New estimation of the curvature effect for the X-ray vacuum diffraction induced by an intense laser field

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Quantum electrodynamics predicts X-ray diffractions under a high-intensity laser field via virtual charged particles, and this phenomenon is called vacuum diffraction (VD). In this paper, we derive a new formula to describe VD in a head-on collision geometry of an X-ray free-electron laser (XFEL) pulse and a laser pulse. The wavefront curvature of the XFEL pulse is newly considered in this formula. With this formula, we also discuss the curvature effect on VD signals based on realistic parameters at the SACLA XFEL facility.

Subject Index B30, C09, C33, C41, J25

1. Introduction

Photon–photon scattering is a nonlinear interaction between photons, intermediated by a virtual electron loop at the lowest order of quantum electrodynamics (QED). To observe this process in a real vacuum, where the Coulomb fields of pre-existing charges do not contribute, many experimental approaches have been considered and carried out to excite this virtual loop. A traditional way to stimulate the vacuum over a macroscopic scale, ∼1 m, is to use permanent or electromagnets that “magnetize” the vacuum [1–3]. Another way, which does not apply such macroscopic-scale external fields, is to use a high-intensity laser that gives a very strong electromagnetic field to “pump” the vacuum [4–6]. The pump laser locally and anisotropically changes the refractive index of the vacuum in its field and provides a kind of lens. In materials, this lens focuses the pump laser itself. This effect has been investigated as self-focusing [7,8]. In a vacuum, in order to detect this lens, another probe light is collided. The probe light is diffracted by this lens, and its polarization is also changed. These phenomena are called “vacuum diffraction (VD)” and “vacuum birefringence (VB)”. X-rays are more suitable as the probe lights than optical lights since the probability of VD and VB is proportional to the square of photon energy [9]. Though GeV photons have higher photon energy than X-rays, the optical technology for X-rays, such as the polarizer [10,11] and the focusing optics [12,13], is more sophisticated than that for GeV photons [14]. Though the detection of VB is implied by the observation of neutron stars [15], this detection is not model-independent, which means that a terrestrial measurement is desirable.
Fig. 1. Schematic drawings of the experimental condition where the XFEL pulse and the laser pulse collide in the head-on geometry. (a) Magnified view around the collision point from the side. The beam axis of the XFEL pulse is on the $Z$-axis, and that of the laser pulse is on the negative direction of the $z$-axis. The laser focus has displacements $(x_L, y_L, z_L)$ from the origin of each axis. The XFEL pulse has incident angles, $\Theta_1^x, \Theta_1^y$, from the $z$-axis, and its beam size has been modeled to be constant. (b) Bird’s-eye view. $\hat{e}_X (\hat{e}_L)$ is the polarization vector of the XFEL (laser) pulse. $\hat{e}_L \theta = 0$, which is on the $x$–$y$ plane, is the polarization vector of the XFEL pulse without the incident angle. $\delta$ is an angle between $\hat{e}_L$ and $\hat{e}_L \theta = 0$. $\vec{k}'$ is the wavevector of the signal X-ray with a diffraction angle, $\theta$, from the $z$-axis. $\theta_x (\theta_y)$ is an angle of $\theta$ in the $x$–$z$ ($y$–$z$) plane.

VD and VB signals in various situations, e.g., different pump laser setups and collision geometries, have been calculated [16–26]. Because X-ray free-electron laser (XFEL) facilities with high-power laser stations [27–29] are appropriate to perform VD and VB experiments, some calculations [16,17,22–25] are specialized to experiments at these facilities. These calculations show the feasibility of the observation of VD and VB [22–25] though the divergence of the XFEL pulse is not considered. The angular divergence of typical XFEL pulses, about 1 $\mu$rad [30,31], is much smaller than the divergence of the VD signal. For example, a laser focal spot of 1 $\mu$m gives a signal divergence of 40 $\mu$rad with an XFEL pulse of 13 keV, and the signal divergence is inversely proportional to the laser focal spot size, approximately [32]. In realistic experiments, however, the XFEL pulse needs to be focused by optics to obtain enough signal. Therefore, the divergence of the XFEL pulse increases to $O(10–1000)\mu$rad and becomes not negligible. This divergence causes a curvature on the wavefront of the XFEL pulse. Since the wavefront curvature affects the signal distribution, consideration of the effect of the curvature is important.

