HOMOLOGICAL STABILITY FOR UNLINKED EUCLIDEAN CIRCLES IN $\mathbb{R}^3$

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Abstract. We prove a homological stability theorem for unlinked Euclidean circles in $\mathbb{R}^3$ and use a theorem of Hatcher and Brendle to deduce homological stability for unlinked embedded circles. We discuss generalizations to unlinked Euclidean spheres with additional structure and applications to homological stability for various groups of diffeomorphisms of connected sums of $D^2 \times S^1$.

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1. Introduction

1.1. Statements. We start by introducing our objects of study, the spaces $C_k$ and $E_k$.

Definition 1.1. For every open subset $U \subset \mathbb{R}^3$ we define a space of unlinked circles in $U$ by

$$C(U) = \bigsqcup_{k \geq 1} \left( \text{Emb}^{\text{unl}} \left( \bigsqcup_{k} S^1, U \right) / \text{Diff} \left( \bigsqcup_{k} S^1 \right) \right)$$

where $\text{Emb}(-, -)$ is the space of embeddings in the $C^\infty$-topology, $\text{Emb}^{\text{unl}}(-, -)$ is the subspace of unknotted unlinked circles, and finally $\text{Diff}(-)$ is the topological group of diffeomorphisms in the $C^\infty$-topology.

We will be interested in the case $U = I^3$ with $I = (0, 1)$ the open unit interval. Note that $C(I^3)$ is a deformation retract of $C(\mathbb{R}^3)$. We denote $C(I^3)$ by $C$ and we define $C_k$ to be the connected component consisting of $k$ circles, so that we get a decomposition

$$C = \bigsqcup_{k \geq 1} C_k$$

In $\mathbb{R}^3$ we have a notion of an Euclidean circle. This is a subset of $\mathbb{R}^3$ given by the image of a standard circle $\{(x, y, z) \in \mathbb{R}^2 | \sqrt{x^2 + y^2} = r \text{ and } z = 0\}$ under an isometry of $\mathbb{R}^3$ with the standard Euclidean metric.

Definition 1.2. The space $E_k$ of $k$ unlinked Euclidean circles in $I^3$ is the subspace of $C_k$ consisting of Euclidean circles.
Though the space \( \bigsqcup_{k \geq 1} C_k \) is an algebra over the operad \( D_3 \) of little 3-cubes, the space \( \bigsqcup_{k \geq 1} E_k \) is only an algebra over the suboperad \( D_3^{conf} \) of conformally embedded little 3-cubes. Elements of \( D_3 \) that are also conformal embeddings are the same as embeddings of \([0,1]^3 \) into \([0,1]^3 \) that are given as a composition of translations and multiplications by a scalar. In particular, fixing elements \( a \in E_1 \) and \( \phi \in D_3^{conf}(2) \), we get a stabilization map

\[
t := \phi(-,a) : E_k \to E_{k+1}
\]

These are unique up to homotopy, as both \( E_1 \) and \( D_3^{conf}(2) \) are path-connected. One way to see the latter is by noting that \( D_3^{conf} \) is a deformation retract of \( D_3 \) and that \( D_3(2) \) is path-connected. We prefer to take \( \phi \) to be the element of \( D_3^{conf}(2) \) given by the two embeddings \((x,y,z) \mapsto (\frac{1}{2}x,\frac{1}{2}y,\frac{1}{2}z)\) and \((x,y,z) \mapsto (\frac{1}{2} + \frac{1}{2}x,\frac{1}{2} + \frac{1}{2}y,\frac{1}{2} + \frac{1}{2}z)\), so that stabilization adds a new circle in the right-back-top corner of the cube. This induces a well-defined stabilization map on homology

\[
t_\ast : H_\ast(E_k) \to H_\ast(E_{k+1})
\]

The following is our main theorem, which we generalize in theorem 4.2 to unlinked Euclidean \( m \)-spheres in \( \mathbb{R}^n \) for \( n \geq 2 \) and \( 0 \leq m \leq n - 1 \).

**Theorem 1.3.** The map \( t_\ast : H_\ast(E_k) \to H_\ast(E_{k+1}) \) is an isomorphism for \( * < \lfloor \frac{k}{2} \rfloor \) and a surjection for \( * = \lfloor \frac{k}{2} \rfloor \).

**Remark 1.4.** Corollary 1.2 of [HW10] proves homological stability for the sequence of groups \( \pi_1(E_k) \). These groups are known as string motion groups. Wilson also proves representation stability for pure string motion groups [Wil12] [Wil13], implying rational homological stability for string motion groups, and proves vanishing of their rational cohomology in positive degrees. Griffin gives a method to compute the integral cohomology in [Gri10]. It is known that \( E_k \) is not a \( K(\pi_1(E_k),1) \) because \( E_k \) is homotopy equivalent to a finite CW complex while \( \pi_1(E_k) \) contains torsion, see the remarks preceding theorem 3 of [BH13].

We now discuss two corollaries of this result. For the first corollary, we use theorem 1.3 to deduce homological stability for the spaces \( C_k \). This requires theorem 4.1 of [BH13]:

**Theorem 1.5** (Brendle-Hatcher). The inclusion \( E_k \hookrightarrow C_k \) is a homotopy equivalence.

Using the same \( \phi \in D_3^{conf}(2) \) and \( a \in E_1 \subset C_1 \) to add a new circle in the corner induces a stabilization map \( t_\ast : H_\ast(C_k) \to H_\ast(C_{k+1}) \) on homology, which is compatible with the inclusions \( E_k \hookrightarrow C_k \) and \( E_{k+1} \hookrightarrow C_{k+1} \). Hence the previous two theorems imply the following corollary:

**Corollary 1.6.** The map \( t_\ast : H_\ast(C_k) \to H_\ast(C_{k+1}) \) is an isomorphism for \( * < \lfloor \frac{k}{2} \rfloor \) and a surjection for \( * = \lfloor \frac{k}{2} \rfloor \).

Martin Palmer, in his thesis [Pal12], has proved a closely related result; for \( M \) any open manifold of dimension \( m \) and \( N \) a compact manifold of dimension \( n \), he proved homological stability for unlinked embeddings of \( k \) copies of \( N \) in \( M \) when \( m \geq 2n + 3 \) and the stabilization comes from an unlinked copy of the manifold in the boundary. Here unlinked is defined to mean that there exist disjoint Euclidean neighborhoods of \( M \) each containing a single of copy of \( N \) isotopic to the standard copy in the boundary. One expects that this result can be improved for particular \( N \), for example where \( N \) is a sphere, as mentioned in [Pal12].

For the second corollary, we use Hatcher’s proof of the Smale conjecture to deduce that \( C_k \) and various decorated variations of it are classifying spaces for groups of diffeomorphisms of the 3-manifold \( \#_k D^2 \times S^1 \) satisfying various types of boundary conditions. In particular, see theorem 5.3 for homological stability of \( BDiff(\#_k D^2 \times S^1; G; D) \), the classifying space of diffeomorphisms fixing a disk in the interior and permuting the \( k \) boundary components, preserving a given identification for each of these boundary components with a torus \( T^2 \).

These results prompt two obvious follow up questions. Firstly, what is the stable homology? Secondly, does a similar homological stability result hold for other 3-manifolds than \( \mathbb{R}^3 \)?
1.2. What is the stable homology? There are two candidates for spaces computing the stable homology, in the sense that their homology is equal to that of $E_k$ in a range increasing as $k \to \infty$. More precisely, we have a pair of maps

$$C_k(\mathbb{R}^3; \mathbb{R}P^2) \to E_k \to \Omega^3\text{Thom}(T\mathbb{R}P^2)$$

both of which one might hope are a homology equivalence in a range.

First, we will describe the space $C_k(\mathbb{R}^3; \mathbb{R}P^2)$ and its map into $E_k$. Let $C_k(\mathbb{R}^3; \mathbb{R}P^2)$ be the configuration space of $k$ points labeled by points in $\mathbb{R}P^2$, i.e. $C_k(\mathbb{R}^3; \mathbb{R}P^2) = \{(x_1, \ldots, x_k, L_1, \ldots, L_k) \in (\mathbb{R}^3)^k \times (\mathbb{R}P^2)^k | x_i \neq x_j \text{ for } i \neq j\}/\mathcal{S}_k$. These spaces satisfy homological stability when increasing the number of points in the configuration, see for example [RW13], and the stable homology can be determined by a scanning argument to be equal to the homology of a connected component of $\Omega^3\Sigma^3\mathbb{R}P^2$ (note that all components are homotopy equivalent since it is a grouplike H-space). There is a map $C_k(\mathbb{R}^3; \mathbb{R}P^2) \to E_k$ sending a labeled configuration to the configuration of $k$ circles $\{C_i\}$ with $C_i$ having its center at the point $x_i$, lying in the affine plane through $x_i$ orthogonal to $L_i$ and having radius $\frac{\min_{i \neq j}(|x_i - x_j|)}{\psi}$.

The following lemma shows that this first map does not compute the stable homology of $E_k$.

**Lemma 1.7.** For $k \geq 2$ we have that $H_1(C_k(\mathbb{R}^3; \mathbb{R}P^2)) = (\mathbb{Z}/2\mathbb{Z})^2$ and $H_1(E_k) = (\mathbb{Z}/2\mathbb{Z})^3$.

**Sketch of proof.** Homological stability for labeled configuration spaces tells us that $H_1(C_k(\mathbb{R}^3; \mathbb{R}P^2)) = H_1(C_k(\mathbb{R}^3; \mathbb{R}P^2))$ for all $k \geq 2$, see e.g. theorem A of [RW13]. Let $\tilde{C}_2(\mathbb{R}^3; \mathbb{R}P^2)$ denote the labeled configuration space, defined as $\{(x_1, x_2, L_1, L_2) \in (\mathbb{R}^3)^2 \times (\mathbb{R}P^2)^2 | x_1 \neq x_2\}$. This space is part of two fiber sequences

$$\mathbb{R}P^2 \times \mathbb{R}P^2 \to \tilde{C}_2(\mathbb{R}^3; \mathbb{R}P^2) \to S^2$$

$$\tilde{C}_2(\mathbb{R}^3; \mathbb{R}P^2) \to C_2(\mathbb{R}^3; \mathbb{R}P^2) \to B\mathcal{S}_2$$

Doing the corresponding two Serre spectral sequences, the second of which involves local coefficients, gives the desired result. More precisely, one resolves the differential in the first spectral sequence by comparing to the fiber sequence $S^2 \times S^2 \to C_2(\mathbb{R}^3, S^2) \to S^2$ and resolves the extension issue in the second spectral sequence by comparing to $\tilde{C}_2(\mathbb{R}^3; S^2) \to C_2(\mathbb{R}^3, S^2) \to B\mathcal{S}_2$.

For $H_1(E_k)$ one abelianizes the presentation of Brendle and Hatcher for $\pi_1(E_k)$ as given in proposition 3.7 of [BH13].

