FURTHER CRITERIA FOR THE EXISTENCE OF STEADY LINE-DRIVEN WINDS

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ABSTRACT

In an earlier paper we showed that steady line-driven disk wind solutions can exist by using “simple” models that mimic the disk environment. Here I extend the concepts introduced in the earlier paper and discuss many details of the analysis of the steady or unsteady nature of one-dimensional line-driven winds. This work confirms the results and conclusions of the earlier paper and is thus consistent with the steady nature of the one-dimensional streamline line-driven disk wind models of Murray and collaborators and the 2.5 dimensional line-driven disk wind models of Pereyra and collaborators. When including gas pressures effects, as is routinely done in time-dependent numerical models, I find that the spatial dependence of the nozzle function continues to play a key role in determining the steady or unsteady nature of supersonic line-driven wind solutions. I show here that the existence or nonexistence of local wind solutions can be proved through the nozzle function without integrating the equation of motion. This work sets a detailed framework with which we will analyze, in a following paper, more realistic models than the “simple” models of the earlier paper.

Subject headings: accretion, accretion disks — hydrodynamics — novae, cataclysmic variables — quasars: absorption lines

1. INTRODUCTION

As discussed in Pereyra et al. (2004, hereafter Paper I), accretion disks are commonly believed to be present in both cataclysmic variables (CVs) and quasi-stellar objects (QSOs) and active galactic nuclei (AGNs). In both types of objects, blueshifted absorption troughs in UV resonance lines are sometimes present, giving direct observational evidence for an outflowing wind. Another property that CVs and QSOs have in common is the existence of a persistent velocity structure in their absorption troughs (when present) over significantly long time scales (Paper I). In order for a line-driven disk wind to account for the wide and/or broad resonance line absorption structures observed in many CVs and QSOs, it must be able to account for the steady velocity structure that is observed. The 2.5 dimensional time-dependent line-driven disk wind models of Pereyra and collaborators, both for CVs and QSOs, have steady disk wind solutions (Pereyra 1997; Pereyra et al. 1997, 2000; Hillier et al. 2002; Pereyra & Kallman 2003). The earlier one-dimensional line-driven disk wind models of Murray and collaborators also find steady disk wind solutions (Murray et al. 1995; Murray & Chiang 1996, 1998; Chiang & Murray 1996).

However, reports that line-driven disk winds are “intrinsically unsteady” (see Paper I) persist, based on the argument that the “unsteadiness” is physically reasonable because of the increasing gravity along the streamlines at the wind base that is characteristic of disk winds. Since the steady nature of CV and QSO wind flows is an observational constraint, whether line-driven disk winds are steady or not is a significant issue. In Paper I we showed that an increase in gravity at the wind base does not imply an unsteady wind solution. We also developed mathematically “simple” models that mimic the disk environment and we showed that line-driven disk winds can be steady.

In the past, evidence in favor of and against steady line-driven disk wind solutions has come from numerically intensive 2.5 dimensional disk wind models. However, model differences can be numerical in nature. Therefore, a detailed analysis that develops well-defined criteria and applies methods independent of the previous numerically intensive 2.5 dimensional models becomes important. The goal of this paper is to extend the concepts of Paper I to establish such methods and criteria. In a following paper (N. Pereyra et al. 2005, in preparation, hereafter Paper III), we will apply the criteria developed here to the exact flux distribution of a standard (Shakura & Sunyaev 1973) accretion disk.

I have included gas pressure effects throughout this paper, rather than assuming gas pressure to be zero as we did in most of Paper I, for reasons of simplicity. It has been argued many times in the literature that the assumption of neglecting gas pressure effects (i.e., zero gas pressure) in a line-driven wind is reasonable, since the main driving force is radiation pressure. However, there are four reasons why it may be important to include gas pressure effects.

First, as was discussed by Castor et al. (1975, hereafter CAK75) for the stellar case and as we discuss in Paper I and here, an important property of steady supersonic line-driven winds, under the Sobolev approximation for the line radiation force, is that it must have a “critical point.” In turn, the position of the critical point determines the exact wind mass-loss rates and velocity laws within the model. The critical point is not just a mathematical construct that facilitates the calculation of basic wind parameters. As was shown by Abbott (1980) for the stellar case, the critical point is the point where the flow velocity equals the backward velocity propagation of density perturbations, referred to as radiative-acoustic waves or Abbott waves (rather than sound waves). Thus, in a line-driven wind the critical point plays a role physically equivalent to the sonic point in a temperature-driven wind (Parker 1960).

However, if one assumes that gas pressure is zero, one finds an infinite family of supersonic solutions of which only one actually has a critical point. The question that naturally arises from this specific result is: Does a steady line-driven wind necessarily have to present a critical point? The answer is yes (for a wind that reaches supersonic speeds under the Sobolev approximation for the line force). The argument is relatively simple. As was shown
by CAK75 for stellar winds, no matter how small the gas pressure may be, the wind solution must have a critical point. Therefore, since the zero gas pressure case is a limiting case of the gas pressure included case when gas pressure tends to zero, it follows that only solutions with a critical point are “real” or “physical” solutions. Therefore, the requirement of a critical point is a gas pressure effect. That is, gas pressure is necessary to identify a unique physical solution, which further analysis shows it to be a solution that presents a critical point.

To illustrate the significance of this, if one assumes that gas pressure is zero (rather than considering the limiting case in which gas pressure tends to zero), the infinite family of possible solutions results in wind mass-loss rates that may vary from any value arbitrarily close to zero up to the wind mass-loss rate of the solution that contains the critical point. Therefore, the fact that the wind mass-loss rate in a supersonic line-driven wind arrives at the maximum possible value (that of the critical point solution) is a gas pressure effect.2

This means that although the exact value of the gas pressure may not significantly affect the actual value of the wind mass-loss rate of a steady wind, the existence of gas pressure causes the wind mass-loss rate to have a unique value that corresponds to the critical point type wind solution. Thus, without gas pressure effects, the wind mass-loss rate would not be uniquely determined, possibly having arbitrarily low values, which in turn could have important effects on the evolution of astrophysical systems for which line-driven winds are present.

