Gauge coupling renormalization in RS1

K. Agashe∗, A. Delgado† and R. Sundrum‡

Department of Physics and Astronomy
Johns Hopkins University
3400 North Charles St.
Baltimore, MD 21218-2686

Abstract

We compute the 4D low energy effective gauge coupling at one-loop order in the compact Randall-Sundrum scenario with bulk gauge fields and charged matter, within controlled approximations. While such computations are subtle, they can be important for studying phenomenological issues such as grand unification. Ultraviolet divergences are cut-off using Pauli-Villars regularization so as to respect 5D gauge and general coordinate invariance. The structure of these divergences on branes and in the bulk is elucidated by a 5D position-space analysis. The remaining finite contributions are obtained by a careful analysis of the Kaluza-Klein spectrum. We comment on the agreement between our results and expectations based on the AdS/CFT correspondence, in particular logarithmic sensitivity to the 4D Planck scale.

∗email: kagashe@pha.jhu.edu
†email: adelgado@pha.jhu.edu
‡email: sundrum@pha.jhu.edu
1 Introduction

Extra dimensions provide a theoretically and phenomenologically interesting avenue for addressing the Planck-Weak hierarchy problem \[1\]. In this paper we focus on the Randall-Sundrum (RS1) proposal based on warped compactifications \[2\]. The geometry of the RS1 vacuum is a compact slice of AdS$_5$,

\[
ds^2 = e^{-2k|\theta|}r_c \eta_{\mu\nu} dx_\mu dx_\nu + r_c^2 d\theta^2, \quad -\pi \leq \theta \leq \pi,
\]

where the extra-dimensional interval is realized as an orbifolded circle of radius $r_c$. The two orbifold fixed points, $\theta = 0, \pi$, correspond to the “UV” (or “Planck”) and “IR” (or “TeV”) branes respectively. In warped spacetimes 5D mass scales do not directly correspond to 4D mass scales in an effective 4D description, rather the relationship depends on location in the extra dimension through the warp factor, $e^{-k|\theta|} r_c$. This allows large 4D mass hierarchies to naturally arise without large hierarchies in the defining 5D theory, whose mass parameters are taken to be of order the observed Planck scale, $M_4 \sim 10^{18}$ GeV. For example, the 4D massless graviton mode is localized near the UV brane \[2\] while Higgs physics is taken to be localized on the IR brane. In the 4D effective theory one then finds

\[\text{Weak Scale} \sim M_4 e^{-k\pi r_c}.\]  

(1.2)

A modestly large radius can then accommodate a TeV-size weak scale. Such a value of the radius can be naturally stabilized using the Goldberger-Wise (GW) mechanism \[3\]. There is a striking associated phenomenology of TeV-scale Kaluza-Klein (KK) graviton resonances since their wave functions are also localized near the IR brane \[2, 4\].

The AdS/CFT correspondence \[5\] casts further light on the RS1 mechanism \[6, 7, 8, 9\]. Effective gravitational field theories in AdS$_5$ encode the constraints of four-dimensional conformal field theory (CFT) Ward identities and unitarity, the AdS$_5$ loop expansion encodes the constraints of a large-$N$ expansion for the CFT, while basic CFT data, such as the scaling dimensions of the most relevant operators, are encoded in the 5D masses and couplings of the AdS$_5$ theory. This is quite analogous to the manner in which chiral lagrangians for pions encode the general constraints of chiral symmetry (breaking) and unitarity in a transparent and useful way. In terms of the correspondence, the RS1 scenario can be viewed as an effective description of a strongly-coupled 4D large-$N$ theory coupled to 4D gravity, which is nearly conformal over the Planck-weak hierarchy. For phenomenological purposes however, it is more useful to employ the AdS$_5$ picture.

Quantum loops in warped spacetimes are rather subtle because they are non-local, and as they span the extra dimension are sensitive to greatly varying 4D mass scales. However, their
effects can be important to compute, for instance in considering radiative stability of radius stabilization \[10,11\] or in studying grand unification in the RS scenario \[12,13\]. In a future paper \[14\] we will further the exploration of grand unification in the RS context. In the present paper we study the loop computation of particular relevance to grand unification, namely the one-loop correction to the effective 4D gauge coupling when bulk gauge fields and bulk charged matter are incorporated into the model \[12,13,15\]. While this paper was in preparation, reference \[16\] appeared which has some overlap with our results. We restrict attention to the effective 4D gauge coupling at energies below the lightest KK mass (\(\sim\)TeV). We verify that the effective 4D gauge coupling robustly contains a logarithmic sensitivity to the 4D Planck scale, “as if” it had been run down from that scale within a purely 4D renormalization group flow \[12,13\]. All detailed phenomenological issues will be postponed until reference \[14\]. For previous phenomenological studies of bulk gauge fields in RS1, see references \[17\].

Several of the results derived here have appeared earlier \[12,13\]. We have tried to give a rather complete treatment here, as simple as possible consistent with rigor and independence of numerical analysis \[1\] and we have specified the parametric size of corrections to our approximations.

