THE N-EIGENVALUE PROBLEM AND TWO APPLICATIONS

MICHAEL J. LARSEN, ERIC C. ROWELL, AND ZHENGHAN WANG

Abstract. We consider the classification problem for compact Lie groups $G \subset U(n)$ which are generated by a single conjugacy class with a fixed number $N$ of distinct eigenvalues. We give an explicit classification when $N = 3$, and apply this to extract information about Galois representations and braid group representations.

1. Introduction

Consider the following two questions:

(1) When does a compact Lie group $G \subset U(n)$ have an element $g \in G$ possessing exactly two eigenvalues.

(2) When does a compact Lie group $G \subset U(n)$ have a cocharacter $U(1) \to G$ such that the composition $U(1) \to U(n)$ is a representation of $U(1)$ with exactly two weights.

A solution to the second problem gives a family of solutions to the first, by choosing $g$ to be almost any element of the image of $U(1)$. The converse is not true. For one thing, any non-central element of order 2 in $G$ has exactly two eigenvalues. To eliminate these essentially trivial solutions, we can insist that the ratio between the two eigenvalues is not $-1$. There remain interesting cases of finite groups $G$ satisfying the first (but obviously not the second) condition, especially when the ratio of eigenvalues is a third or fourth root of unity (see [Bl], [Ko], and [W] for classification results). On the other hand, when $G$ is infinite modulo center, the solutions of the two problems are essentially the same, though the historical reasons for considering them were quite different. The first problem was recently solved in the infinite-mod-center case by M. Freedman, M. Larsen, and Z. Wang [FLW] with an eye toward understanding representations of Hecke algebras. The second problem was solved by J-P. Serre [Ser] nearly thirty years ago in order to classify representations arising from Hodge-Tate modules of weight 1.

This paper is primarily devoted to an effort to understand the analogue of the first problem (the “$N$-eigenvalue problem” of the title) when the number $N \geq 3$ of eigenvalues is fixed and $G$ is infinite modulo its center. As a consequence, we also say something about the second problem. We are especially interested in the case $N = 3$, both because the results can be

The authors are partially supported by NSF grant DMS-034772, and the third-named author is also supported by NSF grant EIA 0130388.
made quite explicit and because it is especially relevant to the applications
we have in mind. To specify our problem more precisely, we make the
following definitions.

A pair \((G, V)\) consists of a compact Lie group \(G\) and a faithful irreducible
complex representation \(\rho: G \to \text{GL}(V)\). Let \(N\) be a positive integer. We say
a pair \((G, V)\) satisfies the \(N\)-eigenvalue property if there exists a generating
element, i.e., an element \(g \in G\) such that the conjugacy class of \(g\) generates
\(G\) topologically and the spectrum \(X\) of \(\rho(g)\) has \(N\) elements and satisfies
the no-cycle property: for all roots of unity \(\zeta_n, n \geq 2\), and all \(u \in \mathbb{C}^\times\),

\[
    u\langle \zeta_n \rangle \not\subset X
\]

(1.1)

Our goal is to classify pairs satisfying the \(N\)-eigenvalue property.

From the perspective of [FLW], the most obvious reason to consider the
\(N\)-eigenvalue property is that certain naturally occurring representations of
the Artin braid groups \(B_n\) satisfy this condition. The braid generators (half-
twists) in the braid group form a generating conjugacy class, and given any
braided tensor category \(\mathcal{C}\) and any object \(x \in \mathcal{C}\), we get a representation of
\(\rho_{n,x}: B_n \to \text{GL}(V_{n,x})\). When \(\rho_{n,x}\) is unitary with respect to a hermitian form
on \(V_{n,x}\), the closure of \(\rho_{n,x}\) is a compact Lie group endowed with a natural
faithful representation and a generating conjugacy class. It is often possible
to control the eigenvalues of half-twists, to guarantee the \(N\)-eigenvalue
condition, and to guarantee irreducibility. In the case when the braided
tensor category \(\mathcal{C}\) is modular, we obtain in addition representations of the
mapping class groups \(\mathcal{M}(\Sigma_g)\) of closed oriented surfaces \(\Sigma_g\) for each genus
\(g\). It is well-known that \(\mathcal{M}(\Sigma_g)\) is generated by the (mutually conjugate)
Dehn twists \(D_c\) on \(3g - 1\) non-separating simple closed curves \(c\) on \(\Sigma_g\) (see [I]). If \(\mathcal{C}\) has \(m\) simple object types, then each \(D_c\) has at most \(m\) distinct
eigenvalues as the eigenvalues of \(D_c\) consist of twists \(\theta_i\) of the simple objects.
When the values \(\theta_i\) satisfy the no-cycle condition, it follows that each irre-
ducible constituent of the representation of \(\mathcal{M}(\Sigma_g)\) arising from \(\mathcal{C}\) defines a
pair satisfying the \(N\)-eigenvalue property for some \(N \leq m\).

The original motivation for the work of [FLW] was for applications to
quantum computing. In [FKLW], topological models of quantum computing
based on unitary topological quantum field theories (TQFTs) are proposed.
Given a topological model of quantum computing, an important issue is
whether or not this topological model is capable of simulating the universal
circuit model of quantum computing [NC]. This question actually depends
on the specific design of the topological quantum computer. But for the
models based on braiding anyons in [FKLW], the universality question is
translated into a question about the closures of the braid group represen-
tations. Quantum computing is the processing of information encoded in
quantum state vectors in certain Hilbert spaces \(V_n\) by unitary transfor-
mations. Universality is the ability to efficiently move any state vector \(v \in V_n\)
sufficiently close to any other state vector in $V_n$. A theorem of Kitaev-Solovay (see [NC]) guarantees efficiency if the available unitary transformations in $U(V_n)$ form a dense subset of $SU(V_n)$. Therefore, universality of topological models in $[FKLW]$ is equivalent to the density of braid group representations.

The unitary Witten-Reshetikhin-Turaev Chern-Simons TQFTs based on the gauge groups $SU(N)$ and $SO(N)$ are of particular interests due to their relevance to braid statistics in condensed matter physics. It was discovered in the 1980s that in dimension 2, there are quasi-particles which are neither fermions nor bosons [Wi]. The most interesting of these anyons are non-abelian: when two such quasi-particles are exchanged, their wave function is changed by a unitary matrix, rather than a complex number, which depends on the exchanging paths (braiding). It is predicted by physicists that the braid statistics of quasi-particles in certain fractional quantum Hall liquids are described by Jones’ unitary braid groups representations or equivalently the braid representations coming from the SU(2) TQFTs. Physicists have also proposed models of braid statistics based on the SO(3) [FF] and SO(5) [Wn] TQFTs. Therefore, it may well be the case that both the Jones and the BMW braid group representations describe braid statistics of quasi-particles in nature. Experiments are proposed to confirm those predictions [DFN].

Problem (2) is significant partly because of its relation to problem (1), but in addition, there are number-theoretic applications, in the spirit of [Ser]. We mention a global one: assuming the Fontaine-Mazur conjecture, we can prove that if $K$ is a number field, $\bar{K}$ is an algebraic closure of $K$, $G_K = \text{Gal}(\bar{K}/K)$, and $X$ is a non-singular projective variety over $K$, then $E_8$ does not occur as a factor of the identity component of the Zariski-closure of $G_K$ in the second étale cohomology group of $\bar{X}$.

The paper is organized as follows. The second and third sections treat the infinite-mod-center case of the $N$-eigenvalue problem. The second section gives the general shape of the solution for all $N$, and the third section gives an actual list for $N = 3$. The fourth section shows that a fairly weak hypothesis on the actual eigenvalues is enough to guarantee that $G$ is infinite modulo its center. The fifth section gives applications to number theory, and the sixth section gives applications to braid group representations. We conclude with a discussion of future applications to topology and quantum computing.

Acknowledgements. The second-named author would like to thank Hans Wenzl for many helpful correspondences.

2. Infinite groups

In this section, we consider the general $N$-eigenvalue problem for infinite compact groups. Our methods come directly from [FLW] and [LW].
Lemma 2.1. Let $V = V_1 \oplus \cdots \oplus V_k$ be a complex vector space and $T: V \to V$ a linear transformation permuting the summands $V_i$ non-trivially. Then the spectrum of $V$ does not satisfy (1.1).

Proof. Renumbering if necessary, we may assume $V$ permutes $V_1, V_2, \ldots, V_r$ cyclically, where $r \geq 2$. Let $W = V_1 \oplus \cdots \oplus V_r$, let $\zeta_r = e^{2\pi i r}$, and let $S: W \to W$ act as the scalar $\zeta_r^i$ on $V_i$. Then

$$ST|_W S^{-1} = \zeta_r T|_W,$$

so the spectrum of $T|_W$ is invariant under multiplication by $\zeta_r$. It is therefore a union of $\langle \zeta_r \rangle$-cosets. \qed

Lemma 2.2. Let $(G_1, V_1)$ and $(G_2, V_2)$ be pairs and let $G$ denote the image of $G_1 \times G_2$ in $\text{GL}(V_1 \otimes V_2)$. If $G$ satisfies the $N$-eigenvalue property, then there exist integers $N_1$ and $N_2$ such that $N_1 + N_2 - 1 \leq N$, and subgroups $G'_1 < G_1$ and $G'_2 < G_2$ such that $(G'_1, V_i)$ satisfies the $N_i$-eigenvalue property and $G_i' Z(G_i) = G_i$ for $i = 1, 2$.

Proof. Let $g \in G$ be a generating element, and let $(g_1, g_2) \in G_1 \times G_2$ map to $G$. Let $G'_i$ denote the subgroup of $G_i$ generated by the conjugacy class of $g_i$. As the conjugacy class of $g$ generates $G$, $(\ker G_i \to G)G'_i = G_i$. By construction, $\ker G_i \to G \subset Z(G_i)$.

The spectrum of $\rho(g)$ is the product of the spectra of $\rho_i(g_i)$. So the lemma reduces to the following claim: if $X_1$ and $X_2$ are finite subgroups of an abelian group $A$ such that $X_1 + X_2$ does not contain a coset of a non-trivial subgroup of $A$, then $|X_1 + X_2| \geq |X_1| + |X_2| - 1$. This is well-known (see, e.g., [Ke]). \qed

If $(G, V)$ arises in this way, we say it is decomposable; otherwise, it is indecomposable. Note that the tensor product of pairs which satisfy the $N_1$ and $N_2$-eigenvalue conditions need not satisfy the $N_1 + N_2 - 1$-eigenvalue condition. For one thing, the product of sets of cardinality $N_1$ and $N_2$ could be as large as $N_1 N_2$. For another, the product of sets satisfying the no-cycle property may itself fail to satisfy the no-cycle property.

Proposition 2.3. Let $(G, V)$ be an indecomposable pair. If $G$ is infinite modulo its center and $(G, V)$ satisfies the $N$-eigenvalue property for some $N$, then $G = G^o Z(G)$.

Proof. Let $g \in G$ be a generating element. Then $g \in G^o$ implies $G = G^o$, in which case there is nothing to prove. If $V|_{G^o}$ is not isotypic, then $g$ acts non-trivially on the isotypic factors, and by Lemma 2.1 the spectrum of $g$ fails to satisfy property (1.1). If $V|_{G^o} = W^n = W \otimes U$, where $G^o$ acts trivially on $U$ and irreducibly on $W$, then the span of $\rho(G^o)$ is $\text{End}(W) \otimes \text{Id}_U \subset \text{End}(V)$, so $\rho(G)$ lies in the normalizer of $\text{End}(W) \otimes \text{Id}_U$, which is $\text{End}(W) \text{End}(V)$. 

4 MICHAEL J. LARSEN, ERIC C. ROWELL, AND ZHENGHAN WANG
Thus \( \rho \) maps \( G \) to \((\text{GL}(W) \times \text{GL}(U))/\mathbb{C}^\times\). Let \( \tilde{G} \) denote the cartesian square
\[
\begin{array}{ccc}
\tilde{G} & \xrightarrow{\tilde{\rho}} & \text{GL}(W) \times \text{GL}(U) \\
\downarrow & & \downarrow \\
G & \xrightarrow{\rho} & (\text{GL}(W) \times \text{GL}(U))/\mathbb{C}^\times.
\end{array}
\]

If \( \tilde{g} \in \pi^{-1}(g) \), then the projections of \( \tilde{\rho}(\tilde{g}) \) to \( \text{GL}(W) \) and \( \text{GL}(U) \) have spectra satisfying the no-cycle property, since the product of these spectra is the spectrum of \( \tilde{\rho}(\tilde{g}) \). If \( \dim W \) and \( \dim U \) are both \( \geq 2 \), then \((G, V)\) is decomposable, contrary to hypothesis. As \( G^o \) is not in the center of \( G \), \( \dim W \geq 2 \). It follows that \( \dim U = 1 \), i.e., the restriction of \( V \) to \( G^o \) is irreducible. Thus every element of \( G \) which commutes with \( G^o \) lies in \( Z(G) \).