A second motive for considering gas pressure effects is that for cases in which a steady line-driven disk wind solution is not possible (when assuming gas pressure to be zero), inclusion of gas pressure effects may allow a uniquely well-determined solution. An example of this is the “S model” of Paper I. If one assumes gas pressure to be zero for the S model, then the corresponding nozzle function is monotonically decreasing, and thus the critical point is at infinity. Since information, in principle, cannot travel an infinite distance in a finite time, a steady physical solution is not realizable for the zero gas pressure case. However, when gas pressure effects are included, the corresponding small corrections to the nozzle function produce a minimum in the nozzle function at a finite distance (Paper I). Thus, once gas pressure effects are considered, the critical point is no longer at infinity but rather at a finite well-defined distance, allowing for a unique physical solution to be found. The fact that the inclusion of gas pressure effects may lead a system from not presenting the existence of a steady solution to presenting the existence of a steady solution is an obvious and more than sufficient motivation to include gas pressure effects.

Third, since evidence in favor and against the existence of steady line-driven disk wind solutions comes from numerically intensive 2.5 dimensional disk wind models that include gas pressure, I am unavoidably led to include gas pressure. If I do not include gas pressure, for example, I leave open the possibility that some subtle gas pressure effect at the wind base could generate strong fluctuations downstream, causing apparent “intrinsic unsteadiness.”

Fourth, and finally, gas pressure is physically present. For example, in their line-driven stellar wind paper, CAK75 did not discuss nor justify why they included gas pressure in their model; they simply included it, since pressure must be present in any hydrodynamic system.

The specific form of plots that would further illustrate the general criteria developed in this paper will depend on the specific models being analyzed. However, it is probably best to consider a specific model after the general framework is discussed. For this reason, I show a series of plots corresponding to the CAK75 model, so as to illustrate and apply the concepts and results of this work to a well-known, well-studied model. In a following paper (Paper III), we shall apply the concepts discussed here to the case of an exact Shakura-Sunyaev disk flux distribution (Shakura & Sunyaev 1973) and compare the results with the cataclysmic variable disk wind models of Pereyra (1997) and Pereyra et al. (2000).

I present a brief discussion of the steady or unsteady versus the stable or unstable wind characteristics in § 2. In § 3 I present the one-dimensional hydrodynamic equations that pertain to the line-driven wind models of this work. I extend the definition of the nozzle function given in Paper I to include temperature gradients in § 4 and discuss the critical point conditions in § 5. In § 6 I develop general criteria for the existence of local steady wind solutions in one-dimensional models and discuss the existence of global solutions in § 7. In § 8 I apply the criteria to the well-known and studied CAK75 stellar wind. In § 9 I briefly discuss ongoing efforts and refer to a following paper (Paper III) in which we will apply the criteria developed here to more realistic disk wind systems than those previously analyzed in Paper I. A summary and conclusions are presented in § 10.

2. STEADY WINDS VERSUS STABLE WINDS

In general, a dynamical physical system is said to be steady if and only if there exists stationary or time-independent solutions to the equations of motion. A system is said to be stable if, in addition to it being steady, arbitrarily small perturbations to the steady solution will either be damped or transported as a wave with constant amplitude by the system. Thus, systems with steady solutions that amplify arbitrarily small perturbations are not stable.

Since, in general, physical systems are in practice subjected to small perturbations, the issue of stability is relevant. In particular, if a hydrodynamic flow presents a steady solution that is however unstable, then it will be virtually impossible to find in nature such a system in its steady state.

Abbott (1980) showed that for the case of a one-dimensional line-driven wind, under the Sobolev approximation for the line force, small perturbations will travel as radio-acoustic waves or Abbot waves (rather than sound waves). The velocity of the radio-acoustic wave is subsonic in the direction of the flow and supersonic in the backward direction. Thus, perturbations to the steady solution of a one-dimensional line-driven wind will not be amplified but rather will be transported through the wind, implying in turn that the wind is stable.3 Therefore, for the specific case of a one-dimensional line-driven wind under the Sobolev approximation, if a steady solutions exists, then it is also stable.4

3 In Paper I, as an illustrative alternative to the above argument and as a consistency check, we showed that the steady one-dimensional line-driven winds analyzed were stable through numerical simulations. Numerical simulations, due to the finite precision of the calculations, inherently present small numerical perturbations.

4 Several authors have pointed out that more realistic treatments of line force that go beyond the Sobolev approximation may lead to flow instabilities (e.g., Owocki & Puls 1999). However, the Sobolev approximation for the line force is a standard working assumption in line-driven wind models, which is currently being implemented by us and has been implemented in the numerical models that have reported “intrinsic unsteadiness.”
3. EQUATION OF MOTION

Following the notation of Paper I, the one-dimensional hydrodynamic equation of motion for a stationary line-driven flow is

\[
\left(1 - \frac{b^2}{2W}\right) A \frac{dW}{dz} = -BA + \gamma A \left(\frac{A}{M} \frac{dW}{dz}\right)^{\alpha} + b^2 \frac{dA}{dz} - A \frac{db^2}{dz},
\]

which is the equation of motion of Paper I corrected for temperature gradient effects, where \( z \) is the independent spatial coordinate, \( W = \frac{V^2}{2} \) is the kinetic energy per mass, \( V \) is the velocity, \( A \) is the area that depends on \( z \), \( M = \rho VA \) is the wind mass-loss rate, \( \rho \) is the density, \( B \) represents the body force that corresponds to the gravitational plus continuum radiation force per mass, \( b \) is the isothermal sound speed, and \( \gamma \) is the “line opacity weighted flux” that also depends on \( z \).

