The Shear Viscosity to Entropy Density Ratio of Hagedorn States

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The fireball concept of Rolf Hagedorn, developed in the 1960’s, is an alternative description of hadronic matter. Using a recently derived mass spectrum, we use the transport model GiBUU to calculate the shear viscosity of a gas of such Hagedorn states, applying the Green-Kubo method to Monte-Carlo calculations. Since the entropy density is rising ad infinitum near \(T_H\), this leads to a very low shear viscosity to entropy density ratio near \(T_H\). Further, by comparing our results with analytic expressions, we find a nice extrapolation behavior, indicating that a gas of Hagedorn states comes close or even below the boundary \(1/4\pi\) from AdS-CFT.

I. INTRODUCTION

The properties of hot and dense matter, created experimentally in heavy-ion collision performed at accelerators like RHIC or CERN, are usually extracted by applying relativistic hydrodynamics or kinetic transport theory. Doing hydrodynamics, transport coefficients like heat or electric conductivity, or shear- or bulk viscosity, are extrinsic inputs which should be calculated from an underlying field theory, as it is Quantum Chromodynamics (QCD) for the for the quark gluon plasma (QGP). The shear viscosity, as one of those transport coefficients, can be calculated employing two-particle scattering processes. Dealing with QGP, there is almost a perfect liquid can be calculated employing two-particle scattering processes. A perfect liquid

\[
\eta(s) \approx 0.13 \text{ in a pure gluon gas. This is as expected, because } \eta/s \text{ increases with decreasing } T, \text{ which goes hand in hand with a decrease of the relevant hadronic cross section in the hadronic phase [3, 4]. On the other hand, asymptotic freedom dictates that } \eta/s \text{ increases with } T \text{ in the deconfined phase. Here the coupling between quarks and gluons decreases logarithmically [5].}
\]

There have been several efforts to study this transport coefficient \(\eta/s\) in microscopic models using the Green-Kubo formalism, as e.g. in UrQMD [6, 7], in SMASH [8] and in pHSD [9]. On the partonic side, either pHSD [9], PCM [10], and BAMPS have been used [11], while within the latter model also a critical test of the Green-Kubo method itself has been performed [12]. Very recently, there was an attempt using a S-matrix based Hadron Resonance Model via the Chapman-Enskog method [13].

Before QCD made the calculation of phase transition and QGP possible, an alternative theory describing hadrons was developed by Rolf Hagedorn in the 1960’s

\[
\rho(m) = \text{const. } m^{-a} \exp[m/T_H],
\]

with \(T_H\) being the so-called “Hagedorn temperature”. Later, Frautschi invented a reformulation [13], yielding a bootstrap equation,

\[
\rho(m) = \rho_0(m) + \sum_N \frac{1}{N!} \left[ \frac{V_0}{(2\pi)^3} \right]^{N-1} \times \int \prod_{i=1}^N [d^3 p_i] \delta^{(4)} \left( \sum p_i - p \right).
\]

Here \(V_0\) is the volume of the Hagedorn states. In general, this equation can not be solved analytically. For the easiest inhomogeneity, \(\rho_0(m) = \delta(m - m_0)\) with \(m_0\) the mass of some initial state, Nahm [16] found a solution with \(a \approx 3\). With \(V_0 \approx (4\pi/3)m_0^3\), one achieves a slope \(T_H \approx 150\text{ MeV}\).

For more realistic inhomogenities of eq. [2], the solution has to be found numerically. Recently, our group developed a prescription with \(N = 2\), where the quantum numbers \(B\) (baryon number), \(S\) (strangeness) and \(Q\) (electric charge) are conserved explicitly [17, 18]. Summing over all quantum numbers, one gets for the prescription given in [19] a Hagedorn spectrum, eq. [1], which is characterized by \(T_H = 165\text{ MeV}\) and \(a = 2.98\) (for a Hagedorn state radius \(R = 1.0\text{ fm}\)). As will be discussed below, these numbers can only be used for analytic estimates with restrictions. For real calculations the detailed, tabulated spectra are used. Nevertheless, the fitted distribution may be used to estimate some quantities in the vicinity of \(T_H\).

The aim of this paper is to study how the shear viscosity over entropy density of a Hagedorn gas behaves...
as a function of the temperature of the gas (cf. also [20]). Thus analytical estimates are compared to Monte Carlo results obtained from box calculations based on the Green-Kubo formalism. In order to check the validity of the results, also results for a pion gas are analyzed, where three different charge states may interact elastically according an isotropic constant cross section $\sigma = 30\text{ mb}$. Thus the interaction is direct comparable to that of the Hagedorn gas.

