MATRIX METHOD FOR PERSISTENCE MODULES ON COMMUTATIVE LADDERS OF FINITE TYPE

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Abstract. The theory of persistence modules on the commutative ladders $CL_n(\tau)$ provides an extension of persistent homology. However, an efficient algorithm to compute the generalized persistence diagrams is still lacking. In this work, we view a persistence module $M$ on $CL_n(\tau)$ as a morphism between zigzag modules, which can be expressed in a block matrix form. For the representation finite case ($n \leq 4$), we provide an algorithm that uses certain permissible row and column operations to compute a normal form of the block matrix. In this form an indecomposable decomposition of $M$, and thus its persistence diagram, is obtained.

Keywords. Persistence modules, Commutative ladders, Computational topology, Algorithms

1. Introduction

Recently, the paper [6] introduced the study of persistence modules on the commutative ladders of finite type. This was motivated in part by a need to study simultaneously robust and common topological features using the ideas of persistent homology [5]. Let us first give an overview of this background and motivation.

One way to construct persistent homology is the following. Let $X$ be a filtration, a non-decreasing sequence of spaces

$$X : X_1 \subset X_2 \subset \ldots \subset X_n.$$ 

Applying a homology functor $H(-)$ with coefficient field $K$, we obtain a sequence

$$H(X) : H(X_1) \rightarrow H(X_2) \rightarrow \cdots \rightarrow H(X_n)$$

of $K$-vector spaces and induced linear maps between them, called the persistent homology of the filtration.

Diagram (1) above can be interpreted in the language of the representation theory of (bound) quivers. This leads one to considering persistence modules in general, of which $H(X)$ in Diagram (1) is an example. With this point of view, a persistence module can be taken to be synonymous to a representation of a bound quiver.

Assuming that $H(X_i)$ is finite dimensional for $i \in \{1, \ldots, n\}$, it is known that the persistence module $H(X)$ can be decomposed into the so-called interval representations. The decomposition into intervals can be used to study the persistent, robust, or multiscale topological features in $X$. The length of each interval (its lifetime) can be interpreted as a measure of persistence or robustness of the topological feature.

More generally, different classes of persistence modules may be used to study, using similar ideas, diagrams of spaces that are not filtrations. As an example, zigzag persistent homology [3] can be used to analyze common topological features in a collection of spaces. Here, let us consider the following simple example of zigzag persistence. Given two spaces
X and Y, we can form the diagram

\[ \mathcal{X} : X \to X \cup Y \leftarrow Y. \]

Applying \( H(-) \), we obtain the diagram

(2) \[ H(\mathcal{X}) : H(X) \to H(X \cup Y) \leftarrow H(Y) \]

of homology vector spaces and induced linear maps. Similar to the classical persistent homology case, it is known that a zigzag module, for example \( H(\mathcal{X}) \) in Diagram (2), can be decomposed into interval zigzag modules. Those that are nonzero from the left (at \( X \)), through the middle, and to the right (at \( Y \)) correspond to topological features that are common to \( X \) and \( Y \).

A shortcoming of the above is that only robust features or only common features can be studied, but not both simultaneously. A motivation for using persistence modules on commutative ladders [6] is to deal with simultaneously common and robust topological features. This can be thought of as a partial generalization towards multidimensional persistence [4].

Let us review how we use commutative ladders to treat simultaneously common and robust features. Suppose that \( X_1 \subset X_2 \) and \( Y_1 \subset Y_2 \) are two-step filtrations of spaces \( X \) and \( Y \). To study the robust and common features shared between them, form the following commutative diagram of homology vector spaces and linear maps:

(3) \[
\begin{array}{ccc}
H(X_2) & \to & H(X_2 \cup Y_2) \leftarrow H(Y_2) \\
\uparrow & & \uparrow \\
H(X_1) & \to & H(X_1 \cup Y_1) \leftarrow H(Y_1)
\end{array}
\]

where the linear maps are induced from the respective inclusions. In this diagram, the vertical direction captures the robust features, while the horizontal direction captures the common features between \( X \) and \( Y \). Indecomposable direct summands isomorphic to

\[
\begin{array}{ccc}
K & \to & K \\
\downarrow & & \downarrow \\
K & \to & K
\end{array}
\]

if any, represent the simultaneously robust and common features.

The above discussion provides some motivations for our interest in persistence modules. In this work, we shall not discuss what particular class of spaces and which homology functor \( H(-) \) are to be used. Instead, we take a persistence module as our starting point. In particular, we consider persistence modules on the commutative ladders \( CL_n(\tau) \), which we define in Section 2.2. Diagram (3) is an example of a persistence module on the bound quiver

(4) \[
\begin{array}{ccc}
\circ & \to & \circ \\
\uparrow & & \uparrow \\
\circ & \to & \circ
\end{array}
\]

As in classical persistence, an indecomposable decomposition of a persistence module plays a key role in understanding its different types of persistent topological features. In the general case however, the indecomposable summands are not completely given by intervals or analogues of intervals.
The algorithm provided in [6] computes an indecomposable decomposition by performing changes of bases on the individual vector spaces in a given persistence module and extracting direct summands. This involves working with the persistence module by its collection of linear maps.

Here, we take a different point of view and reconsider a persistence module on a commutative ladder as a morphism from its bottom row to its top row, via Theorem 2 in Subsection 3.1. Note that the bottom and top rows are nothing but zigzag modules, and thus can be decomposed into interval zigzag modules. Using this fact, the morphism can be written in a block matrix form with respect to these decompositions. In essence, we treat the persistence module as one matrix, but with certain restrictions induced from the structure of homomorphism spaces between interval zigzag modules. We make these ideas precise in Subsections 3.2 and 3.3.

We then provide a procedure for computing an indecomposable decomposition using the above described matrix formalism. The idea is to use column and row operations, as in elementary linear algebra, to find normal forms. While the matrix has entries given by homomorphisms between zigzag modules, the procedure can be reinterpreted to involve only $K$-matrices, provided certain restrictions on the permissible operations on the matrices are respected. These restrictions are also derived from the structure of the homomorphism spaces between the intervals.

The procedure is formalized in Algorithm 1 in Section 4.2.2. The main theorem of this paper is the following.

**Theorem 1.** Assume Algorithm 1 is called with the block matrix problem corresponding to a persistence module $M$ on a commutative ladder of finite type. Then Algorithm 1 terminates and the input matrix is transformed to an isomorphic block matrix consisting only of identity, zero, and strongly zero blocks, and whose indecomposable decomposition corresponds to an indecomposable decomposition of $M$.

Finally, we note that our problem of computing a normal form of a block matrix under certain permissible operations falls under a more general class of problems called “matrix problems”. Matrix problems can be given a theoretical framework via the representation theory of bocses [2, 8]. In this framework, the matrix reductions can be interpreted as reduction functors that induce equivalences of representation categories of bocses. In this work, however, we have kept the necessary theoretical background to a minimum and expressed Algorithm 1 in terms of block matrices and permissible operations.

## 2. Background

### 2.1. Quivers and Persistent Homology.

A *quiver* $Q = (Q_0, Q_1)$ is a directed graph with set of vertices $Q_0$ and set of arrows $Q_1$. An arrow $\alpha \in Q_1$ from a vertex $a \in Q_0$ to a vertex $b \in Q_0$ is denoted by $\alpha : a \to b$. In this case, $a$ is called the source of $\alpha$, and $b$ is its target. A *path* $p = (a | \alpha_1 \ldots \alpha_\ell | b)$ of length $\ell$ from a vertex $a$ to a vertex $b$ is a sequence of $\ell$ arrows $\alpha_1, \ldots, \alpha_\ell$, where the source of $\alpha_1$ is $a$, the target of $\alpha_\ell$ is $b$, and the target of $\alpha_i$ is equal to the source of $\alpha_{i+1}$ for all $i \in \{1, \ldots, \ell - 1\}$. Note that paths of length 0 are allowed. These are the paths $e_a = (a | a)$, called the stationary path at $a$, for each vertex $a$. Moreover, for each arrow $\alpha : a \to b$, we use the same symbol to denote the corresponding path $\alpha = (a | \alpha | b)$.

Let $K$ be a field, which we fix throughout this work. The *path algebra* $KQ$ of a quiver $Q$ is the following $K$-algebra. As a $K$-vector space, it is freely generated by all paths in $Q$. 
The multiplication in $KQ$ is defined by setting

$$(a \mid \alpha_1 \cdots \alpha_r \mid b)(c \mid \beta_1 \cdots \beta_m \mid d) = \begin{cases} (a \mid \alpha_1 \cdots \alpha_r \beta_1 \cdots \beta_m \mid d) & b = c, \\ 0 & \text{otherwise}; \end{cases}$$

and extending $K$-linearly. In this work, we consider only finite quivers $(|Q_0|, |Q_1| < \infty)$ without any oriented cycles\(^1\). With this assumption, $KQ$ is a finite-dimensional $K$-algebra.

Let $\{w_1, \ldots, w_m\}$ be a finite set of $m$ paths that share a common source $s \in Q_0$ and a common target $t \in Q_0$. A linear combination

$$\rho = \sum_{i=1}^{m} c_i w_i \in KQ,$$

is called a relation in $Q$.

A bound quiver $(Q, P)$ is a pair of a quiver $Q$ together with a set of relations $P = \{\rho_i\}_{i \in T}$. The two-sided ideal of $KQ$ generated by a set of relations $P = \{\rho_i\}_{i \in T}$ is denoted by $\langle P \rangle$.

The algebra of a bound quiver $(Q, P)$ is the quotient $A = KQ / \langle P \rangle$.

A representation of a quiver $Q$, denoted $M = (M_a, \varphi_a)_{a \in Q_0, a \in Q_1}$, is a collection of a finite dimensional vector space $M_a$ for each $a \in Q_0$ and a linear map $\varphi_a : M_a \to M_b$ for each arrow $\alpha : a \to b$.

Let $M = (M_a, \varphi_a)_{a \in Q_0, a \in Q_1}$ be a representation $Q$, and $w = (a \mid \alpha_1 \cdots \alpha_r \mid b)$ a path in $Q$. Define the evaluation of $M$ on the path $w$ to be $\varphi_w = \varphi_{\alpha_1} \circ \cdots \circ \varphi_{\alpha_r} : M_a \to M_b$. The representation $M$ is said to be a representation of a bound quiver $(Q, P)$ if $\varphi_{\rho} = \sum_i c_i \varphi_{w_i} = 0$ for all relations $\rho = \sum_i c_i w_i \in P$.

