Comparison of relatively unipotent log de Rham fundamental groups

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Abstract

In this paper, we prove compatibilities of various definitions of relatively unipotent log de Rham fundamental groups for certain proper log smooth integral morphisms of fine log schemes of characteristic zero. Our proofs are purely algebraic. As an application, we give a purely algebraic calculation of the monodromy action on the unipotent log de Rham fundamental group of a stable log curve. As a corollary we give a purely algebraic proof to the transcendental part of Andreatta-Iovita-Kim’s article: obtaining in this way a complete algebraic criterion for good reduction for curves.

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Introduction

Unipotent fundamental groups have been used to obtain several results from number theory to algebraic geometry of hyperbolic curves: from finiteness of integral points ([Ki05], [Had11]) to the study of good/bad reduction in families ([Od95], [AIK15]). In all these studies, the motivic nature of unipotent fundamental groups plays an important role. Namely, the realizations (classical, étale, de Rham and crystalline) of unipotent fundamental groups and comparison theorems between them appear in crucial way in the proofs. Moreover, in the case of semistable reduction, the comparison theorems are ‘logarithmic and relative in nature’ in the sense that, in the formulation of comparison theorems, the log scheme defined by the reduction naturally appears as well as the monodromy action on unipotent fundamental group.

To explain the motivation of our study, let us recall the results in [Od95], [AIK15].

In [Od95], Oda proved the following two results.

(1) For a family of proper hyperbolic complex curves $f : X \rightarrow \Delta^*$ over a punctured disc $\Delta^*$ with semistable reduction at the center, the outer monodromy action of $\pi_1(\Delta^*) \cong \mathbb{Z}$ on the classical unipotent fundamental group of the generic fiber of $f$ is trivial if and only if $f$ has good reduction at the center.

(2) For a proper hyperbolic curve $f : X \rightarrow \text{Spec} K$ over a discrete valuation field $K$ of mixed characteristic $(0,p)$ with semistable reduction, the outer monodromy action of $\pi_1(\text{Spec} K)$ on the $l$-adic unipotent fundamental group $(l \neq p)$ of the geometric generic fiber of $f$ is unramified if and only if $f$ has good reduction.

Also, in [AIK15], Andreatta-Iovita-Kim proved the following result.

(3) For a proper hyperbolic curve $f : X \rightarrow \text{Spec} K$ over a discrete valuation field $K$ of mixed characteristic $(0,p)$ with semistable reduction and a section $\iota$, the monodromy action of $\pi_1(\text{Spec} K)$ on the $p$-adic unipotent fundamental group of the geometric generic fiber of $f$ induced by $\iota$ is crystalline if and only if $f$ has good reduction.

Oda proved the claim (2) by reducing to (1). In fact, he considers a semistable family of hyperbolic curves over $R[[z]]$ (where $R$ is a certain discrete valuation ring) which contains the original semistable family as the locus $z = \pi$ (where $\pi$ is a uniformizer of the valuation ring $O_K$ of $K$) and then he compares the $l$-adic and classical unipotent

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fundamental groups. As a matter of fact, Andreatta-Iovita-Kim proved (3) again by reducing to (1): they use relative $p$-adic Hodge theory (for the above semistable family over $\mathbb{R}[[z]]$), to get the claim on the unipotent log de Rham fundamental group and comparing it with the classical unipotent fundamental group. Thus the main computation of the monodromy action is done in the situation given in (1), i.e. via transcendental methods.

The final purpose of this article, which will be completed in Section 6, is to show that Oda’s proof of (1) is, in some sense, motivic. Namely, (a suitably modified version of) Oda’s proof can be done for unipotent log de Rham fundamental group in purely algebraic way. Thus, by combining with the $p$-adic Hodge part of [AIK15], we can give a purely algebraic proof of the result (3) of Andreatta-Iovita-Kim.

In order to achieve such a result we have to study unipotent log de Rham fundamental group. There are several methods to define such a group, and each method has its own advantage. In the situation of (1), Oda described the fundamental group of the generic fiber of $f$ in terms of graph of groups associated to the dual graph and the fundamental groups of irreducible components of the central fiber, and he proved (1) by using this description and some concrete computation of the monodromy action on certain paths. In order to do a similar argument for unipotent log de Rham fundamental groups, we need the notion of paths, and for this purpose, the definition of them via Tannakian category ([D89], [Laz15, 2.1]) is suitable. On the other hand, we need to calculate concretely the unipotent (log) de Rham fundamental group of curves in purely algebraic way, and for this purpose, the definition of them via minimal model ([GM13], [NA87], [NA93]) is suitable. Also, to finish a purely algebraic proof of the result (3) of Andreatta-Iovita-Kim, we need to use their definition of fundamental group, which is based on the construction of unipotent fundamental group given by Hadian ([Had11]).

So we need to compare the three definitions of unipotent log de Rham fundamental group above. Note also that we need to prove the comparison in relative setting because we need the compatibility of monodromy actions. This comparison results, which are interesting in themselves, are the subject of the first five sections of this article. We prove them in much more general setting than it will be needed in the last section.

We give a more detailed description of each sections. In Section 1, we give some preliminaries on modules with integrable log connection: we will introduce the geometric setting which we consider in the first five sections and we will define various categories related to the category of modules with integrable log connection which will be used later. In particular, we will show the relations between modules with integrable log connection, stratifications and crystals in this relative setting. We will see later how this relative stratification structure will induce the monodromy action on the unipotent log de Rham fundamental groups. In Section 2, we will use the categories defined in the previous section to give our first definition (via Tannakian formalism) of the unipotent log de Rham fundamental group together with the monodromy action (as a log connection). We also prove that the definition
of monodromy action is the same as that one coming from the unipotent version of homotopy exact sequence, although a part of the proof will be postponed in the next section. This is essentially a log version of a result of Lazda [Laz15, Theorem 1.6]. In Section 3, we will introduce the second definition of unipotent log de Rham fundamental group: we will use the notion of pointed universal object. This is a generalization to higher dimension of the definition of Hadian [Had11] and Andreattta-Iovita-Kim [AIK15], as well as a log version of the construction which appeared in Lazda’s paper [Laz15, pp.13–15]. We prove that this new definition is equivalent to that one given in Section 2. In Section 4, we will give the third definition of unipotent log de Rham fundamental group: we will use a generalization of the relative minimal model theory of Navarro-Aznar (NA93). In Section 5, we will prove that this third definition is equivalent to the previous ones. This will be given by introducing the fourth definition, which will be shown to be isomorphic to the third and the first in an independent way. This fourth definition is given by means of relative bar construction. In the absolute case, the equivalence between the construction via minimal model and that one via bar construction has been proved by Bloch-Kriz [BK94] and the equivalence between the construction via bar construction and that one via Tannakian formalism has been proved by Terasoma [Te10]. The equivalence of monodromy actions follows from the fact that they are defined via relative stratification. Finally in Section 6, we restrict ourselves to the case of stable log curves over standard log point and we calculate the monodromy action on unipotent log de Rham fundamental groups of such curves. We prove that this monodromy action is non-trivial (as an outer action) when the given stable log curve is not smooth in classical sense. Using the result of this type instead of Oda’s one, we end Section 6 with a purely algebraic proof of the aforementioned result of Andreattta-Iovita-Kim.

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Convention

We use freely the notions concerning log (formal) schemes which are written in [Kk88, I02, AGT16 II 5], [Og18], [Ts19]. In particular (formal) schemes and
sheaves are considered in the etale site, unless otherwise stated. A (formal) scheme is naturally regarded as a fine log (formal) scheme endowed with trivial log structure. For example, for a ring \( R \), \( \text{Spec } R \) means the fine log scheme \((\text{Spec } R, \text{triv. log str.})\). For a fine log (formal) scheme \( X \), denote its underlying (formal) scheme by \( X^\circ \), the log structure of \( X \) by \( \mathcal{M}_X \) and the structure homomorphism of monoids \( \mathcal{M}_X \to \mathcal{O}_X \) by \( \alpha_X \). A log (formal) scheme is called Noetherian if so is its underlying scheme. Throughout this paper, all the log (formal) schemes are assumed to be Noetherian, unless otherwise stated. A morphism of fine log schemes is called of finite type/proper if so is its underlying morphism of schemes. A morphism of fine log schemes \( f : X \to Y \) is called strict if the morphism of log structures \( f^* \mathcal{M}_Y \to \mathcal{M}_X \) induced by \( f \) is an isomorphism. Cartesian diagrams of fine log (formal) schemes are considered in the category of fine log (formal) schemes, unless otherwise stated.

For a scheme or a ring \( T \) and \( r, s \in \mathbb{N} \), we denote the log structure on \( \mathbb{A}^r_T = \text{Spec}_T \mathcal{O}_T[[t_1, \ldots, t_r]] \) associated to the monoid homomorphism \( \mathbb{N}^s \to \mathcal{O}_T[[t_1, \ldots, t_r]] \) sending \( e_i \) \((1 \leq i \leq \min(r, s))\) to \( t_i \) and sending \( e_i \) \((r + 1 \leq i \leq r + s)\) to \( 0 \) if \( s > r \) (where \( \{e_i\}_{i=1}^s \) is the canonical basis of \( \mathbb{N}^s \)) by \( \mathcal{M}_{r,s} \), and denote the log scheme \((\mathbb{A}^r_T, \mathcal{M}_{r,s})\) by \( \hat{\mathbb{A}}^{r,s}_T \). For a field \( k \), we call \( \mathbb{A}_k^{0,1} \) the standard log point over \( k \).

For a morphism \( h : Y \to Z \) of fine log schemes of finite type, we denote the category of locally free \( \mathcal{O}_Y \)-modules of finite rank (resp. coherent \( \mathcal{O}_Y \)-modules) with log connection on \( Y/Z \) by \( \mathrm{MC}(Y/Z) \) (resp. \( \tilde{\mathrm{MC}}(Y/Z) \)), and denote the full subcategory consisting of modules with integrable connection by \( \mathrm{MIC}(Y/Z) \) (resp. \( \tilde{\mathrm{MIC}}(Y/Z) \)). When \( Z = \text{Spec } R \) for some ring \( R \), we denote it simply by \( \mathrm{MIC}(Y/R) \) (resp. \( \tilde{\mathrm{MIC}}(Y/R) \)). Note that, when \( Y = Z \), \( \mathrm{MIC}(Y/Z) \) (resp. \( \tilde{\mathrm{MIC}}(Y/Z) \)) is just the category of locally free \( \mathcal{O}_Y \)-modules of finite rank (resp. coherent \( \mathcal{O}_Y \)-modules). Note also that, even when \( \Omega^1_{Y/Z} \) is a locally free \( \mathcal{O}_Y \)-module of finite rank (which is the case we are interested in), \( \mathrm{MIC}(Y/Z) \) is a rigid tensor category but not necessarily an abelian category, due to the possible existence of non-trivial log structure on \( Y \). On the other hand, under the same assumption on \( Y/Z \), \( \tilde{\mathrm{MIC}}(Y/Z) \) is an abelian category, but not necessarily a rigid tensor category ([DPS18, Remark 2.10]).

1 Preliminaries

In this section, after giving some preliminaries on modules with integrable log connection, we give a geometric setting which we consider throughout this paper, and we define various categories related to the category of modules with integrable log connection which we use later.

For a morphism \( h : Y \to Z \) of fine log schemes of finite type, we denote the category of locally free \( \mathcal{O}_Y \)-modules of finite rank (resp. coherent \( \mathcal{O}_Y \)-modules) with log connection on \( Y/Z \) by \( \mathrm{MC}(Y/Z) \) (resp. \( \tilde{\mathrm{MC}}(Y/Z) \)), and denote the full subcategory consisting of modules with integrable connection by \( \mathrm{MIC}(Y/Z) \) (resp. \( \tilde{\mathrm{MIC}}(Y/Z) \)). When \( Z = \text{Spec } R \) for some ring \( R \), we denote it simply by \( \mathrm{MIC}(Y/R) \) (resp. \( \tilde{\mathrm{MIC}}(Y/R) \)). Note that, when \( Y = Z \), \( \mathrm{MIC}(Y/Z) \) (resp. \( \tilde{\mathrm{MIC}}(Y/Z) \)) is just the category of locally free \( \mathcal{O}_Y \)-modules of finite rank (resp. coherent \( \mathcal{O}_Y \)-modules). Note also that, even when \( \Omega^1_{Y/Z} \) is a locally free \( \mathcal{O}_Y \)-module of finite rank (which is the case we are interested in), \( \mathrm{MIC}(Y/Z) \) is a rigid tensor category but not necessarily an abelian category, due to the possible existence of non-trivial log structure on \( Y \). On the other hand, under the same assumption on \( Y/Z \), \( \tilde{\mathrm{MIC}}(Y/Z) \) is an abelian category, but not necessarily a rigid tensor category ([DPS18, Remark 2.10]).
For a commutative diagram

$$
Y' \xrightarrow{a} Y \\
\downarrow h' \downarrow h \\
Z' \xrightarrow{b} Z,
$$

(1.1)

the pull-back functors

$$a^*_\text{dr} : \text{MIC}(Y/Z) \rightarrow \text{MIC}(Y'/Z'), \quad a^*_\text{dr} : \tilde{\text{MIC}}(Y/Z) \rightarrow \tilde{\text{MIC}}(Y'/Z')$$

are defined in natural way. In particular, the functors

$$h^*_\text{dr} : \text{MIC}(Z/Z) \rightarrow \text{MIC}(Y/Z), \quad h^*_\text{dr} : \tilde{\text{MIC}}(Z/Z) \rightarrow \tilde{\text{MIC}}(Y/Z)$$

are defined.

An object $E := (E, \nabla)$ in $\tilde{\text{MIC}}(Y/Z)$ gives rise to the de Rham complex $E \otimes_{O_Y} \Omega^\bullet_{Y/Z}$. Using it, we define the $i$-th relative de Rham cohomology by $R^i h^*_\text{dr} E := R^i h_* (E \otimes_{O_Y} \Omega^\bullet_{Y/Z})$. When $h$ is proper, it is a coherent $O_Z$-module (but not necessarily locally free), and so we have the functor

$$R^i h^*_\text{dr} : \tilde{\text{MIC}}(Y/Z) \rightarrow \tilde{\text{MIC}}(Z/Z).$$

In the following, $R^0 h^*_\text{dr} E$ is denoted simply by $h^*_\text{dr} E$.

The adjointness of the functors $h^*, h_*$ for $O_?$-modules ($? = Y$ or $Z$) induces the morphisms of functors $h^*_\text{dr} h^*_\text{dr} \rightarrow \text{id}, \text{id} \rightarrow h^*_\text{dr} h^*_\text{dr}$. For $E \in \text{MIC}(Z/Z)$ and $(E', \nabla') \in \tilde{\text{MIC}}(Y/Z)$, we have the projection formula

$$R^i h^*_\text{dr}(h^*_\text{dr} E \otimes (E', \nabla')) \cong E \otimes_{O_Z} R^i h^*_\text{dr}(E', \nabla').$$

Also, when we have a Cartesian diagram

$$
Y' \xrightarrow{a} Y \\
\downarrow h' \downarrow h \\
Z' \xrightarrow{b} Z
$$

and $E := (E, \nabla) \in \tilde{\text{MIC}}(Y/Z)$, $h$ is separated and integral (so the diagram is a cartesian diagram also for the underling schemes) and when $b$ is flat or when $E \otimes_{O_Y} \Omega^i_{Y/Z} (i \in \mathbb{N})$ and $R^i h^*_\text{dr} E (i \in \mathbb{N})$ are flat over $O_Z$, then we have the base change isomorphism

$$b^* R^i h^*_\text{dr} E \cong R^i h^*_\text{dr} a^*_\text{dr} E.$$

Indeed, we have the base change quasi-isomorphism in derived category

$$Lb^* Rh_* (E \otimes_{O_Y} \Omega^\bullet_{Y/Z}) \cong Rh'_*(a^*_\text{dr} E \otimes_{O_Y} \Omega^\bullet_{Y'/Z'}).$$
by the proof similar to [BO78, Theorem 7.8], namely, by the base change result in affine case and cohomological descent. Then, by taking the $i$-th cohomology on both hand sides, we obtain the required base change isomorphism.

For an object $(E, \nabla)$ in $\text{MIC}(Y/Z)$ and a geometric point $y$ of $Y$, the map $\nabla : E \to E \otimes_{\mathcal{O}_Y} \Omega^n_{Y/Z}$ induces a $k(y)$-linear map $\nabla_y : E|_y \to E|_y \otimes_{\mathcal{O}_{Y,y}} \mathcal{M}^p_{Y,Y,y} / \mathcal{M}^p_{Z,h(y)}$, where $E|_y := E \otimes_{\mathcal{O}_Y} k(y)$. For a $\mathbb{Z}$-linear map $\psi : \mathcal{M}^p_{Y,Y,y} / \mathcal{M}^p_{Z,h(y)} \to \mathbb{Z}$, the composite $(\text{id} \otimes \psi) \circ \nabla_y : E|_y \to E|_y$ is called the residue map of $(E, \nabla)$ at $y$ along $\psi$. We say that $(E, \nabla)$ has nilpotent residues at $y$ if, for any $\mathbb{Z}$-linear map $\psi$ as above, the residue map $\psi \circ \nabla_y$ is nilpotent as endomorphism on $E|_y$ [DPS18, Definition 2.11]. Also, in this article, we say that $(E, \nabla)$ has nilpotent residues if it has nilpotent residues at any geometric point $y$ of $Y$. We denote the full subcategory of $\text{MIC}(Y/Z)$ which consists of objects having nilpotent residues by $\text{MIC}^n(Y/Z)$.

Note that, when we are given the commutative diagram (1.1), the pull-back functor $a^*_{\text{DR}}$ induces the functor $\text{MIC}^n(Y/Z) \to \text{MIC}^n(Y'/Z')$, which is also denoted by $a^*_{\text{DR}}$. Note also that, when the log structures of $Y$ and $Z$ are defined in Zariski topology, the residue map can be defined at closed points $y$ (not geometric points) and so the nilpotence of residues can be checked at closed points.

**Remark 1.1.** In [DPS18, Definition 2.11], the definition of ‘having nilpotent residues’ is different from the one given above: In fact, $(E, \nabla)$ has nilpotent residues in the sense in [DPS18] if it has nilpotent residues at any geometric point $y$ over any closed point of $Y$. In this article, we denote the full subcategory of $\text{MIC}(Y/Z)$ which consists of objects having nilpotent residues in the sense of [DPS18] by $\text{MIC}^n(Y/Z)'$.

The definition in this article has an advantage that the functoriality for pullbacks is valid for any commutative diagram (1.1). On the other hand, the definition in [DPS18] has an advantage that it would be easier to prove that an object has nilpotent residues.

By definition, there exists the canonical inclusion $\text{MIC}^n(Y/Z) \subseteq \text{MIC}^n(Y/Z)'$, and it is an interesting problem if this inclusion is an equality. We do not know the answer to this problem in general case, but in some cases, the above question has the affirmative answer. See [DPS18, Remark 2.12] and Remark 1.16 below.

So far, we worked in the category of fine log schemes, but sometimes we need to work with fine log formal schemes. For a morphism $h : Y \to Z$ topologically of finite type from a (Noetherian) fine log formal scheme $Y$ to a fine log formal scheme $Z$, we can define the category $\text{MIC}(Y/Z)$ (resp. $\widetilde{\text{MIC}}(Y/Z)$) of locally free $\mathcal{O}_Y$-modules of finite rank (resp. coherent $\mathcal{O}_Y$-modules) with integrable log connection on $Y/Z$. (In the definition of integrable log connection, we use the continuous log differential module $\Omega^1_{Y/Z}$.) We can define the pull-back functor and the $i$-th relative de Rham cohomology $R^ih_{\text{DR}*}E := R^ih_{*}(E \otimes_{\mathcal{O}_Y} \Omega^*_{Y/Z})$ ($E \in \widetilde{\text{MIC}}(Y/Z)$) also in this situation. We often use the following proposition later.

**Proposition 1.2.** Let $Y, Z$ be fine log schemes over $\mathbb{Q}$, $Y'$ a fine log formal scheme over $\mathbb{Q}$ and let $Y' \xrightarrow{h'} Y \xrightarrow{h} Z$ be morphisms over $\mathbb{Q}$ such that $h$ is of finite type. 


with $\Omega^1_{Y/Z}$ locally free and that, etale locally on $Y$, $h'$ is isomorphic to the projection $\hat\Lambda^d_Y \to Y$ for some $d$. (For the definition of $\hat\Lambda^d_Y$, see Convention.) Then, for any $E \in \text{MIC}(Y/Z)$, we have the canonical isomorphism

$$R^i h_{\text{dR},*}E \cong R^i(h \circ h')_{\text{dR},*}(h'^*E) \quad (i \in \mathbb{N}).$$

**Proof.** Let us denote by $E \otimes_{\mathcal{O}_Y} \Omega^*_{Y/Z}$ (resp. $h'^*E \otimes_{\mathcal{O}_Y} \Omega^*_{Y'/Z}$) the de Rham complex associated to $E$ (resp. $h'^*E$). Then it suffices to prove that the canonical morphism

$$E \otimes_{\mathcal{O}_Y} \Omega^*_{Y/Z} \to R\pi'_*(h'^*E \otimes_{\mathcal{O}_Y} \Omega^*_{Y'/Z})$$

is a quasi-isomorphism. Since we can work etale locally on $Y$, we may assume that $h'$ is the projection $\hat\Lambda^d_Y \to Y$. Then we have the equality

$$h'^*E \otimes_{\mathcal{O}_Y} \Omega^*_{Y'/Z} = \text{Tot}(h'^*E \otimes_{\mathcal{O}_Y} \Omega^*_{Y/Z}) \otimes_{\hat\Lambda^d_Y} \Omega^*_{\hat\Lambda^d_Y/Y},$$

and by using the projection formula, we see that it suffices to prove that the canonical injective homomorphism

$$\iota : \mathcal{O}_Y \to h'_*\Omega^*_{\hat\Lambda^d_Y/Y}$$

is a quasi-isomorphism. This is standard, but we give a proof for the convenience of the reader. Let $\pi : \Omega^*_{\hat\Lambda^d_Y/Y} \to \mathcal{O}_Y$ be the homomorphism of ‘taking constant term’. Then $\pi \circ \iota$ is the identity, and so it suffices to prove that $\iota \circ \pi$ is homotopic to the identity map on $\Omega^*_{\hat\Lambda^d_Y/Y}$. A homotopy is given by

$$t_1^{i_1} \cdots t_d^{i_d} dt_{j_1} \wedge \cdots \wedge dt_{j_m} \mapsto \begin{cases} (i_1 + 1)^{-1} t_1^{i_1 + 1} \cdots t_d^{i_d} dt_{j_2} \wedge \cdots \wedge dt_{j_m} & \quad \text{(if } j_1 = 1), \\
0 & \quad \text{(otherwise),} \end{cases}$$

where $t_1, \ldots, t_d$ are the coordinate of $\hat\Lambda^d_Y$ and $i_1, \ldots, i_d \geq 0$, $1 \leq j_1 < \cdots < j_m \leq d$. Note that the first element on the right hand side is well-defined because the characteristic of $Y$ is zero. So the proof of the proposition is finished. \qed

Next we introduce the notion of stratifications and crystals, and relate them to modules with integrable connections.

**Definition 1.3.** Let $k$ be a field of characteristic zero and let $h : Y \to Z$ be a morphism of fine log schemes of finite type over $k$. For $m, j \in \mathbb{N}$, let $Y^m(j)$ be the $m$-th log infinitesimal neighborhood of $Y$ in $Y \times_Z \cdots \times_Z Y$ ($(j + 1)$ times) and let $p^m_j : Y^m(1) \to Y$ ($j = 1, 2, \ldots, j', \ldots, 3$) be projections. Then a stratification on a coherent $\mathcal{O}_Y$-module $E$ (with respect to $h$) is a compatible family of $\mathcal{O}_{Y^m(1)}$-linear isomorphisms $\{e^m : p^m_2 E \to p^m_1 E\}_m$ with $e_0 = \text{id}$ and $p^m_{1,2}(e^m) = p^{m'}_{1,2}(e^m) \circ p^m_{2,3}(e^m)$. We denote the category of locally free $\mathcal{O}_Y$-modules of finite rank (resp. coherent $\mathcal{O}_Y$-modules) endowed with stratification (with respect to $h$) by $\text{Str}(Y/Z)$ (resp. $\text{Str}(Y/Z)$).
Definition 1.4. Let \( k, h : Y \to Z \) be as in Definition 1.3.

1. We define the infinitesimal site \((Y/Z)_{\inf}\) of \(Y/Z\) in the following way: An object is given by a triple \((U, T, i)\), where \( U \) is a (Noetherian) fine log scheme over \( Y \), \( T \) is a (Noetherian) fine log scheme of finite type over \( Z \) and \( i \) is a nilpotent exact closed immersion \( U \to T \) over \( Z \). A morphism is defined in natural way and a covering is the one naturally induced from the etale topology on \( T \).

2. We define the stratifying site \((Y/Z)_{\str}\) of \(Y/Z\) as the full subcategory of \((Y/Z)_{\inf}\) consisting of triples \((U, T, i)\) which admits etale locally a morphism \( T \to Y \) over \( Z \) extending \( U \to Y \).

3. We define the sheaf \( \mathcal{O}_{Y/Z} \) on \((Y/Z)_{\inf}\) or on \((Y/Z)_{\str}\) by \( \mathcal{O}_{Y/Z}((U, T, i)) := \Gamma(T, \mathcal{O}_T) \).

4. For a sheaf \( E \) on \((Y/Z)_{\inf}\) (resp. \((Y/Z)_{\str}\)) and an object \((U, T, i)\) in \((Y/Z)_{\inf}\) (resp. \((Y/Z)_{\str}\)), we denote the sheaf on \( T_{\et} \) induced by \( E \) by \( E_T \).

5. We call a sheaf \( E \) of \( \mathcal{O}_{Y/Z}\)-modules on \((Y/Z)_{\inf}\) (resp. \((Y/Z)_{\str}\)) a crystal if, for any morphism \( \varphi : (U', T', i) \to (U, T, i) \) in \((Y/Z)_{\inf}\) (resp. \((Y/Z)_{\str}\)), the morphism \( \varphi^* E_T := \mathcal{O}_T \otimes_{\varphi^{-1} \mathcal{O}_T} \varphi^{-1} E_T \to E_{T'} \) induced by the definition of \( E \) is an isomorphism.

6. We call a crystal \( E \) on \((Y/Z)_* (\ast \in \{\inf, \str\}) \) locally free (resp. coherent) if \( E_T \) is a locally free \( \mathcal{O}_T \)-module of finite rank (resp. coherent \( \mathcal{O}_T \)-module) for any \((U, T, i)\). We denote the category of locally free (resp. coherent) crystals on \((Y/Z)_* \) by \( \text{Crys}(Y/Z)_* \) (resp. \( \widehat{\text{Crys}}(Y/Z)_* \)).

When \( h : Y \to Z \) is log smooth, \((Y/Z)_{\str} = (Y/Z)_{\inf}\) by the infinitesimal lifting property of log smooth morphisms, and so we have an equality \( \widehat{\text{Crys}}(Y/Z)_{\str} = \text{Crys}(Y/Z)_{\inf} \) (resp. \( \widehat{\text{Crys}}(Y/Z)_{\str} = \text{Crys}(Y/Z)_{\inf} \)). In general, the inclusion \((Y/Z)_{\str} \to (Y/Z)_{\inf}\) induces the natural functor \( \text{Crys}(Y/Z)_{\inf} \to \text{Crys}(Y/Z)_{\str} \) (resp. \( \widehat{\text{Crys}}(Y/Z)_{\inf} \to \widehat{\text{Crys}}(Y/Z)_{\str} \)). The categories \( \text{Crys}(Y/Z)_* \), \( \widehat{\text{Crys}}(Y/Z)_* \) \((\ast \in \{\inf, \str\})\) are contravariantly functorial with respect to \( h : Y \to Z \): Indeed, if we are given a commutative diagram (I.1) and an object \((U, T, i)\) in \((Y'/Z')_* \), it is also regarded as an object in \((Y/Z)_* \). Hence, if we are given an object \( E \) in \( \text{Crys}(Y/Z)_* \) (resp. \( \widehat{\text{Crys}}((Y/Z)_*) \)), we can define a sheaf \( h^* E \) on \((Y'/Z')_* \) by putting \( (h^* E)((U, T, i)) := E((U, T, i)) \), and we can check that \( h^* E \) is in fact an object in \( \text{Crys}((Y'/Z')_* \) (resp. \( \widehat{\text{Crys}}(Y'/Z')_* \)).

We explain the relation among modules with integrable log connection, stratifications and crystals. Let \( k, h : Y \to Z \) be as in as in Definition 1.3 and fix the symbol \( \ast \in \{\str, \inf\} \). Let us take an object \((U, T, i)\) in \((Y/Z)_* \). If we denote the the \( m \)-th log infinitesimal neighborhood of \( T \) in \( T \times_Z \cdots \times_Z T \) (\((j + 1)\) times) by \( T^m(j) \), the triple \((U, T^m(j), U \to T \xrightarrow{\Delta} T^m(j)) \) (where \( \Delta \) is the morphism induced by the diagonal morphism \( T \to T \times_Z \cdots \times_Z T \)) is an object in \((Y/Z)_* \). Therefore, if we are given an object \( E \) in \( \widehat{\text{Crys}}(Y/Z)_* \), we obtain naturally an object of the form \( (E_T, \{e^m\}_m) \) in \( \text{Str}(T/Z) \) by defining \( e^m \) to be the isomorphism

\[
p_2^m \ast E_T \xrightarrow{\cong} E_{T^m(1)} \xleftarrow{\cong} p_1^m \ast E_T
\]
induced by the structure of the crystal $E$. (Here $p_j^n : T^m(1) \to T$ are the projections.) When $E$ is locally free, $E_T$ is locally free. Thus we obtain functors

\[(1.2) \quad \text{Crys}((Y/Z)_*) \to \text{Str}(T/Z), \quad \widetilde{\text{Crys}}((Y/Z)_*) \to \widetilde{\text{Str}}(T/Z).\]

Also, when we are given an object $(E, \{\epsilon^m\}_m)$ in $\widetilde{\text{Str}}(T/Z)$, we can define a structure of a log connection $\nabla : E \to E \otimes \mathcal{O}_T \Omega^1_{T/Z}$ by putting

$$\nabla(e) := \epsilon^1(p_2^1(e)) - p_1^*(e) \in \text{Ker}(p_1^*E \to E) \cong E \otimes \mathcal{O}_T \Omega^1_{T/Z},$$

where the last isomorphism is induced by the canonical isomorphism $\text{Ker}(\mathcal{O}_T \to \mathcal{O}_T) \cong \Omega^1_{T/Z}$ [Sh00, Proposition 3.2.5]. (One can prove the Leibniz rule for $\nabla$ in the same way as the proof of [BO78, Proposition 2.9].) Thus we obtain functors

\[(1.3) \quad \text{Str}(T/Z) \to \text{MC}(T/Z), \quad \widetilde{\text{Str}}(T/Z) \to \widetilde{\text{MC}}(T/Z).\]

The functors we defined above are contravariantly functorial with respect to $h : Y \to Z$ and $(U, T, i)$. Concerning the above functors in the case $U = Y$, we have the following proposition.

**Proposition 1.5.** Let $k, h : Y \to Z$, be as in Definition 1.3 and $(U, T, i) \in (Y/Z)_*$. Assume that $U = Y$ and that $\Omega^1_{T/Z}$ is a locally free $\mathcal{O}_T$-module of finite rank. Then the functors (1.2) for $* = \text{str}$ and the functors (1.3) induce the equivalences of categories

\[(1.4) \quad \text{Crys}((Y/Z)_{\text{str}}) \xrightarrow{\cong} \text{Str}(T/Z) \to \text{MIC}(T/Z), \]

\[(1.5) \quad \widetilde{\text{Crys}}((Y/Z)_{\text{str}}) \xrightarrow{\cong} \widetilde{\text{Str}}(T/Z) \to \widetilde{\text{MIC}}(T/Z).\]

Furthermore, when $T \to Z$ is log smooth, the functors (1.2) for $* = \text{inf}$ induce the equivalences of categories

\[(1.6) \quad \text{Crys}((Y/Z)_{\text{int}}) \xrightarrow{\cong} \text{Str}(T/Z), \quad \widetilde{\text{Crys}}((Y/Z)_{\text{int}}) \xrightarrow{\cong} \widetilde{\text{Str}}(T/Z).\]

**Proof.** We see that (1.4) follows from (1.5) and that the first equivalence in (1.6) follows from the second equivalence by restricting to the subcategories of locally free objects. So it suffices to prove the equivalences (1.5) and the second equivalence in (1.6). The strategy of the proof is similar to that in [BO78, §2], [Sh00, §3.2] and [Sh02, §1.2].

First we prove the first equivalence in (1.5). To do so, it suffices to construct a quasi-inverse

\[(1.7) \quad \widetilde{\text{Str}}(T/Z) \to \widetilde{\text{Crys}}((Y/Z)_{\text{str}})\]

of the second functor in (1.2) for $* = \text{str}$. Let $(E, \{\epsilon^m\}_m)$ be an object in $\widetilde{\text{Str}}(T/Z)$ and take an object $(U', T', i')$ in $(Y/Z)_{\text{str}}$. By definition, etale locally on $T'$, there
exists a morphism \( \varphi_1 : T' \to Y \) over \( Z \) extending \( U' \to Y \). Then we define the etale sheaf \( E_{T'} \) on \( T' \) by \( E_{T'} := (i \circ \varphi_1)^* E \). This definition works only etale locally on \( T' \) and depends on the choice of \( \varphi_1 \). If there exists another morphism \( \varphi_2 : T' \to Y \) as \( \varphi_1 \), we see that the morphism \((i \circ \varphi_1) \times (i \circ \varphi_2) : T' \to T \times_Z T\) factors as

\[
T' \xrightarrow{\varphi_{12}} T^m(1) \to T \times_Z T
\]

for some \( m \), where the second morphism is induced by the diagonal map \( T \to T \times_Z T \). Thus we have the isomorphism

\[
\varphi_{12}^* \epsilon^m : (i \circ \varphi_2)^* E \xrightarrow{\epsilon^m} (i \circ \varphi_1)^* E;
\]

and we see that it satisfies the cocycle condition. Hence we can define the etale sheaf \( E_{T'} \) globally on \( T' \) by gluing the local definition by etale descent. Also, we see that the definition of \( E_{T'} \) does not depend on any choice, and that the sheaf on \((Y/Z)_{str}\) defined by \((U', T', i') \mapsto E_{T'}(T')\) is a crystal. Thus we have defined the functor (1.7), and we can check that this is in fact a quasi-inverse of (1.2) for \( * = str \).

Next we prove the second equivalence in (1.5). To do so, first we prove that the essential image of the second functor in (1.3) is contained in \( \widetilde{MIC}(T/Z) \), thus induces the functor

\[
(1.8) \quad \widetilde{Str}(T/Z) \to \widetilde{Crys}((Y/Z)_{inf})
\]

of the functor (1.2) for \( * = \inf \). The construction is similar to that in the previous paragraph: For an object \((E, \{\epsilon^m\}_m)\) in \( \widetilde{Str}(T/Z) \) and an object \((U', T', i')\) in \( (Y/Z)_{inf} \), there exists a morphism \( \varphi_1 : T' \to T \) over \( Z \) extending \( U' \to Y \xrightarrow{i} T \) etale locally on \( T' \), by the log smoothness of \( T \) over \( Z \). Then we define the etale sheaf \( E_{T'} \) on \( T' \) by \( E_{T'} := \varphi_1^* E \) locally on \( T' \), and define \( E_{T'} \) globally by gluing local definition using \( \epsilon^m \) for some \( m \) as above. Then the sheaf on \((Y/Z)_{str}\) defined by \((U', T', i') \mapsto E_{T'}(T')\) is a crystal and thus we have defined the functor (1.8). We can check also that this is a quasi-inverse of (1.2) for \( * = \inf \).

Finally we prove the second equivalence in (1.5). To do so, first we prove that the essential image of the second functor in (1.3) is contained in \( \widetilde{MIC}(T/Z) \), thus induces the functor

\[
(1.9) \quad \widetilde{Str}(T/Z) \to \widetilde{MIC}(T/Z).
\]

To prove this, we may work etale locally. So we assume that \( \Omega_{T/Z}^1 \) has a basis of the form \( \{d\log x_i\}_{i=1}^r \) for some \( x_1, \ldots, x_r \in \mathcal{M}_T \). Then there exists a unique element \( u_i \in \lim_m \ker(\mathcal{O}_{T^m(1)}^* \to \mathcal{O}_T^*) \) for each \( 1 \leq i \leq r \) satisfying

\[
(1.10) \quad (p_2^{m*}(x_i))_m = (p_1^{m*}(x_i))_m \cdot u_i \in \lim_m \mathcal{M}_{T^m(1)}.
\]

We put \( \xi_i := \sum_{n=1}^{\infty} (-1)^{n-1}(u_i - 1)^n/n \in \lim_m \mathcal{O}_{T^m(1)} \) and we will denote the images of \( u_i, \xi_i \) in \( \mathcal{O}_{T^m(1)} \) by the same letters, by abuse of notation. Recall that the canonical isomorphism \( \ker(\mathcal{O}_{T^1(1)} \to \mathcal{O}_T) \cong \Omega_{T/Z}^1 \) in [Sh00, Proposition 3.2.5] sends \( \xi_i =
$u_i - 1$ on the left hand side to $d\log x_i$ on the right hand side. Hence $\{\xi_i\}_{i=1}^r$ is a basis of $\text{Ker}(\mathcal{O}_{T^m(1)} \rightarrow \mathcal{O}_T)$ and so $\{1, \xi_1, \ldots, \xi_r\}$ is a basis of $\mathcal{O}_{T^m(1)}$.

We prove that, for any $m \geq 1$, the elements $\xi^q (|q| \leq m)$ form an $\mathcal{O}_T$-basis of $\mathcal{O}_{T^m(1)}$. (Here we use multi-index notation.) To do so, first recall that the diagonal morphism $T \rightarrow T \times_T T$ factors etale locally as

$$T \rightarrow (T \times_T T) \rightarrow T \times_T T$$

with the first arrow (resp. the second arrow) an exact closed immersion (resp. a log etale morphism) and that, if we put $\mathcal{I} := \text{Ker}(\mathcal{O}_{T \times_T T'} \rightarrow \mathcal{O}_T)$, $\mathcal{O}_{T^m(1)}$ is defined as $\mathcal{O}_{T^m(1)} := \mathcal{O}_{(T \times_T T')/\mathcal{I}^{m+1}}$. From this description, we see easily that the set $\{\xi^q\}_{|q| \leq m}$ generates $\mathcal{O}_{T^m(1)}$ as $\mathcal{O}_T$-module. So it suffices to show that, assuming the linear independence of the elements $\xi^q (|q| \leq m)$ in $\mathcal{O}_{T^m(1)}$ over $\mathcal{O}_T$, the elements $\xi^q (|q| \leq 2m)$ in $\mathcal{O}_{T^{2m}(1)}$ are also linearly independent over $\mathcal{O}_T$. By the universality of log infinitesimal neighborhood, there exist canonical morphisms $\delta^m : T^m(1) \times_T T^m(1) \rightarrow T^{2m}(2) (m \in \mathbb{N})$ such that $\{\delta^m\}_m$ is an isomorphism as inductive system of log schemes. Denote the composite $T^m(1) \times_T T^m(1) \xrightarrow{\delta^m} T^{2m}(2) \xrightarrow{\delta^{2m}} T^{2m}(1)$ by $\overline{\delta^m}$. Thus $\overline{\delta^m}$ is the morphism induced by the composite

$$(T \times_T T)(T \times_T T) = T \times_T T \times_T T \rightarrow T \times_T T,$$

where the last morphism is the projection to the first and the third factors.

Then the morphism

$$\overline{\delta^m} : \mathcal{O}_{T^m(1)} \rightarrow \mathcal{O}_{T^m(1)} \otimes_{\mathcal{O}_T} \mathcal{O}_{T^m(1)}$$

sends $u_i$ to $u_i \otimes u_i$: This fact is proven in [Sh00, p.562] but we provide a proof here for the convenience of the reader. If we denote the map $\varprojlim_m \mathcal{M}_{T^m(1)} \rightarrow \varprojlim_m \mathcal{M}_{T^m(1) \times_T T^m(1)}$ induced by the $i$-th projections $T^m(1) \times_T T^m(1) \rightarrow T^m(1) (m \in \mathbb{N})$ by $\pi_i$ (for $i = 1, 2$), we see from the characterization of $\overline{\delta^m}$ given above that the map of projective limit of log structures

$$(\overline{\delta^m})^*_m : \varprojlim_m \mathcal{M}_{T^m(1)} \rightarrow \varprojlim_m \mathcal{M}_{T^{2m}(1) \times_T T^{2m}(1)}$$

induced by $\overline{\delta^m}_m (m \in \mathbb{N})$ sends $(p_1^{2m}(x_i))_m$ to $\pi_1^*(p_1^{2m}(x_i))_m$ and $(p_2^{2m}(x_i))_m$ to $\pi_2^*(p_2^{2m}(x_i))_m$. Then, by pulling back the equality (1.10) via $(\overline{\delta^m})_m$, we obtain the equality

$$\pi_2^*(p_2^{2m}(x_i))_m = \pi_1^*(p_1^{2m}(x_i))_m \cdot (\overline{\delta^m}_m)_m \in \varprojlim_m \mathcal{M}_{T^m(1)}.$$

On the other hand, by pulling back the equality (1.10) via the first and the second projections $T^m(1) \times_T T^m(1) \rightarrow T^m(1) (m \in \mathbb{N})$, we obtain the equalities

$$\pi_1^*(p_2^{2m}(x_i))_m = \pi_1^*(p_1^{2m}(x_i))_m \cdot (u_i \otimes 1) \in \varprojlim_m \mathcal{M}_{T^m(1)},$$

$$\pi_2^*(p_2^{2m}(x_i))_m = \pi_2^*(p_1^{2m}(x_i))_m \cdot (1 \otimes u_i) \in \varprojlim_m \mathcal{M}_{T^m(1)}.$$
Combining these three equalities, we obtain the equalities $\delta^{m*}(u_i) = u_i \otimes u_i$ ($m \in \mathbb{N}$), as required.

Since $\delta^{m*}$ sends $u_i$ to $u_i \otimes u_i$, it sends $\xi_i$ to $1 \otimes \xi_i + \xi_i \otimes 1$. (This follows from the equality of formal power series

$$\sum_{n=1}^{\infty} (-1)^{n-1} (xy + x + y)^n/n = \sum_{n=1}^{\infty} (-1)^{n-1} x^n/n + \sum_{n=1}^{\infty} (-1)^{n-1} y^n/n.$$  

So, for $q = (q_1, \ldots, q_r) \in \mathbb{N}^r$ with $|q| \leq 2m$, the morphism $\delta^{m*}$ sends $\xi_i^q := \prod_{i=1}^{r} \xi_i^{q_i}$ to $(1 \otimes \xi + \xi \otimes 1)^q := \prod_{i=1}^{r} (1 \otimes \xi_i + \xi_i \otimes 1)^{q_i}$, which is an $\mathbb{N}_{>0}$-linear combination of elements of the form $\xi^a \otimes \xi^b$ ($a, b \in \mathbb{N}^r, a + b = q, |a|, |b| \leq m$). If we put $S_q := \{(a, b) \in \mathbb{N}^r \times \mathbb{N}^r \mid a + b = q, |a|, |b| \leq m\}$, it is non-empty and $S_q$'s ($q \in \mathbb{N}^r, |q| \leq 2m$) are all disjoint. Thus the elements $(1 \otimes \xi + \xi \otimes 1)^q (q \in \mathbb{N}^r, |q| \leq 2m)$ in $\mathcal{O}_{T^m(1)} \otimes_{\mathcal{O}_T} \mathcal{O}_{T^m(1)}$ are linearly independent over $\mathcal{O}_T$ and thus the elements $\xi_i^q (|q| \leq 2m)$ in $\mathcal{O}_{T^2(1)}$ are also linearly independent over $\mathcal{O}_T$, as required.

Note that the argument in the previous paragraph shows that the map $\delta^{m*}$ is an injection into a direct summand as $\mathcal{O}_T$-module.

Let $(E, \{e^n\}_m)$ be an object in Str$(T/Z)$, let $e \in E$ and write the image of $1 \otimes e \in \mathcal{O}_{T^2(1)} \otimes_{\mathcal{O}_T} E = p_2^{e*}E$ by $e^2$ as

$$e^2(1 \otimes e) = e \otimes 1 + \sum_{i=1}^{r} \partial_i(e) \otimes \xi_i + \sum_{1 \leq i \leq j \leq r} \partial_{ij}(e) \otimes (\xi_i \otimes \xi_j + \xi_j \otimes \xi_i) \in E \otimes_{\mathcal{O}_T} \mathcal{O}_{T^2(1)} = p_1^{e*}E.$$  

The cocycle condition for $\{e^m\}_m$ implies that the morphism

$$\delta^{1*}(e^2) : \mathcal{O}_{T^1(1)} \otimes_{\mathcal{O}_T} \mathcal{O}_{T^1(1)} \otimes_{\mathcal{O}_T} E \longrightarrow E \otimes_{\mathcal{O}_T} \mathcal{O}_{T^1(1)} \otimes_{\mathcal{O}_T} \mathcal{O}_{T^1(1)}$$

is equal to the composite

$$\mathcal{O}_{T^1(1)} \otimes_{\mathcal{O}_T} \mathcal{O}_{T^1(1)} \otimes_{\mathcal{O}_T} E \overset{\text{id} \otimes e^1}{\longrightarrow} \mathcal{O}_{T^1(1)} \otimes_{\mathcal{O}_T} E \otimes_{\mathcal{O}_T} \mathcal{O}_{T^1(1)} \overset{\text{id} \otimes \text{id}}{\longrightarrow} E \otimes_{\mathcal{O}_T} \mathcal{O}_{T^1(1)} \otimes_{\mathcal{O}_T} \mathcal{O}_{T^1(1)}.$$  

The image of $1 \otimes e$ by the morphism $\delta^{1*}(e^2)$ in (1.11) is equal to

$$e \otimes 1 \otimes 1 + \sum_{i=1}^{r} \partial_i(e) \otimes (1 \otimes \xi_i + \xi_i \otimes 1) + \sum_{1 \leq i \leq j \leq r} \partial_{ij}(e) \otimes (\xi_i \otimes \xi_j + \xi_j \otimes \xi_i),$$

while that by the composite morphism in (1.12) is calculated as

$$(e^1 \otimes \text{id}) \circ (\text{id} \otimes e^1)(1 \otimes 1 \otimes e)
= (e^1 \otimes \text{id})(1 \otimes e \otimes 1 + \sum_{i=1}^{r} 1 \otimes \partial_i(e) \otimes \xi_i)
= e \otimes 1 \otimes 1 + \sum_{i=1}^{r} \partial_i(e) \otimes \xi_i \otimes 1 + \sum_{i=1}^{r} \partial_i(e) \otimes 1 \otimes \xi_i + \sum_{i,j=1}^{r} \partial_j(\partial_i(e)) \otimes \xi_j \otimes \xi_i.$$  

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Thus we have $\partial_i(\partial_j(e)) = \partial_j(\partial_i(e))$ for $1 \leq i < j \leq r$. On the other hand, if we denote the image of $(E, \{e^m\}_m)$ by the second functor in (1.3) by $(E, \nabla)$, we have $\nabla(e) = \sum_{i=1}^r \partial_i(e) \otimes \text{dlog } x_i$. So we can calculate its curvature as

$$\nabla \circ \nabla(e) = \nabla(\sum_{i=1}^r \partial_i(e) \otimes \text{dlog } x_i) = \sum_{i,j} \partial_j(\partial_i(e)) \otimes \text{dlog } x_j \wedge \text{dlog } x_i$$

$$= \sum_{i<j} (\partial_i(\partial_j(e)) - \partial_j(\partial_i(e))) \otimes \text{dlog } x_i \wedge \text{dlog } x_j = 0.$$

So $(E, \nabla)$ is integrable and hence the functor (1.9) is defined.

Finally we prove that the functor (1.9) is an equivalence. To do so, it suffices to construct its quasi-inverse etale locally. If we are given an object $(E, \nabla) \in \widehat{\text{MIC}}(T/Z)$ and write $\nabla$ locally as $\nabla(e) = \sum_{i=1}^r \partial_i(e) \otimes \text{dlog } x_i$, we define the maps

$$\{e^m : \mathcal{O}_{T^m(1)} \otimes_{\mathcal{O}_T} E \rightarrow E \otimes_{\mathcal{O}_T} \mathcal{O}_{T^m(1)}\}_m$$

by

$$e^m(1 \otimes e) = \sum_{|q| \leq m} (\partial^q(e)/q!) \otimes \xi^q.$$

They are isomorphisms because the inverses are defined by $e \otimes 1 \mapsto \sum_q \xi^q \otimes ((-\partial)^q(e)/q!)$. Also, we can check the coincidence of the maps

$$\overline{\delta^m}(\epsilon^{2m}) : \mathcal{O}_{T^m(1)} \otimes_{\mathcal{O}_T} \mathcal{O}_{T^m(1)} \otimes_{\mathcal{O}_T} E \rightarrow E \otimes_{\mathcal{O}_T} \mathcal{O}_{T^m(1)} \otimes_{\mathcal{O}_T} \mathcal{O}_{T^m(1)},$$

$$\mathcal{O}_{T^m(1)} \otimes_{\mathcal{O}_T} \mathcal{O}_{T^m(1)} \otimes_{\mathcal{O}_T} E \xrightarrow{\text{id} \otimes \epsilon^m} \mathcal{O}_{T^m(1)} \otimes_{\mathcal{O}_T} E \otimes_{\mathcal{O}_T} \mathcal{O}_{T^m(1)}$$

$$\xrightarrow{e^m \otimes \text{id}} E \otimes_{\mathcal{O}_T} \mathcal{O}_{T^m(1)} \otimes_{\mathcal{O}_T} \mathcal{O}_{T^m(1)}.$$

Since $\{\delta^m : T^m(1) \times_T T^m(1) \rightarrow T^{2m}(2)\}_m$ is an isomorphism as inductive system of log schemes, the coincidence of the above maps implies the cocycle condition for the maps $\{e^m\}_m$. Hence it defines a stratification on $E$. Moreover, we see from the coincidence of the above two maps that the map $\epsilon^{2m}$ is determined uniquely by the map $\epsilon^m$, because the map $\overline{\delta^m}$ is an injection into a direct summand as $\mathcal{O}_T$-module. So we see that the above stratification $\{e^m\}_m$ is the unique one on $E$ which gives rise to the connection $\nabla$. Hence the functor $(E, \nabla) \mapsto (E, \{e^m\}_m)$ gives a local quasi-inverse of (1.9). So (1.9) is an equivalence and the proof of the proposition is now finished. □

Remark 1.6. Let $k, h : Y \rightarrow Z$ be as in Definition 1.3 let $U$ be a fine log scheme over $Y$ and let $i : U \hookrightarrow T$ be an exact closed immersion over $Z$ into a Noetherian fine log formal scheme $T$ such that $U$ is a scheme of definition of $T$. If we denote by $T_n$ the closed fine log scheme of $T$ defined by $(\text{Ker }i^*)^n$ and by $i_n$ the canonical exact closed immersion $U \hookrightarrow T_n$, $(U, T_n, i_n)$ is an object in $(Y/Z)_{\text{inf}}$ and thus we have the functors

$$\text{Crys}((Y/Z)_{\text{inf}}) \rightarrow \text{MC}(T_n/Z), \quad \widehat{\text{Crys}}((Y/Z)_{\text{inf}}) \rightarrow \widehat{\text{MC}}(T_n/Z)$$
by \((\ref{1.2}) \circ \ref{1.3}\). Because this functor is compatible with respect to \(n\), they induce the functors
\[
\text{Crys}((Y/Z)_{\text{inf}}) \rightarrow \lim_{\leftarrow n} \text{MC}(T_n/Z) \cong \text{MC}(T/Z),
\]
\[
\tilde{\text{Crys}}((Y/Z)_{\text{inf}}) \rightarrow \lim_{\leftarrow n} \tilde{\text{MC}}(T_n/Z) \cong \tilde{\text{MC}}(T/Z).
\]
If \(T\) admits étale locally a morphism \(T \rightarrow Y\) over \(Z\) compatible with \(U \rightarrow Y\), we can also define the functors
\[
\text{Crys}((Y/Z)_{\text{str}}) \rightarrow \lim_{\leftarrow n} \text{MC}(T_n/Z) \cong \text{MC}(T/Z),
\]
\[
\tilde{\text{Crys}}((Y/Z)_{\text{str}}) \rightarrow \lim_{\leftarrow n} \tilde{\text{MC}}(T_n/Z) \cong \tilde{\text{MC}}(T/Z).
\]
in the same way. The functors we constructed in this remark are contravariantly functorial with respect to \(h : Y \rightarrow Z\) and \((U, T, i)\).

