Abstract

Interpreting the cosmological constant as a thermodynamic pressure and its conjugate quantity as a thermodynamic volume, we reconsider the investigation of $P-V$ critical behaviors of $(1 + n)$-dimensional topological AdS black holes in Lovelock-Born-Infeld gravity. In particular, we derive an explicit expression of the universal number $\chi = \frac{P_c}{V_c}$ in terms of the space dimension $n$. Then, we examine the phase transitions at the critical points of such topological black holes for $6 \leq n \leq 11$ as required by the physical condition of the thermodynamical quantities. More precisely, the Ehrenfest equations have been checked revealing that the black hole system undergoes a second phase transition at the critical points.

Keywords: $P-V$ criticality, topological AdS black holes, Lovelock-Born-Infeld gravity, Ehrenfest thermodynamical equations.
1 Introduction

Black holes in various dimensions have received an increasing attention in connection with higher dimensional supergravity embedded either in superstrings or in M-theory moving on non trivial geometric backgrounds including Calabi-Yau manifolds. A particular interest has been on the extremal black hole solutions using the attractor mechanism developed in \cite{[1,2,3,4]}. In this approach, the corresponding effective potentials and the entropy functions have been computed using the U-duality group theory applied to the black hole charge invariants. Models based on Calabi-Yau manifolds have been elaborated using complex and quaternionic geometries \cite{[4]}.  

Recently, many efforts have devoted to study thermodynamical properties of the black holes using techniques explored in statistical physics and fluids\cite{[5]}. In particular, critical behaviors have been obtained for several black holes in various dimensions using either numerical or analytic calculations \cite{[5,6,7,8,9,10,11,12]}. A special interest has been on Ads black holes in arbitrary dimensions \cite{[13]}. More precisely, the state equations $P = P(T, V)$ have been worked out by considering the cosmological constant as the thermodynamic pressure and its conjugate as the thermodynamic volume. This activity has revealed a nice interplay between the behavior of the RN-AdS black hole systems and the Van der Waals fluids which has been seriously investigated in many places. In fact, it has been shown that the corresponding $P-V$ criticality can be related to the liquid-gas systems of statistical physics. Moreover, the criticality varies nicely in terms of the dimension of the spacetime in which the black holes live. This subject has been extensively investigated producing interesting results \cite{[14-21]}.  

More recently, a special emphasis has put on the thermodynamical properties of topological AdS black holes in Lovelock-Born-Infeld Gravity \cite{[22]}. A seven dimensional critical behavior have been obtained for uncharged and charged black holes. In particular, the $P-V$ diagram has been elaborated for such black holes in three different topologies associated with the curvature configuration.

Motivated by black objects in string theory and the above activities, we reconsider the study of the critically behaviors of $(1+n)$ dimensional topological AdS black holes in Lovelock-Born-Infeld gravity. Interpreting the cosmological constant as a thermodynamic pressure and its conjugate quantity as a thermodynamic volume, we investigate such behaviors in terms of the dimension of the spacetime and other parameters specified later on. Among others, we get an explicit expression of the universal number $\chi = \frac{Pc}{Tc}$, for $6 \leq n \leq 11$ as required by the reality of the thermodynamical quantities. Then, we discuss the phase transitions at the cortical points of these black hole solutions. In particular, the Ehrenfest thermodynamical equations have been verified showing that the black hole system undergoes a second phase
2 Thermodynamics of higher dimensional topological black holes in Lovelock-Born-Infeld gravity

In this section, we reconsider the study of thermodynamics of higher dimensional topological black holes in Lovelock-Born-Infeld gravity. In particular, we give an explicit expression for the corresponding state equation at critical points in \((1 + n)\) dimensions. Then, we discuss the critical behaviors in next sections. The analysis will be made in terms of three parameters: the space dimension \(n\), the Born-Infeld parameter \(\beta\) and the curvature constant \(k\).

