Universal Properties of the Four-Body System with Large Scattering Length

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Abstract

Few-body systems with large scattering length have universal properties that do not depend on the details of their interactions at short distances. We study the universal bound state properties of the four-boson system with large scattering length in an effective quantum mechanics approach. We compute the four-body binding energies using the Yakubovsky equations for positive and negative scattering length. Moreover, we study the correlation between three- and four-body energies and present a generalized Efimov plot for the four-body system. These results are useful for understanding the cluster structure of nuclei and for the creation of weakly-bound tetramers with cold atoms close to a Feshbach resonance.

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I. INTRODUCTION

Effective theories are a powerful framework to exploit a separation of scales in a physical system. The long-distance degrees of freedom must be included dynamically in the effective theory, while short-distance physics enters only through the values of a few coupling constants, often called low-energy constants. Effective theories are widely used in many areas of physics. Recently, a considerable effort was devoted to applying effective field theories in nuclear and atomic physics [1, 2, 3]. If there is no exchange of massless particles, any interaction will appear short ranged at sufficiently low energy. One can then use a very general effective theory with short-range interactions only to describe the universal low-energy properties of the system. Such a theory can be applied to a wide range of systems from nuclear and particle physics to atomic and molecular physics.

In this paper, we focus on few-boson systems with a large two-body scattering length. They are characterized by an unnaturally large two-body scattering length \( a \) which is much larger than the typical low-energy length scale \( l \) given by the range of the interaction. Such systems display a number of interesting effects and universal properties that are independent of the details of the interaction at short distances of order \( l \). The simplest one is the existence of a shallow two-body bound state (called dimer) with universal binding energy \( B_2 = \hbar^2/(ma^2) + O(l/a) \) for positive \( a \) and where \( m \) is the mass of the particles. Low-energy observables can generally be described in a controlled expansion in \( l/|a| \). In the two-boson system, the effective theory reproduces the effective range expansion but the structure of the three-boson system is much richer [3]. It has universal properties that include a geometric spectrum of three-body bound states (so-called Efimov trimers), log-periodic dependence of three-body observables on the scattering length, and a discrete scaling symmetry [4, 5]. These features can be understood as the consequence of a renormalization group limit cycle in the three-body system [6, 7, 8].

In the effective field theory (EFT) formulation of Bedaque et al. [6, 7], the limit cycle is manifest in the renormalization group behavior of a contact three-body interaction required for consistent renormalization. This implies that at leading order in \( l/|a| \), the properties of the three-boson system with large scattering length are not determined by two-body data alone and one piece of three-body information (such as a trimer energy) is required as well. This three-body information can be specified in various ways [3, 6, 7, 8, 9, 11, 11, 12]. In the following, we will use the parameter \( L_3 \) introduced in [12]. As an alternative to the EFT formulation, one can construct an effective theory in a quantum mechanical framework [13]. The contact operators in the field theory are then replaced by an “effective potential” built from smeared out \( \delta \)-function potentials and derivatives thereof. The qualitative features of the renormalization are very similar to the EFT formulation and also show the limit cycle behavior. In the case of positive scattering length, this approach has successfully been applied to the three- and four-boson systems [8, 12, 14] and to the four-nucleon system [15]. Here we extend this work to negative scattering lengths.

The four-body problem has previously been studied in more traditional approaches. Early studies include the Schrödinger equation with separable potentials [17] and field-theoretical models with separable expansions of the three-body T-matrix [18]. For a review of these and other early studies, see Refs. [19, 20]. The nuclear four-body problem has recently been benchmarked by comparing various modern calculational approaches [21]. An overview of recent calculations for the four-body system of \(^4\text{He} \) atoms can be found in Ref. [22].

