COMMUTING SQUARES AND PLANAR SUBALGEBRAS

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ABSTRACT. We show a close relationship between non-degenerate smooth commuting squares of $II_1$-factors with all inclusions of finite index and inclusions of subfactor planar algebras by showing that each leads to a construction of the other. One direction of this uses the Guionnet-Jones-Shlyakhtenko construction.

1. Introduction

The goal of this paper is to establish a relationship between $*$-planar subalgebras of a planar algebra and commuting squares. While results of this nature have been known for a while in the language of standard $\lambda$-lattices - see [Ppa1995] and [Ppa2002] - we employ the Guionnet-Jones-Shlyakhtenko (GJS) construction - see [GnnJnsShl2010] and [JnsShlWlk2010] - to simplify the proofs considerably. Needless to say, the proofs are very pictorial.

We begin with a review of the GJS construction in the version described in [KdySnd2009] in §2. In §3 we start with a non-degenerate commuting square of $II_1$-factors

$$
L \subseteq M \\
N \subseteq K
$$

with all inclusions extremal of finite index and that is smooth in the sense of [Ppa1994] and then show that the planar algebra of $N \subseteq K$ is a $*$-planar subalgebra of that of $L \subseteq M$. In §4, we establish two ‘algebra to analysis’ results. The first deals with going from an inclusion of finite pre-von Neumann algebras to the associated inclusion of their von Neumann algebra completions. The second deals with a sufficiently nice square of finite pre-von Neumann algebras and their corresponding completions, which are shown to form a commuting square. In §5, beginning with a $*$-planar subalgebra $Q$ of a subfactor planar algebra $P$, we appeal to the GJS construction to obtain a smooth non-degenerate commuting square, as above, such that the planar algebras of $N \subseteq K$ and $L \subseteq M$ are identified with $Q$ and $P$ respectively.

2. Basics of GJS-construction

In this section we recall the GJS construction from [KdySnd2009]. Throughout this section, $P$ will be a subfactor planar algebra of modulus $\delta > 1$. The GJS construction associates to $P$, a basic construction tower $M_0 = N \subseteq M = M_1 \subseteq \ldots$
of II$_1$-factors with all inclusions extremal of finite index $\delta^2$ and such that $P$ is the planar algebra of $N \subseteq M$. For all preliminary material on planar algebras we refer the reader to [Jns1999] and for any unspecified notation to [KdySnd2009].

Let $P$ be a subfactor planar algebra of modulus $\delta > 1$. Therefore, we have finite-dimensional $C^*$-algebras $P_n$ for $n \in \text{Col}$ with appropriate inclusions between them. For $k \geq 0$, let $F_k(P)$ be the vector space direct sum $\bigoplus_{n=k}^\infty P_n$ (where $0 = 0_+$, here and in the sequel). A typical element $a \in F_k(P)$ looks like $a = \sum_{n=k}^\infty a_n = (a_k, a_{k+1}, \cdots)$, here of course only finitely many $a_n$’s are non-zero. According to [KdySnd2009], each $F_k(P)$ is equipped with a filtered, associative, unital $\ast$-algebra structure with normalised trace $t_k$ and there are trace preserving filtered $\ast$-algebra inclusions $F_0(P) \subseteq F_1(P) \subseteq F_2(P) \subseteq \cdots$, as well as conditional expectation-like maps $F_k(P) \rightarrow F_{k-1}(P)$. Since we will need these structures explicitly in this paper, we briefly recall them. First, note that an arbitrary element $a \in P_m \subseteq F_k(P)$ is depicted as in Figure 1.

**Figure 1.** Arbitrary element $a \in P_m \subseteq F_k(P)$

**Multiplication:** Consider two elements $a = a_m \in P_m \subseteq F_k(P)$ and $b = b_n \in P_n \subseteq F_k(P)$ for $m, n \geq k$. Then the multiplication in $F_k(P)$, denoted by $(a \# b)$, is defined to be $\sum_{t=|n-m|+k}^{n+m-k} (a \# b)_t$, where $(a \# b)_t$ is defined as in Figure 2. As usual, extend the map bilinearly to the whole of $F_k(P) \times F_k(P)$. This multiplication map $\#$ makes each vector space $F_k(P)$ an associative, unital and filtered algebra.

**Involution:** Define the involution map $\dagger : F_k(P) \rightarrow F_k(P)$ as follows. For $a = a_m \in P_m \subseteq F_k(P)$ define $a^\dagger$ as in Figure 3. In other words, $a^\dagger = Z_{R_k}(a^*)$. Here, of course ‘*’ denotes the usual involution on $P_n$’s. This involution map makes each $F_k(P)$ a $\ast$-algebra.

**Figure 2.** Definition of the $P_t$ component of $a \# b$
Trace: For \( a = (a_k, a_{k+1}, \cdots) \in F_k(P) \) we define a linear functional \( t_k \) on \( F_k(P) \) by \( t_k(a) = \tau(a_k) \) which defines a normalized trace on \( F_k(P) \) that makes \( \langle a, b \rangle := t_k(b^* \# a) \) an inner-product on \( F_k(P) \). Note that the trace of \( a \) is the trace of its \( P_k \)-component.

Inclusion map: \( F_{k-1}(P) \) is included in \( F_k(P) \) in such a way that the restriction takes \( P_{m-1} \subseteq F_{k-1}(P) \) to \( P_m \subseteq F_k(P) \) by taking \( a \in P_{m-1} \) to the element in Figure 4.

