Higher-Rank Fields and Currents

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Abstract

$Sp(2M)$ invariant field equations in the space $\mathcal{M}_M$ with symmetric matrix coordinates are classified. Analogous results are obtained for Minkowski-like subspaces of $\mathcal{M}_M$ which include usual $4d$ Minkowski space as a particular case. The constructed equations are associated with the tensor products of the Fock (singleton) representation of $Sp(2M)$ of any rank. The infinite set of higher-spin conserved currents multilinear in rank-one fields in $\mathcal{M}_M$ is found. The associated conserved charges are supported by $(\kappa M - \frac{\kappa(\kappa - 1)}{2})$-dimensional differential forms in $\mathcal{M}_M$, that are closed by virtue of the rank-$2r$ field equations.
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References
1 Introduction

In [1], $Sp(2M)$ invariant unfolded field equations corresponding to rank-$r$ tensor products of the Fock (singleton) representation of $Sp(2M)$ were introduced. These equations were shown to describe “branes” of different dimensions in the $Sp(2M)$ invariant generalized space-time $\mathcal{M}_M$ with local coordinates $X^{AB}$ which are symmetric $M \times M$ matrices. In [1], the case of rank-two was considered in detail. In particular, all rank-two dynamical (primary) fields and field equations were found and it was shown that dynamical equations for most of the rank-two fields have the form of conservation conditions for conserved currents found in [2], which give rise to the full set of bilinear conserved charges in the rank-one theory.

Here, this analysis is extended to the fields and equations of arbitrary rank. Namely we find all dynamical fields, which are primary fields from the conformal field theory perspective, along with the explicit form of their field equations. It is shown that, similarly to the rank-two case, some of these fields give rise to differential forms that are closed by virtue of their field equations thus generating conserved currents.

Also we consider a similar problem in the Minkowski-like reduction of the $Sp(2M)$ covariant setup in which the generalized Minkowski space has local coordinates $x^{\alpha,\alpha'}$ with $\alpha, \alpha' = 1, 2, \ldots, K$. In particular, at $M = 4$, this allows us to derive all conformal primary currents in the four-dimensional Minkowski space that are built from 4d massless fields of all spins. (These results have been already announced and used in [3] for the analysis of the operator algebra and correlators of conserved currents in four dimensions.)

We expect that results of the present paper may have applications in the context of AdS/CFT holography [4, 5, 6] and especially, higher-spin holography (see, e.g., [7, 8, 9, 10, 11, 12, 13, 14, 15, 16, 17] and references therein) because, as was emphasized in [18], the duality between fields in higher dimensions and currents in lower dimensions to large extent amounts in the language of this paper to the duality between lower-rank fields in $\mathcal{M}_M$ with higher $M$ and higher-rank fields in $\mathcal{M}_M$ with lower $M$.

The rest of the paper is organized as follows.

Section 2 contains summary of our results, presenting the full lists of dynamical fields of any rank-$r$ in $\mathcal{M}_M$ and generalized Minkowski space $\mathcal{M}_M^{nk}$ (including usual four-dimensional Minkowski space), along with their field equations. The form of multilinear conserved currents in $\mathcal{M}_M$ is also presented here. Section 3 contains details of the derivation of equations of motion and the analysis of the sigma-minus cohomology for rank-$r$ fields and field equations in $\mathcal{M}_M$. In Section 4 a structure of differential forms in $\mathcal{M}_M$ and multilinear conserved currents in $\mathcal{M}_M$ are analyzed. Section 5 contains details of the analysis of the sigma-minus cohomology associated with the dynamical rank-$r$ fields and field equations in generalized Minkowski space. In Section 6 some perspectives are briefly discussed. Appendix contains our Young diagram conventions.
2 Results

In this section we summarize our results leaving technical details for the rest of the paper.

2.1 \( \text{Sp}(2M) \) invariant space

2.1.1 Fields and equations

Rank-one \( \text{Sp}(2M) \) invariant unfolded equation is \([13]\)

\[
\left( dX^{AB} \frac{\partial}{\partial X^{AB}} + \sigma_1 \right) C(Y|X) = 0, \quad \sigma_1 = dX^{AB} \frac{\partial^2}{\partial Y^A \partial Y^B}, \tag{2.1}
\]

where \( X^{AB} = X^{BA} \) \((A, B = 1, \ldots, M)\) are matrix coordinates of \( M_M \), \( Y^A \) are some auxiliary commuting variables, that will be referred to as twistor variables. To simplify formulae we will use notation

\[
\xi^{MN} = dX^{MN} : \quad \xi^{MN} = \xi^{NM}, \quad \xi^{MN} \xi^{AD} = -\xi^{AD} \xi^{MN}. \tag{2.2}
\]

Note that Eq. (2.1) admits an interesting interpretation in terms of world-like particle models of \([20, 21]\), providing also a field-theoretical realization of the observation of Fronsdal \([22]\) that the infinite tower of all 4d massless fields enjoys \( \text{sp}(8) \) symmetry.

From (2.1) it follows that most of the component fields in the expansion

\[
C(Y|X) = \sum_{b=0}^{\infty} Y^{A_1} \ldots Y^{A_n} C_{A_1 \ldots A_n}(X) \tag{2.3}
\]

are reconstructed in terms of \((X\text{-derivatives of})\) the primary fields that satisfy

\[
\sigma_1 C(Y|X) = 0. \tag{2.4}
\]

In the rank-one case, the primary fields are \([19]\)

\[
C(X), \quad C_A(X) Y^A. \tag{2.5}
\]

The symmetry properties of \( C \) and \( C_A \) can be represented by the Young diagrams \( \bullet \) and \( \Box \), respectively. (For more detail on Young diagrams see Appendix.)

As a consequence of unfolded equations, the following equations hold

\[
\left( \frac{\partial}{\partial X^{AE}} \frac{\partial}{\partial X^{BD}} - \frac{\partial}{\partial X^{BE}} \frac{\partial}{\partial X^{AD}} \right) C(Y|X) = 0, \tag{2.6}
\]

\[
\left( \frac{\partial}{\partial X^{BD}} \frac{\partial}{\partial Y^A} - \frac{\partial}{\partial X^{AD}} \frac{\partial}{\partial Y^B} \right) C(Y|X) = 0
\]

and, in particular,

\[
\left( \frac{\partial}{\partial X^{AE}} \frac{\partial}{\partial X^{BD}} - \frac{\partial}{\partial X^{BE}} \frac{\partial}{\partial X^{AD}} \right) C(X) = 0, \tag{2.7}
\]

\[
\frac{\partial}{\partial X^{BD}} C_A(X) - \frac{\partial}{\partial X^{AD}} C_B(X) = 0. \tag{2.8}
\]
The symmetry properties of the left-hand-sides of the respective field equations are represented by the Young diagrams $\begin{array}{c} \require{AMScd} \begin{CD} \cdot @>>> \square \\ \square @>>> \square \end{CD} \end{array}$ and $\begin{array}{c} \require{AMScd} \begin{CD} \square @>>> \square \\ \square @>>> \square \end{CD} \end{array}$ in the language of $\sigma^-$ cohomology (see also [24]) convenient for the analysis of the pattern of zero-form higher-rank fields, primary fields and their field equations are represented by the cohomology groups $H^0(\sigma_1^1)$ and $H^1(\sigma_1^1)$, respectively. Hence, the structure of rank-one fields and field equations is represented by the following diagrams

\[ H^0(\sigma_1^1) : Y^0 \quad \text{and} \quad H^1(\sigma_1^1) : Y^1 \]

Note that Eqs. \((2.7),(2.8)\) are the only independent equations obeyed by the primary (=dynamical) fields as a consequence of \((2.1)\). Such equations will be referred to as dynamical.

Rank-two unfolded equations are

\[ \left( \xi^{AB} \frac{\partial}{\partial X^{AB}} + \sigma^2_- \right) C(Y|X) = 0, \tag{2.10} \]

where

\[ \sigma^2_- = \xi^{AB} \frac{\partial^2}{\partial Y_1^A \partial Y_1^B} + \xi^{AB} \frac{\partial^2}{\partial Y_2^A \partial Y_2^B} \tag{2.11} \]

and $Y_j^A \ (j = 1, 2)$ represents a doubled set of twistor variables.

In terms of variables

\[ \sqrt{2} z^A = y_1^A + iy_2^A, \quad \sqrt{2} \bar{z}^A = y_1^A - iy_2^A, \tag{2.12} \]

$\sigma^2_-$ takes the form

\[ \sigma^2_- = 2 \xi^{AB} \frac{\partial^2}{\partial z^A \partial \bar{z}^B}. \tag{2.13} \]

The following homogeneous differential equations hold as a consequence of \((2.13)\)

\[ \varepsilon^{B_1,B_2,B_3}_{B_1,B_2,B_3} \frac{\partial^3}{\partial X^{B_1D_1} \partial X^{B_2D_2} \partial X^{B_3D_3}} C(Y|X) = 0, \quad \varepsilon^{B_1,B_2,B_3}_{B_1,B_2,B_3} \frac{\partial^3}{\partial X^{B_1D_1} \partial \bar{z}^{B_2} \partial \bar{z}^{B_3}} C(Y|X) = 0, \tag{2.14} \]

where $\varepsilon^{A_1...A_m}$ are independent rank-$m$ totally antisymmetric tensors introduced to impose appropriate antisymmetrizations.