As in Paper I, I scale the physical parameters by defining a value of \( r_0, B_0, A_0, \) and \( \gamma_0 \) as the characteristic distance, gravitational acceleration, area, and line opacity weighted flux, respectively. The explicit expressions for the normalized parameters are given in Paper I.

Introducing the scaling equations, the equation of motion becomes

\[
\left(1 - \frac{s}{\omega}\right) a \frac{d\omega}{dx} = -ga + fa \left(\frac{a d\omega}{\dot{m} dq}\right)^{\alpha} + 2s \frac{da}{dx} - 2a \frac{ds}{dx},
\]

which is the normalized equation of motion of Paper I corrected for temperature gradient effects, where \( x \) is the normalized independent spatial coordinate, \( \omega \) is the kinetic energy per mass, \( a \) is the area that depends on \( x \), \( \dot{m} \) is the wind mass-loss rate, \( g \) is the gravitational plus continuum radiation force per mass, \( s \) is the sound speed squared, and \( f \) is the “line opacity weighted flux” that also depends on \( x \). In equation (2) all physical variables are normalized.

Equivalent to the CAK75 independent spatial variable \( u \), I define the variable \( q \) as

\[
q \equiv \int_{x_0}^{x} \frac{1}{a(x')} dx' + q_0,
\]

where \( x_0 \) is an arbitrary position and \( q_0 \) is an arbitrary value.

The equation of motion, in terms of variables \( q \) and \( \omega \), becomes

\[
\left(1 - \frac{s}{\omega}\right) \frac{d\omega}{dq} = -ga + fa \left(\frac{1}{\dot{m}} \frac{d\omega}{dq}\right)^{\alpha} + 2s \frac{da}{dq} - 2 \frac{ds}{dq}.
\]

Similar to the CAK75 notation, I define the function \( h(q) \) as the sum of the independent terms in the equation of motion (i.e., the terms that are independent of \( \omega \) and \( d\omega/dq \)) as

\[
h(q) \equiv -ga + 2s \frac{da}{dq} - 2 \frac{ds}{dq} = -ga + 2s \frac{da}{dx} - 2a \frac{ds}{dx}.
\]

The equation of motion now becomes

\[
\left(1 - \frac{s}{\omega}\right) \frac{d\omega}{dq} = h(q) + fa \left(\frac{1}{\dot{m}} \frac{d\omega}{dq}\right)^{\alpha}.
\]

The problem of existence of a steady solution is thus reduced to determining whether a value of \( \dot{m} \) and a normalized function \( \omega(q) \) exists such that it satisfies the boundary conditions and equation (6). A steady solution for the hydrodynamic one-dimensional model exists if and only if equation (6) is integrable while simultaneously satisfying the boundary conditions. Typically the boundary condition is the position of the sonic point. That is, the equation of motion must not only be integrable, but additionally there must exist a solution to the equation of motion that presents a sonic flow speed at a predefined sonic position (boundary condition).

4. NOZZLE FUNCTION AND CRITICAL POINT

To continue the discussion, I recall the definition of the \( \beta \) function given in Paper I,

\[
\beta(\omega) \equiv 1 - \frac{s}{\omega}.
\]

I extend the definition of the nozzle function to include gradient temperature terms,

\[
n(q) \equiv \alpha (1 - \alpha)^{(1 - \alpha)/\alpha} \frac{(fa)^{1/\alpha}}{(-h)^{(1 - \alpha)/\alpha}}
\]

\[
= \alpha (1 - \alpha)^{(1 - \alpha)/\alpha} \frac{(fa)^{1/\alpha}}{((ga - 2s/a)[da/dq] + 2[ds/dq])^{(1 - \alpha)/\alpha}}
\]

\[
= \frac{(fa)^{1/\alpha}}{(ga - 2s[da/dx] + 2a[ds/dx])^{(1 - \alpha)/\alpha}} \quad \text{for} \; h(q) < 0.
\]

The question of existence of a steady solution is reduced to the question of whether or not, upon integration, one can always determine

\[
\frac{d\omega}{dq} = \frac{d\omega}{dq} (q, \omega).
\]

Viewing \( d\omega/dq \) as a function of variables \( q \) and \( \omega \) that satisfies equation (6), one can divide the \( q-\omega \) plane into five regions depending on whether a solution for \( d\omega/dq \) exists. I do this below following the notation of CAK75 for early-type stars:

Region I : \( \omega < s \) and \( h(q) < 0 \): one solution;
Region II : \( \omega > s \) and \( h(q) < 0 \) and \( \beta(\omega)\dot{m} < n(q) \): two solutions;
Region III : \( \omega > s \) and \( h(q) > 0 \): one solution;
Region IV : \( \omega > s \) and \( h(q) < 0 \) and \( \beta(\omega)\dot{m} > n(q) \): no solution;
Region V : \( \omega < s \) and \( h(q) > 0 \): no solution.

As in Paper I, I make the following five assumptions with respect to the solution \( \omega(q) \): \( \omega(q) \) increases monotonically, the wind starts subsonic, the wind ends supersonic, the wind extends toward infinity, and \( d\omega/dq \) is continuous. Following the
arguments presented in Paper I, and with the additional assumption that asymptotically
\[
q \to \int_{x_0}^{\infty} \frac{1}{a} \, dx + q_0, \quad \text{i.e., } x \to \infty \quad \Rightarrow \quad a(q) \to x^2(q),
\]
and it satisfies the condition
\[
g(q)a(q) \to \text{const.}
\]
(11)
a solution to the one-dimensional equations must then have the following sequence in the \(q-\omega\) plane as \(q\) increases:

Region I : subsonic, \(h(q) < 0\);
Region II : supersonic, \(h(q) < 0\),
lower branch, \(\beta(\omega) \dot{m} < n(q)\);
Region II/Region IV boundary : supersonic, \(h(q) < 0\),
critical point, \(\beta(\omega) \dot{m} = n(q)\);
Region II : supersonic, \(h(q) > 0\),
upper branch, \(\beta(\omega) \dot{m} < n(q)\);
Region III : supersonic, \(h(q) > 0\).
(12)

I extend the definition of a critical point type solution discussed in Paper I to solutions of the equation of motion that satisfy the above conditions (eq. [12]). Thus, as in the isothermal wind case (Paper I), I have shown for the assumptions given here that a steady solution must be a critical point type solution of the equation of motion.

