Global well-posedness for the fractional Boussinesq–Coriolis system with stratification in a framework of Fourier–Besov type

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Abstract
We establish the global well-posedness of the 3D fractional Boussinesq–Coriolis system with stratification in a framework of Fourier type, namely spaces of Fourier–Besov type with underlying space being Morrey spaces (FBM-spaces, for short). Under suitable conditions and rescaled density fluctuation, the result is uniform with respect to the Coriolis and stratification parameters. We cover the critical case of the dissipation, namely half-Laplacian, in which the nonlocal dissipation has the same differential order as the nonlinearity and balances critically the scaling of the quadratic nonlinearities. As a byproduct, considering trivial initial temperature and null stratification, we also obtain well-posedness results in FBM-spaces for the fractional Navier–Stokes–Coriolis system as well as for the Navier–Stokes equations with critical dissipation. Moreover, since small conditions are taken in the weak norm of FBM-spaces, we can consider some initial data with arbitrarily large $H^s$-norms, $s \geq 0$.

Keywords Boussinesq–Coriolis system · Rotating fluids · Stratification · Fractional dissipation · Global well-posedness · Fourier–Besov–Morrey spaces

Mathematics Subject Classification 76D03 · 35A01 · 35Q35 · 35Q86 · 76U05 · 76D50 · 76D05 · 76E06

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1 Introduction

We are concerned with the initial value problem (IVP) for the 3D fractional Boussinesq–Coriolis equations with stratification (FBCS)

\[
\begin{aligned}
\frac{\partial u}{\partial t} + \nu(-\Delta)^{\alpha} u + \Omega \times u + (u \cdot \nabla) u + \nabla p &= g \theta e_3, \\
\frac{\partial \theta}{\partial t} + \eta(-\Delta)^{\alpha} \theta + (u \cdot \nabla) \theta &= -\mathcal{N}^2 u_3, \\
\text{div} u &= 0 \quad \text{for} \quad (x, t) \in \mathbb{R}^3 \times (0, \infty) \quad \text{and} \\
u(0, x) &= u_0(x), \quad \theta(0, x) = \theta_0(x) \quad \text{for} \quad x \in \mathbb{R}^3,
\end{aligned}
\]

(1.1)

where \( u = (u_1(x, t), u_2(x, t), u_3(x, t)) \), \( \theta = \theta(x, t) \) and \( p = p(x, t) \) stand for the fluid velocity, the density fluctuation and the pressure of the fluid, respectively. The kinetic viscosity, the thermal diffusivity and the gravity are respectively represented by the positive constants \( \nu, \eta \) and \( g \). The term \( \Omega \times u \) denotes the so-called Coriolis force where the parameter \( \Omega \neq 0 \) is the speed of rotation of the fluid around the vertical unit vector \( e_3 \).

The term \( g \theta e_3 \) comes from the Boussinesq approximation (see [12]) in which density variations influence proportionally in the gravitational term, while \( \mathcal{N}^2 u_3 \) carries information about the stratification effects where the stratification parameter \( \mathcal{N} > 0 \) represents the Brunt–Väisälä wave frequency related to the buoyancy of the fluid. Moreover, the divergence-free vector \( u_0(x) = (u_{0,1}, u_{0,2}, u_{0,3}) \) is the initial velocity and the scalar function \( \theta_0 = \theta_0(x) \) is the initial density disturbance.

The operator \((-\Delta)^{\alpha}\) is the fractional power of the minus Laplacian and we consider the range \( \frac{1}{2} \leq \alpha < \frac{5}{2} \). In the half-Laplacian case \( \alpha = 1/2 \), we have the critical dissipation in the sense that the nonlocal dissipation has the same differential order as the nonlinearity and balances critically the scaling of the quadratic nonlinearities in (1.1), which introduces further difficulties.

The two effects, one arising from the Coriolis force related to the parameter \( \Omega \) and the other from the stable stratification related to the parameter \( \mathcal{N} \), play an important role in large scale atmospheric dynamics in which the density fluctuation \( \theta \) is considered to depend only on the potential temperature. They are also used in the study of large-scale motions of the oceans. For further details, we refer the reader to [10,12,22]. In view of the analysis of asymptotic regimes as \( \Omega \) and \( \mathcal{N} \) go to infinity (fast oscillating/strongly stratified limit), an important subject is to find frameworks in which system (1.1) is well-posed with bound of the solution and initial-data size (or existence-time) independent of those parameters, that is, uniform with respect to \( \Omega \) and \( \mathcal{N} \) (see, e.g., [4,6,19]), at least uniformly for large \( \Omega \) and \( \mathcal{N} \) (say \(|\Omega|, |\mathcal{N}| > c_0\) where \( c_0 \) is a positive universal constant). Moreover, motivated by the study of statistical properties of turbulence, spaces containing functions nondecaying at infinity are of interest (see, e.g., [17–19]). For these purposes, we employ a framework of Fourier type introduced in [16] to analyze active scalar equations, namely Fourier–Besov–Morrey spaces \( \mathcal{F}N^{s}_{\mu, r, q} \) (FBM-spaces, for short). These spaces are of Fourier–Besov type with underlying space being Morrey spaces and, considering the same scaling, are larger than classical Fourier–Besov spaces \( \mathcal{F}B^{s}_{q, r} \) [24] (see Sect. 2.1, p. 4, for the definition and details). Throughout this paper, spaces of scalar and vector functions are denoted in the same way.

Considering \( \theta \equiv 0, \mathcal{N} = 0 \) and \( \Omega = 0 \) in (1.1), we have the 3D fractional Navier–Stokes equations (3DFNS) for which there is a wide literature about existence of global mild solutions in different critical frameworks. A Banach space \( X \) is said to be critical for (3DFNS) if \( \| f(x) \|_X \approx \| \lambda^{2\alpha-1} f(\lambda x) \|_X \) for all \( \lambda > 0 \), that is, the norm is invariant under the scaling \( f(x) \to \lambda^{2\alpha-1} f(\lambda x) \). Let us start by briefly reviewing some results for the case \( \alpha = 1 \) which
corresponds to the celebrated 3D Navier–Stokes equations. Without making a complete list, we would like to mention the global well-posedness results with small initial data in critical spaces such as Lebesgue \(L^n\) [25], Besov \(\dot{B}^{\mu}_{q,\infty}\) [8], Morrey \(M_{q,n-q}\) [20, 26], Fourier–Besov \(\dot{F}^{\mu}_{q,\infty}\) [24, 28], and BMO\(^{-1}\) [29], among others. For further details, we refer the reader to the book [31]. Also, there are some results about ill-posedness, for instance, see [7, 35] and their references for results in the space \(\dot{B}^{1}_{\infty,r}(\mathbb{R}^3)\) with \(1 \leq r \leq \infty\).

Still for (3DFNS), but with the fractional dissipation \(\alpha\), the author of [32] obtained the global existence of classical solutions for \(\alpha \geq \frac{5}{4}\), without any smallness condition. Nevertheless, the global well-posedness in the case \(\alpha < \frac{5}{4}\) is more subtle and an outstanding open problem. For small initial data in critical spaces, there are global well-posedness results in the Besov space \(\dot{B}^{1-2\alpha+\frac{3}{q}}_{\infty,\frac{3}{q}}(\mathbb{R}^3)\) [38], in the largest critical space \(\dot{B}^{1-2\alpha}_{\infty,\infty}(\mathbb{R}^3)\) with \(\frac{1}{2} < \alpha < 1\) [40], in the Triebel-Lizorkin space \(\dot{F}^{\alpha}_{\frac{3}{1+\alpha},2}(\mathbb{R}^3)\) with \(1 < \alpha < \frac{5}{4}\) [13], in the Fourier–Besov space \(\dot{F}^{4-2\alpha-\frac{3}{q}}_{q,\infty}(\mathbb{R}^3)\) for \(1 \leq q \leq r \leq 2\) and \(\frac{1}{2} < \alpha < \frac{5}{4} - \frac{3}{2q}\), and for \(\alpha = \frac{1}{2}\) with \(r = 1\) and \(1 \leq q \leq \infty\) [39], and in the Fourier–Besov–Morrey space \(\mathcal{F}^{4-2\alpha-\frac{3}{q}}_{\mu,\frac{3}{q}}(\mathbb{R}^3)\) for \(\frac{1}{2} < \alpha < \frac{5}{4} - \frac{3}{2q}\), \(0 \leq \mu < 3\), \(1 \leq q < \infty\) and \(1 \leq r \leq \infty\) (see [14] with \(\Omega = 0\)). Moreover, we mention ill-posedness results in the largest critical space \(\dot{B}^{1-2\alpha}_{\infty,\infty}(\mathbb{R}^3)\) for \(1 < \alpha < \frac{5}{4}\) [11] and in the Triebel-Lizorkin space \(\dot{F}^{\frac{\alpha}{3}}_{\frac{3}{1+\alpha},r}(\mathbb{R}^3)\) for \(r > 2\) and \(1 < \alpha < \frac{5}{4}\) [13].

