THE GÁLVEZ–KOCK–TONKS CONJECTURE FOR DISCRETE DECOMPOSITION SPACES

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Abstract

Gálvez-Carrillo, Kock, and Tonks [17] constructed a decomposition space $U$ of all Möbius intervals, as a recipient of Lawvere’s interval construction for Möbius categories, and conjectured that $U$ enjoys a certain universal property: for every Möbius decomposition space $X$, the space of CULF functors from $X$ to $U$ is contractible. In this paper, we work at the level of homotopy 1-types to prove the first case of the conjecture, namely for discrete decomposition spaces. This provides also the first substantial evidence for the general conjecture.

The discrete case is general enough to cover all locally finite posets, Cartier–Foata monoids, and Möbius categories. The proof is 2-categorical. First, we construct a local strict model of $U$, which is then used to show by hand that the Lawvere interval construction, considered as a natural transformation, does not admit other self-modifications than the identity.

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Introduction

Incidence algebras and Möbius inversion form a cornerstone of combinatorics. It has important applications in many areas of mathematics. Beyond the original applications in number theory (see Hardy and Wright [19]) and group theory ([36] and [18]), one can cite applications in probability theory [31] and algebraic topology [20], and it is also closely related to Hopf-algebraic renormalisation [24] in quantum field theory.

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Since Rota formalised the theory \([33], [22]\) (on the grounds of previous contributions \([36], [18]\)) the standard framework for the theory has been that of posets, but the theory has also been used in the context of monoids \([7]\), and in the more general framework of certain categories called Möbius categories, introduced by Leroux \([29]\). The uniform appearance of the Möbius inversion formula across all application areas led Lawvere \([27]\) in the 1980s to discover that there is a universal Möbius function which induces all other Möbius functions. It is an ‘arithmetic function’ on a certain Hopf algebra of Möbius intervals. A category is an interval if it has an initial and a terminal object and the Möbius condition is a certain finiteness condition \([26]\). This Hopf algebra has the property that it receives a canonical coalgebra homomorphism from every incidence coalgebra of a Möbius category. This includes all locally finite posets and all the monoids considered in \([7]\). Lawvere’s work remained unpublished for some decades, but it is cited in influential texts from that time, such as Joyal \([23]\) and Joni–Rota \([22]\). Independently, Ehrenborg \([10]\) constructed a closely related Hopf algebra, but less universal. It only accounts for intervals in posets. In both cases, the universal object can be interpreted as the colimit of all incidence coalgebras of intervals. The possibility of this is closely related to the local nature of coalgebras, expressed for example in the well-known fact that every coalgebra is the colimit of its finite-dimensional subcoalgebras, see Sweedler \([35]\).

Lawvere’s discovery did not appear in print until Lawvere–Menni \([28]\) in 2010. In that work the authors took an important step towards explaining the universal property by lifting the construction of the Hopf algebra of Möbius intervals to the objective level. This means that its comultiplication is realised as something called a pro-comonoidal structure on certain extensive categories. The original Hopf algebra is exhibited as being only a numerical shadow of this categorical construction. There are at least two precursors to the idea of a more objective approach to incidence algebras. One is given by Joyal \([23]\). In his foundational paper on species, there is a final section where he considers certain decomposition structures on categories (that final section has little to do with species). Another is in the work of Dür \([8]\) who constructed incidence coalgebras of certain categorical and simplicial structures.

However,

many coalgebras, bialgebras and Hopf algebras in combinatorics are not of incidence type,

meaning that they cannot arise directly as the incidence coalgebra of any Möbius category. In fact the Lawvere–Menni Hopf algebra is not of incidence type. This gives the somewhat unsatisfactory situation that the universal object is not of the same type as the objects it is universal for.

A solution to this problem was found by Gálvez, Kock, and Tonks \([15, 16, 17]\). They discovered that the incidence coalgebra construction and Möbius inversion make sense for objects more general than Möbius categories (recall that Möbius categories include locally finite posets and Cartier–Foata monoids). These are completely new objects in this context which they call decomposition spaces. They are certain simplicial objects subject to an axiom that expresses decomposition, in the same way as the Segal condition (which characterises categories among simplicial sets) expresses composition. Decomposition spaces are the same thing as the 2-Segal spaces of Dyckerhoff and Kapranov \([9]\) (see \([11]\) for the last piece of this equivalence). It seems likely that all combinatorial coalgebras, bialgebras and Hopf algebras (with nonnegative section coefficients) arise from the incidence coalgebra construction of decomposition spaces. This has been shown for most of Schmitt’s examples \([34]\) (restriction species in Gálvez–Kock–Tonks \([14]\) and hereditary species in Carlier \([5]\)). Gálvez–Kock–Tonks (as also Dyckerhoff–Kapranov) work in the fully homotopical setting of simplicial
∞-groupoids, but already the discrete case of the notion is very rich, as exemplified by work of Bergner et al. [2] and Kock–Spivak [25], who relate the notion to constructions in algebraic topology and category theory.

Gálvez, Kock and Tonks [17] showed that the Lawvere–Menni Hopf algebra is the incidence coalgebra of a decomposition space $U$. With this discovery the universal property could be stated, showing its nature as a moduli space: For any decomposition space $X$ the mapping space $\text{map}(X,U)$ is contractible. This statement is the Gálvez–Kock–Tonks conjecture, which is the objective of the present paper. The mapping space is the space of CULF maps, as detailed further below. CULF maps were identified to play a key role already in the work of Lawvere and Menni [28].

Lawvere’s original work (suitably upgraded to the new context) shows that $\text{map}(X,U)$ is not empty: it contains $I : X \to U$, which is essentially Lawvere’s interval construction. Gálvez, Kock and Tonks [17] were able to establish one further ingredient of the conjecture, namely that $\text{map}(X,U)$ is connected, meaning that every map is homotopy equivalent to $I$. The finer property of being contractible is the full homotopy uniqueness statement, that not only is every map equivalent to $I$: it is so uniquely (in a coherent homotopy sense).

The homotopy content was one of the reasons for Gálvez, Kock and Tonks to develop the whole theory in a homotopy setting: decomposition spaces are defined to be certain simplicial ∞-groupoids, and everything is fully homotopy invariant. It is an important insight of higher category theory (see for example Lurie [30]) that a universal object cannot exist in any truncated situation. Most famous is the fact that the topos of sets (0-types) contains a classifier for monomorphisms ((−1)-types) but cannot contain a classifier for sets (0-types), and that for these to be classified one needs the 2-topos of groupoids (1-types), and to classify 1-types one needs to 3-topos of 2-types, and so on. Only in the limit is it possible to find a classifier for general homotopy types (∞-groupoids) in the ∞-topos of ∞-groupoids.

At the moment, the technical difficulties of the general Gálvez–Kock–Tonks conjecture are too big.

In the present paper, the first case of the conjecture is proved. Working at the level of 1-types, we define the simplicial groupoid $U$ of intervals, and show that:

**Theorem.** (Theorem 6.7.) $\text{map}(X,U)$ is a contractible 1-groupoid for every 0-truncated decomposition space $X$.

This is the first substantial evidence for the full conjecture. At the same time this level of generality is general enough to cover all the classical theory of incidence algebras and Möbius inversion in combinatorics, since locally finite posets, Cartier–Foata monoids, Möbius categories, and Schmitt’s examples are all 0-truncated simplicial spaces. In particular it gives finally a firm formalisation of Lawvere’s intuition that the interval construction should be universal in some sense. As a particular case it establishes also the universal property of the Ehrenborg Hopf algebra.

The idea of the proof is the following. The theorem, namely the contractibility of the 1-groupoids $\text{map}(X,U)$, is a 2-categorical statement. The proof we give is based on 2-category theory. However, a direct verification of the statement seems intractable, due to coherence problems. The difficulty is that $U : \Delta^{op} \to \text{Grpd}$ is only a pseudo-simplicial groupoid. Jardine [21] has identified all the 2-cell structure and the 17 coherence conditions for pseudo-simplicial groupoids. The definition of modification in this context requires compatibility with all that. The strategy to overcome this difficulty is to build a local strict model, a kind of neighbourhood $U_X \subset U$ around the intervals of a given discrete decomposition space $X$. The bulk of the paper is concerned with setting up this local model and show that it is strict.
To construct this, we introduce a stricter algebraic notion of interval, where the initial and terminal objects are not just given as properties of a discrete decomposition space, but are carried around as data, in the notion of chosen initial and terminal objects. This focus is inspired by the work in another context of Batanin and Markl on operadic categories [1]. This is quite technical, but the benefit is to achieve a strict local model $U_X$ which is shown to be a strict simplicial groupoid and a complete decomposition groupoid, and to receive a strict version of the interval construction. With this strict local model in place, the local version of the contractibility of $\text{map}(X, U_X)$ can be established with 2-category theory by showing that $I : X \to U_X$, interpreted as a natural transformation, does not admit other self-modifications than the identity modification. In the end this check is not so difficult.

At this point it is natural to ask whether the techniques developed here can be applied or refined to prove the conjecture in full generality. Unfortunately this is not very likely, or it would require new conceptual simplifications and new technical tools. The point is that the proof relies on explicit strictification through strict models constructed through explicit data standing in for universal properties (initial and terminal objects, at the level of the objects involved, and strict simplicial objects at the higher level). The level of 2-categories is in practice the highest level where this kind of technique can be applied, and already at this level it is quite tricky to find the balance between properties and property-like structures. The next level, which would be the contractibility of the 2-groupoid $\text{map}(X, U)$ for $X$ a 1-truncated decomposition space, and $U$ the universal simplicial 2-groupoid of 2-intervals, seems out of reach. It seems more promising to pass directly to the homotopical setting of the full conjecture, aiming at using the theory of $(\infty, 2)$-categories. But it seems quite daunting to carry over the explicit strictification strategies to this setting.

In conclusion, the present contribution may be seen as only a small step towards the full conjecture, but it is nevertheless an important step (and actually the first step ever carried out), and enough to cover all the cases envisioned by Lawvere, which includes essentially all the examples from classical combinatorics. It is also already a striking example of how higher category theory (in this case 2-categories) serves to solve problems even in discrete mathematics.

**Organisation of the paper**

We begin in Section 1 with a brief review of basic notions and some results on groupoids. In 1.2, we recall from [15] some basic notions and results of the theory of decomposition groupoids. In 1.9 we review the notion of incidence coalgebra of a decomposition space. In 1.12, we briefly explain the CULF condition for a simplicial map. In 1.15, we review the notion of decalage.

In Section 2, we develop some material required about the notion of slice and coslice Segal groupoids of decomposition groupoids.

In Section 3, we explain the concept of chosen initial object (3.10) and chosen terminal object (3.8). Furthermore, we define what an interval is (3.12) and develop some results for intervals, in particular the standard factorisation (3.17) and a lifting property (3.20) for simplices in an interval.

In Section 4, we construct the stretched-CULF factorisation system in the category of intervals. Furthermore, we developed two important working tools (4.5 and 4.7) that will be useful in next sections.

In Section 5, we define the complete decomposition groupoid of all intervals $U$ [17]. In 5.2, we construct a simplicial groupoid $U_X$ (5.6) that only contains the information of the
intervals of a discrete decomposition groupoid $X$ and prove that $U_X$ is a strict complete decomposition groupoid (5.12 and 5.13). Furthermore, we define a simplicial map $I : X \to U_X$ and prove that $I$ is CULF (5.14). In 5.15, we explain the equivalence between the Grothendieck construction of $U_X$ and Gálvez-Kock-Tonks construction of $\mathcal{U}$ for a fixed discrete decomposition groupoid $X$ (5.18), and we also show that $U_X$ can be seen as a subsimplicial groupoid of $U$ (5.20).

In Section 6, we talk about the Gálvez–Kock–Tonks conjecture formulated in [17] and we cite a partial result (6.1) about the connectedness of the mapping space $\text{map}_{\text{cd}(X,U)}$. In 6.2, we use the concept of modification (6.4) to study a truncated version of the conjecture, the case of discrete decomposition groupoids. We first show that $\text{map}_{\text{cd}(X,U)}$ is contractible (6.6) and after we prove that the groupoid map $\text{map}_{\text{cd}(X,U)}$ is contractible (6.7) which is the statement of the Gálvez–Kock–Tonks conjecture.

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1 Preliminaries

For the convenience of the reader, this section recalls a few background facts and establish notation. These results are not new.

Given a map of groupoids $p : X \to S$ and an object $s \in S$, the fibre $X_s$ of $p$ over $s$ is the pullback

$$
\begin{array}{ccc}
X_s & \longrightarrow & X \\
\downarrow & & \downarrow^p \\
1 & \longrightarrow & S.
\end{array}
$$

A map of groupoids $f : X \to Y$ is a monomorphism when its fibres are $(-1)$-groupoids, that is, are either empty or contractible. The main tool used throughout this paper are (homotopy) pullbacks. We use the following standard lemma many times.

**Lemma 1.1** (See [6]). A square of groupoids

$$
\begin{array}{ccc}
P & \longrightarrow & Y \\
\downarrow & & \downarrow \\
X & \longrightarrow & S
\end{array}
$$

is a homotopy pullback diagram if and only if for each $x \in X$ the induced comparison map $u_x : P_x \to Y_{fx}$ is an equivalence.

If $f$ is an isofibration, the strict pullback of the diagram is homotopy equivalent to the homotopy pullback.

1.2. Decomposition groupoids
We will study the discrete case of the Gálvez–Kock–Tonks conjecture in Section 6. For that we only have to deal with simplicial groupoids rather than simplicial spaces. For this reason we prefer to use the word decomposition groupoid rather than decomposition space in all the paper.

The simplex category $\Delta$ is the category whose objects are the nonempty finite ordinals and whose morphisms are the monotone maps. These are generated by coface maps $d^i : [n - 1] \to [n]$, which are the monotone injective functions for which $i \in [n]$ is not in the image, and codegeneracy maps $s^i : [n + 1] \to [n]$, which are monotone surjective functions for which $i \in [n]$ has a double preimage. We write $d^\perp := d^0$ and $d^{\top} := d^n$ for the outer coface maps.

An arrow of $\Delta$ is termed active, and written $g : [m] \to [n]$, if it preserves end-points, $g(0) = 0$ and $g(m) = n$. An arrow is termed inert, and written $f : [m] \to [n]$, if it is distance preserving, $f(i + 1) = f(i) + 1$ for $0 \leq i < m$.

**Definition 1.3** ([15], Definition 3.1). A decomposition groupoid is a simplicial groupoid $X : \Delta^{\text{op}} \to \text{Grpd}$ such that the image of any pushout diagram in $\Delta$ of an active map $g$ along an inert map $f$ is a pullback of groupoids,

$$
\begin{array}{c}
X_p \xrightarrow{g^*} X_m \\
\downarrow f^* \downarrow \\
X_q \xrightarrow{g^*} X_n \\
\end{array}
$$

This is equivalent to requiring that for each $n \geq 2$ the following diagrams are pullbacks for $0 < i < n$:

$$
\begin{array}{c}
X_{n+1} \xrightarrow{d_{i+1}} X_n \xleftarrow{d_i} X_{n-1} \\
\downarrow d_{\perp} \downarrow d_{\perp} \downarrow \\
X_n \xrightarrow{d_i} X_{n-1} \xleftarrow{d_{\top}} X_{n-1}.
\end{array}
$$

**Definition 1.4** ([15], §2.9). A simplicial groupoid $X : \Delta^{\text{op}} \to \text{Grpd}$ is called Segal groupoid if it satisfies the Segal condition,

$$
\xrightarrow{\simeq} X_1 \times X_0 \cdots \times X_0 \times X_1 \text{ for all } n \geq 0.
$$

This is equivalent to requiring that for each $n > 0$ the following diagram is a pullback

$$
\begin{array}{c}
X_{n+1} \xrightarrow{d_{\top}} X_n \\
\downarrow d_{\perp} \downarrow d_{\perp} \downarrow \\
X_n \xrightarrow{d_{\top}} X_{n-1}.
\end{array}
$$

**Example 1.5.** A rooted plane tree is a rooted tree such that for each node there is a linear order on the set of incoming edges; a forest is a disjoint union of rooted plane trees.

We can define the discrete decomposition groupoid of rooted plane trees as: $\text{RPT}_1$ denotes the set of isoclasses of forests, and $\text{RPT}_2$ denotes the set of isoclasses of forests with
an admissible cut. More generally, \( \text{RPT}_0 \) is defined to be a point, and \( \text{RPT}_k \) is the set of isoclasses of forests with \( k - 1 \) compatible admissible cuts. These form a simplicial discrete groupoid in which the inner face maps forget a cut, and the outer face maps project away stuff: \( d_\perp \) deletes the crown and \( d_\top \) deletes the bottom layer. It is readily seen that \( \text{RPT} \) is not a Segal groupoid: a tree with a cut cannot be reconstructed from its crown and its bottom tree, which is to say that \( \text{RPT}_2 \) is not equivalent to \( \text{RPT}_1 \times \text{RPT}_0 \text{RPT}_1 \). It is straightforward to check that it is a decomposition groupoid [15].

