Research Article
Fractional Entropy-Based Test of Uniformity with Power Comparisons

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In the present paper, we use the fractional and weighted cumulative residual entropy measures to test the uniformity. The limit distribution and an approximation of the distribution of the test statistic based on the fractional cumulative residual entropy are derived. Moreover, for this test statistic, percentage points and power against seven alternatives are reported. Finally, a simulation study is carried out to compare the power of the proposed tests and other tests of uniformity.

1. Introduction

Rao et al. [1] suggested a nonnegative measure of uncertainty and called it the cumulative residual entropy (CRE). For any nonnegative continuous random variable (RV) X with a cumulative distribution function (CDF) \( F(x) = P(X < x) \), the CRE is defined by

\[
\text{CRE}(F) = -\int_0^\infty F(x) \ln(F(x)) \, dx,
\]

(1)

where \( F(x) = 1 - F(x) \) is the reliability function. Rao et al. [1] revealed many salient features of the CRE. For example, the CRE possesses more general mathematical properties than the Shannon entropy, and it can be easily computed from sample data, and these computations asymptotically converge to the true values. Moreover, the CRE deals with the quantity of information in residual life. For the standard uniform distribution, denoted by \( U(0,1) \), Rao et al. [1] showed that the value of the CRE is 1/4. The literature abounds with many different results for Shannon’s entropy and its modifications. Interested readers may refer to [1–17].

Xiong et al. [16] suggested the fractional cumulative residual entropy (FCRE) to extend the CRE to the case of fractional order. For any \( 0 \leq q \leq 1 \), the FCRE for the RV X is defined by

\[
\text{CRE}^q(F) = \int_0^\infty F(x)[-\ln(F(x))]^q \, dx.
\]

(2)

The measure \( \text{CRE}^q(F) \) is a nonadditive and nonnegative. Moreover, it is a convex function of the parameter \( q \), \( \text{CRE}^q(F) = E(X) \), and \( \text{CRE}^1(F) = \text{CRE}(F) \). Xiong et al. [16] derived the FCRE for some well-known distributions; for example, FCRE of the CDF \( U(0,1) \) is \( \Gamma(q+1)/2^q \).

Misagh et al. [15] proposed a weighted form of CRE, which is shift-dependent. This information-theoretic uncertainty measure is called the weighted cumulative residual entropy (WCRE), and it is defined by
Later, Mirali et al. [12] and Mirali and Baratpour [13] studied some of its properties. Moreover, he reported the distribution function of the WCRE and other tests. In [14], the power comparison is performed between the FCRE and CRE by using the Monte Carlo simulation. Moreover, under the null hypothesis \( H_0 \), we have \( F_{CRE} \sim \Gamma(q + 1)/2\), where \( q \) is fixed. Moreover, under the alternative hypothesis that \( F \) is any continuous CDF with support \([0, 1]\), which is not the uniform, we have \( CRE(F_n) \sim \Gamma(q + 1)/2\), where \( r \) is a smaller or larger number than \( \Gamma(q + 1)/2 \).

\section{2. Theoretical Aspects and Test Statistic}

To establish our test of the null hypothesis \( H_0 \), we need the following theorem, which shows that, for a CDF with support \([0, 1]\), one always has \( 0 \leq CRE(F) \leq e^{-q} \), and for the distribution \( U(0, 1) \), we have \( FCRE = \Gamma(q + 1)/2\), and this value is uniquely attained by the uniform distribution, whenever \( q \) is fixed.

**Theorem 1.** Let \( X \) be a nonnegative RV with an absolutely continuous CDF \( F \) with a support \([0, 1]\). From (2), it holds \( 0 \leq CRE(F) \leq e^{-q} \), and \( CRE(F) = \Gamma(q + 1)/2 \) is uniquely acquired by the distribution \( U(0, 1) \).

Proof. Since \( 0 \leq F(x) - \ln(F(x)) \leq 1 \), and the function \( f(x) = x \ln(x) \) has a maximum at \( x = e^{-q} \), we have \( 0 \leq CRE(F) \leq e^{-q} \). On the other hand, using the strict convexity of \( f(x) = x \ln(x) \), it is easy to see that FCRED is a concave function of distribution (with support \([0, 1]\)). This shows that \( CRE(F) = \Gamma(q + 1)/2 \) is uniquely acquired by the distribution \( U(0, 1) \). This completes the proof.