Another relevant model covered by (1.1) corresponds to the case \(\theta = 0\), \(\mathcal{N} = 0\) and general \(\Omega\), namely the fractional Navier–Stokes–Coriolis system. For the value \(\alpha = 1\), that is, the classical Navier–Stokes–Coriolis system, we have \(\Omega\)-uniform global well-posedness results for small initial data in critical Fourier transform-based functional spaces, for clarity, critical with respect to the (3DFNS)-scaling. For instance, Hieber and Shibata [21] showed existence of a unique global mild solution in \(H^\frac{1}{2}(\mathbb{R}^3)\), where the smallness condition is uniform w.r.t. \(\Omega\). After, Giga et al [19] proved \(\Omega\)-uniform global well-posedness in \(FM_0^{-1}(\mathbb{R}^3)\) which can be identified with \(\dot{F}B^{-1}_{1,1}\) and permits to consider spatially nondecaying and almost periodic initial-data. The \(\Omega\)-uniform well-posedness in \(\dot{F}B^{\frac{2}{2}}_{q,\infty}(\mathbb{R}^3)\) with \(1 < q \leq \infty\) and \(\dot{F}B^{-1}_{1,2}(\mathbb{R}^3)\) were proved by Konieczny and Yoneda [28] and Iwabuchi and Takada [24], respectively. Employing the framework of FBM-spaces, Almeida et al. [1] obtained the \(\Omega\)-uniform global well-posedness in \(\mathcal{F}^{2-\frac{1}{q},\infty}_{q,\mu}(\mathbb{R}^3)\) where \(1 \leq q < \infty\) and \(0 \leq \mu < 3\) with \(\mu \neq 0\) when \(q = 1\). Moreover, we have ill-posedness in \(\dot{F}B^{-1}_{1,r}(\mathbb{R}^3)\) when \(2 < r \leq \infty\) (see [24]). For general index \(\alpha\), we have \(\Omega\)-uniform global well-posedness results in the Lei-Lin-type space \(\lambda^{1-2\alpha}(\mathbb{R}^3)\) with \(\frac{1}{2} \leq \alpha \leq 1\) [36], in the Fourier–Besov space \(\dot{F}B^{4-2\alpha-\frac{3}{q}}_{q,\infty}(\mathbb{R}^3)\) with \(\frac{3}{2} \leq \alpha < \frac{1}{3} - \frac{q}{4}\), \(\frac{2}{3} \leq q \leq \infty\) and \(1 < r \leq \infty\) [37], and in the FBM-space \(\mathcal{F}^{4-2\alpha-\frac{1}{q}}_{\mu,\frac{3}{q}}(\mathbb{R}^3)\) with \(\frac{1}{2} < \alpha \leq \frac{5}{6} - \frac{3-\mu}{2q}\), \(0 \leq \mu < 3\), \(1 \leq q < \infty\) and \(1 \leq r \leq 2\) [14].

For (1.1) with \(\alpha = 1\), Sun and Cui [34] showed the global well-posedness with small initial-data in the critical space \(\dot{F}B^{2-\frac{3}{q}}_{q,\infty}(\mathbb{R}^3)\) for \(1 < q \leq \infty\) and \(1 \leq r < \infty\) and in \(\dot{F}B^{-1}_{1,r}(\mathbb{R}^3)\) for \(1 \leq r \leq 2\). Moreover, they also proved the ill-posedness in the Fourier–Besov space \(\dot{F}B^{-1}_{1,r}(\mathbb{R}^3)\) for \(2 < r \leq \infty\).
Finally, it is worth to mention that another class of study investigates the dispersive effect of the associated linear semigroup in order to show global well-posedness, without smallness conditions, for geophysical (and related) models with rotation and/or stratification sufficiently large in comparison with the initial-data norm (see, e.g., [5,6,9,10,22,23,27]). More precisely, they show global well-posedness and asymptotic results in $H^s$ for large-enough values of the parameters $\Omega$ and/or $N$, where the bounds from below for them depend on the initial-data size. See also [15] for an extension of these ideas to obtain global well-posedness in the context of Besov spaces for the case $\theta \equiv 0$, $\mathcal{N} = 0$ and $\alpha = 1$ in (1.1) (Navier–Stokes–Coriolis equations).

In view of previous references, even in the classical dissipation case $\alpha = 1$, there are much less well-posedness results for system (1.1) than Navier–Stokes and Navier–Stokes–Coriolis equations. Motivated by that and bearing in mind the papers [1,16], we consider the framework of FBM-spaces and obtain a global well-posedness result for (1.1) with $\nu = \eta$ (see Theorem 3.1, Remarks 3.2, 3.3), which provides a larger critical initial-data class for global well-posedness that allows to consider singular and nondecaying initial data. Since small conditions are taken in the weak norm of FBM-spaces, we can consider some initial data with arbitrarily large $H^s$-norms, $s \geq 0$, e.g., suitable cut-offs in Fourier variables of homogeneous functions (see Lemma 2.1(v)). Choosing suitable indexes of the spaces, we cover the fractional range $\frac{1}{2} \leq \alpha < \frac{3}{2}$ including the critical dissipation $\alpha = 1/2$. For that, besides the norm of the persistence space, we employ an auxiliary norm of Chemin-Lerner type based on $\mathcal{F}N^s_{q,\mu,r}$-spaces in order to estimate the linear and bilinear terms in (1.1). Considering the rescaled variable $\nu = (u, \sqrt{g}^\theta/N)$ and assuming, for instance, $\mathcal{N}\sqrt{g}/2 \leq |\Omega| \leq 2\mathcal{N}\sqrt{g}$, the well-posedness result is uniform with respect to the parameters $\mathcal{N}$ and $\Omega$. The present paper is part of Ph.D. thesis [3]. After concluding this work, we became aware of the preprint [2] in which the authors independently proved a well-posedness result with $\alpha = 1$ and in a more restricted family of $\mathcal{F}N^s_{q,\mu,r}$-spaces, where $s = 2 - \frac{3-\mu}{q}$, $1 \leq q < \infty$, $0 \leq \mu < 3$ and $1 \leq r \leq 2$.

The outline of this manuscript is as follows. In Sect. 2, we first recall some definitions, notations, basic tools and the definition of Fourier–Besov–Morrey spaces. After, we define the fractional Boussinesq–Coriolis stratification semigroup, suitable time-dependent spaces, and mild solutions for (1.1). In Sect. 3 we state our main result and make some comments about it. Section 4 is devoted to prove linear and bilinear estimates in our functional setting as well as the proof of the main result.

2 Preliminaries

In this section we recall some basic properties about Fourier–Besov–Morrey spaces, the fractional Boussinesq–Coriolis stratification semigroup and other analysis tools we shall use throughout this work.

2.1 Fourier–Besov–Morrey spaces

Fourier–Besov–Morrey spaces (FBM-spaces) are constructed by mean of a kind of localization procedure in Fourier variables on the well-known Morrey spaces (see [16]). Morrey spaces are defined as follows. Let $B_{d}(x_0)$ be the open ball in $\mathbb{R}^n$ centered at $x_0$ and with radius $d > 0$. For $1 \leq q < \infty$ and $0 \leq \mu < n$, the homogeneous Morrey space $\mathcal{M}_{q,\mu} = \mathcal{M}_{q,\mu}(\mathbb{R}^n)$
is the space of all \( f \in L_{loc}^q \) such that
\[
\| f \|_{q, \mu} = \sup_{x_0 \in \mathbb{R}^n, \epsilon > 0} \left( \epsilon^{-\mu} \| f \|_{L^q(B_d(x_0))} \right) < \infty. 
\] (2.1)

In the case \( q = 1, \mathcal{M}_{1, \mu} \) is a subspace of Radon measures and the \( L^1 \)-norm in (2.1) should be understood as the total variation of the measure \( f \) on \( B_d(x_0) \). The space \( \mathcal{M}_{q, \mu} \) endowed with \( \| \cdot \|_{q, \mu} \) is a Banach space. For more details, we refer the reader to [26] and their references.

Next we recall some notations, definitions and properties about Littlewood–Paley decomposition. First recall the notations \( \mathcal{S}(\mathbb{R}^n) \) and \( \mathcal{S}'(\mathbb{R}^n) \) to stand for the Schwartz class and the space of tempered distributions, respectively. Also, the Fourier transform of \( u \) in \( \mathcal{S}'(\mathbb{R}^n) \) is denoted by \( \hat{u} \).

