CHEEGER CONSTANT, $p$-LAPLACIAN, AND GROMOV-HAUSDORFF CONVERGENCE

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Abstract. We discuss the behavior of $(\lambda_{1,p}(M))^{1/p}$ with respect to the Gromov-Hausdorff topology and the variable $p$, where $\lambda_{1,p}(M)$ is the first positive eigenvalue of the $p$-Laplacian on a compact Riemannian manifold $M$. Applications include new estimates for the first eigenvalues of the $p$-Laplacian on Riemannian manifolds with lower Ricci curvature bounds, and isoperimetric inequalities on Gromov-Hausdorff limit spaces. We also establish a new Lichnerowicz-Obata type theorem.

1. Introduction

Let $X$ be a compact metric space and let $\nu$ be a Borel probability measure on $X$ (we call such a pair $(X, \nu)$ a compact metric measure space in this paper). We define Minkowski’s exterior boundary measure $\nu^+(A)$ of a Borel subset $A$ of $X$ by

$$\nu^+(A) := \liminf_{r \to 0} \frac{\nu(B_r(A)) - \nu(A)}{r},$$

where $B_r(A)$ is the open $r$-neighborhood of $A$. Let us define the Cheeger constant $h(X)$ by

$$h(X) := \inf_A \frac{\nu^+(A)}{\nu(A)},$$

where the infimum runs over Borel subsets $A$ of $X$ with $0 < \nu(A) \leq 1/2$ (if $A$ as above does not exist, then we put $h(X) := \infty$). It is known that if $X$ is an $n$-dimensional compact Riemannian manifold and $\nu$ is the canonical Riemannian probability measure on $X$ (we call such a pair an $n$-dimensional compact smooth metric measure space in this paper), then $h(X)$ coincides Cheeger’s original one [8]:

$$h(X) = \inf_\Omega \frac{H^{n-1}(\partial \Omega)}{H^n(\Omega)},$$

where the infimum runs over open subsets $\Omega$ of $X$ having the smooth boundaries $\partial \Omega$ with $H^n(\Omega) \leq H^n(X)/2$ and $H^k$ is the $k$-dimensional Hausdorff measure. See for instance Section VI in [7] and subsection 2.2.4.

For every $1 < p < \infty$, we define the first eigenvalue $\lambda_{1,p}(X)$ of the $p$-Laplacian by

$$\lambda_{1,p}(X) := \inf_f \int_X (\text{Lip} f)^p \, d\nu,$$
where the infimum runs over Lipschitz functions $f$ on $X$ with
\[
\int_X |f|^p \, d\nu = 1 \quad \text{and} \quad \int_X |f|^{p-2} f \, d\nu = 0
\]
(if $f$ as above does not exist, i.e., $X$ is a single point, then we put $\lambda_{1,p}(X) := \infty$). See subsection 2.1 for the definition of the pointwise Lipschitz constant $\text{Lip}_f$ of $f$. It is also known that if $(X, \nu)$ is an $n$-dimensional compact smooth metric measure space, then $\lambda_{1,p}(X)$ coincides the first positive eigenvalue of the following PDE:
\[
\Delta_p f = \lambda |f|^{p-2} f
\]
on $X$, where $\Delta_p f := -\text{div}(\nabla f|^{p-2}\nabla f)$.

For every $n \in \mathbb{N}$, every $K \in \mathbb{R}$, and every $d > 0$, let $M(n, K, d)$ be the set of isometry classes of $n$-dimensional compact smooth metric measure spaces $(M, \text{Vol})$ with $\text{diam} M \leq d$ and
\[
\text{Ric}_M \geq K(n - 1),
\]
where $\text{diam} M$ is the diameter of $M$. We denote by $\overline{M(n, K, d)}$ the Gromov-Hausdorff compactification of $M(n, K, d)$, i.e., every $(Y, \nu) \in \overline{M(n, K, d)}$ is the measured Gromov-Hausdorff limit compact metric measure space of a sequence $\{(X_i, \nu_i)\}_i$ of $(X_i, \nu_i) \in M(n, K, d)$.

The main purpose of this paper is to study the behavior of
\[
(\lambda_{1,p}(X))^{1/p}
\]
with respect to the Gromov-Hausdorff topology and the variable $p$.

In order to state the main result of this paper, let $F$ be the function from $\overline{M(n, K, d)} \times [1, \infty]$ to $(0, \infty]$ defined by
\[
F((Y, \nu), p) := \begin{cases} 
2(\text{diam} Y)^{-1} & \text{if } p = \infty, \\
(\lambda_{1,p}(Y))^{1/p} & \text{if } 1 < p < \infty, \\
h(Y) & \text{if } p = 1.
\end{cases}
\]

The main result of this paper is the following:

**Theorem 1.1.** We have the following:

1. $F$ is upper semicontinuous on $\overline{M(n, K, d)} \times [1, \infty]$.
2. $F$ is continuous on $\overline{M(n, K, d)} \times (1, \infty]$.
3. $F$ is continuous on $\{(Y, \nu)\} \times [1, \infty]$ for every $(Y, \nu) \in \overline{M(n, K, d)}$.

Note that Theorem 1.1 is a generalization of the following results:

1. Cheeger-Colding proved in [14] that the eigenvalues of the Dirichlet Laplacian on spaces with lower Ricci curvature bounds are continuous with respect to the Gromov-Hausdorff topology (this was conjectured by Fukaya in [23]). In particular, this yields that $F$ is continuous on $\overline{M(n, K, d)} \times \{2\}$. 


(2) Grosjean proved in [28] that
\[
\lim_{p \to \infty} (\lambda_{1,p}(M))^{1/p} = \frac{2}{\text{diam } M}
\]
(1.3)
holds for every compact Riemannian manifold \(M\).

Note that it is essential to study \((\lambda_{1,p}(M))^{1/p}\) instead of \(\lambda_{1,p}\). See Remark 3.7 for a reason.

In order to introduce an application of Theorem 1.1, we recall the following Matei’s estimates [52]:
\[
\frac{h(M)}{p} \leq (\lambda_{1,p}(M))^{1/p}
\]
(1.4)
and
\[
(\lambda_{1,p}(M))^{1/p} \leq C(n, K) (h(M) + h(M)^{p})^{1/p}
\]
(1.5)
hold for every \(1 < p < \infty\) and every \(n\)-dimensional compact Riemannian manifold \(M\) with (1.1), where \(C(n, K)\) is a positive constant depending only on \(n\) and \(K\). Note that for \(p = 2\), (1.4) and (1.5) correspond to Cheeger’s isoperimetric inequality [8] and Buser’s one [6], respectively.

For positive numbers \(a, b \in \mathbb{R}_{>0}\), we now use the notation: \(a \asymp b\) if there exists a positive number \(C := C(n, K) > 1\) depending only on \(n\) and \(K\) such that \(C^{-1}b \leq a \leq Cb\) holds. Theorem 1.1 yields the following scale invariant estimates. Compare with (1.4) and (1.5).

**Theorem 1.2.** Let \(M\) be an \(n\)-dimensional compact Riemannian manifold with
\[
(diam M)^2 \text{Ric}_M \geq K(n - 1).
\]
(1.6)
Then, we have
\[
(\lambda_{1,p}(M))^{1/p} \asymp n^{K} h(M) \asymp (diam M)^{-1}
\]
(1.7)
for every \(1 < p < \infty\).

Note that the scale invariant assumption (1.6) is essential. See Colbois-Matei’s result [15] for a reason. There are many important works on lower bounds of \(\lambda_{1,p}\) [39, 52, 63, 59, 70, 71, 76]. It is important that (1.7) is a two-sided bound and is independent of the exponent \(p\). Note that roughly speaking, (1.7) implies that if \((\lambda_{1,p})^{1/p}\) is small (or big) for some \(p\), then \((\lambda_{1,q})^{1/q}\) is also small (or big) for every \(q\), quantitatively. See Corollary 3.8.

Other applications of Theorem 1.1 include the estimates (1.4), (1.5) and (1.7) on limit spaces, a quantitative version of Grosjean’s result (1.3), and a new Lichnerowicz-Obata type theorem. See Corollary 3.3, Remarks 3.4, 3.5 and Theorem 4.1 for the precise statements.

We now give an outline of the proof of Theorem 1.1. It consists of the following two steps:
(1) The first step is to establish a compact embedding of Sobolev spaces \( H_{1,p} \hookrightarrow L^r \) with respect to the Gromov-Hausdorff topology via \((q,p)\)-Poincaré inequality. See Theorem 2.18. This is a key to prove (3) of Theorem 1.1.

(2) The second step is to apply Grosjean’s argument in [28] to our setting with the Rellich type compactness with respect to the Gromov-Hausdorff topology given in [38]. This argument with several results given in [36, 38] allows us to prove Theorem 1.1.

Theorem 1.2 is proven by a compactness argument via Theorem 1.1.

The organization of this paper is as follows:

In Section 2, we will recall several fundamental notion and properties of metric measure spaces. We will also establish the first step above.

Section 3 corresponds to the second step above, i.e., we will finish the proof of Theorem 1.1. We will also prove Theorem 1.2.

