ON CONTINUOUS FIELDS OF JB-ALGEBRAS

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Dedicated to the memory of Professor George Bachman, Polytechnic University, Brooklyn, NY, USA.

Abstract. We introduce and study continuous fields of JB-algebras (which are real non-associate analogues of C*-algebras). In particular, we show that for the universal enveloping C*-algebra $C^*_u(B)$ for the JB-algebra $B$ defined by a continuous field of JB-algebras $A_t$, $t \in T$, on a locally compact space $T$ there exists a decomposition of $C^*_u(B)$ into a continuous field of C*-algebras $C^*_u(A_t)$, $t \in T$, on the same space $T$, composed entirely of the universal enveloping C*-algebras of the corresponding JB-algebras from the aforementioned decomposition of the algebra $B$.

1. Introduction and Preliminaries

Banach associative regular *-algebras over $\mathbb{C}$, so called $C^*$-algebras, were first introduced by Gelfand and Naimark in the paper [5]. Since then these algebras were studied extensively by various authors. This theory now is a big subdomain of the of Functional Analysis as a subject which found applications in almost all branches of Modern Mathematics and Physics. For the basics of the theory of C*-algebras, see for example Pedersen’s monograph [9]. The basic theory of real associative analogues of C*-algebras, so called real C*-algebras, is presented in Li’s monograph [8].

In order to obtain a topological non-commutative version of Gelfand’s characterization of commutative C*-algebras, Dixmier and Douady in [4] introduced a notion of continuous fields of Banach spaces and C*-algebras, which found important applications in classification of C*-algebras (see [3]) and Theoretical Physics (see [7]). According to them, a continuous field of C*-algebras

$$\mathfrak{B}, \{\mathfrak{A}_t, \varphi_t\}_{t \in T},$$

over a locally compact Hausdorff space $T$ consists of a C*-algebra $\mathfrak{B}$, a collection of C*-algebras

$$\{\mathfrak{A}_t\}_{t \in T},$$

and a set

$$\{\varphi_t : \mathfrak{B} \to \mathfrak{A}_t\}_{t \in T},$$

of surjective morphisms, such that:

1). The function

$$t \mapsto \|\varphi_t(x)\|,$$
is in $C_0(T)$ for all $x \in \mathfrak{B}$;
2). The norm of any $x \in \mathfrak{B}$ is
\[ \|x\|_{\mathfrak{B}} = \sup_{t \in T} \|\varphi_t(x)\|_{\mathfrak{A}_t}; \]
3). For any $f \in C_0(T)$ and $x \in \mathfrak{B}$, there is an element
\[ fx \in \mathfrak{B}, \]
for which
\[ \varphi_t(fx) = f(t)\varphi_t(x), \]
for all $t \in T$.

A **section** of the field is an element \( \{x_t\}_{t \in T} \) of
\[ \prod_{t \in T} \mathfrak{A}_t, \]
for which there is an element $x \in \mathfrak{B}$ such that
\[ x_t = \varphi_t(x), \]
for all $t \in T$.

One can see that $\mathfrak{B}$ can be identified with the space of sections of the field, seen as a C*-algebra under pointwise scalar multiplication, addition, adjointing, and operator multiplication, by means
\[ \{\varphi_t(x)\}_{t \in T} \leftrightarrow x. \]

In particular,
\[ x = y, \]
iff
\[ \varphi_t(x) = \varphi_t(y), \]
for all $t$. It is natural that algebra $\mathfrak{B}$ is called a the **C*-algebra of the continuous field of C*-algebras**.