To start, consider the physical action describing the third order Lovelock gravity in the presence of a nonlinear Born-Infeld electromagnetic gauge field studied in [23]. This action, which has been investigated in different context, takes the following form

\[
I_G = \frac{1}{16\pi} \int d^{n+1}x \sqrt{-g} \left(-2\Lambda + \mathcal{L}_1 + \alpha_2 \mathcal{L}_2 + \alpha_3 \mathcal{L}_3 + L(F)\right),
\]

where \(\Lambda\) is the cosmological constant. \(\mathcal{L}_1, \mathcal{L}_2, \mathcal{L}_3\) and \(L(F)\) are Einstein-Hilbert, Gauss-Bonnet, the third order Lovelock and the Born-infeld lagrangians respectively. They are given by the following expressions

\[
\mathcal{L}_1 = R,
\]

\[
\mathcal{L}_2 = R_{\mu\nu\gamma\delta}R^{\mu\nu\gamma\delta} - 4R_{\mu\nu}R^{\mu\nu} + R^2,
\]

\[
\mathcal{L}_3 = 2R^{\mu\nu\sigma\tau}R_\mu R_\nu R_\sigma R_\tau + 8R_{\mu\nu}R_\rho R_\nu R_\rho R_\mu + 24R^{\mu\nu\sigma\tau\rho\xi}R_\mu R_\nu R_\rho R_\xi + 3 \left(1 - \alpha_2 R^{\mu\nu}\right) + 24 \left(1 - \alpha_3 R^{\mu\nu}\right) + 16R_{\mu\nu}R_\rho R_\sigma R_\nu - 12R R_{\mu\nu}R_{\mu\nu} + R^3,
\]

\[
L(F) = 4\beta^2 \left(1 - \sqrt{1 + \frac{F^2}{2\beta^2}}\right).
\]

where the \(\beta\) is the Born-Infeld parame.

The constants \(\alpha_1, \alpha_2\) read as

\[
\alpha_2 = \frac{\alpha}{(n - 2)(n - 3)},
\]

\[
\alpha_3 = \frac{\alpha^2}{72(n - 2)/4},
\]

where \(\alpha\) is the Lovelock coupling constant. It has been shown that the \((n + 1)\)-dimensional static solution takes the form

\[
ds^2 = -f(r)dt^2 + \frac{dr^2}{f(r)} + r^2d\Omega_{n-1}^2
\]
In this solution, the black hole function \( f(r) \) takes the following form

\[
f(r) = k + \frac{r^2}{\alpha} (1 - g(r)^{1/3})
\]

where

\[
g(r) = 1 + \frac{3\alpha m}{r} - \frac{12\alpha^2}{n(n-1)} \left[1 - \sqrt{1 + \eta - \frac{\Lambda}{2\beta^2} + \frac{(n-1)\eta}{(n-2)} F(\eta)}\right]
\]

\( F(\eta) \) is the hypergeometric function given by

\[
F(\eta) = _2 F_1 \left( \frac{1}{2}, \frac{n-2}{2n-2}, \frac{3n-4}{2n-2}, -\eta \right)
\]

where

\[
\eta = \frac{(n-1)(n-2)q^2}{2\beta^2 r^{2n-2}}
\]

It is worth noting that the \((n-1)\) dimensional line element \(d\Omega_{n-1}^2\) depends on the geometry in question. In fact, three situations are present which are described by

\[
d\Omega^2 = \left\{ \begin{array}{ll}
\sum_{i=2}^{n-1} \prod_{j=1}^{i-1} \sin^2 \theta_i d\theta_i^2 & k = 1 \\
d\theta_1^2 + \sin^2 \theta_1 d\theta_1^2 + \sum_{i=3}^{n-1} \prod_{j=2}^{i-1} \sin^2 \theta_j d\theta_j^2 & k = -1 \\
\sum_{i=1}^{n-1} d\phi_i^2 & k = 0
\end{array} \right.
\]

for \((n-1)\) dimensional hypersurfaces with the constant curvature \((n-1)(n-2)k\). For these topological configurations, the Hawking temperature reads as

\[
T = \frac{(n-1)k[3(n-2)r_+^4 + 3(n-4)kr_+^2 + (n-6)k^2\alpha^2] + 12r_+^5 \beta^2(1 - \sqrt{1 + \eta - \frac{\Lambda}{2\beta^2} + \frac{(n-1)\eta}{(n-2)} F(\eta)})}{12\pi(n-1)r_+(r_+^2 + k\alpha)^2}.
\]

