta_N) = \zeta_N^d$. We extend $\sigma_d$ to an automorphism of $\mathcal{F}_N$ by taking a function with $q$-expansion $\sum_n c_n q^{n/N}$ to $\sum_n \sigma_d(c_n) q^{n/N}$. We let $\text{SL}_2(\mathbb{Z})$ act on $\mathcal{F}_N$ by taking a modular function $f \in \mathcal{F}_N$ and a matrix $\gamma \in \text{SL}_2(\mathbb{Z})$ to $f \circ \gamma^t$, i.e., the function $(f \circ \gamma^t)(\tau) = f(\gamma^t(\tau))$ where $\gamma^t$ is the transpose of $\gamma$.

\textbf{Proposition 3.1.} The extension $\mathcal{F}_N$ of $\mathbb{Q}(j)$ is Galois. There is a unique isomorphism

$$\theta_N: \text{GL}_2(\mathbb{Z}/N\mathbb{Z})/\{\pm I\} \cong \text{Gal}(\mathcal{F}_N/\mathbb{Q}(j))$$

such that the following holds for all $f \in \mathcal{F}_N$:

(a) For $A \in \text{SL}_2(\mathbb{Z}/N\mathbb{Z})$, we have $\theta_N(A)f = f \circ \gamma^t$, where $\gamma$ is any matrix in $\text{SL}_2(\mathbb{Z})$ that is congruent to $A$ modulo $N$.

(b) For $A = \begin{pmatrix} 1 & 0 \\ 0 & d \end{pmatrix} \in \text{GL}_2(\mathbb{Z}/N\mathbb{Z})$, we have $\theta_N(A)f = \sigma_d(f)$.

The field $\mathbb{Q}(\zeta_N)$ is the algebraic closure of $\mathbb{Q}$ in $\mathcal{F}_N$ and corresponds to the subgroup $\text{SL}_2(\mathbb{Z}/N\mathbb{Z})/\{\pm I\}$.

We will sketch Proposition 3.1 in §3.4. Throughout the paper, we will let $\text{GL}_2(\mathbb{Z}/N\mathbb{Z})$ act on $\mathcal{F}_N$ via the isomorphism $\theta_N$ (with $-I$ acting trivially).

\textbf{Remark} 3.2. There are other choices for an isomorphism $\text{GL}_2(\mathbb{Z}/N\mathbb{Z})/\{\pm I\}$; for example, one could instead replace the transpose by an inverse in (a). Our choice is explained by our application to modular curves. As a warning, there are several places in the literature where incompatible choices are made with respect to modular curves.
3.2. Modular curves. Let $G$ be a subgroup of $\text{GL}_2(\mathbb{Z}/N\mathbb{Z})$ containing $-I$ that satisfies $\det(G) = (\mathbb{Z}/N\mathbb{Z})^\times$. Denote by $\mathcal{F}_N^G$ the subfield of $\mathcal{F}_N$ fixed by the action of $G$ from Proposition 3.1. Using Proposition 3.1 and the assumption $\det(G) = (\mathbb{Z}/N\mathbb{Z})^\times$, we find that $\mathbb{Q}$ is algebraically closed in $\mathcal{F}_N^G$.

Let $X_G$ be the smooth projective curve with function field $\mathcal{F}_N^G$; it is defined over $\mathbb{Q}$ and is geometrically irreducible. The inclusion of fields $\mathcal{F}_N^G \supseteq \mathbb{Q}(j)$ gives rise to a non-constant morphism
\[ \pi_G: X_G \to \text{Spec } \mathbb{Q}[j] \cup \{\infty\} = \mathbb{P}^1_{\mathbb{Q}}. \]

The morphism $\pi_G$ is non-constant and we have
\[ \deg(\pi_G) = [\text{GL}_2(\mathbb{Z}/N\mathbb{Z})/\{\pm I\} : G/\{\pm I\}] = [\text{GL}_2(\mathbb{Z}/N\mathbb{Z}) : G]. \]

We will also denote the function field $\mathcal{F}_N^G$ of $X_G$ by $\mathbb{Q}(X_G)$. A point in $X_G$ is a cusp or a CM point if $\pi_G$ maps it to $\infty$ or to the $j$-invariant of a CM elliptic curve, respectively.

The following property of the curve $X_G$ is key to our application; we will give a proof in §3.5.

Proposition 3.3. Let $G$ be a subgroup of $\text{GL}_2(\mathbb{Z}/N\mathbb{Z})$ that contains $-I$ and satisfies $\det(G) = (\mathbb{Z}/N\mathbb{Z})^\times$. Let $E$ be an elliptic curve defined over $\mathbb{Q}$ with $j_E \notin \{0,1728\}$. Then $\rho_{E,N}(\text{Gal}_\mathbb{Q})$ is conjugate in $\text{GL}_2(\mathbb{Z}/N\mathbb{Z})$ to a subgroup of $G$ if and only if $j_E$ belongs to $\pi_G(X_G(\mathbb{Q}))$.

The following lemma will be key to finding modular curves of genus 0 with rational points.

Lemma 3.4. Fix a modular function $h \in \mathcal{F}_N - \mathbb{Q}(j)$ such that $J(h) = j$ for a rational function $J(t) \in \mathbb{Q}(t)$. Let $G$ be the subgroup of $\text{GL}_2(\mathbb{Z}/N\mathbb{Z})$ that fixes $h$ under the action on $\mathcal{F}_N$ from Proposition 3.1.

(i) The subgroup $G$ of $\text{GL}_2(\mathbb{Z}/N\mathbb{Z})$ is applicable.
(ii) The modular curve $X_G$ has function field $\mathbb{Q}(h)$. In particular, it is isomorphic to $\mathbb{P}^1_{\mathbb{Q}}$.
(iii) Let $E/\mathbb{Q}$ be an elliptic curve with $j_E \notin \{0,1728\}$. The group $\rho_{E,N}(\text{Gal}_\mathbb{Q})$ is conjugate in $\text{GL}_2(\mathbb{Z}/N\mathbb{Z})$ to a subgroup of $G$ if and only if $j_E = J(t)$ for some $t \in \mathbb{Q} \cup \{\infty\}$.

Proof. By the Galois correspondence coming from the isomorphism $\theta_N$ of Proposition 3.1, the field $\mathbb{Q}(h)$ equals $\mathcal{F}_N^G$ and is an extension of $\mathbb{Q}(j)$ of degree
\[ [\text{GL}_2(\mathbb{Z}/N\mathbb{Z})/\{\pm I\} : G/\{\pm I\}] = [\text{GL}_2(\mathbb{Z}/N\mathbb{Z}) : G]. \]

The field $\mathbb{Q}$ is algebraically closed in $\mathcal{F}_N^G = \mathbb{Q}(h)$, so $\det(G) = (\mathbb{Z}/N\mathbb{Z})^\times$ by Proposition 3.1. Therefore, $\mathbb{Q}(h)$ is the function field of $X_G$ and the field extension $\mathbb{Q}(h)/\mathbb{Q}(j)$ given by $j = J(h)$ corresponds to the morphism $\pi_G: X_G \to \mathbb{P}^1_{\mathbb{Q}}$. The modular curve $X_G$ is thus isomorphic to $\mathbb{P}^1_{\mathbb{Q}}$ and we have $\pi_G(X_G(\mathbb{Q})) = J(\mathbb{Q} \cup \{\infty\})$. This proves (ii). Part (iii) follows from Proposition 3.3.

Finally, we prove that $G$ is applicable. We have $G \neq \text{GL}_2(\mathbb{Z}/N\mathbb{Z})$ since the extension $\mathbb{Q}(h)/\mathbb{Q}(j)$ is non-trivial by our assumption on $h$. Using part (iii) and Proposition 2.2, we find that $G$ contains an applicable subgroup. Since $G \neq \text{GL}_2(\mathbb{Z}/N\mathbb{Z})$ and $G$ contains an applicable subgroup, we deduce that $G$ is applicable.

If $X_G$ has genus 0 and has rational points, then there are in fact curves $E/\mathbb{Q}$ with $\pm \rho_{E,N}(\text{Gal}_\mathbb{Q})$ conjugate to $G$.

Lemma 3.5. Suppose that $X_G$ is isomorphic to $\mathbb{P}^1_{\mathbb{Q}}$; equivalently, the function field of $X_G$ is of the form $\mathbb{Q}(h)$. We have $j = J(h)$ for a unique $J(t) \in \mathbb{Q}(t)$ because of the inclusion $\mathbb{Q}(h) \supseteq \mathbb{Q}(j)$. Then for “most” $u \in \mathbb{Q}$ (more precisely, outside a set of density 0 with respect to height), the groups $\pm \rho_{E,N}(\text{Gal}_\mathbb{Q})$ and $G$ are conjugate in $\text{GL}_2(\mathbb{Z}/N\mathbb{Z})$ for any elliptic curve $E/\mathbb{Q}$ with $j$-invariant $J(u)$. 

Proof. Let $\mathcal{G}$ be the (finite) set of applicable subgroups $H$ of $\text{GL}_2(\mathbb{Z}/N\mathbb{Z})$ satisfying $H \subseteq G$. For each $H \in \mathcal{G}$, let $\pi_{H,G}$ be the natural morphism $X_H \rightarrow X_G$; it has degree $[G : H] > 1$. To prove the lemma, it suffices to show that the set $S := \cup_{H \in \mathcal{G}} \pi_{H,G}(X_H(\mathbb{Q}))$ has density 0 (with respect to the height) in $X_G(\mathbb{Q}) \cong \mathbb{P}^1(\mathbb{Q})$. This is a consequence of Hilbert irreducibility; in the language of [Ser97, §9], the set $S$ is thin and hence has density 0. \hfill $\square$

3.3. The modular curve $X_0(N)$. Let $X_0(N)/\mathbb{Q}$ be the modular curve $X_{B(N)^t}$, where $B(N)^t$ is the transpose of $B(N)$; it consists of the lower triangular matrices and is conjugate to $B(N)$ in $\text{GL}_2(\mathbb{Z}/N\mathbb{Z})$. Let $\Gamma_0(N)$ be the group of matrices in $\text{SL}_2(\mathbb{Z})$ whose image modulo $N$ is upper triangular. A function $f \in F_N$ belongs to $\mathbb{Q}(X_0(N))$ if and only if it has rational Fourier coefficients and $f \circ \gamma = f$ for all $\gamma \in \Gamma_0(N)$. Define the modular curve $X_s(N) := X_C_s(N)$, where $C_s(N)$ is the subgroup of diagonal matrices in $\text{GL}_2(\mathbb{Z}/N\mathbb{Z})$.

Lemma 3.6. The map $\mathbb{Q}(X_0(N^2)) \rightarrow \mathbb{Q}(X_s(N))$, $f(\tau) \mapsto f(\tau/N)$ is an isomorphism of fields. This isomorphism induces an isomorphism between the modular curves $X_s(N)$ and $X_0(N^2)$ which gives a bijection between their cusps.

Proof. Let $\Gamma_s(N)$ be the group of matrices in $\text{SL}_2(\mathbb{Z})$ whose image modulo $N$ is diagonal. The function field of $X_s(N)$ then consist of the $f \in F_N$ with rational Fourier coefficients for which $f \circ \gamma = f$ for all $\gamma \in \Gamma_s(N)$.

Define $w = \left( \begin{smallmatrix} 1 & 0 \\ N & 1 \end{smallmatrix} \right)$; it acts on $\mathfrak{h}$ by linear fractional transformation, i.e., $w_s(\tau) = \tau/N$. Take any $f \in F_N$ whose Fourier coefficients are rational. We have $f \circ w \in \mathbb{Q}(X_s(N))$ if and only if $f \circ w \circ \gamma = f \circ w$ for all $\gamma \in \Gamma_s(N)$. Since $w \Gamma_s(N)w^{-1} = \Gamma_0(N^2)$, we deduce that $f \circ w \in \mathbb{Q}(X_s(N))$ if and only if $f \in \mathbb{Q}(X_0(N^2))$. It is now straightforward to show that the map of fields is well-defined and an isomorphism. The isomorphism of function fields of course induces an isomorphism of the corresponding curves. That the cusps are in correspondence is a consequence of the map $\Gamma_0(N^2) \backslash \mathfrak{h} \rightarrow \Gamma_s(N) \backslash \mathfrak{h}$, $\tau \mapsto w_s(\tau) = \tau/N$ being an isomorphism of Riemann surfaces. \hfill $\square$

Lemma 3.7. Let $\eta(\tau)$ be the Dedekind eta function.

(i) We have $\mathbb{Q}(X_0(4)) = \mathbb{Q}(h)$, where $h(\tau) = \eta(\tau)/\eta(4\tau)^8$.

(ii) We have $\mathbb{Q}(X_0(9)) = \mathbb{Q}(h)$, where $h(\tau) = \eta(\tau)^3/\eta(9\tau)^3$.

Proof. This is well-known; for example see [Elk01]. \hfill $\square$

3.4. Proof of Proposition 3.1. For $\tau \in \mathfrak{h}$, let $\Lambda_\tau$ be the lattice $\mathbb{Z}\tau + \mathbb{Z}$ in $\mathbb{C}$. Set $g_2(\tau) = g_2(\Lambda_\tau)$ and $g_3(\tau) = g_3(\Lambda_\tau)$, and let $\wp(z; \tau)$ be the Weierstrass $\wp$-function relative to $\Lambda_\tau$, cf. [Sil09, §VI.3] for background on elliptic functions. For each pair $a = (a_1, a_2) \in N^{-1}\mathbb{Z}^2 - \mathbb{Z}^2$, define the function

$$f_a(\tau) := \frac{g_2(\tau) g_3(\tau)}{g_2(\tau)^3 - 27g_3(\tau)^2} \cdot \wp(a_1 \tau + a_2; \tau)$$

of the upper half plane. The function $f_a$ is modular of level $N$. Moreover, Proposition 6.9(1) of [Shi94] says that

$$F_N = \mathbb{Q}(j, f_a \mid a \in N^{-1}\mathbb{Z}^2 - \mathbb{Z}^2).$$

For $a, b \in N^{-1}\mathbb{Z}^2 - \mathbb{Z}^2$, we have $f_a = f_b$ if and only if $a$ lies in the same coset of $\mathbb{Q}^2/\mathbb{Z}^2$ as $b$ or $-b$, cf. equation (6.1.5) of [Shi94]. So for any $A \in M_2(\mathbb{Z})$ with determinant relatively prime to $N$, the function $f_{aA}$ depends only on the image $\tilde{A}$ of $A$ in $\text{GL}_2(\mathbb{Z}/N\mathbb{Z})/\{\pm 1\}$. By abuse of notation, we shall denote $f_{aA}$ by $f_{a\tilde{A}}$.

By Theorem 6.6 of [Shi94], there is a unique isomorphism

$$\theta_N : \text{GL}_2(\mathbb{Z}/N\mathbb{Z})/\{\pm 1\} \sim \text{Gal}(F_N/\mathbb{Q}(j))$$

such that $\theta_N(A)f_a = f_{aA}$ for all $A \in \text{GL}_2(\mathbb{Z}/N\mathbb{Z})/\{\pm 1\}$ and $a \in N^{-1}\mathbb{Z}^2 - \mathbb{Z}^2$; we have added the transpose so the map is a homomorphism (and not an antihomomorphism).
Fix any \( \gamma \in \text{SL}_2(\mathbb{Z}) \) and let \( A \in \text{SL}_2(\mathbb{Z}/N\mathbb{Z}) \) be its image modulo \( N \). For any \( a \in N^{-1}\mathbb{Z}^2 - \mathbb{Z}^2 \), the function \( f_a \circ \gamma^t \) agrees with \( f_a \circ \gamma^t \) by equation (6.1.3) of \([\text{Shi94}]\). Using (3.1), we deduce that \( \theta_N(A)f = f \circ \gamma^t \) for all \( f \in \mathcal{F}_N \); this shows that (a) holds.

Now take integer \( d \) relatively prime to \( N \) and let \( A \) be the image of \( \begin{pmatrix} 1 & d \\ 0 & 1 \end{pmatrix} \) in \( \text{SL}_2(\mathbb{Z}/N\mathbb{Z}) \). Take any \( a \in N^{-1}\mathbb{Z}^2 - \mathbb{Z}^2 \); we have \( a = (r/N, s/N) \) with \( r, s \in \mathbb{Z} \). Since \( f_a = f_b \) when \( a \equiv b \) (mod \( \mathbb{Z}^2 \)), we may assume that \( 0 \leq r < N \). We have \( \theta_N(A)f = f_{aA^t} = f_{(r/N, ds/N)} \). By equation (6.2.1) of \([\text{Shi94}]\), we have

\[
(2\pi)^{-2} \varphi(a_1\tau + a_2; \tau) = -1/12 + 2 \sum_{n=1}^{\infty} \frac{aq^n/(1-q^n)}{\zeta_N^a q^{r/N}/(1-\zeta_N^r q^{r/N})^2} - \sum_{n=1}^{\infty} (\zeta_N^{ns} q^{nr/N} + \zeta_N^{-ns} q^{-nr/N}) \cdot q^n/(1-q^n);
\]

applying \( \sigma_d \) to this series gives the same thing with \( s \) replaced by \( ds \). The Fourier coefficients of the expansion of \( g_2(\tau)/g_3(\tau) \) are all \( \pi^{-2} \) times a rational number. Therefore, \( \sigma_d(f_a) = \sigma_d(f_{(r/N, ds/N)}) \) equals \( f_{(r/N, ds/N)} = f_{aA^t} \). Using (3.1), we deduce that \( \theta_N(A)f = \sigma_d(f) \) for all \( f \in \mathcal{F}_N \); this shows that (b) holds.

This explains the existence of an isomorphism \( \theta_N \) as in the statement of Proposition 3.1. The uniqueness if immediate since \( \text{GL}_2(\mathbb{Z}/N\mathbb{Z}) \) is generated by \( \text{SL}_2(\mathbb{Z}/N\mathbb{Z}) \) and matrices of the form \( \begin{pmatrix} 1 & d \\ 0 & 1 \end{pmatrix} \). Theorem 6.6 of \([\text{Shi94}]\) implies that \( \mathbb{Q}(\zeta_N) \) is the algebraic closure of \( \mathbb{Q} \) in \( \mathcal{F}_N \) and that \( \theta_N(A)\zeta_N = \zeta_N^A \).

### 3.5. Proof of Proposition 3.3.

We first construct the inverse of \( \theta_N \) using elliptic curves; we shall freely use definitions from §3.4. Let \( E \) be an elliptic curve defined over an algebraically closed field \( k \) of characteristic 0. Take any non-zero \( N \)-torsion point \( P \in E(k) \). If \( P = (x_0, y_0) \) with respect to some Weierstrass model \( y^2 = 4x^3 - c_2x - c_3 \) of \( E/k \), define \( h_E(P) := c_2c_3/(c_2^3 - 27c_3^2) \cdot x_0 \). If \( j_E \notin \{0, 1728\} \), then one can show that \( h_E(P) \) does not depend on the choice of model.

Let \( E \) be the elliptic curve over \( \mathcal{F}_1 = \mathbb{Q}(j) \) defined by the Weierstrass equation

\[
y^2 = 4x^3 - \frac{27j}{j - 1728}x - \frac{27j}{j - 1728};
\]

it has \( j \)-invariant \( j \). Fix an algebraic closed field \( K \) that contains \( \mathcal{F}_N \supseteq \mathbb{Q}(j) \) and let \( \mathcal{E}[N] \) be the \( N \)-torsion subgroup of \( \mathcal{E}(K) \).

**Lemma 3.8.** There is a basis \( \{P_1, P_2\} \) of the \( \mathbb{Z}/N\mathbb{Z} \)-module \( \mathcal{E}[N] \) such that \( h_E(rP_1 + sP_2) = f_{(r/N, s/N)} \) for all \( (r, s) \in \mathbb{Z}^2 - N\mathbb{Z}^2 \).

**Proof.** Let \( K_0 \) be the extension of \( \mathcal{F}_N \) generated by the functions \( g_2(\tau), g_3(\tau), \varphi(\tau/N; \tau), \varphi'(\tau/N; \tau), \varphi(1/N; \tau), \varphi'(1/N; \tau) \). We may assume that \( K \supseteq K_0 \). Let \( E \) be the elliptic curve over \( K_0 \) defined by \( y^2 = 4x^3 - g_2(\tau)x - g_3(\tau) \); its \( j \)-invariant is \( j = j(\tau) \). The curves \( E \) and \( \mathcal{E} \) are isomorphic over \( K \) since they both have \( j \)-invariant \( j \). Since \( j \notin \{0, 1728\} \), it suffices to prove the lemma for \( E \) instead of \( \mathcal{E} \).

Define the pairs

\( P_1 := (\varphi(\tau/N; \tau), \varphi'(\tau/N; \tau)) \) and \( P_2 := (\varphi(1/N; \tau), \varphi'(1/N; \tau)) \).

We claim that \( P_1 \) and \( P_2 \) form a basis of the \( \mathbb{Z}/N\mathbb{Z} \)-module of \( N \)-torsion in \( E(K) \). To prove the claim it suffices to prove the analogous results after specializing the coefficients of \( E \) and the entries of \( P_1 \) and \( P_2 \) by an arbitrary \( \tau_0 \in \mathfrak{h} \) (since the claim comes down to verifying certain polynomial equations whose variables are the coefficients of the model of \( E \) and the entries of the points). So fix an arbitrary \( \tau_0 \in \mathfrak{h} \). Specializing the model of \( E \) at \( \tau_0 \) gives an elliptic curve \( E_{\tau_0} \) over \( C \) defined by \( y^2 = 4x^3 - g_2(\tau_0)x - g_3(\tau_0) \). From Weierstrass, we know that the map

\[
\mathbb{C}/\Lambda_{\tau_0} \to E_{\tau_0}(\mathbb{C}), \quad z \mapsto (\varphi(z; \tau_0), \varphi'(z; \tau_0)),
\]
with 0 mapping to the point at infinity, gives an isomorphism of complex Lie groups. In particular, the points \( P_1, r_0 = (\varphi(1/N; \tau_0), \varphi'(1/N; \tau_0)) \) and \( P_2, r_0 = (\varphi(1/N; \tau_0), \varphi'(1/N; \tau_0)) \) give a basis for the \( N \)-torsion in \( E_\tau_0(C) \). This is enough to prove our claim. Moreover, we have \( rP_1, r_0 + sP_2, r_0 = (\varphi(r/N \cdot \tau_0 + s/N; \tau_0), \varphi'(r/N \cdot \tau_0 + s/N; \tau_0)) \) for all \( (r, s) \in \mathbb{Z}^2 - N\mathbb{Z}^2 \). Therefore,

\[
h_{E_\tau_0}(rP_1 + sP_2) = g_2(\tau_0)g_3(\tau_0)/(g_2(\tau_0)^3 - 27g_3(\tau_0)^2) \cdot \varphi(r/N \cdot \tau_0 + s/N; \tau_0) = f(r/N, s/N)(\tau_0).
\]

for all \( (r, s) \in \mathbb{Z}^2 - N\mathbb{Z} \). Since this holds for all \( \tau_0 \in \mathfrak{h} \), we deduce that \( h_{E}(rP_1 + sP_2) = f(r/N, s/N) \).

Let \( \rho_N : \text{Gal}(K/Q(j)) \to \text{GL}_2(\mathbb{Z}/N\mathbb{Z}) \) be the representation describing the Galois action on \( \mathcal{E}[N] \) with respect to the basis \( \{P_1, P_2\} \) of Lemma 3.8. The fixed field of the kernel of \( \text{Gal}(K/Q(j)) \) \( \rho_N \) \( \text{GL}_2(\mathbb{Z}/N\mathbb{Z}) \to \text{GL}_2(\mathbb{Z}/N\mathbb{Z})/\{\pm I\} \) is generated by the \( x \)-coordinates of the non-zero points in \( \mathcal{E}[N] \). By (3.1) and Lemma 3.8, the extension \( \mathcal{F}_N \) of \( \mathcal{E}(j) \) is generated by the \( x \)-coordinates of the non-zero points in \( \mathcal{E}[N] \). Therefore, the representation \( \rho_N \) induces an injective homomorphism

\[
\overline{\rho}_N : \text{Gal}(\mathcal{F}_N/Q(j)) \to \text{GL}_2(\mathbb{Z}/N\mathbb{Z})/\{\pm I\}.
\]

In fact, (3.3) is an isomorphism since the groups have the same cardinality by Proposition 3.1.

**Lemma 3.9.** The homomorphism \( \overline{\rho}_N \) is an isomorphism. Moreover, the inverse of \( \overline{\rho}_N \) is the homomorphism \( \theta_N : \text{GL}_2(\mathbb{Z}/N\mathbb{Z})/\{\pm I\} \to \text{Gal}(\mathcal{F}_N/Q(j)) \).

**Proof.** Take any \( \sigma \in \text{Gal}(K/Q(j)) \) and set \( \tilde{\sigma} := \sigma|_{\mathcal{F}_N} \). There are integers \( a, b, c, d \in \mathbb{Z} \) such that \( \sigma(P_1) = aP_1 + cP_2 \) and \( \sigma(P_2) = bP_1 + dP_2 \), so \( \rho_N(\sigma) = A \), where \( A \in \text{GL}_2(\mathbb{Z}/N\mathbb{Z}) \) is the image of \( \left( \begin{array}{cc} a & b \\ c & d \end{array} \right) \) modulo \( N \). Therefore, \( \overline{\rho}_N(\tilde{\sigma}) \) is the class of \( A \) in \( \text{GL}_2(\mathbb{Z}/N\mathbb{Z})/\{\pm I\} \). We need to show that \( \theta_N(A) = \tilde{\sigma} \).