We have the following topological invariance property for the categories \(\text{Crys}((Y/Z)_{\text{inf}})\), \(\tilde{\text{Crys}}((Y/Z)_{\text{inf}})\).

**Proposition 1.7.** Let \(k, h : Y \rightarrow Z\) be as in Definition \(\ref{1.3}\) such that \(h\) is log smooth, and let \(i : Y' \hookrightarrow Y\) be a nilpotent exact closed immersion. Then the natural functors
\[
\text{Crys}((Y/Z)_{\text{inf}}) \rightarrow \text{Crys}((Y'/Z)_{\text{inf}}), \quad \tilde{\text{Crys}}((Y/Z)_{\text{inf}}) \rightarrow \tilde{\text{Crys}}((Y'/Z)_{\text{inf}})
\]
are equivalences.

**Proof.** The triple \((Y', Y, i)\) is an object in \((Y'/Z)_{\text{inf}}\) and the triple \((Y, Y, \text{id})\) is an object in \((Y/Z)_{\text{inf}}\). So, by Proposition \(\ref{1.3}\), \(\text{Crys}((Y'/Z)_{\text{inf}})\) is equivalent to \(\text{Str}(Y/Z)\) and it is equivalent to \(\text{Crys}((Y/Z)_{\text{inf}})\). The same argument works for the categories with tilde. \(\square\)

Now, let \(k\) be a field of characteristic zero and consider the diagram of fine log schemes
\[
(1.13) \quad X \xrightarrow{f} S \xrightarrow{g} \text{Spec} k
\]
satisfying the following conditions.

(A) \(f\) is a proper log smooth integral morphism and \(g\) is a separated morphism of finite type. \(i\) is a section of \(f\).

(B) \(\Omega^1_{S/k}\) is a locally free \(\mathcal{O}_S\)-module of finite rank.
Then, by assumption (B) and Proposition 1.5 we have the equivalences

\[(1.14) \quad \text{MIC}(S/k) \cong \text{Str}(S/k), \quad \tilde{\text{MIC}}(S/k) \cong \tilde{\text{Str}}(S/k).\]

The same equivalences hold also for \(X/k\), because \(\Omega^1_{X/k}\) is locally free and there exist equivalences

\[(1.15) \quad \text{MIC}(X/k) \cong \text{Crys}((X/k)_{\text{str}}), \quad \tilde{\text{MIC}}(X/k) \cong \tilde{\text{Crys}}((X/k)_{\text{str}}),\]

again by Proposition 1.5. Also, since \(f\) is log smooth, there exist equivalences

\[(1.16) \quad \text{MIC}(X/S) \cong \text{Crys}((X/S)_{\text{int}}), \quad \tilde{\text{MIC}}(X/S) \cong \tilde{\text{Crys}}((X/S)_{\text{int}}).\]

For \(m, j \in \mathbb{N}\), let \(S^m(j)\) be the \(m\)-th log infinitesimal neighborhood of \(S\) in \(S \times_k \cdots \times_k S\) ((\(j + 1\)) times). Then we have the category \(\text{Crys}((X/S^m(j))_{\text{int}})\) (resp. \(\tilde{\text{Crys}}((X/S^m(j))_{\text{int}})\)) of locally free crystals (resp. coherent) on \((X/S^m(j))_{\text{int}}\). Let \(p^m_j : S^m(1) \to S\) (\(j = 1, 2\)), \(p^m_{j,j'} : S^m(2) \to S^m(1)\) (\(1 \leq j < j' \leq 3\)) be the projections and let

\[
p^m_{j,\text{inf}} : \text{Crys}((X/S)_{\text{inf}}) \to \text{Crys}((X/S^m(1))_{\text{inf}}),
\]
\[
p^m_{j,j',\text{inf}} : \text{Crys}((X/S^m(1))_{\text{inf}}) \to \text{Crys}((X/S^m(2))_{\text{inf}})
\]

(resp. \(p^m_{j,\text{inf}} : \tilde{\text{Crys}}((X/S)_{\text{inf}}) \to \tilde{\text{Crys}}((X/S^m(1))_{\text{inf}})\),
\[
\tilde{p}^m_{j,j',\text{inf}} : \tilde{\text{Crys}}((X/S^m(1))_{\text{inf}}) \to \tilde{\text{Crys}}((X/S^m(2))_{\text{inf}})
\]

be the associated pull-back functors. Using these, we define the categories \(\text{StrCrys}(X/k)\), \(\text{StrCrys}(X/k)\) as follows:

**Definition 1.8.** Let the notations be as above. Then we define \(\text{StrCrys}(X/k)\) (resp. \(\text{StrCrys}(X/k)\)) to be the category of pairs \((E, \{e^m\}_m)\), where \(E \in \text{Crys}((X/S)_{\text{inf}})\) (resp. \(E \in \tilde{\text{Crys}}((X/S)_{\text{inf}})\)) and \(\{e^m : p^m_{2,\text{inf}}E \to p^m_{1,\text{inf}}E\}_m\) is a compatible family of isomorphisms in \(\text{Crys}((X/S^m(1))_{\text{inf}})\) (resp. \(\tilde{\text{Crys}}((X/S^m(1))_{\text{inf}})\)) (\(m \in \mathbb{N}\)) (resp. \(\text{Crys}((X/S^m(1))_{\text{inf}})\) (\(m \in \mathbb{N}\)) with \(e^0 = \text{id}\) and \(p^m_{1,3,\text{inf}}(e^m) = p^m_{1,2,\text{inf}}(e^m) \circ p^m_{2,3,\text{inf}}(e^m)\).

Then we have the following.

**Proposition 1.9.** Let the notations be as above. Then we have equivalences of categories \(\text{MIC}(X/k) \cong \text{StrCrys}(X/k)\), \(\tilde{\text{MIC}}(X/k) \cong \tilde{\text{StrCrys}}(X/k)\).

**Proof.** We only prove the second equivalence \(\tilde{\text{MIC}}(X/k) \cong \tilde{\text{StrCrys}}(X/k)\), because then we can deduce the first one by checking that the equivalence respects the local freeness. Also, by the equivalence \(\text{(1.15)}\), it suffices to prove the equivalence \(\text{Crys}((X/k)_{\text{str}}) \cong \text{StrCrys}(X/k)\).

First we construct the functor \(\text{Crys}((X/k)_{\text{str}}) \to \text{StrCrys}(X/k)\). Let \(E\) be an object of \(\text{Crys}((X/k)_{\text{str}})\). Then it naturally induces an object \(E/S\) in \(\text{Crys}((X/S)_{\text{int}})\)
by the contravariant functoriality. (Note that \((X/S)_{str} = (X/S)_{inf}\) because \(X\) is log smooth over \(S\).) Let us take an object \(T := (U \hookrightarrow T \overset{i}{\rightarrow} S^{m}(1))\) of \((X/S^{m}(1))_{inf}\).

Denote the object \((U \hookrightarrow T \overset{p_{j}}{\rightarrow} S)\) of \((X/S)_{inf}\) by \(T_{j}\) and the object \((U \hookrightarrow T \rightarrow \text{Spec} \ k)\) in \((X/k)_{str}\) by \(T_{0}\). Then the value of \(p_{j,inf}^{m}(E/S)\) at \(T\) is equal to that of \(E/S\) at \(T_{j}\) and this is further equal to that of \(E\) at \(T_{0}\), which is independent of \(j\). So we have the natural isomorphism \(p_{j,inf}^{m}(E/S) \overset{\simeq}{\longrightarrow} p_{1,inf}^{m}(E/S)\). If we denote it by \(e_{m}\), we see easily that \(E \mapsto (E/S, \{e_{m}\})\) defines the functor \(\widehat{\text{Crys}}((X/k)_{inf}) \longrightarrow \text{StrCrys}(X/k)\).

Next we construct the functor \(\widehat{\text{StrCrys}}(X/k) \longrightarrow \text{Crys}((X/k)_{str})\). Let \((E, \{e_{m}\})\) be an object of \(\widehat{\text{StrCrys}}(X/k)\) and let \(T := (U \hookrightarrow T \overset{\varphi}{\rightarrow} \text{Spec} \ k)\) be an object in \((X/k)_{str}\). Then, there exists a morphism \(\psi : T \rightarrow S\) compatible with \(i, \varphi\) in suitable sense, etale locally on \(T\). So we can define the value \(E_{T}\) of \(E\) at \(T\) locally. Also, if we have two morphisms \(\psi, \psi' : T \rightarrow S\) as above, \(\psi \times \psi' : T \rightarrow S \times \text{Spec} \ S\) factors through \(S^{m}(1)\) for some \(m\), and so \(e_{m}\) defines the isomorphism of the two definitions of \(E_{T}\) (via \(\psi\) and \(\psi'\)) which satisfies the cocycle condition. So we can define \(E_{T}\) globally on \(T\) by etale descent. This construction for objects \(T\) in \((X/k)_{str}\) induces the functor \(\widehat{\text{StrCrys}}(X/k) \longrightarrow \text{Crys}((X/k)_{str})\).

Finally, it is easy to see that the two functors we constructed are the inverses of each other.

Note that there exist the canonical pullback functors

\[
\begin{align*}
    f^{*} : \text{Str}(S/k) & \longrightarrow \text{StrCrys}(X/k), \\
    i^{*} : \text{StrCrys}(X/k) & \longrightarrow \text{Str}(S/k)
\end{align*}
\]

which are equivalent to the functors

\[
\begin{align*}
    f^{*}_{dR} : \text{MIC}(S/k) & \longrightarrow \text{MIC}(X/k), \\
    i^{*}_{dR} : \text{MIC}(X/k) & \longrightarrow \text{MIC}(S/k).
\end{align*}
\]

We give a definition of Gauss-Manin connection in our setting. For \(j = 1, 2\), let us denote by \(X_{j}^{m}\) the fiber product \(X \times_{S^{m}(1)} S^{m}(1)\) and denote by \(q_{j}^{m}, f_{j}^{m}\) the projections \(X_{j}^{m} \longrightarrow X, X_{j}^{m} \longrightarrow S^{m}(1)\), respectively. Also, let \(\tilde{X}^{m}\) be the log formal tube of \(X\) in \(\tilde{X}^{m}(1)_{X}^{2}\), namely, the direct limit of the \(l\)-th log infinitesimal neighborhoods of \(X\) in \(X_{l}^{m} \times_{S^{m}(1)} X_{2}^{m}\) with respect to \(l\) (cf. [CF06, 0.9]). This is a Noetherian fine log formal scheme. Let \(\tilde{f}^{m} : \tilde{X}^{m} \longrightarrow S^{m}(1)\) be the map induced by \(f_{j}^{m}, f_{2}^{m}\) and let \(\tilde{q}_{j}^{m} : \tilde{X}^{m} \longrightarrow X\) be the composite of the projection \(\tilde{X}^{m} \longrightarrow X_{j}^{m}\) and \(q_{j}^{m}\).

Let \((E, \nabla)\) be an object in \(\text{MIC}(X/k)\) and denote its image in \(\text{MIC}(X/S)\) by \((\overline{E}, \overline{\nabla})\). Thanks to hypothesis (B) and the proof of Proposition \ref{prop:gerbe} we know that \(p_{j}^{m}\) is flat, hence we have the diagram

\[
(1.17) \quad p_{j}^{m*}R^{i}f_{dR}*\overline{E} \longrightarrow R^{i}f_{j}^{m*}q_{j}^{m*}dR^{*}\overline{E} \longrightarrow R^{i}\tilde{f}_{dR}*\tilde{\psi}_{j}^{m*}dR^{*}\overline{E}
\]

in which the first map is the base change isomorphism for \(j = 1, 2\). Then we have the following:
Proposition 1.10. Let the notations be as above.

(1) We have the canonical isomorphism \((\hat{q}_2^m)^* \mathcal{E} \cong (\hat{q}_1^m)^* \mathcal{E}\).

(2) The morphisms in the diagram (1.17) are isomorphisms.

Proof. (1) The functor \((\hat{q}_2^m)^* : \hat{\text{MIC}}(X/S) \to \hat{\text{MIC}}(\hat{X}^m/S^m(1)) \to \hat{\text{MIC}}(\hat{X}^m/S^m(1))\) is rewritten as the composite

\[\hat{\text{MIC}}(X/S) \cong \hat{\text{Crys}}((X/S)_{\text{inf}}) \xrightarrow{p_{2,\text{inf}}^*} \hat{\text{Crys}}((X/S^m(1))_{\text{inf}}) \to \hat{\text{MC}}(\hat{X}^m/S^m(1)),\]

where the last arrow is the one in Remark 1.6. So it suffices to prove the existence of canonical isomorphism \(p_{2,\text{inf}}^* \mathcal{E} \cong p_{1,\text{inf}}^* \mathcal{E}\). It follows from the fact that the composite

\[\hat{\text{MIC}}(X/k) \cong \hat{\text{Crys}}((X/k)_{\text{str}}) \to \hat{\text{Crys}}((X/S)_{\text{inf}}) \xrightarrow{p_{2,\text{inf}}^*} \hat{\text{Crys}}((X/S^m(1))_{\text{inf}})\]

is independent of \(j\) and that \(p_{j,\text{inf}}^* \mathcal{E}\) is nothing but the image of \(\mathcal{E}\) by this composite.

(2) It suffices to prove that the second map is an isomorphism. Since both \(f_1^m, f_2^m\) are log smooth lift of \(f\), they are locally isomorphic [Kk88, Proposition 3.14]. Also, \(X \to X^m_j\) is nilpotent. So \(\hat{X}^m\) is locally isomorphic to the log formal tube of \(X^m_j\) in \(X^m_j \times_{S^m(1)} X^m_j\). Therefore, the projection \(\hat{X}^m \to X^m_j\) is etale locally isomorphic to the projection \(\hat{A}^m_{X^m_j} \to X^m_j\), where \(d\) is the rank of \(\Omega^1_{X/S}\). So, by Proposition 1.2, the second morphism in (1.17) is also an isomorphism. \(\Box\)

If we denote the composite of the morphisms in (1.17) for \(j = 2\) and the inverse of morphisms in (1.17) for \(j = 1\) by \(\eta^m : p_{2,\text{inf}}^* R^i f_{\text{dr}*} \mathcal{E} \xrightarrow{\alpha} p_{1,\text{inf}}^* R^i f_{\text{dr}*} \mathcal{E}\), we see that the pair \(R^i f_{\text{dr}*} \mathcal{E} := (R^i f_{\text{dr}*} \mathcal{E}, \{\eta^m\})\) defines an object in \(\hat{\text{Str}}(S/k) \cong \hat{\text{MIC}}(S/k)\). (To check the cocycle condition, we need to work on pullbacks of \(f : X \to S\) to \(S^m(2)\), and we leave the reader to write out the detail.) This is our definition of the Gauss-Manin connection on relative de Rham cohomology.

Remark 1.11. In this remark, we explain the compatibility of several morphisms and the construction of Gauss-Manin connection.

(1) Suppose that we are given a Cartesian diagram

\[
\begin{array}{ccc}
X' & \xrightarrow{g'} & X \\
\downarrow f' & & \downarrow f \\
S' & \xrightarrow{g} & S
\end{array}
\]

and let \(\mathcal{E} := (E, \nabla)\) be an object in \(\hat{\text{MIC}}(X/k)\), whose image in \(\hat{\text{MIC}}(X/S)\) we denote by \(\mathcal{E} := (\mathcal{E}, \nabla)\). If \(g\) is flat or if \(\mathcal{E}\) and all \(R^i f_{\text{dr}*} \mathcal{E}\) \((i \in \mathbb{N})\) are flat over \(\mathcal{O}_S\), the base change isomorphism

\[g_{\text{dr}*} R^i f_{\text{dr}*} \mathcal{E} \to R^i f'_{\text{dr}*} g'^* R^i f_{\text{dr}*} \mathcal{E}\]
in $\widetilde{\text{MIC}}(S'/S')$ is compatible with the maps as in (1.17) by functoriality. So the above morphism induces the isomorphism
\[ g_{dR}^* R^i f_{dR*} E \to R^i f_{dR*} g_{dR}^* E \]
in $\widetilde{\text{MIC}}(S'/k)$.

(2) Let $E := (E, \nabla) \in \text{MIC}(S/k)$, $E' := (E', \nabla') \in \widetilde{\text{MIC}}(X/k)$ and denote the image of $E$ (resp. $E'$) in $\text{MIC}(S/S)$ (resp. $\widetilde{\text{MIC}}(X/S)$) by $E$ (resp. $E' := (E', \nabla)$). Then the isomorphism
\[ R^i f_{dR*} (f_{dR*} E \otimes E') \cong E \otimes R^i f_{dR*} E' \]
of projection formula in $\widetilde{\text{MIC}}(S/S)$ is compatible with the maps in (1.17) by functoriality. So the above morphism induces the isomorphism
\[ (1.18) \quad R^i f_{dR*} (f_{dR*} E \otimes E') \cong E \otimes R^i f_{dR*} E' \]
in $\widetilde{\text{MIC}}(S/k)$.

(3) Let $0 \to E' \to E \to E'' \to 0$ be an exact sequence in $\widetilde{\text{MIC}}(X/k)$ and denote its image in $\widetilde{\text{MIC}}(X/S)$ by $0 \to E' \to E \to E'' \to 0$. Then we have the associated long exact sequence
\[ \cdots \to R^i f_{dR*} E' \to R^i f_{dR*} E \to R^i f_{dR*} E'' \to R^{i+1} f_{dR*} E' \to \cdots \]
in $\widetilde{\text{MIC}}(S/S)$ and it is compatible with the maps in (1.17) by functoriality. Thus the above exact sequence is enriched to the long exact sequence of Gauss-Manin connections
\[ \cdots \to R^i f_{dR*} E' \to R^i f_{dR*} E \to R^i f_{dR*} E'' \to R^{i+1} f_{dR*} E' \to \cdots \]
in $\widetilde{\text{MIC}}(S/k)$.

Remark 1.12. Note that our definition of Gauss-Manin connection here, which has crystalline flavor, is not a priori the same as the usual definition given by the Katz-Oda spectral sequence. Berthelot proved in [B74, V Proposition 3.6.4] the equivalence of two definitions under certain assumption, when the base scheme is killed by a power of some prime number $p$. We will prove later the coincidence of two definitions in certain cases, partly using Berthelot’s result. (See Remarks 1.17, 3.7, 3.8.) We expect that the two definitions coincide in general, but we will not prove it.

In the sequel, we assume the following conditions on the morphism $f$ in (1.13).

(C) For any $i$, $R^i f_{dR*} (\mathcal{O}_X, d)$ (endowed with Gauss-Manin connection) belongs to $\text{MIC}^n(S/k)$. 
(D) \( f_{\text{dR}*}(\mathcal{O}_X, d) = (\mathcal{O}_S, d), \ g_{\text{dR}*}(\mathcal{O}_S, d) = k. \)

Also, let \( N_f\text{MIC}(X/k) \) (resp. \( N_f\text{MIC}^n(X/k) \)) be the full subcategory of \( \text{MIC}(X/k) \) consisting of the objects which are iterated extensions of objects in \( f_{\text{dR}*}\text{MIC}(S/k) \) (resp. \( f_{\text{dR}*}\text{MIC}^n(S/k) \)). Note that, since \( \text{MIC}(S/k) \) is not necessarily an abelian category due to the possible existence of non-trivial log structure on \( S \), we cannot expect that \( N_f\text{MIC}(X/k) \) is an abelian category when the log structure on \( S \) is non-trivial.

For any \((E, \nabla) \in N_f\text{MIC}(X/k) \) (resp. \( N_f\text{MIC}^n(X/k) \)), the morphism of functors \( f_{\text{dR}*}f_{\text{dR}*} \rightarrow \text{id} \) induces the injection

\[
(1.19) \quad f_{\text{dR}*}f_{\text{dR}*}(E, \nabla) \hookrightarrow (E, \nabla)
\]

onto the maximal subobject of \((E, \nabla)\) belonging to the category \( f_{\text{dR}*}\text{MIC}(S/k) \) (resp. \( f_{\text{dR}*}\text{MIC}^n(S/k) \)): Indeed, if \((E, \nabla)\) belongs to \( f_{\text{dR}*}\text{MIC}(S/k) \) (resp. \( f_{\text{dR}*}\text{MIC}^n(S/k) \)), \((1.19)\) is an isomorphism by \((1.18)\) for \( i = 0 \) and the assumption (D), and the injectivity of the map \((1.19)\) in general case is proven by induction on the number of iterations of extension. Moreover, if we have \( f_{\text{dR}*}(F, \nabla_F) \hookrightarrow (E, \nabla) \) for some \((F, \nabla_F) \in \text{MIC}(S/k) \) (resp. \( \text{MIC}^n(S/k) \)), we have

\[
f_{\text{dR}*}(F, \nabla_F) = f_{\text{dR}*}f_{\text{dR}*}f_{\text{dR}*}(F, \nabla_F) \hookrightarrow f_{\text{dR}*}f_{\text{dR}*}(E, \nabla).
\]

The above result remains true if we replace \( X/k \) by \( X/S \): We define \( N_f\text{StrCrys}(X/S) \) as the full subcategory of \( \text{StrCrys}(X/S) \) consisting of the objects which is an iterated extension of objects in \( f_{\text{dR}*}\text{MIC}(S/S) \). Then, for any \((E, \nabla) \in N_f\text{MIC}(X/S) \), we have the injection

\[
f_{\text{dR}*}f_{\text{dR}*}(E, \nabla) \hookrightarrow (E, \nabla)
\]

onto the maximal subobject of \((E, \nabla)\) belonging to the category \( f_{\text{dR}*}\text{MIC}(S/S) \).

Next, we define the full subcategories \( N_f\text{StrCrys}(X/k) \), \( N_f\text{StrCrys}^n(X/k) \) of \( \text{StrCrys}(X/k) \) as follows.

**Definition 1.13.** Let the notations be as above.

1. Let \( N_f\text{StrCrys}(X/k) \) be the full subcategory of \( \text{StrCrys}(X/k) \) consisting of the objects which are iterated extensions of objects in \( f^*\text{Str}(S/k) \).
2. Let \( N_f\text{StrCrys}^n(X/k) \) be the full subcategory of \( N_f\text{StrCrys}(X/k) \) consisting of the objects which are iterated extension of objects in the essential image of \( \text{MIC}^n(S/k) \hookrightarrow \text{MIC}(S/k) \cong \text{Str}(S/k) \overset{f^*}{\rightarrow} \text{StrCrys}(X/k) \).

It is obvious from the definition that \( N_f\text{StrCrys}(X/k) \) (resp. \( N_f\text{StrCrys}^n(X/k) \)) corresponds to \( N_f\text{MIC}(X/k) \) (resp. \( N_f\text{MIC}^n(X/k) \)) via the equivalence \( \text{MIC}(X/k) \cong \text{StrCrys}(X/k) \) in Proposition \([1.9]\).

Now we introduce notation and setting which will be in force all along the rest of the paper.
Notation 1.14. We fix a field $k$ of characteristic zero and the diagram of fine log schemes (1.13) satisfying the conditions (A), (B), (C), (D) above and moreover the following condition (E):

(E) $\text{MIC}^n(S/k)$ is an abelian subcategory of $\widehat{\text{MIC}}(S/k)$.

Also, let $s \hookrightarrow S$ be an exact closed immersion over $k$ from a fine log scheme $s$ such that the structure morphism $s \longrightarrow \text{Spec} \, k$ at the level of underlying schemes is the identity. We denote the morphism $s \hookrightarrow S$ also by $s$ by abuse of notation, and denote the composite $\iota \circ s$ by $x : s \hookrightarrow X$. We denote the fiber product $s \times_S X$ by $X_s$ and the projection $X_s \longrightarrow s$ by $f_s$. Then $x$ induces the closed immersion $s \hookrightarrow X_s$, which we denote also by $x$.

Under the notation above, $R^i f_{dR^*}$ defines the functor

(1.20) $R^i f_{dR^*} : N_f \text{MIC}^n(X/k) \longrightarrow \text{MIC}^n(S/k)$.

Indeed, if $(E, \nabla)$ belongs to $f_{dR}^{*} \text{MIC}^n(S/k)$, $R^i f_{dR^*} E$ belongs to $\text{MIC}^n(S/k)$ by (1.18) and the assumption (C), and one can prove it for general $(E, \nabla) \in N_f \text{MIC}^n(X/k)$ by induction on the number of iterations of extensions, using the exact sequence in Remark 1.11(3), the assumption (E) and the fact that $\text{MIC}^n(S/k)$ is closed under extension in $\widehat{\text{MIC}}(S/k)$. In particular, for $E := (E, \nabla) \in \text{MIC}^n(S/k)$ and $E' := (E', \nabla') \in N_f \text{MIC}^n(X/k)$, the isomorphism (1.18) of projection formula is an isomorphism in $\text{MIC}^n(S/k)$.

Moreover, under the notation above, $\text{MIC}^n(S/k)$ is actually a neutral Tannakian category with fiber functor $s_{dR}^* : \text{MIC}^n(S/k) \longrightarrow \text{MIC}^n(s/s) = \text{Vec}_k$. Also, we have $g_{dR^*}(\mathcal{O}_S, d) = k$ by the assumption (D) and this and the assumption (E) imply, by a similar argument to [DP09, Proposition 1.4.3] that $N_f \text{MIC}^n(X/k)$ is a neutral Tannakian category with fiber functor $x_{dR}^* : N_f \text{MIC}^n(X/k) \longrightarrow \text{MIC}^n(s/s) = \text{Vec}_k$.

Let $N_f \text{MIC}(X_s/s)$ be the subcategory of $\text{MIC}(X_s/s)$ consisting of objects which are iterated extensions of objects in $f_{s, dR}^* \text{MIC}(s/s) = f_{s, dR}^* \text{Vec}_k$. Then we have

$$f_{s, dR^*}(\mathcal{O}_{X_s}, d) = s_{dR}^* f_{dR^*}(\mathcal{O}_X, d) = k$$

by assumptions (C) and (D), and we see that it is also a neutral Tannakian category with fiber functor $x_{dR}^* : N_f \text{MIC}(X_s/s) \longrightarrow \text{MIC}^n(s/s) = \text{Vec}_k$.

Finally, we give a useful sufficient condition for the diagram (1.13) to satisfy the conditions (A), (B), (C), (D) and (E).

Proposition 1.15. Suppose that we are given a diagram (1.13) satisfying the following conditions:

(1) $S$ is geometrically connected, $f$ is a proper log smooth integral saturated ([AGT16 II.5.18], [Ts19, Definition II.2.10]) morphism between $f_s$ log schemes with geometrically reduced, geometrically connected fibers and $g$ is a separated morphism of finite type. $\iota$ is a section of $f$. 

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(2) Etale locally around any closed point $t$ of $S$, there exists a strict etale morphism $S \longrightarrow \mathbb{A}^{r,s}_k$ over $k$ for some $r, s \in \mathbb{N}$.

Then it satisfies the conditions (A), (B), (C), (D) and (E).

**Remark 1.16.** In this remark, we prove that, in the situation of Proposition [1.13], the inclusion of categories $\text{MIC}^n(S/k) \subseteq \text{MIC}^n(S/k)'$ is, actually, an equality ($\text{MIC}^n(S/k)'$ is defined in Remark [1.1]). The proof given here is a modification of the proof in [DPS18, Remark 2.12].

We need to prove that, for an object $(E, \nabla)$ in $\text{MIC}^n(S/k)'$ and a geometric point $y$ of $S$ which is not necessarily over a closed point, $(E, \nabla)$ has nilpotent residues at $y$. Since the assertion is etale local, we may assume that there exists an strict etale morphism $\varphi : S \longrightarrow \mathbb{A}^{r,s}_k$. For $I \subseteq \{1, \ldots, \min(r, s)\}$, let $\mathbb{A}^{r,s}_I$ be the closed log subscheme $\bigcap_{i \in I} \{ t_i = 0 \}$ of $\mathbb{A}^{r,s}$ where $t_1, \ldots, t_r$ are the coordinates of $\mathbb{A}^{r,s}$, and put $\mathbb{A}^{r,s,o}_I := \mathbb{A}^{r,s}_I \setminus \bigcup_{j \not\in I} \mathbb{A}^{r,s}_j$, $S^o_I := \varphi^{-1}(\mathbb{A}^{r,s,o}_I)$. Then $S = \bigcup_{I \subseteq \{1, \ldots, \min(r, s)\}} S^o_I$ set-theoretically. Also, we have the equality $\mathcal{M}_S^{\text{gp}}|_{S^o_I} = \mathbb{Z}^{r,t}_I$, where $\mathcal{M}_S^{\text{gp}} := \mathcal{M}_S^{\text{gp}}/\mathcal{O}_S^{\text{gp}}$ and $r_I := |I| + \max(s - r, 0)$.

Now take any object $(E, \nabla)$ and $y$ as above. Let $I$ be the subset of $\{1, \ldots, \min(r, s)\}$ such that the image of $y$ in $S$ belongs to $S^o_I$. Then $\nabla$ induces a linear map

$$\rho_I : E|_{S^o_I} \longrightarrow E|_{X^o_I} \otimes \mathcal{M}_S^{\text{gp}}|_{S^o_I} = E|_{S^o_I} \otimes \mathbb{Z}^{r,t}_I.$$ 

For a geometric point $z$ of $S^o_I$, we denote the specialization of $\rho_I$ to $z$ by

$$\rho_z : E|_z \longrightarrow E|_z \otimes \mathcal{M}_S^{\text{gp}}|_{S^o_I} = E|_z \otimes \mathbb{Z}^{r,t}_I.$$ 

To prove the assertion, we should prove that, for any $\mathbb{Z}$-linear map $\psi : \mathcal{M}_S^{\text{gp}}|_{S^o_I} = \mathbb{Z}^{r,t} \longrightarrow \mathbb{Z}$, the map $(\text{id} \otimes \psi) \circ \rho_y : E|_y \longrightarrow E|_y$ is nilpotent. By replacing $S$ by a small affine open log subscheme containing the image of $y$ in $S$, we may assume that $E|_{S^o_I}$ is free of rank $c$. For $1 \leq i \leq r_I$, we denote the $i$-th projection $\mathcal{M}_S^{\text{gp}}|_{S^o_I} = \mathbb{Z}^{r,t}_I \longrightarrow \mathbb{Z}$, the map $(\text{id} \otimes \pi_i) \circ \rho_y : E|_y \longrightarrow E|_y$ for any $i$. Since this map is the specialization of the map

$$\rho_{I,i} := (\text{id} \otimes \pi_i) \circ \rho_I : E|_{S^o_I} \longrightarrow E|_{S^o_I},$$ 

we are reduced to proving the nilpotence of the map $\rho_{I,i}$. Choose for any closed point of $S^o_I$ a geometric point over it and let $T$ be the set of such geometric points. Then, by assumption that $(E, \nabla)$ belongs to $\text{MIC}^n(S/k)'$, for any $z \in T$ and for any $1 \leq i \leq r_I$, the map $\rho_{z,i}$ is nilpotent. Hence, as an element in $\text{End}(E|_z)$, $\rho_{z,i}^{-1}$ is equal to 0. On the other hand, since $S^o_I$ is reduced and Jacobson and $E|_{S^o_I}$ is free, the natural map $\text{End}(E|_{S^o_I}) \longrightarrow \prod_{z \in T} \text{End}(E|_z)$ is injective. Hence $\rho_{z,i}^{-1}$ is equal to 0 in $\text{End}(E|_{S^o_I})$. So the proof of the assertion is finished.
Proof. (of Proposition 1.15) It is easy to see that the conditions (A), (B) are satisfied. The first condition in (D) follows from the assumption that \( f \) is proper and has geometrically reduced, geometrically connected fibers. We can prove the second condition in (D) using the geometric connectedness of \( S \) and the local description of \( S \) in (2).

We prove the condition (E). To do so, it suffices to work etale locally on \( S \) and thus we may assume the existence of a strict etale morphism \( \varphi : S \to \mathbb{A}^{r,s}_k \) over \( k \) as in (2). Let \( D := \bigcup_{i=1}^{\min(s,r)} \varphi^{-1}(\{t_i = 0\}) \), where \( t_1, \ldots, t_r \) is the canonical coordinate of \( \mathbb{A}^{r,s}_k \) and let \( S' \) be the log scheme whose underlying scheme is the same as that of \( S \) and whose log structure is associated to the normal crossing divisor \( D \). Then, if \( s \leq r \), we have \( S' = S \) and it is well-known that \( \text{MIC}^n(S'/k) \) is an abelian subcategory of \( \text{MIC}(S/k) \). (This follows from Gérard-Levelt theory \cite{GL76} §3, as shown in \cite{C02} Lemma 3.1.6, Corollary 3.1.7 for example. Also, one can prove it directly in the same way as \cite{Ke07} Lemma 3.2.14.) When \( s > r \), the log structure \( M_S \) of \( S \) is the one associated to the monoid homomorphism \( M_S' \to O_{S'} \) induced by \( \alpha_S : M_S' \to O_{S'} \) and the monoid homomorphism \( \mathbb{N}^{s-r} \to O_{S'} \) sending the standard basis of \( \mathbb{N}^{s-r} \) to zero. From this, we see that the category \( \text{MIC}^n(S/k) \) (resp. \( \text{MIC}(S'/k) \)) is the category of objects in \( \text{MIC}^n(S'/k) \) (resp. \( \text{MIC}(S'/k) \)) endowed with \( (s-r) \) commuting nilpotent endomorphisms (resp. commuting endomorphisms). Thus \( \text{MIC}^n(S/k) \) is an abelian subcategory of \( \text{MIC}(S/k) \) also in this case. So the condition (E) is satisfied.

In the following, we prove the condition (C). By standard argument, we may reduce to the case where \( k \) is finitely generated over \( \mathbb{Q} \). Take a closed point \( t \) of \( S \) and we want to prove that \( R^i f_{dR}^*(O_X, d) \) is locally free around \( t \) and has nilpotent residues at a geometric point over \( t \). Since we may work etale locally around \( t \) and enlarge \( k \), we can assume that \( S \) is affine, \( t \) is \( k \)-rational and that \( g \) factors as \( S \to A \to \text{Spec} k, \) where \( \varphi \) is a strict etale morphism such that the inverse image of the origin in \( A^{r,s}_k \) by \( \varphi \) is equal to \( t \). Hence we have a diagram

\[
\begin{array}{ccc}
X & \xrightarrow{f} & S \\
\varphi & \downarrow & \mathbb{A}^{r,s}_k \\
\text{Spec} k & \to & \text{Spec} k.
\end{array}
\]

Take an affine connected scheme \( \text{Spec} A' \) smooth of finite type over \( \mathbb{Z} \) with \( \text{Frac} A' = k \) and a diagram

\[
\begin{array}{ccc}
X_{A'} & \xrightarrow{f_{A'}} & S_{A'} \\
\varphi_{A'} & \downarrow & A^{r,s}_{A'} \\
\text{Spec} A' & \to & \text{Spec} A'
\end{array}
\]

which satisfy the following:

- The pull-back of the diagram (1.22) by the map \( \alpha : \text{Spec} k \to \text{Spec} A' \) induced by the inclusion \( A' \hookrightarrow \text{Frac} A' = k \) is the diagram (1.21).

- The inverse image \( t_{A'} \) of the origin in \( A^{r,s}_{A'} \) by \( \varphi_{A'} \) is isomorphic to \( \text{Spec} A' \).

- The morphism \( f_{A'} \) is proper log smooth, integral and saturated.
The morphism $\varphi_{A'}$ is strict étale.

Take a prime number $p$ such that $\text{Spec} (A' \otimes \mathbb{Z} F_p)$ is connected and non-empty, let $A$ be the $p$-adic completion of $A' \otimes \mathbb{Z} F_p$ and denote the pullback of the diagram (1.22) by the canonical morphism $\beta : \text{Spec} A \rightarrow \text{Spec} A'$ by

$$X_A \xrightarrow{f_A} S_A \xrightarrow{\varphi_A} A_A^{r,s} \rightarrow \text{Spec} A.$$ (1.23)

Also, let us take a morphism $F : \text{Spec} A \rightarrow \text{Spec} A$ over $\text{Spec} \mathbb{Z}_p$ which is a lift of the absolute Frobenius on $\text{Spec} (A \otimes \mathbb{Z} F_p)$. Let $k_0$ be the perfection of $\text{Frac} (A \otimes \mathbb{Z} F_p)$ and put $W := W(k_0)$. Then there exists a morphism $\gamma : \text{Spec} W \rightarrow \text{Spec} A$ which is compatible with Frobenius morphism on $\text{Spec} W$ and the morphism $F$ on $\text{Spec} A$. Denote the pullback of the diagram (1.23) by $\gamma$ by

$$X_W \xrightarrow{f_W} S_W \xrightarrow{\varphi_W} A_W^{r,s} \rightarrow \text{Spec} W.$$ (1.24)

Denote its special fiber by

$$X_0 \xrightarrow{f_0} S_0 \xrightarrow{\varphi_0} A_0^{r,s} \rightarrow \text{Spec} k_0.$$ (1.25)

generic fiber by

$$X_{k_W} \xrightarrow{f_{k_W}} S_{k_W} \xrightarrow{\varphi_{k_W}} A_{k_W}^{r,s} \rightarrow \text{Spec} k_W$$ (1.26)

and its $p$-adic completion by

$$\widehat{X}_W \xrightarrow{\widehat{f}_W} \widehat{S}_W \xrightarrow{\widehat{\varphi}_W} \widehat{A}_W^{r,s} \rightarrow \text{Spf} W.$$ (1.27)

We denote the canonical open immersion $\text{Spec} k_W \hookrightarrow \text{Spec} W$ by $\delta$.

Since $k$ is the fraction field of $A'$, we have a morphism $\epsilon : \text{Spec} k_W \rightarrow \text{Spec} k$ with $\alpha \circ \epsilon = \beta \circ \gamma \circ \delta$.

Denote the inverse image of the origin by $\varphi_A, \varphi_W, \varphi_{k_W}$ by $t_A, t_W, t_{k_W}$ respectively. Then we have the following:

$$\begin{array}{cccccccc}
t & \longrightarrow & t_{A'} & \longleftarrow & t_A & \longleftarrow & t_W & \longleftarrow & t_{k_W} \\
&&&&&&&&
\text{Spec} k & \xrightarrow{\alpha} & \text{Spec} A' & \xleftarrow{\beta} & \text{Spec} A & \xleftarrow{\gamma} & \text{Spec} W & \xleftarrow{\delta} & \text{Spec} k_W.
\end{array}$$

Let $\psi : B := \Gamma(S_W, \mathcal{O}_{S_W}) \rightarrow W$ be the homomorphism induced by the inclusion $t_W \hookrightarrow S_W$ and let $m$ be the kernel of $\psi_\mathbb{Q} : B_\mathbb{Q} \rightarrow k_W$, where $\mathbb{Q}$ stands for $\otimes \mathbb{Q}$. Then we can regard $R^i f_{k_W, dR}(\mathcal{O}_{X_{k_W}}, d)$ as a finitely generated $B_\mathbb{Q}$-module with integrable connection, and we should prove that it is free at $m$ and that it has nilpotent residues at the closed point of $S_{k_W}$ corresponding to $m$. (Indeed, by Remark 1.16 the nilpotence of residues can be checked at geometric points over closed points.
Also, noting that the log structure on \( S_W \) is the pullback of that on \( \mathbb{A}^{r,s}_W \), the log structure of \( S_W \) is defined in Zariski topology and so the nilpotence of residues can be checked at the closed point corresponding to \( \mathfrak{m} \), not at a geometric point above it. See the second paragraph before Proposition 1.22.

Put \( \hat{B} := \Gamma(S_W, \mathcal{O}_{S_W}) \). Then \( \psi : B = \Gamma(S_W, \mathcal{O}_{S_W}) \to W \) induces the homomorphism \( \hat{\psi} : \hat{B} \to W \). Let \( \hat{\mathfrak{m}} \) be the kernel of \( \hat{\psi}_Q : \hat{B}_Q \to k_W \). Then, to show the claim in the previous paragraph, it suffices to prove that \( E := R^f \mathcal{O}_{X_0/S_W} \otimes \mathbb{Q} \) (where \( f_0, \text{crys} \) denotes the morphism of topoi \( (X_0/S_W)_{\text{crys}} \to (\hat{S}_W)_{\text{et}} \)), regarded as a finitely generated \( \hat{B}_Q \)-module with integrable connection (in rigid analytic sense), is free at \( \hat{\mathfrak{m}} \) and that it has nilpotent residues at the point corresponding to \( \hat{\mathfrak{m}} \), by the comparison theorem of log de Rham and log crystalline cohomology [HK94, Proposition 2.20].

Let \( F : \hat{S}_W \to \hat{S}_W \) be a lift of Frobenius on \( S_0 \) compatible with the morphism \( A^{r,s}_W \to A^{r,s}_W \) over the Frobenius on \( W \) which sends the coordinates of \( A^{r,s}_W \) to their \( p \)-th powers. By the condition (1) on \( f \), we see that \( f_0 \) is a proper log smooth integral morphism of Cartier type [Ts19, Proposition II.2.14]. So, by [HK94], the endomorphism \( F^* \) on \( E \) induced by \( F \) is an isomorphism. To prove the freeness at \( \hat{\mathfrak{m}} \), it suffices to prove that \( E/\hat{\mathfrak{m}}^n E \) is a free \( \hat{B}_Q/\hat{\mathfrak{m}}^n \)-module for any \( n \). Take an surjective homomorphism

\[ (\hat{B}_Q/\hat{\mathfrak{m}}^n)^{\oplus r} \to E/\hat{\mathfrak{m}}^n E \]

which is isomorphic modulo \( \hat{\mathfrak{m}} \), and denote the kernel of it by \( K \). Then it induces the isomorphism

\[ ((\hat{B}_Q/\hat{\mathfrak{m}}^n)^{\oplus r} / K) \otimes_{\hat{B}_Q/\hat{\mathfrak{m}}^n,(F^*)^n} (\hat{B}_Q/\hat{\mathfrak{m}}^n) \xrightarrow{\cong} (E/\hat{\mathfrak{m}}^n E) \otimes_{\hat{B}_Q/\hat{\mathfrak{m}}^n,(F^*)^n} (\hat{B}_Q/\hat{\mathfrak{m}}^n) \]

\[ \cong E/\hat{\mathfrak{m}}^n E ; \]

where the last isomorphism is induced by the isomorphism \( F^* \) on \( E \). Because \( K \subset (\hat{\mathfrak{m}}/\hat{\mathfrak{m}}^n)^{\oplus r} \) and \( (F^*)^n(\hat{\mathfrak{m}}) \subset \hat{\mathfrak{m}}^{p^n} \subset \hat{\mathfrak{m}}^n \), the source of this map is isomorphic to \( (\hat{B}_Q/\hat{\mathfrak{m}}^n)^{\oplus r} \). Hence \( E/\hat{\mathfrak{m}}^n E \) is a free \( \hat{B}_Q/\hat{\mathfrak{m}}^n \)-module for any \( n \), as required.

Now we prove the nilpotence of residues of the connection \( \nabla \) defined on \( E \) at \( \hat{\mathfrak{m}} \). Because of the functoriality of crystalline cohomologies, the morphism \( F : \hat{S}_W \to \hat{S}_W \) induces an endomorphism on \( (E, \nabla) \). Thus, any residue map \( N : E/\hat{\mathfrak{m}} E \to E/\hat{\mathfrak{m}} E \) at \( \hat{\mathfrak{m}} \) satisfies the equality \( N \circ F^* = pF^* \circ N \), and this implies the nilpotence of \( N \). Hence the proof of the proposition is finished. \( \square \)

**Remark 1.17.** In this remark, we give several sufficient conditions for the coincidence of the definition of Gauss-Manin connection in this paper and the usual definition of Gauss-Manin connection via Katz-Oda spectral sequence. (We will use the settings in this remark later in this paper, but we will not use the coincidence results.)

1. Let the situation be as in Proposition 1.15 and assume that we can always take \( s \leq r \) in the condition (2) there. (This implies that \( S \) is log smooth over \( k \).)
Let \((E, \nabla)\) be an object in \(\text{MIC}(X/k)\). Also, assume the following condition \((\ast)\):

\(\ast\) Etale locally on \(S\), there exist diagrams \((1.21), (1.22), (1.23), (1.24), (1.26)\) (with \(s \leq r\)) such that the restriction of \((E, \nabla)\) to \(\text{MIC}(X_{k_W}/k_W)\) by \(\epsilon : \text{Spec} \ k_W \to \text{Spec} \ k\) comes from an object \((E_W, \nabla_W)\) in \(\text{MIC}(X_W/W)\) whose mod \(p^n\) reductions \((n \in \mathbb{N})\) are quasi-nilpotent in the sense of \([BO78]\).

Then the definition of Gauss-Manin connection on \(R^if_{dR*}(E, \nabla)\) in this paper coincides with that via Katz-Oda spectral sequence.

We prove this claim. If we denote the mod \(p^n\) reduction of \(f_W\) and \((E, \nabla)\) by \(f_n : X_n \to S_n, (E_n, \nabla_n)\) respectively, it suffices to prove the coincidence of two definitions of Gauss-Manin connection on \(R^if_{n,dR*}(E_n, \nabla_n)\). The condition \((\ast)\) implies that \((E_W, \nabla_W)\) comes from a locally free crystal \(E\) on \((\hat{X}/\hat{S})_{\text{crys}}\) and so \((E_n, \nabla_n)\) comes from the mod \(p^n\) reduction of \(E\), which is a locally free crystal on \((X_n/S_n)_{\text{crys}}\). Then the coincidence of two definitions of Gauss-Manin connection follows from \([B74, \text{V Proposition 3.6.4}]\), which works without any change in the case \(S\) is log smooth over \(k\).

We expect that the argument in the proof of \([B74, \text{V Proposition 3.6.4}]\) will work also in the case \(s > r\), but we will not pursue this topic in this paper.

(2) Let \(f : X \to S\) be as in Notation 1.14 and assume that we have a Cartesian diagram

\[
\begin{array}{ccc}
X & \longrightarrow & X' \\
\downarrow f & & \downarrow f' \\
S & \longrightarrow & S',
\end{array}
\]

with \(f'\) proper log smooth integral. Also, let \((E, \nabla)\) be an object in \(\text{MIC}(X/k)\) and assume the following.

(a) \((E, \nabla)\) is the pullback of an object \((E', \nabla')\) in \(\text{MIC}(X'/k)\).
(b) For the object \((E', \nabla')\) in (a) and for any \(i\), the underlying \(O_{S'}\)-module of \(R^if_{dR*}(E', \nabla')\) is locally free.
(c) The two definitions of Gauss-Manin connection on \(R^if_{dR*}(E', \nabla')\) coincide.

Then two definitions of Gauss-Manin connection on \(R^if_{dR*}(E', \nabla')\) coincide, because it is the pullback of \(R^if_{dR*}(E', \nabla') \in \text{MIC}(S'/k)\) to \(\text{MIC}(S/k)\) by remark 1.11 (1).

(3) We can apply (1) and (2) in the following setting. Let \(\mathcal{S}\) be the log scheme \((\text{Spec} \ k[[t]], \mathcal{M}_S)\), where \(\mathcal{M}_S\) is the log structure associated to the monoid homomorphism \(\mathbb{N} \to k[[t]]; 1 \mapsto t\). (Note that \(\mathcal{M}_S = O_{S^s} \setminus \{0\}\).) Let \(\tilde{f} : \mathcal{X} \to \mathcal{S}\) be a regular semistable model of a proper smooth curve of genus \(g \geq 2\) over \(\text{Spec} \ k((t))\), endowed with log structure associated to the special fiber. Let \(f : X \to S\) be the special fiber of \(\tilde{f}\) with pullback log structure.

Let \(\mathcal{X}^{\circ}\) be the stable model of \(\mathcal{X}\) with blow-down \(\varphi : \mathcal{X} \to \mathcal{X}^{\circ}\) of schemes (without log structures). If we denote by \(\overline{M}_g\) the moduli log stack of stable log
curves of genus $g$ over $k$ and by $\overline{C}_g$, the universal stable log curve over $\overline{M}_g$, we have a morphism $S^\circ \longrightarrow \overline{M}_g$ such that $\overline{C}_g \times_{\overline{M}_g} S^\circ \cong \mathcal{X}^\circ$. By pulling back the log structures on $\overline{M}_g$ and $\overline{C}_g$, we obtain a morphism of log schemes $(\mathcal{X}^\circ, \mathcal{N}_{\mathcal{X}^\circ}) \longrightarrow (\mathcal{S}^\circ, \mathcal{N}_{\mathcal{S}^\circ})$. Because the generic fiber of $\mathcal{X}^\circ \longrightarrow S^\circ$ is smooth, $\mathcal{N}_{\mathcal{S}^\circ}$ is trivial on the generic point. Thus the structure morphism $\mathcal{N}_{\mathcal{S}^\circ} \longrightarrow \mathcal{O}_{\mathcal{S}^\circ}$ of the log structure $\mathcal{N}_{\mathcal{S}^\circ}$ factors through $\mathcal{O}_{\mathcal{S}^\circ} \setminus \{0\} = \mathcal{M}_S$ and so we have a morphism $\mathcal{S} \longrightarrow (\mathcal{S}^\circ, \mathcal{N}_{\mathcal{S}^\circ})$. By base change, we obtain the morphism $\tilde{f} : \mathcal{X}^\prime := (\mathcal{X}^\circ, \mathcal{N}_{\mathcal{X}^\circ}) \times_{(\mathcal{S}^\circ, \mathcal{N}_{\mathcal{S}^\circ})} \mathcal{S} \longrightarrow \mathcal{S}$. According to $\textbf{[K00] 1.1}$, the log scheme $\mathcal{X}^\prime$ has the form $(k[[t]][x, y]/(xy - t^n), \mathbb{N}^2 \oplus \Delta_{N,n}, \mathbb{N})$ (where $\Delta : \mathbb{N} \longrightarrow \mathbb{N}^2$ is the diagonal map and $n : \mathbb{N} \longrightarrow \mathbb{N}$ is the multiplication by $n$) étale locally around double points. Because $\varphi^0$ is the composition of blow-ups at double points, it is realized as the composition of log blow-ups $\textbf{([N06, §4])}$ $\varphi : \mathcal{X}^\prime \longrightarrow \mathcal{X}$. Thus we have the following commutative diagram

$$
\begin{array}{ccc}
X & \longrightarrow & \mathcal{X} \\
\downarrow f & & \downarrow \varphi \\
\mathcal{S} & \longrightarrow & \mathcal{X}^\prime \\
\downarrow & & \downarrow \\
\overline{C}_g & \longrightarrow & \overline{M}_g,
\end{array}
$$

(1.28)

where the left and the right squares are Cartesian.

Let $(E, \nabla)$ be an object in $\text{MIC}(X/k)$ which comes from an object in $(E^\prime, \nabla^\prime)$ in $\text{MIC}(\overline{C}_g/k)$ which satisfies the condition (b) in (2). Moreover, assume that $(E^\prime, \nabla^\prime)$ satisfies the condition (*) in (1) étale locally on $\overline{M}_g$. We denote the pullback of $(E^\prime, \nabla^\prime)$ to $\mathcal{X}^\prime/S$, $\mathcal{X}/S$ by $(E_{\mathcal{X}^\prime}, \nabla_{\mathcal{X}^\prime})$, $(E_{\mathcal{X}}, \nabla_{\mathcal{X}})$, respectively. Then we can apply (1), (2) to see that the two definitions of Gauss-Manin connection on $R^i \tilde{f}_{\text{dr}*}(E_{\mathcal{X}^\prime}, \nabla_{\mathcal{X}^\prime})$ coincide. Then, using the quasi-isomorphism

$$
R^j \varphi_* \Omega^j_{\mathcal{X}/S} = \Omega^j_{\mathcal{X}^\prime/S} \quad (j \geq 0)
$$

which we prove in Proposition $\textbf{[L18]}$ below and the projection formula, we obtain the isomorphism

$$
R^i \tilde{f}_*(E_X \otimes_{\mathcal{O}_X} \Omega^j_{\mathcal{X}/S}) \cong R^i \tilde{f}^!_* (E_{\mathcal{X}^\prime} \otimes_{\mathcal{O}_{\mathcal{X}^\prime}} \Omega^j_{\mathcal{X}^\prime/S}) \quad (i, j \geq 0)
$$

and so we obtain the isomorphism $R^i \tilde{f}_{\text{dr}*}(E_X, \nabla_X) \cong R^i \tilde{f}^!_{\text{dr}*}(E_{\mathcal{X}^\prime}, \nabla_{\mathcal{X}^\prime})$ for $i \geq 0$ by spectral sequence argument. Thus the two definitions of Gauss-Manin connection on $R^i \tilde{f}_{\text{dr}*}(E_X, \nabla_X)$ coincide. Then, since $R^i f_{\text{dr}*}(E, \nabla)$ is the pullback of $R^i \tilde{f}_{\text{dr}*}(E_X, \nabla_X)$ to $S$ by the base change isomorphism, we conclude that the two definitions of Gauss-Manin connection on $R^i f_{\text{dr}*}(E, \nabla)$ coincide. We can apply this result to the case $(E, \nabla) = (\mathcal{O}_X, d)$.