Similar calculations reveal that the entropy reads as

\[
S = \frac{\Sigma_k (n-1)r_+^{n-5}}{4} \left( \frac{r_+^4}{n-1} + \frac{2kr_+^2 \alpha}{n-3} + \frac{k^2\alpha^2}{n-5} \right).
\]

It follows that this physical solution requires that the integer \(n\) should respond to \(n \geq 6\). It is recalled that \(\Sigma_k\) describes the volume of the \((n-1)\) dimensional hypersurface while the thermodynamic volume takes the following form

\[
V = \frac{\Sigma_k r_+^n}{n}.
\]
Considering the pressure of the black hole as the cosmological constant

\[ P = -\frac{\Lambda}{8\pi}. \quad (17) \]

we get the pressure as a function of the temperature and the horizon radius. Performing numerical calculation, we obtain the expression for the pressure

\[
P = \frac{T}{v} - \frac{k((n - 2)(n - 1)v - 32\pi\alpha T)}{\pi(n - 1)^2v^3} - \frac{16\alpha k^2((n - 4)(n - 1)v - 16\pi\alpha T)}{\pi(n - 1)^4v^3} - \frac{256\alpha^2 k^3(n - 6)}{3\pi(n - 1)^5v^6} + \beta^2 \left( \frac{\sqrt{2n+1(n-2)(n-1)^3\pi^2(n-1)v^{-2n}}}{\beta^2} + 64 - 8 \right). \quad (18)\]

Comparing this equation with the Van der Waals equation of states, we can find the specific volume as follows

\[ v = \frac{4r_+}{n - 1}. \quad (19) \]

In the following, critical behaviors will be investigated to establish an explicit expression of the universal number \( \chi = \frac{P_{cv}}{T_{cv}} \) in \((1 + n)\) dimensions. Then, we study the phase diagram transition using the classical thermodynamical physics.

### 3 Critical behavior description

As mentioned before, we reconsider here the study of the critical behaviors of the above black hole solutions. The general study is beyond the scope of the present work, though we will consider an explicit example corresponding to the uncharged solution. The charged solution will be given in the appendix section. Roughly speaking, the computation leads to the following state equation

\[
P = \frac{(n - 1)^2v^2(\pi(n - 1)Tv - k(n - 2)) - 16\alpha k^2(n - 4) - 2\pi(n - 1)Tv}{\pi(n - 1)^3v^4} - \frac{256\alpha^2 k^2(n - 6) - 3\pi(n - 1)Tv}{3\pi(n - 1)^5v^6}. \quad (20)\]

In fact, numerical calculations give the \( P-V \) diagrams in terms of the space dimension \( n \). This result is represented in figure 1.

It is observed from this figure that the behavior is similar to the Van der Waals one. This allows one to drive the critical point coordinates. To get such critical points, one should solve the following system equations

\[
\frac{\partial P}{\partial v} = 0, \quad \frac{\partial^2 P}{\partial v^2} = 0. \quad (21)\]
Figure 1: The $P - V$ diagrams in arbitrary dimensions where $T_c$ is the critical temperature for $k = \alpha = 1$.

Computations can generate the explicit thermodynamical expressions for the critical values. They are given by

\[
T_c = -\frac{\sqrt{k \left( (n-1)^2(n(2n-29) + 2) - \sqrt{(11 - n)(n-1)^5(n+14)} \right)}}{2\pi(n-2)(n-1)^7} \left( \frac{(n+4)(n-1)^2 + 2\sqrt{(11-n)(n-1)^5}}{(n-2)(n-1)^4} \right)^{5/2} \sqrt{\alpha}
\]  

(22)

\[
v_c = 4 \sqrt{k} \sqrt{\frac{(n+4)(n-1)^2 + 2\sqrt{(11-n)(n-1)^5}}{(n-2)(n-1)^4} \sqrt{\alpha}}
\]  

(23)

\[
P_c = \frac{(n-2)^2(n-1)^5C(n)}{48\pi \left( (n+4)(n-1)^2 + 2\sqrt{(11-n)(n-1)^5} \right)^5 \alpha}
\]  

