Weak Lie 2-Bialgebras* 

**ZHOU CHEN**
Department of Mathematics, Tsinghua University
zchen@math.tsinghua.edu.cn

**MATHIEU STIÉNON**
Department of Mathematics, Pennsylvania State University
stienon@math.psu.edu

**PING XU**
Department of Mathematics, Pennsylvania State University
ping@math.psu.edu

Abstract
We introduce the notion of weak Lie 2-bialgebra. Roughly, a weak Lie 2-bialgebra is a pair of compatible 2-term $L_\infty$-algebra structures on a vector space and its dual. The compatibility condition is described in terms of the big bracket. We prove that (strict) Lie 2-bialgebras are in one-one correspondence with crossed modules of Lie bialgebras.

Contents

1 Introduction 1

2 Lie 2-bialgebras 2

2.1 The big bracket . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . 2

2.2 Weak Lie 2-algebras, coalgebras and bialgebras . . . . . . . . . . . . . . . . . . . 4

3 Lie bialgebra crossed modules 7

3.1 Definition . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . 7

3.2 Main theorem . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . 8

1 Introduction

The main purpose of the paper is to develop the notion of weak Lie 2-bialgebras.

A Lie bialgebra is a Lie algebra endowed with a compatible Lie coalgebra structure. Lie bialgebras can be regarded as the classical limits of quantum groups. A celebrated theorem of Drinfeld establishes a bijection between Lie bialgebras and connected, simply connected Poisson Lie groups. Poisson 2-groups [3] are a natural first step in the search for an appropriate notion of

*Research partially supported by NSF grants DMS-0605725, DMS-0801129 and NSFC grant 11001146.
quantum 2-groups, which is motivated by the recent categorification trend. Their infinitesimal counterparts are called Lie 2-bialgebras or crossed modules of Lie bialgebras.

Recall that a Lie algebra crossed module consists of a pair of Lie algebras \( \theta \) and \( g \) together with a linear map \( \phi : \theta \to g \) and an action of \( g \) on \( \theta \) by derivations satisfying a certain compatibility condition. A Lie bialgebra crossed module is a pair of Lie algebra crossed modules in duality: \((\theta, \theta^*) \) and \((g, g^*) \) are both Lie algebra crossed modules, and \((g \ltimes \theta, \theta^* \ltimes g^*) \) is a Lie bialgebra.

It is well known that Lie algebra crossed modules are a special case of weak Lie 2-algebras (i.e. two-term \( L_\infty \) algebras) [1]. It is natural to wonder what is a weak Lie 2-bialgebra. Such an object ought to be a weak Lie 2-algebra as well as a weak Lie 2-coalgebra, both structures being compatible with one another in a certain sense. Several notions of \( L_\infty \) bialgebras can be found in the existing literature, among which we can mention Kravchenko’s homotopy Lie bialgebras [10] and Merkulov’s homotopy Lie [1] bialgebras [13]. However, none of them serves our purpose. Although it is a weak Lie 2-algebra, a two-term homotopy Lie bialgebra in the sense of Kravchenko is, for instance, not a weak Lie 2-coalgebra due to the degree convention. To obtain the correct compatibility condition, it turns out that one must shift the degree on the underlying \( \mathbb{Z} \)-graded vector space \( V \) so as to modify the “big bracket,” which is a Gerstenhaber bracket on \( S^\bullet (V[2] \oplus V^*[1]). \) Identifying \( S^\bullet (V[2] \oplus V^*[1]) \) with the space \( \Gamma_{\text{poly}}(\wedge^\bullet T[2]M) \) of multivector fields on \( M = V^*[-2] \) with polynomial coefficients, the big bracket can be simply described as the Schouten bracket of multivector fields. 

**In terms of the big bracket, a weak Lie 2-bialgebra on a graded vector space \( V \) is a degree 4 element \( t \) of \( S^\bullet (V[2] \oplus V^*[1]) \) such that \( \{t, t\} = 0. \) (Strict) Lie 2-bialgebras arise as a special case where certain homotopy terms vanish. Our main theorem established a bijection between (strict) Lie 2-bialgebras and crossed modules of Lie bialgebras.

This is the first of a series of papers devoted to the study of Poisson 2-groups [3] and their quantization. We are grateful to the organizers of “Journée Quantique” (June 2010), “WAGP 2010” (June 2010), and “Poisson 2010” (July 2010), where we had the pleasure to present our work. Drafts of this work and slides of our conference talks have circulated in the community. We would like to thank several institutions for their hospitality while work on this project was being done: Penn State University (Chen), Université du Luxembourg (Chen and Stiénon), Institut des Hautes Études Scientifiques and Beijing International Center for Mathematical Research (Xu). We would also like to thank Anton Alekseev, Benjamin Enriquez, Yvette Kosmann-Schwarzbach, Henri Strohmayer, and Alan Weinstein for useful discussions and comments.

Some notations are in order.

**Notations:** Given a graded vector space \( V = \bigoplus_{k \in \mathbb{Z}} V^{(k)} \), \( V[i] \) denotes the graded vector space obtained by shifting the grading on \( V \) according to the rule \((V[i])^{(k)} = V^{(i+k)} \), and \( V^\ast \) denotes the dual vector space, which is graded according to the rule \((V^\ast)[(k) = (V^{(k)})^\ast \). Note in particular that \((V[i])^\ast = (V^\ast)[[-i]. \) We write \( |e| \) for the degree of a homogeneous vector \( e \in V. \) The symbol \( \odot \) is used for the symmetric tensor product: for any homogeneous vectors \( e, f \in V, \)

\[
e \odot f = \frac{1}{2}(e \otimes f + (-1)^{|e||f|} f \otimes e).
\]

The symmetric algebra over \( V \) will be denoted by \( S^\bullet (V). \)

### 2 Lie 2-bialgebras

#### 2.1 The big bracket

We will introduce a graded version of big bracket \([7][17]\) involving graded vector spaces.
Let $V = \bigoplus_{k \in \mathbb{Z}} V^{(k)}$ be a graded vector space. Consider the $\mathbb{Z}$-graded manifold $M = V^*[−2]$ and the shifted tangent space

$$T[2]M \cong (M \times V^*[−2])[2] \cong M \times V^*.$$ 

Consider the space of multi-vector fields on $M$ with polynomial coefficients:

$$\Gamma_{\text{poly}}(\wedge^* T[2]M) \cong S^*(M^*) \otimes (\wedge^* V^*) \cong S^*(V[2]) \otimes S^*(V^*[1]) \cong S^*(V[2] + V^*[1]).$$

In the sequel, let us denote $S = S^*(V[2] + V^*[1])$. The symmetric tensor product on $S$ will be denoted by $\otimes$.

