Entangled states and entropy remnants of a photon–electron system

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Abstract
In the present paper, an example of entanglement between two different kinds of interacting particles, photons and electrons is analysed. The initial-value problem of the Schrödinger equation is solved non-perturbatively for the system of a free electron interacting with a quantized mode of electromagnetic radiation. Wave packets of the dressed states so obtained are constructed in order to describe the spatio-temporal separation of the subsystems before and after interaction. The joint probability amplitudes are calculated for the detection of the electron at some space–time location and the detection of a definite number of photons. The analytical study of the time evolution of entanglement between the initially separated electron wave packet and the radiation mode leads to the conclusion that in general there are non-vanishing entropy remnants in the subsystems after the interaction. On the basis of the simple model to be presented here, the calculated values of the entropy remnants crucially depend on the character of the assumed switching-on and -off of the interaction.

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(Some figures in this article are in colour only in the electronic version.)

1. Introduction

The concept of entanglement has long been introduced by Schrödinger (1935), who was motivated by the criticism expressed by Einstein et al (1935) on the conceptual foundation of quantum mechanics, which at that time drew attention to some seemingly paradoxical features associated with the new theory. Concerning this kind of fundamental questions we refer the reader, for example, to the book by Bohm (1951) and the collection of excellent papers by Bell (2004). More than 30 years after the first reliable experiments by Wu and Shaknov (1950), Aspect et al (1982) measured correlations between entangled photon pairs, proving a strong violation of Bell’s inequalities, in agreement with the predictions of quantum theory. In the meantime, it turned out that entanglement plays a crucial role in the current rapidly developing branches of science, namely in quantum information theory (see e.g. Alber et al (2001), Bouwmeester et al (2001), Stenholm and Suominen (2005)) and in quantum communication and quantum computing (see e.g. Benjabballah et al (1990), Williams (1999), Nielsen and Chuang (2000)).

Besides entanglement between particles of the same kind with some discrete degrees of freedom, there has recently been growing interest in the study of continuous-variable entanglement between different kinds of particles. For instance, Fedorov and co-workers in a series of papers (see Fedorov et al (2004), (2005), (2006), (2007)) have thoroughly analysed from this point of view the wave packet dynamics in breakup processes, such as the ionization of atoms and dissociation of molecules. To some extent, the subject of the present paper belongs to the category of processes, in that we shall deal with the dynamics of free electron wave packets accompanied by photon exchange. The photon–electron interaction in free space is the most fundamental phenomenon of quantum electrodynamics (QED), and the nature of the very-high-order multiphoton processes is still the subject of extensive research today. In the following, we shall study entanglement in high-order photon emission and absorption processes at the simplest imaginable level, namely, we stay in the framework of non-relativistic quantum mechanics, where the interaction is modelled by the minimal coupling term in dipole approximation. In the case of a free electron the single-mode description of the interacting radiation may be justified in that we shall take a highly occupied mode (which, in turn, may also be considered as representative of an assembly of modes in a narrow spectral

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range). It should also be stressed that the discussion on the spatial separation of the two subsystems before and after the interaction will rely, to a large extent, on introducing a phenomenological switching-on and -off of the interaction.

Besides quantum optics, the subject of the present paper belongs also to the 'physics of strong-field phenomena'. Perhaps it is not superfluous to quote some basic references in this field, at least concerning the mathematical description of high-order interactions of quantized radiation and an electron. The first reference to be quoted is undoubtedly the paper by Bloch and Nordsieck (1937), in which they have given the first non-perturbative analytic treatment of the interaction of a free electron with the whole assembly of the quantized radiation field (in the 'soft-photon limit') and in this way solved the problem of infrared divergences appearing in perturbative QED (see also Bogoliubov and Shirkov (1959)). The exact solution of the Dirac equation of the system consisting of an electron and a single quantized plane wave mode of the radiation field was first presented by Bersons (1969), which meant that he constructed a generalization of the famous Volkov (1935) states, which are exact solutions of the Dirac equation of an electron in the presence of an external plane wave of the radiation field (of arbitrarily high intensity). These states have played a very important role in the theory of strong-field interactions from the beginning (just to mention some early basic works, see e.g. Brown and Kibble (1964) on nonlinear Compton scattering, Bunkin and Fedorov (1965) on optical tunnelling and Ritus and Nikishov (1979) on pair creation in a strong laser field). Concerning the strong-field physics today in general, we refer the reader to recent review papers by Mourou et al (2006) on relativistic optics, Ehlotzky et al (2009) on relativistic scattering in strong laser fields and Krausz and Ivanov (2009) on attosecond physics.

The important point for us here is that, although the initial and final dressed states of the free electron embedded in the quantized radiation field have been used so far, these dressed states have exclusively been taken as product states of an electron momentum eigenstate and the state of the photon field. In the frame of such a description, one cannot consider entanglement. One immediately encounters entangled photon–electron states if one constructs wave packets of the dressed states in order to describe more realistically the spatio-temporal dynamics of the interaction process. In a recent paper (Varró (2008), henceforth referred to as I), we have constructed Gaussian wave packets (with respect to the electron’s momentum) of the stationary states of the system of an electron and a quantized mode of the electromagnetic radiation. On the basis of this exact analytic treatment, the essential characteristics of the system (e.g. photon number distribution, von Neumann entropy and linear entropy) were determined. However, these states are not solutions of a true initial value problem; rather they are merely joint stationary states, leaving the following question unanswered: How could these states be generated? The present work is devoted to the discussion of this problem in the simplest framework.

