LOCAL REGULARITY OF THE MAGNETOHYDRODYNAMICS EQUATIONS NEAR THE CURVED BOUNDARY

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Abstract. We study a local regularity condition for a suitable weak solutions of the magnetohydrodynamics equations near the curved boundary.

1. Introduction. We study the local regularity problem for suitable weak solutions \((u, b, \pi) : Q_T \to \mathbb{R}^3 \times \mathbb{R}^3 \times \mathbb{R}\) to the magnetohydrodynamics equations (MHD) in dimensions three

\[
\begin{align*}
&MHD \quad \begin{cases}
    u_t - \Delta u + (u \cdot \nabla)u - (b \cdot \nabla)b + \nabla\pi = 0 \\
    b_t - \Delta b + (u \cdot \nabla)b - (b \cdot \nabla)u = 0 \\
    \text{div } u = 0 \quad \text{and} \quad \text{div } b = 0, \\
    u(x, 0) = u_0(x), \quad b(x, 0) = b_0(x)
\end{cases}
\end{align*}
\]

where \(\Omega\) is a bounded domain in \(\mathbb{R}^3\). Here \(u\) is the flow velocity vector, \(b\) is the magnetic vector and \(\pi = p + \frac{|b|^2}{2}\) is the magnetic pressure. The boundary conditions of \(u\) and \(b\) are given as no-slip and slip conditions, respectively, namely

\[
\begin{align*}
u = 0 \quad \text{and} \quad b \cdot \nu = 0, \quad (\nabla \times b) \times \nu = 0, \quad \text{on } \partial \Omega \times [0, T),
\end{align*}
\]

where \(\nu\) is the outward unit normal vector along boundary \(\Omega\).

The MHD equations describe the macroscopic dynamics of the interaction of moving highly conducting fluids with electro-magnetic fields such as plasma liquid metals, two-phase mixtures (see e.g. [2]).

For the existence of weak solutions for MHD equations, it is well known that it is globally in time. Moreover, in the two-dimensional case, it become regular in [3]. On the other hand, the existence of weak solution for MHD equations with boundary condition (2) in dimension three is proved in [8] and it is shown in [10] that if weak solutions become regular under some condition. However, a regularity question remains open in dimension three not yet as in Navier-Stokes equations.

I will briefly list known results for MHD equations (1) relevant to the regularity criteria in terms of the scaled invariant quantities.

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Numerous works of a regularity criteria for suitable weak solutions have been also studied in terms of the scaled norms. In particular, in [6], the following regularity criteria for a velocity vector for a half space was proved following as:

$$\limsup_{r \to 0} \frac{1}{r^3} \left\| u \right\|_{L^p(B_{x,r})} \left\| u \right\|_{L^q(t-r^2,t)} \leq \epsilon, \quad \frac{3}{p} + \frac{2}{q} = 2, \quad 2 < q < \infty. \quad (3)$$

Other types of conditions in terms of scaled invariant norms near boundary are also found in [15] and [16] (compare to [4, 14, 5] and [18] in the interior cases) also, refer to papers [11, 12, 1] and [9] for Navier-Stokes equations.

This paper is to establish the regularity criteria for the domain near the curved boundary (cf [7] for Navier-Stokes equations). To be more precise, our main result is that Hölder continuity of suitable weak solution $u$ is ensured near sufficiently regular curved boundary provided that the scaled mixed $L^{p,q}$--norm of the velocity field $u$ is small (see Theorem 1.1 for the details).

For notational convenience, we denote for a point $x = (x', x_3) \in \mathbb{R}^3$ with $x' \in \mathbb{R}^2$

$$B_{x,r} = \{ y \in \mathbb{R}^3 : |y - x| < r \}, \quad D_{x,r} = \{ y' \in \mathbb{R}^2 : |y' - x'| < r \}.$$

For $x \in \bar{\Omega}$, we use the notation $\Omega_{x,r} = \Omega \cap B_{x,r}$ for some $r > 0$. If $x = 0$, we drop $x$ in the above notations, for instance $\Omega_{0,r}$ is abbreviated to $\Omega_r$. A solution $u$ and $b$ to magnetohydrodynamics equations (1) is said to be regular at $z = (x, t) \in \Omega \times I$ if $u \in L^\infty(\Omega_{x,r} \times (t-r^2, t))$ for some $r > 0$. In such case, $z$ is called a regular point. Otherwise we say that $u$ is singular at $z$ and $z$ is a singular point.

Next, we give the assumption and remark on the boundary of $\Omega$ (see [7, 17]).

**Assumption 1.** Suppose that $\Omega$ be a class of $C^2 \cap W^{3,\infty}$--boundary (which is its second derivatives are Lipschitz continuous) such that the following is satisfied: For each point $x = (x', x_3) \in \partial \Omega$ there exist absolute positive constants $L$, $\mu$ and $r_0$ independent of $x$ such that we can find a Cartesian coordinate system $\{y_i\}_{i=1}^3$ with the origin at $x$ and a function $\varphi : D_{r_0} \to \mathbb{R}$ satisfying

$$\Omega_{r_0} = \Omega \cap B_{x,r_0} = \{ y = (y', y_3) \in B_{x,r_1} : y_3 > \varphi(y') \}$$

and

$$\varphi(0) = 0, \quad \nabla_y \varphi(0) = 0, \quad \sup_{D_{r_0}} |\nabla^2_y \varphi| \leq L, \quad \| \varphi \|_{W^{3,\infty}(D_{r_0})} \leq \mu.$$

**Remark 1.** The main condition on the Assumption 1 is the uniform estimate of the $C^2$--norms of the function $\varphi$ for each $x \in \partial \Omega$. More precisely, there exists a sufficiently small $r_1$ with $r_1 < r_0$, where $r_0$ is the number in the Assumption 1 such that for any $r < r_1$

$$\sup_{x \in \partial \Omega} \| \varphi \|_{C^2(D_r)} \leq L(1 + r + r^2). \quad (4)$$

This can be easily shown by the Taylor formula.

