Near-optimal Approaches for Binary-Continuous Sum-of-ratios Optimization

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Abstract. In this paper, we investigate a class of non-convex sum-of-ratios programs relevant to decision-making in key areas such as product assortment and pricing, facility location and cost planning, and security games. These optimization problems, characterized by both continuous and binary decision variables, are highly non-convex and challenging to solve. To the best of our knowledge, no existing methods can efficiently solve these problems to near-optimality with arbitrary precision. To address this challenge, we explore a piecewise linear approximation approach that enables the approximation of complex nonlinear components of the objective function as linear functions. We then demonstrate that the approximated problem can be reformulated as a mixed-integer linear program, a second-order cone program, or a bilinear program, all of which can be solved to optimality using off-the-shelf solvers like CPLEX or GUROBI. Additionally, we provide theoretical bounds on the approximation errors associated with the solutions derived from the approximated problem. We illustrate the applicability of our approach to competitive joint facility location and cost optimization, as well as product assortment and pricing problems. Extensive experiments on instances of varying sizes are conducted to assess the efficiency of our method.

Key words: Sum-of-Ratios; Discrete Choice Model; Discretization; Bilinear; Mixed-Integer Linear; Second-Order Cone; Outer-Approximation

Subject classifications: Assortment Optimization; Facility Location; Fractional Programming

Area of review: Optimization

Notation: Boldface characters represent matrices (or vectors), and \( a_i \) denotes the \( i \)-th element of vector \( a \) if it is indexiable. We use \([m]\), for any \( m \in \mathbb{N} \), to denote the set \( \{1, \ldots, m\} \).

1. Introduction

We study the following non-convex optimization problem with binary and continuous variables

\[
\max_{y \in \mathcal{Y}, x \in \mathcal{X}} \left\{ f(y,x) = \sum_{t \in [T]} \left( \frac{a_t + \sum_{i \in [m]} y_i g_t^i(x_i)}{b_t + \sum_{i \in [m]} y_i h_t^i(x_i)} \right) \right\} \text{ s.t. } Ax + By \leq D
\]

(SoR)

where \( y \) are binary and \( x \) are continuous variables, \( \mathcal{Y} \subset \{0,1\}^m \) is the feasible set of \( y \) and \( \mathcal{X} \subset \bigotimes_{i \in [m]} [l_i, u_i] \) is the feasible set of \( x \), \( g_t^i(x), h_t^i(x) \) are univariate functions, i.e., \( g_t^i(x), h_t^i(x) : \mathbb{R} \rightarrow \mathbb{R}, \forall t \in [T], i \in [m], \)
noting that $g^t_i(x), h^t_i(x), t \in [T], i \in [m]$, are univariate functions and are not necessarily convex (or concave), and $Ax + By \leq D$ are some linear constraints on $x, y$. Such a sum-of-ratios problem arises from the use of discrete choice models (McFadden 1981, Train 2003) to predict customer/adversary’s behavior in decision-making and is known to be highly non-convex and challenging to solve, even when the binary variables $y$ are fixed (Li et al. 2019, Duong et al. 2022). As far as we know, this is a first attempt to solve the aforementioned non-convex problems to near global optimality. The problem formulation above has several applications in revenue management, facility location and security games, as described below.

**Competitive facility location and cost optimization.** The formulation (SoR) can be found along an active line of research on competitive maximum covering (or maximum capture) facility location problem with customers’ random utilities (Benati and Hansen 2002; Hasse 2009; Mai and Lodi 2020; Lin and Tian 2021; Dam et al. 2022). The problem refers to maximizing an expected customer demand, in a competitive market, by locating new facilities and making decisions of the budget to spend on each opening facility, assuming that customers make choice decisions according to a discrete choice model. When the costs are fixed, which is the focus of most of the works in the relevant literature, researchers have shown that the facility location problem can be formulated as a mixed-integer linear program (MILP) (Benati and Hansen 2002; Freire et al. 2016; Haase and Müller 2014), or can be solved efficiently by outer-approximation algorithms (Mai and Lodi 2020; Ljubić and Moreno 2018). When the cost optimization is considered but the cost variables only take values from a discrete set, then it has been shown that the joint location and cost optimization problem can be converted to an equivalent facility location problem with binary variables and existing methods can apply (Qi et al. 2022). In contrast, if the cost variables are continuous, the joint problem becomes highly non-convex and may have several local optima (Duong et al. 2022). As far as we know, Duong et al. (2022) is the only work to consider both facility location and cost optimization (with continuous cost variables). In this work, the authors state that the use of the standard mixed-logit model leads to an intractable optimization problem with several local optimal solutions, and they instead propose to use a less-popular discrete choice framework, i.e., the multiplicative random utility maximization framework (Fosgerau and Bierlaire 2009). So, the joint location and cost optimization under the standard logit and mixed-logit model is still an open problem in the respective literature and we deal with it in this work.

**Product assortment and pricing optimization.** This problem refers to the problem of selecting a set of products and making pricing decisions to maximize an expected revenue, assuming that customers make choice decisions according to a discrete choice model. Product assortment and pricing has been one of the most essential problems in revenue management and has received remarkable attention over the recent decades (Talluri and Van Ryzin 2004; Vulcano et al. 2010; Rusmevichientong et al. 2014; Wang and Sahin 2018). The joint assortment and price optimization problem under a (general) mixed-logit model (i.e., one of the most popular and general choice models in the literature) can be formulated in the form of (SoR). When the variables $x$ are fixed and the objective function contains only one ratio, the optimization problem
can be solved in polynomial time under some simple settings, e.g. the problem is unconstrained or with a cardinality constraint (Rusmevichientong et al. 2010; Talluri and Van Ryzin 2004). When the objective function is a sum of ratios and the variable $x$ are fixed, the problem is generally NP-Complete even when there are only two fractions (Rusmevichientong et al. 2014). Approximate solutions, MILP and mixed-integer second order cone programming (MISOCP) reformulations have been developed for this setting (Bront et al. 2009; Méndez-Díaz et al. 2014; Sen et al. 2018). When only the pricing decisions are considered (i.e., the variables $y$ are fixed) and the objective function contains multiple ratios, the problem is highly non-convex and may have several local optima, with respect to both the prices and market shares (Li et al. 2019). Joint assortment and price optimization has been also studied in the literature (Wang 2012), but just under some simple settings (e.g., unconstrained on the prices and a cardinality constraint on the assortment, and the objective functions involves only one ratio). In general, as far as we know, in the context of assortment and price optimization, there is no global solution method to handle the joint problem with multiple ratios and general constraints. Our work is the first attempt to fill this literature gap.

**Stackelberg game under quantal response adversary.** Stackelberg security games (SSG) (Tambe 2011) are a prominent class of security models that have been widely studied in the Artificial Intelligent and Operations Research community. Such model have many real world applications, for example, airport security (Pita et al. 2008), wide-life protection (Fang et al. 2017), and coast guard (An et al. 2012). In a SSG model, the objective is to allocate some security resources to some defending targets, aiming at maximizing an expected defending utility function. Quantal response (or multinomial logit) model has been popularly used to predict adversary’s behavior (Yang et al. 2011, 2012). Solution methods developed for SSG problems under quantal response include convex optimization or piece-wise linear approximation (Yang et al. 2012; Mai and Sinha 2022; Bose et al. 2022). To the best of our knowledge, existing works only focus on single-ratio objective functions and continuous decision variables, offering room for exploring and improvement, e.g., using mixed-logit models to predict adversary’s behavior, or include binary decisions for the selection of facilities to operate. Our work provides a global and general solution method to deal with such general settings.

**Linear fractional programming.** Our work also relates to the literature of binary fractional programming and general fractional programming. In the context of binary fractional programming, the problem is known to be NP-hard, even when there is only one ratio (Prokopyev et al. 2005). The problem is also hard to approximate (Prokopyev et al. 2005). Rusmevichientong et al. (2014) show that for the unconstrained multi-ratio problem, there is no poly-time approximation algorithm that has an approximation factor better than $O(1/m^{1-\delta})$ for any $\delta > 0$, where $m$ is the number of products. Exact solution methods for binary fractional programs include MILP reformulations (Méndez-Díaz et al. 2014; Haase and Müller 2014), or Conic quadratic reformulations (Mehmanchi et al. 2019; Sen et al. 2018). In fact, such MILP and Conic reformulations cannot be directly applied to our context due to the inclusion of continuous variables. Conversely,
when the fractional program primarily deals with continuous variables (with fixed binary variables), it takes on a notably non-convex nature, leading to multiple local optima (Freund and Jarre 2001, Gruzdeva and Strekalovskiy 2018). Consequently, handling it exactly becomes challenging. The general fractional program we are tackling, involving a combination of both binary and continuous variables, presents a particularly intricate problem to solve. To the best of our knowledge, there are currently no exact methods available in the respective literature for achieving (near) optimal solutions in this context.

**Piece-wise Linear Approximation (PWLA):** Our work leverages a PWLA approach to simplify the objective function, leading to more tractable problem formulations. The literature on PWLA is extensive (Lin et al. 2013, Lundell and Westerlund 2013, Westerlund et al. 1998, Lundell et al. 2009), with various techniques integrated into state-of-the-art solvers for mixed-integer nonlinear programs (GUROBI 2024). Although GUROBI’s PWLA techniques offer methods to linearize certain types of nonlinear univariate functions, they are not directly applicable to solve the fractional program in \((\text{SoR})\). In our experiments, by reformulating \((\text{SoR})\) as a bilinear program, we demonstrate that GUROBI’s PWLA can be applied. However, this approach is generally outperformed by our proposed approximation methods across most benchmark instances. It is worth noting that PWLA techniques have also been used to address MNL-based pricing problems (Mai and Sinha 2023, Bose et al. 2022), but these studies focus only on single-ratio programs, making them unsuitable for our context.

**Our contributions.** We make the following contributions:

(i) We explore a discretization mechanism to approximate the nonlinear numerators and denominators of the objective functions by linear functions. We further show that the approximate problem can be reformulated as a MILP or Bilinear program. Moreover, we show that, under some mild conditions that hold in the aforementioned applications, the approximate problem can be reformulated as mixed-integer second-order cone program (MISOCP). The MILP, MISOCP and Bilinear programs then can be handled conveniently by commercial software (e.g., CPLEX or GUROBI). Furthermore, in the context of facility location and cost optimization, we show that the approximated problem features a concave objective function, which allows it to be optimally addressed using an outer-approximation algorithm (Duran and Grossmann 1986, Mai and Lodi 2020).

(ii) We also establish bounds of \(O(1/K)\) for the approximation errors yielded by the approximate solutions, where \(K\) is the number of discretization points. We also provide estimates for \(K\) to guarantee a desired approximation bound. Note that most of previous works rely on the assumption that the customers’ or adversaries’ utility functions have linear structures (e.g., linear in the prices, costs or security resources), our approach can handle any non-linear structures (e.g., the utility function is modelled by a deep neural network) while securing the same performance guarantees.

(iii) We conduct numerical experiments on instances of varying sizes, comparing our methods against several baselines, including a general mixed-integer nonlinear solver and an approach utilizing GUROBI’s
Lipschitzness of \( g \) the class on non-convex problems in (SoR). The results clearly demonstrate the efficiency of our approximation method in providing near-optimal solutions to the non-convex problem.

In short, we develop a solution method that uses commercial software to provide near-optimal solutions to the class on non-convex problems in (SoR). The advantage of this approach is clear; commercial software allows the inclusion of complex business constraints without the need for redesigning algorithms, and is continually improved with new optimization methods and advanced hardware. We appear to be the first to explore and develop global optimization methods for several important optimization problems, including constrained product assortment and pricing, facility location planning, and security games.

**Paper Outline.** Section 2 presents our discretization approach and the MILP, MISOCP and Bilinear reformulations. Section 3 provides performance guarantees for our approximation approach. Section 4 presents applications to the joint assortment and price optimization, and joint facility location and cost optimization problems. Section 5 presents numerical experiments and finally, Section 6 concludes. Proofs, if not present in the main paper, are presented in the appendix.

2. Towards a Globally Optimal Solution: a PWLA Approach

In this section, we present our discretization approach to approximate the non-convex problem (SoR) into a form that can be solved to optimality by an off-the-shelf solver. To begin, let us introduce the following mild assumptions which generally holds in all the aforementioned applications.