Since the beginning of the theory of complex C*-algebras, there were numerous attempts to extend this theory to non-associative algebras which are close to associative, in particular to Jordan algebras. In fact, Alfsen, Shultz and Størmer in [2] defined so called **JB-algebras** as the real Banach–Jordan algebras satisfying for all pairs of elements $x$ and $y$ the inequality of fineness
\[ \|x^2 + y^2\| \geq \|x\|^2, \]
and regularity condition
\[ \|x^2\| = \|x\|^2. \]
If $\mathfrak{A}$ is a C*-algebra, or a real C*-algebra, then the self-adjoint part $\mathfrak{A}_{sa}$ of $\mathfrak{A}$ is a JB-algebra under the Jordan product
\[ x \circ y = \frac{(xy + yx)}{2}. \]
Closed subalgebras of $\mathfrak{A}_{sa}$, for some C*-algebra or real C*-algebra $\mathfrak{A}$, become relevant examples of JB-algebras, and are called **JC-algebras**.

The basic theory of JB-algebras is fully treated in monograph of Hanche-Olsen and Størmer [6]. In particular, in this monograph there is the following theorem which was for the first time presented by Alfsen, Hanche-Olsen and Shultz in the paper [1].
Theorem 1 (Alfsen, Hanche-Olsen, Shultz [1]). For an arbitrary JB-algebra $A$ there exists a unique up to an isometric $*$-isomorphism a C*-algebra $C^*_u(A)$ (the universal enveloping C*-algebra for the JB-algebra $A$), and a Jordan homomorphism

$$\psi_A : A \to C^*_u(A)_{sa},$$

from $A$ to the self-adjoint part of $C^*_u(A)$, such that:

1). $\psi_A(A)$ generates $C^*_u(A)$ as a C*-algebra;
2). for any pair composed of a C*-algebra $\mathfrak{A}$ and a Jordan homomorphism $ho : A \to \mathfrak{A}_{sa}$, from $A$ into the self-adjoint part of $\mathfrak{A}$, there exists a $*$-homomorphism $\hat{\rho} : C^*_u(A) \to \mathfrak{A}$, from the C*-algebra $C^*_u(A)$ into C*-algebra $\mathfrak{A}$, such that

$$\rho = \hat{\rho} \circ \psi_A;$$
3). there exists a $*$-antiautomorphism $\Phi$ of order 2 on the C*-algebra $C^*_u(A)$, such that

$$\Phi(\psi_A(a)) = \psi_A(a),$$

$\forall a \in A$. □

Our plan is to define a continuous field of JB-algebras, the JB-algebra of the continuous field of JB-algebras, and be able in the spirit of Theorem 1 above to associate in a universal sense with each JB-algebra of the continuous field of JB-algebras a C*-algebra of the continuous field of C*-algebras.

2. CONTINUOUS FIELDS OF JB-ALGEBRAS

Let us first introduce a continuous field of JB-algebras.

Definition 1. A continuous field of JB-algebras

$$(B, \{A_t, \varphi_t\}_{t \in T}),$$

over a locally compact Hausdorff space $T$ consists of a JB-algebra $B$, a collection of JB-algebras $\{A_t\}_{t \in T}$, and a set

$$\{\varphi_t : B \to A_t\}_{t \in T},$$

of surjective morphisms, such that:

1). The function

$$t \mapsto \|\varphi_t(x)\|,$$

is in $C_0(T)$ for all $x \in B$;
2). The norm of any $x \in B$ is

$$\|x\|_B = \sup_{t \in T} \|\varphi_t(x)\|_{A_t};$$
3). For any $f \in C_0(T)$ and $x \in B$, there is an element

$$fx \in B,$$

for which

$$\varphi_t(fx) = f(t)\varphi_t(x),$$

for all $t \in T$. 
A section of the field is an element \( \{x_t\}_{t \in T} \) of \( \prod_{t \in T} A_t \), for which there is an element \( x \in B \) such that
\[
x_t = \varphi_t(x),
\]
for all \( t \in T \).

We identify \( B \) with the space of sections of the field, seen as a JB-algebra under pointwise scalar multiplication, addition, operator multiplication, by means
\[
\{\varphi_t(x)\}_{t \in T} \leftrightarrow x.
\]
In particular,
\[
x = y,
\]
iff
\[
\varphi_t(x) = \varphi_t(y),
\]
for all \( t \). It is natural that algebra \( B \) is called a the \textbf{JB-algebra of the continuous field of JB-algebras}.