Now we are ready to state the main part of our main results which is local regularity criteria for a suitable weak solution for MHD equations.

**Theorem 1.1.** Let $(u, b, \pi)$ be a suitable weak solution of the MHD equations (1) according to Definition 2.1. Suppose that for a one pair $p, q$ satisfying $\frac{3}{p} + \frac{2}{q} \leq 2$, $2 < q \leq \infty$ and $(p, q) \neq (\frac{3}{2}, \infty)$, there exists $\epsilon > 0$ depending only on $p, q$ such that for some point $z = (x, t) \in \partial \Omega_{x,r} \times (0, T)$ $u$ is locally in $L^{p,q}_{x,t}$ near $z$ and

$$\limsup_{r \to 0} r^{-\frac{3}{p} + \frac{2}{q} - 1} \left\| u \right\|_{L^p(\Omega_{x,r})} \left\| u \right\|_{L^q(t-r^2,t)} < \epsilon. \quad (5)$$
Then, $u$ and $b$ are regular at $z$.

This paper is organized as follows. In Section 2 we introduce the definition of suitable weak solutions and give some known results for our proof of Theorems. In Section 3 we present the proofs of Theorem 1.1.

2. Preliminaries. In this section we introduce some scaling invariant functionals and the notion of the suitable weak solutions.

We first start with some notations. Let $\Omega$ be a bounded domain in $\mathbb{R}^3$. For $1 \leq q \leq \infty$, we denote the usual Sobolev spaces by $W^{k,q}(\Omega) = \{ u \in L^q(\Omega) : D^\alpha u \in L^q(\Omega), 0 \leq |\alpha| \leq k \}$. As usual, $W^{k,q}_0(\Omega)$ is the completion of $C_0^\infty(\Omega)$ in the $W^{k,q}(\Omega)$ norm. We also denote by $W^{-k,q}(\Omega)$ the dual space of $W^{k,q}_0(\Omega)$, where $q$ and $q'$ are Hölder conjugates. We write the average of $W$ on $\Omega$.

For vector fields $u, v$ we write $\langle u, v \rangle_{1, j=1,2,3}$ as $u \otimes v$. We denote by $C = C(\alpha, \beta, ...)$ which may change from line to line.

As defined earlier, We also denote $\Omega_\epsilon = \Omega \cap B_{\epsilon}$ and $Q_\epsilon = \Omega_\epsilon \times (-r^2, 0)$. Let $r_0$ and $r_1$ be the numbers in the Assumption 1 and the Remark 1, respectively. For any $r < r_1$, we introduce

$$A_u(r) := \sup_{t-r \leq s < t} \frac{1}{r} \int_{\Omega_r} |u(y,s)|^2 dy, \quad E_u(r) := \frac{1}{r} \int_{Q_r} |\nabla u(y,s)|^2 dyds,$$

$$A_b(r) := \sup_{t-r \leq s < t} \frac{1}{r} \int_{\Omega_r} |b(y,s)|^2 dy, \quad E_b(r) := \frac{1}{r} \int_{Q_r} |\nabla b(y,s)|^2 dyds,$$

$$M_u(r) := \frac{1}{r^3} \int_{Q_r} |u(y,s)|^3 dyds, \quad K_b(r) := \frac{1}{r^3} \int_{Q_r} |b(y,s)|^2 dyds,$$

$$Q(r) := \frac{1}{r} \left( \int_{t-r^2}^t \left( \int_{\Omega_r} |\pi(y,s)|^{\kappa} dy \right)^{\frac{1}{\kappa}} ds \right)^{\frac{1}{2}},$$

$$Q_b(r) := \frac{1}{r} \left( \int_{t-r^2}^t \left( \int_{\Omega_r} |\nabla \pi(y,s)|^{\kappa} dy \right)^{\frac{1}{\kappa}} ds \right)^{\frac{1}{2}},$$

where $\kappa, \kappa^*$ and $\lambda$ are numbers satisfying

$$\frac{3}{\kappa} + \frac{2}{\kappa^*} = 4, \quad \frac{1}{\kappa^*} = \frac{1}{\kappa} - \frac{1}{3}, \quad 1 < \lambda < 2. \quad (6)$$

Let $B^+_{r} = B_r \cap \{ x \in \mathbb{R}^3 : x_3 > 0 \}$ and $Q^+_{r} = B^+_{r} \times (-r^2, 0)$. For a suitable weak solution $(v, h, \tilde{\pi})$ and $B^+_{r} \subset \psi(\Omega_{r_1})$,

$$\tilde{A}_v(r) := \sup_{t-r \leq s < t} \frac{1}{r} \int_{B^+_{r}} |v(y,s)|^2 dy, \quad \tilde{E}_v(r) := \frac{1}{r} \int_{Q^+_{r}} |\tilde{\nabla} v(y,s)|^2 dyds,$$

$$\tilde{A}_h(r) := \sup_{t-r \leq s < t} \frac{1}{r} \int_{B^+_{r}} |h(y,s)|^2 dy, \quad \tilde{E}_h(r) := \frac{1}{r} \int_{Q^+_{r}} |\tilde{\nabla} h(y,s)|^2 dyds,$$

$$\tilde{M}_v(r) := \frac{1}{r^3} \int_{Q^+_{r}} |v(y,s)|^3 dyds, \quad \tilde{K}_h(r) := \frac{1}{r^3} \int_{Q^+_{r}} |h(y,s)|^2 dyds,$$

$$\tilde{Q}(r) := \frac{1}{r} \left( \int_{t-r^2}^t \left( \int_{B^+_{r}} |\tilde{\pi}(y,s)|^{\kappa} dy \right)^{\frac{1}{\kappa^*}} ds \right)^{\frac{1}{2}},$$
to (1) if the following conditions are satisfied:

where \(\kappa, \kappa^*\) and \(\lambda\) are numbers satisfying (6).

