The radius of convergence of the Lamé function

Yoon-Seok Choun

Department of Physics, Hanyang University, Seoul, 133-791, South Korea

Abstract

We consider the radius of convergence of a Lamé function, and we show why Poincaré-Perron theorem is not applicable to the Lamé equation.

Keywords: Lamé equation; 3-term recursive relation; Poincaré-Perron theorem

2000 MSC: 33E05, 33E10, 34A25, 34A30

1. Introduction

A sphere is a geometric perfect shape, a collection of points that are all at the same distance from the center in the three-dimensional space. In contrast, the ellipsoid is an imperfect shape, and all of the plane sections are elliptical or circular surfaces. The set of points is no longer the same distance from the center of the ellipsoid. As we all know, the nature is nonlinear and geometrically imperfect. For simplicity, we usually linearize such systems to take a step toward the future with good numerical approximation. Indeed, many geometric spherical objects are not perfectly spherical in nature. The shape of those objects are better interpreted by the ellipsoid because they rotate themselves. For instance, ellipsoidal harmonics are shown in the calculation of gravitational potential [11]. But generally spherical harmonics are preferable to mathematically complex ellipsoid harmonics.

Lamé functions (ellipsoidal harmonic functions) [7] are applicable to diverse areas such as boundary value problems in ellipsoidal geometry, Schrödinger equations for various periodic and anharmonic potentials, the theory of Bose-Einstein condensates, group theory to quantum mechanical bound states and band structures of the Lamé Hamiltonian, etc. As we apply a power series into the Lamé equation in the algebraic form or Weierstrass’s form, the 3-term recurrence relation starts to arise and currently, there are no general analytic solutions in closed forms for the 3-term recursive relation of a series solution because of their mathematical complexity [1, 5, 13]. However, most of well-known special functions (Hypergeometric, Bessel, Legendre, Kummer functions, etc) are well-defined and their mathematical structure of series expansions are well-represented because their series solutions consist of a 2-term recursion relation.

It has been established as a fact that the radius of convergence for a power series solution of Lamé equation is obtained by applying the Poincaré–Perron (P–P) theorem [9, 10]. Recently, the author shows that the uniqueness theorem and an invariant property are violated if the P–P theorem is applied into linear ODEs [3]. Here, we present that the convergent domain of a Lamé
function by applying the P–P theorem is not absolutely convergent but conditionally one. We also show the reason why the P–P theorem causes trouble if it is applied into a Lamé function.

2. The radius of convergence for an infinite series of the Lamé equation in the algebraic form

Lamé equation is a second-order linear ordinary differential equation of the algebraic form

\[
\frac{d^2y}{dx^2} + \frac{1}{2} \left( \frac{1}{x-a} + \frac{1}{x-b} + \frac{1}{x-c} \right) \frac{dy}{dx} + \frac{-\alpha(x+1)x+q}{4(x-a)(x-b)(x-c)} y = 0
\]  

(2.1)

and it has four regular singular points: \( a, b, c \) and \( \infty \). Assume that its solution is

\[
y(z) = \sum_{n=0}^{\infty} d_n z^n + \lambda
\]

where \( z = x-a \)

(2.2)

Substituting (2.2) into (2.1) gives for the coefficients \( c_n \) the recurrence relations

\[
d_{n+1} = A_n d_n + B_n d_{n-1} \\
\text{with } A_n = A, B_n = A B_n
\]

(2.3)

We have two indicial roots which are \( \lambda = 0 \) and \( \frac{1}{2} \). Parameters \( b \) and \( c \) are identical to each other in (2.4).

**Definition 1.** A homogeneous linear ordinary differential equation of order \( j \) with variable coefficients is of the form

\[
a_j(z) y^{(j)} + a_{j-1}(z) y^{(j-1)} + \cdots + a_1(z) y' + a_0(z) y = 0
\]

(2.5)

Assuming its solution as a power series in the form

\[
y(z) = \sum_{n=0}^{\infty} d_n z^n
\]

(2.6)

where \( d_0 \neq 0 \). We obtain the three-term recurrence relation putting (2.6) in (2.3)

\[
d_{n+1} = A_n d_n + B_n d_{n-1}
\]

(2.7)

where \( d_1 = A_0 d_0 \).