As shown in $\begin{array}{c} \require{AMScd} \begin{CD} \square @>>> \square \\ \square @>>> \square \end{CD} \end{array}$, the rank-two primary fields are

\[ C_{A_1...A_m}(X) z^{A_1} \ldots z^{A_m}, \quad \bar{C}_{A_1...A_m}(X) \bar{z}^{A_1} \ldots \bar{z}^{A_m}, \quad C_{A,B}(X) z^A \bar{z}^B \quad (C_{A,B} = -C_{B,A}). \tag{2.15} \]
Dynamical equations for the primary fields are

\[
\varepsilon^{B_1, B_2, B_3} \frac{\partial}{\partial X^{B_1 D_1}} \frac{\partial}{\partial X^{B_2 D_2}} \frac{\partial}{\partial X^{B_3 D_3}} C(X) = 0, \\
\varepsilon^{B_1, B_2, B_3} \frac{\partial}{\partial X^{B_1 D_1}} \frac{\partial}{\partial X^{B_2 D_2}} C_{B_3}(X) = 0, \\
\varepsilon^{B_1, B_2, B_3} \frac{\partial}{\partial X^{B_1 D_1}} \frac{\partial}{\partial X^{B_2 D_2}} \overline{C}_{B_3}(X) = 0, \\
\varepsilon^{E, B} \varepsilon^{C, D} \frac{\partial}{\partial X^{B C}} C_{E D A(m-2)}(X) = 0, \\
\varepsilon^{E, B} \varepsilon^{C, D} \frac{\partial}{\partial X^{B C}} \overline{C}_{E D A(m-2)}(X) = 0.
\]

Hence, the list of the Young diagrams associated with \( H^0(\sigma^2) \) and \( H^1(\sigma^2) \) is

\[
\begin{array}{c|c|c}
H^0(\sigma^2) : Y^0 & \bullet & \square \\
H^1(\sigma^2) : Y^1 & \boxed{\square} & \boxed{\square}
\end{array}
\]

Fields (2.13) and equations (2.16) are in one-to-one correspondence with the basic elements of \( H^0(\sigma^2) \) and \( H^1(\sigma^2) \), respectively. The latter are tensorial spaces with respect to indices \( A, B, \ldots = 1, \ldots M \), characterized by the Young diagrams sharing various columns of the table (2.17).

One observes that in the examples of ranks one and two, dynamical fields associated with \( H^0(\sigma^r) \) and \( H^1(\sigma^r) \) are such that the total number of indices in the first two columns of the respective Young diagrams does not exceed \( r \).

Another property illustrated by these two examples is that all columns starting from the third one of the Young diagrams \( Y^0 \) and \( Y^1 \) associated with \( H^0(\sigma^r) \) and \( H^1(\sigma^r) \) are equal, while the first two columns are such that together they form a rectangular two-column block of height \( r + 1 \), i.e.,

\[
h_1(Y^0) + h_2(Y^1) = h_2(Y^0) + h_1(Y^1) = r + 1, \quad h_k(Y^0) = h_k(Y^1) \quad \forall k \geq 3,
\]

where \( h_k(Y) \) is the height of the \( k^{th} \) column of \( Y \).

We say that a pair of Young diagrams are rank-\( r \) two-column dual if they obey (2.18). From the examples of ranks one and two we observe that dynamical fields and their field equations are described by rank-\( r \) two-column dual Young diagrams. It turns out that this is true for general rank-\( r \) fields.

Rank-\( r \) unfolded equations are

\[
\left( \xi^{AB} \frac{\partial}{\partial X^{AB}} + \sigma_- \right) C(Y|X) = 0,
\]

where

\[
\sigma_- = \xi^{AB} \frac{\partial^2}{\partial Y_{k}^{A} \partial Y_{j}^{B}} \delta_{kj} C(Y|X) = 0 \quad (k, j = 1, \ldots, r).
\]

According to the general argument of [1], these equations are \( Sp(2M) \) symmetric.

Indeed, it is well known (for more detail see e.g. [19]) that any system of equations of the form

\[
(d + \omega) C(X) = 0, \quad d = dX^{A} \frac{\partial}{\partial X^{A}},
\]

\( 6 \)
where $C(X)$ is some set of $p$-forms taking values in a $\mathfrak{g}$-module $\mathcal{V}$ and the 1-form $\omega(X) = dX^A \omega_A(X)$ is some fixed connection of $\mathfrak{g}$ obeying the flatness condition

$$d\omega + \frac{1}{2} [\omega, \wedge \omega] = 0 \quad (2.22)$$

($[.,.]$ denotes the Lie product in $\mathfrak{g}$), is invariant under the global symmetry $\mathfrak{g}$.

Eq. (2.19) provides a particular case of Eq. (2.21) with $\mathfrak{g} = \mathfrak{sp}(2M)$. The generators of $\mathfrak{sp}(2M)$ can be realized as bilinears built from oscillators $Y^A_i$ and $Y^i_A = \frac{\partial}{\partial Y^i}$:

$$[Y^i_A, Y^j_B] = \delta^i_j \delta_A^B, \quad [Y^i_A, Y^B_j] = 0, \quad [Y^i_A, Y^i_B] = 0, \quad i, j = 1 \ldots r.$$  

$$T^{AB} = Y^A_i Y^B_j \delta^{ij}, \quad T_{AB} = Y^i_A Y^j_B \delta^{ij}, \quad T^A_B = Y^A_i Y^j_B,$$  

$$[T_{AB}, T^{CD}] = \delta_A^C T^{D}_B + \delta_A^D T^{C}_B + \delta_B^C T^{D}_A + \delta_B^D T^{C}_A + 2r T^A_A, \quad (2.23)$$

$$[T^B_A, T^{CD}] = \delta_A^C T^{B}_D + \delta_A^D T^{B}_C, \quad [T^A_B, T_{CD}] = -\delta_B^C T_{AD} - \delta_B^D T_{AC}. \quad (2.24)$$

Note that the oscillator representation provides a standard tool of the study of representations of $\mathfrak{sp}(2M)$ [23, 26] (and references therein).

Rank-$r$ equations (2.19) have the form (2.21). The operator $\sigma^r$ (2.20) obeys

$$(\sigma^r)^2 = 0, \quad \{d, \sigma^r\} = 0,$$

which implies that the corresponding connection is flat.

The system (2.19) as well as the generators (2.23) are invariant under the action of $\mathfrak{o}(r)$ on color indices $i, j, \ldots = 1, \ldots r$, that leaves invariant $\delta^{ij}$. Generators of $\mathfrak{o}(r)$

$$\tau_{mk} = t_{mk} - t_{km}, \quad t_{mj} = \delta_{ij} t_{mi}, \quad t_{m}^i = Y^A_i Y^i_A,$$  

obey the standard commutation relations

$$[\tau_{mp}, \tau_{qj}] = \delta_{pq} \tau_{mj} + \delta_{mj} \tau_{pq} - \delta_{mq} \tau_{pj} - \delta_{pj} \tau_{mq}. \quad (2.26)$$

Being mutually commuting, $\mathfrak{o}(r)$ and $\mathfrak{sp}(2M)$ form a Howe-dual pair [27].

As in the lower-rank cases, rank-$r$ primary fields obey the condition

$$\sigma^r C(Y|X) = 0.$$  

(2.27)

In terms of the expansion

$$C(Y|X) = \sum_n C^{i_1 \ldots i_n}_{A_1 \ldots A_n}(X) Y^{A_1} \cdots Y^{A_n},$$  

(2.28)

Eq. (2.27) implies tracelessness of the component fields with respect to color indices

$$\delta_{i_1 i_2} C^{i_1 \ldots i_n}_{A_1 \ldots A_n}(X) = 0 \quad \forall n \geq 2.$$  

(2.29)

Since $Y^A_i$ commute, the tensors $C^{i_1 \ldots i_n}_{A_1 \ldots A_n}(X)$ are symmetric with respect to permutation of any pair of upper and lower indices. Hence $C^{i_1 \ldots i_n}_{A_1 \ldots A_n}(X)$ can be decomposed into a direct sum of tensors described by some irreducible representation
of $\mathfrak{o}(r)$ as well as of $\mathfrak{gl}_M$ with the same symmetry properties described by one or another Young diagram $Y(s_1, ..., s_m)$ ($s_1 + ... + s_m = n$).

Recall that if a tensor with respect to color indices is traceless and has symmetry properties of $Y[h_1, ..., h_m]$, it is nonzero only if

$$h_1 + h_2 \leq r. \quad (2.30)$$

As shown in Section 5, for any rank-$r$ primary field $C \in H^0(\sigma^r)$ associated with a Young diagram $Y^0(n_1, ..., n_m, \underbrace{1, ..., 1}_q)$ ($n_m \geq 2$) that obeys the tracelessness condition, hence obeying

$$2m + q \leq r, \quad (2.31)$$

the left-hand-sides of the dynamical equation has the symmetry properties of the two-column dual Young diagram $Y^1(n_1, ..., n_m, 2, ..., 2, 1, ..., 1)$. Pictorially,

$$\begin{array}{ll}
Y^0 & \quad \text{Rank}-r \text{ two-column duality.} \\
Y^1 & \\
\end{array}$$

These are $\mathfrak{gl}_M$ Young diagrams with respect to indices $A, B = 1, ..., M$. It should be stressed that for $M < r$ some of them may be zero which would mean either that the corresponding primary field is absent ($Y^0 = 0$) or that it does not obey any dynamical equations ($Y^1 = 0$), i.e., the system is off-shell.

Dynamical equations are most conveniently presented in terms of Young diagrams with manifest antisymmetrization. For nonnegative integers $m_1 \geq m_2$, $m_1 + m_2 \leq r$ consider

$$\mathcal{E}^{A_1[r-m_2+1], A_2[r-m_1+1], A_3[m_3], ..., A_k[m_n]}_{i_1[m_1], i_2[m_2], i_3[m_3], ..., i_k[m_n]} \quad (2.33)$$

with the symmetry properties described by $Y^0[m_1, m_2, m_3, ..., m_n]$ in the lower indices and $Y^1[r-m_2+1, r-m_1+1, m_3, ..., m_n]$ in the upper ones. So defined $Y^0$ and $Y^1$ obey the two-column duality condition (2.18). Let $\mathcal{E}$, which plays a role of projector to appropriate tensor representations, be traceless with respect to the color indices. As shown in Section 3, for any $\mathcal{E}$ (2.33), the following equations hold.
by virtue of rank-\(r\) unfolded equation (2.19)

\[
\hat{E}_0 = \epsilon_i^{A_1[r-m_2+1], A_2[r-m_1+1], A_3[m_3], \ldots, A_m[m_n]} \frac{\partial}{\partial X_1^{A_1[r-m_2+1]}} \cdots \frac{\partial}{\partial X_1^{A_2[r-m_1+1]}} \cdots \frac{\partial}{\partial X_1^{A_m[m_n]}} = 0 ,
\]

(2.34)

Since \(\hat{E}_0\) is a homogenous differential operator, the primary fields also satisfy (2.34). The parameter (2.33) is designed in such a way that Eq. (2.34) is non-trivial only for primaries \(C(Y|X)\) with the symmetry properties of \(Y^0[m_1, \ldots, m_n]\).