Conditional expectation-like map: One defines a map \( E_{k-1} : F_k(P) \mapsto F_{k-1}(P) \) (for \( k \geq 1 \)) in such a way that for any arbitrary \( a \in P_m \subseteq F_k(P) \) the element \( \delta E_{k-1}(a) \) of \( P_{m-1} \subseteq F_{k-1}(P) \) is given by the tangle in Figure 5. Then the map \( E_k \) is a \( * \)- and trace-preserving \( F_k(P) - F_k(P) \) bimodule retraction for the inclusion map of \( F_k(P) \) into \( F_{k+1}(P) \).

We need a little bit more terminology.

**Definition 1** ([KdySnd2009]). A finite pre-von Neumann algebra is a complex unital \( * \)-algebra \( A \) that comes equipped with a normalized trace \( t \) such that:
the sesquilinear form defined by \( \langle a, b \rangle = t(b^*a) \) defines an inner product on \( A \). Denote the inner product by \( \langle \ldots \rangle_A \).

- for each \( a \in A \), the left multiplication map \( \lambda_A(a) : A \to A \) is bounded for the trace-induced norm of \( A \).

Examples are \( F_k(P) \) with their natural traces \( t_k \) for \( k \geq 0 \).

**Notation 2.** Let \( \mathcal{H}_A \) be the Hilbert space completion of \( A \) for the associated norm. As usual there exists a natural one-one and linear map \( \Gamma : A \to \mathcal{H}_A \) such that \( \langle \Gamma(a), \Gamma(b) \rangle_{\mathcal{H}_A} = \langle a, b \rangle_A \) for all \( a, b \in A \) and \( \Gamma(A) \) is dense in \( \mathcal{H}_A \). Let us denote by \( \mathcal{H}_k(P) \) the Hilbert space completion of \( F_k(P) \). The Hilbert space \( \mathcal{H}_k(P) \) is nothing but the orthogonal direct sum \( \oplus_{n=k}^\infty P_n \).

The following two results that we quote without proof are taken from [KdySnd2009].

**Lemma 3** (Lemma 4.4 of [KdySnd2009]). Let \( A \) be a finite pre-von Neumann algebra with trace \( t_A \), and \( \mathcal{H}_A \) be the Hilbert space completion of \( A \) for the associated norm, so that the left regular representation \( \lambda_A : A \to B(\mathcal{H}_A) \) is well defined, i.e. for each \( a \in A \), \( \lambda_A(a) : A \to A \) extends to a bounded operator on \( \mathcal{H}_A \). Let \( M_A = \lambda_A(A)' \). Then,

1. The ‘vacuum vector’ \( \Omega_A \in \mathcal{H}_A \) (corresponding to \( 1 \in A \subseteq \mathcal{H}_A \)) is cyclic and separating for the von-Neumann algebra \( M_A \).
2. The trace \( t_A \) extends to faithful, normal, tracial states \( t_A \) on \( M_A \).

Let us denote by \( M_k(P) \subseteq B(\mathcal{H}_k(P)) \) the von Neumann algebra corresponding to the finite pre-von Neumann algebra \( F_k(P) \). Let \( \Omega_k \in \mathcal{H}_k(P) \) be the cyclic and separating vector for \( M_k(P) \). It is easy to see that under the appropriate identifications, \( \Omega_0 = \Omega_1 = \Omega_2 = \cdots \).

**Theorem 4** (Theorem 6.2 of [KdySnd2009]). Let \( P \) be a subfactor planar algebra of modulus \( \delta > 1 \). Then \( M_{k-1}(P) \subseteq M_k(P) \subseteq M_{k+1}(P) \) is (isomorphic to) a basic construction tower of type \( II_1 \)-factors with finite index \( \delta^2 \). Moreover, the subfactor \( M_0(P) \subseteq M_1(P) \) constructed from \( P \) is a finite index and extremal subfactor with planar algebra isomorphic to \( P \).

### 3. Commuting squares to planar subalgebras

Throughout this section, we will deal with a non-degenerate commuting square \( \mathcal{C} \) of \( II_1 \)-factors

\[
\begin{array}{ccc}
L & \subseteq & M \\
\cup & \cup & \cup \\
N & \subseteq & K
\end{array}
\]

with all inclusions extremal of finite index. Suppose that the associated basic construction towers are given by

\[
\begin{array}{ccc}
M_0 = L & \subseteq & M = M_1 & \subseteq & M_2 & \subseteq & \cdots \\
\cup & \cup & \cup & \cup & \cup & \cup & \cdots \\
K_0 = N & \subseteq & K = K_1 & \subseteq & K_2 & \subseteq & \cdots
\end{array}
\]

The following definition is from [Ppa1994].

**Definition 5.** The commuting square \( \mathcal{C} \) is said to be smooth if

\[
N' \cap K_k \subseteq L' \cap M_k
\]

for all \( k \geq 0 \).
Before we prove the main result of this section, we note a result of [Ppa1994] - see Proposition 2.3.2 - which implies that if $\mathcal{C}$ is smooth, then each square in the basic construction tower is also a smooth non-degenerate commuting square. We also recall without proof two elementary facts about general commuting squares (i.e., not necessarily non-degenerate and not necessarily factors).

