Evolution of Black Holes in Brans-Dicke Cosmology

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Abstract

We consider a modified “Swiss cheese” model in the Brans-Dicke theory, and discuss the evolution of black holes in the expanding universe. We define the black hole radius by the Misner-Sharp mass and find the time evolution for dust and vacuum universes.

1 Introduction

The evolution of primordial black holes in scalar-tensor theories has been studied in the literature [1, 2, 3, 4]. The question of what happens to a black hole in an expanding universe in these theories was posed by Barrow, who discussed two scenarios [1]: (a) The effective gravitational “constant” $G(t)$ at a black hole changes along with its cosmological evolution so that the size of a black hole is approximated by $R = 2G(t)M$. (b) $G$ remains constant at the black hole event horizon while it evolves on larger scales; a large inhomogeneity in $G$ is therefore generated. The case (b) was called “gravitational memory” because the black hole remembers the value of $G$ at its formation time. In either case constraints on primordial black holes would be modified [2].

Later Jacobson claimed that there is no “gravitational memory” effect, analyzing the evolution of a scalar field $\phi(t, r)$ in Schwarzschild background [3]. His analytic solution showed that $\phi$ at the event horizon evolves along with its asymptotic value $\phi(t, \infty)$. However, the solution used has a special form with very fast variation of $G \propto t^{-1}$ in the background universe. It was also argued that even if the black hole mass in the Einstein frame is constant the mass in the Jordan frame is time-dependent.

In order to investigate the time-dependence of the black hole mass, Saida and Soda constructed a “cell lattice” universe in the Brans-Dicke (BD) theory [4]. In their model the universe are tessellated by identical polyhedrons, which are replaced by Schwarzschild-like black holes. It was shown that the black hole mass has adiabatic time dependence, and that its time dependence is qualitatively different according to the sign of the curvature of the universe.

As an extension of Saida and Soda’s work, we consider a “Swiss cheese” model [5] in the BD theory and discuss the evolution of the radius and mass of black holes. The ordinary Swiss cheese model refers to a cosmological model in which spherical regions in the Friedmann-Robertson-Walker (FRW) universe are replaced by Schwarzschild black spacetimes. Here we construct such a model in BD cosmology.

2 Background Universe in Brans-Dicke Theory

The BD theory is described by the action,

$$S = \int d^4x \sqrt{-g} \left[ \frac{\phi}{16\pi} R - \frac{\omega}{16\pi\phi}(\nabla_{\mu}\phi)^2 + L_m \right],$$

(1)
where $\phi$ is the BD field, $\omega$ is the BD parameter, and $L_m$ is the matter Lagrangian. The variation of Eq. (8) with respect to $g_{\mu\nu}$ and $\phi$ yield the field equations:

$$\mathcal{R}_{\mu\nu} - \frac{1}{2}g_{\mu\nu}\mathcal{R} = \frac{8\pi}{\phi}T_{\mu\nu} + \frac{\omega}{\phi}\left[\nabla_\mu\phi\nabla_\nu\phi - \frac{1}{2}g_{\mu\nu}(\nabla\phi)^2\right] + \nabla_\mu\nabla_\nu\phi - g_{\mu\nu}\Box\phi,$$

$$\Box\phi = \frac{8\pi}{2\omega + 3}T g_{\mu\nu} R^\mu g_{\nu\rho} R^\rho.$$

As a background universe we assume the FRW spacetime:

$$ds^2 = -dt^2 + a(t)^2 \left\{ \frac{dr^2}{1 - kr^2} + r^2(d\theta^2 + \sin^2 \theta d\varphi^2) \right\},$$

where $k$ is the sign of the spatial curvature. As an energy-momentum tensor we consider a dust fluid:

$$T_{\mu\nu} = \rho u_\mu u_\nu,$$

where $\rho$ and $u_\mu$ are the density and the four velocity of dust, respectively. Then the field equations (8) and (9) reduce the following equations for the background universe:

$$H^2 + \frac{k}{a^2} - \frac{8\pi}{3\phi} - \frac{1}{3}\frac{\dot{\phi}}{\phi} + \frac{\omega}{6}\left(\frac{\dot{\phi}}{\phi}\right)^2 = 0,$$

$$\ddot{\phi} + 3H\dot{\phi} = -\frac{8\pi\rho}{2\omega + 3},$$

where an overdot denotes $d/dt$ and $H = \dot{a}/a$ is the Hubble parameter.

