Parametric Analysis of Gumbel Type-II Distribution under Step-stress Life Test

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Abstract

In this paper, we focus on the parametric inference based on the Tampered Random Variable (TRV) model for simple step-stress life testing (SSLT) using Type-II censored data. The baseline lifetime of the experimental units under normal stress conditions follows Gumbel Type-II distribution with $\alpha$ and $\lambda$ being the shape and scale parameters, respectively. Maximum likelihood estimator (MLE) and Bayes estimator of the model parameters are derived based on Type-II censored samples. We obtain asymptotic intervals of the unknown parameters using the observed Fisher information matrix. Bayes estimators are obtained using Markov Chain Monte Carlo (MCMC) method under squared error loss function and LINEX loss function. We also construct highest posterior density (HPD) intervals of the unknown model parameters.

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Extensive simulation studies are performed to investigate the finite sample properties of the proposed estimators. Finally, the methods are illustrated with the analysis of a real data set.

**Keywords:** Baseline lifetime, Tampering times, Tampering coefficients, Bayesian Analysis, MCMC method, Metropolis-Hastings algorithm, Highest posterior density credible interval.

## 1 Introduction

In reliability analysis, there are many problems with life-testing experiments which require a long time to acquire the test data such as for highly reliable products at the specified use condition. There may occur very few failures or even no failures within a limited testing time under normal operating conditions. In such problems, to induce early failures an accelerated life test (ALT) is often used. If all test units are exposed to accelerated conditions than usual normal conditions, then the test is called ALT. A special class of the ALT, known as step-stress life testing (SSLT), which allows the experimenter to gradually increase the stress levels at some pre-fixed time points during the experiment. In such a life-testing experiment with two stress levels $s_1$ and $s_2$, $n$ identical units are placed on test initially under normal stress level $s_1$. The stress level is changed from $s_1$ to $s_2$, at pre-fixed time $\tau$, known as the tampering time. There may be more stress levels and corresponding to each stress change there would be more than one number of tampering time points we call the multiple step-stress life testing. If there are only two stress levels with one tampering time, then it is known as the simple step-stress life-testing, or simple SSLT. The lifetime distribution under the initial normal stress level is termed as the baseline lifetime distribution. Some key references on ALT model are [1], [2], [3], [4].

There are many situations in life testing and reliability experiments in which the experimental units are lost or removed from test before failure. Therefore, the experimenter
may not obtain complete information of failure times of all the units put on a life testing experiment. Data observed from such experiments are called censored data. The most commonly used censoring schemes are Type-I and Type-II censoring schemes. Briefly, these can be described as follows. Consider \( n \) units are placed under observations on a particular experiment. In the conventional Type-I censoring scheme, the experiment continues up to a pre-specified time \( \eta \), which is pre-fixed. The main drawback of this censoring scheme is one may get very few failures or in worst case no failure till time \( \eta \). Therefore, estimating the unknown parameters can not be done efficiently. On the other hand, the conventional Type-II censoring scheme requires the experiment to continue until a pre-specified number \( r \) (say) of failures \( (r \leq n) \) occurs. However, the Type-II censoring always ensures \( r \) number of failures during the life testing experiment. Here, also the major drawback is the experimental time. If one uses a highly reliable product for the life testing experiment then it will take more time to get a pre-specified number of failures and which leads to a cost constraint for the experimenter.

Goel \cite{Goel1979} first introduced the tampered random variable (TRV) modeling in the context of a simple SSLT (see also \cite{Goel1986}), which assumes that the effect of change of the stress level at time \( \tau \) is equivalent to changing the remaining life of the experimental unit by an unknown positive factor, say \( \beta \) (usually, less than 1). Let \( T \) be the random variable representing the baseline lifetime under normal stress condition. Then, the overall lifetime, denoted by the random variable \( T_{TRV} \), is defined as

\[
T_{TRV} = \begin{cases} 
T, & 0 < T \leq \tau \\
\tau + \beta(T - \tau), & T > \tau,
\end{cases}
\]

where the scale factor \( \beta \), called the tampering coefficient, depends on both the stress levels \( s_1 \) and \( s_2 \) and possibly on \( \tau \) as well. The time point \( \tau \) is called the tampering time. In literature many authors considered this model for estimating different lifetime distributions.
Abdel-Ghaly et al. [7] considered the estimation problem of the Weibull distribution in ALT. Maximum likelihood estimators (MLEs) are obtained for the distribution parameters and the acceleration factor in both Type-I and Type-II censored samples. The modified quasilinearization method is used to solve the nonlinear maximum likelihood equations. Also, the confidence intervals of the estimators are obtained. Wang et al. [8] studied the estimation of the parameters of the Weibull distribution in step-stress ALT under multiply censored data. The MLEs are used to obtain the parameters of the Weibull distribution and the acceleration factor under multiply censored data. Additionally, the confidence intervals for the estimators are also obtained. Ismail [9] obtained the MLEs of Weibull distribution parameters and the acceleration factor under adaptive Type-II progressively hybrid censored data. The method has been extended for an adaptive Type-I progressive hybrid censored data by Ismail [10].

In this paper, we consider the problem of estimation for a Gumbel Type-II distribution based on TRV modeling under simple SSLT using Type-II censoring. The baseline lifetime $T$ follows Gumbel Type-II distribution whose probability density function (PDF) and the cumulative distribution function (CDF) are, respectively,

$$f_T(t) = \alpha \lambda t^{-(\alpha+1)} e^{-\lambda t^{-\alpha}}, \quad t > 0, \quad \alpha > 0, \quad \lambda > 0,$$

(1.1)

and

$$F_T(t) = e^{-\lambda t^{-\alpha}}, \quad t > 0,$$

(1.2)

where $\alpha$, $\lambda$ are the shape and scale parameters, respectively. The hazard rate function of Gumbel Type-II distribution is decreasing or upside-down bathtub (UTB) shape depends on the parameters values. Due to these shapes of the hazard rate function, the Gumbel Type-II distribution is very flexible to model meteorological phenomena such as floods, earthquakes,
and natural disasters, also in medical and epidemiological applications. In recent years many authors have studied statistical properties of the estimators of the model parameters of Gumbel Type-II distribution. Abbas et al. [11] discussed Bayesian estimation of the model parameters of Gumbel Type-II distribution. Then E-Bayesian estimation of the unknown model shape parameter has been studied by Reyad and Ahmed [12]. Sindhu et al. [13] obtained Bayes estimates and corresponding risk of the model parameters based on left-censored data.

Due to importance of the Gumbel Type-II distribution and simple SSLT, in this paper, we have considered the problem of estimation of the parameters of Gumbel Type-II distribution using TRV modeling under simple SSLT. To the best of our knowledge, this problem has not been studied yet. The rest of the article is organized as follows. In Section 2, we introduce the TRV modeling under simple SSLT and derive the corresponding CDF and PDF for Gumbel Type-II baseline lifetimes. Also, the MLEs of the unknown parameters, \( \alpha \), \( \lambda \), and \( \beta \) are derived using Type-II censored samples. We also construct asymptotic intervals of unknown parameters based on the observed Fisher information matrix. Bayes estimates are obtained under the squared error loss function as well as LINEX loss function in Section 3. We compute these estimates using the MH-algorithm of MCMC method. The HPD credible intervals of unknown parameters are discussed as well. Section 4 presents some simulation studies to investigate the finite sample properties of the MLEs. We illustrate the proposed methods through the analysis of a real life data set in Section 5, while Section 6 ends with some concluding remarks.

## 2 Model Description and MLEs

In a simple SSLT model, let us consider \( n \) number of experimental units are placed with initial stress \( s_1 \). After a prefixed time \( \tau \), the initial stress level is changed from \( s_1 \) to \( s_2 \).
The experiment will be terminated when $r^{th}$ failure occurs, where $r$ is a pre-fixed integer. Therefore, the time of failures $t_{1:n} < t_{2:n} < \ldots < t_{r:n}$, are denoted as the observed data. The following are all possible types of data we can get from the Type-II censoring under simple SSLT:

**Case-I:** $t_{1:n} < t_{2:n} < \cdots < t_{r:n} < \tau$,

**Case-II:** $t_{1:n} < t_{2:n} < \cdots < t_{N:n} < \tau < t_{N+1:n} < t_{N+2:n} < \cdots < t_{r:n}$,

**Case-III:** $\tau < t_{1:n} < t_{2:n} < \cdots < t_{r:n}$,

where $N$ is the number of failures at stress level $s_1$. Note that, for Case-I, $N = r$ and for Case-III, $N = 0$. Basically Case-I and Case-III are the special cases of Case-II thereafter we will only focus on Case-II in the remaining part of this paper.

Let us assume that the baseline lifetime follows Gumbel Type-II distribution with $\alpha$, $\lambda$ are shape and scale parameters, respectively. Also, assume that the experimental units are independent and identically distributed (i.i.d.) in the life testing experiment. Now, under the assumption of TRV model, the CDF $F_{TRV}(.)$, of a $T_{TRV}$ is given by

$$F_{TRV}(t) = \begin{cases} 
F_T(t), & \text{if } 0 \leq t < \tau \\
F_T \left( \tau + \frac{t - \tau}{\beta} \right), & \text{if } t \geq \tau \\
e^{-\lambda t^{-\alpha}}, & \text{if } 0 \leq t < \tau \\
e^{-\lambda(\tau + \frac{t - \tau}{\beta})^{-\alpha}}, & \text{if } t \geq \tau.
\end{cases} \quad (2.1)$$
Therefore, the corresponding PDF is given by

\[
\begin{aligned}
f_{TRV}(t) &= \begin{cases} 
  f_T(t), & \text{if } 0 \leq t < \tau \\
  \frac{1}{\beta} f_T \left( \tau + \frac{t-\tau}{\beta} \right), & \text{if } t \geq \tau 
\end{cases} \\
&= \begin{cases} 
  \alpha \lambda t^{-(\alpha+1)} e^{-\lambda t - \alpha}, & \text{if } 0 \leq t < \tau \\
  \frac{\alpha \lambda}{\beta} (\tau + \frac{t-\tau}{\beta})^{-(\alpha+1)} e^{-\lambda (\tau + \frac{t-\tau}{\beta}) - \alpha}, & \text{if } t \geq \tau
\end{cases}
\end{aligned}
\]  

(2.2)

Next we will discuss about the MLEs of the unknown model parameters.