The RS1 effective field theory is of course non-renormalizable, or more accurately it must be renormalized order by order in the derivative-loop expansion. We discuss how this is done in detail in our one-loop calculation. We employ a manifestly 5D gauge-invariant and generally covariant UV regulator, namely Pauli-Villars, as previously used in reference \[12\]. We use a 5D position space analysis to straightforwardly understand the structure of the local UV divergences on the branes and in the bulk by relating them to the much simpler case of flat space compactifications. This was also discussed in reference \[16\]. Renormalization to this order is straightforwardly accomplished.

The dominant contributions to the finite parts of our calculation are then deduced within a mode analysis by careful consideration of the KK spectrum. While our focus is on a particular bulk loop we believe that our methodology has broader applicability.

Finally, we discuss the CFT interpretation and compatibility of our loop-computation as subleading large-\(N\) corrections, in particular, the logarithmic sensitivity to the 4D Planck scale.

The paper is organized as follows. In section 2 we detail the simple model to be studied, namely scalar QED in the bulk. In section 3 we review the classical approximation to this model and its CFT interpretation. In Section 4 we give a 5D position space analysis of UV divergences and renormalization. Section 5 contains a summary of the finite pieces in the 4D effective gauge coupling, followed by a detailed derivation using KK mode analysis and the results of section 4. Section 6 discusses the CFT interpretation of the RS1 loop corrections.

\[1\]For example, we give a purely analytic account of the Pauli-Villars “zero-mode” of reference \[12\].
The central results of our paper are contained in Eqs. (4.6), (4.7), (5.1), (5.2) and (5.3).

2 The Model

We will consider the simplest model which allows us to focus on one-loop radiative corrections to the 4D effective gauge coupling. For this purpose we can fix spacetime to be a non-dynamical background of RS1 form, Eq. (1.1). The gauge theory we consider is 5D scalar QED with a scalar mass $m_5$. We will study this case in this paper because scalar loops are technically more transparent than loops of higher spin particles. Brane-localized charged fields are omitted in this paper; their loop effects are straightforward to compute.

The dominant part of the action is given by

$$S_{\text{bulk}} = \frac{1}{g_5^2} \int d^4x dy \sqrt{-G} \left( -\frac{1}{4} F_{MN} F^{MN} + D_M \phi \left( D^M \phi \right)^\dagger - m_5^2 |\phi|^2 \right), \quad (2.1)$$

where $g_5^2$ has dimension $(\text{mass})^{-1}$ and $y = r_c \theta$. We will take $A_\mu$ and $\phi$ to be orbifold-even while $A_5$ is taken orbifold-odd.

In addition we can add brane localized terms for our bulk fields,

$$S_{UV(IR)} = \int d^4x \sqrt{-g_{UV(IR)}} \left( -\frac{1}{4} \tau_{UV(IR)} F_{\mu\nu} F^{\mu\nu} + \sigma_{UV(IR)} (D_\mu \phi)^\dagger D^\mu \phi \right), \quad (2.2)$$

where $\tau_{UV(IR)}, \sigma_{UV(IR)}$ are small dimensionless couplings. We will consider them to be perturbations which we take to be of the same order as one-loop processes involving bulk interactions. Thus, working to one-loop order the brane-localized vertices do not appear in one-loop graphs, only in tree-level graphs. For simplicity, we omit further potential terms for $\phi$ which would not appear in one-loop graphs.

3 Review of Classical Approximation

We will match our 5D model onto a 4D effective theory at a scale provided by the mass of the lightest Kaluza-Klein (KK) excitation. We will denote this mass scale by $m_{KK}$, which is $O(ke^{-k\pi r_c}) \sim O(\text{TeV})$ \cite{2}. Clearly, the scalar does not affect the gauge coupling classically. The gauge field zero-mode appearing in the 4D effective theory is given by

$$A_\mu = A_\mu(x),$$
$$A_5 = 0.$$  \quad (3.1)$$
Plugging Eq. (3.1) into the action, Eqs. (2.1) and (2.2), one finds

\[ L_{\text{eff}} \ni -\frac{1}{4g_4^2} F_{\mu\nu} F^{\mu\nu}, \]  

(3.2)

where the effective 4D gauge coupling is

\[ \frac{1}{g_4^2} = \tau_{UV} + \tau_{IR} + \frac{\pi r_c}{g_5^2}. \]  

(3.3)

This coupling can be expressed in a suggestive form,

\[ \frac{1}{g_4^2} = \tau_{UV} + \tau_{IR} + \log \left[ \frac{O(M_4)/\text{TeV}}{k g_5^2} \right], \]  

(3.4)

once one puts in the RS-GW mechanism [2,3] for generating the Planck-Weak hierarchy without fundamental 5D hierarchies:

\[ k \pi r_c = \log \left[ \frac{O(M_4)}{\text{TeV}} \right], \]  

(3.5)

where \( M_4 \) is the observed 4D Planck scale. The logarithmic dependence on \( M_4 \) (treated as a variable) is a remarkable feature of the RS1 scenario. For example, flat extra dimensions do not automatically possess this feature (however see the proposal of [18]): in flat 5D, we also get \( 1/g_4^2 \sim \pi r_c/g_5^2 \), but unlike in RS1, \( r_c \) has no relation with the Planck-weak hierarchy.