The paper is organized as follows. In section II, analytic expressions for the thermodynamical quantities of the considered Hagedorns state gas are given. Also, an analytic expression for the shear viscosity of a gas of particles, which interpolates the non-relativistic regime to the relativistic regime necessary for pions is stated. Section III describes the numerical Green-Kubo method used in this analysis and shows intermediate results. The final results for $\eta$ and $\eta/s$ are presented in section IV and discussed in section V.

II. ANALYTIC EXPRESSIONS

A. Thermodynamical quantities

Knowing the general resonance gas partition function in Boltzmann approximation,

$$\ln Z(T,V) = \frac{VT}{2\pi^2} \int dm \rho(m) m^2 K_2 \left( \frac{m}{T} \right),$$

with $K_\nu$ being a modified Bessel function, one may derive all necessary thermodynamical quantities (cf. e.g. [21]), as e.g. particle density $n$, energy density $e$, and entropy density $s$, as

$$n(T) = \frac{T}{2\pi^2} \int dm \rho(m) m^2 K_2 \left( \frac{m}{T} \right),$$

$$e(T) = \frac{T}{2\pi^2} \int dm \rho(m) m^3 \left[ 3 \frac{T}{m} K_2 \left( \frac{m}{T} \right) + K_1 \left( \frac{m}{T} \right) \right],$$

$$s(T) = \frac{1}{2\pi^2} \int dm \rho(m) m^3 K_3 \left( \frac{m}{T} \right),$$

where all chemical potentials have been neglected, $\mu = 0$. Since for Boltzmann statistics the pressure is given by $p = Tn$, it is easily observed using the recurrence relations of the Bessel functions, that the well known Gibbs-Duhem relation,

$$s = \frac{e + p}{T},$$

is fulfilled.

It is important to note, that a simple minded insertion of eq. (1) into the thermodynamical integrals eqs. (4) to (6) leads to faulty results: The Hagedorn spectrum fitted to a function according eq. (1) only describes the high mass ($m \gtrsim 2\text{ GeV}$) contribution, but totally fails below. Here the full Hagedorn spectrum is a sum of the known hadron states and the pure Hagedorn states, thus showing all the mass structures of the known hadrons, fig. 1. Therefore we will use the tabulated spectrum of hadrons and Hagedorn states instead of an analytic approximation in all what follows, except the extrapolations described below. The tabulation stops at Hagedorn state masses $m = 10\text{ GeV}$.

The resulting entropy as a function of temperature is shown in fig. 2. The entropy density for pions is simply calculated by replacing $\rho(m)$ by a properly scaled delta peak according the degeneracy at the pion mass. While the entropy density of the pion gas increases very slowly, the entropy of the Hagedorn gas increases exponentially and gets very steep for $T \gtrsim T_H$. Since the Hagedorn state

\begin{figure}[h]
\centering
\includegraphics[width=0.5\textwidth]{fig1.png}
\caption{The Hagedorn spectrum and a fit according eq. (1) ($T_H = 0.165\text{ GeV}, a = 2.98$) as function of the mass $m$.}
\end{figure}

\begin{figure}[h]
\centering
\includegraphics[width=0.5\textwidth]{fig2.png}
\caption{The entropy density $s$, eq. (6), normalized to $T^3$ as function of the temperature $T$. Also the estimate for $m \to \infty$ (see text for details) is shown.}
\end{figure}
tabulations only extends up to masses $m = 10$ GeV, the (expected) divergence is weakened, showing only a constant increase on a logarithmic scale. This holds true for all thermodynamical quantities mentioned above, as e.g. energy and particle density. For some quantities, it is now possible to add the missing contribution by using the analytic fit function eq. (1) getting the real divergence. Inserting approximations for the Bessel function $K_0$ for large arguments, integrals like e.g. eqs. (4) to (6) may be expressed in terms of the incomplete gamma function. The corresponding result for the entropy density is also shown in fig. 2.

It is obvious that one has to abstain from this procedure, when directly comparing to the Monte Carlo simulations.

We have to mention, that we consider the gas particles to be pointlike, such that there is no volume correction. Since the Hagedorn spectrum generates more and more particles, this also influences the space in a given box volume. Therefore it would be instructive to introduce volume corrections, as e.g. in (21) (22) in future studies.