5. CRITICAL POINT CONDITIONS

As discussed in § 4, if a steady solution exists for a supersonic one-dimensional line-driven wind (with the five assumptions on the \(\omega(q)\) function described in § 4), it must present a critical point. The critical point is in the boundary between Region II and Region IV, that is, it is supersonic,
\[
\omega_c > s.
\]
(13)
The sum of the independent terms of the equation of motion (see eqs. [5] and [6]) is negative,
\[
h(q_c) < 0,
\]
and it satisfies the condition
\[
\beta(\omega_c) \dot{m} = n(q_c),
\]
(15)
where the subscript \(c\) denotes critical point.
I define
\[
\omega' \equiv \frac{d\omega}{dq},
\]
(16)
and following the notation of CAK75, I define
\[
F(q, \omega, \omega') \equiv \left( 1 - \frac{s}{\omega} \right) \omega' - h(q) - fa \left( \frac{1}{\dot{m}} \right) \omega' \left( \frac{1}{\dot{m}} \right).
\]
(17)

Then the following three conditions must hold at the critical point:

1. the equation of motion
\[
F = 0;
\]
(18)
2. the critical point at the boundary between Regions II and IV (§ 4)
\[
\frac{\partial F}{\partial \omega'} = 0;
\]
(19)
3. the continuity of \(\omega'(velocity gradients) (dF/dq = 0 combined with eq. [19])
\[
\frac{\partial F}{\partial q} + \omega' \frac{\partial F}{\partial \omega} = 0.
\]
(20)

One thus finds three equations (eqs. [18]–[20]) with four unknowns, namely, \(q_c, \omega_c, \omega'_c\), and \(\dot{m}\).

Therefore, the critical point cannot be uniquely determined by \(a(q), f(q), g(q),\) and \(s(q)\). Thus, when gas pressure effects are included, the nozzle function \(n(q)\) cannot by itself determine the exact position of the critical point, contrary to the case where gas pressure effects are neglected (Paper I).

The position of the critical point is determined with an additional model constraint that is normally the sonic point position. That is, the equation of motion, upon integration from the critical point to lower velocities, must obtain the correct sonic point.

In the Appendix I analyze in detail the critical point conditions and derive explicit expressions for \(\omega_c, \omega'_c,\) and \(\dot{m}\) as functions of the critical point \(q_c\) (eqs. [A15]–[A17]). Additionally, in the Appendix I also find that in order for \(q_c\) to be a critical point (i.e., in order for \(\omega_c, \omega'_c,\) and \(\dot{m}\) to be determinable), it must hold that
\[
h(q_c) < 0 \quad \text{and} \quad \frac{\alpha}{1 - \alpha \frac{s_c}{\dot{m}}} \frac{\partial n}{\partial q_c} > 0.
\]
(21)

For an isothermal wind \((ds/dq = 0)\), these two conditions for the existence of a critical point reduce to
\[
h(q_c) < 0 \quad \text{and} \quad \frac{\partial n}{\partial q_c} > 0.
\]
(22)

Although the nozzle function \(n\) cannot by itself determine a unique value for the critical point, it can constrain the location of the critical point, and in some cases it can be shown without additional calculations that a steady solution does not exist (e.g., in an isothermal wind with a monotonically decreasing nozzle function).

The existence of a critical point (i.e., the determination of values for \(\omega_c, \omega'_c,\) and \(\dot{m}\) [eqs. (A15)–(A17)] such that the critical point conditions hold [eqs. (18)–(20)]) does not imply that the equation of motion (eq. [6]) is locally or globally integrable. That is, the existence of a point that satisfies the critical point conditions does not ensure that a local (in the vicinity of the point) steady solution exists or that a global (throughout the spatial range being considered) steady solution exists.

In the work of CAK75 for line-driven stellar winds, the existence of a steady solution was proved by finding a solution for the particular case of stellar winds. In the following subsection I show that general criteria for the existence or nonexistence of local solutions for arbitrary geometries can be established through the nozzle function without integrating the equation of motion.
6. REQUIREMENTS FOR THE EXISTENCE OF LOCAL STEADY SOLUTIONS

A local solution in the vicinity of the critical point exists if, upon integration, the solution maintains itself in Region II (see eqs. [10] and [12]). This is equivalent to the condition that at the critical point

$$\beta(\omega_c + \Delta \omega) \dot{m} < n(q_c + \Delta q), \tag{23}$$

where $\Delta q$ is a variation of $q$ in the vicinity of $q_c$, and $\Delta \omega$ is the corresponding variation of $\omega$ determined upon the integration of the equation of motion.

