An exactly soluble non-perturbative model of the pure gauge QCD is derived as a weak coupling limit of the lattice theory in plaquette formulation \(^1\). The model represents QCD as a theory of the weakly interacting field strength fluxes. The area law behavior of the Wilson loop average is a direct result of this representation: the total flux through macroscopic loop is the additive (due to the weakness of the interaction) function of the elementary fluxes. The compactness of the gauge group is shown to be the factor which prevents the elementary fluxes contributions from cancellation. There is no area law in the non-compact theory.

1 Introduction.

It is well understood that the confinement of quarks can be described by the non-perturbative QCD only. The discovery of asymptotic freedom \(^2\) within perturbative approach to the Yang-Mills theory unambiguously peaks QCD as a genuine model of strong interaction. However, the growth of the effective coupling constant towards infrared region, which is the opposite prediction of asymptotic freedom, makes perturbative theory unreliable in infrared. Since we cannot make sense of the path integrals except the gaussian case, the derivation of the non-perturbative solution of the Yang-Mills model, which is equivalent to the practically impossible summation of all Feynman diagrams, cannot be considered as a realistic goal.

The unique known opportunity to solve the problem non-perturbatively is provided by the Wilson lattice model \(^3\) which is exactly soluble\(^1\) for any coupling constant (i.e., non-perturbatively) on any finite lattice. In two-dimensions, for example, the lattice model easily gives an exact solution \(^4\) \(^5\) in continuum. In \(D > 2\) the problem is much more complicated. However, since the lattice model already contains all of the non-perturbative information, the remaining problem of taking continuum limit of the lattice results, in spite of being also a non-trivial task, is essentially more attractive.

It is very important, in this respect, to formulate the lattice model in the most adequate terms. We shall argue that such formulation is the plaquette formulation performed for \(D = 3\) in \(^6\) (this was given in the form different from that of the pioneering works \(^6\)). This formulation is the starting point for present analysis, and will be briefly reviewed in the next section. The motivation (given also in \(^6\)) is as follows.

The partition function of the Wilson pure gauge model in \(D\) dimensions can be written as

\[
Z = \int \prod_l dU_l \prod_p dU_p e^{\lambda_0 \text{tr}(U_p + U_p^\dagger) \delta(U_p, \prod_{l \in p} U_l)} ,
\]

where \(\lambda_0\) is the bare coupling constant, \(l\) and \(p\) denote links and plaquettes of \(D\)-dimensional lattice, \(U_l\) is the unitary matrix (\(U(N)\) or \(SU(N)\)) attached to \(l\)-th link. The products of \(U_l\)'s are ordered according to the geometrical order of links.

\(^1\)At least, numerically.
The gauge-invariant $\delta$-function is

$$\delta(U, V) = \sum_r \chi_r(U) \chi_r(V),$$

where $r$ is an irreducible representation with the character $\chi_r(U)$ and dimension $d_r = \chi_r(I)$.

The only non-zero observables in this model are invariant ordered products of link variables along closed loops, such as

$$W(C) = \langle \frac{1}{N} \text{tr} \prod_{l \in C} U_l \rangle$$

($C$ is the closed contour). It is understood for a long time that the plaquette variables \[6, 7\] are relevant to solution of the quark confinement problem \[3\]. As argued in \[3\], the area-law behavior of (3),

$$W(C) \sim e^{-\sigma A},$$

where $A$ is the area of the minimal surface $S$ bounded by $C$, and $\sigma$ is the positive parameter (string tension), means confinement of quarks, since it corresponds to the linear potential between colored objects.

The practical use of the idea of dealing with only the loop variables has been first explored by Halpern and Batrouni \[6\] in their field strength and plaquette variable formulation of gauge theory and by Makeenko and Migdal \[7\] in relation to the loop equation (see also \[8\] and references therein). However, in spite of all efforts and the progress achieved in these directions, the solution to the confinement problem has not been found.

We follow here a variant \[1\] of approach \[6\] which is based on such formulation of the model in which the only independent variables of the model are the plaquette matrices $U_p$. This reformulation requires calculation of the integral

$$\int \prod_l dU_l \prod_p \delta(U_p, \prod_{l \in p} U_l)$$

in (1), which is responsible for the interaction between plaquettes.

Physically, the plaquette matrices are nothing but extended to the compact group manifold field strength fluxes (just like link matrices are the group-extended vector-potentials). Confinement, or area law, emerges as the consequence of the weakness of the interaction between fluxes – the total flux through macroscopic loop is the additive function of the elementary fluxes (plaquettes), i.e. proportional to the minimal area. The additiveness of elementary fluxes is the necessary but not sufficient condition for the confinement: one has to find a reason why these contributions are accumulated and not canceled in the total flux. We will find that the reason is the compactness of the gauge group. We will also show that the additiveness is indeed emerges in the $\lambda_o \rightarrow 0$ limit (which is the only interesting limit as it corresponds to the continuum). In this limit, as it is easy to see from (1), the leading contribution comes from the saddle point $U_p = I$, and indeed a certain factorization property has to emerge. This leads to a new model (described below) where the average (3) takes the form

$$W(C) = \langle \prod_{p \in S} \frac{\text{tr} U_p}{N} \rangle$$

where $S$ is arbitrary surface bounded by $C$. In the model we will present, this average is surface-invariant. We emphasize that the average (3) considered in the model \[1\] is not surface-invariant, except abelian case (where \[1\] coincides with \[3\]).