The map out of $E_k$ is a scanning map, for example used in [GTMW09]. That scanning map is defined for 1-dimensional compact submanifolds of $\mathbb{R}^3$ in general, but one can of course restrict it to configurations of $k$ unlinked Euclidean circles.

We will give a quick sketch of its definition. Consider the universal bundle $\gamma$ over $\mathbb{R}P^2$, the projective space of lines in $\mathbb{R}^3$. Its orthogonal complement $\gamma^\perp$ is the two-dimensional vector bundle obtained by taking for each line $L \in \mathbb{R}P^2$ the orthogonal complement in $\mathbb{R}^3$. This is isomorphic to $T\mathbb{R}P^2$. The space $\text{Thom}(T\mathbb{R}P^2)$ is the one-point compactification of the total space of $T\mathbb{R}P^2$. It should be thought of as the space of affine lines in $\mathbb{R}^3$, including a line at infinity. Then the map $E_k \to \Omega^3\text{Thom}(T\mathbb{R}P^2)$ is roughly defined as follows. We think of $S^3$ as $\mathbb{R}^3$ with a point at infinity. For a configuration of Euclidean circles $\{C_i\}$ in $\mathbb{R}^3$ there exists an $\epsilon > 0$ such that for all $x \in \mathbb{R}^3$ the ball $B_\epsilon(x)$ either has empty intersection with the configuration of circles or has intersection with a unique point closest to $x$. This $\epsilon$ can be chosen continuously in $\{C_i\}$ and allows us to define a continuous family of diffeomorphisms $B_\epsilon(x) \cong \mathbb{R}^3$. Then $\{C_i\}$ is sent to the map that sends $x$ to either (i) the point at infinity if $B_{\epsilon x}(x) \cap \{C_i\} = \emptyset$ or (ii) to the affine line tangent to the circle at the unique closest point in $B_{\epsilon x}(x) \cap \{C_i\}$ otherwise, after reparametrizing $B_{\epsilon x}(x) \cong \mathbb{R}^3$. The point at infinity in $S^3$ is sent to the point at infinity in $\text{Thom}(T\mathbb{R}P^2)$.

It is unlikely that this map is a homology equivalence in a range, as that statement can be shown to be false for Euclidean circles in Euclidean spaces of sufficiently high dimension. By the lemma 4.4 and the remarks following it, this is equivalent to comparing the spaces $\Omega^\infty\Sigma^\infty BO(2)_+$ and $\Omega^\infty MTO(1)$ by an infinite loop map, equivalently a map of spectra $\Sigma^\infty BO(2)_+ \to MTO(1)$ which we also denote by $s$. By proposition 3.1 of [GTMW09] there is a map $\tilde{s} : \Sigma^\infty BO(2)_+ \to MTO(1)$ whose definition we
will recall in a minute. Its fiber is \( MTO(2) \), which is well-known to have non-trivial homology. This is enough to prove that map \( s \) coming from scanning is not a homology equivalence, if \( s \approx s' \). To see this, note that the long exact sequence in rational homology for a fiber sequence of spectra coincides with the long exact sequence of primitives in the rational homology of the associated fiber sequence of infinite loop spaces.

**Lemma 1.8.** The maps \( s, s': \Omega^n \Sigma^\infty BO(2)_+ \to \Omega^n MTO(1) \) are homotopy equivalent.

**Sketch of proof.** Consider the Grassmannian \( Gr_2(\mathbb{R}^n) \) of unoriented 2-planes in \( \mathbb{R}^n \). Both maps are in fact induced by either (i) maps \( \Sigma^n Gr_2(\mathbb{R}^n)_+ \to \text{Thom}(\gamma_n^+) \), in the case of \( s \), or (ii) by zigzags \( \Sigma^n Gr_2(\mathbb{R}^n)_+ \leftarrow \cdots \to \text{Thom}(\gamma_n^+) \), in the case of \( s' \). Here \( \gamma_n^+ \) is the orthogonal complement to the canonical bundle over \( \mathbb{R}P^{n-1} \).

We already described the scanning map for 1-manifolds in \( \mathbb{R}^3 \) and the same construction in \( \mathbb{R}^n \) gives a map from 1-manifolds in \( \mathbb{R}^n \) to \( \Omega^n \text{Thom}(\gamma_n^+) \). The map \( \Sigma^n Gr_2(\mathbb{R}^n)_+ \to \text{Thom}(\gamma_n^+) \) which is our candidate for the computation of the stable homology is given sending a pair \((P, z) \in \Sigma^n Gr_2(\mathbb{R}^n)_+ \) of a 2-plane \( P \) in \( Gr_2(\mathbb{R}^n) \) and \( z \in \mathbb{R}^n \) to the circle \( C_P \) of radius 1 with center at the origin and lying in the affine plane through \( z \) parallel to \( P \), and scanning this 1-manifold at \( z \in \mathbb{R}^n \). The point at infinity in \( \Sigma^n Gr_2(\mathbb{R}^n)_+ \) gets mapped to the point at infinity in \( \text{Thom}(\gamma_n^+) \). In other words, it is the adjoint of the composite \( Gr_2(\mathbb{R}^n) \to E_n \hookrightarrow M_1(n) \to \Omega^n \text{Thom}(\gamma_n^+) \), where \( E_n \) is the space of single Euclidean circles in \( \mathbb{R}^n \) of radius one and centered at the origin and \( M_1(n) \) is the moduli space of compact 1-manifolds in \( \mathbb{R}^n \). Note that the map \( Gr_2(\mathbb{R}^n) \to E_n \) is a homeomorphism, so we like to think of the circles in \( E_n \) as being described by the plane in which they lie.

The map \( s' \) involves the following general construction. Let \( V \) and \( W \) be vector bundle over a finite CW complex \( B \), then there is a map \( \text{Thom}(V) \to \text{Thom}(V \oplus W) \), whose cofiber is \( \Sigma\text{Thom}(p^*V)_+ \), where \( p \) is the projection from the sphere bundle \( S(W) \) of \( W \) to \( B \). Here the suspension coordinate is naturally given in \((0, \infty)\), which we identify with \( \mathbb{R} \) using the logarithm.

Taking \( B = Gr_2(\mathbb{R}^n) \), \( V = \theta_n^+ \), \( W = \theta_n \), where \( \theta_n \) is the universal 2-plane bundle over \( B \), this construction gives us a map \( \Sigma^n Gr_2(\mathbb{R}^n)_+ \to \Sigma\text{Thom}(p^*\theta_n^+) \). Consider the inclusion \( \mathbb{R}P^{n-2} \to S(W) \) induced by mapping a line \( L \) to the pair \([L \oplus e_n, e_n]\) of the 2-plane \([L \oplus e_n]\) spanned by \( L \) and \( e_n \) and the element \( e_n \) of the sphere in \([L \oplus e_n]\). This pulls \( p^*\theta_n \) back to \( \gamma_n^{-1} \) and is highly connected, so we get a zig-zag of highly connected maps \( \Sigma^n Gr_2(\mathbb{R}^n)_+ \leftarrow \cdots \to \Sigma\text{Thom}(\gamma_n^{-1}) \). Finally, the inclusion \( \mathbb{R}P^{n-1} \to \mathbb{R}P^n \) pulls \( \gamma_n^{-1} \) back to \( \mathbb{R} \oplus \gamma_n^{-1} \) and is highly connected, so we get a zigzag of highly connected maps

\[
\Sigma^n Gr_2(\mathbb{R}^n)_+ \leftarrow \cdots \to \Sigma\text{Thom}(\gamma_n^+)
\]

It follows from the argument in proposition 3.1 of [GTMW09] that this zigzag of maps induces \( s': \Sigma^\infty BO(2)_+ \to MTO(1) \).

The proof that \( s \) and \( s' \) are homotopy equivalent now boils down to the statement that the following diagram commutes up to homotopy

\[
\begin{array}{ccc}
\Sigma^n Gr_2(\mathbb{R}^n)_+ & \longrightarrow & \Sigma^n(E_n)_+ \\
\downarrow & & \downarrow \\
\Sigma^n Gr_2(\mathbb{R}^n)_+ & \longrightarrow & \Sigma^n M_1(n)_+ \rightarrow \Sigma\text{Thom}(\gamma_n^+)
\end{array}
\]

\[
\begin{array}{ccc}
\Sigma^n Gr_2(\mathbb{R}^n)_+ & \longrightarrow & \Sigma\text{Thom}(p^*\theta_n^+) \\
\downarrow & & \downarrow \\
\Sigma^n Gr_2(\mathbb{R}^n)_+ & \longrightarrow & \Sigma\text{Thom}(\gamma_n^{-1}) \rightarrow \Sigma\text{Thom}(\gamma_n^+)
\end{array}
\]

Here the dotted vertical map sends a pair \((P, z) \in \Sigma^n(E_n)_+ \) with \( P \in Gr_2(\mathbb{R}^n) \) and \( z_P \neq 0 \in \mathbb{R}^n \) to the 4-tuple \((P, \frac{z}{|z|}, z_P, \log(|z_P|))\) of \( P \in Gr_2(\mathbb{R}^n), \frac{z}{|z|} \in S(P), z_P \in P^n \) and \( \log(|z_P|) \in \mathbb{R} \). Here \( z_P \) and \( z_P \) are the components of \( z \) in the \( P \)-plane and orthogonal to the \( P \)-plane respectively. If \( z_P = 0 \) the value is the point at infinity in \( \Sigma\text{Thom}(p^*\theta_n^+) \), which is also where the point at infinity in \( \Sigma^n(E_1)_+ \) is sent. This makes the left square commutative

The dotted diagonal map sends a 4-tuple \((P, s, v^+, t) \) to \(([s^+], v^+ + ts) \) where \([s^+]\) is the line in \( \mathbb{R}^n \) spanned by the vector in \( P \) orthogonal to \( s \). We claim this makes the top-left triangle commute up to homotopy. The composite of the two dotted maps sends \((P, z) \) to \(([|z_P|^2], z_P + \frac{\log(|z_P|)}{|z_P|^2}|z_P|) \) if \( z_P \neq 0 \), which is the pair consisting of the line through the component of \( z \) in the \( P \)-plane and the point
then the stabilization map $C_k$ we can find an embedding $\phi$ unlinked with better properties. To be able to define these, we need to describe three canonical subspaces of the standard circle. Following the results of Palmer’s thesis \[ \text{Remark 1.9.} \] similarly, Euclidean $d$-spheres give models for the maps $\Sigma^\infty BO(d)_+ \to MTO(d-1)$ and oriented Euclidean $d$-spheres give models for the maps $\Sigma^\infty BSO(d)_+ \to MTSO(d-1)$.