**Theorem 1.** Assuming \( \lim_{n \to \infty} A_n = A < \infty \) and \( \lim_{n \to \infty} B_n = B < \infty \) in (2.7), the domain of convergence of (2.6) is written by

\[
D := \left\{ z \in \mathbb{C} \mid |A z| + |B z| < 1 \right\}
\]

(2.8)
Therefore, a Lamé series for the 3-term recurrence relation is convergent for

$$D := \left\{ z \in \mathbb{C} \mid |Az| + |Bz^2| = \left| \frac{(2a - b - c)z}{(a - b)(a - c)} \right| + \left| \frac{z^2}{(a - b)(a - c)} \right| < 1 \right\}$$

(2.9)

where $z = x - a$. The coefficients $a$, $b$ and $c$ decide the range of an independent variable $z$ as we see (2.9). With more precision, all possible ranges of $a$, $b$, $c$ and $z$ are given in Table 1 where $a$, $b$, $c$, $z \in \mathbb{R}$.

| Range of the coefficients $a$, $b$ and $c$ | Range of the independent variable $z$ |
|------------------------------------------|-------------------------------------|
| $a = b$ or $a = c$                       | no solution                         |
| $a - b > 0$, $a - c > 0$                 | $|z| < \frac{-(2a - b - c)^2 + 4(a - b)(a - c)}{2}$ |
| $a - b < 0$, $a - c < 0$                 | $|z| < \frac{(2a - b - c)^2 + 4(a - b)(a - c)}{2}$ |
| $a - b < 0$, $a - c > 0$, $2a - b - c > 0$| $|z| < -(a - b)$                     |
| $a - b > 0$, $a - c < 0$, $2a - b - c < 0$| $|z| < (a - b)$                      |
| $b = c > 0$                             | $|z| < (-1 + \sqrt{2})(a - b)$       |
| $b = c < 0$                             | $|z| < (1 - \sqrt{2})(a - b)$        |

Table 1: Boundary condition of $z$ for an infinite series of Lamé equation in the algebraic form about $z = 0$

3. The radius of convergence for an infinite series of the Lamé equation in Weierstrass’s form

Lamé equation in Weierstrass’s form and Heun equation are of Fuchsian types with the four regular singularities. Lamé equation is derived from Heun equation by changing all coefficients $\gamma = \delta = \epsilon = \frac{1}{4}$, $\alpha = \rho^{-2}$, $\alpha = \frac{1}{4}(\alpha + 1)$, $\beta = -\frac{1}{4} \alpha$, $q = -\frac{1}{2} \rho^{-2}$ and an independent variable $x = sn^2(z, \rho)$ \[4, 12\]. Its equation in Weierstrass’s form is represented as we apply the method of separation of variables to Laplace’s equation in an ellipsoidal coordinate system \[7\].

The Lamé equation in Weierstrass’s form is referred as

$$\frac{d^2y}{dz^2} = (\alpha(\alpha + 1) \rho^2 \cdot sn^2(z, \rho) - h)y(z)$$

(3.1)

where $\rho$ and $\alpha$ are real parameters such that $0 < \rho < 1$ and $\alpha \geq -\frac{1}{2}$ in general. The Jacobian elliptic function $sn(z, \rho)$ is represented to be the in the inversion of Legendre’s elliptic integral of the first kind

$$z = \int_{0}^{\sin(\varphi)} \frac{d\varphi}{\sqrt{1 - \rho^2 \sin^2 \varphi}}$$

\[3\]
which gives $sn(z, \rho) = \sin(\text{am}(z, \rho))$. $\text{am}(z, \rho)$ is the Jacobi amplitude and $\rho$ is the modulus of the elliptic function $sn(z, \rho)$. If we take $sn^2(z, \rho) = \xi$ as an independent variable, (3.1) becomes

$$\frac{d^2y}{d\xi^2} + \frac{1}{2} \left( \frac{1}{\xi} + \frac{1}{\xi - 1} + \frac{1}{\xi - \rho^{-2}} \right) \frac{dy}{d\xi} + \frac{-\alpha(\alpha + 1)\xi + h\rho^{-2}}{4\xi(\xi - 1)(\xi - \rho^{-2})^2} y(\xi) = 0$$

(3.2)

This is an equation of Fuchsian type with the four regular singularities: $\xi = 0, 1, \rho^{-2}, \infty$. The first three, namely 0, 1, $\rho^{-2}$, have the property that the corresponding exponents are 0, $\frac{1}{2}$ which is the same as the case of Lamé equation in the algebraic form.