Define the \( \mathbb{Q} \)-variety

\[
U := \mathbb{A}_\mathbb{Q}^1 - \{0, 1728\} = \text{Spec } \mathbb{Q}[j, j^{-1}, (j - 1728)^{-1}];
\]

note that we are now viewing \( j \) as simply a transcendental variable. The equation (3.2) defines a (relative) elliptic curve \( \pi : \mathcal{E} \to U \). The fiber of \( \mathcal{E} \to U \) over the generic fiber of \( U \) is the elliptic curve \( \mathcal{E}/Q(j) \).

Let \( \overline{\eta} \) be the geometric generic point of \( U \) corresponding to the algebraically closed extension \( K \) of \( \mathcal{F}_N \). Let \( \mathcal{E}[N] \) be the \( N \)-torsion subscheme of \( \mathcal{E} \). We can identify the fiber of \( \mathcal{E}[N] \to U \) at \( \overline{\eta} \) with the group \( \mathcal{E}[N] \). Let \( \pi_1(U, \overline{\eta}) \) be the étale fundamental group of \( U \). We can view \( \mathcal{E}[N] \) as a rank 2 lisse sheaf of \( \mathbb{Z}/N\mathbb{Z} \)-modules \( U \) and it hence corresponds to a representation

\[
\mathcal{G}_N : \pi_1(U, \overline{\eta}) \to \text{Aut}(\mathcal{E}[N]) \cong \text{GL}_2(\mathbb{Z}/N\mathbb{Z})
\]

where the isomorphism uses the basis \( \{P_1, P_2\} \) of Lemma 3.8. Taking the quotient by the group generated by \(-I\), we obtain a homomorphism

\[
\overline{\mathcal{G}}_N : \pi_1(U, \overline{\eta}) \to \text{GL}_2(\mathbb{Z}/N\mathbb{Z})/\{\pm I\}.
\]

Note that the representation \( \text{Gal}(K/Q(j)) \to \text{GL}_2(\mathbb{Z}/N\mathbb{Z})/\{\pm I\} \) coming from \( \overline{\mathcal{G}}_N \) factors through the homomorphism \( \overline{\rho}_N \). So by Proposition 3.1 and Lemma 3.9, the representation \( \overline{\mathcal{G}}_N \) is surjective and satisfies \( \overline{\mathcal{G}}_N(\pi_1(U, \overline{\eta})) = \text{SL}_2(\mathbb{Z}/N\mathbb{Z})/\{\pm I\} \).
Now take any subgroup $G$ of $GL_2(\mathbb{Z}/N\mathbb{Z})$ that satisfies $-I \in G$ and $\det(G) = (\mathbb{Z}/N\mathbb{Z})^\times$. Using $\overline{\mathbb{F}}_N$, the group $G/(\pm I)$ corresponds to an étale morphism $\pi: Y_G \to U$. The smooth projective closure of $Y_G$ is thus $X_G$ and the morphism $X_G \to \mathbb{P}^1_U$ arising from $\pi$ is simply $\pi_G$.

Take any rational point $u \in U(\mathbb{Q}) = \mathbb{Q} - \{0, 1728\}$. Viewed as a morphism $\text{Spec} \mathbb{Q} \to U$, the point $u$ induces a homomorphism $u_\ast: \text{Gal} \mathbb{Q} \to \pi_1(U)$; we are suppressing base points so everything is uniquely defined only up to conjugacy. Composing $u_\ast$ with $\overline{\mathbb{F}}_N$ we obtain a homomorphism $\beta_u: \text{Gal} \mathbb{Q} \to GL_2(\mathbb{Z}/N\mathbb{Z})/(\pm I)$. Observe that the group $\beta_u(\text{Gal} \mathbb{Q})$ is conjugate to a subgroup of $G/(\pm I)$ if and only if $u$ lies in $\pi_1(Y_G(\mathbb{Q})) = \pi_G(X_G(\mathbb{Q})) = \{0, 1728, \infty\}$.

The fiber of $\mathcal{E} \to U$ over $u$ is the elliptic curve $\mathcal{E}_u/\mathbb{Q}$ obtained by setting $j$ to $u$ in (3.2). Composing $\rho_{\mathcal{E}_u,N}$: $\text{Gal} \mathbb{Q} \to GL_2(\mathbb{Z}/N\mathbb{Z})$ with the quotient map $GL_2(\mathbb{Z}/N\mathbb{Z}) \to GL_2(\mathbb{Z}/N\mathbb{Z})/(\pm I)$ gives a homomorphism that agrees with $\beta_u$ up to conjugation. Since $-I \in G$, we find that $\rho_{\mathcal{E}_u,N}(\text{Gal} \mathbb{Q})$ is conjugate in $GL_2(\mathbb{Z}/N\mathbb{Z})$ to a subgroup of $G$ if and only if $u \in \pi_G(X_G(\mathbb{Q}))$.

Finally, let $E/\mathbb{Q}$ be any elliptic curve with $j$-invariant $u$. The curve $\mathcal{E}_u/\mathbb{Q}$ also has $j$-invariant $u$. As noted in the introduction, since $E$ and $\mathcal{E}_u$ are elliptic curves over $\mathbb{Q}$ with common $j$-invariant $u \notin \{0, 1728\}$, the groups $\pm \rho_{E,N}(Gal \mathbb{Q})$ and $\pm \rho_{\mathcal{E}_u,N}(Gal \mathbb{Q})$ must be conjugate. This completes the proof of Proposition 3.3.

4. Classification up to a sign

In this section, we prove the parts of the theorems of §1 that involve the groups $\pm \rho_{E,\ell}(Gal \mathbb{Q})$ for an elliptic curve $E/\mathbb{Q}$. In the notation of §2, the group $\pm \rho_{E,\ell}(Gal \mathbb{Q})$ is either applicable or is the full group $GL_2(\mathbb{F}_\ell)$. We consider the primes $\ell$ separately and keep the notation of the relevant subsection of §1.

One of the main tasks is to construct modular curves of genus 0. We will do this by finding functions $h \in \mathcal{F}_\ell - \mathbb{Q}(j)$ such that $j = J(h)$ for some $J \in \mathbb{Q}(t)$. Let $H$ be the subgroup of $GL_2(\mathbb{F}_\ell)$ consisting of elements that fix $h$ under the action from Proposition 3.1. By Lemma 3.4, the group $H$ is an applicable subgroup of $GL_2(\mathbb{F}_\ell)$. Furthermore, $X_H$ has function field $\mathbb{Q}(h)$ and the morphism $\pi_H: X_H \to \mathbb{P}^1_\mathbb{Q}$ is described by the inclusion $\mathbb{Q}(h) \supseteq \mathbb{Q}(j)$. So if $E/\mathbb{Q}$ is a non-CM elliptic curve, then $\rho_{E,1}(Gal \mathbb{Q})$ is conjugate to a subgroup of $H$ if and only if $j_E$ belongs to $\pi_H(X_H(\mathbb{Q})) = J(\mathbb{Q} \cup \{\infty\})$.

We will need to recognize $H$ as a conjugate of one of our applicable subgroups $G_i$ of $GL_2(\mathbb{F}_\ell)$. The degree of $\pi_H$, which is the same as the degree of $\pi(1)$, is equal to the index $[GL_2(\mathbb{F}_\ell) : H]$; this observation will immediately rule out most candidates. We will also make use of Proposition 3.3; observe that the set $\pi_H(X_H(\mathbb{Q}))$ depends only on the conjugacy class of $H$.

Most of this section involves basic algebraic verifications (which are straightforward to check with a computer, see the link in §1.10 for many such details); much of the work, which we will not touch on, is finding the various equations in the first place.

4.1. $\ell = 2$. Fix notation as in §1.1. Up to conjugacy, $G_1$, $G_2$ and $G_3$ are the proper subgroups of $GL_2(\mathbb{F}_2)$.

- Define the function
  \[ h_1(\tau) := 16\eta(2\tau)^8/\eta(\tau/2)^8 = 16(q^{1/2} + 8q + 44q^{3/2} + 192q^2 + 718q^{5/2} + \cdots) \]

  By Lemmas 3.6 and 3.7(i), we have $Q(X_s(2)) = Q(h_1)$. We have $C_4(2) = G_1$, so $Q(X_{C_4}) = Q(h_1)$. The extension $Q(h_1)/Q(j)$ has degree 6, so there is a unique rational function $J(t) \in Q(t)$ such that $j = J(h_1)$. We have $J(t) = f_1(t)/f_2(t)$ for relatively prime $f_1, f_2 \in \mathbb{Q}[t]$ of degree at most 6. Expanding the $q$-expansion of $j f_2(h_1) - f(h_1) = 0$ gives many linear equations in the coefficients of $f_1$ and $f_2$. Using enough terms of the $q$-expansion, we can compute the coefficients of $f_1$ and $f_2$ (they are unique up to scaling $f_1$ and $f_2$ by some constant in $\mathbb{Q}^\times$). Doing this, we found that $J_1(h_1) = j$.

- Define $h_2 := h_1^2/(h_1 + 1)$. Since $J_2(t^2/(t+1)) = J_1(t)$, we have $J_2(h_2) = j$. 

Define $h_3 := F(h_1)$ where $F(t) = (-16t^3 - 24t^2 + 24t + 16)/(t^2 + t)$. Since $J_3(F(t)) = J_1(t)$, we have $J_3(h_3) = j$.

For each integer $1 \leq i \leq 3$, let $H_i$ be the subgroup of $GL_2(\mathbb{F}_2)$ that fixes $h_i$. By Lemma 3.4, $H_i$ is an applicable subgroup of $GL_2(\mathbb{F}_2)$ with index equal to the degree of $J_i(t)$. By comparing the degree of $J_i(t)$ with our list of proper subgroups, we deduce that $H_i$ is conjugate to $G_i$ in $GL_2(\mathbb{F}_2)$.

Theorem 1.1 now follows from Lemma 3.4(iii); we can ignore $t = \infty$ since $J_i(\infty) = \infty$.

4.2. $\ell = 3$. Fix notation as in §1.2. Up to conjugacy, the groups $G_i$ with $1 \leq i \leq 4$ are the applicable subgroups of $GL_2(\mathbb{F}_3)$.

Define the function $h_1 := 1/3 \cdot \eta(\tau)^2/\eta(3\tau)^3$. By Lemmas 3.6 and 3.7(ii), we have $\mathbb{Q}(X_4(3)) = \mathbb{Q}(h_1)$. We have $C_3(3) = G_1$, so $\mathbb{Q}(X_{G_1}) = \mathbb{Q}(h_1)$. The extension $\mathbb{Q}(h_1)/\mathbb{Q}(j)$ has degree 12, so there is a unique rational function $J(t) \in \mathbb{Q}(t)$ such that $j = J(h_1)$. We have $J(t) = f_1(t)/f_2(t)$ for relatively prime $f_1, f_2 \in \mathbb{Q}[t]$ of degree at most 12. Expanding the $q$-expansion of $Jf_2(h_1) = J(h_1)f_2(h_2) - f_1(h_1) = 0$ gives many linear equations in the coefficients of $f_1$ and $f_2$. Using enough terms of the $q$-expansion, we can compute the coefficients of $f_1$ and $f_2$ (they are unique up to scaling $f_1$ and $f_2$ by some constant in $\mathbb{Q}^\times$). Doing this, we found that $J_1(h_1) = j$.

Define $h_2 = F_1(h_1)$ where $F_1(t) = (t^2 + 3t + 3)/t$. Since $J_2(F_1(t)) = J_1(t)$, we have $J_2(h_2) = j$.

Define $h_3 = F_2(h_1)$ where $F_2(t) = t(t^2 + 3t + 3)$. Since $J_3(F_2(t)) = J_1(t)$, we have $J_3(h_3) = j$.

Define $h_4 = F_3(h_2)$ where $F_3(t) = 3(t + 1)(t - 3)/t$. Since $J_4(F_3(t)) = J_2(t)$, we have $J_4(h_4) = j$.

Fix an integer $1 \leq i \leq 4$, and let $H_i$ be the subgroup of $GL_2(\mathbb{F}_3)$ that fixes $h_i$. By Lemma 3.4, we find that $H_i$ is an applicable subgroup and the morphism $\pi_{H_i} : X_{H_i} \to \mathbb{P}_1^1$ is described by the rational function $J_i(t)$. The index $[GL_2(\mathbb{F}_3) : H_i]$ agrees with the degree of $J_i(t)$. By comparing the degree of $J_i(t)$ with our list of applicable subgroups, we deduce that $H_i$ is conjugate to $G_i$ in $GL_2(\mathbb{F}_3)$.

Theorem 1.2(ii) now follows from Lemma 3.4(iii); we can ignore $t = \infty$ since $J_i(\infty) = \infty$. A computation shows that if $H$ is a proper subgroup of $G_i$ satisfying $\pm H = G_i$, then $i \in \{1, 3\}$ and $H$ is one of the groups $H_{i,j}$; this proves Theorem 1.2(i).

4.3. $\ell = 5$. Fix notation as in §1.3. Up to conjugacy, the applicable subgroups of $GL_2(\mathbb{F}_5)$ are the groups $G_i$ with $1 \leq i \leq 9$. Recall that the Rodgers-Ramanujan continued fraction is

$$r(\tau) := q^{1/5} \cdot \frac{1}{1 + \frac{q}{1 + \frac{q^2}{1 + \frac{q^3}{1 + \frac{q^4}{1 + \cdots}}}}}. $$

The function

$$h_1(\tau) := 1/r(\tau) = q^{-1/5}(1 + q - q^3 + q^5 + q^6 - q^7 - 2q^8 + 2q^{10} + 2q^{11} + \cdots)$$

is a modular function of level 5 and satisfies $J_1(h_1) = j$; we refer to Duke [Duk05] for an excellent exposition. An expression for $h_1(\tau)$ in terms of Klein forms can be found in [CC06].

Set $w := (1 + \sqrt{5})/2 \in \mathbb{Q}(\zeta_5)$.

- Define the function $h_2 = h_1 - 1 - 1/h_1$. We have $J_2(t - 1 - 1/t) = J_1(t)$, so $J_2(h_2) = j$. (As noted in equation (7.2) of [Duk05], $h_2$ equals $\eta(\tau)/\eta(5\tau)$.)
- Define $h_3 = F_1(h_2)$ where

$$F_1(t) = \frac{(-3 + w)t - 5}{t + (3 - w)}.$$  

We have $J_3(F_1(t)) = J_2(t)$ and hence $J_3(h_3) = j$.

- Define $h_4 = h_2 + 5/h_2$. We have $J_4(t + 5/t) = J_2(t)$ and hence $J_4(h_4) = j$.  

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Define \( h_5 = h_5^5 \). We have \( J_5(t^5) = J_1(t) \) and hence \( J_5(h_5) = j \).

Define \( h_6 = F_2(h_5) \) where
\[
F_2(t) = \frac{-(w-1)^5t+1}{t+(w-1)^5}.
\]

We have \( J_6(F_2(t)) = J_5(t) \) and hence \( J_6(h_6) = j \). (In the notation of [Duk05, §8], we have \( b = h_6 \).)

Define \( h_7 = F_3(h_3) \) where
\[
F_3(t) = -\frac{t^3 + 10t^2 + 25t + 25}{2t^3 + 10t^2 + 25t + 25}.
\]

We have \( J_7(F_3(t)) = J_3(t) \) and hence \( J_7(h_7) = j \).

Define \( h_8 = h_5 - 11 - h_5^{-1} \). We have \( J_8(t - 11 - t^{-1}) = J_5(t) \) and hence \( J_8(h_8) = j \). (As noted in equation (7.7) of [Duk05], \( h_8 \) equals \((\eta(\tau)/\eta(5\tau))^6\).)

Define \( h_9 = F_4(h_4) \) where
\[
F_4(t) = \frac{(t + 5)(t^2 - 5)}{t^2 + 5t + 5}.
\]

We have \( J_9(F_4(t)) = J_4(t) \) and hence \( J_9(h_9) = j \).

Fix an integer \( 1 \leq i \leq 9 \). Let \( H_i \) be the subgroup of \( \text{GL}_2(\mathbb{F}_5) \) that fixes \( h_i \). By Lemma 3.4, we find that \( H_i \) is an applicable subgroup and the morphism \( \pi_{H_i} : X_{H_i} \to \mathbb{P}_Q^1 \) is described by the rational function \( J_i(t) \).

**Lemma 4.1.** The groups \( H_i \) and \( G_i \) are conjugate in \( \text{GL}_2(\mathbb{F}_5) \) for each \( 1 \leq i \leq 9 \).

**Proof.** The index \([\text{GL}_2(\mathbb{F}_5) : H_i]\) agrees with the degree of \( J_i(t) \). By comparing the degree of \( J_i(t) \) with our list of applicable subgroups, we deduce that \( H_i \) is conjugate to \( G_i \) in \( \text{GL}_2(\mathbb{F}_5) \) for all \( i \in \{1, 4, 7, 8, 9\} \).

The groups \( H_5 \) and \( H_6 \) are not conjugate since one can check that the image of \( \mathbb{P}^1(\mathbb{Q}) = \mathbb{Q} \cup \{\infty\} \) under \( J_5(t) \) and \( J_6(t) \) are different. The groups \( H_5 \) and \( H_6 \) have index 12 in \( \text{GL}_2(\mathbb{F}_5) \) and are not conjugate, so they are conjugate to \( G_5 \) and \( G_6 \) (though we need to determine which is which). The elliptic curve given by the Weierstrass equation \( y^2 + (1 - t)xy - ty = x^3 - tx^2 \) has \( j \)-invariant \( J_6(t) \) and the point \((0,0)\) has order 5. Therefore, \( H_6 \) is conjugate to \( G_6 \) and thus \( H_5 \) is conjugate to \( G_5 \).

The groups \( H_2 \) and \( H_3 \) are not conjugate since one can check that the image of \( \mathbb{P}^1(\mathbb{Q}) = \mathbb{Q} \cup \{\infty\} \) under \( J_2(t) \) and \( J_3(t) \) are different. The groups \( H_2 \) and \( H_3 \) have index 30 in \( \text{GL}_2(\mathbb{F}_5) \) and are not conjugate, so they are conjugate to \( G_2 \) and \( G_3 \) (though we need to determine which is which). Since \( h_7 = F_3(h_3) \) and \( F_3(t) \) belongs to \( \mathbb{Q}(t) \), we find that \( H_3 \) is a subgroup of \( H_7 \). We already know that \( H_7 \) is conjugate to \( N_{ns}(5) \) and one can check that \( G_2 = C_4(5) \) is not conjugate to a subgroup of \( N_{ns}(5) \). Therefore, \( H_3 \) is conjugate to \( G_3 \) and thus \( H_2 \) is conjugate to \( G_2 \).

Theorem 1.4(ii) now follows from Lemma 3.4(iii); we have \( J_i(\infty) = \infty \) for \( i \notin \{3, 7\} \) and we can ignore the values \( J_3(\infty) = 0 \) and \( J_7(\infty) = 8000 \) since they are the \( j \)-invariants of CM elliptic curves. A direct computation shows that if \( H \) is a proper subgroup of \( G_i \) satisfying \( \pm H = G_i \), then \( i \in \{1, 5, 6\} \) and \( H \) is one of the groups \( H_{i,j} \); this proves Theorem 1.4(i).

4.4. \( \ell = 7 \). Fix notation as in §1.4. Up to conjugacy, the applicable subgroups of \( \text{GL}_2(\mathbb{F}_7) \) are the groups \( G_i \) with \( 1 \leq i \leq 7 \) from §1.4 and the groups:

- Let \( G_8 \) be the subgroup of \( \text{GL}_2(\mathbb{F}_7) \) consisting of matrices of the form \( \pm \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \).
- Let \( G_9 \) be the subgroup of \( \text{GL}_2(\mathbb{F}_7) \) consisting of matrices of the form \( \begin{pmatrix} 0 & 0 \\ 0 & \pm a \end{pmatrix} \).
- Let \( G_{10} \) be the subgroup of \( \text{GL}_2(\mathbb{F}_7) \) generated by \( \begin{pmatrix} 0 & -2 \\ 2 & 0 \end{pmatrix} \) and \( \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \).
- Let \( G_{11} \) be the subgroup \( C_8(7) \) of \( \text{GL}_2(\mathbb{F}_7) \).
• Let $G_{12}$ be the subgroup of $\text{GL}_2(\mathbb{F}_7)$ generated by \(egin{pmatrix} 1 & 1 \\ -1 & 1 \end{pmatrix}\) and \(egin{pmatrix} 1 & 0 \\ 1 & 1 \end{pmatrix}\).

For $i = 8, 9, 10, 11$ and 12, the index $[\text{GL}_2(\mathbb{F}_7) : G_i]$ is 168, 168, 84, 56 and 42, respectively.

The Klein quartic is the curve $\mathcal{X}$ in $\mathbb{P}^3_{\mathbb{Q}}$ defined by the equation $x^3y + y^3z + z^3x = 0$; it is a non-singular curve of genus 3. The relevance to us is that $\mathcal{X}$ is isomorphic to the modular curve $X(7) := X_{G_5}$; we refer to Elkies [Elk99] for a lucid exposition. In §4 of [Elk99], Elkies defines a convenient basis $x, y$ and $z$ for the space of cusp forms of $\Gamma(7)$ which satisfy the equation of the Klein quartic and have product expansions

\[ x, y, z = \varepsilon q^{a/7} \prod_{n=1}^{\infty} (1 - q^n)^3 (1 - q^7^n) \prod_{n=0, n \equiv 0 \mod 7} (1 - q^n) \]

where $(\varepsilon, a, n_0)$ is $(-1, 4, 1), (1, 2, 2)$ or $(1, 1, 4)$ for $x, y$ or $z$, respectively. The coordinates $(x : y : z)$ then give the desired isomorphism $X(7) \to \mathcal{X}$.

Define

\[ h_4 := -(y^2z)/x^3 = q^{-1} + 3 + 4q + 3q^2 - 5q^4 - 7q^5 + \ldots; \]

it is a modular function of level 7. Define $h_7 := F_1(h_4)$ where

\[ F_1(t) = t + \frac{1}{1 - t} + \frac{t - 1}{t - 8}. \]

From equations (4.20) and (4.24) of [Elk99], with a correction in the sign of (4.23) of loc. cit., we have $J_7(h_7) = j$. Since $J_7(F_1(t)) = J_4(t)$, we have $J_4(h_4) = j$.

Define $h_3 := F_2(h_4)$ and $h_5 := F_3(h_4)$, where

\[ F_2(t) = \frac{\beta t - (\beta - 1)}{t - \beta} \quad \text{and} \quad F_3(t) = \frac{t - \gamma}{\gamma t - (\gamma - 1)} \]

with $\beta = 4 + 3\zeta_7 + 3\zeta_7^{-1} + \zeta_7^2 + \zeta_7^{-2}$ and $\gamma = \zeta_7^3 + \zeta_7^4 + \zeta_7^8 + \zeta_7^{-2} + 1$. Since $J_3(F_2(t)) = J_4(t)$ and $J_5(F_3(t)) = J_4(t)$, we have $J_3(h_3) = j$ and $J_5(h_5) = j$.

For $i \in \{3, 4, 5, 7\}$, let $H_i$ be the subgroup of $\text{GL}_2(\mathbb{F}_7)$ that fixes $h_i$. We have shown that $J_7(h_i) = j$. By Lemma 3.4, we find that $H_i$ is an applicable subgroup and that the morphism $\pi_{H_i} : X_{H_i} \to \mathbb{P}^1_{\mathbb{Q}}$ is described by the rational function $J_i(t)$.

**Lemma 4.2.** The groups $H_i$ and $G_i$ are conjugate in $\text{GL}_2(\mathbb{F}_7)$ for all $i \in \{3, 4, 5, 7\}$.

**Proof.** The index of $H_i$ in $\text{GL}_2(\mathbb{F}_7)$ agrees with the degree of $J_i(t)$ which is 24 or 8 if $i \in \{3, 4, 5\}$ or $i = 7$, respectively. By our list of applicable subgroups, we deduce that $H_7$ is conjugate to $G_7$ in $\text{GL}_2(\mathbb{F}_7)$. The groups $H_3, H_4$ and $H_5$ are not conjugate in $\text{GL}_2(\mathbb{F}_7)$ (since one can show that the images of $\mathbb{P}^1(\mathbb{Q}) = \mathbb{Q} \cup \{\infty\}$ under $J_3, J_4$ and $J_5$ are pairwise distinct). By our list of applicable subgroups, the groups $H_3, H_4$ and $H_5$ are conjugate to the three subgroups $G_3, G_4$ and $G_5$; we still need to identify $H_3$ with $G_3$, etc.

