UNIFORM MOTIONS IN CENTRAL FIELDS

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Abstract. We present a theoretical problem of uniform motions, i.e. motions with constant magnitude of the velocity in central fields as a nonholonomic system of one particle with a nonlinear constraint. The concept of the article is in analogy with the recent paper [21]. The problem is analysed from the kinematic and dynamic point of view. The corresponding reduced equation of motion in the Newtonian central gravitational field is solved numerically. Appropriate trajectories for suitable initial conditions are presented. Symmetries and conservation laws are investigated using the concept of constrained Noetherian symmetry [9] and the corresponding constrained Noetherian conservation law. Isotachytonic version of the conservation law of mechanical energy is found as one of the corresponding constraint Noetherian conservation law of this nonholonomic system.

1. Introduction. From the mechanics it is known that motions are classified with respect to two basic aspects. The first aspect is the shape of a trajectory, we distinguish between rectilinear and curvilinear motions. The second one is the magnitude of the velocity, we distinguish between uniform and non-uniform motions. In the case of the uniform motion the magnitude of the velocity remains constant during the entire motion, \(|v| = \text{const}\). In the case of non-uniform motion the magnitude of the velocity changes during the motion. Evidently, the motions in central fields are non-uniform motions (except the circular motions). To keep the motions in central fields uniform, it is necessary to consider these motions as constrained motions subjected to a certain additional condition called constraint, which ensures the desired character of the motions.

In this paper we deal with the correction of non-uniform motions in central fields on uniform motions. The issue is solved as a classical initial value problem for a

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constrained mechanical system arisen from a system of one particle moving in a central force field subjected to one constraint

\[ |\mathbf{v}| = \sqrt{v \cdot v} = \sqrt{\dot{x}^2 + \dot{y}^2 + \dot{z}^2} = v_0 = \text{const.}, \]

called \textit{isotachytonic constraint}. The word isotachytonic was introduced in [21] from Greek; \textit{iso} = uniform; \textit{ταχυτητα} = velocity. Since isotachytonic constraint represents the \textit{nonholonomic} constraint \textit{nonlinear} with respect to components of the velocity, it is necessary to adopt a modern approach based on the geometric concept of nonholonomic mechanical systems [6, 7, 9, 11, 13].

This problem possesses some interesting aspects concerning proper kinematics (specific trajectories), understanding the role of the Chetaev constrained force and its active contribution to the energetic balance of the system, understanding of the constraint Noetherian conservation laws as a direct consequence of certain symmetries of the studied constrained system. It seems that the problem can be useful for some practical applications concerning regulation and optimal control of the space satellites and different maneuvers of spacecrafts as a suitable theoretical background [4].

The bibliography in a branch of nonholonomic mechanics is very extensive and includes many important contributions and alternative geometric concepts, see [1]-[21] and references therein. It should be stressed that almost all the papers deal with the case of linear nonholonomic constraints except the several ones, e.g. [10, 12, 20, 21].

In this paper we apply a geometric theory of nonholonomic mechanical systems which was developed for the first order in [7] and then was generalized for the higher order case in [8]. The theory enables an investigation of different aspects of these systems; \textit{reduced equations} of motion of constrained systems (equations of motion on the constraint submanifold), \textit{deformed equations} of motion (deformation of original unconstrained motion equations by adding Chetaev forces), constraint symmetries and corresponding conservation laws [9] and represents a reasonable concept for an alternative mathematical treatment of concrete examples of such systems, either with linear constraints [5, 20] or with nonlinear constraints [10, 20, 21] and even with the higher order constraints [19].

Theoretical problems concerning various modifications of the motion of a particle in the gravitational field have been considered and solved in the late of the 17th century using the infinitesimal calculus, which was recently discovered. The development of mathematical analysis and methods of solving differential equations enabled mathematicians and physicists to study various geometric curves of prescribed properties and to solve complicated problems from mechanics. They were looking for various trajectories along which the particle would move in the gravitational field, if the particle was subjected to some additional constraint.

In 1687 Gottfried Wilhelm Leibniz (1646–1716) called mathematics to find a shape of a curve along it the particle falls in the homogeneous gravitational field \( \mathbf{F} = -mg = (0, -mg) \) such that the vertical component of the velocity remains constant during the motion. The solution was found by Jacob Bernoulli (1654–1705) and he showed that the solution is a semi-cubic parabola called \textit{Leibniz isochrone}. The result was published in Acta Eruditorum Journal in 1690 in Leipzig and it is interesting to remark that in this article first appeared the word integral.

We briefly recall his original solution within the contemporary notation used in mechanics. Let is given the Cartesian coordinate system \( Oxy \) in the vertical plane,
where the $x$-axis is horizontal and the $y$-axis is directed vertically down. The particle of the mass $m$ is located in the homogeneous (constant) gravitational field, which is oriented vertically down, i.e. vector of the gravitational force $\mathbf{F} = (0, mg)$, where $g$ is the constant gravitational acceleration. Without loss of generality, we assume that the particle starts from the origin $O$ with the vector of the initial velocity $\mathbf{v}_0$, which has only the vertical component and is also oriented vertically down, i.e. $\mathbf{v}_0 = (0, v_0)$, $v_0 > 0$. The motion of the particle is subjected to one additional condition that the vertical component of the velocity of the particle remains constant during the motion, i.e.

$$
vy = \dot{y} = v_0 = \text{const.} \quad (2)
$$

The air resistance is neglected. Jacob Bernoulli based his solution on validity of the classical conservation law of the mechanical energy $E$. The potential energy of the particle in the homogeneous gravitational field at the height $y$ is $U(y) = -mgy$.

The mechanical energy $E$ of the particle at the beginning of the motion has the value $E_0 = \frac{1}{2}mv_0^2$, since the potential energy of the particle at the origin $O$ is $U(0) = 0$. The conservation law of the mechanical energy has the form

$$
E = E_k + U(y) = \frac{1}{2}mv^2 - mgy = E_0 = \frac{1}{2}mv_0^2 = \text{const}. \quad (3)
$$

Hence $v^2 - 2gy = v_0^2$. On the other hand $v^2 = v_x^2 + v_y^2 = \dot{x}^2 + \dot{y}^2 = \dot{x}^2 + v_0^2$. Thus,

$$
\dot{x} = \sqrt{2gy}. \quad (4)
$$

After the integration of the constraint condition (2) and with respect the initial condition $y(0) = y_0 = 0$ we get

$$
y(t) = v_0t. \quad (5)
$$

By the substitution into (4) we have $dx/dt = \sqrt{2g v_0 t}$. After the integration and with respect to the initial condition $x(0) = x_0 = 0$ we get

$$
x(t) = \frac{2}{3} \sqrt{2g v_0 t^2}. \quad (6)
$$

The equations (5) and (6) represents the parametric expression for searched trajectory, called Leibniz isochrone. After eliminating of the time parameter from (5) we obtain the analytic equation of Leibniz isochrone in the Cartesian coordinates

$$
x(y) = \frac{2\sqrt{2g}}{3v_0} y^{\frac{3}{2}} = \Lambda y^{\frac{3}{2}}. \quad (7)
$$

Later, in 1699 Pierre Varignon (1654–1722) modified the problem of Leibniz isochrone on the problem of finding such a curve along it the particle would move in the uniform (constant) central gravitational field, i.e. the central field with the constant magnitude of the gravitational force, under the same constraint condition as Leibniz, i.e. that the vertical (radial) component of the velocity of the particle remains constant during the motion. Varignon analysed the problem within the Leibniz formulation of the infinitesimal calculus and found the curve called Varignon’s isochrone. This result was published in Mémoires de l’Académie des Sciences de Paris in 1699.

We present the original Varignon’s solution in the contemporary notation. Varignon considered the special central gravitational field, which is constant at every point and is oriented to the center of the Earth, which is understood as a single massive mass point. This force field is expressed mathematically as follows:

$$
\mathbf{F}(\mathbf{r}) = -\mathbf{F}^{T} = -mg \mathbf{e}_r, \quad (8)
$$
where \( \mathbf{F}(\mathbf{r}) \) is the vector of the gravitational force at the point characterized by the position vector \( \mathbf{r} \), \( F \) is the magnitude of the gravitational force \( \mathbf{F} \). The magnitude \( F \) is constant at every point and equals to \( mg \), where \( m \) is the mass of the particle and \( g \) is the constant gravitational acceleration. The gravitational force vector \( \mathbf{F}(\mathbf{r}) \) lies on the line connecting the center of the field (located in the center of Earth) with this point and is oriented directly to the center. The vector \( \mathbf{e}_r = \mathbf{r}/r \) represents a unit vector in the direction of \( \mathbf{r} \) and \( r \) denotes the distance from the center. Since the position vector \( \mathbf{r} \) is oriented from the center, therefore we write the sign minus for the arbitrary attractive force field. The potential energy of the particle in such field at the distance \( r \) from the center is \( U(r) = mg r \). It is convenient to solve this problem in the polar coordinates \((r, \varphi)\).

The particle of the mass \( m \) is located in the field described above and starts from the initial point \((r_0, \varphi_0)\). The vector of the initial velocity \( \mathbf{v}_0 \) has only the radial component and is oriented to the center, i.e. \( v_r(0) = \dot{r}(0) = -v_0 \). The angular component of the initial velocity equals to zero, \( v_\varphi(0) = 0 \). The motion of the particle is subjected to one additional condition that the radial component of the velocity remains constant during the motion, i.e.

\[
\dot{r} = -v_0 = \text{const.}
\]  

The air resistance is neglected. Varignon as well as Bernoulli before based the solution of this problem on the validity of the classical conservation law of the mechanical energy \( E \). The mechanical energy \( E \) of the particle at the beginning of the motion has the value \( E_0 = (1/2)mv_0^2 + mgr_0 \). The conservation law of the mechanical energy has the form

\[
E = E_k + U(r) = \frac{1}{2}mv^2 + mgr = E_0 = \frac{1}{2}mv_0^2 + mgr_0 = \text{const.,}
\]  

or

\[
v^2 - v_0^2 = 2g(r_0 - r).
\]  

The square of the magnitude of the velocity vector in polar coordinates is \( v^2 = \dot{r}^2 + r^2 \dot{\varphi}^2 \). Due to the constraint condition \(9\), we can write

\[
\dot{r}^2 + r^2 \dot{\varphi}^2 - \dot{r}^2(0) = v_0^2 + r^2 \dot{\varphi}^2 - v_0^2 = 2g(r_0 - r).
\]  

After the integration of the constraint condition \(9\) we obtain

\[
r(t) = -v_0 t + r_0.
\]  

Substituting into \(12\) we get

\[
\dot{\varphi}^2 = \frac{2g v_0 t}{(r_0 - v_0 t)^2} \quad \Rightarrow \quad \dot{\varphi} = \pm \frac{\sqrt{2g v_0 t}}{r_0 - v_0 t}.
\]  

After separation of variables and the integration we obtain

\[
\varphi(t) - \varphi_0 = \pm \sqrt{\frac{2g}{v_0}} \left( \sqrt{\frac{r_0}{v_0}} \arctanh \sqrt{\frac{v_0 t}{r_0}} - \sqrt{t} \right).
\]  

The equation \(13\) and \(15\) represent the parametric expression of the Varignon’s isochrone in the polar coordinates. After the elimination of the time parameter from the equation \(15\) we obtain the analytic equation of Varignon’s isochrone in the polar coordinates

\[
\varphi(r) = \varphi_0 \pm \sqrt{\frac{2g}{v_0}} \left( \sqrt{\frac{r_0}{v_0}} \arctanh \sqrt{\frac{r_0 - r}{r_0}} - \sqrt{\frac{r_0 - r}{v_0}} \right).
\]
2. Nonholonomic Lagrangian systems on fibered manifolds. We briefly recall basic concepts of a theoretical background of nonholonomic mechanical systems on fibered manifolds. Let \( \pi: Y \to X \) be a fibered manifold, \( \dim Y = m + 1 \), \( \pi_1: J^1Y \to X \) and \( \pi_2: J^2Y \to X \) its jet prolongations and \( \pi_{1,0}: J^1Y \to Y \) the jet projection. Coordinate systems are denoted by \((t, q^\sigma, \dot{q}^\sigma)\), \(1 \leq \sigma \leq m \) on \(Y\), \((t, q^\sigma, \dot{q}^\sigma, \ddot{q}^\sigma)\) and \((t, q^\sigma, \dot{q}^\sigma, \ddot{q}^\sigma, \dddot{q}^\sigma)\) on \(J^1Y, J^2Y\), respectively. A mapping \( \gamma: X \ni I \to Y \) is called a (local) section of \( \pi \) if \( \pi \circ \gamma = \text{id}_I \). Sections of \( \pi_1 \) are denoted by \( \delta \). The mappings \( J^1\gamma, J^2\gamma \) are called the first and second jet prolongations of the section \( \gamma \), they are the sections of \( \pi_1 \) and \( \pi_2 \) respectively.

A differential form \( \eta \) on \( J^1Y \) is called contact if \( J^1\gamma^*\eta = 0 \) for every section \( \gamma \) of \( \pi \). The 1-forms \( \omega^\sigma = dq^\sigma - \dot{q}^\sigma dt \) generate a basis of contact forms on \( J^1Y \). Fibered manifolds are naturally endowed with the structure of projectable and vertical vector fields and horizontal and contact forms. For details we refer to [2].

The local 1-form \( \lambda = L(t,q^\sigma,\dot{q}^\sigma)dt \) on \( J^1Y \) represents Lagrangian of the first order, \( \theta_\lambda \) its Poincaré-Cartan form, \( \lambda = Ldt + (\partial L/\partial \dot{q}^\sigma) \omega^\sigma \), and \( E_\lambda = E_\sigma(L) dq^\sigma \wedge dt \) its Euler-Lagrange form, where \( E_\sigma = (\partial L/\partial \dot{q}^\sigma) - (d/dt)(\partial L/\partial \ddot{q}^\sigma) \) and these functions are called Euler-Lagrange expressions.

A section \( \gamma \) of \( \pi \) is called a path of the Euler-Lagrange form \( E_\lambda \) if \( E_\sigma(L) \circ J^2\gamma = 0 \). These equations are called the Euler-Lagrange equations or motion equations and can be written in an intrinsic geometric form \( J^1\gamma^*\xi_\lambda d\theta_\lambda = 0 \), where \( \xi \) is a \( \pi_1 \)-vertical vector field on \( J^1Y \), or quite equivalently in the form \( J^1\gamma^*\xi_\alpha = 0 \), where \( \alpha \) is any 2-form defined on an open subset \( W \subset J^1Y \), such that \( p_1\alpha = E_\lambda (p_1\alpha \) is 1-contact part of \( \alpha \)). The family of all such (local) 2-forms \( \alpha = d\theta_\lambda + F = \)
\[ A_\tau \omega^\tau \wedge dt + B_\tau \omega^\tau \wedge dq^\tau + F, \] where \( F \) runs over the \( \pi_{1,0} \)-horizontal 2-contact 2-forms, is called the first order Lagrangian system, and is denoted by \([\alpha]\).