As anticipated, there is precise matching between the primaries and field equations. This means that the respective subspaces of \(H^0(\sigma^r)\) and \(H^1(\sigma^r)\) form isomorphic \(\mathfrak{o}(r)\) modules. Here one should not be confused that the two-column dual diagram \(Y^1\) does not respect the condition (2.30). The point is that, as an \(\mathfrak{o}(r)\)-tensor, \(Y^1\) is not traceless, containing explicitly a number of \(\mathfrak{o}(r)\) metric tensors \(\delta_{ij}\) adding additional \(\mathfrak{o}(r)\) indices to \(Y^1\) compared to \(Y^0\).

The nontrivial part of the analysis is to prove that the resulting list of dynamical equations is complete. This follows from the analysis of cohomology groups \(H^1(\sigma^r)\) in Section 3.2.

### 2.1.2 Multi-linear currents in \(\mathcal{M}_M\)

The construction of conserved currents in terms of rank-one fields proposed in [2, 1] admits a generalization to higher-rank fields. Recall, that the \(M\)-form

\[
\Omega_2(C) = \epsilon_{E_1 \ldots E_M} \xi^{E_1 A_1} \wedge \ldots \wedge \xi^{E_M A_M} \frac{\partial}{\partial z^{A_1}} \cdots \frac{\partial}{\partial z^{A_M}} C(z, \bar{z}|X)
\]

is closed provided that \(C(z, \bar{z}|X)\) satisfies [2]

\[
\epsilon^{E:B} \frac{\partial^3}{\partial X^{BC} \partial z^{A_i} \partial z^{B}} C^2(Y|X) = 0 .
\]

(2.36)

Now, we are in a position to introduce differential forms in \(\mathcal{M}_M\) closed by virtue of rank-\(r\) unfolded equations (2.13) for \(\kappa = 1, \ldots, M - 1\).

Let \(\mathcal{D}_{A_1 \ldots A_N}\) be the following (\(\kappa M - \frac{\kappa(\kappa - 1)}{2}\))-form

\[
\mathcal{D}_{A_1 \ldots A_N} = \epsilon_{D_1^1 \ldots D_1^M} \cdots \epsilon_{D_n^1 \ldots D_n^M} \\
\xi^{D_1^1 D_2^1} \wedge \ldots \wedge \xi^{D_1^1 D_1^1} \wedge \ldots \wedge \xi^{D_1^1 A_1^1} \wedge \ldots \wedge \xi^{D_1^1 A_1^1} \\
\ldots \\
\xi^{D_n^1 D_n^1} \wedge \ldots \wedge \xi^{D_n^1 D_n^1} \wedge \ldots \wedge \xi^{D_n^1 A_1^1} \wedge \ldots \wedge \xi^{D_n^1 A_1^1}
\]

(2.37)
where \( N = M - \kappa + 1 \) and \( \epsilon_{A_1 \ldots A_M} \) is the totally antisymmetric tensor. In Section 4 it is shown that \( D (2.37) \) has the symmetry properties of the rectangular Young diagram \( Y(N, \ldots, N) \),

\[
D^{A_1; \ldots; A_N} = D^{A_1(N), A_2(N), \ldots, A_N(N)}, \quad N = M - \kappa + 1 \tag{2.38}
\]

and that the \((\kappa M - \frac{\kappa(\kappa-1)}{2})\)-form

\[
\Omega_{2\kappa}(J) = D^{A_1(N), A_2(N), \ldots, A_N(N)} \frac{\partial^N}{\partial z_1^{A_1} \ldots \partial z_1^{A_N}} \ldots \frac{\partial^N}{\partial z_\kappa^{A_1} \ldots \partial z_\kappa^{A_N}} J(z, \bar{z}|X) \big|_{z=\bar{z}=0} \tag{2.39}
\]

where

\[
\sqrt{2} z_j = Y_j + i Y_{j+\kappa}, \quad \sqrt{2} \bar{z}_j = Y_j - i Y_{j+\kappa},
\]

is closed provided that \( J(z, \bar{z}|X) \) satisfies the rank-2\(\kappa\) unfolded equations characterized by the Young diagram \( Y^0[\kappa, \kappa] \). As a result, it generates conserved charges built from those bilinear combinations of rank-\(\kappa\) fields, that are described by \( Y^0 \), hence obeying appropriate rank-2\(\kappa\) equations.

2.2 Minkowski-like reduction

2.2.1 Fields and equations in \( \mathcal{M}_{M}^{Mnk} \)

As shown in \([28, 19]\), the equations of motion for massless fields of all spins in 4d Minkowski space can be formulated in the unfolded form

\[
\xi^{\alpha\beta'} \left( \frac{\partial}{\partial x^{\alpha\beta'}} C(y, \bar{y}|x) + i \frac{\partial^2}{\partial y^\alpha \partial \bar{y}^{\beta'}} \right) C(y, \bar{y}|x) = 0. \tag{2.41}
\]

Here \( y^\alpha \) and \( \bar{y}^{\beta'} \) are auxiliary commuting complex conjugated two-component spinor coordinates \((\alpha, \beta = 1, 2; \alpha', \beta' = 1, 2)\), \( x^{\alpha\beta'} \) are Minkowski coordinates in two-component spinor notations, and \( \xi^{\alpha\alpha'} = dx^{\alpha\alpha'} \).

The system \(2.41\) is a subsystem of \(2.19\) with \( M = 4, r = 2 \) and \( A = (\alpha, \alpha'), B = (\beta, \beta') \) etc, i.e.,

\[
Y^A = (y^\alpha, \bar{y}^{\beta'}), \quad X^{AB} = (x^{\alpha\beta'}, x^{\alpha\beta}, \bar{x}^{\alpha', \beta'}). \tag{2.42}
\]

More generally, consider \( X^{AB} \) \(2.42\) with \( \alpha = 1, \ldots, K \) and \( \alpha' = 1, \ldots, K \) for any \( K \). The space with coordinates \( x^{\alpha\beta'} \) we call \( \mathcal{M}_{M}^{Mnk} \). The rank-\(r\) generalization of \(2.41\), which is a subsystem of \(2.19\) with \( M = 2K, r = 2 \) is

\[
\xi^{\alpha\beta'} \left( \frac{\partial}{\partial x^{\alpha\beta'}} C(y, \bar{y}|x) + i \frac{\partial^2}{\partial y^\alpha \partial \bar{y}^{\beta'}} \eta^{kj} \right) C(y, \bar{y}|x) = 0 \tag{2.43}
\]

for any metric \( \eta^{kj} \), \( k, j = 1, \ldots, r \), \( \alpha, \alpha' = 1, \ldots, K \). We set \( \eta^{kj} = \delta^{kj} \), denoting

\[
\sigma^{r}_{Mnk} = i \xi^{\alpha\beta'} \frac{\partial}{\partial y^\alpha_k} \frac{\partial}{\partial \bar{y}^{\beta'}_k}, \quad \bar{y}^{\beta'}_k = \bar{y}^{\beta'}_j \delta^{kj}. \tag{2.44}
\]
Primary fields in $\mathcal{M}^{Mnk}_M$
\[ C(y, \bar{y}|x) = \sum_{p,q} C^{\alpha_1, \ldots, \alpha_p; \alpha'_1, \ldots, \alpha'_q}_{\beta_1, \ldots, \beta_q}(x) y^{\alpha_1}_i \ldots y^{\alpha_p}_i \bar{y}^{\alpha'_1}_{j_1} \ldots \bar{y}^{\alpha'_q}_{j_q} \] (2.45)

satisfy mutual tracelessness condition
\[ \sigma^{Mnk}_r C(y, \bar{y}|x) = 0. \] (2.46)

In these terms, the algebra $\mathfrak{o}(r)$ extends to $u(r)$ acting on conjugated representations carried by lower and upper color indices, while $\mathfrak{sp}(2M)$ reduces to $u(K, K)$ that acts on spinor indices and commutes to the $u(r)$. Note that $u(r)$ and $u(K, K)$ contain the common central element.

Since $y^\alpha_i$, $\bar{y}^{\alpha'}_j$ commute, the tensors $C^{\alpha_1, \ldots, \alpha_p; \alpha'_1, \ldots, \alpha'_q}_{\beta_1, \ldots, \beta_q}(x)$ are symmetric with respect to permutation of any pair of upper and lower indices. Hence it can be decomposed into a direct sum of tensors described by some irreducible representation of $u(r)$ as well as of $\mathfrak{gl}_K \otimes \mathfrak{gl}_K$ with the symmetry properties described by one or another pair of Young diagram $\mathbf{Y}(n_1, \ldots, n_m)$ and $\mathbf{Y}^\dagger(n_1, \ldots, n_s)$ ($n_1 + \ldots + n_m = p, n_1 + \ldots + n_s = q$).