None of these previous works focused on the universal properties of the four-body system.
with large scattering length which remain poorly understood. A first step towards understanding the four-body system with large positive $a$ was taken in Ref. [12]. By means of an explicit calculation, it was demonstrated that the renormalization of the three-body system automatically ensures the renormalization of the four-body system in this case. Therefore no four-body parameter should enter at leading order. These results were applied to calculate the $^4$He tetramer ground and excited state energies and good agreement with the Monte Carlo results of Blume and Greene was found [22]. Within the renormalized zero-range model, however, Yamashita et al. [23] recently found a strong sensitivity of the deepest four-body energy to a four-body subtraction constant in their equations. They motivated this observation from a general model-space reduction of a realistic two-body interaction close to a Feshbach resonance. The results of Ref. [12] for the $^4$He tetramer were also reproduced. Yamashita et al. concluded that a four-body parameter should generally enter at leading order. They argued that four-body systems of $^4$He atoms and nucleons (where this sensitivity is absent [12, 13, 16]) are special because repulsive interactions strongly reduce the probability to have four particles close together. However, the renormalization of the four-body problem was not explicitly verified in their calculation. Another drawback of their analysis is the focus on the deepest four-body state only. Therefore, it remains to be seen whether their findings are universal or are an artefact of their particular regularization scheme.

The purpose of this paper is to extend the study of Ref. [12] in two important ways. First, we carry out calculations for positive and negative scattering length and study the universal correlations between the three- and four-body binding energies. Second, we map out the scattering length dependence of the shallowest two four-body states and summarize the spectrum in a generalized Efimov-plot. We will follow Ref. [12] and work at leading order in $1/|a|$ using the framework of non-relativistic quantum mechanics to construct an effective interaction potential. The universal properties of the four-body spectrum will be useful for understanding the cluster structure of nuclei [24] as part of the planned experimental program at FAIR and for the creation of weakly-bound tetramers with cold atoms close to a Feshbach resonance. A first step in this direction was already taken in Ref. [25].

The organization of the paper is as follows: In Sec. II, we briefly describe our formalism for the two-, three-, and four-body bound state equations. The discussion of the universal correlations and the four-body spectrum follows in Sec. III. Finally, we summarize and present an outlook in Sec. IV.

## II. FEW-BODY BOUND STATE EQUATIONS IN EFFECTIVE THEORY

The effective low-energy interaction potential generated by a non-relativistic EFT with short-range interactions can be written down in a momentum expansion. In the two-body S-wave sector, it takes the general form

$$\langle k' | V | k \rangle = \lambda_2 + \lambda_{2,2} (k^2 + k'^2)/2 + \ldots ,$$

where $k$ and $k'$ are the relative three-momenta of the incoming and outgoing particles, respectively. Similar expressions can be derived for three- and higher-body interactions. The exact form of the potential depends on the specific regularization scheme used. The low-energy observables, however, are independent of the regularization scheme (up to higher order corrections) and one can choose a convenient scheme for practical calculations.

We regularize the potential in Eq. (1) by multiplying with a Gaussian regulator function, $\exp[-(k^2 + k'^2)/\Lambda^2]$, with the cutoff parameter $\Lambda$. This factor strongly suppresses high-
momentum modes in the region $k, k' \gtrsim \Lambda$ where the effective potential is not valid. The cutoff dependence of the coefficients $\lambda_2(\Lambda), \lambda_{2,2}(\Lambda), \ldots$ is determined by the requirement that low-energy observables are independent of $\Lambda$.

The expansion in Eq. (1) is only useful in conjunction with a power counting scheme that determines the relative importance of the various terms at low energy. The leading order is given by the $\lambda_2$ term which must be iterated to all orders, while the other terms give rise to higher-order corrections that can be included perturbatively \cite{26, 27}. In this paper, we will restrict ourselves to leading order in the expansion in $l/|a|$ and include only the $\lambda_2$ term. In the three-body system, a momentum-independent three-body interaction term $\lambda_3$ must be included together with $\lambda_2$ at leading order in $l/|a|$ \cite{6, 7}. Effective range effects and other higher-order corrections can be included as well \cite{9, 10, 11, 28, 29, 30}. In the four-body system, no new parameter enters at leading order in $l/|a|$ and only $\lambda_2$ and $\lambda_3$ contribute \cite{12}.

