Undecidability of Network Coding, Conditional Information Inequalities, and Conditional Independence Implication

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Abstract

We resolve three long-standing open problems, namely the (algorithmic) decidability of network coding, the decidability of conditional information inequalities, and the decidability of conditional independence implication among random variables, by showing that these problems are undecidable. The proof utilizes a construction inspired by Herrmann’s arguments on embedded multivalued database dependencies, a network studied by Dougherty, Freiling and Zeger, together with a novel construction to represent group automorphisms on top of the network.

Index Terms

Network coding, conditional information inequalities, conditional independence implication, index coding, Turing undecidability.

I. INTRODUCTION

Network coding [1], [2] is a setting in which each node in a network can perform encoding and decoding operations, in order to transmit some messages through the network. There are numerous research works on algorithms for network coding, e.g. [3], [4], [5], [6]. However, it was shown that various problems about network coding are NP-hard, e.g. [7], [8], [9], [10].

It was uncertain if network coding is even decidable, that is, if there exists an algorithm that can determine whether a network is solvable (i.e., admits a coding scheme satisfying the decoding constraints) [7], [11]. Network coding would be decidable if there is a computable upper bound on the alphabet size, as observed by Rasala Lehman [7]. For partial undecidability results, Kühne and Yashfe [12] proved that determining whether a network admits a vector linear network code is undecidable. A potential approach to show undecidability was proposed by Dougherty [13], which involves a reduction from Rhodes’ problem, i.e., the identity problem for finite groups, which is conjectured to be undecidable [14]. If a subset of the messages and edges have a fixed size different from the common size of other messages/edges, then Li [15] showed that the solvability of a network is undecidable. For other related works and discussions on this open problem, refer to [16], [17], [18], [11], [19], [20], [21].

A related problem is the conditional independence implication problem [22], [23], [24], [25], which is to decide whether a statement on the conditional independence among some random variables follows from a list of other such statements. Pearl and Paz [25] introduced a set of axioms, called the semi-graphoid axioms, as a proposed axiomization of conditional independence. This set of axioms was shown to be incomplete by Studený [26], [27]. Partial decidability/undecidability results has been obtained in [28], [29], [30], [31], [32], [33], [34], [35]. In particular, it was noted by Geiger and Meek [30] and Niepert [31] that if all random variables have bounded cardinalities, then the implication problem is decidable. If only a subset of the random variables have bounded cardinalities, then Li [35] has shown that the problem is undecidable. If we consider the whole first-order theory of random variables with the conditional independence relation (instead of only the implication problem), then this theory is undecidable [36]. Nevertheless, the decidability of the conditional independence implication problem remained open [30], [37], [38], [39], [40], [41].

Another closely related problem is conditional information inequalities [41], [42], [43], which is to decide whether a linear inequality involving entropy terms among some random variables follows from a list of other such inequalities. This problem generalizes the conditional independence implication problem since the conditional independence $X \perp Y | Z$ can be expressed as $I(X; Y | Z) \leq 0$. The connection to network coding was studied in [44], [45], [46]. Zhang and Yeung showed the first conditional non-Shannon-type inequality (i.e., cannot be proved using only the fact $I(X; Y | Z) \geq 0$) in [41], and the first unconditional non-Shannon-type inequality in [43]. Also see [47], [48], [49], [50], [51], [52] for more non-Shannon-type inequalities. While Shannon-type inequalities can be verified algorithmically [42], [53], there are also algorithms capable of verifying some non-Shannon-type information inequalities [50], [54], [55], [56]. If we allow affine (instead of only linear) inequalities, then it was shown by Li [55] that the problem is undecidable. The decidability of linear conditional (and unconditional) information inequalities remained open [21], [57], [58], [59], [60].

In this paper, we resolve these open problems by proving the undecidability of the three aforementioned problems. The proof of the undecidability of conditional independence implication (Theorem 3) is inspired by Herrmann’s proof [60], [61] on the
undecidability of embedded multivalued database dependencies (EMVD) [62], which uses a reduction from the uniform word problem for finite semigroups/monoids. While EMVD shares several similarities with probabilistic conditional independence among random variables (e.g., they both satisfy the semi-graphoid axioms [25]), the valid implications in probabilistic conditional independence is neither a subset nor a superset of the valid implications in EMVD [22]. Therefore, the undecidability of conditional independence implication is not a direct corollary of the undecidability of EMVD, and arguments specific to random variables are needed to show the undecidability of conditional independence implication.

The proof of the undecidability of network coding (Theorem 22) utilizes a reduction from the uniform word problem for finite groups, where the groups are embedded as subgroups of the automorphism group of an abelian group. We utilizes a network studied in [63], which captures the structure of an abelian group, together with a novel construction to represent automorphisms of the abelian group using subnetworks.

As a result, the minimum alphabet size needed to solve a network can be uncomputably large, i.e., not upper-bounded by any computable function (Corollary [30]). This is a direct corollary of the undecidability of network coding and the observation in [71] that network coding would be decidable if there is a computable upper bound on the alphabet size. Comparing to the result in [64] which showed the existence of networks which are solvable only for an alphabet size double exponential in the number of nodes and messages, in this paper we show that there are solvable networks whether even double exponential (or triple exponential, etc) would not be sufficient.

Another corollary is that network coding for multiple unicast networks [65] (i.e., each source message is available to one source node and demanded by one receiver node) is also undecidable, due to the result in [66] that a general network coding problem can be reduced to a multiple unicast setting. Index coding [67], [68] is undecidable as well, which is due to the equivalence between network coding and index coding [69], [70].

The paper is organized as follows. In Section II we introduce the Fano-non-Fano condition. In Section III we completes the proof of the undecidability of conditional independence implication by showing a reduction from the uniform word problem for finite monoids. In Section IV we prove the undecidability of network coding.

Remark 1. The proof of the undecidability of conditional independence implication (Theorem 3) originated as an attempt to adapt the proof in [60] (which concerns EMVD instead of random variables) into an argument on random variables. While the final proof of Theorem 3 in Sections II and III still follows the high-level approach in [60] (i.e., showing a reduction from the uniform word problem for finite semigroups/monoids, via an embedding into the endomorphism monoid of an abelian group), the details became rather different from [60], and many of the proof techniques in this paper are novel. The parts that are similar to [60] are marked explicitly.

The constructions in Section IV for the undecidability of network coding, except the use of the network studied in [63], are novel. Due to the design constraints of a network coding setting (network coding is more restrictive than conditional independence in the sense that some conditional independence relations cannot be enforced by network coding), we require a reduction from the uniform word problem for finite groups [71], instead of the uniform word problem for finite semigroups/monoids as in [65].

Remark 2. The main difference between the proof of the undecidability of network coding (Theorem 22) in this paper and the approach proposed by Dougherty [13] is that [13] attempts to use edges to represent words and identities in groups (an identity is in the form \( w = e \) where \( w \) is a word, that is, the equality \( w = e \) is true for all substitution of letters in \( w \) by group elements), whereas we use edges to represent elements in groups. Therefore, the approach in [13] relies on the identity problem for finite groups (i.e., whether a list of identities implies another identity), which is not known to be decidable or undecidable [14]. This contributes to one of the gaps in the approach in [13]. On the other hand, the approach in this paper relies on the uniform word problem for finite groups, which is known to be undecidable [71].

Notations

Throughout this paper, all random variables are assumed to have finite support (i.e., finite random variables). The condition that two random variables \( X, Y \) are independent is written as \( X \perp Y \). The condition that two random variables \( X, Y \) are independent conditional on \( Z \) is written as \( X \perp Y | Z \). For random variables \( X, Y \), we use juxtaposition \( XY \) to denote the joint random variable \( (X, Y) \). For a sequence of random variables \( X_1, \ldots, X_k \) and a set \( U \subseteq \{1, \ldots, k\} \), write \( X_U = (X_{u_1}, \ldots, X_{u_{|U|}}) \), where \( u_1, \ldots, u_{|U|} \) are the elements of \( U \) in ascending order. When we write \( X = Y \) for random variables \( X, Y \), this means \( X = Y \) holds with probability 1. The logical conjunction (i.e., “AND”) between two statements \( P, Q \) is denoted as \( P \land Q \). The logical conjunction between \( P_1, \ldots, P_n \) is denoted as \( \bigwedge_{i=1}^n P_i \).

II. THE FANO-NON-FANO CONDITION

We will prove the first main result in this paper about conditional independence implication.
Theorem 3. The following problem is undecidable: Given \( k, l \in \mathbb{Z}_{\geq 0} \), \( U_i, V_i, W_i \subseteq \{1, \ldots, k\} \) for \( i = 0, \ldots, l \) satisfying \( U_i \cap V_i = U_i \cap W_i = V_i \cap W_i = \emptyset \), determine whether the implication
\[
\left( \bigwedge_{i=1}^{l} X_{U_i} \perp X_{V_i}|X_{W_i} \right) \rightarrow X_{U_0} \perp X_{V_0}|X_{W_0}
\]
holds for all jointly-distributed random variables \( X_1, \ldots, X_k \) with finite support.\(^1\)

Since the problem of conditional information inequalities is a generalization of the conditional independence implication problem, it is undecidable as well. We state the result formally as follows. For a sequence of finite random variables \( X_1, \ldots, X_k \), its entropic vector \(^{[41]}\) is defined as
\[ h(X_1, \ldots, X_k) = h \in \mathbb{R}^{2^k - 1}, \]
where the entries of \( h \) are indexed by nonempty subsets of \( \{1, \ldots, k\} \), and \( h_S := H(X_S) \) (where \( S \subseteq \{1, \ldots, k\} \)) is the joint entropy of \( \{X_i\}_{i \in S} \). The following is a direct corollary of Theorem 3 and the fact that \( X \perp Y|Z \Leftrightarrow -I(X; Y|Z) = H(X, Y, Z) + H(Z) - H(X, Z) - H(Y, Z) \geq 0 \).

Corollary 4. The following problem is undecidable: Given \( k \in \mathbb{Z}_{\geq 0} \), \( a, b \in \mathbb{Q}^{2^k - 1} \), determine whether the implication
\[ a^T h(X_1, \ldots, X_k) \geq 0 \Rightarrow b^T h(X_1, \ldots, X_k) \geq 0 \]
holds for all jointly-distributed random variables \( X_1, \ldots, X_k \) with finite support.\(^2\) The problem remains undecidable if either one or both of the “\( \geq \)” marked with (i) and (ii) are replaced by “\( = \)”.