In section 2, we present the exact solutions of the Schrödinger equation of the interacting photon–electron system developing from a product state parametrized by the electron’s initial momentum and by the initial occupation number of the quantized mode. In section 3, we shall construct Gaussian wave packets of these solutions, which are normalized entangled states developing from an uncorrelated product state of an electron wave packet and a number state. In section 4, the reduced density operators shall be derived for these states, with the help of an analysis of the time evolution of the entropies. In section 5, a brief summary of the results and conclusions closes our paper.

2. Solution of the initial value problem of the interacting photon–electron system

Let us consider the joint system consisting of a quantized mode of the radiation field and a Schrödinger electron. For the sake of simplicity, we take a circularly polarized plane wave for the mode, because in this case the \( A^2 \) term of the Hamiltonian is diagonal in the photon number state basis. Moreover, we shall use dipole approximation, and consider electronic motions in the \( x-y \) plane (the electron’s motion along the \( z \)-direction is a simple free propagation). The Hamiltonian of the system reads

\[
H = \left[ \frac{1}{2m} \left( \frac{\hat{\mathbf{p}}^2}{c} + eA \right)^2 + \hbar \omega \left( A^+ A + 1/2 \right) \right]
\]

\[
= \frac{\hat{\mathbf{p}}^2}{2m} + \hbar \omega \left( A^+ A + 1/2 \right) + \frac{ea}{mc} \cdot \hat{\mathbf{p}} \cdot \left( \hat{\mathbf{A}} + \hat{\mathbf{A}}^+ \right),
\]

\[
\hat{\mathbf{A}} = a(\hat{\mathbf{A}} + \hat{\mathbf{A}}^+), \quad \mathbf{a} \equiv (2\pi \hbar c^2/\omega L^3)^{1/2},
\]

\[
\Omega \equiv \omega \left( 1 + \frac{\omega_p^2}{2\omega^2} \right), \quad \omega_p^2 = 4\pi e^2/mL^3.
\]

In the above equations, \( \hat{\mathbf{A}} = (\hat{\mathbf{E}}_x + i\hat{\mathbf{E}}_y)/\sqrt{2} \) is the complex polarization vector (for right circular polarization, when the field is assumed to propagate in the negative \( z \)-direction), \( \omega \) is the circular frequency of the mode, and \( L^3 \) is the quantization volume. The photon creation and annihilation operators are denoted by \( A^\dagger \) and \( A \), respectively, and \( A A^\dagger - A^\dagger A = \hat{\mathbf{1}} \) photon, where the right-hand side is the unit operator of the Hilbert space of the photon states. The parameters \( -e, m \) and \( c \) are the electron’s charge, mass and the velocity of light in vacuum, respectively, and \( \hbar \) denotes Planck’s constant divided by \( 2\pi \). Note that \( \omega_p \) is formally nothing else but the plasma frequency of an electron gas of density \( 1/L^3 \). Even in the case \( L \sim \lambda = 2\pi c/\omega \), the ratio \( \omega_p/\omega \) is much smaller than unity for usual electron beams and for optical frequencies; however, in the case of terahertz radiation it may not be neglected. In obtaining the right-hand side of equation (1) we have taken into account that \( \hat{\mathbf{E}}^\dagger \cdot \hat{\mathbf{E}} = 1 \) and \( \hat{\mathbf{E}} \cdot \hat{\mathbf{E}}^\dagger = \hat{\mathbf{E}} \cdot \hat{\mathbf{E}} = 0 \). The solutions to the Schrödinger equation \( i\hbar \partial_t \Psi(t) = H \Psi(t) \), describing the time evolution of the system with the above Hamiltonian, have been presented by Bergou and Varró (1981a, 1981b). On the basis of this result, the explicit form of the solutions can be brought to the form

\[
|\Psi(t)\rangle = \exp \left[ -\frac{i}{\hbar} \frac{\hat{\mathbf{p}}^2}{2m(t)} t - i\Omega (A^+ A + 1/2) t \right]
\]

\[
\cdot D[\hat{\mathbf{A}}(\hat{\mathbf{p}}, t)] \cdot |\Psi(0)\rangle,
\]

where

\[
D[\hat{\mathbf{A}}(\hat{\mathbf{p}}, t)] = \left[ \frac{\hbar}{2\omega(t)} \right]^{1/2} \cdot \exp \left[ -\frac{i}{\hbar} \frac{\hat{\mathbf{p}}^2}{2m(t)} \right] \cdot \exp \left[ -\frac{i}{\hbar} \frac{e}{mc} \cdot \hat{\mathbf{p}} \cdot \left( A^+ A^\dagger + A A^\dagger \right) \right].
\]
where

\[ D(\vec{p}, t) \equiv \exp[\hat{\sigma}^* (\vec{p}, t) A - \hat{\sigma} (\vec{p}, t) A^*], \]

\[ m(t) \equiv \frac{1 + \omega_p^2/2\omega^2}{1 + (\omega_p^2/2\omega^2)(2\sin \Omega t)/\Omega} m, \quad (4) \]

and

\[ \hat{\sigma} (\vec{p}, t) \equiv -\left( \vec{p} \cdot \vec{e}^* \right) \frac{ea}{mch\Omega} (\hat{e}^{\Omega t} - 1). \quad (5) \]