There is a standard way to endow $S = \Gamma_{\text{poly}}(\wedge^* T[2]M)$ with a graded Lie bracket, i.e. the Schouten bracket, denoted by $\{\cdot, \cdot\}$. It is a bilinear map $\{\cdot, \cdot\}: S \otimes S \rightarrow S$ satisfying the following rules:

1) $\{v, v\}' = \{e, e\}' = 0, \{v, e\} = (-1)^{|v||e|} \langle v, e \rangle, \forall v, v' \in V[2], e, e' \in V^*[1]$;
2) $\{e_1, e_2\} = -(-1)^{|e_1|+1}(|e_2|+1) \{e_2, e_1\}, \forall e_i \in S$;
3) $\{e_1, e_2 \otimes e_3\} = \{e_1, e_2\} \otimes e_3 + (-1)^{|e_1|+1} |e_2| \{e_2 \otimes e_3\}$.

It is clear that $\{\cdot, \cdot\}$ is of degree 3, i.e.,

$$|\{e_1, e_2\}| = |e_1| + |e_2| + 3,$$

for all homogeneous $e_i \in S$, and the following graded Jacobi identity holds:

$$\{\{e_1, e_2\}, e_3\} = \{e_1, \{e_2, e_3\}\} - (-1)^{|e_1|+1}(|e_2|+1) \{e_2, \{e_1, e_3\}\}.$$ 

Hence $(S, \otimes, \{\cdot, \cdot\})$ is a Schouten algebra, also known as an odd Poisson algebra, or a Gerstenhaber algebra [15].

**Remark 2.1.** Due to our degree conversion, when $V$ is a vector space considered as a graded vector space concentrated at degree 0, the bracket above is different from the usual big bracket in the literature [17].

Any element $F \in S^p(V[2]) \otimes S^q(V^*[1])$ can be considered as a $q$-multivector field on $M = V^*[−2]$. Thus we obtain a multilinear map:

$$D_F: \underbrace{S^*(V[2]) \otimes \cdots \otimes S^*(V[2])}_{q \text{-tuples}} \rightarrow S^*(V[2]),$$

by applying $F$ to $q$-tuples of functions on $M = V^*[−2]$.

It is clear that $D_F$ can be expressed as the successive Schouten brackets of $F$ with functions on $M = V^*[−2]$. Therefore, we have

$$D_F(x_1, \cdots, x_q) = \{\{\cdots \{F, x_1\}, x_2\}, \cdots, x_{q-1}\}, x_q\},$$

for all $x_i \in S^*(V[2])$.

It is easy to see that

$$D_F(x_1, \cdots, x_i, x_{i+1}, \cdots, x_q) = (-1)^{(|x_i|+1)(|x_{i+1}|+1)} D_F(x_1, \cdots, x_{i+1}, x_i, \cdots, x_q). \quad (1)$$
For any $E \in S^k(V[2]) \otimes S^l(V[2])$, and $F \in S^p(V[2]) \otimes S^q(V^*[1])$, we have
\[ D_{(E,F)}(x_1, \ldots, x_n) = \sum_{\sigma \in S_{q,l-1}} \epsilon(\bar{\sigma}) D_E(D_F(x_{\sigma(1)}, \ldots, x_{\sigma(q)}), x_{\sigma(q+1)}, \ldots, x_{\sigma(n)}) \]
\[ -(-1)^{|E|+1}|F|+1 \sum_{\sigma \in S_{l,q-1}} \epsilon(\bar{\sigma}) D_F(D_E(x_{\sigma(1)}, \ldots, x_{\sigma(l)}), x_{\sigma(l+1)}, \ldots, x_{\sigma(n)}), \]

for all $x_1, \ldots, x_n \in S^*(V[2])$, where $n = q + l - 1$. Here $S_{j,n-j}$ denotes the collection of $(j,n-j)$-shuffles and $\epsilon(\bar{\sigma})$ denotes the Koszul sign: switching any two successive elements $x_i$ and $x_{i+1}$ leads to a sign change $(-1)^{|x_i+1|(|x_{i+1}|+1)}$.

### 2.2 Weak Lie 2-algebras, coalgebras and bialgebras

Following Baez-Crans [1], a weak Lie 2-algebra is an $L_\infty$-algebra on the 2-term graded vector space $V = \theta \oplus g$, where $\theta$ is of degree 1 and $g$ is of degree 0. Unfolding the $L_\infty$-structure, we can equivently define a weak Lie 2-algebra as a pair of vector spaces $\theta$ and $g$ endowed with the following structures:

(a) a linear map $\varphi \colon \theta \to g$;
(b) a bilinear skewsymmetric map $[\cdot, \cdot] : g \otimes g \to g$;
(c) a bilinear map $\cdot \cdot \cdot : g \otimes \theta \to \theta$;
(d) a trilinear skewsymmetric map $h : g \otimes g \otimes g \to \theta$, called the homotopy map.

These maps are required to satisfy the following compatibility conditions: for all $w, x, y, z \in g$ and $u, v \in \theta$,

(a) $[[x, y], z] + [[y, z], x] + [[z, x], y] + (\varphi \circ h)(x, y, z) = 0; \quad (3)$
(b) $y \cdot x > (x \cdot u) - x \cdot (y \cdot u) + [x, y] \cdot u + h(\varphi(u), x, y) = 0; \quad (4)$
(c) $\varphi(u) \cdot v + \varphi(v) \cdot u = 0; \quad (5)$
(d) $\varphi(x \cdot u) = [x, \varphi(u)]; \quad (6)$
(e) $-w \cdot h(x, y, z) - y \cdot h(x, z, w) - z \cdot h(x, y, w) + x \cdot h(y, z, w) = h([x, y], z, w) - h([x, z], y, w) + h([x, w], y, z) + h([y, z], x, w) - h([y, w], x, z) + h([z, w], x, y). \quad (7)$

If $h$ vanishes, we call it a strict Lie 2-algebra, or simply a Lie 2-algebra.

Now consider the degree-shifted vector spaces $V[2]$ and $V^*[1]$. Under such a degree convention, the degrees of $g$, $\theta$, $g^*$ and $\theta^*$ are specified as follows:
Remark 2.2. Readers may wonder why we are using such degree conventions. The advantage can be seen in the sequel that under such assumptions, the elements $s$ in Proposition 2.3 and $c$ in Proposition 2.5 and $t$ in Definition 2.6 are all homogenously of degree $-4$.

We will maintain this convention throughout this section. Let $S^* = S^*(V^*[1] \oplus V[2])$.