The modular function $h_4 \in \mathcal{F}_7$ is a Laurent series in $q$ and has rational coefficients. Using Proposition 3.1, this implies that $H_4$ contains the group of matrices of the form $\pm \begin{pmatrix} 1 & 0 \\ * & 1 \end{pmatrix}$ in $\text{GL}_2(\mathbb{F}_7)$. Therefore, $H_4$ must be conjugate to $G_4$ in $\text{GL}_2(\mathbb{F}_7)$. The elliptic curve given by the Weierstrass equation $y^2 + (1 + t - t^2)xy + (t^2 - t^3)y = x^3 + (t^2 - t^3)x^2$ has $j$-invariant $J_3(t)$ and the point $(0, 0)$ has order 7, so $H_3$ is conjugate to $G_3$. Therefore, $H_5$ is conjugate to $G_5$. \( \Box \)

Following Elkies ([Elk99, p.68]), we multiply the equation of the Klein curve to obtain $(x^3y + y^3z + z^3x)(x^3z + z^3y + y^3x) = 0$. Noting that the left hand side is a symmetric polynomial in $x, y$ and
z, one can show that \(s_2^3 + s_3(s_1^2 - 5s_1s_2 + s_1s_2^2 + 7s_1^2s_3) = 0\) where \(s_1 = x + y + z\), \(s_2 = xy + yz + zx\) and \(s_3 = xyz\). We now deviate from Elkies’ treatment. Divide by \(s_1^2s_3^2\) and rearrange to obtain
\[
\left(\frac{s_2}{s_1s_3}\right)^2 + \left(\frac{s_1}{s_2}\right)^2 - \frac{s_2}{s_1s_3} + \frac{s_2^2}{s_1s_3} + 7 = 0.
\]
We thus have \(v^2 + (h_2^2 - 5h_2 + 1)v + 7 = 0\), where
\[
h_2 := s_2^2/s_2 = q^{1/7} + 2 + 2q^{1/7} + 2q^{2/7} + 2q^{3/7} + 3q^{4/7} + 4q^{5/7} + 5q^{6/7} + 7q + 8q^{8/7} + \cdots
\]
and \(v := s_2^2/(s_1s_3)\) are modular functions. We claim that
\[
(4.1) \quad h_7 + (h_2^3 - 4h_2^2 + 3h_2 + 1)((h_2^2 - 5h_2 + 1)v + 7) = 0.
\]
This can be verified algebraically: In the left-hand side of (4.1), replace \(h_7\) by \(F_1(-y^2z/x^3)\), \(h_2\) by \((x + y + z)^2/xyz + xx\), and \(v\) by \((yz + xz)^2/((x + y + z)xyz)\); the numerator of the resulting rational function is divisible by \(xyz + yz^3 + xz^3\).

Completing the square in the equation \(v^2 + (h_2^2 - 5h_2 + 1)v + 7 = 0\), we have
\[
(4.2) \quad w^2 = h_2^4 - 10h_2^3 + 27h_2^2 - 10h_2 - 27,
\]
where \(w = 2v + (h_2^2 - 5h_2 + 1)\). From (4.1), we find that
\[
(4.3) \quad h_7 = \frac{1}{2}(h_2^3 - 4h_2^2 + 3h_2 + 1)((h_2^2 - 10h_2^3 + 27h_2^2 - 10h_2 - 13) - (h_2^2 - 5h_2 + 1)w).
\]
We have \(j = J_7(h_7)\), so (4.2) and (4.3) imply that \(j\) can be written in the form \(\alpha(h_2) + \beta(h_2)w\) for rational functions \(\alpha(t)\) and \(\beta(t)\). A direct computation shows that \(\alpha(t) = J_2(t)\) and \(\beta(t) = 0\), and hence \(J_2(h_2) = j\).

Let \(H_2\) be the subgroup of \(GL_2(\mathbb{F}_7)\) that fixes \(h_2\). We have \(J_2(h_2) = j\), so Lemma 3.4 implies that \(H_2\) is an applicable subgroup and that the morphism \(\pi_{H_2}: X_{H_2} \to \mathbb{P}^1_Q\) is described by the rational function \(J_2(t)\). The index of \(H_2\) in \(GL_2(\mathbb{F}_7)\) is 28 since it agrees with the degree of \(J_2(t)\). By our list of applicable subgroups, we deduce that \(H_2\) is conjugate to \(G_2\) in \(GL_2(\mathbb{F}_7)\).

Let \(H_{11}\) be the subgroup of \(GL_2(\mathbb{F}_7)\) that fixes \(h_2\) and \(w\). The group \(H_{11}\) is an index 2 subgroup of \(H_2\) since the extension \(\mathbb{Q}(h_2, w)/\mathbb{Q}(h_2)\) has degree 2. The group \(H_{11}\) contains \(G_8\) since \(\mathbb{Q}(h_2, w)\) is contained in \(\mathbb{Q}(x/z, y/z)\) which is the function field of \(X(7)\); in particular, \(H_{11}\) is applicable. From our classification of applicable subgroups, we find that \(H_{11}\) is conjugate to \(G_{11}\). The modular curve \(X_{G_{11}}\) thus has function field \(\mathbb{Q}(h_2, w)\) and is hence isomorphic to the smooth projective curve over \(\mathbb{Q}\) with affine model
\[
(4.4) \quad y^2 = x^4 - 10x^3 + 27x^2 - 10x - 27.
\]
The only rational points for the smooth model of (4.4) are the two points at infinity (one can show that it is isomorphic to the quadratic twist by \(-7\) of the curve \(E_{7,1}\) from §1.9, and that this curve has only two rational points). Using that \(J_{2}(\infty) = \infty\), we find that the two rational points of \(X_{H_{11}}\), and hence of \(X_{G_{11}}\), are cusps. Therefore, there is no non-CM elliptic curve \(E/\mathbb{Q}\) for which \(\rho_{E,7}(\text{Gal}_{\mathbb{Q}})\) is conjugate to a subgroup of \(G_{11}\); the same holds for the group \(G_8\) since \(G_8 \subseteq G_{11}\).

Now consider the subfield \(K := \mathbb{Q}(h_2, w/\sqrt{-7})\) of \(\mathbb{F}_7\). Let \(H_1\) be the subgroup of \(GL_2(\mathbb{F}_7)\) that fixes \(K\). From the inclusions \(K \supseteq \mathbb{Q}(h_2) \supseteq \mathbb{Q}(j)\) and (4.2), we find that \(K\) is the function field of the geometrically irreducible curve
\[
(4.5) \quad -7y^2 = x^4 - 10x^3 + 27x^2 - 10x - 27
\]
defined over \(\mathbb{Q}\) (with \((x, y) = (h_2, w/\sqrt{-7})\)). The curve \(X_{H_1}\) is defined over \(\mathbb{Q}\) since \(\mathbb{Q}\) is algebraically closed in \(K\). The only rational points of the smooth projective model of (4.5) are \((x, y) = (5/2, \pm 1/4)\) (one can show that it is isomorphic to the curve \(E_{7,1}\) from §1.9, and that this curve has only two rational points). These two rational points on \(X_{H_1}\) lie over the \(j\)-invariant
\[ J_2(5/2) = 3^3 \cdot 5 \cdot 7^5 / 2^7. \] This shows that for an elliptic curve \( E / \mathbb{Q} \), \( \rho_{E,7}(\text{Gal}_{\mathbb{Q}}) \) is conjugate to a subgroup of \( H_1 \) in GL\(_2(\mathbb{F}_7) \) if and only if \( j_E = 3^3 \cdot 5 \cdot 7^5 / 2^7. \) Since \( X_{H_1} \) has a rational point that is not a cusp, the group \( H_1 \) must be applicable and not conjugate to \( G_{11} \). The group \( H_1 \) is an index 2 subgroup of \( H_2 \) since \( [\mathbb{Q}(h_2, w/\sqrt{-7}) : \mathbb{Q}(h_2)] = 2. \) From our description of applicable groups, we deduce that \( H_1 \) is conjugate to \( G_1. \)

**Remark 4.3.** The rational points on \( X_{H_1} \) were first described by A. Sutherland in [Sut12]. An elliptic curve \( E / \mathbb{Q} \) with \( j \)-invariant \( 3^3 \cdot 5 \cdot 7^5 / 2^7 \) has the distinguished property of not having a 7-isogeny, yet its reduction at primes of good reduction all have a 7-isogeny.

From equation (4.35) of [Elk99], the modular curve \( X_{ns}^+(7) := X_{G_6} \) has function field of the form \( \mathbb{Q}(x) \) and the morphism down to the \( j \)-line is given by \( J_6(x) \); note that there is a small typo in the numerator of equation (4.35) of [Elk99] though the given expression for \( j - 1728 \) is correct.

**Lemma 4.4.** The rational points of the modular curve \( X_{G_{12}} \) are all CM.

**Proof.** The fiber in \( X_{ns}^+(7) \) over \( j = 1728 \) is the (non-reduced) subscheme given by

\[
(2x^4 - 14x^3 + 21x^2 + 28x + 7)(x - 3)((x^4 - 7x^3 + 14x^2 - 7x + 7)(x^4 - 14x^2 + 56x + 21))^2 = 0;
\]

this can be found by factoring \( J_6(x) - 1728. \) Define the modular curve \( X_{ns}(7) := X_{C_{ns}(7)}. \) One can show that the morphism \( X_{ns}(7) \to X_{ns}^+(7) \) is ramified at precisely four points lying over \( j = 1728. \) Since it is defined over \( \mathbb{Q}, \) these four ramification points are the ones given by \( 2x^4 - 14x^3 + 21x^2 + 28x + 7 = 0. \) Therefore, \( X_{ns}(7) \) is defined by an equation

\[
y^2 = c(2x^4 - 14x^3 + 21x^2 + 28x + 7)
\]

for some squarefree \( c \in \mathbb{Z}. \)

We claim that \( c = -1. \) Consider an elliptic curve \( E / \mathbb{Q} \) with \( j \)-invariant \(-2^{15}. \) The value \( x = 1 \) is the unique rational solution to \( J(x) = -2^{15}. \) Setting \( x = 1, \) we have \( y^2 = 44c. \) Therefore, \( K = \mathbb{Q}(\sqrt{11c}) \) is the unique quadratic extension of \( \mathbb{Q} \) for which \( \rho_{E,7}(\text{Gal}_K) \subseteq C_{ns}(7). \) Since \( j_E = -2^{15}, \)

the curve \( E \) has CM by \( \mathbb{Q}(\sqrt{-11}) \) and hence \( \rho_{E,7}(\text{Gal}_{\mathbb{Q}(\sqrt{-11})}) = C_{ns}(7) \) and \( \rho_{E,7}(\text{Gal}_{\mathbb{Q}}) = N_{ns}(7); \)

see §7. Therefore, \( K = \mathbb{Q}(\sqrt{-11}) \) and hence \( c = -1 \) as claimed. (The above argument comes from Schoof.)

Define the subfield \( L = \mathbb{Q}(x, v) \) of \( \mathbb{F}_7 \) where \( v := y/\sqrt{-7}; \) we have

\[
(4.6) \quad 7v^2 = 2x^4 - 14x^3 + 21x^2 + 28x + 7.
\]

Let \( G \) be the subgroup of \( \text{GL}_2(\mathbb{F}_7) \) that fixes \( L; \) it is an index 2 subgroup of \( G_6 = N_{ns}(11) \) since \( L/\mathbb{Q}(x) \) has degree 2. The field \( \mathbb{Q} \) is algebraically closed in \( L \) since \( L/\mathbb{Q}(x) \) is a geometric extension. Therefore, \( \det(G) = \mathbb{F}_7^2. \) There are only two index 2 subgroups of \( G_6 \) with full determinant; they are \( G_{12} \) and \( C_{ns}(7). \) The group \( G \) is thus \( G_{12} \) since \( C_{ns}(7) \) corresponds to the field \( \mathbb{Q}(x, y). \)

Therefore, \( X_{G_{12}} \) has function field \( \mathbb{Q}(x, v) \) with \( x \) and \( v \) related by (4.6). The smooth projective curve defined by (4.6) has genus 1 and a rational point \( (x, v) = (0, 1) \); it is thus an elliptic curve. A computation shows that this elliptic curve is isomorphic to the curve \( E_{7,2} \) of §1.9. The curve \( E_{7,2} \) has only two rational points, so \( (x, v) = (0, \pm 1) \) are the only rational points of the curve defined by (4.6). The lemma follows since \( J_6(0) = 0. \)

If \( E/\mathbb{Q} \) is a non-CM elliptic curve, Lemma 4.4 shows that \( \rho_{E,7}(\text{Gal}_{\mathbb{Q}}) \) is not conjugate to a subgroup of \( G_{12}. \) The same holds for \( G_9 \) and \( G_{10} \) since they are both subgroups of \( G_{12}. \)

Suppose that \( H \) is a proper subgroup of \( G_i \) satisfying \( \pm H = G_i \) for a fixed \( 1 \leq i \leq 7. \) If \( i \neq 1, \)

then \( i \in \{3, 4, 5, 7\} \) and \( H \) is one of the groups \( H_{1,j}. \) If \( i = 1, \) the \( H \) is either \( H_{1,1} \) or another subgroup that is conjugate to \( H_{1,1} \) in \( \text{GL}_2(\mathbb{F}_7). \) This completes the proof of Theorem 1.5(i) and
(ii); we can ignore \( t = \infty \) for \( 2 \leq i \leq 7 \) since \( J_i(\infty) \) is either \( \infty \) or the \( j \)-invariant of a CM elliptic curve.

4.5. \( \ell = 11 \). Fix notation as in §1.5. Up to conjugacy, the group \( \text{GL}_2(\mathbb{F}_{11}) \) has four maximal applicable subgroups: \( B(11), N_s(11), N_{ns}(11) \) and a group \( H_{\mathbb{S}_4} \) whose image in \( \text{PGL}_2(\mathbb{F}_{11}) \) is isomorphic to \( \mathbb{S}_4 \).

4.5.1. Exceptional case. The curve \( X_{H_{\mathbb{S}_4}}(11) := X_{H_{\mathbb{S}_4}} \) has no rational points corresponding to a non-CM elliptic curve; it is isomorphic to an elliptic curve which has only one rational point \( \text{Lig77, Prop. 4.4.8.1} \) and this point corresponds to an elliptic curve with CM by \( \sqrt{-3} \).

4.5.2. Split case. The curve \( X_{N_s}(11) := X_{N_s(11)} \) has no rational points corresponding to a non-CM elliptic curve; see [BPR11] for a more general result. Therefore, there are no non-CM elliptic curves \( E/Q \) such that \( \rho_{E,\ell}(\text{Gal}_Q) \) is conjugate to a subgroup of \( N_s(11) \).

4.5.3. Non-split case. The modular curve \( X_{N_{ns}}(11) := X_{G_3} = X_{N_{ns}(11)} \) has genus 1. Halberstadt [Hal98] showed that the function field of \( X_{N_{ns}}(11) \) is of the form \( K := \mathbb{Q}(x, y) \) with \( y^2 + y = x^3 - x^2 - 7x + 10 \) such that the inclusion \( \mathbb{Q}(j) \subseteq \mathbb{Q}(x, y) \) is given by \( j = J(x, y) \). Therefore, if \( E/Q \) is a non-CM elliptic curve, then \( \rho_{E,11}(\text{Gal}_Q) \) is conjugate to a subgroup of \( N_{ns}(11) \) if and only if \( j_E = J(P) \) for some point \( P \in \mathcal{E}(\mathbb{Q}) \). We only need consider \( P \neq O \) since, as noted in [Hal98], \( J(O) \) is the \( j \)-invariant of a CM elliptic curve.

Let \( G_4 \) be the subgroup of \( G_3 \) consisting of \( g \in G_3 = N_{ns}(11) \) such that \( g \in C_{ns}(11) \) and \( \text{det}(g) \in (\mathbb{F}_{11}^\times)^2 \), or \( g \notin C_{ns}(11) \) and \( \text{det}(g) \notin (\mathbb{F}_{11}^\times)^2 \).

Lemma 4.5. The modular curve \( X_{G_4} \) has no rational points.

Proof. Define the modular curve \( X_{ns}(11) := X_{C_{ns}(11)} \). Proposition 1 of [DFGS14] shows that \( X_{ns}(11) \) can be defined by the equations \( y^2 + y = x^3 - x^2 - 7x + 10 \) and \( u^2 = -(4x^2 + 7x^2 - 6x + 19) \), where \( K = \mathbb{Q}(x, y) \).

Define the field \( L := K(v) \) with \( v = u/\sqrt{-11} \). We have \( L \subseteq \mathbb{F}_{11} \) since \( \sqrt{-11} \in \mathbb{Q}(\zeta_{11}) \). Let \( G \) be the subgroup of \( \text{GL}_2(\mathbb{F}_{11}) \) that fixes \( L \); it is an index 2 subgroup of \( G_3 \) since \( L/K \) has degree 2. The field \( \mathbb{Q} \) is algebraically closed in \( L \) since it is algebraically closed in \( K \) and \( L/K \) is a geometric extension. Therefore, \( \text{det}(G) = \mathbb{F}_{11}^\times \). There are only two index 2 subgroups of \( G_3 \) with full determinant; they are \( G_4 \) and \( C_{ns}(11) \). The group \( G \) is thus \( G_4 \) since \( C_{ns}(11) \) corresponds to the field \( K(u) \).

Therefore, \( X_{G_4} \) has function field \( \mathbb{Q}(x, y, v) \) where \( y^2 + y = x^3 - x^2 - 7x + 10 \) and \( v^2 = 11(4x^3 + 7x^2 - 6x + 19) \). We now homogenize our equations:

\[
y^2z + yz^2 = x^3 - x^2 - 7xz^2 + 10z^3, \quad 11v^2z = (4x^3 + 7x^2z - 6xz^2 + 19z^3).
\]

Combining the two equations (4.7) to remove the \( x^3 \) term, we find that \( 11v^2z = (4y^2z + 4yz^2 + 11x^2z + 22xz^2 - 21z^3) \). Factoring off \( z \), we deduce that the following equations give a model of \( X_{G_4} \) in \( \mathbb{P}_{\mathbb{Q}}^3 \):

\[
y^2z + yz^2 = x^3 - x^2 - 7xz^2 + 10z^3, \quad 11v^2 = (4y^2 + 4yz + 11x^2 + 22xz - 21z^2).
\]

Suppose \( (x, y, z, v) \in \mathbb{P}_{\mathbb{Q}}^3 \) is a solution to (4.8). If \( z = 0 \), then we have \( 0 = x^3 \) and \( 11v^2 = 4y^2 \), which is impossible since 44 is not a square in \( \mathbb{Q} \). So assume that \( z = 1 \). We can then recover the equation \( v^2 = 11(4x^3 + 7x^2 - 6x + 19) \) which has no solutions \( (x, v) \in \mathbb{Q}^2 \); it defines an elliptic curve and a computation shows that its only rational point is the point at \( \infty \). Therefore, \( X_{G_4}(\mathbb{Q}) = \emptyset \). \( \square \)

Let \( E/Q \) be a non-CM elliptic curve for which \( \rho_{E,11}(\text{Gal}_Q) \) is conjugate to a subgroup of \( G_3 \). Suppose that \( \rho_{E,11}(\text{Gal}_Q) \) is conjugate to a subgroup of \( G_3 \). The group \( G_3 \) has no index 2 subgroups \( H \) that satisfy \( \pm H = G_3 \). Therefore, \( \rho_{E,11}(\text{Gal}_Q) \) is conjugate to a subgroup of a maximal applicable subgroup of \( G_3 \). Up to conjugacy, there are two maximal applicable subgroups of \( G_3 \); one is \( G_4 \) and
the other is a subgroup $G_5$ of index 3 in $G_3$. The image \( \overline{G_5} \) of $G_5$ in PGL$_2(\mathbb{F}_{11})$ has order 8 and is hence a 2-Sylow subgroup of PGL$_2(\mathbb{F}_{11})$. Therefore, $G_5$ lies in a subgroup of PGL$_2(\mathbb{F}_{11})$ that is isomorphic to \( \mathfrak{S}_4 \) and hence $G_5$ is conjugate to a subgroup of $H_{\mathfrak{S}_4}$. However, we saw in §4.5.1 that \( \rho_{E,11}(\text{Gal}_Q) \) cannot be conjugate to a subgroup of $H_{\mathfrak{S}_4}$. This implies that \( \rho_{E,11}(\text{Gal}_Q) \) is conjugate to a subgroup of $G_4$ which is impossible by Lemma 4.5. Therefore, \( \rho_{E,11}(\text{Gal}_Q) \) must be conjugate to $G_3$.

4.5.4. Borel case. The modular curve $X_{B(11)}$ is known to have exactly three rational points that are not cusps; they lie above the $j$-invariants $-2^{15}$, $-11^2$ and $-11 \cdot 31^3$, cf. [BK75, p. 79]. An elliptic curve with $j$-invariant $-2^{15}$ has CM, so we need only consider the other two.

Consider the elliptic curve $E/\mathbb{Q}$ defined by $y^2 + xy + y = x^3 + x^2 - 305x + 7888$; it has $j$-invariant $-11^2$ and conductor $11^2$. The division polynomial at 11 of $E$ factors as the product of the irreducible polynomial $f(x) = x^5 - 129x^4 + 800x^3 + 81847x^2 - 421871x - 4132831$ and an irreducible polynomial $g(x)$ of degree 55. Since 11 divides the degree of $g(x)$, we find that \( \rho_{E,11}(\text{Gal}_Q) \) contains an element of order 11. Therefore, there are unique characters $\chi_1, \chi_2$: \( \text{Gal}_Q \to F_{11}^\times \) such that with respect to an appropriate change of basis we have

$$\rho_{E,11}(\sigma) = \left( \begin{array}{cc} \chi_1(\sigma) & 0 \\ 0 & \chi_2(\sigma) \end{array} \right).$$

We have $\chi_1 \chi_2 = \omega$ where \( \omega : \text{Gal}_Q \to F_{11}^\times \) is the character describing the Galois action on the 11-th roots of unity (we have $\omega(p) \equiv p$ (mod 11) for primes $p \neq 11$). The characters $\chi_1$ and $\chi_2$ are unramified at primes \( p \nmid 11 \), so $\chi_1 = \omega^a$ and $\chi_2 = \omega^{11-a}$ for a unique integer $0 \leq a < 10$. Let $w \in \overline{\mathbb{Q}}$ be a fixed root of $f(x)$. One can show that

$$P = (w, -(w^4 - 79w^3 - 3150w^2 + 12193w + 1520110)/11^4)$$

is an 11-torsion point of $E(\overline{\mathbb{Q}})$. The field $\mathbb{Q}(w)$ is a Galois extension of $\mathbb{Q}$ and that the group generated by $P$ is stable under the action of $\text{Gal}_Q$. We thus have $\sigma(P) = \chi_1(\sigma) \cdot P$ for all $\sigma \in \text{Gal}_Q$, and hence $\chi_1(\text{Gal}_Q)$ is a group of order $[\mathbb{Q}(w) : \mathbb{Q}] = 5$.

We have $a_2(E) = -1$, so the roots of the polynomial $\det(xI - \rho_{E,11}(\text{Frob}_2)) \equiv x^2 - (-1)x + 2$ (mod 11) are $4 = 2^2$ and $6 \equiv 2^9$ (mod 11). Since $\chi_1(\text{Frob}_2) \equiv 2^a$ and $\chi_2(\text{Frob}_2) \equiv 2^{11-a}$ are the roots of $\det(xI - \rho_{E,11}(\text{Frob}_2))$ and 2 is a primitive root modulo 11, we have $a \in \{2, 9\}$ and hence $\{\chi_1, \chi_2\} = \{\omega^2, \omega^9\}$. Since $\chi_1(\text{Gal}_Q)$ has cardinality 5, we have $\chi_1 = \omega^2$ and $\chi_2 = \omega^9$. Since 2 is a primitive root modulo 11, the group $\rho_{E,11}(\text{Gal}_Q)$ is generated by $\left( \begin{array}{cc} 2^a & 0 \\ 0 & 2^9 \end{array} \right) = \left( \begin{array}{cc} 4 & 0 \\ 0 & 4 \end{array} \right)$ and $(1 0 1)$, i.e., it equals $H_{11,1}$. In particular, $\pm \rho_{E,11}(\text{Gal}_Q) = G_1$.

Consider the elliptic curve $E/\mathbb{Q}$ defined by $y^2 + xy + y = x^3 + x^2 - 3632x + 82757$; it has $j$-invariant $-11 \cdot 31^3$ and conductor $11^2$. The division polynomial at 11 of $E$ factors as the product of the irreducible polynomial $f(x) = x^5 - 129x^4 + 4793x^3 + 9973x^2 - 3694800x + 52660939$ and an irreducible polynomial $g(x)$ of degree 55. Since 11 divides the degree of $g(x)$, we find that $\rho_{E,11}(\text{Gal}_Q)$ contains an element of order 11. Therefore, there are unique characters $\chi_1, \chi_2$: \( \text{Gal}_Q \to F_{11}^\times \) such that with respect to an appropriate change of basis we have (4.9). The characters $\chi_1$ and $\chi_2$ are unramified at primes \( p \nmid 11 \) and $\chi_1 \chi_2 = \omega$, so $\chi_1 = \omega^a$ and $\chi_2 = \omega^{11-a}$ for a unique integer $a \in \{0, 1, \ldots, 9\}$. Let $w \in \overline{\mathbb{Q}}$ be a fixed root of $f(x)$. One can show that

$$P = (w, (w^4 - 79w^3 + 843w^2 + 45468w - 722625)/11^3)$$

is an 11-torsion point of $E(\overline{\mathbb{Q}})$. The field $\mathbb{Q}(w)$ is a Galois extension of $\mathbb{Q}$ and that the group generated by $P$ is stable under the action of $\text{Gal}_Q$. We thus have $\sigma(P) = \chi_1(\sigma) \cdot P$ for all $\sigma \in \text{Gal}_Q$, and hence $\chi_1(\text{Gal}_Q)$ is a group of order $[\mathbb{Q}(w) : \mathbb{Q}] = 5$. We have $a_2(E) = 1$, so the roots of the polynomial $\det(xI - \rho_{E,11}(\text{Frob}_2)) \equiv x^2 - 1 \cdot x + 2$ (mod 11) are $5 \equiv 2^1$ and $7 \equiv 2^7$ (mod 11). Since
\(\chi_1(\text{Frob}_2) \equiv 2^a\) and \(\chi(\text{Frob}_2) \equiv 2^{11-a}\) are the roots of \(\det(xI - \rho_{E,11}(\text{Frob}_2))\) and 2 is a primitive root modulo 11, we have \(a \in \{4,7\}\) and hence \(\{\chi_1, \chi_2\} = \{\omega^4, \omega^7\}\). Since \(\chi_1(\text{Gal}_Q)\) has cardinality 5, we have \(\chi_1 = \omega^4\) and \(\chi_2 = \omega^7\). Since 2 is a primitive root modulo 11, the group \(\rho_{E,11}(\text{Gal}_Q)\) is generated by \(\left(\begin{smallmatrix} 1 & 0 \\ 0 & 2 \end{smallmatrix}\right)\) and \(\left(\begin{smallmatrix} 0 & 1 \\ 1 & 1 \end{smallmatrix}\right)\), i.e., it equals \(H_{2,1}\). In particular, \(\pm \rho_{E,11}(\text{Gal}_Q) = G_2\).