In order to set up our conventions and formalism, we will briefly review the bound state equations for the two-, three-, and four-body systems. For a more detailed discussion including explicit equations in momentum space, we refer the reader to Refs. \cite{12, 31}.

A. The Two-Body Sector

We write the leading order two-body effective potential in momentum space as:

\[ \langle p | V | q \rangle = \langle p | g \rangle \lambda_2 \langle g | q \rangle , \] (2)

where $\lambda_2$ denotes the two-body coupling constant and $q (p)$ are the relative three-momenta in the incoming (outgoing) channel. The regulator functions

\[ \langle p | g \rangle \equiv g(p) = \exp(-p^2/\Lambda^2) , \] (3)

suppress the contribution from high momentum states. In the few-body literature, they are often called “form factors”. For convenience, we will work in units where Planck’s constant $\hbar$ is set to unity: $\hbar = 1$.

The interaction (2) is separable and the Lippmann-Schwinger equation for the two-body problem can be solved analytically. The two-body t-matrix can be written as \cite{32}:

\[ t(E) = |g\rangle \tau(E) \langle g| , \] (4)

where $E$ denotes the total energy. The two-body propagator $\tau(E)$ is then given by

\[ \tau(E) = \left[ 1/\lambda_2 - 4\pi \int_0^\infty dq q^2 g(q)^2 mE - q^2 \right]^{-1} . \] (5)

A two-body bound-state appears as a simple pole in the two-body propagator $\tau$ at energy $E = -B_2$. Thus for $a > 0$, the two-body coupling constant $\lambda_2(B_2, \Lambda)$ can be fixed from the binding energy $B_2$, which is directly related to the scattering length by $a = 1/\sqrt{mB_2}$ up to higher order corrections in $l/|a|$. For negative scattering length, there is no shallow dimer state and the coupling constant $\lambda_2$ is fixed from matching to the effective range expansion for positive energies.
B. The Three-Body Sector

The low-energy properties of the three-body system for a given effective potential can be obtained by solving the Faddeev equations. The full wave function can be decomposed into four components: one for each two-body subcluster and one for the three-body cluster \([33]\). For identical bosons, the three-body wave function is fully symmetric under exchange of particles and the Faddeev equations simplify considerably. In this case, one only needs to solve equations involving one of the two-body Faddeev components and the three-body component. The two remaining two-body components can be obtained by permutations of particles.

We follow Glöckle and Meier \([33]\) and decompose the full three-body wave function as
\[
\Psi = (1 + P)\psi + \psi_3 , \quad \text{where} \quad P = P_{13}P_{23} + P_{12}P_{23}
\] (6)
is a permutation operator that generates the two not explicitly included Faddeev components from \(\psi\). The operator \(P_{ij}\) simply permutes particles \(i\) and \(j\). Since we are interested only in the binding energies and not in the full wave function, we can eliminate the component \(\psi_3\) and obtain an equation for \(\psi\) alone:
\[
\psi = G_0 t P \psi + G_0 t G_0 t_3 (1 + P) \psi .
\] (7)
where \(G_0\) denotes the free three-particle propagator. Moreover, \(t\) is the two-body t-matrix for the two-body subsystem described by the component \(\psi\) and \(t_3\) is a auxiliary t-matrix defined by the solution of the three-body Lippmann-Schwinger equation with the three-body effective interaction
\[
V_3 = |\xi\rangle \lambda_3 \langle \xi | ,
\] (8)
only. The three-body regulator function \(|\xi\rangle\) is defined as
\[
\langle u_1, u_2| \xi \rangle = \exp \left( - (u_1^2 + 3u_2^2/4)/\Lambda^2 \right) ,
\] (9)
where \(u_1\) and \(u_2\) are the usual Jacobi momenta. Note that \(t_3\) is only a technical construct that is generally cutoff dependent and not observable. For the derivation of an explicit representation of Eq. (7) in momentum space, we refer the reader to Refs. \([12, 31]\).