**Proposition 1.6** ([15], Proposition 3.7). Any Segal groupoid is a decomposition groupoid.

A simplicial map \( F : X \to Y \) is called a right fibration if it is cartesian on all bottom face maps \( d_\perp \). Similarly, \( F \) is called a left fibration if it is cartesian on \( d_\top \).

**Lemma 1.7.** Let \( Y \) be a Segal groupoid and let \( F : X \to Y \) be a simplicial map that is a left or a right fibration, then also \( X \) is a Segal groupoid.

Certain pullbacks in \( \Delta^{\text{op}} \) are preserved by general decomposition groupoids.

**Lemma 1.8** ([15], Lemma 3.10). Let \( X \) be a decomposition groupoid. For all \( n \geq 3 \) and all \( 0 < i < j < n \), the following squares of active face and degeneracy maps are pullbacks

\[
\begin{array}{ccc}
X_{n+1} & \xrightarrow{d_i} & X_n \\
\downarrow d_{i+1} & & \downarrow d_j \\
X_n & \xrightarrow{d_i} & X_{n-1}
\end{array}
\quad
\begin{array}{ccc}
X_{n-3} & \xrightarrow{s_{i-1}} & X_{n-2} \\
\downarrow s_{j-2} & & \downarrow s_{j-1} \\
X_{n-2} & \xrightarrow{s_{i-1}} & X_{n-1}
\end{array}
\]

A decomposition groupoid \( X \) is complete when \( s_0 : X_0 \to X_1 \) is a monomorphism (i.e. is (-1)-truncated). It follows from the decomposition groupoid axiom that in this case all degeneracy maps are monomorphisms ([16], Lemma 2.5).

1.9. The incidence coalgebra of a decomposition groupoid

The span

\[
X_1 \xleftarrow{d_1} X_2 \xrightarrow{(d_2, d_0)} X_1 \times X_0 X_1
\]

defines a linear functor, the comultiplication

\[
\Delta : \text{Grpd}_{/X_1} \to \text{Grpd}_{/X_1 \times X_0 X_1}
\]

\[
f \mapsto (d_2, d_0) \circ d_1'(f).
\]

The morphism \( (d_2, d_0) \circ d_1'(f) \) is defined by the diagram

\[
\begin{array}{ccc}
1 & \xleftarrow{h\text{Fib}_f(d_1')} & \h\text{Fib}_f(d_1) \\
\downarrow r_f & & \downarrow \phi_1(f) \\
X_1 & \xleftarrow{d_i} & X_2 \xrightarrow{(d_2, d_0)} X_1 \times X_0 X_1.
\end{array}
\]

Likewise, the span

\[
X_1 \xleftarrow{s_0} X_0 \xrightarrow{t} 1
\]
defines a linear functor, the counit
\[
\delta : \text{Grpd}_{/X_1} \rightarrow \text{Grpd}
\]
\[
f \mapsto t ! \circ s^*_0(f).
\]
The decomposition groupoid axioms gives the tools to prove that \(\Delta\) satisfies the coassociativity comultiplication property and \(\delta\) the counital property on \(\text{Grpd}_{/X_1}\) ([15], §5.3). We obtain a coalgebra \((\text{Grpd}_{/X_1}, \Delta, \delta)\) called the incidence coalgebra.

**Example 1.10.** The Hopf algebra of rooted plane trees [12] is the free commutative algebra on the set of isomorphism classes of rooted plane trees, with comultiplication defined by summing over certain admissible cuts \(c\):
\[
\Delta(T) = \sum_{c \in \text{admi.cuts}(T)} P_c \otimes R_c.
\]
An admissible cut \(c\) is a splitting of the set of nodes into two subsets, such that the second forms a subtree \(R_c\) containing the root node (or is the empty forest); the first subset, the complement crown, then forms a subforest \(P_c\). We obtain the coalgebra of rooted plane trees directly from the discrete decomposition groupoid of rooted plane trees of Example 1.5.

**Example 1.11** ([15], §5.1). If \(X\) is the nerve of a category (for example, a poset) then \(X_2\) is the set of all composable pairs of arrows. The comultiplication is just defined by:
\[
\Delta(f) = \sum_{b \circ a = f} a \otimes b
\]
and the counit sends identity arrows to 1 and other arrows to 0.

1.12. CULF maps

Recall that a simplicial map \(F : Y \rightarrow X\) is cartesian on an arrow \([n] \rightarrow [k]\) in \(\Delta\), if the naturality square for \(F\) with respect to this arrow is a pullback.

A simplicial map \(h : X \rightarrow Y\) is conservative if it is cartesian with respect to codegeneracy maps, and ULF (unique lifting of factorisations) if it is cartesian with respect to inner coface maps.

**Definition 1.13** ([15], Lemma 4.1). A simplicial map \(F : X \rightarrow Y\) is called CULF if \(F\) is cartesian on each active map.

The notion of CULF can be seen as an abstraction of coalgebra homomorphism. The conservative condition corresponds to counit preservation, ULF corresponds to comultiplicativity.

**Proposition 1.14** ([15], Lemma 4.3). A simplicial map between decomposition groupoids is CULF if and only if it is cartesian on \(d^1 : [1] \rightarrow [2]\).

1.15. Decalage

Given a simplicial groupoid \(X\), the lower dec \(\text{Dec}_\bot X\) is a new simplicial groupoid obtained by deleting \(X_0\) and shifting everything one place down, deleting also all \(d_0\) face maps and all \(s_0\) degeneracy maps. It comes equipped with a simplicial map, called the lower dec map, \(d_\bot : \text{Dec}_\bot X \rightarrow X\) given by the original \(d_0\). Similarly, the upper dec \(\text{Dec}_\top X\) is obtained by instead deleting, in each degree, the top face map \(d_\top\) and the top degeneracy map \(s_\top\). The deleted top face maps becomes the upper dec map \(d_\top : \text{Dec}_\top X \rightarrow X\).
Proposition 1.16 ([15], Proposition 4.9). If \( X \) is a decomposition groupoid then the dec maps \( d_\top : \text{Dec}_\top X \to X \) and \( d_\bot : \text{Dec}_\bot X \to X \) are CULF.

The decomposition property can be characterized in terms of decalage.

Theorem 1.17 ([9], [11], ([15], Theorem 4.10)). For a simplicial groupoid \( X : \Delta^{\text{op}} \to \text{Grpd} \), the following are equivalent

1. \( X \) is a decomposition groupoid
2. both \( \text{Dec}_\bot X \) and \( \text{Dec}_\top X \) are Segal groupoids.

2 Slices

In this section, we develop some constructions with slice and coslice Segal groupoids of decomposition groupoids required to develop the concept of interval in Section 3.

Definition 2.1 ([32], Definition 2.2). A simplicial groupoid \( X : \Delta^{\text{op}} \to \text{Grpd} \) is called upper 2-Segal when \( \text{Dec}_\top X \) is a Segal groupoid, and is called lower 2-Segal when \( \text{Dec}_\bot X \) is a Segal groupoid.

In equivalent terms, for \( 0 < i < n \), a simplicial groupoid is upper 2-Segal when squares as to the left are required to be pullbacks, and lower 2-Segal when this is only required for squares as to the right,

\[
\begin{array}{ccc}
X_{n+1} & \xrightarrow{d_{i+1}} & X_n \\
\downarrow{d_i} & & \downarrow{d_\top} \\
X_n & \xrightarrow{d_i} & X_{n-1}
\end{array}
\quad \text{and} \quad
\begin{array}{ccc}
X_{n+1} & \xrightarrow{d_i} & X_n \\
\downarrow{d_\bot} & & \downarrow{d_\tau} \\
X_n & \xrightarrow{d_i} & X_{n-1}
\end{array}
\]

Lemma 2.2 ([11], Proposition 2.1). Let \( X \) be an upper 2-Segal groupoid. For all \( 0 \leq i \leq n \) the following square is a pullback

\[
\begin{array}{ccc}
X_{n+1} & \xrightarrow{s_{i+1}} & X_{n+2} \\
\downarrow{d_0} & & \downarrow{d_0} \\
X_n & \xrightarrow{s_i} & X_{n+1}
\end{array}
\]

The pullbacks of Lemma 2.2 are called the upper unital condition.

Lemma 2.3 ([11]). Let \( X \) be a lower 2-Segal groupoid. For all \( 0 \leq i \leq n \) the following square is a pullback

\[
\begin{array}{ccc}
X_{n+1} & \xrightarrow{s_i} & X_{n+2} \\
\downarrow{d_{n+1}} & & \downarrow{d_{n+2}} \\
X_n & \xrightarrow{s_i} & X_{n+1}
\end{array}
\]

The pullbacks of Lemma 2.3 are called the lower unital condition.

Definition 2.4. Let \( X \) be an upper 2-Segal groupoid and let \( y \) be an object in \( X_0 \). The slice \( X/y \) is defined as the pullback of the upper dec \( \text{Dec}_\top X \) along \( ^\tau y^\top : 1 \to X_0 \).
We write $u : X/y \to \text{Dec}_\top X$ for the canonical map of Definition 2.4.

**Remark 2.5.** $\Delta_+$ is the category whose objects are finite ordinals (including the empty ordinal) and whose morphisms are the monotone maps. Since $\Delta_+^{op}$ has a terminal object, it makes sense to pull back the lower dec $\text{Dec}_\top X$ along an object in $X_0$. Furthermore, when $X$ is an upper 2-Segal groupoid, we have that $\text{Dec}_\top X$ is a Segal groupoid. For each $y$ in $X_0$ we have that $X/y$ is a cartesian fibration over $\text{Dec}_\top X$, and is therefore a Segal groupoid by Lemma 1.7.

**Definition 2.6.** Let $X$ be an upper 2-Segal groupoid. An object $b \in X_0$ is called *terminal* if the projection map $X/b \to X$ is a levelwise equivalence.

**Proposition 2.7.** Let $X$ be an upper 2-Segal groupoid and let $y$ be an object in $X_0$. Then $s_0(y)$ is a terminal object in $X/y$.

**Proof.** Let $\gamma t^\gamma : 1 \to (X/y)_0$ be defined by the pullback diagram:

$$
\begin{array}{cccc}
X_0 & \leftrightarrow & X_1 & \leftrightarrow & X_2 & \leftrightarrow & X_3 & \leftrightarrow & \text{Dec}_\top X \\
| & & | & & | & & | & & \\
\gamma y & \downarrow d_0 & \gamma u_0 & \downarrow d_0 & \gamma u_1 & \downarrow d_0 & \gamma u_2 & \downarrow u & \\
X/y & \leftrightarrow & (X/y)_1 & \leftrightarrow & (X/y)_2 & \leftrightarrow & (X/y)_3 & \leftrightarrow & X/y \\
\end{array}
$$

In the diagram

$$
\begin{array}{cccc}
X_0 & \leftrightarrow & (X/y)_0 & \leftrightarrow & (X/y)_1 & \leftrightarrow & (X/y)_2 & \leftrightarrow & (X/y)_3 \\
\gamma y & \downarrow \gamma t^\gamma & \downarrow s_0 & \downarrow s_0 & \downarrow s_0 & \downarrow s_1 & \downarrow s_2 & \downarrow s_3 & \downarrow u_0 & \downarrow u_1 & \downarrow u_2 & \downarrow u_3 & \downarrow u_4 & \downarrow u_5 & \downarrow u_6 \\
1 & \leftrightarrow & (X/y)_0 & \leftrightarrow & (X/y)_1 & \leftrightarrow & (X/y)_2 & \leftrightarrow & (X/y)_3 \\
\end{array}
$$

the square (1) commutes by definition of $t$ and (2) commutes by construction of $X/y$. Combining (1) and (2), we have that $d_1 u_1 s_0 t = s_0 y$. The squares (3) and (5) are pullbacks by construction of $(X/y)_{t_1}$ and $X/y$. The squares (4) and (6) are pullbacks since $X$ is an upper 2-Segal groupoid. Combining these squares, we have that $(X/y)_{t_1}$ is the pullback of $d_0 : X_2 \to X_1$ along $\gamma s_0 y^\gamma : 1 \to X_1$. On the other hand, in the diagram
the square (7) is a pullback by construction of $X/y$. Since $X$ is an upper 2-Segal groupoid, we have that (8) is a pullback by Lemma 2.2. Combining (7) and (8), we have that $(X/y)_0$ is the pullback of $d_0 : X_2 \to X_1$ along $\sim s_0 y \triangleright : 1 \to X_1$. Therefore, $((X/y)/s_0(y))_0$ and $(X/y)_0$ are pullbacks over the same diagram, so that $((X/y)/t)_0 \simeq (X/y)_0$. By analogous arguments for each $n \geq 0$, we have that $((X/y)/t)_n \simeq (X/y)_n$, hence altogether $(X/y)_t \simeq X/y$. □

**Remark 2.8.** In the proof of Proposition 2.7, the morphism $\sim t \triangleright : 1 \to (X/y)_0$ is determined by $s_0 y$. We will abuse of notation by identifying $t$ with the morphism $s_0 y$.

**Definition 2.9.** Let $X$ be a lower 2-Segal groupoid and let $x$ be an object in $X_0$. The **coslice** $X_{x/}$ is defined as the pullback of the lower dec $\text{Dec}_- X$ along $\sim x \triangleright : 1 \to X_0$. We write $v : X_{x/} \to \text{Dec}_- X$ for the canonical map of Definition 2.9. Note that for each $x$ in $X_0$, the coslice $X_{x/}$ is Segal. Indeed, $\text{Dec}_- X$ is Segal and $X_{x/}$ is a cartesian fibration over $\text{Dec}_- X$, and is therefore Segal too by Lemma 1.7.

**Definition 2.10.** Let $X$ be a lower 2-Segal groupoid. An object $a \in X_0$ is called **initial** if the projection map $X_{a/} \to X$ is a levelwise equivalence.

**Proposition 2.11.** Let $X$ be a lower 2-Segal groupoid and let $x$ be an object in $X_0$. Then $s_0(x)$ is an initial object in $X_{x/}$.

**Proof.** The proof is analogous to Proposition 2.7, using Lemma 2.3 instead of Lemma 2.2. □

**Lemma 2.12.** Let $\mathcal{C}$ be a decomposition groupoid with an initial object. Then for each object $y$ in $\mathcal{C}$, the slice $\mathcal{C}_{y/}$ has an initial object.

**Proof.** Since $\mathcal{C}$ has an initial object $\bot \mathcal{C}$, for each $y \in \mathcal{C}_0$ there exists a map $f_\bot : \bot \mathcal{C} \to y$. We will prove that $(\mathcal{C}_{y/})_{f_\bot/} \simeq (\mathcal{C}_{\perp/})_{f_\bot/}$ and $(\mathcal{C}_{\perp/})_{f_\bot/} \simeq \mathcal{C}_{y/}$. In the diagram

$$
\begin{array}{ccc}
1 & \xleftarrow{\sim} & (\mathcal{C}_{y/})_{f_\bot/} \\
\downarrow f_\bot & & \downarrow v_0 \\
(\mathcal{C}_{y/})_0 & \xleftarrow{\sim} & (\mathcal{C}_{\perp/})_{f_\bot/} \\
\downarrow u_0 & & \downarrow u_1 \\
\mathcal{C}_1 & \xleftarrow{\sim} & \mathcal{C}_2 \\
\end{array}
$$

the squares (1) and (2) are pullbacks by construction of $((\mathcal{C}_{y/})_{f_\bot/})_0$ and $(\mathcal{C}_{\perp/})_{f_\bot/}$. It follows that the left outer rectangle is a pullback. The squares (3) are (4) pullbacks by construction of $((\mathcal{C}_{\perp/})_{f_\bot/})_0$ and $(\mathcal{C}_{\perp/})_{f_\bot/}$, Therefore, the right outer rectangle is a pullback. This implies that $((\mathcal{C}_{y/})_{f_\bot/})_0$ and $((\mathcal{C}_{\perp/})_{f_\bot/})_0$ are pullbacks over the same diagram. Therefore, $((\mathcal{C}_{y/})_{f_\bot/})_0 \simeq ((\mathcal{C}_{\perp/})_{f_\bot/})_0$. By analogous arguments for each $n \geq 0$, the equivalence $((\mathcal{C}_{y/})_{f_\bot/})_n \simeq ((\mathcal{C}_{\perp/})_{f_\bot/})_n$ implies that $(\mathcal{C}_{y/})_{f_\bot/} \simeq (\mathcal{C}_{\perp/})_{f_\bot/}$. Furthermore, in the diagram
the square (5) is a pullback by construction of \(((\mathcal{C}_\perp/)_0/f_\perp)_0\) and the square (6) is a pullback since \(d_\perp \circ u : \mathcal{C}_\perp/ \to \mathcal{C}\) is an equivalence. This implies that the outer rectangle is a pullback. Since \(((\mathcal{C}_\perp/)_0/f_\perp)_0\) and \((\mathcal{C}/y)_0\) are pullbacks over the same diagram, we have that \(((\mathcal{C}_\perp/)_0/f_\perp)_0 \simeq (\mathcal{C}/y)_0\). Therefore, \(((\mathcal{C}_\perp/)_0/f_\perp)_n \simeq (\mathcal{C}/y)_n\).