Next we recall suitable weak solutions for the magnetohydrodynamics equations (1) in three dimensions.

**Definition 2.1.** Let \(\Omega \subset \mathbb{R}^3\) be a bounded domain satisfying the Assumption 1 and \(I = [0, T]\). We denote \(Q_T = \Omega \times I\). A pair of \((u, b, \pi)\) is a suitable weak solution to (1) if the following conditions are satisfied:

(a) The functions \(u, b : Q_T \to \mathbb{R}^3\) and \(p : Q_T \to \mathbb{R}\) satisfy

\[
Q(r) := \frac{1}{r} \left( \int_{t-r^2}^{t} \left( \int_{B_r^+} \|\tilde{\pi}(y, s)\|^\kappa dy \right)^{\frac{1}{\kappa}} ds \right)^{\frac{1}{2}},
\]

\[
\dot{Q}(r) := \frac{1}{r} \left( \int_{t-r^2}^{t} \left( \int_{B_r^+} \|\nabla \tilde{\pi}(y, s)\|^\kappa dy \right)^{\frac{1}{\kappa}} ds \right)^{\frac{1}{2}},
\]

where \(\kappa, \kappa^*\) and \(\lambda\) are numbers satisfying (6).

(b) \((u, b, \pi)\) solves the MHD equations in \(Q_T\) in the sense of distributions and \(u\) and \(b\) satisfy the boundary conditions (2) in the sense of traces.

(c) \(u, b\) and \(\pi\) satisfy the local energy inequality

\[
\int_{t_0}^{t} \left( \int_{\Omega} \left( |u(x, t)|^2 + |b(x, t)|^2 \right) \phi(x, t) dx \right) dt
+ 2 \int_{t_0}^{t} \int_{\Omega} \left( \int_{t_0}^{t} \left( |\nabla u(x, t')|^2 + |\nabla b(x, t')|^2 \right) \phi(x, t') dx dt' \right)
\leq \int_{t_0}^{t} \int_{\Omega} \left( |u|^2 + |b|^2 \right) \left( \partial_x \phi + \Delta \phi \right) dx dt'
+ \int_{t_0}^{t} \int_{\Omega} \left( |u|^2 + |b|^2 \right) \pi \cdot \nabla \phi dx dt'
- 2 \int_{t_0}^{t} \int_{\Omega} \left( b \cdot u \right) \left( b \cdot \nabla \phi \right) dx dt'.
\]

for all \(t \in I = (0, T)\) and for all nonnegative function \(\phi \in C_0^\infty(\mathbb{R}^3 \times R)\).

Let \(x_0 \in \partial \Omega\). Under the Assumption 1, we can represent \(\Omega_{x_0,r_0} = \Omega \cap B_{x_0,r_0} = \{ y = (y', y_3) \in B_{x_0,r_0} : y_3 > \varphi(y') \}\), where \(\varphi\) is the graph of \(C^2\) in the Assumption 1.

Flattening the boundary near \(x_0\), we introduce new coordinates \(x = \psi(y)\) by formulas

\[
x = \psi(y) \equiv (y_1, y_2, y_3 - \varphi(y_1, y_2)).
\]

We note that the mapping \(y \mapsto x = \psi(y)\) straightens out \(\partial \Omega\) near \(x_0\) such that \(\Omega_{x_0,\rho}\) is transformed onto a subdomain \(\psi(\Omega_{x_0,\rho})\) of \(\mathbb{R}^3_+ = \{ x \in \mathbb{R}^3 \ : x_3 > 0 \}\). We define \(v = u \circ \psi^{-1}\), \(h = b \circ \psi^{-1}\) and \(\hat{\pi} = \pi \circ \psi^{-1}\). Then using the change of variables (8), the equations (1) result in the following equations for \(v, h\) and \(\hat{\pi}\):

\[
\begin{cases}
v_t - \Delta v + (v \cdot \hat{\nabla}) v - (h \cdot \hat{\nabla}) h + \hat{\nabla} \hat{\pi} = 0, & \hat{\nabla} \cdot v = 0, \\
h_t - \Delta h + (v \cdot \hat{\nabla}) h - (h \cdot \hat{\nabla}) v = 0, & \hat{\nabla} \cdot h = 0 \text{ in } \psi(\Omega_{x_0,\rho}), \\
v = 0 & \text{ and} \\
h_3 = 0, & h_{1,x_3} = \psi_{x_1} h_{3,x_3}, \quad h_{2,x_3} = \psi_{x_2} h_{3,x_3} \text{ on } \partial \psi(\Omega_{x_0,\rho}) \cap \{ x_3 = 0 \},
\end{cases}
\]

where \(\hat{\nabla}\) and \(\hat{\Delta}\) are differential operators with variable coefficients defined by

\[
\hat{\nabla} = (\partial_{x_1} - \varphi_{x_1} \partial_{x_3}, \partial_{x_2} - \varphi_{x_2} \partial_{x_3}, \partial_{x_3}),
\]