From the physical point of view by constraints of a mechanical system we understand a given condition or set of conditions, which restrict the possible geometric positions of the mechanical system. We distinguish between holonomic and nonholonomic constraints. From the geometric point of view holonomic constraints represent submanifolds in the configuration space \( Y \), while geometric concept of nonholonomic constraints is such that the nonholonomic constraints represent certain submanifolds in the phase space \( J^1Y \).

By a constraint submanifold \( Q \subset J^1Y \) we shall mean a fibered submanifold \( \pi_{1,0}|Q: Q \to Y \) of the fibered manifold \( \pi_{1,0} \). We denote by \( \iota: Q \hookrightarrow J^1Y \) the canonical embedding of \( Q \) into \( J^1Y \), and suppose \( \dim Q = 2m + 1 - k \). Locally, \( Q \) is given by equations

\[ f^i(t, q^\sigma, \dot{q}^\sigma) = 0, \quad \mathrm{rank} \left( \frac{\partial f^i}{\partial \dot{q}^\sigma} \right) = k, \quad 1 \leq i \leq k, \quad (17) \]

or equivalently in the normal form

\[ \dot{q}^{m-k+i} = g^i(t, q^\sigma, q^1, q^2, \ldots, q^{m-k}), \quad 1 \leq i \leq k, \quad (18) \]
called a system of \( k \) nonholonomic constraints.

Recall, \( Q \)-admissible section is a local section \( \bar{\gamma} \) of the fibered manifold \( \pi \) such that \( J^1\bar{\gamma}(t) \in Q \) for every \( t \in \text{dom} \bar{\gamma} \) and \( Q \)-valued section is a local section \( \bar{\delta} \) of the fibered manifold \( \pi_{1,0} \) such that \( \bar{\delta}(t) \in Q \) for every \( t \in \text{dom} \bar{\delta} \).

The submanifold \( Q \) is naturally endowed with a distribution \( \mathcal{C} \), called canonical distribution \([2]\). The distribution \( \mathcal{C} \) is generated by the following vector fields

\[ \xi_0 = \frac{\partial}{\partial t} \sum_{i=1}^{k} \left( g^i - \frac{\partial g^i}{\partial q^j} \dot{q}^j \right) \frac{\partial}{\partial q^{m-k+i}}, \xi_l = \frac{\partial}{\partial q^l} \sum_{i=1}^{k} \frac{\partial g^i}{\partial q^l} \frac{\partial}{\partial q^{m-k+i}}, \xi_\ell = \frac{\partial}{\partial \dot{q}^\ell}, \quad (19) \]
or it is annihilated by the system of \( k \) linearly independent 1-forms called canonical constraint 1-forms

\[ \phi^i = -\sum_{l=1}^{m-k} \frac{\partial g^i}{\partial q^l} \omega^l + \iota^* \omega^{m-k+i}, \quad 1 \leq i \leq k. \quad (20) \]

The ideal in the exterior algebra of forms on \( Q \) generated by the annihilator \( \mathcal{C}^0 \) of \( \mathcal{C} \) is called the constraint ideal, and denoted by \( \mathcal{I} \); its elements are called constraint forms. The pair \((Q, \mathcal{C})\) is then called a nonholonomic constraint structure.

Consider an unconstrained Lagrangian system \([\alpha] = [d\theta_\lambda]\) defined on \( J^1Y \). For every \( \alpha \in [\alpha] \) we put

\[ \alpha_Q = \iota^* \alpha \mod \mathcal{I} = \iota^* d\theta_\lambda + \bar{F} + \phi_2, \quad (21) \]

where \( \bar{F} \) is a 2-contact 2-form on \( Q \) and \( \phi_2 \in \mathcal{I} \) is any constraint 2-form. The arising class \([\alpha_Q]\) is called the nonholonomic Lagrangian constrained system related to the Lagrangian system \([\alpha]\) on \( J^1Y \) and the constraint structure \((Q, \mathcal{C})\). In fibered coordinates \([\alpha_Q]\) is represented by the following 2-forms

\[ \alpha_Q = A_\tau^l \omega^l \wedge dt + B_\tau^l \omega^l \wedge dq^l + \bar{F}_l \omega^l \wedge \omega^* + \phi_2, \quad (22) \]
where
\[ A'_i = \begin{bmatrix} A_i + A_{m-k+i} \frac{\partial q^i}{\partial q^j} + \left( B_{l,m-k+j} + B_{m-k+i,m-k+j} \frac{\partial q^j}{\partial q^i} \right) \frac{\dot{q}^j}{dt} \end{bmatrix}, \]
\[ B'_{ls} = \left[ B_{ls} + \left( B_{l,m-k+i,s} \frac{\partial q^i}{\partial q^s} + B_{m-k+i,s} \frac{\partial q^i}{\partial q^s} \right) + B_{m-k+i,m-k+j} \frac{\partial q^j}{\partial q^s} \right]_i, \]
\[ 1 \leq i, j \leq k, 1 \leq l, r, s \leq m - k, F_{ls} = F_{ls}(t, q^s, \dot{q}^s) \] are approximated by means of the two mass points
\[ M^1 \text{ and } M^2, \]
subjected to nonholonomic constraints (17). Equations of motion of the nonholonomic Lagrangian system \([\alpha_Q] \) take the following form
\[ (A'_i + B'_{ls} \dot{q}^s) \circ J^2 \tilde{\gamma} = 0, \quad f^i \circ J^1 \tilde{\gamma} = 0. \] (27)

An alternative approach to dynamics of nonholonomic system is based on the existence of an additional force \( \Phi \), called Chetaev (constraint) force which replaces the influence of constraints,
\[ \Phi = (\Phi^i) = \left( \sum_{i=1}^{k} \mu_i \frac{\partial f^i}{\partial q^s} \right), \] (28)
where \( \mu_i \) are Lagrange multipliers and \( f^i \) are left-hand sides in (17). Equations of motion of the Lagrangian system \([\alpha] \) subjected to nonholonomic constraints (17) take the form
\[ \frac{\partial L}{\partial q^s} - \frac{d}{dt} \frac{\partial L}{\partial \dot{q}^s} = \sum_{i=1}^{k} \mu_i \frac{\partial f^i}{\partial q^s}, \quad 1 \leq \sigma \leq m, \] (29)
and are called deformed equations of motion since they arise by a deformation of the original unconstrained system \([\alpha] \) by means of Chetaev force \( \Phi \) (28).

3. Classical motions in central fields.

3.1. Central fields. In the Newtonian reference frame we consider two bodies, which are approximated by means of the two mass points \( P_1 \) and \( P_2 \) with the masses \( M \) and \( m \), respectively. Denote \( \mathbf{r}_1 \) and \( \mathbf{r}_2 \) the position vectors of the mass points \( P_1 \) and \( P_2 \), respectively. Furthermore, denote \( \mathbf{r} = \mathbf{r}_2 - \mathbf{r}_1 \) the vector of their relative position and its magnitude \( |\mathbf{r}| = r \) represents their distance. One can suppose that the bodies interact by means of the force, which depends only on
their relative position and their distance. Such a force, called central force of the interaction, it can be written as follows

$$ F(r) = F(r) \frac{r}{r} = F(r)e_r, \quad (30) $$

where the vector $e_r = \frac{r}{r}$ represents the unit vector oriented from $P_1$ to $P_2$. Evidently, the force vector $F$ lies on the line connecting $P_1$ and $P_2$ and its magnitude $F$ depends only on their distance by means of the monotonic differentiable function $F(r)$.

In accordance with the law of action and reaction one can state: if the particle $P_2$ exerts on the particle $P_1$ by the force $F_{21} = F(r)e_r$ then also the particle $P_1$ exerts on the particle $P_2$ by the force $F_{12} = -F_{21} = -F(r)e_r$, see the figure below.

![Figure 1. Interaction of two bodies](image)

The motion equations for a system of two interacting particles in the Newtonian reference frame then takes a familiar form

$$ M\ddot{r}_1 = F_{21} = F(r) \frac{r}{r} = F(r)e_r, $$

$$ m\ddot{r}_2 = F_{12} = -F(r) \frac{r}{r} = -F(r)e_r. \quad (31) $$

The interaction between particles is mediated through the central force fields generated by the particles. In the framework of the classical Newtonian mechanics, we assume that the interaction is spreading infinitely fast, i.e. immediately (action at distance). Therefore the particle $P_1$ generates a force field $F_{12}$ which immediately acts on the particle $P_2$ and causes a change of its motion state. Similarly, the particle $P_2$ is the source of the force field $F_{21}$ which causing immediately the change of the motion state of the particle $P_1$. In consequence of this interaction both particles move and therefore also their force fields move.

In particular, if the particle $P_1$ has much greater mass than the particle $P_2$, i.e. $M \gg m$, then the mass $M$ of $P_1$ represents almost the mass of the entire system $M + m$, i.e. $M + m \approx M$. The position of the mass center $r_C$ of the system of two interacting particles $P_1$, $P_2$ approximately coincides with the position of the massive particle $P_1$,

$$ r_C = \frac{Mr_1 + mr_2}{M + m} = \frac{M}{M + m} r_1 + \frac{m}{M + m} r_2 \approx r_1. \quad (32) $$

The sum of the motion equations (31) gives us

$$ M\ddot{r}_1 + m\ddot{r}_2 = 0 \quad \text{or} \quad \ddot{r}_1 = -\frac{m}{M} \ddot{r}_2. \quad (33) $$

The relation $m/M \approx 0$ enables us to omit the influence of the force $F_{21}$ on the particle $P_1$, and we obtain $\dot{r}_1 \approx 0$, i.e. the massive particle $P_1$ is at rest or in the
uniform rectilinear motion. Hence, we can investigate the motion of the particle \( P_2 \) in the Newtonian (inertial) reference frame with the origin that is located at the massive particle \( P_1 \) and is rigidly connected with it. In such a way we regard the massive particle \( P_1 \) as approximately fixed source of the force field \( F_{12} \) in which the second lighter particles \( P_2 \) moves. Its motion is governed by the motion equation

\[
mr_2'' = F_{12} = -F(r)e_r. \tag{34}
\]

In the more general case when both particles have comparable masses it is well known from the classical mechanics, that the problem of two interacting particles one can separate into the uniform rectilinear motion of the mass center of the system and into the relative motions of particles \( P_1 \) and \( P_2 \) in the Newtonian (inertial) reference frame connected with the mass center of the system. The relative motion of two particles in this reference frame is mathematically equivalent with the motion of a single particle with the reduced mass \( \bar{m} = Mm/(M + m) \) in the central force field \([30]\). The reduced mass \( \bar{m} \) is approximately equals to \( m \), when \( M \gg m \).

A central field is such a vector field that in arbitrary point (characterized by the position vector \( r \)) the force vector \( F \) lies on the line connecting the center of the field (located in the origin) with this point. The center of the field is placed at the origin, that coincides with the massive particle or with the mass center of the system and is rigidly connected with it.

The force \( F \) is oriented directly from the center (repulsive field) or to the center (attractive field).

Since the position vector \( r \) is oriented from the center then for an attractive field we have \( F(r) = -F(r)e_r \) and \( F(r) = F(r)e_r \) for a repulsive field respectively. The most typical examples of central fields in the classical physics are:

- **Newtonian gravitational field**

  \[
  F(r) = \frac{\kappa Mm \ r}{r^2} = \frac{\alpha}{r^2} e_r, \tag{35}
  \]

  \( \alpha = -\kappa Mm \), \( M \) is the mass of the center, \( m \) is the mass of the moving particle and \( \kappa \) is the universal gravitational constant.

- **Coulomb’s electrostatic field**

  \[
  F(r) = \frac{1}{4\pi \varepsilon} \frac{Qq \ r}{r^2} = \frac{a}{r^2} e_r, \tag{36}
  \]

  \( a = (Qq)/(4\pi \varepsilon) \), \( Q \) is the electric charge of the center, \( q \) is the electric charge of the particle and \( \varepsilon \) is the permittivity of a medium.

- **Elastic force field**

  \[
  F(r) = -kr \frac{r}{r} = -k r e_r, \tag{37}
  \]

  \( k > 0 \) is the spring constant.

It is also possible to consider the following force field which do not have the physical interpretation. These fields have only the theoretical meaning.

- **Constant central gravitational field**

  \[
  F = -mg \frac{r}{r} = -mge_r. \tag{38}
  \]

  The field was taken into account by the Varignon in the context of the Varignon’s isochrone problem.

- **Inverse proportional gravitational field**

  \[
  F(r) = -\frac{b}{r^2} \frac{r}{r} = -\frac{b}{r^2} e_r, \tag{39}
  \]
i.e. the gravitational field which is proportional to the reciprocal value of the distance \( r, \ b > 0. \)

A force field \( \mathbf{F}(\mathbf{r}) \) is called *conservative* (or potential) if \( \text{rot} \mathbf{F}(\mathbf{r}) = 0. \) If \( \mathbf{F}(\mathbf{r}) \) is a conservative field then there exists a scalar function \( U(\mathbf{r}) \) such that
\[
\mathbf{F}(\mathbf{r}) = -\nabla U(\mathbf{r}). \tag{40}
\]

To every point \( X \) (characterized by a position vector \( \mathbf{r} \)) of a conservative field \( \mathbf{F}(\mathbf{r}) \) one can assign a real number \( U(\mathbf{r}) \), called the *potential energy* or the *potential*, given by the following curve integral
\[
U(\mathbf{r}) = -\int_{\bar{X}}^{\mathbf{r}} \mathbf{F}(\mathbf{r}') \cdot d\mathbf{r}', \tag{41}
\]
which represents the work of the field needed to displace a particle along some trajectory from the reference point \( \bar{X} \) to \( X \). The reference point \( \bar{X} \) is usually some significant point of the field. Since in the conservative fields the work is independent of an integration path, one can integrate simply along the line \( \bar{XX} \). Recall that any central field is a conservative field. For the central fields given by \( \mathbf{F}(\mathbf{r}) \sim r^n \) we set \( \bar{X} = 0 \) (center) and for the central fields given by \( \mathbf{F}(\mathbf{r}) \sim 1/r^n \) we set \( \bar{X} = \infty \) (infinity). The potential energy of a central field is given by
\[
U(\mathbf{r}) = -\int_{\bar{X}}^{\mathbf{r}} \mathbf{F}(\mathbf{r}') \cdot e_r \cdot dr' = -\int_{\bar{X}}^{\mathbf{r}} \mathbf{F}(\mathbf{r}') \cdot dr', \tag{42}
\]
where we integrate along the line \( \bar{XX} (\text{dr}' = e_r \cdot dr') \). Hence, the potential energy of the particle in a central field depends on a position of the particle only by means of its distance from the origin (center), thus \( U(\mathbf{r}) = U(r) \). Finally, a relation between the potential energy \( U(\mathbf{r}) \) and the magnitude \( \mathbf{F}(\mathbf{r}) \) of the central force \( \mathbf{F}(\mathbf{r}) \) can be also expressed by the formula
\[
-\frac{dU(\mathbf{r})}{dr} = F(\mathbf{r}). \tag{43}
\]

In particular, the potential energy of the Newtonian gravitational field \( [35] \), the Coulomb’s electrostatic field \( [36] \) and the elastic force field \( [37] \) are expressed by
\[
U(r) = -\frac{\alpha M\mathbf{m}}{r} = \frac{\alpha}{r}, \quad U(\mathbf{r}) = \frac{1}{4\pi \varepsilon} \frac{Qq}{r} = \frac{a}{r}, \quad U(\mathbf{r}) = \frac{1}{2} kr^2, \tag{44}
\]
respectively. The potential energy of the constant gravitational field \( [38] \) and the inverse proportional gravitational field \( [39] \) are expressed by
\[
U(\mathbf{r}) = mgr, \quad U(\mathbf{r}) = b \ln r, \tag{45}
\]
respectively.