In the case of $k = 0$ we know the analytic solution for the dust universe:

$$a(t) = a_0(t - t_+)^{\lambda_+}(t - t_-)^{\lambda_-}, \quad \phi(t) = \phi_0(t - t_+)^{\kappa_+}(t - t_-)^{\kappa_-},$$

with

$$\lambda_\pm = \frac{\omega + 1 + \sqrt{1 + 2\omega/3}}{3\omega + 4}, \quad \kappa_\pm = \frac{1 + 3\sqrt{1 + 2\omega/3}}{3\omega + 4},$$

where $a_0$, $\phi_0$ and $t_\pm$ are arbitrary constants. If we take $t_+ = t_-$, the general solution (8) reduces to the special power-law solution,

$$a(t) = a_0(t - t_0)^{\frac{3\omega + 4}{2\omega}}, \quad \phi(t) = \phi_0(t - t_0)^{\frac{\omega + 1}{2\omega}}.$$

In the limit of $t \to \infty$, the general solution (8) converges to the special power-law solution (11). If the present cosmic age $t$ is large enough, observations cannot constrain the relation between $t_+$ and $t_-$. Therefore, we keep $t_+ - t_- = 0$ a free parameter.

In vacuum case ($\rho = 0$) we have the analytic solutions for all $k$. The vacuum solution for $k = 0$ is expressed as

$$a(t) = a_0 t^{\frac{3\omega + 4}{2\omega}}, \quad \phi = \phi_0 t^{\frac{3\omega + 4}{2\omega}},$$

with

$$\alpha = \frac{1 + \sqrt{1 + 2\omega/3}}{\omega}.$$

where we have omitted the arbitrary constant $t_0$ by fixing the origin of the time coordinate $t$. Introducing the conformal time $\eta = \int dt/a$, the vacuum solutions for $k = \pm 1$ are expressed as

$$k = +1: \quad a(\eta) = (\sinh \eta)^{\frac{1 + \lambda}{3 + 2\omega}}(\cosh \eta)^{\frac{1 - \lambda}{3 + 2\omega}}, \quad \phi(\eta) = (\tanh \eta)^{\lambda},$$

$$k = -1: \quad a(\eta) = (\sin \eta)^{\frac{1 + \lambda}{3 + 2\omega}}(\cos \eta)^{\frac{1 - \lambda}{3 + 2\omega}}, \quad \phi(\eta) = (\tan \eta)^{\lambda},$$

with

$$\lambda = \pm \frac{3}{3 + 2\omega}.$$
3 Modified Swiss-Cheese Model

We now consider the model of a black hole embedded in the FRW universe. We replace a sphere in the FRW spacetime with a vacuum region which contains a black hole. Here “vacuum” means $T_{\mu\nu} = 0$, and does not imply that $R_{\mu\nu} = 0$ due to the existence of the BD field.

Extending Israel’s junction conditions for a singular (or regular) hypersurface \[6\], Sakai and Maeda have studied bubble dynamics in the inflationary universe \[7\]. It was found that one can solve the equations of motion for the boundary without knowing the interior metric if the interior is vacuum, $T_{\mu\nu} = 0$, or has only vacuum energy (a cosmological constant), $T_{\mu\nu} = -\rho g_{\mu\nu}$. Applying this method to the present model, we find the mass and the radius of a black hole without specifying the interior metric, as we shall show below.

Let us consider a spherical hypersurface $\Sigma$ which divides a spacetime into two regions, $V^+$ (outside) and $V^-$ (inside). We define a unit space-like vector, $N^\mu$, which is orthogonal to $\Sigma$ and points from $V^-$ to $V^+$. To describe the behavior of the boundary, we introduce a Gaussian normal coordinate system, $(n, x^i) = (n, \tau, \theta, \phi)$, where $\tau$ is chosen to be the proper time on the boundary. Hereafter, we denote by $\Psi^\pm$ the value of any field variable $\Psi$ defined on $\Sigma$ by taking limits from $V^\pm$.