Consider the ring \( C = \{ \xi \in \mathbb{R}^n; \frac{3}{4} \leq \| \xi \| \leq \frac{5}{3} \} \) and \( \varphi \) a smooth function supported in \( C \) satisfying \( 0 \leq \varphi \leq 1 \) and
\[
\sum_{j \in \mathbb{Z}} \varphi_j(\xi) = 1, \quad \text{for all } \xi \neq 0, \text{ where } \varphi_j(\xi) = \varphi(2^{-j} \xi). 
\] (2.2)

For each \( j \in \mathbb{Z} \), the localization operator \( \Delta_j \) is defined via Fourier transform as
\[
[\Delta_j f](\xi) = \varphi_j(\xi) \hat{f}(\xi) \text{ in } \mathcal{S}'(\mathbb{R}^n),
\]
and the operator \( S_j \) as
\[
S_j f = \sum_{k \leq j-1} \Delta_k f \text{ in } \mathcal{S}'(\mathbb{R}^n). 
\]

Since \( \text{supp} \varphi_j \subset 2^j C \) we verify that
\[
\Delta_j \Delta_k f = 0, \text{ if } |j - k| \geq 2,
\]
and
\[
\Delta_j [S_{k-1} f \Delta_k g] = 0, \text{ if } |j - k| \geq 5.
\]

Let \( 1 \leq q < \infty, 0 \leq \mu < n, 1 \leq r < \infty \) and \( s \in \mathbb{R} \). The Fourier–Besov–Morrey space \( \mathcal{F}N^s_{q, \mu, r} \) is the set of all distributions \( f \in \mathcal{S}'/\mathcal{P} \), where \( \mathcal{P} \) is the set of all polynomials in \( \mathbb{R}^n \), such that \( \varphi_j \hat{f} \in \mathcal{M}_{q, \mu} \), for all \( j \in \mathbb{Z} \), and
\[
\| f \|_{\mathcal{F}N^s_{q, \mu, r}} = \begin{cases} 
\left( \sum_{j \in \mathbb{Z}} (2^{js} \| \varphi_j \hat{f} \|_{q, \mu})^r \right)^{1/r} < \infty, & 1 \leq r < \infty \\
\sup_{j \in \mathbb{Z}} 2^{js} \| \varphi_j \hat{f} \|_{q, \mu} < \infty, & r = \infty.
\end{cases}
\] (2.3)

The pair \( (\mathcal{F}N^s_{q, \mu, r}, \| \cdot \|_{\mathcal{F}N^s_{q, \mu, r}}) \) is a Banach space. The next lemma contains some basic properties of Morrey (see [26]) and FBM-spaces (see [1,16]).

**Lemma 2.1** Let \( s_1, s_2 \in \mathbb{R}, 1 \leq p_1, p_2, p_3 < \infty \) and \( 0 \leq \mu_1, \mu_2, \mu_3 < n \).

(i) (Hölder’s inequality) Let \( \frac{1}{p_3} = \frac{1}{p_1} + \frac{1}{p_2} + \frac{\mu_3}{p_3} = \frac{\mu_1}{p_2} + \frac{\mu_2}{p_1} \). If \( f_i \in \mathcal{M}_{p_i, \mu_i} \) for \( i = 1, 2 \), then \( f_1 f_2 \in \mathcal{M}_{p_3, \mu_3} \) and
\[
\| f_1 f_2 \|_{p_3, \mu_3} \leq \| f_1 \|_{p_1, \mu_1} \| f_2 \|_{p_2, \mu_2}. 
\]
(ii) (Young’s inequality) If \( \varphi \in L^1(\mathbb{R}^n) \) and \( g \in \mathcal{M}_{p_1,\mu_1} \) then
\[
\| \varphi \ast g \|_{p_1,\mu_1} \leq \| \varphi \|_1 \| g \|_{p_1,\mu_1},
\]
where \( \ast \) denotes the standard convolution operator.

(iii) (Bernstein-type inequality) Let \( p_2 \leq p_1 \) be such that \( \frac{n-\mu_1}{p_1} \leq \frac{n-\mu_2}{p_2} \). If \( A > 0 \) and \( \text{supp} (\hat{f}) \subset \{ \xi \in \mathbb{R}^n; |\xi| \leq A2^j \} \), then
\[
\| \xi^\beta \hat{f} \|_{p_2,\mu_2} \leq C 2^{j|\beta|+j\left(\frac{n-\mu_2}{p_2} - \frac{n-\mu_1}{p_1}\right)} \| \hat{f} \|_{p_1,\mu_1},
\]
where \( \beta \) is the multi-index, \( j \in \mathbb{Z} \), and \( C > 0 \) is a constant independent of \( j, \xi \) and \( f \).

(iv) (Sobolev-type embedding) For \( p_2 \leq p_1 \) and \( s_2 \leq s_1 \) satisfying \( s_2 + \frac{n-\mu_2}{p_2} = s_1 + \frac{n-\mu_1}{p_1} \), we have the continuous inclusion
\[
\mathcal{F}^{s_1}_{p_1,\mu_1,\tau_1} \subset \mathcal{F}^{s_2}_{p_2,\mu_2,\tau_2},
\]
for all \( 1 \leq \tau_1 \leq \tau_2 \leq \infty \).

(v) The space \( \mathcal{F}^{s}_{q,\mu,r} \) contains homogeneous functions of degree \( -h = s - n + \frac{n-\mu}{q} \).

### 2.2 Fractional Boussinesq–Coriolis stratification semigroup and mild solutions

In this subsection, by following \cite{27} (see also \cite{22,34}), we recall how to rewrite system (1.1) as an integral equation. Considering \( N = N\sqrt{g} \), \( v = (v^1, v^2, v^3, v^4) = (u^1, u^2, u^3, \sqrt{g}\theta/N) \), \( v_0 = (v_0^1, v_0^2, v_0^3, v_0^4) = (u_0^1, u_0^2, u_0^3, \sqrt{g}\theta_0/N) \), and \( \tilde{\nabla} \equiv (\partial_1, \partial_2, \partial_3, 0) \), we can convert system (1.1) to
\[
\begin{aligned}
\partial_t v + Av + Bv + \tilde{\nabla} p &= -(v \cdot \tilde{\nabla}) v, \quad \text{in } \mathbb{R}^3 \times (0, \infty) \\
\tilde{\nabla} \cdot v &= 0, \quad \text{in } \mathbb{R}^3 \times (0, \infty) \\
v(x, 0) &= v_0(x), \quad \text{in } \mathbb{R}^3,
\end{aligned}
\]
where
\[
A = \begin{pmatrix}
\nu(-\Delta)^{\alpha} & 0 & 0 & 0 \\
0 & \nu(-\Delta)^{\alpha} & 0 & 0 \\
0 & 0 & \nu(-\Delta)^{\alpha} & 0 \\
0 & 0 & 0 & \eta(-\Delta)^{\alpha}
\end{pmatrix}
\quad \text{and } B = \begin{pmatrix}
0 & -\Omega & 0 & 0 \\
\Omega & 0 & 0 & 0 \\
0 & 0 & 0 & -N \\
0 & 0 & N & 0
\end{pmatrix}.
\]

The fractional Boussinesq–Coriolis stratification semigroup \((\text{FBCS})\)-semigroup, namely the semigroup associated to the linear part of (2.5), is denoted by \( \{S^{q}_{\Omega,N}(t)\}_{t \geq 0} \). For \( v = \eta \), we have the following expression for \( \{S^{q}_{\Omega,N}(t)\}_{t \geq 0} \) in Fourier variables
\[
S^{q}_{\Omega,N}(t) f = \cos \left( \frac{|\xi|}{\eta} \right) e^{-\nu t|\xi|^{2\alpha}} M_1(\xi) \hat{f} + \sin \left( \frac{|\xi|}{\eta} \right) e^{-\nu t|\xi|^{2\alpha}} M_2(\xi) \hat{f} + e^{-\nu t|\xi|^{2\alpha}} M_3(\xi) \hat{f},
\]
where, for each \( \xi = (\xi_1, \xi_2, \xi_3) \in \mathbb{R}^3 \),
\[
|\xi| = \sqrt{\xi_1^2 + \xi_2^2 + \xi_3^2}, \quad |\xi'| = \sqrt{N^2\xi_1^2 + N^2\xi_2^2 + \Omega^2\xi_3^2}.
\]
formally convert system (2.5) to the integral equation

\[
M_1(\xi) = \frac{1}{|\xi|'^2} \begin{pmatrix}
\Omega^2\xi_3^2 & 0 & -N^2\xi_1\xi_3 & \Omega N\xi_2\xi_3 \\
0 & \Omega^2\xi_3^2 & -N^2\xi_2\xi_3 & -\Omega N\xi_1\xi_3 \\
-\Omega^2\xi_1\xi_3 & -\Omega^2\xi_2\xi_3 & N^2(\xi_1^2 + \xi_2^2) & 0 \\
\Omega N\xi_2\xi_3 & -\Omega N\xi_1\xi_3 & 0 & N^2(\xi_1^2 + \xi_2^2)
\end{pmatrix},
\]

\[
M_2(\xi) = \frac{1}{|\xi|'/\xi|'} \begin{pmatrix}
0 & -\Omega\xi_3^2 & \Omega\xi_2\xi_3 & N\xi_1\xi_2 \\
-\Omega\xi_3^2 & 0 & -\Omega\xi_1\xi_3 & N\xi_2\xi_3 \\
-\Omega\xi_1\xi_3 & -\Omega\xi_2\xi_3 & 0 & -N(\xi_1^2 + \xi_2^2) \\
N(\xi_1^2 + \xi_2^2) & -N(\xi_1^2 + \xi_2^2) & 0 & 0
\end{pmatrix},
\]

and

\[
M_3(\xi) = \frac{1}{|\xi|'^2} \begin{pmatrix}
N^2\xi_2^2 & -N^2\xi_1\xi_2 & 0 & -\Omega N\xi_2\xi_3 \\
-N^2\xi_1\xi_2 & N\xi_1^2 & 0 & \Omega N\xi_1\xi_3 \\
0 & 0 & 0 & 0 \\
-\Omega N\xi_2\xi_3 & \Omega N\xi_1\xi_3 & 0 & \Omega^2\xi_3^2
\end{pmatrix}.
\]