**Lemma 6.** Let $A_{10} \subseteq A_{11}$ be a pair of finite von Neumann algebras with a faithful normal trace $\text{tr}$ on $A_{11}$ and let $S$ be a self-adjoint subset of $A_{10}$. Then

$$A_{10} \subseteq A_{11} \subseteq S \cap A_{10} \subseteq S \cap A_{11}$$

is a commuting square. \hfill $\blacksquare$

**Lemma 7.** Consider a tower of quadruples of finite von Neumann algebras with a faithful normal trace $\text{tr}$ on $A_{12}$

$$A_{10} \subseteq A_{11} \subseteq A_{12} \subseteq A_{13}$$

such that the following two squares are commuting squares with respect to $\text{tr}$

$$A_{10} \subseteq A_{12} \quad A_{11} \subseteq A_{12}$$

and

$$A_{00} \subseteq A_{02} \quad A_{01} \subseteq A_{02}$$

Then,

$$A_{10} \subseteq A_{11} \subseteq A_{00} \subseteq A_{01}$$

is also a commuting square. \hfill $\blacksquare$

**Theorem 8.** If $\mathcal{C}$ is a smooth non-degenerate commuting square, then the planar algebra of $(N \subseteq K)$ is a planar subalgebra of the planar algebra of $(L \subseteq M)$.

**Proof.** First, denote by $P = P(N \subseteq K)$ (respectively, $Q = P(L \subseteq M)$) the planar algebra of $N \subseteq K$ (respectively, $L \subseteq M$). Figure shows the standard invariants of $N \subseteq K$ and $L \subseteq M$. Each arrow in this figure represents an inclusion map. The dotted arrows on the top level are well defined maps by the assumption of smoothness on $\mathcal{C}$ while those on the bottom level are so because of Proposition 2.3.2 of [Ppa1994].

In particular, for $n \geq 0$, the spaces $P_n = N' \cap K_n$ of the planar algebra $P$ are subspaces of the spaces $Q_n = L' \cap M_n$ of the planar algebra $Q$ - as observed from the dotted arrows on the top level. Also $P_0 = K' \cap K_0$ is the whole of $Q_0 = M' \cap M_0$ - both being $\mathbb{C}$. Hence to see that $P$ is a planar subalgebra of $Q$, it suffices to see that for any tangle $T = T_{k_0, \ldots, k_b}^{k_0, \ldots, k_b}$ in a class of ‘generating tangles’, and inputs $x_i \in P_{k_i}$, $Z_T^Q(x_1 \otimes \cdots \otimes x_b) \in P_{k_0}$.

We will use the following collection of generating tangles - see Theorem 3.3 of [KdySnd2004] but with notation for the tangles as in [KdySnd2009] - $1^{0+}, 1^{0-}, E^{n+2}$ for $n \geq 0$, $EL(1)^{n+1}_n$ for $n \geq 0$, $I^{n+1}_n$ for $n \in \text{Col}$, $ER^{n+1}_n$ for $n \in \text{Col}$ and $M^n_{n,m}$ for $n \in \text{Col}$. In order to be self-contained we illustrate these tangles in Figures 6 and 7. Note that the tangles $1^{0+}$ and $1^{0-}$ are identical except for the shading which is omitted.
We begin by observing that since $P_n$ are unital subalgebras of $Q_n$ and form an increasing chain, if $T$ is one of the tangles $1^{0\pm}, I_{n+1}^{n+1}, EL_{n+1}^{n+1}, ER_{n+1}^{n}, M_{n,n}^n$ for $n \geq 0$ then the output for $T$ lies
in \( P \) whenever the inputs do. We will now verify that this holds for the remaining three generating tangles \( E_{n+2} \) for \( n \geq 0 \), \( EL(1)_{n+1}^{n+1} \) for \( n \geq 0 \), \( ER_{n+1} \) for \( n \in \text{Col} \).

Case I: \( T = E_{n+2} \) for \( n \geq 0 \): What needs to be seen is that \( Z_T^Q(1) \) lies in \( P_{n+2} \). However, \( Z_T^Q(1) \) is a scalar multiple of the Jones projection for the inclusion \( M_n \subseteq M_{n+1} \) which also is the Jones projection for the inclusion \( K_n \subseteq K_{n+1} \) (since these form a non-degenerate commuting square) and hence lies in \( P_{n+2} = N' \cap K_{n+2} \).

Case II: \( T = EL(1)_{n+1}^{n+1} \) for \( n \geq 0 \): What needs to be seen is that for \( x \in P_{n+1} = N' \cap K_{n+1} \), \( Z_T^Q(x) \) also lies in \( P_{n+1} \). A moment’s thought shows that this will follow if the ‘side faces’ of the cubes in Figure 8 are commuting squares. Thus we need to see that for any \( n \geq 0 \), the square

\[
\begin{array}{c}
N' \cap K_{n+1} \\
\cup \cup \\
K' \cap K_{n+1}
\end{array}
\subseteq
\begin{array}{c}
L' \cap M_{n+1} \\
\cup \cup \\
M' \cap M_{n+1}
\end{array}
\]

is a commuting square.

