On regular algebraic surfaces of $\mathbb{R}^3$ with constant mean curvature

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Abstract

We consider regular surfaces $M$ that are given as the zeros of a polynomial function $p : \mathbb{R}^3 \to \mathbb{R}$, where the gradient of $p$ vanishes nowhere. We assume that $M$ has non-zero mean curvature and prove that there exist only two examples of such surfaces, namely the sphere and the circular cylinder.

1 Introduction

An algebraic set in $\mathbb{R}^3$ will be the set

$$M = \{(x, y, z) \in \mathbb{R}^3; p(x, y, z) = 0\}$$

of zeros of a polynomial function $p : \mathbb{R}^3 \to \mathbb{R}$. An algebraic set is regular if the gradient vector $\nabla p = (p_x, p_y, p_z)$ vanishes nowhere in $M$; here $p_x$, $p_y$ and $p_z$ denotes the derivative of $p$ with respect to $x$, $y$ or $z$ respectively.

The condition of regularity is essential in our case. It allows to parametrize the set $M$ locally by differentiable functions $x(u, v)$, $y(u, v)$, $z(u, v)$ (not necessarily polynomials), so that $M$ becomes a regular surface in the sense of differential geometry (see [2] chapter 2 section 2.2, in particular Proposition 2); here $(u, v)$ are coordinates in an open set of $\mathbb{R}^2$.

Since $M$ is a closed set in $\mathbb{R}^3$, it is a complete surface. In addition, being a regular surface, it is properly embedded, i. e., the limit set of $M$ (if any)

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does not belong to $M$ (cf. [15], chapter IV, A.1 p. 113). In particular, regular algebraic surfaces are locally graphs over their tangent planes.

From now on, $M$ will denote a regular algebraic surface in $\mathbb{R}^3$. Due to the regularity condition, one can define on $M$ the basic objects of Differential Geometry of surfaces and pose some differential-algebraic questions within this algebraic category.

For instance, in the last 60 years (namely after the seminal work [4] of Heinz Hopf in 1951), many questions have been worked out on differentiable surfaces of non-zero constant mean curvature $H$. See also [5].

In our case, we have two examples of algebraic regular surfaces that have non-zero constant mean curvature, namely,

1. Spheres, $(x-x_0)^2 + (y-y_0)^2 + (z-z_0)^2 = r^2$ with center $(x_0, y_0, z_0) \in \mathbb{R}^3$ and radius $r = 1/H$.

2. Circular right cylinders, $(x-x_0)^2 + (y-y_0)^2 = r^2$, whose basis is a circle in the plane $xy$ with center $(x_0, y_0)$ and whose axis is a straight line passing through the center and parallel to the $z$ axis.

A first natural question is: Are there further examples?

The first time we heard about this question was in a preprint of Oscar Perdomo (recently published in [13]) where he proves that for polynomials of degree three there are no such surfaces.

In this note, we prove the following general result:

**Theorem:** Let $M$ be a regular algebraic surface in $\mathbb{R}^3$. Assume that it has constant mean curvature $H \neq 0$. Then $M$ is a sphere or a right circular cylinder.

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## 2 Preliminaries

We first observe that, in the compact case, this theorem follows immediately from Alexandrov’s well-known result: An embedded compact surface in $\mathbb{R}^3$ with constant mean curvature is isometric to a sphere.
The second observation is that the total curvature of an algebraic surface is finite. This was first proved by Osserman [11] in the case that the surface is an immersion parametrized by polynomials in two variables.

Here we give a proof of the finiteness of the total curvature for a (implicitly defined) regular algebraic surface $M$.

**Proposition 1**: Let $M$ be a regular algebraic surface in $\mathbb{R}^3$. Assume that it is complete non-compact. Then its total curvature is finite. More precisely,

$$\int_M |K| dM \leq 4\pi C(d),$$

where $K$ is the Gaussian curvature of $M$ and $C(d)$ is a constant that depends only on the degree $d$ of the polynomial defining $M$.