In this paper, we derive a new formula for VD and VB signals in which the XFEL curvature effect is taken into account for the first time. The assumed setup for the calculation is a realistic condition in XFEL facilities; i.e., a focused pump laser pulse and a focused XFEL pulse collide in a head-on geometry. Misalignments between both pulses are also included.

In Sect. 2, before considering the curvature effect, we derive a new simplified formula of the VD and VB signals for XFEL experiments. This approximated formula is useful to understand the parameter dependences of the VD and VB signals. This new formula is validated in Sect. 2.3 by comparison with the previous calculation [22] with real setup parameters at the SACLA XFEL facility [27]. In Sect. 3, we calculate the curvature effect with this new formula. The effect is estimated as the convolution of the XFEL angular distribution and the signal one. The impacts of the curvature effect on the experiments are also discussed in Sect. 3.

2. Simple formula for the VD signal

2.1. Assumed setup for the calculation

A so-called “pump–probe” setup illustrated in Fig. 1(a) is considered. The probe XFEL pulse and the pump laser pulse collide in the head-on geometry. The XFEL pulse has the following field amplitude:
\[ J(x) = J_0 \cos(k(Z - ct) + \psi_0) \exp \left[ -\frac{(Z/c - t)^2}{(\tau_X/2)^2} \right] \]
\[ \times \exp \left[ -\frac{\lambda^2 + \gamma^2}{w_X^2} \right], \]  
(1)

with
\[ \begin{pmatrix} \mathcal{X} \\ \mathcal{Y} \\ \mathcal{Z} \end{pmatrix} = \cos \Theta \begin{pmatrix} 1 + \frac{\tan^2 \Theta_y}{\tan^2 \Theta_x} \left( \frac{1}{\cos \Theta_x} - 1 \right) & -\frac{\tan \Theta_x \tan \Theta_y}{\tan^2 \Theta_x} \left( \frac{1}{\cos \Theta_x} - 1 \right) & -\tan \Theta_x \\ -\frac{\tan \Theta_x \tan \Theta_y}{\tan^2 \Theta_x} \left( \frac{1}{\cos \Theta_x} - 1 \right) & 1 + \frac{\tan^2 \Theta_y}{\tan^2 \Theta_x} \left( \frac{1}{\cos \Theta_x} - 1 \right) & -\tan \Theta_y \\ \frac{\tan \Theta_y}{\tan \Theta_x} & \frac{\tan \Theta_y}{\tan \Theta_x} & 1 \end{pmatrix} \begin{pmatrix} x \\ y \\ z \end{pmatrix}, \]  
(2)

where \( J_0 \) is the peak field strength, \( c \) is the speed of light, \( \tau_X \) is the pulse duration, \( k \) is the wavenumber of the X-ray, and \( \psi_0 \) is a constant phase. The XFEL pulse propagates in the \( Z \)-direction and has an incident angle, \( \Theta \), against the \( z \)-axis. The \( \mathcal{X}, \mathcal{Y}, \mathcal{Z} \)-axis is a rotated \( x,y,z \)-axis by \( \Theta \). \( \Theta_x (\Theta_y) \) denotes the incident angle between the \( Z \)-axis and the \( y-z \) (\( x-z \)) plane, and satisfies \( \frac{1}{\cos \Theta} = 1 + \tan^2 \Theta_x + \tan^2 \Theta_y \). Equation (1) corresponds to a Gaussian beam with infinite Rayleigh length and constant beam size, \( w_X \). This constant assumption on \( w_X \) is reasonable for the calculation of the diffraction because the beam size is almost constant during the interaction due to the long Rayleigh length of the XFEL pulse. Typically, an XFEL beam has a Rayleigh length of 0.01 m with a divergence of 100 \( \mu \)rad and a focal spot of 1 \( \mu \)m. This XFEL pulse does not have the curvature because of the constant beam size.