Now, comparing (3.2) with (2.1), two equations are equivalent to each other with correspondence

$$a \longrightarrow 0$$

$$b \longrightarrow 1$$

$$c \longrightarrow \rho^{-2}$$

$$q \longrightarrow h\rho^{-2}$$

$$x \longrightarrow \xi = sn^2(z, \rho)$$

(3.3)

Putting (3.3) in (2.9), the radius of convergence for an infinite series of the Lamé equation at $\xi = sn^2(z, \rho)$ is given by

$$|\rho^2 sn^4(z, \rho)| + |(1 + \rho^2)sn^2(z, \rho)| < 1$$

(3.4)

For the case of $z, sn(z, \rho) \in \mathbb{R}$ where $0 < \rho < 1$, (3.4) turns to be

$$0 \leq sn^2(z, \rho) < \frac{-1 + \rho^2 + \sqrt{\rho^4 + 6\rho^2 + 1}}{2\rho^2}$$

(3.5)

![Figure 1](image.png)

Figure 1: Domain of convergence of the series where $z, sn(z, \rho) \in \mathbb{R}$ and $0 < \rho < 1$

Fig. 1 represents a graph of (3.5) in the $\rho$-$sn^2(z, \rho)$ plane; the shaded area represents the domain of convergence of the series for the Lamé equation around $sn^2(z, \rho) = 0$; it does not include a dotted line.
4. Poincaré-Perron theorem and its applications to Frobenius solutions

**Theorem 2.** Poincaré-Perron theorem [8]: If the coefficients $\alpha_{i,n}$ where $i = 1, 2, \cdots, k$ of a linear homogeneous difference equation

$$u(n + 1) + \alpha_{1,n}u(n) + \alpha_{2,n}u(n - 1) + \alpha_{3,n}u(n - 2) + \cdots + \alpha_{k,n}u(n - k + 1) = 0$$

have limits $\lim_{n \to \infty} \alpha_{i,n} = \alpha_i$ with $\alpha_{k,n} \neq 0$ and if the roots $\lambda_1, \ldots, \lambda_k$ of the characteristic equation

$$t^k + \alpha_1 t^{k-1} + \alpha_2 t^{k-2} + \cdots + \alpha_k = 0$$

have distinct absolute values.

H. Poincaré suggested that

$$\lim_{n \to \infty} u(n + 1) / u(n)$$

is equal to one of the roots of the characteristic equation in 1885 [10]. And a more general theorem has been extended by O. Perron in 1921 [9] such that

$$\lim_{n \to \infty} u_i(n + 1) / u_i(n) = \lambda_i$$

where $i = 1, 2, \cdots, k$ and $\lambda_i$ is a root of the characteristic equation, and $n \to \infty$ by positive integral increments.

Putting (3.3) in (2.4), we have

$$\lim_{n \to \infty} A_n = A = 1 + \rho^2 \quad \lim_{n \to \infty} B_n = B = -\rho^2$$

(4.1)

From Thm.2, a characteristic equation of (2.3) is obtained by

$$r^2 - Ar - B = 0$$

(4.2)

The asymptotic recurrence relation of (4.2) is given by

$$\overline{d}_{n+1} = A \overline{d}_n + B \overline{d}_{n-1} \quad ; n \geq 1$$

(4.3)

with seed values $\overline{d}_1 = A$ and $\overline{d}_0 = 1$ is chosen for simplicity.

The roots of a polynomial (4.2) have two different moduli such as

$$r_1 = A - \sqrt{A^2 + 4B} / 2 \quad r_2 = A + \sqrt{A^2 + 4B} / 2$$

(4.4)

We know $\lim_{n \to \infty} |d_{n+1}/d_n| = \lim_{n \to \infty} |\overline{d}_{n+1}/\overline{d}_n|$. If $|r_1| < |r_2|$, then $\lim_{n \to \infty} |\overline{d}_{n+1}/\overline{d}_n| \to |r_2|$, so that the radius of convergence for a 3-term recursion relation (4.3) is $|r_2|^{-1}$; as if $|r_2| < |r_1|$, then $\lim_{n \to \infty} |\overline{d}_{n+1}/\overline{d}_n| \to |r_1|$, and its radius of convergence is increased to $|r_1|^{-1}$; and if $|r_1| = |r_2|$ and $r_1 \neq r_2$, $\lim_{n \to \infty} |\overline{d}_{n+1}/\overline{d}_n|$ does not exist; if $r_1 = r_2$, $\lim_{n \to \infty} |\overline{d}_{n+1}/\overline{d}_n|$ is convergent. More details are explained in Appendix B of part A [12] and various text books [1, 6, 14].

