Shortest Odd Paths in Conservative Graphs: Connections and Complexity

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Abstract

We present some reductions between optimization problems for undirected conservative graphs, that is, edge-weighted graphs without negative cycles. We settle the complexity of some of them, and exhibit some remaining challenges. Our key result is that the shortest odd path problem between two given vertices, and its variants, such as the shortest odd cycle problem through a given vertex, turn out to be NP-hard, deciding a long-standing question by Lovász (Open Problem 27 in Schrijver’s book [13]), in the negative. The complexity of finding a shortest odd cycle for conservative weights or of finding an odd $T$-join of minimum cardinality remains open. We finally relate these problems to relevant, solved or hopeful variants.

1 Introduction

Given an undirected graph $G = (V, E)$, for some $F \subseteq E$ and $v \in V$ let $d_F(v)$ denote the degree of $v$ in $F$, i.e., the number of edges in $F$ incident to $v$. Let $V(F)$ denote the set of vertices that are incident to some edge in $F$. A cycle in an undirected graph $G = (V, E)$ is a set $C$ of edges such that the subgraph $(V(C), C)$ is connected and $d_C(v) = 2$ for each vertex $v \in V(C)$. In a digraph $G = (V, E)$, a (directed) cycle additionally satisfies that all in- and out-degrees in $(V(C), C)$ are equal to 1. For two distinct vertices $s$ and $t$ in a graph, an $(s, t)$-path has the same definition except that the two endpoints, $s$ and $t$, have degree 1 in the undirected case; in the directed case, $s$ has in-degree 0 and out-degree 1, while $t$ has out-degree 0 in-degree 1. A cycle $C$ with $s \in V(C)$ is also considered to be an $(s, t)$-path with $s = t$. For two sets of vertices $S, T \subseteq V$ of a graph, an $(S, T)$-path is an $(s, t)$-path for some $s \in S$ and $t \in T$.

Two paths are said to be vertex-disjoint if they do not have a common vertex, and they are said to be openly disjoint if they may have common endpoints but no other common vertices.
A T-join in an undirected graph \( G = (V, E) \) for some \( T \subseteq V \) is a subset of edges \( J \subseteq E \), such that \( d_J(v) \) is odd if \( v \in T \), and even if \( v \in V \setminus T \). An \( \emptyset \)-join is the disjoint union of cycles; inclusionwise minimal, non-empty \( \emptyset \)-joins are exactly the cycles. A T-join with \( |T| = 2 \), that is, with \( T = \{s, t\} \subseteq V \) is the disjoint union of an \((s, t)\)-path and some cycles, so the inclusionwise minimal ones are \((s, t)\)-paths.

Consider the following two problems, which differ only in that the first one restricts the searched cycle to contain a given vertex:

**SHORTEST ODD CYCLE IN CONSERVATIVE GRAPHS THROUGH A POINT** (SOCp)

**Input:** An undirected, conservative graph \( G = (V, E) \) with \( w : E \to \{1, -1\} \), \( p \in V \), and \( k \in \mathbb{Z}_+ \).

**Question:** Is there an odd cycle \( C \) in \( G \) with \( p \in V(C) \) whose weight is at most \( k \)?

**SHORTEST ODD CYCLE IN CONSERVATIVE GRAPHS** (SOC)

**Input:** An undirected, conservative graph \( G = (V, E) \) with \( w : E \to \{1, -1\} \) and \( k \in \mathbb{Z}_+ \).

**Question:** Is there and odd cycle \( C \) in \( G \) whose weight is at most \( k \)?

The key result of this note is the NP-completeness of SOCp (Theorem 2), in contrast with SOC, which remains open. As an immediate corollary, the Shortest Odd Path (SOP) problem between two given vertices in a conservative graph – Problem 27 in Schrijver’s book [13], attributed to Lovász – is also NP-complete as well as some other, closely related problems. After these surprising but not difficult complexity results, various compromises, special cases and weakenings may get more focus, and these are discussed in Section 3. We finish this introduction by some general remarks situating our problems.

The problem SOP already contains the undirected shortest path problem in conservative graphs. Indeed, to find a shortest \((s, t)\)-path, simply add a new vertex \( t' \), and add both an edge and a path of length two from \( t \) to \( t' \), each with weight 0, then find a shortest odd \((s, t')\)-path. More interestingly, even the SOC problem contains the undirected shortest path problem in conservative graphs, as we will see in Section 3.2.