\[
\hat{\Delta} = a_{ij}(x) \partial^2_{x_i,x_j} + b_i(x) \partial_{x_i},
\]

where \(a_{ij}(x)\) and \(b_i(x)\) are appropriate functions.
Lemma 2.2. Let
\[ a_{ij}(x) = \delta_{ij}, \quad a_{33}(x) = a_{3i}(x) = -\varphi_{x_i}, \quad b_i(x) = 0 \quad i = 1, 2, \]
and
\[ a_{33}(x) = 1 + \sum_{i=1}^{2} (\varphi_{x_i})^2, \quad b_3(x) = -\sum_{i=1}^{2} \varphi_{x_i} x_i. \]
As mentioned in Remark 1, if we take a sufficiently small \( r_1 \) with \( r_1 < r_0 \), then (4) holds for any \( r < r_1 \). In addition, the followings are satisfied: for a sufficiently small \( r < r_0 \),
\[ \frac{1}{2} |\nabla v(x, t)| \leq \left| \nabla v(x, t) \right| \leq 2 |\nabla v(x, t)| \quad \text{for all} \ x \in \psi(\Omega(x_0), 2r), \]
\[ B^+_{\psi(x_0), r} \subset \psi(\Omega_{x_0, r}) \subset B^+_{\psi(x_0), 2r}, \quad \psi^{-1}(B^+_{\psi(x_0), r}) \subset \Omega_{x_0, r} \subset \psi^{-1}(B^+_{\psi(x_0), 2r}). \]

From now on, we fix \( x_0 = 0 \) without loss of generality. We suppose that, as above, \( \psi \) is a coordinate transformation so that \( v, \pi \) satisfies (9) in \( \psi(\Omega_{x_0}) \).

Remark 2. Due to the suitability of \( u, h, \pi \) (see Definition 2.1), \( (v, \pi) \) solve (9) in a weak sense and satisfies the following local energy inequality: There exists \( r_2 \) with \( r_2 < r_0 \) where \( r_0 \) is the number in the Assumption 1 such that
\[ \int_{\psi(\Omega_{x_0})} (|v(x, t)|^2 + |h(x, t)|^2) \eta(x, t) dx \\
+ 2 \int_{t_0}^{t} \int_{\psi(\Omega_{x_0})} \left( \left| \nabla v(x, t') \right|^2 + \left| \nabla h(x, t') \right|^2 \right) \eta(x, t') dt' dx t' \\
\leq \int_{t_0}^{t} \int_{\psi(\Omega_{x_0})} \left( |v(x, t')|^2 + |h(x, t')|^2 \right) (\partial_t \eta(x, t') + \hat{\Delta} \eta(x, t')) dx t' dt' \\
+ \int_{t_0}^{t} \int_{\psi(\Omega_{x_0})} \left( |v(x, t')|^2 + |h(x, t')|^2 + 2 \hat{\pi}(x, t') \right) v(x, t') \cdot \nabla \eta(x, t') dx t' dt' \\
- 2 \int_{t_0}^{t} \int_{\psi(\Omega_{x_0})} (h(x, t') \cdot v(x, t'))(h(x, t') \cdot \nabla \eta(x, t')) dx t' dt', \]
where \( \eta \in C^\infty_0(B_r) \) with \( r < r_2 \) and \( \eta \geq 0 \), and \( \hat{\nabla} \) and \( \hat{\Delta} \) are differential operators in (10). \( \square \)

Next lemma shows relations between scaling invariant quantities above (see [7]).

Lemma 2.2. Let \( \Omega \) be a bounded domain satisfying the Assumption 1 and \( x_0 \in \partial \Omega \). Suppose that \( (u, p) \) and \( (v, \pi) \) are suitable weak solutions of (1) in \( \Omega \times I \) and (9) in \( \psi(\Omega_{x_0}) \times I \), respectively, where \( \psi \) is the mapping flattening the boundary in the Assumption 1. Let \( x = \psi(x_0) \). Then there exist sufficiently small \( r_1 \) and an absolute constant \( C \) such that for any \( 4r < r_1 \) the followings are satisfied:
\[ \frac{1}{C} E_v(r) \leq \hat{E}_v(2r) \leq CE_v(4r), \quad \frac{1}{C} A_v(r) \leq \hat{A}_v(2r) \leq CA_v(4r), \]
\[ \frac{1}{C} M_v(r) \leq \hat{M}_v(2r) \leq CM_v(4r), \quad \frac{1}{C} K_v(r) \leq \hat{K}_v(2r) \leq CK_v(4r), \]
\[ \frac{1}{C} Q_v(r) \leq \hat{Q}_v(2r) \leq CQ_v(4r). \]

Also, Lemma 2.2 holds for the quantities \( \hat{E}_h(2r), \hat{A}_h(2r), \hat{M}_v(2r) \) and \( \hat{K}_v(2r) \).
3. **Proof of Theorem.** In this section, we present the proof of the Theorem 1.1. We first show a $\epsilon$-regularity criterion for the suitable weak solution of MHD equations (1) near the boundary. Next we prove a local regularity integration condition for the velocity vector $u$ near boundary. For simplicity, we write $\Psi(r)$ := $A_v(r) + A_h(r) + E_v(r) + E_h(r)$. Let $z = (x,t) \in \Gamma \times I$ and from now on, without loss of generality, we assume $x = 0$ by translation. We first recall that the local energy estimate.

$$\Psi \left( \frac{r}{2} \right) \leq C \left( \tilde{M}_v^2(r) + \tilde{K}_h(r) + \tilde{M}_v(r) + \frac{1}{r^2} \int_{Q_{r}^+} |v| |h| \, dz \right) \left( \frac{r}{2} \right) \leq C \left( \frac{r}{2} \right) \frac{1}{r^2} \int_{Q_{r}^+} |v| |\tilde{h}| \, dz \right). \tag{14}$$

Next we prove a local $\epsilon$-regularity condition near boundary for MHD equations. Its proof is similar to the proof in [6, Proposition 3.1] and so we omit.