The providers of required surface-invariance are the $\delta$-functions of the integral (3). In paper \[1\] this integral has been computed exactly (for $D = 3$), thus, providing the starting point for present analysis.

The paper organized as follows. In Section 2, we briefly repeat the plaquette formulation of \[1\] and write the result as a statistical model. In Section 3, we suggest factorization of characters in the weak...
coupling limit of the lattice theory and derive the effective model of QCD in arbitrary $D$. In Section 4, we discuss compactness and suggest a new method to compute the averages in the compact theory. We compute the Wilson loop average and string tension in general form and give a new criterion for confinement. We also show that in the mean field approximation our solution indeed gives confinement and reproduces the known results in both strong coupling and perturbative regimes. In Section 5, we summarize and discuss the results. We claim that our results mean the rigorous proof of confinement in QCD. In Appendix, we give some useful group-theoretical formulas relevant to the heat kernel analysis.

2 Lattice QCD$_{3}$ as a statistical model.

After taking integral (5) and substituting it into (1) one obtains the model which is defined on the two-dimensional infinite genus lattice formed by hexagons $h$ as shown in Fig.1.

![Figure 1:](image)

Here, we again temporarily introduce link unitary matrices, which are the plaquette matrices of the original model (1). The partition function is

$$Z = \int \prod_{l} dU_{l} \, e^{i \lambda_{0} \text{tr}(U_{l} + U_{l}^{\dagger})} \prod_{h} \delta\left( \prod_{l_{1} \in h} U_{l_{1}}, \prod_{l_{2} \in h} U_{l_{2}} \right).$$

The $\delta$-function is again given by (2). It is important that the order of hexagon’s links in the product (argument of $\delta$-function) is fixed (as $l_{1}l_{2}l_{1}l_{2}$) so that each character contains either $l_{1}$ or $l_{2}$-type link matrices (in Fig.1(a), they are denoted by either thick or dotted lines respectively). A fragment of the resulting 2d lattice is shown in Fig.1(b). The whole lattice is easy to imagine as obtained from the regular 3d lattice (the lattice spacing is doubled) by replacing links by the tubes (handles) and vertices by the smooth connections of tubes.

Besides this two-dimensional property of the model, there is another important consequence of the plaquette formulation which has not been emphasized in paper [1].

Namely, we notice that the partition function (7) can be written in the form of the statistical model of the integer valued $N$-component field $r$:

$$Z = \sum_{\{r\}} \prod_{j} F(\lambda_{0}; r_{1}(j), ..., r_{8}(j)),$$

where $\{r\}$ means a sum over all configurations of $r$’s with each $r$ being assignment of hexagon; index $j$
labels the 8-hexagon configurations shown in Fig.1(a) and described by

\[ F(\lambda_0; r_1, \ldots, r_8) = \int \prod_{k=1}^{12} dU_k \ e^{i \lambda_0 \text{tr}(U_k^{1}U_k^{\dagger})} \prod_{i=1}^{8} \chi_{r_i}(U_{i,1}U_{i,2}U_{i,3}) \]  

(9)

The integral is taken over 12 links corresponding to the dotted lines in Fig.1(a). Only these links enter the characters under integral. Index \(i\) in the second product of (9) labels the hexagons in the configuration. One should emphasize that the surface of Fig.1(b) separates the three-dimensional space into two identical subspaces, and \(j\) labels the configurations from both subspaces (otherwise there would be no interaction in the model).

This representation unambiguously demonstrates the local character of the interaction between the plaquette matrices \(U_p\) (we remind that the matrices of (9) are originally the plaquette matrices \(U_p\)). The integrals over them enter the partition function via the function \(F\) only. Nothing similar to this result can be observed in the link variables formulation of the model. This locality means that the interaction between plaquette matrices \(U_p\) gives rather weak influence on a long-distance (continuum) phenomena which are, instead, defined by the interaction between \(N\)-component fields \(r\) (irreducible representations). The contribution due to \(U_p\)'s affects only the form of the function \(F\), or eventually, the form of the action, and has to be irrelevant in the continuum, especially for such a phenomenon as confinement.

3 Weak coupling limit of the lattice QCD.

It is well known that the continuum limit corresponds to the weak coupling, \(\lambda_0 \to 0\), limit of the lattice model. This has to appear as a result of the renormalization-group analysis on a lattice, but also can be used in advance. Namely, we take the weak coupling limit at the fixed lattice size, while the refinement of the lattice will be postponed for the later stages.

