Analyticity of nonsymmetric Ornstein-Uhlenbeck semigroup with respect to a weighted Gaussian measure

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Abstract

In this paper we show that the realization in $L^p(X, \nu_\infty)$ of a nonsymmetric Ornstein-Uhlenbeck operator $L_p$ is sectorial for any $p \in (1, +\infty)$ and we provide an explicit sector of analyticity. Here, $(X, \mu_\infty, H_\infty)$ is an abstract Wiener space, i.e., $X$ is a separable Banach space, $\mu_\infty$ is a centred non degenerate Gaussian measure on $X$ and $H_\infty$ is the associated Cameron-Martin space. Further, $\nu_\infty$ is a weighted Gaussian measure, that is, $\nu_\infty = e^{-U} \mu_\infty$ where $U$ is a convex function which satisfies some minimal conditions. Our results strongly rely on the theory of nonsymmetric Dirichlet forms and on the divergence form of the realization of $L_2$ in $L^2(X, \nu_\infty)$.

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1 Introduction

In this paper we prove that the realization in $L^p(X, \nu_\infty)$ of the nonsymmetric perturbed Ornstein-Uhlenbeck operator $L_p$ defined on smooth functions $f$ by

$$L_p f(x) = \frac{1}{2} \text{Tr}[D^2 f(x)]_H + \langle x, A^* Df(x) \rangle_{X \times X^*} + [BD_H f(x), D_H U(x)]_H, \quad x \in X,$$

where $U$ is a suitable function (see Hypothesis 2.15), is sectorial in $L^2(X, \nu_\infty)$ and we provide an explicit sector of analyticity.

In finite dimension, the Ornstein-Uhlenbeck operator is the uniformly elliptic second order differential operator $\mathcal{L}$ defined on smooth functions $\varphi$ by

$$\mathcal{L}\varphi(\xi) = \sum_{i,j=1}^n q_{ij} D^2_{ij} \varphi(\xi) + \sum_{i,j=1}^n a_{ij} \xi_j D_i \varphi(\xi), \quad \xi \in \mathbb{R}^n,$$

where $Q = (q_{ij})_{i,j=1}^n$ is a positive definite matrix and $A = (a_{ij})_{i,j=1}^n$. It is well known (see [23, 24]) that $\mathcal{L}$ may fail to generate an analytic semigroup on $L^p(\mathbb{R}^n)$. The additional assumption $\sigma(A) \subseteq \{ z \in \mathbb{C} : \text{Re}z < 0 \}$ implies that the integral

$$Q_\infty := \int_0^{+\infty} e^{tA} Q e^{tA^*} dt,$$

implies that

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is well defined. The centred Gaussian measure $\mu_\infty$ with covariance $Q_\infty$ is an invariant measure for $\mathcal{L}$, i.e.,

$$\int_{\mathbb{R}^n} \mathcal{L} f d\mu_\infty = 0, \quad f \in D(\mathcal{L}).$$

$\mathcal{L}$ behaves well in $L^p(\mathbb{R}^n, \mu_\infty)$. Indeed, the realization $\mathcal{L}_p$ of $\mathcal{L}$ in $L^p(\mathbb{R}^n, \mu_\infty)$ generates an analytic semigroup for any $p \in (1, +\infty)$. In [6] the authors explicitly provide a sector

$$\Sigma_{\theta_p} := \{ re^{i\phi} \in \mathbb{C} : r > 0, \quad |\phi| \leq \theta_p\}, \quad (1.2)$$

where $\theta_p \in (0, \pi/2)$ is an angle which depends on $Q, A$ and $p$, such that $\mathcal{L}_p$ is sectorial in $\Sigma_{\theta_p}$. This sector is optimal, in the sense that if $\theta \in (0, \pi/2)$ is an angle such that $\mathcal{L}_p$ is sectorial in $\Sigma_{\theta}$, then $\theta \leq \theta_p$. In [7] the same authors extend this result to nonsymmetric sub-Markovian semigroups.

In infinite dimension the situation is much more complicated. We consider an abstract Wiener spaces $(X, \mu_\infty, H_\infty)$, where $X$ is a separable Banach space, $\mu_\infty$ is a centred nondegenerate Gaussian measure on $X$ and $H_\infty$ is the associated Cameron-Martin space (see e.g. [8]). It is well known that $H_\infty \subseteq X$ is a Hilbert space with inner product $[\cdot, \cdot]_{H_\infty}$. Let us denote by $Q_\infty : X^* \to X$ the covariance operator of $\mu_\infty$. In this setting, the definition of the Ornstein-Uhlenbeck operator can be given in terms of bilinear forms: given $f, g \in C^1_b(X)$ we set

$$\mathcal{E}(f, g) := \int_X [DH_\infty f, DH_\infty g]_{H_\infty} d\mu_\infty,$$

where $DH_\infty = Q_\infty D$ is the gradient along the directions of $H_\infty$. Following [20, Chapter 1] it follows that there exists an operator $\mathcal{L}_2 : D(\mathcal{L}_2) \subset L^2(X, \mu_\infty) \to X$ such that for any $f \in D(\mathcal{L}_2)$ and any $g \in C^1_b(X)$ we have

$$\mathcal{E}(f, g) = -\int_X \mathcal{L}_2 f g d\mu_\infty.$$

The operator $\mathcal{L}_2$ is self-adjoint and it generates an analytic contraction $C_0$-semigroup on $L^2(X, \mu_\infty)$. Moreover, if $f = \varphi(x_1^*, \ldots, x_n^*)$ for some smooth function $\varphi$ and $x_i^* \in X^*$, $i = 1, \ldots, n$, then the operator $\mathcal{L}_2$ reads as

$$\mathcal{L}_2 f := \sum_{i,j=1}^n q^0_{ij} \frac{\partial^2 \varphi}{\partial \xi_i \partial \xi_j} - \sum_{i=1}^n x_i^* \frac{\partial \varphi}{\partial \xi_i},$$

where $q^0_{ij} = (Q_\infty x_j^*, x_i^*)_{X \times X^*}$. In [17] the authors provide a generalization of $\mathcal{L}_2$, defining the Wiener space $(X, \mu_\infty, H_\infty)$ as follows. They consider two operators $Q : X^* \to X$ and $A : D(A) \subset X \to X$ such that $Q$ is a linear, bounded, nonnegative and symmetric operator (see Hypothesis 2.1) and $A$ is the infinitesimal generator of a strongly continuous semigroup. Let us denote by $(e^{tA})_{t \geq 0}$ the semigroup generated by $A$. They assume that the integral

$$\int_0^\infty e^{tA} Q e^{tA^*} dt,$$

with values in $\mathcal{L}(X^*; X)$, exists as a Pettis integral and the operator $Q_\infty : X^* \to X$ defined by

$$Q_\infty x^* := \int_0^\infty e^{tA} Q e^{tA^*} dt x^*,$$

is the covariance operator of the Gaussian measure $\mu_\infty$. In such a way they can define the Reproducing Kernel Hilbert Space $H$ associated to $Q$, and they prove the closability of a gradient operator $DH = QD$. Thanks to a stochastic representation, the authors define a semigroup $P(t)$ and its infinitesimal
generator \( L \) on \( L^p(X, \mu_\infty) \) which on smooth functions \( f \) (with \( f = \phi(x_1^*, \ldots, x_n^*) \), for some \( \phi \in C^2_b(\mathbb{R}^n) \), \( n \in \mathbb{N} \) and \( x_i^* \in D(A^*) \), \( i = 1, \ldots, n \)) reads as

\[
\mathbb{L}f := \sum_{i,j=1}^{n} \tilde{q}_{ij} \frac{\partial^2 \phi}{\partial \xi_i \partial \xi_j} + \sum_{i=1}^{n} A^* x_i^* \frac{\partial \phi}{\partial \xi_i},
\]

with \( \tilde{q}_{ij} = \langle Q x_i^*, x_j^* \rangle_{X \times X} \). From the results in [16], the authors deduce that the set

\[
\mathcal{F}_0 := \{ f \in \mathcal{F} : \langle \cdot, A^* D f \rangle_{X \times X^*} \in C_b(X) \},
\]

is a core for \( \mathbb{L} \). Here \( \mathcal{F} \) is the set of functions \( f \in C^2_b(X) \) such that there exists \( \phi \in C^2_b(\mathbb{R}^n) \) and \( x_1^*, \ldots, x_n^* \in D(A^*) \) such that \( f(x) = \phi(\langle x, x_1^* \rangle_{X \times X^*}, \ldots, \langle x, x_n^* \rangle_{X \times X^*}) \) for any \( x \in X \). Finally, arguing as in [14], the authors show different characterizations of the analyticity of \( P(t) \). In particular, they prove that \( P(t) \) is analytic in \( L^p(X, \mu_\infty) \) if and only if \( Q_\infty A^* x^* \in H \) for any \( x^* \in D(A^*) \) and there exists a positive constant \( c \) such that

\[
|Q_\infty A^* x^*|_H \leq c |Q x^*|, \quad x^* \in D(A^*).
\]

This characterization is the starting point of [21], where the authors generalize the results in [6] to the infinite dimensional case. To begin with, they prove that the operator \( B \in \mathcal{L}(H) \), which is the extension of \( Q_\infty A^* \) to the whole \( H \), satisfies \( B + B^* = -I_H \). Let

\[
\mathcal{E}_B(u, v) := -\int_X [BD_H u, D_H v]_H d\mu_\infty,
\]

on \( u, v \in C_b^1(X) \), and let \( \mathcal{L} : D(\mathcal{L}) \subset X \to X \) be the operator associated to \( \mathcal{E}_B \) in \( L^2(X, \mu_\infty) \) in the sense of [20, Chapter 1], i.e.,

\[
\mathcal{E}_B(u, v) = -\int_X Luvd\mu_\infty,
\]

for any \( u \in D(\mathcal{L}) \) and \( v \in C_b^1(X) \). The authors show that \( \mathcal{L} = \mathcal{L} \), where \( \mathcal{L} \) is the infinitesimal generator of \( P(t) \). By means of the the numerical range theorem (see [18]) the authors prove that for any \( p \in (1, +\infty) \) the semigroup \( P(t) \) is analytic in \( L^p(X, \mu_\infty) \) with sector of analyticity \( \Sigma_{\theta_p} \) defined in (1.2). Also in this case, this sector is optimal. We remark that, differently from \( \mathcal{L}_2 \), in general the operator \( \mathcal{L} \) is not self-adjoint and therefore it is not possible to use the theory of self-adjoint operators to prove the analyticity of \( \mathcal{L} \).

In this paper we consider the operator \( L_2 \) associated in \( L^2(X, \nu_\infty) \) to the nonsymmetric bilinear form

\[
\mathcal{E}_B^\nu(u, v) := -\int_X [BD_H u, D_H v]_H d\nu_\infty,
\]

in the sense of [20], where

\[
\nu_\infty := e^{-U} \mu_\infty.
\]

On smooth functions the operator \( L_2 \) has the form (1.1). By taking advantage of the definition of \( L_2 \) and its adjoint operator \( L_2^* \) in \( L^2(X, \nu_\infty) \), we extend \( L_2 \) and the associated semigroup to \( L^p(X, \nu_\infty) \), \( p \in (1, +\infty) \). Finally, we prove that the semigroup associated to \( L_p \) is analytic in \( L^p(X, \nu_\infty) \) with sector of analyticity \( \Sigma_{\theta_p} \), and we provide an example to which our results apply.

