CESÀRO SUMMABILITY AND LEBESGUE POINTS OF HIGHER DIMENSIONAL FOURIER SERIES

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Abstract. We give four generalizations of the classical Lebesgue’s theorem to multi-dimensional functions and Fourier series. We introduce four new concepts of Lebesgue points, the corresponding Hardy-Littlewood type maximal functions and show that almost every point is a Lebesgue point. For four different types of summability and convergences investigated in the literature, we prove that the Cesàro means \( \sigma_n^xf \) of the Fourier series of a multi-dimensional function converge to \( f \) at each Lebesgue point as \( n \to \infty \).

1. Introduction. The almost everywhere convergence of the Fejér means \([17]\) of trigonometric Fourier series of a one-dimensional integrable was proved by Lebesgue \([34]\), more exactly,

\[
\sigma_n f(x) := \sum_{k=-n}^{n} \left(1 - \frac{|k|}{n}\right) \hat{f}(k) e^{ikx} \rightarrow f(x)
\]

for almost every \( x \in T \) as \( n \to \infty \), where \( T \) denotes the torus and \( \hat{f}(k) \) the \( k \)th Fourier coefficient. Moreover, he proved that the convergence holds at every Lebesgue point of \( f \). Some years later M. Riesz \([41]\) generalized this theorem for the so called Cesàro means of one-dimensional integrable functions.

For two-dimensional functions, the Fejér means

\[
\sigma_{n_1,n_2} f(x,y) = \sum_{k_1=-n_1}^{n_1} \sum_{k_2=-n_2}^{n_2} \left(1 - \frac{|k_1|}{n_1}\right) \left(1 - \frac{|k_2|}{n_2}\right) \hat{f}(k_1,k_2) e^{ik_1x_1} e^{ik_2x_2}
\]

of a function \( f \in L_1(\log L)(T^2) \) converge almost everywhere to \( f \) as \( n \to \infty \), i.e., as \( n_1,n_2 \to \infty \) (see Zygmund \([60]\)). Considering the convergence on the diagonal or on a cone, only, we get better convergence results. Marcinkiewicz and Zygmund \([37]\) proved that for \( f \in L_1(T^2) \), \( \sigma_{n_1,n_2} f \to f \) almost everywhere provided that \( n \) is in a cone and \( n \to \infty \), i.e., \( \tau^{-1} \leq n_1/n_2 \leq \tau \) for some \( \tau \geq 1 \). The almost everywhere convergence was generalized for \( d \)-dimensional functions and for Cesàro means in \([50]\), however, the set of convergence was unknown.

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In this paper, we give four generalizations of the one-dimensional Lebesgue theorem to Cesàro means \( \sigma_n^\alpha f \) of multi-dimensional functions. The Cesàro summability includes the Fejér summation (when \( \alpha = 1 \)) and was investigated in a great number of papers and books (see e.g. Gát [20, 21, 22], Goginava [23, 24, 25], Simon [43, 44], Nagy, Persson, Tephnadze and Wall [39, 40], Weisz [51, 53] and Zygmund [60]). We generalize the Lebesgue points in four different ways and introduce the so called strong Lebesgue points and the (strong) \( \omega \)-Lebesgue points, where \( \omega > 0 \).

For every type of Lebesgue point, we introduce a different maximal function and study its boundedness on \( L^p \) spaces. This boundedness result shows that almost every point is a Lebesgue point of \( f \). We consider four different types of Cesàro summability, the unrestricted and restricted rectangular summability, the \( \ell_\infty \) and \( \ell_1 \)-summability. The main result is that for each generalization

\[
\lim_{n \to \infty} \sigma_n^\alpha f(x) = f(x)
\]

for each corresponding Lebesgue point and \( f \in L_1(\mathbb{T}^d) \) (in case of the unrestricted rectangular summability \( f \in L_1(\log L)^{d-1}(\mathbb{T}^d) \)). This implies the convergence of the Cesàro means almost everywhere.

2. One-dimensional Fourier series. A measurable function \( f \) is in \( L^p(\mathbb{T}) \) if the norm

\[
\|f\|_p := \left( \int_{\mathbb{T}} |f|^p \, d\lambda \right)^{1/p} < \infty \quad (1 \leq p < \infty),
\]

with the usual modification for \( p = \infty \), where \( \lambda \) denotes the Lebesgue measure and we identify the torus \( \mathbb{T} \) with \([-\pi, \pi]\). The measure of a set \( H \) is also denoted by \(|H|\).

The \( k \)th Fourier coefficient and the \( n \)th partial sum of an integrable function \( f \in L_1(\mathbb{T}) \) are defined by

\[
\hat{f}(k) = \frac{1}{2\pi} \int_{\mathbb{T}} f(x) e^{-ikx} \, dx \quad (k \in \mathbb{Z})
\]

and

\[
s_n f(x) := \sum_{k=-n}^{n} \hat{f}(k) e^{ikx},
\]

respectively. The following theorem due to Riesz [41] is a fundamental result and it can be found in most books about trigonometric Fourier series (e.g., Zygmund [60], Bary [2], Torchinsky [49] or Grafakos [26]).

**Theorem 2.1.** If \( f \in L_p(\mathbb{T}) \) for some \( 1 < p < \infty \), then

\[
\sup_{n \in \mathbb{N}} \|s_n f\|_p \leq C_p \|f\|_p
\]

and

\[
\lim_{n \to \infty} s_n f = f \quad \text{in the } L_p(\mathbb{T})\text{-norm}.
\]

In this paper the constants \( C \) and \( C_p \) may vary from line to line. One of the deepest results in harmonic analysis is Carleson’s theorem, that the partial sums of the Fourier series converge almost everywhere to \( f \in L_p(\mathbb{T}) \) (1 < \( p \leq \infty \)) (see e.g. Carleson [7], Hunt [29], Arias de Reyna [1], Grafakos [26], Muscalu and Schlag [38], Lacey [33] or Demeter [10]).
Theorem 2.2. If \( f \in L_p(T) \) for some \( 1 < p < \infty \), then
\[
\left\| \sup_{n \in \mathbb{N}} |s_n f| \right\|_p \leq C_p \|f\|_p
\]
and if \( 1 < p \leq \infty \), then
\[
\lim_{n \to \infty} s_n f = f \quad \text{a.e.}
\]

Du Bois Reymond [11] and Fejér [18] proved the existence of a continuous function \( f \in C(T) \) and a point \( x_0 \in \mathbb{T} \) such that the partial sums \( s_n f(x_0) \) diverge as \( n \to \infty \). Kolmogorov gave an integrable function \( f \in L_1(T) \), whose Fourier series diverges almost everywhere or even everywhere (see Kolmogorov [31, 32], Zygmund [60] or Grafakos [26]). Theorems 2.1 and 2.2 do not hold for \( p = 1 \) or \( p = \infty \). In these endpoint cases, we have to consider a summability method. The most known summability method is the Fejér summation. In this paper, we study its generalization, the so called Cesàro summability.