Hence, primary fields in $\mathcal{M}^{Mnk}_M$ are
\[ C^{\alpha_1, \ldots, \alpha_p; \alpha'_1, \ldots, \alpha'_q}_{\beta_1, \ldots, \beta_q}(x) \]
\[ y^{\alpha_1}_i \ldots y^{\alpha_m}_{i_1} \ldots y^{\alpha_m}_{i_m} \bar{y}^{\alpha'_1}_{j_1} \ldots \bar{y}^{\alpha'_{s_1}}_{j_{s_1}} \ldots \bar{y}^{\alpha'_{s_n}}_{j_{s_n}} \]
with tensors $C$ that obey the mutual tracelessness condition (2.46), i.e.,
\[ \delta_{ab} C^{(a \alpha_1(n_1-1)) \ldots, \alpha_m(n_m); (b \bar{\beta}_1(n_1-1)) \ldots, \bar{\beta}_{s_1}(n_{s_1})}(x) = 0 \] (2.48)

and have definite symmetry properties in $y$ and $\bar{y}$ described by Young diagrams

\[ \mathbf{Y}_m^0 = \begin{array}{c}
\alpha_1 \alpha_2 \ldots \alpha_p \\
\beta_1 \beta_2 \ldots \beta_q \\
n_1 \ldots n_m > 0,
\end{array} \]
\[ \mathbf{Y}_s^0 = \begin{array}{c}
\alpha'_1 \alpha'_2 \ldots \alpha'_{s_1} \\
\beta'_1 \beta'_2 \ldots \beta'_{s_2} \\
\bar{n}_1 \ldots \bar{n}_{s_n} > 0,
\end{array} \] (2.49)

that have to obey the condition
\[ r \geq s + m \geq 0 \] (2.50)
as a consequence of (2.48).

In the Minkowski case, a pair of two Young diagrams (2.49) is called rank-$r$ two-column dual to the pair $\mathbf{Y}^1(n_1, \ldots, n_m, 1, 1, \ldots, 1)$ and $\mathbf{Y}^\dagger(n_1, \ldots, n_s, 1, 1, \ldots, 1)$

\[ \mathbf{Y}^1 = \begin{array}{c}
\alpha_1 \alpha_2 \ldots \alpha_p \\
\beta_1 \beta_2 \ldots \beta_q \\
n_1 \ldots n_m
\end{array} \]
\[ \mathbf{Y}^\dagger = \begin{array}{c}
\alpha'_1 \alpha'_2 \ldots \alpha'_{s_1} \\
\beta'_1 \beta'_2 \ldots \beta'_{s_2} \\
\bar{n}_1 \ldots \bar{n}_{s_n}
\end{array} \] (2.51)

\[ q = r + 1 - s - m \]
Equations of motion in $\mathcal{M}_{M}^{Mnk}$ are most conveniently represented with the help of the projecting tensor

$$E_{\alpha_{1}[r-s_{1}+1], \alpha_{2}[m_{2}], \ldots, \alpha_{k}[m_{k}]; \alpha'_{1}[r-m_{1}+1], \alpha'_{2}[s_{2}], \ldots, \alpha'_{n}[s_{n}]}_{i_{1}[m_{1}], \ldots, i_{k}[m_{k}]; j_{1}[s_{1}], \ldots, j_{n}[s_{n}]} = (2.52)$$

that obeys the mutual tracelessness condition (2.48) and has the symmetry properties described by two pairs of Young diagrams with manifest antisymmetrization: $Y^{0}[m_{1}, m_{2}, m_{3}, \ldots, m_{k}], \bar{Y}^{0}[s_{1}, s_{2}, \ldots, s_{n}]$ in the color indices and their two-column dual $Y^{1}[r-s_{1}+1, m_{2}, \ldots, m_{k}], \bar{Y}^{1}[r-m_{1}+1, s_{2}, \ldots, s_{n}]$ in the spinor ones.

Rank-$r$ unfolded equations (2.43) have the following consequences

$$\hat{E}_{Y^{0}}\bar{Y}^{0}C(y, \bar{y}|x) = 0,$$

$$\hat{E}_{Y^{0}}\bar{Y}^{0} = E_{\alpha_{1}[r-s_{1}+1], \alpha_{2}[m_{2}], \ldots, \alpha_{k}[m_{k}]; \alpha'_{1}[r-m_{1}+1], \alpha'_{2}[s_{2}], \ldots, \alpha'_{n}[s_{n}]}_{i_{1}[m_{1}], \ldots, i_{k}[m_{k}]; j_{1}[s_{1}], \ldots, j_{n}[s_{n}]} = (2.53)$$

$$\frac{\partial}{\partial y_{i_{1}a}} \ldots \frac{\partial}{\partial y_{i_{k}k}} \frac{\partial}{\partial y_{j_{1}n}} \ldots \frac{\partial}{\partial y_{j_{n}n}} = (2.54)$$

Eq. (2.53) is analogous to Eq. (2.34) and is derived in Section 3.

Since $\hat{E}$ (2.54) is a homogeneous differential operator, Eq. (2.53) is obeyed by the primary fields (2.47). The nontrivial part of the analysis is to show that Eq. (2.53) gives the full list of dynamical equations. This follows from the analysis of the cohomology groups $H^{0}(\sigma^{+}_{a}Mnk)$ and $H^{1}(\sigma^{+}_{a}Mnk)$ of $\sigma^{+}_{a}Mnk$ (2.43) for arbitrary $M = 2K$ in Section 3.

### 2.2.2 4d Minkowski space

The dictionary between tensor and two-component spinor notations is based on

$$A^{\alpha\beta'} = A^{\alpha}_{a}\sigma_{a}^{\alpha\beta'},$$

where $\sigma_{a}^{\alpha\beta'} (a = 0, 1, 2, 3)$ are four Hermitian $2 \times 2$ matrices.

In the 4d Minkowski case with various $r$ the list of primary fields is as follows.

$r = 1$

As follows from the analysis of [28], for $r = 1$ the primary fields are

$$C(x), C(y|x), C(\bar{y}|x).$$

These have symmetry properties described by the following pairs of Young diagrams

$$Y^{0} = \bullet, \quad \bar{Y}^{0} = \bullet,$$

$$Y^{0} = \bullet, \quad \bar{Y}^{0} = \bullet,$$

$$Y^{0} = \bullet, \quad \bar{Y}^{0} = \bullet.$$
The following consequences of (2.43)
\[ \varepsilon^{\alpha\beta} \varepsilon^{\alpha'\beta'} \frac{\partial^2}{\partial x^{\alpha\alpha'} \partial y^{\beta\beta'}} C(y, \bar{y}|x) = 0, \]
\[ \varepsilon^{\alpha\beta} \frac{\partial^2}{\partial x^{\alpha\alpha'} \partial y^{\beta\beta'}} C(y, \bar{y}|x) = 0, \]
\[ \varepsilon^{\alpha'\beta'} \frac{\partial^2}{\partial x^{\alpha\alpha'} \partial y^{\beta\beta'}} C(y, \bar{y}|x) = 0 \]
impose the equations on the primaries (2.56)
\[ \varepsilon^{\alpha\beta} \varepsilon^{\alpha'\beta'} \frac{\partial^2}{\partial x^{\alpha\alpha'} \partial y^{\beta\beta'}} C(x) = 0, \]
\[ \varepsilon^{\alpha\beta} \frac{\partial}{\partial x^{\alpha\alpha'}} \frac{\partial}{\partial y^{\beta\beta'}} C(y|x) = 0, \]
\[ \varepsilon^{\alpha'\beta'} \frac{\partial}{\partial x^{\alpha\alpha'}} \frac{\partial}{\partial y^{\beta\beta'}} \bar{C}(y|x) = 0. \]
The symmetry properties of the left-hand-sides of these equations are described by the following pairs of Young diagrams
\[ Y^{1} = \emptyset, \quad \bar{Y}^{1} = \emptyset; \]
\[ Y^{1} = \Box, \quad \bar{Y}^{1} = \bar{\Box}; \]
\[ Y^{1} = \underbrace{\Box_{n}}, \quad \bar{Y}^{1} = \emptyset. \]

As mentioned in the previous Section, rank-\( r \) primary fields (2.47) obey the symmetry properties of pairs of Young diagrams (2.49), that in the case of \( M = 4, r = 2 \) form the following list
\[ Y^{0}(n) = \underbrace{\Box_{n}}, \quad \bar{Y}^{0}(\bar{n}) = \underbrace{\Box_{\bar{n}}}, \]
\[ Y^{0} = \bullet, \quad \bar{Y}^{0}(\bar{n}, \bar{m}) = \underbrace{\Box_{\bar{n}, \bar{m}}}, \]
\[ Y^{0}(n, m) = \underbrace{\Box_{n, m}}, \quad \bar{Y}^{0} = \bullet. \]

Since the algebra \( u(2) \) spanned by
\[ s_{m}^{n} = g_{m}^{\alpha} \frac{\partial}{\partial y_{n}^{\alpha}} - \bar{y}^{\alpha' n} \frac{\partial}{\partial \bar{y}^{\alpha' m}} \]
commutes to \( \sigma_{M}^{Mnk} \) (2.44), then all primary fields can be described as higher weight representations of \( u(2) \). It is evident, that \( S_{2} \) can be chosen as a positive generator of \( u(2) \). As it is easy to see, higher weight primaries are singlets \( C(x) \), and higher weight fields
\[ C_{0}^{11}(y_{1}, \bar{y}_{2}|x), \quad C_{0}^{20}(y|x), \quad \bar{C}_{0}^{02}(y|x), \]
with symmetry properties of diagrams (2.60) with \( \bar{n}n \neq 0 \), (2.61) and (2.62), respectively.
Hence the full list of primaries consists of all descendants of \( C^0_{i j}(y, \bar{y}|x) \) (2.64), i.e.,
\[
C(x), \quad C^0_{i j}(y, \bar{y}|x) = \begin{pmatrix} s^2 \end{pmatrix}^k C^0_{i j}(y, \bar{y}|x) \quad (i + j = 2).
\]
(2.65)

Note, that these results were used in [3] to describe 4d free conformal primary currents.