**Proposition 1.** There exist $\epsilon^* > 0$ and $r_0 > 0$ such that if $(u,b,\pi)$ is a suitable weak solution of MHD equations satisfying Definition 2.1, $z = (x,t) \in \Gamma \times I$, and

$$\tilde{M}_v(r) + \tilde{M}_h(r) + \tilde{Q}(r) < \epsilon^* \quad \text{for some } r \in (0, r_0), \tag{15}$$

then $z$ is regular point.

The proof of Proposition 1 is based on the following lemma 3.1, which shows a decay estimate of $(u,b,\pi)$.

**Lemma 3.1.** Let $0 < \theta < \frac{1}{2}$. There exist $\epsilon_1 > 0$ and $r_*$ depending on $\lambda$ and $\theta$ such that if $(u,b,\pi)$ is a suitable weak solution of the MHD equations satisfying Definition 2.1, $z = (x,t) \in \Gamma \times (0, T)$, and $M_v^2(r) + M_h^2(r) + \tilde{Q}(r) < \epsilon_1$ for some $r \in (0, r_*)$, then

$$\tilde{M}_v^2(\theta r) + \tilde{M}_h^2(\theta r) + \tilde{Q}(\theta r) < C\theta^{1+\alpha} \left( \tilde{M}_v^2(r) + \tilde{M}_h^2(r) + \tilde{Q}(r) \right),$$

where $0 < \alpha < 1$ and $C > 0$ are constants.

We estimate the scaled norm for suitable weak solutions.

**Lemma 3.2.** Let $z = (x,t) \in \Gamma \times I$.

1. Suppose that $v \in L^{p,q}_{x,t}(Q_{2r}^+)$ with $3/p + 2/q = 2$, $3/2 \leq p \leq \infty$. Then for $0 < r < \rho/4$,

$$\tilde{M}_v(r) \leq C \left( \frac{r}{T} \right) \Psi(\rho) \epsilon, \tag{16}$$

$$\frac{1}{r^2} \int_{Q_{r}^+} |v| |h| \, dz \leq C \left( \frac{r}{T} \right) \Psi(\rho) \epsilon. \tag{17}$$

**Proof.** In the proof of [6, Lemma 3.2], we can obtain the Lemma 3.2 by replacing $u$, $b$ and operator $\nabla$ by $v$, $h$ and operator $\tilde{\nabla}$, respectively. So we omit the proof.

Next, we may continue with scaled norm of $L^{2,2}_{x,t}(Q_{2r}^+)$ estimate of $b$.

**Lemma 3.3.** Let $z = (x,t) \in \Gamma \times I$.

1. Suppose that $u \in L^{p,q}_{x,t}(Q_{2r}^+)$ with $3/p + 2/q = 2$ and $3/2 \leq p < 3$. Then for $0 < r < \rho/4$

$$\tilde{K}_h(r) \leq C \left( \frac{r}{T} \right)^3 \epsilon^2(\rho) \Psi(\rho) + C \left( \frac{r}{T} \right)^2 \tilde{K}_h(\rho). \tag{18}$$

**Proof.** Although the process of proof is similar as in [6, Lemma 3.3], we will give its details for the convenience of readers. For convenience, we write $x = (x_1, x_2, x_3) = (x', x_3)$ and by translation, we assume that without loss of generality, $z = (0, 0) \in \Gamma \times I$. For simplicity, we write $\Psi(r)$ := $A_v(r) + A_h(r) + E_v(r) + E_h(r)$.
Let \( \zeta(x,t) \) be a standard cut off function supported in \( Q_\rho \) such that \( \zeta(x,t) = 1 \) in \( Q_{\rho/2} \). We set \( g(x,t) := -\nabla \cdot (v \otimes h - h \otimes v) \zeta \) in \( Q_{\rho}^+ \), and we then define \( \tilde{g}(x,t) \), an extension of \( g \) from \( Q_{\rho}^+ \) onto \( Q_\rho \), in the following way: \( \tilde{g}(x,t) = g(x,t) \) if \( x_3 \geq 0 \). On the other hand, if \( x_3 < 0 \), then

\[
\tilde{g}_i(x',x_3,t) = g_i(x',-x_3,t), \quad i = 1, 2
\]

\[
\tilde{g}_3(x',x_3,t) = -g_3(x',-x_3,t).
\]

This can be done by extending tangential components of \( v \) and \( h \) as even functions and normal components of \( v \) and \( h \) as odd functions, respectively. We denote such extensions by \( \tilde{v} \) and \( \tilde{h} \) for simplicity. Here we also used the fact that \( \zeta \) and \( \nabla' \zeta \) are even and \( \partial_{x_3} \zeta \) is odd with respect to \( x_3 \)-variable, where \( \nabla' = (\partial_{x_1}, \partial_{x_2}) \).