**Assumption 1 (A1).** The following assumptions hold:
1. \( b_i + \sum_{i \in [m]} y_i h_i(x) > 0 \) for all \( y \in Y, x \in X \)
2. \( h_i(x) \) and \( g_i(x) \) are Lipschitz continuous, i.e., there exist \( L_i^h, L_i^g > 0 \) such that:
   \[
   |h_i(x_1) - h_i(x_2)| \leq L_i^h|x_1 - x_2| \quad \text{and} \quad |g_i(x_1) - g_i(x_2)| \leq L_i^g|x_1 - x_2|, \forall x_1, x_2 \in [l_i, u_i]
   \]

2.1. Discretizing Univariate Components

We first introduce an approach to approximate an univariate function by a linear function of binary variables. Let \( g(x) : \mathbb{R} \to \mathbb{R} \) be an univariate function. Assuming that \( g(x) \) is Lipschitz continuous in \([l, u]\) with Lipschitz constant \( L > 0 \), i.e., \(|g(x_1) - g(x_2)| \leq L|x_1 - x_2| \) for all \( x_1, x_2 \in [l, u] \). For any \( K \in \mathbb{N} \), we discretize the interval \([l, u]\) into \( K \) equal intervals of length \( \Delta = (u - l)/K \) and approximate \( x \) as \( x \approx \bar{x} = l + \Delta\lfloor(x - l)/\Delta \rfloor \). To use this to approximate the original problem as a mixed-integer nonlinear program, we introduce binary variables \( z_k \in \{0, 1\}, k \in [K] \) and approximate \( x \) as \( x \approx \bar{x} = l + \Delta \sum_{k \in [K]} z_k \), where \( z_k \geq z_{k+1} \) and for \( k^* = \lfloor(x - l)/\Delta \rfloor \) we have \( z_{k^*} = 1 \) and \( z_{k^*+1} = 0 \). We then can approximate \( g(x) \) by the following discrete linear function. \( g(x) \approx g(\bar{x}) = g(l) + \Delta \sum_{k \in [K]} \gamma_k^{g} z_k \), where \( \gamma_k^{g} \) is the slope of function \( g(x) \) in the interval \([l + (k-1)\Delta; l + k\Delta] \), i.e., \( \gamma_k^{g} = \frac{1}{\Delta}(g(l + k\Delta) - g(l + \Delta(k - 1))) \), \( \forall k \in [K] \). From the Lipschitzness of \( g(x) \), we have the following bound for the gap between \( g(x) \) and \( g(\bar{x}) \). We will use this for establishing performance guarantees for the approximate problem.
LEMMA 1. For any $x \in [l, u]$ and $\hat{x}$ from the discretization mechanism above: $|g(x) - g(\hat{x})| \leq L\Delta$.

Thus, it can be seen that $\hat{x}$ and $g(\hat{x})$ will approach $x$ and $g(x)$ linearly, as $K$ increases.

2.2. Discretizing the Sum-of-ratios Problem

We now show how to use the discretization technique above to approximate the binary-continuous sum-of-ratios program in $\mathcal{SOR}$. For ease of notation, let us first denote $\Delta_i = (u_i - l_i)/K$. Given any $x \in \mathcal{X}$, we approximate $x_i$, $i \in [m]$, by $l + \Delta_i [K(x_i - l_i)]/(u_i - l_i)$] and approximate the functions $g_i^t(x_i)$ and $h_i^t(x_i)$ by binary variables $z_{ik} \in \{0, 1\}$, $\forall i \in [m], k \in [K]$ as

$$g_i^t(x_i) \approx \tilde{g}_i^t(x_i) = g_i^t(l_i) + \Delta_i \sum_{k \in [K]} \gamma_{ik}^{gt} z_{ik}; \quad h_i^t(x_i) \approx \tilde{h}_i^t(x_i) = h_i^t(l_i) + \Delta_i \sum_{k \in [K]} \gamma_{ik}^{ht} z_{ik},$$

where $\gamma_{ik}^{gt}$ and $\gamma_{ik}^{ht}$ are the slopes of $g_i^t(x_i)$ and $h_i^t(x_i)$ in $[l_i + (k - 1)\Delta_i; l_i + k\Delta_i]$, $\forall k \in [K]$, i.e.,

$$\gamma_{ik}^{gt} = \frac{1}{\Delta_i} (g_i^t(l_i + k\Delta_i) - g_i^t(l_i + (k - 1)\Delta_i)), \forall k \in [K]$$

$$\gamma_{ik}^{ht} = \frac{1}{\Delta_i} (h_i^t(l_i + k\Delta_i) - h_i^t(l_i + (k - 1)\Delta_i)), \forall k \in [K]$$

Then, each $x \in \mathcal{X}$ can be written as $x_i = l_i + \Delta_i \sum_{k \in [K]} z_{ik} + r_i$, where $r_i \in [0, \Delta_i]$ is used to capture the gap between $x_i$ and the binary approximation $l_i + \Delta_i \sum_{k \in [K]} z_{ik}$. We now approximate $\mathcal{SOR}$ by the following mixed-integer nonlinear program

$$\max_{x, y} \quad \hat{f}(y, z, x) = \sum_{t \in [T]} a_t + \sum_{i \in [m]} b_{it} y_i g_i^t(l_i) + \sum_{i \in [m]} \Delta_i \sum_{k \in [K]} \gamma_{ik}^{gt} y_i z_{ik}$$

s.t. $z_{ik} \geq z_{i,k+1}, \forall k \in [K - 1], i \in [m]$  
\hspace{1cm} $x_i = l_i + \Delta_i \sum_{k \in [K]} z_{ik} + r_i, \forall i \in [m]$  
\hspace{1cm} $r_i \in [0, \Delta_i], \forall i \in [m]$  
\hspace{1cm} $Ax + By \leq D$  
\hspace{1cm} $x \in \mathcal{X}, \ z \in \{0, 1\}^{m \times K}, \ y \in \mathcal{Y}.$

The objective function of $\mathcal{APPROX}$ involves bi-linear terms $y_i z_{ik}$ which can be linearized by introducing some additional binary variables $s_{ik} = y_i z_{ik}$ and linear constraints $s_{ik} \leq z_{ik}$, $s_{ik} \leq y_i$, $s_{ik} \geq y_i z_{ik}$.
As a result, (Approx-1) is equivalent to the following mixed-integer nonlinear program.

\[
\begin{align*}
    \max_{z,s,y} & \quad \sum_{i \in [T]} a_i + \sum_{i \in [m]} y_i g_i^l(l_i) + \sum_{i \in [m]} \Delta_i \sum_{k \in [K]} g_{ik} s_{ik} \\
    \text{s.t.} & \quad z_{ik} \geq z_{i,k+1}, \forall k \in [K-1], i \in [m] \\
    & \quad s_{ik} \leq z_{ik}, s_{ik} \leq y_i, s_{ik} \geq z_{ik} + y_i - 1, \forall i \in [m], k \in [K] \\
    & \quad x_i = l_i + \Delta_i \sum_{k \in [K]} z_{ik} + r_i, \forall i \in [m] \\
    & \quad r_i \in [0, \Delta_i], \forall i \in [m] \\
    & \quad Ax + By \leq D \\
    & \quad z, s \in \{0, 1\}^{m \times K}, x \in \mathcal{X}, y \in \mathcal{Y}.
\end{align*}
\]

In the Proposition 1 below, we show that under a condition that often holds in the applications mentioned above, the bi-linear terms \(y_i z_{ik}\) in (Approx) can be simplified without introducing new binary variables \(s_{ik}\). To facilitate this point, we first let \(S\) be the feasible set of the original problem \(\text{SoR}\) i.e., \(S = \{(y, x) | (y, x) \in \mathcal{Y} \times \mathcal{X}; Ax + By \leq D\}\), and \(S(y)\) be the feasible set of the original problem \(\text{SoR}\) with fixed \(y \in \mathcal{Y}\), i.e., \(S(y) = \{x | x \in \mathcal{X}; Ax + By \leq D\}\). We first introduce the following assumption that is needed for the result.

**Assumption 2 (A2).** For any \((x, y) \in S\) we have \((x', y) \in S\) for all \((x', y) \in \mathcal{X} \times \mathcal{Y}\) and \(x' \leq x\).

The above assumption is not restrictive in our applications of interests. For example, in the context of joint assortment and price optimization, one may only require that the prices can vary between some lower and upper bounds or a weighted sum (with non-negative weight parameters) of the prices is less than an upper bound. In the context of facility location and SSG, one may impose an upper bound for the total cost/security resources, i.e., \(\sum_{i \in [m]} x_i \leq C\). With such constraints, Assumption (A2) indeed holds.

Under Assumption (A2), we show, in Proposition 1, that (Approx) can be simplified by removing the variables \(r_i, i \in [m]\), and replacing \(y_i z_{ik}\) by only \(z_{ik}\) and adding constraints \(y_i \geq z_{i1}\), which implies that if \(y_i = 0\) then \(z_{ik} = 0\) for all \(i \in [m], k \in [K]\).
**Proposition 1.** Suppose (A1) and (A2) hold, we have that (Approx) is equivalent to the following mixed-integer program.

\[
\begin{align*}
\max_{\bar{z}, \bar{y}, \bar{x}} \quad & \bar{f}(\bar{y}, \bar{z}, \bar{x}) = \sum_{t \in T} a_t + \sum_{i \in [m]} b_t + \sum_{i \in [m]} y_i h_i^t(l_i) + \sum_{i \in [m]} \Delta_i \sum_{k \in [K]} \gamma_{ik} z_{ik} \quad \text{(Approx-2)} \\
\text{s.t.} \quad & z_{ik} \geq z_{i,k+1}, \forall k \in [K-1], i \in [m] \quad (1) \\
& y_i \geq z_{i,1}, \forall i \in [m] \quad (2) \\
& x_i = l_i + \Delta_i \sum_{k \in [K]} z_{ik}, \forall i \in [m] \quad (3) \\
& Ax + By \leq D \quad (4) \\
& z \in \{0,1\}^{m \times K}, \ y \in \mathcal{Y}, \ x \in \mathcal{X}.
\end{align*}
\]

The approximate programs in (Approx-1) and (Approx-2) take the form of linear fractional programs with binary variables present in both the numerators and denominators. Prior studies [Méndez-Díaz et al. 2014; Haase and Müller 2014] have shown that such programs can be converted into MILPs by introducing additional continuous variables, using big-M or McCormick inequalities. In the context of assortment optimization under the mixed-logit model, Sen et al. (2018) demonstrated that a linear fractional program can also be equivalently reformulated as a second-order cone program (SOCP). However, the approach in Sen et al. (2018) is not directly applicable for reformulating (Approx-1) and (Approx-2), as it assumes that all coefficients in the denominators are non-negative, which is not the case in our context when the function \( h_i^t \) is increasing. In the following section, we address this issue and show how (Approx-1) and (Approx-2) can be reformulated as a mixed-integer second-order cone program. Additionally, we explore a bilinear reformulation that can be solved to optimality using off-the-shelf solvers, while requiring fewer additional variables and constraints compared to the MILP or SOCP reformulations.