Eq. (3.4), though the result of 5D classical physics, looks like a quantum gauge coupling in a purely 4D theory which has been run down from a Planckian value, \( 1/g_4^2(\sim O(M_4)) \sim \tau_{UV} \), with a renormalization group equation,

\[ \mu \frac{d}{d\mu} \frac{1}{g_4^2(\mu)} = -b, \]  

(3.6)

where \( b = 1/(k g_5^2) \) [7]. The running appears to stop at a TeV threshold, with a threshold correction \( \delta (1/g_4^2) = \tau_{IR} \). Note that the \( \beta \)-function coefficient, \( b \), necessarily has an infrared-free sign and that this would happen whether the bulk gauge field was non-abelian or abelian (as it is here for the sake of simplicity).

These observations are not coincidental, but are in accord with the AdS/CFT correspondence [5] applied to the RS1 scenario [6,7,8,9]. The CFT interpretation of the RS1 model with bulk gauge field is indeed a purely 4D theory consisting of a strongly-coupled large-\( N \) CFT. The conformal invariance is not exact, the central deformations being that a global symmetry of the CFT is gauged by a gauge field external to the CFT, \( A_\mu(x) \), and there is a fundamental charged scalar field, \( \phi(x) \), which has a coupling to the CFT at the Planck scale of the form, \( \delta \mathcal{L} = \phi(x) \mathcal{O}_{\text{CFT}}(x) \), where \( \mathcal{O}_{\text{CFT}}(x) \) is an irrelevant (marginal) CFT operator if \( m_5 \neq 0 \) (\( m_5 = 0 \)). The Planckian value of the gauge coupling is \( 1/g_4^2 = \tau_{UV} \). The CFT is
also spontaneously broken at the TeV scale. The leading large-$N$ effects of this dual CFT picture, including the running of the gauge coupling of $A_\mu$, are captured by classical effects in the RS1 picture. In particular Eq. (3.4) describes the running, valid between the Planck and TeV scales, due to CFT charged matter. Despite the fact that the CFT itself is strongly self-coupled (although not strongly coupled to the external gauge field), the running can be summarized by a single constant, $b$, as is familiar in one-loop perturbation theory. Here, this fact follows directly from the conformal invariance of the charged matter dynamics. The fact that $b$ has infrared-free sign (whether or not $A_\mu$ is Abelian or non-Abelian, so long as the rank of the gauge group is fixed in the large-$N$ limit) follows from the fact that the charged matter comes in complete large-$N$ representations which always overwhelm asymptotically free contributions to $b$. At the TeV threshold where scale invariance is broken, there is a threshold correction to the gauge coupling $\delta \left(1/g_4^2\right) = \tau_{IR}$.

Note that although the CFT interpretation of Eq. (3.4) is the gauge coupling at momenta $q \ll \text{TeV}$, the log $q$ dependence one would expect from $\phi$ loops (if the $\phi$ mass is tuned to be sufficiently small) is absent. This is because $\phi$ loops are subleading in the large-$N$ expansion, while the classical approximation in the RS1 scenario corresponds to leading order. That is, in large-$N$ the external gauge coupling is scaled to be of order $1/\sqrt{N}$ so that running only arises above $\sim\text{TeV}$ where this suppression is compensated by large-$N$ multiplicities. Below $\sim\text{TeV}$ the dual of our RS1 theory has at most a single charged scalar and so there is no multiplicity enhancement. We will see the running effects due to $\phi$ loops when we include subleading large-$N$ effects, corresponding to loop effects in our model. We now turn to these RS1 loop effects.

4 Quantum Divergence Structure and Renormalization

The most straightforward way to understand the divergence structure of the one-loop vacuum polarization is to view the Feynman diagrams in position space. They can be formally expressed in terms of the $\phi$-propagator, $G(x, y; x', y')$. It satisfies the defining equation,

$$\Box_{(x,y)}G(x, y; x', y') = \delta^4(x - x')\delta(y - y'),$$

(4.1)

where $\Box$ is the d’Alembertian in the 5D warped background, subject to the orbifold boundary conditions,

$$\partial_y G_{\text{boundaries}} = 0.$$  

(4.2)

[^2]: Slightly irrelevant perturbations at the Planck scale can stabilize the resulting Goldstone boson of the scale symmetry. This is the dual of the Goldberger-Wise mechanism. [8]
Recall that we are taking brane-localized terms to be negligible within one-loop diagrams. The Feynman diagrams can only diverge when the initial and final points in a propagator coincide in the Feynman position-integral. Thus all divergences must be local and must correspond to either local bulk terms or local brane terms. To determine what the relevant local divergence structures are it is useful to use the fact that for short-distance propagation in the vicinity of divergences the finite AdS radius of curvature is irrelevant and therefore the propagators can be approximated by their flat space equivalents. For bulk divergences we need only consider the 5D gauge theory in 5D Minkowski spacetime, while for brane-localized divergences we need only consider orbifolds of 5D Minkowski spacetime. Once the flat space divergences are determined their warped equivalents are obtained by inserting the metric dependence using general covariance (also see the discussion in reference \[16\]). It is simpler to euclideanize the whole problem and replace 5D Minkowski space by 5D Euclidean space, where the bulk propagator (away from any orbifolds) is $1/|X - X'|^3$ for short distances.