For the matter field, we consider dust (or vacuum) for $V^+$ and vacuum for $V^-$: \[16\]
\[T_{\mu\nu}^+ = \rho u^\mu u^\nu, \quad T_{\mu\nu}^- = 0\]
Although we assume a smooth boundary at which there is no surface density, it is not obvious that this matching is possible at all times. Therefore, we introduce the surface energy-momentum tensor on the boundary surface, \[17\]
\[S_{ij} \equiv \lim_{\epsilon \to 0} \int_{-\epsilon}^{\epsilon} dn T_{ij} = \text{diag}(-\sigma, \varpi, \varpi),\]
where $\sigma$ and $\varpi$ are the surface energy density and the surface pressure of $\Sigma$, respectively.

Introducing the extrinsic curvature tensor of the world hypersurface $\Sigma$, $K_{ij} \equiv N_i \nabla_j$, we can write the junction conditions on $\Sigma$ as \[18\]
\[K_{ij}^\pm = \frac{4\pi}{\phi} \left( S_{ij} - \frac{\omega}{2\omega + 3} \text{Tr} S \gamma_{ij} \right),\]
\[2\gamma S^i_j |_{ij} = [T^i_n]^\pm,\]
\[K^+_ij S^j_i + \frac{2\pi}{\phi} \left( S^i_j S^j_i - \frac{\omega}{2\omega + 3} (\text{Tr} S)^2 \right) = [T^i_n]^\pm,\]
where we have defined the bracket as $[\Psi]^\pm \equiv \Psi^+ - \Psi^-$ and the vertical bar $|$ as the three-dimensional covariant derivative. The junction condition for the BD field is derived from Eq.\[3\] as \[21\]
\[[\phi_n]^\pm = -\frac{24\pi}{3 + 2\omega} \text{Tr} S, \quad \phi^+ = \phi^-,\]
which implies that $\phi$ is continuous at $\Sigma$ and inhomogeneous in $V_-$. The extrinsic curvature tensor of $\Sigma$ in the homogeneous side $V^+$ is expressed as \[22\]
\[K^+_\tau = \gamma^3 \frac{dv}{dt} + \gamma v H,\]
\[K^+_\theta = \frac{\gamma(1 + v HR)}{R} = \frac{\epsilon}{R} \sqrt{1 + \left( \frac{dR}{d\tau} \right)^2} - R^2 \left( H^2 + \frac{k}{a^2} \right),\]
where \[24\]
\[R = a(t)r|\Sigma, \quad v \equiv \frac{dr}{dt}|_\Sigma, \quad \gamma \equiv \frac{\partial t}{\partial \tau}|_\Sigma = \frac{1}{\sqrt{1 - v^2}}, \quad \text{and} \quad \epsilon \equiv \text{sign}(K^+_\theta) = \text{sign} \left( \frac{\partial R}{\partial n} \right).\]
From Eqs.(16), (17), (19), (20), (22)-(24), we obtain the equations of motion:

\[ \frac{dR}{dt} = \frac{dr}{d\chi} v + HR, \quad (25) \]

\[ \gamma^3 \frac{dv}{dt} = -\gamma \left\{ \left(1 - 2w\right) vH - \frac{2w dr}{R} \frac{d\chi}{d\chi} \right\} + \frac{2\pi \sigma}{\phi} \left\{ 1 + 4w + \frac{(1 - 2w)^2}{(2w + 3)} \right\} - \frac{\gamma^2 v^2 \rho}{\sigma}, \quad (26) \]

\[ \frac{d\sigma}{dt} = -2\sigma (1 + w) \frac{dR}{dt} + \gamma v \rho, \quad (27) \]

where \( w \equiv \varpi / \sigma \).

Once initial values of \( R, v, \) and \( \sigma \) are given, the equations of motion (25)-(27) determine their evolution.

