Sigma, tau and Abelian functions of algebraic curves

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Abstract
We compare and contrast three different methods for the construction of the differential relations satisfied by the fundamental Abelian functions associated with an algebraic curve. We realize these Abelian functions as logarithmic derivatives of the associated sigma function. In two of the methods, the use of the tau function, expressed in terms of the sigma function, is central to the construction of differential relations between the Abelian functions.

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1. Introduction

In the mid-1970s, the results of Its, Matveev, Dubrovin and Novikov (see [9]) led to the discovery of a remarkable \( \theta \)-functional formula to solve the KdV equation

\[
    u_t = 6u u_x - u_{xxx}.
\]

This solution was given as a second logarithmic derivative of a Riemann theta-function:

\[
    u(x, t) = -\frac{\partial^2}{\partial x^2} \ln \theta(Ux + Vt + W) + C \tag{1.1}
\]

with \( U, V, W = \text{const} \in \mathbb{C}^g \) and \( C \in \mathbb{C} \). The theta function in that formula was constructed from a hyperelliptic curve \( X_g \), while the ‘winding vectors’ \( U, V \) are periods of Abelian differentials of the second kind on \( X_g \). Further, this formula is in a sense universal; it was generalized by Krichever [20] to other integrable hierarchies—whose solutions were associated with other algebraic curves. In this paper we consider the converse problem.

Given an algebraic curve \( X_g \), of genus \( g \), its Riemann period matrix \( \tau \) and its Jacobi variety \( \text{Jac}(X_g) = \mathbb{C}/(1_g \oplus \tau) \), we may construct \( \theta \)-functions \( \theta(u; \tau) \), \( u \in \text{Jac}(X_g) \); then the fundamental Abelian functions on \( \text{Jac}(X_g) \) may be realized as the second logarithmic

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derivatives of $\theta(u, \tau)$, $\varphi_j(u) = -\frac{\partial^2 \ln \theta(u, \tau)}{\partial u_j \partial u_j}$. We wish to construct all differential relations between these Abelian functions on $\text{Jac}(X_g)$.

The simplest case, the Weierstrass cubic, $y^2 = 4x^3 - g_2x - g_3$, is an algebraic curve of genus one. This is uniformized by the Weierstrass elliptic functions, $x = \varphi(u)$, $y = \varphi'(u)$, and the differential relations are

\begin{align*}
\varphi'' &= 6\varphi^2 - \frac{g_2}{2}, \\
\varphi'^2 &= 4\varphi^3 - g_2\varphi - g_3. \quad (1.2)
\end{align*}

In the case of higher genera, $g > 1$, the derivation of analogous equations becomes much more complicated. In particular, the fundamental Abelian functions are now the partial derivatives of a function of $g$ variables. The different approaches to this problem form the main content of the paper. We restrict our analysis to the case of $(n, s)$-curves introduced and investigated in this context by Buchstaber, Leykin and Enolski [3, 4]:

\begin{equation}
X_g: \quad y^n = x^d + \sum_{n_i + s_j < n} \lambda_{ij} x^i y^j. \tag{1.3}
\end{equation}

These represent a natural generalization of elliptic curves to higher genera, and include the general hyperelliptic curve ($n = 2$).

With any such curve we may associate an object which is fundamental to all our treatments of this problem, the fundamental bi-differential. This is the unique symmetric meromorphic 2-form on $X_g \times X_g$, whose only second-order pole lies on the diagonal $Q = S$, and which satisfies

\[
\omega(Q, S) - \frac{d\xi(Q) d\xi(S)}{(\xi(Q) - \xi(S))^2} = \Phi(\xi(S), \xi(Q)) \frac{d\xi(Q) d\xi(S)}{\xi(Q) - \xi(S)},
\]

where $\Phi(\xi(S), \xi(Q))$ is holomorphic, and $\xi(Q), \xi(S)$ are local coordinates in the vicinity of a base point $P, \xi(P) = 0$. Usually $\omega(Q, S)$ is realized as the second logarithmic derivative of the prime form or theta function [14]. However, in our development we use an alternative representation of $\omega(Q, S)$ in the algebraic form that goes back to Weierstrass and Klein, and which was well documented by Baker [1]:

\[
\omega(Q, S) = \frac{\mathcal{F}(Q, S)}{f_z(Q) f_w(S)(x - z)^2} \, dz \, dx + 2 du(Q)^T \times du(S). \tag{1.4}
\]

where $Q = (x, y)$, $S = (z, w)$, and the function $\mathcal{F}(Q, S) = \mathcal{F}((x, y), (z, w))$ is a polynomial of its arguments with coefficients depending on the parameters of the curve $X_g$, $du = (du_1, \ldots, du_g)^T$ is the vector of basic holomorphic differentials. The factor 2 in the final term is introduced so that this theory reduces to the Weierstrass theory in the elliptic case. Here, $x$ is a symmetric matrix expressed in terms of the first and second period matrices, $2\omega$ and $2\eta$, respectively, as $x = \omega^{-1}\eta$. This provides the normalization of $\omega(Q, S)$. We will refer to the first term on the right-hand side of (1.4), which involves the polynomial $\mathcal{F}(Q, S)$, as the algebraic part, so that

\[
\omega(Q, S) = \omega^{al}(Q, S) + 2 du(Q)^T \times du(S). \tag{1.5}
\]

This representation was revisited and developed by Buchstaber, Leykin and Enolski [3] and more recently by Nakayashiki [24].

The algebraic representation of the fundamental differential, as described above, lies behind the definition of the multivariate sigma function in terms of the theta function. This differs from $\theta$ by an exponential and modular factor:

\[
\sigma(u) = C(\tau) \exp \left[ \frac{1}{2} u^T \omega^{-1}\eta u \right] \theta \left( \frac{1}{2} \omega^{-1}u; \tau \right). \tag{1.6}
\]

Here the $g \times g$ matrices $2\omega, 2\eta$ are the first and second period matrices, and $\tau = \omega^{-1}\omega'$. The modular constant $C(\tau)$ is known explicitly for hyperelliptic curves and a number of other
cases. However its explicit form is not necessary here, for the fundamental Abelian functions are independent of \( C(\tau) \). These modifications make \( \sigma(u) \) invariant with respect to the action of the symplectic group, so that for any \( \gamma \in Sp(2g, \mathbb{Z}) \), we have

\[
\sigma(u; \gamma \tau) = \sigma(u; \tau).
\] (1.7)

The multivariate sigma function is the natural generalization of the Weierstrass sigma function to algebraic curves of higher genera, i.e. \((n, s)\)-curves in this context. In his lectures [31], Weierstrass started by defining the sigma function in terms of series with coefficients given recursively, which was the key point of the Weierstrass theory of elliptic functions. A generalization of this result to the genus-two curve was started by Baker [2] and recently completed by Buchstaber and Leykin [6], who obtained recurrence relations between the coefficients of the sigma series in closed form. In addition, Buchstaber and Leykin recently found an operator algebra that annihilates the sigma function of a higher genera \((n, s)\)-curve [7]. The recursive definition of the higher genera sigma functions remains a challenging problem to solve, with [7] providing a definite step. We believe that the future theory of the sigma and corresponding Abelian functions can be formulated on the basis of sigma expansions that will complete the extension of the Weierstrass theory to curves of higher genera.

In this paper we study the interrelation of the multivariate sigma and Sato tau functions. The \( \tau \)-function was introduced by Sato [29, 30] in the much more general context of integrable hierarchies. But it seems there are few results studying algebraic curves in Sato theory. However we should mention recent work by Konopelchenko and Ortenzi analysing algebro-geometric structure in Birkhoff strata of the Sato Grassmannian [19]. The recent papers of Matsutani and Previato [22, 23], studying Jacobi inversion on Jacobian strata of \((r, s)\) curves, are also relevant to our work, relating stratification of the Sato Grassmannian to partitions.

