Scale Symmetry Spontaneously Broken by Asymptotic Behavior

E.I. Guendelman

Physics Department, Ben-Gurion University, Beer-Sheva 84105, Israel

November 2, 2021

Abstract

Conserved quantities are obtained and analyzed in the new models with global scale invariance recently proposed. Such models allow for non trivial scalar field potentials and masses for particles, so that the scale symmetry must be broken somehow. We get to this conclusion by showing that the infrared behavior of the conserved currents is singular so that there are no conserved charges associated with the global scale symmetry. The scale symmetry plays nevertheless a crucial role in determining the structure of the theory and it implies that in some high field regions the potentials become flat.

1 Introduction

The idea of scale invariance, which implies that physics at all scales must be the same is an appealing thought. A straightforward application of scale invariance to the construction of physical theories leads however to consequences that are very much against what we observe in the universe, since in physics certain scales appear, through the appearance of typical atomic sizes, typical masses for particles, etc.

One can attempt to construct scale invariant theories where the scale invariance is spontaneously broken in the usual sense. This situation is also
not satisfactory, since then Goldstone’s theorem\textsuperscript{1} tell us that there must be a massless boson, associated with scale invariance\textsuperscript{2}, which is not observed so far.

It is known however that violent infrared behavior can invalidate Goldstone’s theorem in certain instances. For example the famous chiral \text{U}(1) problem in QCD is solved, in spite of the fact that even in the presence of the chiral \text{U}(1) anomaly, one can define a locally conserved chiral \text{U}(1) current. Local current conservation nevertheless does not guarantee conservation of the integrated charge if the current has a singular infrared behavior, as it is the case for QCD instantons. The absence of a conserved charge does not allow us therefore to apply Goldstone’s theorem and therefore the chiral \text{U}(1) problem in QCD is solved\textsuperscript{3}.

Here we will see that something like this can happen in the case of dilatation symmetry, although the effect is already evident at the classical level and there is no need to consider instantons. Indeed a model is found where dilatation symmetry is exact, a local conservation law follows, but the infrared behavior of the spatial components of the conserved dilatation current is so singular in the infrared that charge can easily "leak out" to infinity, thus invalidating the possibility of a conserved scale charge. Thus we obtain, non conservation of the scale charge without need to introduce any scale symmetry breaking terms in the action.

The model is constructed by allowing, in addition to the usual volume element used for integration in the action $\sqrt{-g}d^4x$, where $g = \text{det} g_{\mu\nu}$, another one\textsuperscript{4}, $\Phi d^4x$, where $\Phi$ is a density built out of degrees of freedom independent of that of $g_{\mu\nu}$. To achieve global scale invariance, also a "dilaton" $\phi$ has to be introduced\textsuperscript{5}. For example, given 4-scalars $\varphi_a$ ($a = 1,2,3,4$), one can construct the density

$$\Phi = \varepsilon^{\mu\nu\alpha\beta}\varepsilon_{abcd}\partial_\mu \varphi_a \partial_\nu \varphi_b \partial_\alpha \varphi_c \partial_\beta \varphi_d$$

and we allow global scale transformations for the $\varphi_a$ to be independent to those of the metric $g_{\mu\nu}$. The infrared behavior of the $\varphi_a$ fields plays an essential role in the singular infrared behavior of the dilatation current and in the global non conservation of the dilatation charge, although there is a locally conserved dilatation current.

Although no dilatation charge exists, the dilatation symmetry has important consequences in the structure of the theory, the form of the scalar field
potentials and their interactions to other fields, etc.

2 The Model

As mentioned in the introduction, we look at an action which uses both measures of integration $\sqrt{-gd^4x}$ and $d^4x$ and consider therefore the form,

$$S = \int Ld^4x$$

(2)

where

$$L = L_1\Phi + L_2\sqrt{-g}$$

(3)

One can notice that $\Phi$ is the total derivative of something, for example, one can write

$$\Phi = \partial_\mu (\varepsilon^{\mu \nu \alpha \beta} \varepsilon_{abcd} \varphi_a \partial_\nu \varphi_b \partial_\alpha \varphi_c \partial_\beta \varphi_d).$$

(4)

This means that there is a shift symmetry that can be applied on $L_1$

$L_1 \rightarrow L_1 + \text{constant}$. Since such shift just adds the integral of a total divergence to the action (2)-(3) and it does not affect therefore the equations of motion of the theory. In the action (2)-(3) the measure carries degrees of freedom independent of that of the metric and that of the matter fields. The most natural and successful formulation of the theory is achieved when the connection coefficients are also treated as an independent degrees of freedom. This is what is usually referred to as the first order formalism.