Proposition 2.3. Under the above degree convention, a weak Lie 2-algebra structure is equivalent to a solution to the equation:
\[
\{s, s\} = 0,
\]
where $s = \tilde{\varphi} + \tilde{b} + \tilde{a} + \tilde{h}$ in $S(-4)$ such that
\[
\begin{align*}
\tilde{\varphi} &\in \theta^* \circ \mathfrak{g}, \\
\tilde{b} &\in (\circ^2 \mathfrak{g}^*) \circ \mathfrak{g}, \\
\tilde{a} &\in \mathfrak{g}^* \circ \theta^* \circ \theta, \\
\tilde{h} &\in (\circ^3 \mathfrak{g}^*) \circ \theta.
\end{align*}
\]

Here the bracket stands for the big bracket as in Section 2.1.

Proof. There is a bijection between the structure maps $\varphi$, $[\cdot, \cdot]$, $\cdot \triangleright \cdot$ and $h$ and the data $\tilde{\varphi}$, $\tilde{b}$, $\tilde{a}$ and $\tilde{h}$. They are related by the following equations:
\[
\begin{align*}
\varphi(u) &= D_{\tilde{\varphi}}(u), \\
[x, y] &= D_{\tilde{b}}(x, y), \\
x \triangleright u &= D_{\tilde{a}}(x, u), \\
h(x, y, z) &= D_{\tilde{h}}(x, y, z),
\end{align*}
\]
\forall x, y, z \in \mathfrak{g}, u \in \theta.

Since $s = \tilde{\varphi} + \tilde{b} + \tilde{a} + \tilde{h} \in S(-4)$, a simple computation leads to
\[
\{s, s\} = \{\tilde{b}, \tilde{b}\} + \{\tilde{a}, \tilde{a}\} + 2\{\tilde{\varphi}, \tilde{b}\} + 2\{\tilde{\varphi}, \tilde{a}\} + 2\{\tilde{\varphi}, \tilde{h}\} + 2\{\tilde{b}, \tilde{a}\} + 2\{\tilde{b}, \tilde{h}\} + 2\{\tilde{a}, \tilde{h}\}.
\]

By using Eq. (2), we have, $\forall x, y, z \in \mathfrak{g}$,
\[
D_{\{s, s\}}(x, y, z) = D_{\{\tilde{b}, \tilde{b}\}}(x, y, z) + 2D_{\{\tilde{\varphi}, \tilde{h}\}}(x, y, z) = 2D_{\tilde{b}}(D_{\tilde{b}}(x, y), z) + \text{c.p.} + 2D_{\tilde{\varphi}}(D_{\tilde{h}}(x, y, z) = [[x, y], z] + \text{c.p.} + \varphi \circ h(x, y, z),
\]
\text{Similarly,}
\[
\begin{align*}
D_{\{s, s\}}(x, y, u) &= 2(y \triangleright (x \triangleright u) - x \triangleright (y \triangleright u) + h(\varphi(u), x, y) + [x, y] \triangleright u), \\
D_{\{s, s\}}(u, x) &= \varphi(x \triangleright u) + [\varphi(u), x], \\
D_{\{s, s\}}(u, v) &= 2(\varphi(u) \triangleright v + \varphi(v) \triangleright u), \\
D_{\{s, s\}}(x, y, z, w) &= 2(h([x, y], z, w) + w \triangleright h(x, y, z) + \text{c.p.}.
\end{align*}
\]
It is simple to see that $\{s, s\}$ vanishes if and only if the LHS of Eqs. (9)-(13) vanish. The latter is equivalent to the compatibility conditions of a weak Lie 2-algebra. This concludes the proof.
In the sequel, we denote a weak Lie 2-algebra by \((\theta \to g, l)\) in order to emphasize the map from \(\theta\) to \(g\). Sometimes we will omit \(l\) and denote a weak Lie 2-algebra by \((\theta \to g)\). If \((g^* \to \theta^*)\) is a weak Lie 2-algebra, then \((\theta \to g)\) is called a weak Lie 2-coalgebra.

**Remark 2.4.** Equivalently, a weak Lie 2-coalgebra underlying \((\theta \to g)\) is a 2-term \(L_\infty\)-structure on \(g^* \oplus \theta^*\), where \(g^*\) has degree 1 and \(\theta^*\) has degree 0.

Similarly, we have the following

**Proposition 2.5.** A weak Lie 2-coalgebra is equivalent to a solution to the equation:

\[\{c, c\} = 0,\]

where \(c = \check{\phi} + \check{\epsilon} + \check{\alpha} + \check{\eta} \in S^{(-4)}\) such that

\[
\begin{align*}
\check{\phi} &\in \theta^* \odot g, \\
\check{\epsilon} &\in \theta^* \odot (\odot^2 \theta), \\
\check{\alpha} &\in g^* \odot \theta, \\
\check{\eta} &\in g^* \odot (\odot^3 \theta).
\end{align*}
\]

We denote such a weak Lie 2-coalgebra by \((\theta \to g, c)\).

Now we are ready to introduce the main object of this section.

**Definition 2.6.** A weak Lie 2-bialgebra consists of a pair of vector spaces \(\theta\) and \(g\) together with a solution \(t = \check{b} + \check{a} + \check{h} + \check{\phi} + \check{\epsilon} + \check{\alpha} + \check{\eta} \in S^{(-4)}\) to the equation:

\[\{t, t\} = 0.\]

Here \(\check{b}, \check{a}, \check{h}, \check{\phi}, \check{\epsilon}, \check{\alpha}, \check{\eta}\) are as in Eqs. [8] and [14].

If, moreover, \(\check{h} = 0\), it is called a quasi-Lie 2-bialgebra. If both \(\check{h}\) and \(\check{\eta}\) vanish, we say that the Lie 2-bialgebra is strict, or simply a Lie 2-bialgebra.

**Remark 2.7.** Note that, in the literature, there exist notions of homotopy Lie bialgebras [16] and Lie 2-bialgebras [13]. However, weak Lie 2-bialgebras in our sense are neither of them. This is explained as follows.

The pattern is that any of these notions are saying a homogenous element \(h\) in a (even or odd) Poisson algebra satisfying \(\{h, h\} = 0\), whose bracket \(\{\ , \}\) are defined analogue to the usual big bracket of Kosmann-Schwarzbach [17].

In fact, the big bracket in [16] is defined on \(S^*(V \oplus V^*)\) (without any degree shifting) and generalizes the usual big bracket. Our big bracket in Section 2.7 does not reduce to the usual big bracket when \(V\) is a vector space. Thus the usual Lie bialgebras in the sense of Drinfeld [4] is not a special case of our Lie 2-bialgebras. But Homotopy Lie bialgebras in the sense of Kravchenko includes them.