The one-loop vacuum polarization is given by the sum of the two diagrams in Fig. 1. We will first examine potential bulk divergences by considering the contributions of spacetime vertices away from orbifolds. Fig. 1B contains a singular factor $G(x, y; x, y)$. We can write this as a limit of point-splitting, \[\lim_{x', y' \rightarrow x, y} G(x, y; x', y') \], whereupon we see that the diagram is cubically divergent. It is also straightforward to see that as the two vertices in Fig. 1A approach each other in the Feynman integral, this diagram is also cubically divergent. We can regulate the divergences gauge-invariantly using the standard method of Pauli-Villars (PV) regularization, with a set of scalar regulator fields $\psi$, some with fermionic statistics, having the same form of action as $\phi$,

$$S_{\text{bulk}} = \int d^4x dy \sqrt{-G} \left( D^M \psi (D_M \psi) \right) - \Lambda_5^2 |\psi|^2 \right). \quad (4.3)$$

This cut-off represents the unknown physics due to bulk states with large 5D masses which have been integrated out to yield our model as an effective field theory. We will consider $\Lambda_5 \gg k, m_5$. We must understand the extent to which our results are sensitive to this unknown physics. By standard power-counting it is clear that the local cubic bulk divergences in Fig. 1 multiply $A_M^2$. Since our regulator is gauge-invariant such divergences cancel between the two diagrams as usual. Power-counting and gauge invariance now shows that there remains a single linear divergence in Fig. 1 multiplying $F^2_{MN}$.

Now let us consider what divergences emerge when the Feynman vertices approach an orbifold fixed hyperplane \[19\],\[20\]. We can write the propagator between two points $X, X'$ for an orbifolded 5D Euclidean space as a superposition of a pure Euclidean space propagator for those two points and a pure Euclidean space propagator from the $Z_2$ image of $X$ in the orbifold.
"mirror" to $X'$,

$$G(X, X') = \frac{1}{2|X - X'|^3} + \frac{1}{2|X_P - X'|^3},$$

(4.4)

where $X_P$ is just the "mirror image" of $X$ in the orbifold fixed hyperplane. We then find that beyond the expected bulk divergence discussed above, which persists in the vicinity of the brane, Fig. II gives rise to a new type of divergence from cross terms involving the product (of derivatives) of $1/|X - X'|^3$ with $1/|X_P - X'|^3$. By power-counting, these can only lead to logarithmic or quadratic divergences in the Feynman integral when both $X$ and $X'$ coincide on the orbifold fixed hyperplane. The Pauli-Villars regularization cuts off these divergences as well and gauge invariance again forces the quadratic divergences multiplying $A_\mu^2 \delta(y)$ to vanish, leaving only a logarithmic divergence multiplying $F_{\mu\nu}^2 \delta(y)$.

Technically, cubic and quadratic divergences require three regulators fields. However, since these divergences are gauge non-invariant and cancel we need only use a single PV field to regulate the remaining gauge invariant linear and logarithmic divergences. We will do this from now on.

It is now simple to translate the 5D orbifolded Euclidean space divergence structures to the RS1 spacetime. To this we can add the finite parts of the effective action for the gauge field up to one-loop order. The result is

$$\Gamma = \int d^4 x d\delta \sqrt{-G} \Lambda F_{MN} F^{MN} + \int d^4 x \sqrt{-g_{UV}} \left(-\frac{b_1 \log \Lambda}{4} F_{\mu\nu} F^{\mu\nu}\right)$$

$$+ \int d^4 x \sqrt{-g_{IR}} \left(-\frac{b_1 \log \Lambda}{4} F_{\mu\nu} F^{\mu\nu}\right) + \text{finite non-local one-loop corrections}. \quad (4.5)$$

We have not computed the precise coefficient $c$ of the linear divergence because it is highly regularization-scheme dependent. In any scheme it is a number of $O(1/24\pi^3)$, a 5D loop factor. Brane-localized logarithmic divergences are however not regularization dependent since the logarithms must contain finite mass/energy scales in order to balance dimensions. Thus the coefficients must be physical and we have written the results in terms of the $\beta$-function coefficient for purely 4D massless scalar QED, $b_4 = 1/(24\pi^2)$. 

Figure 1: One-loop diagrams contributing to the vacuum polarization

(A) 

(B)
Eq. (4.5) demonstrates that all one-loop cut-off sensitivity can be eliminated by renormalization of couplings we have already included,

\[
\frac{1}{g_5^2 R(k)} \equiv c(\Lambda - k) + \frac{1}{g_5^2},
\]

\[
\tau_{UV(IR)} R(k) \equiv \frac{b_k}{4} \log \frac{\Lambda}{k} + \tau_{UV(IR)}.
\] (4.6)

The only sensitivity to the unknown massive physics represented by the cut-off is parameterized by these renormalized values.