The upshot of this discussion is that the question of the stable homology is open for now:

**Question 1.10.** What is the stable homology of the $E_k$?

1.3. **Does a similar homological stability result hold for other 3-manifolds than $\mathbb{R}^3$?** Unlinked Euclidean circles don’t make sense in this generality, but unlinked embedded circles do. More precisely, for every 3-manifold $M$ one can define a space of unlinked circles in $M$ by

$$ C(M) = \bigsqcup_{k \geq 1} \left( \text{Emb}^{\text{unl}} \left( \bigsqcup_k S^1, M \right) / \text{Diff} \left( \bigsqcup_k S^1 \right) \right) $$

where $\text{Emb}^{\text{unl}}(-,-)$ is now the subspace consisting of those embeddings $f$ having the property that we can find an embedding $\phi : \bigsqcup_k \mathbb{R}^3 \to M$ such that $f$ is equal to $\phi$ restricted to $\bigsqcup_k S^1$ with $S^1 \subset \mathbb{R}^3$ the standard circle. Following the results of Palmer’s thesis [Pal12], we conjecture:

**Conjecture 1.11.** Let $M$ be any connected 3-manifold that is the interior of a manifold with boundary. Then the stabilization map $C_k(M) \to C_{k+1}(M)$ obtained by adding a new circle along a chosen disk in the boundary is a homology equivalence in a range increasing to $\infty$ as $k$ goes to $\infty$.

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2. **Reductions to more structured subspaces**

In this section we give several statements about subspaces of $E_k$ consisting of configurations of circles with better properties. To be able to define these, we need to describe three canonical subspaces of $\mathbb{R}^3$ associated to an Euclidean circle $C$ in $I^3$.

(i) The first is its **center**: if $C$ is the image of $\{(x,y,z) \in \mathbb{R}^2 | \sqrt{x^2 + y^2} = r \text{ and } z = 0\}$ under an isometry $f$, then its center is the image of the origin under the same isometry. We call the number $r$ appearing in the previous definition the **radius** of the disk. These two definitions should not be surprising.

(ii) The second is its **disk**: if $C$ is the image of $\{(x,y,z) \in \mathbb{R}^2 | \sqrt{x^2 + y^2} = r \text{ and } z = 0\}$ under an isometry $f$, then its disk is the image of $\{(x,y,z) \in \mathbb{R}^3 | \sqrt{x^2 + y^2} < r \text{ and } z = 0\}$ under that isometry.

(iii) The third is its **microcosm**: if $c$ is the center of $C$ and $r$ its radius, then it is the subspace $\{(x,y,z) \in \mathbb{R}^3 | ||(x,y,z) - c|| \leq 2r\}$.

Given a configuration $\{C_i\}$ of unlinked Euclidean circles we let $\{c_i\}$, $\{r_i\}$, $\{D_i\}$ and $\{M_i\}$ denote the corresponding collections of circles, radii, disks and microcosms respectively. Microcosms were defined by Brendle and Hatcher for the notion of complexity. The complexity of a configuration of Euclidean spaces is defined as follows: if $r_i$ is the radius of $C_i$, then we take the
maximum of all numbers $r_i/r_j$ with $r_i \leq r_j$ for all pairs of circles $C_i$ and $C_j$ whose microcosms intersect. It is 0 if no microcosms intersect. For $c \in (0, 1]$, let $E_k^{\leq c}$ be the subspace of $E_k$ of configurations of complexity $\leq c$. These configurations have the same homotopy type as $E_k$ by theorem 2.1 of [BH13]:

**Theorem 2.1** (Brendle-Hatcher). The inclusion $E_k^{\leq c} \hookrightarrow E_k$ is a homotopy equivalence for any $c \in (0, 1]$.

**Sketch of proof.** One thinks of a configuration of Euclidean circles in $\mathbb{R}^3$ as a configuration of half-spheres in $\mathbb{R}^3 \times \mathbb{R}_{\geq 0}$, such that the intersection of this configuration with $\mathbb{R}^3 \times \{0\}$ gives the original configuration of Euclidean circles. By intersecting $\mathbb{R}^3 \times \{t\}$ one obtains a canonical shrinking of the configuration of circles. This procedure replaces a quotient $r_i/r_j$ by $(r_i-t)/(r_j-t)$, which is strictly smaller if $r_i < r_j$. If $r_i = r_j$, then instead the circles will become smaller at the same rate while their centers stay at the same points, so that for some sufficiently large time $t$ their microcosms will no longer intersect. Thus by applying this procedure for sufficiently large amounts of time we can make the complexity arbitrarily small.

Unfortunately this procedure has the problem that the circles will eventually contract to a point and disappear. So what one does instead is slow down this shrinking procedure just before a circle disappears, so that the circle will never contract to a point, while at the same time moving the circle along with the isotopy induced by the shrinking of the other circles. The amount at which one slows down the shrinking procedure must be such that even though the circle doesn’t disappear, it is still shrinking fast enough to guarantee that the complexity doesn’t increase and the circles never intersect.

Doing this procedure in families allows one to the prove that the inclusion is a weak homotopy equivalence and since it is a map between spaces having the homotopy type of CW-complexes it is a homotopy equivalence.

We will need two subspaces of $E_k$ which consist of configurations of circles with additional properties.

**Definition 2.2.**

(i) Let $F_{k,s}$ be the subspace of $E_k$ such that at least $s$ circles $C_i$ have the following property: if another circle $C_j$ intersects its disk $D_i$, then the center $c_j$ of $C_j$ doesn’t intersect $D_i$.

We set $E_{k,s}^{\leq c} = F_{k,s} \cap E_k^{\leq c}$.

(ii) Let $D_{k,s}$ be the subspace of $E_k$ such that at least $s$ circles have no other circle intersecting their disk. We set $D_{k,s}^{\leq c} = D_{k,s} \cap E_k^{\leq c}$.

Note that $D_{k,s} \subset F_{k,s}$ and $D_{k,s}^{\leq c} \subset F_{k,s}^{\leq c}$.

**Lemma 2.3.** The inclusion $D_{k,s}^{\leq c} \hookrightarrow D_{k,s}$ is a homotopy equivalence for any $c \in (0, 1]$.

**Proof.** The procedure of the proof of theorem 2.1 of [BH13] (theorem 2.1 in this note) can be made to preserve the property that at least $s$ circles have no other circle intersecting their disk.

**Lemma 2.4.** The inclusion map $D_{k,k-s} \hookrightarrow E_k$ is $s$-connected.

**Proof.** It suffices to prove that

$$D_{k,k-s}^{\leq 1/3k} \hookrightarrow E_k^{\leq 1/3k}$$

is $s$-connected. To do this we must prove that for $0 \leq n \leq s$ we can find a dotted lift in the following diagram, making it commute up to homotopy ($S^{−1}$ is the empty set):

$$\begin{array}{ccc}
S^{n-1} & \longrightarrow & D_{k,k-s}^{\leq 1/3k} \\
\downarrow & & \downarrow \\
D^n & \longrightarrow & E_k^{\leq 1/3k}
\end{array}$$

We will construct an intermediate lift $\tilde{g}$ as below before finally constructing $G$:
The important thing to note is that $\mathcal{F}_{k,k-s}^{\leq 1/3k}$ is the complement of a finite union of submanifolds of codimension at least $s + 1$ and contains $\mathcal{D}_{k,k-s}^{\leq 1/3k}$. Using general position arguments, this means that as long as $n \leq s$ (by Sard’s lemma generically a map from a manifold of dimension $\leq s$ doesn’t intersect a codimension $\geq s + 1$ submanifold) we can modify $D^n \to \mathcal{E}_k^{\leq 1/3k}$ by a small perturbation supported away from the boundary $S^{n-1}$ so that it lands in $\mathcal{F}_{k,k-s}^{\leq 1/3k}$. This gives us the intermediate lift $\tilde{g}$.

Next we will construct $G$. To do this we define a topological category $\mathbf{D}$ of disjunction data. This is the topological category with space of objects equal to pairs $\{(C_i), (a_i)\}$ of a configuration $\{C_i\} \in \mathcal{F}_k^{\leq 1/3k}$ of circles and for each circle $C_i$ of radius $r_i$ and center $c_i$ a real number $a_i \in (r_i, 2r_i)$ such that $\{x \in \mathbb{R}^3 ||x - c_i| = a_i\}$ does not intersect any circle. The topology is given by the original topology on the configurations of circles and the discrete topology on the $a_i$. There is a unique morphism from $\{(C_i), (a_i)\}$ to $\{(C'_i), (a'_i)\}$ if and only if $\{C_i\} = \{C'_i\}$ and for each circle $C_i$ we have that $a_i \leq a'_i$.

For a given configuration $\{C_i\}$ the set of objects is non-empty by our choice of $c = \frac{1}{3k}$. To see this note that the complexity being smaller than $\frac{1}{3k}$ means it if a circle intersects the microcosm of $C_i$, its radius is at most $\frac{t}{3k}$. Since there are at most $k - 1$ circles that intersect the microcosm of $C_i$, they can’t fill up all the possible radii between $r_i$ and $2r_i$.

Let $B\mathbf{D}$ denote the geometric realization of the nerve of this category and note that forgetting the $a_i$’s gives a continuous map $B\mathbf{D} \to \mathcal{F}_k^{\leq 1/3k}$. Lemma 3.4 of [Gal11] says this map is a homotopy equivalence if the maps $\text{Ob}(\mathbf{D}) \to \mathcal{F}_k^{\leq 1/3k}$ and $\text{Mor}(\mathbf{D}) \to \mathcal{F}_k^{\leq 1/3k}$ are étale, i.e. local homeomorphism and for each $y \in \mathcal{F}_k^{\leq 1/3k}$ the fiber $B\mathbf{D}(y)$ is contractible. The étale condition follows from our choice of discrete topology on the real numbers and the condition on the fibers follow from the fact that for a fixed configuration $\{R_i\}$ the category is a product of totally ordered categories (the ordering coming from the standard ordering on the $a_i$ as elements of $\mathbb{R}$).

As a consequence we can lift $D^n$ from $\mathcal{E}_k^{\leq 1/3k}$ to $B\mathbf{D}$, obtaining $\bar{g}$ in the process:

We will give a map $H : \mathbb{R}_{\geq 0} \times B\mathbf{D} \to \mathcal{E}_k$ and a function $\tau : D^n \to \mathbb{R}_{\geq 0}$ with support away from $S^{n-1}$ such that $H(\tau(-), \bar{g}(-))$ lands in the subspace $\mathcal{D}_{k,k-s}$ of $\mathcal{E}_k$ and is the identity near $S^{n-1}$. We will also show that this map preserves configurations of complexity $\leq c$ and hence will actually land $\mathcal{D}_{k,k-s}^{\leq 1/3k}$.
To define $H$ we define a standard family of non-continuous maps $\Phi : \mathbb{R}_{\geq 0} \times \mathbb{R}^3 \to \mathbb{R}^3$ by

$$\Phi_t(x) = \begin{cases} \frac{1}{1+t^2}x & \text{if } ||x|| \leq 1 \\ x & \text{otherwise} \end{cases}$$

Given $a > 0$ and $c \in \mathbb{R}^3$ we give a one-parameter family of maps $\Phi_{a,c} : \mathbb{R}_{\geq 0} \times \mathbb{R}^3 \to \mathbb{R}^3$ by defining $\lambda_{a,c} : \mathbb{R}^3 \to \mathbb{R}^3$ to be $x \mapsto ax + c$ and then setting $\Phi_{a,c,t}(-) = \lambda_{a,c} \circ \Phi_t(\lambda_{a,c}(-))$.

