Deformed Boson Condensate as a Model of Dark Matter

Mahnaz Maleki,† Hosein Mohammadzadeh,† Zahra Ebadi,† and Morteza Nattagh Najafi†

1 Department of Physics, University of Mohaghegh Ardabili, P.O. Box 179, Ardabil, Iran

We consider the condensate of \( q \)-deformed bosons as a model of dark matter. Our observations demonstrate that for all \( q \) values, the system condenses below a \( q \)-dependent critical temperature \( T_c^q \). The critical temperature interestingly tends to infinity when \( q \to 0 \), so that the \( q \)-deformed boson gas is always in the condensed phase in this limit irrespective to the temperature. We argue that this has remarkable outcomes, e.g., on the entropy of the system, and also the fraction of the particles in the ground state. Especially, by direct evaluation of the entropy of the system we reveal that it tends to zero at this limit for all temperatures, and also the fraction of particles in the ground state becomes unity. These observations prove the consistency of the model, put it in the list of appropriate candidates for the dark matter.

I. INTRODUCTION

One half of the Nobel prize in physics 2019 was awarded to Phillip James Edwin Peebles for contributions to understanding of the evolution of the universe and for theoretical discoveries in physical cosmology. These works involve the Big Bang nucleosynthesis, dark matter, and dark energy and the cosmic structure formation [1, 2]. In fact, the standard model of cosmology is indebted to his efforts.

Although such studies have opened new avenues in the physical cosmology, we are suffering yet a big lack of knowledge in this area. The standard model of cosmology suggests that approximately 4% of the total energy content of the universe is made up of ordinary baryonic matter, and the remaining 96% is composed of an unknown form of matter or energy, that is, 22% of this content is an unknown matter that it is a non-baryonic matter called dark matter, and 74% of the remaining content is composed of unknown energy called dark energy [3–5]. One of the important issues in the standard model of cosmology is about the nature of dark matter.

It is known that to explain the velocity of the galaxies in a cosmic cluster, a missing mass is required other than the luminous matter. Also, the cosmology nucleosynthesis investigation and observed anisotropies in Cosmic Microwave Background (CMB) confirms the existence of an unknown matter in galaxy and cosmic clusters formation and consequently affects the motion of galaxies [6]. By studying the observational data about the galaxies rotation profile and gravitational lensing, it is concluded that an important part of the galaxies disk is made up of dark matter [6–9]. For explaining these cosmological observations, various models for the dark part of the universe were proposed. One of these models is called (ΛCDM) which is based on the assumption that the particles have no charge, and are cold and long lived [3]. Also, the astrophysical observations demonstrate that the dark matter is composed of Weakly Interacting Massive Particles (WIMP). In fact, the theoretical considerations imply that the particles of the dark matter should be discovered in beyond the baryonic matter, such as supersymmetric particles or extra high energetic particles. Although the direct detection of dark matter seems not to be possible, its detection indirectly via annihilation (to standard matter in the sun, or in the experiments in the large Hadron collider (LHC)), and upcoming electron-positron linear collider is possible [10].

Another popular model for the dark matter is Bose-Einstein condensate (BEC). The thought that dark matter is in the form of BEC was proposed initially in [12–13], and restudied in seminal works [14–20]. At very low temperature, all particles of an ideal Bose gas condense into the same quantum ground state, making a BEC. The BEC transition takes place at a temperature \( T_c \) under which a considerable fraction of particles condense into the ground state [21]. Different properties of the BEC model of dark matter have been investigated recently in [21–22]. Using the Gross-Pitaevskii equation, some characteristics of dark matter such as density, velocity of rotation and mass profile is considered [28–31]. Of course, BEC has been utilized in some other aspects of gravitation and cosmology [32–33].

In the BEC model of dark matter, boson particles are considered as the main component of dark matter and the Bose-Einstein condensation occurs at low temperature. Nowadays, the average temperature of the universe is low and about 2.7 k. If Big Bang model is a valid model for the creation of the universe, it is expected that in the far past epochs, the temperature of the universe should be more than the current situation. Therefore, the models describing dark matter based on BEC, lose their reliability. In fact, the BEC model of dark matter can not be a valid model in all epochs. Using deformed statistics, we will introduce a generalized statistics condensate as a good model of dark matter for all epochs.