In flat 5D space, a similar linear $\Lambda$ dependent correction to $1/g_5^2$ is usually referred to as “power-law running” [21]. In contrast, in RS1, this leads to $\delta (1/g_5^2) \sim c\Lambda r_c \sim c\Lambda/k \log [M_4/\text{TeV}]$, which appears as a one-loop Planckian logarithm; this logarithm has the same origin as in the case of the tree-level coupling.

We can now write a general form for the one-loop corrected gauge coupling in the effective 4D theory below $m_{KK}$:

\[
\frac{1}{g_5^2(q)} = \tau_{UV} R(k) + \tau_{IR} R(k) + \frac{\pi r_c}{g_5^2 R(k)} + f(q, r_c, m_5, k),
\] (4.7)

where $f$ is finite as $\Lambda \to \infty$. This limit is a reasonable approximation since we are considering $\Lambda \gg k, m_5$.

We now turn to calculating the dominant behavior of $f$. Since it is finite, we can no longer use 5D locality and the simplifications of relating AdS$_5$ locally to 5D Minkowski space. Therefore it is no longer profitable to use a position space analysis. Instead we make use of the KK decomposition of all states to exploit the preserved 4D Poincare symmetry of the RS1 background.

## 5 KK mode analysis

### 5.1 Set-up

To calculate the renormalized gauge coupling in the 4D effective theory at one-loop order, we consider the sum of 4D vacuum polarization diagrams with charged (physical + PV) KK states in the loop and the zero-mode of the gauge field on external legs (with $q \ll \text{TeV}$) [12]. The couplings in each diagram are completely fixed by 4D gauge invariance, and the signs of each diagram fixed by the Bose (Fermi) statistics of the physical (PV) charged fields. Thus, only the 4D mass spectrum is needed to compute the diagrams. There are two potential sources of UV divergence. Each 4D diagram has standard divergences, while there can be a further
divergence in the sum over KK towers. However, these divergences must be cut off by the PV regularization as seen in section 4. The basic idea is to pair up each physical mode contribution with a PV mode contribution: this pair gives a finite one-loop contribution depending only on the masses of the modes. The fact that PV regularization provides a complete cut-off in position space then implies that the sum over all such pairs also converges.

We first summarize the results for three regimes of $m_5$: $m_5 \ll q \ll m_{KK} \sim O(\text{TeV})$, $m_{KK} \ll m_5 \ll k$ and $m_5 \gg k$. We then derive these results by combining a detailed mode analysis with our earlier study of UV divergences in section 4. While the results are very simple in form (for example, the terms sensitive to $r_c$ are linear), they do not follow entirely from simple general considerations. Our detailed mode analysis appears necessary for proof.

5.2 Summary of results

For $m_5 \ll q \ll m_{KK} \sim O(\text{TeV})$, we get (cf. Eq. (4.17))

$$f(q,r_c,m_5,k) = b_4 \left( \log \frac{k}{q} + \xi k \pi r_c + O(1) \right),$$

(5.1)

where $\xi$ is a constant of order one. Note that the $\xi$-dependent effect can be renormalized away by a straightforward modification of first line of Eq. (4.6). By “$O(1)$” we refer to terms which are insensitive to any of our formal large parameters, for example these terms are bounded as $\Lambda/k \to \infty$ or $k \pi r_c \to \infty$.

For $\text{TeV} \ll m_5 \ll k$, we get

$$f(q,r_c,m_5,k) = b_4 \left( \log \frac{k}{m_5} - \frac{m_5^2}{8k} \pi r_c + \xi k \pi r_c + O(1) \right).$$

(5.2)

Note that the “$\xi$” appearing in Eqs. (5.1) and (5.2) are the same, but the $O(1)$ terms are different.

This case will be very important when we study GUTs because we will encounter $X$, $Y$ gauge bosons with bulk masses $m_5 \sim k$. Reference [12] considered the case where the PV mass $\Lambda \ll k$ and found the same dependence on $\Lambda$ as the $m_5$ dependence in Eq. (5.2). The relevance of such a mass scale smaller than $k$ for grand unification, although in a different scenario than [14], was also pointed out in [12].

The case $m_5 \gg k$ is straightforward. By reasoning along the same lines as in section 4 for the cut-off dependence, we get

$$f(q,r_c,m_5,k) = \frac{b_4}{2} \log \frac{k}{m_5} + c (k - m_5) \pi r_c.$$

(5.3)

We will not discuss this case any further.
5.3 Derivation of results

We first need to discuss the KK spectrum for physical and PV charged modes.

For a scalar with 5D mass \( m_5 \), the classical wave equation of motion (neglecting \( \sigma_{UV,IR} \) terms in Eq. (2.2) at this order) determines the spectrum of KK mass eigenvalues \[22, 23\]:

\[
b_\nu \left( \frac{m_n}{k} \right) = b_\nu \left( \frac{m_n}{k} e^{k\pi r_c} \right),
\]

(5.4)

where

\[
b_\nu (x) = \frac{(2 - \nu)J_\nu (x) + xJ_{\nu-1} (x)}{(2 - \nu)Y_\nu (x) + xY_{\nu-1} (x)}
\]

(5.5)

and \( \nu = \sqrt{4 + \frac{m_5^2}{k^2}} \). For the PV KK spectrum, \( \Lambda \) replaces \( m_5 \).