### 2.3. MILP Reformulation

As previously discussed, (Approx-1) and (Approx-2) take the form of linear fractional programs, which can be reformulated as MILPs. Although converting (Approx-1) and (Approx-2) into MILPs is relatively straightforward given the existing literature, we include these reformulations here for the sake of completeness. Our focus will be on (Approx-2), with the understanding that (Approx-1) can be reformulated in a
similar manner. First, let \( w_t = \frac{1}{b_t + \sum_{i \in [m]} y_i h_i^L(l_i) + \sum_{i \in [m]} \Delta_i \sum_{k \in [K]} \gamma_{ik}^L z_{ik}} \). We can write \( \text{Approx-2} \) as follows:

\[
\begin{align*}
\max_{z, w, x, y} & \quad \sum_{t \in [T]} a_t w_t + \sum_{i \in [m]} y_i w_t g_i^L(l_i) + \sum_{i \in [m]} \Delta_i \sum_{k \in [K]} \gamma_{ik}^L w_t z_{ik} \\
\text{s.t.} & \quad b_t w_t + \sum_{i \in [m]} y_i w_t h_i^L(l_i) + \sum_{i \in [m]} \Delta_i \sum_{k \in [K]} \gamma_{ik}^L w_t z_{ik} = 1 \\
& \quad z \in \{0, 1\}^{m \times K}, \quad y \in \mathcal{Y}, \quad w \in \mathbb{R}^T, \quad x \in \mathcal{X}.
\end{align*}
\]

The above formulation involves some bi-linear terms: \( w_t z_{ik} \) and \( w_t y_i \), \( \forall t \in [T], i \in [m], k \in [K] \), that can be linearized using the Big-M technique or McCormick inequalities \( \text{McCormick} \, [1976] \). We will employ the McCormick inequalities since they offer tighter continuous relaxations. To this end, let \( U^w_t \) and \( L^w_t \) are an upper bound and lower bound of \( w_t \). By introducing new variables \( u_{tik} \) and \( v_{ti} \) to represent \( w_t z_{ik} \) and \( w_t y_i \), we can linearize the bi-linear terms as

\[
\begin{align*}
u_{tik} & \leq U^w_t z_{ik}; \quad u_{tik} \geq L^w_t z_{ik} & \quad (5) \\
u_{tik} & \leq w_t - L^w_t (1 - z_{ik}); \quad u_{tik} \geq w_t - U^w_t (1 - z_{ik}) & \quad (6) \\
v_{ti} & \leq U^w_t y_i; \quad u_{tik} \geq L^w_t y_i & \quad (7) \\
v_{ti} & \leq w_t - L^w_t (1 - y_i); \quad u_{tik} \geq w_t - U^w_t (1 - y_i) & \quad (8)
\end{align*}
\]

We then can reformulate \( \text{Approx-2} \) as the following MILP:

\[
\begin{align*}
\max_{x, y, z, w, u, v} & \quad \sum_{t \in [T]} a_t w_t + \sum_{i \in [m]} v_{ti} g_i^L(l_i) + \sum_{i \in [m]} \Delta_i \sum_{k \in [K]} \gamma_{ik}^L u_{tik} \\
\text{s.t.} & \quad b_t w_t + \sum_{i \in [m]} v_{ti} h_i^L(l_i) + \sum_{i \in [m]} \Delta_i \sum_{k \in [K]} \gamma_{ik}^L u_{tik} = 1 \\
& \quad z \in \{0, 1\}^{m \times K}, \quad y \in \mathcal{Y}, \quad w \in \mathbb{R}^T, \quad x \in \mathcal{X}, \quad u \in \mathbb{R}^{T \times m \times K}, \quad v \in \mathbb{R}^{T \times m}.
\end{align*}
\]

2.4. MISOCP Reformulation

We reformulate \( \text{Approx-2} \) as MISOCP programs, which would offer tighter relaxations, as compared the MILP reformulation presented above. The idea is to introduce additional variables and make use of conic quadratic inequalities of the form \( x_1^2 \leq x_2 x_3 \) with \( x_1, x_2, x_3 \geq 0 \). Here, we note that MISOCP reformulations for 0-1 fractional programs have been studied in prior works \( \text{Sen et al.} \, [2018] \, \text{Mehmanchi et al.} \, [2019] \), but
these reformulations rely on the assumption that all the coefficients of the binary variables in the numerators and denominators are non-negative. In our context, this may not be the case, depending on the behaviors of functions \( h_t^i(x_i) \) within the denominators. In the following sections, we present two MISOCP reformulations for the nonlinear program in (Approx-2) in two scenarios: one where \( h_t^i(x_i) \) exhibits a monotonically increasing behavior (relevant in the context of facility and cost optimization), and the other where \( h_t^i(x_i) \) exhibit a monotonically decreasing pattern (pertinent to assortment and price optimization).

**PROPOSITION 2.** If \( h_t^i(l_i) > 0 \) and \( h_t^i(x_i) \) are monotonic increasing for \( x_i \in [l_i, u_i] \), for all \( t \in [T] \) and \( i \in [m] \), then (Approx-2) can be reformulated as the following MISOCP

\[
\begin{align*}
\min_{y, w, \theta, \lambda} & \left\{ \sum_{t \in [T]} w_t (\lambda_t b_t - a_t) + \sum_{i \in [m]} v_t (\lambda_t h_t^i(l_i) - g_t^i(l_i)) + \sum_{i \in [m]} \Delta_i \sum_{k \in [K]} (\lambda_t \gamma_{ik}^{ht} - \gamma_{ik}^{gt}) u_{tik} - \sum_{t \in [T]} \lambda_t \right\} \\
\text{s.t.} & \quad \theta_t = b_t + \sum_{i \in [m]} y_i h_t^i(l_i) + \sum_{i \in [m]} \Delta_i \sum_{k \in [K]} \gamma_{ik}^{ht} z_{ik}, \forall t \in [T] \\
& \quad w_t \theta_t \geq 1; \forall t \in [T] \\
& \quad v_t \theta_t \geq y_t^2 \forall t \in [T], i \in [m] \\
& \quad u_{tik} \theta_t \geq z_{ik}^2 \forall t \in [T], i \in [m], k \in [K] \\
\text{Constraints:} & \quad \text{(1)} - \text{(2)} - \text{(3)} - \text{(4)} \\
\end{align*}
\]

\[ z \in \{0, 1\}^{m \times K}, y \in Y, w, \theta \in \mathbb{R}^T, x \in X, u \in \mathbb{R}^{T \times m \times K}, v \in \mathbb{R}^T \times m \]

where \( \lambda_t > 0, \forall t \in [T] : \lambda_t > \max \left\{ \max_{i \in [m]} \left\{ \frac{g_t^i(l_i)}{h_t^i(l_i)} \right\} ; \max_{x} \left\{ \frac{a_t + \sum_{i \in [m]} y_i g_t^i(x_i)}{b_t + \sum_{i \in [m]} y_i h_t^i(x_i)} \right\} ; \max_{t, i, k} \left\{ \frac{\gamma_{ik}^{gt}}{\gamma_{ik}^{ht}} \right\} \right\} \).

If \( h_t^i(x_i) \) is not monotonic increasing as in the case of the joint assortment and price optimization problem, some bi-linear terms cannot be converted into rotated cones, thus we need to make use of McCormick inequalities to linearize them. We state this result the proposition below.
PROPOSITION 3. If \( h_i^t(l_i) > 0 \) for all \( t \in [T] \) and \( i \in [m] \), then (Approx-2) can be reformulated as the following MISOCP:

\[
\begin{align*}
\min_{y, x, z, \theta} & \quad \sum_{t \in [T]} w_t (\lambda_t b_t - a_t) + \sum_{i \in [m]} v_{ti} (\lambda_t h_i^t(l_i) - g_i^t(l_i)) + \sum_{i \in [m]} \Delta_i \sum_{k \in [K]} (\lambda_t \gamma_{ik}^t - \gamma_{ikt}^t) u_{tik} - \sum_{i \in [m]} \lambda_t \\
\text{s.t.} & \quad \theta_t = b_t + \sum_{i \in [m]} y_i h_i^t(l_i) + \sum_{i \in [m]} \Delta_i \sum_{k \in [K]} \gamma_{ikt}^t z_{ik}, \forall t \in [T] \\
& \quad w_i \theta_t \geq 1; \forall t \in [T] \\
& \quad v_{ti} \theta_t \geq y_i^2 \forall t \in [T], i \in [m] \\
& \quad u_{tik} \leq U_i^t w_i z_{ik}; u_{tik} \geq L_i^t w_i z_{ik} \quad \text{Constraints: (1) - (2) - (3) - (4)} \\
& \quad z \in \{0, 1\}^{m \times K}, y \in \mathcal{Y}, w, \theta \in \mathbb{R}^T, x \in \mathcal{X}, u \in \mathbb{R}^{T \times m \times K}, v \in \mathbb{R}^{t \times m}
\end{align*}
\]

where \( \lambda_t > 0, \forall t \in [T] : \lambda_t > \max \left\{ \max_{i \in [m]} \left\{ \frac{g_i^t(l_i)}{b_i} \right\}; \max_{x \in \mathcal{X}} \left\{ \frac{a_i + \sum_{i \in [m]} y_i g_i^t(x_i)}{b_i + \sum_{i \in [m]} y_i h_i^t(x_i)} \right\} \right\} \).

2.5. Bilinear Reformulation

A drawback of the MILP and MISOCP reformulations presented above is that they require a significant number of additional variables and constraints. In particular, the inclusion of McCormick inequalities can lead to weak continuous relaxations, making these reformulations slow to solve for large instances. In the following, we demonstrate that (Approx-2) can also be reformulated as a bilinear program, offering several advantages: (i) the bilinear reformulation requires significantly fewer additional variables and constraints compared to the MILP and MISOCP versions; (ii) it can be efficiently handled by modern solvers such as GUROBI; (iii) this approach also enables the application of GUROBI’s PWLA technique to solve our continuous-binary program, as we will illustrate in the experimental section.

The bilinear reformulation can be obtained by defining:

\[
\begin{align*}
n_t &= a_t + \sum_{i \in [m]} y_i g_i^t(l_i) + \sum_{i \in [m]} \Delta_i \sum_{k \in [K]} \gamma_{ikt}^t z_{ik}; \\
d_t &= b_t + \sum_{i \in [m]} y_i h_i^t(l_i) + \sum_{i \in [m]} \Delta_i \sum_{k \in [K]} \gamma_{ikt}^t z_{ik}
\end{align*}
\]
Then we can reformulate (Approx-2) as the following bilinear version:

\[
\max_{x,y,z,n,o} \sum_{t \in [T]} o_t \quad \text{ (Bilinear)}
\]

s.t. \( o_t d_t \leq n_t \)

\[
n_t = a_t + \sum_{i \in [m]} y_i g_t^i(l_i) + \sum_{i \in [m]} \sum_{k \in [K]} \gamma_{ik}^t z_{ik}, \quad \forall t \in [T]
\]

\[
d_t = b_t + \sum_{i \in [m]} y_i h_t^i(l_i) + \sum_{i \in [m]} \sum_{k \in [K]} \gamma_{ik}^t z_{ik}, \quad \forall t \in [T]
\]

Constraints: (1) - (2) - (3) - (4)

\( z \in \{0, 1\}^{m \times K}, \ y \in Y, \ n \in \mathbb{R}^T, \ d \in \mathbb{R}^T, \ o \in \mathbb{R}^T, \ x \in X. \)

It can be observed that the bilinear reformulation in (Bilinear) requires only 3\(T\) additional variables and 2\(T\) additional constraints. On the other hand, the MILP reformulation in (MILP) introduces approximately \(T + TmK + Tm\) additional variables and 2\((TmK + Tm)\) McCormick constraints, while the MISOCP requires around \(2T + Tm + TmK\) variables and \(2T + Tm + TmK\) additional constraints. Although the bilinear constraints in (Bilinear) are non-convex, they can be efficiently managed using modern techniques in bilinear programming (Misener and Floudas 2014, Burer 2009, GUROBI Optimization, LLC 2023).

3. Performance Guarantees

In this section, we analyze the approximation errors associated with the approximate program (Approx), considering them as functions of the number of linear segments \(K\). These analyses provide insights into how the parameter \(K\) (the number of pieces) affects the quality of the approximation, as well as guidelines on selecting an appropriate \(K\) to achieve the desired solution accuracy. This helps in balancing the trade-off between computational efficiency and approximation quality.

In the following, we first establish conditions and guarantees for the convergence of the approximate problem (Approx) to the original problem (SoR) as the number of discretization points \(K\) increases. We further extend the results for a general stochastic fractional program arising from the use of the mixed-logit discrete choice model (McFadden 1981) with continuous-distributed random parameters, for which one needs to approximate the expectation by sample average approximation (SAA) before the discretization techniques can be applied.

3.1. Main Results

In this section, we establish a bound for the approximation errors of the solutions returned by (Approx). For ease of notation, given any \(x \in X\), let \(\delta^K(x)\) be a vector of size \(m\) with elements

\[
\delta^K(x)_i = l_i + \Delta_i \lfloor (x_i - l_i)/\Delta_i \rfloor, \quad \forall i \in [m].
\]
Let us also define \( F^t = \max_{(y, x) \in \mathcal{S}} \left\{ (a_t + \sum_{i \in [m]} y_i g_i(x_i))/ (b_t + \sum_{i \in [m]} y_i h_i(x_i)) \right\} \) and \( H^t = \min_{(y, x) \in \mathcal{S}} \left\{ b_t + \sum_{i \in [m]} y_i h_i(x_i) \right\} \). We also denote \((x^*, y^*)\) as an optimal solution to the original problem \((\text{SoR})\) and \((\bar{x}, \bar{y})\) as an optimal solution to the approximate problem \((\text{Approx})\). We start with the following lemma.

**Lemma 2.** For any \( y \in \mathcal{Y} \) and \( x, x' \in \mathcal{X} \) and \( \epsilon > 0 \) such that \( ||x - x'||_{\infty} \leq \epsilon \), we have \( |f(y, x) - f(y, x')| \leq \epsilon C \), where \( C = \sum_{t \in [T]} \left( \frac{\sum_{i \in [m]} t_i^q}{H_t} + \frac{\sum_{i \in [m]} L_t^i}{H_t} \right) \).