There are different generalized statistics, each of which has different origins. Infinite statistic, \( q \)-bosons and \( q \)-fermion, \( \mu \)-bosons, fractional exclusion statistics and non-extensive statistics are examples of such statistics [34–39]. Recently, it has been shown that condensate of infinite statistic can be a good alternative to the particles
of dark matter\cite{40}. Also, condensate of $\mu$-Bose gas as a model of dark matter has been proposed \cite{41} and more recently the galaxy rotation curves has been considered based on the $\mu$-deformation approach \cite{42}. Also, the effective dark matter theory has been studied in electron-positron annihilation in a typical supernova explosion using deformed statistics \cite{43}.

The rest of the paper is organized as follows. In section \[II\] we review the $q$-deformed algebra. We derive the internal energy and particle number of $q$-deformed statistics. In section \[III\] we obtain the critical condensation temperature of $q$-bosons. Also, we consider the $q$-boson condensate as an effective model for dark matter and show that for small values of deformation parameter, it can be a valid model for all epochs due to the very high critical condensation temperature in section \[IV\]. Also, we study the equation of state and entropy of $q$-boson condensate and show that the results are compatible with properties of dark matter. Finally, we conclude the paper in section \[V\].

II. $q$-DEFORMED STATISTICS

In this section we briefly review the $q$-deformed statistics concentrating mainly on the average occupation number. The symmetric $q$-oscillator algebra is defined in terms of the creation operator ($a^\dagger$), annihilation operator ($a$) and the $q$-number operator ($N = a^\dagger a$) as follows\cite{44-46}

\[
a a^\dagger - kqa^\dagger a = q^{-N}, \quad [a, a]_k = [a^\dagger, a^\dagger]_k = 0, \quad (1)
\]

\[
[N, a^\dagger] = a^\dagger, \quad [N, a] = -a, \quad (2)
\]

\[
a^\dagger a = [N], \quad a a^\dagger = [1 + kN], \quad (3)
\]

where $[a, b]_k = ab - kba$ and $k = 1(k = -1)$ is for $q$-bosons ($q$-fermions). Working in the $q$-based numbers facilitates much the calculations in this algebra. The numbers in base of $q$ are written as

\[
[x] = \frac{q^x - q^{-x}}{q - q^{-1}}. \quad (4)
\]

Also, the Jackson derivative (JD) is of great help in the following calculations for $q$-algebra, defined as follows

\[
D_x^{(q)} f(x) = \frac{f(qx) - f(q^{-1}x)}{(q - q^{-1})x}. \quad (5)
\]

Now, we consider the following Hamiltonian of non-interacting $q$-deformed oscillators

\[
H = \sum_i (\epsilon_i - \mu) N_i, \quad (6)
\]

where, the index $i$ is the state label, $\mu$ is the chemical potential and $\epsilon_i$ is the kinetic energy in the state $i$ with the number operator $N_i$. We can define $q$-basic mean occupation number by

\[
[n_i] = \frac{Tr(e^{-\beta H} a_i^\dagger a_i)}{Tr(e^{-\beta H})}, \quad (7)
\]

where, $\beta = \frac{1}{k_B T}$ and $k_B$ is the Boltzmann constant. It is straightforward to obtain the mean value of the occupation number as follows \cite{35-36}

\[
n_i = n(\epsilon) = \frac{1}{q - q^{-1}} \ln \left( \frac{z^{-1}e^{\beta \epsilon_i} - q^{-k}}{z^{-1}e^{\beta \epsilon_i} - q^k} \right), \quad (8)
\]

where, $z = e^{\beta \mu}$ is the fugacity. Therefore, the internal energy and particle number of an ideal gas with $q$-deformed statistic in a $d$ dimensional box are given by \cite{36}

\[
U = \int_0^\infty e^\epsilon n(\epsilon) \Omega(\epsilon) d\epsilon
\]

\[
\frac{d}{2} V \frac{2\pi m k_B T}{h^2} (q - q^{-1}) \frac{2\pi m k_B T}{h^2} h_{d/2+2}(z, k, q), \quad (9)
\]