(24)
Figure 2: The universal number in term of space-time dimension.

where the $C(n)$ is a quantity depending on the space dimension $n$. It is given by

$$C(n) = \frac{(n(n(n(9n - 536) + 4772) + 3328) - 6048)(n - 1)^2 + 2\sqrt{(11 - n)(n - 1)^5(n((68 - 11n)n + 1420) - 432)}}{2.}

$$

Combining these obtained critical expressions, we can get an explicit form of the universal number $\chi = \frac{P \varphi}{T_c}$. Indeed, this number is given by

$$\chi = \frac{(n - 1)^4C(n)}{6 \left( (n + 4)(n - 1)^2 + 2\sqrt{(11 - n)(n - 1)^5(n + 14)} \right)} D(n)

$$

where

$$D(n) = \frac{(n - 1)^2(n(2n - 29) + 2) - \sqrt{(11 - n)(n - 1)^5(n + 14)}}{2.}

$$

It is worth noting to comment this expression involving interesting features. First, it recovers the result given in [23]. Indeed, taking $n = 6$, we get the critical coordinates

$$T_c = \frac{1}{\pi \sqrt{5\alpha}}, \quad v_c = \frac{4\sqrt{\alpha}}{\sqrt{5}}, \quad P_c = \frac{17}{200\pi \alpha}, \quad P_c v_c = \frac{17}{50}.

$$

The universal number $\chi$ behaves nicely in terms of the space dimension. It is illustrated in figure 2. It follows from this figure the the critical behaviors appear only in the range $6 \leq n \leq 11$.

However, it is observed that $n$ must be lower than 11 as required by the physical condition of the critical volume. This is not a surprising feature in higher energy physics. In fact, a close inspection in higher dimensional theory shows that this critical dimension appears naturally in string theory and related topics. Indeed, $n = 11$ corresponds to a non perturbative limit of
type IIB superstring. This limit is interpreted in terms of 12 dimensional theory. As proposed by Vafa, it is known by F-theory built from a geometric realisation of $SL(2, Z)$ duality [24]. This observation motivates us to think about a string theory realization of these black holes in terms of the brane physics. We hope to comeback to this issue in future.

The last remark that one should make concerns the topological configuration of the space-time. In fact, we have three cases classified as follows:

- $k = 1$: in this case, the black hole involves a critical behavior controlled by the parameter $\alpha$ and the space dimension $n$.
- $k = 0$: flat topology produces the ideal gas state equation without showing any any critical behavior.
- $k = -1$: the hyperbolic geometry of the uncharged black hole does not show any critical behavior.

More details on this finding will be given in the appendix section.

4 Ehrenfest scheme

Having discussed the $P-V$ criticality of $(1 + n)$ dimensional topological AdS black holes in Lovelock-Born-Infeld Gravity, we move now to study the corresponding phase transitions using classical thermodynamics principals. It is recalled that the classification of such phases associated with the first order and the higher orders can be done in terms of the Clausius-Clapeyron-Ehrenfest equations. Indeed, the first order transition is ensured by the fact the Clausius-Clapeyron equation is satisfied at the critical points. However, the second order transition arises when the Ehrenfest thermodynamical equations are verified. In this section, we examine such equations using results obtained in the classical thermodynamics [25, 26, 27]. In fact, the Ehrenfest equations read as

\[
\left( \frac{\partial P}{\partial T} \right)_S = \frac{C_{p_2} - C_{p_1}}{VT(\Theta_2 - \Theta_1)} = \frac{\Delta C_p}{VT\Delta \Theta}. \tag{29}
\]

\[
\left( \frac{\partial P}{\partial T} \right)_V = \frac{\Theta_2 - \Theta_1}{\kappa_{T_2} - \kappa_{T_1}} = \frac{\Delta \Theta}{\Delta \kappa_T}. \tag{30}
\]

