Asymptotic stability of solitons of the gKdV equations with general nonlinearity

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Abstract

We consider the generalized Korteweg-de Vries equation

\[ \partial_t u + \partial_x (\partial_x^2 u + f(u)) = 0, \quad (t, x) \in [0, T) \times \mathbb{R}, \]  
(0.1)

with general \( C^3 \) nonlinearity \( f \). Under an explicit condition on \( f \) and \( c > 0 \), there exists a solution in the energy space \( H^1 \) of (0.1) of the type \( u(t, x) = Q_c(x - x_0 - ct) \), called soliton.

In this paper, under general assumptions on \( f \) and \( Q_c \), we prove that the family of soliton solutions around \( Q_c \) is asymptotically stable in some local sense in \( H^1 \), i.e. if \( u(t) \) is close to \( Q_c \) (for all \( t \geq 0 \)), then \( u(t) \) locally converges in the energy space to some \( Q_{c+} \) as \( t \to +\infty \). Note in particular that we do not assume the stability of \( Q_c \). This result is based on a rigidity property of equation (0.1) around \( Q_c \) in the energy space whose proof relies on the introduction of a dual problem. These results extend the main results in [12], [13], [15] and [11], devoted to the pure power case.

1 Introduction

We consider the generalized Korteweg-de Vries (gKdV) equations:

\[ \partial_t u + \partial_x (\partial_x^2 u + f(u)) = 0, \quad (t, x) \in [0, T) \times \mathbb{R}, \]  
(1.1)

for \( u(0) = u_0 \in H^1(\mathbb{R}) \), with a general \( C^3 \) nonlinearity \( f \). We assume that there exists an integer \( p \geq 2 \) such that

\[ f(u) = au^p + f_1(u) \quad \text{where } a > 0 \text{ and } \lim_{u \to 0} \frac{f_1(u)}{u^p} = 0. \]  
(1.2)

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This is the only assumption on \( f \) in this paper. Denote \( F(s) = \int_0^s f(s')ds' \).

Note that the local Cauchy problem is well-posed in \( H^1 \), using the arguments of Kenig, Ponce and Vega \([7], [8]\), see Remark 3 below. Moreover, the following conservation laws holds for \( H^1 \) solutions:

\[
\int u^2(t) = \int u_0^2, \quad E(u(t)) = \frac{1}{2} \int (\partial_x u(t))^2 - F(u(t)) = \frac{1}{2} \int (\partial_x u_0)^2 - F(u_0)
\]

Recall that if \( Q_c \) is a solution of

\[
Q''_c + f(Q_c) = c Q_c, \quad x \in \mathbb{R}, \quad Q_c \in H^1(\mathbb{R}), \tag{1.3}
\]

then \( R_{c,x_0}(t,x) = Q_c(x-x_0-ct) \) is solution of (1.1). We call soliton such nontrivial traveling wave solution of (1.1).

By well-known results on equation (1.3) (see section 2), there exists \( c_*(f) > 0 \) (possibly, \( c_*(f) = +\infty \)) defined by

\[
c_*(f) = \sup\{c > 0 \text{ such that } \forall c' \in (0,c), \exists Q_{c'} \text{ positive solution of (1.3)}\}.
\]

See Section 2 for another characterization of \( c_*(f) \) related to \( f \).

Recall that if a solution \( Q_c > 0 \) of (1.3) exists then \( Q_c \) is the unique (up to translation) positive solution of (1.3) and can be chosen even on \( \mathbb{R} \) and decreasing on \( \mathbb{R}^+ \).

The main result of this paper is the following:

**Theorem 1 (Asymptotic stability)** Assume that \( f \) is \( C^3 \) and satisfies (1.2). Let \( 0 < c_0 < c_*(f) \). There exists \( \alpha_0 > 0 \) such that if \( u(t) \) is a global \((t \geq 0)\) \( H^1 \) solution of (1.1) satisfying

\[
\forall t \geq 0, \quad \inf_{r \in \mathbb{R}} \|u(t, \cdot + r) - Q_{c_0}\|_{H^1} < \alpha_0, \tag{1.4}
\]

then the following hold.

1. Asymptotic stability in the energy space. There exist \( t \mapsto c(t) \in (0,c_*(f)), t \mapsto \rho(t) \in \mathbb{R} \) such that

\[
u(t) - Q_{c(t)}(\cdot - \rho(t)) \to 0 \quad \text{in } H^1(x > \frac{\rho(t)}{c_0}) \text{ as } t \to +\infty. \tag{1.5}\]

2. Convergence of the scaling parameter. Assume further that there exists \( \sigma_0 > 0 \) such that \( c \mapsto \int Q_c^2 \) is not constant in any interval \( I \subset [c_0 - \sigma_0,c_0 + \sigma_0] \). Then, by possibly taking a smaller \( \alpha_0 > 0 \), there exists \( c_+ \in (0,c_*(f)) \) such that \( c(t) \to c_+ \) as \( t \to +\infty \).

The main ingredient of the proof of Theorem 1 is a rigidity theorem around \( Q_{c_0} \).

**Theorem 2 (Nonlinear Liouville Property around \( Q_{c_0} \))** Assume that \( f \) is \( C^3 \) and satisfies (1.2). Let \( 0 < c_0 < c_*(f) \). There exists \( \alpha_0 > 0 \) such that if \( u(t) \) is a global \((t \in \mathbb{R})\) \( H^1 \) solution of (1.1) satisfying, for some function \( t \mapsto \rho(t) \), \( C, \sigma > 0 \),

\[
\forall t \in \mathbb{R}, \quad \|u(t, \cdot + \rho(t)) - Q_{c_0}\|_{H^1} \leq \alpha_0, \tag{1.6}
\]

\[
\forall t, x \in \mathbb{R}, \quad |u(t, x + \rho(t))| \leq Ce^{-\sigma|x|}, \tag{1.7}
\]

then there exists \( c_1 > 0, x_1 \in \mathbb{R}, \) such that

\[
\forall t, x \in \mathbb{R}, \quad u(t,x) = Q_{c_1}(x - x_1 - c_1t).
\]
Theorem 1 above is fundamental in proving the main results of [17], concerning the problem of collision of two solitary waves for general KdV equations. Indeed, as a corollary of the proofs of Theorem 1, Theorem 2 and [19], we obtain asymptotic stability of multi-solitons, see Section 5 for a precise result and more details on the proofs. See also [18] for more qualitative properties.

The arguments of [16] and [17] allow to describe the collision of two solitary waves in a large but fixed interval of time. Large time asymptotics are necessary to preserve the soliton structure after the collision as \( t \to +\infty \) (Theorem 1 and Theorem 3 in the present paper and Proposition 2 in [18]). This is especially important in Theorem 1.1 of [17], where we describe the behavior after the interaction of a solution which is as \( t \to -\infty \) exactly a 2-soliton solution.

Recall that the first result concerning asymptotic stability for solitons of (1.1) was proved by Pego and Weinstein [22], for the power case in some weighted spaces (with exponential decay at infinity in space) under spectral assumptions, checked only for the nonlinearities \( u^2 \) and \( u^3 \). This was extended by Mizumachi [21] under the same spectral assumptions with the condition \( \int_{x > 0} u^2(x) dx < +\infty \) on the solution.

Then, in [12] and [13], we have proved asymptotic stability in the energy space of the solitons of (1.1) in the power case respectively for \( p = 5 \) (critical) and \( p = 2, 3 \) and 4 (subcritical). In these papers, Theorem 2 was also the main ingredient of the asymptotic stability proof. These results have been improved and simplified in [15] in the subcritical power case. The proof is direct, with no reduction to an Liouville property such as Theorem 2. The proof uses a Virial identity which was verified only for \( u^2, u^3 \) and \( u^4 \) using the explicit expression of \( Q(x) \). Finally, in [11] the proof of the linear Liouville property (which is the main ingredient of the proof of Theorem 2) was simplified and extended in the power case for any \( p > 1 \).

Theorems 1 and 2 above present the first result of asymptotic stability of solitary waves for (1.1) with any nonlinearity, thus in cases where \( Q_c(x) \) have no explicit expression. In particular, the proof of Theorem 2 contains an intrinsic argument for any solitary wave satisfying \( 0 < c < c_\ast(f) \), which does not depend on a specific potential.

We also point out that with respect to [13], the arguments to prove Theorem 1 from Theorem 2 have been much simplified and extended. Instead of relying on the Cauchy theory in \( H^s \) for \( 0 < s < 1 \) as in [13], this reduction uses only localized energy type arguments (see proof of Proposition 4 and Appendix A). Moreover, the proof of Theorem 2 is direct, introducing a nonlinear dual problem.

Remark 1. We focus on the case \( Q_c > 0 \) (other solutions are negative and can be addressed by changing \( f(u) \) into \(-f(-u)\) in equation (1.1)). The exponential decay assumption (1.7) can be replaced by an assumption of compactness of \( u(t, \cdot + \rho(t)) \) in \( L^2 \), for \( t \in \mathbb{R} \) (see [12, 13]).

Remark 2. Note that if \( Q_{c_0} \) is nonlinearly stable (in the sense that \( \frac{d}{dt} \int_{|x| = c_0} |Q_{c_0}|^2 \) > 0, see Weinstein [27]), then assumption (1.6) can be replaced by the same assumption only at \( t = 0 \). However, the main point is that such a stability assumption is not needed to have asymptotic stability, which means that these two properties are not related. For example, in the power case for any \( p \geq 2 \), \( c_\ast(u^p) = +\infty \), and thus Theorem 1 holds in the subcritical \( (p = 2, 3, 4) \), critical \( (p = 5) \) and super critical case \( (p \geq 6) \), for any soliton.

In the super critical and critical cases, the soliton is unstable (Bona et al. [2, 14]). In Theorem 1, we make a global assumption on the solution (i.e. formally \( u_0 \) does not belong
to the instable manifold of the solitons). Whether or not such solutions exist in this case is an open question, however, the motivation of Theorem 1 in this case is to remove the possibility of any other dynamic around $Q$ (such as for example quasi-periodic solutions close to $Q$ or solutions oscillating between close solitons). In the case of the super critical nonlinear Schrödinger equation in dimension one, Krieger and Schlag [9] have constructed a subspace of codimension 5 of initial data in which a solution close to a soliton converges to the soliton.

Remark 3. In the case $f(u) = u^p - au^q$, where $2 \leq p < q$ are integers and $a > 0$ is a constant, $c_*$ is explicit: $c_* = s_0^{-1} - as_0^{-q-1}$, where $s_0 = \left( \frac{1}{2} \left( \frac{q+1}{q-1} \right) \right)^{1/p}$. Moreover, there is no soliton $Q_c$ for any $c > c_*$. Thus, Theorem 1 applies to any existing soliton in this case. For example, physical applications of this kind of nonlinearity in the context of the NLS equation are discussed in Sulem and Sulem [25]. See also Grillakis [5].

Note that the condition $f \in C^3$ can be relaxed. Indeed, all the arguments in this paper hold for $f \in C^2$. The condition $f \in C^3$ is only assumed to obtain well-posedness of the Cauchy problem in $H^1$ by [7], [8]. More precisely, for $p \geq 3$, well-posedness in $H^1$ for $f \in C^2$ follows directly from Theorem 3.6 in [7], and thus Theorems 1 and 2 hold for $f \in C^2$. If $p = 2$, one has to rely on the estimates and the norms introduced in the proof of Theorem 2.1 of [8] for $f(u) = u^2$, in the case $f \in C^3$ (we expect that a compactness argument should work in this case for $f \in C^2$).

Remark 4. It is clear that if $\frac{d}{dc} \int_{Q_c}^2 |c_{c_0} \neq 0$ ($c$ is not critical for stability) then $c(t)$ has a limit by Theorem 1. Our condition in Theorem 1 is more general (for example, if $f$ is analytic, then the assumption holds unless $f(u) = u^5$). Moreover, in the case $f(u) = u^5$, we do not expect convergence of $c(t)$ for general initial data.

Remark 5. One important tool in our analysis is a property of monotonicity of $L^2$ mass at the right in space for solutions of the KdV equation (see Appendix A). For the nonlinear Schrödinger equation, such a monotonicity property has been introduced in [20] to prove the stability of $N$ solitary waves for a class of suitable nonlinearities, but so far not for proving asymptotic stability. The question of asymptotic stability of solitary waves for the NLS equation (nonlinear Schrödinger equation) is mostly open, see results for special nonlinearities by Buslaev and Perleman [3], Perelman [23] and Rodnianski, Schlag and Soffer [24]. It is a promising direction of research.

Remark 6. In the integrable case ($f(u) = u^2$), using the inverse scattering transform, a general decomposition result has been proved by Eckaus and Schuur [4]: any smooth ($C^4$) and sufficiently decaying solution decomposes as $t \to +\infty$ in a finite sum of solitons plus a dispersive part that converges to zero in some sense. This implies the result of Theorem 1 for this nonlinearity and such initial data. Such questions for the integrable NLS equation (cubic NLS equation in one space dimension) are open.

The paper is organized as follows. In Section 2, we recall some preliminary results concerning solutions of (1.3). In Section 3, we prove Theorem 2 and in Section 4, we prove Theorem 1. Section 5 is devoted to the multi-soliton case.

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2 Preliminary results

2.1 Stationary equation (1.3)

First, we recall the necessary and sufficient condition for existence of a solution of (1.3), and some properties of the solution. Let $f$ be $C^2$ and satisfy (1.2) (so that for any $c > 0$, $\frac{c}{2}s^2 - F(s) > 0$ for small positive $s$).

**Claim 2.1** Let $c > 0$. There exists a nontrivial solution $Q_c \in H^1(\mathbb{R})$ ($Q_c(x_0) > 0$ for $x_0 \in \mathbb{R}$) of (1.3) if and only if there exists $s_0 > 0$ the smallest positive zero of $s \mapsto \frac{c}{2}s^2 - F(s)$ and $s_0$ satisfies $c s_0 - f(s_0) < 0$.

