Abstract. We study the enhanced algebraic group $G$ of $G = \text{GL}(V)$ over $C$, which is a product variety $\text{GL}(V) \times V$, endowed with an enhanced cross product. Associated with a natural tensor representation of $G$, there is naturally an enhanced Schur algebra $E$. We precisely investigate its structure, and study the dualities on the enhanced tensor representations for variant groups and algebras. In this course, an algebraic model of so-called degenerate double Hecke algebras (DDHA) is produced, and becomes a powerful implement. The applications of $E$ and DDHA give rise to two results for the classical representations of $\text{GL}(V)$: (i) A duality between $\text{GL}(V)$ and DDHA; (ii) A branching duality formula. With aid of the above, we further obtain an enhanced Schur-Weyl duality for $G$. What is more, the enhanced Schur algebra $E$ turns out to have only one block. The Cartan invariants for $E$ are precisely determined.

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Introduction

0.1. An algebraic group $G$ is called a semi-reductive group if $G$ is a semi-direct product of a reductive closed subgroup $G_0$ and the unipotent radical $U$. The study of semi-reductive...
algebraic groups and their Lie algebras becomes very important to lots of cases. The following examples are what we are concerned.

(i) Suppose \( k \) is an algebraically closed field of characteristic \( p \geq 7 \). Then finite-dimensional simple Lie algebras over \( k \) are isomorphically classified into two types, one is of classical type from the analogues of complex simple Lie algebras, the other one is of Cartan type (see [23]). For a given restricted Cartan-type simple Lie algebra \( \mathcal{L} \) over \( k \), there is a distinguished maximal subalgebra \( \mathfrak{g} \) which is not only stable under the action of \( G := \text{Aut}(\mathcal{L}) \), but also essentially coincides with \( \text{Lie}(G) \). This \( G \) turns out to be a semi-reductive algebraic group (see [18], [33], etc.). What is the most important, the study of irreducible representations for \( \mathcal{L} \) can be reduced, to a large extent, to the one for \( \mathfrak{g} \) (see [20], [24], [25], [28], [29], [35], [36], etc.).

(ii) Parabolic subgroups of a reductive algebraic group are semi-reductive.

(iii) Suppose \( G \) is a reductive algebraic group over an algebraically closed field \( F \) with \( \mathfrak{g} = \text{Lie}(G) \), and the characteristic of \( F \) is good for \( G \) in the sense of [14, §3.9]. For a nilpotent element \( X \in \mathfrak{g} \), denote by \( G_X \) the centralizer subgroup of \( X \) in \( G \). Then \( G_X \) is a semi-reductive algebraic group (see [16, Proposition 5.10]).

(iv) For a connected reductive algebraic group \( G \) and a given finite-dimensional representation \((M, \rho)\) of \( G \), one can naturally construct a new algebraic group from \( G \times M \) associated with \( \rho \) (see Example [14]). This new algebraic group is a semi-reductive group, which is particularly called an enhanced reductive group of \( G \).

One of our motivations to raise this question is the study of representations of non-classical simple Lie algebras in prime characteristic. Apart from this motivation, it has its own interest to investigate the question. So we raise such a question, and initiate this investigation.

Semi-reductive algebraic groups keep many good structural features (see §1.2–§1.3). A preliminary investigation on semi-reductive groups shows some interesting phenomenon and some good applications (see [22]). In this paper, we will be mostly concerned with problems arising from the above (iv).

0.2. Let \( G = \text{GL}(V) \) and \( \nu \) be the natural representation of \( G \) on \( V \). Then we have a typical enhanced reductive algebraic group \( \overline{G} = G \times \nu V \), which is a closed subgroup of \( \text{GL}(V) \) with \( V \) being a one-dimensional extension of \( V \). The enhanced reductive group \( \overline{G} \) will be the main topic of the present paper.

By the classical Schur-Weyl duality, the study of polynomial representations of general linear groups produces Schur algebras. Precisely, for a given infinite filed \( F \), and \( G = \text{GL}(n, F) \) the general linear group over \( F \), the Schur algebra \( S_F(n, r) \) is exactly \( \text{End}_F(E^{\otimes r})^E \) for \( E = F^n \). The clear structure makes Schur algebras become powerful and tractable in the study of polynomial representations of \( \text{GL}(n, F) \) (see [12], [19], etc.). By analogy of this, the tensor representations of an enhanced group \( \overline{G} \) naturally produce the so-called enhanced Schur algebra \( \mathcal{E}(n, r) \) which is the algebra generated by the image of \( \overline{G} \) in the \( r \)th tensor representation \( V^{\otimes r} \) (see [25]).

For simplicity, we work with \( F = \mathbb{C} \) in the main body of the text. Then we can realise \( \mathcal{E}(n, r) \) as a subalgebra of \( S_{\mathbb{C}}(n + 1, r) \).

0.3. The main purposes are double. One is to develop the enhanced Schur algebras and their representations. The another one is to investigate dualities of variant groups and algebras
in the enhanced tensor representations. In this course, an important algebraic model of so-called degenerate double Hecke algebras (DDHAs for short) is introduced, in the same spirit of degenerate affine Hecke algebras and the related (see [6] and [2]). Roughly speaking, the DDHA $\mathcal{H}_r$ of type $A_{r-1}$ is a combination of varieties of subalgebras generated compatibly by the group algebras $\mathbb{C}\mathfrak{S}_r$ and $\mathbb{C}\mathfrak{S}_i$ for all positive integers $l$ not bigger than $r$. The $l$th DDHA $\mathcal{H}_r^l$ is an associative algebra defined by generators $s_i$, and $x_\sigma$, $i = 1, \ldots, r - 1$, and $\sigma \in \mathfrak{S}_l$, and by relations:

$$s_i x_\sigma = x_{s_i \sigma}, \quad x_\sigma s_i = x_{s_\sigma s_i} \quad \text{for } \sigma \in \mathfrak{S}_l, i < l;$$

$$s_i x_\sigma = x_\sigma s_i \quad \text{for } \sigma \in \mathfrak{S}_l, i > l.$$ 

Together with the defining relations of $\mathfrak{S}_r$ and of $\mathfrak{S}_l$ (see (3.1)-(3.6) for the complete defining relations of a DDHA). This is an infinite-dimensional algebra. Nevertheless, the $l$th DDHA $\mathcal{H}_r^l$ naturally arises from the tensor representation $(\mathbb{C}^{n+1})^{\otimes r}$ over $\text{GL}_n$. The core operators in $\text{End}_\mathbb{C}((\mathbb{C}^{n+1})^{\otimes r})$ for $l < r$ and $\sigma \in \mathfrak{S}_l$, are presented as below

$$x_\sigma = \Psi_l(\sigma) \otimes \text{id}^{\otimes r-l}$$

where $\Psi_l(\sigma)$ is a position permutation by $\sigma$ in the first $l$ tensor factors (see the paragraph around (3.7) for the precise meaning). From those $x_\sigma$ and all usual operators $s_i$ interchanging the $i$th and $(i + 1)$th factors in $(\mathbb{C}^{n+1})^{\otimes r}$, $\mathcal{H}_r^l$ naturally arises. Correspondingly, $(\mathbb{C}^{n+1})^{\otimes r}$ becomes a natural module over $\mathcal{H}_r^l$. This representation is denoted by $\Xi$. Then $\Xi(\mathcal{H}_r^l)$ becomes finite-dimensional, denote by $D(n, r)$. On the enhanced-tensor representation space $V^{\otimes r}$, the role of pair $(\mathcal{H}_r^l, D(n, r))$ is somewhat a counterpart of the one of pair $(\text{GL}(V), S_C(n, r))$ when we consider the usual tensor representation space $V^{\otimes r}$.

The above DDHAs are powerful ingredients in the course of establishing dualities for the enhanced tensor representations.

0.4. Below are the main results.

**Theorem 0.1.** (1) (Theorem 2.11) The enhanced Schur algebra $\mathcal{E}(n, r)$ has a basis $\{\xi_{\bar{s}_{\bar{a}}, \bar{j}} \mid (\bar{s}, \bar{a}, j) \in E\}$, and

$$\dim \mathcal{E}(n, r) = \sum_{k=0}^{r} \binom{n + k - 1}{k} \binom{n + k}{k}.$$ 

There are a set of canonical generators $\{\theta_{s, t}, \xi_{i, j} \mid (s, t) \in \Lambda, (i, j) \in D/\mathfrak{S}_r\}$ for $\mathcal{E}(n, r)$.

(2) (Theorems 6.2 and 6.4) The enhanced algebra $\mathcal{E}(n, r)$ has one block. The isomorphism classes of irreducible modules and the indecomposable projective modules (PIM for short) are parameterized by the set of dominant weights $\Lambda^+ := \bigcup_{l=1}^{r} \Lambda^+(n, l)$, respectively. For an irreducible module $D_\gamma^l$ and a PIM $P_\gamma$ with $\gamma, \gamma' \in \Lambda^+$, set $a_{\gamma, \gamma'} := (P_\gamma : D_\gamma^l)$, and $\ell(\gamma) = \#\mathfrak{S}_n \cdot \gamma$. Then the Cartan invariants are presented as

$$a_{\gamma, \gamma'} = \begin{cases} 1, & \text{if } \gamma' = \gamma; \\ \ell(\gamma'), & \text{if } |\gamma'| < |\gamma|; \\ 0, & \text{otherwise.} \end{cases}$$

1In the literature, double affine Hecke algebras (DAHA) arising from the study of KZ equations are familiar to authors (ref. [7]). Our DDHA have no direct relation with the former.
With aid of the structural description of $E(n, r)$, we establish a duality between DDHA and $\text{GL}_n$ in $\text{End}_C((\mathbb{C}^{n+1})^{\otimes r})$ and its consequence a “duality branching formula” in dimensions. As its application, we finally obtain the enhanced Schur-Weyl duality stated below.

**Theorem 0.2.** For $V = \mathbb{C}^{n+1}$, denote by $(V^{\otimes r}, \Phi)$ the tensor representations of $\text{GL}_{n+1} = \text{GL}(V, \mathbb{C})$, and by $(V^{\otimes r}, \Psi)$ the permutation representations of the symmetric group $\mathfrak{S}_r$.

(1) (Theorem 4.3) Let $\mathcal{H}_r$ be the degenerate double Hecke algebra associated with $\mathfrak{S}_r$. Then the following restricted Schur-Weyl duality holds.

$$\text{End}_C(\mathcal{H}_r(V^{\otimes r})) = \Xi(\mathcal{H}_r);$$
$$\text{End}_C(\mathcal{H}_r(V^{\otimes r})) = \mathcal{C} \Phi(\text{GL}_n).$$

(2) (Corollary 4.5) For $\lambda \in \Lambda^+(n, l)$ (a partition of $l$ with $n$ parts), denote by $S^\lambda_l$ the irreducible module of $\mathfrak{S}_l$ corresponding to $\lambda$. The following dimension formula can be regarded as a duality to the classical branching rule. For $l \in \{1, 2, \ldots, r\}$, and $\lambda \in \Lambda^+(n, l)$,

$$\sum_{\mu \in \Lambda^+(n+1, r)} \dim S^\mu_r = \binom{r}{l} \dim S^\lambda_l$$

(3) (Theorem 5.3) $\text{End}_B(G)(V^{\otimes r}) = \Xi(\mathcal{H}_r)^G$ for $G = \text{GL}(V)$.

It is worth mentioning that the study of invariants beyond reductive groups is challenging (see [8]-[10]), consequently the invariant property of $G$ as above has its own interest.

0.5. Our paper is organized as follows. In the first section, we introduce some basic notions and notations for semi-reductive groups. Then we give some fundamental properties of semi-reductive groups. In the second section, we introduce the enhanced Schur algebras, and investigate their structure. In the third section, we introduce the degenerate double Hecke algebras and demonstrate their representation meaning in the enhanced tensor products. The forth section is devoted to the proof of Theorem 4.3 and Corollary 4.5. With the above, in the fifth section we first prove the enhanced Schur Weyl duality and then give some tensor invariants. In the concluding section, we investigate the representations of the enhanced Schur algebras, obtaining the results on the blocks and Cartan invariants.

0.6. As to other aspects, we will make investigations somewhere else (to see [26], [27], [31], etc.). The enhanced Schur algebras can be defined in prime characteristic. Their modular representations will be an interesting topic in the future. It is worth mentioning that the nilpotent cone of $\mathfrak{gl}(V) := \text{Lie}(G)$ for $G = \text{GL}(V)$ is the same as the enhanced nilpotent cone studied by Achar-Henderson in [1]. Adjoint nilpotent orbits of $G$ in $\mathfrak{gl}(V)$ are compatible with enhanced nilpotent orbits studied in [1]. With respect to those, there are some interesting phenomenon parallel to the $\text{GL}(V)$-theory in prime characteristic, including Jantzen’s realization of support varieties of Weyl modules by the closures of some nilpotent orbits (see [17], [21]).

1. **Semi-reductive groups and semi-reductive Lie algebras**

In this section, all vector spaces and varieties are over a field $\mathbb{F}$ which stands for either the complex number field $\mathbb{C}$, or an algebraically closed field $k$ of characteristic $p > 0$. 

1.1. Notions and notations.

Definition 1.1. An algebraic group \( G \) over \( \mathbb{F} \) is called semi-reductive if \( G = G_0 \ltimes U \) with \( G_0 \) being a reductive subgroup, and \( U \) the unipotent radical. Let \( g = \text{Lie}(G) \), and \( g_0 = \text{Lie}(G_0) \) and \( u = \text{Lie}(U) \), then \( g = g_0 \oplus u \).

In the following, we list some examples of semi-reductive Lie algebras (resp. semi-reductive algebraic groups).

Example 1.2. (The non-negative graded part of a restricted Lie algebra of Cartan type over \( \mathbb{k} \)) Let \( \mathcal{A}_n = \mathbb{k}[x_1, \ldots, x_n]/(x_1^p, \ldots, x_n^p) \), a quotient of the polynomial ring by the ideal generated by \( x_i^p \), \( 1 \leq i \leq n \). Set \( \mathcal{L} = \text{Der}(\mathcal{A}_n) \). Then \( \mathcal{L} \) is a simple Lie algebra unless \( n = 1 \) and \( p = 2 \) (this simple Lie algebra is usually called a generalized Witt algebra, denoted by \( W_n \)). Associated with the degrees of polynomial quotients, \( \mathcal{L} \) becomes a graded algebra \( \mathcal{L} = \sum_{d \geq -1} \mathcal{L}_d \). Let \( g = \sum_{d \geq 0} \mathcal{L}_d \). Then \( g = \text{Lie}(G) \) for \( G := \text{Aut}(\mathcal{L}) \) (see [33]). Furthermore, \( G = GL(n, \mathbb{k}) \ltimes U \) and \( g = \text{Lie}(G) \) with \( \mathcal{L}_0 \cong \mathfrak{gl}(n, \mathbb{k}) \), \( \text{Lie}(U) := \sum_{d \geq 1} \mathcal{L}_d \). So \( G \) is a semi-reductive group and \( g \) is a semi-reductive Lie algebra.

More generally, apart from \( W_n \), there are another three series of Cartan type restricted simple Lie algebras \( S_n, H_n, K_n \) (see [31, 32]). Each of them is endowed with the corresponding graded structure. Similarly, one can consider \( \mathfrak{g} = \sum_{d \geq 0} \mathcal{L}_d \) for \( \mathcal{L} = X_n \) for \( X \in \{ S, H, K \} \), and \( G = \text{Aut}(X_n) \). The corresponding semi-reductive groups and Lie algebras also appear as above.

Example 1.3. Let \( G \) be a connected reductive algebraic group over \( \mathbb{F} \). For any given nilpotent element \( X \in \mathfrak{g} \), let \( G_X \) be the centralizer of \( X \) in \( G \), and \( g_X = \text{Lie}(G_X) \). By [16 §5.10–§5.11], \( G_X = C_X \ltimes R_X \) is semi-reductive.

Example 1.4. (Enhanced reductive algebraic groups) Let \( G_0 \) be a connected reductive algebraic group over \( \mathbb{F} \), and \( (M, \rho) \) be a finite-dimensional rational representation of \( G_0 \) with representation space \( M \) over \( \mathbb{F} \). Consider the product variety \( G_0 \times M \). Regard \( M \) as an additive algebraic group. The variety \( G_0 \times M \) is endowed with an enhanced cross product structure denoted by \( G_0 \times_\rho M \), by defining for any \( (g_1, v_1), (g_2, v_2) \in G_0 \times M \)

\[
(g_1, v_1) \cdot (g_2, v_2) := (g_1g_2, \rho(g_1)v_2 + v_1). \tag{1.1}
\]

Then by a straightforward computation it is easily known that \( G_0 := G_0 \times_\rho M \) becomes a group with identity \( (e, 0) \) for the identity \( e \in G_0 \), and \( (g, v)^{-1} = (g^{-1}, -\rho(g)^{-1}v) \). And \( G_0 \times_\rho M \) has a subgroup \( G_0 \) identified with \( (G_0, 0) \) and a subgroup \( M \) identified with \( (e, M) \). Furthermore, \( G_0 \) is connected since \( G_0 \) and \( M \) are irreducible varieties. We call \( G_0 \) an enhanced reductive algebraic group associated with the representation space \( M \). What is more, \( G_0 \) and \( M \) are closed subgroups of \( G_0 \), and \( M \) is a normal closed subgroup. Actually, we have \( (g, w)(e, v)(g, w)^{-1} = (e, \rho(g)v) \) for any \( (g, w) \in G_0 \). From now on, we will write down \( \hat{g} \) for \( (g, 0) \) and \( e^v \) for \( (e, v) \) unless other appointments. It is clear that \( e^v \cdot e^w = e^{v+w} \) for \( v, w \in V \).

Suppose \( g_0 = \text{Lie}(G_0) \). Then \( (M, d(\rho)) \) becomes a representation of \( g_0 \). Naturally, \( \text{Lie}(G_0) = g_0 \oplus M \), with Lie bracket

\[
[(X_1, v_1), (X_2, v_2)] := ([X_1, X_2], d(\rho)(X_1)v_2 - d(\rho)(X_2)v_1),
\]

which is called an enhanced reductive Lie algebra.

Clearly, \( G_0 \) is a semi-reductive group with \( M \) being the unipotent radical.
1.2. In the remainder of this section, we assume that $G = G_0 \ltimes U$ is a connected semi-reductive algebraic group over an algebraically closed field $\mathbb{F}$ where $G_0$ is a connected reductive subgroup and $U$ the unipotent radical. Let $\mathfrak{g} = \text{Lie}(G)$ be the Lie algebra of $G$. In the following we will illustrate the structure of Borel subgroups of $G$.

**Lemma 1.5.** The following statements hold.

1. Suppose $B$ is a Borel subgroup of $G$. Then $B \supset U$. Furthermore, $B_0 := B \cap G_0$ is a Borel subgroup of $G_0$, and $B = B_0 \ltimes U$.