### B. Shear viscosity

To investigate the shear viscosity of pion or Hagedorn states gas, it is important to ensure that the underlying formulae are valid for the desired range of the variable $z = m/T$, while $m$ is the mass of the particle and $T$ the temperature of the system. For relevant temperatures $T = 100 - 200$ MeV and masses $m > 138$ MeV, the covered range is $z = 10^{-3} - 1.5$. Thus one needs a non-relativistic prescription, which reaches till $m \sim T$. For this the expression valid for all masses and all temperatures is selected as (24)

\[
\eta = \frac{15}{16} \frac{T}{\sigma} \frac{K_5^2(z)}{(az^2 + b) K_2(2z) + (cz^3 + d) K_3(2z)}.
\]  
(8)

Choosing the values of the constants as

\[a = 15, \ b = 2, \ c = 3, \ d = 49,\]

(9)
yields the well known first order approximations (24)3

By slightly adjusting these constants to

\[a = 14.55, \ b = 1.13, \ c = 2.95, \ d = 46.85,\]

(10)
eq (8) gives a nice interpolation of numerical results (26) and yields the also well known higher order limiting formulae (27), (28)

\[
\eta \approx \frac{1.2654 T}{\sigma},
\]

(11)

\[
\eta \approx \frac{0.3175 \sqrt{\pi} T}{\sigma} \sqrt{\frac{m}{T}} \left(1 + 1.6349 \frac{T}{m}\right).
\]

Thus, eq. (8) with the modified factors eq. (10) will be used further-on in this work.

Another expression covering all values of $z$ may be found in (23). This expression differs from the given one by more than 20 % for $z > 0.1$ and is therefore not covered here.

The overall shear viscosity of a mixture of particles is given by the weighted sum of the viscosities of each particle species (29), which in the given case of the Hagedorn gas converts into a integration over all masses,

\[
\eta(T) = \frac{T}{2\pi^2 n(T)} \int \mathrm{d}m \rho(m) m^2 K_2 \left(\frac{m}{T}\right) \eta(m)
\]

(13)

with $\eta(m)$ given by eqs. (8) and (10).

### III. NUMERICAL CONTEMPLATION

#### A. Implementation into GiBUU

The Gießen Boltzmann-Uehling-Uhlenbeck (GiBUU) project (30) simulates nuclear reactions as $e + A, \pi + A, \nu + A, \text{hadron} + A$ (i.e. $p + A, \pi + A$) or $A + A$ at energies of 10 MeV to 100 GeV. Here the BUU equation

\[
[\partial_t (\nabla_r H_i) \nabla_r - (\nabla_r H_i) \nabla_r] f_i(r, p, t) = C [f_i, f_j, ...]
\]

(14)
is solved, where $i = N, \Delta, \pi, \rho, ...$. The collision term $C$ conventionally involves the decay and scattering of 1-, 2- and 3-body processes, $C = C_{1 \to 2} + C_{2 \to 3} + C_{3 \to 2}$, which splits into a resonance model for low energies and the string model for high energies. In the actual implementation (19), all interactions (even elastic scattering) are replaced by Hagedorn state creation and decay processes, i.e. by $2 \to 1$ and $1 \to 2$ processes alone. The Hagedorn spectrum tabulation limits the available energy range to be below 10 GeV.

In the simulations, the particles are thermally initialized in a box (non-reflecting boundaries) with fixed volume of (10 fm)$^3$. The interaction is according a constant cross section of $\sigma = 30$ mb for the pion gas and $\sigma = \pi R^2 = 31.4$ mb for the Hagedorn gas. For the pion gas a time step size of $\Delta t = 0.1$ fm and $N_t = 30000$ timesteps was chosen, while the Hagedorn gas where calculated at a lower time step size of $\Delta t = 0.01$ fm and $N_t = 25000$ timesteps, which is justified because of the less steady correlation function at higher time, however bypassing too long calculation times. These values are extracted from the error estimation via the colored noise studies described below.

It is checked, that detailed balance is fully respected and the mass and quantum number distributions are constant over the full simulation time.