Equations of motion depend on different tensors of the form (2.52). To be nonzero for \( M = 4 \) and \( r = 2 \), tensor (2.52) should have \( m_1 = \bar{s}_1 = 1 \) and \( m_j \leq 1, \bar{s}_j \leq 1 \), since \( \alpha_j \) and \( \alpha_j' \) take just two values. In particular, Eqs. (2.53) with \( m_1 = \bar{s}_1 = 1 \) and \( m_j = \bar{s}_j = 0 \) at \( j > 1 \) give
\[
\varepsilon^{\alpha\beta} \varepsilon^{\alpha'\beta'} \frac{\partial^3}{\partial x^{\alpha\alpha'} \partial y_1^\beta \partial y_2^\beta} C(y, \bar{y}|x) = 0, \quad \varepsilon^{\alpha\beta} \varepsilon^{\alpha'\beta'} \frac{\partial^3}{\partial x^{\alpha\alpha'} \partial y_1^\beta \partial y_2^\beta} C(y, \bar{y}|x) = 0, \quad \varepsilon^{\alpha\beta} \varepsilon^{\alpha'\beta'} \frac{\partial}{\partial x^{\alpha\alpha'}} \left( \frac{\partial^2}{\partial y_1^\beta \partial y_2^\beta} - \frac{\partial^2}{\partial y_2^\beta \partial y_1^\beta} \right) C(y, \bar{y}|x) = 0,
\]
where \( \varepsilon^{\alpha\beta} \) and \( \varepsilon^{\alpha'\beta'} \) are any antisymmetric tensors. These equations have symmetry properties of \( Y^1 = \emptyset \) and \( Y^1 = \emptyset \).

For \( r \geq 3 \), dynamical fields (2.47) are described by Young diagrams (2.49) with at most two rows. Since \( H^1(\sigma_r^{Mnk}) \) is empty for \( r \geq 3 \), the rank-\( r \geq 3 \) primary fields are off shell, not obeying any field equations.

## 3 Derivation of equations in \( \mathcal{M}_M \)

### 3.1 Derivation of equations

By virtue of (2.19), Eq. (2.34) is equivalent to
\[
\varepsilon^{A_1[r-m_2+1], A_2[r-m_1+1], A_3[m_3], \ldots, A_k[m_k]}_{i_1[m_1], i_2[m_2], i_3[m_3], \ldots, i_k[m_k]}, \quad \frac{\partial^2}{\partial Y_{j_1}^{A_{m_1+1}} \partial Y_{j_2}^{A_{m_2+1}} \ldots \partial Y_{j_N}^{A_{m_N+1}}} \cdot \frac{\partial}{\partial Y_{i_1}^{A_{m_1}} \partial Y_{i_2}^{A_{m_2}} \ldots \partial Y_{i_k}^{A_{m_k}}} C(Y|X) = 0,
\]
where \( N = r + 1 - m_1 - m_2 \). Since differential operators \( \frac{\partial}{\partial Y_{j_m}} \) commute, while the indices in columns of the both of the Young diagrams, \( Y[m_1, m_2, m_3, \ldots, m_k] \) and its two column dual \( Y[m_1, m_2, m_3, \ldots, m_k] \), are antisymmetrized, the indices \( j_1 \ldots j_N \) are antisymmetrized with the indices \( i_1[m_1] \), while the indices \( j_2 \ldots j_N \) are antisymmetrized with the indices \( i_2[m_2] \). Such antisymmetrizations gives zero by virtue of the following
Lemma 1
Let a tensor \( F_{[m], [n]} \) be traceless and antisymmetric both in indices \( i \) and in \( j \) taking \( r \) values (it is not demanded that \( F_{[m], [n]} \) has properties of a Young diagram). Consider the tensor
\[
G_{[m+p], [n+p]} = F_{[m], [m]} \delta_{i_{m+1} j_{n+1}} \cdots \delta_{i_{m+p} j_{n+p}}
\]
resulting from the total antisymmetrization of indices \( i \) and \( j \). Then
\[
G_{[m+p], [n+p]} = 0 \quad \text{at} \quad m + n > r - p.
\]

The proof follows from the determinant formula applied to the double dual tensor \( \tilde{G} \)
\[
\tilde{G}^{k[r-m-p], l[r-n-p]} := \epsilon^{k[r-m-p] i[m+p]} \epsilon^{l[r-n-p] j[n+p]} G_{i[m+p], j[n+p]}.
\]
Indeed,
\[
\tilde{G}^{k[r-m-p], l[r-n-p]} := \epsilon^{k[r-m-p] i[m]} \epsilon^{l[r-n-p] j[n]} r[p] F_{i[m], j[n]}.
\]
Expressing the product of two totally antisymmetric symbols in terms of Kronecker symbols one observes that for \( m + n > r - p \) at least one pair of indices of the traceless tensor \( F_{[m], [n]} \) is contracted hence giving zero. □

Thus Eq. (3.3) indeed follows from Eq. (2.34). Note that Eq. (3.3) at \( p = 0 \) gives Eq. (2.30).

3.2 Details of \( \sigma_- \)-cohomology analysis
As shown in [23] (for more detail see [1, 24]), dynamical content of (2.19) is encoded by the cohomology group \( H^0(\sigma_+) = \text{Ker}(\sigma_+^r) / \text{Im}(\sigma_+^r) \big|_{p'} \), where \( |_{p'} \) denotes the restriction to the subspace of \( p \)-forms. In particular, independent dynamical fields contained in the set of 0-forms \( C(Y|X) \) take values in \( H^0(\sigma_+) \), and there are as many independent differential equations on the dynamical 0-forms in (2.19) as the dimension of \( H^1(\sigma_+) \).

Our analysis generalizes those of [1], where the case of rank-2 was considered, and of [29], where conformal field equations in Minkowski space of any dimension were obtained. The main tool is the standard homotopy trick.

Let a linear operator \( \Omega \) act in a linear space \( V \) and satisfy \( \Omega^2 = 0 \). By definition, \( H(\Omega) = \ker \Omega / \text{Im} \Omega \) is the cohomology space. Let \( \Omega^* \) be some other nilpotent operator, \( (\Omega^*)^2 = 0 \). Then the operator
\[
\Delta = \{ \Omega, \Omega^* \}
\]
satisfies \( [\Omega, \Delta] = [\Omega^*, \Delta] = 0 \). From (3.6) it follows that \( \Delta \ker \Omega \subset \text{Im} \Omega \). Therefore \( H \subset \ker \Omega / \Delta(\ker \Omega) \). Suppose now that \( V \) is a Hilbert space in which \( \Omega^* \) and \( \Omega \) are conjugated. Then \( \Delta \) is semi-positive. If also the operator \( \Delta \) is quasifinite-dimensional, i.e. \( V = \sum V_A \) with finite-dimensional subspaces \( V_A \) such that \( \Delta(V_A) \subset V_A \) and \( V_A \) is orthogonal to \( V_B \) for \( A \neq B \), then \( \Delta \) can be diagonalized and it is easy to see that \( \ker \Omega / \Delta(\ker \Omega) = \ker \Delta \cap \ker \Omega \). Therefore, in this case,
\[
H \subset \ker \Delta \cap \ker \Omega.
\]
This formula is particularly useful for the practical analysis.

Defining
\[
\frac{\partial}{\partial \xi_{BD}} \xi^{MN} = \frac{1}{2} \left( \delta^M_B \delta^N_D + \delta^N_B \delta^M_D \right) - \xi^{MN} \frac{\partial}{\partial \xi_{BD}},
\]
we observe that the operators
\[
\chi^A_B = 2\xi^{AD} \frac{\partial}{\partial \xi_{BD}},
\]
form \(\mathfrak{gl}(M)\) with the commutation relations
\[
[\chi^A_B, \chi^N_M] = \chi^A_M \delta^N_B - \chi^N_M \delta^A_B, \quad [\chi^A_B, \xi^{MN}] = (\xi^{AN} \delta^M_B + \xi^{AM} \delta^N_B), \quad [\chi^A_B, \xi_{MN}] = - (\xi_{BN} \delta^A_M + \xi_{BM} \delta^A_N).
\]

In this notation,
\[
\Omega^M = \sigma^r \xi^{AB} = T_{AB} \xi^{AB}
\]
with \(T_{AB}\) (2.23). Setting
\[
\Omega^* = \mathcal{T}_{CD} \frac{\partial}{\partial \xi_{CD}},
\]
we introduce
\[
\Delta^r = \{\Omega^M, \Omega^*\}.
\]

An important property of so defined \(\Delta^r\) is that it is semi-positive because \(\Omega^M\) and \(\Omega^*\) are conjugated with respect to the positive-definite Fock space generated by \(Y_i^A\) and \(\xi^{AB}\) treated as creation operators.

By virtue of (2.24), (3.10) and (3.11), \(\Delta^r\) can be represented in the form
\[
\Delta^r = (T_A^B + \chi^B_A)(T_A^B + \chi^B_A) + \left( T^{AB} T_{AB} - T_A^B T_B^A \right) + (r - (M + 1)) \chi^A_A.
\]

An elementary computation gives by virtue of (2.23)-(2.26)
\[
2 \left( T^{AB} T_{AB} - T_A^B T_B^A + (M + 1 - r)T_A^A \right) = \tau_{mn} \tau^{mn} - \tau_{mn} \tau^{nm}.
\]

Hence
\[
\Delta^r = \nu_B^A \nu^A_B + \frac{1}{2} \tau_{mk} \tau^{mk} - (M + 1 - r) \nu^A_A,
\]
where
\[
\nu^A_B = \chi^A_B + T^A_B
\]
are generators of \(\mathfrak{gl}^\text{tot}_M\) that acts on all indices \(A, B, \ldots\) independently on whether they are carried by \(Y_i^A\) or by \(\xi^{AB}\). The first and second terms on the r.h.s. of (3.17) are the quadratic Cazimir operators of the algebras \(\mathfrak{gl}^\text{tot}_M\) and \(\mathfrak{o}(r)\), respectively.

