Linearized potential vorticity mode and its role in transition to baroclinic instability

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Abstract. Stratified shear flows have been studied using Rapid Distortion Theory (RDT) and DNS. If this flow is in addition subjected to vertical rotation, a slaved horizontal stratification is forced and baroclinic instability can occur. In this context, the RDT analysis shows an extension of the unstable domain up to a Richardson number \( \text{Ri} \) of 1. This work is completed here with new results on transition to baroclinic instability. Especially, the role of \( k_x \approx 0 \) modes (small streamwise wavenumbers) and the importance of coupling with the potential vorticity mode \( u^{(\Omega_{pot})} \) is shown to be determinant for dramatic transient growth at intermediate times.

1 Introduction

The baroclinic context is characterized by a superposition of three coupled phenomena: (a) the Coriolis force, caused by earth rotation in geophysical flows; (b) stable stratification due to density gradients in the atmosphere, which lead to buoyancy forces in the vertical direction; (c) high vertical velocity gradients, at the altitude of the tropopause in atmospheric flows, in the form of jet streams, which, in first approximation, are modelled by homogeneous shear. Linear theory, or Rapid Distortion Theory (RDT), was applied to this problem by Salhi & Cambon (2006). It was shown that the gyroscopic torque resulting from the misalignment of system vorticity, in the vertical direction, and shear-induced vorticity, in the spanwise direction, (see figure 1 right) can be exactly balanced by an additional density gradient in the horizontal direction. This mechanism is close to the ‘geostrophic front adjustment’ in real geophysical flows Pedlosky (1987), and allows to use the simplified formalism of RDT for homogeneous turbulence (and DNS, presented in ETC13 Pieri et al. (2011)). The additional mean density gradient tilts the mean isopycnal lines with respect to the horizontal direction, so that baroclinic instability is triggered. For a mean flow characterized by a shear rate \( S \), a vertical stabilizing mean buoyancy gradient resulting in a constant Brunt-Väisälä frequency \( N \), and a Coriolis parameter \( f \) in the \( f \)-plane approximation, the baroclinicity parameter \( \varepsilon_B = fS/N^2 \) is also the above-mentioned angle of isopycnal lines and is a crucial instability parameter. The other relevant independent external parameter is the Richardson number \( \text{Ri} = N^2/S^2 \). Linear analysis is here referred to as RDT, following the nomenclature used in engineering and geophysics by Salhi & Cambon (2006), but there exist many studies based on the same formalism in other disciplines that use different terminologies, as in Mamatsashvili et al. (2010) using the shearing sheet approximation from astrophysicists. The reader is referred to Salhi & Cambon (2010) for a survey of this linear theory in engineering, applied mathematics and astrophysics. Our strategy for using RDT and DNS in Pieri et al. (2011) is generic, with the following essential components:
(i) Systematic use of a mean (or base) flow which is an exact solution of Euler Boussinesq equations:
This “admissibility condition” allows us to balance the gyroscopic torque in a physical way, looking at the conservation of (mean) absolute vorticity derived from Euler equations.

(ii) Decomposition of the fluctuating flow in terms of advected Fourier modes with time-dependent wave-vectors. These modes are called “Kelvin modes” (Applied Mathematics, probably following H. K. Moffatt in RDT) and “shear waves” (Astrophysics) and corresponds to “Rogallo space” (engineering). In addition, fluctuating velocity modes are split using the Craya-Herring frame of reference, resulting in a minimal number of two solenoidal components.

(iii) The linear solution is generated by a complete deterministic Green’s function, applied to the velocity/buoyancy fluctuating field, for arbitrary initial data and possibly arbitrary forcing, prior to any calculation of statistics.

(iv) Classical conservation of potential vorticity is applied. In the “RDT” context, this results in an invariant of the motion, as the linearized absolute potential vorticity, in which the vorticity of the mean shear is involved.

(v) Prediction from both random turbulence realizations of the velocity field, as in KS (Kinematic simulation), and prediction of statistical quantities, as in conventional RDT, are given, in order to investigate the parameter range, before applying costly DNS.

Formally, ii reduces the basic linear problem, for stability analysis or RDT and KS prediction, to a 3-rank system of ordinary equations with time-dependent coefficients. The fluctuating field in 5 components in physical space, 3 components for the velocity, one for the buoyancy and one for the pressure, reduces to a 3-component one in Fourier space, using the Craya-Herring decomposition into toroidal kinetic, poloidal kinetic, and potential energy. The rank of the system is still reduced to a non-homogeneous system of two equations, thanks to iv. In addition to the exact equations, which are given here, a generalized “vortex-wave” decomposition is derived, with physical interpretation.