Don’t be confused: the shortest path problem for conservative undirected graphs cannot be solved with the shortest path algorithms you know for conservative digraphs. In order to use them, you would have to include each edge in both directions, and negative edges would lead to negative cycles consisting of two edges. The algorithms for directed graphs are all based on the fact that subpaths of shortest paths are shortest and the triangle inequality holds, which is not
true in the undirected case: the undirected case contains minimum-weight matching problems, and conversely, finding a shortest \((s, t)\)-path with conservative weights is equivalent to finding a shortest \(s, t\)-join \([13, \text{Section 29.2}]\), a problem that can be easily reduced, for arbitrary weights, to weighted matching problems \([7]\). This is a polynomial-time algorithm, but much more difficult than the analogous algorithms for directed graphs.

Requiring odd parity from the path will lead, as mentioned, to \(\text{NP}\)-completeness even for closed paths (SOCp). However, the SOC problem has been studied from multiple aspects (and under various names), and has lead to deep results that may lead to polynomial algorithms: Minimax theorems for the equivalent problem of finding a shortest odd \(T\)-join hold if some “bad minors” are excluded; see the works by Guenin \([11]\) and Abdi \([1]\). However, on the one hand, no polynomial-time algorithm has been exhibited, not even for this special case in \(\text{NP} \cap \text{coNP}\). On the other hand, a randomized polynomial-time algorithm has been given for SOC by Geelen and Kapadia \([8]\), making polynomial-time solvability plausible. In Section 3.2 we discuss these connections.

2 \(\text{NP}\)-completeness

We present now a well-known \(\text{NP}\)-complete problem that will be reduced to SOCp. Its planar special case is known to be one of the simplest open disjoint paths problems.

**BACK AND FORTH PATHS (BFP)**

**Input:** A digraph \(\hat{G} = (\hat{V}, \hat{E})\) and \(s \neq t \in \hat{V}\).

**Question:** Are there two openly disjoint paths, one from \(s\) to \(t\), the other from \(t\) to \(s\)?

**Theorem 1** ([4, Theorem 2]). **BFP is \(\text{NP}\)-complete.**

**Theorem 2.** \(\text{SOCp}\) is \(\text{NP}\)-complete, even when the negative edges form a matching, \(k = 1\), and there exists \(t \in V\) so that \(G - t\) is bipartite.

**Proof.** SOCp is clearly in \(\text{NP}\). Let the digraph \(\hat{G} = (\hat{V}, \hat{E})\) with vertices \(s, t \in \hat{V}\) be an instance of BFP, and construct from it an undirected graph as follows. Split each vertex \(v \in \hat{V} \setminus \{t\}\) to an out-copy \(v_1\) and an in-copy \(v_2\), except for leaving \(t\) as it is, but defining \(t_1 := t_2 := t\). For each arc \(uv \in \hat{E}\) define an edge \(u_1v_2\) with \(w(u_1v_2) := 1\). Furthermore, add an edge \(v_1v_2\) for each \(v \in \hat{V} \setminus \{t\}\) with \(w(v_1v_2) := -1\).

Denote \(V_i := \{v_i : v \in \hat{V}\}\) for \(i = 1, 2\), and \(E := \{u_1v_2 : uv \in \hat{E}\} \cup \{v_1v_2 : v \in \hat{V} \setminus \{t\}\}\), so that the constructed (undirected) graph is \(G = (V_1 \cup V_2, E)\), and let \(k := 1\). Clearly, the negative edges form a matching, and thus the weight function \(w\) is conservative. Note that \(G - t\) is bipartite, so all odd cycles contain \(t\).

**Claim:** \(\forall \hat{C} \subseteq \hat{E}\) cycle, \(s, t \in V(\hat{C})\), \(\exists C\) cycle in \(G\), \(w(C) = 1\), \(s_1 \in V(C)\), and vice versa.

Indeed, let \(\hat{C} \subseteq \hat{E}\) be a (directed) cycle in \(\hat{G}\) with \(s, t \in V(\hat{C})\), and let us associate with it the (undirected) cycle \(C := \{u_1v_2 : uv \in \hat{C}\} \cup \{v_1v_2 : v \in V(C) \setminus \{t\}\}\) in \(G\). The cycle \(C\) alternates between edges of weight 1 and \(-1\) in every vertex but \(t\), so \(w(C) = 1\), and \(s_1 \in V(C)\).
Conversely, a cycle $C \subseteq E$ in $G$ with $w(C) = 1$ and $s_1 \in V(C)$ must be an odd cycle due to its weight, so $t \in V(C)$ follows as noted earlier. Moreover, $C - t$ must alternate between edges of weight $-1$ and $1$, so $C$ corresponds to a directed cycle $\hat{C} \subseteq \hat{E}$. Since $s_1 \in V(C)$ by definition, and we know $t \in V(C)$ as well, the cycle $\hat{C}$ contains both $s$ and $t$, so the claim is proved.