2. Any maximal torus $T$ of $G$ is conjugate to a maximal torus $T_0$ of $G_0$.

**Proof.** (1) As $U$ is the unipotent radical of $G$, $BU$ is still a closed subgroup containing $B$. We further assert that $BU$ is solvable. Firstly, by a straightforward computation we have that the $i$th derived subgroup $\mathcal{D}^i(BU)$ is contained in $\mathcal{D}^i(B)U$. By the solvableness of $B$, there exists some positive integer $t$ such that $\mathcal{D}^t(B)$ is the identity group $\{e\}$. So $\mathcal{D}^t(BU) \subset U$. Secondly, as $U$ is unipotent, and then solvable. So there exists some positive integer $r$ such that $\mathcal{D}^r(U) = \{e\}$. Hence $\mathcal{D}^{t+r}(BU) = \{e\}$. The assertion is proved.

The maximality of the solvable closed subgroup $B$ implies that $BU = B$, this is to say, $U$ is contained in the unipotent radical $B_0$ of $B$. Set $B_0 = B \cap G_0$ which is clearly a closed solvable subgroup of $G_0$. By the same token as above, $B_0U$ is a closed solvable subgroup of $B$. On the other hand, for any $b \in B$, by the definition of semi-reductive groups we have $b = b_0u$ for some $b_0 \in G_0$ and $u \in U$. As $U \subset B$, we have further that $b_0 \in B_0$. From this it follows that $B$ is contained in $B_0 \ltimes U$. The remaining thing is to prove that $B_0$ is a Borel subgroup of $G_0$. It is clear that $B_0$ is a solvable closed subgroup of $G_0$. If $B_0$ is not a maximal solvable closed subgroup of $G_0$, then the solvable closed subgroup $B_0'U$ of $G$ contains $B$ properly. It contradicts with the maximality of $B$. Hence $B_0$ is really maximal. Summing up, the statement in (1) is proved.

(2) Note that the maximal tori in $G$ are all conjugate (see [4, 11.3]). This statement follows from (1). \qed

For a semi-reductive Lie algebra, the following is clear.

**Lemma 1.6.** For a semi-reductive group $G$ and $\mathfrak{g} = \text{Lie}(G)$, all maximal tori of $\mathfrak{g}$ are conjugate under adjoint action of $G$.

By Lemmas 1.5 and 1.6, we can choose a maximal torus $T$ of $G$, which lies in $G_0$ without loss of generality. By [4, §8.17], we have the following decomposition of root spaces associated with $T$

$$\mathfrak{g} = \mathfrak{t} + \sum_{\alpha \in \Phi(G_0,T)} (\mathfrak{g}_0)_\alpha + \sum_{\alpha \in \Phi(U,T)} \mathfrak{u}_\alpha$$

(1.2)

where $\Phi(U,T)$ is the set of roots of $U$ relative to $T$, and $\mathfrak{u} = \text{Lie}(U)$ admits a decomposition

$$\mathfrak{u} = \sum_{\alpha \in \Phi(U,T)} \mathfrak{u}_\alpha.$$

In the sequent arguments, the root system $\Phi(G_0,T)$ will be denoted by $\Delta$ for simplicity, which is actually independent of the choice of $T$. We fix a positive root system $\Delta^+$. The corresponding Borel subgroup will be denoted by $B$ which contains $T$, and the corresponding simple system is denoted by $\Pi$. 
1.3. The following facts are clear.

**Lemma 1.7.** Let \( G \) be a connected semi-reductive group, and \( T \) a maximal torus of \( G_0 \).

1. Set \( \mathcal{W}(G,T) := \text{Nor}_G(T)/\text{Cent}_G(T) \). Then \( \mathcal{W}(G,T) \cong \mathcal{W} \), where \( \mathcal{W} \) is the Weyl group of \( G_0 \). This is to say, \( G \) admits the Weyl group \( \mathcal{W} \) coinciding with the one of \( G_0 \).

2. Let \( \{ w \mid w \in W \} \) be a set of representatives in \( \text{Nor}_G(T) \) of the elements of \( \mathcal{W} \). Denote by \( C(w) \) the double coset \( BwB \), and by \( C_0(w) \) the corresponding component \( B_0wB_0 \) of the Bruhat decomposition of \( G_0 \). Then for any \( w \in \mathcal{W} \)

\[
C(w) = C_0(w) \ltimes U. \tag{1.3}
\]

3. Let \( w = s_1 \cdots s_h \) be a reduced expression of \( w \in \mathcal{W} \), with \( s_i = s_{\gamma_i} \) for \( \gamma_i \in \Pi \). Then \( C(w) \cong \mathbb{A}^h \ltimes B \) for the affine \( h \)-space \( \mathbb{A}^h \).

4. For \( w \in \mathcal{W} \), set \( \Phi(w) := \{ \alpha \in \Delta^+ \mid w \cdot \alpha \in -\Delta^+ \} \), and \( U_w := \prod_{\alpha \in \Phi(w)} U_{\alpha} \), then \( U_{w^{-1}} \times B \cong C(w) \) via sending \((u,b)\) onto \( wb \).

5. (Bruhat Decomposition) We have \( G = \bigcup_{w \in \mathcal{W}} BwB \). Therefore, for any \( g \in G \), \( g \) can be written uniquely in the form \( wib \) with \( w \in \mathcal{W} \), \( u \in U_{w^{-1}} \) and \( b \in B \).

**Proof.** (1) Note that \( U \) is the unipotent radical of \( G \), and \( G = G_0 \ltimes U \). We have \( \text{Nor}_G(T) = \text{Nor}_{G_0}(T) \) and \( \text{Cent}_G(T) = \text{Cent}_{G_0}(T) \). The statement is proved.

2. Note that \( Uw = wU \) and \( B = B_0U = UB_0 \) for \( B_0 = B \cap G_0 \). We have

\[
C(w) = B_0UwB_0U = B_0wUB_0U = B_0wB_0U = C_0(w) \ltimes U. \tag{1.4}
\]

(3) and (4) follow from (2) along with [30, Lemma 8.3.6].

(5) follows from (2) along with the Bruhat decomposition Theorem for \( G_0 \) ([30, Theorem 8.3.8]).

**Proposition 1.8.** Let \( G \) be a connected semi-reductive group. Keep the notations as in Lemma 1.7. The following statements hold.

1. The group \( G \) admits a unique open double coset \( C(w_0) \), where \( w_0 \) is the longest element of \( \mathcal{W} \).

2. For any given Borel subgroup \( B \), let \( \mathcal{B} \) denote the homogeneous space \( G/B \), and \( \pi : G \rightarrow \mathcal{B} \) be the canonical morphism. Then \( \pi \) has local sections.

3. For a rational \( B \)-module \( M \), there exists a fibre bundle over \( \mathcal{B} \) associated with \( M \), denoted by \( G \times^B M \).

**Proof.** (1) Note that \( C(w_0) \) is a \( B \times B \)-orbit in \( G \) under double action, and \( C(w_0) \) is open in its closure. Thanks to Lemma 1.7(4), we have

\[
\dim C(w_0) = \#\{ \Delta^+ \} + \dim B = \dim G.
\]

On the hand, \( G \) is irreducible, so \( G = C(w_0) \). It follows that \( C(w_0) \) is an open subset of \( G \). By the uniqueness of the longest element in \( \mathcal{W} \) and Lemma 1.7(3)(5), we have that \( C(w_0) \) is the unique open double coset.
(2) According to [30] §5.5.7, we only need to certify that (a) \( \mathcal{B} \) is covered by open sets \( \{ U \} \); (b) each of such open sets has a section, i.e. a morphism \( \sigma : U \to \mathcal{B} \) satisfying \( \pi^{-1} \circ \sigma = \text{id}_U \). Thanks to [30] Theorem 5.5.5, \( \pi \) is open and separable. Let \( X(w) = \pi(C(w)) \). By (1), \( X(w_0) \) is an open set of \( \mathcal{B} \). By Lemma [30](4), \( \pi \) has a section on \( X(w_0) \). Note that \( \{ gX(w_0) \mid g \in G \} \) constitute of open covering of \( \mathcal{B} \). Using the translation, the statement follows from the argument on \( X(w_0) \).

(3) follows from (2) and [30] Lemma 5.5.8. \qed

2. Enhanced tensor representations and enhanced Schur algebras

From now on, the ground field \( \mathbb{F} \) will be \( \mathbb{C} \). We suppose \( G = \text{GL}(V) \) over \( \mathbb{C} \), and suppose that \( \underline{G} := G \times, V \) is an enhanced group of \( G \) associated with the natural representation \( \nu \) on \( V \). All representations for algebraic groups are always assumed to be rational ones.

2.1. Enhanced natural modules. An irreducible \( G \)-module becomes naturally an irreducible \( \underline{G} \) with trivial \( V \)-action. The isomorphism classes of irreducible rational representations of \( \underline{G} \) coincide with the ones of \( G \).

Denote by \( \underline{V} \) the one-dimensional extension of \( V \simeq \mathbb{C}^n \) via the vector \( \eta \), i.e. \( \underline{V} = V \oplus \mathbb{C} \eta \simeq \mathbb{C}^{n+1} \). Then \( \underline{V} \) naturally becomes a \( \underline{G} \)-module which is defined for any \( g := (g, v) \in \underline{G} \) and \( u = u + a\eta \in \underline{V} \) with \( g \in G, v \in V \) and \( a \in \mathbb{C} \), via

\[
g \cdot u = \nu(g)u + av + a\eta. \tag{2.1}
\]

It is not hard to see that this module is a rational module of \( \underline{G} \). The corresponding rational representation of \( \underline{G} \) is denoted by \( \underline{\nu} \), which gives rise to a short exact sequence of \( \underline{G} \)-modules

\[
V \hookrightarrow \underline{V} \twoheadrightarrow \mathbb{C} \tag{2.2}
\]

where \( \mathbb{C} \) means the one-dimensional trivial \( \underline{G} \)-module. The following fact is clear.

**Lemma 2.1.** The enhanced reductive algebraic group \( \underline{\text{GL}(V)} \) is a closed subgroup of \( \text{GL}(V) \).

2.2. Classical Schur-Weyl duality. Recall the classical Schur-Weyl duality. The natural representation of \( \text{GL}(V) \) on \( V \) gives rise to a \( G = \text{GL}(V) \)-module on the tensor product \( V^{\otimes r} \) for any given positive integer \( r \), with diagonal \( g \)-action for any \( g \in G \). The corresponding representation is denoted by \( \phi \). This means

\[
\phi(g)(v_1 \otimes v_2 \otimes \cdots \otimes v_r) = \nu(g)v_1 \otimes \nu(g)v_2 \otimes \cdots \otimes \nu(g)v_r
\]

for any \( g \in \text{GL}(V) \) and any monomial tensor product \( v_1 \otimes \cdots \otimes v_r \in V^{\otimes r} \).

In the meanwhile, \( V^{\otimes r} \) naturally becomes a \( \mathfrak{S}_r \)-module with permutation action. The corresponding \( \mathfrak{S}_r \)-representation on \( V^{\otimes r} \) is denoted by \( \psi \). This means that for any \( \sigma \in \mathfrak{S}_r \),

\[
\psi(\sigma)(v_1 \otimes v_2 \otimes \cdots \otimes v_r) = v_{\sigma^{-1}(1)} \otimes v_{\sigma^{-1}(2)} \otimes \cdots \otimes v_{\sigma^{-1}(r)}. \tag{2.3}
\]

The classical Schur-Weyl duality shows that the images of \( \phi \) and \( \psi \) are double centralizers in \( \text{End}_C(V^{\otimes r}) \), i.e. for \( G = \text{GL}(V) \)

\[
\text{End}_C(V^{\otimes r}) = \mathbb{C}\psi(\mathfrak{S}_r)
\]

\[
\text{End}_{\mathfrak{S}_r}(V^{\otimes r}) = \mathbb{C}\phi(G) \tag{2.4}
\]

Here \( \mathbb{C}\psi(\mathfrak{S}_r) \) and \( \mathbb{C}\phi(G) \) stand for the subalgebras of \( \text{End}_C(V^{\otimes r}) \) generated by \( \psi(\mathfrak{S}_r) \) and \( \phi(G) \) respectively.
2.3. **Enhanced tensor representations.** Now consider the $r$th tensor product $V^\otimes r$ for a fixed positive integer $r$, which becomes a $G$-modules by diagonal action.

From the classical Schur-Weyl duality (see (2.4)), $V^\otimes r$ has the following decomposition as $GL(V) \times S_r$-modules

\[ V^\otimes r = \bigoplus_\lambda L^\lambda \otimes D^\lambda \]  \hspace{1cm} (2.5)

where $\lambda$ in the sum ranges over the set of partitions of $r$ with $n$ parts (zero parts are allowed), both $L^\lambda$ and $D^\lambda$ are the irreducible highest weight module of $GL(V)$ and the irreducible module of $S_r$ respectively, associated with the partition $\lambda$ (see [11]).

We now consider the composition factors of $V^\otimes r$ as a $G$-module. Keeping (2.2) and (2.5) in mind, one easily has the following.

**Proposition 2.2.** As a $G$-module, the following formula holds in the Grothendieck group of the rational $G$-module category

\[ [V^\otimes r] = \sum_\mu c_\mu [L^\mu] \]

where the sum is over partitions $\mu$ of $l \in \{0, 1, \ldots, r\}$ of length $\leq n$, and

\[ c_\mu = \binom{r}{l} \dim S^\mu_l. \]

2.4. **A preliminary to enhanced Schur-Weyl duality.** Denote by $\Phi$ the representation of $G$ on the $r$-tensor product module $V^\otimes r$. Then the following question is naturally raised.

**Question 2.3.** What is the centralizer of $C_{\Phi(G)}$ in $\text{End}_C(V^\otimes r)$? What role does $S_r$ play in the enhanced case as in the classical Schur-Weyl duality?

It is not trivial to give a complete answer which is left till §5 (see Theorem 5.3). Here we give some preliminary investigation. Note that $GL(V)$ contains a closed subgroup $G$ (Lemma 2.1). According to the classical Schur-Weyl duality, $GL(V)$ and $S_r$ are dual in the sense of the double centralizers in $\text{End}_C(V^\otimes r)$ (we will still denote by $\Phi$ the natural representation of $GL(V)$ on $V^\otimes r$ if the context is clear). So the significant investigation of Question 2.3 is to decide the centralizer of $C_{\Phi(G)}$ in $\text{End}_C(V^\otimes r)$. For this, we first have the following observation

**Lemma 2.4.** For any $\xi \in \text{End}_C(V^\otimes r)$, $\xi(V^\otimes r) \subseteq V^\otimes r$.

**Proof.** For any nonzero $w \in V^\otimes r$, we want to show that $\xi(w) \in V^\otimes r$. We might as well suppose $\xi(w) \neq 0$. Then we write

\[ \xi(w) = \sum_{i=1}^t u_{i1} \otimes u_{i2} \otimes \cdots \otimes u_{ir} \in V^\otimes r \]  \hspace{1cm} (2.6)

with $u_{ij} = u_{ij} + a_{ij} \eta$, where $u_{ij} \in V$, and $a_{ij} \in \mathbb{C}$, and the number $t$ of summands in (2.6) is minimal among all possible expression of $\xi(w)$.

By the assumption, $\xi(\Phi(g)w) = \Phi(g)\xi(w)$ for any $g := (g, v) \in G$ with $g \in G$ and $v \in V$. In particular, we take some special element $(e, v) \in G$, denoted by $e_v$. Then we have an equation

\[ \xi(\Phi(e_v)w) = \Phi(e_v)\xi(w). \]  \hspace{1cm} (2.7)
By (2.1), \( \nu(e^v)(u_{ij}) = u_{ij} + a_{ij}v + a_{ij}\eta \). So we have
\[
\Phi(e^v)\xi(w) = \sum_i (u_{i1} + a_{i1}v + a_{i1}\eta) \otimes \cdots \otimes (u_{ir} + a_{ir}v + a_{ir}\eta).
\]

On the other hand, by (2.1) again we have \( \nu(e^v)u = u \) for any \( u \in V \). Hence, \( \Phi(e^v)w = w \). We have
\[
\xi(\Phi(e^v)w) = \sum_i u_{i1} \otimes \cdots \otimes u_{ir}.
\]

By comparing both sides of (2.7), the arbitrariness of \( v \) leads to all \( a_{ij} \) being equal to zero. The proof is completed. \( \square \)

**Proposition 2.5.** Denote by \( \Psi \) the permutation representation of \( \mathfrak{S}_r \) on \( V^{\otimes r} \) defined as in (2.3). The following statements.

1. \( C\Psi(\mathfrak{S}_r) \) is a subalgebra of \( \text{End}_C(V^{\otimes r}) \).
2. There is a surjective homomorphism of algebras
\[
\text{End}_C(V^{\otimes r}) \rightarrow C\Psi(\mathfrak{S}_r).
\]

**Proof.** (1) Note that \( G \) is a subgroup of \( \text{GL}(V) \) and then \( \text{End}_{\text{GL}(V)}(V^{\otimes r}) \subset \text{End}_C(V^{\otimes r}) \). This statement follows from the classical Schur-Weyl duality with respect to \( \text{GL}(V) \) and \( \mathfrak{S}_r \).

(2) By Lemma 2.4, we can define a map \( \text{Res} \) from \( \text{End}_G(V^{\otimes r}) \) to \( \text{End}_C(V^{\otimes r}) \) by sending \( \xi \) to \( \xi|_{V^{\otimes r}} \). Then it is easily seen that \( \text{Res} \) is an algebra homomorphism. Furthermore, we assert that \( \text{Res} \) is surjective. Actually, by the classical Schur-Weyl duality we have \( \text{End}_G(V^{\otimes r}) = C\Psi(\mathfrak{S}_r) \). So any element of \( \text{End}_G(V^{\otimes r}) \) can be expressed as \( \sum_i a_i\sigma_i \) which is a finite \( \mathbb{C} \)-linear combination of some \( \sigma \in \mathfrak{S}_r \). Take \( \xi \) to be a morphism in \( \text{End}_C(V^{\otimes r}) \) sending \( u_1 \otimes \cdots \otimes u_r \in V^{\otimes r} \) to
\[
\sum_i a_iu_{i1}^{-1}(1) \otimes \cdots \otimes u_{ir}^{-1}(r).
\]

Then \( \xi \) is \( G \)-equivariant. And \( \text{Res}(\xi) \) is exactly \( \sum_i a_i\sigma_i \in \text{End}_G(V^{\otimes r}) \). So the assertion is proved. \( \square \)

### 2.5. Enhanced Schur algebras.

From now on, we always assume \( \text{dim} V = n \). Set \( A := \text{End}_C(V^{\otimes r}) \), and \( S = C\Phi(\text{GL}(V)) \), \( \mathcal{E} := C\Phi(G) \) and \( \mathcal{S} := C\Psi(\mathfrak{S}_r) \) for \( G = \text{GL}(V) \times_{\nu} V \). Call \( \mathcal{E} \) an enhanced Schur algebra which reflects the polynomial representations of degree \( r \) for the enhanced reductive algebraic group \( G \). Obviously, \( \mathcal{E} \) is a subalgebra of the semi-simple algebra \( S \). Note that \( C\Phi(\text{GL}(V)) = \text{End}_C(V^{\otimes r})^\mathfrak{S}_r \) is actually the classical Schur algebra \( S(n+1, r) \) (see [12]). Lemma 2.1 still holds in any case. Correspondingly, we denote \( \mathcal{E} \) by \( \mathcal{E}(n, r) \) more precisely.

