The hypersurfaces with conformal normal Gauss map in $H^{n+1}$ and $S^{n+1}_1$

Shuguo Shi

Abstract. In this paper we introduce the fourth fundamental form for the hypersurfaces in $H^{n+1}$ and the space-like hypersurfaces in $S^{n+1}_1$ and discuss the conformality of the normal Gauss maps of the hypersurfaces in $H^{n+1}$ and $S^{n+1}_1$. Particularly, we discuss the surfaces with conformal normal Gauss maps in $H^3$ and $S^3_1$ and prove a duality property. We give the Weierstrass representation formula for the space-like surfaces in $S^3_1$ with conformal normal Gauss maps. We also state the similar results for the time-like surfaces in $S^3_1$.

1 Introduction

It is well known that the classical Gauss map has played an important role in the study of the surface theory in $\mathbb{R}^3$ and has been generalized to the submanifold of arbitrary dimension and codimension immersed into the space forms with constant sectional curvature( see [15] in detail).

Particularly, for the $n$-dimensional submanifold $x: M \to V$ in space $V$ with constant sectional curvature, Obata[13] introduced the generalized Gauss map which assigns to each point $p$ of $M$ the totally geodesic $n$-subspace of $V$ tangent to $x(M)$ at $x(p)$. He defined the third fundamental form of the submanifold in constant curvature space as the pullback of the metric of the set of all the totally geodesic $n$-subspaces in $V$ under the generalized Gauss map. He derived a relationship among the Ricci form of the immersed submanifold and the first, the second and the third fundamental forms of the immersion. Meanwhile, Lawson[10] discussed the generalized Gauss map of the immersed

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surfaces in $S^3$ and prove a duality property between the minimal surfaces in $S^3$ and their generalized Gauss map image. Epstein[4] and Bryant[3] defined the hyperbolic Gauss map for the surfaces in $H^3$ and Bryant[3] obtained a Weierstrass representation formula for the constant mean curvature one surfaces with conformal hyperbolic Gauss map. Using the Weierstrass representation formula, Bryant also studied the properties of constant mean curvature one surfaces. Using the hyperbolic Gauss map, Gálvez and Martínez and Milán[6] studied the flat surfaces in $H^3$ with conformal hyperbolic Gauss map with respect to the second conformal structure on surfaces (see [7] for the definition) and obtained a Weierstrass representation formula for such as surfaces.

Kokubu[8] considered the $n$-dimensional hyperbolic space $H^n$ as a Lie group $G$ with a left-invariant metric and defined the normal Gauss maps of the surfaces which assigns to each point of the surface the tangent plane translated to the Lie algebra of $G$. He also gave a Weierstrass representation formula for minimal surfaces in $H^n$. On the other hand, Gálvez and Martínez[5] studied the properties of the Gauss map of a surface $\Sigma$ immersed into the Euclidean 3-space $\mathbb{R}^3$ by using the second conformal structure on surface and obtained the Weierstrass representation formula for the surfaces with prescribed Gauss map. Motivated by their work, the author[16] gave a Weierstrass representation formula for the surfaces with prescribed normal Gauss map and Gauss curvature in $H^3$ by using the second conformal structure on surfaces. From this, the surfaces whose normal Gauss maps are conformal have been found and the translational surfaces with conformal normal Gauss maps locally are given. In [17], the author classified locally the ruled surfaces with conformal normal Gauss maps within the Euclidean ruled surfaces and studied some global properties of the ruled surfaces and translational surfaces with conformal normal Gauss maps.

Aiyama and Akutagawa [1] defined the normal Gauss map for the space-like surfaces in the de Sitter 3-space $S^3$ and gave the Weierstrass representation formula for the space-like surfaces in $S^3$ with prescribed mean curvature and normal Gauss map.

The purpose of this paper is to study the conformality of the normal Gauss maps for the hypersurfaces in $H^{n+1}$ and the space-like hypersurfaces in $S^3$ and to prove a duality property between the surfaces in $H^3$ and the space-like surfaces in $S^3$ with conformal normal Gauss maps. The rest of this paper is organized as follows. In the second section, we describe the generalized definition of the normal Gauss map for the hypersurfaces in $H^{n+1}$ and the space-like hypersurfaces in $S^3$ (cf.[1][8]). The third section introduces the fourth fundamental form for the hypersurfaces in $H^{n+1}$ and $S^3$ and obtains a relation among the first, the second, the third and the fourth fundamental
forms of the hypersurfaces. As a application, we discuss the conf ormality
of the normal Gauss map for the hypersurfaces in $H^{n+1}$ and the space-like
hypersurfaces in $S^{n+1}_1$. By means of the generalized Gauss map of the surfaces
in $H^3$ and $S^3_1$, the fourth one proves a duality property between the surfaces
in $H^3$ and the space-like surfaces in $S^3_1$ with conformal normal Gauss maps.
The fifth one gives the Weierstrass representation formula for the space-like
surfaces in $H^3$ and $S^3_1$, the fourth one proves a duality property between the surfaces
in $H^3$ and the space-like surfaces in $S^3_1$ with conformal normal Gauss maps.
In the last section, we state the similar
results for time-like surfaces in $S^3_1$ with conformal normal Gauss map.

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## 2 Preliminaries

Take the upper half-space models of the hyperbolic space $H^{n+1}(-1)$ and the
de Sitter space $S^{n+1}_1(1)$

$$R^{n+1}_+ = \{ (x_1, x_2, \cdots, x_{n+1}) \in R^{n+1} | x_{n+1} > 0 \}$$

with respectively the Riemannian metric $ds^2 = \frac{1}{x_{n+1}^2}(dx_1^2 + dx_2^2 + \cdots + dx_{n+1}^2)$
and the Lorentz metric $ds^2 = \frac{1}{x_{n+1}^2}(dx_1^2 + dx_2^2 + \cdots + dx_n^2 - dx_{n+1}^2)$ (cf.[1]).