Since in realistic cases \( \omega_p/\omega \ll 1 \), the dressed mass \( m(t) \) is practically identical to the bare mass \( m \). Anyway, it is interesting to note that, in principle, this effective mass \( m(t) \) can be much larger than the bare mass (if \( \omega \) approaches \( \omega_p/\sqrt{2} \) from above); moreover, it can also be negative (if \( \omega < \omega_p/\sqrt{2} \)), which corresponds to negative energies or, formally, imaginary momenta. As a result, instead of the usual spreading of an electronic wave packet, a contraction would take place. Similar phenomena can certainly be more realistic, e.g. in the case of the scattering of resonant atoms off the vacuum field at the entrance to a microwave cavity, as was discussed by Scully et al. (1996). In the present paper we shall not discuss the question of the possible physical relevance of the formal appearance of the negative mass, and henceforth (after equation (6)) we shall denote \( m(t) \) and \( \Omega \) simply by \( m \) and \( \omega \), respectively. As is well known, the displacement operators of the kind displayed in equation (4) have an important role in the quantum theory of optical coherence and coherent states, as was first shown by Glauber (1963).

The solution given by equation (3) looks particularly simple if the initial state is a product of a momentum eigenstate and a generalized coherent state of the quantized mode, i.e. if \( |\Psi(0)\rangle = |\vec{p}\rangle \otimes D(\sigma|n_0\rangle |\alpha\rangle \), then

\[ |\Psi(t)\rangle = |\vec{p}\rangle e^{-i\hbar (p^2/2m)(t)} \otimes D(\sigma|n_0\rangle \otimes D(\sigma|\alpha\rangle e^{-i\Omega^2 t} \cdot D(\alpha|n_0\rangle e^{-i(n_0t^2/2\Omega)} , \quad (6) \]

where \( \sigma \) is proportional to the initial complex amplitude of the expectation value of the electric field strength of the mode. Because of the property \( D(\sigma)D(\alpha) = D(\sigma + \alpha)\exp[i\hbar(\sigma\alpha^*)] \) of the displacement operators, the state has the structure \( |\vec{p}\rangle \otimes D(\beta)|n_0\rangle \), thus it is still a product state for \( t > 0 \), too. If \( \alpha = 0 \) and \( n_0 = 0 \), then the initial vacuum state \( |0\rangle \) goes over to the ordinary coherent state \( |\sigma\rangle \); thus one may say that (at least, according to the present very simplified description) the self-radiation field of the free electron is in a coherent state. We also note that, owing to the unitarity of the displacement operators, the exact solutions given by equation (6) form a complete orthogonal set on the product Hilbert space \( H_{\text{photon}} \otimes H_{\text{electron}} \) for any parameter \( \alpha \), i.e.

\[ \{\Psi(t)\} \quad (\forall \alpha) \]

\[ \int d^3p \sum_{n=0} \langle \Psi(t)|\vec{p}, n, \alpha\rangle \langle \Psi(t)|\vec{p}, n, \alpha\rangle = 1_{\text{photon}} \otimes 1_{\text{electron}} \quad (7a) \]

The photon statistics of the generalized coherent state of the type \( D(\sigma|n\rangle \) is determined by the matrix elements

\[ c_{l,n},n = \langle k|D(\sigma)|n\rangle = \begin{cases} \langle n!k!\rangle^{1/2} \kappa^{-n} L_n^{\kappa-n}(\sigma^2) e^{-|\sigma|^2/2} & (k \geq n), \\ \langle k!n!\rangle^{1/2} (-\sigma^2)^n L_n^{\kappa-n}(\sigma^2) e^{-|\sigma|^2/2} & (0 \leq k < n), \end{cases} \quad (8) \]

where \( L_n^\kappa \) denotes generalized Laguerre polynomials (see e.g. Gradsteyn and Ryzhik (2000), formula 8.970.1). To our knowledge, the matrix elements of the type given by equation (8) were first obtained by Bloch and Nordsieck (1937), who applied them to the case of \( n = 0 \), i.e. all modes were considered initially in a vacuum state, and then the now well-known coherent states with the associated Poisson photon number distributions \( |\sigma|^{2n}\exp((-|\sigma|^2)/k! \) resulted. In the opposite case of large excitations \( (n \gg 1) \) these matrix elements can be brought to a more tractable form

\[ \langle n_0 + k | D(\vec{p}, t) e^{-i\omega t} | n_0 \rangle \]

\[ = e^{-i(k\omega t - \omega n)} J_k \left( \sqrt{2m\sigma|\vec{p}| t} \right) + O(n_0^{-3/4}), \]

\[ = e^{-i(k\omega t - \omega n)} J_k \left( \sqrt{2m\sigma|\vec{p}| t} \right) + O(n_0^{-3/4}), \quad (9) \]