We stress that, at the best of our knowledge, in the case of perturbed Ornstein-Uhlenbeck operator no explicit core of \( L_p \) is known. However, for \( p \geq 2 \) we identify a set of smooth functions which allows us to overcome this difficulty, and a we obtain the desired result. In the case \( p \in (1, 2) \) we take advantage of the fact that \( D(L_2) \) is a core for \( L_p \).
It would be interesting to provide more examples to which apply our results and to understand some features of the covariance operator $Q_\infty$. Indeed, if one consider the classical Wiener space, i.e., the case $X = L^2(0, 1)$, $Q$ as in (5.1) and $A = -Id$, then $Q_\infty = \frac{d}{2}Q$ and a function $f$ is an eigenvector of $Q$ with eigenvalue $\lambda$ if and only if $f$ solves on $(0, 1)$ the problem

$$\lambda f'' + f = 0, \quad f(0) = 0, \quad f'(1) = 0.$$  

However, also in apparently friendly contexts the situation is far to be well understood. In the example which we provide in Section 5 we have an explicit formula for $Q_\infty$, but we don’t know how to get more informations on $Q_\infty$ and $L$. We devote these and other stimulating questions to future papers.

The paper is organized as follows. In Section 2 we uniform the notations used in the symmetric and in the nonsymmetric case, which are different and sometimes may give rise to confusion and misunderstandings. Then, we prove that $D_H$ is closable on smooth functions in $L^p(X, \nu_\infty)$ for any $p \in (1, +\infty)$ and define the Sobolev spaces as the domain of the closure of $D_H$. Section 3 is devoted to define the nonsymmetric Ornstein-Uhlenbeck operator and semigroup in $L^p(X, \nu_\infty)$. At first, thanks to the theory of nonsymmetric Dirichlet forms, we provide the definition of the Ornstein-Uhlenbeck operator and semigroup in $L^2(X, \nu_\infty)$. Later, we extend both the operator $L_2$ and the semigroup $(T_2(t))_{t \geq 0}$ to any $L^p(X, \nu_\infty)$, $p \in (1, +\infty)$. We conclude the section by showing the inclusion $D(L_p) \subset D(L_q)$ for any $p, q \in (1, +\infty)$ and $q > p$. These results allow us to overcome the fact that we don’t know a core for $L_p$. In Section 4 we use the numerical range theorem to show that $L_p$ generates an analytic semigroup in $L^p(X, \nu_\infty)$ with sector $\Sigma_{\theta_p}$ for any $p \in (1, +\infty)$. Finally, in Section 5 we provide a explicit example of operators $Q$ and $A$ and of function $U$ which satisfy our assumptions.

### 1.1 Notations

Let $X$ be a separable Banach space. We denote by $\langle \cdot, \cdot \rangle_{X \times X^*}$ the duality, by $\| \cdot \|_X$ its norm and by $\| \cdot \|_{X^*}$ the norm of its dual. Further, for a general Banach space $V$ we denote by $\mathcal{L}(V)$ the space of linear bounded operators from $V$ onto $V$ endowed with the operator norm. For any $n \in \mathbb{N}$ and any $p \in [1, +\infty]$ let $C^k_b(\mathbb{R}^n)$ the continuous and bounded functions on $\mathbb{R}^n$ whose derivatives up to the order $k$ are continuous and bounded. We denote by $C^k_b(X)$ the set of Fréchet-differentiable functions on $X$ up to order $k$ with bounded Fréchet derivative.

Let $Y$ be a separable Hilbert space with inner product $\langle \cdot, \cdot \rangle_{Y \times Y}$ and let $\gamma$ be a Borel measure on $X$. For any $p \in [1, +\infty]$ let us set

$$\|f\|_{L^p(X, \gamma; Y)} := \left(\int_X |f(x)|^p \gamma(dx)\right)^{1/p},$$

for any measurable function $f : X \to Y$. We denote by $L^p(X, \gamma; Y)$ the space of equivalence classes of Bochner integrable functions $f$ with $\|f\|_{L^p(X, \gamma; Y)} < +\infty$.

For any $y, z \in Y$ we denote by $y \otimes z : Y \times Y \to \mathbb{R}$ the map defined by

$$\langle y \otimes z \rangle(x, w) = \langle y, x \rangle_Y \langle z, w \rangle_Y, \quad x, w \in Y.$$

### 2 Preliminaries and Sobolev spaces

We state the following assumptions on the operators $Q$ and $A$.

**Hypothesis 2.1.**

(i) $Q : X^* \to X$ is a linear and bounded operator which is symmetric and nonnegative, i.e.,

$$\langle Q x^*, y^* \rangle_{X \times X^*} = \langle Q y^*, x^* \rangle_{X \times X^*}, \quad \langle Q x^*, x^* \rangle_{X \times X^*} \geq 0, \quad \forall x^*, y^* \in X^*.$$

(ii) $A : D(A) \subseteq X \to X$ is the infinitesimal generator of a strongly continuous contraction semigroup $(e^{tA})_{t \geq 0}$ on $X$. 


The following definition shows that given a nonnegative and symmetric operator \( F : X^* \to X \) we can define a Hilbert space \( K \subset X \), which is called the Reproducing Kernel Hilbert Space associated to \( F \).

**Definition 2.2.** Let \( F : X^* \to X \) be a linear, bounded, nonnegative and symmetric operator. On \( FX^* \) we define the inner product \( \langle Fx^*, y^* \rangle_K := \langle x^*, F^*y^* \rangle_{X \times X^*} \) for any \( x^*, y^* \in X^* \). We denote by \( |Kx^*|^2_K := \langle x^*, x^* \rangle_{X \times X^*} \) the associated norm. We set \( K := FX^2 \subset X \) and we call \( K \) the Reproducing Kernel Hilbert Space (RKHS) associated with \( F \).

From [27, Proposition 1.2] the function \( s \mapsto e^{sA}Qe^{sA^*} \) is strongly measurable and we may define, for any \( t > 0 \), the nonnegative symmetric operator \( Q_t \in \mathcal{L}(X^*; X) \) by

\[
Q_t := \int_0^t e^{sA}Qe^{sA^*} ds.
\]

Further, we denote by \( H_t \) the Reproducing Kernel Hilbert Space associated to \( Q_t \). We assume that the family of operators \( (Q_t)_{t \geq 0} \) satisfies the following hypotheses (see e.g. [17, Sections 2 & 6]).

**Hypothesis 2.3.** (i) The operator \( Q_t \) is the covariance operator of a centred Gaussian measure \( \mu_t \) on \( X \) for any \( t > 0 \).

(ii) For any \( x^* \in X^* \), there exists weak \(-\lim_{t \to +\infty} Q_t x^* =: Q_\infty x^* \) and \( Q_\infty \) is the covariance operator of a centred nondegenerate Gaussian measure \( \mu_\infty \).

Hypothesis 2.3(ii) implies that

\[
\widehat{\mu}_\infty(f) = \exp \left( -\frac{1}{2} \langle Q_\infty f, f \rangle_{X \times X^*} \right), \quad f \in X^*.
\]

We follow [3, Chapter 2] to construct the Cameron-Martin space \( H_\infty \) associated to \( \mu_\infty \), which gives the abstract Wiener space \( (X, \mu_\infty, H_\infty) \). We conclude by showing that \( H_\infty \) is the Reproducing Kernel Hilbert Space associated with \( Q_\infty \).

From [3, Fernique Theorem 2.8.5] it follows that \( X^* \subset L^2(X, \mu_\infty) \), and we denote by \( j : X^* \to L^2(X, \mu_\infty) \) the injection of \( X^* \) in \( L^2(X, \mu_\infty) \). From [3, Theorem 2.2.4] we have

\[
\langle Q_\infty f, g \rangle_{X \times X^*} = \int_X fg d\mu_\infty, \quad f, g \in X^*.
\]

We denote by \( X^*_\mu_\infty \) the closure of \( j(X^*) \) in \( L^2(X, \mu_\infty) \) and we define \( R : X^*_\mu_\infty \to (X^*)' \) by

\[
R(f)(g) := \int_X fg d\mu_\infty, \quad f \in X^*_\mu_\infty, \ g \in X^*.
\]

For any \( f \in X^*_\mu_\infty \), the map \( g \mapsto R(f)(g) \) is weak*-continuous on \( X^* \), and therefore \( R(X^*_\mu_\infty) \subset X \). We still denote by \( R(f) \) the unique element \( y \in X \) such that for any \( g \in X^* \) we have \( R(f)(g) = \langle y, g \rangle_{X \times X^*} \).

Further, the injection \( j \) is the adjoint operator of \( R \). The Cameron-Martin space \( H_\infty \) associated to \( \mu_\infty \) is defined as follows (see e.g. [3, Chapter 2, Section 2]):

\[
|h|_{H_\infty} := \sup \left\{ \langle h, \ell \rangle_{X \times X^*} : \ell \in X^*, \ R(\ell)(\ell) = \|R^*\ell\|_{L^2(X, \mu_\infty)}^2 \leq 1 \right\},
\]

\[
H_\infty := \{ h \in X : |h|_{H_\infty} < +\infty \}.
\]

From [3, Lemma 2.4.1] it follows that \( h \in H_\infty \) if and only if there exists \( \widehat{h} \in X^*_\mu_\infty \) such that \( R(\widehat{h}) = h \).

\( H_\infty \) is a Hilbert space if endowed with inner product

\[
[h, k]_{H_\infty} = \langle \widehat{h}, \widehat{k} \rangle_{L^2(X, \mu_\infty)}, \quad h, k \in H_\infty.
\]
We stress that for any \( f \in X^* \), from (2.1) and (2.2) we have \( Q_\infty f \in H_\infty \) and that \( R(R^* f) = Q_\infty f \), i.e., \( \overline{Q_\infty f} = R^* f \). Further, from (2.3) we deduce that

\[
\langle Q_\infty f, g \rangle_{X \times X^*} = \langle Q_\infty f, Q_\infty g \rangle_{L_\infty}, \quad f, g \in X^*.
\]

We get the following characterization of \( H_\infty \).

**Lemma 2.4.** \( H_\infty = \overline{Q_\infty X^*}^{\| \cdot \|_{L_\infty}} \), that is, the Cameron-Martin space \( H_\infty \) is the Reproducing Kernel Hilbert Space associated to \( Q_\infty \).