Let \( X_1, X_2, \ldots, X_n \) be a random sample with a continuous CDF \( F \), with support \([0, 1]\). Moreover, under the null hypothesis \( H_0 \), we have \( F_{CRE} \sim \Gamma(q + 1)/2\). On the other hand, under the alternative hypothesis (that \( F \) is any continuous CDF with support \([0, 1]\)), which is not the uniform, we have \( CRE(F_n) \sim \Gamma(q + 1)/2\), where \( r \) is a smaller or larger number than \( \Gamma(q + 1)/2 \).

\section{Theorem 2. The test based on the sample estimate \( R_{n}^{q} \) is consistent.}

Proof. From Glivenko–Cantelli theorem (see Tucker [19]), we have \( \sup_{\theta} |F_n(t) - F(t)| \xrightarrow{p} 0 \). On the other hand, Theorem 3 in Xiong et al. [16] asserts that \( CRE(F_n) \xrightarrow{p} CRE(F) \), which proves the theorem.

**Theorem 3.** Suppose that the random sample \( X_1, X_2, \ldots, X_n \) has been drawn from an unknown continuous CDF \( F \) defined on \([0, 1]\). Then, from (6), we have \( 0 \leq R_n^{q} \leq e^{-q} \).

Proof. Since the function \( f(p) = p \ln(p) \), \( 0 < p < 1 \), has a maximum value at \( e^{-q} \), \( 0 \leq q \leq 1 \); therefore,
\[0 \leq R_n^q = CRE_q(F_n) = \sum_{i=1}^{n-1} (1 - \frac{1}{n}) \left( -\ln \left( 1 - \frac{1}{n} \right) \right)^q (X_{(i+1)} - X_{(i)})\]
\[\leq \sum_{i=1}^{n-1} e^{-q}(X_{(i+1)} - X_{(i)}) = e^{-q}(X_{(n)} - X_{(1)}) \leq e^{-q}.\]

This completes the proof of the theorem. □

**Theorem 4.** Under \(H_0\), from (6), the mean and the variance of \(R_n^q\) are, respectively,
\[\mathbb{E}(R_n^q) = \frac{1}{n+1} \sum_{i=1}^{n-1} A_i,\]
\[\text{Var}(R_n^q) = \frac{n}{(n + 1)^2(n + 2)} \sum_{i=1}^{n-1} A_i^2.\]

**Proof.** The proof directly follows by noting that, for any \(i = 1, 2, \ldots, n-1\), the RV \(W_i = (X_{(i+1)} - X_{(i)})\), based on the CDF \(U(0, 1)\), has beta distribution, i.e., \(W_i \sim \text{Beta}(1, n)\) (cf. [20]). This completes the proof. □

**Remark 1.** Under \(H_0\), from (6), (8), and (9), we have \(\lim_{n \to \infty} \mathbb{E}(R_n^q) = q) \frac{1}{2(q+1)} = CRE_q(U)\) and \(\lim_{n \to \infty} \text{Var}(R_n^q) = 0\), where \(CRE_q(U)\) is the FCRE of the CDF \(U(0, 1)\).

The critical region, which describes the test procedure, is given by the following two inequalities:
\[CRE_q(F_n) \leq CRE_q^* \Leftarrow \text{lower} \text{ or } CRE_q^*(F_n) \geq CRE_q^*(a/2) \Leftarrow \text{upper},\]

where \(\alpha\) is the desired level of significance, and \(CRE_q^*(a)\) is the \(a\)-quantile of the asymptotic, or approximated, CDF of the test statistic \(CRE_q^*(F_n)\), under \(H_0\). In the next section, we derive the asymptotic and approximated CDF of the test statistic \(CRE_q^*(F_n)\). These quantiles are computed by using the Monte Carlo method.