Here we deal with the ‘algebro-geometric \( \tau \)-function’ (AGT) associated with an algebraic curve. The AGT of the genus-\( g \) curve \( X_g \) is defined, following Fay [15, 16], as a function of the ‘times’ \( t = (t_1, \ldots, t_g, t_{g+1}, \ldots) \), a point \( u \in Jac(X_g) \), as well as a point \( P \in X_g \); it is given by the formula

\[
\tau(t; u, P) = \theta \left( \sum_{k=1}^{\infty} U_k(P) t_k + \frac{1}{2} \omega^{-1} u \right) \exp \left\{ \frac{1}{2} \sum_{m, n \geq 1} \omega_{mn}(P) t_m t_n \right\}.
\]

Here the ‘winding vectors’ \( U_k(P) \) appear in the expansion of the normalized holomorphic integral \( v \), and the quantities \( \omega_{mn}(P) \) define the holomorphic part of the expansion of the fundamental differential of the second kind \( \omega(Q, S) \) near point \( P \). Rather than the above, we then introduce the \( \tau \)-function by the equivalent formula

\[
\frac{\tau(t; u, P)}{\tau(0; u, P)} = \sigma \left( \sum_{k=1}^{\infty} A^{-1} U_k(P) t_k + u \right) \frac{1}{\sigma(u)} \exp \left\{ \frac{1}{2} \sum_{k, l=0}^{\infty} \omega^{alg}_{kl}(P) t_k t_l \right\}. \tag{1.8}
\]

This modular invariant representation of \( \tau \) in terms of \( \sigma \) was used by Harnad and Enolski [13] to analyse the Schur function expansion of \( \tau \) for the case of algebraic curves. Recently Nakayashiki [26] has independently suggested a similar expression for the AGT in terms of multivariate \( \sigma \)-functions. In this paper we concentrate on the application of this representation to the derivation of the differential relations between the Abelian functions of the \((n, s)\)-curve, continuing and developing the work of [13].

Developing a further analogy with the Weierstrass theory of elliptic functions, we represent the Abelian functions, that is, the \( 2g \)-periodic functions

\[
F(u + 2n\omega + 2n'\omega') = F(u) \quad \forall n, n' \in \mathbb{N}
\]
on the Jacobian
\[ \widetilde{\text{Jac}}(X_g) = \mathbb{C}/2\omega \oplus 2\omega' = A^{-1}\text{Jac}(X_g), \quad 2\omega = A, \]
as second and higher logarithmic derivatives:
\[ \zeta_i(u) = \frac{\partial}{\partial u_i} \ln \sigma(u), \quad (1.9) \]
\[ \varphi_{ij}(u) = -\frac{\partial^2}{\partial u_i \partial u_j} \ln \sigma(u), \quad \varphi_{ijk}(u) = -\frac{\partial^3}{\partial u_i \partial u_j \partial u_k} \ln \sigma(u), \quad \text{etc.}, \quad (1.10) \]
where \( i, j, k, \ldots = 1, \ldots, g \). We should remark that \( \zeta(u) \) are the non-Abelian functions. In this notation, the genus-one Weierstrass equations (1.2) become
\[ \varphi_{1111} = 6\varphi_{111}^2 - g_2, \quad \varphi_{1111}^2 = 4\varphi_{111}^3 - g_2\varphi_{111} - g_3. \]
For general \( g \), \( \varphi_{ij}, \varphi_{ijk} \ldots \) are called the Kleinian \( \varphi \)-functions. They are convenient coordinates to represent the dependent variables in the hierarchy of integrable systems.

In this paper we compare and contrast three approaches to obtain the partial differential relations for the Abelian functions associated with the \((n, s)\)-curve \( X_g \), using Kleinian \( \varphi \)-functions as coordinates.

The first of these, and the best known, is the classical approach of comparing two different expansions of the fundamental bi-differential; this yields first the solution of the Jacobi inversion problem for the curve, and in higher orders, a sequence of differential relations involving \( \varphi_{ij} \).

The \( \tau \)-function approach to the derivation of completely integrable systems of KP type has led to two different ways [8] to obtain the relations between the Taylor coefficients of the \( \tau \)-function expansion. The first of these specifically exploits the fact that these Taylor coefficients are determinants, i.e. the Plücker coordinates in the Grassmanian, and they hence satisfy the Plücker relations [15]. The second is based on the bilinear identity, which leads to the residue formula [16, 21], giving a family of differential polynomials in \( \tau \) which must vanish—these give the partial differential equations (PDEs) we need.

We further consider and compare the two techniques based on the \( \tau \)-function method, which give a derivation of the required differential relations; specializing to a particular algebraic curve, we consider its AGT. The special feature of our development is that we define this \( \tau \)-function in terms of the multidimensional \( \sigma \)-function of the curve, leading to coordinates that are explicitly written in terms of Kleinian \( \varphi \)-functions. The differential relations we find between these functions can be understood as arising from special solutions of integrable hierarchies of KP type, associated with the given curve (see for example [1, 3, 24]). We will describe the correspondence between individual differential equations for \( \varphi \)-functions with Young diagrams defining Plücker relations. We illustrate these approaches by considering two particular examples: the genus-two hyperelliptic curves [2] and the genus-three trigonal curve, which can be found in [3, 5, 12].

We remark that in similar context, Nimmo [27] in 1982 applied the methods of symmetric function theory to describe the KP hierarchy, and studied the action of the recursion operator. The general approach based on the Plücker coordinates for deriving KP-flows in terms of Kleinian \( \sigma \)-functions was recently discussed in [13]. Here we develop these ideas and consider some non-trivial examples to clarify the interrelation of the \( \tau \)-functional formulation of integrable hierarchies and the \( \sigma \)-functional approach. We will also consider the relationship between this derivation, which is based on the Plücker relations, and that based on the residue formula. Both give a systematic way of generating the required relations, but the differences between the two approaches are instructive.
2. Algebraic curves

Let \( X_g \) be a genus-\( g \geq 1 \) algebraic curve given by the polynomial equation
\[
f(x, y) = 0, \quad f(x, y) = y^g + y^{g-1}\alpha_1(x) + \cdots + \alpha_0(x).
\]
We shall consider in what follows two relatively simple curves of the class (2.1),

**Example I:** the hyperelliptic genus-two curve
\[
f(x, y) = y^2 - (4x^5 + \alpha_4x^4 + \cdots + \alpha_0);
\]

**Example II:** the cyclic trigonal genus-three curve
\[
f(x, y) = y^3 - (x^4 + \mu_3x^3 + \mu_6x^2 + \mu_9x + \mu_{12}).
\]

We equip \( X_g \) with a canonical basis of cycles \((a_1, \ldots, a_g; b_1, \ldots, b_g) \in H_1(X, \mathbb{Z})\). We denote by \( du = (du_1, \ldots, du_g)\) the vector whose entries are independent holomorphic differentials of the curve \( X_g \) as well as their \( a \) and \( b \)-periods,
\[
2\omega = \begin{pmatrix}
\int_{a_i} du_i \\
\int_{b_j} du_i
\end{pmatrix}_{i,j=1,\ldots,g}, \quad 2\omega' = \begin{pmatrix}
\int_{a_i} du_i \\
\int_{b_j} du_i
\end{pmatrix}_{i,j=1,\ldots,g}.
\]
The period matrix \((2\omega, 2\omega')\) is the first period matrix, and the matrix \( \tau = \omega^{-1}\omega' \) belongs to the upper Siegel half-space, \( \mathcal{G} : \tau^T = \tau, \Im \tau > 0 \).

