The Kugo-Ojima color confinement criterion, which is based on the BRST symmetry of the continuum QCD is numerically tested by the lattice Landau gauge simulation. We first discuss the Gribov copy problem and the BRST symmetry on the lattice. The lattice Landau gauge can be formulated with options of the gauge field definition, $U$-linear type or log $U$ type. The Kugo-Ojima parameter $u^{ab}$ which is expected to be $-\delta^{ab}$ in the continuum theory is found to be $-0.76^{ab}$ in the strong coupling region, and the magnitude is a little less in the weak coupling region in log $U$ type simulation. Those values are weakened even further in $U$-linear type. The horizon function defined by Zwanziger is evaluated in both types of gauge field and compared. The horizon function in the log $U$ version is larger than the other, but in the weak coupling region, the expectation value of the horizon function is suggested to be zero or negative.

1 Introduction

There are essentially two aspects in the manifestation of color confinement in the Landau gauge QCD. One aspect is the linear potential between quarks, which was conjectured by Gribov as a consequence of an enhancement of the singularity of the ghost propagator due to the restriction of the gauge field $A$ on the transverse plane. Another aspect is the absence of free single colored particle state in the asymptotic Hilbert space, which culminates in the Kugo and Ojima color confinement criterion based on the BRST(Becchi-Rouet-Stora-Tyutin) symmetry: i.e. in the Landau gauge, a coefficient in the two-point function produced by the ghost, the antighost and the gauge field becomes $-\delta^{ab}$, where $a$ and $b$ specify the color in the adjoint representation. Analytical
calculation of this value is extremely difficult and so far no verification has been performed. In 1994, Zwanziger developed a lattice QCD theory for Gribov ambiguity. He claimed that, if the restriction to the fundamental modular region is achieved, the gluon propagator at 0 momentum should vanish in the continuum limit.

As the matter of first principle, it is still unclear whether the bases of these two theories are consistent or not, that is, 'Does the Kugo-Ojima theory properly resolve the problem of Gribov copy?' and/or 'Is the Zwanziger theory satisfactorily well defined in the continuum limit?' Considering this fundamental problem on one side, we make numerical verifications on conjectures of these theories on the other. In the lattice QCD test, we address the following problems: whether the gluon propagator is infrared finite, how singular the ghost propagator is in the infrared region, whether the Kugo-Ojima color confinement criterion is satisfied, and how close to zero from negative the Zwanziger horizon function is.

2 The Gribov problem and the lattice simulation of the Landau gauge QCD

2.1 The path integral formulation of the gauge fixed theory in the presence of the Gribov copy (Fujikawa, Hirschfeld)

First we give a brief review of the path integral formulation of the gauge fixed theory by Fujikawa and Hirschfeld. They discussed a possible situation which may give a way out of the gauge fixing degeneracy problem (the Gribov problem). In the following, all indices representing multi-degree of freedom are suppressed. Let us define a gauge unfixed partition function as,

$$ Z = \int dU e^{-\beta S(U)} .$$

In derivation of the Faddeev-Popov formula of the gauge fixed theory $f(U) = 0$, one considers the determinant function $\Delta(U^g)$,

$$ \Delta(U^g) = \det \left( \frac{\partial f(U^g)}{\partial U^g} \right) = \det \left( \frac{\partial f(U^g)}{\partial A^g} \right) $$

where $A_\mu = A_\mu(U)$, and one finds that $\Delta(U^g)$ is indeed a function of $U^g$, and looks at the integral on the gauge orbit, $U^g$,

$$ N(U) = \int dq \Delta(U^g) \delta(f(U^g)).$$
Obviously $N(U)$ is an orbit function, that is, $N(U^g) = N(U)$. At the intersection points of the gauge orbit $U^g$ with the surface $f(U) = 0$, i.e., $g = g_i(U)$, Gribov’s copies, the above delta function is transcribed to give

$$N(U) = \int dg \sum_i \frac{\Delta(U^g)}{|\Delta(U^g)|} \delta(g - g_i(U)) = \int dg \sum_i \text{sign}(\Delta(U^g))\delta(g - g_i(U)).$$

(2)

These delta functions contribute 0 or $\pm 1$, and thus, if the orbit function $N(U)$ is non vanishing over all orbits, $N(U) \neq 0$, then in use of the identity

$$1 = \frac{1}{N(U)} \int dg \Delta(U^g)\delta(f(U^g)),$$

the standard FP procedure applies, and one factors out the gauge volume and obtains the formula for expectation values of functions, $F(U)$,

$$\langle F(U) \rangle_{\text{gauge}} = \frac{\int dU \Delta(U)\delta(f(U))F(U)e^{-\beta S(U)}/N(U)}{\int dU \Delta(U)\delta(f(U))e^{-\beta S(U)}/N(U)},$$

(3)

provided $F(U)$ is gauge invariant. In the case when $F(U)$ is gauge non-invariant, $F(U)$ in (3) should be formally replaced with a gauge invariant function,

$$\tilde{F}(U) = \frac{\int dg F(U^g)}{\int dg},$$

(4)

if one really wishes to obtain the Boltzmann average of the gauge non-invariant function $F(U)$. The formula (3) allows the BRST formulation as in the following, and if $N(U)$ is particularly a constant, then the standard Lagrangian of the continuum theory is justified. The formula (3) also derives a natural lattice simulation algorithm of the gauge fixed theory. The treatment of the gauge fixed theory here is, however, somewhat too formal for its practical use, particularly with respect to gauge non-invariant function in the presence of Gribov copies. We discuss this point in the following section.