### 3. One-dimensional Cesàro summability

For \( \alpha \neq -1, -2, \ldots \) and \( n \in \mathbb{N} \), let
\[
A_n^\alpha := \binom{n + \alpha}{n} = \frac{(\alpha + 1)(\alpha + 2) \cdots (\alpha + n)}{n!}.
\]
It is easy to see that \( A_n^0 = 1, A_n^{-1} = 1 \) and \( A_n^1 = n + 1 \) (\( n \in \mathbb{N} \)).

For \( f \in L_1(T) \), \( n \in \mathbb{N} \) and \( \alpha > 0 \), let us define the Cesàro means \( \sigma_n^\alpha f \) of the Fourier series of \( f \) and the Cesàro kernel \( K_n^\alpha \) by
\[
\sigma_n^\alpha f(x) := \frac{1}{A_n^{\alpha - 1}} \sum_{k=1}^{n} A_{n-1}^{\alpha - 1} \hat{f}(k) e^{ikx}
\]
and
\[
K_n^\alpha(t) := \frac{1}{A_n^{\alpha - 1}} \sum_{k=1}^{n} A_{n-1}^{\alpha - 1} e^{ikt},
\]
respectively. The Cesàro means are also called \((C, \alpha)\)-means. It is easy to see that
\[
\sigma_n^\alpha f(x) = \frac{1}{2\pi} \int_{\mathbb{T}} f(x - t) K_n^\alpha(t) \, dt
\]
and
\[
\sigma_n^\alpha f(x) = \frac{1}{A_n^{\alpha - 1}} \sum_{j=0}^{n-1} A_{n-1}^{\alpha - 1} s_j f(x).
\]
If \( \alpha = 1 \), then we get back the Fejér means:
\[
\sigma_n f(x) := \sigma_n^1 f(x) = \sum_{k=-n}^{n} \left( 1 - \frac{|k|}{n} \right) \hat{f}(k) e^{ikx} = \frac{1}{n} \sum_{j=0}^{n-1} s_j f(x).
\]

The next theorem extends Theorem 2.1.

**Theorem 3.1.** Suppose that \( 0 < \alpha < \infty \) and \( 1 \leq p < \infty \). If \( f \in L_p(T) \), then
\[
\sup_{n \in \mathbb{N}} \|\sigma_n^\alpha f\|_p \leq C_\alpha \|f\|_p
\]
and
\[
\lim_{n \to \infty} \sigma_n^\alpha f = f \quad \text{in the } L_p(T)\text{-norm.}
\]
If \( f \) is a continuous function, then the theorem holds also for \( p = \infty \).
In the proof, we use the important inequality
\[ K_\alpha^n(t) \leq C \min \left( n, \frac{1}{n^{\alpha}|t|^{\alpha+1}} \right) =: g_\alpha^n(t), \quad (n \in \mathbb{N}, t \in (-\pi, \pi), t \neq 0) \quad (1) \]
(see Lebesgue [34], Riesz [41] or Zygmund [60]). From this and Minkowski’s inequality, it follows that
\[ \sup_{n \in \mathbb{N}} \int_{\mathbb{T}} |K_\alpha^n| \, d\lambda \leq C_\alpha, \]
which implies the proof.

The almost everywhere convergence is much more complicated. To this end, first we introduce a maximal function. For an integrable function \( f \) of one variable, the Hardy-Littlewood maximal function is defined by
\[ Mf(x) := \sup_{h>0} \frac{1}{2h} \int_{-h}^{h} |f(x-t)| \, dt. \]
It is well known (see e.g. Stein [47]) that the maximal operator \( M \) is of weak type \((1,1)\) and bounded on \( L^p(\mathbb{T}) \) \((1 < p \leq \infty)\), i.e.,

**Theorem 3.2.** We have
\[ \sup_{\rho>0} \rho \lambda(Mf > \rho) \leq 3\|f\|_1 \quad (f \in L_1(\mathbb{T})). \]
Moreover, if \( 1 < p \leq \infty \), then
\[ \|Mf\|_p \leq C_p \|f\|_p \quad (f \in L_p(\mathbb{T})). \]

Now we introduce the concept of Lebesgue points. Theorem 3.2 implies that
\[ \lim_{h \to 0} \frac{1}{2h} \int_{-h}^{h} f(x-t) \, dt = f(x) \]
for almost every \( x \in \mathbb{T} \) and \( f \in L_1(\mathbb{T}) \). Thus
\[ \lim_{h \to 0} \frac{1}{2h} \left| \int_{-h}^{h} (f(x-t) - f(x)) \, dt \right| = 0 \]
for almost every \( x \in \mathbb{T} \). Though the definition of Lebesgue points is a stronger condition, we can see in the next theorem that almost every point is a Lebesgue point. A point \( x \in \mathbb{T} \) is called a Lebesgue point of \( f \in L_1(\mathbb{T}) \) if
\[ \lim_{h \to 0} \frac{1}{2h} \int_{-h}^{h} |f(x-t) - f(x)| \, dt = 0. \]

**Theorem 3.3.** Almost every point \( x \in \mathbb{T} \) is a Lebesgue point of \( f \in L_1(\mathbb{T}) \).

One can see easily that if \( x \) is a Lebesgue point of \( f \in L_1(\mathbb{T}) \), then \( f(x) \) and \( Mf(x) \) are both finite. The next theorem can be found e.g. in Riesz [41] or Zygmund [60]. Here we give a simple proof for this theorem.