For example, let $Q$ and $M$ be the following quiver and representation:

$$Q: \begin{array}{c}
\circ \quad \circ \\
\circ \quad \circ \\
\circ \quad \circ \\
\end{array} \begin{array}{c}
\begin{array}{c}
\alpha \\
\beta \\
\gamma \\
\end{array} \\
\begin{array}{c}
0 \\
2 \\
1 \\
\end{array} \\
\begin{array}{c}
3 \\
4 \\
5 \\
\end{array} \\
\end{array} \quad \begin{array}{c}
\begin{array}{c}
M_1 \\
M_2 \\
M_3 \\
M_4 \\
\end{array} \\
\begin{array}{c}
\varphi_1 \\
\varphi_2 \\
\varphi_3 \\
\varphi_4 \\
\end{array} \\
\begin{array}{c}
\varphi_a \\
\varphi_b \\
\varphi_c \\
\varphi_d \\
\end{array} \\
\end{array}$$

respectively. If $P = \{\rho = \gamma \delta - \beta \alpha\}$, then $M$ is a representation of $(Q, P)$ if and only if $\varphi_{\rho} = \varphi_{\delta} \varphi_{\gamma} - \varphi_{\alpha} \varphi_{\beta} = 0$. In other words, this implies that $M$ in Diagram [5] forms a commutative diagram of $K$-vector spaces and linear maps. In general, we define the set of commutative relations $C$ of a quiver $Q$ to be the set of relations of the form $p - p'$ where $p$ and $p'$ are any two different paths from vertices $a$ to $b$, for any pair of vertices $a$ and $b$.

**Definition 1.** The representation category $\text{rep} Q$ of $Q$ is the following category.

- **Objects:** finite-dimensional representations of the quiver $Q$.
- **Morphisms:** Let $M = (M_a, \varphi_a)_{a \in Q_0, a \in Q_1}$ and $N = (N_a, \psi_a)_{a \in Q_0, a \in Q_1}$ be representations of $Q$. A morphism $f : M \to N$ is a collection of $K$-linear maps $f_a : M_a \to N_a$ such that for all arrows $\alpha : a \to b$ in $Q$, the diagram

$$\begin{array}{ccc}
M_a & \xrightarrow{\varphi_a} & M_b \\
\downarrow{f_a} & & \downarrow{f_b} \\
N_a & \xrightarrow{\psi_a} & N_b \\
\end{array}$$

is commutative. The collection of morphisms from $M$ to $N$ is denoted by $\text{Hom}(M, N)$.

\(^1\text{An oriented cycle is a path with nonzero length whose source is equal to its target.}\)
\begin{itemize}
  \item Composition: for \( f = \{ f_a \}_{a \in Q_0} : V \to W \) and \( g = \{ g_a \}_{a \in Q_0} : W \to U \), \( g \circ f = \{ g_a f_a \}_{a \in Q_0} \).
\end{itemize}

The representation category \( \text{rep}(Q, P) \) of a bound quiver \((Q, P)\) is the full subcategory of \( \text{rep} \) \( Q \) with objects consisting of the representations of \((Q, P)\).

The direct sum \( M \oplus N \) of representations \( M = (M_a, \varphi_a) \) and \( N = (N_a, \psi_a) \) of \((Q, P)\) is the representation with the vector space \( M_a \oplus N_a \) for each vertex \( a \in Q_0 \) and the linear map \( \varphi_a \oplus \psi_a : M_a \oplus N_a \to M_b \oplus N_b \) for each arrow \( a : a \to b \).

A representation \( M \neq 0 \) is indecomposable if \( M \cong N \oplus N' \) implies \( N = 0 \) or \( N' = 0 \). From the Krull-Remak-Schmidt theorem, every representation \( M \) can be decomposed into a sum of indecomposable representations \( M \cong W_1 \oplus \cdots \oplus W_s \), unique up to isomorphism and permutation of terms. A quiver \( Q \) or a bound quiver \((Q, P)\) is said to be finite type (representation-finite) if the number of isomorphism classes of its indecomposable representations is finite, and is infinite type (representation-infinite) otherwise. For more details on the representation theory, see for example [1].

Let \( f \) and \( b \) be symbols, representing “forward” and “backward”. An orientation \( \tau \) is a sequence \( \tau = (\tau_1, \ldots, \tau_{n-1}) \) where \( \tau_i \) is either \( f \) or \( b \) for each \( 1 \leq i \leq n - 1 \). Given \( n \geq 1 \) and an orientation \( \tau \), define the quiver

\[
A_n(\tau) : \begin{array}{c}
1 \leftarrow 2 \leftarrow \cdots \leftarrow n,
\end{array}
\]

where the \( i \)-th arrow \( \begin{array}{c} 1 \\
0 \end{array} \leftarrow \begin{array}{c} i + 1 \\
i \end{array} \) is \( \tau_i \) is \( f \), and is \( \tau_i \) if \( \tau_i = b \). We say that a quiver \( A_n(\tau) \) is \( A_n \)-type.

From Gabriel’s theorem [2], any \( A_n \)-type quiver is representation-finite. For \( 1 \leq b \leq d \leq n \), define the interval representation

\[
\mathbb{I}[b, d] : 0 \leftarrow \cdots \leftarrow 0 \leftarrow K \leftarrow K \leftarrow \cdots \leftarrow K \leftarrow 0 \leftarrow \cdots \leftarrow 0,
\]

in \( \text{rep} A_n(\tau) \), which consists of copies of the vector space \( K \) from indices \( b \) to \( d \) and \( 0 \) elsewhere, and where the maps between the vector spaces \( K \) are identity maps and zero otherwise. It is known that \( \{ \mathbb{I}[b, d] \}_{1 \leq b \leq d \leq n} \) gives a complete list of indecomposable representations of \( A_n(\tau) \) up to isomorphism. Thus, any representation \( M \) of \( A_n(\tau) \) can be decomposed as a direct sum

\[
M \cong \bigoplus_{1 \leq b \leq d \leq n} \mathbb{I}[b, d]^{m_{b,d}}
\]

where the numbers \( m_{b,d} \in \mathbb{Z}_{\geq 0} \) are multiplicities.

Classical persistent homology can be viewed as a representation (a persistence module) \( M \) of \( A_n(\tau) \) with the orientation \( \tau = ff \cdots f \). Each interval representation \( \mathbb{I}[b, d] \) that appears as a direct summand in a given persistence module tracks a homology class which is born in \( H(X_b) \) and persists up to \( H(X_d) \). The lengths of these intervals can be taken as encoding the persistence or robustness of the homological features of the filtration.

The persistence diagram \( D_M \) of a persistence module \( M \) on \( A_n(\tau) \) is the multiset

\[
D_M = \{ (b, d) \text{ with multiplicity } m_{b,d} \mid 1 \leq b \leq d \leq n \},
\]

where the multiplicities \( m_{b,d} \) are determined by an indecomposable decomposition of \( M \) as in Eq. (6). The persistence diagram can be visualized by plotting the points \( (b, d) \) together with multiplicities \( m_{b,d} \) on a plane, and provides a compact way to represent the presence and lifespans of the persistent topological features.
The ideas of persistent homology have been extended to a wide variety of underlying quivers. For example, consider a collection $X_1, X_2, \ldots, X_M$ of spaces $X_j$ that do not form a filtration. Instead, one can form the diagram

$$
\begin{array}{cccccccccc}
X_1 & \longrightarrow & X_1 \cup X_2 & \leftarrow & X_2 & \longrightarrow & X_2 \cup X_3 & \leftarrow & \cdots & \leftarrow & X_M
\end{array}
$$

and obtain the persistence module

$$(7) \quad H(X_1) \longrightarrow H(X_1 \cup X_2) \leftarrow H(X_2) \longrightarrow H(X_2 \cup X_3) \leftarrow \cdots \leftarrow H(X_M)$$

which is a representation of $\mathbb{A}_n(fbfb \cdots fb)$. Since the underlying quiver is $\mathbb{A}_n$-type, the indecomposable representations are given by the intervals. An indecomposable decomposition of Eq. (7), gives the persistent homological features in the collection $X_1, \ldots, X_M$. In this case, the interval representations can be interpreted as features common among certain spaces. This is one example of a persistence module over a quiver of $\mathbb{A}_n$-type, which in general are called zigzag persistence modules. For more details, see [3].

2.2. Persistence Modules on Commutative Ladders.

**Definition 2.** Let $\tau = (\tau_1, \ldots, \tau_{n-1})$ be an orientation. The ladder quiver $L_n(\tau)$ is

$$
L_n(\tau) : \quad \circ^{1'} \leftarrow \circ^{2'} \leftarrow \cdots \leftarrow \circ^{n'}
$$

where the directions of the arrows on both the top and bottom rows are determined by the orientation $\tau$. The commutative ladder $CL_n(\tau)$ is the ladder quiver $L_n(\tau)$ bound by commutative relations. A persistence module on the commutative ladder $CL_n(\tau)$ is a representation of $CL_n(\tau)$.

Recall that the Auslander-Reiten quiver $\Gamma = (\Gamma_0, \Gamma_1)$ of a bound quiver $(Q, P)$ is another quiver whose vertices $\Gamma_0$ are given by all isomorphism classes of indecomposable representations of $(Q, P)$, and whose arrows are given by the following. For every pair of vertices $[M], [N] \in \Gamma_0$, $\Gamma$ has an arrow $[M] \rightarrow [N]$ if and only if there exists an irreducible morphism\(^2\) $f : M \rightarrow N$.

The paper [6] shows that for any orientation $\tau$, $CL_n(\tau)$ is representation-finite if and only if $n \leq 4$. The Auslander-Reiten quivers of the representation-finite cases are listed in the paper [6]. Figure 1 here shows the Auslander-Reiten quiver of $CL_3(fb)$.

The vertices of the Auslander-Reiten quiver in Figure 1 are denoted by their dimension vectors. Recall that the dimension vector $\dim M$ of a representation $M$ is the vector of dimensions (as $K$-vector spaces) of $M(a)$ for vertices $a \in Q_0$. It is helpful to write the dimension numbers $\dim_K M(a)$ corresponding to the positions of the vertices $a \in Q_0$. For example, the dimension vector of the indecomposable representation

$$
\begin{array}{cccccc}
K & \longrightarrow & 0 & \longleftrightarrow & K
\end{array}
$$

\[^2\text{An irreducible morphism is a morphism satisfying the following two conditions: (i) $f$ is neither a retraction nor a section. (ii) For any factorization $f = f_1 \circ f_2$, either $f_1$ is a retraction or $f_2$ is a section.}\]
Figure 1. Auslander-Reiten quiver of $\text{CL}_3(\mathbf{f} \mathbf{b})$ is denoted as $101_{111}$. While the dimension vector is invariant under isomorphism, nonisomorphic representations may have the same dimension vector in general.

Moreover, the entries of the dimension vectors of indecomposable representations may exceed 1. For example, Figure 1 has vertices $0_{010}$ and $1_{121}$ representing the indecomposable representations

\[ (8) \]

\[ \begin{array}{c}
K & \rightarrow & K^2 \\
\uparrow & & \uparrow \\
0 & \rightarrow & K
\end{array} \quad \text{and} \quad \begin{array}{c}
K & \rightarrow & K \\
\uparrow & & \uparrow \\
K & \rightarrow & K^2
\end{array} \]

respectively.

