The Ehrenfest Theorem in Quantum Field Theory

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Abstract

The validity of the Ehrenfest’s theorem in Abelian and non-Abelian quantum field theories is examined. The gauge symmetries are taken to be unbroken. By suitably choosing the physical subspace, the above validity is proven in both the cases.

Key words: Ehrenfest’s theorem - Schrödinger equation - expectation values - Dirac equation - Abelian field theory - gauge fixing - quantum lagrangian - physical subspace - Non-Abelian field theory - Quantum Chromo Dynamics - path integral approach - Faddeev-Popov ghosts - quantum equations - BRS transformation - Global gauge and scale transformations - physical states.

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(Dedicated to the memory of Prof. Alladi Ramakrishnan)

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Professor Alladi Ramakrishnan founded the Institute of Mathematical Sciences (MATSCIENCE) in 1962 and attracted bright young students interested in theoretical physics. His contributions to the theory of Stochastic
processes, elementary particle physics and Generalized Clifford Algebras will be remembered forever. He was instrumental in my joining MATSCIENCE in 1977 and encouraged me till his end in my research work. I consider it my duty to dedicate this article in his memory.

1. Quantum Mechanics

In quantum mechanics, it is reasonable to expect the motion of a wave packet to agree with the motion of the corresponding classical particle whenever the potential energy changes by a small amount over the dimensions of the wave packet. If we mean by the ‘position’ and the ‘momentum’ vectors of the wave packet, their expectation values, then we can show that the classical and the quantum motions agree. This important result is known as Ehrenfest’s theorem [1,2]. To illustrate this theorem, let us first consider non-relativistic quantum mechanics. We have the Schrödinger equation

\[ \begin{align*}
    i\hbar \frac{\partial \psi(\vec{x}, t)}{\partial t} &= -\frac{\hbar^2}{2m} \nabla^2 \psi(\vec{x}, t) + V(\vec{x}) \psi(\vec{x}, t), \\
    -i\hbar \frac{\partial \psi(\vec{x}, t)^\dagger}{\partial t} &= -\frac{\hbar^2}{2m} \nabla^2 \psi(\vec{x}, t)^\dagger + V(\vec{x}) \psi(\vec{x}, t)^\dagger,
\end{align*} \]

(1)

where \( m \) is the mass of the particle and \( V(\vec{x}) \) is the real potential.

We shall take the wave function \( \psi(\vec{x}, t) \) in (1) as normalized. Then the expectation value of the \( x \)-component of the position operator and its time derivative are

\[ \begin{align*}
    \langle x \rangle &= \int \psi^\dagger x \psi \, d\tau, \\
    \frac{d}{dt} \langle x \rangle &= \int (\frac{d\psi^\dagger}{dt}) x \psi \, d\tau + \int \psi^\dagger x (\frac{d\psi}{dt}) \, d\tau.
\end{align*} \]

(2)

Using (1), it follows

\[ \frac{d}{dt} \langle x \rangle = -\frac{i\hbar}{m} \int \psi^\dagger \frac{\partial}{\partial x} \psi \, d\tau = \frac{1}{m} \langle p_x \rangle. \]

(3)

Similarly starting from \( \langle p_x \rangle = -i\hbar \int \psi^\dagger \frac{\partial}{\partial x} \psi \, d\tau \), it is easy to find

\[ \frac{d}{dt} \langle p_x \rangle = \langle -\frac{\partial V(\vec{x})}{\partial x} \rangle. \]

(4)
From (2) and (4), we note that the classical equations of motion
\[
\frac{d\vec{x}}{dt} = \frac{\vec{p}}{m}; \quad \frac{d\vec{p}}{dt} = -\vec{\nabla}V(\vec{x}),
\] (5)
are satisfied by their expectation values in quantum mechanics. The wave packet moves like a classical particle whenever the expectation value gives a good representation of the classical variable. They provide an example of the correspondence principle [1,2].