### 2.1 Maximum Likelihood Estimation

In this section we will determine MLEs for the unknown model parameters under Type-II censored sample using TRV modeling based on simple SSLT. Let \( T_{1:n}, T_{2:n}, \ldots, T_{n:n} \) be the Type-II censored random sample of size \( n \) from the Gumbel Type-II distribution described in (2.1) with the unknown model parameters \( \alpha, \lambda, \) and \( \beta \). Therefore, the likelihood function for TRV modeling under Type-II censoring can be written as

\[
L(\alpha, \lambda, \beta|data) \propto \prod_{i=1}^{N} f_T(t_{i:n}) \prod_{i=N+1}^{r} f_T \left( \tau + \frac{t_{i:n} - \tau}{\beta} \right) \left[ 1 - F_T \left( \tau + \frac{t_{r:n} - \tau}{\beta} \right) \right]^{n-r}. 
\]

Thus, the likelihood function for Gumbel Type-II distribution can be written as

\[
L(\alpha, \lambda, \beta|data) \propto \alpha^r \lambda^r \beta^{-N} \prod_{i=1}^{N} t_{i:n}^{-(\alpha+1)} e^{-\lambda t_{i:n} - \alpha} \prod_{i=1}^{r-N} \left( \tau + \frac{t_{i:n} - \tau}{\beta} \right)^{-(\alpha+1)} \\
\times e^{-\lambda \sum_{i=1}^{r-N} \left( \tau + \frac{t_{i:n} - \tau}{\beta} \right)^{-\alpha}} \left[ 1 - e^{-\lambda \left( \tau + \frac{r-n-\tau}{\beta} \right)^{-\alpha}} \right]^{n-r}. 
\]

(2.3)
Therefore, the log-likelihood function can be written as

\[
l(\alpha, \lambda, \beta) = r \log \alpha + r \log \lambda + (r - N) \log \beta - (\alpha + 1) \left( \sum_{i=1}^{N} \log t_i + \sum_{i=1}^{r-N} \log z_i \right) \\
- \lambda \left( \sum_{i=1}^{N} t_i^{-\alpha} + \sum_{i=1}^{r-N} z_i^{-\alpha} \right) + (n - r) \log \left( 1 - e^{-\lambda z_r^{-\alpha}} \right),
\]

where, \( z_i = \tau + \frac{t_i - \tau}{\beta} \) and \( z_r = \tau + \frac{t_r - \tau}{\beta} \). Note that, for notational simplicity, rest of the paper we write \( t_{i:n} \) and \( t_{r:n} \) as \( t_i \) and \( t_r \), respectively.

Now, taking partial derivatives of \( l(\alpha, \lambda, \beta) \) with respect to unknown parameters we get likelihood equations as given below

\[
\frac{\partial l}{\partial \alpha} = \frac{r}{\alpha} - \left( \sum_{i=1}^{N} \log t_i + \sum_{i=1}^{r-N} \log z_i \right) + \lambda \left( \sum_{i=1}^{N} t_i^{-\alpha} \log t_i + \sum_{i=1}^{r-N} z_i^{-\alpha} \log z_i \right) \\
- (n - r) \frac{\lambda z_r^{-\alpha} \log z_r}{(1 - e^{-\lambda z_r^{-\alpha}})} = 0,
\]

\[
\frac{\partial l}{\partial \lambda} = \frac{r}{\lambda} - \sum_{i=1}^{N} t_i^{-\alpha} - \sum_{i=1}^{r-N} z_i^{-\alpha} + (n - r) \frac{z_r^{-\alpha}}{\left( e^{\lambda z_r^{-\alpha}} - 1 \right)} = 0,
\]

and

\[
\frac{\partial l}{\partial \beta} = \frac{r - N}{\beta} - (\alpha + 1) \sum_{i=1}^{r-N} \frac{z_i'}{z_i} - \alpha \lambda \sum_{i=1}^{r-N} \frac{z_i^{-(\alpha+1)} z_i'}{z_i} \\
- (n - r) \frac{\alpha \lambda z_r^{-(\alpha+1)} z_r'}{\left( e^{\lambda z_r^{-\alpha}} - 1 \right)} = 0,
\]

where \( z_i' = \frac{\tau - t_i}{\beta^2} \) and \( z_r' = \frac{\tau - t_r}{\beta^2} \).

As the likelihood equations are in implicit form of the unknown parameters, thus we cannot solve it explicitly to determine the MLEs of \( \alpha, \lambda, \) and \( \beta \) as \( \hat{\alpha}, \hat{\lambda}, \) and \( \hat{\beta} \). Therefore solve the equations (2.5), (2.6) and (2.7) numerically by using some numerical method, such as Newton-Raphson.
2.2 Approximate Confidence Intervals

In this section we want to construct the asymptotic confidence intervals for the unknown model parameters. To obtain $100(1 - \gamma)\%$ confidence interval of the unknown parameters of Gumbel Type-II distribution under simple SSLT, we have to calculate the asymptotic variance-covariance matrix. Using asymptotic normality properties of MLEs of the parameters, an asymptotic variance-covariance matrix can be obtained. In doing so, the variance of $\hat{\alpha}$, $\hat{\lambda}$, and $\hat{\beta}$ are required. These can be obtained from the diagonal elements of the inverse of the observed Fisher information matrix, $\hat{I}^{-1}(\hat{\alpha}, \hat{\lambda}, \hat{\beta})$, where

$$
\hat{I}(\hat{\alpha}, \hat{\lambda}, \hat{\beta}) = \begin{bmatrix}
-l_{11} & -l_{12} & -l_{13} \\
-l_{12} & -l_{22} & -l_{23} \\
-l_{13} & -l_{23} & -l_{33}
\end{bmatrix}_{(\alpha, \lambda, \beta)=(\hat{\alpha}, \hat{\lambda}, \hat{\beta})}, 
$$

(2.8)

and $l_{ij} = \frac{\partial^2 \log L}{\partial \theta_i \partial \theta_j}$ for $i, j = 1, 2, 3$, where, $\Theta = (\theta_1, \theta_2, \theta_3) = (\alpha, \lambda, \beta)$. So here,

$$
l_{11} = -\frac{r}{\alpha^2} - \lambda \sum_{i=1}^{N} \frac{(\log t_i)^2}{t_i^\alpha} - \lambda \sum_{i=1}^{N} \frac{(\log z_i)^2}{z_i^\alpha} - (n - r) \frac{\lambda [z_r^\alpha + e^{\lambda z_r^{-\alpha}(\lambda - z_r^\alpha)}]}{z_r^{2\alpha(e^{\lambda z_r^{-\alpha}} - 1)^2} (\log z_r)^2},
$$

$$
l_{12} = l_{21} = \sum_{i=1}^{N} \frac{\log t_i}{t_i^\alpha} + \sum_{i=1}^{N} \frac{\log z_i}{z_i^\alpha} - (n - r) \frac{\lambda z_r^\alpha (e^{\lambda z_r^{-\alpha}} - 1) + \alpha (z_r^\alpha + e^{\lambda z_r^{-\alpha}}(\lambda - z_r^\alpha)) \log z_r}{z_r^{2\alpha+1(e^{\lambda z_r^{-\alpha}} - 1)^2}},
$$

$$
l_{13} = l_{31} = -(n - r) \frac{\lambda z_r^\alpha (e^{\lambda z_r^{-\alpha}} - 1) + \alpha (z_r^\alpha + e^{\lambda z_r^{-\alpha}}(\lambda - z_r^\alpha)) \log z_r}{z_r^{2\alpha+1(e^{\lambda z_r^{-\alpha}} - 1)^2}} - \sum_{i=1}^{r-N} \frac{z_i^\alpha}{z_i} - \lambda \sum_{i=1}^{r-N} \frac{z_i^\alpha z_i'}{z_i} (\alpha \log z_i - 1).
$$
\[ l_{22} = \frac{r}{\lambda^2} - (n - r) \frac{z_r^{-2\alpha} e^{\lambda z_r^{-\alpha}}}{(e^{\lambda z_r^{-\alpha}} - 1)^2}. \]

\[ l_{23} = l_{32} = \sum_{i=1}^{r-N} \alpha z_i^{-\alpha} \frac{z_i'}{z_i} - (n - r) \frac{z_r'}{z_r} \frac{z_r^{-2\alpha}}{e^{\lambda z_r^{-\alpha}} - 1} \left( e^{\lambda z_r^{-\alpha}} (z_r^{\alpha} - \lambda) - z_r^{\alpha} \right). \]

\[ l_{33} = -\frac{r - N}{\beta^2} + (\alpha + 1) \sum_{i=1}^{r-N} \left( \frac{z_i'}{z_i} + \frac{2}{\beta} \right) - \alpha \lambda \sum_{i=1}^{r-N} \left( z_i^{-\alpha} \frac{z_i'}{z_i} \left[ (\alpha + 1) \frac{z_i'}{z_i} - \frac{2}{\beta} \right] \right) \]

\[ - \frac{\alpha \lambda}{\beta^2} \left( \frac{z_r'}{z_r} \right)^2 \frac{z_r^{-2\alpha}}{(e^{\lambda z_r^{-\alpha}} - 1)^2} \left[ (\alpha - 1) t_r z_r^{\alpha} - (2\beta + \alpha - 1) \tau z_r^{\alpha} + \tau e^{\lambda z_r^{-\alpha}} (2\beta - 1) z_r^{\alpha} \right. \]

\[ + \alpha (z_r^{\alpha} - \lambda) \right) + t_r e^{\lambda z_r^{-\alpha}} \left( z_r^{\alpha} + \alpha (\lambda - z_r^{\alpha}) \right). \]

Then the 100(1 - \gamma)\% approximate confidence intervals for \( \alpha, \lambda, \) and \( \beta \) are given by,

\[ \left( \hat{\alpha} \pm z_{\frac{\gamma}{2}} \sqrt{Var(\hat{\alpha})} \right), \left( \hat{\lambda} \pm z_{\frac{\gamma}{2}} \sqrt{Var(\hat{\lambda})} \right), \text{ and } \left( \hat{\beta} \pm z_{\frac{\gamma}{2}} \sqrt{Var(\hat{\beta})} \right), \]

where \( z_{\frac{\gamma}{2}} \) is the upper \( \frac{\gamma}{2} \)-th percentile of a standard normal distribution.