Finally we recall the definition of the persistence diagram of a representation $M$ of $\text{CL}_n(\tau)$. By the above considerations, $M$ has

\[ M \cong \bigoplus_{\ell \in \Gamma_0} I_{\ell}^{k_{\ell}} \text{ for some } k_{\ell} \in \mathbb{Z}_{\geq 0}, \]

where $\Gamma = (\Gamma_0, \Gamma_1)$ is the Auslander-Reiten quiver of $\text{CL}_n(\tau)$. The persistence diagram of $M$ is the map

\[ D_M : \Gamma_0 \rightarrow \mathbb{Z}_{\geq 0} \]

\[ |\ell| \mapsto k_{\ell}. \]

In the representation finite case, $\Gamma$ is a finite quiver, and we draw $D_M$ by labelling the vertices $|\ell|$ of $\Gamma$ with the numbers $k_{\ell}$.

3. Main Results

We provide a decomposition algorithm for persistence modules on commutative ladders of finite type by reinterpreting the modules as matrices of homomorphisms between interval representations.

3.1. From Representations to Arrows.

**Definition 3.** The arrow category $\text{arr}(\text{rep } Q)$ of $\text{rep } Q$ is the following category.

- **Objects:** All morphisms $\phi : V \rightarrow W$ of $\text{rep } Q$, for all objects $V$ and $W$ of $\text{rep } Q$.
- **Morphisms:** A morphism $F = (F_V, F_W) : \phi_1 \rightarrow \phi_2$ from an object $\phi_1 : V_1 \rightarrow W_1$ to $\phi_2 : V_2 \rightarrow W_2$ is a pair of morphisms $(F_V : V_1 \rightarrow V_2, F_W : W_1 \rightarrow W_2)$ of $\text{rep } Q$, such
that

\[
\begin{array}{ccc}
V_2 & \xrightarrow{\phi_2} & W_2 \\
F_V \uparrow & \searrow F_W & \\
V_1 & \xrightarrow{\phi_1} & W_1
\end{array}
\]

commutes.

- Composition: Given \( F = (F_V, F_W) : \phi_1 \to \phi_2 \) and \( G = (G_V, G_W) : \phi_2 \to \phi_3 \)

\[
G \circ F = (G_V F_V, G_W F_W)
\]

In this context, we call objects of the arrow category as arrows to distinguish them from objects of the base category \( \text{rep}Q \).

**Theorem 2.** Let \( \tau \) be an orientation. There is an isomorphism of \( K \)-categories

\[
\text{rep}\, \text{CL}_n(\tau) \cong \text{arr}(\text{rep} \, \mathbb{A}_n(\tau)).
\]

**Proof.** An isomorphism functor \( F : \text{rep}\, \text{CL}_n(\tau) \to \text{arr}(\text{rep} \, \mathbb{A}_n(\tau)) \) can be constructed by taking a persistence module \( M \in \text{rep}\, \text{CL}_n(\tau) \) to the morphism defined by \( M \) from its bottom row to its top row. Similarly, a morphism between two persistence modules \( \lambda : M \to N \) defines a morphism \( F(\lambda) \) between the corresponding arrows \( F(M), F(N) \) in the obvious way.

\( \square \)

The isomorphism \( F : \text{rep}\, \text{CL}_n(\tau) \to \text{arr}(\text{rep} \, \mathbb{A}_n(\tau)) \) constructed above allows us to identify a persistence module \( M \) on \( \text{CL}_n(\tau) \) with the corresponding arrow \( F(M) \).

### 3.2. Arrows to a Matrix Formalism

Fix an orientation \( \tau \). For ease of notation, we define the following.

**Definition 4.** The relation \( \geq \) is defined on the set of interval representations of \( \mathbb{A}_n(\tau) \), \( \{ \llbracket b, d \rrbracket : 1 \leq b \leq d \leq n \} \), by setting \( \llbracket a, b \rrbracket \geq \llbracket c, d \rrbracket \) if and only if \( \text{Hom}(\llbracket a, b \rrbracket, \llbracket c, d \rrbracket) \) is nonzero.

It can be checked that \( \geq \) is reflexive and antisymmetric but in general is not transitive. While we use the same symbols \( \llbracket b, d \rrbracket \) for the intervals of any \( \mathbb{A}_n(\tau) \), note that these intervals and thus \( \geq \) depend on the underlying orientation \( \tau \). We write \( \llbracket a, b \rrbracket \triangleright \llbracket c, d \rrbracket \) if \( \llbracket a, b \rrbracket \geq \llbracket c, d \rrbracket \) and \( \llbracket a, b \rrbracket \not= \llbracket c, d \rrbracket \).

**Lemma 1.** Let \( \llbracket a, b \rrbracket, \llbracket c, d \rrbracket \) be interval representations of \( \mathbb{A}_n(\tau) \).

1. The dimension of \( \text{Hom}(\llbracket a, b \rrbracket, \llbracket c, d \rrbracket) \) as a \( K \)-vector space is either 0 or 1.
2. A \( K \)-vector space basis \( \{ f_{a,b}^{c,d} \} \) can be chosen for each nonzero \( \text{Hom}(\llbracket a, b \rrbracket, \llbracket c, d \rrbracket) \) such that if \( \llbracket a, b \rrbracket \triangleright \llbracket c, d \rrbracket \), \( \llbracket c, d \rrbracket \triangleright \llbracket e, f \rrbracket \) and \( \llbracket a, b \rrbracket \triangleright \llbracket e, f \rrbracket \), then

\[
\begin{align*}
\phi\triangleright f_{a,b}^{c,d} = f_{c,d}^{e,f} f_{a,b}^{c,d}.
\end{align*}
\]

**Proof.**

1. Let us use the notation \( \llbracket a, b \rrbracket = \{ a, a+1, \ldots, b \} \) to denote the interval of integers \( i \) with \( a \leq i \leq b \) and consider \( g_i = (g_i)_{i=1}^n \in \text{Hom}(\llbracket a, b \rrbracket, \llbracket c, d \rrbracket) \). Suppose that \( \text{Hom}(\llbracket a, b \rrbracket, \llbracket c, d \rrbracket) \) is nonzero.

Note that \( g_i = 0 \) for \( i \not\in [a, b] \cap [c, d] \). It follows that if \( \llbracket a, b \rrbracket \cap [c, d] = \emptyset \), then \( \text{Hom}(\llbracket a, b \rrbracket, \llbracket c, d \rrbracket) = \{ 0 \} \), a contradiction. Therefore \( \llbracket a, b \rrbracket \cap [c, d] \neq \emptyset \).

\( \llbracket a, b \rrbracket \triangleright \llbracket c, d \rrbracket \) and \( \llbracket c, d \rrbracket \triangleright \llbracket a, b \rrbracket \) imply \( \llbracket a, b \rrbracket = \llbracket c, d \rrbracket \).
Fix an index \(j \in [a, b] \cap [c, d] \neq \emptyset\). We claim that \(g_i = g_j\) for any \(i \in [a, b] \cap [c, d]\), by the commutativity requirement on morphisms. To see this, suppose that \(i = j + 1\) with \(i \in [a, b] \cap [c, d]\). Then \(g_i = g_j\) follows from the commutativity of

\[
\begin{array}{ccc}
K & \xrightarrow{\text{id}_K} & K \\
\downarrow{g_i} & & \downarrow{g_i} \\
K & \xleftarrow{\text{id}_K} & K
\end{array}
\quad \text{or} \quad
\begin{array}{ccc}
K & \xleftarrow{\text{id}_K} & K \\
\downarrow{g_i} & & \downarrow{g_i} \\
K & \xrightarrow{\text{id}_K} & K
\end{array}
\]

for \(\tau^f = f\) or \(\tau^i = b\), respectively. A similar argument shows that the above claim holds for \(i = j - 1\) with \(i \in [a, b] \cap [c, d]\). Repeating this argument, we get that \(g_i = g_j\) as long as \(i \in [a, b] \cap [c, d]\).

Thus, any morphism \(g\) is uniquely determined by its value \(g_j\) for some \(j \in [a, b] \cap [c, d]\). This provides an isomorphism of \(K\)-vector spaces

\[
\text{Hom}(\mathbb{I}[a, b], \mathbb{I}[c, d]) \cong \text{Hom}_K(K, K)
\]

by taking \(g\) to \(g_j\). Since \(\text{Hom}_K(K, K) \cong K\), we conclude that if \(\text{Hom}(\mathbb{I}[a, b], \mathbb{I}[c, d])\) is nonzero, then its dimension is 1.

(2) For all pairs of intervals with \(\mathbb{I}[a, b] \supseteq \mathbb{I}[c, d]\) define \(f_{a,b}^{c,d}\) by

\[
(f_{a,b}^{c,d})_i = \begin{cases} 
\text{id}_K & \text{if } i \in [a, b] \cap [c, d], \\
0 & \text{otherwise.}
\end{cases}
\]

The above discussion shows that \(f_{a,b}^{c,d}\) is in \(\text{Hom}(\mathbb{I}[a, b], \mathbb{I}[c, d])\), and any \(g \in \text{Hom}(\mathbb{I}[a, b], \mathbb{I}[c, d])\) can be written as \(g = g_j(1)f_{a,b}^{c,d}\) for any \(j \in [a, b] \cap [c, d]\). Moreover, this choice of \(f_{a,b}^{c,d}\) satisfies Eq. (9) by construction.

\[\square\]

**Example 1.** With orientation \(\tau = f f \cdots f\), the homomorphism spaces are

\[
\text{Hom}(\mathbb{I}[a, b], \mathbb{I}[c, d]) = \begin{cases} 
Kf_{a,b}^{c,d}, & c \leq a \leq d \leq b, \\
0, & \text{otherwise.}
\end{cases}
\]

The basis functions \(f_{a,b}^{c,d}\) are given by

\[
(f_{a,b}^{c,d})_i = \begin{cases} 
\text{id}_K, & a \leq i \leq d, \\
0, & \text{otherwise.}
\end{cases}
\]

With \(n = 2\), \(\mathbb{I}[2, 2] \supseteq \mathbb{I}[1, 2]\) and \(\mathbb{I}[1, 2] \supseteq \mathbb{I}[1, 1]\) but \(\mathbb{I}[2, 2] \not\supseteq \mathbb{I}[1, 1]\). This also provides an example to illustrate that \(\supseteq\) may not be transitive.