It follows that for every \( g \notin G^o Z(G) \), conjugation by \( g \) induces an automorphism of \( G^o \) which is not inner. By [St, 7.5], this implies that there exists a maximal torus \( T \) of \( G^o \) such that \( gTg^{-1} = T \) but conjugation by \( g \) induces a non-trivial automorphism of \( T \). The characters of \( T \) appearing in \( V|_T \) span \( X^*(T) \otimes \mathbb{R} \) since \( V \) is a faithful representation. Therefore, a non-trivial automorphism of \( T \) must permute the weights of \( V \) non-trivially. By Lemma 2.1 this implies that the spectrum of \( g \) violates the no-cycle property, contrary to hypothesis. Thus \( g \in G^o Z(G) \), and since the conjugacy class of \( g \) generates \( G \), it follows that \( G = G^o Z(G) \). \( \square \)

**Proposition 2.4.** Let \((G, V)\) be as in Proposition 2.3. Then \( G \) is the product of the derived group \( D \) of \( G^o \) and a group of scalar matrices in \( V \). The group \( D \) is simple modulo its center, and the restriction of \( V \) to \( D \) is irreducible. If the highest weight \( \lambda \) of \( V|_D \) is written as a linear combination \( \sum_i a_i \varpi_i \), where \( \varpi_i \) are the fundamental weights, then \( \sum_i a_i b_i \leq N-1 \), where the \( b_i \) are positive integers determined by the root system of \( D \).

**Proof.** As \( G^o \) is connected, \( G^o = DZ(G^o) \). As \( V|_{G^o} \) is irreducible, \( Z(G^o) \) contains only scalars, as does \( Z(G) \). Thus \( G = DZ(G^o)Z(G) \), and the product \( Z(G^o)Z(G) \) is scalar in \( \text{GL}(V) \). The centralizer of \( D \) in \( \text{GL}(V) \) equals the centralizer of \( G^o = DZ(G^o) \) since \( Z(G^o) \) is scalar. It follows that \( V|_D \) is irreducible. Any tensor decomposition of \( V|_D \) extends to \( G \) since scalars respect any tensor decomposition; it follows that \( V|_D \) is tensor indecomposable and therefore that \( D \) is simple modulo its center. Let \( \lambda \) denote its highest weight.

Let \( g \) be a generating element, and let \( t \in D \) be such that \( g^{-1}t \) is a scalar. \( T \) be a maximal torus of \( D \) containing \( t \), \( R \) the root system of \( D \) with respect to \( T \), and \( \langle \cdot, \cdot \rangle \) the Killing form on \( X^*(T) \otimes \mathbb{R} \). Let
\[
\langle \beta, \alpha \rangle = \frac{2(\beta, \alpha)}{(\alpha, \alpha)},
\]
and fixing a Weyl chamber, let \( \gamma \) denote the root dual to the highest root in \( R \). Thus \( \gamma \) is the highest short root. By [Bo, VIII, §7, Prop. 3(i)], the
maximal arithmetic progression of the form \( \lambda, \lambda - \gamma, \lambda - 2\gamma, \ldots \) contained in the set of weights of \( V \) has length
\[
1 + \langle \lambda, \gamma \rangle = 1 + \sum_i a_i b_i,
\]
where the positive integers \( b_i \) are the coefficients in the representation of the highest root in \( R \) in terms of the simple roots. If this sum exceeds \( N \), then the geometric progression of values
\[
\lambda(t), (\lambda - \gamma)(t), (\lambda - 2\gamma)(t), \ldots
\]
must either take \( \geq N + 1 \) distinct values, or fail \( \square \), or be constant. The first two possibilities are ruled out by hypothesis, and it follows that \( \gamma(t) = 1 \).

If \( w \) belongs to the Weyl group, the same considerations apply to the weight sequence \( w(\lambda), w(\lambda) - w(\gamma), w(\lambda) - 2w(\gamma), \ldots \), so \( w(\lambda)(t) = 1 \). On the other hand, the short weights in a simple root system form a single Weyl orbit and generate the root lattice, so \( \alpha(t) = 1 \) for all roots. This implies that \( t \) lies in the center of \( G \) and therefore that \( \rho(t) \) is scalar, contrary to hypothesis. \( \square \)

One can also formulate the \( N \)-eigenvalue property for complex Lie groups:

**Definition 2.5.** Let \( G_C \) be a reductive complex Lie group and \((\rho, V)\) a faithful irreducible complex representation of \( G_C \). Then \((G_C, V)\) satisfies the \( N \)-eigenvalue property if there exists a semisimple generating element \( g_C \in G_C \) whose conjugacy class generates a Zariski-dense subgroup of \( G_C \), and such that the spectrum of \( \rho(g_C) \) consists of \( N \) eigenvalues satisfying the no-cycle condition.

**Lemma 2.6.** Let \( G_C \) be a reductive complex Lie group and \((\rho, V)\) a faithful irreducible complex representation of \( G_C \). Let \( G \) be a maximal compact subgroup of \( G_C \). Then \((G, V)\) satisfies the \( N \)-eigenvalue property.

**Proof.** Let \( T_C \) denote the Zariski-closure of the cyclic group \((g_C)\) and \( T \subset T_C \) the (unique) maximal compact subgroup. As \( T \) can be regarded as the set of (real) points of a real algebraic group whose complex points give \( T_C \), \( T \) is Zariski-dense in \( T_C \). We can decompose the restriction of \( V \) to \( T_C \) as a direct sum of eigenspaces \( V_\chi \) associated to characters \( \chi \) of \( T_C \). There must be exactly \( N \) such eigenspaces, since any coincidence among \( \chi_1(g_C), \ldots, \chi_{N+1}(g_C) \) gives the same coincidence for the characters on all of \( T_C \). The condition that \( \chi_i(t) \neq \chi_j(t) \) is open and non-empty in \( T_C \) as is the condition that \( \{\chi_1(t), \ldots, \chi_N(t)\} \) satisfy the no-cycle condition. It follows that \( T \) contains an element \( g \) which satisfies both conditions.

As all maximal compact subgroups of \( G_C \) are conjugate, without loss of generality we may assume \( T \subset G \). We can regard \( G \) as the group of real points of a real linear algebraic group whose complex points give \( G_C \) and \( T \subset G \) as a Zariski-closed subgroup. Let \( H \subset G \) denote the smallest normal Zariski-closed subgroup of \( G \) containing \( g \), or equivalently, \( T \). Thus \( H \) can be regarded as the group of real points of an algebraic group which is a normal subgroup of the algebraic group with real locus \( G \). Let \( H_C \) denote the group.
of \( \mathbb{C} \)-points of this subgroup. If \( H \neq G \), then \( H_{\mathbb{C}} \neq G_{\mathbb{C}} \), so \( g_{\mathbb{C}} \in T_{\mathbb{C}} \subset H_{\mathbb{C}} \) is contained in a proper normal subgroup of \( g_{\mathbb{C}} \), contrary to hypothesis. It follows that \( g \) is a generating element for \((G, V)\).

\( \Box \)

3. The 3-eigenvalue problem

In this section, we give an explicit solution of the \( \leq 3 \)-eigenvalue problem, assuming throughout that \( G \) is a compact Lie group which is infinite modulo center.

**Proposition 3.1.** If \((G, V)\) is a pair satisfying the 2-eigenvalue property, and \( \Phi \) denotes the root system of \( G \) and \( \varpi \) the highest weight of \( V \) in the notation of [FLW, 1.1], then \((\Phi, \varpi)\) is one of the following:

1. \((A_r, \varpi_i)\), \(1 \leq i \leq r\).
2. \((B_r, \varpi_r)\).
3. \((C_r, \varpi_1)\).
4. \((D_r, \varpi_i)\), \(i = 1, r - 1, r\).

**Proof.** This is the statement of [FLW, 1.1]. \( \Box \)

Before treating the general 3-eigenvalue problem, we make a detailed study of the \( A_r \) case.

**Lemma 3.2.** Let \((\rho, V)\) be an irreducible representation of \( \text{SU}(n) \) with highest weight \( \varpi \), and \( t \) a non-central element of \( \text{SU}(n) \). Suppose there are at most three eigenvalues of \( \rho(t) \) and they satisfy the no-cycle property. Then one of the following is true:

1. For \( 1 \leq i \leq n - 1 \), \( \varpi = \varpi_i \), and \( t \) has characteristic polynomial \( (x - \lambda_j)^{n-1}(x - \lambda_i^{1-n}) \); the eigenvalues of \( \rho(t) \) are \( \lambda_i, \lambda_i^{1-n} \).
2. For \( 1 \leq i \leq n - 1 \), \( \varpi = \varpi_i \), and \( t \) has characteristic polynomial \( (x - \lambda_j)^{n-2}(x - \lambda_2)^2 \); the eigenvalues of \( \rho(t) \) are \( \lambda_1, \lambda_2, \lambda_2^{-1} \).
3. For \( i \in \{1, 2, n-2, n-1\} \), \( \varpi = \varpi_i \), and \( t \) has eigenvalues \( \lambda_1, \lambda_2 \); the spectrum of \( \rho(t) \) is \( \{\lambda_1, \lambda_2\}, \{\lambda_1^2, \lambda_1 \lambda_2, \lambda_2^2\}, \{\lambda_1^{1-2}, \lambda_1^{-1} \lambda_2^{-1}, \lambda_2^{-2}\} \), or \( \{\lambda_1^{-1}, \lambda_2^{-1}\} \), if \( i = 1, 2, n-2, \) or \( n-1 \) respectively.
4. For \( 1 \leq i \leq n - 1 \), \( \varpi = \varpi_i \), and \( t \) has characteristic polynomial \( (x - \lambda_i^{n-2})(x - \lambda \mu)(x - \lambda \mu^{-1}) \); the eigenvalues of \( \rho(t) \) are \( \lambda_1, \lambda_1 \mu, \) and \( \lambda_1 \mu^{-1} \).
5. For \( i = 1 \) or \( i = n - 1 \), \( \varpi = \varpi_i \), and \( t \) has eigenvalues \( \lambda_1, \lambda_2, \lambda_3 \); the eigenvalues of \( \rho(t) \) are the \( \lambda_j \) or the \( \lambda_j^{-1} \) if \( i = 1 \) or \( i = n - 1 \) respectively.
6. For \( i = 1 \) or \( i = n - 1 \), \( \varpi = 2 \varpi_i \), and \( t \) has eigenvalues \( \lambda_1 \) and \( \lambda_2, \) each of multiplicity at least 2; the eigenvalues of \( \rho(t) \) are \( \{\lambda_1^2, \lambda_1 \lambda_2, \lambda_2^2\} \) or \( \{\lambda_1^{-2}, \lambda_1^{-1} \lambda_2^{-1}, \lambda_2^{-2}\} \) if \( i = 1 \) or \( n - 1 \) respectively.
(7) The highest weight $\varpi$ is $\varpi_1 + \varpi_{n-1}$, and $t$ has eigenvalues $\lambda_1$ and $\lambda_2$, each of multiplicity at least 2; the eigenvalues of $\rho(t)$ are $\lambda_1/\lambda_2$, 1, and $\lambda_2/\lambda_1$.

(8) For $1 \leq i \leq j \leq n - 1$, $\varpi = \varpi_i + \varpi_j$, and $t$ has characteristic polynomial $(x - \lambda)^{n-1}(x - \lambda^{1-n})$; the eigenvalues of $\rho(t)$ are $\lambda_i + j$, $\lambda_i + j - n$, and $\lambda_i + j - 2n$.

In particular, only case (2) can give three eigenvalues not in geometric progression.