### 3 Bayesian Estimation

In this section, we will determine the Bayes estimates of the unknown parameters \( \alpha, \lambda, \) and \( \beta \) based on Type-II censored data from Gumble Type-II distribution. The most commonly used symmetric loss function is squared error loss (SEL) function and an asymmetric loss function is LINEX loss (LL) function. These loss functions are defined as

\[ L_{SE}(h(\theta), \hat{h}(\theta)) = (h(\theta) - \hat{h}(\theta))^2, \]

and

\[ L_{LL}(h(\theta), \hat{h}(\theta)) = e^u(\hat{h}(\theta) - h(\theta)) - u(\hat{h}(\theta) - h(\theta)) - 1, \] \( u \neq 0, \)
where $\hat{h}(\theta)$ is an estimate of a parametric function $h(\theta)$ and $u$ is a real number. All the parameters $\alpha$, $\lambda$, and $\beta$ are unknown. In this case, there doesn’t exist any natural joint conjugate prior distribution. Thus, according to Kundu and Pradhan [14] we assume independent priors for $\alpha$, $\lambda$, and $\beta$ as Gamma($a, b$), Gamma($c, d$), and Beta($p, q$) distributions, respectively. We recall that $X \sim$ Gamma($a, b$), if its PDF is given by

$$f_1(x) \propto x^{a-1}e^{-bx}, \; x > 0, \; a, \; b > 0. \tag{3.1}$$

Further, if $X \sim$ Beta($p, q$), then its PDF is

$$f_2(x) \propto x^{p-1}(1-x)^{q-1}, \; 0 < x < 1, \; p, \; q > 0. \tag{3.2}$$

Now, the joint prior distribution of the unknown parameters is obtained as

$$\pi^*(\alpha, \lambda, \beta) \propto \alpha^{a-1}e^{-ba}\lambda^{c-1}e^{-d\lambda}\beta^{p-1}(1-\beta)^{q-1}, \; \alpha > 0, \; \lambda > 0, \; 0 < \beta < 1, \tag{3.3}$$

where, $a$, $b$, $c$, $d$, $p$, and $q$ are the hyper parameters. Note that the hyper parameters reflect the prior knowledge about the unknown parameters, and can take the value from any positive real numbers. After some calculations, the joint posterior PDF of the unknown parameters $\alpha$, $\lambda$, and $\beta$ can be obtained as

$$\pi(\alpha, \lambda, \beta|data) = k \frac{\alpha^{r+a-1}\lambda^{r+c-1}}{\beta^{N+1-r-p}}(1-\beta)^{q-1}e^{-(b\alpha+d\lambda)}e^{-\left[(\alpha+1)\sum_{i=1}^{N}\log(t_i)+\lambda\sum_{i=1}^{N}t_i^{-\alpha}\right]}$$

$$\times e^{-\left[(\alpha+1)\sum_{i=1}^{r-N}\log(\tau+t_i)\lambda+\sum_{i=1}^{r-N}\tau^{-\alpha}\right]}\left[1-e^{-\lambda(\tau+t_i)^{-\alpha}}\right]^{n-r}, \tag{3.4}$$
where

\[
k^{-1} = \int_0^1 \int_0^\infty \int_0^\infty \left[ \frac{\alpha r^a \lambda^{r+c-1}}{\beta^{N+1-r-p}} (1 - \beta)^{q-1} e^{-(b\alpha + d\lambda)} e^{-[(\alpha+1) \sum_{i=1}^N \log(t_i) + \lambda \sum_{i=1}^N t_i^{-\alpha}]} \right. \\
\times \left. e^{-[(\alpha+1) \sum_{i=1}^r \log(1 + \frac{t_i - \tau}{\beta}) + \lambda \sum_{i=1}^r (\tau + \frac{t_i - \tau}{\beta})^{-\alpha}] \left[ 1 - e^{-\lambda (\tau + \frac{t_i - \tau}{\beta})^{-\alpha}} \right]^{(n-r)}} \right] \, d\alpha \, d\lambda \, d\beta.
\]

Under the loss functions SEL and LL, the Bayes estimates of \( h(\alpha, \lambda, \beta) \) can be written as \( \hat{h}_{SE}(\alpha, \lambda, \beta) \) and \( \hat{h}_{LI}(\alpha, \lambda, \beta) \), respectively, where

\[
\hat{h}_{SE}(\alpha, \lambda, \beta) = \int_0^1 \int_0^\infty \int_0^\infty h(\alpha, \lambda, \beta) \pi(\alpha, \lambda, \beta | \text{data}) \, d\alpha \, d\lambda \, d\beta,
\]

(3.5)

and

\[
\hat{h}_{LI}(\alpha, \lambda, \beta) = - \left( \frac{1}{u} \right) \log \left[ \int_0^1 \int_0^\infty \int_0^\infty e^{-uh(\alpha, \lambda, \beta)} \pi(\alpha, \lambda, \beta | \text{data}) \, d\alpha \, d\lambda \, d\beta \right].
\]

(3.6)

Since (3.5) and (3.6) can not be solved explicitly, hence we use a numerical method to solve these equations.

### 3.1 MCMC Method

In this subsection, Markov Chain Monte Carlo (MCMC) method is adopted to enumerate the Bayes estimates of unknown parameters \( \alpha, \lambda, \) and \( \beta \) under both the loss functions SEL and LL. In addition, HPD intervals are also composed by using the generated MCMC samples. From the posterior density function given by (3.4), the conditional posterior densities can
be written as

\[
\pi_1(\alpha|\lambda, \beta, data) \propto \alpha^{r+\alpha-1} e^{-\beta \lambda} e^{-\left[(\alpha+1) \sum_{i=1}^{N} \log(t_i)+\lambda \sum_{i=1}^{N} \tau_i\right]} \left[1 - e^{-\lambda(\tau + \frac{\tau r}{\beta})^{-\alpha}}\right]^{(n-r)}
\]

\[
\pi_2(\lambda|\alpha, \beta, data) \propto \lambda^{r+\alpha-1} e^{-\beta \lambda} \left[\sum_{i=1}^{N} t_i^{\alpha} + \sum_{i=1}^{N} \left(\tau + \frac{\tau r}{\beta}\right)^{-\alpha}\right] \left[1 - e^{-\lambda(\tau + \frac{\tau r}{\beta})^{-\alpha}}\right]^{(n-r)},
\]

and

\[
\pi_3(\beta|\alpha, \lambda, data) \propto \frac{1}{\beta^{N+1-r-p}} e^{-\left[(\alpha+1) \sum_{i=1}^{N} \log(t_i)+\lambda \sum_{i=1}^{N} \tau_i\right]} \left[1 - e^{-\lambda(\tau + \frac{\tau r}{\beta})^{-\alpha}}\right]^{(n-r)}.
\]

The above density functions \(\pi_1(\alpha|\lambda, \beta, data)\), \(\pi_2(\lambda|\alpha, \beta, data)\), and \(\pi_3(\beta|\alpha, \lambda, data)\) can not be written in the form of any well known distributions. Therefore, the MCMC samples can not be generated from these densities given in (3.7), (3.8), and (3.9) directly. So, the Metropolis-Hastings algorithm is used to generate MCMC samples from the conditional densities. Then the Bayes estimates can be obtained by using the following steps:

**Step 1:** Choose initial values as \(\alpha^{(1)} = \hat{\alpha}\), \(\lambda^{(1)} = \hat{\lambda}\) and \(\beta^{(1)} = \hat{\beta}\) and set \(i = 1\).

**Step 2:** Generate \(\alpha^{(i)}, \lambda^{(i)}\), and \(\beta^{(i)}\) with normal distribution as \(\alpha^{(i)} \sim N(\alpha^{(i-1)}, \text{var}(\hat{\alpha}))\), \(\lambda^{(i)} \sim N(\lambda^{(i-1)}, \text{var}(\hat{\lambda}))\), and \(\beta^{(i)} \sim N(\beta^{(i-1)}, \text{var}(\hat{\beta}))\).

**Step 3:** Compute \(\Omega_{\alpha} = \min\left(1, \frac{\pi_1(\alpha^{(i)}|\lambda^{(i-1)}, \beta^{(i-1)}, data)}{\pi_1(\alpha^{(i-1)}|\lambda^{(i-1)}, \beta^{(i-1)}, data)}\right)\), \(\Omega_{\lambda} = \min\left(1, \frac{\pi_2(\lambda^{(i)}|\alpha^{(i-1)}, \beta^{(i-1)}, data)}{\pi_2(\lambda^{(i-1)}|\alpha^{(i-1)}, \beta^{(i-1)}, data)}\right)\), and \(\Omega_{\beta} = \min\left(1, \frac{\pi_3(\beta^{(i)}|\alpha^{(i-1)}, \lambda^{(i-1)}, data)}{\pi_3(\beta^{(i-1)}|\alpha^{(i-1)}, \lambda^{(i-1)}, data)}\right)\).

**Step 4:** Generate samples for \(\tau_1, \tau_2\), and \(\tau_3\), where \(\tau_1 \sim \text{Uniform}(0, 1)\), \(\tau_2 \sim \text{Uniform}(0, 1)\), and \(\tau_3 \sim \text{Uniform}(0, 1)\).