The laser pulse is modeled as a pulsed Gaussian beam with a field amplitude:
\[ E(x) = E_0 \cos(\Phi(x)) \exp \left[ -\frac{(z/c + t)^2}{(\tau_L/2)^2} \right] \]
\[ \times \frac{w_{L0}}{w_L(z)} \exp \left[ -\frac{(x - x_L)^2 + (y - y_L)^2}{w_L(z)^2} \right], \]  
(3)

where \( E_0 \) is the peak field strength, \( \Phi(x) \) is a term that denotes the phase of a Gaussian beam, and \( \tau_L \) is the pulse duration. \( w_L(z) = w_{L0} \sqrt{1 + \left( \frac{z - z_L}{z_{RL}} \right)^2} \) is the beam size as a function of \( z \) with a beam waist, \( w_{L0} \), and Rayleigh length, \( z_{RL} = \frac{\pi w_{L0}^2}{\lambda} \). \( x_L, y_L, \) and \( z_L \) are displacements from the origin of the \( x-y-z \) coordinate to the laser focus. Since the photon exchange between the two pulses is well suppressed, the oscillating term \( \cos(\Phi(x)) \), which depends on the photon energy of the laser, can be averaged over the oscillation of the laser field [22,32]. This gives \( \cos^2(\Phi(x)) \rightarrow \frac{1}{2} \).

We calculate the pulse energy of the laser, \( W \), to express the peak intensities, \( \mathcal{E}_0^2 \) and \( J_0^2 \), with measurable quantities:
\[ W = \int dx dy dz \epsilon_0 E^2(x) \simeq \epsilon_0 \left( \frac{2\pi}{\tau_L} \right)^{3/2} \mathcal{E}_0^2 w_{L0}^2 c \tau_L, \]  
(4)

where \( \epsilon_0 \) is the dielectric constant. From this, the peak intensities can be written as
\[ \mathcal{E}_0^2 \simeq \frac{32W}{(2\pi)^{3/2}\epsilon_0 w_{L0}^2 c \tau_L} \quad \text{and} \quad J_0^2 \simeq \frac{32N h c k}{(2\pi)^{3/2}\epsilon_0 w_X^2 c \tau_X}, \]  
(5)

where \( N \) is the number of X-rays in the XFEL pulse, \( h c k \) is the photon energy of the X-ray, and \( h \) is the reduced Planck constant.
2.2. Simplification of the signal formula

In the assumed setup, the interaction probability of VD for each X-ray, \( P \), is written as \[22\]

\[
\frac{d^3P}{dk^3} = \frac{1}{N} \frac{\alpha^4 h^5 e_0^3}{m_e c^9} k'(1 + \cos \Theta)^2 (1 + \cos \theta)^2 (16 + 33 \sin^2 \delta) \\
\times J_0^2 \mathcal{E}_0^4 |\mathcal{M}|^2,
\]

where \( \mathcal{X} = (ct, x, y, z) \) and \( \mathcal{X}' = (k', \vec{k}') \) are four-vectors with the metric \((-1, 1, 1, 1)\), and \( k' = |\vec{k}'| \). \( \vec{k}' \) is the wavevector of the signal X-ray, \( \theta \) is the diffraction angle of the signal X-ray from the \( z \)-axis. \( \alpha \) is the fine-structure constant, \( m_e \) is the electron mass. \( \delta \), which is illustrated in Fig. 1(b), is the angle between the polarization vector of the laser pulse, \( \hat{e}_L \), and a vector, \( \hat{e}_{\Theta=0} \), where \( \hat{e}_{\Theta=0} \) is a polarization vector of the XFEL pulse without the incident angle.

To simplify Eq. (6), we apply the following approximations and assumptions, which are reasonable in practical experiments:

(i) A short-pulse approximation where both pulse durations satisfy \( c\tau_L \ll z_{RL} \) and \( c\tau_X \ll z_{RL} \). This condition is reasonable since the pulse length of typical femtosecond lasers is shorter than the Rayleigh length. For example, the pulse length of typical femtosecond lasers satisfies \( c\tau_L < 30 \mu m (\tau_L < 100 \text{ fs}) \), and \( c\tau_X \) is much shorter than that, whereas \( z_{RL} \approx 400 \mu m \) for \( w_{L0} = 10 \mu m \) and \( \lambda = 800 \text{ nm} \). Under these conditions, we can approximate \( w_L(z) \) to be constant during the interaction as

\[
w_L(z) \approx w_L = w_{L0} \sqrt{1 + \left( \frac{z_L}{z_{RL}} \right)^2}.
\]

(ii) A small-incident-angle approximation that the incident angle satisfies \( \Theta \ll 1 \). We take only the leading order of \( \Theta \).