The claim shows that our construction reduces BFP to SOCp, since a solution of BFP is exactly a cycle $\hat{C} \subseteq \hat{E}$ in $\hat{G}$ with $s, t \in V(\hat{C})$, and according to the claim such a cycle exists if and only if there exists an odd cycle $C$ in $G$ of weight at most $1$ containing $s_1$; note that an odd cycle of weight at most $1$ can have neither weight $0$ (due to its parity) nor negative weight (due to conservativeness), so must have weight exactly $1$. The instance $(G, p := s_1, k := 1)$ of SOCp to which BFP is reduced satisfies the additional assertions, as checked above, so we can conclude that SOCp is NP-complete and already for the family of the claimed particular instances.

By simply inspecting the instances of the above proof, the NP-hardness of the following problem of Lovász ([13, Problem 27, pp. 517, 1456]) is an immediate corollary.

**SHORTEST ODD PATH IN CONSERVATIVE GRAPHS (SOP)**

**Input:** An undirected, conservative graph $G = (V, E)$ with $w : E \to \{1, -1\}$, $s, t \in V$, and $k \in \mathbb{Z}$.

**Question:** Is there and odd $(s, t)$-path with total weight at most $k$?

**Corollary 3.** SOP is NP-complete, even when the negative edges form a matching, $k = 1$, and there exists $t \in V$ so that $G - t$ is bipartite.

**Proof.** SOCp is the special case of SOP where $s = t$, so we are done. If we want to require $s \neq t$, then with the notation of the proof of Theorem 2, observe that the instance $(G, s_1, k = 1)$ of SOCp has a ‘yes’ answer if and only if there exists an odd $(s_1, s')$-path of weight $k = 1$ in the graph $G'$ obtained from $G$ by replacing the edge $s_1s_2$ with an $s's_2$ edge of weight $-1$ for a new vertex $s'$.

It is tempting to use the following version DISP for polynomial algorithms in new cases:

**DISJOINT SHORTEST PATHS IN CONSERVATIVE GRAPHS (DISP)**

**Input:** An undirected, conservative graph $G = (V, E)$ with $w : E \to \{1, -1\}$, $s_1, s_2, t_1, t_2 \in V$, and $k \in \mathbb{Z}_+$.

**Question:** Does $G$ contain two vertex-disjoint $\{(s_1, s_2), (t_1, t_2)\}$-paths with total weight at most $k$?

However, DISP also turns out to be NP-complete:

**Corollary 4.** DISP is NP-complete, even when the negative edges form a matching, and $G$ is bipartite.

**Proof.** We reduce from BFP using the same construction as in the proof of Theorem 2 with the only difference that we split all vertices of the input digraph $\hat{G} = (\hat{V}, \hat{E})$, including $t$, add the edge $t_1t_2$ to $E$, and define $w(t_1t_2) := -1$. Then the resulting graph $G$ is bipartite, and $(\hat{G}, s, t)$ is
Figure 1: An example where edges of the path $P^-$ have weight $-1$, shown as red, bold lines, with all remaining edges having weight 1. An odd cycle containing $s$ must also contain $t$, and the unique such cycle yields also a solution for DISP (with vertices $s_1, s_2, t_1, t_2$), and also a shortest odd $(s_1, s_2)$-path.

a ‘yes’-instance of BFP if and only if there exists a cycle $C$ of weight 0 in $G$ containing both $s_1$ and $t_1$, which in turn holds if and only if there exist two vertex-disjoint $(\{s_1, s_2\}, \{t_1, t_2\})$-paths of total length $k = 2$ in $G$.

Corollary 4 contrasts the well-known fact that finding two disjoint $(\{s_1, s_2\}, \{t_1, t_2\})$-paths for some vertices $s_1, s_2, t_1$, and $t_2$ with minimum total weight in an undirected graph with non-negative edge weights is a standard classical minimum-cost flow problem [13]. The example depicted in Figure 1 gives some intuition on the strong connection between DISP and our problems SOCp and SOP.