**Step 5:** Set

- \(\alpha = \alpha^{(i)}\), if \(\tau_1 \leq \Omega_{\alpha}\); otherwise \(\alpha = \alpha^{(i-1)}\),
- \(\lambda = \lambda^{(i)}\), if \(\tau_2 \leq \Omega_{\lambda}\); otherwise \(\lambda = \lambda^{(i-1)}\),
- \(\beta = \beta^{(i)}\), if \(\tau_3 \leq \Omega_{\beta}\); otherwise \(\beta = \beta^{(i-1)}\).
Step 6: Set $i = i + 1$.

Step 7: Repeat steps 1 to 6, $M$ times to get $\alpha^{(1)}, \ldots, \alpha^{(M)}; \lambda^{(1)}, \ldots, \lambda^{(M)}$, and $\beta^{(1)}, \ldots, \beta^{(M)}$.

Then, the Bayes estimates of $\alpha$, $\lambda$, and $\beta$ under SEL function are given as

$$
\hat{\alpha}_{SE} = \frac{1}{M} \sum_{i=1}^{M} \alpha^{(i)}, \quad \hat{\lambda}_{SE} = \frac{1}{M} \sum_{i=1}^{M} \lambda^{(i)}, \quad \text{and} \quad \hat{\beta}_{SE} = \frac{1}{M} \sum_{i=1}^{M} \beta^{(i)}.
$$

Further, the Bayes estimates of $\alpha$, $\lambda$, and $\beta$ under LL function are given as

$$
\hat{\alpha}_{LI} = -\frac{1}{u} \log \left( \frac{1}{M} \sum_{i=1}^{M} e^{-u\alpha^{(i)}} \right),
$$

$$
\hat{\lambda}_{LI} = -\frac{1}{u} \log \left( \frac{1}{M} \sum_{i=1}^{M} e^{-u\lambda^{(i)}} \right),
$$

and

$$
\hat{\beta}_{LI} = -\frac{1}{u} \log \left( \frac{1}{M} \sum_{i=1}^{M} e^{-u\beta^{(i)}} \right).
$$

Furthermore, to construct $100(1 - \gamma)\%$ HPD credible intervals for $\alpha$, $\lambda$, and $\beta$, we use the method given by [15]. According to the method, the samples are re-arranged in increasing order and these are obtained as $\alpha^{[1]}, \ldots, \alpha^{[M]}; \lambda^{[1]}, \ldots, \lambda^{[M]}$ and $\beta^{[1]}, \ldots, \beta^{[M]}$. Then, the $100(1 - \gamma)\%$ HPD credible intervals are obtained as

$$
\left( \alpha^{[M\gamma/2]}, \alpha^{[M(1-\gamma/2)]} \right), \quad \left( \lambda^{[M\gamma/2]}, \lambda^{[M(1-\gamma/2)]} \right), \quad \text{and} \quad \left( \beta^{[M\gamma/2]}, \beta^{[M(1-\gamma/2)]} \right),
$$

where $\gamma$ is the nominal significance level.
4 Simulation Study

In this section, a simulation study is carried out to compare the performance of different estimates of parameters for Gumbel Type-II distribution under simple SSLT based on Type-II censoring. The performance of estimates are compared on the basis of the average estimate (AE) values and mean squared error (MSE). To observe the changes in the values of the parameters, the simulation study is constructed based on 10,000 Type-II censored samples under simple SSLT. Three different choices of the sample size $n = 50, 150, \text{ and } 250$ are taken to study the behavior of the estimates with change in the sample size. Also, corresponding to each sample size we choose some moderate value for $r$. We have also considered the Gumbel Type-II baseline lifetime distribution with two choices of the shape parameter $\alpha = 1, 1.5$ and the common scale parameter $0.75$. In the case of LL function, we consider the values of $u$ as $-0.05$ and $1$. In Table 1 and Table 2, the AE, MSEs, and average length (AL) of the corresponding asymptotic confidence intervals with coverage probabilities (CP) based on classical estimation are presented with $\tau = 0.6$ and $0.75$, respectively. In Table 3 and Table 4, the average values of Bayesian estimates under SEL and LL function with corresponding MSEs, and AL of confidence intervals with corresponding CP of HPD credible intervals are presented when $\alpha = 1$ along with $\tau = 0.6$ and $0.75$, respectively. In Table 5 and Table 6, the Bayesian estimates under SEL and LL function are presented when $\alpha = 1.5$ along with $\tau = 0.6$ and $0.75$, respectively. From Table 1 and Table 2, the following conclusions have been made:

- For fixed values of $\tau, \beta, \alpha$, and $n$, when values of $r$ increase then in most of the cases the values of MSEs, average length of asymptotic confidence intervals and corresponding coverage probabilities decrease.

- For fixed values of $\tau, \beta, \alpha$, when values of $n$ increase then the average values of estimates, MSEs, and average length of asymptotic confidence intervals decrease.
• For fixed values of $\tau$, $\beta$, $n$, and $r$, when values of $\alpha$ increases then the average value of estimates for $\alpha$ and $\beta$ increase but the average values of estimates of $\lambda$ decrease.

• For fixed values of $\tau$, $\alpha$, $n$, and $r$, when values of $\beta$ increase then the average values of estimates, MSEs and average length of asymptotic confidence intervals decrease but the corresponding coverage probabilities increase.

• For fixed values of $\beta$, $\alpha$, $n$, and $r$, when values of $\tau$ increase then in most of the cases the MSEs, average length of asymptotic confidence intervals and corresponding coverage probabilities decrease.

• For fixed values of $\beta$, $\alpha$, $n$, and $r$, when values of $\tau$ increase then in most of the cases the average values of the estimates of $\alpha$ and $\beta$ decrease but the average values of estimates of $\lambda$ increase.

Now, from Tables 3, 4, 5 and 6 the following conclusions can be made:

• For fixed values of $\tau$, $\beta$, $\alpha$, $n$, and $r$, LL function under $u = 1$ performs better than other estimates. Also in that case, HPD intervals perform better than asymptotic confidence intervals.

• For fixed values of $\tau$, $\beta$, $\alpha$, and $n$, when values of $r$ increase then in most of the cases the values of MSEs and the average length of HPD intervals decrease but the corresponding coverage probability increases.

• For fixed values of $\tau$, $\beta$, and $\alpha$, when values of $n$ increase then MSEs and the average length of HPD intervals decrease.

• For fixed values of $\tau$, $\beta$, $n$ and $r$, when values of $\alpha$ increase then the average values of the Bayes estimates, MSEs and the average length of HPD intervals increase.
For fixed values of $\tau$, $\alpha$, $n$, and $r$, when values of $\beta$ increase then the average values of Bayes estimates, MSEs and average length of HPD credible intervals decrease.

For fixed values of $\beta$, $\alpha$, $n$, and $r$, when values of $\tau$ increase then in most of the cases the MSEs, average length of HPD credible intervals and corresponding coverage probabilities decrease.

5 Real Data Analysis

In this section, we consider a real data set from Lee and Wang [16] to illustrate the estimation methods developed in this paper. The data set contains remission time (in months) of a random sample of 124 bladder cancer patients and the corresponding failure times are given in Table 7. To check the goodness-of-fit of this data to the Gumbel Type-II distribution, K-S test has been employed. From this test, we observe the K-S distance is 0.14243 and the corresponding p-value is 0.01306. Also, for the purpose of goodness-of-fit test, different plots are considered in Figure 1 and Figure 2. Figure 1 represents the comparison between the theoretical CDF of Gumbel Type-II distribution and the empirical CDF and QQ-plot of the given real data set. If $(X_1, \cdots, X_n)$ are $n$ number of i.i.d. random variables with CDF $F(t)$, then the empirical CDF (ECDF) is given as $F_n(t) = \frac{1}{n} \sum_{i=1}^{n} I_{X_i \leq t}$, where $I_W$ denotes the indicator of the event $W$. Here Q-Q plot represents the points $(F^{-1}(i/(n + 1)), x_{(i)})$, where $x_{(i)}$ denotes the ordered data for $i = 1, \cdots, n$. Then Figure 2 represents the comparison between the theoretical density of Gumbel Type-II distribution and the histogram and box-plot of given real data. From the box-plot, it can be concluded that the given distribution is right-skewed.

Different simple step-stress samples based Type-II censoring scheme are considered by using different values of $\tau$, $\beta$ and $r$ when $\alpha = 0.75$ and $\lambda = 2.5$. In Table 8 computed values of the MLEs, Bayes estimates based on SEL and LL functions, average length of asymptotic
Table 1: Simulation results for the Gumbel Type-II baseline lifetime distribution with scale parameter $\lambda = 0.75$ and shape parameter $\alpha = 1.5$ for different choices of $\beta$ when $\tau = 0.6$ under simple SSLT.