Proof. By Proposition 2.4 if $\rho(t)$ has $N \leq 3$ eigenvalues, $\varpi$ is a sum of at most $N - 1$ fundamental weights. If $\varpi = \varpi_i$ and $t$ has eigenvalues $\lambda_1, \ldots, \lambda_n$, the eigenvalues of $\rho(t)$ are

$$\left\{ \prod_{s \in S} \lambda_s \mid S \subset \{1, \ldots, n\}, |S| = i \right\}.$$

Duality exchanges $\varpi_i$ and $\varpi_{n-i}$ so without loss of generality we may assume $i \leq n/2$. If $\lambda_1, \ldots, \lambda_4$ are all distinct, and $n \geq i + 3$ (in particular, this holds if $n \geq 5$), then

$$\{\lambda_j \lambda_5 \lambda_6 \cdots \lambda_{3+i} \mid 1 \leq j \leq 4\}$$

already contains four distinct elements. If $n = 4$ and $i = 2$, two products $\lambda_k \lambda_j$ and $\lambda_k \lambda_l$ are distinct unless $\{i, j\}$ and $\{k, l\}$ are complementary sets, in which case the equality implies $\lambda_i \lambda_j = \pm 1$. At least one of $\lambda_1 \lambda_j$, $2 \leq j \leq 4$ is neither 1 nor $-1$, so there must be at least four elements in the set $\{\lambda_1 \lambda_2, \ldots, \lambda_3 \lambda_4\}$. If

$$\lambda_1 = \lambda_2 = \lambda_3 = \lambda_4 \neq \lambda_5 = \lambda_6,$$

and $i \geq 3$, then

$$\{\lambda_1^2 \lambda_2^3 \lambda_4 \cdots \lambda_{3+i} \mid 0 \leq j \leq 3\}$$

contains a non-constant 4-term geometric progression in the spectrum of $\rho(t)$, contrary to hypothesis. If

$$\lambda_1 = \lambda_2 \neq \lambda_3 = \lambda_4 \neq \lambda_5 \neq \lambda_1,$$

then

$$\{\lambda_2^2 \lambda_3 \lambda_6 \cdots \lambda_{2+i}, \lambda_1 \lambda_2^3 \lambda_6 \cdots \lambda_{2+i}, \lambda_1 \lambda_2 \lambda_3 \lambda_6 \cdots \lambda_{2+i}, \lambda_1 \lambda_2 \lambda_3 \lambda_6 \cdots \lambda_{2+i}\}$$

contains at least four distinct elements unless $\lambda_1 \lambda_3 = \lambda_2^2$ and $\lambda_2 \lambda_3 = \lambda_1^2$, in which case it does not satisfy (1). The remaining possibilities are that $t$ has two distinct eigenvalues, one of multiplicity 1; two distinct eigenvalues, one of multiplicity 2; two distinct eigenvalues of arbitrary multiplicity, and $i$ (or $n - i$) is $\leq 2$; three distinct eigenvalues, two of them of multiplicity 1; or three distinct eigenvalues of arbitrary multiplicity, and $i$ (or $n - i$) is 1.

These give rise to cases (1), (2), (3), (4), and (5) respectively. If $\lambda = \varpi_i + \varpi_j$, $i \leq j$, is among the weights appearing in $V_{\varpi}$, then $\varpi_i - 1 + \varpi_j + 1$ also appears, where we define $\varpi_0 = \varpi_n = 0$. Thus if $\varpi = \varpi_i + \varpi_j$, $i \leq j$,
then either \( \varpi_{i+j}, \varpi_{2n-i-j} \) or \( \varpi + \varpi_{n-1} \) is among the weights of \( V_\varpi \), as \( i+j \) is less than, greater than, or equal to \( n \).

First we consider the case \( i+j = n \). If \( t \) has three distinct eigenvalues \( \lambda_1, \lambda_2, \lambda_3 \), then

\[
|\{\lambda_1/\lambda_2, 1/\lambda_1, 1/\lambda_2, 1/\lambda_3, \lambda_3/\lambda_1, \lambda_1/\lambda_3, \lambda_2/\lambda_3, \lambda_3/\lambda_2\}| \leq 3
\]

implies that the set violates (1.1) with \( n = 3 \). Thus, \( t \) has exactly two eigenvalues \( \lambda_1 \) and \( \lambda_2 \). If \( i \geq 2 \) and \( \lambda_1 \) and \( \lambda_2 \) each occurs with multiplicity \( \geq 2 \), then

\[
\{\lambda_1^2/\lambda_2^2, 1/\lambda_1, \lambda_2/\lambda_1, \lambda_2^2/\lambda_1^2\}
\]

is contained in the spectrum of \( \rho(t) \) since the Weyl orbits of \( \varpi_1 + \varpi_{n-1} \) and \( \varpi_2 + \varpi_{n-2} \) are subsets of the weights of \( V_\varpi \). As \( \lambda_1 \neq \lambda_2 \), either this set contains 5 distinct elements or it violates (1.1). The remaining cases are (7) and the \( i+j = n \) case of (6).

If \( i+j \neq n \), replacing \( V_\varpi \) by its dual if necessary, we can assume that \( i+j < n \). If \( 3 \leq i+j \leq n-3 \), then \( \varpi_{i+j} \) is a weight of \( V_\varpi \), so by the analysis above, \( t \) has two eigenvalues, one with multiplicity one, and we are in case (8). If \( i+j = 2 \), we see that \( 2\varpi_1 \) and \( \varpi_2 \) are both weights of \( V_\varpi \), so if \( \lambda_1, \lambda_2, \lambda_3 \) are eigenvalues of \( t \),

\[
\{\lambda_1^2/\lambda_2^2, \lambda_2^2/\lambda_3 \}
\]

is contained in the spectrum of \( \rho(t) \), contrary to assumption. If there are exactly two eigenvalues, we get (6) and the \( i+j = 1 \) case of (8).

If \( i+j = n-2 \), then \( V_\varpi \) contains all the weights of \( V_{\varpi_{n-2}} \), so \( t \) may have only two eigenvalues, \( \lambda_1 \) and \( \lambda_2 \), by the analysis of the case that \( \varpi \) is a fundamental weight, above. If each occurs with multiplicity \( \geq 2 \) and (without loss of generality) \( \lambda_1 \) occurs with multiplicity \( \geq 3 \), then

\[
\{\lambda_2/\lambda_1^2, 1/\lambda_1, 1/\lambda_2, 1/\lambda_3\}
\]

is a 4-term geometric progression contained in the spectrum of \( \rho(t) \) contrary to hypothesis. If \( i+j = n-1 \), then \( V_\varpi \) contains all the weights of \( V_{\varpi_{n-1}} \). If \( \lambda_1 \) and \( \lambda_2 \) are eigenvalues of \( t \) of multiplicity \( \geq 2 \), then the spectrum of \( \rho(t) \) contains the 4-term geometric progression

\[
\{\lambda_2/\lambda_1^2, 1/\lambda_1, 1/\lambda_2, 1/\lambda_3\}.
\]

If \( t \) has three distinct eigenvalues \( \lambda_1, \lambda_2, \lambda_3 \), then the spectrum of \( \rho(t) \) contains

\[
\{1/\lambda_1, 1/\lambda_2, 1/\lambda_3, \lambda_1/\lambda_2\lambda_3, \lambda_2/\lambda_1\lambda_3, \lambda_3/\lambda_1\lambda_2\}
\]

which either violates the no-cycle condition or contains more than 3 elements. It follows that \( t \) has exactly two eigenvalues, one of multiplicity \( n-1 \). So all of these possibilities are subsumed in case (8).

\[\Box\]

**Theorem 3.3.** If \((G, V)\) is an indecomposable pair satisfying the 3-eigenvalue property, \( \Phi \) denotes the root system of the derived group \( D \) of \( G^0 \), and \( \varpi \).
the highest weight of $V$, then $(\Phi, \varpi)$ is either one of the pairs enumerated in Proposition 2.4 or one of the following:

1. $(A_r, \varpi_i + \varpi_j)$, $1 \leq i \leq j \leq r$.
2. $(B_r, \varpi_i)$, $1 \leq i \leq r - 1$.
3. $(B_r, 2\varpi_1)$.
4. $(C_r, \varpi_i)$, $2 \leq i \leq r$.
5. $(C_r, 2\varpi_1)$.
6. $(D_r, \varpi_i)$, $2 \leq i \leq r - 2$.
7. $(D_r, \varpi)$, $\varpi \in \{2\varpi_{r-1}, \varpi_{r-1} + \varpi_r, 2\varpi_r\}$.
8. $(E_6, \varpi_i)$, $i = 1, 3, 6$.
9. $(E_7, \varpi_i)$, $i = 1, 7$.
10. $(F_4, \varpi_4)$.
11. $(G_2, \varpi_2)$.

If there exists a generating element with three eigenvalues which do not form a geometric progression, then $(\Phi, \varpi)$ is $(A_r, \varpi_1)$ or $(A_r, \varpi_r)$.

Proof. By Proposition 2.4, the root system is simple and if $\varpi = \sum_i a_i \varpi_i$ and the highest root is $\sum_i b_i \alpha_i$, then $\sum_i a_i b_i \leq 2$. By [Bo, Planches], this reduces the possibilities to those listed, together with:

12. $(D_r, \varpi)$, $\varpi \in \{2\varpi_1, \varpi_1 + \varpi_{r-1}, \varpi_1 + \varpi_r\}$.
13. $(E_6, \varpi)$, $\varpi \in \{2\varpi_1, \varpi_2, \varpi_5, 2\varpi_6, \varpi_1 + \varpi_6\}$.
14. $(E_7, \varpi)$, $\varpi \in \{\varpi_2, \varpi_6, 2\varpi_7\}$.
15. $(E_8, \varpi)$, $\varpi \in \{\varpi_1, \varpi_8\}$
16. $(F_4, \varpi_1)$.

To see that the classical cases (1)–(7) above are achieved, we let $G = D$ and $V$ the indicated representation, and we choose the generating element as follows. For $A_r$, we let $g$ be the image of the diagonal element diag$(\lambda^- \alpha, \ldots, \lambda^{-r} \alpha) \in SU(r + 1)$ in $G$. For $B_r$, we let $g$ denote the image of an element in Spin$(2r + 1)$ whose image in SO$(2r + 1)$ is diag$(\lambda, \lambda^{-1}, 1, \ldots, 1)$. For $C_r$, we let $g$ denote the image of the element $(\alpha, \lambda^{-1}, 1, \ldots, 1)$ in Sp$(2r)$. For $D_r$, we let $g$ denote the image of an element in Spin$(2r)$ whose image in SO$(2r)$ is diag$(\lambda, \lambda^{-1}, 1, \ldots, 1)$.

Next we show that the excluded cases (12)–(16) above do not occur. For $D_r$, we consider an element $g$ whose image in SO$(2r)$ has eigenvalues $\lambda_1^{\pm 1}$, $\ldots, \lambda_r^{\pm 1}$. In $V_{2\varpi}$, the eigenvalues of $g$ are $\lambda_1^{\pm 2}$, $\lambda_i^{\pm 1} \lambda_j^{\pm 1}$, and 1. It is easy to see these represent at least 5 distinct values. A similar analysis rules out the remaining cases in (12).

For $E_6$ and $F_4$ we use the existence of equal rank semisimple subgroups of the form $A_2^2$. As these subgroups share a maximal torus with their ambient groups, every generating element $g$ can be conjugated into the subgroup. We use the branching rules tabulated in [MP] to compute the restrictions of $G$-representations via SU$(3)^k \to G$; since the center of SU$(3)^k$ has exponent 3, and since we know that there are no 2-eigenvalue solutions for $F_4$ and $E_6$, there can be no $\leq 3$-eigenvalue solutions coming from central elements of
SU(3)\(^k\) and satisfying (1.1). If \(M(\lambda) \in \text{SU}(3)\) has eigenvalues \(\lambda, \lambda, \lambda^{-2}\), then \(M(\lambda) \times M(\lambda^{-1})\) maps to an element of \(F_4\) which has eigenvalues \(\lambda^{-3}, 1, \lambda^3\) for \(V_{\varpi_4}\). The restriction of \(F_4\) to \(\text{SU}(3)^2\) is

\[
V_{2\mu_2} \boxtimes V_{\mu_1} \oplus V_{2\mu_1} \boxtimes V_{\mu_2} \oplus V_{\mu_1+\mu_2} \boxtimes V_0 \oplus V_0 \boxtimes V_{\mu_1+\mu_2};
\]

the image of any element non-central in both factors has at least four eigenvalues from the first summand; the image of any element central in the second factor but not the first has at least four eigenvalues from the first two summands; the image of any element central in the first factor but not in the second has at least four eigenvalues or the eigenvalues \(\{1, e^{\pm 2\pi i/3}\}\) from the first two summands. For \((E_6, \varpi_1)\), the image of \(M(\lambda) \times M(\lambda) \times 1\) has eigenvalues \(\{\lambda^{-2}, \lambda, \lambda^4\}\), and it is not difficult to see that this is essentially the only way to get three eigenvalues. For \((E_6, \varpi_2)\), the image of \(M(\lambda) \times M(\lambda) \times 1\) has eigenvalues \(\{\lambda^{-3}, 1, \lambda^3\}\). To see that the excluded cases (13) do not give solutions to the 3-eigenvalue problem, we note that

\[
V_{\varpi_2|A_2^3} = V_{\mu_1+\mu_2} \boxtimes V_{\mu_2} \boxtimes V_{\mu_2} \oplus V_{\mu_2} \boxtimes V_{\mu_1+\mu_2} \boxtimes V_{\mu_1} \oplus \cdots ;
\]

\[
V_{2\varpi_1|A_2^3} = V_{\mu_1+\mu_2} \boxtimes V_{\mu_2} \boxtimes V_{\mu_2} \oplus V_{\mu_2} \boxtimes V_{\mu_1+\mu_2} \boxtimes V_{\mu_1} \oplus \cdots ;
\]

\[
V_{\varpi_1+\varpi_6|A_2^3} = V_{2\mu_1} \boxtimes V_{\mu_1} \boxtimes V_{\mu_1} \oplus V_{\mu_1} \boxtimes V_{2\mu_1} \boxtimes V_{\mu_1} \oplus \cdots .
\]

These summands are already enough to guarantee that if \((E_6, \varpi_2), (E_6, 2\varpi_1)\), or \((E_6, \varpi_1 + \varpi_6)\) satisfies the 3-eigenvalue condition, any generating element in \(A_2^3\) must be central in two of the three factors and have eigenvalues \(\lambda, \lambda, \lambda^{-2}, \lambda^3 \neq 1\), in the third. However, if \(\omega^3 = 1\), neither

\[
\{\lambda^{-3}, 1, \lambda^3, \omega\lambda, \omega\lambda^{-2}\}
\]

nor

\[
\{\lambda^2, \lambda^{-1}, \lambda^{-4}, \omega\lambda, \omega\lambda^{-2}\}
\]

can have order \(\leq 3\) and satisfy the no-cycle property.