2.2 The simulation algorithm for the gauge fixed theory (Mandula-Ogilvie) and the Gribov problem

Multiplying by the gauge volumes, the denominator and the numerator of the formula (3), respectively, one recovers the $Z$ in the denominator, and has in the numerator,

$$\int dU \int dg \Delta(U^g)\delta(f(U^g))F(U^g)e^{-\beta S(U)}/N(U).$$

(5)
As in (2), one obtains the numerator as

\[ \int dU \sum_i \text{sign} (\Delta(U^g_i)) F(U^g_i) e^{-\beta S(U)} / N(U) \]  

(6)

where \( U^g_i \) is the \( i \)-th Gribov copy on the orbit \( U^g \). This gives us the algorithm in the simulation

\[ \langle F(U) \rangle |_{f \text{ gauge}} = \frac{1}{Z} \int dU \sum_i \frac{\text{sign}(\Delta(U^g_i)) F(U^g_i)}{N(U)} e^{-\beta S(U)} = \langle \bar{F}(U) \rangle \]  

(7)

where the last averaging \( \langle \rangle \) is that of simulation, i.e., that with respect to the Boltzmann weight \( e^{-\beta S} \), and \( \bar{F}(U) \) is a sign-weighted average of \( F(U^g_i) \) on the gauge orbit \( U^g \). Readers should have already noticed that (7) gives trivial results when \( F(U) \) is gauge invariant, that is, we do not need any gauge transformation at all. Necessity of the function \( \bar{F}(U) \) is rather impractical as well, and therefore we start from the formula (3) with a gauge non-invariant function \( F(U) \) for work purpose. Then the result (7) is highly nontrivial, and \( \bar{F}(U) \) can be viewed as being defined as a new gauge invariant function from \( F(U) \) and \( f(U) \). The sign-weighted average is impractical and a simple average may be similarly considered as such. If the averaging has, however, a large fluctuation this formula would rather be interpreted as representing Gribov noise apart from the original function \( F(U) \), although (7) could be considered as giving a sort of gauge invariant function out of \( F(U) \) and \( f(U) \). A modification of the gauge such that one chooses a unique copy among others on the orbit is favored, that is, a new gauge without Gribov copy, and in that case, the above formula is useful in practice, and this procedure may naturally change the result in the above formula (7) without Gribov noise.

2.3 The BRST formulation and the Gribov problem

The obvious standard FP formula allows BRST formulation,

\[ \langle F(U) \rangle |_{f \text{ gauge}} = \frac{1}{Z'} \int d\mu \exp(\delta \int \bar{c} f(U)) F(U) e^{-\beta S(U)} / N(U), \]  

(8)

\[ Z' = \int d\mu \exp(\delta \int \bar{c} f(U)) e^{-\beta S(U)} / N(U), \]  

(9)

where \( \delta \) stands for BRST transformation,

\[ \delta \bar{c} = iB, \quad \delta A_\mu(U) = D_\mu(U)c, \quad \delta c = -cc, \quad \delta B = 0, \]  

(10)
and the measure is defined as $d\mu = dUdBd\bar{c}d\bar{c}$, where $dB$ integration is performed on $R$ and $d\bar{c}d\bar{c}$ is suitably defined differentiations with respect to Grassmann numbers, $c$ and $\bar{c}$, such that

$$\int d\bar{c}d\bar{c} e^{\bar{c}M} = \det M, \quad (11)$$

$$\int d\bar{c}d\bar{c} cxye^{\bar{c}M} = (\det M)(-M^{-1})_{xy}. \quad (12)$$

It was shown by Neuberger that if the gauge fixing function $f(U)$ is a smooth function of compact variables $U$, then the expectation value of gauge invariant function $F(U)$ becomes an indefinite form,

$$\langle F(U) \rangle |_{f \text{ gauge}} = \frac{0}{0}. \quad (13)$$

which implies that all Gribov copies contribute to give total cancellation, in other words, the assumption that the above $N(U)$ is non vanishing on all orbits, does not hold, and the formula (3) is totally meaningless.