The big bracket in [13] is called an odd Poisson structure, and is defined on \(S^*(V \oplus V^*[1])\). Although it could be identified with our big bracket in Section 2.7 under some proper degree adjustments, an element \(h\) that form a Lie 2-bialgebra in the sense of Merkulov does not define a weak Lie 2-bialgebra in our sense because it has degree 2 as an element in \(S^*(V \oplus V^*[1])\), whereas in \(S^*(V^*[1] \oplus V[2])\) it is not even homogenous.

**Proposition 2.8.** Let \((\theta, g, t)\) be a weak Lie 2-bialgebra as in Definition 2.6. Then \((\theta \to g, s)\), where \(s = \check{\phi} + \check{b} + \check{a} + \check{h}\), is a weak Lie 2-algebra, while \((\theta \to g, c)\), where \(c = \check{\phi} + \check{\epsilon} + \check{\alpha} + \check{\eta}\), is a weak Lie 2-coalgebra.
Proof. It is easy to see, by examining each component, that \( \{t, t\} = 0 \) implies \( \{s, s\} = 0 \) and \( \{c, c\} = 0 \). Hence \( (\theta \to g, s) \) is a weak Lie 2-algebra and \( (\theta \to g, c) \) is a weak Lie 2-coalgebra by Proposition 2.3 and Proposition 2.5.

Remark 2.9. Of course there are other compatibility conditions between \( s \) and \( c \) and these conditions can be written in terms of the structure maps. We omit them in the current paper.

Example 2.10. Assume that \( g \) is a semisimple Lie algebra. Let \( (\cdot, \cdot)^{\theta} \) be its Killing form. Then \( h(x, y, z) = h([y, z])^{\theta}, \forall x, y, z \in g, \) is a Lie algebra 3-cocycle, where \( h \) is a constant. Let \( \theta = \mathbb{R} \). Then \( \mathbb{R} \to g \) together with \( h \) becomes a weak Lie 2-algebra, called string Lie 2-algebra \( \text{II} \). More precisely, the string Lie 2-algebra is as follows:

(a) \( \theta = \mathbb{R} \) is the abelian Lie algebra;
(b) \( g \) is a semisimple Lie algebra;
(c) \( \phi = 0 : \theta \to g; \)
(d) \( g \triangleright \theta = 0; \)
(e) \( h = h(\cdot, \cdot) : \wedge^3 g \to \theta, \) where \( h \) is a fixed constant.

Now fix an element \( x \in g \). We equip a Lie 2-coalgebra on \( \mathbb{R} \to g \) as follows:

(a) \( g^* \) is abelian Lie algebra;
(b) \( \theta^* \cong \mathbb{R} \) is abelian Lie algebra;
(c) \( \phi = 0 : g^* \to \theta^*; \)
(d) \( \tilde{\eta} = 0 : \wedge^3 \theta^* \to g^*; \)
(e) The \( \theta^* \)-action on \( g^* \) is given by

\[
1 \triangleright \xi = \text{ad}^*_x \xi, \quad \forall \xi \in g^*.
\]

(Here \( x \in g \) is fixed.)

One can verify directly that this is indeed a weak Lie 2-bialgebra.

3 Lie bialgebra crossed modules

3.1 Definition

Definition 3.1. A Lie algebra crossed module consists of a pair of Lie algebras \( \theta \) and \( g \), and a linear map \( \phi : \theta \to g \) such that \( g \) acts on \( \theta \) by derivations and satisfies, for all \( x, y \in g, u, v \in \theta, \)

1) \( \phi(u) \triangleright v = [u, v]; \)
2) \( \phi(x \triangleright u) = [x, \phi(u)], \)

where \( \triangleright \) denotes the \( g \)-action on \( \theta \).

Remark 3.2. Note that 1) and 2) imply that \( \phi \) must be a Lie algebra homomorphism.
We write \( (\theta \to g) \) to denote a Lie algebra crossed module. The associated semidirect product Lie algebra is denoted by \( g \ltimes \theta \).

The following proposition indicates that crossed modules of Lie algebras are in one-one correspondence with Lie 2-algebras. We refer the reader to [1] for a detailed exposition.

**Proposition 3.3.** Lie algebra crossed modules are equivalent to (strict) Lie 2-algebras.

We are now ready to introduce the following

**Definition 3.4.** A Lie bialgebra crossed module is a pair of Lie algebra crossed modules in duality: \( (\theta \to g) \) and \( (g^* \to \theta^*) \), where \( \phi^T = -\phi^* \), are both Lie algebra crossed modules such that \( (g \ltimes \theta, \theta^* \ltimes g^*) \) is a Lie bialgebra.

**Proposition 3.5.** If \( ((\theta \to g), (g^* \to \theta^*)) \) is a Lie bialgebra crossed module, so is \( ((g^* \to \theta^T, \theta \to g)) \).

The following proposition justifies our terminology.

**Proposition 3.6.** If \( ((\theta \to g), (g^* \to \theta^*)) \) is a Lie bialgebra crossed module, then both pairs \( (\theta, \theta^*) \) and \( (g, g^*) \) are Lie bialgebras.

**Proof.** Since \( \theta \) and \( \theta^* \) are Lie subalgebras of \( g \ltimes \theta \) and \( \theta^* \ltimes g^* \), respectively, and \( (g \ltimes \theta, \theta^* \ltimes g^*) \) is a Lie bialgebra, it follows that \( (\theta, \theta^*) \) is a Lie bialgebra. Similarly, \( (g, g^*) \) is also a Lie bialgebra. \( \square \)

**Example 3.7.** We can construct a Lie bialgebra crossed module from a usual Lie bialgebra. Given a Lie bialgebra \( (\theta, \theta^*) \), consider the trivial Lie algebra crossed module \( (\theta \to \theta) \), where the second \( \theta \) acts on the first \( \theta \) by the adjoint action. In the mean time, consider the dual Lie algebra crossed module \( (\theta^* \to \theta^*) \), where the second \( \theta^* \) takes the opposite Lie bracket on \( \theta^* \): \(-[\cdot, \cdot]_s\), and the action of the second \( \theta^* \) on the first \( \theta^* \) is by \( \kappa_2 \triangleright \kappa_1 = -[\kappa_2, \kappa_1]_s \), \( \forall \kappa_1, \kappa_2 \in \theta^* \). It is simple to see that \( ((\theta \to \theta), (\theta^* \to \theta^*)) \) is a Lie bialgebra crossed module.

### 3.2 Main theorem

**Theorem 3.8.** There is a bijection between Lie bialgebra crossed modules and (strict) Lie 2-bialgebras.

We need a few lemmas before proving this theorem.

The following lemma is standard. For instance, see [1].