Thus, denoting the symbol of \(S_{\Omega,N}^\alpha(t)\) in (2.6) by \([S_{\Omega,N}^\alpha(t)]^\prime(\xi)\), we can write

\[
[S_{\Omega,N}^\alpha(t)f]^\prime(\xi) = [S_{\Omega,N}^\alpha(t)]^\prime(\xi)\hat{f}(\xi).
\]

As pointed out by Sun and Cui [34], the components \(M^j_{j,k}\) of the matrix \(M_j(\xi)\) can be estimated as

\[
|M^j_{j,k}(\xi)| \leq L =: \max\left\{2, \frac{|\Omega|}{\mathcal{N} \sqrt{\mathcal{R}}}, \frac{\mathcal{N} \sqrt{\mathcal{R}}}{|\Omega|}\right\}
\]

for \(1 \leq j, k \leq 4, 1 \leq l \leq 3\) and for all \(\xi \in \mathbb{R}^3\). (2.7)

This bound will be used often in the estimates for the (FBCS)-semigroup \(\{S_{\Omega,N}^\alpha(t)\}_{t>0}\).

Now we recall the extended Helmholtz projection operator \(\tilde{\mathcal{P}} = (\tilde{P}_{jk})_{4 \times 4}\)

\[
\tilde{P}_{jk} = \begin{cases}
\delta_{jk} + R_j R_k, & 1 \leq j, k \leq 3 \\
\delta_{jk}, & \text{otherwise},
\end{cases}
\]

where \(\delta_{jk}\) denotes the Kronecker delta and \(R_j\) is the Riesz transform on \(\mathbb{R}^3\) for each \(j = 1, 2, 3\). After applying this operator on (2.5), we can employ Duhamel’s principle in order to formally convert system (2.5) to the integral equation

\[
v(t) = S_{\Omega,N}^\alpha(t)v_0 + B(v, v),
\]

where the bilinear operator \(B(v, w)\) is defined via the Fourier transform as

\[
[B(v, w)]^\prime(\xi, t) = -\int_0^t [S_{\Omega,N}^\alpha(t - \tau)]^\prime(\xi, [\tilde{P}]^\prime(\xi, i\xi, 0)^T \cdot [v \otimes w]^\prime(\xi, \tau)] d \tau.
\]

(2.10)

Observe that the projection \(\tilde{P}\) has the symbol

\[
[\tilde{P}_{mk}]^\prime(\xi) = \begin{cases}
\delta_{mk} - (\xi_m \xi_k) / |\xi|^2, & 1 \leq m, k \leq 3 \\
\delta_{mk}, & \text{otherwise}
\end{cases}.
\]
Throughout the paper, solutions for the integral equation (2.9) are called mild solutions for system (1.1).

Expression (2.13) is known as Bony’s paraproduct formula (see [31]). For an abstract quadratic equation, after obtaining the needed estimates, Lemma 2.2

Let \( \| \cdot \| \) be a Banach space and \( B : X \times X \rightarrow X \) a bilinear operator satisfying \( \| B(x_1, x_2) \| \leq K \| x_1 \| \| x_2 \| \) for all \( x_1, x_2 \), where \( K > 0 \) is a constant. If \( 0 < \varepsilon < \frac{1}{4K} \) and \( \| y \| \leq \varepsilon \), then the equation \( x = y + B(x, x) \) has a solution in \( X \). Moreover, this solution is unique in the closed ball \( \{ x \in X; \| x \| \leq 2\varepsilon \} \) and \( \| x \| \leq 2 \| y \| \). The solution depends continuously on \( y \); in fact, if \( \| \tilde{y} \| \leq \varepsilon, \tilde{x} = \tilde{y} + B(\tilde{x}, \tilde{x}) \) and \( \| \tilde{x} \| \leq 2\varepsilon \), then

\[
\| x - \tilde{x} \| \leq (1 - 4K\varepsilon)^{-1} \| y - \tilde{y} \|. \tag{2.14}
\]

In order to avoid extensive fixed point computations, we shall use the following lemma (see [31]) for an abstract quadratic equation, after obtaining the needed estimates.

**Lemma 2.2** Let \( (X, \| \cdot \|) \) be a Banach space and \( B : X \times X \rightarrow X \) a bilinear operator satisfying \( \| B(x_1, x_2) \| \leq K \| x_1 \| \| x_2 \| \) for all \( x_1, x_2 \), where \( K > 0 \) is a constant. If \( 0 < \varepsilon < \frac{1}{4K} \) and \( \| y \| \leq \varepsilon \), then the equation \( x = y + B(x, x) \) has a solution in \( X \). Moreover, this solution is unique in the closed ball \( \{ x \in X; \| x \| \leq 2\varepsilon \} \) and \( \| x \| \leq 2 \| y \| \). The solution depends continuously on \( y \); in fact, if \( \| \tilde{y} \| \leq \varepsilon, \tilde{x} = \tilde{y} + B(\tilde{x}, \tilde{x}) \) and \( \| \tilde{x} \| \leq 2\varepsilon \), then

\[
\| x - \tilde{x} \| \leq (1 - 4K\varepsilon)^{-1} \| y - \tilde{y} \|. \tag{2.14}
\]

Observe that, by writing \( y = S^p_{I, N}(t)v_0 \) and \( B(v, w) \) as in (2.10), the integral equation (2.9) presents the form \( v = y + B(v, v) \) required in the above lemma.

Finally we define two time-dependent spaces. Let \( 1 \leq p \leq \infty, 0 < T \leq \infty \) and \( I = (0, T) \). The Banach spaces \( L^p(I; \mathcal{F}N^s_{q, \mu, r}) \) and \( \mathcal{L}^p(I; \mathcal{F}N^s_{q, \mu, r}) \) are the set of Bochner measurable functions from \( I \) to \( \mathcal{F}N^s_{q, \mu, r} \) with respective norms given by

\[
\| f \|_{L^p(I; \mathcal{F}N^s_{q, \mu, r})} = \left\| \left\| f(\cdot, t) \right\|_{\mathcal{F}N^s_{q, \mu, r}} \right\|_{L^p(I)} = \left\| \left( \sum_{j \in \mathbb{Z}} (2^{js} \| \varphi_j \hat{f} \|_{\mu, r}) \right)^{1/r} \right\|_{L^p(I)} \tag{2.15}
\]
and

\[
\| f \|_{L^p(I; \mathcal{F}N^r_q, \mu, r)} = \left( \sum_{j \in \mathbb{Z}} (2^{js} \| \varphi_j \hat{f} \|_{L^p(I; \mathcal{M}_{q, \mu})})^r \right)^{1/r} = \left( \sum_{j \in \mathbb{Z}} (2^{js} \| \varphi_j \hat{f} \|_{q, \mu, L^p(I)})^r \right)^{1/r}.
\]

(2.16)

We use the notation

\[X_r := \mathcal{L}^\infty(I; \mathcal{F}N^r_q, \mu, r) \cap \mathcal{L}^1(I; \mathcal{F}N^r_{q,+2\mu, r}).\]

### 3 Main result

In this section we state the main result of this work.

**Theorem 3.1** Let \( v = \eta, I = (0, \infty), 0 \leq \mu < 3, 1 \leq q < \infty \) and \( s = 4 - 2\alpha - \frac{3-\mu}{q} \). Assume that \( \alpha \) and \( r \) satisfy either the following conditions:

(i) \( \frac{1}{2} < \alpha < \frac{5}{2} - \frac{3-\mu}{2q} \) and \( 1 \leq r \leq \infty \), or

(ii) \( \alpha = \frac{5}{2} - \frac{3-\mu}{2q}, \mu = 0 \) and \( q \leq r \leq 2 \), or

(iii) \( \alpha = \frac{1}{2} \) and \( r = 1 \).