An essential point of his argument is as follows. Let us consider a general expression

$$Z'(t) = \int d\mu \exp(t\delta \int \bar{c}f(U))G(U) \quad (14)$$

with a gauge invariant function $G(U)$ as the Boltzmann weight. Then one finds that

$$\frac{dZ'}{dt}(t) = \int d\mu(\delta \int \bar{c}f(U)) \exp(t\delta \int \bar{c}f(U))G(U) \quad (15)$$

can be written from nilpotency of $\delta$, i.e., $\delta^2 = 0$, and from $\delta G(U) = 0$ as,

$$\frac{dZ'}{dt}(t) = \int d\mu \delta \left( \int \bar{c}f(U) \right) \exp(t\delta \int \bar{c}f(U))G(U). \quad (16)$$

If $W(U, B, c, \bar{c})$ is an analytic function of the compact variables $U$, then one can show that

$$\int d\mu \delta W(U, B, c, \bar{c}) = 0. \quad (17)$$

Thus it follows that Together with $Z'(0) = \int \cdots d\bar{c}G(U) = 0$, one finds that $Z'(t) = 0$, so $Z'(1) = 0$. This concludes total cancellation of the Gribov copies of gauges given by analytic gauge functions of $U$. Thus one is forced to consider non-analytic gauge functions of $U$ as desired gauge functions. As a one dimensional $U(1)$ toy example which avoids (17), one can consider $U = e^{iA}$,
A = \Im \log U$, where $A$ is not continuous at $U = -1$, and with definition $\delta A = c$, one finds that

$$\int dU dB d\bar{c}d\bar{\bar{c}} \delta (e^{-B^2 \bar{c}A}) = \int_{-\pi}^{\pi} dA \int_{-\infty}^{\infty} dB d\bar{c}d\bar{\bar{c}} \ e^{-B^2 \bar{c}\bar{\bar{c}}} \neq 0.$$  \hfill (18)

Although this problem is still open in the lattice BRST formulation, use of non compact variables in gauge fixing functions may be helpful.

2.4 The Landau gauge and the Gribov problem

Now we focus on the Landau gauge in $SU(3)$ lattice QCD, that is, $f(U) = \partial_\mu A_\mu(U)$, where there are some options of definition $A_\mu(U)$ as

1. $U$-linear one: $A_{x,\mu} = \frac{1}{2} (U_{x,\mu} - U_{x,\mu}^\dagger)$, traceless part;

2. use of exponential map: $U_{x,\mu} = \exp A_{x,\mu}$, $A_{x,\mu}^\dagger = -A_{x,\mu}$, where absolute values of all eigenvalues of $A_{x,\mu}$ do not exceed $4\pi/3$.

In the latter definition, $A_{x,\mu}(U)$ is not analytic with respect to compact variable $U_{x,\mu}$ contrary to the former one. In both cases the Landau gauge, $\partial A^g = 0$, can be characterized in use of optimizing functions $F_U(g)$ of $g$, such that $\delta F_U(g) = 0$ for any $\delta g$.

1. $U$-linear definition: $F_U(g) = \frac{2N}{dV(N^2-1)} \sum_{x,\mu} (1 - \frac{1}{N} \text{Retr} U_{g,x,\mu}^g)$.

2. use of exponential map: $F_U(g) = ||A^g||^2 = \frac{1}{dV(N^2-1)} \sum_{x,\mu} \text{tr} \left( A^g_{x,\mu} A^g_{x,\mu}^\dagger \right)$, where $N$ is the number of colors and $d$ is the dimension. It is noteworthy that $||A^g||^2$ is a continuous function of compact variables $U^g$ in spite of non analytic property of $A_{x,\mu}(U^g)$.

In both options of the gauge field definition, the variation of the optimizing function, $F_U(g)$, under infinitesimal gauge transformation $g^{-1} \delta g = \epsilon$, reads as

$$\Delta F_U(g) = -2 \langle \partial A^g | \epsilon \rangle + \langle \epsilon | - \partial D(A^g) | \epsilon \rangle + \cdots,$$  \hfill (19)

where $D_\mu(A)$ denotes the covariant derivative in each definition;

1. in $U$-linear version,

$$D_\mu(U_{x,\mu})\phi = \frac{1}{2} \left\{ \frac{U_{x,\mu} + U_{x,\mu}^\dagger}{2}, \partial_\mu \phi \right\} + [A_{x,\mu}, \phi].$$  \hfill (20)
\[ \partial_\mu \phi = \phi(x + \mu) - \phi(x), \quad \bar{\phi} = \frac{1}{2} (\phi(x + \mu) + \phi(x)), \quad (21) \]

2. in \( A = \log U \) version,

\[ D_\mu (A_{x,\mu}) \phi = S(A_{x,\mu}) \partial_\mu \phi + [A_{x,\mu}, \bar{\phi}], \quad (22) \]

where \( A_{x,\mu} = \text{adj} A_{x,\mu} = [A_{x,\mu}, \cdot] \), and

\[ S(x) = \frac{x/2}{\text{th}(x/2)}. \quad (23) \]

Gribov copy is a generic phenomenon in both definitions as well as in the continuum\(^\text{4}\), there exist a lot of local minima of \( F_U(g) \) along the gauge orbit \( U^g \). Thus the naive Landau gauge loses its solid basis both in the theoretical and in the simulation viewpoint for examination of gauge non-invariant quantities such as gluon propagator, ghost propagator, etc.