**Lemma 3.9.** Given a Lie algebra \( g \), a \( g \)-module \( \theta \) and a linear map \( \phi : \theta \to g \) satisfying the following two conditions:

\[
\phi(x \triangleright u) = [x, \phi(u)],
\]

\[
\phi(u) \triangleright v = -\phi(v) \triangleright u,
\]

for all \( u, v \in \theta, x \in g \), then there exists a unique Lie algebra structure on \( \theta \) such that \( (\theta \to g) \) is a Lie algebra crossed module.
Proof. Define the Lie bracket on $\theta$ by
\[ [u, v] = \phi(u) \triangleright v, \quad \forall \; u, v \in \theta. \]

The rest of the claim can be easily verified directly. \hfill\Box

For $k \geq 1$, write
\[ W_k = \left\{ w \in g \wedge (\wedge^{k-1}\theta) \mid \iota_{\zeta_1} \iota_{\phi^*\zeta_2} w = -\iota_{\zeta_2} \iota_{\phi^*\zeta_1} w \; \forall \zeta_1, \zeta_2 \in g^* \right\}. \tag{17} \]

Let
\[ D_\phi : \wedge^* (g \ltimes \theta) \rightarrow \wedge^* (g \ltimes \theta) \]
denote the degree-0 derivation with respect to the wedge product such that $D_\phi(x + u) = \phi(u)$, $\forall x \in g, u \in \theta$.

**Lemma 3.10.** A Lie algebra crossed module structure on $(g^* \xrightarrow{\phi^*} \theta^*)$, where $\phi^* = -\phi^*$, is equivalent to a pair of linear maps $\delta : g \rightarrow W_2 \subset g \wedge \theta$ and $\omega : \theta \rightarrow \wedge^2\theta$ satisfying the following conditions
\begin{enumerate}
  \item $D_\phi \omega = \delta \phi$;
  \item $\omega^2 = 0$;
  \item $(\omega + \delta) \circ \delta = 0$.
\end{enumerate}

Here, we consider both $\omega$ and $\delta$ as degree 1 derivations on the exterior power $\wedge^* (g \ltimes \theta)$, by letting $\omega(g) = 0$ and $\delta(\theta) = 0$.

Moreover in this case, the cobracket $\partial : g \ltimes \theta \rightarrow \wedge^2(g \ltimes \theta)$ corresponding to the Lie algebra structure on $\theta^* \ltimes g^*$ is given by:
\[ \partial(x + u) = \omega(u) + \delta(x) + \pi(x), \quad \forall x \in g, u \in \theta, \tag{18} \]

where $\pi : g \rightarrow \wedge^2 g$ is the map $\pi = -\frac{1}{2} D_\phi \circ \delta$.

Proof. According to Lemma 3.9, a Lie algebra crossed module structure underlying $(g^* \xrightarrow{\phi^*} \theta^*)$ is equivalent to assigning a Lie algebra structure $[\cdot, \cdot]_s$ on $\theta^*$ and an action $\triangleright$ of $\theta^*$ on $g^*$ such that
\begin{align*}
\phi^*(\kappa \triangleright \xi) &= [\kappa, \phi^*(\xi)]_s, \tag{19} \\
\phi^*(\xi) \triangleright \zeta &= -\phi^*(\zeta) \triangleright \xi, \tag{20}
\end{align*}

for all $\xi, \zeta \in g^*$, $\kappa \in \theta^*$. Introduce linear maps $\delta$ and $\omega$ by
\begin{align*}
\langle \delta(x) | \xi \wedge \kappa \rangle &= \langle x | \kappa \triangleright \xi \rangle; \\
\langle \omega(u) | \kappa_1 \wedge \kappa_2 \rangle &= -\langle u | [\kappa_1, \kappa_2]_s \rangle,
\end{align*}

$\forall x \in g, u \in \theta, \xi \in g^*, \kappa, \kappa_1, \kappa_2 \in \theta^*$. It is simple to see that $\omega^2 = 0$ is equivalent to the Jacobi identity for $[\cdot, \cdot]_s$, and $(\omega + \delta) \circ \delta = 0$ is equivalent to that $\triangleright$ is an action of $\theta^*$ on $g^*$. Moreover, Eq. (19) is equivalent to the condition $D_\phi \omega = \delta \phi$ and Eq. (20) is equivalent to the condition that $\delta$ takes values in $W_2$. 

9
To prove Eq. (13), \( \forall x \in \mathfrak{g}, u \in \theta, \xi, \zeta \in \mathfrak{g}^*, \kappa, \kappa_1, \kappa_2 \in \theta^* \), we have

\[
\langle \partial(u)|\kappa_1 \wedge \kappa_2 \rangle = -\langle u|[\kappa_1, \kappa_2]\rangle = \langle \omega(u)|\kappa_1 \wedge \kappa_2 \rangle, \\
\langle \partial(x)|\kappa \wedge \xi \rangle = -\langle x|[\kappa, \xi]\rangle = -\langle x|\kappa \triangleright \xi \rangle = \langle \delta(x)|\kappa \wedge \xi \rangle,
\]

and

\[
\langle \partial(x)|\xi \wedge \zeta \rangle = -\langle x|[\xi, \zeta]\rangle \\
= -\langle x|\phi^T(\xi) \triangleright \zeta \rangle \\
= \langle \delta(x)|\phi^T(\xi) \wedge \zeta \rangle \\
= \langle -\frac{1}{2}(D_\phi \circ \delta)(x)|\xi \wedge \zeta \rangle.
\]

This concludes the proof of the lemma. \( \square \)

**Proposition 3.11.** Let \( (\theta \xrightarrow{\phi} \mathfrak{g}) \) be a Lie algebra crossed module. It is a Lie bialgebra crossed module if and only if there is a pair of linear maps \((\delta, \omega)\) as in Lemma [3, II] that, in addition, satisfies the following conditions:

1) \( \delta \) is a Lie algebra 1-cocycle;
2) \( x \triangleright \omega(u) - \omega(x \triangleright u) = \Pr_{\lambda^2\theta}([u, \delta(x)]), \) for all \( x \in \mathfrak{g}, u \in \theta. \)

**Proof.** Assume that \( (\theta \xrightarrow{\phi} \mathfrak{g}) \) and \( (\mathfrak{g}^* \xrightarrow{\phi^T} \theta^*) \) are Lie algebra crossed modules. It suffices to prove that Conditions 1)-2) are equivalent to \((\mathfrak{g} \ltimes \theta, \theta^* \ltimes \mathfrak{g}^*)\) being a Lie bialgebra. The latter is equivalent to

\[
\partial[E, F] = [E, \partial(F)] - [F, \partial(E)], \quad \forall E, F \in \mathfrak{g} \ltimes \theta, \tag{21}
\]

where \( \partial : \mathfrak{g} \ltimes \theta \to \wedge^2(\mathfrak{g} \ltimes \theta) \) is the cobracket as given in Eq. (18).