Lemma 2.13. Let \(\mathcal{C}\) be a decomposition groupoid with a terminal object. Then for each object \(x\) in \(\mathcal{C}\), the coslice \(\mathcal{C}_x/\) has a terminal object.

Proof. The proof is analogous to Lemma 2.12.

3 The interval construction

In this section we continue to work at the level of simplicial groupoids, but the results in this section will actually only be applied in the case of simplicial sets in Section 5.

Let \(X\) be a decomposition groupoid. Applying lower and upper decalage to \(X\), we obtain a new decomposition groupoid \(\text{Dec}_\top \text{Dec}_\perp X\) and a map \(\epsilon : \text{Dec}_\top \text{Dec}_\perp X \to X\) which is CULF by Proposition 1.16.

Definition 3.1. Let \(X\) be a decomposition groupoid and let \(f\) be an object in \(X_1\). The Segal groupoid \(I_f\) is defined as the pullback of the upper-lower dec \(\text{Dec}_\top \text{Dec}_\perp X\) along \(\llcorner f\lrcorner : 1 \to X_1:\)

\[
\begin{array}{cccccc}
0 & \xleftarrow{d_1} & 1 & \xrightarrow{d_0} & 2 & \xrightarrow{d_4} & X \\
\downarrow{d_0} & & \downarrow{d_0} & & \downarrow{d_0} & & \uparrow{\epsilon} \\
X_0 & \xrightarrow{s_0} & X_1 & \xrightarrow{s_1} & X_2 & \xrightarrow{s_2} & X_3 & \xrightarrow{s_3} & X_4 \\
\downarrow{d_0} & & \downarrow{d_0} & & \downarrow{d_0} & & \downarrow{d_0} & & \downarrow{d_0} \\
X_1 & \xrightarrow{d_1} & X_2 & \xrightarrow{d_1} & X_3 & \xrightarrow{d_1} & X_4 & \xrightarrow{d_1} & \text{Dec}_\top \text{Dec}_\perp X \\
\uparrow{\llcorner f\lrcorner} & & \uparrow{\llcorner f\lrcorner} & & \uparrow{\llcorner f\lrcorner} & & \uparrow{\llcorner f\lrcorner} & & \uparrow{\llcorner f\lrcorner} \\
1 & \xleftarrow{w_0} & (I_f)_0 & \xrightarrow{w_1} & (I_f)_1 & \xrightarrow{w_2} & (I_f)_2 & \xrightarrow{w_3} & I_f \\
\end{array}
\]

We write \(w : I_f \to \text{Dec}_\top \text{Dec}_\perp X\) for the simplicial map obtained in this way. From its construction as a pullback, it is clear that \(w\) is CULF. Furthermore, applying lower and upper decalage to \(X\) gives two sections on \(\text{Dec}_\top \text{Dec}_\perp X\). For \(n \geq 0\), the first one is induced by the map \(s_\perp : X_n \to X_{n+1}\). Using the split induced by \(s_\perp\), we deduce that \(s_\perp(f)\) is an initial object in \(I_f\). By analogous arguments, we deduce that \(s_\top(f)\) is a terminal object in \(I_f\). These affirmations will be proven later.

Remark 3.2. When \(X\) is a decomposition groupoid, we have that \(\text{Dec}_\top \text{Dec}_\perp X\) is a Segal groupoid. By construction we have that \(I_f\) is cartesian over \(\text{Dec}_\top \text{Dec}_\perp X\), and therefore satisfies the Segal condition by Lemma 1.7.
Example 3.3. Dür [8] gave an incidence-coalgebra construction of the Butcher–Connes–Kreimer coalgebra by starting with the category of forests and root-preserving inclusions, generating a coalgebra and imposing the equivalence relation that identifies two root-preserving forest inclusions if their complement crowns are isomorphic forests.

Consider the discrete decomposition groupoid of rooted plane trees $\text{RPT}_1$ of Example 1.5. We can consider a tree $T$ as an object in $\text{RPT}_1$. The interval $I_T$ can be described as follows: $(I_T)_0$ is the set of all isoclasses of compatible admissible cuts over $T$, and $(I_T)_k$ is the set of isoclasses of all $k + 1$ compatible admissible cuts over $T$.

We can relate the construction of Dür with the interval construction of a rooted plane tree as follows: note that admissible cuts are essentially the same thing as root-preserving forest inclusions: then the cut is interpreted as the division between the included forests and the forest induced on the nodes in its complement. In this way we see that $(I_T)_k$ is the discrete groupoid of $k + 1$ consecutive root-preserving inclusions ending in $T$.

When $X$ is the ordinary nerve of a category, the construction is due to Lawvere [28]: the objects of $I_f$ are two-step factorisations of $f$. The 1-cells are arrows between such factorisations, or equivalently 3-step factorisations, and so on.

Remark 3.4. Let $X$ be a decomposition set and $f \in X_1$. The Segal set $I_f$ is described as follows:

1. An object of $I_f$ is any $\sigma \in X_2$ such that $d_1(\sigma) = f$.
2. Given two objects $\sigma$ and $\overline{\sigma}$ in $I_f$, a morphism $\gamma : \sigma \to \overline{\sigma}$ in $I_f$ is any object $\gamma \in X_3$, such that $d_2(\gamma) = \sigma$ and $d_1(\gamma) = \overline{\sigma}$.
3. Given two morphisms $\gamma : \sigma \to \sigma'$ and $\overline{\gamma} : \sigma' \to \overline{\sigma}$ of $I_f$, the composition is defined by $\overline{\gamma} \circ \gamma := d_2(\eta)$, where $\eta \in X_4$ satisfies that $d_1(\eta) = \overline{\gamma}$ and $d_3(\eta) = \gamma$. The unique existence of $\eta$ is a consequence of the decomposition-groupoid axioms in the form of Lemma 1.8. Associativity also follows by Lemma 1.8.

The double decalage construction induces a CULF map $M_f : I_f \to X$, defined by the composition of $w$ and $\epsilon$ in Definition 3.1. We write $u : X/y \to \text{Dec}_\top X$ for the canonical map of Definition 2.4 and $v : X/x/ \to \text{Dec}_\bot X$ for the canonical map of Definition 2.9. When further (co)slicing is used we decorate the $u$ or $v$ with a prime.

Lemma 3.5. Let $X$ be a decomposition groupoid. For each $x, y$ in $X_0$ and $f$ in $X_1$ such that $d_0(f) = y$ and $d_1(f) = x$, there are canonical equivalences $(X/x)/f \to I_f$ and $(X/y)/f \to I_f$ such that the following diagram commutes

\[
\begin{array}{ccc}
(X/x)/f & \xrightarrow{\sim} & I_f \\
\downarrow_{d_\top \circ u} & & \downarrow_{M_f} \\
X_{x/} & \xleftarrow{\sim} & X_{y/}
\end{array}
\]

\[
\begin{array}{ccc}
(X/y)/f & \xrightarrow{\sim} & I_f \\
\downarrow_{d_\bot \circ v} & & \downarrow_{d_\bot \circ v'} \\
X_{x/} & \xleftarrow{\sim} & X_{y/}
\end{array}
\]

Proof. In the diagram

\[
\begin{array}{ccc}
1 & \xleftarrow{(1)} & ((X/x)/f)_0 \\
\downarrow_{\alpha_0} & & \downarrow_{\beta_0} \\
(X/x)_0 & \xleftarrow{\alpha_0} & (X/x)_1 \\
\downarrow_{\beta_1} & & \downarrow_{\alpha_1} \\
X_1 & \xleftarrow{\beta_1} & X_2
\end{array}
\]

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the squares (1) and (2) are pullbacks by construction of \(((X_x/)/f)_0\) and \((X_x/)_1\). Therefore, 
\(((X_x/)/f)_0\) is the pullback of \(d_1 : X_2 \to X_1\) along \(\cap f : 1 \to X_1\). But this is precisely
the pullback characterisation of \((I_f)_0\). This implies that 
\(((X_x/)/f)_0 \simeq (I_f)_0\). By analogous arguments for each \(n \geq 0\), we have that
\(((X_x/)/f)_n \simeq (I_f)_n\), hence altogether \(((X_x/)/f) \simeq I_f\). Therefore, the left square of the lemma commutes. The right side of the square of the lemma
commutes by analogous arguments. \(\square\)

When \(X\) is the ordinary nerve of a category, Lemma 3.6 is the same as Lemma 3.2 in [28].

**Lemma 3.6.** Let \(X\) be a decomposition groupoid with an initial object \(\perp\) and a terminal
object \(\top\). Let \(h : \perp \to \top\) be a map from \(\perp\) to \(\top\), then \(X \simeq I_h\).

**Proof.** We will prove that \((X_\perp)/h \simeq I_h\) and \((X_\perp)_0 \simeq X\). In the diagram

\[
\begin{array}{ccc}
1 & \xleftarrow{} & ((X_\perp)/h)_0 \\
\downarrow h & & \downarrow s_0 \\
(X_\perp)_0 & \xleftarrow{} & (X_\perp)_1 \\
\downarrow v_0 & & \downarrow v_1 \\
X_1 & \xleftarrow{} & X_2 \\
\downarrow d_1 & & \downarrow d_1 \\
\end{array}
\]

the squares (1) and (2) are pullbacks by construction of \(((X_\perp)/h)_0\) and \((X_\perp)_1\). This implies that 
\(((X_\perp)/h)_0\) is the pullback of \(d_1 : X_2 \to X_1\) along \(h : 1 \to X_1\). Since \(((X_\perp)/h)_0\) and \((I_h)_0\) are pullbacks over the same diagram, we have that \(((X_\perp)/h)_0 \simeq (I_h)_0\). By analogous arguments for each \(n \geq 0\), we have that \(((X_\perp)/h)_n \simeq (I_h)_n\). Therefore, \((X_\perp)/h \simeq I_h\).

In the proof of Lemma 2.12, we showed that when \(X\) is a decomposition groupoid with an initial object, for each \(y \in X\), we have that \(X/y \simeq (X_\perp)//f\). Here \(f\perp\) is the unique map from \(\perp\) to \(y\). When \(y = \top\), we have that \(X/\top \simeq (X_\perp)/h\). Furthermore, \(X/\top \simeq X\) since \(\top\) is a terminal object. Combining these equivalences, we have that \(X \simeq I_h\). \(\square\)

When \(X\) is the ordinary nerve of a category, Lemma 3.6 is the same as Lemma 3.3 in [28].

**Proposition 3.7.** Let \(X\) be a complete decomposition groupoid. Then for each \(f \in X_1\), the
Segal groupoid \(I_f\) is complete in the sense of decomposition groupoids.

**Proof.** By construction of \(I_f\), we have the following diagram

\[
\begin{array}{ccc}
(I_f)_0 & \xrightarrow{s_0} & (I_f)_1 \\
w_0 & \downarrow & \downarrow d_0 \\
X_1 & \xrightarrow{s_1} & X_2 \\
\downarrow s_1 & & \downarrow d_1 \\
& & X_1. \\
\end{array}
\]

Applying the prism lemma, the left square is a pullback and since \(s_1 : X_1 \to X_2\) is a
monomorphism, the morphism \(s_0 : (I_f)_0 \to (I_f)_1\) is a monomorphism. Therefore, \(I_f\) is complete. \(\square\)

The intervals are main objects of this work. In order to keep everything as strict as possible, we need to fix their initial and terminal objects:

**Definition 3.8.** Let \(X\) be an upper 2-Segal groupoid. A pair \((b, s)\) is called chosen terminal
if \(b\) is a terminal object in \(X\) and \(s : X \to X/b\) is a strict section for the canonical map \(d_{\top u} : X/b \to X\).
Example 3.9. For an upper 2-Segal groupoid $X$ and for any $y \in X_0$, we already know from Proposition 2.7 that $s_0(y)$ is a terminal object in $X_{/y}$. Furthermore, the section $d_\top : X \to \Dec\top X$ induces a section $t : X \to X_{/y}$. In this way $X_{/y}$ acquires a canonical chosen terminal.

Definition 3.10. Let $X$ be a lower 2-Segal groupoid. A pair $(a, r)$ is called chosen initial if $a$ is an initial object in $X$ and $r : X \to X_{a/}$ is a strict section for the canonical map $d_1 v : X_{a/} \to X$.

Example 3.11. For a lower 2-Segal groupoid $X$ and for any $x \in X_0$, we already know from Proposition 2.11 that $s_0(x)$ is an initial object in $X_{x/}$. Furthermore, the section $d_\bot : X \to \Dec\bot X$ induces a section $p : X \to X_{x/}$. In this way $X_{x/}$ acquires a canonical chosen initial.

Definition 3.12. An interval is a Segal groupoid $\mathcal{C}$ with a chosen initial object $(\bot, p)$ and a chosen terminal object $(\top, t)$ such that the morphism $r : \bot \to \top$ induces by the section map $p$ and the morphism $r' : \bot \to \top$ induces by the section map $t$ are equal. We always denote this map $\varpi : \bot \to \top$.

Remark 3.13. Batanin and Markl [1] used the notion of a category with a chosen local terminal objects, meaning a category which in each connected component is provided with a chosen terminal object. This notion plays an important role in their theory of operadic categories. Garner, Kock and Weber [13] observed that the structure of chosen local terminal objects is precisely to be a coalgebra for the upper-Dec comonad. This in turn amounts to having an extra top degeneracy map for the nerve of the category. When we insist on having the strict section $s$, it is inspired by this decalage viewpoint on chosen terminals. Indeed, decalage plays a crucial role in our theory, and provides such sections. Similarly of course, the notion of chosen local initial object amounts to coalgebra structure for the lower-Dec comonad, via extra bottom degeneracy maps, as the strict section $r$ in our definition. Finally, the main point here is of course the combination of the two ideas. An interval structure is in particular a coalgebra for the two-sided-Dec comonad. This is very much in line with the notion of flanking of Gálvez–Kock–Tonks [17]. We note finally that the insistence on the coincidence of the two choices of map $\bot \to \top$ is natural from the viewpoint of the two-sided decalage of a simplicial object. It is in this case the simplicial identity $s_\top s_\bot = s_\bot s_\top$.

When $X$ is an upper 2-Segal groupoid and $f \in X_1$. It is straightforward to see that $(s_1(f), s_{s_\top})$ is a chosen terminal object in $I_f$. On the other hand, when $X$ is a lower 2-Segal groupoid. The pair $(s_0(f), s_{s_\bot})$ is a chosen initial object in $I_f$. A decomposition groupoid is a lower and an upper 2-Segal groupoid. Therefore, the Segal groupoid $I_f$ has a canonical chosen terminal object $(s_1(f), s_{s_\top})$ and a canonical chosen initial object $(s_0(f), s_{s_\bot})$. Furthermore, the double decalage construction of $X$ helps us to deduce that the morphism $r : s_0(f) \to s_1(f)$ induces by the section map $s_{s_\bot}$ and the morphism $r' : s_0(f) \to s_1(f)$ induces by the section map $s_{s_\top}$ are equal. Hence altogether, we have the following result.

Lemma 3.14. Let $X$ be a decomposition groupoid and $f \in X_1$. The Segal groupoid $I_f$ has a canonical structure of interval.

Definition 3.15. A simplicial map between intervals is termed stretched, and written $S : \mathcal{C} \to \mathcal{D}$, if it preserves the chosen initial object $(\bot_C, p)$ and the chosen terminal object $(\top_C, t)$, and the map $\varpi_C : \bot_C \to \top_C$.

Example 3.16. When $\mathcal{C}$ is an interval, the map $M_{\varpi_C} : I_{\varpi_C} \to \mathcal{C}$ is stretched.
For $\lambda : \Delta^n \to X$, we denote by $\text{long}(\lambda)$ the 1-simplex $\Delta^1 \dashv \Delta^n \overset{\lambda}{\to} X$.

**Proposition 3.17.** Let $X$ be a decomposition groupoid. Given $\lambda \in X_n$ and $f = \text{long}(\lambda)$, there exists a unique $\phi : \Delta^n \to I_f$ such that the diagram

$$
\begin{array}{c}
\Delta^n \\
\phi \downarrow \\
I_f
\end{array}
\xymatrix{
\Delta^n \\
\phi \ar[r]^\lambda \\
X \\
\ar[r]_{M_f} \\
X
}
$$

commutes.