In Section 4, we will apply Theorem 1.1 to establish a new Lichnerowicz-Obata type theorem for limit spaces and give an application.

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2. Preliminaries

In this section, we fix several notation and prepare several tools on metric measure spaces in order to prove Theorem 1.1.

2.1. Notation. For real numbers \( a, b \in \mathbb{R} \) and a positive number \( \epsilon > 0 \), throughout this paper, we use the following notation:

\[
    a = b \pm \epsilon \iff |a - b| < \epsilon.
\]

Let us denote by \( \Psi(\epsilon_1, \epsilon_2, \ldots, \epsilon_k; c_1, c_2, \ldots, c_l) \) some positive valued function on \( \mathbb{R}_{>0}^k \times \mathbb{R}^l \) satisfying

\[
    \lim_{\epsilon_1, \epsilon_2, \ldots, \epsilon_k \to 0} \Psi(\epsilon_1, \epsilon_2, \ldots, \epsilon_k; c_1, c_2, \ldots, c_l) = 0
\]

for fixed real numbers \( c_1, c_2, \ldots, c_l \). We often denote by \( C(c_1, c_2, \ldots, c_l) \) some positive constant depending only on fixed real numbers \( c_1, c_2, \ldots, c_l \).

Let \( X \) be a metric space and let \( x \in X \). For every \( r > 0 \), put \( B_r(x) := \{ w \in X; \overline{x, w} < r \} \), where \( \overline{x, w} \) is the distance between \( x \) and \( w \). We say that \( X \) is a geodesic space if for every \( p, q \in X \) there exists an isometric embedding \( \gamma : [0, \overline{p, q}] \to X \) such that \( \gamma(0) = p \) and \( \gamma(\overline{p, q}) = q \) hold (we call \( \gamma \) a minimal geodesic from \( p \) to \( q \)). For every Lipschitz function \( f \) on \( X \), let

\[
    \text{Lip} f := \sup_{a \neq b} \frac{|f(a) - f(b)|}{a, b},
\]
let
\[ \text{Lip} f(x) := \lim_{r \to 0} \sup_{y \in B_r(x) \setminus \{x\}} \frac{|f(x) - f(y)|}{x, y} \]
if \( x \) is not isolated in \( X \), and let \( \text{Lip} f(x) := 0 \) otherwise.

2.2. Metric measure spaces. Throughout subsection 2.2, we always discuss about a compact metric measure space \((X, \nu)\). Let \( \kappa, \tau \), and \( \lambda \) be positive numbers.

2.2.1. Doubling condition and Poincaré inequality. We recall the definitions of doubling conditions and Poincaré inequalities (for Lipschitz functions) on metric measure spaces:

**Definition 2.1.** Let \( p, q \in [1, \infty) \).

1. We say that \((X, \nu)\) satisfies the doubling condition for \( \kappa \) if \( \nu(B_{2r}(x)) \leq 2^\kappa \nu(B_r(x)) \) holds for every \( r > 0 \) and every \( x \in X \).
2. We say that \((X, \nu)\) satisfies the \((q, p)\)-Poincaré inequality for \( \tau \) if
   \[ \left( \frac{1}{\nu(B_r(x))} \int_{B_r(x)} \left| f - \frac{1}{\nu(B_r(x))} \int_{B_r(x)} f \, d\nu \right|^q \, d\nu \right)^{1/q} \leq \tau r \left( \frac{1}{\nu(B_r(x))} \int_{B_r(x)} (\text{Lip} f)^p \, d\nu \right)^{1/p} \]
   holds for every \( x \in X \), every \( r > 0 \), and every Lipschitz function \( f \) on \( X \).

**Remark 2.2.**
1. The Hölder inequality yields that if \((X, \nu)\) satisfies the \((q, p)\)-Poincaré inequality for \( \tau \), then for every \( \hat{p} \geq p \) and every \( \hat{q} \leq q \), \((X, \nu)\) satisfies the \((\hat{q}, \hat{p})\)-Poincaré inequality for \( \tau \).
2. Assume that \((X, \nu)\) satisfies the doubling condition for \( \kappa \). For every \( f \in L^1(X) \), let \( \text{Leb} f \) be the set of points \( x \in X \) with
   \[ \lim_{r \to 0} \frac{1}{\nu(B_r(x))} \int_{B_r(x)} |f(y) - f(x)| \, d\nu = 0. \]
   Then, we have
   \[ \nu(X \setminus \text{Leb} f) = 0. \]
   See for instance [32] for the proof.
3. Let \( 1 < p < \infty \). Assume that \((X, \nu)\) satisfies the doubling condition for \( \kappa \) and that \((X, \nu)\) satisfies the \((1, p)\)-Poincaré inequality for \( \tau \). Then, we can define the Sobolev space \( H_{1,p}(X) \). See for instance [9, 29, 66] for the definition. It is known that the space of Lipschitz functions on \( X \) is dense in \( H_{1,p}(X) \). See [9, Theorem 4.47].

We now recall Hajlasz-Koskela’s Poincaré-Sobolev inequality:

**Theorem 2.3.** [30, Theorem 1] Let \( 1 \leq p < \infty \). Assume that \( X \) is a geodesic space, that \((X, \nu)\) satisfies the doubling condition for \( \kappa \), and that \((X, \nu)\) satisfies the \((1, p)\)-Poincaré inequality for \( \tau \). Then, we see that \((X, \nu)\) satisfies the \((\hat{p}, p)\)-Poincaré inequality for some \( C := C(\kappa, \tau, p) > 0 \) and some \( \hat{p} := \hat{p}(\kappa, \tau, p) > p \).
Remark 2.4. We can assume that $C$ and $\hat{p}$ as in Theorem 2.3 are continuous with respect to the valuables $\kappa$, $\tau$, and $p$. See [30, Theorem 1] (or [32, Theorem 4.18]).

We end this subsection by introducing the following Colding-Minicozzi’s result:

**Proposition 2.5.** [18, Lemma 3.3] Assume that $X$ is a geodesic space and that $(X, \nu)$ satisfies the doubling condition for $\kappa$. Then, we see that

$$v(B_r(x) \setminus B_{(1-\delta)r}(x)) \leq \Psi(\delta; \kappa)v(B_r(x))$$

holds for every $r > 0$, every $x \in X$, and every $0 < \delta < 1/2$. In particular, if $X$ is not a single point, then $\nu$ is atomless, i.e., $\nu(\{x\}) = 0$ holds for every $x \in X$.

2.2.2. **Segment inequality.** Throughout subsection 2.2.2, we always assume that $X$ is a geodesic space.

For every nonnegative valued Borel function $f$ on $X$ and any $x, y \in X$, let

$$F_f(x, y) := \inf_{\gamma} \int_{[0, x, y]} f(\gamma) ds,$$

where the infimum runs over minimal geodesics $\gamma$ from $x$ to $y$. We now recall the definition of the segment inequality (on balls) for $\lambda$ by Cheeger-Colding (see also [10, Theorem 2.15]):

**Definition 2.6.** [14] We say that $(X, \nu)$ satisfies the segment inequality for $\lambda$ if

$$\int_{B_r(x) \times B_r(x)} F_f(y, z) d(\nu \times \nu) \leq \lambda r v(B_r(x)) \int_{B_r(x)} f d\nu$$

holds for every $x \in X$, every $r > 0$, and every nonnegative valued Borel function $f$ on $X$.

In [14], Cheeger-Colding proved the following:

**Proposition 2.7.** [14] Assume that $(X, \nu)$ satisfies the doubling condition for $\kappa$ and that $(X, \nu)$ satisfies the segment inequality for $\lambda$. Then we see that $(X, \nu)$ satisfies the $(1, 1)$-Poincaré inequality for some $\tau(\kappa, \lambda) > 0$.

See Section 2 in [14] for the proof.

The following proposition will be used in the proof of Theorem 1.1.

**Proposition 2.8.** Under the same assumptions as in Proposition 2.7, we see that

$$\text{Lip} f = ||\text{Lip} f||_{L^\infty(X)}$$

holds for every Lipschitz function $f$ on $X$.

**Proof.** It suffices to check

$$\text{Lip} f \leq ||\text{Lip} f||_{L^\infty(X)}. \quad (2.1)$$

There exists a Borel subset $A$ of $X$ such that $\nu(X \setminus A) = 0$ holds and that $\text{Lip} f \leq ||\text{Lip} f||_{L^\infty(X)}$ holds on $A$. By applying the segment inequality to the indicator function
1 of \( X \setminus A \), there exists a Borel subset \( V \) of \( X \times X \) such that \((u \times v)((X \times X) \setminus V) = 0\) holds and that for every \((x, y) \in V\) and every \( \epsilon > 0 \), there exists a minimal geodesic \( \gamma \) from \( x \) to \( y \) such that
\[
\int_{[0, s_0]} 1_{X \setminus A}(\gamma(s))ds < \epsilon
\]
holds. Therefore, since \( \text{Lip} f \) is an upper gradient of \( f \) (see \cite{9, 33} for the definition), we have
\[
|f(x) - f(y)| \leq \int_{[0, s_0]} \text{Lip} f(\gamma(s))ds = \int_{[0, s_0]} 1_A(\gamma(s))\text{Lip} f(\gamma(s))ds + \int_{[0, s_0]} 1_{X \setminus A}(\gamma(s))\text{Lip} f(\gamma(s))ds \leq ||\text{Lip} f||_{L^\infty(X, \overline{xy})} + \text{Lip} f\epsilon.
\]
(2.2)
Since \( \epsilon \) is arbitrary and \( V \) is dense in \( X \times X \), (2.2) yields (2.1). \( \square \)

2.2.3. First eigenvalue of \( p \)-Laplacian. For every \( 1 \leq p < \infty \) and every \( f \in L^p(X) \), let
\[
c_p(f) := \inf_{c \in \mathbb{R}} \left( \int_X |f - c|^p du \right)^{1/p}.
\]
We omit the proof of next proposition because it is easy to check it.