**Theorem 3.4.** If \( 0 < \alpha < \infty \), then
\[ \lim_{n \to \infty} \sigma_n^\alpha f(x) = f(x) \]
for all Lebesgue points of \( f \in L_1(\mathbb{T}) \).
Proof. First suppose that $0 < \alpha \leq 1$. Set

$$G(u) := \int_{-u}^{u} |f(x - t) - f(x)| \, dt \quad (u > 0).$$

Since $x$ is a Lebesgue point of $f$, for all $\epsilon > 0$, there exists a negative integer $m$ such that

$$\frac{G(u)}{2u} \leq \epsilon \quad \text{if} \quad 0 < u \leq 2^m \pi.$$

We fix this integer $m$. Since

$$\sigma_n^\alpha f(x) - f(x) = \frac{1}{(2\pi)^d} \int_T (f(x - t) - f(x)) K_n^\alpha(t) \, dt,$$

we have

$$|\sigma_n^\alpha f(x) - f(x)| \leq C \int_{-2^m \pi}^{2^m \pi} |f(x - t) - f(x)| |K_n^\alpha(t)| \, dt$$

$$+ C \int_{T \setminus (-2^m \pi, 2^m \pi)} |f(x - t) - f(x)| |K_n^\alpha(t)| \, dt$$

$$=: A_1(x) + A_2(x).$$

Using that $g_n^\alpha$ is non-increasing and its integral is independent of $n$ (see (1)), we estimate $A_1(x)$ by

$$A_1(x) = C \sum_{k = -\infty}^{m} \int_{(-2^k \pi, 2^k \pi) \setminus (-2^{k-1} \pi, 2^{k-1} \pi)} |f(x - t) - f(x)| |K_n^\alpha(t)| \, dt$$

$$\leq C \sum_{k = -\infty}^{m} \sup_{(-2^k \pi, 2^k \pi) \setminus (-2^{k-1} \pi, 2^{k-1} \pi)} |K_n^\alpha| G(2^k \pi)$$

$$\leq C \epsilon \sum_{k = -\infty}^{m} \sup_{(-2^k \pi, 2^k \pi) \setminus (-2^{k-1} \pi, 2^{k-1} \pi)} |g_n^\alpha| \leq C \epsilon \int_T |g_n^\alpha(t)| \, dt \leq C \epsilon.$$

On the other hand, (1) implies

$$A_2(x) \leq C \sup_{T \setminus (-2^m \pi, 2^m \pi)} |K_n^\alpha| \int_{T \setminus (-2^m \pi, 2^m \pi)} |f(x - t) - f(x)| \, dt$$

$$\leq \frac{C}{n^{\alpha 2^m (\alpha + 1)}} (\|f\|_1 + |f(x)|) < \epsilon$$

if $n$ is large enough. The case $1 < \alpha < \infty$ can be handled in the usual way.

Since every point of continuity is a Lebesgue point, we get

**Corollary 1.** Suppose that $0 < \alpha < \infty$, $f \in L_1(T)$ and $f$ is continuous at a point $x$. Then

$$\lim_{n \to \infty} \sigma_n^\alpha f(x) = f(x).$$

Now we generalize these results to higher dimensional Fourier series.
4. Higher dimensional Fourier series. For a set $\mathbb{Y} \neq \emptyset$, let $\mathbb{Y}^d$ be its Cartesian product $\mathbb{Y} \times \ldots \times \mathbb{Y}$ taken with itself $d$ times. We briefly write $L_p(\mathbb{T}^d)$ instead of the $L_p(\mathbb{T}^d, \lambda)$ space equipped with the norm

$$\|f\|_p := \left(\int_{\mathbb{T}^d} |f|^p d\lambda\right)^{1/p} \quad (1 \leq p < \infty),$$

with the usual modification for $p = \infty$, where $\lambda$ denotes the $d$-dimensional Lebesgue measure. Let $\log^+ u := \max(0, \log u)$. The set $L_p(\log L)^k(\mathbb{T}^d)$ $(k \in \mathbb{N}, 1 \leq p < \infty)$ consists all measurable functions $f$ for which

$$\|f\|_{L_p(\log L)^k} := \left(\int_{\mathbb{T}^d} |f|^p (\log^+ |f|)^k d\lambda\right)^{1/p} < \infty.$$ 

Obviously, we get back the $L_p(\mathbb{T}^d)$ spaces if $k = 0$. We have for all $k \in \mathbb{P}$ and $1 \leq p < r \leq \infty$ that

$$L_p(\mathbb{T}^d) \supset L_p(\log L)^k(\mathbb{T}^d) \supset L_p(\log L)^{k-1}(\mathbb{T}^d) \supset L_p(\mathbb{T}^d) \supset L_r(\mathbb{T}^d).$$

We introduce the following notations. For $x = (x_1, \ldots, x_d) \in \mathbb{R}^d$ and $u = (u_1, \ldots, u_d) \in \mathbb{R}^d$ set

$$u \cdot x := \sum_{k=1}^d u_k x_k, \quad \|x\|_q := \left(\sum_{k=1}^d |x_k|^q\right)^{1/q} \quad (1 \leq p < \infty)$$

and

$$\|x\|_\infty := \sup_{k=1,\ldots,d} |x_k|.$$ 

For higher dimensional integrable functions $f$, we define the Fourier coefficients by

$$\hat{f}(k) = \frac{1}{(2\pi)^d} \int_{\mathbb{T}^d} f(x) e^{-i k \cdot x} \, dx \quad (k \in \mathbb{Z}^d).$$

We generalize the one-dimensional partial sums for higher dimensional functions basically in two ways. In the first generalization, we take the sum over the indices $\|k\|_q \leq n$ instead of $k = -n, \ldots, n$, where $1 \leq q \leq \infty$. These sums are called $\ell_q$-partial sums. In the second generalization, we take the sum in each dimension, i.e., over the indices $|k_1| \leq n_1, \ldots, |k_d| \leq n_d$. Here, we call the sums rectangular partial sums. The most natural choices, $q = 2, q = 1, q = \infty$ and the rectangular partial sums are investigated in several papers and books (for $q = 2$, see e.g. Stein and Weiss [48, 47], Davis and Chang [9], Grafakos [26, 27, 28], Lu and Yan [35], Feichtinger and Weisz [15, 16], for $q = 1$, Berens, Li and Xu [5, 6, 4, 58], Weisz [51, 53], for $q = \infty$, Marcinkiewicz [36], Zhizhiashvili [59], Weisz [51, 53], for the rectangular sums, Zygmund [60] and Weisz [51, 53]). Since Cesàro summability is considered only for $q = 1, q = \infty$ and for the rectangular version, in this paper we focus on these cases (see Figure 1).