If both \( E \) and \( F \) are in \( \mathfrak{g} \), it is simple to see that Eq. (21) is equivalent to \( \delta : \mathfrak{g} \to \mathfrak{g} \ltimes \theta \) being a Lie algebra 1-cocycle. On the other hand, we claim that when \( E = x \in \mathfrak{g} \) and \( F = u \in \theta \), Eq. (21) is equivalent to the second condition in the statement of the proposition. First of all, note that Eq. (21) implies that

\[
\partial[x, u] = [x, \partial(u)] - [u, \partial(x)] \\
= x \triangleright \omega(u) - [u, \delta(x)] - \frac{1}{2} D_\phi \circ \delta(x). \tag{22}
\]

Since \( \partial[x, u] = \omega(x \triangleright u) \), it suffices to prove that \([u, \delta(x)] - \frac{1}{2} D_\phi \circ \delta(x) = \Pr_{\lambda^2\theta}([u, \delta(x)])\).

For this purpose, let us assume that \( \delta(x) = \sum_i y_i \wedge v_i \), where \( y_i \in \mathfrak{g} \) and \( v_i \in \theta \). The condition that \( \delta(x) \in W_2 \) is essentially equivalent to

\[
\sum_i \langle \phi(v_i)|\xi \rangle y_i + \langle y_i|\xi \rangle \phi(v_i) = 0, \quad \forall \xi \in \mathfrak{g}^*. \tag{23}
\]

The latter implies that, for any \( u \in \theta \),

\[
\sum_i \left( (y_i \triangleright u) \wedge \phi(v_i) - y_i \wedge (\phi(y_i) \triangleright u) \right) = 0.
\]

Indeed, for all \( \xi \in \mathfrak{g}^* \),

\[
\tau_\xi \left( \sum_i \left( (y_i \triangleright u) \wedge \phi(v_i) - y_i \wedge (\phi(y_i) \triangleright u) \right) \right) = -\sum_i \left( (y_i \triangleright u) \langle \phi(v_i)|\xi \rangle + \langle y_i|\xi \rangle \langle \phi(y_i) \triangleright u \rangle \right) \\
= -\sum_i \left( \langle \phi(v_i)|\xi \rangle y_i + \langle y_i|\xi \rangle \phi(v_i) \right) \triangleright u \\
= 0.
\]
Thus, we have
\[
[u, \delta(x) - \frac{1}{2} D_\phi \circ \delta(x)] = [u, \sum_i y_i \wedge v_i - \frac{1}{2} \sum_i y_i \wedge \phi(v_i)]
\]
\[
= \sum_i [(u, y_i) \wedge v_i + y_i \wedge [u, v_i] + \frac{1}{2} (y_i \triangleright u) \wedge \phi(v_i) + \frac{1}{2} y_i \wedge (\phi(v_i) \triangleright u)]
\]
\[
= \sum_i [u, y_i] \wedge v_i + \frac{1}{2} \sum_i \left( (y_i \triangleright u) \wedge \phi(v_i) - y_i \wedge (\phi(y_i) \triangleright u) \right)
\]
\[
= \sum_i [u, y_i] \wedge v_i + \frac{1}{2} \sum_i \left( (y_i \triangleright u) \wedge \phi(v_i) - y_i \wedge (\phi(y_i) \triangleright u) \right)
\]
\[
= \sum_i [u, y_i] \wedge v_i + \frac{1}{2} \sum_i \left( (y_i \triangleright u) \wedge \phi(v_i) - y_i \wedge (\phi(y_i) \triangleright u) \right)
\]
\[
= \Pr_{\Lambda, \beta\theta}([u, \delta(x)]).
\]

Finally, if both \( E \) and \( F \) are in \( \theta \), Eq. (21) is equivalent to \( \omega : \theta \rightarrow \Lambda^2 \theta \) being a Lie algebra 1-cocycle. However, the latter follows from Conditions 1)-2). To see this, for any \( u, v \in \theta \) we have
\[
\omega[u, v] = \omega(\phi(u) \triangleright v)
\]
\[
= \phi(u) \triangleright \omega(v) - \Pr_{\Lambda, \beta\theta}([v, \delta(\phi(u))])
\]
\[
= [u, \omega(v)] - \Pr_{\Lambda, \beta\theta}([v, D_\phi(\omega(u))])
\]
\[
= [u, \omega(v)] - [v, \omega(u)],
\]
where, in the last equality, we used the following identity
\[
\Pr_{\Lambda, \beta\theta}([v, D_\phi(\zeta)]) = [v, \zeta], \quad \forall \zeta \in \Lambda^2 \theta,
\]
which can be proved by a direct verification.

This concludes the proof. \( \square \)

Now we are ready to prove Theorem 3.8.

**Proof of Theorem 3.8.** Let \( t = \tilde{b} + \tilde{\alpha} + \tilde{\varphi} + \tilde{\epsilon} + \tilde{\alpha} \in \mathcal{S}(-4) \), where \( \tilde{b}, \tilde{\alpha}, \tilde{\varphi}, \tilde{\epsilon}, \tilde{\alpha} \) are given as in Eqs. (8) and (11). It is simple to see that the equation \{\( t, t \)\} = 0 is equivalent to the following three equations:
\[
\left\{ \tilde{b} + \tilde{\alpha} + \tilde{\varphi} + \tilde{\epsilon} + \tilde{\alpha} \right\} = 0; \quad (24)
\]
\[
\left\{ \tilde{\varphi} + \tilde{\epsilon} + \tilde{\alpha} \right\} = 0; \quad (25)
\]
\[
\left\{ \tilde{b}, \tilde{\alpha} \right\} + \left\{ \tilde{\alpha}, \tilde{\epsilon} \right\} + \left\{ \tilde{\alpha}, \tilde{\alpha} \right\} = 0. \quad (26)
\]