The three-body binding energies are given by those values of \(E\) for which Eq. (7) has a nontrivial solution. By expressing the two-body coupling constant \(\lambda_2\) in terms of the binding energy of the shallow two-body bound state, we have already renormalized the two-body problem. The three-body problem can therefore be renormalized by requiring one of the three-body binding energies to be fixed as one varies the cutoff. All other low-energy three-body observables can then be predicted. This renormalization procedure determines the three-body coupling constant \(\lambda_3(B_3, \Lambda)\) uniquely.

The dimensionless coupling constant \(\Lambda^4 \lambda_3(\Lambda)\) shows a limit cycle behavior \([12]\). For large values of the cutoff \(\Lambda\), \(\Lambda^4 \lambda_3\) flows towards an ultraviolet limit cycle. For \(\Lambda \to \infty\), it has the limiting behavior
\[
\lambda_3(\Lambda) = \frac{c}{\Lambda^4} \frac{\sin(s_0 \ln(\Lambda/L_3) - \arctan(1/s_0))}{\sin(s_0 \ln(\Lambda/L_3) + \arctan(1/s_0))} ,
\] (10)
where \(s_0 \approx 1.00624\) is a transcendental number that determines the period of the limit cycle. The constant \(c = 0.074 \pm 0.003\) was determined in Ref. \([12]\). The discrete scaling symmetry associated with a limit cycle is manifest in Eq. (10). If the cutoff \(\Lambda\) is multiplied by a factor...
\[ \exp(n\pi/s_0) \approx (22.7)^n \] with \( n \) an integer, the three-body coupling \( \lambda_3 \) is unchanged. \( L_3 \) is a three-body parameter generated by dimensional transmutation. It can be determined by fixing a three-body binding energy \( B_3 \). Alternatively, one could use a three-body binding energy directly to characterize the value of the three-body coupling \( \lambda_3 \) at a given cutoff \[ [7, 10] \].

C. The Four-Body Sector

We now turn to the four-body sector. The four-body binding energies are given by the Yakubovsky equations which are a generalization of the Faddeev equations to the four-body system. The full four-body wave function \( \Psi \) is first decomposed into Faddeev components, followed by a second decomposition into Yakubovsky components. For identical bosons, one has two Yakubovsky components \( \psi_A \) and \( \psi_B \). We start from the Yakubovsky equations including a general three-body force in the form written down by Glöckle and Kamada \[ [34] \]. The full four-body bound state wave function is decomposed into the Yakubovsky components \( \psi_A \) and \( \psi_B \) via

\[ \Psi = (1 + (1 + P)P_{34})(1 + P)\psi_A + (1 + P)(1 + \tilde{P})\psi_B , \] (11)

where \( P_{ij} \) exchanges particles \( i \) and \( j \), \( P \) is defined in Eq. (6), and \( \tilde{P} \) is given by

\[ \tilde{P} = P_{13}P_{24} \] (12)

The equations for the two wave function components read:

\[ \psi_A = G_0 t_{12}P[(1 + P_{34})\psi_A + \psi_B] + \frac{1}{3}(1 + G_0 t_{12})G_0 V_3 \Psi , \]

\[ \psi_B = G_0 t_{12}\tilde{P}[(1 + P_{34})\psi_A + \psi_B] , \] (13)

where \( t_{12} \) denotes the two-body t-matrix for particles 1 and 2 and \( V_3 \) is the three-body force defined in Eq. (8). Note that the three-body force couples to the full four-body wave function \( \Psi \). The factor of one third in front of the three-body force term arises because we insert the full three-body interaction for \( V_3 \). This is possible since we consider three-body contact interactions which are symmetric under the exchange of any pair of particles.

The renormalization analysis of the four-body system is complicated by the cutoff dependence of the number of bound states in the three-body subsystems. The further the cutoff \( \Lambda \) is increased, the more three-body bound states appear. While this has no influence on low-energy three-body observables, it creates an instability in the four-body system which can collapse into a deep three-body bound state plus another particle. This limits cutoff variations to an interval \( \Lambda_0 < \Lambda < 22.7 \Lambda_0 \) for some \( \Lambda_0 \), in which the number of three-body bound states remains constant. Despite this restriction, the cutoff can be varied by more than a factor of ten which is sufficient to study the renormalization properties and obtain converged numerical results \[ [12] \].