Now, let \(M\) be a representation of \(CL_n(\tau)\) with \(n \leq 4\). By the isomorphism \(F: \text{rep}CL_n(\tau) \cong \text{arr}(\text{rep}A_n(\tau))\) in Theorem 2, we identify \(M\) with its corresponding arrow \(F(M): V \to W\) in \(\text{arr}(\text{rep}A_n(\tau))\). Note that \(V\) is in \(\text{rep}A_n(\tau)\) and thus can be decomposed as

\[
\eta_V: V \cong \bigoplus_{1 \leq a \leq b \leq n} \mathbb{I}[a, b]^{m_{a,b}}, \quad (m_{a,b} \in \mathbb{Z}_{\geq 0}: \text{multiplicity})
\]

as in Eq. (9). A similar isomorphism \(\eta_W\) can be obtained for \(W\). Through these isomorphisms, define

\[
\Phi = \eta_W F(M) \eta_V^{-1}: \bigoplus_{1 \leq a \leq b \leq n} \mathbb{I}[a, b]^{m_{a,b}} \to \bigoplus_{1 \leq c \leq d \leq n} \mathbb{I}[c, d]^{m'_{c,d}}.
\]

In fact, \((\eta_V, \eta_W): F(M) \to \Phi\) is an isomorphism in \(\text{arr}(\text{rep}A_n(\tau))\).
Moreover, $\Phi$ can be written in a block matrix form

\begin{equation}
\Phi = \begin{bmatrix} \Phi_{a,b} \end{bmatrix}
\end{equation}

where each block matrix entry $\Phi_{a,b}^{c,d}$: $I[a,b] \to I[c,d]$ is obtained from $\Phi$ by the appropriate inclusion and projection. That is, $\Phi_{a,b}^{c,d}$ is the composition of

\begin{equation}
I[a,b]_{m_{a,b}} \xrightarrow{1} \bigoplus_{1 \leq a \leq b \leq n} I[a,b]_{m_{a,b}} \xrightarrow{\Phi} \bigoplus_{1 \leq c \leq d \leq n} I[c,d]_{m_{c,d}} \xrightarrow{\pi} I[c,d]_{m_{c,d}}.
\end{equation}

In a similar manner, each block $\Phi_{a,b}^{c,d}$ can be further expressed as a matrix of homomorphisms

\begin{equation}
\Phi_{a,b}^{c,d} = [g_{i,j}], \quad (1 \leq i \leq m_{a,b}, 1 \leq j \leq m_{c,d}).
\end{equation}

where each $g_{i,j} \in \text{Hom}(I[a,b], I[c,d])$.

For intervals $I[a,b] \supseteq I[c,d]$, part two of Lemma 1 shows that for each $i$, $j$ we can write $g_{i,j} = \mu_{i,j}f_{a,b}^{c,d}$ for some $\mu_{i,j} \in K$. Factoring out $f_{a,b}^{c,d}$ from $\Phi_{a,b}^{c,d}$ with $I[a,b] \supseteq I[c,d]$, we get

\begin{equation}
\Phi_{a,b}^{c,d} = \begin{cases} M_{a,b}^{c,d}f_{a,b}^{c,d} & \text{if } I[a,b] \supseteq I[c,d], \\
0 & \text{otherwise,}
\end{cases}
\end{equation}

where each $M_{a,b}^{c,d}$ is an $m_{a,b} \times m_{c,d}$ matrix with entries in $K$. To summarize, we define the following.

**Definition 5.** Let $M$ be a persistence module on $\text{CL}_a(\tau)$. The block matrix form $\Phi(M)$ of $M$ is

\begin{equation}
\Phi(M) = \begin{bmatrix} \Phi_{a,b}^{c,d} \end{bmatrix} = \begin{bmatrix} M_{a,b}^{c,d} f_{a,b}^{c,d} \end{bmatrix}_{I[a,b] \supseteq I[c,d]},
\end{equation}

where each $\Phi_{a,b}^{c,d}$ is as defined in Eq. (11).

In the matrix formalism, we label the rows and columns of the block matrix corresponding to the summand $I[a,b]_{m_{a,b}}$ by $a:b$. We say that the block $\Phi_{a,b}^{c,d}$ is in row $c:d$ and column $a:b$.

### 3.3 Permissible operations.

While we have written $\Phi$ in a block matrix form, not all of the usual row and column operations on $K$-matrices correspond to a meaningful change of basis. The fact that there exist some pairs of intervals where $\text{Hom}(I[a,b], I[c,d])$ is zero leads to some complications.

If $(R, S): \Phi' \cong \Phi$ is an isomorphism, then

\begin{align*}
\bigoplus_{1 \leq a \leq b \leq n} I[a,b]_{m_{a,b}} \xrightarrow{\Phi} \bigoplus_{1 \leq a \leq b \leq n} I[a,b]_{m'_{a,b}} \\
\xrightarrow{R} \bigoplus_{1 \leq a \leq b \leq n} I[a,b]_{m_{a,b}} \xrightarrow{\Phi'} \bigoplus_{1 \leq a \leq b \leq n} I[a,b]_{m'_{a,b}}
\end{align*}

commutes and $\Phi' = S^{-1} \Phi R$. Observing that the domain and codomain of $R$ are direct summations, $R$ can be written in a matrix form $R = \left[ R_{a,b}^{c,d} f_{a,b}^{c,d} \right]_{I[a,b] \supseteq I[c,d]}$ relative to them, by an argument similar to that done for $\Phi$. Similarly, $S$ can be written in a matrix form. It can be checked that $(R, S): \Phi' \to \Phi$ is an isomorphism if and only if all the diagonal entries $R_{a,b}^{a,b}$ and $S_{a,b}^{a,b}$ are invertible.
Let us discuss column operations and assume $S$ is the identity. In analogy to usual linear algebra, column operations on $\Phi$ correspond to a change of interval summands induced by $R$. To see this, let us choose a column $a:b$ and suppose that $I[a, b] \supseteq I[c, d]$.

The block entry at row $c:d$ in column $a:b$ of $\Phi = \Phi R$ is

$$
[\Phi R]_{a:b}^{c:d} = \sum_{I[a, b] \supseteq I[c, f], I[c, d]} (M_{e:f}^{c:d} f_{e:f}^{c:d}) (M_{a:b}^{c:d} R_{a:b}^{c:d})
$$

$$
= \left( \sum_{I[a, b] \supseteq I[c, f], I[c, d]} M_{e:f}^{c:d} R_{a:b}^{c:d} \right) f_{e:f}^{c:d}
$$

where $\Phi R$ is computed as a usual multiplication of block matrices. In the last step, we used the property that $f_{e:f}^{c:d} f_{e:f}^{c:d} = f_{e:f}^{c:d}$ as guaranteed by Lemma $2$. Note that the resulting coefficient of $f_{e:f}^{c:d}$ above involves only addition and multiplication of $K$-matrices. Furthermore, since $I[a, b] \supseteq I[a, b]$, it is equal to

$$
\sum_{I[a, b] \supseteq I[c, f], I[c, d]} M_{e:f}^{c:d} R_{a:b}^{c:d} = \left( M_{a:b}^{c:d} R_{a:b}^{c:d} + \sum_{I[a, b] \supseteq I[c, f], I[c, d]} M_{e:f}^{c:d} R_{a:b}^{c:d} \right).
$$

In this form, we see that apart from a change of basis $R_{a:b}$ within the column $a:b$, we also permit addition of multiples of columns $e:f$ with $I[a, b] \supseteq I[e, f]$. A similar analysis can be performed for row operations.

For example, let us consider a persistence module $M$ on $CL_2(f)$ corresponding to

$$
\Phi = \begin{bmatrix}
2:2 & 1:2 & 1:1 \\
M_{2:2}^{1:2} id_{[2,2]} & 0 & 0 \\
M_{2:2}^{1:2} f_{1:2}^{2:2} & M_{1:2}^{1:1} id_{[1,2]} & 0 \\
1:1 & \begin{bmatrix} 0 & M_{1:2}^{1:1} f_{1:2}^{1:1} & M_{1:1}^{1:1} id_{[1,1]} \end{bmatrix} \\
\end{bmatrix}
$$

Since $1:1 \nsubseteq 1:2$, $1:1 \nsubseteq 2:2$, $1:2 \nsubseteq 2:2$, and $2:2 \nsubseteq 1:1$, the blocks in the corresponding positions are zero. An automorphism $R$ on $V = [2,2]^{m_{2:2}} \oplus [1,2]^{m_{1:2}} \oplus [1,1]^{m_{1:1}}$ as defined above can be written as

$$
R = \begin{bmatrix}
2:2 & 1:2 & 1:1 \\
R_{2:2}^{1:2} id_{[2,2]} & 0 & 0 \\
R_{2:2}^{1:2} f_{1:2}^{2:2} & R_{1:2}^{1:1} id_{[1,2]} & 0 \\
1:1 & \begin{bmatrix} 0 & R_{1:2}^{1:1} f_{1:2}^{1:1} & R_{1:1}^{1:1} id_{[1,1]} \end{bmatrix} \\
\end{bmatrix}
$$

Thus, $\Phi R$ is

$$
\begin{bmatrix}
M_{2:2}^{1:2} R_{2:2}^{1:2} id_{[2,2]} & 0 & 0 \\
(M_{2:2}^{1:2} R_{2:2}^{1:2} + M_{1:2}^{1:1} R_{1:2}^{1:1}) f_{1:2}^{2:2} & M_{1:2}^{1:1} R_{1:1}^{1:1} id_{[1,2]} & 0 \\
(M_{1:2}^{1:1} R_{2:2}^{1:2} (f_{1:2}^{1:1} f_{2:2}^{2:2})) & (M_{1:1}^{1:1} R_{1:2}^{1:1} + M_{1:1}^{1:1} R_{1:1}^{1:1}) f_{1:2}^{1:1} & M_{1:1}^{1:1} R_{1:1}^{1:1} id_{[1,1]} \\
\end{bmatrix}
$$

Since $f_{1:2}^{1:1} f_{2:2}^{2:2} = 0$, the lower left corner is still a zero block, as can be expected.
4. Algorithm

4.1. Input and Notation. Let $n \leq 4$, $\tau$ be an orientation, and $M$ be a persistence module on $\mathcal{C}L_n(\tau)$. In the previous section, we constructed the block matrix $\Phi(M) = [\Phi^{c,d}_{a,b}] = [M^{c,d}_{a,b}]_{a,b}$. Two blocks at distinct entries are said to be column (row) neighbors if they are in the same column (row). Neighbors do not have to be directly adjacent to each other in the block matrix. By an abuse of notation, we let $(a:b, c:d)$ referring to row $a:b$ and column $c:d$ also refer to the block (submatrix) located at that entry.

Recall that the vertices of the Auslander-Reiten quiver $\Gamma(h_n(\tau))$ of $h_n(\tau)$ are in bijective correspondence to the interval representations. Hence, the quiver structure of $\Gamma(a_n(\tau))$ naturally induces a partial order on the set of intervals, by going from source vertices to sink vertices. We fix a total order $\prec$ extending this by resolving ambiguities using reverse lexicographic order on the pairs $(b, d)$. For example, since $A_n(f b)$ has Auslander-Reiten quiver

```
1:1 \rightarrow 1:2 \rightarrow 2:2 \rightarrow 2:3 \rightarrow 1:3 \rightarrow 3:3
```

we get the order $2:2 \prec 2:3 \prec 1:2 \prec 1:3 \prec 3:3 \prec 1:1$. Here, the ambiguities are resolved as $2:3 \prec 1:2$, and $3:3 \prec 1:1$. We shall use $\prec$ to order the columns and rows of the block matrix.

Finally, we define the data that serves as input to Algorithm 1.

**Definition 6.** Let $M$ be a persistence module on $\mathcal{C}L_n(\tau)$. The block matrix problem of $M$ is the block matrix $\Phi(M)$, together with permissible operations, and with rows and columns ordered as below.

1. If $I[a,b] \supseteq I[c,d]$ then operations from row $a:b$ to row $c:d$ are permissible.
2. If $I[a,b] \supseteq I[c,d]$ then operations to column $a:b$ from column $c:d$ are permissible.