**Proposition 2.9.** We have the following:

1. The function \( f \mapsto c_p(f) \) on \( L^p(X) \) is 1-Lipschitz.
2. \( c_p(f + k) = c_p(f) \) and \( c_p(kf) = |k|c_p(f) \) hold for every \( f \in L^p(X) \) and every \( k \in \mathbb{R} \).
3. For any \( 1 \leq p \leq \hat{p} < \infty \) and every \( f \in L^\hat{p}(X) \), we have \( c_p(f) \leq c_{\hat{p}}(f) \leq ||f||_{L^\hat{p}(X)}. \)

In order to give another formulation of \( \lambda_{1,p}(X) \), we introduce the following lemma by Wu-Wang-Zheng given in \cite{74}.

**Lemma 2.10.** \cite[Lemma 2.2]{74} Let \( 1 < p < \infty \) and let \( f \in L^p(X) \). Then, for a number \( t \in \mathbb{R} \), the following two conditions are equivalent:

1. \[
\left( \int_X |f - t|^p du \right)^{1/p} = c_p(f),
\]
2. \[
\int_X |f - t|^{p-2}(f - t)du = 0.
\]
Moreover, there exists a unique \( s_0 \in \mathbb{R} \) such that
\[
\left( \int_X |f - s_0|^p du \right)^{1/p} = c_p(f)
\]
holds. We denote by \( a_p(f) \) \( s_0 \).
The following formulation of $\lambda_{1,p}(X)$ is necessary to prove Theorem 1.1.

**Corollary 2.11.** For every $1 < p < \infty$, we have

$$\lambda_{1,p}(X) = \inf_f \int_X (\text{Lip} f)^p dv,$$

where the infimum runs over Lipschitz functions $f$ on $X$ with $c_p(f) = \|f\|_{L^p} = 1$.

**Proof.** This is a direct consequence of Proposition 2.9 and Lemma 2.10. 

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2.2.4. *Cheeger constant.* For every $f \in L^1(X)$, we say that a number $M_f \in \mathbb{R}$ is a median of $f$ if

$$v(f \geq M_f) \geq \frac{1}{2}$$

and

$$v(f \leq M_f) \geq \frac{1}{2}$$

hold. It is easy to check the following (see also Section VI in [7]):

**Lemma 2.12.** For every $f \in L^1(X)$, there exists a median of $f$. Moreover, we see that

$$\int_X |f - M_f| dv = c_1(f)$$

holds for every median $M_f$ of $f$ (thus, we often denote by $a_1(f)$ a median $M_f$ of $f$ in this paper).

**Proposition 2.13.** Assume that $v$ is atomless. Then, we have

$$h(X) = \inf_f \int_X \text{Lip} f dv,$$

where the infimum runs over Lipschitz functions $f$ on $X$ with $c_1(f) = \|f\|_{L^1} = 1$.

**Proof.** This is a direct consequence of Proposition 2.9, Lemma 2.12 and [57, Lemma 2.2]. See also [4, 21, 56].

Thus, in this paper, we often use the following notation:

$$\lambda_{1,1}(X) := h(X).$$

**Remark 2.14.** Let $1 \leq p < \infty$. By Corollary 2.11 and Proposition 2.13, if $(X, \nu)$ satisfies the $(p, p)$-Poincaré inequality for $\tau$ and $\nu$ is atomless, then, we have

$$\lambda_{1,p}(X) \geq \frac{1}{\tau \text{diam } X}$$

In particular, by Propositions 2.3 and 2.4 if $X$ is a geodesic space, $(X, \nu)$ satisfies the doubling condition for $\kappa$, and $(X, \nu)$ satisfies the $(1, p)$-Poincaré inequality for $\tau$, then,

$$\lambda_{1,p}(X) \geq \frac{1}{C \text{diam } X}$$

holds, where $C := C(\kappa, \tau, p) > 0$. 
2.3. Gromov-Hausdorff convergence. In this subsection 2.3, we discuss Gromov-
Hausdorff convergence. In particular, we introduce Cheeger-Colding's works and the
notion of $L^p$-convergence with respect to the Gromov-Hausdorff convergence which are
perform key roles to prove Theorem 1.1.

2.3.1. Gromov-Hausdorff convergence and $L^p$-convergence of functions. Let \{$(X_i, \nu_i)$\}$_{i \leq \infty}$
be a sequence of compact metric measure spaces. We say that $(X_i, \nu_i)$ Gromov-Hausdorff
converges to $(X_\infty, \nu_\infty)$ if there exist a sequence of Borel maps $\phi_i : X_i \to X_\infty$ and a
sequence of positive numbers $\epsilon_i \searrow 0$ such that the following three conditions hold:

1. $X_\infty = B_{\epsilon_i}(\phi_i(X_i))$ holds,
2. $x, y = \phi_i(x), \phi_i(y) \pm \epsilon_i$ holds for every $i < \infty$ and any $x, y \in X_i$,
3. $(\phi_i)_*\nu_i$ converges weakly to $\nu_\infty$ on $X_\infty$.

Then, we denote by $(X_i, \nu_i) \to (X_\infty, \nu_\infty)$ the convergence for short. See [12] [23] [27].

Moreover, for a sequence \{$x_i$\}_{i \leq \infty} of points $x_i \in X_i$, we say that $x_i$ converges to $x_\infty$
with respect to the convergence $(X_i, \nu_i) \to (X_\infty, \nu_\infty)$ if $\phi_i(x_i)$ converges to $x_\infty$. Then we
also denote by $x_i \to x_\infty$ the convergence for short.

We introduce the notion of $L^p$-convergence of functions with respect to the Gromov-
Hausdorff topology by Kuwae-Shioya given in [43] [44]. We give an equivalent version of
the original definition given in [38] because it is useful to compare with the case of tensor
fields which will be discussed in subsection 2.3.2.

Assume that $(X_i, \nu_i) \to (X_\infty, \nu_\infty)$, that every $X_i$ is a geodesic space, and that there
exists a positive number $\kappa > 0$ such that for every $i \leq \infty$, $(X_i, \nu_i)$ satisfies the doubling
condition for $\kappa$. Note that $\nu_i(B_r(x_i)) \to \nu_\infty(B_r(x_\infty))$ holds for every $x_i \to x_\infty$ and every
$r > 0$. Let $1 < p < \infty$ and let \{f$_i$\}$_{i \leq \infty}$ be a sequence of $L^p$-functions $f_i \in L^p(X_i)$.

**Definition 2.15.** [43] [44]

1. We say that $f_i$ $L^p$-converges weakly to $f_\infty$ on $X_\infty$ if $\sup_i \|f_i\|_{L^p} < \infty$ and
\[
\lim_{i \to \infty} \int_{B_r(x_i)} f_i d\nu_i = \int_{B_r(x_\infty)} f_\infty d\nu_\infty \tag{2.3}
\]
hold for every $x_i \to x_\infty$ and every $r > 0$.

2. We say that $f_i$ $L^p$-converges strongly to $f_\infty$ on $X_\infty$ if $f_i$ $L^p$-converges weakly to $f_\infty$ on $X_\infty$ and
$\limsup_{i \to \infty} \|f_i\|_{L^p} \leq \|f_\infty\|_{L^p}$ holds.