A Lagrange function of a particle in a central field characterized by the potential energy \( U(\mathbf{r}) \) represented in the Cartesian coordinates with the origin at the center has the standard form
\[
L = E_k - U(\mathbf{r}) = \frac{1}{2} m|\dot{\mathbf{r}}|^2 - U(\mathbf{r}) = \frac{1}{2} m(\dot{x}^2 + \dot{y}^2 + \dot{z}^2) - U(\mathbf{r}) = \frac{1}{2} mv^2 - U(\mathbf{r}), \tag{46}
\]
where \( \mathbf{v} = \dot{\mathbf{r}} = (\dot{x}, \dot{y}, \dot{z}) \) is a vector of the velocity of the particle. The motion equations of the particle (Euler-Lagrange equations) one can write in the vector form
\[
m\ddot{\mathbf{r}} = \frac{F(\mathbf{r})}{r} \mathbf{r}. \tag{47}
\]
3.2. Conservation laws. Let us consider the Lagrangian \( L = L(t, q^\sigma, \dot{q}^\sigma) \) of the first order in term of generalized coordinates \( q^\sigma \) and generalized velocities \( \dot{q}^\sigma \) in the classical form
\[
L(t, q^\sigma, \dot{q}^\sigma) = E_k(t, q^\sigma, \dot{q}^\sigma) - U(t, q^\sigma, \dot{q}^\sigma),
\]
where \( E_k \) is the kinetic energy and \( U \) is the generalized potential. The most general expression for the kinetic energy \( E_k \) in terms of generalized coordinates is
\[
E_k(t, q^\sigma, \dot{q}^\sigma) = A_{\sigma\nu}(t, q^\sigma)\dot{q}^\sigma \dot{q}^{\sigma'} + B_{\sigma
u}(t, q^\sigma)\dot{q}^\sigma + C(t, q^\sigma) = T_2 + T_1 + T_0,
\]
i.e. the kinetic energy \( E_k \) is a homogeneous at most quadratic function of the generalized velocities \( \dot{q}^\sigma \). The \( T_0, T_1 \) and \( T_2 \) are those parts of \( E_k \) that are homogeneous functions of degree 0, 1 and 2 in the generalized velocities, respectively, \([2]\).

The expression \([49]\) for the kinetic energy simplifies if the mechanical system does not explicitly depend on time, e.g. if considered constraints are scleronomic. In this case the kinetic energy \( E_k \) reduces only into
\[
E_k = T_2 = A_{\sigma\nu}(q^\sigma)\dot{q}^\sigma \dot{q}^{\sigma'},
\]
i.e. the kinetic energy \( E_k \) is a homogeneous function of degree 2 in generalized velocities.

Suppose that the potential \( U \) does not depend on the generalized velocities, and does not depend on time, i.e.
\[
U = U(q^\sigma),
\]
then the Lagrangian \( L \) takes the form
\[
L = E_k(q^\sigma, \dot{q}^\sigma) - U(q^\sigma) = T_2(q^\sigma, \dot{q}^\sigma) - U(q^\sigma) = A_{\sigma\nu}(q^\sigma)\dot{q}^\sigma \dot{q}^{\sigma'} - U(q^\sigma),
\]
i.e. the Lagrangian \( L \) does not depend explicitly on time,
\[
\frac{\partial L}{\partial \dot{q}^\mu} = 0.
\]
One can find
\[
\frac{\partial L}{\partial \dot{q}^\mu} \dot{q}^\mu + \frac{\partial E_k}{\partial \dot{q}^\mu} \dot{q}^\mu \frac{\partial T_2}{\partial \dot{q}^\mu} \dot{q}^{\mu'} = 2A_{\mu\nu}(q^\sigma)\dot{q}^{\mu'} \dot{q}^{\nu'} = 2T_2 = 2E_k.
\]
Consequently, the function
\[
\frac{\partial L}{\partial \dot{q}^\mu} \dot{q}^\mu - L = 2E_k - (E_k - U(q^\sigma)) = E_k + U(q^\sigma) = E,
\]
represents the mechanical energy, which remains constant. Indeed,
\[
\frac{d}{dt} \left( \frac{\partial L}{\partial \dot{q}^\mu} \dot{q}^\mu - L \right) = \frac{d}{dt} \left( \frac{\partial L}{\partial \dot{q}^\mu} \dot{q}^\mu \right) - \frac{dL}{dt} = \frac{d}{dt} \left( \frac{\partial L}{\partial \dot{q}^\mu} \right) \dot{q}^\mu + \frac{\partial L}{\partial q^\nu} \ddot{q}^\nu - \frac{dL}{dt} = \dot{q}^\mu \left[ \frac{d}{dt} \left( \frac{\partial L}{\partial \dot{q}^\mu} \right) - \frac{\partial L}{\partial q^\nu} \right] = 0,
\]
since the Euler-Lagrange equations hold.

Evidently, the kinetic part of the Lagrangian \([46]\) of the particle moving in the central field is a homogeneous quadratic function in Cartesian components \( \dot{x}, \dot{y}, \dot{z} \) of the velocity \( \mathbf{v} \), thus the conservation law of the mechanical energy \( E \) of the particle holds
\[
E = E_k + U(r) = \frac{1}{2} m v^2 + U(r) = E_0 = \frac{1}{2} m v_0^2 + U(r_0) = \text{const},
\]
\( v_0 \) is the magnitude of the initial velocity \( \mathbf{v}_0 \) and \( r_0 \) is the initial distance of the particle from the center.
Moreover, it is well known that the motion of the particle in an arbitrary central field is always planar. This fact is a direct consequence of the conservation law of the angular momentum \( l \) of the particle,

\[
l = r \times mv = l_0 = r_0 \times mv_0 = \text{const.},
\]

(58)

\( r_0 \) represents a starting position of the particle, \( v_0 \) is a vector of the initial velocity of the particle. Thus, at each time the position vector \( r \) and the velocity vector \( v \) of the particle lie in the plane perpendicular to the constant angular momentum vector \( l_0 \).

Without loss of generality we consider that the motion of the particle takes place e.g. in the \( xy \)-plane (\( z = 0 \)). The position vector \( r \) and the velocity vector \( v \) can be expressed in the polar coordinates \((r, \varphi)\) as follows

\[
\begin{align*}
  r &= re_r, \\
  v &= v_r e_r + v_\varphi e_\varphi = \dot{r}e_r + r\dot{\varphi}e_\varphi.
\end{align*}
\]

(59)

Consequently for the angular momentum vector \( l \) we obtain

\[
l = r \times mv = re_r \times m(\dot{r}e_r + r\dot{\varphi}e_\varphi) = mr^2\dot{\varphi}(e_r \times e_\varphi) = mr^2\dot{\varphi}e_z = (0, 0, mr^2\dot{\varphi}),
\]

(60)

and with respect to (58) one can write

\[
l = (0, 0, mr^2\dot{\varphi}) = l_0 = (0, 0, l_0),
\]

(61)

where \( l_0 = \pm |l_0| \). On the other side the conservation law (58) can be rewritten in the scalar form for the magnitude of the instantaneous angular momentum vector as follows

\[
|l| = mr(t)v(t)\sin \beta(t) = |l_0| = mr_0v_0\sin \beta_0 = \text{const.},
\]

(62)

where \( \beta(t) \in (0, \pi) \) is the instantaneous angle between vectors \( r(t) \) and \( v(t) \), and \( |l_0| \) is the magnitude of the initial angular momentum \( l_0 \), \( \beta_0 \in (0, \pi) \) is the initial angle between vectors \( r_0 \) and \( v_0 \).

3.3. Motions in central fields. From the previous subsection we know that a particle in a central field can be treated as a simple mechanical system with the corresponding Lagrangian \( L \) expressed in the polar coordinates \((r, \varphi)\) with the origin placed in the center of the force field,

\[
L = \frac{1}{2}m(\dot{r}^2 + r^2\dot{\varphi}^2) - U(r).
\]

(63)

Evidently, the kinetic part of the Lagrangian (63) is a homogeneous quadratic function of time derivatives \( \dot{r}, \dot{\varphi} \), of the polar coordinates \( r, \varphi \), therefore the conservation law of the mechanical energy in the form

\[
\frac{1}{2}m(\dot{r}^2 + r^2\dot{\varphi}^2) + U(r) = E_0,
\]

(64)

holds.

The movement of the particle is then represented by the solution of the corresponding Euler-Lagrange equations

\[
\begin{align*}
  m\ddot{r} - mr\dot{\varphi}^2 &= -\frac{dU}{dr} = F(r), \\
  \frac{d}{dt}(mr^2\dot{\varphi}) &= 0 \quad \Rightarrow \quad mr^2\dot{\varphi} = l_0 = \text{const.},
\end{align*}
\]

(65)

(66)
where \( l_0 = mr_0^2 \dot{\varphi}(0) = mr_0 v_r(0) = mr_0 v_0 \sin \beta_0 \). Using the conservation law of angular momentum (66) the equation (65) describing the radial component of the motion takes the following form

\[
m \ddot{r} = F(r) + mr_0^2 \dot{\varphi}^2 = F(r) + \frac{m v_0^2}{r} = F(r) + F_c(r).
\] (67)

It means that the influence of the "centrifugal" term \( m v_0^2 / r \) onto the radial component of the motion one can replace by means of the action of an additional repulsive central force

\[
F_c = F_c(r) e_r,
\] (68)

where

\[
F_c(r) = \frac{l_0^2}{mr_0^3} = \frac{m r_0^2 v_r^2(0)}{r^3}.
\] (69)

The differential equations (65) and (66) can be solved numerically for the concrete force field \( F(r) \) and for the given initial conditions. The particle can start from the initial position \( r(0) = r_0 \), \( \varphi(0) = \varphi_0 \), with the initial velocity vector \( \mathbf{v}(0) = (v_r(0), v_\varphi(0)) \) by the various possible ways, see the Fig 2.

\[
v_r(0) > 0, \quad v_\varphi(0) > 0
\]
\[
v_r(0) > 0, \quad v_\varphi(0) < 0
\]
\[
v_r(0) < 0, \quad v_\varphi(0) > 0
\]
\[
v_r(0) < 0, \quad v_\varphi(0) < 0
\]
\[
\beta_0 = \pi / 2
\]
\[
\beta_0 = \pi
\]

**Figure 2. The initial conditions scheme**

For the various values of the magnitude \( v_0 \) of the initial velocity \( \mathbf{v}_0 \) and its radial and angular components certain numerical solutions are obtained. Since we have

\[
v_r^2(0) + v_\varphi^2(0) = v_0^2,
\] (70)

one can change e.g. only the angular component \( v_\varphi(0) \) and consequently \( v_r(0) \) is then determined by

\[
v_r(0) = \pm \sqrt{v_0^2 - v_\varphi^2(0)} = \pm \sqrt{v_0^2 - r_0^2 \dot{\varphi}^2(0)} = \pm \sqrt{v_0^2 - v_0^2 \sin^2 \beta_0} = v_0 \cos \beta_0.
\] (71)

In particular, if \( v_\varphi(0) = 0 \) then \( v_r(0) = \dot{r}(0) = v_0 \) and it is a radial motion. If \( v_r(0) = 0 \) then \( v_\varphi(0) = v_0 \) and it is a motion with zero radial component of the initial velocity. Investigated motions with non-zero angular component of the initial velocity will be considered only for the case \( v_\varphi(0) > 0 \), i.e. anticlockwise motions, thus \( l_0 > 0 \).

As an alternative starting point for the investigation of motions of the particle in central fields can serve the conservation laws. In the polar coordinates the
conservation laws (57) and (62) take the following form
\[
\frac{1}{2}m(r^2 + r^2 \dot{\phi}^2) + U(r) = E_0 = \frac{1}{2}m(v_r^2(0) + v_\phi^2(0)) + U(r_0) = \frac{1}{2}mv_0^2 + U(r_0), 
\]
(72)

\[ mv \sin \beta = m v_\phi = mr^2 \dot{\phi} = l_0 = mr_0 v_\phi(0) = mr_0 v_\phi(0) = mr_0 v_\phi(0) = m v_\phi(0) \sin \beta, \]

\[ v_\phi = v \sin \beta = r \dot{\phi} \]
is the angular component of \( v \). The system (72) represents the system of two first order differential equations for unknown functions \( r = r(t) \), \( \phi = \phi(t) \). We substitute \( \dot{\phi} \), \( \ddot{\phi} = \frac{l_0}{mr^2} \)
into the first equation of the system (72),
\[
\frac{1}{2} m \left( \dot{r}^2 + \frac{l_0^2}{mr^2} \right) + U(r) = E_0, 
\]
(74)
or
\[
\dot{r}^2 = \frac{2}{m} \left( E_0 - U(r) - \frac{l_0^2}{2mr^2} \right), 
\]
(75)
and after the separation of variables we get
\[
\dot{t} = \frac{dr}{dt} = \pm \sqrt{\frac{2E_0 - 2U(r)}{m} - \frac{l_0^2}{m^2r^2}} \Rightarrow t = \pm \int_{r_0}^{r} \frac{dr'}{\sqrt{\frac{2E_0}{m} - \frac{2U(r')}{m} - \frac{l_0^2}{m^2r'^2}}}, 
\]
(76)
The sign plus is valid in the time interval in which the particle moves away from the center and the sign minus is valid for the approaching to the center.

In general, integration of (76) leads to a numerical quadrature. If it is possible (after the integration of (76)) to express \( r = r(t) \) as the inverse function to \( t = t(r) \) given by (76), then we get the time dependence of the angle coordinate \( \phi = \phi(t) \),
\[
\phi(t) - \phi_0 = l_0 \int_{r_0}^{r} \frac{dt'}{mr^2(t')}, 
\]
where for \( l_0 > 0 \) is \( \phi(t) \) an increasing function of time (anticlockwise motion) and for \( l_0 < 0 \) is \( \phi(t) \) an decreasing function of time (clockwise motion). Relations (76) and (77) represent a parametric expression of the trajectory of a motion in a central field.