Here $L_1$ and $L_2$ are taken to be $\varphi_a$ independent.

There is a good reason not to consider mixing of $\Phi$ and $\sqrt{-g}$, like for example using

$$\frac{\Phi^2}{\sqrt{-g}}$$

(5)

this is because (2)-(3) is invariant (up to the integral of a total divergence) under the infinite dimensional symmetry

$$\varphi_a \rightarrow \varphi_a + f_a(L_1)$$

(6)

where $f_a(L_1)$ is an arbitrary function of $L_1$ if $L_1$ and $L_2$ are $\varphi_a$ independent. Such symmetry (up to the integral of a total divergence) is absent if mixed terms (like (5)) are present. Therefore (2)-(3) is considered for the case when no dependence on the measure fields $\varphi_a$ appears in $L_1$ or $L_2$.
3 The Action Principle for a Scalar Field

We will study now the dynamics of a scalar field $\phi$ interacting with gravity as given by the following action

$$S_\phi = \int L_1 \Phi^4 d^4x + \int L_2 \sqrt{-g} d^4x$$  \hspace{1cm} (7)

$$L_1 = \frac{-1}{\kappa} R(\Gamma, g) + \frac{1}{2} g^{\mu\nu} \partial_\mu \phi \partial_\nu \phi - V(\phi)$$  \hspace{1cm} (8)

$$L_2 = U(\phi)$$  \hspace{1cm} (9)

$$R(\Gamma, g) = g^{\mu\nu} R_{\mu\nu}(\Gamma), R_{\mu\nu}(\Gamma) = R^\lambda_{\mu\nu\lambda}$$  \hspace{1cm} (10)

$$R^\lambda_{\mu\nu\sigma}(\Gamma) = \Gamma^\lambda_{\mu\nu,\sigma} - \Gamma^\lambda_{\mu\sigma,\nu} + \Gamma^\lambda_{\alpha\sigma} \Gamma^\alpha_{\mu\nu} - \Gamma^\lambda_{\alpha\nu} \Gamma^\alpha_{\mu\sigma}.$$  \hspace{1cm} (11)

In the action the measure carries degrees of freedom independent of that of the metric and that of the matter fields. The most natural and successful formulation of the theory is achieved when the connection coefficients are also treated as an independent degrees of freedom. This is what is usually referred to as the first order formalism. Therefore, in the variational principle $\Gamma^\lambda_{\mu\nu}, g_{\mu\nu}$, the measure fields scalars $\varphi_a$ and the scalar field $\phi$ are all to be treated as independent variables although the variational principle may result in equations that allow us to solve some of these variables in terms of others.

4 Global Scale Invariance

If we perform the global scale transformation ($\theta = \text{constant}$)

$$g_{\mu\nu} \rightarrow e^\theta g_{\mu\nu}$$  \hspace{1cm} (12)

then (9) is invariant provided $V(\phi)$ and $U(\phi)$ are of the form

$$V(\phi) = f_1 e^{\alpha \phi}, U(\phi) = f_2 e^{2 \alpha \phi}$$  \hspace{1cm} (13)

and $\varphi_a$ is transformed according to

$$\varphi_a \rightarrow \lambda_a \varphi_a$$  \hspace{1cm} (14)
(no sum on a) which means

$$\Phi \rightarrow \left( \prod_a \lambda_a \right) \Phi \equiv \lambda \Phi$$  \hspace{1cm} (15)

such that

$$\lambda = e^\theta$$  \hspace{1cm} (16)

and

$$\phi \rightarrow \phi - \frac{\theta}{\alpha}.$$  \hspace{1cm} (17)

In this case we call the scalar field $\phi$ needed to implement scale invariance ”dilaton”.