### 3. Percentage Points of the Test Statistic

In this section, we obtain the asymptotic distribution of \(R_n^q\) under \(H_0\). From (6), we can write \(R_n^q = \sum_{i=1}^{n-1} T_i\), where \(T_i = A_iW_i, i = 1, 2, \ldots, n-1, \text{ and } W_i \sim \text{Beta}(1, n)\). Thus, we can see that \(T_i\)‘s have the following probability density function (PDF):
\[f_{T_i}(t) = \frac{n}{A_i} \left(1 - \frac{t}{A_i} \right)^{n-1}, \quad i = 1, 2, \ldots, n-1.\]

The mean and variance of \(T_i\) are, respectively,
\[\mu_i = \mathbb{E}(T_i) = A_i\mathbb{E}(W_i) = \frac{A_i}{n + 1},\]
\[\sigma_i^2 = \text{Var}(T_i) = A_i^2\text{Var}(W_i) = \frac{nA_i^2}{(n + 1)^2(n + 2)}.\]

According to Lyapunov central limit theorem (see Billingsley [21]), we have \(\sum_{i=1}^{n-1} (T_i - \mu_i) / \sqrt{\sum_{i=1}^{n-1} \sigma_i^2} = (R_n^q - \mathbb{E}(R_n^q)) / \sqrt{\text{Var}(R_n^q)} \xrightarrow{d} N\), where \(N\) is the standard normal RV (in the sequel, the standard normal distribution will be denoted by \(N(0, 1)\)). Therefore, under \(H_0\), the percentage point (\(a\)-quantile) \(CRE_q^*(a)\) is approximated according to the asymptotic normality of \(R_n^q\) for large \(n\) by
\[CRE_q^*(a) = \mathbb{E}(R_n^q) + \sqrt{\text{Var}(R_n^q)}Z_a,\]
where \(Z_a\) corresponds to the quantile \((a \times 100)\) of the CDF \(N(0, 1)\).

Johannesson and Giri [22] proposed an approximation of the CDF of linear combination of the finite number of beta RVs. Noughabi [14] used this approximation to obtain approximately the percentage points of the CRE for finite \(n\).

By adopting the same procedure of Noughabi [14], we can obtain an approximation of \(R_n^q\) for finite \(n\) as follows:
\[R_n^q \approx \left( \sum_{i=1}^{n-1} A_i \right)Y,\]

where the RV \(Y\) has Beta \((a, b)\) distribution,
\[a = \frac{(n + 2)(\sum_{i=1}^{n-1} A_i)^2}{(n + 1)^2}, \quad b = \frac{n}{n + 1} \left( \frac{(n + 2)(\sum_{i=1}^{n-1} A_i^2)}{(\sum_{i=1}^{n-1} A_i^2)} - 1 \right),\]

and \(A_i = (1 - (i/n))(-\ln(1 - (i/n)))\), \(0 \leq q \leq 1, i = 1, 2, \ldots, n-1\). According to (14), the mean and variance of \(R_n^q\) are, respectively,
\[\mathbb{E}(R_n^q) = \left( \sum_{i=1}^{n-1} A_i \right)\frac{a}{a + b},\]
\[\text{Var}(R_n^q) = \left( \sum_{i=1}^{n-1} A_i \right)^2 \frac{ab}{(a + b)^2(a + b + 1)}.\]

Now, by using this approximation of \(R_n^q\), the quantiles of order \(a/2\) and \(1 - (a/2)\) of the approximated CDF of the test statistic \(CRE_q^*(F_n)\) under \(H_0\) are, respectively,
\[\text{lower} = \left( \sum_{i=1}^{n-1} A_i \right) F^{-1}\left( \frac{a}{2} \right),\]
\[\text{upper} = \left( \sum_{i=1}^{n-1} A_i \right) F^{-1}\left( \frac{1 - a}{2} \right),\]
where \( F^{-1}(.) \) is the quantile function of the CDF \( F \), \( F \) is the Beta\((a,b)\) distribution, and \( a \) and \( b \) are defined in (15).

