NULL STRINGS AND MEMBRANES IN DEMIANSKI-NEWMAN BACKGROUND

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We consider null bosonic $p$-branes moving in curved space-times. Some exact solutions of the classical equations of motion and of the constraints for the null string and the null membrane in Demianski-Newman background are found.

1 Introduction

The null $p$-branes are the zero tension limit of the usual $p$-branes, the 1-brane being a string. This relationship between them may be regarded as a generalization of the correspondence between massless and massive particles. On the other hand, the characteristic scale of string theory is given by the string tension $T = (2\pi\alpha')^{-1}$. Then one may view the high energy limit of string theory as a zero tension limit, since only the energy measured in string units, $E/T^{1/2}$, is relevant.

The investigations in the domain of classical and quantum string propagation in curved space-times are relevant for the quantum gravity as well as for the understanding of the cosmic string models in cosmology. As we have already mentioned, strings are characterized by an energy scale $T^{1/2}$ and the length of the string scales like $1/T^{1/2}$. The gravitational field provides another length scale, the curvature radius of the space-time $R_c$. For a string moving in a gravitational field an appropriate parameter is the dimensionless constant $C = R_c T^{1/2}$. Large values of $C$ imply weak gravitational field. One may reach large values of $C$ by letting $T \to \infty$. In this limit the string shrinks to a point. In the opposite limit, small values of $C$, one encounters strong gravitational fields and it is appropriate to consider $T \to 0$, i.e. null or tensionless strings.

The motion of classical null bosonic strings in curved backgrounds have been considered recently in [1, 2, 3, 4, 5, 6, 7]. The dynamics of null $p$-branes living in Friedmann-Robertson-Walker space-time with flat space-like section ($k = 0$) have been investigated in [8].

In this letter we describe the classical evolution of the null strings and membranes in Demianski-Newman background [9]. In Sec. 2 we begin with the general case of tensionless $p$-branes moving in $D$-dimensional curved space-times. In Sec. 3 we obtain some exact solutions of the equations of motion and of the constraints for the null string ($p = 1$) and the null membrane ($p = 2$) in the above mentioned background. Sec. 4 is devoted to our concluding remarks.

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2 Null $p$-Branes in Curved Background

To begin with, let us write down one of the possible actions for the null bosonic $p$-brane living in a $D$-dimensional curved space-time with metric tensor $g_{\mu\nu}(x)$:

$$S = \int d^{p+1}\xi \mathcal{L}, \quad \mathcal{L} = V^J V^K \partial_J x^\mu \partial_K x^\nu g_{\mu\nu}(x), \quad (1)$$

$$\partial_J = \partial/\partial \xi^J, \quad \xi^J = (\xi^0, \xi^1) = (\tau, \sigma^j), \quad J, K = 0, 1, \ldots, p, \quad j, k = 1, \ldots, p, \quad \mu, \nu = 0, 1, \ldots, D - 1.$$ 

It is an obvious generalization of the flat space-time action given in [10]. One easily verifies that if $x^\mu(\xi), g_{\mu\nu}(\xi)$ are world-volume scalars and $V^J(\xi)$ is a world-volume contravariant vector density of weight $q = 1/2$, then $\mathcal{L}(\xi)$ is a scalar density of weight $q = 1$. As a consequence, the variation of the action (1) is

$$\delta_\varepsilon S = \int d^{p+1}\xi \partial_J (\varepsilon^J \mathcal{L})$$

and thus this action is invariant under world-volume reparametrizations if suitable boundary conditions are assumed.

Varying (1) with respect to $x^\nu$ and $V^J$, one obtains the following equations of motion:

$$\begin{align*}
\partial_J \left( V^J V^K \partial_K x^\lambda \right) + \Gamma^\lambda_{\mu\nu} V^J V^K \partial_J x^\mu \partial_K x^\nu &= 0, \\
V^J \partial_J x^\mu \partial_K x^\nu g_{\mu\nu}(x) &= 0,
\end{align*}$$

where $\Gamma^\lambda_{\mu\nu}$ is the connection compatible with the metric $g_{\mu\nu}(x)$:

$$\Gamma^\lambda_{\mu\nu} = \frac{1}{2} g^{\lambda\rho} (\partial_\mu g_{\nu\rho} + \partial_\nu g_{\mu\rho} - \partial_\rho g_{\mu\nu}).$$