\[
N = \int_0^\infty n(\epsilon) \Omega(\epsilon) d\epsilon
\]

\[
\frac{d}{2} V \frac{2\pi m k_B T}{h^2} d^{d/2} H_{d/2+1}(z, k, q), \quad (10)
\]

where,

\[
H_n(a z, k, q) = Li_n(a z q^k) - Li_n(a z q^{-k}), \quad (11)
\]

and $Li_n(x)$ denotes the polylogarithm function. We note that the density of the single particle state is

\[
\Omega(\epsilon) = \frac{V}{\Gamma(\frac{d}{2}) \frac{2\pi m}{h^d} c^{d/2-1}}, \quad (12)
\]

where $V$ is the volume of the system, $m$ is the mass of particle, $\Gamma(n)$ is the well known gamma function and $h$ is the planck constant. Also, we suppose that the dispersion relation is $\epsilon = p^2/2m$.

III. $q$-BOSON CONDENSATION

Thermodynamic behaviours of $q$-bosons and $q$-fermion vastly has been investigated by several authors \cite{36-47,49}. Using thermodynamic geometry method, it has been shown that the intrinsic statistical interaction of an ideal gas with $q$-bosons is attractive. Also, it has been shown that the thermodynamic curvature is singular at a critical value of fugacity which is given by \cite{36}

\[
z_q = \begin{cases} 
q^2 & q < 1 \\
q^{-2} & q > 1
\end{cases}.
\]

It is well-known that the thermodynamic curve of ordinary boson gas is singular at $z = 1$, where the Bose-Einstein condensation (BEC) occurs. Similar to BEC, the critical fugacity could be related to the condensation of $q$-deformed boson gas. We restrict ourselves to the
range that the deformation parameter belongs to $0 \leq q \leq 1$. In fact, in the limit of $q \to 1$, the problem reduces to the ordinary bosons and the small values of deformation parameter correspond to the more deformed cases. Using the particle number find the conditions under which the $q$-boson system condensates. Interestingly we found that for all considered $q$ values, the condensation occur under a critical temperature ($T_c^q$) satisfying the relation

$$n = \frac{N}{V} = \frac{1}{(q - q^{-1})} \left(\frac{2\pi m_b k_B T_c^q}{\hbar^2}\right)^{d/2} H_{d/2+1}(q^2, 1, q)$$

from which we obtain

$$k_B T_c^q = \frac{2\pi \hbar^2}{m_q} \left(\frac{n(q - q^{-1})}{H_{d/2+1}(q^2, 1, q)}\right)^{2/d},$$

(15)

where, $m_q$ refers to the mass of $q$-bosons. Also, for an ideal boson gas, the critical temperature of Bose-Einstein condensation is evaluated as follows [50]

$$k_B T_c^b = \frac{2\pi \hbar^2}{m_b} \left(\frac{n}{\zeta(d/2)}\right)^{2/d}.$$

(16)

where, $m_b$ is the mass of bosons and $\zeta(x)$ is the Riemann zeta function. Comparing the ordinary BEC and $q$-deformed condensates, we deduce some interesting results. We evaluate the ratio of critical condensation temperature of $q$-bosons to condensation temperature of ordinary bosons as follows

$$\frac{T_c^q}{T_c^b} = \frac{m_b}{m_q} \left(\frac{q - q^{-1}\zeta(d/2)}{H_{d/2+1}(q^2, 1, q)}\right)^{2/d}.$$

(17)

Eq. (17) shows that the critical temperature depends on the value of deformation parameter. We suppose that $m_q = m_b$ and $d = 3$ and depict the critical condensation temperature as a function of deformation parameter in Fig. 1.

It is obvious that the critical temperature of $q$-bosons goes to infinity for enough small values of deformation parameter, while it tends to the critical temperature of ordinary bosons ($T_c^b$) at the limit $q \to 1$. We can argue that $q$-deformed boson gas with enough small deformation parameter will be in condensate phase at any finite temperature because of the large value of critical temperature.