3.1. Empirical Weighted Cumulative Residual Entropy. From (3), Misagh et al. [15] proposed the empirical WCRE by

\[
\text{CRE}_w(F_n) = - \sum_{i=1}^{n-1} \left( \frac{X_{(i+1)^2}^2 - X_{(i)}^2}{2} \right) \left( 1 - \frac{i}{n} \right) \ln \left( 1 - \frac{i}{n} \right)
\]

\[
= \sum_{i=1}^{n-1} A_i U_i,
\]

where \( A_i = X_{(i+1)^2}^2 - X_{(i)}^2/2 \), \( U_i = -(1 - (i/n)) \ln (1 - (i/n)) \), \( i = 1, 2, \ldots, n - 1 \).

We suggest the following statistic of a consistent test based on (18):

\[
T^n_w = \text{CRE}_w(F_n) = \sum_{i=1}^{n-1} A_i U_i.
\]

**Theorem 5.** The test based on the sample estimate \( T^n_w \) is consistent. \( \square \)

**Proof.** From Mirali et al. [12] and by using Glivenko–Cantelli theorem, (see Tucker [19]), we have \( \text{CRE}_w(F_n) \xrightarrow{a.s.} \text{CRE}_w(F) \), which proves the theorem. \( \square \)

**Theorem 6.** Let \( X_1, X_2, \ldots, X_n \) be a random sample drawn from an unknown continuous CDF \( F \) defined on \([0,1] \). Then, from (18), we get \( 0 \leq T^n_w \leq 1/2e \).

**Proof.** Since the function \( f(p) = -p \ln p, \) \( 0 < p < 1 \), has a maximum value at \( 1/e \), therefore,

\[
0 \leq T^n_w = \text{CRE}_w(F_n) = - \sum_{i=1}^{n-1} \left( \frac{X_{(i+1)^2}^2 - X_{(i)}^2}{2} \right) \left( 1 - \frac{i}{n} \right) \ln \left( 1 - \frac{i}{n} \right)
\]

\[
\leq \frac{1}{2} \sum_{i=1}^{n-1} \frac{1}{e} \left( X_{(i+1)^2}^2 - X_{(i)}^2 \right) \leq \frac{1}{2e} \sum_{i=1}^{n-1} X_{(i+1)^2}^2 - X_{(i)}^2 \leq \frac{1}{2e}
\]

(20)

This completes the proof. \( \square \)

3.2. Percentage Points. We generate 50,000 samples of size \( n \), where \( n = 10, 20, 30, 40, 50, 70, 100 \), from \( U(0,1) \). Using (6), the test statistic \( R^n_q \) is estimated by the empirical \( R^n_q \) for each sample and the same for \( T^n_w \). Moreover, we can see that \( \text{CRE}^{0.1}_{10}(U) = 0.4438 \), \( \text{CRE}^{0.05}_{10}(U) = 0.3133 \), \( \text{CRE}^{0.0}1(U) = 0.2576 \) and \( \text{CRE}_w(U) = 0.1388 \), where \( \text{CRE}^q(U) \) and \( \text{CRE}_w(U) \) are the FCRE and WCRE of the CDF \( U(0,1) \), respectively. Consequently, for \( R^n_q \), we present the percentage points of the Monte Carlo method, asymptotic normality, and beta approximation by using (10), (13), and (17), respectively. The result of this study is given in Table 1, where we note that the difference between the percentage points decreases when \( n \) increases. Besides, for \( R^n_0 \), the accuracy of the Monte Carlo method is more than the other two methods.

Figures 1–4 represent the empirical PDF’s of the test statistics using Monte Carlo samples with \( n = 10, 20, 30, 50, 100 \). When \( n \) increases, it turned out that the test statistics are nearer to the exact values, which implies that the bias and the variance decrease with increasing \( n \).