In the \(\lambda_0 \to 0\) limit, the leading contribution to the function \(F\) comes from the abelian saddle point, \(U_p = I\). In the vicinity of this point, the model still preserves its non-abelian nature since the restriction \(U_p = I\) for the plaquette matrix (which is the product of link matrices, \(U_p = \prod_{l \in p} U_l\)), imposes a slight constraint on each link matrix \(U_l\), and leaves its non-abelian property unbroken.

Technically, near this saddle point, the character of the product can be replaced by the product of characters,

\[ \chi_r(ABC) \to d_r^{-2} \chi_r(A)\chi_r(B)\chi_r(C) \]  

(10)

where factor \(d_r^{-2}\) provides a proper normalization. Several initial terms of the saddle-point expansion of \(F\) are exactly given by the substitution (10). In abelian theory, in particular, both sides of (10) are identical to each other.

After replacement (10), \(D = 3\) partition function takes the form

\[ Z = \sum_\{r\} \prod_c d_r^{2} \prod_p f_p \]  

(11)

where sum goes over all configurations of \(r\)'s, product goes over all cubes \(c\) of the original lattice, and interaction between neighboring cubes \(c_1\) and \(c_2\), on their common plaquette \(p\) is defined by

\[ f_p^{D=3} = \int dU \ e^{i \lambda_0 \text{tr}(U+U^{\dagger})} \chi_{r_1}(U)\chi_{r_2}(U^{\dagger}) d_{r_1} d_{r_2} \]  

(12)

Here and below, we do not distinguish in notations between cubes and representations attached to them. For example, \(r_1\) means representation attached to the cube \(c_1\).

Applying (10) to the Wilson loop average (3), we realize that it takes the form of (6). Such an average, if considered in the model (1), is not surface-invariant (except the \(U(1)\) case). This is easy to see in the large-\(N\) limit where this average factorizes into the one-plaquette averages \(W_o\) with the result...
\[ W(C) = W^A_S \] (\( S \) is arbitrary surface). The reason for this is simple: factorization \( \Pi \) deforms (except the \( U(1) \) case) the \( \delta \)-functions which are the only providers of the surface-invariance in \( \Pi \). Nevertheless, in the model \( \Pi \) (which is obtained from \( \Pi \) by the same deformation \( \Pi \)), as we will check below, the average \( \Pi \) is surface-invariant.

The physics behind the replacement \( \Pi \) is quite simple. Let us remind that the plaquette variables are the group-extended field strength fluxes. The replacement \( \Pi \) manifests the abelian character of interaction between them. Let us emphasize that this property is clearly presents already in the original model \( \Pi \). In the weak coupling limit, this property becomes more transparent. It is this property which provides the first important ingredient of the area law, as it leads to the additiveness (in terms of the algebra) of the elementary fluxes. To make this qualitative picture more precise, we will perform the group Fourier-transform of the model, i.e., complete the formulation in terms of the irreducible representations \( r \) only (\( r \) is the Fourier-image of the unitary matrix \( U \)).

In arbitrary \( D \), even without a step-by-step repetition of the procedure of \( \Pi \), it is easy to understand that after the factorization \( \Pi \), the partition function takes the same form \( \Pi \), with the only difference being in the function \( f \). In \( D = 4 \), \( f \) is the integral of four characters (instead of two in \( D = 3 \)) corresponding to four cubes \( c_p \) sharing plaquette \( p \). In the arbitrary-\( D \) case, \( f \) is the integral of \( 2D - 4 \) characters corresponding to \( 2D - 4 \) cubes sharing each plaquette:

\[
\begin{align*}
 f^D_p &= \int dU \ e^{\frac{\chi}{\alpha} \text{tr} (U + U^\dagger)} \prod_{c_p=1}^{2D-4} \frac{\chi_{r_p}(U)}{d_{r_p}},
\end{align*}
\]

As a remark, it is fruitful to understand how the model \( \Pi \) emerges in the link-variables formulation. Considering cube as a frame (set of its links), one writes its two-dimensional functional of boundaries according to the general formula \( \Pi \):

\[
\begin{align*}
\sum_r d^2_r e^{-\frac{\chi}{2\pi} \text{tr} C_2(r)} \prod_{p=1}^m \chi_{r_p}(U_p),
\end{align*}
\]

where \( C_2(r) \) is the quadratic Casimir eigenvalue. For the cubic frame, \( m = 6 \), and \( A = 0 \). Thus, \( \Pi \) takes the form

\[
\begin{align*}
\sum_r d^{-4}_r \prod_{p=1}^6 \chi_{r_p}(U_p).
\end{align*}
\]

Then, the product of these factors over all cubes, integrated over independent variables \( U_p \)'s with the weights \( e^{\frac{\chi}{\alpha} \text{tr} (U_p + U_p^\dagger)} \), i.e. \( \Pi \),

\[
\begin{align*}
Z = \int \prod_p dU_p \ e^{\frac{\chi}{\alpha} \text{tr} (U_p + U_p^\dagger)} \prod_r \sum_{\rho_c} d^{-4}_{r_c} \prod_{p_c=1}^6 \chi_{r_c}(U_{p_c}),
\end{align*}
\]

exactly coincides with the partition function \( \Pi \).

