Direct mode summation for the Casimir energy of a solid ball

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The Casimir energy of a solid ball placed in an infinite medium is calculated by a direct frequency summation using the contour integration. It is assumed that the permittivity and permeability of the ball and medium satisfy the condition \( \varepsilon_1 \mu_1 = \varepsilon_2 \mu_2 \). Upon deriving the general expression for the Casimir energy, a dilute compact ball is considered \((\varepsilon_1 - \varepsilon_2)^2/\varepsilon_1^2 \ll 1\). In this case the calculations are carried out which are of the first order in \( \xi^2 \) and take account of the five terms in the Debye expansion of the Bessel functions involved. The implication of the obtained results to the attempts of explaining the sonoluminescence via the Casimir effect is shortly discussed.

12.20.-m, 12.20.Ds, 78.60.Mq

I. INTRODUCTION

The Casimir energy, determined by the first quantum correction to the ground state of a quantum field system with allowance for nontrivial boundary conditions, proves to be essential in many problems of the elementary particle theory, in quantum cosmology, and in physics of condensed matter. However, up to now there is no universal method for calculating the Casimir effect for arbitrary boundary conditions. This has been done only for simple field configurations of high symmetry: gap between two plates, sphere, cylinder, wedge and so on. The curvature of the boundary and account of the dielectric and magnetic properties of the medium lead to considerable complications. While the attractive force between two uncharged metal plates has been calculated by Casimir as far back as 1948 [1], this effect for perfectly conducting spherical shell in vacuum was computed by Boyer only in 1968 [2] (see also the latter calculations [3–6]). If an infinitely thin spherical shell separates media with arbitrary dielectric \((\varepsilon_1, \varepsilon_2)\) and magnetic \((\mu_1, \mu_2)\) characteristics, this problem is not solved till now [8]. The main drawback here is the lack of a consistent method for removing the divergences. Besides an attempt to revive the quasiclassical model of an extended electron proposed by Casimir [11], interest in this problem was also initiated by investigations of the bag models in hadron physics [12, 13] and recently by search for the mechanism of sonoluminescence [15].

In this paper we calculate the Casimir energy of a solid ball by making use of the direct summation of eigenfrequencies of vacuum electromagnetic field by contour integration [16, 17]. A definite advantage of this method, compared with the Green’s function technique employed in [7–9, 18], is its simplicity and visualization. We consider a compact ball placed in an infinite medium when \( \varepsilon_1 \mu_1 = \varepsilon_2 \mu_2 \). This condition enables one to treat the divergencies analogously to the case of a perfectly conducting spherical shell [6]. Upon deriving the general expression for the Casimir energy, we address ourselves to the case of a dilute ball \( \xi^2 \ll 1 \), \( \xi = (\varepsilon_1 - \varepsilon_2)/(\varepsilon_1 + \varepsilon_2) \). The calculations here are of the first order in \( \xi^2 \) and take account of the five terms in the Debye expansion of the Bessel functions involved. In this way we attain some generalization and refinement of the results obtained in this problem earlier [18].

The layout of the paper is as follows. In Sect. II we derive a general expression for the Casimir energy of a solid ball in an infinite surrounding under the condition \( \varepsilon_1 \mu_1 = \varepsilon_2 \mu_2 = c^{-2} \), where \( c \) is an arbitrary constant not necessary equal to one (it is the light velocity in the medium), the mode-by-mode summation of eigenfrequencies being used. In Sect. III the Casimir energy of a dilute ball \( \xi^2 \ll 1 \) is calculated. The implication of the obtained result to the Schwinger attempt to explain the sonoluminescence via the Casimir effect is also considered. In Conclusion (Sect. IV) the results of the paper are briefly discussed. Dispersive effects are ignored in our paper.