Therefore, a local steady solution exists if

$$\left\{ \frac{d^2}{dq^2} [\beta(\omega) \dot{m} - n(q)] \right\}_{q_c} < 0. \tag{24}$$

I define

$$\beta' = \frac{d\beta}{d\omega}, \quad \beta'' = \frac{d^2\beta}{d\omega^2},$$

$$n' = \frac{dn}{dq}, \quad n'' = \frac{d^2n}{dq^2},$$

$$\omega' = \frac{d\omega}{dq}, \quad \omega'' = \frac{d^2\omega}{dq^2}. \tag{25}$$

From equation (24), if the critical point conditions hold for a given $q$, then a local solution in the vicinity of that point exists providing

$$\beta''(\omega) m(\omega')^2 + \beta'(\omega) \dot{m} \omega'' - n''(q) < 0. \tag{26}$$

For a given $q$, the variables $\omega$, $\omega'$, and $\dot{m}$ can be determined through equations (A15), (A16), and (A17), respectively. Considering the definition of $\beta(\omega)$ (eq. [7]), $\beta'(\omega)$ and $\beta''(\omega)$ are given by the following equations:

$$\beta'(\omega) = \frac{s}{\omega^3}, \tag{27}$$

$$\beta''(\omega) = -2 \frac{s}{\omega^7}. \tag{28}$$

The parameter $n''(q)$ can be calculated from the functions that define the specific model, namely $a(q)$, $s(q)$, $g(q)$, and $f(q)$, through equations (5) and (8). However, the evaluation of equation (26) also requires the determination of $\omega''$. This can be obtained as follows. A solution to the equation of motion must, of course, be such that the equation of motion $F = 0$ holds (eq. [6]; eq. [18]). Therefore,

$$\frac{dF}{dq} = 0. \tag{29}$$

That is,

$$\frac{\partial F}{\partial q} + \omega' \frac{\partial F}{\partial \omega} + \omega'' \frac{\partial F}{\partial \omega'} = 0. \tag{30}$$

Thus, in general,

$$\omega'' = -\frac{\omega' (\partial F / \partial \omega)}{\partial F / \partial \omega'}. \tag{31}$$

But at the critical point, both the denominator and the numerator of equation (31) are equal to zero (eqs. [19] and [20], respectively). Therefore,

$$\omega_c'' = \lim_{q \rightarrow q_c} \frac{-(\partial F / \partial q) - \omega' (\partial F / \partial \omega)}{(\partial F / \partial \omega')} \tag{32}$$

or

$$\omega_c'' = -\frac{d}{dq}(\partial F / \partial q) \left[ \frac{(d/q)(\partial F / \partial \omega)}{(d/q)(\partial F / \partial \omega')} \right]. \tag{33}$$

Thus, at the critical point

$$\frac{d}{dq} \left( \frac{\partial F}{\partial q} \right) + \frac{d}{dq} \left( \omega' \frac{\partial F}{\partial \omega} \right) + \omega'' \frac{d}{dq} \left( \frac{\partial F}{\partial \omega'} \right) = 0. \tag{34}$$

Therefore,

$$\frac{\partial^2 F}{\partial q^2} + \omega' \frac{\partial^2 F}{\partial \omega \partial q} + \omega'' \frac{\partial^2 F}{\partial \omega' \partial q} + \omega'' \frac{\partial^2 F}{\partial \omega \partial q} + \omega' \frac{\partial^2 F}{\partial \omega^2} + \omega'' \frac{\partial^2 F}{\partial \omega^2 \partial q}$$

$$+ \omega'' \left( \frac{\partial^2 F}{\partial \omega^2 \partial q} + \omega' \frac{\partial^2 F}{\partial \omega^2} + \omega' \frac{\partial^2 F}{\partial \omega^2} \right) = 0, \tag{35}$$

which is a second-order equation with respect to $\omega''$. The partial derivatives in equation (35) can be calculated through equation (17). The values of $\omega_c$, $\omega_c''$, and $\dot{m}$ can be calculated through equations (A15)–(A17).

Given one-dimensional line-driven wind models with arbitrary geometries, gravitational fields, flux distributions, and temperature structures, within the approach of this work, the first step in analyzing the existence of steady solutions is to determine for which spatial points the critical point conditions as discussed in § 5 hold.

If there are no points where the critical point conditions hold, then a steady solution does not exist. On the other hand, if these conditions hold for some of the points, then the next step is to determine which subset of points allow a local solution (i.e., satisfy eq. [26]). If there are no points where local solutions of the equation of motion are possible, then a steady solution does not exist.

However, the existence of local solutions does not ensure the existence of a global solution. For example, if a global minimum of the nozzle function exists farther out than the set of points that allow a local solution [and assuming $h(q) < 0$ from the set of local solution points out to the global minimum], then a global solution does not exist; this is because the condition $\beta(\omega) \dot{m} < n(q)$ for being in Region II of the $q-\omega$ plane (rather than the non-solution Region IV) breaks down before the solution is extended to the nozzle global minimum point. At the critical point $\beta(\omega_c) \dot{m} = n(q_c)$, and $\beta(\omega)$ is a monotonically increasing function (eq. [7]).

As indicated above, the local solutions in this work refer to local solutions in the vicinity of a critical point. The requirement
of a critical point for the existence of a steady solution was discussed in § 4.

Thus, the requirements for the existence of local steady solutions are

\[
\begin{align*}
    h(q_*) &< 0, \\
    \left( \frac{1}{2s_*} \frac{ds}{dq} \right)^2 + \frac{\alpha}{1 - \alpha} \frac{1}{n(q_*)} \frac{dn}{dq} &> 0, \\
    \beta''(\omega) n(\omega) - n''(q_*) &< 0.
\end{align*}
\]

7. REQUIREMENTS FOR THE EXISTENCE OF A GLOBAL STEADY SOLUTION

The existence of a global solution (a solution that spans throughout the spatial range of the model) requires the existence of a critical point (§ 4), which in turn implies that the critical point conditions must hold at the given point (§ 5), and requires, of course, the existence of a local solution (§ 6) that can be extended from the critical point toward infinity in the outward direction and toward the sonic point in the inward direction.

In addition, it is required that the global solution be such that upon integration in the inward direction the wind reaches sound speed at the sonic point. The exact position of the sonic point is a boundary condition of the model (e.g., in the CAK75 stellar wind model the position of the sonic point is assumed to be approximately equal to the stellar photospheric radius).