The \( \theta \)-function \( \theta(\alpha \mid z; \tau) \) with characteristics \([\alpha] = [a_{\tau}^T b_{\tau}^T]\) and \([2\alpha] \in \mathbb{Z}^g \times \mathbb{Z}^g\) of the algebraic curve \( X_g \) of genus \( g \) is defined through its Fourier series
\[
\theta \begin{bmatrix} a_{\tau}^T \\ b_{\tau}^T \end{bmatrix} (z; \tau) = \sum_{m \in \mathbb{Z}^g} \exp\{i\pi (m + a)^T \tau (m + a) + 2i\pi (m + a)^T (z + b)\}
\]
and possesses the periodicity property: for arbitrary \( a, b \in \mathbb{Q}^g \) and arbitrary \( a', b' \in \mathbb{Q}^g \), the following formula holds
\[
\theta \begin{bmatrix} a_{\tau}^T \\ b_{\tau}^T \end{bmatrix} (z + \tau a' + b'; \tau) = \theta(\alpha + \alpha' \mid (b + b')^T (z + b)) \times \exp(-2i\pi a'_{\tau} a' + 2i\pi (b + b')^T z - 2i\pi (b + b')^T a').
\]

We introduce the associated meromorphic differentials \( dr = (dr_1, \ldots, dr_g)^T \) and their periods
\[
2\eta = -\begin{pmatrix}
\int_{a_i} dr_i \\
\int_{b_j} dr_i
\end{pmatrix}_{i,j=1,\ldots,g}, \quad 2\eta' = -\begin{pmatrix}
\int_{a_i} dr_i \\
\int_{b_j} dr_i
\end{pmatrix}_{i,j=1,\ldots,g}
\]
which form the second period matrix \((2\eta, 2\eta')\). The period matrices satisfy the condition
\[
\begin{pmatrix}
\omega & \omega' \\
\eta & \eta'
\end{pmatrix}
\begin{pmatrix}
0 & 1_g \\
-1_g & 0
\end{pmatrix}
\begin{pmatrix}
\omega & \omega' \\
\eta & \eta'
\end{pmatrix}^T
= -\frac{i\pi}{2}
\begin{pmatrix}
0 & 1_g \\
-1_g & 0
\end{pmatrix}.
\]

Here we denote the half-periods of the holomorphic and meromorphic differentials by \((\omega, \omega')\) and \((\eta, \eta')\), respectively, in order to emphasize the analogy with the Weierstrass theory. We will also use the notation \( \mathcal{A} = 2\omega \) and \( \mathcal{B} = 2\omega' \) for the periods of holomorphic differentials. Further we denote
\[
\nu(Q) = (dv_1(Q), \ldots, dv_g(Q))^T = \mathcal{A}^{-1}du(Q)
\]
as the vector of normalized holomorphic differentials.

The explicit calculation of canonical holomorphic differentials and the meromorphic differentials conjugate to them is well understood; in particular we have
Example I
\[
\begin{align*}
\text{du}_1 &= \frac{\text{dx}}{y}, \\
\text{du}_2 &= \frac{\text{dx}}{y}, \\
\text{dr}_1 &= \frac{x^2 \text{dx}}{y}, \\
\text{dr}_2 &= \frac{x(\alpha_3 + 2\alpha_4 x + 12x^2)}{4y} \text{dx}.
\end{align*}
\]

Example II
\[
\begin{align*}
\text{du}_1 &= \frac{\text{dx}}{3y}, \\
\text{du}_2 &= \frac{x \text{dx}}{3y^2}, \\
\text{du}_3 &= \frac{\text{dx}}{3y^2}, \\
\text{dr}_1 &= \frac{x^2 \text{dx}}{3y^2}, \\
\text{dr}_2 &= \frac{2x \text{dx}}{3y}, \\
\text{dr}_3 &= -\frac{(5x^2 + 3\mu_3 x + \mu_6)}{3y} \text{dx}.
\end{align*}
\]

Remark 2.1. Note that our labelling of the differentials is the reverse of [12], with the interchange 1 ↔ 2 in Example I, and (1, 2, 3) ↔ (3, 2, 1) in Example II.

We introduce the fundamental bi-differential \(\omega(Q, S)\) on \(X \times X\) which is uniquely defined by the following conditions:

(i) it is symmetric:
\[
\omega(Q, S) = \omega(S, Q);
\]

(ii) it has its only pole along the diagonal \(Q = S\), in which neighbourhood it can be expanded in a power series as
\[
\omega(Q, S) = \frac{\text{d}\xi(Q) \text{d}\xi(S)}{\xi(Q) - \xi(S)} = \sum_{m,n \geq 1} \omega_{mn}(P) \xi(Q)^{n-1} \xi(S)^{n-1} \text{d}\xi(Q) \text{d}\xi(S); \quad (2.7)
\]

(iii) it is normalized such that
\[
\oint_{a_j} \omega(Q, S) = 0, \quad j = 1, \ldots, g. \quad (2.8)
\]

The well-known realization of the differential \(\omega(Q, S)\) involves the Schottky–Klein prime form \(E(Q,S)\), which is a \((-1/2, -1/2)\)-differential defined for arbitrary points \(Q, S \in X\),
\[
E(Q, S) = \theta([\alpha])(\int_Q^S \text{dv})/h_{\omega}(Q)h_{\omega}(S), \quad (2.9)
\]
where \(\theta(\alpha)(\alpha)\) is a \(\theta\)-function with non-singular odd characteristics \([\alpha]\), \(\text{dv}\) is the vector of normalized holomorphic differentials and
\[
h_{\omega}(Q)^2 = \sum_{k=1}^g \frac{\partial}{\partial z_k} \theta([\alpha]) \text{du}_k(Q).
\]
The bi-differential \(\omega(Q, S)\) is then given by [14]
\[
\omega(Q, S) = \text{d}Q \text{d}s \ln E(Q, S). \quad (2.10)
\]

We emphasize that in this paper, we will instead rely on alternative ‘algebraic’ constructions of the differential \(\omega(Q, S)\). By following classical works such as [17, 18], together with results documented in [1] we express the differential \(\omega(Q, S)\) in the form
\[
\omega(Q, S) = \frac{\mathcal{F}(Q, S)}{f_y(Q)f_w(S)(x - z)^2} \text{dx} \text{dz} + 2d u(Q)^T \times du(S), \quad (2.11)
\]
where \( Q = (x, y), S = (z, w) \) and the function \( \mathcal{F}(Q, S) = \mathcal{F}((x, y), (z, w)) \) is a polynomial of its arguments, with coefficients depending on the moduli of the curve \( X_g \). Finally, \( \kappa \) is a symmetric matrix \( \kappa^T = \kappa \) that is chosen to provide a normalization of \( \omega(Q,S) \); it is expressible in terms of the first and second period matrices \( \kappa = \omega^{-1} \eta \). We will refer to the term of \( \omega(Q,S) \) including the polynomial \( F(Q, S) \) as its algebraic part,

\[
\omega(Q,S) = \omega_{\text{alg}}(Q, S) + 2 \mathbf{d} u(Q)^T \kappa \mathbf{d} u(S).
\]

(2.12)

In the vicinity of a base point \( P \), where points \( Q \) and \( S \) are represented by the local coordinates \( \xi(Q) \) and \( \xi(S) \), respectively, the holomorphic part of \( \omega_{\text{alg}}(Q, S) \) is expanded in the series

\[
\omega_{\text{alg}}(Q, S) = \sum_{k,l=0}^{\infty} \omega_{k,l}(P) \xi(Q)^k \xi(S)^l \mathbf{d} \xi(Q) \mathbf{d} \xi(S).
\]

(2.14)

An algorithm to construct the polynomial \( \mathcal{F}(Q, S) \) is known, see e.g. [1], and therefore the functions such as \( \omega_{k,l}(P) \) are needed for our construction and are considered as known. In all that follows we will take the fixed base point \( P \) to be \((\infty, \infty)\). We shall present below some explicit expressions for \( \omega_{\text{alg}}(Q, S) \) in the simplest cases.

**Example I:**

\[
\omega_{\text{alg}}((x, y), (z, w)) = \frac{F(x, z) + 2yw^2}{4(x-z)^2} \frac{dx}{y} \frac{dz}{w}, \quad Q = (x, y), S = (z, w),
\]

(2.15)

where

\[
F(x, z) = 4x^2z^2(x+z) + 2\alpha_4x^2z^2 + \alpha_3xz(x+z) + 2\alpha_2xz + \alpha_1(x+z) + 2\alpha_0.
\]

The polynomial \( F(x,z) \), with the properties \( F(x, z) = F(z, x) \) and \( F(x, x) = 2y^2 \), is sometimes called the ‘Kleinian polar’.