According to Proposition 2.3, Eq. (24) is equivalent to that \( (\theta \rightarrow_{\tilde{\beta}} \mathfrak{g}) \) is a Lie algebra crossed module, where \( \phi(u) = D_{\tilde{\beta}}(u) \), \( \forall u \in \theta \), the Lie bracket on \( \mathfrak{g} \) is given by \([x, y] = D_{\tilde{b}}(x, y)\), \( \forall x, y \in \mathfrak{g} \), and the action of \( \mathfrak{g} \) on \( \theta \) is given by \( x \triangleright u = D_{\tilde{\alpha}}(x, u) \), \( \forall x \in \mathfrak{g}, u \in \theta \). According to Proposition 2.3, Eq. (26) is equivalent to that \( (\theta \rightarrow_{\tilde{\beta}} \mathfrak{g}) \) is a Lie 2-coalgebra, or \( (\mathfrak{g}^* \hat{\triangleright}_{\tilde{\beta}^*} \theta^*) \) is a Lie algebra crossed module. It is simple to show, by a straightforward computation, that the linear maps \( \delta \) and \( \omega \) associated to the Lie algebra crossed module \( (\mathfrak{g}^* \hat{\triangleright}_{\tilde{\beta}^*} \theta^*) \) are related to \( \tilde{\epsilon} \) and \( \tilde{\alpha} \) by the following relations:
\[
\langle \delta(x) | \kappa \wedge \kappa \rangle = \left\{ x, \left\{ \{ \tilde{\alpha}, \kappa \} , \kappa \right\} \right\} = - \left\{ \left\{ D_{\tilde{\alpha}}(x), \kappa \right\} , \kappa \right\}; \\
\langle \omega(u) | \kappa_1 \wedge \kappa_2 \rangle = \left\{ u, \left\{ \{ \tilde{\epsilon}, \kappa_1 \} , \kappa_2 \right\} \right\} = \left\{ \left\{ D_{\tilde{\epsilon}}(u), \kappa_1 \right\} , \kappa_2 \right\}.
\]
\( \forall \xi \in g^*, \kappa, \kappa_1, \kappa_2 \in \theta^* \).

Since the left hand side of Eq. \((24)\) belongs to \((\odot^2 g^*) \otimes g \circ \theta + (\odot^2 \theta) \otimes g^* \otimes \theta^*\), hence we have

\[
\begin{aligned}
\left\{ \left\{ D_{\{b,a\}}(x,y) + D_{\{\tilde{a},\tilde{a}\}}(x,y), \xi \right\}, \kappa \right\} &= \left\{ \left\{ D_{\{\tilde{a},\tilde{a}\}}(x,y) + D_{\{\tilde{a},\tilde{a}\}}(x,y), \xi \right\}, \kappa \right\} \\
&= \left\{ D_{\{\tilde{a},\tilde{a}\}}(x,y) + D_{\{\tilde{a},\tilde{a}\}}(x,y), \xi \right\}, \kappa \\
&= (-\delta[x,y] + [x, \delta(y)] - [y, \delta(x)] \mid \xi \wedge \kappa).
\end{aligned}
\]

and

\[
\begin{aligned}
\left\{ \left\{ D_{\{b,a\}}(x,u), \kappa_1 \right\}, \kappa_2 \right\} &= \left\{ \left\{ D_{\{\tilde{a},\tilde{a}\}}(x,u), \kappa_1 \right\}, \kappa_2 \right\} \\
&= \left\{ D_{\{\tilde{a},\tilde{a}\}}(x,u), \kappa_1 \right\}, \kappa_2 \\
&= (-x \triangleright \omega(u) + \omega(x \triangleright u) + \text{Pr}_{\lambda \geq 0}(u, \omega(u))[\kappa_1 \wedge \kappa_2]).
\end{aligned}
\]

Therefore it follows that Eq. \((24)\) is equivalent to that the pair \((\delta, \omega)\) satisfies the two compatibility conditions in Proposition \(3.11\) Hence we conclude that \(\{t, t\} = 0\) is equivalent to that the couple \((\theta \overset{\phi}{\circ} g)\) and \((g^* \overset{\phi^T}{\circ} \theta^*)\) is a Lie bialgebra crossed module.

**Theorem 3.12.** Let \((\theta \overset{\phi}{\circ} g)\) and \((g^* \overset{\phi^T}{\circ} \theta^*)\) be Lie algebra crossed modules. Then they form of a Lie bialgebra crossed module if and only if \((g, \theta^*)\) is a matched pair of Lie algebras, where the \(g\)-module structure on \(\theta^*\) is the dual to the given \(g\)-module on \(\theta\), while the \(\theta^*\)-module structure on \(g\) is dual to the given on \(g^*\).

**Proof.** By definition, \((g, \theta^*)\) is a matched pair of Lie algebras if and only if

\[
\begin{aligned}
\kappa \triangleright [x, y] &= [x, \kappa \triangleright y] - [y, \kappa \triangleright x] + (y \triangleright \kappa) \triangleright x - (x \triangleright \kappa) \triangleright y, \\
x \triangleright [\kappa_1, \kappa_2] &= [\kappa_1, x \triangleright \kappa_2] - [\kappa_2, x \triangleright \kappa_1] + (\kappa_2 \triangleright x) \triangleright \kappa_1 - (\kappa_1 \triangleright x) \triangleright \kappa_2,
\end{aligned}
\]

(27)
(28)

for all \(x, y \in g, \kappa_1, \kappa_2 \in \theta\).

We claim that Eq. \((24)\) is equivalent to \(\delta\) being a Lie algebra 1-cocycle, while Eq. \((25)\) is equivalent to Condition (2) in Proposition \(3.11\).

Indeed, a tedious computation leads to

\[
\langle \delta([x,y]) | \xi, \kappa \rangle = \langle x \triangleright \delta(y) | \xi, \kappa \rangle + \langle y \triangleright \delta(x) | \xi, \kappa \rangle,
\]

for all \(\xi \in g^*, \kappa \in \theta^*\).

And for all \(\kappa_1, \kappa_2 \in \theta^*\),

\[
\langle \omega(x \triangleright u) | \kappa_1, \kappa_2 \rangle - \langle x \triangleright \omega(u) | \kappa_1, \kappa_2 \rangle + \langle \text{Pr}_{\lambda \geq 0}(u, \omega(u))[\kappa_1, \kappa_2] \rangle.
\]

This concludes the proof.\qed
Corollary 3.13. Let \((\theta, \theta^*)\) be a Lie bialgebra. Assume that \(\mathcal{J}\) is contained in the center of \(\theta\), i.e. \([\theta, \mathcal{J}] = 0\), and \(\omega(\mathcal{J}) \subset \lambda^2 \mathcal{J}\), where \(\omega : \theta \rightarrow \lambda^2 \theta\) is the cobracket on \(\theta\). Then there is an induced Lie bialgebra crossed module structure underlying \((\theta \xrightarrow{\phi} g)\), where \(g = \theta / \mathcal{J}\) is the quotient Lie algebra and \(\phi\) is the projection.

Proof. Identifying \(g^*\) with \(\mathcal{J}^0 \subset \theta^*\), we see that \(g^*\) is an ideal of \(\theta^*\), and the map \(\phi^T : g^* \rightarrow \theta^*\) is the composition of the inclusion with \(-I\). Hence \((g^* \xrightarrow{\phi^T} \theta^*)\) is a Lie algebra crossed module.