The set of points for which the critical point conditions hold, and that allow a local solution, define the range in which the critical point must be if a global solution exists. The exact position of the critical point is determined by initially guessing with a value within this range and iteratively adjusting the critical point position until the correct sonic point is achieved when integrating inward. This iterative process to determine the exact critical point position in a one-dimensional line-driven wind was originally applied by CAK75 in the study of stellar winds. Although CAK75 did not present an independent proof of the existence of local solutions as I do here, in § 8 I show that in the isothermal CAK75 model, for the stellar parameters used by CAK75, the points that allow local solutions extend from the photospheric radius to beyond 100 times the photospheric radius. In the original CAK75 model the iterative process was done over a range of a few integers of the photospheric radius.

Thus, the requirements for the existence of a global steady solution are (in addition to the requirements of existence of a local steady solution) that a critical point exists such that upon integration of the equation of motion, the following two conditions hold:

\[
\omega(q_*) = s, \\
\beta'(\omega) n < n(q_*).
\]

8. APPLICATION TO THE ISOThERMAL CAK75 STELLAR WIND

It is well known that when a Sobolev treatment is used for the line radiation force, a steady wind solution for early-type stars is obtained (e.g., Owocki & Puls 1999). In Paper I we illustrated concepts and notations by applying them to the well-known well-studied CAK75 stellar wind, neglecting effects of gas pressure. Here I apply the concepts and notations that I present in this second paper, but now including gas pressure effects under the assumption of an isothermal wind.

The purpose of this section in not to shed new light on the CAK75 model, which is already a well-studied model, but rather to illustrate the new concepts introduced in this paper by applying them to a problem familiar to a reader of the Astrophysical Journal. However, a new result is presented, that of the proof of existence of a local solution by analytical means rather than by numerical means. This is significant because the critical point, by definition, is in the boundary between a solution region and a non-solution region, and therefore numerical methods that typically average parameter values in the vicinity of a point to extrapolate a function will present difficulties around the critical point and will have to be adapted (at the critical point) to the specific case of line-driven winds.

In other words, subtle numerical methods have to be correctly and consistently introduced, geared specifically to the case of line-driven winds in order to integrate the equation of motion from the critical point. Since the purpose of this series of papers is to analyze the steady nature of line-driven disk winds in a form independent of previous numerical efforts, the fact that I have found an analytical method to determine the existence of local solutions, without integrating the equation of motion, becomes relevant. In particular, CAK75 did not prove the existence of local solutions but rather reported that they had obtained one through numerical integration. Since the difference between our models and those claiming “intrinsic unsteadiness” is a numerical one, how far our analytical conclusions can be taken becomes important.

As in Paper I, I introduce the following characteristic scales for the CAK75 model:

\[
r_0 = R, \quad B_0 = \frac{GM}{R^2} (1 - \Gamma), \quad A_0 = 4\pi R^2, \\
\gamma_0 = \frac{\kappa_\text{e} L}{c 4\pi R^2} \frac{k}{\kappa_\text{e} V_\text{in}^4},
\]

where \( R \) is the photospheric radius, \( G \) is the gravitational constant, \( M \) is the stellar mass, \( \kappa_\text{e} \) is the Thomson cross section per mass, \( c \) is the speed of light, \( L \) is the stellar luminosity, \( k \) and \( \alpha \) are the CAK75 line-force parameters, \( V_\text{in} \) is the ion thermal velocity, and \( \Gamma \) is the Eddington ratio given by

\[
\Gamma = \frac{\kappa_\text{e} L}{4\pi G M c^2}.
\]

For the CAK75 model the independent spatial variable is the distance \( r \) to the center of the star, and thus \( x = r/R \).

Taking \( x_0 = 1 \) and \( q_0 = -1 \), the variable \( q \) (eq. [3]) becomes

\[
q = -\frac{1}{x}.
\]

Equation (6) becomes the equation of motion, that is,

\[
\left( 1 - \frac{s}{\omega} \right) \frac{d\omega}{dq} = h(q) + f a \left( \frac{1}{\dot{m}} \frac{d\omega}{dq} \right) \alpha,
\]

where the function \( h(q) \) is given by

\[
h(q) = -g a - \frac{4s}{q} - 2 \frac{ds}{dq}.
\]
and in turn,
\[ ga = 1 \quad \text{and} \quad fa = \frac{1}{\alpha^\alpha (1 - \alpha)^{1-\alpha}}. \]  

(47)

Since I am assuming an isothermal wind \((ds/dq = 0)\),
\[ h(q) = -1 - \frac{4s}{q}. \]  

(48)

For the isothermal CAK75 stellar wind I implement a set of parameters corresponding to an O5f star, namely,
\[ M = 60 \, M_\odot, \quad \Gamma = 0.4, \quad R = 13.8 \, R_\odot, \]  

and I use the line-force parameters also used by CAK75, namely,
\[ k = 1/30 \quad \text{and} \quad \alpha = 0.7. \]  

(50)

To study the existence of steady solutions, I first consider the \(h\) function. As discussed in §5, \(h\) must be negative at the critical point. From Figure 1 I find that \(h\) is negative from the photospheric height to beyond 100 times the photospheric radius. The sonic radius is assumed here to be equal to the photospheric radius. The maximum possible value for the critical point position, as constrained by \(h(q)\), can be determined by equations (44) and (48). However, given the additional assumption that the sonic radius is equal to the photospheric radius, I do not expect the critical point to be beyond 100 times the photospheric radius.

Since I am assuming an isothermal wind, it follows that at the critical point the nozzle function must be increasing with position (eq. [22]). I show in Figure 2 the nozzle function and find that the nozzle function is monotonically increasing from the photospheric height to beyond 100 times the photospheric height. Thus, the critical point conditions (§5) hold for all spatial points between the photospheric radius and beyond 100 times this radius. In other words, for all these spatial points, there are well-defined, well-determined values for \(\omega_c\), \(\omega'_c\), and \(\dot{m}\).