### 4.5.5. Polynomials for \(X_{\text{ns}}^{(11)}\). This subsection is dedicated to sketching Remark 1.7 and making the polynomials explicit; fix notation as in §1.5. Define the polynomials:

\[ A(x) = (x^5 - 9x^4 + 17x^3 + 20x^2 - 73x + 43)^{11}, \]

\[ B(x) = -(x^3 + 3x^2 - 6)^6(108000x^{49} + 23793840x^{48} - 413223722x^{47} - 5377010368x^{46} + 230799738529x^{45} - 3137869050351x^{44} + 2320591172335x^{43} - 90936268647246x^{42} + 33563647471596x^{41} + 1631415220074871x^{40} - 7744726079195413x^{39} - 3218815397602111x^{38} + 236712051437217644x^{37} - 1686428698022253344x^{36} + 798480400202603554x^{35} - 30444781435263809696x^{34} + 9684982650401032248x^{33} - 23206439488359673213x^{32} + 21017553541339535857x^{31} + 16096958063249464826x^{30} - 31768533668978648360109x^{29} + 4829119612282659771817x^{28} - 143949381999306373170039x^{27} + 315827025781563232420857x^{26} - 42159697972048599269121x^{25} - 234929885880162547645306x^{24} + 366824143755302280195917x^{23} - 14221091463553801024770599x^{22} + 39148264563215734730610917x^{21} - 87534472061810348609315974x^{20} + 166474240219619575379485393x^{19} - 27504071573054834247036345x^{18} + 399144725377223909937142938x^{17} - 511840960382358144595839458x^{16} + 581656165533421466571816x^{15} - 586206578906981243989668654x^{14} + 523465655841901079370457175x^{13} - 413200824632802503354807972x^{12} + 282720832775316643952335709x^{11} - 175049577131269087795781453x^{10} + 9291657262697367901485620x^{9} - 42636417323385892254033027x^{8} + 167542924567377814457709x^{7} - 5570911068111167263502302x^{6} + 1542648801995330874184236x^{5} - 34781905342429836793068x^{4} + 6168475328903338239178x^{3} - 8117065260720937228985x^{2} + 7083187430491449799x - 308573604602631018655), \]

\[ C(x) = (4x - 5)(x^2 + 3x - 6)^6(9x^2 - 28x + 23)(x^4 - 5x^3 + 74x^2 - 245x + 223)^3 \cdot (4x^4 - 9x^3 - x^2 + 21x - 32)^3(25x^4 - 114x^3 + 167x^2 - 86x + 20)^3. \]

#### Proposition 4.6. For \(j \in \mathbb{Q}\), we have \(J(P) = j\) for some point \(P \in E(\mathbb{Q}) - \{O\}\) if and only if \(A(x)j^2 + B(x)j + C(x) \in \mathbb{Q}[x]\) has a rational root.

**Proof.** Take \((x, y) \in \mathcal{E} - \{O\}\). Using the equation \(y^2 + y = x^3 - x^2 - 7x + 10\), a direct computation shows that \(J(x, y)A(x) = a(x)y + b(x)\) for unique \(a, b \in \mathbb{Q}[x]\). Multiplying \(y^2 + y = x^3 - x^2 - 7x + 10\) by \(x^2\), we deduce that \((JA - b)^2 + a(JA - b) - a^2(x^3 - x^2 - 7x + 10) = 0\). Therefore, \(A^2J^2 + (-2b + a)AJ + b^2 - ba - a^2(x^3 - x^2 - 7x + 10) = 0\). Our polynomials \(B\) and \(C\) satisfy \(B = -2b + a\) and \(C = (b^2 - ba - a^2(x^3 - x^2 - 7x + 10))/A\). We thus have

\[(4.10) \quad A(x)J(x, y)j^2 + B(x)J(x, y) + C(x) = 0\]

for all \((x, y) \in \mathcal{E} - \{O\}\).

First suppose that \(j = J(x_0, y_0)\) for some \((x_0, y_0) \in \mathcal{E}(\mathbb{Q}) - \{O\}\). Then \(0 = A(x_0)J(x_0, y_0)^2 + B(x_0)J(x_0, y_0) + C(x_0) = A(x_0)j^2 + B(x_0)j + C(x_0)\) and hence \(A(x)j^2 + B(x)j + C(x)\) has a rational root.

Now fix \(j \in \mathbb{Q}\) and suppose that there is an \(x_0 \in \mathbb{Q}\) such that \(A(x_0)j^2 + B(x_0)j + C(x_0) = 0\). Define \(\Delta(x) := B(x)^2 - 4A(x)C(x)\). A computation shows that \(\Delta(x) = D(x)^2(x^3 - x^2 - 7x + 41/4) - 4A(x)C(x)\).

\[ \Delta(x) = (x - 2.3)(x - 5.7)(x - 7.5) \]

Thus, \(\Delta(x)\) has roots in \(\mathbb{Q}\) and \(A(x)j^2 + B(x)j + C(x)\) has a rational root.
for a polynomial $D \in \mathbb{Q}[x]$ that has no rational roots. The rational number $\Delta(x_0) = D(x_0)^2(x_0^3 - x_0^2 - 7x_0 + 41)/4$ is a square since $j$ is a root of $A(x_0)X^2 + B(x_0)X + C(x_0) \in \mathbb{Q}[X]$. Therefore, $v^2 = x_0^3 - x_0^2 - 7x_0 + 41/4$ for some $v \in \mathbb{Q}$. With $y_0 = v - 1/2$, we have $y_0^2 + y_0 = x_0^3 - x_0^2 - 7x_0 + 10$ and hence $P := (x_0, y_0)$ is a point in $E(\mathbb{Q}) - \{O\}$. We could have chosen $v$ with a different sign, so $P' := (x_0, -v - 1/2) = (x_0, -y_0 - 1)$ also belongs to $E(\mathbb{Q}) - \{O\}$.

We claim that $J(P) \neq J(P')$. Suppose that they are in fact equal. Using that $J(x, y)A(x) = a(x)y + b(x)$, we find that $a(x_0)y_0 = a(x_0)(-y_0 - 1)$. Since $a(x)$ has no rational roots, we must have $y_0 = -1/2$ and hence $v = 0$. However, this is impossible since $x^3 - x^2 - 7x + 41/4$ has no rational roots, so the claim follows. From (4.10), we find that $J(P)$ and $J(P')$ are distinct roots of $A(x_0)X^2 + B(x_0)X + C(x_0)$. Since $j$ is also a root of this quadratic polynomial, we deduce that $j = J(P)$ or $j = J(P')$. \hfill \square

4.6. $\ell = 13$. We shall prove parts (i) and (ii) of Theorem 1.8 (part (iv) was explained in the introduction); so we will focus on $B(13)$ and its subgroups. We first rule out subgroups of $C_s(13)$.

**Lemma 4.7.** There are no non-CM elliptic curves $E/\mathbb{Q}$ for which $\rho_{E,13}(\text{Gal}_{\mathbb{Q}})$ is conjugate in $\text{GL}_2(\mathbb{F}_{13})$ to a subgroup of $C_s(13)$.

**Proof.** Kenku has proved that the only rational points of $X_0(13^2)$ are cusps, cf. [Ken80,Ken81]. By Lemma 3.6, we deduce that the only rational points of the modular curve $X_{C_s(13)}$ are cusps. \hfill $\square$

One can show that the applicable subgroups of $B(13) = G_6$ that are not subgroups of $C_s(13)$ are $G_1$, $G_2$, $G_3$, $G_4$, $G_5$, and $G_i \cap G_j$ with $i \in \{1, 2\}$ and $j \in \{3, 4, 5\}$. Note that these subgroups are normal in $B(13)$.

We now describe several modular function constructed by Lecacheux [Lec89, p.56]. Define

$$f(\tau) = \frac{\wp(1/13; \tau) - \wp(2/13; \tau)}{\wp(1/13; \tau) - \wp(3/13; \tau)} \quad \text{and} \quad g(\tau) = \frac{\wp(1/13; \tau) - \wp(2/13; \tau)}{\wp(2/13; \tau) - \wp(3/13; \tau)}$$

where $\wp(z; \tau)$ is the Weierstrass $\wp$-function at $z$ of the lattice $\mathbb{Z}\tau + \mathbb{Z} \subseteq \mathbb{C}$. Define the functions

$$h_5 := \frac{(g - 1)(g - 1) + 1 - f}{(f - 1)(g - f)} \quad \text{and} \quad h_2 := \frac{f - 1}{g - 1}$$

The functions $h_5$ and $h_2$ belong to $F_{13}$ and satisfy $F_2(h_2) = F_5(h_5)$, where

$$F_2(t) = t + (t - 1)/t - 1/(t - 1) - 4 = (t^3 - 4t^2 + 1)/(t^2 - t) \quad \text{and} \quad F_5(t) = t - 1/t - 3 = (t^2 - 3t - 1)/t;$$

this follows from [Lec89, p.56–57] with $H = h_5$ and $h = h_2$.

Let $h_6$ be the function $F_2(h_2) = F_5(h_5)$; it is called $a - 3$ in [Lec89] and satisfies $J_6(h_6) = j$, cf. [Lec89, p.62]. Since $J_2(t) = J_6(F_2(t))$ and $J_5(t) = J_6(F_5(t))$, we have $J_2(h_2) = j$ and $J_5(h_5) = j$.

Define $\alpha := -\zeta_{13}^{11} - \zeta_{13}^{10} - \zeta_{13}^3 - \zeta_{13}^2 + 1$. Define the rational functions

$$F_1(t) = 13(t^2 - t)/(t^3 - 4t^2 + t + 1) \quad \text{and} \quad \phi_1(t) = (at + 1 - \alpha)/(t - \alpha);$$

Define the modular function $h_1 := \phi_1(h_2) \in F_{13}$. One can check that $F_1(\phi_1(t)) = F_2(t)$ and hence $F_1(h_1) = F_2(h_2) = h_6$. Since $J_1(t) = J_6(F_1(t))$, we have $J_1(h_1) = j$.

Define $\beta := \zeta_{13}^4 + \zeta_{13}^{10} + \zeta_{13}^9 + \zeta_{13}^7 + \zeta_{13}^6 + \zeta_{13}^5 + \zeta_{13}^3 + \zeta_{13}^2 + 2$. Define the rational functions

$$F_3(t) = (5t^3 + 7t^2 + 8t - 5)/(t^3 - 4t^2 + t + 1) \quad \text{and} \quad \phi_3(t) = (\beta t - 1)/(t - \beta) - 1).$$

Define the modular function $h_3 := \phi_3(h_2) \in F_{13}$. One can check that $F_3(\phi_3(t)) = F_2(t)$ and hence $F_3(h_3) = F_2(h_2) = h_6$. Since $J_3(t) = J_6(F_3(t))$, we have $J_3(h_3) = j$. 28
Define $\gamma = (1 + \sqrt{13})/2$; it belongs to $\mathbb{Q}(\zeta_{13})$ and moreover equals $\gamma = -\zeta_{13}^{11} - \zeta_{13}^{8} - \zeta_{13}^{7} - \zeta_{13}^{6} - \zeta_{13}^{5} - \zeta_{13}^{2}$. Define the rational functions

$$F_4(t) = 13t/(t^2 - 3t - 1) \quad \text{and} \quad \phi_4(t) = ((2 - \gamma)t + 1)/(t - 2 + \gamma)).$$

Define the modular function $h_4 := \phi_4(h_5) \in \mathcal{F}_{13}$. One can check that $F_4(\phi_4) = F_5(t)$ and hence $F_4(h_4) = F_5(h_5) = h_4$. Since $J_5(t) = J_6(F_5(t))$, we have $J_5(h_4) = j$.

For $1 \leq i \leq 6$, let $H_i$ be the subgroup of $\text{GL}_2(\mathbb{F}_7)$ that fixes $h_i$. We have shown that $J_i(h_i) = j$.

By Lemma 3.4, we find that $H_i$ is an applicable subgroup and that the morphism $\pi_{H_i, \mathbb{Q}} : X_{H_i} \to \mathbb{P}^1_{\mathbb{Q}}$ is described by the rational function $J_i(t)$.

**Lemma 4.8.** The groups $H_i$ and $G_i$ are conjugate in $\text{GL}_2(\mathbb{F}_{13})$ for all $1 \leq i \leq 6$.

**Proof.** The index of $H_6$ in $\text{GL}_2(\mathbb{F}_{13})$ is equal to 14, i.e., the degree of $J_6$ as a morphism. Therefore, $H_6$ must be conjugate to $B(13)$. The index $[H_6 : H_i]$ equals the degree of $F_i(t)$, and is thus 3 if $i \in \{1, 2, 3\}$ and 2 if $i \in \{4, 5\}$.

The groups $H_1$, $H_2$ and $H_3$ are not conjugate in $\text{GL}_2(\mathbb{F}_{13})$ since one can show that the images of $\mathbb{P}^1(\mathbb{Q}) = \mathbb{Q} \cup \{\infty\}$ under $J_1$, $J_2$ and $J_3$ are distinct. Therefore, $H_1$, $H_2$ and $H_3$ are conjugate to $G_1$, $G_2$ and $G_3$, which are the applicable subgroups of $B(13)$ of index 2; however, we still need to determine which group is conjugate to which.

Let $E/\mathbb{Q}$ be the elliptic curve defined by $y^2 = x^3 - 338x + 2392$. The group $\rho_{E, 13}(\text{Gal}_{\mathbb{Q}})$ is conjugate to a subgroup of $H_3$ since $J_E = J_3(0)$. One can check that $E/\mathbb{Q}$ has good reduction at 3 and that $a_3(E) = 0$. Since $x^2 - a_3(E) + 3 \equiv (x - 6)(x + 6) \pmod{13}$, we deduce that the eigenvalues of the matrix $\rho_{E, 13}(\text{Frob}_3)$ are 6 and $-6$. For every matrix in $G_1$ or $G_2$ has an eigenvalue in $(\mathbb{F}_{13}^\times)^3 = \{\pm 1, \pm 5\}$. Since 6 and $-6$ do not belong to $(\mathbb{F}_{13}^\times)^3$, we deduce that $H_3$ is not conjugate to $G_1$ and $G_2$. Therefore, $H_3$ is conjugate to $G_3$.

Let $E/\mathbb{Q}$ be the elliptic curve defined by $y^2 = x^3 - 2227x - 59534$. We have $J_E = J_2(2)$ and $J_E \notin J_1(\mathbb{Q} \cup \{\infty\})$. Therefore, $\rho_{E, 13}(\text{Gal}_{\mathbb{Q}})$ is conjugate to a subgroup of $H_2$ and not conjugate to a subgroup of $H_1$. By computing the division polynomial of $E$ at the prime 13, we find that $E$ has a point $P$ of order 13 whose $x$-coordinate is $17 + 8\sqrt{17}$. So with respect to a basis of $E[13]$ whose first element is $P$, we find that $\rho_{E, 13}(\text{Gal}_{\mathbb{Q}})$ is a subgroup of $G_2$. Therefore, $H_2$ is conjugate to $G_2$, and hence $H_1$ is conjugate to $G_1$.

The groups $H_4$ and $H_5$ are not conjugate in $\text{GL}_2(\mathbb{F}_{13})$ since one can show that the images of $\mathbb{P}^1(\mathbb{Q}) = \mathbb{Q} \cup \{\infty\}$ under $J_4$ and $J_5$ are distinct. Therefore, $H_4$ and $H_5$ are conjugate to $G_4$ and $G_5$, which are the applicable subgroups of $B(13)$ of index 3; however, we still need to determine which group is conjugate to which.

Let $E/\mathbb{Q}$ be the elliptic curve defined by $y^2 = x^3 - 3024x - 69552$. We have $J_E = J_5(2)$ and $J_E \notin J_4(\mathbb{Q} \cup \{\infty\})$. Therefore, $\rho_{E, 13}(\text{Gal}_{\mathbb{Q}})$ is conjugate to a subgroup of $H_5$ and not conjugate to a subgroup of $H_4$. By computing the division polynomial of $E$ at the prime 13, we find that $E$ has a point $P$ of order 13 whose $x$-coordinate $w$ is a root of $x^3 - 3024x + 12096$. The cubic extension $\mathbb{Q}(w)$ of $\mathbb{Q}$ is Galois, so with respect to a basis of $E[13]$ whose first element is $P$, we find that $\rho_{E, 13}(\text{Gal}_{\mathbb{Q}})$ is a subgroup of $G_5$. Therefore, $H_5$ is conjugate to $G_5$, and hence $H_4$ is conjugate to $G_4$.

We have thus completed the proof of Theorem 1.8(ii); we can ignore $t = \infty$ since $J_i(\infty) = J_5(0)$ if $H$ is a proper subgroup of $G_i$ satisfying $\pm H = G_i$, then one can show that $i \in \{4, 5\}$ and $H$ is one of the groups $H_{i,j}$. To complete the proof of Theorem 1.8(i), we need only show that the modular curves $X_{G_{i,j}}$, with fixed $i \in \{1, 2\}$ and $j \in \{3, 4, 5\}$, have no rational points other than cusps. It suffices to prove the same thing for the modular curves $X_{H_{i,j}}$. 


The function field of $X_{H_i \cap H_j}$ is $\mathbb{Q}(h_i, h_j)$ and the generators $h_i$ and $h_j$ satisfy the relation $F_i(h_i) = h_6 = F_j(h_j)$. The smooth projective (and geometrically irreducible) curve over $\mathbb{Q}$ arising from the equation $F_i(x) = F_j(y)$ is thus a model of $X_{H_i \cap H_j}$.

The following Magma code shows that if $(x, y) \in \mathbb{Q}^2$ is a solution of $F_i(x) = F_j(y)$ (where we say that both sides equal $\infty$ if the denominators vanish), then $y = 0$. The code considers the projective (and possibly singular) curve $C_{i,j}$ in $\mathbb{P}_\mathbb{Q}^2$ defined by the affine equation $F_i(x) = F_j(y)$ (we first clear denominators and homogenize). We then find a genus 2 curve $C$ that is birational with $C_{i,j}$ and is defined by some Weierstrass equation $y^2 = f(x)$ with $f(x) \in \mathbb{Q}[x]$ a separable polynomial of degree 5 or 6. We then check that the Jacobian $J$ of $C$ has rank 0, equivalently, that $J(\mathbb{Q})$ is a finite group (Magma accomplishes this by computing the 2-Selmer group of $J$). Using that $J(\mathbb{Q})$ has rank 0, the function Chabauty0 finds all the rational points on $C$. Using the birational isomorphism between $C$ and $C_{i,j}$, we can determine the rational points of $C$.

```
K<=>FunctionField(Rationals());
F:=[13*(t^2-t)/(t^3-4*t^2+t+1), (t^3-4*t^2+t+1)/(t^2-t),
   (-5*t^3+7*t^2+8*t-5)/(t^3-4*t^2+t+1), 13*t/(t^2-3*t-1), (t^2-3*t-1)/t ];
P2<x,y,z>:=ProjectiveSpace(Rationals(),2);
for i in [1,2,3] do
  f:=(Numerate(Evaluate(F[i],x/z)- Evaluate(F[j],y/z)));
  while Evaluate(f,z,0) eq 0 do f:= f div z; end while;
  Co:=Curve(P2,f);
b,C1,f1:=IsHyperelliptic(Co); C2,f2:=SimplifiedModel(C1);
Jac:=Jacobian(C2); RankBound(Jac) eq 0;
S:=Chabauty0(Jac);
b,g1:=IsInvertible(f1); b,g2:=IsInvertible(f2);
T:=g1(g2(S) join SingularPoints(C1)) join SingularPoints(Co);
{P: P in T | P[2] ne 0 and P[3] ne 0} eq {};
end for;
```

We find that if $F_i(x) = F_j(y)$ for some $x, y \in \mathbb{Q} \cup \{\infty\}$, then $y = 0$ or $y = \infty$. Thus the only rational points of $X_{H_i \cap H_j}$ are cusps since $J_j(0) = J_j(\infty) = \infty$ for $j \in \{4, 5\}$.

4.7. $\ell = 17$. We now prove Theorem 1.10(i). Let $E/\mathbb{Q}$ be the elliptic curve defined by the Weierstrass equation $y^2 + xy + y = x^3 - 19089x - 36002922$; it has $j$-invariant $-17 \cdot 373^3/2^{17}$ and conductor $2 \cdot 5^2 \cdot 17^2$. The division polynomial of $E$ at 17 factors as a product of $f(x) = x^4 + 482x^3 + 1144x^2 - 1580984x - 958623689$ with irreducible polynomials of degree 4 and $8 \cdot 17$. Fix a point $P \in E(\overline{\mathbb{Q}})$ whose $x$-coordinate $w$ is a root of $f(x)$; it is a 17-torsion point. Let $C$ be the cyclic group of order 17 generated by $P$; it is stable under the $\text{Gal}_\mathbb{Q}$ action. Let $\chi_1: \text{Gal}_\mathbb{Q} \to \mathbb{F}_{17}^\times$ be the homomorphism such that $\sigma(P) = \chi_1(\sigma) \cdot P$ for $\sigma \in \text{Gal}_\mathbb{Q}$. One can show that the degree 4 extension $\mathbb{Q}(w)/\mathbb{Q}$ is Galois, so $\chi_1(\text{Gal}_\mathbb{Q})$ has cardinality 4 or 8. There is a second character $\chi_2: \text{Gal}_\mathbb{Q} \to \mathbb{F}_{17}^\times$ such that, with respect to an appropriate change of basis, we have

$$
\rho_{E,17}(\sigma) = \begin{pmatrix}
\chi_1(\sigma)^* & 0 \\
0 & \chi_2(\sigma)
\end{pmatrix}.
$$

The cardinality of $\rho_{E,17}(\text{Gal}_\mathbb{Q})$ is divisible by 17 since the division polynomial of $E$ at 17 has an irreducible factor whose degree is divisible by 17. We have $\chi_1 \chi_2 = \omega$ where $\omega: \text{Gal}_\mathbb{Q} \to \mathbb{F}_{17}^\times$ is the character describing the Galois action on the 17-th roots of unity (we have $\omega(\text{Frob}_p) = p$ for primes $p \neq 17$). The characters $\chi_1$ and $\chi_2$ are unramified at primes $p \nmid 2 \cdot 5 \cdot 17$, so $\chi_1 = \omega^a \chi$ and $\chi_2 = \omega^{17-a} \chi^{-1}$ for some integer $0 \leq a < 16$ and some character $\chi: \text{Gal}_\mathbb{Q} \to \mathbb{F}_{17}^\times$ unramified at $p \nmid 2 \cdot 5$.

Let $H_1$ and $H_2$ be the subgroup of $\text{GL}_2(\mathbb{F}_\ell)$ consisting of matrices of the form

$$
\begin{pmatrix}
\omega(\sigma)^a & 0 \\
0 & \omega(\sigma)^{17-a}
\end{pmatrix}
$$

and

$$
\begin{pmatrix}
\chi_1(\sigma) & 0 \\
0 & \chi_1(\sigma)^{-1}
\end{pmatrix}.
$$
respectively, with $\sigma \in \text{Gal}_Q$. Since $\omega$ and $\chi$ are ramified at different primes, we find that the image of $\rho_{E, \ell}$ is generated by $\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$ and the groups $H_1$ and $H_2$.

The character $\chi$ is unramified at primes $p \nmid 2 \cdot 5$ and has image in a cyclic group of order 16. Therefore, $\chi$ must factor through the group $\text{Gal}(Q(\zeta_{64}, \zeta_{5})/Q)$. Since $641 \equiv 1 \pmod{64 \cdot 5}$, we have $\chi(\text{Frob}_{641}) = 1$. Therefore, $\chi_1(\text{Frob}_{641}) = \omega(\text{Frob}_{641})^3 \cdot 1 \equiv 641^a \pmod{17}$ is a root of

$$x^2 - a_{641}(E)x + 641 = x^2 - (-9)x + 641 \equiv (x - 641^6)(x - 641^{11}) \pmod{17},$$

and hence $a \in \{6, 11\}$ since 641 is a primitive root modulo 17. If $a = 11$, then $\chi_1(\text{Gal}_Q) = \mathbb{F}_1^x$ which is impossible since the cardinality of $\chi_1(\text{Gal}_Q)$ is 4 or 8. Therefore, $a = 6$. The group $H_1$ thus consists of matrices of the form $\begin{pmatrix} c^6 & 0 \\ 0 & c^{11} \end{pmatrix}$ with $c \in \mathbb{F}_1^x$, and in particular is generated by $\begin{pmatrix} 5 & 0 \\ 0 & 5^{11} \end{pmatrix} = \begin{pmatrix} 2 & 0 \\ 0 & 1 \end{pmatrix}$.