The proof of Theorem 3 is divided into Sections 1 and 1. We begin with introducing some notations. The condition that \( X, Y, Z \) are mutually independent is written as
\[ X \perp Y \perp Z \Leftrightarrow X \perp Y \land XY \perp Z. \]
The condition that \( Y \) contains no more information than \( X \), i.e., \( Y \) is a function of \( X \), is written as
\[ Y \leq X \Leftrightarrow Y \perp X. \]
Also, we write
\[ X \leq Y \Leftrightarrow X \leq Y \land Y \leq X. \]
We use the fact in \(^{[41]}\) that if \( Y_1, Y_2, Y_3 \) satisfy that any one is a function of the other two, and they are pairwise independent, then \( Y_1 \) is uniformly distributed over its support (also true for \( Y_2, Y_3 \) and \( Y_1, Y_2, Y_3 \) have the same cardinality. Also see the coordinatization via 3-net in \(^{[72]}\), \(^{[60]}\). Define the predicate \( \text{tri}(Y_1, Y_2, Y_3) \) over the variables \( Y_1, Y_2, Y_3 \) as
\[
\text{tri}(Y_1, Y_2, Y_3) :
Y_1 \leq Y_2 Y_3 \land Y_2 \leq Y_1 Y_3 \land Y_3 \leq Y_1 Y_2
\land Y_1 \perp Y_2 \land Y_1 \perp Y_3 \land Y_2 \perp Y_3.
\]
One example of \( Y_1, Y_2, Y_3 \) satisfying \( \text{tri}(Y_1, Y_2, Y_3) \) is that \( Y_1, Y_2 \sim \text{Unif}(Y) \) i.i.d., and \( Y_3 = Y_1 + Y_2 \), where \( Y \) is a finite abelian group.

\(^1\)Theorem 3 continues to hold if \( X_1, \ldots, X_k \) are discrete random variables (with finite or countably infinite support). This is because if some discrete random variables satisfy the \text{tri} condition \(^{[41]}\), then they must be finite.

\(^2\)Corollary 4 continues to hold if \( X_1, \ldots, X_k \) are discrete random variables (with finite or countably infinite support) with finite entropy.
Our construction requires the Fano matroid and the non-Fano matroid \([73]\). Let \( \mathcal{E} = \{1, 2, 3, 12, 13, 23, 123\} \) be the ground set. In the non-Fano matroid \([73, 74]\), the set of dependent sets of size 3 is given by

\[
\mathcal{D}_N := \{\{1, 2, 12\}, \{1, 3, 13\}, \{2, 3, 23\}, \{1, 23, 123\}, \{2, 13, 123\}, \{3, 12, 123\}\},
\]

which are the solid lines in Figure 1. In the Fano matroid \([73]\), the set of dependent sets of size 3 is given by

\[
\mathcal{D}_F := \mathcal{D}_N \cup \{12, 13\},
\]

which are the solid lines together with the dotted circle in Figure 1. Also write

\[
\mathcal{I}_F := \{\{i, j, k\} \in 2^\mathcal{E} : |\{i, j, k\}| = 3 \land \{i, j, k\} \notin \mathcal{D}_F\}
\]

for the set of independent sets of size 3 in the Fano matroid.

The **Fano-non-Fano condition** on the seven random variables \(A_1, A_2, A_3, A_{12}, A_{13}, A_{23}, A_{123}\) is defined as

\[
\text{fnf}(A_1, A_2, A_3, A_{12}, A_{13}, A_{23}, A_{123}) := \bigwedge_{\{i,j,k\} \in \mathcal{D}_N} \text{tri}(A_i, A_j, A_k) \land \bigwedge_{\{i,j,k\} \in \mathcal{I}_F} A_i \perp A_j \perp A_k,
\]

i.e., we enforce the dependent sets of size 3 in the non-Fano matroid, and the independent sets of size 3 in the Fano matroid.

Note that \(A_{12} \neq \{A_1, A_2\}\), and we treat the subscript 12 as the concatenation of two symbols 1 and 2 (or simply the number 12) instead of the set \(\{1, 2\}\).

Note that \(\bigwedge_{\{i,j,k\} \in \mathcal{I}_F} A_i \perp A_j \perp A_k\) in (2) can be replaced simply by \(A_1 \perp A_2 \perp A_3\), as shown below.

**Proposition 5.** The Fano-non-Fano condition holds if and only if

\[
\bigwedge_{\{i,j,k\} \in \mathcal{D}_N} \text{tri}(A_i, A_j, A_k) \land A_1 \perp A_2 \perp A_3.
\]

**Proof:** We only have to prove the “if” direction. Since \(\text{tri}(A_i, A_j, A_k)\) ensures \(A_i, A_j, A_k\) are uniform with the same cardinality, we know that \(A_i\) (\(i \in \mathcal{E}\)) are all uniform with the same cardinality (let it be \(q\)). Since \(A_2 \leq A_{12}\) by \(\text{tri}(A_1, A_2, A_{12})\), and \(A_3 \leq A_{13}\) by \(\text{tri}(A_1, A_3, A_{13})\), we have \(A_1A_2A_3 \equiv A_1A_{12}A_{13}\), and hence there is a one-to-one correspondence between the tuple \(A_1, A_{12}, A_{13}\) and \((A_1, A_2, A_3)\), and the tuple \((A_1, A_{12}, A_{13})\) is also uniformly distributed over a set of size \(q^3\), implying \(A_1 \perp A_{12} \perp A_{13}\). Similarly, we have

\[
\begin{align*}
A_1A_2A_3 & \leq A_1A_2A_{23} \leq A_1A_2A_{123} \\
& \leq A_1A_{12}A_{13} \leq A_1A_{12}A_{23} \leq A_1A_2A_3,
\end{align*}
\]

hence all terms above contain the same information. For each term above, the three random variables are independent. This covers all cases in \(\mathcal{I}_F\) by symmetry.

The Fano-non-Fano condition plays a similar role as the permuting frame of equivalences in \([75, 60]\). The rest of the proof of Theorem 3 is inspired by the arguments in \([60]\), but with different presentation and proofs.

Given \(A_1, A_2, A_3, A_{12}, A_{13}, A_{23}, A_{123}\) (denote this collection as \(\{A_i\}_{i \in \mathcal{E}}\)) satisfying the Fano-non-Fano condition, we call \((\mathcal{A}, \{\theta_i\}_{i \in \mathcal{E}})\) an abelian group labeling, where \(\mathcal{A}\) is an abelian group, and \(\theta_i\) is a bijective function mapping values of \(A_i\) to \(\mathcal{A}\), if we have

\[
\begin{align*}
\theta_{12}(A_{12}) &= \theta_1(A_1) + \theta_2(A_2), \\
\theta_{13}(A_{13}) &= \theta_1(A_1) + \theta_3(A_3), \\
\theta_{23}(A_{23}) &= \theta_2(A_2) + \theta_3(A_3), \\
\theta_{123}(A_{123}) &= \theta_1(A_1) + \theta_2(A_2) + \theta_3(A_3)
\end{align*}
\]

with probability 1. The goal of this section is to show the following proposition.

**Proposition 6.** If \(\{A_i\}_{i \in \mathcal{E}}\) satisfy the Fano-non-Fano condition, then there exists an abelian group labeling \((\mathcal{A}, \{\theta_i\}_{i \in \mathcal{E}})\).

An equivalent form of \([5]\) has appeared in \([63]\) Def. 6]. The proof of Proposition 6 also shares a number of similarities with \([63]\) Prop. 5]. While it is possible to use the arguments in \([63]\) Prop. 5] to prove Proposition 5 in this paper, we include our proof of Proposition 6 for the sake of completeness.

\(^3\)Very loosely speaking, the \(\alpha_1\) in \([60]\) corresponds to \(A_2A_3\) in this paper, whereas \(\varepsilon_{12}\) in \([60]\) corresponds to \(A_{12}A_3\) in this paper.
In the remainder of this section, we assume the Fano-non-Fano condition holds, and construct an abelian group labeling. Since \( \text{tri}(A_i, A_j, A_k) \) ensures \( A_i, A_j, A_k \) are uniform with the same cardinality, we know that \( \{A_i\}_{i \in E} \) are all uniform with the same cardinality. For any \( \{i, j, k\} \in \mathcal{D}_N \), since \( \text{tri}(A_i, A_j, A_k) \) holds, \( A_k \) is a function of \( A_i, A_j \), and hence we can let this function be \( f_k^{i,j} \). We use the notation \( f_{ij}^{i,j} \) for a function mapping values of \( (A_i, A_j) \) to values of \( A_k \), where the superscript \( \text{"i, j"} \) denotes the “domain” and the subscript \( \text{"k"} \) denotes the “codomain”. The choice of using superscript for domain and subscript for codomain is due to the usual notation \( B^A \) for the set of functions with domain \( A \) and codomain \( B \).

Note that there is a bijection between \( \{A_i, A_j, A_k\} \) and \( \{A_i\}_{i \in E} \) since \( A_1, A_2, \ldots \) can be determined from \( A_1, A_2, \ldots \) and \( A_1A_23 \) can be determined from \( A_1, A_2, A_3 \). Since \( \{1, 2, 3\} \in \mathcal{I}_E \), \( A_1 \perp A_2 \perp A_3 \), and there are \( k^3 \) possible values of the tuple \( (A_1, A_2, A_3) \), and hence there are \( k^3 \) possible values of the tuple \( \{A_i\}_{i \in E} \). For any \( i,j,k,l \) such that \( \{i, j, k, l\} \in \mathcal{I}_E \), since there are \( k^4 \) possible values of the tuple \( (A_i, A_j, A_k, A_l) \) (the same as the number of tuples \( \{A_i\}_{i \in E} \), there is a bijection between \( \{A_i, A_j, A_k, A_l\} \) and \( \{A_i\}_{i \in E} \), we can let the function from \( \{A_i, A_j, A_k\} \) to \( A_l \) be \( f_{i,j,k}^{i,j,k}(a_i, a_j, a_k) \).

We first prove some properties of these functions.

**Proposition 7.** The following holds as long as all the functions \( f \) involved are defined\(^4\):

1) \( f_k^{i,j}(a, b) = f_k^{i,j}(b, a) \), and \( f_k^{i,j,k}(a, b, c) = f_k^{i,j,k}(b, c, a) \) (similar for any permutation of arguments).

2) \( f_k^{i,j}(a, b) = a \).

3) \( f_k^{i,j,k}(a, b, c) = f_k^{m,k}(f_k^{i,j}(a, b), c) \).

**Proof:** The first statement follows directly from the definition. For the second statement, if \( A_i = a \) and \( A_j = b \) (note that this has positive probability since \( f_k^{i,j} \) is defined), meaning that \( \{i, j, k\} \in \mathcal{D}_N \), which implies \( A_i \perp A_j \) implies \( A_k = f_k^{i,j}(a, b) \), then \( P((A_i, A_j, A_k) = (a, b, f_k^{i,j}(a, b))) > 0 \), giving \( f_k^{i,j}(f_k^{i,j}(a, b), b) = a \) since \( f_k^{i,j}(A_i, A_j) = A_i \) must hold with probability 1. For the third statement, if \( A_i = a, A_j = b, A_k = c \) (note that this has positive probability since \( f_k^{i,j,k} \) is defined), meaning that \( \{i, j, k\} \in \mathcal{I}_E \), which implies \( A_i \perp A_j \perp A_k \), then \( A_m = f_m^{i,j}(a, b) \) and \( A_1 = f_1^{i,j,k}(a, b, c) \), giving \( f_1^{m,k}(f_1^{i,j}(a, b), c) = f_1^{i,j,k}(a, b, c) \).