5 The conserved dilatation current

Since there is the symmetry (12), (14), (15), (16), (17), according to Noether’s theorem, there is a conserved current given by (since the variation of the lagrangian density vanishes under such symmetry),

$$j^\mu = \partial L \partial (\partial_\mu \varphi_a) \delta \varphi_a + \partial L \partial (\partial_\mu \phi) \delta \phi$$  \hspace{1cm} (18)

since in the first order formalism $\frac{\partial L}{\partial \varphi_a} = 0$ and $\delta \Gamma^\lambda = 0$ under the symmetry (12), (14), (15), (16), (17).

Let us now consider what we should take for $\delta \varphi_a$. As part of the dilatation symmetry, we have that $\varphi_a \rightarrow \lambda_a \varphi_a$ (no sum on a) and since $\left( \prod_a \lambda_a \right) \equiv \lambda = e^\theta$, we have, taking a transformation infinitesimally close to the identity, i.e. $\lambda_a = 1 + \epsilon_a$, with $\epsilon_a << 1$ and all $\epsilon_a$ equal, so that $\epsilon_a = \theta/4$ and since also $\delta \phi = -\frac{\theta}{\alpha}$, that the conserved dilatation current is,

$$j^\mu_\theta = -\frac{\theta}{\alpha} \Phi \partial^\mu \phi + \theta \varepsilon^{\mu\nu\alpha\beta} \varepsilon_{abcd} \varphi_a \varphi_b \varphi_c \varphi_d \partial_\nu \varphi_\alpha \partial_\alpha \varphi_\beta \partial_\beta \varphi_\delta \partial_\delta L_1 \equiv \theta j^\mu_D$$  \hspace{1cm} (19)

Notice that in the derivation of the conserved current (19) we have taken a very particular choice of the parameters $\lambda_a$ (all of them equal). One can ask the question: what if we relax this assumption?, noticing that one can retain the scale symmetry even when we add to the transformations of the $\varphi_a$ transformations where the volume $\Phi$ is not transformed.
Indeed one can study the effect of the volume preserving diffeomorphisms\(^6,7\)

\[ \varphi'_a = \varphi'_a(\varphi_a) \]  

(20)

such that

\[ \varepsilon_{abcd} \frac{\partial \varphi'_a}{\partial \varphi'_f} \frac{\partial \varphi'_b}{\partial \varphi'_g} \frac{\partial \varphi'_c}{\partial \varphi'_h} \frac{\partial \varphi'_d}{\partial \varphi'_i} = \varepsilon_{fghi} \]  

(21)

In this case, it has been shown that\(^7\) the "composite gauge field" (which enters in (19))

\[ A_{\mu\nu\alpha} = \frac{1}{4} \varphi_a \varepsilon_{abcd} \frac{\partial \varphi_b}{\partial x^\mu} \frac{\partial \varphi_c}{\partial x^\nu} \frac{\partial \varphi_d}{\partial x^\alpha} \]  

(22)

transforms under a diffeomorphism (21) as

\[ A_{\mu\nu\alpha} \to A_{\mu\nu\alpha} + \partial [\nu \Lambda_{\nu\alpha}] \]  

(23)

and in the case the transformation is infinitesimal

\[ \varphi'_a = \varphi_a + \Gamma_a(\varphi_b), \frac{\partial \Gamma_a}{\partial \varphi_a} = 0 \]  

(24)

the divergence free condition in \( \varphi_a \) space for \( \Gamma_a \) implies that

\[ \Gamma_a(\varphi_b) = \frac{1}{2} \varepsilon_{abcd} \frac{\partial \Gamma_{cd}}{\partial \varphi_b} \]  

(25)

for some \( \Gamma_{cd} \) and \( \Lambda_{\nu\alpha} \) in (23) is then given by\(^7\)

\[ \Lambda_{\nu\alpha} = [(1 - \frac{1}{4} \varphi_b \frac{\partial}{\partial \varphi_b}) \Gamma_{cd} - \frac{3}{2} \varphi_b \frac{\partial}{\partial \varphi_c} \Gamma_{b|d|}] \frac{\partial \varphi_c}{\partial x^\nu} \frac{\partial \varphi_d}{\partial x^\alpha} \]  

(26)

In this case, it is a straightforward problem to show that if we perform a volume preserving diffeomorphism in the internal scalar space, to the original dilatation current we add a piece that is separately conserved (one has to use eq. (27) of next section for this elementary proof).