#### B. Green-Kubo formalism

The Green-Kubo method is the common method to compute transport coefficients like shear viscosity, elec-
tric or heat conductivity etc. assuming, that the probability distribution of the time-averaged dissipative flux is Gaussian \[31\]. It can be derived from the dissipation-fluctuation theorem \[32, 33\] and reads (see e.g. \[11\])

\[
\eta = \frac{1}{T} \int_V d^3r \int_0^\infty dt \langle \pi^{xy}(\vec{r}, t) \pi^{xy}(0, 0) \rangle .
\]  

(15)

Here \(\pi^{xy}\) indicates a fixed spatial component of the volume averaged shear tensor \(3\) and \(\langle \rangle\) denotes the ensemble average of the argument. The shear stress component, defined as

\[
\pi^{xy}(\vec{r}, t) = \int \frac{g d^3p}{(2\pi)^3 E} p^x p^y f(\vec{r}, t; \vec{p}) ,
\]  

(16)

is in the simulation replaced by a discretized version,

\[
\pi^{xy} = \frac{1}{V} \sum_{i=1}^{N_{\text{part}}} \frac{p^x_i p^y_i}{p^t_i} ,
\]  

(17)

summing up all particles in the box with volume \(V\) at a given time \(t\). The correlator is obtained by the time and ensemble average in the limit \(t_{\text{corr}} \to \infty\),

\[
C^{xy}(t) = \frac{\langle \pi^{xy}(t) \pi^{xy}(0) \rangle}{\langle \pi^{xy}(0) \rangle} = \frac{1}{t_{\text{corr}}} \int_0^{t_{\text{corr}}} \pi^{xy}(t + t') \pi^{xy}(t') dt'.
\]

(18)

where \(N_{\text{corr}} = t_{\text{corr}}/\Delta t\) and \(i = t/\Delta t\) and \(\mathcal{F}_\omega\) denotes the Fourier-transformed of its argument, applying the Wiener-Khinchin theorem. Here, \(\pi^{xy}_\omega\) stands for the Fourier-transformed of \(\pi\). If the system fluctuates around the equilibrium state, one finds \(1\)

\[
C^{xy}(t) = C^{xy}(0) e^{-t/\tau} .
\]  

(19)

Therefore one obtains

\[
\eta = \frac{V}{T} \int_0^\infty dt \frac{C^{xy}(t)}{T} = \frac{C^{xy}(0) V T}{T} .
\]  

(20)

This procedure is illustrated in fig. 3 showing an example of the oscillating \(\pi^{xy}(t)\), and in fig. 4 where the exponential decaying slopes are clearly visible.

The value \(C^{xy}(0)\) is of special interest because the analytic expression can be calculated easily noticing, that \(C^{xy}(0) = \text{Var}[\pi^{xy}]\). Thus, using the continuous formulation eq. \(16\), one obtains for one single particle species with mass \(m\) and degreenacy \(g \lesssim 11\)

\[
C^{xy}_m(0) = \frac{g}{30\pi^2 V} \int_0^\infty dp \frac{p^6}{E^2} \exp \left( -\frac{E}{T} \right) .
\]  

(21)

\[3\] We use internally the three combinations \(\pi^{xy}, \pi^{xz}, \pi^{yz}\) as an additional possibility to estimate the statistical error.

With \(E = \sqrt{m^2 + p^2}\). This integral has to be performed numerically. Finally, to get a result for the Hagedorn gas, one has to sum over all masses,

\[
C^{xy}(0) = \frac{1}{30\pi^2 V} \int dm \rho(m) \int_0^\infty dp \frac{p^6}{E^2} \exp \left( -\frac{E}{T} \right) .
\]  

(22)

Irrespective of the numerical integrations, we will call these results still 'analytical' to contrast them from the results obtained via the Monte Carlo calculations. One observes a very nice agreement of analytical, eq. \(22\), and numerical results, eq. \(18\), as shown in fig. 5.

If \(C^{xy}(0)\) is one value of interest one gets out of the Green-Kubo formalism, the other one is the relaxation time \(\tau\), the slope of the correlator, shown in fig. 4. One observes, that the \(\tau\) parameter of the pion gas decreases smoothly and less rapid than that of the Hagedorn gas. Here, no analytic estimator is available at the moment.
FIG. 5. A comparison of analytical, eq. (22), and numerical results, eq. (18), of $C^{xy}(0)$.

FIG. 6. Results for the slope parameter $\tau$ from the fitting procedure for varying temperature $T$. The error bars indicate the statistical error (see text for details).

mate, the results of the fits for the relaxation time $\tau$ vary drastically between different runs. Therefore also the statistical error of this quantity as obtained by calculating the Jackknife variance (for a review see [34]) is shown in fig. 6.