The existence of local solutions in the vicinity of each critical point can be determined through equation (26). The expressions \(\beta'\) and \(\beta''\) in equation (26) depend on the critical point position through the condition \(\omega = \omega_c\), and \(\dot{m}\) also depends on the critical point (eq. [A17]). Figure 3 shows that all the spatial points up to 100 times the photospheric radius present local solutions in their respective vicinities. That is, throughout the
The global solution is found by determining the exact critical point position by the constraint of the sonic point position, as discussed in §7, and then integrating the equation of motion inward and outward. Figure 4 presents the velocity versus position in the CAK75 stellar wind for the assumed parameters. As a consistency check for the global solution, in Figure 5 I plot functions $n$ and $\beta m$. In the supersonic region while $h < 0$, for a steady solution to exist, it must hold that $\beta m < n$ for points other than the critical point, and $\beta m = n$ at the critical point.

By comparing the analysis in Paper I of a zero gas pressure CAK75 wind with this analysis of an isothermal CAK75 wind, I find the following two effects of including gas pressure (in addition to unavoidably complicating the analysis). First, in the zero gas pressure case the position of the critical point is undetermined and all points above the photospheric radius become critical points because the nozzle function is constant; however, when gas pressure effects are included, corrections to the nozzle function arise and a unique critical point is determined through the conditions discussed in §7. Second, in the zero gas pressure CAK75 wind one has to introduce the additional condition of requiring a critical point in order to uniquely determine the wind mass-loss rate.

Without gas pressure effects there are infinite possible solutions, each with a different wind mass-loss rate. They vary between the value corresponding to the solution that contains a critical point, down to wind mass-loss rates arbitrarily close to zero. When gas pressure effects are included, only one solution exists, which is the one that contains the critical point. Thus, when gas pressure effects are included, the physical solution for...
The equation of motion is found without additional conditions other than the equation of motion itself.

9. FUTURE APPLICATION TO LINE-DRIVEN DISK WINDS

This paper is part of an ongoing group effort to model and analyze the line-driven accretion disk wind scenario for QSOs. Our first 2.5 dimensional model results (Hillier et al. 2002) are encouraging in that they are roughly consistent with QSO observational constraints. Among the model-observational agreements is that we are finding steady/stable winds within our models and, therefore, steady wind line profiles.

However, reports that line-driven disk winds are “intrinsically unsteady” (see Paper I) persist, and therefore it becomes relevant to present clear straightforward evidence, independent of previous numerically intensive methods, on whether or not steady and stable line-driven disk wind solutions exist. If the line-driven disk winds were “intrinsically unsteady,” then one would have to discard the line-driven disk wind scenario on observational grounds.

After considerable efforts and mathematical analysis of the equations, we have concluded that, under the Sobolev approximation, a line-driven wind off a standard accretion disk (Shakura & Sunyaev 1973) is steady.

Given the extensive work that has been done in order to arrive at this conclusion, we have decided that it would be best to present it in a systematic form, placing emphasis on the physical ideas on which the results are based. In Paper I, using simplified models, we showed that the increase in gravity at the wind base along the streamlines, which is characteristic of accretion disk winds, does not imply an unsteady wind. In addition, we showed in Paper I that under reasonable (but simplified) conditions line-driven disk winds can be steady. The motivation for using the simplified models was, in turn, to simplify the mathematical analysis and thus focus on the physical principles.

However, Paper I left one important question unanswered: If one established an equivalent analysis to more-realistic disk models, models such as those currently used to numerically simulate 2.5 dimensional line-driven disk winds, would there continue to exist steady disk wind solutions? The answer is yes.

But in order to derive the answer, one obviously has to analyze a more realistic and mathematically more complex system. Our approach to presenting the results of the more realistic models has been to first establish a more detailed theoretical and mathematical framework that would in turn allow us to analyze the more realistic models.

The objective of this paper is to present this theoretical framework and to illustrate it by applying it to the well-known CAK75...
stellar wind. In a forthcoming paper (Paper III), we will apply the framework developed to the flux distribution corresponding to a standard Shakura & Sunyaev (1973) accretion disk.

10. SUMMARY AND CONCLUSIONS

In Paper I we used “simple” models that mimic the disk environment to show that steady wind solutions can exist. Paper I emphasized the underlying physics behind the steady nature of line-driven disk winds. The goal of this paper has been to extend the concepts introduced in Paper I and discuss important aspects of the analysis of steady or unsteady one-dimensional line-driven winds that were mentioned in Paper I but not discussed in detail.

Specifically, I show that when including gas pressures effects, the spatial dependence of the nozzle function continues to play a key role in determining the steady or unsteady nature of supersonic line-driven wind solutions. The existence or non-existence of local wind solutions can be determined through the nozzle function without integrating the equation of motion, as I discuss in detail in § 6. This provides a useful numerical test for models aimed at simulating line-driven disk winds.

This work sets a detailed framework with which we will analyze more-realistic models than the “simple” models of Paper I. In a following paper (Paper III), we shall apply the framework discussed here to the case of an exact Shakura-Sunyaev disk flux distribution and show that if the accretion disk is capable of sustaining the corresponding wind mass flow, the wind driven off a standard disk is steady. This is important in that, in turn, it shows that the likely scenario for the formation of absorption troughs in CVs is a line-driven disk wind. It also shows that a line-driven accretion disk wind continues to be a promising scenario to explain the broad absorption lines in QSOs. In the following paper we shall also compare the results with the cataclysmic variable disk wind models of Pereyra (1997) and Pereyra et al. (2000).