| $\tau$ | $\beta$ | $n$ | $r$ | Parameters | $\alpha = 1$ | $\alpha = 1.5$ |
|-------|--------|----|----|------------|--------------|--------------|
|       |        |    |    | AE        | MSE          | AL           | CP          | AE        | MSE          | AL           | CP          |
| 0.6   | 0.35   | 50 | 30 | $\alpha$ | 1.143 0.072 | 0.851 0.943  | 1.806 0.301 | 1.637 0.944 |
|       |        |    |    | $\lambda$ | 0.684 0.032 | 0.678 0.936  | 0.646 0.053 | 0.833 0.901 |
|       |        |    |    | $\beta$  | 0.486 0.061 | 0.839 0.978  | 0.495 0.063 | 0.829 0.981 |
| 40    |        |    |    | $\alpha$ | 1.143 0.068 | 0.816 0.940  | 1.788 0.270 | 1.570 0.943 |
|       |        |    |    | $\lambda$ | 0.682 0.031 | 0.660 0.937  | 0.651 0.051 | 0.817 0.903 |
|       |        |    |    | $\beta$  | 0.496 0.065 | 0.837 0.981  | 0.496 0.064 | 0.829 0.976 |
| 150   | 60     |    |    | $\alpha$ | 1.042 0.017 | 0.468 0.944  | 1.593 0.061 | 0.870 0.949 |
|       |        |    |    | $\lambda$ | 0.729 0.011 | 0.412 0.933  | 0.716 0.018 | 0.515 0.923 |
|       |        |    |    | $\beta$  | 0.406 0.024 | 0.543 0.961  | 0.395 0.014 | 0.435 0.966 |
| 120   |        |    |    | $\alpha$ | 1.046 0.015 | 0.438 0.944  | 1.590 0.054 | 0.812 0.944 |
|       |        |    |    | $\lambda$ | 0.727 0.010 | 0.396 0.926  | 0.716 0.017 | 0.495 0.914 |
|       |        |    |    | $\beta$  | 0.397 0.013 | 0.406 0.961  | 0.395 0.013 | 0.399 0.956 |
| 250   | 100    |    |    | $\alpha$ | 1.025 0.009 | 0.360 0.947  | 1.557 0.033 | 0.663 0.948 |
|       |        |    |    | $\lambda$ | 0.738 0.006 | 0.322 0.939  | 0.729 0.010 | 0.403 0.930 |
|       |        |    |    | $\beta$  | 0.381 0.011 | 0.377 0.956  | 0.378 0.007 | 0.322 0.959 |
| 200   |        |    |    | $\alpha$ | 1.027 0.008 | 0.334 0.943  | 1.555 0.029 | 0.618 0.945 |
|       |        |    |    | $\lambda$ | 0.735 0.006 | 0.309 0.934  | 0.729 0.010 | 0.387 0.928 |
|       |        |    |    | $\beta$  | 0.378 0.007 | 0.303 0.957  | 0.377 0.007 | 0.298 0.953 |
| 0.6   | 0.7    | 50 | 30 | $\alpha$ | 1.097 0.044 | 0.825 0.975  | 1.694 0.144 | 1.557 0.986 |
|       |        |    |    | $\lambda$ | 0.714 0.025 | 0.693 0.946  | 0.691 0.034 | 0.866 0.949 |
|       |        |    |    | $\beta$  | 0.820 0.056 | 1.381 0.978  | 0.831 0.056 | 1.383 0.983 |
| 40    |        |    |    | $\alpha$ | 1.085 0.035 | 0.773 0.980  | 1.665 0.112 | 1.457 0.987 |
|       |        |    |    | $\lambda$ | 0.720 0.023 | 0.674 0.953  | 0.699 0.031 | 0.842 0.960 |
|       |        |    |    | $\beta$  | 0.826 0.056 | 1.377 0.978  | 0.832 0.057 | 1.384 0.980 |
| 150   | 60     |    |    | $\alpha$ | 1.036 0.014 | 0.469 0.956  | 1.576 0.050 | 0.864 0.966 |
|       |        |    |    | $\lambda$ | 0.734 0.010 | 0.469 0.942  | 0.724 0.015 | 0.518 0.948 |
|       |        |    |    | $\beta$  | 0.769 0.040 | 0.966 0.961  | 0.763 0.033 | 0.829 0.972 |
| 120   |        |    |    | $\alpha$ | 1.038 0.012 | 0.435 0.966  | 1.574 0.040 | 0.804 0.972 |
|       |        |    |    | $\lambda$ | 0.732 0.009 | 0.397 0.951  | 0.723 0.014 | 0.497 0.954 |
|       |        |    |    | $\beta$  | 0.769 0.032 | 0.785 0.974  | 0.771 0.032 | 0.779 0.977 |
| 250   | 100    |    |    | $\alpha$ | 1.022 0.008 | 0.359 0.955  | 1.551 0.029 | 0.661 0.958 |
|       |        |    |    | $\lambda$ | 0.740 0.006 | 0.323 0.953  | 0.731 0.010 | 0.403 0.944 |
|       |        |    |    | $\beta$  | 0.745 0.027 | 0.726 0.964  | 0.745 0.023 | 0.632 0.971 |
| 200   |        |    |    | $\alpha$ | 1.025 0.008 | 0.334 0.958  | 1.549 0.025 | 0.615 0.962 |
|       |        |    |    | $\lambda$ | 0.737 0.006 | 0.309 0.946  | 0.731 0.009 | 0.387 0.947 |
|       |        |    |    | $\beta$  | 0.748 0.022 | 0.599 0.976  | 0.747 0.021 | 0.591 0.973 |
Table 2: Simulation results for the Gumbel Type-II baseline lifetime distribution with scale parameter $\lambda = 0.75$ and shape parameter $\alpha = 1.5$ for different choices of $\beta$ when $\tau = 0.75$ under simple SSLT.

| $\tau$ | $\beta$ | $n$ | $r$ | Parameters | $\alpha = 1$ | $\alpha = 1.5$ |
|-------|--------|-----|-----|------------|--------|-------------|
|       |        |     |     | AE     | MSE    | AL     | CP | AE     | MSE    | AL     | CP |
| 0.75  | 0.35   | 50  | 30  | $\alpha$ | 1.107  | 0.047 | 0.734 | 0.947 | 1.683 | 0.137 | 1.215 | 0.942 |
|       |        |     |     | $\lambda$ | 0.705 | 0.024 | 0.605 | 0.918 | 0.699 | 0.029 | 0.659 | 0.908 |
|       |        |     |     | $\beta$  | 0.478 | 0.061 | 0.869 | 0.971 | 0.458 | 0.046 | 0.742 | 0.972 |
| 40    | $\alpha$ | 1.108 | 0.046 | 0.712 | 0.942 | 1.694 | 0.139 | 1.197 | 0.938 |
|       | $\lambda$ | 0.705 | 0.024 | 0.596 | 0.912 | 0.694 | 0.028 | 0.649 | 0.903 |
|       | $\beta$  | 0.476 | 0.055 | 0.783 | 0.977 | 0.464 | 0.045 | 0.699 | 0.979 |
| 150   | 60     |     |     | $\alpha$ | 1.036 | 0.014 | 0.408 | 0.945 | 1.564 | 0.038 | 0.663 | 0.941 |
|       | $\lambda$ | 0.731 | 0.009 | 0.360 | 0.936 | 0.726 | 0.033 | 0.388 | 0.932 |
|       | $\beta$  | 0.407 | 0.028 | 0.599 | 0.953 | 0.409 | 0.033 | 0.664 | 0.942 |
| 120   |       |     |     | $\alpha$ | 1.040 | 0.012 | 0.393 | 0.940 | 1.571 | 0.033 | 0.654 | 0.942 |
|       | $\lambda$ | 0.728 | 0.008 | 0.352 | 0.928 | 0.724 | 0.010 | 0.385 | 0.925 |
|       | $\beta$  | 0.387 | 0.011 | 0.381 | 0.958 | 0.382 | 0.009 | 0.344 | 0.958 |
| 250   | 100    |     |     | $\alpha$ | 1.024 | 0.009 | 0.311 | 0.941 | 1.538 | 0.020 | 0.512 | 0.944 |
|       | $\lambda$ | 0.736 | 0.007 | 0.279 | 0.938 | 0.735 | 0.006 | 0.304 | 0.936 |
|       | $\beta$  | 0.388 | 0.007 | 0.511 | 0.944 | 0.383 | 0.015 | 0.427 | 0.954 |
| 200   |       |     |     | $\alpha$ | 1.024 | 0.007 | 0.302 | 0.943 | 1.538 | 0.020 | 0.304 | 0.942 |
|       | $\lambda$ | 0.736 | 0.005 | 0.275 | 0.935 | 0.735 | 0.006 | 0.304 | 0.936 |
|       | $\beta$  | 0.372 | 0.006 | 0.287 | 0.956 | 0.384 | 0.015 | 0.433 | 0.953 |

| 0.75  | 0.7    | 50  | 30  | $\alpha$ | 1.079 | 0.035 | 0.721 | 0.966 | 1.635 | 0.096 | 1.190 | 0.968 |
|       |        |     |     | $\lambda$ | 0.726 | 0.021 | 0.614 | 0.943 | 0.717 | 0.024 | 0.666 | 0.942 |
|       |        |     |     | $\beta$  | 0.807 | 0.055 | 1.382 | 0.971 | 0.801 | 0.051 | 1.250 | 0.975 |
| 40    | $\alpha$ | 1.074 | 0.030 | 0.692 | 1.311 | 1.630 | 0.083 | 1.160 | 0.976 |
|       | $\lambda$ | 0.726 | 0.020 | 0.630 | 0.945 | 0.720 | 0.023 | 0.661 | 0.948 |
|       | $\beta$  | 0.814 | 0.035 | 1.311 | 0.975 | 0.809 | 0.050 | 1.206 | 0.978 |
| 150   | 60     |     |     | $\alpha$ | 1.030 | 0.011 | 0.407 | 0.955 | 1.554 | 0.030 | 0.669 | 0.956 |
|       | $\lambda$ | 0.735 | 0.008 | 0.361 | 0.941 | 0.732 | 0.009 | 0.393 | 0.944 |
|       | $\beta$  | 0.754 | 0.035 | 0.892 | 0.961 | 0.749 | 0.032 | 0.813 | 0.961 |
| 120   |       |     |     | $\alpha$ | 1.038 | 0.010 | 0.393 | 0.955 | 1.562 | 0.029 | 0.650 | 0.961 |
|       | $\lambda$ | 0.731 | 0.007 | 0.353 | 0.942 | 0.727 | 0.009 | 0.386 | 0.934 |
|       | $\beta$  | 0.756 | 0.030 | 0.742 | 0.968 | 0.753 | 0.026 | 0.678 | 0.972 |
| 250   | 100    |     |     | $\alpha$ | 1.020 | 0.007 | 0.314 | 0.952 | 1.534 | 0.018 | 0.513 | 0.952 |
|       | $\lambda$ | 0.740 | 0.005 | 0.282 | 0.947 | 0.738 | 0.006 | 0.305 | 0.945 |
|       | $\beta$  | 0.742 | 0.027 | 0.712 | 0.958 | 0.737 | 0.024 | 0.660 | 0.962 |
| 200   |       |     |     | $\alpha$ | 1.025 | 0.006 | 0.302 | 0.953 | 1.539 | 0.017 | 0.499 | 0.951 |
|       | $\lambda$ | 0.736 | 0.004 | 0.275 | 0.942 | 0.735 | 0.006 | 0.301 | 0.942 |
|       | $\beta$  | 0.740 | 0.020 | 0.568 | 0.970 | 0.733 | 0.017 | 0.517 | 0.969 |
Table 3: Simulation results of Bayesian estimates for the Gumbel Type-II baseline lifetime distribution with scale parameter $\lambda = 0.75$ and shape parameter $\alpha = 1$ for different choices of $\beta$ when $\tau = 0.6$ under simple SSLT.