Nevertheless, considering a relaxation time as the inverse of an interaction rate, one may express (in a low density approximation) $\tau = 1/T = 1/\langle n\sigma v_{\text{rel}} \rangle$. Thus, the exponential behavior of $\tau$ as function of the temperature $T$ is mainly dictated by the increase of the particle density $n$. It may be matter of debate, if the factor $\langle \sigma v_{\text{rel}} \rangle$ really directly translates into the transport cross section $\sigma_{\text{tr}} = 2/3\sigma_{\text{tot}}$. Here further investigations are at order.

It is very instructive to check the Green-Kubo method against some known input. For this, an implementation of the algorithm of generating random numbers with memory [35] enables to dial in specific values for the correlation and compare with the results of the Green-Kubo method. Error estimates according a Jacknife method show clearly, that the error scales as usual with $1/\sqrt{N_{\text{run}}}$, if $N_{\text{run}}$ independent runs are performed, but with $1/N_{\text{timestep}}$ in a single run. Thus it is more preferable, to perform long runs, than doing multiple short runs. In addition, having an estimate for the correlation time $\tau$, the effect of the timestep size may be estimated correctly.

IV. RESULTS

The final results for the shear viscosity using the Green-Kubo method eq. (20) are shown in fig. 7 and compared to analytic estimates. The agreement is very well; while there is some tiny underestimation for $T < 100$ MeV, Monte Carlo results coincide very well with the analytic estimates for higher temperatures. Here one can also see, that $\eta$ stays more or less the same for both species at lower temperature and starts to diverge the more particles are created in the box in the Hagedorn case. The values explode, if the particle number density increases ad infinitum near $T_H$. Nevertheless, it increases less rapidly than the entropy density as shown in fig. 2.

It is interesting to observe, that the competing differences in the intermediate result of $C(0)$ and $\tau$ cancel each other at low temperatures and only for $T \gtrsim 140$ MeV, a different behavior between the pion gas and the Hagedorn state gas may be observable.

Combining both the results of the thermodynamical quantities (the entropy density), and the shear viscosity, fig. 8 shows the final result, the shear viscosity to entropy density ratio. As expected, at low temperatures the results for the pion gas and the Hagedorn gas coincide. Since the entropy density $s$ very rapidly starts to diverge with increasing temperature, also the fraction $\eta/s$ diverges. Finally, all the calculated results via the Monte
FIG. 8. The final result is $\eta$ normalized to the entropy density for numerical and analytical estimates. Error bars are as in fig. 7. The KSS bound $1/4\pi$ is indicated. The Hagedorn extrapolation $M \rightarrow \infty$ contains both separate $\eta$ and $s$ extrapolations.

Carlo/Green Kubo approach a stop at values above the KSS bound of $1/4\pi$. The analytic estimates indicate, that the results drop below this boundary and go to zero when temperature increases further.

Using the statistical error for $\tau$, one can compute the errors for $\eta$ and $\eta/s$. In fig. 7 and fig. 8 one observes, that the numerical results including the errorbars do not match the analytical curve.

This leads us to the finding, that there are some systematic error in the Green-Kubo formalism, which are underestimated in the current work.

V. CONCLUSIONS

In the present work, the transport coefficient $\eta/s$ has been calculated for a gas of Hagedorn resonances. Using the usual way of doing Monte Carlo simulations with a Green-Kubo analysis, it has been shown, that these results coincide very well with some analytic estimates. In addition, the same analysis has been performed for a single pion gas with elastic interactions. This, on one hand allows to check the used analysis routines and also on the second hand side indicates the differences of the interactions.

Here, while the MC calculations only consider Hagedorn states with masses $m < 10$ GeV, the analytic estimates allow to extrapolate to a Hagedorn spectrum up to infinite masses. Interestingly, the influence of high mass Hagedorn states with $m > 10$ GeV is only visible in the present analysis at temperatures $T > 160$ MeV, which are very close to the underlying Hagedorn temperature $T_H = 165$ MeV.

Finally, the main result of this study is the finding, that the fraction $\eta/s$ drops while approaching the limiting Hagedorn temperature. While $\eta$ itself increases with increasing temperature, the growth of $s$ overwhelms it and dominates the overall behavior. The KSS bound is violated at $T_H$.

This singular behavior may be cured by a phase transition to some other phase with increasing $\eta/s$, being beyond the Hagedorn picture, since the Hagedorn temperature is a limiting temperature.

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