For \(E_n, n \geq 7\), we use the equal rank subgroups \(A_n\). Again, \([MP]\) gives the restriction of \(V_{\varpi}\) to \(\text{SU}(n+1)\). The following table lists all irreducible components of these restrictions for all possible \(\varpi\). It also specifies the
eigenvalues in $V_\varpi$ for the image of the scalar matrix $\zeta I$ and the matrix $M(\lambda)$:

| $\Phi$ | $\varpi$ | $\{\mu_i\}$ | $\zeta I$ e-values | $M(\lambda)$ e-values |
|--------|----------|---------------|---------------------|-----------------------|
| $E_7$  | $\varpi_1$ | $\mu_1 + \mu_7, \mu_4$ | $\pm 1$ | $\lambda^{-8}, \lambda^{-4}, 1, \lambda^4, \lambda^8$ |
| $E_7$  | $\varpi_2$ | $\mu_1 + \mu_5, \mu_3 + \mu_7, 2\mu_1, 2\mu_7$ | $\pm i$ | $\lambda^{-14}, \lambda^{-10}, \lambda^{-6}, \lambda^{-2}, \lambda^2, \lambda^6, \lambda^{10}, \lambda^{14}$ |
| $E_7$  | $\varpi_6$ | $\mu_1 + \mu_3, \mu_5 + \mu_7, \mu_1 + \mu_7, \mu_2 + \mu_6$ | $\pm 1$ | $\lambda^{-12}, \lambda^{-8}, \lambda^{-4}, 1, \lambda^4, \lambda^8, \lambda^{12}$ |
| $E_7$  | $\varpi_7$ | $\mu_2, \mu_6$ | $\pm i$ | $\lambda^{-6}, \lambda^{-2}, \lambda^2, \lambda^6$ |
| $E_7$  | $2\varpi_7$ | $0, \mu_4, \mu_2 + \mu_6, 2\mu_2, 2\mu_6$ | $\pm 1$ | $\lambda^{-12}, \lambda^{-8}, \lambda^{-4}, 1, \lambda^4, \lambda^8, \lambda^{12}$ |
| $E_8$  | $\varpi_1$ | $\mu_3, \mu_6, \mu_1 + \mu_8$ | $1, e^{\pm 2\pi i/3}$ | $\lambda^{-9}, \lambda^{-6}, \lambda^{-3}, 1, \lambda^3, \lambda^6, \lambda^9$ |
| $E_8$  | $\varpi_8$ | $\mu_1 + \mu_2, \mu_1 + \mu_5, \mu_1 + \mu_8, \mu_2 + \mu_7, \mu_4 + \mu_8, \mu_7 + \mu_8$ | $1, e^{\pm 2\pi i/3}$ | $\lambda^{-15}, \lambda^{-12}, \lambda^{-9}, \lambda^{-6}, \lambda^{-3}, 1, \lambda^3, \lambda^6, \lambda^9, \lambda^{12}, \lambda^{15}$ |

It follows that neither scalar matrices nor matrices of the form $M(\lambda)$ give rise to 3-eigenvalue solutions. By Lemma 3.2, the only possible solutions to the 3-eigenvalue problem for $E_7$ and $E_8$ are the pairs $(E_7, \varpi_1)$, $(E_7, \varpi_7)$, and $(E_8, \varpi_1)$. For the first, an element of SU(8) with eigenvalues $\lambda$, $\lambda$, $\lambda$, $\lambda$, $\lambda$, $\lambda$, $\lambda^{-3}$, $\lambda^{-3}$ maps to an element of $E_7$ with eigenvalues $\lambda^{-4}, 1, \lambda^4$. For the second, an element of SU(8) with eigenvalues $\lambda$, $\lambda$, $\lambda$, $\lambda$, $\lambda^{-1}$, $\lambda^{-1}$, $\lambda^{-1}$, $\lambda^{-1}$ maps to an element of $E_7$ with eigenvalues $\lambda^{-2}, 1, \lambda^2$. For $(E_8, \varpi_1)$, the only possibility is an element of SU(9) with $\lambda_1$ of multiplicity 7 and $\lambda_2$ of multiplicity 2. This maps to an element of $E_8$ with two three-term geometric progressions of eigenvalues: $\lambda_1^3, \lambda_2^3, \lambda_1 \lambda_2^2$, and $\lambda_1 \lambda_2^{-1}, 1, \lambda_1^{-1} \lambda_2$. To have three eigenvalues in all, we must have $\lambda_1^3 = \lambda_1 \lambda_2^{-1}$, which together with $\lambda_1^3 \lambda_2^{-1} = 1$ implies that the eigenvalues are all equal, which we have already seen is not a possibility.

The case of $G_2$ is trivial.

When there are three eigenvalues not in geometric progression, the representations cannot be self-dual, and if $\phi = A_r$, then $\varpi \in \{\varpi_1, \varpi_r\}$ by Lemma 3.2. The only remaining cases for which $V_\varpi$ is not self-dual are $(D_r, V_{2\varpi_{r-1}})$ and its dual (when $r$ is odd) and $(E_6, \varpi_1)$ and its dual. In the first case, as $r$ is odd, the Weyl orbit of $\varpi_1$ lies in the set of weights of both $V_{2\varpi_{r-1}}$ and $V_{2\varpi_r}$. The eigenvalues contributed by these weights come in mutually inverse pairs; if there are $\leq 3$ but not three in geometric progression, then there must be two: $\lambda$ and $\lambda^{-1}$, which are distinct from one another. Then the Weyl orbit of $\varpi_3$ also lies in the set of weights of $V_{\varpi}$, so $\lambda^3, \lambda, \lambda^{-1}, \lambda^{-3}$ are all eigenvalues of $\rho(t)$, which is absurd. In the second case, restricting from $E_6$ to SU(6) $\times$ SU(2), we get

$$V_{\varpi_1} \otimes V_{\varpi_1} \oplus V_{\varpi_2} \otimes V_6.$$ 

The second summand contributes $\leq 2$ eigenvalues or 3 eigenvalues not in geometric progression, so an inverse image $(g_1, g_2)$ of the generating element
must be a scalar $\zeta$ in $SU(6)$ (and therefore $\zeta^6 = 1$). The eigenvalues of $g$ in the first summand are $\{\zeta \lambda, \zeta^4 \lambda^{-1}\}$ which are in geometric progression, contrary to assumption.

4. The asymptotic $N$-eigenvalue condition

In this section we consider what can be said when the eigenvalues of a generating element are sufficiently general. One hypothesis which is strong enough for our purposes is that the eigenvalues are distinct $r$th roots of unity where $r$ is a sufficiently large prime. We consider a somewhat more general condition.

**Proposition 4.1.** Let $T \cong U(1)^d$ be a torus and $U$ an open neighborhood of the identity in $T$. There exists a finite set $S$ of characters $\chi : T \to U(1)$ and an integer $m$ such that if $n$ is a positive integer and $t \in T$ an $n$-torsion point, at least one of the following must be true:

1. There exists $\chi \in S$ such that $\chi(t) \neq 1$ has order $\leq m$.
2. There exists an integer $k$ relatively prime to $n$ such that $t^k \in U$.

**Proof.** We use induction on dimension, the proposition being trivial in dimension 0.

By Urysohn’s lemma there exists a continuous function $f : T \to [0, 1]$ such that $f(x) = 0$ for $x \not\in U$ and $f(x) = 1$ in some neighborhood of the identity. It is well-known (see, e.g. [SW, VII Th. 1.7]) that finite linear combinations of characters are dense in the $L^\infty$ norm on the set of continuous functions on $T$. It follows that there exists a real-valued finite sum $f(x) := \sum_{\chi \in S} a_{\chi} \chi(x)$ such that $f(x) < 0$ for all $x \in T \setminus U$ and $a_0 = \int f(x) \, dx > 0$. Enlarging $S$ if necessary, we may assume without loss of generality that if $n\chi \in S$ for some positive integer $n$, then $\chi \in S$.

Suppose $\chi(t) = 1$ for some non-trivial character $\chi \in S$. Let $\lambda \in S$ denote a primitive character in $S$ and $k$ a positive integer such that $\chi = k\lambda$. If $m$ is taken greater than the value of $k$ associated with any character in $S$, either (1) is satisfied or $\lambda(t) = 1$. As $\lambda$ is primitive, ker $\lambda$ is a subtorus of $T$. As there are only finitely many subtori arising in this way, the proposition follows by induction.

We may therefore assume that the order of $\chi(t)$ is greater than $m$ for each $\chi \in S$. We have

$$\sum_{\{k \in [0, n] \cap \mathbb{Z} : (k, n) = 1\}} f(t^k) = a_0 \phi(n) + \sum_{\chi \in S \setminus \{0\}} \sum_{\{k \in [0, n] \cap \mathbb{Z} : (k, n) = 1\}} a_{\chi} \chi(t^k).$$

If $n_{\chi}$ is the order of $\chi(t)$, then

$$\sum_{\{k \in [0, n] \cap \mathbb{Z} : (k, n) = 1\}} \chi(t^k) = \frac{\phi(n)}{\phi(n_{\chi})} \sum_{\{k \in [0, n_{\chi}] \cap \mathbb{Z} : (k, n_{\chi}) = 1\}} \chi(t^k) = \frac{\mu(n) \phi(n)}{\phi(n_{\chi})}.$$
Choosing \( m \) large enough that for all \( n_\chi > m \),
\[
\sum_{\chi \in S \setminus \{0\}} |a_\chi| \leq \phi(n_\chi)a_0,
\]
we conclude that
\[
\sum_{\{k \in [0,n] \cap \mathbb{Z} | (k,n)=1\}} f(t^k) \geq 0
\]
and therefore that \( t^k \in U \) for some \( k \) prime to \( n \).

\[\square\]

**Theorem 4.2.** For every integer \( N \geq 2 \) there exists an integer \( m \) such that if \((G,V)\) satisfies the \( N \)-eigenvalue property with a generator \( g \) with eigenvalues \( \lambda_1, \ldots, \lambda_N \), and \( G \) is finite modulo its center, then the group \( \langle \lambda_i\lambda_j^{-1} \rangle \) generated by ratios of eigenvalues of \( \rho(g) \) contains a non-trivial root of unity of order less than \( m \).

**Proof.** If \( G \) is finite modulo its center and acts irreducibly on \( V \), then either \( G^\circ \) is trivial or it consists of all scalars of absolute value 1. In the latter case, we can replace \( g \) by \( \text{det}(g)^{1/\dim(V)}g \) for any choice of root, and the resulting conjugacy class still satisfies the \( N \)-eigenvalue property, generates a subgroup of \( G \cap \text{SL}(V) \) (which is finite), and determines the same group of eigenvalue ratios \( \langle \lambda_i\lambda_j^{-1} \rangle \). Without loss of generality, therefore, we may assume \( G \) is finite.

Any automorphism of \( \mathbb{C} \) determines an automorphism of the abstract group \( \text{GL}_n(\mathbb{C}) \) for each \( n \). Consider the quotient \( T = U(1)^n/U(1) \) of the diagonal unitary matrices by the unitary scalar matrices. Let \( U \subset T \) denote the image of \( A^n \) in \( T \), where \( A \) is the arc from \(-\pi/6\) to \( \pi/6 \), and let \( n \) be the order of the group generated by the eigenvalues of \( g \). We apply Proposition 4.1 to obtain \( m \) large enough that our hypotheses imply the existence of a field automorphism \( \sigma \) of \( \mathbb{C} \) such that all the eigenvalues of \( \sigma(\rho(g)) \) lie in an arc of length \( \leq \pi/3 \) on the unit circle. By [B] Theorem 8, this implies that the representation representation \( \sigma \circ \rho \) is imprimitive. As the conjugacy class of \( g \) generates \( G \), the element \( g \) itself must satisfy the hypothesis of Lemma 2.1 and therefore the spectrum of \( \sigma(\rho(g)) \) does not satisfy (1.1). As this property is stable under Galois action, the spectrum of \( \rho(g) \) fails to satisfy (1.1), contrary to hypothesis.

\[\square\]

**Corollary 4.3.** For every integer \( N \geq 2 \) there exists an integer \( m \) such that if \((G,V)\) satisfies the \( N \)-eigenvalue property with a generator \( g \) of prime order \( r \), then \( r < m \) or \( G \) is infinite modulo its center.