Now we compute the function \( f \). Irreducible representation \( r \) is parametrized by the Young table parameters (representation’s highest weight components) \( n_\mu (\mu = 1, \ldots, N) \) with the dominance condition \( n_1 \geq n_2 \geq \ldots \geq n_N \). We switch to \( h_\mu = n_\mu - \mu + N \) (\( h_1 > h_2 > \ldots > h_N \)). The formula for dimension of representation is

\[
\begin{align*}
 d_r &= \frac{\Delta(h)}{\Delta(h^0)}, \\
 \Delta(h) &= \prod_{\mu < \nu} (h_\mu - h_\nu),
\end{align*}
\]

(\( \Delta(h) \) is the Vandermonde determinant, \( h^0 = N - \mu \), and the character is:

\[
\begin{align*}
\chi_r(U) &= \frac{\det_{\mu \nu} e^{i\phi_{h_\mu}}}{\Delta(e^{i\phi})} = d_r \frac{\Delta(\phi)}{\Delta(e^{i\phi})} \int dV \ e^{i\text{tr} V \phi V^\dagger h},
\end{align*}
\]

\[3\text{This model, in a different context, was considered in [10].}\]
where $e^{i\phi}$ are eigenvalues of $U$. The second equality in (18) is due to the Itzykson-Zuber formula (11).

Substituting this formulas to the equation (12), we have

$$ f_p^{D=3} = \frac{\det_{\mu\nu}I_{h_\nu(r_1) - h_\mu(r_2)}(2N/\lambda_0)}{d_{r_1}d_{r_2}}. $$

where $I_n$ is the modified Bessel function. Replacing $I_n(2N/\lambda_0)$ by its $\lambda_0 \to 0$ asymptotics,

$$ I_n\left(\frac{2N}{\lambda_0}\right) \sim e^{-\frac{2N}{\lambda_0}}, $$

we write (19) as

$$ f_p^{D=3} = e^{-\frac{\lambda_0}{2N} tr(h_1^2 + h_2^2)} \det_{\mu\nu} e^{-\frac{\lambda_0}{N} h_{1,\mu}h_{2,\nu}} \frac{\Delta(h_1)\Delta(h_2)}{\Delta(h)} . $$

In the Appendix, we show that substitution (20) is equivalent to the replacing Wilson action $\frac{N}{\lambda_0} tr(U + U^\dagger)$ by the heat kernel $\sum_{\rho} d\rho e^{-\frac{\lambda_0}{2N} C_2(\rho)} \chi_\rho(U)$. This replacement is also equivalent to the saddle point $(\lambda_0 \to 0)$ expansion around $\phi = 0$ ($U = I$) in (12).

In arbitrary $D$, substituting (17) and (18) to (13), we have

$$ Z = \sum_{\{h\}} \prod_c \Delta^2(h(c)) \prod_p \int dV_p \ e^{-\frac{\lambda_0}{N} tr(\sum_c V_p h_{cp} V_p^\dagger)^2} . $$

(we have switched to the notation $h_{ij} \equiv h(c_{ij})$). Geometrically, formula (24) can be represented as a tetrahedron with vertices $j$, links $(i,j)$ and triangles $k$, with corresponding assignment of matrices $h_{ij}$, $U_{ij} = V_i V_j^\dagger$ and the conditions at the triangles.

As one can see from this example, the reduction of (23) to the eigenvalues, like (21), is highly non-trivial in $D > 3$. Exactly the same problem appears in the matrix models of 2d quantum gravity in the case when embedding target space corresponds to the physical situation of central charge $c > 1$. In spite of the efforts made in recent years the solution to such problems is unknown, though some important physical information can be extracted in the simplest physical case of only one constraint (this corresponds to $c = 1$), considered in [12] (see also review [13] and references therein).

In spite of the highly non-gaussian character of $f_p$ (23) in terms of eigenvalues $h$, as one can see in examples of (21) and (24), $f_p$ is the gaussian function in terms of the full hermitian matrices $H = VH^\dagger$.

We shall use $f_p$ in the general form (23). The resulting partition function is

$$ Z = \sum_{\{h\}} \prod_c \Delta^2(h(c)) \prod_p \int dV_p \ e^{-\frac{\lambda_0}{N} tr(\sum_c h_{cp})^2} . $$

($c_p$ is the cube containing plaquette $p$). The compactness of the gauge group is reflected in the integrerness, or rather discreteness, of eigenvalues $h$. 

6
4 Compactness, loop averages and string tension.