As discussed in Ref.[7], initial values should satisfy the angular component of (18),

\[ \gamma (1 + vHR) - \epsilon \sqrt{1 + \left( \frac{dR}{dt} \right)^2} - \frac{R_{MS}}{R} = -\frac{8\pi \sigma R \omega + 1 + w}{\phi} \frac{2\omega + 3}{2\omega + 3}, \quad (28) \]

where we choose \( \epsilon = +1 \). \( R_{MS} \) is defined as

\[ R_{MS} \equiv R^-(1 - g^{\mu \nu} R^-_{\mu \nu}), \quad (29) \]

where \( R^- \) is defined as \( R^- \equiv \sqrt{g_{\mu \nu}} \) at \( \Sigma \) on the side of \( V^- \). Because the Misner-Sharp mass is defined as \[ M_{MS} \equiv \frac{R^-}{2G} (1 - g^{\mu \nu} R^-_{\mu \nu} = R^-_{\mu \nu}), \quad (30) \]

we call \( R_{MS} \) the “Misner-Sharp radius”. Note that \( R_{MS} \) is a purely geometrical quantity and independent of theories of gravitation.

If we considered a bubble in which there is no black hole (or singularity), we would have to solve the field equations with regularity condition at the centre and the boundary condition (28), as done in Ref.[7]. However, because we are interested in black hole solutions, we do not have to take such a regularity condition into account. Thus we can use Eq.(28) to determine \( R_{MS} \).

At the initial time we suppose \( v = 0 \) and \( \sigma = \varpi = 0 \). Then \( R_{MS} \) is expressed as

\[ R_{MS} = R^3 \left( H^2 + \frac{k}{a^2} \right). \quad (31) \]

Let us now discuss whether \( v \) and \( \sigma \) remain zero during the ensuing evolution. Suppose \( w = 0 \), then the only nontrivial term in Eq.(24) is \( \gamma^2 v^2 \rho / \sigma \). If \( v \) and \( \sigma \) evolved from zero, Eq.(28) shows \( vH \sim \sigma / \phi \), so that \( \rho v^2 / \sigma \sim \rho v / H \phi \). Therefore, Eqs.(24) and (27) guarantee that, if \( v = \sigma = 0 \) at a certain time, \( v = \sigma = 0 \) at all time. Interestingly, this result is true only for the dust case, \( \varpi / \sigma = 0 \); otherwise \((2w/R)(dr/d\chi)\) in Eq.(26) would shift \( v \) from zero.

In the case of Schwarzschild spacetime, the Misner-Sharp radius coincides with the event horizon. Although it is not always true for general spacetimes, we speculate that the Misner-Sharp radius is a well-defined measure of the size of a black hole. In the next section, we show the evolution of \( R_{MS} \) for black holes in several background cosmological models.

### 4 Evolution of Black Holes

The evolution of the Misner-Sharp radius for the \( k = 0 \) dust universe is given by Eqs.(8) and (31),

\[ R_{MS} = a_0^3 \varpi^3 \left( \frac{\lambda_+}{t - t_+} + \frac{\lambda_-}{t - t_-} \right)^2 (t - t_+)3 \lambda_+ (t - t_-)3 \lambda_-, \quad (32) \]

where \( r_0 \) is the comoving radius of the vacuum region. Equation (32) shows that the black hole size decreases with time.
If we define the black hole mass as
\[ M_{MS} \equiv \frac{\phi R_{MS}}{2}, \] (33)
it coincides with the mass defined by Saida and Soda [4]. For the \( k = 0 \) dust universe, we obtain
\[ M_{MS} = \frac{a_0^3\phi_0}{2} \left( \frac{\lambda_+}{t - t_+} + \frac{\lambda_-}{t - t_-} \right)^2 (t - t_+)^{3\lambda_+ + \kappa_+} (t - t_-)^{3\lambda_- + \kappa_-}. \] (34)

It is easy to see that Eq.(34) reduces to \( M_{MS} = \text{constant} \), if we choose \( t_+ = t_- \), which is the same result as that found by Saida and Soda [4]. They also showed \( M_{MS} \) increases for \( k = +1 \) and decreases for \( k = -1 \), and concluded that the evolution of the mass depends qualitatively on the sign of the curvature of the universe. We should note, however, that their conclusion is true only for the special case \( t_+ = t_- \), or equivalently, only for the asymptotic behavior of \( M_{MS} \) at \( t \to \infty \).