2.5 Sophisticated Landau gauge

Zwanziger devised various regions of the transverse plane of \( A^g \), i.e., \( \partial A^g = 0 \), depending on properties of a point \( U^g \) on the plane. For example, one defines Gribov region \( \Omega \) as

\[ \Omega = \{ A | -\partial D(A) \geq 0, \partial A = 0 \} , \quad (24) \]

where \( D_\mu (A) \) denotes the covariant derivative, and \( -\partial D \geq 0 \) implies that the Faddeev-Popov operator \( -\partial D \) is positive definite. A point on the Gribov region is a local minimum of \( F_U(g) \), but it is known that some points on the Gribov region can be gauge copies of each other. Thus one defines the fundamental modular region \( \Lambda \) as the absolute minimum along the gauge orbits.

\[ \Lambda = \{ A | \| A \|^2 = \text{Min}_g \| A^g \|^2 \}, \quad \Lambda \subset \Omega . \quad (25) \]

Then one defines a corresponding region \( \mathcal{U}_\Lambda \) of configuration \( U \), as

\[ \mathcal{U}_\Lambda = \{ U | A(U)^T \in \Lambda \} \]

\[ (A^T \text{ denotes transverse component of } A(U)) \quad (26) \]

Putting an indicator function \( \theta_\Lambda \) of the set \( \mathcal{U}_\Lambda \) as

\[ \theta_\Lambda(U) = 1 \text{ if } U \in \mathcal{U}_\Lambda, \quad \theta_\Lambda(U) = 0 \text{ if } U \notin \mathcal{U}_\Lambda, \]
and putting \( \bar{\theta}_\Lambda(U) = 1 - \theta_\Lambda(U) \), one can define a corresponding gauge function as

\[
f_\Lambda(U) = \partial A \cdot \theta_\Lambda(U) + \bar{\theta}_\Lambda(U)
\]  

(27)

The arguments in the preceding subsections formally applies for a corresponding gauge function containing non-analytic Heaviside function. The gauge-fixing algorithm in the simulation is required to attain the \textbf{absolute minimum} of the \( F_U(g) \) along the gauge orbit. But the global minimization is difficult in general and developing the efficient algorithm is still an open problem.\[10\]

3 The Kugo-Ojima confinement criterion and the Gribov-Zwanziger’s theory

3.1 Kugo-Ojima’s theory

A sufficient condition of the color confinement given by Kugo and Ojima is that \( u_{ab} \) defined by the two-point function of the FP ghost fields, \( c(x), \bar{c}(y) \), and \( A_\nu(y) \),

\[
\int e^{ip(x-y)} \langle 0|TD_\mu c^a(x)g(A_\nu \times \bar{c})^b(y)|0\rangle d^4x = (g_{\mu\nu} - \frac{p_\mu p_\nu}{p^2})u_{ab}(p^2)
\]  

(28)

satisfies \( u_{ab}(0) = -\delta_{ab} \).

Brief survey of Kugo-Ojima’s argument that \( u_{ab} = -\delta_{ab} \) is a sufficient condition of the colour confinement is the following.

1. The BRS transformation \( \delta_B \) is given by replacing gauge transformation parameters \( \theta^a \) as \( \theta^a(x) \rightarrow \lambda c^a(x) \), where \( \lambda \) is an imaginary Grassmann number and \( c^a(x) \) is the ghost field. Putting \( C(x) = gc^a(x)\Lambda_a \) where \( \Lambda_a \) is antihermitian such that \( [\Lambda_a, \Lambda_b] = f_{abc}\Lambda_c \) and \( \delta_B = \lambda \delta_B \), and antihermitian matrix \( A_\mu(x) = gA^a_\mu(x)\Lambda_a \), the BRS transformation reads

\[
\delta_B \phi = -C \phi, \quad \delta_B A_\mu = D_\mu(A)C = \partial_\mu C + [A_\mu, C].
\]

The nilpotency requirement of \( \delta_B \) derives,

\[
\delta_B C = -C^2.
\]

For each \( c^a(x) \), one introduces an anti-ghost \( \bar{c}^a(x) \), and similarly the matrix \( \bar{C} \), and defines,

\[
\delta_B \bar{C} = iB,
\]
then the nilpotency gives \( \delta_B B = 0 \). Then the gauge-fixing and Faddeev-Popov Lagrangian can be written as

\[
\mathcal{L}_{\text{GF}+\text{FP}} = -i\delta_B [\bar{c}^a (\partial^\mu A_\mu^a + \frac{1}{2}\alpha B^a)]
= B^a \partial^\mu A_\mu^a + \frac{1}{2}\alpha B^a B^a + i\bar{c}^a \partial^\mu D_\mu c^a.
\]  

(29)

Together with the original gauge invariant Lagrangian, the BRS invariance of the total Lagrangian follows from the nilpotency \( \delta_B^2 = 0 \). Thus the BRS symmetry is a symmetry of a gauge fixed Lagrangian, so to say, a quantum gauge symmetry. The corresponding conserved BRS charge reads as

\[
Q_B = \int d^3x \left[ B^a D_0 c^a - \partial_0 B^a \cdot c^a + \frac{i}{2} g \partial_0 \bar{c}^a \cdot (c \times c)^a \right]
\]

where \((F \times G)^a = f_{abc} F^b G^c\). Thus one defines physical space \( \mathcal{V}_{\text{phys}} = \{|\text{phys}\rangle\} \) as

\[
Q_B |\text{phys}\rangle = 0.
\]

It is to be noted that both ghost and anti-ghost fields are considered as hermitian fermi fields so that the derived Hamiltonian should be hermitian. This assignment is important for derivation of the unitarity, but due to this assignment, there necessarily involves an indefinite metric in a space of ghost and anti-ghost fields.