Expanding this about the base point gives

\[
\omega_{0,0}^{\text{alg}} = -\frac{\alpha_4}{8},
\]
\[
\omega_{1,0}^{\text{alg}} = \omega_{1,0}^{\text{alg}} = 0,
\]
\[
\omega_{2,0}^{\text{alg}} = \omega_{2,0}^{\text{alg}} = -\frac{16\alpha_3 - 3\alpha_2^2}{128}, \quad \omega_{1,1}^{\text{alg}} = 0,
\]

etc.

**Remark 2.2.** For hyperelliptic curves, the coefficients \( \omega_{i,j}^{\text{alg}} \) vanish if either of \( i \) or \( j \) is odd.

**Example II:**

\[
\omega_{\text{alg}}((x, y), (z, w)) = \frac{\mathcal{F}((x, y), (z, w))}{(x-z)^2 f_y(x, y) f_w(z, w)} dxdz
\]

(2.16)

with the polynomial \( \mathcal{F}((x, y); (z, w)) \) given by the formula

\[
\mathcal{F}((x, y), (z, w)) = w^2 y^2 + w \left[ \frac{f(x, y)}{y} \right] + T(x, z) + y \left[ \frac{f(z, w)}{w} \right] + T(z, x)
\]

(2.17)
\[ T(x, z) = 3\mu_{12} + (z + 2x)\mu_9 + x(x + 2z)\mu_6 + 3\mu_3x^2z + x^2z^2 + 2x^3z. \] (2.18)

Expanding this about the base point gives
\[
\omega_{0,0}^{\text{alg}} = 0,
\omega_{0,1}^{\text{alg}} = \omega_{1,0}^{\text{alg}} = -\frac{2}{3}\mu_3,
\omega_{0,4}^{\text{alg}} = \omega_{4,0}^{\text{alg}} = -\frac{2}{7}\mu_6 + \frac{5}{7}\mu_3^2,
\omega_{1,3}^{\text{alg}} = \omega_{3,1}^{\text{alg}} = -\frac{2}{3}\mu_6 + \frac{4}{7}\mu_3^2,
\omega_{2,2}^{\text{alg}} = 0,
\vdots
\]

**Remark 2.3.** \( \omega_{i,j}^{\text{alg}} = 0 \) unless \( i + j \equiv 1 \mod 3 \). This is a consequence of the cyclic symmetry of the curve.

### 3. Algebro-geometric \( \theta, \sigma, \) and \( \tau \)-functions

Let \( v \in \text{Jac}(X_g) \) and \( \theta(v) \) be a canonical \( \theta \)-function, that is, a \( \theta \)-function with zero characteristics:

\[
\theta(v) = \sum_{m \in \mathbb{Z}^g} \exp\{i\pi m^T \tau m + 2i\pi v^T m\}. 
\]

The starting point of this paper is the transition from \( \theta \)- to \( \sigma \)-functions. For any point \( u \in \text{Jac}(X) \) we define

\[
\sigma(u) = C(\tau)\theta(A^{-1}u)\exp\left\{\frac{1}{2}u^T \tau u\right\},
\]

where \( \theta(v) \) is the canonical \( \theta \)-function, and \( C(\tau) \) is a certain modular constant that we do not need for the results that follow. We note that this \( \sigma \)-function differs from the ‘fundamental \( \sigma \)-function’ described in the publications mentioned in the introduction by the absence of a shift of the \( \theta \)-argument by the vector of Riemann constants. Thus \( \sigma(0) \neq 0 \). The sigma-function inherits a quasi-periodicity property from the corresponding \( \theta \)-function.

We introduce the Kleinian multi-variable \( \zeta \) and \( \wp \)-functions as above (1.9, 1.10). These functions are suitable coordinates to describe Abelian functions and the KP-type hierarchies of differential relations between them.

In higher genera, the relations between Abelian functions become somewhat lengthy. These relations can often be summarized concisely by developing a matrix formulation of the theory, as was done [3] in the hyperelliptic case.

The Sato–Fay AGT of the genus-\( g \) curve \( X_g \) of arguments \( t = (t_1, \ldots, t_g, t_{g+1}, \ldots)^T \), \( u \in \text{Jac}(X) \), \( P \in X_g \) is defined as

\[
\tau(t; u, P) = \theta \left( \sum_{k=1}^{\infty} U_k(P) t_k + A^{-1}u \right) \exp\left\{ \frac{1}{2} \sum_{m, n \geq 1} \omega_{mn}(P) t_m t_n \right\}. 
\]

Here \( A \) is the matrix of periods of canonical holomorphic differentials and the winding vectors defined in (3.3), \( U_k(P) \), appear in the expansion of the normalized holomorphic integral \( v \) in the vicinity of the given point \( P \in X \):

\[
\int_{p_0}^{Q} dv(Q') = \int_{p_0}^{P} dv(Q') + \sum_{k=1}^{\infty} U_k(P) \xi(Q)^k.
\]
with $\xi(Q)$ being the local coordinate of point $Q$ in the vicinity of the given point $P$, so that $\xi(P) = 0$. The quantities $\omega_{mn}(P)$ define the holomorphic part of the expansion of the fundamental second kind differential $\omega_Q(S)$ near point $P$ according to (2.7).

We restrict ourselves to the case of algebraic curves with a branch point at infinity, and take $P = (\infty, \infty)$ to be the base point where we expand all our functions. The winding vectors are in this case $U_k(\infty) \equiv U_k = A^{-1} R_k$, $k = 1, \ldots, g$. (3.3)

where $R_1, R_2, \ldots$ are the residues of canonical holomorphic integrals multiplied by the differentials of the second kind with poles of order $k$ at infinity, giving

$$
R_k = \frac{1}{k} \left. \frac{d^{k-1}}{d\xi(Q)^{k-1}} \frac{du(Q)}{d\xi(Q)} \right|_{Q=\infty}^{-1}.
$$

Using the above definitions, we can see that the AGT is given by the formula, equivalent to (1.8),

$$
\tau(t; u) = \sigma \left( \sum_{k=1}^{\infty} R_k t_k + u \right) \exp \left\{ \frac{1}{2} \sum_{k,l=0}^{\infty} \sigma_{k,l} t_k t_l \right\}, \quad \text{etc.}
$$

In (3.4), $\sigma_{k,l}^{alg}$ is the algebraic part of the holomorphic part $\omega_{k,l}$ of the expansion of the bi-differential as defined in (2.14). One can see that the non-algebraic part, i.e. the normalizing bi-linear form, is absorbed into the $\sigma$-function.

Once we have an expression involving the derivatives of the sigma function, we need to convert this to an expression involving the derivatives of the $\wp$-function. To do this we start with the definition of the $\zeta$-function (1.9), which we write in the form

$$
\sigma_i(u) = \zeta_i(u) \sigma(u), \quad i = 1, \ldots, g;
$$

then repeated differentiation gives us a ladder of relations which enable us to recursively express any derivative of $\sigma$ in terms of $\wp_{ij,k}, \zeta_i, \text{and } \sigma$, all evaluated at $u$:

$$
\sigma_{ij}(u) = \sigma_j(u) \zeta_i(u) - \sigma(u) \wp_{ij}(u)
$$

$$
\sigma_{ijk}(u) = \sigma_{jk}(u) \zeta_i(u) - \sigma(u) \wp_{ijk}(u) - \sigma(u) \wp_{i}(u) - \sigma_i(u) \wp_{ij}(u) - \sigma_j(u) \wp_{ik}(u),
$$

etc.

In the following sections, we will look at different methods for constructing such relations between the derivatives of $\sigma$, and hence between the Abelian functions associated with the curve.