where we have used an asymptotic formula of Hüb's type (see e.g. Erdélyi (1953), formula 10.15(2), or equations (A14) and (A15) in I). In equation (9) \( J_k(z) \) denotes ordinary Bessel functions of the first kind of order \( k \), and we have introduced the azimuth angle \( \chi \) of the momentum vector by the definition \( \vec{p} = p(\cos \chi, \sin \chi) \). We have also introduced the quantity \( A_0 = (c/\omega)\sqrt{2\pi \rho \hbar \omega} \), by taking the definition of \( \sigma \) in equation (5) into account. In fact, \( A_0 \) is equivalent to the amplitude of the classical vector potential \( A_0 = A_0(\vec{e} e^{-i\omega t} + \vec{e}^* e^{i\omega t}) \) associated with the photon density \( \rho = n_0/L^3 \), if we make the identification \( u = E_0^2/4\pi = \rho \hbar \omega \). Here \( u \) denotes the energy density of the mode, with \( E_0 = -\hat{\mathcal{A}}_{cl}/\sqrt{\hbar c} = F_0(\vec{E}_0, \sin \omega t - \vec{E}_0 \cos \omega t) \) being the electric field strength with amplitude \( F_0 = \langle \langle 0| c\rangle A_0 \rangle \sqrt{2} = \sqrt{4\pi \rho \hbar \omega} \) (which may contain a slow time dependence). In equation (9) the function \( |h(t)| \) represents a slow time dependence of the modulus of \( \sigma(\vec{p}, t) \). More precisely, \( h(t) \) is defined by the equation

\[ h(t) \equiv |h(t)|e^{i\omega t} \equiv \omega^2 \int_0^t dt' \int_0^{t'} dt'' f(t'')e^{i\omega(t'' - t')}, \quad (10) \]

where \( f(t) \) is a dimensionless switching function of the electric field, i.e. \( E_0(t) = F_0 f(t) \). In obtaining equation (10) we have taken into account the classical relationship \( E = -\partial A/\partial ct \).

In fact, the present analysis to follow can also be applied for the consideration of the more general case when the electric field strength has the form \( E = F_0 f(t + z/c)\hat{E}_0, \sin \omega(t + z/c) + \hat{E}_0 \cos \omega(t + z/c) \), where \( f \) is an envelope function. For motions of the electron restricted to the \( z = 0 \) plane the envelope function may be considered as modelling the switching-on and -off of the interaction. Of course, with the use of a phenomenological switching function, one has to keep in mind that this procedure, according to the usual Fourier description, implicitly assumes excitations of an assembly of modes in some spectral range \( \Delta \nu \) around the central frequency.
At the end of the present section, we note that the matrix elements given by equation (9) can be derived from the semiclassical Schrödinger equation of an electron interacting with an external field \( E_0(t) \) given above. Really, if we take as a solution a momentum eigenstate, then this contains a factor with periodic phase modulation, and this factor can be decomposed into the Fourier series, by using the Jacobi–Anger formula (see e.g. Erdélyi (1953), formula 7.2.4 (26), or equation (A4) in I),

\[
\exp \left\{ -i(\mu(t)p + \hbar k)\sin(\omega t - \chi - \eta) \right\} = \sum_{k=-\infty}^{\infty} J_k(\mu(t)p) e^{-ik(\omega t - \chi - \eta)},
\]

where

\[
\begin{align*}
\mu(t) & \equiv \mu_0|b(t)|, \\
\mu_0 & \equiv eA_0\sqrt{2/mc^2} \equiv eF_0/mc\omega, \\
F_0 & \equiv \sqrt{4\pi(n_0/L^3)}\hbar\omega.
\end{align*}
\]

In equation (12) we have defined the ‘dimensionless intensity parameter’ \( \mu_0 \), which plays an important role in strong-field physics. Its numerical value can be expressed in terms of the peak intensity \( I \) of the radiation field measured in \( \text{W cm}^{-2} \), and of the photon energy \( E_\text{ph} \) measured in eV. According to the above discussion, we have also introduced the amplitude of the electric field strength \( F_0 \equiv (\omega/c)A_0\sqrt{2} = \sqrt{4\pi(n_0/L^3)}\hbar\omega \), where the photon density is defined as \( \rho = n_0/L^3 \) (kept fixed as both \( n_0 \) and \( L \) go to infinity). In this way, the Fourier coefficients in equation (11) coincide, with an accuracy of order \( n_0^{-3/4} \), with the matrix elements given by equation (9). Consequently, in the large-intensity limit we are allowed to use the ‘quasi-classical’ formula

\[
\langle n_0 + k | D[\sigma(\vec{\rho}, t)e^{-i\omega t}] | n_0 \rangle \to J_k \left( 2\sqrt{n_0} |\sigma(\vec{\rho}, t)| \right) e^{-ik(\omega t - \chi - \eta)} = J_k(\mu(t)p) e^{-ik(\omega t - \chi - \eta)},
\]

where the notations are the same as in equations (12) and (9).