Let us rewrite for further convenience the Lagrangian density (1) in the form ($\partial_\tau = \partial/\partial \tau, \partial_j = \partial/\partial \sigma^j$):

$$\mathcal{L} = \frac{1}{4 \mu^0} g_{\mu\nu}(x) (\partial_\tau - \mu^j \partial_j) x^\mu (\partial_\tau - \mu^k \partial_k) x^\nu, \quad (2)$$

where the connection between $V^J$ and $(\mu^0, \mu^j)$ is given by

$$V^J = (V^0, V^j) = \left( -\frac{1}{2 \sqrt{\mu^0}}, \frac{\mu^j}{2 \sqrt{\mu^0}} \right).$$

Now the Euler-Lagrange equation for $x^\nu$ takes the form:

$$\partial_\tau \left[ \frac{1}{2 \mu^0} (\partial_\tau - \mu^k \partial_k) x^\lambda \right] - \partial_j \left[ \frac{\mu^j}{2 \mu^0} (\partial_\tau - \mu^k \partial_k) x^\lambda \right]$$

$$+ \frac{1}{2 \mu^0} \Gamma^\lambda_{\mu\nu} (\partial_\tau - \mu^j \partial_j) x^\mu (\partial_\tau - \mu^k \partial_k) x^\nu = 0. \quad (3)$$
The equations of motion for the Lagrange multipliers $\mu^0$ and $\mu^i$ which follow from (2) give the constraints:

$$
\begin{align*}
g_{\mu\nu}(x)(\partial_\tau - \mu^j \partial_j)x^\mu(\partial_\tau - \mu^k \partial_k)x^\nu &= 0, \\
g_{\mu\nu}(x)(\partial_\tau - \mu^k \partial_k)x^\mu \partial_j x^\nu &= 0.
\end{align*}
$$

(4)\hspace{1cm}(5)

We note (and it is easy to check) that in the gauge $\mu^i = \text{const}$, $x^\nu(\tau, \sigma_1) = x^\nu(\mu^k \tau + \sigma^k)$ is a nontrivial solution of the equations of motion (3) and of the constraints (4), (5), depending on $D$ arbitrary functions of $p$ variables for the null $p$-brane moving in arbitrary $D$-dimensional gravity background.

In terms of $x^\nu$ and the conjugated momentum $p_\nu$ the constraints (4) can be written as:

$$
T_0 = g^{\mu\nu}(x)p_\mu p_\nu = 0, \quad T_j = p_\nu \partial_j x^\nu = 0.
$$

Then one can show that the Hamiltonian which corresponds to the Lagrangian density (2) is a linear combination of $T_0$ and $T_j$

$$
H = \int d^p\sigma (\mu^0 T_0 + \mu^j T_j).
$$

Computation of the Poisson brackets between $T_0$ and $T_j$ results in the following constraint algebra

$$
\begin{align*}
\{T_0(\sigma_1), T_0(\sigma_2)\} &= 0, \\
\{T_0(\sigma_1), T_j(\sigma_2)\} &= [T_0(\sigma_1) + T_0(\sigma_2)] \partial_j \delta^p(\sigma_1 - \sigma_2), \\
\{T_j(\sigma_1), T_k(\sigma_2)\} &= [\delta^j_k T_k(\sigma_1) + \delta^j_k T_j(\sigma_2)] \partial_\tau \delta^p(\sigma_1 - \sigma_2), \\
\sigma &= (\sigma^1, ... , \sigma^p).
\end{align*}
$$

Comparing the equalities (3) with the zero tension limit of the bosonic $p$-brane flat space-time constraint algebra [11] we see that they coincide. However this is not the case with the Hamiltonian equations of motion of course. These are:

$$
\begin{align*}
(\partial_\tau - \mu^k \partial_k)x^\nu &= 2\mu^0 g^{\nu\lambda}(x)p_\lambda, \\
\partial_\nu p_\rho - \partial_\rho (\mu^k p_\rho) + \mu^0 \partial_\rho g^{\nu\lambda}(x)p_\nu p_\lambda &= 0.
\end{align*}
$$

(7)

General solutions of the equations (3) can be easily found only in flat space-times. For example, the general solution for the null string case in an arbitrary gauge is [12]:

$$
\begin{align*}
x^\nu(\tau, \sigma_1) &= g^\nu(\nu) - 2 \int^{\sigma_1} \frac{\mu^0(s)}{[\mu^1(s)]^2} ds f^\nu(\nu), \\
p_\nu(\tau, \sigma_1) &= f_\nu(\nu)/\mu^1(\sigma_1),
\end{align*}
$$