Let \( (\Omega, \mathcal{N}) \in (\mathbb{R} - \{0\})^2 \) and \( v_0 = (u_0, \sqrt{g} \theta_0 / \mathcal{N}) \in \mathcal{F}N^r_q, \mu, r \) be such that \( u_0 \) is a divergence-free vector field and \( \theta_0 \) a scalar field. Then, there are two constants \( \varepsilon = \varepsilon(v, \Omega, \mathcal{N}) > 0 \) and \( C = C(v, \Omega, \mathcal{N}) > 0 \) such that if \( \| v_0 \|_{\mathcal{F}N^r_q, \mu, r} \leq \varepsilon \) then (1.1) has a unique global mild solution \( v = (u, \sqrt{g} \theta / \mathcal{N}) \in X_r \) such that \( u \) is divergence-free and \( v \) is unique in the closed ball \( \{v \in X_r; \| v \|_{X_r} \leq C\varepsilon\} \). Moreover, the solution \( (u, \sqrt{g} \theta / \mathcal{N}) \) is weakly time-continuous from \( [0, \infty) \) to \( S'(\mathbb{R}^3) \) and depends continuously on the initial data \((u_0, \sqrt{g} \theta_0 / \mathcal{N})\).

Assuming further that \( \mathcal{N}, \sqrt{g}/2 \leq |\Omega| \leq 2\mathcal{N}\sqrt{g} \), we can take the constants \( \varepsilon \) and \( C \) independent of \( \Omega \) and \( \mathcal{N} \). So, for the rescaled unknown variable \( v = (u, \sqrt{g} \theta / \mathcal{N}) \), we obtain the uniform global well-posedness with respect to both \( \Omega \) and \( \mathcal{N} \).

**Remark 3.2** Similar computations, using the fractional Stokes–Coriolis semigroup \( S^\varepsilon_{\Omega}(t) \), instead of the fractional Boussinesq–Coriolis stratification semigroup, which is defined via the Fourier transform as

\[
\left[ S^\varepsilon_{\Omega}(t) f \right] \hat{\xi}(\xi) = \left[ e^{-\varepsilon|\xi|^2 t} \cos \left( \frac{\Omega \xi_3}{|\xi|} t \right) I + e^{-\varepsilon|\xi|^2 t} \sin \left( \frac{\Omega \xi_3}{|\xi|} t \right) \hat{M}(\xi) \right] \hat{f}(\xi),
\]

where \( I \) is the identity matrix of order \( 3 \times 3 \) and \( \hat{M}(\xi) \) is the matrix

\[
\hat{M}(\xi) = \frac{1}{|\xi|} \begin{pmatrix}
0 & \xi_3 & -\xi_2 \\
-\xi_3 & 0 & \xi_1 \\
\xi_2 & -\xi_1 & 0
\end{pmatrix},
\]

give us similar estimates in FBM-spaces. Moreover, the fact that functions \( \cos \) and \( \sin \) are bounded, and that \( \| \hat{M}(\xi) \| \leq 2 \) for all \( \xi \in \mathbb{R}^3 \), yield estimates independent of the Coriolis parameter \( \Omega \).

Thus, one also can prove the following result for the fractional Navier–Stokes–Coriolis system (FNSC), that is, system (1.1) with \( \theta \equiv 0 \) and \( \mathcal{N} = 0 \). Let the interval \( I \) and the
parameters $\mu$, $q$ and $s$ be as in Theorem 3.1. Let $\Omega \in \mathbb{R}$ and $u_0 \in \mathcal{F}\mathcal{N}_{q,\mu,r}^s$ be a divergence-free vector field. Assume also the conditions (i), (ii) and (iii) in Theorem 3.1. There are two constants $\varepsilon = \varepsilon(v, \alpha) > 0$ and $C = C(v, \alpha) > 0$ (independent of $\Omega$) such that (FNSC) has a unique global mild solution $u$ in the closed ball $\{w \in \mathcal{X}_r; \|w\|_{\mathcal{X}_r} \leq C\varepsilon\}$ satisfying $\nabla \cdot u = 0$ provided that $\|u_0\|_{\mathcal{F}\mathcal{N}_{q,\mu,r}^s} \leq \varepsilon$. In particular, considering the critical dissipation $\alpha = 1/2$, we have the well-posedness in critical Fourier–Besov–Morrey spaces for the Navier–Stokes system with critical dissipation, that is, (FNSC) with the Coriolis parameter $\Omega = 0$ and $\alpha = 1/2$, which extends the corresponding result of [39] in Fourier–Besov spaces.

**Remark 3.3** By technical reasons from the employed method and estimates for the (FBCS)-semigroup (2.6) (see (2.7) and Lemmas 4.1 and 4.2 below), we consider Remark 3.3. For each pair $(/\Omega_1, /\Omega_2)$, which extends the corresponding result of [39] in Fourier–Besov spaces.

**4 Proofs**

In this section we prove Theorem 3.1. For that, we shall establish some estimates for the fractional Boussinesq–Coriolis stratification semigroup (2.6) and the bilinear term (2.10).

**4.1 Linear and bilinear estimates**

**Lemma 4.1** Let $I = (0, \infty)$, $0 \leq \mu < 3$, $1 \leq q_2 \leq q_1 < \infty$, $1 \leq r \leq \infty$ and $s \in \mathbb{R}$. For each pair $(\Omega, N) \in (\mathbb{R} - \{0\})^2$, consider $L = \max\{q_1, N \sqrt{r}/|\Omega|^2\}$. Then, there is a constant $C > 0$, independent of the parameters $\Omega$ and $N$, such that

\[
\begin{align*}
\| S^q_{\Omega_1, N}(t) v_0 \|_{\mathcal{F}\mathcal{N}_{q,\mu,r}^s} &\leq CL(v) \frac{1}{|q|} \left( \frac{1 - \mu}{q_2} - \frac{3 - \mu}{q_1} \right) \| v_0 \|_{\mathcal{F}\mathcal{N}_{q_1,\mu,r}^s}, \quad (4.1) \\
\| S^q_{\Omega_2, N}(t) v_0 \|_{\mathcal{L}_{\infty}(I; \mathcal{F}\mathcal{N}_{q_1,\mu,r}^s)} &\leq CL \| v_0 \|_{\mathcal{F}\mathcal{N}_{q_1,\mu,r}^s} \quad \text{and} \quad (4.2) \\
\| S^q_{\Omega_1, N}(t) v_0 \|_{\mathcal{L}^1(I; \mathcal{F}\mathcal{N}_{q_1,\mu,r}^{s+2\alpha})} &\leq CL^{-1} \| v_0 \|_{\mathcal{F}\mathcal{N}_{q_1,\mu,r}^s}, \quad (4.3)
\end{align*}
\]

for all $v_0 \in \mathcal{F}\mathcal{N}_{q_1,\mu,r}^s$.

**Proof** Consider $p \geq q_1 \geq 1$ such that $1/q = 1/p + 1/q_1$. From the expression of the semigroup (2.6), the bound (2.7) of the matrix $M_j$ for $j = 1, 2, 3$, Hölder’s inequality and the scaling property $\|f(\lambda \cdot)\|_{p,\mu} = \lambda^{-\frac{q-p}{p}} \|f\|_{p,\mu}$, it follows that

\[
\begin{align*}
\| \varphi_j [S^q_{\Omega_1, N}(t)]^* \hat{v}_0 \|_{q_2, \mu} &= \left\| \varphi_j e^{-\nu |\xi'|^{2\alpha}} \left[ \cos \left( \frac{|\xi'|}{|\xi'|} \right) M_1(\xi) \hat{v}_0(\xi) + \sin \left( \frac{|\xi'|}{|\xi'|} \right) M_2(\xi) \hat{v}_0(\xi) + M_3(\xi) \hat{v}_0(\xi) \right] \right\|_{q_2, \mu} \\
&\leq CL \| e^{-\nu ||\xi'||^{2\alpha}} \varphi_j \hat{v}_0(\xi) \|_{q_2, \mu} \\
&= CL \| e^{-\nu |t|} \frac{1}{|\xi'|^{2\alpha}} \varphi_j \hat{v}_0(\xi) \|_{q_2, \mu}
\end{align*}
\]
\[ C \in \text{estimate.} \]

Using (2.10) and defined in Fourier variables by after multiplying by 2 after \( v \) we get (4.1). Observe that, when \( q_1 = q_2 \), we have

\[ 2^j \| \varphi_j \|_{L^2} \| \hat{u}_0 \|_{q_1, \mu} \leq C L 2^j \| \varphi_j \hat{u}_0 \|_{q_1, \mu} \]

and thus, taking the \( L^\infty \) norm on \( I \) and then the \( L^r(\mathbb{Z}) \) norm, we obtain (4.2). The last inequality (4.3) follows from the estimates

\[
\| \varphi_j [S_{\Omega, N}(t)]^* \hat{u}_0 \|_{q_1, \mu} \leq C L \| e^{-v_{t+2}2\alpha t} \|_{L^1(\mathbb{R})} \| \varphi_j \hat{u}_0 \|_{q_1, \mu} \\
\leq C L v^{-1} 2^{-2\alpha} \| \varphi_j \hat{u}_0 \|_{q_1, \mu},
\]

after multiplying by \( 2^{-2\alpha} \) and then applying the \( L^r(\mathbb{Z}) \) norm on both sides of (4.4).