(iii) A forward approximation that the diffraction angle satisfies \( \theta \ll 1 \). This condition is reasonable in experiments where an X-ray is used as a probe beam because the diffraction of an X-ray is small due to its short wavelength. The approximation enables us to take only the leading order of \( \theta \).

(iv) A cosine term approximation. The cosine term in Eq. (1) is expressed as \( \cos(k(Z - ct) + \psi_0) = \sum_{q=\pm 1} \left\{ \frac{1}{2} \exp[iq(k(Z - ct) + \psi_0)] \right\} \), and the \( q = +1 \) term can be ignored because it is substantially suppressed in the \( dk' \) integration \[22\].

By applying (i)–(iv), Eq. (6) becomes
\[
\frac{d^3 P}{dk'^3} \approx \frac{1}{2\pi} \frac{2^6}{45^2 \alpha^6} m_e^6 c^{10} (16 + 33 \sin^2 \delta) kk' W^2 \\
\times \exp \left[ -\frac{4w^2 (x_1^2 + y_1^2)}{w_L^2 w_X^2} \right] \\
\times w^2 \exp \left[ -\frac{1}{2} w^2 \left( (k' \theta_x - k \Theta_x)^2 + (k' \theta_y - k \Theta_y)^2 \right) \right] \\
\times \frac{1}{\sqrt{2\pi}} \frac{\alpha^4}{\alpha^2} \exp \left[ -\frac{1}{2} \left( \frac{c^2}{X} \right)^2 (k' - k)^2 \right], \quad (9)
\]

with

\[
w^2 = \frac{w_L^2 w_X^2}{w_L^2 + 2w_X^2}, \quad (10)
\]

where \( \theta_x (\theta_y) \) is the diffraction angle of the signal X-ray from the \( y-z \) \( (x-z) \) plane. The exponential term in the fourth line in Eq. (9) is also expressed as \( \exp \left[ -\frac{1}{2} \left( \frac{c^2}{X} \right)^2 (ch k' - chk)^2 \right] \). This term states that the energy width of the signal X-rays is \( \frac{2\hbar}{\tau_X} \). The energy width, e.g., 0.1 eV given by \( \tau_X = 10 \) fs, is small compared to the photon energy of the signal X-rays, \( chk = \mathcal{O}(1-10) \) keV. Thus, we can practically neglect the energy spread of the signal X-rays and can further apply the following;

(v) A delta function approximation that the fourth line of Eq. (9) is approximated to be \( \delta(k' - k) \) in the \( dk' \) integration.

Executing the \( dk' \) integration with (v), we obtain the fully simple formula of VD as

\[
\frac{dP}{d\cos \theta} \approx \frac{2^6}{45^2 \alpha^6} m_e^6 c^{10} (16 + 33 \sin^2 \delta) k^2 W^2 \\
\times \frac{1}{w_L^2 (w_L^2 + 2w_X^2)} \times \frac{2}{\pi w^2} \int I_{IV}(x,y) dx dy \\
\times (kw)^2 \exp \left[ -\frac{1}{2} (kw)^2 \left( (\theta_x - \Theta_x)^2 + (\theta_y - \Theta_y)^2 \right) \right], \quad (11)
\]

with

\[
I_{IV}(x,y) = \exp \left[ -\frac{2x^2 + y^2}{w_X^2} \right] \\
\times \left( \exp \left[ -\frac{2(x - x_L)^2 + (y - y_L)^2}{w_L^2} \right] \right)^2. \quad (12)
\]

It consists of three parts: the parameter dependence part (the first line), the form part of the two pulses (the second line), and the angular distribution part (the third line). It provides a useful approximation to understand the phenomenon of VD and to optimize experimental setups since the dependence of the interaction probability on each parameter is clear.