The columns from left to right are ordered in increasing $\prec$ while rows from top to bottom are in decreasing $\prec$.

Note that the permissible operations are the rules derived in Subsection 3.3 and that applying these operations result in a block matrix isomorphic to $\Phi(M)$. For convenience, we distinguish the permissible operations that operate only within a fixed row or column block. An inner row (column) operation is a row (column) operation that only affects $K$-vector rows (columns) within some fixed row (column) $a:b$.

Recall that if $I[a,b] \not\supseteq I[c,d]$, then the block $(c:d, a:b)$ is always zero, even after the application of any permissible operations. To distinguish them from other blocks that just happen to be numerically zero, we denote them by $\emptyset$ and call these blocks strongly zero blocks.

Otherwise, for $I[a,b] \supseteq I[c,d]$, we use the symbol $*$ to abbreviate the block $M^{c,d}_{a,b}$ at $(c:d, a:b)$. This indicates that these blocks are so far unprocessed. As we operate on the block matrix, their status as unprocessed will be changed to either an identity matrix $E$ or a zero matrix $0$.

**Notation 1.** To denote the possible block statuses, we use:

- $*$ for unprocessed blocks,
- $\emptyset$ for strongly zero blocks,
The blocks marked as $\emptyset$, $E$, and $0$ are considered processed.

Note that the block matrix may have numerically identity or zero blocks, even though we label their status as being unprocessed $\ast$. This status only reflects the fact that they have not yet been examined and fixed through the course of the algorithm.

**Example 2.** The block matrix problem corresponding to a persistence module on $CL_3(fb)$ has the form

\[
\begin{bmatrix}
2:2 & 2:3 & 1:2 & 3:3 & 1:1 \\
1:1 & \emptyset & \emptyset & \ast & \emptyset & \ast \\
3:3 & \emptyset & \ast & \emptyset & \ast & \emptyset \\
1:3 & \ast & \ast & \ast & \emptyset & \emptyset \\
1:2 & \ast & \emptyset & \ast & \emptyset & \emptyset \\
2:3 & \ast & \emptyset & \emptyset & \emptyset & \emptyset \\
2:2 & \ast & \emptyset & \emptyset & \emptyset & \emptyset
\end{bmatrix}
\]

(14)

4.2. **Algorithm.** Given a persistence module $M$ on $CL_n(\tau)$ where $n \leq 4$, the input to Algorithm 1 is the block matrix problem of $M$. Below, we shall also use the notation $M$ to denote the block matrix problem associated to the persistence module $M$.

The algorithm uses the following two facts. Given a usual $K$-matrix $N$, there exist invertible matrices $R$ and $S$ (of appropriate sizes) such that $RNS = \begin{bmatrix} E & 0 \\ 0 & 0 \end{bmatrix}$, a Smith normal form. Thus, by using appropriate inner row and column operations, a block $\ast$ can be transformed into the form $\begin{bmatrix} E & 0 \\ 0 & 0 \end{bmatrix}$. Meanwhile, using some identity submatrix $E$, a row (column) neighbor $\ast$ can be zeroed out using appropriate permissible column (row) operations. Complications come from the side effects of these operations.

**Algorithm 1** Main Algorithm. Input: a block matrix problem $M$

1: procedure **MATRIX\_REDUCTION**($M$)  
2: while $M$ has unprocessed submatrices do
3: \hspace{1em} $v_\ast$ ← the bottommost $\ast$ block of the rightmost column with $\ast$ blocks in $M$
4: \hspace{1em} $F_R$ ← **ROW\_SIDE\_EFFECT**($v_\ast$)
5: \hspace{1em} $F_C$ ← **COL\_SIDE\_EFFECT**($v_\ast$)
6: \hspace{1em} Transform $v_\ast$ to Smith normal form by inner operations on $M$.
7: \hspace{2em} for all $v' \in F_R$ do **COL\_FIX**($v'$)
8: \hspace{2em} for all $v' \in F_C$ do **ROW\_FIX**($v'$)
9: \hspace{1em} Update the partitioning of blocks in block matrix $M$.
10: \hspace{1em} while there exist blocks $v_t$ with ($p ← \text{ERASABLE}(v_t)$) not null do
11: \hspace{2em} Zero out $v_t$ via the procedure indicated by $p$.

The main while loop of Algorithm 1 can be divided broadly into four main parts.

1. Transform one appropriate block $v_\ast$ into a Smith normal form (line 6) by inner row and column operations on $M$.
2. The operations performed in the previous part may affect the forms of neighboring identity blocks. We transform them back to identity blocks (lines 7 and 8).
3. Update the partitioning of the blocks (line 9). After obtaining the Smith normal form $\begin{bmatrix} E & 0 \\ 0 & 0 \end{bmatrix}$, we split up columns and rows so that each identity matrix $E$ is its own block in $M$. 
(4) Greedily zero out erasable blocks $v$ by addition of multiples of identity blocks. We first illustrate parts one to three by an example. Suppose that

$$M = \begin{bmatrix}
* & * & v_s \\
* & \emptyset & E
\end{bmatrix}$$

where operations from row 1 to row 2 and vice versa are impermissible. We get:

$$M = \begin{bmatrix}
* & * & v_s \\
* & \emptyset & E
\end{bmatrix} \xrightarrow{1} \begin{bmatrix}
* & * & E \ 0 \\
* & \emptyset & S
\end{bmatrix} \xrightarrow{2} \begin{bmatrix}
* & * & E \ 0 \\
* & \emptyset & E
\end{bmatrix},$$

where the numbers above the isomorphisms indicate the procedures being performed.

In the first part, the block $v_s$ is chosen by the heuristic given in Algorithm 1, line 3. Note that $v_s$ is therefore dependent on the ordering of the rows and columns, which we have fixed in Definition 6. By inner operations on $M$, the block $v_s$ is transformed to Smith normal form. In particular, there are invertible matrices $R$ and $S$ such that $Rv_sS = \begin{bmatrix} E & 0 \\ 0 & 0 \end{bmatrix}$.

Algorithm 2

1: function COL_SIDE_EFFECT($v$)
2: return $\{ v' \mid v' \text{ is an identity column neighbor of } v \}$

1: function ROW_SIDE_EFFECT($v$)
2: return $\{ v' \mid v' \text{ is an identity row neighbor of } v \}$

Next, the block $E$ below $v_s$ becomes $ES = S$, possibly not an identity matrix. This is recorded as a side effect. Since $S$ is invertible, it can be transformed back by using only inner row operations in ROW_FIX. In general there may be other identity blocks in the same row as $S$ whose forms are affected by these row operations. To fix them, we recursively call ROW_FIX and COL_FIX in Algorithm 3. Checking that this does not lead to an infinite recursion for the cases we consider is part of the proof of Theorem 1.

Algorithm 3

1: function COL_FIX($v$)
2: $V' \leftarrow$ COL_SIDE_EFFECT($v$)
3: Transform $v$ to an identity by inner column operations on $M$.
4: for all $v' \in V'$ do ROW_FIX($v'$)

1: function ROW_FIX($v$)
2: $V' \leftarrow$ ROW_SIDE_EFFECT($v$)
3: Transform $v$ to an identity by inner row operations on $M$.
4: for all $v' \in V'$ do COL_FIX($v'$)

Next is part three, where we update the block matrix partitioning to isolate the identity blocks $E$. Both the row and column of $v_s$ are split into two. We get

$$M \cong \ldots \cong \begin{bmatrix}
* & * & E & 0 \\
* & * & 0 & 0 \\
* & \emptyset & E & 0 \\
* & \emptyset & 0 & E
\end{bmatrix}$$

Since $v_s$ has a column neighbor $E$, the bottom row also needs to be split to isolate the parts of the old identity block.
Finally, we discuss part four. A simple case for a target block $v_t$ to be erasable is when $v_t = (r,c)$ has a column neighbor identity block $v_E = (r',c)$ that has no nonzero row neighbors, and such that row operations from row $r'$ to row $r$ are permissible. Using permissible row operations, the block $v_t$ can be zeroed out by addition of a multiple of the identity block $v_E$. A similar statement holds if there exists a row neighbor identity block $v_E$ satisfying similar conditions.

The above cases present no side effects. In general, zeroing out the target block $v_t$ by addition of multiples of a row (column) may change the forms of other processed blocks. We separate the cases of row and column erasability in Algorithm 4.

**Algorithm 4**

```plaintext
1: function ERASABLE($v_t$, $v_f = null$, visited = {}) 
2: if ROW_ERASABLE($v_t$, $v_f$, visited) is not null then 
3: return ROW_ERASABLE($v_t$, $v_f$, visited) 
4: else if COL_ERASABLE($v_t$, $v_f$, visited) is not null then 
5: return COL_ERASABLE($v_t$, $v_f$, visited) 
6: else 
7: return null
```

In zeroing out the target $v_t$, we avoid changing the forms of any previously obtained identity blocks. It is also possible that a zero block $v'_t$ may become nonzero as a side effect. The algorithm ensures that if this happens, then $v'_t$ can and will be transformed back to 0 again. Iteratively, repairing these side effects may introduce more side effects. Thus, we recursively call on our check for erasability on each side effect. To avoid any infinite recursion, we keep track of the targets $v_t$ visited, and visit each block as a target at most once for each top-level call to ERASABLE.

If the above conditions can be satisfied, the function ERASABLE returns a finite directed tree, called the process tree, that records the procedure to zero out $v_t$. Each vertex in a process tree is labelled with a pair $(v_t, v_E)$ of a target block and an identity block that can be used to zero out $v_t$. The successor vertices $(v'_t, v'_E)$ of a vertex $(v_t, v_E)$ consist of all $v'_t$ that appear as side effects in the operation to zero out $v_t$ using $v_E$.

If no such procedure can be found, then ERASABLE returns a null (empty) process tree. This means that the block in question is declared as not being erasable in the current step of the algorithm.

Let us discuss ROW_ERASABLE in Algorithm 5 in detail. In line 2 we use the function COL_SIDE_EFFECT($v_t$) to get candidate identity blocks $v_E$. We consider only unvisited blocks $v_E$ where the row operation from $v_E$ to $v_t$ is permissible, and where $v_E$ is not the flagged block $v_f$. Its purpose will become clear below.

Now, NONZERO_ROW_NEIGHBORS($v_E$) is defined to return the set of row neighbors $u$ of $v_E$ that are not zero nor strongly zero. Each $u$ can potentially induce a side effect, which we check one by one. To illustrate, consider the following arrangement

\[
\begin{bmatrix}
\vdots & \vdots & \vdots \\
\cdots & u & \cdots & v_E & \cdots \\
\cdots & \vdots & \vdots \\
\cdots & v'_t & \cdots & v \cdots \\
\vdots & \vdots & \vdots
\end{bmatrix}
\]
Algorithm 5 Check whether or not $v_t$ is row erasable without using block $v_f$.