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II. CASIMIR ENERGY OF A SOLID BALL UNDER THE CONDITION $\varepsilon_1\mu_1 = \varepsilon_2\mu_2$

Let us consider the Casimir of a solid ball of radius $a$, consisting of a material which is characterized by permittivity $\varepsilon_1$ and permeability $\mu_1$. We assume that the ball is placed in an infinite medium with permittivity $\varepsilon_2$ and permeability $\mu_2$. We also suppose that the conductivity of the ball material and its surroundings is equal to zero.

In our consideration the main part will be played by equations determining the eigenfrequencies $\omega$ of the electromagnetic oscillations for this configuration \[19\]. It is convenient to rewrite these equations in terms of the Riccati-Bessel functions

$$\tilde{s}_i(x) = xj_i(x), \quad \tilde{e}_i(x) = xh_i^{(1)}(x), \quad (2.1)$$

where $j_i(x) = \sqrt{\pi/2x}J_{i+1/2}(x)$ is the spherical Bessel function and $h_i^{(1)}(x) = \sqrt{\pi/2x}H_{i+1/2}^{(1)}(x)$ is the spherical Hankel function of the first kind. For the TE-modes the frequency equation reads

$$\Delta_{TE}^i(\omega) \equiv \sqrt{\varepsilon_1 \mu_2} \tilde{s}_i(k_1 a)\tilde{e}_i(k_2 a) - \sqrt{\varepsilon_2 \mu_1} \tilde{s}_i(k_1 a)\tilde{e}_i'(k_2 a) = 0, \quad (2.2)$$

where $k_i = \sqrt{\varepsilon_i \mu_i} \omega_i$, $i = 1, 2$ are the wave numbers inside and outside the ball, respectively; prime stands for the differentiation with respect to the argument ($k_1 a$ or $k_2 a$) of the corresponding Riccati-Bessel function. The frequencies of the TM-modes are determined by

$$\Delta_{TM}^i(\omega) \equiv \sqrt{\varepsilon_2 \mu_2} \tilde{s}_i'(k_1 a)\tilde{e}_i(k_2 a) - \sqrt{\varepsilon_1 \mu_1} \tilde{s}_i'(k_1 a)\tilde{e}_i'(k_2 a) = 0. \quad (2.3)$$

The orbital quantum number $l$ in (2.2) and (2.3) assumes the values 1, 2, \ldots. Under mutual change $\varepsilon_i \leftrightarrow \mu_i$, $i = 1, 2$ frequency equations (2.2) and (2.3) transform into each other.

It is worth noting that the frequencies of the electromagnetic oscillations determined by Eq. (2.2) and (2.3) are the same inside and outside the ball. The physical reason for this is that photons do not perform work when passing through the boundary at $r = a$. This is in contrast to the case of perfectly conducting spherical shell in vacuum \[3\], where eigenfrequencies inside the shell and outside it are determined by different equations \[19\].

As usual we define the Casimir energy by the formula

$$E = \frac{1}{2} \sum_s (\omega_s - \bar{\omega}_s), \quad (2.4)$$

where $\omega_s$ are the roots of Eqs. (2.2) and (2.3) and $\bar{\omega}_s$ are the same roots under condition $a \to \infty$. Here $s$ is a collective index that stands for a complete set of indices for the roots of Eqs. (2.2) and (2.3). Denoting the roots of Eqs. (2.2) and (2.3) by $\omega_{nl}^{(1)}$ and $\omega_{nl}^{(2)}$, respectively, we can cast Eq. (2.4) in the explicit form

$$E = \frac{1}{2} \sum_{\alpha=1}^{2} \sum_{l=1}^{\infty} \sum_{m=-l}^{l} \sum_{n=1}^{\infty} (\omega_{nl}^{(\alpha)} - \bar{\omega}_{nl}^{(\alpha)}) = \sum_{l=1}^{\infty} E_l, \quad (2.5)$$

where the notation

$$E_l = (l + 1/2) \sum_{\alpha=1}^{2} \sum_{n=1}^{\infty} (\omega_{nl}^{(\alpha)} - \bar{\omega}_{nl}^{(\alpha)}) \quad (2.6)$$

