QCD STRINGS AS CONSTRAINED
GRASSMANNIAN SIGMA MODEL

K.S. Viswanathan\textsuperscript{1} and R. Parthasarathy\textsuperscript{2}
Department of Physics
Simon Fraser University
Burnaby B.C., Canada V5A 1S6.

\textsuperscript{1}e-mail address: kviswana@sfu.ca
\textsuperscript{2}Permanent address: The Institute of Mathematical Sciences, Madras 600113, India.
e-mail address: sarathy@imsc.ernet.in
Abstract

We present calculations for the effective action of string world sheet in $R^3$ and $R^4$ utilizing its correspondence with the constrained Grassmannian $\sigma$-model. Minimal surfaces describe the dynamics of open strings while harmonic surfaces describe that of closed strings. The one-loop effective action for these are calculated with instanton and anti-instanton background, representing N-string interactions at the tree level. The effective action is found to be the partition function of a classical modified Coulomb gas in the confining phase, with a dynamically generated mass gap.
I. INTRODUCTION

Two dimensional non-linear sigma models share many features with 4-d Yang-Mills theories. They both are scale-invariant, asymptotically free and possess multi-instanton solutions [1]. In spite of these similarities, not much has been done to explore if there is a deeper relation between these theories. Over the years string models [2] have been proposed to describe QCD flux tubes, deemed responsible for quark confinement. It has been widely recognized that QCD strings should take into account the extrinsic geometry of the string world sheet [3]. General properties of strings with extrinsic curvature action

$$S = \frac{2}{\kappa_0} \int \sqrt{-g} \ H \ |d^2\xi|^2$$

have been analyzed [3]. For example, it was shown that this term is asymptotically free [3,4]. However, so far it has not been established if these rigid string theories are appropriate to model QCD flux tubes. It is not known if, for example, they are confining theories.

The present authors have in a series of publications [4-7] studied the extrinsic geometry of string world sheet immersed in background n-dimensional space from the point of view of Grassmannian \(\sigma\)-models. The set of all tangent planes to the world sheet of strings immersed in \(\mathbb{R}^n\) and regarded as a 2-dimensional Riemann surface endowed with the induced metric, is equivalent to the Grassmannian manifold \(G_{2,n} \simeq SO(n)_{\times} SO(n-2) \approx Q_{n-2} \subset CP^{n-1}\). Note that \(G_{2,n}\) can be realized as a quadric \(Q_{n-2}\) in \(CP^{n-1}\). It is this representation that we use throughout our work. However, it is not an ordinary \(\sigma\)-model, since not every field in \(G_{2,n}\) forms tangent plane to the world sheet. This forces the \(G_{2,n}\) fields to satisfy \((n-2)\) integrability conditions which have been derived explicitly in [4,6,8,9] by the use of Gauss mapping [8,9]. The Gauss map is a mapping of the tangent planes to the world sheet \(X^\mu(z, \bar{z})\) into the Grassmannian \(G_{2,n}\) realized as a quadric in \(CP^{n-1}\). There is one third order differential constraint on the \(G_{2,n}\) fields and \((n-3)\) algebraic constraints on the derivative of the Gauss map. Note that the integrability conditions on the \(G_{2,n}\) fields allow us to study the string model in terms of a constrained \(G_{2,n}\) \(\sigma\)-model. These constraints ensure that the \(\sigma\)-model does in fact represent a string world sheet. We are stressing this point, because most other authors incorporate the constraint by requiring that the normals \(N^\mu\) (\(\sigma\)-model fields) to the surface satisfy the condition \(\partial_z X^\mu \cdot N^\mu = 0\), thereby making it very difficult to implement this constraint without dealing with \(X^\mu\)
coordinates. Both the Nambu-Goto (NG) action and the extrinsic curvature action can be written in terms of their images in $G_{2,n}$ through Gauss maps and the integrability conditions can be implemented by Lagrange multipliers. Thus the problem of string dynamics visualized as the dynamics of the world sheet immersed in background $R^n$, can be transformed into, at least at the classical level, that of a constrained Grassmannian $\sigma$-model. To complete the picture, the immersion coordinates $X^\mu(z, \bar{z})$ can be reconstructed from the constrained $G_{2,n}$ $\sigma$-model fields [4].

From the above discussion, the advantages of studying QCD strings as a constrained $\sigma$-model should be clear. The extrinsic curvature action, which usually leads to fourth derivative theory in $X^\mu$, becomes $\sigma$-model action in terms of $G_{2,n}$ fields; the higher derivatives arising only through the differential integrability condition. In quantizing this theory, we need to know the proper measure to use for functional integral over $G_{2,n}$ fields. We recall that already in the theory of QCD flux tubes, the standard string quantization cannot be correct [10]. Thus the measure for $X^\mu$ integration is not completely known. Because of this uncertainty, we take the viewpoint that we can describe QCD flux tubes by constrained $G_{2,4}$ $\sigma$-model (underlying field theory) with the usual sigma model measure. We shall show in this article that the resulting theory has features similar to the unconstrained $G_{2,4}$ theory, thereby establishing through string description the connection between QCD and 2-dimensional $\sigma$-models. It will further be seen in this article that a major advantage of formulating the QCD flux tubes through Gauss mapping is that it allows one to do path integrals over a select class of surfaces having prescribed extrinsic geometric properties.