2. The BRS algebra is given by BRS charge, \( Q_B \), and FP ghost charge, \( Q_c \), as

\[
Q_B^2 = 0, \quad [iQ_c, Q_B] = Q_B, \quad [Q_c, Q_c] = 0.
\]

Since these \( Q_B, Q_c \) are commuting with other conserved charge, all asymptotic one particle states can be classified by irreducible representations. Due to the nilpotency of BRS charge, \( Q_B \), there only exist BRS singles and doublets. From hermiticity of ghosts, \( Q_c \) is defined as a generator of scale transformation of FP ghosts. It is to be noted that while \( Q_c \) being hermite, FP ghost number is counted by \( N_{FP} = iQ_c \), and that among FP ghost number eigenstates, \( \langle M|N\rangle \neq 0 \) only if \( M = -N \). Due to this metric structure, BRS doublets always appear in pair of opposite sign FP ghost numbers, and this pair is called a BRS 4-tet (quartet). Under the assumption that BRS singlets have positive metric, it is proved that \( \mathcal{V}_{\text{phys}} \) has positive semidefinite in such a way that BRS 4-tet particles appear only in zero norm.
3. From the Ward-Takahashi identities,

\[
F.T.[\langle 0 | T D_\mu c^a(x) \bar{c}^b(y) | 0 \rangle] = -i \langle 0 | T A_\mu^a(x) B^b(y) | 0 \rangle \]

\[
= i \delta^{ab} \frac{P_\mu}{p^2},
\]

(30)

where F.T. implies the Fourier transformation, it follows that Heisenberg operators, \( D_\mu c^a, \bar{c}^a, A_\mu^a, B^a \) necessarily have massless asymptotic fields when \( x_0 \to \pm \infty \),

\[
A_\mu^a(x) \to \partial_\mu \chi^a(x) + \cdots,
\]

\[
B^a(x) \to \beta^a(x) + \cdots,
\]

\[
D_\mu c^a(x) \to \partial_\mu \gamma^a(x) + \cdots,
\]

\[
\bar{c}^a(x) \to \bar{\gamma}^a(x) + \cdots.
\]

One finds from the BRS transformation that for each colour \( a \), a set of the above massless asymptotic fields form a BRS 4-tet (quartet) such that

\[
[iQ_B, \chi^a(x)] = \gamma^a(x), \quad \{iQ_B, \bar{\gamma}^a(x)\} = i\beta(x)
\]

\[
\{iQ_B, \gamma^a(x)\} = 0, \quad [iQ_B, \beta^a(x)] = 0
\]

and

\[
\{\gamma^a(x), \bar{\gamma}^b(y)\} = -i[\gamma^a(x), \beta^b(y)] = -\delta^{ab} D(x - y)
\]

4. With respect to symmetry breaking in general, the following statements are equivalent;

(a) Charge of symmetry, \( Q = \int d^3 x j_0 \), is well-defined.

(b) Symmetry is not broken, \( Q |0\rangle = 0 \)

(c) There exist no massless one particle states in \( j_\mu \) spectrum,

\[
\langle 0 | j_\mu(x) | p^2 = 0 \rangle = 0
\]

5. The Noether current corresponding to the conservation of the colour symmetry is

\[
g J_\mu^a = \partial^\nu F_{\mu\nu}^a + \{Q_B, D_\mu \bar{c}\},
\]

where its ambiguity by divergence of antisymmetric tensor should be understood, and this ambiguity is utilised so that massless contribution may be eliminated for the charge, \( Q^a \), to be well defined.
6. Denoting \( g(A_\mu \times \bar{c})^a \rightarrow u_\mu^b \partial_\mu \bar{c}^b \), and then \( D_\mu \bar{c}^a \rightarrow (1 + u)^b_\mu \partial_\mu \bar{c}^b \), one obtains that

\[
F.T.(0)TD_\mu c^a(x)g(A_\nu \times \bar{c})^b(y)|0\rangle = (g_{\mu\nu} - \frac{p_\mu p_\nu}{p^2})u_\mu^a(p^2),
\]

provided \( A_\mu \) has a vanishing expectation value. The current \( \{Q_B, D_\mu \bar{c}\} \) contains the massless component, \((1 + u)^b_\mu \partial_\mu \bar{c}^b(x)\). We can modify the Noether current for colour charge \( Q^a \) such that

\[
gJ^a_\mu = gJ^a_\mu - \partial^\nu \Gamma^a_{\mu\nu} = \{Q_B, D_\mu \bar{c}\}.
\]

In the case of \( 1 + u = 0 \), massless component in \( gJ^a_0 \) is vanishing and the colour charge

\[
Q^a = \int d^3x \{Q_B, g^{-1}D_0 \bar{c}^a(x)\}
\]

becomes well defined.