Moreover, $Q_c$ is $C^4$, unique up to translation and can be chosen so that $Q_c(0) = s_0$, $Q_c(x) = Q_c(-x)$, $Q_c(x) > 0$ for all $x > 0$. Finally, there exists $K > 0$ such that

$$\forall x \in \mathbb{R}, \quad \frac{1}{K}e^{-\sqrt{c}|x|} \leq Q_c(x) \leq Ke^{-\sqrt{c}|x|}, \quad |Q_c'(x)| \leq Ke^{-\sqrt{c}|x|}. \quad (2.1)$$

**Proof.** We refer to Berestycki and Lions [11], Theorem 5 and Remark 6.3 in section 6 for the proof of these results.

By assumption (1.2) and Claim 2.1 there exists $\bar{c} > 0$ such that for any $0 < c < \bar{c}$, $Q_c$ exists with $\|Q_c\|_{L^\infty} \to 0$ as $c \to 0$. Thus we may define

$$c_* = \sup\{c > 0 \text{ such that } \forall c' \in [0, c], \exists Q_{c'} \text{ positive solution of (1.3)}\}.$$ 

In the power case, we have $c_* = +\infty$ by scaling property. Note also that if $c_* < +\infty$ then from Claim 2.1 there exists no nontrivial solution of (1.3) for $c = c_*$. Let us give another characterization of $c_*$, which is the one used in the proofs.

**Claim 2.2** A unique even positive solution $Q_c$ of (1.3) exists and satisfies

$$\forall x \in \mathbb{R}, \quad Q_c(x)f(Q_c(x)) - 2F(Q_c(x)) > 0$$

if and only if $0 < c < c_*$. 

Note that this property is related to the Palais-Smale condition for the corresponding variational problem in dimension greater or equal than 2.

**Proof.** First, let $c > 0$ and assume the existence of $Q_c > 0$ solution of (1.3) satisfying (2.2). Let $s_c = Q_c(0)$. Since $Q_c(\mathbb{R}) = (0, s_c)$, by (2.2), we have:

$$\forall s \in (0, s_c), \quad sf(s) - 2F(s) > 0. \quad (2.3)$$

Let $0 < c' < c$. Let us prove that there exists a positive solution of (1.3) for $c'$. Since $\frac{c'}{2}s_c^2 - F(s_c) < \frac{c}{2}s^2 - F(s) = 0$ and (1.2), there exists $0 < s_{c'} < s_c$ the first zero of $\frac{c'}{2}s^2 - F(s)$, and by (2.3), $s_{c'}f(s_{c'}) - 2F(s_{c'}) > 0$. Together with $\frac{c'}{2}s_{c'}^2 - F(s_{c'}) = 0$ this implies that $c's_{c'} - f(s_{c'}) < 0$ and thus by Claim 2.1, there exists $Q_{c'}$ solution of (1.3) with $c = c'$. Since $0 < c' < c$ is arbitrary, we have proved $0 < c < c_*$. Second, let us consider $0 < c < c_*$. For the sake of contradiction assume that for some $0 < s \leq Q_c(0)$, $sf(s) - 2F(s) \leq 0$. Let $0 < s_1 \leq Q_c(0)$ be the smallest such $s$, so that by (1.2), $s_1f(s_1) - 2F(s_1) = 0$ and $sf(s) - 2F(s) > 0$ on $(0, s_1)$. Let $c' = \frac{F(s_1)}{s_1^2}$. Since $s \mapsto \frac{F(s)}{s^2}$ is strictly increasing on $[0, s_1]$, $s_1$ is the first zero of $s \mapsto \frac{c}{2}s^2 - F(s)$. Using $s_1f(s_1) - 2F(s_1) = 0$, we obtain that $c's_1 - f(s_1) = 0$, which implies that equation (1.3) has no solution for $c = c'$, a contradiction with the definition of $c_*$. 

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2.2 Linearized operator around $Q_c$

Let $\varphi(x)$ be a $C^2$ even function such that $0 \leq \varphi \leq 1$, $|\varphi'| + |\varphi''| \leq 4$ on $\mathbb{R}$, $\varphi \equiv 1$ on $[0, 1]$, $\varphi \equiv 0$ on $[2, +\infty)$. Let $\langle f, g \rangle$ denote the $L^2$ scalar product of $f$ and $g$. We consider the linearized operator around $Q_{c_0}$:

$$\mathcal{L}_{c_0} = -\partial_x^2 + c_0 - f'(Q_{c_0}).$$

(2.4)

**Claim 2.3** Let $0 < c_0 < c_*(f)$. The following properties hold

1. There exist unique $\lambda_0 > 0$, $\tilde{\chi}_{c_0} \in H^1(\mathbb{R})$, $\tilde{\chi}_{c_0} > 0$ such that $\mathcal{L}_{c_0}\tilde{\chi}_{c_0} = -\lambda_0\tilde{\chi}_{c_0}$, $\langle \tilde{\chi}_{c_0}, \tilde{\chi}_{c_0} \rangle = 1$.

2. For all $u \in H^1$, $\mathcal{L}_{c_0}u = 0$ is equivalent to $u = \lambda Q'_{c_0}$, $\lambda \in \mathbb{R}$.

3. For all $h \in L^2$, if $\langle h, Q'_{c_0} \rangle = 0$ then there exists a unique $u \in H^2$ such that $\langle u, Q'_{c_0} \rangle = 0$ and $\mathcal{L}_{c_0}u = h$.

Moreover, $\mathcal{L}_{c_0}S_{c_0} = -Q_{c_0}$ where $S_{c_0} = \frac{d}{dc}Q_{c=\cdot}.

4. There exist $B, \lambda_1, \sigma_1 > 0$ such that for all $c \in [c_0 - \sigma_1, c_0 + \sigma_1]$, the function $\chi_c(x) = \tilde{\chi}_c(x)\varphi(\frac{x}{B})$ satisfies

$$\int \chi_c Q_c > 0, \quad \frac{\lambda_0}{2} \leq -\frac{\langle \mathcal{L}_c \chi_c, \chi_c \rangle}{\langle \chi_c, \chi_c \rangle} \leq \lambda_0, \quad (2.5)$$

$$\forall u \in H^1(\mathbb{R}), \quad \int uQ'_c = \int u\mathcal{L}_c \chi_c = 0 \implies \langle \mathcal{L}_c u, u \rangle \geq \lambda_1 \langle u, u \rangle. \quad (2.6)$$

**Proof.** The first three properties follow from classical arguments. See Weinstein [26], proof of Proposition 2.8b for $N = 1$ and proof of Proposition 2.10. Note that letting $S_{c_0} = \frac{d}{dc}Q_{c=\cdot}$, then by differentiating the equation of $Q_c$ with respect to $c$, we obtain $\mathcal{L}_{c_0}S_{c_0} = -Q_{c_0}$. Note also that $\tilde{\chi}_c(x) \leq Ke^{-\sqrt{\lambda_0}x}$ and (2.6) holds for $\tilde{\chi}$.

Now, let $\chi_c(x) = \tilde{\chi}_c(x)\varphi(\frac{x}{B})$. By index theory of quadratic form, it is enough to check (2.5). We have $\chi_c \geq 0$ and $\chi_c \neq 0$, so that $\int \chi_c Q_c > 0$, $\langle \chi_c, \chi_c \rangle = 1 + O(e^{-\sqrt{\lambda_0}B})$ and $\mathcal{L}_c \chi_c = (\mathcal{L}_c \tilde{\chi}_c)\varphi(\frac{x}{B}) - \frac{1}{B^2} \varphi'\varphi(\frac{x}{B}) \tilde{\chi}_c(\frac{x}{B}) - \frac{1}{B^2} \varphi''\varphi(\frac{x}{B}) \tilde{\chi}_c(\frac{x}{B})$ so that $\langle \mathcal{L}_c \chi_c, \chi_c \rangle = -\lambda_0 + O(e^{-\sqrt{\lambda_0}B})$. Thus, (2.5) follows by taking $B$ large enough.

From now on, $B$ is fixed to such value. Note that $\chi_c$ has support in $[-2B, 2B]$, uniform in $c \in [c_0 - \sigma_1, c_0 + \sigma_1]$.

2.3 Decomposition of a solution close to $Q_{c_0}$

**Lemma 2.1** (Modulation of a solution close to $Q_{c_0}$) Let $0 < c_0 < c_*$, $K_0 > 0$ and $\alpha_0 > 0$ such that for any $0 < \alpha < \alpha_0$ and $T_0 > 0$, if $u(t)$ solution of (1.1) satisfies

$$\forall t \in [0, T_0], \quad \inf_{r \in \mathbb{R}} \|u(t) - Q_{c_0}(\cdot - r)\|_{H^1} \leq \alpha, \quad (2.7)$$

then there exist $c(t) > 0$, $\rho(t) \in C^1([0, T_0])$ such that

$$\eta(t, x) = u(t, x) - Q_{c(t)}(x - \rho(t)), \quad (2.8)$$
satisfies, for all $t \in [0, T_0]$,

$$
\int \tilde{\chi}_c(t)(x - \rho(t))\eta(t,x)dx = \int Q'_c(t)(x - \rho(t))\eta(t,x)dx = 0,
$$

(2.9)

$$
|c(t) - c_0| + \|\eta(t)\|_{H^1} \leq K_0 \alpha,
$$

(2.10)

$$
|\eta(t)| + |\eta'(t) - c(t)| \leq K_0 \left( \int \eta^2(t,x)e^{-|x-\rho(t)|}dx \right)^{\frac{1}{2}}.
$$

(2.11)

Proof. This is a standard application of the implicit function theorem. See for example [14], Proposition 1 for details. Note that $d\frac{dc}{dc}Q'_c|_{c'=c}$ and $d\frac{dx}{dx}Q'_c|_{x'=0} = Q'_c(x)$. Thus, the nondegeneracy conditions are (by Claim 2.3),

$$
\int S_c \tilde{\chi}_c = -\frac{1}{\lambda_0} \int L_c(S_c) \tilde{\chi}_c = \frac{1}{\lambda_0} \int Q_c S_c > 0, \quad \int \tilde{\chi}_c Q'_c = 0,
$$

$$
\int Q'_c S_c = 0, \quad \int (Q'_c)^2 > 0.
$$

3 Rigidity results

This section is devoted to the proof of the rigidity theorem (Theorem 2), see Section 3.2. First, in section 3.1, we give a linear version of the result to present the main idea in the simplest case.

3.1 Linear Liouville property

In this section, under the assumptions of Theorem 2, we prove a rigidity result for $H^1$ solutions of the following linearized equation

$$
\partial_t \eta = \partial_x(L_{c_0}\eta), \quad (t, x) \in \mathbb{R} \times \mathbb{R}, \quad \text{where} \quad L_{c_0}\eta = -\eta_{xx} + c_0 \eta - f'(Q_{c_0})\eta.
$$

(3.1)

Note that the arguments of Lemma 9 in [12] (based on linear estimates of Kenig, Ponce and Vega [8]) prove that the Cauchy problem for (3.1) is globally well-posed in $H^1(\mathbb{R})$ (in a certain sense). By $H^1$ solution, we mean a solution constructed in this way. Any such solution can be approached by regular solutions which allows to justify formal computations.

Proposition 1 (Linear Liouville property) Let $0 < c_0 < c_4(f)$. Let $\eta \in C(\mathbb{R}, H^1(\mathbb{R}))$ be solution of (3.1). Assume that there exist $K > 0, \sigma > 0$ such that

$$
\forall (t, x) \in \mathbb{R} \times \mathbb{R}, \quad |\eta(t, x)| \leq Ke^{-\sigma|x|}.
$$

(3.2)

Then, there exists $b_0 \in \mathbb{R}$ such that for all $t \in \mathbb{R}$, $\eta(t) \equiv b_0 Q'_{c_0}$.

Remark. Note that since $Q'_{c_0}$ verifies $L_{c_0} Q'_{c_0} = 0$ and has exponential decay, $\eta(t) \equiv b_0 Q'_{c_0}$ is solution of (3.1)–(3.2).

Let $\eta(t)$ be an $H^1$ solution of (3.1) satisfying (3.2). As in [11], we introduce a dual problem related to $\eta$. 


Lemma 3.1 (Introduction of the dual problem) Let
\[ v(t, x) = \mathcal{L}_{c_0} \eta(t, x) + \alpha(t)Q_{c_0} \] where \[ \alpha(t) = -\frac{\int \eta \mathcal{L}_{c_0} \chi_{c_0}}{\int \chi_{c_0} Q_{c_0}}. \]

Then, \( v \in C(\mathbb{R}, H^1(\mathbb{R})) \) and \( v(t) \) satisfies:

1. Equation of \( v \).
   \[ \partial_t v = \mathcal{L}_{c_0} (\partial_x v) + \alpha'(t)Q_{c_0}, \quad (t, x) \in \mathbb{R} \times \mathbb{R}. \] (3.3)

2. Exponential decay. There exists \( K > 0 \) such that
   \[ \forall (t, x) \in \mathbb{R} \times \mathbb{R}, \quad |v(t, x)| \leq Ke^{-\frac{\sqrt{c_0}}{8}|x|}. \] (3.4)

3. Orthogonality relations.
   \[ \forall t \in \mathbb{R}, \quad \int v(t, x) \chi_{c_0}(x) dx = \int v(t, x) Q'_{c_0}(x) dx = 0. \] (3.5)

4. Virial identity on the dual problem. Let
   \[ \mu_{c_0}(x) = -\frac{Q'_{c_0}(x)}{Q_{c_0}(x)} \] then
   \[ \frac{1}{2} \frac{d}{dt} \int v^2(t, x) \mu_{c_0}(x) dx = \int \partial_x v \mathcal{L}_{c_0}(v \mu_{c_0}). \] (3.6)

Proof of Lemma 3.1:
1. We have \( \eta_t = \partial_x (\mathcal{L}_{c_0} \eta) = v_x - \alpha(t)Q'_{c_0} \), thus using \( \mathcal{L}_{c_0} Q'_{c_0} = 0 \), we obtain
   \[ v_t = \mathcal{L}_{c_0} \eta_t + \alpha'(t)Q_{c_0} = \mathcal{L}_{c_0} v_x + \alpha'(t)Q_{c_0}. \]

2. From monotonicity arguments on \( \eta(t) \) and on \( v(t) \), we claim that there exists \( K > 0 \) such that for all \( t \in \mathbb{R} \),
   \[ \int (v^2_x + v^2)(t) \exp \left( \frac{\sqrt{c_0}}{4} |x| \right) dx \leq K. \] (3.7)
   (proof in Appendix A). By \( \|v \exp \left( \frac{\sqrt{c_0}}{8} |x| \right)\|_{L^\infty} \leq K \|v \exp \left( \frac{\sqrt{c_0}}{8} |x| \right)\|_{H^1} \), this implies (3.4).

The fact that \( v \in C(\mathbb{R}, H^1(\mathbb{R})) \) then follows from the equation.