Note that $\Phi_{a,c,0}$ is the identity. An element of $(t,y) \in \mathbb{R}_{\geq 0} \times BD$ is represented by a $t \in \mathbb{R}_{\geq 0}$, an element of a simplex $(s_0, \ldots, s_n) \in \Delta^n$, a configuration $\{C_j\}$ and a $(n+1)$-tuple of collections of $k$ real numbers $\{a_j\}$ (where $0 \leq j \leq n$ and $1 \leq i \leq k$). We order the circles $C_j$ by their radii $r_j$ (note that microcosms of $C_i$ and $C_j$ only intersect if their radii are different by our restriction on the complexity) and we denote the center of $C_j$ by $c_j$. We define $H$ on this element as follows:

$$H_t(\{C_j\}) = (\Phi_{a_1^0,c_1,t,s_0} \circ \cdots \circ \Phi_{a_n^0,c_n,t,s_0}) \circ (\Phi_{a_1^0,c_1-1,t,s_0} \circ \cdots \circ \Phi_{a_n^0,c_n-1,t,s_0}) \circ \cdots \circ (\Phi_{a_1,t,c_1,t,s_0} \circ \cdots \circ \Phi_{a_n,t,c_n,t,s_0}(\{C_j\}))$$

See figure 1 for an example of $H_T$ applied to a 0-simplex. This is a continuous map, because even though the maps $\Phi_{a,c}$ are not continuous on the spheres $\{x \in \mathbb{R}^3 \mid ||x-c|| = a\}$, by construction no circle intersects any of these spheres.

Let us look at what each "vertical" composite $\Phi_{a_1^m,c_k,t,s_m} \circ \cdots \circ \Phi_{a_n^m,c_1,t,s_m}$ for $0 \leq m \leq n$ does to the complexity. Every circle $C_i$ is at least being shrunk by its own map, the $i$th one $\Phi_{a_i^m,c_i,t,s_m}$, by a factor $\frac{1}{1+t^2s_m}$. These factors will all cancel out in the calculation of complexity. What about circles being shrunk by maps belonging to other circles? If a circle $C_i$ is contained in the microcosm of $C_j$, then it is either (i) contained the ball of radius $a_j^m$ around the center $c_j$ of $C_j$ and shrunk by an additional factor $\frac{1}{1+t^2s_m}$ or (ii) not contained in that ball. In first case $r_i/r_j$ is after time $t$ either (i) multiplied by $\frac{1}{1+t^2s_m}$ or (ii) the same. Thus if $r_i \leq r_j$ the quotient $r_i/r_j$ either decreases or stays constant. We conclude that the vertical composites preserve complexity $\leq c$. The full map $H_t$ is a composite of $n+1$ such maps, so $H_t$ preserves the complexity $\leq c$ subspaces.

For each $x \in D^n$ there exists a real number $T_x \geq 0$ with $H(T, \vec{g}(x))$ lying in $D_{k,k-s}$ for all $T \leq T_x$. This is true because if a circle $C_j$ intersects the disk of $C_i$, but its center doesn’t, $H$ will shrink $C_j$ faster than $C_i$ and eventually its radius will be smaller than the distance of its center to the disk of $C_i$. In a neighborhood of $S^{n-1}$ in $D^n$ we already land in $D_{k,k-s}$ and hence take $T_x = 0$. By compactness of $D^n$ and the fact that the $T_x$ can be chosen locally bounded above, we can find a continuous function $\tau : D^n \to \mathbb{R}_{\geq 0}$ that $\tau(x) \geq T_x$ for all $x \in D^n$. Using this we can define the lift $G : D^n \to D_{k,k-s}^{\leq c}$ as $G(x) = H(\tau(x), \vec{g}(x))$.

We end this section by introducing some notation.

**Definition 2.5.** The circles in a configuration $\{C_i\}$ in $D_{k,k-s}$ that have no other circle intersecting their disk will be called the **good circles**.

Using this definition, $D_{k,k-s}$ is the subspace of $E_k$ consisting of configurations with at least $k-s$ good circles.

### 3. The proof of homological stability

To finish the proof of homological stability we do a semisimplicial resolution, followed by a standard spectral sequence argument. We will freely use the language of semisimplicial sets and spaces, and of simplicial complexes. Definitions and examples are given in appendix A.
3.1. A semisimplicial space of arcs. Suppose one starts with a configuration of circles \( \{C_i\} \in D_{k,k-s} \), then we will define a semisimplicial space of arcs connecting the centers of good circles to the boundary and prove it is highly connected. This will be special case of theorem A.7 – a generalization of an argument of Galatius and Randall-Williams in [GRW12] – which we prove in the appendix.
Definition 3.1. Fix a \( \{C_i\} \in D_{k,k-s} \), then the semisimplicial space \( A_*([C_i]) \) has 0-simplices the space of 4-tuples \( (t, \lambda, \eta, \gamma) \) of \( t \in (0, 1) \), \( \lambda \in (0, \infty) \), \( \eta \in (0, \lambda] \) and embedded arcs \( \gamma : [0, \lambda] \to [0, 1]^3 \) with the following two properties:

(i) They must start at a point \((t, t, t, 0) \in [0,1]^3\), look like \( s \mapsto (t, t, s) \) for \( s \in (0, \eta) \) and end at a center \( c_i \) of a good circle \( C_i \), such that the tangent vector at \( \gamma_i \) at the endpoint is orthogonal to the plane containing the disk \( D_i \) of \( C_i \).

(ii) The arc \( \gamma_i \) is disjoint from all circles \( C_i \) and disjoint from the disk \( D_j \) of \( C_j \), except at its endpoints.

A \( p \)-simplex is a \((p+1)\)-tuple of \((t_i, \lambda_i, \eta_i, \gamma_i)\) for \( 0 \leq i \leq p \) as above, additionally satisfying the following two properties:

(i) They must have \( t_0 < t_1 < \ldots < t_p \).

(ii) For each \( i \neq j \) the arc \( \gamma_i \) is disjoint from \( \gamma_j \) and the disk \( D_j \) of the circle \( C_j \) to which \( \gamma_j \) connects.

The \( p \)-simplices are topologized as a subspace of \( (0,1)^{p+1} \times (0,\infty)^{p+1} \times \Emb([0,1], [0,1]^3)^{p+1} \), where the \( \gamma_i \) are rescaled by \( \lambda_i \) to give an embedding \([0,1] \to [0,1]^3\) and the embeddings have the \( C^\infty \)-topology.

Note that \( A_*([C_i]) \) is a Hausdorff ordered flag space, with ordering coming from the ordering of the \( t_i \)'s.

Proposition 3.2. The realization \( ||A_*([C_i])|| \) is at least \((k-s-2)\)-connected.

Proof. The associated discrete semiplex complex admits a map to the simplicial complex \( \Inj_n(S) \) of injective words on the set \( S \) of good circles. This simplicial complex has as vertices the elements of the set \( S \) of good circles. The \((p+1)\)-simplices are distinct \((p+1)\)-tuples of good circles. The map is given by sending each arc \( \gamma \) to the circle \( C_i \), whose center it is connected to. Recall that the complex \( \Inj_n(S) \) is weakly Cohen-Macaulay of dimension \(|S| - 1 \), see example A.2 and \(|S| \geq k-s \).

To check that the conditions of theorem A.7 hold for this map first that note that disjoint arcs have disjoint endpoints, so (i) holds. For (ii) we must check the following: suppose we are given a collection \( \{(t_1, \lambda_1, \eta_1, \gamma_1), \ldots, (t_n, \lambda_n, \eta_n, \gamma_n)\} \) and a good circle \( C_0 \) whose center \( c_0 \) is not hit by any of the arcs. Then we must show there exists an arc \( \gamma \in A_0([C_i]) \) in \( I^3 \) connecting the point \((t, t, 0) \) for some \( (t_1, \ldots, t_n) \) to \( c_0 \), which must be disjoint from all the \( \gamma_i \)'s and corresponding \( D_i \)'s. It will have some length \( \lambda \) and must be standard for some time \( \eta \). To see that such an arc exists note that \( \gamma \) must in particular be an arc in the complement in \( I^3 \) of the disks \( D_0 \) to which the \( \gamma_i \) connect, the disk \( D_0 \) of \( C_0 \) and the images of the \( \gamma_i \). This complement is path-connected and open, so there is some arc connecting \((t, t, 0) \) and \( c_0 \). The rest of the conditions on \( \gamma \) may easily be arranged. Thus we conclude that \( ||A_*([C_i])|| \) is at least \((k-s-2)\)-connected. \( \square \)

3.2. A semisimplicial resolution by arcs. Let \( D_{k,s}^* \) be the augmented semisimplicial space with \( p \)-simplices pairs of a configuration \( \{C_i\} \in D_{k,k-s} \) and a \( p \)-simplex \( \{(t_0, \lambda_0, \eta_0, \gamma_0), \ldots, (t_p, \lambda_p, \eta_p, \gamma_p)\} \) of \( A_*([C_i]) \). The topology is given as a subspace of \( D_{k,k-s}^* \times (0,1)^{p+1} \times (0,\infty)^{p+1} \times \Emb([0,1], [0,1]^3)^{p+1} \). The augmentation is the map to \( E_k \) forgetting the arcs.

This induces map \( ||D_{k,s}^*|| \to E_k \), where \(|-|\) is the geometric realisation as defined in the appendix. The connectivity of this map is the subject of the next lemma. The proof uses the notion of a Serre microfibration. The definition is similar to that of a Serre fibration, but lifting is only required for an initial segment. More precisely, a map \( f : X \to Y \) is a Serre microfibration is for each map \( D^n \to X \) and extension to \( D^n \times [0,1] \to Y \) making the diagram commute, there is an \( \epsilon > 0 \) and a lift \( D^n \times [0, \epsilon) \to X \). By proposition 2.5 of [GRW12] a Serre microfibration with \( n \)-connected fibers is \((n+1)\)-connected.

Lemma 3.3. The map \( ||D_{k,s}^*|| \to E_k \) induced by the augmentation is \( \min(s,k-s-1) \)-connected.

Proof. The augmentation map factors as \( ||D_{k,s}^*|| \to D_{k,k-s} \to E_k \). This latter is \( s \)-connected by lemma 2.4, so it suffices to investigate the connectivity of the former.

The map \( ||D_{k,s}^*|| \to D_{k,k-s} \) is a Serre microfibration with \((k-s-2)\)-connected fibers, essentially using the fact that good circles stay good under small perturbations and the parametrized isotopy extension theorem, so it is \((k-s-1)\)-connected. \( \square \)
Next we investigate the homotopy type of the space of $p$-simplices of $D_{p, s}^{k,s}$.

**Lemma 3.4.** We have that $D_{p, s}^{k,s}$ is homotopy equivalent to $D_{k-p-1,k-p-1-s}$. 

**Proof.** For a point in $D_{p, s}^{k,s}$ we call the circle $C_0$ connected to the boundary by $\gamma_0$ the first circle, etc.