Note that if the spaces are the same, i.e., $(X_i, \nu_i) \equiv (X_\infty, \nu_\infty)$, then the notions above coincide
the ordinary sense of $L^p$-convergence. In [38] [43] [44], several fundamental properties
on $L^p$-convergence were proven. We now introduce two fundamental properties on
$L^p$-weak convergence only which are well-known in the case that the spaces are the same.
Roughly speaking:

1. (WC) Every $L^p$-bounded sequence has an $L^p$-weak convergent subsequence.
2. (LS) $L^p$-norms are lower semicontinuous with respect to the $L^p$-weak topology.
Remark 2.16. Moreover, we can define a more general notion \( \{L^p_i\}_{i \leq \infty}\)-convergence for a convergent sequence of positive numbers \( p_i \to p_\infty \in (1, \infty) \) and a sequence \( \{g_i\}_{i \leq \infty} \) of \( L^p_i \)-functions \( g_i \in L^p_i(X_i) \) as follows: We say that \( g_i \{L^p_i\}_{i \leq \infty}\)-converges weakly to \( g_\infty \) on \( X_\infty \) if \( \sup_i ||g_i||_{L^p_i} < \infty \) and (2.3) hold. We also say that \( g_i \{L^p_i\}_{i \leq \infty}\)-converges strongly to \( g_\infty \) on \( X_\infty \) if \( g_i \{L^p_i\}_{i \leq \infty}\)-converges weakly to \( g_\infty \) on \( X_\infty \) and \( \limsup_{i \to \infty} ||g_i||_{L^p_i} \leq ||g_\infty||_{L^p_\infty} \) holds. Note that we can also define \( \{L^p_i\}_{i \leq \infty}\)-strong convergence by a different way even if the case of \( p_i \in [1, \infty) \). See [38] Remark 3.36. We here introduce a property on this convergence: If \( f_i \{L^p_i\}_{i \leq \infty}\)-converges strongly to \( f_\infty \) on \( X_\infty \), then \( f_i \{L^p_i\}_{i \leq \infty}\)-converges strongly to \( f_\infty \) on \( X_\infty \) for every \( \{p_i\}_{i \leq \infty} \subset [1, p] \) with \( p_i \to p_\infty \). In particular, we have
\[
\lim_{i \to \infty} ||f_i||_{L^{p_i}} = ||f_\infty||_{L^{p_\infty}}. \tag{2.4}
\]
See [38] for the details.

Proposition 2.17. Assume that \( f_i \{L^p_i\}_{i \leq \infty}\)-converges strongly to \( f_\infty \) on \( X_\infty \). Then for every \( \{p_i\}_{i \leq \infty} \subset [1, p] \) with \( p_i \to p_\infty \), we have
\[
\lim_{i \to \infty} c_{p_i}(f_i) = c_{p_\infty}(f_\infty).
\]

Proof. [2,4] yields
\[
\lim_{i \to \infty} ||f_i - a_{p_\infty}(f_\infty)||_{L^{p_i}} = ||f_\infty - a_{p_\infty}(f_\infty)||_{L^{p_\infty}}.
\]
Since \( c_{p_i}(f_i) \leq ||f_i - a_{p_\infty}(f_\infty)||_{L^{p_i}} \) holds for every \( i \), we have
\[
\limsup_{i \to \infty} c_{p_i}(f_i) \leq c_{p_\infty}(f_\infty). \tag{2.5}
\]
On the other hand, since
\[
|a_{p_i}(f_i)| \leq ||f_i - a_{p_i}(f_i)||_{L^{p_i}} + ||f_i||_{L^{p_i}} \leq 2||f_i||_{L^{p_i}} \leq 2\sup_i ||f_i||_{L^p} < \infty,
\]
without loss of generality, we can assume that there exists a number \( a_\infty \in \mathbb{R} \) such that \( a_{p_i}(f_i) \to a_\infty \) holds. [2,4] yields
\[
\lim_{i \to \infty} ||f_i - a_{p_i}(f_i)||_{L^{p_i}} = ||f_\infty - a_\infty||_{L^{p_\infty}}.
\]
Since \( ||f_\infty - a_\infty|| \geq c_{p_\infty}(f_\infty) \), we have
\[
\liminf_{i \to \infty} c_{p_i}(f_i) \geq c_{p_\infty}(f_\infty). \tag{2.6}
\]
(2.5) and (2.6) yield the assertion. \( \square \)

Theorem 2.18. Let \( q \in (1, \infty) \) and let \( \tau > 0 \). Assume that for every \( i < \infty \), \( (X_i, v_i) \) satisfies the \( (q, p)\)-Poincaré inequality for \( \tau \). Then, for every sequence \( \{f_i\}_{i \leq \infty} \) of Sobolev functions \( f_i \in H_{1,p}(X_i) \) with \( \sup_i ||f_i||_{H_{1,p}} < \infty \), there exist a subsequence \( \{f_{i(j)}\}_j \) and an \( L^q \)-function \( f_\infty \in L^q(X_\infty) \) such that \( f_{i(j)} \{L^r_i\}_{i \leq \infty}\)-converges strongly to \( f_\infty \) on \( X_\infty \) for every \( 1 < r < q \).
Proof. By (WC), without loss of generality, we can assume that there exists an \( L^q \)-function \( f_\infty \in L^q(X_\infty) \) such that \( f_i \) \( L^q \)-converges weakly to \( f_\infty \) on \( X_\infty \). It suffices to check that \( f_i \) \( L^r \)-converges strongly to \( f_\infty \) on \( X_\infty \) for every \( 1 < r < q \).

Let \( 1 < r < q, t \geq 1, d := \sup_i \text{diam} X_i, L := \sup_i \|f_i\|_{H_1} \), and let \( K_{i,t} \) denotes the set of points \( x \in X_i \) satisfying that

\[
\frac{1}{v_i(B_r(x))} \int_{B_r(x)} (g_{fi})^p dv_i \leq t^p
\]

holds for every \( r > 0 \), where \( g_{fi} \) is the generalized minimal upper gradient of \( f_i \) (see for instance [9] for the definition). It is not difficult to check that \( v_i(X_i \setminus K_{i,t}) \leq C(\kappa, d, L)t^{-p} \) holds (cf. [32] Theorem 2.2]). Then by a ‘telescope argument using the (1, \( p \))-Poincaré inequality for \( \tau \) on \( X_i \), without loss of generality, we can assume that \( f_i \) is \( C(\kappa, \tau) \)-Lipschitz on \( K_{i,t} \) (cf. [9] Theorem 4.14]). By Macshane’s lemma (cf. [9] (8.2)), there exists a \( C(\kappa, \tau) \)-Lipschitz function \( f_{i,t} \) on \( X_i \) such that \( f_{i,t} \mid_{K_{i,t}} \equiv f_i \) holds.

Let \( x_{i,t} \in K_{i,t} \cap \text{Leb} f_i \). Applying a telescope argument again yields

\[
|f_i(x_{i,t})| \leq \int_{X_i} |f_i| dv_i + C(\kappa, \tau, d)t \leq C(\kappa, \tau, d, L)t.
\]

Thus, for every \( y_i \in X_i \), we have

\[
|f_{i,t}(y_i)| \leq |f_{i,t}(x_{i,t})| + \text{Lip} f_{i,t} y_i, x_{i,t} \\
\leq |f_i(x_{i,t})| + C(\kappa, \tau, d)t \leq C(\kappa, \tau, d, L)t.
\]

In particular,

\[
\int_{X_i} |f_{i,t}| dv_i = \int_{X_i \setminus K_{i,t}} |f_{i,t}| dv_i + \int_{K_{i,t}} |f_i| dv_i \\
\leq C(\kappa, \tau, d, L)t v_i(X_i \setminus K_{i,t}) + L \leq C(\kappa, \tau, d, L).
\]

On the other hand, by [9] Corolalry 2.25 and Theorem 5.1], we have

\[
\left( \int_{X_i} (\text{Lip} f_{i,t})^p dv_i \right)^{1/p} \leq \left( \int_{X_i \setminus K_{i,t}} (\text{Lip} f_{i,t})^p dv_i \right)^{1/p} + \left( \int_{K_{i,t}} (g_{fi})^p dv_i \right)^{1/p} \\
\leq (C(\kappa, \tau)^{p} v_i(X_i \setminus K_{i,t}))^{1/p} + L \leq C(\kappa, \tau, d, L).
\]

Therefore, (2.7), (2.8), and the \( (q, p) \)-Poincaré inequality for \( \tau \) on \( X_i \) give \( ||f_{i,t}||_{L^q} \leq C(\kappa, \tau, d, L) \). Thus, the Hölder inequality yields that

\[
\left( \int_{X_i} |f_i - f_{i,t}|^r dv_i \right)^{1/r} = \left( \int_{X_i} 1_{X_i \setminus K_{i,t}} |f_i - f_{i,t}|^r dv_i \right)^{1/r} \\
\leq v_i(X_i \setminus K_{i,t})^{1/(\alpha r)} ||f_i - f_{i,t}||_{L^q} \leq \Psi
\]

holds for every \( i \), where \( \beta := q/r > 1, \Psi = \Psi(t^{-1}; \kappa, \tau, d, L, q, r) \) and \( \alpha \) is the conjugate exponent of \( \beta \).
On the other hand, without loss of generality, we can assume that there exists a $C(κ, τ)t$-Lipschitz function $f_{∞, t}$ on $X_∞$ such that $f_{i, t}(z_i) → f_{∞, t}(z_i)$ holds for every sequence $\{z_i\}_{i ≤ ∞}$ of points $z_i ∈ X_i$ with $z_i → z_∞$ (cf. [38, Proposition 3.3]). Note that for every $1 < s < ∞$, $f_{i, t}$ $L^s$-converges strongly to $f_{∞, t}$ on $X_∞$ (cf. [38, Proposition 3.32]). Then (LS) yields

$$||f_∞ - f_{∞, t}||_{L^r} ≤ \liminf_{i→∞} ||f_i - f_{i, t}||_{L^r} ≤ Ψ.$$ (2.10)

Since $t$ is arbitrary, (2.9) and (2.10) yield that $f_i L^r$-converges strongly to $f_∞$ on $X_∞$. □

We give an application of Theorem 2.18.