To complete the proof that $\rho_{E, 17}(\text{Gal}_Q)$ is $G_1$, it suffices to show that $H_2$ is generated by $\begin{pmatrix} 4 & 0 \\ 0 & 4 \end{pmatrix}$; equivalently, to show that the image of $\chi$ is cyclic of order 4. As noted earlier, $\chi$ factors through the group $\text{Gal}(Q(\zeta_{64}, \zeta_{5})/Q) \approx (\mathbb{Z}/64 \cdot 5\mathbb{Z})^\times$. One can then show that $\text{Gal}(Q(\zeta_{64}, \zeta_{5})/Q)$ is generated by $\text{Frob}_{103}$, $\text{Frob}_{137}$ and $\text{Frob}_{307}$. The primes $p \in \{103, 137, 307\}$ were chosen to be congruent to 1 modulo 17, and hence $\chi(\text{Frob}_p) = \chi_1(\text{Frob}_p)$ is a root of $x^2 - a_p(E)x + p$ modulo 17. It is then straightforward to check that $\chi(\text{Frob}_{103})$, $\chi(\text{Frob}_{137})$ and $\chi(\text{Frob}_{307})$ all have order 4.

The elliptic curve $E'/Q$ defined by the Weierstrass equation $y^2 + xy + y = x^3 - 3041x + 64278$ has $j$-invariant $-17^2 \cdot 101^3/2$. One can show that $E'/C$ is isomorphic to $E'$. The group $\rho_{E'/17}(\text{Gal}_Q)$ is thus conjugate to $G_2$ in $\text{GL}_2(\mathbb{F}_1)$.

Finally, we note that $G_1$ and $G_2$ have no index 2 subgroups that do not contain $-I$.

4.8. $\ell = 37$. We now prove Theorem 1.10(ii). Let $E/Q$ be the elliptic curve defined by the equation $y^2 + xy + y = x^3 + x^2 - 8x + 6$; it has $j$-invariant $-7 \cdot 11^3$ and conductor $5^2 \cdot 7^2$. The division polynomial of $E$ at 17 factors as a product of $f(x) := x^9 - 15x^5 - 90x^4 - 50x^3 + 225x^2 + 125x - 125$ with irreducible polynomials of degree 6, 6 and $18 \cdot 37$. Fix a point $P \in E(\mathbb{Q})$ whose $x$-coordinate $u$ is a root of $f(x)$; it is a 37-torsion point. Let $C$ be the cyclic group of order 37 generated by $P$; it is stable under the Galois action.

Let $\chi_1: \text{Gal}_Q \to \mathbb{F}_{37}^x$ be the homomorphism such that $\sigma(P) = \chi_1(\sigma) \cdot P$ for $\sigma \in \text{Gal}_Q$. One can show that the degree 6 extension $Q(w)/Q$ is Galois, so $\chi_1(\text{Gal}_Q)$ has cardinality 6 or 12; in particular $\chi_1(\text{Gal}_Q)$ is a subgroup of $(\mathbb{F}_{37}^x)^3$. There is a second character $\chi_2: \text{Gal}_Q \to \mathbb{F}_{37}^x$ such that, with respect to an appropriate change of basis, we have

$$\rho_{E, 37}(\sigma) = \begin{pmatrix} \chi_1(\sigma) & * \\ 0 & \chi_2(\sigma) \end{pmatrix}.$$ 

The cardinality of $\rho_{E, 37}(\text{Gal}_Q)$ is divisible by 37 since the division polynomial of $E$ at 37 has an irreducible factor whose degree is divisible by 37. So to prove that $\rho_{E, 37}(\text{Gal}_Q) = G_3$, it suffices to show that the homomorphism $\chi_1 \times \chi_2: \text{Gal}_Q \to \mathbb{F}_{37}^x \times \mathbb{F}_{37}^x$ is surjective.

The characters $\chi_1$ and $\chi_2$ are unramified at primes $p \nmid 5 \cdot 7 \cdot 37$. By Proposition 11 of [Ser72], we have $\{\chi_1, \chi_2\} = \{\alpha, \alpha^{-1} \cdot \omega\}$ where $\alpha: \text{Gal}_Q \to \mathbb{F}_{37}^x$ is a character unramified at primes $p \nmid 5 \cdot 7$ and $\omega: \text{Gal}_Q \to \mathbb{F}_{37}^x$ is the character describing the Galois action on the 37-th roots of unity. Since $\alpha$ is unramified at 37, we find that the character $\alpha^{-1} \cdot \omega$ is surjective and that $(\alpha \times (\alpha^{-1} \cdot \omega))(\text{Gal}_Q) = (\alpha(\text{Gal}_Q) \times \mathbb{F}_{37}^x).$ Since $\chi_1$ is surjective, we must have $\chi_1 = \alpha$ and $\chi_2 = \alpha^{-1} \cdot \omega$. It thus suffices to show that the image of $\alpha$ contains an element of order 12. The fixed field of the kernel of $\alpha$ is contained in $Q(\zeta_{5}, \zeta_7)$ since it is unramified at $p \nmid 5 \cdot 7$ and has image relatively prime to $5 \cdot 7$. Since $107 \equiv 2 \pmod{35}$, we have $\alpha(\text{Frob}_2) = \alpha(\text{Frob}_{107})$. Therefore, $\alpha(\text{Frob}_2)$ is a common root of $x^2 - a_2(E)x + 2 = x^2 + x + 2$ and $x^2 - a_{107}(E)x + 107 = x^2 + 11x + 107$ modulo 37. This implies that $\alpha(\text{Frob}_2)$ equals 8 in $\mathbb{F}_{37}$ which has order 12.
One can show that the quotient of $E$ by $C$ is the elliptic curve $E'/\mathbb{Q}$ defined by $y^2 + xy + y = x^3 + x^2 - 208083x - 36621194$; it has $j$-invariant $-7 \cdot 137^3 \cdot 2083^3$. The group $\rho_{E',37}(\text{Gal}_\mathbb{Q})$ is thus conjugate in $\text{GL}_2(\mathbb{F}_{37})$ to $G_4$.

Finally we note that $G_3$ and $G_4$ have no index 2 subgroups that do not contain $-I$.

5. QUADRATIC TWISTS

Fix an elliptic curve $E/\mathbb{Q}$ with $j_E \notin \{0, 1728\}$ and an integer $N \geq 3$.

Define the group $G := \pm \rho_{E,N}(\text{Gal}_\mathbb{Q})$ and let $\mathcal{H}$ be the set of proper subgroups $H$ of $G$ that satisfy $\pm H = G$. For each group $H \in \mathcal{H}$, we obtain a character

$$\chi_{E,H} : \text{Gal}_{\mathbb{Q}} \to \{\pm 1\}$$

by composing $\rho_{E,N}$ with the quotient map $G \to G/H \cong \{\pm 1\}$. The fixed field of the kernel of the character $\chi_{E,H}$ is of the form $\mathbb{Q}(\sqrt{d_{E,H}})$ for a unique squarefree integer $d_{E,H}$. Define the set

$$\mathcal{D}_E := \{d_{E,H} : H \in \mathcal{H}\}.$$ 

Using $\pm \rho_{E,N}(\text{Gal}_\mathbb{Q}) = G$, we find that different groups $H \in \mathcal{H}$ give rise to distinct characters $\chi_{E,H}$ and thus $|\mathcal{D}_E| = |\mathcal{H}|$.

5.1. Twists with smaller image. For a squarefree integer $d$, let $E_d/\mathbb{Q}$ be a quadratic twist of $E/\mathbb{Q}$ by $d$. By choosing an appropriate basis of $E_d[\ell]$, we may assume that $\rho_{E_d,N} : \text{Gal}_\mathbb{Q} \to \text{GL}_2(\mathbb{Z}/N\mathbb{Z})$ satisfies

$$\rho_{E_d,N} = \chi_d \cdot \rho_{E,N},$$

where $\chi_d : \text{Gal}_{\mathbb{Q}} \to \{\pm 1\}$ is the character corresponding to the extension $\mathbb{Q}(\sqrt{d})/\mathbb{Q}$. We have $\pm \rho_{E_d,N}(\text{Gal}_\mathbb{Q}) = \pm \rho_{E,N}(\text{Gal}_\mathbb{Q}) = G$. Therefore, $\rho_{E_d,N}(\text{Gal}_\mathbb{Q})$ is equal to either $G$ or to one of the subgroups $H \in \mathcal{H}$.

We now show that $\mathcal{D}_E$ is precisely the set of squarefree integers $d$ for which the image of $\rho_{E_d,N}$ is not conjugate to $G$.

Lemma 5.1. Take any squarefree integer $d$.

(i) We have $d \in \mathcal{D}_E$ if and only if the group $\rho_{E_d,N}(\text{Gal}_\mathbb{Q})$ is conjugate in $\text{GL}_2(\mathbb{Z}/N\mathbb{Z})$ to a proper subgroup of $G$.

(ii) If $d = d_{E,H}$ for some $H \in \mathcal{H}$, then $\rho_{E_d,N}(\text{Gal}_\mathbb{Q})$ is conjugate in $\text{GL}_2(\mathbb{Z}/N\mathbb{Z})$ to $H$.

Proof. Take any group $H \in \mathcal{H}$. Composing $\rho_{E_d,N} : \text{Gal}_\mathbb{Q} \to G$ with the quotient map $G \to G/H \cong \{\pm 1\}$ gives the character $\chi_d \cdot \chi_{E,H}$. Therefore, $\rho_{E_d,N}(\text{Gal}_\mathbb{Q})$ is a subgroup of $H$ (and hence equal to $H$) if and only if $\chi_{E,H} = \chi_d$; equivalently, $d = d_{E,H}$. Parts (i) and (ii) are now immediate. \square

Since $|\mathcal{D}_E| = |\mathcal{H}|$, we deduce from Lemma 5.1 that the map

$$\mathcal{H} \to \mathcal{D}_E, \quad H \mapsto d_{E,H}$$

is a bijection.

Remark 5.2. Observe that $\rho_{E_d,N}(\text{Gal}_\mathbb{Q})$ being conjugate to $H$ in $\text{GL}_2(\mathbb{Z}/N\mathbb{Z})$ need not imply that $d = d_{E,H}$. For example, it is possibly for distinct groups in $\mathcal{H}$ to be conjugate in $\text{GL}_2(\mathbb{Z}/N\mathbb{Z})$.

5.2. Computing $\mathcal{D}_E$. Now assume that $N \geq 3$ is odd; we shall explain how to compute $\mathcal{D}_E$ (we will later be interested in the case where $N$ is an odd prime). Let $M_E$ be the set of squarefree integers that are divisible only by primes $p$ such that $p|N$ or such that $E$ has bad reduction at $p$.

For each $r \geq 1$, let $\mathcal{D}_r$ be the set of $d \in M_E$ such that

$$a_p(E) \not\equiv -2 \left(\frac{d}{p}\right) \pmod{N}$$

holds for all primes $p \leq r$ for which $E$ has good reduction and $p \equiv 1 \pmod{N}$. 32
Lemma 5.3. Suppose that $N$ is odd. We have $\mathcal{D}_E \subseteq \mathcal{D}_r$ with equality holding for all sufficiently large $r$.

Proof. Define $\mathcal{D} := \cap_r \mathcal{D}_r$; it is the set of $d \in M_E$ such that (5.1) holds for all primes $p \equiv 1 \pmod{N}$ for which $E$ has good reduction. We have $\mathcal{D}_r \subseteq \mathcal{D}_r'$ if $r \geq r'$, so it suffices to prove that $\mathcal{D} = \mathcal{D}_E$.

Take any $d \in \mathcal{D}$. We have $a_p(E_d) = \left(\frac{2}{p}\right) a_p(E) \neq -2$ (mod $N$) for all primes $p \equiv 1 \pmod{N}$ for which $E$ has good reduction. By the Chebotarev density theorem, there are no elements $g \in \rho_{E_d,N}(\text{Gal}_Q)$ satisfying $\det(g) = 1$ and $\text{tr}(g) = -2$. In particular, the group $\rho_{E_d,N}(\text{Gal}_Q)$ does not contain $-I$ and hence $d \in \mathcal{D}_E$ by Lemma 5.1(i). Therefore, $\mathcal{D} \subseteq \mathcal{D}_E$.

We have $\mathcal{D}_E \subseteq M_E$ since each character $\chi_{E,H}$ factors through $\rho_{E,N}$ (and is hence unramified at all primes $p \not| N$ for which $E$ has good reduction).

Now take any $d \in \mathcal{D}_E - \mathcal{D}$. We have a $p \equiv 1 \pmod{N}$ for which $E$ has good reduction and $a_p(E_d) = \left(\frac{2}{p}\right) a_p(E) \equiv -2$ (mod $N$). Define $g := \rho_{E_d,N}(\text{Frob}_p)$; it has trace $-2$ and determinant $1$. Since $N$ is odd, some power of $g$ is equal to $-I$. Therefore, $\rho_{E_d,N}(\text{Gal}_Q) = \pm \rho_{E_d,N}(\text{Gal}_Q) = G$ which contradicts that $d \in \mathcal{D}_E$. Therefore, $\mathcal{D}_E - \mathcal{D}$ is empty and hence $\mathcal{D}_E \subseteq \mathcal{D}$. \hfill \Box

One can compute the finite sets $\mathcal{D}_r$ for larger and larger values of $r$ until $|\mathcal{D}_r| = |\mathcal{H}|$ and then $\mathcal{D}_E = \mathcal{D}_r$. This works since we always have an inclusion $\mathcal{D}_E \subseteq \mathcal{D}_r$ by Lemma 5.3, and equality holds when $|\mathcal{D}_r| = |\mathcal{H}|$ since $|\mathcal{D}_E| = |\mathcal{H}|$.

When $N$ is a prime, the integers in $\mathcal{D}_E$ come in pairs.

Lemma 5.4. Suppose $N = \ell$ is an odd prime. Let $\mathcal{D}_E^\ell$ be the set of $d \in \mathcal{D}_E$ for which $\ell \mid d$. Then

$$\mathcal{D}_E = \bigcup_{d \in \mathcal{D}_E^\ell} \{d, (-1)^{(\ell-1)/2} \ell \cdot d\}.$$  

Proof. Define $\ell^* := (-1)^{(\ell-1)/2} \ell$. Take any $d \in \mathcal{D}_E$. We need to show that $d \ell^*$ or $d/\ell^*$ belong to $\mathcal{D}_E$ (whichever one is a squarefree integer). After possibly replacing $E$ by $E_d$, we may assume that $d = 1$ and hence we need only verify that $\ell^* \in \mathcal{D}_E$.

So assume that $\rho_{E,\ell}(\text{Gal}_Q)$ is a proper subgroup of $G$ and hence is equal to one of the $H \in \mathcal{H}$. We need to show that $\rho_{E',\ell}(\text{Gal}_Q)$ is also a proper subgroup of $G$, where $E' := E_{\ell^*}$.

The field $Q(\sqrt{\ell^*}) \subseteq Q(\zeta_\ell)$ is a subfield of both $Q(E[\ell])$ and $Q(E'[\ell])$. Since $E$ and $E'$ are isomorphic over $Q(\sqrt{\ell^*})$, we deduce that $[Q(E'[\ell]) : Q] = [Q(E[\ell]) : Q]$. Therefore, $|\rho_{E',\ell}(\text{Gal}_Q)| = |Q(E'[\ell]) : Q| = |Q(E[\ell]) : Q| = |\rho_{E,\ell}(\text{Gal}_Q)| = |H| = |G|/2$. By cardinality assumption, we deduce that $\rho_{E',\ell}(\text{Gal}_Q)$ is conjugate to a proper subgroup of $G$. \hfill \Box

Remark 5.5. One could also use the methods of this section to help determine $\mathcal{H}$. For example, if $\mathcal{D}_r = \emptyset$ for some $r$, then $\mathcal{H} = \emptyset$. Suppose we are in the setting, like what happens often in the introduction, where we know that $|\mathcal{H}| \geq 2$ because we have two explicit elements of $\mathcal{H}$. Then to verify that $|\mathcal{H}| = 2$, one need only find an $r$ such that $|\mathcal{D}_r| = 2$.

5.3. Some examples.

5.3.1. Take $\ell = 7$. Let $E/Q$ be the elliptic curve defined by $y^2 = x^3 - 5^3 7^4 x - 5^4 7^2 106$; it has $j$-invariant $3^3 \cdot 5 \cdot 7^5/2^7$ and conductor $2 \cdot 5^2 \cdot 7^2$. From the part of Theorem 1.5 proved in §4.4, we know that $\pm \rho_{E,7}(\text{Gal}_Q)$ is conjugate to the group $G_1$ of §1.4. Let $\mathcal{H}$ be the set of proper subgroups $H$ of $G_1$ such that $\pm H = G_1$. The set $\mathcal{H}$ consists of two groups; they are both conjugate in $\text{GL}_2(\mathbb{F}_7)$ to the group $H_{1,1}$ of §1.4. The curve $E$ is denoted by $E_1$ in §1.4.

We have $\mathcal{D}_E \subseteq M_E = \{\pm 1, \pm 2, \pm 5, \pm 7, \pm 10, \pm 14, \pm 35, \pm 70\}$. The primes 211, 239 and 337 are congruent to 1 modulo $\ell$. One can check that

$$a_{211}(E) = 16 \equiv 2 \pmod{7}, \hspace{1cm} a_{239}(E) = -5 \equiv 2 \pmod{7}, \hspace{1cm} a_{337}(E) = -5 \equiv 2 \pmod{7}.$$
So if \( d \in \mathcal{D}_{337} \), then \( \left( \frac{d}{311} \right) = 1 \), \( \left( \frac{d}{339} \right) = 1 \) and \( \left( \frac{d}{337} \right) = 1 \). Checking the \( d \in M_E \), we find that \( \mathcal{D}_{337} \subseteq \{1, -7\} \). Since \( |H| = 2 \), we deduce that \( \mathcal{D}_E = \{1, -7\} \).

Now let \( E'/\mathbb{Q} \) be any elliptic curve with \( j \)-invariant \( 3^3 \cdot 5 \cdot 7^5 / 2^7 \). Using Lemma 5.1, we deduce that \( \rho_{E',7}(\text{Gal}_\mathbb{Q}) \) is conjugate to \( G_1 \) if and only if \( E' \) is not isomorphic to \( E \) or its quadratic twist by \(-7\). When \( \rho_{E',7}(\text{Gal}_\mathbb{Q}) \) is not conjugate to \( G_1 \) it must be conjugate to \( H_{1,1} \) in \( \text{GL}_2(\mathbb{F}_7) \).

### 5.3.2. Take \( \ell = 11 \)

Let \( G_1, H_{1,1} \) and \( H_{1,2} \) be the groups from §1.5. The set \( \mathcal{H} \) of proper subgroups \( H \) of \( G_1 \) for which \( \pm H = G_1 \) is equal to \( \{H_{1,1}, H_{1,2}\} \).

Let \( E/\mathbb{Q} \) be the elliptic curve defined by \( y^2 + xy + y = x^3 + x^2 - 305x + 7888 \); it has \( j \)-invariant \(-11^2 \) and is isomorphic to the curve \( E_1 \) of §1.5. In §4.5.4, we showed that \( \rho_{E,11}(\text{Gal}_\mathbb{Q}) \) and \( \pm \rho_{E,11}(\text{Gal}_\mathbb{Q}) \) are conjugate in \( \text{GL}_2(\mathbb{F}_{11}) \) to \( H_{1,1} \) and \( G_1 \), respectively.

Using Lemma 5.4 and \( |\mathcal{D}_E| = |\mathcal{H}| \), we deduce that \( \mathcal{D}_E = \{1, -11\} \). Lemma 5.1 implies that if \( E'/\mathbb{Q} \) has \( j \)-invariant \(-11^2 \), then \( \rho_{E',11}(\text{Gal}_\mathbb{Q}) \) is not conjugate to \( G_1 \) if and only if \( E' \) is isomorphic to \( E \) or its quadratic twist by \(-11\). If \( E' \) is isomorphic to \( E \) or its twist by \(-11\), then \( \rho_{E',11}(\text{Gal}_\mathbb{Q}) \) is conjugate in \( \text{GL}_2(\mathbb{F}_{11}) \) to \( H_{1,1} \) or \( H_{1,2} \), respectively.

### 5.3.3. Take \( \ell = 11 \)

Let \( G_2, H_{2,1} \) and \( H_{2,2} \) be the subgroups of \( \text{GL}_2(\mathbb{F}_{11}) \) from §1.5. The set \( \mathcal{H} \) of proper subgroups \( H \) of \( G_2 \) for which \( \pm H = G_2 \) is equal to \( \{H_{2,1}, H_{2,2}\} \).

Let \( E/\mathbb{Q} \) be the elliptic curve defined by \( y^2 + xy + y = x^3 + x^2 - 363x + 82757 \); it has \( j \)-invariant \(-11 \cdot 131^3 \) and is isomorphic to the curve \( E_2 \) of §1.5. In §4.5.4, we showed that \( \rho_{E,11}(\text{Gal}_\mathbb{Q}) \) and \( \pm \rho_{E,11}(\text{Gal}_\mathbb{Q}) \) are conjugate in \( \text{GL}_2(\mathbb{F}_{11}) \) to \( H_{2,1} \) and \( G_2 \), respectively.

Using Lemma 5.4 and \( |\mathcal{D}_E| = |\mathcal{H}| \), we deduce that \( \mathcal{D}_E = \{1, -11\} \). Lemma 5.1 implies that if \( E'/\mathbb{Q} \) has \( j \)-invariant \(-11 \cdot 131^3 \), then \( \rho_{E',11}(\text{Gal}_\mathbb{Q}) \) is not conjugate to \( G_2 \) if and only if \( E' \) is isomorphic to \( E \) or its quadratic twist by \(-11\). If \( E' \) is isomorphic to \( E \) or its twist by \(-11\), then \( \rho_{E',11}(\text{Gal}_\mathbb{Q}) \) is conjugate in \( \text{GL}_2(\mathbb{F}_{11}) \) to \( H_{2,1} \) or \( H_{2,2} \), respectively.

### 6. Quadratic twists of families

In this section, we complete the proof of the theorems from §1.

#### 6.1. General setting

Fix an integer \( N \geq 3 \) and an applicable subgroup \( G \) of \( \text{GL}_2(\mathbb{Z}/N\mathbb{Z}) \). Let \( \mathcal{H} \) be the set of proper subgroups \( H \) of \( G \) that satisfy \( \pm H = G \).

Assume that the morphism \( \pi_G : X_G \to \mathbb{P}^1_{\mathbb{Q}} \) arises from a rational function \( J(t) \in \mathbb{Q}(t) \), i.e., the function field of \( X_G \) is of the form \( \mathbb{Q}(h) \) where \( j = J(h) \).

Let \( g(t) \) be a rational function \( \mathbb{Q}(t) \) such that

\[
\begin{align*}
a(t) := -3g(t)^2 J(t)/(J(t) - 1728) \\
b(t) := -2g(t)^3 J(t)/(J(t) - 1728)
\end{align*}
\]

belong to \( \mathbb{Z}[t] \), and for which there is no irreducible element \( \pi \) of the ring \( \mathbb{Z}[t] \) such that \( \pi^2 \) divides \( a \) and \( \pi^3 \) divides \( b \). After possibly changing \( g \) by a sign, we may assume that \( g \) is the quotient of two polynomials with positive leading coefficient; the function \( g(t) \) is now uniquely determined. Define \( \Delta := -16(4a^3 + 27b^2) \); it is a polynomial in \( \mathbb{Z}[t] \) and equals \( 2^{12}3^6 J(t)^2/(J(t) - 1728)^3 g(t)^6 \).

Let \( \mathscr{M} \) be the set of squarefree \( f(t) \in \mathbb{Z}[t] \) which divide \( N\Delta(t) \).

Take any \( u \in \mathbb{Q} \) for which \( J(u) \notin \{0, 1728, \infty\} \). We have \( \Delta(u) \neq 0 \) and hence \( f(u) \neq 0 \) for all \( f \in \mathscr{M} \). Let \( E_u/\mathbb{Q} \) be the elliptic curve defined by the Weierstrass equation \( y^2 = x^3 + a(u)x + b(u) \); note that \( \Delta(u) \neq 0 \) since \( J(u) \notin \{0, 1728, \infty\} \). One can readily check that the curve \( E_u \) has \( j \)-invariant \( J(u) \). Warning: this is not to be confused with the quadratic twist notation we used in §5.
Proposition 6.1. There is an injective map 
\[ \mathcal{H} \to \mathcal{M}, \quad H \mapsto f_H \]
such that for any \( u \in \mathbb{Q} \) with \( J(u) \notin \{0, 1728, \infty\} \) and \( \pm \rho_{E_u,N}(\text{Gal}_Q) \) conjugate to \( G \) in \( \text{GL}_2(\mathbb{Z}/N\mathbb{Z}) \), the following hold:

(a) If \( E'/\mathbb{Q} \) is an elliptic curve with \( j \)-invariant \( J(u) \), then \( \rho_{E',N}(\text{Gal}_Q) \) is conjugate to \( G \) in \( \text{GL}_2(\mathbb{Z}/N\mathbb{Z}) \) if and only if \( E' \) is not isomorphic to the quadratic twist of \( E_u \) by \( f_H(u) \) for all \( H \in \mathcal{H} \).