We consider below strings in background $R^3$ and $R^4$ only. The corresponding Grassmannian manifolds are $G_{2,3} \simeq CP^1$ and $G_{2,4} \simeq CP^1 \times CP^1$ respectively. Two classes of surfaces, minimal and harmonic, are considered. Minimal surfaces are non-compact and have zero scalar mean curvature $h$ [8,9]. They correspond to minimum action solution to the NG action (area term) i.e. $\Box X^\mu = 0 \Leftrightarrow h = 0$. Minimal surfaces describe the dynamics of open strings. In the $\sigma$-model language, minimal surfaces are described by instantons. In the instanton configuration the $\sigma$-model fields are holomorphic. An $N$-instanton solution that is meromorphic arises as the Gauss map of a world sheet with $2N$ punctures. This describes $N$-open string interac-
tions at the tree level (genus zero). Harmonic surfaces correspond to solutions to the equations of motion of the image of the extrinsic curvature action in the Grassmannian $G_{2,n}$ [8,9]. However they do not generally minimize the extrinsic curvature action when expressed in terms of $X^\mu$ as $\int \sqrt{g} (\Box X^\mu)^2 d^2 \xi$ with $g_{\alpha\beta}$ as the induced metric. The equation of motion following from this action is,

$$\sqrt{g} \Box^2 X^\mu + \sqrt{g} \Box X^\mu (\Box X^\nu)^2 + 2(\partial_\nu X^\mu)(\partial_\lambda \Box X^\nu) + 2(\partial_\nu X^\mu)(\partial_\lambda \Box X^\nu) = 0. \quad (1)$$

For immersion in $R^3$ there is only one normal to the surface defined through $\Box X^\mu = h N^\mu$. The above equation can be easily generalized to immersion in $R^4$, by using $\Box X^\mu = h_1 N_1^\mu + h_2 N_2^\mu$ where $h_1, h_2$ are the projections of $H^\mu$ on to the two normals [4]. Returning to immersion in $R^3$, for constant $h$ surfaces (1) reads as $\Box N^\mu + h_2 N^\mu = 0$. Expressing $N^\mu$ in terms of $G_{2,3}$ fields [4], we find that the this is satisfied only when the $G_{2,3}$ fields are anti-holomorphic. Similar conclusion is reached for immersion in $R^4$. So, for harmonic surfaces the choice of Grassmannian fields as anti-holomorphic minimizes the extrinsic curvature action whether it is written in terms of $X^\mu$ or the Grassmannian fields. The Gaussian curvature for these surfaces is a constant and the principal curvatures are same, thus the world sheet topologically corresponds to a 2-sphere. Harmonic surfaces describe the dynamics of closed strings. In the language of the $\sigma$ - model these surfaces correspond to anti-instantons i.e. $\sigma$ - model fields that are anti-holomorphic. An N-anti-instanton solution arises as the Gauss map of a world sheet with 2N punctures and describes N-closed string interactions at the tree level.

We compute the quantum fluctuations following Fateev, Frolov and Schwarz [11] around instantons (minimal surfaces) and anti-instantons (harmonic surfaces) in $R^3$ in section II and in $R^4$ in section III. The resulting effective action is found to be the partition function for a modified two dimensional classical Coulomb gas (MCGS) in the plasma phase for immersion of both minimal and harmonic surfaces in $R^3$ while in $R^4$, it is a MCGS for immersion of minimal surfaces and a MCGS with interaction between the two $CP^1$- anti-instantons for immersion of harmonic surfaces; both being in the confining plasma phase. This means that there is a mass gap for both open and closed QCD strings which is dynamically generated.
II. QUANTUM FLUCTUATIONS - IMMERSION IN $R^3$

The Gauss map of a 2-d Riemann surface conformally immersed in $R^3$ has been considered in detail in Ref. 4, 8 and 9. By conformal immersion it is meant that the induced metric is in the conformal gauge ($g_{zz} = g_{\bar{z}\bar{z}} = 0; g_{z\bar{z}} \neq 0$), where $z = \xi_1 + i\xi_2$ with $(\xi_1, \xi_2)$ as coordinates on the world sheet. The Gauss map is described by,

$$\partial_z X^\mu = \psi \{1 - f^2, i(1 + f^2), 2f\},$$

(2)

where $f$ is the $CP^1$ field and the complex function $\psi$ is determined by the extrinsic geometry and $f$ [4]. The NG and extrinsic curvature actions in terms of $G_{2,3} \simeq CP^1 \sigma$-model field $f$ [4] are,

$$S = \sigma \int \frac{1}{h^2(z, \bar{z})} \frac{|f_{\bar{z}}|^2}{(1 + |f|^2)^2} \frac{i}{2} dz \wedge d\bar{z}$$

$$+ \frac{2}{\alpha_0} \int \frac{|f_{\bar{z}}|^2}{(1 + |f|^2)^2} \frac{i}{2} dz \wedge d\bar{z},$$

(3)

where $\sigma$ is the string tension and $\alpha_0$ is a dimensionless coupling whose renormalized expression is asymptotically free [4]. The integrability condition on $f$ is,

$$Im \left( \frac{f_{z\bar{z}}}{f_{\bar{z}}} - \frac{2f_{\bar{z}}^2}{1 + |f|^2} \right)_{\bar{z}} = 0,$$

(4)

whenever $f_{\bar{z}} \neq 0$. The notation used throughout in this paper is: $f_{\bar{z}} \equiv \partial_{\bar{z}} f$. For minimal surfaces, there is no integrability condition on $f$ and for harmonic surfaces, $f$ satisfies a stronger integrability condition given in (7).