Next, consider the heat type equation for \( \tilde{w}(x,t) \) for \( (x,t) \in \mathbb{R}^3 \times (-\infty, 0) \) by

\[
\tilde{w}_t - \tilde{\Delta} \tilde{w} = \tilde{g} \quad \text{in} \quad \mathbb{R}^3 \times (-\infty, 0).
\]

Moreover, we can see that \( \partial_{x_3} \tilde{w}_i = \psi \tilde{w}_i, \tilde{w}_{3,x_3} \) for \( i = 1, 2 \) and \( \tilde{w}_3 = 0 \) on \( \{x_3 = 0\} \). Let \( \tilde{h} = h - \tilde{w} \) in \( Q_{\rho}^+ \). Then \( \tilde{h} \) satisfies

\[
\tilde{h}_t - \tilde{\Delta} \tilde{h} = 0 \quad \text{in} \quad Q_{\rho}^+.
\]

and \( \partial_{x_3} \tilde{h}_i = \psi \tilde{h}_i, \tilde{h}_{3,x_3} \) for \( i = 1, 2 \) and \( \tilde{h}_3 = 0 \) on \( \{x_3 = 0\} \cap Q_\rho \). Now we extend \( \tilde{w} \) by the same manner as \( g \), denoted by \( \tilde{h} \), from \( Q_{\rho/2}^+ \) onto \( Q_{\rho/2} \). We then see that

\[
\tilde{h}_t - \tilde{\Delta} \tilde{h} = 0 \quad \text{in} \quad Q_{\rho}^+.
\]

Via the regularity theory, we have

\[
\int_{Q_\rho} |\tilde{h}|^2 dz \leq C\left(\frac{\rho}{\rho'}\right)^5 \int_{Q_{\rho}^+} |\tilde{h}|^2 dz.
\]  

(19)

On the other hand, due to Sobolev embedding, we have \( \|\tilde{w}\|_{L^2(B_{\rho})} \leq \|\tilde{w}\|_{L^2(\mathbb{R}^3)} \leq C\|\nabla \tilde{w}\|_{L^\infty(\mathbb{R}^3)} \) and we then take \( L^2 \) integration for the above in time interval \( (\rho^2, 0) \) such that we obtain

\[
\|\tilde{w}\|_{L^2_{\rho}(Q_\rho)} \leq C\|\nabla \tilde{w}\|_{L^{\infty}_{\rho}(\mathbb{R}^3 \times (-\rho^2, 0))} \leq C\|\nabla h \zeta\|_{L^{\infty}_{\rho}(\mathbb{R}^3 \times (-\rho^2, 0))}
\]

\[
\leq C\|\nabla \tilde{h}\|_{L^{\infty}_{\rho}(\mathbb{R}^3 \times (-\rho^2, 0))} \leq C\|\tilde{v}\|_{L^2_{\rho}(Q_\rho)} \|\tilde{h}\|_{L^2_{\rho}(Q_\rho)}.
\]  

(20)

where \( 3/\alpha + 2/\beta = 3/2 \) and \( 2 \leq \alpha < 6 \), since \( 3/2 \leq p < 3 \). Using the estimate (20) and Sobolev inequality, we have

\[
\frac{1}{\rho^2} \|\tilde{w}\|^2_{L^2_{\rho}(Q_\rho)} \leq C\frac{1}{\rho^2} \|\tilde{h}\|^2_{L^2_{\rho}(Q_\rho)} \frac{1}{\rho} \|\tilde{h}\|^2_{L^2_{\rho}(Q_\rho)} \leq C\frac{1}{\rho^2} \|\tilde{v}\|^2_{L^2_{\rho}(Q_\rho)} \frac{1}{\rho} \|\tilde{h}\|^2_{L^2_{\rho}(Q_\rho)} \leq \frac{C}{\rho^2} \|\tilde{v}\|^2_{L^2_{\rho}(Q_\rho)} (\hat{A}_h(\rho) + \hat{E}_h(\rho)) \leq C\epsilon^2 \Psi(\rho).
\]  

(21)

Combining estimates (19) and (21), we obtain

\[
\hat{K}_h(r) = \frac{1}{r^3} \|\tilde{h}\|^2_{L^2_{\rho}(Q_\rho)} \leq \frac{1}{r^3} \|\tilde{w}\|^2_{L^2_{\rho}(Q_\rho)} + \frac{1}{r^3} \|\tilde{h}\|^2_{L^2_{\rho}(Q_\rho)} \leq \frac{C}{r^3} \|\tilde{v}\|^2_{L^2_{\rho}(Q_\rho)} + \frac{C}{r^2} \hat{K}_h(\rho).
\]

This completes the proof.

In next lemma we show an estimate of the gradient of pressure .
Lemma 3.4. Let \( z = (x, t) \in \Gamma \times I \). Then for \( 0 < r < \rho/4 \),

\[
\hat{Q}_i(r) \leq C \left( \frac{\rho}{r} \right) \left( \tilde{A}^{\frac{3}{2}\kappa}_{v,0} (\rho) \hat{E}^\frac{1}{2}_v (\rho) + \tilde{A}^{\frac{3}{2}\kappa}_{h,0} (\rho) \hat{E}^\frac{1}{2}_h (\rho) \right) + C \left( \frac{r}{\rho} \right) \left( \hat{E}^\frac{1}{2}_v (\rho) + \hat{Q}_1 (\rho) \right),
\]

(22)

where \( \kappa \) and \( \lambda \) are numbers in (6).

Proof. Although the process of proof is similar as in [6, Lemma 3.4], we will give its details under our circumstances. We assume, via translation, that \( z = (x, t) = (0, 0) \). We choose a domain \( \hat{B}^+ \) with a boundary such that \( B^+_{\frac{3}{2}} \subset \hat{B}^+ \subset B^+_{\rho} \), and we denote \( \hat{Q}^+ := \hat{B}^+ \times (-\rho^2, 0) \).