is introduced. Here we have taken into account that the eigenfrequencies $\omega_{nl}^{(\alpha)}$ do not depend on the azimuthal quantum number $m$. For partial energies $E_l$ we use representation in terms of the contour integral provided by the Cauchy theorem \[20\]

$$E_l = \frac{l + 1/2}{2\pi i} \oint_C dz z \frac{d}{dz} \ln \frac{\Delta_{TE}^l(az)\Delta_{TM}^l(az)}{\Delta_{TE}^l(\infty)\Delta_{TM}^l(\infty)}, \quad (2.7)$$

where the contour $C$ surrounds, counterclockwise, the roots of the frequency equations in the right half-plane. Location of the roots of Eqs. (2.2) and (2.3) enables one to deform the contour $C$ into a segment of the imaginary axis ($-i\Lambda, i\Lambda$) and a semicircle of radius $\Lambda$ in right half-plane. At a given value of $\Lambda$ a finite number of the roots of frequency equations is taken into account. Thus $\Lambda$ plays the role of a regularization parameter for the initial sum in Eq. (2.6) which should be subsequently removed to infinity. In this limit the contribution of the semicircle of radius $\Lambda$ into integral (2.7)
vanishes. From physical considerations it is clear that multiplier $z$ in (2.7) is understood to be the \( \lim_{\mu \to a} \sqrt{\varepsilon^2 + \mu^2} \), where $\mu$ is the photon mass. Therefore in the integral along the segment \((-i\Lambda, i\Lambda)\) we can integrate once by parts, the nonintegral terms being canceled. As a result Eq. (2.7) acquires the form

$$E_l = \frac{l + 1/2}{\pi a} \int_0^\infty dy \ln \frac{\Delta_{TE}^l(iy)\Delta_{TM}^l(iy)}{\Delta_{TE}^l(i\infty)\Delta_{TM}^l(i\infty)}.$$  

(2.8)

Now we need the modified Riccati-Bessel functions

$$s_l(x) = \sqrt{\frac{x}{2}} I_\nu(x), \quad e_l(x) = \sqrt{\frac{2x}{\pi}} K_\nu(x), \quad \nu = l + 1/2,$$

(2.9)

where $I_\nu(x)$ and $K_\nu(x)$ are the modified Bessel functions [21]. With allowance for the asymptotics of $s_l(x)$ and $e_l(x)$ at $x \to \infty$ and fixed $l$

$$s_l(x) \simeq \frac{1}{2} e^{x}, \quad e_l(x) \simeq e^{-x}$$

(2.10, 2.11)

equation (2.8) can be rewritten as

$$E_l = \frac{l + 1/2}{\pi a} \int_0^\infty dy \ln \left\{ \frac{4 e^{-2(\varepsilon_1 - \varepsilon_2)}}{(\sqrt{\varepsilon_1 \mu_2^2} + \sqrt{\varepsilon_2 \mu_1^2})^2} \times \left[ \sqrt{\varepsilon_1 \varepsilon_2 \mu_1 \mu_2} \left( (s_l(q_1)e_l(q_2))^2 + (s_l(q_1)e'_l(q_2))^2 \right) - (\varepsilon_1 \mu_2 + \varepsilon_2 \mu_1) s_l(q_1)s'_l(q_1)e_l(q_2)e'_l(q_2) \right] \right\},$$

(2.12)

where $q_i = \sqrt{\varepsilon_i \mu_i} y$, $i = 1, 2$. We shall use this general equation in the next Section but here we address ourselves to the special case when the condition

$$\varepsilon_1 \mu_1 = \varepsilon_2 \mu_2 = c^{-2}$$

(2.13)

is fulfilled. Here $c$ is an arbitrary positive constant (the light velocity in medium). Physical implications of this condition at $c = 1$ can be found in [22]. Now Eq. (2.13) is simplified considerably

$$E_l = \frac{c(l + 1/2)}{\pi a} \int_0^\infty dy \ln \left\{ \frac{4}{\varepsilon + \varepsilon^{-1} + 2} \left[ (s_l(y)e_l(y))^2 + (s_l(y)e'_l(y))^2 \right] - (\varepsilon + \varepsilon^{-1}) s_l(y)s'_l(y)e_l(y)e'_l(y) \right\},$$