In these equations, $\Theta = \frac{1}{V} \left( \frac{\partial V}{\partial T} \right)_P$ is the volume expansion and $\kappa_T = \frac{1}{T} \left( \frac{\partial T}{\partial P} \right)_V$ defines the isothermal compressibility coefficient.
In what follows, we compute the relevant thermodynamical quantities involved in the above equations for \((1 + n)\) dimensional topological AdS black holes in Lovelock-Born-Infeld gravity. Indeed, combining equations (14), (15) and (17), we get a general expression of the temperature

\[
T = \frac{k(n - 1) \left(k^2(n - 6)\alpha^2 + 3k(n - 4)\alpha\zeta(S)^2 + 3(n - 2)\zeta(S)^4 + 48\pi P\zeta(S)^6\right)}{12\pi(n - 1)\zeta(S) \left(k\alpha + \zeta(S)^2\right)^2}
\]  

(31)

where \(\zeta(S)\) is the real positive root solving the entropy function equation (15). Performing similar calculations, we can get the specific heat at constant pressure and the volume expansion coefficient. They are given respectively by

\[
C_P = \frac{\zeta(S) \left(k\alpha + \zeta(S)^2\right) \left(k(n - 1) \left(k^2(n - 6)\alpha^2 + 3k(n - 4)\alpha\zeta(S)^2 + 3(n - 2)\zeta(S)^4 + 48\pi P\zeta(S)^6\right)\right)}{B(k, \alpha, S, P)\zeta'(S)}
\]

\[
\Theta = \frac{12\pi(n - 1)n\zeta(S) \left(k\alpha + \zeta(S)^2\right)^3}{B(k, \alpha, S, P)}
\]  

(32)

where the function \(B(k, \alpha, S, P)\) reads as

\[
B(k, \alpha, S, P) = -k^4(n - 6)(n - 1)\alpha^3 - 2k^3(n - 9)(n - 1)\alpha^2\zeta(S)^2 + 18k^2(n - 1)\alpha\zeta(S)^4 - 3k\zeta(S)^6((n - 3)n - 80\pi P\alpha + 2) + 48\pi P\zeta(S)^8.
\]  

(33)

Using the famous thermodynamic relation

\[
\left(\frac{\partial V}{\partial P}\right)_T \left(\frac{\partial P}{\partial T}\right)_V \left(\frac{\partial T}{\partial V}\right)_p = -1,
\]  

(34)

we obtain the expression of the isothermal compressibility coefficient

\[
K_T = \frac{48\pi n\zeta(S)^6 \left(k\alpha + \zeta(S)^2\right)}{B(k, \alpha, S, P)}.
\]  

(35)

It follows from these equations the existence of a special factor appearing in their dominators. This factor can be explored to bring critical behaviors for the above calculated thermodynamic quantities. In fact, the constraint \(B(k, \alpha, S, P) = 0\) leading to a divergence of the heat capacity can be easily verified for the critical points.

To discuss the validity of Ehrenfest equations at the critical points, one should explore the expression of the volume expansion coefficient \(\Theta\). The calculation gives

\[
V\Theta = \left(\frac{\partial V}{\partial T}\right)_p \left(\frac{\partial V}{\partial S}\right) \left(\frac{\partial S}{\partial T}\right)_p = \left(\frac{\partial V}{\partial S}\right)_p \left(\frac{C_P}{T}\right).
\]  

(36)
Moreover, the right handed side of eq. (29) can be converted to

\[
\frac{\Delta C_p}{TV \Delta \Theta} = \left[ \left( \frac{\partial S}{\partial V} \right)_p \right]_c, \tag{37}
\]

where the index \(c\) indicates the values of the thermodynamical variables at the critical points. Exploring eqs. (15), (16) and (37), we obtain

\[
\frac{\Delta C_p}{TV \Delta \Theta} = \left[ \frac{\pi \frac{1}{2} (-n-1) \Gamma \left( \frac{n+1}{2} \right) \zeta(S)^{1-n}}{2 \zeta'(S)} \right]_c. \tag{38}
\]

Using eq. (31), the left handed side of eq. (29) becomes

\[
\left[ \left( \frac{\partial P}{\partial T} \right)_{S,c} \right] = \frac{(n - 1) (k \alpha + \zeta(S_c)^2)^2}{4 \zeta(S_c)^5}. \tag{39}
\]