3
where \( g^\nu (v) \), \( f^\nu (v) \) are arbitrary functions of the variable
\[
v = \tau + \int^s \frac{ds}{\mu^1(s)}.
\]
In curved space-times one can try to find backgrounds in which the system of partial differential equations (4) essentially simplifies. An example of this type are the conformally flat spaces, in which the equation of motion for \( p^\nu \) (in the gauge \( \mu^j = \text{constants} \)) reduces to \( (\partial_\tau - \mu^j \partial_j)p^\nu = 0 \) if we take into account the constraint \( T_0 = 0 \), and has as a general solution \( p^\nu (\tau, \sigma^k) = p^\nu (\mu^k \tau + \sigma^k) \).

3 Null Strings and Membranes in Demianski-Newman Space-Time

In this section we will work in the gauge \( \mu^0, \mu^j = \text{constants} \), in which the Euler-Lagrange equations (3) have the form:
\[
(\partial_\tau - \mu^j \partial_j)(\partial_\tau - \mu^k \partial_k)\partial^\lambda + \Gamma^\lambda_{\mu\nu}(\partial_\tau - \mu^j \partial_j)\partial^\mu(\partial_\tau - \mu^k \partial_k)\partial^\nu = 0.
\]
We are going to look for solutions of the equations of motion (8) and constraints (4), (5), for the null string (\( p = 1 \)) and the null membrane (\( p = 2 \)) moving in Demianski-Newman background [9]. The metric for this space-time is of the following type:
\[
ds^2 = \ g_{00}(dx^0)^2 + 2g_{01}dx^0dx^1 + 2g_{03}dx^0dx^3 + 2g_{13}dx^1dx^3 + g_{22}(dx^2)^2 + g_{33}(dx^3)^2.
\]
It can be put in a form in which the manifest expressions for \( g_{\mu\nu} \) are given by the equalities (all other components are zero) [13]:
\[
g_{00} = -\exp(+2U), \quad g_{11} = \left(1 - \frac{a^2 \sin^2 \theta}{R^2}\right) \exp(-2U),
\]
\[
g_{22} = \left(R^2 - a^2 \sin^2 \theta\right) \exp(-2U),
\]
\[
g_{33} = R^2 \sin^2 \theta \exp(-2U) - \left[\frac{2(Mr + l)a \sin^2 \theta + 2R^2 l \cos \theta}{R^2 - a^2 \sin^2 \theta}\right]^2 \exp(+2U),
\]
\[
g_{03} = -\frac{2(Mr + l)a \sin^2 \theta + 2R^2 l \cos \theta}{R^2 - a^2 \sin^2 \theta} \exp(+2U),
\]
where
\[
\exp(\pm 2U) = \left[1 - 2\frac{Mr + l(a \cos \theta + l)}{r^2 + (a \cos \theta + l)^2}\right]^{\pm 1}, \quad R^2 = r^2 - 2Mr + a^2 - l^2,
\]
\( M \) is a mass parameter, \( a \) is an angular momentum per unit mass and \( l \) is the NUT parameter. The metric (9) is a stationary axisymmetric metric. It belongs to the vacuum solutions of the Einstein field equations, which are of type D under Petrov’s classification. The Kerr and NUT space-times are particular cases of the considered
metric and can be obtained by putting \( l = 0 \) or \( a = 0 \) in (9). The case \( l = 0, a = 0 \) obviously corresponds to the Schwarzschild solution.

Starting with the string case, we introduce the ansatz
\[
\begin{align*}
x^0(\tau, \sigma^1) &= C_0 f(z^1) + t(\tau), \\
x^1(\tau, \sigma^1) &= r(\tau), \\
x^2(\tau, \sigma^1) &= \theta(\tau), \\
x^3(\tau, \sigma^1) &= C_3 f(z^1) + \phi(\tau), \\
z^1 &= \mu^1 \tau + \sigma^1,
\end{align*}
\]

where \( f(z^1) \) is an arbitrary function of \( z^1 \) and \( C_0, C_3 \) are constants. The substitution of (10) in (4), (5) and (8), for the given metric, allows us to obtain a solution of the equations for \( \dot{t} \equiv \partial t / \partial \tau \), \( \dot{\phi} \equiv \partial \phi / \partial \tau \) and constraints (5) in the form (\( C_1 = const \)):
\[
\begin{align*}
\dot{t}(\tau) &= -C_1 \left( C_0 g_{03} + C_3 g_{33} \right) \exp(-\mathcal{H}), \\
\dot{\phi}(\tau) &= +C_1 \left( C_0 g_{00} + C_3 g_{03} \right) \exp(-\mathcal{H}), \\
\mathcal{H} &= \int \left( g_{00} g_{00} + 2g_{03} g_{03} + g_{33} g_{33} \right).
\end{align*}
\]