**Proposition 1.18.** Suppose that we are given a diagram of fs log schemes

$$
\begin{array}{ccc}
\mathcal{X} & \longrightarrow & \mathcal{X}^\prime \\
\downarrow & & \downarrow \\
\mathcal{S} & \longrightarrow & \mathcal{S}
\end{array}
$$

$$
\text{Proposition 1.18.}$$
such that $X'$ is log regular, $\varphi$ is a log blow-up and $\Omega^1_{X'/S}$ is locally free. Then, for each $j \geq 0$, we have a quasi-isomorphism
\[ R\varphi_* \Omega^j_{X/S} = \Omega^j_{X'/S}. \]

Proof. Since $X'$ is log regular, we can apply [N06, Theorem 4.7] and [Kk94, Theorem 11.3] to have a quasi-isomorphism
\[ R\varphi_* \mathcal{O}_X = \mathcal{O}_{X'}. \]
Because $\varphi$ is log etale, $\varphi^* \Omega^j_{X'/S} = \Omega^j_{X'/S}$. From the fact that $\Omega^1_{X'/S}$ is locally free, by the projection formula, we have a quasi-isomorphism
\[ R\varphi_* \Omega^j_{X'/S} = \Omega^j_{X'/S} \]
for each $j$.

2 Relative Tannakian theory

In this section, we recall the definition of relatively unipotent log de Rham fundamental group of Lazda [Laz15], which is based on the theory of Tannakian categories due to Deligne [D89], [D90]. Let $\mathcal{C}, \mathcal{D}$ be Tannakian categories endowed with exact faithful $k$-linear tensor functors $t : \mathcal{C} \to \mathcal{D}, \omega : \mathcal{D} \to \mathcal{C}$ with $\omega \circ t = \text{id}$. (We regard $\mathcal{D}$ as a Tannakian category over $\mathcal{C}$ (via $t$) endowed with a fiber functor $\omega$ to $\mathcal{C}$.) Denote $\pi(\mathcal{C}) := \text{Spec} \mathcal{O}_{\pi(\mathcal{C})}$, $\pi(\mathcal{D}) := \text{Spec} \mathcal{O}_{\pi(\mathcal{D})}$ be the fundamental group of $\mathcal{C}, \mathcal{D}$, which is a group scheme in $\mathcal{C}, \mathcal{D}$ respectively. $\mathcal{O}_{\pi(\mathcal{C})}$ is defined as
\[ \mathcal{O}_{\pi(\mathcal{C})} := \text{Ker}(\bigoplus_{\alpha: V \to W \in \mathcal{C}} V \otimes W^\vee \to \bigoplus_{V \in \mathcal{C}} V \otimes V^\vee), \]
where the arrow is the difference of the map induced by $V \otimes W^\vee \xrightarrow{\alpha \otimes \text{id}} W \otimes W^\vee$ ($\alpha : V \to W \in \mathcal{C}$) and the map induced by $V \otimes W^\vee \xrightarrow{\text{id} \otimes \alpha^\vee} V \otimes V^\vee$ ($\alpha : V \to W \in \mathcal{C}$), $\mathcal{O}_{\pi(\mathcal{D})}$ is defined in the same way (see [D90, 8.13]). Then the map of group schemes in $\mathcal{D}$
\[ (2.1) \quad t^* : \pi(\mathcal{D}) \to \pi(\mathcal{C}) \]
is defined by the map $t|_{\mathcal{O}_{\pi(\mathcal{C})}} : \mathcal{O}_{\pi(\mathcal{C})} \to \mathcal{O}_{\pi(\mathcal{D})}$, which is induced by the map $\bigoplus_{V \in \mathcal{C}} t(V) \otimes t(V)^\vee \to \bigoplus_{W \in \mathcal{D}} W \otimes W^\vee$ of “the inclusion into $t(V)$-component ($V \in \mathcal{C}$)”. By applying $\omega$, we obtain the map
\[ (2.2) \quad \omega(t^*) : \omega(\pi(\mathcal{D})) \to \pi(\mathcal{C}), \]
defined by $\mathcal{O}_{\pi(\mathcal{C})} \to \omega(\mathcal{O}_{\pi(\mathcal{D})})$, which is induced by the map $\bigoplus_{V \in \mathcal{C}} V \otimes V^\vee \to \bigoplus_{W \in \mathcal{D}} \omega(W) \otimes \omega(W)^\vee$ of “the inclusion into $t(V)$-component ($V \in \mathcal{C}$)”. We define the
group scheme \( G(D, \omega) := \text{Spec} \mathcal{O}_{G(D, \omega)} \) in \( C \) by \( G(D, \omega) := \ker \omega(t^*) \). By definition, \( \mathcal{O}_{G(D, \omega)} \) is defined as the pushout of the diagram

\[
1_C \leftarrow \mathcal{O}(\pi(C)) \rightarrow \omega(\mathcal{O}(\pi(D))).
\]

For a group scheme \( G \) in \( C \), we consider the category \( \text{Rep}_C G \) of representations of \( G \) in \( C \). For \( V \in D \), the map \( \omega(V) \rightarrow \omega(V) \otimes \mathcal{O}(\pi(D)) \) induced by the map \( V \otimes V^\vee \rightarrow \bigoplus_{W \in D} W \otimes W^\vee \) of ‘the inclusion into \( V \)-component’ gives a coaction of \( \omega(\mathcal{O}(\pi(D))) \) on \( \omega(V) \), hence a representation of \( \omega(\pi(D)) \) on \( \omega(V) \). This induces a representation of \( G(D, \omega) \) on \( \omega(V) \) and so the functor \( \omega \) induces the functor

\[
(2.3) \quad D \rightarrow \text{Rep}_C(G(D, \omega)).
\]

On the other hand, for a group scheme \( G = \text{Spec} \mathcal{O}_G \) in \( C \), we have the functors

\[
t : C \rightarrow \text{Rep}_C G, \quad \omega : \text{Rep}_C G \rightarrow C,
\]

where \( t \) is the functor that embeds \( C \) in \( \text{Rep}_C G \) as trivial representations and \( \omega \) is the forgetful functor. Using these, we can define the group scheme \( G(\text{Rep}_C G, \omega) \). The coactions \( \omega(V) \rightarrow \omega(V) \otimes \mathcal{O}_G \) (\( V \in \text{Rep}_C G \)) induce the map \( \omega(\mathcal{O}(\pi(\text{Rep}_C G))) \rightarrow \mathcal{O}_G \). Then we can see that it induces the map \( \mathcal{O}_G(\text{Rep}_C G, \omega) \rightarrow \mathcal{O}_G \), thus the map

\[
(2.4) \quad G \rightarrow G(\text{Rep}_C G, \omega).
\]

In [Laz15, Prop. 2.3], Lazda proved the following.

**Proposition 2.1.** The functor \( (2.3) \) is an equivalence and the morphism \( (2.4) \) is an isomorphism.

Assume that we are given group schemes \( G, G' \) in \( C \) and a \( k \)-linear exact tensor functor

\[
\alpha : \text{Rep}_C G \rightarrow \text{Rep}_C G'
\]

compatible with the forgetful functors \( \omega_G, \omega_{G'} \) to \( C \). Then we have a morphism \( \alpha^* : \pi(\text{Rep}_C G') \rightarrow \alpha(\pi(\text{Rep}_C G)) \), and it is easy to see that this induces the morphism

\[
G' = G(\text{Rep}_C G', \omega_{G'}) \rightarrow G(\text{Rep}_C G, \omega_G) = G,
\]

where the first and the last equality are due to Proposition 2.1.

Let \( C \) be a Tannakian category and let \( \eta : C \rightarrow \text{Vec}_k \) be a fiber functor. Then it is well-known that, if we put \( G := \eta(\pi(C)) \), we have the equivalence \( C \xrightarrow{\cong} \text{Rep}_k(G) \) by Tannaka duality. For \( V \in C \), the \( G \)-action on \( \eta(V) \) is given in the following way: Let \( R \) be a \( k \)-algebra and take \( g \in G(R) \) which is a ring homomorphism \( g : \mathcal{O}_G = \eta(\mathcal{O}(\pi(C))) \rightarrow R \). Then, by definition of \( \mathcal{O}_C \), we have a natural map \( \text{[D90, 8.4]} \):

\[
\eta(V) \otimes \eta(V)^\vee \rightarrow \eta(\mathcal{O}(\pi(C))) \xrightarrow{g} R.
\]
and this induces the map
\[ g_V : \eta(V) \otimes R \to \eta(V) \otimes R, \]
which is defined to be the action of \( g \in G(R) \) on \( \eta(V) \otimes R \).

If we consider the previous definition of \( \mathcal{O}_{G(\mathcal{D}, \omega)} \), which considers it as a quotient of \( \bigoplus_{V \in \mathcal{D}} \omega(V) \otimes \omega(V)^\vee \), we see the following: for \( g \in G(R) \) as above, the action of \( g \) on \( \eta(\mathcal{O}_{G(\mathcal{D}, \omega)}) \otimes R \) is defined to be the morphism induced by

\[ (2.5) \quad (\eta \omega(V) \otimes \eta \omega(V)^\vee) \otimes R \to (\eta \omega(V) \otimes \eta \omega(V)^\vee) \otimes R \quad (V \in \mathcal{D}), \]

which is the action of \( g \) on \( \omega(V) \otimes \omega(V)^\vee \) where \( g_{\omega(V) \otimes \omega(V)^\vee} = \omega(V) \otimes g_{\omega(V)^\vee} \). If we identify \( (\eta \omega(V) \otimes \eta \omega(V)^\vee) \otimes R \) with \( \text{End}(\eta \omega(V) \otimes R) \), the action \(2.5) \) is just the conjugate action \( g_{\omega(V)} \circ - \circ g_{\omega(V)^\vee}^{-1} \).

Let us consider the following split exact sequence of group schemes
\[
1 \longrightarrow \eta(G(\mathcal{D}, \omega)) \longrightarrow \eta(\omega(\pi(\mathcal{D}))) \longrightarrow \eta(\omega(\pi(\mathcal{C}))) = G \longrightarrow 1, \]

Then the right action of \( g \in G(R) \) on \( \eta(G(\mathcal{D}, \omega))(R) \) induced by the action on \( \eta(\mathcal{O}_{G(\mathcal{D}, \omega)}) \otimes R \) above is equal to the conjugate action \( \eta(\omega^*)(R)(g)^{-1} \circ - \circ \eta(\omega^*)(R)(g) \).

Now, let the notation be as in Notation 1.14. Then we have the functors between Tannakian categories
\[
f_{dR}^* : \text{MIC}^n(S/k) \to \text{N}_f \text{MIC}^n(X/k), \quad t_{dR}^* : \text{N}_f \text{MIC}^n(X/k) \to \text{MIC}^n(S/k). \]
If we apply the construction above to the above functors, we obtain the first definition of relatively unipotent de Rham fundamental group, which is due to Lazda [Laz15, Remark 2.4]:

**Definition 2.2 (First definition of relatively unipotent \( \pi_1 \)).** Let the notations be as above. We define the relatively unipotent de Rham fundamental group \( \pi_1^{dR}(X/S, \iota) \) by
\[
\pi_1^{dR}(X/S, \iota) := G(\text{N}_f \text{MIC}^n(X/k), t_{dR}^*). \]
This is a group scheme in \( \text{MIC}^n(S/k) \).

We define \( \pi_1^{dR}(S, s), \pi_1^{dR}(X, x), \pi_1^{dR}(X_s, x) \) as the Tannaka dual of \( (\text{MIC}^n(S/k), s_{dR}^*), (\text{N}_f \text{MIC}^n(X/k), x_{dR}^*), (\text{N}_f \text{MIC}(X_s/s), x_{dR}^*), \) respectively. Then, applying the previous construction, we have the split exact sequence
\[
1 \longrightarrow s_{dR}^* \pi_1^{dR}(X/S, \iota) \longrightarrow \pi_1^{dR}(X, x) \longrightarrow \pi_1^{dR}(S, s) \longrightarrow 1, \]
where \( f_* , \iota_* \) are the morphisms induced by \( f , \iota \) respectively. Moreover, the morphism \( \pi^\text{dR}(X_s, x) \rightarrow \pi^\text{dR}(X, x) \) induced by the exact closed immersion \( X_s \hookrightarrow X \) induces the morphism

\[
\pi^\text{dR}(X_s, x) \rightarrow s^* \pi^\text{dR}(X/S, \iota).
\]

(2.6)

Then we have the following:

**Proposition 2.3.** The morphism (2.6) is an isomorphism. So we have the split exact sequence

\[
1 \rightarrow \pi^\text{dR}(X_s, x) \rightarrow \pi^\text{dR}(X, x) \rightarrow \pi^\text{dR}(S, s) \rightarrow 1
\]

Proof. This is due to Lazda ([Laz15, Corollary 1.20]) when the log structures on \( X, S \) are trivial and \( S \) is smooth over \( k \), and the same proof works also in our case. We provide a proof for the convenience of the reader.

Denote the functors of restriction \( N^f \text{MIC}^n(X/k) \rightarrow N^f \text{MIC}(X_s/s), \text{MIC}^n(S/k) \rightarrow \text{MIC}(s/s) \) by \( (E, \nabla_E) \rightarrow (E, \nabla_E)|_{X_s}, (E, \nabla_E) \rightarrow (E, \nabla_E)|_s \), respectively. By [W97, I Proposition 1.4] and [EHS08, Appendix A], it suffices to prove the following three claims.

1. Any object in \( N^f \text{MIC}(X_s/s) \) is a quotient of an object of the form \( (E, \nabla_E)|_{X_s}, ((E, \nabla_E) \in N^f \text{MIC}^n(X/k)) \).

2. For any object \( (E, \nabla_E) \) in \( N^f \text{MIC}^n(X/k) \) with \( (E, \nabla_E)|_{X_s} \) trivial, there exists an object \( (V, \nabla_V) \) in \( \text{MIC}^n(S/k) \) with \( (E, \nabla_E) = f^*_{\text{dR}}(V, \nabla_V) \).

3. Let \( (E, \nabla_E) \) be an object in \( N^f \text{MIC}^n(X/k) \) and let \( (F_0, \nabla_{F_0}) \in N^f \text{MIC}(X_s/s) \) be the largest trivial subobject of \( (E, \nabla_E)|_{X_s} \). Then there exists a subobject \( (E_0, \nabla_{E_0}) \) of \( (E, \nabla_E) \) with \( (F_0, \nabla_{F_0}) = (E_0, \nabla_{E_0})|_{X_s} \).

(In fact, the claim (1) implies the injectivity of the map (2.6) and the claims (2), (3) imply the surjectivity of the map (2.6).)

Note that we have seen before Definition 1.13 that, for an object \( (E, \nabla_E) \) in \( N^f \text{MIC}^n(X/k) \), the map

\[
f^*_{\text{dR}} f^*_{\text{dR}}(E, \nabla_E) \rightarrow (E, \nabla_E)
\]

(2.7)

gives an injection onto the maximal subobject of \( (E, \nabla_E) \) which belongs to the category \( f^*_{\text{dR}} \text{MIC}^n(S/k) \). Using this, we prove the claim (2). Let \( (E, \nabla_E) \) be an object in \( N^f \text{MIC}^n(X/k) \) with \( (E, \nabla_E)|_{X_s} \) trivial. By restricting (2.7) to \( X_s \) and using the base change property (see remark 1.11 and 1.20), we obtain the morphism

\[
f^*_{s, \text{dR}} f^*_{s, \text{dR}}((E, \nabla_E)|_{X_s}) \cong f^*_{s, \text{dR}}((f^*_{\text{dR}}(E, \nabla_E))|_s) = (f^*_{\text{dR}} f^*_{\text{dR}}(E, \nabla_E))|_{X_s} \rightarrow (E, \nabla_E)|_{X_s},
\]

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which is an isomorphism due to the triviality of $(E, \nabla_E)|_{X_s}$. Thus the map (2.7) is an isomorphism (the two objects have the same rank) and so the claim holds if we put $(V, \nabla_V) := f_{\text{dR}*}(E, \nabla_E)$.

Next we prove the claim (3). Let $(E, \nabla_E)$ and $(F_0, \nabla_{F_0}) \subseteq (E, \nabla_E)|_{X_s}$ be as in the statement of the claim. Then we have $(F_0, \nabla_{F_0}) = f^*_{s, \text{dR}}f_{s, \text{dR}'}((E, \nabla_E)|_{X_s})$. So, by the base change property, we obtain the isomorphism

$$(F_0, \nabla_{F_0}) = f^*_{s, \text{dR}}f_{s, \text{dR}'}((E, \nabla_E)|_{X_s}) \cong f^*_{s, \text{dR}}((f_{\text{dR}'}(E, \nabla_E))|_{X_s}) = (f_{\text{dR}*}f_{\text{dR}'}(E, \nabla_E))|_{X_s}.$$ 

So the claim holds if we put $(E_0, \nabla_{E_0}) = f_{\text{dR}*}f_{\text{dR}'}(E, \nabla_E)$.

Finally we explain the proof of the claim (1). We will construct a projective system of objects $\{W_n\}_n$ in $\text{MIC}^n(X/k)$ in the next section (Remark 3.2) such that, for any object $(F, \nabla_F)$ in $\text{MIC}(X/s)$, there exist $n, N \in \mathbb{N}$ and a surjection $W_n^{\oplus N}|_{X_s} \rightarrow (F, \nabla_F)$. This implies (1). Hence the proof of proposition will be finished modulo the construction of the projective system $\{W_n\}_n$. 

Since $\pi^\text{dR}_1(X/S, \iota)$ is a group scheme in $\text{MIC}^n(S/k)$, $s^*_{\text{dR}}\pi^\text{dR}_1(X/S, \iota)$ is endowed with the action of $\pi^\text{dR}_1(S, s)$ (called the monodromy action) by Tannaka duality. By Proposition 2.3 and by the results of this section we have the following corollary:

**Corollary 2.4.** Let the notations be as above. Then the monodromy action of $\pi^\text{dR}_1(S, s)$ on $s^*_{\text{dR}}\pi^\text{dR}_1(X/S, \iota)$ is equal to the action of $\pi^\text{dR}_1(S, s)$ on $\pi^\text{dR}_1(X_s, s)$ defined by the conjugate action $g \mapsto \iota_*(g^{-1} \circ - \circ \iota_*(g) (g \in \pi^\text{dR}_1(S, s))$ via the isomorphism $\pi^\text{dR}_1(X_s, s) \cong s^*_{\text{dR}}\pi^\text{dR}_1(X/S, \iota)$ of Proposition 2.3.

**Remark 2.5.** In [DPS18], the second and the third author proved the homotopy exact sequence for maximal geometrically pro-trigonalizable quotients of de Rham fundamental groups, in the case where $S$ is the standard log point $\mathbb{A}^n_{\mathbb{C}}$ and $f : X \rightarrow S$ is a quasi-projective normal crossing log schemes.

This implies the split exact sequence

$$1 \rightarrow \pi^\text{dR}_1(X_s, s) \rightarrow \pi^\text{dR}_1(X, x) \xrightarrow{f_*} \pi^\text{dR}_1(S, s) \cong \mathbb{G}_{a,k} \rightarrow 1.$$ 

In the introduction of [DPS18], they defined the monodromy action of $\pi^\text{dR}_1(S, s) = \mathbb{G}_{a,k}$ on $\pi^\text{dR}_1(X_s, s)$ by the conjugate $g \mapsto \iota_*(g^{-1} \circ - \circ \iota_*(g) (g \in \pi_1(S, s))$. Hence, by Corollary 2.4, their definition of monodromy action is the same as the one given in this section.

### 3 Construction of Hadian, Andreata-Iovita-Kim and Lazda

Let the notations be as in Notation 1.14. In this section, we give the second definition of relatively unipotent de Rham fundamental group along the lines of the
construction of Hadian [Had11], Andreatta-Iovita-Kim [AIK15] and Lazda [Laz15], moreover we will prove the coincidence of this new definition with the one given in the previous section.

The following theorem (construction) is the key ingredient of the definition. The notations are those of the previous sections.

**Theorem 3.1.** There exists a projective system \((W, e) := \{(W_n, e_n)\}_{n \geq 1}\) consisting of

- \(W_n := (\overline{W}_n, \{e_n^m\}_m) \in N_f\text{StrCrys}^n(X/k) \cong N_f\text{MIC}^n(X/k), \) and
- A morphism \(e_n : (\mathcal{O}_S, d) \rightarrow \iota^*_{\text{dr}}W_n \) in \(\text{MIC}^n(S/k)\)

which satisfies the following conditions:

(W0) For any \(n \geq 2\), the morphism \(f_{\text{dr}*}(\overline{W}_{n-1}^\vee) \rightarrow f_{\text{dr}*}^\vee(W_n)\) induced by the transition map \(\overline{W}_n \rightarrow \overline{W}_{n-1}\) is an isomorphism.

(W1) For any \(E \in N_f\text{MIC}(X/S)\) of index of unipotence \(\leq n\) and a morphism \(v : \mathcal{O}_S \rightarrow \iota^*_{\text{dr}}E\) in \(\text{MIC}(S/S)\), there exists a unique morphism \(\varphi : \overline{W}_n \rightarrow E\) in \(N_f\text{MIC}(X/S)\) with \(\iota^*_{\text{dr}}(\varphi) \circ \tau_n = v\), where \(\tau_n : \mathcal{O}_S \rightarrow \iota^*_{\text{dr}}\overline{W}_n\) is the underlying morphism of \(e_n\) in \(N_f\text{MIC}(S/S)\). Moreover, the same universality holds after base change by any morphism \(S' \rightarrow S\) of finite type.

(W2) For any \(n\), there exists an exact sequence

\[
0 \rightarrow f^*_{\text{dr}}R^1f_{\text{dr}*}(W_n^\vee) \rightarrow W_{n+1} \rightarrow W_n \rightarrow 0
\]

in \(N_f\text{MIC}^n(X/k)\), where on \(f^*_{\text{dr}}R^1f_{\text{dr}*}(W_n^\vee)\) we put the log connection induced by the Gauss-Manin connection.

Here, we say that an object in \(N_f\text{MIC}(X/S)\) is of index of unipotence \(\leq n\) if it can be written as an iterated extension of at most \(n\) objects in \(f^*_{\text{dr}}\text{MIC}(S/S)\).

**Proof.** For \(n = 1\), we put \(W_1 := (\mathcal{O}_X, d)\) and define \(e_1\) to be the canonical isomorphism \((\mathcal{O}_S, d) \xrightarrow{e} \iota^*_{\text{dr}}(\mathcal{O}_X, d)\). Then we can check the condition (W1) for \(n = 1\): Indeed, if we are given an object \(E = f^*_{\text{dr}}V \in f^*_{\text{dr}}\text{MIC}(S/S)\) and a morphism \(v : \mathcal{O}_S \rightarrow \iota^*_{\text{dr}}E = V\) in \(\text{MIC}(S/S)\), the map \(\varphi : \mathcal{O}_X \rightarrow E\) in \(N_f\text{MIC}(X/S)\) satisfies \(\iota^*_{\text{dr}}(\varphi) = v\) if \(\varphi = f^*_{\text{dr}}v\), and if we have a morphism \(\varphi\) as above with \(\iota^*_{\text{dr}}(\varphi) = v\), we necessarily have

\[
\varphi = f^*_{\text{dr}}f^*_{\text{dr}*}\varphi = f^*_{\text{dr}}\iota^*_{\text{dr}}f^*_{\text{dr}*}f_{\text{dr}*}\varphi = f^*_{\text{dr}}\iota^*_{\text{dr}}\varphi = f^*_{\text{dr}}v.
\]

Now we construct \((W_{n+1}, e_{n+1})\) from \((W_i, e_i) (i \leq n)\). Since \(W_n\) belongs to \(N_f\text{MIC}^n(X/k)\), \(R^1f^*_{\text{dr}}(\overline{W}_n^\vee)\) belongs to \(\text{MIC}^n(S/k)\). In particular, \(R^1f^*_{\text{dr}}(\overline{W}_n^\vee)\) belongs to \(\text{MIC}(S/S)\). When \(S\) is affine, we define \(\overline{W}_{n+1} \in N_f\text{MIC}(X/S)\) as the extension

\[
0 \rightarrow f^*_{\text{dr}}R^1f^*_{\text{dr}*}(\overline{W}_n^\vee) \rightarrow \overline{W}_{n+1} \xrightarrow{\pi} \overline{W}_n \rightarrow 0
\]
whose extension class, which is regarded as an element in
\[ \Gamma(S, R^1 f_{\text{dR}*}(\overline{W}_n^\vee) \otimes R^1 f_{\text{dR}*}(\overline{W}_n^\vee) = \text{End}(R^1 f_{\text{dR}*}(\overline{W}_n^\vee)), \]
is the identity. Also, let \( \tau_{n+1} : \mathcal{O}_S \to \iota_{\text{dR}*} \overline{W}_{n+1} \) be a morphism which lifts the map \( \tau_n : \mathcal{O}_S \to \iota_{\text{dR}*} \overline{W}_n \).

We check the properties (W0) and (W1) for \((\overline{W}_n+1, \epsilon_{n+1})\) (assuming \(S\) and \(S'\) affine). The method is the same as [Had11, Proposition 2.6], [AIK15, Proposition 3.3] and [Laz15, Proposition 1.17], but we write down the argument for the convenience of the reader. Since the proof on \(X \times_{S} S'/S'\) is the same as that on \(X/S\) (by base change property (W1)), we prove the property only on \(X/S\). In the proof, we denote the map \( \overline{W}_n^\vee \to \overline{W}_n \) in (3.2) by \( \pi \). One has the long exact sequence
\[
0 \to f_{\text{dR}*}(\overline{W}_n^\vee) \to f_{\text{dR}*}(\overline{W}_{n+1}^\vee) \to R^1 f_{\text{dR}*}(\overline{W}_n^\vee) \to \cdots
\]
associated to the dual of the exact sequence (3.2). By definition of \( \overline{W}_{n+1} \) and the standard argument in homological algebra, we see that the map \( \delta \) is the identity (see [Laz15, Proposition 1.15]). Thus \( \gamma \) is an isomorphism and \( \epsilon \) is zero. In particular, we have checked the property (W0). Also, any extension in \( N_f \text{MIC}(X/S) \) of the form
\[
0 \to f_{\text{dR}*}V \to E \to \overline{W}_n \to 0 \quad (V \in \text{MIC}(S/S))
\]
splits if we pull it back by \( \pi : \overline{W}_{n+1} \to \overline{W}_n \).

We take an object \( E \in N_f \text{MIC}(X/S) \) of index of unipotence \( \leq n + 1 \) and a morphism \( v : \mathcal{O}_S \to \iota_{\text{dR}*} E \). Then there exists an exact sequence of the form
\[
(3.3) \quad 0 \to f_{\text{dR}*}V \xrightarrow{\alpha} E \xrightarrow{\beta} E' \xrightarrow{\epsilon} 0
\]
in \( N_f \text{MIC}(X/S) \), where \( E' \) is of index of unipotence \( \leq n \). By induction hypothesis, there exists a unique morphism \( \psi : \overline{W}_n \to E' \) with \( \iota_{\text{dR}}(\psi) \circ \tau_n = \iota_{\text{dR}}(\beta) \circ v \). By considering the pullback of the above exact sequence by \( \psi \circ \pi \), we obtain the following diagram
\[
\begin{array}{ccc}
0 & \to & f_{\text{dR}*}V \\
\phi' \downarrow & & \downarrow \psi \circ \pi \\
E & \xrightarrow{\beta} & E'
\end{array}
\]
where \( \phi' \) is a section. If we denote the composite \( \overline{W}_{n+1} \xrightarrow{\phi'_1} \overline{E} \to E \) by \( \varphi_1 \), we have the equality
\[
\iota_{\text{dR}}(\beta) \circ (\iota_{\text{dR}}(\varphi_1) \circ \tau_{n+1} - v) = \iota_{\text{dR}}(\psi) \circ \iota_{\text{dR}}(\pi) \circ \tau_{n+1} - \iota_{\text{dR}}(\beta) \circ v
= \iota_{\text{dR}}(\psi) \circ \tau_n - \iota_{\text{dR}}(\beta) \circ v = 0.
\]
Hence there exists a map \( w : O_S \rightarrow \iota_{\text{dr}}^* f_{\text{dr}}^* V = V \) such that \( \iota_{\text{dr}}^* (\alpha) \circ w = \iota_{\text{dr}}^* (\varphi_1) \circ \overline{\tau}_{n+1} - v \). On the other hand, there exists a map \( \varphi_2 : \overline{W}_{n+1} \rightarrow f_{\text{dr}}^* (V) \) with \( \iota_{\text{dr}}^* (\varphi_2) \circ \overline{\tau}_{n+1} = w \) by the property (W1) in the case \( n = 1 \) (and the compatibility of \( (\overline{W}_{n+1}, \overline{\tau}_{n+1}) \) with \( (\overline{W}_1, \overline{\tau}_1) \)). Then, if we put \( \varphi := \varphi_1 \circ \varphi_2 \), we obtain that

\[
\iota_{\text{dr}}^* (\varphi) \circ \overline{\tau}_{n+1} = \iota_{\text{dr}}^* (\varphi_1 - \alpha \circ \varphi_2) \circ \overline{\tau}_{n+1} = \iota_{\text{dr}}^* (\varphi_1) \circ \overline{\tau}_{n+1} - \iota_{\text{dr}}^* (\alpha) \circ w = v,
\]
as required.

We prove the uniqueness of \( \varphi \) by induction on the index of unipotence of \( E \). When the index of unipotence is \( \leq 1 \), \( E \) has the form \( f_{\text{dr}}^* V \) and to give a map \( \varphi : \overline{W}_{n+1} \rightarrow f_{\text{dr}}^* V \) is equivalent to giving a map \( V^\vee \rightarrow f_{\text{dr}}^* (\overline{W}_{n+1}) \), which we denote by \( \psi \). Because the map \( f_{\text{dr}}^* (\overline{W}_{n+1}) \rightarrow f_{\text{dr}}^* (\overline{W}_{n+1}) \) induced by the transition map \( \overline{W}_{n+1} \rightarrow \overline{W}_1 \) is an isomorphism by the property (W0) for \( \overline{W}_i (2 \leq i \leq n + 1) \) which we have already shown, \( \psi \) factors through \( f_{\text{dr}}^* (\overline{W}_{n+1}) \), namely, \( \varphi \) factors through \( \overline{W}_1 \).

Then the uniqueness of \( \varphi \) in this case follows from the property (W1) for \( (\overline{W}_1, \overline{\tau}_1) \). In general case, we take an exact sequence \((\ref{count})\) and assume the existence of another map \( \varphi' : \overline{W}_{n+1} \rightarrow E \) with \( \iota_{\text{dr}}^* (\varphi') \circ \overline{\tau}_{n+1} = v \). By the induction hypothesis applied to \( E' \), we have \( \beta \circ \varphi = \beta \circ \varphi' \) and so there exists a map \( \psi : \overline{W}_{n+1} \rightarrow f_{\text{dr}}^* V \) with \( \varphi - \varphi' = \alpha \circ \psi \). Also, we have equalities

\[
0 = \iota_{\text{dr}}^* (\varphi - \varphi') \circ \overline{\tau}_{n+1} = \iota_{\text{dr}}^* (\alpha) \circ \iota_{\text{dr}}^* (\psi) \circ \overline{\tau}_{n+1}.
\]

Thus we see that \( \iota_{\text{dr}}^* (\psi) \circ \overline{\tau}_{n+1} = 0 \), and this implies (again by induction hypothesis) that \( \psi = 0 \). So \( \varphi = \varphi' \) and the uniqueness is proved.

So far, we defined \( (\overline{W}_{n+1}, \overline{\tau}_{n+1}) \) when \( S \) is affine and checked the properties (W0), (W1) for it when \( S \) and \( S' \) are affine. When \( S \) is not necessarily affine, we take an affine open covering \( \{ S_\alpha \}_\alpha \) of \( S \) and define \( (\overline{W}_{n+1}, \overline{\tau}_{n+1}) \) on each \( S_\alpha \), which we denote by \( (\overline{W}_{n+1}, \overline{\tau}_{n+1})_\alpha \). Then, on each \( S_\alpha \cap S_\beta \), the universality in (W1) implies that there exists a unique isomorphism \( (\overline{W}_{n+1}, \overline{\tau}_{n+1})_\alpha |_{S_\alpha \cap S_\beta} \cong (\overline{W}_{n+1}, \overline{\tau}_{n+1})_\beta |_{S_\alpha \cap S_\beta} \). Hence \( (\overline{W}_{n+1}, \overline{\tau}_{n+1})_\alpha \)'s glue and give the pair \( (\overline{W}_{n+1}, \overline{\tau}_{n+1}) \) on \( S \). One can check the properties (W0), (W1) of this pair for general \( S \) and \( S' \) by reducing to the affine case.

In the next step we are going to enhance the relative connection structure of \( \overline{W}_{n+1} \) to an absolute one, defining in this way \( \overline{W}_{n+1} \). Let \( f_j^m : X_j^m \rightarrow S^m(1) \), \( p_j^m : S^m(1) \rightarrow S \), \( q_j^m : X_j^m \rightarrow X \), \( \hat{f}_j^m : \hat{X}_j^m \rightarrow S^m(1) \) and \( \hat{q}_j^m : \hat{X}_j^m \rightarrow X \) be as in Section 1, (before Definition \ref{conn} and Proposition \ref{pushout}), and let \( \iota_j^m : S^m(1) \rightarrow X_j^m \) be the base change of \( \iota \), which is a section of \( f_j^m \). Then, by the property (W1), there exists a unique isomorphism \( \epsilon_j^m : (q_2^m \circ \overline{W}_{n+1}, p_2^m \circ \overline{\tau}_{n+1}) \cong (q_1^m \circ \overline{W}_{n+1}, p_1^m \circ \overline{\tau}_{n+1}) \) (\( m \in \mathbb{N} \)) in the categories (see Proposition \ref{iso})

\[
\text{MIC}(X_1^m / S^m(1)) \xrightarrow{\cong} \text{Crys}((X / S^m(1))_{\text{int}}) \xrightarrow{\cong} \text{MIC}(X_2^m / S^m(1))
\]

which satisfies the cocycle condition. (To prove the cocycle condition, we need to work on pullbacks of \( f : X \rightarrow S \) to \( S^m(2) \). We leave the reader to write the detailed
argument.) So the object \( W_{n+1} := (\overline{W}_{n+1}, \{\epsilon_{n+1}^m\}_m) \) in \( \text{StrCrys}(X/k) \cong \text{MIC}(X/k) \) is defined, and the isomorphisms \( p_{2,\text{dr}}^m \epsilon^m_{n+1} \cong p_{1,\text{dr}}^m \epsilon^m_{n+1} \) \((m \in \mathbb{N})\) induce the map \( \epsilon^m_{n+1} : (\mathcal{O}_S, d) \rightarrow t^e_{\text{dr}} W_{n+1} \).

Let us prove that \( W_{n+1} \) belongs to \( N_f \text{MIC}^n(X/k) \) and it satisfies the property (W2). By the commutative diagram

\[
\begin{array}{ccc}
\text{MIC}(X_1^m / S^m(1)) & \cong & \text{Crys}((X / S^m(1))_{\text{inf}}) \\
\downarrow & & \downarrow \\
\text{MIC}(\tilde{X}^m / S^m(1)),
\end{array}
\]

the isomorphisms \( \epsilon^m_n : (q_{2,\text{dr}}^m \overline{W}_n, p_{2,\text{dr}}^m \overline{\epsilon}_n) \cong (q_{1,\text{dr}}^m \overline{W}_n, p_{1,\text{dr}}^m \overline{\epsilon}_n) \) \((m \in \mathbb{N})\) induce the isomorphisms \( (\overline{q}_2^m)^* \overline{W}_n \cong (\overline{q}_1^m)^* \overline{W}_n \) \((m \in \mathbb{N})\) in \( \text{MIC}(\tilde{X}^m / S^m(1)) \), and by the argument explained after Proposition 1.10 we obtain the isomorphisms

\[
\delta^m_n : p_{2,\text{dr}}^m f^* R^1 f_{\text{dr}*}(\overline{W}_n) \cong p_{1,\text{dr}}^m f^* R^1 f_{\text{dr}*}(\overline{W}_n) \quad (m \in \mathbb{N})
\]

in \( \text{MIC}(S^m(1)/S^m(1)) \) \((m \in \mathbb{N})\). The object \( (R^1 f_{\text{dr}*}(\overline{W}_n), \{\delta^m_n\}) \), regarded as an object in \( \text{Str}(S/k) \cong \text{MIC}(S/k) \), is nothing but the object endowed with the Gauss-Manin connection (which we have denoted by \( R^1 f_{\text{dr}*}(W^n) \)). If we pull them back to the categories

\[
\text{MIC}(X_1^m / S^m(1)) \cong (X / S^m(1))_{\text{inf}} \cong \text{MIC}(X_2^m / S^m(1)) \quad (m \in \mathbb{N}),
\]

we obtain the isomorphisms

\[
\tilde{\delta}^m_n : q_{2,\text{dr}}^m f^* R^1 f_{\text{dr}*}(\overline{W}_n) = f_{2,\text{dr}}^* f_{2,\text{dr}}^* R^1 f_{\text{dr}*}(\overline{W}_n) \cong \tilde{f}_{1,\text{dr}}^* R^1 f_{\text{dr}*}(\overline{W}_n) = \tilde{q}_{1,\text{dr}}^m f^* R^1 f_{\text{dr}*}(\overline{W}_n) \quad (m \in \mathbb{N}).
\]

Then the object \( (f^* R^1 f_{\text{dr}*}(W_n), \{\tilde{\delta}^m_n\}) \), regarded as an object in \( \text{StrCrys}(X/k) \cong \text{MIC}(X/k) \), is \( f^* R^1 f_{\text{dr}*}(W_n) \).

When \( S \) is affine, the extension group for the exact sequence of the form \( q_{j,\text{dr}}^m \) is calculated as the group of global sections of

\[
R^1 f_{\text{dr}*}(q_{j,\text{dr}}^m (\overline{W}_n) \otimes q_{j,\text{dr}}^m R^1 f_{\text{dr}*}(\overline{W}_n))
= R^1 f_{\text{dr}*}(q_{j,\text{dr}}^m (\overline{W}_n) \otimes f_{j,\text{dr}}^* p_{j,\text{dr}}^m R^1 f_{\text{dr}*}(\overline{W}_n))
\cong p_{j,\text{dr}}^m R^1 f_{\text{dr}*}(\overline{W}_n) \otimes p_{j,\text{dr}}^m R^1 f_{\text{dr}*}(\overline{W}_n),
\]

where the last isomorphism follows from the projection formula and the base change isomorphism. If we identify \( q_{j,\text{dr}}^m (W_n) \) \((j = 1, 2)\) via \( \epsilon^m_n \) and \( q_{j,\text{dr}}^m f^* R^1 f_{\text{dr}*}(\overline{W}_n) \) \((j = 1, 2)\) via the dual of \( \tilde{\delta}^m_n \), the sheaves \( p_{j,\text{dr}}^m R^1 f_{\text{dr}*}(\overline{W}_n) \otimes p_{j,\text{dr}}^m R^1 f_{\text{dr}*}(\overline{W}_n) \) \((j = 1, 2)\) are identified via \( \delta^m_n \otimes \delta^m_{n'} \). (Note that \( \delta^m_n \) is the isomorphism induced by \( \epsilon^m_n \), and
\(\tilde{\delta}_n^m\) is just the pullback of \(\delta_n^m\). Therefore, the identification \(\delta_n^m \otimes \delta_n^{m\vee}\) induces the isomorphism of extension groups

\[
\text{End}(p_{2,dr}^m R^1f_{dr*}((W')^\vee)) = \Gamma(S, p_{2,dr}^m R^1f_{dr*}((W')^\vee) \otimes p_{2,dr}^m R^1f_{dr*}((W')^\vee))
\]

\(\cong \Gamma(S, p_{1,dr}^m R^1f_{dr*}((W')^\vee) \otimes p_{1,dr}^m R^1f_{dr*}((W')^\vee)) = \text{End}(p_{1,dr}^m R^1f_{dr*}((W')^\vee))\)

which sends the identity map in \(\text{End}(p_{2,dr}^m R^1f_{dr*}((W')^\vee))\) to the identity map in \(\text{End}(p_{1,dr}^m R^1f_{dr*}((W')^\vee))\). Hence the extension class of the exact sequence \(q_{2,dr}^m (3.2)\) is identified with that of \(q_{1,dr}^m (3.2)\) and so there exists an isomorphism \(\phi_{n+1}^m : q_{2,dr}^m W_{n+1} \to q_{1,dr}^m W_{n+1}\) which makes the following diagram commutative:

\[
\begin{array}{ccccccc}
0 & \to & q_{2,dr}^m f_{dr}^* R^1f_{dr*}((W')^\vee) & \to & q_{2,dr}^m W_{n+1} & \to & q_{2,dr}^m W_n & \to & 0 \\
& \downarrow & (\tilde{\delta}_n^m)^\vee & & \downarrow & \phi_{n+1}^m & \downarrow & \psi_{n+1}^m & & \\
0 & \to & q_{1,dr}^m f_{dr}^* R^1f_{dr*}((W')^\vee) & \to & q_{1,dr}^m W_{n+1} & \to & q_{1,dr}^m W_n & \to & 0.
\end{array}
\]

Moreover, by using the property (W1) for \(W_n\), we can modify the isomorphism \(\phi_{n+1}^m\) (by some morphism \(q_{2,dr}^m W_n \to q_{1,dr}^m f_{dr}^* R^1f_{dr*}((W')^\vee)\)) to the unique isomorphism such that \(q_{2,dr}^m (\phi_{n+1}^m) \circ q_{1,dr}^m (\psi_{n+1}^m) = q_{1,dr}^m (\psi_{n+1}^m)\). The uniqueness allows us to define the isomorphism \(\phi_{n+1}^m\) globally in the non affine case too, and by the property (W1) for \(W_{n+1}\), we see that the map \(\phi_{n+1}^m\) is equal to \(\psi_{n+1}^m\). Hence the diagram (3.4) tells us that \(W_{n+1}\) fits in the exact sequence as in (3.1). Hence \(W_{n+1}\) belongs to \(N_f \text{MIC}^n(X/k)\) and we have the property (W2) for \(W_{n+1}\), as required.

**Remark 3.2.** By the base change property in (W1), for any object \(E\) in \(N_f \text{MIC}(X_s/s)\) and any morphism \(v : k \to x^*_s E\) in \(\text{MIC}(s/s) = \text{Vec}_k\), there exists a unique morphism \(\varphi_v : W_n|_{X_s} \to E\) in \(N_f \text{MIC}(X_s/s)\) with \(x^*_s (\varphi_v) \circ (e_n|_{X_s}) = v\) for some \(n\). By considering maps \(v_1, \ldots, v_N : k \to x^*_d E\) whose direct sum \(k^\oplus N \to x^*_d E\) is surjective, we obtain a surjective map \(\bigoplus_{i=1}^N \varphi_n : W_n^\oplus|_{X_s} \to E\) for some \(n\). This is the property we used in the proof of Proposition 2.3 and so its proof is now finished.

**Remark 3.3.** The definition of \(\{(W_n, e_n)\}_{n \geq 1}\) is functorial in the following sense:

If we are given a commutative diagram

\[
\begin{array}{ccccccc}
S & \to & X & \to & S & \to & \text{Spec} \ k \\
\phi_S & & \phi_X & & \phi_S & & \phi_k \\
S' & \to & X' & \to & S' & \to & \text{Spec} \ k'
\end{array}
\]

(3.5)

such that \(f', g'\) and \(\iota'\) also satisfy the conditions in Notation 1.14 and if we denote the objects \(\{(W_n, e_n)\}_{n \geq 1}\) for \(f', g', \iota'\) by \(\{(W_n', e'_n)\}_{n \geq 1}\), we have the canonical morphism in \(N_f \text{MIC}^n(X/k)\)

\[
\{(W_n', e'_n)\}_{n \geq 1} \to \{(\varphi_{X,dr} W_n, \varphi_{S,dr} e_n)\}_{n \geq 1}.
\]

(3.6)
In fact, the underlying morphism
\[(3.7) \quad \{(W_n', e_n')\}_{n \geq 1} \longrightarrow \{(\varphi_{X, dR}^* W_n, \varphi_{S, dR}^* e_n)\}_{n \geq 1}.\]
of (3.6) in \(N_{MIC}(X/S)\) is defined by the property (W1) and it is upgraded to a morphism (3.6) in \(N_{MIC}^n(X/k)\), because the construction of \(\{(W_n', e_n')\}_{n \geq 1}\) as a pointed projective system in \(N_{MIC}^n(X/k)\) is done again via the universal property (W1).

Also, the definition of \(\{(W_n, e_n)\}_{n \geq 1}\) is compatible with base change in the following sense: if we are given the commutative diagram (3.5) as above and if the left square of (3.5) is Cartesian, then the morphism (3.6) is an isomorphism. In fact, in our construction by induction of \(W_{n+1}, W'_n\) (using the base change property first and (W1)) we get the isomorphisms \(\varphi_{S, dR}^* R^1 f_{dR*}(W'_n) \cong R^1 f_{dR*}(\varphi_{X, dR}^* W'_n) \cong R^1 f_{dR*}(\varphi_{S, dR}^* W_n)\); moreover the local definition of \(W_{n+1}, W'_n\) as extension classes is compatible via \(\varphi_{S, dR}^*\). So we can define the isomorphism (3.7) locally, and this local definition glues by the property (W1). Then the isomorphism (3.7) can be upgraded to the isomorphism (3.6) in \(N_{MIC}^n(X/k)\) as before.

Also, if we are given the commutative diagram (3.5) as above (with the left square not necessarily Cartesian) with \(\varphi_S = \text{id}_S\) such that the induced morphism \(R^i f_{dR*}(O_X, d) \longrightarrow R^i f_{dR*}(O_X, d)\) is an isomorphism for \(i = 0, 1\) and injective for \(i = 2\), then the morphism (3.6) is an isomorphism. In fact, we see that \(R^1 f_{dR*}(W'_n) \cong R^1 f_{dR*}(\varphi_{X, dR}^* W'_n) \cong R^1 f_{dR*}(\varphi_{S, dR}^* W_n)\) in this case, and the local definitions of \(W_{n+1}, W'_n\) as extension classes are compatible. So we can define the isomorphism (3.7) locally, which glues and which is upgraded to the isomorphism (3.6) as before.

**Remark 3.4.** The property (W1) implies that, for any \(\overline{E} \in N_{MIC}(X/S)\), the morphism
\[(3.8) \quad f_{dR*} \mathcal{H}om(W, \overline{E}) \longrightarrow \iota_{dR}^* E; \quad \varphi \mapsto \iota_{dR}^*(\varphi(\overline{e}))\]
(where \(\overline{e} := \{e_n\}_n\) is an isomorphism in \(MIC(S/S)\), where \(\mathcal{H}om(-, -)\) denotes the internal hom object in the ind-category. This is true also after a base change by any morphism \(S' \longrightarrow S\). Hence we obtain by considering stratification that, for any \(E \in N_{MIC}^n(X/k)\), the morphism
\[(3.9) \quad f_{dR*} \mathcal{H}om(W, E) \longrightarrow \iota_{dR}^* E; \quad \varphi \mapsto \iota_{dR}^*(\varphi(e))\]
is an isomorphism in \(MIC^n(S/k)\). This property characterizes \((W, e)\).

**Remark 3.5.** Assume that the log structures on \(X\) and \(S\) are trivial and \(S\) is smooth over \(k\). In [W97 I Theorem 3.5], Wildeshaus constructed the variation of Hodge structure on \(X\) called the generic pro-sheaf which satisfies the property analogous to the isomorphism (3.9). Also, he remarked in [W97 I Remark in p.61] that its
underlying module with integrable connection satisfies the isomorphism (3.9). So the underlying module with integrable connection of his generic pro-sheaf is the same as \( W \) in this paper. Note that he starts the argument in the case \( k = \mathbb{C} \) and uses the homotopy exact sequence

\[
1 \longrightarrow \pi_1^{\text{top}}(X_s, x) \longrightarrow \pi_1^{\text{top}}(X, x) \longrightarrow \pi_1^{\text{top}}(S, s) \longrightarrow 1
\]

of topological fundamental groups in the construction of generic pro-sheaf (see [W97, p.58]). So his construction is not purely algebraic.

**Remark 3.6.** By the property (W0), we have the isomorphism

\[
\mathcal{O}_S = f_{\text{dR}*}(W_1^\vee) \xrightarrow{\cong} \cdots \xrightarrow{\cong} f_{\text{dR}*}(W_n^\vee)
\]

in \( \text{MIC}^n(S/k) \), which we denote by \( e'_n \). We see that the composite

\[
(3.10) \quad \mathcal{O}_S \xrightarrow{\epsilon'_n} f_{\text{dR}*}(W_n^\vee) = \iota_{\text{dR}}^* f_{\text{dR}*} f_{\text{dR}*}(W_n^\vee) \longrightarrow \iota_{\text{dR}}^* (W_n^\vee) \xrightarrow{\epsilon'_n} \mathcal{O}_S
\]

is an isomorphism.

**Remark 3.7.** Assume that the log structures on \( X \) and \( S \) are trivial and \( S \) is smooth over \( k \). In [Laz15], Lazda gives a different, purely algebraic construction of the projective system \( \{(W_n, e_n)\}_{n \geq 1} \) in \( \text{NfMIC}^n(X/k) \), which we explain here.

We denote by \( f_{\text{dR}*}(-), R^1 f_{\text{dR}*}(-) \) the relative zeroth and first de Rham cohomology endowed with the Gauss-Manin connection induced by Katz-Oda filtration. (It is the same as the higher direct image in the sense of \( D \)-modules [DMSS00, Proposition 1.4] up to shift of index.) Lazda constructs the projective system of triples \( \{(W_n, e_n, e'_n)\}_{n \geq 1} \) with \( e'_n \) is a map \( \mathcal{O}_S \longrightarrow f_{\text{dR}*}(W_n^\vee) \) in \( \text{MIC}^n(S/k) \) and \( e_n \) is as in Theorem 3.1, such that the composite (3.10) (with \( f_{\text{dR}*} \) replaced by \( f_{\text{dR}*}^D \)) is an isomorphism. When \( n = 1, W_1 := (\mathcal{O}_X, d) \), \( e_1 \) is the identity \( \mathcal{O}_S \longrightarrow \iota_{\text{dR}}^* \mathcal{O}_X = \mathcal{O}_S \) and \( e'_1 \) is also the identity \( \mathcal{O}_S \longrightarrow f_{\text{dR}*}^D \mathcal{O}_X = f_{\text{dR}*} \mathcal{O}_X = \mathcal{O}_S \).

He constructs \( (W_{n+1}, e_{n+1}, e'_{n+1}) \) from \( (W_n, e_n, e'_n) \) in the following way: he considers the Leray spectral sequence for \( W_n^\vee \otimes f_{\text{dR}}^D R^1 f_{\text{dR}*}^D(W_n^\vee) \)

\[
(3.11) \quad 0 \longrightarrow H^1_{\text{dR}}(S, R^1 f_{\text{dR}*}^D(W_n^\vee)) \xrightarrow{\alpha_1} H^1_{\text{dR}}(X, W_n^\vee \otimes f_{\text{dR}*}^D R^1 f_{\text{dR}*}^D(W_n^\vee)) \\
\xrightarrow{\beta} \text{End}(R^1 f_{\text{dR}*}^D(W_n^\vee)) \\
\longrightarrow H^2_{\text{dR}}(S, R^1 f_{\text{dR}*}^D(W_n^\vee)) \xrightarrow{\alpha_2} H^2_{\text{dR}}(X, W_n^\vee \otimes f_{\text{dR}*}^D R^1 f_{\text{dR}*}^D(W_n^\vee)).
\]

For \( i = 1, 2 \), we have a section

\[
\gamma_i : H^i_{\text{dR}}(X, W_n^\vee \otimes f_{\text{dR}}^D R^1 f_{\text{dR}*}^D(W_n^\vee)) \longrightarrow H^i_{\text{dR}}(S, \iota_{\text{dR}}^* W_n^\vee \otimes R^1 f_{\text{dR}*}^D(W_n^\vee)) \longrightarrow H^i_{\text{dR}}(S, R^1 f_{\text{dR}*}^D(W_n^\vee)).
\]
of \( \alpha_i \) induced by \( e_i^\vee \). Thus we see that there exists a unique element \( e \in H^1_{dR}(X, W_n^\vee \otimes f_{dR}^1 f_{dR*}(W_n^\vee))^\vee \) with \( \beta(e) = \text{id} \in \text{End}(R^1 f_{dR*}(W_n^\vee)) \) and \( \gamma_1(e) = 0 \). Then \( W_{n+1} \) is defined to be the object in the extension

\[
0 \longrightarrow f_{dR}^1 f_{dR*}(W_n^\vee) \longrightarrow W_{n+1} \longrightarrow W_n \longrightarrow 0
\]

(3.12)

whose extension class is \( e \). The condition \( \beta(e) = \text{id} \) implies that the dual of the exact sequence (3.12) induces the isomorphism \( f_{dR}^1 f_{dR*}(W_n^\vee) \cong f_{dR*}(W_n^\vee) \) as we proved in the proof of Theorem 3.1, thus we define

\[
e'_{n+1} : \mathcal{O}_S \longrightarrow f_{dR*}(W_n^\vee) \cong f_{dR*}(W_{n+1}^\vee).
\]

The condition \( \gamma_1(e) = 0 \) implies that the dual of \( \iota_{dR}^* \) splits after we push it out by \( e_n^\vee \). Thus we have a morphism \( \iota_{dR}^* W_n^\vee \longrightarrow \mathcal{O}_S \) which extends \( e_n^\vee \). We define \( e_{n+1} \) to be the dual of this map. By construction, the property (W0) is satisfied.