6 The Equations of Motion

We will now work out the equations of motion for arbitrary choice of \( V(\phi) \) and \( U(\phi) \). We study afterwards the choice (15) which allows us to obtain
the results for the scale invariant case and also to see what differentiates this from the choice of arbitrary $U(\phi)$ and $V(\phi)$ in a very special way.

Let us begin by considering the equations which are obtained from the variation of the fields that appear in the measure, i.e. the $\varphi_a$ fields. We obtain then

$$A^\mu_a \partial_\mu L_1 = 0 \quad (27)$$

where $A^\mu_a = \varepsilon^{\mu\alpha\beta\gamma} \varepsilon_{abcd} \partial_\nu \varphi_a \partial_\alpha \varphi_b \partial_\beta \varphi_c \partial_\gamma \varphi_d$. Since it is easy to check that $A^\mu_a \partial_\mu \varphi_a' = \frac{\delta \Phi}{\delta \varphi_a}$ it follows that $\det (A^\mu_a) = 4 \Phi^3 \neq 0$ if $\Phi \neq 0$. Therefore if $\Phi \neq 0$ we obtain that $\partial_\mu L_1 = 0$, or that

$$L_1 = -\frac{1}{\kappa} R(\Gamma, g) + \frac{1}{2} g^{\mu\nu} \partial_\mu \varphi \partial_\nu \varphi - V = M \quad (28)$$

where $M$ is constant.

Let us study now the equations obtained from the variation of the connections $\Gamma^\lambda_{\mu\nu}$. We obtain then

$$-\Gamma^\lambda_{\mu\nu} - \Gamma^\alpha_{\beta\mu} g^{\beta\lambda} \sigma^{\mu\nu} + \delta^\lambda_{\mu} g^{\alpha\beta} \Gamma^\alpha_{\beta\nu} - g^{\alpha\nu} \partial_\mu g^{\alpha\beta} \sigma_{\beta\nu} = 0 \quad (29)$$

If we define $\Sigma^\lambda_{\mu\nu}$ as $\Sigma^\lambda_{\mu\nu} = \Gamma^\lambda_{\mu\nu} - \{^\lambda_{\mu\nu}\}$ where $\{^\lambda_{\mu\nu}\}$ is the Christoffel symbol, we obtain for $\Sigma^\lambda_{\mu\nu}$ the equation

$$-\sigma_{,\lambda} g_{\mu\nu} + \sigma_{,\mu} g_{\nu\lambda} - g_{\nu\alpha} \Sigma^\alpha_{,\lambda\mu} - g_{\mu\alpha} \Sigma^\alpha_{,\nu\lambda} + g_{\mu\nu} \Sigma^\alpha_{,\lambda\alpha} + g_{\nu\lambda} g_{\alpha\mu} g^{\beta\gamma} \Sigma^\alpha_{,\beta\gamma} = 0 \quad (30)$$

where $\sigma = \ln \chi, \chi \equiv \frac{\Phi}{\sqrt{-g}}$.

The general solution of (30) is

$$\Sigma^\alpha_{\mu\nu} = \delta^\alpha_{\mu} \lambda_{,\nu} + \frac{1}{2} \left( \sigma_{,\mu} \delta^\alpha_{\nu} - \sigma_{,\beta} g_{\mu\nu} g^{\alpha\beta} \right) \quad (31)$$

where $\lambda$ is an arbitrary function due to the $\lambda$ - symmetry of the curvature $^8 R^\lambda_{\mu\nu\alpha} (\Gamma), \Gamma^\alpha_{\mu\nu} \rightarrow \Gamma^\alpha_{\mu\nu} = \Gamma^\alpha_{\mu\nu} + \delta^\alpha_{\mu} Z_{,\nu}$ (32)

$Z$ being any scalar (which means $\lambda \rightarrow \lambda + Z$).