Similar calculations can be done using eqs. (15), (16) and (31). In this way, the left handed side of eq. (30) can be obtained as

\[
\left[ \left( \frac{\partial P}{\partial T} \right)_{V,c} \right] = \frac{(n - 1) (k \alpha + \zeta(S_c)^2)^2}{4 \zeta(S_c)^5}. \tag{40}
\]

Exploring the expressions of the isothermal compressibility coefficient \(K_T\) and volume expansion coefficient \(\Theta\), we get

\[
V K_T = - \left( \frac{\partial V}{\partial P} \right)_T = \left( \frac{\partial T}{\partial V} \right)_V \left( \frac{\partial V}{\partial T} \right)_p = \left( \frac{\partial T}{\partial P} \right)_V V \Theta. \tag{41}
\]

The right handed side of eq. (30) produces the following formulæ

\[
\frac{\Delta \Theta}{\Delta K_T} = \left[ \left( \frac{\partial P}{\partial T} \right)_{V,c} \right] = \frac{(n - 1) (k \alpha + \zeta(S_c)^2)^2}{4 \zeta(S_c)^5} \tag{42}
\]

showing the validity of the second Ehrenfest’s equation.

It is worth noting that the Prigogine-Defay (PD) ratio [28, 29] can be also computed. Indeed, the calculation shows the following expression

\[
\Pi = \frac{\Delta C_p \Delta K_T}{TV(\Delta \Theta)^2} = \left[ \frac{\pi \frac{1}{2} (-n-1) \Gamma \left( \frac{n+1}{2} \right) \zeta(S)^{6-n}}{\zeta'(S) (k \alpha + \zeta(S)^2)^2} \right]_c. \tag{43}
\]

To illustrate the above calculation, we consider the case of \(n = 7\) associated with an eight
dimensional black solution. In this case, the expression of the $\zeta$ is reduced to

$$
\zeta(S) = \sqrt[\frac{3}{4}]{\frac{\pi^4 k^3 \alpha^3 + 12S}{\pi^{4/3}}} - k\alpha. \tag{44}
$$

Moreover, eqs. (38) and (39) become

$$
\frac{\Delta C_P}{TV \Delta \Theta} = \left[ \left( \frac{\partial S}{\partial V} \right) P \right]_{\delta J} = \left[ \left( \frac{\partial P}{\partial T} \right) S \right]_{\delta J} = \frac{3 \left( \pi^5 k^3 \alpha^3 + 12\pi S_c \right)^{2/3}}{2 \left( \sqrt[\frac{3}{4}]{\pi^4 k^3 \alpha^3 + 12S_c - \pi^{4/3}k\alpha} \right)^{5/2}}. \tag{45}
$$

It is obvious to see the validity of Ehrenfest first equation at the critical point. Eq. (40) and (42) give the relation

$$
\frac{\Delta \Theta}{\Delta K_T} = \left[ \left( \frac{\partial P}{\partial T} \right) V \right]_{\delta J} = \frac{3 \left( \pi^5 k^3 \alpha^3 + 12\pi S_c \right)^{2/3}}{2 \left( \sqrt[\frac{3}{4}]{\pi^4 k^3 \alpha^3 + 12S_c - \pi^{4/3}k\alpha} \right)^{5/2}}. \tag{46}
$$

It follows that the Ehrenfest second equation is also valid at the critical point. Moreover, the Prigogine-Defay (PD) ratio reduces to

$$
\Pi = 1, \tag{47}
$$

showing a phase transition although the existence of the divergence near the critical point. It is noted that this matches perfectly with the second order equilibrium transition discussed in [30, 31]. For more detail on this case, we plot all quantities in figure 3. It is observed that such quantities diverge at the critical points.
Figure 3: $C_p$, $\Theta$ and $k_T$ in terms of the entropy for $k = \alpha = 1$.

5 Conclusion and open questions

In this paper, we have reconsidered criticality of $(n+1)$ dimensional topological AdS black holes in Lovelock-Born-Infeld gravity. Interpreting the cosmological constant as a thermodynamic pressure and its conjugate quantity as a thermodynamic volume, we have studied thermodynamical behaviours in terms of the space dimension $n$. More precisely, we have derived an explicit expression of the universal number $\chi = \frac{B_{\Omega}}{T_c^2}$ in terms of $n$. Then, we have discussed the phase transitions at the cortical points. In particular, the Ehrenfest thermodynamical equations have been verified showing that the black hole system undergoes a second phase transition.