| $\tau$ | $\beta$ | $n$ | $r$ | Parameters | SEL (u = 0.05) | LL (u = 0.05) | LL (u = 1) | HPD |
|-------|--------|-----|-----|------------|----------------|----------------|-------------|-----|
| 0.6   | 0.35   | 50  | 30  | $\alpha$  | 1.021 0.015    | 1.022 0.015    | 1.015 0.014  | 0.443 0.962 |
|       |        |     |     | $\lambda$ | 0.737 0.008    | 0.736 0.008    | 0.734 0.008  | 0.338 0.956 |
|       |        |     |     | $\beta$   | 0.393 0.002    | 0.393 0.002    | 0.392 0.002  | 0.133 0.986 |
| 40    |        |     |     | $\alpha$  | 1.027 0.014    | 1.028 0.014    | 1.022 0.013  | 0.358 0.968 |
|       |        |     |     | $\lambda$ | 0.734 0.007    | 0.734 0.007    | 0.732 0.008  | 0.220 0.962 |
|       |        |     |     | $\beta$   | 0.392 0.002    | 0.392 0.002    | 0.391 0.002  | 0.124 0.988 |
| 150   | 60     |     |     | $\alpha$  | 1.014 0.008    | 1.015 0.008    | 1.010 0.007  | 0.293 0.974 |
|       |        |     |     | $\lambda$ | 0.739 0.006    | 0.739 0.006    | 0.737 0.005  | 0.202 0.932 |
|       |        |     |     | $\beta$   | 0.391 0.002    | 0.391 0.002    | 0.389 0.001  | 0.129 0.983 |
| 120   |        |     |     | $\alpha$  | 1.042 0.007    | 1.042 0.007    | 1.039 0.006  | 0.279 0.983 |
|       |        |     |     | $\lambda$ | 0.727 0.005    | 0.726 0.005    | 0.724 0.005  | 0.193 0.957 |
|       |        |     |     | $\beta$   | 0.388 0.001    | 0.388 0.001    | 0.386 0.001  | 0.114 0.989 |
| 250   | 100    |     |     | $\alpha$  | 1.029 0.006    | 1.029 0.006    | 1.026 0.005  | 0.365 0.945 |
|       |        |     |     | $\lambda$ | 0.732 0.004    | 0.732 0.004    | 0.730 0.004  | 0.186 0.944 |
|       |        |     |     | $\beta$   | 0.394 0.002    | 0.394 0.002    | 0.392 0.002  | 0.117 0.973 |
| 200   |        |     |     | $\alpha$  | 1.052 0.006    | 1.052 0.006    | 1.049 0.005  | 0.259 0.947 |
|       |        |     |     | $\lambda$ | 0.721 0.004    | 0.721 0.004    | 0.719 0.004  | 0.188 0.949 |
|       |        |     |     | $\beta$   | 0.435 0.007    | 0.435 0.007    | 0.433 0.007  | 0.208 0.994 |
| 0.6   | 0.7    | 50  | 30  | $\alpha$  | 1.014 0.014    | 1.015 0.014    | 1.008 0.013  | 0.368 0.942 |
|       |        |     |     | $\lambda$ | 0.743 0.007    | 0.744 0.007    | 0.741 0.008  | 0.221 0.931 |
|       |        |     |     | $\beta$   | 0.793 0.009    | 0.793 0.009    | 0.789 0.008  | 0.257 0.967 |
| 40    |        |     |     | $\alpha$  | 1.020 0.012    | 1.020 0.012    | 1.014 0.011  | 0.356 0.944 |
|       |        |     |     | $\lambda$ | 0.739 0.008    | 0.739 0.008    | 0.736 0.008  | 0.225 0.939 |
|       |        |     |     | $\beta$   | 0.792 0.009    | 0.793 0.009    | 0.789 0.008  | 0.256 0.977 |
| 150   | 60     |     |     | $\alpha$  | 1.013 0.008    | 1.013 0.008    | 1.009 0.007  | 0.292 0.952 |
|       |        |     |     | $\lambda$ | 0.743 0.005    | 0.743 0.005    | 0.741 0.005  | 0.201 0.931 |
|       |        |     |     | $\beta$   | 0.788 0.008    | 0.788 0.008    | 0.785 0.007  | 0.249 0.966 |
| 120   |        |     |     | $\alpha$  | 1.2024 0.006  | 1.023 0.006    | 1.020 0.005  | 0.276 0.941 |
|       |        |     |     | $\lambda$ | 0.737 0.005    | 0.738 0.005    | 0.735 0.005  | 0.195 0.956 |
|       |        |     |     | $\beta$   | 0.787 0.008    | 0.787 0.008    | 0.785 0.007  | 0.246 0.973 |
| 250   | 100    |     |     | $\alpha$  | 1.020 0.006    | 1.020 0.006    | 1.017 0.005  | 0.255 0.953 |
|       |        |     |     | $\lambda$ | 0.738 0.004    | 0.738 0.004    | 0.736 0.004  | 0.185 0.946 |
|       |        |     |     | $\beta$   | 0.782 0.007    | 0.783 0.007    | 0.779 0.006  | 0.238 0.983 |
| 200   |        |     |     | $\alpha$  | 1.029 0.004    | 1.029 0.004    | 1.027 0.003  | 0.224 0.963 |
|       |        |     |     | $\lambda$ | 0.732 0.003    | 0.732 0.003    | 0.730 0.003  | 0.176 0.927 |
|       |        |     |     | $\beta$   | 0.782 0.007    | 0.782 0.007    | 0.779 0.006  | 0.237 0.991 |
Table 4: Simulation results of Bayesian estimates for the Gumbel Type-II baseline lifetime distribution with scale parameter $\lambda = 0.75$ and shape parameter $\alpha = 1$ for different choices of $\beta$ when $\tau = 0.75$ under simple SSLT.

| $\tau$ | $\beta$ | $n$ | $r$ | Parameters | SEL | LL ($u = -0.05$) | LL ($u = 1$) | HPD |
|--------|---------|-----|-----|------------|-----|----------------|--------------|-----|
| 0.75   | 0.35    | 50  | 30  | $\alpha$  | 1.009 | 1.010          | 1.004        | 0.366 | 0.938 |
|        |         |     |     | $\lambda$ | 0.743 | 0.743          | 0.740        | 0.222 | 0.949 |
|        |         |     |     | $\beta$   | 0.383 | 0.383          | 0.383        | 0.134 | 0.943 |
| 40     | $\alpha$| 1.013 | 1.014 | 1.007 | 0.359 | 0.946 |
|        | $\lambda$| 0.744 | 0.744 | 0.741 | 0.220 | 0.955 |
|        | $\beta$ | 0.384 | 0.384 | 0.383 | 0.134 | 0.962 |
| 150    | 0.60    | 60  |     | $\alpha$  | 1.027 | 1.028          | 1.023        | 0.310 | 0.941 |
|        | $\lambda$| 0.732 | 0.732 | 0.730 | 0.212 | 0.934 |
|        | $\beta$ | 0.381 | 0.381 | 0.379 | 0.128 | 0.953 |
| 120    | $\alpha$| 1.017 | 1.018 | 1.014 | 0.265 | 0.963 |
|        | $\lambda$| 0.742 | 0.742 | 0.740 | 0.194 | 0.930 |
|        | $\beta$ | 0.380 | 0.380 | 0.379 | 0.127 | 0.964 |
| 250    | 0.100   | 100 |     | $\alpha$  | 1.020 | 1.021          | 1.017        | 0.262 | 0.939 |
|        | $\lambda$| 0.736 | 0.736 | 0.734 | 0.193 | 0.926 |
|        | $\beta$ | 0.378 | 0.378 | 0.377 | 0.124 | 0.947 |
| 200    | $\alpha$| 1.014 | 1.014 | 1.012 | 0.220 | 0.946 |
|        | $\lambda$| 0.741 | 0.741 | 0.739 | 0.177 | 0.957 |
|        | $\beta$ | 0.377 | 0.377 | 0.376 | 0.122 | 0.962 |
| 0.75   | 0.7     | 50  | 30  | $\alpha$  | 1.023 | 1.024          | 1.017        | 0.367 | 0.922 |
|        | $\lambda$| 0.738 | 0.738 | 0.735 | 0.227 | 0.935 |
|        | $\beta$ | 0.756 | 0.756 | 0.754 | 0.175 | 0.957 |
| 40     | $\alpha$| 1.016 | 1.016 | 1.010 | 0.352 | 0.936 |
|        | $\lambda$| 0.742 | 0.742 | 0.739 | 0.223 | 0.942 |
|        | $\beta$ | 0.755 | 0.756 | 0.754 | 0.174 | 0.962 |
| 150    | 0.60    | 60  |     | $\alpha$  | 1.020 | 1.021          | 1.010        | 0.303 | 0.940 |
|        | $\lambda$| 0.737 | 0.737 | 0.735 | 0.211 | 0.932 |
|        | $\beta$ | 0.750 | 0.750 | 0.747 | 0.165 | 0.956 |
| 120    | $\alpha$| 1.023 | 1.023 | 1.019 | 0.264 | 0.958 |
|        | $\lambda$| 0.742 | 0.743 | 0.741 | 0.193 | 0.957 |
|        | $\beta$ | 0.751 | 0.751 | 0.750 | 0.166 | 0.961 |
| 250    | 0.100   | 100 |     | $\alpha$  | 1.015 | 1.014          | 1.012        | 0.254 | 0.931 |
|        | $\lambda$| 0.744 | 0.744 | 0.742 | 0.192 | 0.945 |
|        | $\beta$ | 0.745 | 0.745 | 0.744 | 0.165 | 0.974 |
| 200    | $\alpha$| 1.017 | 1.017 | 1.015 | 0.208 | 0.934 |
|        | $\lambda$| 0.740 | 0.740 | 0.738 | 0.175 | 0.966 |
|        | $\beta$ | 0.746 | 0.746 | 0.745 | 0.155 | 0.983 |
Table 5: Simulation results of Bayesian estimates for the Gumbel Type-II baseline lifetime distribution with scale parameter $\lambda = 0.75$ and shape parameter $\alpha = 1.5$ for different choices of $\beta$ when $\tau = 0.6$ under simple SSLT.