We remark that it is probably possible to prove a stronger version of this corollary, in which a good bound is given for \( m \), using [Z] as a starting point.

5. **Application to Hodge-Tate theory**

Let \( \overline{\mathbb{Q}}_p \) be an algebraic closure of \( \mathbb{Q}_p \), and \( \mathbb{C}_p \) denote the completion of \( \overline{\mathbb{Q}}_p \). Let \( K \) and \( L \) be subfields of \( \overline{\mathbb{Q}}_p \) finite over \( \mathbb{Q}_p \), and let \( \Gamma_K := \text{Gal}(\overline{\mathbb{Q}}_p/K) \).
Let $V_L \cong L^d$ be a finite-dimensional $L$-vector space and $\rho_L: \Gamma_K \rightarrow \text{GL}(V_L)$ a continuous representation. Then $\Gamma_K$ acts on both factors of $V_{C_p} := V_L \otimes_{\mathbb{Q}_p} \mathbb{C}_p$. The representation is said to be Hodge-Tate if $V_{C_p}$ decomposes as a direct sum of factors $V_{iC_p}$ such that $\Gamma_K$ acts on $V_i$ through the $i$th tensor power of the cyclotomic character. If $X$ is a complete non-singular variety over $K$ and $\tilde{X}$ is obtained from $X$ by extending scalars to $\mathbb{Q}_p$, then $V_L := H^k(\tilde{X}, L)$ is Hodge-Tate for all non-negative integers $k$, and the factors $V_{iC_p}$ are non-zero only if $0 \leq i \leq k$ (shown).

Let $G_L$ denote the Zariski-closure of the image of $\rho_L(\Gamma_K)$ in $\text{GL}_d$. By the axiom of choice, any two uncountable algebraically closed fields of characteristic zero whose cardinalities are the same are isomorphic. Therefore, $\mathbb{C} \cong \mathbb{C}_p$, and extending scalars, we can view $G_C$ as a complex algebraic group. Let $G$ denote a maximal compact subgroup of $G_C$. The inclusion $G_C \subset \text{GL}(V_C)$ gives $G$ a complex representation which we denote $(\rho, V)$. If $\rho_L$ is absolutely irreducible, then $V_C$ is an irreducible representation of $G_C$ and therefore of $G$.

Although $G_L$ need not be connected, by passing to a finite extension $K'$ of $K$ (i.e., replacing $\Gamma_K$ by a normal open subgroup) we can replace $G_L$ by its identity component. Therefore, in trying to understand what Lie algebras and Lie algebra representations can arise from Hodge-Tate structures with specified weights, without loss of generality we may assume that $G_L$ is connected.

**Definition 5.1.** Let $G_C$ be a connected reductive algebraic group over $\mathbb{C}$, and $V$ a faithful complex representation of $G_C$. We say that $(G_C, V)$ is of $N$-eigenvalue type if for every almost simple normal subgroup $H_C$ of $G_C$ and every irreducible factor $W$ of $V|_{H_C}$, the image of $H_C$ in $\text{GL}(W)$ satisfies the $N_W$-eigenvalue property for some $N_W \leq N$.

**Lemma 5.2.** Let $G_C$ be a connected reductive complex Lie group and $(\rho, V)$ a faithful representation. Let $g_C \in G_C$ be semisimple elements generating a Zariski-dense subgroup of $G_C$, such that the spectrum of $\rho(g_C)$ has $N$ eigenvalues satisfying the no-cycle condition. Then $(G_C, V)$ is of $N$-eigenvalue type.

**Proof.** Let $D_C$ denote the derived group of $G_C$. The universal cover $\tilde{D}_C$ factors into simply connected, almost simple groups $G_{jC}$. Every irreducible factor $W$ of $V$ restricts to an irreducible representation of $\tilde{D}_C$ which decomposes as $W_1 \oplus \cdots \oplus W_k$, where $W_j$ is an irreducible representation of $G_{jC}$.

Each $g_C$ in our generating set factors as $d_C z_C$, where $z_C$ lies in the center of $G_C$. We choose $d_C \in D_C$ lying over $d_C$, and let $g_{ijC}$ denote the $G_{jC}$ coordinate of $d_C$. For each $j$, there exists $W$ such that $W_j$ is non-trivial and $i$ such that $g_{ijC}$ does not lie in the center of $G_{jC}$. As $g_C$ is semisimple, the same is true of $d_C$ and therefore $d_C$ and therefore $g_{ijC}$. Moreover, it has at most $N$ eigenvalues on $W_j$ and they satisfy (1.4), since if
Theorem 5.3. If $\rho$ is an absolutely irreducible Hodge-Tate representation of $G_K$ with $N$ distinct weights, then $(G^\circ_c, V)$ is of $N$-eigenvalue type.

Proof. The grading of $V$ which assigns $V$ degree $i$ uniquely determines a cocharacter $h: G_m \to G^\circ_c$ such that $\rho h$ acts isotypically on $V$ by the $i$th power character. By Sen, $G^\circ_L$ is the smallest $L$-algebraic subgroup of $GL_d$ which contains $h(G_m)$. Thus $\{h^\sigma(G_m) \mid \sigma \in \text{Aut}_L(C)\}$ generates $G^\circ_c$. If $u \in C^\times$ is of infinite order, then any element $g \in h^\sigma(u)$ (Zariski-topologically) generates $h^\sigma(G_m)$. Together, the $g$ generate $G^\circ_c$. There are exactly $N$ distinct eigenvalues of $\rho(g)$ and they satisfy the no-cycle condition. The theorem now follows from Lemma 5.2. □

Theorem 5.4. Assume that the Fontaine-Mazur conjecture [PM Conj. 5a] holds. If $X$ is a complete non-singular variety over a number field $K$, $k$ is a non-negative integer, $G_C$ is the complexification of the Zariski closure of $\text{Gal}(K/K)$ in $\text{Aut}(H^k(X, \mathbb{Q}_p))$, and $V = \text{Aut}(H^k(X, \mathbb{Q}_p)) \otimes_{\mathbb{Q}_p} \mathbb{C}$, then $(G^\circ_c, V)$ is of $k$-eigenvalue type.

Proof. As $X$ has good reduction over $K$, there exists a rational integer $M$ such that $X$ is the generic fiber of a smooth proper scheme $\mathcal{X}$ over $\mathcal{O}_K[1/M]$, where $\mathcal{O}_K$ is the ring of integers of $K$. Thus, the homomorphism $\text{Gal}(\bar{K}/K) \to \text{Aut}(H^k(X, \mathbb{Q}_p))$ factors through $\rho: \Gamma_{K,Mp} \to \text{GL}_n(\mathbb{Q}_p)$, the Galois group over $K$ of the maximal subfield of $\bar{K}$ unramified over any prime of $\mathcal{O}_K$ not dividing $Mp$.

For each prime $v$ of $\mathcal{O}_K$ dividing $Mp$, we fix an embedding $K \hookrightarrow \bar{K}_v$ and therefore an embedding $\Gamma_{G_v} \hookrightarrow \Gamma_{K,Mp}$. Let $G_v$ be the Zariski-closure of $\rho(\Gamma_{K,Mp})$ in $\text{GL}_n$, $G_v$ is the Zariski-closure of $\rho(\Gamma_{G_v})$, and $G_p$ the normal subgroup of $G$ generated by $G_v$ for all $v$ lying over $p$. Replacing $K$ by a finite extension, we may assume that $G_v$ is connected for all such $v$, so $G_p$ is generated by conjugates of the $G_v$. By Theorem 5.3, the complexification $G_p\mathbb{C}$, together with its $n$-dimensional representation, is of $k$-eigenvalue type. If $G_p$ is of finite index in $G$, the theorem follows. Otherwise, there exists a homomorphism $\Gamma_{K,Mp} \to G(\mathbb{Q}_p)/G_p(\mathbb{Q}_p)$ with Zariski-dense, and therefore infinite $p$-adic analytic image. By construction, this homomorphism is unramified at all primes over $v$. Such a homomorphism cannot exist according to the Fontaine-Mazur conjecture. □

Corollary 5.5. If the Fontaine-Mazur conjecture is true, then for every complex non-singular variety $X$ over a number field $K$, the Zariski closure of the image of $\text{Gal}(K/K)$ in $\text{Aut}(H^2(X, \mathbb{Q}_p))$ has no factor of type $E_8$. 

6. Application to braid group representations

Artin’s braid group $B_m$ is generated by $\sigma_1, \ldots, \sigma_{m-1}$ subject to relations
\[ \sigma_i \sigma_j = \sigma_j \sigma_i \quad \text{if } |i - j| \geq 2, \quad \sigma_i \sigma_{i+1} \sigma_i = \sigma_{i+1} \sigma_i \sigma_{i+1} \quad \text{for } 1 \leq i \leq m - 1. \]

In [FLW], the closed images of the unitary $q = e^{2\pi i/\ell}$ Hecke algebra representations of the braid groups are completely analyzed (completing a program initiated by Jones) for $\ell \geq 5$ and $\ell \neq 6$. In this section, we will carry out a similar analysis. We also discuss the situations in which the braid group representations arising from quantum groups at roots of unity satisfy the 3-eigenvalue condition.

6.1. Set-up. Given an irreducible unitary representation $(\rho, V)$ of $B_m$ there are three distinct possibilities for $G = \rho(B_m)$

(1) $G/Z(G)$ is finite
(2) $SU(V) \subset G$
(3) $G/Z(G)$ is infinite, but $SU(V) \not\subset G$.

While the first (finite group) and third (non-dense) possibilities are interesting, we will focus on the second. There are a number of reasons for doing this. Firstly, we will see that $SU(V) \subset G$ is the generic situation, while the other (non-dense) cases require a case-by-case analysis that we will carry out in a separate work. Also, density is crucial for applications to quantum computing—our original motivation. Lastly, the application of Theorem 3.3 leads most directly to the conclusion $SU(V) \subset G$, i.e., by showing that $(G, V)$ is an indecomposable pair satisfying the 3-eigenvalue property for which the three eigenvalues do not form a geometric progression. Nearly all of the finite group/non-dense examples come from pairs having eigenvalues in geometric progression which will be considered in a forthcoming paper by the first two authors. We proceed with the following program:

(1) Determine which representations have exactly three eigenvalues.
(2) Determine conditions for the representations from (1) to be unitary.
(3) Determine when the three eigenvalues from (1) and (2) satisfy the no-cycle condition. This will give us all pairs $(G, V)$.
(4) Determine when the three eigenvalues from (1) and (2) are not in geometric progression. Although this does not ensure density, it does guarantee the pair $(G, V)$ is indecomposable, as three eigenvalues coming from a decomposable pair must be in geometric progression by Lemma 2.2.
(5) Determine when $G$ is infinite modulo the center for the cases not excluded by (1)-(4).

6.2. BMW-algebra representations of the braid groups. We apply the strategy outlined above to BMW-algebras, first recalling what is well-known and then proceeding to the subsequent steps.
6.2.1. Definitions and combinatorial results. Most of the material here can be found in [Wz1], and we summarize the details germane to the problem, carrying out steps (1) and (2) in the above program.

The Birman-Wenzl-Murakami (BMW) algebras are a sequence of finite dimensional algebras equipped with Markov traces. They can be described as quotients of the group algebra \( \mathbb{C}(r,q)B_m \) of Artin’s braid group where \( r \) and \( q \) are complex parameters. The precise definition of the BMW-algebra \( C_m(r,q) \) is:

**Definition 6.1.** Let \( g_1, g_2, \ldots, g_{m-1} \) be invertible generators satisfying the braid relations (B1) and (B2) above and:

(R1) \((g_i - r^{-1})(g_i - q)(g_i + q^{-1}) = 0\)

(R2) \(e_i g_{i-1}^\pm e_i = r^\pm e_i\), where

(E) \((q - q^{-1})(1 - e_i) = g_i - g_i^{-1}\) defines \(e_i\).

The relations (R2) can be best understood by pictures where \( g_i \) is the the braid generator \( \sigma_i \) and \( e_i \) is the \( i \)-th generator of the Temperley-Lieb algebra. Relation (R1) shows that the image of \( g_i \) in any representation of \( C_m(r,q) \) has 3 eigenvalues: \( r^{-1}, q \) and \( -q^{-1} \). When \( r \neq \pm q^n \) and \( q \) is not a root of unity, each BMW-algebra \( C_m(r,q) \) is finite-dimensional and semisimple with simple components labeled by Young diagrams with \( m - 2j \geq 0 \) boxes for \( j \in \mathbb{N} \). In other words, the BMW-algebra is a direct sum of full matrix algebras. For each simple component \( C_{m,\lambda} \) let \( V_{m,\lambda} \) be the unique non-trivial simple \( C_{m,\lambda} \)-module. Then the branching rule for restricting \( V_{m,\lambda} \) to \( C_{m-1}(r,q) \) is:

\[
V_{m,\lambda} \cong \bigoplus_{\mu \leftrightarrow \lambda} V_{m-1,\mu}
\]

where \( V_{m-1,\mu} \) is a simple \( C_{m-1}(r,q) \)-module and \( \mu \) is a Young diagram with \( m - 1 - 2j \geq 0 \) boxes obtained from \( \lambda \) by adding/removing a box to/from \( \lambda \). This description of inclusions among to BMW-algebras can be neatly encoded in a graph called the *Bratteli diagram*. The graph consists of vertices labelled by \( (m, \lambda) \) with \( |\lambda| = m - 2k \) arranged in rows (labelled by integers \( m \)). Vertices in adjacent rows are connected if the their labels differ by 1 in the first entry and by one box in the second. The dimension of \( V_{m,\lambda} \) can thus be computed by adding up the dimensions of the \( V_{m-1,\mu} \) whose labels are connected to \( (m, \lambda) \) by an edge. We obtain representations of \( B_m \) on \( \bigoplus \lambda V_{m,\lambda} \) via the map \( \sigma_i \rightarrow g_i \in C_m(r,q) \).