In Ref. \[ [23] \], it was pointed out that the calculation of \[ [12] \] for the \(^4\)He tetramer was limited to repulsive three-body forces \( \lambda_3 > 0 \). The ability to renormalize without a four-body parameter was attributed to a very small probability of the four particles to be close together due to the repulsive three-body force. We note that the sign of the three-body force is cutoff dependent because of the limit cycle behavior of \( \lambda_3(\Lambda) \) in Eq. (10). In Ref. \[ [12] \],
we have performed calculations with both attractive and repulsive three-body forces. In particular, the three-body force is attractive for cutoffs close to $\Lambda_0$. However, in this case $\Lambda \sim (mB_4^{(0)})^{1/2}$ and the ground state energy $B^{(0)}_4$ is far from the converged value. The final converged value for $B^{(0)}_4$ was indeed obtained with a repulsive three-body force. The excited state energy $B^{(1)}_4$, however, is already converged at smaller cutoffs where the three-body force is attractive. Increasing the cutoff beyond $22.7 \Lambda_0$, where the three-body force is attractive again would require to project out the unphysical deep three-body state that appears in this case. Such a subtraction is beyond the scope of this work. In the remainder of the paper, we will follow [12] and not include a four-body parameter in our calculations.

III. FOUR-BODY UNIVERSALITY

We now apply this effective theory to calculate the universal properties of the 4-boson bound state spectrum and universal scaling functions. We do not discuss the 2- and 3-boson spectrum in detail, since such a discussion can be found in Refs. [3, 7, 12].

A. Bound State Spectrum

The four-body spectrum is most conveniently discussed in a generalized Efimov plot. This plot was introduced by Vitaly Efimov to summarize the universal properties of the three-body spectrum [3, 4]. The set of all possible low-energy three-body states can be represented as points $(a^{-1}, K)$ on the plane whose horizontal axis is $1/a$ and whose vertical axis is the momentum variable

$$K = \text{sign}(E) \sqrt{m|E|}.$$  \hspace{1cm} (14)

It is convenient to introduce polar coordinates consisting of a radial variable $H$ and an angular variable $\xi$ defined by

$$1/a = H \cos \xi, \quad K = H \sin \xi.$$  \hspace{1cm} (15)

In terms of these polar coordinates, the discrete scaling symmetry in the three-body system is simply a rescaling of the radial variable: $H \rightarrow \exp(n\pi/s_0) H$.

The $a^{-1}$–$K$ plane for three identical bosons is shown in Fig. 1. We will refer to the bosons simply as particles in the following. The possible states are three-particle scattering states (PPP), particle-dimer scattering states (PD), and Efimov trimers (T). The threshold for scattering states is indicated by the hatched area. The Efimov trimers are represented by the heavy lines below the threshold.\(^1\) There are infinitely many branches of Efimov trimers close to threshold, but only a few are shown. A given physical system has a specific value of the scattering length and can be represented by a vertical line. The infinite scattering length limit corresponds to tuning this line to the $K$ axis. For a real physical system, not all states will behave as depicted in the figure. With momenta of order $1/l$ (or energies of

\(^1\) The curves for the trimer binding energies in Fig. 1 actually correspond to plotting $H^{1/4} \sin \xi$ versus $H^{1/4} \cos \xi$. This effectively reduces the discrete symmetry factor 22.7 down to $22.7^{1/4} = 2.2$, allowing a greater range of $a^{-1}$ and $K$ to be shown in the Figure.
order $1/(ml^2)$ one is able to probe details of the short-distance mechanism leading to the large scattering length and the effective theory does no longer apply. As a consequence, only states with $Kl \ll 1$ and systems with $l/|a| \ll 1$ will show the universal behavior.