1: function ROW_ERASABLE($v_t, v_f, visited$)
2: $V' = \text{COL\_SIDE\_EFFECT}(v_t)$
3: $visited \leftarrow visited \cup \{ v_t \}$
4: for all $v_E \in V'$ not in visited, $v_E \neq v_f$, and row operation from $v_E$ to $v_t$ permissible do
5: usable $\leftarrow$ true; subtrees $\leftarrow \{ \}$
6: for all $u \in \text{NONZERO\_ROW\_NEIGHBORS}(v_E)$ do
7: $v'_t \leftarrow$ the block in same row as $v_t$ and same column as $u$.
8: if $v'_t$ is in visited or $v'_t = E$ then
9: usable $\leftarrow$ false; break
10: if ($v'_t = 0$ and ($p \leftarrow \text{ERASABLE}(v'_t, u, visited) = \text{null}$)) then
11: usable $\leftarrow$ false; break
12: subtrees $\leftarrow$ subtrees $\cup \{ p \}$.
13: if usable then
14: return process tree with root $(v_t, v_E)$ and arrows to the roots of subtrees.
15: return null

where $v_E$ is the identity block under consideration. Here, $u$ is a nonzero row neighbor of $v_E$. Since we want to add multiples of row $r_1$ to $r_2$ to zero out $v_t$, the block $v'_t$ in same row $r_2$ as $v_t$ and same column as $u$ (Line 7) may possibly have its form affected.

The next few lines handle the checking of block $v'_t$. If the block $v'_t$ is an identity block, or if it has already been visited previously, then we do not use row $r_1$. If the block $v'_t$ is zero, we need to check whether or not it can be transformed back to zero again. Here, the flag $v_f$ comes into play. We set the flagged block as $v_f = u$ in the call to ERASABLE in Line 10, since we do not want to use $u$ to zero out $v'_t$, thereby undoing the operations to zero out $v_t$.

If a nonempty process tree is returned by the top-level call to ERASABLE($v_t$) in Algorithm 1 then $v_t$ is erasable. By construction, it suffices to traverse the process tree and do the operations indicated to zero out $v_t$ and fix all side effects.

Let us reproduce here Theorem 1 concerning Algorithm 1.

**Theorem 1.** Assume Algorithm 1 is called with the block matrix problem corresponding to a persistence module $M$ on a commutative ladder of finite type. Then Algorithm 1 terminates and the input matrix is transformed to an isomorphic block matrix consisting only of identity, zero, and strongly zero blocks, and whose indecomposable decomposition corresponds to an indecomposable decomposition of $M$.

Whether or not Algorithm 1 terminates depends not on the particular persistence module, but on the statuses of the blocks and the status changes brought about by the operations. Moreover, the operations to be performed only depends on the arrangement of the statuses. All these depend only on the initial arrangement, which in turn depends on the orientation $\tau$ and the ordering chosen for the intervals.

From a result in [6], a commutative ladder $CL_n(\tau)$ is finite type if and only if $n \leq 4$, so that there are only a finite number of cases to check. Below, we provide proofs for Theorem 1 with orientations $f$, $fb$, and $fff$. The proofs for the other orientations are similar.
Algorithm 6 Check whether or not $v_t$ is column erasable without using block $v_f$.

1: function COL_ERASABLE($v_t$, $v_f$, visited)
2: $V' = \text{ROW\_SIDE\_EFFECT}(v_t)$
3: visited $\leftarrow$ visited $\cup \{ v_t \}$
4: for all $v_E \in V'$ not in visited, $v_E \neq v_f$, and column operation from $v_E$ to $v_t$ permissible do
5:     usable $\leftarrow$ true; subtrees $\leftarrow \{ \}$
6:     for all $u \in \text{NONZERO\_COL\_NEIGHBORS}(v_E)$ do
7:         $v_t' \leftarrow$ the block in same column as $v_t$ and same row as $u$.
8:         if $v_t'$ is in visited or $v_t' = E$ then
9:             usable $\leftarrow$ false; break
10:        if ($v_t' = 0$ and ($p \leftarrow \text{ERASABLE}(v_t', u, visited)) = \text{null}$) then
11:           usable $\leftarrow$ false; break
12:        subtrees $\leftarrow$ subtrees $\cup \{ p \}$.
13:     if usable then
14:        return process tree with root $(v_t, v_E)$ and arrows to the roots of subtrees.
15: return null

Furthermore, an indecomposable decomposition can easily be read off the resulting normal form consisting of only identity, zero, and strongly zero block, and the correspondence to an indecomposable decomposition of the persistence module $M$ is provided by Theorem 2.

We were unable to find a proof that does not involve manually checking each possible orientation. Given a particular persistence module, it is clear for each completed iteration of the main while loop in Algorithm 1 the total number of scalar entries in unprocessed blocks strictly decreases. Moreover, the procedure ERASABLE avoids any infinite recursion by construction. The difficulty comes from the use of Algorithm 1 line 3 for choosing $v_*$ and subsequently showing that all side effects can always be resolved.

4.2.1. Case CL$_2(f)$. The input block matrix problem is of the form

\[
\begin{bmatrix}
1:1 & 2:2 \\
1:1 & 1:1 \\
1:2 & 2:2 \\
\end{bmatrix}
\]

in general. Initially, all top to bottom and left to right operations are impermissible. The red arrows show the additional impermissible operations.

First the unprocessed block at $v_* = (1:1, 1:1)$ is transformed by inner elementary operations to Smith normal form $\begin{bmatrix} E & 0 \\ 0 & 0 \end{bmatrix}$. Note that $v_*$ has no identity neighbors so that there are no side effects to undo.
Updating the block partitioning, the matrix is now in the form

\[
\begin{bmatrix}
0 & * & E & 0 \\
0 & * & 0 & 0 \\
* & * & 0 & 0 \\
* & 0 & 0 & 0
\end{bmatrix}.
\]

For convenience, we use subscripts to distinguish the two columns and rows corresponding to 1:1 obtained after the repartitioning. Additions from the columns in 1:1 to the columns in 1:2 are permitted, and the unprocessed submatrix \((1:1, 1:2)\) is erasable using the newly processed \(E\), without any side effects. We get the form

\[
\begin{bmatrix}
0 & * & 0 & E & 0 \\
0 & * & 0 & 0 & 0 \\
* & * & 0 & 0 & 0 \\
* & 0 & 0 & 0 & 0
\end{bmatrix}
\]

(15) \(\cong\)

\[
\oplus\begin{bmatrix}
1:1
\end{bmatrix}
\]

which we have expressed as a direct sum of block matrices.

Here, we can extract two indecomposable representations of \(CL_2(f)\). The identity submatrix \(E\) in \((1:1, 1:1)\) is

\[
E_{1:1}^f = \begin{bmatrix}
1f_{1:1}^f & 0 & \ldots & 0 \\
0 & 1f_{1:1}^f & \ldots & 0 \\
\vdots & \vdots & \ddots & \vdots \\
0 & 0 & \ldots & 1f_{1:1}^f
\end{bmatrix}
\]

where

\[
f_{1:1}^f = \begin{bmatrix}
K \\
\vdots \\
K
\end{bmatrix}
\]

(16)

as in the proof of Lemma[1]

Via the isomorphism functor \(F\) in Theorem[2] the arrow \(f_{1:1}^f\) in Eq. (16) can be regarded as the corresponding representation \(F^{-1}(f_{1:1}^f)\). This is indecomposable. Thus, \(Ef_{1:1}^f\) corresponds to a direct sum of \(m\) copies of the representation in Eq. (16), where \(m\) is the size of \(E\).

The third term in Eq. (15) is an empty matrix with 0 rows, and represents the arrow \(0 : \mathbb{I}[1, 1]^{m_1} \rightarrow 0\) in rep \(\mathbb{A}_d(f)\), where \(m_1\) is the number of \(K\)-vector columns in 1:1. By the isomorphism, this corresponds to a direct sum of \(m_1\) copies of the indecomposable representation

\[
\begin{bmatrix}
0 \\
K
\end{bmatrix}
\]

Now, the row 1:1 and columns 1:1, 1:2 in the block matrix Eq. (15) will not affect nor be affected by subsequent operations, so we hide them from the block matrix. The
unprocessed block $(1:2, 1:2)$ is next transformed to Smith normal form to get

\[
\begin{pmatrix}
1:1 & 1:2 & 2:2 \\
\emptyset & * & * \\
* & 0 & \emptyset \\
\end{pmatrix}
\sim
\begin{pmatrix}
1:1 & 1:2 & 2:2 \\
\emptyset & * & * \\
* & E & 0 \\
\end{pmatrix}.
\]

We see that $(1:1, 1:2_1)$ is erasable using $(1:2_1, 1:2_1)$. The checking via \textsc{row\_erasable} in Algorithm 5 proceeds as follows. While \(u = (1:2_1, 2:2_1)\) is a nonzero row neighbor of \(E\), the computed potential side effect is \(v' = (1:1, 2:2_1)\). Since \(v'\) is strongly zero, addition from row 1:2 will not affect it.

Similarly, \((1:2_1, 2:2_1)\) is erasable. After zeroing out erasable blocks, we get

\[
\begin{pmatrix}
1:1 & 1:2 & 2:2 \\
\emptyset & * & * \\
* & E & 0 \\
\end{pmatrix}
\]

and then

\[
\begin{pmatrix}
2:1 & 2:2 \\
\emptyset & 0 & 0 \\
0 & E & 0 \\
\end{pmatrix}.
\]

The identity submatrix \(E\) in \((1:2_1, 1:2_1)\) corresponds to copies of the indecomposable representation

\[
\begin{array}{c}
K \\
\Rightarrow \\
K
\end{array}
\]

as direct summands.

Once again we abbreviate the block matrix:

\[
\begin{pmatrix}
1:1 & 1:2 \\
\emptyset & * \\
* & 0 \\
\end{pmatrix}
\]

and then

\[
\begin{pmatrix}
1:1 & 1:2 \\
\emptyset & E \\
0 & 0 \\
\end{pmatrix}.
\]

after transforming the next target \((1:1, 1:2_1)\) to Smith normal form. The identity submatrix in \((1:1, 1:2_1)\), the row 1:1_2, and the column 1:2_2 correspond to copies of the indecomposable representations with dimension vectors \(1_0\), \(0_0\), and \(1_1\) respectively, as direct summands.

What remains is the form

\[
\begin{pmatrix}
1:2 \\
* \\
* \\
\end{pmatrix}
\]

from which we get

\[
\begin{pmatrix}
2:2 & 2:2 \\
0 & * \\
E & 0 \\
0 & 0 \\
\end{pmatrix}
\]

after transforming the next target \((2:2_1, 2:2_2)\) to normal form, and zeroing out erasable blocks. The identity submatrix \((2:2_1, 2:2_1)\) and the row 2:2_2 correspond respectively to copies of the indecomposable representations with dimension vectors \(0_1\) and \(0_0\).
Abbreviating again, we are left with $1:2\begin{bmatrix} 2 & 2 \\ \end{bmatrix}^*$.
Transforming the last target $(1:2, 2:2)$ to normal form yields

$\begin{bmatrix} E & 0 \\ 0 & 0 \\ \end{bmatrix}_{1:2}$.