The explicit expression \( r = r(\phi) \) of the trajectory in the polar coordinates can be obtained immediately from (75) using the formula \( mr^2 \dot{\phi} = l_0 \). Since
\[
\dot{r} = \frac{dr}{d\phi} \dot{\phi} = \frac{l_0}{mr^2} \frac{dr}{d\phi}, 
\]
(78)
we obtain
\[
\frac{dr}{d\phi} = \pm \frac{mr^2}{l_0} \sqrt{\frac{2}{m} \left( E_0 - U(r) - \frac{l_0^2}{2mr^2} \right)} 
\]
(79)
and after the separation and integration
\[
\phi(r) - \phi(r_0) = \pm l_0 \int_{r_0}^{r} \frac{dr'}{r'^2 \sqrt{2m(E_0 - U(r')) - \frac{l_0^2}{m}\frac{1}{r'^2}}}.
\]
(80)
Evidently, the relation (80) represents two functions \( \varphi = \varphi(r) \), i.e. the motion of the particle is reversible. The particle can move anticlockwise or clockwise. The last integral is suitable to compute using the substitution \( r = 1/u \),

\[
\varphi(1/u) - \varphi(1/u_0) = \mp l_0 \int_{1/u_0}^{1/u} \frac{du'}{\sqrt{2m(E - U(1/u') - l_0^2 u'^2}}. \tag{81}
\]

The presented formulas can be found almost in every classical books of theoretical mechanics, e.g. in [2]. It is known that the integral (81) can be expressed by means of elementary functions for power function types potentials \( U(r) = kr^n \) for which

\( n = -2, \quad n = -1 \) (Newtonian or Coulomb’s potential), \( n = 0 \) (constant potential) and \( n = 2 \) (potential of the elastic force).

3.4. Motions in Newtonian gravitational field, Kepler’s problem. In particular, for the Newtonian gravitational potential \( U(r) = \alpha/r, \quad \alpha = -\pi Mm, \) the integral (81) leads to the well known Kepler’s trajectories represented in polar coordinates by the formula

\[
r(\varphi) = \frac{p}{1 + \epsilon \cos(\varphi - \varphi_0)}, \tag{82}
\]

where

\[
p = \frac{l_0^2}{m|\alpha|}, \quad \epsilon = \sqrt{1 + \frac{2l_0^2E_0}{m|\alpha|^2}}. \tag{83}
\]

The equation (82) represents the conic sections with the focus placed in the center of the Newtonian gravitational field with the focal parameter \( p \) and with the numerical eccentricity \( \epsilon \).

Geometric parameters \( p \) and \( \epsilon \) given by (83) are determined only by the initial value \( E_0 \) of the mechanical energy \( E \),

\[
E_0 = \frac{1}{2} mv_0^2 - \frac{|\alpha|}{r_0}, \tag{84}
\]

and by means of the initial value \( l_0 \) of the angular momentum \( l \),

\[
l_0 = mr_0 v_\varphi(0) = mr_0 v_0 \sin \beta_0. \tag{85}
\]

The appropriate trajectory is given by the relations between the initial values \( E_0 \) and \( l_0 \) as follows:

For the circular trajectory \( \epsilon = 0 \) \( \Rightarrow \) \( E_0 = -\frac{m\epsilon_0^2}{2l_0^2} \),

For the elliptic trajectory \( 0 < \epsilon < 1 \) \( \Rightarrow \) \( E_0 \in \left(-\frac{m\epsilon_0^2}{2l_0^2}, 0\right) \),

For the parabolic trajectory \( \epsilon = 1 \) \( \Rightarrow \) \( E_0 = 0 \),

For the hyperbolic trajectory \( \epsilon > 1 \) \( \Rightarrow \) \( E_0 > 0 \).

Conditions (86) are specific conditions for particular trajectories of the moving particle in the Newtonian gravitational field.

If the initial velocity has zero angular component, \( v_\varphi = 0 \), then with respect to (85) the magnitude of the angular momentum \( l = l_0 = 0 \). Hence the motion is realized only in the radial direction. If \( v_r(0) = v_0 > 0 \), this is the radial motion from the center (upward throw), if \( v_r(0) = -v_0 < 0 \), this is the radial motion to the center (drop) and if \( v_r(0) = v_0 = 0 \), this is the free fall in the Newtonian gravitational field.
Especially, if the particle starts with zero radial velocity component i.e. \( v_r(0) = v_0 \), then \( l_0 = mr_0 v_0 \). Moreover, if the initial energy \( E_0 \) is given by the first relation in (86), one can derive the formula for the circular velocity (the first space velocity),

\[
\frac{1}{2} m v_0^2 + \frac{\alpha}{r_0} = - \frac{m a^2}{2 l_0} \quad \Rightarrow \quad v_0 = v_{\text{circ}} = \sqrt{\frac{2 M}{r_0}}. \quad (87)
\]

If the initial energy \( E_0 = 0 \) (the parabolic trajectory) then one can derive the formula for the escape velocity,

\[
\frac{1}{2} m v_0^2 + \frac{\alpha}{r_0} = 0 \quad \Rightarrow \quad v_0 = v_{\text{esc}} = \sqrt{\frac{2 M}{r_0}} = \sqrt{2} v_{\text{circ}}. \quad (88)
\]

3.5. Effective potential energy. The conditions (86) for the values of the initial mechanical energy \( E_0 \) can also be derived graphically from the so-called diagram of the effective potential energy. A diagram of the effective potential energy enables us to make a qualitative analysis of the trajectories of motions in the central fields for arbitrary course of the potential energy \( U(r) \) of the given force field \([2]\).

In the equation (74) for total mechanical energy of the particle moving in the central field

\[
E = \frac{1}{2} m r^2 + \frac{l_0^2}{2 m r^2} + U(r) = E_0, \quad (89)
\]

the first term \( (1/2) m r^2 \) represents the kinetic energy of the radial motion, the second term \( l_0^2/(2 m r^2) = (1/2) m u_r^2 \) is called the centrifugal energy. The third term \( U(r) \) represents the potential energy of the given force field \( F(r) \).

The function

\[
U_{\text{ef}}(r) = U(r) + \frac{l_0^2}{2 m r^2} = U(r) + \frac{m r_0^2 v_r^2(0)}{2 r^2} = U(r) + U_c(r), \quad (90)
\]

is called the effective potential energy of the particle. Notice that initial angular momentum \( l_0 = m r_0 v_r(0) = m r_0 v_0 \sin \beta_0 \) plays here the role of a parameter, therefore we sometimes write \( U_{\text{ef}}(r, l_0) \). The centrifugal energy \( U_c(r) \) represents the potential energy of an additional repulsive central force \( F_c \) \([90]\),

\[
F_c = -\text{grad} U_c = -\frac{l_0^2}{m r^3} e_r. \quad (91)
\]

In particular in the case of the radial motions the parameter \( l_0 = 0 \), i.e. \( v_r(0) = 0 \), there is no repulsive central force, and the effective potential energy coincides with the potential energy of the central field, i.e. \( U_{\text{ef}}(r) = U(r) \). On the contrary the parameter \( l_0 \) is maximal, \( l_0 = m r_0 v_0 \), if the radial component \( v_r(0) = 0 \), i.e. \( v_r(0) = v_0 \), then the influence of the repulsive central force \( F_c \) is maximal.

Introducing the effective potential \([90]\) the equation (75) takes the form

\[
\dot{r}^2 = \frac{2}{m} \left( E_0 - U_{\text{ef}}(r) \right). \quad (92)
\]

Since \( \dot{r}^2(t) \) is a non-negative function the right-hand side of the equation \([92]\) must also be non-negative, i.e. it holds the relation

\[
E_0 \geq U_{\text{ef}}(r). \quad (93)
\]

This condition is principal for a qualitative analysis of trajectories in a central field, since it restricts admissible intervals of radial distances from the center. It is useful to solve the inequality \([93]\) graphically. In the diagram of the effective potential energy one can draw a constant function corresponding to a certain value of the initial energy \( E_0 \). Intersection points with the graph \( U_{\text{ef}}(r) \) determine the limits
of possible radial distances from the center. These points are called turning points and since they are solutions of the equation \( E_0 - U_{\text{ef}} = 0 \), the radial component \( v_r = \dot{r} \) of the velocity at these points is null. In these points the function \( \dot{r} = \dot{r}(t) \) changes its sign, i.e. either the approaching of the particle to the center turns into the motion away from the center, such turning point is called the pericenter (the particle is located in the closest distance from the center), or the motion away from the center turns into the approaching to the center, such turning point called the apocenter (the particle is located in the maximal distance from the center).

If there exist two such points, say \( r_1, r_2 \), then the motion is realized in the bounded region around the center and is called the finite motion. If the admissible interval of distances of the motion is bounded only from the left, the particle can approach the center up to a distance \( r_1 \) and then the particle escapes away from the center, the motion is called infinite or escaping motion. If the admissible interval of distances of the motion is bounded only from the right, e.g. by \( r_2 \), the particle can move away from the center up to a distance \( r_2 \) and then it approaches the center.

In particular case if the line of the constant value \( E_0 \) is a tangent line to the graph of the effective potential then the appropriate contact point represents an extremum \( r_{\text{ext}} \) and the motion proceeds in a constant distance \( r = r_{\text{ext}} \) from the center, and is called circular motion. If at the point \( r_{\text{ext}} \) is minimum then the corresponding circular orbit is stable, i.e. a small change of the initial conditions leads to a finite motion in the interval of possible distances \( \langle r_1, r_2 \rangle \). If at the point \( r_{\text{ext}} \) is maximum then the corresponding circular orbit is unstable, i.e. a small change of the initial conditions leads either to an approaching to the center or to an infinite motion.

3.6. Classification of trajectories in Newtonian gravitational field from the diagram of the effective potential energy. The effective potential energy for the Newtonian gravitational field takes the form

\[
U_{\text{ef}}(r) = \frac{\alpha}{r} + \frac{l_0^2}{2mr^2} = -\frac{\kappa Mm}{r} + \frac{mr_0^2v_0^2(0)}{2r^2},
\]

and the course of this function is drawn on the following figure.

![Figure 3](image)

**Figure 3.** The effective potential of the Newtonian gravitational field

We see that for small distances \( r \) the centrifugal potential \( l_0^2/(2mr^2) \) is dominant and on the contrary for long distances the Newtonian gravitational potential
\(-\alpha Mm/r\) prevails. The function \(U_{\text{ef}}(r)\) has a minimum value \(U_{\text{ef}}^{\text{min}} = -(ma^2)/(2l_0^2)\) at the point \(r_{\text{min}} = -l_0^2/(ma) = l_0^2/(\alpha Mm^2)\) and is null at the point \(r^0 = -l_0^2/(2ma) = l_0^2/(2\alpha Mm^2)\). Notice that \(r_{\text{min}} = 2r^0\). In the Fig. 3 the restricted area in which the particle cannot be located is displayed.

Now we perform the complete classification of motions in the Newtonian gravitational field with zero radial component of the initial velocity using the diagram of the effective potential energy \(U_{\text{ef}}\). In the following figures one can see the typical situations concerning selected values of the initial energy \(E_0\) related to the appropriate courses of the effective potential energy \(U_{\text{ef}}(r, l_0)\) depending on the parameter \(l_0 = mr_0v_0\) and the corresponding trajectories. The testing particle starts in the fixed distance \(r_0\) from the center with the initial velocity vector \(v_0\) in the “horizontal” direction, i.e. in the direction perpendicular to the connecting line of the starting point and the center (motions with zero radial component of the initial velocity). The only variable quantity is the magnitude of the initial velocity \(v_0\). However the value of the initial velocity \(v_0\) determines the value of the initial energy \(E_0(v_0) = (1/2)mv_0^2 + \alpha/r_0\), the value of the initial angular momentum \(l_0(v_0) = mr_0v_0\) and consequently also the graph of the effective potential energy \(U_{\text{ef}}(r, v_0) = -(\alpha mM)/r + mr_0^2v_0^2/(2r^2)\). Recall that in this case the effective potential energy \(U_{\text{ef}}\) takes the null value at the point \(r^0 = r_0^2v_0^2/(2\alpha M)\) and has minimum \(U_{\text{ef}}^{\text{min}} = -(\alpha^2mM^2)/(2r_0^2v_0^2)\) at the point \(r_{\text{min}} = r_0^2v_0^2/(\alpha M)\).

\[
\begin{align*}
U_{\text{ef}} & \quad \vdots \\
E_0(v_0) & \quad rE_0(v_0) \\
U_{\text{ef}}(r, l_0) & \quad E_0 = 0 \Rightarrow v_0 = v_{\text{esc}} \\
hyperbolic\ trajectory & \quad \text{parabolic trajectory}
\end{align*}
\]

**Figure 4. Infinite motions in the Newtonian gravitational field**

For \(E_0 > 0\) (i.e. the corresponding magnitude of the initial velocity \(v_0 > v_{\text{esc}}\)) there exists just one intersection with the graph, the motion is infinite and proceeds in the interval of the possible distances \(r \in (r_0, \infty)\), and the trajectory of the particle is hyperbolic. The case \(E_0 = 0\) is formally the same as the preceding situation but it represents a limit case when \(v_0 = v_{\text{esc}}\), for which the motion becomes infinite. The particle escapes away from the distance \(r_0\) along the parabolic trajectory, see Fig. 4.

For \(E_0 < 0\) and \(E_0 > U_{\text{ef}}^{\text{min}}\) (i.e. \(v_{\text{circ}} < v_0 < v_{\text{esc}}\)) there exist two intersection points, \(r_0, r_1\). The motion is finite and proceeds in the interval of distances \(r \in (r_0, r_1)\), the trajectory is elliptical with the focus at the center and the starting point \(r_0\) is the pericenter of this trajectory, see Fig. 5.

If \(E_0 = U_{\text{ef}}^{\text{min}}\) (i.e. \(v_0 = v_{\text{circ}}\)) then the motion proceeds in a constant distance \(r_0 = r_{\text{min}}\) from the center, the trajectory is circular, Fig. 6.
UNIFORM MOTIONS IN CENTRAL FIELDS

\[ r_0 < r_1 \]

**Figure 5.** Finite motion in the interval of distances \( r \in (r_0, r_1) \)

\[ E_0 = U_{\text{ef}}^{\text{min}} \Rightarrow v_0 = v_{\text{circ}} \]

circular trajectory

\[ E_0 = U_{\text{ef}} \]

**Figure 6.** Finite motion at the constant distance \( r_0 \)

**Figure 7.** Finite motion in the interval of distances \( r \in (r_1, r_0) \)
It would seem that in the case \( E_0 < U_\text{ef}^{\text{min}} \) (i.e. \( v_0 < v_{\text{circ}} \)) the particle is located in the restricted area, thus the motion would not have been possible. However, it is necessary to realize that for these values of the initial energy \( E_0 \) resp. the magnitude of the initial velocity \( v_0 \) the course of effective potential changes in such a way that minimum is at the smaller distance \( r_{\text{min}} \) than in the previous case and its absolute value \( |U^{\text{min}}_\text{ef}| \) increases. Consequently, the line of the constant value \( E_0 \) has in this case again two intersection points, \( r_1, r_0 \). The motion is finite and proceeds again along the elliptical trajectory with the focus at the center. Unlike Fig. 5 the initial distance \( r_0 \) is the apocenter of this trajectory, see Fig. 7. By that the classification of the trajectories of motions in the Newtonian gravitational field with zero radial component of the initial velocity is completed.