The projective system \( \{(W_n, e_n)\}_{n \geq 1} \) Lazda constructed satisfies the property (W1) (the proof is the same as that in our case), and satisfies the exact sequence (3.12). But it does not immediately imply the isomorphism (3.9) because the structure of \( W_n \) as an object in \( \text{N}_f \text{MIC}^n(X/k) \) \( = \text{N}_f \text{StrCrys}^n(X/k) \) is not induced by stratification.

Now we compare our definition of the projective system \( \{(W_n, e_n)\}_{n \geq 1} \) with that by Ladza (which we denote by \( \{(W_n^L, e_n^L)\}_{n \geq 1} \) in the sequel). First, by (3.9), we have a unique projective system of morphisms \( \{\varphi_n : (W_n, e_n) \longrightarrow (W_n^L, e_n^L)\}_{n \geq 1} \) in \( \text{N}_f \text{MIC}^n(X/k) \). Because both \( (W_n, e_n)|_{X_s}, (W_n^L, e_n^L)|_{X_s} \) satisfies the property (W1) (after the pullback by \( s \longrightarrow S \)), the map \( \varphi_n|_{X_s} \) is an isomorphism. Hence \( \varphi_n \) is an isomorphism and so we have shown that our definition and Lazda’s definition are the same.

Moreover, we are going to prove that the isomorphism \( \overline{\varphi}_n : R^1 f_{dR*}(W_n^\vee)^\vee \cong R^1 f_{dR*}(W_n^\vee)^\vee \) in \( \text{MIC}(S/S) \) induced by the underlying isomorphism \( \overline{\varphi}_n : \overline{W}_n \longrightarrow \overline{W}_n^L \) of \( \varphi_n \) in \( \text{N}_f \text{MIC}(X/S) \) can be enriched to an isomorphism \( R^1 f_{dR*}(W_n^\vee)^\vee \cong R^1 f_{dR*}(W_n^\vee)^\vee \) in \( \text{MIC}^n(S/k) \). In fact, consider the following diagram in \( \text{MIC}(X/S) \)

\[
0 \longrightarrow f_{dR}^* R^1 f_{dR*}(W_n^\vee) \longrightarrow \overline{W}_{n+1} \longrightarrow \overline{W}_n \longrightarrow 0
\]

(3.13)

\[
0 \longrightarrow f_{dR}^* R^1 f_{dR*}(W_n^\vee) \longrightarrow \overline{W}_{n+1}^L \longrightarrow \overline{W}_n^L \longrightarrow 0,
\]

where the vertical arrows are isomorphisms. When \( S \) is affine, the extension class defined by \( \overline{W}_{n+1} \) is equal to that by \( \overline{W}_{n+1}^L \) via the above diagram by construction. Thus there exists a map \( \phi_{n+1} : \overline{W}_{n+1} \longrightarrow \overline{W}_{n+1}^L \) which fits into the diagram (3.13), and we can adjust it uniquely using the property (W1) for \( \overline{W}_n \) so that \( \iota_{dR}^* (\phi_{n+1}) \circ \overline{\varphi}_{n+1} = \overline{\varphi}_{n+1}^L \) (as in the last lines of the proof of Theorem 3.1). So the isomorphism \( \phi_{n+1} \) is defined globally and by the property (W1), we see that \( \overline{\varphi}_{n+1} = \phi_{n+1} \).
The isomorphisms $\varphi_n$, $\varphi_{n+1}$ and the exact sequences (3.1), (3.12) induce the isomorphism $R^1 f_{\text{dR}*}(W_n^\vee)^{\vee} \cong R^1 f_{\text{dR}*}(W_n^L)^{\vee}$ in $\text{MIC}^n(S/k)$. When we consider its restriction $R^1 f_{\text{dR}*}(\overline{W}_n)^{\vee} \cong R^1 f_{\text{dR}*}(\overline{W}_n^L)^{\vee}$ to $\text{MIC}(S/S)$, we see from the diagram (3.13) together with the morphism $\phi_{n+1}$ and the equality $\overline{\varphi}_{n+1} = \phi_{n+1}$ that it coincides with $\overline{\varphi}_n$. Hence we have the enriched structure as we wanted.

The claim we proved above can be rephrased to the fact that the two definitions of Gauss-Manin connections on $R^1 f_{\text{dR}*}(W_n^\vee)$ are the same.

**Remark 3.8.** In Remark [3.7](#) we proved the coincidence of two definitions of Gauss-Manin connections on $R^1 f_{\text{dR}*}(W_n^\vee)$ when the log structures on $X$ and $S$ are trivial and $S$ is smooth over $k$. In this remark, we extend this result to some other cases.

1. Let the situation be as in Proposition [1.15](#) and assume that we can always take $s \leq r$ in the condition (2) there. Assume moreover that $f^{-1}(S_{\text{triv}}) = X_{\text{triv}}$, where $S_{\text{triv}}, X_{\text{triv}}$ is the locus in $S, X$ on which the log structure is trivial. In this case, the coincidence holds on $S_{\text{triv}}$ by Remark [3.7](#) and this implies the coincidence on $S$, because the canonical map $E \to j_*j^*E$ is injective if $E$ is a locally free sheaf on $S_{\text{triv}}$ and $j$ is the open immersion $S_{\text{triv}} \hookrightarrow S$.

2. Let the situation be as in Remark [1.17][1](#) in the base change property and the log blow-up invariance of Katz-Oda spectral sequence on the diagram in (1.28).

**Remark 3.9.** When we are given a projective system $\{([W_n, \tau_n])\}$ of objects $W_n (n \geq 1)$ in $\text{N}_f \text{MIC}(X/S)$ and morphisms $\tau_n : O_S \to \iota_{\text{dR}*}W_n (n \geq 1)$ in $\text{MIC}(S/S)$ with the property (W1) and the exact sequences (3.2) (for $n \geq 1$), there is at most one way to upgrade it to a projective system $\{([W_n, e_n])\}$ of objects $W_n (n \geq 1)$ in $\text{N}_f \text{MIC}^n(X/k)$ and morphisms $e_n : O_S \to \iota_{\text{dR}*}W_n (n \geq 1)$ in $\text{MIC}^n(S/k)$ which satisfies one of the following conditions:

1. For any $n \geq 1$, the exact sequence (3.2) in $\text{MIC}(X/S)$ can be up-graded to an exact sequence (3.1) in $\text{N}_f \text{MIC}^n(X/k)$, where we endow $R^1 f_{\text{dR}*}(W_n^\vee)^{\vee}$ with the Gauss-Manin connection (in our sense).

2. The same as (1) holds when we endow $R^1 f_{\text{dR}*}(W_n^\vee)^{\vee}$ with the Gauss-Manin connection induced by Katz-Oda filtration.

The proof is essentially the same as [AIK15 Prop 6.1(2)], but we give a detailed argument here. The uniqueness of $e_n (n \in \mathbb{N})$ is clear, because $e_n$ is uniquely determined by $\tau_n$. Assume we have the required uniqueness up to $W_n$ and denote by $\nabla_1, \nabla_2 : \overline{W}_{n+1} \to \overline{W}_{n+1} \otimes \Omega^1_{X/k}$ two integrable connections which upgrade the one on $\overline{W}_{n+1}$. Then their difference $\nabla_1 - \nabla_2$ induces a morphism $\phi : \overline{W}_n \to f_{\text{dR}}^* R^1 f_{\text{dR}*}((\overline{W}_n^\vee)^{\vee}) \otimes_{O_S} \Omega^1_{S/k}$ as $O_S$-modules.

We check that $\phi$ is a morphism in $\text{MIC}(X/S)$ by local computation. Let $\text{dlog } x_1, \ldots, \text{dlog } x_r (x_i \in \mathcal{M}_S)$ be a basis of $\Omega^1_{S/k}$ and let $\text{dlog } x_1, \ldots, \text{dlog } x_s (x_i \in \mathcal{M}_X)$ be
a basis of $\Omega^1_{X/k}$ extending it. We denote the image of $\text{dlog } x_{r+1}, \ldots, \text{dlog } x_s$ in $\Omega^1_{X/S}$ by $\text{dlog } x_{r+1}, \ldots, \text{dlog } x_s$, respectively. We define $\nabla^{(i)}_1, \nabla^{(i)}_2$ by

$$\nabla_l(m) = \sum_{i=1}^{s} \nabla^{(i)}_l(m) \text{dlog } x_i \quad (l = 1, 2, m \in \overline{W}_{n+1}).$$

Note that $\nabla^{(i)}_1 = \nabla^{(i)}_2$ for $r + 1 \leq i \leq s$ because both $\nabla_1, \nabla_2$ upgrades the connection on $\overline{W}_{n+1}$. Let $\nabla', \nabla''$ be the connection of $\overline{W}_n, f_{\text{dr}}^* R^1 f_{\text{dr}*}(\overline{W}_n)^{\vee} \otimes_{\mathcal{O}_S} \Omega^1_{S/k}$, respectively. For an element $m \in \overline{W}_n$, if we denote its local lift to $\overline{W}_{n+1}$ by $\tilde{m}$, $\phi(m)$ is equal to $\sum_{i=1}^{r} (\nabla^{(i)}_1 - \nabla^{(i)}_2)(\tilde{m}) \text{dlog } x_i$ and so

$$(\nabla'' \circ \phi)(m) = \sum_{j=r+1}^{s} \sum_{i=1}^{r} \nabla^{(j)}_i \circ (\nabla^{(i)}_1 - \nabla^{(i)}_2)(\tilde{m}) \text{dlog } x_i \otimes \text{dlog } x_j.$$ 

On the other hand, $\sum_{j=r+1}^{s} \nabla^{(j)}_i \text{dlog } x_j$ is a lift of $\nabla'(m)$ to $\overline{W}_{n+1} \otimes \Omega^1_{X/S}$ and so

$$(\phi \circ \nabla')(m) = \sum_{i=1}^{r} \sum_{j=r+1}^{s} (\nabla^{(i)}_1 - \nabla^{(i)}_2) \circ \nabla^{(j)}_i \text{dlog } x_i \otimes \text{dlog } x_j.$$ 

Because

$$\nabla^{(j)}_1 \circ (\nabla^{(i)}_1 - \nabla^{(i)}_2) = \nabla^{(j)}_1 \circ \nabla^{(i)}_1 - \nabla^{(j)}_2 \circ \nabla^{(i)}_1 = \nabla^{(j)}_1 \circ \nabla^{(i)}_1 - \nabla^{(j)}_1 \circ \nabla^{(i)}_2$$

$$= \nabla^{(j)}_1 \circ \nabla^{(j)}_2 - \nabla^{(j)}_2 \circ \nabla^{(j)}_2 = \nabla^{(j)}_1 \circ \nabla^{(j)}_1 - \nabla^{(j)}_2 \circ \nabla^{(j)}_1 = (\nabla^{(j)}_1 - \nabla^{(j)}_2) \circ \nabla^{(j)}_1$$

by the integrability of $\nabla_1, \nabla_2$ and the equality $\nabla^{(j)}_1 = \nabla^{(j)}_2$ ($r + 1 \leq j \leq s$), we see the equality $\nabla'' \circ \phi = \phi \circ \nabla'$. So $\phi$ is a morphism in MIC($X/S$), as required.

Since $\phi$ is a morphism in MIC($X/S$), and by the property (W1), it is determined by the morphism $t_{\text{dr}}^*(\nabla_1 - \nabla_2) \circ \overline{\tau}_n$. It is zero because it is equal to $t_{\text{dr}}^*(\nabla_1) \circ \overline{\tau}_{n+1} - i_{\text{dr}}^*(\nabla_2) \circ \overline{\tau}_{n+1}$ and the section $\overline{\tau}_{n+1}(1) = e_{n+1}(1)$ is horizontal with respect to both $i_{\text{dr}}^*(\nabla_1)$ and $i_{\text{dr}}^*(\nabla_2)$. Hence $\nabla_1 = \nabla_2$, as required.

**Remark 3.10.** In this remark, we compare our definition of the projective systems $\{(\overline{W}_n, \epsilon_n)\}_{n}, \{(W_n, e_n)\}_{n}$ and that of Andreatta-Iovita-Kim [AIK15].

Let the situation be as in Remark [1.17(3)] and we work on $X/S$ or on $X/S'$. (To simplify the description, we only deal with $X/S'$: the same argument works also for $X/S$.) We put $H^i := R^i f_{\text{dr}*}(\mathcal{O}_X, d) \in \text{MIC}(S/S)$ and define

$$R^n, \quad \gamma_n : R^n \otimes H^1 \longrightarrow R^{n-1} \otimes H^2 \ (n \geq 1), \quad i_n : R^n \hookrightarrow R^{n-1} \otimes H^1 \ (n \geq 2)$$

inductively in the following way: First, put $R^1 := H^1$ and let $\gamma_1 : R^1 \otimes H^1 = H^1 \otimes H^1 \stackrel{\cup}{\longrightarrow} H^2$ be the cup product. If we have defined $R^n$ and $\gamma_n$, let $R^{n+1} := \text{Ker} \gamma_n$, let $i_{n+1}$ be the canonical inclusion $R^{n+1} \hookrightarrow R^n \otimes H^1$ and let $\gamma_{n+1}$ be the composite

$$R^{n+1} \otimes H^1 \xrightarrow{i_{n+1} \otimes \text{id}} R^n \otimes H^1 \otimes H^1 \xrightarrow{\text{id} \otimes \cup} R^n \otimes H^2.$$
Andreatta-Iovita-Kim gave a definition of the projective system \( \{ (\overline{W}_n, \overline{e}_n) \} \) with exact sequences

\[
(*)_n 0 \rightarrow \overline{W}^\vee_n \rightarrow \overline{W}^\vee_{n+1} \rightarrow f^*_{dR} R^n \rightarrow 0
\]

for \( n \geq 1 \) (where the map \( \overline{W}^\vee_n \rightarrow \overline{W}^\vee_{n+1} \) is the dual of the transition map \( \overline{W}_{n+1} \rightarrow \overline{W}_n \)) such that the connecting map \( a_n : R^n \rightarrow R^1 f^*_{dR} (\overline{W}^\vee_n) \) associated to \( (*)_n \) is an isomorphism and that the map \( b_n : R^1 f^*_{dR} (\overline{W}^\vee_{n+1}) \rightarrow R^n \otimes H^1 \) on the first cohomology groups associated to \( (*)_n \) is the identity map on \( R^1 = R^1 f^*_{dR} \mathcal{O}_X \). Their method is the following: first, they put \( (\overline{W}_1, \overline{e}_1) := (\mathcal{O}_X, \mathcal{O}_S \xrightarrow{=} \iota_{dR}^* \mathcal{O}_X) \) and define \( a_1 \) to be the identity map on \( R^1 = R^1 f^*_{dR} \mathcal{O}_X \). Next, once defined \( (\overline{W}_i, \overline{e}_i) \) and the isomorphisms \( a_i \) for \( 1 \leq i \leq n \), they introduce \( (\overline{W}_{n+1}, \overline{e}_{n+1}) \) fitting in the exact sequence \( (*)_n \) by identifying \( (\overline{W}_{n+1}, \overline{e}_{n+1}) \) with the dual of \( (\overline{W}_n, \overline{e}_n) \) via \( a_n \) and arguing in the same way as our definition, meaning that the extension corresponds to the identity as in \( (3.2) \). Then \( a_n \) becomes the connecting map of \( (*)_n \). Then they consider the diagram

\[
\begin{array}{ccc}
R^1 f^*_{dR} (\overline{W}^\vee_{n+1}) & \xrightarrow{b_n} & R^n \otimes H^1 \\
\downarrow \scriptstyle{a_n \otimes \text{id}} & & \downarrow \scriptstyle{a_n \otimes \text{id}} \\
R^1 f^*_{dR} (\overline{W}^\vee_n) \otimes H^1 & \xrightarrow{b_{n-1} \otimes \text{id}} & R^{n-1} \otimes H^1 \otimes H^1 \\
\end{array}
\]

where the top horizontal line is a part of the long exact sequence associated to \( (*)_n \), the long vertical arrow comes from \( (*)_{n-1} \) and the lower horizontal arrow is the cup product. Then they prove that the rectangle is anti-commutative. Since we have \( b_{n-1} \circ a_n = i_n \), then \( \cup \circ (b_{n-1} \otimes \text{id}) \circ (a_n \otimes \text{id}) = \gamma_n \); moreover the long vertical arrow of the above diagram composed with the top horizontal arrow is the zero map, because they come form a cohomological long exact sequence. Hence the image of \( b_n \) should be in the kernel of \( \gamma_n \) which is \( R^{n+1} \); thus we obtain a map \( R^1 f^*_{dR} (\overline{W}^\vee_{n+1}) \rightarrow R^{n+1} \).

They prove that this map is an isomorphism and denote its inverse by \( a_{n+1} \), and so the induction works.

So their construction is the same as ours except that they build isomorphisms \( a_n (n \geq 1) \) in the process of construction. We expect the existence of the isomorphisms \( a_n (n \geq 1) \) in more general case than in the framework of Remark \( 1.17(3) \), but we will not pursue this topic in this paper.

After giving the definition of the projective system \( \{ (\overline{W}_n, \overline{e}_n) \} \), Andreatta-Iovita-Kim upgrade it to the projective system \( \{ (W_n, e_n) \} \) satisfying the condition (2) in Remark \( 3.9 \) by using \( p \)-adic Hodge theory. Because our projective system \( \{ (W_n, e_n) \} \) satisfies the condition (1) in Remark \( 3.9 \) and the conditions (1) and (2) are equivalent in the hypothesis of this remark by Remark \( 3.8(2) \), we see that the projective system \( \{ (W_n, e_n) \} \) of Andreatta-Iovita-Kim is the same as ours.
We give the second definition of relatively unipotent de Rham fundamental group \( \pi(X/S, \iota) \) by using \( W = \{(W_n, e_n)\}_n \), following the method of [Had11] and [AIK15].

By (3.9), we have the isomorphism
\[
(3.14) \quad f_{\text{dR}}^* \text{Hom}(W, W) \cong \iota_{\text{dR}}^* W,
\]
and so \( \iota_{\text{dR}} W \) has a structure of a ring. Moreover, if we denote the projective system \( \{W_m \otimes W_n\}_{m,n} \) by \( W \widehat{\otimes} W \), we have the isomorphism
\[
(3.15) \quad f_{\text{dR}}^* \text{Hom}(W, W \widehat{\otimes} W) \cong \iota_{\text{dR}}^* (W \widehat{\otimes} W)
\]
again by (3.9), and the morphisms
\[
e_m \otimes e_n : \mathcal{O}_S \rightarrow \iota_{\text{dR}}^* (W_m \otimes W_n)
\]
in \( \text{MIC}^n(S/k) \) induce the morphism \( W \rightarrow W \widehat{\otimes} W \), thus the coproduct structure \( \iota_{\text{dR}} W \rightarrow \iota_{\text{dR}}^* W \widehat{\otimes} \iota_{\text{dR}} W \) on \( \iota_{\text{dR}} W \). We see that, with these structures, \( (\iota_{\text{dR}} W)^\vee := \lim_n (\iota_{\text{dR}} W_n)^\vee \) forms a commutative Hopf algebra object in the ind-category of \( \text{MIC}^n(S/k) \). So we can give the following definition:

**Definition 3.11 (Second definition of relatively unipotent \( \pi_1 \)).** Let the notations be as above. We define the relatively unipotent de Rham fundamental group \( \pi_{1,\text{dR}}^\text{dR}(X/S, \iota) \) by
\[
\pi_{1,\text{dR}}^\text{dR}(X/S, \iota) := \text{Spec} (\iota_{\text{dR}}^* W)^\vee,
\]
which is a group scheme in \( \text{MIC}^n(S/k) \).

Then we have the following comparison theorem on two definitions of relatively unipotent de Rham fundamental groups:

**Theorem 3.12.** The first and the second definitions of \( \pi_{1,\text{dR}}^\text{dR}(X/S, \iota) \) are canonically isomorphic.

**Proof.** In this proof, we denote the second definition of \( \pi_{1,\text{dR}}^\text{dR}(X/S, \iota) \) by \( \pi_{1,\text{dR}}^\text{dR}(X/S, \iota)' \).

For an object \( E \) in \( \text{N}_f \text{MIC}^n(X/k) \), we have the canonical map
\[
f_{\text{dR}}^* \text{Hom}(W, W) \otimes f_{\text{dR}}^* \text{Hom}(W, E^\vee) \rightarrow f_{\text{dR}}^* \text{Hom}(W, E^\vee)
\]
and this corresponds to the map
\[
\iota_{\text{dR}}^* W \otimes \iota_{\text{dR}}^* E^\vee \rightarrow \iota_{\text{dR}}^* E^\vee
\]
by (3.9). This map defines the representation of \( \pi_{1,\text{dR}}^\text{dR}(X/S, \iota)' \) on \( \iota_{\text{dR}}^* E \) in \( \text{MIC}^n(S/k) \). Hence we have defined the functor
\[
\text{N}_f \text{MIC}^n(X/k) \rightarrow \text{Rep}_{\text{MIC}^n(S/k)}(\pi_{1,\text{dR}}^\text{dR}(X/S, \iota')).
\]
Since we have $N_t \text{MIC}^n(X/k) = \text{Rep}_{\text{MIC}^n(S/k)}(\pi_{1}^{\text{dR}}(X/S, \iota))$ by Proposition 2.11 we obtain the morphism

$$\pi_{1}^{\text{dR}}(X/S, \iota)' \longrightarrow \pi_{1}^{\text{dR}}(X/S, \iota)$$

of group schemes in $\text{MIC}^n(S/k)$ by the argument after Proposition 2.11. To prove that this is an isomorphism, it suffices to prove that the restriction of this morphism to $\text{MIC}(s/s)$, which is the morphism

(3.16) $$\pi_{1}^{\text{dR}}(X_{s/s}, x)' \longrightarrow \pi_{1}^{\text{dR}}(X_{s}, x),$$

is an isomorphism, because we can check the claim after applying the functor $\text{MIC}^n(S/k) \longrightarrow \text{MIC}(s/s)$ which is exact and faithful.

The proof of the isomorphism (3.16) is the same as that in [Had11, Theorem 2.9], which we explain here for the convenience of the reader. To simplify the notation, we denote the category $N_t \text{MIC}(X_{s/s})$ by $C$, the fundamental group $\pi_{1}^{\text{dR}}(X_{s}, x)'$ by $G$, the base change of $W = \{W_n\}$ to $X_{s}/s$ by $W^C = \{W_n^C\}$ and the fiber functor $x_{\text{dR}} : C \longrightarrow \text{Vec}_k$ by $E \mapsto E|_{x}$. Then $G$ is equal to $\text{Spec}(W_{C}|_{x})^\vee$ and the morphism (3.16) is induced by the functor

(3.17) $$C \longrightarrow \text{Rep}_k(G)$$

defined by $E \mapsto E|_{x}$ with $E|_{x}$ endowed with a $G$-action. It suffices to prove that the functor (3.17) is an equivalence.

First we prove the full-faithfulness by checking that the map

$$\text{Hom}_C(E, F) \longrightarrow \text{Hom}_G(E|_{x}, F|_{x})$$

induced by (3.17) is an isomorphism. By Remark 3.2 any $E \in C$ admits an exact sequence of the form $W^{C}_{m} \longrightarrow W^{C}_{n} \longrightarrow E \longrightarrow 0$ for some $n, m, r, s \in \mathbb{N}$, and we can take $n, m$ arbitrarily large. Thus it suffices to prove that the map

$$\text{Hom}_C(W^{C}_{n}, F) \longrightarrow \text{Hom}_G(W^{C}_{n}|_{x}, F|_{x})$$

induced by (3.17) is an isomorphism for sufficiently large $n$. This is true because $\text{Hom}_C(W^{C}_{n}, F)$ is equal to $F|_{x}$ by Remark 3.2 and $\text{Hom}_G(W^{C}_{n}|_{x}, F|_{x}) = \text{Hom}_G(W^{C}_{n}|_{x}, F|_{x})$ is also equal to $F|_{x}$. (Note that, since $G$ is equal to $\text{Spec}(W_{C}|_{x})^\vee$, giving a $G$-equivariant map $W^{C}_{n}|_{x} \longrightarrow F|_{x}$ is equivalent to giving an image of the unit element $1 \in W_{n}|_{x}$.)

Next we prove the essential surjectivity. For any $V \in \text{Rep}_k(G)$ and $v \in V$, there exists a unique $G$-equivariant map $W^{C}_{n}|_{x} \longrightarrow V$ with $1 \mapsto v$, and this map factors through some $W^{C}_{n}|_{x}$. Thus we see that there exists a $G$-equivariant surjection $W^{C}_{n}|_{x} \longrightarrow V$ for some $n, r \in \mathbb{N}$. Repeating this argument, we see that there exists a $G$-equivariant exact sequence $W^{C}_{n}|_{x} \longrightarrow W^{C}_{n}|_{x} \longrightarrow V \longrightarrow 0$ for some $n, m, r, s \in \mathbb{N}$. Since the functor (3.17) is fully faithful, there exists a morphism $\tilde{\alpha} : W^{C}_{m} \longrightarrow W^{C}_{n}$ which is sent to $\alpha$ by the functor (3.17). Then $\text{Coker} \tilde{\alpha}$ is sent to $V$ and so the essential surjectivity is proved. Hence the functor (3.17) is an equivalence and the proof is finished.

\[\square\]
4 Relative minimal model of Navarro-Aznar

In this section we introduce a third definition of relatively unipotent de Rham fundamental group by using the theory of relative minimal model developed by Navarro-Aznar [NA93]. Because our definition is different from his original one, we will give the steps in detail. We will state the compatibility with the previous definitions, but the proof will be postponed to Section 5 (by introducing a fourth definition of the relatively unipotent de Rham fundamental group). We review now the theory of differential graded commutative algebras (dgca’s), following [NA93]. Let $R$ be a commutative algebra over a field $k$ of characteristic zero.

A dgca is a graded algebra $A = \bigoplus_{i=0}^{\infty} A^i$ endowed with a differential $d : A^i \rightarrow A^{i+1}$ ($i \geq 0$) with $d \circ d = 0$ satisfying
\[
xy = (-1)^{pq}yx \quad (x \in A^p, y \in A^q),
\]
\[
d(xy) = (dx)y + (-1)^px(dy) \quad (x \in A^p, y \in A^q).
\]

The $k$-algebra $R$ is regarded as an dgca with $R^0 = R, R^i = 0$ ($i > 0), d = 0$. A morphism of dgca’s is a morphism of algebras compatible with grading and differential.

An $R$-dgca is a dgca $A$ endowed with a morphism $R \rightarrow A$ of dgca’s. An augmented $R$-dgca is an $R$-dgca $(A, i : R \rightarrow A)$ endowed further with another morphism $e : A \rightarrow R$ of dgca’s (called an augmentation) such that $e \circ i = \text{id}$. An augmented morphism of (augmented) $R$-dgca’s is a morphism of $R$-algebras compatible with grading, differential (and augmentation). (In [NA93], an augmented morphism is called a pointed morphism, and the adjective ‘pointed’ is often used for several notions with augmentation. In this paper, we will always use the adjective ‘augmented’ in place of ‘pointed’.)

A Hirsch extension of an (augmented) $R$-dgca is an (augmented) inclusion $A \hookrightarrow A \otimes \wedge(E)$ of (augmented) $R$-dgca’s, where $E$ is a free $R$-module homogeneous of degree 1, $\wedge(E)$ is the free graded commutative algebra generated by $E$ and the differential of $A \otimes \wedge(E)$ is induced by a map which sends $E$ to $A$. (It is called a Hirsch extension of degree 1 in [NA93, (2.1)], but we will omit the term ‘of degree 1’ because we will not treat the case of higher degree.)

An (augmented) $R$-dgca $A$ is called $(1, q)$-minimal if there exists a sequence of Hirsch extensions
\[
R = A(0) \subseteq A(1) \subseteq \cdots \subseteq A(q) = A.
\]

An (augmented) $R$-dgca $A$ is called 1-minimal if $A$ is the union of $(1, q)$-minimal (augmented) $R$-dgca’s ($q \in \mathbb{N}$) with respect to Hirsch extensions as above. (as a matter of fact it is indicated as $(2, 0)$-minimal in [NA93, (2.1)], but we call it 1-minimal, following [GM13].)

A $(1, q)$-minimal model of a(n augmented) $R$-dgca $A$ is a(n augmented) morphism
\[
\rho_q : M(q) \rightarrow A
\]
from a $(1, q)$-minimal (augmented) $R$-dgca $M(q)$ (endowed with filtration $\{M(q')\}_{q'=0}^q$ as \((4.1)\)) such that the induced maps $H^i(M(q')) \to H^i(A)$ ($i = 0, 1$) are isomorphisms for $1 \leq q' \leq q$ and the induced map $H^2(M(q' - 1), A) \to H^2(M(q'), A)$ is zero for $2 \leq q' \leq q$. A $1$-minimal model of an (augmented) $R$-dgca $A$ is an augmented morphism

$$\rho : M \to A$$

which is the union of $(1, q)$-minimal models $(q \in \mathbb{N})$ as above with respect to Hirsch extensions $M(q - 1) \subseteq M(q)$. By definition, we see that $H^i(\rho) : H^i(M) \to H^i(A)$ is an isomorphism for $i = 0, 1$ and injective for $i = 2$. Note that the composition of a $(1, q)$-minimal model $M(q) \to A$ and a quasi-isomorphism of (augmented) $R$-dgca $A \to B$ is again a $(1, q)$-minimal model and the same is true also for $1$-minimal models.

For two morphisms $f, g : A \to B$ of $R$-dgca’s, a Sullivan homotopy \cite[Lemma 2.3]{NA93} between them is a morphism of $R$-dgca’s $h : A \to R(t, dt) \otimes B$ with $p_0 \circ h = f, p_1 \circ h = g$, where $R(t, dt)$ is the $R$-dgca $R[t] \oplus R[t]dt$ and $p_i (i = 0, 1)$ is the morphism $R(t, dt) \otimes B \to B$ defined by $t \mapsto i, dt \mapsto 0$. For two morphisms $f, g : A \to B$ of augmented $R$-dgca’s, a Sullivan homotopy between them is a morphism $h : A \to R(t, dt) \otimes B$ as above such that the diagram

$$
\begin{array}{ccc}
A & \xrightarrow{h} & R(t, dt) \otimes B \\
\downarrow & & \downarrow \\
R & \longrightarrow & R(t, dt)
\end{array}
$$

is commutative, where vertical arrows are augmentations and the lower horizontal map is the canonical inclusion. We write a Sullivan homotopy $h$ between $f$ and $g$ by $h : f \simeq_{Su} g$. Then, following \cite[Lem. 2.7]{NA93} and \cite[10.7]{GM13}, we have:

**Lemma 4.1.** If $A$ is (augmented and) $(1, q)$-minimal, the relation on the set of (augmented) morphisms $A \to B$ defined by Sullivan homotopy is an equivalence relation.

**Proof.** We only recall the proof of transitivity given in \cite[Prop. 12.7]{EH10} which we use later. Assume that we are given (augmented) morphisms $f_0, f_1, f_2 : A \to B$, a Sullivan homotopy $h_1 : A \to R(t_1, dt_1) \otimes B$ between $f_0$ and $f_1$, and a Sullivan homotopy $h_2 : A \to R(t_2, dt_2) \otimes B$ between $f_1$ and $f_2$. Then $h_1$ and $h_2$ naturally induce the morphism

$$h_1 \times h_2 : A \to (R(t_1, dt_1) \times_R R(t_2, dt_2)) \otimes_R B,$$

where $R(t_1, dt_1) \times_R R(t_2, dt_2)$ denotes the fiber product with respect to the map $t_1 \mapsto 1, dt_1 \mapsto 0$ and the map $t_2 \mapsto 0, dt_2 \mapsto 0$. Define the $R$-dgca $C$ by

$$C := (R(t_0, dt_0) \otimes_R R(t_1, dt_1) \otimes_R R(t_2, dt_2))/\left(\sum_{i=0}^2 t_i - 1, \sum_{i=0}^2 dt_i\right)$$

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and consider the morphism

\[ p : C \longrightarrow R(t_1, dt_1) \times_R R(t_2, dt_2) \]

defined by \( t_0 \mapsto (1 - t_1, 1 - t_2), t_1 \mapsto (t_1, 0), t_2 \mapsto (0, t_2) \). Then we have the diagram

\[
\begin{array}{c}
C \otimes_R B \\
p \otimes \text{id} \\
\downarrow \\
A \xrightarrow{h_1 \times h_2} (R(t_1, dt_1) \times_R R(t_2, dt_2)) \otimes_R B,
\end{array}
\]

and one sees that the map \( p \otimes \text{id} \) is a surjective quasi-isomorphism. Using this fact, one proves ([GM13, 10.4], [NA93, Lem. 2.5], [FHT01, Lem. 12.4]) that there exists a morphism \( q : A \longrightarrow C \otimes_R B \) which makes the above diagram commutative. Then the composition of \( q \) with the morphism \( C \otimes_R B \longrightarrow R(t, dt) \otimes_R B \) given by \( t_0 \mapsto 1 - t, t_1 \mapsto 0, t_2 \mapsto t \) gives a Sullivan homotopy between \( f_0 \) and \( f_2 \). \( \square \)

The following proposition ([NA93, Prop. 2.9]) assures the uniqueness of \((1, q)\)-minimal model.

**Proposition 4.2.** Let \( A, A' \) be augmented \( R \)-dgca’s and let us assume given the following augmented diagram of augmented \( R \)-dgca’s

\[
\begin{array}{ccc}
A & \xrightarrow{f} & A' \\
\rho \uparrow & & \rho' \uparrow \\
M(q) & & M'(q),
\end{array}
\]

where \( \rho, \rho' \) are \((1, q)\)-minimal models. Then there exists a unique morphism \( \varphi : M(q) \longrightarrow M'(q) \) which admits a Sullivan homotopy \( h : f \circ \rho \simeq_{\text{Su}} \rho' \circ \varphi \), and such Sullivan homotopy \( h \) is unique. Moreover, if \( f \) is a quasi-isomorphism, \( \varphi \) is an isomorphism.

**Proof.** The claims except the last one is proven in [NA93, Prop. 2.9], and the last claim is proven in [NA93, Prop. 2.9] when \( A = A' \) and \( f = \text{id} \). We can reduce the last claim to the case with \( A = A' \) and \( f = \text{id} \) by considering the diagram (4.2) as the diagram

\[
\begin{array}{ccc}
A' & \xrightarrow{\text{id}} & A' \\
f \circ \rho \uparrow & & \rho' \uparrow \\
M(q) & & M'(q).
\end{array}
\]

\( \square \)

Next we recall some properties of sheaves of dgca’s [NA93, (2.14)]. Let \( S \) be a scheme over a field \( k \) of characteristic zero. Then one can define the notion of a sheaf
of (augmented) $\mathcal{O}_S$-dgca’s and that of a(n augmented) morphism between sheaves of $\mathcal{O}_S$-dgca’s in usual way.

We will need a notion of ho-morphism. A covering sieve of opens of $S$ is an open covering $\mathcal{U} = \{U\}_{a}$ of $S$ such that, for any $U \in \mathcal{U}$ and any open $U' \subseteq U$, $U'$ belongs to $\mathcal{U}$. Then, for sheaves of (augmented) $\mathcal{O}_S$-dgca’s $\mathcal{A}, \mathcal{B}$, a(n augmented) ho-morphism $f : \mathcal{A} \to \mathcal{B}$ is defined to be a family of (augmented) morphisms

$$f(U) : \mathcal{A}(U) \to \mathcal{B}(U)$$

of $\mathcal{O}_S(U)$-dgca’s for every affine $U \in \mathcal{U}$ such that, for any two affines $U' \subseteq U$ in $\mathcal{U}$, the map $\mathcal{B}(U' \subseteq U) \circ f(U) : \mathcal{A}(U) \to \mathcal{B}(U')$ admits a Sullivan homotopy with $f(U') \circ \mathcal{A}(U' \subseteq U)$. (Here $\mathcal{A}(U' \subseteq U) : \mathcal{A}(U) \to \mathcal{A}(U')$, $\mathcal{B}(U' \subseteq U) : \mathcal{B}(U) \to \mathcal{B}(U')$ are restriction morphisms.) Note that our definition of ho-morphism is different from that in [NA93] in the sense that, in our definition, the morphisms $f(U)$ are defined only on affine $U$’s in $\mathcal{U}$. A morphism is a ho-morphism, and a ho-morphism naturally induces maps $H^i(\mathcal{A}) \to H^i(\mathcal{B})$ ($i \in \mathbb{N}$) between cohomology sheaves. Hence the notion of quasi-isomorphism makes sense for ho-morphisms.

A Hirsch extension of sheaves of (augmented) $\mathcal{O}_S$-dgca’s is an injective (augmented) morphism $\mathcal{A} \hookrightarrow \mathcal{B}$ such that there exists a covering sieve of opens $\mathcal{U}$ of $S$ and that, for each $U \in \mathcal{U}$, the map $\mathcal{A}(U) \hookrightarrow \mathcal{B}(U)$ is a Hirsch extension in the previous sense. A sheaf of (augmented) $\mathcal{O}_S$-dgca’s $\mathcal{A}$ is called $(1, q)$-minimal if there exists a sequence of Hirsch extensions

$$(4.3) \quad \mathcal{O}_S = \mathcal{A}(0) \subseteq \mathcal{A}(1) \subseteq \cdots \subseteq \mathcal{A}(q) = \mathcal{A}.$$ 

Also, a sheaf of (augmented) $\mathcal{O}_S$-dgca’s $\mathcal{A}$ is called 1-minimal if $\mathcal{A}$ is the union of $(1, q)$-minimal sheaves of (augmented) $\mathcal{O}_S$-dgca’s ($q \in \mathbb{N}$) with respect to Hirsch extensions as above.

A $(1, q)$-minimal model of a(n augmented) sheaf of $\mathcal{O}_S$-dgca’s $\mathcal{A}$ is a(n augmented) ho-morphism $\rho_q : \mathcal{M}(q) \to \mathcal{A}$ from a $(1, q)$-minimal sheaf of (augmented) $\mathcal{O}_S$-dgca’s $\mathcal{M}(q)$ such that there exists a covering sieve of opens $\mathcal{U}$ of $S$ with any $\rho_q(U) : \mathcal{M}(q)(U) \hookrightarrow \mathcal{A}(U)$ with affine $U \in \mathcal{U}$ a $(1, q)$-minimal model in the previous sense. A 1-minimal model of a(n augmented) sheaf of $\mathcal{O}_S$-dgca’s $\mathcal{A}$ is a 1-minimal (augmented) sheaf of $\mathcal{O}_S$-dgca’s $\mathcal{M} = \bigcup_q \mathcal{M}(q)$ endowed, for each $q \in \mathbb{N}$, with a ho-morphism $\rho_q : \mathcal{M}(q) \to \mathcal{A}$ with respect to a covering sieve of opens $\mathcal{U}_q$ of $S$ such that $\mathcal{U}_q \subseteq \mathcal{U}_{q-1}$ and that $\rho_q(U)|_{\mathcal{M}(q-1)(U)} = \rho_{q-1}(U)$ for any $U \in \mathcal{U}_q$ affine.

For two ho-morphisms $f, g : \mathcal{A} \to \mathcal{B}$ of sheaves of (augmented) $\mathcal{O}_S$-dgca’s, a Sullivan homotopy between them is a family of Sullivan homotopies $f(U) \simeq_{\text{Su}} g(U)$ ($U \in \mathcal{U}$ affine) for some covering sieve of opens $\mathcal{U}$ of $S$. We denote a (sheaf version of) Sullivan homotopy $h$ between $f$ and $g$ also by $h : f \simeq_{\text{Su}} g$. 

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Then Lemma 4.1 implies that, when $\mathcal{A}$ is (augmented and) $(1, q)$-minimal, the relation on the set of (augmented) ho-morphisms $\mathcal{A} \to \mathcal{B}$ defined by Sullivan homotopy is an equivalence relation.

To prove a sheaf version of Proposition 4.2, we need to impose the quasi-coherence condition on sheaves of $\mathcal{O}_S$-dgca’s.

**Proposition 4.3.** Let $\mathcal{A}, \mathcal{A}'$ be quasi-coherent sheaves of augmented $\mathcal{O}_S$-dgca’s and let us assume given the following augmented diagram of ho-morphisms

$$
\begin{array}{c}
\mathcal{A} \\
\rho
\end{array}
\xymatrix{
\ar[r]^f & \\
\rho' \\
\mathcal{M}(q) \ar[u] & \mathcal{M}'(q),
}
$$

where $\rho, \rho'$ are $(1, q)$-minimal models. We assume moreover that $\mathcal{M}(q), \mathcal{M}'(q)$ are quasi-coherent, as well as all the sheaves of $\mathcal{O}_S$-dgca’s appearing in their filtration. Then there exists a unique morphism $\phi : \mathcal{M}(q) \to \mathcal{M}'(q)$ which admits a Sullivan homotopy $h : f \circ \rho \simeq_S \rho' \circ \phi$, and such a Sullivan homotopy $h$ is unique. Moreover, if $f$ is a quasi-isomorphism, $\phi$ is an isomorphism.

**Proof.** We can suppose that all the ho-morphisms in the diagram are defined with respect to the same sieve of opens $\mathcal{U}$. Take an affine $U \in \mathcal{U}$. By the quasi-coherence, $H^i(\mathcal{A}(U)) = H^i(\mathcal{A})(U)$ and the same holds for $\mathcal{M}(q)$. Thus the morphisms $\rho(U) : \mathcal{M}(q)(U) \to \mathcal{A}(U), \rho'(U) : \mathcal{M}'(q)(U) \to \mathcal{A}'(U)$ are $(1, q)$-minimal models. Hence we are in the situation of Proposition 4.2 and so we have the unique morphism $\varphi(U) : \mathcal{M}(q)(U) \to \mathcal{M}'(q)(U)$ satisfying the conclusion of that proposition. By unicity in Proposition 4.2 the morphisms $\varphi(U)$’s glue to give a required morphism $\varphi : \mathcal{M}(q) \to \mathcal{M}'(q)$ of sheaves. \qed

We prove a theorem on the existence of $(1, q)$-minimal model and 1-minimal model, the idea of the proof is based on the methods of [NA93, Thm. 5.6]. Let $g : S \to \text{Spec} \, k$ be as in Notation 1.14, and let $p^m_j : S^m(1) \to S (j = 1, 2, m \in \mathbb{N}), p^m_{jj'} : S^m(2) \to S^m(1) (1 \leq j < j' \leq 3, m \in \mathbb{N})$ be as in Section 1. Let $\mathcal{A}$ be a quasi-coherent sheaf of augmented $\mathcal{O}_S$-dgca’s and for $r = 1, 2$, let $\{A^m(r)\}_m$ be a projective system of quasi-coherent sheaves of augmented $\mathcal{O}_{S^m(r)}$-dgca’s. Assume that $H^0(\mathcal{A}) = \mathcal{O}_S, H^0(A^m(r)) = \mathcal{O}_{S^m(r)}$. Also, assume that we are given a morphism

$$
\eta : A^0(1) \to \mathcal{A}
$$

and a projective system of morphisms

$$
\begin{array}{c}
\eta^m_j : (p^m_j)^* A \to A^m(1) \\
\eta_{jj'}^m : (p^m_{jj'})^* A^m(1) \to A^m(2)
\end{array}
\quad (j = 1, 2),
\quad (1 \leq j < j' \leq 3)
which are all quasi-isomorphisms such that \( \eta \circ \eta^0_1 = \eta \circ \eta^0_2 = \text{id} \) (note that \( p^0_1 = p^0_2 = \text{id} \)) and that, for any \( m \), the diagram (4.4)

\[
\begin{array}{ccc}
(p^m_{2,3})^* (p^m_2)^* \mathcal{A} & \xrightarrow{(p^m_{2,3})^* \eta^m_2} & (p^m_{2,3})^* \mathcal{A}^m(1) \\
(p^m_{1,3})^* \eta^m_2 & \downarrow \eta^m_{1,3} & \downarrow \eta^m_{1,2} \\
(p^m_{1,3})^* \mathcal{A} & \xrightarrow{(p^m_{1,2})^* \eta^m_1} & (p^m_{1,2})^* \mathcal{A}^m(1) \\
(p^m_{1,3})^* (p^m_1)^* \mathcal{A} & \xrightarrow{(p^m_{1,2})^* \eta^m_2} & (p^m_{1,2})^* (p^m_1)^* \mathcal{A}
\end{array}
\]

is commutative. This commutativity implies that, for any \( i \), the composites

\[
H^i(\eta^m_i)^{-1} \circ H^i(\eta^m_2) : (p^m_1)^* H^i(\mathcal{A}) \xrightarrow{\sim} H^i(\mathcal{A}^m(1)) \xrightarrow{\sim} (p^m_1)^* H^i(\mathcal{A}) \quad (m \in \mathbb{N})
\]

define a stratification on \( H^i(\mathcal{A}) \). (Remember that, thanks to the hypothesis (B) and the proof of Proposition 1.5, the maps \( p^m_j \) are flat for \( j = 1, 2 \).) We assume moreover that, with this stratification, \( H^i(\mathcal{A}) \) belongs to \( \text{MIC}^n(S/k) \).

**Theorem 4.4.** Let the situation be as above. Then there exist unique \((1,q)\)-minimal models

\[
\rho : \mathcal{M}(q) \rightarrow \mathcal{A}, \quad \rho^m(r) : \mathcal{M}^m(r)(q) \rightarrow \mathcal{A}^m(r) \quad (r = 1, 2, m \in \mathbb{N})
\]

and isomorphisms

\[
\zeta^m_j : (p^m_j)^* \mathcal{M}(q) \xrightarrow{\sim} \mathcal{M}^m(1)(q) \quad (j = 1, 2, m \in \mathbb{N}),
\]

\[
\zeta^m_{j,j'} : (p^m_{j,j'})^* \mathcal{M}^m(1)(q) \xrightarrow{\sim} \mathcal{M}^m(2)(q) \quad (1 \leq j < j' \leq 3, m \in \mathbb{N})
\]

which satisfy the following conditions:

1. For each \( m' \leq m \), there exists a unique morphism \( \mathcal{M}^m(r)(q) \rightarrow \mathcal{M}^m(r)(q) \) which is compatible with \( \rho^m(r) \) up to unique Sullivan homotopy and compatible with \( \zeta^m_j \), \( \zeta^m_{j,j'} \), \( \zeta^m_{j,j'} \) as morphisms.
2. For \( m \in \mathbb{N} \), the augmented diagrams

\[
\begin{array}{ccc}
(p^m_j)^* \mathcal{A} & \xrightarrow{\eta^m_j} & \mathcal{A}^m(1) \\
(p^m_j)^* \rho & \uparrow & \rho^m(1) \\
(p^m_{j,j'})^* \mathcal{M}(q) & \xrightarrow{\zeta^m_{j,j'}} & \mathcal{M}^m(1)(q) \\
(p^m_{j,j'})^* \rho^m(1) & \uparrow & (p^m_{j,j'})^* \rho^m(2)
\end{array}
\]

\[
\begin{array}{ccc}
(p^m_{j,j'})^* \mathcal{M}(q) & \xrightarrow{\zeta^m_{j,j'}} & \mathcal{M}^m(1)(q) \\
(p^m_{j,j'})^* \rho^m(1) & \uparrow & (p^m_{j,j'})^* \rho^m(2)
\end{array}
\]

\[
\begin{array}{ccc}
(p^m_{j,j'})^* \mathcal{M}(q) & \xrightarrow{\zeta^m_{j,j'}} & \mathcal{M}^m(1)(q) \\
(p^m_{j,j'})^* \rho^m(1) & \uparrow & (p^m_{j,j'})^* \rho^m(2)
\end{array}
\]
are commutative up to unique Sullivan homotopy.

(3) \( \zeta_1^0 = \zeta_2^0 \) via the canonical identification \( p_1^0 = p_2^0 = \text{id} \), and there exists a commutative diagram as \( (1.4) \) with \( A, A^m(r), \eta^m_{ij}, \eta^m_{ij'} \) replaced by \( M, M^m(r), \zeta^m_{ij}, \zeta^m_{ij'} \) reflectively. (Hence, for each \( i \in \mathbb{N} \), the isomorphisms \( \{ (\zeta_1^m)^{-1} \circ \zeta_2^m \}_m \) induce a structure of stratification on the degree \( i \) part \( M(q)^i \) of \( M(q) \).

(4) With respect to the structure of stratification defined in (3), \( M(q)^i \) belongs to \( \text{MIC}^n(S/k) \).

Proof. Since \( H^1(A) \) belongs to \( \text{MIC}^n(S/k) \), we can define the covering sieve of opens \( \mathcal{U}_1 \) in \( S \) by

\[
\mathcal{U}_1 := \{ U \subseteq S \mid H^1(A) \text{ is free on } U \}.
\]

Take an affine \( U \in \mathcal{U}_1 \) and put \( A_U := A(U), R_U := O_S(U) \). By the quasi-coherence of \( A, H^1(A_U) \) is a free \( R_U \)-module. Let \( \sigma : E := H^1(A_U) \to Z^1(A_U) \) be a section of the canonical projection \( Z^1(A_U) \to H^1(A_U) \), and define \( M(1)_U \) by \( M(1)_U := \bigwedge(E) \) with zero differential. Also, define the morphism \( M(1)_U \to A_U \) as the one induced by \( E \xrightarrow{\sigma} Z^1(A_U) \to A_U \). Then \( M(1)_U \) is a \((1,1)\)-minimal model of \( A_U \).