Analogous computation in terms of fermionic oscillator realization of generators of the orthogonal group, that makes the antisymmetrization manifest, gives the well-known formula used e.g. in [29]
\[
\tau_{mk} \tau^{mk} = 2 \sum_j h_i(h_i - r - 2(i - 1)),
\]

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where \( i \) enumerates columns of the \( \mathfrak{o}(r) \) traceless Young diagram \( Y[h_1, \ldots, h_k] \) and \( h_i \) are heights of the respective columns. Analogously,

\[
\nu^A_B \nu^A_B = - \sum_i H_i(H_i - M - 1 - 2(i - 1)), \tag{3.20}
\]

where \( i \) enumerates columns of the \( \mathfrak{gl}_M \) Young diagram \( Y'[H_1, \ldots, H_m] \) and \( H_i \) are heights of the respective columns.

Note that, in addition to indices associated with the indices of \( Y \), the diagram \( Y' \) can contain indices carried either by differentials \( \xi^{AB} \) or by the \( \mathfrak{o}(r) \) invariant “tracefull” combinations

\[
U^{AB} = \delta^{ij} Y'^A_i Y'^B_j, \tag{3.21}
\]

which are symmetric in the indices \( A, B \). As a result \( Y' \) belongs to \( Y \otimes (2 \otimes \ldots \otimes 2) \) for some \( N \geq 1 \). Here pairs of indices carried by \( U^{AB} \) are symmetrized, while those carried by \( \xi^{AB} \) are antisymmetrized. This may impose additional constraints on the shape of \( Y'[h_1 + s_1, h_2 + s_2, \ldots, h_m + s_m] \) with \( \sum s_i = 2N \), \( r + 1 \geq h_1 + s_1 \geq \ldots \geq h_k + s_k \), where \( N \) is not smaller than the degree of the differential form.

Alternatively, the bosonic oscillator realization of the generators, that makes symmetries manifest, gives

\[
\nu^A_B \nu^A_B = \sum_i L_i(L_i + M + 1 - 2i), \tag{3.22}
\]

\[
\tau_{mn} \tau^{mn} = -2 \sum_{i=1}^h l_i(l_i + r - 2i),
\]

where summations are over all rows of the respective Young diagrams, while \( L_i \) and \( l_i \) are lengths of the rows of the respective diagrams. Hence, Eq. \((3.13)\) gives the following two equivalent formulae

\[
\Delta^r = - \sum_i H_i(H_i - 2(i - 1)) + \sum_j h_j(h_j - 2(i - 1)) + r \sum_i (H_i - h_i), \tag{3.23}
\]

\[
\Delta^r = \sum_i L_i(L_i - 2i) - \sum_i l_i(l_i - 2i) + r \sum_i (L_i - l_i). \tag{3.24}
\]

The simplest way to see that this formulae are equivalent is to use the following useful observation. Any Young diagram is the unification of its elementary cells \( S_{i,j} \) on the intersection of its \( i-th \) row and \( j-th \) column, i.e., \( Y = \bigcup_{S_{i,j} \in Y} S_{i,j} \). Then for \( Y[H_1, \ldots, H_m] \) equivalent to \( Y(L_1, \ldots, L_k) \), depending on the order of summation we obtain

\[
\sum_{S_{pm} \in Y} (p - m) = \sum_p \sum_{S_{pm} \in H_p} (p - m) = -\frac{1}{2} \sum_p H_p - 2p + 1 \tag{3.25}
\]
and
\[ \sum_{s_{pm} \in Y} (p - m) = \frac{1}{2} \sum_j L_j (L_j - 2j + 1). \tag{3.26} \]

A pair of Young diagrams
\[ Y[h_1, h_2, \ldots, h_k], \quad Y'[H_1, H_2, \ldots, H_k] \tag{3.27} \]
will be called rank-\(r\) \textit{n-column dual} if the condition (2.30) holds and
\[ H_i = r + n - 1 - h_{n-j+1} \quad \text{for} \quad i = 1, \ldots, n \leq k, \quad H_i = h_i \quad \text{for} \quad i = n+1, \ldots, k. \tag{3.28} \]
At \(n = 2\) this amounts to (2.18).

Our analysis of \(\sigma_-\) cohomology is based on the following

Lemma 2

Any rank \(r\) \(n\)-column dual Young diagrams give \(\Delta^r = 0\). Indeed, Eq. (3.23) gives
\[ \Delta^r = -\sum_i s_i (s_i + 2h_i - 2(i-1) - r), \tag{3.29} \]
where \(s_i = H_i - h_i\). For any \(n\)-column dual diagrams (3.28) \(s_i = s_{n-i+1} = r + n - 1 - (h_i + h_{n-i+1})\). Then
\[ \Delta^r = -\frac{1}{2} \sum_i (r + n - 1 - (h_i + h_{n-i+1}))(2(n - 1) - 2(i - 1) - 2(n - i)) = 0 \tag{3.30} \]

Now it is elementary to check that the diagrams associated with primary fields and their field equations belong to \(\text{Ker} \Delta^r\). In the case of primary fields associated with \(H_0\), \(Y = Y'\). Hence Eq. (3.23) gives zero, as anticipated. In the case of field equations associated with \(H_1\), \(\Delta^r\) is also zero because the diagram for fields is two-column dual to that of their equations.

To show that this list of solutions is complete, analogously to [29], one can use the semi-positive definiteness of \(\Delta^r\). Indeed, the representation (3.23) for Cazimir operators tells us that for a given set of cells of a \(GL(M)\) diagram, the minimal value of the operator \(\Delta^r\) is reached when all cells are situated maximally southwest, \(i.e.,\), possible values of \(k\) and \(m\) in (3.25) are respectively maximized and minimized. For a given field (\(i.e., Y[h_1, h_2, \ldots, h_k]\)) this optimization applies to the cells associated with the indices of the differentials and the traceful blocks \(U^{AB}\) (3.21). From the semi-positivity of \(\Delta^r\), (3.25) and Lemma 2 it follows that \(\Delta^r\) can only be zero for rank \(r\) \(n\)-column dual Young diagrams. However, not all of them can be nontrivial for \(H^p(\sigma_-)\) at given \(p\). Indeed, any rank \(r\) \(n\)-column dual Young diagram where some two columns are occupied by the indices of \(U^{AB}\) are zero by Lemma 1. Therefore, for \(H^0(\sigma_-)\) and \(H^1(\sigma_-)\) the solutions are exhausted by the rank-\(r\) 0-column dual and two-column dual Young diagrams, respectively.
4 Differential forms and higher-rank currents in $\mathcal{M}_M$

That dynamical degrees of freedom associated with the rank-one equations (2.1) live on a $M$-dimensional surface $S \subset \mathcal{M}_M \otimes \mathbb{R}^M$ suggests that conserved charges associated with these equations have to be built in terms of $M$-forms that are closed as a consequence of the rank-two field equations (2.19). As shown in [30], the following $M-$form in $\mathcal{M}_M \otimes \mathbb{R}^{2M}$

$$\left(\frac{1}{2} \xi^{AB} \frac{\partial}{\partial z^B} - d \bar{z}^A\right)^M J^2(z, \bar{z} | X) \bigg|_{\bar{z}=0} \tag{4.1}$$

is closed provided that $J^2(z, \bar{z} | X)$ solves (2.19) at $r = 2$. Bilinears in solutions $C_j$ of (2.19) with $r = 1$ and parameters $\eta$ that commute to $\sigma^2$ (2.11)

$$J^2_\eta(z, \bar{z} | X) = \eta \delta^{kj} C_k(z + \bar{z} | X) C_j(i (z - \bar{z}) | X)$$

give conserved currents [1, 18, 3].

Since, modules of solutions of the rank-$\kappa$ equations in $\mathcal{M}_M \otimes \mathbb{R}^{\kappa M}$ are functions of $\kappa M$ variables $y_i^\alpha$ one might guess that in the rank-$r$ case the dimension of a "local Cauchy bundle" [31] on which initial data should be given to determine a solution everywhere in $\mathcal{M}_M$ is $\kappa M$. This is however not quite the case because the $O(\kappa)$ relates different solutions to each other not affecting evolution of any given solution. This suggests that the dimension of the true local Cauchy bundle is $\kappa M - \frac{1}{2} \kappa (\kappa - 1)$. Correspondingly, the respective currents should be represented by closed $\kappa(M - \frac{1}{2}(\kappa - 1))$-forms which is just the degree of the form (2.33).

On the other hand, one can see that the straightforward generalization of (4.1) to conserved currents in $\mathcal{M}_M \otimes \mathbb{R}^{2\kappa M}$ for arbitrary $\kappa$ which is,

$$\hat{D}^\kappa J^{2\kappa}(z, \bar{z} | X) \bigg|_{z_1 = \ldots = z_{\kappa} = 0}, \tag{4.2}$$

$$\hat{D}^\kappa = \prod_{j=1}^\kappa \xi^{A_1} \ldots A_M \left\{ \left(\frac{1}{2} \xi^{A_1B} \frac{\partial}{\partial z^B} - d \bar{z}^A A_1^j\right) \ldots \left(\frac{1}{2} \xi^{A_MB} \frac{\partial}{\partial z^B} - d \bar{z}^A A_M^j\right) \right\}$$

where $\xi^{A_1 \ldots A_M}$ is the rank-$M$ totally antisymmetric tensor, does not work. The point is that, although these forms are closed by virtue of rank-$2\kappa$, unfolded equations, their pullback to $\mathcal{M}_M$ is zero for $\kappa > 1$ thus obstructing the construction of conserved charges in the form of integrals over $\mathcal{M}_M$. Indeed, up to a numerical factor, the pull-back of the form (4.2) to $\mathcal{M}_M$ is

$$\left\{ \prod_{j=1}^\kappa \xi^{A_1} \ldots A_M \xi^{A_1B_1^j} \ldots \xi^{A_MB_M^j} \right\} \left(\frac{\partial}{\partial z^j} B_1^j \ldots \frac{\partial}{\partial z^j} B_M^j \right) J^{2\kappa}(z, \bar{z} | X) \bigg|_{\bar{z}=0}. \tag{4.3}$$

Any Young diagrams associated with the combination of differentials in the brackets can only contain rows of lengths $l \leq M + 1$ since otherwise it would contain either antisymmetrization over more than $M$ indices or symmetrization of some two
anticommuting differentials. On the other hand, because of $\varepsilon$-symbols, it contains $r$ columns of the maximal height $M$ associated with the indices $A$ in (4.3). Anticommutativity of the differentials implies that the part of the diagram associated with the indices $B$ should contain $r$ total symmetrizations. However, since the first $r$ columns are occupied by the indices $A$, this is not possible for $r > 1$ as at least two symmetrized indices $B$ will belong to the same column, which implies antisymmetrization. Thus (4.3) is zero for $\kappa > 1$.