Now we will establish a few properties of the continuous field of JB-algebras. The first one is about locally uniform closedness of the continuous field of JB-algebras.

**Proposition 1.** The JB-algebra \( B \) of sections of a continuous field of JB-algebras is \textit{locally uniformly closed}, i.e. if
\[
x \in \prod_{t \in T} A_t,
\]
is such that for every
\[
s \in T,
\]
and every \( \varepsilon > 0 \) there exists
\[
y_s \in B,
\]
and a neighborhood
\[
V_s \subset T,
\]
of \( s \) in which
\[
\|x_t - \varphi_t(y_s)\| < \varepsilon,
\]
for all
\[
t \in V_s,
\]
and also
\[
\lim_{t \to \infty} \|x_t\| = 0,
\]
then
\[
x \in B.
\]
Alternatively, if the function
\[
t \mapsto \|x_t - z_t\|,
\]
lies in \( C_0(T) \) for each
\[
z \in B,
\]
then
\[
x \in B.
\]
Proof. Under conditions of the first part of the Proposition, there exists a compact
set
\[ K \subseteq T, \]
for which
\[ \| x_t \| < \varepsilon, \]
outside of \( K \), as well as a finite cover
\[ \{ V_{t_1}, \ldots, V_{t_n} \}, \]
of \( K \),
\[ K \subseteq \{ V_{t_1}, \ldots, V_{t_n} \}. \]

Now we have to recall a notion of a partition of unity on \( K \) subordinate to this
cover (see for example [10] and [7]). Let \( K \) be a Hausdorff space, and
\[ \{ V_\alpha \}_{\alpha \in \Lambda}, \]
be a \textbf{locally finite open cover} of \( K \), i.e. each point of of \( K \) has a neighborhood
that intersects only with a finite number of the sets \( V_\alpha \). A \textbf{partition of unity
subordinate to the given cover} is a collection of positive functions
\[ \{ u_\alpha \}_{\alpha \in \Lambda}, \]
such that each \( u_\alpha \) is a compactly supported continuous real-valued function with
\[ \sum_{\alpha \in \Lambda} u_\alpha = 1. \]
A partition of unity always exists when \( K \) is paracompact (see [10]). So, let us take
a partition of unity
\[ \{ u_i \}_{i=1}^n, \]
on \( K \) subordinate to the aforementioned finite cover
\[ \{ V_{t_1}, \ldots, V_{t_n} \}. \]
Let us consider the
\[ y = \sum_{i=1}^n u_i y_{t_i}. \]
From Definition 1.3 it follows that
\[ y \in B, \]
and satisfies the condition
\[ \sup_{t \in T} \| x_t - y_t \| < \varepsilon. \]
Therefore, from Definition 1.2 and completeness of \( B \) it follows that
\[ x \in B. \]
Now, given any
\[ x \in \prod_{t \in T} A_t, \]
and
\[ s \in T, \]
because \( \varphi_s \) is surjective, there exists an element
\[ y_s \in B, \]
such that
\[ x_s = \varphi_s(y_s). \]
The assumption of the second part of Proposition 1 then implies that the conditions in the first part of this Proposition are satisfied, such that
\[ x \in B. \]

The following Proposition gives conditions for the existence and uniqueness of a continuous field of JB-algebras whose collection of sections contains a subset possessing some natural properties.

**Proposition 2.** Let
\[ \{A_t\}_{t \in T}, \]
be a family of JB-algebras indexed by a locally compact Hausdorff space \( T \), and a subset
\[ \tilde{B} \subseteq \prod_{t \in T} A_t, \]
that satisfies the following properties:
1). The set
\[ \{x_t : x \in \tilde{B}\}, \]
is dense in \( A_t \) for each \( t \in T \);
2). The function
\[ t \mapsto \|x_t\|, \]
lies in \( C_0(T) \) for each \( x \in \tilde{B} \);
3). The set \( \tilde{B} \) is a Jordan algebra under pointwise operations.