Suppose now \( U' \subseteq U \) be affine opens in \( \mathcal{U}_1 \) and consider the following diagram:

\[
\begin{array}{ccc}
A_U \otimes_{R_U} R_{U'} & \longrightarrow & A_{U'} \\
\uparrow & & \uparrow \\
M(1)_U \otimes_{R_U} R_{U'} & & M(1)_{U'}.
\end{array}
\]

Because \( R_{U'} \) is flat over \( R_U \) and \( A \) is quasi-coherent, the horizontal map is a quasi-isomorphism, and the left vertical map is a \((1,1)\)-minimal model. Hence, by Proposition \( 4.2 \) there is a unique isomorphism \( M(1)_U \otimes_{R_U} R_{U'} \to M(1)_{U'} \) which makes the diagram commutative up to Sullivan homotopy. By unicity, this isomorphism is compatible if we consider affine \( U'' \subseteq U' \subseteq U \) in \( \mathcal{U}_1 \). To this data, we can associate a sheaf \( \mathcal{M}(1) \) of \( O_S \)-dgca’s which is locally free of finite rank on each degree (hence quasi-coherent) and we have naturally a ho-morphism of \((1,1)\)-minimal model \( \rho : \mathcal{M}(1) \to A \) with respect to \( \mathcal{U}_1 \). We can define the \((1,1)\)-minimal models \( \rho(r)^m : M^m(r)(1) \to A^m(r) (r = 1, 2, m \in \mathbb{N}) \) in the same way. We can define the isomorphisms

\[
\zeta^m_j : (p_j^m)^*\mathcal{M}(1) \xrightarrow{\sim} \mathcal{M}^m(1)(1) \quad (j = 1, 2, m \in \mathbb{N}),
\]

\[
\zeta^m_{jj'} : (p_{jj'}^m)^*\mathcal{M}(1)(1) \xrightarrow{\sim} \mathcal{M}^m(2)(1) \quad (1 \leq j < j' \leq 3, m \in \mathbb{N})
\]

by Proposition \( 4.3 \) and the conditions (1), (2), (3) follow also from Proposition \( 4.3 \). (We have the equality \( \zeta_1^0 = \zeta_2^0 \) because we also have the isomorphism \( \zeta : \mathcal{M}^0(1)(1) \to \mathcal{M}(1) \) compatible with \( \eta \) up to unique Sullivan homotopy and we see that \( \zeta \circ \zeta_1^0 = \zeta \circ \zeta_2^0 = \text{id} \) by Proposition \( 4.3 \).) The condition (4) for \( i = 1 \) is immediate because \( M(1)^1 \) (endowed with the stratification \( \{ (\zeta_1^m)^{-1} \circ \zeta_2^m \}_m \) is isomorphic to \( H^1(A) \) via \( \rho \), and this implies the condition (4) for general \( i \) because \( M(1)^i \)'s \( (i \in \mathbb{N}) \) are wedge products of \( M(1)^1 \). So the theorem holds for \( q = 1 \).
Next, we prove the theorem for general $q$, assuming the existence of $(1, q - 1)$-minimal models $\mathcal{M}(q - 1) \to A$, $\mathcal{M}^m(r)(q - 1) \to A^m(r)$ ($r = 1, 2, m \in \mathbb{N}$) and isomorphisms

$$
(p_j^m)^* \mathcal{M}(q - 1) \cong \mathcal{M}^m(1)(q - 1) \quad (j = 1, 2, m \in \mathbb{N}),
$$

$$
(p_{j,j'}^m)^* \mathcal{M}(1)(q - 1) \cong \mathcal{M}^m(2)(q - 1) \quad (1 \leq j < j' \leq 3, m \in \mathbb{N})
$$
as in the statement of the theorem. (Note that $\mathcal{M}(q - 1)$ is locally free of finite rank on each degree and hence quasi-coherent.) Moreover, by minimality, for the second cohomology group of the mapping fiber we get the equality $H^2(\mathcal{M}(q - 1), \mathcal{A}) = \text{Ker}(H^2(\mathcal{M}(q - 1)) \to H^2(\mathcal{A}))$ and it is locally free by induction, because both $H^2(\mathcal{M}(q - 1)), H^2(\mathcal{A})$ belong to $\text{MIC}^n(S/k)$. So we can define the covering sieve of opens $\mathcal{U}_q$ in $S$ by

$$
\mathcal{U}_q := \{ U \in \mathcal{U}_{q-1} \mid H^2(\mathcal{M}(q - 1), A) \text{ is free on } U \}.
$$

Take $U \in \mathcal{U}_q$ affine and put $A := A(U)$, $\mathcal{M}(q - 1) := \mathcal{M}(q - 1)(U)$ and $R := \mathcal{O}_S(U)$. By the quasi-coherence of $\mathcal{A}$ and $\mathcal{M}(q - 1)$, $H^2(\mathcal{M}(q - 1), A)$ is free. Let

$$
\sigma : E := H^2(\mathcal{M}(q - 1), A) \to Z^2(\mathcal{M}(q - 1), A)
:= \{(m, a) \in M(q - 1)^2 \oplus A^1 \mid dm = 0, \rho'(m) = da\}
$$

(where $d$ denotes the differential of $M(q - 1)$ and $A$ and $\rho'$ denotes the map $M(q - 1) \to A$) be a section of the canonical projection $Z^2(\mathcal{M}(q - 1), A) \to H^2(\mathcal{M}(q - 1), A)$, and define $M(q)$ by $M(q) := M(q - 1) \otimes \bigwedge(E)$, with differential induced by that on $M(q - 1)$ and the map

$$
E \xrightarrow{\sigma} Z^2(\mathcal{M}(q - 1), A) \hookrightarrow M(q - 1)^2 \oplus A^1 \xrightarrow{\text{proj}} M(q - 1)^2.
$$

Also, define the morphism $M(q) \to A$ extending $M(q - 1) \to A$ as the one induced by the map

$$
E \xrightarrow{\sigma} Z^2(\mathcal{M}(q - 1), A) \hookrightarrow M(q - 1)^2 \oplus A^1 \xrightarrow{\text{proj}} A^1.
$$

Then $M(q)$ is a $(1, q)$-minimal model of $A$. By sheafifying as before using Propositions 4.2, we see that $M(q)$'s for every affine $U \in \mathcal{U}_q$ form a sheaf $\mathcal{M}(q)$ and that the morphisms $M(q) \to A$ for every affine $U \in \mathcal{U}_q$ form a ho-morphism of $(1, q)$-minimal model $\rho : \mathcal{M}(q) \to A$ with respect to $\mathcal{U}_q$. The $(1, q)$-minimal models

$$
\rho^m(r) : \mathcal{M}^m(r)(q) \to A^m(r) \quad (r = 1, 2, m \in \mathbb{N})
$$

are defined in the same way, and the isomorphisms

$$
\zeta_j^m : (p_j^m)^* \mathcal{M}(q) \cong \mathcal{M}^m(1)(q) \quad (j = 1, 2, m \in \mathbb{N}),
$$

$$
\zeta_{j,j'}^m : (p_{j,j'}^m)^* \mathcal{M}(1)(q) \cong \mathcal{M}^m(2)(q) \quad (1 \leq j < j' \leq 3, m \in \mathbb{N})
$$

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are defined by Proposition 4.3, and the conditions (1), (2), (3) follow also from Proposition 4.3.

Finally we check the condition (4). By construction, we have the commutative diagram

\[
\begin{array}{ccc}
E & \xrightarrow{\sigma} & Z^2(M(q - 1), A) \\
& \downarrow d & \downarrow \\
Z^2(M(q - 1)) & \rightarrow & H^2(M(q - 1))
\end{array}
\]

where the horizontal arrows other than \(\sigma\) are the canonical surjections, the right vertical line is exact the left vertical arrow \(d\) is the differential on \(M(q)\). Since \(E \cong M(q)^1/M(q - 1)^1\), the differential on \(M(q)\) induces the exact sequence

\[
0 \rightarrow M(q)^1/M(q - 1)^1 \rightarrow H^2(M(q - 1)) \rightarrow H^2(A),
\]

which is compatible with the stratification. Thus \(M(q)^1/M(q - 1)^1\) belongs to \(\text{MIC}^n(S/k)\). Hence so does \(M(q)^1\), and so \(M(q)^i\) because \(M(q)^i\)'s \((i \in \mathbb{N})\) are wedge products of \(M(q)^1\). So the proof of the theorem is finished.

\[\square\]

**Corollary 4.5.** Let the situation be as above. Then there exist unique 1-minimal models \(M = \bigcup_q M(q), M^m(r) = \bigcup_q M^m(r)(q)\) and isomorphisms

\[
\zeta^m_j : (p_j^m)^* \mathcal{M} \cong M^m(1) \quad (j = 1, 2, m \in \mathbb{N}),
\]

\[
\zeta^m_{j,j'} : (p_{j,j'}^m)^* \mathcal{M}^m(1) \cong M^m(2) \quad (1 \leq j, j' \leq 3, m \in \mathbb{N})
\]

which satisfy the conditions (1), (2), (3) in Theorem 4.4 and the following condition:

(4) With respect to the structure of stratification defined in (3), the degree \(i\) part \(\mathcal{M}^i\) of \(\mathcal{M}\) belongs to the ind-category of \(\text{MIC}^n(S/k)\).

**Proof.** The assertion immediately follows from Theorem 4.4 by taking the union with respect to \(q\).

\[\square\]

Now we apply Corollary 4.5 to our geometric situation and give the third definition of relatively unipotent de Rham fundamental group. Let the notations be as in Notation 1.14 and we use the methods and definitions in of [NA87]. Let \(X = \bigcup_{i \in I} U_i\) be an affine open covering of \(X\) with \(I\) is finite. Fix an order on \(I\) and for \(n \in \mathbb{N}\), define \(X_n\) by \(X_n := \prod_{i_0 < \cdots < i_n \in I} U_{i_0} \cap \cdots \cap U_{i_n}\). Then \(X_n\)'s form a strict simplicial scheme \(X_\bullet\) and we have a morphism \(\pi : X_\bullet \rightarrow X\). Let \(\pi_S : S_\bullet \rightarrow S\) be
the pull-back of \( \pi \) by \( \iota \). Then the section \( \iota \) induces a surjective morphism of strict cosimplicial sheaves of \( \mathcal{O}_S \)-dgca’s

\[
\mathcal{A}_{\bullet \ast} := \pi_{\ast} \Omega_{X/S} \xrightarrow{\iota} \pi_{\ast} \mathcal{O}_S.
\]

By applying the Thom-Whitney functor \( \mathbf{s}_{\text{TW}} \) ([NA87 §3]), we obtain the surjective morphism of sheaves of \( \mathcal{O}_S \)-dgca’s \( \mathbf{s}_{\text{TW}}(\iota^\ast) : \mathbf{s}_{\text{TW}}(\mathcal{A}_{\bullet \ast}) \to \mathbf{s}_{\text{TW}}(\pi_{\ast} \mathcal{O}_S) \) such that \( H^0(\mathbf{s}_{\text{TW}}(\pi_{\ast} \mathcal{O}_S)) = \mathcal{O}_S \).

**Definition 4.6.** Let the notations be as above. We define the sheaf of \( \mathcal{O}_S \)-dgca’s \( \mathcal{A}_{X/S} \subseteq \mathbf{s}_{\text{TW}}(\mathcal{A}_{\bullet \ast}) \) as the inverse image of \( \mathcal{O}_S = H^0(\mathbf{s}_{\text{TW}}(\pi_{\ast} \mathcal{O}_S)) \subseteq \mathbf{s}_{\text{TW}}(\pi_{\ast} \mathcal{O}_S) \) by \( \mathbf{s}_{\text{TW}}(\iota^\ast) \).

Then, \( \mathcal{A}_{X/S} \) is in fact a sheaf of augmented \( \mathcal{O}_S \)-dgca’s.

If we denote the functor which associates the single complex to a strict cosimplicial complex by \( \mathbf{s} \), we have functorial quasi-isomorphisms ([NA87 Thm. 3.3])

\[
\mathcal{A}_{X/S} \to \mathbf{s}_{\text{TW}}(\mathcal{A}_{\bullet \ast}) \to \mathbf{s}(\mathcal{A}_{\bullet \ast}).
\]

Note that each term in \( \mathcal{A}_{\bullet \ast} \) is quasi-coherent. Then we can conclude that \( \mathbf{s}_{\text{TW}}(\mathcal{A}_{\bullet \ast}) \) is quasi-coherent in each degree: in fact, our covering \( \mathcal{X} = \bigcup_{i \in I} U_i \) is finite and so the end functor which is used to define \( \mathbf{s}_{\text{TW}} \) involves finitely many nonzero quasi-coherent sheaves. For the same reason, \( \mathbf{s}_{\text{TW}}(\pi_{\ast} \mathcal{O}_S) \) is quasi-coherent in each degree. Hence \( \mathcal{A}_{X/S} \) is quasi-coherent in each degree.

\( H^i(\mathbf{s}(\mathcal{A}_{\bullet \ast})) \) is nothing but \( R^i \mathcal{f}_{\text{dR}*}(\mathcal{O}_X, d) \), we have the functorial isomorphism

\[
H^i(\mathcal{A}_{X/S}) \xrightarrow{\sim} R^i \mathcal{f}_{\text{dR}*}(\mathcal{O}_X, d).
\]

Let \( p_j^m : S^m(1) \to S, q_j^m : X_j^m \to S^m(1), f_j^m : X_j^m \to S^m(1), \hat{f}^m : \hat{X}^m \to S^m(1), \hat{q}_j^m : \hat{X}^m \to X \) be as in Section 1, before Definition 1.8 and before Proposition 1.10 and let \( \tilde{\gamma} : S^m(1) \to \hat{X}^m \) be the section of \( \hat{f}^m \) induced by \( \iota \). Also, let \( (X_j^m) \) be \( X \times_X X_j^m \), let \( \hat{X}_j^m \) be the log formal tube of \( X_j^m \) in \( X \times_S \mathcal{O}_S(\mathcal{O}_S(1)) \) and let \( S^m(1) \) be the inverse image of \( \hat{X}_j^m \to \hat{X}^m \) by \( \tilde{\gamma} \). Then the section \( \tilde{\gamma} \)

induces the surjective morphism of strict cosimplicial sheaves of \( \mathcal{O}_{S^m(1)} \)-dgca’s

\[
f_{\ast} \pi_{\ast} \Omega_{\hat{X}_j^{m}/S^m(1)} \to \pi_{\ast} \mathcal{O}_{S^m(1)}.
\]

and we can define the augmented sheaf of \( \mathcal{O}_{S^m(1)} \)-dgca’s from it, which we denote by \( \mathcal{A}_{X/S^m(1)} \). Working on \( S^m(2) \) and considering the formal log tube in triple products, we can define the augmented sheaf of \( \mathcal{O}_{S^m(2)} \)-dgca’s in a similar way, which we denote by \( \mathcal{A}_{X/S^m(2)} \). By functoriality of the construction, we have the canonical morphisms

\[
\eta : \mathcal{A}_{X/S^m(1)} \to \mathcal{A}_{X/S},
\]

\[
\{ \eta_{j}^m : (p_j^m)^{\ast} \mathcal{A}_{X/S} \to \mathcal{A}_{X/S^m(1)} \}_{m} \quad (j = 1, 2),
\]

\[
\{ \eta_{j,j'}^m : (p_{j,j'}^m)^{\ast} \mathcal{A}_{X/S^m(1)} \to \mathcal{A}_{X/S^m(2)} \}_{m} \quad (1 \leq j < j' \leq 3)
\]

\(^1\text{Navarro-Aznar uses directly} \mathbf{s}_{\text{TW}}(\mathcal{A}_{\bullet \ast}), \text{pretending it is augmented over} \mathcal{O}_S \text{ ([NA93 §6]). Because it is not the case in general, we need to make this construction.}

\(^{55}\text{Navarro-Aznar uses directly} \mathbf{s}_{\text{TW}}(\mathcal{A}_{\bullet \ast}), \text{pretending it is augmented over} \mathcal{O}_S \text{ ([NA93 §6]). Because it is not the case in general, we need to make this construction.}
with $\eta \circ \eta_1^0 = \eta \circ \eta_2^0 = \text{id}$ which fit into the commutative diagram like (1.4). Moreover, $\eta$, $\eta_j^m$’s are quasi-isomorphisms because the morphisms of cohomologies

$$H^i(\eta_j^m) : (p_j^m)^*H^i(\mathcal{A}_{X/S}) \longrightarrow H^i(\mathcal{A}_{X/S^{m(1)}})$$

(note that $p_j^m$’s are flat) are identified with the isomorphisms

$$p_j^m R^i f_{\text{dR}*}(\mathcal{O}_X, d) \cong R^i \hat{f}_{\text{dR}*}(\mathcal{O}_{\mathcal{X}_m}, d)$$

via (4.5) and the variant of it for $\mathcal{A}_{X/S^{m(1)}}$. By the same reason, $\eta_j^m, j$’s are also quasi-isomorphisms. Thus the composition

$$H^i(\eta_1^m)^{-1} \circ H^i(\eta_2^m) : (p_2^m)^*H^i(\mathcal{A}_{X/S}) \cong H^i(\mathcal{A}_{X^{m(1)}}) \cong (p_1^m)^*H^i(\mathcal{A}_{X/S}) \quad (m \in \mathbb{N})$$

defines a stratification on $H^i(\mathcal{A}_{X/S})$. Moreover, since it is identified with the composition

$$p_2^m R^i f_{\text{dR}*}(\mathcal{O}_X, d) \cong R^i \hat{f}_{\text{dR}*}(\mathcal{O}_{\mathcal{X}_m}, d) \cong p_1^m R^i f_{\text{dR}*}(\mathcal{O}_X, d) \quad (m \in \mathbb{N})$$

giving the Gauss-Manin connection on $R^i f_{\text{dR}*}(\mathcal{O}_X, d)$, $H^i(\mathcal{A}_{X/S})$ belongs to $\text{MIC}^m(S/k)$. Therefore, our objects fit the hypotheses in Corollary 4.5 and so there exists a unique 1-minimal model $M_{X/S}$ of $\mathcal{A}_{X/S}$ endowed with the stratification

$$(\zeta_1^m)^{-1} \circ \zeta_2^m : (p_2^m)^*M_{X/S} \cong (p_1^m)^*M_{X/S}$$

with respect to which $M_{X/S}^i (i \in \mathbb{N})$ belongs to the ind-category of $\text{MIC}^m(S/k)$ and the differentials $M_{X/S}^i \longrightarrow M_{X/S}^{i+1}$ are morphisms in the ind-category of $\text{MIC}^m(S/k)$ because $\zeta_j^m (j = 1, 2)$ are morphisms of sheaves of dgca’s. In particular, each $M_{X/S}^i$ is flat over $\mathcal{O}_S$. Note that $M_{X/S}$ is the union of (1, $q$)-minimal models, which we denote by $M_{X/S}(q)$. (In particular each $M_{X/S}(q)^i$ belongs to $\text{MIC}^m(S/k)$.)

Now put $(M_{X/S}^1)^\vee := \lim_{\rightarrow q}(M_{X/S}(q)^1)^\vee$ and denote the isomorphism

$$p_2^m(M_{X/S}^1)^\vee \cong p_1^m(M_{X/S}^1)^\vee$$

induced by $(\zeta_1^m)^{-1} \circ \zeta_2^m$ by $\epsilon^m$. Then the pair $((M_{X/S}^1)^\vee, \{\epsilon^m\})$ is an object in the pro-category of $\text{MIC}^m(S/k)$. The minus of the differential $-d : M_{X/S}^1 \longrightarrow M_{X/S}^2 = M_{X/S}^1 \wedge M_{X/S}^1$ endows it with a structure of pro-nilpotent Lie algebra in the pro-category of $\text{MIC}^m(S/k)$; $(M_{X/S}(q)^1)^\vee$ is a pro-nilpotent Lie algebra for all $q$. (The minus sign comes from the fact that $M_{X/S}$ is in degree 1 in the complex $M_{X/S}$, while we regard $M_{X/S}^1$ as a Lie coalgebra sitting in degree 0.) So we can give the third definition of relatively unipotent de Rham fundamental groups in the following way:

**Definition 4.7 (Third definition of relatively unipotent $\pi_1$).** Let the notations be as above. We define the relatively unipotent de Rham fundamental group $\pi_1^{\text{dR}}(X/S, \iota)$ as the pro-unipotent algebraic group in $\text{MIC}^m(S/k)$ corresponding to the pro-nilpotent Lie algebra $((M_{X/S}^1)^\vee, \{\epsilon^m\})$. 

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Then we have the following comparison theorem.

**Theorem 4.8.** The first and the third definitions of \( \pi_{1}^{dR}(X/S, \iota) \) are canonically isomorphic.

The proof of this theorem will be given in Section 5.

**Remark 4.9.** We can apply to the morphism \( X_s \to s \) the same construction we used in order to define the sheaf of augmented \( \mathcal{O}_S \)-dgca’s \( \mathcal{A}_{X/S} \). So we can define the augmented \( k \)-dgca \( \mathcal{A}_{X/s} \). Then, we can define the 1-minimal model of it, which we denote by \( M_{X/s} = \bigcup_q M_{X/s}(q) \). (To define \( M_{X/s} \), the classical theory of 1-minimal model in [GM13, §13] is enough and we do not need the relative version.)

Because we have a family of ho-morphisms \( \{ M_{X/S}(q) \to \mathcal{A}_{X/S} \}_q \), there exists a decreasing sequence of affine open subschemes \( U_q \) \( (q \in \mathbb{N}) \) containing \( s \) such that the above ho-morphisms induce morphisms \( M_{X/S}(q')(U_q) \to \mathcal{A}_{X/S}(U_q) \) \( (q', q \in \mathbb{N}, q' \leq q) \) of \( (1, q') \)-minimal models which is compatible with respect to \( q' \). By taking tensor product with respect to the map \( \mathcal{O}_S(U_q) \to k \) induced by \( s \) and composing it with the map \( \mathcal{A}_{X/S} \otimes_{\mathcal{O}_S, s^*} k \to \mathcal{A}_{X/s} \) \( (\text{where } s^* \text{ denotes the map } \mathcal{O}_S \to \mathcal{O}_s = k \text{ induced by } s) \), we obtain the compatible family of morphisms \( M_{X/S}(q') \otimes_{\mathcal{O}_S, s^*} k \to \mathcal{A}_{X/s} \) \( (1 \leq q' \leq q) \). We can prove that the morphism \( M_{X/S}(q) \otimes_{\mathcal{O}_S, s^*} k \to \mathcal{A}_{X/s} \) we obtained is the \( (1, q) \)-minimal model of \( \mathcal{A}_{X/s} \). To see it, it suffices to prove that \( H^i(M_{X/S}(q') \otimes_{\mathcal{O}_S, s^*} k) \to H^i(\mathcal{A}_{X/s}) \) is an isomorphism for \( i = 0, 1, 1 \leq q' \leq q \) and that \( H^2(M_{X/S}(q') \otimes_{\mathcal{O}_S, s^*} k, \mathcal{A}_{X/s}) \to H^2(M_{X/S}(q') \otimes_{\mathcal{O}_S, s^*} k, \mathcal{A}_{X/s} \otimes_{\mathcal{O}_S, s^*} k) \to H^2(M_{X/S}(q') \otimes_{\mathcal{O}_S, s^*} k, \mathcal{A}_{X/s}) \) is the zero map. Since each term of \( M_{X/S}(q') \) are objects in \( \text{MIC}^n(S/k) \), they and their cohomologies are locally free.

Hence we have \( H^i(M_{X/S}(q') \otimes_{\mathcal{O}_S, s^*} k) = H^i(M_{X/S}(q')) \otimes_{\mathcal{O}_S, s^*} k = H^i(\mathcal{A}_{X/s}) \otimes_{\mathcal{O}_S, s^*} k \) for \( i = 0, 1 \). Since \( H^i(\mathcal{A}_{X/S}) \) (resp. \( H^i(\mathcal{A}_{X/s}) \)) is the (relative) de Rham cohomology \( R^i f_{\text{dR}}^!(\mathcal{O}_X, d) \) (resp. \( R^i f_{\text{dR}}(\mathcal{O}_X, d) \)), we have the base change isomorphism \( H^i(\mathcal{A}_{X/S}) \otimes_{\mathcal{O}_S, s^*} k = H^i(\mathcal{A}_{X/s}) \) and so the first condition to be the \( (1, q) \)-minimal model is proved. Because the first condition is fulfilled, we have

\[
H^2(M_{X/S}(q') \otimes_{\mathcal{O}_S, s^*} k, \mathcal{A}_{X/s}) = \text{Ker}(H^2(M_{X/S}(q') \otimes_{\mathcal{O}_S, s^*} k)) \to H^2(\mathcal{A}_{X/s}),
\]

and this is the base change of \( H^2(M_{X/S}(q'), \mathcal{A}_{X/s}) = \text{Ker}(H^2(M_{X/S}(q')) \to H^2(\mathcal{A}_{X/s})) \) by its local freeness as above. So the second condition to be the \( (1, q) \)-minimal model follows from that for \( M_{X/S}(q')(U_q) \) \( (1 \leq q' \leq q) \).

So, by Proposition 4.2, we have a unique isomorphism of \( (1, q) \)-minimal models \( M_{X/S}(q) \otimes_{\mathcal{O}_S, s^*} k \to M_{X/s}(q) \). We can do the same construction with \( U_q \) replaced by \( U_{q+1} \), and we obtain another isomorphism \( M_{X/S}(q) \otimes_{\mathcal{O}_S, s^*} k \to M_{X/s}(q) \), which is compatible with the restriction. By construction, the latter isomorphism is the restriction of the isomorphism \( M_{X/S}(q+1) \otimes_{\mathcal{O}_S, s^*} k \to M_{X/s}(q+1) \) to \( M_{X/S}(q) \). So we can take the union of this isomorphisms with respect to \( q \) and obtain the isomorphism \( M_{X/S} \otimes_{\mathcal{O}_S, s^*} k \to M_{X/s} \). In other words, \( M_{X/s} \) is canonically isomorphic to the image of \( M_{X/S} \) by the ind-version of the restriction functor \( \text{MIC}^n(X/S) \to \text{MIC}(s/s) \).
Remark 4.10. The definition of the connection on \((M^1_{X/S})^\vee\) given above is not the same as the original one of Navarro-Aznar in [NA93, §5, §6]. We expect that his definition is the same as our definition in Definition 4.7, but we will not pursue this topic in this paper.

5 Relative bar construction

In this section, we give the fourth definition of relatively unipotent de Rham fundamental groups by using the relative version of bar construction. Then we prove the coincidence of this definition with the first one given in Section 2. The proof of Theorem 4.8 will be achieved by showing that the fourth definition is equivalent to the third one. This will complete the circle: all four definitions are equivalent.

First we give a brief explanation on bar construction in relative setting: the construction is the one given in [Hai87], but we work over an arbitrary commutative ring \(R\). (In [Hai87], \(R\) is assumed to be a field.) Let \(M\) be an augmented \(R\)-dgca. Assume that, for any \(i \in \mathbb{N}\), the degree \(i\) part \(M^i\) of \(M\) and the \(i\)-th cohomology \(H^i(M)\) are flat \(R\)-modules and that \(H^0(M) = R\). (The flatness assumption here is automatic in the situation in [Hai87], but we need to impose this assumption in order that the K"unneth formula, e.g. the middle equality in (5.1) below, holds in our setting.) Let \(\overline{M} := \text{Ker}(M \to R)\) be the augmentation ideal.

For \(s, t \in \mathbb{N}\), let \(B^{-s,t}(M)\) be the set of elements in \(\bigotimes R^s \overline{M}\) which are homogeneous of degree \(t\). We can define the combinatorial differential \(d_C : B^{-s,t}(M) \to B^{-s+1,t}(M)\) and the internal differential \(d_I : B^{-s,t}(M) \to B^{-s,t+1}(M)\) satisfying \(d_Cd_I + d_Id_C = 0\), as in [Hai87, p. 275]. Then we define \(B(M)\) as the single complex associated to \((B^{-s,t}(M), d_C, d_I)\).

We define the bar filtration \(F^{-s}B(M)\) by \(F^{-s}B(M) := \bigoplus_{u \leq s} B^{-u,v}(M)\). The spectral sequence associated to it is called the Eilenberg-Moore spectral sequence. The \(E_1\)-term of the Eilenberg-Moore spectral sequence is written as

\[
E_1^{-s,t} = H^t(\bigotimes_R^s \overline{M}) = (\text{degree } t \text{ part of } \bigotimes_R^s H^\bullet(\overline{M})) = B^{-s,t}(H^\bullet(M)).
\]

We have the morphisms

\[
i : B(R) \to B(M), \quad e : B(M) \to B(R)
\]

induced by the inclusion \(R \hookrightarrow M\) and the augmentation \(M \to R\), and we can define the morphisms

\[
\wedge : B(M) \otimes_R B(M) \to B(M), \quad \Delta : B(M) \to B(M) \otimes_R B(M)
\]
called the shuffle product and the diagonal ([Hai87, pp. 275–276]). These morphisms induce morphisms on 0-th cohomology

\[ i_H : R \rightarrow H^0(B(M)), \quad e_H : H^0(B(M)) \rightarrow R, \]

\[ \wedge_H : H^0(B(M)) \otimes_R H^0(B(M)) \rightarrow H^0(B(M)), \]

\[ \Delta_H : H^0(B(M)) \rightarrow H^0(B(M)) \otimes_R H^0(B(M)) \]

and we can check that \( H^0(B(M)) := (H^0(B(M)), i_H, e_H, \wedge_H, \Delta_H) \) forms a Hopf algebra over \( R \).

Now we go back to our geometric situation. Assume that we are in the framework of Notation [1.14]. Let \( \mathcal{A}_{X/S}, \mathcal{A}_{X/S^{m(r)}} (r = 1, 2, m \in \mathbb{N}) \) be the sheaf of augmented dgca’s defined in the previous section (Definition 4.6 and what follows), and let \( M_{X/S}, M_{X/S^{m(r)}} \) be their 1-minimal models. Recall that we also have isomorphisms

\[ \zeta_j^m : (p_j^m)^*M_{X/S} \xrightarrow{\cong} M_{X/S^{m(1)}} \quad (j = 1, 2, m \in \mathbb{N}), \]

\[ \zeta_{j,j'}^m : (p_{j,j'}^m)^*M_{X/S^{m(1)}} \xrightarrow{\cong} M_{X/S^{m(2)}} \quad (1 \leq j, j' \leq 3, m \in \mathbb{N}) \]

with \( \zeta_1^0 = \zeta_2^0 \) which fits into a commutative diagram like (1.4). In particular, the composite

\[ (\zeta_1^m)^{-1} \circ \zeta_2^m : (p_2^m)^*M_{X/S} \xrightarrow{\cong} (p_1^m)^*M_{X/S} \]

defines a stratification, and we saw in the previous section that \( M_{X/S}^i (i \in \mathbb{N}) \) belongs to the ind-category of \( \text{MIC}^n(S/k) \). Also, \( H^i(M_{X/S}) = R^if_{dR*}(O_X, d) (i \in \mathbb{N}) \) belongs to \( \text{MIC}^n(S/k) \). Hence the conditions required for performing the bar construction above are satisfied for \( M_{X/S} \) and \( M_{X/S^{m(r)}} \). Therefore, we can define the 0-th cohomologies of bar constructions \( H^0(B(M_{X/S})), H^0(B(M_{X/S^{m(r)}})) \), and we have the isomorphisms

\[ H^0(B(\zeta_j^m)) : (p_j^m)^*H^0(B(M_{X/S})) \xrightarrow{\cong} H^0(B(M_{X/S^{m(1)}})) \quad (j = 1, 2, m \in \mathbb{N}), \]

\[ H^0(B(\zeta_{j,j'}^m)) : (p_{j,j'}^m)^*H^0(B(M_{X/S^{m(1)}})) \xrightarrow{\cong} H^0(B(M_{X/S^{m(2)}})) \quad (1 \leq j, j' \leq 3, m \in \mathbb{N}) \]

which fit into a commutative diagram like (4.4). Hence the composite

\[ H^0(B(\zeta_1^m))^{-1} \circ H^0(B(\zeta_2^m)) : (p_2^m)^*H^0(B(M_{X/S})) \xrightarrow{\cong} (p_1^m)^*H^0(B(M_{X/S})) \]

defines a stratification on \( H^0(B(M_{X/S})) \). Moreover, the Eilenberg-Moore spectral sequences for \( B(M_{X/S}), B(M_{X/S^{m(r)}}) \) are compatible with the maps \( B(\zeta_j^m), B(\zeta_{j,j'}^m) \). Since their \( E_1 \)-terms are objects in \( \text{MIC}^n(S/k) \), the Eilenberg-Moore spectral sequences can be understood as a spectral sequence in the ind-category of \( \text{MIC}^n(S/k) \). Thus we see that \( H^0(B(M_{X/S})) \) is a Hopf algebra object in the ind-category of \( \text{MIC}^n(S/k) \). Using this, we give the fourth definition of relatively unipotent de Rham fundamental groups.
Definition 5.1 (Fourth definition of relatively unipotent $\pi_1$). Let the notations be as above. Then we define the relatively unipotent de Rham fundamental group $\pi_{1}^{\text{dR}}(X/S, \iota)$ by

$$\pi_{1}^{\text{dR}}(X/S, \iota) := \text{Spec} \, H^{0}(B(M_{X/S})), $$

which is a group scheme in $\text{MIC}^{n}(S/k)$.

Remark 5.2. One may be tempted to consider the bar construction of dgca’s $A_{X/S}, A_{X/S}^{m}(r)$. Note that we cannot perform the same construction as before because we do not know the flatness of $A_{X/S}, A_{X/S}^{m}(r)$.

Remark 5.3. We can also define the bar construction $B(M_{X/S}, A_{X/S}^{m}(r))$ of the minimal augmented $k$-dgca $M_{X/S}$ associated to the morphism $X_{s} \longrightarrow s$, and $H^{0}(B(M_{X/s}))$ has a structure of Hopf algebra over $k$. Note that each cohomology group $H^{n}(B(M_{X/S})(n \in \mathbb{N}))$ has a structure of an object in the ind-category of $\text{MIC}^{n}(S/k)$ defined by the composite $H^{n}(B(\zeta^{m}))/2 \cdot H^{n}(B(\zeta_{2}^{m}))$, hence it is flat over $O_{S}$. This fact and the flatness of each term of $B(M_{X/S})$ over $O_{S}$ imply that the canonical morphism

$$H^{0}(B(M_{X/S})) \otimes_{O_{S}, s^{\ast}} k \longrightarrow H^{0}(B(M_{X/s}))$$

(where $s^{\ast} : O_{S} \longrightarrow O_{s} = k$ is the morphism induced by $s$) is an isomorphism. In other words, $H^{0}(B(M_{X/s}))$ is the image of $H^{0}(B(M_{X/S}))$ by the ind-version of the restriction functor $\text{MIC}^{n}(S/k) \longrightarrow \text{MIC}(s/s)$.

Then we have the following comparison result.

Theorem 5.4. The first and the fourth definitions of $\pi_{1}^{\text{dR}}(X/S, \iota)$ are canonically isomorphic.

Before proving Theorem 5.4, we give a proof of Theorem 4.8 assuming Theorem 5.4

Proof of Theorem 4.8 assuming Theorem 5.4. We prove that the pro-nilpotent Lie algebra in the pro-category of $\text{MIC}^{n}(S/k)$ associated to $\text{Spec} \, H^{0}(B(M_{X/S}))$ is naturally isomorphic to $(M_{X/S}^{V})$. We put

$$QH^{0}(B(M_{X/S})) := \text{Ker}(e_{H})/ \wedge_{H} (\text{Ker}(e_{H}) \otimes \text{Ker}(e_{H})), $$

which is called the module of indecomposables of $H^{0}(B(M_{X/S}))$. Then it has a structure of a Lie coalgebra

$$QH^{0}(B(M_{X/S})) \longrightarrow \bigwedge^{2} QH^{0}(B(M_{X/S}))$$

induced by the map

$$\Delta_{H} - \tau \circ \Delta_{H} : H^{0}(B(M_{X/S})) \longrightarrow H^{0}(B(M_{X/S})) \otimes H^{0}(B(M_{X/S})), $$
where $\tau$ is the endomorphism on $H^0(B(M_X/S)) \otimes H^0(B(M_X/S))$ defined by $x \otimes y \mapsto y \otimes x$. On the other hand, the minus of the differential $-d : M^1_{X/S} \rightarrow M^2_{X/S} = \Lambda^2 M^1_{X/S}$ defines a structure of Lie coalgebra on $M^1_{X/S}$. Then it suffices to prove that these two Lie coalgebras (in the ind-category of $\text{MIC}^n(S/k)$) are isomorphic, because the Lie algebra associated to the group scheme $\text{Spec} H^0(B(M_X/S))$ (in the ind-category of $\text{MIC}^n(S/k)$) is the dual of $QH^0(B(M_X/S))$.

One can check that the projection $B(M_X/S) \rightarrow B(M_X/S)^{-1,1} = M^1_{X/S}[1]$ (where $[1]$ denotes the shift of the degree by 1) induces the morphism $QH^0(B(M_X/S)) \rightarrow M^1_{X/S}$ of Lie coalgebras on $S$, and since this construction is functorial, this is actually a morphism of Lie coalgebra objects in the ind-category of $\text{MIC}^n(S/k)$. Therefore, to prove that it is isomorphic, it suffices to prove that the restriction of this morphism to $\text{MIC}(S/s)$, which is the morphism $QH^0(B(M_{X/s}/s)) \rightarrow M^1_{X/s}$, is isomorphic, because the functor $\text{MIC}^n(S/k) \rightarrow \text{MIC}(s/s)$ is a fiber functor, hence exact and faithful. This is proven by Bloch-Kriz ([BK94, Theorem 2.30], see also [Hai87, Theorem 2.6.2]). So we are done.

**Remark 5.5.** The definition of Lie coalgebra on $QH^0(B(M_X/S))$ follows [Hai87, 2.6] (hence follows [Hai84, (6.24)]), because the one given in [BK94, Lemma 2.29(a)] is not correct. (The coproduct $\Delta_H$ itself does not induce the Lie coalgebra structure.) Also, our definition of the Lie coalgebra structure on $M^1_{X/S}$ involves the minus sign. We can check that the morphism $QH^0(B(M_X/S)) \rightarrow M^1_{X/S}$ in the proof is compatible with these corrected definitions.

To prove Theorem 5.4, we define the category of $A_{X/S}$-connections and prove some of its basic properties. Let $\text{MIC}(A_{X/S})$ be the category of locally free modules with integrable $A_{X/S}$-connection, that is, the category of pairs $(E, \nabla)$ consisting of a locally free $A^0_{X/S}$-module $E$ of finite rank and an integrable connection $\nabla : E \rightarrow E \otimes A^0_{X/S} A^1_{X/S}$ (an additive map satisfying the Leibniz rule $\nabla(\omega) = a\nabla(e) + e \otimes da$ ($a \in A^0_{X/S}, e \in E$) such that the composite

$$E \xrightarrow{\nabla} E \otimes A^0_{X/S} A^1_{X/S} \xrightarrow{\nabla_1} E \otimes A^0_{X/S} A^2_{X/S},$$

where $\nabla_1$ is the map defined by $e \otimes \omega \mapsto \nabla(e) \wedge \omega + e \otimes d\omega$, is zero). We can also define the category $\text{MIC}(A_{X/S/SM(r)})$ for $r = 1, 2, m \in \mathbb{N}$. The categories $\text{MIC}(A_{X/S})$, $\text{MIC}(A_{X/S/SM(r)})$ are exact tensor categories in natural way. We have ‘the pullback functors’

$$f^* : \text{MIC}(S/S) \rightarrow \text{MIC}(A_{X/S}),$$

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\[ f^{m*} : \text{MIC}(S^m(r)/S^m(r)) \rightarrow \text{MIC}(\mathcal{A}_{X/S^m(r)}) \quad (r = 1, 2, m \in \mathbb{N}), \]
\[ q^{m*}_j : \text{MIC}(\mathcal{A}_{X/S}) \rightarrow \text{MIC}(\mathcal{A}_{X/S^m(1)}) \quad (j = 1, 2, m \in \mathbb{N}), \]
\[ q^{m*}_{j,j'} : \text{MIC}(\mathcal{A}_{X/S^m(1)}) \rightarrow \text{MIC}(\mathcal{A}_{X/S^m(2)}) \quad (1 \leq j < j' \leq 3, m \in \mathbb{N}), \]
\[ \eta^* : \text{MIC}(\mathcal{A}_{X/S^0(1)}) \rightarrow \text{MIC}(\mathcal{A}_{X/S}) \]

which are induced by the morphisms \( \mathcal{O}_S \rightarrow \mathcal{A}_{X/S}, \mathcal{O}_{S^m(r)} \rightarrow \mathcal{A}_{X/S^m(r)}, \mathcal{A}_{X/S} \rightarrow \mathcal{A}_{X/S^m(1)}, \mathcal{A}_{X/S^m(1)} \rightarrow \mathcal{A}_{X/S^m(2)} \) and \( \mathcal{A}_{X/S^0(1)} \rightarrow \mathcal{A}_{X/S} \) defined by \( f, f^m, \hat{q}^m_j \), the analogue of \( \hat{q}^m_j \) for the map \( q^m_{j,j'} \) and the diagonal \( X \rightarrow \hat{X}^0 \), respectively. These functors are exact tensor functors.

Let \( N_f \text{MIC}(\mathcal{A}_{X/S}) \) be the full subcategory of \( \text{MIC}(\mathcal{A}_{X/S}) \) whose objects are iterated extensions of objects in \( f^* \text{MIC}(S/S) \subseteq \text{MIC}(\mathcal{A}_{X/S}) \). It is also an exact tensor category in a natural way.

Also, we define a category \( \mathcal{E} \) and its full subcategory \( N_f \mathcal{E}^n \) which will play an important role in this section.

**Definition 5.6.** Let \( \mathcal{E} \) be the category of pairs \((E, \{e^n\})\) consisting of an object \( E \) in \( \text{MIC}(\mathcal{A}_{X/S}) \) and a compatible family of isomorphisms (called stratifications) \( \{e^n : q_2^m*E \rightarrow q_1^m*E\} \) in \( \text{MIC}(\mathcal{A}_{X/S^m(1)}) \) with \( e^0 = \text{id} \) which satisfies the cocycle condition \( q_{1,2}^m*(e^m) \circ q_{2,3}^m*(e^m) = q_{1,3}^m*(e^m) \) in \( \text{MIC}(\mathcal{A}_{X/S^m(2)})(m \in \mathbb{N}) \). Then the pullback functors \( f^*, f^{m*} \) above induce the pullback functor

\[ f^* : \text{MIC}(S/k) \cong \text{Str}(S/k) \rightarrow \mathcal{E}. \]

Let \( N_f \mathcal{E}^n \) be the full subcategory of \( \mathcal{E} \) whose objects are iterated extensions of objects in the essential image of the functor

\[ \text{MIC}^n(S/k) \hookrightarrow \text{MIC}(S/k) \xrightarrow{f^*} \mathcal{E}. \]

Note that \( N_f \mathcal{E}^n \) is also an exact tensor category in a natural way.

We call an object in \( N_f \text{MIC}(\mathcal{A}_{X/S}) \) (resp. \( N_f \mathcal{E}^n \)) is of index of unipotence \( \leq n \) if it can be written as an iterated extension of at most \( n \) objects in \( f^* \text{MIC}(S/S) \) (resp. \( f^* \text{MIC}^n(S/k) \)).

An object \((E, \nabla)\) in \( \text{MIC}(\mathcal{A}_{X/S}) \) naturally induces the de Rham complex \( E \otimes_{\mathcal{A}_{X/S}} \mathcal{A}_{X/S} \). We define the \( i \)-th relative de Rham cohomology \( R^if_*(E) \) of \((E, \nabla)\) by

\[ R^if_*(E) := H^i(E \otimes_{\mathcal{A}_{X/S}} \mathcal{A}_{X/S}), \]

and we put \( f_*(E) := R^0f_*(E) \). For \( E \in \text{MIC}(S/S) \) and \( E' \in \text{MIC}(\mathcal{A}_{X/S}) \), we have the projection formula

\[ R^if_*(f^*E \otimes E') = E \otimes R^if_*E'. \]

When \( E' = (\mathcal{A}_{X/S}^0, d) \), it is written as

\[ R^if_*(f^*E) = E \otimes H^i(\mathcal{A}_{X/S}) = E \otimes R^if_{dR*}(\mathcal{O}_X, d). \]
We define the $i$-th relative de Rham cohomology $R^if_*^nE$ for $E \in \text{MIC}(\mathcal{A}_{X/S^n(r)})$ in the same way, and the projection formula holds also in this case.

When we are given an object $(E, \{\epsilon^m\})$ in $\mathcal{E}$, we have functorial diagrams

$$
\begin{align*}
(5.3) & \quad p_2^{m*}R^if_*^nE \rightarrow R^if_*^nq_2^{m*}E \xrightarrow{R^if_*^n\epsilon^m} R^if_*^nq_1^{m*}E \leftarrow p_1^{m*}R^if_*^nE \quad (m \in \mathbb{N})
\end{align*}
$$

in which the second arrow is an isomorphism. We do not know whether the first and the third arrows are isomorphisms in general, but they are isomorphisms when $E$ belongs to $f^*\text{MIC}^n(S/k)$, by projection formula and the isomorphism \((\ref{tr17})\) for $(E, \nabla) = (\mathcal{O}_X, d)$. Hence we see by induction that the arrows in \((\ref{5.3})\) are isomorphisms when $(E, \{\epsilon^m\})$ belongs to $N_f\mathcal{E}^n$, and the diagrams \((\ref{5.3})\) induce the structure of an object in $\text{MIC}^n(S/k) \subseteq \text{MIC}(S/k) \cong \text{Str}(S/k)$. So we see that the functor $R^if_*$ induces the functor

$$
R^if_* : N_f\mathcal{E}^n \rightarrow \text{MIC}^n(S/k).
$$

When $i = 0$, we denote this functor by $f_*$. We have a morphism of functors $f^*f_* \rightarrow \text{id}$ from $N_f\mathcal{E}^n$ to itself, and we can check (by using projection formula) that, for any $E \in N_f\mathcal{E}^n$, $f^*f_*E \rightarrow E$ gives an injection onto the maximal subobject of $E$ which belongs to $f^*\text{MIC}^n(S/k)$.

**Proposition 5.7.** $N_f\mathcal{E}^n$ is a neutral Tannakian category.

**Proof.** We prove that $N_f\mathcal{E}^n$ is an abelian category. (Then it is easy to see that $N_f\mathcal{E}^n$ is a rigid tensor abelian category and a fiber functor is defined by the composition

$$
N_f\mathcal{E}^n \xrightarrow{\iota^*} \text{MIC}^n(S/k) \rightarrow \text{MIC}(s/s) = \text{Vec}_k
$$

of ‘the pullback functor’ $\iota^*$ by the augmentations $\mathcal{A}_{X/S} \rightarrow \mathcal{O}_S, \mathcal{A}_{X/S^n}= \mathcal{O}_{S^n}$ and the functor ‘taking fiber at $s$’.)

Let $\alpha : E \rightarrow F$ be a morphism in $N_f\mathcal{E}^n$ with $E$ (resp. $F$) of index of unipotence $\leq n$ (resp. $\leq m$). We prove that the kernel of $\alpha$ exists and it is of index of unipotence $\leq n$ and that the cokernel of $\alpha$ exists and it is of index of unipotence $\leq m$.

When $n = m = 1$, $E, F$ belong to $f^*\text{MIC}^n(S/k)$ and so $\alpha$ is equal to $f^*f_*\alpha : f^*f_*E \rightarrow f^*f_*F$. Because $f^*$ is exact on $\text{MIC}^n(S/k)$ and $\text{MIC}^n(S/k)$ is assumed to be abelian by condition (E) of \(\text{Notation 1.14}\) we see that $f^*\text{Ker}(f_*\alpha)$ is the kernel of $\alpha$ and that $f^*\text{Coker}(f_*\alpha)$ is the cokernel of $\alpha$. This shows the claim in this case.

Next we prove by induction the claim for general $n$ and $m = 1$. We have an exact sequence in $N_f\mathcal{E}^n$

$$
0 \rightarrow E' \rightarrow E \rightarrow E'' \rightarrow 0
$$

with $E'$ (resp. $E''$) of index of unipotence $\leq n - 1$ (resp. $\leq 1$). So we have the commutative diagram

$$
\begin{array}{ccc}
0 & \rightarrow & E' \\
\downarrow{\alpha'} & & \downarrow{\alpha} \\
0 & \rightarrow & F
\end{array}
$$
in $N_fE^n$ with horizontal lines exact. It induces the following exact sequence of $A^0_{X/S}$-modules:

$$0 \to \operatorname{Ker} \alpha' \to \operatorname{Ker} \alpha \to E'' \xrightarrow{\delta} \operatorname{Coker} \alpha' \to \operatorname{Coker} \alpha \to 0.$$  

By induction hypothesis, $\operatorname{Coker} \alpha'$ is an object in $N_fE^n$ of index of unipotence $\leq 1$. We see also, as above, that $\delta$ is enriched to a morphism in $N_fE^n$ by functoriality of the connecting map $\delta$: in fact, there are analogous diagrams and exact sequences of $A^0_{X/S(m(r))}$-modules which are compatible with respect to stratifications. Thus $\operatorname{Coker} \alpha$ is an object in $N_fE^n$ of index of unipotence $\leq 1$. Moreover, by induction hypothesis, the kernel of $\delta$ is an object in $N_fE^n$ of index of unipotence $\leq 1$ and $\operatorname{Ker} \alpha'$ is an object in $N_fE^n$ of index of unipotence $\leq n-1$. Thus Ker $\alpha$ is an object in $N_fE^n$ of index of unipotence $\leq n$. Hence the claim is proved in this case.

Finally we prove the claim in general case by induction on $m$. We have an exact sequence in $N_fE^n$

$$0 \to F' \to F \to F'' \to 0$$

with $F'$ (resp. $F''$) of index of nilpotence $\leq 1$ (resp. $\leq m-1$). So we have the commutative diagram

$$
\begin{array}{c}
0 \to 0 \to E \xrightarrow{\alpha} E \xrightarrow{\alpha'} 0 \\
0 \downarrow \downarrow \downarrow \downarrow \\
0 \to F' \to F \to F'' \to 0
\end{array}
$$

in $N_fE^n$ with horizontal lines exact. It induces the following exact sequence of $A^0_{X/S}$-modules:

$$0 \to \operatorname{Ker} \alpha \to \operatorname{Ker} \alpha' \xrightarrow{\delta} F' \to \operatorname{Coker} \alpha \to \operatorname{Coker} \alpha' \to 0.$$  

By induction hypothesis, Ker $\alpha'$ is an object in $N_fE^n$ of index of unipotence $\leq n$, and we see that $\delta$ is enriched to a morphism in $N_fE^n$ by functoriality of the connecting map $\delta$. Thus Ker $\alpha$ is an object in $N_fE^n$ of index of unipotence $\leq n$. Moreover, by induction hypothesis, the cokernel of $\delta$ is an object in $N_fE^n$ of index of unipotence $\leq 1$ and Coker $\alpha'$ is an object in $N_fE^n$ of index of unipotence $\leq m-1$. Thus Coker $\alpha$ is an object in $N_fE^n$ of index of unipotence $\leq m$. Hence the claim is proved, and so the proof of the proposition is finished.

By using the functors between Tannakian categories

$$f^* : \operatorname{MIC}^n(S/k) \to N_fE^n, \quad i^* : N_fE^n \to \operatorname{MIC}^n(S/k),$$

we can define the group scheme $G(N_fE^n, i^*)$ in $\operatorname{MIC}^n(S/k)$ by the method explained in Section 2.

Also, recall that we defined the augmented $k$-dgca $A_{X,s}$ in Remark [9]. Hence we can define the category $\operatorname{MIC}(A_{X,s})$ of locally free $A_{X,s}$-modules of finite rank.
endowed with integrable $A_{X,s}$-connection and the pullback functor $f^*_s : \text{MIC}(A_{X,s}/s) \to \text{MIC}(s/s)$ induced by the canonical morphism $k \hookrightarrow A_{X,s}$. Let $N_{f_s} \text{MIC}(A_{X,s}/s)$ be the full subcategory whose objects are iterated extensions of objects in $f^*_s \text{MIC}(s/s)$. An object $(E, \nabla)$ in $\text{MIC}(A_{X,s}/s)$ naturally induces the de Rham complex $E \otimes_{A_{X,s}} A_{X,s}$, and we define the $i$-th de Rham cohomology $R^i f_{ss} E$ of $(E, \nabla)$ by

$$R^i f_{ss} E := H^i(E \otimes_{A_{X,s}/s} A_{X,s}).$$

We put $f_{ss} E := R^0 f_{ss} E$. As in the previous case, for $E \in \text{MIC}(s/s)$ and $E' \in \text{MIC}(A_{X,s}/s)$, we have the projection formula

$$R^i f_{ss}(f^*_s E \otimes E') = E \otimes R^i f_{ss} E'$$

and when $E' = (A_{X,s}/s, d)$, it is written as

$$R^i f_{ss} f^*_s E = E \otimes H^i(A_{X,s}/s) = E \otimes R^i f_{ss} \text{dR}_s(\mathcal{O}_{X,s}, d).$$

We have a morphism of functors $f^*_s f_{ss} \to \text{id}$ from $N_{f_s} \text{MIC}(A_{X,s}/s)$ to itself, and we can check (by using projection formula) that, for any $E \in N_{f_s} \text{MIC}(A_{X,s}/s)$, $f^*_s f_{ss} E \to E$ gives an injection onto the maximal trivial subobject of $E$.

We have the restriction functor $N_f \mathcal{E}^n \xrightarrow{|X_s|} N_{f_s} \text{MIC}(A_{X,s}/s)$ induced by the morphism $A_{X/S} \to A_{X,s}$ defined by the closed immersion $X_s \hookrightarrow X$. By using the projection formula and induction on index of unipotence, we see that, for $E \in N_f \mathcal{E}^n$, there exists the base change isomorphism

$$\tag{5.5} (R^i f_{s} E)|_s \cong R^i f_{ss}(E|_{X_s}),$$

where we denote the restriction $\text{MIC}^n(S/k) \to \text{MIC}(s/s)$ by $|_s$. By the same reason, we have the similar base change isomorphism with respect to any morphism $S' \to S$ of finite type for $E \in N_f \mathcal{E}^n$.