II.1 MINIMAL SURFACES IN $R^3$

As pointed out in the introduction, minimal surfaces in $R^3$ represent the world sheet of open strings in background 3-d Euclidean space. For these surfaces the area is minimum and $h = 0$. The first term in (3) is the area term $\int \sqrt{g} d^2 \xi$. The classical action for the underlying $G_{2,3} \sigma$-model is the second term (3) and classically there is no integrability on the Gauss
map. We consider as a classical background for the $CP^1$-field, the instanton configuration [11],

$$
  f(z) = c \frac{\prod_{i=1}^{N}(z-a_i)}{\prod_{i=1}^{N}(z-b_i)},
$$

(5)

where $\{c, a_i, b_i\}$ are the (complex) instanton parameters. The quantum fluctuations $\nu(z, \bar{z})$ around (5) are defined by,

$$
  f(z) \rightarrow f(z) + \nu(z, \bar{z}).
$$

(6)

The fluctuated field in (6) also arises as the Gauss map of a surface obtained from the given (classical) minimal surface. We need the integrability condition on $\nu(z, \bar{z})$. Since classically there is no integrability condition for minimal surfaces, the Lagrange multiplier field to implement the condition on $\nu$ must be quantum. We restrict the fluctuation (6) to correspond to harmonic surfaces with small but constant $h$. A Gauss map is said to be harmonic if it satisfies the Euler-Lagrange equation,

$$
  f_{z\bar{z}} - \frac{2\bar{f}f_z f_{\bar{z}}}{1 + |f|^2} = 0,
$$

(7)

which is also the equation of motion of the extrinsic curvature action. It can be shown [4] that if (7) is satisfied then $h$ is constant. The Gauss map thus satisfies the above stronger integrability condition whose linearized version with (5) as the classical background is,

$$
  \nu_{z\bar{z}} - \frac{2\bar{f}f_z \nu_{\bar{z}}}{1 + |f|^2} = 0.
$$

(8)

This is the integrability condition on $\nu$ to be implemented by quantum multiplier $\lambda$. So the classical minimal surface ($h = 0$) is fluctuated to represent a (quantum) harmonic surface with small but constant $h$. Expanding the classical action of the underlying $G_{2,3}$ $\sigma$-model using (5) and implementing the condition (8), we find,

$$
  S = \frac{2\pi}{\alpha_0} N + \frac{4}{\alpha_0} \int \bar{v} \Delta f \bar{v} \sqrt{\mathcal{G}} d^2 z \\
  + \int \lambda \left( \nu_{z\bar{z}} - \frac{2\bar{f}f_z \nu_{\bar{z}}}{1 + |f|^2} \right) d^2 z + h.c.,
$$

(9)
where,

\[ \Delta_f = -\frac{1}{\sqrt{G}} \rho \partial_z \rho^{-2} \partial_z \rho, \]

\[ \tilde{\nu} = \frac{2 \nu}{\rho} \prod_{i=1}^{N} (z - b_i)^2, \]

\[ \rho = \rho_0 \prod_{i=1}^{N} |z - b_i|^2, \]

\[ \rho_0 = 1 + |f|^2, \]  \hspace{1cm} (10)

where, \( G_{\alpha\beta} \) is a metric introduced to avoid infra-red divergences, which will eventually be taken as \( \delta_{\alpha\beta} \). Thus the effective action of open strings in background \( R^3 \) is given by (9). The third term in (9) can be simplified to,

\[ \int \tilde{\lambda} \Delta_f \tilde{\nu} d^2 z + h.c., \]  \hspace{1cm} (11)

where,

\[ \tilde{\lambda} = \frac{\rho \lambda}{2} \prod_{i=1}^{N} (z - b_i)^2. \]

The partition function is obtained by the functional integral of the exponential of \( S \) in (9) over \( \tilde{\nu}, \tilde{\nu}, \lambda, \tilde{\lambda} \) with the instanton measure arising from the change of the sigma model measure \([df][d\bar{f}]\) to a measure in the instanton manifold. By the standard shift of the integration variables, the functional integration gives the partition function as,

\[ Z = \sum_{N=0}^{\infty} (N!)^{-2} \exp\left( -\frac{2 \pi N}{\alpha_0} \right) \int d\mu_0 \det \left( \frac{4}{\alpha_0} \Delta_f \right)^{-1}, \]  \hspace{1cm} (12)

where \( d\mu_0 \) is the instanton measure [11] and a sum over instantons of all winding numbers is introduced. The effect of the integrability condition is to produce additional \((\det \Delta_f)^{-\frac{1}{2}}\) from (11). The determinant \( \Delta_f \) and the instanton measure \( d\mu_0 \) have been evaluated and as this procedure is given in detail in [11], we give only the results here,

\[ Z = \sum_{N=0}^{\infty} (N!)^{-2} \left[ \left( \frac{4}{\alpha_0} \right)^2 \exp\left( -\frac{\pi}{\alpha_0} \right) \exp(2 \log \Lambda) \right]^{2N} \]
\[
\int \frac{d^2 c}{(1 + |c|^2)^2} \prod_{j=1}^N \frac{d^2 a_j d^2 b_j}{(1 + |c|^2)^2} (detM)^{-1} \exp\left[\sum_{i<j} \log |a_i - a_j|^2 + \sum_{i<j} \log |b_i - b_j|^2 - 3 \sum_{i,j} \log |a_i - b_j|^2\right].
\]

This is the partition function for open strings in background \(R^3\). In (13), \(\Lambda\) is the effect of ultra-violet cutoff in regularizing the determinant of \(\Delta f\). This is removed by renormalizing the coupling constant \(\alpha_0\) as,

\[
\alpha_R(\mu) = \frac{\alpha_0}{1 - 2(\frac{\alpha_0}{\pi}) \log \frac{\Lambda}{\mu}},
\]

where \(\mu\) is the renormalization point. For unconstrained \(CP^1\) model [11], the factor in front of \((\frac{\alpha_0}{\pi})\) is unity. The role of the integrability condition is thus to change the above factor, from \((d - 2)\) to \((d - 1)\), for immersion in \(d\)-dimensional space. This is in agreement with our earlier calculations [4]. (13) represents the classical grand partition function of a two dimensional modified Coulomb gas system (MCGS), modified in the sense that the potential energy between oppositely charged particles is threefold stronger than the repulsive energy between like charges. In the unconstrained \(CP^1\) model, the corresponding partition function is precisely that of a 2-d Coulomb gas. The modification here is due to the integrability condition. Furthermore, there is an extra factor of \(detM\) where, \(M_{ij} = \int z^i z^j \rho^{-2} d^2 z\). We have checked that the (finite) contribution of \(detM\) is independent of the instanton parameters. Absorbing this and other factors involving \(c\), and denoting them by \(\kappa\), we rewrite (13) as,

\[
Z = \sum_{\kappa} \int \frac{\kappa^N}{(N!)^2} \exp(-E_N(a,b)) \prod_{j=1}^N d^2 a_j d^2 b_j,
\]

where,

\[
E_N(a,b) = -\sum_{i<j} \log |a_i - a_j|^2 - \sum_{i<j} \log |b_i - b_j|^2 + 3 \sum_{i,j} \log |a_i - b_j|^2.
\]