4. The ‘classical’ method

The starting point for this approach is the Klein formula, which compares two different expressions for the fundamental bi-differential

**Theorem 4.1.** Let the canonical holomorphic differentials of the curve $f(x, y) = 0$ be represented in the form

$$
\frac{du_i(x, y)}{f_j(x, y)} dx, \quad i = 1, \ldots, g,
$$

where $U_i(x, y)$ are the monomials of their variables. Then the following $g$ relations hold

$$
\sum_{i,j=1}^{g} \wp_{ij} \left( \int_{P_0}^{(x_1, y_1)} du - \sum_{k=1}^{g} \int_{P_0}^{(x_k, y_k)} du + K_{P_0} \right) U_i(x, y) U_j(x_k, y_k) = \frac{F(x, y; x_k, y_k)}{(x - x_k)^2}, \quad k = 1, \ldots, g.
$$

(4.1)
with the polynomial \( F(P, Q) = F(x, y; z, w) \) defined in (2.12), an arbitrary base point of the Abel map \( P_0 \) and the corresponding vector of Riemann constants, \( K_{P_0} \).

As an application of this theorem consider the case of a curve with a branch point at \( P_0 = (\infty, \infty) \). Let us tend \( P = (x, y) \rightarrow (\infty, \infty) \). Both the right- and left-hand sides of (4.1) have poles at infinity. Equating the principal part of poles, we get a set of relations between \( \wp \) and the multi-indexed \( \wp \)-symbols which can be interpreted as differential equations for the \( \wp_{k,l} \)-functions. In this way, the solution of the Jacobi inversion problem can also be derived in terms of \( \wp \)-functions.

4.1. Example: hyperelliptic curve of genus two

The \( \sigma \)-functional realization of hyperelliptic functions of a genus-two curve has already been discussed in many places; see e.g. [1, 3]. But we shall briefly describe here the principal points of the construction to convey the structure of the theory that we wish to develop for higher genera curves.

We consider the genus-two hyperelliptic curve of Example I. The algebraic part of the fundamental bi-differential is given as above in equation (2.15) [1].

Following the procedure in those papers, we equate the principal parts of the highest (second-order) pole to obtain

\[
\wp_{12} + x \wp_{11} - x^2 = 0,
\]

which is the \( x \)-part of the Jacobi inversion problem for this curve. Then equating the principal parts of the next, lower pole and using (4.2) leads to the relation

\[
y_k = -\wp_{112} - x_k \wp_{111},
\]

which completes the solution of the Jacobi inversion problem.

We can use these relations to eliminate \( y \) and quadratic and higher terms in \( x \), and we get from the next term

\[
(\frac{1}{2} \wp_{1111} - 3 \wp_{11}^2 - \frac{1}{2} \alpha_4 \wp_{11} - 2 \wp_{12} - \frac{1}{2} \alpha_3) x_k + \frac{1}{2} \alpha_4 \wp_{112} - \frac{1}{2} \alpha_4 \wp_{112} - 3 \wp_{11} \wp_{12} + \wp_{22} = 0.
\]

This equation holds for both \( x_k \), so the coefficients of different powers of \( x_k \) must be identically zero, and solving for \( \wp_{1111} \) and \( \wp_{1112} \) we find

\[
\wp_{1111} = 6 \wp_{11}^2 + \alpha_4 \wp_{11} + 4 \wp_{12} + \frac{1}{2} \alpha_3,
\]

\[
\wp_{1112} = 6 \wp_{11} \wp_{12} + \alpha_4 \wp_{12} - 2 \wp_{22},
\]

the first two 4-index relations in the genus-two case. A relation for 3-index functions \( \wp_{i,j,k} \) can be obtained by cross-derivation of (4.4) and (4.5):

\[
\wp_{122} + \wp_{11} \wp_{112} - \wp_{12} \wp_{111} = 0.
\]

Other equations are found from higher order terms. At every stage we need to substitute for higher derivative terms (such as \( \wp_{11111} \), for example) by using the derivatives of previously derived relations. In addition, multiplication by a 3-index \( \wp_{i,j,k} \) is sometimes useful, followed by substitution of known relations which are quadratic in \( \wp_{i,j,k} \). The first such relation is found to be

\[
\text{Jac}_6 : \quad \wp_{11}^2 = 4 \wp_{11}^3 + \alpha_3 \wp_{11} + \alpha_4 \wp_{11}^2 + 4 \wp_{12} \wp_{11} + \alpha_2 + 4 \wp_{22}.
\]

This is one of the relations that describe the Jacobi variety of the curve as algebraic variety. These relations are well documented in [2]; see also [3]. Below we show an alternative way to derive these relations.
We remark briefly that one variation on the classical method is to invoke the use of the sigma function expansion in the \(u_i\) variables. The first few terms in the sigma expansion are derived from the lowest order expansion terms of (4.1). These can then be used in a bootstrap fashion to derive higher order PDEs for the \(\wp\)-functions. This approach was essential in the genus-six (4,5) case considered in [10], which is discussed in more detail in that paper.

5. Derivation of integrable hierarchies via Plücker relations

The key to this approach is Sato’s formula, theorem (5.1) below. This gives an expansion of a ratio of \(\tau\)-functions, \(\tau(t;u)/\tau(0;u)\), as a series of rational expressions in the \(\tau\)-function and its derivatives at \(t = 0\). The \(t\)-dependence is given in terms of the Schur polynomials \(s_i(t)\) in the times \(t_i\). The coefficients of these polynomials are the determinants of differential expressions of \(\tau\), which are Plücker coordinates on a Grassmannian. Such coordinates satisfy the Plücker relations—each partition \(\lambda\) can be expanded in hooks, and the corresponding Plücker coordinates are expressible, analogously to Giambelli’s formula, as determinants of single-hook partitions. These relations give the differential relations for the Abelian functions which we seek.

For any partition \(\lambda : \alpha_1 \geq \alpha_2 \geq \ldots \geq \alpha_n\) of \(|\lambda| = \sum_i \alpha_i\), the Schur polynomial of \(n\) variables \(x_1, \ldots, x_n\) is defined by

\[
s_\lambda(x) = \det(p_{\alpha_i-(i+j)}(x))_{i,j=1,\ldots,n},
\]

where the elementary Schur functions \(p_m(x)\) are generated by the series

\[
\sum_{m=0}^{\infty} p_m(x) t^m = \exp \left\{ \sum_{n=1}^{\infty} x_n t^n \right\}.
\]

The first few Schur polynomials are

\[
\begin{align*}
s_1(x) &= x_1, \\
s_2(x) &= x_2 + \frac{1}{2} x_1^2, \quad s_{1,1}(x) = -x_2 + \frac{1}{2} x_1^2, \\
s_3(x) &= x_3 + x_1 x_2 + \frac{1}{6} x_1^3, \quad s_{2,1}(x) = -x_3 + \frac{1}{5} x_1^3, \quad s_{1,1,1}(x) = x_3 - x_1 x_2 + \frac{1}{6} x_1^3, \\
s_4(x) &= x_4 + x_1 x_3 + \frac{1}{2} x_2^2 + \frac{1}{4} x_1^2 x_2 + \frac{1}{2} x_1^4, \quad \text{etc.}
\end{align*}
\]

The Cauchy–Littlewood formula

\[
\exp \left\{ \sum_{n=1}^{\infty} x_n y_n \right\} = \sum_\lambda s_\lambda(x) s_\lambda(y),
\]

where \(s_\lambda(x)\) is the Schur function of the partition \(\lambda : \alpha_1 \geq \alpha_2 \geq \ldots \geq \alpha_n\), leads to the Taylor expansion in the form

\[
f(x) = \exp \left\{ \sum_{n=1}^{\infty} x_n \frac{\partial}{\partial y_n} \right\} f(y) \bigg|_{y=0} = \sum_\lambda s_\lambda(x) s_\lambda \left( \frac{1}{n} \frac{\partial}{\partial y_n} \right) f(y) \bigg|_{y=0}.
\]

We now introduce the Frobenius notation for partitions. Single-hook partitions \((\alpha, \beta)\) are denoted \((\alpha + 1, 1^\beta)\). All partitions can be decomposed into finitely many hooks:

\[
(\lambda) = (\alpha_1, \ldots, \alpha_r | \beta_1, \ldots, \beta_r).
\]

The total number of hooks \(r\) is called the rank of the partition \(\lambda\), [28]. In particular, a partition which decomposes into two hooks (rank 2) is written as

\[
(n, m, 2^k, 1^l) = (n-1, m-2|k+l+1, k).
\]
where \( n > m > 1, k \geq 0 \) and \( l \geq 0 \). Most of the formulae below are derived using rank 2 partitions.