The following corollary is then a direct result of Lemma 2 and the fact that \( ||x - \delta(x)||_{\infty} \leq \max_{i \in [m]} \{ (u_i - l_i)/K \} \).

**Corollary 1.** For any \( (y, x) \in \mathcal{S} \) we have \( |f(y, x) - f(y, \delta^K(x))| \leq \frac{C}{K} \max_{i \in [m]} \{ u_i - l_i \} \)

Lemma 3 further simplifies \((\text{Approx})\) by excluding variables \( z \) and rewriting the objective function of \((\text{Approx})\) in the form of the original objective function in \((\text{SoR})\). This will be useful for the proof of Theorem 1 below.

**Lemma 3.** The approximate problem \((\text{Approx})\) is equivalent to

\[
\begin{align*}
\max_{x, y, r} & \quad f(y, x - r) \\
\text{s.t.} & \quad Ax + By \leq D \\
& \quad x_i - r_i \in \{ l_i + \Delta i k; \ k = 0, 1, 2, \ldots, K \}, \forall i \in [m] \\
& \quad r_i \in [0, \Delta_i], \forall i \in [m] \\
& \quad (x) \in \mathcal{X}, y \in \mathcal{Y}, r \in \mathbb{R}^m
\end{align*}
\]

As a result, \((\text{Approx})\) is equivalent to \( \max_{y \in \mathcal{Y}, x \in \mathcal{X}} \left\{ \left| f(y, \delta^K(x)) \right| \middle| Ax + By \leq D \right\} \).

We are now ready to establish a performance guarantee for any solution returned by solving the approximation problem \((\text{Approx})\). To facilitate our exposition, let \((x^*, y^*)\) be an optimal solution to the original problem \((\text{SoR})\) and \((x^K, y^K)\) be optimal to the approximate problem \((\text{Approx})\). Theorem 1 below shows that solutions to the approximation problem \((\text{Approx})\) converge linearly to optimal ones as \( K \) increases.

**Theorem 1.** Suppose that \((A1)\) hold, then \( |f(y^*, x^*) - f(y^K, x^K)| \leq \frac{2C}{K} \max_{i \in [m]} \{ u_i - l_i \} \).

### 3.2. Sampling and Discretization

We have so far consider sum-of-ratios programs which arises from using the mixed-logit model with random choice parameters supported by a finite set of parameters. In a more general setting, the choice parameters can follow a continuous distribution (McFadden 1981), for which the objective function would involve an expectation over an infinite set of random parameters. To handle this, one can approximate the expectation.
by sequential sampling approximation (SSA) and then use the discretization techniques developed above. In the following we establish approximation bounds for such a 2-level approximation approach. We first consider a stochastic sum-of-ratios optimization problem of the following form.

\[
\max_{y, x} \mathcal{F}(y, x) = \mathbb{E}_{\xi} \left[ \frac{a(\xi) + \sum_{i \in [m]} y_i g_i(x_i, \xi)}{b(\xi) + \sum_{i \in [m]} y_i h_i(x_i, \xi)} \right]
\]

\[
\text{s.t. } Ax + By \leq D
\]

\[
y \in \mathcal{Y}, \ x \in \mathcal{X}
\]

where the expectation can be taken over a continuous distribution of \(\xi\). Let us define an SAA version of \((\text{Stoc-SoR})\), obtained by taking \(T\) sample of \(\xi\).

\[
\max_{y, x} \hat{\mathcal{F}}^T(y, x) = \sum_{t \in [T]} \frac{a(\xi^t) + \sum_{i \in [m]} y_i g_i(x_i, \xi^t)}{b(\xi^t) + \sum_{i \in [m]} y_i h_i(x_i, \xi^t)}
\]

\[
\text{s.t. } Ax + By \leq D
\]

\[
y \in \mathcal{Y}, \ x \in \mathcal{X}
\]

where \(\{\xi^1, \ldots, \xi^T\}\) are \(T\) i.i.d. samples of \(\xi\). \((\text{SAA-SoR})\) is of the same structure with the sum-of-ratios problem considered in \((\text{SoR})\). Let \((y^*, x^*)\), \((\hat{y}, \hat{x})\) and \((\hat{y}^K, \hat{x}^K)\) be optimal solution to the stochastic problem \((\text{Stoc-SoR})\), its SAA version \((\text{SAA-SoR})\) and the discretization version of \((\text{SAA-SoR})\), respectively. We aim to establish a bound for \(|\mathcal{F}(y^*, x^*) - \mathcal{F}(\hat{y}^K, \hat{x}^K)|\) which reflects the performance of a solution given by our SAA and discretization approach. Before presenting the main result, we need assumptions that are similar to those used in the sum-of-ratios case.

**Assumption 3 (A3).** The following holds almost surely

- The ratio \(\frac{a(\xi) + \sum_{i \in [m]} y_i g_i(x_i, \xi)}{b(\xi) + \sum_{i \in [m]} y_i h_i(x_i, \xi)}\) is lower and upper bounded
- \(b(\xi) + \sum_{i \in [m]} y_i h_i(x_i, \xi) > 0\)
- \(h_i(x_i, \xi)\) and \(g_i(x_i, \xi)\) are Lipschitz continuous, i.e., there exist \(L_i^h, L_i^b > 0\) such that

\[
|h_i(x_1, \xi) - h_i(x_2, \xi)| \leq L_i^h|x_1 - x_2|, \forall x_1, x_2 \in [l_i, u_i]
\]

\[
|g_i(x_1, \xi) - g_i(x_2, \xi)| \leq L_i^b|x_1 - x_2|, \forall x_1, x_2 \in [l_i, u_i]
\]

Under (A3), let us assume that there are two constants \(\mathcal{F}^\mathcal{E}\) and \(\mathcal{F}\) such that, with probability one,

\[
\mathcal{F} \leq \frac{a(\xi) + \sum_{i \in [m]} y_i g_i(x_i, \xi)}{b(\xi) + \sum_{i \in [m]} y_i h_i(x_i, \xi)} \leq \mathcal{F}^\mathcal{E}.
\]
Let $H$ be a constant such that, with probability one, $H \leq b(\xi) + \sum_{i \in [m]} y_i h_i(x_i, \xi)$ for all $(y, x \in S)$, and $C^* = \left( \frac{\sum_{i \in [m]} L_i^2}{H} + \frac{H}{H} \right)$. We also let $\Psi = \mathcal{F} - \mathcal{F}$ for notation brevity. We state our main results in Theorem 2 below.

**Theorem 2.** Given any $\epsilon > 0$, suppose (A3) holds, then the inequality $|\mathcal{F}(y^*, x^*) - \mathcal{F}(y^K, x^K)| \leq \epsilon$ holds with a probability at least $1 - 6 \exp \left( - \frac{2\epsilon^2}{25\Psi} \right)$ if we choose $K$ such that $K \geq \frac{5T C^*}{\epsilon} \max_{i \in [m]} (u_i - l_i)$.

The above theorem provides a relation between the number of samples $T$ of the SSA and the number of discretization points $K$. It generally implies that, to have a high probability of getting near-optimal solutions, one would need to select a sufficiently large $T$, and $K$ should be at least proportional to the number of samples $K$. In the following corollary, we further leverage this point to establish some estimates for $T$ and $K$ that secure a high probability of getting near-optimal solutions. The estimates are direct results from the proof of Theorem 2.

**Corollary 2.** For any $\gamma \in (0, 1)$ and $\epsilon > 0$, we have $\mathbb{P}[|\mathcal{F}(y^*, x^*) - \mathcal{F}(y^K, x^K)| \leq \epsilon] \geq 1 - \gamma$ if $K$ and $T$ are chosen such that $T \geq \frac{25\Psi \ln(6/\gamma)}{2\epsilon^2}$; $K \geq \frac{5T C^*}{\epsilon} \max_{i \in [m]} ((u_i - l_i))$.

The estimates provided in Corollary 2 are likely to be on the conservative side. In practice, achieving the desired performance may require significantly smaller values of $K$ and $T$. Nevertheless, these estimates provide valuable insights into how the performance criteria $\epsilon$ and $\gamma$ influence the selection of the sample size $T$ and the number of pieces $K$ of the discretization approach.

### 4. Applications

We briefly discuss the applications of our discretization approach to two prominent classes of decision-making problems, i.e., maximum capture facility location, and assortment and price optimization. The SSG formulation is similar to that of the price optimization and we refer the reader to (Yang et al., 2012, Fang et al., 2017) for more details.

#### 4.1. Joint Facility Location and Cost Optimization

Let $[m]$ be the set of available locations to setup new facility. For each customer segment $t \in [T]$, let $v_{ti} = x_i \eta_{ti} + \kappa_{ti}$ be the corresponding utility function of location $i \in [m]$, where $x_i$ is the cost to spend on location $i$, $\eta_{ti}$ is a cost sensitivity parameter that reflects the impact of the cost $x_i$ on the choice probabilities, and $\kappa_{ti}$ are other parameters that affect the choices. We should have $\eta_{ti} > 0$ since an increase in the cost $x_i$ should add more value to the corresponding utility of facility $i$. The probability that a facility located at location $i$ will be chosen by a customer $t \in [T]$, under the logit model, is given as

$$P^t \left( \left[ m \cup \{0\} \right] \right) = \sum_{i \in [T]} \frac{\exp{(x_i \eta_{ti} + \kappa_{ti})}}{U_C^t + \sum_{i \in [m]} \exp{(x_i \eta_{ti} + \kappa_{ti})}}$$

where 0 refers to the option of selecting a facility of the competitor and $U_C^t$ captures the utilities of all the competitor’s facilities. The objective now is to maximize an expected captured demand, i.e., the expected
number of customers attracted by the opening facilities (the problem typically referred to as the maximum capture problem - MCP). We can write the problem as

$$
\max_{y \in \mathcal{Y}, x \in \mathcal{X}} \left\{ f(y, x) = \sum_{t \in [T]} q_t \sum_{i \in [m]} y_i \exp \left( \frac{x_i \theta_i + \kappa_t}{U_C^t} \right) \right\}
$$

(MCP)

where \( q_t \) is the proportion of customers in segment \( t \) in the market. The maximum capture problem (MCP) differs from the joint assortment and price optimization problem presented below as the objective function of (MCP) can be simplified into a sum-of-ratios program in which the variables \( y, x \) only appear in the denominators:

$$
f(y, x) = \sum_{t \in [T]} q_t - \sum_{t \in [T]} \frac{q_t U_C^t}{U_C^t + \sum_{i \in [m]} y_i \exp \left( \frac{x_i \theta_i + \kappa_t}{U_C^t} \right)}.
$$

As a result, to apply the approximation methods above, we can let \( g_i^t(x_i) = 0 \), \( a_t = q_t U_C^t \), \( b_t = U_C^t \) and \( h_t^t(x_i) = \exp(x_i \eta_i t + \kappa_t) \). Since \( \eta_i t > 0 \), \( h_t^t(x_i) \) is monotonic increasing in \( x_i \), thus the MISOCOCP reformulation in Proposition 2 can be used. Moreover, the approximate problem (Approx-2) can be written as

$$
\max_{x, y, z} \tilde{f}(y, z, x) = \sum_{t \in [T]} q_t - \sum_{t \in [T]} b_t + \sum_{i \in [m]} y_i h_t^t(\eta_i t) + \sum_{i \in [m]} \Delta_t \sum_{k \in [K]} \gamma_{ik} h^t \zeta_{ik}
$$

s.t. Constraints: (1) - (2) - (3) - (4)

\( z \in \{0, 1\}^{m \times K}, y \in \mathcal{Y}, x \in \mathcal{X} \).