**Proof.** The proof is quite simple but we provide it for reader’s convenience. Let \( h \in H_\infty \). Then, there exists \( \hat{h} \in X_{\mu_\infty}^* \) such that \( R_{\mu_\infty} \hat{h} = h \). In particular, there exists \( (f_n) \subset X^* \) such that \( R^* f_n \to \hat{h} \) in \( L^2(X, \mu_\infty) \). We claim that \( Q_\infty f_n \to h \) in \( H_\infty \). Indeed, from (2.3) and recalling that \( \overline{Q_\infty f_n} = R^* f_n \) for any \( n \in \mathbb{N} \), it follows that

\[
|Q_\infty f_n - h|_{L_\infty}^2 = \int_X |R^* f_n - \hat{h}|_{\mu_\infty}^2 d\mu_\infty \to 0, \quad n \to +\infty.
\]

This means that \( H_\infty \subseteq \overline{Q_\infty X^*}^{\| \cdot \|_{L_\infty}} \). The converse inclusion follows from analogous arguments. \( \Box \)

The continuous injection of \( Q_\infty X^* \) into \( X \) can be continuously extended to \( H_\infty \). We denote by \( i_\infty \) the extension of this injection. If we denote by \( i_\infty^* : X^* \to H_\infty^* \) the adjoint operator and we identify \( H_\infty^* \) with \( H_\infty \) by means of the Riesz Representation Theorem, then \( Q_\infty = i_\infty \circ i_\infty^* \). Indeed, for any \( f, g \in X^* \) we have

\[
i_\infty \circ i_\infty^* f, g \rangle_{X \times X^*} = [i_\infty^* f, i_\infty^* g]_{H_\infty} = \langle R^* f, R^* g \rangle_{L^2(X, \mu_\infty)} = \langle Q_\infty f, g \rangle_{X \times X^*},
\]

which gives \( Q_\infty = i_\infty \circ i_\infty^* \).

We introduce the following spaces of functions, which have been already considered in [21, 22].

**Definition 2.5.** For any \( k \in \mathbb{N} \cup \{ \infty \} \) we set

\[
\mathcal{F} \mathcal{C}^{k,1}_b(X) := \{ f(x) = \varphi(x,x_1^1)_{X \times X^*}, \ldots, (x,x_n^1)_{X \times X^*} : n \in \mathbb{N}, \varphi \in C^k_b(\mathbb{R}^n), x_i \in D(A^*), \quad i = 1, \ldots, n, \ x \in X \}.
\]

**Remark 2.6.** We stress that the spaces \( \mathcal{F} \mathcal{C}^{k,1}_b(X) \) are different from those considered in [1, 5, 8, 15]. Indeed, in these papers the authors consider the spaces \( \mathcal{F} \mathcal{C}^k_b(X) \), that is, the spaces of cylindrical functions \( f \) such that \( f(x) = \varphi(x,y_1^1)_{X \times X^*}, \ldots, (x,y_n^1)_{X \times X^*} \) for any \( x \in X \), for some \( \varphi \in C^k_b(\mathbb{R}^n) \) and \( y_i, \ldots, y_n \in X^* \). Even if the space \( \mathcal{F} \mathcal{C}^{k,1}_b(X) \) is smaller than \( \mathcal{F} \mathcal{C}^k_b(X) \) it is ”good” in the sense that it is big enough. Indeed, from [19, Theorem 2.2] it follows that \( D(A^*) \) is weak*-dense in \( X^* \). Since \( \mathcal{F} \mathcal{C}^{k,1}_b(X) \) is dense in \( L^p(X, \mu_\infty) \) for any \( p \in [1, +\infty) \) and any \( k \in \mathbb{N} \) (see [3, Corollary 3.5.2]), we get that \( \mathcal{F} \mathcal{C}^{k,1}_b(X) \) is dense in \( L^p(X, \mu_\infty) \) for any \( p \in [1, +\infty) \) and any \( k \in \mathbb{N} \).

**Example 2.7.** We provide a construction of the classical Wiener space by means of special operators \( A \) and \( Q \). We consider the classical Wiener space \( (X, H_\infty, \nu_\infty) \), where \( X = \mathbb{L}^2(0,1), H_\infty = \{ f \in W^{1,2}(0,1) : f(0) = 0 \} \) and \( \mu_\infty = P^W \) is the classical Wiener measure, see e.g. [3, Example 2.3.11 & Remark 2.3.13]. Let us denote by \( Q_\infty \) its covariance operator and \( Q := Q_\infty^{1/2} \). Then, if we set

\[
D(A) := qx, \quad A := -Q,
\]

\((A, D(A))\) is a closed operator with dense domain satisfying \( \langle Af, f \rangle_{L^2(X, \mu_\infty)} \leq 0 \) for any \( f \in D(A) \). Therefore, \( A \) generates an analytic semigroup which is also strongly continuous. Further, we have

\[
Q_t = Q_\infty (Id_X - e^{tA}), \quad t > 0,
\]

which implies that \( Q_t \) is a trace class operator for any \( t > 0 \) and the covariance operator \( Q_\infty \) coincides with the integral

\[
\int_0^{+\infty} e^{tA} Q e^{tA} dt.
\]
2.1 Reproducing Kernel associated to $Q$ and Sobolev Spaces

We recall that $Q$ is a bounded, linear, nonnegative and symmetric operator. From Definition 2.2 we can define a scalar product on $QX^*$ and we denote by $H$ the Reproducing Kernel Hilbert Space associated to $Q$. $H$ is a Hilbert space endowed with the scalar product $[\cdot, \cdot]_H$. The inclusion $QX^* \hookrightarrow X$ can be extended to the injection $i : H \to X$ and we consider the adjoint operator $i^* : X^* \to H$, where again we have identify $H^*$ and $H$. Arguing as for $i^*$ and $i^*$, we infer that $Q = i \circ i^*$.

The following hypothesis is very important since [17, Theorem 8.3] states that it is equivalent to the analyticity in $L^p(X, \mu_\infty)$ of the Ornstein-Uhlenbeck semigroup $P(t)$ defined on $C_b(X)$ by

$$(P(t)f)(x) := \int_X f(e^{tA}x + y)\mu_t(dy), \quad f \in C_b(X),$$

and extended to $L^p(X, \mu_\infty)$ for any $p \in (1, +\infty)$.

**Hypothesis 2.8.** For any $x^* \in D(A^*)$ we have $i^*_\infty A^* x^* \in H$ and there exists a positive constant $c$ such that

$$|i^*_\infty A^* x^*|_H \leq c|i^* x^*|_H, \quad x^* \in D(A^*). \quad (2.6)$$

$i^*$ is continuous with respect to the weak* topology on $X^*$ and to the weak topology on $H$. Since $D(A^*)$ is weak*-dense in $X^*$, it follows that $i^*$ maps $D(A^*)$ onto a dense subspace of $H$. Then, there exists an operator $B \in \mathcal{L}(H)$ such that $B i^* x^* = i^*_\infty A^* x^*$ for any $x^* \in D(A^*)$ and $\|B\|_{\mathcal{L}(H)} \leq c$. The operator $B$ enjoys the following properties.

**Lemma 2.9.** [21, Lemma 2.2] $B + B^* = -Id_H$ and $[Bh, h]_H = -\frac{1}{2}\|h\|^2_H$ for any $h \in H$.

We now introduce two operators which are crucial for the definition of Sobolev spaces in our context. The first one is the gradient along the directions of the Reproducing Kernel Hilbert Space $H$, while the second one allows us to prove an integration by parts formula with respect to suitable directions in $H$ (see e.g. [15, Section 3]).

**Definition 2.10.** We define the operator $D_H : \mathcal{F}C^1_b(X) \to L^p(X, \mu_\infty; H)$ by

$$D_H(f)(x) := i^* D_f(x) = \sum_{j=1}^n \frac{\partial \varphi}{\partial \xi_j}((x, x_1^*, \ldots, x_n^*)_{X \times X^*})i^* x_j^*, \quad x \in X,$$

where $f \in \mathcal{F}C^1_b(X)$ and $f(x) = \varphi((x, x_1^*)_{X \times X^*}, \ldots, (x, x_n^*)_{X \times X^*})$ for some $n \in \mathbb{N}$, $\varphi \in C^1_b(\mathbb{R}^n)$, $x_i^* \in X^*$ for $i = 1, \ldots, n$ and for any $x \in X$.

**Definition 2.11.** We define the operator $V : D(V) \subseteq H_\infty \to H$ as follows:

$$D(V) := \{i^* x^* : x^* \in X^*\}, \quad V(i^* x^*) = i^* x^*, \quad x^* \in X^*. \quad (2.7)$$

$V$ is densely defined on $H_\infty$, then it is possible to consider the adjoint operator $V^* : D(V^*) \subseteq H \to H_\infty$. Thanks to Hypothesis 2.8 and [17, Theorems 8.1, 8.3 & Proposition 8.7] it follows that $D_H$ is closable in $L^p(X, \mu_\infty)$ and [15, Theorem 3.5] gives that the operator $V$ is closable. We still denote by $D_H$ the closure of $D_H$ and by $W^{1,p}_H(X, \mu_\infty)$ the domain of the closure. We set

$$\|f\|_{W^{1,p}_H(X, \mu_\infty)} := \|f\|_{L^p(X, \mu_\infty)} + \|D_H f\|_{L^p(X, \mu_\infty; H)}, \quad f \in W^{1,p}_H(X, \mu_\infty).$$

The following lemma shows that $\mathcal{F}C^{1,1}_b(X)$ is dense in $W^{1,p}_H(X, \mu_\infty)$ for any $p \in (1, +\infty)$.

**Lemma 2.12.** Let $f \in \mathcal{F}C^1_b(X)$. Then, for any $p \in (1, +\infty)$ there exists a sequence $(f_n) \subset \mathcal{F}C^{1,1}_b(X)$ such that $f_n \to f$ in $W^{1,p}_H(X, \mu_\infty)$ as $n \to +\infty$. In particular, this gives that $\mathcal{F}C^{1,1}_b(X)$ is dense in $W^{1,p}_H(X, \mu_\infty)$ for any $p \in (1, +\infty)$.
Proof. We recall that \( D(A^*) \) is weak*-dense in \( X^* \) (see [19, Theorem 2.2]). This implies that for any \( x^* \in X^* \) there exists a sequence \( (x_{n})_{n} \subseteq D(A^*) \) which weak* converges to \( x^* \) as \( m \rightarrow +\infty \), i.e., \( (x, x_{n})_{X \times X^*} \rightarrow (x, x^*)_{X \times X^*} \) as \( m \rightarrow +\infty \) for any \( x \in X \).

We claim that for any \( h \in H \) and any \( x^* \in X^* \) we have \( [i^*x^*, h]_{H} = (h, x^*)_{X \times X^*} \). From the definition of \( H \), this is true when \( h = i^*y^* \) for some \( y^* \in X^* \). For a generic \( h \in H \), let \( (x_{n})_{n} \subseteq X^* \) be such that \( i^*x_{n} \rightarrow h \) in \( H \) as \( n \rightarrow +\infty \). Since \( H \subseteq X \) with continuous embedding, it follows that \( (i \circ i^*)x_{n} \rightarrow h \) in \( X \) as \( n \rightarrow +\infty \). Then,

\[
[i^*x^*, h]_{H} = \lim_{n \rightarrow +\infty} [i^*x^*, i^*x_{n}]_{H} = \lim_{n \rightarrow +\infty} \langle (i \circ i^*)x_{n}, x^* \rangle_{X \times X^*} = \langle h, x^* \rangle_{X \times X^*},
\]

and the claim is so proved.

Let \( f \in \mathcal{F}C_{b}^{1}(X) \). We only consider \( f(x) = \varphi((x, x^*)_{X \times X^*}) \) with \( \varphi \in C_{b}^{0}(\mathbb{R}), x^* \in X^* \) and \( x \in X \), the general case easily follows from this one. We set \( \tilde{f}_{n} := \varphi((x, x_{n})_{X \times X^*}) \), where \( (x_{n})_{n} \subseteq D(A^*) \) is a sequence which weak* converges to \( x^* \) as \( n \rightarrow +\infty \). Then, \( \tilde{f}_{n} \rightarrow f \) pointwise, and the dominated convergence theorem gives that \( \tilde{f}_{n} \rightarrow f \) in \( L^{p}(X, \mu_{\infty}) \) as \( n \rightarrow +\infty \) for any \( p \in (1, +\infty) \).