Another view point about the $q$-bosons seems sound and interesting. Eq. (16) indicates that the heavier the bosons are, the lower the critical condensation temperature is. In fact, by increasing the mass of bosons, the condensation temperature tends to the zero temperature. Therefore, possibility of condensation of heavy bosons is less than the light ones. However, using Eq. (17) we can argue that at the same value of critical temperature for ordinary and $q$-boson condensation, more heavier deformed bosons in comparison with ordinary bosons could exist in condensed phase, specially for small value of deformation parameter.

Up to the authors’ knowledge, this is the first observation of such a phenomenon, i.e. the infinite condensation critical temperature. As we will see in the following sections, this opens new possibilities for explaining the less-known features of the dark matter.

\section*{IV. $q$-DEFORMED CONDENSATE AND DARK MATTER}

Recently, it has been proposed that BEC could be an appropriate candidate for describing the dark matter. Some useful information about the velocity profile of different galaxies is extracted by BEC model [28–30]. Also, condensate of infinite statistics as a candidate for dark matter has been investigated [40]. An important question exists about the BEC models for dark matter. Are such models valid for all epochs? Once such models are built, the important question arises concerning the time evolution of the condensates. By this we don’t mean the time dependent evaporation of BEC, but the time evolution of the equation of state of the condensate. This evolution contains, but is not restricted only to the thermodynamic quantities, like the temperature and the volume etc., but involves the time evolution of the intrinsic parameters like $q$. As outlined in Ref. [28–51], the deformation parameters can be related to the interaction between ordinary Bosons, so that by the expansion of the system one expects that this quantity is variable. At the moment, there is no equation describing the time evolution of $q$, but by some general considerations, some informa-
tion can be obtained, which is the aim of the present paper.

Although, the current effective temperature of universe is low, we expect a high temperature for early universe. Therefore, condensate of boson gas as a candidate for dark matter fails down for past epochs, more specially for early universe. In other words, since the temperature of the early universe was very high, BEC could not be stable for that stage, and the BEC theory of Ref. [28–30] fails for hot universe. Also if the dark matter has minimal interaction with luminous matter, which means that it is nearly conservative the problem of entropy of dark matter remains unanswered: what happens to the lost entropy, when the dark matter condensates at the time that the temperature of the universe was suitable to BEC?

We considered the condensation of a $q$-deformed boson gas in previous section. We showed that the critical condensation temperature becomes very high for a deformed bosons with small deformation parameter. In fact, the critical temperature goes to infinity at the limit of $q \to 0$. Therefore, a deformed gas with small deformation parameter will be in condensate phase and can be a suitable model for dark matter in all time durations.

Recently, it has been shown that if dark matter is assumed to consist of an ideal gas of ordinary bosons of mass $m_b$, then for $m_b \leq 1\text{eV}$, the critical condensation temperature below which they will form a Bose-Einstein condensate exceeds the temperature of the universe at all times [20]. According to the arguments of last paragraph of pervious section, there are no restriction on the mass of deformed bosons to form a condensate at all times for enough small values of deformation parameter. Our model, in addition to relaxing this condition, has other strengths and benefits that are explained in the following.

A. Equation of state in condensed phase

Experimental results and theoretical arguments suggest that the interaction between particles of dark matter should be very weak, so we expect that the equation of state of it is just the same as the ideal gas. Therefore, in this subsection we show that our model satisfies this, i.e. the equation of state of the condensate of $q$-bosons is of the same for as the ideal gas, and then in the next subsection we turn to the key observations and explanations. By a simple evaluation, we can show that

$$PV = \frac{2}{3}U,$$

and using Eq.(9), we work out that in condensed phase the equation of states reads

$$PV = \gamma Nk_BT,$$

where,

$$\gamma = \frac{H_2(q^2,1,q)}{H_2(q^2,1,q)}.$$  

We plot the coefficient $\gamma$ as a function of deformation parameter in Fig. [2]. It is obvious that that at the limit of small values of deformation parameter ($q \to 0$), the coefficient $\gamma \to 1$. Therefore, the equation of state of an ideal gas with $q$-deformed bosons becomes the equation of state of an ideal classical gas without any interaction, $PV = Nk_BT$ for small $q$s, justifying that the condensate of $q$-bosons is compatible with non-interacting dark matter.