Since the labels of random variables do not matter in conditional independence statements, we can assign any labels to the random variables. We now assign labels to the random variables in \( \{A_i\}_{i \in E} \). Our goal is to assign labels such that (5) holds without the functions \( \theta \)'s, i.e., \( A_{12} = A_1 + A_2, A_{13} = A_1 + A_3, A_{23} = A_2 + A_3, \) and \( A_{123} = A_1 + A_2 + A_3 \). Without loss of generality, assume 0 is an element in the supports of \( A_1, A_2, \) and \( A_3 \). Let the support of \( A_{123} \) be \( \mathcal{A} \). We label the value of \( A_{123} \) when \( A_1 = A_2 = A_3 = 0 \) as \( A_{123} = 0 \), i.e.,

\[
f_{123}^{1,2,3}(0, 0, 0) = 0.
\]

The other labels in \( \mathcal{A} \) are arbitrary.

We now assign labels to the values of \( A_{12} \) such that for any \( a \in \mathcal{A} \),

\[
f_{12}^{123,3}(a, 0) = a,
\]

i.e., we label the value \( f_{12}^{123,3}(a, 0) \) of \( A_{12} \) as \( a \). After this labeling, the support of \( A_{12} \) is also \( \mathcal{A} \). This is a valid labeling since \( a \mapsto f_{12}^{123,3}(a, 0) \) is a bijective function that maps values of \( A_{123} \) to values of \( A_{12} \) (it is bijective since the inverse of this function is \( a \mapsto f_{12}^{123,3}(a, 0) \) by (4)). Similarly, we label \( A_{13} \) and \( A_{23} \) such that \( f_{13}^{132,2}(a, 0) = f_{23}^{23,1}(a, 0) = a \).

We then label \( A_1, A_2 \), and \( A_3 \) such that

\[
f_{1}^{123,23}(a, 0) = f_{23}^{23,13}(a, 0) = f_{3}^{23,12}(a, 0) = a
\]

for any \( a \in \mathcal{A} \). Note that 0 is in the support of \( A_{23} \) since the support of \( A_{23} \) is \( \mathcal{A} \). While we have previously fixed an element 0 in the support of \( A_1 \), we do not change the label of this element since

\[
f_{1}^{123,23}(0, 0) = f_{23}^{123,23}(0, 0) = 0,
\]

by (4).

The labeling is constructed such that the function \( a \mapsto f_k^{i,j}(a, 0) \) is the identity function for some triples \( i, j, k \). We now show that this holds for some more triples \( i, j, k \).

**Proposition 8.** For any distinct \( i, j, k \in \{1, 2, 3\} \),

\[
f_{ij}^{i,j}(a, 0) = f_{ij}^{ij,k}(a, 0) = a,
\]

where the \( ij \) in the subscripts and superscripts denote concatenation, where the order of \( i, j \) is ignored (e.g. \( f_{ij}^{i,j} \) is \( f_{12}^{2,1} \) when \( i = 2, j = 1 \)).

\(^4\)Recall that \( f_k^{i,j} \) is defined if and only if \( \{i, j, k\} \in \mathcal{D}_N \), and \( f_k^{i,j,k} \) is defined if and only if \( \{i, j, k\} \in \mathcal{I}_E \).
Proof: We have
\[ f_{123}^{12,3}(a,0) = f_{123}^{12,3}(f_{12}^{12,3}(a,0),0) = a \] (10)
by (7) and (4). Also,
\[ f_{12}^{1,2}(a,0) = f_{123}^{12,3}(f_{12}^{1,2}(a,0),0) = f_{123}^{1,2}(a,0) \]
where equalities marked with (a) are by Proposition 8, and (b) is by (5). Similarly
\[ f_{123}^{12,3}(a,0) = f_{123}^{12,3}(f_{12}^{1,2}(a,0),0) \]
where the lines marked by (a) are by (10), (b) are by (5), (c) is by (6), (d) is by (8), and (e) is by (4). The result follows from
symmetry.

Moreover, it turns out that many of \( f_{k}^{1,3} \) are actually the same function under this labeling.

**Proposition 9.** The following functions are the same:
\[ f_{12}^{1,2} = f_{12}^{2,1} = f_{13}^{1,3} = f_{13}^{3,1} = f_{23}^{2,3} = f_{23}^{3,2} = f_{123}^{123} = f_{123}^{2,13} = f_{123}^{3,12} = f_{123}^{23,1} = f_{123}^{13,2} = f_{123}^{12,3} \]

**Proof:** Using Proposition 8,
\[ f_{12}^{1,2}(a,b) = f_{123}^{12,3}(f_{12}^{1,2}(a,b),0) = f_{123}^{1,2}(a,0) \]
where equalities marked with (a) are by Proposition 8 and (b) is by 5. Similarly \( f_{12}^{1,2}(a,b) = f_{123}^{12,3}(a,b) \). Hence \( f_{12}^{1,2}(a,b) = f_{123}^{1,2}(a,b) \). By repeated use of this fact and Proposition 7.1, we have \( f_{12}^{1,2}(a,b) = f_{123}^{12,3}(a,b) \). Hence \( f_{12}^{1,2}(a,b) = f_{123}^{12,3}(a,b) \).

We define an abelian group over \( \mathcal{A} \) by
\[ a + b := f_{12}^{1,2}(a,b), \quad \neg a := f_{2}^{1,2}(a,0), \]
and the identity element is 0. A consequence of Proposition 7.1 and Proposition 9 is that + is commutative, i.e., \( a + b = b + a \).

To check that \( (\mathcal{A},+,0) \) is an abelian group, we first prove + is associative.

**Proposition 10.** We have \( (a + b) + c = a + (b + c) \), i.e.,
\[ f_{12}^{1,2}(f_{12}^{1,2}(a,b),c) = f_{12}^{1,2}(a,f_{12}^{1,2}(b,c)). \]

**Proof:** Using Proposition 9,
\[ f_{12}^{1,2}(f_{12}^{1,2}(a,b),c) = f_{123}^{12,3}(f_{12}^{1,2}(a,b),c) = f_{123}^{1,2}(a,f_{123}^{1,2}(b,c)) = f_{12}^{1,2}(a,f_{12}^{1,2}(b,c)), \]
where (a) is by (5).

We then prove \( \neg a \) is the additive inverse of \( a \).

**Proposition 11.** We have \( a + (\neg a) = 0 \), i.e.,
\[ f_{12}^{1,2}(a,f_{2}^{1,12}(a,0)) = 0. \]
Proposition 12. For any finite abelian group \( A \), there exist \( \{A_i\}_{i \in E} \) satisfying the Fano-non-Fano condition and an abelian group labeling \((A, \{\theta_i\}_{i \in E})\).

Proof: Take \( A_1, A_2, A_3 \) to be i.i.d. uniformly drawn elements in \( A \), and \( A_{12} = A_1 + A_2, A_{13} = A_1 + A_3, A_{23} = A_2 + A_3, A_{123} = A_1 + A_2 + A_3 \). We have \( \text{tri}(A_1, A_2, A_{12}) \) since \( A_1 = A_{12} - A_2, A_2 = A_{12} - A_1 \). Similarly \( \text{tri}(A_1, A_{23}, A_{123}) \) holds. By Proposition 5 the Fano-non-Fano condition is satisfied, and \((A, \{\theta_i\}_{i \in E})\), with \( \theta_i \) being the identity function, is an abelian group labeling.

III. REDUCTION FROM THE WORD PROBLEM FOR FINE MONOIDS

A. The Word Problem and the Endomorphism Monoid

This section follows the same high-level ideas as [60, Section 6] (i.e., reduction from the uniform word problem for finite monoids, and the use of the endomorphism monoid), but with different arguments concerning random variables (instead of equivalence relations in [60]).

The uniform word problem for groups/semigroups/monoids [76, 77, 78] is to determine whether two words represent the same element in the group/semigroup/monoid, given a presentation of the group/semigroup/monoid. In this paper, we will utilize the uniform word problem for finite monoids in [79]. Recall that a finite monoid \((M, \cdot, e)\) is a finite set \( M \) with an associative binary operation \( \cdot \) (i.e., \( a \cdot (b \cdot c) = (a \cdot b) \cdot c \)) and an identity element \( e \in M \). A sequence of elements \( x_1 \cdots x_k \) is called a word, and it corresponds to the product \( \prod_{i=1}^{k} x_i = x_1 \cdots x_k \) in the monoid. The word problem is to decide, given a list of equalities between the product represented by words, whether another such equality follows from the list. More formally, given \( l, k \in \mathbb{Z}_{>0}, a_{i,j} \in \{1, \ldots, k\} \) for \( i = 0, \ldots, l \) and \( j = 1, \ldots, m_i \), and \( b_{i,j} \in \{1, \ldots, k\} \) for \( i = 0, \ldots, l \) and \( j = 1, \ldots, n_i \), the uniform word problem for finite monoids is to determine whether the implication

\[
\prod_{i=1}^{l} \left( \prod_{j=1}^{m_i} x_{a_{i,j}} = \prod_{j=1}^{n_i} x_{b_{i,j}} \right) \Rightarrow \prod_{j=1}^{m_i} x_{a_{i,j}} = \prod_{j=1}^{n_i} x_{b_{i,j}}
\]

(11)

holds for all finite monoid \((M, \cdot, e)\) and all \( k \)-tuples \( x_1, \ldots, x_k \in M \). It was shown in [79] that the uniform word problem for finite monoids is undecidable, i.e., there does not exist an algorithm that, given \( l, k, \{a_{i,j}\}, \{b_{i,j}\} \) as input, outputs whether the implication holds for all \((M, \cdot, e)\) and \( x_1, \ldots, x_k \in M \).

Note that we can restate the uniform word problem in the following equivalent form: Given \( l, k \in \mathbb{Z}_{>0}, a_i, b_i, c_i \in \{1, \ldots, k\} \) for \( i = 0, \ldots, l \), determine whether the implication

\[
\bigwedge_{i=1}^{l} (x_{a_i} \cdot x_{b_i} = x_{c_i}) \Rightarrow x_{a_0} = x_{c_0}
\]

(12)

holds for all finite monoid \((M, \cdot, e)\) and all \( k \)-tuples \( x_1, \ldots, x_k \in M \). To show how we can translate (11) into (12), define intermediate variables \( y_{i,j} \) for \( i = 0, \ldots, l \) and \( j = 1, \ldots, m_i \) subject to the constraints \( y_{i,1} = x_{a_{i,1}} \) and \( y_{i,j} = y_{i,j-1} \cdot x_{a_{i,j}} \) for \( j = 2, \ldots, m_i \). Then we can replace \( \prod_{j=1}^{m_i} x_{a_{i,j}} \) by \( y_{i,m_i} \). Similarly define \( z_{i,j} \) for \( i = 0, \ldots, l \) and \( j = 1, \ldots, n_i \) and replace \( \prod_{j=1}^{n_i} x_{b_{i,j}} \) by \( z_{i,n_i} \). Finally, for any equality constraint (e.g. \( y_{i,m_i} = z_{i,n_i} \)), replace all occurrences of one of them by the other.