(b) If \( E'/\mathbb{Q} \) is isomorphic to the quadratic twist of \( E_u \) by \( f(u) \) for some \( H \in \mathcal{H} \), then \( \rho_{E',N}(\text{Gal}_Q) \) is conjugate to \( H \) in \( \text{GL}_2(\mathbb{Z}/N\mathbb{Z}) \).

The sets \( \{f_H(u) : H \in \mathcal{H}\} \) and \( \mathcal{D}_{E_u} \) represent the same cosets in \( Q^\times/(Q^\times)^2 \), with \( \mathcal{D}_{E_u} \) defined as in §5.

Proof. Define the scheme \( U := \text{Spec} \mathbb{Z}[t, N^{-1}, \Delta(t)^{-1}] \). By taking the square root of a polynomial \( f \in \mathcal{M} \), we obtain an étale extension of \( U \) of degree 1 or 2; we denote the corresponding quadratic character by \( \chi_f : \pi_1(U) \to \{\pm 1\} \). Conversely, every (continuous) character \( \pi_1(U) \to \{\pm 1\} \) is of the form \( \chi_f \) for a unique \( f \in \mathcal{M} \). (Note that 2 always divides \( \Delta(t) \)).

The Weierstrass equation
\[ y^2 = x^3 + a(t)x + b(t) \]
defines a relative elliptic curve \( E \to U \). Let \( E[N] \) be the \( N \)-torsion subscheme of \( E \). The morphism \( E[N] \to U \) allows us to view \( E[N] \) as a lisse sheaf of \( \mathbb{Z}/N\mathbb{Z} \)-modules on \( U \) that is free of rank 2. The sheaf \( E[N] \) then gives rise to a representation
\[ \rho_N : \pi_1(U) \to \text{GL}_2(\mathbb{Z}/N\mathbb{Z}) \]
that is uniquely defined up to conjugacy (we will suppress the base point in our fundamental group since we are only interested in \( \rho_N \) up to conjugacy).

We now consider specializations of \( E \). Take any \( u \in U(\mathbb{Q}) \), i.e., an element \( u \in \mathbb{Q} \) with \( \Delta(u) \neq 0 \). One can show that elements \( u \in U(\mathbb{Q}) \) can also be described as those \( u \in \mathbb{Q} \) for which \( J(u) \notin \{0, 1728, \infty\} \). We can specialize \( E \) at \( u \) to obtain the elliptic curve that we have denoted \( E_u/\mathbb{Q} \); it is defined by \( y^2 = x^3 + a(u)x + b(u) \) and has \( j \)-invariant \( J(u) \).

Let \( \rho_{u,N} : \text{Gal}_Q \to \text{GL}_2(\mathbb{Z}/N\mathbb{Z}) \) be the specialization of \( \rho_N \) at \( u \); it is obtained by composing the homomorphism \( u_0 : \text{Gal}_Q \to \pi_1(U) \) coming from \( u \in U(\mathbb{Q}) \) with \( \rho_N \). The homomorphism \( u_0 \) agrees, up to conjugacy, with the representation \( \rho_{E_u,N} \) that describes the Galois action on the \( N \)-torsion points of \( E_u \). So taking \( \rho_{E_u,N} = \rho_{u,N} \), specialization gives an inclusion \( \rho_{E_u,N}(\text{Gal}_Q) \subseteq \rho_N(\pi_1(U)) \).

We claim that \( \pm \rho_N(\pi_1(U)) \) and \( G \) are conjugate in \( \text{GL}_2(\mathbb{Z}/N\mathbb{Z}) \). By Lemma 3.5, the group \( \pm \rho_{E_u,N}(\text{Gal}_Q) \) is conjugate to \( G \) in \( \text{GL}_2(\mathbb{Z}/N\mathbb{Z}) \) for “most” \( u \in \mathbb{Q} \). By Hilbert’s irreducibility theorem, the group \( \pm \rho_{E_u,N}(\text{Gal}_Q) \) equals \( \pm \rho_N(\pi_1(U)) \) for “most” \( u \in \mathbb{Q} \). This proves the claim.

We may thus assume that \( G = \pm \rho_N(\pi_1(U)) \) and hence we have a representation \( \rho_N : \pi_1(U) \to G \). Specializations thus give inclusions \( \rho_{E_u,N}(\text{Gal}_Q) \subseteq G \). Take any \( H \in \mathcal{H} \) and let \( \chi_H : \pi_1(U) \to \{\pm 1\} \) be the character obtained by composing \( \rho_N \) with the quotient map \( G \to G/H \cong \{\pm 1\} \). We thus have \( \chi_H = \chi_{f_H} \) for a unique polynomial \( f_H \in \mathcal{M} \).

Specializing \( \chi_H \) at \( u \), we obtain the character \( \chi_{E_u,H} : \text{Gal}_Q \to \{\pm 1\} \) from §5. With notation as in §5, we find that the integer \( d_{E_u,H} \) lies in the same class in \( Q^\times/(Q^\times)^2 \) as \( f_H(u) \). Therefore, the classes of \( \mathcal{D}_{E_u} \) in \( Q^\times/(Q^\times)^2 \) are represented by the set \( \{f_H(u) : H \in \mathcal{H}\} \). Parts (a) and (b) are now immediate consequences of Lemma 5.1. \( \Box \)

We claim that the set of polynomials
\[ \mathcal{F} := \{f_H : H \in \mathcal{H}\} \]
is uniquely determined and has cardinality \(|\mathcal{H}|\). By Hilbert irreducibility, one can chose \(u \in U(\mathbb{Q})\) such that \(\pm \rho_{E,u,N}(\text{Gal}_{\mathbb{Q}}) = G\) and such that the map \(\mathcal{M} \to \mathbb{Q}^\times/(\mathbb{Q}^\times)^2, f \mapsto f(u) \cdot (\mathbb{Q}^\times)^2\) is injective. The uniqueness of \(\mathcal{F}\) then follows from part (a) of Proposition 6.1.

### 6.2 Computing \(\mathcal{F}\).

We now focus on the case where \(N\) is a prime \(\ell \in \{3, 5, 7, 13\}\). Fix notation as in the subsection of §1 for the given \(\ell\).

Let \(G\) be one of the subgroups \(G_i\) of \(\text{GL}_2(\mathbb{F}_\ell)\) in §1 for which there is a corresponding rational function \(J(t) := J_i(t) \in \mathbb{Q}(t) - \mathbb{Q}\). The group \(G\) is applicable and in particular contains \(-I\).

We take notation as in §6.1. In particular, \(\mathcal{H}\) is the set of proper subgroups \(H\) of \(G\) such that \(\pm H = G\). We shall assume that \(\mathcal{H} \neq \emptyset\) (otherwise \(\mathcal{F} = \emptyset\)); this holds when

\[
(\ell, i) \in \{(3, 1), (3, 3), (5, 1), (5, 5), (5, 6), (7, 1), (7, 3), (7, 4), (7, 5), (7, 7), (13, 4), (13, 5)\}.
\]

In each of these cases, one can check that \(|\mathcal{H}| = 2\).

We now explain how to compute the set \(\mathcal{F} = \{f_H : H \in \mathcal{H}\}\); it has cardinality \(|\mathcal{H}| = 2\). Take any \(u \in \mathbb{Q}\) with \(J(u) \notin \{0, 1728, \infty\}\) such that \(J(u) \notin J_j(\mathbb{Q})\) for all \(j < i\). From the parts of the main theorems proved in §4, this implies that \(\pm \rho_{E,u,\ell}(\text{Gal}_{\mathbb{Q}})\) is conjugate to \(G\). Let \(D_{E,u}\) be the (computable!) set from §5. From Proposition 6.1, we find that

\[
\mathcal{F} \subseteq \{f \in \mathcal{M} : f(u) \in d(\mathbb{Q}^\times)^2 \text{ for some } d \in D_{E,u}\}.
\]

By considering (6.1) with many such \(u \in \mathbb{Q}\), one is eventually left with only \(|\mathcal{H}|\) candidates \(f \in \mathcal{F}\) to be of the form \(f_H\); this then produces the set \(\{f_H : H \in \mathcal{H}\}\) of order \(|\mathcal{H}|\) (for our examples, one only needs to check \(u \in \{1, 2, 3, 4\}\)). One could also work with a single \(u \in \mathbb{Q}\) chosen so that the map \(\mathcal{F} \to \mathbb{Q}^\times/(\mathbb{Q}^\times)^2, f \mapsto f(u) \cdot (\mathbb{Q}^\times)^2\) is injective. This method thus produces \(\mathcal{F}\).

Doing the above computations, we find that

\[
\{f_H : H \in \mathcal{H}\} = \{f_1, \ell^* f_1\}
\]

for a unique polynomial \(f_1 \in \mathcal{F}\), where \(\ell^* := (-1)^{(\ell-1)/2} \cdot \ell\); this can also be deduced from \(|\mathcal{H}| = 2\) and Lemma 5.4. We thus have \(f_1 = f_{M_1}\) and \(\ell^* f_1 = f_{M_2}\), where \(\mathcal{H} = \{M_1, M_2\}\).

Let \(h\) be the largest element of \(\mathbb{Z}[t]\), in terms of divisibility, with positive leading coefficient such that \(h^4\) divides \(a f_1^2\) and \(h^6\) divides \(b f_1^3\); define \(A := (af_1^3)/h^4\) and \(B := (bf_1^3)/h^6\) in \(\mathbb{Z}[t]\). The Weierstrass equation

\[
y^2 = x^3 + A(t)x + B(t)
\]

is precisely the equation given for \(E_{i,u}\) in the subsection of §1 corresponding to the prime \(\ell\). (For code verifying these claims, see the link given in §1.10.)

For \(u \in \mathbb{Q}\) with \(J(u) \notin \{0, 1728, \infty\}\), let \(E_{i,u}\) be the elliptic curve over \(\mathbb{Q}\) defined by setting \(t\) equal to \(u\). Let \(E'/\mathbb{Q}\) be any elliptic curve with \(j_{E'} \notin \{0, 1728\}\) for which \(\pm \rho_{E',\ell}(\text{Gal}_{\mathbb{Q}})\) is conjugate to \(G\) in \(\text{GL}_2(\mathbb{F}_\ell)\). From the parts of the main theorems proved in §4, we have \(j_{E'} = J(u)\) for some \(u \in \mathbb{Q}\). The curve \(E_u/\mathbb{Q}\) also has \(j\)-invariant \(J(u)\). The twist of \(E_u\) by \(f_1(u)\) is isomorphic to the the curve \(E_{i,u}/\mathbb{Q}\). By Proposition 6.1, we deduce that \(\rho_{E',\ell}(\text{Gal}_{\mathbb{Q}})\) is conjugate to \(G\) if and only if \(E'\) is not isomorphic to \(E_{i,u}\) and not isomorphic to the quadratic twist of \(E_{i,u}\) by \(\ell^*\). By Proposition 6.1, \(\rho_{E',\ell}(\text{Gal}_{\mathbb{Q}})\) is conjugate to \(M_1\) or \(M_2\) when \(E'\) is isomorphic to \(E_{i,u}\) or the quadratic twist of \(E_{i,u}\) by \(\ell^*\), respectively.

It thus remains to determine \(M_1\) and \(M_2\).

If \((\ell, i) \in \{(3, 1), (7, 1)\}\), then \(M_1\) and \(M_2\) are both conjugate to \(H_{i,1}\) since the two groups in \(\mathcal{H}\) are conjugate in \(\text{GL}_2(\mathbb{F}_\ell)\). We shall now assume that \((\ell, i) \notin \{(3, 1), (7, 1)\}\). We then have
\( \mathcal{H} = \{H_{i,1}, H_{i,2}\} \). It thus remains to prove that \( M_1 = H_{i,1} \) (and hence \( M_2 = H_{i,2} \)).

Suppose that \((\ell, i) \in \{(5, 1), (5, 5), (5, 6), (7, 3), (7, 4), (13, 4), (13, 5)\}\). Take \( u, p \) and \( a \) as in Table 2 below for the pair \((\ell, i)\).

| \((\ell, i)\) | (5, 1) | (5, 5) | (5, 6) | (7, 3) | (7, 4) | (13, 4) | (13, 5) |
|---|---|---|---|---|---|---|---|
| \( u \) | 1 | 2 | 1 | 2 | 2 | 1 | 1 |
| \( p \) | 2 | 3 | 2 | 3 | 3 | 2 | 2 |
| \( a \) | −2 | −1 | −2 | −3 | −3 | 2 | 2 |

Table 2.

The element \( u \in \mathbb{Q} \) is chosen so that \( J_i(u) \notin \{0, 1728, \infty\} \) and such that \( J_i(u) \notin J_j(\mathbb{Q} \cup \{\infty\}) \) for all \( j < i \). Define the elliptic curve \( E := \mathcal{E}_{i,u}/\mathbb{Q} \). By our choice of \( u \), the group \( \rho_{E,\ell}(\text{Gal}_\ell) \) is conjugate in \( \text{GL}_2(\mathbb{F}_\ell) \) to \( M_1 \).

The curve has good reduction at the prime \( p \) and we have \( a = a_p(E) \). Let \( t_p \) be the image of \((a, p)\) in \( \mathbb{F}_\ell \); it equals \((\text{tr}(A), \det(A))\) with \( A := \rho_{E,\ell}(\text{Frob}_p) \in M_1 \). A direct computation shows that \( t_p \notin \{(\text{tr}(A), \det(A)) : A \in H_{i,2}\} \). Therefore, \( M_1 \) is not conjugate to \( H_{i,2} \). So \( M_1 \) must be conjugate to \( H_{i,1} \) and hence \( M_2 \) is conjugate to \( H_{i,2} \).

Finally, consider the remaining pairs \((\ell, i) \in \{(3, 3), (7, 5), (7, 7)\}\).

Consider \((\ell, i) = (3, 3)\). The pair \((3(u + 1)^2, 4(u(u + 1)^2)\) is a point of order 3 of \( \mathcal{E}_{3,u} \) for all \( u \). This implies that \( M_1 \) is conjugate in \( \text{GL}_2(\mathbb{F}_3) \) to a subgroup of \((1, \ast)\). So \( M_1 \neq H_{i,2} \) and hence \( M_1 = H_{i,1} \).

We may now suppose that \( \ell = 7 \) and \( i \in \{5, 7\} \).

Take \( i = 5 \). Let \( E'/\mathbb{Q} \) be the elliptic curve defined by \( y^2 = x^3 - 2835(-7)^2x - 71442(-7)^3 \); it is the quadratic twist of \( \mathcal{E}_{5,0} \) by \(-7\). Using Theorem 1.5(ii), which we proved in §4, we find that \( \pm \rho_{E',\ell}(\text{Gal}_\ell) \) is conjugate to \( G_5 \). The group \( \rho_{E',\ell}(\text{Gal}_\ell) \) is thus conjugate to \( M_2 \). So to prove that \( M_2 = H_{i,2} \), and hence \( M_1 = H_{i,1} \), we need only verify that \( E' \) has a 7-torsion point defined over some cubic field. Let \( w \in \mathbb{Q} \) be a root of the irreducible polynomial \( x^3 - 441x^2 - 83349x + 22754277 \). The pair \((w, 21w - 1323)\) is a point of order 7 on \( E' \).

Finally, take \( i = 7 \). Let \( E'/\mathbb{Q} \) be the elliptic curve defined by \( y^2 = x^3 - 17870609043(-7)^2x + 19567237639563291 \). The pair \((w, 1323w - 714884373)\) is a point of order 7 on \( E' \).

7. Proof of Propositions from §1.9

Let \( E \) be an elliptic curve defined over \( \mathbb{Q} \) that has complex multiplication. Let \( R \) be the ring of endomorphisms of \( E_{\overline{\mathbb{Q}}} \). Let \( k \subseteq \overline{\mathbb{Q}} \) be the minimal extension of \( \mathbb{Q} \) over which all the endomorphisms of \( E_{\overline{\mathbb{Q}}} \) are defined; it is an imaginary quadratic field. Moreover, we can identify \( k \) with \( R \otimes_{\mathbb{Z}} \mathbb{Q} \) (the action of \( R \) on the Lie algebra of \( E_k \) gives a ring homomorphism \( R \to k \) that extends to an isomorphism \( R \otimes_{\mathbb{Z}} \mathbb{Q} \to k \)). The field \( k \) has discriminant \(-D\).

Take any odd prime \( \ell \). For each integer \( n \geq 1 \), let \( E[\ell^n] \) be the \( \ell^n \)-torsion subgroup of \( E(\overline{\mathbb{Q}}) \). The \( \ell \)-adic Tate module \( T_\ell(E) \) of \( E \) is the inverse limit of the groups \( E[\ell^n] \) with multiplication by \( \ell \) giving transition maps \( E[\ell^{n+1}] \to E[\ell] \); it is a free \( \mathbb{Z}_\ell \)-module of rank 2. The natural Galois action
on $T_{\ell}(E)$ can be expressed in terms of a representation
\[ \rho_{E,\ell^*} : \text{Gal}_k \to \text{Aut}_{\mathbb{Z}_\ell}(T_{\ell}(E)). \]
The ring $R$ acts on each of the $E[\ell^n]$ and this induces a faithful action of $R$ on $T_{\ell}(E)$.

The Tate module $T_{\ell}(E)$ is actually a free module over $R_{\ell} := R \otimes_{\mathbb{Z}} \mathbb{Z}_\ell$ of rank 1 (see the remarks at the end of §4 of [ST68]). We can thus make an identification $\text{Aut}_{R_{\ell}}(T_{\ell}(E)) = R_{\ell}^\times$. The actions of $\text{Gal}_k = \text{Gal}(\overline{\mathbb{Q}}/k)$ and $R_{\ell}$ on $T_{\ell}(E)$ commute, so the restriction of $\rho_{E,\ell^*}$ to $\text{Gal}_k$ gives a representation
\[ \text{Gal}_k \to \text{Aut}_{R_{\ell}}(T_{\ell}(E)) = R_{\ell}^\times. \]

**Lemma 7.1.**

(i) If $E$ has good reduction at $\ell$, then $\rho_{E,\ell^*}(\text{Gal}_k) = R_{\ell}^\times$.

(ii) If $j_E \neq 0$, then $\rho_{E,\ell^*}(\text{Gal}_k)$ is an open subgroup of $R_{\ell}^\times$ whose index is a power of 2.

**Proof.** Since $R_{\ell}^\times$ is commutative, we can factor $\rho_{E,\ell^*}|_{\text{Gal}_k}$ through the maximal abelian quotient of $\text{Gal}_k$. Composing with the reciprocity map of class field theory, we obtain a continuous representation $\rho_{E,\ell^*} : \mathbb{A}_k^\times \to R_{\ell}^\times$, where $\mathbb{A}_k^\times$ is the group of ideles of $k$. Define $k_{\ell} := k \otimes_{\mathbb{Z}} \mathbb{Q}_{\ell} = \prod_{\nu \mid \ell} k_{\nu}$, where the product is over the places $\nu$ of $k$ lying over $\ell$ and $k_{\nu}$ is the completion of $k$ at $\nu$. For an idele $a \in \mathbb{A}_k^\times$, let $a_{\ell}$ be the component of $a$ in $k_{\ell}^\times$. From [ST68, Theorems 10 & 11], there is a unique homomorphism $\varepsilon : \mathbb{A}_k^\times \to k^\times$ such that $\varepsilon(\rho_{E,\ell^*}(a)) = \varepsilon(a)a_{\ell}^{-1}$ for $a \in \mathbb{A}_k^\times$. The homomorphism $\varepsilon$ satisfies $\varepsilon(x) = x$ for all $x \in k^\times$ and its kernel is open in $\mathbb{A}_k^\times$. We identify $R_{\ell}^\times = \prod_{\nu \mid \ell} \mathcal{O}_\nu^\times$, where $\mathcal{O}_\nu$ is the valuation ring of $k_{\nu}$, with a subgroup of $\mathbb{A}_k^\times$ (by letting the coordinates at the places $\nu \nmid \ell$ of $k$ be 1). Let $B$ be the kernel of $\varepsilon|_{R_{\ell}^\times}$.

First suppose that $E$ has good reduction at $\ell$, and hence at all places $\nu \mid \ell$ of $k$. By the first corollary of Theorem 11 in [ST68], we deduce that $\varepsilon$ is unramified at all $\nu \mid \ell$. Therefore, $B = R_{\ell}^\times$ and hence $\rho_{E,\ell^*}(R_{\ell}^\times) = R_{\ell}^\times$. Therefore, $\rho_{E,\ell^*}(\text{Gal}_k)$ contains, and hence is equal to, $R_{\ell}^\times$.

Now suppose that $j_E \neq 0$. Since $\ell$ is odd and $j_E \neq 0$, the subgroup of $R[\ell^{-1}]^\times$ consisting of roots of unity has order 2 or 4. By Theorem 11(ii) and Theorem 6(b) in [ST68], we find that $B$ is an open subgroup of $R_{\ell}^\times$ with index a power of 2. So $\rho_{E,\ell^*}(B) = B$ and hence $\rho_{E,\ell^*}(\text{Gal}_k) \supseteq B$. Therefore, $\rho_{E,\ell^*}(\text{Gal}_k)$ is an open subgroup of $R_{\ell}^\times$ whose index is a power of 2. \(\square\)

The following gives constraints on the elements of $\rho_{E,\ell^*}(\text{Gal}_Q - \text{Gal}_k)$. Since $R$ is a quadratic order, there is an element $\beta \in R - \mathbb{Z}$ such that $\beta^2 \in \mathbb{Z}$; note that $\beta$ is not defined over $\mathbb{Q}$. We can view $\beta$ as an endomorphism of $T_{\ell}(E)$.

**Lemma 7.2.** For any $\sigma \in \text{Gal}_Q - \text{Gal}_k$, we have $\rho_{E,\ell^*}(\sigma)\beta = -\beta\rho_{E,\ell^*}(\sigma)$ and $\text{tr}(\rho_{E,\ell^*}(\sigma)) = 0$.

**Proof.** Take any $\sigma \in \text{Gal}_Q - \text{Gal}_k$. The group $\text{Gal}_Q$ acts on $R$ and we have $\sigma(\beta) = -\beta$ since $\beta^2 \in \mathbb{Z}$ and $\beta$ is not defined over $\mathbb{Q}$ (but is defined over $k$). So for each $P \in E[\ell^n]$, we have $\sigma(\beta(P)) = \sigma(\beta)(\sigma(P)) = -\beta(\sigma(P))$. Taking an inverse limit, we deduce that $\rho_{E,\ell^*}(\sigma)\beta = -\beta\rho_{E,\ell^*}(\sigma)$. In $\text{Aut}_{\mathbb{Q}_\ell}(T_{\ell}(E) \otimes_{\mathbb{Z}_\ell} \mathbb{Q}_\ell) \cong \text{GL}_2(\mathbb{Q}_\ell)$, we have $\rho_{E,\ell^*}(\sigma) = -\beta\rho_{E,\ell^*}(\sigma)^{-1}$. Taking traces we deduce that $\text{tr}(\rho_{E,\ell^*}(\sigma)) = -\text{tr}(\rho_{E,\ell^*}(\sigma))$ and hence $\text{tr}(\rho_{E,\ell^*}(\sigma)) = 0$. \(\square\)

**Lemma 7.3.** Suppose that $\ell \nmid D$ and that $E$ has good reduction at $\ell$.

(i) If $\ell$ splits in $k$, then $\rho_{E,\ell}(\text{Gal}_Q)$ is conjugate in $\text{GL}_2(\mathbb{F}_\ell)$ to $N_{\alpha}(\ell)$.

(ii) If $\ell$ is inert in $k$, then $\rho_{E,\ell}(\text{Gal}_Q)$ is conjugate in $\text{GL}_2(\mathbb{F}_\ell)$ to $N_{\alpha}(\ell)$.

**Proof.** Lemma 7.1 implies that the group $C := \rho_{E,\ell}(\text{Gal}_k)$ is isomorphic to $(R/\ell R)^\times$. The ring $R/\ell R$ is isomorphic to $\mathbb{F}_\ell \times \mathbb{F}_\ell$ or $\mathbb{F}_{\ell^2}$ when $\ell$ splits or is inert in $k$, respectively. Therefore, $C$ is a Cartan subgroup of $\text{GL}_2(\mathbb{F}_{\ell^2})$; it is split if and only if $\ell$ splits in $k$. Let $N$ be the normalizer of $C$ in $\text{GL}_2(\mathbb{F}_\ell)$. The group $C = \rho_{E,\ell}(\text{Gal}_k)$ is normal in $\rho_{E,\ell}(\text{Gal}_Q)$ since $k/\mathbb{Q}$ is a Galois extension, so $\rho_{E,\ell}(\text{Gal}_Q) \subseteq N$. 38
It remains to show that $\rho_{E,\ell}(\text{Gal}_Q) = N$. Suppose that $\rho_{E,\ell}(\text{Gal}_Q) \neq N$, and hence $\rho_{E,\ell}(\text{Gal}_Q) = C = \rho_{E,\ell}(\text{Gal}_k)$. This implies that the actions of $\text{Gal}_Q$ and $R$ on $E[\ell]$ commute. However, this contradicts Lemma 7.2 which implies that the actions of $\sigma \in \text{Gal}_Q - \text{Gal}_k$ and $\beta$ on $E[\ell]$ anti-commute. Therefore, $\rho_{E,\ell}(\text{Gal}_Q) = N$. 