3. By the choice of \( \alpha(t) \) and \( \mathcal{L}_{c_0} Q'_{c_0} = 0 \), we have
   \[ \int v \chi_{c_0} = \int \mathcal{L}_{c_0} \eta \chi_{c_0} + \alpha(t) \int Q_{c_0} \chi_{c_0} = 0, \quad \int v Q'_{c_0} = \int \mathcal{L}_{c_0} \eta Q'_{c_0} + \alpha(t) \int Q_{c_0} Q'_{c_0} = 0. \]

4. From the equation of \( v \) and \( \int Q_{c_0} v(t) \mu_{c_0} = -\int v(t) Q'_{c_0} = 0 \), we have
   \[ \frac{1}{2} \frac{d}{dt} \int v^2(t) \mu_{c_0} = \int v \partial_t v \mu_{c_0} = \int \mathcal{L}_{c_0} (\partial_x v) \mu_{c_0} + \alpha'(t) \int Q_{c_0} v \mu_{c_0} = \int \partial_x v \mathcal{L}_{c_0}(v \mu_{c_0}). \]

Now, we claim the following.
Lemma 3.2 (Positivity of the quadratic form) For any $0 < c < c_*(f)$, there exists $\lambda_2(c) > 0$ continuous such that
\begin{equation}
\forall x \in \mathbb{R}, \quad \frac{\lambda_2}{\cosh^{p-1}(\sqrt{c}x)} \leq \mu'_c(x) \leq \frac{1}{\lambda_2 \cosh^{p-1}(\sqrt{c}x)},
\end{equation}

\begin{equation}
\forall w \in H^1, \quad -\int \partial_x w \mathcal{L}_c (w \mu_c) = \frac{3}{2} \int (\partial_x (\frac{w}{c}))^2 Q'_c \mu'_c \geq \lambda_2 \int w^2 \mu'_c - \frac{1}{\lambda_2} \left( \int w \chi_c \right)^2.
\end{equation}

Proof of Lemma 3.2. First, by $Q'_c = cQ_c - f(Q_c)$ and $(Q'_c)^2 = cQ^2_c - 2F(Q_c)$, we have by Claim 2.2 and $0 < c < c_*$, $x \neq 0$,
\begin{equation}
\mu'_c = \frac{1}{Q'_c} ((Q'_c)^2 - Q_c Q''_c) = \frac{1}{Q'_c} (Q_c f(Q_c) - 2F(Q_c)) > 0
\end{equation}
and we obtain (3.10) from (1.2) and (2.1) and continuity arguments.

Next, let $z = \frac{w}{Q_c}$ so that $w \mu_c = -z Q'_c$. We claim
\begin{equation}
-\int \partial_x w \mathcal{L}_c (w \mu_c) = \frac{3}{2} \int (\partial_x z)^2 Q'_c \mu'_c.
\end{equation}
Using
\begin{equation}
\mathcal{L}_c (z Q'_c) = z \mathcal{L}_c Q'_c - 2\partial_x z Q''_c - \partial^2_x z Q'_c = -2\partial_x z Q'_c - \partial^2_x z Q'_c,
\end{equation}
we have
\begin{align*}
-\int \partial_x w \mathcal{L}_c (w \mu_c) &= -\int \partial_x (Q'_c z) \mathcal{L}_c (z Q'_c) = \int (Q'_c z + Q_c \partial_z z)(-2\partial_x z Q''_c - \partial^2_x z Q'_c) \\
&= \int (z Q'_c Q''_c + (\partial_x z)^2 Q'_c)^2 - \frac{1}{2} (Q'_c)^2 - 2(\partial_x z)^2 Q_c Q''_c + \frac{1}{2} (\partial_x z)^2 (Q_c Q'_c) \\
&= \frac{3}{2} \int (\partial_x z)^2 Q'_c Q''_c = \frac{3}{2} \int (\partial_x z)^2 Q'_c \mu'_c,
\end{align*}
by (3.10), which proves (3.11).

Let $Z(x) = z(x) \cosh^{-\frac{p+1}{2}}(\sqrt{c}x)$. By (3.8) and direct computations, we have ($\delta > 0$)
\begin{equation}
\int (\partial_x z)^2 \mu'_c Q'_c \geq \delta \int (\partial_x z)^2 \cosh^{-p-1}(\sqrt{c}x) = \delta \langle \tilde{\mathcal{L}}_c Z, Z \rangle,
\end{equation}
where $\tilde{\mathcal{L}}_c Z = -Z_{xx} + \frac{c}{4} (p + 1)^2 Z - \frac{c}{4} (p + 1)(p + 3) Z \cosh^{-2}(\sqrt{c}x)$. From standard arguments, since the function $Q_c \chi_c \cosh^{-\frac{p+1}{2}}(\sqrt{c}x)$ is nonnegative, not zero and belongs to $L^2$ (this is where we use that $\chi_c$ is compactly supported), there exists $\lambda > 0$ such that
\begin{equation}
-\frac{2}{3} \int \partial_x w \mathcal{L}_c (w \mu_c) = \int (\partial_x z)^2 \mu'_c Q'_c \geq \delta \langle \tilde{\mathcal{L}}_c Z, Z \rangle \geq \lambda \int Z^2 - \frac{1}{\lambda} \left( \int Z Q_c \chi_c \cosh^{-\frac{p+1}{2}}(\sqrt{c}x) \right)^2.
\end{equation}
Since $w = z Q_c = Z Q_c \cosh^{-\frac{p+1}{2}}(\sqrt{c}x)$, from (3.8), we obtain ($\lambda_2 > 0$)
\begin{equation}
-\int \partial_x w \mathcal{L}_c (w \mu_c) \geq \lambda_2 \int w^2 \mu'_c - \frac{1}{\lambda_2} \left( \int w \chi_c \right)^2.
\end{equation}
Proof of Proposition 7. By (3.6), Lemma 3.2 and (3.5), we have

\[-\frac{1}{2} \frac{d}{dt} \int v^2(t) \mu_{c_0} \geq \lambda_2 \int v^2 \mu'_{c_0}.\] (3.13)

Since \(|\mu_{c_0}(x)| \leq C\) on \(\mathbb{R}\) and \(v(t)\) is uniformly bounded in time in \(L^2\), \(\lim_{t \to \pm \infty} \int v^2(t) \mu_{c_0} = l_\pm\) exist and by integrating (3.13),

\[\int_{-\infty}^{+\infty} \int v^2(t,x) \mu'_{c_0}(x) dx dt \leq \frac{1}{2 \lambda_2} (l_- - l_+) < +\infty.\] (3.14)

By (3.8), it follows that for a sequence \(t_n \to +\infty\), we have \(v(t_n) \to 0\) in \(L^2_{loc}(\mathbb{R})\) and thus by (3.4), \(v(t_n) \to 0\) in \(L^2(\mathbb{R})\) as \(n \to +\infty\) and \(l_+ = 0\). Similarly, \(l_- = 0\). Thus, by (3.14) and \(v \in C(\mathbb{R}, H^1)\), we obtain

\[\forall (t,x) \in \mathbb{R} \times \mathbb{R}, \quad v(t,x) = 0.\]

It follows that \(L_{c_0} \eta(t) = -\alpha(t) Q_{c_0}\). Thus, by Claim 2.3, we obtain, for some bounded function \(\beta(t)\),

\[\eta(t) = \alpha(t) S_{c_0} + \beta(t) Q'_{c_0}.\]

By the equation of \(\eta(t)\) (3.1), and the orthogonality of \(S_{c_0}\) and \(Q'_{c_0}\), we obtain \(\beta'(t) = -\alpha(t)\) and \(\alpha'(t) = 0\). Since \(\beta(t)\) and \(\alpha(t)\) are bounded, we deduce \(\alpha(t) \equiv 0\) and \(\beta(t) \equiv b_0\).

3.2 Nonlinear Liouville property - Proof of Theorem 2

The proof of Theorem 2 follows the same lines as the proof Proposition 1. Consider now \(u(t)\) as in Theorem 2. We first decompose \(u(t,x)\) similarly as in Lemma 2.1 using modulation theory. We obtain, for all \(t \geq 0\),

\[\eta(t,x) = u(t,x + \rho(t)) - Q_{c(t)}(x),\] (3.15)

where \(c(t), \rho(t)\) are \(C^1\) functions chosen so that

\[\int \eta(t,x) L_{c(t)} \chi_{c(t)}(x) dx = \int \eta(t,x) Q'_{c(t)}(x) dx = 0.\] (3.16)

(The nondegeneracy conditions in this case are \(\int S_c L_c \chi_c = \int L_c(S_c) \chi_c = -\int Q_c \chi_c < 0\) and \(\int (Q'_c)^2 > 0\).) Recall that

\[\|\eta(t)\|_{H^1} + |c(t) - c_0| \leq K \alpha_0.\] (3.17)

Thus, we can choose \(\alpha_0 > 0\) small enough so that, for all \(t \geq 0\), \(c(t) \in [c_0 - \sigma_0, c_0 + \sigma_0] \subset (0, c_*)\), for \(\sigma_0 > 0\) small enough so that Claim 2.3 and Lemma 3.2 apply to \(c = c(t)\).

As for the linear equation, we introduce a dual problem.

Lemma 3.3 (Dual problem for the nonlinear equation) Let

\[v(t,x) = -\eta_{xx} + c \eta - (f(Q_c + \eta) - f(Q_c)) = L_c \eta - (f(Q_c + \eta) - f(Q_c) - f'(Q_c) \eta).\]

Then, \(v \in C(\mathbb{R}, H^1(\mathbb{R}))\) and \(v(t)\) satisfies
1. Equation of $v$.

\[
v_t = -v_{xxx} + cv_x - v_x f'(Q_c + \eta) + (\rho' - c)v_x + c'(Q_c + \eta)
= \mathcal{L}_c(v_x) - v_x (f'(Q_c + \eta) - f'(Q_c)) + (\rho' - c)v_x + c'(Q_c + \eta).
\] (3.18)

2. Exponential decay. There exists $K > 0$ such that,

\[
\forall (t, x) \in \mathbb{R} \times \mathbb{R}, \quad |\eta(t, x)| + |v(t, x)| \leq Ke^{-\frac{\alpha}{L_2} |x|}.
\] (3.19)

3. Estimates and almost orthogonality relations. There exists $K > 0$ such that, $\forall t \in \mathbb{R},$

\[
|\eta'| + |\rho' - c| \leq K \|\eta\|_{L^2}, \quad \left| \int v\eta' \right| + \left| \int v\chi_c \right| \leq K \|\eta\|^2_{L^2}, \quad \|\eta\|_{L^2} \leq K \|v\|_{L^2}.
\] (3.20)

4. Virial type estimates. There exists $\lambda_3, B > 0$ such that, $\forall t \in \mathbb{R},$

\[
-\frac{1}{2} \frac{d}{dt} \int v^2 \mu_c \geq \lambda_3 \int v^2 \mu_c' - \frac{1}{\lambda_3} \|v\|^2_{H^1} \|\eta\|_{L^2}
\] (3.21)

\[
-\frac{1}{2} \frac{d}{dt} \int v^2 \geq \lambda_3 \int (v_x^2 + v^2) - \frac{1}{\lambda_3} \int_{|x| \leq B} v^2
\] (3.22)

Remark. Note that at the first order, we have $v(t) \sim \mathcal{L}_c \eta(t)$, $\int v\chi_c \sim 0$ and $\int vQ'_c \sim 0$ as in the proof of the linear Liouville property.

**Proof of Lemma 3.3**. 1. First, we write the equation of $\eta(t)$, from (3.15), (1.1) and (1.3)

\[
\eta_t = u_t + \rho' u_x - c' S_c = -(u_{xx} + f(\eta))_x + \rho' u_x - c' S_c
= (-\eta_{xx} + c\eta - (f(Q_c + \eta) - f(Q_c)))_x + (\rho' - c)(Q_c + \eta)_x - c' S_c
\] (3.23)

where $v = -\eta_{xx} + c\eta - (f(Q_c + \eta) - f(Q_c))$. Now, we compute $v_t$:

\[
v_t = -\eta_{xxx} + c\eta_t - \eta_t f'(Q_c + \eta) + c' \eta - c' S_c (f'(Q_c + \eta) - f'(Q_c))
= -v_{xxx} + cv_x - v_x f'(Q_c + \eta) + (\rho' - c)(-f(Q_c + \eta))_{xxx} + c(Q_c + \eta)_x - (Q_c + \eta)_x f'(Q_c + \eta)
- c' (-S_{xxx} + cS_c - S_c f'(Q_c + \eta)) + c' \eta - c' S_c (f'(Q_c + \eta) - f'(Q_c)).
\]

Since $v_x = -\eta_{xxx} + c\eta_x - \eta_x f'(Q_c + \eta) - Q'_c (f'(Q_c + \eta) - f'(Q_c))$, we obtain

\[
v_t = -v_{xxx} + cv_x - v_x f'(Q_c + \eta) + (\rho' - c)(v_x + \mathcal{L}_c Q'_c) - c' \mathcal{L}_c S_c + c' \eta.
\]

Thus, by $\mathcal{L}_c Q'_c = 0$ and $\mathcal{L}_c S_c = -Q_c$ (see Claim 2.3), we obtain (3.18).

2. By monotonicity arguments, we claim that there exists $K > 0$ (independent of $\alpha_0$) such that for all $t \in \mathbb{R},$

\[
\int (v_x^2 + v^2) \exp \left( \frac{\sqrt{c_0}}{4} |x| \right) dx \leq K.
\] (3.24)

See the proof of (3.24) in Appendix A. Note that (3.24) implies (3.19) (see proof of Lemma 3.1).
3. By classical arguments (multiply \(3.23\) by \(\chi_c\) (respectively, by \(Q^\prime_c\)) and integrate on \(\mathbb{R}\), we obtain \(|c'| + |\rho' - c| \leq K\|\eta\|_{L^2}\). See \(1.14\) for example.

Next, \(\int vQ^\prime_c = \int \mathcal{L}_c\eta Q^\prime_c - \int (f(Q_c + \eta) - f(Q_c))Q^\prime_c\) and since \(\mathcal{L}_cQ^\prime_c = 0\) and \(|f(Q_c + \eta) - f(Q_c)| \leq K\eta^2\) (\(f\) is \(C^2\)), we obtain \(\int v\mathcal{L}_c\eta \leq K\int \eta^2\). Since \(\int \eta\mathcal{L}_c\chi_c = 0, \int v\chi_c = \int (\mathcal{L}_c\eta - f(Q_c + \eta) - f(Q_c)\chi_c - f'(Q_c)\eta)\chi_c\) implies \(\int v\chi_c \leq K\int \eta^2\).