Let $Y \subset D_{p, s}^{k,s}$ be the subspace consisting of $k$ circles, such that (i) there exist real numbers $0 = s_{-1} < s_0 < s_1 < s_p < 1$ so that there is a single circle with arc in each subspace $(s_{i-1}, s_i)^2 \times (0, s_i)$ lying in an horizontal plane, and the remaining $k-p-1$ circles are in the subspace $(s_p, 1)^3 \subset I^3$, and (ii) the arcs for the first $p+1$ circles are of the form $s \mapsto (t_i, t_i, s)$. By evening out the size of the boxes, translating and moving the circles such that their center and radius is fixed, one sees that this is homotopy equivalent to $D_{k-p-1,k-p-1-s}$. We will describe a retraction of $D_{p, s}^{k,s}$ onto $Y$.

We move the ordered circle labeled with an arc to the correct position one at a time, starting with the first one, which we denote $C_0$. We will describe this procedure in words for the first circle. There are three steps:

(i) We first define the number $\tilde{r}_0$ to be half of the minimum of the following numbers: (a) the radius $r_0$ of $C_0$, (b') $t_1 - t_0$, (b'') $t_0$, (c) the minimum over $i$ of the distance of $\gamma_0$ to the $\gamma_i$ and the disk $D_i$ of the circle $C_i$ whose center they connect to, (c') the minimum of the distance of $\gamma_0$ to remaining $(k-p-1)$ circles, (d) the largest real number $\varrho$ such that the $\varrho$-neighborhood of $\gamma_0$ is actually a tubular neighborhood of $\gamma_0$. The assignment of this number to a point in $D_{p, s}^{k,s}$ is continuous.

(ii) We first decrease the radius $r_0$ of $C_0$ to $\tilde{r}_0$ while keeping its center and orientation fixed. This procedure is well-defined because there is no other circle intersecting the disk of $C_0$, so it can’t hit any circles while shrinking.

(iii) Move the circle along $\gamma_0$, in the sense that its center lies on $\gamma_0$ and its orientation remains orthogonal to the tangent vector to $\gamma_0$. While doing this shrink the radius to the minimum of $\tilde{r}_0$ and the distance of the center to the plane $\mathbb{R}^2 \times \{0\} \subset \mathbb{R}^3$. Our choice of $\tilde{r}_0$ in the first step implies that by (b') and (b'') it can’t go out of $I^3$ (this also uses the additional decrease of radius mentioned before), by (c') it can’t hit other arcs or the disks of the circles they connect to, by (c'') it can’t hit other circles, and by (d) it can’t hit $\gamma_0$ itself. Thus we get a well-defined path in $G_{p, s}^{k,s}$, if while doing this decrease $\gamma_0$ and possibly $\eta_0$ in length so the center of $\gamma_0$ stays at its endpoint and $\eta_0$ still is $\leq \gamma_0$.

We stop as soon as both the following conditions are met: (a) $\lambda_0 = \eta_0$ and (b) the maximal $z$-coordinate of $C_0$ is smaller than the smallest $z$-coordinate of any of the circles without arcs, with we denote by $Z$. Note that $Z$ is continuous in $D_{p, s}^{k,s}$.

Let us also give a formula for this. To do this, let $r_0$, $c_0$ and $L_0$ denote the radius, center and direction orthogonal to the plane containing $C_0$ respectively. Let $\gamma_0$ and $\lambda_0$ be the arc attaching the center of $C_0$ with its length. We will give formulas for $r_0(t)$, $c_0(t)$, $L_0(t)$, $\lambda_0(t)$, $\eta_0(t)$ and $\gamma_0(t)$ for $t \in [0, 2]$ describing the procedure outlined above:

\[
\begin{align*}
r_0(t) &= \begin{cases} (1-t)r_0 + t\tilde{r}_0 & \text{if } 0 \leq t \leq 1 \\ \min(\tilde{r}_0, d(c_0(t), \mathbb{R}^2 \times \{0\}) & \text{if } 1 \leq t \leq 2 \end{cases} \\
c_0(t) &= \begin{cases} c_0 & \text{if } 0 \leq t \leq 1 \\ \gamma_0((2-t)\lambda_0 + (t-1)\min(\eta_0, Z)) & \text{if } 1 \leq t \leq 2 \end{cases} \\
L_0(t) &= \begin{cases} L_0 & \text{if } 0 \leq t \leq 1 \\ \left[\frac{\partial}{\partial t}(2-t)\lambda_0(2-t) + (t-1)\min(\eta_0, Z)\gamma(s) \right] & \text{if } 1 \leq t \leq 2 \end{cases}
\end{align*}
\]
\[ \lambda_0(t) = \begin{cases} 
\lambda_0 & \text{if } 0 \leq t \leq 1 \\
(2-t)\lambda_0 + (t-1)\min(\eta_0, Z) & \text{if } 1 \leq t \leq 2 
\end{cases} \]

\[ \eta_0(t) = \begin{cases} 
\eta_0 & \text{if } 0 \leq t \leq 1 \\
\min(\eta_0, \lambda_0(t)) & \text{if } 1 \leq t \leq 2 
\end{cases} \]

\[ \gamma_0(t) = \begin{cases} 
\gamma_0 & \text{if } 0 \leq t \leq 1 \\
\gamma_0((2-t)\lambda_0 + (t-1)\min(\eta_0, Z)) & \text{if } 1 \leq t \leq 2 
\end{cases} \]

These formulas should make the continuity in \( D_{k,s}^{k,s} \) obvious, because as \( \gamma_0 \) and \( Z \) are. We repeat the same procedure to the next circle \( C_1 \), changing all numbers in step (i) appropriately, etc. After we have applied it to all \( p+1 \) labeled circles, we get a point that almost lies in \( Y \). The only thing that’s left to the do is scale the remaining \( k-p-1 \) circles without arcs until they lie in the subspace (\( s_p, 1 \))³.

The resulting map is the retraction \( r : D_{k,s} \rightarrow Y \).

If \( i : Y \rightarrow D_{k,s}^{k,s} \) denotes the inclusion, then since \( r \) is at the end of a homotopy starting at the identity, \( i \circ r \simeq id_{D_{k,s}^{k,s}} \). If we start with a point in \( Y \), this homotopy stays inside \( Y \) and thus gives a homotopy \( r \circ i \simeq id_Y \).

\[ \square \]

3.3. A spectral sequence argument. We will now give the spectral sequence argument that completes the proof of theorem 1.6.

The stabilization map \( \phi \) adding a standard circle \( a \in E_1 \) using \( \phi \) induces a well-defined semisimplicial map \( t_* : D_{k,s}^{k,s} \rightarrow D_{k+1,s}^{k+1,s+1} \) of augmented semisimplicial spaces. This uses our choice of \( \phi \); a different one might interfere with the arcs. We could investigate in which range it is an isomorphism. To do this we would be running the spectral sequence for the realization of an augmented semisimplicial space, which has \( E_1 \)-page in a range equal to \( E_{p,q}^1 = H_*(\mathcal{E}_{k-1}) \) for \( p \leq -1 \) and converges to zero in a range by the connectivity of the augmentation map. Its \( d_1 \)-differentials are alternatively 0 and \( t_* \) and homological stability follows by the standard argument.

However, for ease of generalization we want to work in the relative situation and end up giving a proof very similar to that in [RW13]. In particular suppose hypothetically that in lemma 3.4 instead of getting \( D_{k,s}^{k,s} \simeq D_{k-p-1,k-p-1-s} \) we would have had as a result that \( D_{k,s}^{k,s} \simeq D_{k-p-1,k-p-1-s} \times X^{p+1} \) for some fixed space \( X \). Then we claim that as long as \( X \) is path-connected, the same proof as given below almost works verbatim. This claim is justified by footnotes pointing out which minor modifications one must make.

To work in the relative setting we define \( C_{k,s}^{k,s} \) to be the pointed augmented semisimplicial space with \( p \)-simplices equal to \( \operatorname{Cone}(D_{k,s}^{k,s} \rightarrow D_{k+1,s+1}^{k+1,s+1}) \). Since the stabilization map is compatible with the retractions as in the lemma, we have that

\[ C_{k,s}^{k,s} \simeq \operatorname{Cone}(D_{k-p-1,k-p-1-s} \rightarrow D_{k-p,k-p-s}) \]

We now need to relate these spaces to the \( \mathcal{E}_k \), so that we can apply induction over \( k \). To do this, we note that the inclusions \( D_{k-p-1,k-p-1-s} \rightarrow \mathcal{E}_{k-p-1} \) and \( D_{k-p,k-p-s} \rightarrow \mathcal{E}_{k-p} \) are \( s \)-connected by lemma 2.4. Hence the map \( C_{k,s}^{k,s} \rightarrow \operatorname{Cone}(\mathcal{E}_{k-p-1} \rightarrow \mathcal{E}_{k-p}) \) is homologically \((s+1)\)-connected.¹

We set \( C_{k,s}^{k,s} \) to be \( C_{k,s}^{k,s} \) with \( s = \lfloor \frac{k}{2} \rfloor \). The previous argument tells us that the maps \( C_k \rightarrow \operatorname{Cone}(\mathcal{E}_{k-p-1} \rightarrow \mathcal{E}_{k-p}) \) are homologically \( \lfloor \frac{k}{2} \rfloor \)-connected. Furthermore the augmentation map induces a map \( \|C_{k,s}^{k,s}\| \rightarrow \operatorname{Cone}(\mathcal{E}_k \rightarrow \mathcal{E}_{k+1}) \) which is \( \lfloor \frac{k}{2} \rfloor \)-connected as a consequence of lemma 3.3.

Let \( X_\bullet \) be a pointed augmented semisimplicial space, then there is an reduced augmented spectral sequence coming filtering the reduced fat geometric realisation \( \|X_\bullet\| \rightarrow \|X_\bullet\| \) by \( \|X_\bullet\| \leq \bigvee_{0 \leq q \leq p} X_p \wedge \Delta^p / \sim : \)

\[ E_{1,q}^1 = \tilde{H}_q(X_p) \Rightarrow \tilde{H}_{p+q+1}(\operatorname{cone}(\|X_\bullet\| \rightarrow X_{-1})) = H_{p+q+1}(X_{-1}, \|X_\bullet\|) \]

[1]In the case with \( X \)'s the map would be \( C_{p,q}^{k,s} \rightarrow \operatorname{Cone}(\mathcal{E}_{k-p-1} \rightarrow \mathcal{E}_{k-p}) \wedge (X^{p+1})_+ \) and one makes similar modifications later on in the argument.
Theorem 3.5. The map \( t_* : H_*(\mathcal{E}_k) \to H_*(\mathcal{E}_{k+1}) \) is an isomorphism for \( * < \left\lfloor \frac{k}{2} \right\rfloor \) and a surjection for \( * = \left\lfloor \frac{k}{2} \right\rfloor \).