For $f \in L_1(\mathbb{T}^d)$, the $n$th $\ell_q$-partial sum $s_n^q f$ $(1 \leq q \leq \infty)$ and the $n$th rectangular partial sum of the Fourier series of $f$ are given by

$$s_n^q f(x) := \sum_{k \in \mathbb{Z}^d, \|k\|_q \leq n} \hat{f}(k) e^{i k \cdot x}, \quad (n \in \mathbb{N})$$

and

$$s_nf(x) := \sum_{|k_1| \leq n_1} \cdots \sum_{|k_d| \leq n_d} \hat{f}(k) e^{i k \cdot x}, \quad (n = (n_1, \ldots, n_d) \in \mathbb{N}^d),$$
Figure 1. Regions of the $\ell_q$-partial sums for $d = 2$.

respectively.

**Theorem 4.1.** Let $f \in L_p(\mathbb{T}^d)$ for some $1 < p < \infty$. Then

$$\sup_{n \in \mathbb{N}^d} \|s_n f\|_p \leq C_p \|f\|_p$$

and

$$\lim_{n \to \infty} s_n f = f \quad \text{in the } L_p(\mathbb{T}^d)-\text{norm.}$$

If $q = 1, \infty$, then

$$\sup_{n \in \mathbb{N}} \|s_n^q f\|_p \leq C_p \|f\|_p$$

and

$$\lim_{n \to \infty} s_n^q f = f \quad \text{in the } L_p(\mathbb{T}^d)-\text{norm.}$$

For $n \in \mathbb{N}^d$, $n \to \infty$ means that $n_j \to \infty$ for each $j = 1, \ldots, d$. If $d \geq 2$ and $q = 2$, then the same result holds only for $p = 2$ (see Fefferman [14] or Grafakos [26, p. 743] or Lu and Yan [35, p. 743]).

The analogue of Carleson’s theorem holds also for the $\ell_q$-partial sums with $q = 1, \infty$ in higher dimensions (see Fefferman [12, 13] and Grafakos [26, p. 231]), but it does not hold for the rectangular partial sums.

**Theorem 4.2.** If $q = 1, \infty$ and $f \in L_p(\mathbb{T}^d)$ for some $1 < p < \infty$, then

$$\left\|\sup_{n \in \mathbb{N}} |s_n^q f|\right\|_p \leq C_p \|f\|_p$$

and if $1 < p \leq \infty$, then

$$\lim_{n \to \infty} s_n^q f = f \quad \text{a.e.}$$

For a general function in $L_p(\mathbb{T}^d)$ ($p < 2$), almost everywhere convergence of the $\ell_2$-partial sums is not true if the dimension is sufficiently large (see Stein and Weiss [48, p. 268]). It is an open problem, whether Theorem 4.2 holds for $p = 2$ and $q = 2$. A counterexample, which proves the next result, can be found in Fefferman [13].
Theorem 4.3. There exists a continuous function $f$ such that for the rectangular partial sums $s_n f$,
\[
\lim_{n \to \infty} s_n f(x) = f(x)
\]
does not hold for any $x \in \mathbb{T}^d$.

Since Theorems 4.1 and 4.2 are not true for $p = 1$, we consider again Cesàro summability in the endpoint case.

5. Higher dimensional Cesàro summability. For $f \in L_1(\mathbb{T}^d)$, $n \in \mathbb{N}$, $\alpha \geq 0$ and $q = 1$ or $q = \infty$, the $n$th $\ell_q$-Cesàro means $\sigma_{n}^{q, \alpha} f$ of the Fourier series of $f$ and the corresponding $\ell_q$-Cesàro kernel $K_{n}^{q, \alpha}$ are introduced by
\[
\sigma_{n}^{q, \alpha} f(x) := \frac{1}{A_{n-1}^{\alpha}} \sum_{k \in \mathbb{Z}^d, \|k\|_q \leq n} A_{n-1-\|k\|_q}^\alpha \hat{f}(k) e^{ik \cdot x}
\]
and
\[
K_{n}^{q, \alpha}(t) := \frac{1}{A_{n-1}^{\alpha}} \sum_{k \in \mathbb{Z}^d, \|k\|_q \leq n} A_{n-1-\|k\|_q}^\alpha e^{ikt},
\]
respectively. We can show that
\[
\sigma_{n}^{q, \alpha} f(x) = \frac{1}{(2\pi)^d} \int_{\mathbb{T}^d} f(x-t) K_{n}^{q, \alpha}(t) \, dt = \frac{1}{A_{n-1}^{\alpha}} \sum_{j=0}^{n-1} A_{n-1-j}^{\alpha} s_{q} f(x).
\]