We are interested in obtaining unitary representations of \( B_m \) from BMW-algebras, so we must consider semisimple quotients with \( r \) and \( q \) specialized at roots of unity. Specifically, we let \( r = q^n \) for \(-1 \neq n \in \mathbb{Z} \) and \( q = e^{\pi i/\ell} \) \((\ell \neq 1)\), i.e., a primitive \( 2\ell \)th root of unity. If a given irreducible representation is unitary for \( q = e^{\pi i/\ell} \), it will remain so for \( q = e^{-\pi i/\ell} \). For other choices of primitive roots of unity we cannot expect to have unitarity. The quotient of each specialized BMW-algebra by the annihilator of the trace \( A_m := \{ a \in C_m(r,q) : tr(ab) = 0 \quad \text{for all } b \} \) is semisimple and we denote
it $C_m(q^n, q)$ (where $q$ is understood to be $e^{\pi i/\ell}$). The branching rules and simple decomposition described above for the generic case still essentially apply to $C_m(q^n, q)$, except that some components no longer appear, and fewer Young diagrams are needed to describe the persisting components (for all $m$). Precisely which components survive depends on the values $\ell$ and $n$, and the derivation can be found in [Wz1], the results of which we will describe below. For now it is enough to note that each simple component (sector) that does survive the quotient gives us an irreducible representation of $\mathcal{B}_m$. Let $\rho_{(m,\lambda)}^{(n,\ell)}$ acting on $V_{(m,\lambda)}^{(n,\ell)}$ be the representation of $\mathcal{B}_m$ corresponding to the simple component of $C_m(q^n, q)$ labeled by $\lambda$. Since the conjugacy class of $\rho_{(m,\lambda)}^{(n,\ell)}(\sigma_1)$ generates the closed image of $\mathcal{B}_m$ topologically, there is a chance that the pair

$$
(\rho_{(m,\lambda)}^{(n,\ell)}(\mathcal{B}_m), V_{(m,\lambda)}^{(n,\ell)})
$$

satisfies the 3-eigenvalue property.

As a first step we need to know the conditions under which the image of $\sigma_1 \in \mathcal{B}_m$ under $\rho_{(m,\lambda)}^{(n,\ell)}$ has 3 distinct eigenvalues. The answer is well-known to experts (see [Wz1]): for $m \geq 3$, the image of $\sigma_1$ under the irreducible representation $\rho_{(m,\lambda)}^{(n,\ell)}$ has 3 distinct eigenvalues precisely when $|\lambda| < m$ and $C_{3,\square}$ is three dimensional. This is equivalent to the requirement that the corresponding simple component $C_{m,\lambda}$ contains the simple component $C_{3,\square}$. This is most easily seen by considering the Bratteli diagram as described above. It is shown in [Wz1] that $C_m(q^n, q)/A_m \cong \mathcal{I}_m \oplus \mathcal{I}_m(q^2)$ where $\mathcal{I}_m(q^2)$ is a quotient of the Iwahori-Hecke algebra of type $A_{m-1}$, and $\mathcal{I}_m$ is the ideal generated by $e_{m-1}$ (see [Wz1]). The Young diagrams labeling simple components of $\mathcal{I}_m(q^2)$ have $m$ boxes, whereas those of $\mathcal{I}_m$ have $m - 2j$ boxes for some $j \geq 1$. The representations of $\mathcal{B}_m$ corresponding to the Iwahori-Hecke algebra part of $C_m(q^n, q)$ have been studied in [FWL], where they are analyzed using the solution to the 2-eigenvalue problem. Thus the image of $\sigma_1$ on the irreducible representation $V_{m,\lambda}$ ($m \geq 3$) has (exactly) 3 distinct eigenvalues precisely when $|\lambda| < m$ and $C_{3,\square}$ is 3-dimensional in which case the eigenvalues are $\{q^{-n}, q, -q^{-1}\}$. We can eliminate many redundant cases using isomorphisms (see [TBW2]):

$$
C_m(q^n, q) \cong C_m(-q^{-n}, q) \cong C_m(-q^n, -q) \cong C_m(q^{-n}, q^{-1}).
$$

We describe the restrictions more precisely in the following, which is a reformulation of several results in [Wz1] and [R]. Denote by $\lambda_i$ (resp. $\lambda'_i$) the number of boxes in the $i$th row (resp. column) of the Young diagram $\lambda$.

**Proposition 6.2.** Let $q = e^{\pi i/\ell}$ and $m \geq 3$.

1. The matrix algebra $C_{3,\square}$ is a simple 3-dimensional subalgebra of $C_m(q^n, q)$, provided one of the following conditions holds:
   a. $n = 1$ and $\ell \geq 3$
   b. $n = 2$ and $\ell \geq 4$
Proof. The only way \( \pm 6.2 \) that \( \ell \) of \( \rho \) are unitary was proved in \([Wz1]\). The full (reducible) representations of Lemma 6.4. accomplished with simple computations. We have:

- reducible representations satisfying the conditions of Proposition 6.2 are \( \mathbb{C} \) factoring over \( 6.2.2 \).
- irreducible sector is unitary.

get unitarizable representations, but not uniformly, so that for each fixed \( q \), \( - \pi i / \ell \) fails the no-cycle property if and only if \( (1) \) and \( \lambda \) is in the corresponding set of admissible Young diagrams in (2). These representations are unitary except possibly in case (d).

Remark 6.3. Observe that the set in 2(a) is infinite and independent of \( \ell \). The other four labeling sets are finite, and it is easy to see that the corresponding Bratteli diagrams are periodic. In the case \( n = 2 \) there is a slight exception to the rule for constructing the Bratteli diagram: the diagrams labeled by \( [\ell - 1, 1] \) and \( [\ell - 1] \) are not connected by an edge (see \([Wz1]\), Prop. 6.1). The fact that the representations in (a), (b), (c) and (e) are unitary was proved in \([Wz1]\). The full (reducible) representations of \( B_m \) factoring over \( \mathbb{C}_m(q^n, q) \) corresponding to case (d) were shown in \([R1]\) to be non-unitarizable for any \( q \) when \( \ell > 2(-n + 1) \). This leaves only finitely many possible \( \ell \) for each fixed \( n \), and even in these cases one can use the techniques of \([R1]\) to show that for \( q = e^{\pi i / \ell} \) one does not get unitarity except in degenerate cases. Restricting to the irreducible sectors one may get unitarizable representations, but not uniformly, so that for \( m \gg 0 \) no irreducible sector is unitary.

6.2.2. Cycles and geometric progressions. The eigenvalues of any of the irreducible representations satisfying the conditions of Proposition 6.2 are \( \{q, -q^{-1}, q^n\} \), with \( q = e^{\pi i / \ell} \). Steps (3) and (4) of the program can be accomplished with simple computations. We have:

Lemma 6.4. Let \( n, \ell \) and \( \lambda \) be as in Proposition 6.2. Then the eigenvalues of \( \rho_{(m, \lambda)}(\sigma_1) \):

1. fail the no-cycle property if and only if \( n = 1 \) or \( (n, \ell) = (3, 6) \) and
2. are in geometric progression if and only if \( n \in \{3, \ell - 3, \pm \ell / 2\} \).

Proof. The only way \( \{q, -q^{-1}, q^n\} \) can fail the no-cycle condition is if it contains a coset of \( \{\pm 1\} \) or \( \{1, e^{2\pi i / 3}, e^{4\pi i / 3}\} \). With the restrictions in Prop. 6.2 that \( \ell \geq n + 2 \) for \( n > 0 \) and \( \ell \geq 4 - n \) for \( n < 0 \) as well as \( \ell \geq 3 \) one
checks that only \( n = 1 \) and \((n, \ell) = (3, 6)\) fail no-cycle. For the eigenvalues to be in geometric progression (still satisfying the conditions of Prop. 6.2) we check the solutions of \( \lambda_1 \lambda_2 - (\lambda_3)^2 = 0 \) for the three possible assignments of \( \lambda_3 \). These yield the three solutions for \( n \) above.

\[ \square \]

Remark 6.5. All of the exceptional cases \( n \in \{1, 3, \ell - 3, \pm \ell/2\} \) will be considered in a future work. As we remarked above the case \( n = 1 \) is unique in that the labelling set of irreducible sectors is infinite. In fact, it is not hard to see, using the classification of \( m \)-dimensional irreducible representations of \( B_m \) found in [FLSV], that one obtains some finite group images for every \( m \) when \( n = 1 \). By the isomorphisms of BMW-algebras corresponding to \( r \leftrightarrow -r^{-1} \) we see that the two cases \( n = 3 \) and \( n = \ell - 3 \) are actually the same.

Moreover, it can be shown that the (specialized quotient) BMW-algebras \( \mathcal{C}_m(q^2, q) \) can be embedded (diagonally) in quotients of the tensor squares of Iwahori-Hecke algebras \( \mathcal{H}_m(q^2) \). This indicates that the corresponding pairs may be tensor decomposable. In the subcase \((n, \ell) = (3, 6)\) work of Jones in [J1] shows that the images are all finite groups (essentially \( \text{PSL}(2m, 3) \)). The case \( n = -\ell/2 \) sometimes also have finite group images e.g. when \((n, \ell) = (-5, 10)\), see [J2].

6.2.3. Infinite images and density. Finally, we need to determine, for representations not excluded by the steps (1)-(4) above, the values of \( m, \ell, n, \) and \( \lambda \) for which the image of \( B_m \) under the unitary irreducible representation \( \rho_{(m, \lambda)}^{(n, \ell)} \) in \( \mathcal{C}_m(q^n, q) \) with \( q = e^{\pi i/\ell} \) is infinite modulo the center. Proposition 6.2 implies that a sufficient condition for \( \rho_{(m, \lambda)}^{(n, \ell)} \) to have infinite image is that the 3-dimensional representation \( \rho_{(3, \Omega)}^{(n, \ell)} \) have infinite image. So as a first step, we study this condition. For convenience of notation we denote this representation simply by \( \rho \) despite its dependence on the parameters. A non-unitary realization of \( \rho \) is given by:

\[
\begin{align*}
\sigma_1 \to A := \begin{pmatrix}
\frac{1}{q^n} & \frac{q^2-1}{q} & 0 \\
0 & \frac{q}{q^n} & i \\
0 & -i & 0
\end{pmatrix}, \\
\sigma_2 \to B := \begin{pmatrix}
0 & 0 & -i \\
0 & \frac{1}{q^n} & \frac{-i(q^2-1)}{q^{n+1}} \\
i & 0 & \frac{q^2-1}{q^n}
\end{pmatrix}
\end{align*}
\]

found in [BW].

Blichfeldt [Bl] has determined the irreducible finite subgroups of \( \text{PSL}(3, \mathbb{C}) \).
Six are primitive groups of orders 36, 60, 72, 168, 216, and 360, and the imprimitive subgroups come in two infinite families isomorphic to extensions of \( S_3 \) and \( Z_3 \) by abelian groups.

Definition 6.6. A group \( \Gamma \) is primitive if \( \Gamma \) has a faithful irreducible representation which cannot be expressed as a direct sum of subspaces which \( \Gamma \) permutes nontrivially.

By Lemma 2.1 a sufficient condition for \( G = \rho(B_3) \) to be primitive is that the spectrum of \( \rho(\sigma_1) \) satisfies the no-cycle property. So by Lemma
the image of ρ is only imprimitive in the excluded cases n = 1 and (n, ℓ) = (3, 6). So we may assume that the G is primitive. We wish to determine when G is infinite modulo the center. By rescaling the images of the generators σ_i by the cube root of the determinant of ρ(σ_i) we may assume that G ⊂ SL(3, C), and to determine the image modulo the center it suffices to consider the projective image. Thus G/Z(G) ⊂ PSL(3, C), and we may apply Blichfeldt’s classification. We state his result and include some useful information about orders of elements in:

**Proposition 6.7.** The primitive subgroups of PSL(3, C) are:

1. The Hessian group H of order 216 or a normal subgroup of H of order 36 or 72. The Hessian group is the subgroup of A_9 generated by (124)(568)(397) and (456)(798), and has elements of order \{1, 2, 3, 4, 6\}.
2. The simple group PSL(2, 7) ⊂ A_7 of order 168. The orders of elements are \{1, 2, 3, 4, 7\}.
3. The simple group A_5 having elements of orders \{1, 2, 3, 5\}.
4. The simple group A_6 having elements of orders \{1, 2, 3, 4, 5\}.