3. Entangled photon–electron states in the case of high initial photon excitations

The entangled photon–electron states developing from a pure initial state being the product of a number state and an electron wave packet is taken in the form

\[
|\Psi_g(t)\rangle \equiv \int d^2p g(\vec{p} - \vec{p}_0) e^{-i(\hbar/h)p \cdot \vec{r}_0} |\Psi(t) |\Psi_g(t), n_0, 0\rangle, \\
g(\vec{p}) \equiv g(p) = (w/h\sqrt{\pi}) \exp \left( -p^2 w^2/2h^2 \right),
\]

where \( g \) has been specialized to a Gaussian weight function of width \( w \), and \( |\Psi(t) |\Psi_g(t), n_0, 0\rangle \) is a special case of the state given by equation (6) with \( \alpha = 0 \). Owing to the orthonormality property displayed in equation (7a), these time-dependent packets are normalized, too. The physical situation with which the state given by equation (14) may be associated is the following. At time \( t = 0 \), the electron is located sharply around the central position \( \vec{r}_0 \) with an initial drift momentum \( \vec{p}_0 \), and it is injected into the interaction region (or exposed to the radiation) which is swept by a light pulse propagating along the \( z \)-direction. The longitudinal motion of the electron is assumed to be very slow, and it is also assumed that it stays close to the \( z = 0 \) plane. Since the coupling of the \( z \)-motion is negligible in this case, we consider simply a planar motion in the \( x-y \) plane.

In order to have an explicit form of the state defined by equations (14) and (6), we express it in the electron’s coordinate representation and, at the same time, expand it in terms of the photon number eigenstates. Thus we write

\[
|\Psi_g(t)\rangle = \int d^2r |\Xi(\vec{r}, t)\rangle \langle \vec{r} | = \sum_{k=-\infty}^{\infty} |\Phi_k(t)\rangle |n_0 + k\rangle,
\]

where

\[
|\Xi(\vec{r}, t)\rangle \equiv \sum_{k=-\infty}^{\infty} |\Phi_k(t)\rangle |n_0 + k\rangle, \\
|\Phi_k(t)\rangle = \int d^2r |\Psi_k(\vec{r}, t)\rangle \langle \vec{r} |.
\]

It is clear that the states \( |\Xi(\vec{r}, t)\rangle \in H_{\text{photon}} \) and \( |\Phi_k(t)\rangle \in H_{\text{electron}} \) satisfy the normalization conditions

\[
\int d^2r \langle \Xi(\vec{r}, t) | \Xi(\vec{r}, t) \rangle = 1, \quad \sum_{k=-\infty}^{\infty} \langle \Phi_k(t) | \Phi_k(t) \rangle = 1,
\]

where the scalar products, of course, are meant in the Hilbert spaces \( H_{\text{photon}} \) and \( H_{\text{electron}} \), respectively. The summation index in the above equations has been shifted merely for later convenience. According to equations (15) and (16), the wave packet solution \( |\Psi(t)\rangle \) can be expressed as a joint sum and an integral,

\[
|\Psi_g(t)\rangle = \sum_{k=-\infty}^{\infty} \int d^2r \Psi_k(\vec{r}, t) \langle \vec{r} | \Psi_g(t)\rangle |n_0 + k\rangle,
\]

and thus, this solution describes mixed continuous and discrete entanglement.

In order to characterize the entangled photon–electron states \( |\Psi_g(t)\rangle \), defined in equations (14) and (6), we have to determine the expansion coefficients expressed by scalar products in the second equation of equation (18). Having shifted the momentum variable by \( \vec{p}_0 \), the integral becomes

\[
\Psi_k(\vec{r}, t) \equiv \langle n_0 + k | \vec{r} | \Psi_g(t)\rangle \\
= \int_0^{\infty} dp g(p) \frac{1}{2\pi\hbar} \exp \left\{ -i[p^2 t/2m\hbar] + \phi \right\} \\
\times \int_0^{2\pi} d\chi \exp \left\{ i[p/\hbar] \cos(\chi - \psi(t)) \right\} \\
\times \langle n_0 + k | D[\sigma(\vec{\rho}_0 + \vec{p}_0, t)e^{-i\omega t}] | n_0 \rangle,
\]

where \( \Phi \equiv \vec{p}_0 \cdot (\vec{r} - \vec{r}_0)/\hbar - p_0^2 t/2m\hbar - (n_0 + 1/2)\omega t \), and we have introduced the polar decomposition of the shifted position by the relation \( r(t)(\cos \psi(t), \sin \psi(t)) \equiv \vec{r} - \vec{r}_0 - (\vec{p}_0/m)t \). The displacement operator in equation (19) can be factorized into the product of the displacement operators (the
additional phase factor goes to zero in the quasi-classical limit) containing only \( \vec{p}_0 \) and \( \vec{p} \). Then, by inserting the unit operator of the Hilbert space of the mode in the suitable form, applying the general formula given by equation (13), and by using the same integration method as in appendix A in I, we obtain

\[
\Psi_\text{SI}(\vec{r}, t) = \frac{e^{i\delta}}{w\sqrt{\pi}} \exp \left\{ -\left( \frac{(\mu(t)\vec{\lambda}/w)^2 + (r(t)/w)^2}{2(1 + it/\tau)} \right) \right\} \\
\times \sum_{l=-\infty}^{\infty} \frac{I_k(l)}{(1 + it/\tau)} \\
\times J_l[|\mu(t)\vec{p}_0c/h\omega|]e^{il[\phi + \varphi(t)]},
\]

where

\[
1/\tau \equiv \hbar/m w^2, \quad \mu(t) \equiv \mu_0 |h(t)|, \\
r(t)\cos(\varphi(t)), \sin(\varphi(t)) \equiv \vec{r} - \vec{r}_0 = (\vec{p}_0/m)t, \\
\vec{\lambda} \equiv \lambda/2\pi = c/\omega.
\]