Giambelli’s formula now shows how to expand a Schur function \( s_\lambda \) in hooks:

\[
s_\lambda(x) = \det(s_{(\alpha_i, \beta_j)}(x))_{1 \leq i, j \leq r}.
\]

**Theorem 5.1** (Sato formula). Let \( \tau(t; \, u) \) be any function of vector arguments

\[
U_1 t_1, U_2 t_2, \ldots, \ t = (t_1, t_2, \ldots) \in \mathbb{C}^\infty,
\]

where \( U_1, U_2, \ldots \) is an infinite set of constant complex vectors from \( \mathbb{C}^k \) and \( u \in \mathbb{C}^k \) is a parameter. Suppose that \( \tau(0; \, u) \neq 0 \). Then for any partition \( (\lambda) = (\alpha_1, \ldots, \alpha_j | \beta_1, \ldots, \beta_l) \) and any \( u \in \mathbb{C}^k \),

\[
\frac{\tau(t; \, u)}{\tau(0; \, u)} = \sum_k s_k(t) \det((-1)^{\beta_i + 1} A_{(\alpha_i | \beta_j)}(u)),
\]

where \( A_{(m|n)}(u) \) with \( m > 0, n > 0 \) form a linear basis of the Grassmannian:

\[
A_{(m|n)}(u) = -A_{(n|m)}(-u) = (-1)^{m+1} s_{m+1} \frac{\partial}{\partial t} \tau(t; \, u)|_{t=0} \tau(0; \, u)^{-1}
\]

\[
= \sum_{\alpha=0}^m p_{m+n+1} (-\partial_\alpha) p_{m-\alpha} (\partial_t) \tau(t; \, u)|_{t=0} \tau(0; \, u)^{-1}
\]

and

\[
\partial_t = \left( \frac{\partial}{\partial t_1}, \frac{1}{2} \frac{\partial}{\partial t_2}, \frac{1}{3} \frac{\partial}{\partial t_3}, \ldots \right).
\]

**Theorem 5.2** (Plücker relations). For any partition \( (\lambda) = (\alpha_1, \ldots, \alpha_j | \beta_1, \ldots, \beta_l) \) and any \( u \in \mathbb{C}^k \), the \( \tau \)-function satisfies

\[
\frac{\tau(0; \, u)}{\tau(t; \, u)} = \det(s_{(\alpha_i | \beta_j)}(\partial_t) \tau(t; \, u)|_{t=0}).
\]

In particular, for any curve \( X \), its Sato AGT has the expansion

\[
\frac{\tau(t; \, u)}{\tau(0; \, u)} = 1 + A_{(00)}(u) + A_{(10)}(u) + A_{(20)}(u) + \cdots + A_{(m|n)}(u) + A_{(01)}(u) s_1(t) + A_{(11)}(u) s_2(t) + A_{(21)}(u) \left( t_2 + \frac{1}{2} t_1 \right) + \cdots,
\]

where \( A_{(m|n)}(u) \) as defined in (5.2) are the elements of the semi-infinite matrix \( A \),

\[
A = \begin{pmatrix}
A_{(00)}(u) & A_{(01)}(u) & A_{(02)}(u) & \cdots \\
A_{(10)}(u) & A_{(11)}(u) & A_{(12)}(u) & \cdots \\
A_{(20)}(u) & A_{(21)}(u) & A_{(22)}(u) & \cdots \\
\vdots & \vdots & \vdots & \ddots
\end{pmatrix} = (A_0, A_1, \ldots)
\]

with infinite vectors \( A_i, \ i = 1, 2, \ldots \). With more complicated partitions such as \( (m_1, \ldots, m_k | n_1, \ldots, n_k) \) we associate according to the Giambelli formula certain minors of \( A \),

\[
A_{(m_1, \ldots, m_k | n_1, \ldots, n_k)}(u) = \det(A_{(m|n)}(u)), \ i, j = 1, \ldots, k.
\]

Note that these \( A \)'s are not independent—they satisfy the Plücker relations. These are a family of differential equations satisfied by \( \tau \), that represent a completely integrable hierarchy of KP type.
The definitions and relations given above are valid for any multivariate function \( \tau(t; u) \). Below we consider the \( \tau \)-functions constructed on the Jacobi varieties of algebraic curves.

To each symbol we shall put in correspondence its weight \( \wp_1, \ldots, \wp_{1k_1}, \ldots, \wp_{2k_2}, \ldots, \wp_{gkg} \) \( \Leftrightarrow \sum_{j=1}^{g} k_j w_j \), where \( w_i \) is the order of vanishing of the holomorphic integral \( \int d\omega_i \) at infinity. In other words, if the curve has a Weierstrass point at infinity, then \( w_i = 1 < w_2 < \cdots < w_g \) is the Weierstrass gap sequence at infinity.

For a given weight \( W \) consider all Young diagrams which decompose into only two hooks, and write the corresponding relations between multi-index symbols \( \wp_i \). The first non-trivial Young diagram corresponds to the partition \( \lambda = (2, 2) \). There is only one multi-index symbol of weight 4, that is \( \wp_{1111}(u) \), and its corresponding Plücker relation for any curve is of the form

\[
A(m+1, k+1+i, l) = A(m+1, k+i+1, l) A(m+1, k, l+1) A(n, k+1, l) A(n, k, l+1),
\]

These equations are all bilinear PDEs in the \( \tau \)-function. They may be expanded in terms of the Kleinian \( \wp_{ij} \) and \( \zeta_i \) functions.

The next group of diagrams are of weight 5, and correspond to the partitions \( \lambda = (3, 2) \) and \( \lambda = (2, 2, 1) \), with their transposes, which give the same equations, in the hyperelliptic case. Both these Plücker relations lead to the equation

\[
\left( \xi_1(u) + \frac{\partial}{\partial u_1} \right) \text{KdV}_4 = 0,
\]
i.e. the Young diagram of weight 5 gives no new equations, and we conclude that the following correspondence is valid:

\[
\begin{align*}
\lambda &= (2, 2) \\
\lambda &= (3, 3), \quad \lambda = (2, 2, 1)
\end{align*}
\]  \iff \text{KdV}_4

At weight 6 we have three independent Young diagrams with (2, 2) centres:

\[
\begin{array}{ccc}
& & \\
& & \\
& & \\
\end{array}, \quad
\begin{array}{ccc}
& & \\
& & \\
& & \\
\end{array}, \quad \text{and} \quad
\begin{array}{ccc}
& & \\
& & \\
& & \\
\end{array}
\]

The other two diagrams of weight 6

\[
\begin{array}{ccc}
& & \\
& & \\
& & \\
\end{array}, \quad
\begin{array}{ccc}
& & \\
& & \\
& & \\
\end{array}
\]

again give the same results as their transposes.

**Remark 5.3.** The Plücker relations associated with any Young diagram and its transpose are always the same for a hyperelliptic curve; however, for general curves, this is no longer true.

These three independent diagrams give an overdetermined system of three equations. After substituting for higher derivatives of the known relation (5.6), we can solve for the two unknowns \(\wp_{1112}\) and \(\wp_{111}^2\) to get

\[
\text{KdV}_6 : \quad \wp_{1112} = 6\wp_{112}\wp_{11} - 2\wp_{22} + \alpha_4\wp_{12} \tag{5.7}
\]

\[
\text{Jac}_6 : \quad \wp_{111}^2 = 4\wp_{111}^3 + \alpha_3\wp_{111} + \alpha_4\wp_{111}^2 + 4\wp_{112}\wp_{11} + \alpha_2 + 4\wp_{22}. \tag{5.8}
\]

The relation (5.7) is another generalization of the equation for \(\wp''\) in the genus-one case, and is identical to (4.5) found using the classical method. The relation (5.8) corresponds to Weierstrass’ equation for \((\wp')^2\) in the genus-one case. In what follows, we will always assume that higher derivatives of the known relations at a lower weight have been eliminated.