In other words, we need to show that for any \( x \in M' \cap M_{n+1} \), we must have \( E_{N' \cap K_{n+1}}^{L' \cap M_{n+1}}(x) \in K' \cap K_{n+1} \). First observe that the following quadruple, say \( D \), is a commuting square:

\[
\begin{array}{c}
K' \cap M_{n+1} \\
\cup \cup \\
K' \cap K_{n+1}
\end{array}
\subseteq
\begin{array}{c}
N' \cap M_{n+1} \\
\cup \cup \\
N' \cap K_{n+1}
\end{array}
\]

Indeed, this follows once we apply Lemma 6 and Lemma 7 to the following tower of quadruples:

\[
\begin{array}{c}
K' \cap M_{n+1} \\
\cup \cup \\
K' \cap K_{n+1}
\end{array}
\subseteq
\begin{array}{c}
N' \cap M_{n+1} \\
\cup \cup \\
N' \cap K_{n+1}
\end{array}
\subseteq
\begin{array}{c}
M_{n+1} \\
\cup \cup \\
K_{n+1}
\end{array}
\]

Now since \( x \in M' \cap M_{n+1} \subseteq K' \cap M_{n+1} \) and \( D \) is a commuting square it follows that \( E_{N' \cap K_{n+1}}^{L' \cap M_{n+1}}(x) \in K' \cap K_{n+1} \). This completes the proof of case II.

Case III: \( T = ER_{n+1} \) for \( n \in \text{Col} \): If \( n = 0 \), the verification is trivial so we will treat the case \( n \geq 0 \). What needs to be seen is that for \( x \in P_{n+1} = N' \cap K_{n+1} \), \( Z_T^Q(x) \) lies in \( P_n = N' \cap K_n \). A moment’s thought shows that this will follow if the ‘top faces’ of the cubes in Figure 8 are commuting squares. Thus we need to see that for any \( n \geq 0 \), the square \( O \)

\[
\begin{array}{c}
L' \cap M_{n} \\
\cup \cup \\
N' \cap K_{n}
\end{array}
\subseteq
\begin{array}{c}
L' \cap M_{n+1} \\
\cup \cup \\
N' \cap K_{n+1}
\end{array}
\]

is a commuting square. Consider \( x \in N' \cap K_{n+1} \). By Lemma 6

\[
\begin{array}{c}
M_{n} \\
\cup \cup \\
L' \cap M_{n}
\end{array}
\subseteq
\begin{array}{c}
M_{n+1} \\
\cup \cup \\
L' \cap M_{n+1}
\end{array}
\]

is a commuting square. Thus, \( E_{L' \cap M_{n}}^{L' \cap M_{n+1}}(x) = E_{M_{n+1}}^{M_{n+1}}(x) \). Since,

\[
\begin{array}{c}
M_{n} \\
\cup \cup \\
K_{n}
\end{array}
\subseteq
\begin{array}{c}
M_{n+1} \\
\cup \cup \\
K_{n+1}
\end{array}
\]

are commuting squares. Thus, \( E_{L' \cap M_{n}}^{L' \cap M_{n+1}}(x) = E_{M_{n+1}}^{M_{n+1}}(x) \). Since,
This proves that $O$ is a commuting square as desired. \hfill $\square$

4. Compatible pairs and quadruples of finite pre-von Neumann algebras

Definition 9. [KdySnd2009] A compatible pair of finite pre-von Neumann algebras is a pair $(A, t_A)$ and $(B, t_B)$ of finite pre-von Neumann algebras such that $A \subseteq B$ is a unital inclusion and $t_B|_A = t_A$. Given a such pair of compatible pre-von Neumann algebras, identify $\mathcal{H}_A$ with a subspace of $\mathcal{H}_B$ so that $\Omega_A = \Omega_B = \Omega$.

Some of the following results may be implicit in [KdySnd2009].

Theorem 10. Let $(A, t_A) \subseteq (B, t_B)$ be a compatible pair of finite pre-von Neumann algebras. Let $E_A : B \to A$ be a $*$-and trace-preserving $A \to A$ bimodule retraction for the inclusion map of $A$ into $B$. Let $\lambda_A : A \to \mathcal{B}(\mathcal{H}_A)$ and $\lambda_B : B \to \mathcal{B}(\mathcal{H}_B)$ be the left regular representations of $A$ and $B$ respectively and let $M_A = \lambda_A(A)''$ and $M_B = \lambda_B(B)''$. Then,

1. $t_B^\dagger(t_A(x)) = t_A^\dagger(x)$ for $x \in M_A$, where $t_A$ is the normal inclusion of $M_A$ into $M_B$ as in Proposition 4.6 of [KdySnd2009].
2. The map $E_A$ extends continuously to an orthogonal projection, call it $e_A$, from $\mathcal{H}_A$ onto the closed subspace $\mathcal{H}_A$.
3. $J_B e_A J_B = e_A$, where $J_B$ is the modular conjugation operator which is the unique bounded extension, to $\mathcal{H}_B$, of the involutive, conjugate-linear, isometry defined on the dense subspace $\Gamma(B) \subseteq \mathcal{H}_B$ by $\gamma(b) \mapsto \gamma(b^*)$.
4. The map $E_A$ extends to the unique trace preserving conditional expectation map $E_A : M_B \to M_A$. It is continuous for the $SOT^*$-topologies on the domain and range.
5. $E_A(x)(\Gamma(a)) = J_A(\lambda_A(a^*)e_A x^* \Omega)$ for $x \in M_B$ and $a \in A$.