In the second line of Eq. (11), \( I_{IV}(x,y) \) is the product of intensity distributions of the two pulses, \( (J(x)^2) \times (E(x)^2)^2 \), at the collision point. The exponents agree with the number of photons (one X-ray and two laser photons), which comprises the leading-order Feynman diagram for VD (the
Table 1. Two sets of parameters used for calculations. One is the parameter set of the prototype experiment at SACLA. The other is the set for the future experiment planned at SACLA.

| Parameters                  | Prototype experiment | Future experiment |
|-----------------------------|----------------------|-------------------|
| XFEL photon energy $\hbar c$ | 9.8 keV              | 9.8 keV           |
| Laser                       | 0.6 TW laser         | 500 TW laser      |
| Laser pulse energy $W$      | 0.21 mJ              | 12.5 J            |
| XFEL beam size $w_X$        | 57 μm                | 2 μm              |
| Laser beam size $w_L$       | 9.8 μm               | 1 μm              |
| XFEL pulse duration $\tau_X$| 17 fs                | 17 fs             |
| Laser pulse duration $\tau_L$| 40 fs                | 42 fs             |
| Collision rate              | 30 Hz                | 1 Hz              |

Table 2. Comparison of VD signals obtained by Eq. (11) and by the previous calculation [22]. The X-ray flux at SACLA has been assumed to be $3 \times 10^{11}$ photons/pulse. For the comparison of divergences, the divergence of the XFEL pulse is also shown.

| Parameters                  | Prototype experiment | Future experiment |
|-----------------------------|----------------------|-------------------|
| Signal divergence $\Phi$    | Eq. (11)             | 5.84 μrad         | 60.5 μrad        |
|                             | previous calculation | 5.84 μrad         | 57.0 μrad        |
| Interaction probability $P$ | Eq. (11)             | $3.9 \times 10^{-26}$ | $9.4 \times 10^{-12}$ |
|                             | previous calculation | $3.9 \times 10^{-26}$ | $7.9 \times 10^{-12}$ |
| Expected signal rate (in $18 < \theta_y < 58$ μrad) | Eq. (11) | $8 \times 10^{-23}$ photons/s | 0.7 photons/s |
|                             | previous calculation | $8 \times 10^{-23}$ photons/s | 0.6 photons/s |
| $c\tau_L/z_{RL}$            | 0.032                | 3.2               |
| Divergence of XFEL pulse    | 65.0 μrad            | 20.2 μrad         |

QED box diagram). This overlap of the two pulses can be thought of as an effective interaction volume. It works as a weight on $P$, showing how well the two pulses overlap in space-time, and is decreased by large displacements, $(x_L, y_L)$. $w$ gives the $2\sigma$ size of this interaction volume.

The third line of Eq. (11) shows that the angular distribution has a Gaussian profile with the signal divergence, $\Phi = \frac{2}{kw}$, and the center of the distribution is the incident angle. The uncertainty principle binds a relation between the signal divergence and the size of the interaction volume in the transverse direction: $\Delta p \Delta x = \frac{\hbar k}{w^2} \times \frac{\pi}{2} = \frac{\hbar}{2}$.

Equation (11) is also useful to calculate the VB signal, which has polarization orthogonal to the incident X-ray because the VB signal can be inferred by substituting $(16 + 33 \sin^2 \delta) \rightarrow \frac{9}{4} \sin(2\delta)$ [22].

2.3. Validation of the simple formula

We validate Eq. (11) by comparing the calculated results with the numerical results from the previous calculation [22]. For the calculations, we use two sets of experimental parameters summarized in Table 1. One parameter set is obtained by a prototype experiment, which was performed in 2017 (Y. Seino et al., manuscript in preparation) at SACLA. The other is for a future experiment achievable at SACLA. In the calculations, perfect alignment for both cases is assumed ($\Theta = 0$, $x_L = y_L = z_L = 0$), and the calculated results are summarized in Table 2. The good agreement of the signal divergence in both parameter sets shows that the angular distribution is well approximated by
Fig. 2. Evaluation of the applicability of the short-pulse approximation for the long-pulse condition. The signal divergence and interaction probability calculated by Eq. (11), $\Phi_{\text{Eq}(11)}$ and $P_{\text{Eq}(11)}$, are compared with those calculated by the previous calculation, $\Phi_{\text{Previous}}$ and $P_{\text{Previous}}$, respectively. The parameters of the future experiment in Table 1 are used, and the Rayleigh length of the laser is $z_{RL} = 3.9 \mu$m. Left: $\Phi_{\text{Eq}(11)}/\Phi_{\text{Previous}}$. Right: $P_{\text{Eq}(11)}/P_{\text{Previous}}$.