In the case of the \(CP^1\) model, it is known that the CGS at the inverse temperature \(\beta = 1\) is in the plasma phase and has a mass gap. The critical
temperature for transition to molecular phase has been determined using an iterated mean field approximation by Kosterlitz and Thouless [12]. It is given as solution to \( q^2 \beta_c \simeq 2 + 2.6\pi \exp(-\mu' \beta_c) \), where \( q^2 \) is the strength of the attractive interaction. For \( \mu' = 0 \), \( \beta_c \simeq 10 \), and so the CGS is in the plasma phase. The MCGS in (16) is again at \( \beta = 1 \), but with the strength of the attractive interaction \( q^2 = 3 \). The critical temperature, is not known for MCGS. However, we can argue that the effect of the stronger attractive interaction would be to increase the critical temperature. This qualitative feature is in agreement with the estimate of \( \beta_c \) from the above expression which gives \( \beta_c \simeq 3 \) for \( q^2 = 3 \) for zero chemical potential. This suggests that the MCGS in (16) is in the plasma phase with a mass gap.

II.2. HARMONIC SURFACES IN \( R^3 \)

The Gauss map of surfaces immersed in \( R^3 \) is said to be harmonic if \( f \) satisfies (7). For harmonic Gauss maps [8,9], the mean curvature scalar \( h \) is constant and the given surface is compact. Harmonic maps thus describe closed string dynamics [2]. The integrability condition for harmonic Gauss maps is also the equation of motion (7). In order to minimize the extrinsic curvature action in terms of \( X^\mu \) (see (1)), \( f \) in (7) should be anti-holomorphic. This result is also the content of Chern’s theorem [14]. In this case, as mentioned in the introduction, the surface is actually a Riemann sphere and thus we are considering closed string dynamics at the tree level. As a solution to harmonic Gauss map (7), we consider the anti-instanton configuration,

\[
f(\bar{z}) = \bar{c} \prod_{i=1}^{N} \left( \bar{z} - \bar{a}_i \right) \prod_{i=1}^{N} \left( \bar{z} - \bar{b}_i \right),
\]

(17)

where \( \{\bar{c}, \bar{a}_i, \bar{b}_i\} \) are the anti-instanton parameters. The points \((\bar{a}_i, \bar{b}_i)\) at which the Gauss map has zeroes and poles represent punctures on the sphere and thus we can think of inclusion of such configurations as representing \( N \)-string interactions at the tree level. Classically we have \( h \) constant and so redefining \( \frac{\sigma}{\kappa} + \frac{2}{\alpha_0} \) as \( \frac{2}{\alpha_0} \), the classical action is,

\[
S = \frac{2}{\alpha_0} \int \frac{|f_{\bar{z}}|^2}{(1 + |f|^2) \frac{i}{2}} dz \wedge d\bar{z}
\]
\[
+ \int \lambda \{f_{z\bar{z}} - \frac{2\bar{f}fzf_{\bar{z}}}{1 + |f|^2}\} \frac{i}{2} dz \wedge d\bar{z} + h.c.,
\]

(18)
where the integrability condition (7) for harmonic surfaces is implemented in (18) by a Lagrange multiplier field $\lambda$. For the classical background (17), the integrability condition (7) is identically satisfied. The equations of motion of the total classical action contain in addition to (7) for $f$, a homogeneous equation for $\lambda$. The trivial solution $\lambda_{cl} = 0$ is chosen as this choice is reasonable, for, at the classical level, the equation of motion for $f$ already ensures the integrability condition and thus no multiplier field would be needed. The quantum fluctuation $\nu$,

$$f(\bar{z}) \rightarrow f(\bar{z}) + \nu(z, \bar{z}),$$

(19)

describes a surface which is Gauss mapped into $G_{2,3}$. To ensure that $f(z, \bar{z})$ in (19) also arises as a Gauss map of a surface, we implement the constraint in (18) in its linearized version (linear in $\nu$) with $\lambda$ as quantum. Thus the fluctuated surface is represented by harmonic Gauss map and so its scalar mean curvature $h$ is also constant. Then expanding (18) using (17), we obtain,

$$S = \frac{2\pi}{\alpha_0} N + \frac{4}{\alpha_0} \int \tilde{\nu} \Delta f \tilde{\nu} \sqrt{G} d^2 z$$

$$+ \int \tilde{\lambda} \{ \nu_{z\bar{z}} - \frac{2\tilde{f}_z \nu_z}{1 + |f|^2} \} d^2 z + h.c.,$$

(20)

where $\tilde{\nu}$ and $\Delta f$ are same quantities that appear in (10) with $z \rightarrow \bar{z}$, $b \rightarrow \bar{b}$, and $\frac{i}{2} dz \wedge d\bar{z}$ is denoted by $d^2 z$. The evaluation of the partition function proceeds exactly the same way as in II.1. The system is represented by a modified Coulomb gas in the plasma phase.

In both the cases, the theory of world sheet immersed in $R^3$ is a modified CGS at an inverse temperature $\beta = 1$ in the plasma phase with a dynamically generated mass gap $m = \mu \exp(-\pi/\alpha_R(\mu))$.