Then there exists a unique continuous field of JB-algebras
\[ (B, \{A_t, \varphi_t\}_{t \in T}), \]
whose collection of sections contains \( \tilde{B} \). Namely, \( B \) consists of all
\[ x \in \prod_{t \in T} A_t, \]
for which the function
\[ x \mapsto \|x_t - z_t\|, \]
lies in \( C_0(T) \) for each \( z \in \tilde{B} \), regarded as JB-algebra under pointwise operations, and the norm of Definition 1.2. Finally,
\[ \varphi_t(x) = x_t, \]
\( t \in T \), is the evaluation map.

**Proof.** We show first that the algebra \( B \) defined above is locally uniformly closed. With the objects \( x, s, \varepsilon, y_s \) and \( V \) as specified in Proposition 1, take \( z \in \tilde{B} \) arbitrary, and define the functions
\[ f_{xz} : t \mapsto \|x_t - z_t\|, \]
and
\[ f_{yz} : t \mapsto \|\varphi_t(y_s) - z_t\|. \]
Using the triangle inequality for the norm in Banach space, we get that
\[ |(|\|x\| - \|y\|)| \leq \|x - y\|, \]
and that gives us that
\[ |f_{xz}(t) - f_{yz}(t)| < \varepsilon, \]
for all \( t \in V \). By assumption, the function \( f_{yz} \) is continuous, so that
\[ |f_{yz}(t) - f_{yz}(s)| < \varepsilon, \]
for all \( t \)'s in some neighborhood \( V' \) of \( s \). Combining the inequalities, we get
\[ |f_{xz}(t) - f_{xz}(s)| < 3\varepsilon, \]
for all
\[ t \in V \cap V'. \]
Therefore \( f_{xz} \) is continuous at \( s \), which was arbitrary, so that \( x \in B \) by the definition of \( B \).

Now, we show uniqueness of \( B \). Using this property one can easily see that \( B \) is a JB-algebra, and that the condition 3 in Definition 1 is satisfied. It is clear from Definition 1.1 and the definition of \( B \) in Proposition 2 that \( B \) is maximal. On the other hand, according to the second part of Proposition 1, \( B \) is minimal, so, \( B \) is unique. □

We are ready now to present the main result of the paper.

**Definition 2.** An *-isomorphism (resp. Jordan isomorphism) of continuous fields of C*-algebras (resp. JB-algebras) over the same base Hausdorff locally compact topological space \( T \) is the isometric *-isomorphism (Jordan isometric isomorphism) of the C*-algebras (resp. JB-algebras) of the continuous fields via a map respecting the fibers.