**Proof.** Since $f = \text{long}(\lambda)$, the pullback property of $(I_f)_{n-2}$ gives a unique map $\gamma : 1 \to (I_f)_{n-2}$ such that the diagram

$$
\begin{array}{c}
1 \\
\xymatrix{
\ar[r]^\phi \\
(I_f)_{n-2} \\
\ar[r]_{\gamma} \\
X_n \\
\ar[r]^\ell \\
X_1.
}
\end{array}
$$

commutes. For each $n \geq 2$, the diagram

$$
\begin{array}{c}
(I_f)_{n-2} \\
\xymatrix{
\ar[r]^{\gamma} \\
X_n \\
\ar[r]_{\ell} \\
X_1.
}
\end{array}
$$

commutes. Therefore, $w_{n-2} = (M_f)_n \circ s \circ \perp$. In the diagram

$$
\begin{array}{c}
(I_f)_{n-2} \\
\xymatrix{
\ar[r]^{s \circ \perp} \\
Fib_{f}(\ell \circ (M_f)_n) \\
\ar[r]_{\perp} \\
(I_f)_n \\
\ar[r]_{M_f} \\
X_n \\
\ar[r]^\ell \\
X_1.
}
\end{array}
$$

the pullback property of $Fib_f(\ell \circ (M_f)_n)$ gives a unique map $\phi$ such that the diagram commutes. The groupoid $Fib_{\perp}(\ell \circ (M_f)_n)$ is defined by the pullback of $Fib_f(\ell \circ (M_f)_n)$ along $\perp : 1 \to (I_f)_{n-2}$. Since $(M_f)_n \perp \lambda = \lambda$, the pullback property of $Fib_{\perp}(\ell \circ (M_f)_n)$ gives a unique map $\phi : 1 \to Fib_{\perp}(\ell \circ (M_f)_n)$ such that the following diagram commutes.
With \( \phi := r \circ r' \circ \phi \), it is straightforward to check that \( \phi \) satisfies \( (M_f)_n \phi = \lambda \). The uniqueness of \( \phi \) follows from the uniqueness of \( \phi \). The \( n \)-simplex \( \phi \) is determined by \( \lambda \), so we will denote it by \( \phi_\lambda \).

**Remark 3.18.** The uniqueness of \( \phi \) guaranteed in the proof 3.17 is only up to unique isomorphism (as it depends on universal properties of pullbacks of groupoids). With a bit more care, it can be refined to a strict uniqueness. This requires using the explicit model of groupoid pullback given by the comma category, and exploiting the stricter universal property of this model. We prefer to avoid this subtlety here. In any case, our main application of this proposition is in the case where \( X \) is a discrete decomposition groupoid, and in this case the uniqueness is strict in any case.

**Lemma 3.19.** Let \( \mathcal{C} \) and \( \mathcal{D} \) be two decomposition groupoids. Let \( F : \mathcal{C} \to \mathcal{D} \) be a CULF map. Let \( \lambda : \Delta^n \to \mathcal{C} \) and \( \mu : \Delta^n \to \mathcal{D} \) be two \( n \)-simplices of \( \mathcal{C} \) and \( \mathcal{D} \) such that \( F \lambda = \mu \). For \( 0 \leq i \leq n \), the map \( p : \Delta^m \to \Delta^n \) induces a unique equivalence \( \overline{F} : I_f \to I_g \) with \( f = \text{long}(\lambda p) \) and \( g = \text{long}(\mu p) \) such that the diagram commutes

\[
\begin{array}{ccc}
\Delta^m & \xrightarrow{\phi_{\lambda p}} & \Delta^n \\
I_f & \xrightarrow{\overline{F}} & I_g \\
\end{array}
\]

where \( \phi_{\lambda p} \) and \( \phi_{\mu p} \) are the \((n - 1)\)-simplices induce by \( \lambda p \) and \( \mu p \).

**Proof.** In the diagram

\[
\begin{array}{ccc}
(I_f)_0 & \xrightarrow{w_0} & 1 \\
\downarrow & & \downarrow \overline{r f} \\
\mathcal{C}_2 & \xrightarrow{d_1} & \mathcal{C}_1 \\
F_2 & \downarrow \overline{F_1} & \downarrow F_1 \\
\mathcal{D}_2 & \xrightarrow{d_1} & \mathcal{D}_1 \\
\end{array}
\]

the square (1) is a pullback by construction of \( I_f \) and the square (2) is a pullback since \( F \) is CULF. By hypothesis \( F \lambda = \mu \). This implies that \( \text{long}(F \lambda p) = \text{long}(\mu p) \), in other words \( Ff = g \). Therefore, \((I_f)_0\) is the pullback of \( d_1 : \mathcal{D}_2 \to \mathcal{D}_1 \) along \( \overline{g} \). By the pullback property of \((I_f)_0\), we have that \((I_f)_0 \simeq (I_g)_0\). By analogous arguments, for each \( n \geq 0 \) we have that \((I_f)_n \simeq (I_g)_n\). Hence altogether, we have an equivalence \( \overline{F} : I_f \to I_g \). For \( n \geq 0 \), in the diagram

\[
\begin{array}{ccc}
(I_f)_n & \rightarrow & \text{Fib}_\lambda((M_f)_n) \\
\downarrow & \downarrow & \downarrow \\
(I_f)_{n-2} & \xrightarrow{r} & \text{Fib}_\ell((M_f)_n) \\
\downarrow & \downarrow & \downarrow \\
(I_f)_n & \xrightarrow{(M_f)_n} & X_n \\
\end{array}
\]
the square (3) commutes by construction of $\overline{F}$ and (4) commutes since $F$ is a simplicial map. Combining (3) and (4), the square

\begin{equation}
\begin{array}{ccc}
(I_f)_n & \overline{F}_n & (I_g)_n \\
\downarrow w_n & \downarrow & \downarrow w'_n \\
\mathcal{C}_{n+2} & \xrightarrow{F_{n+2}} & \mathcal{D}_{n+2}
\end{array}
\end{equation}

commutes. Since (3) and (4) commute, and $Ff = g$, it follows that $\overline{F}$ is stretched. Furthermore, in the diagram

\begin{equation}
\begin{array}{ccc}
I_f & \xrightarrow{\overline{F}} & I_g \\
\downarrow M_f & & \downarrow M_g \\
\mathcal{C} & \xrightarrow{F} & \mathcal{D}
\end{array}
\end{equation}

the triangle (6) commutes by Proposition 3.17. Since the squares (5) and (6) commute, and $F\lambda p = \mu p$, it follows that the diagram

\begin{equation}
\begin{array}{ccc}
\Delta^{n-1} & \xrightarrow{\overline{F}_{\phi_{\mu d^i}}} & I_g \\
\downarrow \lambda d^i & & \downarrow M_g \\
\mathcal{C} & \xrightarrow{F} & \mathcal{D}
\end{array}
\end{equation}

commutes but $\phi_{\mu d^i}$ also satisfies that $M_g \phi_{\mu d^i} = \mu d^i$. Therefore, $\overline{F}\phi_{\mu d^i} \simeq \phi_{\mu d^i}$ by Proposition 3.17.

When we consider decomposition sets, the equivalence $\overline{F}$ of Lemma 3.19 is an isomorphism.

\textbf{Lemma 3.20.} Let $\mathcal{C}$ be an interval. Let $\lambda : \Delta^n \rightarrow \mathcal{C}$ be an $n$-simplex in $\mathcal{C}$. There exists a unique $(n + 2)$-simplex $\eta : \Delta^{n+2} \rightarrow \mathcal{C}$ such that $d_\perp d_\top(\eta) \simeq \lambda$ and $\text{long}(\eta) = \varpi_c$.

\textbf{Proof.} Consider $a = d_\top(\text{long}(\lambda))$ and $b = d_\perp(\text{long}(\lambda))$. Since $\mathcal{C}$ is an interval, we have a chosen edge $f_\top : \Delta^1 \rightarrow \mathcal{C}$ such that $d_\top(f_\top) = \perp_c$ and $d_\perp(f_\top) = a$. By the same argument, we have a chosen edge $f_\tau : \Delta^1 \rightarrow \mathcal{C}$ such that $d_\perp(f_\tau) = \tau_c$ and $d_\top(f_\tau) = b$. In the diagram
the squares (1) and (2) are pullbacks since $\mathcal{C}$ is a Segal groupoid. Therefore, the outer rectangle is a pullback. By the pullback property of $\mathcal{C}_{n+1}$, there exists a unique map $\mu : 1 \to \mathcal{C}_{n+1}$ such that the diagram commutes. Since the square (1) commutes and $d_\perp \mu \simeq \lambda$, we have that $d_\perp \text{long}(\mu) \simeq b$. Furthermore, in the diagram

the squares (3) and (4) are pullbacks since $\mathcal{C}$ is a Segal groupoid. Therefore, the outer rectangle is a pullback. By the pullback property of $\mathcal{C}_{n+2}$, there exists a unique map $\eta : 1 \to \mathcal{C}_{n+2}$ such that the diagram commutes. It is straightforward to check that $d_\perp d_\top \eta \simeq \lambda$. Note that $d_\perp (\text{long}(\eta)) = \top_\mathcal{C}$ and $d_\top (\text{long}(\eta)) = \perp_\mathcal{C}$. It follows that $\text{long}(\eta) = \omega_\mathcal{C}$ since $\mathcal{C}$ is an interval.

The following lemmas will be used in Section 5.

**Lemma 3.21.** Let $\mathcal{C}$ and $\mathcal{D}$ be two decomposition groupoids. Let $F : \mathcal{C} \to \mathcal{D}$ be a CULF map and let $f$ be a map in $\mathcal{C}$. Then there is induced an equivalence $I_{Ff} \simeq I_f$ such that the diagram

\[
\begin{array}{ccc}
I_f & \xrightarrow{M_f} & \mathcal{C} \\
\simeq & & \\
I_{Ff} & \xrightarrow{M_{Ff}} & \mathcal{D}
\end{array}
\]

commutes.

**Proof.** In the diagram
the square (1) is a pullback by construction of $I_f$, and (2) is a pullback since $F$ is CULF. Combining (1) and (2), we have that $(I_f)_0$ is the pullback of $d_1 : \mathcal{D}_2 \to \mathcal{D}_1$ along $\gamma F f^{-} : 1 \to \mathcal{D}_1$. But this is precisely the pullback characterisation of $I_{F f}$. Since $(I_f)_0$ and $(I_{F f})_0$ are pullbacks over the same diagram, it follows that $(I_{F f})_0 \simeq (I_f)_0$ and the triangle (3) commutes. Furthermore, since $F$ is a simplicial map, the right square

\[
\begin{array}{cccc}
(I_f)_0 & \rightarrow & \mathcal{C}_2 & \rightarrow \mathcal{C}_0 \\
\downarrow^{(3)} w_0 & & \downarrow^{(4)} F_0 \ \\
(I_{F f})_0 & \rightarrow & \mathcal{D}_2 & \rightarrow \mathcal{D}_0 \\
\end{array}
\]

commutes. By analogous arguments for each $n \geq 0$, we have that $(I_{F f})_n \simeq (I_f)_n$, hence altogether $I_{F f} \simeq I_f$. Remember that $M_f = \epsilon \circ w$ and $M_{F f} = \epsilon' \circ w'$. Combining (3) and (4), it is straightforward to see that the diagram

\[
\begin{array}{ccc}
I_f & \rightarrow & \mathcal{C} \\
\downarrow^{I_{F f}} & & \downarrow^F \\
\mathcal{D} & \rightarrow & \mathcal{D} \\
\end{array}
\]

commutes.

\[\square\]

4 Stretched-CULF factorisation system

In this section, we will work with simplicial sets rather than simplicial groupoids. The word interval thus means 1-Segal sets with a chosen initial object and a chosen terminal object. For an alternative approach in the context of Segal spaces, see Gálvez-Carrillo, Kock, and Tonks ([17], §1).

A factorisation system in a category $\mathcal{D}$ consists of two classes $E$ and $F$ of maps, that we shall depict as $\rightarrow$ and $\twoheadrightarrow$, such that

1. The class $F$ is closed under equivalences.

2. The classes $E$ and $F$ are orthogonal, $E \perp F$. That is, given $e \in E$ and $f \in F$, for every solid square

\[
\begin{array}{ccc}
e & \rightarrow & f \\
\downarrow & & \downarrow \\
\end{array}
\]

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\[
\begin{array}{ccc}
e & \rightarrow & f \\
\downarrow & & \downarrow \\
\end{array}
\]
the set of fillers is contractible.

3. Every map $h$ admits a factorisation

\[
\begin{array}{c}
h \\
\downarrow e \\
f
\end{array}
\]

with $e \in E$ and $f \in F$.

**Remark 4.1.** The classical notion of orthogonal factorisation system requires that $E$ be closed under equivalence. In our case it is not required. In case $E$ is not closed under equivalence we can always saturate it.

Let $\text{Ar}^E(\mathcal{D}) \subset \text{Ar}(\mathcal{D})$ denote the full subcategory spanned by the arrows in the left-hand class $E$.

**Lemma 4.2** ([17], Lemma 1.3). The domain projection $\text{Ar}^E(\mathcal{D}) \to \mathcal{D}$ is a cartesian fibration. The cartesian arrows in $\text{Ar}^E(\mathcal{D})$ are given by squares of the form

\[
\begin{array}{c}
. \\
| \\
| \\
. \\
\end{array}
\]

Let $\text{Int}$ be the category whose objects are intervals and whose morphisms are functors. We need some preliminary results to prove that the stretched functors as left-hand class and the CULF functors as right-hand class form a factorisation system in $\text{Int}$.

**Lemma 4.3.** Let $\mathcal{A}$, $\mathcal{B}$ and $\mathcal{C}$ be three intervals. Let $S : \mathcal{A} \to \mathcal{B}$ be a stretched map. Let $F : \mathcal{A} \to \mathcal{C}$ and $G : \mathcal{B} \to \mathcal{C}$ be two simplicial maps such that $G \circ S = F$. Then $F$ is stretched if and only if $G$ is stretched.

**Proof.** Let $\varpi_\mathcal{A} : \bot_\mathcal{A} \to \top_\mathcal{A}$ be the chosen map from $\bot_\mathcal{A}$ and $\top_\mathcal{A}$. Consider $\varpi_\mathcal{B} : \bot_\mathcal{B} \to \top_\mathcal{B}$ and $\varpi_\mathcal{C} : \bot_\mathcal{C} \to \top_\mathcal{C}$ the chosen maps in $\mathcal{B}$ and $\mathcal{C}$. It straightforward to check that $F$ is stretched when $G$ is stretched since $F = G \circ S$. For the other direction, consider $F$ stretched, we have that

\[
G(\varpi_\mathcal{B}) = G(S(\varpi_\mathcal{A})) = F(\varpi_\mathcal{A}) = \varpi_\mathcal{C}.
\]

This means that $G$ is stretched.

We already know from Lemma 3.6 that $\mathcal{C}$ is equivalent to $I_{\varpi_\mathcal{C}}$ but it does not guarantee the preservation of the chosen initial object and the chosen terminal object. However, the map $M$ is a special case of Lemma 3.6 which preserves the interval structure by the following result.

**Lemma 4.4.** Let $\mathcal{C}$ be an interval. The simplicial map $M_{\varpi_\mathcal{C}} : I_{\varpi_\mathcal{C}} \to \mathcal{C}$ has an inverse $W : \mathcal{C} \to I_{\varpi_\mathcal{C}}$.  

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Proof. By Lemma 3.20, for each \( n \)-simplex \( \lambda : \Delta^n \to \mathcal{C} \), there exists a unique \((n+2)\)-simplex \( \eta_{\lambda} : \Delta^{n+2} \to \mathcal{C} \) such that

\[
d_{-1}d_+ (\eta_{\lambda}) = \lambda \tag{1}
\]

and

\[
\text{long}(\eta) = \varpi_c \tag{2}
\]

On the other hand, the outer diagram

\[
\begin{array}{ccc}
1 & \xrightarrow{\eta_{\lambda}} & \mathcal{I}_c^n \\
\downarrow & \searrow \downarrow \mathcal{I}_c^n & \downarrow \mathcal{I}_c^n \\
\mathcal{C}_{n+2} & \rightarrow & \mathcal{C}_1
\end{array}
\]

commutes by Eq. (2), so the pullback property of \( (\mathcal{I}_c^n)_n \) gives us a unique map \( \eta_{\lambda} : 1 \to (\mathcal{I}_c^n)_n \) such that the diagram commutes. This gives a simplicial map \( W : \mathcal{C} \to \mathcal{I}_c^n \) that is defined for each \( n \)-simplex \( \lambda : \Delta^n \to \mathcal{C} \) as \( W(\lambda) = \eta_{\lambda} \). For each \( n \)-simplex \( \lambda : \Delta^n \to \mathcal{C} \), the following diagram commutes by (3), Eq. (1) and the definition of \( \mathcal{M}_{\varpi_c} \).

\[
\begin{array}{ccc}
\Delta^n & \xrightarrow{W(\lambda)} & \mathcal{I}_c^n \\
\downarrow \mathcal{M}_{\varpi_c} & \searrow \downarrow \mathcal{M}_{\varpi_c} & \downarrow \mathcal{M}_{\varpi_c} \\
\mathcal{C} & \xrightarrow{\mathcal{M}_{\varpi_c}} & \mathcal{C}
\end{array}
\]

This means that \( \mathcal{M}_{\varpi_c} \circ W = \text{id}_\mathcal{C} \). Consider \( \mu : \Delta^n \to \mathcal{I}_c^n \). Since \( \mathcal{M}_{\varpi_c} \circ W = \text{id}_\mathcal{C} \), we have that the diagram

\[
\begin{array}{ccc}
\Delta^n & \xrightarrow{W_{\varpi_c} \mu} & \mathcal{I}_c^n \\
\downarrow \mathcal{M}_{\varpi_c} & \searrow \downarrow \mathcal{M}_{\varpi_c} & \downarrow \mathcal{M}_{\varpi_c} \\
\mathcal{C} & \xrightarrow{\mathcal{M}_{\varpi_c}} & \mathcal{C}
\end{array}
\]

commutes but \( \mu \) also satisfies this commutative condition. Therefore, we have that \( W \circ \mathcal{M}_{\varpi_c}(\mu) = \mu \) by Proposition 3.17. Hence, \( W \circ \mathcal{M}_{\varpi_c} = \text{id}_{\mathcal{I}_c^n} \).