At weight 7 we have five independent Young diagrams with (2,2) centres. The resulting overdetermined system of equations, of weight 7, contains \(\xi_1\) multiplied by linear combinations of the two weight-6 relations (5.8,5.7). In addition we get a relation at weight 7, namely

\[
\wp_{112}\wp_{111} - \wp_{122} - \wp_{112}\wp_{111} = 0,
\]

as noted above. We refer to relations of this type, linear in the three-index \(\wp_{ijk}\), as quasilinear. These have no counterpart in the genus-one theory. They can also be derived by cross-differentiation between the two relations (5.6), (5.7), since

\[
\frac{\partial}{\partial t_2} \wp_{1111} = \frac{\partial}{\partial t_1} \wp_{1112}.
\]

Equations \text{KdV}_4 and \text{KdV}_6 describe the genus-two solutions of the KdV hierarchy associated with the given curve. To describe the Jacobi variety and the Kummer variety, we must consider diagrams of higher weights.

For the diagram of weight 8, we find a set of eight overdetermined equations with solution given by

\[
\text{Jac}_8 : \quad \wp_{111}\wp_{111} = \frac{1}{2}\alpha_3\wp_{112}^2 + 2\wp_{112}^2 - 2\wp_{112}\wp_{22} + \frac{1}{2}\wp_{11} + 4\wp_{112}\wp_{12} + \alpha_4\wp_{112}^2
\]

\[
\text{KdV}_8 : \quad \wp_{1122} = 2\wp_{11}\wp_{22} + 4\wp_{12}^2 + \frac{1}{2}\alpha_3\wp_{12}.
\]
At weight 9 the only new relation is the quasilinear relation
\[ 8 \wp_{122} \wp_{111} - 4 \wp_{112} \wp_{112} - 4 \wp_{222} + 2 \alpha_4 \wp_{122} - \alpha_3 \wp_{112} - 4 \wp_{111} \wp_{22} = 0. \]

At weight 10 we have 18 overdetermined systems of equations, which can be solved for the three functions of weight 10, \( \wp_{1222}, \wp_{1122}, \wp_{1112}, \wp_{122}, \) giving \( KdV_{10} : \)
\[ \wp_{1222} = 6 \wp_{122} \wp_{11} + \wp_{112} - \frac{3}{2} \wp_{111} - \alpha_0. \]
\[ \text{Jac}_{10}^{(1)} : \wp_{1111} \wp_{112} = -\frac{1}{2} \wp_{111} \wp_{112} + 2 \wp_{22} \wp_{111} + 2 \wp_{111} \wp_{112} + \alpha_3 \wp_{122} \wp_{11}. \]
\[ \text{Jac}_{10}^{(2)} : \wp_{1112} = \alpha_0 - 4 \wp_{22} \wp_{11} + \alpha_4 \wp_{12} + 4 \wp_{11} \wp_{22}. \]

The equations \( \text{Jac}_8, \text{Jac}_{10}^{(1)}, \text{Jac}_{10}^{(2)} \) represent an embedding of the Jacobi variety as a three-dimensional algebraic variety into the complex space \( \mathbb{C}^5 \) whose coordinates are \( \wp_{111}, \wp_{112}, \wp_{122}, \wp_{1112}, \wp_{1222}. \)

At weight 12 we get the final 4-index relation for \( \wp_{2222} \) and two quadratic 3-index relations for \( \wp_{1122}, \wp_{1112} \). We can continue in this manner at weight 14 to get more quadratic 3-index relations [3]. At the odd weights 11, 13, 15, we get quasilinear relations which can also be found by cross-differentiation. As a practical point we note that the equations we derive can often contain ideals generated by the lower weight relations, and some work is required to identify genuinely new relations.

At weight 16 we have 117 independent relations giving an overdetermined system of equations (we have checked only a selection of these). At this weight a new feature occurs. As well as the equations expressing the quadratic 3-index term \( \wp_{1222} \wp_{1122} \) in terms of cubics in \( \wp_{ij} \), we have terms which are quartic in \( \wp_{ij} \). We can pick one of these quartic terms, say \( \wp_{122} \), and solve for this and for \( \wp_{1222} \) to give us two relations. The relation involving a quartic in \( \wp_{ij} \) is just the Kummer variety of the curve. This is the quotient of the Jacobi variety, \( \text{Kum}(X) = \text{Jac}(X)/\{u \to -u\} \). In the case \( g = 2 \), the Kummer variety is a surface in \( \mathbb{C}^3 \) which is given analytically by a quartic equation. This relation can also be found from the identity
\[ (\wp_{111}^2)(\wp_{112}^2) - (\wp_{1111} \wp_{112})^2 = 0. \]

The same quartic also appears, multiplied by various factors, at higher weights.

5.1. Example: trigonal curve of genus three

As before we consider Example II, (2.3) whose holomorphic differentials are given in section 2 [5, 11, 12]. Here we generally follow the notation of [12]. The fundamental second kind differential is (2.16)–(2.18).

The first Young diagram leading to a non-trivial Plücker relation, as in the genus-two case, corresponds to the partition \( \lambda = (2, 2) \). Writing (5.3) in this case we obtain, after simplification,
\[ \wp_{1111} = 6 \wp_{11}^2 - 3 \wp_{22}. \]

The weight of the diagram is again 4 in this case, which is the same as the weight of the equation, defined as the weighted sum of indices in which the index ‘1’ has weight 1, index ‘2’ has weight 2 and index ‘3’ has weight 5. We will use this weight correspondence in other cases too, and will denote the weight of the object by subscript \( i + 2j + 5k \), where \( i, j \) and \( k \) are respectively the numbers of 1, 2 and 3 in the multi-index relation.

In the trigonal case we no longer have the symmetry about the diagonal of the diagram that we have in the genus-two case, but we can restrict ourselves by taking the symmetric or
antisymmetric combination of the two diagrams related by transposition. In the weight 5 case we have the antisymmetric combination

\[
\begin{array}{c}
\begin{array}{c}
\vdots \\
\end{array}
\end{array}
- 
\begin{array}{c}
\begin{array}{c}
\vdots \\
\end{array}
\end{array}
\]

which gives the weight 5 trigonal PDE

\[\varphi_{112} = 6\varphi_{11}\varphi_{12} + 3\mu_3\varphi_{11}.\]

For the symmetric case we get a derivative of the weight 4 equation, plus \(\xi_1\) multiplied by the same equation. With even (odd) weights, the symmetric (antisymmetric) combinations give the 4-index \(\varphi_{ijkl}\) relations.

At weight 6 we have the three symmetric diagrams

\[
\begin{array}{c}
\begin{array}{c}
\vdots \\
\end{array}
\end{array}
+ 
\begin{array}{c}
\begin{array}{c}
\vdots \\
\end{array}
\end{array}
+ 
\begin{array}{c}
\begin{array}{c}
\vdots \\
\end{array}
\end{array}
\]

which give a set of three overdetermined equations with the unique solution

\[\varphi_{111} = 4\varphi_{11}^2 + \varphi_{12}^2 + 4\varphi_{13} - 4\varphi_{11}\varphi_{22},\]

\[\varphi_{122} = 4\varphi_{13} + 2\mu_6 + 4\varphi_{12}^2 + 2\varphi_{11}\varphi_{22} + 3\mu_3\varphi_{12}.\]

Continuing in this way we recover the strictly trigonal versions of the full set of equations given in [12].

6. Derivation of the integrable hierarchies via residues

This method of deriving integrable equations on the Jacobians of algebraic curves is based on the following relation due to Sato, [29], [30], as applied to the algebro-geometric context by Fay [15].

The Hirota bilinear equations

\[
\text{Res}_{\xi=0} \frac{1}{\xi^2} \left[ \exp \left( \sum_{n=1}^{\infty} t_n \xi^{-n} \right) \exp \left\{ -\sum_{n=1}^{\infty} \frac{\xi^n}{n} \frac{\partial}{\partial t_n} \right\} \tau(t; u) \right. \\
\times \exp \left( \sum_{n=1}^{\infty} t'_n \xi^{-n} \right) \exp \left\{ -\sum_{n=1}^{\infty} \frac{\xi^n}{n} \frac{\partial}{\partial t'_n} \right\} \tau(t'; -u) \left. \right] = 0, \quad (6.1)
\]

where \(\xi = \xi(Q)\) is the local coordinate of point \(Q\) near \(P, \xi(P) = 0\) defining the KP hierarchy, are equivalent to the following bilinear identity.