If we choose the gauge $\lambda = \frac{\sigma}{2}$, we obtain

$$\Sigma^\alpha_{\mu\nu} (\sigma) = \frac{1}{2} \left( \delta^\alpha_{\mu} \sigma_{,\nu} + \delta^\alpha_{\nu} \sigma_{,\mu} - \sigma_{,\beta} g_{\mu\nu} g^{\alpha\beta} \right). \quad (33)$$

7
Considering now the variation with respect to $g^{\mu\nu}$, we obtain

$$\Phi(\frac{-1}{\kappa} R_{\mu\nu}(\Gamma) + \frac{1}{2} \phi_{,\mu} \phi_{,\nu} - \frac{1}{2} \sqrt{-g} U(\phi) g_{\mu\nu} = 0$$

(34)

Solving for $R = g^{\mu\nu} R_{\mu\nu}(\Gamma)$ and introducing in (28), we obtain a constraint,

$$M + V(\phi) - \frac{2U(\phi)}{\chi} = 0$$

(35)

that allows us to solve for $\chi$,

$$\chi = \frac{2U(\phi)}{M + V(\phi)}.$$  

(36)

To get the physical content of the theory, it is convenient to go to the Einstein conformal frame where

$$\overline{g}_{\mu\nu} = \chi g_{\mu\nu}$$

(37)

and $\chi$ given by (36). In terms of $\overline{g}_{\mu\nu}$ the non Riemannian contribution $\Sigma_{\mu\nu}^{\alpha}$ disappears from the equations, which can be written then in the Einstein form ($R_{\mu\nu}(\overline{g}_{\alpha\beta}) = \text{usual Ricci tensor}$)

$$R_{\mu\nu}(\overline{g}_{\alpha\beta}) - \frac{1}{2} \overline{g}_{\mu\nu} R(\overline{g}_{\alpha\beta}) = \frac{\kappa}{2} T_{\mu\nu}^{\text{eff}}(\phi)$$

(38)

where

$$T_{\mu\nu}^{\text{eff}}(\phi) = \phi_{,\mu} \phi_{,\nu} - \frac{1}{2} \overline{g}_{\mu\nu} \phi_{,\alpha} \phi_{,\beta} \overline{g}^{\alpha\beta} + \overline{g}_{\mu\nu} V_{\text{eff}}(\phi)$$

(39)

and

$$V_{\text{eff}}(\phi) = \frac{1}{4U(\phi)} (V + M)^2.$$  

(40)

In terms of the metric $\overline{g}^{\alpha\beta}$, the equation of motion of the Scalar field $\phi$ takes the standard General - Relativity form

$$\frac{1}{\sqrt{-\overline{g}}} \partial_{\mu}(\overline{g}^{\mu\nu} \sqrt{-\overline{g}} \partial_{\nu} \phi) + V'_{\text{eff}}(\phi) = 0.$$

(41)

Notice that if $V + M = 0$, $V_{\text{eff}} = 0$ and $V'_{\text{eff}} = 0$ also, provided $V'$ is finite and $U \neq 0$ and regular there. This means the zero cosmological constant state
is achieved without any sort of fine tuning. This is the basic feature that characterizes theories with the additional measure $\Phi$ where $L_1$ and $L_2$ are $\varphi_a$ independent and allows them to solve the cosmological constant problem[^4]. It should be noticed that the equations of motion in terms of $\bar{g}_{\mu \nu}$ are perfectly regular at $V + M = 0$ although the transformation (37) is singular at this point. In terms of the original metric $g_{\mu \nu}$, the equations do have a singularity at $V + M = 0$. The existence of the singular behavior in the original frame implies the vanishing of the vacuum energy for the true vacuum state in the bar frame, but without any singularities there.

In what follows we will study (40) for the special case of global scale invariance, which as we will see displays additional very special features which makes it attractive in the context of cosmology.