the Gaussian profile. The calculated results for the prototype experiment show good agreement with the previous calculation because the Rayleigh length and the pulse duration of the laser satisfy the condition of the short-pulse approximation (i) ($c\tau_{X}/z_{RL} = 0.032 \ll 1$). The interaction probability, $P$, for the future experiment also has enough agreement for practical use though the condition of the short-pulse approximation is not sufficiently satisfied ($c\tau_{L}/z_{RL} = 3.2$). The small discrepancy between $P$ for Eq. (11) and that for the previous calculation is no more than 20%.

The applicability of the short-pulse approximation for the long-pulse condition ($c\tau_{L}/z_{RL} > 1$) is evaluated by comparison with the previous calculations. The ratios of the signal divergence and the interaction probability are shown in Fig. 2 as the dependence on $c\tau_{L}/z_{RL}$. In this evaluation, the parameters are based on the future experiment, and the same pulse durations ($\tau = \tau_{X} = \tau_{L}$) and perfect alignment are assumed for simplicity. From the laser beam size (1 $\mu$m) and the laser wavelength (800 nm), the Rayleigh length is calculated as $z_{RL} = 3.9 \mu$m, and $c\tau/z_{RL} = 1$ corresponds to $\tau = 13$ fs. As shown in Fig. 2, our simple formula still shows good agreement for the long-pulse condition, and the discrepancy is less than 20% for the $c\tau/z_{RL} = 2.6$ case.

The expected signal rates in Table 2 are calculated with an angle of acceptance ($18<\theta_{y}<58$ $\mu$rad), which is the same angle as the detection system used in the prototype experiment. The expected signal rate reaches 0.7 photons/s in the future experiment by assuming $3 \times 10^{11}$ photons/pulse of the X-ray flux, which shows that VD can be observed.

3. VD formula with XFEL curvature effect

So far, we have considered the simple formula of VD without the curvature. However, the XFEL pulse has the curvature in practical experiments, and the curvature effectively gives the incident angle to the X-ray. We consider the XFEL pulse as a Gaussian beam, as shown in Fig. 3, and the beam size is given as $w_{X} = w_{X0}\sqrt{1 + (\frac{z_{X}}{z_{RX}})^2}$, where $z_{X}$ is the distance from the collision point to the focus along the $z$-axis, $w_{X0}$ is a minimum beam size, and $z_{RX}$ is the Rayleigh length. The curvature radius of the wavefront is

\[ r = \frac{z_{RX}}{2}. \]
Fig. 3. Schematic view of the focused XFEL pulse at the collision point. The XFEL pulse is away from the focus by \( z_X \). The wavefront is drawn as a thick line. \( \Theta_n \) (\( \Theta_{ny} \)) is an angle between the normal direction of the wavefront, \( \vec{\Theta}_n \), and the \( y-z \) (\( x-z \)) plane. These angles effectively give the incident angles to the X-ray.

Table 3. Summary of parameters used to estimate the curvature effect for each case in Fig. 4. The fourth column corresponds to the parameters of the prototype experiment.

| Parameters | Without curvature | With curvature and displacement |
|------------|-------------------|--------------------------------|
| \( y_L \)  | 0 \( \mu m \) | 0 \( \mu m \) | 3.7 \( \mu m \) |
| \( z_X \)  | - | 0.85 \( m \) | 0.85 \( m \) |
| \( z_{RX} \) | \( \infty \) | 0.09 \( m \) | 0.09 \( m \) |
| Signal fraction | \( 2 \times 10^{-10} \) | \( 1 \times 10^{-4} \) | \( 3 \times 10^{-6} \) |

\( (\text{in } 18 < \theta_y < 58 \, \mu \text{rad}) \)

\[
R(z_X) = z_X \left(1 + \left( \frac{z_{RX}}{z_X} \right)^2 \right). \tag{13}
\]