If $f \in L_1(\mathbb{T}^d)$, $n = (n_1, \ldots, n_d) \in \mathbb{N}^d$ and $\alpha = (\alpha_1, \ldots, \alpha_d) \in \mathbb{R}^d_+$, we define the $n$th rectangular Cesàro mean $\sigma_{n}^{\alpha} f$ and the Cesàro kernel $K_{n}^{\alpha}$ by
\[
\sigma_{n}^{\alpha} f(x) := \frac{1}{\prod_{i=1}^{d} A_{n_i-1}^{\alpha_i}} \sum_{|k_1| \leq n_1} \ldots \sum_{|k_d| \leq n_d} \prod_{i=1}^{d} A_{n_i-1-|k_i|}^{\alpha_i} \hat{f}(k) e^{ik \cdot x}
\]
and
\[
K_{n}^{\alpha}(t) := \frac{1}{\prod_{i=1}^{d} A_{n_i-1}^{\alpha_i}} \sum_{|k_1| \leq n_1} \ldots \sum_{|k_d| \leq n_d} \prod_{i=1}^{d} A_{n_i-1-|k_i|}^{\alpha_i} e^{ikt},
\]
respectively. We can see that
\[
\sigma_{n}^{\alpha} f(x) = \frac{1}{(2\pi)^d} \int_{\mathbb{T}^d} f(x-t) K_{n}^{\alpha}(t) \, dt = \frac{1}{\prod_{i=1}^{d} A_{n_i-1}^{\alpha_i}} \sum_{k_1=0}^{n_1-1} \ldots \sum_{k_d=0}^{n_d-1} \prod_{i=1}^{d} A_{n_i-1-k_i}^{\alpha_i} \hat{f}(k) e^{ik \cdot x}
\]
and
\[
K_{n}^{\alpha} = K_{n_1}^{\alpha_1} \otimes \cdots \otimes K_{n_d}^{\alpha_d},
\]
where the functions $K_{n_i}^{\alpha_i}$ are the one-dimensional Cesàro kernels.

If $\alpha = 1$, resp. all $\alpha_i = 1$, then we get back the $\ell_q$-, resp. rectangular, Fejér means,
\[
\sigma_{n}^{q} f(x) := \sum_{k \in \mathbb{Z}^d, \|k\|_q \leq n} \left( 1 - \frac{\|k\|_q}{n} \right) \hat{f}(k) e^{ik \cdot x} = \frac{1}{n} \sum_{k=0}^{n-1} s_{q} f(x),
\]
\[ \sigma_n f(x) := \sum_{|k_1| \leq n_1} \cdots \sum_{|k_d| \leq n_d} \prod_{i=1}^d \left( 1 - \frac{|k_i|}{n_i} \right) \hat{f}(k) e^{ik \cdot x} \]

resp.

\[ = \frac{1}{\prod_{i=1}^d n_i} \sum_{k_1=1}^{n_1-1} \cdots \sum_{k_d=1}^{n_d-1} s_k f(x). \]

We generalize Theorem 3.1 as follows.

**Theorem 5.1.** Let \( 1 \leq p < \infty \) and \( f \in L_p(\mathbb{T}^d) \). If \( 0 < \alpha_j < \infty \) for all \( j = 1, \ldots, d \), then

\[ \sup_{n \in \mathbb{N}^d} \| \sigma_n^\alpha f \|_p \leq C \| f \|_p \]

and

\[ \lim_{n \to \infty} \sigma_n^\alpha f = f \quad \text{in the } L_p(\mathbb{T}^d)-\text{norm}. \]

If \( 0 < \alpha < \infty \) and \( q = 1 \) or \( q = \infty \), then

\[ \sup_{n \in \mathbb{N}} \| \sigma_n^{\alpha, q} f \|_p \leq C \| f \|_p \]

and

\[ \lim_{n \to \infty} \sigma_n^{\alpha, q} f = f \quad \text{in the } L_p(\mathbb{T}^d)-\text{norm}. \]

If \( f \) is a continuous function, then the theorem holds also for \( p = \infty \).

The proof for the rectangular summability follows by iteration. Moreover, the proof for the \( \ell_q \) summation is again based on the inequality

\[ \sup_{n \in \mathbb{N}} \int_{\mathbb{T}^d} | K_n^\alpha f | \, d\lambda \leq C_\alpha. \]

The proof of this inequality is much more complicated than in the one-dimensional case (see [51]). In the remaining part of this paper, we will generalize Lebesgue’s theorem (Theorem 3.4) to higher dimensional Fourier series in four different ways. We study four different type of summability and for each summation we have to investigate different maximal operators and Lebesgue points.

6. **Unrestricted rectangular Cesàro summability.** As a first generalization of the maximal operator \( M \), we introduce the strong Hardy-Littlewood maximal function by

\[ M_s f(x) := \sup_{h \in \mathbb{R}^d_+} \prod_{j=1}^d \int_{-h_j}^{h_j} |f(x-t)| \, dt, \]

where \( f \in L_1(\mathbb{T}^d) \). For \( d > 1 \) it is known that there is a function \( f \in L_1(\mathbb{T}^d) \) such that \( M_s f = \infty \) almost everywhere (see Jessen, Marcinkiewicz and Zygmund [30] and Saks [42]). Thus, in contrary to the one-dimensional case, \( M_s \) cannot be of weak type \((1,1)\) if \( d > 1 \). However, we know the following weak type inequality (see Chang and Fefferman [8]).

**Theorem 6.1.** If \( f \in L(\log L)^{d-1}(\mathbb{T}^d), \) then

\[ \sup_{\rho > 0} \rho \lambda(M_s f > \rho) \leq C + C \left\| f \right\|_{L(\log L)^{d-1}}. \]
Moreover, for $1 < p \leq \infty$, we have

$$
\|M_s f\|_p \leq C_p \|f\|_p \quad (f \in L_p(\mathbb{T}^d)).
$$

If we take the supremum in $M_s$ only over the diagonal,

$$
M f(x) := \sup_{h > 0} \frac{1}{(2h)^d} \int_{-h}^h \cdots \int_{-h}^h |f(x - t)| dt,
$$

then $M$ is again of weak type $(1, 1)$. Unfortunately we cannot use this operator in the theory of Cesàro summability.

The convergence

$$
\lim_{h \to 0} \frac{1}{\prod_{j=1}^d (2h_j)} \int_{-h_1}^{h_1} \cdots \int_{-h_d}^{h_d} |f(x - t) - f(x)| dt = 0.
$$

for $f \in L_1(\log L)^{d-1} (\mathbb{T}^d)$ and almost every $x \in \mathbb{T}^d$, follows from Theorem 6.1 by a usual density argument. Here $h \to 0$ means that $h_j \to 0$ for all $j = 1, \ldots, d$. Note that this result does not hold for all $f \in L_1(\mathbb{T}^d)$ if $d > 1$ (see Jessen, Marcinkiewicz and Zygmund [30] and Saks [42]). Motivated by this convergence result, a point $x \in \mathbb{T}^d$ is called a strong Lebesgue point of $f \in L_1(\mathbb{T}^d)$ if

$$
\lim_{h \to 0} \frac{1}{\prod_{j=1}^d (2h_j)} \int_{-h_1}^{h_1} \cdots \int_{-h_d}^{h_d} |f(x - t) - f(x)| dt = 0.
$$

For the concept of the Lebesgue and strong Lebesgue points, see e.g. Feichtinger and Weisz [16] and the references therein.