### III. QUANTUM FLUCTUATIONS-IMMERSION IN $R^4$

#### QCD-STRINGS

The Grassmannian $\sigma$-model approach to the string dynamics presented in section II, is extended to string world sheet immersed in $R^4$. We have two
normals $N^\mu_i (\mu = 1, 2, 3, 4; i = 1, 2)$ at each point on the surface and so there are two extrinsic curvature tensors $H^\mu_{\alpha\beta i}$ for the surface. The scalar mean curvature $h = \sqrt{h_1^2 + h_2^2}$ and the detailed expressions for $h_1, h_2$ and the extrinsic curvature action in terms of Gauss map are derived in Ref.4. In the case of $R^4$, $Q_2$ which is equivalent to $CP^1 \times CP^1$ is parameterized by $f_1$ and $f_2$ and the Gauss map is given by,

$$\partial_z X^\mu = \psi [1 + f_1 f_2, i(1 - f_1 f_2), f_1 - f_2, -i(f_1 + f_2)],$$

(21)

where the complex function $\psi$ is determined by the extrinsic geometry and $f_1$ and $f_2$ [4]. The two integrability conditions are,

$$\text{Im} \left( \sum_{i=1}^2 f_{iz\bar{z}} - \frac{2\bar{f}_i f_{iz}}{1 + |f_i|^2} \right) = 0,$$

(22)

and,

$$|F_1| = |F_2|,$$

(23)

where $F_i = f_{iz}/(1 + |f|^2)$ and whenever $f_{iz} \neq 0$. In our considerations [4] the world sheet is described locally by $X^\mu(z, \bar{z})$ and the Gauss map allows us to express NG and extrinsic curvature actions as $G_{2,4} \sigma$-model action,

$$S = \int \left( \frac{\sigma}{h^2(z, \bar{z})} + \frac{2}{\alpha_0} \right) \sum_{i=1}^2 \frac{|f_{iz}|^2}{1 + |f_i|^2} d^2z.$$

(24)

This together with (22) and (23) describes the dynamics of the string world sheet in background $R^4$. The scalar mean curvature $h$ is given by [4],

$$(\log h)_z = \sum_{i=1}^2 \left( f_{iz\bar{z}} - \frac{2\bar{f}_i f_{iz}}{1 + |f_i|^2} \right),$$

(25)

and the Gauss map (21) is said to be harmonic if [8,9],

$$f_{iz\bar{z}} - \frac{2\bar{f}_i f_{iz}}{1 + |f_i|^2} = 0; \quad i = 1, 2.$$

(26)

From (25) and (26) it follows that when the Gauss map is harmonic, $h$ is constant. But all surfaces of constant $h$ are not necessarily harmonic. This observation will be used here.
III.1 MINIMAL IMMERSION IN $R^4$

The Gauss map of minimal surfaces ($h = 0$) in $R^4$ has been studied in detail [8,9,14] and accordingly, if $F_1 = F_2 \equiv 0$, then the Gauss map represents a minimal surface in $R^4$, provided the surface is non-compact. A solution to $F_1 = F_2 \equiv 0$ is given by holomorphic functions $f_1(z)$ and $f_2(z)$. There are no integrability conditions classically. The holomorphic functions $f_1(z)$ and $f_2(z)$ are chosen as,

$$ f_i(z) = c_i \frac{\prod_{j=1}^{N_i} (z - a_{ij})}{\prod_{j=1}^{N_i} (z - b_{ij})}; \quad i = 1, 2, \ldots \tag{27} $$

representing background instantons, with parameters $\{c_i, a_{ij}, b_{ij}\}$ for $i = 1, 2$.

Quantum fluctuations $\nu_i(z, \bar{z})$ around the instanton background are defined through,

$$ f_i(z) \to f_i(z) + \nu_i(z, \bar{z}); \quad i = 1, 2. \tag{28} $$

The fluctuated surface also arises as the Gauss map of a surface obtained from the given classical minimal surface. So we need integrability conditions on $\nu_i$. Since classically there are no integrability conditions for minimal surface, the Lagrange multiplier fields to implement the conditions on $\nu_i$ must be quantum. In view of the similar situation in II.1, we restrict the fluctuations (28) to represent harmonic surfaces (i.e. the fluctuated surface has constant scalar mean curvature). Thus the fields (28) are required to satisfy (26) in its linearized version, linear in $\nu_i$. For the background (27), the linearized version of (26) is,

$$ \nu_{iz\bar{z}} - \frac{2\bar{f}_i f_{iz} \nu_{i\bar{z}}}{1 + |f_i|^2} = 0; \quad i = 1, 2, \ldots \tag{29} $$

which are implemented by two multipliers. For surfaces immersed in $R^4$, we need to examine the algebraic integrability condition (23) as well, for the fields (28). It can be readily checked that there are no linear terms in $\nu_i$ arising from the constraint (23) in the instanton background.
As a classical action, we consider the action for the underlying $G_{2,4}$ $\sigma$-model which can be rewritten as,

$$S = \frac{2\pi}{\alpha_0} (N_1 + N_2) + \frac{4}{\alpha_0} \int \sum_{i=1}^{2} \frac{|f_{iz}|^2}{(1 + |f_i|^2)^2} d^2z,$$  \hspace{1cm} (30)

where $N_1$, $N_2$ are the winding numbers of the two $CP^1$ instantons. Expanding (30) using (27) and (28), and implementing (29), the effective action is found to be,

$$S = \frac{2\pi}{\alpha_0} (N_1 + N_2) - \frac{4}{\alpha_0} \int \left( \bar{\nu}_1 \triangle f_1 \nu + \bar{\nu}_2 \triangle f_2 \nu \right) d^2z + h.c$$

$$+ \int \bar{\lambda}_1 \{\nu_{1z\bar{z}} - \frac{2\bar{f}_1 f_{1z} \nu_{1\bar{z}}}{1 + |f_1|^2}\} d^2z + h.c$$

$$+ \int \bar{\lambda}_2 \{\nu_{2z\bar{z}} - \frac{2\bar{f}_2 f_{2z} \nu_{2\bar{z}}}{1 + |f_2|^2}\} d^2z + h.c.,$$  \hspace{1cm} (31)

where

$$\triangle f_i = -\frac{1}{\sqrt{G}} \rho_i \partial_2 \rho_i^{-2} \partial_{\bar{z}} \rho_i,$$

$$\bar{\nu}_i = \frac{2\nu_i}{\rho_i} \prod_{j=1}^{N_i} (z - b_{ij})^2,$$

$$\rho_i = \rho_{0i} \prod_{j=1}^{N_i} |z - b_{ij}|^2,$$

$$\rho_{0i} = 1 + |f_i|^2.$$  \hspace{1cm} (32)