By the above arguments, we have the following sequence of subalgebras in \( A \):
\[
S(n, r) \subset \mathcal{E}(n, r) \subset S(n+1, r).
\]

(2.8)
So \( (n^2 + r - 1) \leq \text{dim} \mathcal{E}(n, r) \leq (n^2 + 2n + r) \).

**Question 2.6.** What about basis of the enhanced Schur algebra \( \mathcal{E}(n, r) \) and how to describe their multiplication formulas? what about this algebra?

For this, by (2.8) we first need to describe a set of generators of \( \mathcal{E}(n, r) \).
2.5.1. Let us first recall some facts on the classical Schur algebras. The readers refer to [12] or [9] for the details.

For a given positive integer $m$, set $m = \{1, 2, \ldots, m\}$. Denote by $A(m, r)$ the space consisting of the elements expressible as polynomials which are homogeneous of degree $r$ in the polynomial function $c_{i,j}$ $(i,j \in m)$ on $\text{GL}(m, \mathbb{C})$. Then $A(m, r)$ has a basis (modulo the order of factors in the monomials)

$$c_{i,j} = c_{i_1j_1}c_{i_2j_2} \cdots c_{i_rj_r}$$

for $i = (i_1, \ldots, i_r), j = (j_1, \ldots, j_r) \in m^r$. The symmetric group $S_r$ acts on the left on $m^r$ by $\sigma.i = (i_{\sigma^{-1}(1)}, \ldots, i_{\sigma^{-1}(r)})$. Furthermore, $S_r$ acts also on the set $m^r \times m^r$ by $\sigma.(i,j) = (\sigma.i, \sigma.j)$. So we can define an equivalence relation $\sim$ on $m^r \times m^r$, $(i,j) \sim (k,l)$ if and only if $(k,l) = \sigma.(i,j)$ for some $\sigma \in S_r$. The number of the equivalence classes (or the orbits) in $m^r$ under such a $S_r$-action is just $d_{m,r} := \binom{m^2 + r - 1}{r}$. Then $A(m, r)$ becomes a coalgebra of dimension $d_{m,r}$ with the coproduct $\delta$ and the counit $\varepsilon$ as below

$$\Delta : A(m, r) \rightarrow A(m, r) \otimes A(m, r),$$

$$c_{i,j} \mapsto \sum_k c_{i,k} \otimes c_{k,j}; \tag{2.9}$$

and $\varepsilon(c_{i,j}) = \delta_{i,j}$.

Alternatively, the classical Schur algebra $S(m, r)$ can be defined as the dual of the $A(m, r)$. So $S(m, r)$ has basis $\{\xi_{i,j} | (i,j) \in m^r \times m^r / \sim\}$ dual to the basis $\{c_{i,j} | (i,j) \in m^r \times m^r / \sim\}$ of $A(m, r)$. This means

$$\xi_{i,j}(c_{k,l}) = \begin{cases} 1, & \text{if } (i,j) \sim (k,l); \\ 0, & \text{otherwise.} \end{cases}$$

Furthermore, $S(m, r)$ is an associative algebra of dimension $d_{m,r}$ with the following multiplication rule

\begin{align*}
\text{(S1)} & \quad \xi_{i,j}\xi_{k,l} = \sum_{p,q} a_{i,j,k,l,p,q} \xi_{p,q}, \text{ where } a_{i,j,k,l,p,q} \text{ equals the number of the elements } s \in m^r
\text{ satisfying } (i,j) \sim (p,s) \text{ and } (k,l) \sim (s,q). \\
\text{(S2)} & \quad \xi_{i,j}\xi_{k,l} = 0 \text{ unless } j \sim k. \\
\text{(S3)} & \quad \xi_{i,i}\xi_{j,j} = \xi_{i,j} = \xi_{j,i}\xi_{j,j}. 
\end{align*}

2.5.2. A key lemma for Schur algebras. Let $\Phi : \text{GL}(m, \mathbb{C}) \rightarrow \text{End}_{\mathbb{C}}(V^\otimes r)$ for $V = \mathbb{C}^m$. By the classical Schur algebra theory, we can identify the image of $\Phi$ with $S(m, r)$. So we can write the image precisely.

In general, take $g$ from $\text{GL}(m, \mathbb{C})$ with $g = (g_{pq})_{m \times m}$. Then

$$\Phi(g) = \sum_{(p,q) \in m^r \times m^r / S_r} a_{pq} \xi_{p,q} \text{ with } a_{pq} = \prod_{i=1}^r g_{p_iq_i} \text{ for } p = (p_1, \ldots, p_r), q = (q_1, \ldots, q_r). \tag{2.10}$$

Set $I_g := \{(p,q) \in m^r \times m^r / S_r | a_{pq} \neq 0\}$. Then $\Phi(g)$ can be expressed as

$$\Phi(g) = \sum_{(p,q) \in I_g} a_{pq} \xi_{p,q}. \tag{2.11}$$
Set $I^0(g) = \{(p, q) \in m \times m \mid g_{pq} \neq 0\}$, and set $\mathbb{C}^x = \mathbb{C} \setminus \{0\}$. Denote $B(g) := \{h \in \text{GL}(m, \mathbb{C}) \mid I^0(h) = I^0(g)\}$. Then $B(g)$ can be regarded as an intersection of a non-empty open subset in $\mathbb{C}^{\#I^0(g)}$ with $(\mathbb{C}^x)^{\#I^0(g)}$. So it is still a non-empty open subset of $\mathbb{C}^{\#I^0(g)}$.

Fix an order for all elements of the set $I_g$. Set $\lambda(g) = \#I_g$. Then we can talk about the matrix for $\Phi(g)$ for $g \in \text{GL}(m, \mathbb{C})$. The following observation is fundamental, which will be important to the sequel arguments.

**Lemma 2.7.** Suppose $S(m, r)$ is the classical Schur algebra associated with $\text{GL}(m, \mathbb{C})$ and degree $r$. Keep the notations as above. The following statements hold.

1. For any given $g \in \text{GL}(m, \mathbb{C})$, there exist $\lambda(g)$ elements $h^{(i)} \in B(g)$, $i = 1, \ldots, \lambda(g)$ such that the matrix $(a^{(i)}_{pq})_{\lambda(g) \times \lambda(g)}$ is invertible, where $a^{(i)}_{pq}$ is defined in the same sense as in (2.11) with respect to $h^{(i)}$.

2. Consequently, for any $(p, q) \in I_g$, $\xi_{p, q} = \sum_{i=1}^{\lambda(g)} c_i \Phi(h^{(i)})$ for some $c_i \in \mathbb{C}$.

3. Furthermore, for any $(p, q) \in m^r \times m^r$, there exists $g \in \text{GL}(m, \mathbb{C})$ such that $(p, q) \in I_g$. Therefore, (2) is valid for any basis element $\xi_{p, q}$ of $S(m, r)$.

**Proof.** (1) For $g = (g_{ij})_{m \times m} \in \text{GL}(m, \mathbb{C})$, we set $l = \lambda(g)$ and $I_g = \{\tau_1, \ldots, \tau_l\}$. Then

$$\Phi(g) = \sum_{k=1}^{l} a_{\tau_k} \xi_{\tau_k} \quad \text{with} \quad a_{\tau_k} = \prod_{i=1}^{r} g_{p_i q_i} \text{ if } \tau_k = (p, q) \in m^r \times m^r/\mathfrak{S}_r.$$ 

For $s \in \mathbb{N}$, set

$$\mathfrak{P}_s = \{(c_{ij})_{m \times m} \in (\mathbb{C}^x)^{m^2} \mid (c^s_{ij}g_{ij})_{m \times m} \in \text{GL}(m, \mathbb{C})\}.$$

Each $\mathfrak{P}_s$ is a non-empty open subset of $\mathbb{A}^{m^2}$. In particular, $\bigcap_{s=1}^{l-1} \mathfrak{P}_s \neq \emptyset$. For $c = (c_{ij})_{m \times m} \in \bigcap_{s=1}^{l-1} \mathfrak{P}_s$, let $g_{c, s} = (c^s_{ij}g_{ij})_{m \times m} \in \text{GL}(m, \mathbb{C})$ for $s = 1, \ldots, l-1$. Then $\Phi(g_{c, s}) = \sum_{k=1}^{l} f_k(c_{ij})^s a_{\tau_k} \xi_{\tau_k}$ where $f_k(x_{ij})$ is a monomial of degree $r$ over the $n^2$ variables $x_{ij}$ ($1 \leq i, j \leq n$) and $f_s(x_{ij}) \neq f_t(x_{ij})$ for any $s \neq t$. Therefore,

$$\bigcap_{j=1}^{l-1} \mathfrak{P}_j \cap \bigcap_{1 \leq s \neq t < l} \mathfrak{X}_{st} \neq \emptyset,$$

where

$$\mathfrak{X}_{st} = \{(c_{ij})_{m \times m} \in (\mathbb{C}^x)^{m^2} \mid f_s(c_{ij}) \neq f_t(c_{ij})\} \quad \text{for} \quad 1 \leq s \neq t \leq l.$$ 

Now take

$$c = (c_{ij})_{m \times m} \in \left( \bigcap_{j=1}^{l-1} \mathfrak{P}_j \right) \cap \left( \bigcap_{1 \leq s \neq t < l} \mathfrak{X}_{st} \right).$$

Set $h^{(1)} = g$ and $h^{(i)} = g_{c, i-1}$ ($2 \leq i \leq l$). Then the corresponding matrix $(a^{(i)}_{pq})_{\lambda(g) \times \lambda(g)}$ forms a Vandermonde one. It is desired.

(2) follows directly from (1).

(3) Take $c_1, \ldots, c_m \in \mathbb{C} \setminus \{0\}$ such that $c_i \neq c_j$ for any $1 \leq i \neq j \leq m$, and $g = (g_{ij})$ with $g_{ij} = c_j^{i-1}$ for $1 \leq i, j \leq m$. Then $g \in \text{GL}(m, \mathbb{C})$ and $(p, q) \in I_g$ for any $(p, q) \in m^r \times m^r$. \qed
2.5.3. Now we look for a set of generators of $\mathcal{E}(n, r)$ modulo $S(n, r)$. Generally, we denote by $\mathcal{F}(\text{GL}_m)$ the set of functions on $\text{GL}_m := \text{GL}(m, \mathbb{C})$ for a positive integer $m$. Then we can naturally regard $\mathcal{F}(\text{GL}_n)$ as a subset of $\mathcal{F}(\text{GL}_{n+1})$, $A(n, r)$ as a sub-coalgebra of $A(n+1, r)$ and $S(n, r)$ as a subalgebra of $S(n+1, r)$. Consider a map

$$\theta : \mathcal{E}(n, r) \to S(n + 1, r)$$

sending $\Phi(g)$ to $\theta_g$ for $g = (g, v)$. This is an algebra homomorphism. Note that $(g, v) = (g, 0)(e, g^{-1}v)$. So $\theta_g = \theta_{(g, 0)}\theta_{(e, g^{-1}v)}$. By the definition of enhanced groups (see (1.11)), under $\theta$ we can identify $\Phi(\text{GL}(V))$ with $S(n, r)$. Still set $e^v = (e, v)$ for $v \in V$. Denote $\Omega(n, r)$ the subalgebra of $\mathcal{E}(n, r)$ generated by $\theta_{e^v}$ with $v$ ranging over $V$. Then the first question is to understand $\theta_{e^v}$ and the subalgebra $\Omega(n, r)$ generated by them.

In the standard basis elements $c_{ij}$ of $A(m, r)$, the factor $c_{i,j}$ stands for the function which associates to each $g \in \text{GL}_m$ its $(i, j)$-coefficient $g_{ij}$.

Let us turn to $\text{GL}(V)$. Firstly, from now on we will always fix a basis

$$\{\eta_i \mid i = 1, 2, \ldots, n + 1\}$$

for $V$ with $\eta_{n+1} = \eta; \eta_1, \ldots, \eta_n \in V$,

and then identify $\text{GL}(V)$ with $\text{GL}_{n+1}$. In particular, $e^v$ becomes the following $(n+1) \times (n+1)$ matrix

$$
\begin{pmatrix}
1 & 0 & 0 & \cdots & 0 & 0 & a_1 \\
0 & 1 & 0 & \cdots & 0 & 0 & a_2 \\
0 & 0 & 1 & \cdots & 0 & 0 & a_3 \\
\vdots & \vdots & \vdots & \ddots & \vdots & \vdots & \vdots \\
0 & 0 & 0 & \cdots & 0 & 1 & a_n \\
0 & 0 & 0 & \cdots & 0 & 0 & 1
\end{pmatrix}
$$

where $v = \sum_{i=1}^n a_i \eta_i \in V$.

In the sequel, we always set $N := \{1, 2, \ldots, n + 1\}$ (note that, in general, we set $1 := \{1, 2, \ldots, l\}$ for the positive integer $l$ throughout the paper, only with an exception for $n + 1$ because we intend to stress this special situation in our paper). For $s = (s_1, \ldots, s_n) \in \mathbb{N}^n$, $t = (t_1, \ldots, t_{n+1}) \in \mathbb{N}^{n+1}$ we set $|s| = \sum s_i, |t| = \sum t_i$. Furthermore, we denote by $(\ldots i^d \ldots)$ an $r$-tuple in $N^r$, where $i$ appears $d$ times continuously. By a straightforward computation we have

$$\theta_{e^v} = \sum_{(s, t)} (\prod_{k=1}^n a_k^{s_k}) \xi_{(1^s 2^s \ldots n^s n+1^t (n+1)^t)}((n+1)^{|s|} 1^{d} 2^{d} ... n^{d} (n+1)^{d+1}) \cdot \quad (2.12)$$

Here $(s, t)$ in the sum runs through the range $\Lambda := \{(s, t) \in \mathbb{N}^n \times \mathbb{N}^{n+1} \mid |s| + |t| = r\}$, and we appoint $0^0$ to be 1 if it appears in the coefficient $a_s := \prod_{k=1}^n a_k^{s_k}$. Denote by $\theta_{s, t}$ the summand in the expression in (2.12) corresponding to $(s, t)$.

Set $\mathbb{N}_r^n := \{s \in \mathbb{N}^n \mid |s| \leq r\}$. For $s \in \mathbb{N}_r^n$, set $\Lambda_s := \{t \in \mathbb{N}^{n+1} \mid (s, t) \in \Lambda\}$, and set

$$\Theta_s := \sum_{t \in \Lambda_s} \theta_{s,t}.$$ 

Then

$$\theta_{e^v} = \sum_{s \in \mathbb{N}_r^n} a_s \Theta_s.$$
Lemma 2.8. The subalgebra $\Omega(n, r)$ has a basis $\{\Theta_s \mid s \in \mathbb{N}^n_r\}$. Therefore, $\dim \Omega(n, r) = \sum_{k=0}^{r} \binom{n+k-1}{k}$.

Proof. For $a \in \mathbb{C}$, take $v_a = \sum_{i=1}^{n} a^i \eta_i \in V$. Then
$$\theta_{e^v_a} = \sum_{s \in \mathbb{N}^n_r} a^{||s||} \Theta_s,$$
where $||s|| = \sum_{i=1}^{n} i^{i-1} s_i$. Then $||s|| \neq ||s'||$ for any distinct $s, s' \in \mathbb{N}^n_r$. Denote $\lambda(n, r) = \# \mathbb{N}^n_r$. Take $c \in \mathbb{C}^\times$ and $c$ is not a root of unit, and $a_i = c^i$ for $0 \leq i \leq \lambda(n, r) - 1$. This implies that $\Theta_s$ is in the subspace spanned by $\{\theta_{e^v_a} \mid 0 \leq i \leq \lambda(n, r) - 1\}$ for any $s \in \mathbb{N}^n_r$. In particular, $\Theta_s$ is in the subalgebra generated by $\theta_{e^v}$ for $v \in V$. Consequently, $\Omega(n, r)$ coincides with the subalgebra of $E(n, r)$ generated by $\theta_{e^v}$ for $v \in V$. And we have $\Omega(n, r) = \sum_{s \in \mathbb{N}^n_r} \mathbb{C} \Theta_s$.

Next we need to show that all $\Theta_s$ are linearly independent. For any $s \in \mathbb{N}^n_r$, take $t \in \mathbb{N}^{n+1}$ such that $(s, t) \in \Lambda$. Denote by $c_{s,t}$ the basis element in $A(n+1, r)$ corresponding to $\xi_{s,t}$. Then $\Theta_s(c_{s,t}) = 1$, and $\Theta_s(c_{s',t}) = 0$ whenever $s \neq s'$. For any given combinator equation $\sum_s k_s \Theta_s = 0$, taking the value at $c_{s,t}$ we have $k_s = 0$. Hence, those $\Theta_s$ are proved to be linearly independent, so that the dimension formula follows.

Thanks to Lemma 2.7, we have another result.

Lemma 2.9. The space $T(n, r)$ spanned by $\theta_{s,t}$ for all $(s, t) \in \Lambda$ is just the subalgebra generated by $\theta_{(t,v)}$ with $t$ running through the subgroup $T(n) \subset \text{GL}(n, \mathbb{C})$ consisting of diagonal matrices, and $v$ ranging over $V$. Consequently, all $\theta_{s,t}$ are contained in $E(n, r)$.

2.5.4. Restricted Schur algebras. Secondly, denote by $\Delta(n, r)$ the subalgebra generated by $\theta_{\hat{g}}$ for $\hat{g} := (g, 0)$ with $g$ ranging over $GL(V)$. We call $\Delta(n, r)$ a restricted Schur subalgebra of $E(n, r)$. Let us investigate this subalgebra. We first consider the set
$$D := \mathfrak{S}_r. \{(p_l(n+1)^{r-l}), (q_r(n+1)^{r-l}) \in \mathbb{N}^r \times \mathbb{N}^r \mid p_l, q_r \in \mathbb{Z}^l, l = 0, 1, \ldots, r\}.$$
By (S1)-(S3), the span of $\xi_{i,j}$ with $(i,j)$ ranging over $D$ forms a subalgebra of $S(n+1, r)$, denoted by $\Delta(n, r)$. On the other hand, by definition we know $\Delta(n, r) \subset \Delta(n, r)'$. By definition, $S(n, r) = \sum_{(p,q) \in \mathbb{N}^r \times \mathbb{N}^l} \mathbb{C} \xi_{p,q}$.