Proof. (1) First recall (see the proof of Lemma 3) that for a finite pre-von Neumann algebra $B$, the faithful, normal, tracial state $t_B^\dagger$ is the restriction to the von Neumann algebra $M_B$ of the linear functional $\tilde{t}_B$ on $\mathcal{B}(\mathcal{H}_B)$ defined by $\tilde{t}_B(x) = \langle x \Omega, \Omega \rangle$ for $x \in \mathcal{B}(\mathcal{H}_B)$. Note that $\tilde{t}_B(\lambda_B(b)) = t_B(b)$ for $b \in B$. Now, for any $x \in M_A$ we have, $t_B^\dagger(t_A(x)) = \tilde{t}_B(t_A(x)) = \langle t_A(x) \Omega, \Omega \rangle = \langle J_B(x^* \Omega), \Omega \rangle = \langle J_B(x \Omega, \Omega \rangle = \langle x \Omega, \Omega \rangle = t_A^\dagger(x)$. (2) The continuous extension from $\mathcal{H}_B$ onto $\mathcal{H}_A$ of $E_A$ is defined by $e_A(\Gamma(b)) = \Gamma(E_A(b))$ for $b \in B$. We show that $e_A$ is nothing but the orthogonal projection onto the closed subspace $\mathcal{H}_A$ of $\mathcal{H}_B$. Observe that the range of the operator $e_A \in \mathcal{B}(\mathcal{H}_B)$ is precisely $\mathcal{H}_A$. Since, $E_A$ is a $A \to A$ bimodule retraction map for the inclusion of $A$ into $B$ it is clear that $e_A^2 = e_A$. Another routine calculation proves that...
Thus by \[Mk1954\] \(E\) expectation map \(\{E\}_{\lambda}^{\xi}\) and hence \(e_{A}^{*} = e_{A}\). This proves that \(e_{A}\) is the orthogonal projection onto \(\mathcal{H}_{A}\).

(3) First it should be clear that \(J_{B}|_{A} = J_{A}\). Now for any \(b \in B\) we have \(J_{B}e_{A}J_{B}(\Gamma(b)) = J_{B}e_{A}(\Gamma(b^{*})) = J_{B}\Gamma(\tilde{E}_{A}(b^{*})) = \Gamma(E_{A}(b)) = e_{A}(\Gamma(b))\). Since \(J_{B}e_{A}J_{B}\) and \(e_{A}\) agree on a dense set we get the desired equality.

(4) To obtain a formula for a condition expectation from \(M_{B}\) onto \(M_{A}\) we follow the standard trick. As a first step we prove that for any \(x \in M_{B}, e_{A}xe_{A} \in \mathcal{M}_{e_{A}}\).

The reader should observe that \(e_{A} \in \mathcal{M}_{e_{A}}\). Further, \(J_{A}M_{A}J_{A} = (M_{A})'\) (see the proof of Lemma 4.4 (item (2)) of \[KdySn2009\]). Therefore, it is sufficient to prove that \(e_{A}xe_{A} \in (J_{B}M_{A}J_{B})'e_{A}\). Indeed, it is routine to check that \(e_{A}xe_{A} \in (e_{A}J_{B}M_{A}J_{B}e_{A})'\) and hence the conclusion follows. Suppose, \(e_{A}xe_{A} = \tilde{E}(x)e_{A}\) for some \(\tilde{E}(x) \in M_{A}\). Then, \(\tilde{E}(x) \in M_{A}\) is uniquely determined since \(\Omega\) is separating for \(M_{A}\) by Lemma 3. Define, \(E_{A} : M_{B} \rightarrow M_{A}\) by \(E_{A}(x) = \tilde{E}(x)\) for \(x \in M_{B}\). Next we show that for \(y \in M_{A}\), \(t_{A}^{*}(\tilde{E}(x)y) = t_{B}^{*}(x\iota_{A}(y))\). This follows from the following array of equations.

\[
\begin{align*}
t_{A}^{*}(\tilde{E}(x)y) &= t_{A}^{*}(\tilde{E}(x)y) \\
&= \langle \tilde{E}(x)y, \Omega \rangle_{A} \\
&= \langle \tilde{E}(x)e_{A}y, \Omega \rangle_{A} \\
&= \langle e_{A}xe_{A}y, \Omega \rangle_{B} \quad \text{[Since } e_{A}xe_{A} = \tilde{E}(x)e_{A}] \\
&= \langle y, \Omega \rangle_{B} \\
&= t_{B}^{*}(x\iota_{A}(y)).
\end{align*}
\]

Thus by \[Mk1954\] \(E_{A}\) is the unique trace preserving conditional expectation from \(M_{B}\) onto \(M_{A}\) with respect to the trace \(t_{B}^{*}\). We want to show that the conditional expectation map \(E_{A} : M_{B} \rightarrow M_{A}\) is continuous for the SOT*-topologies on the domain and range. Suppose, a net \(\{x_{\alpha}\}\) converges to \(x \in M_{B}\) in SOT*-topology. Take an arbitrary element \(\xi \in \mathcal{H}_{A}\). There exists \(\eta \in \mathcal{H}_{B}\) such that \(e_{A}\eta = \xi\). Now observe that \(E_{A}(x_{\alpha})\xi = E_{A}(x_{\alpha})e_{A}\eta = e_{A}x_{\alpha}e_{A}\eta\). But since \(x_{\alpha}\) converges to \(x\) in SOT* topology we see that \(e_{A}x_{\alpha}e_{A}\eta\) converges to \(e_{A}xe_{A}\eta\). In other words, the map \(E_{A}(x_{\alpha})\xi\) converges to \(E_{A}(x)\xi\). Now the continuity of \(E_{A}\) in SOT* topology follows from the fact that \(E_{A}(x^{*}) = (E_{A}(x))^{*}\).