Notice that in terms of the variables $\phi, \bar{g}_{\mu \nu}$, the "scale" transformation becomes only a shift in the scalar field $\phi$, since $\bar{g}_{\mu \nu}$ is invariant (since $\chi \to \lambda^{-1}\chi$ and $g_{\mu \nu} \to \lambda g_{\mu \nu}$)

$$\bar{g}_{\mu \nu} \to \bar{g}_{\mu \nu}, \phi \to \phi - \frac{\theta}{\alpha}. \quad (42)$$

If $V(\phi) = f_1 e^{\alpha \phi}$ and $U(\phi) = f_2 e^{2\alpha \phi}$ as required by scale invariance (14), (16), (17), (18), (19), we obtain from (40)

$$V_{\text{eff}} = \frac{1}{4f_2^2} (f_1 + M e^{-\alpha \phi})^2 \quad (43)$$

Since we can always perform the transformation $\phi \to -\phi$ we can choose by convention $\alpha > 0$. We then see that as $\phi \to \infty, V_{\text{eff}} \to \frac{f_1^2}{4f_2^2} = \text{const.}$ providing an infinite flat region. For the interpretation of this flat region of the potential in terms of the restoration of a conserved charge and therefore the possibility of interpreting it as the appearance of a Goldstone boson, see the discussion section. A minimum of this effective potential is achieved at zero without fine tuning for any case where $\frac{f_1}{M} < O$

### 7 The Equation of Motion of the Scalar Field from the Conservation Law

We can derive the equations of motion for $\phi$ directly from the variation with respect to $\phi$, this gives eq.(41). It is interesting to see however that from the
local conservation \( \partial_\mu j^\mu_D = 0 \), with \( j^\mu_D \) given by (19) gives exactly the same equation. Indeed, demanding that \( \partial_\mu j^\mu_D = 0 \), with \( j^\mu_D \) given by (19) implies,

\[
\Phi L_1 - \frac{1}{\alpha} \partial_\mu (\Phi g^{\mu\nu} \partial_\nu \phi) = 0
\]  

(44)

but \( \Phi = \chi \sqrt{-g} = \chi^{-1} \sqrt{-g} \) and \( g^{\mu\nu} \Phi = g^{\mu\nu} \chi \sqrt{-g} = g^{\mu\nu} \sqrt{-g} \), so that (44) becomes (using that \( L_1 = M \)) that,

\[
- \alpha \chi^{-1} M + \frac{1}{\sqrt{-g}} \partial_\mu (g^{\mu\nu} \sqrt{-g} \phi) = 0
\]

(45)

but for the form (43) of the effective potential it is satisfied that \( V'_{\text{eff}} = -\alpha \chi^{-1} M \), so that the current conservation is nothing but the scalar field equation when the choice (13) is made.

8 The infrared behavior of the dilaton current and the reason for Goldstone’s theorem breakdown

To see the basic reasons why the dilatation current (19) has an infrared singular behavior, let us consider the spatial behavior of the \( \phi_a \) fields for the case of a simple spatially flat Robertson-Walker solution of the form

\[
d s^2 = -dt^2 + R^2(t)(dx^2 + dy^2 + dz^2), \phi = \phi(t)
\]

(46)

From eq.(35) and (46), we see also that \( \chi = \chi(t) \). Then, since \( \chi = \chi(t) = \frac{\Phi}{R^3(t)} \), we get that,

\[
\Phi = R^3(t) \chi(t) = \varepsilon^{\mu
\nu\alpha\beta} \varepsilon_{abcd} \partial_\mu \phi_a \partial_\nu \phi_b \partial_\alpha \phi_c \partial_\beta \phi_d
\]

(47)

(47) can be solved by taking

\[
\phi_1 = x, \phi_2 = y, \phi_3 = z, \phi_4 = -\frac{1}{4!} \int \chi(t') R^3(t') dt'
\]

(48)

For this case, with a time dependent scalar field \( \phi(t) \) and with \( \phi_a \) given by (48), the spatial components of the current \( j^\mu_D \), as given by (19) diverge
linearly as $x^i \to \infty$ ($x^1 = x, x^2 = y, x^3 = z$). In fact $j_D^i \to M x^i \chi(t) R^3(t)$ as $x^i \to \infty$.