This work comes up with many open questions. An interesting one concerns the space dimension $n$. It has been realized that the integer $n$ is constrained to lie within the range

$$6 \leq n \leq 11$$

as required with the reality of the values of the physical quantities at the critical points. A
fast inspection shows that dimensions 6 and 11 appear naturally in the study of higher dimensional theories including superstrings and F-theory. This observation may provide a new challenge on such black holes and theirs connections with string theory compactification. We believe that the above range can be explored to investigate possible realizations in terms of brane physics. This issue deserves a more deeper study to be addressed in coming works.

Appendix

In this appendix, we summarize the calculation for the charged black holes in the asymptotic limit of $\beta$ ($\beta \to \infty$). In this limit, all topologies show critical behaviours for $6 \leq n \leq 11$. The numerical calculations are listed herebelow.

| $q$ | $\alpha$ | $T_c$ | $v_c$ | $P_c$ | $\frac{P_c}{v_c}$ |
|-----|----------|-------|-------|-------|-----------------|
| 0.5 | 1        | 0.1855| 1.3731| 0.0452| 0.3349          |
| 2   | 1        | 0.1839| 1.4728| 0.0438| 0.3509          |
| 1   | 1        | 0.1851| 1.4036| 0.0448| 0.3401          |
| 1   | 0.5      | 0.2537| 1.1748| 0.0807| 0.3737          |
| 1   | 2        | 0.1313| 1.9254| 0.0226| 0.3328          |

Table 1: Critical values for $k = 1, n = 7, \beta \to \infty$

| $q$ | $\alpha$ | $T_c$ | $v_c$ | $P_c$ | $\frac{P_c}{v_c}$ |
|-----|----------|-------|-------|-------|-----------------|
| 0.5 | 1        | 0.2292| 1.1308| 0.0684| 0.3374          |
| 2   | 1        | 0.2287| 1.1561| 0.0677| 0.3426          |
| 1   | 1        | 0.2297| 1.1071| 0.0689| 0.3323          |
| 1   | 0.5      | 0.3157| 0.9437| 0.1243| 0.3716          |
| 1   | 2        | 0.1627| 1.5188| 0.0347| 0.3245          |

Table 2: Critical values for $k = 1, n = 8, \beta \to \infty$

| $q$ | $\alpha$ | $T_c$ | $v_c$ | $P_c$ | $\frac{P_c}{v_c}$ |
|-----|----------|-------|-------|-------|-----------------|
| 0.5 | 1        | 0.2747| 0.9182| 0.0987| 0.3298          |
| 2   | 1        | 0.2742| 0.9389| 0.0979| 0.3352          |
| 1   | 1        | 0.2751| 0.8990| 0.0993| 0.3245          |
| 1   | 0.5      | 0.3786| 0.7844| 0.1786| 0.3701          |
| 1   | 2        | 0.1949| 1.2253| 0.0500| 0.3149          |

Table 3: Critical values for $k = 1, n = 9, \beta \to \infty$
| q | α | $T_c$ | $v_c$ | $P_c$ | $\frac{P_c}{T_c}$ |
|---|---|---|---|---|---|
| 0.5 | 1 | 0.3210 | 0.7631 | 0.1358 | 0.3229 |
| 2 | 1 | 0.3205 | 0.7816 | 0.1348 | 0.3287 |
| 1 | 1 | 0.3214 | 0.7453 | 0.1367 | 0.3170 |
| 1 | 0.5 | 0.4422 | 0.6688 | 0.2439 | 0.3689 |
| 1 | 2 | 0.2277 | 0.9944 | 0.0691 | 0.3018 |