For a given $g \in \text{GL}(n, \mathbb{C})$ with $g = (g_{pq})_{n \times n}$, by the classical Schur algebra theory,
$$\Phi(\hat{g})|_{V^{\oplus r}} = \sum_{(p,q) \in \mathbb{N}^r \times \mathbb{N}^l} a_{pq} \xi_{p,q} \text{ with } a_{pq} = \prod_{i=1}^{r} g_{piq_i}. \tag{2.13}$$
Recall the notation $I_g = \{(p, q) \in \mathbb{Z}^r \times \mathbb{N}^l / \mathfrak{S}_r \mid a_{pq} \neq 0\}$. Then $\Phi(\hat{g})$ can be expressed as
$$\Phi(\hat{g}) = \sum_{(p,q) \in I_g} T^g_{p,q} \tag{2.14}$$
where $T^g_{p,q} = \sum_{(i,j) \in D/\mathfrak{S}_r} a_{ij} \xi_{i,j}$ with
$$a_{ij} := \prod_{k=1}^{t} g_{piq_k} \tag{2.15}$$
Lemma 2.10. The restricted Schur algebra $\Delta(n, r)$ coincides with $\Delta(n, r)'$, this is to say, it is spanned by $\xi_{i, j}$ with $(i, j)$ ranging over $D/\mathcal{G}_r$.

Proof. It suffices to prove that for any $(i, j) \in D$, $\xi_{i, j}$ lies in $\Delta(n, r)$. Indeed, it is obvious that there exists $g \in \text{GL}(n, \mathbb{C}) \hookrightarrow \text{GL}(n + 1, \mathbb{C})$ such that $(i, j) \in I_g$ where $\hookrightarrow$ is the canonical imbedding upon the left-up $n \times n$-block with the right-down most entry equal to 1, i.e., $g \mapsto \text{Diag}(g, 1)$. By applying Lemma 2.7, $\xi_{i, j} = \sum C\Phi(h^{(i)})$ for some $h^{(i)} \in B(g) \subset \text{GL}(n, \mathbb{C})$. This is desired. \hfill \Box

2.6. The structure of $\mathcal{E}(n, r)$. Now it is a position to investigate the structure of $\mathcal{E}(n, r)$. For any given element in $N^r$, modulo the order we can write it in a standard form:

$$(1^{s_1}2^{s_2} \ldots n^{s_n}(n + 1)^{r - |s|})$$

where $s := (s_1, \ldots, s_n) \in N^n$. So for any basis element $\xi_{i, j} \in S(n + 1, r)$ we can write

$$\xi_{i, j} = \xi_{(1^{s_1}2^{s_2} \ldots n^{s_n}(n + 1)^{r - |s|})j} \quad \text{(2.16)}$$

with $j \in N^r$. Set

$$E := \{ (\tilde{s}, j) := ((1^{s_1}2^{s_2} \ldots n^{s_n}(n + 1)^{r - |s|}), (j_1 \ldots j_{|s|}(n + 1)^{r - |s|})) \in N^r \times N^r \mid s \in N^n, (j_1 \ldots j_{|s|}) \in N^{|s|} \}.$$ 

and $\tilde{E} = \mathcal{G}_r.E$.

Theorem 2.11. Keep the above notations. The following statements hold.

1. The enhanced Schur algebra $\mathcal{E}(n, r)$ is generated by $\Omega(n, r)$ and $\Delta(n, r)$ with product axioms as (S1)-(S3), this is to say, $\mathcal{E}(n, r)$ is generated by $\Theta_s$ and $\xi_{i, j}$, with $s \in N^n$ and $(i, j) \in D/\mathcal{G}_r$.

2. The enhanced Schur algebra $\mathcal{E}(n, r)$ has a basis $\{ \xi_{\tilde{s}, j} \mid (\tilde{s}, j) \in \tilde{E} \}$, and

$$\dim \mathcal{E}(n, r) = \sum_{k=0}^{r} \binom{n + k - 1}{k} \binom{n + k}{k}.$$ 

3. Set $\mathcal{E}(n, r)_i = \mathbb{C} \text{-span} \{ \xi_{\tilde{s}, j} \in E \mid |s| \geq i \}$, $i = 0, 1, \ldots, r$. Then $\mathcal{E}(n, r)_i$ is an ideal of $\mathcal{E}(n, r)$, $i = 0, 1, \ldots, r - 1$. Moreover, $\mathcal{E}(n, r)_i/\mathcal{E}(n, r)_{i+1} \cong S(n, i) \ltimes \mathfrak{a}_i$ as algebras, where $\mathfrak{a}_i$ is a nonzero abelian ideal for $i = 0, \ldots, r - 1$, while $\mathcal{E}(n, r)_r \cong S(n, r)$.

4. Furthermore, $\mathcal{E}(n, r)$ has another set of generators $\{ \theta_{s, t}, \xi_{i, j} \mid (s, t) \in \Lambda, (i, j) \in D/\mathcal{G}_r \}$.

Proof. The statement (1) follows from Lemmas 2.8 and 2.10.

(2) Denote by $\mathcal{E}(n, r)'$ the subspace of $S(n + 1, r)$ spanned by $\{ \xi_{\tilde{s}, j} \mid (\tilde{s}, j) \in E \}$. By definition, $\mathcal{E}(n, r) \subset \mathcal{E}(n, r)'$. What remains is to show that for any $z = \xi_{\tilde{s}, j}$ with $(\tilde{s}, j) \in E$, $z$ must fall in $\mathcal{E}(n, r)$. Modulo the order, we can write $(\tilde{s}, j) \in E$ as $((p_1p_2 \ldots p_n(n + 1)^{r - s}), (q_1 \ldots q_t(n + 1)^{s-t}(n + 1)^{r-s}))$ with $(p_1p_2 \ldots p_t, q_1 \ldots q_t) \in N^t \times N^t$ for $t \leq s$. Then

$$z = \xi_{p_1p_2 \ldots p_t(n + 1)^{r-s}, q_1 \ldots q_t(n + 1)^{r-t}}.$$
Consider
\[ x := \xi_{p_1p_2...p_s(n+1)^{r-s}}, q_1...q_{l+r-2}s(n+1)^{r-s} \in \Delta(n,r) \]
and
\[ y := \xi_{q_1...q_{l+r-2}s(n+1)^{r-s}}, q_1...q_{l+r-2}s(n+1)^{r-t} \in T(n,r). \]

Thanks to (S1), it follows that \( xy = (c+1)z \) where \( c = \# \{ \sigma \in S_l(p_1,...,p_l) \} \) for \( (S_l)^p := \{ \sigma \in S_l \mid \sigma(p_1,...,p_l) = (p_1,...,p_l) \} \). Consequently, by Lemmas 2.9 and 2.10 we have that \( z \) really lies in \( \mathcal{E}(n,r) \). The dimension formula is obvious.

(3) By (S1)-(S3), it is readily known that \( \mathcal{E}(n,r)_{i+1} \) is an ideal of \( \mathcal{E}(n,r)_i \), \( i = 0,...,r \).

And
\[ \mathcal{E}(n,r)_i/\mathcal{E}(n,r)_{i+1} = s_i \oplus a_i \]
where
\[ s_i = \mathbb{C}\text{-span}\{\xi_{s_j} \mid s_j \in E_i(1) \}, a_i = \mathbb{C}\text{-span}\{\xi_{s_k} \mid s_k \in E_i(2) \} \]
with
\[ E_i(1) = \left\{ (\tilde{s}_j) : = (1^{s_1}2^{s_2}...n^{s_n}(n+1)^{r-i}), (j_1...j_l(n+1)^{r-i}) \mid s \in \mathbb{N}^n, |s| = i, (j_1...j_l) \in n^i \right\} \]
and
\[ E_i(2) = \left\{ (\tilde{s}_k) : = (1^{s_1}2^{s_2}...n^{s_n}(n+1)^{r-i}), (k_1...k_l(n+1)^{r-i}) \mid s \in \mathbb{N}^n, |k| = i, (k_1...k_l) \in n^i \right\} \]
The following mapping
\[ \Xi : s_i \rightarrow S(n,i) \]

\[ \xi_{s_j} \rightarrow \xi_{((1^{s_1}2^{s_2}...n^{s_n})(j_1...j_l))} \]
gives an algebra isomorphism between \( s_i \) and \( S(n,i) \). Furthermore, \( a_i \) is an ideal of \( \mathcal{E}(n,r)/\mathcal{E}(n,r)_{i+1} \). In particular, \( a_r = 0 \), and \( \mathcal{E}(n,r)_{i+1} \cong S(n,r) \). \qed

3. Degenerate double Hecke algebras

In this section, we introduce degenerate double Hecke algebras which will be important in the sequel arguments. For the symmetric group \( S_l \), we denote by \( s_l = (l,l+1) \) for \( l = 1,...,r-1 \), the transposition just interchanging \( l \) and \( l+1 \), and fixing the others.

3.1. Degenerate double Hecke algebras.

3.1.1. For a given positive integers \( r \) and \( l \) with \( r > l \), we consider the following algebra \( \mathcal{H}_l^r \) defined by generators \( \{ x_\sigma \mid \sigma \in S_l \} \cup \{ s_i \mid i = 1,2,...,r \} \) and relations as below.

\[ s_i^2 = 1, \quad s_is_j = s_js_i \text{ for } 0 < i \neq j \leq r-1, |j-i| > 1; \quad (3.1) \]
\[ s_is_j s_i = s_js_i s_j \text{ for } 0 < i \neq j \leq r-1, |j-i| = 1; \quad (3.2) \]
\[ x_\sigma x_\mu = x_{\sigma \mu} \text{ for } \sigma, \mu \in S_l; \quad (3.3) \]
\[ s_is_\sigma = x_{s_i \sigma}, \quad x_\sigma s_i = x_{\sigma s_i} \text{ for } \sigma \in S_l, i < l; \quad (3.4) \]
\[ s_is_\sigma = x_\sigma x_is_i \text{ for } \sigma \in S_l, i > l. \quad (3.5) \]
This is an infinite-dimensional associative algebra. We call $\mathcal{H}_l^r$ the $l$th degenerate double Hecke algebra of $\mathfrak{S}_r$. By (3.3), the subalgebra $X_l$ generated by $x_\sigma$ for $\sigma \in \mathfrak{S}_l$ is isomorphic to $\mathbb{C}\mathfrak{S}_l$. As well as being a subalgebra of $\mathcal{H}_l^r$, $\mathbb{C}\mathfrak{S}_r$ is also a quotient, via the homomorphism $\mathcal{H}_l^r \rightarrow \mathbb{C}\mathfrak{S}_r$ mapping $s_i \mapsto s_i$ and $x_\sigma \mapsto 0$ for each $i = 1, \ldots, r - 1$ and $\sigma \in \mathfrak{S}_l$. Additionally, we make an appointment that $\mathcal{H}_0^r := \langle \mathfrak{S}_r, X_0 \rangle$ with $X_0 = \mathbb{C}x_\emptyset$ satisfying $f = f.x_\emptyset = x_\emptyset.f$ for $f \in \mathcal{H}_0^r$, and $\mathcal{H}_r^r := \langle \mathbb{C}s_i, x_\sigma \rangle$ with all $s_i, x_\sigma, i = 1, \ldots, r - 1$ and $\sigma \in \mathfrak{S}_r$ satisfying (3.4). Here and after, (□) stands for a $\mathbb{C}$-algebra generated by $\emptyset$. Then $\mathcal{H}_r^r \cong \mathbb{C}\mathfrak{S}_r$, and $\mathcal{H}_r^r \cong \mathbb{C}\mathfrak{S}_r^{[2]}$ with $\mathfrak{S}_r^{[2]}$ being a group giving rise to a non-split extension $\mathfrak{S}_r \hookrightarrow \mathfrak{S}_r^{[2]} \twoheadrightarrow \mathfrak{S}_r$.

3.1.2. Full degenerate double Hecke algebras. Now we combine all $\mathcal{H}_l^r$ ($l = 0, 1, \ldots, r$) into a full degenerate double Hecke algebras.

**Definition 3.1.** The degenerate double Hecke algebra $\mathcal{H}_r$ of $\mathfrak{S}_r$ is an associative algebra with generators $s_i$ ($i = 1, \ldots, r - 1$), and $x_\sigma^{(l)}$ for $\sigma \in \mathfrak{S}_l$, $l = 0, 1, \ldots, r$, and with relations as (3.1)-(3.4) in which $x_\sigma, x_i$ are replaced by $x^{(l)}_\sigma, x^{(l)}_i$, and additional ones:

$$x^{(l)}_i x^{(k)}_j = 0 \text{ for } \delta \in \mathfrak{S}_l, \gamma \in \mathfrak{S}_k \text{ with } k \neq l. \quad (3.6)$$

Naturally, as well as being a subalgebra of $\mathcal{H}_r$, $\mathbb{C}\mathfrak{S}_r$ is also a quotient of $\mathcal{H}_r$, via the homomorphism $\mathcal{H}_r \rightarrow \mathbb{C}\mathfrak{S}_r$ mapping $s_i \mapsto s_i$ and $x^{(l)}_\sigma \mapsto 0$ for each $i = 1, \ldots, r - 1$ and $\sigma \in \mathfrak{S}_l$, and $l = 0, 1, \ldots, r$.

3.2. Degenerate double Hecke algebras arise from the following question:

**Question 3.2.** For the $\text{GL}(V)$-tensor representation on $V^{\otimes r}$, $\text{End}_{\mathbb{C}}(V_r^{\otimes r})^{\text{GL}(V)} = ?$

Let us begin the arguments with turning to $V_r^{\otimes r}$ which is regarded as a $\text{GL}(V)$-module by fixing $\eta$.

Keep the notations as before. In particular, we fix a basis $\{\eta_1, \ldots, \eta_n; \eta_{n+1} := \eta\}$ for $V = V \oplus \mathbb{C}\eta$ with $V = \sum_{i=1}^n \mathbb{C}\eta_i$. Associated with this basis, $\text{GL}(V) = \text{GL}(n, \mathbb{C})$, and $\text{GL}(V) = \text{GL}(n + 1)$. And $\text{GL}(n, \mathbb{C})$ is canonically regarded as a subgroup of $\text{GL}(n + 1, \mathbb{C})$ by the established imbedding $\text{GL}(V) \hookrightarrow \text{GL}(V) \hookrightarrow \text{GL}(V)$ sending $g \in \text{GL}(V)$ to $(g, 0) \in \text{GL}(V)$ (see Example 1.4 for the notations).

In this view, we already know that the image of $\text{GL}(V)$ by $\Phi$ is exactly $\Delta(n, r)$ (\$2.5.4$ and Lemma 2.10). In the following we will exactly determine $\text{End}_{\Delta(n, r)}(V^{\otimes r})$ and then decide the restricted version of Schur-Weyl duality from $\text{GL}(n + 1, \mathbb{C})$ to $\text{GL}(n, \mathbb{C})$.

3.3. Recall that all $\eta_k = \eta_{j_1} \otimes \eta_{j_2} \otimes \cdots \otimes \eta_{j_r}$ for $i = (i_1, \ldots, i_r) \in \mathbb{N}^r$ form a basis of $V^{\otimes r}$. For a given $j = (j_1, \ldots, j_r) \in \mathbb{N}^r$, there exists a unique $l \in \{0, 1, \ldots, r\}$ and $j'_l = (j'_1, \ldots, j'_l) \in \mathbb{N}^l$ such that $j \sim (j'_l (n + 1)^{-l})$, $l$ is called the $n$-rank of $j$, denoted by $\text{rk}_n(j)$. All elements with $n$-rank equal to $l$ constitute a subset of $\mathbb{N}^r$, denoted by $\mathbb{N}_l^r$. Clearly, $\mathbb{N}^r = \bigcup_{l=0}^r \mathbb{N}_l^r$. Correspondingly, $V^{\otimes r}$ can be decomposed into a direct sum of subspaces: $V^{\otimes r} = \bigoplus_{l=0}^r V_l^{\otimes r}$ with $V_l^{\otimes r} := \sum_{i \in \mathbb{N}_l^r} \mathbb{C}\eta_i$, $l = 0, 1, \ldots, r$.

Clearly, each $V_l^{\otimes r}$ is stabilized under $\mathfrak{S}_r$-action. Hence this action gives rise to a representation of $\mathfrak{S}_r$ on $V_l^{\otimes r}$, denoted by $\Psi[l]$. For a subset $I = \{i_1, \ldots, i_l\} \subset [r]$ whose elements are assumed to be ordered increasingly, we can write a subspace $V_{ij}^{\otimes r}$ of $V_l^{\otimes r}$ as

$$V_{ij}^{\otimes r} := \mathbb{C}\text{-span}\{\eta_j = \eta_{j_1} \otimes \cdots \otimes \eta_{j_r} \mid j_{ik} \in \mathbb{N}, k = 1, \ldots, l; j_d = n + 1 \text{ for } d \neq i_k\}.$$
This means, for any $\eta_j \in V_I^{\otimes r}$, there exists
\[
\tau_j = \begin{pmatrix} 1 & 2 & \cdots & r \\ i_1 & i_2 & \cdots & i_r \end{pmatrix} \in \mathcal{S}_r,
\]
such that $\Psi(\tau_j)\eta_j = \eta_{j_{(n+1)^{r-1}}} \in V_I^{\otimes r}$ for some $j_i \in \Pi'$, where $\{i_{d+1}, \ldots, i_n\} = \{j_d \mid d \in \mathbb{R} \setminus I\}$.
Conversely, for any $\eta_j \in V_I^{\otimes r}$, there exists $\tau \in \mathcal{S}_r$ and $\eta_j \in V_I^{\otimes r}$ such that $\eta_j = \Psi(\tau)(\eta_j')$.
What is more, as a vector space
\[
V_I^{\otimes r} = \bigoplus_{I \subset \mathcal{I}} V_I^{\otimes r}
\]
where $I$ in the sum ranges over all subsets of $\mathcal{I}$ consisting of $l$ elements. In the above, each $V_I^{\otimes r}$ is a GL($V$)-module. The corresponding representation is a subrepresentation of GL($V$) on $V_I^{\otimes r}$, the latter of which is also denoted by $\Phi$ for brevity.

For $I \subset \mathcal{I}$ with $\# I = l$, denote by $\text{Sym}(I)$ the symmetric group of $I$ consisting all permutation of the $I$, which is isomorphic to $\mathcal{S}_I$. Naturally, any $\sigma \in \text{Sym}(I)$ gives rise to a transformation on $V_I^{\otimes r}$ which just permutes the set $\{\eta_j \in V_I^{\otimes r}\}$ via changing the position of factor $\eta_{j_i}$ into the position of $\eta_{j_{(\sigma^{-1})}}(i)$ for all $i \in I$ and fixing the other factors. This gives rise to an representation of Sym($I$) on $V_I^{\otimes r}$. For $I = \mathcal{I}$, the corresponding symmetric group is directly denoted by $\mathcal{S}_I$. The corresponding representation is denoted by $\Psi|_I$.