We can check in the same way as the proof of Proposition 5.7 that $N_{f_s} \text{MIC}(A_{X,s}/s)$ is a neutral Tannakian category with the fiber functor $x^* : N_{f_s} \text{MIC}(A_{X,s}/s) \to \text{Vec}_k$ induced by the augmentation map $A_{X,s} \to k$. So we can define the Tannakian dual of $(N_{f_s} \text{MIC}(A_{X,s}/s), x^*)$, which we denote by $G(N_{f_s} \text{MIC}(A_{X,s}/s), x^*)$. Moreover, since the composite of pullback functors

$$\text{MIC}^n(S/k) \xrightarrow{f^*} N_f \mathcal{E}^n \xrightarrow{|X_s|} N_{f_s} \text{MIC}(A_{X,s}/s)$$

factors through $\text{MIC}(s/s) = \text{Vec}_k$, the functor $|_{X_s}$ induces the morphism

$$\tag{5.6} G(N_{f_s} \text{MIC}(A_{X,s}/s), x^*) \to s^*_{\text{dR}} G(N_f \mathcal{E}^n, x^*).$$

Then we have the following:

**Proposition 5.8.** The morphism (5.6) is an isomorphism.
Proof. We imitate the proof of Proposition 2.3. It suffices to prove the following three claims.

(1) Any object in $N_f \text{MIC}(\mathcal{A}_{X/s})$ is a quotient of an object of the form $E|_{X_s}$ ($E \in N_f \mathcal{E}^n$).

(2) For any object $E$ in $N_f \mathcal{E}^n$ with $E|_{X_s}$ trivial, there exists an object $V$ in $\text{MIC}^n(S/k)$ with $E = f^*V$.

(3) Let $E$ be an object in $N_f \mathcal{E}^n$ and let $F_0 \in N_f \text{MIC}(\mathcal{A}_{X/s})$ be the largest trivial subobject of $E|_{X_s}$. Then there exists a subobject $E_0$ of $E$ with $F_0 = E_0|_{X_s}$.

First we prove the claim (2). By restricting the map $f^*E \to E$ to $\text{MIC}(\mathcal{A}_{X/s})$ and using base change property (5.5), we obtain the morphism

$$f^*_s f_{ss}(E|_{X_s}) \cong f^*_s((f^*_sE)|_s) = (f^*_sE)|_{X_s} \to E|_{X_s},$$

which is an isomorphism due to the triviality of $E|_{X_s}$. Thus the map $f^*_sE \to E$ is an isomorphism and so the claim holds if we put $V := f^*_sE$.

Next we prove the claim (3). If $E$ and $F_0 \subseteq E|_{X_s}$ be as in the statement of the claim, we have $F_0 = f^*_s f_{ss}(E|_{X_s})$. So, by the base change property (5.5), we obtain the isomorphism

$$F_0 = f^*_s f_{ss}(E|_{X_s}) \cong f^*_s((f^*_sE)|_s) = (f^*_sE)|_{X_s},$$

So the claim holds if we put $E_0 = f^*_s f_{ss}E$.

Finally, in order to prove the claim (1), it suffices to construct a projective system of objects $\{W_n\}_n$ in $N_f \mathcal{E}^n$ such that, for any object $F$ in $N_f \text{MIC}(\mathcal{A}_{X/s})$, there exist $n, N \in \mathbb{N}$ and a surjection $W_n^\oplus \to F$. This will be done in Proposition 5.9 and Remark 5.10 below.

Proposition 5.9. There exists a projective system $(W, e) := \{(W_n, e_n)\}_{n \geq 1}$ consisting of

- $W_n := (\overline{W}_n, \{e_n^m\}_m) \in N_f \mathcal{E}^n$, and
- A morphism $e_n : (\mathcal{O}_S, d) \to \tau^*W_n$ in $\text{MIC}^n(S/k)$

which satisfies the following conditions:

(W0)' For any $n \geq 2$, the morphism $f_s(\overline{W}_n^\vee) \to f_s(\overline{W}_n)$ induced by the transition map $\overline{W}_n \to \overline{W}_{n-1}$ is an isomorphism.

(W1)' For any $E \in N_f \text{MIC}(\mathcal{A}_{X/S})$ of index of unipotence $\leq n$ and a morphism $\nu : \mathcal{O}_S \to \tau^*E$ in $\text{MIC}(S/S)$, there exists a unique morphism $\varphi : \overline{W}_n \to E$ in $N_f \text{MIC}(\mathcal{A}_{X/S})$ with $\tau^*(\varphi) \circ \overline{e}_n = \nu$, where $\overline{e}_n : \mathcal{O}_S \to \tau^*\overline{W}_n$ is the underlying morphism of $e_n$ in $N_f \text{MIC}(\mathcal{A}_{X/S})$. Moreover, the same universality holds after base change by any morphism $S' \to S$ of finite type.
(W2) For any $n$, there exists an exact sequence

\[(5.7) \quad 0 \longrightarrow f^*R^1f_*(W_n^\vee) \longrightarrow W_{n+1} \longrightarrow W_n \longrightarrow 0\]

in $N_f\mathcal{E}^n$, where $R^1f_* : N_f\mathcal{E}^n \longrightarrow \text{MIC}^n(S/k)$ is the functor defined in (5.4).

Proof. We repeat the proof of Theorem 3.1 with some mild modifications. For $n = 1$, we put $W_1 := (\mathcal{A}_{X/S}^1, d)$ and define $e_1$ to be the canonical isomorphism $(\mathcal{O}_S, d) \rightarrow \iota^*(\mathcal{A}_{X/S}^0, d)$. Then we can check the condition (W1)' for $n = 1$ in the same way as the proof of Theorem 3.1.

We construct $(W_{n+1}, e_{n+1})$ from $(W_i, e_i)\ (i \leq n)$. By induction hypothesis, we see that $R^1f_*(\overline{W}_i^\vee)$ belongs to MIC$(S/S)$. So, when $S$ is affine, we can define $\overline{W}_{n+1} \in N_f\text{MIC}(\mathcal{A}_{X/S})$ as the extension

\[(5.8) \quad 0 \longrightarrow f^*R^1f_*(\overline{W}_n^\vee) \longrightarrow \overline{W}_{n+1} \longrightarrow \overline{W}_n \longrightarrow 0\]

whose extension class, which is regarded as an element in

$$
\Gamma(S, R^1f_*(\overline{W}_n^\vee) \otimes R^1f_*(\overline{W}_n^\vee)) = \text{End}(R^1f_*(\overline{W}_n^\vee)),
$$

is the identity. Also, let $\overline{\tau}_{n+1} : \mathcal{O}_S \rightarrow \iota^*\overline{W}_{n+1}$ be a morphism which lifts the map $\overline{\tau}_n : \mathcal{O}_S \rightarrow \iota^*\overline{W}_n$.

We need to check the properties (W0)' and (W1)' for $(\overline{W}_{n+1}, \overline{\tau}_{n+1})$, assuming $S$ and $S'$ affine. By using the base change property after (5.5), we see that the proof on $X \times_X S'/S'$ is the same as that on $X/S$ and the proof of the properties on $X/S$ can be done in the same way as that of Theorem 3.1. Then we can define $(\overline{W}_{n+1}, \overline{\tau}_{n+1})$ for general $S$ by gluing and check the properties (W0)', (W1)' for general $S$ and $S'$ in the same way as in the proof of Theorem 3.1.

Next, let $f_j^m : X_j^m \rightarrow S^m(1), p_j^m : S^m(1) \rightarrow S, q_j^m : X_j^m \rightarrow X, \hat{f}_j^m : \hat{X}_j^m \rightarrow S^m(1)$ and $\hat{g}_j^m : \hat{X}_j^m \rightarrow X$ be as in Section 1, and let $\iota_j^m : S^m(1) \rightarrow X_j^m$ be the base change of $\iota$, which is a section of $f_j^m$. Then we can define the sheaves of augmented dgca’s $\mathcal{A}_{X_j^m/S^m(1)}$ and the categories $N_f\text{MIC}(\mathcal{A}_{X_j^m/S^m(1)})$ from the morphism $f_j^m$. Also, we can define the category $\text{N}_f\text{MIC}(\mathcal{A}_{X/S^m(1)})$ from the morphism $\hat{f}_j^m$. Then we have the quasi-isomorphisms

$$
\mathcal{A}_{X_j^m/S^m(1)} \cong \mathcal{A}_{X/S^m(1)} \cong \mathcal{A}_{X_1^m/S^m(1)}
$$

by Proposition 1.2 and these induce the equivalences of categories

$$
N_f\text{MIC}(\mathcal{A}_{X_j^m/S^m(1)}) \cong N_f\text{MIC}(\mathcal{A}_{X/S^m(1)}) \cong N_f\text{MIC}(\mathcal{A}_{X_1^m/S^m(1)}).
$$

Then, by the property (W1)', there exists uniquely the isomorphism

$$
\epsilon_{n+1}^m : (q_2^m\overline{W}_{n+1}, p_2^m\overline{\tau}_{n+1}) \cong (q_1^m\overline{W}_{n+1}, p_1^m\overline{\tau}_{n+1}) \ (m \in \mathbb{N})
$$
in the framework of the categories in the above diagram, which satisfies the cocycle condition. (To prove the cocycle condition, we need to work on pullbacks of \( f : X \to S \) to \( S^m(2) \). We leave the reader to write out the detailed argument.) So the object \( W_{n+1} := (\tilde{W}_{n+1}, \{ e_{m+1} \}_m) \) in the category \( \mathcal{E} \) is defined, and the isomorphisms \( p^m_2 \tau_{n+1} \cong p^m_1 \tau_{n+1} \) \((m \in \mathbb{N})\) induces the map \( e_{n+1} : (\mathcal{O}_S, d) \to \epsilon W_{n+1} \).

Finally we need to prove that \( W_{n+1} \) belongs to \( N_f \mathcal{E}^n \) and that it satisfies the property \((W2)'\). Since \( W_n \) belongs to \( N_f \mathcal{E}^n \), we obtain the isomorphisms of Gauss-Manin connection

\[
\delta^n_m : p^n_2 R^1 f_* (W^n_s) \cong p^n_1 R^1 f_* (W^n_s) \quad (m \in \mathbb{N}).
\]

by \([5.4]\). Using this, the rest of the proof can be done in the same way as the proof of Theorem \([3.1]\). So the proof of the proposition is finished. \( \square \)

Remark 5.10. It corresponds to Remark \([3.2]\) and it furnishes the last step of the proof of Proposition \([5.8]\). By \((W1)'\), for any object \( E \) in \( N_f \text{MIC}(A_{X,s}) \) and any morphism \( v : k \to x^*E \) in \( \text{MIC}(s/s) = \text{Vec}_k \), there exists a unique morphism \( \varphi_v : W_n|_{X_s} \to E \) in \( N_f \text{MIC}(A_{X,s}) \) with \( x^*(\varphi) \circ (e_n|_{X_s}) = v \) for some \( n \). By considering maps \( v_1, \ldots, v_N : k \to x^*E \) whose direct sum \( k^\oplus N \to x^*E \) is surjective, we obtain a surjective map \( \oplus_{i=1}^N \varphi_v : W_n^\oplus N|_{X_s} \to E \) for some \( n \).

Now we compare the categories \( N_f \text{MIC}^n(X/k) \) and \( N_f \mathcal{E}^n \), which is the first step of the proof of Theorem \([5.4]\).

Proposition 5.11. There exists an equivalence of categories \( N_f \text{MIC}^n(X/k) \cong N_f \mathcal{E}^n \) over \( \text{MIC}^n(S/k) \). Equivalently, there exists an isomorphism

\[
G(N_f \mathcal{E}^n, \epsilon^*) \cong \pi^1^{\text{dR}}(X/S, \iota)
\]

of group schemes in \( \text{MIC}^n(S/k) \), where \( \pi^1^{\text{dR}}(X/S, \iota) \) denotes the first definition of relatively unipotent de Rham fundamental group.

Proof. First we define the functor \( N_f \text{MIC}^n(X/k) \to N_f \mathcal{E}^n \). Let \( E := (E, \{ e^n \}) \) be an object in \( N_f \text{MIC}^n(X/k) \cong N_f \text{StrCrys}^n(X/k) \).

Let \( \pi : X_* \to X, \pi_S : S_* \to S \) be as in Section 4. Consider the surjective morphism of strict cosimplicial sheaves

\[
\mathcal{A}^\bullet(\epsilon)(E) := f_* \pi_\ast ((E \otimes \Omega^\bullet_X)|_{X_*}) \xrightarrow{\epsilon} \pi_S \ast (\epsilon^*E|_{S_*})
\]

induced from \( \iota \), where \( E \otimes \Omega^\bullet_X \) denotes the de Rham complex associated to \( E \), regarded as an object in \( N_f \text{MIC}(X/S) \). By applying the Thom-Whitney functor \( s_{\text{TW}} (\text{[NA87], 3}) \), we obtain the surjective morphism of complexes of sheaves \( s_{\text{TW}}(\epsilon^*) : s_{\text{TW}}(\mathcal{A}^\bullet(\epsilon)(E)) \to s_{\text{TW}}(\pi_S \ast (\epsilon^*E|_{S_*})) \) such that \( H^0(s_{\text{TW}}(\pi_S \ast (\epsilon^*E|_{S_*}))) = \epsilon^*E \). We define the complex of sheaves \( \mathcal{A}_{X/S}(E) \subseteq s_{\text{TW}}(\mathcal{A}^\bullet(\epsilon)(E)) \) as the inverse image of \( \epsilon^*E \to s_{\text{TW}}(\pi_S \ast (\epsilon^*E|_{S_*})) \) by \( s_{\text{TW}}(\epsilon^*) \). By construction, \( \mathcal{A}_{X/S}(E) \) has a structure of differential graded module over \( \mathcal{A}_{X/S} \).
We prove that the pair \((A_{X/S}(E)^0, A_{X/S}(E)^0) \rightarrow A_{X/S}(E)^1\) defines an object in \(\text{MIC}(A_{X/S})\). To do so, it suffices to prove that \(A_{X/S}(E)^0\) is a locally free \(A_{X/S}\) module and that the natural maps \(A_{X/S}(E)^0 \otimes A_{X/S}^o A_{X/S}^i \rightarrow A_{X/S}(E)^i \) \((i \in \mathbb{N})\) are isomorphisms. Because the functor \(N_f \text{MIC}(X/S) \ni E \mapsto A_{X/S}(E)\) is exact by the exactness of \(s_{\text{TW}}\) \((\text{[NA87] \S 2})\), it suffices to check the claim when \(E \in N_f \text{MIC}(X/S)\) is of the form \(f_{\text{dr}}^* F\) \((F \in \text{MIC}(S/S))\). Also, since it suffices to check the claim locally on \(S\), we may assume that \(F\) is free and the claim is trivially true in this case. So we have defined an object

\[
A_{X/S}(E)^0 := (A_{X/S}(E)^0, A_{X/S}(E)^0) \rightarrow A_{X/S}(E)^0 \otimes A_{X/S}^o A_{X/S}^1
\]

in \(\text{MIC}(A_{X/S})\).

Also, we can apply the same construction to \((\overline{q}_2^m)^* E \cong (\overline{q}_1^m)^* E \in \text{MIC}(\hat{X}^m/S^m(1))\)
(\(\text{where the notation is as in Section 1, before Proposition \[1.10\]}\) and obtain an object in \(\text{MIC}(A_{X/S^m(1)})\), which we denote by

\[
A_{X/S^m(1)}(E)^0 := (A_{X/S^m(1)}(E)^0, A_{X/S^m(1)}(E)^0) \rightarrow A_{X/S^m(1)}(E)^0 \otimes A_{X/S^m(1)}^o A_{X/S^m(1)}^1
\]

We can define the object

\[
A_{X/S^m(2)}(E)^0 := (A_{X/S^m(2)}(E)^0, A_{X/S^m(2)}(E)^0) \rightarrow A_{X/S^m(2)}(E)^0 \otimes A_{X/S^m(2)}^o A_{X/S^m(2)}^1
\]

in the same way and we can check the compatibility of \(A_{X/S}(E)^0\) and \(A_{X/S^m(r)}(E)^0\) \((r = 1, 2, m \in \mathbb{N})\) by induction of index of unipotence of \(E\). So they form an object in \(N_f \mathcal{E}^n\) and so we have defined a functor \(N_f \text{MIC}^n(X/k) \rightarrow N_f \mathcal{E}^n\).

This functor induces a morphism of group schemes \(G(N_f \mathcal{E}^n, \iota^*) \rightarrow \pi_1(X/S, \iota)\) in \(\text{MIC}^n(S/k)\). To prove this is an isomorphism, it suffices to prove it after we apply the restriction functor \(\text{MIC}^n(S/k) \rightarrow \text{MIC}(s/s)\) which is exact and faithful. By Propositions \[2.3\] and \[5.8\] this is equivalent to the claim that the functor \(N_{f_s} \text{MIC}(X_s/s) \rightarrow N_{f_s} \text{MIC}(A_{X_s/s})\), which is defined from the morphism \(X_s \rightarrow s\) as above, is an equivalence of categories. Because objects on both hand sides are iterated extensions of trivial objects, it suffices to check that Hom and Ext groups of such objects are isomorphic. This is true because Hom and Ext can be understood as zeroth and first cohomology groups and cohomology groups with trivial coefficient are isomorphic by construction of \(A_{X_s/s}\) (to obtain the equivalence, one actually needs isomorphism on zeroth and first cohomology groups and the injectivity on the second cohomology groups).

By using the 1-minimal models \(M_{X/S}, M_{X/S^m(r)}\) \((\text{resp. the } (1, q)\)-minimal models \(M_{X/S}, M_{X/S^m(r)}\)) instead of \(A_{X/S}, A_{X/S^m(r)}\), we can define the category similar to \(N_f \mathcal{E}^n\), which we denote by \(N_f \mathcal{E}_M^n\) \((\text{resp. } N_f \mathcal{E}_M^{n(q)})\). Next we compare the categories \(N_f \mathcal{E}^n, N_f \mathcal{E}_M^n\), which is the second step of the proof of Theorem \[5.4\].

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Proposition 5.12. There exists an equivalence of categories $N_f \mathcal{E}_M^n \to N_f \mathcal{E}_M^n$ over $MIC^n(S/k)$. Equivalently, there exists an isomorphism

$$G(N_f \mathcal{E}_M^n, \iota^*) \cong G(N_f \mathcal{E}_M^n, \iota^*)$$

of group schemes in $MIC^n(S/k)$.

Proof. Recall that we have families of ho-morphisms

$$\rho_q : M_{X/S}(q) \to A_{X/S}, \quad \rho^m(r)_q : M_{X/S^m(r)}(q) \to A_{X/S^m(r)} \quad (r = 1, 2, m \in \mathbb{N}, q \in \mathbb{N})$$

which are compatible in suitable sense up to unique Sullivan homotopy. One problem is that $\rho_q (q \in \mathbb{N})$ are not necessarily morphisms and so it is not immediate that $\rho_q (q \in \mathbb{N})$ induce the functor $N_f MIC(M_{X/S}) \to N_f MIC(A_{X/S})$. We will construct this functor by constructing it locally on the level of $(1, q)$-minimal case, patching the local constructions, and taking the union with respect to $q \in \mathbb{N}$.

Fix $q \in \mathbb{N}$. Then there exists an affine open covering $S = \bigcup_{\alpha} U_{\alpha}$ of $S$ such that the above ho-morphism induces for each $U_{\alpha}$ the morphism $\rho_{q, \alpha}(U_{\alpha}) : M_{X/S}(q)(U_{\alpha}) \to A_{X/S}(U_{\alpha})$. We write by $\rho_{q, \alpha} : M_{X/S}(q)|_{U_{\alpha}} \to A_{X/S}|_{U_{\alpha}}$ the associated morphism of sheaves. Then we can define the corresponding functor

$$\rho_{q, \alpha, *} : N_f MIC(M_{X/S}(q)|_{U_{\alpha}}) \to N_f MIC(A_{X/S}|_{U_{\alpha}}).$$

We prove that $\rho_{q, \alpha, *}$'s are compatible with respect to $\alpha$. On $U_{\alpha'} := U_{\alpha} \cap U_{\alpha'}$, there exists a unique Sullivan homotopy

$$h : M_{X/S}(q)|_{U_{\alpha'}} \to \mathcal{O}_s(t, dt) \otimes A_{X/S}|_{U_{\alpha'}}$$

between $\rho_{q, \alpha}|_{U_{\alpha'}}$ and $\rho_{q, \alpha'}|_{U_{\alpha'}}$. Then $\rho_{q, \alpha, *}|_{U_{\alpha'}}$ is written as the composite

$$N_f MIC(M_{X/S}(q)|_{U_{\alpha'}}) \xrightarrow{h_*} N_f MIC(\mathcal{O}_s(t, dt) \otimes A_{X/S}|_{U_{\alpha'}}) \xrightarrow{p_{0, *}} N_f MIC(A_{X/S}|_{U_{\alpha'}})$$

(where $h_*$ is the functor induced by $h$ and $p_{0, *}$ is the functor induced by the morphism $p_0 : \mathcal{O}_s(t, dt) \otimes A_{X/S}|_{U_{\alpha'}} \to A_{X/S}|_{U_{\alpha'}}$ appeared in the definition of Sullivan homotopy) and $\rho_{q, \alpha', *}|_{U_{\alpha'}}$ is written as a similar composite, with $p_0$ replaced by $p_1$. On the other hand, if we denote the inclusion $A_{X/S}|_{U_{\alpha'}} \hookrightarrow \mathcal{O}_s(t, dt) \otimes A_{X/S}|_{U_{\alpha'}}$ by $i$, it induces the functor

$$i_* : N_f MIC(A_{X/S}|_{U_{\alpha'}}) \to N_f MIC(\mathcal{O}_s(t, dt) \otimes A_{X/S}|_{U_{\alpha'}})$$

and it is an equivalence because $i$ is a quasi-isomorphism. Since we have $p_0 \circ i = \text{id} = p_1 \circ i$, we see that

$$\rho_{q, \alpha, *}|_{U_{\alpha'}} = p_{0, *} h_* \cong i_* h_* \cong p_{1, *} h_* = \rho_{q, \alpha', *}|_{U_{\alpha'}}. \tag{5.9}$$

We should check that the isomorphism (5.9), which we denote by $\gamma_{\alpha\alpha'}$, satisfies the cocycle condition. The isomorphism $\gamma_{\alpha\alpha'}$ is defined via the Sullivan homotopy $h$, and
the isomorphism \( \gamma_{\alpha',\alpha} \) is defined in the same way via the unique Sullivan homotopy of the form
\[
h': M_X/S(q)|_{U_{\alpha'\alpha'}} \longrightarrow \mathcal{O}_S(t, dt) \otimes A_X/S|_{U_{\alpha'\alpha'}}.
\]
Then, on \( U_{\alpha\alpha'\alpha''} := U_{\alpha} \cap U_{\alpha'} \cap U_{\alpha''} \), we have the Sullivan homotopy \( h|_{U_{\alpha\alpha'\alpha''}} \) between \( \rho_{q,\alpha}|_{U_{\alpha\alpha'\alpha''}} \) and \( \rho_{q,\alpha'}|_{U_{\alpha\alpha'\alpha''}} \) and the Sullivan homotopy \( h'|_{U_{\alpha'\alpha''}} \) between \( \rho_{q,\alpha}|_{U_{\alpha'\alpha''}} \) and \( \rho_{q,\alpha''}|_{U_{\alpha'\alpha''}} \). Then, we can construct the Sullivan homotopy \( H \) between \( \rho_{q,\alpha}|_{U_{\alpha\alpha'\alpha''}} \) and \( \rho_{q,\alpha''}|_{U_{\alpha'\alpha''}} \) by the construction in the proof of Lemma [1.1] and we see from the construction that the isomorphism \( \rho_{q,\alpha,\ast}|_{U_{\alpha\alpha'\alpha''}} \cong \rho_{q,\alpha''\ast}|_{U_{\alpha'\alpha''}} \) we obtain from \( H \) is the same as the composite \( \gamma_{\alpha\alpha'} \circ \gamma_{\alpha\alpha''} \). On the other hand, by the uniqueness of Sullivan homotopy in Proposition [1.2] we see that \( H \) is equal to the restriction of the unique Sullivan homotopy
\[
h'': M_X/S(q)|_{U_{\alpha\alpha'}} \longrightarrow \mathcal{O}_S(t, dt) \otimes A_X/S|_{U_{\alpha\alpha'}}
\]
between \( \rho_{q,\alpha}|_{U_{\alpha\alpha'}} \) and \( \rho_{q,\alpha''}|_{U_{\alpha\alpha'}} \) to \( U_{\alpha\alpha'\alpha''} \). Thus we see the equality \( \gamma_{\alpha\alpha'} \circ \gamma_{\alpha\alpha''} = \gamma_{\alpha\alpha''} \), namely, we proved the required cocycle condition. Therefore, the isomorphisms [5.9] for \( \alpha, \alpha' \) glue to give a functor
\[
\rho_{q,\ast}: N_fMIC(M_X/S(q)) \longrightarrow N_fMIC(A_{X/S}).
\]
By the same argument, we can construct the functors
\[
\rho^m(r)_{q,\ast}: N_fMIC(M_X/S^m(r)(q)) \longrightarrow N_fMIC(A_{X/S^m(r)}) \quad (r = 1, 2, m \in \mathbb{N})
\]
and the functors \( \rho_{q,\ast}, \rho^m(r)_{q,\ast} \) are compatible. (To prove the compatibility, we argue again with Sullivan homotopy because the diagram in Theorem [4.12] is commutative only up to unique Sullivan homotopy.) Hence these functors induce the compatible functors \( N_fE^n_M(q) \longrightarrow N_fE^n(q) \in \mathbb{N} \). (The compatibility here follows again from the argument, with Sullivan homotopy.) Because \( N_fE^n_M \) is the inductive limit of \( N_fE^n_M(q) \) \( q \in \mathbb{N} \), we obtain the functor \( N_fE^n_M \longrightarrow N_fE^n \), thus the morphism
\[
(5.10) \quad G(N_fE^n, \iota^*) \longrightarrow G(N_fE^n_M, \iota^*)
\]
of group schemes in \( \text{MIC}^n(S/k) \). To prove this morphism is an isomorphism, it suffices to check it after we apply the restriction functor \( \text{MIC}^n(S/k) \longrightarrow \text{MIC}(S/s) \) which is exact and faithful. So, by Proposition [5.8] (and its analogue for \( M_X/S \)), it suffices to prove that the morphism
\[
G(N_f MIC(A_{X/s}), x^*) \longrightarrow G(N_f MIC(M_X/s), x^*)
\]
is an isomorphism. Thus it suffices to prove that the functor
\[
N_f MIC(M_X/s) \longrightarrow N_f MIC(A_{X/s})
\]
is an equivalence. This is true because the map on cohomologies \( H^i(M_X/s) \longrightarrow H^i(A_{X/s}) \) is an isomorphism for \( i = 0, 1 \) and injective for \( i = 2 \) by construction of \( M_X/s \). So we finished the proof of proposition. \( \square \)
The third step of the proof is to compare the category \( N_f \mathcal{E}_M \) and the category \( \text{Rep}_{\text{MIC}^n(S/k)}(\text{Spec} \, H^0(B(M_X/S))) \) of representations of \( H^0(B(M_X/S)) \) in \( \text{MIC}^n(S/k) \).

**Proposition 5.13.** There exists an equivalence of categories

\[
(5.11) \quad \text{Rep}_{\text{MIC}^n(S/k)}(\text{Spec} \, H^0(B(M_X/S))) \xrightarrow{\cong} N_f \mathcal{E}_M^n
\]

over \( \text{MIC}^n(S/k) \). Equivalently, there exists an isomorphism

\[
(5.12) \quad G(N_f \mathcal{E}_M^n, \iota^*) \xrightarrow{\cong} \text{Spec} \, H^0(B(M_X/S))
\]

of group schemes in \( \text{MIC}^n(S/k) \).

**Proof.** The proof is inspired by the argument of [Te10, Thm 7.6] in the absolute case and we will reduce to it. The composition

\[
\Delta' : B(M_X/S) \rightarrow B(M_X/S) \otimes \mathcal{O}_S \xrightarrow{\text{proj.}} B(M_X/S) \otimes \mathcal{O}_S \rightarrow B(M_X/S)
\]

of the map \( x \mapsto \Delta(x) - x \otimes 1 \) and the canonical projection induce an integrable \( M_{X/S} \)-connection

\[
\nabla_H : H^0(B(M_X/S)) \rightarrow H^0(B(M_X/S)) \otimes \mathcal{O}_S M_{X/S}^1.
\]

Because \( \nabla_H \) can be written as the inductive limit of the maps

\[
H^0(F^{-s}B(M_X/S)) \rightarrow H^0(F^{-s+1}B(M_X/S)) \otimes \mathcal{O}_S M_{X/S}^1,
\]

we see that \( (H^0(B(M_X/S)), \nabla_H) \) actually belongs to the ind-category of \( N_f \text{MIC}(M_X/S) \). Moreover, since this construction is also possible on \( S^m(r) \) \((r = 1, 2, m \in \mathbb{N})\), \( (H^0(B(M_X/S)), \nabla_H) \) is naturally regarded as an object in the ind-category of \( N_f \mathcal{E}_M^n \).

An object in the category \( \text{Rep}_{\text{MIC}^n(S/k)}(\text{Spec} \, H^0(B(M_X/S))) \) induces a pair \((V, \Delta_V)\) of a locally free \( \mathcal{O}_S \)-module of finite rank endowed with a comodule structure \( \Delta_V : V \rightarrow V \otimes \mathcal{O}_S H^0(B(M_X/S)) \), and similar pairs on \( S^m(r) \) \((r = 1, 2, m \in \mathbb{N})\) which are compatible. We define the integrable \( M_{X/S} \)-connection corresponding to \((V, \Delta_V)\) as the kernel of the map

\[
(5.13) \quad \Delta_V \otimes \text{id} - \text{id} \otimes \Delta_H : V \otimes H^0(B(M_X/S)) \rightarrow V \otimes H^0(B(M_X/S)) \otimes H^0(B(M_X/S))
\]

endowed with a structure of connection induced from that on \( H^0(B(M_X/S)) \)'s on the right. Since this construction is possible also on \( S^m(r) \) \((r = 1, 2, m \in \mathbb{N})\), we can define the above object as an object in the ind-category of \( N_f \mathcal{E}_M^n \). However, it actually belongs to \( N_f \mathcal{E}_M^n \) because the kernel of the map \((5.13)\) is equal to the image of \( \Delta_V : V \rightarrow V \otimes \mathcal{O}_S H^0(B(M_X/S)) \). In this way we define the functor \((5.11)\), hence the morphism \((5.12)\). To see that it is an isomorphism, it suffices to prove it after we
apply the restriction functor $\text{MIC}^n(S/k) \longrightarrow \text{MIC}(s/s)$ which is exact and faithful. So it suffices to prove the functor

$$\text{Rep}_k(\text{Spec} H^0(B(M_{X/s}))) \xrightarrow{\cong} N_f \text{MIC}(M_{X/s})$$

defined in a similar way as above is an equivalence, and this is shown in the proof of [Te10, Thm. 7.6]. So we are done.

Now we can prove Theorem 5.4:

**Proof of Theorem 5.4.** By Propositions 5.11, 5.12 and 5.13, we have isomorphisms

$$\pi_1^{\text{dR}}(X/S, i) \xleftarrow{\cong} G(N_f \mathcal{E}^n, t^*) \xrightarrow{\cong} G(N_f \mathcal{E}^n_M, t^*) \xleftarrow{\cong} \text{Spec} H^0(B(M_{X/S}))$$

of group schemes in $\text{MIC}^n(S/k)$. So we are done.

**Remark 5.14.** Assume that the log structures on $X$ and $S$ are trivial and that $S$ is smooth over $k$. Then, as we explained in Remark 4.10, there is a priori another way to define an integrable connection on $S/k$ on $M^1_{X/S}$, hence on $M_{X/s}$, which is due to Navarro-Aznar. This induces the structure of an integrable connection on the bar construction $H^0(B(M_{X/S}))$, and so we can regard $\text{Spec} H^0(B(M_{X/S}))$ as a group scheme in $\text{MIC}^n(S/k)$. So we obtain yet another definition of the relatively unipotent de Rham fundamental group $\pi_1^{\text{dR}}(X/S, i)$.

We expect that this definition is compatible with our definition in Definition 5.1, but we will not pursue this topic in this paper.

### 6 Calculation of monodromy for stable log curves

In this section we calculate the monodromy action on the relatively unipotent log de Rham fundamental group in the case of stable log curves: we will use all the various definitions we gave in the previous paragraphs. As an application, we will have a purely algebraic proof of the result of Andreatta-Iovita-Kim [AIK15].

**6.1 Statement of main result and first reductions**

Throughout this section, $k$ will be a field of characteristic zero and $S$ will be the standard log point over $k$. First we introduce the notion of log curve, which is due to F. Kato [Kf00].

**Definition 6.1.** A morphism $f : X \longrightarrow S$ of fs log schemes is a log curve if it is log smooth, integral and if its geometric fiber is a reduced and connected curve.

Then we have the following local description by [Kf00 1.1]:
Proposition 6.2. Let \( f : X := (X^\circ, \mathcal{M}_X) \rightarrow S := (S^\circ, \mathcal{M}_S) \) be a log curve and assume \( k \) is algebraically closed. Then the underlying scheme \( X^\circ \) of \( X \) has at worst ordinary double points. Moreover, there exists a finite set of closed points \( \{s_1, \ldots, s_r\} \) in the smooth locus \( X^{\circ, \text{sm}} \) of \( X^\circ \) (called marked points) such that the log structure of \( f : X \rightarrow S \) is described in the following way:

1. Etale locally around a double point, the underlying morphism \( f^\circ : X^\circ \rightarrow S^\circ = \text{Spec } k \) factors as

   \[ X^\circ \rightarrow \text{Spec } k[x, y]/(xy) \rightarrow \text{Spec } k \]

   with the first morphism etale, and the log structure of \( f \) is associated to the chart

   \[
   \mathbb{N}^2 \oplus \Delta, m \mathbb{N} \rightarrow \mathcal{O}_X, \quad \mathbb{N} \rightarrow k, \quad \mathbb{N} \rightarrow \mathbb{N}^2 \oplus \Delta, m \mathbb{N}.
   
   \begin{align*}
   ((1,0),0) &\mapsto x & 1 &\mapsto 0 & 1 &\mapsto ((0,0),1) \\
   (0,1),0) &\mapsto y & \\
   ((0,0),1) &\mapsto 0 
   \end{align*}

   for some \( m \geq 1 \), where \( \Delta : \mathbb{N} \rightarrow \mathbb{N}^2 \) is the diagonal map and \( m : \mathbb{N} \rightarrow \mathbb{N} \) is the multiplication by \( m \).

2. Etale locally around each \( s_i \), \( f^\circ \) factors as

   \[ X^\circ \rightarrow \text{Spec } k[x] \rightarrow \text{Spec } k \]

   with the first morphism etale, and the log structure of \( f \) is associated to the chart

   \[
   \mathbb{N}^2 \rightarrow \mathcal{O}_X, \quad \mathbb{N} \rightarrow k, \quad \mathbb{N} \rightarrow \mathbb{N}^2.
   
   \begin{align*}
   (1,0) &\mapsto 0 & 1 &\mapsto 0 & 1 &\mapsto (1,0) \\
   (0,1) &\mapsto x 
   \end{align*}

3. Etale locally at other points, \( f^\circ \) is smooth and the log structure on \( f \) is associated to the chart

   \[
   \mathbb{N} \rightarrow \mathcal{O}_X, \quad \mathbb{N} \rightarrow k, \quad \mathbb{N} \rightarrow \mathbb{N}.
   
   \begin{align*}
   1 &\mapsto 0 & 1 &\mapsto 0 & 1 &\mapsto 1 
   \end{align*}

Moreover, the locus \( X_{\text{triv}} := \{x \in X \mid \mathcal{M}_{S, \overline{f(x)}} \twoheadrightarrow \mathcal{M}_{X, \overline{s}}\} \) is equal to \( X^{\circ, \text{sm}} \backslash \{s_1, \ldots, s_r\} \).

For a log curve \( f : X \rightarrow S \), the reduced divisor of marked points of its geometric fiber descends to a reduced divisor on \( X \), which we denote by \( D \). We say that \( f : X \rightarrow S \) is marked (resp. unmarked) if \( D \) is nonempty (resp. empty).

Remark 6.3. Let \( f : X := (X^\circ, \mathcal{M}_X) \rightarrow S := (S^\circ, \mathcal{M}_S) \) be a log curve with \( k \) not necessarily algebraically closed, and let \( s \) be a \( k \)-rational point of \( X \) contained in the divisor \( D \) of marked points defined above. Let \( \overline{s} \cong \text{Spec } \overline{k} \) (where \( \overline{k} \) is an algebraic closure of \( k \)) be a geometric point over \( s \) and let \( \mathcal{M}_s, \mathcal{M}_{\overline{s}} \) be the pullback
of the log structure $M_X$ to $s$, $\mathfrak{s}$, respectively. Then Proposition 6.2(2) implies that $M_\mathfrak{s}$ is associated to the chart

$$N^2 \to \mathcal{O}_\mathfrak{s}; \ (1,0) \mapsto 0, \ (0,1) \mapsto 0.$$ 

Thus $M_\mathfrak{s}/\mathcal{O}_\mathfrak{s}^\times \cong \mathbb{N}^2$. Since $M_\mathfrak{s}/\mathcal{O}_\mathfrak{s}^\times$ is the stalk at $\mathfrak{s}$ of the etale sheaf $M_s/\mathcal{O}_s^\times$ on $s$, the monoid $M_\mathfrak{s}/\mathcal{O}_\mathfrak{s} \cong \mathbb{N}^2$ is endowed with the canonical action of $\text{Aut}(\mathfrak{S}/s) \cong \text{Gal}(\overline{k}/k)$.

We prove that this action is trivial. To show it, it suffices to prove that the action of any element of $\text{Aut}(\mathfrak{S}/s)$ on the set $\{(1,0), (0,1)\}$ is trivial. This holds because the element $(1,0)$, which is the pullback of an element of the log structure $M_S$, is fixed by any element of $\text{Aut}(\mathfrak{S}/s)$ because $S$ is the standard log point over $k$.

The triviality of the above action implies that $M_s/\mathcal{O}_s^\times$ is the constant sheaf $\mathbb{N}^2$ on $s$. Also, by [HN17, Proposition 1.3], we have an isomorphism

$$M_s \cong \mathcal{O}_s^\times \times (M_s/\mathcal{O}_s^\times) \cong \mathcal{O}_s^\times \times \mathbb{N}^2.$$ 

Hence we conclude that $M_s$ is associated to the chart

$$N^2 \to \mathcal{O}_s; \ (1,0) \mapsto 0, \ (0,1) \mapsto 0,$$

namely, the log structure $M_s$ admits the chart (6.1) globally on $s$, without taking any etale extension of $s$.

A similar property holds also for $k$-rational smooth points of $X$, but it does not necessarily hold for $k$-rational double points of $X$.

In this section we apply the results of the previous sections to the diagram

$$X \xrightarrow{f} S \xrightarrow{g} \text{Spec } k,$$

where $f$ is a proper log curve, $g$ is the structure morphism and $\iota$ is a section of $f$. Also, let $s \hookrightarrow S$ be the identity map and let $x$ be the composition $s \xrightarrow{\iota} S \xrightarrow{g} X$. Then we are in the situation of [1.14] (The conditions (A), (B) (C), (D) (E) follows from Proposition [1.15]). We translate in this situation various objects we have studied in the last sections. The category $\text{MIC}^n(S/k)$ is canonically equivalent to the category of pairs $(V, N)$ consisting of a finite-dimensional vector space $V$ and a nilpotent endomorphism $N$, and so we identify them in the following. The fiber functor $s^*_{\text{dR}}: \text{MIC}^n(S/k) \to \text{Vec}_k$ is the forgetful functor $(V, N) \mapsto V$. The Tannaka dual $\pi^\text{dR}(S, s)$ of $(\text{MIC}^n(S/k), s^*_{\text{dR}})$ is isomorphic to $\mathbb{G}_{a,k}$ and an object $(V, N) \in \text{MIC}^n(S/k)$ corresponds to the representation $\mathbb{G}_{a,k} \to GL(V); t \mapsto \exp(tN)$ by the equivalence $\text{MIC}^n(S/k) \cong \text{Rep}(\mathbb{G}_{a,k}) = \text{Rep}(\mathbb{G}_{a,k})$ of Tannaka duality. We denote the Tannaka dual of $(N_f, \text{MIC}^n(X/k), x^*_{\text{dR}})$ by $\pi^\text{dR}(X, x)$, as in the previous sections. However, unlike the previous sections, we abusively denote the group scheme $s^*_{\text{dR}}\pi^\text{dR}(X/S, \iota) = \pi^\text{dR}(X_s, x)$ over $k$, which is the Tannaka dual of

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When $m$ is globally, without taking any etale extension of $\alpha$, there exists a section $\iota$ of the monodromy action on $N \rightarrow S$, endowed with the conjugate action of $\pi^dR(X,S,\iota)$ over $k$ induced by $\epsilon := \iota_{\epsilon}(1)$ (see Section 2). The action of $1 \in G_{a,k} = \pi^dR(S,s)$ on $\pi^dR(X,S,\iota)$ (the conjugation by $\epsilon := \iota_{\epsilon}(1)$) is called the monodromy action.

To state the main result in this section, we introduce the notion of stable log curve (due to F. Kato [Kf00, Cor. 1.1]).

**Definition 6.4.** When $k$ is algebraically closed, a proper log curve $f : X \rightarrow S$ is called a stable log curve if the underlying morphism of schemes $f^\circ : X^\circ \rightarrow S^\circ$ endowed with marked points $\{s_1, \ldots, s_r\}$ in Proposition 6.2 is a pointed stable curve in the sense of [Kn83], namely, if it is not a smooth elliptic curve with no marked point and, for any irreducible component $Y$ of $X^\circ$ birational to $\mathbb{P}_k^1$, the cardinality of double points of $X$ in $Y$ (counted doubly when a double point is a self intersection point) plus that of marked points in $Y$ is $\geq 3$. For general $k$, a proper log curve $f : X \rightarrow S$ is called a stable log curve if so is its geometric fiber.

Then the main result of this section is the following:

**Theorem 6.5.** Let $f : X \rightarrow S$ be a stable log curve and $\iota : S \rightarrow X$ be its section. Then the monodromy action on $\pi^dR(X,S,\iota)$ is trivial as an element in $Out(\pi^dR(X,S,\iota))$ if and only if the underlying morphism of schemes of $f$ is smooth.

First we prove the ‘if’ part of Theorem 6.5. To state it in a more precise form, we introduce the notion of good section.

**Definition 6.6.** The section $\iota : S \rightarrow X$ is called good if its image is in $X_{triv}$.

**Remark 6.7.** If $s$ is a $k$-rational double point of $X$ and assume that the the pullback log structure on $s$ from that on $X$ admits a chart of the form

$$\mathbb{N}^2 \oplus_{\Delta,N,m} \mathbb{N} \rightarrow \mathcal{O}_s, \quad ((1,0),0) \mapsto 0, \quad ((0,1),0) \mapsto 0, \quad ((0,0),1) \mapsto 0$$

globally, without taking any etale extension of $s$ (cf. Proposition 6.2(1)). Then, when $m = 1$, there does not exist a section $\iota : S \rightarrow X$ of $f$ whose image is $s$: such a section would induce a section $\alpha : \mathbb{N}^2 \oplus_{\Delta,N,1} \mathbb{N} = \mathbb{N}^2 \rightarrow \mathbb{N}$ of the diagonal map $\mathbb{N} \rightarrow \mathbb{N}^2$ with $\alpha^{-1}(0) = \{0\}$, and this is impossible. On the other hand, when $m \geq 2$, there exists a section $\iota : S \rightarrow X$ of $f$ whose image is $s$. (For example, the map

$$\mathbb{N}^2 \oplus_{\Delta,N,m} \mathbb{N} \rightarrow \mathbb{N}; \quad ((a,b),c) \mapsto a(m-1) + b + c$$
is a section of the map \( N \to N^2 \oplus_{\Delta,N,m} N \) in Proposition 6.2(1), and this induces the required section \( \iota \) of \( f \). But this section is not good by definition.

Moreover, if \( s \) is a \( k \)-rational point in the marked divisor, we have a section \( \iota : S \to X \) of \( f \) whose image is \( s \). Indeed, we always have a global chart \((6.1)\) by Remark 6.3 and there exists a section \( \beta : N^2 \to N \) of the map \( N \to N^2 \) in Proposition 6.2 with \( \beta^{-1}(0) = \{0\} \), which induces the required section \( \iota \) of \( f \). (Such a map \( \beta \) is necessarily of the form \((1,0) \mapsto 1, (0,1) \mapsto b \) for some \( b \geq 1 \).) This section is not good either.

Then the ‘if’ part of Theorem 6.3 follows from the following proposition.

**Proposition 6.8.** Let \( f : X \to S \) be a stable log curve such that underlying morphism of schemes of \( f \) is smooth. Then we have the following:

1. If the section \( \iota \) is good, the monodromy action on \( \pi^\text{dR}_1(X/S, \iota) \) is trivial as an element of \( \text{Aut}(\pi^\text{dR}_1(X/S, \iota)) \).
2. If the section \( \iota \) is not good, the monodromy action on \( \pi^\text{dR}_1(X/S, \iota) \) is trivial as an element of \( \text{Out}(\pi^\text{dR}_1(X/S, \iota)) \).

**Proof.** Let \( \epsilon := \iota_*(1) \in \pi^\text{dR}_1(X,x) \) be as before Definition 6.4. Take \( \gamma \in \pi^\text{dR}_1(X/S, \iota) \) and for \((E,\nabla) \in N_f \text{MIC}^n(X/k)\), we calculate the action of \( \epsilon^{-1} \circ \gamma \circ \epsilon \) on \( \iota^*E \).

1. If we denote the image of \((E,\nabla)\) by \( \eta^\text{dR}_* : N_f \text{MIC}^n(X/k) \to \text{MIC}^n(S/k) \) by \((\iota^*E,N)\), the action of \( \epsilon \) on \( \iota^*E \) is given by \( \exp(N) \). Also, since \( \epsilon^{-1} \circ \gamma \circ \epsilon \) is an element in \( \pi^\text{dR}_1(X/S, \iota) \), the action of it on \( \iota^*E \) depends only on the image \((E,\nabla)\) of \((E,\nabla)\) in \( N_f \text{MIC}(X/S) \). Since the underlying morphism of \( f \) is smooth, if we denote the log structure on \( X^\circ \) associated to the marked divisor \( D \) by \( \mathcal{N} \) and denote the log scheme \((X^\circ,\mathcal{N})\) by \( X' \), we have the equivalence \( N_f \text{MIC}(X/S) \cong N_f \text{MIC}(X'/k) \). (Here \( f' : X' \to Spec k \) is the structure morphism.) Denote the object on the right hand side corresponding to \((E,\nabla)\) by \((E,\nabla')\). Then, to calculate the action of \( \epsilon^{-1} \circ \gamma \circ \epsilon \) on \( \iota^*E \), we may replace \((E,\nabla)\) by the image of \((E,\nabla')\) by the functor \( N_f \text{MIC}(X'/k) \to N_f \text{MIC}(X/k) \). If we do so, we see that \( N = 0 \) by construction.

Hence the action of \( \epsilon^{-1} \circ \gamma \circ \epsilon \) on \( \iota^*E \) is the same as that of \( \gamma \) and so \( \epsilon^{-1} \circ \gamma \circ \epsilon = \gamma \).

2. If we endow \( x(:= \text{the image of } S^\circ \text{ by } \iota) \) with the pullback log structure from \( X \), the log structure on \( x \) is induced by the map \( N^2 \to k; (1,0) \mapsto 0, (0,1) \mapsto 0 \) by Remark 6.3 and the restriction of \((E,\nabla)\) to \( x \) is identified with the triple \((E|_x, N_1, N_2)\) consisting of a finite-dimensional \( k \)-vector space endowed with two commuting nilpotent endomorphisms. The map \( \iota' : S \to x \) induced by \( \iota \) is induced by the identity map on \( Spec k \) and the monoid homomorphism \( \beta : N^2 \to N \) as in Remark 6.7 for some \( b \geq 1 \). By pulling back the triple \((E|_x, N_1, N_2)\) by \( \iota' \), we see that the image of \((E,\nabla)\) by \( \iota^* \text{dR} \) is \((\iota^*E, N_1 + bN_2)\). Hence the action of \( \epsilon \) on \( \iota^*E \) is given by \( \exp(N_1 + bN_2) \). As in the proof of (1), to know the action of \( \epsilon^{-1} \circ \gamma \circ \epsilon \) on \( \iota^*E \), we may replace \((E,\nabla)\) by the image of \((E,\nabla')\) in \( N_f \text{MIC}^n(X/k) \). If we do so, we see that \( N_1 = 0 \). Also, the functorial action \( \exp(bN_2) \) on all \( \iota^*E \)'s defines an element \( \eta \) in \( \pi^\text{dR}_1(X/S, \iota) \), because \( N_2 \) only depends on the image of \((E,\nabla)\) in \( N_f \text{MIC}(X/S) \). Then the action of \( \epsilon^{-1} \circ \gamma \circ \epsilon \) on \( \iota^*E \) is the same as that of \( \eta^{-1} \circ \gamma \circ \eta \).
and so $\epsilon^{-1} \circ \gamma \circ \epsilon = \eta^{-1} \circ \gamma \circ \eta$. Therefore, the monodromy action on $\pi_1^{dR}(X/S, \iota)$ is trivial as an element of $\text{Out}(\pi_1^{dR}(X/S, \iota))$.

We deal now with the ‘only if’ part of Theorem 6.5, we will introduce some propositions in order to reduce the proof to simpler cases.

**Proposition 6.9.** Let $k \subseteq k'$ be an extension of fields. Then the ‘only if’ part of Theorem 6.5 is true if it is true for the base extension

$$f' : X' := X \times_{\text{Spec } k} \text{Spec } k' \longrightarrow S' := S \times_{\text{Spec } k} \text{Spec } k'$$

$$\iota' : S' \longrightarrow X'$$

of $f, \iota$ by the morphism $\text{Spec } k' \longrightarrow \text{Spec } k$.

**Proof.** Note that we have the isomorphism $\pi_1^{dR}(X'/S', \iota') \cong \pi_1^{dR}(X/S, \iota) \times_{\text{Spec } k} \text{Spec } k'$ which is compatible with monodromy action. This follows from the second definition (Definition 3.11) of $\pi_1^{dR}(X/S, \iota)$ and the fact that $W_n$ in Section 3 is compatible with base change by the morphism $\text{Spec } k' \longrightarrow \text{Spec } k$ (Remark 3.3). Also, it follows from the third definition (Definition 4.7) of $\pi_1^{dR}(X/S, \iota)$ and the fact that the construction of 1-minimal model $M_{X/S}$ is compatible with base change by the morphism $\text{Spec } k' \longrightarrow \text{Spec } k$. Also, the underlying morphism $f^\circ$ of $f$ is smooth if and only if that $f'^\circ$ of $f'$ is smooth. Hence, if $f^\circ$ is not smooth, $f'^\circ$ is not smooth either and so the monodromy action is not trivial as an element in $\text{Out}(\pi_1^{dR}(X'/S', \iota'))$. Hence it is not trivial as an element in $\text{Out}(\pi_1^{dR}(X/S, \iota))$ either. So we are done.

To prove the next lemma we need an explicit description of certain log blow-ups of a proper log curve $X \longrightarrow S$. Assume that $k$ is algebraically closed and let $s$ be a double point of $X$. Then the local description of $X$ around $s$ is given in Proposition 6.2(1). Assume that $m \geq 2$ in the local description there. If we consider the ideal $J := (N^2 + \Delta_{m, n} N) \setminus \{0\} \subseteq N^2 + \Delta_{m, n} N$ and glue the ideal (defined etale locally around $s$) in $\mathcal{M}_X$ generated by $J$ and $\mathcal{M}_X|_{X\setminus\{s\}}$, we obtain the coherent ideal $\mathcal{J}$ of $\mathcal{M}_X$. We denote the log blow-up of $X$ with respect to $\mathcal{J}$ by $h : X' \longrightarrow X$.