**Proof.** Let $g : M \to S^2(1)$ be the Gauss map of $M$. It is well known the Gauss curvature $K = \det(dg)$, where $dg$ is the differential of the map $g$. Of course the result is true if $M$ is either a plane or a circular cylinder. So, we ruled out these two cases from our proof. Let $M^*$ be the set of points in $M$ where $K \neq 0$. Then, restricted to $M^*$, $g$ is a local diffeomorphism; that is, given $q \in g(M^*)$ and $m_\alpha \in \{g^{-1}(q)\}$, $\alpha$ belonging to a set of indices $A$, there exist neighborhoods $U$ of $q$ and $V_\alpha$ of $m_\alpha$ such that, for each $\alpha$, $g$ maps $V_\alpha$ diffeomorphically onto $U$. In fact, $g$ restrict to $M^*$ is a covering of $N(M^*)$ without ramification points. Since

$$\int_M |K| dM = \int_{M^*} |K| dM$$

the theorem is proved if we show that the mentioned covering has only a finite number of leaves.

Now, fix a plane $P$ passing through the origin of $\mathbb{R}^3$ and in $P$ fix an orthonormal basis $\{e_1, e_2\}$. Then, there exists a point $m \in M$ such that, up to translations, $P = T_m(M)$ if and only if

$$\frac{\nabla_p}{|\nabla_p|}(m) \perp P$$

what is equivalent to the system of equations

$$\begin{align*}
\langle \frac{\nabla_p}{|\nabla_p|}(m), e_1 \rangle &= 0 \\
\langle \frac{\nabla_p}{|\nabla_p|}(m), e_2 \rangle &= 0
\end{align*}$$

(1)
Set $e_1 = \sum a_i U_i$ and $e_2 = \sum b_i U_i$ where $U_1 = (1, 0, 0)$, $U_2 = (0, 1, 0)$, $U_3 = (0, 0, 1)$ is the canonical basis of $\mathbb{R}^3$. Then, since $|\nabla p| \neq 0$, then (1) takes the form

$$
\begin{align*}
& \left\{ \begin{array}{ll}
px a_1 + py a_2 + pz a_3 = 0 \\
px b_1 + py b_2 + pz b_3 = 0
\end{array} \right.
\end{align*}
$$

(2)

where the $a_i, b_i$, $i = 1, 2, 3$ are real numbers.

The equations in (2) describe algebraic surfaces, $\Sigma_1$ and $\Sigma_2$, which correspond to the coefficients $a_i$ and $b_i$ respectively, and whose degrees are $\leq (d-1)$, where $d$ is the degree of $p$.

The surfaces $\Sigma_1$ and $\Sigma_2$, together with the original surface $M$ determines points $m \in M$ as follows:

$\Sigma_j$ interset $M$, $j = 1, 2$, in a curve $C_j$. If the intersection $C_1 \cap C_2$ contains a point $m \in M^*$, since $K(m) \neq 0$, such intersection is unique in a neighborhood of $m$. By Bezout’s theorem the total number of intersections is bounded above by $(d-1)^2$, as we wished. This proves the proposition.

Since $\int_M |K| \, dM$ is finite, it follows from a theorem of Huber [6] that the surface $M$ of Proposition 1 is finitely connected, i.e., it is a compact surface with a finite number of ends.

The proof of our Theorem uses in a crucial way the structure theory for embedded, complete finitely connected surfaces with non-zero constant mean curvature developed by Korevaar, Kusner and Solomon in [7] after some preliminary work by Meeks [9]. The statement that we need from these papers is as follows:

**Theorem A:** ([9] and [7].) Let $M$ be a complete, non-compact, properly embedded surface in $\mathbb{R}^3$ with non-zero constant mean curvature. Assume that $M$ is finitely connected. Then, the ends of $M$ are cylindrically bounded. Furthermore, for each end $E$ of $M$, there exists a Delaunay surface $\Sigma \subset \mathbb{R}^3$ such that $E$ and $\Sigma$ can be expressed as cylindrical graphs $\rho_E$ and $\rho_\Sigma$ so that, near infinity, $|\rho_E - \rho_\Sigma| < Ce^{-\lambda x}$ where $C \geq 0$ and $\lambda > 0$ are constants.