In Fig. 2, we generalize this plot to the four-body system and plot the square root of the four-body energy versus the inverse scattering length. The lines representing the trimer energies in Fig. 1 now become the scattering thresholds for trimer-particle scattering. The vertical dotted line indicates the limit of infinite scattering length. Our results for the four-body ground and excited state energies $B_4^{(0)}$ and $B_4^{(1)}$ are given by the circles and triangles, respectively. Four-body states (tetramers) can only be stable if their energy is below all scattering thresholds, otherwise they become resonances and acquire a width from the decay into the corresponding scattering states. The lower (upper) solid lines indicate the thresholds for scattering of a ground state (excited state) trimer and another particle. For positive $a$, there are also scattering thresholds for scattering of two dimers and a dimer plus two particles indicated by the dash-dotted and dashed lines, respectively. These thresholds apply to both tetramer states. A given physical system is again characterized by a vertical line corresponding to a particular value of the scattering length. Depending on the value of the scattering length, the ground state trimer-particle threshold or the dimer-dimer threshold can be closest to the tetramer states. In this paper, we focus on the region where the closest threshold is given by ground state trimer-particle scattering. If the plot was extended further to the right, the dimer-dimer threshold will eventually become the lowest one.

We have obtained the dependence of the four-body energies on the scattering length $a$ for fixed three-body parameter $L_3$ from explicit solutions of the Yakubovsky equations (13). All quantities are given in units of $L_3$. Note that $L_3$ is only defined up to a rescaling by the discrete scaling factor $\exp(n\pi/s_0) \approx (22.7)^n$, so the absolute scale of Fig. 2 is arbitrary up to

FIG. 1: The $a^{-1}$–$K$ plane for the three-body problem. The allowed regions for three-particle scattering states and particle-dimer scattering states are labeled $PPP$ and $PD$, respectively. The heavy lines labeled $T$ are two of the infinitely many branches of Efimov states. The cross-hatching indicates the threshold for scattering states.
such factors. The circles and triangles show our results for the four-body ground and excited state energies $B_4^{(0)}$ and $B_4^{(1)}$ as a function of the scattering length, respectively. The tetramer excited state remains close to the threshold for decay into a trimer and another particle for essentially all calculated values of $1/a$. It moves away from this threshold near the point where the threshold disappears at $K = 0$. The tetramer ground state is considerably deeper. It stays essentially parallel to the excited state but starts to move away for small positive scattering lengths. Due to numerical instabilities, we were not able to reach the region where the two-dimer threshold becomes relevant.

For the limits of universal behavior the same restrictions as in the three-body system apply. With momenta of order $1/l$ (or energies of order $1/(ml^2)$) one is able to probe details of the short-distance mechanism leading to the large scattering length and the effective theory does no longer apply. As a consequence, only states with $Kl \ll 1$ and systems with $l/|a| \ll 1$ will show the universal behavior in Fig. 2. Which states are within the universal window then depends on the actual values of $a$ and $l$ in the system under consideration. For $^4$He atoms there is some evidence that both tetramer states are within this range [12].

It is interesting to note that the universal theory always produces two four-body states [31]. The number of four-body states is independent of the number of three-body states. The latter can be expected since the excited three-body states are shallower by at least a factor of 515 and therefore should have little influence on the four-body states. The discrete scaling symmetry then suggests that there are exactly two four-body resonances between any two three-body states with the shallower of the four-body states having a slightly lower energy.

FIG. 2: The $a^{-1}$–$K$ plane for the four-body problem. The circles and triangles indicate the four-body ground and excited state energies $B_4^{(0)}$ and $B_4^{(1)}$, while the lower (upper) solid lines give the thresholds for decay into a ground state (excited state) trimer and a particle. The dash-dotted (dashed) lines give the thresholds for decay into two dimers (a dimer and two particles). The vertical dotted line indicates infinite scattering length. All quantities are given in units of the three-body parameter $L_3$. 
FIG. 3: The correlation between the three-body ground state energy $B_3^{(0)}$ and the four-body ground and excited state energies $B_4^{(0)}$ (circles) and $B_4^{(1)}$ (triangles) in units of $1/(ma^2)$ for $a > 0$. The inset shows the threshold region in more detail.

than the shallower of the two three-body states. These states are stable for some range of cutoffs, but become resonances with the appearance of a deeper three-body state when the cutoff is increased beyond the critical value $22.7\Lambda_0$ discussed in section II C. Checking this conjecture would require finding resonance poles in the complex plane which is beyond the scope of this work.