Let $M$ be a $n$-dimensional Riemannian manifold and $x : M^n \to H^{n+1}$ (resp. $x : M^n \to S^{n+1}_1$) be an immersed hypersurface (resp. space-like hypersurface)
with the local coordinates $u_1, u_2, \cdots, u_n$. In this paper, we agree with the fol-
lowing ranges of indices: $1 \leq i, j, k, \cdots \leq n$ and $1 \leq A, B, C, \cdots \leq n+1$. The first and the second fundamental forms are given, respectively, by $I = g_{ij} du_i du_j$ and $II = h_{ij} du_i du_j$. The unit normal vector (resp. time-like unit normal vec-
tor) of $x(M)$ is $\hat{N} = x_{n+1} \eta_1 \frac{\partial}{\partial x_1} + x_{n+1} \eta_2 \frac{\partial}{\partial x_2} + \cdots + x_{n+1} \eta_{n+1} \frac{\partial}{\partial x_{n+1}}$, where $\eta_1^2 + \eta_2^2 + \cdots + \eta_{n+1}^2 = 1$ (resp. $\eta_1^2 + \eta_2^2 + \cdots + \eta_n^2 - \eta_{n+1}^2 = -1$).

We have the Weingarten formula

$$\frac{\partial \eta_A}{\partial u_k} = \frac{1}{x_{n+1}} \left( \eta_{n+1} \frac{\partial x_A}{\partial u_k} - g^{ij} h_{kl} \frac{\partial x_A}{\partial u_j} \right)$$

(resp. $\frac{\partial \eta_A}{\partial u_k} = \frac{1}{x_{n+1}} \left( \eta_{n+1} \frac{\partial x_A}{\partial u_k} + g^{ij} h_{kl} \frac{\partial x_A}{\partial u_j} \right)$).


Identifying $H^{n+1}$ and $S_1^{n+1}$ with the Lie group (cf.[8])

$$G = \begin{pmatrix}
1 & 0 & \cdots & 0 & \log x_{n+1} \\
0 & x_{n+1} & \cdots & 0 & x_1 \\
\vdots & \vdots & \ddots & \vdots & \vdots \\
0 & 0 & \cdots & x_{n+1} & x_n \\
0 & 0 & \cdots & 0 & 1
\end{pmatrix} : (x_1, x_2, \cdots, x_{n+1}) \in \mathbb{R}^{n+1}_+,$$

the multiplication is defined as the matrix multiplication and the identity is $e = (0, 0, \cdots, 0, 1)$. The Riemannian metric of $H^{n+1}$ and the Lorentz metric of $S_1^{n+1}$ are left-invariant and \(\hat{X}_1 = x_{n+1} \frac{\partial}{\partial x_1}, \hat{X}_2 = x_{n+1} \frac{\partial}{\partial x_2}, \cdots, \hat{X}_{n+1} = x_{n+1} \frac{\partial}{\partial x_{n+1}}\) are the left-invariant unit orthonormal vector fields. Now, the unit normal vector (resp. time-like unit normal vector) field of $x(M)$ can be written as $N = \eta_1 \hat{X}_1 + \eta_2 \hat{X}_2 + \cdots + \eta_{n+1} \hat{X}_{n+1}. \tilde{N}$. Left translating $N$ to $T_e(R^{n+1}_+)$, we obtain

$$\tilde{N} : M \to S^n(1) \subset T_e(R^{n+1}_+)(\text{resp.} \tilde{N} : M \to H^{n}(1) \subset T_e(R^{n+1}_+)),$$

$$\tilde{N} = L_{x^{-1}*}(N) = \eta_1 \frac{\partial}{\partial x_1}(e) + \eta_2 \frac{\partial}{\partial x_2}(e) + \cdots + \eta_{n+1} \frac{\partial}{\partial x_{n+1}}(e).$$

Call $\tilde{N}$ the normal Gauss map of the immersed hypersurface $x : M \to H^{n+1}(\text{resp.} \text{space-like hypersurface } x : M \to S_1^{n+1})(\text{cf.[1][8]}).$

### 3 The fourth fundamental form

**Definition.** Let $M$ be a $n$-dimensional Riemannian manifold. Call $IV = \langle d\tilde{N}, d\tilde{N} \rangle$ the fourth fundamental form of the immersed hypersurface $x : M \to H^{n+1}(\text{resp.} \text{space-like hypersurface } x : M \to S_1^{n+1})$, where the scalar product $\langle \cdot, \cdot \rangle$ is induced by the Euclidean metric of $R^{n+1}$ (resp. the Lorentz-Minkowski metric of $L^{n+1}$).

**THEOREM 3.1.** Let $M$ be a $n$-dimensional Riemannian manifold with Ricci form $Ric$. Let $x : M \to H^{n+1}(\text{resp.} x : M \to S_1^{n+1})$ be an immersed hypersurface (resp. space-like hypersurface) with mean curvature $H = \frac{1}{n} tr(II)$. Then

$$IV = \eta_{n+1}^2 I - 2\eta_{n+1} II + III$$

(3.1)

$$\text{(resp.} IV = \eta_{n+1}^2 I + 2\eta_{n+1} II + III\text{)},$$

(3.2)

where $III = nHII - (n-1)I - Ric$ (resp. $III = nHII - (n-1)I + Ric$) is Obata’s third fundamental form of $x(M)$ (see [13]).
Proof. At first we prove the Theorem for \(H^{n+1}\). Choose the normal coordinates \(u_1, u_2, \cdots, u_n\) near \(p \in M\). By the Weingarten formula, we get

\[
IV = \langle d\tilde{N}, d\tilde{N} \rangle = \frac{\partial \eta_A}{\partial u_i} \frac{\partial \eta_A}{\partial u_j} du_i du_j
\]

\[
= \frac{1}{x_{n+1}^2} \left( \eta_{n+1} \frac{\partial x_A}{\partial u_i} - h_{ik} \frac{\partial x_A}{\partial u_k} \right) \left( \eta_{n+1} \frac{\partial x_A}{\partial u_j} - h_{ji} \frac{\partial x_A}{\partial u_i} \right) du_i du_j
\]

\[
= (\eta_{n+1}^2 \delta_{ij} - 2\eta_{n+1} h_{ij} + h_{ik} h_{jk}) du_i du_j.
\]

(3.3)

\(III = h_{ik} h_{jk} du_i du_j\) is the third fundamental form [13] and by the Gauss equation, \(III = nHII - (n-1)I - Ric\). (3.1) is proved.

Next, similar to the above proof, for \(S^{n+1}_1\), we have

\[
IV = (\eta_{n+1}^2 \delta_{ij} + 2\eta_{n+1} h_{ij} + h_{ik} h_{jk}) du_i du_j.
\]

(3.4)

Similar to the proof of (3.1), we can prove (3.2).

Next, we consider the applications of these formulas (3.1) – (3.4). In the following of this paper, that the normal Gauss map is conformal means that the fourth fundamental form is proportional to the second fundamental form, i.e. \(IV = \rho II\) for some smooth function \(\rho\) on \(M\).