Let us fix \( p \in (1, +\infty) \). We show that there exists a sequence \( (f_{n})_{n} \subseteq \mathcal{F}C_{b}^{1}(X) \) such that \( f_{n} \rightarrow f \) in \( L^{p}(X, \mu_{\infty}) \) and \( D_{H}f_{n} \rightarrow D_{H}f \) in \( L^{p}(X, \mu_{\infty}; H) \) as \( n \rightarrow +\infty \). From the definition of \( D_{H} \) we have

\[
D_{H}f_{n}(x) = \varphi((x, x_{n})_{X \times X^*})i^*x_{n}, \quad x \in X, \quad n \in \mathbb{N}.
\]

From (2.8), for any \( h \in H \) we get

\[
[i^*x_{n}^{*}, h]_{H} = (h, x_{n}^{*})_{X \times X^*} \rightarrow (h, x^*)_{X \times X^*} = [i^*x^*, h]_{H}, \quad n \rightarrow +\infty.
\]

This implies that \( (i^*x_{n}^{*})_{n} \subseteq H \) weakly converges in \( H \) to \( i^*x^* \) as \( n \rightarrow +\infty \) and so the sequence \( (i^*x_{n}^{*}) \) is bounded in \( H \). Therefore, there exists a positive constant \( c_{p} \) such that \( \|\tilde{f}_{n}\|_{W_{H}^{p}(X, \mu_{\infty})} \leq c_{p} \) for any \( n \in \mathbb{N} \). From [11, Chapter 3] we deduce that \( L^{p}(X, \nu_{\infty}; H) \) is uniformly convex for any \( p \in (1, +\infty) \), and so \( L^{p}(X, \nu_{\infty}; H) \) has the Banach-Saks property (see e.g., [11, Theorem 1, pag. 78]). We apply this property to the bounded sequence \( (D_{H}\tilde{f}_{n})_{n} \), hence there exists a subsequence \( (D_{H}\tilde{f}_{k_{n}}) \subseteq (D_{H}\tilde{f}_{n}) \) such that if we set

\[
f_{n} := \frac{\sum_{i=1}^{n} \tilde{f}_{k_{i}} + \ldots + \tilde{f}_{k_{n}}}{n}, \quad n \in \mathbb{N},
\]

the sequence

\[
D_{H}f_{n} := \frac{\sum_{i=1}^{n} D_{H}\tilde{f}_{k_{i}} + \ldots + D_{H}\tilde{f}_{k_{n}}}{n}, \quad n \in \mathbb{N},
\]

converges to a function \( \Psi \) in \( L^{p}(X, \nu_{\infty}; H) \) as \( n \rightarrow +\infty \). Clearly, \( f_{n} \rightarrow f \) as \( n \rightarrow +\infty \) in \( L^{p}(X, \mu_{\infty}) \).

From the fact that \( D_{H} \) is a closed operator on \( L^{p}(X, \nu_{\infty}) \), we infer that \( \Psi = D_{H}f \). To conclude, we notice that \( f_{n} \in \mathcal{F}C_{b}^{1}(X) \) for any \( n \in \mathbb{N} \).

\[\square\]

**Lemma 2.13.** For any \( x^* \in D(A^*) \), we have \( Bi^*x^* \in D(V^*) \) and \( V^*(Bi^*x^*) = i_{x}^{**}A^*x^* \).

**Proof.** The result is contained in the proof of [21, Theorem 2.3], but for reader’s convenience we provide the simple proof. Let \( x^* \in D(A^*) \). From the definition of \([\cdot, \cdot]_{H}\), that of \([\cdot, \cdot]_{H_{\infty}}\) and that of \( V \), for any \( y^* \in X^* \) we have

\[
[Bi^*x^*, V(i_{x}^{**}y^*)]_{H} = [Bi^*x^*, i^*y^*]_{H} = [i_{x}^{**}A^*x^*, i^*y^*]_{H} = (i_{x}^{**}A^*x^*, y^*)_{X \times X^*} = [i_{x}^{**}A^*x^*, i_{x}^{**}y^*]_{H_{\infty}},
\]

which means that \( Bi^*x^* \in D(V^*) \) and \( V^*(Bi^*x^*) = i_{x}^{**}A^*x^* \). \[\square\]
Remark 2.14. If $Q = Q_\infty$, i.e., the Malliavin setting, $D_H$ is the Malliavin derivative, $V$ is the identity operator and for any $p \in [1, +\infty)$ the space $W_{H}^{1,p}(X, \mu_\infty)$ is the Sobolev space considered in [3, Chapter 5].

We are now ready to state the hypotheses on the weighted function $U$.

**Hypothesis 2.15.** $U$ is a proper $\|\cdot\|_X$-lower semi-continuous convex function which belongs to $W_{H}^{1,p}(X, \mu_\infty)$ for any $p \in [1, +\infty)$.

It is useful to notice that Hypothesis 2.15 and [2, Lemma 7.5] imply that $e^{-U} \in W_{H}^{1,p}(X, \mu_\infty)$ for any $p \in [1, +\infty)$. This allows us to introduce the bounded measure

$$\nu_\infty := e^{-U}d\mu_\infty. \quad (2.9)$$

We prove that $D_H : \mathcal{F}\mathcal{G}_{b}^{1}(X) \to L^p(X, \nu_\infty; H)$ is closable in $L^p(X, \nu_\infty)$. To this aim we prove an intermediate result, which is the extension of [15, Lemma 3.3] for the weighted measure $\nu_\infty$.

**Lemma 2.16.** Let $f \in \mathcal{F}\mathcal{G}_{b}^{1}(X)$ and let $h \in D(V^*)$. Then,

$$\int_X [D_Hf, h]_H d\nu_\infty = \int_X fV^*h d\nu_\infty + \int_X f[D_HU, h]_H d\nu_\infty. \quad (2.10)$$

**Proof.** From [15, Lemma 3.3] we know that

$$\int_X [D_Hg, h]_H d\mu_\infty = \int_X gV^*hd\mu_\infty, \quad (2.11)$$

for any $g \in \mathcal{F}\mathcal{G}_{b}^{1}(X)$ and any $h \in D(V^*)$. We would like to apply (2.11) with $g = fe^{-U}$. The density of $\mathcal{F}\mathcal{G}_{b}^{1}(X)$ in $W_{H}^{1,p}(X, \mu_\infty)$ for any $p \in [1, +\infty)$ implies that (2.11) holds true for any $g \in W_{H}^{1,p}(X, \mu_\infty)$ and $p \in [1, +\infty)$. From Hypothesis 2.15 and [22, Lemma 3.3], we infer that $D_H(fe^{-U}) = (D_Hf)e^{-U} - (D_HU)fe^{-U}$. Then, $fe^{-U} \in W_{H}^{1,p}(X, \mu_\infty)$ for any $p \in [1, +\infty)$ and we can apply (2.11) with $g = fe^{-U}$. We get

$$\int_X [D_Hf, h]_H d\nu_\infty = \int_X [D_Hf, h]_H e^{-U} d\mu_\infty = \int_X [D_H(fe^{-U}), h]_H d\mu_\infty + \int_X f[D_HU, h]_H e^{-U} d\mu_\infty$$

$$= \int_X fe^{-U}V^*hd\mu_\infty + \int_X f[D_HU, h]_H d\nu_\infty$$

$$= \int_X fV^*hd\nu_\infty + \int_X f[D_HU, h]_H d\nu_\infty.$$ 

Integration by parts formula (2.10) is the key tool to prove the closability of $D_H$ in $L^p(X, \nu_\infty)$ with $p \in (1, +\infty)$.

**Proposition 2.17.** $D_H : \mathcal{F}\mathcal{G}_{b}^{1}(X) \to L^p(X, \nu_\infty; H)$ is closable in $L^p(X, \nu_\infty)$ for any $p \in (1, +\infty)$. We still denote by $D_H$ the closure of $D_H$ in $L^p(X, \nu_\infty)$ and we denote by $W_{H}^{1,p}(X, \nu_\infty)$ the domain of its closure. Finally, for any $p \in (1, +\infty)$ the space $W_{H}^{1,p}(X, \nu_\infty)$ endowed with the norm

$$\|f\|_{1,p,H} := \|f\|_{L^p(X, \nu_\infty)} + \|D_Hf\|_{L^p(X, \nu_\infty; H)}, \quad f \in W_{H}^{1,p}(X, \nu_\infty),$$

is a Banach space, and for $p = 2$ it is a Hilbert space with inner product

$$\langle f, g \rangle_{W_{H}^{1,2}(X, \nu_\infty)} := \int_X fg d\nu_\infty + \int_X [D_Hf, D_Hg]_H d\nu_\infty, \quad f, g \in W_{H}^{1,2}(X, \nu_\infty).$$
Proof. Let us fix $p \in (1, +\infty)$. $(V, D(V))$ is closable from $H_{\infty}$ onto $H$, then from [15, Theorem 3.4] it follows that $D(V^*)$ is weak dense in $H$ and there exists an orthonormal basis $\{v_n : n \in \mathbb{N}\} \subset D(V^*)$ of $H$. To show that $D_H$ is closable, let us consider a sequence $(f_n) \subset \mathcal{F}C^1_b(X)$ such that $f_n \to 0$ and $D_H f_n \to F$ in $L^p(X, \nu_\infty)$ and in $L^p(X, \nu_\infty; H)$, respectively. If we show that $F = 0$ we infer the closability of $D_H$. To prove that $F = 0$ let us consider $g \in \mathcal{F}C^1_b(X)$. From (2.10) applied to the function $f_n := f_n g \in \mathcal{F}C^1_b(X)$ we have

$$\int_X [D_H f_n, v_j]_{H} gd\nu_\infty = \int_X [D_H(f_n g), v_j]_{H} d\nu_\infty - \int_X [D_H g, v_j]_{H} f_n d\nu_\infty$$

$$= \int_X f_n \hat{g} v_j d\nu_\infty + \int_X [D_H U, v_j]_{H} f_n g d\nu_\infty - \int_X [D_H g, v_j]_{H} f_n d\nu_\infty,$$

for any $j \in \mathbb{N}$. Letting $n \to +\infty$ in (2.12) we infer that

$$\int_X [F, v_j]_{H} gd\nu_\infty = \lim_{n \to +\infty} \int_X [D_H f_n, v_j]_{H} gd\nu_\infty = 0,$$

for any $j \in \mathbb{N}$ and any $g \in \mathcal{F}C^1_b(X)$. The density of $\mathcal{F}C^1_b(X)$ in $L^p(X, \nu_\infty)$ implies that $[F(x), v_j]_{H} = 0$ for $\nu_\infty$-a.e. $x \in X$ for any $j \in \mathbb{N}$. This gives that $F(x) = 0$ for $\nu_\infty$-a.e. $x \in X$. The second part of the statement follows from standard arguments.  \(\Box\)

Remark 2.18. Arguing as in Lemma 2.12, it follows that the space $\mathcal{F}C^{k,1}_b(X)$ is dense in $W^{1,p}_H(X, \nu_\infty)$ for any $k \in \mathbb{N} \cup \{\infty\}$ and any $p \in (1, +\infty)$.