To prove that \(((\theta \xrightarrow{\phi} g), (g^* \xrightarrow{\phi^T} \theta^*))\) is a Lie bialgebra crossed module, it suffices to prove that \((g, \theta^*)\) is a matched pair of Lie algebras according to Theorem 3.12. Note that \((\theta, \theta^*)\) is a Lie algebra matched pair since it is a Lie bialgebra. Therefore it remains to show that \(\mathcal{J} \oplus 0\) is an ideal of the double Lie algebra \(D = \theta \triangleright \theta^*\). In fact, for any \(X \in \mathcal{J}\) and \(\kappa \in \theta^*\), we have \(ad_x^* \kappa = 0\) and \(ad_j^* \kappa \in \mathcal{J}\). Hence \([X, \kappa]_D = ad_X^* \kappa - ad_j^* X \in \mathcal{J} \oplus 0\).

This concludes the proof.

By this corollary, we construct the following example.

Example 3.14. Consider the Lie subalgebra \(u(n)\) of \(gl_n(\mathbb{C})\). Let \(\theta \subset gl_n(\mathbb{C})\) be the Lie subalgebra consisting of upper triangular matrices whose diagonal elements are real numbers. It is standard that \((\theta, u(n))\) is a Lie bialgebra. Indeed \(\theta \oplus u(n) \cong gl_n(\mathbb{C})\), and both \(\theta\) and \(gl_n(\mathbb{C})\) are lagrangian subalgebras of \(gl_n(\mathbb{C})\) under the nondegenerate pairing \(\langle X|Y \rangle = \text{Im} \langle \text{Tr}(XY) \rangle\), for \(X, Y \in gl_n(\mathbb{C})\). Hence \((gl_n(\mathbb{C}), \theta, u(n))\) is a Manin triple and thus \((\theta, u(n))\) forms a Lie bialgebra.

Let \(\mathcal{J} = \mathbb{R}I\). It is clear that \(\mathcal{J}\) is the center of \(\theta\) and \(\omega(\mathcal{J}) = 0\). Hence \(g = \theta / \mathcal{J}\) is the Lie algebra of traceless upper triangular matrices whose diagonal elements are real. As a consequence, \((\theta \xrightarrow{\phi} g)\), where \(\phi\) is the map \(A \rightarrow A - \text{tr}A\), is a Lie bialgebra crossed module.

References

[1] John C. Baez and Alissa S. Crans, Higher-dimensional algebra. VI. Lie 2-algebras, Theory Appl. Categ. 12 (2004), 492–538 (electronic). MR2068522 (2005m:17039)
[2] John C. Baez and Aaron D. Lauda, Higher-dimensional algebra. V. 2-groups, Theory Appl. Categ. 12 (2004), 423–491 (electronic). MR2068521 (2005m:18005)
[3] Zhuo Chen, Mathieu Stienon, and Ping Xu, On quasi-Poisson Lie 2-groups, In preparation.
[4] V. G. Drinfel’d, Quantum groups, (Berkeley, Calif., 1986), Amer. Math. Soc., Providence, RI, 1987, pp. 798–820. MR934283 (89f:17028)
[5] David Iglesias Ponte, Camille Laurent-Gengoux, and Ping Xu, Universal lifting theorem and quasi-Poisson groupoids, available at arXiv:math/0507396v1
[6] Tom Lada and Jim Stasheff, Introduction to SH Lie algebras for physicists, Internat. J. Theoret. Phys. 32 (1993), no. 7, 1087–1103, DOI 10.1007/BF00671791. MR1235010 (94g:17059)
[7] Pierre B. A. Lecomte and Claude Roger, Modules et cohomologies des bigèbres de Lie, C. R. Acad. Sci. Paris Sér. I Math. 310 (1990), no. 6, 405–410 (French, with English summary). MR1046522 (91c:17013)
[8] Yvette Kosmann-Schwarzbach, Grand crochet, crochets de Schouten et cohomologies d’algèbres de Lie, C. R. Acad. Sci. Paris Sér. I Math. 312 (1991), no. 1, 123–126 (French, with English summary). MR1086516 (92b:17028)
[9] , Jacobian quasi-bialgebras and quasi-Poisson Lie groups, Mathematical aspects of classical field theory (Seattle, WA, 1991), Contemp. Math., vol. 132, Amer. Math. Soc., Providence, RI, 1992, pp. 459–489. MR1188453 (94b:17025)
[10] K. C. H. Mackenzie, On symplectic double groupoids and the duality of Poisson groupoids, Internat. J. Math. 10 (1999), no. 4, 435–456, DOI 10.1142/S0129167X99000185. MR1697617 (2000g:58029)
[11] Kirill C. H. Mackenzie and Ping Xu, Integration of Lie bialgebroids, Topology 39. (2000), no. 3, 445–467, DOI 10.1016/S0040-9383(98)00069-X. MR1746902 (2001b:53104)
[12] S. A. Merkulov, *PROP profile of Poisson geometry*, Comm. Math. Phys. **262** (2006), no. 1, 117–135, DOI 10.1007/s00220-005-1385-7. MR2200884 (2006j:53122)

[13] _____, *Wheeled Pro(p)file of Batalin-Vilkovisky formalism*, Comm. Math. Phys. **295** (2010), no. 3, 585–638, DOI 10.1007/s00220-010-0987-x. MR2600029 (2011d:17038)

[14] Tahar Mokri, *Matched pairs of Lie algebroids*, Glasgow Math. J. **39** (1997), no. 2, 167–181, DOI 10.1017/S0017089500032055. MR1460632 (99a:58165)

[15] Theodore Voronov, *Graded manifolds and Drinfeld doubles for Lie bialgebroids*, Quantization, Poisson brackets and beyond (Manchester, 2001), Contemp. Math., vol. 315, Amer. Math. Soc., Providence, RI, 2002, pp. 131–168. MR1958834 (2004f:53098)

[16] Olga Kravchenko, *Strongly homotopy Lie bialgebras and Lie quasi-bialgebras*, Lett. Math. Phys. **81** (2007), no. 1, 19–40, DOI 10.1007/s11005-007-0167-x. MR2327020 (2008c:17020)

[17] Yvette Kosmann-Schwarzbach, *Quasi, twisted, and all that... in Poisson geometry and Lie algebroid theory*, The breadth of symplectic and Poisson geometry, Progr. Math., vol. 232, Birkhäuser Boston, Boston, MA, 2005, pp. 363–389. MR2103012 (2005g:53157)

[18] J. H. C. Whitehead, *Note on a previous paper entitled “On adding relations to homotopy groups.”*, Ann. of Math. (2) **47** (1946), 806–810. MR0017537 (8,167a)

[19] _____, *Combinatorial homotopy, II*, Bull. Amer. Math. Soc. **55** (1949), 453–496. MR0030760 (11,48c)