The identity submatrix $(1:2_1, 2:2_1)$, the row $1:2_2$, and the column $2:2_2$ correspond respectively to copies of the indecomposable representations with dimension vectors $11, 11, 00$, and $00$.

It is clear that we have obtained all possible indecomposable representations of $CL_2(f)$. This can be confirmed for example by checking with the Auslander-Reiten quiver of $CL_2(f)$.

Given a particular persistence module $M$ on $CL_2(f)$, the algorithm gives the multiplicities of each of these indecomposables in an indecomposable decomposition of $M$.

4.2.2. Case $CL_3(fb)$. The input block matrix is given in Example 2 [2].

While the presence of impermissible operations did not cause any noticeable complications in the case of $CL_2(f)$, in general this is not so. For better readability, we indicate only the relevant impermissible operations at each step below. Below, each numbered step corresponds to one pass of the outer while loop in Algorithm 1.

(1) In this step, $v_\ast$ is $(1:1, 1:1)$.

(a) Transform $(1:1, 1:1)$ to Smith normal form, giving the block matrix

$\begin{bmatrix} 2 & 2 & 3 & 1:2 & 1:3 & 3:3 & 1:1 & 1:1 \\ 1:1 & 0 & 0 & 0 & E & 0 \\ 1:2 & 0 & 0 & 0 & 0 & 0 \\ 3:3 & 0 & 0 & 0 & 0 & 0 \\ 1:3 & 0 & 0 & 0 & 0 & 0 \\ 1:2 & 0 & 0 & 0 & 0 & 0 \\ 2:3 & 0 & 0 & 0 & 0 & 0 \\ 2:2 & 0 & 0 & 0 & 0 & 0 \\ \end{bmatrix}$.

(b) Zero out the blocks $(1:1_1, 1:2)$ and $(1:1_1, 1:3)$ by additions from the identity submatrix at $(1:1_1, 1:1)$:

$\begin{bmatrix} 2 & 2 & 3 & 1:2 & 1:3 & 3:3 & 1:1 & 1:1 \\ 1:1 & 0 & 0 & 0 & E & 0 \\ 1:2 & 0 & 0 & 0 & 0 & 0 \\ 3:3 & 0 & 0 & 0 & 0 & 0 \\ 1:3 & 0 & 0 & 0 & 0 & 0 \\ 1:2 & 0 & 0 & 0 & 0 & 0 \\ 2:3 & 0 & 0 & 0 & 0 & 0 \\ 2:2 & 0 & 0 & 0 & 0 & 0 \\ \end{bmatrix}$.

(c) The identity submatrix $(1:1_1, 1:1)$ and the columns in $1:1_2$ give copies of indecomposable representations isomorphic to

$K \rightarrow 0 \leftarrow 0 \quad 0 \rightarrow 0 \leftarrow 0$.

$K \rightarrow 0 \leftarrow 0 \times$ and $K \rightarrow 0 \leftarrow 0$.
corresponding to the vertices 100 and 100 in Figure 1.

(d) We are left with

\[
\begin{pmatrix}
1 & 1 & 0 & 0 & 0 & 0 \\
2 & 3 & 1 & 2 & 1 & 3 \\
2 & 2 & 1 & 2 & 1 & 3 \\
3 & 3 & 0 & 0 & 0 & 0 \\
1 & 3 & 0 & 0 & 0 & 0 \\
1 & 2 & 0 & 0 & 0 & 0 \\
1 & 2 & 0 & 0 & 0 & 0 \\
2 & 3 & 0 & 0 & 0 & 0 \\
2 & 2 & 0 & 0 & 0 & 0 \\
\end{pmatrix}
\]

(2) We combine the next two steps. Here, \( v^* \) is \((3:3, 3:3)\), and then...

(3) ... \( v^* \) is \((1:3, 1:3)\).

The direct summands with dimension vectors 001 and 000, and then 111 can be extracted without any extra complications:

\[
\begin{pmatrix}
1 & 1 & 0 & 0 & 0 & 0 \\
2 & 3 & 1 & 2 & 1 & 3 \\
2 & 2 & 1 & 2 & 1 & 3 \\
1 & 3 & 0 & 0 & 0 & 0 \\
1 & 2 & 0 & 0 & 0 & 0 \\
2 & 3 & 0 & 0 & 0 & 0 \\
2 & 2 & 0 & 0 & 0 & 0 \\
\end{pmatrix}
\]

(4) After transforming \( v^* = (3:3, 1:3) \) to Smith normal form, the block matrix is now

\[
\begin{pmatrix}
1 & 1 & 0 & 0 & 0 & 0 \\
2 & 3 & 1 & 2 & 1 & 3 \\
2 & 2 & 1 & 2 & 1 & 3 \\
3 & 3_1 & 0 & 0 & 0 & E \\
3 & 3_2 & 0 & 0 & 0 & 0 \\
3 & 3_1 & 0 & 0 & 0 & 0 \\
3 & 3_2 & 0 & 0 & 0 & 0 \\
1 & 3 & 0 & 0 & 0 & 0 \\
1 & 2 & 0 & 0 & 0 & 0 \\
2 & 3 & 0 & 0 & 0 & 0 \\
2 & 2 & 0 & 0 & 0 & 0 \\
\end{pmatrix}
\]

Row operations from 3:3 to 1:1 are impermissible, and so the only candidate \( E \) cannot be used to zero out \((1:1, 1:3_1)\). Block \((1:1, 1:3_1)\) is therefore not erasable. The block \((3:3_1, 2:3)\) however, is erasable and is zeroed out. No direct summands are identified at this step.

(5) Next, \( v^* = (1:1, 1:3_2) \) and direct summands 001 and 000 can be extracted. We are left with the block matrix form:

\[
\begin{pmatrix}
1 & 1 & 0 & 0 & 0 & 0 \\
2 & 3 & 1 & 2 & 1 & 3 \\
2 & 2 & 1 & 2 & 1 & 3 \\
3 & 3_1 & 0 & 0 & 0 & E \\
3 & 3_2 & 0 & 0 & 0 & 0 \\
3 & 3_1 & 0 & 0 & 0 & 0 \\
3 & 3_2 & 0 & 0 & 0 & 0 \\
1 & 3 & 0 & 0 & 0 & 0 \\
1 & 2 & 0 & 0 & 0 & 0 \\
2 & 3 & 0 & 0 & 0 & 0 \\
2 & 2 & 0 & 0 & 0 & 0 \\
\end{pmatrix}
\]
(6) After transforming $v_\ast = (1:1, 1:3)$ to Smith normal form, the column neighbor $(3:3_1, 1:3)$ may no longer be the identity. After ROW FIX it is transformed back to an identity. The block matrix is now in the following form:

$$
\begin{bmatrix}
1:1 & 0 & 0 & E & 0 \\
1:1 & 0 & 0 & * & 0 \\
3:3_1 & 0 & 0 & E & 0 \\
3:3_2 & 0 & 0 & 0 & E \\
3:3_1 & 0 & * & 0 & 0 \\
1:3 & * & * & 0 & 0 \\
1:2 & * & 0 & * & 0 \\
2:3 & * & * & 0 & 0 \\
2:2 & * & * & 0 & 0 \\
\end{bmatrix}
$$

after repartitioning to isolate identity submatrices into their own blocks.

After zeroing out $(1:1_1, 1:2_2)$, direct summands with dimension vectors $101_111$ and $001_111$ can be extracted, as follows. We get the decomposition

$$
\begin{bmatrix}
1:1 & 0 & 0 & 0 & E & 0 \\
1:1 & 0 & 0 & * & 0 \\
3:3_1 & 0 & 0 & E & 0 \\
3:3_2 & 0 & 0 & 0 & E \\
3:3_1 & 0 & * & 0 & 0 \\
1:3 & * & * & 0 & 0 \\
1:2 & * & 0 & * & 0 \\
2:3 & * & * & 0 & 0 \\
2:2 & * & * & 0 & 0 \\
\end{bmatrix}
\oplus
\begin{bmatrix}
1:1_1 & E \\
3:3_1 & E \\
3:3_2 & E \\
\end{bmatrix},
$$

where the second term corresponds to $101_111$, and the third term to $001_111$. The first term is sent to the next step.

(7) We combine the next two steps. First, $v_\ast$ is $(1:2_1, 1:2_2)$ and $110_110$ is extracted.

(8) Subsequently, $v_\ast$ is $(1:3_1, 1:2_2)$. These two steps yield the block matrix forms

$$
\begin{bmatrix}
2:2 & 2:3 & 1:2 & 1:3_1 & 1:3_2 \\
1:1 & 0 & 0 & E & 0 \\
1:1 & 0 & 0 & * & 0 \\
3:3_1 & 0 & 0 & E & 0 \\
3:3_2 & 0 & 0 & 0 & E \\
3:3_1 & 0 & * & 0 & 0 \\
1:3 & * & * & 0 & 0 \\
1:2 & * & 0 & * & 0 \\
2:3 & * & * & 0 & 0 \\
2:2 & * & * & 0 & 0 \\
\end{bmatrix}
\oplus
\begin{bmatrix}
2:2 & 2:3 & 1:2 \\
1:1 & 0 & 0 & * \\
3:3 & 0 & 0 & 0 \\
1:3_1 & 0 & * & 0 \\
1:3_2 & * & 0 & 0 \\
\end{bmatrix}
$$

and then

$$
\begin{bmatrix}
2:2 & 2:3 & 1:2_1 & 1:2_2 \\
1:1 & 0 & 0 & 0 \\
3:3_1 & 0 & 0 & 0 \\
1:3_1 & 0 & E & 0 \\
1:3_2 & * & 0 & 0 \\
1:2 & * & 0 & 0 \\
2:3 & * & 0 & 0 \\
2:2 & * & 0 & 0 \\
\end{bmatrix}.
$$

Note that column operations from 1:2 to 2:3 are impermissible.

(9) Combining steps again for brevity, $v_\ast$ is $(1:1, 1:2_2)$ and then...

(10) ... $v_\ast$ is $(2:3, 2:3)$.
(11) Here, \(v_s\) is \((1:3_2, 2:3)\). After transforming \(v_s\) to Smith normal form, the matrix is of the form

\[
\begin{pmatrix}
3:3 & 0 & * & 0 \\
1:3_1 & 0 & * & E \\
1:3_2 & * & E & 0 & 0 \\
1:3_3 & * & 0 & 0 & 0 \\
1:2 & * & 0 & 0 & 0 \\
2:3 & * & 0 & 0 & 0 \\
2:2 & * & 0 & 0 & 0
\end{pmatrix}
\]

First of all, note that \((3:3, 2:3)\) is erasable, without any concerns of side effects.

Moreover, \(v_t = (1:3_1, 2:3_1)\) is also erasable. Zeroing it out by additions from the identity \(E\) at \((1:3_2, 2:3_1)\) may cause the 0 block at \(v'_t = (1:3_1, 2:2)\) to become nonzero, but \(v'_t\) can be zeroed out again by additions from \((1:3_1, 1:2)\). In other words, the side effect \(v'_t\) is erasable, and thus \(v_t\) is, too. This illustrates the idea behind Algorithm 5 and the recursive checking of erasability. Similarly, \((1:3_2, 2:2)\) is also erasable.