**Theorem 2.19.** Let $1 ≤ q < ∞$ and let $(X, v)$ be a compact metric measure space. Assume that $X$ is a geodesic space, that $(X, v)$ satisfies the doubling condition for some $κ > 0$, and that $(X, v)$ satisfies $(1, q)$-Poincaré inequality for some $τ > 0$. Then, we see that the function $r → λ_{1, r}(X)$ is right-continuous at $q$.

**Proof.** Let $ε > 0$ and let $f$ be a Lipschitz function on $X$ with $||\text{Lip} f||_{L^q} ≤ λ_{1, q}(X) + ε$ and $||f||_{L^q} = c_q(f) = 1$. Then, since $λ_{1, r}(X) ≤ (c_q(f))^{-r} ||\text{Lip} f||_{L^q}^{q/r}$ holds for every $r > q$, by letting $r \searrow q$, Proposition 2.17 yields $\limsup_{r \searrow q} λ_{1, r}(X) ≤ ||\text{Lip} f||_{L^q}^{q/r} ≤ λ_{1, q}(X) + ε$. Since $ε$ is arbitrary, we have

$$\limsup_{r \searrow q} λ_{1, r}(X) ≤ λ_{1, q}(X).$$ (2.11)

On the other hand, let $\{q_i\}_i$ be a sequence of positive numbers $q_i \searrow q$, and let $\{f_i\}_i$ be a sequence of Lipschitz functions $f_i$ on $X$ with $||\text{Lip} f_i||_{L^{q_i}}^{q_i} ≤ λ_{1, q_i}(X) + ε$ and $||f_i||_{L^{q_i}} = c_{q_i}(f_i) = 1$. By Theorems 2.3, 2.18 and (2.11), without loss of generality, we can assume that there exist a number $r > p$ and an $L^r$-function $f_∞ ∈ L^r(X)$ such that $f_i L^r$-converges strongly to $f_∞$ on $X$. Thus, by Proposition 2.17 we have $c_q(f_i) → c_q(f_∞)$.

Since

$$λ_{1, q}(X) ≤ c_q(f_i)^{-q}||\text{Lip} f_i||_{L^q}^{q} ≤ c_q(f_i)^{-q}||\text{Lip} f_i||_{L^{q_i}}^{q_i} ≤ c_q(f_i)^{-q}(λ_{1, q_i}(X) + ε)^{q/q_i},$$

by letting $i → ∞$ and $ε → 0$, we have $\liminf_{i→∞} λ_{1, q_i}(X) ≥ λ_{1, q}(X)$. □

**Remark 2.20.** It is known that finite dimensional Alexandrov spaces, $CD(K, ∞)$-metric measure spaces, and locally strongly doubling metric measure spaces satisfy the $(1, 1)$-Poincaré inequalities. See [45, Theorem 7.2], [62, Theorem 1], and [63, Theorem 4.1]. Thus, by Theorem 2.19, for every such a metric measure space $(X, v)$, we see that the function $r → λ_{1, r}(X)$ is right continuous on $[1, ∞)$. See also for instance [51, 42, 44, 51, 60, 61, 67, 68, 73] for these topics.
2.3.2. Cheeger-Colding’s works and $L^p$-convergence of tensor fields. In [11, 12, 13, 14], Cheeger-Colding developed the structure theory on limit spaces of Riemannian manifolds with lower Ricci curvature bounds. We introduce several their results we need to prove Theorem 1.1.

**Theorem 2.21.** [11, 12, 13, 14] Let $(X, v) \in \overline{M(n, K, d)}$. Then we have the following:

1. $(X, v)$ satisfies the doubling condition for $\kappa := \kappa(n, K, d)$.
2. $(X, v)$ satisfies the segment inequality for $\lambda := \lambda(n, K, d)$.
3. There exist a topological space $TX$ (called the tangent bundle of $X$) and a Borel map $\pi : TX \to X$ such that the following three conditions hold:
   - (a) $v(X \setminus \pi(TX)) = 0$ holds,
   - (b) For every $x \in \pi(TX)$, the fiber $T_xX := \pi^{-1}(x)$ is a finite dimensional Hilbert space. Let us denote by $\langle \cdot, \cdot \rangle$ the inner product (called the Riemannian metric of $X$) for short,
   - (c) For every Lipschitz function $f$ on $X$, there exist a Borel subset $X_f$ of $X$ and a section $\nabla f : X_f \to TX$ such that $v(X \setminus X_f) = 0$ holds and that
     \[ |\nabla f|(x) = \text{Lip} f(x) \quad (2.12) \]
     holds for every $x \in X_f$, where $|\nabla f| := \sqrt{\langle \nabla f, \nabla f \rangle}$. Moreover, for every $g \in H_{1,p}(X)$, there exists a section $\nabla g$ such that
     \[ ||g||_{H_{1,p}(X)} = \left( \int_X |g|^p dv \right)^{1/p} + \left( \int_X |\nabla g|^p dv \right)^{1/p} \]
     holds.

See [11, 12, 13, 14] for the details. Let us denote by $L^p(TX)$ the set of $L^p$-sections from $\pi(TX)$ to $TX$.

In [38], we discussed $L^p$-convergence of tensor fields with respect to the Gromov-Hausdorff topology (see also [37]). We recall it in the case of vector fields only. Let $(X_i, v_i) \to (X_\infty, v_\infty)$ in $\overline{M(n, K, d)}$ with $\text{diam } X_\infty > 0$, and let $1 < p < \infty$.

**Definition 2.22.** [38] Definition 1.1] Let $V_i \in L^p(TX_i)$ for every $i \leq \infty$.

1. We say that $V_i$ $L^p$-converges weakly to $V_\infty$ on $X_\infty$ if $\sup_{i \leq \infty} ||V_i||_{L^p} < \infty$ and
   \[ \lim_{i \to \infty} \int_{B_r(x_i)} \langle V_i, \nabla r_{y_i} \rangle dv_i = \int_{B_r(x_\infty)} \langle V_\infty, \nabla r_{y_\infty} \rangle dv_\infty \]
   hold for every $x_i \to x_\infty$, every $y_i \to y_\infty$, and every $r > 0$, where $r_{y_i}$ is the distance function from $y_i$, i.e., $r_{y_i}(w) := \overline{y_i, w}$.
2. We say that $V_i$ $L^p$-converges strongly to $V_\infty$ on $X_\infty$ if $V_i$ $L^p$-converges weakly to $V_\infty$ on $X_\infty$ and $\limsup_{i \to \infty} ||V_i||_{L^p} \leq ||V_\infty||_{L^p}$ holds.
Compare with Definition 2.15. We can also get several fundamental properties of this convergence, e.g., (WC) and (LS) in this setting. See [38] Propositions 3.50 and 3.64.

We end this subsection by introducing two results given in [38]. One of them is a Rellich type compactness. The other is the continuity of the first eigenvalues of the $p$-Laplacian with respect to the Gromov-Hausdorff topology. In Section 3, they will play crucial roles in the proof of Theorem 1.1.

**Theorem 2.23.** [38, Theorem 4.9] Let $\{f_i\}_{i<\infty}$ be a sequence of Sobolev functions $f_i \in H_{1,p}(X_i)$ with $\sup_i \|f_i\|_{H_{1,p}(X_i)} < \infty$. Then, there exist a Sobolev function $f_{\infty} \in H_{1,p}(X_{\infty})$ and a subsequence $\{f_{i(j)}\}_j$ such that $f_{i(j)}$ } $L^p$-converges strongly to $f_{\infty}$ on $X_{\infty}$ and that $\nabla f_{i(j)}$ $L^p$-converges weakly to $\nabla f_{\infty}$ on $X_{\infty}$. In particular, we have

$$\liminf_{j \to \infty} \int_{X_{i(j)}} |\nabla f_{i(j)}|^p d\nu_{i(j)} \geq \int_{X_{\infty}} |\nabla f_{\infty}|^p d\nu_{\infty}. \quad (2.13)$$

**Theorem 2.24.** [38, Theorem 4.20] The function $((X, \nu), p) \mapsto \lambda_{1,p}(X)$ from $\overline{M(n, K, d)} \times (1, \infty)$ to $(0, \infty]$ is continuous.

3. Proof of main theorems

We are now in a position to prove Theorem 1.1.

**Proof of (2) of Theorem 1.1**

By Theorem 2.24, it suffices to check that if $p_j \to \infty$ and $(X_j, \nu_j) \to (X_{\infty}, \nu_{\infty})$ in $\overline{M(n, K, d)}$, then

$$\lim_{j \to \infty} \lambda_{1,p_j}(X_j) = \frac{2}{\text{diam } X_{\infty}} \quad (3.1)$$

holds under the assumption that $(X_j, \nu_j) \in M(n, K, d)$ holds for every $j < \infty$. We give a proof of (3.1) by separating the following two cases:

The case of $\text{diam } X_{\infty} > 0$.

Note that the following argument is essentially due to Grosjean [28], however, in our setting, it is little more delicate than the proof of [28, Theorem 1.1] because we need several results stated in Section 2.