3 The perturbed nonsymmetric Ornstein-Uhlenbeck operator

3.1 The Ornstein-Uhlenbeck operator in $L^2(X, \nu_\infty)$

We introduce the nonsymmetric Ornstein-Uhlenbeck operator by means of the theory of bilinear Dirichlet forms. Let

$$\mathcal{E}(u, v) := -\int_X [BD_{H} u, D_{H} v]_{H} d\nu_\infty, \quad u, v \in \mathcal{D},$$

with domain $\mathcal{D} = W^{1,2}_H(X, \nu_\infty)$. From Lemma 2.9 we get

$$\mathcal{E}(u, u) = -\int_X [BD_{H} u, D_{H} u]_{H} d\nu_\infty = \frac{1}{2} \int_X [D_{H} u, D_{H} u]_{H} d\nu_\infty = \frac{1}{2} \|D_{H} u\|_{L^2(X, \nu_\infty; H)}^2, \quad u \in \mathcal{D},$$

which implies that $\mathcal{E}$ is positive definite. If we consider the symmetric part $\mathcal{E}(u, v) := \frac{1}{2}(\mathcal{E}(u, v) + \mathcal{E}(v, u))$ of $\mathcal{E}$, with $u, v \in \mathcal{D}$, we have

$$\mathcal{E}(u, v) = -\frac{1}{2} \int_X ([BD_{H} u, D_{H} v]_{H} + [BD_{H} v, D_{H} u]_{H}) d\nu_\infty$$

$$= -\frac{1}{2} \int_X ([BD_{H} u, D_{H} v]_{H} + [B^* D_{H} u, D_{H} v]_{H}) d\nu_\infty = \frac{1}{2} \int_X [D_{H} u, D_{H} v] d\nu_\infty.$$

Proposition 2.17 implies that $(\mathcal{E}, \mathcal{D})$ is a symmetric closed form on $L^2(X, \nu_\infty)$. Finally, for any $u, v \in \mathcal{D}$, from Hypothesis 2.8 we have

$$|\mathcal{E}(u, v)| \leq \int_X \|BD_{H} u, D_{H} v\|_{H} d\nu_\infty \leq \|B\|_{L(H)} \int_X \|D_{H} u\|_{H} \|D_{H} v\|_{H} d\nu_\infty$$

$$\leq c \|D_{H} u\|_{L^2(X, \nu_\infty; H)} \|D_{H} v\|_{L^2(X, \nu_\infty; H)} = 4c \mathcal{E}(u, u)^{1/2} \mathcal{E}(v, v)^{1/2}.$$
This implies that \((E,D)\) satisfies the strong (and hence the weak) sector condition (see [20, Chapter 1, Section 2 and Exercise 2.1]) and therefore \((E,D)\) is a coercive closed form on \(L^2(X,\nu_\infty)\). According to [20, Chapter 1] we define a densely defined operator \(L_2\) as follows:

\[
\begin{align*}
D(L_2) := & \left\{ u \in W^{1,2}_H(X,\nu_\infty) : \text{there exists } g \in L^2(X,\nu_\infty) \text{ such that} \right. \\
& \left. \mathcal{E}(u,v) = -\int_X g v d\nu_\infty, \ \forall v \mathcal{F}\mathcal{C}_b^1(X) \right\}, \\
L_2u := & g.
\end{align*}
\] (3.3)

Remark 3.1. From [20, Chapter 1, Sections 1&2] it follows that \(L_2\) generates a strongly continuous contraction semigroup on \(L^2(X,\nu_\infty)\) which we denote by \((T_2(t))_{t \geq 0}\). In particular, \(1 \in \rho(L_2)\). The operator \(L_2\) is called perturbed Ornstein-Uhlenbeck operator in \(L^2(X,\nu_\infty)\) and the associated semigroup \((T_2(t))_{t \geq 0}\) is called perturbed Ornstein-Uhlenbeck semigroup in \(L^2(X,\nu_\infty)\).

In the following we will need of the adjoint operator \(L^*_2\) of \(L_2\). We recall that formally \(L^*_2\) is defined as follows:

\[
\begin{align*}
D(L^*_2) := & \left\{ v \in L^2(X,\nu_\infty) : \exists g \in L^2(X,\nu_\infty) \text{ such that} \right. \\
& \left. \int_X g u d\nu_\infty = \int_X v L_2 u d\nu_\infty, \ u \in D(L_2) \right\}, \\
L^*_2v := & g.
\end{align*}
\]
Moreover, let us consider the adjoint semigroup \((T^*_2(t))_{t \geq 0}\) of \((T_2(t))_{t \geq 0}\). Even if in general it is not a strongly continuous semigroup, [20, Chapter 1, Theorem 2.8] ensures that \((T^*_2(t))_{t \geq 0}\) is strongly continuous and \(L^*_2\) is its generator. Further, [20, Chapter 1, Corollary 2.10] implies that \(D(L^*_2) \subset D = W^{1,2}_H(X,\nu_\infty)\).

We give a characterization of \(L^*_2\) in terms of bilinear form on \(L^2(X,\nu_\infty)\). Let us introduce the nonsymmetric bilinear form

\[
\tilde{\mathcal{E}}(u,v) := -\int_X [B^*D_H u, D_H v] H d\nu_\infty, \ u,v \in \mathcal{D},
\] (3.4)

with domain \(\mathcal{D} := W^{1,2}_H(X,\nu_\infty)\). Arguing as for \(\mathcal{E}\) it is possible to prove that \(\tilde{\mathcal{E}}\) is a coercive closed form on \(L^2(X,\nu_\infty)\) and therefore the operator \(\tilde{L}_2\) defined as

\[
\begin{align*}
D(\tilde{L}_2) := & \left\{ u \in W^{1,2}_H(X,\nu_\infty) : \text{there exists } g \in L^2(X,\nu_\infty) \text{ such that} \right. \\
& \left. \tilde{\mathcal{E}}(u,v) = -\int_X g u d\nu_\infty, \ \forall v \mathcal{F}\mathcal{C}_b^1(X) \right\}, \\
\tilde{L}_2u := & g,
\end{align*}
\] (3.5)
generates a strongly continuous semigroup \((\tilde{T}_2(t))_{t \geq 0}\) on \(L^2(X,\nu_\infty)\). The next result shows that \(\tilde{L}_2\) is indeed the adjoint operator of \(L_2\) and \((\tilde{T}_2(t))_{t \geq 0}\) is the adjoint semigroup of \((T_2(t))_{t \geq 0}\).

**Proposition 3.2.** \(D(\tilde{L}_2) = D(L^*_2)\) and \(\tilde{L}_2 u = L^*_2 u\) for any \(u \in D(L^*_2)\). Therefore, \(\tilde{T}_2(t) = T^*_2(t)\) for any \(t \geq 0\).

**Proof.** Let \(u \in D(\tilde{L}_2)\). For any \(v \in D(L_2)\) we have

\[
\int_X \tilde{L}_2 u v d\nu_\infty = \int_X [B^*D_H u, D_H v] H d\nu_\infty = \int_X [BD_H v, D_H u] d\nu_\infty = \int_X L_2 u v d\nu_\infty.
\]

From the definition of \(L^*_2\) it follows that \(u \in D(L^*_2)\) and \(L^*_2 u = \tilde{L}_2 u\). To prove the converse inclusion, let \(u \in D(L_2)\). We recall that \(u \in W^{1,2}_H(X,\nu_\infty)\). For any \(v \in D(L_2)\) we have

\[
\int_X L^*_2 u v d\nu_\infty = \int_X u L_2 v d\nu_\infty = \int_X [BD_H v, D_H u] H d\nu_\infty = \int_X [B^* D_H u, D_H v] H d\nu_\infty = -\tilde{\mathcal{E}}(u,v).
\] (3.6)
From [20, Chapter 1, Theorem 2.13(ii)] it follows that \( D(L_2) \) is dense in \( D = W_{H}^{1,2}(X, \nu_\infty) \). Therefore, (3.6) gives \( u \in D(\tilde{L}_2) \) and \( \tilde{L}_2 u = L^*_u u \).

We conclude this subsection by showing that \( \mathcal{F} \mathcal{C}^{2,1}_b(X) \subset D(L_2) \) and for any \( u \in \mathcal{F} \mathcal{C}^{2,1}_b(X) \) an explicit formula for \( L_2 u \) is available. To this aim, we recall the definition of Trace class operator on \( \mathcal{L}(H) \): given a nonnegative operator \( \Phi \in \mathcal{L}(H) \), we say that \( \Phi \) is a trace class operator if

\[
\sum_{n=1}^{\infty} [\Phi h_n, h_n]_H < +\infty,
\]

where \( \{h_n : n \in \mathbb{N}\} \) is any orthonormal basis of \( H \). We define the Trace \( \text{Tr}[\Phi] \) of \( \Phi \) as

\[
\text{Tr}[\Phi]_H := \sum_{n=1}^{\infty} [\Phi h_n, h_n]_H.
\]

For any \( f \in \mathcal{F} \mathcal{C}^{2,1}_b(X) \) such that \( f(x) = \varphi(x, x_1^\ast)_{X \times X^*} \cdots , (x, x_n^\ast)_{X \times X^*} \) for some \( \varphi \in C^2_b(\mathbb{R}^n) \), \( x_i^\ast \in D(A^*) \), \( i = 1, \ldots, n \) and \( x \in X \), we define the second order derivative along \( H \) as

\[
D_H^2 f(x) := \sum_{j,k=1}^{n} \frac{\partial^2 \varphi}{\partial \xi_j \partial \xi_k} (x, x_1^\ast)_{X \times X^*} \cdots , (x, x_n^\ast)_{X \times X^*} Q x_j^\ast \otimes Q x_k^\ast.
\]

\( D_H^2 f(x) \) is a trace class operator for any \( x \in X \) and

\[
\text{Tr}[D_H^2 f(x)]_H = \sum_{j,k=1}^{n} (Q x_j^\ast, x_k^\ast)_{X \times X^*} \frac{\partial^2 \varphi}{\partial \xi_j \partial \xi_k} (x, x_1^\ast)_{X \times X^*} \cdots , (x, x_n^\ast)_{X \times X^*}, \quad x \in X.
\]

**Proposition 3.3.** \( \mathcal{F} \mathcal{C}^{2,1}_b(X) \subset D(L_2) \) and for any \( u \in \mathcal{F} \mathcal{C}^{2,1}_b(X) \) we have

\[
L_2 u(x) = \frac{1}{2} \text{Tr}[D_H^2 u(x)]_H + \langle x, A^* D u(x) \rangle_{X \times X^*} + [BD_H u(x), D_H U(x)]_H, \quad \nu_\infty - \text{a.e.} \ x \in X. \quad (3.7)
\]