4. Uniform motions in central fields - three dimensional situation. Let us consider the mechanical system of one particle moving in a central field. The Lagrangian of the system is given by (46). The particle is subject to isotachytonic constraint

\[
|\mathbf{v}|^2 = v^2 = \mathbf{v} \cdot \mathbf{v} = \dot{x}^2 + \dot{y}^2 + \dot{z}^2 = v_0^2 = \text{const.} \neq 0, \tag{95}
\]

i.e. the magnitude of the velocity \( v \) remains constant during the motion. The constraint (95) can also be represented by

\[
f(\dot{x}, \dot{y}, \dot{z}) = \dot{x}^2 + \dot{y}^2 + \dot{z}^2 - v_0^2 = 0. \tag{96}
\]

The existence of the constraint (95) or (96) gives rise the constrained force called Chetaev force

\[
\Phi = (\Phi_x, \Phi_y, \Phi_z) = \mu \left( \frac{\partial f}{\partial \dot{x}}, \frac{\partial f}{\partial \dot{y}}, \frac{\partial f}{\partial \dot{z}} \right) = \mu (2\dot{x}, 2\dot{y}, 2\dot{z}) = 2\mu \mathbf{v}, \tag{97}
\]

\( \mu = \mu(t) \) is a Lagrange multiplier. The motion equations of this constrained mechanical system are obtained by a deformation of the original unconstrained motion equations (47) by means of the Chetaev force (97),

\[
m\ddot{r} = m\dot{v} = \frac{F(r)}{r} r + 2\mu \mathbf{v}. \tag{98}
\]

From the modifications of the equation (98) follows

\[
\mathbf{v} \cdot \dot{\mathbf{v}} = \frac{F(r)}{m} \mathbf{v} \cdot \mathbf{r} + \frac{2\mu}{m} \mathbf{v} \cdot \mathbf{v}. \tag{99}
\]

By the differentiating the constraint condition (95) we obtain \( \mathbf{v} \cdot \dot{\mathbf{v}} = 0 \), consequently for the Lagrange multiplier \( \mu \) we get

\[
2\mu = -\frac{F(r)}{rv_0^2} \mathbf{v} \cdot \mathbf{r} = -\frac{F(r)}{v_0^2} \mathbf{v}_r = -\frac{F(r)}{v_0^2} \dot{r}. \tag{100}
\]

Since

\[
\frac{dl}{dt} = \dot{l} = \frac{d}{dt}(\mathbf{r} \times \mathbf{p}) = m(\mathbf{r} \times \dot{\mathbf{v}}), \tag{101}
\]

the motion equations (98) (after the left vector multiplication by \( \mathbf{r} [4] \)) take the form

\[
m(\mathbf{r} \times \dot{\mathbf{v}}) = \frac{F(r)}{r} (\mathbf{r} \times \mathbf{r}) + 2\mu (\mathbf{r} \times \mathbf{v}) \Rightarrow \dot{l} = 2\mu (\mathbf{r} \times \mathbf{v}), \tag{102}
\]

and finally

\[
\dot{l} = \frac{2\mu}{m} l, \tag{103}
\]
thus vectors $l$ and $\dot{l}$ are collinear. Hence, uniform motions in central fields are always planar, the motions proceed in the plane perpendicular to the angular momentum $l$. In particular, if the initial value $l_0$ of the angular momentum $l$ is zero (initial vectors $r_0$ and $v_0$ are collinear) then the angular momentum remains null during the motion (the uniform radial motion either from the center or to the center).

Notice that in the case $\dot{r}(t) = 0$, i.e. when the particle moves in the constant distance $r_0$ from the center (the uniform circular motion) Lagrange multiplier $\lambda$ becomes null and (103) gives us the classical conservation law of the angular momentum $l = l_0 = \text{const}$. As we will see at the beginning of the subsection 5.3 in this extraordinary case the Chetaev force (97) disappears and therefore the uniform circular motion can be considered as well as the classical (non constrained) circular motion or the constrained circular motion in the Newtonian gravitational field.

The differential equation (103) can be easily integrated and we obtain

$$l = l_0 \exp \left( \int_{0}^{t} \frac{2\mu(t')}{m} dt' \right).$$

(104)

Now, we substitute the Lagrange multiplier (100) into (104) and after some rearrangements

$$l = l_0 \exp \left( - \int \frac{F(r')}{mv_0^2} \dot{r}' dt' \right) = l_0 \exp \left( - \int_{r_0}^{r} \frac{F(r')}{mv_0^2} dr' \right),$$

(105)

we get the dependence of the angular momentum $l$ of the particle on its distance $r$ from the center during the uniform motion in a central field. The last relation in (105) can be rewritten in the form

$$l(t) e^{-\frac{U(r(t))}{mv_0^2}} = l_0 e^{-\frac{U(r_0)}{mv_0^2}} = \text{const}. \quad (106)$$

The conservation law (106) of the angular momentum $l$ can be rewritten in the scalar form as follows

$$l(t) e^{-\frac{U(r(t))}{mv_0^2}} = mr(t)v(t) \sin \beta(t) e^{-\frac{U(r(t))}{mv_0^2}},$$

$$= mr_0v_0 \sin \beta_0 e^{-\frac{U(r_0)}{mv_0^2}} = l_0 e^{-\frac{U(r_0)}{mv_0^2}} = \text{const}, \quad (107)$$

which represents the isochronous version of the conservation law of the angular momentum instead (62).

5. Uniform motions in Newtonian gravitational field - isochronous Kepler’s problem.

5.1. Motion equations. Similarly as in the Subsection 3.3, we will assume that the uniform motions proceed in the $xy$-plane, i.e. $l_0 = (0,0,l_0)$ and the problem will be solved in the polar coordinates $(r, \varphi)$. For Lagrangian we have

$$L = \frac{1}{2} m (\dot{r}^2 + r^2 \dot{\varphi}^2) - U(r),$$

(108)

and the isochronous constraint is now expressed by the equation

$$\ddot{r}(r,\dot{r},\dot{\varphi}) = \dot{r}^2 + r^2 \ddot{\varphi}^2 - v_0^2 = 0.$$
The reduced (constrained) Lagrangian \( \bar{L} \) has the expression
\[
\bar{L} = \frac{1}{2} mv_0^2 - U(r). \tag{110}
\]
The motions we are going to study will be considered as the anticlockwise motions \( \dot{\phi} \geq 0 \). Hence the equation (109) can be rewritten into the explicit form
\[
\dot{\phi} = \sqrt{v_0^2 - \dot{r}^2} \frac{r}{r}. \tag{111}
\]
The reduced motion equation (27) of the reduced Lagrangian (110) takes the form
\[
mv_0^2 \dot{r} - \alpha r^2 = F(r) + \frac{mv_0^2}{r}, \tag{112}
\]
in particular in the Newtonian gravitational field
\[
mv_0^2 \dot{r} - \frac{\alpha r^2}{r^2} = \alpha \frac{r^2}{r} + \frac{mv_0^2}{r}, \tag{113}
\]
which is in accordance with the motion equation presented in \cite{4}. The second term on the right-hand side of (112), respective (113) we interpret as a certain additional repulsive central force
\[
\tilde{F}_c = \left( \frac{mv_0^2}{r} \right) e_r. \tag{114}
\]

An alternative approach to the dynamics offers us deformed equations of motion, which give rise from the original unconstrained motion equations by adding Chetaev force. The deformed equations of the uniform motions in the Newtonian gravitational field in the polar coordinates are expressed by
\[
m\ddot{r} - m\dot{r}^2 - \frac{\alpha}{r^2} = \mu \frac{\partial f}{\partial \dot{r}} = 2\mu \dot{r},
2m\dot{r}\ddot{\phi} + m\dot{r}^2\dot{\phi} = \mu \frac{\partial f}{\partial \dot{\phi}} = 2\mu r^2 \dot{\phi}. \tag{115}
\]
Previous equations can be rewrite as follows
\[
ma_r = \frac{\alpha}{r^2} + \Phi_r,
ma_\phi = \Phi_\phi, \tag{116}
\]
where
\[
a_r = \ddot{r} - r^2 \dot{\phi}^2, \quad a_\phi = r \ddot{\phi} + 2r \dot{\phi} \dot{r}, \quad \Phi_r = 2\mu \dot{r}, \quad \Phi_\phi = 2\mu r \dot{\phi}, \quad F = \frac{\alpha}{r^2} e_r, \tag{117}
\]
a\(_r\), a\(_\phi\), \Phi\(_r\) and \Phi\(_\phi\) are radial and angular components of the acceleration \( a \) and radial and angular components of the Chetaev force \( \Phi \), respectively.

The deformed equations (116) can be reformulated in the classical Newtonian vector form
\[
ma = F + \Phi = F(r)e_r + \Phi = \frac{\alpha}{r^2} e_r + 2\mu \dot{v}. \tag{118}
\]

The procedure of the elimination of the Lagrange multiplier \( \mu \) from (115) together with the equation of constraint (111) leads to the reduced equation (113).

By the lifting of the constraint equation (95) written in the vector form \( \dot{v} \cdot v - v_0^2 = 0 \) onto the space of accelerations \( J^2Y \) by means of the total time differentiating, we get \( a \cdot 2\dot{v} = 0 \), i.e. \( a \perp \dot{v} \). Such acceleration \( \tilde{a} \) which satisfies this condition, i.e. the constraint admissible acceleration, is called in [18] partial non-holonomic...
acceleration. The condition $\mathbf{a} \cdot 2\mathbf{v} = 0$, where $\mathbf{v} = v_r \mathbf{e}_r + v_\varphi \mathbf{e}_\varphi = \mathbf{v}_r + r \mathbf{\dot{v}}_r \mathbf{e}_\varphi$, can be satisfied for the following expression of the acceleration $\mathbf{a}$,

$$\mathbf{a} = \mathbf{\ddot{v}}_r + \mathbf{\dot{v}}_r \mathbf{e}_\varphi = \mathbf{e}_r - \frac{\mathbf{r}}{r \mathbf{e}_\varphi}.$$  \hspace{1cm} (119)

Indeed,

$$\mathbf{a} \cdot \mathbf{v} = \mathbf{\ddot{v}}_r v_r + \mathbf{\dot{v}}_r v_\varphi = \dot{r} - \frac{\mathbf{r}}{r \mathbf{e}_\varphi} \dot{\mathbf{r}} = 0.$$  \hspace{1cm} (120)

By the scalar multiplication of the deformed equation (118) in the vector form by the non-holonomic partial acceleration (119) and by the application of the constraint condition (111) and its lift

$$\mathbf{\ddot{v}}_r = -\frac{\sqrt{v_0^2 - \mathbf{r}^2}}{r^2} \dot{r} - \frac{\dot{r}}{r \sqrt{v_0^2 - \mathbf{r}^2}} \dot{\mathbf{r}},$$  \hspace{1cm} (121)

we derive after some manipulations the reduced equation (112) or (113). The same result can be obtained by the scalar multiplication the undeformed equations of the motion (47) by the partial non-holonomic acceleration (119).

If we suppose that $\dot{\varphi} \neq 0$, then after certain manipulations of the second deformed equation (115) we get the expression for the Lagrange multiplier $\mu$ in the form of the total time derivative,

$$\mu = \frac{m}{2} \frac{d}{dt} \left( \ln r + \ln \sqrt{v_0^2 - \mathbf{r}^2} \right).$$  \hspace{1cm} (122)

Dynamical equations for uniform motions in central fields admit special solutions, which are uniform radial motions ($v_\varphi(0) = 0$), i.e. when the particle moves uniformly rectilinearly in a radial direction even from the center or to the center and uniform circular motions ($v_r(0) = 0$), when the particle moves uniformly around the center at the constant distance $r_0$. Except these special motions, when $l_0 = l = 0$ or $\dot{r}(t) = 0$, the problem of uniform motions in central fields one can solve formally analytically by means of quadratures.

Let us suppose that $l_0 \neq 0$ and $\dot{r} \neq 0$. Using the isotachytonic version of the conservation law of the angular momentum (107), where $l = m r^2 \dot{\varphi}$, we get,

$$\dot{\varphi} = \frac{l_0}{mr^2} e^{\frac{U(r_0)}{m \varphi}} \frac{U(r)}{m \varphi}.$$  \hspace{1cm} (123)

Taking into account the equation of the constraint (111) we have

$$r^2 = v_0^2 - \frac{l_0^2}{m^2 r^2} e^{\frac{2U(r_0)}{m \varphi}} e^{\frac{2U(r)}{m \varphi}}.$$  \hspace{1cm} (124)

After the separation of variables and integration we obtain

$$t = \pm \int_{r_0}^{r} \frac{dr'}{\sqrt{v_0^2 - \frac{l_0^2}{m^2 r'^2} e^{\frac{2U(r_0)}{m \varphi}} e^{\frac{2U(r')}{m \varphi}}}},$$  \hspace{1cm} (125)

which especially in the Newtonian gravitational field gives us

$$t - t_0 = \pm \int_{r_0}^{r} \frac{mr'dr'}{\sqrt{m^2 v_0^2 r'^2 - l_0^2 e^{\frac{2U(r)}{m \varphi}} \left( \frac{1}{r} - \frac{1}{r_0} \right)}}.$$  \hspace{1cm} (126)
in our case we set \( t_0 = 0 \). Since \( \dot{\varphi} = \frac{d\varphi}{dt} \) then

\[
\frac{d\varphi}{dr} = \frac{\varphi}{\dot{r}} = \pm \frac{l_0 e^{-\frac{1}{m_0^2}(U(r_0)-U(r))}}{r \sqrt{m^2 r^2 v_0^2 - l_0^2 e^{-\frac{2}{m_0^2}(U(r_0)-U(r))}}}.
\] (127)

In the case of the Newtonian gravitational field we get after the integration the following

\[
\varphi - \varphi_0 = \pm \int_{r_0}^{r} \frac{l_0 e^{-\frac{1}{m_0^2}(\frac{1}{r} - \frac{1}{r_0})}}{r' \sqrt{m^2 r'^2 v_0^2 - l_0^2 e^{-\frac{2}{m_0^2}(\frac{1}{r'} - \frac{1}{r_0})}}} dr'.
\] (128)

However, presented expression of the solution, which can be found also in [4], is not suitable for the visualization of the trajectories. Therefore the differential equations (111) and (113) are solved numerically by the Runge-Kutta method. The possible motions can be classified with respect to initial conditions, especially with respect to different values of the radial and angular components of the initial velocity as follows: uniform motions with zero angular component of the initial velocity - uniform radial motions, uniform motions with zero radial component of the initial velocity and uniform motion with non-zero radial and non-zero angular component of the initial velocity. The above mentioned special solutions, uniform circular motions, evidently belong to the second class. We are going to investigate only the first two classes because the third class is just a composition of them.

