Anderson localization in a two-dimensional random gap model

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Abstract:
We study the properties of the spinor wavefunction in a strongly disordered environment on a two-dimensional lattice. By employing a transfer-matrix calculation we find that there is a transition from delocalized to localized states at a critical value of the disorder strength. We prove that there exists an Anderson localized phase with exponentially decaying correlations for sufficiently strong scattering. Our results indicate that suppressed backscattering is not sufficient to prevent Anderson localization of surface states in topological insulators.

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1 Introduction

The classical approach to randomly scattered particles leads to diffusion, where random scattering originates either from particle-particle collisions (e.g., in a gas) or from collisions with (static) impurity scatterers. In quantum systems, however, diffusion appears only for weak disorder whereas it is destroyed due to Anderson localization at stronger randomness [1, 2]. This effect is particularly strong in low-dimensional systems, such as two-dimensional graphene sheets or the surface of topological insulators. The scaling approach to generic random scattering [2] states that diffusion is entirely suppressed by Anderson localization for dimension \( d \leq 2 \). On the other hand, it has been argued that Anderson localization is prevented on the surface of topological insulators due to suppressed backscattering [3, 4].

Inspired by the recent observation of metallic behavior (i.e. diffusive or even ballistic transport) in disordered two-dimensional systems (graphene) [5, 6], a general discussion of diffusion and localization of quantum particle is required, which takes into account a spinor structure of the wavefunction. Two possibilities have been considered, namely ballistic transport for finite systems [7, 8] and diffusive transport for infinite systems [9]. Diffusion is related to long ranged correlations, which is usually caused by spontaneous symmetry breaking [9, 11]. This behavior might be restricted to the regime of weak scattering, since strong scattering is capable to localize particles. The aspect of weak localization is ignored here on purpose because it has its own problems [11, 12]. This will be discussed in a separate paper. Instead, we will focus in the following mostly on the case of strong scattering. This is motivated by recent numerical studies, which have indicated that there is a transition to a localized phase at sufficiently strong disorder [13, 14]. Here we will analyze details of the transition in terms of the scaling behavior of the localization length for strips of finite width. Moreover, the infinite system will be treated analytically within a strong scattering expansion. The latter provides a rigorous proof for exponential localization, supporting the numerical results at strong disorder. We study a random gap model with linear spectrum (2D Dirac fermions), but our methods can be easily applied to other systems as well.
We consider the surface Hamiltonian of a topological insulator with bulk inversion symmetry of momentum $\mathbf{k}$ \cite{3,14,15,16}:

$$H = \hbar \begin{pmatrix} h(\mathbf{k}) & 0 \\ 0 & h^*(-\mathbf{k}) \end{pmatrix}, \quad h(\mathbf{k}) = \begin{pmatrix} C + M - (D + \delta)k^2 & v_F(k_x + ik_y) \\ v_F(k_x - ik_y) & C - M - (D - \delta)k^2 \end{pmatrix}$$

(1)

This Hamiltonian consists of a pair of massive Dirac Hamiltonians $h(\mathbf{k})$, $h^*(-\mathbf{k})$. We include disorder by a random variable $M$ with mean $\bar{M}$. For our numerical transfer-matrix calculation we use a box distribution with width $W$. For simplicity we choose the Dirac point, where $C = 0$ and $D = 0$. The main feature is that there are two bands that touch each other at a spectral node $k = 0$ if $M = 0$, whereas $M \neq 0$ opens a gap $\Delta = 2|\bar{M}|$. Thus, a random $M$ creates a random gap. Our aim is to calculate the localization length $\Lambda$ of the eigenstate $\psi$ at energy $E = 0$ which satisfies $h\psi = 0$ and the transition probability of a moving particle. The two block Hamiltonians $h(\mathbf{k})$, $h^*(-\mathbf{k})$ act on two separate spaces with the same localization properties. Therefore, it is sufficient to study just one of them.