Now we shall establish some estimates for the linear operator \( \xi_{\Omega, N}^\alpha \), which is linked to (2.10) and defined in Fourier variables by

\[
\left[ \xi_{\Omega, N}^\alpha(f) \right]^*(\xi, t) = \int_0^t \left[ S_{\Omega, N}(t - \tau) \right]^*(\xi)(\hat{f}) \hat{f}(\xi, \tau) d \tau.
\]

**Lemma 4.2** Let \( I = (0, \infty), 0 \leq \mu < 3 \), \( 1 < q < \infty \), \( 1 \leq r \leq \infty \), and \( \alpha = \frac{4 - 2\alpha - \frac{3\mu}{q}}{q} \). Assume that \( (\Omega, N) \in (\mathbb{R} - \{0\})^2 \) and consider \( L = \max\{2, \frac{|\Omega|}{N'}, \frac{N}{\|\hat{f}\|_{\Omega}} \} \). There is a constant \( C = C(\alpha) > 0 \) such that

\[
\| \xi_{\Omega, N}^\alpha(f) \|_{L^r(I; \mathcal{F}^\alpha_{q, \mu, r})} \leq C L f \| L^1(I; \mathcal{F}^\alpha_{q, \mu, r}),
\]

\[
\| \xi_{\Omega, N}^\alpha(f) \|_{L^r(I; \mathcal{F}^\alpha_{q, \mu, r} + 2\alpha)} \leq C L v^{-1} \| f \| L^\infty(I; \mathcal{F}^\alpha_{q, \mu, r}),
\]

\[
\| \xi_{\Omega, N}^\alpha(f) \|_{L^1(I; \mathcal{F}^\alpha_{q, \mu, r})} \leq C L v^{-1} \| f \| L^1(I; \mathcal{F}^\alpha_{q, \mu, r}),
\]

for all \( f \in L^\infty(I; \mathcal{F}^\alpha_{q, \mu, r}) \) or \( f \in L^1(I; \mathcal{F}^\alpha_{q, \mu, r}) \), according to the corresponding estimate.

**Proof** Using \( \text{supp}(\varphi_j) \subset D_j = \{ \xi \in \mathbb{R}^3; 2^{j-1} \leq |\xi| \leq 2^{j+2} \}, \| \hat{f} \|^r(\xi) \leq 2 \) and the fact that \( L \) is a bound of the matrix \( M_j \) for \( j = 1, 2, 3 \), we can use Young’s inequality in the time-variable in order to estimate

\[
\| \varphi_j [\xi_{\Omega, N}(f)]^*(\cdot, t) \|_{q_1, \mu} \leq \left\| \int_0^t \| \left[ S_{\Omega, N}(t - \tau) \right]^*(\xi)(\hat{f}) \hat{f}(\xi, \tau) d \tau \right\|_{L^\infty(I)} \leq C L \left\| \int_0^t e^{-v_{t+2}2\alpha t} \|_{L^\infty(I)} \| \varphi_j \hat{f}(\xi, \tau) d \tau \right\|_{L^\infty(I)} \leq C L \left\| e^{-v_{t+2}2\alpha t} \|_{L^\infty(I)} \| \varphi_j \hat{f}(\xi, \tau) \right\|_{L^1(I; \mathcal{M}_{q, \mu})}
\]
To conclude we are going to show that \( \alpha \), which yields (4.6) after multiplying by 2, yields the estimate (4.5). For (4.6), we use Young’s inequality in the time-variable in order to obtain

\[
\| \varphi_j [z_{\Omega, N}]^{\alpha} (\cdot, t) \|_{q, \mu, \nu} L^\infty(I) \leq C \left\| \int_{t_0}^t e^{-v(t-\tau)2^{2(\gamma(j-1))}} \| \varphi_j (\cdot, \tau) \|_{q, \mu, \nu} d \tau \right\|_{L^\infty(I)}
\]

\[
\leq C \| e^{-v/2^{2(\gamma(j-1))}} \|_{L^1(I)} \| \varphi_j \hat{f} \|_{L^\infty(I) ; M_{q, \mu}}
\]

\[
\leq C \| 2^{-2\alpha j} \| \varphi_j \hat{f} \|_{L^\infty(I) ; M_{q, \mu}, (4.8)}
\]

which yields (4.6) after multiplying by \( 2^{j(s+2\alpha)} \) and then applying the \( l^r(\mathbb{Z}) \)-norm on both sides. For the last estimate, we have that

\[
\| \varphi_j [z_{\Omega, N}]^{\alpha} (\cdot, t) \|_{q, \mu, \nu} L^1(I) \leq C \left\| \int_{t_0}^t e^{-v(t-\tau)2^{2(\gamma(j-1))}} \| \varphi_j (\cdot, \tau) \|_{q, \mu, \nu} d \tau \right\|_{L^1(I)}
\]

\[
\leq C \| e^{-v/2^{2(\gamma(j-1))}} \|_{L^1(I)} \| \varphi_j \hat{f} \|_{L^1(I) ; M_{q, \mu}}
\]

\[
\leq C L v^{-1} 2^{-2\alpha j} \| \varphi_j \hat{f} \|_{L^1(I) ; M_{q, \mu}} . (4.9)
\]

Now it is sufficient to multiply by \( 2^{j(s+2\alpha)} \) and then take the \( l^r(\mathbb{Z}) \)-norm on both sides of (4.9) to obtain (4.7).

\[\square\]

**Lemma 4.3** Let \( I = (0, \infty) \), \( 0 \leq \mu < 3 \), \( 1 \leq q < \infty \), and \( s = 4 - 2\alpha - \frac{3\mu}{q} \). Assume that \( \alpha \) and \( r \) satisfy the following conditions

(i) \( \frac{1}{2} < \alpha < \frac{\mu}{2} - \frac{3\mu}{2q} \) and \( 1 \leq r \leq \infty \), or

(ii) \( \frac{1}{2} < \alpha = \frac{\mu}{2} - \frac{3\mu}{2q} \), \( \mu = 0 \) and \( q \leq r \leq 2 \), or

(iii) \( \alpha = \frac{1}{2} \) and \( r = 1 \).

Let \( (\Omega, \mathcal{N}) \in (\mathbb{R} - \{0\})^2 \) and \( L = \max\{2, \|\Omega\|_{\mathcal{N}^r}, \frac{\sqrt{\gamma}}{\|\mathcal{N}\|_{\mathcal{N}^r}}\} \). Then, there is a constant \( C = C(\alpha) > 0 \) such that

\[
\| B(v, w) \|_{X_r} \leq C L \max\{1, v^{-1}\} \| v \|_{X_r} \| w \|_{X_r}, \quad (4.10)
\]

for all \( v, w \in X_r = L^\infty(I; \mathcal{F}_{q, \mu}^3) \cap L^1(I; \mathcal{F}_{q, \mu}^{3+2\alpha}) \).

**Proof** First we shall establish estimate (4.10) for the assumption (i). Since \( \text{supp } \varphi_j \subset \{ \xi \in \mathbb{R}^3; | \xi | \leq \frac{1}{2} 2^{j+1} \} \) and \( \| [\hat{\mathcal{N}}]^{\alpha}(\xi) \| \leq 2 \), Bernstein-type inequality yields

\[
\| \varphi_j [\xi_\theta^\alpha, \theta, 0]^T[v \otimes w]^{\alpha}(\xi, t) \|_{q, \mu} \leq C 2^{j+1} \| \varphi_j [v \otimes w]^{\alpha}(\xi, t) \|_{q, \mu} .
\]

Next, we use Lemma 4.2 with \( f = \hat{\mathcal{N}} \cdot (v \otimes w) \) to obtain

\[
\| B(v, w) \|_{L^1(I; \mathcal{F}_{q, \mu}^{3+2\alpha})} + \| \hat{B}(v, w) \|_{L^\infty(I; \mathcal{F}_{q, \mu}^{3+2\alpha})}
\]

\[
\leq C L \max\{1, v^{-1}\} \| f \|_{L^1(I; \mathcal{F}_{q, \mu}^{3+2\alpha})}
\]

\[
\leq C L \max\{1, v^{-1}\} \| v \otimes w \|_{L^1(I; \mathcal{F}_{q, \mu}^{3+2\alpha})} .
\]

To conclude we are going to show that

\[
\| v \otimes w \|_{L^1(I; \mathcal{F}_{q, \mu}^{3+2\alpha})} \leq C \| v \|_{X_r} \| w \|_{X_r} .
\]
For that, recalling that \( \hat{\varphi}_k = \varphi_{k-1} + \varphi_k + \varphi_{k+1} \), we consider Bony’s decomposition

\[
\varphi_j [v w]^* = \sum_{|k-j| \leq 4} \varphi_j [(S_{k-1} v)^* (\varphi_k \hat{w})] + \sum_{|k-j| \leq 4} \varphi_j [(S_{k-1} w)^* (\varphi_k \hat{v})]
\]

\[
+ \sum_{k \geq j-2} \varphi_j [(\varphi_k \hat{w}) * (\varphi_k \hat{v})]
\]

\[
= I_1 + I_2 + I_3.
\]