7. The physical state condition \( Q_B V_{\text{phys}} = 0 \) together with the equation (31) implies that all BRS singlet one particle states \( |f\rangle \in V_{\text{phys}} \) are colour singlet states. This statement implies that all coloured particles in \( V_{\text{phys}} \) belong to BRS 4-tet and have zero norm. This is the colour confinement.

8. In the course of their derivation, they assume Lorentz invariance and that the colour symmetry is not broken.

9. They also proved that if the vector massless asymptotic field is missing in a channel \( a \), and if the channel \( a \) belongs to the image of \( 1 + u \) then the massless 4-tet in \( j^a_\mu \) can not be cancelled, and the colour symmetry with charge \( Q^a \), is spontaneously broken. (Inverse Higgs mechanism theorem)

The corresponding Euclidian expression is as follows,

\[
\int e^{-ip(x-y)} \langle D_\mu c^a g(A_\nu \times \bar{c})^b \rangle d^4x = (\delta_{\mu\nu} - \frac{p_\mu p_\nu}{p^2})u^{ab}(p^2),
\]

which can be calculated by

\[
\frac{1}{V} \sum_{x,y} e^{-ip(x-y)} \left\langle \text{tr} \left( \lambda^a D_\mu \frac{1}{i\partial D} \left[ -A_\nu \lambda^b \right] \right)_{xy} \right\rangle = (\delta_{\mu\nu} - \frac{p_\mu p_\nu}{p^2})u^{ab}(p^2),
\]

where \( \lambda^a \) is a normalized antihermitian basis of Lie algebra, \( V \) a lattice volume, and the ghost propagator is given by

\[
\langle \bar{c}^a_{x} c^b_{y} \rangle = \left\langle \text{tr} \left( \lambda^a \frac{1}{i\partial D} \lambda^b \right)_{xy} \right\rangle.
\]
3.2 Zwanziger’s theory

The fundamental modular region $\Lambda$ is specified by the absolute minimum along the gauge orbits in the Gribov region $\Omega$.

$$\Lambda = \{ A \| A \| ^2 = \text{Min}_g \| A^g \| ^2 \}, \quad \Lambda \subset \Omega = \{ A \leftarrow \partial D \geq 0 , \partial A = 0 \} \ . \ (34)$$

Zwanziger relaxes the periodicity restriction on the gauge transformation $g$, and imposes larger periods than the original. Then some two points in the fundamental modular region $\Lambda$ may be bridged to be Gribov copies of each other, and one of them is not the absolute minimum of the minimizing function along the gauge orbit anymore. Surviving points as the absolute minimum consist of core region $\Xi (\Xi \subset \Lambda)$. In the so defined core region $\Xi$, a horizon function $H(U)$ given below is negative.

The Horizon function is defined as follows. Let two point tensor $G_{\mu\nu xy}^{ab}$ be

$$G_{\mu\nu xy}^{ab} = \text{tr} \left( \lambda^a \dagger D_\mu \frac{1}{-\partial D} (-D_\nu) \lambda^b \right) _{xy} \ . \ (35)$$

Then $H(U)$ is given as

$$H(U) = \sum _{x,y,a} G_{\mu\nu xy}^{aa} - (N^2 - 1)E(U) \ (36)$$

where $E(U)$ reads as follows;

1. in $U$-linear version, $E(U) = \sum _l \frac{1}{N} \text{Re} \ \text{tr} U_l$,

2. in $A = \log U$ version, $E(U) = \frac{1}{N^2 - 1} \sum _{l,a} \text{tr} \left( \lambda^a \dagger S(A_l) \lambda^a \right)$, $\text{tr} \lambda^a \dagger S(A_l) \lambda^a$

where $A_l = \text{adj} A_l$, and $S(x) = \frac{x/2}{\text{th}(x/2)}$.

Let us define an average tensor $G_{\mu\nu xy}$ be $G_{\mu\nu xy}\theta ^{ab} = \langle G_{\mu\nu xy}^{ab} \rangle$, provided color symmetry is not broken. One sees that a Fourier transform of the average tensor,

$$G_{\mu\nu} (p) = \frac{1}{V} \sum _{x,y} e^{-ip(x-y)} G_{\mu\nu xy}$$

takes a form

$$G_{\mu\nu} (p)\delta ^{ab} = \left( \frac{e}{d} \right) \frac{p_\mu p_\nu}{p^2} \delta ^{ab} - \left( \delta ^{\mu\nu} - \frac{p_\mu p_\nu}{p^2} \right) u ^{ab} , \ (37)$$
where \( e = \langle E(U) \rangle / V \), and that it is related with the horizon function as

\[
\langle H(U) \rangle / V = (N^2 - 1) \left[ \lim_{p \to 0} G_{\mu\nu}(p) - e \right].
\]