To show that, as announced in Section 2, the $(\kappa M - \kappa(\kappa-1)/2)$-form $\Omega_{2\kappa}(J^{2\kappa})$ (2.39) is closed by virtue of (2.19), generating nontrivial conserved charges in $\mathcal{M}_M$ we have to analyze the structure of differential forms in $\mathcal{M}_M$ in some more detail.

Let

$$Q_{A_1B_1;\ldots;A_nB_n} \xi^{A_1B_1} \wedge \ldots \wedge \xi^{A_nB_n}$$

(4.4)

be a $n$–form. Anticommutativity of the differentials implies

$$Q_{A_1B_1;\ldots;A_iB_i;\ldots;A_jB_j;\ldots;A_nB_n} = -Q_{A_1B_1;\ldots;A_jB_j;\ldots;A_iB_i;\ldots;A_nB_n}.$$  

(4.5)

This property along with the symmetry within every pair $A_iB_i$ heavily restricts possible symmetry types of the coefficients $Q$ (4.4). Namely, $\mathfrak{gl}(M)$ irreducible tensor decomposition of (4.4) consists of all tensors described by almost symmetric Young diagrams as defined below, each in one copy.

Consider any Young diagram, that is invariant under reflection with respect to the diagonal. To obtain an almost symmetric diagram add one cell to each row that intersects the diagonal. Pictorially, shading the added cells,

```
Symmetric ⇒ Almost symmetric
```

(4.6)

For example, for $n \leq 5$ the full list of almost symmetric Young diagrams is

- $n=1$: $Y(2)$
- $n=2$: $Y(3,1)$
- $n=3$: $Y(4,1,1)$ and $Y(3,3)$
- $n=4$: $Y(5,1,1,1)$ and $Y(4,3,1)$
- $n=5$: $Y(6,1,1,1,1)$, $Y(5,3,1,1)$ and $Y(4,4,2)$.

For the proof consider first $Q$ (4.4) obeying (4.5) in the symmetric basis of Young diagrams $Y(l_1,\ldots,l_s)$ with $l_1 + \ldots + l_s = 2n$. Let $N_k$ count a number of pairs $AB$ that contribute to the $k$-th row and some $k+n$-th row with $n > 0$. 
From (4.5) it follows that all pairs of the $k$-th class can contribute either $N_k$ or $1 + N_k$ to the $k$-th row and either 0 or 1 cell different $p$-th rows with $p \geq k + 1$. This gives the inequality $l_k \leq k + N_k$, because for $j < k$ each of $j$-th classes can contribute to the $k$-th row either 0 or 1 cell, while the $N_k$-th class can contribute there either $N_k$ or $N_k + 1$ cells. Note, that the inequality becomes the equality if and only if each of $j$-th classes contributes just one cell for $j < k$, while the $N_k$-th class contributes $N_k + 1$ cells.

This implies $l_k^+ \leq N_k$, where $l_k^+$ is the number of cells of the $k^{th}$ row above the diagonal. Therefore the number of cells above the diagonal is dominated by $\sum_k N_k$. On the other hand, since the total number of cells is twice the number of pairs $n$, $2n = 2 \sum_k N_k$. \hfill (4.7)

Hence, the number of cells above the diagonal is dominated by $n$.

Now consider $Q$ (4.4) obeying (4.5) in the antisymmetric basis of Young diagrams $Y[h_1; \ldots; h_m]$ with $h_1 + \ldots + h_m = 2n$. Let $\tilde{N}_m$ count a number of pairs $AB$ that contribute to the $m$-th column and some $m + n$-th column with $n > 0$ ($n = 0$ is impossible because on the antisymmetrization). From (4.4) it follows that all pairs of this class can only contribute no more then one cell to different $p$-th columns with $p \geq k + 1$. This gives the inequality $h_m \leq m + \tilde{N}_m - 1$, implying $h_m^+ \leq \tilde{N}_m - 1$, where $h_m^+$ is the number of cells of the $m$-th columns that are not above the diagonal (under and on the diagonal). As above this inequality becomes the equality if and only if for each $j < k$, the $j$-th class contributes to the $k$-th column just one cell, while the $\tilde{N}_k$-th class contributes $\tilde{N}_k$ cells.

Therefore the number of cells that are not above the diagonal is dominated by $\sum_m \tilde{N}_m$. As above, $2n = 2 \sum_m \tilde{N}_m$. \hfill (4.8)

Hence the number of cells that are not above the diagonal is dominated by $n$.

Therefore, the number of cells above the diagonal equals to the number of cells that are not above the diagonal, i.e., each of constructed classes contributes equals number of cells above the diagonal and not above the diagonal.

Then $\sum_k l_k - k = \sum_k N_k$ and $\sum_m h_m - (m-1) = \sum_m \tilde{N}_m$. This gives $l_k = k + N_k$ and $h_m = m + \tilde{N}_m - 1$ for any $k$ and $m$.

Now it is not difficult to show that the number $L$ of rows of length $l_k > k$ is equal to the number of columns of height $h_k \geq k$, and $h_k + 1 = l_k$ for $1 \leq k \leq L$. Indeed, the first column height $h_1$ is not smaller than $N_1 = l_1 - 1$, while the first row length $l_1$ is not smaller than $\tilde{N}_1 = h_1 + 1$, etc. This completes the proof.

Now we are in a position to prove that $\Omega_{2\kappa}(J)$ (2.39) is closed by virtue of...
Eq. (2.19). Consider any differential $(\kappa M - \frac{\kappa (\kappa - 1)}{2})$-form

\[
\Omega_{2\kappa}(A) = \xi^D_1 D_1^\kappa \wedge \xi^D_2 D_2^\kappa \wedge \ldots \wedge \xi^D_{\ell} D_\ell^\kappa \wedge \xi^D_{\kappa+1} A_{\kappa+1}^\kappa \wedge \ldots \wedge \xi^D_M A_M^{\kappa+1} \wedge \ldots \wedge \xi^D_M A_M^{\kappa+1}
\]

\begin{equation}
\tag{4.9}
\end{equation}

where $A$ has symmetry properties of the almost symmetric Young diagram $Y(M + 1, \ldots, M + 1, \kappa, \ldots, \kappa)$. Then the only non-zero component of the differential form $d\Omega_{2\kappa}(A)$ can be represented by the following almost symmetric Young diagram

\[
Y(M + 1, \ldots, M + 1, \kappa, \ldots, \kappa) = \]

\begin{equation}
\tag{4.10}
\end{equation}

However, the dynamical equation (2.34) with $Y_0 = Y_0[\kappa, \kappa]$, being the first order differential equation with respect to $X$ derivatives, just implies that the projection to this diagram is zero. Hence, the form $\Omega_{2\kappa}(J)$ (2.39) is shown to be closed on shell.

## 5 $\sigma^r_{-Mnk}$ cohomology

Here we consider cohomology of the operator $\sigma^r_{-Mnk}$ (2.44). Operators

\[
\phi^\alpha_\beta = \xi^{\alpha\gamma} \frac{\partial}{\partial \xi^{\beta\gamma}} , \quad \bar{\phi}^{\alpha'}_{\beta'} = \xi^{\alpha\alpha'} \frac{\partial}{\partial \xi^{\alpha'\beta'}} ,
\]

\begin{equation}
\tag{5.1}
\end{equation}

form $\mathfrak{gl}_K(\mathbb{C})$.

Operators

\[
T^{\alpha\beta'} = y_\beta^- y_\beta' , \quad T^{\alpha}_{\beta'} = \frac{\partial}{\partial y_\alpha^-} \frac{\partial}{\partial y^{\beta'}}, \quad T^\alpha_{\beta} = y_\beta^- \frac{\partial}{\partial y^\alpha}, \quad T^{\alpha'}_{\beta'} = y^{\alpha'}\beta' \frac{\partial}{\partial y^{\beta'}}
\]

\begin{equation}
\tag{5.2}
\end{equation}

form $u(K, K)$.

Operators $\varsigma^m_n$ (2.63) form $u(r)$ with the commutation relations

\[
[\varsigma^m_n, \varsigma^j_q] = \delta^m_q \varsigma^j_n - \delta^j_q \varsigma^m_n.
\]

\begin{equation}
\tag{5.3}
\end{equation}

Since the operators $\varsigma^m_p$ (2.63) commute to $u(K, K)$ (5.2) $u(r)$ acts on solutions of the higher-rank equations that form $u(r)$-modules. Irreducible $u(r)$-modules satisfy the mutually tracelessness conditions (2.46) equivalent to

\[
T^{\alpha\beta'} P(y, \bar{y}, \xi|x) = 0.
\]

\begin{equation}
\tag{5.4}
\end{equation}
By virtue of (5.2)-(2.63) this can be rewritten as
\[
\Delta^\text{r}_{Mnk} = \{\Omega^M, \Omega^M_M\} = \frac{1}{2} \left( T^\alpha_\beta + \phi^\alpha_\beta \right) \left( T^\beta_\alpha + \phi^\beta_\alpha \right) + \frac{1}{2} \left( T^\alpha_\nu + \phi^\alpha_\nu \right) \left( T^\nu_\alpha + \phi^\nu_\alpha \right) + \left( T^\alpha_\nu + \frac{1}{2} T^\nu_\alpha + \frac{1}{2} T^\alpha_\nu \right) \left( T^\nu_\alpha + \frac{1}{2} T^\alpha_\nu \right) + \left( r - K \right) \phi^\alpha_\alpha.
\]

where
\[
\nu^算是 \phi^\alpha_\beta + T^\alpha_\beta, \phi^\alpha_\nu = T^\alpha_\nu = \phi^\alpha_\nu + T^\alpha_\nu.
\]

The first three terms on the r.h.s. of (3.17) are the quadratic Cazimir operators of the algebras \(u(K)\) and \(u(r)\).