Let \( (v, \pi_1) \) be the unique solution of the following the Stokes system

\[
v_t - \hat{\Delta} v + \nabla \pi_1 = -(v \cdot \nabla) v + (b \cdot \nabla) h, \quad \text{div} v = 0 \quad \text{in} \; \hat{Q}^+, \quad (\pi_1)_{B^+_{\rho}} = t_{B^+_{\rho}} (y, t) dy = 0, \quad t \in (-\rho^2, 0),
\]

\[
v = 0 \quad \partial B^+ \times [-\rho^2, 0], \quad v = 0 \quad \hat{B}^+ \times \{ t = -\rho^2 \}.
\]

Using the perturbed Stokes estimate [17], we have the following estimate

\[
\frac{1}{\rho^2} \| v \|_{L^2_{\lambda, i} (\hat{Q}^+)} + \frac{1}{\rho} \| \nabla v \|_{L^2_{\lambda, i} (\hat{Q}^+)} + \| v_t \|_{L^2_{\lambda, i} (\hat{Q}^+)} + \| \hat{\nabla}^2 v \|_{L^2_{\lambda, i} (\hat{Q}^+)} + \| \hat{\nabla} \pi_1 \|_{L^2_{\lambda, i} (\hat{Q}^+)}
\]

\[
\leq C \left[ \| (v \cdot \hat{\nabla}) v \|_{L^2_{\lambda, i} (\hat{Q}^+)} + \| (h \cdot \hat{\nabla}) h \|_{L^2_{\lambda, i} (\hat{Q}^+)} \right]
\]

\[
\leq C \left( \| v \|_{L^2_{\lambda, i} (\hat{Q}^+)} + \| h \|_{L^2_{\lambda, i} (\hat{Q}^+)} \right)
\]

\[
\leq C \left( \rho \tilde{A}^{\frac{3}{2}\kappa}_{v,0} (\rho) \hat{E}^\frac{1}{2}_v (\rho) + \rho \tilde{A}^{\frac{3}{2}\kappa}_{h,0} (\rho) \hat{E}^\frac{1}{2}_h (\rho) \right),
\]

where we used the following estimates in last inequality above:

\[
\| (v \cdot \hat{\nabla}) v \|_{L^2_{\lambda, i} (\hat{Q}^+)} \leq \| v \|_{L^2_{\lambda, i} (\hat{Q}^+)} \| \hat{\nabla} v \|_{L^2_{\lambda, i} (\hat{Q}^+)} + \| h \|_{L^2_{\lambda, i} (\hat{Q}^+)} \| \hat{\nabla} h \|_{L^2_{\lambda, i} (\hat{Q}^+)}
\]

\[
\leq C \rho \tilde{A}^{\frac{3}{2}\kappa}_{v,0} (\rho) \hat{E}^\frac{1}{2}_v (\rho) + C \rho \tilde{A}^{\frac{3}{2}\kappa}_{h,0} (\rho) \hat{E}^\frac{1}{2}_h (\rho).
\]

Next, let \( w = u - v \) and \( \pi_2 = \pi - (\pi)_{B^+_{\frac{3}{2}} - \pi_1} \). Then \( (w, \pi_2) \) solves the following boundary value problem:

\[
w_t - \hat{\Delta} w + \nabla \pi_2 = 0, \quad \text{div} w = 0 \quad \text{in} \; \hat{Q}^+, \quad w = 0 \quad \text{on} \; (\partial \hat{B}^+ \cap \{ x_3 = 0 \}) \times [-\rho^2, 0].
\]

Now we take \( \kappa' \) such that \( 3/\kappa' + 2/\lambda = 2 \). Then from the local estimate near the boundary for the Stokes systems (see [13]), we obtain

\[
\| \hat{\nabla}^2 w \|_{L^2_{\lambda, i} (\hat{Q}^+)} + \| \nabla \pi_2 \|_{L^2_{\lambda, i} (\hat{Q}^+)}
\]

\[
\leq C \left( \frac{1}{\rho^2} \| w \|_{L^2_{\lambda, i} (\hat{Q}^+)} + \frac{1}{\rho} \| \nabla w \|_{L^2_{\lambda, i} (\hat{Q}^+)} + \frac{1}{\rho} \| \pi_2 \|_{L^2_{\lambda, i} (\hat{Q}^+)} \right)
\]

\[
\leq C \left( \frac{1}{\rho^2} \| \hat{\nabla} w \|_{L^2_{\lambda, i} (\hat{Q}^+)} + \| \hat{\nabla} \pi \|_{L^2_{\lambda, i} (\hat{Q}^+)} + \frac{1}{\rho} \| \hat{\nabla} v \|_{L^2_{\lambda, i} (\hat{Q}^+)} + \frac{1}{\rho} \| \pi_1 \|_{L^2_{\lambda, i} (\hat{Q}^+)} \right),
\]
Lemma 3.5. sketch of proof for the convenience of readers. We note first that via Hölder’s inequality, the proof is same as in [6, Theorem 1.1], we provide the proof of Theorem 1.1.

Proof. In the proof of [6, Lemma 3.2], we can obtain the Lemma 3.2 by replacing \( \nabla \pi \) and operator \( \hat{\nabla} \) by \( \hat{\nabla} \pi \) and operator \( \nabla \), respectively. So we omit the proof.

We remark that, via Young’s inequality, (22) can be estimated as follows:

\[
\hat{Q}(r) \leq C \left( \left( \frac{p}{r} \right) + \left( \frac{r}{\rho} \right) \right) \Psi(\rho) + C \left( \frac{r}{\rho} \right) \left( \hat{Q}(\rho) + 1 \right). \tag{23}
\]

Next lemma shows an estimate of a scaled norm of pressure.