**Theorem 6.2.** If $f \in L_1(\log L)^{d-1} (\mathbb{T}^d)$, then almost every point $x \in \mathbb{T}^d$ is a strong Lebesgue point of $f$.

This is not true for $f \in L_1(\mathbb{T}^d)$ if $d > 1$. Note that $L_1(\log L)^{d-1} (\mathbb{T}^d) \supset L_p(\mathbb{T}^d)$ for all $1 < p \leq \infty$.

Here is the first generalization of Lebesgue’s theorem (see [57]).

**Theorem 6.3.** Suppose that $0 < \alpha_j \leq 1$ for all $j = 1, \ldots, d$. If $M_s f(x)$ is finite and $x$ is a strong Lebesgue point of $f \in L_1(\log L)^{d-1} (\mathbb{T}^d)$, then

$$
\lim_{n \to \infty} \sigma_n^a f(x) = f(x).
$$

Since by Theorems 6.1 and 6.2 almost every point is a strong Lebesgue point of $f$ and $M_s f$ is almost everywhere finite if $f \in L_1(\log L)^{d-1} (\mathbb{T}^d)$, we obtain that the convergence in Theorem 6.3 holds almost everywhere. In the one-dimensional case, if $x$ is a (strong) Lebesgue point, then $M_s f(x)$ is finite and $L_1(\log L)^{d-1} (\mathbb{T}^d) = L_1(\mathbb{T})$, hence we get back Theorem 3.4. Recall that $L_1(\log L)^{d-1} (\mathbb{T}^d) \supset L_p(\mathbb{T}^d)$ for $1 < p \leq \infty$ and $d > 1$. If $f$ is continuous at a point $x$, then $x$ is also a strong Lebesgue point. So we obtain

**Corollary 2.** Suppose that $0 < \alpha_j \leq 1$ for all $j = 1, \ldots, d$. If $M_s f(x)$ is finite and $f \in L_1(\log L)^{d-1} (\mathbb{T}^d)$ is continuous at a point $x$, then

$$
\lim_{n \to \infty} \sigma_n^a f(x) = f(x).
$$
7. Restricted rectangular Cesàro summability. Based on the result of Marcinkiewicz and Zygmund mentioned in the introduction, we can hope that we get convergence results for all integrable functions if we consider the Cesàro means and the convergence on a cone, only. To this, we have to find a maximal operator that is of weak type \((1,1)\). As we mentioned in the preceding section, \(Mf\) is not suitable. So we start from the definition of the strong maximal operator \(Msf\) and introduce its weighted version.

For some \(\omega \geq 0\) and \(f \in L_1(\mathbb{T}^d)\), we define the Hardy-Littlewood type maximal function

\[
M^{\omega,1}f(x) := \sup_{i \in \mathbb{R}^d, h > 0} \frac{2^{-\omega ||i||_1}}{(2h)^{d||i||_1}} \int_{-2^ih}^{2^ih} \cdots \int_{-2^{d}h}^{2^{d}h} |f(x-t)| \, dt,
\]

where \(||i||_1 := \sum_{k=1}^{d} |i_k| \ (i = (i_1, \ldots, i_d))\). Here \(2^{-\omega ||i||_1}\) is the weight, \((2h)^{d||i||_1}\) is the measure of the rectangle we integrate on. If \(\omega = 0\), we obtain the strong Hardy-Littlewood maximal function, moreover, if \(\omega = 0\) and \(i_1 = \cdots = i_d\), then the Hardy-Littlewood maximal function \(Mf\). The operator \(M^{\omega,1}\) is of weak type \((1,1)\) if \(\omega > 0\) (see [52]).

**Theorem 7.1.** Let \(\omega > 0\). We have

\[
\sup_{\rho > 0} \rho \lambda (M^{\omega,1}f > \rho) \leq C \|f\|_{L_1(\mathbb{T}^d)} \quad (f \in L_1(\mathbb{T}^d)).
\]

If \(1 < p \leq \infty\), then

\[
\|M^{\omega,1}f\|_p \leq C_p \|f\|_{L_p(\mathbb{T}^d)} \quad (f \in L_p(\mathbb{T}^d)).
\]

The definition of the strong Lebesgue points presented in the previous section is equivalent to

\[
\lim_{r \to 0} \sup_{0 < h < r} \frac{1}{\prod_{j=1}^{d} (2h)^j} \int_{-h}^{h} \cdots \int_{-h}^{h} |f(x-t) - f(x)| \, dt = 0.
\]

Based on this observation and on the definition of \(M^\omega\), let

\[
U^{\omega,1}_r f(x) := \sup_{i \in \mathbb{N}^d, h > 0} \frac{2^{-\omega ||i||_1}}{(2h)^{d||i||_1}} \int_{-2^ih}^{2^ih} \cdots \int_{-2^dih}^{2^dih} |f(x-t) - f(x)| \, dt.
\]

For \(\omega > 0\), a point \(x \in \mathbb{T}^d\) is called an \(\omega\)-Lebesgue point of \(f \in L_1(\mathbb{T}^d)\) if

\[
\lim_{r \to 0} U^{\omega,1}_r f(x) = 0.
\]

Different versions of Lebesgue points were considered in Gabisoniya [19] and Skopina [45, 46] for two dimensions. If \(\omega = 0\), then we get back the definition of the strong Lebesgue points. Every \(\omega_2\)-Lebesgue point is a \(\omega_1\)-Lebesgue point \((0 < \omega_2 < \omega_1 < \infty)\), because \(U^{\omega_1,1}_r f \leq U^{\omega_2,1}_r f\). If \(f\) is continuous at \(x\), then \(x\) is an \(\omega\)-Lebesgue point of \(f\). The next two theorem were proved in [52, 55].