2.1 Localization length

The localization length $\Lambda$ of the eigenstates of Hamiltonian (1) can be calculated numerically within a transfer-matrix approach. For this purpose the continuous Hamiltonian must be discretized in space (cf. Appendix A). Then the transfer-matrix $T_l$ of the eigenvalue problem $\psi_{l+1} = h^Y \psi_l + h^D \psi_{l-1}$ (cf. Eqs. (25), (26)) reads

$$T_l = \begin{pmatrix} h^Y & h^D \\ 1 & 0 \end{pmatrix},$$

(2)

which enables us to evaluate the Lyapunov exponents of the wavefunction $\psi$ \cite{17,18}. With the initial values $\psi_0$ and $\psi_1$ the iteration of Eq. (26) provides the wavefunction $\psi_L$ at site $L$ by applying the product matrix

$$M_L = \prod_{l=1}^L T_l.$$  

(3)

For a random Hamiltonian this is a product of random matrices that satisfies Oseledec’s theorem \cite{19}. The latter states that there exists a limiting matrix

$$\Gamma = \lim_{L \to \infty} (M_L^1 M_L)^{1/2L}.$$  

(4)

The eigenvalues of $\Gamma$ are usually written as a diagonal matrix with exponential functions $\exp(\gamma_i)$, where $\gamma_i$ is the Lyapunov exponent (LE). Adapting the numerical algorithm described in \cite{18}, the whole Lyapunov spectrum can be calculated and the smallest LE is identified with the inverse localization length $1/\Lambda$ \cite{17}. $\Lambda$ increases with the system width $M$ according to a power law $\Lambda \propto M^\alpha$, where $\alpha > 1$ ($\alpha < 1$) in the regime of extended (localized) states, and $\alpha = 1$ in the critical regime. For the exponentially localized regime we expect $\Lambda \propto const$. According to the one-parameter scaling theory by MacKinnon \cite{20}, the normalized localization length $\tilde{\Lambda} = \Lambda/M$, being a function of disorder strength $W$ and system width $M$, depends only on a single parameter:

$$\tilde{\Lambda}(M,W) = f(\xi(W)/M),$$  

(5)

where $\xi$ is a characteristic length of the system generated by disorder. Thus, any change of disorder strength $W$ can be compensated by a change of the system width $M$. If there is a scale-invariant point $W_c$ we can expand $\tilde{\Lambda}$ in its vicinity by assuming a power law with critical exponent $\nu$ of the correlation length as $\xi = |W - W_c|^{-\nu}$. Then we have \cite{18}

$$\ln \tilde{\Lambda} = \ln \tilde{\Lambda}_c + \sum_{s=1}^S A_s \left( |W - W_c| M^{1/\nu} \right)^s = \ln \tilde{\Lambda}_c + \sum_{s=1}^S A_s \left( \frac{\xi}{M} \right)^{-s/\nu}.$$  

(6)
2.2 Transition probability

The motion of a quantum particle from site $r'$ to site $r$ with frequency $i\epsilon$ is described by the transition probability

$$P_{r,r'}(i\epsilon) = \frac{K_{r,r'}(i\epsilon)}{\sum_r K_{r,r'}(i\epsilon)}$$  \hspace{1cm} (7)

with

$$K_{r,r'}(i\epsilon) = \langle \text{Tr}_4 \left[ G_{r,r'}(i\epsilon)G_{r',r}(i\epsilon) \right] \rangle_v = \langle \text{Tr}_4 \left[ G_{r,r'}(i\epsilon)G_{r',r}(-i\epsilon) \right] \rangle_v,$$  \hspace{1cm} (8)

where $G(i\epsilon) = (i\epsilon + H)^{-1}$ is the one-particle Green’s function of the Hamiltonian $H$ and $\langle \ldots \rangle_v$ is the average with respect to random scatterers. $\text{Tr}_4(\ldots)$ is the trace with respect to the 4 spinor components. The last equation in Eq. (8) follows from the fact that the Hamiltonian is Hermitian ($H^\dagger = H$).

After Fourier transformation of the average two-particle Green’s function $K_{r,r'}(i\epsilon) \to k_{r,r'}(t)$ we study the motion of the quantum particle with the mean-square displacement of the coordinate $r_k$

$$\langle r_k^2 \rangle = \frac{\sum_r r_k^2 k_{r,0}(t)}{\sum_r k_{r,0}(t)}.$$  \hspace{1cm} (9)

This expression grows linearly with time $t$ in the case of diffusion, whereas it is finite for large times in the case of Anderson localization. Thus, diffusion requires a long range correlation $K_{r,r'}(i\epsilon)$ for small $\epsilon$ and short range correlations in the case of Anderson localization. A natural approach to study the latter for strong randomness would be a hopping expansion of $K_{r,r'}(i\epsilon)$. Unfortunately, such an expansion is plagued by poles on both sides of the real axis. This problem can be avoided if we focus on the most relevant contributions of the randomly fluctuating product of Green’s functions $G_{r,r'}(i\epsilon)G_{r',r}(-i\epsilon)$. They are associated with the underlying chiral symmetry.