We now describe the commutator subgroup of the normalizer $N$ of a Cartan subgroup $C$ of $\text{GL}_2(\mathbb{F}_\ell)$. Let $\varepsilon: N \to N/C \cong \{\pm 1\}$ be the quotient map, and define the homomorphism 

$$\varphi: N \to \{\pm 1\} \times \mathbb{F}_\ell^\times, \ A \mapsto (\varepsilon(A), \det(A));$$

it is surjective.

**Lemma 7.4.**

(i) The commutator subgroup of $N$ is ker $\varphi$, i.e., the subgroup of $C$ consisting of matrices with determinant 1.

(ii) If $H$ is a subgroup of $N$ satisfying $\pm H = N$, then $H = N$.

**Proof.** The kernel of $\varphi$ contains the commutator subgroup of $N$ since the image of $\varphi$ is abelian. It suffices to show that every element in ker $\varphi$ is a commutator. If $N$ is conjugate to $N_s(\ell)$, this is immediate since 

$$\left( \begin{array} {cc} 1 & \varepsilon \\ 0 & 1 \end{array} \right) \left( \begin{array} {cc} 0 & \beta \\ a & 0 \end{array} \right) \left( \begin{array} {cc} 1 & \varepsilon \\ 0 & 1 \end{array} \right)^{-1} \left( \begin{array} {cc} 0 & \beta \\ a & 0 \end{array} \right)^{-1} = \left( \begin{array} {cc} a & 0 \\ 0 & a^{-1} \end{array} \right).$$

We now consider the non-split case. We may take $C = C(\ell), N = N(\ell)$ and with the explicit $\varepsilon \in \mathbb{F}_\ell^\times$ as given in the notation section of §1. Fix $\beta \in \mathbb{F}_\ell^\times$ for which $\beta^2 = \varepsilon$. The map $C(\ell) \to \mathbb{F}_\ell^\times, \ (a b \varepsilon^2) \to a + b\beta$ is a group isomorphism. Fix any $B \in N(\ell) - C(\ell)$. One can check that the map 

$$C(\ell) \to C(\ell), \ A \mapsto BAB^{-1}A^{-1}$$

corresponds to the homomorphism $\mathbb{F}_\ell^\times \to \mathbb{F}_\ell^\times, \ a \to a^{\ell-1}$. In particular, the image of the map (7.1) is the unique (cyclic) subgroup of $C(\ell)$ of order $\ell + 1$; these are the matrices in $C(\ell)$ with determinant 1. This completes the proof of (i).

Finally, let $H$ be a subgroup of $N$ satisfying $\pm H = N$. The group $H$ is normal in $N$ and $N/H$ is abelian, so $H$ contains the commutator subgroup of $N$. From (i), the commutator subgroup of $N$, and hence $H$, contains $-I$. Therefore, $H = \pm H = N$. 

**7.1 Proof of Proposition 1.14(i) and (ii).** Let $E/\mathbb{Q}$ be an CM elliptic curve with $j_E \neq 0$. The curve $E$ is thus a twist of one of the curves $E_{D,f}/\mathbb{Q}$ from Table 1. Take any odd prime $\ell \nmid D$. The curve $E_{D,f}$ has good reduction at $\ell$. By Lemma 7.3, the group $\rho_{E_{D,f},\ell}(\text{Gal}_Q)$ is the normalizer $N$ of a Cartan subgroup $C$ of $\text{GL}_2(\mathbb{F}_\ell)$.

Also the Cartan subgroup $C$ is split or non-split if $\ell$ is split or inert, respectively, in $k$.

First suppose that $j_E \neq 1728$. Since $j_E \notin \{0, 1728\}$, the curve $E$ is a quadratic twist of $E_{D,f}$. As noted in the introduction, this implies that $\pm \rho_{E,\ell}(\text{Gal}_Q)$ and $\pm \rho_{E_{D,f},\ell}(\text{Gal}_Q) = N$ are conjugate in $\text{GL}_2(\mathbb{F}_\ell)$. After first conjugating $\rho_{E,\ell}(\text{Gal}_Q)$, we may assume that $N = \pm \rho_{E,\ell}(\text{Gal}_Q)$. By Lemma 7.4, we have $\rho_{E,\ell}(\text{Gal}_Q) = N$.

Now suppose that $j_E = 1728$. Let $\mu_4$ be the group of 4-th roots of unity in $R$. The elliptic curve $E/\mathbb{Q}$ can be defined by an equation of the form $y^2 = x^3 + dx$ for some non-zero integer $d$, i.e., $E$ is a quartic twist of $E_{4,1}$. There is thus a character $\alpha: \text{Gal}_k \to \mu_4 \subseteq R^\times$ such that the representations $\rho_{E,\ell}\alpha$ and $\alpha \cdot \rho_{E_{4,1},\ell}\alpha: \text{Gal}_k \to R^\times$ are equal. We have $\rho_{E_{4,1},\ell}\alpha(\text{Gal}_k) = R^\times_\ell$ by Lemma 7.1(i), so the image of $\rho_{E,\ell}\alpha(\text{Gal}_k)$ in $R^\times_\ell/\{\pm 1\}$ has index 1 or 2. Therefore, the image of $\rho_{E,\ell}(\text{Gal}_k)$ in $C/\{\pm I\}$ has index 1 or 2. We have $\rho_{E,\ell}(\text{Gal}_Q) \not\subseteq C$ since otherwise the actions of $\text{Gal}_Q$ and $R$ on $E[\ell]$ would commute (which is impossible by Lemma 7.2). Therefore, the image of $\rho_{E,\ell}(\text{Gal}_Q)$ in $N/\{\pm I\}$ is an index 1 or 2 subgroup.

The group $G := \pm \rho_{E,\ell}(\text{Gal}_Q)$ thus has index 1 or 2 in $N$. Since $\rho_{E,\ell}(\text{Gal}_k) \subseteq C$ and $\rho_{E,\ell}(\text{Gal}_Q) \not\subseteq C$, the quadratic character $\varepsilon \circ \rho_{E,\ell}: \text{Gal}_Q \to \{\pm 1\}$ corresponds to the extension $k = Q(i)$ of $Q$. The homomorphism $\det \circ \rho_{E,\ell}: \text{Gal}_Q \to \mathbb{F}_\ell^\times$ is surjective and factors through $\text{Gal}(Q(\zeta_\ell)/Q)$. We have
The group \( G \) contains the commutator subgroup of \( N \). By Lemma 7.4, we deduce that \( G \) contains the kernel of \( \varphi \). Since \( G \) contains the kernel of \( \varphi \) and \( \varphi(G) = \{ \pm 1 \} \times \mathbb{F}_q^\times \), we have \( G = N \). By Lemma 7.4, we conclude that \( \rho_{E,\ell}(\mathbb{Q}) = N \).

7.2. Proof of Proposition 1.14(ii). We first consider the elliptic curve \( E = E_{D,\ell} \) over \( \mathbb{Q} \) from Table 1 with \( D = \ell \), where \( \ell \) is an odd prime and \( j_E \neq 0 \). We have \( k = Q(\sqrt{-D}) \).

Lemma 7.5. The group \( \pm \rho_{E,\ell}(\mathbb{Q}) \) is conjugate to \( G \).

Proof. Let \( \overline{\beta} \) be the image of \( f \sqrt{-D} \) in \( R/\ell R \). Since \( \ell \) is odd, the \( \mathbb{F}_\ell \)-module \( R/\ell R \) has basis \( \{ \overline{\beta}, 1 \} \) and \( \overline{\beta}^2 = 0 \). Using this basis, we find that \( R/\ell R \) is isomorphic to the subring \( A := \mathbb{F}_\ell \left( \begin{smallmatrix} 0 & 1 \\ 0 & 0 \end{smallmatrix} \right) \oplus \mathbb{F}_\ell \left( \begin{smallmatrix} 1 & 0 \\ 0 & 1 \end{smallmatrix} \right) \) of \( M_2(\mathbb{F}_\ell) \). Using that \( \rho_{E,\ell}(\mathbb{Q}) \subseteq \mathbb{R}^\times \), we deduce that \( \rho_{E,\ell}(\mathbb{Q}) \) is conjugate in \( GL_2(\mathbb{F}_\ell) \) to a subgroup of \( \mathbb{R}^\times \). We may thus assume that

\[
\rho_{E,\ell}(\mathbb{Q}) \subseteq \mathbb{R}^\times = \left\{ \left( \begin{array}{cc} a & b \\ 0 & a \end{array} \right) : a \in \mathbb{F}_\ell^\times, b \in \mathbb{F}_\ell \right\}.
\]

By Lemma 7.1(ii), we deduce that \( \left[ \mathbb{R}^\times : \rho_{E,\ell}(\mathbb{Q}) \right] \) is a power of 2 and hence \( \rho_{E,\ell}(\mathbb{Q}) \) contains the order \( \ell \) group \( \left( \begin{smallmatrix} 1 & 1 \\ 0 & 1 \end{smallmatrix} \right) \). The order of \( \rho_{E,\ell}(\mathbb{Q}) \) is divisible by \((\ell - 1)/2 \) since \( \det(\rho_{E,\ell}(\mathbb{Q})) = (\mathbb{F}_\ell^\times)^2 \), so \( \rho_{E,\ell}(\mathbb{Q}) \) contains \( \left( \begin{array}{cc} a & b \\ 0 & a \end{array} \right) \) for some \( a \in \mathbb{F}_\ell^\times \) and \( b \in \mathbb{F}_\ell \). Therefore, \( \pm \rho_{E,\ell}(\mathbb{Q}) = \mathbb{R}^\times \) since \(-1\) is not a square in \( \mathbb{F}_\ell \) (we have \( \ell \equiv 3 \mod 4 \)).

Fix any \( \sigma \in \mathbb{Q} - \mathbb{Q}_\ell \). The matrix \( g = \rho_{E,\ell}(\sigma) \) is upper triangular since the Borel subgroup \( B(\ell) \) is the normalizer of \( \mathbb{R}^\times = \pm \rho_{E,\ell}(\mathbb{Q}) \) in \( GL_2(\mathbb{F}_\ell) \). We have \( \text{tr}(g) = 0 \) by Lemma 7.2, so \( g = \left( \begin{array}{cc} a & b \\ 0 & -a \end{array} \right) \) for some \( a \in \mathbb{F}_\ell^\times \) and \( b \in \mathbb{F}_\ell \). The group \( \pm \rho_{E,\ell}(\mathbb{Q}) \) is generated by \( g \) and \( \mathbb{R}^\times \) and is thus \( G \).

The subgroups \( H \) of \( G \) that satisfy \( \pm H = G \) are \( H_1, H_2 \) and \( G \).

Lemma 7.6. The groups \( \rho_{E,\ell}(\mathbb{Q}) \) and \( H_1 \) are conjugate in \( GL_2(\mathbb{F}_\ell) \).

Proof. By Lemma 7.5, we may assume that \( \pm \rho_{E,\ell}(\mathbb{Q}) = G \). There are thus unique characters \( \psi_1, \psi_2 : \mathbb{Q} \to \mathbb{F}_\ell^\times \) such that \( \rho_{E,\ell} = \left( \begin{array}{cc} \psi_1 & * \\ 0 & \psi_2 \end{array} \right) \). Let \( f \in \mathbb{Q}[x] \) be the \( \ell \)-th division polynomial of \( E/\mathbb{Q} \); it is a polynomial of degree \((\ell^2 - 1)/2 \) whose roots in \( \overline{\mathbb{Q}} \) are the \( x \)-coordinates of the non-zero points in \( E[\ell] \). Since \( \pm \rho_{E,\ell}(\mathbb{Q}) = G \), we find that \( f = f_1f_2 \) where the polynomials \( f_1, f_2 \in \mathbb{Q}[x] \) are irreducible, and \( f_1 \) has degree \((\ell - 1)/2 \). We may take \( f_1 \) so that it is monic. Take any root \( a \in \overline{\mathbb{Q}} \) of \( f_1 \) and choose a point \( P = (a, b) \in E[\ell] \). We have \( \sigma(P) = \psi_1(\sigma)P \) for all \( \sigma \in \mathbb{Q}_\ell \). Therefore, \( Q(a,b) \) is the fixed field in \( \overline{\mathbb{Q}} \) of \( \ker \psi_1 \).

Suppose that \( \ell = 3 \). The point \((3, -2)\) of \( E_{3,2} \) has order 3. The point \((12, -4)\) of \( E_{3,3} \) has order 3. Therefore, \( Q(a,b) = \mathbb{Q} \).

Suppose that \( \ell = 7 \). For the curve \( E_{7,1} \) we have computed that \( f_1 = x^3 - 441x^2 + 59339x - 2523451 \). If \( a \in \overline{\mathbb{Q}} \) is a root of \( f_1 \), then one can check that \( (a, -7a + 49) \) belongs to \( E \). For the curve \( E_{7,2} \) we have computed that \( f_1 = x^3 - 49x^2 - 1029x + 31213 \). If \( a \in \overline{\mathbb{Q}} \) is a root of \( f_1 \), then one can check that \( (21a - 2107) \) belongs to \( E \). In both cases, we have \( [Q(a,b) : \mathbb{Q}] = 3 \).

Suppose that \( \ell > 7 \). Dieulefait, González-Jiménez and Jiménez-Urrzo have computed \( Q(a,b) \) and found it to be equal to the maximal totally real subfield \( Q(\zeta_\ell)^+ \) of \( Q(\zeta_\ell) \), cf. Lemma 4 of [DGJJUU11]. They also give a link to files containing an explicit polynomial \( f_1 \). In particular, \( [Q(a,b) : \mathbb{Q}] = (\ell - 1)/2 \). (However, note that the conclusions on the image of \( \rho_{E,\ell} \) in Proposition 9 of [DGJJUU11] are not correct.) In all cases, the image of \( \psi_1 \) has order \([Q(a,b) : \mathbb{Q}] = (\ell - 1)/2 \), so the group \( \rho_{E,\ell}(\mathbb{Q}) \) cannot be \( G \) or \( H_2 \). Therefore, \( \rho_{E,\ell}(\mathbb{Q}) = H_1 \).
Take any elliptic curve $E'/\mathbb{Q}$ with the same $j$-invariant as $E = E_{D,f}$; it is a quadratic twist. Now take $D_E$ as in §5. Since $#D_E = # \{H_1, H_2 \} = 2$, we deduce from Lemma 5.4 (and $\ell \equiv 3 \pmod{4}$) that $D_E = \{1, -\ell\}$.

Since $D_E = \{1, -\ell\}$, we deduce from Lemma 7.6 that if $E'/\mathbb{Q}$ is not isomorphic to $E$ or its quadratic twist by $-\ell$, then $\rho_{E',\ell}(\text{Gal}_{\mathbb{Q}})$ is conjugate to $\pm \rho_{E,\ell}(\text{Gal}_{\mathbb{Q}}) = \pm H_1 = G$. If $E'$ is isomorphic to $E$ or its quadratic twist by $-\ell$, then $\rho_{E,\ell}(\text{Gal}_{\mathbb{Q}})$ is conjugate to $H_1$ or $H_2$, respectively.

7.3. Proof of Proposition 1.15. If $E/\mathbb{Q}$ is given by $y^2 = f(x)$ with $f(x) \in \mathbb{Q}[x]$ a separable cubic, then $\rho_{E,2}(\text{Gal}_{\mathbb{Q}})$ is isomorphic to the groups $\text{Gal}(f)$, i.e., the Galois group of the splitting field of $f$ over $\mathbb{Q}$. Observe that $\text{GL}_2(\mathbb{F}_2) \cong \mathbb{S}_3$. It thus suffices to compute $\text{Gal}(f)$ since the cardinality of a subgroup of $\mathbb{S}_3$ determines it up to conjugacy.

For the $j_E = 1728$ case, we have $f(x) = x^3 - dx = x(x^2 - d)$. We have $\text{Gal}(f) = \text{Gal}(\mathbb{Q}(\sqrt{d})/\mathbb{Q})$ which has order 1 or 2 when $d$ is a cube or non-cube, respectively.

For the $j_E = 0$ case, we have $f(x) = x^3 + d$. We have $\text{Gal}(f) = \text{Gal}(\mathbb{Q}(\sqrt{d}, \zeta_3)/\mathbb{Q}) = \text{Gal}(\mathbb{Q}(\sqrt{d}, \sqrt{-3})$ which has order 2 or 6 when $d$ is a cube or non-cube, respectively.

For $j_E \notin \{0, 1728\}$, the group $\rho_{E,2}(\text{Gal}_{\mathbb{Q}})$ does not change if we replace $E$ by a quadratic twist (since $-I \equiv I \pmod{\ell}$), so one need only consider the specific curve $E = E_{D,f}$. Using the $f(x)$ of Table 1, one can check that $\text{Gal}(f)$ has order 2 for the $j$-invariants listed in (i) and otherwise has order 6.

Proposition 1.15 is now a direct consequence of the above computations.

7.4. Proof of Proposition 1.16. Take any prime $\ell \geq 5$; we will deal with $\ell = 3$ in §7.4.1. We first consider an elliptic curve $E_d/\mathbb{Q}$ defined by the equation

$$y^2 = x^3 + 16d^2$$

for a fixed cube-free integer $d \geq 1$. We have $R = \mathbb{Z}[\omega]$ and $k = \mathbb{Q}(\omega)$, where $\omega := (-1 + \sqrt{-3})/2$ is a cube root of unity in $k$. The ring $R$ is a PID.

If $\ell$ is congruent to 1 or 2 modulo 3, define $C(\ell)$ be the Cartan subgroup $C_n(\ell)$ or $C_{ns}(\ell)$, respectively. Let $N(\ell)$ be the normalizer of $C(\ell)$ in $\text{GL}_2(\mathbb{F}_\ell)$.

**Lemma 7.7.** After replacing $\rho_{E_d,\ell}$ by a conjugate representation, we will have $\rho_{E_d,\ell}(\text{Gal}_{\mathbb{Q}}) \subseteq N(\ell)$ and $\rho_{E_d,\ell}(\text{Gal}_k) \subseteq C(\ell)$ with

$$[N(\ell) : \rho_{E_d,\ell}(\text{Gal}_{\mathbb{Q}})] = [C(\ell) : \rho_{E_d,\ell}(\text{Gal}_k)] \in \{1, 3\}.$$

**Proof.** We have $E_1 = E_{3,1}$. By Lemma 7.3, we have $\rho_{E_1,\ell}(\text{Gal}_{\mathbb{Q}}) = N(\ell)$. The curves $E_d$ and $E_1$ are isomorphic over $\mathbb{Q}(\sqrt{d})$, so $\rho_{E_d,\ell}(\text{Gal}_{\mathbb{Q}(\sqrt{d})})$ is conjugate to a subgroup of $N(\ell)$ of index 1 or 3. Therefore, $\rho_{E_d,\ell}(\text{Gal}_{\mathbb{Q}})$ is conjugate to a subgroup of $N(\ell)$ of index 1 or 3. Since $\rho_{E_d,\ell}(\text{Gal}_{\mathbb{Q}}) \not\subseteq C(\ell)$ and $\rho_{E_d,\ell}(\text{Gal}_k) \subseteq C(\ell)$, we deduce that $[N(\ell) : \rho_{E_d,\ell}(\text{Gal}_{\mathbb{Q}})] = [C(\ell) : \rho_{E_d,\ell}(\text{Gal}_k)]$. \hfill $\Box$

To determine the index in Lemma 7.7, we first compute some cubic residue symbols. Recall that we have already defined a representation $\rho_{E_d,\ell^\infty} : \text{Gal}_k \to \mathbb{R}_\ell^\times$.

**Lemma 7.8.** Let $\lambda$ be a prime of $R$ dividing $\ell$ that satisfies $\lambda \equiv 2 \pmod{3\ell}$. Take any non-zero prime ideal $\mathfrak{p} \mid 6d\ell$ of $R$. We have $\mathfrak{p} = R\pi$ for some $\pi \equiv 2 \pmod{3\ell}$. Then

$$\left( \frac{\rho_{E_d,\ell^\infty}(\text{Frob}_\mathfrak{p})}{\lambda} \right) \equiv \left( \frac{d^{\frac{\ell(\ell^2 - 1)}{3}} \lambda}{\pi} \right)_3,$$

where we are using cubic residue characters and the field $R/\lambda R$ has order $\ell^e$.
Proof. By Example 10.6 of [Sil94, II §10], we have $\rho_{E_d,\ell,\infty}(\text{Frob}_p) = -\left(\frac{4 \cdot 16d^2}{\pi}\right)_6 \cdot \pi$, where we are using the 6-th power residue symbol. Therefore,

$$
\rho_{E_d,\ell,\infty}(\text{Frob}_p) = -\left(\frac{d}{\pi}\right)_6^2 = -\left(\frac{d}{\pi}\right)_3 = -\left(\frac{d}{\pi}\right)_3^2 \cdot \pi
$$

and hence

$$
\left(\frac{\rho_{E_d,\ell,\infty}(\text{Frob}_p)}{\lambda}\right)_3 = \left(-\frac{(d^2)}{\lambda}\right)_3 \cdot \pi = \left(d^2\right)_3 \left(\frac{\ell - 1}{\pi}\right)_3 \left(\frac{\pi}{\lambda}\right)_3 = \left(d^2\right)_3 \left(\frac{\ell - 1}{\pi}\right)_3 \left(\frac{\lambda}{\pi}\right)_3 = \left(d^2\right)_3 \left(\frac{\ell - 1}{\pi}\right)_3 \left(\frac{\lambda}{\pi}\right)_3,
$$

where we have used cubic reciprocity.

□

Lemma 7.9. Suppose that $\ell \equiv 2 \pmod{3}$. Then the group $\rho_{E_d,\ell}(\text{Gal}_k)$ has index 3 in $C(\ell)$ if and only if $\ell \equiv 2 \pmod{9}$ and $d = \ell$, or $\ell \equiv 5 \pmod{9}$ and $d = \ell^2$. Note that $C(\ell)$ has a unique index 3 subgroup.

Proof. Using Lemma 7.7 and $\ell \geq 5$, we find that $\rho_{E_d,\ell}(\text{Gal}_k)$ is an index 3 subgroup of $C(\ell)$ if and only if $\rho_{E_d,\ell,\infty}(\text{Gal}_k)$ lies in a closed subgroup of $R_{\ell,\infty}^\chi$ of index 3. We have $C(\ell) = C_{ns}(\ell)$ since $\ell \equiv 2 \pmod{3}$, so $R_{\ell,\infty}^\chi$ has a unique index 3 closed subgroup, i.e., the group of $a \in R_{\ell,\infty}^\chi$ with $\left(\frac{a}{\ell}\right)_3 = 1$.

By the Chebotarev density theorem and Lemma 7.8 with $\lambda = \ell$, we deduce that $\rho_{E_d,\ell}(\text{Gal}_k)$ is an index 3 subgroup of $C(\ell)$ if and only if $d^2(\ell - 1)/3\ell$ is a cube in $R/p$ for all primes $p \nmid 6d\ell$ of $R$; equivalently, $d^2(\ell - 1)/3\ell$ is a cube in $R$. Since $d^2(\ell - 1)/3\ell$ is a rational integer, it is a cube in $R$ if and only if it is a cube in $\mathbb{Z}$.

We have $2(\ell - 1)/3 \equiv 2(\ell + 1)/3 \pmod{3}$, so we need only determine when the integer $d^2(\ell + 1)/3\ell$ is a cube. In the following, we use that $d \geq 1$ is cube-free and that $\mathbb{Z}$ has unique factorization. If $\ell = 2 + 9m$, then $d^2 + 6d\ell$ is a cube if and only if $d = \ell$. If $\ell = 5 + 9m$, then $d^2 + 6d\ell$ is a cube if and only if $d = \ell^2$. If $\ell = 8 + 9m$, then $d^6 + 6d\ell$ is never a cube.

□

Lemma 7.10. Suppose that $\ell \equiv 1 \pmod{3}$. Then the group $\rho_{E_d,\ell}(\text{Gal}_k)$ has index 3 in $C(\ell)$ if and only if $\ell \equiv 4 \pmod{9}$ and $d = \ell^2$, or $\ell \equiv 7 \pmod{9}$ and $d = \ell$.

The group $\rho_{E_d,\ell}(\text{Gal}_k)$ is conjugate to $C(\ell) = C_s(\ell)$ or the subgroup consisting of matrices of the form $\begin{pmatrix} \alpha & 0 \\ \beta & \ell \end{pmatrix}$ with $a/b \in \mathbb{F}_\ell^\times$ a cube.

Proof. Using Lemma 7.7 and $\ell \geq 5$, we find that $\rho_{E_d,\ell}(\text{Gal}_k)$ is an index 3 subgroup of $C(\ell)$ if and only if $\rho_{E_d,\ell,\infty}(\text{Gal}_k)$ lies in a closed subgroup of $R_{\ell,\infty}^\chi$ of index 3. Let us describe the index 3 subgroups of $R_{\ell,\infty}^\chi$. Since $\ell \equiv 1 \pmod{3}$, we have $\ell = \lambda_1\lambda_2$ for irreducibles $\lambda_i \in R$ that we may choose to be congruent to 2 modulo 3$R$. We have $R_{\ell,\infty}^\chi = R_{\lambda_1}^\chi \times R_{\lambda_2}^\chi$. The cubic residue symbol $\left(\frac{\cdot}{\lambda}\right)$ defines a homomorphism $\varphi_1: R_{\ell,\infty}^\chi \to \mu_3 := \langle \omega \rangle$. Since $\ell \geq 5$, we find that every non-trivial homomorphism $R_{\ell,\infty}^\chi \to \mu_3$ is of the form $\varphi_2 := \varphi_1^{e_1} \varphi_2^{e_2}$ with $e = (e_1,e_2) \in \{0,1,2\}^2 - \{(0,0)\}$. Therefore, $\rho_{E_d,\ell}(\text{Gal}_k)$ is an index 3 subgroup of $C(\ell)$ if and only if $\rho_{E_d,\ell,\infty}(\text{Gal}_k) \subseteq \ker \varphi_e$ for some $e \neq (0,0)$.