We will show the desired undecidability result via a reduction from the uniform word problem for finite monoids. In Section 11 we have shown that random variables satisfying the Fano-non-Fano condition correspond to a finite abelian group. We then consider the endomorphism monoid of an abelian group. Recall that a homomorphism \( g : A \rightarrow B \), where \( A, B \) are
abelian groups, is a function satisfying $g(a + b) = g(a) + g(b)$. An endomorphism in $\mathcal{A}$ is a homomorphism $g : \mathcal{A} \to \mathcal{A}$. The endomorphism monoid of an abelian group $\mathcal{A}$, denoted as $\text{End}(\mathcal{A})$, is the set of endomorphisms in $\mathcal{A}$, equipped with the operation $g \cdot h : \mathcal{A} \to \mathcal{A}$ where $g \cdot h(a) = g(h(a))$.

The following proposition in [80] and [60] Prop. 19] shows that there is no loss of generality in considering endomorphism monoids instead of general finite monoids, in the sense that any finite monoid can be embedded into the endomorphism monoid of a finite abelian group. Refer to [60] Prop. 19] for the proof. Recall that an embedding from a monoid to another monoid is an injective function $h : \mathcal{M} \to \mathcal{N}$ satisfying $h(a \cdot b) = h(a) \cdot h(b)$ and $h(e_{\mathcal{M}}) = e_{\mathcal{N}}$, where $e_{\mathcal{M}}$ is the identity element in $\mathcal{M}$.

**Proposition 13 ([80]).** For any finite monoid, there exists an embedding from that monoid into the endomorphism monoid of a finite abelian group.

Therefore, the implication (12) holds for all finite monoids if and only if (12) holds for all finite abelian group $\mathcal{A}$ and all $k$-tuples $x_1, \ldots, x_k \in \text{End}(\mathcal{A})$. To show this, note that if (12) holds for all finite monoids, then it clearly holds for all endomorphism monoids of finite abelian groups. If (12) holds for all endomorphism monoids of finite abelian groups, then for any finite monoid $\mathcal{M}$, find an embedding $h : \mathcal{M} \to \text{End}(\mathcal{A})$ where $\mathcal{A}$ is a finite abelian group. For $x_1, \ldots, x_k \in \mathcal{M}$, if $\bigwedge_{i=1}^k \prod_{j=1}^{m_i} x_{a_{i,j}} = \prod_{j=1}^{n_i} x_{b_{i,j}}$, then $\bigwedge_{i=1}^k \prod_{j=1}^{m_i} h(x_{a_{i,j}}) = \prod_{j=1}^{n_i} h(x_{b_{i,j}})$, implying $\prod_{j=1}^{m_i} h(x_{a_{0,j}}) = \prod_{j=1}^{n_i} h(x_{b_{0,j}})$, which gives $h(\prod_{j=1}^{m_i} x_{a_{0,j}}) = h(\prod_{j=1}^{n_i} x_{b_{0,j}})$, and $\prod_{j=1}^{m_i} x_{a_{0,j}} = \prod_{j=1}^{n_i} x_{b_{0,j}}$ since $h$ is injective. Hence (12) also holds for all finite monoids.

### B. Representing Endomorphisms as Random Variables

The next step is to represent an endomorphism in $\mathcal{A}$ using a random variable. If $(\mathcal{A}, \{\theta_i\}_{i \in \mathcal{E}})$ is an abelian group labeling with $\theta_i$ being identity functions (i.e., $A_i \in \mathcal{A}$), then we would represent an endomorphism $g : \mathcal{A} \to \mathcal{A}$ by a random variable $U = A_1 - g(A_2)$. To check whether $U$ corresponds to an endomorphism (up to relabeling) using conditional independence relations, define the predicate $\text{end}_{1,2}(\{A_i\}_{i \in \mathcal{E}}, U)$ by

$$\text{end}_{1,2}(\{A_i\}_{i \in \mathcal{E}}, U) :$$

$$\forall V, W : \text{fnf}(\{A_i\}_{i \in \mathcal{E}})$$

$$\land \text{ueq}(U, A_1) \land \text{ueq}(V, A_1) \land \text{ueq}(W, A_1)$$

$$\land U \triangleq A_1 | A_2 \land V \triangleq A_1 | A_{23} \land U \triangleq V | A_3$$

$$\land W \triangleq A_3 | A_2 \land U \triangleq W | A_3,$$

where we define $X \triangleq Y | Z \iff X \triangleq Z \land Y \triangleq Z \land X \triangleq Y$, i.e., if we are given $Z$, then $X$ and $Y$ have the same cardinality as $Z$, and $\text{ueq}(X, Y)$ is a predicate that checks whether $X, Y$ are both uniform and have the same cardinality (this expression was given in [33]; refer to [33] for the proof):

$$\text{ueq}(X, Y) :$$

$$\exists U_1, U_2, U_3 : \text{tri}(X, U_1, U_2) \land \text{tri}(Y, U_1, U_3).$$

We show that a random variable $U$ satisfying $\text{end}_{1,2}(\{A_i\}_{i \in \mathcal{E}}, U)$ corresponds to an endomorphism in $\mathcal{A}$.

**Proposition 14.** Given $\{A_i\}_{i \in \mathcal{E}}, U$ satisfying $\text{end}_{1,2}(\{A_i\}_{i \in \mathcal{E}}, U)$, and an abelian group labeling $(\mathcal{A}, \{\theta_i\}_{i \in \mathcal{E}})$ of $\{A_i\}_{i \in \mathcal{E}}$. Then there exists a unique endomorphism $g : \mathcal{A} \to \mathcal{A}$, and a unique bijective function $\phi$ mapping the values of $U$ to $\mathcal{A}$, satisfying

$$\phi(U) = \theta_1(A_1) - g(\theta_2(A_2))$$

with probability 1.

**Proof:** Without loss of generality, assume $\{\theta_i\}_{i \in \mathcal{E}}$ are identity functions. Let the supports of $U, V, W$ be $\mathcal{U}, \mathcal{V}, \mathcal{W}$ respectively, with $|\mathcal{U}| = |\mathcal{V}| = |\mathcal{W}| = |\mathcal{A}|$ since $\text{ueq}(U, A_1) \land \text{ueq}(V, A_1) \land \text{ueq}(W, A_1)$. Since $U \triangleq A_1 | A_2$, we can find injective functions $\mu_{a_2} : A \to \mathcal{U}$ for $a_2 \in A$ such that $U = \mu_{A_2}(A_1)$. Note that $\mu_{a_2}$ is bijective since $|A| = |\mathcal{U}|$. Without loss of generality, assume $\mu_0(a) = a$ for $a \in A$, i.e., we assign the label $a$ to the value $\mu_0(a)$ of $U$. After this labeling, we have $U = A$. Similarly, we can assume $V = A$, and define the bijective functions $\nu_{a_3} : A \to \mathcal{V}$ such that $V = \nu_{A_3}(A_1)$ and $\nu_0(a) = a$. Similarly, we can assume $W = A$, and define the bijective functions $\omega_{a_2} : A \to \mathcal{A}$ such that $W = \omega_{A_2}(A_1 + A_3)$ and $\omega_0(a) = a$.

Consider $U \triangleq W | A_3$. Let $\tau : A^2 \to A$ such that $W = \tau(A_3, U)$. We have, for any $a_1, a_2, a_3 \in A$,

$$\omega_{a_2}(a_1 + a_3) = \tau(a_3, \mu_{a_2}(a_1)).$$

This step is inspired by [60] Lemma 31].
Substituting $a_2 = 0$, we have $a_1 + a_3 = \tau(a_3, a_1)$. Hence,

$$
\omega_a(a_1 + a_3) = a_3 + \mu_a(a_1).
$$

Substituting $a_3 = -a_1$,

$$
\omega_a(0) = -a_1 + \mu_a(a_1).
$$

Hence $\mu_a(a_1) = a_1 + \omega_a(0)$. Substituting $a_1 = 0$, we have $\omega_a(0) = \mu_a(0)$. Hence,

$$
\mu_a(a_1) = a_1 + \mu_a(0).
$$

Consider $U \doteq V|A_3$. Let $\kappa : \mathcal{A}^2 \to \mathcal{A}$ such that $V = \kappa(A_3, U)$. We have, for any $a_1, a_2, a_3 \in \mathcal{A}$,

$$
\nu_{a_2 + a_3}(a_1) = \kappa(a_3, \mu_a(a_1)) = \kappa(a_3, a_1 + \mu_a(0)).
$$

Substituting $a_2 = 0$, we have $\nu_{a_3}(a_1) = \kappa(a_3, a_1)$. Hence,

$$
\nu_{a_2 + a_3}(a_1) = \nu_{a_3}(a_1 + \mu_a(0)).
$$

Substituting $a_3 = 0$, we have $\nu_{a_2}(a_1) = a_1 + \mu_a(0)$. Therefore,

$$
a_1 + \mu_a + a_3(0) = a_1 + \mu_a(0) + \mu_a(0),
$$

and hence $\mu_{a + b}(0) = \mu_a(0) + \mu_b(0)$. As a result, if end\(_{1,2}(\{A_i\}_i \in \mathcal{E}), U\) is satisfied, and (\(\mathcal{A}, \{\theta_i\}_i \in \mathcal{E}\)) where $\{\theta_i\}_i \in \mathcal{E}$ are identity functions is an abelian group labeling of $\{A_i\}_i \in \mathcal{E}$, then we can find an endomorphism $g(a) = -\mu_a(0)$ in $\mathcal{A}$ satisfying $U = \mu_{A_3}(A_1) = A_1 - g(A_2)$ up to relabeling.

For uniqueness, assume $\phi'$ and $g'$ satisfy \([14]\), i.e., $\phi'(U) = A_1 - g'(A_2)$. We have

$$
\phi'(a_1 - g(a_2)) = a_1 - g'(a_2).
$$

Substituting $a_2 = 0$, since $g, g'$ are endomorphisms, we have $\phi'(a_1) = a_1$. Substituting back to \([15]\),

$$
a_1 - g(a_2) = a_1 - g'(a_2),
$$

$$
g'(a_2) = g(a_2).
$$

Hence the choice of $\phi, g$ is unique.