As we already noticed, the Wilson loop average (3) takes the form (3) in the model (27). At the plaquettes \( p \in S \), \( f_p \) has to be replaced by

\[
f'_p = \int \prod_{c_p} dV_{c_p} e^{-\frac{\lambda}{N} \text{tr} \left( \sum_{c_p} H_{c_p} + H(f) \right)^2}.
\]

(26)

The eigenvalues of matrix \( H(f) \) are the fundamental representation components; \( \text{tr} H^2(f) = C_2(f) \) where \( C_2(f) \) is quadratic Casimir of the fundamental representation: \( C_2(f) = N \) for \( U(N) \) and \( C_2(f) = N - \frac{1}{N} \) for \( SU(N) \). Then,

\[
W(C) = \langle \prod_{p \in S} f'_p \rangle = e^{-\frac{\lambda C_2(f)}{2N}} A(S) \langle \prod_{p \in S} \prod_{c_p} e^{-\frac{\lambda}{N} \text{tr} H_{c_p} H(f)} \rangle,
\]

(27)

where averaging \( \langle ... \rangle \) is understood in the sense of (25), \( A \) is the lattice area of \( S \).

The average (27) is manifestly surface-invariant. To check this, one chooses another surface \( S' \) and makes the shift \( H \rightarrow H - H(f) \) in the 3-volume connecting surfaces \( S \) and \( S' \) (in \( D = 3 \) this is simply a volume bounded by the compact surface \( S \cup S' \). The result is again (27) but with the surface \( S' \). The global minimum of the action corresponds to the \textit{minimal} surface. However, one cannot vary over integer-valued variables.

We apply now the generalization of the idea [14] realized in two-dimensional large-\( N \) theory. In the non-compact theory, \( h \) is continuous variable. Then, the sum over \( h \)-configurations in (23) turns into the path integral \( \int \mathcal{D}h(x) \) with the measure \( \mathcal{D}h = \Delta^2(h) \prod_k dh_k \). The dominance constraint is automatically satisfied due to the antisymmetry of the Vandermonde determinant. In the compact theory, we still can redefine the nature of variables assuming that the effect of compactness can be accumulated into the new coupling constant \( \lambda \). The definition of \( \lambda \) is

\[
\sum_{\{h\}} \prod_c \Delta^2(h(c)) \prod_p \int \prod_{c_p} dV_{c_p} e^{-\frac{\lambda}{N} \text{tr} \left( \sum_{c_p} H_{c_p} \right)^2} = \int \prod_c \mathcal{D}h(c) \prod_p \int \prod_{c_p} dV_{c_p} e^{-\frac{\lambda}{N} \text{tr} \left( \sum_{c_p} H_{c_p} \right)^2}.
\]

(28)

In other words, the model takes the form

\[
Z = \int \prod_c \mathcal{D}h(c) \prod_p \int \prod_{c_p} dV_{c_p} e^{-\frac{\lambda}{N} \text{tr} \left( \sum_{c_p} H_{c_p} \right)^2}.
\]

(29)

Since in the theory defined by (29) one can vary over \( H \), the minimal area (which corresponds to the maximum of the integrand) becomes distinguished. We have

\[
W'(C) = e^{-\frac{(\lambda_0 - \lambda) C_2(f)}{2N} A_{\text{min}}} \langle \prod_{p \in S_{\text{min}}} \prod_{c_p} e^{-\frac{\lambda}{N} \text{tr} H_{c_p} H(f)} \rangle_{\text{non-compact}} \equiv e^{-\frac{(\lambda_0 - \lambda) C_2(f)}{2N} A_{\text{min}}} W'(C),
\]

(30)

where \( W'(C) \) is the average in the non-compact theory with coupling constant \( \lambda \) defined by (28).

In \( D = 2 \) case of the infinite plane,

\[
W'(C) = e^{-\frac{\lambda C_2(f)}{2N} A} \quad \Rightarrow \quad W(C) = e^{-\frac{\lambda_0 C_2(f)}{2N} A}
\]

(31)

for any \( N \), including abelian theory, in agreement with the known result [4]. It is also not difficult to check that in the topologically non-trivial \( D = 2 \) cases, one can reproduce the results of [5].

In \( D > 2 \), \( W'(C) \) contains no area term. According to the definition \( W(C) \sim e^{-\sigma A_{\text{min}}} \), equation (30) gives the dimensionless (since we have a dimensionless area) string tension

\[
\sigma = \frac{\lambda_0 - \lambda C_2(f)}{2N},
\]

(32)
where \( \lambda \) is defined by equation (28). In general, \( \lambda \) is the function of all parameters of the theory: \( \lambda_0, N \) and \( D \).