Proof. It suffices to prove that \( \tilde{H}_0(\text{Cone}(\mathcal{E}_k \to \mathcal{E}_{k+1})) = 0 \) for \( * < \left\lfloor \frac{k}{2} \right\rfloor \). We do this by induction on \( k \).

The case \( k = 1 \) is trivially true, as both \( \mathcal{E}_1 \) and \( \mathcal{E}_2 \) are connected.

Applying the spectral sequence \( 1 \) to the pointed augmented semisimplicial space \( C_k^* \) gives a spectral sequence with \( E^1 \)-page

\[
E^1_{p,q} = \tilde{H}_q(\text{Cone}(\mathcal{E}_{k-p-1} \to \mathcal{E}_{k-p})) = \tilde{H}_q(\mathcal{E}_{k-p}, \mathcal{E}_{k-p-1})
\]

in the range \( q \leq \left\lfloor \frac{k}{2} \right\rfloor \) and converging to \( 0 \) in the range \( p + q + 1 \leq \left\lfloor \frac{k}{2} \right\rfloor \). By induction \( E^1_{p,q} = 0 \) for \( p \geq 0 \) and \( q \leq \left\lfloor \frac{k-p}{2} \right\rfloor \). This implies that \( E^1_{-1,q} = 0 \) for \( q \leq \left\lfloor \frac{k}{2} \right\rfloor \).

If \( k \) is odd then \( \left\lfloor \frac{k}{2} \right\rfloor = \left\lfloor \frac{k-1}{2} \right\rfloor \) and we are done. If \( k \) is even, write \( K = \left\lfloor \frac{k}{2} \right\rfloor \), and then an inspection of the range in which the spectral sequence is zero tells us it suffices to prove that \( d^1 : E^1_{0,K} \to E^1_{-1,K} \) is zero. We have that \( E^1_{0,K} = \tilde{H}_K(\mathcal{E}_k, \mathcal{E}_{k-1}), E^1_{-1,K} = \tilde{H}_K(\mathcal{E}_{k+1}, \mathcal{E}_k) \) and the differential between them is induced by the augmentation, i.e. the stabilization map.\(^2\) An isotopy moving \( \phi \) from the right-top-back corner to the bottom-left-front corner gives us a map diagonal in the a square

\[
\begin{array}{ccc}
\mathcal{E}_{k-1} & \xrightarrow{t} & \mathcal{E}_k \\
\downarrow & & \downarrow \\
\mathcal{E}_k & \xrightarrow{id} & \mathcal{E}_{k+1}
\end{array}
\]

where the square is commutative and the triangles commute up to homotopy. As in section 6 of [RW13], from this and the Puppe sequence we get a factorization of \( \tilde{H}_K(\mathcal{E}_k, \mathcal{E}_{k-1}) \to \tilde{H}_K(\mathcal{E}_{k-1}, \mathcal{E}_k) \) as

\[
H_K(\mathcal{E}_k, \mathcal{E}_{k-1}) \to \tilde{H}_K(\Sigma(\mathcal{E}_{k-1})_+) \to H_K(\Sigma(\mathcal{E}_{k-1})_+, \mathcal{E}_k)
\]

with the map \( \tau : \Sigma(\mathcal{E}_{k-1})_+ \to \text{Cone}(\mathcal{E}_k \to \mathcal{E}_{k+1}) \) determined by the nullhomotopies. Note that \( \tilde{H}_K(\Sigma(\mathcal{E}_{k-1})_+) \cong H_K(\mathcal{E}_{k-1}) \) and in those terms the map \( \tilde{H}_K(\Sigma(\mathcal{E}_{k-1})_+) \to \tilde{H}_K(\mathcal{E}_{k+1}, \mathcal{E}_k) \) is given as follows. Let \( a \in H_{K-1}(\mathcal{E}_{k-1}) \), then \([S^1] \otimes a \in H_K(S^1 \times \mathcal{E}_{k-1})\) and we can send this to \( H_K(\mathcal{E}_{k+1}) \) via \( \tau \) and then to \( H_K(\mathcal{E}_{k+1}, \mathcal{E}_k) \). Now consider the diagram

\[
\begin{array}{ccc}
H_{K-1}(\mathcal{E}_{k-2}) & \xrightarrow{H_K(t)} & H_K(\mathcal{E}_k) \\
\downarrow & & \downarrow \\
H_{K-1}(\mathcal{E}_{k-1}) & \xrightarrow{H_K(\mathcal{E}_k)} & H_K(\mathcal{E}_{k+1}) \quad 0 \\
\downarrow & & \downarrow \\
H_{K-1}(\mathcal{E}_{k-1}) & \xrightarrow{H_K(\mathcal{E}_{k+1})} & H_K(\mathcal{E}_{k+1}, \mathcal{E}_k)
\end{array}
\]

By the definition of relative homology the diagonal map is zero and by the inductive hypothesis the left hand vertical map is surjection. Hence \( \tau \) is zero and we are done.

\[\square\]

4. Generalizations

We next discuss two straightforward generalizations.

4.1. Higher dimension and codimension. For any \( n \geq 2 \) and \( 0 \leq m \leq n-1 \) there is a notion of unlinked embeddings of a \( k \)-tuple of \( m \)-spheres in an \( n \)-dimensional manifold \( M \), using which we define

\[
C^{n,m}(M) = \bigcup_{k \geq 1} \left( \text{Emb}^{\text{unl}} \left( \bigsqcup_k S^m, M \right) / \text{Diff} \left( \bigsqcup_k S^m \right) \right)
\]

This has connected components \( C^{n,m}_k(M) \). Also note that by general position arguments every embedding is unlinked if \( m < \frac{n-1}{2} \).

\(^2\)This is also true in the case of \( X \)'s, but one would need to use here that there is no other homology than \( H_0 = \mathbb{Z} \) coming from \( X \) in \( E_{0,K}^1 \), by the vanishing of homology of the cone in degrees \( * < K \) and Künneth.
We are only interested in these spaces for \( M = \mathbb{R}^n \) (or equivalently \( M = I^n \)). In this case we can make sense of Euclidean spheres. This is a subset of \( \mathbb{R}^n \) given by an isometric embedding of a standard sphere \( \{ z \in \mathbb{R}^{m+1} \mid \|z\| = r \} \) into \( \mathbb{R}^n \), where both \( \mathbb{R}^{m+1} \) and \( \mathbb{R}^n \) are given the Euclidean metric.

**Definition 4.1.** Let \( \mathcal{E}^{n,m}_k \) be the subspace of \( O^{n,m}_k \) consisting of \( k \)-tuples of Euclidean \( m \)-spheres in \( \mathbb{R}^n \).

It is unlikely that \( \mathcal{E}^{n,m}_k \hookrightarrow O^{n,m}_k \) is a homotopy equivalence unless \( n \leq 3 \). All the steps of the proof of theorem 1.5 rely on results that are only true in low dimension: parametrized surgery along 2-disks and Smale’s conjecture as proven by Hatcher. Indeed, for \( m \) odd the calculation in [FH78] of the rational homotopy groups of \( BDiff_+(S^m) \) can be used to show that this inclusion is not a homotopy equivalence for \( n \gg m \) and \( m \gg 0 \). This means that the proof of corollary 1.6 doesn’t generalize, but the proof of the main theorem 1.3 does.

**Theorem 4.2.** The map \( t_* : H_*(\mathcal{E}^{n,m}_k) \to H_*(\mathcal{E}^{n,m}_{k+1}) \) is an isomorphism for \( * < \lfloor \frac{k}{2} \rfloor \) and a surjection for \( * \leq \lfloor \frac{k}{2} \rfloor \).

**Proof.** If \( m = n - 1 \), then unlinked implies that all spheres have disjoint disks and hence the map \( \mathcal{E}^{n,m-1}_k \to C_k(\mathbb{R}^n) \), where \( C_k(\mathbb{R}^n) \) is the configuration space of \( k \) unordered distinct points in \( \mathbb{R}^n \), sending a sphere to its center is a homotopy equivalence. The result then follows from theorem A of [RW13], which proves homological stability for configuration spaces (in fact with a slightly better range).

For \( m \leq n - 2 \) exactly the same proof as in the case \( n = 3, m = 1 \) works, if theorem 2.1 holds in higher dimensions. But an inspection of the proof shows it generalizes to other dimensions. \( \square \)

Let us comment how some other cases other than \( m = 1, n = 3 \) recover known results.

**Example 4.3 (Homological stability for hyperoctahedral groups).** The space \( \mathcal{E}^{n,0}_k \) is homotopy equivalent to a particular type of configuration space. Let \( F_{2k}(\mathbb{R}^n) = \{(x_1, \ldots, x_{2k}) \in [\mathbb{R}^n]^{2k} \mid x_i \neq x_j \text{ for } i \neq j\} \) be the configuration space of \( 2k \) ordered points in \( \mathbb{R}^n \), then \( \mathcal{E}^{n,m}_k \) is homotopy equivalent to \( F_{2k}(\mathbb{R}^m)/(\mathbb{S}_2 \wr \mathbb{S}_k) \). The space \( F_{2k}(\mathbb{R}^m) \) is \((m-2)\)-connected by general position arguments and thus \( F_{2k}(\mathbb{R}^m)/(\mathbb{S}_2 \wr \mathbb{S}_k) \) admits a \((m-2)\)-connected map to \( B(\mathbb{S}_2 \wr \mathbb{S}_k) \). Hence our results implies homological stability for the hyperoctahedral groups \((\mathbb{S}_2 \wr \mathbb{S}_k)\) when increasing \( k \).

This was already proven by Hatcher and Wahl in much greater generality; proposition 1.6 of [HW10] gives homological stability when increasing \( k \) for \( G \wr \mathbb{S}_k \) where \( G \) is any discrete group. There is a higher-dimensional generalization of this, which replaces \( \mathbb{S}_2 \wr \mathbb{S}_k \cong O(1) \) with \( O(m) \).

**Example 4.4 (Homological stability for \( O(m) \wr \mathbb{S}_k \)).** Consider the subspace of \( Emb^\text{unl}(\prod_k S^m, \mathbb{R}^n) \) of Euclidean embeddings. This is \((n-2m-2)\)-connected by general position arguments and admits a free proper action of \( O(m) \wr \mathbb{S}_k \) with quotient \( \mathcal{E}^{n,m}_k \). This means that \( \mathcal{E}^{n,m}_k \) is equivalent to \( B(O(m) \wr \mathbb{S}_k) \) in a range and our results imply homological stability for \( O(m) \wr \mathbb{S}_k \) when increasing \( k \).

This can also be deduced from homological stability for labeled configuration spaces, by labeling with \( BG \) for \( G = O(n) \). This allows one to compute the stable homology: it is \( H_*(Q_0BG_+) \), where \( Q_0 X_+ \) denotes a component of \( \Omega^\infty \Sigma^\infty X_+ \) (all are homotopy equivalent).

### 4.2. Circles with additional structure.

In our original definitions we took the quotient of the space of ordinary embeddings of circles by the group \( \text{Diff}(\bigcup_k S^1) \cong \text{Diff}(S^1) \wr \mathbb{S}_k \) of all diffeomorphisms.