Then, it can be seen that the objective function of (9) is concave in \( z, y \). Thus, the approximated problem can also be solved by a multi-cut outer-approximation algorithm [Mai and Lodi 2020]. The idea is to introduce variable \( \theta_t \) to represent each component of the objective function and write (9) as

$$
\max_{x, y, z} \tilde{f}(y, z, x) = \sum_{t \in [T]} q_t - \sum_{t \in [T]} q_t U_C^t \theta_t
$$

(10)

s.t. \( \theta_t \geq g_t(y, z) \)

(11)

Constraints: (1) - (2) - (3) - (4)

\( z \in \{0, 1\}^{m \times K}, y \in \mathcal{Y}, x \in \mathcal{X} \),

where \( g_t(y, x) = \frac{\eta_t + \sum_{i \in [m]} y_i h_t^t(\eta_i t) + \frac{1}{2} \sum_{i \in [m]} \Delta_t \sum_{k \in [K]} \gamma_{ik} h^t \zeta_{ik}}{U_C^t + \sum_{i \in [m]} y_i \exp \left( \frac{x_i \theta_i + \kappa_t}{U_C^t} \right)} \) is convex in \( y, z \). This convexity allows us to replace Constraints (11) by valid subgradient cuts of the form \( \theta_t \geq g_t(\bar{y}, \bar{z}) + \nabla_y g_t(\bar{y}, \bar{z})^T(y - \bar{y}) + \nabla_z g_t(\bar{y}, \bar{z})^T(z - \bar{z}) \), where \( (\bar{\theta}, \bar{y}, \bar{z}) \) is a candidate solution such that Constraints (11) are not satisfied. An
outer-approximation algorithm can be executed in an iterative manner. At each iteration $j$, we solve (10) where Constraints (11) are replaced by subgradient cuts and obtain $(\theta^j, y^j, z^j, \mathbf{z}^j)$. For each $t$ such that $\theta^j_t > g_t(y^j, z^j) + \epsilon$, we add subgradient cuts to (11). If $\theta^j_t \leq g_t(y^j, z^j) + \epsilon$ for all $t \in [T]$ ($\epsilon$ is a chosen stopping threshold), we stop the algorithm and return the current solution candidate. It can be guaranteed that such an outer-approximation will converge to an optimal solution to (10) (Duran and Grossmann [1986], Mai and Lodi [2020]). So, in summary, besides using an off-the-shelf solver (e.g., GUROBI) to solve the MILP, MISOCP and Bilinear reformulations in Section 2, the approximate problem can be exactly handled by an outer-approximation algorithm. \textbf{It is worth mentioning that, our work marks the first time the cost optimization aspect (with continuous cost variables) is addressed in the context of the MCP under random utilities.}

\subsection{4.2. Joint Assortment and Price Optimization}

We briefly describe the joint assortment and prove optimization problem under the mixed-logit model, noting that the same notations are reused as in the previous section to reduce the notational burden, although these notations may carry different meanings in this particular section. Let $[m]$ be the set of products. There is a non-purchase item indexed by 0. Let $x_i$ be the price of product $i \in [m]$. For each customer segment $t \in [T]$, each product is associated with an utility $v_{ti} = x_i \eta_{ti} + \kappa_{ti}$, where $\eta_{ti}$ is a price sensitivity parameter of product $i$, which should take a negative value, and $\kappa_{ti}$ are other parameters that affect the choices (e.g., values that capture other product features). The purchasing probability of a product $i$ can be computed as

$$P \left( i \mid [m] \cup \{0\} \right) = \frac{\exp(x_i \eta_{ti} + \kappa_{ti})}{1 + \sum_{i \in [m]} \exp(x_i \eta_{ti} + \kappa_{ti})},$$

where the value 1 in the denominator refers to the utility of the non-purchase option. The joint assortment and price optimization problem under the mixed logit model has the following form

$$\max_{y \in Y, x \in \mathcal{X}} \left\{ f(y, x) = \sum_{t \in [T]} \sum_{i \in [m]} y_i x_i \exp \left( x_i \eta_{ti} + \kappa_{ti} \right) \right\}, \quad \text{(Assort-Price)}$$

where the objective $f(y, x)$ is an expected revenue, $y_i, x_i$ are assortment and price decisions, respectively. To apply the approximation methods above, we can simply let $g^*_t(x_i) = x_i \exp \left( x_i \eta_{ti} + \kappa_{ti} \right)$ and $h^*_t(x_i) = \exp \left( x_i \eta_{ti} + \kappa_{ti} \right)$, noting that the assumptions in Proposition 3 holds as $h^*_t(x_i)$ is monotonic decreasing, thus MISOCP-2 can be used. Some relevant constraints would be $\sum_{i \in [m]} y_i \leq M$ (i.e., a cardinality constraint on the number of products offered), lower and upper bounds ($l_i, u_i$) for the prices. In some contexts one may require that a total price of a product bundle to be less than an upper bound, i.e., $\sum_{i \in S} y_i x_i \leq W$ for some $S \subset [m]$. Under such constraints, we can see that Assumption (A2) satisfies, thus the simplified
approximation version in \textit{Approx-1} can be used. Note that such a constraint $\sum_{i \in S} y_i x_i \leq W$ can be easily linearized using McCormick inequalities \cite{McCormick1976}. Moreover, in this context, the objective function of the approximation problem is not concave in $y, z$, thus the outer-approximation framework does not apply. We also note that while joint assortment and price optimization has been extensively studied in the literature, most relevant existing work primarily focuses on single-ratio formulations with relatively simple constraints or without any constraints on prices \cite{Wang2012, GallegoWang2014}. In contrast, our work addresses a broader and more general setting by tackling the joint problem under the mixed-logit model, allowing for any linear constraints to be imposed on both the assortment and prices.

5. Numerical Experiments

5.1. Experimental settings

The experiments are conducted on a PC with processors Intel(R) Core(TM) i7-9700 CPU @ 3.00GHz, RAM of 16 gigabytes, and operating system Window 10. The code is in C++ and links to GUROBI 11.0.3 (under default settings) to solve the MILP, MISOCP and Bilinear programs. We set the CPU time limit for each instance as 3600 seconds, i.e., we stop the algorithms/solver if they exceed the time budget and report the best solutions found. We provide numerical experiments based on two applications: assortment and price optimization (A&P), and maximum capture problem (MCP). The SSG problem shares the same structure with that of the A&P, thus we do not consider it in these experiments.

We will compare our approaches (the MILP, MISOCP, and bilinear reformulations of our approximate problem, solver by GUROBI) against the following two baselines:

- **GUROBI’s PWLA**: Recent developments in GUROBI have introduced several built-in functions for approximating commonly-used nonlinear univariate functions, such as $e^x, a^x, x^a, \sin(x), \cos(x)$, using PWLA techniques. While these features do not directly apply to our problem, as GUROBI cannot directly handle fractional objective functions, we have found a way to leverage GUROBI’s PWLA components by using a bilinear reformulation in \textit{Bilinear}. This approach involves converting the fractional objective in \textit{SoR} into a bilinear form and utilizing GUROBI’s built-in function to approximate the exponential components. Details of this approach for the A&P and MCP can be found in Appendix B.

The key difference between this approach and our PWLA method in \textit{Approx} is that GUROBI’s PWLA approximates each exponential function via separate sets of built-in piecewise linear constraints, rather than using a single set of variables $\{z_{ik}, i \in [m], k \in [K]\}$ as we do. As a result, the number of additional binary variables in GUROBI’s PWLA is proportional to both the number of original variables $x_i$ (for all $i \in [m]$) and the number of exponential terms in the objective function. In contrast, our PWLA approach only discretize the original continuous variables $\{x_i, i \in [m]\}$, resulting in a more compact formulation and fewer binary variables than the GUROBI-based PWLA.
- **SCIP (Bolusani et al. 2024):** The second baseline method is using SCIP - one of the best solvers for mixed-integer nonlinear programs (Bolusani et al. 2024). We use this solver to directly solve the original nonlinear problems (MCP) and (Assort-Price). The formulation is generated by C++ and linked to SCIP version 9.1.1 for solving. To guarantee the fairness of comparison, we set the minimum number of threads used in SCIP to 8 (SCIP uses a single thread under the default setting) and unlimited number of threads can be parallelly used.

To streamline the presentation of comparisons, we denote the approaches based on the MISOCP reformulation, Bilinear reformulation in (Bilinear), and GUROBI’s PWLA as MIS, BL, and GP, respectively. Additionally, the SCIP-based approach is labeled as SCIP. To provide an overview of the formulation sizes across different approaches, Table 1 summarizes the number of variables and constraints based on $T$ and $m$. It is evident that the MILP and MIS formulations are the most complex, requiring the highest number of variables and constraints. In contrast, the formulation used in the SCIP solver introduces minimal additional elements, adding only one extra variable and constraint to (SoR) to convert the objective function into a nonlinear constraint. Although the GUROBI’s PWLA-based formulation appears to involve fewer variables and constraints compared to the MILP, MIS, and BL methods, the execution of the built-in function addGenConstrExp() introduces several new binary variables and constraints. This results in an overall formulation that can become significantly larger than that of the MILP, MIS, or BL approaches.

| Table 1  | Problem formulation sizes used in MILP, MIS, BL, GP, and SCIP approaches |
|----------|--------------------------------------------------------------------------|
|          | #Variables | #Constraints       | #Variables | #Constraints       |
|          | Binary     | Continuous         | Binary     | Continuous         |
| MILP     | $m + mK$   | $T + m + Tm + TmK$ | $m + mK$   | $T + 2m + mK + 4Tm + 4TmK + 2$ |
| MIS      | $m + mK$   | $2T + m + Tm + TmK$ | $m + mK$   | $2T + m + Tm + TmK$ |
| BL       | $m + mK$   | $2T + m + 3m + mK + 2$ | $m + mK$   | $3T + m + mK$ |
| GP       | $m$        | $3T + m + 2Tm$     | $m$        | $3T + m + 3Tm$    |
| SCIP     | $m$        | $m + 1$            | $m$        | $m + 1$           |

The experimental results for the two baselines were obtained using the same PC that was used to run the MILP, MISOCP, and Bilinear programs. We use the same number of PWLA segments for both our methods and the GP approach. The code and datasets used for these experiments are available at: [https://github.com/giangph-or/BCSO](https://github.com/giangph-or/BCSO).

### 5.2. Solution Quality as $K$ Increases.

In this first experiment, we study the performance of our discretization approach as a function of $K$, aiming to explore how we should select $K$ to practically achieve desirable solutions. Note that the sizes of the MILP, MISOCP and Bilinear reformulations are proportional to $K$, so it is important to have a good selection for $K$ to balance the trade-off between computational cost and solution quality. To assess the solution quality with different values of $K$, we randomly generate A&P instances of $T = 5$, $m = 10$, $C \in [4, 6]$ and $M \in [3, 5]$. For each instance, we vary $K$ from 1 to 50 and solve the approximate problem to get solutions.
(y^K, x^K). We then compute the percentage gaps between \( f(y^K, x^K) \) and \( f(y^{K_0}, x^{K_0}) \), where \( K_0 \) is chosen to be sufficiently large to ensure that \( f(y^{K_0}, x^{K_0}) \) is almost equal to the optimal value. Here, we choose \( K_0 = 100 \), noting that the values of \( f(y^K, x^K) \) remain almost the same for any \( K > K_0 \). In Figure 1 we plot the means and standard errors of the percentage gaps (%) over 10 random instances, where the solid curve shows the means, and the shaded area represents the standard errors. We can see that the percentage gaps become about less than 0.4% with \( K = 10 \), and are just below 0.03% with \( K = 25 \). This indicates that \( K = 25 \) would be sufficient to achieve very good approximations. According to this experiment, we will use \( K = 25 \) for the next experiments.

![Figure 1](image)

**Figure 1** Performance as \( K \) increases.

### 5.3. Comparison Results

We start with the joint location and cost optimization problem (i.e., MCP). We introduce two constraints, namely, a cardinality constraint on the number of selected locations, which can be represented as \( \sum_{i \in [m]} y_i \leq M \), and an upper limit on the overall cost spent for the new facilities \( \sum_{i \in [m]} y_i x_i \leq C \). As stated earlier, apart from the MILP, MISOCP and Bilinear reformulations, the problem can also be solved by an outer-approximation algorithm (denoted as OA). We therefore conduct numerical comparisons of the six approaches: OA, MILP, MIS, BL, GP, and SCIP solving the original formulation (denoted by SCIP). For each set \((T, m, C, M)\), we generate 3 independent instances and solve them by the six methods, then report the number of times each method returns the optimal objective values, as well as optimality gaps reported when the solver terminates. The average runtime is calculated based on the runtime of independent instances which are optimally solved in each set \((T, m, C, M)\), while the average gap is taken from all of 3 instances. The comparison results are shown in Tables 2 and 3 where “-” indicates that the solver exceeds the time budget (i.e., 1 hour) and was forced to stop, and the best results are shown in bold. In Table 2, we present the results for instances with a large value of \( T \) (up to 100), while maintaining a small or medium...
number of locations $m$. In contrast, Table 3 reports the results for instances with a large number of locations ($m$ varies from 200 to 1000), while keeping $T$ at a small value of 10. Note that the MILP and SCIP solver cannot solve any instance with $T = 10$, therefore, the results of these methods are not included in Table 3.