5.2. Uniform radial motions - motions with zero angular component of the initial velocity. Let the initial conditions of the particle are the following:

\[
r(0) = r_0, \quad v_r(0) = \dot{r}(0) = v_0, \quad v_\varphi(0) = \varphi(0) = 0.
\] (129)

Then the magnitude \( l_0 = m r_0^2 \dot{\varphi}(0) \) of the initial angular momentum \( l_0 \) is 0. It is easy to see from (103) that the magnitude \( l \) remains null during the motion. So, \( \dot{\varphi}(t) = 0 \) for every \( t \), thus \( \varphi(t) = \varphi_0 = \text{const} \). Hence, the motion is radial. Furthermore, the constraint equation (109) reduces to

\[
\dot{r}^2(t) = \dot{r}^2(0) = v_0^2,
\] i.e. \( \dot{r}(t) = \pm v_0 \),

and can be easily integrated,

\[
r(t) = \pm v_0 t + r_0.
\] (131)

The particle moves uniformly rectilinearly in the radial direction either from the center \((+v_0)\) or to the center \((-v_0)\). The reduced equation of motion (113) degenerates. There is only one non-zero component of the Chetaev force \( \Phi \), the radial component

\[
\Phi_r = 2 \mu \dot{r}(0) = 2 \mu v_0.
\] (132)

While the second deformed equation of (115) becomes trivial, the first one represents the equilibrium condition between the central gravitational and Chetaev force, \( |F| = |\Phi| \):

\[
-\frac{\alpha}{r^2} = \Phi_r, \quad \alpha = -\kappa M m,
\] (133)

which is provided by the Lagrange multiplier \( \mu \),

\[
\mu = -\frac{\alpha}{2 v_0 r^2} = -\frac{\alpha}{2 v_0 (v_0 t + r_0)^2} \quad \text{for the motion from the center},
\]
\[
\mu = \frac{\alpha}{2 v_0 r^2} = \frac{\alpha}{2 v_0 (-v_0 t + r_0)^2} \quad \text{for the motion to the center}.
\] (134)
5.3. Uniform motions with zero radial component of the initial velocity. Let \( v_r(0) = \dot{r}(0) = 0 \), hence \( v_\varphi(0) = r_0 \dot{\varphi}(0) = v_0 \). First of all we analyze one special motion - uniform circular motion. As it will be shown in the Section 7 this motion is unstable. In the unconstrained case the uniform circular motion arises directly as the solution of differential equations (72) with respect to the specific initial conditions typical for this case:

\[
\begin{align*}
    r(0) &= r_0, \quad \varphi(0) = \varphi_0, \quad v_r(0) = \dot{r}(0) = 0, \quad v_\varphi(0) = r_0 \dot{\varphi}(0) = v_\text{circ},
\end{align*}
\]

where \( v_\text{circ} \) is given by (57). On the other side, the uniform circular motion can also be treated as constrained motion. Indeed, it can be obtained as the solution of the reduced equation (113) or deformed equations (116) with the same specific initial conditions (135). It shows that the presence of the isotachytonic constraint is redundant in this case, i.e. the Chetaev force vanishes.

The uniform circular motion is described by the parametric equations

\[
\begin{align*}
    r(t) &= r_0, \quad \varphi(t) = \dot{\varphi}(0)t + \varphi_0.
\end{align*}
\]

Then

\[
\begin{align*}
    \dot{r}(t) &= \ddot{r}(t) = 0, \quad \dot{\varphi}(t) = \dot{\varphi}(0) \quad \text{and} \quad \ddot{\varphi}(t) = 0,
\end{align*}
\]

hence the isotachytonic constraint (109) becomes

\[
\begin{align*}
    r_0^2 \dot{\varphi}^2(t) &= r_0^2 \dot{\varphi}^2(0) = v_0^2 = v_\text{circ}^2.
\end{align*}
\]

The reduced equation (113) or the first deformed equation (116) reduces to the equilibrium condition \( |F| = |F_c| \) between the central gravitational \( F \) and the repulsive central force \( F_c \) (114), which here plays the role of the classical centrifugal force \( mv_0^2/r_0 \),

\[
\begin{align*}
    -\frac{\alpha}{r_0^2} &= \frac{mv_0^2}{r_0}, \quad \alpha = -\kappa M m,
\end{align*}
\]

from which follows the admissible value for the magnitude of the initial velocity (57). Evidently, the radial component \( \Phi_r \) of the Chetaev force \( \Phi \) is 0. Moreover, since the second deformed equation (115) becomes trivial, it follows that \( \mu = 0 \) and immediately \( \Phi_\varphi = 0 \), the Chetaev constrained force vanishes.

On the following figures the selected trajectories of the uniform motions in the Newtonian gravitational field with zero radial component of the initial velocity obtained by the numerical solutions of the differential equations (111) and (113) under the following initial conditions

\[
\begin{align*}
    r(0) &= r_0 = 5, \quad \varphi(0) = \varphi_0 = \frac{\pi}{2}, \quad v_\varphi(0) = r_0 \dot{\varphi}(0) = v_0,
\end{align*}
\]

are visualized. For the numerical calculations we choose \( M = 100, m = 1, \kappa = 1 \), hence \( \alpha = -100 \). Just as in the Subsection 3.6 the particle starts from the fixed distance \( r_0 \) from the center with the magnitude of the initial velocity \( v_0 \) in the direction perpendicular to the connecting line of the starting point and the center (motions with zero radial component of the initial velocity). The figures illustrate the trajectories of uniform motions in the same typical situations, i.e. for the specific values of the magnitude of the initial velocity: \( 0 < v_0 < v_\text{circ}, \quad v_0 = v_\text{circ}, \quad v_\text{circ} < v_0 < v_\text{esc}, \quad v_0 = v_\text{esc} = v_\text{circ} \sqrt{2}, \quad v_0 > v_\text{esc} \) as well as the figures of the trajectories of the classical motions presented in the Subsection 3.6. The dashed curves represent the trajectories of the classical motions in the Newtonian gravitational field under the same initial conditions.
6. **Energetic balance of the uniform motions in central fields.** In this section we find the first integral - integral of energy - of the reduced equation of motion.
and will be discussed the energetic balance of the uniform motions. We omit uniform circular motions, when the energetic balance is trivial, and the radial motions since the reduced equation degenerates.

At first, we multiply the reduced equation (112) by $\dot{r}$, $\dot{r} \neq 0$. We obtain

$$F(r)\ddot{r} + m v_0^2 \dot{r} - \frac{m v_0^2 \dot{r}}{v_0^2 - r^2} = 0,$$

which can be expressed (after the multiplying the both sides of the previous equation by $-1$) in the form of the total time derivative

$$\frac{d}{dt} \left( - \int F(r)dr + m v_0^2 \ln \frac{1}{r} + m v_0^2 \ln \frac{1}{\sqrt{v_0^2 - \dot{r}^2}} \right) = 0.$$

Hence, the function

$$\psi_1(r, \dot{r}) = - \int F(r)dr + m v_0^2 \ln \frac{1}{r} + m v_0^2 \ln \frac{1}{\sqrt{v_0^2 - \dot{r}^2}}$$

is the first integral, i.e. it remains constant along every isotachytonic trajectory (except the circular and radial ones). First term in (143),

$$U(r) = - \int F(r)dr,$$

corresponds to the potential energy of the central force defined by (42). Second one,

$$\tilde{U}_c(r) = m v_0^2 \ln \frac{1}{r},$$

is called the modified centrifugal potential, the potential of the additional repulsive central force $\tilde{F}_c$ (114),

$$\tilde{F}_c = -\text{grad} \tilde{U}_c(r) = \frac{m v_0^2}{r} \mathbf{e}_r.$$

By the relation

$$\tilde{U}_{ef}(r) = U(r) + \tilde{U}_c(r),$$

the modified effective potential energy is defined. The last term in (143),

$$\tilde{E}_k = m v_0^2 \ln \frac{1}{v_0^2 - \dot{r}^2} = \frac{m v_0^2}{2} \ln \frac{1}{v_0^2 \dot{\varphi}^2} = \frac{m v_0^2}{2} \ln \frac{1}{v_0^2 \sin^2 \beta(t)} = E_k \kappa(t),$$

replaces the constant kinetic energy $E_k = m v_0^2 / 2$ in the energetic balance and is called the isotachytonic compensation of the kinetic energy. The function

$$\kappa = \kappa(t) = \ln \frac{1}{v_0^2 - \dot{r}^2} = \ln \frac{1}{r^2 \dot{\varphi}^2} = \ln \frac{1}{v_0^2 \sin^2 \beta(t)}$$

is called the isotachytonic compensation coefficient. Finally, the expression (143) becomes

$$\psi_1(r, \dot{r}) = U(r) + \tilde{U}_c(r) + \tilde{E}_k = U(r_0) + \tilde{U}_c(r_0) + E_k \kappa(0) = \text{const.},$$

where $\kappa(0) = \ln(1/(v_0^2 \sin^2 \beta_0))$, and can be interpreted as the isotachytonic version of the conservation law of the mechanical energy in central fields. The function $\psi_1$ is called the modified mechanical energy and is denoted by $\tilde{E}$.

The conservation law (150) can be reformulated in the following way:

$$\tilde{U}_{ef}(r) - \tilde{U}_{ef}(r_0) = \tilde{E}_k(0) - \tilde{E}_k(t) = E_k \cdot (\kappa(0) - \kappa(t)),$$
respectively
\[ \Delta U_{ef} = -E_k \Delta \kappa. \] (152)

The conservation law (152) can be expressed in the following differential form
\[ \frac{d}{dt}(\tilde{U}_{ef} + \tilde{E}_k) = \frac{d\dot{U}_{ef}}{dt} + E_k \frac{d\kappa}{dt} = 0 \] (153)
along every isotachytonic trajectory except the circular and radial ones.

If we join logarithm terms in (143) and add the conservation law (106), we get
\[ U(r) + \frac{mv_0^2}{2} \ln \frac{1}{r^2(v_0^2 - r^2)} = U(r_0) + \frac{mv_0^2}{2} \ln \frac{1}{r_0^2(v_0^2 - r^2(0))} = \text{const.}, \] (154)
the isotachytonic analog of the classical conservation laws (72). If we substitute the expression (111) of the constraint into the second equation of (154) then after some arrangements we obtain the isotachytonic conservation law of the mechanical energy (the first equation of (154)), thus on the constraint manifold both conservation laws (154) merge into the each other.

Evidently, the work \( W_{\Phi} \) of the Chetaev constraint force \( \Phi \) along constrained path \( J^{1,2} \) balances the changes of the potential energy \( U(r) \) of the moving particle instead of the kinetic energy \( E_k \), which remains constant. A computation of the work \( W_{\Phi} \) of the Chetaev force along a small piece \( \ell \) of the constrained trajectory \( J^{1,2} \) from the given position \( r_1 \) to \( r_2 \) in the infinitesimal small time interval \( \Delta t = t_2 - t_1 \) in the polar coordinates gives us,
\[
\Delta W_{\Phi} = \int_{\ell} \Phi \cdot dr = \int_{\ell} (\Phi_r h_r \, dr + \Phi_\varphi h_\varphi \, d\varphi) = \int_{\ell} (\mu \dot{r} \, dr + 2\mu r^2 \dot{\varphi} \, d\varphi)
= 2 \int_{t_1}^{t_2} \left( \mu \frac{dr}{dt} \, dt + \mu r^2 \, d\varphi \right) dt = 2 \int_{t_1}^{t_2} \mu (r^2 + r^2 \varphi^2) \, dt
= 2\mu v_0^2 \int_{t_1}^{t_2} \mu dt \int_{r_1}^{r_2} \frac{d}{dt} \left( \ln r + \ln \frac{1}{\sqrt{v_0^2 - r^2}} \right) \, dt,
\]
where \( h_r = 1 \) and \( h_\varphi = r \) are the Lamé coefficients for the polar coordinates. After the integration we get
\[
\Delta W_{\Phi} = - \left[ m v_0^2 \ln \frac{1}{r} \right]_{r_1}^{r_2} - \left[ \frac{m v_0^2}{2} \ln \frac{1}{v_0^2 - r^2} \right]_{t_1}^{t_2}
= - (\Delta \tilde{U}_c + \Delta \tilde{E}_k) = - (\Delta \tilde{U}_c + E_k \Delta \kappa).
\]
Finally, the isotachytonic version of the conservation law of the mechanical energy along the entire trajectory \( J^{1,2} \) can be expressed by the following alternatives:
\[
\frac{d}{dt}(U + \tilde{U}_c + \tilde{E}_k)_{J^{1,2}} = \frac{d}{dt}(\tilde{U}_{ef} + \tilde{E}_k)_{J^{1,2}} = \frac{d}{dt}(U - W_{\Phi})_{J^{1,2}} = 0.
\] (157)

We conclude this section by the discussion concerning the energetic balance for concrete particular cases of the uniform motions in the Newtonian gravitational field:

- **general motions (the isotachytonic version of conservation law of energy)**
\[
\frac{\alpha}{r} + \frac{mv_0^2}{2} \ln \frac{1}{r} + \frac{mv_0^2}{2} \ln \frac{1}{v_0^2 - r^2} = \text{const.},
\] (158)
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- circular motions (the classical version of conservation law of energy)
  \[ \frac{\alpha}{r_0} + \frac{mv_{circ}^2}{2} = \text{const.}, \]  
  \[ (159) \]

- radial motions
  \[ \frac{\alpha}{r_0} - \frac{\alpha}{r} = \Delta W_\Phi. \]  
  \[ (160) \]

7. Classification of uniform motions in Newtonian gravitational field from the diagram of the modified effective potential energy. Recall, that for uniform non-radial motions (\( \dot{\phi}(0) \neq 0 \) i.e. \( \dot{r}(0) \neq v_0 \)) in central fields it holds the isotachytonic version of the conservation law of the mechanical energy, which has the form

\[ \tilde{E} = \tilde{U}_{ef}(r) + \tilde{E}_k(t) = \tilde{U}_{ef}(r_0) + \tilde{E}_k(0) = \tilde{E}_0 = \text{const.}, \]  
\[ (161) \]

where

\[ \tilde{U}_{ef}(r) = U(r) + \tilde{U}_c(r) \]  
\[ (162) \]

is the modified effective potential energy, which is the sum of the potential energy \( U(r) \) of the considered central field and the modified centrifugal potential energy \( \tilde{U}_c(r) = mv_0^2 \ln(1/r) \) and where the function

\[ \tilde{E}_k(t) = \frac{mv_0^2}{2} \ln \frac{1}{v_0^2 - \dot{r}^2} = \frac{mv_0^2}{2} \ln \frac{1}{r^2 \dot{\varphi}^2} = \frac{mv_0^2}{2} \ln \frac{1}{v_0^2 \sin^2 \beta(t)} \]  
\[ (163) \]

is the isotachytonic compensation of the constant kinetic energy \( E_k = (1/2)mv_0^2 \). Notice that the value of the magnitude of the initial velocity \( v_0 \) plays here the role of a parameter, therefore we sometimes write \( \tilde{U}_{ef}(r, v_0) \).