(38)

He defined the **augmented core region** \( \Psi = \{ U : H(U) \leq 0 \} \cap \Omega \) (\( \Xi \subset \Psi \subset \Omega \)). \( \Psi \) and \( \Lambda \) are qualitatively similar, and he defined the partition function \( Z_{\Psi} \) in the path integral in use of the corresponding Landau gauge function \( f_{\Psi}(U) \), and concluded in the infinite volume limit that

\[
\lim_{V \to \infty} \langle H(U) \rangle / V = 0.
\]

Putting Kugo-Ojima parameter as \( u_{ab}(0) = -\delta_{ab}c \), one finds from (37), (38), that

\[
\left( \frac{e}{d} \right) + (d - 1)c - e = (d - 1) \left( c - \frac{e}{d} \right) = 0,
\]

which is called horizon condition. Since we can measure \( c \) and \( e \) by the lattice simulation, we can check to what extent Zwanziger’s horizon condition holds in our simulation. With respect to the value \( e/d \), note that the classical vacuum is characterized by \( e/d = 1 \).

### 4 The numerical results

#### 4.1 The algorithms of lattice Landau gauge fixing

In the definition \( A = \log U \), our Landau gauge fixing algorithm is as follows. We define the gauge field on links as an element of \( SU(3) \) Lie algebra as,

\[
e^{A_{x,\mu}} = U_{x,\mu}, \quad \text{where} \quad A_{x,\mu}^I = -A_{x,\mu}^I.
\]

(39)

We perform the gauge transformation as \( e^{A_{x,\mu}} = g^{\dagger}_x e^{A_{x,\mu}} g_x + \mu \) and define \( |\partial A| = \max_{x,\mu,a} |\partial A^a_x|, \quad \| \partial A \|^2 = \frac{1}{V(N^2 - 1)} \sum tr \partial A^I_x \partial A_x \). The Landau gauge is realized by minimizing \( \| A^a \|^2 \) via a gauge transformation \( g^I U g \), where \( g = e^\epsilon \).

In order to obtain \( \epsilon \), we switch the following two methods, depending on the current value of \( |\partial A| \) in comparison to some critical parameter \( |\partial A|_{cr} \).

1. When \( |\partial A| > |\partial A|_{cr} \), \( \epsilon_x = \frac{\eta'}{|\partial A|} \partial A_x \) with suitable parameter \( 1 < \eta' < 2.2 \)

2. When \( |\partial A| < |\partial A|_{cr} \), \( \epsilon = (-\partial^\mu D_{\mu}(A))^{-1} \eta \partial A \) where \( 1 < \eta < 2 \) is a parameter.

The restriction to the fundamental modular region is not always achieved. But, we observed that the obtained norm \( \| A \| \) is larger or smaller than that obtained after the smeared gauge fixing within 1% accuracy.
In case of the $U$-linear definition of gauge field $A$, we perform the site-local exact algorithm\cite{15}, with suitable over-relaxation parameter, $\eta = 1.6$, starting from gauge fixed configurations of $A = \log U$. It is found that the exact over-relaxation $g = W^{1.6}$ is faster than the stochastic over-relaxation, where $W$ is the site-local exact solution obtained by solving the nonlinear equation for finding the best fit gauge transformation on even-(odd)-site.

4.2 The Kugo-Ojima two-point function and the ghost propagator

The FP operator is

$$M[U] = -(\partial \cdot D(A)) = -(D(A) \cdot \partial), \quad (40)$$

and we define $Ad(A_\mu)$ by putting $D_\mu(A) = \partial_\mu + Ad(A_\mu)$. The inverse, $M^{-1}[U] = (M_0 - M_1[U])^{-1}$, is calculated perturbatively by using the Green function of the Poisson equation $M_0^{-1} = (-\partial^2)^{-1}$ and $M_1 = \partial_\mu Ad(A_\mu)$, as

$$M^{-1} = M_0^{-1} + \sum_{k=0}^{N_{\text{end}}} (M_0^{-1} M_1)^k M_0^{-1}. \quad (41)$$

The ghost propagator\cite{33} is infrared divergent and its singularity can be parameterized as $p^{-2(1+\alpha)}$, where $p^2 = \sum_{k\mu} (4 \sin^2 \frac{\pi k_\mu}{L})$, $(-L/2 < k_\mu \leq L/2)$. It depends on $\beta$ slightly, but its finite-size effect is small\cite{16}. These qualitative features are in agreement with the analysis of the Dyson-Schwinger equation\cite{17}.