It is not surprising to see that MILP and GP are outperformed by the other methods, in terms of solution quality. The three best approaches are BL, OA and MIS, respectively. These methods solve all instances
to optimal, and the BL provides the shortest runtime, followed by OA and MIS. Interestingly, MIS clearly outperforms MILP in terms of both solution quality and computing time — MIS is able to return best objective values for all the instances and the maximum running time is just about 37.68 seconds, while MILP cannot return the best objectives for several large-sized instances and always exceeds the time budget of 3600 seconds. This would be due to the concavity of the objective function and the fact that the function $h^i(x_i)$ is monotonic increasing and the MIS reformulation (MISOCP-1) can be used, instead of MISOCP-2. It can be seen that (MISOCP-1) has no McCormick inequalities, thus yields tighter relaxations than MISOCP-2.

| $T$ | $m$ | $C$ | $M$ | MILP | MIS | BL | GP | SCIP | #Solved to optimal | Average runtime (s) | Average gap (%) |
|-----|-----|-----|-----|------|-----|-----|-----|------|-------------------|--------------------|-----------------|
| 10  | 4   | 3   | 3   | 3   | 3   | 3   | 3   | 3    | 1.84              | 1.63               | 0.36            |
| 6   | 3   | 3   | 3   | 3   | 3   | 3   | 3   | 3    | 2.31              | 2.09               | 1.12            |
| 3   | 3   | 3   | 3   | 3   | 3   | 3   | 3   | 3    | 1.79              | 2.32               | 0.36            |
| 5   | 3   | 3   | 3   | 3   | 3   | 3   | 3   | 3    | 3.56              | 3.87               | 1.53            |
| 8   | 6   | 3   | 3   | 3   | 3   | 3   | 3   | 3    | 28.96             | 32.83              | 4.97            |
| 10  | 6   | 3   | 3   | 3   | 3   | 3   | 3   | 3    | 135.42            | 174.53             | 30.83           |
| 12  | 6   | 3   | 3   | 3   | 3   | 3   | 3   | 3    | 39.17             | 38.50              | 8.48            |
| 10  | 10  | 3   | 3   | 3   | 3   | 3   | 3   | 3    | 515.15            | 226.25             | 99.99           |
| 20  | 16  | 0   | 0   | 3   | 3   | 3   | 3   | 3    | -                 | 60.25              | 13.69           |
| 25  | 0   | 0   | 3   | 3   | 3   | 3   | 3   | 3    | -                 | 71.74              | 13.24           |
| 16  | 0   | 0   | 3   | 3   | 3   | 3   | 3   | 3    | -                 | 141.85             | 62.84           |
| 25  | 0   | 0   | 3   | 3   | 3   | 3   | 3   | 3    | -                 | 117.25             | 39.79           |
| 42  | 23  | 0   | 0   | 3   | 3   | 3   | 3   | 3    | -                 | 423.85             | 316.58          |
| 35  | 0   | 0   | 3   | 3   | 3   | 3   | 3   | 3    | -                 | 590.52             | 299.46          |
| 23  | 0   | 0   | 3   | 3   | 3   | 3   | 3   | 3    | -                 | 297.64             | 552.75          |
| 35  | 0   | 0   | 3   | 3   | 3   | 3   | 3   | 3    | -                 | 204.49             | 1034.48         |
| 100 | 33  | 0   | 0   | 3   | 2   | 0   |    |      | -                 | 312.04             | 295.18          |
| 50  | 0   | 0   | 3   | 0   | 0   |    |      |      | -                 | 167.83             | -               |
| 60  | 0   | 0   | 3   | 0   | 0   |    |      |      | -                 | 46.12              | -               |
| 50  | 0   | 0   | 3   | 0   | 0   |    |      |      | -                 | 66.77              | -               |
| 80  | 66  | 0   | 0   | 3   | 0   | 0   |    |      | -                 | 185.37             | -               |
| 100 | 0   | 0   | 3   | 0   | 0   |    |      |      | -                 | 252.35             | -               |
| 66  | 0   | 0   | 3   | 0   | 0   |    |      |      | -                 | 406.01             | -               |
| 100 | 0   | 0   | 3   | 0   | 0   |    |      |      | -                 | 113.27             | -               |
| 120 | 0   | 0   | 3   | 0   | 0   |    |      |      | -                 | 24.22              | 0.02            |
| 10  | 4   | 3   | 3   | 3   | 3   | 3   | 3   | 3    | 4.32              | 4.66               | 0.46            |
| 5   | 3   | 3   | 3   | 3   | 3   | 3   | 3   | 3    | 12.91             | 11.60              | 2.95            |
| 3   | 3   | 3   | 3   | 3   | 3   | 3   | 3   | 3    | 4.77              | 4.97               | 0.63            |
| 6   | 3   | 3   | 3   | 3   | 3   | 3   | 3   | 3    | 10.29             | 9.30               | 6.86            |
| 8   | 6   | 3   | 3   | 3   | 3   | 3   | 3   | 3    | 272.58            | 284.30             | 23.02           |
| 10  | 6   | 3   | 3   | 3   | 3   | 3   | 3   | 3    | 2539.87           | 2518.96            | 324.10          |
| 12  | 6   | 3   | 3   | 3   | 3   | 3   | 3   | 3    | 142.63            | 147.38             | 36.05           |
| 10  | 10  | 2   | 2   | 1   | 3   | 0   |    |      | 1687.05           | 2370.60            | 2057.87         |
| 16  | 0   | 0   | 1   | 2   | 0   |    |      |      | -                 | 702.08             | 368.35          |
| 25  | 0   | 0   | 0   | 3   | 0   |    |      |      | -                 | -                 | 1411.17         |
| 16  | 0   | 0   | 2   | 0   | 0   |    |      |      | -                 | 219.66             | -               |
| 25  | 0   | 0   | 2   | 0   | 0   |    |      |      | -                 | 787.94             | -               |
| 40  | 33  | 0   | 0   | 0   | 0   | 0   |    |      | -                 | -                 | -               |
| 50  | 0   | 0   | 2   | 0   | 0   | 0   |    |      | -                 | 2331.63            | -               |
| 60  | 0   | 0   | 0   | 0   | 0   | 0   |    |      | -                 | 2402.24            | -               |
| 100 | 0   | 0   | 0   | 0   | 0   | 0   |    |      | -                 | 2402.24            | -               |

Summary: 46 46 101 75 21

Let us now shift our attention to the A&P problem. These A&P instances pose a significantly greater challenge in terms of solving, especially when compared to the MCP ones. This heightened complexity...
primarily arises from the non-convex nature of the approximated problem. We, therefore, adopt a small value of $T$ small, specifically setting it to 2 and 5, while varying the number of products $m$ up to 200. Similar to the MCP instances, we introduce two constraints, i.e., a cardinality constraint on the size of the selected assortment $\sum_{i \in [m]} y_i \leq M$, and an upper bound constraint on a weighted sum of the prices $\sum_{i \in [m]} \alpha_i y_i x_i \leq C$, where $\alpha_i$ take random values in $[0.5, 1]$. The second constraint can be described as one that mandates the total price of a given set of offered products to remain below a specified upper limit. For each group of $(m, C, M)$, we randomly generate 3 instances and report the numbers of instances that are solved to optimality.

![Figure 2](image)

**Figure 2** Average gap of objective value provided by two best approaches, BL and GP, compared to SCIP solver on the A&P instances; instances are grouped by $(T, m, C, M)$.

It is clear that **BL** outperforms other methods in terms of solution quality, and the **MILP** and **MIS** perform worse than Gurobi’s PWLA (i.e., **GP**) in returning good solutions. In cases of **MILP** and **MIS**, large number of McCormick inequalities limits their performances when $m \geq 50$. The **GP** is the fastest approach for solving instances with $m \leq 50$, however, it cannot handle instances with large $m$ because of the increment of complexity inside the built-in approximation process as we mentioned at the beginning of this section. Figure 2 also shows that the global optimal objective value provided by **SCIP** is better than the approximated objective of the **BL** and **GP** when $m \leq 20$, however, the maximum gap is insignificant ($\leq 0.21\%$). For sets of instances with $m > 20$, the objective values of **BL** and **GP** are $0.1\%-3.5\%$ higher than ones given by **SCIP** when time limit is exceeded.
6. Conclusion

We studied a class of non-convex binary-continuous sun-of-ratios programs that can be used for decision-making in prominent application domains such as assortment and price optimization or maximum capture facility location, or security games. We developed a discretization technique that allows us to linearize the nonlinear numerators and denominators of the objective function, which further opens up the possibility of reformulating the nonlinear problem as MILP, MISOCP or Bilinear programs. Subsequently, we furnished performance guarantees for solutions derived from resolving the approximated problem and provided estimates for $K$ (i.e., the number of discretization points) to ensure near-optimality. Numerical experiments based on assortment and price optimization, and facility and cost optimization instances show the efficiency of our approximation approaches. Notably, our PWLA approach combined with the bilinear reformulation demonstrated a significant performance advantage over several baseline methods, including those based on GUROBI’s PWLA and the general mixed-integer nonlinear solver SCIP. Future directions would be to explore other classes of problems, e.g., those resulting from the use of other discrete choice models such as the nested, cross-nested logit (Train 2003) or the network-based Generalized Extreme Value model (Daly and Bierlaire 2006, Mai et al. 2017).

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Appendix A: Missing Proofs

Appendix A provides detailed proofs that were omitted from the main paper, while Appendix B presents reformulations that facilitate the use of GUROBI’s PWLA to solve the A&P and MCP problems to near-optimality.

A.1. Proof of Proposition 1

Let \((\hat{y}, \hat{z}, \hat{x})\) be an optimal solution to \((\text{Approx})\). We define \((\bar{y}, \bar{z}, \bar{x})\) as

\[
\bar{y} = \hat{y},
\]

\[
\bar{z}_{ik} = \begin{cases} 
0 & \text{if } \hat{y}_i = 0 \\
\hat{z}_{ik} & \text{if } \hat{y}_i = 1
\end{cases} \forall i \in [m], k \in [K],
\]

\[
\bar{x}_i = l_i + \Delta_i \sum_{k \in [K]} \hat{z}_{ik}; \forall i \in [m].
\]

Let \((\bar{y}^*, \bar{z}^*, \bar{x}^*)\) be an optimal solution to \((\text{Approx-2})\). We will prove that this solution is also optimal to \((\text{Approx})\). Since \((\bar{y}^*, \bar{z}^*, \bar{x}^*)\) is feasible to \((\text{Approx})\), we have

\[
\hat{f}(\bar{y}^*, \bar{z}^*, \bar{x}^*) \leq \hat{f}(\bar{y}, \bar{z}, \bar{x})
\]

From Assumption (A2), we see that \(\bar{x}_i \leq \bar{z}_i\) for all \(i \in [m]\), thus \((\bar{y}, \bar{z}, \bar{x})\) is also feasible to \((\text{Approx-2})\). Thus,

\[
\hat{f}(\bar{y}^*, \bar{z}^*, \bar{x}^*) \geq \hat{f}(\bar{y}, \bar{z}, \bar{x}).
\]

We now see that if \(\hat{y}_i = 0\) then \(\bar{z}_{ik} = 0\) for all \(i \in [m], k \in [k]\). Thus \(\hat{y}_i \bar{z}_{ik} = 0\). Same for \((\bar{y}^*, \bar{z}^*, \bar{x}^*)\) we also have \(\hat{y}_i \bar{z}_{ik} = 0\). Moreover, from the selection of \((\bar{y}, \bar{z}, \bar{x})\) we have \(\hat{y}_i \bar{z}_{ik} = \bar{y}_i \bar{z}_{ik}\). Putting all together we have

\[
\hat{f}(\bar{y}^*, \bar{z}^*, \bar{x}^*) = \hat{f}(\bar{y}, \bar{z}, \bar{x})
\]

\[
\hat{f}(\bar{y}, \bar{z}, \bar{x}) = \hat{f}(\bar{y}, \bar{z}, \bar{x})
\]

\[
\hat{f}(\bar{y}, \bar{z}, \bar{x}) = \hat{f}(\bar{y}, \bar{z}, \bar{x}).
\]

Combine (12)-(16) we get

\[
\hat{f}(\bar{y}^*, \bar{z}^*, \bar{x}^*) = \hat{f}(\bar{y}, \bar{z}, \bar{x}),
\]

which implies that \((\bar{y}^*, \bar{z}^*, \bar{x}^*)\) is also an optimal solution to \((\text{Approx})\), as desired. Q.E.D.