**Proof.** Let \( u \in \mathcal{F} \mathcal{C}^{2,1}_b(X) \) be such that \( u(x) = \varphi(x, x_1^\ast)_{X \times X^*} \cdots , (x, x_n^\ast)_{X \times X^*} \), with \( \varphi \in C^2_b(\mathbb{R}^m) \), \( x_i^\ast \in D(A^* \) for \( i = 1, \ldots, m \) and \( x \in X \), and let \( v \in \mathcal{F} \mathcal{C}^{2,1}_b(X) \). From Lemma 2.13 for any \( x^\ast \in D(A^*) \) we have \( B_i x^\ast \in D(V^*) \) and \( V^*(B_i x^\ast) = i^*_{\nu} A^* x^\ast \). The form of \( u \), integration by parts formula (2.10) with \( f = uv \) and the computations in the proof of [21, Theorem 2.3] give

\[
\mathcal{E}(u, v) = - \int_X [BD_H u(x), D_H v(x)]_H \nu_\infty(dx)
\]

\[
= - \sum_{n=1}^{m} \int_X [D_H v(x), B_i^* x_n^\ast]_H \frac{\partial \varphi}{\partial \xi_n} (x, x_1^\ast)_{X \times X^*} \cdots , (x, x_n^\ast)_{X \times X^*} \nu_\infty(dx)
\]

\[
= \sum_{n=1}^{m} \int_X v(x) \left( \sum_{j=1}^{n} \frac{\partial^2 \varphi}{\partial \xi_j \partial \xi_n} (x, x_1^\ast)_{X \times X^*} \cdots , (x, x_n^\ast)_{X \times X^*} V^* B_i^* x_n^\ast - [D_H U(x), B_i^* x_n^\ast]_H \frac{\partial \varphi}{\partial \xi_n} (x, x_1^\ast)_{X \times X^*} \cdots , (x, x_n^\ast)_{X \times X^*} \nu_\infty(dx)
\]

\[
= - \int_X v(x) \left( \frac{1}{2} \text{Tr}[D_H^2 u(x)]_H + \langle x, A^* D u(x) \rangle_{X \times X^*} + [BD_H u(x), D_H U(x)]_H \nu_\infty(dx). \right)
\]

Since

\[
x \mapsto \frac{1}{2} \text{Tr}[D_H^2 u(x)]_H + \langle x, A^* D u(x) \rangle_{X \times X^*} + [BD_H u(x), D_H U(x)]_H \in L^2(X, \nu_\infty),
\]

it follows that \( u \in D(L_2) \) and

\[
L_2 u(x) = \frac{1}{2} \text{Tr}[D_H^2 u(x)]_H + \langle x, A^* D u(x) \rangle_{X \times X^*} + [BD_H u(x), D_H U(x)]_H, \quad \nu_\infty - \text{a.e.} \ x \in X.
\]
3.2 The nonsymmetric Ornstein-Uhlenbeck operator in $L^p(X, \nu_{\infty})$

In this subsection we consider the realization of the semigroup $(T_2(t))_{t \geq 0}$ in $L^p(X, \nu_{\infty})$ with $p \in (1, +\infty)$, and we show some important properties of the perturbed Ornstein-Uhlenbeck semigroup in $L^p(X, \nu_{\infty})$. We need a technical lemma, which is the analogous of [8, Lemma 2.7] in our setting, about the differentiability of the positive and negative part of a function $u \in W^{1,2}_H(X, \nu_{\infty})$.

**Lemma 3.4.** Let $u \in W^{1,2}_H(X, \nu_{\infty})$. Then, $|u|, u^+, u^- \in W^{1,2}_H(X, \nu_{\infty})$ and $D_Hu = \text{sign}(u)D_Hu$. Further, $D_Hu$ vanishes on $u^{-1}(0) \nu_{\infty}$-a.e.; $D_H(u^+) = 1_{\{u>0\}}D_Hu$ and $D_H(u^-) = -1_{\{u<0\}}D_Hu$.

**Proof.** The proof is analogous to that of [8, Lemma 2.7] and we omit it. We simply remark that, to prove that second part, as in the proof of Proposition 2.17, we consider the basis $\{\eta_n : n \in \mathbb{N}\}$ of $H$ of elements of $D(V^*)$ and we show that

$$\int_{\{u=0\}} [D_Hu, v_i]_H \varphi \nu_{\infty} = 0,$$

for any $u \in W^{1,2}_H(X, \nu_{\infty})$ and any $\varphi \in \mathcal{P}C^1_c(X)$. \qed

Thanks to Lemma 3.4 we can prove that both $L_2$ and $L_2^*$ are Dirichlet operators and therefore that both $(T_2(t))_{t \geq 0}$ and $(T_2^*(t))_{t \geq 0}$ are sub-Markovian operators on $L^2(X, \nu_{\infty})$. For reader’s convenience, we recall the definitions of Dirichlet and sub-Markovian operators and their main properties (see e.g. [20, Chapter 1, Definition 4.1 & Proposition 4.3]).

**Definition 3.5.** Let $(E, B, \mu)$ be a measure space and let $\mathcal{H} := L^2(E, \mu)$ be a Hilbert space.

(i) A semigroup $(S(t))_{t \geq 0}$ on $\mathcal{H}$ is called sub-Markovian if for any $t \geq 0$ and any $f \in \mathcal{H}$ with $0 \leq f \leq 1 \mu$-a.e., we have $0 \leq S(t)f \leq 1 \mu$-a.e.

(ii) A closed linear densely defined operator $A$ on $\mathcal{H}$ is called Dirichlet operator on $\mathcal{H}$ if

$$\int_E Au(u - 1)^+ d\mu \leq 0, \quad u \in D(A).$$

**Proposition 3.6.** Let $(S(t))_{t \geq 0}$ be a strongly continuous contraction semigroup on $L^2(E, \mu)$ with generator $A$. Then, the following are equivalent:

(i) $(S(t))_{t \geq 0}$ is a sub-Markovian semigroup on $L^2(E, \mu)$.

(ii) $A$ is a Dirichlet operator on $L^2(E, \mu)$.

We prove that it is possible to extend the semigroup $(T_2(t))_{t \geq 0}$ to a strongly continuous contraction semigroup on $L^p(X, \nu_{\infty})$ for any $p \in [1, +\infty)$. We follow the proof of [10, Theorem 1.4.1].

**Proposition 3.7.** The semigroup $(T_2(t))_{t \geq 0}$ can be uniquely extended to a positive contraction semigroup $(T_p(t))_{t \geq 0}$ on $L^p(X, \nu_{\infty})$ for any $p \in [1, +\infty)$. These semigroups are strongly continuous and are consistent in the sense that if $q \geq p$ then $T_p(t)f = T_q(t)f$ for any $f \in L^q(X, \nu_{\infty})$.

**Proof.** For reader’s convenience, we split the proof into different steps.

**Step 1.** We prove that both $L_2$ and $L_2^*$ are Dirichlet operators on $L^2(X, \nu_{\infty})$. Let $u \in D(L_2)$. Then, $u \in W^{1,2}_H(X, \nu_{\infty})$ and from Lemma 3.4 we infer that $(u-1)^+ \in W^{1,2}_H(X, \nu_{\infty})$ and $D_H(u-1)^+ = 1_{u \geq 1}D_Hu$. Therefore,

$$\int_X L_2u(u-1)^+ d\nu_{\infty} = \int_X |BD_Hu, D_H(u-1)^+|_H d\nu_{\infty} = \int_{\{u>1\}} [BD_Hu, D_Hu]_H d\nu_{\infty} \leq 0,$$
thanks to Lemma 2.9. The computations for $L^2_\nu$ are analogous. Hence, both $L_2$ and $L^2_\nu$ are Dirichlet operators on $L^2(X, \nu_\infty)$, which means that $(T_2(t))_{t \geq 0}$ and $(T^2_\nu(t))_{t \geq 0}$ are sub-Markovian semigroups on $L^2(X, \nu_\infty)$.

**Step 2.** We claim that $L^1(X, \nu_\infty)$ and $L^\infty(X, \nu_\infty)$ are invariant for $T_2(t)$, for any $t \geq 0$. From Step 1 we know that for any $f \in L^2(X, \nu_\infty)$ such that $0 \leq f \leq 1$ $\nu_\infty$-a.e. we have $0 \leq T_2(t)f \leq 1$ $\nu_\infty$-a.e. Then, it follows that $L^\infty(X, \nu_\infty)$ is invariant under $(T_2(t))_{t \geq 0}$. Obviously, the same holds true for $(T^2_\nu(t))_{t \geq 0}$. Let $f \in L^2(X, \nu_\infty)$. For any $g \in L^\infty(X, \nu_\infty)$, we have

$$
\left| \int_X T_2(t)f g d\nu_\infty \right| = \left| \int_X fT^\ast_2(t)g d\nu_\infty \right| \leq \|f\|_{L^1(X, \nu_\infty)} \|g\|_{L^\infty(X, \nu_\infty)}, \quad t \geq 0,
$$

(3.8)

since also $T^\ast_2(t)$ is a contraction on $L^\infty(X, \nu_\infty)$. (3.8) and the density of $L^2(X, \nu_\infty)$ in $L^1(X, \nu_\infty)$ implies that for any $f \in L^1(X, \nu_\infty)$ we have $T_2(t)f \in L^1(X, \nu_\infty)$ for any $t \geq 0$ and

$$||T_2(t)f||_{L^1(X, \nu_\infty)} \leq ||f||_{L^1(X, \nu_\infty)}, \quad t \geq 0.
$$

The claim is so proved. By applying the Riesz-Thorin Interpolation Theorem [25, Section 1.18.7, Theorem 1] we conclude that $(T_2(t))_{t \geq 0}$ extends to a positive contraction semigroup $(T_2(t))_{t \geq 0}$ on $L^p(X, \nu_\infty)$ for any $p \in [1, +\infty)$. Uniqueness follows by density.

**Step 3.** Now we show that $(T_p(t))_{t \geq 0}$ is strongly continuous if $p \in [1, +\infty)$. Let $f \in C_0(X)$. We have

$$
\lim_{t \to 0} ||T_1(t)f - f||_{L^1(X, \nu_\infty)} = \lim_{t \to 0} \int_X |T_1(t)f - f| d\nu_\infty \leq \lim_{t \to 0} \nu_\infty(X)^{1/2} ||T_2(t)f - f||_{L^2(X, \nu_\infty)} = 0.
$$

The density of continuous bounded functions in $L^1(X, \nu_\infty)$ implies that $(T_1(t))_{t \geq 0}$ is strongly continuous on $L^1(X, \nu_\infty)$. By interpolation, we infer the strong continuity of $(T_p(t))_{t \geq 0}$ on $L^p(X, \nu_\infty)$ for any $p \in (1, 2)$. Finally, the reflexivity of $L^p(X, \nu_\infty)$ (see e.g. [12, Section 4, Theorem 1]) for any $p \in (1, +\infty)$ and [9, Theorem 1.34] allow us to conclude that $(T_p(t))_{t \geq 0}$ is strongly continuous on $L^p(X, \nu_\infty)$ for any $p \in (2, +\infty)$. \hfill $\Box$

For any $p \in [1, +\infty)$ let us denote by $L_p$ the infinitesimal generator of $(T_p(t))_{t \geq 0}$. Since $(T_p(t))_{t \geq 0}$ is a positive strongly continuous semigroup for any $p \in [1, +\infty)$, we get $1 \in \rho(L_p)$ for any $p \in [1, +\infty)$. The following result holds true.

**Proposition 3.8.** For any $p, q \in (1, +\infty)$ with $q > p$, we have $D(L_q) \subset D(L_p)$ with continuous embedding and for any $u \in D(L_q)$ we have $L_q u = L_p u$. In particular, $D(L_p) \subset W^{1, 2}_H(X, \nu_\infty)$ with continuous embedding for any $p \geq 2$.