An elementary computation in terms of bosonic oscillator realization of \(u(K)\), that makes the symmetrization manifest, by virtue of mutually tracelessness conditions (3.4) gives
\[
\zeta_m^m = \sum_i l_i (l_i - 2i + 1) + \sum_j \bar{l}_j (\bar{l}_j - 2j + 1),
\]

where \(i\) enumerates rows of the \(gl_i\) Young diagram \(Y(l_1, \ldots, l_i)\) and \(l_i\) are lengths of the respective rows, \(j\) enumerates rows of the \(gl_i\) Young diagram \(\bar{Y}(\bar{l}_1, \ldots, \bar{l}_i)\) and \(\bar{l}_i\) are lengths of the respective rows. Analogous formula resulting from fermionic oscillator realization is
\[
\zeta^m_m = -\sum_i h_i (h_i - 2(i - 1) - 1) - \sum_j \bar{h}_j (\bar{h}_j - 2(j - 1) - 1),
\]

where \(h_i\) and \(\bar{h}_i\) are heights of the respective columns of \(Y(h_1, \ldots, h_i)\) and \(\bar{Y}(\bar{h}_1, \ldots, \bar{h}_k)\).

Using analogous formulas for the \(u(K)\) Cazimir operators \(\nu^\alpha_\beta\) and \(\nu^\alpha_\nu\) introduced in Section 3.2, Eq. (5.7) gives
\[
\Delta^\text{r}_{Mnk} = \frac{1}{2} \left( -\sum_m H_m (H_m - 2(m - 1)) - \sum_n \bar{H}_n (\bar{H}_n - 2(n - 1)) \right) + \sum_i h_i (h_i - 2(i - 1)) + \sum_j \bar{h}_j (\bar{h}_j - 2(j - 1)) + (1 + r) \sum_p \left( H_p + \bar{H}_p - h_p - \bar{h}_p \right),
\]

where \(m\) enumerates columns of the \(u(K)\) Young diagram of the variables \(y Y[H_1, \ldots, H_i]\) and \(H_i\) are heights of the respective columns, while \(n\) enumerates columns of the \(u(K)\) Young diagram \(\bar{Y}[\bar{H}_1, \ldots, \bar{H}_i]\) of the variables \(\bar{y}\).

As in Section 3.2, in addition to indices associated with \(Y\) and \(\bar{Y}\), the diagrams \(Y'\) and \(\bar{Y'}\) can contain additional indices carried either by differentials \(\xi^\alpha_\beta\) or by the \(u(r)\) invariant tracefull combinations
\[
y^\alpha_\beta \bar{y}^\beta_i = \delta^{ij} y^\alpha_i \bar{y}^\beta_j.
\]
As a result $\mathbf{Y}' \otimes \mathbf{Y}'$ belongs to $\bigotimes_{N}^{\mathbf{Y}} \otimes \bigotimes_{N}^{(1)} \otimes \cdots \otimes \bigotimes_{N}^{(1)} \bigotimes_{N}^{(1)}$ for some

\[ N \geq 1. \]

Hence $H_i = h_i + s_i, \overline{H}_i = \bar{h}_i + p_i$ with some $s_i \geq 0, p_i \geq 0, \sum s_i = \sum p_i = N$. Consider the following two pairs of Young diagrams

\[ \mathbf{Y}[h_1, h_2, \ldots, h_l], \quad \overline{\mathbf{Y}}[\bar{h}_1, \bar{h}_2, \ldots, \bar{h}_k], \quad (5.12) \]

obeying the tracelessness condition $h_1 + \bar{h}_1 \leq r$ (2.50) and

\[ \mathbf{Y}'[H_1, H_2, \ldots, H_l], \quad \overline{\mathbf{Y}'}[\overline{H}_1, \overline{H}_2, \ldots, \overline{H}_k] \quad (5.13) \]

such that

\[ H_i = r + n - \bar{h}_{n-i+1} \quad \text{for} \quad i = 1, \ldots, n \leq \max(l, k), \quad H_i = h_i \quad \text{for} \quad i = n + 1, \ldots, l, \quad (5.14) \]

\[ \overline{H}_i = r + n - h_{n-i+1} \quad \text{for} \quad i = 1, \ldots, n \leq \max(l, k), \quad \overline{H}_i = \bar{h}_i \quad \text{for} \quad i = n + 1, \ldots, k. \]

Such pair will be referred to as rank $r$ 2$n$-column dual pairs of Young diagrams. For $n = 1$ it is equivalent to (2.51).

Lemma 3

Any rank $r$ 2$n$-column dual pair of Young diagrams gives zero $\Delta^r_{Mnk}$ (5.10). Indeed, Eq. (5.10) gives

\[ \Delta^r_{Mnk} = \frac{1}{2} \left( - \sum s_i (s_i + 2h_i - 2(i - 1) - (r + 1)) - \sum p_i (p_i + 2\bar{h}_i - 2(i - 1) - (r + 1)) \right), \quad (5.15) \]

where $s_i = H_i - h_i$, $p_i = \overline{H}_i - \bar{h}_i$. For any 2$n$-column dual pair (5.13), (5.14) $s_i = p_{n-i+1} = r + n - (\bar{h}_i + h_{n-i+1})$. Hence

\[ \Delta^r_{Mnk} = - \frac{1}{2} \left( \sum (r + n - a - (\bar{h}_i + h_{n-i+1}))(2(r + n - a - (\bar{h}_i + h_{n-i+1}))) \right) \]

\[ + 2h_i - 2(i - 1) - 2(r + 1) + 2\bar{h}_{n-i+1} - 2(n - i + 1)) = 0. \quad (5.16) \]

Now we observe that, in the case of $H^0$, $\mathbf{Y} = \mathbf{Y}'$, $\overline{\mathbf{Y}} = \overline{\mathbf{Y}'}$. Hence, Eq. (5.10) gives zero, as anticipated. In the case of $H^1$, the diagrams (5.12) (5.13), that form two-column dual pairs, send $\Delta^r$ (3.23) to zero as well.

Using semi-positive definiteness of $\Delta^r_{Mnk}$ along with Lemma 3 and formula (3.25) along the lines of Section 3.2, it is not difficult to show that this gives complete list of solutions.

6 Conclusion

There is a number of problems raised by the analysis of this paper. One of the most interesting is to study conserved charges generated by the constructed currents. A single conserved current is expected to generate many different charges upon integration with different global symmetry parameters $\eta$. (For example, stress tensor
generates the full conformal algebra being integrated with the parameters of trans-
lations, Lorentz rotations, dilatations and special conformal transformations.) See,
e.g., [3] for the analysis of this issue for rank-two conserved currents. It is there-
fore important to find the full space of the corresponding symmetry parameters \( \eta \)
leading to independent charges. An interesting peculiarity of this analysis is that,
as shown in [30] for the case of rank two, to obtain non-zero charges in Minkowski
subspace of \( \mathcal{M}_M \) it is necessary to consider parameters \( \eta \) that are singular in some
of coordinates in \( \mathcal{M}_M \) transversal to Minkowski space. It is therefore necessary to
find what is an appropriate singularity of \( \eta \) in the general case of any rank that gives
rise to non-zero conserved charges.

Another peculiarity is that, being multilinear in the dynamical fields, the charges
resulting from the proposed currents cannot be represented as integrals in usual
Minkowski space, requiring integration over a larger space like \( \mathcal{M}_M \) or its tensor
product with the twistor space. Nevertheless, being nonlocal from the perspective
of Minkowski space, the charges are well-defined and should form some algebra. An
interesting question is what is this algebra and, more specifically, what is its relation
with the multiparticle algebra proposed recently in [32] and [3] where it was shown,
in particular, that the usual bilinear (i.e., rank-two) currents give rise to a set of
charges that forms the higher-spin algebra.

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Appendix. Young diagrams

A tensor \( A_{B_1^{l_1} \ldots B_{l_m}^{m}} \) obeys symmetry properties of the Young
diagram \( Y(l_1, \ldots, l_m) \) with manifest symmetrization provided that \( A \) is symmetric
with respect to permutations of \( l_k \) indices \( B_1^k, \ldots, B_{l_k}^k \) of any \( k \)-th row, while sym-
metrization over \( l_k + 1 \) indices \( B_1^k, \ldots, B_{l_k}^k, B_j^p \) gives zero for any \( p > k \). Such tensors
are conventionally denoted as \( A_{B(l_1), \ldots, B(l_m)} \).

Analogously, a tensor \( A_{B_1^{h_1} \ldots B_{h_p}^p} \) obeys symmetry properties of Young diagram \( Y[h_1, \ldots, h_p] \) with manifest antisymmetrization provided
that it is antisymmetric with respect to permutations of \( h_k \) indices \( B_1^{h_k}, \ldots, B_{l_k}^{h_k} \) of any \( k \)-th column while the antisymmetrization over \( h_k + 1 \) indices \( B_1^{h_k}, \ldots, B_{l_k}^{h_k}, B_j^q \) gives zero for any \( q > k \). Such tensors are conventionally denoted as \( A_{B[h_1], \ldots, B[h_p]} \).

A height of the \( k \)-th column of \( Y \) is denoted as \( h_k(Y) \) while a length of the \( k \)-th row of \( Y \) is denoted as \( l_k(Y) \). Let us stress that components of the tensor with sym-
metry properties of any Young diagram with manifest symmetrization \( Y(l_1, \ldots, l_m) \)
can be represented as a linear combination of components of the tensor with sym-
metry properties of the same Young diagram with manifest antisymmetrization.
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