As an example, we demonstrate that in the case of the compact QED\(_3\), (32) reproduces the result of \[15\]. Equation (28) takes the form

\[
\sum_{\{n\}} e^{-\frac{\lambda_0}{2} \sum_{(c_i,c_j)} (n_i-n_j)^2} = \int \prod_c dh_c e^{-\frac{\lambda}{2} \sum_{(c_i,c_j)} (h_i-h_j)^2} \cdot \tag{33}
\]

In the mean-field approximation we have

\[
\int dh e^{-\frac{\lambda}{2} h^2} = \sum_n e^{-\frac{\lambda_0}{2} n^2} = \int dh e^{-\frac{\lambda}{2} h^2} \sum_m e^{-\frac{\lambda_0}{2} m^2} \tag{34}
\]

(the second equality is the Poisson resummation formula; \( m \) is integer). Thus,

\[
\frac{1}{\sqrt{\lambda}} = \frac{1}{\sqrt{\lambda_0}} \sum_m e^{-\frac{\lambda_0}{2} m^2} \tag{35}
\]

and

\[
\sigma_{U(1)} = \frac{\lambda_o - \lambda}{2} = \frac{\lambda_0}{2} \left(1 - \frac{1}{(\sum_m e^{-\frac{\lambda_0}{2} m^2})^2}\right), \tag{36}
\]

i.e., gives both small-\( \lambda_0 \) result of \[15\]: \( \sigma \sim \lambda_0 e^{-\frac{\lambda_0^2}{2}} \), and the known large-\( \lambda_0 \) behavior: \( \sigma \sim \frac{1}{\lambda_0} \).

It is also clear that the qualitatively similar \( \lambda_0 \to 0 \) and \( \lambda_0 \to \infty \) behaviors take place in the non-abelian case and thus coincide with the known results of perturbative analysis and strong coupling expansion.

Similarly to (30), the loop average in representation \( R \) is

\[
W_R(C) \equiv \langle \frac{\chi_R(U(C))}{d_R} \rangle \sim e^{-\sigma_{U(1)}^R C_R^2 A_{\text{min}}}. \tag{37}
\]

This gives a possibility to compute the arbitrary average,

\[
\xi(C) \equiv \langle \xi(U) \rangle = \sum_R \xi_R W_R(C), \tag{38}
\]

where

\[
\xi_R = \int dU \xi(U) \chi_R(U^\dagger) \tag{39}
\]

is the Fourier coefficient of \( \xi(U) \).

5 Summary and discussion.

Thus, taking the \( \lambda_0 \to 0 \) limit of QCD in plaquette formulation \[1\], we derived the non-perturbative model \[23\], which is formulated in terms of observables only (loop variables) and therefore is much more suitable than the model \[1\] for the analysis of physics of QCD. This model is shown to be exactly soluble in some particularly interesting cases relevant to the continuum. Although the model \[23\] is obtained as the \( \lambda_0 \to 0 \) limit of the Wilson model \[1\], we suggest that it also describes the lattice QCD in the non-empty interval \( \lambda_0 \in [0, \lambda^*_0] \). By definition, \( \lambda^*_0 \) is the upper limit where the model \[25\] is still equivalent to \[1\]. In the \( U(1) \) case, for example, \( \lambda^*_0 = \infty \) since \( 23 \) coincides with \[1\].

We computed continuum Wilson loop average and obtained the area law. Since equation (28) gives \( \lambda_0 > \lambda \), the string tension \( (32) \) is non-vanishing. We consider this result as eventual proof of confinement in QCD.
Qualitatively, the picture of confinement can be given in terms of elementary fluxes, which are the plaquette variables. Then, QCD is a theory of weakly interacting fluxes. The total flux through any surface is the additive function of elementary fluxes, which provides the necessary ingredient for the area law. The interaction between fluxes provides the surface-invariance of the averages like (6), which is the eventual result of the global gauge invariance.

The reason which turns this additiveness into the area law (and thus provides the sufficient ingredient) is compactness of the gauge group (reflected in $\lambda_0 > \lambda$ solution of the equation (28) and thus into the non-vanishing string tension). A solution of the equation (28) is non-trivial problem in general. Here, we only demonstrated that it gives proper behavior for the string tension both in the $\lambda_0 \to 0$ and in the $\lambda_0 \to \infty$ limits, and thus provides $\lambda_0 > \lambda$ for any $\lambda_o$.

The additiveness of fluxes takes place also in the model (1) and becomes more transparent in the model (25) due to the factorization of characters (10). The factorization also provides the crucial simplification of the model (as one might expect in the continuum limit) and its solubility. As we already mentioned, the reason why the model remains non-trivial under this replacement lays in the nature of plaquette formulation: the condition $U_p \equiv \prod U_l = I$ imposes a slight restriction on matrices $U_l$. Thus one learns that even within the gaussian model the non-trivial results can be obtained. Recently, similar idea was argued by I.Kogan and A.Kovner [16].

There are several aside remarks.