One can generalize the main theorem to many examples of circles with additional structure by changing the words “ordinary” and “all” in the previous sentence. For example, one can replace \( \text{Diff}(S^1) \) by the subgroup \( \text{Diff}_+(S^1) \) of orientation preserving diffeomorphisms. By similarly replacing \( \text{Diff}(S^1) \) with particular subgroups one can obtain spaces of parametrized (but unlabeled) circles or circles with level structures. Alternatively, one can endow circles with additional structure like a parametrization. As before, this should generalize to higher dimensions and codimensions. For example, one can obtain homological stability for \( SO(n) \wr \mathbb{S}_k \) for any \( n \geq 1 \) as a consequence. Let’s now spell out in more detail the two most important examples:
(i) If one replaces Diff(S^1) with Diff^+(S^1) as suggested above, the result is a space C^+ of collections of unlinked oriented circles in R^3, or equivalently T^3. This has components C^+_k and adding a new circle gives a stabilization map t : C^+_k → C^+_k+1. Our proof for homological stability for C_k generalizes to this setting, as long as one is careful enough to replace A_k(\{C_i\}) with A^+_k(\{C_i\}), where one demands that the arc are attached to the "bottom" of the disk, in the sense that the orientation of the circles is such that it goes around the tangent vector of the arc in counterclockwise direction. This semisimplicial space is highly connected for the same reasons; it is an ordered flag complex with a nice map to a weakly Cohen-Macaulay flag complex.

**Corollary 4.5.** The map t_* : H_* (C^+_k) → H_* (C^+_k+1) is an isomorphism for * < \lfloor \frac{k}{2} \rfloor and a surjection for * = \lfloor \frac{k}{2} \rfloor.

(ii) Let’s now define spaces of parametrized circles with a trivializations of their normal bundle. For an element \{C_i\} of \text{Emb}^{\text{uni}}(\bigsqcup_i S^1, D^3) let \text{Triv}(\{\nu_i\}) be the space of trivializations of the normal bundles of the \{C_i\}, i.e. a product of the spaces of vector bundles maps \bR^2 × S^1 → \nu_i. This forms a bundle \text{Triv}(\nu) over \text{Emb}^{\text{uni}}(\bigsqcup_i S^1, D^3), which has a \mathfrak{S}_k\text{-action extending that on the unlinked embeddings. We define } C^\nu_\nu to be the subspace of \text{Triv}(\{\nu\})/\mathfrak{S}_k of those unlinked parametrized circles with trivialization of their normal bundles that is non-winding, in the sense that the embedding \phi : \bigsqcup_i \mathbb{R}^3 → D^3 used in the definition of unlinked can be chosen in such a way that the standard trivialization of the standard S^1 ⊂ R^3 is identified under \phi with the trivializations of the circles.

By combining the modification in the case of oriented circles with the more general spectral sequence argument involving a space X (here coming from the parametrizations and the trivializations of the normal bundle), one obtains the following corollary:

**Corollary 4.6.** The map t_* : H_* (C^\nu_\nu) → H_* (C^\nu_{\nu+1}) is an isomorphism for * < \lfloor \frac{\nu}{2} \rfloor and a surjection for * = \lfloor \frac{\nu}{2} \rfloor.

5. **Homological stability for diffeomorphisms of connected sums of D^2 × S^1**

In this section we prove homological stability for diffeomorphisms of connected sums of D^2 × S^1, fixing a disk and permuting the boundary components. This is the most restricted version of the diffeomorphism group for which we can prove homological stability. Indeed, we believe that the group of diffeomorphisms of connected sums of D^2 × S^1 fixing a disk and the entire boundary is unlikely to satisfy homological stability, being a more likely candidate for representation stability.

Let \#_g D^2 × S^1 denote the connected sum of g copies of D^2 × S^1. This has g boundary components diffeomorphic to a torus T^2 and we pick a diffeomorphism with a torus for each of these, so that the boundary components are now parametrized. We furthermore pick a copy of D^3 in the interior of \#_g D^2 × S^1.

**Definition 5.1.** Let \text{Diff}(\#_g D^2 × S^1; \mathfrak{S}_\partial; D^3) be the topological group of diffeomorphisms of \#_g D^2 × S^1 which permute the boundary components diffeomorphic to T^2 preserving their identifications and fix the disk D^3, in the C^∞-topology.

**Proposition 5.2.** The space C^\nu_g is a classifying space for Diff(\#_g D^2 × S^1; \mathfrak{S}_\partial; D^3).

**Proof.** Consider the map of \text{Diff}(D^3; \partial D^3) → C^\nu_g obtained by acting on a fixed configuration \{C_i\} of embedded circles and trivializations of their normal bundle by the diffeomorphism of its ambient space. Since \text{Diff}(D^3; \partial D^3) is path-connected – in fact contractible by Hatcher’s proof of Smale’s conjecture [Hat83] – and the trivializations being non-winding is preserved by isotopies, this action is well-defined. By the isotopy extension theorem, it is surjective and furthermore a fiber bundle. The fiber is given by \text{Diff}(D^3; \mathfrak{S}_\{C_i\}_\nu; \partial D^3), the diffeomorphisms of D^3 fixing the boundary \partial D^3, permuting \{C_i\} preserving their identification and fixing the trivialization of the normal bundle of each C_i. Since \text{Diff}(D^3; \partial D^3) is contractible we conclude that C^\nu_g ≃ B\text{Diff}(D^3; \mathfrak{S}_\{C_i\}_\nu; \partial D^3) and it now suffices to prove that Diff(D^3; \mathfrak{S}_\{C_i\}_\nu; \partial D^3) ≃ Diff(\#_g D^2 × S^1; \mathfrak{S}_\partial; D^3) as topological groups.
This is true because the diffeomorphisms fixing \( \{ \tilde{C}_i \} \) pointwise and fixing their normal bundle deformation retract onto the subgroup fixing a particular choice of tubular neighborhood \( U \) up to permutation, i.e. \( \text{Diff}(D^3; \mathcal{S}_{\{ \tilde{C}_i \}, s}; \partial D^3) \simeq \text{Diff}(D^3; \mathcal{S}_U; \partial D^3) \). But the latter is homeomorphic to \( \text{Diff}(D^3 \setminus U; \mathcal{S}_U; \partial D^3) \) by cutting out the interior of the tubular neighborhoods. We now simply note that \( D^3 \setminus \text{int} U \) is diffeomorphic to \( \# gD^2 \times S^1 \) with a disk removed and put this disk back in. \( \square \)

**Theorem 5.3.** The map \( t_* : H_\ast(B\text{Diff} (\# gD^2 \times S^1; \mathcal{S}_g; D^3)) \to H_\ast(B\text{Diff} (\# g+1D^2 \times S^1; \mathcal{S}_g; D^3)) \) is an isomorphism for \( * < \left\lfloor \frac{n}{2} \right\rfloor \) and a surjection for \( * = \left\lfloor \frac{n}{2} \right\rfloor \).

**Proof.** Since \( B\text{Diff} (\# g+1D^2 \times S^1; \mathcal{S}_g; D^3) \simeq C_g^\nu \), this follows directly from corollary 4.6, after noting that the two stabilization maps coincide under this identification. \( \square \)

**Remark 5.4.** Using other spaces of circles one can prove similar homological stability results for classifying spaces of diffeomorphisms of \( \# gD^2 \times S^1 \). We list several of these:

(i) The space \( C_g \) is a classifying space for diffeomorphisms of \( \# gD^2 \times S^1 \) which fix a disk and extend over the boundary. This is equivalent to preserving the boundary setwise and preserving one of the meridians around each of the boundary tori up to isotopy. Hence one can deduce homological stability for the classifying space of these diffeomorphisms.

(ii) The space \( C_g^\nu \) is a classifying space for diffeomorphisms of \( \# gD^2 \times S^1 \) which fix a disk, preserve the boundary setwise, the orientation of the boundary and one of the meridians around each of the boundary tori up to isotopy.

**Appendix A. Ordered flag spaces**

In this section we give a general theorem for proving that semisimplicial spaces are highly connected. This is a generalization of the argument used in [GRW12] to prove that their semisimplicial space \( K_\ast(W) \) is highly connected.

First we recall the difference between semisimplicial sets and simplicial complexes. Their definitions are similar, but the main difference between them is that the vertices, 1-dimensional faces, etc. up to \( (k-1) \)-dimensional faces of a \( k \)-simplex in a semisimplicial set are ordered, while vertices, 1-dimensional faces, etc. up to \( (k-1) \)-dimensional faces of a \( k \)-simplex in a simplicial complex are not. Let’s give the precise definitions:

(i) Semisimplicial sets are functors \( X_\bullet : \Delta^{\text{op}}_{\text{fin}} \to \text{Set} \), which is equivalent to a collection of set \( X_n \) for \( n \geq 0 \) and face maps \( d_i : X_n \to X_{n-1} \) satisfying the well-known face relations. The set \( X_k \) is called the set of \( k \)-simplices. A related definition is that of a semisimplicial space; this is a functor \( X_\bullet : \Delta^{\text{op}}_{\text{fin}} \to \text{Top} \).

(ii) A simplicial complex \( Y_\circ \) is a set \( Y_0 \) of vertices and for \( n \geq 1 \) collections \( Y_n \) of \( (n+1) \) element subsets of \( Y_0 \). These must have the property that each \( n \)-element subset is in \( Y_{n-1} \). The set \( Y_k \) is called the set of \( k \)-simplices.

To distinguish between them, we use \((-)\_\bullet \) for semisimplicial sets or spaces and \((-)\_\circ \) for simplicial complexes. Both semisimplicial sets or spaces and simplicial complexes have a geometric realisation by glueing together simplices using the face maps or face relations, which we denote by \( \| X_\bullet \| \) and \( | Y_\circ | \) respectively:

\[
\| X_\bullet \| := \bigsqcup_{p \geq 0} X_p \times \Delta^p / \sim_\bullet \quad | Y_\circ | := \bigsqcup_{p \geq 0} Y_p \times \Delta^p / \sim_\circ
\]

We will be interested in using certain nice simplicial complexes, the *weakly Cohen-Macaulay flag complexes*, to prove that certain semisimplicial spaces, the *ordered flag spaces* with a map to a weakly Cohen-Macaulay flag complex having a certain lifting property, have highly connected geometric realisation. We will now define these notions.

**Definition A.1.** A simplicial complex \( X_\circ \) is said to be *weakly Cohen-Macaulay of dimension* \( \geq n \) if \( | X_\bullet | \) is \( (n-1) \)-connected and the link of each \( p \)-simplex is \( (n-p-2) \)-connected.
Example A.2. An example of a weakly Cohen-Macauley complex is the simplicial complex $\text{Inj}_p(S)$ of injective words on a set $S$. The vertices are the elements of $S$ and a collection $\{s_0, \ldots, s_p\}$ is a $p$-simplex if $s_i \neq s_j$ for $i \neq j$. Note that $|\text{Inj}_p(S)| \cong \Delta^{(p+1)}$. It is thus contractible, as are links of all $p$-simplices except for the $(|S| - 1)$-simplex, whose link is empty. From this one deduces $\text{Inj}_p(S)$ is weakly Cohen-Macauley of dimension $|S| - 1$.