Such current does indeed give flux at infinity. The current grows linearly with distance, so that the total flux is proportional to the volume enclosed and obviously the total dilatation charge is not conserved here.

In the context of theories with additional measure $\Phi$, there are other instances where Goldstone’s theorem can fail. For example in (7)-(11), take the model, without scale invariance where $U(\phi) = \Lambda = \text{constant}$ and $V(\phi) = J\phi$, the model has a symmetry up to the integral of a total divergence, $\phi \to \phi + c, c = \text{constant}$. In this case, since $V_{\text{eff}} = \frac{1}{2\Lambda}(J\phi + M)^2$, we see again that no Goldstone boson is present in the particle spectrum, as was observed in the last paper of Ref.4, although the conserved currents were not obtained there. Working out the conserved current associated with this symmetry, we see that it is $j_{\text{shift}}^\mu = \Phi \partial^\mu \phi + J\varepsilon^{\mu\nu\alpha\beta} \varepsilon_{abcd} \phi_\alpha \partial_\nu \phi_\beta \partial_\alpha \phi_\delta \partial_\beta \phi_\gamma$ which has a singular infrared behavior, exactly for the same reasons the dilatation current has (i.e. because of the singular behavior of the $\phi_a$ fields at spatial infinity).

Notice that the potential $V_{\text{eff}} = \frac{1}{4\Lambda}(J\phi + M)^2$ is contained in the class of potentials being discussed here, i.e. $V_{\text{eff}} = \frac{1}{4f_2}(f_1 + M e^{-\alpha\phi})^2$, for the limit $\alpha \to 0, \alpha M \to \text{constant}, \alpha^2 M, \alpha^3 M, \ldots \to 0$, so that $V_{\text{eff}} = \frac{1}{4f_2}(f_1 + M - M\alpha\phi)^2$, so that if $f_1 + M$ is kept fixed in this limit, we obtain a purely quadratic potential. The flat region has, in this limit been pushed out and has gone away.

9 Discussion and Conclusions

We have studied the structure of the conserved quantities in the scale invariant theories where an additional measure $\Phi$ is introduced.

In this case, we have seen that although the global scale invariance plays a crucial role in determining the structure of the theory, allowing only a very specific type of effective potential for a scalar field, there are however no Goldstone Bosons associated with the spontaneous breaking of this symmetry.

The reason for this is that although a locally conserved dilatation current can be defined, the spatial part of the current has a singular infrared behavior, making it impossible then to prove that the dilatation charge will be
conserved. As we have seen this also happens in other global shift-like sym-
metries, which can be understood as singular limits of the scale symmetries
discussed in the body of the paper.

An interesting phenomena that takes place in these scale invariant models
is that as long as we go to a regime where the constant of integration $M$ can
be ignored, a dilatation symmetry charge appears to be conserved and a
flat potential, which means an associated Goldstone boson appears in this
regime.

In fact, we see that the infrared singular part of the current (19) is that
proportional to $M$ (recall that $L_1 = M$) and if $M$ can be ignored for some
reason and set to zero, such infrared problems go away. In terms of the
effective potential $V_{\text{eff}}$, we see that as long as $M$ can be ignored, for $\alpha \phi \to \infty$, $V_{\text{eff}}$ becomes a flat potential and $\phi$ becomes in this regime a true Goldstone
boson.

We see therefore that potentials with flat regions, as required in the cos-
mological models of new inflation\textsuperscript{9,10} for example can appear from a first
principle, scale invariance. Such flat potentials may be of interest also in the
present stages of the Universe\textsuperscript{5}.

For the discussion we have limited ourselves to the simplest case where
we have discussed only a scalar field. The introduction of scale invariant
masses for other fields does not change in any way the qualitative features
like the singular infrared behavior and associated non conservation of the
scale charge which is the focus of this paper.