3 Connections, Questions and Conclusion

In this section we establish further connections between the problems we have been studying to some known results and open questions.

3.1 Classical Results

Forgetting the parity: We mentioned in Section 1 the simple fact that a shortest $(s, t)$-path in a conservative graph can be determined by finding an inclusionwise minimal shortest $\{s, t\}$-join, which is in fact equivalent to the minimum-weight $T$-join problem for arbitrary, not necessarily conservative weights. The first, well-known solution of the latter problem reduces it to non-negative weights, and then solves it as a weighted matching problem on $T$ [7], [13, Section 29.2].

Non-negative weights: A shortest odd (or, equivalently, even) path between two vertices in an undirected graph with non-negative edge weights can also be determined by a well-known simple polynomial-time reduction to minimum-weight matchings. (The algorithm is “Waterloo folklore”, related to Edmonds’ classical work on matchings [9, 13]). Strangely, the Odd Path Polyhedron (the “dominant” of odd paths, and the related integer minimax theorem [14], see also [13]) have been determined much later.
**Odd T-joins:** They have been intensively studied in terms of the “idealness” (integrality) of their blocking polyhedra. Idealness roughly means that good characterization (minimax) theorems hold for the minimization of odd T-joins for non-negative weight functions, equivalent to general weight functions in this case. We point only at some of the most recent references, which may send the interested reader to further connections, results and conjectures:

Guenin [11] characterized in terms of two small excluded minors when inclusionwise minimal odd T-joins are “ideal” in the \(|T| = 2\) special case, and Abdi [1] proved that actually a stronger minimax theorem holds then. However, no polynomial-time algorithm has been deduced for these privileged cases either, and also no \(NP\)-hardness result has been proved for the shortest odd T-join problem in general.

**MINIMUM-WEIGHT ODD T-JOIN (MOTJ)**

**Input:** An undirected graph \(G = (V, E)\) with \(w : E \to \mathbb{Q}\), \(T \subseteq V\), and \(k \in \mathbb{Q}\).

**Question:** Is there an odd T-join with total weight at most \(k\)?

A minimum-weight odd T-join consists of a T-join and some cycles. For conservative weights, only inclusionwise minimal odd T-joins play a role for minimization. Such an odd T-join \(J\) contains at most one cycle; moreover, this cycle, if it exists, is odd; if \(|T| = 0\), then in particular, \(J\) is an odd cycle. If \(|T| = 2\), say \(T = \{s, t\}\), then even if weights are positive, \(J\) may consist of an even \((s, t)\)-path and an odd cycle \(C\); then \(C\) can have at most one common vertex with \(J \setminus C\), as observed in [10]. Minimum-weight odd \(\{s, t\}\)-joins may actually be more easily tractable than the NP-complete SOP:

A randomized polynomial-time algorithm by Geelen and Kapadia [8] places MOTJ in the class RP, saving the problem from being suspected to be NP-hard (which would imply \(NP \subseteq RP\)), and suggesting the following conjecture (for a full range of equivalent conjectures, cf. Theorem 7):

**Conjecture 5.** MOTJ and SOC can be solved in polynomial time.

The “max side” of the mentioned minimax theorems concerns transversals of odd T-joins which in the simplest case of odd cycles (i.e., \(T = \emptyset\)) are easily seen to be exactly the complements of cuts: their minimization is equivalent to the Maximum Cut problem, one of the sample \(NP\)-hard problems. However, for planar graphs the duality between faces and vertices reduces this problem to the shortest T-join problem [2], solving Maximum Cut for planar graphs; for graphs embeddable into the projective plane the corresponding reduction is to MOTJ, and only a partial solution could be given to the corresponding special case of MOTJ [6].

SOC can clearly be reduced to SOCp but the opposite reduction seems to organically resist. This is an analogous situation to the problem of finding a minimum-weight odd hole (an induced cycle of length at least four) through a given vertex is NP-complete [5], while without the requirement of containing a given vertex it has been recently proved to be polynomially solvable [3].

The applications and relevance of this family of problems and the mentioned signs of its tractability leading to the above conjecture makes it interesting to clarify the polynomial equivalence of closely related problems, as we do in Section 3.2.
Proposition 6. Let \( w : E \to \mathbb{R} \) be arbitrary, and \( T \subseteq V \) with \( |T| \) even. If \( F \) is a \( w \)-minimum \( T \)-join and \( |F| \) is even, then \( F_{\text{odd}} \) is a \( w \)-minimum odd \( T \)-join if and only if \( F_{\text{odd}} = F \Delta C \) for some \( w[F] \)-minimum odd \( \emptyset \)-join \( C \).