**Remark:** The first assertion in Theorem A comes from [9]. The final assertion is from [7], theorem 5.18.
3 Proof of the Theorem

We can assume that $M$ is complete and non-compact; otherwise it is a sphere. Thus, by Proposition 1, $M$ has finite total curvature, and hence, by Huber’s theorem, $M$ is compact with finitely many ends. By Theorem A, each end $E$ of $M$ converges exponentially to a Delaunay surface $\Sigma$. Since $M$ is embedded, the Delaunay surface $\Sigma$ to which an end $E$ converges has to be an onduloid or a right circular cylinder.

We first claim that the Delaunay surface $\Sigma$ towards which $E$ converges is actually a cylinder.

Suppose it is not. By a rigid motion, we can assume that the axis of $\Sigma$ is parallel to the $y$ axis and meets the $z$ axis. Then, there is a value $z_0$ of $z$ such that the line $y \to (0, y, z_0)$ intersects $\Sigma$ infinitely often. Since $E$ approaches $\Sigma$ at infinity, the algebraic equation $p(0, y, z_0) = 0$ has infinitely many solutions. This is impossible. So $\Sigma$ is a cylinder as we claimed.

We claim now that $E$ contains a open set of the cylinder $\Sigma$.

To see this, we take a rigid motion so that one of the straight lines of the cylinder $\Sigma$ agrees with the coordinate $y$-axis. Thus, one of the intersection curves of $E$ with the plane $x = 0$ is a curve $\beta$ that converges to the $y$-axis. If $y$ is large enough, $\beta$ is given by

$$\beta(y) = (0, y, z(y)),$$

where $z(y)$ is a function that satisfies

$$\lim_{y \to \infty} z(y) = 0$$

Since the curve $\beta$ belongs to the end $E$, we have

$$p(0, y, z(y)) = 0 \quad (3)$$

Observe that the polynomial $p$ can be written as

$$p(x, y, z) = a_n z^n + a_{n-1} z^{n-1} + \ldots + a_1 z + a_0$$

where $a_k = a_k(x, y)$ is a polynomial in $x$ and $y$ of degree $\leq n$. By Theorem A, we have that

$$\lim_{y \to \infty} z(y) = \lim_{y \to \infty} Ce^{-\lambda y} = 0$$

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By a known result in Calculus, we have, for any integer $k$,
\[ \lim_{y \to \infty} y^k e^{-\lambda y} = 0 \]
for any integer $k$.

Thus, by computing the limit in the equation $p(0, y, z(y)) = 0$ as $y \to \infty$ along the curve $\beta$, we obtain that $a_0$ does not depend on $Y$, and $a_0 = 0$. This means that, for any $y$, the equation $p(0, y, z) = 0$ has $z = 0$ as a root, i.e., the straight line $y \to (0, y, 0)$ is contained in $E$.

The above argument applies to an arbitrary straight line of $\Sigma$. It follows that an open set in $E$ is a cylinder. This proves our claim.

Thus, there exists an open set $U$ in $M$ with the property that the Gaussian curvature $K$ vanishes in $M$. Since $M$ is analytic, $K$ vanishes identically in $M$. It is then well known (see e.g. [8]) that $M$ is a cylinder. Since $H$ is constant, this is a circular cylinder. This proves the Theorem.

**Remark:** A crucial point in the proof is that the convergence in [7] is exponential. It allows us to prove that not only an arbitrary line in the cylinder $\Sigma$ converges to $E$ but that actually it is contained in $E$.

### 4 Final Remarks

**The case** $H = 0$. There are many algebraic minimal surfaces in $\mathbb{R}^3$ (see p. 161 of the English translation of Nitsche’s book [10]). However, the examples we are most familiar with, namely, the Enneper surface and the Hennenberg surface, are not embedded; thus they are not regular algebraic surfaces.