4. Power Analysis

In this section, we study the power test of Monte Carlo study under alternative distributions. The power of \( R^n_q \) is estimated by the proportion of the generated samples falling into the critical region. Under seven alternative distributions, the power of the test statistic \( R^n_q \) is calculated by the Monte Carlo study of generating 50,000 samples each of size \( n \), where \( n = 20, 30, 50 \). The alternative CDFs proposed by Stephens [18] in power study of uniformity tests are as follows:

\[ A_l: F(y) = 1 - (1 - y)^l, \quad 0 \leq y \leq 1, l = 1, 2, 3, \]

\[ B_l: F(y) = \begin{cases} 2^{-1}(1 - y)^l, & 0 \leq y \leq 0.5, \\ 1 - 2^{-1}(1 - y)^l, & 0.5 \leq y \leq 1, l = 1, 2, 3, \end{cases} \]

\[ C_l: F(y) = \begin{cases} 0.5 - 2^{-1}(0.5 - y)^l, & 0 \leq y \leq 0.5, \\ 0.5 + 2^{-1}(y - 0.5)^l, & 0.5 \leq y \leq 1, for l = 1, 2. \end{cases} \]

(21)

In Table 2, based on the Monte Carlo study, we recorded the power values of the proposed test statistics \( R^n_q, T^n_w \), Kolmogorov–Smirnov (K-S), Kuiper (V), Cramer-von Mises (W²), Watson (U²), and Anderson-Darling (A²), for \( n = 10, 20, 30 \) and \( \alpha = 0.05 \). From Table 2, we can conclude the following:

(1) If \( q \) increases and tends to 1 (\( q \to 1 \)), the power of \( \text{CRE}^q \) test, for alternative \( A_l(B_l)(C_l) \), decreases (increases) (increases), and vice versa, if \( q \) decreases and tends to 0 (\( q \to 0 \)).

(2) If \( q \to 1 \), the \( \text{CRE}^q \) test, for alternative \( A_l(B_l) \), gives the worst (best) performance compared with the other tests.

(3) To compare the performance between \( \text{CRE}^q \) and \( \text{CRE}_w \) tests, we observe that:

(a) For the alternative \( A_l, q \to 1 \), \( \text{CRE}_w \) performs better than \( \text{CRE}^q \) and vice versa if \( q \to 0, n \) increases.

(b) For the alternative \( B_l, q \to 1 \), \( \text{CRE}^q \) performs better than \( \text{CRE}_w \), and vice versa, if \( q \to 0, n \) increases.

(c) For the alternative \( C_l, q \to 0 \), \( \text{CRE}_w \) performs better than \( \text{CRE}^q \), and vice versa, if \( q \to 1 \).

Stephens [18] noted that \( V \) and \( U^2 \) tests will reveal a change at variance. Therefore, we observe the following:

(1) For alternative \( A_l, q \to 0 \), \( \text{CRE}^q \) performs better than \( V \) and \( U^2 \), and vice versa, if \( q \to 1 \).
Table 1: Percentage points of the proposed test statistics $R^q_n$ and $T^w_n$ at level $\alpha = 0.05$.

| $n$ | $q$ | Monte Carlo method | Normal approximation | Beta approximation | $T^w_n$ |
|-----|-----|---------------------|----------------------|-------------------|---------|
|     |     | Upper   | Lower   | Upper   | Lower   | Upper   | Lower   |
| 10  | 0.1 | 0.5131  | 0.2282  | 0.6165  | 0.1293  | 0.6495  | 0.1672  |
|     | 0.5 | 0.3522  | 0.1818  | 0.4481  | 0.1065  | 0.47003 | 0.1315  | 0.1574  | 0.0669  |
| 20  | 0.1 | 0.50608 | 0.3041  | 0.6012  | 0.2144  | 0.6217  | 0.2368  |
|     | 0.5 | 0.3458  | 0.2341  | 0.4282  | 0.1632  | 0.4415  | 0.1777  | 0.1544  | 0.0957  |
| 30  | 0.1 | 0.4995  | 0.3357  | 0.58407 | 0.2556  | 0.5987  | 0.2715  |
|     | 0.5 | 0.3422  | 0.2544  | 0.4135  | 0.19007 | 0.423   | 0.2002  | 0.15275 | 0.1068  |
| 40  | 0.1 | 0.4955  | 0.3145  | 0.5709  | 0.2809  | 0.5824  | 0.2931  |
|     | 0.5 | 0.3393  | 0.2646  | 0.40309 | 0.2064  | 0.4104  | 0.2142  | 0.1515  | 0.1126  |
| 50  | 0.1 | 0.4905  | 0.3653  | 0.5607  | 0.2983  | 0.5701  | 0.3082  |
|     | 0.5 | 0.3374  | 0.2712  | 0.3953  | 0.2177  | 0.4013  | 0.2241  | 0.1506  | 0.1163  |
| 70  | 0.1 | 0.4854  | 0.3801  | 0.5461  | 0.3213  | 0.553   | 0.3285  |
|     | 0.5 | 0.3345  | 0.2796  | 0.3844  | 0.2326  | 0.3888  | 0.2372  | 0.1492  | 0.1206  |
| 100 | 0.1| 0.4799  | 0.3919  | 0.5317  | 0.3418  | 0.5367  | 0.3469  |
|     | 0.5 | 0.3317  | 0.2859  | 0.37404 | 0.2459  | 0.3772  | 0.2492  | 0.14805 | 0.1242  |

