Random Initialization Solves Shapley’s Fictitious Play Counterexample

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In 1964 Shapley [5] devised a family of games for which fictitious play (FP) fails to converge to Nash equilibrium. The games are two player non-zero-sum and depicted in Equation 1. The payoffs satisfy $a_i > b_i > c_i$ and $\alpha_i > \beta_i > \gamma_i$, for $i = 1, 2, 3$. One particular instance from this family that has been singled out [6] is given in Equation 2. This game can be viewed as a generalized version of Rock-Paper-Scissors [6]. Note that we must permute the first two columns then the last two rows in this game to obtain a game that satisfies Shapley’s conditions—the permuted game is given in Equation 3. Shapley’s argument is as follows. He assumes that both players play their first pure strategy in the initial round, and then follow a standard simultaneous version of fictitious play (Algorithm 1). He then shows that play will cycle in such a way that the runs of pure strategy profiles increase exponentially, failing convergence to the Nash equilibrium frequencies. Note that all games in Shapley’s class contain a unique Nash equilibrium, and for the example we have singled out it selects each pure strategy with equal probability. Shapley states,

The argument we have given is independent of the tie-breaking rule. With minor modifications it can also handle the case of alternating moves, as well as the case of nonintegral run lengths. The latter implies that the differential-equation version of FP (see [2]) will also fail to converge to the solution [5].

$$ \begin{bmatrix}
(a_1, \beta_1) & (c_2, \gamma_1) & (b_3, \alpha_1) \\
(b_1, \alpha_2) & (a_2, \beta_2) & (c_3, \gamma_2) \\
(c_1, \alpha_3) & (b_2, \alpha_3) & (a_3, \beta_3)
\end{bmatrix} \quad (1) $$

$$ \begin{bmatrix}
(0, 0) & (2, 1) & (1, 2) \\
(1, 2) & (0, 0) & (2, 1) \\
(2, 1) & (1, 2) & (0, 0)
\end{bmatrix} \quad (2) $$

$$ \begin{bmatrix}
(2, 1) & (0, 0) & (1, 2) \\
(1, 2) & (2, 1) & (0, 0) \\
(0, 0) & (1, 2) & (2, 1)
\end{bmatrix} \quad (3) $$

We confirm Shapley’s finding empirically for the example game by running fictitious play with the given initialization for a large number of iterations. Using $T = 100,000$ we obtain strategy frequencies of player 1 of $(0.3097, 0.5981, 0.0922)$, and for player 2 of $(0.7290, 0.2348, 0.0362)$, which are both clearly very far from the equilibrium frequencies. For $T = 1,000,000$ we obtain strategy frequencies for player 1 of $(0.3870, 0.0598, 0.5522)$, and for player 2 of $(0.1523, 0.0235, 0.8242)$. As Shapley described, the strategies cycle between exponentially-increasing runs of pure strategy profiles, and convergence is not obtained.

\[1\] It was previously known that fictitious play converges to Nash equilibrium in two-player zero-sum games, and in two-player non-zero-sum games with 2 strategies per player. Subsequently it had been shown to converge to Nash equilibrium in certain other game classes such as $2 \times N$ two-player games with generic payoffs [1].
Algorithm 1 Classic fictitious play for $n$-player games

**Inputs:** Game $G$, initial mixed strategies $\sigma^0_i$ for $i \in N$, number of iterations $T$

for $t = 1$ to $T$ do
  for $i = 1$ to $n$ do
    $\sigma'_t i = \arg \max_{\sigma_i} u_i(\sigma_i, \sigma_{t-1}^{t-1} - i)$
    $\sigma_t i = (1 - \frac{1}{t+1}) \sigma_{t-1}^{t-1} i + \frac{1}{t+1} \sigma'_t i$
  return $(\sigma_T^1, \ldots, \sigma_T^n)$

The algorithm also fails to converge for any pure-strategy initializations, and therefore Shapley’s selection was not special. In fact, we verify that fictitious play also fails to converge to equilibrium in this game if we initialize all pure strategies to be played with probability $\frac{1}{3}$ for each player, despite the fact that this constitutes the unique Nash equilibrium strategy profile.