3.4. Representations of $\mathcal{H}_r$ on $V_I^{\otimes r}$. On $V^{\otimes l}$, there is a permutation representation $\Psi_I^V$ of $\mathcal{S}_I$ defined via $\Psi_I^V(\sigma)$ sending $v_1 \otimes v_2 \otimes \cdots \otimes v_l$ onto $v_{\sigma^{-1}(1)} \otimes v_{\sigma^{-1}(2)} \otimes \cdots \otimes v_{\sigma^{-1}(l)}$ for $\sigma \in \mathcal{S}_I$.
Keep in mind the notations $l = \{1, 2, \ldots, l\} \subset \Pi$, and $V_I^{\otimes r} = V^{\otimes l} \otimes \eta^{r-l}$. Extending $\Psi_I^V$, we define the following linear operator on $V_I^{\otimes r}$
\[
x_\sigma = \Psi_I^V(\sigma) \otimes \text{id}^{r-l} \in \text{End}_C(V_I^{\otimes r})
\]
(3.7)
for $\sigma \in \mathcal{S}_I$. Next we extend $x_\sigma$ to an element $x_\sigma^I$ of $\text{End}_C(V_I^{\otimes r})$ by annihilating any other summand $V_I^{\otimes r}$ with $I \neq \mathcal{I}$.

Now let us look at the representation meaning of $s_i \in \mathcal{H}_r$ in $\text{End}_C(V_I^{\otimes r})$. This one keeps the role of $\Psi(s_i)$ such that the conjugation of $x_\sigma^I$ by $\Psi(s_i)$ will be an operator translating $x_\sigma^I \in \text{End}_C(V_I^{\otimes r})$ to the forthcoming parallel one $x_{\sigma}^{J} \in \text{End}_C(V_J^{\otimes r})$ for $I = \{1, 2, \ldots, l-1, l+1\}$.

In general, for $\eta_j \in V_I^{\otimes r}$ with $I \subset \mathcal{I}$ and $\# I = l$, we can write $\eta_j = \Psi(\tau_I)\eta_{j_{(n+1)^{r-1}}}$ for some $j_i \in \Pi'$. Then $\Psi(\tau_I) \circ x_\sigma \circ \Psi(\tau_I^{-1})$ lies in $\text{End}_C(V_I^{\otimes r})$ for any $\sigma \in \mathcal{S}_I$, which extends to an element of $\text{End}_C(V_I^{\otimes r})$ by annihilating any other summand $V_J^{\otimes r}$ with $J \neq I$. This elements is denoted by $x_\sigma^I$. All $x_\sigma^I$ ($\sigma \in \mathcal{S}_I$) generate a subalgebra in $\text{End}_C(V_I^{\otimes r})$, denoted by $E_I$ which is isomorphic to $\mathbb{C}\Psi_I^V(\mathcal{S}_I)$. Set $x_e^I := \sum_{I} x_\sigma^I$, where $I$ in the sum ranges over all subsets of $\mathcal{I}$ containing $l$ elements, and $e$ represents the identity element in $\mathcal{S}_I$. Then $x_e^I$ is just the identity mapping on $V_I^{\otimes r}$. Sometimes, $e$ also indicates the identity element in $\mathcal{S}_r$ if the context is clear.

Now it is a position to demonstrate a representation of the degenerate double Hecke algebra $\mathcal{H}_r$ on $V_I^{\otimes r}$.

**Lemma 3.3.** The following statements hold.

1. For $1 \leq l \leq r$, there is an algebra homomorphism $\Xi_I^l : \mathcal{H}_r \rightarrow \text{End}_C(V_I^{\otimes r})$ defined by sending $s_i \mapsto \Psi|_I(s_i)$ and $x_\sigma \mapsto x_\sigma^I$. 

(2) For \( l = 0 \), there is an algebra homomorphism \( \Xi_0 : \mathcal{H}_r^0 \rightarrow \text{End}_C(V_r^{\otimes r}) \) defined by sending \( s_i \mapsto \Psi_l(s_i) = \text{id} \).

(3) For any \( l \in \{0,1,\ldots,r\} \), for any \( g \in \text{GL}(V) \) commutes with any elements from \( \Xi_l(\mathcal{H}_r^l) \) in \( \text{End}_C(V_r^{\otimes r}) \).

(4) Set \( E_l := \bigoplus_{I \in \mathcal{S}_l} E_l \) with \( \mathcal{S}_l := \{ I \subset \mathcal{L} \mid \#I = l \} \), correspondingly \( E_0 := E_0 \). Then \( E_l \) is a left \( \Psi_l(\mathcal{G}_r) \)-module under the conjugation, i.e. \( \Psi_l(\tau).x_i^l = \Psi_l(\tau)x_i^l\Psi_l(\tau)^{-1} \) for \( \tau \in \mathcal{G}_r \). Furthermore, \( \Xi_l(\mathcal{H}_r^l) = \Psi_l(\mathcal{G}_r)E_l \).

(5) On the enhanced tensor space \( V_r^{\otimes r} \), there is a representation \( \Xi \) of \( \mathcal{H}_r \) defined via:

\[ \Xi_l(\mathcal{H}_r^l) = \Psi ; \]

\[ \Xi_l(x_r)\mid_{V_r^{\otimes r}} = \Xi_l(x_r) \quad \text{and} \quad \Xi_l(x_r)\mid_{V_r^{\otimes r}} = 0 \quad \text{for} \quad k \neq l. \]

**Proof.** (1) For \( l \leq r \), we need to show that \( \Xi_l \) keeps the relations (3.11)-(3.13).

Recall that for any \( 1 \leq i \leq r-1, j \in \mathbb{N}_r \), we have

\[ \Xi_l(s_i)(\eta_j) = \Psi_l(s_i)(\eta_j) = \eta_{\sigma(i)} = \eta_{j_1} \otimes \cdots \otimes \eta_{j_{i+1}} \otimes \eta_{j_i} \otimes \cdots \otimes \eta_{j_n}. \]

Hence, it is readily known that

\[ \Xi_l(s_i)^2 = \text{id}, \quad \Xi_l(s_i)\Xi_l(s_j) = \Xi_l(s_j)\Xi_l(s_i) \quad \text{for} \quad 0 \leq i \neq j \leq r-1, |j-i| > 1; \]

(3.8)

and

\[ \Xi_l(s_i)\Xi_l(s_j)\Xi_l(s_i) = \Xi_l(s_j)\Xi_l(s_i)\Xi_l(s_i) \quad \text{for} \quad 0 \leq i \neq j \leq r-1, |j-i| = 1. \]

(3.9)

For any \( \sigma, \mu \in \mathcal{S}_l \), and \( j = (j_1(n+1)^{r-l}) \) with \( j_i \in n^l \), we have

\[ \Xi_l(x_{r}) \circ \Xi_l(x_{\mu})(\eta_j) = \eta_{\sigma(\mu)(j)} = \Xi_l(x_{\mu})(\eta_j). \]

And \( \Xi_l(x_{r}) \circ \Xi_l(x_{\mu})(\eta_j) = 0 = \Xi_l(x_{\sigma})(\eta_j) \) for any \( k \in \mathbb{N}_r \) with \( \eta_k \notin V_r^r \). Hence,

(3.10)

For any \( \sigma, \mu \in \mathcal{S}_l \), and \( j = (j_1(n+1)^{r-l}) \) with \( j_i \in n^l \), we have

\[ \Xi_l(s_i) \circ \Xi_l(x_{\sigma})(\eta_j) = \eta_{\sigma^{-1}(j_1)\cdots j_{\sigma^{-1}(i+1)}j_{\sigma^{-1}(i)}j_{\sigma^{-1}(i)}\cdots j_{\sigma^{-1}(n+1)\cdots j_{\sigma^{-1}(n+1)^{r-l}}}} = \Xi_l(x_{s_{\sigma\bar{a}a}})(\eta_j). \]

And \( \Xi_l(s_i) \circ \Xi_l(x_{\sigma})(\eta_j) = 0 = \Xi_l(x_{s_{\sigma\bar{a}a}})(\eta_k) \) for any \( k \in \mathbb{N}_r \) with \( \eta_k \notin V_r^r \). Hence,

(3.11)

Moreover, similar arguments yield that

\[ \Xi_l(x_{\sigma}) \circ \Xi_l(s_i) = \Xi_l(x_{\sigma}s_i), \sigma \in \mathcal{S}_l, i < l. \]

(3.12)

and

\[ \Xi_l(s_i) \circ \Xi_l(x_{\sigma}) = \Xi_l(x_{\sigma}s_i), \sigma \in \mathcal{S}_l, i < l. \]

(3.13)

Now it follows from (3.11)-(3.13) that \( \Xi_l \) is an algebra homomorphism from \( \mathcal{H}_r^l \) to \( \text{End}_C(V_r^{\otimes r}) \).

(2) In this situation, \( \Xi_0 \) obviously keeps the relations (3.11)-(3.13). Hence, \( \Xi_0 \) is an algebra homomorphism from \( \mathcal{H}_r^0 \) to \( \text{End}_C(V_r^{\otimes r}) \).
(3) It suffices to show that for any \( g \in \text{GL}(V) \), \( 1 \leq i \leq r - 1 \), \( \sigma \in \mathcal{S}_l \),

\[
\Phi(\dot{g})\Xi_l(s_i) = \Xi_l(s_i)\Phi(\dot{g}) \quad \text{and} \quad \Phi(\dot{g})\Xi_l(x_\sigma) = \Xi_l(x_\sigma)\Phi(\dot{g}).
\]

Indeed,

\[
\Phi(\dot{g})\Xi_l(s_i)(\eta_j) = \Phi(\dot{g})(\eta_{j_1} \otimes \cdots \otimes \eta_{j_{l+1}} \otimes \eta_{j_{l}} \cdots \otimes \eta_{j_r})
\]

\[
= \Phi(\dot{g})(\eta_{j_1}) \otimes \cdots \otimes \Phi(\dot{g})(\eta_{j_{l+1}}) \otimes \Phi(\dot{g})(\eta_{j_{l}}) \cdots \otimes \Phi(\dot{g})(\eta_{j_{r}})
\]

\[
= \Xi_l(s_i)\Phi(\dot{g})(\eta_j).
\]

Hence, \( \Phi(\dot{g})\Xi_l(s_i) = \Xi_l(s_i)\Phi(\dot{g}) \).

Furthermore, for any \( j = (j_1, \ldots, j_n) = (j_l(n+1)^{r-l}) \) with \( j_l = (j_1, \ldots, j_l) \in \mathcal{U}_l \),

\[
\Phi(\dot{g})\Xi_l(x_\sigma)(\eta_j) = \Phi(\dot{g})(\eta_{j_{l-1}}(i) \otimes \cdots \otimes \eta_{j_{l-1}(n)})
\]

\[
= (\Phi(\dot{g})\eta_{j_{l-1}}(i)) \otimes \cdots \otimes (\Phi(\dot{g})\eta_{j_{l-1}(n)})
\]

\[
= \Xi_l(x_\sigma)\Phi(\dot{g})(\eta_j).
\]

And \( \Phi(\dot{g})\Xi_l(x_\sigma)(\eta_k) = \Xi_l(x_\sigma)\Phi(\dot{g})(\eta_k) = 0 \) for any \( k \in \mathbb{N}^r \) with \( k_s = n + 1 \) for some \( s \leq l \). This implies that \( \Phi(\dot{g})\Xi_l(x_\sigma) = \Xi_l(x_\sigma)\Phi(\dot{g}) \), as desired.

(4) The first part follows from the second one. We only need to prove the latter by different steps.

(i) First of all, by a direct check, \( \Psi|_{\mathcal{L}(\mathbb{C}\mathcal{G}_r)}x_i^l \) is an associative algebra because \( \Psi|_{\mathcal{L}(\tau)}x_i^l = \Psi|_{\mathcal{L}((\tau)x_i^l)} = x_i^l \tau \) for \( \tau \in \mathcal{G}_r \). Correspondingly, it is a left module of \( \Psi|_{\mathcal{L}(\mathbb{C}\mathcal{G}_r)} \).

(ii) By the above, it is not hard to see that \( \Xi_l(\mathcal{H}_r^\sigma) \) is spanned by \( \Psi|_{\mathcal{L}(\tau)}x_i^l \) with \( \tau \in \mathcal{G}_r \), \( \sigma \in \mathcal{G}_l \) and \( I \in \mathcal{L}_r \). Hence \( \Xi_l(\mathcal{H}_r^\sigma) = \Psi|_{\mathcal{L}(\mathbb{C}\mathcal{G}_r)}E_l \).

(iii) As to the algebra homomorphism, it can be directly verified.

(5) We need to show that \( \Xi \) keeps the relations \( (3.1)-(3.3) \). Since \( \Psi|_{\mathcal{G}_r} \) is a representation of \( \mathcal{G}_r \), \( \Xi \) keeps the relations \( (3.1)-(3.2) \). Moreover, note that \( \Xi|_{\mathcal{G}_l} = \Xi_l \) for \( 0 \leq l \leq r \), we have

\[
\Xi(x_\sigma) \circ \Xi(x_\mu) = \Xi_l(x_\sigma) \Xi_l(x_\mu) = \delta_{kl} \Xi_l(x_{\sigma \mu}) = \delta_{kl} \Xi_l(x_\sigma x_\mu) \quad \text{for} \quad \sigma, \mu \in \mathcal{G}_l, k, l \in \mathbb{R}.
\]

\[
(3.14)
\]

\[
\Xi(s_i) \circ \Xi(x_i^l) = \Psi(s_i) \Xi_l(x_i^l) = \Xi_l(x_{s_i \sigma}) = \Xi_l(x_i^l) \quad \text{for} \quad \sigma \in \mathcal{G}_l, i < l.
\]

\[
(3.15)
\]

\[
\Xi(x_i^l) \circ \Xi(s_i) = \Xi_l(x_i^l) \Psi(s_i) = \Xi_l(x_{\sigma s_i}) = \Xi_l(x_i^l) \quad \text{for} \quad \sigma \in \mathcal{G}_l, i < l.
\]

\[
(3.16)
\]

\[
\Xi(s_i) \circ \Xi(x_i^l) = \Psi(s_i) \Xi_l(x_i^l) = \Xi_l(x_i^l) = \Xi_l(x_i^l) \quad \text{for} \quad \sigma \in \mathcal{G}_l, i > l.
\]

\[
(3.17)
\]

\[
\Xi(x_i^l) \circ \Xi(s_i) = \Xi_l(x_i^l) \Psi(s_i) = \Xi_l(x_i^l) = \Xi_l(x_i^l) \quad \text{for} \quad \sigma \in \mathcal{G}_l, i > l.
\]

\[
(3.18)
\]

So \( \Xi \) is an algebra homomorphism, thereby a representation of \( \mathcal{H}_r \).

We will further investigate this representation in the next section.
3.5. Finite dimensional DDHAs. Keep the notations as above.

**Definition 3.4.** Set \( D(n, r) := \Xi(\mathcal{H}_r) \), which we call the finite-dimensional degenerate double Hecke algebra of \( \mathcal{S}_r \) (f.d. DHHA for short).

Set \( D(n, r)_l := \Xi_l(\mathcal{H}_r^l) \) for \( l = 0, 1, \ldots, r \). We can extend the action of \( D(n, r)_l \) on the whole of \( \bigoplus_l V_r^{\otimes r} \) as follows. Set
\[
\Psi_l(\sigma) := \Psi|_{V_r^{\otimes r}}(\sigma) \circ x_e^l.
\]
Then \( \Psi_l \) defines a representation of \( \mathcal{S}_r \) on \( V_r^{\otimes r} \). Now \( \Psi_l \) extends a representation of \( \mathcal{S}_r \) on \( V_r^{\otimes r} \) by letting \( \Psi_l(\mathcal{S}_r)(\eta_j) = 0 \) for \( \eta_j \in V_r^{\otimes r} \) with \( k \neq l \). Similarly, each \( x_e^l \) extends to an element in \( \text{End}_c(V_r^{\otimes r}) \) with trivial action on \( V_r^{\otimes r} \) for \( k \neq l \). So \( D(n, r)_l \) annihilates \( V_r^{\otimes r} \) for \( k \neq l \). Thus, each \( D(n, r)_l \) becomes a subalgebra of \( A \), and two different such subalgebras are commutative.

**Lemma 3.5.** Keep the notations as above. Then the following statements hold.

1. The f.d. DDHA \( D(n, r) \) is a direct sum of all \( D(n, r)_l \) \( (l = 0, \ldots, r) \), i.e.,
\[
D(n, r) = \bigoplus_{l=0}^r D(n, r)_l.
\]

2. Set \( d(n, l) := \sum_{\lambda \in P(l, n)} (\dim S^\lambda)^2 \) with \( S^\lambda \) denoting the irreducible Specht module of \( \mathcal{S}_l \) corresponding to \( \lambda \in \text{Par}(l, n) \) (see (4.1) for the notation). Then \( \dim D(n, r)_l = d(n, l) \binom{r}{l}^2 \) and
\[
\dim D(n, r) = \sum_{l=0}^r d(n, l) \binom{r}{l}^2.
\]

3. For \( l \in \{0, 1, \ldots, r\} \) there is a basis of \( E_l \): \( \{x_e^l_{\sigma_{l,i}} \mid I \in \mathcal{J}_l, \sigma_{l,i} \in \mathcal{S}_l, i = 1, \ldots, d(n, l)\} \)
with \( \sigma_{l,1} = \text{id} \), and \( \binom{r}{l} \) elements \( c_{l, I} \in \Psi|_l(\mathcal{S}_r) \) such that \( D(n, r)_l \) has a basis \( \{c_{l, I} x_{\sigma_{l, i} e}^l \mid (I, J) \in \mathcal{J}_l^2, i = 1, \ldots, d(n, l)\} \). In particular, \( x_e = \sum_{l=1}^r \sum_{I \in \mathcal{J}_l} x_e^l + x_\emptyset = \text{just the identity of } D(n, r) \).

4. The following decomposition of \( D(n, r) \) into a direct sum of subspaces holds:
\[
D(n, r) = \mathbb{C} \Psi(\mathcal{S}_r) x_e \oplus \bigoplus_{l=1}^r \bigoplus_{(j, I) \in \mathcal{J}_l^2, i=1, \ldots, d(n, l)} \mathbb{C} c_{l, I} x_{\sigma_{l, i} e}^l.
\]

**Proof.** (1) By definition, \( \bigoplus_{l=0}^r D(n, r)_l \) is really a direct sum in \( \text{End}_c(V_r^{\otimes r}) \). In the following, we prove (3.19). Let \( \tau \in \mathcal{S}_r \). Then for any \( i \in \mathbb{N}^r \), we have
\[
\Psi(\tau)(\eta_i) = \Psi|_{\mathbb{C}^r}(\tau)(\eta_i) = \sum_{l=0}^r \sum_{J \subseteq \mathbb{N}^r} \Psi|_l(\tau) x_e^l(\eta_i).
\]
where \( s = \text{rk}_\mathbb{C}(i) \). This implies that
\[
\Psi(\tau) = \sum_{l=0}^r \sum_{J \subseteq \mathbb{N}^r} \Psi|_l(\tau) x_e^l.
\]
Hence,
\[ D(n, r) \subset \bigoplus_{l=0}^{r} D(n, r)_l. \]

On the other hand, since
\[ \Psi_i(\sigma)(\eta_i) = \delta_{t, r_k(\iota)} \Psi(\sigma)(\eta_i) = \Psi(\sigma)x_e(\eta_i), \forall i \in \mathbb{N}^r, \]
it follows that
\[ \Psi_i(\sigma) = \Psi(\sigma)x_e. \]
This implies that
\[ \bigoplus_{l=0}^{r} D(n, r)_l \subset D(n, r). \]

We complete the proof for (3.19).