**Claim 3.1.** We have

$$\limsup_{j \to \infty} (\lambda_{1,p_j}(X_j))^{1/p_j} \leq \frac{2}{\text{diam } X_{\infty}}.$$ 

The proof is as follows. For every $j \leq \infty$, let $d_j := \text{diam } X_j$, let $x_{j,1}$ and $x_{j,2}$ be points in $X_j$ with $x_{j,1}, x_{j,2} = d_j$, and let $\delta_{j,1}$ and $\delta_{j,2}$ be the 1-Lipschitz functions on $X_j$ defined by

$$\delta_{j,i}(x) := \max \left\{ \frac{d_j}{2} - \frac{x_{j,i} - x}{d_j}, 0 \right\}.$$
Without loss of generality, we can assume that \( x_{j,i} \to x_{\infty,i} \) holds for every \( i = 1, 2 \). Then by an argument similar to the proof of \([28\text{, Theorem 1.1}]\), we see that

\[
\left( \lambda_{1,p_j}(X_j) \right)^{1/p_j} \leq \max_i \left\{ \left( \frac{1}{v_j(B_{d_j/2}(x_{j,i}))} \int_{B_{d_j/2}(x_{j,i})} |\delta_{j,i}|^p dv_j \right)^{-1/p_j} \right\}
\]

holds for every \( j < \infty \). Therefore, since \( \delta_{j,i} \), \( L^p \)-converges strongly to \( \delta_{\infty,i} \) on \( X_\infty \) for every \( i = 1, 2 \) and every \( 1 < p < \infty \) (see for instance \([38\text{, Proposition 3.32}]\)), the Hölder inequality yields that for every \( 1 < p < \infty \), we have

\[
\limsup_{j \to \infty} \left( \lambda_{1,p_j}(X_j) \right)^{1/p_j} \leq \max_i \left\{ \left( \frac{1}{v_\infty(B_{d_\infty/2}(x_{\infty,i}))} \int_{B_{d_\infty/2}(x_{\infty,i})} |\delta_{\infty,i}|^p dv_\infty \right)^{-1/p} \right\}.
\]

By letting \( p \to \infty \), we have

\[
\limsup_{j \to \infty} \left( \lambda_{1,p_j}(X_j) \right)^{1/p_j} \leq \max_i \{ ||\delta_{\infty,i}||_{L^1(X_\infty)} \} = \frac{2}{d_\infty}.
\]

Thus, we have Claim 3.1.

**Claim 3.2.** We have

\[
\liminf_{j \to \infty} \left( \lambda_{1,p_j}(X_j) \right)^{1/p_j} \geq \frac{2}{\text{diam } X_\infty}.
\]

The proof is as follows. For every \( j < \infty \), let \( f_j \) be a first eigenfunction for \( \lambda_{1,p_j}(X_j) \), let \( \Omega_j^+ := f_j^{-1}(R_{>0}) \), and let \( \Omega_j^- := f_j^{-1}(R_{<0}) \). Let \( f_j^\pm \in H_{1,p_j}(\Omega_j^\pm) \) be positive valued eigenfunctions associated to the first Dirichlet eigenvalues \( \lambda_{1,p_j}^D(\Omega_j^\pm) \) of the \( p \)-Laplacian on \( \Omega_j^\pm \) with

\[
||f_j^\pm||_{L^\infty(\Omega_j^\pm)} = 1,
\]

respectively (we will omit to write ‘respectively’ below for simplicity).

We extend \( f_j^\pm \) by 0 outside \( \Omega_j^\pm \). Thus, we have \( f_j^\pm \in H_{1,p_j}(X_j) \). For every \( 1 < p < \infty \), by an argument similar to the proof of \([28\text{, Theorem 1.1}]\), we have

\[
\left( \int_{X_j} (\text{Lip } f_j^\pm)^p dv_j \right)^{1/p} \leq \lambda_{1,p_j}(X_j)^{1/p_j}
\]

for every sufficiently large \( j \).

Thus, by Theorem 2.23 and Claim 3.1, without loss of generality, we can assume that there exist Borel functions \( f_\infty^\pm \) on \( X_\infty \) such that \( f_\infty^\pm \in H_1(X_\infty) \) hold for every \( 1 < p < \infty \), that \( f_\infty^\pm \) \( L^p \)-converge strongly to \( f_\infty^\pm \) on \( X_\infty \) for every \( 1 < p < \infty \), and that \( \nabla f_\infty^\pm \) \( L^p \)-converge weakly to \( \nabla f_\infty^\pm \) on \( X_\infty \) for every \( 1 < p < \infty \). Therefore, by \([2.12], [2.13], \) and \([3.3]\), we see that

\[
\left( \int_{X_\infty} (\text{Lip } f_\infty^\pm)^p dv_\infty \right)^{1/p} \leq \liminf_{j \to \infty} \left( \int_{X_j} (\text{Lip } f_j^\pm)^p dv_j \right)^{1/p} \leq \liminf_{j \to \infty} \left( \lambda_{1,p_j}(X_j) \right)^{1/p_j}
\]
hold for every $1 < p < \infty$. Thus, by letting $p \to \infty$, we have

$$
||\text{Lip} f_\infty^\pm||_{L^\infty(X_\infty)} \leq \liminf_{j \to \infty} \left( \lambda_{1,p_j}(X_j) \right)^{1/p_j}.
$$

(3.4)

By Proposition 2.7 and Theorem 2.21, since

$$
\frac{1}{v_\infty(B_r(w))} \int_{B_r(w)} \left| f_\infty^\pm - \frac{1}{v_\infty(B_r(w))} \int_{B_r(w)} f_\infty^\pm dv_\infty \right| dv_\infty \leq C(n, K, d) r ||\text{Lip} f_\infty^\pm||_{L^\infty(X_\infty)}
$$

holds for every $w \in X_\infty$ and every $r > 0$, applying a telescope argument yields that there exists a Borel subset $A$ of $X_\infty$ such that $v_\infty(X_\infty \setminus A) = 0$ holds and that $f_\infty^\pm$ are Lipschitz on $A$. In particular, since $A$ is dense, we see that $f_\infty^\pm$ are Lipschitz on $X_\infty$.

Thus, Proposition 2.8 and (3.4) give

$$
\text{Lip} f_\infty^\pm \leq \liminf_{j \to \infty} \left( \lambda_{1,p_j}(X_j) \right)^{1/p_j}.
$$

(3.5)

On the other hand, it is easy to check that Hajlasz-Koskela’s quantitative Sobolev-embedding theorem (to Hölder spaces) [31, (25) of Theorem 5.1] yields that $\{f_j^\pm\}_{j < \infty}$ are asymptotically uniformly equicontinuous on $X_\infty$ (see [38, Definition 3.2] for the definition of asymptotically uniformly equicontinuous). In particular, by [38, Remark 3.8], we see that

$$
f_j^+(x_j) \to f_\infty^+(x_\infty)
$$

(3.6)

hold for every $x_j \to x_\infty$.

Let $\Omega^\pm := (f_\infty^\pm)^{-1}(R_{>0})$. Then by (3.2) and (3.6), we have $\Omega^\pm \neq \emptyset$. Since

$$
f_j^+ f_j^- \equiv 0
$$

holds on $X_j$ for every $j < \infty$, by letting $j \to \infty$, we see that

$$
f_\infty^+ f_\infty^- \equiv 0
$$

holds on $X_\infty$. In particular, we see that $\Omega^+$ and $\Omega^-$ are pairwise disjoint. Thus, we see that

$$
\frac{2}{\text{diam } X_\infty} \leq \max \left\{ \frac{1}{r(\Omega^+)} , \frac{1}{r(\Omega^-)} \right\}
$$

(3.7)

holds, where

$$
r(\Omega^\pm) := \max_{x \in \Omega^\pm} x, \partial \Omega^\pm \text{ and } \frac{x, \partial \Omega^\pm}{y \in \partial \Omega^\pm} := \inf \frac{x, y}{x, y}.
$$

Let $x^\pm$ be points in $X_\infty$ with $f_\infty^+(x^\pm) = 1$ and let $y^\pm$ be points in $\partial \Omega^\pm$ with $x^\pm, y^\pm = x^\pm, \partial \Omega^\pm$. Then since $f_\infty^+(y^\pm) = 0$, we have

$$
1 = |f_\infty^+(x^\pm) - f_\infty^+(y^\pm)| \leq \text{Lip} f_\infty^+ x^\pm, y^\pm \leq \text{Lip} f_\infty^+ r(\Omega^\pm).
$$

Thus, (3.5) yields

$$
\max \left\{ \frac{1}{r(\Omega^+)} , \frac{1}{r(\Omega^-)} \right\} \leq \liminf_{j \to \infty} \left( \lambda_{1,p_j}(X_j) \right)^{1/p_j}.
$$

(3.8)

Therefore, by (3.7) and (3.8), we have Claim 3.2.
Thus, we have (3.1) in this case.

The case of $\text{diam } X_\infty = 0$.