(5) Define \(P = \{x \in M_{B} : E_{A}(x)(\Gamma(a)) = J_{A}(\lambda_{A}(a^{*})e_{A}x^{*}\Omega)\forall a \in A\}\). Simple calculations show that for each \(b \in B, \lambda_{B}(b) \in P\). In fact, for any \(a \in A, J_{A}(\lambda_{A}(a^{*})e_{A}\lambda_{B}(b)^{*}\Omega) = \Gamma(E_{A}(b)) = E_{A}(\lambda_{B}(b))\Gamma(a)\).

Next we show that \(P\) is a SOT* closed subspace of \(M_{B}\). For this, consider a net \(\{x_{\alpha}\} \subseteq P\) converging to \(x\) in SOT* topology. As we have already seen that \(E_{A}\) is SOT* continuous, \(E_{A}(x_{\alpha})\) also converges to \(E_{A}(x)\) in SOT* topology. Thus, \(E_{A}(x_{\alpha})(\Gamma(a)) \rightarrow E_{A}(x)(\Gamma(a))\). But since each \(x_{\alpha}\) belongs to \(P\) we see that \(E_{A}(x_{\alpha})(\Gamma(a)) = J_{A}(\lambda_{A}(a^{*})e_{A}x_{\alpha}^{*}\Omega)\). But as \(\{x_{\alpha}\}\) converges to \(x\) in SOT* topology we see that \(J_{A}(\lambda_{A}(a^{*})e_{A}x_{\alpha}^{*}\Omega) \rightarrow J_{A}(\lambda_{A}(a^{*})e_{A}x^{*}\Omega)\). Therefore, \(E_{A}(x)(\Gamma(a)) = J_{A}(\lambda_{A}(a^{*})e_{A}x^{*}\Omega)\). Thus we have proved that \(P\) is a SOT* closed subspace of \(M_{B}\). Furthermore, \(\lambda_{B}(B) \subseteq P\). Hence, \(P = M_{B}\). This completes the proof.

Let us record two easy facts for future reference.
Remark 11. The inclusion map $\iota_A$ (of $M_A$ into $M_B$) is given by the formula 
$\iota_A(x)(\Gamma(b)) = J_B(\lambda_B(b^*x^*)\Omega)$ for $x \in M_A$ and $b \in B$. Therefore, $\iota_A(\lambda_A(a)) = \lambda_B(a)$ for any $a \in A$. See [KdySnd2009] for details.

Remark 12. For any $b \in B$, $E_A(\lambda_B(b)) = \lambda_A(E_A(b))$. This can be easily verified using the formula of $E_A$ given in Theorem 10.

Definition 13. A commuting square of finite pre-von Neumann algebras
is a quadruple

\[
\begin{array}{c}
A_{10} \\ \cup \\
A_{00}
\end{array} \quad \begin{array}{c}
A_{11} \\
\cup \\
A_{01}
\end{array}
\]

satisfying the following three properties:

- each pair of inclusions in the quadruple is a compatible pair of finite pre-von Neumann algebras; that is, $\iota_{A_{1i}}|_{A_{ij}} = \iota_{A_{ij}}$ for $i, j \in \{0, 1\}$.
- there exist $*$- and trace-preserving $A_{ij} - A_{ij}$ bimodule retractions $E_{A_{ij}}$, corresponding to each $i, j \in \{0, 1\}$, for the inclusion map of $A_{ij}$ into $A_{1i}$.
- $E_{A_{10}} E_{A_{01}}(a_{11}) = E_{A_{00}}(a_{11}) = E_{A_{01}} E_{A_{10}}(a_{11})$ for $a_{11} \in A_{11}.$

Theorem 14. Consider a commuting square of finite pre-von Neumann algebras

\[
\begin{array}{c}
A_{10} \\ \cup \\
A_{00}
\end{array} \quad \begin{array}{c}
A_{11} \\
\cup \\
A_{01}
\end{array}
\]

Following the notation of Theorem 10, the quadruple

\[
\begin{array}{c}
M_{A_{10}} \\ \cup \\
M_{A_{00}}
\end{array} \quad \begin{array}{c}
M_{A_{11}} \\
\cup \\
M_{A_{01}}
\end{array}
\]

is a commuting square of von Neumann algebras with respect to the inclusions $\iota_{A_{ij}}$ of $M_{A_{ij}}$ into $M_{A_{1i}}$ and conditional expectations $E_{A_{ij}} : M_{A_{1i}} \to M_{A_{ij}}$.

Proof. To see that the quadruple of von Neumann algebras is a commuting square the equation needed to be verified (for any $x \in M_{A_{11}}$) is the following:

$\iota_{A_{10}} E_{A_{10}} \iota_{A_{01}} E_{A_{01}}(x) = \iota_{A_{00}} E_{A_{00}}(x) = \iota_{A_{01}} \iota_{A_{10}} E_{A_{10}}(x)$.