B. Universal Correlations

We now consider the universal correlations in the three-body system for negative scattering length. In particular, we focus on the correlation between the binding energies in the three- and four-body systems. Such correlations were first observed in nuclear physics and are known as Tjon lines [35]. For recent discussions of the nuclear Tjon line in the context of the EFT for short-range interactions and low-momentum nucleon-nucleon potentials, see Refs. [15, 16]. The Tjon lines for spinless bosons with positive scattering length were discussed in [12] for the range of binding energies relevant to $^4$He atoms.

Here, we discuss the correlation between the three-body ground state energy $B_3^{(0)}$ and the four-body ground and excited state energies $B_4^{(0)}$ and $B_4^{(1)}$ for values of $B_3^{(0)}$ from threshold up to $B_3^{(0)} \approx 200/(ma^2)$. In Fig. 3, we show the correlation between the ground state energy in the three-body system and the ground and excited state energies in the four-body system for $a > 0$. The corresponding plot for negative scattering length is shown in Fig. 4.

The inset in both figures shows the threshold region in more detail. The nonlinear behavior of the Tjon lines in this region is evident. For positive scattering length, we cannot calculate all the way down to the three-body threshold at $B_3 = 1/(ma^2)$ due to numerical instabilities. The correlations have positive curvature for $a > 0$ and negative curvature for
FIG. 4: The correlation between the three-body ground state energy $B_{3}^{(0)}$ and the four-body ground and excited state energies $B_{4}^{(0)}$ (circles) and $B_{4}^{(1)}$ (triangles) in units of $1/(ma^2)$ for $a < 0$. The inset shows the threshold region in more detail.

$a < 0$. Outside the threshold region, the correlation is approximately linear independent of the sign of $a$. A similar linear relation holds for the correlation between different three-body energies (e.g. ground and excited states) [3]. This is a direct consequence of the discrete scaling symmetry in the three-body system. For $1/a = 0$, this symmetry ensures that the binding energies of subsequent three-body states differ by factors of approximately 515. If the scattering length is finite, the correlation is still linear to a good approximation. Only for states close to threshold the linearity breaks down. In this case, there is still an exact scaling symmetry, but it relates states corresponding to different values of the scattering length. Since no new parameter enters at leading order in the four-body system, the above arguments immediately generalize to the four-body case. Away from the thresholds, the binding energies of the ground and excited four-body states are related to the deepest three-body state by factors of 5 and 1.01, respectively.

Similar linear correlations between the binding energies of $N$- and $(N-1)$-body systems with positive scattering length and $N > 4$ were recently found by Hannah and Blume [36]. They studied bosonic clusters with up to $N = 40$ atoms interacting additively through two-body van der Waals potentials. Using exact Monte Carlo methods, they determined the energies as well as the interparticle distances. They found approximately linear relations for the correlation between the ground state energy ratios $B_{N}^{(0)}/B_{N-2}^{(0)}$ and $B_{N-1}^{(0)}/B_{N-2}^{(0)}$ for $N$ up to 10. This observation suggests that the discrete scaling symmetry holds up to $N = 10$. Consequently, no new parameters would enter into the effective theory at leading order for up to 10 particles. If true, it would open up the exciting possibility to predict the universal properties of $N$-body system up to $N = 10$ from the scattering length $a$ and the three-body parameter $L_3$ alone.
IV. SUMMARY AND OUTLOOK