Lemma 4.5. Let \( \mathcal{A}, \mathcal{C} \) be two intervals. Let \( \mathcal{D} \) be a discrete decomposition groupoid. Let \( F : \mathcal{C} \to \mathcal{D} \) be a CULF functor. Let \( W, V : \mathcal{A} \to \mathcal{C} \) be two functors such that \( FW = FV \). Then \( W = V \).

Proof. Let \( \lambda \) be an \( n \)-simplex in \( \mathcal{A} \). By Lemma 3.20, there exists a unique \((n+2)\)-simplex \( \eta_{W\lambda} : \Delta^{n+2} \to \mathcal{C} \) such that \( \text{long}(\eta_{W\lambda}) = \varpi_c \) and \( d_{-1}d_+ (\eta_{W\lambda}) = W\lambda \). Furthermore, by the same argument, there exists a unique \((n+2)\)-simplex \( \eta_{V\lambda} : \Delta^{n+2} \to \mathcal{C} \) such that \( \text{long}(\eta_{V\lambda}) = \varpi_c \) and \( d_{-1}d_+ (\eta_{V\lambda}) = V\lambda \). Since \( \text{long}(F\eta_{W\lambda}) = F\varpi_c \), the outer diagram
commutes. Therefore, the pullback property of \((I_{F \bowtie C})_n\) gives a unique \(n\)-simplex \(F \eta_{V \lambda}: \Delta^n \to I_{F \bowtie C}\), such that the diagram commutes. By analogous arguments, we get \(n\)-simplices \(F \eta_{W \lambda}: \Delta^n \to I_{F \bowtie C}\), \(\eta_{V \lambda}: \Delta^n \to I_{F \bowtie C}\), and \(\eta_{W \lambda}: \Delta^n \to I_{F \bowtie C}\) such that \(w_n(F \eta_{W \lambda}) = F \eta_{W \lambda}\), \(w_n(\eta_{V \lambda}) = \eta_{V \lambda}\) and \(w_n(\eta_{W \lambda}) = \eta_{W \lambda}\). Furthermore, the diagram

\[
\Delta^n \xrightarrow{F \eta_{V \lambda}} \Delta^n \xrightarrow{M_{F \bowtie c}} \Delta^n \xrightarrow{F V \lambda} \mathcal{D}
\]

commutes since

\[
M_{F \bowtie c} F \eta_{V \lambda} = d_1 d_\top w_n F \eta_{V \lambda} = d_1 d_\top F(\eta_{V \lambda}) = F d_1 d_\top (\eta_{V \lambda}) = F V \lambda.
\]

By analogous arguments with \(W\) instead of \(V\), we have the commutative diagram

\[
\Delta^n \xrightarrow{F \eta_{W \lambda}} \Delta^n \xrightarrow{M_{F \bowtie c}} \Delta^n \xrightarrow{F W \lambda} \mathcal{D}.
\]

Since \(F W = F V\), it follows that the lower part of the triangle (3) coincide with the lower part of (2), so by Proposition 3.17 also the horizontal sides coincide, in other words \(F \eta_{W \lambda} = F \eta_{V \lambda}\). By hypothesis \(F\) is CULF, so we have an isomorphism \(K : I_{\bowtie c} \to I_{F \bowtie C}\) such that \(F w = w^K\) by Lemma 3.21. Since \(F w = w^K\), the diagram

\[
\Delta^n \xrightarrow{K \eta_{V \lambda}} \Delta^n \xrightarrow{M_{F \bowtie c}} \Delta^n \xrightarrow{F \eta_{V \lambda}} \mathcal{D}
\]

commutes, but if we substitute \(K \eta_{V \lambda}\) by \(F \eta_{V \lambda}\) the diagram also commutes. Therefore, by the pullback property of \((I_{F \bowtie C})_n\), we have that \(K \eta_{V \lambda} = F \eta_{V \lambda}\). By analogous arguments, we have that \(K \eta_{W \lambda} = F \eta_{W \lambda}\). Hence altogether \(K \eta_{W \lambda} = K \eta_{V \lambda}\), and since \(K\) is an isomorphism, we have that \(\eta_{W \lambda} = \eta_{V \lambda}\). By hypothesis \(\mathcal{C}\) is an interval, so we have that \(I_{\bowtie c} \simeq \mathcal{C}\) by Lemma 4.4. Since \(I_{\bowtie c} \simeq \mathcal{C}\) and \(\eta_{W \lambda} = \eta_{V \lambda}\), it follows that \(\eta_{W \lambda} = \eta_{V \lambda}\). Applying \(d_1 d_\top\), we have that \(W \lambda = V \lambda\). \(\square\)
Lemma 4.6. Let \( \mathcal{C} \) and \( \mathcal{D} \) be two intervals. Let \( F : \mathcal{C} \to \mathcal{D} \) be a functor. Then \( F \) admits a stretched-CULF factorisation.

Proof. By Lemma 4.4, we have an inverse map \( W : \mathcal{C} \to I_{\mathcal{C}} \) of \( M_{\mathcal{C}} : I_{\mathcal{C}} \to \mathcal{C} \). Consider \( n \geq 0 \). For the moment, fix \( n \geq 0 \). In the diagram

\[
\begin{array}{ccc}
\mathcal{C}_n & \xrightarrow{W_n} & (I_{\mathcal{C}})_n \\
\downarrow w_n & & \downarrow\alpha \\
\mathcal{C}_{n+2} & \xrightarrow{\mathcal{C}_{n+2}} & (I_{\mathcal{C}})_n \\
\downarrow& & \downarrow\gamma \\
\mathcal{C}_n & \xrightarrow{\mathcal{C}} & (I_{\mathcal{C}})_n \\
\downarrow& & \downarrow\delta \\
\mathcal{D}_n & \xrightarrow{\mathcal{D}_{n+2}} & (I_{\mathcal{C}})_n \\
\downarrow& & \downarrow\epsilon \\
\mathcal{D}_n & \xrightarrow{\mathcal{D}} & (I_{\mathcal{C}})_n \\
\end{array}
\]

consider first the pullback region. For each \( \eta \in (I_{\mathcal{C}})_n \), we have that \( \ell F_n w_n (\eta) = F \mathcal{C} \), so the pullback property of \( (I_{\mathcal{C}})_n \) gives a unique map \( S_n : (I_{\mathcal{C}})_n \to (I_{\mathcal{C}})_n \) such that the square (1) commutes. The square (2) commutes since \( F \) is a simplicial map. Furthermore, by definition, we have that \( M_{\mathcal{C}} = \epsilon \circ w \) and \( M_{\mathcal{C}} = \epsilon ' \circ w ' \) are CULF maps. Hence altogether, if we put \( S_n := S_n \circ W_n \), the diagram

\[
\begin{array}{ccc}
\mathcal{C}_n & \xrightarrow{F_n} & (D)_n \\
\downarrow S_n & & \downarrow (M_{\mathcal{C}})_n \\
(\mathcal{C})_n & \xrightarrow{(\mathcal{C})_n} & (D)_n \\
\end{array}
\]

commutes. It is easy to check that the maps \( S_n \), for varying \( n \), assemble into a simplicial map \( S : I_{\mathcal{C}} \to I_{\mathcal{C}} \) and altogether, we get a simplicial map \( S : \mathcal{C} \to I_{\mathcal{C}} \) and a factorization \( F = M_{\mathcal{C}} \circ S \).

Remember that the chosen terminal objects in \( I_{\mathcal{C}} \) and \( I_{\mathcal{C}} \) are \( s_1(\mathcal{C}) \) and \( s_1(F \mathcal{C}) \). Since \( S \) is a simplicial map and \( F w = w ' S \), it is straightforward to see that \( S s_1(\mathcal{C}) = s_1(F \mathcal{C}) \). This implies that \( S \) respects the chosen terminal object. By analogous arguments, \( S \) respects the chosen initial object and the chosen map. Therefore, \( S \) is stretched. \( W \) is stretched by Lemma 4.4. Hence altogether, \( S \) is stretched.

Lemma 4.7. Let \( \mathcal{E} \), \( \mathcal{E}' \) and \( \mathcal{C} \) be three intervals. Let \( \mathcal{D} \) be a discrete decomposition groupoid. Let \( S : \mathcal{E} \to \mathcal{E}' \) be a stretched map and let \( F : \mathcal{C} \to \mathcal{D} \) be a CULF map. Let \( G : \mathcal{E} \to \mathcal{C} \) and \( H : \mathcal{E}' \to \mathcal{D} \) be simplicial maps. For the commutative square

\[
\begin{array}{ccc}
\mathcal{E} & \xrightarrow{G} & \mathcal{C} \\
\downarrow S & & \downarrow F \\
\mathcal{E}' & \xrightarrow{H} & \mathcal{D} \\
\end{array}
\]

the category of fillers is contractible.

Proof. To prove that the category of fillers is contractible, we will first construct a map \( L : \mathcal{E}' \to \mathcal{C} \) such that \( L \circ S = G \) and \( F \circ L = H \) and then prove that this map is unique.
We assumed that $S$ is stretched, so $S(\varpi_\xi) = \varpi_\xi'$. For each $n$-simplex $\lambda : \Delta^n \to \mathcal{E}'$, Lemma 3.20 gives an $(n + 2)$-simplex $\eta_\lambda : \Delta^{n+2} \to \mathcal{E}'$ such that

$$\text{long}(\eta_\lambda) = \varpi_\xi' \quad (1)$$

and

$$d_1 d_\top(\eta_\lambda) = \lambda. \quad (2)$$

Since $HS = FG$ and $S$ is stretched, we have that $H(\varpi_\xi') = FG(\varpi_\xi)$. Since $H(\varpi_\xi') = FG(\varpi_\xi)$ and by Eq. (1), it follows that $H\eta_\lambda$ induces a unique map $H\eta_\lambda : \Delta^n \to I_{FG}\varpi_\xi$ such that the diagram

\[
\begin{array}{ccc}
1 & \xrightarrow{id} & 1 \\
\downarrow H(\eta_\lambda) & & \downarrow F\varpi_\xi \\
(3) (I_{FG}\varpi_\xi)_n & \xrightarrow{w'_n} & I_{FG}\varpi_\xi \\
\downarrow D_{n+2} & \xrightarrow{\ell} & D_1
\end{array}
\]

commutes. We define the map $L' : \mathcal{E}' \to I_{FG}\varpi_\xi$ by $L'(\lambda) = \overline{H(\eta_\lambda)}$, for each $n$-simplex $\lambda : \Delta^n \to \mathcal{E}'$. It is straightforward to check that $L'$ is a simplicial map. Since $F$ is CULF, Lemma 3.21 provides an isomorphism $K : I_{FG}\varpi_\xi \to I_G\varpi_\xi$. On the other hand, in the diagram

\[
\begin{array}{ccc}
\mathcal{E} & \xrightarrow{G} & \mathcal{C} \\
\downarrow S & & \downarrow F \\
\mathcal{E}' & \xrightarrow{L'} & \mathcal{D} \\
\downarrow H & & \downarrow M_{FG}\varpi_\xi \\
\mathcal{C} & \xrightarrow{K} & \mathcal{C} \\
\downarrow & & \downarrow \\
\mathcal{D} & \xrightarrow{M_{FG}\varpi_\xi} & \mathcal{D}
\end{array}
\]

the square (4) commutes by Lemma 3.21. The commutativity of (5) is the following calculation:

\[
M_{FG}\varpi_\xi L'(\lambda) = d_1 d_\top u'_n H(\eta_\lambda) \quad \text{(by def. of } M_{FG}\varpi_\xi \text{ and } L')
\]

\[
= d_1 d_\top H(\eta_\lambda) \quad \text{(by triangle (3))}
\]

\[
= Hd_1 d_\top(\eta_\lambda) \quad \text{(since } H \text{ is a sim. map)}
\]

\[
= H(\lambda). \quad \text{(by Eq. (2))}
\]

We define $L : \mathcal{E}' \to \mathcal{C}$ as $L := M_G\varpi_\xi \circ K \circ L'$. Since (4) and (5) commutes, we have that $H = F \circ L$ and $L \circ S = G$.

Suppose $V : \mathcal{E}' \to \mathcal{D}$ is another map such that $H = F \circ V$ and $V \circ S = G$. Since $F$ is CULF and $FV = FL$, we can apply Lemma 4.5 to conclude that $V = L$.

\[\square\]

Remark 4.8. In Lemma 4.7, we have that the diagram
commutes. By hypothesis $F$ is CULF. Therefore, $L$ is a CULF if and only if $H$ is CULF. On the other hand, by hypothesis $S$ is stretched and applying Lemma 4.3, we have that $L$ is a stretched if and only if $G$ is stretched.

Remark 4.9. If we had required $\mathcal{D}$ to be an interval, then Lemma 4.7 would say that the stretched and CULF maps are orthogonal classes of maps in the category of intervals and simplicial maps, as exploited in the the following proposition. It will be important later in 5.2 that we allow $\mathcal{D}$ to be more general than just an interval.

Proposition 4.10. The stretched functors as left-hand class and the CULF functors as right-hand class form a factorisation system in $\text{Int}$.

Proof. The CULF functors are closed under equivalence. We have that every functor $F$ in $\text{Int}$ admits a stretched-CULF factorisation by Lemma 4.6. Therefore, we only have to prove that the classes are orthogonal, which follows from Lemma 4.7.  

5 The decomposition groupoid $\mathcal{U}$

In Section 4, we define a factorisation system in $\text{Int}$ that we can use to define a fibration that encode the pseudo-simplicial groupoid of intervals.

Let $\text{Ar}^s(\text{Int}) \subset \text{Ar}(\text{Int})$ denote the full subcategory spanned by the stretched functors. $\text{Ar}^s(\text{Int})$ is a cartesian fibration over $\text{Int}$ via the domain projection by Lemma 4.2. We now restrict this cartesian fibration to $\Delta \subset \text{Int}$

$$\text{Ar}^s(\text{Int})_{|\Delta} \xrightarrow{f.f.} \text{Ar}^s(\text{Int})$$

$$\Delta \xrightarrow{f.f.} \text{Int.}$$

We put

$$\mathcal{U} := \text{Ar}^s(\text{Int})_{|\Delta}.$$ 

$\mathcal{U} \rightarrow \Delta$ is the cartesian fibration of subdivided intervals. By Lemma 4.2, the cartesian maps in $\mathcal{U}$ are squares

$$\Delta^k \xrightarrow{\text{CULF}} \Delta^n$$

$$\mathcal{E} \xrightarrow{\text{CULF}} \mathcal{D}.$$ 

The cartesian fibration $\mathcal{U} \rightarrow \Delta$ determines a right fibration $\mathcal{U}^{\text{cart}} \rightarrow \Delta$, and hence by straightening ([4], Theorem. 8.3.1) a simplicial groupoid

$$U : \Delta^{op} \rightarrow \text{Grpd}$$

where $\text{Grpd}$ is the 2-category of large groupoids, functors and natural transformations.
Theorem 5.1 ([15], Theorem 4.8). The simplicial groupoid $U : \Delta^{\text{op}} \to \hat{\text{Grpd}}$ is a complete decomposition groupoid.

5.2. The complete decomposition groupoid $U_X$

The decomposition groupoid $U : \Delta^{\text{op}} \to \text{Grpd}$ is not a strict simplicial object but only a pseudo-simplicial object. In a famous paper [21], Jardine figured out all the 2-cell data and 17 coherence laws for pseudo-simplicial objects in terms of face and degeneracy maps. This makes it very hard to work with. We overcome this difficulty by building a local strict model, a kind of neighbourhood $U_X \subset U$ around the intervals of a given discrete decomposition groupoid $X$. Furthermore, we will use the interval construction to define a map $I : X \to U_X$ and prove that it is CULF. For this we need some preliminaries.

Lemma 5.3. Let $X$ be a discrete decomposition groupoid and let $p : [m] \to [n]$ be a morphism in $\Delta$. Given a commutative triangle

$$
\begin{array}{ccc}
\Delta^m & \xrightarrow{p} & \Delta^n \\
\downarrow{\phi_{\lambda'}} & & \downarrow{\phi_{\lambda}} \\
X & \xleftarrow{\lambda} & \\
\end{array}
$$

the standard factorisation (3.17) as in the solid square

$$
\begin{array}{ccc}
\Delta^m & \xrightarrow{p} & \Delta^n \\
\downarrow{\phi_{\lambda'}} & & \downarrow{\phi_{\lambda}} \\
I_f' & \xrightarrow{H} & I_f \\
\downarrow{M_f'} & & \downarrow{M_f} \\
X & \xleftarrow{f} & \\
\end{array}
$$

induces a unique dashed map $H$ as indicated, and $H$ is CULF.