Using this result we have the following:

**Theorem 6.8.** Let n and ℓ be chosen so that ρ_{(n, ℓ)}^{(3, □)} is a 3-dimensional unitary irreducible representation of B_3 with eigenvalues not in geometric progression and satisfying the no-cycle condition. That is, n and ℓ satisfy the hypotheses of Proposition 6.2(1)(b), (c) or (e) in addition to n \notin \{3, ℓ – 3, ±ℓ/2\}. Let m ≥ 3 and |λ| < m with λ (n, ℓ)-admissible. The closure of the group ρ_{(m, λ)}^{(n, ℓ)}(B_m) is infinite modulo the center with two exceptions: if (n, ℓ) ∈ \{(-5, 14), (-9, 14)\} with (m, λ) ∈ \{(3, □), (4, [0])\} then the projective images are isomorphic to PSL(2, 7). Excluding these cases, if the dimension of the representation ρ_{(m, λ)}^{(n, ℓ)} is k, then the closure of the image of B_m contains SU(k).

**Proof.** Knowing the specific eigenvalues of ρ(σ_1) we compute its projective order t(n, ℓ) as a function of ℓ and n to be:

\[
t(n, ℓ) = \begin{cases} 
\frac{ℓ}{2} & \text{if } ℓ \equiv 2 \pmod{4} \text{ and } n \equiv 3 \pmod{4} \\
ℓ & \text{if } ℓ \equiv 0 \pmod{4} \text{ and } n \text{ even or} \\
ℓ \equiv 2 \pmod{4} \text{ and } n \equiv 1 \pmod{4} \\
2ℓ & \text{otherwise}
\end{cases}
\]

Under the stated hypotheses on n and ℓ we consider cases, comparing with the list of possible orders of elements in Blichfeldt’s classification.

1. If ℓ is odd, then ℓ ≥ 5 in which case t(n, ℓ) ≥ 10 which is too large.
2. If ℓ ≡ 0 (mod 4) then ℓ = 8 is the smallest value not yet excluded which gives t(n, ℓ) ≥ 8 which is again too large.
(3) If $\ell \equiv 2 \pmod{4}$ then $\ell \geq 6$ and $t(n, \ell) \geq 12$ unless $n$ is odd. If $n \equiv 1 \pmod{4}$ then $\ell \geq 10$ which gives us $t(n, \ell) = \ell \geq 10$ which does not appear on the list. When $\ell \equiv 2 \pmod{4}$ and $n \equiv 3 \pmod{4}$ with $n > 0$ we must have $n \geq 7$ which forces $\ell \geq 18$ since $n \neq \ell/2$. For $\ell \equiv 2 \pmod{4}$ and $n \equiv 3 \pmod{4}$ with $n < 0$ we must have $\ell \geq 14$ (since $\ell = 10$ leads to $n = -5 = -\ell/2$), which has the two possible values $n = -5$ or $n = -9$ which we claim gives rise to finite images.

Observe that $t(-5, 14) = t(-9, 14) = 7$.

To show that the projective images for $(-5, 14)$ and $(-9, 14)$ are both PSL(2, 7) we first observe that it is enough by the isomorphism of 6.1 with $r \leftrightarrow -r^{-1}$ and $q^{-5} \leftrightarrow q^{-14+5} = q^{-9}$ so these two cases give the same images. Then we use the explicit matrices $A$ and $B$ above to define $S = B^{-1}$ and $T = BAB$ which then (projectively) satisfy the relations $S^7 = (S^4T)^4 = (ST)^3 = T^2 = I_{3 \times 3}$ defining PSL(2, 7). It is immediate from the Bratteli diagram that the representation of $B_4$ corresponding to $(4, [0])$ is irreducible and isomorphic to the that of $(3, \square)$ when restricted to $B_3$. Moreover, the representations of $B_4$ corresponding to diagrams $[1^2]$ and $[2]$ each contain the representation of $B_3$ corresponding to the Young diagram $[1^2, 1]$ which was shown in [LW] to have infinite image (modulo the center). For all of the infinite image cases the hypotheses of Theorem 6.3 are satisfied and the eigenvalues are not in geometric progression so density follows. 

6.3. Quantum Groups. In this subsection we consider braid group actions on centralizer algebras of representations of quantum groups at roots of unity. We find and analyze examples in which we may apply Theorem 5.3. We follow the general strategy in Subsection 6.1, but we note that as the representation spaces available to us are not necessarily simple subquotients of braid group algebras (unlike BMW-algebras) there is a subtlety regarding irreducibility.

6.3.1. Braid group action on centralizer algebras. The Drinfeld-Jimbo quantum group $U := U_q g$ associated to a simple Lie algebra $g$ is a ribbon Hopf-algebra. The so-called universal $R$-matrix that intertwines the coproduct with the opposite coproduct on $U$ can be used to construct representations of the braid group $B_n$ on the morphism space $\text{End}_U(V^\otimes n)$ for any finite dimensional highest weight $U$-module $V$ as follows. Fix such a $U$-module $V$ and define $\check{R} = P_V \circ R \mid_{V \otimes V} \in \text{End}_U(V^\otimes 2)$ to be the $U$-isomorphism afforded us by composing the image of the universal $R$-matrix acting on $V \otimes V$ with the “flip” operator $P_V: v_1 \otimes v_2 \rightarrow v_2 \otimes v_1$. Then define isomorphisms for each $1 \leq i \leq n-1$:

$$\check{R}_i := 1^{\otimes (i-1)} \otimes \check{R} \otimes 1^{\otimes (n-i-1)} \in \text{End}_U(V^\otimes n)$$

so that the $\check{R}_i$ satisfy the braid group relations. Then define a representation of $B_n$ on $\text{End}_U(V^\otimes n)$ by $\sigma_i f = \check{R}_i \circ f$.

Lusztig has defined a modified form of the quantum group $U$ so that one may specialize the quantum parameter $q$ to $e^{\pm \pi i/\ell}$. In fact, one may choose
any \( q \) so that \( q^2 \) is a primitive \( \ell \)th root of unity, but we will restrict our attention to \( q = e^{\pi i/\ell} \) since these values (sometimes) yield unitary representations (see [Wz2]), which remain unitary for \( q \). The full representation category of \( U \) at roots of unity is not semisimple, but has a semisimple subquotient category. This process is essentially due to Andersen and his coauthors (see [A] and references therein). This yields a semisimple ribbon category \( \mathcal{F} \) (see [T] for the definitions) with finitely many simple objects labeled by highest weights in a truncation of the dominant Weyl chamber, called the Weyl alcove. The braid group still acts on \( \text{End}_U(V \otimes^n) \) for any object \( V \) as above, and for each quantum group we look for simple objects \( V_\lambda \) so that the images of the braid generators on the irreducible subrepresentations of \( \text{End}_U(V_\lambda \otimes^n) \) have 3 eigenvalues. Because the tensor product rules for objects labelled by weights near the upper wall of the Weyl alcove depends on \( \ell \), we do not explicitly determine all \( V_\lambda \) giving rise to pairs with the 3-eigenvalue property, and restrict our attention to weights near 0. As in the BMW algebra setting, we will always have an irreducible 3-dimensional representation of \( B_3 \) to which we may reduce most questions. We sketch the idea (see e.g. [TbW1] Section 3): If \( V \) is a simple object in (a finite semisimple ribbon category) \( \mathcal{F} \) such that \( V \otimes V \cong V_0 \oplus V_1 \oplus V_2 \) is the decomposition into 3 inequivalent simple objects then \( \text{End}_U(V \otimes^3) \) has a 3-dimensional irreducible subrepresentation isomorphic to \( \text{Hom}_U(V \otimes^3, W) \) for a simple object \( W \) appearing in \( V \otimes^3 \) with multiplicity three. Provided the (categorical) \( \alpha \)-dimension of each of \( W \), \( V \) and \( V_i \) are non-zero then this representation is irreducible and the image of \( \sigma_1 \) will have three distinct eigenvalues. As in the BMW-algebra situation we can construct a Bratteli diagram encoding the containments of the semisimple finite dimensional algebras:

\[
\text{End}_U(V) \subset \text{End}_U(V \otimes V) \subset \cdots \subset \text{End}_U(V \otimes^n) \cdots
\]

The simple components of \( \text{End}_U(V \otimes^n) \) will be isomorphic to \( \text{Hom}_U(V \otimes^n, V_\mu) \) where \( V_\mu \) is a simple object appearing in the decomposition of \( V \otimes^n \). The edges of the Bratteli diagram are determined by decomposing \( V \otimes V \) where \( V_\gamma \) is a simple subobject of \( V \otimes^{(n-1)} \). There are techniques known for obtaining these decompositions, for example Littelman’s path basis technique [L], or crystal bases. However, when we consider the action of the braid group \( B_n \) on the spaces \( \text{Hom}_U(V \otimes^n, V_\mu) \) we have no guarantee that the action is irreducible. This is because \( \text{End}_U(V \otimes^n) \) might not be generated by the image of \( B_n \).

6.3.2. Density results. We proceed to find pairs \((X_r, \lambda)\) so that the ribbon category corresponding to the quantum group \( U_qg(X_r) \) of Lie type \( X_r \) has simple object \( V_\lambda \) with \( V_\lambda \otimes^3 \cong V_0 \oplus V_1 \oplus V_2 \) as above. We find that \((A_r, \varpi_2), (A_r, 2\varpi_1), (B_r, \varpi_1), (C_r, \varpi_1), (D_r, \varpi_1)\) and \((E_6, \varpi_1)\) do satisfy these conditions (where the weights \( \varpi_i \) are labeled as in [Bo]). With these in hand we compute the eigenvalues of the images of \( \sigma_i \) in the corresponding representations. We use the following result found in [LR], Corollary 2.22(3) originally
due to Reshetikhin. The form $\langle \cdot, \cdot \rangle$ is the symmetric inner product on the root lattice normalized so that the square lengths of short roots is 2, and the weight $\rho$ is the half sum of the positive roots.

**Proposition 6.9.** Suppose that $V = V_\varpi$ is an irreducible representation of the quantum group $U_q\mathfrak{g}$ and that $V \otimes V_\lambda$ is multiplicity free for all $V_\lambda$ appearing in some $V^{\otimes n}$. Then for any $V_\nu$ appearing in $V^{\otimes 2}$ we have:

$$R_i |_{V_\nu} = \pm q^{(1/2)(\nu, \varpi+2\rho)-(\varpi, \varpi+2\rho)}1_{V_\nu}$$

where the sign is +1 if $V_\nu$ appears in the symmetrization of $V^{\otimes 2}$ and −1 if $V_\nu$ appears in the anti-symmetrization of $V^{\otimes 2}$.

We record the results in Table 1, where the notation follows [Bo]. The braid group representations corresponding to Lie types $B, C$ and $D$ are the same as those factoring over specializations of BMW-algebras, due to $q$-Schur-Weyl-Brauer duality, see [WZ]. For this reason we ignore these cases in the following weaker version of Theorem 6.8.

**Theorem 6.10.** Let $(X_r, \lambda)$ be as in Table 1 with $X = A_r$ or $E_6$. Then

1. For $(A_r, \varpi_2)$: provided $r \geq 3$ and $\ell \geq \max(r+3, 7)$, $\text{Hom}_U((V_\lambda)^{\otimes 3}, V_{\varpi_2+\varpi_4})$ is unitary, irreducible and 3-dimensional and the image of $\sigma_1$ has 3 distinct eigenvalues. If $V_\mu$ appears in $V_{\varpi_2+\varpi_4} \otimes V_\lambda^{\otimes 3}$ then $\text{Hom}_U(V^{\otimes n}, V_\mu)$ contains an irreducible unitary representation of $B_n$ with the 3-eigenvalue property. When $\ell \notin \{10, 14\}$, the eigenvalues of the image of $\sigma_1$ are not in geometric progression and the images of $B_n$ are infinite modulo the center and so are dense in these cases.