These fluctuations have been studied previously in Ref. [21], where the large scale behavior was reduced to the Grassmann integral

$$K_{r,r'} \approx K_0 \int \varphi_r \varphi_{r'} J D[\varphi, \varphi']$$  \hspace{1cm} (10)

with $D[\varphi, \varphi'] = \prod_r d\varphi_d\varphi'$ and with the Jacobian

$$J = \frac{\det(\mathbf{1} + \gamma_+ \varphi \varphi' - \varphi \gamma_- \varphi' + \gamma_+ \varphi \gamma_- \varphi')}{\det(H_0 + i\eta')} = 1,$$  \hspace{1cm} (11)

where we have used the average one-particle Green’s functions $\gamma_\pm$.

$J$ is an expression that does not depend on the integration variables.

The relation between the expectation value in Eq. (12) and the integral in Eq. (10) is based on two facts. Firstly, we have a large freedom to choose a distribution of the random Green’s function with the same expectation value. Secondly, by choosing a proper distribution we find a saddle-point approximation for the corresponding integration. This procedure was described in detail in Refs. [21] [22], leading eventually to Eq. (10). As a result we have been able to avoid the spurious singularities, which appear when we apply a hopping expansion and integrate with respect to the random term of the Hamiltonian.

In the specific case of a random gap we have [22]

$$\gamma_\pm = \pm 2i\eta(H_0 + i\eta')^{-1} \quad (\eta' = \eta + \epsilon).$$  \hspace{1cm} (12)

with $H_0 = \langle H \rangle_v$ and

$$J = \frac{\det(-H_0 + i\eta')}{\det(H_0 + i\eta')} = 1,$$  \hspace{1cm} (13)

where the last equation followed from the symmetry $\sigma_3(h)^T \sigma_2 = -\langle h \rangle$ of the Hamiltonian (11) if $C = D = 0$. The parameter $\eta$ is the scattering rate, which can be considered as an external parameter that is either calculated in self-consistent Born approximation [22] or is taken from experimental measurements [23]. In any case, the scattering rate increases with increasing disorder.

Eq. (10) is a convenient starting point to study transport properties with the Jacobian

$$J = \exp \left\{ -\text{Tr} \left[ \log (\mathbf{1} + \gamma_+ \varphi \varphi' - \varphi \gamma_- \varphi' + \gamma_+ \varphi \gamma_- \varphi') \right] \right\},$$  \hspace{1cm} (14)

where the trace $\text{Tr}$ is with respect to the four-dimensional spinor space and the position $r$. This expression has been treated previously for weak scattering. In this case it leads to diffusion, where the correlation function is a diffusion propagator [9]. In Sect. 4 we will employ it for strong scattering.
3 Numerical Results: scaling of the localization length

Now we return to the method described in Sect. 2.1 and calculate the localization length \( \Lambda \). Our calculation for strong randomness (i.e. large \( W \)) provides a critical value \( W_c \), where the system is delocalized (localized) for \( W < W_c \) (\( W > W_c \)). Around the critical value \( W_c \) we observe one-parameter scaling behavior for the normalized localization length \( \bar{\Lambda} \), as described in Sect. 2.1. Some results are depicted in Fig. 1 and the results of the fitting procedure are listed in Table 1. This behavior is indicative of an Anderson transition.

Figure 1: Numerical evidence for a localization transition in two dimensions. The scaling behavior of the normalized localization length \( \bar{\Lambda} \) as a function of increasing disorder \( W \) is plotted here for \( \bar{m} = 0.8 \) and \( \delta = 0.5 \). Left panel: Fit to Eq. (6) near the critical point. Right panel: Rescaled normalized localization length \( \bar{\Lambda} \) near the critical point.

| Average gap \( \bar{m} \) | 0       | 0.2     | 0.8     |
|------------------------|---------|---------|---------|
| Exponent \( \nu \)     | 1.299 ± 0.066 | 1.397 ± 0.069 | 1.451 ± 0.024 |
| Critical disorder \( W_c \) | 7.668 ± 0.008 | 7.629 ± 0.015 | 7.727 ± 0.011 |
| Disorder range          | 7.35 ≤ W ≤ 7.8 | 7.1 ≤ W ≤ 8.0 | 6.6 ≤ W ≤ 8.4 |
| System sizes            | 30 ≤ M ≤ 80  | 20 ≤ M ≤ 80  | 20 ≤ M ≤ 80  |

Table 1: Critical values for \( \delta = 0.5 \) obtained from fitting the data to Eq. (6).