Proof. Since $\phi_{\lambda'}$ is stretched and $M_f$ is CULF, the required map $H$ is given by Lemma 4.7, and it is CULF by Remark 4.8.

Lemma 5.4. Let $X$ be a discrete decomposition groupoid. Let $p : [m] \to [n]$ be a morphism in $\Delta$. Let $\lambda$ and $\mu$ be two $n$-simplices in $X$. Consider $f = \text{long}(\lambda)$ and $g = \text{long}(\mu)$. Let $F : I_f \to I_g$ be an invertible CULF functor such that $F \phi_{\lambda} = \phi_{\mu}$. Then there exists a unique isomorphism $F' : I_{f'} \to I_{g'}$ such that $F' \phi_{\lambda p} = \phi_{\mu p}$ where $f' = \text{long}(\lambda p)$ and $g' = \text{long}(\mu p)$.

Proof. Proposition 3.17 gives the standard factorisations

$$
M_f \phi_{\lambda} = \lambda, \quad M_{f'} \phi_{\lambda p} = \lambda p, \quad M_g \phi_{\mu} = \mu, \quad \text{and} \quad M_{g'} \phi_{\mu p} = \mu p.
$$

On the other hand, Lemma 5.3 provides a unique CULF map $H : I_{f'} \to I_f$ such that the diagram

$$
\begin{array}{ccc}
\Delta^m & \xrightarrow{\phi_{\lambda p}} & I_f \\
\downarrow{\phi_{\lambda p}} & & \downarrow{M_f} \\
I_{f'} & \xrightarrow{M_{f'}} & X
\end{array}
$$

(1)
commutes. Furthermore,

\[ M_gF H\phi_{\lambda p} = M_g\phi_{\mu p} = \mu p. \]

(by square (1))

(since \( F\phi_{\lambda} = \phi_{\mu} \))

(since \( M_g\phi_{\mu} = \mu \))

Since \( M_gF H\phi_{\lambda p} = \mu p \) and \( M_g'\phi_{\mu p} = \mu p \), we have that the diagram

\[
\begin{array}{c}
\Delta^m \\
\downarrow_{\phi_{\lambda p}} \\
I_f' \rightarrow I_g' \\
\downarrow_{M_g F H} \downarrow \downarrow_{\phi_{\lambda p}} \\
X \\
\end{array}
\]

commutes. Since \( \phi_{\lambda p} \) is stretched and \( M_g' \) is CULF, we can apply Lemma 4.7 and Remark 4.8 to (2) to get a unique CULF-stretched map \( F' : I_f' \rightarrow I_g' \) such that the diagram

\[
\begin{array}{c}
\Delta^m \\
\downarrow_{\phi_{\lambda p}} \\
I_f' \rightarrow I_g' \\
\downarrow_{M_g F H} \downarrow \downarrow_{F'} \\
X \\
\end{array}
\]

commutes. By Proposition 4.10, \( F' \) is an isomorphism since it is stretched and CULF. □

**Remark 5.5.** In Lemma 5.4, the diagram

\[
\begin{array}{c}
\Delta^m \\
\downarrow_{\phi_{\lambda p}} \\
I_f' \rightarrow I_g' \\
\downarrow_{M_g F H} \downarrow \downarrow_{F'} \\
X \\
\end{array}
\]

commutes. When \( p \) is active, we have that \( f' = f \) and \( g' = g \). Furthermore, if we substitute \( F' \) by \( F \), the diagram also commutes. By Lemma 5.4, we have that \( F' \simeq F \). Therefore, when we will work with an active map \( p \), we will use \( F \).

Let \( X : \Delta^{op} \rightarrow \text{Set} \) be a discrete decomposition groupoid. The simplicial groupoid (we will see that it satisfies the simplicial identities in 5.6)

\[ U_X : \Delta^{op} \rightarrow \widehat{\text{Grpd}} \]

is defined as follows:

1. On objects, \((U_X)_n\) is the groupoid whose objects are pairs \((I_f, \phi_{\lambda})\). Here \( \lambda \in X_n \) with \( \text{long}(\lambda) = f \) and \( \phi_{\lambda} \rightarrow I_f \) is the corresponding stretched \( n \)-simplex in \( I_f \) of 3.17. A morphism \( F : (I_f, \phi_{\lambda}) \rightarrow (I_g, \phi_{\mu}) \) is a stretched isomorphism functor \( F : I_f \rightarrow I_g \) such that \( F\phi_{\lambda} = \phi_{\mu} \).

2. On face maps \( d_i : (U_X)_n \rightarrow (U_X)_{n-1} \) is the functor defined as:

   (a) Let \( \phi_{\lambda} : \Delta^n \rightarrow I_f \) be an object in \((U_X)_n\). Here \( \lambda \in X_n \). Consider \( f' = \text{long}(\lambda d^i) \).

   The \((n - 1)\)-simplex \( \lambda d^i \) induces an \((n - 1)\)-simplex \( \phi_{\lambda d^i} : \Delta^{n-1} \rightarrow I_{f'} \) by Proposition 3.17. We define \( d_i \) on objects by \( d_i (I_f, \phi_{\lambda}) := (I_{f'}, \phi_{\lambda d^i}) \).
(b) Let \( F : (I_f, \phi_\lambda) \to (I_g, \phi_\mu) \) be a morphism in \((U_X)_n\). We define \( d_i \) on morphisms by: \( F' : (I_f, \phi_{\lambda d^i}) \to (I_g, \phi_{\mu d^i}) \). Here \( f' = \text{long}(\lambda d^i) \) and \( g' = \text{long}(\mu d^i) \). The functor \( F' \) was defined in Lemma 5.4.

3. On degeneracy maps, \( s_j : (U_X)_{n-1} \to (U_X)_n \) is defined on objects by \( s_j(I_f, \phi_\lambda) := (I_f, \phi_{\lambda s^j}) \). For a morphism \( F \) in \((U_X)_{n-1}\), the functor is defined by \( s_j F = F \).

**Proposition 5.6.** Let \( X \) be a discrete decomposition groupoid. Then \( U_X : \Delta^{\text{op}} \to \text{Grpd} \) is a strict simplicial groupoid.

**Proof.** We will prove that \( d_i \circ d_j = d_{j-1} \circ d_i \) for \( i < j \). We will first prove this for objects. Consider \( \lambda \in X_n \) and \( \phi_\lambda : \Delta^n \to I_f \). By definition of \( U_X \), we have that \( d_j(I_f, \phi_\lambda) = (I_f', \phi_{\lambda d^j}) \).

Proposition 3.17 gives the standard factorisation \( M_f' \phi_{\lambda d^j} = \lambda d^j \). (1)

Here \( f' = \text{long}(\lambda d^j) \). Applying \( d_i \) to \( d_j(I_f, \lambda) \), we have that

\[
d_i d_j(I_f, \lambda) = d_i(I_f', \phi_{\lambda d^j}) = (I_{f''}, \phi_{M_f' \phi_{\lambda d^j} d^i}), \tag{2}
\]

where \( f'' = \text{long}(M_f' \phi_{\lambda d^j} d^i) \) and \( \phi_{M_f' \phi_{\lambda d^j} d^i} : \Delta^{n-2} \to I_{f''} \) is given by Proposition 3.17 to form the commutative diagram

\[
\begin{array}{ccc}
\Delta^{n-2} & \xrightarrow{\phi_{M_f' \phi_{\lambda d^j} d^i}} & I_{f''} \\
\downarrow & & \downarrow \\
M_f' \phi_{\lambda d^j} d^i & \xrightarrow{(3)} & M_{f''} \\
\downarrow & & \downarrow \\
X. & \xrightarrow{(4)} & X.
\end{array}
\]

Proposition 3.17 also gives the triangle

\[
\begin{array}{ccc}
\Delta^{n-2} & \xrightarrow{\phi_{\lambda d^j d^i}} & I_{f''} \\
\downarrow & & \downarrow \\
\lambda d^j d^i & \xrightarrow{(4)} & M_{f''} \\
\downarrow & & \downarrow \\
X. & \xrightarrow{(7)} & X,
\end{array}
\]

and by Eq. (1), the diagonal maps of (3) and (4) are the same. Therefore, by the uniqueness part of Proposition 3.17 we conclude that

\[
\phi_{M_f' \phi_{\lambda d^j} d^i} = \phi_{\lambda d^j d^i}
\]

Combining this with (2) we conclude that

\[
d_i d_j(I_f, \phi_\lambda) = (I_{f''}, \phi_{\lambda d^j d^i}). \tag{5}
\]

By analogous arguments,

\[
d_{j-1} d_i(I_f, \phi_\lambda) = (I_\vec{f'}, \phi_{\lambda d^j d^{i-1}}) \tag{6}
\]

where \( \vec{f'} = \text{long}(\lambda d^i d^{i-1}) \) and \( \phi_{\lambda d^j d^{i-1}} : \Delta^{n-2} \to I_{\vec{f'}} \) is given by Proposition 3.17 to form the commutative diagram

\[
\begin{array}{ccc}
\Delta^{n-2} & \xrightarrow{\phi_{\lambda d^j d^{i-1}}} & I_{\vec{f'}} \\
\downarrow & & \downarrow \\
\lambda d^j d^{i-1} & \xrightarrow{(7)} & M_{\vec{f'}} \\
\downarrow & & \downarrow \\
X. & \xrightarrow{(7)} & X.
\end{array}
\]

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Since $d^j d^i = d^i d^{j-1}$, we have that $\lambda d^j d^i = \lambda d^i d^{j-1}$. This implies that $f'' = \overline{f''}$ and $M_{f''} = M_{\overline{f''}}$. Furthermore, the diagram

$$
\begin{array}{ccc}
\Delta_n^{n-2} & \xrightarrow{\phi_{\lambda d^j d^i}} & I_{f''} \\
\downarrow{\lambda d^i d^{j-1}} & & \downarrow{M_{f''}} \\
I & & X
\end{array}
$$

commutes since

$$
\lambda d^j d^i = \lambda d^i d^{j-1} = M_{f''} \phi_{\lambda d^j d^i} = M_{\overline{f''}} \phi_{\lambda d^j d^i}.
$$

(by triangle (4))

Since $M_{g''} F H \phi_{\lambda d^j d^i} = \mu d^j d^i$, we can apply Lemma 4.7 and Remark 4.8 to (10) to get an unique invertible map $\overline{F} : I_{f''} \to I_{g''}$ such that the diagram

$$
\begin{array}{ccc}
\Delta_n^{m} & \xrightarrow{\phi_{\mu d^j d^i}} & I_{g''} \\
\downarrow{\phi_{\lambda d^j d^i}} & & \downarrow{M_{g''}} \\
I & & X
\end{array}
$$

commutes. By analogous arguments, we have a unique invertible map $\overline{G} : I_{f''} \to I_{g''}$ such that the diagram

$$
\begin{array}{ccc}
\Delta_n^{m} & \xrightarrow{\phi_{\mu d^j d^i}} & I_{g''} \\
\downarrow{\phi_{\lambda d^j d^i}} & & \downarrow{M_{g''}} \\
I & & X
\end{array}
$$

(11)

commutes. Since $\phi_{\lambda d^j d^i}$ is stretched and $M_{g''}$ is CULF, we can apply Lemma 4.7 and Remark 4.8 to (10) to get an unique invertible map $\overline{F} : I_{f''} \to I_{g''}$ such that the diagram

$$
\begin{array}{ccc}
\Delta_n^{m} & \xrightarrow{\phi_{\mu d^j d^i}} & I_{g''} \\
\downarrow{\phi_{\lambda d^j d^i}} & & \downarrow{M_{g''}} \\
I & & X
\end{array}
$$

(12)
commutes. Since $d_i d_j (I_f, \phi_\lambda) = d_{j-1} d_i (I_f, \phi_\lambda)$, we have that

\[ \phi_{\lambda d^i d^j} = \phi_{\lambda d^j d^{i-1}}. \]  \hfill (9)

By an analogous argument,

\[ \phi_{\mu d^i d^j} = \phi_{\mu d^j d^{i-1}}. \]  \hfill (13)

Combining (9), (11) and (13), we have that the diagram

\[
\begin{array}{ccccc}
\Delta^n & \phi_{\mu d^i d^j} & \rightarrow & I_{f^\nu} & \\
\downarrow & & & \downarrow & \\
I_{f^\nu} & \phi_{\lambda d^i d^j} & \rightarrow & M_{f^\nu} & \\
& & & & \phi_{\lambda d^j d^{i-1}} \rightarrow X
\end{array}
\]  \hfill (14)

commutes, but if we substitute $\overline{F}$ by $\overline{G}$ the diagram also commutes (by square (12)). Therefore, by Lemma 4.7, we have that $d_i d_j F = d_{j-1} d_i F$. Analogous arguments can be used with the other identities. Altogether we have established that $U_X$ is a simplicial groupoid.

\[ \square \]

**Remark 5.7.** Let $(I_f, \phi_\lambda)$ be an object in $(U_X)_n$. Here $\lambda \in X_n$. When $0 < i < n$, the $(n-1)$-simplex $\lambda d^i$ satisfies $\text{long}(\lambda d^i) = f$. This implies that $d_i (I_f, \phi_\lambda) = (I_f, \phi_{\lambda d^i})$. Therefore, $d_i$ only affects the inside of $\lambda$ and respects the chosen initial object and the chosen terminal object in $I_f$.

**Lemma 5.8.** Let $p : [n] \rightarrow [m]$ be an active map in $\Delta$. Then $p^* : (U_X)_m \rightarrow (U_X)_n$ is a discrete fibration.

**Proof.** Let $(I_f, \phi_\lambda)$ be an object in $(U_X)_m$ and let $F : (I_h, \phi_\eta) \rightarrow p^* (I_f, \phi_\lambda)$ be an isomorphism in $(U_X)_n$. By Remark 5.7, since $p$ is active, we have that $p^* (I_f, \phi_\lambda) = (I_f, \phi_{\lambda p})$. In the diagram

\[
\begin{array}{ccc}
\Delta^n & \phi_\eta & \rightarrow & I_h & \\
\downarrow & & \phi_{\lambda p} & \rightarrow & F \\
\Delta^m & \phi_\lambda & \rightarrow & I_f & \\
\end{array}
\]

to give a lift to $F$ in $(U_X)_m$ is to produce a dotted arrow $\phi_{\eta_\varphi}$. This dotted arrow produces an object $(I_h, \phi_{\eta_\varphi})$ in $(U_X)_m$ and $F$ also produces an isomorphism in $(U_X)_m$ from $(I_h, \phi_{\eta_\varphi})$ to $(I_f, \phi_{\lambda p})$. Since $F$ is invertible, the dotted arrow exists and is uniquely determined. Therefore $p^*$ is a discrete fibration. \[ \square \]

**Example 5.9.** In general, the image of an inert map of $\Delta^{op}$ under $U_X$ is not a discrete fibration. Let $C$ be the category pictured by the following diagram

\[
\begin{array}{ccc}
x & \xrightarrow{a} & y \\
\downarrow{f_1} & & \downarrow{f_2} \\
z & \xrightarrow{c} & z' \\
\end{array}
\]

\[
\begin{array}{ccc}
y' & \xrightarrow{a'} & b' \\
\downarrow{y} & & \downarrow{y'} \\
b & \xrightarrow{b} & z \\
\end{array}
\]

\[
\begin{array}{ccc}
c' & \xrightarrow{c'} & d' \\
\downarrow{f_2} & & \downarrow{f_2} \\
d & \xrightarrow{d} & z' \\
\end{array}
\]

\[\text{Diagram for Example 5.9.} \]

\[31\]
Consider $f = f_2 f_1$ and $dc = f_2 = d' c'$. Since $x$ is an initial object and $z'$ is a terminal object, we have that $N(\mathfrak{C}) \cong I_f$ by Lemma 4.4. Let $\lambda_{f_1,c,d}$ be the $3$-simplex induced by the morphism $f_1$, and $c$, and $d$ in $(N(\mathfrak{C}))-3$. Let $(N(\mathfrak{C}), \lambda_{f_1,c,d}) \in (U_N(\mathfrak{C}))-3$ be the interval construction of $\lambda_{f_1,c,d}$. Applying $d_0$ to $\lambda_{f_1,c,d}$, we have that $d_0(N(\mathfrak{C}), \lambda_{f_1,c,d}) = (I_{dc}, \lambda_{c,d})$. Similarly, applying $d_0$ to $\lambda_{f_1,c,d}'$, we have that $d_0(N(\mathfrak{C}), \lambda_{f_1,c,d}') = (I_{dc}', \lambda_{c,d}')$. Let $\mathcal{F} : (I_{dc}, \lambda_{c,d}) \rightarrow (I_{dc}, \lambda_{c,d})$ be the functor, defined for each object $t$ in $I_{dc}$ as:

$$\mathcal{F}(t) = \begin{cases} \bar{y} & \text{if } t = \bar{y} \\ \bar{y} & \text{if } t = \bar{y}' \\ t & \text{else} \end{cases}$$

We can construct two lifts of $\mathcal{F}$ in $(U_N(\mathfrak{C}))$. Let $F_1 : (N(\mathfrak{C}), \lambda_{f_1,c,d}) \rightarrow (N(\mathfrak{C}), \lambda_{f_1,c,d})$ be the functor that fixes all the objects in $\mathfrak{C}$ except $\bar{y}$ and $\bar{y}'$. It is easy to check that $d_0 F_1 = \mathcal{F}$. On the other hand, let $F_2 : (N(\mathfrak{C}), \lambda_{f_1,c,d}) \rightarrow (N(\mathfrak{C}), \lambda_{f_1,c,d})$ be the functor defined by objects by

$$F_2(t) = \begin{cases} y' & \text{if } t = y \\ y & \text{if } t = y' \\ \bar{y} & \text{if } t = \bar{y} \\ \bar{y}' & \text{if } t = \bar{y}' \\ t & \text{else} \end{cases}$$

It is straightforward to see that $d_0 F_2 = \mathcal{F}$. Therefore, $F_1$ and $F_2$ are two lifts of $\mathcal{F}$.