5.3.1 \( m_5 \ll q \ll m_{KK} \)

The spectrum

There are four distinct regions of \( m_n \) for physical and PV fields: (a) \( m_n \sim \Lambda e^{-k\pi r_c} \), (b) \( \Lambda e^{-k\pi r_c} \ll m_n \ll k \), (c) \( k \sim m_n \sim \Lambda \) and (d) \( m_n \gg \Lambda \). In the following, \( \nu \approx 2 \) for physical modes and \( \nu = \sqrt{4 + \Lambda^2/k^2} \) for PV modes.

(a) \( m_n \sim \Lambda e^{-k\pi r_c} \)

In this region, because \( e^{k\pi r_c} \gg 1 \), we get \( m_n \ll k \). So, the number of eigenvalues in this region can only depend on \( \Lambda/k \) and not on \( k\pi r_c \), because the LHS of Eq. (5.4) is approximately zero. We denote this number by \( N^{\text{phys}}_{(a)} \) and \( N^{\text{PV}}_{(a)} \), respectively.

It is straightforward to check that there is a single mode with 4D mass \( \ll m_{KK} \), where the arguments of both LHS and RHS of Eq. (5.4) are small. This mode is the lightest physical state (which is a zero mode for \( m_5 = 0 \)). For \( 0 < m_5 \ll m_{KK} \) its mass is

\[
m_0 \approx \frac{m_5}{\sqrt{2}}.
\]

(5.6)

(b) \( \Lambda e^{-k\pi r_c} \ll m_n \ll k \)

Here, the argument of the LHS of Eq. (5.4) \( \ll 1 \), where we use the approximation

\[
b_\nu (x) \approx \frac{2(x/2)^{2\nu} (1 + 2/\nu) (1 - \nu) \sin \nu \pi \Gamma(1 - \nu)}{\Gamma(\nu) [x^2 - 2(2 - \nu)(1 - \nu)]}.
\]

(5.7)

The RHS of Eq. (5.4) can be approximated by \( \cot \left[ m_n/k e^{k\pi r_c} - \pi/2 + \pi/4 + O \left(k e^{-k\pi r_c}/m_n\right)\right] \); where the error term is \( \nu \)-dependent. The KK mass eigenvalues are therefore given by \[23\]

\[
m_n = \left( n - \frac{3}{4} + \frac{1}{2} \nu + O \left(n^2 e^{-2k\pi r_c}\right) + O \left(\frac{1}{n}\right)\right) \pi ke^{-k\pi r_c},
\]

(5.8)
where the error terms are \( \nu \)-dependent. Note that in this region \( n \, e^{-k \pi r_c} \ll 1 \).

(c) \( k \sim m_n \sim \Lambda \)

Here, the LHS of Eq. (5.4) is a piecewise smooth function, i.e., it has discontinuities at zeroes of the denominator of \( b(\nu/m_n/k) \), but is otherwise smooth. Since it is independent of \( k \pi r_c \), the number of discontinuities in each tower in this region depends only on \( \Lambda/k \). The RHS of Eq. (5.4) is approximated \( \cot \left( m_n/k \, e^{k \pi r_c} - \pi/2 \, \nu + \pi/4 \right) \) as in (b).

Let us divide the 4D mass spectrum in this region into “bins” of size equal to the period of the RHS of Eq. (5.4), i.e., \( \pi k e^{-k \pi r_c} \). We see that each tower has one eigenvalue per bin, except possibly at discontinuities of the LHS, where there may be zero or two eigenvalues in a bin. We will refer to bins with such a discontinuity (for either physical or PV case) as “exceptional bins”, and the modes in these bins as “exceptional modes”. Thus, the number of exceptional modes, denoted by \( N^{\text{phys}}_{(\nu)} \) and \( N^{\text{PV}}_{(\nu)} \), respectively, depend on \( \Lambda/k \) and not on \( k \pi r_c \). Note that this one-eigenvalue-per-bin rule holds in region (b), with no exceptions, by Eq. (5.8).

(d) \( m_n \gg \Lambda \)

Here, \( b_{\nu}(m_n/k) \approx \cot (m_n/k - \pi/2 \, \nu + \pi/4) \) while \( b_{\nu}(m_n/k \, e^{k \pi r_c}) \) is approximately \( \cot \left( m_n/k \, e^{k \pi r_c} - \pi/2 \, \nu + \pi/4 \right) \). Thus, the eigenvalues are given by

\[
m_n \approx \frac{k \pi n}{e^{k \pi r_c} - 1}.
\]

The corrections to Eq. (5.9) are \( O(1/n) \) for large \( n \) and are \( \nu \)-dependent.

Pairing modes

Having discussed the mass spectrum in the four regions of \( m_n \), we now turn to the question of how to pair one-loop contributions of physical and PV modes in each region. From our analysis in section 4 we know that the one-loop contribution summed over PV and physical modes in all regions is UV finite.

In region (d), we pair the one-loop contributions of physical and PV modes with the same \( n \). Using Eq. (5.9) and approximating the sum over modes as an integral, we get a finite and \( r_c \)-independent contribution to \( f \) (cf. Eq. (4.7)) from the infinite number of modes in region (d). The error in this approximation is \( O(e^{-k \pi r_c}) \).