**Theorem 7.2.** If \(\omega > 0\), then almost every point \(x \in \mathbb{T}^d\) is an \(\omega\)-Lebesgue point of \(f \in L_1(\mathbb{T}^d)\).
Theorem 7.3. Suppose that $0 < \omega < \alpha_j \leq 1$ for all $j = 1, \ldots, d$. If $M^{\omega,1}f(x)$ is finite and $x$ is an $\omega$-Lebesgue point of $f \in L_1(\mathbb{T}^d)$, then

$$\lim_{n \to \infty, n \in \mathbb{R}^d} \sigma_n^\omega f(x) = f(x).$$

By Theorems 7.1 and 7.2, the convergence holds almost everywhere. If $f$ is continuous at a point $x$, then $x$ is also an $\omega$-Lebesgue point. So we obtain

Corollary 3. Suppose that $0 < \omega < \alpha_j \leq 1$ for all $j = 1, \ldots, d$. If $M^{\omega,1}f(x)$ is finite and $f \in L_1(\mathbb{T}^d)$ is continuous at a point $x$, then

$$\lim_{n \to \infty, n \in \mathbb{R}^d} \sigma_n^\omega f(x) = f(x).$$

8. $\ell_\infty$-Cesàro summability. In this section, we generalize the maximal operator $M^{\omega,1}f$ investigated in the previous section. Under a diagonal, we understand a diagonal of the cube $[0, \pi]^d$. Let us denote by $P_{2^{i_1}h, \ldots, 2^{i_d}h}$ a parallelepiped, whose center is the origin and whose sides are parallel to the axes and/or to the diagonals and whose $k$th side length is $2^{i_k+1}h$ if the $k$th side is parallel to an axis and $\sqrt{2^{i_k+1}}h$ if the $k$th side is parallel to a diagonal ($i \in \mathbb{N}^d, h > 0, k = 1, \ldots, d$). More exactly, at least one side of $P_{2^{i_1}h, \ldots, 2^{i_d}h}$ is parallel to one of the axes and the other sides are parallel to the axes and/or to the diagonals.

For $\omega > 0$ and $f \in L_1(\mathbb{T}^d)$, the Hardy-Littlewood type maximal function $M^\omega f$ is given by

$$M^\omega f(x) := \sup_{P_{2^{i_1}h, \ldots, 2^{i_d}h}, i \in \mathbb{N}^d, h > 0} 2^{-\omega \|i\|_1} \left| \int_{P_{2^{i_1}h, \ldots, 2^{i_d}h}} |f(x - t)| dt \right|,$$

where the supremum is taken over all parallelepipeds $P_{2^{i_1}h, \ldots, 2^{i_d}h}$ ($i \in \mathbb{N}^d, h > 0$) just defined.

If we take the supremum only over all rectangles with sides parallel to the axes, we get back the definition of the maximal operator $M^{\omega,1}_p f$ from the previous section. Thus $M^\omega f \geq M^{\omega,1}_p f$. We point out the definition in the two-dimensional case.
Besides\( M^{\omega,1}f \), we introduce
\[
M^{\omega,2} f(x_1, x_2) := \sup_{i_1, i_2 \in \mathbb{N}, h > 0} \frac{2^{-\omega(i_1 + i_2)}}{4 \cdot 2^{i_1+2i_2} h^2} \int_{-2^{i_1} h}^{2^{i_1} h} \int_{-2^{i_2} h}^{2^{i_2} h} |f(x_1 - t_1, x_2 - t_2)| \, dt_2 \, dt_1,
\]
\[
M^{\omega,3} f(x_1, x_2) := \sup_{i_1, i_2 \in \mathbb{N}, h > 0} \frac{2^{-\omega(i_1 + i_2)}}{4 \cdot 2^{i_1+2i_2} h^2} \int_{-2^{i_1} h}^{2^{i_1} h} \int_{-2^{i_2} h}^{2^{i_2} h} |f(x_1 - t_1, x_2 - t_2)| \, dt_2 \, dt_1,
\]
\[
M^{\omega,4} f(x_1, x_2) := \sup_{i_1, i_2 \in \mathbb{N}, h > 0} \frac{2^{-\omega(i_1 + i_2)}}{4 \cdot 2^{i_1+2i_2} h^2} \int_{-2^{i_2} h}^{2^{i_2} h} \int_{-2^{i_1} h}^{2^{i_1} h} |f(x_1 - t_1, x_2 - t_2)| \, dt_1 \, dt_2.
\]
as well as
\[
M^{\omega,5} f(x_1, x_2) := \sup_{i_1, i_2 \in \mathbb{N}, h > 0} \frac{2^{-\omega(i_1 + i_2)}}{4 \cdot 2^{i_1+2i_2} h^2} \int_{-2^{i_1} h}^{2^{i_1} h} \int_{-2^{i_2} h}^{2^{i_2} h} |f(x_1 - t_1, x_2 - t_2)| \, dt_1 \, dt_2.
\]
Note that in \( M^{\omega,1} f \), we take the supremum over rectangles with sides parallel to the axes and in \( M^{\omega,j} f \) \((j = 2, 3, 4, 5)\), over parallelograms with at most one side parallel to one of the axes and with the other sides parallel to the diagonals of the square \([0, \pi]^2\). Then we have
\[
M^\omega f(x_1, x_2) = \sum_{j=1}^{5} M^{\omega,j} f(x_1, x_2)
\]
for all \(\omega > 0\). Though the operator \( M^\omega \) is larger than \( M^{\omega,1} \), it is of weak type \((1, 1)\) (see [52]).