**THEOREM 3.2.** Let \(M\) be a \(n\)-dimensional Riemannian manifold and \(x : M \to H^{n+1}\) (resp. \(x : M \to S^{n+1}_1\)) be an immersed hypersurface (resp. space-like hypersurface) without umbilics. Then the normal Gauss map of \(x(M)\) is conformal if and only if at each point of \(M\), there exists exactly two distinct principal curvatures and the sectional curvature \(R(X \wedge Y) = -1 + \eta_{n+1}^2\) (resp. \(R(X \wedge Y) = 1 - \eta_{n+1}^2\)), where the vectors \(X\) and \(Y\) belong to different principal direction spaces.

Proof. The case of \(H^{n+1}\). For any point \(p \in M\), let \(\{e_1, e_2, \cdots, e_n\}\) be a local frame field so that \((h_{ij})\) is diagonalized at this point, i.e. \(h_{ij}(p) = \lambda_i \delta_{ij}\). By \(IV = \rho II\) and (3.3), we get, for \(i = 1, 2, \cdots, n\), that

\[
\eta_{n+1}^2 - 2\eta_{n+1} \lambda_i + \lambda_i^2 = \rho \lambda_i,
\]

(3.5)

i.e.

\[
\lambda_i^2 - (\rho + 2\eta_{n+1}) \lambda_i + \eta_{n+1}^2 = 0.
\]

(3.6)

Because \(x(M)\) has no umbilics, the equation (3.6) with respect to \(\lambda_i\) has exactly two distinct solutions \(\lambda\) and \(\mu\) and \(\lambda \mu = \eta_{n+1}^2\). By the Gauss equation, one may prove \(R(X \wedge Y) = -1 + \lambda \mu = -1 + \eta_{n+1}^2\).
Conversely, choose the local tangent frame \( \{ e_1, e_2, \cdots, e_n \} \) and the dual frame \( \{ \omega_1, \omega_2, \cdots, \omega_n \} \) near \( p \), such that \( h_{ij} = 0, i \neq j \) and \( h_{11} = h_{22} = \cdots = h_{rr} = \lambda \neq \mu = h_{r+1r+1} = \cdots = h_{nn} \). Then \( \eta_{n+1}^2 = \lambda \mu \). By (3.3),

\[
IV = (\eta_{n+1}^2 - 2\eta_{n+1} \lambda + \lambda^2)(\omega_1^2 + \cdots + \omega_r^2)
  + (\eta_{n+1}^2 - 2\eta_{n+1} \mu + \mu^2)(\omega_{r+1}^2 + \cdots + \omega_n^2)
  = (\mu - 2\eta_{n+1} + \lambda)(\omega_1^2 + \cdots + \omega_r^2)
  + (\lambda - 2\eta_{n+1} + \mu)(\omega_{r+1}^2 + \cdots + \omega_n^2)
  = (\lambda - 2\eta_{n+1} + \mu)\II.
\]

The sufficiency has been proved for \( H^{n+1} \). Similarly, we can prove Theorem 3.2 for \( S_1^{n+1} \).

**Remark.** By (3.5), we know that the normal Gauss maps of all totally umbilics hypersurfaces except the totally geodesic hyperspheres in \( H^{n+1} \) are conformal. Similarly, for the space-like hypersurfaces in \( S_1^{n+1} \), since \( \eta_{n+1} \neq 0 \), the normal Gauss maps of all totally umbilic space-like hypersurfaces except totally geodesic space-like hypersurfaces are conformal.

For \( H^3 \) and \( S_3^1 \), by Theorem 3.2, we immediately get

**THEOREM 3.3.** Let \( M \) be a 2-dimensional Riemannian manifold and \( x : M \to H^3 \) (resp. \( x : M \to S^3_1 \)) be an immersed surface (resp. space-like surface) without umbilics. Then the normal Gauss map of \( x(M) \) is conformal if and only if the Gauss curvature \( K = -1 + \eta_3^2 \) (resp. \( K = 1 - \eta_3^2 \)).

**Remark.** In [16][17], we assume that the second fundamental form is positive definite and induces the conformal structure on the surfaces in \( H^3 \). Here, the assumption with respect to the positive definite second fundamental form is dropped.

**THEOREM 3.4.** Let \( M \) be a \( n \)-dimensional Einstein manifold and \( x : M \to H^{n+1} \) (resp. \( x : M \to S_1^{n+1} \)) be an immersed hypersurface (resp. space-like hypersurface) with the non-degenerate second fundamental form and without umbilics. If the normal Gauss map of \( x(M) \) is conformal map, i.e. \( IV = \rho\II \), then \( n = 2 \) and \( \rho = 2(H - \eta_3) \) (resp. \( \rho = 2(H + \eta_3) \)).

**Proof.** We only prove the Theorem for \( H^{n+1} \). \( M \) is an Einstein manifold, so \( Ric = \frac{\bar{S}}{n}I \), where \( \bar{S} \) is the scalar curvature of \( M \). (3.1) becomes

\[
\left( \eta_{n+1}^2 - (n - 1) - \frac{\bar{S}}{n} \right)I + (nH - 2\eta_{n+1} - \rho)\II = 0.
\]

Because \( x(M) \) has no umbilics, we have

\[
nH = 2\eta_{n+1} + \rho.
\]
By Theorem 3.2 and its proof, we assume that \( \lambda_1 = \cdots = \lambda_r = \lambda \neq \mu = \lambda_{r+1} = \cdots \lambda_n \), then
\[
 r\lambda + (n-r)\mu = 2\eta_{n+1} + \rho.
\]
By (3.6),
\[
 \lambda + \mu = 2\eta_{n+1} + \rho.
\]
So \((r-1)\lambda + (n-r-1)\mu = 0\). By Theorem 3.2, \( \lambda \) and \( \mu \) have same signature. So \( r = 1 \) and \( n = 2 \). Hence \( \rho = 2H - 2\eta_3 \).