(2) Due to (1), it suffices to prove the formula \( \dim D(n, r)_l = d(n, l)(\binom{r}{l})^2 \). Firstly, for any given \( I \subset \mathcal{L}_l \), the subalgebra \( D_I \) generated by all \( x^I_\sigma (\sigma \in \mathcal{S}_l) \) is isomorphic to \( \mathbb{C}\Psi_I^1(\mathcal{S}_l) \) which is of dimension equal to \( d(n, l) \) by the representation theory of symmetric groups. So the dimensions of all those \( D_I \) are the same. Secondly, by Lemma 3.3(5) \( D(n, r)_l \) is spanned by \( \Psi(\tau)x^I_\sigma \) for all \( \tau \in \mathcal{S}_r, \sigma \in \mathcal{S}_l \) and \( I \subset \mathcal{L}_r \) with \( \#I = l \). On the other hand, for any subsets \( I = \{i_1 < i_2 < \cdots < i_l\}, J = \{j_1 < j_2 < \cdots < j_l\} \) of \( \mathcal{L}_r \) with \( \#I = \#J = l \), take \( \tau_{IJ} \in \mathcal{S}_r \) such that \( \tau_{IJ}(j_k) = i_k \) for \( k = 1, \cdots, l \). For any two \( \tau_1 \) and \( \tau_2 \in \mathcal{S}_r \), if \( \tau_1^{-1} \circ \tau_2 \) stabilizes \( I \), then \( \Psi(\tau_1)x^I_\sigma = \Psi(\tau_2)x^I_\sigma' \) for some \( \sigma' \in \mathcal{S}_l \). Actually, for \( \tau \in \mathcal{S}_r \) satisfying \( \tau|_I \in \text{Sym}(I) \) which is isomorphic to \( \mathcal{S}_l \), say by \( \theta \), we have \( \Psi(\tau)x^I_\sigma = x^J_{\theta(\sigma|_I)} \) for any \( \sigma \in \mathcal{S}_l, I \subset \mathcal{L}_r \).

By the above arguments, we have that for any ordered pair \((I, J)\), the above \( \tau_{IJ} \) changes a set of basis of \( D_I \) into the ones of \( D_J \). There are \( \binom{r}{l}^2 \) such ordered pairs \((I, J)\). Summing up, along with the definition of \( D_I \) we have \( \dim D(n, r)_l = d(n, l)(\binom{r}{l})^2 \), as desired.

(3) This statement follows from the arguments in (2).

(4) Note that the chosen basis elements for every \( E_I \) in the statement (3) include \( x_e^I \). We make a proper adjustment of the basis such that the new basis contains \( x_e \). Then the corresponding decomposition follows.

**Remark 3.6.** If \( n > l \), then the following set
\[ \{ \Psi_I(\tau_{JI})x^I_\sigma \mid I \subset \mathcal{L}_r, J \subset \mathcal{L}_r, \#I = \#J = l, \sigma \in \mathcal{S}_l \} \]
is a basis of \( D(n, r)_l \). Moreover, the algebra structure of \( D(n, r)_l \) is given by
\[
(\Psi_I(\tau_{LK})x^K_\mu)(\Psi_I(\tau_{JI})x^I_\sigma) = \begin{cases} 
\Psi_I(\tau_{LI})x^I_{\mu\sigma}, & \text{if } J = K; \\
0, & \text{otherwise.}
\end{cases} \tag{3.20}
\]

**Proof of Remark 3.6.** Note that in this case, the elements \( x^I_\sigma \) for all \( \sigma \in \mathcal{S}_l \) and for any given \( I \) in the remark are linearly independent in \( D_I \). The first statement follows from the arguments in the proof of Lemma 3.5(3). To show (3.20), we can assume that \( I = l \) without loss of generality. We note that the non-vanishing range of the operator \( \Psi_I(\tau_{JI})x^I_\sigma \)
is contained in $V_l^\otimes r$ for $J \neq K$. Hence, $(\Psi_l(\tau_{LK})x^K_\mu)(\Psi_l(\tau_{JI})x^I_\sigma)(\eta_i) = 0$. For the case $J = K$, we have

$$(\Psi_l(\tau_{LI})x^I_\mu)(\Psi_l(\tau_{JI})x^I_\sigma)(\eta_i) = \begin{cases} \Psi_l(\tau_{LI})\eta_{(\mu \circ \sigma)i}, & \text{if } i \in N_l^r, i_k \in n, \forall k \in I; \\ 0, & \text{otherwise.} \end{cases}$$

Hence, (3.20) follows.

4. Duality related to the degenerate double Hecke algebra and the branching duality formula

In this section, with aid of some structural property of enhanced Schur algebras, we establish a duality between the degenerate double Hecke algebra $H_r$ and $GL(n, \mathbb{C})$ on the tensor space $(\mathbb{C}^{n+1})^\otimes r$. We will keep the nations as before.

4.1. Let us first recall the classical branching law for general linear groups. For positive integers $m, r$, set

$$\text{Par}(r, m) = \{\mu = (\mu_1, \cdots, \mu_m) \in \mathbb{N}^m \mid |\mu| = r, \mu_1 \geq \mu_2 \geq \cdots \geq \mu_m\}.$$

An element from $\text{Par}(r, m)$ is usually called a dominant weight, which is actually a partition of $r$ into $m$ parts (zero parts are allowed). For another weight $\lambda \in \text{Par}(l, n)$ with $l \leq r$ and $n \leq m$, we call $\lambda$ interlaces $\mu$ if $\mu_1 \geq \lambda_1 \geq \mu_2 \geq \lambda_2 \geq \cdots \geq \mu_{m-1} \geq \lambda_{m-1} \geq \mu_m$ where $\lambda_{n+1} = \cdots = \lambda_{m-1} = \lambda_{m-2} = 0$. We denote $\lambda \preceq \mu$ if $\lambda$ interlaces $\mu$.

For a given $\mu \in \text{Par}(r, n+1)$, let $L^\mu_{n+1}$ be an irreducible $GL(n+1, \mathbb{C})$-module with highest weight $\mu$. Then as a module over $GL(n, \mathbb{C})$, there is a unique decomposition

$$L^\mu_{n+1} = \bigoplus_{l=0}^r \bigoplus_{\lambda \in \text{Par}(l, n)} \delta_{\lambda \preceq \mu} L^\lambda_l$$

with

$$\delta_{\lambda \preceq \mu} = \begin{cases} 1, & \text{if } \lambda \preceq \mu; \\ 0, & \text{otherwise.} \end{cases}$$

There is a similar branching law for symmetric groups (see (4.8)).

4.2. In the following, we give the decomposition of $V_l^\otimes r$ into a direct sum of irreducible modules, as a $H_l^r$-modules. For this, denote by $L^\lambda_n$ the irreducible $GL(n, \mathbb{C})$-module with highest weight $\lambda$ for $\lambda \in \text{Par}(l, n)$. Keep denoting by $S^\lambda_l$ the irreducible Specht module over $\mathfrak{S}_l$ corresponding to $\lambda$ as before.

Recall that $GL(n, \mathbb{C})$ has a standard Borel subgroup consisting of all upper triangular invertible matrices, denoted by $\mathfrak{B}$. Denote by $\mathfrak{N}$ the unipotent radical of $\mathfrak{B}$ which consists of all unipotent upper triangular matrices.

**Lemma 4.1.** Keep the notations as above. As a $H_l^r$-module, $V_l^\otimes r$ decomposes into a direct sum of irreducible modules as follows

$$\bigoplus_{\lambda \in \text{Par}(l, n)} (D^\lambda_l)^{\oplus \dim L^\lambda_n}$$

(4.1)
where as a \( C\mathfrak{S}_l \)-module,
\[
D^\lambda_l = \bigoplus_{J \in \mathcal{P}(l)} S^\lambda_I \oplus S^\lambda_2 \oplus \cdots \oplus S^\lambda_{l} = (i_i)^{1} \text{ times}
\]

Proof. For any given \( J = \{j_1, j_2, \ldots, j_l\} \subset \mathfrak{r} \), we have a subspace \( V_J^r \) in \( V^r \), which is spanned by \( \eta_i = \eta_{i_1} \otimes \cdots \otimes \eta_{i_l} \) with \( i_1, \ldots, i_l \in \eta \). Then
\[
V^r_J = \bigoplus_{\#J = l} V^r_J
\]
where in the sum, \( J \) ranges over all subsets of \( r \) consisting of \( l \)-elements. The number of such \( J \) is exactly \( \binom{r}{l} \). For a fixed \( J \), \( V^r_J \) admits an \( \mathfrak{S}_l \)-action, which just permutates the position indicated by \( J \), this is to say, for any \( \sigma_i \in \mathfrak{S}_l \), \( \sigma_i \eta_i = \eta_{\sigma_i i} \) if defining \( \sigma_i \eta = \delta((\sigma_i \cdot i_i))(n + 1)^{-l} \) for \( i = \delta(i_i(n + 1)^{-l}) \). On the other hand, \( V^r_J \) becomes a \( GL(V) \)-module with every factor \( \eta \) fixed.

Thus \( V^r_J \) can be regarded as an \( r \)-tensor space of \( V \) with \( (GL_n \times \mathfrak{S}_l) \)-action. Thanks to the classical Schur-Weyl duality, as a \( (GL_n \times \mathfrak{S}_l) \)-module we have the following decomposition
\[
V^r_J = \bigoplus_{\lambda \in \mathcal{P}(l, n)} L_\lambda^n \otimes S^\lambda_I.
\]

Let us show the meaning of \( S^\lambda_I \) in the above decomposition by demonstrating the standard representatives in the isomorphism class of \( S^\lambda_I \). For the simplicity of arguments, we might as well suppose \( J = I \) which is equal to \( \{1, 2, \ldots, l\} \) without any loss of generality. Take \( \lambda = (\lambda_1, \lambda_2, \ldots, \lambda_n) \in \mathcal{P}(l, n) \). Set
\[
\eta(\lambda) = \eta_1 \otimes \cdots \otimes \eta_l \otimes \eta_2 \otimes \cdots \otimes \eta_2 \otimes \cdots \otimes \eta_n \otimes \cdots \otimes \eta_{\lambda_i} \otimes \cdots \otimes \eta.
\]
Consider the orbit \( \mathfrak{S}_l \cdot \eta(\lambda) \). The space spanned by this orbit forms an \( \mathfrak{S}_l \)-module, which is denoted by \( I^\lambda_l \). Then \( I^\lambda_l \) is actually the \( \lambda \)-weighted space of \( GL(n, \mathbb{C}) \)-module \( V^r_I \). In particular, \( I^\lambda_l \) contains all maximal \( \lambda \)-weighted vectors in \( GL(n, \mathbb{C}) \)-module \( V^r_I \), all of which by definition, are \( \mathfrak{H}_r \)-invariants in \( I^\lambda_l \). Furthermore, the invariant space in \( I^\lambda_l \) under the action of \( \mathfrak{H}_r \) is exactly the unique direct summand isomorphic to \( S^\lambda_I \), in the complete reducible decomposition of \( I^\lambda_l \) as \( \mathfrak{S}_l \)-module (see [11, §9.1.1]). This summand is denoted by \( S^\lambda_I \). More generally, each \( V^r_I \) for \( I \subset r \) (with \( \#I = l \)) admits the corresponding \( I^\lambda_l \) which contains a unique direct summand \( S^\lambda_I \) parallel to \( S^\lambda_l \). This means, there exists \( \sigma \in \mathfrak{S}_r \) such that the conjugation by \( \Psi_I(\sigma) \) maps \( S^\lambda_I \) onto \( S^\lambda_I \). Obviously,
\[
D^\lambda_l := \bigoplus_{\#I = l} S^\lambda_I
\]
is a \( \Psi_I(\mathfrak{S}_r) \)-module, which is as an \( \mathfrak{S}_l \)-module, isomorphic to the direct sum of \( \binom{r}{l} \) copies of \( S^\lambda_I \).

Next we prove that \( D^\lambda_l \) is an irreducible module over \( \mathfrak{H}_r^l \). For this, we only need to show that for any given nonzero vector \( w \in D^\lambda_l \), the cyclic \( \mathfrak{H}_r^l \)-submodule \( W \) generated by \( w \) must coincide with \( D^\lambda_l \) itself. Now, we write \( w = \sum_{\text{some } i} w_I \) with some nonzero \( w_I \in V^r_I \). Fix one
$w_1$. Recall $\mathcal{H}_r^l$ has a subalgebra $E_l$ by abuse of the notation (see §3.4), which is isomorphic to $\mathbb{C}[\mathfrak{h}]^l(\mathfrak{g}_l)$. So the cyclic module $E_l w_1$ of $E_l$ coincides with the irreducible $\mathbb{C}[\mathfrak{g}_l]$-module $S^\lambda_l$. This implies that there is some $x^I_{\sigma}$ for $\sigma \in \mathfrak{g}_l$ such that $x^I_{\sigma} w_1 \neq 0$. On the other hand, by definition (see §3.4) $x^I_{\sigma} \sum_{j \neq I} w_j = 0$. Therefore, $x^I_{\sigma} w = x^I_{\sigma} w_1$ is a nonzero vector in $S^\lambda_l$.

Hence the submodule $W$ contains an irreducible $E_l$-module $S^\lambda_l$. By $\mathbb{C}[\mathfrak{g}_l]$-action, we finally have that the $\mathcal{H}_r^l$-submodule $W$ coincides with $\bigoplus_{\lambda \in \text{Par}(l,n)} S^\lambda_l$, which is equal to $D^\lambda_l$.

According to the previous analysis, $D^\lambda_l$ is really an irreducible $\mathcal{H}_r^l$-module. From the above arguments, along with (4.2) and (4.3), the lemma follows. \hfill $\square$

4.3. Let us first notice that $V^{\otimes r}$ naturally becomes a representation space of $E(n, r)$ (and of $S(n+1, r)$ more generally) by defining

$$\xi(\eta) = \sum_{k \in \mathbb{N}^r} \xi(c_{k,i}) \eta_k$$

(4.4)

for any $\xi \in E(n, r)$, and any basis elements $\eta_k = \eta_{i_1} \otimes \cdots \otimes \eta_{i_r} \in \text{of } V^{\otimes r}$ ($i \in \mathbb{N}^r$). This representation is denoted by $\Upsilon$. We are actually considering $\Upsilon : E(n, r) \rightarrow \text{End}_\mathbb{C}(V^{\otimes r})$.

As before, let $\Phi : \text{GL}(n+1) \rightarrow \mathcal{A} := \text{End}_\mathbb{C}(V^{\otimes r})$ be the natural representation. By the classical Schur algebra theory, we can identify the image of $\Phi$ with $S(n+1, r)$, in aid of $\Upsilon$.

Consequently, we have the following corollary to Lemma 4.1.

**Corollary 4.2.** Keep the notations as above. As a $\mathcal{H}_r$-module, $V^{\otimes r}$ decomposes into a direct sum of irreducible modules as follows

$$\bigoplus_{l=0}^r \bigoplus_{\lambda \in \text{Par}(l,n)} (D^\lambda_l)^{\oplus \dim L^\lambda_l}.$$  

(4.5)

4.4. We have the following restricted version of Schur-Weyl duality from $\text{GL}(n+1, \mathbb{C})$ to $\text{GL}(n, \mathbb{C})$.

**Theorem 4.3.** (Restricted Schur-Weyl duality) Keep the above notations. The following statements hold.

1. $\text{End}_{D(n,r)}(V^{\otimes r}) = \mathbb{C}[\Phi(\text{GL}(V))].$
2. The following duality holds:

$$\text{End}_{\Delta(n,r)}(V^{\otimes r}) = D(n, r);$$

$$\text{End}_{D(n,r)}(V^{\otimes r}) = \Delta(n, r).$$

(4.6)

**Proof.** (1) By Lemma 3.3(4), $\mathbb{C}[\Phi(\text{GL}(V))] \subset \text{End}_{D(n,r)}(V^{\otimes r})$. We need to show the opposite inclusion.

Note that $D(n, r) = \Xi(\mathcal{H}_r)$. So $\text{End}_{D(n,r)}(V^{\otimes r}) \subset \text{End}_{\mathbb{C}[\mathfrak{g}_l]}(V^{\otimes r}) = \mathbb{C}[\Phi(\text{GL}(V))]$. The last equation is due to the classical Schur-Weyl duality for $\text{GL}(V)$ and $\mathfrak{g}_r$. Thus, for any $\phi \in \text{End}_{D(n,r)}(V^{\otimes r})$, we can write $\phi = \sum_{h \in \text{GL}(V)} a_h \Phi(h)$ for $a_h \in \mathbb{C}$. In the following, we will show that $\phi$ must lie in $\mathbb{C}[\Phi(\text{GL}(V))]$.

We identify $\text{GL}(V)$ with $\text{GL}(n+1, \mathbb{C})$, and $\Phi(\text{GL}(V))$ with $S(n+1, r)$ by the classical theory of Schur algebras. Thus, we rewrite $\phi = \sum_{(i,j) \in \mathbb{N}^r \times \mathbb{N}^l} a_{ij} \xi_{ij}$ for $a_{ij} \in \mathbb{C}$ satisfying
\[\pi_l = \pi_{r,l} \text{ for any } r \in \mathcal{S}_r.\] Furthermore, \(\phi = \sum_{l=0}^r \phi_l\) with
\[
\phi_l = \sum_{(i,j) \in N_r \times N_r \rightarrow \mathbb{R}} a_{ij} \xi_{ij}.
\]
In this view, by Lemma \(2.10\) we only need to prove that all \(\phi_l\) lie in the restricted Schur algebra \(\Delta(n, r)\). The arguments proceed in steps.

(1-1) By the assumption \(\phi \in \text{End}_{D(n,r)}(V^{\otimes r})\), \(\phi \circ x_{\sigma}^l = x_{\sigma}^l \circ \phi\) for all \(\sigma \in \mathcal{S}_l\), and \(l = 0, 1, \ldots, r\). We first claim that \(\phi\) stabilises \(V^{\otimes r}_l\) for every \(l\).