By Claim \(2.3\) and \(3.16\), we have \(\langle \mathcal{L}_c\eta, \eta \rangle \geq \lambda_1\int \eta^2\). Thus, since \(f\) is \(C^2\),

\[
\langle v, \eta \rangle = \langle \mathcal{L}_c\eta, \eta \rangle = \int (f(Q_c + \eta) - f(Q_c) - f'(Q_c)\eta)\eta \geq \lambda_1\int \eta^2 - K\|\eta\|_{L^\infty}\int \eta^2 \geq \frac{1}{2}\lambda_1\int \eta^2,
\]

for \(\alpha_0\) small enough using \(3.17\). Thus \(\int \eta^2 \leq K\int v^2\) by Cauchy-Schwartz inequality.

4. Proof of \((3.21)\). By the equation of \(v\)

\[
-\frac{1}{2}\frac{d}{dt}\int v^2\mu_c = -\int v_t v\mu_c - \frac{c'}{2}\int v^2\frac{d\mu_c}{dc} = -\int v_x\mathcal{L}_c(v\mu_c) + R_1
\]

where

\[
R_1 = \int (f'(Q_c + \eta) - f'(Q_c))\mu_c v_x v + \frac{1}{2}\int v^2\mu_c' + c'\int vQ^\prime_c - c'\int \eta v\mu_c - \frac{c'}{2}\int v^2\frac{d\mu_c}{dc}
\]

By Lemma \(3.2\) and \(3.20\), we have

\[
-\int v_x\mathcal{L}_c(v\mu_c) \geq \lambda_2 \int v^2\mu_c' - K\|\eta\|_{L^2}^4 \geq \lambda_2 \int v^2\mu_c' - K\|\eta\|_{L^2}\|v\|_{L^2}^2.
\]

Now, we prove \(|R_1| \leq K\|\eta\|_{L^2}\|v\|^2_{H^1}\) and \((3.21)\) will follow.

Since \(f\) is \(C^2\), we have \(\|f'(Q_c + \eta) - f'(Q_c)\| \leq K|\eta|\) and so

\[
\int (f'(Q_c + \eta) - f'(Q_c))\mu_c v_x v \leq K\|v\|_{L^\infty}\int |\eta|v_x \leq K\|v\|^2_{H^1}\|\eta\|_{L^2}.
\]

By \(3.20\) and since \(\mu_c, \mu_c', \frac{d\mu_c}{dc}\) are bounded, we have

\[
|(\rho' - c)\int v^2\mu_c'| + |c'\int vQ^\prime_c| + |c'\int \eta v\mu_c| + \frac{c'}{2}\int v^2\frac{d\mu_c}{dc} \leq K\|\eta\|_{L^2}\|v\|^2_{L^2}.
\]

Proof of \((3.22)\). By the equation of \(v\), we have

\[
-\frac{1}{2}\frac{d}{dt}\int v^2 = -\int v_t v = -\int v_x\mathcal{L}_c(vx) + R_2
\]

where

\[
R_2 = \int (f'(Q_c + \eta) - f'(Q_c))vx v + \frac{1}{2}(\rho' - c)\int v^2 - c'\int vxQ^\prime_c - c'\int vx\eta.
\]

First, by straightforward calculations, and using \((2.21), 1.2\)

\[
-2\int v_x\mathcal{L}_c(vx) = \int (3v_x^2 + v^2 - f'(Q_c)v^2 - xQ'^2 v^2) \geq \int (3v_x^2 + v^2) - K\int v^2 e^{-\frac{\sqrt{K}}{2}|x|}.
\]

Now, we estimate \(R_2\):

\[
\left|\int (f'(Q_c + \eta) - f'(Q_c))vx v\right| \leq K\|v\|_{L^\infty}\int |x\eta|v_x \leq K\|v\|^2_{H^1}\|\eta\|_{L^2}^2
\]

\[
\leq K\|v\|^2_{H^1}\|x^2\eta\|^\frac{3}{2}_{L^2}\|\eta\|^\frac{1}{2}_{L^2} \leq \frac{1}{10}\|v\|^2_{H^1},
\]

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for \( \alpha_0 \) small enough, using (3.17) and (3.19) (the constant in (3.19) does not depend on \( \alpha_0 \)). We also have for \( \alpha_0 \) small, from (3.17) and (3.20), \( \frac{1}{7} |\rho' - \rho| \leq \frac{1}{10} \int v^2 \). From (3.20) and (2.1), \( |c' \int xvQc| \leq \frac{1}{10} \int v^2 + K \int v^2 e^{-\frac{\sqrt{B}}{2|x|}} \). Next, \( |c' \int x\eta| \leq K ||v||_{L^2} \int |\eta||v| \leq \frac{1}{10} ||v||_{L^2}^2 \) is controlled as above. In conclusion, we have proved:

\[
-\frac{1}{2} \frac{d}{dt} \int xv^2 \geq \frac{1}{2} \left( (v_x^2 + v^2) - K_0 \int v^2 e^{-\frac{\sqrt{B}}{2|x|}} \right).
\]

Now, fix \( B > 0 \) such that \( K_0 e^{-\frac{\sqrt{B}}{2}} \leq \frac{1}{4} \). Then, we obtain

\[
-\frac{1}{2} \frac{d}{dt} \int xv^2 \geq \frac{1}{4} \int (v_x^2 + v^2) - K_0 \int_{|x| < B} v^2.
\]

**Proof of Theorem 2.** Consider \( u(t) \) as in Theorem 2 with \( \alpha_0 > 0 \), small enough so that, for all \( t \geq 0, c(t) \in [c_0 - \sigma_0, c_0 + \sigma_0] \subset (0, c_*), \) for \( \sigma_0 > 0 \) small enough so that Claim 2.3 and Lemma 3.3 apply to \( c(t) \). Let

\[
V(t) = -\frac{1}{2} \int (\mu_c + \varepsilon_0 x)v^2,
\]

where \( \varepsilon_0 = \frac{1}{2} \lambda_3^2 \inf \{ \mu'_c (x); |x| < B, c \in [c_0 - \sigma_0, c_0 + \sigma_0] \} > 0 \).

Then, from Lemma 3.3 and the definition of \( \varepsilon_0 \) we have for all \( t \),

\[
V'(t) \geq \lambda_3 \int v^2 \mu'_c + \lambda_3 \varepsilon_0 \int (v_x^2 + v^2) - \frac{1}{\lambda_3} ||v||_{H^1}^2 ||x||_{L^2}^2 - \frac{1}{\lambda_3} \varepsilon_0 \int_{|x| < B} v^2
\]

\[
\geq \lambda_3 \varepsilon_0 \int (v_x^2 + v^2) - \frac{1}{\lambda_3} ||v||_{H^1}^2 ||x||_{L^2}^2.
\]

Now, we choose \( \alpha_0 > 0 \) small enough so that by (3.17), \( \frac{1}{\lambda_3} ||\eta(t)||_{L^2} \leq \frac{1}{2} \lambda_3 \varepsilon_0 \). Thus,

\[
V'(t) \geq \varepsilon_1 \int (v_x^2 + v^2), \quad \varepsilon_1 = \frac{1}{2} \lambda_5 \varepsilon_0.
\]

By (3.19), \( V(t) \) is uniformly bounded on \( \mathbb{R} \), \( \lim_{t \to \pm \infty} V(t) = V_{\pm \infty} \) and thus

\[
\int_{-\infty}^{+\infty} \int (v_x^2 + v^2) \leq \frac{1}{\varepsilon_1} (V_+ - V_-).
\]

Thus, there exist \( t_n \to +\infty \) such that \( v(t_n) \to 0 \), as \( n \to +\infty \) in \( H^1(\mathbb{R}) \) and from (3.19), \( V_+ = \lim_{n \to +\infty} V(t_n) = 0 \). Similarly, \( V_- = 0 \). Using (3.26) again, we obtain

\[
\forall t, x \in \mathbb{R}, \quad v(t, x) \equiv 0.
\]

From (3.20), \( \forall t \in \mathbb{R}, \eta(t) = 0, c'(t) = 0, \rho'(t) = c(t) \). Thus, by (3.15), \( u(t, x) = Q_{c_0}(x - c(0)t - \rho(0)) \) is a soliton solution. This concludes the proof of Theorem 2.
4 Asymptotic stability - Proof of Theorem 1

The proof of the asymptotic stability is based on the nonlinear Liouville property as in [13].

For a general nonlinearity, we do not use the direct approach used in [15]. Indeed, for this approach, we would need spectral information on an linear operator related to $\mathcal{L}$, which we are not able to prove in general. In contrast, the dual problem introduced in Section 3 can be understood for general nonlinearity, since the underlying linear operator is always nonnegative (see Lemma 3.2). This is an intrinsic property of the dual problem.

Since working with the dual problem requires more regularity on the solution, we cannot work directly on the original $H^1$ solution. Thus the proof of Theorem 1 consists in using Theorem 2 on limiting objects, which are more regular than the solution itself.

However, we point out that the proof presented here is simpler than the one in [13]. Indeed, the convergence of $u(t_n)$ to an asymptotic object $\tilde{u}(t)$ is obtained by monotonicity properties (such as Lemma A.1) and not by the arguments of well-posedness for the Cauchy problem for (1.1) in $H^s$ ($0 < s < 1$) and localization as in [13].

We claim the following

Proposition 2 (Convergence to a compact solution) Under the assumptions of Theorem 1, for any sequence $t_n \to +\infty$, there exists a subsequence $(t_{\phi(n)})$ and $\tilde{u}_0 \in H^1(\mathbb{R})$ such that for all $A > 0$,

\[ u(t_{\phi(n)}, x + \rho(t_{\phi(n)})) \to \tilde{u}_0 \quad \text{in} \quad H^1(x > -A) \quad \text{as} \quad n \to +\infty, \tag{4.1} \]

where $c(t)$, $\rho(t)$ are associated to the decomposition of $u(t)$ as in Lemma 2.1.

Moreover, the solution $\tilde{u}(t)$ of (1.1) corresponding to $\tilde{u}(0) = \tilde{u}_0$ is global ($t \in \mathbb{R}$) and there exists $K > 0$ such that

\[
\forall t \in \mathbb{R}, \quad \|\tilde{u}(t, \cdot + \tilde{\rho}(t)) - Q_{c_0}\|_{H^1} \leq \alpha_0, \\
\forall t, x \in \mathbb{R}, \quad |\tilde{u}(t, x + \tilde{\rho}(t))| \leq K \exp \left(-\frac{\sqrt{\alpha_0}}{10}|x|\right), \tag{4.2} \]

where $\tilde{c}(t)$, $\tilde{\rho}(t)$ are associated to the decomposition of $\tilde{u}(t)$ as in Lemma 2.1 and $\tilde{\rho}(0) = 0$.

Let us first prove Theorem 1 assuming Proposition 2 and then prove Proposition 2.

Proof of Theorem 1 assuming Proposition 2 Let $u(t)$ satisfy the assumptions of Theorem 1 and $\alpha_0 > 0$ small enough so that Theorem 2 holds.

From Proposition 2 for any sequence $t_n \to +\infty$ there exists a subsequence $t_{n'}$ and $\tilde{c}_0$ such that $c(t_{n'}) \to \tilde{c}_0$, and $\tilde{u}_0 \in H^1(\mathbb{R})$ such that $u(t_{n'}, x + \rho(t_{n'})) - \tilde{u}_0 \to 0$ in $H^1(x > -A)$, for any $A > 0$. Moreover, the solution $\tilde{u}(t)$ associated to $\tilde{u}(0) = \tilde{u}_0$ satisfies (1.2) and $\tilde{c}(0) = \tilde{c}_0$, $\tilde{\rho}(0) = 0$.

Now we apply Theorem 2 to the solution $\tilde{u}(t)$. It follows that $\tilde{u}(t) = Q_{c_1}(x - x_1 - c_1 t)$. By uniqueness of the decomposition in Lemma 2.1 applied to $\tilde{u}(0)$, we have $c_1 = \tilde{c}_0$ and $x_1 = 0$.

Therefore, $u(t_{n'}, x + \rho(t_{n'})) - Q_{\tilde{c}_0} \to 0$ in $H^1(x > -A)$, for any $A > 0$, or equivalently, $u(t_{n'}, x + \rho(t_{n'})) - Q_{c(t_{n'})} \to 0$ in $H^1(x > -A)$ for any $A > 0$. Thus, this being true for any sequence $t_n \to +\infty$, it follows that, for any $A > 0$,

\[ u(t, \cdot + \rho(t)) - Q_{c(t)} \to 0 \quad \text{in} \quad H^1(x > -A) \quad \text{as} \quad t \to +\infty. \]
Now, we observe that \( \int Q_{c(t)}^2 \to M_+ > 0 \) as \( t \to +\infty \). This follows from monotonicity arguments. See proof of Proposition 3 in [13] and also step 3 of the proof of Proposition 2.

Assuming now that there exists \( \sigma_0 > 0 \) such that \( c \mapsto \int Q_c^2 \) is not constant in any interval \( I \subset [c_0 - \sigma_0, c_0 + \sigma_0] \). By possibly taking a smaller \( \alpha_0 > 0 \) so that \( c(t) \in [c_0 - \sigma_0, c_0 + \sigma_0] \) for all \( t \), it follows from the continuity of \( c(t) \) that \( c(t) \) has a limit as \( t \to +\infty \).

Finally, using the arguments of the proof of Proposition 3 in [13], we improve the convergence result to finish the proof of Theorem 1.

**Proof of Proposition 2**. We consider a solution \( u(t) \) satisfying the assumptions of Theorem 1. First, we apply Lemma 2.1 to \( u(0) \) to finish the proof of Theorem 1.

Let \( \tilde{u} \) be the solution of (1.1) corresponding to \( \tilde{u}(0) = \tilde{u}_0 \) and defined on the maximal time interval \( (-\bar{T}_-, \bar{T}_+) \).