| $\tau$ | $\beta$ | $n$ | $r$ | Parameters | SEL | LL($u = -0.05$) | LL($u = 0.5$) | HPD |
|--------|--------|-----|-----|------------|-----|----------------|----------------|-----|
| 0.6    | 0.35   | 50  | 30  | $\alpha$  | 1.505 | 0.026         | 1.506          | 0.026 | 1.497 | 0.025 | 0.422 | 0.958 |
|        |        |     |     | $\lambda$ | 0.737 | 0.007         | 0.737          | 0.007 | 0.735 | 0.007 | 0.225 | 0.963 |
|        |        |     |     | $\beta$   | 0.394 | 0.002         | 0.394          | 0.002 | 0.393 | 0.002 | 0.125 | 0.975 |
| 40     |        |     |     | $\alpha$  | 1.511 | 0.022         | 1.512          | 0.022 | 1.503 | 0.021 | 0.415 | 0.972 |
|        |        |     |     | $\lambda$ | 0.742 | 0.007         | 0.743          | 0.007 | 0.741 | 0.007 | 0.219 | 0.975 |
|        |        |     |     | $\beta$   | 0.394 | 0.002         | 0.394          | 0.002 | 0.393 | 0.002 | 0.114 | 0.990 |
| 150    | 60     |     |     | $\alpha$  | 1.519 | 0.017         | 1.520          | 0.017 | 1.513 | 0.016 | 0.378 | 0.947 |
|        |        |     |     | $\lambda$ | 0.736 | 0.005         | 0.736          | 0.005 | 0.733 | 0.006 | 0.204 | 0.954 |
|        |        |     |     | $\beta$   | 0.392 | 0.002         | 0.392          | 0.002 | 0.391 | 0.002 | 0.131 | 0.987 |
| 120    |        |     |     | $\alpha$  | 1.538 | 0.012         | 1.539          | 0.012 | 1.533 | 0.011 | 0.352 | 0.962 |
|        |        |     |     | $\lambda$ | 0.735 | 0.005         | 0.735          | 0.004 | 0.733 | 0.005 | 0.196 | 0.943 |
|        |        |     |     | $\beta$   | 0.392 | 0.002         | 0.392          | 0.002 | 0.391 | 0.002 | 0.130 | 0.989 |
| 250    | 100    |     |     | $\alpha$  | 1.522 | 0.012         | 1.523          | 0.012 | 1.517 | 0.011 | 0.334 | 0.965 |
|        |        |     |     | $\lambda$ | 0.740 | 0.004         | 0.741          | 0.004 | 0.738 | 0.004 | 0.185 | 0.946 |
|        |        |     |     | $\beta$   | 0.381 | 0.001         | 0.381          | 0.001 | 0.380 | 0.001 | 0.129 | 0.981 |
| 200    |        |     |     | $\alpha$  | 1.531 | 0.008         | 1.531          | 0.008 | 1.526 | 0.007 | 0.325 | 0.957 |
|        |        |     |     | $\lambda$ | 0.736 | 0.003         | 0.736          | 0.003 | 0.734 | 0.003 | 0.189 | 0.936 |
|        |        |     |     | $\beta$   | 0.381 | 0.001         | 0.381          | 0.001 | 0.379 | 0.001 | 0.128 | 0.983 |
| 0.6    | 0.7    | 50  | 30  | $\alpha$  | 1.511 | 0.025         | 1.511          | 0.025 | 1.503 | 0.024 | 0.434 | 0.935 |
|        |        |     |     | $\lambda$ | 0.733 | 0.008         | 0.733          | 0.008 | 0.731 | 0.008 | 0.229 | 0.947 |
|        |        |     |     | $\beta$   | 0.793 | 0.009         | 0.793          | 0.009 | 0.789 | 0.008 | 0.258 | 0.953 |
| 40     |        |     |     | $\alpha$  | 1.516 | 0.021         | 1.516          | 0.021 | 1.507 | 0.019 | 0.419 | 0.942 |
|        |        |     |     | $\lambda$ | 0.732 | 0.007         | 0.732          | 0.007 | 0.729 | 0.008 | 0.221 | 0.939 |
|        |        |     |     | $\beta$   | 0.794 | 0.009         | 0.794          | 0.009 | 0.790 | 0.008 | 0.253 | 0.974 |
| 150    | 60     |     |     | $\alpha$  | 1.522 | 0.016         | 1.522          | 0.016 | 1.516 | 0.015 | 0.367 | 0.948 |
|        |        |     |     | $\lambda$ | 0.736 | 0.005         | 0.737          | 0.005 | 0.735 | 0.005 | 0.169 | 0.947 |
|        |        |     |     | $\beta$   | 0.789 | 0.008         | 0.789          | 0.008 | 0.786 | 0.007 | 0.251 | 0.958 |
| 120    |        |     |     | $\alpha$  | 1.544 | 0.013         | 1.543          | 0.013 | 1.538 | 0.012 | 0.362 | 0.953 |
|        |        |     |     | $\lambda$ | 0.728 | 0.005         | 0.727          | 0.005 | 0.725 | 0.005 | 0.201 | 0.929 |
|        |        |     |     | $\beta$   | 0.790 | 0.008         | 0.790          | 0.008 | 0.787 | 0.007 | 0.251 | 0.965 |
| 250    | 100    |     |     | $\alpha$  | 1.529 | 0.011         | 1.529          | 0.011 | 1.524 | 0.010 | 0.336 | 0.937 |
|        |        |     |     | $\lambda$ | 0.733 | 0.004         | 0.733          | 0.004 | 0.731 | 0.004 | 0.188 | 0.933 |
|        |        |     |     | $\beta$   | 0.787 | 0.008         | 0.788          | 0.008 | 0.784 | 0.007 | 0.246 | 0.971 |
| 200    |        |     |     | $\alpha$  | 1.549 | 0.009         | 1.549          | 0.009 | 1.544 | 0.008 | 0.332 | 0.940 |
|        |        |     |     | $\lambda$ | 0.724 | 0.003         | 0.724          | 0.003 | 0.721 | 0.004 | 0.186 | 0.947 |
|        |        |     |     | $\beta$   | 0.786 | 0.008         | 0.786          | 0.008 | 0.783 | 0.007 | 0.245 | 0.984 |
Table 6: Simulation results of Bayesian estimates for the Gumbel Type-II baseline lifetime distribution with scale parameter $\lambda = 0.75$ and shape parameter $\alpha = 1.5$ for different choices of $\beta$ when $\tau = 0.75$ under simple SSLT.

| $\tau$ | $\beta$ | $n$ | $r$ | Parameters | SEL $\alpha$ | SEL $\lambda$ | SEL $\beta$ | LL $(u = -0.05)$ $\alpha$ | LL $(u = -0.05)$ $\lambda$ | LL $(u = -0.05)$ $\beta$ | LL $(u = 1)$ $\alpha$ | LL $(u = 1)$ $\lambda$ | LL $(u = 1)$ $\beta$ | HPD $\alpha$ | HPD $\lambda$ | HPD $\beta$ |
|-------|--------|-----|-----|------------|--------------|--------------|-------------|-----------------|-----------------|-----------------|-----------------|-----------------|-----------------|-------------|-------------|-------------|
| 0.75  | 0.35   | 50  | 30  | $\alpha$  | 1.510 0.025 | 1.511 0.025 | 1.502 0.024 | 0.489 0.956 | 0.739 0.007 | 0.739 0.007 | 0.736 0.008 | 0.219 0.916 | 0.384 0.001 | 0.384 0.001 | 0.383 0.001 | 0.135 0.948 |
| 40    | $\alpha$  | 1.513 0.021 | 1.514 0.022 | 1.505 0.021 | 0.413 0.969 | 0.739 0.008 | 0.740 0.008 | 0.737 0.008 | 0.224 0.939 | 0.382 0.001 | 0.382 0.001 | 0.383 0.001 | 0.133 0.967 |
| 150   | 60     | $\alpha$  | 1.513 0.016 | 1.514 0.016 | 1.508 0.015 | 0.365 0.943 | 0.746 0.005 | 0.746 0.005 | 0.744 0.005 | 0.199 0.932 | 0.382 0.001 | 0.382 0.001 | 0.381 0.001 | 0.130 0.948 |
| 120   | $\alpha$  | 1.528 0.012 | 1.529 0.012 | 1.523 0.011 | 0.351 0.952 | 0.738 0.005 | 0.738 0.005 | 0.736 0.005 | 0.197 0.958 | 0.381 0.001 | 0.381 0.001 | 0.380 0.001 | 0.129 0.959 |
| 250   | 100    | $\alpha$  | 1.509 0.012 | 1.510 0.012 | 1.505 0.011 | 0.328 0.955 | 0.746 0.004 | 0.747 0.004 | 0.744 0.004 | 0.184 0.933 | 0.379 0.001 | 0.379 0.001 | 0.378 0.001 | 0.124 0.948 |
| 200   | $\alpha$  | 1.526 0.007 | 1.526 0.007 | 1.523 0.007 | 0.311 0.962 | 0.740 0.003 | 0.740 0.003 | 0.738 0.003 | 0.179 0.942 | 0.378 0.001 | 0.378 0.001 | 0.377 0.001 | 0.121 0.987 |
| 0.75  | 0.7    | 50  | 30  | $\alpha$  | 1.507 0.027 | 1.507 0.027 | 1.499 0.026 | 0.428 0.939 | 0.744 0.008 | 0.744 0.008 | 0.741 0.007 | 0.223 0.935 | 0.755 0.003 | 0.755 0.003 | 0.753 0.003 | 0.175 0.958 |
| 40    | $\alpha$  | 1.516 0.022 | 1.517 0.022 | 1.508 0.021 | 0.402 0.941 | 0.741 0.008 | 0.741 0.008 | 0.738 0.008 | 0.221 0.946 | 0.756 0.003 | 0.756 0.003 | 0.754 0.003 | 0.173 0.966 |
| 150   | 60     | $\alpha$  | 1.525 0.016 | 1.526 0.016 | 1.520 0.015 | 0.360 0.962 | 0.742 0.005 | 0.742 0.005 | 0.740 0.005 | 0.198 0.938 | 0.752 0.003 | 0.752 0.003 | 0.750 0.003 | 0.167 0.969 |
| 120   | $\alpha$  | 1.531 0.013 | 1.532 0.013 | 1.526 0.012 | 0.325 0.934 | 0.734 0.005 | 0.734 0.005 | 0.732 0.005 | 0.196 0.955 | 0.752 0.003 | 0.752 0.003 | 0.750 0.003 | 0.163 0.981 |
| 250   | 100    | $\alpha$  | 1.518 0.012 | 1.518 0.012 | 1.513 0.011 | 0.342 0.933 | 0.743 0.004 | 0.743 0.004 | 0.741 0.004 | 0.183 0.941 | 0.748 0.002 | 0.748 0.002 | 0.746 0.002 | 0.161 0.968 |
| 200   | $\alpha$  | 1.529 0.007 | 1.530 0.007 | 1.526 0.007 | 0.296 0.944 | 0.739 0.003 | 0.740 0.003 | 0.738 0.003 | 0.174 0.964 | 0.748 0.003 | 0.748 0.003 | 0.746 0.002 | 0.160 0.981 |
Table 7: Lifetimes of the failure units are presented for Real Data.