There is a complete algebraic similarity with the principal chiral field model (PCF) defined in the lower dimensions than QCD. The above considerations can be easily carried on for PCF: the PCF partition function is given by $\text{QCD}_{D-1}$ (25) where the cubes and plaquettes should be respectively replaced by the plaquettes and links 4. The Wilson loop (3) corresponds to the two-point PCF correlator $G_{xy}$. The plaquette formulation [1] becomes the link formulation for PCF, and $G_{xy}$ takes the form of (27), where plaquettes of the surface $S$ should be replaced by the links of arbitrary curve $L_{xy}$ connecting points $x$ and $y$. To prove that another curve $L'_{xy}$ in $D = 2$ produces the same result, one makes the constant shift $H \to H + H(f)$ in all plaquettes of the surface enclosed by $L_{xy} \cup L'_{xy}$. To prove the invariance in $D = 3$, one has to make such a shift in all plaquettes of the arbitrary surface enclosed by $L_{xy} \cup L'_{xy}$. We actually used the analogy between QCD and PCF to check the surface-invariance of the loop average (27) in QCD4.

It is easy to recognize in $D = 3$ version of (29) the Kazakov-Migdal model [17]. Even though the coincidence takes place only in $D = 3$, the observation is quite peculiar. The idea of [17] to interpret the diagonalizing unitary matrices $U_{xy}$ as a prototype of the QCD gauge field was dismissed due to the observation [3, 18] that $U_{xy}$-built averages are trivial due to the local $Z_N$ symmetry of the model. However, as we now understand, the KM master field (i.e., solution of the large-$N$ saddle point equation for the partition function) is indeed true QCD master field in $D = 3$ (and only in $D = 3$), while $U_{xy}$ has nothing to do with the QCD gauge field (the actual gauge field of QCD is represented by its Fourier-images, $h_x$ fields). Thus, the KM master field could be directly used in the large-$N$ limit of the model (29) in $D = 3$. Unfortunately, this is not found.

It is still a challenging problem to extend the present analysis to the full version of the model which includes the matter fields.

The highly non-trivial problem is the renormalization-group analysis and computation of the non-perturbative beta-function. I suppose this can be achieved in the computational direction provided by the equation (28).

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Appendix: Heat kernel and $\lambda_0$-expansion.

In this Appendix we give some useful formulas and establish a connection between the technique of Section 3 and the heat kernel technique.

In derivation of (21), the Bessel function has been replaced by its $\lambda_0 \to 0$ asymptotics. We now demonstrate that this is an equivalent of the replacement of the Wilson action $\exp \frac{1}{\lambda_0} \text{tr} (U + U^\dagger)$ by the heat kernel [4],

$$\sum_R d_R e^{-\frac{\lambda_0}{2N} C_2(R)} \chi_R (U) .$$  \hfill (40)

Expression (12) after substitution (40) takes the form

$$f_p = \frac{1}{d_{r_1} d_{r_2}} \sum_R d_R e^{-\frac{\lambda_0}{2N} C_2(R)} D^R_{r_1 r_2} ,$$  \hfill (41)

where $D^R_{r_1 r_2}$ is the multiplicity of representation $R$ in the tensor product $r_1 \otimes r_2$,

$$D^R_{r_1 r_2} = \int dU \chi_R (U^\dagger) \chi_{r_1} (U) \chi_{r_2} (U^\dagger) ,$$  \hfill (42)

$$\chi_{r_1} (U) \chi_{r_2} (U^\dagger) = \sum_R D^R_{r_1 r_2} \chi_R (U) .$$  \hfill (43)
The equivalence between (21) and (41) can be easily established in their $\lambda_o$-expansions. Expansion of Itzykson-Zuber integral entering (21) is

$$\int du \, e^{-\frac{1}{N} \sum_{\mu \nu} a_{\mu \nu} b_{\mu \nu}} = \sum_{n} \frac{(-\lambda_o)^n}{n! N^n} \int du \, tr^n(AUBU)$$

(44)

(where $a$ and $b$ are the diagonal matrices of the eigenvalues of $A$ and $B$). The low orders of this expansion can be derived using explicit expressions for powers of traces $tr^n(AUBU)$ via the characters $\chi_r(AUBU)$ and formula

$$\int du \chi_r(AUBU) = \frac{1}{d_r} \chi_r(A) \chi_r(B).$$

(45)

We have

$$\sum_{\mu \nu} a_{\mu \nu} b_{\mu \nu} = 1 - \lambda_o \sum_{\mu \nu} a_{\mu \nu} b_{\mu \nu} +$$

$$\sum_{\mu \nu} \frac{\lambda^2}{2N^3(N-1)} \left( N tr^2 A tr^2 B + N tr A tr B - tr^2 A tr B^2 - tr A^2 tr B^2 \right) + O(\lambda_o^3),$$

(46)

where to derive $O(\lambda_o^3)$ term we have used

$$\chi_{20\ldots 0}(V) = \frac{1}{2} (tr^2 V + tr V^2), \quad \chi_{110\ldots 0}(V) = \frac{1}{2} (tr^2 V - tr V^2),$$

(47)

and thus,

$$tr^2 V = \chi_{20\ldots 0}(V) + \chi_{110\ldots 0}(V).$$

(48)

The higher order terms can be derived similarly. Expansion (46) is then to be substituted into (21).