In fact it is simple to prove the following proposition

**Proposition:** There are no regular algebraic minimal surfaces in $\mathbb{R}^3$ except the plane.

**Proof:** Let $M$ be an algebraic surface in $\mathbb{R}^3$. As we have seen, such surface it is finitely connected, i.e., it is a compact surface with a finite number of ends. We also know that $M$ is properly embedded.

Let $E$ be one of its ends. Parametrically $E$ can be described by a map $x : D - \{O\} \to \mathbb{R}^3$, where $D$ is an open disk of $\mathbb{R}^2$ centered at the origin and $O$ is the origin.
We may assume, after a rotation if necessary, that the Gauss map, which extends to \( O \) (see Osserman [12]), takes on the value \((0, 0, 1)\) at \( O \). The two simplest examples of such ends are the plane and (either end of) the catenoid.

Now we use a result proved by R. Schoen [14]. He showed that such an end is the graph of the function \( x_3 \) defined over the \((x_1, x_2)\)-plane and

When \( a \neq 0 \) the end is of catenoid type. When \( a = 0 \) the end of the planar type. In fact, if \( a \neq 0 \) the function \( x_3 \) will be asymptotic to the graph of the function \( \log \rho \); if \( a = 0 \) it will be asymptotic to the graph of a constant function (equal to \( \beta \)).

Let’s assume that \( E \) is of the catenoid type. Consider the curve \( \alpha \) intersection of the \( E \) with the plane \( x_2 = 0 \) in the region \( x_1 > 0 \). Since \( M \) is given by the equation \( p(x_1, x_2, x_3) = 0 \), then the curve \( \alpha \) is algebraic, given by \( p(x_1, 0, x_3) = 0 \). This curve must be asymptotic to the graph of the function \( x_3 = a \log x_1 \). But this is impossible. Hence, \( M \) can not have end of the catenoid type.

Thus, all the ends of \( M \) are of the planar type. But they are in finite number. Since \( M \) is embedded, the planes asymptotic to \( M \) must be parallel. It follows that there are two parallel planes such that \( M \) is contained in the region bounded by them. It follows by the halfspace theorem for minimal surfaces [3] that \( M \) must be a plane.

**Hypersurfaces in** \( \mathbb{R}^{n+1}, n \geq 3 \). In this case we consider the zeros of a polynomial function \( p(x_0, x_1, \ldots, x_n) \), \( n \geq 3 \), with \( \nabla p \neq 0 \) everywhere, and call it regular algebraic hypersurfaces \( M^n \) of \( \mathbb{R}^{n+1} \). Similar to the case \( n = 2 \), the only compact example of such hypersurfaces are spheres. This follows immediately from Alexandrov theorem. So, we are left to consider the complete non-compact case. A generalized cylinder \( C^k \) in \( \mathbb{R}^{n+1} \) is a product \( B^k \times \mathbb{R}^{n-k} \), where the basis \( B^k \subset \mathbb{R}^{k+1} \subset \mathbb{R}^{n+1} \) is a hypersurface of \( \mathbb{R}^{k+1} \) and the product is embedded in \( \mathbb{R}^{n+1} \) in the canonical way, i.e., \( B^k \times \mathbb{R}^{n-k} \subset \mathbb{R}^{k+1} \times \mathbb{R}^{n-k} \). It is easily checked that when \( B \) is a \( k \)-sphere, \( C^k \) has nonzero constant mean curvature. The following lemma is again a consequence of Alexandrov’s theorem.

**Lemma.** Let \( C^k \) be an algebraic regular generalized cylinder in \( \mathbb{R}^{n+1} \) whose basis \( B \) is a compact hypersurface. If \( C^k \) has constant mean curvature then the bases \( B^k \) is a \( k \)-sphere.

We do not know any further examples of a regular algebraic hypersurfaces
in $\mathbb{R}^{n+1}$, $n > 2$, with nonzero constant mean curvature. We can ask a question similar to the one we answered for $n = 2$. The possible extension of our proof, however, needs new ideas. Although the total Gauss-Kronecker curvature is again finite, there is no Huber theorem for $n > 2$, and the proof of the structure theorem of [7] does not work for hypersurfaces in $\mathbb{R}^{n+1}$, $n > 2$.

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