The condition for compatibility of this solution with the other equations and constraints (3) reads:

\[
\exp(\mathcal{H}) \equiv h = g_{00} g_{33} - g_{03}^2.
\]

Then one can show that the solutions for \( \dot{r} = \partial r / \partial \tau \) and \( \dot{\theta} = \partial \theta / \partial \tau \) are expressed by the equalities:
\[
\begin{align*}
\dot{r}^2 &= -g_{11} \left[ C_1^2 \frac{G}{h} + g_{22} \left( g_{22} \dot{\theta}^2 \right) \right], \\
g_{22} \dot{\theta}^2 &= C_2(r) + C_2^2 \int d\theta h^{-2} \left[ g_{22} G \frac{\partial h}{\partial \theta} - h \frac{\partial g_{22} G}{\partial \theta} \right].
\end{align*}
\]

In obtaining (12), we have used that for the metric (9) the following condition is fulfilled: \( \partial g_{22} / \partial z_{11} = 0 \).

In the particular case when \( x^2 = \theta = \theta_0 = const \), one can integrate the equations of motion and constraints completely to obtain an exact solution for the null string in the gravity background (9). This solution is given by (10), where
\[
\begin{align*}
t - t_0 &= \pm \int_{r_0}^r dr \left[ \frac{(C^0 + C^3 A_0) A_0}{R^2 \sin^2 \theta_0} \exp(+2U_0) - C^3 \exp(-2U_0) \right] W^{-1/2}, \\
\varphi - \varphi_0 &= \pm \int_{r_0}^r dr \frac{C^0 + C^3 A_0}{R^2 \sin^2 \theta_0} \exp(+2U_0) W^{-1/2}, \\
C_1(\tau - \tau_0) &= \pm \int_{r_0}^r dr W^{-1/2},
\end{align*}
\]
\[ W = \left( R^2 - a^2 \sin^2 \theta_0 \right)^{-1} \left[ \left( C^3 \right)^2 R^2 - \frac{(C^0 + C^3 A_0)^2}{\sin^2 \theta_0} \exp(+4U_0) \right], \]

\[ A_0 = \frac{2(Mr + l)a \sin^2 \theta_0 + 2R^2 l \cos \theta_0}{R^2 - a^2 \sin^2 \theta_0}, \quad U_0 = U|_{\theta=\theta_0}, \]

\[ t_0, r_0, \varphi_0, \tau_0 = \text{constants}. \]

For the null membrane, we use the ansatz

\[
\begin{align*}
    x^0(\tau, \varphi) &= C^0 F[w(z_1^1, z_2^1)] + t(\tau), \\
    x^1(\tau, \varphi) &= r(\tau), \quad x^2(\tau, \varphi) = \theta(\tau), \\
    x^3(\tau, \varphi) &= C^3 F[w(z_1^2, z_2^2)] + \varphi(\tau), \\
    z^j &= \mu^j \tau + \sigma^j, \quad j = 1, 2,
\end{align*}
\]  

(14)

where the arbitrary function \( F \) depends on \( z^j \) only by the one variable \( w \). Then one proceeds as in the null string case to obtain an exact solution for the null membrane in the Demianski-Newman background. It is given by (14) with \( \theta = \theta_0 = \text{const} \) and \( t, \varphi \) and \( \tau \) taken from (13).

4 Conclusions

In this letter we perform some investigations on the classical dynamics of the null bosonic branes in curved space-times. In the second section we give the action, show that it is reparametrization invariant and write down the equations of motion and constraints in an arbitrary gauge. Then we construct the corresponding Hamiltonian and compute the constraint algebra. It coincides with the flat space-time algebra. The Hamiltonian equations of motion are also given. In the third section we consider the dynamics of the null strings and membranes in the Demianski-Newman background. Some exact solutions of the equations of motion and of the constraints are found. In particular, our null string solution (10), (13) generalizes the one obtained in [5] for the Kerr space-time.

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