*Proof.* Let $u \in D(L_q)$. Then, we have

$$
||t^{-1}(T_p(t)u - u) - L_q u||_{L^p_t(X, \nu_\infty)}^p = \int_X \left| \frac{t^{-1}(T_p(t)u - u)}{t} - L_q u \right|^p d\nu_\infty
$$

$$
\leq (\nu_\infty(X))^{1/r'} ||t^{-1}(T_q(t)u - u) - L_q u||_{L^r_t(X, \nu_\infty)}^{1/r} \to 0,
$$

as $t \to 0$, where $r = \frac{q}{p}$ and $r' = \frac{q}{q-p}$ . Hence, $u \in D(L_p)$ and $L_p u = L_q u$.

The last part follows from the fact that $D(L_2) \subset W^{1, 2}_H(X, \nu_\infty)$ with continuous injection. \hfill $\Box$

## 4 Analyticity of the semigroup associated to $L_p$

In this section we show that $L_p$ is sectorial in $L^p(X, \nu_\infty)$ for any $p \in (1, +\infty)$, i.e., $(T_p(t))_{t \geq 0}$ is an analytic semigroup on the sector $\Sigma_{\theta_p} := \{ re^{i\phi} : r > 0, \phi < \theta_p \}$, where

$$
cotg(\theta_p) = \frac{\sqrt{(p-2)^2 + p^2 \gamma^2}}{2\sqrt{p-1}}, \quad \gamma := \|B - B^*\|_{L(H)}.
$$

(4.1)
To this aim we follow the approach of [21, Section 3]. We introduce the following spaces of functions.

**Definition 4.1.** For any $p \in (1, +\infty)$ we set $L^p_{\mathbb{C}}(X, \nu_\infty) := L^p(X, \nu_\infty) + iL^p(X, \nu_\infty)$ with dual product $\langle f, g \rangle := \int_X f\overline{g} d\nu_\infty$ for any $f \in L^p_{\mathbb{C}}(X, \nu_\infty)$ and $g \in L^p_{\mathbb{C}}(X, \nu_\infty)$. For any $k \in \mathbb{N} \cup \{\infty\}$ we denote by $\mathcal{F}\mathcal{C}^k(X; \mathbb{C})$ the functions $f = u + iv$ such that $u, v \in \mathcal{F}\mathcal{C}^k(X)$. We set $W^{1,p}_{\mathbb{C}}(X, \nu_\infty) := W^{1,p}_H(X, \nu_\infty) + iW^{1,p}_H(X, \nu_\infty)$ for any $p \in (1, +\infty)$.

We consider the operator $L^p_{\mathbb{C}}$, on $D(L^p_{\mathbb{C}}) := D(L_p) + iD(L_p)$ endowed with the complexified norm of $D(L_p)$, defined by $L^p_{\mathbb{C}}f := L_p u + iL_p v$, where $f := u + iv \in D(L^p_{\mathbb{C}})$.

**Remark 4.2.** It is not hard to prove that all the results in Section 2 and Section 3 can be extended by complexification to the complex case.

**Remark 4.3.** We recall the definition of duality map. Given a Banach space $Y$ and given a duality $(\cdot, \cdot)_{Y \times Y^*}$ between $Y$ and $Y^*$, the duality map $\partial(y) \subset Y^*$ of $y \in Y$ is given by $\partial(y) := \{y^* \in Y^* : (y, y^*)_{Y \times Y^*} = \|y\|^2_Y = \|y^*\|^2_{Y^*}\}$. For any $p \in (1, +\infty)$ and any $f \in L^p_{\mathbb{C}}(X, \nu_\infty)$, with respect to the duality $(f, g) := \int_X fg d\nu_\infty$, we have $\partial(f) = \{\|f\|^2_{-p} f^*\}$, with

$$f^*(x) := \begin{cases} \overline{f(x)} |f(x)|^{p-2}, & f(x) \neq 0, \\ 0, & f(x) = 0. \end{cases}$$

In particular, $f^*$ is well defined also for $p \in (1, 2)$.

For any $\theta \in [0, \pi/2)$ we set $C_\theta := \cot\theta$. We will apply the following proposition, which is an adaptation of [21, Proposition 3.2] to our situation.

**Proposition 4.4.** Let $\mathcal{A}$ be a densely defined operator on $L^p(X, \nu_\infty)$ and assume that $1 \in \rho(\mathcal{A})$. Then, the following are equivalent:

(i) $\mathcal{A}$ generates an analytic $C_0$-semigroup on $L^p(X, \nu_\infty)$ which is contractive on $\Sigma_\theta$;

(ii) for any $f \in D(\mathcal{A})$ we have

$$\left| \text{Im} \left( \int_X \mathcal{A} f \overline{f} d\nu_\infty \right) \right| \leq -C_\theta \text{Re} \left( \int_X \mathcal{A} f \overline{f} d\nu_\infty \right).$$

(4.2)

**Remark 4.5.** Let $f \in \mathcal{F}\mathcal{C}^1(X; \mathbb{C})$ and let $p \geq 2$. Then, $f^* \in W^{1,2}_{\mathbb{C}}(X, \nu_\infty)$ and we have

$$D_H f^* = D_H (\overline{f}) |f|^{p-2} = |f|^{p-2} D_H \overline{f} + (p-2)|f|^{p-4} f \overline{f} \text{D}_H f,$$

where $f = u + iv$. In particular, $D_H f^*$ is bounded. It is enough to consider the sequence $(f_n) \subset \mathcal{F}\mathcal{C}^1(X)$ given by $f_n := \mathcal{J}(\theta_n \circ f)$, with $\theta_n(\xi) = (\xi^2 + \frac{1}{n})^{(p-2)/2}$ for any $\xi \in \mathbb{R}$ and $n \in \mathbb{N}$.

Finally, we recall [21, Lemma 3.3], which is obtained by repeating the computations of [6, Lemma 5].

**Lemma 4.6.** For any $f \in \mathcal{F}\mathcal{C}^1(X; \mathbb{C})$ and any $p \in [2, +\infty)$ we have

$$-\text{Re}[B_D H f, D_H f^*]_H = -\text{Re}[B^* D_H f, D_H f^*]_H$$

$$= \frac{1}{2} |f|^{p-4} \left( (p-1) |\text{Re}(\mathcal{J} D_H f)|^2_H + |\text{Im}(\mathcal{J} D_H f)|^2_H \right),$$

(4.3)

and

$$\text{Im}[B_D H f, D_H f^*]_H = p|f|^{p-4} \left( B + \frac{1}{2} I_H \right) \text{Im}(\mathcal{J} D_H f), \text{Re}(\mathcal{J} D_H f),$$

$$\text{Im}[B^* D_H f, D_H f^*]_H = p|f|^{p-4} \left( B^* + \frac{1}{2} I_H \right) \text{Im}(\mathcal{J} D_H f), \text{Re}(\mathcal{J} D_H f).$$

(4.4)
Following the arguments of [21, Theorem 3.4] we obtain the analyticity of the semigroup \((T_p(t))_{t \geq 0}\) for any \(p \in (1, +\infty)\).

**Proposition 4.7.** \((T_p(t))_{t \geq 0}\) is analytic in \(L^p(X, \nu_\infty)\) on the sector \(\Sigma_{\theta_p}\).

**Proof.** We show that Proposition 4.4(ii) is satisfied with \(\mathcal{A} = L_p\) and \(\theta = \theta_p\). To begin with, the positivity of \((T_p(t))_{t \geq 0}\) implies that \(1 \in \rho(L_p)\) for any \(p \in (1, +\infty)\). At first we consider \(p \in [2, +\infty)\) and then we deal with the case \(p \in (1, 2)\).

**Step 1.** Let \(p \in [2, +\infty)\), let \(f \in \mathcal{F} \mathcal{E}^{2,1}_b(X; \mathbb{C})\) and let \(f^* := \overline{f} |f|^{p-2} \in C_b(X)\). Let us set

\[
a := |\text{Re}(\overline{f} D_H f)|_H, \quad b := |\text{Im}(\overline{f} D_H f)|_H.
\]

From (4.3) we infer that

\[
-\text{Re}[BD_H f, D_H f^*]_H = \frac{1}{2} |f|^{p-4} ((p-1) a^2 + b^2).
\]

Since \(B + B^* = -I_H\) we easily get

\[
\left| B + \frac{1}{2} I_H \right|_{L(H)} = \left| \frac{1}{2} B - \frac{1}{2} B^* \right|_{L(H)} = \frac{1}{4} \gamma^2 + \left( \frac{1}{2} - \frac{1}{p} \right)^2,
\]

where \(\gamma\) has been introduced in (4.1). The Cauchy-Schwarz inequality and (4.4) give

\[
|\text{Im}[BD_H f, D_H f^*]|_H \leq |f|^{p-4} C_{\theta_p} \sqrt{p-1}.
\]

Thanks to the Young’s inequality \(2ab\sqrt{p-1} \leq (p-1)a^2 + b^2\) we deduce that

\[
|\text{Im}[BD_H f, D_H f^*]|_H \leq \frac{1}{2} |f|^{p-4} C_{\theta_p} ((p-1)a^2 + b^2) = -\text{Re}[BD_H f, D_H f^*]_H,
\]

for any \(f \in \mathcal{F} \mathcal{E}^{2,1}_b(X)\).

Let \(f = u + iv \in D(L^p_{\mathcal{C}})\) and let us consider a sequence \((f_n := u_n + iv_n) \in \mathcal{F} \mathcal{E}^{2,1}_b(X; \mathbb{C})\) such that \(u_n \to u\) and \(v_n \to v\) in \(W^{1,2}_H(X, \nu_\infty)\), and \(u_n \to u\) and \(v_n \to v\) \(\nu_\infty\)-a.e. in \(X\). These sequences exists thanks to Remark 2.18, to Proposition 3.8 and thanks to Remark 4.2.