In equation (20), \( I_n(z) \) denotes a modified Bessel function of the first kind of order \( n \) (see section 7.2 in Erdélyi (1953)), and the polar representation \( p_0(c\cos(\chi_0), \sin(\chi_0)) \equiv \vec{p}_0 \) has been introduced. The characteristic time \( \tau \) of the spreading of the wave packet strongly depends on the initial width \( w \) as is well known. According to the definition given by equation (19), the space–time functions \( \Psi_\text{SI}(\vec{r}, t) \) have a clear physical meaning, namely, they are joint probabilities of the simultaneous events when the electron is at position \( \vec{r} \) and at the same time \( k \) ‘excess photons’ are excited or de-excited around the large mean photon number \( n_0 \). At the start of the interaction at \( t = 0 \) the coupling is zero, i.e. \( \mu(0) = 0 \), and from equation (20) we see that these functions reduce the simple form \( \Psi_\text{SI}(\vec{r}, 0) = \delta_{k0}e^{i\delta}\phi_\text{SI}(\vec{r}, t) \), where \( \phi_\text{SI} \) is a freely evolving electron wave packet. This means that we recover the initial state \( |\Psi(0)\rangle \) in accord with the definition in equation (18). The expansion coefficients given by equation (20) are time-dependent generalizations of the stationary ones, presented in equation (22) in I. Here we allow non-vanishing \( \vec{p}_0 \neq 0 \) and \( \vec{r}_0 \neq 0 \), which is needed to describe localized propagation into the interaction region and the separation in space and time from that region. We may say that we have derived above a mathematical background on the basis of which we can take over the classical intuition in modelling the scattering process. From equation (20) we see that the coupling between the photons and the electron is governed by the intensity parameter \( \mu_0 = 10^{-6}f^{1/2}/E_{\text{ph}} \), which has already been defined in equation (12). The (presumably slowly varying) envelope function \( h(t) \) (for the definition see equation (10)) describes the details of the switching-on and -off of the interaction between the electron and the radiation field. It can be shown that the normalization conditions in equation (17) are satisfied if the expansion coefficients are that in equation (20), just obtained. This means that the proper normalization ‘survives’ the quasi-classical limit.

In the special case when \( \vec{p}_0 = 0 \), \( |\hbar(t)| = 1 \) and \( \varphi = 0 \), the photon states \( |\Sigma(\vec{r}, t)\rangle \) determined with the expansion coefficients in equation (20) have a close connection with a class of number-phase minimum uncertainty states. The states of the mode that minimize the uncertainty product of the photon number and the Susskind and Glogower (1964) cosine operator have been obtained by Jackiw (1968). Concerning this point we refer the reader to sections 2 and 4 of I (in particular see equation (27) of I).

At the end of the present section we would like to note that the coupling constant \( \mu \) has very transparent physical meanings. If one solves the Newton equations \( m\dot{x} = -eE_x \) and \( m\dot{y} = -eE_y \) of the electron in the presence of the external electric field \( \vec{E} = E_0(\vec{e}_x \sin\omega t + \vec{e}_y \cos\omega t) \), then it is easily seen that \( \mu \) is just the ratio of the amplitude of spatial oscillation to the wavelength of the radiation. If this ratio approaches unity, then the dipole approximation is not applicable in the case of a free electron. On the other hand, since \( \mu \) is the ratio of the velocity amplitude of oscillation to the velocity of light, it is clear that the nonrelativistic description is not justified if \( \mu \) gets close to (or above) unity. The dimensionless quantity \( \mu\lambda/\omega \) in the expression of the joint probability \( \Psi_\text{SI}(\vec{r}, t) \) in equation (20) is the ratio of the amplitude of classical oscillation of the electron to the initial transverse width of the wave packet. The classical oscillations mentioned above could have been derived and incorporated into the wave packet dynamics, if we had used the ‘\( \vec{r}, \vec{E}\)-gauge’ for the interaction, but from the point of view of entanglement this is not needed here. Concerning the choice of gauges in quantum optics we refer the reader to the excellent books by Loudon (2000), Scully and Zubairy (1997) and Schleich (2001) from the extensive literature.

4. Time evolution of the entropies of the photon–electron system and entropy remnants after the interaction

In this section, we shall determine the reduced density operator of the quantized mode of the radiation field interacting with the electron, and follow the dynamics of the interaction by assuming two kinds of switching.

In order to obtain the reduced density operator \( \hat{\rho} \) of the quantized mode associated with the entangled state \( |\Psi_\text{SI}(t)\rangle \), we use the expansion coefficients \( \Psi_\text{SI}(\vec{r}, t) \) given by equation (20), and the electron’s position representation to calculate the trace. The partial trace (denoted by \( \text{Tr}' \)) of the dyad \( |\Psi_\text{SI}(\vec{r}, t)\rangle\langle\Psi_\text{SI}(\vec{r}, t)| \) can be expressed as

\[
\hat{\rho} \equiv \text{Tr}' \left\{ \left| \Psi_\text{SI}(t) \right\rangle \langle \Psi_\text{SI}(t) \right\} = \sum_{k=-\infty}^{\infty} \sum_{l=-\infty}^{\infty} P_{kl} |n_0+k \rangle \langle n_0+l|,
\]

\[
P_{kl} \equiv \int d^3r \Psi_\text{SI}(\vec{r}, t)\Psi_\text{SI}^\dagger(\vec{r}, t).
\]