Thus, the one-loop contribution from the remaining finite number of (physical + PV) modes in the regions (a), (b) and (c) must also be UV finite, i.e., the logarithmic divergences have to cancel between physical and PV diagrams. This is possible if and only if the number of modes in the regions (a)-(c) is the same for the two towers. Since the one-eigenvalue-per-bin rule is valid in region (b), this region has the same number of physical and PV modes which can be paired up. Also, the modes in non-exceptional bins in region (c) can be paired up. Thus, the
sum of the number of modes in region (a) and the number of exceptional modes in region (c) has to be the same for physical and PV towers, i.e., $N_{\text{phys}}^{(a)} + N_{\text{phys}}^{(c)} = N_{\text{PV}}^{(a)} + N_{\text{PV}}^{(c)}$.

In region (b), we pair the contribution of physical and PV modes in the same bin. Using Eq. (5.8) and approximating the sum over these bins as an integral gives

$$\delta_{\text{1-loop}} \left(1/g_4^2(q)\right) \approx \frac{\Lambda}{k^2} b_4 \left( k \pi r_{c} \, \text{Frac} \left( \frac{\Lambda}{2k} - 1 \right) + s \left( \frac{\Lambda}{k}, k \pi r_{c} \right) \right),$$

(5.10)

where $\text{Frac}(x)$ denotes the fractional part of $x$ and $s$ can be bounded by a function of $\Lambda/k$ alone.

In region (c), we do not have a good analytic approximation for the spectrum other than the one-eigenvalue-per-bin rule in non-exceptional bins. Using this rule, the sum over these bins can be easily bounded:

$$\delta_{\text{1-loop}} \left(1/g_4^2(q)\right) < b_4 \left( \log \frac{\Lambda}{k} + O(1) \right).$$

(5.11)

Finally, we consider exceptional modes in region (c) and all modes in region (a) together. The number of physical modes is $N_{\text{phys}}^{(a)} + N_{\text{phys}}^{(c)}$ and number of PV modes is $N_{\text{PV}}^{(a)} + N_{\text{PV}}^{(c)}$. Recall that they are equal and we can pair them arbitrarily. If both modes in a pair come from the same region, the one-loop logarithm is $r_{c}$ independent. If not, the one-loop logarithm is $\propto r_{c}$. The lightest physical mode (the zero-mode as $m_{5} \to 0$) is an exception to this rule because, unlike other modes in region (a), $m_{0} \ll q \ll m_{KK}$ (see Eq. (5.6)). Therefore, we get

$$\delta_{\text{1-loop}} \left(1/g_4^2(q)\right) \approx b_4 \left( k \pi r_{c} \, u \left( \frac{\Lambda}{k} \right) + v \left( \frac{\Lambda}{k} \right) + \log k/q \right),$$

(5.12)

where $u, v$ are $r_{c}$ independent because the $N$’s are independent of $r_{c}$. We have dropped power-suppressed $q$-dependence in all contributions. Similarly, we have dropped power suppressed $m_{5}$ dependence since $m_{5} \ll q$. For the lightest mode, $q$ dependence is non-analytic and is therefore retained. This log $q$ dependence is entirely determined within the low energy 4D effective theory where we have a single light charged scalar.

Putting together contributions from all regions and using the $\Lambda$ sensitivity already determined in Eqs. (4.6) and (4.7), Eq. (5.1) follows.

### 5.3.2 TeV $\ll m_{5} \ll k$

**The spectrum**

The PV spectrum is as before.

The physical spectrum in the four regions is as follows (in the following $\nu = \sqrt{4 + m_{5}^2/k^2} \approx 2 + m_{5}^2/(4k^2)$).
(a) \( m_n \sim \Lambda e^{-k \pi r_c} \)

The eigenvalues are shifted relative to those in the case \( m_5 \ll \text{TeV} \) by \( \sim m_5^2/k e^{-k \pi r_c} \ll k e^{-k \pi r_c} \sim \text{spacing between eigenvalues} \), except that there is no analog of the light mode (cf. Eq. (5.6)) in this region. Thus,

\[
\begin{align*}
N_{\text{phys}}^{(a)} (\text{TeV} \ll m_5 \ll k) &= N_{\text{phys}}^{(a)} (m_5 \ll \text{TeV}) - 1, \\
N_{\text{PV}}^{(a)} (\text{TeV} \ll m_5 \ll k) &= N_{\text{PV}}^{(a)} (m_5 \ll \text{TeV}).
\end{align*}
\]

(5.13)

(b) \( \Lambda e^{-k \pi r_c} \ll m_n \ll k \)

Previously, for \( m_5 \ll \text{TeV} \), there were no exceptional bins in this region. Now, however, the LHS of Eq. (5.4) has a single discontinuity as can be seen from Eq. (5.7) and the RHS of Eq. (5.4) is approximately \( \cot \left[ m_n/k e^{k \pi r_c} - \pi/2 + \pi/4 + O \left( k e^{-k \pi r_c}/m_n \right) \right] \). This discontinuity gives rise to a single exceptional physical mode near \( m_5/\sqrt{2} \). Far away from the discontinuity, \( m_5 \ll m_5 \) or \( m_n \gg m_5 \), Eq. (5.8) continues to hold. For \( m_n \sim O(m_5) \), other than the exceptional mode, all we will use is that the one-eigenvalue-per-bin rule holds.