Etale locally, $X'$ is strict etale over the log blow-up of the log scheme

$$X_0 := (\text{Spec } k[x, y]/(xy), (N^2 + \Delta_{m, n} N)^n)$$

in Proposition 6.2(1) by the ideal $J$. By [N06 Proposition 4.3, Corollary 4.8], it is the closed subscheme $X'_0$ defined by $t$ of the normalization $Y'$ of the blow up of $Y := \text{Spec } k[x, y, t]/(xy - t^m)$ along the ideal $(x, y, t)$. Because $Y'$ is covered by the schemes

(6.3) $\text{Spec } k[x, t/x], \text{ Spec } k[y, t/y], \text{ Spec } k[x/t, y/t, t]/((x/t)(y/t) - t^{m-2}),$

$X'_0$ is covered by the schemes

(6.4) $\text{Spec } k[x, t/x]/(x(t/x)), \text{ Spec } k[y, t/y]/(y(t/y)), \text{ Spec } k[x/t, y/t]/((x/t)(y/t))$
in the case \(m \geq 3\), and they are endowed with log structures associated to \(\mathbb{N}^2, \mathbb{N}^2 \oplus_{\Delta, \mathbb{N}, m-2} \mathbb{N}\), respectively, i.e. the cases (1) with \(m = 1\), (1) with \(m = 1\) and (1) with \(m\) replaced by \(m-2\) in the notation introduced in Proposition 6.2. In the case \(m = 2\), the third affine scheme in (6.4) should be replaced by \(\text{Spec } k[x/t, y/t]/((x/t)(y/t)−1)\) endowed with the log structure associated to \(\mathbb{N}^2 \oplus \Delta, \mathbb{N}, 0\). Because the basis of the first factor \(\mathbb{N}^2\) are sent to \(x/t\) and \(y/t\) which are now invertible, the log structure is associated to the first factor \(\mathbb{N}\). Hence we are in the case (3) of Proposition 6.2.

From this description, we see that \(X' \to S\) is again a log curve. Then we have the following:

**Lemma 6.10.** Let \(f : X \to S\) be a proper log curve with a section \(\iota : S \to X\) and let \(h : X' \to X\) be the log blow-up at the double point \(s\) as above. Also, let \(f' : X \to S\) be the composite \(f \circ h\) and let \(\iota' : S \to X'\) be the section induced by \(\iota\). (Note that \(h\) is an isomorphism on \(X\) \(\{s\}\) and that the image of \(\iota\) cannot be \(r\) due to Remark 6.7.) Then we have the isomorphism

\[
h_* : \pi^\text{dR}_1(X'/S, \iota') \xrightarrow{\cong} \pi^\text{dR}_1(X/S, \iota)
\]

which is compatible with monodromy action.

**Proof.** First we prove the isomorphism \(H^n_{\text{dR}}(X/S) \xrightarrow{\cong} H^n_{\text{dR}}(X'/S)\) of log de Rham cohomologies for any \(n\). Let \(X_0, X'_0, Y, Y'\) be as in the notation before the lemma. Then we have the Cartesian diagram

\[
\begin{array}{ccc}
X_0' & \to & Y' \\
\downarrow h_0 & & \downarrow h_Y \\
X_0 & \to & Y,
\end{array}
\]

where \(h_0, h_Y\) are log blow-ups and horizontal maps are exact closed immersions. Because the log blow-up \(h\) is log etale, we have the isomorphism \(h^*\Omega^1_{X/S} \cong \Omega^1_{X'/S}\).

By using this, the projection formula and a spectral sequence, we see that, to prove the isomorphism in the lemma, it suffices to prove that the morphism

\[
\mathcal{O}_X \to Rh_*\mathcal{O}_{X'}
\]

is an isomorphism. (See the proof of Proposition 1.18). This is further reduced to proving the isomorphism

\[
\mathcal{O}_{X_0} \xrightarrow{\cong} Rh_{0,*}\mathcal{O}_{X'_0}.
\]

Thanks to [N06, Corollary 4.7] and [Kk94, Theorem 11.3], we have the isomorphism

\[
(6.5) \quad \mathcal{O}_Y \xrightarrow{\cong} Rh_{Y,*}\mathcal{O}_{Y'}.
\]

Moreover, since the multiplication by \(t\) is injective on the rings appearing in (6.3), \(\mathcal{O}_{X'_0}\) and \(\mathcal{O}_{Y'}\) are Tor-independent over \(\mathcal{O}_{X'}\). Hence, by [ST] Tag 08IB and (6.5), we have the isomorphism

\[
\mathcal{O}_{X_0} = \mathcal{O}_{X_0} \otimes_{\mathcal{O}_{Y'}} \mathcal{O}_Y = \mathcal{O}_{X_0} \otimes_{\mathcal{O}_{Y'}} Rh_{Y,*}\mathcal{O}_{Y'} \cong Rh_{0,*}L(\mathcal{O}_{X_0} \otimes_{\mathcal{O}_{Y'}} \mathcal{O}_Y, \mathcal{O}_{Y'}) = Rh_{0,*}\mathcal{O}_{X'_0},
\]

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as required. So we have the isomorphism $H^n_{dR}(X/S) \cong H^n_{dR}(X'/S)$.

The above isomorphism of cohomologies implies the equivalence $h^*_n : \text{NfMIC}(X/S) \cong \text{NfMIC}(X'/S)$, hence the isomorphism $h^*_n : \pi^n_{dR}(X'/S, \iota') \cong \pi^n_{dR}(X/S, \iota)$.

Noting that this isomorphism fits into the commutative diagram

$1 \rightarrow \pi^1_{dR}(X'/S, \iota') \rightarrow \pi^1_{dR}(X', \iota') \stackrel{f_*}{\rightarrow} \pi^1_{dR}(S, \iota) \cong \mathbb{G}_{a,k} \stackrel{\text{id}}{\rightarrow} 1$

of split exact sequences, we see that the vertical arrows are isomorphisms and that the isomorphism in the lemma is compatible with monodromy action. So the lemma is proved.

Let us introduce the notion of (minimal) semistable log curve.

**Definition 6.11.** When $k$ is algebraically closed, a proper log curve $X \rightarrow S$ is called a semistable log curve if, in the local description around any double point of $X$ given in Proposition 6.2, we can take $m = 1$. A semistable log curve $X \rightarrow S$ is called minimal if it is not a smooth elliptic curve with no marked point and, for any irreducible component $Y$ of $X^o$ birational to $\mathbb{P}^1_k$ which intersects with other components at $\leq 1$ point, the cardinality of double points of $X$ in $Y$ (counted doubly when a double point is a self intersection point) plus that of marked points in $Y$ is $\geq 3$. For general $k$, a proper log curve $f : X \rightarrow S$ is called a (minimal) semistable log curve if so is its geometric fiber.

What we actually prove from the next subsection is the following theorem:

**Theorem 6.12.** Let $f : X \rightarrow S$ be a minimal semistable log curve and let $\iota : S \rightarrow X$ be its section. Then the monodromy action on $\pi^1_{dR}(X/S, \iota)$ is nontrivial as an element in $\text{Out}(\pi^1_{dR}(X/S, \iota))$ if the underlying morphism of schemes of $f$ is not smooth.

The following proposition allows us to reduce the ‘only if’ part of Theorem 6.5 to the above theorem:

**Proposition 6.13.** The ‘only if’ part of Theorem 6.5 follows from Theorem 6.12.

**Proof.** We assume the validity of Theorem 6.12 and prove the ‘only if’ part of Theorem 6.5. By Proposition 6.9 we may assume that $k$ is algebraically closed. Assume that we are given a stable log curve $f : X \rightarrow S$ and its section $\iota : S \rightarrow X$ such that $X^o$ is not smooth over $k$. By the local description (6.4) of log blow-up as above, we see that there exists a composition of log blow-ups $X' \rightarrow X$ as before Lemma 6.10 (possibly identity) such that $X'$ is a semistable log curve with $X'^o$ not
smooth over $k$. Moreover, since $X$ is stable and the new components appearing in $X'$ are isomorphic to $\mathbb{P}_k^1$ and intersects with other components at 2 points, $X'$ is minimal. Thus, by Theorem 6.12 the monodromy action is nontrivial as an element of $\text{Out}(\pi^\text{dR}_1(X'/S, \iota'))$ (where $\iota' : S \to X'$ is the section induced by $\iota$). Then, by Lemma 6.10 we conclude that the monodromy action is nontrivial as an element of $\text{Out}(\pi^\text{dR}_1(X/S, \iota))$.

Finally we prove a proposition which allows us to change the base points for log de Rham fundamental groups. Let $f : X \to S$ be a stable log curve or a minimal semistable log curve with section $\iota : S \to X$ and suppose that we are given a diagram of exact faithful $k$-linear tensor functors

\[
\begin{array}{ccc}
N_f \text{MIC}(X/S) & \xrightarrow{\omega} & N_f \text{MIC}_n(X/k) \\
\downarrow{\tilde{\omega}} & & \downarrow{f^*} \\
\text{Vec}_k & \xrightarrow{s^*_\text{dR}} & \text{MIC}_n(S/k)
\end{array}
\]

(where $r$ is the restriction functor) with $\tilde{\omega} \circ f^*_\text{dR} = \text{id}, s^*_\text{dR} \circ \tilde{\omega} = \omega, \omega \circ f^*_\text{dR} = s^*_\text{dR}, \omega \circ r = \omega$, and denote the Tannaka dual of $(N_f \text{MIC}(X/S), \omega), (N_f \text{MIC}_n(X/k), \omega)$ by $\pi^\text{dR}_1(X/S, \omega), \pi^\text{dR}_1(X, \omega)$, respectively. Then we have the following:

**Proposition 6.14.** Let the notation be as above. Then the diagram (6.6) of functors induces the split exact sequence

\[
\begin{array}{ccc}
1 & \to & \pi^\text{dR}_1(X/S, \omega) \\
\downarrow{f_\omega} & & \downarrow{f_\omega} \\
\pi^\text{dR}_1(X, \omega) & \to & \pi^\text{dR}_1(S, s) \\
\downarrow{\tilde{\omega}^*} & & \downarrow{\tilde{\omega}^*} \\
1 & & 1
\end{array}
\]

Moreover, the induced monodromy action on $\pi^\text{dR}_1(X/S, \omega)$ by the diagram is nontrivial as an element in $\text{Out}(\pi^\text{dR}_1(X/S, \omega))$ if and only if the monodromy action on $\pi^\text{dR}_1(X/S, \iota)$ is nontrivial as an element in $\text{Out}(\pi^\text{dR}_1(X/S, \iota))$. In particular, the validity of ‘only if’ part of Theorem 6.5 and Theorem 6.12 is independent of the choice of fiber functors as in the diagram (6.6).

**Proof.** First note that the functor

\[
(k\text{-algebras}) \to (\text{Sets}); \quad R \mapsto \begin{cases} \text{tensor isomorphisms from} \\
N_f \text{MIC}(X/S) \xrightarrow{s^*_\text{dR}} \text{Vec}_k \to \text{Proj}_R \\
N_f \text{MIC}(X/S) \xrightarrow{\tilde{\omega}} \text{Vec}_k \to \text{Proj}_R
\end{cases}
\]

(where $\text{Proj}_R$ is the category of finitely generated projective $R$-modules) is representable by an affine $k$-scheme $\pi^\text{dR}_1(X/S, x^*_\text{dR}, \omega)$ which is faithfully flat over $k$ ([DM82, Theorem 3.2]). Since we may enlarge $k$ to prove the proposition, we can assume that $\pi^\text{dR}_1(X/S, x^*_\text{dR}, \omega)$ has a $k$-rational point $\sigma$, namely, that there exists
Let the notations be as above.

Proposition 6.15.\(\sigma\) be the isomorphism

\[
\pi^\text{dR}_1(X/S,\iota) \overset{\cong}{\rightarrow} \pi^\text{dR}_1(X/S,\overline{\omega}); \quad \alpha \mapsto \sigma \circ \alpha \circ \sigma^{-1}.
\]

Also, let \(\sigma'\) be the isomorphism from the fiber functor \(x^*_{\text{dr}} : N_f\text{MIC}^n(X/k) \rightarrow \text{Vec}_k\) to the fiber functor \(\omega : N_f\text{MIC}^n(X/k) \rightarrow \text{Vec}_k\) induced by \(\sigma\) and let \(s(\sigma)\) be the isomorphism

\[
\pi^\text{dR}_1(X,x) \overset{\cong}{\rightarrow} \pi^\text{dR}_1(X,\omega); \quad \alpha \mapsto \sigma' \circ \alpha \circ \sigma'^{-1}.
\]

Then we have the commutative diagram

\[
\begin{array}{cccccccc}
1 & \rightarrow & \pi^\text{dR}_1(X/S,\iota) & \rightarrow & \pi^\text{dR}_1(X,x) & \rightarrow & \pi^\text{dR}_1(S,s) & \rightarrow & 1 \\
\downarrow{\pi(\sigma)} & & \downarrow{s(\sigma)} & & \downarrow{id} & & & \\
1 & \rightarrow & \pi^\text{dR}_1(X/S,\overline{\omega}) & \rightarrow & \pi^\text{dR}_1(X,\omega) & \rightarrow & \pi^\text{dR}_1(S,s) & \rightarrow & 1,
\end{array}
\]

where the top horizontal line is the exact sequence in Proposition 2.3. Indeed, the left square is commutative by construction, and for the right square, \(s(\sigma)\) induces a conjugation on \(\pi^\text{dR}_1(S,s)\) by some element, which is necessarily trivial because \(\pi^\text{dR}_1(S,s) \cong \mathbb{G}_{a,k}\) is commutative. So the bottom horizontal line is also exact and so we obtain the required exact sequence.

As for the monodromy action, it is easy to see that, via the isomorphisms \(\pi(\sigma)\), the conjugate action of \(\iota_*\) in \(\pi^\text{dR}_1(X,x)\) on \(\pi^\text{dR}_1(X/S,\iota)\) and that of \(\overline{\omega}^*\) in \(\pi^\text{dR}_1(X,\omega)\) on \(\pi^\text{dR}_1(X/S,\overline{\omega})\) differ by some inner action on \(\pi^\text{dR}_1(X/S,\overline{\omega})\). Thus we can identify them in \(\text{Out}(\pi^\text{dR}_1(X/S,\overline{\omega}))\) and so we obtain the latter assertion of the proposition.\(\square\)

6.2 Calculation of Lie algebra

Assume \(k\) algebraically closed and let \(f : X \rightarrow S\) be a semistable log curve. In this subsection, we give an explicit calculation of the pro-nilpotent Lie algebra \(L(X/S)\) associated to the de Rham fundamental group \(\pi^\text{dR}_1(X/S,\iota)\). Although the results in this subsection are well-known through transcendental arguments, we give here purely algebraic proofs.

Put \(g := \dim H^1(X,\mathcal{O}_X)\) and call it the genus of \(f : X \rightarrow S\). Also, let \(r\) be the number of marked points of \(f : X \rightarrow S\). First recall the following result on the de Rham cohomologies of \(X\) over \(S\).

**Proposition 6.15.** Let the notations be as above.

1. If \(f : X \rightarrow S\) is unmarked \((r = 0)\), we have \(\dim H^0_\text{dR}(X/S) = 1, \dim H^1_\text{dR}(X/S) = 2g, \dim H^2_\text{dR}(X/S) = 1, \dim H^3_\text{dR}(X/S) = 0\ (n \geq 3)\).
2. If \(f : X \rightarrow S\) is marked \((r > 0)\), we have \(\dim H^0_\text{dR}(X/S) = 1, \dim H^1_\text{dR}(X/S) = 2g + r - 1, \dim H^2_\text{dR}(X/S) = 0\ (n \geq 2)\).
Proof. In both cases, $\dim H^0_{\text{dr}}(X/S) = 1$ by geometric reducedness and geometric connectedness of $f$. Also, $\dim H^n_{\text{dr}}(X/S) = 0$ for $n \geq 3$ because the cohomological dimension for coherent cohomology of $X$ is 1 (which is well-known in the case of smooth curves and we can reduce to this case by considering normalization).

(1) In this case $\Omega^1_{X/S}[1]$ is a dualizing sheaf by [Ts99, Theorem 2.21]; in fact using [Ts99, Corollary 2.6] one can find that the ideal $I_f$ in [Ts99, Theorem 2.21] is the whole structure sheaf $\mathcal{O}_X$. We have $\dim H^0(X, \Omega^1_{X/S}) = g$ by Grothendieck-Serre duality (see [Ts99, Theorem 1.5] for example), and by $E_1$-degeneration of Hodge-de Rham spectral sequence [Kk88], we see that $\dim H^1_{\text{dr}}(X/S) = 2g$. By Tsuji’s Poincaré duality [Ts99, §3], we see that $\dim H^2_{\text{dr}}(X/S) = 1$.

(2) Let $D$ denote the divisor given by the marked points. In this case $\Omega^1_{X/S}(-D)[1]$ is the dualizing sheaf by [Ts99, Theorem 2.21]: in this case the ideal $I_f$ in [Ts99, Theorem 2.21] is $\mathcal{O}_X(-D)$. By Tsuji’s Poincaré duality, $\dim H^2_{\text{dr}}(X/S) \leq \dim H^0(X, \mathcal{O}(-D)) = 0$.

Also, if we denote the semistable log curve which we obtain by erasing one marked point $P$ of $X$ by $X'$, we have the exact sequence of de Rham complex

$$0 \longrightarrow \Omega^\bullet_{X'/S} \longrightarrow \Omega^\bullet_{X/S} \longrightarrow k_P[-1] \longrightarrow 0.$$ 

Then the claim for $\dim H^1_{\text{dr}}(X/S)$ follows from the result for $\dim H^n_{\text{dr}}(X/S)$ ($n = 0, 2$), the long exact sequence associated to the above sequence and induction on $r$.  

The following proposition, which calculates $L(X/S)$ in the marked case, is the first main result in this subsection.

**Proposition 6.16.** When $r > 0$, $L(X/S)$ is a free pro-nilpotent Lie algebra of rank $2g + r - 1$.

Proof. We construct explicitly the 1-minimal model of the $k$-dgca $\mathcal{A}_{X/S}$ as in Section 4. First note that $H^n(\mathcal{A}_{X/S}) = H^n_{\text{dr}}(X/S)$ ($n \in \mathbb{N}$). Put $V := H^1(\mathcal{A}_{X/S})$ and take a section $\sigma : V \longrightarrow Z^1(\mathcal{A}_{X/S})$ of the canonical projection $Z^1(\mathcal{A}_{X/S}) \longrightarrow H^1(\mathcal{A}_{X/S})$. Then define the $k$-dgca $B_{X/S}$ as

$$B^0_{X/S} := k, \quad B^1_{X/S} := V, \quad B^n_{X/S} = 0 \quad (n \geq 2)$$

with zero differential, and define the morphism $i : B_{X/S} \longrightarrow \mathcal{A}_{X/S}$ by the maps $k \hookrightarrow \mathcal{A}_{X/S}, \sigma \hookrightarrow Z^1(\mathcal{A}_{X/S}) \hookrightarrow \mathcal{A}_{X/S}$. Then $i$ is a quasi-isomorphism by Proposition 6.15(2). Hence, to construct the 1-minimal model of $\mathcal{A}_{X/S}$, it suffices to construct that of $B_{X/S}$.

Let $L$ be the free Lie algebra associated to the dual $V^\vee$ of $V$, and let $\{\text{Fil}^n L\}_n$ be the central descending filtration of $L$. (We follow the convention $L = \text{Fil}^1 L$.) For $q \in \mathbb{N}$, we put $L_q := L/\text{Fil}^{q+1} L$, denote its dual by $L^\vee_q$ and put $L^\vee := \lim_{\longrightarrow} L^\vee_q$. Define the
$k$-dgca $\bigwedge L_q^\vee$ (where $L_q^\vee$ is sitting in degree 1) by introducing the differential induced by the map $L_q^\vee \to \bigwedge^2 L_q^\vee$ dual to $[-, , ] : L_q \wedge L_q \to L_q$ (the minus sign comes from the shift of the degree). Also, we define the map of $k$-dgca’s $\rho_q : \bigwedge L_q^\vee \to B_{X/S}$ as the one induced by the $k$-linear map $\rho^1_q : L_q^\vee \to V$, which is defined inductively in the following way: If $q = 1$, we define the map $\rho^1_q : L_q^\vee \to V$ as the canonical map $L_1^\vee \xrightarrow{\cong} (V^\vee)^\vee = V$. For general $q$, we define the map $\rho^1_q : L_q^\vee \to V$ as any chosen map extending $\rho^1_{q-1}$.

To prove the proposition, it suffices to prove that the map $\rho_q$ is the $(1, q)$-minimal model for any $q$ (and so $\varinjlim \rho_q : \bigwedge L_q^\vee \to B_{X/S}$ is the 1-minimal model and $L(X/S) = \varinjlim q L_q$). By definition of the $(1, q)$-minimal model, we have to prove that $\bigwedge L_q^\vee \hookrightarrow \bigwedge L_{q+1}$ is a Hirsch extension, that $H^0(\bigwedge L_q^\vee) = k$, that $H^1(\bigwedge L_q^\vee) = H^1(B_{X/S}) = V$ and that the map

$$H^2(\bigwedge L_q^\vee, B_{X/S}) = H^2(\bigwedge L_q^\vee) \to H^2(\bigwedge L_{q+1}^\vee, B_{X/S}) = H^2(\bigwedge L_{q+1}^\vee)$$

is zero.

By using the fact that the map $[-, , ] : L_{q+1} \wedge L_{q+1} \to L_{q+1}$ factors through $L_q \wedge L_q$, we see that the inclusion $\bigwedge L_q^\vee \hookrightarrow \bigwedge L_{q+1}$ is a Hirsch extension. It is easy to see that $H^0(\bigwedge L_q^\vee) = k$. It is also easy to see that the homology of

$$L_q \wedge L_q \xrightarrow{-[, ,]^{-1}} L_q \xrightarrow{0} k$$

is $L_1$ and so $H^1(\bigwedge L_q^\vee) = L_1^\vee = V$. Hence, to see that $\rho_q$ is the $(1, q)$-minimal model for any $q \in \mathbb{N}$, it suffices to see that the map $H^2(\bigwedge L_q^\vee) \to H^2(\bigwedge L_{q+1}^\vee)$ is zero. To prove this, we embed the complex

$$\bigwedge L_{q+1}^\vee := [k \to L_{q+1}^\vee \to \bigwedge L_{q+1}^\vee \to \bigwedge L_{q+1}^\vee \to \cdots]$$

in the complex

$$(\bigwedge L)^* := [k \to L^* \to (\bigwedge L)^* \to (\bigwedge L)^* \to \cdots]$$

calculating the Lie algebra cohomology $H^n_{\text{Lie}}(L, k)$. (Here, for a possibly infinite-dimensional vector space $W$, we denoted its dual by $W^*$ to distinguish from the notation $L^\vee$, which is the ind-finite dimensional dual of the pro-finite dimensional vector space $L$.) Because $L$ is a free Lie algebra of rank $2g + r − 1$, the category of $L$-modules is identified with the category of modules over a free group $G$ of rank $2g + r − 1$. Because the cohomological dimension of $G$ is 1, we see that $H^n((\bigwedge L)^*) = H^n_{\text{Lie}}(L, k) = H^n(G, k) = 0$ for $n \geq 2$. Now let $\varphi$ be a 2-cocycle of the complex $\bigwedge L_q^\vee$, regarded as the linear form $\varphi : \bigwedge^2 L_q \to k$. Since it is a 2-coboundary in the complex $(\bigwedge L)^*$, we see that there exists a linear form $\psi : L \to k$ such that the composite $\bigwedge^2 L \to \bigwedge^2 L_q \xrightarrow{\varphi} k$ is written as $\bigwedge^2 L \xrightarrow{[ , , ]^{-1}} L \xrightarrow{\psi} k$.  

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Since $\text{Fil}^{q+1} L \otimes L$ is sent to zero by the former map, so is by the latter map and so $\psi$ factors through $L_{q+1}$. Thus we see that $\varphi$ is a 2-coboundary in the complex $\bigwedge L_q^\vee$. So we have shown that the map $H^2(\bigwedge L_q^\vee) \to H^2(\bigwedge L_{q+1}^\vee)$ is zero, as required. So the proof is finished.

Next we calculate $L(X/S)$ in the unmarked case by comparing with the marked case. In the following, we denote the free pro-nilpotent Lie algebra of rank $n$ with generator $w_1, \ldots, w_n$ by $L(w_1, \ldots, w_n)$ and denote the ideal of $L(w_1, \ldots, w_n)$ generated by an element $y \in L(w_1, \ldots, w_n)$ by $\langle y \rangle$. Then we have the following proposition, which is the second main result of this subsection:

**Proposition 6.17.** When $r = 0$, we have the isomorphism of pro-nilpotent Lie algebras

$$L(X/S) \cong L(w_1, \ldots, w_2g)/\langle \sum_{i=1}^g [w_{2i-1}, w_{2i}] \rangle.$$  

**Proof.** Let $P^o$ be the image of the section $\iota : S \to X$ (with trivial log structure). Let $f' : X' \to S$ be the proper log curve we obtain by adding a marked point at $P^o$ to $X$, and let $P$ be the log scheme $(P^o, \mathcal{M}_{X'|P^o})$. Also, let $f_P : P \to S$ be the composite $P \hookrightarrow X' \xrightarrow{f'} S$. Then we can define the category $\text{MIC}_n^a(P/S)$, which is identified with the category of pairs $(V, N)$ of a finite-dimensional $k$-vector space $V$ and a nilpotent endomorphism $N$ on $V$. Thus the Lie algebra $L(P/S)$ associated to its Tannaka dual, $\mathbb{G}_{a,k}$, is the abelian Lie algebra $k$. Also, we have the commutative diagram

$$\begin{array}{ccc}
\text{MIC}_n^a(P/S) & \longrightarrow & \text{MIC}(S/S) \\
\uparrow & & \uparrow \\
\text{N}f\text{MIC}(X'/S) & \longrightarrow & \text{N}f\text{MIC}(X/S)
\end{array}$$

of categories. We can associate to it the corresponding commutative diagram

$$\begin{array}{ccc}
L(P/S) & \longrightarrow & 0 \\
\downarrow & & \downarrow \\
L(X'/S) & \longrightarrow & L(X/S)
\end{array}$$

(6.8)

of pro-nilpotent Lie algebras, by taking the pro-nilpotent Lie algebras associated to the Tannaka duals of the categories in the diagram. Since an object in $\text{N}f\text{MIC}(X/S)$ is nothing but an object in $\text{N}f\text{MIC}(X'/S)$ trivial in $\text{N}f\text{MIC}(P/S)$, the diagram (6.8) is a push-out diagram in the category of pro-nilpotent Lie algebras. Thus $L(X/S)$ is the quotient of $L(X'/S)$ by the ideal generated by one element, denoted with $z$, which is the image of 1 by $k = L(P/S) \to L(X'/S)$.

To identify the element $z$ (up to scalar), we investigate the construction of $(1, q)$-minimal models of $\mathcal{A}_{X/S}, \mathcal{A}_{X'/S}$ for $q = 1, 2$. First we put $V_1 := H^1(\mathcal{A}_{X/S}) = ...$
\(H^1(A_{X/S})\) (the latter equality follows from the computation in the proof of Proposition 6.15) and take sections \(\sigma_1 : V_1 \rightarrow Z^1(A_{X/S}), \sigma'_1 : V_1 \rightarrow Z^1(A_{X'/S})\) of the canonical projections \(Z^1(A_{X/S}) \rightarrow H^1(A_{X/S}), Z^1(A_{X'/S}) \rightarrow H^1(A_{X'/S})\) in a compatible way. Using the maps \(\sigma_1, \sigma'_1\), we obtain the diagram of \(k\)-dgca’s

\[
\begin{array}{ccc}
\wedge V_1 & \longrightarrow & A_{X/S} \\
\downarrow & & \downarrow \\
\wedge V_1 & \longrightarrow & A_{X'/S}
\end{array}
\]

(with zero differentials on \(\wedge V_1\)) in which the horizontal arrows are \((1,1)\)-minimal models. Next we put

\[
V_2 := H^2(\wedge V_1, A_{X/S}) = \text{Ker}(\bigwedge^2 V_1 \longrightarrow H^2(A_{X/S})),
\]

\[
V'_2 := H^2(\wedge V_1, A_{X'/S}) = \bigwedge^2 V_1
\]

and take sections \(\sigma_2 : V_2 \rightarrow Z^2(\wedge V_1, A_{X/S}), \sigma'_2 : V'_2 \rightarrow Z^2(\wedge V_1, A_{X'/S})\) of the canonical projections \(Z^2(\wedge V_1, A_{X/S}) \rightarrow H^2(\wedge V_1, A_{X/S}), Z^2(\wedge V_1, A_{X'/S}) \rightarrow H^2(\wedge V_1, A_{X'/S})\) in a compatible way. Using the \((1,1)\)-minimal models and the maps \(\sigma_2, \sigma'_2\), we obtain the diagram of \(k\)-dgca’s

\[
\begin{array}{ccc}
\wedge(V_1 \oplus V_2) & \longrightarrow & A_{X/S} \\
\downarrow & & \downarrow \\
\wedge(V_1 \oplus V'_2) & \longrightarrow & A_{X'/S}
\end{array}
\]

(where the differential on \(\wedge(V_1 \oplus V_2)\) is induced by \(V_2 \hookrightarrow \bigwedge^2 V_1\) and that on \(\wedge(V_1 \oplus V'_2)\) is similarly defined) in which the horizontal arrows are \((1,2)\)-minimal models. From this construction, we see that the surjection of Lie algebras

\[
(6.9) \quad L(X'/S)/\text{Fil}^3L(X'/S) \longrightarrow L(X/S)/\text{Fil}^3L(X/S)
\]

is identified with the surjection

\[
(6.10) \quad (V_1 \oplus V'_2)^\vee \longrightarrow (V_1 \oplus V_2)^\vee.
\]

Let \(v_1, ..., v_{2g}\) be a symplectic basis of \(V_1\) with respect to the map \(\bigwedge^2 V_1 \rightarrow H^2(A_{X/S})\) so that \(v_{2i-1} \wedge v_{2i}\) \((1 \leq i \leq g)\) is sent to a non-zero element of \(H^2(A_{X/S})\), and let \(w_1, ..., w_{2g}\) be the dual basis on \(V_1^\vee\). (Note that \(\bigwedge^2 V_1 \rightarrow H^2(A_{X/S})\) is a surjective map to 1-dimensional \(k\)-vector space because it is identified with the cup product pairing \(\bigwedge^2 H^1_{dr}(X/S) \rightarrow H^2_{dr}(X/S)\) which is perfect by Poincaré duality.) Then \(V'_2\) has a basis \(w_j \wedge w_l (1 \leq j < l \leq 2g)\), and \(-\sum_{i=1}^{g} w_{2i-1} \wedge w_{2i} \in V'_2^\vee \subseteq (V_1 \oplus V'_2)^\vee\) is a generator (as \(k\)-vector space) of the kernel of the surjection
Because this element is equal to $\sum_{i=1}^{g}[w_{2i-1}, w_{2i}]$ by definition of the Lie bracket on $(V_1 \oplus V_2)^* = L(X'/S)/\Fil^3 L(X'/S)$ and $L(X'/S)$ is a free pro-nilpotent Lie algebra with generator $w_1, \ldots, w_{2g}$ by Proposition 6.16 we see that there exists an isomorphism

$$L(X'/S)/\Fil^3 L(X'/S) \cong L(w_1, \ldots, w_{2g})/\Fil^3 L(w_1, \ldots, w_{2g})$$

such that the image of $z$ in $L(X'/S)/\Fil^3 L(X'/S)$ is equal to $\sum_{i=1}^{g}[w_{2i-1}, w_{2i}]$ up to scalar.

To prove the proposition, it suffices to prove that there exists a compatible family of isomorphisms

$$L(X'/S)/\Fil^{q+1} L(X'/S) \cong L(w_1, \ldots, w_{2g})/\Fil^{q+1} L(w_1, \ldots, w_{2g}) \quad (q \geq 2)$$

such that the image of $z$ in $L(X'/S)/\Fil^{q+1} L(X'/S)$ is equal to $\sum_{i=1}^{g}[w_{2i-1}, w_{2i}]$ up to scalar, by induction on $q$. Assume the claim for $q$ and take a lift of $w_i \in L(X'/S)/\Fil^{q+1} L(X'/S)$ to $L(X'/S)/\Fil^{q+2} L(X'/S)$, which we denote by the same letter. Then the image of $z$ in $L(X'/S)/\Fil^{q+2} L(X'/S)$ has the form

$$\sum_{i=1}^{g}[w_{2i-1}, w_{2i}] + \sum_{i=1}^{g}[w_{2i-1}, y_{2i}] + \sum_{i=1}^{g}[y_{2i-1}, w_{2i}]$$

with $y_i \in \Fil^q L(X'/S) \ (1 \leq i \leq 2g)$ up to scalar. Since the above element can be rewritten as

$$\sum_{i=1}^{g}[w_{2i-1} + y_{2i-1}, w_{2i} + y_{2i}]$$

in $L(X'/S)/\Fil^{q+2} L(X'/S)$ and by denoting $w_i + y_i$ again by $w_i$, we obtain the isomorphism

$$L(X'/S)/\Fil^{q+2} L(X'/S) \cong L(w_1, \ldots, w_{2g})/\Fil^{q+2} L(w_1, \ldots, w_{2g})$$

such that the image of $z$ in $L(X'/S)/\Fil^{q+2} L(X'/S)$ is equal to $\sum_{i=1}^{g}[w_{2i-1}, w_{2i}]$ up to scalar. So the proof of the proposition is finished. 

6.3 Tangential sections

In this subsection, we assume that $k$ is algebraically closed, and let $f : X \to S$ be a semistable log curve and $x$ be a double point of $X$ endowed with the pullback log structure. Then, as we saw in Remark 6.7 there is no section of $f$ with image in $x$. However, in this subsection, we define two canonical ‘sections with image $x$’ in Tannaka theoretic sense, which we will call the **tangential sections**. Tangential sections will play an important role in the calculation of monodromy.
Let $f : X \rightarrow S$, $x$ be as above and let $f_x$ be the composite $x \mapsto X \rightarrow S$. Then $f_x$ is associated to a chart

$$
\begin{align*}
\mathbb{N}^2 & \rightarrow k, \quad \mathbb{N} \rightarrow k, \quad \mathbb{N} \rightarrow \mathbb{N}^2. \\
(1, 0) & \mapsto 0, \quad 1 \mapsto 0, \quad 1 \mapsto (1, 1) \\
(0, 1) & \rightarrow 0
\end{align*}
$$

We fix such a chart. The categories $\text{MIC}^n(x/k)$, $\text{MIC}^n(x/S)$ are defined as before and we have the natural functors

$$
\begin{align*}
\text{MIC}^n(x/k) & \rightarrow \text{MIC}^n(x/S) \quad (\text{restriction functor}), \\
x^*_\text{dR} : & \text{MIC}^n(x/k) \rightarrow \text{MIC}(x/x) = \text{Vec}_k, \\
x^*_\text{dR} : & \text{MIC}^n(x/S) \rightarrow \text{MIC}(x/x) = \text{Vec}_k, \\
f^*_x\text{dR} : & \text{MIC}^n(S/k) \rightarrow \text{MIC}^n(x/k).
\end{align*}
$$

Also, by the above expression of the log scheme $x$, we see that the category $\text{MIC}^n(x/k)$ is equivalent to that of triples $(V, N, N')$ consisting of a finite-dimensional $k$-vector space $V$ and two commuting nilpotent endomorphisms $N, N'$ on $V$.

Now, let $\omega, \omega' : \text{MIC}^n(x/k) \rightarrow \text{MIC}^n(S/k)$ be the functors defined by $(V, N, N') \mapsto (V, N)$ and $(V, N, N') \mapsto (V', N')$, respectively. Then we have the following commutative diagram of functors

$$
\begin{array}{ccc}
\text{MIC}^n(x/S) & \xrightarrow{r} & \text{MIC}^n(x/k) \\
\downarrow{x^*_\text{dR}} & & \downarrow{x^*_\text{dR}} \\
\text{Vec}_k & \xrightarrow{f^*_x\text{dR}} & \text{MIC}^n(S/k)
\end{array}
$$

and a similar one with $\omega$ replaced by $\omega'$. If we denote the Tannaka dual of $(\text{MIC}^n(x/k), x^*_\text{dR})$ and $(\text{MIC}^n(x/S), x^*_\text{dR})$ by $\pi^\text{dR}_1(x, x)$ and $\pi^\text{dR}_1(x/S, x)$, respectively, we see easily from the above description of the category $\text{MIC}^n(x/k)$ that we have the split exact sequence

$$
1 \rightarrow \pi^\text{dR}_1(x/S, x) \rightarrow \pi^\text{dR}_1(x, x) = \mathbb{G}^2_{a,k} \xrightarrow{f^*_x\text{dR}} \pi^\text{dR}_1(S, s) = \mathbb{G}_{a,k} \rightarrow 1.
$$

and a similar split exact sequence with $\omega^*$ replaced by $\omega'^*$. We see also that the map $f^*_{x,\text{dR}}$ is the sum and the map $\omega^*$ (resp. $\omega'^*$) is the inclusion of $\mathbb{G}_{a,k}$ into the first (resp. the second) component of $\mathbb{G}^2_{a,k}$. Hence, if we put $\epsilon := \omega^*(1)$ and $\epsilon' := \omega'^*(1)$, $\epsilon'^{-1} \circ \epsilon$ is equal to a non-zero (and non-torsion) element $\eta$ of $\pi^\text{dR}_1(x/S, x)$.

A concrete description of the element $\eta$ is given in the following way. An object in $\text{MIC}^n(x/S)$ is given by a finite-dimensional vector space $V$ endowed with a nilpotent map $N := (N_1, N_2) : V \rightarrow V \otimes \mathbb{Z} (\mathbb{Z}^2/\Delta(\mathbb{Z}))$, where $\Delta : \mathbb{Z} \rightarrow \mathbb{Z}^2$ is the diagonal map. Then the action of $\eta$ on such object is given by $\exp(N_1 - N_2)$.
Now we compose the above construction with the canonical inclusion $x \to X$. Let $\omega_x$ be the composite $N_f^{\text{MIC}}(X/k) \to \text{MIC}^n(x/k) \xrightarrow{\omega} \text{MIC}^n(S/k)$ and define $\omega'_x$ similarly, with $\omega$ replaced by $\omega'$. The functors $\omega_x, \omega'_x$ are called tangential sections of $X$ at $x$. Then we have commutative diagram of functors

\[
\begin{array}{ccc}
N_f^{\text{MIC}}(X/S) & \xrightarrow{\omega_x} & \text{MIC}^n(S/k) \\
\downarrow \text{Vec}_k & & \\
N_f^{\text{MIC}}(X/k) & \xrightarrow{\omega'_x} & \text{MIC}^n(S/k)
\end{array}
\]

and a similar commutative diagram with $\omega_x$ replaced by $\omega'_x$. If we denote the Tannaka dual of $(N_f^{\text{MIC}}(X/k), x_{\text{dr}}), (N_f^{\text{MIC}}(X/S), x'_{\text{dr}})$ by $\pi_1^{\text{dR}}(X, x), \pi_1^{\text{dR}}(X/S, x)$ respectively, we obtain the split exact sequence

\[
1 \to \pi_1^{\text{dR}}(X/S, x) \xrightarrow{\omega_x} \pi_1^{\text{dR}}(X, x) \xrightarrow{f_x} \pi_1^{\text{dR}}(S, s) = G_{a,k} \to 1.
\]

and a similar split exact sequence with $\omega_x^* \to \omega'_x^*$, by Proposition 6.14. If we put $\epsilon_x := \omega_x^*(1)$ and $\epsilon'_x := \omega'_x^*(1)$, $\epsilon'_x^{-1} \circ \epsilon_x$ is equal to the image of $\eta \in \pi_1^{\text{dR}}(x/S, x)$ in $\pi_1^{\text{dR}}(X/S, x)$.

**Remark 6.18.** As we see above, there are two tangential sections at a double point $x$ of $X$. We will use the following geometric description to indicate precisely one choice out of the two. Let $\tilde{X} \to X$ be the normalization of $X^\circ$, endowed with the pullback log structure from that on $X$. Then the inverse image of $x$ consists of two points $x_1, x_2$ (endowed with pullback log structure) which are both isomorphic to $x$ (as log scheme). If we take one point $x_1$, then there is a unique connected component $X_1$ of $\tilde{X}$ containing $x_1$; in our fixed chart $\mathbb{N}^2 \to \mathcal{O}_x = \mathcal{O}_{x_1}$, there is exactly one copy of $\mathbb{N}$ which extends to the zero map $\mathbb{N} \to \mathcal{O}_{X_1}$ in the log structure $\mathcal{M}_{X_1} \to \mathcal{O}_{X_1}$. Also, if we take the other point $x_2$, the corresponding copy of $\mathbb{N}$ in $\mathbb{N}^2$ is different from the one we obtain from $x_1$. Thus, by saying the tangential section at $x_1$ or at $x_2$, we can identify precisely one of the two tangential sections at $x$.

In the rest of this subsection, we give some calculation of local monodromy. Let $f : X \to S$ be a semistable log curve and let $X_1$ be a connected component of the normalization of $X^\circ$, endowed with the pullback log structure from that on $X$. Let $C_1 \subseteq X_1$ be the inverse image of the double points of $X$, let $D_1 \subseteq X_1$ be the inverse image of the marked points of $X$. Then, etale locally around a point in $C_1$, the underlying morphism of the map $f_1 : X_1 \to S$ is given as

\[
X_1^\circ \to \text{Spec } k[t] \to \text{Spec } k
\]
with first morphism etale, and the log structure of \( f_1 \) is associated to the chart

(6.13) \[
\begin{align*}
N^2 & \to O_{X_1}, \quad N \to k, \quad N \to N^2. \\
(1, 0) & \mapsto 0 \quad 1 \mapsto 0 \quad 1 \mapsto (1, 1) \\
(0, 1) & \mapsto t
\end{align*}
\]

Let \( P \) be the submonoid of \( N^2 \) generated by \((1, 1), (0, 1)\) and define the morphisms of log schemes \( f'_1 : X'_1 \to S \) by changing the log structure around \( C_1 \) to the one associated to the chart

(6.14) \[
\begin{align*}
P & \to O_{X_1}, \quad N \to k, \quad N \to P. \\
(1, 1) & \mapsto 0 \quad 1 \mapsto 0 \quad 1 \mapsto (1, 1) \\
(0, 1) & \mapsto t
\end{align*}
\]

Changing the log structure in this way has the effect that the points in \( C_1 \) are changed to marked points of \( \mathbb{P} \) with first morphism etale, and the log structure of \( f \) of log schemes we extend the field \( x \) \( \text{Vec} \) \( P \) the other hand, since \( P^{\text{sp}} = \mathbb{Z}^2 \), we have the equality of log differential modules \( \Omega^1_{X_1/S} = \Omega^1_{X'_1/S} \) and so we have the equivalences

(6.15) \[
N_{f'_1}\text{MIC}(X_1/S) = N_{f'_1}\text{MIC}(X'_1/S), \quad N_{f'_1}\text{MIC}^n(X_1/k) = N_{f'_1}\text{MIC}^n(X'_1/k).
\]

Next we define the log scheme \( X''_1 \) to be the scheme \( X''_1 = X'_1 \) endowed with the log structure associated to the normal crossing divisor \( C_1 \cup D_1 \). Then we have the equivalence

(6.16) \[
N_{f'_1}\text{MIC}(X'_1/S) = N_{f''_1}\text{MIC}(X''_1/k),
\]

where \( f''_1 : X''_1 \to k \) is the structure morphism (no log structure on \( k \)).

For closed points \( x, y \) of \( X \), let \( \pi^\text{dR}_1(X/S, x, y) \) be the \( k \)-scheme representing the \( k \)-scheme representing the set of tensor isomorphisms from the functor \( x^\text{dR}_f : N_{f}\text{MIC}(X/S) \to \text{MIC}(x/x) = \text{Vec}_k \) to the functor \( y^\text{dR}_f : N_{f}\text{MIC}(X/S) \to \text{MIC}(y/y) = \text{Vec}_k \). For closed points \( x, y \) of \( X_1 \), we define the \( k \)-scheme \( \pi^\text{dR}_1(X_1/S, x_1, y_1) \) in a similar way, using the category \( N_{f}\text{MIC}(X_1/S) \) instead of \( N_{f}\text{MIC}(X/S) \). They have a \( k \)-rational point after we extend the field \( k \). If \( x_1, y_1 \) are sent to \( x, y \) respectively by the map \( X_1 \to X \), we have the corresponding morphism \( \pi^\text{dR}_1(X_1/S, x_1, y_1) \to \pi^\text{dR}_1(X/S, x, y) \).

Then we have the following:

**Proposition 6.19.** Let the notations be as above.

1. Let \( x \) be a double point of \( X \) with fixed inverse image \( x_1 \) in \( X_1 \), and let \( \alpha \) be an element of \( \pi^\text{dR}_1(X/S, x) \) coming from \( \pi^\text{dR}_1(X_1/S, x_1) \). Let \( \omega \) be the tangential section at \( x_1 \) and put \( \epsilon := \omega^*(1) \in \pi^\text{dR}_1(X, x) \). Then we have \( \epsilon^{-1} \circ \alpha \circ \epsilon = \alpha \).

2. Let \( x, y \) be double points (possibly equal) of \( X \) with fixed distinct inverse images \( x_1, y_1 \) in \( X_1 \), and let \( \alpha \) be an element of \( \pi^\text{dR}_1(X/S, x, y) \) coming from \( \pi^\text{dR}_1(X_1/S, x_1, y_1) \). Let \( \omega_x, \omega_y \) be the tangential section at \( x_1, y_1 \) respectively and put \( \epsilon_x := \omega_x^*(1) \in \pi^\text{dR}_1(X, x), \epsilon_y := \omega_y^*(1) \in \pi^\text{dR}_1(X, y) \). Then we have \( \epsilon_y^{-1} \circ \alpha \circ \epsilon_x = \alpha \).
Proposition 6.20. Let the notations be as above. Assume that \( X_1 \) is isomorphic to \( \mathbb{P}^1_k \) with \( D_1 = \emptyset, C_1 = \{x_1, y_1\} \) and that the image of \( x_1, y_1 \) in \( X \), which we denote by \( x, y \), are different. Let \( \alpha \) be an element of \( \pi_1^{\text{DR}}(X_1/S, x_1, y_1) \) defined by
\[
E|_x \xleftarrow{\sim} \Gamma(X_1, E) \xrightarrow{\sim} E|_y \quad ((E, \nabla) \in N_{f_1, \text{MIC}}(X_1/S))
\]
and denote its image in \( \pi_1^{\text{DR}}(X/S, x, y) \) by the same letter. On the other hand, let \( \omega_x \) (resp. \( \omega_y \)) be the tangential section at \( x_1 \) (resp. \( y_1 \)) and let \( \omega'_x \) (resp. \( \omega'_y \)) be the tangential section at \( x \) (resp. \( y \)) which is different from \( \omega_x \) (resp. \( \omega_y \)). Also, let \( \eta_x \) (resp. \( \eta_y \)) be the image of \( \omega'^*_x(1)^{-1} \circ \omega'^*_y(1) \) (resp. \( \omega'^*_y(1)^{-1} \circ \omega'^*_y(1) \)) in \( \pi_1^{\text{DR}}(X/S, x) \) (resp. \( \pi_1^{\text{DR}}(X/S, y) \)). Then we have \( \eta^{-1}_y \circ \alpha = \alpha \circ \eta_x \).

Remark 6.21. Note that, when \( X_1 \) is isomorphic to \( \mathbb{P}^1_k \), \( E \) is a finite direct sum of \( \mathcal{O}_{X_1} \) for any \((E, \nabla) \in N_{f_1, \text{MIC}}(X_1/S)\) because \( H^1(X_1, \mathcal{O}_{X_1}) = 0 \) (see [D89, Proposition 12.3]). Thus the element \( \alpha \) of \( \pi_1^{\text{DR}}(X_1/S, x_1, y_1) \) in the statement of Proposition 6.20 is well-defined.

Proof of Proposition 6.19  
(1) First, note that we have the diagram

\[
\begin{array}{c}
1 \longrightarrow \pi_1^{\text{DR}}(X_1/S, x_1) \longrightarrow \pi_1^{\text{DR}}(X_1, x_1) \xrightarrow{f_1_*} \pi_1^{\text{DR}}(S, s) = \mathbb{G}_{a, k} \longrightarrow 1.
\end{array}
\]

Since \( X_1 \) is not log smooth over \( S \), it is not immediate that the previous sequence is exact. However, because of the equivalences (6.15), the sequence can be identified with the sequence

\[
1 \longrightarrow \pi_1^{\text{DR}}(X'_1/S, x_1) \longrightarrow \pi_1^{\text{DR}}(X'_1, x_1) \xrightarrow{f_1_*} \pi_1^{\text{DR}}(S, s) = \mathbb{G}_{a, k} \longrightarrow 1,
\]

which is known to be exact because \( X'_1 \) is log smooth over \( S \). Thus the diagram (6.17) is a split exact sequence. Hence it suffices to prove the required equality in \( \pi_1^{\text{DR}}(X_1/S, x_1) \). For an object \((E, \nabla) \) in \( N_{f_1, \text{MIC}}(X_1/k) \), if we denote its image in \( \text{MIC}^a(x_1/k) \) by \((E|_{x_1}, N, N')\), the action of \( \epsilon \) on \( E|_{x_1} \) is given by \( \exp(N) \). Also, since \( \epsilon^{-1} \circ \alpha \circ \epsilon \) is an element in \( \pi_1^{\text{DR}}(X_1/S, x_1) \), the action of it on \( E|_{x_1} \) depends only on the image \((E, \nabla) \) of \((E, \nabla) \) in \( N_{f_1, \text{MIC}}(X_1/S) \). So we can take an object \((E, \nabla') \) in \( N_{f_1, \text{MIC}}(X'_1/k) \) corresponding to \((E, \nabla) \) by the equivalences of categories (6.15), (6.16), and replace \((E, \nabla) \) by the image of \((E, \nabla') \) in \( N_{f_1, \text{MIC}}(X_1/k) \). If we do so, we have \( N = 0 \). So we obtain that \( \epsilon^{-1} \circ \alpha \circ \epsilon = \alpha \).

(2) By the same argument as in (1), it suffices to prove the equality \( \alpha^{-1} \circ \epsilon^{-1} \circ \alpha \circ \epsilon \circ x = \text{id} \) in \( \pi_1^{\text{DR}}(X_1/S, x_1) \). For an object \((E, \nabla) \) in \( N_{f_1, \text{MIC}}(X_1/k) \), if we denote the image of it in \( \text{MIC}^a(x_1/k) \) (resp. \( \text{MIC}^a(y_1/k) \)) by \((E|_{x_1}, N_x, N'_x)\) (resp. \((E|_{y_1}, N_y, N'_y)\)), the action of \( \epsilon_x \) on \( E|_{x_1} \) (resp. \( \epsilon_y \) on \( E|_{y_1} \)) is given by \( \exp(N_x) \) (resp. \( \exp(N_y) \)). Also, since \( \alpha^{-1} \circ \epsilon^{-1} \circ \alpha \circ \epsilon \) is an element in \( \pi_1^{\text{DR}}(X_1/S, x_1) \), the action of it on \( E|_{x_1} \) depends only on the image \((E, \nabla') \) of \((E, \nabla) \) in \( N_{f_1, \text{MIC}}(X_1/S) \). If we replace \((E, \nabla) \) as in (1), we have \( N_x = 0, N_y = 0 \) and so we obtain that \( \alpha^{-1} \circ \epsilon^{-1} \circ \alpha \circ \epsilon = \text{id} \). \( \square \)
Proof of Proposition 6.20. As in the proof of Proposition 6.19 it suffices to prove the equality \( \alpha^{-1} \circ \eta_y \circ \alpha \circ \eta_x = \text{id} \) in \( \pi_1^{\text{dR}}(X_1/S, x_1) \). If we define \((E|_{x_1}, N_x, N'_x)\), \((E|_{y_1}, N_y, N'_y)\) as in the proof of Proposition 6.19(2), the action of \( \eta_x \) on \( E|_{x_1} \) (resp. \( \epsilon_y \) on \( E|_{y_1} \)) is given by \( \exp(N_y - N'_y) \) (resp. \( \exp(N_y - N'_y) \)), and if we replace \((E, \nabla)\) as in the proof of Proposition 6.19(2), we have \( N_x = 0, N_y = 0 \). Also, via the identification by \( \alpha \), we have the equality of operators \( N'_x + N'_y = 0 \) by the residue theorem. (Indeed, \( N'_x \) (resp. \( N'_y \)) is the residue of \( \nabla \) at \( x \) (resp. at \( y \)).) Thus \( \exp(-N'_y) = \exp(-N'_x)^{-1} \) and so we have the required equality \( \alpha^{-1} \circ \eta_y \circ \alpha \circ \eta_x = \text{id} \). \( \square \)

6.4 Calculation of monodromy (I)

In this subsection, we prove Theorem 6.12 in the case where the dual graph of \( X \) has a loop.

Theorem 6.22. Theorem 6.12 holds when the dual graph of the geometric fiber of \( f : X \to S \) has a loop.

In fact, we can prove the non-triviality of the monodromy action on the abelianization \( \pi_1^{\text{dR}}(X/S, i)_{\text{ab}} \). Although there exists a proof which uses only the cohomology theory (see [M84], for example), we give a proof using our theory of the relative de Rham fundamental groups.