(2.14)

where $\varepsilon = \varepsilon_1 / \varepsilon_2$. The argument of the logarithm in (2.14) can be transformed, if the following two equalities for the functions $s_l(y)$ and $e_l(y)$

$$s'_l(y)e_l(y) - s_l(y)e'_l(y) = 1, \quad (2.15)$$

$$s'_l(y)e_l(y) + s_l(y)e'_l(y) = (s_l(y)e_l(y))'. \quad (2.16)$$

are taken into account. It gives

$$E_l = \frac{c(l + 1/2)}{\pi a} \int_0^\infty dy \ln \left\{ 1 - \xi^2 \left[ (s_l(y)e_l(y))' \right]^2 \right\},$$

(2.17)

where

$$\xi = \frac{\mu_2 - \mu_1}{\mu_2 + \mu_1} = \frac{\varepsilon_1 - \varepsilon_2}{\varepsilon_1 + \varepsilon_2}, \quad \varepsilon_i \mu_i = c^{-2}, \quad i = 1, 2.$$
Thus, for a ball with a vacuum on the outside, \( \xi = (1 - \mu)/(1 + \mu) = (\varepsilon - 1)/(\varepsilon + 1) \) and \( c = 1 \). The expression (2.17) agrees with the results obtained in [23], if one performs a partial integration of the expression for \( E \) given in these references and puts the cutoff parameter \( \delta \) equal to zero. If \( \varepsilon = 0 \) or \( \infty \) and \( c = 1 \) then, as one could expect, Eq. (2.17) turns into the analogous expression for the perfectly conducting spherical shell in vacuum [3,6].

We remark that in a previous paper [18, (1982)] an expression for the Casimir energy was calculated that seemingly is in conflict with Eq. (2.17). Namely, Eq. (2.42) in that paper corresponds to the following expression for these references and puts the cutoff parameter \( \delta \) equal to zero. If \( \varepsilon = 0 \) or \( \infty \) and \( c = 1 \) then, as one could expect, Eq. (2.17) turns into the analogous expression for the perfectly conducting spherical shell in vacuum [3,6].

It turns out, however, that these two equations (2.17) and (2.18) are in agreement. To show the equivalence is not quite trivial, but follows after some algebra taking into account the derivatives of the logarithms and the Wronskian (2.15). We note that in a previous paper [18, (1982)] an expression for the Casimir energy was calculated that seemingly is in conflict with Eq. (2.17). Namely, Eq. (2.42) in that paper corresponds to the following expression for these references and puts the cutoff parameter \( \delta \) equal to zero. If \( \varepsilon = 0 \) or \( \infty \) and \( c = 1 \) then, as one could expect, Eq. (2.17) turns into the analogous expression for the perfectly conducting spherical shell in vacuum [3,6].

We now invoke the following useful expansion, which was worked out by one of us some time ago [24].

We now address ourselves to the consideration of the Casimir energy of a dilute compact ball when

\[ |\xi| \ll 1. \] (3.1)

That means, to the lowest order in \( \xi \),

\[ \ln(1 - \xi^2 \lambda_t^2) \simeq -\xi^2 \lambda_t^2, \] (3.2)

which reflects a general property of all Casimir calculations in dilute media: the lowest order correction for all physical quantities is proportional to the square of the susceptibility (electric or magnetic). We shall henceforth work only to the second order in \( \xi \). From (2.17) we get for the Casimir energy

\[ E = -\frac{\xi^2}{\pi a} \sum_{\ell=1}^\infty \nu \int_0^\infty dx \lambda_t^2(x). \] (3.3)

For simplicity we have put here the constant \( c \) in condition (2.13) to be equal 1.