| $\tau$ | $r$ | Stress level | Failure time                                    |
|--------|-----|--------------|------------------------------------------------|
| 2.5    | 50  | 1            | 0.080 0.200 0.400 0.500 0.510 0.810 0.900 1.050 1.190 1.260 1.350 1.400 1.460 1.760 2.020 2.090 2.220 2.260 2.460 |
|        |     | 2            | 2.520 2.560 2.570 2.595 2.615 2.625 2.665 2.685 2.760 2.875 2.905 2.930 2.990 3.010 3.035 3.070 3.100 3.160 3.190 3.340 3.365 3.380 3.415 3.420 3.450 3.500 3.505 3.685 3.740 |
| 60     | 1   |              | 0.080 0.200 0.400 0.500 0.510 0.810 0.900 1.050 1.190 1.260 1.350 1.400 1.460 1.760 2.020 2.090 2.220 2.260 2.460 |
|        |     | 2            | 2.520 2.560 2.570 2.595 2.615 2.625 2.665 2.685 2.760 2.875 2.905 2.930 2.990 3.010 3.035 3.070 3.100 3.160 3.190 3.340 3.365 3.380 3.415 3.420 3.450 3.500 3.505 3.685 3.740 3.780 3.795 3.835 3.910 3.910 3.920 3.955 3.955 3.995 4.060 |
| 4      | 50  | 1            | 0.080 0.200 0.400 0.500 0.510 0.810 0.900 1.050 1.190 1.260 1.350 1.400 1.460 1.760 2.020 2.090 2.220 2.260 2.460 2.540 2.620 2.640 2.690 2.690 2.730 2.750 2.830 2.870 3.020 3.250 3.310 3.360 3.480 3.520 3.570 3.640 3.700 3.820 3.880 |
|        |     | 2            | 4.090 4.115 4.130 4.165 4.170 4.200 4.250 4.255 4.435 4.490. |
| 60     | 1   |              | 0.080 0.200 0.400 0.500 0.510 0.810 0.900 1.050 1.190 1.260 1.350 1.400 1.460 1.760 2.020 2.090 2.220 2.260 2.460 2.540 2.620 2.640 2.690 2.690 2.730 2.750 2.830 2.870 3.020 3.250 3.310 3.360 3.480 3.520 3.570 3.640 3.700 3.820 3.880 |
|        |     | 2            | 4.090 4.115 4.130 4.165 4.170 4.200 4.250 4.255 4.435 4.490 4.530 4.545 4.585 4.660 4.660 4.670 4.705 4.705 4.745 4.810. |
confidence intervals and HPD credible intervals based on the real data set are tabulated. It is observed that the Bayes estimates perform better than MLEs. Further it has been noticed that the HPD credible interval performs better than asymptotic confidence intervals.

Figure 1: (a) ECDF and CDF comparison and (b) QQ-Plot for the Gumbel Type-II distribution fitted to given real data set.

Figure 2: (a) Histogram and theoretical density comparison and (b) Box-plot for Gumbel Type-II distribution fitted to given real data set.
Table 8: Simulation results of Classical and Bayesian estimates and length of asymptotic confidence intervals (ACI) and highest posterior density (HPD) confidence intervals for the Gumbel Type-II baseline lifetime distribution under simple SSLT based on given real data.

| $\tau$ | $\beta$ | $r$ | Para. | MLE | SEL | LL($u = -0.05$) | LL($u = 1$) | ACI | HPD |
|---|---|---|---|---|---|---|---|---|---|
| 2.5 | 0.5 | 50 | $\alpha$ | 0.401 | 0.566 | 0.566 | 0.564 | 0.229 | 0.195 |
|   |   |   | $\lambda$ | 2.623 | 2.497 | 2.497 | 2.495 | 1.041 | 0.229 |
|   |   |   | $\beta$ | 0.118 | 0.518 | 0.518 | 0.517 | 0.201 | 0.054 |
| 60 | 0.5 | 0.5 | $\alpha$ | 0.441 | 0.605 | 0.605 | 0.604 | 0.222 | 0.179 |
|   |   |   | $\lambda$ | 2.633 | 2.414 | 2.415 | 2.413 | 1.039 | 0.242 |
|   |   |   | $\beta$ | 0.123 | 0.533 | 0.533 | 0.533 | 0.188 | 0.056 |
| 0.75 | 50 | 0.5 | $\alpha$ | 0.401 | 0.562 | 0.562 | 0.560 | 0.229 | 0.214 |
|   |   |   | $\lambda$ | 2.624 | 2.630 | 2.631 | 2.628 | 1.041 | 0.234 |
|   |   |   | $\beta$ | 0.177 | 0.780 | 0.780 | 0.779 | 0.302 | 0.067 |
| 60 | 0.5 | 0.5 | $\alpha$ | 0.441 | 0.618 | 0.618 | 0.616 | 0.222 | 0.237 |
|   |   |   | $\lambda$ | 2.633 | 2.533 | 2.533 | 2.530 | 1.039 | 0.262 |
|   |   |   | $\beta$ | 0.185 | 0.769 | 0.769 | 0.768 | 0.283 | 0.060 |
| 4 | 0.5 | 50 | $\alpha$ | 0.505 | 0.570 | 0.570 | 0.568 | 0.219 | 0.214 |
|   |   |   | $\lambda$ | 2.452 | 2.537 | 2.537 | 2.536 | 0.921 | 0.117 |
|   |   |   | $\beta$ | 0.254 | 0.578 | 0.578 | 0.578 | 0.415 | 0.140 |
| 60 | 0.5 | 0.5 | $\alpha$ | 0.525 | 0.658 | 0.658 | 0.655 | 0.214 | 0.204 |
|   |   |   | $\lambda$ | 2.454 | 2.479 | 2.479 | 2.478 | 0.923 | 0.117 |
|   |   |   | $\beta$ | 0.181 | 0.565 | 0.565 | 0.564 | 0.242 | 0.118 |
| 0.75 | 50 | 0.5 | $\alpha$ | 0.514 | 0.699 | 0.699 | 0.697 | 0.219 | 0.189 |
|   |   |   | $\lambda$ | 2.446 | 2.550 | 2.550 | 2.549 | 0.921 | 0.125 |
|   |   |   | $\beta$ | 0.381 | 0.784 | 0.784 | 0.781 | 0.623 | 0.073 |
| 60 | 0.5 | 0.5 | $\alpha$ | 0.525 | 0.681 | 0.681 | 0.680 | 0.214 | 0.120 |
|   |   |   | $\lambda$ | 2.454 | 2.528 | 2.528 | 2.526 | 0.923 | 0.167 |
|   |   |   | $\beta$ | 0.271 | 0.789 | 0.789 | 0.789 | 0.363 | 0.088 |
6 Conclusion

In this paper, we obtained estimates of the unknown model parameters of the Gumbel Type-II distribution under both classical and the Bayesian approaches using TRV modeling for simple SSLT. It has been observed that the MLEs can not be obtained explicitly for both the unknown parameters. Therefore, we used the Newton-Raphson iterative method to compute MLEs by using \( R \) software. Also, we obtained the Bayes estimates based on the symmetric and asymmetric loss functions under the assumption of independent priors. A Monte Carlo simulation study is performed to compare the performance of the estimates in terms of the average values and MSEs. It has been noticed that the Bayes estimates under LL function perform better than the other estimates. The asymptotic confidence intervals and HPD intervals are also obtained. It is noticed that the HPD credible intervals perform better than asymptotic confidence intervals, as expected. Further, a real data set on bladder cancer is considered for illustrative purposes.

The authors declare that they do not have any conflict of interests.

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