B. Entropy of $q$-bosons condensate

We can assign some problems about the entropy of ordinary Boltzmann and Bose-Einstein statistics, if constituents of dark matter obey such statistics. For an ideal classical gas in a box with volume $V$, we know that the partition function is

$$Z_N = \frac{1}{N!} \left(\frac{V}{\lambda^3}\right)^N,$$

where $\lambda = h/\sqrt{2\pi mk_BT}$ is the thermal wavelength[50]. Also, we calculate the entropy of the system using the free energy which is given by $A = -k_BT\ln Z_N$, and obtain that

$$S = -\frac{\partial A}{\partial T}_{V,N} = Nk_B \left(\ln\left(\frac{V}{N\lambda^3}\right) + \frac{5}{2}\right).$$

It is obvious that by decreasing the temperature, the thermal wavelength grows up. Clearly at a certain sufficiently low temperature $V \sim \lambda^3$ while $N \gg 1$ and
consequently the entropy of the system will become negative\[^5\]. In fact, the origin of this problem is related to the Gibbs $1/N!$ factor due to the distinguishability of the classical particles. For an ideal boson gas in condensate regime, evaluation of the entropy is straightforward and is given by

$$S^b = k_B \frac{5}{2} \frac{V}{\lambda^3} \xi \left( \frac{5}{2} \right), \quad T \leq T_c^b. \tag{23}$$

for which the negative entropy problem does not further exist. However, the entropy of the system depends on the temperature. Decreasing the temperature causes growing up the thermal wavelength and consequently the entropy of the system is reduced. BEC model of dark matter in a typical galaxy, produces another problem: the entropy of the system gradually decreases from the early universe to current epoch. Compensation of this reduction by the environment around (luminous matter) needs some interactions between dark matter and other constituents of universe, in contrast with the hypothesis that the dark matter should be non interacting.

We show that the entropy of the condensate of a $q$-deformed boson gas, has none of the problems mentioned above. Especially we argue that the entropy is not negative for deformed bosons for any temperature. To find that out, by introducing the Helmholtz free energy $A = \mu N - PV$, we represent the entropy as follows

$$S = \frac{U - A}{T} = \frac{U + PV - \mu N}{T}, \tag{24}$$

where $\mu$ is the chemical potential

$$\mu = k_B T \ln z. \tag{25}$$

In the condensate phase, the fugacity is shown to be $z_q = q^2$, leading $\mu$ to have the form:

$$\mu = k_B T \ln q^2, \quad T \leq T_c^q. \tag{26}$$

Using Eqs. (10), (18) and (26) and incorporating them into Eq. (24), one obtains the entropy of $q$-deformed boson gas in the condensate phase as follows

$$S^q = \frac{k_B}{q - q^{-1}} \frac{V}{\lambda^3} \left( \frac{5}{2} \right) H_{7/2}(q^2, 1, q) - H_{5/2}(q^2, 1, q) \ln(q^2). \tag{27}$$

It is clear in Fig. (3) that the entropy of deformed condensate is equal to the entropy of boson condensate at the limit of $q = 1$. However, it is interesting that the entropy of condensate of $q$-deformed bosons is vanished at the limit of $q \to 0$. Of course, the zero entropy of $q$-boson condensate occurs at any finite temperature. We argued that in the limit of small values of deformation parameter, the critical condensation temperature goes to infinity and the system lives in condensate regime at any finite temperature and the entropy of the system is zero. Therefore, variation of temperature in different time duration, does not affect the entropy of the system. Therefore, $q$-boson condensate as a model of dark matter does not need to interact with surrounding or remnant of the universe to compensate the reduction of entropy.

Another viewpoint to the entropy reduction problems seems interesting. We could interpret the deformation parameter as a dynamical parameter which varies with temperature of each epoch, while the entropy of the system remains fixed. Fig. (4) shows the general behavior of deformation parameter with temperature of each epoch. It is obvious that at early universe which we expect very high temperature, the deformation parameter is small. As time goes on and the temperature decreases, the deformation parameter grows up and tends to unity and the statistics of particles turns to the ordinary bosons while the entropy of the system is fixed.