Actually, if not so, then there must be a basis element \(\eta_j \in V^{\otimes r}_l\) not obeying the stability claimed above for \(\phi\). From \(4.4\) it follows \(\phi(V^{\otimes r}_l) = \phi_l(V^{\otimes r}_l)\). Violation of the stability implies that \(\phi_l(\eta_j) = \eta_l + \eta_k\) with \(\eta_l \in V^{\otimes r}_l\) but \(\eta_k \notin V^{\otimes r}_l\). Suppose \(j = \tau(j'_l(n + 1)^{r-l})\) for \(\tau \in \mathcal{S}_r\), and \(j'_l \in \mathbb{Z}_l^r\). Then \((\tau x_{\id}^l)^{-1}(\eta_j) = \eta_j\). On the other side, \((\tau x_{\id}^l)^{-1} \circ \phi(\eta_j) = \phi \circ (\tau x_{\id}^l)^{-1}(\eta_j)\). This leads to a contradiction that \(\tau x_{\id}^l)^{-1}(\eta_l + \eta_k) = \eta_l + \eta_k\). So the claim is true.

(1-2) By (1-1), \(\phi_l\) is commutative with \(x_{\sigma}^l\) for all \(\sigma \in \mathcal{S}_l\). From this, by the same arguments as in (1-1) it follows that \(\phi_l\) stabilises \(V^{\otimes r}_l\). Hence \(\phi_l\) is actually commutative with \(\mathcal{S}_l\) in \(\text{End}_C(V^{\otimes r}_l)\).

Recall that \(V^{\otimes r}_l = V^{\otimes n} \otimes \eta^{\otimes r-l}\), and \(x_{\sigma}^l = \Psi(\sigma) \otimes \id^{\otimes r-l}\). By applying the classical Schur Weyl duality, we have that \(\phi \big|_{V^{\otimes r}_l} \in \Phi(\text{GL}(V))|_{V^{\otimes r}_l}\). Note that \(\phi \big|_{V^{\otimes r}_l} = \phi_l \big|_{V^{\otimes r}_l}\). From this, it is deduced that \(\phi_l \big|_{V^{\otimes r}_l} \in \Phi(\text{GL}(V))|_{V^{\otimes r}_l}\). By definition, we can further deduce that \(\phi_l \big|_{V^{\otimes r}_l} \in \Phi(\text{GL}(V))|_{V^{\otimes r}_l}\). From this, it follows that \(\phi_l \in \Phi(\text{GL}(V)) \subset S(n+1, r)\) in the same sense as in \(4.3\) because \(\phi_l \big|_{V^{\otimes r}_l} = 0\) for \(k \neq l\). Hence \(\phi_l \in \Delta(n, r)\).

By the analysis in the beginning, we accomplish the proof of the first statement.

(2) The second equation in \(4.6\) follows from (1). Note that \(V^{\otimes r}_l\) is completely reducible over \(D(n, r)\) (Corollary \(4.2\)). The first equation in \(4.6\) follows from the classical double commutant theorem (see \[\text{III} \] \(\S 4.1.5\)). \(\square\)

**Corollary 4.4.** The set consisting of \(D^l_\lambda\) for \(\lambda \in \text{Par}(l, n)\) and \(l = 0, 1, \ldots, r\) form a representative set of the isomorphism class of irreducible \(D(n, r)\)-modules.

**Proof.** Since \(\Delta(n, r)\) is semisimple, so is \(D(n, r)\) by Theorem \(4.3\) and the Double Commutant Theorem. Furthermore, each \(D(n, r)_{\lambda}\) is semisimple for \(l = 0, 1, \ldots, r\) by Lemma \(3.5(1)\). Thanks to Lemma \(4.4\) \(D^l_\lambda\) is an irreducible \(D(n, r)_{\lambda}\)-module for any \(\lambda \in \text{Par}(l, n)\), and \(D^l_\lambda \not\cong D^l_\mu\) for \(\lambda \neq \mu\). Moreover,
\[
\sum_{\lambda \in \text{Par}(l, n)} (\dim D^l_\lambda)^2 = \sum_{\lambda \in \text{Par}(l, n)} \left(\binom{r}{l}\right)^2 (\dim S^l_{\lambda})^2 \\
= \left(\binom{r}{l}\right)^2 \sum_{\lambda \in \text{Par}(l, n)} (\dim S^l_{\lambda})^2 \\
= \left(\binom{r}{l}\right)^2 d(n, l) \\
= \dim D(n, r)_{l} \quad \text{(Lemma \(3.5(4)\))}.
\]
Hence, the set \( \{ D^\lambda_l \mid \lambda \in \text{Par}(l,n) \} \) exhausts all representatives of isomorphism classes of irreducible \( D(n,r)_l \)-modules. Consequently, the desired assertion holds.

4.5. **Branching duality formula.** From Theorem 4.3 we derive a dimension relation between irreducible modules from \((\text{GL}_n, \mathcal{S}_r)\)-duality, and irreducible modules from \((\text{GL}_{n+1}, \mathcal{S}_r)\)-duality.

**Corollary 4.5.** The following statements hold.

(1) Under the action of \( \text{GL}(n, \mathbb{C}) \times \mathcal{H}_r \), the space of \( r \)-tensors over \( \mathbb{C}^{n+1} \) decomposes as

\[
(\mathbb{C}^{n+1})^\otimes r \cong \bigoplus_{l=0}^{r} \bigoplus_{\lambda \in \text{Par}(l,n)} L^\lambda_l \otimes D^\lambda_l.
\]

(2) (Branching duality formula) In the Schur-Weyl duality, the irreducible pairs \( (L^\mu_{n+1}, S^\mu_r) \) \((\mu \in \text{Par}(r,n+1))\) for \((\text{GL}_{n+1}, \mathcal{S}_r)\) and the irreducible pairs \( (L^\lambda_n, S^\lambda_l) \) \((\lambda \in \text{Par}(l,n))\) for \((\text{GL}_n, \mathcal{S}_l)\) with \(l = 1, \ldots, r\) satisfy the following branching duality formula

\[
\sum_{\mu \in \text{Par}(r,n+1)} \delta_{\lambda \leq \mu} \dim S^\mu_r = \binom{r}{l} \dim S^\lambda_l
\]

(4.7)

for any \( \lambda \in \text{Par}(l, n) \) \((l \leq r)\).

**Proof.** (1) Keep Lemma 4.1 in mind. Then this decomposition follows from Theorem 4.3 and the classical double commutant theorem (see [11, §4.16]).

(2) It follows from (1) that as a \( \text{GL}(n, \mathbb{C}) \)-module

\[
(\mathbb{C}^{n+1})^\otimes r \cong \bigoplus_{l=0}^{r} \bigoplus_{\lambda \in \text{Par}(l,n)} (L^\lambda_l) \oplus \binom{r}{l} \dim S^\lambda_l L^\lambda_l.
\]

On the other hand, according to the classical Schur-Weyl duality, \((\mathbb{C}^{n+1})^\otimes r\) has the following decomposition, as a \( \text{GL}(n+1, \mathbb{C}) \)-module:

\[
(\mathbb{C}^{n+1})^\otimes r \cong \bigoplus_{\mu \in \text{Par}(r,n+1)} (L^\mu_{n+1})^\otimes \dim S^\mu_r
\]

\[=
\sum_{\mu \in \text{Par}(r,n+1)} \sum_{\lambda \in \text{Par}(l,n)} (\delta_{\lambda \leq \mu} \dim S^\mu_r) L^\lambda_l
\]

\[=
\sum_{l=0}^{r} \sum_{\lambda \in \text{Par}(l,n)} \sum_{\mu \in \text{Par}(r,n+1)} (\delta_{\lambda \leq \mu} \dim S^\mu_r) L^\lambda_l
\]

Compare both decompositions. The equation (4.7) follows.

**Remark 4.6.** (1) When \( l = r \), the formula (4.7) becomes trivial.

(2) When \( l = r - 1 \), the formula (4.7) becomes

\[
\sum_{\mu \in \text{Par}(r,n+1)} \delta_{\lambda \leq \mu} \dim S^\mu_r = r \dim S^\lambda_{r-1}
\]
for $\lambda \in \text{Par}(r-1,n)$. This formula can be regarded as a duality to the following classical branching rule (see [11 Corollary 9.2.7])

$$\text{Res}_{\mathfrak{S}_r}^E \cdot^{\mu} \mathfrak{S}_r \cong \bigoplus_{\lambda \in \text{Par}(r-1,n)} \delta_{\lambda \leq \mu} S^\lambda_r.$$  \hspace{1cm} (4.8)

5. Enhanced tensor invariants and enhanced Schur-Weyl duality

Keep the notations as before. In this section, we will give the answer to Question 2.3 then establish the enhanced Schur-Weyl duality and give some applications.

5.1. Recall the notations in \[2.5.3\] In particular, $V$ has a basis $\{\eta_1, \ldots, \eta_n; \eta_{n+1}\}$ with $\eta_i \in V$ for $i = 1, \ldots, n$; $\eta_{n+1} = \eta$. Then $V^{\otimes r}$ has a basis $\{\eta_i = \eta_{i_1} \otimes \eta_{i_2} \otimes \cdots \otimes \eta_{i_r} | i = (i_1, i_2, \ldots, i_n) \in N^n\}$. Let us recall that $V^{\otimes r}$ naturally becomes a representation space of $\mathfrak{S}(n,r)$ (and of $S(n+1,r)$ more generally) by definition as in (4.4).

Let us first investigate what those elements look like if $D$ exists. We first have the following basic observation.

Lemma 5.1. Denote by $\mathcal{E}(n,r)_{l,0}$ the subspace of $\mathcal{E}(n,r)$ spanned by $\xi_{\delta^{(l-1)(n+1)}_{-1,l(n+1)r}}$ with $\delta^{(l-1)(n+1)}_{-1,l(n+1)r}$ ranging over $n^l$, and by $\mathcal{E}(n,r)_{\geq 0,0}$ the direct sum of all $\mathcal{E}(n,r)_{l,0}$ with $l = 0, 1, \ldots, r$. Then the image of $\mathcal{E}(n,r)_{\geq 0,0}$ is exactly $(V^{\otimes r})^{\mathfrak{S}_r}$. \hspace{1cm} \Box

Proof. Recall that for any $l = 0, 1, \ldots, r$, $V^{\otimes r}$ has a basis consisting of $\eta_i \in V^{\otimes r}$ with $i$ ranging over $n^l$. Note that $i \sim i_{(l+1)^{-l}}$ with $i_{(l+1)^{-l}} \in n^{l+1}$. By definition, $\xi_{\delta^{(l-1)(n+1)}_{-1,l(n+1)r}}$, $\xi_{\delta^{(l-1)(n+1)}_{-1,l(n+1)r}}$, is a fundamental invariant in $(V^{\otimes r})^{\mathfrak{S}_r}$ arising from the $\mathfrak{S}_r$-orbit of $\eta_i$ (see (4.4)). Correspondingly, the image of $\mathcal{E}(n,r)_{l,0}$ coincides with $(V^{\otimes r})^{\mathfrak{S}_r}$. When $l$ ranges over $\{0, 1, \ldots, r\}$, the lemma follows.

As a result, we have the following crucial lemma.

Lemma 5.2. For any $\phi \in \text{End}_{\mathcal{E}(n,r)}(V^{\otimes r})$, $\phi$ must lie in $\mathbb{C}\Psi(\mathfrak{S}_r)+D(n,r)_{\text{ann}}$ where $D(n,r)_{\text{ann}} := \{\phi \in D(n,r) | \phi((V^{\otimes r})^{\mathfrak{S}_r}) = 0\}$. \hspace{1cm} \Box

Proof. Note that $\text{End}_{\mathcal{E}(n,r)}(V^{\otimes r}) \subset \text{End}_{\mathcal{E}(n,r)}(V^{\otimes r})$. From Theorem [2.5.1], $\text{End}_{\mathcal{E}(n,r)}(V^{\otimes r}) \subset D(n,r)$. In the following, we will show that $\text{End}_{\mathcal{E}(n,r)}(V^{\otimes r}) \cap D(n,r) \subset \mathbb{C}\Psi(\mathfrak{S}_r)+D(n,r)_{\text{ann}}$. By Lemma [2.5.4],

$$D(n,r) = \mathbb{C}\Psi(\mathfrak{S}_r)x_r \oplus \bigoplus_{l=1}^r \bigoplus_{(J,I) \in \mathfrak{S}_r^l} \mathbb{C}c_{J,I}\delta_{x_{(J,I)}}.$$  \hspace{1cm} (5.1)

Thanks to Proposition [2.5] we only need to show that

$$\text{End}_{\mathcal{E}(n,r)}(V^{\otimes r}) \cap \bigoplus_{l=1}^r \bigoplus_{(J,I) \in \mathfrak{S}_r^l} \mathbb{C}c_{J,I}\delta_{x_{(J,I)}} \subset D(n,r)_{\text{ann}}.$$  \hspace{1cm} (5.1)

Actually, for any given nonzero $\phi = \sum_{l=t}^s \phi_l$ with $0 < t \leq s \leq r$, and $\phi_l \in D(n,r)_{l}$, $l = t, t+1, \ldots, s$, such that $\phi \in \text{End}_{\mathcal{E}(n,r)}(V^{\otimes r})$ and both $\phi_s, \phi_t$ are not zero. By the assumption, for any $\xi \in \mathcal{E}(n,r)_{\geq 0,0}$, we have $\xi \circ \phi = \phi \circ \xi$. Then on one side $\xi \circ \phi$ vanishes because $\phi = \sum_{l=t}^s \phi_l$ with $t > 0$. Hence, $\phi \circ \xi$ must be zero for all $\xi \in \mathcal{E}(n,r)_{\geq 0,0}$. Thanks to
Lemma 5.1 the image of $E(n, r)_{\geq 0,0}$ is exactly $(V^{\otimes r})^{S_r}$. This means that $\phi$ annihilates the space $(V^{\otimes r})^{S_r}$.

Summing up, (5.1) is proved. The lemma follows. \hfill \Box

Generally, we have the following theorem.

**Theorem 5.3.** (Enhanced Schur-Weyl duality) Keep the notations as above, in particular $G = \text{GL}(V)$. Set $D(n, r)^V = \{ \phi \in D(n, r) \mid e^v \circ \phi \circ e^{-v} = \phi, \forall v \in V \}$. Then the following statements hold.

1. $\text{End}_{\text{Rep}(G)}(V^{\otimes r}) = D(n, r)^V$.
2. The above $D(n, r)^V$ can be described as

$$\mathbb{C} \Psi(\mathfrak{S}_r) \subset D(n, r)^V \subset \mathbb{C} \Psi(\mathfrak{S}_r) + D(n, r)_{\text{ann}}.$$ 

**Proof.** (1) Keep in mind $E(n, r) = \mathbb{C} \Phi(\mathfrak{G})$, and $\mathfrak{G} = G \times_r V$. Then the statement follows from Lemma 4.3.
(2) The first inclusion follows from Lemma 2.5(1). The second one is due to Lemma 5.2. \hfill \Box

5.2. **Example.** Let us demonstrate the above theorem by an example. Keep the notations as before, and let $I = \{1, 2\}$, $I_i = \{i\}$ for $i = 1, 2$, and $\sigma \in \mathfrak{S}_2$ interchanging 1 and 2. Then we have the following demonstration of $D(n, r)^V$.

1. If $r = 1$ or 2, and $n \geq r$, then $D(n, r)^V = \mathbb{C} \Psi(\mathfrak{S}_r)$.
2. If $r = 2$ and $n = 1$, then $D(n, r)^V = \mathbb{C} \Psi(\mathfrak{S}_r) \oplus \mathbb{C}(x^I_e - x^I_\sigma)$.

**Proof of the demonstration.** Recall that $V$ has a basis $\eta_1, \ldots, \eta_n; \eta_{n+1} := \eta$, where $\eta_1, \ldots, \eta_n$ forms a basis of $V$.

(1) In the case $r = 1$, it is obvious that $D(n, r)_{\text{ann}} = 0$. The assertion is obvious.

Now we suppose that $n \geq r = 2$. Then $D(n, r)$ has a basis $\{x^I_e, x^I_\sigma, \Psi(\sigma)x^{I_1}_e, \Psi(\sigma)x^{I_2}_e, x^{I_2}_e, x^I_e, x^I_\sigma\}$. Take any $\phi \in D(n, r)^V$, by the discussion as in Lemma 5.2 we can assume that

$$\phi = a_1 x^I_e + a_2 x^I_\sigma + a_3 \Psi(\sigma)x^{I_1}_e + a_4 \Psi(\sigma)x^{I_2}_e + a_5 x^I_e + a_6 x^I_\sigma + a_7 x^I_\sigma,$$  \hfill (5.2)

where $a_i \in \mathbb{C}$ for $1 \leq i \leq 7$. By the assumption, $\phi \circ \Phi(e^n) = \Phi(e^n) \circ \phi$. In particular, on one hand,

$$\phi \circ \Phi(e^n)(\eta \otimes \eta)$$

$$= \phi(\eta_1 + \eta) \otimes (\eta_1 + \eta)$$

$$= (a_1 + a_2)\eta_1 \otimes \eta_1 + (a_3 + a_6)\eta \otimes \eta_1 + (a_4 + a_5)\eta_1 \otimes \eta + a_7 \eta \otimes \eta,$$  \hfill (5.3)

and one the other hand,

$$\Phi(e^n) \circ \phi(\eta \otimes \eta)$$

$$= \Phi(e^n)(a_7 \eta \otimes \eta)$$

$$= a_7(\eta_1 + \eta) \otimes (\eta_1 + \eta)$$

$$= a_7 \eta_1 \otimes \eta_1 + a_7 \eta \otimes \eta_1 + a_7 \eta_1 \otimes \eta + a_7 \eta \otimes \eta.$$  \hfill (5.4)

By comparing both sides of (5.3) and (5.4), we have

$$a_1 + a_2 = a_3 + a_6 = a_4 + a_5 = a_7.$$  \hfill (5.5)
Moreover, on one hand,
\[ \phi \circ \Phi(e^n)(\eta_2 \otimes \eta) = \phi(\eta_2 \otimes (\eta_1 + \eta)) = a_1 \eta_2 \otimes \eta_1 + a_2 \eta_1 \otimes \eta_2 + a_3 \eta \otimes \eta_2 + a_5 \eta_2 \otimes \eta. \] (5.6)

On the other hand,
\[ \Phi(e^n) \circ \phi(\eta_2 \otimes \eta) = \Phi(e^n)(a_3 \eta \otimes \eta_2 + a_5 \eta_2 \otimes \eta) = a_3 \eta_1 \otimes \eta_2 + a_3 \eta \otimes \eta_2 + a_5 \eta_2 \otimes \eta_1 + a_5 \eta_2 \otimes \eta. \] (5.7)

By comparing both sides of (5.6) and (5.7), we have
\[ a_1 = a_5, a_2 = a_3. \] (5.8)

It follows from (5.5) and (5.8) that
\[ a_1 = a_5 = a_6 := a, a_2 = a_3 = a_4 = b, a_7 = a + b. \]

Consequently,
\[ \phi = a(x_e^I + x_e^{I_1} + x_e^{I_2} + x_e^\emptyset) + b(x_\sigma^I + \Psi(\sigma)x_e^{I_1} + \Psi(\sigma)x_e^{I_2} + x_e^\emptyset) = a\Psi(e) + b\Psi(\sigma) \in \mathbb{C}\Psi(\mathcal{G}_e). \]

The assertion follows.