From the definition of \(f^*_m\), we have that \(f^*_m \to f^*\ \nu_\infty\)-a.e. in \(X\). Further, \(\|f^*_m\|_{L^p(X, \nu_\infty)} = \|f_m\|_{L^p(X, \nu_\infty)}\) is uniformly bounded with respect to \(m \in \mathbb{N}\). Hence, there exists a function \(g \in L^p(X, \nu_\infty)\) such that, up to a subsequence which we still denote by \((f^*_m)\), \(f^*_m \to g\) as \(m \to +\infty\) in \(L^p(X, \nu_\infty)\). Since \(f^*_m \to f^*\ \nu_\infty\)-a.e. in \(X\), it follows that \(g = f^*\ \nu_\infty\)-a.e. in \(X\), i.e.,

\[
\int_X h f^*_m d\nu_\infty \to \int_X h f^* d\nu_\infty, \quad n \to +\infty, \quad \forall h \in L^p(X, \nu_\infty).
\]

From Remark 4.5 it follows that

\[
\lim_{m \to +\infty} \int_X |\text{Re}[BD_H f_m, D_H f^*_m]|_H - \text{Re}[BD_H f, D_H f^*_m]|_H d\nu_\infty = 0, \quad (4.11)
\]

\[
\lim_{m \to +\infty} \int_X |\text{Im}[BD_H f_m, D_H f^*_m]|_H - \text{Im}[BD_H f, D_H f^*_m]|_H d\nu_\infty = 0. \quad (4.12)
\]

Indeed,

\[
\int_X |\text{Re}[BD_H f_m, D_H f^*_m]|_H - \text{Re}[BD_H f, D_H f^*_m]|_H d\nu_\infty
\]

\[
\leq \|B\|_{L(H)} \int_X |D_H f - D_H f_m|_H |D_H f^*_m|_H d\nu_\infty
\]

\[
\leq \|B\|_{L(H)} \|D_H f - D_H f_m\|_{L^p(X, \nu_\infty; H)} \|D_H f^*_m\|_{L^p(X, \nu_\infty; H)}.
\]
We claim that \( \|D_H f_m^n\|_{L^p(X, \nu; \mathbb{H})} \) is uniformly bounded with respect to \( m \in \mathbb{N} \). Indeed, for any \( m \in \mathbb{N} \) we have

\[
\|D_H f_m^n\|_{L^p(X, \nu; \mathbb{H})} \leq 2^{p'-1} \left( \int_X |f_m|^{p'(p-2)} |D_H f_m^n| d\nu \right) + (p-2) \int_X |f_m|^{p'(p-2)} |D_H f_m^n| d\nu \right).
\]

We recall that \( p' = \frac{p}{p-1} \). By applying the Hölder inequality with \( q = p-1 \) and \( q' = \frac{p-1}{p-2} \), it follows that

\[
\|D_H f_m^n\|_{L^p(X, \nu; \mathbb{H})} \leq 2^{p'-1}(p-1) \|f_m\|_{L^p(X, \nu; \mathbb{H})}^\frac{1}{q'} \|D_H f_m^n\|_{L^p(X, \nu; \mathbb{H})}^\frac{1}{q} \leq c_p, \quad m \in \mathbb{N},
\]

for some positive constant \( c_p \), since both \( \|f_m\|_{L^p(X, \nu; \mathbb{H})} \) and \( \|D_H f_m^n\|_{L^p(X, \nu; \mathbb{H})} \) converge as \( n \to +\infty \). Then, the claim is true and (4.11) and (4.12) follow from the fact that \( D_H f_m \to D_H f \) in \( W^{1,2}_{0}(X, \nu) \) as \( m \to +\infty \). Same arguments also work for (4.12).

From Proposition 3.8, (4.9), (4.11), (4.11) and (4.12) we get

\[
\left| \int_X L_p f f^*_\nu d\nu \right| = \left| \int_X L_2 f f^*_\nu d\nu \right| = \lim_{m \to +\infty} \left| \int_X L_2 f f^*_\nu d\nu \right| = \left| \int_X L_2 f f^*_\nu d\nu \right|
\]

This shows that Proposition 4.4(ii) holds true for any \( f \in D(L^1_p) \), for any \( p \in [2, +\infty) \).

**Step 2.** Let \( p \in (1, 2) \). We claim that \( D(L^2_p) \) is a core for \( D(L^1_p) \). Remark 3.8 with \( q = 2 \) implies that \( D(L^2_p) \subset D(L^1_p) \). From Step 1, we know that \( (T_2(t))_{t \geq 0} \) is analytic in \( L^2(X, \nu_\infty) \) and therefore \( T(t)D(L_2) \subset D(L_2) \) for any \( t \geq 0 \). Since \( T_p(t) = T_2(t) \) on \( L^2(X, \nu_\infty) \), we infer the \( T_p(t)D(L_2) = T_2(t)D(L_2) \subset D(L_2) \). Moreover, \( \mathcal{F} \mathcal{C} \mathcal{B}^2(X) \subset D(L_2) \). This implies that \( D(L_2) \) is dense in \( L^p(X, \nu_\infty) \). From [13, Chapter 1, Proposition 1.7] and Remark 4.2 we deduce that the claim is true.

Let \( f \in D(L^2_p) \) and let \( (f_n) \subset D(L^2_p) \) be a sequence which converges to \( f \) in \( D(L^2_p) \) as \( n \to +\infty \) and \( f_n \to f \) \( \nu_\infty \)-a.e. in \( X \). As in (4.10), we can prove that, up to a subsequence, \( f_n^* \to f^* \) as \( n \to +\infty \) in \( L^p(X, \nu_\infty) \). Then, we have

\[
\left| \int_X L_p f f^*_\nu d\nu \right| = \lim_{n \to +\infty} \left| \int_X L_p f_n f^*_n d\nu \right| = \lim_{n \to +\infty} \left| \int_X L_2 f_n f^*_n d\nu \right| \quad (4.14)
\]

and the last equality follows from Proposition 3.8 with \( p = 2 \). From (4.13) with \( p = 2 \) and \( f \) replaced by \( f_n \) we infer that

\[
\left| \int_X L_2 f_n f^*_n d\nu \right| \leq -C_{\theta_2}Re \left( \int_X L_2 f_n f^*_n d\nu \right), \quad n \in \mathbb{N}. \quad (4.15)
\]
Collecting (4.14) and (4.15) we get

\[
\left| \text{Im} \int_X L_p f f^* dv_\infty \right| \leq - \lim_{n \to +\infty} C_{f_n} \text{Re} \left( \int_X L_2 f_n f_n^* dv_\infty \right) = - \lim_{n \to +\infty} C_{f_n} \text{Re} \left( \int_X L_p f_n f_n^* dv_\infty \right).
\]

This implies that Proposition 4.4(ii) is satisfied for \( p \in (1, 2) \).

\[\square\]

5 Example

We provide an example of operators \( A \) and \( Q \) which satisfy Hypotheses 2.1, 2.3 and 2.8. Let \( X := L^2(0, 1) \), let \( A \) be the realization of the Laplace operator in \( L^2(0, 1) \) with Dirichlet boundary conditions and domain \( W^{2,2}((0, 1), \partial \xi) \cap W_0^{1,2}((0, 1), \partial \xi) \), and let \( Q : X \to X \) be the covariance operator of the Wiener measure on \( X \), i.e.,

\[
Qf(x) := \int_0^1 \min\{x, y\} f(y) dy, \quad x \in (0, 1), \tag{5.1}
\]

for any \( f \in L^2(0, 1) \) (see e.g. [26]). It is well known that \( A \) is self-adjoint and that \( e_k = \sqrt{2} \sin(k\pi \cdot) \), \( k \in \mathbb{N} \), is an orthonormal basis of \( L^2((0, 1), d\xi) \) of eigenvectors of \( A \) with corresponding eigenvalues \( \lambda_k = -k^2 \pi^2 \). We denote by \( (c^tA)_{t \geq 0} \) the semigroup generated by \( A \). \((c^tA)_{t \geq 0} \) is analytic on \( L^2((0, 1), d\xi) \) and \( e^{tA} e_k = e^{-k^2 \pi^2 t} e_k \) for any \( k \in \mathbb{N} \). Then, it is not hard to see that for any smooth function \( f \) we have

\[
(Qe^{tA} f)(x) = \sqrt{2} \sum_{k=1}^{\infty} e^{-k^2 \pi^2 t} \langle f, \sqrt{2} \sin(k\pi \cdot) \rangle_{L^2} \left( \frac{1}{k^2 \pi^2} \sin(k\pi x) + \frac{(-1)^{k+1}}{k\pi} x \right).
\]

Moreover,

\[
(e^{tA}Qe^{tA} f)(x) = \sqrt{2} \sum_{k=1}^{\infty} e^{-2k^2 \pi^2 t} \langle f, \sqrt{2} \sin(k\pi \cdot) \rangle_{L^2} \frac{1}{k^2 \pi^2} \sin(k\pi x)
\]

\[+ 2 \sum_{k,j=1}^{\infty} e^{-(k^2+j^2) \pi^2 t} \langle f, \sqrt{2} \sin(k\pi \cdot) \rangle_{L^2} (-1)^{k+j} \frac{1}{k\pi} \langle x, \sqrt{2} \sin(j\pi \cdot) \rangle_{L^2} \sin(j\pi x)\]

\[= \sqrt{2} \sum_{k=1}^{\infty} e^{-2k^2 \pi^2 t} \langle f, \sqrt{2} \sin(k\pi \cdot) \rangle_{L^2} \frac{1}{k^2 \pi^2} \sin(k\pi x)\]

\[+ 2\sqrt{2} \sum_{k,j=1}^{\infty} e^{-(k^2+j^2) \pi^2 t} \langle f, \sqrt{2} \sin(k\pi \cdot) \rangle_{L^2} \frac{(-1)^{k+2}}{kj\pi^2} \sin(j\pi x).\]

Integrating between 0 and \( t \) we get

\[
(Q_t) f(x) = \sqrt{2} \sum_{k=1}^{\infty} \langle f, e_k \rangle_{L^2} \frac{1 - e^{-2k^2 \pi^2 t}}{2k^4 \pi^4} \sin(k\pi x)
\]

\[+ 2\sqrt{2} \sum_{k,j=1}^{\infty} \langle f, e_k \rangle_{L^2} \frac{(-1)^{k+j+2}(1 - e^{-(k^2+j^2) \pi^2 t})}{kj(k^2 + j^2) \pi^4} \sin(j\pi x).\]
Proposition 5.1. \( Q_t \) is a trace class operator for any \( t > 0 \), \( Q_t \to Q_\infty \) in the operator norm and \( Q_\infty \) is a trace class operator, where

\[
Q_\infty f(x) = \sqrt{2} \sum_{k=1}^{\infty} \langle f, e_k \rangle_{L^2} \frac{1}{2k^4 \pi^4} \sin(k\pi x) + 2\sqrt{2} \sum_{k,j=1}^{\infty} \langle f, e_k \rangle_{L^2} \frac{(-1)^{k+j+2}}{kj(k^2 + j^2)\pi^4} \sin(j\pi x)
\]

\[
= \frac{3\sqrt{2}}{2} \sum_{k=1}^{\infty} \langle f, e_k \rangle_{L^2} \frac{1}{2k^4 \pi^4} \sin(k\pi x) + 2\sqrt{2} \sum_{j \neq k}^{\infty} \langle f, e_k \rangle_{L^2} \frac{(-1)^{k+j+2}}{kj(k^2 + j^2)\pi^4} \sin(j\pi x). \tag{5.2}
\]

Proof. From the above computations we have

\[
\sum_{k=1}^{\infty} \langle Q_t e_k, e_k \rangle_{L^2} = \frac{3\sqrt{2}}{2} \sum_{k=1}^{\infty} \frac{1 - e^{-k^2 \pi^2 t}}{k^4 \pi^4} < +\infty,
\]

and

\[
\sum_{k=1}^{\infty} \langle Q_\infty e_k, e_k \rangle_{L^2} = \frac{3\sqrt{2}}{2} \sum_{k=1}^{\infty} \frac{1}{k^4 \pi^4} < +\infty.
\]

Finally, let us take \( U : X \to \mathbb{R} \) defined by

\[
U(f) := \int_0^1 f(\xi)^2 d\xi, \quad f \in X.
\]

From [5, Subsection 7.1] we infer that \( U \in W^{1,p}_H(X, \mu_\infty) \) for any \( p \in (1, +\infty) \). Hence, the Ornstein-Uhlenbeck operator \( L_p \) is sectorial in \( L^p(L^2(0, 1), e^{-U} \mu_\infty) \) for any \( p \in (1, +\infty) \).

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