We are thus able to extract \(011\), leaving the form:

\[
\begin{pmatrix}
3:3 & 0 & * & 0 \\
1:3_1 & 0 & * & E \\
1:3_2 & * & E & 0 & 0 \\
1:3_3 & * & 0 & 0 & 0 \\
1:2 & * & 0 & 0 & 0 \\
2:3 & * & 0 & 0 & 0 \\
2:2 & * & 0 & 0 & 0
\end{pmatrix}
\]

(12) The procedures from this step on are similar to the ones we have done, and direct summands with dimension vectors \(111\) and \(010\) will be extracted.

The dimension vector \(111\) comes from the direct summand \(1:3 \begin{pmatrix} 2:3 & 1:2 \end{pmatrix} \). This is \(m\) copies of the arrow \([f_1^{1:3} f_2^{1:3} : \mathbb{I}[2, 3] \oplus \mathbb{I}[1, 2] \to \mathbb{I}[1, 3], \text{where } m \text{ is the size of the } E\). The dimension vector \(010\) similarly comes from a summand with two identity blocks that cannot zero out each other. Via the isomorphism \(F\) in Theorem 2 these summands corresponds to the representations given in Eq. (8).
4.2.3. Case $CL_4(fff)$. We have not yet seen an identity block declared erasable in Algorithm $[4]$. For $CL_4(fff)$, this occurs while working with unprocessed blocks in the column 3:3. Below, we quickly go through the procedures leading up to this occurrence.

The input block matrix is of the form

$$
\begin{array}{cccccccc}
1:1 & 1:2 & 1:3 & 2:2 & 1:4 & 2:3 & 1:2 & 1:1 \\
\hline
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
\end{array}
$$

in general. We have chosen not to display the impermissible operations here. For the steps similar to ones already done in previous cases, we only provide the resulting block matrix form after sequences of operations. Each numbered item below expresses the result after a sequence of steps involving $v_3$ taken from a particular column.

(1) By procedures on column 1:1, direct summands with dimension vectors $1000$ and $1000$ can be extracted, leaving us with the block matrix form:

$$
\begin{array}{cccccccc}
1:1 & 1:2 & 1:3 & 2:2 & 1:4 & 2:3 & 1:2 & 1:1 \\
\hline
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
\end{array}
$$

(2) Next, procedures on column 1:2 yield $1100$ and $1100$, giving us the form:

$$
\begin{array}{cccccccc}
1:1 & 1:2 & 1:3 & 2:2 & 1:4 & 2:3 & 1:2 & 1:1 \\
\hline
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
\end{array}
$$
(3) Procedures on column 1:3 extract \(\begin{bmatrix} 1110 & 1000 & 0000 \end{bmatrix}^{1110} \text{ and } 1110^{1110}\). The block matrix is now

\[
\begin{bmatrix}
0 & 0 & 0 & 0 & * & 0 \\
0 & 0 & 0 & 0 & * & E \\
0 & 0 & 0 & * & * & * & 0 \\
* & * & 0 & * & 0 & 0 \\
* & * & 0 & 0 & 0 & 0 \\
* & 0 & 0 & 0 & 0 & 0 \\
* & * & 0 & 0 & 0 & 0 \\
* & 0 & 0 & 0 & 0 & 0
\end{bmatrix}
\]

where we note that column operations from 1:3 to 2:2 are impermissible.

(4) Procedures on column 2:2 yield \(\begin{bmatrix} 0100 & 1100 & 1100 & 0000 \end{bmatrix}^{0100} \text{ and } \begin{bmatrix} 0100 & 1100 & 1210 & 0000 \end{bmatrix}^{0100}\). Similar to what we have seen before, \(1100\) arises from the direct summand \(E\) that can be obtained after the prescribed operations. We then obtain the matrix form:

\[
\begin{bmatrix}
0 & 0 & 0 & 0 & * & 0 \\
0 & 0 & 0 & 0 & * & E \\
0 & 0 & 0 & * & * & * & 0 \\
* & * & 0 & * & 0 & 0 \\
* & 0 & 0 & 0 & 0 & 0 \\
* & * & 0 & 0 & 0 & 0 \\
* & 0 & 0 & 0 & 0 & 0 \\
* & * & 0 & 0 & 0 & 0
\end{bmatrix}
\]

(5) Working on column 1:4 next, we extract \(\begin{bmatrix} 1111 & 1000 & 1000 & 0000 \end{bmatrix}^{1111} \text{ and } 1111^{1111}\). The matrix is now of the form

\[
\begin{bmatrix}
0 & 0 & 0 & 0 & * & 0 \\
0 & 0 & 0 & * & * & 0 \\
0 & 0 & 0 & * & 0 & 0 \\
* & 0 & 0 & 0 & 0 & 0 \\
* & * & 0 & 0 & 0 & 0 \\
* & 0 & 0 & 0 & 0 & 0 \\
* & * & 0 & 0 & 0 & 0 \\
* & 0 & 0 & 0 & 0 & 0
\end{bmatrix}
\]
(6) From column 2:3, \(0110\), \(0100\), \(0110\), \(0100\), \(1111\), \(0110\), \(1221\), \(1221\), and \(0110\) are extracted. The summands with dimension vectors \(1210\) and \(1221\) involve three identity blocks. In particular, \(1210\) corresponds to a direct summand

\[
\begin{bmatrix}
0 & 1 \\
1 & 0
\end{bmatrix}
\]

This is isomorphic to copies of the persistence module

\[
K^2 \xrightarrow{id} K^2 \xrightarrow{[0,1]} K \xrightarrow{E} 0
\]

by a choice of basis. Similarly, \(1221\) comes from a direct summand

\[
\begin{bmatrix}
1 & 1 \\
0 & 2
\end{bmatrix}
\]

which is copies of the persistence module

\[
K \xrightarrow{id} K^2 \xrightarrow{[1,0]} K \xrightarrow{E} 0
\]

We caution the reader that several identity blocks will appear and stay for some iterations before being extracted. The matrix form below is the result of all the operations taken in this step.

\[
\begin{bmatrix}
0 & 0 & 0 & * & 0 & 0 \\
0 & 0 & 0 & 0 & E & 0 \\
0 & 0 & 0 & * & E & 0 \\
0 & 0 & 0 & * & * & 0 \\
0 & 0 & 0 & 0 & E & 0 \\
0 & 0 & 0 & 0 & 0 & 0
\end{bmatrix}
\]

(7) Procedures on column 2:4 provide us with the direct summands with dimension vectors

\[
\begin{bmatrix}
1 & 2 \\
1.3 \\
1.3 \\
1.3 \\
2.2 \\
2.2 \\
1.4 \\
2.3 \\
2.4 \\
3.3 \\
3.4 \\
4.4 \\
\end{bmatrix}
\]

We note that \(1210\) comes from a direct summand with three identity blocks. After all the operations, the matrix form is
(8) In the course of procedures on column 3:3, we detect an erasable identity block. This occurs immediately after we transform \((2:3_1, 3:3)\) to Smith normal form. Prior to this some other blocks in column 3:3 are processed without issue. Below, we show the relevant part of the current status of the matrix before and after taking Smith normal form:

\[
\begin{bmatrix}
1:3_1 & 1:3_2 & 1:3_3 & 1:3_4 & 2:2 & 1:4_1 & 1:4_2 & 2:3_1 & 2:3_2 & 2:3_3 & 2:4_1 & 2:4_2 & 2:4_3 & 3:3 & 3:4 & 4:4 \\
0 & 0 & * & 0 & 0 & 0 & 0 & E & 0 \\
0 & 0 & * & 0 & 0 & 0 & 0 & E & 0 \\
0 & 0 & * & 0 & 0 & 0 & 0 & E & 0 \\
0 & * & 0 & 0 & 0 & 0 & 0 & E & 0 \\
* & * & 0 & 0 & 0 & 0 & 0 & E & 0 \\
0 & 0 & * & 0 & 0 & 0 & 0 & E & 0 \\
0 & 0 & * & 0 & 0 & 0 & 0 & E & 0 \\
0 & 0 & * & 0 & 0 & 0 & 0 & E & 0 \\
* & * & 0 & 0 & 0 & 0 & 0 & E & 0 \\
* & * & 0 & 0 & 0 & 0 & 0 & E & 0 \\
* & 0 & 0 & 0 & 0 & 0 & 0 & E & 0 \\
* & 0 & 0 & 0 & 0 & 0 & 0 & E & 0 \\
\end{bmatrix}
\]

\[
\begin{bmatrix}
3:3_1 & 3:3_2 & 3:3_3 & 3:3_4 & 3:4_1 & 3:4_2 & 3:4_3 & 3:4_4 \\
0 & 0 & E & 0 \\
0 & 0 & E & 0 \\
0 & 0 & E & 0 \\
0 & 0 & E & 0 \\
0 & 0 & E & 0 \\
0 & 0 & E & 0 \\
0 & 0 & E & 0 \\
0 & 0 & E & 0 \\
0 & 0 & E & 0 \\
0 & 0 & E & 0 \\
0 & 0 & E & 0 \\
\end{bmatrix} \approx
\begin{bmatrix}
1:4_1 & 1:4_2 & 2:3_1 & 2:3_2 & 2:3_3 & 2:3_4 \\
0 & 0 & E & 0 & 0 & 0 \\
0 & 0 & E & 0 & 0 & 0 \\
E & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & E & 0 & 0 & 0 \\
E & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & E & 0 & 0 & 0 \\
0 & 0 & E & 0 & 0 & 0 \\
0 & 0 & E & 0 & 0 & 0 \\
0 & 0 & E & 0 & 0 & 0 \\
0 & 0 & E & 0 & 0 & 0 \\
\end{bmatrix}
\]

Note that the other identity submatrices are split after updating the block partitioning.

We see that the identity block \((2:3_1, 3:3)\) is erasable by additions from the rows in 2:3_1 to the rows in 2:3_3 and then additions from the columns in 2:4_3 to the columns in 2:4_1 to zero out the side effects.

Here are extracted. We remark that indecomposables with dimension vectors 1220 and 0110 are extracted a second time. This is not double counting. Rather, the corresponding multiplicities are simply added together to get the correct multiplicities for these indecomposables.
By procedures on column 3:4, we extract:

\[
\begin{bmatrix}
0 & 0 & 0 & 0 \\
0 & E & 0 & 0 \\
* & * & 0 & 0 \\
0 & 0 & E & 0 \\
0 & * & 0 & 0 \\
* & * & 0 & 0 \\
0 & 0 & 0 & 0 \\
0 & * & 0 & 0 \\
* & * & 0 & 0 \\
* & 0 & 0 & 0
\end{bmatrix}
\]

and leave the form:

\[
\begin{bmatrix}
0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0
\end{bmatrix}
\]

Finally, straightforward procedures on column 4:4 yield:

\[
\begin{bmatrix}
0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0
\end{bmatrix}
\]

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