In the case of relativistic quantum mechanics, the manipulations are a little less direct. We consider the Dirac equation [3].
\[
H = \vec{\alpha} \cdot \vec{p} + \beta m, \quad i\hbar \frac{\partial \psi}{\partial t} = (\vec{\alpha} \cdot \vec{p} + \beta m)\psi,
\] (6)
where $\vec{\alpha}$ and $\beta$ are hermitian $4 \times 4$ matrices and $\psi$ is a $4 \times 1$ column vector. We shall set the velocity of light $c$ to unity hereafter. By using the Heisenberg equation of motion $\frac{dx}{dt} = \frac{1}{i\hbar} [x, H]$, it is seen that
\[
\int \psi^\dagger \frac{dx}{dt} \psi d\tau = \int \psi^\dagger \alpha_x \psi d\tau.
\] (7)
First, we recall the plane wave solutions $\psi^{(i)}$ [3] of the Dirac equation,
\[
\psi^1(x) = \sqrt{\frac{E + m}{2m}} \ e^{-ipx} \begin{pmatrix} 1 \\ 0 \\ \frac{p_z}{(E+m)} \\ \frac{p_x}{(E+m)} \end{pmatrix}; \quad \psi^2(x) = \sqrt{\frac{E + m}{2m}} \ e^{ipx} \begin{pmatrix} 0 \\ 1 \\ \frac{p_z}{(E+m)} \\ \frac{p_x}{(E+m)} \end{pmatrix},
\]
\[
\psi^3(x) = \sqrt{\frac{E + m}{2m}} \ e^{ipx} \begin{pmatrix} \frac{p_z}{(E+m)} \\ \frac{p_x}{(E+m)} \\ 1 \\ 0 \end{pmatrix}; \quad \psi^4(x) = \sqrt{\frac{E + m}{2m}} \ e^{ipx} \begin{pmatrix} \frac{p_z}{(E+m)} \\ \frac{p_x}{(E+m)} \\ 0 \\ 1 \end{pmatrix},
\]
corresponding to positive energy ($E > 0$) spin-up, spin-down states and negative energy ($E < 0$) spin-up, spin-down states of the electron respectively, $px = Et - \vec{p} \cdot \vec{x}$ and $p_{\pm} = p_x \pm ip_y$. These solutions satisfy $\psi^{(i)\dagger}(x)\psi^{(j)}(x) =$
\( \frac{E}{m} \delta^{i,j} \ (i, j = 1, 2, 3, 4) \). Using them, we construct the wave packets

\[
\Psi(E > 0) = \sum_{i=1}^{2} \int A_i(\vec{p}) \psi^{(i)} d^3 p,
\]

\[
\Psi(E < 0) = \sum_{i=3}^{4} \int A_i(\vec{p}) \psi^{(i)} d^3 p,
\]

and note

\[
\int \Psi^{\dagger}(E > 0) \Psi(E > 0) d^3 x = \int d^3 p \frac{E}{m} \{|A_1(\vec{p})|^2 + |A_2(\vec{p})|^2\}. \quad (9)
\]

Similar expression can be written for \( \Psi(E < 0) \). Using the explicit representation of the \( \alpha_x \) matrix [3], we have

\[
\int \Psi^{\dagger}(E > 0) \alpha_x \Psi(E > 0) d^3 x = \int d^3 p \left( \frac{p_x}{m} \right) \{|A_1(\vec{p})|^2 + |A_2(\vec{p})|^2\}. \quad (10)
\]

From (7), (9) and (10), it follows \( \frac{d}{dt} \langle x \rangle = \langle \vec{p} \rangle \), showing the validity of the Ehrenfest’s theorem. Further, we consider the Dirac particle in an external electromagnetic field. Setting the vector potential zero (for simplicity), the Dirac Hamiltonian is

\[
H = \vec{\alpha} \cdot \vec{p} + \beta m - e\phi,
\]

where \( \phi \) is the scalar potential. Using the Heisenberg equation of motion for a dynamical variable \( F \), \( \frac{dF}{dt} = \frac{i}{\hbar}[F, H] \), it follows that \( \frac{d\vec{p}}{dt} = -\vec{\nabla}(-e\phi) \) and so \( \langle \frac{d\vec{p}}{dt} \rangle = -\langle \vec{\nabla}(-e\phi) \rangle \), showing the validity of the Ehrenfest’s theorem.

Thus in quantum mechanics, we see that the expectation values of the position and the momentum operators satisfy the classical equations of motion. We would like to extend this to quantum field theory.