Recent work has shown that convergence of fictitious play can be significantly improved by using several initial strategy profiles and selecting the best one [4]. The general approach is depicted in Algorithm 2. The best-performing approach used maximin initialization based on solving a nonconvex quadratic program formulation. However just generating the strategies uniformly at random performed almost as well and is computationally simpler, so we will experiment with that approach, which is described in Algorithm 3.

Algorithm 2 Fictitious play with multiple initializations

**Inputs:** Game $G$, set of $K$ initial mixed strategies $\sigma^0_k, i$ for $i \in N$ $k = 1 \ldots, K$, number of iterations $T$

$\epsilon^* = \infty$

for $k = 1$ to $K$ do
  for $t = 1$ to $T$ do
    for $i = 1$ to $n$ do
      $\sigma'_t k, i = \arg \max_{\sigma_i} u_i(\sigma_i, \sigma_{t-1}^{t-1} k, i)$
      $\sigma_t k, i = (1 - \frac{1}{t+1}) \sigma_{t-1}^{t-1} k, i + \frac{1}{t+1} \sigma'_t k, i$
    $\epsilon_k = \max_i \max_{\sigma_i} \left[ u_i(\sigma_i, \sigma_{T}^{T} k, i) - u_i(\sigma_{T}^{T}, \sigma_{T}^{T} k, i) \right]$
    if $\epsilon_k < \epsilon^*$ then
      $\sigma^* = \sigma_t^T k$
      $\epsilon^* = \epsilon_k$
  return $\sigma^*$

We applied this approach to solve Shapley’s game using 100,000 different initializations generated using Algorithm 3 with $T = 100,000$. We found that 33,403 of them produced values of $\epsilon$ (i.e., $\epsilon_k$ from Algorithm 2) smaller than $10^{-4}$. So fictitious play converged to the equilibrium for approximately $\frac{1}{3}$ of the initializations. So in expectation if we run fictitious play using three random initializations, we expect to converge to the equilibrium in one. Figure 1 shows a heatmap of the initializations producing convergence to the Nash equilibrium out of the first 100 initializations. We suspect that the example game is representative and that similar results would apply to the other games in Shapley’s family as well.

References

[1] Ulrich Berger. Fictitious play in 2xn games. *Journal of Economic Theory*, 120:139–154, 2005.

\[^2\]Note that strict Nash equilibria are absorbing states in fictitious play [3]; however in the game we are considering the Nash equilibrium is not strict, which is obvious since only pure-strategy equilibria can be strict.
Algorithm 3 Algorithm for generating uniform initial strategies

**Inputs:** Game $G$ with $n$ players and $m$ strategies per player

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for $i = 1$ to $n$
    $Z_i = 0$
    for $j = 1$ to $m$
        $u = \text{uniform random number in (0,1)}$
        $\sigma_i(j) = -\ln(u)$
        $Z_i = Z_i + \sigma_i(j)$
    end for
    return $\sigma = (\sigma_1, \ldots, \sigma_n)$
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Figure 1: Heatmap of the strategy profiles out of the first 100 initializations that produce $\epsilon < 10^{-4}$. The first three columns are player 1’s strategy probabilities and next three columns are player 2’s probabilities.

[2] George W. Brown. Iterative solutions of games by fictitious play. In Tjalling C. Koopmans, editor, *Activity Analysis of Production and Allocation*, pages 374–376. John Wiley & Sons, 1951.

[3] Drew Fudenberg and David Levine. *The Theory of Learning in Games*. MIT Press, 1998.

[4] Sam Ganzfried. Fictitious play with maximin initialization. In *IEEE Conference on Decision and Control (CDC)*, 2022.

[5] Lloyd S Shapley. Some topics in two-person games. In M. Drescher, L. S. Shapley, and A. W. Tucker, editors, *Advances in Game Theory*. Princeton University Press, 1964.

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