We now invoke the following useful expansion, which was worked out by one of us some time ago [24].

\[ \lambda_t = \frac{t^3}{2 \nu} \left[ 1 - \frac{1}{8 \nu^2} \left( 2 - 27 t^2 + 60 t^4 - 35 t^6 \right) \right. \]
\[ \left. - \frac{1}{128 \nu^4} \left( 108 t^2 - 3615 t^4 + 21420 t^6 - 47250 t^8 + 44352 t^{10} - 15015 t^{12} \right) + O(1/\nu^6) \right]. \] (3.4)

Here, \( t(z) = (1 + z^2)^{-1/2} \), \( z = x/\nu \). This expression is based upon the Debye expansions for the modified Bessel functions [21]. From (3.4) we calculate

\[ \lambda_t^2 = \frac{t^6}{4 \nu^2} \left[ 1 - \frac{1}{4 \nu^2} \left( 2 - 27 t^2 + 60 t^4 - 35 t^6 \right) \right. \]
\[ \left. + \frac{1}{16 \nu^4} \left( 1 - 54 t^2 + 1146 t^4 - 6200 t^6 + 13185 t^8 - 12138 t^{10} + 4060 t^{12} \right) + O(1/\nu^6) \right]. \] (3.5)
which can now be inserted into Eq. (3.3). From the integral representation of the beta function \[ B(q, p) = \frac{\Gamma(q)\Gamma(p)}{\Gamma(q+p)} \]
we derive the formula
\[ \int_0^\infty t^p(z)\,dz = \sqrt{\pi} \frac{2}{\Gamma \left( \frac{p - 1}{2} \right)} , \]
which is quite useful in the present context. After some calculation we obtain
\[ E = -\frac{3\xi^2}{64a} \left[ \sum_{l=1}^\infty \nu^0 - \frac{9}{128} \sum_{l=1}^\infty \frac{1}{\nu^2} + \frac{423}{16384} \sum_{l=1}^\infty \frac{1}{\nu^4} + O \left( \frac{1}{\nu^6} \right) \right] . \]
Here, the first sum will be regularized by means of the Riemann zeta function \[ (3.8) \]
from which \[ \sum_{l=1}^\infty \nu^0 = -1 \]. Moreover, since the two last sums in \[ (3.7) \] are known,
\[ \sum_{l=1}^\infty \frac{1}{\nu^2} = \frac{1}{2} \pi^2 - 4, \quad \sum_{l=1}^\infty \frac{1}{\nu^4} = \frac{1}{6} \pi^4 - 16 , \]
we get, omitting the remainder in \[ (3.7) \],
\[ E = \frac{3\xi^2}{64a} \left[ 1 + \frac{9}{128} \left( \frac{1}{2} \pi^2 - 4 \right) - \frac{423}{16384} \left( \frac{1}{6} \pi^4 - 16 \right) \right] . \]
The energy is positive, corresponding to a repulsive surface force. Remember, though, that we are working here with the nondispersive theory only.

The structure of the three different terms in \[ (3.10) \] is the following. The first term stems from the order \( 1/\nu \) in the Debye expansion. This term agrees with the results obtained by Milton \[ (7) \] and Milton and Ng \[ (8) \] in the case when the condition \[ (2.13) \] holds. Numerically, the three terms between square brackets in \[ (3.10) \] are \[ \left[ 1 + 0.06573 - 0.00610 \right] \]. Thus the second term, stemming from the order \( 1/\nu^3 \) in the Debye expansion, describes a repulsive correction of about 6.6 per cent. Finally the third term, stemming from the order \( 1/\nu^5 \) in the Debye expansion, describes a 0.6 per cent attractive correction. We have thus improved the calculations in \[ (6,8) \] by four orders in magnitude. The next correction, not included here, is of order \( 1/\nu^7 \) in the Debye expansion.