2. Abelian field theory

We consider the lagrangian density for the electromagnetic field minimally coupled to a source \( j^\mu(x) \) (Dirac current)

\[
\mathcal{L} = -\frac{1}{4} F_{\mu\nu} F^{\mu\nu} + eA_\mu(x) j^\mu(x),
\]

\( \mathcal{E} \& \mathcal{M} \). Using them, we construct the wave packets

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\mathcal{L} = -\frac{1}{4} F_{\mu\nu} F^{\mu\nu} + eA_\mu(x) j^\mu(x),
\]
where $A_\mu(x)$ is the electromagnetic field, $e$ is the coupling strength and
\[ F_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu. \] (13)

The corresponding classical equations (Euler-Lagrange equations) are
\[ \partial_\mu F^{\mu\nu} + ej^\nu = 0. \] (14)

Eqn.14 is the classical equation of motion and gives the Maxwell equations with source.

It is well known that the manifestly covariant theory of massless vector field is to be quantized with indefinite metric [4]. The impossibility of quantizing the electromagnetic field with positive definite metric has been shown by Mathews, Seetharaman and Simon [5]. A physically meaningful theory is constructed by introducing a 'subsidiary condition', which is a condition defining the physical subspace of the indefinite metric Hilbert space of the electromagnetic field. In here, we follow the $B$-field formalism of Nakanishi [6]. In order to quantize the above lagrangian, one has to fix the gauge. This is seen by considering the coefficient of the terms quadratic in $A_\mu$ in the action $S = \int d^4x L$ (after a partial integration). This coefficient is the differential operator $\Box g^{\mu\nu} - \partial^\mu \partial^\nu$. The two-point function $\langle A_\mu(x)A_\nu(y) \rangle$ is governed by the above differential operator.

The Feynman propagator for the photon (quantized electromagnetic field) is the inverse of this differential operator in the momentum space. As this differential operator is not invertible, the photon propagator is not defined. This difficulty is avoided by choosing a gauge. We choose the covariant gauge $\partial^\mu A_\mu = 0$ and implement this gauge fixing in the lagrangian by adding the 'gauge fixing term' $-\frac{1}{2a}(\partial^\mu A_\mu)^2$ where $a$ is a parameter. This modifies the coefficient of the terms quadratic in $A_\mu$ in the action $S$ as $\Box g^{\mu\nu} - \partial^\mu \partial^\nu + \frac{1}{a} \partial^\mu \partial^\nu$. This, in the momentum space is $-p^2 g^{\mu\nu} + (1 - \frac{1}{a})p^\mu p^\nu$ whose inverse is $-\frac{1}{p^2}\{g_{\mu\nu} + \frac{a-1}{p^2}p_\mu p_\nu\}$, which is the Feynman propagator for the photon in the covariant gauge.

We introduce the above covariant gauge fixing via $B(x)$, an auxiliary hermitian scalar field and consider the quantum lagrangian
\[ L = -\frac{1}{4}F_{\mu\nu}F^{\mu\nu} + B(x)\partial^\mu A_\mu + \frac{a}{2}B^2(x) + eA_\mu(x)j^\mu(x), \] (15)
where \( a \) is a parameter. It is important to realize that the gauge field \( A_\mu(x) \) and \( B(x) \) in (15) are operators while the gauge field in (12) is a classical field. The quantum equations of motion from (15) are

\[
\partial_\mu F^{\mu\nu} - \partial^\nu B(x) = -e j^\nu, \\
\partial^\nu A_\mu + a B(x) = 0.
\]

(16)

Using the second equation to eliminate the \( B \)-field in the lagrangian, we recover the gauge fixing term \(-\frac{1}{2a} (\partial^\mu A_\mu)^2\). By taking \( \partial_\nu \) of the first equation and using the conservation of the current \( j^\nu(x) \), namely \( \partial_\nu j^\nu(x) = 0 \), we see that \( B(x) \) satisfies the equation of motion for a massless scalar field, admitting positive and negative frequency solutions. Eqn.16 can be considered to be the quantum Maxwell equations while (14) is the classical equation of motion. The fields in (16) are operators and act on functions (states) in the indefinite metric Hilbert space. For this reason, this method of quantization is called ”operator method of quantization”. In order to ensure that physically meaningful degrees of freedom only contribute (the longitudinal and the time-like photons are unphysical) to the observables, we impose Gupta’s subsidiary condition on the photon states by

\[
B^+(x)|\phi\rangle = 0,
\]

where the superscript + denotes the positive frequency part of \( B(x) \). The physical subspace in the indefinite metric Hilbert space is defined in (17). The physical subspace \( V_{\text{phys}} \) is the totality of the states \( |\phi\rangle \) satisfying (17). Now consider the expectation value of the quantum equations of motion (16) between physical states \( |\phi\rangle \) defined in (17). They are

\[
\langle \phi | \partial_\mu F^{\mu\nu} - \partial^\nu B(x) + e j^\nu |\phi\rangle = 0; \quad |\phi\rangle \in V_{\text{phys}}, \\
\langle \phi | \partial_\mu A^\mu + a B(x) |\phi\rangle = 0.
\]