Lemma 3.5. Let \( z = (x, t) \in \Gamma \times I \). Suppose that \( \hat{\nabla} \hat{\pi} \in L^{\kappa, \lambda}_x(Q_\rho) \) and \( \hat{\pi} \in L^{\kappa, \lambda}_x(Q_\rho) \), where \( 3/\kappa + 2/\lambda = 4 \), \( 1/\kappa^* = 1/\kappa - 1/3 \) and \( 1 < \lambda < 2 \). Then for \( 0 < r < \rho/4 \),

\[
\hat{Q}(r) \leq C \left( \frac{\rho}{r} \right) \hat{Q}(\rho) + C \left( \frac{r}{\rho} \right)^{3-1} \hat{Q}(\rho). \tag{24}
\]

Proof. In the proof of [6, Lemma 3.2], we can obtain the Lemma 3.2 by replacing \( \pi \) and operator \( \nabla \) by \( \hat{\pi} \) and operator \( \hat{\nabla} \), respectively. So we omit the proof.

We are ready to present the proof of Theorem 1.1.

Proof of Theorem 1.1. The proof is same as in [6, Theorem 1.1], we provide the sketch of proof for the convenience of readers. We note first that via Hölder’s inequality, it suffices to show the case that \( 3/p + 2/q = 2 \), \( 2 < q < \infty \). Recalling Lemma 3.4 and Lemma 3.5, we have

\[
\frac{1}{r^2} \int_{Q_{r^2}^+} |u| |\pi| dz \leq \frac{1}{r} \|u\|_{L^{\kappa, \lambda}_x(Q_{r^2})} \frac{1}{r} \||\pi\|_{L^{\kappa, \lambda}_x(Q_{r^2})} \leq C \left( \frac{\rho}{r} \right)^2 \Psi(\rho) + C \left( \frac{r}{\rho} \right) \hat{Q}(\rho) + C \left( \frac{r}{\rho} \right)^{3-1} Q(\rho), \tag{25}
\]
where (23) is also used. With aid of Lemma 3.2, Lemma 16, (23) and (25), we have
\[
\Psi\left(\frac{r}{2}\right) \leq C \left[ \epsilon^2 \frac{b}{r}^3 + \epsilon (\frac{b}{r})^2 + (\frac{\epsilon}{\rho} + \epsilon + \frac{1}{\rho}) (\frac{b}{\rho})^2 + \epsilon + \frac{1}{\rho} \right] \Psi(\rho) + C\epsilon Q_1(\rho) + C\epsilon \left(\frac{r}{\rho}\right)^{\frac{3}{4}} Q(\rho),
\]
where we used the Young’s inequality and \( K_b(\rho) \leq \Psi(\rho) \). Let \( \epsilon_1 \) and \( \epsilon_2 \) be small positive numbers, which will be specified later.

Adding \( \epsilon_1 \tilde{Q}_1(\frac{\epsilon}{2}) \) and \( \epsilon_2 \tilde{Q}(\frac{\epsilon}{2}) \) to both sides in (26), and using (23) and Lemma 3.5, we obtain
\[
\Psi\left(\frac{r}{2}\right) + \epsilon_1 \tilde{Q}_1\left(\frac{r}{2}\right) + \epsilon_2 \tilde{Q}\left(\frac{r}{2}\right) \leq C \left[ \epsilon^2 \frac{b}{r}^3 + \epsilon (\frac{b}{r})^2 + (\frac{\epsilon}{\rho} + \epsilon + \epsilon_1) (\frac{b}{\rho})^2 + \epsilon + \frac{1}{\rho} \right] \Psi(\rho) + C\epsilon Q_1(\rho) + C\epsilon \left(\frac{r}{\rho}\right)^{\frac{3}{4}} Q(\rho) + C \left[ \epsilon + \epsilon_1 \frac{r}{\rho} \right].
\]
We fix \( \theta \in (0, \frac{1}{4}) \) with \( C(\theta + \theta^{\frac{3}{4}} - 1) < \frac{1}{4} \) and then choose \( \epsilon_1, \epsilon_2 \) and \( \epsilon \) satisfying
\[
0 < \epsilon_1 < \min \left\{ \frac{\theta}{64C}, \frac{\tilde{\epsilon}}{64C} \right\}, \quad 0 < \epsilon_2 < \frac{\epsilon_1 \theta}{32C}, \quad 0 < \epsilon < \min \left\{ \frac{\tilde{\epsilon}}{64C}, \epsilon_2, \frac{\theta^6}{64C^2} \right\},
\]
where \( \tilde{\epsilon} \) is a constant, which will be specified later. Therefore, we have
\[
\Psi(\theta r) + \epsilon_1 Q_1(\theta r) + \epsilon_2 Q(\theta r) \leq \frac{\tilde{\epsilon}}{32} + \frac{1}{2} \left( \Psi(r) + \epsilon_1 Q_1(r) + \epsilon_2 Q(r) \right).
\]
Iterating (27), we can see that there exists a sufficiently small \( r_0 > 0 \) such that for all \( r < r_0 \)
\[
\Psi(r) + \epsilon_1 Q_1(r) + \epsilon_2 Q(r) \leq \frac{\tilde{\epsilon}}{8},
\]
Therefore, we obtain \( \Psi(r) + Q(r) \leq \epsilon^* / 2 \) for all \( r < r_1 \), which implies the regularity condition in Proposition 1. This completes the proof. \( \square \)

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