Being weakly Cohen-Macauley is inherited by links. The link of a $p$-simplex in a simplicial complex that is weakly Cohen-Macauley of dimension $\geq n$ is itself weakly Cohen-Macauley of dimension $\geq n - p - 1$, see lemma 2.1 of \cite{GRW12}.

The reason for introducing the notion of weakly Cohen-Macauley is that it implies that maps into realizations of weakly Cohen-Macauley complexes can be homotoped to have nice properties. To make this precise we need the following definition:

Definition A.3. A simplicial map $f_\circ : X_\circ \to Y_\circ$ between simplicial complexes is said to be simplexwise injective if for any $p \geq 1$ and any $p$-simplex $\sigma = \{x_0, \ldots, x_p\}$ in $X_p$ with $x_i \neq x_j$ for $i \neq j$ we have that $f(x_i) \neq f(x_j)$ for $i \neq j$.

Then the following is a part of theorem 2.4 of \cite{GRW12}.

Lemma A.4 (Galatius-Randal-Williams). Let $X_\circ$ be a simplicial complex and $f : S^i \to |X_\circ|$ be map which is simplicial with respect to some PL triangulation of $S^i$. Then, if $X_\circ$ is weakly Cohen-Macauley of dimension $n$ and $i \leq n - 1$, $f$ extends to a simplicial map $g : D^{i+1} \to |X_\circ|$ which is simplexwise injective on the interior of $D^{i+1}$.

We will be interested in particular types of simplicial complexes and semisimplicial spaces, which are exactly the ones that occur most often in nature.

Definition A.5. A flag complex is a simplicial complex with the property that $(k+1)$-tuple $\{x_0, \ldots, x_k\}$ of vertices is a $k$-simplices if and only if for each $i \neq j$ the pair $\{x_i, x_j\}$ is a 1-simplex.

If two vertices are part of a 1-simplex they are said to be disjoint. The complex of injective words considered before is such a flag complex. There is also a version of this for semisimplicial sets or spaces. We give the definition in the case of spaces, as this includes the case of sets by considering these as discrete spaces.

Definition A.6. An ordered flag space is a semisimplicial space $Y_\bullet$ with the following two properties:

(i) The map $Y_k \to Y_0 \times \ldots \times Y_0$ is a homeomorphism onto its image, which is an open subset.

(ii) Given a $(k+1)$-tuple $y_0, \ldots, y_k$ of elements in $Y_0$ there is a unique reordering $\sigma : \{0, \ldots, k\} \to \{0, \ldots, k\}$ such that $(y_{\sigma(0)}, \ldots, y_{\sigma(k)}) \in Y_k \subset Y_0 \times \ldots \times Y_0$ if and only if for each $i < j$ either the pair $(y_i, y_j)$ or the pair $(y_j, y_i)$ lies in $Y_1 \subset Y_0 \times Y_0$.

Note that in particular condition (ii) endows the elements of $Y_0$ with a partial ordering: $y_0 \prec y_1$ if $y_0$ and $y_1$ are disjoint and $(y_0, y_1) \in Y_1$.

To an ordered flag space $Y_\bullet$ one can associate a semisimplicial set $Y_\bullet^\delta$ by forgetting the topology on the spaces $Y_p$ of $p$-simplices. This semisimplicial set is equivalent to a flag complex $Y_\circ^\delta$, which has as vertices the elements of the set underlying $Y_0$. A set $\{y_0, \ldots, y_k\}$ of such elements forms $k$-simplices in $Y_\circ^\delta$ if and only if there is a permutation $\sigma : \{0, \ldots, k\} \to \{0, \ldots, k\}$ such that $(y_{\sigma(0)}, \ldots, y_{\sigma(k)}) \in Y_k$. The uniqueness of the permutation in condition (iii) of the definition of an ordered flag space tells us that $Y_\bullet^\delta$ is recoverable from $Y_\circ^\delta$ and furthermore an inspection of the simplices involved in both geometric realisations tells us there is a homeomorphism

$$||Y_\bullet^\delta|| \cong |Y_\circ^\delta|$$

We are now able to prove that our theorem saying that ordered flag spaces whose associated flag complex which admits a map to a weakly Cohen-Macauley flag complex with a certain lifting property, are highly connected. This is nothing but an abstraction of the arguments in section 4 of \cite{GRW12}.
Theorem A.7. Let $X_o$ be a flag complex that is weakly Cohen-Macaulay of dimension $n$ and $Y_\bullet$ be a Hausdorff ordered flag space. Suppose we are given a map $f : Y_o^\delta \to X_o$ with the following two properties:

(i) We have that $y_0$ and $y_1$ disjoint in $Y_1$ implies that $f(y_0)$ and $f(y_1)$ are disjoint in $X_1$.

(ii) For all finite collections of elements $\{y_1, \ldots, y_k\}$ in $Y_0$ and $x_0$ not equal to and disjoint from all elements of $\{f(y_1), \ldots, f(y_k)\}$, there is a $y_0 \in Y_0$ with $f(y_0) = x_0$ with $y_0$ disjoint from the $y_i$'s and furthermore $y_0 \succ y_i$ for all $i$.

Then $Y_o^\delta$ is weakly Cohen-Macaulay of dimension $n$ and both $|Y_o^\delta|$ and $|Y_\bullet|$ are $(n-1)$-connected.

Proof. We will first prove that $|Y_o^\delta|$ is also weakly Cohen-Macaulay of dimension $n$. We first prove it is $(n-1)$-connected. Let $-1 \leq i \leq n-1$ and consider a diagram

$$
\begin{array}{ccc}
S^i & \to & |Y_o^\delta| \\
\downarrow f & & \downarrow \phi \\
D^{i+1} & \to & |X_o|
\end{array}
$$

then our goal is to find a dotted lift. We can assume that the maps $S^i \to |Y_o^\delta|$ and $D^{i+1} \to |X_o|$ are simplicial with respect some PL triangulation of their domains. Furthermore, by lemma A.4 we can assume that $g$ is simplexwise injective on the interior.

By putting a total order on the vertices $v_0, \ldots, v_N$ in the interior of $D^{i+1}$ it suffices to inductively pick lifts for vertices in the interior of $D^{i+1}$. Suppose we have already lifted $v_0$ up to $v_{l-1}$, and let $L \subset D^{i+1}$ be the subcomplex spanned by $S^l$ and the $v_i$'s for $0 \leq l \leq j-1$. Consider the set of vertices $\{w_1, \ldots, w_k\} \subset L \cap \text{Link}(x_j)$ and the set $\{y_1, \ldots, y_k\}$ of lifts of $\{w_1, \ldots, w_k\}$ to $Y_o^\delta$. Since $g$ is simplexwise injective we have that $x_0 := g(v_j)$ is not equal to any of the $x_i$'s and disjoint from all of them. Using our hypothesis we can lift it to a $y_0$ disjoint from the $y_i$'s. We conclude that this procedure after $N+1$ steps gives us the desired lift.

For the link $\text{Link}(y)$ of a $p$-simplex $y = \{y_0, \ldots, y_p\}$ of $Y_\bullet$, note that the map $f$ restricts to a map $\text{Link}(y) \to \text{Link}(f(y))$ by property (i). The link $\text{Link}(f(y))$ is itself Cohen-Macaulay of dimension at least $n - p - 1$. Hence the same argument as above gives that $\text{Link}(y)$ has the same connectivity of $\text{Link}(f(y))$, i.e. it is $(n - p - 2)$-connected.

This argument can be generalized even more. Define the upward link $\text{Link}^+(y)$ of a $p$-simplex $y = \{y_0, \ldots, y_p\}$ of $Y_\bullet$ to be the subcomplex of $\text{Link}(y)$ spanned by those $y_i$'s satisfying $y_i' \succ y_i$ for all $0 \leq i \leq p$. Then we get a map $\text{Link}^+(y) \to \text{Link}(f(y))$ and the same lifting argument as above tells us that $\text{Link}^+(y)$ is $(n - p - 2)$-connected as well.

The connectivity of $|Y_o^\delta|$ follows from the homeomorphism $|Y_o^\delta| \cong |Y_o^\delta|$. Next we want to prove that $|Y_\bullet|$ is $(n-1)$-connected. There is the identity map $\iota : Y_o^\delta \to Y_\bullet$. Interpolating this map is the bisemisimplicial space $\tilde{Y}_\bullet$ with space of $(p+q)$-simplices is equal as a set to $X_{p+q+1} \subset X_0^{X_0^{p+q+2}}$, but with the ordinary topology on the first $p+1$ copies of $X_0$ and the discrete topology on the last $q+1$ copies of $X_0$. There are two augmentations $\epsilon : \tilde{Y}_\bullet \to Y_o^\delta$ and $\delta : \tilde{Y}_\bullet \to Y_\bullet$ giving by forgetting the first, respectively second, semisimplicial direction. We have that $||\iota|| \leq ||\delta|| \leq ||\epsilon|| : ||\tilde{Y}_\bullet|| \to ||Y_\bullet||$. This homotopy is given by shifting the $(p,q)$-simplices of $\tilde{Y}_\bullet$ into the $(0,p)$-simplices via the $(p,p+q+1)$-simplices using a linear interpolation. The upshot is that the augmentation $||\epsilon||$ factors over the $(n-1)$-connected space $||\tilde{Y}_\bullet||$.

We fix $p$ and consider $||\tilde{Y}_{p,\bullet}||$ as a subspace of $Y_p \times ||Y_o^\delta||$. Our assumptions on $Y$, in particular condition (i) of ordered flag complexes and that it is Hausdorff, imply that corollary 2.8 of [GRW12] applies. This implies we suffice to consider the map $\epsilon_p : ||\tilde{Y}_{p,\bullet}|| \to Y_p$ and prove that its fibers are highly connected. The fibers over $y = \{y_0, \ldots, y_p\} \in Y_p$ are equal to geometric realisation of the subsemisimplicial space of $Y_o^\delta$ of spanned by simplices succeeding all vertices of $y$. Under the homeomorphism between $||Y_o^\delta||$ and $|Y_o^\delta|$ this exactly corresponds to the upward link $\text{Link}^+(y)$, which we proved was $(n - p - 2)$-connected before. The corollary mentioned before now implies that the map $\epsilon_p : ||\tilde{Y}_{p,\bullet}|| \to Y_p$ is $(n - p - 1)$-connected. By the realization theorem for semisimplicial spaces the map
\[ ||\epsilon|| : ||\overline{Y}_n|| \rightarrow ||Y_i|| \text{ is } (n-1)\text{-connected. But } ||\epsilon|| \text{ factors up to homotopy over the } (n-1)\text{-connected space } ||Y^3|| \text{ and we thus conclude that } ||Y_i|| \text{ is } (n-1)\text{-connected.} \]

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