Estimate for the term \( I_1 \): First note that \( \| \varphi_k \hat{u} \|_{L^1} \leq C 2^{k(3 - \frac{3\mu}{q})} \| \varphi_k \hat{u} \|_{q, \mu} \) follows from the Bernstein-type inequality with \( \beta = 0 \), \( (p_2, \mu_2) = (1, 0) \) and \( (p_1, \mu_1) = (q, \mu) \). Now we proceed as follows:

\[
\| I_1 \|_{L^1(I; \mathcal{M}_{q, \mu})} \leq \int_I \left\| \sum_{|k-j| \leq 4} \varphi_j [(S_{k-1} v)^* (\varphi_k \hat{w})] \right\|_{q, \mu} \, dt
\]

\[
\leq C \int_I \left( \sum_{|k-j| \leq 4} \| \varphi_{k'} \hat{v} \|_{L^1} \right) \| \varphi_k \hat{w} \|_{q, \mu} \, dt
\]

\[
\leq C \int_I \sum_{|k-j| \leq 4} 2^k (-1 + 2\alpha) 2^k (4 - 2\alpha - \frac{3\mu}{q}) \| \varphi_{k'} \hat{v} \|_{q, \mu} \| \varphi_k \hat{w} \|_{q, \mu} \, dt
\]

\[
\leq C \| v \|_{L^\infty(I; \mathcal{F}N_{q, \mu, r})} \int_I \sum_{|k-j| \leq 4} 2^k (3 - 3\alpha) \| \varphi_k \hat{w} \|_{q, \mu} \, dt.
\]

Next we can rewrite (4.11) in order to obtain

\[
\| I_1 \|_{L^1(I; \mathcal{M}_{q, \mu})} \leq C \| v \|_{L^\infty(I; \mathcal{F}N_{q, \mu, r})} 2^j \sum_{k \in \mathbb{Z}} 2^k \chi_{|j| \leq 4} (k-j) X_l(I; |l| \leq 4) \sum_{k \in \mathbb{Z}} 2^k \chi_{|j| \leq 4} (k-j)
\]

\[
\times 2^{k(4 - \frac{3\mu}{q})} \| \varphi_k \hat{w} \|_{L^1(I; \mathcal{M}_{q, \mu})}
\]

\[
\leq C \| v \|_{L^\infty(I; \mathcal{F}N_{q, \mu, r})} 2^j \sum_{k \in \mathbb{Z}} 2^k \chi_{|j| \leq 4} (k-j) (a_l * b_l)_j,
\]

where \( a_l = 2^{-l(5 + 2\alpha - \frac{3\mu}{q})} X_l(I; |l| \leq 4) \) and \( b_k = 2^{k(4 - \frac{3\mu}{q})} \| \varphi_k \hat{w} \|_{L^1(I; \mathcal{M}_{q, \mu})} \). Multiplying by \( 2^j(5 - 2\alpha - \frac{3\mu}{q}) \), taking the \( L^r(\mathbb{Z}) \)-norm on both sides of (4.12), and using Young’s inequality for series, we get

\[
\| 2^j(5 - 2\alpha - \frac{3\mu}{q}) \| I_1 \|_{L^1(I; \mathcal{M}_{q, \mu})} \| L^r(\mathbb{Z}) \| \leq C \| v \|_{L^\infty(I; \mathcal{F}N_{q, \mu, r})} \| a_l \|_1 \| b_k \|_{L^r(\mathbb{Z})}
\]

\[
\leq C \| v \|_{L^\infty(I; \mathcal{F}N_{q, \mu, r})} \| w \|_{L^1(I; \mathcal{F}N_{q, \mu, r})}
\]

and then

\[
\| 2^j(s+1) \| I_1 \|_{L^1(I; \mathcal{M}_{q, \mu})} \| L^r(\mathbb{Z}) \| \leq C \| v \|_{L^r(\mathbb{Z})} \| w \|_{L^s(\mathbb{Z}).}
\]

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Similar computations lead us
\[ \| 2^{(s+1)} I_3 \|_{L^1(I; \mathcal{M}_{q,\mu})} \leq C \| v \|_{\mathcal{X}_r} \| w \|_{\mathcal{X}_r}. \]  (4.14)

In what follows, we estimate the parcel \( I_3:\)
\[
\begin{align*}
\| I_3 \|_{L^1(I; \mathcal{M}_{q,\mu})} & \leq \int \sum_{k \geq j-2} \| \phi_j \left( (\varphi_k \hat{w}) \ast (\tilde{\varphi_k \hat{w}}) \right) \|_{q,\mu} \, dt \\
& \leq C \int \sum_{k \geq j-2} \| \varphi_k \hat{w} \|_{q,\mu} \| \tilde{\varphi_k \hat{w}} \|_{L^1} \, dt \\
& \leq C \int \sum_{k \geq j-2} \| \varphi_k \hat{w} \|_{q,\mu} \sum_{|k-k'| \leq 1} \| \varphi_{k'} \hat{w} \|_{L^1} \, dt \\
& \leq C \sum_{k \geq j-2} \int \sum_{|k-k'| \leq 1} 2^k \left( \frac{3-\frac{\mu}{q}}{2} \right) \| \varphi_k \hat{w} \|_{q,\mu} \| \varphi_k \hat{w} \|_{q,\mu} \, dt \\
& \leq C \sum_{k \geq j-2} \int \left( \sum_{|k-k'| \leq 1} 2^k \left( -1+2\alpha \right) 2^k \left( 4-2\alpha - \frac{3-\mu}{q} \right) \right) \| \varphi_k \hat{w} \|_{L^1(I; \mathcal{M}_{q,\mu})} \, dt \\
& \leq C \sum_{k \geq j-2} \int 2^{k} \left( -1+2\alpha \right) 2^{k} \left( 4-2\alpha - \frac{3-\mu}{q} \right) \| \varphi_k \hat{w} \|_{L^1(I; \mathcal{M}_{q,\mu})} \, dt \\
& \leq C \| w \|_{L^\infty(I; \mathcal{F}^{N_q}_{q,\mu})} \sum_{k \geq j-2} 2^k \left( -1+2\alpha \right) \| \varphi_k \hat{w} \|_{L^1(I; \mathcal{M}_{q,\mu})} \\
& \leq C \| w \|_{L^\infty(I; \mathcal{F}^{N_q}_{q,\mu})} \sum_{k \geq j-2} 2^k \left( -1+2\alpha + \frac{\mu}{q} \right) \| \varphi_k \hat{w} \|_{L^1(I; \mathcal{M}_{q,\mu})} \\
& = C \| w \|_{L^\infty(I; \mathcal{F}^{N_q}_{q,\mu})} 2^j \left( -1+2\alpha + \frac{\mu}{q} \right) \sum_{k \in \mathbb{Z}} 2^{-(j-k)} \left( -1+2\alpha + \frac{\mu}{q} \right) \chi_{[l:l \leq l]}(j-k) \\
& \quad \times 2^k \left( -1+2\alpha + \frac{\mu}{q} \right) \| \varphi_k \hat{w} \|_{L^1(I; \mathcal{M}_{q,\mu})}. \\
\end{align*}
\]

Thus
\[ 2^{j(s+1)} \| I_3 \|_{L^1(I; \mathcal{M}_{q,\mu})} \leq C \| w \|_{L^\infty(I; \mathcal{F}^{N_q}_{q,\mu})} (a_l \ast b_k) j, \]
with \( a_l = 2^{l(-1+2\alpha + \frac{\mu}{q})} \chi_{[l:l \leq 2]} \) and \( b_k = 2^{k(4-\frac{3-\mu}{q})} \| \varphi_k \hat{w} \|_{L^1(I; \mathcal{M}_{q,\mu})}. \) After applying the \( \ell^{r}(\mathbb{Z}) \)-norm on both sides of the above inequality, and using Young’s inequality for series, we arrive at
\[
\begin{align*}
\| 2^{j(s+1)} I_3 \|_{L^1(I; \mathcal{M}_{q,\mu})} \|_{\ell^r(\mathbb{Z})} & \leq C \| w \|_{L^\infty(I; \mathcal{F}^{N_q}_{q,\mu})} \| a_l \|_{\ell^1(\mathbb{Z})} \| b_k \|_{\ell^r(\mathbb{Z})} \\
& \leq C \| w \|_{\ell^1(I; \mathcal{F}^{N_q}_{q,\mu})} \| w \|_{L^\infty(I; \mathcal{F}^{N_q}_{q,\mu})} \\
& \leq \| v \|_{\mathcal{X}_r} \| w \|_{\mathcal{X}_r},
\end{align*}
\]
where we used the assumption \(-5+2\alpha + \frac{\mu}{q} < 0\), i.e., \( \alpha < \frac{5}{2} - \frac{3-\mu}{2q} \). Therefore, using (4.13), (4.14) and the last estimate, we obtain
\[
\| v \otimes w \|_{\ell^1(I; \mathcal{F}^{N_q}_{q,\mu})} \leq \| 2^{j(s+1)} I_3 \|_{L^1(I; \mathcal{M}_{q,\mu})} \|_{\ell^r(\mathbb{Z})} + \| 2^{j(s+1)} I_2 \|_{L^1(I; \mathcal{M}_{q,\mu})} \|_{\ell^r(\mathbb{Z})}.
\]
where

These computations give us (4.10) under the conditions of item (i).