4 Analytic Results: strong scattering expansion

The integral representation of the correlation function \( K_{rr'} \) in Eq. (10) with the Jacobian in Eq. (14) enables us to study the regime of strong scattering (i.e. \( \eta \gg 1 \)) by applying a \( 1/\eta \) expansion. For Eq. (12) we then get

\[
\gamma_{\pm} = \frac{2}{\eta} \left( 1 \pm (i/\eta') H_0 \right)^{-1} \sim -2 \mp (i/\eta) H_0
\]

for \( \epsilon \ll \eta \). With \( h_0 = (i/\eta) H_0 \) the Jacobian is approximated by

\[
J \sim \exp \left\{ -\text{Tr} \log \left[ 1 + h_0 \phi \phi' + \phi h_0 \phi' - h_0 \phi h_0 \phi' \right] \right\} \prod_r (1 - 8 \phi_r \phi'_r) .
\]

This allows us to rewrite the correlation function as

\[
K_{rr'} \approx K_0 \int \phi_r \phi'_r \exp \left\{ -\text{Tr} \log \left[ 1 + h_0 \phi \phi' + \phi h_0 \phi' - h_0 \phi h_0 \phi' \right] \right\} \prod_r (1 - 8 \phi_r \phi'_r) D[\phi, \phi']
\]
\[ K_0 \frac{\partial}{\partial \alpha} \int \exp \{ \alpha \varphi r' - \text{Tr} \log (1 + h_0 \varphi + \varphi h_0 - h_0 \varphi h_0') \} \prod_r (1 - 8 \varphi r' \varphi') D[\varphi, \varphi'] \big|_{\alpha=0} \]  

and to expand the exponential function

\[ = K_0 \frac{\partial}{\partial \alpha} \sum_{l \geq 0} \frac{1}{l!} \int (\alpha \varphi r' - \text{Tr} \log (1 + h_0 \varphi + \varphi h_0 - h_0 \varphi h_0'))^l \prod_r (1 - 8 \varphi r' \varphi') D[\varphi, \varphi'] \big|_{\alpha=0} \]

\[ = K_0 \frac{\partial Z}{\partial \alpha} \bigg|_{\alpha=0}. \]  

Here we have used the expression

\[ Z = \sum_{l \geq 0} \frac{1}{l!} \langle (\sum_j A_j)^l \rangle, \]

where \( \sum_j A_j \) is the expansion of \( \alpha \varphi r' - \text{Tr} \log (1 + h_0 \varphi + \varphi h_0 - h_0 \varphi h_0') \):

\[ \sum_j A_j = \alpha \varphi r' + \sum_{j \geq 1} \frac{(-1)^j}{j} \text{Tr} \left( [h_0 \varphi + \varphi h_0 - h_0 \varphi h_0']^j \right) \]

and the average is with respect to the normalized integral:

\[ \langle ... \rangle = \frac{1}{N} \int ... \prod_r (1 - 8 \varphi r' \varphi') D[\varphi, \varphi'] . \]

Using the fact that the factors of the product \( A_1 A_2 \cdots A_j \) can be reorganized as products of connected clusters \( \{B_k\} \) (cf. Appendix [B]), we obtain from the Linked Cluster Theorem

\[ \frac{\partial Z}{\partial \alpha} = Z \frac{\partial}{\partial \alpha} \log Z = Z \frac{\partial}{\partial \alpha} \sum_k \langle B_k \rangle . \]

Thus only those expressions \( \langle B_k \rangle \) contribute that contain \( \alpha \). These contributions form random walks from site \( r \) to site \( r' \) with the discrete hopping term of Eq. (19) (cf. Fig. 2). They can be estimated as

\[ \left| \frac{\partial}{\partial \alpha} \sum_k \langle B_k \rangle \right|_{\alpha=0} \leq \text{const.} (4 \mu / \eta) |r - r'| , \]

where the factor 4 is due to the two dimensional random walk and \( \mu = \max(v_F, \delta) \). Thus, we need \( \eta > 4 \mu \) in order to have an absolutely convergent series and an exponential decay. The latter leads to Anderson localization, according to our discussion in Sect. 2.2.