Let $\mathcal{M}$ be the set of $(X, \nu) \in \overline{M(n, K, 1)}$ with $\text{diam } X = 1$, and let $\mathcal{N} := \mathcal{M} \times [2, \infty]$. By Claims 3.1 and 3.2, we see that $F$ is continuous on $\mathcal{N}$. In particular, since $\mathcal{N}$ is compact, we have $C_1(n, K) := \min_{\mathcal{N}} F > 0$ and $C_2(n, K) := \max_{\mathcal{N}} F < \infty$. Note that the rescaled Riemannian manifolds $(\hat{M}_i, \text{Vol}) := (M_i, (\text{diam } M_i)^{-2} g_{M_i}, \text{Vol})$ are in $\mathcal{M}$.

Thus, we have $C_1(n, K) \leq (\lambda_{1,p_i}(\hat{M}_i))^{1/p_i} \leq C_2(n, K)$. Since $\lambda_{1,p_i}(\hat{M}_i) = \lambda_{1,p_i}(M_i)(\text{diam } M_i)^{p_i}$, we have

$$\lim_{i \to \infty} (\lambda_{1,p_i}(X_i))^{1/p_i} = \infty.$$ 

Therefore, we have also (3.1) in this case. □

Proof of (3) of Theorem 1.1.

This is a direct consequence of (2) of Theorem 1.1, Theorems 2.19 and 2.21. □

Proof of (1) of Theorem 1.1.

By Theorem 2.24 and (2) of Theorem 1.1, it suffices to check that if $(X_i, \nu_i) \to (X_\infty, \nu_\infty)$ in $M(n, K, d)$ and $\{p_i\}_{i<\infty} \subset [1, \infty]$ with $p_i \to 1$, then

$$\limsup_{i \to \infty} \lambda_{1,p_i}(X_i) \leq h(X_\infty)$$

holds. Without loss of generality, we can assume $\text{diam } X_\infty > 0$. Let $\epsilon > 0$ and let $f_\infty$ be a Lipschitz function on $X_\infty$ with $\|f_\infty\|_{L^1} = 1$, $c_1(f_\infty) = 1$, and

$$\left| h(X_\infty) - \int_{X_\infty} \text{Lip } f_\infty \, d\nu_\infty \right| < \epsilon.$$ 

By [36, Theorem 4.2], without loss of generality, there exists a sequence $\{f_i\}_i$ of Lipschitz functions $f_i$ on $X_i$ such that $\text{sup } \text{Lip } f_i < \infty$ and $f_i, df_i$ $L^p$-converge strongly to $f_\infty, df_\infty$ on $X_\infty$ for every $1 < p < \infty$, respectively. Then, by Proposition 2.17, we have

$$\limsup_{i \to \infty} \lambda_{1,p_i}(X_i) \leq \lim_{i \to \infty} (c_{p_i}(f_i))^{-p_i} \int_{X_i} (\text{Lip } f_i)^{p_i} \, d\nu_i = \int_{X_\infty} \text{Lip } f_\infty \, d\nu_\infty \leq h(X_\infty) + \epsilon.$$ 

Since $\epsilon$ is arbitrary, we have (3.9). □

We give a quantitative version of Grojean's result [28, Theorem 1.1]:

**Corollary 3.3.** Let $n \in \mathbb{N}$ and let $K \in \mathbb{R}$. Then for every $\epsilon > 0$, there exists a positive number $p_0 := p_0(n, K, \epsilon) > 1$ such that

$$\left| \text{diam } M (\lambda_{1,p}(M))^{1/p} - 2 \right| < \epsilon$$

holds for every $p > p_0$ and every $n$-dimensional compact Riemannian manifold $M$ with (1.6).

**Proof.** The proof is done by contradiction. Assume that the assertion is false. Then, there exist a positive number $\tau > 0$, a divergent sequence $p_i \to \infty$, and a sequence
(M_i, Vol) → (M_∞, v) in \( \overline{M(n, K, 1)} \) such that \( \text{diam } M_i = 1 \) and
\[
\left| \left( \lambda_{1,p_i}(M_i) \right)^{1/p_i} - 2 \right| \geq \tau
\]
hold for every \( i < \infty \). Since \( \text{diam } M_\infty = 1 \), by letting \( i \to \infty \) in (3.10), Theorem 1.1 yields
\[
0 = |2 - 2| \geq \tau > 0.
\]
This is a contradiction. \( \square \)

**Remark 3.4.** In [53], Matei showed that the function
\[
p \mapsto p(\lambda_{1,p}(M))^{1/p}
\]
is strictly increasing on \((1, \infty)\) for every compact Riemannian manifold \( M \). See [53, Proposition 2.6]. (3) of Theorem 1.1 yields that this holds on \([1, \infty)\). In particular, we can reprove Matei’s isoperimetric inequality (1.4) which is a generalization of Cheeger’s one given in [8] to the first eigenvalue of \( p \)-Laplacian:
\[
h(M) = \lambda_{1,1}(M) < p(\lambda_{1,p}(M))^{1/p}
\]
for every \( 1 < p < \infty \). See [52, Theorem 4.1] for the original proof. Moreover, this argument with Theorem 1.1 allows us to prove a weak version of (3.11) on limit spaces, i.e.,
\[
h(X) \leq p(\lambda_{1,p}(X))^{1/p}
\]
holds for every \( 1 < p < \infty \) and every \((X, v) \in \overline{M(n, K, d)}\).

We are now in a position to prove Theorem 1.2.

**Proof of Theorem 1.2**

First we discuss upper bounds. Let \( M \) be as in the proof of Theorem 1.1, let \( \bar{N} := M \times [1, \infty] \) and let \( M \) be a compact \( n \)-dimensional Riemannian manifold with (1.6). Since \( \bar{N} \) is compact, by (1) of Theorem 1.1 we have \( C_3(n, K) := \max_{X \in \bar{N}} F < \infty \). In particular, we see that
\[
(\lambda_{1,p}(M))^{1/p} \text{diam } M \leq C_3(n, K)
\]
holds for every \( 1 < p < \infty \).

Next we discuss lower bounds. Let \( C_1(n, K) \) be as in the proof of Theorem 1.1. Then, we see that
\[
C_1(n, K) \leq (\lambda_{1,p}(M))^{1/p} \text{diam } M
\]
holds for every \( 2 \leq p < \infty \).

On the other hand, Theorem 2.3, Remarks 2.4, 2.14 and Theorem 2.21 yield that
\[
0 < C_4(n, K) \leq (\lambda_{1,p}(M))^{1/p} \text{diam } M
\]
holds for every \( 1 < p < 2 \). (3.12), (3.13), and (3.14) yields the assertion. \( \square \)
Remark 3.5. Theorem 1.2 also holds on limit spaces. The reason is as follows. Let $(X, \nu)$ be the Gromov-Hausdorff limit compact metric measure space of a sequence of $n$-dimensional compact smooth metric measure spaces $(M_i, \text{Vol})$ with $(\text{diam } M_i)^2 \text{Ric}_{M_i} \geq K(n-1)$. Without loss of generality, we can assume $\text{diam } X > 0$.

Then, by Theorem 1.2 for every $1 < p < \infty$ and every $i < \infty$, we have $(\lambda_{1,p}(M_i))^{1/p} \approx (\text{diam } M_i)^{-1}$. Thus, by letting $i \to \infty$, we have $(\lambda_{1,p}(X))^{1/p} \approx (\text{diam } X)^{-1}$. By letting $p \to 1$, (3) of Theorem 1.1 yields

$$(\lambda_{1,p}(X))^{1/p} \approx h(X)^{-1} \approx (\text{diam } X)^{-1}.$$  

Remark 3.6. In [52, 59, 70], Matei, Naber-Valtorta, and Valtorta gave the sharp lower bounds for the first eigenvalues of the $p$-Laplacian on manifolds with lower Ricci curvature bounds. We discuss here on zero lower bound of Ricci curvature. See Section 4 for positive lower bounds.

In [70], we knew that

$$(\lambda_{1,p}(M))^{1/p} \geq \frac{2\pi (p-1)^{1/p}}{p \sin(\pi/p) \text{diam } M}$$  

(3.15)

holds for every nonnegatively Ricci curved compact Riemannian manifold $M$. Thus, since it is easy to see that the right hand side of (3.15) goes to $2/\text{diam } M$ as $p \to 1$, (3) of Theorem 1.1 and (3.15) allow us to reprove Gallot’s estimate [26]:

$$h(M) \geq \frac{2}{\text{diam } M}$$  

(3.16)

holds for every $M$ as above. Note that the right hand side of (3.15) goes also to $2/\text{diam } M$ as $p \to \infty$ and that by Theorem 1.1, we see that (3.15) and (3.16) hold on the Gromov-Hausdorff limit compact metric measure space of a sequence of $n$-dimensional nonnegatively Ricci curved compact Riemannian manifolds.

Remark 3.7. By Grosjean’s result (1.3), it is easy to check that

$$\lim_{p \to \infty} \lambda_{1,p}(M) = \begin{cases} \infty & \text{if } \text{diam } M < 2, \\ 0 & \text{if } \text{diam } M > 2 \end{cases}$$  

holds for every compact Riemannian manifold $M$ (note that by Theorem 1.1, this also holds for every $(Y, \nu) \in \overline{M(n,K,d)}$). In particular, we have

$$\lim_{p \to \infty} \lambda_{1,p}(S^n(r)) = \begin{cases} \infty & \text{if } r < 2/\pi, \\ 0 & \text{if } r > 2/\pi, \end{cases}$$  

(3.17)

where $S^n(r) := \{ x \in \mathbb{R}^{n+1} : |x| = r \}$.