Expansion of (41) can be derived by technique of (21) which is based on the replacement of $-C_2(R)$ by the group Laplace operator $tr \partial^2_U$,.

$$f_p = \frac{1}{d_{r_1} d_{r_2}} e^{\frac{\lambda_o}{2N} tr \partial^2_U \chi_{r_1}(U) \chi_{r_2}(U^\dagger)} \bigg|_{U=I}.$$

(49)

Applying $tr \partial^2_U$ to both sides of (43) one derives the identity

$$\sum_{r} D_{r_1 r_2 r}^R d_{r} C_2(R) = C_2(r_1) + C_2(r_2) - \frac{2}{N} C_1(r_1) C_1(r_2),$$

(50)

which gives the order $O(\lambda_o)$ of $f_p$. $C_1(r)$ is the first (linear) Casimir eigenvalue. To obtain the next order we apply $tr \partial^2_U$ twice to (43) and get

$$\sum_{r} D_{r_1 r_2 r}^R d_{r} C_2^2(R) = \left( C_2(r_1) + C_2(r_2) - \frac{2}{N} C_1(r_1) C_1(r_2) \right)^2 +$$

$$\frac{\lambda^2}{N^2} \left( C_2(r_1) - \frac{1}{N} C_1^2(r_1) \right) \left( C_2(r_2) - \frac{1}{N} C_1^2(r_2) \right),$$

(51)

etc. Thus, for (41), one has

$$f_p = 1 - \frac{\lambda_o}{N} \left( C_2(r_1) + C_2(r_2) - \frac{2}{N} C_1(r_1) C_1(r_2) \right) +$$

$$\frac{\lambda^2}{N^2} \left( C_2(r_1) + C_2(r_2) - \frac{2}{N} C_1(r_1) C_1(r_2) \right)^2 +$$

$$\frac{\lambda^2}{N^2(N^2-1)} \left( C_2(r_1) - \frac{1}{N} C_1^2(r_1) \right) \left( C_2(r_2) - \frac{1}{N} C_1^2(r_2) \right) + O(\lambda_o^3).$$

(52)

Taking into account explicit expressions for Casimir eigenvalues,

$$C_1(r) = \sum_{k=1}^{N} h_k = tr h,$$

(53)
\[ C_2(r) = \text{tr} h^2 - R \],
\[ (R = \frac{N(N^2-1)}{21}) \text{ and definition of } h_\mu \text{ is changed to } h_\mu = n_\mu - \mu + \frac{N+1}{2} \] one checks order-by-order agreement between \[(54)\] and \[(55)\] (with the substitution of \[(51)\] and \[(52)\])

Thus, the heat-kernel Young table parameters \( h_\mu \) play the role of the eigenvalues of the hermitian matrix of Section 3, and the subsequent equivalence between heat-kernel and matrix-model approaches one establishes by identification: \( C_2(r) \) with \( \text{tr} h^2 \), \( C_1(r) \) with \( \text{tr} h \), \( d_r \) with \( \Delta(h) \) etc. The \( \lambda_0 \)-expansion \[(46)\], or \[(52)\], shows that \( f_p \) is non-singular symmetric function of \( h_\mu \).

There are some changes for \( SU(N) \) group with respect to the general unitary group \( U(N) \). In particular, Casimirs of \( SU(N) \) can be obtained from Casimirs of \( U(N) \) by the replacement \( h_\mu \to h'_\mu = h_\mu - \frac{1}{N} \text{tr} h \). Thus, for \( SU(N) \),
\[ C_1(r) = 0 \] \[ C_2(r) = \text{tr} h^2 - \frac{1}{N} \text{tr}^2 h \] (we ignore the additive constant \( R \)). Thus, hermitian matrix \( h \) of Section 3 is the traceless matrix if the gauge group is \( SU(N) \).

Correspondingly, for \( SU(N) \), the formula \[(51)\] takes the form
\[ \frac{1}{d_{r_1} d_{r_2}} \sum_R D^R_{r_1 r_2} d_R C_2^2(R) = \left( C_2(r_1) + C_2(r_2) \right)^2 + \frac{4}{N^2 - 1} C_2(r_1) C_2(r_2) , \] while instead of \[(52)\] we have

\[ f_p = 1 - \frac{\lambda_0}{2N} \left( C_2(r_1) + C_2(r_2) \right) + \frac{\lambda_0^2}{4N^2} \left( C_2(r_1) + C_2(r_2) \right)^2 + \frac{\lambda_0^2 C_2(r_1) C_2(r_2)}{N^2(N^2 - 1)} + O(\lambda_0^3) . \] In applications, the generalization of the formula \[(50)\] to the case of arbitrary number of characters can be useful:
\[ \sum_R d_R C_2(R) \int dU \prod_j \frac{\chi_j(U)}{d_{r_j}} = \sum_j C_2(r_j) + \frac{2}{N} \sum_{i<j} C_1(r_i) C_1(r_j) . \]