Figure 2: Typical contribution to the \( 1/\eta \) expansion of \( Z \). There are the five connected clusters \( B_1, \ldots, B_5 \) which are disconnected of each other. In particular, there is a random walk \( B_3 \) from site \( r \) to site \( r' \), the only contribution to the correlation function \( K_{rr'} \) in Eq. (17).
5 Discussion and conclusion

Our analytic calculation supports the numerical result of a localized phase at sufficiently strong disorder. Here it should be noticed that the calculations are based on different quantities, namely the localization length $\Lambda$ and the exponential decay of the average transition matrix $K_{rr'}$. Since the localization length is self-averaging according to Eq. (4), it is expected that this quantity should be very robust in a real system. On the other hand, the localization length is difficult to measure directly in an experiment. Therefore, the transition matrix is more accessible because it is related to the conductivity by the relation

$$\sigma_{xx} \approx e^2 \hbar^2 \sum_r x^2 K_{r0},$$

where $x$ is the direction of the position $r$, in which the external electric field is applied. In the DC limit $\epsilon \to 0$ the conductivity vanishes when $K_{r0}$ decays exponentially. This is in stark contrast to the weak scattering case where the expansion in powers of $\eta$ gives a diffusion propagator \[22\]

$$\tilde{K}_q \propto \frac{1}{\epsilon + Dq^2}$$

with the diffusion coefficient $D$. After Fourier transformation $q \to r$ this expression gives a correlation function that decays like $\sim r^{-1/2}$. Moreover, it gives a finite non-vanishing DC conductivity, since

$$\sum_r x^2 K_{r0} = -\frac{\partial \tilde{K}_q}{\partial q_x} \bigg|_{q=0} \propto \frac{2D}{\epsilon^2}.$$

On the surface of a typical topological insulator we expect substantial scattering due to disorder \[3\]. Our results indicate that the suppressed backscattering may not be able to prevent the localization of surface states. Therefore, it might be crucial for the appearance of a metallic behavior to reduce the disorder on the surface. In this case it could even be possible to observe an Anderson transition from extended to localized states, as our results indicate. Our calculation gives a rough estimate for the localized behavior in which the scattering rate must be larger than the bandwidth of the system without disorder.

In conclusion, we have studied a model for surface states on a topological insulator. Contrary to the assumption that suppressed backscattering may always create a metallic phase, we have found that the surface states are localized for strong scattering by disorder. For weak scattering, however, there is a metallic behavior and a phase transition from a delocalized to a localized phase when the disorder strength is increased. The transition is characterized by one-parameter scaling of the normalized localization length with a non-universal exponent.

A Numerical transfer-matrix calculation

A numerical treatment of the Dirac Hamiltonian requires a discretization in space. However, the naive discretization through replacing the differential operator by a difference operator leads to additional new nodes, which is often called fermion doubling or multiplication \[24\]. In real space there are two methods to circumvent this problem \[25, 26, 27\]. One that we will adopt in this section goes back to an idea of Susskind. We start with discretizing the differential operator in an anti-symmetric way

$$\partial_x f(x) \approx \frac{1}{2\Delta}(f_{l+\Delta} - f_{l-\Delta}),$$

where $\Delta$ is the lattice constant which we set to $\hbar v_F$ in the following. The discrete Dirac equation for $m = 0$ then takes the form

$$-\frac{i}{2} \sigma_1 \{\psi_{l+1,n} - \psi_{l-1,n}\} - \frac{i}{2} \sigma_2 \{\psi_{l,n+1} - \psi_{l,n-1}\} = E\sigma_0 \psi_{l,n}$$

with lattice points given by the integer coordinates $(l, n)$. Fourier transformation leads to eigenvalues $E = \pm \sqrt{\sin(k_x)^2 + \sin(k_y)^2}$ which have four Dirac cones in the Brillouin zone corresponding to four
Dirac fermions. In order to open a gap at three of them we introduce a lattice operator which acts on a wave function as \[ \hat{B} \psi_{l,n} = \frac{1}{2} \{ \psi_{l+1,n} + \psi_{l-1,n} + \psi_{l,n+1} + \psi_{l,n-1} \}. \] (21)

The discretized form of the Hamiltonian \[ \text{(1)} \] for uniform gap now reads

\[ h = \sin(k_x)\sigma_1 - \sin(k_y)\sigma_2 + [m + \delta(\cos(k_x) + \cos(k_y) - 2)]\sigma_3, \] (22)

which gives \( h(\mathbf{k}) \) of Eq. \[ \text{(1)} \] in the continuum limit and has the dispersion

\[ E = \pm \sqrt{\sin(k_x)^2 + \sin(k_y)^2 + (m + \delta \cos(k_x) + \delta \cos(k_y) - 2\delta)^2}. \] (23)

For \( m = 0, \delta \neq 0 \) there is a node at \( k_x = k_y = 0 \) and three additional nodes for \( m = 0, \delta = 0 \) at \( k_x, k_y = \pm \pi \). Using this model node degeneracy can be lifted via the parameter \( \delta \).