2 Transient growth and $k_x \approx 0$ modes

![Figure 1](image_url)

**Figure 1.** The Craya-Herring frame (left): the divergence-free condition is implicitly imposed by locally projecting on the solenoidal base formed by planes perpendicular to each wavevector $k$. The baroclinic configuration (right).

The linear system obtained in spectral space using Craya-Herring components of velocity (toroidal $u^{(1)}$, poloidal $u^{(2)}$) as unknowns is given by

$$\frac{d}{dt} \begin{bmatrix} u^{(1)} \\ u^{(2)} \\ u^{(3)} \end{bmatrix} = S \begin{bmatrix} 0 & \cos \delta \sin \phi + \varepsilon_B Ri \sin \delta & 0 \\ -\varepsilon_B Ri \sin \delta & \sin \delta \cos \delta \cos \phi & -Ri \cos \delta \\ -\varepsilon_B \cos \phi & \cos \delta + \varepsilon_B \sin \delta \sin \phi & 0 \end{bmatrix} \begin{bmatrix} u^{(1)} \\ u^{(2)} \\ u^{(3)} \end{bmatrix}$$  \(1\)
where $\delta \in [-\pi/2, \pi/2]$ and $\phi \in [0, 2\pi]$ are respectively the co-latitude $\hat{\delta} = \pi/2 - \theta$ and longitude of the corresponding wavevector $k$ in the spectral space. The variable $u^{(3)}$ is the rescaled buoyancy mode defined by $u^{(3)} = S/N^2 \delta b$. Here $b$ accounts for the Fourier coefficient of the fluctuating part $b'$ of the buoyancy field $b = b + b'$. The advection equation for the vertical component $k_y$ of the wavevector $k$ is

$$\frac{dk_y}{dt} = -Sk_x$$

(2)

and its equivalent for the latitude is

$$\frac{d\delta}{dt} = S \cos^2 \delta \cos \phi$$

(3)

In the particular case $\delta \neq \pm \pi/2$, this equation can be solved by the mean of variables separation to give (taking $t_0 = 0$ and $\delta(t = 0) = \delta_0$):

$$\delta(t) = \tan (\tan \delta_0 - tS \cos \phi)$$

(4)

We observe that for modes $k$ such that $\cos \phi \tan \delta_0 > 0$, there exists $\tau = \tan \delta_0/(S \cos \phi) > 0$ such that for all $t \geq 0$

$$\delta(t + \tau) = \tan (-tS \cos \phi)$$

(5)

Let us take $\phi = \hat{\phi} + \epsilon$ with $\epsilon \ll 1$. Then for small $\epsilon$, $\tau$ is proportional to $1/\epsilon$. Numerical results showing the temporal behaviour of the real part of the poloidal component $u^{(2)}$ of the velocity field are presented on fig. 2 for different values of $\epsilon$. We show that these modes are stable in that they have finite amplitudes but present higher transient growth with $\epsilon$ closer to 0. It is also shown that the time of change of regime is also proportional to $1/\epsilon$ (like $\tau$) meaning that the change of regime is experienced later with $\epsilon$ smaller. This particular behaviour of modes with $k_x \to 0$ is confirmed by our kinematic simulation model. We show that these modes play an important role at intermediate times i.e. in the transition to baroclinic instability. At long times, the results match with the analysis by Salhi & Cambon (2006) and show that modes with $k_x = 0$ are the dominant ones (see fig. 4).

3 Vortex-wave decomposition

In this section we introduce a vortex-wave like decomposition following the work by Sagaut & Cambon (2008) and Tevzadze et al. (2008). In geophysics, the conservation of the potential vorticity (PV hereafter) along the flow motion (referred to as Ertel’s theorem) is a well-known feature. It traduces the advection equation for the vertical component $k_y$ of the wavevector $k$ gives in our context:

$$(k_z + R k_y)u^{(3)} - (\varepsilon B k_z k_y / k_\perp + k_\perp)u^{(1)} - \varepsilon B k_z k_y / k_\perp u^{(2)} = k_\perp u^{(\Pi_{pol})}$$