Strictly speaking our consideration is applicable only when the condition \( \varepsilon_i \mu_i = 1 \) is satisfied. However keeping in mind a smooth dependence of the Casimir energy on the parameters specifying the field configuration, one can expect that the final formula can also be used in the case of a nonmagnetic dilute dielectric ball placed in vacuum (or a spherical cavity in an infinite dielectric surrounding) at least for estimation only. Therefore we present here some numerical estimations relating to sonoluminescence. With a good accuracy one can put in dimensional units
\[ E \simeq 3\hbar c \xi^2 / (64a) \]
Take, as an example, \( |\xi| = 0.1 \), \( a = 4 \cdot 10^{-4} \) cm. Then \( E \simeq 2 \cdot 10^{-5} \) eV. This is immensely smaller than the amount of energy \( (\sim 10 \text{ MeV}) \) emitted in a sonoluminescent flash. Furthermore, the Casimir energy \[ (3.11) \], being positive, increases when the radius of the ball decreases. The latter eliminates completely the possibility of explaining, via the Casimir effect, the sonoluminescence for bubbles in a liquid. As known \[ (20) \] emission of light takes place at the end of the bubble collapse. Recently an important experimental studies have been done to measure the duration of the sonoluminescence flash \[ (27) \]. In view of all this it is difficult to imagine that the Casimir effect, at least in its nondispersive version, should be important for the sonoluminescence phenomenon.

### IV. CONCLUSION

Our method for calculating the Casimir energy \( E \) by means of the contour integral \[ (2.7) \] proves to be very convenient and effective. As known, there are in principle at least two different methods for calculating \( E \): one can follow a
local approach, implying use of the Green’s function to find the energy density (or the surface force density). Or, one can sum the eigenfrequencies directly. Equation (2.7) thus means that we have adopted the latter method here. The Cauchy integral formula turns out to be most useful in other context also, such as in the calculation of the Casimir energy for a piece wise uniform relativistic string [28]. A survey on this subject can be found in [29]. The great advantage of the method is that the multiplicity of zeros in the dispersion function is automatically taken care of, i.e., one does not have to plug in the degeneracy in the formalism by hand.

A remarkable feature of our approach is also the ultimate formula for the Casimir energy having the form of the spectral representation, i.e., of an integral with respect to frequency between the limits \(0, \infty\) of a smooth function, spectral density. Evidently, for physical applications one needs to know the frequency range which gives the main contribution into the spectral density. An example of this representation for the partial energies \(E_l\) is Eq. (2.17), where the substitution \(y = \omega a\) should be done. As shown above, the partial energies \(E_l\) decrease rapidly as \(l\) increases. Therefore the most interesting is a few first values of \(l\). In this case, as one could expect, the spectral density is different from zero when \(\omega a \approx 1\). Keeping in mind the search for the origin of the sonoluminescence we put \(a = 4 \cdot 10^{-4}\) cm. Then the wavelength of the photon in question turns out to be \(2.5 \cdot 10^{-4}\) cm, i.e., this radiation belongs to infrared region, while in experiments on sonoluminescence the blue light is observed [26]. This fact also testifies against the possibility of explaining the sonoluminescence by the Casimir effect.

It is worth noting that the spectral distribution of the Casimir energy is practically not discussed in literature while the space density of this energy has been investigated in detail (see, for example [18] (1983)). From the physical point of view the space density and spectral density of energy in this problem should be treated on the same footing. One can remind here the treatment of the Casimir effect as a manifestation of the fluctuations of the vacuum fields [30], these fluctuations being occurred in space and time simultaneously.

It should be emphasized that in this paper we have neglected the dispersion effects when calculating the Casimir energy. Importance of this point has been demonstrated in [31]. As for the elucidation of the sonoluminescence origin, we have to stress once more that in our consideration we have contented ourselves with the static Casimir effect only.

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