We obtain two different moduli by putting (4.1) in (4.4) such as

$$r_1 = 1 + \rho^2 - |1 - \rho^2| / 2 \quad r_2 = 1 + \rho^2 + |1 - \rho^2| / 2$$

(4.5)
ρ is mostly between 0 and 1, then we have $|r_1 = \rho^2| < |r_2 = 1|$ in (4.5). And Poincaré-Perron theorem tells us that the radius of convergence for an independent variable $sn^2(z, \rho)$ is $|sn^2(z, \rho)| < 1$. If $z, sn(z, \rho) \in \mathbb{R}$, then the boundary condition of $sn^2(z, \rho)$ is

$$0 \leq sn^2(z, \rho) < 1 \quad (4.6)$$

The corresponding domain of convergence in the real axis, given by (4.6), is shown shaded in Fig. 2; it does not include a dotted line, and maximum modulus of $sn^2(z, \rho)$ is the unity.

For the simple numeric computation, we say $||A_n|| - |A|, ||B_n|| - |B| < \epsilon$ for $\forall n$ in (2.4). Simplicity, we treat $A_n$ and $B_n$ terms as the unity in (2.4), and its recurrence relation is equivalent to (4.3). Let assume that the P–P theorem provides us the radius of convergence for a solution in series. Then we know that a solution of its power series is absolutely convergent and we can rearrange of its terms for the series solution. Consider the following summation series such as

$$y(\xi) = \sum_{n=0}^{N} \sum_{m=0}^{N} \frac{(n+m)!}{n! \ m!} \tilde{x}^n \tilde{y}^m$$

where $\tilde{x} = -\rho^2 \xi^2$, $\tilde{y} = (1 + \rho^2) \xi$ and $\xi = sn^2(z, \rho) \quad (4.7)$

If $N \to \infty$, (4.7) turns to be the generating function of the sequence in (4.3).

For instance, put $\rho = 0.8$ in (4.5)

$$0 \leq sn^2(z, 0.8) < -\frac{(1 + 0.8^2) + \sqrt{0.8^4 + 6 \times 0.8^2 + 1}}{2 \times 0.8^2} \approx 0.50875 \quad (4.8)$$

And if Perron’s rule verifies that an infinite series of Lamé equation is absolute convergent and gives the corrected radius of convergence, (4.6) must to be satisfied. Then, we also have a solution in series at $0.50875 \leq sn^2(z, 0.8) < 1$.

Consider $sn^2(z, 0.8) = 0.7$ in (4.7) with various positive integer values $N$ such as $N = 10, 50, 100, 200, 300, \cdots, 1000$ where $\rho = 0.8$ in Mathematica program.
series expansion of (4.3) such as why we take errors of the radius of convergence obtained by Perron’s rule, first of all, we give a P–P theorem, is not absolute convergent but only conditionally one. In order to answer a reason for a power series solution. And a series solution for an infinite series, obtained by applying the Therefore, we conclude that we can not use the P–P theorem to obtain the radius of convergence for a power series solution. And a series solution for an infinite series, obtained by applying the P–P theorem, is not absolute convergent but only conditionally one. In order to answer a reason why we take errors of the radius of convergence obtained by Perron’s rule, first of all, we give a series expansion of (4.3) such as

\[
\sum_{n=0}^{\infty} d_n \xi^n = 1 + A\xi + (A^2 + B)\xi^2 + (A^3 + 2AB)\xi^3 + (A^4 + 3A^2B + A^3)\xi^4 + (A^5 + 4A^3B + 3AB^2)\xi^5 + \cdots
\]

(4.9)

In general, a series \(\sum_{n=0}^{\infty} d_n \xi^n\) is called absolutely convergent if \(\sum_{n=0}^{\infty} |d_n||\xi|^n\) is convergent. Take moduli of each sequence in (4.9)

\[
\sum_{n=0}^{\infty} |d_n||\xi|^n = 1 + |A||\xi| + |A^2 + B||\xi|^2 + |A^3 + 2AB||\xi|^3 + |A^4 + 3A^2B + B^2||\xi|^4 + |A^5 + 4A^3B + 3AB^2||\xi|^5 + \cdots
\]

(4.10)

The Cauchy ratio test tells us that a series is absolute convergent if \(\lim_{n \to \infty} |d_{n+1}/d_n| |\xi| = \lim_{n \to \infty} |\xi|\) is less than the unit. And Poincaré-Perron theorem give us the value of \(\lim_{n \to \infty} |d_{n+1}/d_n|\) in (4.10).