In this paper, we have studied the universal properties of the four-body system with large scattering length. These properties are useful for understanding the cluster structure of nuclei [24], as part of the planned experimental program at FAIR and experiments on the creation of weakly-bound tetramers with cold atoms close to a Feshbach resonance. We have concentrated on the bound state problem of four bosons starting from the Yakubovsky equations. We have constructed an effective interaction potential including both a two- and three-body contact interaction. This is the minimal set of contact interactions required for renormalization of the three-body problem. The two parameters of the effective potential were determined from matching to the binding energy of the dimer \((a > 0)\) or the two-body scattering length \((a < 0)\) and the excited state of the trimer. We have then solved the four-body bound state problem and obtained the scattering length dependence of the four-body bound state spectrum. This dependence was presented in a generalized Efimov plot. For all considered values of \(a\), we have found two four-body states. The tetramer excited state stays close to the trimer-particle threshold, while the ground state is considerably deeper. We have conjectured, that there are always two four-body resonances between any two three-body states. As the cutoff is increased, these states come to life as stable bound states, but turn into resonances as additional deep three-body states appear when the cutoff is increased beyond a critical value.

An important consequence of the large scattering length is the emergence of universal scaling functions. Since there are only two parameters at leading order, \(a\) and \(L_3\), few-body observables normalized to the scattering length \(a\) must be correlated and follow a line parameterized by \(L_3\). We have calculated the universal scaling functions relating the tetramer energies to the trimer ground state energy. A similar correlation between the triton and alpha particle energies is known from nuclear physics as the Tjon line [15, 16, 35]. Close to the three-body thresholds, these correlations have positive curvature for \(a > 0\) and negative curvature for \(a < 0\). Away from the thresholds, they are linear to very high accuracy. This linearity can be understood from the discrete scaling symmetry. Similar linear correlations between the binding energies of \(N\)- and \((N-1)\)-body systems with \(N > 4\) were recently found from exact Monte Carlo calculations using two-body van der Waals potentials [36]. This observation suggests the absence of \(N\)-body parameters for \(N > 4\), which would allow to predict \(N\)-body observables from two- and three-body input alone.

The limits of universal behavior in real systems are determined by the values of \(a\) and \(l\) in those systems. With momenta of order \(1/l\) (or energies of order \(1/(ml^2)\)) one is able to probe details of the short-distance mechanism leading to the large scattering length and our effective theory does no longer apply. As a consequence, only states with \(kl \ll 1\) and systems with \(l/|a| \ll 1\) will show the universal behavior discussed in this paper. Corrections are suppressed by powers of \(l/|a|\) and \(kl\) where \(\kappa\) is the typical momentum scale in the process considered (e.g. the binding momentum for bound states). In Ref. [12], we have shown that our results can be applied to the \(^4\)He tetramer. We are not aware of any data on four-body systems with large negative scattering length.

We also stress that the universality of our results relies on the validity of the renormalization group analysis in Ref. [12]. In this work, renormalization of the four-body equations was achieved without the introduction of a four-body interaction. If this result would not hold, the properties of the four-body system with large scattering length would also depend on a four-body parameter (cf. Ref. [23]).
There are a number of directions that should be pursued in future work. First, it would be interesting to extend the calculation to smaller positive values of \( a \), where the relevant decay threshold for the tetramer states is the dimer-dimer threshold. This would complete the generalized Efimov plot. Our conjecture of two four-body resonances between any two trimer states could be tested by extending the calculation to complex values of the energy. The widths of such states are currently unknown. Moreover, the application of our results to cluster structures in nuclei requires the inclusion of the long-range Coulomb force. Work in this direction is in progress. The general power counting for four-body forces is still not understood. At which order does the leading four-body interaction enter? In the three-body system, e.g., the first and second order correction are due to the two-body effective range \[10\]. If a similar situation holds in the four-body system, it would be possible to predict low-energy four-body observables up to corrections of order \((l/|a|)^3\) from two- and three-body information alone. The extension of the effective theory to calculate four-body scattering observables would be very valuable. The knowledge of the dimer-dimer scattering length, for example, is important for experiments with ultracold atoms. For the simpler problem of fermions with two spin states (where the three-body parameter \( L_3 \) does not contribute), the dimer-dimer scattering length was recently calculated exactly \[37\] and using a perturbative \( \epsilon \)-expansion around \( d = 4 \) \[38\].

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