Table 4: Critical values for $k = 1, n = 10, \beta \rightarrow \infty$

| q | α | $T_c$ | $v_c$ | $P_c$ | $\frac{P_c}{T_c}$ |
|---|---|---|---|---|---|
| 0.5 | 1 | 0.3680 | 0.6465 | 0.1806 | 0.3173 |
| 2 | 1 | 0.3674 | 0.6637 | 0.1790 | 0.3235 |
| 1 | 1 | 0.3685 | 0.6290 | 0.1820 | 0.3106 |
| 1 | 0.5 | 0.5063 | 0.5815 | 0.3206 | 0.3682 |
| 1 | 2 | 0.2615 | 0.7556 | 0.0930 | 0.2689 |

Table 5: Critical values for $k = 1, n = 11, \beta \rightarrow \infty$

| q | α | $T_c$ | $v_c$ | $P_c$ | $\frac{P_c}{T_c}$ |
|---|---|---|---|---|---|
| 0.5 | 1 | 0.5527 | 0.8521 | 0.4747 | 0.7317 |
| 2 | 1 | 0.6328 | 0.8838 | 0.5087 | 0.7104 |
| 1 | 1 | 0.4731 | 0.8136 | 0.4430 | 0.7617 |
| 1 | 0.5 | 2.0124 | 0.7288 | 1.7661 | 0.6396 |
| 1 | 2 | 0.1751 | 0.7352 | 0.2149 | 0.9022 |

Table 6: Critical values for $k = -1, n = 7, \beta \rightarrow \infty$

| q | α | $T_c$ | $v_c$ | $P_c$ | $\frac{P_c}{T_c}$ |
|---|---|---|---|---|---|
| 0.5 | 1 | 0.3808 | 0.6471 | 0.2480 | 0.9570 |
| 2 | 1 | 0.6118 | 0.7506 | 0.6104 | 0.7490 |
| 1 | 1 | 0.4760 | 0.6964 | 0.5506 | 0.8056 |
| 1 | 0.5 | 2.0342 | 0.6343 | 2.0967 | 0.6538 |
| 1 | 2 | 0.2218 | 0.8562 | 0.2480 | 0.9570 |

Table 7: Critical values for $k = -1, n = 8, \beta \rightarrow \infty$

| q | α | $T_c$ | $v_c$ | $P_c$ | $\frac{P_c}{T_c}$ |
|---|---|---|---|---|---|
| 0.5 | 1 | 0.3210 | 0.7631 | 0.1358 | 0.3229 |
| 2 | 1 | 0.3205 | 0.7816 | 0.1348 | 0.3287 |
| 1 | 1 | 0.3214 | 0.7453 | 0.1367 | 0.3170 |
| 1 | 0.5 | 0.4422 | 0.6688 | 0.2439 | 0.3689 |
| 1 | 2 | 0.2277 | 0.9944 | 0.0691 | 0.3018 |

Table 8: Critical values for $k = -1, n = 9, \beta \rightarrow \infty$
| q | α | $T_c$ | $v_c$ | $P_c$ | $P_{cv}$ |
|---|---|---|---|---|---|
| 0.5 | 1 | 0.5380 | 0.6291 | 0.6973 | 0.8153 |
| 2 | 1 | 0.5991 | 0.6496 | 0.7267 | 0.7880 |
| 1 | 1 | 0.4794 | 0.6069 | 0.6711 | 0.8495 |
| 1 | 0.5 | 2.0537 | 0.5597 | 2.4475 | 0.6670 |
| 1 | 2 | 0.1125 | 0.6285 | 0.3209 | 1.7925 |

Table 9: Critical values for $k = -1, n = 10, \beta \to \infty$

| q | α | $T_c$ | $v_c$ | $P_c$ | $P_{cv}$ |
|---|---|---|---|---|---|
| 0.5 | 1 | 0.5359 | 0.5541 | 0.8290 | 0.8572 |
| 2 | 1 | 0.5915 | 0.5710 | 0.8569 | 0.8272 |
| 1 | 1 | 0.4833 | 0.5366 | 0.8046 | 0.8934 |
| 1 | 0.5 | 2.0762 | 0.4997 | 2.8232 | 0.6795 |
| 1 | 2 | 0.1571 | 0.6014 | 0.3658 | 1.3999 |

Table 10: Critical values for $k = -1, n = 11, \beta \to \infty$

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