We then show that the converse of Proposition \([14]\) holds as well. Therefore there is a one-to-one correspondence (up to relabelling) between endomorphisms and random variables satisfying end\(_{1,2}(\{A_i\}_i \in \mathcal{E}), U\).

**Proposition 15.** Given $\{A_i\}_i \in \mathcal{E}$ satisfying $\text{fusc}(\{A_i\}_i \in \mathcal{E})$, an abelian group labeling $(\mathcal{A}, \{\theta_i\}_i \in \mathcal{E})$, and an endomorphism $g : \mathcal{A} \to \mathcal{A}$. Let $U = \theta_1(A_1) - g(\theta_2(A_2))$. Then end\(_{1,2}(\{A_i\}_i \in \mathcal{E}), U\) holds.

**Proof:** Without loss of generality, assume $\{\theta_i\}_i \in \mathcal{E}$ are identity functions, and hence $U = A_1 - g(A_2)$. Take $V = A_1 - g(A_2 + A_3) = A_1 - g(A_2) - g(A_3)$, and $W = A_1 - g(A_2 + A_3)$. It is straightforward to check that the conditions in \([15]\) hold.

Similar to \([15]\), we can define the predicates $\text{end}_{i,j}(\{A_i\}, U)$ for $i \neq j \in \{1, 2, 3\}$, which checks whether there exists an endomorphism $g : \mathcal{A} \to \mathcal{A}$ and a bijective function $\phi$ satisfying $\phi(U) = \theta_i(A_i) - g(\theta_j(A_j))$.

It is left to represent the composition of endomorphisms using random variables.

**Proposition 16.** Given $\{A_i\}_i \in \mathcal{E}, U_1, U_2, U_3$ satisfying end\(_{1,2}(\{A_i\}, U_1), \text{end}_{2,3}(\{A_i\}, U_2), \text{end}_{1,3}(\{A_i\}, U_3)$, and an abelian group labeling $(\mathcal{A}, \{\theta_i\}_i \in \mathcal{E})$. We have $U_3 \leq U_1 U_2$ if and only if $g_3 = g_1 \cdot g_2$, where $g_i$ is the endomorphism corresponding to $U_i$.

**Proof:** Without loss of generality, assume $\{\theta_i\}_i \in \mathcal{E}$ and the mappings $\phi$ in Proposition \([14]\) are identity functions. Hence $U_1 = A_1 - g_1(A_2), U_2 = A_2 - g_2(A_3), U_3 = A_1 - g_3(A_3)$. For the “if” direction, if $g_3 = g_1 \cdot g_2$, then $U_3 = A_1 - g_1(g_2(A_3)) = U_1 + g_1(U_2)$ is a function of $U_1, U_2$.

For the “only if” direction, assume $U_3 \leq U_1 U_2$. Let $h$ be a function such that $U_3 = h(U_1, U_2)$. We have

$$
a_1 - g_3(a_3) = h(a_1 - g_1(a_2), a_2 - g_2(a_3)).
$$

Substituting $a_1 = x + g_1(a_2)$ and $a_3 = 0$,

$$
x + g_1(a_2) = h(x, a_2).
$$

*This step is inspired by \([60]\) Lemma 21.*
Substituting back to (16),

\[ a_1 - g_3(a_3) = a_1 - g_1(a_2) + g_1(a_2 - g_2(a_3)) = a_1 - g_1(g_2(a_3)). \]

Therefore, we have \( g_3 = g_1 \cdot g_2. \)

The problem of Proposition 16 is that \( U_1, U_2, U_3 \) are subject to \( \text{end}_{i,j} \) for different \( i, j \). We want a predicate that checks for composition of endomorphisms using \( \text{end}_{1,2} \) only. Therefore, we require a way to convert between \( \text{end}_{i,j} \) for different \( i, j \). Define

\[
\text{conv}^{1,2}_{1,3}((A_i), U, V) :
\exists W : \text{end}_{1,2}((A_i), U) \land \text{end}_{1,3}((A_i), V) \land \text{end}_{2,3}((A_i), W)
\land A_{13} \leq A_{12} W \land V \leq U W.
\]  

(17)

**Proposition 17.** Given \( \{A_i\}_{i \in \mathcal{E}}, U, V \) satisfying \( \text{end}_{1,2}((A_i), U) \) and \( \text{end}_{1,3}((A_i), V) \), and an abelian group labeling \( (\mathcal{A}, \{\theta_i\}_{i \in \mathcal{E}}) \). We have \( \text{conv}^{1,2}_{1,3}((A_i), U, V) \) if and only if the endomorphism corresponding to \( U, V \) are the same.

**Proof:** Without loss of generality, assume \( \{\theta_i\}_{i \in \mathcal{E}} \) and the mappings \( \phi \) in Proposition 14 are identity functions. Let \( U = A_1 - g(A_2), V = A_1 - h(A_3) \). First show the “only if” direction. Note that \( A_{12} \) satisfies \( \text{end}_{1,2}((A_i), A_{12}) \) and corresponds to the negation endomorphism \( x \mapsto -x \), and \( A_{13} \) satisfies \( \text{end}_{1,3}((A_i), A_{12}) \) and corresponds to the negation endomorphism \( x \mapsto -x \). By Proposition 18, since \( A_{13} \leq A_{12} W \), \( W \) corresponds to the identity endomorphism (which, when composed with the negation endomorphism, gives the negation endomorphism). By Proposition 16, since \( V \leq U W \), \( V \) corresponds to the same endomorphism as \( U \). For the “if” direction, take \( W \) such that \( \text{end}_{2,3}((A_i), W) \) is satisfied, and it corresponds to the identity endomorphism.

Using \( \text{conv}^{1,2}_{1,3} \), we can convert between different values for the second index \( j \) in \( \text{end}_{i,j} \). To convert between different first indices \( i \), define

\[
\text{conv}^{1,2}_{3,2}((A_i), U, V) :
\exists W : \text{end}_{1,2}((A_i), U) \land \text{end}_{3,2}((A_i), V) \land \text{end}_{1,3}((A_i), W)
\land A_{12} \leq W A_{23} \land V \leq W U.
\]  

(18)

**Proposition 18.** Given \( \{A_i\}_{i \in \mathcal{E}}, U, V \) satisfying \( \text{end}_{1,2}((A_i), U) \) and \( \text{end}_{3,2}((A_i), V) \), and an abelian group labeling \( (\mathcal{A}, \{\theta_i\}_{i \in \mathcal{E}}) \). We have \( \text{conv}^{1,2}_{3,2}((A_i), U, V) \) if and only if the endomorphism corresponding to \( U, V \) are the same.

**Proof:** Without loss of generality, assume \( \{\theta_i\}_{i \in \mathcal{E}} \) and the mappings \( \phi \) in Proposition 14 are identity functions. Let \( U = A_1 - g(A_2), V = A_3 - h(A_2) \). First show the “only if” direction. As in the proof of Proposition 17, we can assume \( W = A_1 - A_3 \). Since \( V \leq W U \), we can let \( \kappa \) be a function such that \( V = \kappa(W, U) \). We have

\[
a_3 - h(a_2) = \kappa(a_1 - a_3, a_1 - g(a_2)).
\]  

(19)

Substituting \( a_3 = a_1 - x \) and \( a_2 = 0 \),

\[
a_1 - x = \kappa(x, a_1).
\]

Substituting back to (19),

\[
a_3 - h(a_2) = a_1 - g(a_2) - (a_1 - a_3),
\]

giving \( h = g \). For the “if” direction, take \( W \) such that \( \text{end}_{1,3}((A_i), W) \) is satisfied, and it corresponds to the identity endomorphism.

Combining these constructions, we can use the following predicate to check whether \( U_1, U_2, U_3 \) with \( \text{end}_{1,2}((A_i), U_j) \) for \( j = 1, 2, 3 \) satisfy \( g_3 = g_1 \cdot g_2 \), where \( g_i \) is the endomorphism corresponding to \( U_i \):

\[
\text{comp}^{1,2}_{1,3}((A_i), U_1, U_2, U_3) :
\exists V_1, V_2 : \bigwedge_{j=1}^{3} \text{end}_{1,2}((A_i), U_j)
\land \text{conv}^{1,2}_{1,3}((A_i), U_1, V_1) \land \text{conv}^{1,2}_{3,2}((A_i), U_2, V_2)
\land U_3 \leq V_1 V_2.
\]  

(20)

The following is a direct consequence of Propositions 16, 17, and 18.
**Proposition 19.** Given \( \{A_i\}_{i \in E}, U_1, U_2, U_3 \) satisfying \( \text{end}_{1,2}(\{A_i\}, U_j) \) for \( j = 1, 2, 3 \), and an abelian group labeling \( (A, \{\theta_i\}_{i \in E}) \). We have \( \text{comp}_{1,2}(\{A_i\}, U_1, U_2, U_3) \) if and only if \( g_3 = g_1 \cdot g_2 \), where \( g_i \) is the endomorphism corresponding to \( U_i \).

We then show how to check for equality between two endomorphisms with the same \( i, j \) in \( \text{end}_{i,j} \).

**Proposition 20.** Given \( \{A_i\}_{i \in E}, U, V \) satisfying \( \text{end}_{1,2}(\{A_i\}, U) \) and \( \text{end}_{1,2}(\{A_i\}, V) \), and an abelian group labeling \( (A, \{\theta_i\}_{i \in E}) \). We have \( U \leq V \) if and only if the endomorphism corresponding to \( U, V \) are the same.

**Proof:** Without loss of generality, assume \( \{\theta_i\}_{i \in E} \) and the mappings \( \phi \) in Proposition 14 are identity functions. Let \( U = A_1 - g(A_2), V = A_1 - h(A_2) \). The “if” direction follows from the uniqueness in Proposition 14. For the “only if” direction, assume \( U \leq V \), and let \( \kappa \) be a function such that \( U = \kappa(V) \). We have \( a_1 - g(a_2) = \kappa(a_1 - h(a_2)) \). Substituting \( a_2 = 0 \), we have \( \kappa(a_1) = a_1 \). Hence \( a_1 - g(a_2) = a_1 - h(a_2) \), which gives \( g = h \).

Combining (20) and Proposition 20 we know that (12) holds for all finite abelian group \( A \) and all \( x_1, \ldots, x_k \in \text{End}(A) \) if and only if the implications

\[
\bigcap_{j=1}^{k} \text{end}_{1,2}(\{A_i\}, U_j) \\
\land \bigcap_{j=1}^{k} \text{comp}_{1,2}(\{A_i\}, U_{a_j}, U_{b_j}, U_{c_j}) \\
\rightarrow U_{a_0} \leq U_{c_0}
\]

(21)

holds for all finite random variables \( \{A_i\}, U_1, \ldots, U_k \). The complete proof of this equivalence is given below for the sake of completeness.

**Proposition 21.** The implication (12) holds for all finite abelian group \( A \) and all \( x_1, \ldots, x_k \in \text{End}(A) \) if and only if the implication (21) holds for all finite random variables \( \{A_i\}, U_1, \ldots, U_k \).