The integral on the right-hand side of the second equation in equation (22) can be evaluated analytically, yielding

\[
P_{kl} \equiv e^{-i(k-l)\omega} e^{-q} \sum_{n=-\infty}^{\infty} I_k(n) J_n(k) J_{n-(k-l)}(k),
\]

\[
q \equiv \frac{1}{2} \left[ \mu(t)(\lambda/2\pi w) \right]^2, \quad \kappa \equiv \frac{\mu(t)\vec{p}_0}{\hbar\omega}.
\]
where we have introduced the phase angle \( \alpha \equiv \omega t - \eta - \chi_0 \).
Since we are investigating the quasi-classical limiting case
\( (n_0 \rightarrow \infty) \), henceforth the lower limit of the sum will be taken as \(-\infty\). With the help of the sum rules \( \sum_{k=-\infty}^{\infty} I_k(z) = e^z \) and \( \sum_{n=-\infty}^{\infty} J_n^2(z) = 1 \), it can be easily proved that \( \text{Tr} \hat{P} = 1 \), so our quasi-classical analytic results obtained are consistent with the necessary requirement for probabilities. By taking the time average \( \bar{P}_{kl} \) of \( P_{kl} \) over one cycle, only the diagonal terms survive; thus from equation (23) we receive the result
\[
\bar{P}_{kl} = \delta_{kl} p_k, \quad p_k \equiv \sum_{n=-\infty}^{\infty} I_{k-n}(q)e^{-q} J_n^2(\kappa), \quad \sum_{k=-\infty}^{\infty} p_k = 1.
\] (24)

For small values of either of the arguments, from the general expression in equation (24) the probability distribution of the photon occupation number can be immediately derived
\[
p_k = I_k(q)e^{-q} \quad (\kappa \ll 1) \quad \text{and} \quad p_k = J_k^2(\kappa) \quad (q \ll 1).
\] (25)

We note that each limiting expression in equation (25) satisfies the proper normalization condition. If \( \bar{p}_0 = 0 \), then \( \kappa \) is exactly zero, and the first equation of equation (25) contains no approximation. For simplicity, in the numerical examples we have used the first approximate formula in equation (25).

On the basis of the diagonal expression in equation (24), we are able to write down immediately an explicit form of the von Neumann entropy of the photon field
\[
S_{\text{photon}}[\hat{P}] = -\text{Tr}[\hat{P} \log \hat{P}] = -\text{Tr}[\bar{P} \log \bar{P}] = S_{\text{photon}}[\{p_k\}] = -\sum_{k=-\infty}^{\infty} p_k \log p_k,
\] (26)

which can be calculated numerically. On the other hand, up to now we have not been able to diagonalize the electron’s density operator (which can also be calculated analytically).
A possibility to get around this difficulty and quantify in a simple way the entanglement is provided by the calculation of the linear entropy \( H \). The linear entropy has been used by several authors (see e.g. Zurek et al (1993) and Joos et al (2003)), because it is much easier to calculate, since the diagonalization of the density operator is not needed. The definition of \( H \) and its explicit form for the distribution in equation (24) read
\[
H = 1 - \text{Tr}(\hat{P}^2), \quad K = 1/\text{Tr}(\bar{P}^2),
\]
\[
H_{\text{electron}} = H_{\text{photon}}[\bar{P}] = H_{\text{photon}}[\{p_k\}]
= 1 - \sum_{k=-\infty}^{\infty} p_k^2
\]
\[
H = 1 - e^{-2q} \sum_{n,m=-\infty}^{\infty} I_{n-m}(2q)J_n^2(\kappa)J_m^2(\kappa).
\] (27)

In deriving the last equation, we have used the summation formula \( \sum_{n=-\infty}^{\infty} I_n(z)I_n(z) = I_0(2z) \) for the modified Bessel function. The Schmidt number \( K = 1/(1 - H) \geq 1 \) is closely related to the linear entropy, and in some cases it is better to visualize the degree of bipartite entanglement with the
dependence of this quantity on the input parameters. In the case when \( \kappa \ll 1 \), from equation (27) we obtain
\[
H = 1 - \bar{I}_0(2q)e^{-2q} \quad [p_k = I_k(q)e^{-q} \quad (\kappa \ll 1)],
\] (28)

and, on the other hand, in the case when \( q \ll 1 \), we have from equation (27)
\[
H = 1 - \sum_{k=-\infty}^{\infty} J_k^4(\kappa) \quad [p_k = J_k^2(\kappa) \quad (q \ll 1)].
\] (29)

In the numerical examples we have used the following two envelope functions for the electric field strength of the highly populated photon mode:
\[
f_1(t) = \sin[(\pi/T_1)(t - t_0)],
\]
\[
f_2(t) = \sin^2[(\pi/T_1)(t - t_0)] \quad (t_0 \leq t \leq t_0 + T_1).
\] (30)

In each case the interaction is limited to the interval \( t_0 \leq t \leq t_0 + T_1 \), out of which \( f_1 \) and \( f_2 \) are zero, such that they are continuous at the points of the switching-on and -off. The corresponding ‘interaction functions’ \( h_1(t) \) and \( h_2(t) \), according to the definition in equation (10), can be determined by elementary calculations.