The following results will help us to prove that $U_X$ is a complete decomposition groupoid. When $S$ is a simplicial groupoid, we have a simplicial set induced by the object functor Obj : Grpd $\rightarrow$ Set, which is defined as forgetting the morphisms. We denote Obj $\circ S$ as $S^0$. This means that the simplicial set $S^0 : \Delta^{op} \rightarrow$ Set is defined as follows: for each $n \geq 0$, the set $(S^0)_n := \text{Obj}(S_n)$ is the set of objects in $S_n$.

**Proposition 5.10.** Let $X : \Delta^{op} \rightarrow$ Set be a discrete decomposition groupoid. Then $U_X^0 \cong X$.

**Proof.** We will construct two simplicial maps $O : U_X^0 \rightarrow X$ and $\overline{T} : X \rightarrow U_X^0$ such that $O \circ \overline{T} = \text{id}_X$ and $\overline{T} \circ O = \text{id}_{U_X^0}$. We define the forgetful map $O : U_X^0 \rightarrow X$ as $O(I_f, \phi_\mu) = \mu$. The map $\overline{T} : X \rightarrow U_X^0$ is defined as $\overline{T}(\lambda) = (I_f, \phi_\lambda)$ where $f = \text{long}(\lambda)$ and $\phi_\lambda$ the $n$-simplex induced by $\lambda$ of Proposition 3.17. So that we have $M_f \phi_\lambda = \lambda$. It is easy to check that $O$ and $\overline{T}$ are a simplicial maps.

Consider $f \in X_1$ and $\mu : \Delta^n \rightarrow I_f$. Note that $f = \text{long}(M_f \mu)$ since $\mu$ is stretched. The pair $(I_f, \mu)$ is an $n$-simplex in $U_X^0$. Applying $O$ and $\overline{T}$, we have that $\overline{T} \circ O(I_f, \mu) = (I_f, \phi_{M_f \mu})$ where (by definition of $\overline{T}$) $\phi_{M_f \mu}$ is the map given by Proposition 3.17, and so we have that the diagram

$$\begin{array}{ccc} \Delta^n & \xrightarrow{\phi_{M_f \mu}} & I_f \\ M_f \mu \downarrow & & \downarrow \mu \\ X & \xrightarrow{O} & U_X^0 \end{array}$$

commutes but if we substitute $\phi_{M_f \mu}$ by $\mu$, the triangle (1) also commutes. Therefore, by Proposition 3.17, we have that $\phi_{M_f \mu} = \mu$. This implies that $\overline{T} \circ O(I_f, \mu) = (I_f, \mu)$ and proves that $\overline{T} \circ O = \text{id}_{U_X^0}$. On the other hand, let $\lambda$ be an $n$-simplex in $X$. Applying $\overline{T}$ and $O$, we have that $O \circ \overline{T}(\lambda) = M_f \phi_\lambda$. By Proposition 3.17, we have that $M_f \phi_\lambda = \lambda$. Therefore $O \circ \overline{T} = \text{id}_X$. ∎
Corollary 5.11. Let $X$ be a discrete decomposition groupoid. The simplicial groupoid $U^0_X$ is a discrete decomposition groupoid.

Lemma 5.12. Let $X$ be a discrete decomposition groupoid. The simplicial groupoid $U_X : \Delta^{op} \to \text{Grpd}$ is a decomposition groupoid.

Proof. We need to show that for an active-inert pullback square in $\Delta^{op}$, the image under $U_X$ is a pullback

$$
(U_X)_m \xrightarrow{g} (U_X)_n \\
\downarrow h \\
(U_X)_k \xrightarrow{\bar{g}} (U_X)_s.
$$

Here $g$ and $\bar{g}$ are active maps, $h$ and $\bar{h}$ are inert maps. The map $h$ induces a morphism between fibres that we will denote by $h'$. Since $g$ and $\bar{g}$ are active maps, they are discrete fibrations by Lemma 5.8. Therefore, we can work with strict fibres. By Lemma 1.1, the previous square is a pullback if and only if for each object $(I_f, \phi_0)$ in $(U_X)_n$, the morphism $h' : \text{Fib}_{(I_f, \phi_0)}(g) \to \text{Fib}_{\mathfrak{P}(I_f, \phi_0)}(\bar{g})$ is an equivalence. Here $\text{Fib}_{(I_f, \phi_0)}(g)$ is the strict fibre of $g$ over $(I_f, \lambda)$ and $\text{Fib}_{\mathfrak{P}(I_f, \phi_0)}(\bar{g})$ is the strict fibre of $\bar{g}$ over $\mathfrak{P}(I_f, \phi_0)$.

Since $\text{Fib}_{(I_f, \phi_0)}(g)$ and $\text{Fib}_{\mathfrak{P}(I_f, \phi_0)}(\bar{g})$ are discrete groupoids by Lemma 5.8, we have that $h'$ is an equivalence if and only if $\text{Obj}(h')$ is an isomorphism. By Corollary 5.11, we have that $U^0_X$ is a discrete decomposition groupoid. So the diagram

$$
(U^0_X)_m \xrightarrow{g} (U^0_X)_n \\
\downarrow h \\
(U^0_X)_k \xrightarrow{\bar{g}} (U^0_X)_s
$$

is a pullback. This implies that $\text{Obj}(h') : \text{Obj}(\text{Fib}_{(I_f, \phi_0)}(g)) \to \text{Obj}(\text{Fib}_{\mathfrak{P}(I_f, \phi_0)}(\bar{g}))$ is an isomorphism by Lemma 5.8. Hence, $h' : \text{Fib}_{(I_f, \phi_0)}(g) \to \text{Fib}_{\mathfrak{P}(I_f, \phi_0)}(\bar{g})$ is an equivalence. \qed

Lemma 5.13. Let $X$ be a decomposition groupoid. Then the decomposition groupoid $U_X$ is a complete.

Proof. To establish that $U_X$ is complete, we need to check that the map $s_0 : (U_X)_0 \to (U_X)_1$ is a monomorphism. Since $s_0$ is active, we have that $s_0$ is a discrete fibration by Lemma 5.8. Therefore, we will consider strict fibres. For $f \in X_1$, consider $\phi_f : \Delta^1 \to 

\Delta^1 \phi_{s_0(x)} \to I_f \\
\downarrow f \\
X

(1)

When $d_0(f) \neq d_1(f)$, the fibre is empty. In case $d_0(f) = d_1(f)$, consider $(I_{s_0(x)}, \phi_x)$ and $(I_{s_0(y)}, \phi_y)$ two objects in $\text{Fib}_{(I_f, \phi_f)}(s_0)$. Since $s_0$ is active and $s_0(I_{s_0(x)}, \phi_x) = (I_f, \phi_f)$, we have that the diagram

$$
\Delta^1 \phi_{s_0(x)} \to I_f \\
\downarrow f \\
X

(1)$$

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commutes. Furthermore, since \( s_0 \) is active and \( s_0(I_{s_0(y)}, \phi_y) = (I_f, \phi_f) \), we have that the diagram

\[
\begin{array}{ccc}
\Delta^1 & \xrightarrow{\phi_{s_0(y)}} & I_f \\
\downarrow f & & \downarrow M_f \\
X & & X
\end{array}
\]  

(2)

commutes. The lower part of the triangles (1) and (2) coincide, so by Proposition 3.17 also the horizontal sides coincide, in other words \( \phi_{s_0(x)} = \phi_{s_0(y)} \), and applying \( M_f \) we have that \( s_0(x) = s_0(y) \). Furthermore, since \( X \) is a discrete decomposition groupoid, it follows that \( x = y \). Therefore, \( (I_{s_0(x)}, \phi_x) = (I_{s_0(y)}, \phi_y) \). \( \square \)

We define a simplicial map \( I : X \to U_X \), using the interval construction, which sends an object \( \lambda \in X_n \) to the pair \((I_f, \phi_\lambda)\) with \( f = \text{long}(\lambda) \) and \( \phi_\lambda \) is the \( n \)-simplex induced by \( \lambda \) of Proposition 3.17.

**Proposition 5.14.** Let \( X \) be a discrete decomposition groupoid. The simplicial map \( I : X \to U_X \) is CULF.

**Proof.** Using Proposition 1.14, it is enough to prove that the following diagram is a pullback

\[
\begin{array}{ccc}
X_2 & \xrightarrow{d_1} & X_1 \\
\downarrow I_2 & & \downarrow I_1 \\
U_{X_2} & \xrightarrow{d_1} & U_{X_1}.
\end{array}
\]

Since \( d_1 \) is active, we have that \( d_1 \) is a discrete fibration by Lemma 5.8. Therefore, we can work with strict fibres. On the other hand, by Lemma 1.1, the previous square is a pullback if and only if for each \( f \in X_1 \) the induced map on fibres \( I_2 : \text{Fib}_f(d_1) \to \text{Fib}_{(I_f, \phi_f)}(d_1) \) is an equivalence. Here \( \text{Fib}_f(d_1) = \{ \sigma \in X_2 \mid d_1(\sigma) = f \} \) and \( \text{Fib}_{(I_f, \phi_f)}(d_1) \) is the fibre of \( d_1 \) over \( (I_f, \phi_f) \). Since \( \text{Fib}_{(I_f, \phi_f)}(d_1) \) is a discrete groupoid, it is enough to prove that \( I_2 \) is bijective in objects. First we will see that it is injective. Let \( \sigma \) and \( \overline{\sigma} \) be two objects in \( \text{Fib}_f(d_1) \) such that \( I_2(\sigma) = I_2(\overline{\sigma}) \). This means that \( \phi_\sigma = \phi_{\overline{\sigma}} \). Applying \( M_f \) to \( \phi_\sigma = \phi_{\overline{\sigma}} \), we conclude that \( \sigma = \overline{\sigma} \) (since \( M_f \phi_\sigma = \sigma \) and \( M_f \phi_{\overline{\sigma}} = \overline{\sigma} \)) by Proposition 3.17).

For the surjectivity of \( I_2 \), let \( (I_f, \mu) \) be an object in \( \text{Fib}_{(I_f, \phi_f)}(d_1) \) and consider \( M_f \mu \in \text{Fib}_f(d_1) \). It is straightforward to check that \( I_2(M_f \mu) = (I_f, \mu) \). Hence altogether, \( I_2 : \text{Fib}_f(d_1) \to \text{Fib}_{(I_f, \phi_f)}(d_1) \) is a bijection. \( \square \)

**5.15. Comparison with Gálvez-Carrillo, Kock and Tonks**

Gálvez-Carrillo, Kock, and Tonks [17] defined \( U \) using the cartesian fibration of subdivided intervals \( \text{dom} : \mathcal{U} \to \Delta \). We can also use this approach to define \( U_X \). Let \( X \) be a discrete decomposition groupoid. Let \( \text{Ar}^s(\text{Int}_{|X|})_\lambda \) be the full subcategory of \( \mathcal{U}^{\text{cart}} \) whose objects are stretched simplicial maps \( (\Delta^k \to I_f \text{ where } f \in X_1) \) and whose morphisms are given by squares

\[
\begin{array}{ccc}
\Delta^k & \xrightarrow{p} & \Delta^n \\
\downarrow \lambda & & \downarrow \mu \\
I_f & \xrightarrow{F} & I_g
\end{array}
\]  

(1)

where \( f, g \in X_1 \) and \( F \) is a CULF map. To follow the notation of [17], we denote \( \text{Ar}^s(\text{Int}_{|X|})_\lambda \) as \( \mathcal{U}_X^{\text{cart}} \). The Grothendieck construction \( \text{dom} : \Delta/U_X \to \Delta \) of \( U_X : \Delta^{op} \to \text{Grpd} \) is described as follows:

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- The objects in $\Delta_{/U_X}$ are pairs $([n], (I_f, \phi_\gamma))$. Here $\gamma \in X_n$ and $\text{long}(\gamma) = f$, and $\phi_\gamma : \Delta^n \to I_f$ is the corresponding stretched $n$-simplex in $I_f$ of 3.17.

- The morphisms in $\Delta_{/U_X}$ are pairs $(p, F') : ([n], (I_f, \phi_\gamma)) \to ([m], (I_g, \phi_\eta))$. Here $p : [n] \to [m]$ and $F' : (I_f, \phi_\gamma) \to p^*(I_g, \phi_\eta)$ is an invertible CULF map such that $F'\phi_\gamma = \phi_{np}$.

- The domain projection map $\text{dom} : \Delta_{/U_X} \to \Delta$ is given by $\text{dom}([n], (I_f, \phi_\mu)) = [n]$ and $\text{dom}(p, F') = p$.

We will see that $U^\text{cart}_X$ and $\Delta_{/U_X}$ are equivalent, but we need some preliminaries.

**Remark 5.16.** The square (1) can be rewritten applying results of Sections 3 and 4. Consider $\varpi_{I_f}$ the chosen map from the chosen initial object to the chosen terminal object in $I_f$. Since $F$ is CULF, Lemma 3.21 gives an isomorphism $K : I_{\varpi_{I_f}} \to I_F \varpi_{I_f}$ such that the diagram

$$
\begin{array}{c}
I_{\varpi_{I_f}} \xrightarrow{K} I_F \varpi_{I_f} \\
M_{\varpi_{I_f}} \downarrow \\
I_f \xrightarrow{F} I_g
\end{array}
$$

commutes. Since $I_f$ is an interval, the map $M_{\varpi_{I_f}}$ has an inverse $W : I_f \to I_{\varpi_{I_f}}$ by Lemma 4.4. We thus have $M_{F \varpi_{I_f}} \circ K \circ W = F$. Combining this equality and (1), we have that the diagram

$$
\begin{array}{c}
\Delta^k \xrightarrow{\lambda} I_f \xrightarrow{K \circ W} I_F \varpi_{I_f} \\
p \downarrow \\
\Delta^n \xrightarrow{\mu} I_g \xrightarrow{M_g \circ M_{F \varpi_{I_f}}} X
\end{array}
$$

commutes. Proposition 3.17 applied to the $k$-simplex $\mu p$ gives a $k$-simplex $\phi_{\mu p}$ together with a commutative diagram

$$
\begin{array}{c}
\Delta^k \xrightarrow{\phi_{\mu p}} I_{F \varpi_{I_f}} \\
M_g \mu p \downarrow \\
\Delta^n \xrightarrow{\mu p} I_g \xrightarrow{M_g} X
\end{array}
$$

but if we substitute $\phi_{\mu p}$ by $KW \lambda$ the diagram also commutes (since (3) commutes). Therefore, by Proposition 3.17 we have that

$$
\phi_{\mu p} = KW \lambda.
$$

(5)

Combining (4) and (5), we have a map

$$
KW : (I_f, \lambda) \to p^*(I_g, \mu)
$$

(6)
in $\Delta_{/U_X}$ where $p^*(I_g, \mu) = (I_{F \varpi_{I_f}}, \phi_{\mu p})$. 

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Remark 5.17. A morphism \((p, F') : ([n], (I_f, \phi_\gamma)) \to ([m], (I_g, \phi_\eta))\) in \(\Delta/UX\) can be interpreted as a commutative diagram in the following way: by definition of \(UX\), we have that \(p^*(I_g, \eta) = (I_g', \phi_{np})\) where \(g' = \text{long}(\eta p)\) and \(\phi_{np}\) is the \(n\)-simplex of Proposition 3.17. Since \(M_g \phi_\eta = \eta p\) (by 3.17), we have that the solid rectangle

\[
\begin{array}{ccc}
\Delta^n & \xrightarrow{\phi_{np}} & I_g' \\
p \downarrow & & \downarrow \text{M}_g' \\
\Delta^m & \xrightarrow{\phi_\eta} & I_g
\end{array}
\]  

(7)

commutes. Lemma 5.3 now gives a unique dashed CULF map \(G\) as indicated. Furthermore, since \(F' \phi_\gamma = \phi_{np}\), we have that the square

\[
\begin{array}{ccc}
\Delta^n & \xrightarrow{p} & \Delta^m \\
\phi_\gamma \downarrow & & \downarrow \phi_\eta \\
I_f & \xrightarrow{G \circ F'} & I_g
\end{array}
\]  

(8)

commutes.