\section{A duality for the surfaces in \( H^3 \) and \( S^3_1 \) with conformal normal Gauss maps}

Let \( L^4 \) be the Minkowski 4-space with the canonical coordinates \( X_0, X_1, X_2, X_3 \) and the Lorentz-Minkowski scalar product \(-X_0^2 + X_1^2 + X_2^2 + X_3^2\). The Minkowski model of \( H^3 \) is given by
\[
 H^3 = \{(X_0, X_1, X_2, X_3) | -X_0^2 + X_1^2 + X_2^2 + X_3^2 = -1, X_0 > 0\}
\]
and is identified with the upper half-space model \( \mathbb{R}^3_+ \) of \( H^3 \) by
\[
 (x_1, x_2, x_3) = \left( \frac{X_1}{X_0 - X_3}, \frac{X_2}{X_0 - X_3}, \frac{1}{X_0 - X_3} \right).
\]
Accordingly, the space-like normal vector of the surface in the Minkowski model of \( H^3 \) is \( N = N_0 \partial_{\partial X_0} + N_1 \partial_{\partial X_1} + N_2 \partial_{\partial X_2} + N_3 \partial_{\partial X_3} \), where
\[
 N_0 = \frac{X_1}{X_0 - X_3} \eta_1 + \frac{X_2}{X_0 - X_3} \eta_2 + \frac{1 - X_0(X_0 - X_3)}{X_0 - X_3} \eta_3,
\]
\[
 N_1 = \eta_1 - X_1 \eta_3, \quad N_2 = \eta_2 - X_2 \eta_3,
\]
\[
 N_3 = \frac{X_1}{X_0 - X_3} \eta_1 + \frac{X_2}{X_0 - X_3} \eta_2 + \frac{1 - X_3(X_0 - X_3)}{X_0 - X_3} \eta_3.
\]
We get
\[
 \eta_3 = \frac{N_0 - N_3}{X_3 - X_0} \quad (4.1)
\]
The Minkowski model of the de Sitter 3-space is defined as
\[
 S^3_1 = \{(X_0, X_1, X_2, X_3) | -X_0^2 + X_1^2 + X_2^2 + X_3^2 = 1\} \simeq S^2 \times \mathbb{R}
\]
and can be divided into three components as follows(cf. [1]),
\[
 S_- = \{X \in S^3_1 | X_0 - X_3 < 0\} \simeq \mathbb{R}^3,
\]
Identify $S_-$ and $S_+$ with the upper half-space model $R^3_+$ of the de Sitter 3-space by (cf. [1])

\[
(x_1, x_2, x_3) = \left( \frac{X_1}{|X_0 - X_3|}, \frac{X_2}{|X_0 - X_3|}, \frac{1}{|X_0 - X_3|} \right).
\]

For the space-like surface $X : M \to S^3_1$, let $U_- = X^{-1}(S_-)$ and $U_+ = X^{-1}(S_+)$, then $U_- \cup U_+$ is the open dense subset of $M$. On $U_- \cup U_+$, the time-like unit normal vector is $N = N_0 \frac{\partial}{\partial x_0} + N_1 \frac{\partial}{\partial x_1} + N_2 \frac{\partial}{\partial x_2} + N_3 \frac{\partial}{\partial x_3}$, where

- $N_0 = \frac{X_1}{X_0 - X_3} \eta_1 + \frac{X_2}{X_0 - X_3} \eta_2 - \frac{1 + X_0(X_0 - X_3)}{X_0 - X_3} \eta_3$,
- $N_1 = \eta_1 - X_1 \eta_3$,
- $N_2 = \eta_2 - X_2 \eta_3$,
- $N_3 = \frac{X_1}{X_0 - X_3} \eta_1 + \frac{X_2}{X_0 - X_3} \eta_2 - \frac{1 + X_3(X_0 - X_3)}{X_0 - X_3} \eta_3$.

We get

\[
\eta_3 = \frac{N_0 - N_3}{X_3 - X_0}.
\]

Remark. In [1], the normal Gauss map of the space-like surface $X : M \to S^3_1$ is defined globally on $M$. Because of the density of $U_-$ and $U_+$ in $M$, in this paper, we may consider that the normal Gauss map of the space-like surface $X : M \to S^3_1$ is defined on $U_-$ and $U_+$.

Let $X : M \to H^3$ (resp. $X : M \to S^3_1$) be an immersed surface (resp. space-like surface). Parallel translating the space-like (resp. time-like) unit normal vector $N$ to the origin of $L^4$, one gets the map $N : M \to S^3_1$ (resp. $N : M \to H^3$) which is usually called generalized Gauss map of $X : M \to H^3$ (resp. $X : M \to S^3_1$). The generalized Gauss map image can be considered as the surface in $S^3_1$ (resp. $H^3$).

**THEOREM 4.1** (cf [9], Prop 3.5). (1) Let $X : M \to H^3$ be a 2-dimensional immersed surface. Then its generalized Gauss map $N : M \to S^3_1$ is a branched space-like immersion into $S^3_1$ with branch points where $K = -1$. And, when $K \neq -1$, the curvature of $N : M \to S^3_1$ is $K^* = \frac{K}{K + 1}$ and the volume element is $dV_N = |K + 1| dV_X$.

(2) Let $X : M \to S^3_1$ be a 2-dimensional space-like immersed surface. Then its generalized Gauss map $N : M \to H^3$ is a branched immersion into $H^3$ with branch points where $K = 1$. And, when $K \neq 1$, the curvature of $N : M \to H^3$ is $K^* = \frac{K}{1 - K}$ and the volume element is $dV_N = |1 - K| dV_X$.
Proof. In the context of this paper, we prove (2). For any \( p \in M \), let \( \{e_0, e_1, e_2, e_3\} \) be the orthonormal frame near \( p \), such that \( e_0 = X, e_3 = N \). Let \( \{\omega_0, \omega_1, \omega_2, \omega_3\} \) be the dual frame. The connection 1-forms is \( \omega^\alpha_\beta, \alpha, \beta = 0, 1, 2, 3 \). The coefficients of the second fundamental form of \( X : M \to S_1^3 \) is given by \( \omega^3_i = h_{ij}\omega_j, h_{ij} = h_{ji}, i, j = 1, 2, 3 \). The induced metric of \( N : M \to H^3 \) is \( ds^2 = \langle dN, dN \rangle = h_{ik}h_{jk}\omega_i\omega_j \). Choose the local tangent frame \( \{e_1, e_2\} \) near \( p \), such that \( h_{ij} = \lambda_i\delta_{ij} \). Then \( ds^2 = \lambda_1^2\omega^2_1 + \lambda_2^2\omega^2_2 \). So, when \( \lambda_1\lambda_2 \neq 0 \), i.e. \( K \neq 1 \), \( N(M) \) is an immersed surface into \( H^3 \). Its space-like unit normal vector is \( X \) and the second fundamental form is \( II = -\langle dX, dN \rangle = -\lambda_1\omega^2_1 - \lambda_2\omega^2_2 \). By the Gauss equation, \( K^* = -1 + \frac{1}{\lambda_1\lambda_2} = \frac{K}{1-K} \).