Figure 1: The estimated PDF’s of $R^{0.1}_n$ based on $U(0, 1)$.

Figure 2: The estimated PDF’s of $R^{0.5}_n$ based on $U(0, 1)$.

Figure 3: The estimated PDF’s of $R^{0.9}_n$ based on $U(0, 1)$.

Figure 4: The estimated PDF’s of $T^{w}_n$ based on $U(0, 1)$.
Table 2: Power estimates of the tests at level $\alpha = 0.05$.

| $n$ | Alternative | $R_2^0$ | $R_2^0$ | $T_2^0$ | $K-S$ | $V$ | $W^2$ | $U^2$ | $A^2$ |
|-----|-------------|--------|--------|--------|------|-----|------|------|------|
| 10  | $A_{1.5}$   | 0.10708| 0.0908 | 0.07208| 0.14002| 0.12616| 0.0756 | 0.1456| 0.07776| 0.1877 |
|     | $A_2$       | 0.2771 | 0.2327 | 0.15104| 0.3414 | 0.30298| 0.1631 | 0.3551| 0.16308| 0.4761 |
|     | $B_{1.5}$   | 0.10406| 0.1314 | 0.1302 | 0.0896 | 0.07352| 0.0971 | 0.0741| 0.1017 | 0.1349 |
|     | $B_2$       | 0.2427 | 0.3379 | 0.3357 | 0.21402| 0.1184 | 0.2307 | 0.1104| 0.2481 | 0.3269 |
|     | $B_3$       | 0.5763 | 0.7662 | 0.7723 | 0.5516 | 0.2424 | 0.5394 | 0.2154| 0.5699 | 0.72308|
|     | $C_{1.5}$   | 0.0843 | 0.119  | 0.1217 | 0.0942 | 0.0342 | 0.0974 | 0.0239| 0.1031 | 0.0222 |
|     | $C_2$       | 0.1354 | 0.2478 | 0.2543 | 0.1723 | 0.0402 | 0.2333 | 0.01114| 0.2475 | 0.00924|

(2) For the alternative $B_1, q \rightarrow 1$, CRE performs better than $V$ and $U^2$, and vice versa, if $q \rightarrow 0, n$ increases.

(3) For the alternative $C_1, q \rightarrow 0$, $V$ and $U^2$ performs better than CRE.

(4) CRE performs better than $V$ and $U^2$ against the alternative $A$.

(5) CRE performs better than $V$ and $U^2$ against the alternative $B_1$, $n$ increases. But, $V$ and $U^2$ perform better than CRE against the alternative $C_1$.

5. Conclusion

For the CDFs with support $[0,1]$, we exhibited that the values of CRE and CRE are within $[0,e^{-q}]$ and $[0,1/2e]$, respectively. Moreover, the test of uniformity was proposed by calculating the percentage points and power analysis of CRE and CRE. Besides, for CRE, we obtained the percentage points by using the Monte Carlo method via the simulation and the normality asymptotic, as well as the beta approximation. Moreover, for CRE the percentage points were derived by using the Monte Carlo method via the simulation. A power comparison was performed between the FCRE and WCRE and other tests, where, by changing the value of $q$, we indicated when the test has higher and lower power compared with the other tests.

Data Availability

The simulated data used to support the findings of this study are included within the article.

Conflicts of Interest

The authors declare that they have no conflicts of interest concerning the publication of this article.

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