(6)

with $k_z^2 = k_x^2 + k_z^2$. This result can be recovered by forming a linear combination of the three rows of the previous Kelvin-Moffat matrix (1). The simplified linear system one has to solve for a given mode in the Craya-Herring frame is:

$$\frac{d}{dt} \begin{bmatrix} u^{(1)} \\ u^{(2)} \end{bmatrix} = S \begin{bmatrix} 0 & (\cos \delta \sin \phi + R \sin \delta) \\ \cos \delta \sin \phi + R \sin \delta & -\cos \delta \sin \phi + R \sin \delta \end{bmatrix} \begin{bmatrix} u^{(1)} \\ u^{(2)} \end{bmatrix}$$

$$+ \begin{bmatrix} 0 \\ -R \cos^2 \delta \end{bmatrix} u^{(\Pi_{pol})}$$
This new formulation highlights first that modes $k$ with $\delta \approx \pm \pi/2$ (vertical modes) are not coupled with the PV mode $u^{(\Omega_{pot})}$ and secondly that coupling is maximal for equatorial modes $k$ having $\delta \approx 0$, where $\delta$ is time-dependent in our model. Eq. (3) shows that asymptotically $\delta \to \pm \pi/2$ simply describing that the shear dynamics tends to reduce the coupling with the PV mode. It also shows that the modes satisfying the condition $\cos \phi \tan \delta_0 > 0$ that were previously identified to be transitally dominant are also the modes that experience an angle $\delta = 0$ i.e. a maximal coupling with the PV mode at the critical time $\tau$ defined in section 2. This transitional growth of modes with $k_x \approx 0$ is thus the consequence of potential vorticity conservation and the coupling with the PV mode enforced by shear effects at intermediate times only since eq. (3) shows that at long times the shear inhibits the coupling with $u^{(\Omega_{pot})}$. Setting $k_x = 0$ i.e. $\phi = \pm \pi/2$ makes the coupling with the PV mode constant with time since in this case $\delta$ is not time-dependent anymore (these are modes unaffected by shear). To see the effect of the coupling with the PV mode we set $\phi = \pi/2$ and compare results varying $\delta$. The coupling is maximum at $\delta = 0$ and minimum at $\delta = \pm \pi/2$.

The same RDT simulations are done with zero initial potential vorticity to clearly show what happens without the coupling with $u^{(\Omega_{pot})}$. Results are presented on fig 3. We observe that the solution changes of regime at the same characteristic time $\tau$ but does not experience large variations in amplitude reducing $\varepsilon$. It shows that the transient behaviour of these modes is due to small $k_x$ but that a non-zero potential vorticity is necessary to observe its temporary dominance. An analysis of the transition regime gives a
Figure 3. Real part of the poloidal component of the velocity with time $tS$ for $\phi = \frac{\pi}{2} + \varepsilon$ and $\delta = \frac{\pi}{4}$: $\varepsilon = 0.01$ (a); $\varepsilon = 0.005$ (b); $\varepsilon = 0.001$ (c); $\varepsilon = 0$ (d). $Ri = 1$ and $\varepsilon_B = 2$ with $u^{(\Omega_{pot})}(0) = 0$.

Figure 4. Time-evolution of the kinetic energy distribution as a function of the longitude. Kinematic Simulation results for 1024 random modes averaged over 100 statistical events. $Ri = 0.5$, $\varepsilon_B = 0.2$.

departure from small amplitudes to bigger ones that behaves like

$$f(t - t_0) = \pi \frac{u^{(\Omega_{pot})}}{\varepsilon} \text{erf} \left( \beta \varepsilon (t - t_0) \right) \exp \left( -\gamma \varepsilon (t - t_0)^2 \right)$$  \hspace{1cm} (7)
with

\[ \beta_\varepsilon \propto \sqrt{\varepsilon} \quad (8) \]

\[ \gamma_\varepsilon \propto \varepsilon \quad (9) \]

We illustrate this for one particular case, see fig. 5. It shows that when \( u^{(\Omega_{pot})} \neq 0 \), the solution has an amplitude proportional to \( 1/\varepsilon \) that completes our analysis and demonstrates that modes closer to \( k_x = 0 \) will experience higher transitional growth at longer times \( \tau \) (see fig.4).

**Figure 5.** Left: The RDT solution (real part of the poloidal component of the velocity field) for \( \varepsilon = 10^{-4} \) and \( \| u^{(\Omega_{pot})} \| = 10^2 \). Right: the corresponding fitting using our analytical approximation \( f \) defined by (7). The parameters in \( f \) are set to \( \beta_\varepsilon = 11.10^{-3}, \gamma_\varepsilon = 136.10^{-7} \) and \( t_0 = 10295.5 \).

**Conclusion**

In this paper we have shown that vortex-wave decomposition is essential in transitional stability analysis in the baroclinic context. The coupling with the vortex mode, noted \( u^{(\Omega_{pot})} \) here, is responsible for high transitional growth of modes with \( k_x \approx 0 \). Rapid distortion theory and Kinematic Simulation completed by analytical work are all consistent with this new result. In particular, to observe important transitional growth it is necessary to satisfy both \( k_x \approx 0 \) but \( k_x \neq 0 \) and \( u^{(\Omega_{pot})} \neq 0 \).

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