The constant \( \tilde{E}_0 \) occurring on the right-hand side of \[ (161) \] represents the initial value of the modified mechanical energy \( \tilde{E} \), and is equal to

\[ \tilde{E}_0 = \tilde{U}_{ef}(r_0) + \tilde{E}_k(0) = U(r_0) + mv_0^2 \ln \frac{1}{r_0} + \frac{mv_0^2}{2} \ln \frac{1}{v_0^2 - \dot{r}^2(0)}. \]  
\[ (164) \]

Notice that in the case of motions with zero radial component of the initial velocity we obtain

\[ \tilde{E}_0 = U(r_0) + mv_0^2 \ln \frac{1}{r_0} + \frac{mv_0^2}{2} \ln \frac{1}{v_0^2}. \]  
\[ (165) \]

After the standard manipulations with the equation \[ (161) \] we get

\[ \dot{r}^2 = v_0^2 - e^{-\frac{2\alpha}{mv_0^2}(E_0 - \tilde{U}_{ef}(r))}. \]  
\[ (166) \]

Since \( \dot{r}^2(t) \) is non-negative function, the right-hand side of \[ (166) \] must be also non-negative,

\[ v_0^2 - e^{-\frac{2\alpha}{mv_0^2}(E_0 - \tilde{U}_{ef}(r))} \geq 0. \]  
\[ (167) \]

After further manipulations one can derive the condition

\[ \tilde{E}_0 + \frac{mv_0^2}{2} \ln v_0^2 \geq \tilde{U}_{ef}(r), \]  
\[ (168) \]

which is a certain analog of the condition \[ (93) \]. It enables us to make a qualitative analysis of the uniform non-radial motions in central fields as well as the condition \[ (93) \] provides a possibility for the qualitative analysis of the classical motions in
central fields. In particular, for motions with zero radial component of the initial velocity the relation (168) simplifies to \( \tilde{U}_{\text{ef}}(r_0) \geq \tilde{U}_{\text{ef}}(r) \), where
\[
\tilde{U}_{\text{ef}}(r_0) = U(r_0) + mv_0^2 \ln \frac{1}{r_0}.
\] (169)

Like in the Subsection 3.6 the graphic solution of the inequality (168) respective (169) provides us admissible intervals of radial distances from the center in which the particle can be located under the given initial value \( \tilde{E}_0 \) of the modified mechanical energy, respective the magnitude of the initial velocity \( v_0 \).

The modified effective potential energy for the Newtonian gravitational field is the function
\[
\tilde{U}_{\text{ef}}(r) = \frac{a}{r} + mv_0^2 \ln \frac{1}{r} = -\frac{\kappa M m}{r} + mv_0^2 \ln \frac{1}{r}.
\] (170)

The graph of this function is shown on the following figure.

![Figure 9. Modified effective potential of the Newtonian gr. field](image)

We see that the function \( \tilde{U}_{\text{ef}}(r) \) is negative at its domain of definition. At the point \( r_{\text{max}} = (\kappa M)/v_0^2 \) has maximum value \( \tilde{U}_{\text{ef}}^{\text{max}} = -mv_0^2(1 - \ln(v_0^2/(\kappa M))) \). Restricted area in which the particle cannot be located is displayed. If we consider only motions with zero radial component of the initial velocity for which it is relevant (169), we obtain using the diagram of the modified potential energy \( \tilde{U}_{\text{ef}}(r) \) the following classification of uniform motions in the Newtonian gravitational field.

If \( \tilde{U}_{\text{ef}}(r_0) = \tilde{U}_{\text{ef}}^{\text{max}} \), it represents a limit case, when the line of the constant value \( \tilde{U}_{\text{ef}}(r_0) \) is a tangent line to the graph of \( \tilde{U}_{\text{ef}}(r) \). The motion proceeds at the constant distance \( r = r_{\text{max}} \) and since it is maximum, the appropriate motion is the unstable circular motion.

If \( \tilde{U}_{\text{ef}}(r_0) < \tilde{U}_{\text{ef}}^{\text{max}} \), in this case there exist two intersection points \( r_1 \), \( r_2 \) of the line of the constant value \( \tilde{U}_{\text{ef}}(r_0) \) with the graph \( \tilde{U}_{\text{ef}}(r) \). Then interval of admissible distances is either \( (0, r_1) \) or \( (r_2, \infty) \). In the first subcase it is a finite motion to the center (ending in the center). In the second subcase it is an infinite (escaping) motion. The case which is realized depends on the starting distance \( r_0 \) from the center.

In the following figures one can see these three typical situations for the case of uniform motions with zero radial component of the initial velocity, i.e. when the particle starts in the fixed distance \( r_0 \) from the center with the magnitude of the
initial velocity $v_0$ ($v_0 < v_{\text{circ}}$, $v_0 = v_{\text{circ}}$, $v_0 > v_{\text{circ}}$) in the direction perpendicular to the connecting line of the starting point and the center.

The fact that the circular trajectory is unstable, is illustrated on the following figure. A small change of the value of the magnitude of the initial velocity $v_0$ with respect to the value $v_{\text{circ}}$, i.e. $v_0 = v_{\text{circ}} \pm \varepsilon$, causes a perturbation of the circular trajectory. The motion begins almost as circular one. A deviation of the trajectory of the particle from the circular orbit will increase in time and particle either will approach to the center or will escape from the center. The rate of the deviation of the trajectory is proportional to the value of deviation $\varepsilon$. 
8. Uniform motions in the inverse proportional gravitational field. Let us consider the particle of the mass $m$ moving uniformly with the constant magnitude of the velocity $v_0$ in the inverse proportional gravitational field

$$F(r) = \frac{-b}{r}e_r, \quad b > 0, \quad b \neq mv_0^2,$$

(171)

with the potential

$$U(r) = b \ln r.$$  

(172)

The modified effective potential for this central gravitational field is the function

$$\tilde{U}_e(r) = U(r) + \tilde{U}_c(r) = b \ln r + mv_0^2 \ln \frac{1}{r} = (b - mv_0^2) \ln r.$$  

(173)

Notice that for $b > mv_0^2$ the potential $U(r)$ of the attractive central field (171) is dominant and on the contrary in the case $b < mv_0^2$ the modified centrifugal potential $\tilde{U}_c(r)$ prevails. The course of the modified effective potential for the both cases is illustrated on the Fig. 12.

The particle starts from the initial position $r(0) = r_0$, $\varphi(0) = \varphi_0$ with the initial velocity vector $v_0 = \dot{r}(0)e_r + r_0\dot{\varphi}(0)e_\varphi$. We consider only non-radial motions ($\dot{\varphi}(0) \neq 0$, i.e. $\dot{r}(0) \neq v_0$). The magnitude of the initial angular momentum vector is $l_0 = mrv_0 \sin \beta_0$, where $\beta_0 \in (0, \pi)$ is the initial angle between the vector of the initial position $r_0$ and the initial velocity vector $v_0$.
Recall that it holds the isotachytonic version of the conservation law of the mechanical energy \( E_0 \), where the value of the constant \( E_0 \) represents the initial value of the modified energy \( \tilde{E} \) and is equal to
\[
\tilde{E}_0 = (b - mv_0^2) \ln r_0 + \frac{mv_0^2}{2} \ln \frac{1}{v_0 - r_0(0)} = (b - mv_0^2) \ln r_0 - mv_0^2 \ln(v_0 \sin \beta_0). \tag{174}
\]
Recall that uniform motions in central fields can proceed only on such interval of distances \( r \) for which the inequality \( 168 \) holds. From the course of the modified effective potential it is evident, that in the case \( b > mv_0^2 \) every initial value \( \tilde{E}_0 \) of the modified energy \( \tilde{E} \) leads to motions approaching to the center and falling into the center. On the contrary, in the case \( b < mv_0^2 \), every initial value \( \tilde{E}_0 \) leads to the escaping motions away from the center, cf. Fig. [12].

Let us derive the explicit expression \( r = r(\varphi) \) of trajectories of uniform non-radial motions in the inverse proportional gravitational field. In accordance with the equation \( 127 \) we get
\[
\frac{d\varphi}{dr} = \pm \frac{l_0 e^{-\frac{l}{mv_0}(b \ln r_0 - b \ln r)}}{r \sqrt{m^2 r^2 v_0^2 - l_0^2 e^{-\frac{b}{mv_0}(b \ln r_0 - b \ln r)}}} = \pm \frac{l_0 \left( \frac{r}{r_0} \right)^{\frac{b}{mv_0}}}{r \sqrt{m^2 r^2 v_0^2 - l_0^2 \left( \frac{r}{r_0} \right)^{\frac{mv_0}{2}}}} \tag{175}
\]
One can write
\[
\frac{d\varphi}{dr} = \pm \frac{\Omega r^\sigma}{r \sqrt{(m^2 v_0^2 - \Omega^2 r^2)^2}}, \quad \Omega = l_0 r_0^{-\frac{b}{mv_0}}, \quad \sigma = \frac{b}{mv_0^2} - 1. \tag{176}
\]
After the integration we have
\[
\varphi - \varphi(r_0) = \pm \frac{1}{\sigma} \left[ \arcsin \left( \frac{\Omega}{mv_0} r^\sigma \right) - \arcsin \left( \frac{\Omega}{mv_0} r_0^\sigma \right) \right]. \tag{177}
\]
Following,
\[
\frac{\Omega}{mv_0} r_0^\sigma = \frac{l_0}{mv_0} r_0^\sigma - \frac{b}{mv_0^2} r_0 = \frac{l_0}{mv_0} r_0^\sigma - \frac{b}{mv_0^2} = \frac{l_0}{mv_0 r_0} = \sin \beta_0 \tag{178}
\]
we get
\[
\varphi - \varphi_0 = \pm \frac{1}{\sigma} \left[ \arcsin \left( \frac{\Omega}{mv_0} r_0^\sigma \right) - \beta_0 \right], \tag{179}
\]
which can be expressed explicitly in the polar coordinates
\[
r^\sigma(\varphi) = \frac{mv_0}{\Omega} \sin[\sigma(\varphi - \varphi_0) \pm \beta_0] = R_0^\sigma \sin[\sigma(\varphi - \varphi_0) \pm \beta_0], \tag{180}
\]
where
\[
R_0 = \left( \frac{mv_0}{\Omega} \right)^{\frac{1}{2}} \frac{r_0}{\sqrt{\sin \beta_0}}. \tag{181}
\]
The formula \( 180 \) is the typical expression for the sinusoidal spirals. We distinguish between two cases mentioned above.

- \( b > mv_0^2 \), i.e. \( \sigma > 0 \)
  \[
x(\varphi) = R_0 \sqrt{\sin[\sigma(\varphi - \varphi_0) \pm \beta_0]} \cos \varphi, \]
  \[
y(\varphi) = R_0 \sqrt{\sin[\sigma(\varphi - \varphi_0) \pm \beta_0]} \sin \varphi, \quad R_0 = \frac{r_0}{\sqrt{\sin \beta_0}}. \tag{182}
\]
The particle approaches to the center along a piece of the corresponding sinusoidal spiral and finally falls on the center, Fig. [13].
• $b < mv_0^2$, i.e. $\sigma < 0$

\begin{align*}
  x(\varphi) &= \frac{R_0 \cos \varphi}{\sqrt[4]{\sin(\sigma(\varphi_0 - \varphi) \pm \beta_0)}}, \\
  y(\varphi) &= \frac{R_0 \sin \varphi}{\sqrt[4]{\sin(\sigma(\varphi_0 - \varphi) \pm \beta_0)}}, \quad R_0 = r_0 \sqrt[4]{\sin \beta_0}.
\end{align*}

(183)

The particle escapes away from the center along a piece of the corresponding “reciprocal” sinusoidal spiral, Fig. 13.

In particular, for the motions with zero radial component of the initial velocity ($r_0 \dot{\varphi}(0) = v_0$), i.e. particle starts in the direction perpendicular to the connecting line of the starting point and the center, the relation (168) simplifies into $\tilde{U}_{ef}(r_0) \geq \tilde{U}_{ef}(r_0)$, where $\tilde{U}_{ef}(r_0) = (b - mv_0^2) \ln r_0$. In this case $\beta_0 = \pi/2$, consequently $R_0 = r_0$ and the parametric equations take the following simpler expressions,

• $b > mv_0^2$, i.e. $\sigma > 0$

\begin{align*}
  x(\varphi) &= r_0 \sqrt[4]{\cos(\sigma(\varphi - \varphi_0))} \cos \varphi, \\
  y(\varphi) &= r_0 \sqrt[4]{\cos(\sigma(\varphi - \varphi_0))} \sin \varphi.
\end{align*}

(184)

• $b < mv_0^2$, i.e. $\sigma < 0$

\begin{align*}
  x(\varphi) &= \frac{r_0 \cos \varphi}{\sqrt[4]{\cos(\sigma(\varphi - \varphi_0))}}, \\
  y(\varphi) &= \frac{r_0 \sin \varphi}{\sqrt[4]{\cos(\sigma(\varphi - \varphi_0))}}.
\end{align*}

(185)

Figure 13. Sinusoidal spirals

9. Symmetries and conservation laws for uniform motions in Newtonian gravitational field. The existence of constraint Noetherian symmetries and corresponding conservation laws of the nonholonomic constrained system of one particle in the Newtonian gravitational field with the isotachytonic constraint is investigated from the geometric point of view.

Recall that a vector field $\Xi \in \mathcal{C}$ is the constraint Noetherian symmetry of the nonholonomic constrained system $[\alpha_Q]$ if $\mathcal{L}_\Xi \ast \theta_\lambda$ is a constraint 1-form [9] ($\mathcal{L}_\Xi$ denotes the Lie derivative along a vector field $\Xi$).
The constraint version of the Noether’s theorem is formulated in [9]: Let $\lambda$ be a Lagrangian on $J^1Y$ and $\Xi$ be a constraint symmetry of the nonholonomic constrained system $[\alpha_Q]$. Then along every constrained path $\gamma$ the following identities are satisfied,

$$d(i_\Xi(\iota^*\theta_\lambda) \circ J^1\gamma) = 0, \text{ or } i_\Xi(\iota^*\theta_\lambda) \circ J^1\gamma = \text{const.} \quad (186)$$

The studied isotachytonic system of one particle in the Newtonian gravitational field is geometrically characterized by the Poincaré-Cartan form $\iota^*\theta_\lambda$ on the constraint submanifold $Q$,

$$\iota^*\theta_\lambda = -\left(\frac{mv_0^2}{2} + \frac{\alpha}{r}\right) dt + m\dot{r}dr + mṙ\sqrt{v_0^2 - \dot{r}^2}d\varphi. \quad (187)$$

The canonical distribution $\mathcal{C}$ is generated by the following three vector fields defined on $Q$,

$$\xi_1 = \frac{\partial}{\partial t} + \frac{v_0^2}{r\sqrt{v_0^2 - \dot{r}^2}} \frac{\partial}{\partial \varphi}, \quad \xi_2 = \frac{\partial}{\partial r} - \frac{\dot{r}}{r\sqrt{v_0^2 - \dot{r}^2}} \frac{\partial}{\partial \varphi}, \quad \xi_3 = \frac{\partial}{\partial r}. \quad (188)$$

An arbitrary vector field $\Xi$ belonging to the canonical distribution $\mathcal{C}$ (so called Chetaev vector field) takes the form of the linear combination

$$\Xi = A\xi_1 + B\xi_2 + C\xi_3, \quad (189)$$

where coefficients $A$, $B$, $C$ are functions on the constraint submanifold $Q$. Canonical distribution $\mathcal{C}$ is annihilated by the canonical constraint 1-form $\phi$,

$$\phi = -\frac{v_0^2}{r\sqrt{v_0^2 - \dot{r}^2}} dt + \frac{\dot{r}}{r\sqrt{v_0^2 - \dot{r}^2}} dr + d\varphi. \quad (190)$$

Let Chetaev vector field $\Xi$ (189) be a constraint Noetherian symmetry then the corresponding conservation law for the studied isotachytonic mechanical system in the Newtonian gravitational field has the form

$$\left[ A(t, r, \varphi, \dot{r}) \cdot \left(\frac{1}{2}mv_0^2 - \frac{\alpha}{r}\right) \right] \circ J^1\gamma = \text{const.} \quad (191)$$

The conservation law (191) is independent on the coefficients $B, C$ of the constraint symmetry $\Xi$.