**Step 1.** Exponential decay and strong convergence in \( L^2 \) on the right.

Consider the function \( \psi \) defined on \( \mathbb{R} \) by

\[
\psi(x) = \frac{2}{\pi} \arctan \left( \exp \left( \frac{x}{4} \right) \right), \quad \text{so that } \lim_{x \to \pm \infty} \psi = 1, \lim_{x \to -\infty} \psi = 0.
\]

(4.4)

Following Step 2 of the proof of Theorem 1 in [15] and the monotonicity arguments for (1.1) (see Lemma A.1), we have for all \( x_0 > 0 \),

\[
\limsup_{t \to +\infty} \int (u_x^2 + u^2)(t, x) \psi(\sqrt{c_0}(x - x_0)) dx \leq K \exp \left( -\frac{\sqrt{c_0}}{4} x_0 \right).
\]

(4.5)

Now, we prove the following

for all \( A > 0 \), \( u(t_n, \cdot + \rho(t_n)) \to \tilde{u}_0 \) in \( L^2(x > -A) \),

\[
(4.6)
\]

\[
\forall t_0 \in [0, \bar{T}_+), \quad \sup_{t \in [0, t_0]} \int (\tilde{u}_x^2 + \tilde{u}^2)(t, x) \exp \left( \frac{\sqrt{c_0}}{4} x \right) dx \leq K(t_0) < +\infty, \quad (4.7)
\]

\[
\sup_{t \in [0, t_0]} \| \tilde{u}(t, x) \exp \left( \frac{\sqrt{c_0}}{4} x \right) \|_{L^\infty} \leq K(t_0) < +\infty. \quad (4.8)
\]

**Proof of (4.6).** Since \( u(t_n, \cdot + \rho(t_n)) \to \tilde{u}_0 \) in \( H^1 \) weak, we have \( u(t_n, \cdot + \rho(t_n)) \to \tilde{u}_0 \) in \( L^2_{loc}(\mathbb{R}) \), and thus by (4.5), we obtain,

for all \( A > 0 \), \( u(t_n, \cdot + \rho(t_n)) \to \tilde{u}_0 \) in \( L^2(x > -A) \).

(4.9)

**Proof of (4.8).** From (4.5) and weak convergence in \( H^1 \), we have for all \( x_0 > 0 \),

\[
\int (\tilde{u}_0_x^2 + \tilde{u}_0^2)(x) \psi(\sqrt{c_0}(x - x_0)) dx \leq K \exp \left( -\frac{\sqrt{c_0}}{4} x_0 \right).
\]

(4.10)
Now, we prove a similar estimate for $\tilde{u}(t)$, i.e. \eqref{eq:51} for $t \in [0, \bar{T}_+]$, with a rough constant and without using monotonicity arguments. This kind of property is quite well-known for the gKdV equation (see Kato \cite{Kato60}). Let $0 < t_0 < \bar{T}_+$. Note that $\sup_{[0,t_0]} \|\tilde{u}(t)\|_{L^\infty} \leq 2 \sup_{[0,t_0]} \|\tilde{u}(t)\|_{H^1} \leq K(t_0)$, and so $|\tilde{u}| + |F(\tilde{u})| \leq K(t_0)\tilde{u}^2$. Thus, using $0 < \psi' < K\psi$ and $|\psi''| \leq K\psi$, by the computations of the proof of Lemma \ref{lem:31}, we have, for all $x_0 > 0$,

\begin{equation}
\frac{1}{\sqrt{c_0}} \frac{d}{dt} \int \tilde{u}^2(t,x)\psi(\sqrt{c_0}(x-x_0)) = \int (-3\tilde{u}_x^2 + 2(\tilde{u}f(\tilde{u}) - F(\tilde{u}))\psi'(\sqrt{c_0}(x-x_0)) + c_0 \int \tilde{u}^2\psi''(\sqrt{c_0}(x-x_0)) \leq K(t_0) \int \tilde{u}^2\psi(\sqrt{c_0}(x-x_0)),
\end{equation}

\begin{equation}
\frac{1}{\sqrt{c_0}} \frac{d}{dt} \int \tilde{u}_x^2 - 2F(\tilde{u})\psi(\sqrt{c_0}(x-x_0)) \leq K(t_0) \int \tilde{u}_x^2\psi(\sqrt{c_0}(x-x_0)) + K^2(t_0) \int \tilde{u}^2\psi(\sqrt{c_0}(x-x_0)).
\end{equation}

First, we deduce from \eqref{eq:50} and \eqref{eq:51} that $\forall t \in [0,t_0]$ and $\forall x_0 > 0$, $\int \tilde{u}^2(t,x)\psi(\sqrt{c_0}(x-x_0)) \leq K(t_0)\exp\left(-\frac{\sqrt{c_0}}{4}x_0\right)$. Then, by \eqref{eq:52}, we obtain

\begin{equation}
\int (\tilde{u}_x^2 + \tilde{u}^2)(t,x)\psi(\sqrt{c_0}(x-x_0))dx \leq K(t_0)\exp\left(-\frac{\sqrt{c_0}}{4}x_0\right).
\end{equation}

By \eqref{eq:53}, we have, for $\delta_1 > 0$, $\forall t \in [0,t_0]$ and $\forall x_0 > 0$,

$$
\int_{x < x_0} (\tilde{u}_x^2 + \tilde{u}^2)(t,x)\exp\left(\frac{\sqrt{c_0}}{4}x_0\right)dx \leq \frac{1}{\delta_1} \sqrt{\int (\tilde{u}_x^2 + \tilde{u}^2)(t,x)\psi(x-x_0)dx} \leq \frac{1}{\delta_1} K(t_0),
$$

and thus, passing to the limit $x_0 \to +\infty$, \eqref{eq:51} is proved. Finally, by $\|w\|_{L^\infty(\mathbb{R})} \leq 2\|w\|_{L^2(\mathbb{R})} \|w_x\|_{L^2(\mathbb{R})}$, and \eqref{eq:52} we also obtain the pointwise estimate \eqref{eq:51}.

Step 2. Strong convergence of $u(t_n + t_0 + \rho(t_n))$ to $\tilde{u}(t)$ on the right.

Lemma 4.1 The solution $\tilde{u}(t)$ is global, i.e. $\bar{T}_- = \bar{T}_+ = +\infty$. Moreover, for all $t \in \mathbb{R}$,

$$
\inf_{r \in \mathbb{R}} \|\tilde{u}(t_0 + r) - Q_{c_0}\|_{H^1} \leq Kc_0,
$$

for all $A > 0$, $u(t_n + t_0 + \rho(t_n)) \to \tilde{u}(t)$ in $H^1(\mathbb{R} | x > -A)$ as $n \to +\infty$,

$$
\tilde{p}(0) = 0, \quad \rho(t_n + t) - \rho(t_n) \to \tilde{p}(t) \quad \text{as} \quad n \to +\infty,
$$

where $\tilde{p}(t)$ is associated to the decomposition of $\tilde{u}(t)$ as in Lemma \ref{lem:21}.

The proof of Lemma 4.1 contains the main new arguments.

Proof of Lemma 4.1 For any $t \in (-\bar{T}_-, \bar{T}_+)$, we set

\begin{equation}
v_n(t,x) = u(t_n + t, x + \rho(t_n)) - \tilde{u}(t,x).
\end{equation}

Then, from the equation of $u(t)$ and $\tilde{u}(t)$ and \eqref{eq:51}, $v_n(t)$ satisfies

\begin{equation}
\partial_t v_n = -\partial_x(\partial_x^2 v_n + f(\tilde{u} + v_n) - f(\tilde{u})), \quad t \in (-\bar{T}_-, \bar{T}_+), x \in \mathbb{R}
\end{equation}

\begin{equation}
\int v_n^2(0)\psi(\sqrt{c_0}x) \to 0 \quad \text{as} \quad n \to +\infty.
\end{equation}

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Convergence in $L^2$ at the right for $t \geq 0$. Let $0 < t_0 < T_+$. We prove the following estimate:

$$\sup_{t \in [0, t_0]} \int v^2_n(t) \psi(\sqrt{c_0} x) \leq K(t_0) \int v^2_n(0) \psi(\sqrt{c_0} x). \quad (4.17)$$

Note that $\forall t \in [0, t_0]$, $||\tilde{u}(t)||_{L^\infty} \leq K ||\tilde{u}(t)||_{H^1} \leq K(t_0)$, and since $f$ is $C^2$ and $f(0) = f'(0) = 0$ $|f(\tilde{u} + v_n) - f(\tilde{u})| \leq K|v_n|$ and $|F(v_n)| \leq K|v_n|^2$, and $|f(\tilde{u} + v_n) - f(\tilde{u}) - f(v_n)| \leq K|\tilde{u}||v_n|$.

By computations similar to the ones in the proof of Lemma A.1, we have

$$\frac{1}{\sqrt{c_0}} \frac{d}{dt} \int v^2_n \psi(\sqrt{c_0} x) = -3 \int v^2_n \psi'(\sqrt{c_0} x) + c_0 \int v^2_n \psi''(\sqrt{c_0} x) + 2 \int (f(\tilde{u} + v_n) - f(\tilde{u}))(v_n \psi(\sqrt{c_0} x))_x.$$

We claim the following estimate of the nonlinear term:

$$2 \int (f(\tilde{u} + v_n) - f(\tilde{u}))(v_n \psi(\sqrt{c_0} x))_x \leq \frac{1}{10} \int v^2_n \psi'(\sqrt{c_0} x) + K \int v^2_n \psi(\sqrt{c_0} x) \quad (4.18)$$

Indeed, by direct computations;

$$\int (f(\tilde{u} + v_n) - f(\tilde{u}))(v_n \psi(\sqrt{c_0} x))_x$$

$$= \int (f(\tilde{u} + v_n) - f(\tilde{u}))(\sqrt{c_0} v_n \psi'(\sqrt{c_0} x) + v_{nx} \psi(\sqrt{c_0} x)).$$

$$= \sqrt{c_0} \int ((f(\tilde{u} + v_n) - f(\tilde{u}))v_n - F(v_n)) \psi'(\sqrt{c_0} x) + \int (f(\tilde{u} + v_n) - f(\tilde{u}) - f(v_n)) v_{nx} \psi(\sqrt{c_0} x)$$

$$= \mathbf{I} + \mathbf{II}.$$

We have

$$|\mathbf{I}| \leq K(t_0) \int v^2_n \psi'(\sqrt{c_0} x)$$

$$|\mathbf{II}| \leq K(t_0) \int |\tilde{u}||v_n||v_{nx}| \psi(\sqrt{c_0} x)$$

$$\leq K(t_0) \left\| \tilde{u} \sqrt{\frac{\psi'(\sqrt{c_0} x)}{\psi(\sqrt{c_0} x)}} \right\|_{L^\infty} \int |v_n||v_{nx}| \sqrt{\psi(\sqrt{c_0} x) \psi'(\sqrt{c_0} x)}$$

$$\leq K(t_0) \left\| \tilde{u} \sqrt{\frac{\psi'(\sqrt{c_0} x)}{\psi(\sqrt{c_0} x)}} \right\|_{L^\infty} \left( \int v^2_n \psi'(\sqrt{c_0} x) \right)^{\frac{1}{2}} \left( \int v^2_n \psi(\sqrt{c_0} x) \right)^{\frac{1}{2}}.$$

By the expression of $\psi$, we have $\sqrt{\frac{\psi'(\sqrt{c_0} x)}{\psi(\sqrt{c_0} x)}} \leq K(1 + e^{\frac{x^2}{b^2}})$ and thus from (4.18), we obtain

$$|\mathbf{II}| \leq K(t_0) \left( \int v^2_n \psi'(\sqrt{c_0} x) \right)^{\frac{1}{2}} \left( \int v^2_n \psi(\sqrt{c_0} x) \right)^{\frac{1}{2}}$$

$$\leq \frac{1}{10} \int v^2_n \psi'(\sqrt{c_0} x) + K(t_0) \int v^2_n \psi(\sqrt{c_0} x).$$
Thus, \( (1.18) \) is proved and by \(|\psi'''| \leq K\psi\), we obtain
\[
\frac{1}{\sqrt{c_0}} \frac{d}{dt} \int v_n^2 \psi(\sqrt{c_0} x) \leq -2 \int v_n^2 \psi' (\sqrt{c_0} x) + K(t_0) \int v_n^2 \psi' (\sqrt{c_0} x). \tag{4.19}
\]
This gives \( (1.17) \), and so, by \( (1.16) \), we obtain
\[
\sup_{t \in [0,t_0]} \int v_n^2(t) \psi(\sqrt{c_0} x) \to 0 \quad \text{as } n \to +\infty. \tag{4.20}
\]
In particular, for all \( A > 0 \), as \( n \to +\infty \),
\[
u(t_n + t, + \rho(t_n)) \to \tilde{u}(t) \text{ in } L^2(x > -A) \quad \text{and} \quad u(t_n + t, + \rho(t_n)) \to \tilde{u}(t, .) \text{ in } H^1 \text{ weak,}
\]
by the uniform \( H^1 \) bound on \( u(t) \), and by \( (4.3) \),
\[
\forall t \in [0, \tilde{T}_+), \quad \inf_{r \in \mathbb{R}} \| \tilde{u}(t, . + r) - Q_{c_0} \|_{H^1} \leq \alpha_0, \quad \text{and so } \tilde{T}_+ = +\infty.
\]
Convergence in \( L^2 \) at the right for \( t \leq 0 \). Let \( -\tilde{T}_- < t_1 < 0 \). There exist \( \tilde{u}_1(0) \in H^1 \) and a subsequence \( (t_{\phi(n)}) \) such that
\[
u(t_{\phi(n)} + t_1, . + \rho(t_{\phi(n)})) \to \tilde{u}_1(0) \quad \text{in } H^1 \text{ weak as } n \to +\infty.
\]
We reproduce on \( \tilde{u}_1(0) \) the analysis done so far on \( \tilde{u}(0) \). In particular, let \( \tilde{u}_1(t) \) be the solution of \( (1.1) \) corresponding to \( \tilde{u}_1(0) \) defined on \( (-\tilde{T}_-, \tilde{T}_+) \). It follows that \( \tilde{T}_1+ = +\infty \) and
\[
u(t_{\phi(n)}, . + \rho(t_{\phi(n)})) \to \tilde{u}_1(-t_1) \quad \text{in } H^1 \text{ weak as } n \to +\infty, \quad \text{and thus } \tilde{u}_0 = \tilde{u}_1(-t_1).
\]
By uniqueness of the \( H^1 \) solution of \( (1.1) \), we obtain \( \tilde{u}_1(0) = \tilde{u}(t_1) \) and \( u(t_{\phi(n)} + t_1, . + \rho(t_{\phi(n)})) \to \tilde{u}(t_1) \). In fact, the convergence \( u(t_n + t_1, . + \rho(t_n)) \to \tilde{u}(t_1) \) holds actually for the whole sequence \( (t_n) \).
As before, we obtain
\[
\forall t > -\tilde{T}_-, \quad \inf_{r \in \mathbb{R}} \| \tilde{u}(t, . + r) - Q_{c_0} \|_{H^1} \leq \alpha_0 \quad \text{and so } \tilde{T}_- = +\infty.
\]
Therefore, we are able to define \( \tilde{c}(t), \tilde{\rho}(t) \), associated to the decomposition of \( \tilde{u}(t) \) as in Lemma 2.1. By continuity and uniqueness of the decomposition in \( H^1 \), we have
\[
\tilde{\rho}(0) = 0 \quad \text{and for all } t \in \mathbb{R}, \quad \rho(t_n + t) - \rho(t) \to \tilde{\rho}(t) \text{ as } n \to +\infty. \tag{4.21}
\]
In conclusion, in addition to \( (1.21) \), we have obtained so far, for all \( t \in \mathbb{R} \),
\[
\inf_{r \in \mathbb{R}} \| \tilde{u}(t, . + r) - Q_{c_0} \|_{H^1} \leq K\alpha_0,
\]
for all \( A > 0, \quad u(t_n + t, . + \rho(t_n)) \to \tilde{u}(t) \text{ in } L^2(x > -A) \text{ as } n \to +\infty .
\]
Convergence in \( H^1 \) at the right. From the weak convergence and \( (4.5) \), there exists \( K > 0 \) such that
\[
\forall x_0 > 0, \forall t \in \mathbb{R}, \quad \int (\tilde{u}^2 + \tilde{v}^2)(t, x + \tilde{\rho}(t))c_0(x - x_0)dx \leq K \exp \left(-\frac{\sqrt{K}}{4}(x_0)\right), \tag{4.22}
\]
and thus, as before,
\[\forall t \in \mathbb{R}, \forall x > 0, \quad |\bar{u}(t, x + \rho(t))| \leq K \exp \left(-\frac{\sqrt{c_0}}{8} x \right).\] \hspace{1cm} (4.23)

Let \(v_n(t)\) be defined in (4.14) for all \(t \in \mathbb{R}\). We claim that
\[\forall t \in \mathbb{R}, \quad \int v_{nx}^2(t)\psi(\sqrt{c_0}x)dx \to 0 \quad \text{as} \quad n \to +\infty.\] \hspace{1cm} (4.24)

In particular, by (4.24) and (4.25), this implies that for all \(t \in \mathbb{R}\), for all \(A > 0\), \(u(t_n + t, + \rho(t_n)) \to \bar{u}(t)\) in \(H^1(x > -A)\) and Lemma 4.1 follows.