**Proof:** For the “if” direction, assume (21) holds for all random variables, then for any finite abelian group \( A \) and \( x_1, \ldots, x_k \in \text{End}(A) \) satisfying the left hand side of (12), let \( \{A_i\} \) satisfy \( \text{fnf}(\{A_i\}) \) such that \( (A, \{\theta_i\}_{i \in E}) \) is an abelian group labeling (by Proposition 12). Let \( U_j \) satisfy \( \text{end}_{1,2}(\{A_i\}, U_j) \) corresponding to the endomorphism \( x_j \) (by Proposition 15). Since \( x_{a_j} \cdot x_{b_j} = x_{c_j}, \text{comp}_{1,2}(\{A_i\}, U_{a_j}, U_{b_j}, U_{c_j}) \) holds by Proposition 19, the left hand side of (21) holds, and hence

\( U_{a_0} \leq U_{c_0}, \) and \( x_{a_0} = x_{c_0} \) by Proposition 20.

For the “only if” direction, assume (12) holds for all finite abelian group \( A \) and all \( x_1, \ldots, x_k \in \text{End}(A) \). Fix any \( \{A_i\}, U_1, \ldots, U_k \) satisfying the left hand side of (21). Fix any abelian group labeling \( (A, \{\theta_i\}_{i \in E}) \). By Proposition 14 let \( x_j \) be the endomorphism corresponding to \( U_j \). Since \( \text{comp}_{1,2}(\{A_i\}, U_{a_j}, U_{b_j}, U_{c_j}) \), we have \( x_{a_j} \cdot x_{b_j} = x_{c_j} \) by Proposition 19 and hence the left hand side of (12) holds, implying \( x_{a_0} = x_{c_0} \), which gives \( U_{a_0} \leq U_{c_0} \) by Proposition 20.

We have shown a reduction from the word problem for endomorphism monoids of abelian groups to the conditional independence implication problem, which gives the desired undecidability result. Note that the actual random variables in the conditional independence implication problem (stated in the form in Theorem 3) is considerably more than just \( \{A_i\}, U_1, \ldots, U_k \), since there are many existentially-quantified intermediate random variables (e.g. \( V, W \) in (13)) in the construction (existential quantification becomes universal quantification since all these predicates appear on the left hand side of the implication in (21)).

Also note that Theorem 5 requires \( U_i, \mathcal{V}_i, \mathcal{W}_i \) to be disjoint, though our construction involves \( Y \leq X \iff Y \perp Y|X \), which is a non-disjoint conditional independence condition. Disjointness is not an obstacle, since it was shown in [51, Thm 4] that the conditional independence implication problem for disjoint \( U_i, \mathcal{V}_i, \mathcal{W}_i \) can be reduced from the conditional independence implication problem for general (not necessarily disjoint) \( U_i, \mathcal{V}_i, \mathcal{W}_i \). Also see [51] for a related argument.

This completes the proof of Theorem 4.

### IV. Network Coding

We use the same definition of network as [24] with minor notational differences. A network is a directed acyclic multigraph \((N, E)\), where \( N \) is the vertex set and \( E \) is the edge set. Let \( M_i \) \((i = 1, \ldots, n)\) be the source messages, which are independent uniformly distributed random variables with cardinality \( q \), where \( q \in \mathbb{Z}_{\geq 2} \) is the alphabet size. Each node \( v \in N \) has access to a subset of the messages with indices in the set \( C_v \subseteq \{1, \ldots, n\} \), and demands another subset of the messages \( D_v \subseteq \{1, \ldots, n\} \). In topological order of the multigraph, each node transmits a signal, which is an element in a set of size \( q \), along each outgoing
edge, where the signals can depend on the messages that the node has access to, and the signals along incoming edges to the node. At the end of the transmission, each node must decode the set of messages it demands.

More precisely, we let \( X_f \) be the signal along edge \( f \in \mathcal{L} \), which is a random variable with cardinality at most \( q \). The \textit{coding constraint} is that
\[
(M_{D_v}, X_{\text{out}(v)}) \preceq (MC_v, X_{\text{in}(v)})
\] (22)
for all \( v \in \mathcal{N} \), where \( \text{in}(v) \subseteq \mathcal{L} \) is the set of incoming edges to the node \( v \), and \( \text{out}(v) \) is the set of outgoing edges, and we write \( X_{\text{out}(v)} = \{X_f\}_{f \in \text{out}(v)} \). The problem is to decide whether the network is \textit{solvable}, that is, whether there exists \( q \in \mathbb{Z}_{\geq 2}, M_1, \ldots, M_n \) which are independent uniformly distributed random variables with cardinality \( q \), and \( \{X_f\}_{f \in \mathcal{L}} \) with cardinalities at most \( q \), satisfying the coding constraint (22).

In this section, we will show that network coding is undecidable.

**Theorem 22.** The following problem is undecidable: Given a network, decide whether it is solvable.

We may also be interested in the case where the messages and signals are sequences of symbols in an alphabet of size \( c \in \mathbb{Z}_{\geq 2} \) with the same length (e.g. \( c = 2 \) if bit sequences are being sent). Equivalently, we may consider a sequence to be an element in the overall alphabet of size \( q = c^m \) for some \( m \in \mathbb{Z}_{\geq 1} \). This case is undecidable as well.

**Theorem 23.** For any fixed \( c \in \mathbb{Z}_{\geq 2} \), the following problem is undecidable: Given a network, decide whether there exists \( m \in \mathbb{Z}_{\geq 1} \) such that the network is solvable with alphabet size \( q = c^m \).

The proof is divided into the following subsections.

A. \textit{The Left Regular Representation of a Finite Group}

Fix a finite field \( \mathbb{F} \). For a finite set \( S \), write \( \mathbb{F}^S = \{\{x_a\}_{a \in S} : x_a \in \mathbb{F}\} \) for the \(|S|\)-dimensional vector space where each element \( x = \{x_a\}_{a \in S} \) is a vector with entries indexed by elements in \( S \). Given any finite group \( B \), we consider the \textit{left regular representation} \([82]\), a basic construction in representation theory, which is an embedding from \( B \) into the general linear group \( \text{GL}(\mathbb{F}^B) \), where \( \text{GL}(\mathbb{F}^B) \) consists of invertible linear functions (or automorphisms) \( \mathbb{F}^B \to \mathbb{F}^B \), or equivalently, invertible \(|B| \times |B|\) matrices, with group operation given by function composition or matrix multiplication. For any finite group \( B \), define an embedding \( \lambda_B : B \to \text{GL}(\mathbb{F}^B) \), where \( \lambda_B(b) \) is the function \( \{x_a\}_{a \in B} \mapsto \{x_{b^{-1}a}\}_{a \in B} \). It is straightforward to check that \( \lambda_B \) is an injective homomorphism.

We prove the following key observation about the left regular representation.

**Proposition 24.** If \( b \in B \), \( b \neq e_B \) (the identity element of \( B \)), then the function \( \lambda_B(b) - \text{id}_{\mathbb{F}^B} \) (i.e., the function \( \mathbb{F}^B \to \mathbb{F}^B, x \mapsto \lambda_B(b)(x) - x \)) is a linear function with rank at least \(|B|/2\).

\textbf{Proof:} Let \( b \neq e_B \). Consider the cyclic subgroup \( \langle b \rangle = \{b^n : n \in \mathbb{Z}\} \) of \( B \). We have \(|\langle b \rangle| \geq 2\). Partition \( B \) into cosets in the form \( \langle b \rangle c = \{b^n c : n \in \mathbb{Z}\} \), \( c \in B \). Consider the entries of \( \lambda_B(b)(x) - x \) with indices in \( \langle b \rangle c \).

The entry with index \( b^n c \) is \( x_{b^{n-1}c} - x_{b^{n-1}c} \) for \( n = 0, \ldots, |\langle b \rangle| - 1 \) (note that \( \lambda_B(b)(x) \) is performing a cyclic shift of entries within \( \langle b \rangle c \)). It is straightforward to check that, if we only consider indices in \( \langle b \rangle c \), then the linear function \( \{x_{b^n c} : n = 0, \ldots, |\langle b \rangle| - 1\} \mapsto \{\lambda_B(b)(x) - x_{b^n c} : n = 0, \ldots, |\langle b \rangle| - 1\} \) has rank \(|\langle b \rangle| - 1\). In sum, \( x \mapsto \lambda_B(b)(x) - x \) has rank \(|\langle b \rangle| - 1\). Since \(|\langle b \rangle| \geq 2\), \( \text{rank} \lambda_B(b)(x) - x \geq |B|/2 \).

In this section, instead of showing a reduction from the uniform word problem for finite monoids as in \([60]\), we will be using the uniform word problem for finite groups, which is also undecidable \([71]\). We state an equivalent form of the uniform word problem for finite groups: Given \( l, k \in \mathbb{Z}_{>0}, a_i, b_i, c_i \in \{1, \ldots, k\} \) for \( i = 1, \ldots, l \), determine whether the implication
\[
\bigwedge_{i=1}^{l} (x_{a_i} \cdot x_{b_i} = x_{c_i}) \rightarrow x_1 = e_B
\] (23)
holds for all finite group \( B \) and all \( k \)-tuples \( x_1, \ldots, x_k \in B \). Note that the original word problem allows having inverse (e.g. \( a^{-1} \)) appear in a word, though this can be emulated in (23) by introducing an intermediate variable \( \bar{a} \) satisfying \( \bar{a} \cdot a = e \), where \( e \) is another variable satisfying \( e \cdot e = e \) (which forces \( e \) to be the identity element), and hence we can use \( \bar{a} \) in place of \( a^{-1} \).
B. The Network for Enforcing Abelian Group

We first prove a useful fact about random variables.

**Proposition 25.** If $X, Y, Z$ are finite random variables with supports $\mathcal{X}, \mathcal{Y}, \mathcal{Z}$ respectively, satisfying $X \perp Z$, $Z \leq XY$, and $|\mathcal{Y}| \leq |\mathcal{Z}|$, then we have $|\mathcal{Y}| = |\mathcal{Z}|$ and $Y \leq XZ$. Moreover, if $Z$ is uniformly distributed, then $Y$ is uniformly distributed and independent of $X$ as well.