By Theorem 3.3, (4.1), (4.2) and Theorem 4.1, we get the following duality.

**Theorem 4.2.** Let \( M \) be a connected 2-dimensional manifold. Let \( X : M \to H^3 \) be an immersed surface without umbilics and \( K \neq -1 \) and let \( N : M \to S_1^3 \) be a space-like surface without umbilics and \( K \neq 1 \). Suppose that \( N : M \to S_1^3 \) is the generalized Gauss map of \( X : M \to H^3 \) and vice versa. Then, the normal Gauss map of \( X : M \to H^3 \) is conformal if and only if one of \( N : M \to S_1^3 \) is conformal. And, at this time, \( dV_N = \left( \frac{N_0 - N_3}{X_3 - X_0} \right)^2 dV_X \).

**Remark.** Like [10] for minimal surfaces in \( S^3 \), we call the generalized Gauss map \( N : M \to S_1^3 \) the polar variety of the immersed surface \( X : M \to H^3 \) with conformal normal Gauss map and vice versa.

## 5 Weierstrass representation formula

In this section, we give the Weierstrass representation formula for the space-like surfaces in \( S_1^3 \) with conformal normal Gauss maps. At first, we describe the normal Gauss map and the de Sitter Gauss map of the space-like surfaces in \( S_1^3 \). Take the upper half-space model \( R^3_+ \) of \( S_1^3 \).

The normal Gauss map of the space-like surface \( x : M \to S_1^3 \) is given by \( \tilde{N} = \eta_1 \frac{\partial}{\partial x_1}(e) + \eta_2 \frac{\partial}{\partial x_2}(e) + \eta_3 \frac{\partial}{\partial x_3}(e) : M \to H^2(-1) \subset L^3 \). By means of the stereographic projection from the north pole \((0, 0, 1)\) of \( H^2(-1) \) to the \((x_1, x_2)\)-plane identified with \( C \), we get

\[
g^S = \frac{\eta_1 + i\eta_2}{1 - \eta_3} : M \to C \cup \{ \infty \} \setminus \{|z| = 1\},
\]

which is also called the normal Gauss map of the space-like surface \( x : M \to S_1^3 \).
\( \vec{N} \) can be written as

\[
\vec{N} = \left( -\frac{g + \bar{g}}{|g|^2 - 1}, \frac{i(g - \bar{g})}{|g|^2 - 1}, \frac{1 + |g|^2}{|g|^2 - 1} \right).
\]

Next, we describe the definition of the de Sitter Gauss map for the space-like surfaces in \( S^3_1 \) (in [11], it is still called hyperbolic Gauss map), which is the analogue of Epstein and Bryant’s hyperbolic Gauss map for the surfaces in \( H^3 \) (cf [3][4][16]). The time-like geodesic is either the Euclidean equilateral half-hyperbola consisting of two branches which is orthonormal to the coordinate plane \( \{(x_1, x_2, 0) | (x_1, x_2) \in \mathbb{R}^2 \} \cup \{\infty\} \) two points. Since the geodesic is oriented, we may speak of one of the two points as the initial point and the other one as the final point. Call the final point the image of the de Sitter Gauss map for \( x(M) \) at the point \( x \). Denote the de Sitter Gauss map by \( G^S \). On the coordinate plane \( \{(x_1, x_2, 0) | (x_1, x_2) \in \mathbb{R}^2 \} \), we introduce the natural complex coordinate \( z = x_1 + ix_2 \). Using the Euclidean geometry, as similar as done in the Theorem 5.1 of [16], we get

\[
G^S = x_1 + ix_2 + x_3g^S.
\]  

Let \( x = (x_1, x_2, x_3) : M \to H^3 \) be an immersed surface with unit normal vector \( N = x_3\eta_1\frac{\partial}{\partial x_1} + x_3\eta_2\frac{\partial}{\partial x_2} + x_3\eta_3\frac{\partial}{\partial x_3} \). By the duality given in section 4, the generalized Gauss map of \( x : M \to H^3 \) is given, when \( \eta_3 > 0 \), by

\[
N = \left( \frac{\eta_1}{\eta_3}x_3 - x_1, \frac{\eta_2}{\eta_3}x_3 - x_2, x_3 \right) : M \to S^3_1,
\]  

and when \( \eta_3 < 0 \), by

\[
N = \left( x_1 - \frac{\eta_1}{\eta_3}x_3, x_2 - \frac{\eta_2}{\eta_3}x_3, -\frac{x_3}{\eta_3} \right) : M \to S^3_1
\]  

and in the Minkowski model of the de Sitter 3-space, their time-like unit normal vector is \( X : M \to H^3 \). Again by the duality given in section 4, a straightforward computation shows us that the normal Gauss map of \( N : M \to S^3_1 \) is given by

\[
\tilde{N} = \frac{\eta_1}{\eta_3} \frac{\partial}{\partial x_1}(e) + \frac{\eta_2}{\eta_3} \frac{\partial}{\partial x_2}(e) + \frac{1}{\eta_3} \frac{\partial}{\partial x_3}(e) : M \to H^2(-1).
\]

So,

\[
g^S = \frac{\eta_1 + i\eta_2}{1 - \frac{1}{\eta_3}} = \frac{\eta_1 + i\eta_2}{\eta_3 - 1} = -g^H,
\]  

(5.4)
where \( g^H : M \to C \cup \{\infty\} \) is exactly the normal Gauss map of \( x : M \to H^3(\text{cf}[8][16][17]). \) From this, we also prove the Theorem 4.2.

By (5.1)-(5.4) and the Theorem 5.1 of [16], we get that when \( \eta_3 > 0, \) i.e. \( |g^S| > 1, \)
\[
G^S = -G^H, \tag{5.5}
\]
and when \( \eta_3 < 0, \) i.e. \( |g^S| < 1, \)
\[
G^S = G^H, \tag{5.6}
\]
where \( G^H \) is exactly the hyperbolic Gauss map of \( x : M \to H^3(\text{cf}[3][4][16]). \)

In the following, we write respectively \( g^S \) and \( G^S \) as \( g \) and \( G. \)

By (5.2)-(5.6) and the Weierstrass representation for the surfaces in \( H^3 \) with conformal normal Gauss map[16], we get the Weierstrass representation formula for the space-like surfaces in \( S^3_1 \) with conformal normal Gauss map.