A.2. Proof of Proposition 2

We first convert \((\text{Approx-2})\) to a minimization problem and write it equivalently as

\[
\min_{(y, x) \in \Theta} \left\{ - \sum_{t \in [T]} a_t + \sum_{i \in [m]} y_i g_i^t(l_i) + \sum_{i \in [m]} \sum_{k \in [K]} \gamma_{ik}^t z_{ik} \right\}
\]

\[
\Leftrightarrow \min_{(y, x) \in \Theta} \left\{ \sum_{i \in [m]} \lambda_i b_i - a_t + \sum_{i \in [m]} y_i (\lambda_i h_i^t(l_i) - g_i^t(l_i)) + \sum_{i \in [m]} \sum_{k \in [K]} \gamma_{ik}^t (\lambda_i \gamma_{ik}^t - \gamma_{ik}^t) z_{ik} - \sum_{t \in [T]} \lambda_i \right\}
\]

\[
\text{with } \lambda_i = \frac{1}{\Delta_i}.
\]
where $\Theta$ is the feasible set of $\text{Approx-2}$ for notational simplicity and $\lambda_t$ are selected in such a way that

$$
\lambda_t h^t_i(l_i) > g^t_i(l_i); \forall t \in [T], i \in [m]
$$

$$
\lambda_t \gamma_{ik}^t > \gamma_{ik}^{gt}; \forall t \in [T], i \in [m], k \in [K]
$$

$$
\lambda_t \left( b_t + \sum_{i \in [m]} y_i h^t_i(l_i) + \sum_{i \in [m]} \sum_{k \in [K]} \gamma_{ik}^t z_{ik} \right) \geq a_t + \sum_{i \in [m]} y_i g^t_i(l_i) + \sum_{i \in [m]} \sum_{k \in [K]} \gamma_{ik}^{gt} z_{ik}
$$

which is always possible due to our assumptions. Now given the choices of $\lambda_t$, we let

$$
w_t = \frac{1}{b_t + \sum_{i \in [m]} y_i h^t_i(l_i) + \sum_{i \in [m]} \sum_{k \in [K]} \gamma_{ik}^t z_{ik}}; \text{ and } \theta_t = 1/w_t
$$

We then write (17) as

$$
\begin{align*}
\min_{\mathbf{y}, \mathbf{x} \in \Theta} & \quad \left\{ \sum_{t \in [T]} w_t \left( \lambda_t b_t - a_t + \sum_{i \in [m]} y_i (\lambda_t h^t_i(l_i) - g^t_i(l_i)) + \sum_{i \in [m]} \sum_{k \in [K]} (\lambda_t \gamma_{ik}^t - \gamma_{ik}^{gt}) z_{ik} \right) - \sum_{t \in [T]} \lambda_t \right\} \\
\text{s.t.} & \quad \theta_t = b_t + \sum_{i \in [m]} y_i h^t_i(l_i) + \sum_{i \in [m]} \sum_{k \in [K]} \gamma_{ik}^t z_{ik}, \forall t \in [T] \\
& \quad w_t \theta_t = 1; \forall t \in [T]
\end{align*}
$$

(18)

We first observe that, due to the way we choose $\lambda_t$, we have

$$
\left( \lambda_t b_t - a_t + \sum_{i \in [m]} y_i (\lambda_t h^t_i(l_i) - g^t_i(l_i)) + \sum_{i \in [m]} \sum_{k \in [K]} (\lambda_t \gamma_{ik}^t - \gamma_{ik}^{gt}) z_{ik} \right) \geq 0
$$

which implies that the equalities (19) can be safely replaced by rotated cones $w_t \theta_t \geq 1; \forall t \in [T]$. We also let $v_{ti} = w_t y_i$ and $u_{ti} = w_t z_{ik}$, $\forall t, i, k$ and write (18) as

$$
\begin{align*}
\min_{\mathbf{y}, \mathbf{x} \in \Theta} & \quad \left\{ \sum_{t \in [T]} w_t (\lambda_t b_t - a_t) + \sum_{i \in [m]} v_{ti} (\lambda_t h^t_i(l_i) - g^t_i(l_i)) + \sum_{i \in [m]} \sum_{k \in [K]} (\lambda_t \gamma_{ik}^t - \gamma_{ik}^{gt}) u_{ti} - \sum_{t \in [T]} \lambda_t \right\} \\
\text{s.t.} & \quad \theta_t = b_t + \sum_{i \in [m]} y_i h^t_i(l_i) + \sum_{i \in [m]} \sum_{k \in [K]} \gamma_{ik}^t z_{ik}, \forall t \in [T] \\
& \quad w_t \theta_t \geq 1; \forall t \in [T] \\
& \quad v_{ti} = w_t y_i \forall t \in [T], i \in [m] \\
& \quad u_{ti} = w_t z_{ik} \forall t \in [T], i \in [m], k \in [K]
\end{align*}
$$

(20)

We now see that the equalities (20) can be safely replaced by $v_{ti} \theta_t \geq y_i$, which is possible because the coefficients associated with $v_{ti}$ in the objective function (i.e. $\lambda_t h^t_i(l_i) - g^t_i(l_i)$) is positive, thus we always want $v_{ti}$ to be as small
as possible. Moreover, \( v_t \theta_t \geq y_t \) is also equivalent to the rotated cones \( v_t \theta_t \geq y_t^2 \). Similarly, we can replace (21) by \( u_{tik} \theta_t \geq z_{ik}^2 \). In summary, we can reformulate \( \text{Approx-2} \) as the MISOCP program

\[
\min_{(y,x) \in \Theta} \left\{ \sum_{t \in [T]} w_t (\lambda_t b_t - a_t) + \sum_{i \in [m]} v_t (\lambda_t h_i^t (x_i) - g_i^t (x_i)) + \sum_{i \in [m]} \sum_{k \in [K]} (\lambda_t h^t_{ik} - g_{ikt}^t) u_{tik} - \sum_{i \in [T]} \lambda_t \right\}
\]

s.t. \( \theta_t = b_t + \sum_{i \in [m]} y_i h_i^t (x_i) + \sum_{i \in [m]} \sum_{k \in [K]} g_{ikt}^t z_{ikt} \), \( \forall t \in [T] \)

\( w_t \theta_t \geq 1; \forall t \in [T] \)

\( v_t \theta_t \geq y_t^2 \) \( \forall t \in [T], i \in [m] \)

\( u_{tik} \theta_t \geq z_{ik}^2 \) \( \forall t \in [T], i \in [m], k \in [K] \)

which completes the proof. Q.E.D.

### A.3. Proof of Lemma 2

For ease of notation, let

\[
G^t(x) = a_t + \sum_{i \in [m]} y_i g_i^t(x_i)
\]

\[
H^t(x) = b_t + \sum_{i \in [m]} y_i h_i^t(x_i)
\]

\[
F^t(x) = \frac{G^t(x)}{H^t(x)}
\]

We first note that, due to the lipschitzness of \( g_i^t(x_i) \) and \( h_i^t(x_i) \), for all \( i \in [m] \), we have

\[
\left| G^t(x) - G^t(x') \right| \leq \sum_{i \in [m]} y_i \left| g_i^t(x_i) - g_i^t(x_i') \right| \leq \sum_{i \in [m]} L_i^g \epsilon \overset{\text{def}}{=} \epsilon^G
\]

\[
\left| H^t(x) - H^t(x') \right| \leq \sum_{i \in [m]} y_i \left| h_i^t(x_i) - h_i^t(x_i') \right| \leq \sum_{i \in [m]} L_i^h \epsilon \overset{\text{def}}{=} \epsilon^H
\]

We first bound the gap \( |F^t(y, x) - F^t(y, x')| \) by a function of \( \epsilon \). We write

\[
|F^t(y, x) - F^t(y, x')| = \left| \frac{G^t(x)H^t(x') - G^t(x')H^t(x)}{H^t(x)H^t(x')} \right|
\]

with a note that, from Assumption 1, \( H^t(x) \) and \( H^t(x') \) are positive. We see that \( H^t = \min_{(y,x) \in S} H^t(x) \) and \( \overline{F}^t = \max_{(y,x) \in S} |G^t(x)/H^t(x)| \). We consider the following two cases:
• If $G^i(x)H^i(x') \geq G^i(x')H^i(x)$, then we write

$$|F^i(y, x) - F^i(y, x')| \leq \frac{G^i(x)H^i(x) - G^i(x')H^i(x)}{H^i(x)H^i(x')} \leq \frac{(G^i(x) + \epsilon^G)H^i(x') - G^i(x')H^i(x)}{H^i(x)H^i(x')}$$

$$\leq \frac{\epsilon^G}{H^i(x')} + \frac{G^i(x)}{H^i(x)H^i(x')} (H^i(x') - H^i(x)) \leq \frac{\epsilon^G}{H^i(x')} + \frac{\epsilon^H}{H^i(x')} \epsilon^H$$

$$(b) \leq \epsilon \left( \sum_i L^i_{t^i} + \frac{F'}{H^i} \sum_{i \in [m]} L^i_{t^i} \right)$$

where $(a)$ is because $|H^i(x') - H^i(x)| \leq \epsilon^H$ and $(b)$ is due to the definition of $\epsilon^G, \epsilon^H$ in (22) and (23).

• For the other case when $G^i(x)H^i(x') \leq G^i(x')H^i(x)$, we also write

$$|F^i(y, x) - F^i(y, x')| \leq \frac{G^i(x')H^i(x) - G^i(x)H^i(x)}{H^i(x)H^i(x')} \leq \frac{(G^i(x) + \epsilon^G)H^i(x) - G^i(x)H^i(x')}{H^i(x)H^i(x')}$$

$$\leq \frac{\epsilon^G}{H^i(x')} + \frac{G^i(x)(H^i(x) - H^i(x'))}{H^i(x)H^i(x')} \leq \epsilon \left( \sum_i L^i_{t^i} + \frac{F'}{H^i} \sum_{i \in [m]} L^i_{t^i} \right)$$

Thus, for both cases we have

$$|F^i(y, x) - F^i(y, x')| \leq \epsilon \left( \sum_i L^i_{t^i} + \frac{F'}{H^i} \sum_{i \in [m]} L^i_{t^i} \right)$$

We are now ready to bound $|f(y, x) - f(y, x')|$ as

$$|f(y, x) - f(y, x')| \leq \sum_{i \in [T]} |F^i(y, x) - F^i(y, x')| \leq \epsilon \sum_{i \in [T]} \left( \sum_i L^i_{t^i} + \frac{F'}{H^i} \sum_{i \in [m]} L^i_{t^i} \right)$$

as desired \quad Q.E.D.

A.4. Proof of Lemma 3

It can be seen that if $(y, x, r)$ is feasible to (P1), then we have $k_i \in \{0, 1, \ldots, K\}$ such that $z_i - r_i = l_i + \Delta_i k_i$. Moreover, by selecting $z \in \{0, 1\}^{m \times K}$ such that $z_{ik} = 1$ if $k \leq k_i$ and $z_{ik} = 0$ if $k > k_i$, we see that $(y, x, z, r)$ is feasible to (Approx) and $f(y, x - r) = \hat{f}(y, z, r)$. On the other hand, if $(y, x, z, r)$ is feasible to (Approx), then $(y, x, r)$ is also feasible to (P1) and $f(y, x - r) = \hat{f}(y, z, r)$. All these imply that (P1) is equivalent to (Approx).

To prove that (Approx) is equivalent to $\max_{y \in Y, x \in X} \left\{ f(y, \delta^K(x)) \right\} | Ax + By \leq D \}$, we further see that, for any $(y, x, r)$ feasible to (P1), we have $\delta^K(x) = x - r$, thus $f(y, x - r) = f(y, \delta^K(x))$. On the other hand, for any $(y, x)$ feasible to $\max_{y \in Y, x \in X} \left\{ f(y, \delta^K(x)) \right\} | Ax + By \leq D \}$, by defining $r = x - \delta^K(x)$, we see that $(y, x, r)$ is feasible to (P1) and $f(y, x - r) = f(y, \delta^K(x))$. All these imply that $\max_{y \in Y, x \in X} \left\{ f(y, \delta^K(x)) \right\} | Ax + By \leq D \}$ is equivalent to (P1). It follows that (Approx) is equivalent to $\max_{y \in Y, x \in X} \left\{ f(y, \delta^K(x)) \right\} | Ax + By \leq D \}$ as desired.
A.5. Proof of Theorem 1

Since \((y^K, x^K)\) is an optimal solution to the approximation problem \((\text{Approx})\) and \((\text{Approx})\) is equivalent to \(\max_{y \in \mathcal{Y}, x \in \mathcal{X}} \{f(y, \delta^K(x)) \mid Ax + By \leq D\}\) (Lemma 3), we have

\[
(y^K, x^K) = \arg\max_{y \in \mathcal{Y}, x \in \mathcal{X}} \{f(y, \delta^K(x)) \mid Ax + By \leq D\}
\]

Since \((y^K, x^K)\) is feasible to \((\text{SoR})\), we first have the following inequality:

\[
f(y^*, x^*) \geq f(y^K, x^K)
\]  
(25)

On the other hand, \((y^*, x^*)\) is feasible to \(\max_{y \in \mathcal{Y}, x \in \mathcal{X}} \{f(y, \delta^K(x)) \mid Ax + By \leq D\}\). It then follows that

\[
f(y^K, \delta^K(x^K)) \geq f(y^*, \delta^K(x^*))
\]  
(26)

Moreover, Corollary 1 tells us that

\[
\begin{cases}
f(y^K, x^K) \geq f(y^K, \delta^K(x^K)) - \frac{C}{K} \max_{i \in [m]} (u_i - l_i) \\
f(y^*, \delta^K(x^*)) \geq f(y^*, (x^*)) - \frac{C}{K} \max_{i \in [m]} (u_i - l_i)
\end{cases}
\]

Combine this with (25) and (26) we have

\[
f(y^*, x^*) \geq f(y^K, x^K) \geq f(y^K, \delta^K(x^K)) - \frac{C}{K} \max_{i \in [m]} (u_i - l_i) \geq f(y^*, (x^*)) - \frac{2C}{K} \max_{i \in [m]} (u_i - l_i)
\]

which implies \(f(y^*, x^*) - f(y^K, x^K) \leq \frac{2C}{K} \max_{i \in [m]} (u_i - l_i)\) as desired.
Q.E.D.