**Theorem 2.** For an arbitrary continuous field of JB-algebras
\[(B, \{A_t, \varphi_t\}_{t \in T}),\]
over a locally compact Hausdorff space \( T \), there exists a unique up to an *-isomorphism a continuous field of C*-algebras
\[(C_u^*(B), \{C_u^*(A_t), \widehat{\varphi}_t\}_{t \in T}),\]
(the universal enveloping continuous field of C*-algebras for the continuous field of JB-algebras
\[(B, \{A_t, \varphi_t\}_{t \in T})\]
over the same base space \( T \), and a Jordan homomorphism
\[\psi_B : B \to C_u^*(B)_{sa},\]
from \( B \) to the self-adjoint part of \( C_u^*(B) \), as well as a family of Jordan homomorphisms
\[\psi_{A_t} : A_t \to C_u^*(A_t)_{sa},\]
t \( t \in T \), from \( A_t \) to the self-adjoint part of \( C_u^*(A_t) \), for each \( t \in T \), such that:
1. \( \psi_B(B) \) generates \( C_u^*(B) \) as a C*-algebra, and each \( \psi_{A_t}(B) \) generates each \( C_u^*(A_t) \) as a C*-algebra for each \( t \in T \);
2. for any pair composed of a continuous field of C*-algebras \((B, \{A_t, \varphi_t\}_{t \in T})\) and a family of Jordan homomorphisms
\[\rho : B \to B_{sa},\]
from $B$ into the self-adjoint part of $\mathfrak{B}$, and
\[ \rho_t : A_t \to (\mathfrak{A}_t)_{sa}, \]
for each $t \in T$, from $A_t$ into the self-adjoint part of $\mathfrak{A}_t$, there exist a \*\-homomorphism
\[ \hat{\rho} : C^*_u(B) \to \mathfrak{B}, \]
from the C*-algebra $C^*_u(B)$ into C*-algebra $\mathfrak{B}$, and a family of \*\-homomorphisms
\[ \hat{\rho}_t : C^*_u(A_t) \to \mathfrak{A}_t, \]
for each $t \in T$, from the C*-algebra $C^*_u(A_t)$ into C*-algebra $\mathfrak{A}_t$ such that
\[ \rho = \hat{\rho} \circ \psi_B, \]
and
\[ \rho_t = \hat{\rho}_t \circ \psi_{A_t}, \]
for each $t \in T$;
3). there exists a \*\-antiautomorphism $\Phi$ of order 2 on the C*-algebra $C^*_u(B)$, such that
\[ \Phi(\psi_B(x)) = \psi_B(x), \]
\[ \forall x \in B, \text{ as well as there exists a family of \*\-antiautomorphism $\Phi_t$ of order 2 on the C*-algebra } C^*_u(A_t), \text{ for each } t \in T, \text{ such that } \]
\[ \Phi(\psi_{A_t}(x_t)) = \psi_{A_t}(x_t), \]
\[ \forall x_t \in A_t, \text{ and every } t \in T. \]

Proof. Let $(B, \{A_t, \varphi_t\}_{t \in T})$ be a given continuous field of JB-algebras. Let $C^*_u(B)$ be the universal enveloping C*-algebra for the JB-algebra of the continuous field, and the family of C*-algebras $C^*_u(A_t)$ for each $t \in T$ be the universal enveloping C*-algebra for the JB-algebra $A_t$, $t \in T$. Let
\[ \psi_B : B \to C^*_u(B)_{sa}, \]
and for each $t \in T$,
\[ \varphi_t : B \to A_t, \]
and
\[ \psi_{A_t} : A_t \to C^*_u(A_t)_{sa}. \]

From Theorem 1 it follows that $\psi_B(B)$ is dense in $C^*_u(B)_{sa}$, and $\psi_{A_t}(A_t)$ is dense in $C^*_u(A_t)_{sa}$ for each $t \in T$. So, without a loss of generality we can assume that for each
\[ x, y \in C^*_u(B)_{sa} \]
there exist $a_n, b_n \in B$, such that
\[ x = \lim_{n \to \infty} \psi_B(a_n), \]
and
\[ y = \lim_{n \to \infty} \psi_B(b_n), \]
where the limit is taken in the norm of $C^*_u(B)$, as well as
\[ x_t = \varphi_t(x) = \lim_{n \to \infty} \psi_{A_t}(\varphi_t(a_n)), \]
and
\[ y_t = \varphi_t(y) = \lim_{n \to \infty} \psi_{A_t}(\varphi_t(b_n)). \]
$t \in T$, where the limit is taken in the norm of $C^*_u(A_t)$. For each $t \in T$, we will define
\[ \hat{\varphi}_t : C^*_u(B) \to C^*_u(A_t), \]
the following way:
\[ \hat{\varphi}_t(x + iy) = x_t + iy_t. \]
Because $C^*_u(A_t)_{sa}$ is norm closed for each $t \in T$ (see [9]), the last identity is well defined. Moreover, from the fact that $\varphi_t$ is surjective for each $t \in T$ it follows that $\hat{\varphi}_t$ is surjective for each $t \in T$ as well. Thus,
\[ (C^*_u(B), \{C^*_u(A_t), \hat{\varphi}_t\}_{t \in T}), \]
is in fact a continuous field of $C^*$-algebras. The rest of the Theorem is obtained as a corollary by application of Theorem 1 in fibers, and Propositions 1 and 2.

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