Thus, (3.17) tells us that we can NOT get a $\lambda_{1,p}$-version of Theorem 1.1 i.e., it is essential to study (1.2) instead of $\lambda_{1,p}(X)$ in our setting.
We end this section by giving a direct consequence of Theorem 1.2.

**Corollary 3.8.** Let \( \epsilon > 0 \) and let \( M \) be an \( n \)-dimensional compact Riemannian manifold with (1.6). Assume that

\[
(\lambda_{1,p}(M))^{1/p} < \epsilon
\]

holds for some \( 1 \leq p \leq \infty \). Then we see that

\[
(\lambda_{1,q}(M))^{1/q} < \epsilon C(n,K)
\]

holds for every \( 1 \leq q \leq \infty \).

4. **New Lichnerowicz-Obata type theorem for limit spaces**

In this section, we establish a new Lichnerowicz-Obata type theorem for limit spaces (Theorem 4.1) and give an application (Corollary 4.4). Note that Theorem 4.1 for \( p = 2 \) is a direct consequence of Cheeger-Colding’s result [14, Theorem 7.9], Colding’s result [16, Lemma 1.10], and Croke’s result [19, Theorem B]. See also [1, 3, 34]. We use the following notation for convenience:

\[
(\lambda_{1,\infty}(X))^{1/\infty} := \frac{2}{\text{diam } X}.
\]

**Theorem 4.1.** Let \((X, \nu) \in M(n, 1, \pi)\) and let \( 1 < p \leq \infty \). Then we have

\[
(\lambda_{1,p}(X))^{1/p} \geq (\lambda_{1,p}(S^n))^{1/p}.
\]

(4.1)

Moreover, we see that the equality of (4.1) holds if and only if \( \text{diam } X = \pi \) holds.

**Proof.** The assertion for \( p = \infty \) follows from Myers’s diameter theorem [58]. Thus, we assume \( 1 < p < \infty \). Then, (4.1) follows directly from Matei’s result [52, Theorem 3.1] and Theorem 1.1. On the other hand, Valtorta’s result [71, Theorem 3.2.4] and Theorem 1.1 yield that if \( \lambda_{1,p}(X) = \lambda_{1,p}(S^n) \) holds, then \( \text{diam } X = \pi \) holds.

Therefore, we assume \( \text{diam } X = \pi \). Then, Cheeger-Colding’s warped product theorem [11, Theorem 5.14] yields that there exists a compact geodesic space \( Y \) with \( \text{diam } Y \leq \pi \) such that \( X \) is isometric to the spherical suspension \( S^0 \ast Y \) of \( Y \), where

\[
S^0 \ast Y := ([0, \pi] \times Y) / \{\{0, \pi\} \times Y\}
\]

and the distance is defined by

\[
[(s, y), [(t, z)] := \arccos (\cos s \cos t + \sin s \sin t \cos y, z).
\]

Thus, we identify \( X \) with \( S^0 \ast Y \). Let \( x_0 := [(0, y_0)] \) and let \( x_1 := [(\pi, y_0)] \), where \( y_0 \) is a fixed point in \( Y \).

**Claim 4.2.** For every \( f \in L^1([0, \pi]) \), we see that

\[
\int_X f \circ r_{x_0} d\nu = \int_{S^n} f \circ r_{y_1} d\text{Vol}
\]

holds for every \( w \in S^n \).
The proof is as follows. By [35, Theorem 5.2], there exists a positive valued $L^\infty$-function $g$ on $X$ such that
\[
\int_X h d\nu = \int_0^\pi \int_{\partial B_t(x_0)} ghd\nu_{-1} dt
\]
holds for every $h \in L^1(X)$ (see [13, 35] for the definition of the measure $\nu_{-1}$).

On the other hand, by Colding’s argument in the proof of [17, Lemma 5.10] (or [60]), we have
\[
v(B_R(x_0) \setminus B_r(x_0)) = \text{Vol}(B_R(w) \setminus B_r(w))
\]
for any $0 < r < R \leq \pi$. Thus, (4.3) and (4.4) yield that
\[
\int_R^r \int_{\partial B_t(x_0)} g d\nu_{-1} dt = \text{Vol}(B_R(w) \setminus B_r(w))
\]
holds for any $0 < r < R \leq \pi$. In particular, we see that
\[
\int_{\partial B_t(x_0)} g d\nu_{-1} = \frac{H^{n-1}(\partial B_t(w))}{H^n(S^n)}
\]
holds for a.e. $t \in [0, \pi]$. Claim 4.2 follows from (4.3) and (4.5).

Let $\hat{f}$ be a first eigenfunction for $\lambda_{1,p}(S^n)$. It is known that $\hat{f}$ is in $C^{1,\alpha}$ for some $\alpha > 0$ and that without loss of generality, we can assume that $\hat{f}$ is radial from a point $w \in S^n$, i.e., there exists a function $F$ on $[0, \pi]$ such that $\hat{f} = F \circ r_w$ holds (see for instance [52, Corollary 3.1]). Let $f := F \circ r_{x_0}$.

By Claim 4.2, we have
\[
c_p(f) = c_p(\hat{f}).
\]
On the other hand, by (4.2), it is easy to check that
\[
\text{Lip}_f(x) = \left| \frac{dF}{dt}(x_0, x) \right|
\]
holds for every $x \in X \setminus \{x_0, x_1\}$. In particular, by Claim 4.2 we have
\[
\int_X (\text{Lip}_f)^p dv = \int_X \left| \frac{dF}{dt}(x_0, x) \right| dv = \int_{S^n} \left| \frac{dF}{dt}(w, x) \right| d\text{Vol} = \int_{S^n} |\nabla f|^p d\text{Vol}.
\]
Thus, Corollary 2.11 (4.6), and (4.7) yield
\[
\lambda_{1,p}(X) \leq (c_p(f))^{-p} \int_X (\text{Lip}_f)^p dv = \lambda_{1,p}(S^n).
\]
Thus, by (4.1) and (4.8), we have the assertion. □

**Remark 4.3.** By Lévy-Gromov’s isoperimetric inequality [27] and (1) of Theorem 1.1 we have the inequality $h(X) \geq h(S^n)$ for every $(X, \nu) \in M(n, 1, \pi)$. It is expected that the equality holds if and only if diam $X = \pi$ holds. If $F$ as in Section 1 is continuous on $M(n, K, d) \times [1, \infty]$, then this follows from Bayle’s result [2].

We end this section by giving the following application of Theorem 4.1.
Corollary 4.4. Let $\epsilon > 0$, let $p > 1$, and let $M$ be an $n$-dimensional compact Riemannian manifold with $\text{Ric}_M \geq n - 1$. Assume that there exists $p \leq q \leq \infty$ such that
\[ \left| \left( \lambda_{1,q}(M) \right)^{1/q} - \left( \lambda_{1,q}(S^n) \right)^{1/q} \right| < \epsilon \]
holds. Then, we have
\[ \left| \left( \lambda_{1,q}(M) \right)^{1/q} - \left( \lambda_{1,q}(S^n) \right)^{1/q} \right| < \Psi(\epsilon; n, p) \]
for every $p \leq \hat{q} \leq \infty$.

Proof. The proof is done by contradiction. Assume that the assertion is false. Then there exist a positive number $\tau > 0$, a sequence $p_i \to p_\infty$ in $[p, \infty]$, a sequence $\hat{p}_i \to \hat{p}_\infty$ in $[\hat{p}, \infty]$, and a sequence $(M_i, \text{Vol}) \to (M_\infty, \nu)$ in $\mathcal{M}(n, 1, \pi)$ such that
\[ \lim_{i \to \infty} \left( \frac{1}{p_i} \right) \left( \lambda_{1,p_i}(M_i) \right) = \left( \frac{1}{p_\infty} \right) \left( \lambda_{1,p_\infty}(S^n) \right) \]
holds and that
\[ \left| \left( \lambda_{1,p_i}(M_i) \right)^{1/p_i} - \left( \lambda_{1,p_\infty}(S^n) \right)^{1/p_\infty} \right| \geq \tau \]
holds for every $i < \infty$. Thus, by Theorem 4.1, we see that
\[ \left( \lambda_{1,p_\infty}(M_\infty) \right)^{1/p_\infty} = \left( \lambda_{1,p_\infty}(S^n) \right)^{1/p_\infty} \quad (4.9) \]
and
\[ \left( \lambda_{1,\hat{p}_\infty}(M_\infty) \right)^{1/\hat{p}_\infty} \neq \left( \lambda_{1,\hat{p}_\infty}(S^n) \right)^{1/\hat{p}_\infty} \quad (4.10) \]
hold. However, (4.9) and (4.10) contradict Theorem 4.1. $\square$

Remark 4.5. It is expected that we can also choose $p = 1$ as in Corollary 4.4. If $F$ as in Section 1 is continuous on $\mathcal{M}(n, K, d) \times [1, \infty]$, then this also follows from an argument similar to the proof of Corollary 4.4.

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