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APPENDIX A

ANALYSIS OF THE CRITICAL POINT CONDITIONS

The explicit forms of equations (18)–(20) are, respectively,

\[
\left(1 - \frac{s}{\omega}\right)\omega' - h - fa\left(\frac{1}{m}\right)^{\alpha} (\omega')^{\alpha} = 0,
\]

\[
\left(1 - \frac{s}{\omega}\right) - \alpha fa\left(\frac{1}{m}\right)^{\alpha} (\omega')^{\alpha-1} = 0,
\]

\[
\mathbf{s}\left(\frac{\omega'}{\omega}\right)^{2} - \frac{ds}{dq}\left(\frac{\omega'}{\omega}\right) - \frac{dh}{dq} - \frac{d(fa)}{dq}\left(\frac{1}{m}\right)^{\alpha} (\omega')^{\alpha} = 0.
\]

From equations (A1) and (A2), respectively, one then obtains

\[
fa\left(\frac{1}{m}\right)^{\alpha} (\omega')^{\alpha} = \left(1 - \frac{s}{\omega}\right)\omega' - h;
\]

\[
fa\left(\frac{1}{m}\right)^{\alpha} (\omega')^{\alpha} = \frac{\omega'}{\alpha} \left(1 - \frac{s}{\omega}\right).
\]

Therefore,

\[
\left(1 - \frac{s}{\omega}\right)\omega' - h = \frac{\omega'}{\alpha} \left(1 - \frac{s}{\omega}\right),
\]

which leads to

\[
\frac{\omega'}{\omega} = \frac{\alpha (-h)}{1 - \frac{s}{\omega}}.
\]

Equation (A7) can be rewritten in the form

\[
\frac{\omega'}{\alpha} \left(1 - \frac{s}{\omega}\right) = \frac{(-h)}{1 - \frac{s}{\omega}}.
\]

Comparing equations (A5) and (A8), I have

\[
fa\left(\frac{1}{m}\right)^{\alpha} (\omega')^{\alpha} = \frac{(-h)}{1 - \frac{s}{\omega}},
\]

\[
\frac{(-h)}{1 - \frac{s}{\omega}}.
\]
or

\[
\left(\frac{1}{m}\right)^\alpha (\omega')^\alpha = \frac{1}{fa} \frac{(-h)}{1 - \alpha}.
\] (A10)

Substituting equation (A10) into equation (A3), one finds

\[
\frac{1}{2} \left(\frac{\omega'}{\omega}\right)^2 - \frac{1}{2s} \frac{ds}{dq} \left(\frac{\omega'}{\omega}\right) - \frac{\alpha q}{1 - \alpha} \frac{(-h)}{2s} \frac{d\ln(n)}{dq} \left\{ \ln \left(\frac{(fa)^{1/\alpha}}{(-h)^{(1-\alpha)/\alpha}}\right) \right\} = 0.
\] (A11)

Substituting the nozzle function \( n \) (eq. [8]) into the natural logarithm of the third term of equation (A11), one has

\[
\frac{1}{2} \left(\frac{\omega'}{\omega}\right)^2 - \frac{1}{2s} \frac{ds}{dq} \left(\frac{\omega'}{\omega}\right) - \frac{\alpha - s}{n} \frac{(-h)}{2s} \frac{d\ln(n)}{dq} = 0.
\] (A12)

Assuming that the wind is either isothermal or that the temperature decreases with \( q \) (i.e., decreases with \( x \)) and taking the positive root of the quadratic equation (A12), one finds

\[
\omega' = \frac{1}{2s} \frac{ds}{dq} + \left\{ \left(\frac{1}{2s} \frac{ds}{dq}\right)^2 + \frac{\alpha - s}{n} \frac{(-h)}{2s} \frac{d\ln(n)}{dq} \right\}^{1/2}.
\] (A13)

Comparing equations (A7) and (A13), I have

\[
\frac{\alpha (-h)}{1 - \alpha} \omega = \frac{1}{2s} \frac{ds}{dq} + \left\{ \left(\frac{1}{2s} \frac{ds}{dq}\right)^2 + \frac{\alpha - s}{n} \frac{(-h)}{2s} \frac{d\ln(n)}{dq} \right\}^{1/2},
\] (A14)

and thus

\[
\omega_c = s + \alpha (-h) \frac{1}{1 - \alpha} \left\{ \frac{1}{2s} \frac{ds}{dq} + \left\{ \left(\frac{1}{2s} \frac{ds}{dq}\right)^2 + \frac{\alpha - s}{n} \frac{(-h)}{2s} \frac{d\ln(n)}{dq} \right\}^{1/2} \right\}^{-1}.
\] (A15)

Substituting equation (A15) into equation (A13), one finds

\[
\omega' = s + \alpha (-h) \frac{1}{1 - \alpha} \left\{ \frac{1}{2s} \frac{ds}{dq} + \left\{ \left(\frac{1}{2s} \frac{ds}{dq}\right)^2 + \frac{\alpha - s}{n} \frac{(-h)}{2s} \frac{d\ln(n)}{dq} \right\}^{1/2} \right\} + \alpha (-h) \frac{1}{1 - \alpha}.
\] (A16)

In turn, substituting equation (A16) into equation (A10) results in

\[
m = \left(\frac{fa}{1 - \alpha} \left(-h\right)^{\frac{1}{1 - \alpha}} (1 - \alpha)^{1/\alpha}\right) \left\{ \frac{1}{2s} \frac{ds}{dq} + \left\{ \left(\frac{1}{2s} \frac{ds}{dq}\right)^2 + \frac{\alpha - s}{n} \frac{(-h)}{2s} \frac{d\ln(n)}{dq} \right\}^{1/2} \right\}^{1/\alpha} + \alpha (-h) \frac{1}{1 - \alpha}.
\] (A17)

(cf. eqs. [36], [37], and [39] of CAK75).

Therefore, in order for \( q_c \) to be a critical point (i.e., in order for \( \omega_c, \omega'_c, \) and \( m \) to be determinable), it must hold that

\[
h(q_c) < 0 \quad \text{and} \quad \left. \frac{d\ln(n)}{dq} \right|_{q_c} > 0.
\] (A18)

For an isothermal wind \( (ds/dq = 0) \) these two conditions, for the existence of a critical point, reduce to

\[
h(q_c) < 0 \quad \text{and} \quad \left. \frac{dn}{dq} \right|_{q_c} > 0.
\] (A19)
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