By comparing (31) with (9), it is seen that a doubling corresponding to the two $CP^1$ instantons occur. The evaluation of the partition function then is similar as in section II.1, now with measures for the two $CP^1$ instantons and then the result is,

$$Z = \sum_{N_1,N_2} \frac{k^{N_1+N_2}}{(N_1!)^2(N_2!)^2} \exp \left[ -E_{N_1}(a_1,b_1) - E_{N_2}(a_2,b_2) \right]$$

$$\times \prod_{j=1}^{N_1} d^2a_{1j} d^2b_{1j} \prod_{k=1}^{N_2} d^2a_{2k} d^2b_{2k},$$  \hspace{1cm} (33)
where,

$$E_{N}(a_i, b_i) = - \sum_{k<j}^{N_1} \log | a_{ik} - a_{ij} |^2 - \sum_{k<j}^{N_1} \log | b_{ik} - b_{ij} |^2 + 3 \sum_{k,j}^{N_1} \log | a_{ik} - b_{ij} |^2, \quad i = 1, 2. \quad (34)$$

The partition function (33) thus represents two modified Coulomb gas, arising from the two $CP^1$ instantons. Following the discussion in section II, we conclude that there is a mass gap in the quantum theory of minimal surfaces in $R^4$.

**III.2 SURFACES OF CONSTANT MEAN CURVATURE**

We consider 2-d surfaces in $R^4$ described by harmonic Gauss map as representing closed QCD-strings. For harmonic Gauss map, as can be seen from (25), the scalar mean curvature $h$ is constant. The two $CP^1$-fields satisfy the harmonic map equation (26). When $h$ is constant, the NG and extrinsic curvature actions can be written as,

$$S = \left( \frac{\sigma}{h^2} + \frac{2}{\alpha_0} \right) \int \sum_{i=1}^{2} \frac{|f_i|^2}{(1 + |f_i|^2)^2} d^2z, \quad (35)$$

whose equations of motion are the same as (26). However, as pointed out in the introduction, the extrinsic curvature action expressed in terms of $X^\mu(z, \bar{z})$ acquires a minimum only when $f_1$ and $f_2$ are anti-holomorphic. In this case, the surface is a compact 2-sphere representing the world sheet. The two anti-holomorphic functions are chosen as the two $CP^1$-anti-instanton configurations,

$$f_i(z) = \bar{c}_i \prod_{j=1}^{N_i} (\bar{z} - \bar{a}_{ij}) \prod_{j=1}^{N_i} (\bar{z} - \bar{b}_{ij}), \quad i = 1, 2. \quad (36)$$

where $\{\bar{c}_i, \bar{a}_{ij}, \bar{b}_{ij}\}$ for $i=1,2$ are the anti-instanton parameters. The above background satisfies the harmonic map equation (26). The algebraic integrability condition (23) has to be satisfied at the classical level, in order for (36)
to represent a Gauss map. This puts conditions on the positions of the two \( CP^1 \) instantons.

The quantum fluctuations are defined by,

\[
    f_i(\bar{z}) \to f_i(\bar{z}) + \nu_i(z, \bar{z}); \quad i = 1, 2. \tag{37}
\]

We now examine the integrability conditions on (37). We restrict fluctuations to represent surfaces of constant scalar mean curvature. In this case, the constraint reads as (see (25)),

\[
    \sum_{i=1}^{2} \left( \frac{f_{iz}}{f_{i\bar{z}}} - \frac{2\bar{f}_i f_{iz}}{1 + |f_i|^2} \right) = 0, \tag{38}
\]

whose linearized version is,

\[
    \sum_{i=1}^{2} \left( \frac{\nu_{iz}}{f_{i\bar{z}}} - \frac{2\bar{f}_i \nu_{iz}}{1 + |f_i|^2} \right) = 0. \tag{39}
\]

The second integrability condition \(| F_1 |^2 = | F_2 |^2\) is likewise expanded to give its linearized version. Expanding (35), using (36) and (37), and implementing (39) along with the linearized version of \(| F_1 |^2 = | F_2 |^2\), we obtain,

\[
    S_{eff} = \frac{2\pi}{\beta_0} (N_1 + N_2) - \frac{4}{\beta_0} \int \sum_{i=1}^{2} \left( \bar{\nu}_i \Delta f_i \nu_i \right) d^2 z
    + \int \lambda \sum_{i=1}^{2} \left( \frac{\nu_{iz}}{f_{i\bar{z}}} - \frac{2\bar{f}_i \nu_{iz}}{1 + |f_i|^2} \right) d^2 z + h.c,
    - \int \bar{\nu}_1 (\partial_z \chi) \left( \frac{f_{1\bar{z}}}{1 + |f_1|^2} \right)^2 d^2 z - h.c,
    + \int \bar{\nu}_2 (\partial_z \chi) \left( \frac{f_{2\bar{z}}}{1 + |f_2|^2} \right)^2 d^2 z + h.c, \tag{40}
\]