Now, let us prove (4.24). Let \(t_0 \in \mathbb{R}\). It follows from (4.19) integrated on \([t_0 - 1, t_0]\) and (4.20), that
\[\int_{t_0-1}^{t_0} \int v_{xn}^2(t, x)\psi'(\sqrt{c_0}x)dxdt \to 0 \quad \text{as} \quad n \to +\infty.\]
Thus, by (4.5), we obtain:
\[\int_{t_0-1}^{t_0} \int v_{xn}^2(t, x)\psi(\sqrt{c_0}x)dxdt \to 0 \quad \text{as} \quad n \to +\infty.\] \hspace{1cm} (4.25)

Now, we claim for any \(t_0 - 1 \leq t \leq t_0:\)
\[\int v_{nx}^2(t_0)\psi(\sqrt{c_0}x)dx \leq \int v_{nx}^2(t)\psi(\sqrt{c_0}x)dx + K(t_0) \int_{t_0-1}^{t_0} \int v_{nx}^2(t')\psi(\sqrt{c_0}x)dxdt' \]
\[+ K(t_0) \sup_{t' \in [t_0-1, t_0]} \int v_{nx}^2(t')\psi(\sqrt{c_0}x)dx.\] \hspace{1cm} (4.26)

By (4.20) and (4.25), we find \(\int v_{nx}^2(t_0)\psi(\sqrt{c_0}x)dx \leq \int v_{nx}^2(t)\psi(\sqrt{c_0}x)dx + o(1)\), and thus integrating on \(t \in [t_0 - 1, t_0]\), using (4.25) again, we prove (4.24).

Now, let us prove (4.26). Define
\[J(t) = \int (v_{nx}^2 - 2F(v_n))(t)\psi(\sqrt{c_0}x)dx,\]
so that
\[\frac{1}{\sqrt{c_0}} \frac{d}{dt} J = -3 \int v_{nxx}^2\psi'(\sqrt{c_0}x) + c_0 \int v_{nx}^2\psi''(\sqrt{c_0}x) \]
\[+ 2 \int (f(\bar{u} + v_n) - f(\bar{u}))v_{nx}\psi(\sqrt{c_0}x)x - 2 \int (v_{nx} + f(\bar{u} + v_n) - f(\bar{u}))(v_n)\psi(\sqrt{c_0}x)x \]
\[\leq -2 \int v_{nxx}^2\psi'(\sqrt{c_0}x) + K \int (v_{nx}^2 + v_n^2)\psi(\sqrt{c_0}x) \]
\[+ 2 \int (f(\bar{u} + v_n) - f(\bar{u}) - f(v_n))v_{nx}\psi(\sqrt{c_0}x),\]
by controlling terms as in the proof of (4.18) \( \|v_n \sqrt{\psi(\sqrt{c_0}x)}\|_{L^\infty}^2 \leq K \int (v_{n,x}^2 + v_n^2) \psi(\sqrt{c_0}x)) \).

Now, we control the last term:

\[
\int (f(\bar{u} + v_n) - f(\bar{u}) - f(v_n)) x v_{n,xx} \psi(\sqrt{c_0}x)
\]

\[
= \int \bar{u}_x f'(\bar{u} + v_n) - f'(\bar{u}) v_{n,xx} \psi(\sqrt{c_0}x) + v_{n,x} f'(\bar{u} + v_n) - f'(v_n) v_{n,xx} \psi(\sqrt{c_0}x)
\]

\[
\leq K \int (|\bar{u}_x| |v_n| + |\bar{u}||v_{n,x}|) |v_{n,xx}| \psi(\sqrt{c_0}x)
\]

\[
\leq \int v_{n,xx} \psi'(\sqrt{c_0}x) + K \| \bar{u} \psi(\sqrt{c_0}x) \|^2 \| v_{n,xx} \psi(\sqrt{c_0}x)
\]

\[
+ K \| v_n \sqrt{\psi(\sqrt{c_0}x)} \|^2 \| (\bar{u}_x)^2 \psi(\sqrt{c_0}x)
\]

We have \( \|v_n \sqrt{\psi(\sqrt{c_0}x)}\|_{L^\infty}^2 \leq K \int (v_{n,x}^2 + v_n^2) \psi(\sqrt{c_0}x) \) and \( \frac{\psi(\sqrt{c_0}x)}{\psi(\sqrt{c_0}x)} \leq K(1 + e^{\frac{\sqrt{c_0}}{2}}) \). Thus, using (4.17)–(4.3), we obtain

\[
\frac{d}{dt} J(t) \leq K \int (v_{n,x}^2 + v_n^2) \psi(\sqrt{c_0}x).
\]

Integrating between \( t \) and \( t_0 \) and using \( \int F(v_n) \psi(\sqrt{c_0}x) \leq K \int v_n^2 \psi(\sqrt{c_0}x) \), (4.26) is proved. Thus, Lemma 4.1 is proved.

**Step 3.** Exponential decay of \( \bar{u}(t, x) \). We prove

\[
\forall t, x \in \mathbb{R}, \quad |\bar{u}(t, x + \tilde{\rho}(t))| \leq K \exp \left(-\frac{\sqrt{c_0}}{16} |x| \right).
\]

(4.27)

We claim

\[
\forall x_0 > 0, \forall t \in \mathbb{R}, \quad \int \bar{u}^2(t, x + \tilde{\rho}(t))(1 - \psi(\sqrt{c_0}(x + x_0))) dx \leq K \exp \left(-\frac{\sqrt{c_0}}{4} x_0 \right).
\]

(4.28)

Note that (4.27) is a direct consequence of (4.23), (4.28) and the global \( H^1 \) bound on \( \bar{u}(t) \) using \( \|w\|_{L^\infty(x > x_0)}^2 \leq 2 \|w\|_{L^2(x > x_0)}^2 \|w_x\|_{L^2(x > x_0)} \).

Proof of (4.28). Estimate (4.28) was already proved in the same context in [13] and [19] (see for example [19], Lemma 7). Let us sketch a proof.

We use monotonicity arguments similar to the ones in Lemma 4.1. Let \( m_0 = \int \bar{u}_0^2 \). Let \( x_0 > 0 \) and \( t_0 \in \mathbb{R} \). By \( L^2 \) norm conservation and Lemma 4.1 for \( n(x_0) > 0 \) large enough, we have

\[
m_0 - \int \bar{u}^2(t_0)(1 - \psi(\sqrt{c_0}(x - \tilde{\rho}(t_0) + x_0))) = \int \bar{u}^2(t_0)\psi(\sqrt{c_0}(x - \tilde{\rho}(t_0) + x_0))
\]

\[
\geq \int u^2(t_n + t_0)\psi(\sqrt{c_0}(x - \rho(t_0 + t_n) + x_0)) - \exp \left(-\frac{\sqrt{c_0}}{4} x_0 \right).
\]

By monotonicity properties on \( u(t) \), for \( n' \geq n \) so that \( t_{n'} \geq t_n + t_0 \), it follows that

\[
m_0 - \int \bar{u}^2(t_0)(1 - \psi(\sqrt{c_0}(x - \tilde{\rho}(t_0) + x_0)))
\]

\[
\geq \int u^2(t_{n'})\psi(\sqrt{c_0}(x - \rho(t_{n'}) + x_0 + \frac{\rho}{4}(t_{n'} - (t_n + t_0)))) - K \exp \left(-\frac{\sqrt{c_0}}{4} x_0 \right).
\]
Again from the convergence of $u(t_{n'},+\rho(t_{n'}))$ to $\tilde{u}(0)$, for $n' = n'(n,x_0)$ large enough, we have $\int u^2(t_{n'})\psi(\sqrt{\epsilon_0}(x-\rho(t_{n'})) + x_0 + \frac{m}{4}(t_{n'} - (t_n + t_0))) \geq m_0 - \exp\left(-\frac{\sqrt{\epsilon_0}}{4}x_0\right)$. This proves that $\int \tilde{u}^2(t_0)(1 - \psi(\sqrt{\epsilon_0}(x-\tilde{\rho}(t_0) + x_0))) \leq K \exp\left(-\frac{\sqrt{\epsilon_0}}{4}x_0\right)$, thus (4.28) is proved.

From Lemma 4.1 and (4.27), Proposition 2 is proved.

5 Multi-soliton case

Now, we give a application of our results to the case of solutions containing several solitons. Let $N \geq 1$, $x_1, \ldots, x_N \in \mathbb{R}$, and

$$0 < c_N^0 < \ldots < c_N^1 < c_\ast(f), \quad \forall j, \quad \partial_c \int Q_{c=\epsilon_j}^2 > 0,$$

it was proved in [10] that there exists a unique solution $U(t)$ in $H^1$ of (1.1) such that

$$\left\|U(t) - \sum_{j=1}^{N} Q_{c_j}(\cdot - c_j t - x_j)\right\|_{H^1(\mathbb{R})} \to 0 \quad \text{as} \quad t \to +\infty.$$  

(5.2)

This solution $U(t)$ is called a multi-soliton solution (in [10], the result is proved only for the power case $f(u) = u^p$ for $p = 2, 3, 4, 5$ but the proof is exactly the same for a general $f(u)$ with stable solitons in the sense (5.1)).

The stability of such multi-soliton structures has been studied previously in [19]. Indeed, the main result in [19] is that under assumption (5.1), if

$$\inf_{r_j \in \mathbb{R}} \left\|u(0) - \sum_{j=1}^{N} Q_{c_j}(\cdot - r_j)\right\|_{H^1} < \alpha_0,$$

for $L_0$ large enough and $\alpha_0$ small enough, then the solution $u(t)$ of (1.1) satisfies

$$\forall t \geq 0, \quad \inf_{r_j \in \mathbb{R}} \left\|u(t) - \sum_{j=1}^{N} Q_{c_j}(\cdot - r_j)\right\|_{H^1} < A(\alpha_0 + e^{-\gamma t}).$$

(5.4)

Again the proof of this result in [19] was for the power case ($p = 2, 3, 4$), but the same proof applies to a general $f(u)$ under assumption (5.1).

In [19], the asymptotic stability of such multi-soliton was also proved, but the proof was restricted to $p = 2, 3$ and 4, since it was based on [15] (linear Liouville argument). As a direct consequence of Theorem 2 and the proof of Theorem 1, we now extend the asymptotic stability result by the following.

**Theorem 3 (Asymptotic stability of multi-soliton solution)** Assume that $f$ is $C^3$ and satisfies (1.2). Let $N \geq 1$ and $0 < c_N^0 < \ldots < c_N^1 < c_\ast(f)$. There exist $L_0 > 0$ and $\alpha_0 > 0$ such that if $u(t)$ is a global ($t \geq 0$) $H^1$ solution of (1.1) satisfying (5.4) then the following hold.
1. **Asymptotic stability in the energy space.** There exist \( t \mapsto c_j(t) \in (0, c_*(f)) \), \( t \mapsto \rho_j(t) \in \mathbb{R} \) such that

\[
u(t) - \sum_{j=1}^{N} Q_{c_j(t)}(\cdot - \rho(t)) \to 0 \quad \text{in } H^1(x > \frac{c_j(0)}{10} t) \quad \text{as } t \to +\infty.
\]

(5.5)

2. **Convergence of the scaling parameter.** Assume further that there exists \( \sigma_0 > 0 \) such that \( c \mapsto \int Q_c^\perp \) is not constant in any interval \( I \subset [c_j - \sigma_0, c_j + \sigma_0] \). Then, by possibly taking a smaller \( \alpha_0 > 0 \), there exists \( c_{j,+} \in (0, c_*(f)) \) such that \( c(t) \to c_{j,+} \) as \( t \to +\infty \).

**Sketch of the proof.** The proof of Theorem 3 does not use any new argument with respect to Theorems 1 and 2 and the proof of the main results in 19.

The first observation is that assuming (5.4), we have the analogue of Lemma 2.1: there exist \( \eta(t, x) = u(t, x) - \sum_{j=1}^{N} Q_{c_j(t)}(x - \rho_j(t)) \), (5.6)

satisfies, for all \( t \in [0, T_0] \), for all \( j = 1, \ldots, N \),

\[
\int \tilde{x}_{c_j(t)}(x - \rho_j(t))\eta(t, x)dx = \int Q'_{c_j(t)}(x - \rho_j(t))\eta(t, x)dx = 0,
\]

(5.7)

\[
|c_j(t) - c_j^0| + \|\eta(t)\|_{H^1} \leq K_0 \alpha_0, \quad \rho_j(t) - \rho_{j+1}(t) > \frac{L_0}{2} + \sigma t \quad (\sigma > 0),
\]

(5.8)

\[
|\rho'(t)| + |\rho_j'(t) - c_j| \leq K_0 \left( \int \eta^2(t, x)e^{-|x-\rho_j(t)|}dx \right)^{\frac{1}{2}}.
\]

(5.9)

Now, we prove asymptotic stability by considering various regions related to the position of the solitons.