(18)

Using \( B^- = (B^+)^\dagger \) and (17), (18) becomes

\[
\langle \phi | \partial_\mu F^{\mu\nu} + e j^\nu |\phi\rangle = 0; \quad \forall |\phi\rangle \in V_{\text{phys}}, \\
\langle \phi | \partial_\mu A^\mu |\phi\rangle = 0.
\]

(19)

Comparing (19) with (14), we see that the expectation value of the quantum equation of motion taken with the states in the physical subspace reproduces
the classical equations of motion, generalizing the Ehrenfest’s theorem to Abelian quantum field theory. Since the classical equation of motion is linear in $A_\mu(x)$, one can separate the positive and negative frequency parts and then the second equation above gives $\partial^\mu A^\mu_\mu(x)|\phi\rangle = 0$, subsidiary operator condition of Gupta. This feature is not shared by the non-Abelian theory as there the classical equation of motion for the non-Abelian gauge field is non-linear and a separation into positive and negative frequency parts is not possible.

3. Non-Abelian Field Theory

As an example, we consider $SU(3)$ gauge theory relevant to Quantum Chromo Dynamics (QCD), the gauge theory of the strong interactions of quarks. The classical lagrangian density is given by

$$\mathcal{L}_{YM} = -\frac{1}{4} F^a_{\mu\nu} F^{\mu\nu a} + g A^a_\mu j^{\mu a},$$

(20)

where $j^{\mu a}$ is the external source (color current of the quark), $\mu, \nu$’s are the Lorentz indices, $a, b, c$’s are the $SU(3)$ group indices, $g$ is the coupling strength, and

$$F^a_{\mu\nu} = \partial_\mu A^a_\nu - \partial_\nu A^a_\mu + g f^{abc} A^b_\mu A^c_\nu.$$  

(21)

In above, $f^{abc}$’s are the structure constants of $SU(3)$ and $g$ is also the coupling strength of the self interaction of the non-Abelian gauge fields. The above lagrangian is gauge invariant. This can be verified by using the infinitesimal gauge transformation on the gauge field $A^a_\mu$, namely

$$A^a_\mu \rightarrow A^a_\mu + D^a_{\mu} \omega^b, \quad \omega^a \in SU(3),$$

$$D^a_{\mu} = \partial_\mu \delta^{ab} + g f^{abc} A^c_\mu.$$  

(22)

Consider the first term in the lagrangian. Then it is found, using the Jacobi identity

$$f_{bed} f_{dae} + f_{cad} f_{dbe} + f_{abd} f_{dce} = 0,$$  

(23)

that

$$\delta_{gauge}(F^{\mu a} F^a_{\mu \nu}) = 2 g f^{abc} F^{\mu a} F^{\mu c} \omega^b \equiv 0.$$  

(24)
The classical equations of motion from (20) are

\[ D^{ab}_\mu F^{\mu\nu}_b + gj^{\nu a} = 0. \quad (25) \]

The operator \( D^{ab}_\mu \) in (22) is called the covariant derivative in the adjoint representation and using the Jacobi identity (23), it is found that the commutator \([D_\mu, D_\nu]^{ab} = -gf^{abc}F^a_{\mu\nu}\).

Acting on (25) by \( D^{ca}_\nu \), using the commutator, it is seen that

\[ D^{ab}_\nu j^{\nu b} = 0, \quad (26) \]

that is, the current \( j^{\nu a} \) is covariantly conserved. As the source \( j^{\mu a} \) is gauge invariant, in the action integral the second term in the lagrangian is invariant using (22) and (26) after one partial integration. Thus the lagrangian in (20) is gauge invariant.