**THEOREM 5.1.** Let \( M \) be a simply connected Riemannian surface. Given the map \( G : M \to C \cup \{\infty\} \) and the nonconstant conformal map \( g : M \to C \cup \{\infty\}\{ |z| = 1 \} \).

(1) When the holomorphic map \( g : M \to C \cup \{\infty\}\{ |z| = 1 \} \) satisfies \( |g| > 1 \) and
\[
\frac{G_z}{g_z} > 0, \tag{5.7}
\]
\[
|g|^2 |G_z| > |G_z|, \tag{5.8}
\]
\[
G_{zz} + \frac{\bar{g}_z}{(|g|^4 - 1)\bar{g}} G_z - \frac{|g|^2 \bar{g} g_z}{|g|^4 - 1} G_z = 0, \tag{5.9}
\]
put
\[
x_1 = \text{Re} \left\{ G - \frac{1 + |g|^2}{\bar{g} g_z} G_z \right\}, \tag{5.10}
\]
\[
x_2 = \text{Im} \left\{ G - \frac{1 + |g|^2}{\bar{g} g_z} G_z \right\}, \tag{5.11}
\]
\[
x_3 = \frac{1 + |g|^2}{|g|^2} G_z. \tag{5.12}
\]

Then \( x = (x_1, x_2, x_3) : M \to S^3_1 \) is a space-like surface with de Sitter Gauss map \( G \) and holomorphic normal Gauss map \( g \) and Gauss curvature \( K \) satisfying \( \sqrt{1 - K} = \frac{1 + |g|^2}{|g|^2 - 1}. \) And the conformal structure on \( M \) is induced by the negative definite second fundamental form. Conversely, any surface \( x : M \to S^3_1 \) with \( \sqrt{1 - K} = \frac{1 + |g|^2}{|g|^2 - 1}(= \eta_3) \) can be given by (5.10)(5.11)(5.12) and the de Sitter Gauss map \( G \) and the normal Gauss map \( g \) must satisfy (5.7)(5.8)(5.9),
where the conformal structure on \( M \) is induced by the negative definite second fundamental form.

(2) When the antiholomorphic map \( g : M \to C \cup \{ \infty \} \setminus \{|z| = 1\} \) without holomorphic points satisfies \(|g| < 1\) and

\[
\frac{G_z}{|g|^2 g_z} > 0,
\]

\[
\frac{|g|^2 |G_z|}{|G_z|} < 1,
\]

\[
G_{zz} + \frac{\bar{g}_z}{(|g|^4 - 1)g} G_z - \frac{|g|^2 \bar{g}g_z}{|g|^4 - 1} G_z = 0,
\]

put

\[
x_1 = \text{Re} \left\{ G - \frac{1 + |g|^2}{\bar{g}g_z} G_z \right\},
\]

\[
x_2 = \text{Im} \left\{ G - \frac{1 + |g|^2}{\bar{g}g_z} G_z \right\},
\]

\[
x_3 = \frac{1 + |g|^2}{|g|^2 g_z} G_z.
\]

Then \( x = (x_1, x_2, x_3) : M \to S^3_1 \) is a space-like surface with de Sitter Gauss map \( G \) and antiholomorphic normal Gauss map \( g \) and Gauss curvature \( K \) satisfying

\[
\sqrt{1 - K} = \frac{1 + |g|^2}{1 - |g|^2}.
\]

And the conformal structure on \( M \) is induced by the negative definite second fundamental form. Conversely, any surface \( x : M \to S^3_1 \) with \( \sqrt{1 - K} = \frac{1 + |g|^2}{1 - |g|^2} (= -\eta_b) \) can be given by (5.16)(5.17)(5.18) and the de Sitter Gauss map \( G \) and the normal Gauss map \( g \) must satisfy (5.13)(5.14)(5.15), where the conformal structure on \( M \) is induced by the negative definite second fundamental form.

### 6 Graphs and examples

In this section, we give the examples of surfaces in \( S^3_1 \) with conformal normal Gauss maps within the translational surfaces and the Euclidean ruled surfaces.

In \( H^3 \), the graph \((u, v, f(u, v))\) with conformal normal Gauss map satisfies the following fully nonlinear PDE (cf.[16][17])

\[
f(f_{uu} f_{vv} - f^2_{uv}) + [(1 + f^2_v) f_{uu} - 2 f_u f_v f_{uv} + (1 + f^2_u) f_{uv}] = 0.
\]

Take the upper half-space model of \( S^3_1 \). Consider the space-like graph \((u, v, f(u, v))\) in \( S^3_1 \) with \( f^2_u + f^2_v < 1 \). Its Gauss curvature is given by

\[
K = \frac{f^2 (f_{uu} f_{vv} - f^2_{uv}) - f [(1 - f^2_v) f_{uu} + 2 f_u f_v f_{uv} + (1 - f^2_u) f_{uv}] + (1 - f^2_u - f^2_v)}{(1 - f^2_u - f^2_v)^2}.
\]
So $K = 1 - \eta^2_3$ is equivalent to
\[ f(f_{uu}f_{vv} - f^2_{uv}) - [(1 - f^2_v)f_{uu} + 2f_u f_v f_{uv} + (1 - f^2_u)f_{vv}] = 0, \quad (6.2) \]
where $f^2_u + f^2_v < 1$. This is the fully nonlinear PDE which the space-like graph in $S^3_1$ with $K = 1 - \eta^2_3$ must satisfy.

Remark. There exists a nice duality between the solutions of minimal surface equation
\[ (1 + f^2_v)f_{uu} - 2f_u f_v f_{uv} + (1 + f^2_u)f_{vv} = 0 \]
in $R^3$ and the ones of maximal surface equation
\[ (1 - f^2_v)f_{uu} + 2f_u f_v f_{uv} + (1 - f^2_u)f_{vv} = 0 \]
in Lorentz-Minkowski 3-space $L^3$(cf.[2]). Here, by the duality given by (5.2)(or (5.3)), we know that if $f(u,v)$ is the solution of (6.1), then the local graph of the surface $(-ff_u - u, -ff_v - v, f\sqrt{1 + f^2_u + f^2_v})$ in $S^3_1$ satisfies (6.2). Conversely, if $f(u,v)$ is the solution of (6.2) with $f^2_u + f^2_v < 1$, then the local graph of the surface $(ff_u - u, ff_v - v, f\sqrt{1 - f^2_u - f^2_v})$ in $H^3$ satisfies (6.1).