Define, $Q = \{ x \in M_{A_{11}} : \iota_{A_{10}} E_{A_{10}} \iota_{A_{01}} E_{A_{01}}(x) = \iota_{A_{00}} E_{A_{00}}(x) = \iota_{A_{01}} \iota_{A_{10}} E_{A_{10}}(x) \}$.

Now using Remarks 11 and Remarks 12 it follows easily that for any $\lambda_{A_{11}}(a_{11}) \in \lambda_{A_{11}}(A_{11}) \subseteq M_{A_{11}}$ the following equations hold:

$\iota_{A_{10}} E_{A_{10}} \iota_{A_{01}} E_{A_{01}}(\lambda_{A_{11}}(a_{11})) = \lambda_{A_{11}}(E_{A_{10}} E_{A_{01}}(a_{11}))$,

and

$\iota_{A_{00}} E_{A_{00}}(\lambda_{A_{11}}(a_{11})) = \lambda_{A_{11}}(E_{A_{00}}(a_{11}))$.

Since by assumption we have that $E_{A_{10}} E_{A_{01}}(a_{11}) = E_{A_{00}}(a_{11}) = E_{A_{01}} E_{A_{10}}(a_{11})$ for $a_{11} \in A_{11}$ we conclude that $\lambda_{A_{11}}(A_{11}) \subseteq Q$. Furthermore, since each $\iota_{A_{ij}}$ and each $E_{A_{ij}}$ is SOT*-continuous we conclude that $Q$ is an SOT*-closed subspace of $M_{A_{11}}$. Therefore, $Q = M_{A_{11}}$. This completes the proof. \(\square\)
5. Planar subalgebras and commuting squares

In this section we assume that $Q$ is a $*$-planar subalgebra of a subfactor planar algebra $P$ of modulus $\delta > 1$. Let us denote by $M_k(Q)$ the $II_1$-factor of section §2 corresponding to $F_k(Q)$.

We first observe some simple properties of conditional expectation-like maps associated to a compatible pair of finite pre-von Neumann algebras in the following lemma whose proof is simple and omitted.

Lemma 15. Let $(A, t_A) \subseteq (B, t_B)$ be a compatible pair of finite pre-von Neumann algebras. Given an element $b \in B$ suppose there is an element $a \in A$, such that for any $c \in A$ the following holds true:

\[(5.1) \quad t_A(ac) = t_B(bc).\]

Then, $a$ is necessarily unique, and denoting it by $E_A(b)$, the following relations hold:

1. $E_A(a) = a$ for all $a \in A$;
2. $E_A(b^*) = E_A(b)^*$;
3. $E_A(a_1b_2a_3) = a_1E_A(b)a_2$ for $a_1, a_2 \in A$ and $b \in B$. \hfill $\square$

We prove next that the compatible pair of finite pre-von Neumann algebras $(F_1(Q), t_1) \subseteq (F_1(P), t_1)$ always admits a conditional expectation-like map.

Theorem 16. For the compatible pair of finite pre-von Neumann algebras

\[(F_1(Q), t_1) \subseteq (F_1(P), t_1),\]

there exists a $*$-preserving and $t_1$-preserving $F_1(Q) - F_1(Q)$-bimodule retraction, say $E$, for the usual inclusion of $F_1(Q)$ into $F_1(P)$.

Proof. First observe that there exists a conditional expectation $E_{Q_n}$ from $P_n$ onto $Q_n$ such that

\[\tau(E_{Q_n}(x_n)y_n) = \tau(x_ny_n) \quad \text{for all} \quad x_n \in P_n, y_n \in Q_n.\]

Define $E : F_1(P) \to F_1(Q)$ as follows. For $x = (x_1, x_2, \cdots) \in F_1(P)$, set $E(x) = (E_{Q_1}(x_1), E_{Q_2}(x_2), \cdots) \in F_1(Q)$. Next we claim that the following equation holds true for all $(y_1, y_2, \cdots) \in F_1(Q)$.

\[(5.2) \quad t_1((E_{Q_1}(x_1), E_{Q_2}(x_2), \cdots)\#(y_1, y_2, \cdots)) = t_1((x_1, x_2, \cdots)\#(y_1, y_2, \cdots)).\]

Here, by definition of the trace $t_1$, the left hand side is the trace of the $Q_1$ component of $(E_{Q_1}(x_1), E_{Q_2}(x_2), \cdots)\#(y_1, y_2, \cdots)$ while the right hand side is the trace of the $P_1$ component of $(x_1, x_2, \cdots)\#(y_1, y_2, \cdots)$. A little computation using the definition of the product $\#$ shows that Equation 5.2 will follow once the equation in Figure 9 holds for all $x_n \in P_n$ and $y_n \in Q_n$.\hfill $\square$

The left and right hand sides of this figure represent the traces of $E_{Q_n}(x_n)$ and $x_n$ against $Z_{R^n-1}(y_n)$ respectively. However since $Q$ is a planar subalgebra of $P$, $Z_{R^n-1}(y_n) \in Q_n$, and by definition of the conditional expectation $E_{Q_n}$, the desired equality holds.