Here we propose two main scenarios. The first scenario is that $q$ is fixed for all times (during the expansion of the universe), and it is small enough that for all stages the BEC phase is stable, so that its entropy is fixed and no other scenarios are needed to explain the entropy reduction. The second scenario, is the time-dependent $q$, so that as the universe age goes to zero (the early hot universe), $q \to 0$, where the critical temperature diverges. The other point of view is based on a presumable duality between the free $q$-boson system and the interacting ordinary bosons, in which $q$ controls the interactions. If true, then one expects that the interaction between ordinary bosons (corresponding to the $q$ parameter in the dual $q$-boson system) depends on the universe age, and consequently is time dependent. In this case the latter scenario has the benefit of giving us this effective time dependent $q$, which has not been analyzed before. This
also predicts that the BEC phase of $q$ bosons is conservative, and no complementary theory is needed to explain the entropy.

In the following we find the temperature dependence of the ground state occupation number of $q$ bosons, and show that for sufficiently small $q$, all particles settle down in the ground state irrespective to the system temperature. The number of ground state particles is given by:

$$N_0(T) = N - N_e(T)$$  \hspace{1cm} (28)

where $N_0$ and $N_e$ are the number of particles in the ground and excited states respectively. Also, the number of particles in the excited states in the condensate phase is

$$N_e(T) = \frac{V}{(q - q^{-1})} \left( \frac{2\pi m k_B T}{\hbar^2} \right)^{3/2} H_{5/2}(q^2, 1, q), \hspace{1cm} T \leq T_q^c,$$  \hspace{1cm} (29)

while at $T = T_q^c$, all particles are in excited states, so that:

$$N = N_e(T = T_q^c) = \frac{V}{(q - q^{-1})} \left( \frac{2\pi m k_B T_c^q}{\hbar^2} \right)^{3/2} H_{5/2}(q^2, k, q).$$  \hspace{1cm} (30)

Therefore, it is straightforward to obtain that

$$\frac{N_0}{N} = 1 - \frac{N_e}{N} = 1 - \left( \frac{T}{T_q^c} \right)^{3/2}.$$  \hspace{1cm} (31)

But we showed that the critical condensation temperature goes to infinity for small values of $q$. Therefore, $T/T_q^c$ vanishes at any finite temperature and consequently $N_0/N$ goes to unity, meaning that all $q$-bosons with sufficiently small $q$, live in condensate phase at any finite temperature and occupy the ground state of the system. In such circumstances, the system has only one microstate and the entropy of the system will be zero.

V. CONCLUSION

We considered the condensate of an ideal $q$-boson gas as a model of dark matter. We argue that because of some reasons, $q$-boson condensate with enough small value of deformation parameter is compatible with the properties of dark matter.

It has been shown that the BEC as a model of dark matter gives some useful information about the density profile, rotational velocity and mass profile. In fact a good agreement between theoretically BEC model predicted rotation curves and the observational data. However, we expect that the temperature of universe to be high in past epochs. Thus, BEC model for dark matter is not valid in all time durations. $q$-boson condensate could be a valid and compatible model for all epochs. We obtained the critical condensation temperature of $q$-bosons and found that for small values of deformation parameter, the cortical temperature goes to infinity. Therefore, the system lives in condensate phase at any finite temperature.

The equation of stat of an ideal $q$-boson gas at condensate phase returns to the equation of state on an ideal classical gas for small value of deformation parameter. Considering $q$-boson condensate as a model of dark matter is in consistency with the non interacting nature of dark matter.

We evaluated the entropy of $q$-bosons at condensate phase and showed that it is vanished at in the small value limit of deformation parameter. The entropy of ordinary bosons condensate depends on the temperature and by decreasing the temperature, the entropy will be decreased. In order to compensate of this reduction of entropy, the system have to interact with surroundings or remnant of the galaxy. This is in contrast with the non interacting behaviour of dark matter. Zero entropy of $q$-boson condensate for small values of $q$ at any finite temperature indicates that the proposed model is consistent with the non interacting behaviour of dark matter.

We argued that all $q$-bosons with small value of deformation parameter, occupy the ground state at any finite temperature. In fact, the particles never has been excited and live in ground state for all time duration. Maybe, non detectability of constituents of dark matter is related to the non excitability of such particles.