Recall that inclusionwise minimal odd \( \emptyset \)-joins are cycles. The proposition is illustrated by Figure 2. Note that an optimal odd \( \{s,t\} \)-join may be the disjoint union of an even \( (s,t) \)-path and an odd cycle. Recall also the well-known fact that \( F \) is a \( w \)-minimum \( T \)-join if and only if \( w[F] \) is conservative \([10, 13]\).

Proof of Proposition 6. Observe first that if \( F_{\text{odd}} \subseteq E \) is an odd \( T \)-join, then \( F_{\text{odd}} \Delta F \) is an odd \( \emptyset \)-join, and conversely, if \( C \subseteq E \) is an odd \( \emptyset \)-join, then \( F \Delta C \) is an odd \( T \)-join. Hence, there is a one-to-one correspondence between odd \( T \)-joins and odd \( \emptyset \)-joins, with \( F_{\text{odd}} \) corresponding to \( F \Delta F_{\text{odd}} \) and vice versa. Note further that

\[
w[F](F \Delta F_{\text{odd}}) = w(F_{\text{odd}} \setminus F) - w(F \setminus F_{\text{odd}}) = w(F_{\text{odd}}) - w(F).
\]

Since \( w(F) \) is a fixed value, we obtain that \( F \Delta F_{\text{odd}} \) minimizes \( w[F] \) over all odd \( \emptyset \)-joins exactly if \( F_{\text{odd}} \) minimizes \( w \) over all odd \( T \)-joins.

Theorem 7. The following problems are polynomially equivalent:

(i) MOTJ;
(ii) MOTJ with conservative weights;
(iii) MOTJ with non-negative weights;
(iv) MOTJ with conservative weights for \( T = \emptyset \);
(v) SOC for arbitrary conservative weights.
Proof. A polynomial-time algorithm for (i), i.e., MOTJ in general, clearly implies one for (ii), which, in turn, implies one for (iii).

To prove the polynomial-time solvability of (iv) from that of (iii), consider the input of MOTJ with $T = \emptyset$ consisting of a graph $G = (V, E)$ and a conservative $w$. We can assume that $G$ contains an even number of negative edges, since otherwise we can simply add to $G$ an edge of weight $-1$ incident to a new vertex. Define now a non-negative weighted instance $(G, |w|, T)$ of MOTJ with $T := \{ v \in V : d_{E^-}(v) \text{ is odd} \}$ where $E^- := \{ e \in E : w(e) < 0 \}$. Then $E^-$ is a $|w|$-minimal $T$-join, and it is even. Now by Proposition 6, $J$ is a $|w|$-minimal odd $T$-join if and only if $C := J \Delta E^-$ is a $w = |w||E^-|$-minimal odd $\emptyset$-join. Hence, an algorithm for (iii) applied to $(G, |w|, T)$ yields a solution for our instance $(G, w)$ of (iv).

The claim that polynomial-time solvability of (iv) implies the same for (v) follows by noting that a solution for (v) can be obtained from a solution for (iv) with the same input instance by deleting the 0-weight even cycles, and possibly all but one 0-weight odd cycle.

We have thus asserted the path of implications from the polynomial-time solvability of (i) to that of (v). A polynomial-time algorithm for (i) follows from one for (v) by Proposition 6, since a shortest odd cycle for conservative weights is always a minimum-weight odd $\emptyset$-join. \qed

Solving MOTJ with $|T| \leq 2$ and non-negative weights in polynomial time is relatively easy (Section 3.1). However, if only one of $T$ and $w$ is restricted, then the general problem can be reduced to these (seemingly) more special ones (Section 3.2). The case where the absolute values of the weights are 1 are not proved to be easier than for general weights, for any of the problems.

### 3.3 Conclusion

We proved the NP-completeness of shortest odd path problems in conservative undirected graphs, answering a long-standing question of Lovász [13, Problem 27], and exhibited some related, still open relevant challenges. Another interesting research direction is now to study the parameterized complexity and approximability of the SOP problem and its other NP-hard variants. Some initial FPT results have been achieved by part of our research group formed during the 12th Emléktábla Workshop, Gárdony, Hungary, 2022 [12].

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