**Theorem 8.1.** Let \(\omega > 0\). We have
\[
\sup_{\rho > 0} \rho \lambda(M^\omega f > \rho) \leq C \|f\|_{L_1(\mathbb{T}^d)} \quad (f \in L_1(\mathbb{T}^d)).
\]
If \(1 < p \leq \infty\), then
\[
\|M^\omega f\|_p \leq C_p \|f\|_{L_p(\mathbb{T}^d)} \quad (f \in L_p(\mathbb{T}^d)).
\]

Here we introduce a stronger version of Lebesgue points than the \(\omega\)-Lebesgue points. Let
\[
U^\omega_r f(x) := \sup_{P_{2^{i_1}h, \ldots, 2^{i_d}h} \in \mathbb{N}^d, h > 0} \frac{2^{-\omega i_1}}{2^{i_1+2i_2} h^2} \int_{P_{2^{i_1}h, \ldots, 2^{i_d}h}} \int_{P_{2^{i_1}h, \ldots, 2^{i_d}h}} |f(x - t) - f(x)| \, dt,
\]
where the supremum is taken over all parallelepipeds whose center is the origin and whose sides are parallel to the axes and/or to the diagonals as in the definition of \( M^\omega_r f \). Taking the supremum in the definition of \( U^\omega_r f \) over all parallelepipeds whose sides are parallel to the axes, we obtain the definition of \( U^\omega_r f \). In the
two-dimensional case, similarly to $\mathcal{M}^{\omega,j} f$, we can define $U_r^{\omega,j} f$ for $j = 2, 3, 4, 5$ as follows:

$$U_r^{\omega,2} f(x_1, x_2) := \sup_{i_1, i_2 \in \mathbb{N}} \frac{2^{-\omega(i_1 + i_2)}}{4 \cdot 2^{i_1 + i_2} h^2} \int_{-2^{i_1} h}^{2^{i_1} h} \int_{-2^{i_2} h}^{2^{i_2} h} |f(x - t) - f(x)| \, dt_2 \, dt_1,$$

$$U_r^{\omega,3} f(x_1, x_2) := \sup_{i_1, i_2 \in \mathbb{N}} \frac{2^{-\omega(i_1 + i_2)}}{4 \cdot 2^{i_1 + i_2} h^2} \int_{-2^{i_1} h}^{2^{i_1} h} \int_{-2^{i_2} h}^{2^{i_2} h} |f(x - t) - f(x)| \, dt_2 \, dt_1,$$

$$U_r^{\omega,4} f(x_1, x_2) := \sup_{i_1, i_2 \in \mathbb{N}} \frac{2^{-\omega(i_1 + i_2)}}{4 \cdot 2^{i_1 + i_2} h^2} \int_{-2^{i_2} h}^{2^{i_2} h} \int_{-2^{i_1} h}^{2^{i_1} h} |f(x - t) - f(x)| \, dt_1 \, dt_2,$$

and

$$U_r^{\omega,5} f(x_1, x_2) := \sup_{i_1, i_2 \in \mathbb{N}} \frac{2^{-\omega(i_1 + i_2)}}{4 \cdot 2^{i_1 + i_2} h^2} \int_{-2^{i_2} h}^{2^{i_2} h} \int_{-2^{i_1} h}^{2^{i_1} h} |f(x - t) - f(x)| \, dt_1 \, dt_2.$$

Obviously,

$$U_r^{\omega} f(x_1, x_2) = \sum_{j=1}^{5} U_r^{\omega,j} f(x_1, x_2).$$

For $\omega > 0$, a point $x \in \mathbb{T}^d$ is called an strong $\omega$-Lebesgue point of $f \in L_1(\mathbb{T}^d)$ if

$$\lim_{r \to 0} U_r^{\omega} f(x) = 0.$$

Since $U_r^{\omega,1} f \leq U_r^{\omega} f$, this definition is indeed stronger than the definition of $\omega$-Lebesgue points. Note that every strong $\omega_2$-Lebesgue point is a strong $\omega_1$-Lebesgue point ($0 < \omega_2 < \omega_1 < \infty$), because of $U_r^{\omega_1} f \leq U_r^{\omega_2} f$. If $f$ is continuous at $x$, then $x$ is a strong $\omega$-Lebesgue point of $f$ for all $\omega > 0$.

**Theorem 8.2** ([52]). If $\omega > 0$, then almost every point $x \in \mathbb{T}^d$ is a strong $\omega$-Lebesgue point of $f \in L_1(\mathbb{T}^d)$.

**Theorem 8.3.** If $0 < \alpha < \infty$, $0 < \omega < \min(\alpha, 1)/d$, $\mathcal{M}^{\omega} f(x)$ is finite and $x$ is a strong $\omega$-Lebesgue point of $f \in L_1(\mathbb{T}^d)$, then

$$\lim_{n \to \infty} \sigma_n^{\omega,\alpha} f(x) = f(x).$$

For the proof of this theorem see [54]. Note that Belinsky [3] proved that the convergence does not hold for all Lebesgue points defined by $M f$. Since by Theorems 7.1 and 8.2 almost every point is a strong $\omega$-Lebesgue point and the maximal operator $\mathcal{M}^{\omega} f$ is almost everywhere finite for $f \in L_1(\mathbb{T}^d)$, Theorem 8.3 implies the almost everywhere convergence of $\sigma_n^{\infty,\alpha} f$ to $f$ if $f \in L_1(\mathbb{T}^d)$. 
Corollary 4. Suppose that $0 < \alpha < \infty$, $0 < \omega < \min(\alpha, 1)/d$. If $M_\omega f(x)$ is finite and $f \in L_1(T^d)$ is continuous at a point $x$, then
\[ \lim_{n \to \infty} \sigma_n^{\infty, \alpha} f(x) = f(x). \]

9. $\ell_1$-Cesàro summability. In this short section, we use the same maximal operator $M_\omega f$ and the strong $\omega$-Lebesgue points as in the previous section.

Theorem 9.1 ([56]). Suppose that $0 < \alpha < \infty$, $0 < \omega < \min(\alpha, 1)/d$ and $M_\omega f(x)$ is finite. If $f \in L_1(T^d)$ is periodic with respect to $\pi$ and $x$ is a strong $\omega$-Lebesgue point of $f$, then
\[ \lim_{n \to \infty} \sigma_n^{1, \alpha} f(x) = f(x). \]

Theorems 7.1 and 8.2 mean that almost everywhere convergence holds, too.

Corollary 5. Suppose that $0 < \alpha < \infty$, $0 < \omega < \min(\alpha, 1)/d$ and $M_\omega f(x)$ is finite. If $f \in L_1(T^d)$ is periodic with respect to $\pi$ and $f$ is continuous at a point $x$, then
\[ \lim_{n \to \infty} \sigma_n^{1, \alpha} f(x) = f(x). \]

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