Next, as similar as done in section 6 of [16], we get the following Theorem.

**THEOREM 6.1.** The nontrivial translational space-like surfaces with the form $f(u,v) = \phi(u) + \psi(v)$ in $S^3_1$ with conformal normal Gauss map are given, up to a linear translation of variables, by
\[ f(u,v) = \sqrt{a^2 + u^2} \pm \sqrt{b^2 + v^2} \quad (6.3) \]
with $f^2_u + f^2_v < 1$, where $a$ and $b$ are nonzero constants. The parameter form of these translational surfaces are locally given by
\[ x(u,v) = (a \sinh u, b \sinh v, a \cosh u + b \cosh v). \quad (6.4) \]

Considered as surfaces in 3-dimensional Minkowski space $L^3$, the space-like ruled surfaces in $S^3_1$ can be represented as $x(u,v) = \alpha(v) + u\beta(v) : D \to S^3_1$, where $D(\subset R^2)$ is a parameter domain and $\alpha(v)$ and $\beta(v)$ are two vector value functions into $L^3$ corresponding to two curves in $L^3$. When $\beta$ is locally nonconstant, without loss of generality we can assume that either $\langle \beta, \beta \rangle = 1, \langle \beta', \beta' \rangle = \pm 1$, and $\langle \alpha', \beta' \rangle = 0$ or $\langle \beta, \beta \rangle = 1, \langle \beta', \beta' \rangle = 0$, and $\langle \alpha', \beta \rangle = 0$, where $\langle \cdot, \cdot \rangle$ is the scalar product in $L^3$. As similar as done in Theorem 2 of [16], we have
**THEOREM 6.2.** Up to an isometric transformation
\[
(x_1, x_2, x_3) \rightarrow (x_1 \cos \theta - x_2 \sin \theta + a, x_1 \sin \theta + x_2 \cos \theta + b, x_3)
\] (6.5)
in \(S^3_1\), every space-like ruled surface in \(S^3_1\) with conformal normal Gauss map is locally a part of one of the following,

1. ordinary Euclidean space-like planes in \(S^3_1\),
2. \((ucosh v, c \cdot \sinh v, u \sinh v)\), for a constant \(c \neq 0\),
3. \((c_2 \sinh v + u \cosh v, c_1 \sinh v, c_2 \cosh v + u \sinh v)\), for constant \(c_1 \neq 0\) and \(c_2 \neq 0\).

We should note that in the proof of Theorem 6.2, only when \(\langle \beta', \beta' \rangle = -1\), we may get the nontrivial cases (2) and (3).

Locally, the ruled surfaces (2) and (3) in Theorem 6.2 can be represented as the graph \((u, v, f(u, v))\) as follows,

**COROLLARY.** \(f(u, v) = \pm \frac{c^2 + uv}{\sqrt{c^2 + v^2}}\) is a solution of equation (6.2), where \(c_1 \neq 0\) and \(c_2\) are constants.

**Remark.** In \(H^3\), the translational surfaces
\[
(a \cos u, b \cos v, a \sin u + b \sin v)
\] (6.6)
and the ruled surfaces
\[
(u \cos v, c \cdot \sin v, u \sin v)
\] (6.7)
and
\[
(-c_2 \sin v + u \cos v, c_1 \cdot \sin v, c_2 \cos v + u \sin v)
\] (6.8)
with conformal normal Gauss map have been obtained ([16][17]), where \(a, b, c, c_1\) and \(c_2\) are nonzero constants. Using (5.2) (or (5.3)) and Theorem 4.2, we may check that up to a isometric transformation (6.5) in \(S^3_1\) (\(\theta = \pm \frac{\pi}{2}\)), (6.4) in Theorem 6.1 and (2) and (3) in Theorem 6.2 are, respectively, the polar varieties of (6.6),(6.7) and (6.8) and vice versa.

**Remark.** Every geodesic of \(H^3\), corresponding respectively to \(u = 0, u = \pi\), \(v = 0\) and \(v = \pi\) on surfaces (6.6) and to \(v = \frac{\pi}{2}\) on surfaces (6.7) and to \(v = \pm \frac{\pi}{2}\) on surfaces (6.8) follow which \(K = -1\) is mapped to a simple point in \(S_0\) by the generalized Gauss map.
7 Time-like surfaces in $S^3_1$ with conformal normal Gauss map

In this section, we state the similar results as the aboved for the time-like surfaces in $S^3_1$ without proofs.

Take the upper-half space model of $S^3_1$. Let $M$ be a 2-dimensional Lorentz surface and $x : M \to S^3_1$ be the time-like immersion with the local coordinates $u_1, u_2$. The space-like unit normal vector is $N = x_3 \eta_1 \frac{\partial}{\partial x_1} + x_3 \eta_2 \frac{\partial}{\partial x_2} + x_3 \eta_3 \frac{\partial}{\partial x_3}$, where $\eta_1^2 + \eta_2^2 - \eta_3^2 = 1$. Left-translating $N$ to $T_e(R^3_+)$, we obtain

$$\tilde{N} : M \to S^2_1(1) \subset T_e(R^3_+),$$

$$\tilde{N} = L_{x^{-1}}(N) = \eta_1 \frac{\partial}{\partial x_1}(e) + \eta_2 \frac{\partial}{\partial x_2}(e) + \eta_3 \frac{\partial}{\partial x_3}(e),$$

which is called the normal Gauss map of time-like surface $x : M \to S^3_1$ (cf.[1]).

Call $IV = \langle d\tilde{N}, d\tilde{N} \rangle$ the fourth fundamental form of the time-like surface $x : M \to S^3_1$. We have $IV = (\eta_3^2 g_{ij} - 2 \eta_3 h_{ij} + g^{kl} h_{ij} h_{kl}) du_i du_j$. Of course, we may also define the high-dimensional version of the fourth fundamental form for the time-like hypersurfaces in $S^{n+1}_1$.

**THEOREM 7.1.** Let $M$ be a 2-dimensional Lorentz surface and $x : M \to S^3_1$ be a time-like immersed surface without umbilics. Then the normal Gauss map of $x(M)$ is conformal if and only if the Gauss curvature $K = 1 + \eta_3^2$.

In the Minkowski model of the de Sitter 3-space $S^3_1$, the generalized Gauss map $N : M \to S^3_1$ of the time-like surface $x : M \to S^3_1$ is a branched time-like immersion with branch points where $K = 1$.