![Figure 1: The ghost propagator as function of the lattice momentum. The data are $\beta = 5.5$ (box) and $6.0$ (triangle), \cite{16}. The fitted curve is $1.287/p^{2.779}$ for $\beta = 5.5$ and $1.162/p^{2.545}$ for $\beta = 6.0$ (dashed).](image1)

![Figure 2: The Kugo-Ojima parameter $|u_a^a|$ as the function of the spatial extent of the lattice $aL(fm)$. The data are $\beta = 6$ (triangle), and $\beta = 5.5$ (box) $8^4, 12^4, 16^4$ from left to right, respectively.](image2)
We measured the lattice version of $|u_b^a(0)|$ on $8^3 \times 16, 8^4 \times 12^4$ and $16^4$ for $\beta = 5.5$ and 6. When $\beta = 6$ and the lattice size is small, the Polyakov loop distribution deviates from the uniform distribution. In this case, we perform the $\mathbb{Z}_3$ rotation by multiplying the global phase $e^{\pm \frac{2\pi}{3}i}$ such that the distribution concentrate around one angle, before we measure the Kugo-Ojima two-point function.

We obtained that $u_b^a(0)$ is consistent with $-c\delta^a_b$, $c = 0.7$ in $SU(3)$ quenched simulation, $\beta = 5.5$, on $8^4, 12^4$ and $16^4$.

4.3 The gluon propagator

The gluon propagator is infrared finite. We parameterized the zero-temperature lattice data using the Stingl’s Factorised Denominator Rational Approximant (FDRA) method. The effective mass of the gluon in the analysis of $8^3 \times 16$ is found to be about 600 MeV.

The infrared finiteness is in accordance with the Kugo-Ojima color confinement mechanism. As stated in their inverse Higgs mechanism theorem, if we have no massless vector poles in all channels of the gauge field $A_a^\mu$, and if the color symmetry is not broken at all, it follows that $1 + u = 0$.

5 Discussion and conclusions

We performed the first test of the Kugo-Ojima color confinement criterion in lattice Landau gauge. We observed that the $8^4, 12^4$ and $16^4$ lattice data. The data of $\beta = 5.5$ indicates that $u = -0.7$, and those of $\beta = 6.0$ are smaller by about 10%.

In the Zwanziger’s theory, the two-point function $G_{\mu\nu}(k^2)$ can be expressed in terms of the Kugo-Ojima two-point function as (37). Zwanziger’s horizon condition in the infinite volume limit reads as $G_{\mu\mu}(0) = e = \langle E(U) \rangle$. In terms of the Kugo-Ojima parameter $c$, the left hand side can be written as $(e/4) + 3c$, and the horizon condition is that $c = e/d$. In the table below $e_1$ and $e_2$ stand for $e$ in our $16^4$ lattice simulation of the first and the second option of the gauge fields, respectively.

If the gauge fixing could be performed so that it brings the configuration into the core region or the augmented core region and if the infinite volume limit is considered somehow, then the legitimate check of the horizon condition could be done. The core gauge fixing is, however, difficult, and even impossible in general, which implies that the core gauge is literally not the gauge. A configuration of the core region of period $L$, belongs to a fundamental modular region of larger period $NL$ as well as to a fundamental modular region
of period $L$ by definition. Such a configuration is particular one in the fundamental modular region of period $L$, to which a generic configuration in the fundamental modular region of period $L$ cannot be gauge transformed even by a relaxed gauge transformation of larger period. A relaxed gauge transformation with larger period $NL$ unique up to global gauge transformation can bring the generic configuration above to the fundamental modular region of period $NL$, but at the same time breaks the periodicity of $L$ in general. Thus restriction to core region is neither an argument of 'gauge' nor an argument of trace disappearance of periodicity. It is highly dynamical hypothesis that the core region and the fundamental modular region give the same limiting correlation functions, i.e., dynamics, in the infinite volume limit. Thus we take a standpoint that the horizon condition derived from Zwanziger’s restriction to the core region is simply tested by the dynamics of the fundamental modular region, and give the direct results in the table although obviously not in the infinite volume limit.

Table 1: $\beta$ dependence of the Kugo-Ojima parameter $c$, the tensor $G_{\mu\mu}/4$, trace $e$ divided by the dimension $d$. The suffix 1 corresponds to the $u$-linear and 2 corresponds to the log $U$ definition. Data are those of $16^4$, except $\beta = 5.5$ $U$-linear data, which are those of $8^4$.

| $\beta$ | $c_1$ | $e_1/d$ | $G_{\mu\mu1}(0)/d$ | $c_2$ | $e_2/d$ | $G_{\mu\mu2}(0)/d$ |
|--------|-------|--------|---------------------|-------|--------|---------------------|
| 5.5    | 0.570(58) | 0.780(3) | 0.622(45) | 0.712(18) | 0.657(1) | 0.698(14) |
| 6.0    | 0.576(79) | 0.860(1) | 0.647(57) | 0.628(94) | 0.693(1) | 0.644(70) |

Simulation data show in general that when $\beta$ becomes larger, $e$ becomes larger, while $c$ has an opposite tendency. This fact itself does not necessarily disprove the horizon condition, but our data of $c$ which is calculated in the $A = \log U$ version already gives the zero-intersection of $G_{\mu\mu}/d - e_2/d$ in the increase of $\beta$ from 5.5 to 6.

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