where \(\lambda\) and \(\chi\) are the quantum multipliers and \(\frac{2}{\beta_0} = \frac{\sigma}{\hbar} + \frac{2}{\alpha_0}\). The terms involving the multiplier fields are rewritten using (36) as,

\[
    S_{eff} = \frac{2\pi}{\beta_0} (N_1 + N_2) - \frac{4}{\beta_0} \int \left( \bar{\nu}_1 \Delta f_i \nu_1 + \bar{\nu}_2 \Delta f_i \nu_2 \right) d^2 z
\]
+ \int \tilde{\lambda}_1 \Delta f_1 \tilde{\nu}_1 + h.c + \int \tilde{\lambda}_2 \Delta f_2 \tilde{\nu}_2 + h.c \\
- \int \tilde{\nu}_1 \chi_1 - h.c + \int \tilde{\nu}_2 \chi_2 + h.c., \quad (41)

where,

$$\tilde{\lambda}_i = \frac{\rho_i \lambda}{2 \prod_{j=1}^{N_i} (z - b_{ij})^2 f_{i\bar{z}}},$$

$$\chi_i = \frac{\rho_i f_{i\bar{z}} \chi'}{2 \rho_0 \prod_{j=1}^{N_i} (\bar{z} - \bar{b}_{ij})^2},$$

with $\chi' = \partial_z \chi$. The expressions for $\Delta, \tilde{\nu}$ in (39) are the same as in (31) with $z \to \bar{z}, \quad b \to \bar{b}$. The partition function is obtained by the functional integral of the exponential of $S_{eff}$ over all the quantum fields. In order to perform this, the quantum action is rewritten, by shifting the fields, as,

$$S_q = -\frac{4}{\beta_0} \left[ \int (\tilde{\nu}_1 - \beta_0 \tilde{\lambda}_1 + \beta_0 \xi_1) \Delta f_1 (\tilde{\nu}_1 - \beta_0 \tilde{\lambda}_1 + \beta_0 \xi_1) d^2 z \\
+ \int (\tilde{\nu}_2 - \beta_0 \tilde{\lambda}_2 + \beta_0 \xi_2) \Delta f_2 (\tilde{\nu}_2 - \beta_0 \tilde{\lambda}_2 + \beta_0 \xi_2) d^2 z \\
- \int (\beta_0 (\tilde{\lambda}_1 - \xi_1)) \Delta f_1 (\beta_0 (\tilde{\lambda}_1 - \xi_1)) d^2 z \\
- \int (\beta_0 (\tilde{\lambda}_2 + \xi_2)) \Delta f_2 (\beta_0 (\tilde{\lambda}_2 + \xi_2)) d^2 z \right], \quad (42)$$

where $\xi_i = \Delta_f^{-1} \chi_i$, with the prime denoting the determinant of non-zero eigenvalues of $\Delta_f$. Note that $\tilde{\lambda}_1$ and $\tilde{\lambda}_2$ as well as $\xi_1$ and $\xi_2$ are not independent. The integration over $\{\tilde{\nu}_1, \tilde{\nu}_2, \lambda, \chi\}$ can be transformed into their linear combinations in (42). This introduces the Jacobian of the transformation, which is calculated using (36) and (23) as $\text{det}^{-1}(\Delta_f^{-1} + \Delta_f^{-1})$. Thus the partition function for (41) becomes,

$$Z = \beta_0^{-2(N_1 + N_2)} \exp \left( -\frac{2\pi}{\beta_0} \right) \int \text{det} \Delta_f^{-1} \text{det} \Delta_f^{-1} \text{det}(\Delta_f^{-1} + \Delta_f^{-1})^{-1} d\mu_1 d\mu_2, \quad (43)$$

The role of the integrability conditions is first to produce additional $(\text{det} \Delta_f)^{-\frac{N}{2}}$ as in III.1 and secondly to produce $\text{det}(\Delta_f^{-1} + \Delta_f^{-1})^{-1}$ from the Jacobian, with the classical instanton parameters related by (23).
We now discuss the implications of (43). If the Jacobian is ignored, then we would have obtained MCGS for the two \( CP^1 \)-instantons with no interaction between them. In this way the integrability condition (23) couples the two \( CP^1 \)-instantons at the quantum level as well. \( det\Delta_{f_1}^{-1} \) and \( det\Delta_{f_2}^{-1} \) are evaluated using the methods of Fateev, Frolov and Schwarz [11]. We infer the form of \( det^{-1}(\Delta_{f_1}^{-1} + \Delta_{f_2}^{-1}) \) by the following procedure. Let us note that the interaction term is symmetric in the indices 1 and 2. When \( \Delta_{f_1} = \Delta_{f_2} \), (43) reduces to the results for immersion in \( R^3 \) (as it should be) with one instanton measure removed. The regularized expression for \( logdet\Delta_{f_1}^{-1} \) is,

\[-4 \sum_{j,k} log|a_{1j} - b_{1k}|^2 - 4log|c_1|^2N_1(1 + |c_1|^2),\]

and similar expression for \( logdet\Delta_{f_2}^{-1} \), apart from \( detM \). These observations suggest that \( logdet(\Delta_{f_1} + \Delta_{f_2}^{-1}) \) should be of the form,

\[2 \sum_{j,k} log|a_{1j} - b_{2k}|^2 + 2 \sum_{j,k} log|a_{2j} - b_{1k}|^2 - A \sum_{j,k} log|a_{1j} - a_{2k}|^2 - B \sum_{j,k} log|b_{1j} - b_{2k}|^2, \tag{44}\]

apart from \( c_{1,2} \) factors. Then when \( a_{1j} = a_{2k} ; b_{1j} = b_{2k} \) and with one instanton measure removed, we recover the results for immersion in \( R^3 \), modulo an infinite constant. In (44) we have included the last two terms as possible additional interactions among the two \( CP^1 \) - anti-instantons with arbitrary coefficients. Then, the partition function \( Z \) can be written as,