We absorb the index \( n \) with the help of matrix representation and write for the wave function

\[ \psi_{l+1} = h^Y \psi_{l} + h^D \psi_{l-1}. \] (24)

Each spinor component is now a \( M \)-component vector, where \( M \) is the width of a strip and thus \( n = 1, 2, \ldots, M \). The matrices \( h^Y, h^D \) read

\[
\begin{align*}
    h^Y_{n,n} &= 2S^{-1} [E\sigma_0 + (2\delta - m)\sigma_3] \\
    h^Y_{n,n+1} &= S^{-1} [i\sigma_2 - \delta\sigma_3] \\
    h^Y_{n,n-1} &= -S^{-1} [i\sigma_2 + \delta\sigma_3] \\
    h^D_{n,n} &= -S^{-1} [i\sigma_1 + \delta\sigma_3]
\end{align*}
\] (25)

with \( S = -i\sigma_1 + \delta\sigma_3 \) and where \( h^Y \) has periodic boundary conditions in the \( y \)-direction. This matrix structure allows us to construct a transfer matrix \( T_l \) through the equation \[ \text{(18)} \]

\[
\begin{pmatrix}
    \psi_{l+1} \\
    \psi_l
\end{pmatrix} =
\begin{pmatrix}
    h^Y & h^D \\
    1 & 0
\end{pmatrix}
\begin{pmatrix}
    \psi_l \\
    \psi_{l-1}
\end{pmatrix} = T_l
\begin{pmatrix}
    \psi_l \\
    \psi_{l-1}
\end{pmatrix}.
\] (26)

### B Linked Cluster Theorem

We must organize the \( 1/\eta \) expansion in order to extract the spatial decay of the correlation function \( K_{rr'} \). For this purpose we employ the Linked Cluster Theorem \[ \text{(29)} \]. The latter can be formulated for the expression

\[ \frac{1}{\eta} \langle \left( \sum_j A_j \right)^l \rangle = \frac{1}{\eta} \sum_{j_1, j_2, \ldots, j_l} \langle A_{j_1} A_{j_2} \cdots A_{j_l} \rangle. \] (27)

The product of the \( A_jA_j \) is called disconnected (unlinked) if the two factors do not share any Grassmann variable. This would lead to \( \langle A_jA_j \rangle = \langle A_j \rangle \langle A_j \rangle \). Otherwise they are called connected (linked) and we would have \( \langle A_iA_j \rangle \neq \langle A_i \rangle \langle A_j \rangle \). In the sum \[ \text{(27)} \] we combine for a given set \( j_1, j_2, \ldots, j_l \) all connected factors in products \( \{ B_k \} \) such that

\[
\langle A_{j_1} A_{j_2} \cdots A_{j_l} \rangle = \langle B_{k_1} \rangle \langle B_{k_2} \rangle \cdots \langle B_{k_n} \rangle \quad (n \leq l),
\] (28)

where the new indices \( k_1, \ldots, k_n \) refer to the indices \( j_1, \ldots, j_l \) of the combined factors \( A_j \). Now we must reorganize the summation. A permutation of the \( j_1, j_2, \ldots, j_l \) gives the same expression for \[ \text{(28)} \]. Therefore, the summation with respect to the permutations contributes only a factor \( l! \). On the other hand, we allow also a permutation of the \( k_1, k_2, \ldots, k_n \), which would also leave the expression \[ \text{(28)} \] invariant. Consequently, we must divide the summation with respect to these \( n \) permutations by \( n! \). This gives us eventually

\[
Z = \sum_{l \geq 0} \frac{1}{l!} \langle \left( \sum_j A_j \right)^l \rangle = \sum_{n \geq 0} \frac{1}{n!} \langle \sum_k \langle B_k \rangle \rangle^n,
\]

which is the Linked Cluster Theorem, since the \( B_k \) are connected according to our construction.
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