Using the covariant derivative, the classical equation of motion (22) can be rewritten as

\[ \partial_\mu F^{\mu\nu a} + gf^{acb}A^c_\mu F^{\mu\nu b} + gj^{\nu a} = 0, \]

\[ \partial_\mu F^{\mu\nu a} = -gJ^{\nu a}, \quad \text{where} \]

\[ J^{\nu a} \equiv j^{\nu a} + f^{abc}A^c_\mu F^{\mu\nu b}. \quad (27) \]

The current \( J^{\nu a} \) contains besides the matter contribution, the non-Abelian fields. The non-Abelian fields themselves act as the source (like in gravity). By inspection, we see that \( \partial_\nu J^{\nu a} = 0 \), i.e., the current \( J^{\nu a} \) is ordinarily conserved.

An attempt to quantize (20) along the lines of the Abelian theory i.e., "operator method of quantization", runs into difficulty. The auxiliary fields \( B^a(x) \) in this case do not satisfy \( \Box B^a(x) = 0 \) due to the self-coupling property of the non-Abelian fields. So it is not possible to write down the positive and negative frequency parts. Further the classical equations of motion are non-linear. The proper method is to use the "path integral approach". For the reasons given in the Abelian field theory, here also we need to fix the gauge to obtain the propagator for the gauge fields \( A^a_\mu(x) \). Further, in the "path integral method", one integrates all possible gauge field configurations. As
the lagrangian (20) is gauge invariant, two gauge field configurations related by gauge transformation will give the same lagrangian. This, in the path integral amounts to double counting in the space of gauge fields. This is avoided by fixing the gauge and integrating over the space of gauge fields modulo gauge fixing. We choose the covariant gauge $F_a^\mu = \partial^\mu A_a^\mu(x) = 0$.

The above gauge fixing relation however does change by the gauge transformation and so the gauge variation of the gauge fixing relation is non-trivial in non-Abelian gauge theory. This, in the path integral approach, brings in the Faddeev-Popov ghost (anti-commuting scalars) fields. Using the results from the "path integral approach" [7], the lagrangian density for quantum non-Abelian theory can be written as

$$L = -\frac{1}{4} F_{\mu\nu}^{a} F^{\mu\nu a} - \partial^\mu B^a A_a^\mu + \frac{\alpha}{2} B^a B^a - i \partial^\mu \bar{c}^a \left( D^{ab}_a \bar{c}^b \right) + gj^a_{\nu} A^{\nu a},$$

where $\alpha$ is a gauge parameter and $c$'s are the ghost fields. They are hermitian 

$$c^a = (c^a)^\dagger; \quad \bar{c}^a = (\bar{c}^a)^\dagger,$$

and the ghost fields $c^a$ and $\bar{c}^a$ anti-commute.

A comparison of (28) with (15) reveals that now we have (for $SU(3)$) eight auxiliary fields $B^a$ and a new term involving the Faddeev-Popov ghost fields. One can also quantize the Abelian massless field by the above procedure ("path integral approach") and in that case, the ghosts decouple from the gauge fields. In contrast, in (28), the fourth term contains coupling of the ghost fields with the gauge fields. This is crucial. The second and the third terms in (28) are the gauge fixing part and the fourth term is the Faddeev-Popov ghost part $L_{FP}$. Using (29) and the anti-commuting property of the ghost fields, it is seen that $\bar{L}_{FP} = L_{FP}$. The quantum equations of motion following from (28) are:

$$D^{ab}_\mu F^{\mu\nu b} = \partial^\nu B^a - gj^a_{\nu} - igf^{abc} \left( \partial^\nu \bar{c}^b \right) c^c,$$

$$\partial_\mu A^\mu + \alpha B^a = 0,$$

$$D^{ab}_\mu \left( \partial^\mu \bar{c}^b \right) = 0,$$

$$\partial_\mu \left( D^{\mu ab} c^b \right) = 0.$$

(30)
Before considering the physical states, we recall that the quantum lagrangian (28) is gauge fixed. So, we do not have the local gauge invariance in (28). However, it was found by Becchi, Rouet and Stora (BRS) [8] that (28) is invariant under a special global transformation (First Global Transformation) involving Faddeev-Popov ghosts. This BRS transformation is given by

\[ \delta A^a_\mu = D^a_{\mu b} c^b = [iQ, A^a_\mu], \]
\[ \delta \psi = ig c^a t^a \psi, \]
\[ \delta B^a = 0 = [iQ, B^a], \]
\[ \delta c^a = -\frac{g}{2} f^{abc} c^b c^c = \{iQ, c^a\}, \]
\[ \delta \bar{c}^a = iB^a = \{iQ, \bar{c}^a\}. \]