Denote \(\Omega(f)\) the differential

\[
\Omega(t, P) = \sum_{k=1}^{\infty} k t_k \Omega_k(P), \quad (6.2)
\]

where \(\Omega_k(P)\) are the normalized second kind differentials with \(k\)th-order poles at infinity, so that \(\Omega(t, P)\) has an essential singularity

\[
\Omega(t, P) \sim \left( \sum_{n=1}^{\infty} n t_n \xi^{-n-1} + O(1) \right) d\xi,
\]

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and $\Omega_k(P)$ are defined by

$$
\Omega_k(P) = \frac{d\xi}{\xi^{k+1}} - \sum_{m=1}^{\infty} \frac{1}{m} \omega_{km} \xi^{m-1} d\xi,
$$

(6.3)

\[ \oint_{C_j} \Omega_k(P) = 0, \quad j = 1, \ldots, g. \]

**Theorem 6.1** (bilinear identity). Let $S^1$ be a small circle about the point $p \in X$; then

$$
\int_{S^1} \left( A(Q) - A(P) - \sum_{i=1}^{\infty} U_i t_i - u \right) \theta \left( A(Q) - A(P) - \sum_{i=1}^{\infty} U_i t_i' + u \right)
\times \exp \left\{ \int_{P_0}^{P} \Omega(t + t', P) \right\} E^{-2}(P, Q) d\xi(Q) = 0,
$$

(6.4)

where $\Omega(P) = \int_{P_0}^{P} dv$, Abelian image of a point $P$, $\xi(Q)$ is the local coordinate of point $Q$ near $P$, $\xi(P) = 0$, $E(P, Q)$ is the Schottky–Klein prime form (2.9) and the integration is around the unit circle centred at point $p$, $S^1 = \{ q, |\zeta(q)| = 1 \}$.

**Proof.** Note that the integrand in (6.4) is holomorphic in $q$ in the interior of $S^1$ except at its centre $Q = P$ where it has an isolated essential singularity with zero residue. Therefore this integral vanishes.

To prove the equivalence of (6.1) and (6.4) note first that

$$
\Omega(t + t') = \Omega(t) + \Omega(t').
$$

(6.5)

In (6.4), the factor

$$
\theta \left( A(Q) - A(P) - \sum_{i=1}^{\infty} U_i t_i - u \right) \exp \left\{ \int_{P_0}^{P} \Omega(t) \right\}
$$

(6.6)

can be expressed as

$$
\exp \left\{ \sum_{n=1}^{\infty} t_n \xi^{-n} \right\} \theta \left( A(Q) - A(P) - \sum_{i=1}^{\infty} U_i t_i - u \right) \exp \left\{ \sum_{m,n=1}^{\infty} \omega_{mn} t_n \xi^{-m} \right\}.
$$

Taking into account the form (3.2) for $\tau(t; u)$, the last two factors in the above formula may be expressed as

$$
\theta \left( A(Q) - A(P) - \sum_{i=1}^{\infty} U_i t_i - u \right) \exp \left\{ \sum_{m,n=1}^{\infty} \omega_{mn} t_n \xi^{-m} \right\} = \exp \left\{ -\sum_{n=1}^{\infty} \frac{\xi^n}{n} \frac{\partial^n}{\partial t_n} \right\} \tau(t; u),
$$

(6.7)

which completes the proof of the equivalence of (6.4) and (6.1).

**Remark 6.2.** Equations of this form were first written down in terms of ‘vertex operators’

$$
\exp \left\{ \sum_{l=1}^{\infty} t_{n,x^{-n}} \right\} \exp \left\{ -\sum_{l=1}^{\infty} \frac{x^n}{n} \frac{\partial^n}{\partial t_n} \right\}
$$

for general $\tau$-functions in the work of Sato. They may be understood as the generating functions for the integrable hierarchies of Hirota bilinear equations.

Using this theorem we may obtain PDEs relating the Kleinian symbols, $\wp_{ij}$, $\wp_{ijk}$, etc. To do this we now substitute the $\tau$-function (1.8) into this expression and compute the
residue. This is parameterized by \( t \) and \( e \). The coefficients of monomials in \( t \) are differential-difference expressions in \( \sigma \) at arguments \( +u \) and \( -u \). To get purely differential expressions, we now let \( u \to 0 \); the ‘time’ derivatives act on \( \sigma \) via its \( u \)-dependence, though, so \( \sigma_i(u) = \sigma_i(-u) \to \pm \sigma_i(0) \), etc. We then replace the derivatives of the \( \sigma \)-function by the derivatives of the \( \wp \)-function using the recursive relations (3.5) described earlier. This recovers the PDEs for the Kleinian functions which we need.

**Example I**

In the genus-two hyperelliptic case, the first nonvanishing relations occur at \( t \)-weight 3 in \( t_i \), i.e. the coefficients of \( t_i^3 \) or \( t \). Both these give the first 4-index relation

\[
\wp_{1111} = 6\wp_{11} + \alpha_4\wp_{11} + 4\wp_{12} + \frac{1}{2}\alpha_3.
\]

The next non-zero term is at weight 5, where we recover the relation for \( \wp_{1112} \), etc. In contrast with the previous section, this approach only gives the even derivative relations at odd \( t \)-weights.

The calculations by this approach, because they involve bilinear products of \( \tau \)-functions, soon become very computer-intensive, and we have not pursued them very far.

**Example II**

Within this method we are able to recover all quadratic relations for four indexed symbols \( \wp_{ijkl}, 1 \geq i, j, k, l \leq 3 \), and cubic relations for three indexed symbols \( \wp^{ijk} \). We report here the quartic relation between even variables, which must be one of the relations defining the Kummer variety of the \((3,4)\)-curve,

\[
\begin{align*}
\wp_{12}^4 - \wp_{22}^3 - 2\wp_{11}\wp_{13} + 2\wp_{13}^2 + 4\wp_{12}\wp_{13} + 3\wp_{11}\wp_{22} - 6\wp_{11}\wp_{13}\wp_{22} + 4\wp_{11}\wp_{12}\wp_{23} + \wp_{22}^2 - 2\wp_{11}\wp_{22} - 4\wp_{11}\wp_{13} + 8\wp_{11}\wp_{13} + 2\mu_{12} + 4\mu_6\wp_{11}^2 - 4\mu_6\wp_{11}\wp_{22} + 3\mu_5\wp_{12}\wp_{13} + 3\mu_5\wp_{11}\wp_{23} + \mu_3\wp_{13}^2 + 2\mu_6\wp_{13} + \mu_6\wp_{12} + \mu_9\wp_{12} - \mu_3^2\wp_{11} - 3\mu_3\wp_{11}\wp_{12}\wp_{22} &= 0.
\end{align*}
\]

This relation is of independent interest since it is of weight 12 and cannot be written in the form (5.9). The detailed structure of the Kummer variety in the \((3,4)\) case is receiving further investigation and will be reported elsewhere.

7. Discussion

In view of the ‘compare and contrast’ objectives of our paper, the reader may wish to know which of these methods is the most effective when calculating the required PDEs. Unfortunately this is not an easy question to answer. Currently, the two methods associated with the \( \tau \)-function take rather longer to execute, with the residue method in particular being slow and with large memory overheads. However this may be due in part to the fact that we have extensive experience over 10 years or more in developing computer algebra code for the ‘classical method’ in all its variations. We have much less experience in working with the \( \tau \)-function methods. So it may be that with more study and with more computational experience, the \( \tau \)-function methods become more competitive. We would stress, however, that both \( \tau \)-function methods are more systematic than the classical methods, and may provide a suitable way of calculating the PDEs associated with more complicated curves.

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