Now, we are looking for the explicit expression of the constraint Noetherian symmetries $\Xi$. Using the definition of the constraint Noetherian symmetry $\Xi$ we set

$$\mathcal{L}_\Xi(\iota^*\theta_\lambda) = G\phi, \quad (192)$$

where $G = G(t, r, \varphi, \dot{r})$ is an arbitrary function on $Q$. If we compare the same differentials on left and right-hand sides of (192), we get the following system of equations

$$\frac{\partial A}{\partial t} \left(\frac{1}{2}mv_0^2 - \frac{\alpha}{r}\right) + B\frac{\alpha}{r^2} = -G\frac{v_0^2}{r\sqrt{v_0^2 - \dot{r}^2}},$$

$$\frac{\partial A}{\partial r} \left(\frac{1}{2}mv_0^2 - \frac{\alpha}{r}\right) - A\frac{mv_0^2}{r} + B\frac{m\dot{r}}{r} + Cm = G\frac{\dot{r}}{r\sqrt{v_0^2 - \dot{r}^2}},$$

$$\frac{\partial A}{\partial \varphi} \left(\frac{1}{2}mv_0^2 - \frac{\alpha}{r}\right) + Bm\sqrt{v_0^2 - \dot{r}^2} - C\frac{m\dot{r}}{\sqrt{v_0^2 - \dot{r}^2}} = G,$$

$$\frac{\partial A}{\partial \dot{r}} \left(\frac{1}{2}mv_0^2 - \frac{\alpha}{r}\right) + m\dot{r}(A\frac{v_0^2}{r} - B\dot{\varphi}) - Bm = 0. \quad (193)$$
The system (193) consists of four equations for unknown coefficients $A$, $B$, $C$ of the seeking constraint symmetry $\Xi$, and the parameter $G$. The parameter $G$ can be directly eliminated. The remaining unknown coefficients $B$ and $C$ can be expressed by means of the coefficient $A$ and its partial derivatives as follows,

$$
B = \frac{v_0^2 - \dot{r}^2}{mv_0^2} \left( \frac{1}{2} mv_0^2 - \frac{\alpha}{r} \right) \frac{\partial A}{\partial \dot{r}} + \dot{r} A,
$$

$$
C = -\frac{v_0^2 - \dot{r}^2}{mv_0^2} \left( \frac{1}{2} mv_0^2 - \frac{\alpha}{r} \right) \left( \frac{\partial A}{\partial r} - \frac{\dot{r}}{r} \frac{\partial A}{\partial \varphi} \right) + \frac{v_0^2 - \dot{r}^2}{r} A.
$$

Finally, we obtain one linear partial differential equation of the first order,

$$
\frac{\partial A}{\partial t} + \dot{r} \frac{\partial A}{\partial r} + \sqrt{v_0^2 - \dot{r}^2} \frac{\partial A}{\partial \varphi} + \frac{v_0^2 - \dot{r}^2}{r} \left( \frac{\alpha}{mv_0^2 r} + 1 \right) \frac{\partial A}{\partial \dot{r}} = -\frac{\alpha}{r} \dot{r} A,
$$

for unknown function $A$ depending on four independent variables $t$, $r$, $\varphi$ and $\dot{r}$. The equation (195) can be solved by the method of characteristics. A general solution can be expected in the implicit form

$$
F(\psi_1, \psi_2, \psi_3, \psi_4) = 0,
$$

where $F$ is an arbitrary differentiable function, and $\psi_1$, $\psi_2$, $\psi_3$, $\psi_4$ are independent first integrals of the characteristic system associated with PDE (195),

$$
\frac{dt}{1} = \frac{dr}{\dot{r}} = \frac{r d\varphi}{\sqrt{v_0^2 - \dot{r}^2}} = \frac{d\dot{r}}{\frac{mv_0^2}{r} - \frac{\alpha}{r}} = \frac{\frac{mv_0^2}{r} - \frac{\alpha}{r}}{\frac{\alpha + mv_0^2 r}{mv_0^2 r}} dA.
$$

Only two pairs of (197) are separable. We take into account the second and the forth term in (197) and obtain the following independent first integral of the characteristic system (197)

$$
\psi_1(r, \dot{r}) = \frac{\alpha}{r} + mv_0^2 \ln \frac{1}{r} + \frac{mv_0^2}{2} \ln \frac{1}{v_0^2 - \dot{r}^2} = \text{const}_1,
$$

which is just the isotachytonic version of the conservation law of mechanical energy (143). The second separable pair (second and fifth term in (197)) gives us the other first integral $\psi_2$,

$$
\psi_2(A, \dot{r}) = A \left( \frac{mv_0^2}{2} - \frac{\alpha}{r} \right) = A \cdot \bar{L} = \text{const}_2,
$$

which is just the general form of the conservation law (191). In particular one can suppose the general solution (196) of the equation (195) in the form

$$
F(\psi_1, \psi_2, \psi_3, \psi_4) = \psi_2 - G(\psi_1, \psi_3, \psi_4),
$$

where $G$ is another differentiable function of the remaining first integrals $\psi_1, \psi_3, \psi_4$. In this case we obtain general expression for the first component of the symmetry (189) in the explicit form

$$
A(t, r, \varphi, \dot{r}) = \frac{G(\psi_1, \psi_3, \psi_4)}{\frac{mv_0^2}{2} - \frac{\alpha}{r}} = \frac{G}{\bar{L}},
$$

(201)
where \( \psi_3, \psi_4 \) are the additional independent first integrals which can be expressed by quadratures,

\[
\psi_3(t, r, \dot{r}) = t - \int_0^r \frac{r'dr'}{\sqrt{v_0^2r'^2 - r^2(v_0^2 - \dot{r}^2)e^{\frac{2\alpha}{m_0}(\frac{1}{r} - \frac{1}{r')}}}}
\]

(202)

and

\[
\psi_4(r, \varphi, \dot{r}) = \varphi - \int_0^r \frac{r\sqrt{v_0^2 - \dot{r}^2}e^{\frac{2\alpha}{m_0}(\frac{1}{r} - \frac{1}{r'})}dr'}{r'\sqrt{v_0^2r'^2 - r^2(v_0^2 - \dot{r}^2)e^{\frac{2\alpha}{m_0}(\frac{1}{r} - \frac{1}{r'})}}}.
\]

(203)

If we set in integrands (202) respective (203) \( r = r_0, \dot{r} = \dot{r}(0) \), then the expression \( r^2(v_0^2 - \dot{r}^2) = l_0^2/m^2 \) and consequently these formulas become directly the parametric expressions \( t = t(r) \) and \( \varphi = \varphi(r) \) of the trajectories obtained by means of the quadratures (126) and (128). Hence the first integrals \( \psi_3 \) and \( \psi_4 \) evaluated along a trajectory of the uniform motion in the Newtonian gravitational field take just the values of the integrations constants \( t_0 = 0 \), respective \( \varphi(t_0) = \varphi(0) = \varphi_0 \) prescribed by the initial conditions.

The equation (201) together with the first integrals \( \psi_1, \psi_3 \) and \( \psi_4 \) formally represents the set of all constraint Noetherian symmetries of the considered nonholonomic constrained system.

Notice that the conserved function \( \psi_1 \) is Noetherian conservation law of the form (199). If we take

\[
A(r, \dot{r}) = A_1 = \frac{\frac{a}{r} + mv_0^2 \ln \frac{1}{r} + \frac{m_0^2}{2} \ln \frac{1}{\sqrt{r^2 - r'^2}}}{\frac{2mv_0^2}{r} - \frac{a}{r}} = \frac{\psi_1(r, \dot{r})}{L},
\]

(204)

then from (199) we directly get the isotachytonic version of the conservation law of mechanical energy \( \psi_1 \), respective \( \dot{E} \). Evidently, the function \( A_1 \) is the particular solutions of PDE (195), since it satisfy the implicit relation \( -\psi_1 + \psi_2 = 0 \). Finally, the conservation law (198) corresponds to the following constraint symmetry

\[
\Xi_1 = A_1\xi_1 + B_1\xi_2 + C_1\xi_3 = A_1 \frac{\partial}{\partial t} + B_1 \frac{\partial}{\partial r} + \frac{A_1v_0^2 - B_1\dot{r}}{r\sqrt{v_0^2 - \dot{r}^2}} \frac{\partial}{\partial \varphi} + C_1 \frac{\partial}{\partial \dot{r}}.
\]

(205)

where \( A_1 \) is given by (204) and the corresponding \( B_1, C_1 \) one can compute by means of (194) substituting \( A = A_1 \).

10. **Conclusions.** The modification of the classical motions of the particle in the central fields on uniform motions means substantial changes from the point of view of kinematics, dynamics and energetic balance of the considered constrained motions.

While the trajectories of classical motions in the Newtonian gravitational field one can get in the explicit analytical form (122) describing the Kepler’s trajectories, in the case of uniform motions in this field it is not possible to express the formula (128) describing their trajectories by means of some reasonable elementary functions. The expression (128) is presented also in [3], but it is not suitable for the visualization of the trajectories. Therefore we solve reduced motion equation (113) together with the constraint equation (111) numerically with respect to suitable initial conditions.

In the paper the trajectories of the uniform motions in the Newtonian gravitational field for the typical values of the initial velocity \( 0 < v_0 < v_{\text{circ}}, v_0 = v_{\text{circ}}, v_{\text{circ}} < v_0 < v_{\text{esc}}, v_0 = v_{\text{esc}} = v_{\text{circ}}\sqrt{2}, v_0 > v_{\text{esc}} \) was presented and compared with
the corresponding classical unconstrained motions under the same initial conditions, Fig. 8. From the analysis of trajectories of the uniform motions follows that they have quite different qualitative character in the correspondence with the trajectories of the classical motions in the Newtonian gravitational field. There is only one common special case for both kinds of motions - uniform circular motions with the initial velocity $v_0 = v_{\text{circ}}$ (87). But there is a significant difference: while in the classical case the circular motion is stable, in the constrained (isotachytonic) case the circular motion is unstable, i.e. that it represents a border between the falling motions to the center (for $v_0 < v_{\text{circ}}$) and the escaping motions (for $v_0 > v_{\text{circ}}$).

Described situation is a direct consequence of the course of the modified effective potential energy $\tilde{U}_{\text{ef}}(r)$, Fig. 9, which possesses a global maximum and therefore it does not permit the existence of stable periodic trajectories as the closed Kepler's elliptical trajectories in the classical case for $v_0 < v_{\text{circ}}$ or $v_{\text{circ}} < v_0 < v_{\text{esc}}$.

The dynamics of the uniform motions in the central fields is explained by the presence of the Chetaev constrained force $\Phi$, which ensures the requirement of the uniformity of the motion. Recall the expression of the Chetaev force,

$$\Phi = \Phi_r e_r + \Phi_\phi e_\phi = 2\mu(t) (r e_r + r^2 \dot{e}_\phi) = 2\mu(t) v(t) = 2\mu(t)v_0 e_v(t),$$

(206)

where $\mu(t)$ is the Lagrange multiplier, $v(t)$ is the velocity vector at the time $t$, $e_v(t)$ is the unit vector in the direction of $v(t)$. It is evident that during the motion the Chetaev constraint force has to change its magnitude $|\Phi| = 2v_0|\mu(t)|$ and also its direction $e_v(t)$. However, in this case the time dependence of the Chetaev force can not be explicitly expressed unlike the uniform projectile motion in the homogeneous gravitational field [21].

From the point of view of the conservation laws one can state firstly, that the conservation law of angular momentum $l$ has in this case another modified form than the classical conservation law of the angular momentum, i.e. its magnitude $|l|$ is not constant but it changes during the uniform motions in central fields according to (107). However what remains valid is the fact, that during the uniform motions in central fields the angular momentum $l$ lies still in the same line determined by the initial angular momentum $l_0$, cf. (103) respective (104). This fact allows us to reduce uniform motions in central fields (as well as classical motions) into the planar motions proceeding in the initial plane perpendicular to the initial angular momentum $l_0$ [4].

Secondly, the requirement for uniformity of the motion in central fields completely disturbs the classical conservation law of the mechanical energy, which obviously can not be satisfied. However, one can formulate the isotachytonic version of the conservation law of mechanical energy, in which the role of the standard kinetic energy $E_k$ plays the isotachytonic compensation of the kinetic energy $\tilde{E}_k$ (148), which balances changes of the modified effective potential energy $\tilde{U}_{\text{ef}}$ (147) (the sum of the standard potential energy $U$ of a central field and the modified centrifugal energy $\tilde{U}_{c}$ (145)) instead of the classical kinetic energy $E_k$, which remains constant during the uniform motions. This modified version of the conservation law of energy was identified as one ($\psi_1$, (198)) of the particular conservation laws $\psi_1$, $\psi_2$, $\psi_3$, $\psi_4$ in the set of Noetherian conservation laws (200) for uniform motions in the Newtonian gravitational field arising from the constraint Noetherian symmetry (205) of the considered mechanical system applying the general definition [9] of the constraint Noetherian symmetry of nonholonomic constrained Lagrangian systems.
As an interesting particular result we present in the Sec. 8 an explicit analytical form of trajectories (182) (sinusoidal spirals) of the uniform motions in the special gravitational field, called inverse proportional gravitational field (39). Note that trajectories of classical (unconstrained) motions in this field can not be expressed by means of reasonable elementary functions.

As the main result we obtained in the Sec. 9 a formal expression of all Noetherian conservation laws (200) by means of three base conservation laws $\psi_1$, $\psi_3$ and $\psi_4$ derived as a certain general solution of PDE (195) using the method of characteristics. Physical interpretation of these conversation laws in the context of the solution of the problem of uniform motions in the Newtonian gravitational field was found. The first base conservation law $\psi_1$ represents the isotachytonic version of the conservation law of mechanical energy, as it was mentioned above. The base conservation laws $\psi_3$ and $\psi_4$ represent directly the parametric expressions $t = t(r)$ and $\varphi = \varphi(r)$ of the trajectories of the uniform motions in the Newtonian gravitational field. These conservation laws one can understand as trivial conservation laws, which carry only an information about some initial data of the motion. Indeed, when we evaluate $\psi_3$ and $\psi_4$ along some concrete trajectory of the uniform motion then we obtain just the value $t_0$, the instant of the beginning of the motion (in our case we set $t_0 = 0$) and the value $\varphi(t_0) = \varphi(0) = \varphi_0$, the polar angle of the starting point of the motion.

The results described above contribute to the issue of the kinematics, dynamics, Noetherian symmetries and conservation laws of nonholonomic mechanical systems.

We are aware that the studied problem is only a kind of a theoretical modification of the real situation, whose technical realization would seem difficult to implement.

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