Remark 5.16 gives the tools to construct a functor \(E : \mathcal{U}_X^{\text{cart}} \to \Delta/UX\). For an object \(\lambda : \Delta^n \to I_f\) in \(\mathcal{U}_X^{\text{cart}}\), the functor \(E\) is defined by \(E(\lambda) := (I_f, \lambda)\). For a morphism

\[
\begin{array}{ccc}
\Delta^k & \xrightarrow{p} & \Delta^n \\
\lambda \downarrow & & \downarrow \mu \\
I_f & \xrightarrow{F} & I_g
\end{array}
\]  

(1)

in \(\mathcal{U}_X^{\text{cart}}\), the functor \(E\) sends \((p, F)\) to the morphism \(([p], KW) : (I_f, \lambda) \to (I_g, \mu)\) where \(KW : (I_f, \lambda) \to p^*(I_g, \mu)\) is given by Eq. (6) in Remark 5.16.

Theorem 5.18. Let \(X\) be a discrete decomposition groupoid. Then the functor \(E : \mathcal{U}_X^{\text{cart}} \to \Delta/UX\) is fully faithful and surjective in objects.

Proof. It is easy to see that \(E\) is surjective on objects since the pair \((I_f, \lambda)\) in \(\Delta/UX\) is equal to \(E(\lambda)\). For the faithfulness of \(E\), let

\[
\begin{array}{ccc}
\Delta^k & \xrightarrow{p} & \Delta^n \\
\lambda \downarrow & & \downarrow \mu \\
I_f & \xrightarrow{F} & I_g
\end{array}
\]

and

\[
\begin{array}{ccc}
\Delta^k & \xrightarrow{p} & \Delta^n \\
\lambda \downarrow & & \downarrow \mu \\
I_f & \xrightarrow{\overline{F}} & I_g
\end{array}
\]  

be two morphisms in \(\mathcal{U}_X^{\text{cart}}\) such that \(E(p, F) = E(\overline{p}, \overline{F})\). This means that \((p, KW) = (\overline{p}, KW')\). Here \(KW : (I_f, \lambda) \to p^*(I_g, \mu)\) is given by Eq. (6) in Remark 5.16 and \(KW' : (I_f, \lambda) \to \overline{p}^*(I_g, \mu)\) is also given by Eq. (6) in Remark 5.16 but applied to \((\overline{p}, \overline{F})\). Since \((p, KW) = (\overline{p}, KW')\), we have that \(p = \overline{p}\) and the diagram

\[
\begin{array}{ccc}
\Delta^k & \xrightarrow{\lambda} & I_f \\
p \downarrow & & \downarrow F \\
\Delta^n & \xrightarrow{\mu} & I_g
\end{array}
\]

\[
\begin{array}{ccc}
I_f & \xrightarrow{K \circ W} & I_{F \circ \iota_f} \\
F \downarrow & & \downarrow \text{M}_g \circ M_{F \circ \iota_f} \\
I_g & \xrightarrow{M_g \circ M_{F \circ \iota_f}} & X
\end{array}
\]

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commutes by Remark 5.16. This implies that $M_g \circ F = M_g \circ \overline{F}$. Since $M_g$ is CULF, Lemma 4.5 now implies that $F = \overline{F}$. Hence altogether, we have that $(p, F) = (\overline{p}, \overline{F})$. Therefore, $E$ is faithful. For the fullness condition of $E$, consider $(p, F') : ([n], (I_f, \phi_n)) \to ([m], (I_g, \phi_m))$ a morphism in $\Delta : U_X$. By square (8) in Remark 5.17, we have that $(p, F')$ induces a map $(p, G \circ F')$ in $U_X^\text{ext}$. It is not hard to check that $E(p, G \circ F') = (p, F')$. Therefore, $E$ is full.

Remark 5.19. The decomposition groupoid $U$ also admits a description similar to $U_X$. The differences are the following:

- The intervals do not depend on a fixed discrete decomposition groupoid $X$. This means that an object in $U_n$ is a pair $(C, \lambda)$ where $C$ is an interval and $\lambda : \Delta^n \to C$. A morphism $F : (C, \lambda) \to (D, \mu)$ in $U_n$ is an invertible CULF map $F : C \to D$ such that $F\lambda = \mu$.

- For a map $p : [m] \to [n]$, the map $p^* : (U_X)_n \to (U_X)_m$ is the functor defined as:

1. Let $\lambda : \Delta^n \to C$ be an object in $U_n$. Consider $f' = \text{long}(\lambda p)$. The $(n - 1)$-simplex $\lambda p$ induces an $(n - 1)$-simplex $\phi_{\lambda p} : \Delta^{n - 1} \to I_{f'}$ by Proposition 3.17. We define $p^*$ on objects by $p(C, \lambda) := (I_{f'}, \phi_{\lambda p})$.

2. Let $F : (C, \lambda) \to (D, \eta)$ be a morphism in $U_n$. We define $p^*$ on morphisms by: $F' : (I_{f'}, \phi_{\lambda p}) \to (I_{g'}, \phi_{\mu p})$. Here $f' = \text{long}(\lambda p)$ and $g' = \text{long}(\mu p)$. The functor $F'$ was defined in Lemma 3.19.

- The simplicial identities holds up to isomorphism, this means that $U$ is a pseudo simplicial map. For example consider $(C, \varpi_C)$ in $U_1$, it is easy to check that $d_1 s_0(C, \varpi_C) = (I_{\varpi_C}, \phi_{\varpi_C})$ which is isomorphic to $(C, \varpi_C)$ since $C \cong I_{\varpi_C}$ by Lemma 4.4.

A simplicial map $G : X \to Y$ between decomposition groupoids is full and faithful if for all objects $x, y \in X$ it induces an equivalence on the mapping groupoids

$$G_{x,y} : \text{map}_X(x, y) \to \text{map}_Y(Gx, Gy).$$

When $X$ is a discrete decomposition groupoid, we have a canonical simplicial map $j : U_X \to U$, defined by $j(I_f, \phi_\lambda) = (I_f, \phi_\lambda)$ for $(I_f, \phi_\lambda) \in (U_X)_n$ and $jF = F$ for $F \in (U_X)_n$. It is straightforward to prove the following result.

Lemma 5.20. Let $X$ be a discrete decomposition groupoid. Then the simplicial map $j : U_X \to U$ is full and faithful.

Gálvez-Carrillo, Kock and Tonks [17] defined the CULF classifying map $I' : \Delta / X \to U$. It takes an $n$-simplex $\lambda : \Delta^n \to X$ to an $n$-subdivided interval $\phi_\lambda : \Delta^n \to I_f$ in $U$ (or to the pair $(I_f, \phi_\lambda)$ in $U_n$). Here $f = \text{long}(\lambda)$. The classifying map $I'$ in our work is the map $(j \circ I) : X \to U$, since for each $\lambda \in X_n$, we have that $(j \circ I)(\lambda) = (I_f, \phi_\lambda)$ which is the same as $I'(\lambda)$. We will abuse of notation and denote $I'$ as $I$ in Section 6.

6 Gálvez–Kock–Tonks Conjecture

Let $\mathbf{cDcmp}$ denote the $\infty$-category of complete decomposition spaces and CULF maps. The construction of the complete decomposition groupoid $U$ was motivated by the following
Gálvez–Kock–Tonks Conjecture ([17], §5.4) For each decomposition space \( X \), the space \( \text{map}_{\text{Dcmp}}(X, U) \) is contractible.

A partial result is the following

**Theorem 6.1** ([17], Theorem 5.5). For each complete decomposition space \( X \), the space \( \text{map}_{\text{Dcmp}}(X, U) \) is connected.

We study a truncated version of the Gálvez–Kock–Tonks conjecture, the case of discrete decomposition groupoids \( X : \Delta^{op} \to \text{Set} \). It is enough to consider a 1-truncated version of \( U \), and therefore, the groupoid map \( \text{map}_{\text{Dcmp}}(X, U) \) only has information at two levels: natural transformations from \( X \) to \( U \) and modifications of such natural transformations.

Theorem 6.1 implies that every natural transformation from \( X \) to \( U \) is isomorphic to \( I \). Therefore, to prove the conjecture, we only need to prove that \( I \) does not admit other self-modifications than the identity.

### 6.2. Modifications

A modification between two CULF natural transformations is a family of 2-cells in the 2-category of (small) categories that satisfies some coherence conditions as indicated in the following definition:

**Definition 6.3** ([3], Definition 7.3.1). Let \( \mathcal{C} \) and \( \mathcal{D} \) be two 2-categories. Let \( F, G : \mathcal{C} \to \mathcal{D} \) be two functors and \( \alpha, \beta : F \Rightarrow G \) be two natural transformations from \( F \) to \( G \). A modification \( \Gamma : \alpha \Rightarrow \beta \) assigns to each object \( x \) in \( \mathcal{C} \) a 2-cell \( \Gamma_x : \alpha_x \Rightarrow \beta_x \) of \( \mathcal{D} \) compatibly with the 2-cell components of \( F \) and \( G \) in the sense of equation

\[
\begin{align*}
F(x) & \xrightarrow{\Gamma_x} G(x) \\
F(f) & \downarrow \beta_f \downarrow G(f) = F(f) \downarrow \alpha_f \downarrow G(f) \\
F(y) & \rightarrow G(y)
\end{align*}
\]

We are interesting in the case where \( \mathcal{C} = \Delta \) and \( \mathcal{D} = \text{Grpd} \), and where \( F = X \) and \( G = U_X \), and where \( \alpha \) and \( \beta \) are both equal to \( I \). In this case, Definition 6.3 can be written as follows.

**Definition 6.4.** A modification \( \Gamma : I \to I \) assigns to each \([n] \) in \( \Delta \) a natural transformation \( \Gamma_n : I_n \to I_n \) in \( \text{Grpd} \) such that for each \( n \geq 1 \) the following equations hold for each \( 0 \leq i \leq n \) and \( 0 \leq j < n \)

\[
\begin{align}
X_n & \xrightarrow{\Gamma_n} (U_X)_n \\
\downarrow d_i & \quad \Downarrow s_j \\
X_{n-1} & \rightarrow (U_X)_{n-1}
\end{align}
\]

\[
\begin{align}
X_n & \xrightarrow{\tilde{n}} (U_X)_n \\
\downarrow d_i & \quad \Downarrow s_j \\
X_{n-1} & \rightarrow (U_X)_{n-1}
\end{align}
\]

\[
\begin{align}
X_{n-1} & \xrightarrow{\tilde{n}} (U_X)_{n-1} \\
\downarrow s_j & \quad \Downarrow s_j \\
X_n & \rightarrow (U_X)_n
\end{align}
\]

\[
\begin{align}
X_{n-1} & \xrightarrow{\tilde{n}} (U_X)_{n-1} \\
\downarrow s_j & \quad \Downarrow s_j \\
X_n & \rightarrow (U_X)_n
\end{align}
\]
Remark 6.5. We can define a modification $\Gamma : I \to I$ level by level, so let $\Gamma_n : I_n \to I_n$ be a component of the modification $\Gamma$. Given $\lambda \in X_n$, consider $\phi_\lambda$ the $n$-simplex induced by $\lambda$ constructed in Proposition 3.17 and $f = \text{long}(\lambda)$. The modification $\Gamma$ assigns to $\lambda$ an invertible functor $\Gamma^\lambda_n : (f, \phi_\lambda) \to (f, \phi_\lambda)$ in $(U_X)_n$. The morphism $\Gamma^\lambda_n$ has associated an underlying map $\overline{\Gamma^\lambda_n} : I_f \to I_f$.

Let $p : [m] \to [n]$ be an active map. By Remark 6.7, we have that $p^*\overline{\Gamma_n^\lambda} = \overline{\Gamma_m^\lambda}$. This implies that

$$\overline{\Gamma^\lambda_m} = \overline{\Gamma^\lambda_n}$$

where $\overline{\Gamma^\lambda_m} : I_f \to I_f$ is the underlying map of $\Gamma^\lambda_m$. The difference between $\Gamma^\lambda_n$ and $\Gamma^\lambda_m$ is that the first one respects the $n$-subdivision $\phi_\lambda$ and the other respects the $m$-subdivision $\phi_{xp}$.

Lemma 6.6. Let $X$ be a discrete decomposition groupoid. The mapping groupoid map $\text{map}_{\text{Decomp}}(X, U_X)$ is contractible.

Proof. Theorem 6.1 shows that we only have to prove that $I$ does not admit other self-modifications $\Gamma$ than the identity. Let $\lambda$ be an $n$-simplex in $X$ and put $f = \text{long}(\lambda)$. Let $\Gamma$ a modification, with components $\Gamma_n : I_n \to I_n$ and consider $\overline{\Gamma^\lambda_n} : I_f \to I_f$ the underlying map associated to $\Gamma^\lambda_n : (f, \phi_\lambda) \to (f, \phi_\lambda)$ of Remark 6.5.

Since $\text{long} : [1] \to [n]$ is an active map in $\Delta$, by Remark 6.5, we have that

$$\overline{\Gamma_n^\lambda} = \overline{\Gamma_f^\lambda} \quad \text{(1)}$$

where $\overline{\Gamma_f^\lambda} : I_f \to I_f$ is the underlying map associated to $\Gamma_f^\lambda : (f, \phi_f) \to (f, \phi_f)$. On the other hand, given a morphism $\alpha : \sigma \to \overline{\sigma}$ in $I_f$, Lemma 3.20 gives a stretched 3-simplex $\eta_\alpha : \Delta^3 \to I_f$ such that

$$d_\perp d_\top \eta_\alpha = \alpha \quad \text{(2)}$$

The modification $\Gamma$ assigns to $M_\alpha \eta_\alpha$ an invertible map $\Gamma^{\eta_\alpha}_3 : (f, \eta_\alpha) \to (f, \eta_\alpha)$ such that $\overline{\Gamma^{\eta_\alpha}_3} \eta_\alpha = \eta_\alpha$. Furthermore,

$$\overline{\Gamma^{\eta_\alpha}_3}(\alpha) = \overline{\Gamma^{\eta_\alpha}_3}(d_\perp d_\top \eta_\alpha) = d_\perp d_\top \overline{\Gamma^{\eta_\alpha}_3}(\eta_\alpha) = d_\perp d_\top (\eta_\alpha) = \alpha \quad \text{(by Eq. 2)}$$

$$\text{since } \overline{\Gamma^{\eta_\alpha}_3} \text{ is a sim. map}$$

By Definition 6.4, we have the equality

$$X_3 \xrightarrow{l_1 \Gamma_\lambda^\alpha} (U_X)_3 \xrightarrow{d_1 d_1^\perp} (U_X)_1 = X_3 \xrightarrow{d_1 d_1^\perp} X_1 \xrightarrow{l_1 \Gamma_\lambda^\alpha} (U_X)_1.$$

This equation implies that $d_1 d_1^\perp(\Gamma^{\eta_\alpha}_3) = \Gamma_f^\lambda$. Since $d^1 d^1$ is active, we have that $\overline{\Gamma^{\eta_\alpha}_3} = \overline{\Gamma_f^\lambda}$ by Remark 6.5. Hence altogether, for each $\alpha \in I_f$

$$\overline{\Gamma_n^\lambda}(\alpha) = \overline{\Gamma_f^\lambda}(\alpha) = \overline{\Gamma^{\eta_\alpha}_3}(\alpha) = \alpha \quad \text{(by Eq. (1))}$$

Since $\Gamma^\lambda_n$ is the identity arrow for each $\lambda \in X_n$, we have that $\Gamma$ is the identity modification. \qed

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Theorem 6.7. Let $X$ be a discrete decomposition groupoid. The mapping groupoid $\text{map}_{c\text{Dcmp}}(X,U)$ is contractible.

Proof. Each natural transformation from $X$ to $U$ is isomorphic to $I$ by Theorem 6.1. We can factor $I : X \to U$ as

$$
\begin{array}{ccc}
X & \xrightarrow{I} & U \\
\downarrow{I} & & \downarrow{J} \\
U_X & \xrightarrow{J} & U
\end{array}
$$

Since $J : U_X \to U$ is full and faithful (5.20), we have that $\eta : \text{map}_{c\text{Dcmp}}(X,U_X) \to \text{map}_{c\text{Dcmp}}(X,U)$ is also full and faithful. Since $\text{map}_{c\text{Dcmp}}(X,U_X)$ is contractible (6.6) and $\text{map}_{c\text{Dcmp}}(X,U)$ is connected (6.1), it follows that $\text{map}_{c\text{Dcmp}}(X,U)$ is contractible. \hfill \Box

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