Proof. We may assume that \( k \) is algebraically closed by Proposition 6.9 and we may change the base point as we like by Proposition 6.14.

Take a loop in the dual graph of \( X \). Let \( Y_1, \ldots, Y_n \) be the irreducible components which appear as vertices of the loop, and let \( y_1, \ldots, y_n \) be the double points which appear as edges of the loop. When \( n \geq 3 \), we assume that \( y_i (1 \leq i \leq n - 1) \) is in \( Y_i \cap Y_{i+1} \) and \( y_n \) is \( Y_n \cap Y_1 \). Let \( \bar{X} \to X \) be the normalization of \( X^\circ \) endowed with the pullback log structure of that on \( X \), and let \( X_i (1 \leq i \leq n) \) be the connected component of \( \bar{X} \) corresponding to \( Y_i \). Define \( x_i, x'_i (1 \leq i \leq n) \) in the following way: When \( n = 1 \), let \( x_1 \) be a point of the inverse image of \( y_1 \) in \( X_1 \) and let \( x'_1 \) be another point of the inverse image of \( y_1 \) in \( X_1 \). When \( n \geq 2 \), let \( x_i (1 \leq i \leq n) \) be the inverse image of \( y_i \) in \( X_i \), let \( x'_i (1 \leq i \leq n - 1) \) be the inverse image of \( y_i \) in \( X_{i+1} \) and let \( x'_n \) be the inverse image of \( y_n \) in \( X_1 \). Let \( \alpha_1 \) be an element of \( \pi_{\text{dR}}(X_1/S, x'_1, x_1) \) and denote its image in \( \pi_{\text{dR}}(X/S, y_n, y_1) \) by the same letter. Let \( \alpha_i (2 \leq i \leq n) \) be an element of \( \pi_{\text{dR}}(X_i/S, x'_{i-1}, x_i) \) and denote its image in \( \pi_{\text{dR}}(X/S, y_{i-1}, y_i) \) by the same letter. (We may assume the existence of the elements \( \alpha_i (1 \leq i \leq n) \) by extending \( k \), which is allowed by Proposition 6.9.) We will calculate the monodromy action on the element

\[
\alpha_n \circ \alpha_{n-1} \circ \cdots \circ \alpha_1 \in \pi_1^{\text{dR}}(X/S, y_n).
\]

Let \( \omega_i, \omega'_i \) be the tangential section at \( x_i, x'_i \) respectively and put \( \epsilon_i := \omega_i^*(1), \epsilon'_i := \omega'_{i}^*(1), \eta_i := \epsilon'_{i-1} \circ \epsilon_i \). Then, by applying the monodromy action with respect to \( \omega'_i \)
to the element (6.18), we obtain the element

\[ \epsilon_n^{-1} \circ \alpha_n \circ \alpha_{n-1} \circ \cdots \circ \alpha_1 \circ \epsilon'. \]

It is equal to

\[ (\epsilon_n^{-1} \circ \epsilon_n) \circ (\epsilon_{n-1}^{-1} \circ \epsilon_{n-1}) \circ \cdots \circ (\epsilon_1^{-1} \circ \epsilon_1) \circ (\epsilon_1^{-1} \circ \alpha_1 \circ \epsilon'), \]

and by Proposition 6.19(2), it is equal to

\[ \eta_n \circ \alpha_n \circ \eta_{n-1} \circ \cdots \circ \eta_1 \circ \alpha_1. \]

Since the difference of the elements (6.18), (6.20) in the abelianization \( \pi^\text{dR}_1(X/S, y_n) \) is equal to

\[ \eta_n \circ \eta_{n-1} \circ \cdots \circ \eta_1, \]

it suffices to see that the element (6.21) is nontrivial in \( \pi^\text{dR}_1(X/S, y_n) \).

Let \( \gamma_i \) be an element in \( \Gamma(X_i, \Omega^1_{X_i/S}) \) whose residue at \( x_i \) is 1, whose residue at \( x_i' \) (when \( i = 1 \)) is \(-1\) and whose residues along other marked points and double points are zero. (Such an element exists by the residue theorem and Riemann-Roch.) Also, let \( X_i(n+1 \leq i \leq n') \) be the connected component of \( \tilde{X} \) other than \( X_i(1 \leq i \leq n) \), and for \( n+1 \leq i \leq n' \), put \( \gamma_i := 0 \in \Gamma(X_i, \Omega^1_{X_i/S}) \). Then we see that \( \gamma_i \)'s glue and give an element \( \gamma \in \Gamma(X, \Omega^1_{X/S}) \). We consider the action of the element on the object

\[ (E, \nabla) := \left( \mathcal{O}^2_X, d + \begin{pmatrix} 0 & \gamma \\ 0 & 0 \end{pmatrix} \right) \in N_{f\text{MIC}}(X/S). \]

By definition of \( \eta_i \) (and the concrete description given in the last subsection), it acts on \( E|_{y_i} = k^2 \) by the matrix \( \exp \begin{pmatrix} 0 & 2n \\ 0 & 0 \end{pmatrix} \). Thus the action of (6.21) is given by

\[ \exp \begin{pmatrix} 0 & 2n \\ 0 & 0 \end{pmatrix}, \]

and since it is not the identity, the monodromy action is nontrivial on \( \pi^\text{dR}_1(X/S, y_n) \). So the proof is finished.

\[ \square \]

### 6.5 Calculation of monodromy (II)

In this subsection, we reduce the proof of Theorem 6.12 to the case where the dual graph of the geometric fiber is a line and all the components corresponding to non-terminal vertices are \( \mathbb{P}^1_k \).

To prove it, first we prove the proposition which allows us to reduce the proof of Theorem 6.12 to the case with less irreducible components. Assume \( k \) is algebraically closed, let \( f : X \to S \) be a semistable log curve, let \( X_1, X_2 \) be unions of some irreducible components of \( X \) endowed with pullback log structure from that of \( X \).
and let \( x \) be a double point of \( X \) such that \( X_1 \cup X_2 = X, X_1 \cap X_2 = \{x\} \). Etale locally around \( x \), each \( X_i \) has a log structure of the form \([6.13]\). We define the log scheme \( X_i' \) by changing the log structure around \( x \) to the one induced by the chart \([6.14]\). As before, changing the log structure in this way has the effect that \( x \) is changed to a marked point. Also, let \( Q \) be the submonoid of \( \mathbb{N}^2 \) generated by \((1,1)\) and we define the log scheme \( \overline{X}_i \) by changing the log structure around \( x \) to the one induced by the chart \([6.14]\). Changing the log structure in this way has the effect that \( x \) is changed to a smooth point. We denote the structure morphisms \( X_i \to S, X_i' \to S, \overline{X}_i \to S \) by \( f_i, f_i', \overline{f}_i \) respectively. There are canonical morphisms \( X_i \to X_i' \to \overline{X}_i \) over \( S \) and, as before, we have the equivalences

\[
(6.22) \quad N^f_{\text{MIC}}(X_i/S) = N^f'_{\text{MIC}}(X_i'/S), \quad N^{f^*}_{\text{MIC}}(X_i/k) = N^{f'^*}_{\text{MIC}}(X_i'/k).
\]

Let \( \omega \) be a tangential section at \( x \). As in the proof of Proposition \([6.19]\), we can prove the existence of the split exact sequence

\[
1 \longrightarrow \pi^\text{dR}_1(X_i/S, x) \longrightarrow \pi^\text{dR}_1(X_i, x) \xrightarrow{f_i^*} \pi^\text{dR}_1(S, s) = \mathbb{G}_{a, k} \longrightarrow 1.
\]

by using the equivalence \([6.22]\). Also, since \( \overline{f}_i : \overline{X}_i \to S \) is a semistable log curve, we have the split exact sequence

\[
1 \longrightarrow \pi^\text{dR}_1(\overline{X}_i/S, x) \longrightarrow \pi^\text{dR}_1(\overline{X}_i, x) \xrightarrow{\overline{f}_i^*} \pi^\text{dR}_1(S, s) = \mathbb{G}_{a, k} \longrightarrow 1.
\]

Hence the monodromy action on \( \pi^\text{dR}_1(X_i/S, x), \pi^\text{dR}_1(\overline{X}_i/S, x) \) are defined as the conjugate action by \( \epsilon := \omega^*(1) \). Then we have the following proposition.

**Proposition 6.23.** Let the notations be as above. Then, if the monodromy action on \( \pi^\text{dR}_1(\overline{X}_1/S, x) \) is nontrivial as an element in \( \text{Out}(\pi^\text{dR}_1(\overline{X}_1/S, x)) \), that on \( \pi^\text{dR}_1(X/S, x) \) is nontrivial as an element in \( \text{Out}(\pi^\text{dR}_1(X/S, x)) \).

**Proof.** Note that the monodromy action on the de Rham fundamental groups induces that on the associated Lie algebras. By functoriality, we have the following commutative diagrams of pro-nilpotent Lie algebras compatible with the monodromy action:

\[
\begin{array}{ccc}
L(x/S) & \longrightarrow & L(X_1/S) \\
\downarrow & & \downarrow \\
L(X_2/S) & \longrightarrow & L(\overline{X}_1/S)
\end{array}
\]

\[
(6.23)
\]

\[
\begin{array}{ccc}
L(x/S) & \longrightarrow & L(X_1/S) \\
\downarrow & & \downarrow \\
0 & \longrightarrow & L(\overline{X}_1/S).
\end{array}
\]

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Recall that we indicated by $N_{f MIC}(X/S)$ (resp. $N_{f MIC}(X_1/S)$, $N_{f MIC}(X_2/S)$), $MIC^n(x/S)$ the subcategory of $MIC(X/S)$ (resp. $MIC(X_1/S)$, $MIC(X_2/S)$, $MIC(x/S)$) consisting of iterated extensions of trivial objects. Since to give an object in $MIC(X/S)$ is equivalent to give a pair of objects in $MIC(X_1/S)$ and $MIC(X_2/S)$ which are compatible in $MIC(x/S)$, then we see that the first diagram in (6.23) is a push-out diagram in the category of pro-nilpotent Lie algebras compatible with the monodromy action. By the same reason, the second diagram in (6.23) is also a push-out diagram in the category of pro-nilpotent Lie algebras compatible with the monodromy action. Hence the diagrams in (6.23) induces the surjective morphism of pro-nilpotent Lie algebras

$$L(X/S) \longrightarrow L(\overline{X}_1/S)$$

which is compatible with the monodromy action. Indeed, it is induced by the map between the subdiagrams

$$
\begin{array}{ccc}
L(x/S) & \longrightarrow & L(X_1/S) \\
\downarrow & & \downarrow \\
L(X_2/S), & & 0,
\end{array}
$$

which is compatible with the monodromy action. Hence we obtain the surjective morphism of group schemes

$$\pi^\text{dR}_1(X/S,x) \longrightarrow \pi^\text{dR}_1(\overline{X}_1/S,x)$$

which is compatible with the monodromy action. Therefore, if the monodromy action on $\pi^\text{dR}_1(\overline{X}_1/S,x)$ is nontrivial as an element in $\text{Out}(\pi^\text{dR}_1(\overline{X}_1/S,x))$, that on $\pi^\text{dR}_1(X/S,x)$ is nontrivial as an element in $\text{Out}(\pi^\text{dR}_1(X/S,x))$. \hfill \Box

When $k$ is algebraically closed and $f : X \longrightarrow S$ is a semistable log curve, an irreducible component of $X$ is called a terminal component if the corresponding vertex in the dual graph of $X$ is terminal (meets only one edge). Then the following theorem is the main result in this subsection:

**Theorem 6.24.** To prove Theorem 6.12, it suffices to prove it in the following case: $k$ is algebraically closed, the dual graph of $X$ is a line and the non-terminal components of $X$ are isomorphic to $\mathbb{P}^1_k$ with no marked points.

**Proof.** Let $\Gamma$ be the dual graph of $X$. We may assume that $\Gamma$ is a tree by Theorem 6.22. Then there exist at least two terminal vertices: connect two of them by (the unique) line $L$. Then the line $L$ consists of vertices $v_1, ..., v_n$ and the edges $e_1, ..., e_{n-1}$ such that $e_i (1 \leq i \leq n-1)$ connects $v_i$ and $v_{i+1}$. Denote the irreducible component corresponding to $v_i$ by $X_i$ and the double point corresponding to $e_i$ by $x_i$.

Suppose that there exists an edge $e$ not in $L$ which touches some vertex $v_i$ in $L$. (Because such a vertex $v_i$ cannot be terminal, we have $2 \leq i \leq n-1$.) Since $\Gamma$ is a tree, $\Gamma \setminus \{e\}$ is the disjoint union of subgraphs $\Gamma_1 \sqcup \Gamma_2$ with $L \subseteq \Gamma_1$. Let $Y_1$
(resp. $Y_2$) be the union of irreducible components of $X$ corresponding to the graph $\Gamma_1$ (resp. $\Gamma_2 \cup \{e, v_i\}$). Then we have $X = Y_1 \cup Y_2$, $Y_1 \cap Y_2 = \{x_i\}$. If we define $\overline{Y}_1$ from $Y_1$ as before Proposition 6.23, we see by Proposition 6.23 that the theorem for $X$ is reduced to the theorem for $\overline{Y}_1$. Note that $\overline{Y}_1$ remains to be minimal: The only irreducible component in $\overline{Y}_1$ which is changed is $X_i$, and since $X_i$ intersects with $X_{i-1}$ and $X_{i+1}$, this change does not violate the minimality. Thus we could reduce to the case with fewer irreducible components. Repeating this procedure, we can reduce the theorem to the case that the dual graph of $X$ is equal to the line $L$.

Next, let $\overline{X}$ be the log scheme which we obtain by changing the log structure around the marked points on $X_i$ ($2 \leq i \leq n - 1$) to the pullback log structure from that on $S$. (This change has the effect to erase the marked points from $X_i$ ($2 \leq i \leq n - 1$).) Then, since $\pi_{1,\text{dR}}(X/S, \iota) \to \pi_{1,\text{dR}}(\overline{X}/S, \iota)$ is surjective morphism compatible with monodromy action, we can reduce the theorem for $X$ to the theorem for $\overline{X}$. So we may assume that $X_i(2 \leq i \leq n - 1)$ has no marked points (Proposition 6.23).

Finally, if there is an index $i$ ($2 \leq i \leq n - 1$) with $X_i$ nonisomorphic to $\mathbb{P}^1_k$, we take the minimal $i$ with this condition, let $L_1$ (resp. $L_2$) be the subgraph of $L$ whose vertices are $v_1, ..., v_i$ (resp. $v_{i+1}, ..., v_n$) and let $Y_1$ (resp. $Y_2$) be the union of irreducible components of $X$ corresponding to the graph $L_1$ (resp. $L_2 \cup \{e, v_i\}$). Then, if we define $\overline{Y}_1$ from $Y_1$ as before Proposition 6.23, we see by Proposition 6.23 that the theorem for $X$ is reduced to the theorem for $\overline{Y}_1$ and that $\overline{Y}_1$ is a minimal semistable log curve satisfying the condition described in the statement of the theorem. So we are done. 

\[ \square \]

### 6.6 Calculation of monodromy (III)

In this subsection, assume that $k$ is algebraically closed. Let $f : X \to S$ be a minimal semistable log curve satisfying the condition stated in Theorem 6.24, namely, a minimal semistable log curve such that the dual graph of $X$ is a line and the non-terminal components of $X$ are isomorphic to $\mathbb{P}^1_k$ with no marked point. In this subsection, we calculate the monodromy action on certain elements in the de Rham fundamental group of $X$ over $S$.

Let us fix some notations. Let $Y_i(0 \leq i \leq n)$ be the irreducible components of $X$ and let $y_i(0 \leq i \leq n - 1)$ be the double points of $X$ such that $Y_i \cap Y_{i+1} = \{y_i\}$. Note that, by our assumption, $Y_i(1 \leq i \leq n - 1)$ are isomorphic to $\mathbb{P}^1_k$. We assume that $Y_i$'s, $y_i$'s are endowed with the pullback log structure of that on $X$. Let $\tilde{X} \to X$ be the normalization of $X$ endowed with the pullback log structure of that on $X$ and let $x_i(0 \leq i \leq n - 1)$ be the connected component of $\tilde{X}$ corresponding to $Y_i$. Let $x_i(0 \leq i \leq n - 1)$ be the inverse image of $y_i$ in $X_i$ and let $x_i(0 \leq i \leq n - 1)$ be the inverse image of $y_i$ in $X_{i+1}$. Let $\alpha_0$ (resp. $\alpha_n$) be an element of $\pi_{1,\text{dR}}(X_0/S, x_0)$ (resp. $\pi_{1,\text{dR}}(X_n/S, x_n)$) and denote its image in $\pi_{1,\text{dR}}(X_i/S, y_0)$ (resp. $\pi_{1,\text{dR}}(X_i/S, y_n)$) by the same letter. Also, let $\alpha_i(1 \leq i \leq n - 1)$ be the element of $\pi_{1,\text{dR}}(X_i/S, x_{i-1}, x_i)$
defined by

\[ E|_{x_{i-1}'}, \Gamma(Y_i, E) \xrightarrow{\sim} E|_{x_i}, ((E, \nabla) \in N_{f_i} \text{MIC}(X_i/S), f_i : X_i \to S), \]

and denote its image in \( \pi_1(X/S, y_{i-1}, y_i) \) by the same letter. (The well-definedness of the elements \( \alpha_i(1 \leq i \leq n - 1) \) follows from Remark 6.21.) We put \( \alpha := \alpha_0, \beta = \alpha_1^{-1} \circ \cdots \circ \alpha_{n-1}^{-1} \circ \alpha_n \circ \alpha_{n-1} \circ \cdots \circ \alpha_1 \) and calculate the monodromy action on the element \( \beta \circ \alpha \).

Let \( \omega_i, \omega_i' \) be the tangential section at \( x_i, x_{i}' \) respectively and put \( \epsilon_i := \omega_i^*(1), \epsilon_i' := \omega_{i}'^*(1), \eta_i := \epsilon_i^{-1} \circ \epsilon_i \). Then we have the following:

**Proposition 6.25.** Let the notations be as above. If we apply the monodromy action with respect to \( \omega_0 \) to \( \beta \circ \alpha \), we obtain the element \( \eta_0^{-n} \circ \beta \circ \eta_0^n \circ \alpha \).

**Proof.** By definition of the monodromy action, the element we obtain is

\[ \epsilon_0^{-1} \circ \alpha_1^{-1} \circ \cdots \circ \alpha_{n-1}^{-1} \circ \alpha_n \circ \alpha_{n-1} \circ \cdots \circ \alpha_1 \circ \alpha_0 \circ \epsilon_0, \]

which can be rewritten as

\[
\begin{align*}
(\epsilon_0^{-1} \circ \epsilon_0') & \circ (\epsilon_0^{-1} \circ \alpha_1^{-1} \circ \epsilon_1) \circ (\epsilon_1^{-1} \circ \epsilon_1') \circ \cdots \circ (\epsilon_{n-2}^{-1} \circ \alpha_{n-1}^{-1} \circ \epsilon_{n-1}) \circ (\epsilon_{n-1}^{-1} \circ \epsilon_{n-1}') \\
& \circ (\epsilon_{n-1}^{-1} \circ \alpha_n \circ \epsilon_n') \circ (\epsilon_n^{-1} \circ \epsilon_n) \circ (\epsilon_{n-1}^{-1} \circ \alpha_{n-1} \circ \epsilon_{n-2}) \circ \cdots \circ (\epsilon_1^{-1} \circ \alpha_1 \circ \epsilon_0') \\
& \circ (\epsilon_1^{-1} \circ \epsilon_0) \circ (\epsilon_0^{-1} \circ \alpha_0 \circ \epsilon_0).
\end{align*}
\]

By Proposition 6.19, it is equal to

\[ \eta_0^{-1} \circ \alpha_1^{-1} \circ \eta_1^{-1} \circ \cdots \circ \alpha_{n-1}^{-1} \circ \eta_{n-1}^{-1} \circ \alpha_n \circ \eta_{n-1} \circ \alpha_{n-1} \circ \cdots \circ \alpha_1 \circ \eta_0 \circ \alpha_0 \]

and by Proposition 6.20, it is further equal to

\[ \eta_0^{-n} \circ \alpha_1^{-1} \circ \cdots \circ \alpha_{n-1}^{-1} \circ \alpha_n \circ \alpha_{n-1} \circ \cdots \circ \alpha_1 \circ \eta_0^n \circ \alpha_0 = \eta_0^{-n} \circ \beta \circ \eta_0^n \circ \alpha. \]

(Here we apply Proposition 6.20 by replacing \( X_1, x_1, y_1 \) with \( X_i, x_{i-1}', x_i \) respectively, and the elements \( \eta_x, \eta_y \) in the proposition are now \( \eta_{i-1}^{-1}, \eta_i \) respectively. We then obtain the equality \( \eta_i \circ \alpha_i = \alpha_i \circ \eta_{i-1} \).) \( \square \)

**Remark 6.26.** Let the notation be as above and \( Y := Y_0, Z := \bigcup_{i=1}^n Y_i, y := y_0 \) (endowed with the pullback log structure of that on \( X \)). Then the pro-nilpotent Lie algebra \( L(X/S) \) fits into the following push-out diagram in the category of pro-nilpotent Lie algebras:

\[
\begin{array}{ccc}
L(y/S) & \longrightarrow & L(Y/S) \\
\downarrow & & \downarrow \\
L(Z/S) & \longrightarrow & L(X/S).
\end{array}
\]

To finish the proof of Theorem 6.12 (which will be given in the next subsection) we need to have a concrete description of the maps \( L(y/S) \to L(Y/S), L(y/S) \to L(Z/S) \). This is the subject of this remark.
First, if we construct the log scheme $Y'$ from $Y$ using the methods we introduced at the beginning of Subsection 6.5 ($y$ becomes a marked point), we have $L(Y/S) \cong L(Y'/S)$. But, since $Y'$ has $y$ as a marked point, we conclude that $L(Y/S)$ is a free pro-nilpotent Lie algebra by Proposition 6.16. We denote its rank by $m$ in this remark. (Note that, by minimality assumption on $X$, $m \geq 2$.) Next, if we construct the log scheme $\overline{Y}$ from $Y$ (again following the notation and recipe of Subsection 6.5), we have the push-out diagram of pro-nilpotent Lie algebras

$$
\begin{array}{ccc}
L(y/S) & \longrightarrow & L(Y/S) \\
\downarrow & & \downarrow \\
0 & \longrightarrow & L(\overline{Y}/S),
\end{array}
$$

namely, $L(\overline{Y}/S)$ is the quotient of $L(Y/S)$ by the ideal generated by the image of $1 \in k = L(y/S) \longrightarrow L(Y/S)$.

If $Y$ has a marked point, so does $\overline{Y}$ and the number of marked points in $\overline{Y}$ is one less than that in $Y'$. Hence, by Proposition 6.16 $L(\overline{Y}/S)$ is a free pro-nilpotent Lie algebra whose rank is one less than that of $L(Y/S)$. Thus we see that there exists an isomorphism of free pro-nilpotent Lie algebras

(6.24)

$$L(Y/S) \cong L(v_1, \ldots, v_m)$$

(we use the notation as before Proposition 6.17) and we may suppose that the image of $1 \in k = L(y/S) \longrightarrow L(Y/S)$ is $v_1$. On the other hand, if $Y$ does not have a marked point, neither does $\overline{Y}$. Then we have $m = 2g$ with $g(\geq 1)$ the genus of $\overline{Y}$ and there exist compatible isomorphisms

(6.25)

$$L(Y/S) \cong L(Y'/S) \cong L(v_1, \ldots, v_{2g}), \quad L(\overline{Y}/S) \cong L(v_1, \ldots, v_{2g})/\langle \sum_{i=1}^{g} [v_{2i-1}, v_{2i}] \rangle$$

with the image of $1 \in k = L(y/S) \longrightarrow L(Y/S)$ equal to $\sum_{i=1}^{g} [v_{2i-1}, v_{2i}]$, by the proof of Proposition 6.17.

Next, note that we have an isomorphism $L(Y_n/S) \cong L(Z/S)$. Indeed, if we put $Z_i := \bigcup_{j=i}^{n} Y_j$ (in particular: $Z_n = Y_n, Z_1 = Z$), we have the push-out diagram of pro-nilpotent Lie algebras

$$
\begin{array}{ccc}
L(y_i/S) & \longrightarrow & L(Y_i/S) \\
\downarrow & & \downarrow \\
L(Z_{i+1}/S) & \longrightarrow & L(Z_i/S),
\end{array}
$$

and since we see easily that the upper horizontal arrow is an isomorphism (both are equal to $k$: indeed in the projective line $Y_i$ with the induced log-structure we have two marked points and we can use Proposition 6.16, we see that the lower horizontal
arrow is also an isomorphism and we conclude with an isomorphism $L(Y_n/S) \xrightarrow{\cong} L(Z/S)$. Using this, we can see that the map $L(y/S) \to L(Z/S)$ may be described in a similar way as the map $L(y/S) \to L(Y/S)$. In fact: if $Z$ has a marked point (i.e. so does $Y_n$), there exists an isomorphism of free pro-nilpotent Lie algebras

(6.26) $L(Z/S) \cong L(w_1, \ldots, w_m)$

for some $m \geq 2$ such that the image of $1 \in k = L(y/S) \to L(Z/S)$ can be given equal to $w_1$. When $Z$ does not have a marked point, there exist compatible isomorphisms

(6.27) $L(Z/S) \cong L(w_1, \ldots, w_{2g'})$, $L(Z'/S) \cong L(w_1, \ldots, w_{2g'})/(\sum_{i=1}^{g} [w_{2i-1}, w_{2i}])$

for some $g' \geq 1$ (here $g'$ is the genus of $Y_n$ and $Z'$ is the log scheme we obtain from $Z$ by the recipe in Subsection 6.5) with the image of $1 \in k = L(y/S) \to L(Z/S)$ equal to $-\sum_{i=1}^{g} [w_{2i-1}, w_{2i}]$.

6.7 End of the proof

In this subsection, we finish the proof of the ‘only if’ part of Theorem 6.5 and Theorem 6.12.

Proof of the ‘only if’ part of Theorem 6.5 and Theorem 6.12. By Proposition 6.13, the ‘only if’ part of Theorem 6.5 follows from Theorem 6.12. Then, by Theorem 6.24, it suffices to prove Theorem 6.12 in the case where $k$ is algebraically closed, the dual graph of $X$ is a line and the non-terminal components of $X$ are isomorphic to $\mathbb{P}^1_k$ with no marked point. In the following, we assume that $X$ satisfies these assumptions.

Let the notations be as in Subsection 6.6 and, in particular, the decomposition as in Remark 6.26. Then to such a decomposition we apply Proposition 6.25: the monodromy action on $\pi_{1}^{\text{dR}}(X/S, y)$ sends an element of the particular form as in the proposition $\beta \circ \alpha (\alpha \in \pi_{1}^{\text{dR}}(Y/S, y), \beta \in \pi_{\text{dR}}(Z/S, y))$ to $\eta_0^{-n} \circ \beta \circ \eta_0^n \circ \alpha$, where $\eta_0$ is a nontrivial element of $\pi_{1}^{\text{dR}}(y/S, y) \cong G_{a.k}$. (Note that the isomorphism $L(Y_n/S) \xrightarrow{\cong} L(Z/S)$ in Remark 6.26 induces the isomorphism

$\pi_{1}^{\text{dR}}(Y_n/S, y_{n-1}) \xrightarrow{\cong} \pi_{\text{dR}}(Z/S, y); \quad \alpha_{n} \mapsto \alpha_{1}^{-1} \circ \cdots \circ \alpha_{n-1}^{-1} \circ \alpha_{n} \circ \alpha_{n-1} \circ \cdots \circ \alpha_{1}$.

So the particular element $\beta$ in Proposition 6.25 can be understood as any element of $\pi_{1}^{\text{dR}}(Z/S, y)$!

We assume that the monodromy action on $\pi_{1}^{\text{dR}}(X/S, y)$ is trivial as an element in $\text{Out}(\pi_{1}^{\text{dR}}(X/S, y))$ and deduce a contradiction. By assumption, there exists an element $\delta \in \pi_{1}^{\text{dR}}(X/S, y)$ such that the monodromy action is given by $\gamma \mapsto \delta^{-1} \circ$
the assertions (6.28), (6.29) imply the following assertions:

\[ \eta_0^{-n} \circ \beta \circ \eta_0^n \circ \alpha = \delta^{-1} \circ \beta \circ \alpha \circ \delta \]

for any \( \alpha \in \pi_1^{\text{dR}}(Y/S, y), \beta \in \pi_1^{\text{dR}}(X/S, y) \). By considering the case \( \beta = \text{id} \) and the case \( \alpha = \text{id} \), we obtain the following:

(6.28) For any \( \alpha \in \pi_1^{\text{dR}}(Y/S, y) \), \( \alpha = \delta^{-1} \circ \alpha \circ \delta \).

(6.29) For any \( \beta \in \pi_1^{\text{dR}}(Z/S, y) \), \( \eta_0^{-n} \circ \beta \circ \eta_0^n = \delta^{-1} \circ \beta \circ \delta \).

We write these assertions in the Lie algebra \( L(X/S)/\text{Fil}^4L(X/S) \). Let \( \{\Gamma^n\}_n \) be the lower central series of \( \pi_1^{\text{dR}}(X/S, y) \) (with \( \Gamma^1 = \pi_1^{\text{dR}}(X/S, y) \)) and consider the exponential isomorphism

\[ \exp : L(X/S)/\text{Fil}^4L(X/S) \overset{\simeq}{\longrightarrow} \pi_1^{\text{dR}}(X/S, y)/\Gamma^4. \]

By Baker-Campbell-Hausdorff formula, we have

\[ \exp(v) \exp(w) = \exp(v + w + (1/2)[v, w] + (1/12)([v, [v, w]] + [w, [v, w]])) \]

for \( v, w \in L(X/S)/\text{Fil}^4L(X/S) \). In the following, for \( n \in \mathbb{N} \), we denote the equality in \( L(X/S)/\text{Fil}^nL(X/S) \) by \( \equiv_n \). If we put \( \alpha = \exp(a), \delta = \exp(d) \), the equality \( \alpha = \delta^{-1} \circ \alpha \circ \delta \) implies the equality

\[ a + d + (1/2)[a, d] + (1/12)([a, [a, d]] + [d, [d, a]]) \equiv_4 d + a + (1/2)[d, a] + (1/12)([d, [d, a]] + [a, [a, d]]), \]

hence the equality \( [a, d] \equiv_4 0 \). Also, if we put \( \beta = \exp(b), e = \exp(\eta_0^n) \), the equality \( \eta_0^{-n} \circ \beta \circ \eta_0^n = \delta^{-1} \circ \beta \circ \delta \) implies the equality

\[ \exp(d) \exp(-e) \exp(b) = \exp(b) \exp(d) \exp(-e), \]

which is rewritten as

\[ \exp(d - e - (1/2)[d, e] + \cdots) \exp(b) = \exp(b) \exp(d - e - (1/2)[d, e] + \cdots) \]

in \( \pi_1^{\text{dR}}(X/S, y)/\Gamma^4 \), and this implies the equality \([b, d] \equiv_4 [b, e + (1/2)[d, e]] \). Thus the assertions (6.28), (6.29) imply the following assertions:

(6.30) For any \( a \in L(Y/S) \), \([a, d] \equiv_4 0 \).

(6.31) For any \( b \in L(Z/S) \), \([b, d] \equiv_4 [b, e + (1/2)[d, e]] \).

We will prove that these assertions imply a contradiction.

(1) First we consider the case where both \( Y, Z \) have a marked point. By the calculation in Remark (6.26) we have the isomorphisms

\[ L(Y/S) \cong L(v_1, \ldots, v_m), \quad L(Z/S) \cong L(w_1, \ldots, w_{m'}) \]
for some \( m, m' \geq 2 \) such that the image of \( e \in L(y/S) \) in \( L(Y/S) \) (resp. \( L(Z/S) \)) is equal to \( v_1 \) (resp. \( w_1 \)). Thus

\[
L(X/S) \cong L(v_1, \ldots, v_m, w_2, \ldots, w_{m'}).
\]

We prove that it is impossible to satisfy the conditions \((6.30), (6.31)\) in the quotient \( L(v_1, v_2, w_2) \) of \( L(X/S) \). By \((6.30)\), we have \([v_1, d] \equiv 3 \) 0 and this implies that \( d \), which is a priori of the form \( d \equiv_2 C_1v_1 + C_2v_2 + C_3w_2 \), is indeed of the form \( d \equiv_2 Cv_1 \) for some \( C \in \mathbb{k} \). By \((6.30)\) again, we have \([v_2, d] \equiv 3 \) 0 and so \( C = 0 \), thus \( d \equiv_2 0 \). This implies \([w_2, d] \equiv 3 \) 0, while by \((6.31)\) we have

\[
[w_2, d] \equiv 4 [w_2, v_1 + (1/2)[d, v_1]] \equiv 3 [w_2, v_1].
\]

Because \([w_2, v_1]\) is nonzero in \( L(v_1, v_2, w_2)/\text{Fil}^3L(v_1, v_2, w_2) \), this is a contradiction.

(2) Next we consider the case where \( Y \) has no marked point and \( Z \) has a marked point. In this case, by the calculation in Remark 6.26, we have the isomorphisms

\[
L(Y/S) \cong L(v_1, \ldots, v_{2g}), \quad L(Z/S) \cong L(w_1, \ldots, w_{m'})
\]

for some \( g \geq 1, m \geq 2 \) such that the image of \( e \in L(y/S) \) in \( L(Y/S) \) (resp. \( L(Z/S) \)) is equal to \( \sum_{i=1}^g [v_{2i-1}, v_{2i}] \) (resp. \( w_1 \)). Thus

\[
L(X/S) \cong L(v_1, \ldots, v_{2g}, w_2, \ldots, w_{m'}).
\]

We prove that it is impossible to satisfy the conditions \((6.30), (6.31)\) in the quotient \( L(v_1, v_2, w_2) \) of \( L(X/S) \). By \((6.30)\), we have \([v_1, d] \equiv 3 \) 0 and this implies that \( d \equiv_2 Cv_1 \) for some \( C \in \mathbb{k} \). By \((6.30)\) again, we have \([v_2, d] \equiv 3 \) 0 and so \( C = 0 \), thus \( d \equiv_2 0 \). By \((6.30)\) again, we have \([v_1, d] \equiv 4 \) 0. Because it is well-known that \( \text{Fil}^2L(v_1, v_2, w_2)/\text{Fil}^3L(v_1, v_2, w_2) \) is of dimension 3 with basis

\[
[v_1, v_2], \quad [v_1, w_2], \quad [v_2, w_2],
\]

and that \( \text{Fil}^3L(v_1, v_2, w_2)/\text{Fil}^4L(v_1, v_2, w_2) \) is of dimension 8 with basis

\[
[v_1, [v_1, v_2]], \quad [v_1, [v_1, w_2]], \quad [v_1, [v_2, w_2]], \quad [v_2, [v_1, v_2]], \quad [v_2, [v_2, w_2]], \quad [w_2, [v_1, v_2]], \quad [w_2, [v_1, w_2]], \quad [w_2, [v_2, w_2]],
\]

the map

\[
[v_1, -] : \text{Fil}^2L(v_1, v_2, w_2)/\text{Fil}^3L(v_1, v_2, w_2) \longrightarrow \text{Fil}^3L(v_1, v_2, w_2)/\text{Fil}^4L(v_1, v_2, w_2)
\]

is injective. Thus we see that \( d \equiv_3 0 \). This implies \([w_2, d] \equiv 0 \) 0, while by \((6.31)\) we have

\[
[w_2, d] \equiv 4 [w_2, [v_1, v_2]].
\]

Because \([w_2, [v_1, v_2]]\) is nonzero in \( L(v_1, v_2, w_2)/\text{Fil}^3L(v_1, v_2, w_2) \), this is a contradiction.
(3) Next we consider the case where $Y$ has a marked point and $Z$ has no marked point. In this case, by the calculation in Remark 6.26, we have the isomorphisms

$$L(Y/S) \cong L(v_1, \ldots, v_m), \quad L(Z/S) \cong L(w_1, \ldots, w_{2g'})$$

for some $m \geq 2, g' \geq 1$ such that the image of $e \in L(y/S)$ in $L(Y/S)$ (resp. $L(Z/S)$) is equal to $v_1$ (resp. $-\sum_{i=1}^{g'} [w_{2i-1}, w_{2i}]$). Thus

$$L(X/S) \cong L(v_2, \ldots, v_n, w_1, \ldots, w_{2g'}).$$

We prove that it is impossible to satisfy the conditions (6.30), (6.31) in the quotient $L(v_2, w_1, w_2)$ of $L(X/S)$. By (6.30), we have $[v_2, d] \equiv 3 0$ and this implies that $d \equiv_2 C v_2$ for some $C \in k$. By (6.31), we have

$$[w_1, d] \equiv_4 [w_1, [w_1, w_2] - (1/2) [[w_1, w_2], d]] \equiv_3 0$$

and so $C = 0$, thus $d \equiv_2 0$. By (6.30) again, we have $[v_2, d] \equiv_4 0$. By the same reason as in (2), we see that the map

$$[v_2, -] : \text{Fil}^2 L(v_2, w_1, w_2) / \text{Fil}^3 L(v_2, w_1, w_2) \longrightarrow \text{Fil}^3 L(v_2, w_1, w_2) / \text{Fil}^4 L(v_2, w_1, w_2)$$

is injective and so $d \equiv_3 0$. This implies $[w_1, d] \equiv_4 0$, while by (6.31) we have

$$[w_1, d] \equiv_4 [w_1, [w_1, w_2]].$$

Because $[w_2, [w_1, w_2]]$ is nonzero in $L(v_2, w_1, w_2) / \text{Fil}^4 L(v_2, w_1, w_2)$, this is a contradiction.

(4) Finally we consider the case where both $Y, Z$ have no marked point. The proof in this case is essentially due to [Od95]. In this case, by the calculation in Remark 6.26, we have the isomorphisms

$$L(Y/S) \cong L(v_1, \ldots, v_g), \quad L(Z/S) \cong L(w_1, \ldots, w_{2g'})$$

for some $g, g' \geq 1$ such that the image of $e \in L(y/S)$ in $L(Y/S)$ (resp. $L(Z/S)$) is equal to $\sum_{i=1}^{g} [v_{2i-1}, v_{2i}]$ (resp. $-\sum_{i=1}^{g'} [w_{2i-1}, w_{2i}]$). Thus

$$L(X/S) \cong L(v_1, \ldots, v_g, w_1, \ldots, w_{2g'}) / (\sum_{i=1}^{g} [v_{2i-1}, v_{2i}] + \sum_{i=1}^{2g'} [w_{2i-1}, w_{2i}]).$$

We prove that it is impossible to satisfy the conditions (6.30), (6.31) in the quotient $L := L(v_1, v_2, w_1, w_2) / ([v_1, v_2] + [w_1, w_2])$ of $L(X/S)$. By (6.30), we have $[v_1, d] \equiv_3 0$ and this implies that $d \equiv_2 C v_1$ for some $C \in k$. By (6.30) again, we have $[v_2, d] \equiv_3 0$ and so $C = 0$, thus $d \equiv_2 0$. By (6.30) again, we have $[v_1, d] \equiv_4 0$. By [Lab70, Proposition 4], the dimension of $\text{Fil}^2 L / \text{Fil}^3 L$ (resp. $\text{Fil}^3 L / \text{Fil}^4 L$) is 5 (resp. 16), and a basis is given by

$$[v_1, v_2], \quad [v_1, w_1], \quad [v_1, w_2], \quad [v_2, v_1], \quad [v_2, w_1], \quad [v_2, w_2].$$
Thus the map 
\[ [v_1, -] : \text{Fil}^2 L/\text{Fil}^3 L \rightarrow \text{Fil}^3 L/\text{Fil}^4 L \]
is injective and so we see that \( d \equiv 3 \mod 0 \). This implies \( [w_2, d] \equiv 4 \mod 0 \), while by (6.31) we have 
\[ [w_2, d] \equiv 4 [w_2, [v_1, v_2]] \].
Because \( [w_2, [v_1, v_2]] \) is nonzero in \( L/\text{Fil}^4 L \), this is a contradiction.

Thus we proved that the assertions (6.28), (6.29) imply a contradiction in all the cases. So the proof of the theorems is finished. \( \square \)

### 6.8 Application to Andreatta-Iovita-Kim theorem

In this final part, we review the theorem of Andreatta-Iovita-Kim [AIK15] on \( p \)-adic good reduction criterion for proper hyperbolic curves and we give an outline of its proof. Then we give an alternative, purely algebraic proof for the transcendental part of their theorem: this will be done by using the results of the previous subsections.

First we recall the setting of their article [AIK15] (with different notation). Let \( K \) be a complete discrete valuation field of characteristic zero, with valuation ring \( \mathcal{O}_K \) and perfect residue field \( F \). We fix a uniformizer \( \pi \) of \( \mathcal{O}_K \) and an algebraic closure \( \overline{K} \) of \( K \). Let

\[ f_K : X_K \rightarrow \text{Spec} \ K \]

be a proper smooth geometrically irreducible curve of genus \( g \geq 2 \), and assume that it admits a regular semistable model

\[ f_{O_K}^o : X_{O_K}^o \rightarrow \text{Spec} O_K. \]

We may and do assume that the special fiber does not have \((-1)\)-curve. With this setting, we see that \( f_K \) has good reduction if and only if \( f_{O_K}^o \) is smooth. We endow the source and the target of \( f_{O_K}^o \) with the log structure associated to the special fiber. Then we obtain the proper log smooth morphism of log schemes, which we denote by

\[ f_{O_K} : X_{O_K} \rightarrow S_{O_K}. \]

Also, let \( \iota_{O_K} : S_{O_K} \rightarrow X_{O_K} \) be a section of \( f_{O_K} \). Let

\[ f_{\overline{K}} : X_{\overline{K}} \rightarrow \text{Spec} \ \overline{K}, \ \iota_{\overline{K}} : \text{Spec} \ \overline{K} \rightarrow X_{\overline{K}} \]

be the scalar extension of \( f_{O_K}, \iota_{O_K} \) to \( \overline{K} \). We denote by \( \pi_1^{\text{et}}(X_{\overline{K}}, \iota_{\overline{K}}) \) the Tannaka dual of the category of unipotent smooth \( \mathbb{Q}_p \)-adic etale sheaves on \( X_{\overline{K}} \) (with respect
to the fiber functor of pullback by $\iota_{K}$. Then it has been proved in [AIK15] that $\pi_{1}^{et}(\mathcal{X}_{K}, \iota_{K})$ is semistable as a proalgebraic group over $\mathbb{Q}_{p}$ endowed with the action of $\text{Gal}(\overline{K}/K)$ (see [AIK15, Theorem 1.4]), and the main theorem of that article is [AIK15, Theorem 1.6]:

**Theorem 6.27.** $f_{K} : X_{K} \rightarrow \text{Spec } K$ has good reduction if and only if $\pi_{1}^{et}(\mathcal{X}_{K}, \iota_{K})$ is crystalline.

For the definition of crystalline-ness, see [AIK15, Definition 1.5]. In order to achieve such a result some geometric constructions have been made. Let $W := W(F)$ be the Witt ring of $F$ and let $k$ be its fraction field. We denote the special fiber of $f_{O_{K}}$ (endowed with the pullback log structure) by

$$f_{F} : X_{F} \rightarrow S_{F}.$$

Then, by the absence of $(-1)$-curves in $X_{F}$, $f_{F}$ is a minimal semistable log curve in the sense of Section 6.1. We denote the $p$-adic completion of $f_{O_{K}}$ by

$$\hat{f}_{O_{K}} : \hat{X}_{O_{K}} \rightarrow \hat{S}_{O_{K}}.$$

Let $S_{W[[z]]}$ be the scheme $\text{Spec } W[[z]]$ endowed with the log structure associated to $\mathbb{N} \rightarrow W[[z]]; 1 \mapsto z$, and let $\hat{S}_{W[[z]]}$ be its $(p, z)$-adic completion. We define the exact closed immersion $\hat{S}_{O_{K}} \hookrightarrow \hat{S}_{W[[z]]}$ by $W[[z]] \rightarrow O_{K}; z \mapsto \pi$. By the unobstructedness in the deformation theory for $\hat{f}_{O_{K}}$ [Kk88, Proposition 3.14], there exists a proper log smooth integral lift

$$\hat{f}_{W[[z]]} : \hat{X}_{W[[z]]} \rightarrow \hat{S}_{W[[z]]}$$

of $\hat{f}_{O_{K}}$, and we can algebraize it to the proper log smooth integral morphism

$$f_{W[[z]]} : X_{W[[z]]} \rightarrow S_{W[[z]]}.$$

Let

$$f_{k[[z]]} : X_{k[[z]]} \rightarrow S_{k[[z]]}$$

be the scalar extension of $f_{W[[z]]}$ to $k[[z]]$.

At a non-smooth point of $\hat{X}_{O_{K}}$, the formal local ring has the form $\text{Spf } O'_{K}[[x, y]]/(xy - \pi)$ for some finite etale extension $O'_K$ of $O_K$. By the unicity of the local lift, the formal local ring at this point in $X_{W[[z]]}$ has the form $\text{Spf } W'[[x, y, z]]/(xy - z)$ for some finite etale extension $W'$ of $W$ (see [AIK15, p.2604]). If we denote the fiber at $'z = 0'$ of the morphisms $f_{W[[z]]}, f_{k[[z]]}$ by

$$f_{W} : X_{W} \rightarrow S_{W}, \quad f_{k} : X_{k} \rightarrow S_{k}$$

respectively, we have the Cartesian diagram

\[
\begin{array}{ccc}
X_{F} & \longrightarrow & X_{W} \\
| & f_{F} | & f_{W} | \\
| & f_{k} | & \\
S_{F} & \longrightarrow & S_{W}
\end{array}
\]

\[
\begin{array}{ccc}
& & X_{k} \\
& & \downarrow \quad \quad \quad \downarrow \\
& & f_{k} \\
& & S_{k}
\end{array}
\]
and the formal local ring at a non-smooth point of $X_W$ in the closed fiber has the form $\text{Spf } W'[x, y]/(xy)$ for some finite etale extension $W'$ of $W$. Also, the formal local ring at a smooth point of $X_W$ in the closed fiber has the form $\text{Spf } W[[x]]$ for some finite extension $W'$ of $W$. Thus there exists an etale covering $\sqcup_{i \in I} \tilde{X}_{W,i} \rightarrow X_W$ of $X_W$ as a scheme such that, for each $i$, $\tilde{X}_{W,i}$ is connected and etale (as a scheme) over $\text{Spec } W[x, y]/(xy)$ or $\text{Spec } W[x]$. When $\tilde{X}_{W,i}$ is etale over $\text{Spec } W[x, y]/(xy)$, we denote the inverse image of the locus $\{x = 0\}, \{y = 0\}$ in $\tilde{X}_{W,i}$ by $\tilde{X}_{W,i,x}, \tilde{X}_{W,i,y}$, respectively.

Let $C$ be an irreducible component of $X_W$. Then the inverse image of $C$ in $\tilde{X}_{W,i}$ is either $\tilde{X}_{W,i,x}, \tilde{X}_{W,i,y}$ or empty when $\tilde{X}_{W,i}$ is etale over $\text{Spec } W[x, y]/(xy)$, and equal to $\tilde{X}_{W,i}$ or empty when $\tilde{X}_{W,i}$ is etale over $\text{Spec } W[x]$. Thus we see that $C$ is smooth over $W$ (as a scheme). Also, for two distinct irreducible components $C, C'$ of $X_W$, the inverse image of $C \cap C'$ in $\tilde{X}_{W,i}$ is either $\tilde{X}_{W,i,x} \cap \tilde{X}_{W,i,y}$ or empty when $\tilde{X}_{W,i}$ is etale over $\text{Spec } W[x, y]/(xy)$, and empty when $\tilde{X}_{W,i}$ is etale over $\text{Spec } W[x]$. Thus $C \cap C'$ is etale over $W$ (as a scheme). From this description, we see that the set of irreducible components of $X_F$ corresponds bijectively to that of $X_W$ and then to that of $X_k$ in such a way that the genus of each irreducible component is preserved. We see also that the dual graph of $X_F$ is the same as that of $X_k$. Therefore, $f_k : X_k \rightarrow S_k$ is again a minimal semistable log curve and the underlying morphism of schemes of $f_k$ is smooth if and only if so is that of $f_F$.

Now let $S_W[[x]]$ be the scheme $\text{Spec } W[[x]]$ endowed with the log structure associated to $N \rightarrow W[[x]]; 1 \mapsto px$ and let $S_W[[x]] \rightarrow S_W[[z]]$ be the morphism defined by $z \mapsto px$. Denote the pullback of $f_W[[x]]$ by this morphism by

$$f_W[[x]] : X_W[[x]] \rightarrow S_W[[x]],$$

and let

$$f_k[[x]] : X_k[[x]] \rightarrow S_k[[x]]$$

be the scalar extension of $f_W[[x]]$ to $k[[x]]$. Note that the the fiber at $z = 0$ of the morphism $f_k[[x]]$ is the same as the morphism

$$f_k : X_k \rightarrow S_k$$

we already defined. Also, beginning from $\iota_{O_K}$, we also obtain a section $\iota_{k[[x]]}$ of $f_k[[x]]$. For $f_k[[x]]$, Andreatta-Iovita-Kim defined a projective system $\{(W_n, e_n)\}_{n}$ of connections, which we denote here by $\{(W_{n,k[[x]]}, e_{n,k[[x]]})\}_{n}$, by the method explained in Remark 3.10. We proved in that remark that their definition coincides with our definition (which we gave in Section 3). Recall also that, by our methods in Section 3, we are able to define a projective system of connections $\{(W_n, e_n)\}_{n}$ for $f_k$, which we denote here by $\{(W_{n,k}, e_{n,k})\}_{n}$, and by Remark 3.3 we see that $\{(W_{n,k}, e_{n,k})\}_{n}$ is the restriction of $\{(W_{n,k[[x]]}, e_{n,k[[x]]})\}_{n}$ to $X_k$.

By [AIK15], Proposition 6.2, $\pi_{\ast}^\dagger (X_{\overline{K}}, t_{\overline{K}})$ is crystalline if and only if the residue of $\iota_{k[[x]]}^{\ast}*, \text{dR}(W_{n,k[[x]]})$ at the locus $x = 0$ is zero for every $n$. Until this point, they never
use transcendental method. After that, to prove the triviality/nontriviality of the residue according to having good/bad reduction of \( f[k][x] \), they use transcendental methods reducing the claim to Oda’s result [Od95], hence giving a transcendental proof of Theorem 6.27.

Now we give an alternative, purely algebraic, proof of Theorem 6.27.

An algebraic proof of Theorem 6.27. It suffices to give an algebraic proof of the transcendental part of their proof. We use the notations given above. Since \( \{ W_{n,k}\}_n \) is the restriction of \( \{ W_{n,k}[x]\}_n \) to \( X_k \), the residue of \( \iota_k^*,dR W_{n,k}[x] \) at the locus \( x = 0 \) is nothing but the nilpotent map on \( \iota_k^*,dR W_{n,k} \) which occurs as a part of data as an object in \( \text{MIC}^n(S_k/k) \). Because \( \pi_1^{dR}(X_k/S_k,\iota_k) \) is defined as the spectrum of \( \lim_n (\iota_k^*,dR W_{n,k})^\vee \), the triviality/nontriviality of the residue is equivalent to the triviality/nontriviality of monodromy action on \( \pi_1^{dR}(X_k/S_k,\iota_k) \) as an element in \( \text{Aut}(\pi_1^{dR}(X_k/S_k,\iota_k)) \). Because our \( X_k \) does not have a marked point (because we started with a proper smooth curve \( f_K : X_K \to \text{Spec} K \)), \( \iota_k \) is necessarily a good section by Remark 6.7. Hence, if the underlying morphism of \( f_K \) is smooth, the monodromy action on \( \pi_1^{dR}(X_k/S_k,\iota_k) \) is trivial as an element in \( \text{Aut}(\pi_1^{dR}(X_k/S_k,\iota_k)) \) by Proposition 6.8(1). If the underlying morphism of \( f_k \) is not smooth, the monodromy action on \( \pi_1^{dR}(X_k/S_k,\iota_k) \) is nontrivial as an element in \( \text{Out}(\pi_1^{dR}(X_k/S_k,\iota_k)) \) by Theorem 6.12 and so nontrivial as an element in \( \text{Aut}(\pi_1^{dR}(X_k/S_k,\iota_k)) \). Since we saw, in the construction before the algebraic proof of Theorem 6.27, that the underlying morphism of schemes of \( f_k \) is smooth if and only if so is that of \( f_F \), this finishes the proof.

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