A.6. Proof of Theorem 2

From Hoeffding’s inequality (Boucheron et al. 2013), for any \((y, x) \in \mathcal{S}\) and any \(\kappa > 0\), the following inequality holds with probability at least \(1 - 2 \exp\left(\frac{-2T \kappa^2}{\Psi^2}\right)\)

\[
|\mathcal{F}(y, x) - \mathcal{F}_T(y, x)| \leq \kappa
\]  
(27)

We first bound the gap \(|\mathcal{F}(y^*, x^*) - \mathcal{F}_T(y, \hat{x})|\) by considering two cases

- If \(\mathcal{F}(y^*, x^*) \geq \mathcal{F}_T(\hat{y}, \hat{x})\), the we have

\[
\mathcal{F}(y^*, x^*) \geq \mathcal{F}_T(\hat{y}, \hat{x}) \geq \mathcal{F}_T(y^*, x^*)
\]

Therefore, \(|\mathcal{F}(y^*, x^*) - \mathcal{F}_T(\hat{y}, \hat{x})| \leq |\mathcal{F}(y, x) - \mathcal{F}_T(y, x)|\)

- If \(\mathcal{F}(y^*, x^*) \leq \mathcal{F}_T(\hat{y}, \hat{x})\), then

\[
\mathcal{F}_T(\hat{y}, \hat{x}) \geq \mathcal{F}(y^*, x^*) \geq \mathcal{F}(\hat{y}, \hat{x})
\]

which also leads to \(|\mathcal{F}(y^*, x^*) - \mathcal{F}_T(\hat{y}, \hat{x})| \leq |\mathcal{F}(y, x) - \mathcal{F}_T(y, x)|\).

So, for any \(\kappa > 0\) we have

\[
\mathbb{P}\left(|\mathcal{F}(y^*, x^*) - \mathcal{F}_T(\hat{y}, \hat{x})| \leq \kappa\right) \geq 1 - 2 \exp\left(\frac{-2T \kappa^2}{\Psi^2}\right)
\]  
(28)
Moreover,

\[ |\mathcal{F}(\mathbf{y}^*, \mathbf{x}^*) - \mathcal{F}(\mathbf{y}, \mathbf{x})| \leq |\mathcal{F}(\mathbf{y}^*, \mathbf{x}^*) - \hat{\mathcal{F}}^T(\mathbf{y}, \mathbf{x})| + |\hat{\mathcal{F}}^T(\mathbf{y}, \mathbf{x}) - \mathcal{F}(\mathbf{y}, \mathbf{x})| \]

Thus,

\[
\mathbb{P}[|\mathcal{F}(\mathbf{y}^*, \mathbf{x}^*) - \mathcal{F}(\mathbf{y}, \mathbf{x})| \geq \epsilon] \leq \mathbb{P}
\left[ |\mathcal{F}(\mathbf{y}^*, \mathbf{x}^*) - \hat{\mathcal{F}}^T(\mathbf{y}, \mathbf{x})| \geq \epsilon/2 \right] + \mathbb{P}
\left[ |\hat{\mathcal{F}}^T(\mathbf{y}, \mathbf{x}) - \mathcal{F}(\mathbf{y}, \mathbf{x})| \leq \epsilon/2 \right] \leq 4 \exp\left( -\frac{T \epsilon^2}{2\Psi^2} \right)
\]

Now we bound the gap \(|\mathcal{F}(\mathbf{y}^*, \mathbf{x}^*) - \mathcal{F}(\mathbf{y}^K, \mathbf{x}^K)|\) by triangle inequality as

\[
|\mathcal{F}(\mathbf{y}^*, \mathbf{x}^*) - \mathcal{F}(\mathbf{y}^K, \mathbf{x}^K)| \leq |\mathcal{F}(\mathbf{y}^*, \mathbf{x}^*) - \mathcal{F}(\mathbf{y}, \mathbf{x})| + |\mathcal{F}(\mathbf{y}, \mathbf{x}) - \mathcal{F}(\mathbf{y}^K, \mathbf{x}^K)|
\]

\[
\leq |\mathcal{F}(\mathbf{y}^*, \mathbf{x}^*) - \mathcal{F}(\mathbf{y}, \mathbf{x})| + |\mathcal{F}(\mathbf{y}, \mathbf{x}) - \hat{\mathcal{F}}^T(\mathbf{y}, \mathbf{x})| + |\hat{\mathcal{F}}^T(\mathbf{y}, \mathbf{x}) - \mathcal{F}(\mathbf{y}^K, \mathbf{x}^K)|
\]

(29)

Thus, the large-deviation probabilities can be bounded as

\[
\mathbb{P}
\left[ |\mathcal{F}(\mathbf{y}^*, \mathbf{x}^*) - \mathcal{F}(\mathbf{y}^K, \mathbf{x}^K)| \leq \epsilon \right] \leq \mathbb{P}
\left[ |\mathcal{F}(\mathbf{y}^*, \mathbf{x}^*) - \mathcal{F}(\mathbf{y}, \mathbf{x})| \leq \frac{2\epsilon}{5} \right] + \mathbb{P}
\left[ |\mathcal{F}(\mathbf{y}, \mathbf{x}) - \hat{\mathcal{F}}^T(\mathbf{y}, \mathbf{x})| \leq \frac{\epsilon}{5} \right] + \mathbb{P}
\left[ |\hat{\mathcal{F}}^T(\mathbf{y}, \mathbf{x}) - \mathcal{F}(\mathbf{y}^K, \mathbf{x}^K)| \leq \frac{\epsilon}{5} \right] \leq 6 \exp\left( -\frac{2T \epsilon^2}{25\Psi} \right) + \mathbb{P}
\left[ |\hat{\mathcal{F}}^T(\mathbf{y}, \mathbf{x}) - \mathcal{F}(\mathbf{y}^K, \mathbf{x}^K)| \leq \frac{\epsilon}{5} \right]
\]

(30)

where (a) is obtained by replacing \(\kappa\) by \(\epsilon/5\) and \(2\epsilon/5\) in (27) and (28), respectively. We then can bound the last term of (30) by using Theorem 1. That is, the theorem tells us that

\[
|\hat{\mathcal{F}}^T(\mathbf{y}, \mathbf{x}) - \mathcal{F}(\mathbf{y}^K, \mathbf{x}^K)| \leq \frac{2C^*}{K} \max_{i \in [m]} (u_i - l_i)
\]

which implies

\[
|\hat{\mathcal{F}}^T(\mathbf{y}, \mathbf{x}) - \mathcal{F}(\mathbf{y}^K, \mathbf{x}^K)| \leq \frac{\epsilon}{5}
\]

if we choose

\[
\frac{\epsilon}{5} \geq \frac{2C^*}{K} \max_{i \in [m]} (u_i - l_i), \text{ or } K \geq 10TC^* \max_{i \in [m]} \{(u_i - l_i)\}.
\]

Combine this with (30), we complete the proof. Q.E.D.

A.7. Proof of Corollary 2

This is a direct result of Theorem 2, as we can see that if we choose

\[
T \geq \frac{25\Psi \ln(6/\gamma)}{2\epsilon^2}; \quad K \geq \frac{5TC^*}{\epsilon} \max_{i \in [m]} \{(u_i - l_i)\},
\]

then according to Theorem 2, we see that

\[
\mathbb{P}[|\mathcal{F}(\mathbf{y}^*, \mathbf{x}^*) - \mathcal{F}(\mathbf{y}^K, \mathbf{x}^K)| \leq \epsilon] \geq 1 - 6 \exp\left( -\frac{2T \epsilon^2}{25\Psi} \right) \geq 1 - \gamma
\]

as desired.
Appendix B: Reformulations for GUROBI’s PWLA

We present reformulations that enable the use of GUROBI’s PWLA built-in functions to solve the two problems under consideration (A&P and MCP) to near-optimality.

B.1. Assortment and Price Optimization

Let us first consider the assortment and price optimization problem:

\[
\begin{align*}
\max_{y \in \mathcal{Y}, x \in \mathcal{X}} & \quad f(y, x) = \sum_{t \in [T]} \frac{\sum_{i \in [m]} y_i x_i \exp\left(x_i \eta_{ti} + \kappa_{ti}\right)}{1 + \sum_{i \in [m]} y_i \exp\left(x_i \eta_{ti} + \kappa_{ti}\right)} \\
\text{s.t.} & \quad \sum_{t \in [T]} \theta_t \\ & \quad \theta_t \leq \frac{u_t}{v_t} \\
& \quad v_t = 1 + \sum_{i \in [m]} y_i \exp\left(x_i \eta_{ti} + \kappa_{ti}\right) \\
& \quad u_t = \sum_{i \in [m]} y_i x_i \exp\left(x_i \eta_{ti} + \kappa_{ti}\right) \\
& \quad A x + B y \leq D \\
& \quad z \in \{0, 1\}^{m \times K}, \ y \in \mathcal{Y}, x \in \mathcal{X}.
\end{align*}
\]

The bilinear terms with binary variables \(y_i w_{ti}\) can be linearized by letting \(s_{ti} = y_i w_{ti}\) and using McCormick inequalities as:

\[
\begin{align*}
L_{ti} y_i & \leq s_{ti} \leq U_{ti} y_i \\
w_{ti} - U_{ti} (1 - y_i) & \leq s_{ti} \leq w_{ti} - L_{ti} (1 - y_i)
\end{align*}
\]
where $L^w_{ti}$ and $U^w_{ti}$ are lower and upper bounds of $w_{ti}$. We then write the above formulation as follows:

$$\max_{x,y} \sum_{t \in [T]} \theta_t$$

subject to

$$\theta_t v_t \leq u_t, \quad \forall t \in [T]$$

$$w_{ti} = \exp \left( x_i \eta_{ti} + \kappa_{ti} \right)$$

$$v_t = 1 + \sum_{i \in [m]} s_{ti}$$

$$u_t = \sum_{i \in [m]} x_i s_{ti}$$

$$L^w_{ti} y_i \leq s_{ti} \leq U^w_{ti} y_i$$

$$w_{ti} - U^w_{ti} (1 - y_i) \leq s_{ti} \leq w_{ti} - L^w_{ti} (1 - y_i)$$

$$Ax + By \leq D$$

$$z \in \{0, 1\}^{m \times K}, \quad y \in \mathcal{Y}, \quad x \in \mathcal{X}.$$
Then we can reformulate the problem as follows:

$$\max_{x,y} \sum_{t \in [T]} \theta_t$$

(Bi-Exp-MCP)

s.t. $\theta_t v_t \leq u_t, \ \forall t \in [T]$ (34)

$w_{ti} = \exp(x_i \eta_{ti} + \kappa_{ti})$ (35)

$v_t = U_C^t + \sum_{i \in [m]} y_i w_{ti}$ (36)

$u_t = q_t \sum_{i \in [m]} y_i u_{ti}$ (37)

$Ax + By \leq D$

$z \in \{0, 1\}^{m \times K}, \ y \in Y, \ x \in X.$

Similarly in the (Bi-Exp-AP), the exponential terms in (35) and the bilinear terms in (34), (36) and (37) can be managed through piecewise linear approximation and bilinear programming techniques incorporated in recent versions of this solver.