(a) **Asymptotic stability around the first soliton on the right.**

Here, we follow exactly the proof of Proposition 2. Let \( t_n \to +\infty \), for a subsequence \( t_\phi(n) \), \( u(t_\phi(n)) + \rho_1(t_\phi(n)) \to \tilde{u}_{0,1} \), and \( \tilde{u}_1(t) \) solution of (1.1) corresponding to \( \tilde{u}_1(0) = \tilde{u}_{0,1} \) satisfies (1.2). Indeed, in the proof of Proposition 2 only the behavior of the solution \( u(t) \) at the right of the soliton \( Q_{c_1(t)} \) is concerned, the presence of \( N - 1 \) solitons on the left does not change the argument. Thus, as in the proof of Theorem 1 using Theorem 2 we obtain \( \tilde{u}_1(t) = Q_{c_1,+}(x - c_1,+ t) \), where \( c_1(t_\phi(n)) \to c_{1,+} \). Therefore, for any \( A > 0 \), \( u(t, x + \rho_1(t)) \to Q_{c_{1,+}} \) on \( H^1(x > -A) \). Finally, using only monotonicity arguments, we obtain

\[
u(t) - Q_{c_{1,+}}(\cdot - \rho_1(t)) \to 0 \quad \text{on } H^1(x > \frac{1}{2}(\rho_1(t) + \rho_2(t)))
\]

see 19, Section 4.1 and 15, proof of Theorem 1.

(b) **Asymptotic stability on each solitons by iteration.** We prove the result on the other solitons by iteration on \( j \) from 1 to \( N \) of the following statement:

\[
\exists c_{j,+} \text{ s.t. } u(t) - Q_{c_{j,+}}(\cdot - \rho_j(t)) \to 0 \quad \text{on } H^1(x > \frac{1}{2}(\rho_j(t) + \rho_{j+1}(t))),
\]

(5.10)

(if \( j = N \), the convergence is on \( H^1(x > \frac{1}{10} \rho_N(t)) \)).
Assume that (5.10) holds for \( 1 \leq j_0 < N \). Let us prove it for \( j_0 + 1 \). The only point that differs from the case of \( j = 1 \) is the analogue of Lemma 4.1 to prove strong convergence in \( H^1 \) on the right.

For any \( t_n \to +\infty \), there exists \( c_{j_0+1} \), \( \tilde{u}_{0,j_0+1} \) such that (up to a subsequence still denoted by \( t_n \)):

\[
\begin{align*}
\text{where } \tilde{u}_{0,j_0+1} \text{ has exponential decay on the right}. & \\
\text{Set, for } j = 1, \ldots, j_0, & \\
R_j(t,x) &= R_j^0 C_j^0(t,x) = \sum_{j=1}^{j_0} (x - c_{j_0,0} + t - \rho_j(t_n) + \rho_{j_0+1}(t_n)).
\end{align*}
\]

where \( \tilde{u}_{j_0+1} = \tilde{u} \) is the solution of (1.1) corresponding to \( \tilde{u}_{0,j_0+1} \).

Following Proposition 2, it is enough to prove

\[
\int (v_n^2 + v_{nx}^2)(t,x)\psi(\sqrt{c_{j_0+1}}x)dx \to 0 \quad \text{as } n \to +\infty. \quad (5.11)
\]

Proof of (5.11). Let us just check convergence for \( \int v_n^2(t,x)\psi(\sqrt{c_{j_0+1}}x)dx \), the case of \( v_{nx} \) is treated as in Proposition 2. First, we have

\[
\partial_t v_n = -\partial_x (\partial^2 v_n + f(\tilde{u} + \sum_{j=1}^{j_0} R_j + v_n) - f(\tilde{u}) - \sum_{j=1}^{j_0} f(R_j)), \quad \text{and}
\]

\[
\int v_n^2(0,x)\psi(\sqrt{c_{j_0+1}}x)dx \to 0 \quad \text{as } n \to +\infty.
\]

Computing (energy method) \( \frac{d}{dt} \int v_n^2(t,x)\psi(\sqrt{c_{j_0+1}}x)dx \), as in the proof of Proposition 2 the only term which has to checked is:

\[
\int (f(\tilde{u} + \sum_{j=1}^{j_0} R_j + v_n) - f(\tilde{u}) - \sum_{j=1}^{j_0} f(R_j))v_{nx}\psi(\sqrt{c_{j_0+1}}x) =
\]

\[
\int (f(\tilde{u} + \sum_{j=1}^{j_0} R_j + v_n) - f(\sum_{j=1}^{j_0} R_j + v_n))v_{nx}\psi(\sqrt{c_{j_0+1}}x)
\]

\[
+ \int (f(\sum_{j=1}^{j_0} R_j + v_n) - \sum_{j=1}^{j_0} f(R_j))v_{nx}\psi(\sqrt{c_{j_0+1}}x) = I + II.
\]

\[
|I| \leq C \int |\tilde{u}|(\|v_n\| + \sum_{j=1}^{j_0} |R_j|)\|v_{nx}\|\psi(\sqrt{c_{j_0+1}}x) \leq C \int |\tilde{u}|\|v_n\|\|v_{nx}\|\psi(\sqrt{c_{j_0+1}}x) + Ce^{-\sigma(t_N + t)}.
\]
Define

\[ II = -\sqrt{c_{j0}+1} \int (F(\sum_{j=1}^{j_0} R_j + v_n) - F(\sum_{j=1}^{j_0} R_j) - v_n f(\sum_{j=1}^{j_0} R_j)) \psi'(\sqrt{c_{j0}+1} x) \]

\[ + \int v_n (\sum_{j=1}^{j_0} f(R_j) - f(\sum_{j=1}^{j_0} R_j)) \psi'(\sqrt{c_{j0}+1} x) \]

\[ - \int (R_{jx} f(\sum_{j=1}^{j_0} R_j + v_n) - f(\sum_{j=1}^{j_0} R_j)) - v_n f'(\sum_{j=1}^{j_0} R_j)) \psi'(\sqrt{c_{j0}+1} x) \]

\[ = II_1 + II_2 + II_3. \]

Then \(|II_1| \leq C \int v_n^2 \psi'(\sqrt{c_{j0}+1} x), |II_2| \leq Ce^{-\sigma(t_n+t)}, |II_3| \leq C \int v_n^2 \psi'(\sqrt{c_{j0}+1} x)\) implies

\[ \frac{d}{dt} \int v_n^2 (t, x) \psi'(\sqrt{c_{j0}+1} x) dx \leq C \int v_n^2 (t, x) \psi'(\sqrt{c_{j0}+1} x) dx + Ce^{-\sigma(t_n+t)}, \]

and the conclusion.

A Monotonicity results

Define \( \psi(x) = \frac{2}{\pi} \arctan(\exp(x/4)) \), so that \( \lim_{+\infty} \psi = 1, \lim_{-\infty} \psi = 0 \) and for all \( x \in \mathbb{R}, \psi(-x) = 1 - \psi(x) \). Note also that by direct calculations

\[ \psi'(x) = \frac{1}{4\pi \cosh(x/4)} > 0, \quad \psi'''(x) \leq \frac{1}{16} \psi'(x), \quad (A.1) \]

\[ \exists \delta_1 > 0, \forall x < 0, \psi(x) \geq \delta_1 \exp\left(\frac{x}{4}\right), \quad \psi'(x) \geq \delta_1 \exp\left(\frac{x}{4}\right). \quad (A.2) \]

A.1 Monotonicity arguments on \( u(t) \)

Let \( u(t) \) be a solution of (1.1) satisfying the assumptions of Lemma 2.1 for \( t \in [0, T_0] \). Let \( x_0 > 0 \). We define, for \( 0 \leq t \leq t_0 \leq T_0 \), \( \psi_0(t, x) = \psi(\sqrt{c_0}(x - \rho(t_0) + \frac{\rho}{2}(t_0 - t) - x_0)) \) and

\[ I_{x_0, t_0}(t) = \int u^2(t, x) \psi_0(t, x) dx, \quad J_{x_0, t_0}(t) = \int (\frac{\partial^2}{\partial t^2} - 2F(u + c_0 u^2))(t, x) \psi_0(t, x) dx. \]

Lemma A.1 There exists \( K = K(c_0) > 0 \) such that for \( c_0 \), small enough, for all \( 0 \leq t \leq t_0 \leq T_0 \),

\[ I_{x_0, t_0}(t_0) - I_{x_0, t_0}(t) \leq K \exp\left(-\frac{\sqrt{c_0}}{4} x_0\right), \quad J_{x_0, t_0}(t_0) - J_{x_0, t_0}(t) \leq K \exp\left(-\frac{\sqrt{c_0}}{4} x_0\right). \quad (A.3) \]

Proof of Lemma 4.1 The proof is the same as the one of Lemma 3 in [15] for \( u^p \), we repeat it for a general nonlinearity \( f(u) \). By simple calculations, for \( \phi : \mathbb{R} \to \mathbb{R} \) of class \( C^3 \), we have

\[ \frac{d}{dt} \int u^2 \phi = 2 \int u_t u \phi = -2 \int (u_{xx} + f(u)) u \phi = 2 \int (u_{xx} + f(u))(u_x \phi + u \phi') \]

\[ = \int \left(-3u_x^2 + 2(u f(u) - F(u))\right) \phi' + \int u^2 \phi''' , \]

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\[
\frac{d}{dt} \left( u_x^2 - 2F(u) \right) \phi = 2 \int (u_x u_x - f(u) u_t) \phi = -2 \int u_t (u_x + f(u)) \phi - 2 \int u_x u_x' \\
= - \int (u_{xx} + f(u))^2 \phi' + 2 \int (u_{xx} + f(u)) u_x \phi' \\
= \int \left[ -(u_{xx} + f(u))^2 - 2u_{xx}^2 + 2u_{xx} f'(u) \right] \phi' + \int u_x^2 \phi''.
\]

We obtain from the previous calculations and \((A.1)\), for all \(t \leq t_0\),
\[
\frac{d}{dt} \int u^2 \psi_0 = - \int \left( 3u_x^2 + \frac{c_0}{8} u^2 - 2(u f(u) - F(u)) \right) \psi_0 x + \int u^2 \psi_{0xx} \\
\leq - \int \left( 3u_x^2 + \frac{c_0}{4} u^2 - 2(u f(u) - F(u)) \right) \psi_0 x.
\]

Let \(R_0 > 0\) to be chosen later.

(i) For \(t, x\) such that \(|x - \rho(t)| \geq R_0\), by \((2.1), (2.8)\),
\[
|u(t, x)| \leq Q_{c_0}(x) + \|u(t) - Q_{c_0}\|_{L^\infty} \leq Q_{c_0}(x) + \|u(t) - Q_{c_0}\|_{H^1} \leq K e^{-\frac{x^2}{4} R_0} + c_0.
\]

Therefore, for \(c_0\) small enough and \(R_0\) large enough, we have, for such \(t, x\)
\[
|u(t, x)| \leq K e^{-\frac{x^2}{8} (t_0 - t)} e^{-\frac{x^2}{4} x_0}.
\]

Therefore, since \(||u||_{L^\infty} \leq K\), we obtain
\[
\frac{d}{dt} \int u^2 \psi_0 \leq - \int \left( 3u_x^2 + \frac{c_0}{8} u^2 - 2(u f(u) - F(u)) \right) \psi_0 x - K e^{-\frac{x^2}{8} (t_0 - t)} e^{-\frac{x^2}{4} x_0} \leq -K e^{-\frac{x^2}{8} (t_0 - t)} e^{-\frac{x^2}{4} x_0}. \tag{A.4}
\]

By integration between \(t\) and \(t_0\), we obtain \((A.3)\) for \(I_{t_0, t_0}\).

Similarly, using \((A.1)\), we have
\[
\frac{d}{dt} \int \left( u_x^2 - 2F(u) \right) \psi_0 = \int \left[ -(u_{xx} + f(u))^2 - 2u_{xx}^2 + 2u_{xx} f'(u) \right] \psi_0 x \\
- \frac{c_0}{2} \int (u_x^2 - 2F(u)) \psi_0 x + \int u_x^2 \psi_{0xx} \\
\leq - \int \left[ (u_{xx} + f(u))^2 + 2u_{xx}^2 + \frac{c_0}{4} u_x^2 - 2u_{xx} f'(u) - c_0 F(u) \right] \psi_0 x.
\]

Splitting in two regions \(|x - \rho(t)| \geq R_0, |x - \rho(t)| \leq R_0\) as before, by the same argument, we control the nonlinear terms so that by \((A.4)\)
\[
\frac{d}{dt} \int \left( u_x^2 - 2F(u) + c_0 u^2 \right) \psi_0 \leq - \int \left( 2u_{xx}^2 + \frac{c_0}{8} u_x^2 + \frac{c_0}{16} u^2 \right) \psi_0 x - K e^{-\frac{x^2}{8} (t_0 - t)} e^{-\frac{x^2}{4} x_0}.
\]

Therefore, by integration, we obtain \((A.3)\). Note for future use that for \(0 \leq t \leq t_0 \leq T_0\),
\[
\int u^2(t_0) \psi_0(t_0) + \frac{1}{16} \int_t^{t_0} \int (u_x^2 + c_0 u^2) \psi_0 x dt' \leq \int u^2(t) \psi_0(t) + K \exp \left( -\frac{\sqrt{c_0}}{4} x_0 \right) \tag{A.5}
\]
\[
\int (u_x^2 + c_0 u^2)(t_0) \psi_0(t_0) + \frac{1}{16} \int_t^{t_0} \int (u_{xx}^2 + c_0 u_x^2 + c_0 u^2) \psi_0 x dt' \leq \int (u_x^2(t) + u^2(t)) \psi_0(t) + K \exp \left( -\frac{\sqrt{c_0}}{4} x_0 \right). \tag{A.6}
\]
A.2 Monotonicity arguments on the linearized problem. Proof of (3.7)

Let \( \eta(t) \) be as in Proposition 1. We claim the following preliminary result.