By Lemma 7.8, we have $\left(\frac{\rho_{E_d,\ell,\infty}(\text{Frob}_p)}{\lambda_i}\right)_3 = \left(\frac{2(\ell - 1)}{\pi}\right)_3^2$ and hence

$$
\varphi_e(\rho_{E_d,\ell,\infty}(\text{Frob}_p)) = \left(d^{2(\ell - 1)}\right)_3 \left(\frac{\ell - 1}{\pi}\right)_3 \left(d^{2(\ell - 1)}\right)_3 \left(\frac{\pi}{\lambda}\right)_3 = \left(d^{2(\ell - 1)}\right)_3 \left(\frac{\ell - 1}{\pi}\right)_3 \left(\frac{\lambda}{\pi}\right)_3 = \left(d^{2(\ell - 1)}\right)_3 \left(\frac{\ell - 1}{\pi}\right)_3 \left(\frac{\lambda}{\pi}\right)_3
$$

for all $p \nmid 6d\ell$. Using the Chebotarev density theorem, we deduce that $\rho_{E_d,\ell,\infty}(\text{Gal}_k) \subseteq \ker \varphi_e$ if and only if $\beta := d^{2(\ell - 1)e_1 + e_2} \lambda_1^{e_1} \lambda_2^{e_2}$ is a cube in $R$.

First suppose that $e_1 \neq e_2$. Let $v_{\lambda_i}: R^\times \to \mathbb{Z}$ be the valuation for the prime $\lambda_i$ and let $v_{\ell}: \mathbb{Q}^\times \to \mathbb{Z}$ be the valuation for $\ell$. We have

$$
v_{\lambda_i}(\beta) = v_{\lambda_i}(e_1 + 2(\ell - 1)e_2) = v_{\lambda_i}(d) + v_{\lambda_i}(2(\ell - 1)(e_1 + e_2) + 2(\ell - 1)(e_1 + e_2)) = v_{\lambda_i}(d).
$$
We have $e_1 \neq e_2 \pmod{3}$ since $e_1 \neq e_2$, so $v_{\lambda_i}(\beta) \neq 0 \pmod{3}$ for some $i \in \{1, 2\}$. Therefore, $\beta \in R$ is not a cube.

Now suppose that $e_1 = e_2$. We may assume that $e_1 = e_2 = 1$ since $\varphi(2, 2)$ is the square of $\varphi(1, 1)$ and hence have the same kernel. So $\beta = d^{4(\ell-1)/3} \ell$. Since $\beta$ is a rational integer, it is a cube in $\mathbb{Z}$ if and only if it is a cube in $R$. If $\ell = 1 + 9m$, then $\beta = (d^{4m})^3 \ell$ is not a cube. If $\ell = 4 + 9m$, then $\beta = (d^{4m+1})^3 \cdot d^2 \ell$ which is a cube if and only if $d = \ell$.

Finally, suppose we are in the case where $\rho_{E_d, \ell}(\text{Gal}_k)$ is an index 3 subgroup of $C_s(\ell)$. There are 4 index 3 subgroups of $C_s(\ell)$. Two of the groups consist of the matrices $A := \left( \begin{smallmatrix} a & 0 \\ 0 & b \end{smallmatrix} \right)$ for which $a$ is a cube (or $b$ is a cube); these groups cannot equal $\rho_{E_d, \ell}(\text{Gal}_k)$ since it would correspond to the case where $e_1 = 0$ or $e_2 = 0$ (and hence $e_1 \neq e_2$). Another index 3 subgroup of $C_s(\ell)$ is the subgroup of matrices whose determinant is a cube; this is impossible since $\det(\rho_{E_d, \ell}(\text{Gal}_k)) = \mathbb{R}_\ell^\times$. Therefore, the only possibility for the image of $\rho_{E_d, \ell}$ is the group of $A$ with $a/b$ a cube.

We now complete the proof of the proposition for the curve $E_d/\mathbb{Q}$. From Lemmas 7.7, 7.9 and 7.10, we deduce that $\rho_{E_d}(\text{Gal}_\mathbb{Q})$ has index 1 or 3 in $N(\ell)$, with index 3 occurring if and only if one of the following hold:

- $\ell \equiv 2 \pmod{9}$ and $d = \ell$.
- $\ell \equiv 5 \pmod{9}$ and $d = \ell^2$.
- $\ell \equiv 4 \pmod{9}$ and $d = \ell^2$.
- $\ell \equiv 7 \pmod{9}$ and $d = \ell$.

Set $M := \rho_{E_d, \ell}(\text{Gal}_k)$; we may assume that it is the index 3 subgroup of $C(\ell)$ from Lemma 7.9 or 7.10. The group $M$ is normal in $N(\ell)$. We have $[N(\ell) : M] = 6$ and $\det(M) = \mathbb{R}_\ell^\times$, so $N(\ell)/M$ is non-abelian by Lemma 7.4(i). So $N(\ell)/M$ is isomorphic to $\mathfrak{S}_3$ and hence, up to conjugation, $N(\ell)$ has a unique index 3 subgroup $G'$ satisfying $G' \subseteq M$. Therefore, $G'$ is conjugate in $\text{GL}_2(\mathbb{F}_\ell)$ to both $\rho_{E_d, \ell}(\text{Gal}_\mathbb{Q})$ and the group $G$ from part (iii) or (iv) of Lemma 1.16. This finishes the proof of Proposition 1.16 for the curve $E_d/\mathbb{Q}$ and $\ell > 3$.

Finally suppose that $E/\mathbb{Q}$ is any elliptic curve with $j$-invariant 0; it is defined by a Weierstrass equation $y^2 = x^3 + dm^3$ for some integer $m \neq 0$ and cube-free integer $d$. It suffices to show that $\rho_{E, \ell}(\text{Gal}_\mathbb{Q})$ is conjugate to $\rho_{E_d, \ell}(\text{Gal}_\mathbb{Q})$ in $\text{GL}_2(\mathbb{F}_\ell)$. The curves $E$ and $E_d$ are quadratic twists, so $\pm \rho_{E_d, \ell}(\text{Gal}_\mathbb{Q})$ is conjugate to $\pm \rho_{E_d, \ell}(\text{Gal}_\mathbb{Q})$. The general case of Proposition 1.16 is thus a consequence of the following lemma.

**Lemma 7.11.** There are no proper subgroups $\pm \rho_{E_d, \ell}(\text{Gal}_\mathbb{Q})$ has no proper subgroups $H$ such that $\pm H = \pm \rho_{E_d, \ell}(\text{Gal}_\mathbb{Q})$.

**Proof.** If $\pm \rho_{E_d, \ell}(\text{Gal}_\mathbb{Q})$ is conjugate to $N(\ell)$, then the lemma follows immediately from Lemma 7.4(ii). From the case of Proposition 1.16 we have already proved (i.e., for the curve $E_d$ and prime $\ell \geq 3$), we need only show that the group $G$ from parts (iii) and (iv) of Lemma 1.16 have no proper subgroups $H$ satisfying $\pm H = G$. Equivalently, we need to show that $-I$ is a commutator of such a subgroup $G$. With $G$ as in Lemma 1.16(iii), this follows from $(\begin{smallmatrix} 1 & 0 \\ 0 & 1 \end{smallmatrix}) (\begin{smallmatrix} 0 & 1 \\ 1 & 0 \end{smallmatrix}) (\begin{smallmatrix} 1 & 0 \\ 0 & 1 \end{smallmatrix})^{-1} (\begin{smallmatrix} 1 & 0 \\ 0 & 1 \end{smallmatrix})^{-1} = (\begin{smallmatrix} -1 & 0 \\ 0 & 1 \end{smallmatrix})$. So we may take $G$ as in Lemma 1.16(iv).

Fix any $B \in G - C(\ell)$. As noted in the proof of Lemma 7.4, the map $\varphi: C(\ell) \to C(\ell)$, $A \mapsto BABA^{-1}$ is a homomorphism whose image is cyclic of order $\ell + 1$. Therefore, $\varphi(G \cap C(\ell))$ is the cyclic subgroup of $C(\ell)$ of order $(\ell + 1)/3$. In particular, $\varphi(G \cap C(\ell))$ contains $-I$ which is the unique element of order 2 in $C(\ell)$. Therefore, $-I$ is a commutator of $G$. \qed
7.4.1. $\ell = 3$ case. We now consider the prime $\ell = 3$ with $E/\mathbb{Q}$ defined by the elliptic curve $y^2 = x^3 + d$. The division polynomial of $E/\mathbb{Q}$ at $3$ is $3x(x^3 + 4d)$. The points of order 3 in $E(\overline{\mathbb{Q}})$ are thus $(0, \pm \sqrt{d})$ and $(-\sqrt{4d} \omega^e, \pm \sqrt{-3\sqrt{d}})$ with $e \in \{0, 1, 2\}$. The points $P_1 = (0, \sqrt{d})$ and $P_2 = (-\sqrt{4d}, \sqrt{-3\sqrt{d}})$ form a basis of $E[3]$. With respect to this basis, we have

$$\rho_{E,3} = \begin{pmatrix} \psi_1 * & \psi_2 \\ 0 & \psi_2 \end{pmatrix},$$

with characters $\psi_1, \psi_2 : \text{Gal}_\mathbb{Q} \to \mathbb{F}_3^\times$. The quadratic character $\psi_1$ describes the Galois action on $P_1$ and it thus corresponds to the extension $\mathbb{Q}(\sqrt{d})$ of $\mathbb{Q}$. The quadratic character $\psi_1 \psi_2 = \det \circ \rho_{E,3} : \text{Gal}_\mathbb{Q} \to \mathbb{F}_3^\times$ corresponds to the extension $\mathbb{Q}(\zeta_3) = \mathbb{Q}(\sqrt{-3})$ of $\mathbb{Q}$. Therefore,

$$\langle \psi_1 \times \psi_2 \rangle(\text{Gal}_\mathbb{Q}) = \begin{cases} \{1\} \times \mathbb{F}_3^\times & \text{if } d \text{ is a square}, \\
\mathbb{F}_3^\times \times \{1\} & \text{if } -3d \text{ is a square}, \\
\mathbb{F}_3^\times \times \mathbb{F}_3^\times & \text{otherwise}.
\end{cases}$$

To compute the image of $\rho_{E,3}(\text{Gal}_\mathbb{Q})$ it remains to determine when its cardinality is divisible by 3 or not. From $P_1$ and $P_2$, it is clear that $\rho_{E,3}(\text{Gal}_\mathbb{Q})$ is divisible by 3 if and only if $4d$ is not a cube.

8. Proof of Proposition 1.13

By Theorem 1.11, we may assume that $\rho_{E,\ell}(\text{Gal}_\mathbb{Q})$ is a subgroup of $N_{ns}(\ell)$. Let $I_\ell$ be an inertia subgroup of $\text{Gal}_\mathbb{Q}$ for the prime $\ell$. We will show that $\rho_{E,\ell}$ has large image by showing that the group $\rho_{E,\ell}(I_\ell)$ is large. The cardinality of $\rho_{E,\ell}(I_\ell)$ is not divisible by $\ell$ since it is a subgroup of $N_{ns}(\ell)$. The group $\rho_{E,\ell}(I_\ell)$ is thus cyclic since the tame inertia group at $\ell$ is pro-cyclic, cf. [Ser72, §1.3].

Let $v_\ell$ be the $\ell$-adic valuation on $\mathbb{Q}_\ell$ normalized so that $v_\ell(\ell) = 1$. Let $\mathbb{Q}_\ell^{un}$ be the maximal unramified extension of $\mathbb{Q}_\ell$ in a fixed algebraic closed field $\overline{\mathbb{Q}}_\ell$. An embedding $\overline{\mathbb{Q}} = \overline{\mathbb{Q}}_\ell$ allows us to identify $I_\ell$ with the subgroup $\text{Gal}(\overline{\mathbb{Q}}_\ell/\mathbb{Q}_\ell^{un})$ of $\text{Gal}_\mathbb{Q} := \text{Gal}(\overline{\mathbb{Q}}_\ell/\mathbb{Q}_\ell)$. Let $\Delta_E$ be the minimal discriminant of $E/\mathbb{Q}$.

- First suppose that $v_\ell(j_E) \geq 0$ and that $v_\ell(\Delta_E)$ is not congruent to 2 and 10 modulo 12.

Let $L$ be the smallest extension of $\mathbb{Q}_\ell^{un}$ for which $E$ base extended to $L$ has good reduction. Define $e = [L : \mathbb{Q}_\ell^{un}]$. There is thus a finite extension $K/\mathbb{Q}_\ell$ such that $E$ base extended to $K$ has good reduction and that $v_\ell(K^\times) = e^{-1}\mathbb{Z}$, where $v_\ell$ is the valuation on $K$ that extends $v_\ell$. From [Ser72, §5.6], we find that $e \in \{1, 2, 3, 4\}$; this uses our assumption on $v_\ell(\Delta_E)$.

Let $\mathcal{I}$ be the inertia subgroup of $\text{Gal}_K := \text{Gal}(\overline{\mathbb{Q}}_\ell/K)$; it is a subgroup of $I_\ell$. The action of $\mathcal{I}$ on $E[\ell]$ is semi-simple since the cardinality of $\rho_{E,\ell}(\mathcal{I})$ is relatively prime to $\ell$ (the group $N_{ns}(\ell)$ has this property). Let $\theta_1 : \mathcal{I} \to \mathbb{F}_\ell^\times$ and $\theta_2 : \mathcal{I} \to \mathbb{F}_\ell^\times$ be fundamental characters of level 1 and 2, respectively, cf. [Ser72, §1.7].

**Lemma 8.1.** The representation $\rho_{E,\ell}|_{\mathcal{I}} : \mathcal{I} \to \text{GL}_2(\mathbb{F}_\ell)$ is irreducible.

**Proof.** Suppose that $\rho_{E,\ell}|_{\mathcal{I}}$ is reducible. The representation $\rho_{E,\ell}|_{\mathcal{I}}$ is given by a pair of characters $\theta_{\ell}^{e_1}$ and $\theta_{\ell}^{e_2}$ with $0 \leq e_1 \leq e_2 < \ell - 1$. From Proposition 11 of [Ser72], we can take $e_1 = 0$ and $e_2 = e$. The image of $\rho_{E,\ell}(\mathcal{I})$ in $\text{PGL}_2(\mathbb{F}_\ell)$ is thus isomorphic to $\theta_{\ell}^e(\mathcal{I})$ and hence is cyclic of order $(\ell - 1)/\gcd(\ell - 1, e)$.

The matrix $A^2$ is scalar for all $A \in N_{ns}(\ell) - C_{ns}(\ell)$. Therefore, the order of every element in the image of $N_{ns}(\ell) \to \text{PGL}_2(\mathbb{F}_\ell)$ divides $\ell + 1$. Since $\gcd(\ell + 1, \ell - 1) = 2$, we deduce that $(\ell - 1)/\gcd(\ell - 1, e)$ equals 1 or 2. This is a contradiction since $\ell \geq 17$.

Scalar multiplication and a choice of $\mathbb{F}_\ell$-basis for $\mathbb{F}_\ell^\times$ allows us to identify $\mathbb{F}_\ell^\times$ with a subgroup of $\text{Aut}_{\mathbb{F}_\ell}(\mathbb{F}_\ell^\times) \cong \text{GL}_2(\mathbb{F}_\ell)$. Since $\rho_{E,\ell}|_{\mathcal{I}}$ is irreducible by Lemma 8.1, it is isomorphic to $\theta_{\ell}^{e_1 + e_2\ell} : \mathcal{I} \to \text{GL}_2(\mathbb{F}_\ell)$.
$\mathbb{F}_\ell^\times \hookrightarrow \text{GL}_2(\mathbb{F}_\ell)$ for some $0 \leq e_1, e_2 \leq \ell - 1$. As an $\mathbb{F}_\ell[I]$-module, $E[\ell]$ is isomorphic to the dual of the étale cohomology group $H^1_\text{ét}(E, \mathbb{F}_\ell)$. By Théorème 1.2 of [Car08], we may take $0 \leq e_1, e_2 \leq e$ (when $E$ has good reduction at $\ell$, and hence $e = 1$, this follows from [Ser72, Prop. 12]). We have $e_1 \neq e_2$ since otherwise $\theta_2^e + \theta_2^{e'} = (\theta_2^{e+1})^{e_1}$ is not irreducible.

Let $g$ be the greatest common divisor of $e_1 + e_2 \ell$ and $\ell + 1$. We have $(e_1 + e_2 \ell) - e_2(\ell + 1) = e_1 - e_2 \in \{\pm 1, \pm 2, \pm 3, \pm 4\}$ since $0 \leq e \leq 4$, so $g \in \{1, 2, 3, 4\}$. Therefore, $\rho_{E,\ell}(I)$ contains a cyclic group of order $(\ell + 1)/g$.

**Lemma 8.2.** The group $\rho_{E,\ell}(I)$ is a subgroup of $C_{ns}(\ell)$ with index 1 or 3.

**Proof.** Set $H := \rho_{E,\ell}(I)$; it is cyclic. We claim that $H$ is a subgroup of $C_{ns}(\ell)$. Suppose not, then the order of $H$ divides $2(\ell - 1)$ since $A^2$ is a scalar matrix for any $A \in N_{ns}(\ell) - C_{ns}(\ell)$. Therefore, $(\ell + 1)/g$ divides $2(\ell - 1)$ since $\rho_{E,\ell}(I) \subseteq H$ contains an element of order $(\ell + 1)/g$. Since $\gcd(\ell + 1, \ell - 1) = 2$, we deduce that $(\ell + 1)/g$ divides 4. This is impossible since $\ell \geq 17$ and $g \leq 4$.

It remains to bound the index of $H$ in $C_{ns}(\ell)$. We have $\det(H) = \mathbb{F}_\ell^\times$ since $\det \rho_{E,\ell}$ describes the Galois action on the $\ell$-th roots of unity. Therefore, the group $H$ is cyclic and its order is divisible by $\ell - 1$ and $(\ell + 1)/g$. Since $\gcd(\ell + 1, \ell - 1) = 2$, we deduce that the order of $H$ is divisible by $(\ell - 1)(\ell + 1)/(2g)$. Therefore, the index $b := [C_{ns}(\ell) : H]$ divides $2g$.

Suppose $b$ is even. Since $C_{ns}(\ell)$ is cyclic, the group $H$ must be contained in $\{A \in C_{ns}(\ell) : \det(A) \in (\mathbb{F}_\ell^\times)^2\}$; this is the unique index 2 subgroup of $C_{ns}(\ell)$; however, this is impossible since $\det(H) = \mathbb{F}_\ell^\times$. So $b$ is odd and divides $2g \in \{2, 4, 6, 8\}$. Therefore, $b$ is 1 or 3. \hfill $\square$

Now define $H := \rho_{E,\ell}(\text{Gal}_Q) \cap C_{ns}(\ell)$. We have $\rho_{E,\ell}(\text{Gal}_Q) \nsubseteq C_{ns}(\ell)$ since $C_{ns}(\ell)$ is not applicable; it does not contain an element with trace 0 and determinant $-1$. So if $H = C_{ns}(\ell)$, then $\rho_{E,\ell}(\text{Gal}_Q) = N_{ns}(\ell)$.

We are thus left to consider the case where $H$ is the (unique) index 3 subgroup of $C_{ns}(\ell)$. The group $H$ is a normal subgroup of $N_{ns}(\ell)$ of index 6.

**Lemma 8.3.** We have $\ell \equiv 2 \pmod{3}$ and the quotient group $N_{ns}(\ell)/H$ is isomorphic to $S_3$.

**Proof.** If $\ell \equiv 1 \pmod{3}$, then $\det(H) = (\mathbb{F}_\ell^\times)^3 \subseteq \mathbb{F}_\ell^\times$. This is impossible since $\det(\rho_{E,\ell}(\text{Gal}_Q)) = \mathbb{F}_\ell^\times$ and $[\rho_{E,\ell}(\text{Gal}_Q) : H] = 2$. Therefore, $\ell \equiv 2 \pmod{3}$. One can now verify that $N_{ns}(\ell)$ quotiented out by the scalar matrices is isomorphic to a dihedral group. It is then easy to check that $N_{ns}(\ell)/H$ is the dihedral group of order $2 \cdot 3$; it is thus isomorphic to $S_3$. \hfill $\square$

The index 3 subgroups of $S_3$ are all conjugate so, up to conjugacy, $G$ (as in the statement of Proposition 1.13) is the unique index 3 subgroup of $N_{ns}(\ell)$ that contains $H$. Therefore, $\rho_{E,\ell}(\text{Gal}_Q)$ and $G$ are conjugate subgroups.

- Suppose that $v_\ell(j_E) \geq 0$.
  
  By twisting $E'/Q$ by 1 or $\ell$, we obtain an elliptic curve $E'/Q$ with $v_\ell(\Delta_{E'})$ not congruent to $2$ and $10$ modulo $12$. The group $\pm \rho_{E,\ell}(\text{Gal}_Q)$ is conjugate to $\pm \rho_{E',\ell}(\text{Gal}_Q)$. The previous case applies and shows that $\pm \rho_{E,\ell}(\text{Gal}_Q)$ is conjugate to $\pm G = G$ or $\pm N_{ns}(\ell) = N_{ns}(\ell)$ in $\text{GL}_2(\mathbb{F}_\ell)$.

  It remains to show that $\pm \rho_{E,\ell}(\text{Gal}_Q) = \rho_{E,\ell}(\text{Gal}_Q)$; if not then there is an index 2 subgroup $H$ of $G$ or $N_{ns}(\ell)$ such that $-I \notin H$. The group $H \cap C_{ns}(\ell)$ is then an index 2 or 6 subgroup of $C_{ns}(\ell)$ that does not contain $-I$. However, the cardinality of $H \cap C_{ns}(\ell)$ is even, so it contains an element of order 2 which must be $-I$.

- Finally suppose that $v_\ell(j_E) < 0$.
  
  There exists an element $q \in \mathbb{Q}_\ell$ with $v_\ell(q) = -v_\ell(j_E) > 0$ such that

  $$j_E = (1 + 240 \sum_{n \geq 1} n^3 q^n/(1 - q^n))^3/(q \prod_{n \geq 1} (1 - q^n)^24).$$

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Let $E/Q_\ell$ be the Tate curve associated to $q$, cf. [Sil94, V§3]; it is an elliptic curve with $j$-invariant $j_E$ and the group $E(\overline{Q})$ is isomorphic to $\overline{Q}^\times/\langle q \rangle$ as a $\text{Gal}_{Q_\ell}$-module. In particular, the $\ell$-torsion subgroup $E[\ell]$ is isomorphic as an $F_\ell[\text{Gal}_{Q_\ell}]$-module to the subgroup of $\overline{Q}^\times/\langle q \rangle$ generated by an $\ell$-th root of unity $\zeta$ and a chosen $\ell$-th root $q^{1/\ell}$ of $q$. Let $\sigma: \text{Gal}_{Q_\ell} \rightarrow F_\ell^\times$ and $\rho: \text{Gal}_{Q_\ell} \rightarrow F_\ell$ be the maps defined so that $\sigma(\zeta) = \zeta^{\alpha(\sigma)}$ and $\sigma(q^{1/\ell}) = \zeta^{\beta(\sigma)}q^{1/\ell}$. So with respect to the basis $\{\zeta, q^{1/\ell}\}$ for $E[\ell]$, we have $\rho_{E,\ell}(\sigma) = \left( \begin{array}{cc} \alpha(\sigma) & \beta(\sigma) \\ 1 & 1 \end{array} \right)$ for $\sigma \in \text{Gal}_{Q_\ell}$. The curves $E$ and $E$ are quadratic twists of each other over $Q_\ell$ (the curve $E$ is non-CM since its $j$-invariant is not an integer). So there is a character $\chi: \text{Gal}_{Q_\ell} \rightarrow \{\pm 1\}$ such that, after an appropriate choice of basis for $E[\ell]$, we have 

$$
\rho_{E,\ell}(\sigma) = \chi(\sigma) \left( \begin{array}{cc} \alpha(\sigma) & \beta(\sigma) \\ 1 & 1 \end{array} \right)
$$

for all $\sigma \in \text{Gal}_{Q_\ell}$. Since $\alpha$ is surjective, we find that the image of $\rho_{E,\ell}(\text{Gal}_{Q_\ell})$ in $\text{PGL}_2(F_\ell)$ contains a cyclic group of order $\ell - 1$. However, the image of $N_{ns}(\ell)$ in $\text{PGL}_2(F_\ell)$ has order $2(\ell + 1)$. Since $\rho_{E,\ell}(\text{Gal}_{Q_\ell}) \subseteq N_{ns}(\ell)$, we find that $\ell - 1$ divides $2(\ell + 1)$; this is impossible since $\gcd(\ell - 1, \ell + 1) = 2$ and $\ell \geq 17$.

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