**Proof:** Let $g$ be a function such that $Z = g(X, Y)$. For any fixed $x$, the set $\{g(x, y) : y \in \mathcal{Y}\}$ has size at most $|\mathcal{Y}|$, and it has size $|\mathcal{Y}|$ if and only if the function $y \mapsto g(x, y)$ is injective. Since $Z = g(x, Y)$ conditional on $X = x$, and the number of possible values of $Z$ conditional on $X = x$ is $|\mathcal{Z}| \geq |\mathcal{Y}|$, the number of possible values of $g(x, Y)$ must be $|\mathcal{Z}|$ as well, and hence $|\mathcal{Z}| = |\mathcal{Y}|$, and the function $y \mapsto g(x, y)$ is injective, and hence it is bijective since $|\mathcal{Y}| = |\mathcal{Z}|$. We can find function $h : \mathcal{X} \times \mathcal{Z} \rightarrow \mathcal{Y}$ such that $z \mapsto h(x, z)$ is the inverse of $y \mapsto g(x, y)$. We have $Y = h(X, Z)$. If $Z$ is uniformly distributed, then $h(x, Z)$ is uniformly distributed as well for any fixed $x$ since $z \mapsto h(x, z)$ is bijective, implying that $Y = h(X, Z)$ is uniformly distributed and independent of $X$.

Given a network and $\{M_i\}_{i=1,\ldots,n}$, $\{X_f\}_{f \in \mathcal{E}}$ for a subset $\mathcal{L} \subseteq \mathcal{L}$, we say that $\{M_i\}, \{X_f\}_{f \in \mathcal{L}}$ satisfy the coding constraint if there exists $X_f$ for the remaining $f \in \mathcal{L}\setminus\mathcal{L}$ such that $\{M_i\}, \{X_f\}_{f \in \mathcal{L}}$ satisfy the coding constraint (22). We will use networks and subnetworks to enforce various conditions on the messages and signals.

We utilize the network in [63] (Fig. 1) as the base of our construction, which is given in Figure 2 (also see [83]). It was shown in [63] that this network enforces the abelian group structure in (7). The following proposition follows from [63] Prop. 5 and Propositions 6 and 12, though we include a short proof for the sake of completeness.

**Proposition 26.** Given $A_1, A_2, A_3$ are independent and uniformly distributed with cardinality $q$, and $\{A_i\}_{i \in \mathcal{E}}$ are random variables with cardinality at most $q$. Then $\{A_i\}_{i \in \mathcal{E}}$ satisfy the Fano-non-Fano condition if and only if $\{A_i\}_{i \in \mathcal{E}}$ satisfy the coding constraint in the base network in Figure 2 (where $A_1, A_2, A_3$ are source messages, and $A_{12}, A_{13}, A_{23}, A_{123}$ are signals along edges).

**Proof:** The “only if” direction is straightforward. For the “if” direction, assume the coding constraint in the base network in Figure 2 is satisfied. Since $A_{12} \perp A_3$ (by $A_1A_2 \perp A_3$ and $A_{12} \leq A_1A_2$) and $A_3 \leq A_{12}A_3$, we have $A_{12} \leq A_1A_2A_3$ by Proposition 25. Since $A_1A_2 \perp A_3$ and $A_3 \leq A_{12}A_1A_2$, we have $A_1A_2 \perp A_{12}A_3$ by Proposition 25. Similarly, $A_1A_3 \perp A_{12}A_2$ and $A_2A_3 \perp A_{12}A_3$. Since $A_{12}A_2 \perp A_3$ and $A_3 \leq A_{123}A_{12}$, we have $A_{12} \leq A_{12}A_{123}$ and $A_{123} \perp A_2$ by Proposition 25. Hence $\text{tri}(A_{12}, A_3, A_{123})$ holds. Since $A_1A_3 \perp A_2$ and $A_2 \leq A_{123}A_{123}$, we have $A_1A_3 \perp A_{12}A_{123}$ by Proposition 25. Since $A_1 \perp A_{12}$ and $A_{12} \leq A_1A_2$, we have $A_2 \leq A_1A_2$ by Proposition 25. Similarly, $A_1 \leq A_{12}A_1$. Hence $\text{tri}(A_1, A_2, A_{12})$ holds. The result follows from Proposition 5.

If we want to enforce that the signal along an edge is $A_1$, we can simply require the ending node to decode $A_1$. However, if we want to enforce that the signal is $A_{12}$, then we require a subnetwork given in Figure 3 called the $\text{chk}_{12}$ subnetwork.
The subnetwork has two inputs $U$ and $A_{123}$ (which is always chosen to be $A_{123}$ from the base network in Figure 2 and is omitted later in network diagrams), and checks whether $U \doteq A_{12}$. More explicitly, the subnetwork checks the condition

$$\text{chk}_{12}(\{A_i\}, U) : \quad A_1 \leq A_2 U \wedge A_2 \leq A_1 U \wedge A_3 \leq A_{123} U.$$ 

We now show that the $\text{chk}_{12}$ subnetwork can check if a signal has the same information as $A_{12}$.

**Proposition 27.** Given $\{A_i\}_{i \in E}$ satisfying the Fano-non-Fano condition 2), each with cardinality $q$, and $U$ is a random variable with cardinality at most $q$. We have $\text{chk}_{12}(\{A_i\}, U)$ if and only if $U \doteq A_{12}$.

**Proof:** The “if” direction is straightforward. For the “only if” direction, assume $A_1 \leq A_2 U \wedge A_2 \leq A_1 U \wedge A_3 \leq A_{123} U$. Since $A_2 \perp A_1$ and $A_1 \leq A_2 U$, we have $U \leq A_2 A_1$ and $U \perp A_2$ by Proposition 25. Hence $\text{tri}(A_1, A_2, U)$ holds. Since $A_2 A_3 \perp A_1$ and $A_1 \leq A_2 A_3 U$, we have $A_2 A_3 \perp U$ by Proposition 25. Since $U \perp A_3$ and $A_3 \leq U A_{123}$, we have $A_{123} \leq U A_3$ by Proposition 25. Since $A_{123} \perp A_3$ and $A_3 \leq A_{123} U$, we have $U \leq A_{123} A_3$ and $A_{123} \perp U$ by Proposition 25. Hence $\text{tri}(U, A_3, A_{123})$ holds. Therefore, we have $\text{fnf}(A_1, A_2, A_3, U, A_{13}, A_{23}, A_{123})$ by Proposition 5.

By Proposition 6 let $(A, \{\theta_i\}_{i \in E})$ be an abelian group labeling of $\{A_i\}$, and $(\mathcal{A}, \{\theta_i\}_{i \in E})$ be an abelian group labeling of $(A_1, A_2, A_3, U, A_{13}, A_{23}, A_{123})$. We use $+$ for the group operation of $\mathcal{A}$, and $\perp$ for the group operation of $\mathcal{A}$. We have $\theta_1 A_1 + \theta_2 A_2 + \theta_3 A_3 = 0$ (write $\theta_1 A_1 = \theta_1(A_1)$ for brevity). Conditional on the event $\theta_3 A_3 = 0$, we have

$$A_{123} = \theta_1^{-1} 0 \left( \theta_1 A_1 + \theta_2 A_2 + \theta_3 \theta_3^{-1} 0 \right).$$

Combining this with $\tilde{\theta}_1 A_1 + \tilde{\theta}_2 A_2 = 0$, we have $A_{123} = 0$, and noting that $(A_1, A_2)$ can be any pair of values conditional on $\theta_3 A_3 = 0$, we have, for any $a_1, a_2$,

$$\tilde{\theta}_1 a_1 + \tilde{\theta}_2 a_2 = \tilde{\theta}_{123} \theta_{123}^{-1} \left( \theta_1 a_1 + \theta_2 a_2 + \theta_3 \theta_3^{-1} 0 \right).$$

Hence $\tilde{\theta}_1 U = \tilde{\theta}_1 A_1 + \tilde{\theta}_2 A_2$ contains the same information as $A_{12} = \theta_1 A_1 + \theta_2 A_2$.

---

**C. Enforcing Automorphisms via Subnetworks**

First we introduce a subnetwork that checks the condition $\text{end}_{1,2}(\{A_i\}, U)$ in (13). The subnetwork is given in Figure 4.

**Proposition 28.** Given $\{A_i\}_{i \in E}$ satisfying the Fano-non-Fano condition 2), each with cardinality $q$, and $U$ is a random variable with cardinality at most $q$. We have $\text{end}_{1,2}(\{A_i\}, U)$ if and only if $\{A_i\}, U$ satisfy the coding constraint in the $\text{end}_{12}$ subnetwork in Figure 4.

**Proof:** The “only if” direction follows directly from the definition. We now prove the “if” direction. Since $A_2 \perp A_1$ and $A_1 \leq A_2 U$, we have $U \leq A_2 A_1$ (and hence $U \doteq A_1|A_2$), $U$ is uniform with cardinality $q$, and $U \perp A_2$ by Proposition 25. Similarly, $V \doteq A_1|A_{23}$, $W \doteq A_{13}|A_2$, and $V, W$ are uniform with cardinality $q$. Since $A_2 A_3 \perp A_1$ and $A_1 \leq A_3 V \leq A_2 A_3 V$, we have $A_2 A_3 \perp V$ by Proposition 25. Since $A_3 \perp V$ and $V \leq A_3 U$, we have $U \leq A_3 V$ (and hence $U \doteq V|A_3$). Since
A_2 A_3 \perp A_{13} \text{ and } A_{13} \leq A_2 W \leq A_2 A_3 W, \text{ we have } A_2 A_3 \perp W \text{ by Proposition }^{25} \text{ Since } A_3 \perp W \text{ and } W \leq A_3 U, \text{ we have } U \leq A_3 W \text{ (and hence } U = W|A_3).$

Next, we introduce a subnetwork for the condition $\conv_{1,3}^{1,2}(\{A_i\}, U, V)$ in (17). The subnetwork is given in Figure 5. Note that $U$ is an input to the subnetwork, and $V$ is an output, and $U, V$ satisfy the coding constraint if and only if $\conv_{1,3}^{1,2}(\{A_i\}, U, V)$ holds. We also introduce the id_{2,3} subnetwork that checks whether $W$ satisfies $\end_{2,3}(\{A_i\}, W)$ and corresponds to the identity endomorphism (i.e., $W = A_2 - A_3$). Figure 5 can be obtained directly from (17), so we omit the proof. The $\conv_{3,2}$ subnetwork for (18) can be obtained similarly.

We then introduce a subnetwork for the condition $\comp_{1,2}(\{A_i\}, U_1, U_2, U_3)$ in (19). The subnetwork is given in Figure 6. Figure 6 can be obtained directly from (19), so we omit the proof.

The problem of the subnetwork in Figure 6 is that $U_3$ is an output. This is undesirable since for each random variable $U_3$, we will only be able to enforce one instance of $\comp_{1,2}(\{A_i\}, U_1, U_2, U_3)$, since $U_3$ can only appear once as the output of a subnetwork. To be able to enforce many instances of $\comp_{1,2}$, we will require the four subnetworks given in Figure 7: 1) the inv_{1,2} subnetwork, which checks whether the endomorphisms corresponding to $U, V$ are automorphisms and are inverses of each other (by checking whether their composition is the identity endomorphism, note that an endomorphism on a finite group having a left or right inverse is sufficient for it to be an automorphism); 2) the $\end_{1,2}$ subnetwork, which checks whether the endomorphisms corresponding to $U$ is an automorphism (by checking whether an inverse exists); 3) the $\iota_{1,2}$ subnetwork, which checks whether the endomorphisms corresponding to $U, V$ are automorphisms and are the same (by checking whether they have the same inverse); and 4) the $\iota\comp_{1,2}$ subnetwork, which checks whether the endomorphisms $g_1, g_2, g_3$ corresponding to $U_1, U_2, U_3$ are automorphisms and satisfy $g_3 = g_1 \cdot g_2$ (note that $g_1 \cdot g_2$ being an automorphism implies that $g_1$ and $g_2$ are automorphisms).