**Claim A.1** There exists \( K > 0 \) such that, for all \( t_0 \in \mathbb{R} \),

\[
\int_{t_0}^{t_0} \int_{-\infty}^{t} \frac{3}{4} c_0^2 \eta_t^2 + \frac{3}{4} c_0^2 \eta_t^2 \exp \left( \frac{c_0}{4} \left( x - \frac{c_0}{2} (t_0 - t) \right) \right) dx dt \leq K.
\]

**Remark.** We obtain a gain of regularity on \( \eta(t) \) using the decay assumption (3.2) and monotonicity arguments.

**Proof of Claim A.1.** Let \( t_0 \in \mathbb{R} \), \( x_0 > 0 \), and \( \tilde{x} = \sqrt{c_0} (x - \frac{c_0}{2} (t_0 - t) - x_0) \). Then, by similar calculations as in Lemma A.1 using in particular (A.1), we have

\[
\frac{d}{dt} \int \eta^2 \psi(\tilde{x}) = -3 \sqrt{c_0} \int \eta^2 \psi'(\tilde{x}) - \frac{3}{4} c_0^3 \int \eta^2 \psi''(\tilde{x})
\]

\[
+ \int \eta^2 \left( \sqrt{c_0} f'(Q_{c_0}) \psi'(\tilde{x}) - f''(Q_{c_0}) Q_{c_0} \psi(\tilde{x}) \right)
\]

\[
\frac{d}{dt} \int \eta^2 \psi(\tilde{x}) \leq -3 \sqrt{c_0} \int \eta^2 \psi'(\tilde{x}) - \frac{1}{8} c_0^3 \int \eta^2 \psi''(\tilde{x}) - 2 \int (f'(Q_{c_0}) \eta)(\eta \psi(\tilde{x})).
\]

By (1.2) and (2.1), we have

\[
|f'(Q_{c_0}(x))| + |f''(Q_{c_0}(x)) Q_{c_0}(x)| \leq K \exp(-\sqrt{c_0}(p-1)|x|).
\]

Using (3.2) and arguing as in [11], proof of Lemma 5, we obtain for \( x_0 > 0 \) (considering three regions),

\[
\frac{d}{dt} \int \eta^2 \psi(\tilde{x}) \leq -3 \sqrt{c_0} \int \eta^2 \psi'(\tilde{x}) - \frac{1}{8} c_0^3 \int \eta^2 \psi''(\tilde{x}) + K \exp \left( -\sqrt{c_0} \left( \frac{c_0}{8} (t_0 - t) - \frac{x_0}{4} \right) \right).
\]

Integrating between \( t < t_0 \) and \( t_0 \), we obtain, for all \( t \)

\[
\int_{t}^{t_0} \eta ^2 \psi(\sqrt{c_0} (x - x_0)) \frac{1}{8} \sqrt{c_0} \int_{t}^{t_0} \left( \eta^2 + c_0 \eta^2 \right)^2 \psi'(\tilde{x}) dt
\]

\[
\leq \int_{t}^{t_0} \eta^2 \psi(\sqrt{c_0} (x - \frac{c_0}{2} (t_0 - t) - x_0)) + K \exp \left( -\sqrt{c_0} \left( \frac{c_0}{4} x_0 \right) \right).
\]

Passing to the limit \( t \to -\infty \), using (3.2) and then (A.2), we find, for all \( t_0 \),

\[
\int_{-\infty}^{t_0} \int \left( \eta^2 + c_0 \eta^2 \right) \psi(\tilde{x}) dt \leq K \exp \left( -\sqrt{c_0} \left( x_0 \right) \right).
\]

\[
\int_{-\infty}^{t_0} \int_{x < x_0 + \frac{c_0}{2} (t_0 - t)} \left( \eta^2 + c_0 \eta^2 \right) \exp \left( \frac{c_0}{4} (x - \frac{c_0}{2} (t_0 - t) \right) dx dt
\]

\[
\leq \frac{1}{\tilde{c}_0} \exp \left( \frac{c_0}{4} x_0 \right) \int_{-\infty}^{t_0} \int \left( \eta^2 + c_0 \eta^2 \right) \psi(\tilde{x}) dx dt \leq K.
\]

Let \( x_0 \to +\infty \), we find, for all \( t_0 \in \mathbb{R} \),

\[
\int_{-\infty}^{t_0} \int \left( \eta^2 + c_0 \eta^2 \right) \exp \left( \frac{c_0}{4} (x - \frac{c_0}{2} (t_0 - t) \right) dx dt \leq K.
\]
Now, we use (A.8). We expand the nonlinear term as follows:

\[
2 \int (f'(Q_{c0})\eta_x(x)\psi(x))_x = 2 \int (f'(Q_{c0})\eta_x + f''(Q_{c0})Q'_{c0}\eta)(\eta_{xx}\psi(x) + \sqrt{c_0}\eta_x\psi'(x))
\]

\[
= \int \eta_x^2(-f''(Q_{c0})Q'_{c0}\psi(x) + \sqrt{c_0}f'(Q_{c0})\psi'(x)) + 2 \int \sqrt{c_0}f''(Q_{c0})Q'_{c0}\psi'(x)\eta_x
\]

\[
+ 2 \int f''(Q_{c0})Q'_{c0}\psi(x)\eta_{xx} = I + II + III.
\]

Note that by (A.9), we have

\[
|f'(Q_{c0}(x))| + |f''(Q_{c0}(x))Q'_{c0}(x)|\psi(x) \leq K \exp(-\sqrt{c_0}(p-1)|x|)\psi(x) \leq K\psi(x). \quad (A.12)
\]

Indeed, for \( \tilde{x} \leq 0 \), we have \( \psi(\tilde{x}) \leq K\psi(\tilde{x}) \) and for \( \tilde{x} > 0 \), we have \( 0 < \tilde{x} \leq x \) and so \( \exp(-\sqrt{c_0}(p-1)|x|) \leq K \exp(-\sqrt{c_0}\tilde{x}) \leq K\psi(\tilde{x}) \).

Thus, \( I + II \leq K \int (\eta_x^2 + \eta^2)\psi'(x) \) and

\[
III \leq \sqrt{c_0} \int \eta_{xx}^2\psi'(x) + K \int \eta^2\psi'(x).
\]

From (A.8), we obtain

\[
\frac{d}{dt} \int \eta_x^2\psi(x) + 2\sqrt{c_0} \int \eta_{xx}\psi'(x) \leq K \int (\eta_x^2 + \eta^2)\psi'(x). \quad (A.13)
\]

From (A.11), there exists a sequence \( t_n \to -\infty \) so that \( \int \eta_x^2(t_n)\psi(x) \to 0 \) as \( n \to -\infty \). Thus, integrating (A.13) between \( t_0 \) and \( t_n \) and passing to the limit as \( n \to +\infty \), using (A.10), we obtain, arguing as before, for all \( t_0 \in \mathbb{R} \),

\[
\int_{-\infty}^{t_0} \int \eta_{xx}^2(x)\psi'(x)dx dt \leq K \exp\left(-\frac{\sqrt{c_0}}{4}x_0\right), \quad (A.14)
\]

\[
\int_{-\infty}^{t_0} \int \eta_{xx}^2(x) \exp\left(\pm\frac{\sqrt{c_0}}{4}(x - \frac{c_0}{2}(t_0 - t))\right) dx dt \leq K. \quad (A.15)
\]

and Claim (A.1) is proved.

Assuming that \( f \) is \( C^3 \), one can apply monotonicity arguments again on \( \eta_{xx}(t) \) and conclude the result for \( \psi \). If \( f \) is only \( C^2 \), we use the equation of \( \psi \). Recall that we argue by density again.

*Proof of (3.7).* Setting \( \tilde{\psi}(t) = \psi(t) - \alpha(t)Q_{c0} \), we see that \( \tilde{\psi}(t) \) satisfies

\[
\tilde{\psi}_t = \mathcal{L}_{c_0}\tilde{\psi}_x.
\]

First, by the definition of \( \tilde{\psi}(t) \) and Claim (A.1) (see (A.11), (A.14)), we have

\[
\int_{-\infty}^{t_0} \int \tilde{\psi}_x^2(t^n)\psi'(x)dx dt \leq K \exp\left(-\frac{\sqrt{c_0}}{4}x_0\right), \quad (A.16)
\]

\[
\int \tilde{\psi}_x^2(t^n)\psi(x)dx \to 0 \quad (A.17)
\]
for a sequence \( t_n \to -\infty \), where \( \tilde{x} \) is defined in Claim \( \text{A.1} \). By the equation of \( \tilde{v} \), we have as in the proof of Claim \( \text{A.1} \)

\[
\frac{d}{dt} \int \tilde{v}^2 \psi(\tilde{x}) \leq -\frac{1}{2} \sqrt{c_0} \int (\tilde{v}_x^2 + c_0 \tilde{v}^2) \psi(\tilde{x}) - \int \tilde{v}^2 (\sqrt{c_0} f(Q_{c_0}) \psi'(\tilde{x})) - \int \tilde{v}^2 f'(Q_{c_0}) Q_{c_0}' \psi(\tilde{x})
\]

Integrating on \((-\infty, t_0]\) and combining these estimates with \((\text{A.16}), (\text{A.17})\), arguing as in the proof of Claim \( \text{A.1} \) we obtain for all \( t \in \mathbb{R} \),

\[
\int \left( v_x^2 + c_0 v^2 \right)(t) \exp \left( \frac{\sqrt{c_0}}{4} x \right) dx \leq K.
\]

Using the transformation \( x \to -x \), \( t \to -t \) (the equation of \( v \) and the assumptions are invariant by this transformation), \( (3.7) \) is proved.

### A.3 Proof of \((3.24)\).

We are in the context of Theorem 2. In particular, we assume \((1.6)\) and \((1.7)\) on the solution \( u(t) \). Using the same arguments as in the proof of Claim \( \text{A.1} \) we first claim the following preliminary result of \( u(t) \).

**Claim A.2** There exists \( K > 0 \) such that

\[
\forall t \in \mathbb{R}, \quad \int (u_{xx}^2 + c_0 u_x^2 + c_0^2 u^2)(t) \exp \left( \frac{\sqrt{c_0}}{4} |x - \rho(t)| \right) dx \leq K. \quad (\text{A.18})
\]

**Proof of Claim A.2** From \((1.7)\), letting \( t \to -\infty \) in \((\text{A.5})\), we have

\[
\int_{-\infty}^{t_0} \int (u_x^2 + c_0 u^2) \psi_0 dx dt \leq K \exp \left( -\frac{\sqrt{c_0}}{4} x_0 \right).
\]

By \((\text{A.2})\), and then letting \( x_0 \to +\infty \),

\[
\int_{-\infty}^{t_0} \int_{\rho(t_0) - \frac{c_0}{2}(t_0-t)} \left( u_x^2 + c_0 u^2 \right)(t) \exp \left( \frac{\sqrt{c_0}}{4} (x - \rho(t_0) + \frac{c_0}{2}(t_0-t)) \right) dx dt \\
\leq \frac{1}{\delta_1} \exp \left( \frac{\sqrt{c_0}}{4} x_0 \right) \int_{-\infty}^{t} (u_x^2 + c_0 u^2) \psi_0 dx dt \leq K,
\]

\[
\int_{-\infty}^{t_0} \int \left( u_x^2 + c_0 u^2 \right)(t) \exp \left( \frac{\sqrt{c_0}}{4} (x - \rho(t_0) + \frac{c_0}{2}(t_0-t)) \right) dt \leq K. \quad (\text{A.19})
\]

From \((\text{A.19})\), there exists a sequence \( t_n \to -\infty \) such that \( \int (u_x^2(t_n) + u^2(t_n)) \psi_0(t_n) \to 0 \) as \( n \to +\infty \). Thus, from \((\text{A.6})-\text{A.7})\) applied to \( t = t_n \), and passing to the limit as \( n \to +\infty \), we obtain

\[
\int_{-\infty}^{t_0} \int (u_{xx}^2 + c_0 u_x^2 + c_0^2 u^2) \psi_0 dx dt \leq K \exp \left( -\frac{\sqrt{c_0}}{4} x_0 \right).
\]

Arguing as before with \((\text{A.2})\), we get, for all \( t_0 \),

\[
\int_{-\infty}^{t_0} \int \left( u_{xx}^2 + c_0 u_x^2 + c_0^2 u^2 \right)(t) \exp \left( \frac{\sqrt{c_0}}{4} (x - \rho(t_0) + \frac{c_0}{2}(t_0-t)) \right) dt \leq K. \quad (\text{A.20})
\]
Now, we use a monotonicity argument on $u_{xx}(t)$ to be able to give information on $v(t)$. By similar calculations as in the proof of Lemma A.1, we have
\[
\frac{d}{dt} \int u_{xx}^2 \psi_0 \leq - \int (3u_{xxx}^2 + c_0^2 u_{xx}^2) \psi_0 + \int u_{xx}^2 (f'(u)\psi_0 - f''(u)u_x \psi_0) + 2 \int u_{xx}^2 f''(u) \psi_0 + 2 \int u_{xxx}^2 f''(u) \psi_0.
\]
We control the nonlinear terms as before, and then using (A.12),
\[
\int u_{xx}^2 (f'(u)\psi_0 - f''(u)u_x \psi_0) + 2 \int u_{xx}^2 f''(u) \psi_0 \leq K \int (u_{xx}^2 + u_{xx}^2) \psi_0,
\]
\[
\int u_{xxx}^2 f''(u) \psi_0 \leq \int u_{xxx}^2 \phi_0 + K \int u_{xx}^2 \psi_0.
\]
Arguing as before, we obtain the following conclusion, for all $t_0 \in \mathbb{R}$,
\[
\int (u_{xx}^2 + c_0 u_{xx}^2 + c_0^2 u_{xx}^2) (t_0) \exp \left( \frac{\sqrt{c_0}}{4} (x - \rho(t_0)) \right) + \int_{-\infty}^{t_0} \int (u_{xxx}^2 + c_0 u_{xx}^2 + c_0^2 u_{xx}^2 + c_0^3 u_{xx}^2) (t) \exp \left( \frac{\sqrt{c_0}}{4} (x - \rho(t_0) + \frac{c_0}{2} (t_0 - t)) \right) dt \leq K.
\]
(A.21)
Since equation (1.1) is invariant by the transformation $x \to -x$, $t \to -t$, the claim is proved.

Proof of (3.24). Estimate (A.18) and the decay on $Q_c(t)$ (see (2.1)) imply:
\[
\int v^2(t, x) \exp \left( \frac{\sqrt{c_0}}{4} |x| \right) dx \leq K.
\]
From this estimate, using the equation of $v$ and (A.21), and arguing exactly as for the linear case (proof of (3.7)), we obtain (3.24).

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