At a position \((x_{EP}, y_{EP})\) on the wavefront, the normalized normal direction, \( \vec{\Theta}_n \), is given as follows:

\[
\vec{\Theta}_n = \cos \Theta_n \left( \tan \Theta_{nx}, \tan \Theta_{ny}, 1 \right)
= \left( -\frac{x_{EP}}{R(z_X)}, -\frac{y_{EP}}{R(z_X)}, \cos \Theta_n \right) \approx (\Theta_{nx}, \Theta_{ny}, 1), \quad (\Theta_n \ll 1), \tag{14}
\]

where \( \Theta_{nx} \) (\( \Theta_{ny} \)) is an angle between the normal direction and the \( y-z \) (\( x-z \)) plane and satisfies \( \frac{1}{\cos^2 \Theta_n} = 1 + \tan^2 \Theta_{nx} + \tan^2 \Theta_{ny} \). These angles are approximately given by the \( x, y \) components of the normal direction.

Under the situation that \( \Theta_n \ll 1 \), the new VD formula is given by the convolution of the angle of the normal direction and the original angular distribution, Eq. (11), with the weight \( I_{IV}(x_{EP}, y_{EP}) \). The convolution is given by the integrations over \( x_{EP} \) and \( y_{EP} \) as

\[
\frac{dP_{\text{curve}}}{dxdy} = f \left[ I_{IV}(x_{EP}, y_{EP}) \right] \left| \frac{dP}{d\cos \theta} \right| \bigg|_{\Theta_n = \Theta_{nx}, \Theta_y = \Theta_{ny}}, \tag{15}
\]

where the VB signal can also be inferred as mentioned in Sect. 2.2. Due to the convolution, the original divergence, \( 2/w_d \), is smeared by the divergence caused by the curvature at the collision point, \( w/R(z_X) \), and the signal divergence becomes their root sum square, \( \sqrt{(2/w_d)^2 + (w/R(z_X))^2} \). The center of the angular distribution is also shifted to the normal direction at the center of the interaction volume, and the central angles become \((-2w^2x_L/w_X^2R(z_X), -2w^2y_L/w_Z^2R(z_X))\).
Fig. 4. Angular distributions of the signal X-rays. Solid line: without the curvature effect. Dotted line: with the curvature effect. Dashed line: with the curvature effect and the displacement of the two pulses. The calculations are based on the prototype experiment parameters in Tables 1 and 3. All distributions are normalized to unity.

As an example of the estimation of the curvature effect, we consider the case of the prototype experiment shown in Table 1. In this experiment, the XFEL pulse is focused with an X-ray lens, and the Rayleigh length is measured as $z_{RX} = 0.09$ m. The focal point of the XFEL pulse is adjusted at 0.85 m downstream from the collision point ($z_X = 0.85$ m), and the beam size at the collision point is $w_X = 57 \mu$m. The displacement between the XFEL pulse and the laser pulse is found as $y_L = 3.7 \mu$m. With these parameters, the divergence caused by the curvature is $8.03 \mu$rad, and it smears the original divergence, $5.84 \mu$rad, to $9.94 \mu$rad. In addition to this, the displacement shifts the central angle of the angular distribution to $-4.3 \mu$rad. The parameters are summarized in Table 3, and these changes are shown in Fig. 4. These changes caused by the curvature effect significantly affect the expected signals. In the prototype experiment, we set the acceptance of the signal X-rays as $18 < \theta_y < 58 \mu$rad. As shown in the bottom row of Table 3, the expected signal fractions are changed by more than a few orders of magnitudes. To avoid these uncertainties, precise alignments of the beam positions ($x_L, y_L \ll w$) and the focal position ($z_X \ll z_{RX}$) are required for future high-sensitivity experiments.

4. Conclusion

We derive a new simple VD and VB formula in the head-on collision geometry of the XFEL pulse and the laser pulse. The simple formula is the product of terms with physically different origins and helps us to understand the parameter dependences of the signal easily. The results obtained by the new formula are compared with those by the previous calculation, and we get good agreements. We also derive the new VD and VB formula considering the wavefront curvature of the XFEL, which is taken into account for the first time. The curvature effect is estimated as the convolution of the XFEL angular distribution and the VD one. This new formula shows that the angular distribution of the signal is broadened and that it depends strongly on the displacement between the XFEL and the laser foci.

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