\[Z = \sum_{N_1, N_2} \frac{\kappa^{N_1+N_2}}{(N_1!)^2(N_2!)^2} exp[-E_{N_1}(a_1, b_1) - E_{N_2}(a_2, b_2) - V_{N_1, N_2}] \prod_{j=1}^{N_1} d^2a_{1j} d^2b_{1j} \prod_{k=1}^{N_2} d^2a_{2k} d^2b_{2k}, \tag{45}\]

where,

\[E_{N_1}(a_1, b_1) = - \sum_{i<j}^{N_1} log|a_{1i} - a_{1j}|^2 - \sum_{i<j}^{N_1} log|b_{1i} - b_{1j}|^2 + 3 \sum_{i,j}^{N_1} log|a_{1i} - b_{1j}|^2,\]
\[
E_{N_2}(a_2, b_2) = -\sum_{i<j}^{N_2} \log |a_{2i} - a_{2j}|^2 - \sum_{i<j}^{N_2} \log |b_{2i} - b_{2j}|^2
+ 3 \sum_{i,j}^{N_2} \log |a_{2i} - b_{2j}|^2,
\]
\[
V_{N_1,N_2} = -2 \sum_{i,j}^{N_1,N_2} \log |a_{1i} - b_{2j}|^2 - 2 \sum_{i,j}^{N_1,N_2} |a_{2i} - b_{1j}|^2
+ A \sum_{i,j}^{N_1,N_2} \log |a_{1i} - a_{2j}|^2 + B \sum_{i,j}^{N_1,N_2} \log |b_{1i} - b_{2j}|^2, \quad (46)
\]

where the \(c_1, c_2\) dependent terms and others have been absorbed in \(\kappa\). The partition function (44) thus represents two MCGS with possible interactions between them. In the absence of interactions, the system is in the plasma phase, as can be seen from III.1. The effect of the interactions between the two \(CP^1\)-anti-instantons in (45) is not expected to change the transition temperature. A detailed study of the interaction in (45) will shed further light on the phase transition.

We now comment on the renormalization of the coupling constant \(\beta_0\) for immersion in \(R^4\). Regularization of the determinants in (43) leads to the following result for \(\beta_R(\mu)\),

\[
\beta_R(\mu) = \frac{\beta_0}{1 - 3 \left(\frac{e_0}{\pi}\right) \log \Delta_{\mu}}. \quad (47)
\]

The factor 3 in the denominator in (47) is consistent with our observation following (14). For surfaces immersed in \(R^4\), the self-intersection number, a topological invariant, in terms of the Gauss map (21) is,

\[
I = \frac{1}{2} \int \{ |F_1|^2 - |\hat{F}_1|^2 - |F_2|^2 - |\hat{F}_2|^2 \} d^2 z, \quad (48)
\]

where \(\hat{F}_i = f_{iz}/(1 + |f_i|^2); i=1,2\) and this vanishes identically for the background configuration studied here. The self-intersection number plays the role of the \(\theta\)-term in the QCD lagrangian [13]. In order to study the \(\theta\)-term, we need to consider immersed surfaces of non-zero self-intersection number.
We have presented calculations for the effective action of the string world sheet in $R^3$ and $R^4$ utilizing its correspondence with the constrained Grassmannian $\sigma$-model. Two classes of surfaces, i.e. minimal ($h = 0$) and harmonic ($h = constant \neq 0$) which describe respectively, the dynamics of N-string interactions at the tree level of open and closed strings are studied at the one loop level. The effective action is found to be the classical partition function of a modified 2-d Coulomb gas at an inverse temperature $\beta = 1$ and it is found that the system is in the confining (plasma) phase with a mass gap.

We now make qualitative estimates on the mass ratio of the glue-ball to a rho meson. The mass gap generated in the case of open-string dynamics (III.1) is $m_{open} = \mu \exp(-\pi/\alpha_R(\mu))$ where $\alpha_R(\mu) = \alpha_0/(1 - 2\alpha_0/\pi \log \Lambda)$. The bare coupling $\alpha_0$ is due to the extrinsic curvature action alone, since for minimal surfaces $h = 0$. However, as pointed out in III.1, the quantum fluctuations are restricted to constant scalar mean curvature surfaces. Thus, the effective action is governed by a coupling constant $\frac{\sigma}{h_q^2} + \frac{2}{\alpha_0^2}$. This allows us to write,

$$m_{open} \simeq \mu \exp\left(-\frac{\sigma}{2h_q^2} - \frac{1}{\alpha_0^2} + \frac{1}{4} \log \frac{\Lambda}{\mu}\right), \quad (49)$$

where $h_q$ is small but constant scalar mean curvature of the quantum surface. The first excited state in the case of open QCD-strings corresponds to a pion. However as we do not have chiral symmetry breaking mechanism in our approach, we take rho meson to be the first excited state. In the case of closed strings, the classical surface has a constant non-zero $h$ (III.2) and so the fluctuated surface effectively has constant $h + h_q$ mean curvature. $2/\beta_0^2$ in (40) will effectively be replaced by $\frac{\sigma}{(h + h_q)^2} + \frac{2}{\alpha_0^2}$. The mass gap then is,

$$m_{closed} \simeq \mu \exp\left(-\frac{\sigma}{(h + h_q)^2} - \frac{1}{\alpha_0^2} + \frac{1}{4} \log \frac{\Lambda}{\mu}\right), \quad (50)$$

where the same renormalization point as in (49) is used. As the lowest hadronic state of closed QCD-string corresponds to a glue-ball, we deduce
that,

\[
\frac{m_{\text{glueball}}}{m_{\rho}} \sim \exp\left(\frac{\sigma}{2} \left( \frac{1}{\hbar_q^2} - \frac{1}{(\hbar + h_q)^2} \right) \right) > 1. \tag{51}
\]

This estimate seems to agree with the current thinking on the subject.

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