**THEOREM 7.2.** Let $M$ be a connected 2-dimensional Lorentz surface. Let $X : M \to S^3_1$ be a time-like surface without umbilics and $K \neq 1$. If the normal Gauss map of $X : M \to S^3_1$ is conformal, then the normal Gauss map of its generalized Gauss map $N : M \to S^3_1$ is also conformal and vice versa.

The time-like graph $(u, v, f(u, v))$ in $S^3_1$ with conformal normal Gauss map also satisfies the fully nonlinear PDE (6.2) with $f_u^2 + f_v^2 > 1$.

**THEOREM 7.3.** The nontrivial translational time-like surfaces with the form $f(u, v) = \phi(u) + \psi(v)$ in $S^3_1$ with conformal normal Gauss map are given, up to a linear translation of variables, by

$$(1) f(u, v) = \sqrt{u^2 + a^2} \pm \sqrt{v^2 + b^2},$$
\( f(u, v) = \sqrt{u^2 - a^2} \pm \sqrt{v^2 - b^2}, \)

\( f(u, v) = \sqrt{u^2 + a^2} \pm \sqrt{v^2 - b^2}, \)

\( f(u, v) = \sqrt{u^2 - a^2} - \sqrt{v^2 + b^2}, \)

(5) Flaherty time-like surface in \( S^3_1 \) (cf.[12]) \( f(u, v) = \pm u + \psi(v), \)

where \( a \) and \( b \) are nonzero constants and \( \psi'(v) \neq 0. \)

We may prove that the normal Gauss map of the time-like surfaces (2) and (3) in Theorem 6.2 are also conformal. In addition, for the time-like ruled surface \( x(u, v) = \alpha(v) + u\beta(v) \) in \( S^3_1 \), we may also assume the remained four cases:

(i) \( \langle \beta, \beta \rangle = -1, \langle \beta', \beta' \rangle = 1, \) and \( \langle \alpha', \beta' \rangle = 0, \)

(ii) \( \beta \) is constant null vector,

(iii) \( \beta \) is constant and \( \langle \beta, \beta \rangle = -1, \langle \alpha', \beta \rangle = 0, \)

(iv) \( \langle \beta, \beta \rangle = 0, \langle \beta', \beta' \rangle = 1, \) and \( \langle \alpha', \beta' \rangle = 0, \)

where \( \langle \cdot, \cdot \rangle \) is the scalar product in \( L^3 \). Hence, we have

**THEOREM 7.4.** Up to an isometric transformation (6.5) in \( S^3_1 \), every time-like ruled surface in \( S^3_1 \) with conformal normal Gauss map is locally a part of one of the following,

(1) ordinary Euclidean time-like planes in \( S^3_1 \),

(2) ordinary Euclidean generalized cylinder \( x(u, v) = \alpha(v) + u\beta, \) where \( \beta = (0, 0, 1) \) and \( \alpha(v) \) is arbitrary curve in \( L^3 \) with \( \langle \alpha', \alpha' \rangle > 0, \)

(3) \( (u \cosh v, c \cdot \sinh v, u \sinh v), \) for a constant \( c \neq 0, \)

(4) \( (c_2 \sinh v + u \cosh v, c_1 \sinh v, c_2 \cosh v + u \sinh v), \) for constant \( c_1 \neq 0 \) and \( c_2 \neq 0, \)

(5) \( (u \sinh v, c \cdot \cosh v, u \cosh v), \) for a constant \( c \neq 0, \)

(6) \( (c_2 \cosh v + u \sinh v, c_1 \cosh v, c_2 \sinh v + u \cosh v), \) for constant \( c_1 \neq 0 \) and \( c_2 \neq 0, \)

(7) Flaherty’s time-like surfaces in \( S^3_1 \) (cf.[12]), \( x(u, v) = \alpha(v) + u\beta, \) where \( \beta = (1, 0, 1) \) and \( \alpha(v) \) is arbitrary curve in \( L^3 \) with \( \langle \alpha', \beta \rangle \neq 0. \)

We should note that in the proof of Theorem 7.4, only for case (i) and (ii), we may get the surfaces (5)(6)(7) in Theorem 7.4. For case (iv), we may assume \( \beta(v) = (\rho(v) \cos \theta(v), \rho(v) \sin \theta(v), \rho(v)) \) with \( \rho^2(\theta')^2 = 1. \) Next, we get a contradictory system of equations.

**Remark.** Up to a isometric transformation (6.5) in \( S^3_1 (\theta = \pm \frac{\pi}{2}) \), the time-like surfaces (3) and (4) in Theorem 7.4 are, respectively, the polar varieties of the time-like surfaces (5) and (6) in Theorem 7.4 and vice versa. The similar
result also holds for the time-like surfaces in Theorem 7.3. Generally, if \( f(u, v) \) is the solution of (6.2) with \( f_u^2 + f_v^2 > 1 \), then the local graph of the surface \((ff_u - u, ff_v - v, f\sqrt{f_u^2 + f_v^2} - 1)\) in \(S^3_1\) also satisfies (6.2).

Locally, the ruled surfaces (4) and (5) in Theorem 7.4 can be represented as the graph \((u, v, f(u, v))\) as follows,

**COROLLARY.** \( f(u, v) = \pm \frac{c_1 - uv}{\sqrt{v^2 - c_1^2}} \) is a solution of equation (6.2), where \( c_1 \neq 0 \) and \( c_2 \) are constants.

**Remark.** When we do not assume that \( f > 0 \), (6.3) and \( f(u, v) = \pm \frac{c_1 u + uv}{\sqrt{c_1^2 + v^2}} \) and \( f(u, v) = \pm u + \psi(v), \psi'(v) \neq 0 \), are all nontrivial entire solutions of the equation (6.2) defined on \(R^2\). In addition, the cone \( f(u, v) = \sqrt{u^2 + v^2} \) is also the special solution of the equation (6.2), but its graph is the light-like surface. By Omori-Yau’s Maximum Principle[14][18], there exist no entire solution \( f(u, v) \) of (6.2) satisfying \( f_u^2 + f_v^2 > 1 \) and \( f > 0 \) on \(R^2\). Does there exist nontrivial entire solutions of equation (6.2) defined on \(R^2\) satisfying \( f_u^2 + f_v^2 < 1 \) and \( f > 0 \)?

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School of Mathematics and System Sciences, 
Shandong University, Jinan 250100, 
P.R. China 
E-mail: shishuguo@hotmail.com