(2) Suppose \( r = 2 \) and \( n = 1 \). Take any \( \phi \in D(n, r)^V \). As the arguments in (1), we can write \( \phi \) as the form (5.2). By the assumption, \( \phi \circ \Phi(e^n) = \Phi(e^n) \circ \phi \). In particular, on one hand,
\[ \phi \circ \Phi(e^n)(\eta_1 \otimes \eta) = \phi(\eta_1 \otimes (\eta_1 + \eta)) = (a_1 + a_2)\eta_1 \otimes \eta_1 + a_3 \eta \otimes \eta_1 + a_5 \eta_1 \otimes \eta, \] (5.9)

and one the other hand,
\[ \Phi(e^n) \circ \phi(\eta_1 \otimes \eta) = \Phi(e^n)(a_3 \eta \otimes \eta_1 + a_5 \eta_1 \otimes \eta) = (a_3 + a_5)\eta_1 \otimes \eta_1 + a_3 \eta \otimes \eta_1 + a_5 \eta_1 \otimes \eta. \] (5.10)

By comparing both sides of (5.9) and (5.10), we have
\[ a_1 + a_2 = a_3 + a_5. \] (5.11)

Then it follow from (5.5) and (5.11) that
\[ a_3 = a_4 := c, a_5 = a_6 := d, a_7 = a_1 + a_2 = c + d. \]

Consequently,
\[ \phi = d(x_e^I + x_e^{I_1} + x_e^{I_2} + x_e^\emptyset) + c(x_\sigma^I + \Psi(\sigma)x_e^{I_1} + \Psi(\sigma)x_e^{I_2} + x_e^\emptyset) + (a_1 - c)x_e^I + (a_2 - d)x_\sigma^I \]
\[ = d\Psi(e) + c\Psi(\sigma) + (a_1 - c)(x_e^I - x_\sigma^I) \in \mathbb{C}\Psi(\mathcal{G}_e) \oplus \mathbb{C}(x_e^I - x_\sigma^I). \]

Moreover, it is a routine to check that \( x_e^I - x_\sigma^I \in D(n, r)^V \), and the assertion follows.
5.3. A conjecture. We propose the following conjecture.

**Conjecture 5.4.** When \( n \geq r \), \( D(n,r)^V \) coincides with \( \mathbb{C}\Psi(\mathfrak{S}_r) \).

5.4. Enhanced tensor invariants. Identify \( V^{\otimes r} \) with \( (V^r)^* \). Then there is a natural \( G \)-equivariant isomorphism of vector spaces

\[
T : V^{\otimes r} \otimes V^{*\otimes r} \to \text{End}_\mathbb{C}(V^{\otimes r}).
\]

With the enhanced Schur-Weyl duality, we describe the enhanced tensor invariants.

Still set \( \eta \) invariants in the mixed-tensor product \( V \). We have

\[
\text{Proposition 5.5.} \text{ Let } G = \text{GL}(V). \text{ When } D(n,r)^V = \mathbb{C}\Psi(\mathfrak{S}_r), \text{ then the space of } G \text{-invariants in the mixed-tensor product } V^{\otimes r} \otimes V^{\otimes r} \text{ is generated by these } C_\sigma \text{ with } \sigma \in \mathfrak{S}_r.
\]

**Proof.** By the definition of \( T \) mentioned before, for \( \sigma \in \mathfrak{S}_r \), we have

\[
T(C_\sigma)\eta_j = \sum_{i \in \mathbb{N}^r} \eta_i^* (\eta_j) \eta_{\sigma,i} = \eta_{\sigma,j}.
\]

This means that \( T(C_\sigma) = \Psi(\sigma) \). By the assumption, the \( G \)-invariants in \( V^{*\otimes r} \) is isomorphic to \( \sum_{\sigma \in \mathfrak{S}_r} \mathbb{C}\Psi(\sigma) \) via \( T \). Hence, the proposition is proved. \( \square \)

6. Representations of the enhanced Schur algebra

Keep the notations as above. In this section, we proceed to study representations of the enhanced Schur algebra \( \mathcal{E}(n,r) \). In the whole section, by the term “a module of \( \mathcal{E}(n,r) \)” we will always mean a right module of \( \mathcal{E}(n,r) \).

6.1. Irreducible modules and PIMs. Recall \( \mathcal{E}(n,r) \) has a sequence of right ideals of \( \mathcal{E}(n,r) \):

\[
\mathcal{E}(n,r) = \mathcal{E}(n,r)_0 \supset \mathcal{E}(n,r)_1 \supset \mathcal{E}(n,r)_2 \supset \cdots \supset \mathcal{E}(n,r)_r \supset 0.
\]

with quotients \( \mathcal{E}(n,r)_i/\mathcal{E}(n,r)_{i+1} \cong S(n,i) \rtimes a_i \) (Lemma 2.11(3)). Those right ideals satisfy \( \mathcal{E}(n,r)_i\mathcal{E}(n,r) = \mathcal{E}(n,r)_i \). Naturally, \( S(n,r) \) naturally becomes a \( \mathcal{E}(n,r) \)-module.

Consider the primitive idempotent decomposition of the identity element \( \text{id} \) in \( \mathcal{E}(n,r) \)

\[
\text{id} = \sum_{s \in \mathbb{N}_0^r} \xi_{\tilde{s}_a,\tilde{s}_a}.
\]

We have

\[
\mathcal{E}(n,r) = \sum_{s \in \mathbb{N}_0^r} \xi_{\tilde{s}_a,\tilde{s}_a} \mathcal{E}(n,r).
\]
Furthermore, for $0 \leq l \leq r$ we set

$$\mathcal{E}(n, r)[l] := \mathbb{C}\text{-span}\{\xi_{\tilde{s}, \tilde{s}_l} \in E \mid |s| = l\}.$$ 

Then by (S1)-(S3) again,

$$\mathcal{E}(n, r)[l] = \sum_{s \in \mathbb{N}_l^2 \mid |s| = l} P_{\tilde{s}, \tilde{s}_l} \quad (6.3)$$

6.1.1. Let us diverge by recalling some general results on irreducible modules for the classical Schur algebras. First of all, the Schur algebra $S(g)$ gives rise to an automorphism of $W$ by permutation via $\gamma \in \mathcal{G}_m$. Each $\mathcal{G}_m$-orbit of $\Lambda(m, r)$ contains only one dominant weight $\gamma = (\gamma_1, \ldots, \gamma_m)$ which satisfies $\gamma_1 \geq \gamma_2 \geq \cdots \geq \gamma_m$. Denote by $\Lambda^+(m, r)$ the set of all dominant weights. Then $\Lambda^+(m, r)$ exactly consists of all partition of $r$ admitting at most number $m$ of nonzero parts, coinciding with $\text{Par}(r, m)$. 

**Lemma 6.1.** Suppose $\nu$ is a dominant weight, and shares the same $\mathcal{G}_m$-orbit with $\lambda$. Then two $S(m, r)$-irreducible modules $\tilde{D}_\lambda$ and $\tilde{D}_\nu$ are isomorphic.

**Proof.** Recall that $S(m, r) = \Phi(\text{GL}(m, \mathbb{C}))$ with $\text{GL}(m, \mathbb{C}) = \text{GL}(W)$ associated to a given basis $\eta_i, i = 1, \ldots, m$ of $W \cong \mathbb{C}^m$. Suppose $\gamma = w(\lambda)$ for $w \in \mathcal{W}$ and $\lambda \in \Lambda(m, r)$. Each $\mathcal{W}$-orbit of $\Lambda(m, r)$ contains only one dominant weight $\gamma = (\gamma_1, \ldots, \gamma_m)$ which satisfies $\gamma_1 \geq \gamma_2 \geq \cdots \geq \gamma_m$. Denote by $\Lambda^+(m, r)$ the set of all dominant weights. Then $\Lambda^+(m, r)$ exactly consists of all partition of $r$ admitting at most number $m$ of nonzero parts, coinciding with $\text{Par}(r, m)$.

Thus the isomorphism classes of irreducible $S(m, r)$-modules are parameterized by $\Lambda^+(m, r)$.

For $\lambda \in \Lambda(m, r)$, the representative of the corresponding isomorphism classes of irreducibles is denoted by $\tilde{D}_\gamma$ where $\gamma = w(\lambda) \cap \Lambda^+(m, r)$. Denote by $\ell(\gamma)$ the number of different elements in the $\mathcal{W}$-orbit of $\gamma$. Then there is a direct sum decomposition of irreducible modules for $S(m, r)$:

$$S(m, r) \cong \bigoplus_{\gamma \in \Lambda^+(m, r)} \tilde{D}_{\gamma}^{\oplus \ell(\gamma)}. \quad (6.4)$$

6.1.2. Let us turn back to the enhanced Schur algebra $\mathcal{E}(n, r)$. For a given $s \in \mathbb{N}_r^m$ with $|s| = l \leq r$, $\tilde{s} = (1^{s_1}, \ldots, n^{s_n}(n + 1)^{r-l})$. By (S1)-(S3), $P_{\tilde{s}, \tilde{s}_l} = \mathcal{E}(n, r)[l]$ and $\xi_{\tilde{s}, \tilde{s}_l} \mathcal{E}(n, r)[l+1] = 0$. So $P_{\tilde{s}, \tilde{s}_l}$ can be decomposed into a direct sum of two subspaces

$$P_{\tilde{s}, \tilde{s}_l}' = \mathbb{C}\text{-span}\{\xi_{\tilde{s}, \tilde{s}_l} P_{l} \mid P_{l} \in \mathcal{P}^l\} \quad (6.5)$$

\(^2\text{short for Principal Indecomposable Modules.}\)
This implies that $D$. Furthermore, by the classical theory of Schur algebras, $P$.

The set $\Lambda$ is defined as before, the number of different elements in the $\mathcal{W}$-orbit of $\gamma$. For that, we need the following preliminary result which describes the decomposition $\mathcal{W}$-modules.

**Theorem 6.2.** The set $\Lambda := \{ \gamma \in \Lambda^+ (n, l) \mid l = 0, \cdots, r \}$ parameterize both of the isomorphism classes of irreducible $\mathcal{E}(n, r)$-modules and of indecomposable projective $\mathcal{E}(n, r)$-modules. The corresponding irreducible (resp. indecomposable projective) modules are $D_\gamma$ (resp. $P_\gamma$) with $\gamma \in \Lambda$.

**6.2. Cartan invariants and block degeneracy** for $\mathcal{E}(n, r)$. By the above arguments, the $P_\gamma$'s with $\gamma \in \bigcup_{l=0}^{r} \Lambda^+ (n, l)$ form all PIM's for $\mathcal{E}(n, r)$, up to isomorphisms. In this subsection, we will precisely determine the Cartan invariants for the enhanced Schur algebra $\mathcal{E}(n, r)$. For that, we need the following preliminary result which describes the decomposition factors appearing in $Q_{\pi_{\gamma}}$ defined in (6.6) and their multiplicity.

**Lemma 6.3.** Keep the notations as before. Then for any given $s \in \mathbb{N}_r^n$ with $1 \leq |s| = l \leq r$, the following equality holds in the Grothendieck group for the $\mathcal{E}(n, r)$-module category.

$$[Q_{\pi_{\gamma}}] = \sum_{i=0}^{l-1} \sum_{\gamma \in \Lambda^+ (n, i)} \ell(\gamma) [D_\gamma] = \sum_{i=0}^{l-1} [S(n, i)],$$

where $\ell(\gamma)$ is defined as before, the number of different elements in the $\mathcal{W}$-orbit of $\gamma$.

**Proof.** The second equation of (6.7) is derived from (6.4). We only need to prove the first one.

Recall

$$Q_{\pi_{\gamma}} = \mathbb{C}\text{-span}\{ \xi_{\pi_{\gamma}, q_{(n+1)}^{s-1}} \mid q_{l} \in \mathbb{N}^d, \text{rk}_{\mathbb{Z}}(q_{l}) < l \}. $$

Set

$$Q_{\pi_{\gamma}}^{(l)} = \mathbb{C}\text{-span}\{ \xi_{\pi_{\gamma}, q_{(n+1)}^{s-1}} \mid q_{l} \in \mathbb{N}^d, \text{rk}_{\mathbb{Z}}(q_{l}) \leq l - i \} \text{ for } 1 \leq j \leq l.$$ 

By (S1)-(S3), each $Q_{\pi_{\gamma}}^{(l)}$ is a $\mathcal{E}(n, r)$-submodule of $Q_{\pi_{\gamma}}$, and we have the following decreasing sequence of $\mathcal{E}(n, r)$-modules:

$$Q_{\pi_{\gamma}} = Q_{\pi_{\gamma}}^{(1)} \supset Q_{\pi_{\gamma}}^{(2)} \supset \cdots \supset Q_{\pi_{\gamma}}^{(l)} \supset 0.$$
Furthermore,

\[ \bigoplus_{j=0}^{r} \mathcal{E}(n,r)_j \]

acts trivially on \( Q^{(i)}_{\pi_s}/Q^{(i+1)}_{\pi_s} \) and \( Q^{(i)}_{\pi_s}/Q^{(i+1)}_{\pi_s} \cong S(n, l - i) \) as \( S(n, l - i) \)-modules. Consequently, \( Q^{(i)}_{\pi_s}/Q^{(i+1)}_{\pi_s} \cong S(n, l - i) \) as \( \mathcal{E}(n, l - i) \)-modules because \( S(n, l - i) \) is a \( \mathcal{E}(n, r) \)-module. Furthermore, by (6.4) we have

\[ [Q^{(i)}_{\pi_s}/Q^{(i+1)}_{\pi_s}] = \sum_{\gamma \in \Lambda^+(n, l - i)} \ell(\gamma)[D_{\gamma}] \]

Consequently,

\[ [Q_{\pi_s}] = \sum_{i=1}^{l} [Q^{(i)}_{\pi_s}/Q^{(i+1)}_{\pi_s}] = \sum_{i=1}^{l} \sum_{\gamma \in \Lambda^+(n, l - i)} \ell(\gamma)[D_{\gamma}] = \sum_{i=0}^{l-1} \sum_{\gamma \in \Lambda^+(n,i)} \ell(\gamma)[D_{\gamma}] \]

The proof is completed. \( \Box \)

As usual, we denote by \( (P_{\gamma} : D_{\gamma'}) \) the multiplicity of the irreducible module \( D_{\gamma'} \) appearing in composition series of the indecomposable projective module \( P_{\gamma} \) for \( \gamma, \gamma' \in \bigcup_{i=0}^{r} \Lambda^+(n, l) \). We are now in the position to determine the Cartan invariants for the enhanced Schur algebra \( \mathcal{E}(n, r) \).

**Theorem 6.4.** Let \( a_{\gamma, \gamma'} := (P_{\gamma} : D_{\gamma'}) \) for any \( \gamma, \gamma' \in \Lambda^+ := \bigcup_{i=0}^{r} \Lambda^+(n, l) \). Then

\[
a_{\gamma, \gamma'} = \begin{cases} 
1, & \text{if } \gamma' = \gamma; \\
\ell(\gamma'), & \text{if } |\gamma'| < |\gamma|; \\
0, & \text{otherwise.} 
\end{cases} \tag{6.8}
\]

Consequently, the Cartan matrix \( (a_{\gamma, \gamma'})_{n \times n} \) is an invertible and upper triangular one with diagonal entries being 1, where \( n = \#\Lambda^+ \).

**Proof.** If \( \gamma \in \Lambda^+(n, 0) \), i.e., \( \gamma = 0 = (0, \ldots, 0) \), then \( P_0 = D_0 \) is irreducible, and the assertion (6.8) holds for \( \gamma = 0 \) and any \( \gamma' \in \Lambda \). In the following, we assume \( \gamma \in \Lambda^+(n, l) \) with \( 1 \leq l \leq r \). By the discussion in §6.1.2, \( P_{\gamma} \) has a unique maximal submodule \( Q_{\gamma} \) with \( P_{\gamma}/Q_{\gamma} \cong D_{\gamma} \). Hence, it follows from Lemma 6.3 that

\[ [P_{\gamma}] = [P_{\gamma}/Q_{\gamma}] + [Q_{\gamma}] = [D_{\gamma}] + \sum_{i=0}^{l-1} \sum_{v \in \Lambda^+(n,i)} \ell(\gamma)[D_v]. \]

This implies the assertion (6.8). The other assertions are obvious. The proof is completed. \( \Box \)
As a direct consequence of Theorem 6.4, we have the following result on block structure of the enhanced Schur algebra.

**Corollary 6.5.** The enhanced Schur algebra $E(n, r)$ has only one block.

The block structure is closely related to the center for a finite-dimensional algebra (see [3, §1.8]). As to the latter, we have the following observation.

**Proposition 6.6.** When $D(n, r)^V = C\Psi(\mathfrak{g}_r)$, the center of $E(n, r)$ is one-dimensional.

**Proof.** Denote by $C$ the center of $E(n, r)$. Note that $E(n, r) \subset \text{End}_C(V^\otimes r)$. We have $C \subset \text{Cent}_{E(n, r)}(\text{End}_C(V^\otimes r))$.

Hence $C = \text{End}_{E(n, r)}(V^\otimes r) \cap E(n, r)$. When $D(n, r)^V = C\Psi(\mathfrak{g}_r)$, $C = C\Psi(\mathfrak{g}_r) \cap E(n, r)$ by Theorem 5.3. Recall $E(n, r)$ has a basis $\{\xi_{\tilde{s}, j} \mid (\tilde{s}, j) \in E\}$ (Theorem 2.11). For any given nonzero $\phi \in C$, $\phi$ can be written in two forms

$$\phi = \sum_{(\tilde{s}, j) \in E} a_{\tilde{s}, j} \xi_{\tilde{s}, j} \quad (6.9)$$

and

$$\phi = \sum_{\delta \in \mathfrak{g}_r} b_{\delta} \Psi(\delta). \quad (6.10)$$

We refine (6.10) as $\phi = \sum_{\delta \in S} b_{\delta} \Psi(\delta)$ where $S \subset \mathfrak{g}_r$ and all $b_{\delta} \neq 0$ for $\delta \in S$. Then we finally write (6.10) as

$$\phi = \sum_{\delta \in S} b_{\delta} \sum_{s \in N^p_n} \xi_{\tilde{s}, \delta, \tilde{s}} = \sum_{s \in \mathbb{N}^p_n} \sum_{\delta \in S} b_{\delta} \xi_{\tilde{s}, \delta, \tilde{s}}.$$

If $S \neq \{\text{id}\}$, then we can take $\delta (\neq \text{id}) \in S$. In this case, there certainly exists $s \in \mathbb{N}^p_n$ such that $(\tilde{s}, \delta(\tilde{s})) \notin E$. This contradicts with (6.9).

So it is only possible that $S$ coincides with $\{\text{id}\}$. Consequently, $\phi \in C\text{id}$. Hence $C = C\text{id}$. The proof is completed. 

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