The results of this section are numerically illustrated in some special cases in the following figures. In figure 1, the time dependence of the envelope functions, (a) \( f_1(t) \) and (c) \( f_2(t) \) defined in equation (30), are shown and in (b) \( |h_1(t)| \) and (d) \( |h_2(t)| \), the time dependence of the moduli of the ‘interaction function’ defined in equation (10) are shown.

**Figure 1.** Shows the time dependence of the envelope functions, (a) \( f_1(t) \) and (c) \( f_2(t) \) defined in equation (30), where the unit of time is the period \( T \) of the radiation mode. In each figure the pulse is switched-on at \( t = 10 \times T \), and switched-off at \( t = 31 \times T \); that is, the duration of the interaction is \( T_1 = 21 \times T \). Panels (b) \( |h_1(t)| \) and (d) \( |h_2(t)| \) show the time dependence of the moduli of the ‘interaction function’ defined in equation (10), by specializing the envelope functions to \( f_1(t) \) and \( f_2(t) \), respectively. On average \( |h_1(t)| \) increases with time; on the other hand, \( |h_2(t)| \) roughly follows the shape of its switching function.
Figure 2. The time evolution of the von Neumann entropies (a) $S_1$ and (c) $S_2$, and the Schmidt numbers (b) $K_1$ and (d) $K_2$ of the photon number distribution given by equations (26) and (27), respectively, in the special case when $\mu_0 = 0$, and for the numerical value $q_0 = 2$. The subscripts refer to the switching function used in the calculation. In the defining equation of $q$, equation (23), we have set $\mu(0) = \mu_0 \approx 10^{-1}$ and $\lambda/4\pi w = 10^6$. This means that for the optical radiation we have taken the wavelength $\lambda \approx 10^{-3}$ cm and the intensity $I = 10^{12}$ W cm$^{-2}$. For the initial width of the electron wave packet $w \approx 10^{-8}$ cm, i.e. 1 Å has been assumed. The time scales in this figure are the same as that in figure 1. As can be seen in these figures, in each case of the two different switching functions there are ‘entropy remnants’ left in the subsystems after the interaction was switched-off.

Although both $f_1(t)$ and $f_2(t)$ are continuous, according to the qualitative behaviour of the corresponding $h$-functions, $f_2(t)$ is ‘smooth’ in comparison with $f_1(t)$. At the end of the interaction (in the present case at the time $t = 31$) neither of them vanishes. We have proved that this is true for any rational ratio $T_1/T$. It is interesting to note that $Re[h_2(t)]$ is necessarily zero at $T_1$ if $T_1/T$ is an integer.

In figure 2, we see the time evolution of the von Neumann entropies (a) $S_1$ and (c) $S_2$, and the Schmidt numbers (b) $K_1$ and (d) $K_2$ of the photon number distribution given by equations (26) and (27), respectively, in the special case when $\mu_0 = 0$ and for the numerical value $q_0 = 2$. As can be seen in these figures, in each case of the two different switching functions there are ‘entropy remnants’ left in the subsystems after the interaction was switched-off. The mathematical reason for that is that neither of the functions $h_1(T_1) \neq 0$ and $h_2(T_1) \neq 0$ gets exactly zero at the time of termination of the interaction, as clearly illustrated in figure 1. Moreover, the root of these entropy remnants is the still entangled state operator $|\Psi(T_1)\rangle\langle\Psi(T_1)|$ of the complete system, which, due to the absence of the interaction, evolves freely in later times $t > T_1$, according to the unperturbed free Hamiltonian. The entanglement survives to some extent in each case because the final density operator of the complete state (at $t = T_1$) is not factorized to a simple product of the form $P_{\text{photon}} \otimes P_{\text{electron}}$. This can be immediately seen from the functional form of the joint expansion coefficients $\Psi_k(\hat{r}, t = T_1)$.

5. Summary

In this paper, we have discussed interactions between photons and electrons, and derived exact analytic expressions for the entangled state of the system evolving from an initial product state representing an electronic wave packet and a number eigenstate of the quantized photon mode. The von Neumann entropy of the photon and the Schmidt number of the photon and of the electron have been presented in the quasi-classical limit. Since we have solved the initial value problem exactly, we were able to study the time evolution of the entropies of these simple subsystems and draw some conclusions concerning the question of reversibility and irreversibility of the interaction.

We have made a comparison between the two cases distinguished by the different envelope functions modeling the switching-on and -off of the interaction between a light pulse and a localized electron. Although each of these functions is continuous at the instant of switching, the time evolutions of the system in the two cases are qualitatively different. In one case, there is an accumulation of the entropy by the end of the interaction, whereas in the other case the system is almost recovering to a pure state, i.e. it makes an almost reversible cycle. However, there are always some entropy remnants present at the end of the process, and the photon–electron system gets off the interaction region in an entangled state, although they are already separating from each other. On the basis of our analytic results we have presented a few numerical illustrations of the existence of the mentioned entropy remnants.

Finally, we note that there are more or less straightforward approaches for the generalization of our analysis beyond the dipole approximation and for the relativistic description.

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