(31)

where \( Q \) is the BRS-charge

\[ Q = \int d^3x \{ B^a (D^a b c^b) - \partial_0 B^a c^a + i \frac{g}{2} f^{abc} \partial_0 c^a c^b c^c \}. \]

(see [7] for details) From (31), it is seen that \( \delta F^a_{\mu \nu} = g f^{abc} F^c_{\mu \nu} c^b \) and the invariance of (28) under (31) can be verified.

Though the local gauge invariance is explicitly broken by the gauge fixing, (28) has global gauge symmetry. This global gauge transformation (Second Global Transformation)

\[ \Delta A^a_\mu = f^{abc} \theta^b A^c_\mu, \]
\[ \Delta \psi_i = -i(t^a)_{ij} \theta^a \psi_j, \]
\[ \Delta \bar{\psi}_i = i(t^a)_{ji} \theta^a, \]
\[ \Delta B^a = f^{abc} \theta^b B^c, \]
\[ \Delta c^a = f^{abc} \theta^b c^c, \]
\[ \Delta \bar{c}^a = f^{abc} \theta^b \bar{c}^c. \]

(32)

where \( \theta^a \) is the global gauge parameter, generates the conserved Noether current

\[ J^a_\mu = f^{abc} A^a_\mu b_{\nu c} + j^a_\mu + f^{abc} A^b_\mu B^c - if^{abc} c^b (D^c_d c^d_\mu) + if^{abc} \partial_\mu c^b c^c, \]
\[ = J^a_\mu + f^{abc} A^b_\mu B^c - if^{abc} c^b (D^c_d c^d_\mu) + if^{abc} (\partial_\mu c^b) c^c, \]

(33)

where in the last step we used the third relation in (27).
We now consider the first equation in (30) and rewrite that as
\[ \partial_\mu F^{\mu a} + gf^{abc} A_\mu^c F^{\mu b} = \partial^\nu B^a - g j^{\nu a} - ig f^{abc}(\partial_\nu \bar{c}^b) c^c. \] (34)

This, in view of (33) can be written as
\[ \partial_\mu F^{\mu a} + g J^{\nu a} = (D_\nu A^c) - ig f^{abc} \bar{c}^b (D^{cd} c^d). \] (35)

The right side of (35) can be expressed, using the BRS transformations (31), as \(-i\delta(D^{\nu ab} \bar{c}^b)\) and so (35) becomes
\[ \partial_\mu F^{\mu a} + g J^{\nu a} = \{ Q, D^{\nu ab} \bar{c}^b \}. \] (36)

This quantum equation of motion is to be compared with the classical equation of motion (27). We note that \(J^{\nu a}\) in (27) is replaced by \(J^{\nu a}\) in (36) and the right side is expressed as a BRS-variation. Both \(J^{\nu a}\) and \(J^{\nu a}\) are ordinarily conserved. That the quantum equation (34) can be written in the form (36) was first shown by Ojima [9].

The vector space for the non-Abelian gauge fields, on which the quantum equations act is an indefinite metric space. A physical subspace of this is to be defined. It was shown by Kugo and Ojima [10] that the physical space is defined by the condition
\[ Q|\phi\rangle = 0. \] (37)

Taking the expectation value of (36) between the physical states and using (37) it follows
\[ \langle \phi|\partial_\mu F^{\mu a} + g J^{\nu a}|\phi\rangle = 0. \] (38)

This expression when compared with the classical equation of motion (27) shows that the Ehrenfest theorem is not fully satisfied. The global conserved current \(J^{\nu a}\) differs from the conserved current \(J^{\nu a}\), as seen from (33).