Finally, we appeal to Lemma 15 to complete the proof.\hfill $\square$

In the following theorem we provide an example of a commuting square of finite pre-von Neumann algebras arising from a $*$-planar subalgebra.
Theorem 17. Let $Q$ be a $*$-planar subalgebra of a subfactor planar algebra $P$ of modules $\delta > 1$. The following quadruple, call it $\mathcal{F}$, is a commuting square of finite pre-von Neumann algebras:

$$
F_0(P) \subseteq F_1(P) \\
\cup \\
F_0(Q) \subseteq F_1(Q).
$$

Proof. That each inclusion of finite pre-von Neumann algebras in the quadruple $\mathcal{F}$ is a compatible pair is obvious. Also, we have a $*$- and trace-preserving $F_0(P) - F_0(P)$ bimodule map $E_0 : F_1(P) \to F_0(P)$ which is a retraction for the inclusion of $F_0(P)$ into $F_1(P)$. Moreover, by Theorem 16 there exists a $*$- and $t_1$- preserving bimodule map $E = E_{F_1(Q)}^{F_1(P)}$ which is also a retraction for the inclusion of $F_1(Q)$ into $F_1(P)$. As $F_0(P) \cap F_1(Q) = F_0(Q)$, to show that $\mathcal{F}$ is a commuting square of finite pre-von Neumann algebras, it suffices to show that the following equation holds:

$$(E \circ E_0)(x_n) = (E_0 \circ E)(x_n) \quad \forall x_n \in P_n \subseteq F_1(P).$$

Computing with the definitions of $E$ and $E_0$, we see that it suffices to verify the pictorial equation on the left of Figure 10 for all $x_n \in P_n$, or equivalently, that for all $y_n \in Q_n$, the two elements of $P_n$ on the right of Figure 10 have the same trace.

Finally, this equality of traces holds since $Q$ is a planar subalgebra of $P$, just as in the proof of Theorem 16. $\square$

Before we state and prove the main result of this section, we need a lemma which also follows from Theorem 7.1 of [SnoWtn1994]. For completeness we sketch a simple proof.
Lemma 18. Consider a commuting square $C$ of type $II_1$ factors:

$$
\begin{array}{c}
L \subseteq M \\
\cup \\
N \subseteq K \\
\end{array}
$$

with $[K : N] = [M : L]$. Then it is a nondegenerate commuting square, i.e., $\text{span}(KL) = M = \text{span}(LK)$.

Proof. Suppose that $\Lambda := \{\lambda_i : i \in I = \{1, 2, \ldots, n\}\}$ is a Pimsner-Popa basis for $K/N$. Thus, the matrix $q(K, N, \Lambda) := ((q_{ij}))$, where $q_{ij} = E_K^L(k_i k_j^*) \forall i, j$, is a projection in $M_n(N)$ such that $\text{tr}(q(K, N, \Lambda)) = [K : N]$. Since by assumption $C$ is a commuting square, we see that $E_L^M(\lambda_i k_j^*) = E_L^M E_K^L(\lambda_i k_j^*) = E_N^M(\lambda_i k_j^*) = q_{ij}$. Therefore, $q(M, L, \Lambda) = q(K, N, \Lambda)$ and further $\text{tr}(q(M, L, \Lambda)) = \text{tr}(q(K, N, \Lambda)) = [K : N] = [M : L]$. This proves that $\{\lambda_i : i \in I\}$ is also a basis for $M/L$. Now the non-degeneracy of $C$ follows from [Pa1994]. This completes the proof. 

We are now ready to deduce the main result of this section.

Theorem 19. Suppose $Q$ is a $*$-planar subalgebra of the subfactor planar algebra $P$ (of modulus $\delta > 1$). Then there exists a smooth non-degenerate commuting square of type $II_1$-factors:

$$
\begin{array}{c}
M_0(P) \subseteq M_1(P) \\
\cup \\
M_0(Q) \subseteq M_1(Q) \\
\end{array}
$$

such that planar algebra of $M_0(P) \subseteq M_1(P)$ is isomorphic to $P$ and the planar algebra of $M_0(Q) \subseteq M_1(Q)$ is isomorphic to $Q$.

Proof. By Theorem 17 it follows that the quadruple $\mathcal{F}$ defined as follows

$$
\begin{array}{c}
F_0(P) \subseteq F_1(P) \\
\cup \\
F_0(Q) \subseteq F_1(Q) \\
\end{array}
$$

is a commuting square of finite pre-von Neumann algebras. Next, apply Theorem 14 to obtain a commuting square $\mathcal{G}$ of type $II_1$ factors as follows:

$$
\begin{array}{c}
M_0(P) \subseteq M_1(P) \\
\cup \\
M_0(Q) \subseteq M_1(Q) \\
\end{array}
$$

But by Theorem 14 we know that the planar algebra of the extremal subfactor $M_0(P) \subseteq M_1(P)$ is isomorphic to $P$ and is of index $\delta^2$. Similarly, the planar algebra of $M_0(Q) \subseteq M_1(Q)$ is isomorphic to $Q$ and is also of index $\delta^2$ (since $Q$ is a planar subalgebra of $P$). Therefore, $[M_1(P) : M_0(P)] = [M_1(Q) : M_0(Q)] = \delta^2$. Then we apply Lemma 18 to conclude that $\mathcal{G}$ is a non-degenerate commuting square. Finally recall that (see for example Proposition 5.2 in [KdySnd2009]) $(M_0(Q))' \cap M_k(Q) = Q_k \subseteq P_k = (M_0(P))' \cap M_k(P)$. Thus $\mathcal{G}$ is a smooth non-degenerate commuting square. This completes the proof. 

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