For the conditions in item (ii), we can proceed similarly to the proof of [39, Theorem 5] in order to obtain (4.10). The details are left to the reader.

Finally, assume the conditions in item (iii). For the first term $I_1$, we have

$$
\|I_1\|_{L^1(I; \mathcal{M}_{q,\mu})} \leq C \sum_{|k-j| \leq 4} \sum_{|k'-k| \leq 2} 2^k \left(3 - \frac{3\mu}{q}\right) \|\varphi_{k'} \hat{\psi}\|_{L^\infty(I; \mathcal{M}_{q,\mu})} \|\varphi_k \hat{\psi}\|_{L^1(I; \mathcal{M}_{q,\mu})}
$$

$$
\leq C \|v\|_{L^\infty(I)} \left(1 + \mathcal{F}\mathcal{N}^{3 - \frac{3\mu}{q}}_{q,\mu,1}\right) 2^{-j} \left(4 - \frac{3\mu}{q}\right) (a_j * b_k)_j,
$$

where $a_j = 2^{j(4 - \frac{3\mu}{q})} \chi_{|l| \leq 4}$ and $b_k = 2^{k(4 - \frac{3\mu}{q})} \|\varphi_k \hat{\psi}\|_{L^1(I; \mathcal{M}_{q,\mu})}$. Multiplying by $2^{j(4 - \frac{3\mu}{q})}$ and then taking the $L^1$-norm on both sides of the above estimate, and using Young’s inequality for series, we get

$$
2^j \left(4 - \frac{3\mu}{q}\right) \|I_1\|_{L^1(I; \mathcal{M}_{q,\mu})} \leq C \|v\|_{L^\infty(I)} \left(1 + \mathcal{F}\mathcal{N}^{3 - \frac{3\mu}{q}}_{q,\mu,1}\right) \|w\|_{L^1(I; \mathcal{F}\mathcal{N}^{3 - \frac{3\mu}{q}}_{q,\mu,1})}.
$$

Similar computations lead us to

$$
2^j \left(4 - \frac{3\mu}{q}\right) \|I_2\|_{L^1(I; \mathcal{M}_{q,\mu})} \leq C \|w\|_{L^\infty(I)} \left(1 + \mathcal{F}\mathcal{N}^{3 - \frac{3\mu}{q}}_{q,\mu,1}\right) \|v\|_{L^1(I; \mathcal{F}\mathcal{N}^{3 - \frac{3\mu}{q}}_{q,\mu,1})}.
$$

Now we consider the estimate for the term $I_3$. We have that

$$
\|I_3\|_{L^1(I; \mathcal{M}_{q,\mu})} \leq C \sum_{k \geq j - 2} \int_{|k-k'| \leq 1} \sum_{|k-k'| \leq 1} 2^k \left(3 - \frac{3\mu}{q}\right) \|\varphi_{k'} \hat{\psi}\|_{q,\mu} \|\varphi_k \hat{\psi}\|_{q,\mu} \, dt
$$

$$
\leq C \sum_{k \geq j - 2} \sum_{|k-k'| \leq 1} 2^k \left(3 - \frac{3\mu}{q}\right) \|\varphi_{k'} \hat{\psi}\|_{L^\infty(I; \mathcal{M}_{q,\mu})} \|\varphi_k \hat{\psi}\|_{L^1(I; \mathcal{M}_{q,\mu})}
$$

$$
\leq C \|w\|_{L^\infty(I)} \left(1 + \mathcal{F}\mathcal{N}^{3 - \frac{3\mu}{q}}_{q,\mu,1}\right) 2^{-j} \left(4 - \frac{3\mu}{q}\right) (a_j * b_k)_j,
$$

where $a_j = 2^{j(4 - \frac{3\mu}{q})} \chi_{|l| \leq 2}$ and $b_k = 2^{k(4 - \frac{3\mu}{q})} \|\varphi_k \hat{\psi}\|_{L^1(I; \mathcal{M}_{q,\mu})}$. Multiplying by $2^{j(4 - \frac{3\mu}{q})}$, taking the $L^1$-norm on both sides of the above estimate, and using Young’s inequality for series, we obtain

$$
2^j \left(4 - \frac{3\mu}{q}\right) \|I_3\|_{L^1(I; \mathcal{M}_{q,\mu})} \leq C \|w\|_{L^\infty(I)} \left(1 + \mathcal{F}\mathcal{N}^{3 - \frac{3\mu}{q}}_{q,\mu,1}\right) \|v\|_{L^1(I; \mathcal{F}\mathcal{N}^{3 - \frac{3\mu}{q}}_{q,\mu,1})},
$$

and we are done.

\[\square\]

### 4.2 Proof of Theorem 3.1

In this subsection we shall apply Lemma 2.2. Consider $y = S_{\Omega,s}^\nu(t) v_0$ and $\mathcal{X}_r = \mathcal{L}^{\infty}(I; \mathcal{F}\mathcal{N}^{\frac{3}{2} + 2\alpha}_{q,\mu,r}) \cap L^1(I; \mathcal{F}\mathcal{N}^{\frac{3}{2} + 2\alpha}_{q,\mu,r})$ with $I = (0, \infty)$. By estimates (4.2) and (4.3) with
where \( C \) is a constant. Bilinear estimate (4.10) yields
\[
\| B(v, w) \|_{\mathcal{X}_t} \leq C L \max\{1, v^{-1}\} \| v \|_{\mathcal{X}_t} \| w \|_{\mathcal{X}_t},
\]
and we can choose the initial data \( v_0 \) satisfying
\[
\| v_0 \|_{\mathcal{F}_{q,q,q,r}^q} \leq \varepsilon < \frac{1}{4(CL \max\{1, v^{-1}\})^2}.
\]
Therefore, Lemma 2.2 implies that system (1.1) has a unique global mild solution \( v \in \mathcal{X}_t \) such that \( \| v \|_{\mathcal{X}_t} \leq 2CL \max\{1, v^{-1}\} \varepsilon \). Observe that the constant \( L \) depends on the parameters \( \Omega \), \( \mathcal{N} \) and \( \sqrt{g} \), except when \( \frac{\Omega}{\sqrt{g}} \) is bounded from below and above by positive constants.

In the case \( \mathcal{N} \sqrt{g}/2 \leq \Omega \leq 2 \mathcal{N} \sqrt{g} \), its value is equal to 2. Since \( \tilde{V} \cdot \nu_0 = 0 \) in \( S'((3) \), the mild formulation (2.9) implies that \( \tilde{V} \cdot v = 0 \) for \( \nu = (u^1, u^2, u^3, \sqrt{g} \theta / \mathcal{N}) \). In order to prove Theorem 3.1 in the case of items (ii) and (iii), we use the estimate (4.10) with the conditions (ii) and (iii) of Lemma 4.3, respectively.

Finally, we employ the estimate (2.14) in Lemma 2.2 in order to show the continuous dependence of \( v \) with respect to the initial data \( v_0 \). In fact, we know that
\[
\| v - \tilde{v} \|_{\mathcal{X}_t} \leq \frac{1}{1 - 4(CL \max\{1, v^{-1}\})^2 \varepsilon} \| S^\alpha_{\Omega, N}(t) v_0 - S^\alpha_{\Omega, N}(t) \tilde{v}_0 \|_{\mathcal{X}_t}
\]
\[
\leq \frac{CL \max\{1, v^{-1}\}}{1 - 4(CL \max\{1, v^{-1}\})^2 \varepsilon} \| v_0 - \tilde{v}_0 \|_{\mathcal{F}_{q,q,q,r}^q}.
\]
This inequality gives us the Lipschitz continuity for the data-solution map \( v_0 \mapsto v \) from \( \{ v_0 \in \mathcal{F}_{q,q,q,r}^q; \| v_0 \|_{\mathcal{F}_{q,q,q,r}^q} \leq \varepsilon \} \) to \( \{ v \in \mathcal{X}_t; \| v \|_{\mathcal{X}_t} \leq 2CL \max\{1, v^{-1}\} \varepsilon \} \). Finally, the weak time-continuity statement follows from standard arguments (see, e.g., [30, Lemma 3.3 (p.989) and Lemma 4.8 (p.998)]).

\[\square\]

Data availability Data sharing not applicable to this article as no data sets were generated or analyzed during the current study.

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