\section{Acknowledgments}

I would like to thank J. Bekenstein, A. Davidson, A.Guth, A. Kaganovich,
P.Mannheim, E.Nissimov, S.Pacheva and L.C.R. Wijewardhana for conve-
sations on the subjects discussed here. In particular, I thank P.Mannheim for
strongly encouraging me to look at the reasons why Goldstone’s theorem can
fail for some symmetries in theories with the additional measure $\Phi$ discussed
here and for discussions on scale invariance, a demand we both agree has to
play a fundamental role in gravity, but where he has developed a different
approach\textsuperscript{11} to the one explained here.
References

[1] Y.Nambu and G. Jona-Lasinio, Phys. Rev. 122 (1961) 345; Phys. Rev. 124 (1961) 246; J.Goldstone, Nuovo Cimento 19 (1961) 15.

[2] See for example S.Coleman’s Erice lectures ”Dilatations”, reprinted in S.Coleman, ”Aspects of Symmetry”, Cambridge University Press (1985).

[3] G.t’Hooft, Phys. Rev. Lett. 37 (1976) 8; Phys. Rev. D14 (1976) 3432.

[4] E.I. Guendelman and A.B. Kaganovich, Phys. Rev., D53, (1996) 7020; E.I. Guendelman and A.B. Kaganovich, Proceedings of the third Alexander Friedmann International Seminar on Gravitation and Cosmology, ed. by Yu. N. Gneding, A.A. Grib and V.M. Mostepanenko (Friedmann Laboratory Publishing, St. Petersburg, 1995); E.I. Guendelman and A.B. Kaganovich, Phys. Rev., D55, (1997) 5970; E.I. Guendelman and A.B. Kaganovich, Mod. Phys. Lett., A12, (1997) 2421; E.I. Guendelman and A.B. Kaganovich, Phys. Rev., D56, (1997) 3548; E.I. Guendelman and A.B. Kaganovich, Hadronic Journal, 21, (1998) 19; E.I. Guendelman and A.B. Kaganovich, Mod. Phys. Lett., A13, (1998) 1583; F. Gronwald, U. Muench and F.W. Hehl, Hadronic Journal, 21, (1998) 3; E.I. Guendelman and A.B. Kaganovich, ”Gravity Cosmology and Particle Field Dynamics without the Cosmological Constant Problem”, to appear in the Proceedings of the sixth International Symposium on Particle, Strings and Cosmology, PASCOS-98; E.I. Guendelman, ”Gauge Condensates and Gauge Dynamics, the cosmological and strong CP problems”, to appear in the Int. Journ. of Mod. Phys. A.; E.I. Guendelman and A.B. Kaganovich, ”Field Theory Models without the Cosmological Constant problem”, Plenary talk (given by E.I. Guendelman) at the fourth Alexander Friedmann International Seminar on Gravitation and Cosmology, [gr-qc/9809052](https://arxiv.org/abs/gr-qc/9809052) and extended version of this, [gr-qc/9905029](https://arxiv.org/abs/gr-qc/9905029), to appear in Phys. Rev. D15.

[5] E.I. Guendelman, [gr-qc/9901017](https://arxiv.org/abs/gr-qc/9901017), to appear in Mod. Phys. Lett. A; E.I. Guendelman, [gr-qc/9901064](https://arxiv.org/abs/gr-qc/9901064).

[6] E.I.Guendelman, E.Nissimov and S.Pacheva, Phys. Lett. B360 (1995) 57; C.Castro, Int. Journ. of Mod. Phys. A13 (1998) 1263.
[7] E.I. Guendelman, E. Nissimov and S. Pacheva, [hep-th/9903245].

[8] A. Einstein, "The Meaning of Relativity", MJF books, NY (1956), see Appendix II.

[9] For a non technical review and a good collection of further references on different aspects of inflation see A. Guth, "The Inflationary Universe", Vintage, Random House (1998). For a more technical review see E.W. Kolb and M.S. Turner, "The Early Universe", Addison Wesley (1990).

[10] The original papers on new inflation are A.D. Linde, Phys. Lett, 108B, (1982) 389; A. Albrecht and P.J. Steinhardt, Phys. Rev. Lett, 48, (1982) 1220.

[11] see for example, P. Mannheim, [gr-qc/9903005] and references.