(c) \( k \sim m_n \sim \Lambda \)

As before, the modes follow the one-eigenvalue-per-bin rule with some \( N_{\text{phys}}^{(c)} \), \( N_{\text{PV}}^{(c)} \) “exceptions”.

(d) \( m_n \gg \Lambda \)

The spectrum (Eq. (5.9)) and analysis of corrections is as before.

**Pairing modes**

The pairing in region (d) follows just as before. The sum over non-exceptional modes in region (b) is given by

\[
\delta_{\text{1-loop}} \left( 1/g_4^2(q) \right) \approx b_4 \left( k \pi r_c \left[ \frac{\Lambda}{2k} - 1 \right] - \frac{m_5^2}{8k^2} \right) + t \left( \frac{\Lambda}{k}, \frac{m_5}{k}, k \pi r_c \right),
\]

(5.14)

where \( t \) is bounded by a function of \( \Lambda/k \) alone. Although Eq. (5.8) does not hold for \( m_n \sim O(m_5) \) as discussed above, the contribution of these modes can easily be bounded and shown to affect only \( t \). The bound on the contribution of non-exceptional modes in region (c) is unchanged from the \( m_5 \ll \text{TeV} \) case (Eq. (5.11)).

Now consider modes in region (a) and exceptional modes in regions (c) and (b). The number of physical and PV modes are equal. We will pair the exceptional physical mode in region (b)

\footnote{The eigenvalues in the case \( m_5 \ll \text{TeV} \) are given by \( k e^{-k \pi r_c} \times \text{zeros of } J_1(x) \).}

\footnote{This is analogous to the PV “zero-mode” which plays a central role in the analysis of [12].}
with an exceptional PV mode in region (c). All the modes in region (a) and the remaining exceptional modes in region (c) can be paired arbitrarily. All these pairings yield

\[ \delta_{1\text{-loop}} \left( 1/g_4^2(q) \right) \approx b_4 \left( k\pi r_c \right) \left( \Lambda/k \right) + w \left( \Lambda/k \right) + \log k/m_5 . \]  

(5.15)

Note that because of Eq. (5.13), \( u \) appearing in Eqs. (5.15) and (5.12) is the same. The contribution of the lightest mode for \( m_5 \ll q \ll \text{TeV} \) is replaced by the contribution of the exceptional mode in region (b). The functions \( w \) and \( v \) in Eqs. (5.15) and (5.12) need not be equal. Using the \( \Lambda \) sensitivity determined in Eqs. (4.6) and (4.7) and above observations, Eq. (5.2) follows.

6 The dual CFT: subleading corrections

As discussed earlier, the RS model in classical approximation is dual to a 4D CFT picture at leading order in a large-\( N \) expansion. The one-loop RS corrections are then dual to subleading large-\( N \) corrections [5]. Our one-loop RS results, Eqs. (4.6), (4.7), (5.1) and (5.2) can be reexpressed in the dual form:

\[ \frac{1}{g_4^2(q)} = \tilde{\tau}_{\text{UV}} + \hat{b} \log \left[ \frac{O(M_4)}{\text{TeV}} \right] \right] + \tilde{\tau}_{\text{IR}} + b_4 \log \frac{\text{TeV}}{q}, \text{ for } m_5 \ll q \]

\[ = \tilde{\tau}_{\text{UV}} + \hat{b} \log \left[ \frac{O(M_4)}{\text{TeV}} \right] \right] + \tilde{\tau}_{\text{IR}}, \text{ for } m_5 \gg \text{TeV}. \]  

(6.1)

In the dual interpretation, \( \tilde{\tau}_{\text{UV}} \) or \( \hat{\tau}_{\text{UV}} \) sets the Planckian gauge couplings, the CFT charged matter and \( \phi(x) \) lead to logarithmic running down to the TeV threshold with \( \beta \)-function coefficients \( \hat{b} \) or \( \hat{b} \), \( \tilde{\tau}_{\text{IR}} \) or \( \hat{\tau}_{\text{IR}} \) is a TeV-threshold correction and \( b_4 \log \text{TeV}/q \) is the running due to \( \phi \) in the case where its mass is much smaller than \( q \). All of the coefficients receive sub-leading corrections (compared to the classical approximation/leading order in large-\( N \) expansion in section [3]) except for \( b_4 \) which vanishes at leading-order. It is not yet known how to compute the detailed form of these corrections, for example, the \( m_5 \) dependence (in the case \( m_5 \gg \text{TeV} \)) directly from CFT considerations, but the RS picture allows us to estimate them, as we have done in section [5].

Note that it is significant that the loop corrections in the RS model admit the dual interpretation. For example, if Eqs. (5.11) and (5.12) had contained a contribution \( \propto \sqrt{k\pi r_c} \sim \sqrt{\log [O(M_4)/\text{TeV}]} \), the CFT interpretation would fail.
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