Now we consider (33) and note that this difference is given by \(f^{abc} A_\mu^b B^c - if^{abc} \bar{c}^b (D^{d} c^d) + i f^{abc}(\partial_\mu \bar{c}^b) c^c\). The first two terms can be expressed using (31) as \(\delta(if^{abc}\bar{c}^b A_\mu)\) noting that when the BRS-variation crosses the ghost field it picks up a sign. So the first two terms can be rewritten as
\{-Q, f^{abc} \overline{c}^b A^c_\mu \} \text{ and this when taken between the physical states vanishes. Then, (38) becomes}

\[ \langle \phi | \partial_\mu F^{\mu\nu a} + g J^{\nu a} + i f^{abc} (\partial^\nu \overline{c}^b) c^c | \phi \rangle = 0. \]  

(39)

This still differs from the classical equation of motion by a term involving ghosts only.

We now take up the quantum lagrangian (28) and note it is invariant under the scale transformation (Third Global Transformation)

\[ c^a \rightarrow e^\alpha c^a; \quad \overline{c}^a \rightarrow e^{-\alpha} \overline{c}^a, \]  

(40)

with \( \alpha \) a constant. This global transformation affects only the FP-ghost fields in (28). The Noether current corresponding to this transformation is given by

\[ J^\lambda_{gh} = \delta_{\alpha} c^a \frac{\partial L}{\partial (\partial_\lambda c^a)} + \delta_{\alpha} \overline{c}^a \frac{\partial L}{\partial (\partial_\lambda \overline{c}^a)}, \]

\[ = i \overline{c}^a (D^\lambda_{ab} c^b) - i (\partial^\lambda \overline{c}^a) c^a, \]  

(41)

as \( \alpha \) is arbitrary. The corresponding conserved charge \( Q_{gh} = (Q_{gh})^\dagger \) is called the FP-ghost charge generating the above scale transformation on the ghost fields, leaving other fields invariant [7]. This is given by

\[ \delta_{gh} c^a = [i Q_{gh}, c^a] = c^a; \quad \delta_{gh} \overline{c}^a = [i Q_{gh}, \overline{c}^a] = -\overline{c}^a. \]  

(42)

Using the above, the third term in (39) can be written as

\[ i f^{abc} (\partial_\mu \overline{c}^b) c^c = -\frac{1}{2} \delta_{gh} [i f^{abc} (\partial_\mu \overline{c}^b) c^c], \]

\[ = \frac{1}{2} [Q_{gh}, f^{abc} (\partial_\mu \overline{c}^b) c^c], \]  

(43)

as \( \delta_{gh} \) when crosses a FP-ghost field picks up a sign.

We defined the physical subspace in (37) as the assembly of states in the indefinite metric Hilbert space annihilated by the BRS-Charge. We now restrict the physical subspace further by another subsidiary condition

\[ Q_{gh} | \phi \rangle = 0. \]  

(44)
Then, using (43) in the last term in (39) and in view of the further restriction (44) on the physical states, (39) becomes
\[
\langle \phi | \partial_\mu F^{\mu a} + g J^a | \phi \rangle = 0, \tag{45}
\]
showing that the expectation value of the quantum equation of motion for the non-Abelian gauge fields agrees with the classical equation of motion (27).

Now we examine the other quantum equations of motion in (30). The second equation in (30), in view of the BRS-transformation (31) can be written as \( \partial_\mu A^{\mu a} + \alpha \{ Q, \bar{c}^a \} = 0 \) which when its expectation value between the physical states defined in (37) are taken gives \( \langle \phi | \partial_\mu A^{\mu a} | \phi \rangle = 0 \), giving the gauge fixing condition. The third equation in (30), in view of the third global transformation (42), is written as \( [i Q_{gh}, (D_{\mu}^{ab} (\partial^{\mu} \bar{c}^b))] = 0 \) and its expectation value taken between the physical states vanishes on account of (44). The fourth equation in (30), using the BRS-transformation (first global transformation) becomes \( [i Q, \partial_\mu A^{\mu a}] \) whose expectation value between the physical states vanishes on account of (37). This shows the validity of Ehrenfest’s theorem for the quantum non-Abelian theory. We have made use of three global transformations to arrive at this conclusion.

4. Summary

The Ehrenfest theorem in quantum mechanics is shown to be satisfied in the quantum field theory by suitably taking the physical subspace for the gauge fields. In the Abelian quantum field theory, the one subsidiary condition on the physical states of the photon is enough to show this. In the case of non-Abelian field theory, the subsidiary condition (37) is not enough and one has to further restrict the physical space by (44). Then the expectation value of the quantum equations of motion between the physical states satisfying (37) and (44) agree with the classical equations of motion, including the gauge fixing condition.

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