However, we can not obtain the radius of convergence using this basic principle. We must take all absolute values inside parentheses of (4.9) such as

\[
\sum_{n=0}^{\infty} |d_n||\xi|^n = 1 + |A||\xi| + (|A|^2 + |B|)|\xi|^2 + (|A|^3 + 2|A||B|)|\xi|^3 + (|A|^4 + 3|A|^2|B| + |B|^2)|\xi|^4 + (|A|^5 + 4|A|^3|B| + 3|A||B|^2)|\xi|^5 + \cdots
\]

(4.11)

The more explicit explanation about this mathematical phenomenon is available in Ref.[2].

Take all absolute values of constant coefficients \(a_i\) of the characteristic equation in Thm[2]

\[t^5 + |x_1|t^{d-1} + |x_2|t^{d-2} + \cdots + |x_k| = 0
\]

(4.12)
suggesting the roots of its characteristic equation as $\lambda^*_1, ..., \lambda^*_k$. And $\lim \left| \frac{u(n+1)}{u(n)} \right|$ as $n \to \infty$ in Thm $[2]$ is equivalent to $|\lambda^*_i|$. With this reconsideration, the corrected radius of convergence for a Lamé function is equivalent to (3.5) since $0 < \rho < 1$ where $z, sn(z, \rho) \in \mathbb{R}$.

Fig $[3]$ represents two different shaded areas of convergence in Figs $[1]$ and $[2]$. In the bright shaded area, the domain of absolute convergence of the series for the Lamé equation around $sn^2(z, \rho) = 0$ is not available; it only provides the domain of conditional convergence for it: And in the dark shaded region, two different domains of convergence are equivalent to each other.

Figure 3: Two different domains of absolute and conditional convergence of the Lamé equation

References

[1] Bateman, H., Erdelyi, A., Higher transcendental functions 3. Automorphic functions. McGraw-Hill, 1955.
[2] Choun, Y.S., “The radius of convergence of the Heun function,” arXiv:1803.03115
[3] Choun, Y.S., The violation of a uniqueness theorem and an invariant in the application of Poincaré–Perron theorem to Heun’s equation, arXiv:1605.08960
[4] Heun, K., “Zur Theorie der Riemann’schen Functionen zweiter Ordnung mit vier Verzweigungspunkten.” Mathematische Annalen, 33, 161 (1889).
[5] Hobson, E.W., The theory of spherical and ellipsoidal harmonics. Cambridge Univ. Press, 1931.
[6] Kristensson, G., Second order differential equations: special functions and their classification, (Springer-Verlag New York, 2014).
[7] Lamé, G., “Sur les surfaces isothermes dans les corps homogenes en equilibre de temperature,” J. Math. Pures. Appl., 2 (1837), 147–188 (French).
[8] Milne-Thomson, L.M., The calculus of finite differences, (Macmillan and Co., 1933).
[9] Perron, O., Über Summenungleichungen und Poincarésche Differenzengleichungen, Math. Ann. 84 (1921), 1–15.
[10] Poincaré, H., Sur les Equations Lineaires aux Differentielles Ordinaires et aux Differences Finies, (French) Amer. J. Math. 7(3) (1885), 203–258.
[11] Romain, G. and Jean-Pierre, B., “Ellipsoidal Harmonic expansions of the gravitational potential: Theory and application,” Celest. Mech. Dyn. Astron., 79(4) (2001), 235–275.
[12] Ronveaux, A., Heun’s Differential Equations, (Oxford University Press, 1995).
[13] Whittaker, E.T. and Watson, G.N., A course of modern analysis. Cambridge Univ. Press, 1952.
[14] Wimp, J., Computation with recurrence relations, (Pitman, 1984)