2. For $(A_r, 2\varpi_1)$: $\text{Hom}_U((V_\lambda)^{\otimes 3}, V_{2\varpi_1+2\varpi_2})$ is unitary, irreducible and 3-dimensional provided $r \geq 1$ and $\ell \geq r + 5$. If $V_\mu$ appears in $V_{2\varpi_1+2\varpi_2} \otimes V_\lambda^{\otimes 3}$ then $\text{Hom}_U(V^{\otimes n}, V_\mu)$ contains an irreducible unitary representation of $B_n$ with the 3-eigenvalue property. When $\ell \notin \{6, 10\}$ the eigenvalues of the image of $\sigma_1$ are not in geometric progression and the images of $B_n$ are infinite modulo the center and so are dense in these cases.

| $(X_r, \lambda)$ | $S^2(V_\lambda)$ | $\Lambda^2(V_\lambda)$ | Eigenvalues |
|------------------|------------------|------------------|------------|
| $(A_r, \varpi_2)$ | $V_{2\varpi_2}$ | $V_{\varpi_1+\varpi_4} \oplus V_{\varpi_2}$ | $q^{r+1}\{q, -q^{-1}, -q^{-5}\}$ |
| $(A_r, 2\varpi_1)$ | $V_{2\varpi_2} \oplus V_{2\varpi_1} \oplus V_{2\varpi_2}$ | $V_{2\varpi_1+\varpi_2}$ | $-q^{r-1}\{-q^{-1}, -q^5, q\}$ |
| $(B_r, \varpi_1)$ | $V_{\varpi_2} \oplus V_{\varpi_2}$ | $V_{\varpi_2}$ | $\{q, q^{-1}, -q^{-2}\}$ |
| $(C_r, \varpi_1)$ | $V_{\varpi_2}$ | $V_{\varpi_2} \oplus V_{\varpi_2}$ | $\{q, -q^{-1}, -q^{-2r-1}\}$ |
| $(D_r, \varpi_1)$ | $V_{\varpi_2} \oplus V_{\varpi_2}$ | $V_{\varpi_2}$ | $\{q, q^{-r-1}, -q^{-1}\}$ |
| $(E_6, \varpi_1)$ | $V_{2\varpi_1} \oplus V_{\varpi_6}$ | $V_{\varpi_3}$ | $q^{1/3}\{q, q^{-9}, -q^{-1}\}$ |

**Table 1.** Eigenvalues of $R_i$
(3) for $(E_6, \varpi_1)$: $\text{Hom}_U((V_{\lambda})^\otimes 3, V_{-\varpi_6})$ is unitary, irreducible and 3-dimensional provided $\ell \geq 14$. If $V_{\mu}$ appears in $V_{\varpi_1 + \varpi_6} \otimes V_{\lambda}^\otimes n - 3$ then $\text{Hom}_U(V^\otimes n, V_{\mu})$ contains an irreducible unitary representation of $B_n$ with the 3-eigenvalue property. Provided $\ell \neq 18$, the eigenvalues of the image of $\sigma_1$ are not in geometric progression and the images of $B_n$ are infinite modulo the center and so are dense in these cases.

**Proof.** For the object labelled by $V_{\nu}$ to be in the fundamental alcove, we must have $\langle \nu + \rho, \theta \rangle < \ell$ where $\theta$ is the highest root. This condition together with the requirement that the eigenvalues be distinct yield the first restrictions in each case. The unitarity of the representations is shown in [Wz2]. In each case the representation spaces $\text{Hom}_U(V^\otimes n, V_{\mu})$ described in the theorem contain the 3-dimensional representation spaces, so by restriction to $B_3$ we see that the $B_n$ representations must contain an irreducible unitary subrepresentation with the 3-eigenvalue property. Geometric progressions appear in each of the three cases if and only if $\ell = 10$ in the first case, $\ell = 6$ or 10 in the second case and $\ell = 18$ in the last case. Computing projective orders of the images of $\sigma_1$ and comparing as in the proof of Theorem 6.8 we find that the only finite group image that arises is in the first case with $\ell = 14$. With these exceptions, the hypotheses of Theorem 6.3 are satisfied and we may conclude the images are dense. □

**Remark 6.11.** To get sharper results we would need to describe the decompositions of the $B_n$ representations $\text{Hom}_U(V^\otimes n, V_{\mu})$ that appear in the above theorem. This is in general quite complicated. In fact, the type $E_6$ case appears in an exceptional series discussed in [WZ3] (and extended slightly in [R2]). These give new semisimple finite dimensional quotients of the braid group algebras analogous to BMW-algebras about which little is known.

### 6.4. Concluding Remarks

In comparing this work to the 2-eigenvalue paper, it may be noted that we do not provide applications of our results to the distribution of values of the Kauffman polynomial in analogy with those given for the Jones polynomial in [FLW §5]. That is, we do not consider the set of values $F_L(a, z)$ for fixed $a$ and $z$ and varying $L$. These values can be described as linear combinations of traces of any braid with closure $L$ in the different irreducible factors of a BMW algebra, just as in [FLW]. The difficulty is that our information on the closures of braid groups in BMW algebras is less detailed than the corresponding information for Hecke algebras. In particular, we have not completely determined these closures for the irreducible factors of the BMW-representations which are excluded in the statement of Theorem 6.8. Neither have we determined the equivalences and dualities existing between different irreducible factors in a fixed BMW-algebra. We certainly expect the limiting distributions to be Gaussian as for the Jones polynomial, but we do not yet have enough information to ensure that this is so.
In 1990s, Vertigan (see Theorems 6.3.5 and 6.3.6 of [Wel]) analyzed the classical computational complexity of exactly evaluating various knot polynomials at fixed complex values. With a few exceptions, all evaluations are \#P-hard. At these few exceptional values, the link invariants have classical topological interpretations and can be computed in polynomial time. These results fit very well with the analysis of closed images of the braid group representations. In the case of unitary Jones representations of the braid groups at $q$, the closed image is dense in the corresponding special unitary groups exactly when computing the link invariants is \#P-hard at $q$, while the finite image cases correspond to polynomial time computations. Part of the appeal of working out the exceptions to Theorem [6.8] is the hope of relating these cases to interesting special values of the Kauffman polynomial.
References

[A] H. H. Andersen, Tensor products of quantized tilting modules, Comm. Math. Phys. 149 (1991), 149-159.

[BW] J. Birman and H. Wenzl, Braids, link polynomials and a new algebra, Trans. Amer. Math. Soc. 313 (1989), no. 1, 249–273.

[Bi] H. F. Blichfeldt, Finite Collineation Groups, University of Chicago Press, Chicago, 1917.

[Bo] N. Bourbaki, Groupes et algèbres de Lie, Chap. 4–6 (1968), Chap. 7–8 (1975), Hermann, Paris.

[BFLW] M. Bordewich, M. Freedman, L. Lovsz, and D. Welsh, Approximate Counting and Quantum Computation, Combinatorics, Probability and Computing archive Volume 14 , Issue 5-6 (November 2005).

[DFN] S. Das Sarma, M. Freedman, and C. Nayak, Topologically-Protected Qubits from a Possible Non-Abelian Fractional Quantum Hall State, Phys. Rev. Lett. 94, 166802 (2005).

[Fa] G. Faltings, p-adic Hodge theory, J. Amer. Math. Soc. 1 (1988), no. 1, 255–299.

[FF] P. Fendley and E. Fradkin, Realizing non-Abelian statistics in time-reversal-invariant systems, Phys. Rev. B 72, 024412 (2005).

[FM] J.-M. Fontaine and B. Mazur, Geometric Galois representations. Elliptic curves, modular forms, & Fermat’s last theorem (Hong Kong, 1993), 41–78, Ser. Number Theory, I, Internat. Press, Cambridge, MA, 1995.

[FLSV] E. Formanek, W. Lee, I. Sysoeva, M. Vazirani, The irreducible complex representations of the braid group on n strings of degree \( \leq n \), J. Algebra Appl. 2 (2003), no. 3, 317–333.

[FKLW] M. Freedman, A. Kitaev, M. J. Larsen and Z. Wang, Topological quantum computation, Mathematical challenges of the 21st century (Los Angeles, CA, 2000), Bull. Amer. Math. Soc. (N.S.) 40 (2003), no. 1, 31–38.

[FLW] M. H. Freedman, M. J. Larsen, and Z. Wang, The two-eigenvalue problem and density of Jones representation of braid groups, Comm. Math. Phys. 228 (2002), 177-199, arXiv: [math.GT/0103200]

[I] N. V. Ivanov, Mapping class groups, Handbook of geometric topology, 523–633, North-Holland, Amsterdam, 2002.

[J1] V. F. R. Jones, Braid groups, Hecke algebras and type II \_1 factors, Geometric methods in operator algebras (Kyoto, 1983), 242–273, Pitman Res. Notes Math. Ser., 123, Longman Sci. Tech., Harlow, 1986.

[J2] V. F. F. Jones, On a certain value of the Kauffman polynomial, Comm. Math. Phys. 125 (1989), no. 3, 459–467.

[Ke] J. H. B. Kemperman, On small sumsets in an abelian group, Acta Math. 103 (1960), 63–88.

[Ko] A. V. Korlyukov, Finite linear groups that are generated by quadratic elements of order 4 and 3. (Russian) Problems in algebra, No. 4 (Russian) (Gomel, 1986), 146–151, “Universitet-skoe”, Minsk, 1989.

[LW] M. J. Larsen and Z. Wang, Density of the SO(3) TQFT representation of mapping class groups, Comm. Math. Phys.,Volume 260(2005), Number 3, 641 - 658.

[LR] R. Leduc and A. Ram, A ribbon Hopf algebra approach to the irreducible representations of centralizer algebras: the Brauer, Birman-Wenzl, and type A Iwahori-Hecke algebras, Adv. in Math. 125 (1997) 1-94.

[L] P. Littelman, Paths and root operators in representation theory, Ann. of Math. (2) 142 (1995), no. 3, 499–525.

[MP] W. G. Mckay and J. Patera, Tables of dimensions, indices, and branching rules for representations of simple Lie algebras, Lecture notes in pure and applied math., vol 69, Marcel Dekker, Inc., New York, 1981.
[NC] M. Nielsen and I. Chuang, *Quantum Computation and quantum information*, Cambridge University Press, 2000.

[R1] E. Rowell, *On a family of non-unitarizable ribbon categories*, Math Z. **250** no. 4 (2005) 745–774.

[R2] E. Rowell, *A note on tensor categories of Lie type $E_9$*, J. Algebra **284** no. 1 (2005) 296–309.

[Sen] S. Sen, *Lie algebras of Galois groups arising from Hodge-Tate modules*, Ann. of Math. (2) **97** (1973), 160–170.

[Ser] J.-P. Serre, *Groupes algébriques associés aux modules de Hodge-Tate*. Journées de Géométrie Algébrique de Rennes. (Rennes, 1978), Vol. III, pp. 155–188, Astérisque, 65, Soc. Math. France, Paris, 1979.

[SW] E. Stein and G. Weiss, *Introduction to Fourier Analysis on Euclidean Spaces*, Princeton University Press, Princeton, N.J., 1971.

[St] R. Steinberg, *Endomorphisms of linear algebraic groups*, Memoirs of the American Mathematical Society, No. 80 American Mathematical Society, Providence, R.I., 1968.

[TbW1] I. Tuba and H. Wenzl, *Representations of the braid group $B_3$ and of SL(2, Z)*, Pacific J. Math. **197** (2001), no. 2, 491–510.

[TbW2] I. Tuba and H. Wenzl, *On braided tensor categories of type BCD*, Crelles J., to appear

[T] V. G. Turaev, *Quantum Invariants of Knots and 3-Manifolds*, de Gruyter Studies in Mathematics, 18. Walter de Gruyter & Co., Berlin, 1994.

[TrW1] V. G. Turaev and H. Wenzl, *Quantum invariants of 3-manifolds associated with classical simple Lie algebras*, Int. J Math. **4** (1993), 323-358.

[TrW2] V.G. Turaev and H. Wenzl, *Semisimple and modular tensor categories from link invariants*, Math. Ann. **309** (1997) 411–461.

[W] D. Wales, *Quasiprimitive linear groups with quadratic elements*, J. Algebra **245** (2001), no. 2, 584–606.

[Wel] D. Welsh, *Complexity: Knots, Colourings and Counting*, LMS Lecture Notes Series 186, Cambridge University Press, Cambridge, 1993.

[Wn] X.-C. Wen, *Projective Construction of Non-Abelian Quantum Hall Liquids*, Phys. Rev. B **60**, 8827 (1999).

[Wz1] H. Wenzl, *Quantum groups and subfactors of type $B$, $C$, and $D$*, Comm. Math. Phys. **133** (1990), no. 2, 383–432.

[Wz2] H. Wenzl, *C$^*$ tensor categories from quantum groups*, J. AMS **11** (1998) 261–282.

[Wz3] H. Wenzl, *On tensor categories of Lie type $E_N$, $N \neq 9$*, Adv. Math. **177** (2003) 66–104.

[Wi] F. Wilczek, *Braid statistics and anyon superconductivity*, World Scientific Pub Co Inc (December, 1990).

[Z] A. E. Zalesskii, *Minimal polynomials and eigenvalues of $p$-elements in representations of quasi-simple groups with a cyclic Sylow $p$-subgroup*, J. London Math. Soc. (2) **59** (1999), no. 3, 845–866.

E-mail address: larsen@math.indiana.edu

E-mail address: errowell@indiana.edu

E-mail address: zhewang@indiana.edu
Department of Mathematics, Indiana University, Bloomington, IN 47405, U.S.A.