Next, let us consider the scalar-field dominated (vacuum) universe. The evolution of \( R_{MS} \) and \( M_{MS} \) in flat, open, and closed universes are given by
\[
\begin{align*}
  k = 0 : & \quad R_{MS} = \frac{r_0^3 \phi_0^3}{9(1 + \alpha)^2} \left( \frac{1 - 2\omega}{1 + \omega} \right), \\
  & \quad M_{MS} = \frac{a_0^3 r_0^3 \phi_0}{18(1 + \alpha)^2} t^{-1}, \\
  k = +1 : & \quad R_{MS} = \frac{r_0^3}{2} (\cos \eta)^{\frac{3 + \lambda}{2}} (\sin \eta)^{-\frac{1 + \lambda}{2}} (\cos 2\eta - \sin 2\eta - \lambda), \\
  & \quad M_{MS} = \frac{r_0^3}{4} (\tan \eta)^{\lambda} (\cos \eta)^{-\frac{3 + \lambda}{2}} (\sin \eta)^{-\frac{1 + \lambda}{2}} (\cos 2\eta - \sin 2\eta - \lambda), \\
  k = -1 : & \quad R_{MS} = \frac{r_0^3}{2} (\cosh \eta)^{\frac{3 - \lambda}{2}} (\sinh \eta)^{-\frac{1 + \lambda}{2}} (\cosh 2\eta - \sinh 2\eta - \lambda), \\
  & \quad M_{MS} = \frac{r_0^3}{4} (\tanh \eta)^{\lambda} (\cosh \eta)^{-\frac{3 - \lambda}{2}} (\sinh \eta)^{-\frac{1 + \lambda}{2}} (\cosh 2\eta - \sinh 2\eta - \lambda). \\
\end{align*}
\] (35-40)

If we take \( \omega > 500 \), both \( \alpha \) and \( \lambda \) are negligible. We see that \( R_{MS} \) and \( M_{MS} \) both decrease with increasing time, except in the contracting phase of the \( k = +1 \) universe.

5 Discussion

We have constructed a modified “Swiss cheese” model in the Brans-Dicke theory, and discussed the evolution of black holes for dust and vacuum universes. We have defined the size of a black hole by the Misner-Sharp mass, and found that it always decreases as long as the universe is in an expanding phase.

Although we have not specified the metric around a black hole, the mass and radius of a black hole we have obtained coincide with those of Saida and Soda [4], who assumed a Schwarzschild-like metric. This means that their ansatz of the Schwarzschild-like metric does not introduce a specialization of the problem. One fundamental problem remains. Although the radius \( R_{MS} \) represents a typical size of a black hole, the relation between it and the event horizon has not been demonstrated explicitly and this problem will be considered further in future work.

Acknowledgments

This work was partially supported by JSPS Programs for Research Abroad (the 1999 financial year).
References

[1] J.D. Barrow, Phys. Rev. D 46 (1992) R3227.
[2] J.D. Barrow and B.J. Carr, Phys. Rev. D 54 (1996) 3920.
[3] T. Jacobson, Phys. Rev. Lett. 83 (1999) 2699.
[4] H. Saida and J. Soda, preprint gr-qc/0006058.
[5] C.C. Dyer and C. Oliwa, preprint astro-ph/0004090.
[6] W. Israel, Nuovo Cim. 44B (1966) 1; V.A. Berezin, V.A. Kuzmin and I.I. Tkachev, Phys. Rev. D 36 (1987) 2919.
[7] N. Sakai and K. Maeda, Prog. Theor. Phys. 90 (1993); Phys. Rev. D 48 (1993).
[8] C.W. Misner and D.H. Sharp, Phys. Rev. 136 (1964) B571; S.A. Hayward, Phys. Rev. D 53 (1996) 1938.