The main reason we use the uniform word problem for finite groups instead of finite monoids is that we require the inverse in order to check for equality of automorphisms, and hence to have a subnetwork that checks for composition which takes $U_1, U_2, U_3$ as inputs. Designing a subnetwork that checks for equality on general endomorphisms (not necessarily automorphisms) does not appear to be straightforward.
Figure 6. The $\text{comp}_{1,2}$ subnetwork, where the inputs are $U_1, U_2$ corresponding to endomorphisms $g_1, g_2$, and outputs $U_3$ corresponding to the endomorphism $g_3 = g_1 \cdot g_2$.

Figure 7. Top left: The $\text{inv}_{1,2}$ subnetwork, which checks whether the endomorphisms corresponding to $U,V$ are automorphisms and are inverses of each other. Top right: The $\text{iend}_{1,2}$ subnetwork, which checks whether the endomorphisms corresponding to $U$ is an automorphism. Bottom left: The $\text{ieq}_{1,2}$ subnetwork, which checks whether the endomorphisms corresponding to $U,V$ are automorphisms and are the same. Bottom right: The $\text{icomp}_{1,2}$ subnetwork, which checks whether the endomorphisms $g_1, g_2, g_3$ corresponding to $U_1, U_2, U_3$ are automorphisms and satisfy $g_3 = g_1 \cdot g_2$.

Figure 8. The final network, which contains the the base network in Figure 2. Each $U_j (j = 1, \ldots, k)$ is input to an $\text{iend}_{1,2}$ subnetwork. There are $l$ $\text{icomp}_{1,2}$ subnetwork, where the $j$-th one is connected to $U_{a_j}, U_{b_j}, U_{c_j}$. All solid edges, as well as all edges in all subnetworks and the the base network, are assumed to be two parallel edges. Only the dashed edge carrying $T$ is a single edge.
The final network is given in Figure 8, which contains the base network in Figure 2 (it has edges sending \( A_{12}, A_{13}, A_{23}, A_{123} \) to the other subnetworks, which are omitted for clarity). Each \( U_j \) \((j = 1, \ldots, k)\) is input to an \( \text{id}_1,2 \) subnetwork. There are \( l \) \( \text{icomp}_{1,2} \) subnetwork, where the \( j \)-th one is connected to \( U_{a_j}, b_j, c_j \). All solid edges, as well as all edges in all subnetworks and the base network, are assumed to be two parallel edges (together they can transmit \( q^3 \) possible values, and we can regard the alphabet size of these components to be \( q^3 \)). The source messages are also duplicated accordingly, i.e., we let \( A_i = A_i'' \) for \( i = 1,2,3 \), where each of \( A_i'' \) has cardinality \( q \), so we can treat \( A_i \) as a source message with alphabet size \( q^2 \). The source messages are \( A_1, A_2, A_3 \), or more precisely, \( A_1', A_2', A_2'', A_3', A_3'' \). Only the dashed edge carrying \( T \) (which is a function of \( A_2) \) is a single edge that can only transmit \( q \) different values.

The network is designed to be solvable if and only if there exists \( q \in \mathbb{Z}_{\geq 2} \), and random variables \( A_i \) \((i = 1, \ldots, k)\) with cardinality at most \( q^2 \), \( U_i \) \((i = 1, \ldots, k)\) with cardinality at most \( q^2 \), \( E \) with cardinality at most \( q^3 \), and \( T \) with cardinality at most \( q \), satisfying that \( A_1, A_2, A_3 \) are independent and uniformly distributed with cardinality \( q^2 \), and

\[
\text{id}_1,2(\{A_i\}, E) \wedge T \leq A_2 \wedge A_1 A_2 \leq U_1 E T.
\]

We complete the proof of Theorems 22 and 23 by showing that the network is not solvable if and only if (23) holds.

Proposition 29. For any fixed \( c \in \mathbb{Z}_{\geq 2} \), the following are equivalent:

- The implication (23) holds for all finite group \( B \) and all \( x_1, \ldots, x_k \in B \).
- The network in Figure 8 is unsolvable.
- The network in Figure 8 is unsolvable with alphabet size \( q = c^m \) for all \( m \in \mathbb{Z}_{\geq 1} \).

Proof: First, we show that if the implication (23) does not hold, then the network is solvable, and also solvable for some \( q = c^m \). Assume the implication (23) does not hold. There exists a finite group \( B \) and \( x_1, \ldots, x_k \in B \) satisfying the left hand side of (23), but not the right hand side, i.e., we have \( x_1 \neq e_B \). Fix any prime \( p \) and finite field \( \mathbb{F} \) with \( |\mathbb{F}| = p^2 \). Consider the left regular representation \( \lambda_B : B \to \text{GL}(\mathbb{F}^B) \). By Proposition 12 let \( \{A_i\} \) satisfy \( \text{fnf}(\{A_i\}) \) such that \( \langle \mathbb{F}^B, \{\theta_i\} \rangle \) is an abelian group labeling, where \( \mathbb{F}^B \) is treated as an abelian group under addition here. Assume \( \theta_i \) are identity functions, so \( A_i \in \mathbb{F}^B \). Note that the cardinality of \( A_i \) is \( p^{2|B|} \geq 4 \). Let \( U_j \) satisfy \( \text{id}_1,2(\{A_i\}, U_j) \) corresponding to the endomorphism \( \lambda_B(x_j) \) (by Proposition 15 and that \( \lambda_B(x_j) \) is an automorphism with an inverse). We have \( \text{id}_1,2(\{A_i\}, U_j) \) and \( \text{icomp}_{1,2}(\{A_i\}, U_{a_j}, b_j, c_j) \) by the left hand side of (23). Since \( x_1 \neq e_B \), by Proposition 24 the linear function \( \lambda_B(x_1) - \text{id}_B \) has rank at least \( |B|/2 \), and hence there exists a linear function \( t : \mathbb{F}^B \to \mathbb{F}^{|B|/2} \) such that the function \( x \mapsto (\lambda_B(x_1)(x) - x, t(x)) \) is injective. Let \( T = t(A_2) \) with cardinality at most \( |\mathbb{F}^{|B|/2}| = p^{|B|/2} \). Then \( A_2 \) can be deduced from \( U_1 = A_1 - \lambda_B(x_1)(A_2), E = A_1 - A_2 \) (corresponds to the identity endomorphism) and \( T = t(A_2) \) by considering \((E - U_1, T) = (\lambda_B(x_1)(A_2) - A_2, t(A_2))\), and \( A_1 = E + A_2 \) can be deduced as well. Hence (24) is satisfied with \( q = p^{|B|} \), and the network is solvable.

To show that the network is solvable for some \( q = c^m \), let \( c = \prod_{i=1}^n p_i \), where \( p_i \) are primes (possibly with duplicates). We have shown that the network is solvable with \( q = p_i^{|B|} \). By combining these codes together, the network is solvable with \( q = \prod_{i=1}^n p_i^{|B|} = c^m \).

Next, we show that if the implication (23) holds, then the network is not solvable, and hence not solvable for any \( q = c^m \). Assume the implication (23) holds. Assume the contrary that the network is solvable, and (24) holds. Fix any abelian group labeling \( (A_i, \{\theta_i\}) \). By \( \text{id}_1,2(\{A_i\}, U_j) \), we can find the endomorphism \( x_j \in \text{End}(A) \) corresponding to \( U_j \) (Proposition 14), which is actually an automorphism since \( \text{id}_1,2(\{A_i\}, U_j) \) holds. By \( \text{icomp}_{1,2}(\{A_i\}, U_{a_j}, b_j, c_j) \), we have \( x_{a_j} : x_{b_j} = x_{c_j} \). Applying (23) on the automorphism group \( \text{Aut}(A) \), we have \( x_1 = e_{\text{Aut}(A)} \). Hence \( U_1 \) contains the same information as \( A_1 - A_2 \), which is the same information as \( E \). The tuple \((U_1, E, T)\) can have at most \( q^3 \) different values, and cannot be used to deduce \((A_1, A_2)\) which has \( q^4 \) different values, giving a contradiction. Hence the network is not solvable.

As a corollary of Theorem 22 and the observation in [1] that network coding would be decidable if there is a computable upper bound on the alphabet size, the minimum alphabet size needed to solve a network is not upper-bounded by any computable function. We include a precise statement and a proof for the sake of completeness. For a solvable network, its minimum alphabet size is the smallest \( q \) such that the network is solvable with alphabet size \( q \). Let \( q_{\text{max}}(m) \) be the maximum of the minimum alphabet sizes of solvable networks with at most \( m \) nodes, \( m \) edges and \( m \) source messages. Note that \( q_{\text{max}}(m) \) is

8A computable function is a function that can be computed by an algorithm (or a Turing machine).
finite since there are finitely many networks with at most $m$ nodes/edges/messages. Then $q_{max}(m)$ is not upper-bounded by any computable function.

**Corollary 30.** There does not exist any computable function $f : \mathbb{Z}_{\geq 1} \rightarrow \mathbb{Z}_{\geq 2}$ satisfying that $q_{max}(m) = O(f(m))$ as $m \rightarrow \infty$.

**Proof:** Assume the contrary that there is a computable function $f$ such that $q_{max}(m) = O(f(m))$. Let $q_{max}(m) \leq cf(m)$ for $m \geq m_0$. Let $g(m) := \lceil c \cdot f(m) \rceil$ for $m \geq m_0$, and $g(m) = q_{max}(m)$ for $m < m_0$. Note that $q_{max}(m) \leq g(m)$, and $g$ is computable since $m \mapsto \lceil c \cdot f(m) \rceil$ is computable, and a computable function is still computable after changing finitely many values. Consider an algorithm that, given a network, compute $m$ as the maximum among the number of nodes, number of edges and number of source messages, exhaust all coding schemes (encoding functions at each node) with alphabet size at most $g(m)$, and output “solvable” if any coding scheme satisfies the decoding constraint, and output “unsolvable” otherwise.

If the network is solvable, since $q_{max}(m) \leq g(m)$, the algorithm finds a working coding scheme, and correctly outputs “solvable”. If the network is unsolvable, then the algorithm cannot find any working coding scheme, and correctly outputs “unsolvable”. We have found an algorithm that determines whether a network is solvable, contradicting Theorem 22. \hfill \blacksquare

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