Renormalization group for $\varphi^4$-theory with long-range interaction and the critical exponent $\eta$ of the Ising model

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We calculate the critical exponent $\eta$ of the $D$-dimensional Ising model from a rather simple truncation of the functional renormalization group flow equations for a scalar field theory with long-range interaction. Our approach relies on the smallness of the inverse range of the interaction and on the assumption that the Ginzburg momentum defining the width of the scaling regime in momentum space is larger than the scale where the renormalized interaction crosses over from long-range to short-range; the numerical value of $\eta$ can then be estimated by stopping the renormalization group flow at this scale. In three dimensions our result $\eta = 0.03651$ is in good agreement with recent conformal bootstrap calculations. We extend our calculations to fractional dimensions $D$ and obtain the resulting critical exponent $\eta(D)$ between two and four dimensions. For dimensions $2 \leq D \leq 3$ our result for $\eta$ is consistent with previous calculations. Our approach is generally applicable and expected to be useful to estimate $\eta$ for other universality classes.

The calculation of the precise numerical values of the critical exponents which characterize the power-law singularities of various thermodynamic observables in the vicinity of continuous phase transitions remains one of the big challenges in theoretical physics. Although the renormalization group (RG) theory developed by Wilson, Fisher, and others [1–8] in the 1970s provides a deep understanding of the origin of the universality of the critical exponents, a controlled calculation of their numerical values for systems whose dimensionality $D$ lies below the so-called upper critical dimension $D_u$ is very difficult due to the absence of a small parameter. A successful strategy [9] to solve this problem is to calculate the first few terms in the expansion of the critical exponents in powers of $\epsilon = D_u - D$. For the Ising universality class where $D_u = 4$ this method gives accurate estimates in three dimensions if the asymptotic series generated by the $\epsilon$-expansion is carefully extrapolated [9,12]. Another set of precise results was obtained by means of Monte-Carlo simulations [13–14], while the most accurate results, such as, for example, the value of the anomalous dimension $\eta = 0.0362978(20)$, have been determined by the conformal bootstrap method [15–16]. Recent sophisticated truncations of the exact functional renormalization group (FRG) flow equation for the effective action [17] based on either the derivative expansion [18,19] or the so-called BMW-approximation [20] have produced results for $\eta$ which lie within $10\%$ from the accepted value.

In this work we develop a new method for calculating the exponent $\eta$ of the Ising universality class in arbitrary dimensions. Our method is based on a simple truncation of the exact FRG flow equations of a scalar field theory with long-range quartic interaction. We show that the inverse range of the interaction can be used as a small parameter to control the truncation of the hierarchy of FRG flow equations. Although the range of the renormalized interaction decreases as the RG is iterated, we show that by stopping the RG flow at some finite scale (which will be uniquely defined below) we obtain the critical exponent $\eta$ of the Ising model in $D = 2$ and $D = 3$ within an accuracy of about $10\%$. The fact that in $D = 3$ our result $\eta = 0.03651$ deviates significantly less from the accepted result [14,16] indicates that in this case our method may actually be more accurate than anticipated.

Our starting point is the following action for a real scalar field $\varphi(x)$ in $D$ dimensions with long-range interaction $V_0(x)$,

$$S[\varphi] = \frac{1}{2} \int d^Dx \left[ r_0 \varphi^2(x) + c_0 (\nabla \varphi(x))^2 \right] + \frac{1}{8} \int d^Dx \int d^Dx' \varphi^2(x) V_0(x,x') \varphi^2(x'),$$

(1)

where $r_0$ is proportional to the inverse order-parameter susceptibility in mean-field approximation, the constant $c_0$ is positive, and a short-distance cutoff $1/\Lambda_0$ of the order of the lattice spacing of the underlying Ising model is implicit. We assume that the Fourier transform $V_0(k)$ of the interaction is for wavevectors $k \neq 0$ given by

$$V_0(k) = \int d^Dx e^{-ik\cdot x} V_0(x) = \frac{1}{m_0 + b_0 k^2},$$

(2)

with positive constants $m_0$ and $b_0$. Moreover, precisely for $k = 0$ we set $V_0(k = 0) = 0$, so that the perturbative expansion does not contain tadpole diagrams such as the Hartree contribution to the self-energy. In three dimensions Eq. (2) corresponds to the screened Coulomb interaction $V_0(x) = e^{-\kappa_0|x|)/(4\pi b_0|x|)$, where the wavevector $\kappa_0 = \sqrt{m_0/b_0}$ can be identified with the inverse range of the interaction. We assume that $\kappa_0$ is much smaller than the ultraviolet cutoff $\Lambda_0$. The small parameter $\kappa_0/\Lambda_0$ will play an important role for controlling the accuracy of our truncation of the FRG flow equations. Note that in the usual Ginzburg-Landau-Wilson functional [8,21] describing the long-wavelength order-parameter fluctuations of
the Ising model the interaction is usually assumed to be local, \( V_0(x) \propto \delta(x) \). However, because of universality, we can also obtain the critical exponents of the Ising universality class from the long-range interaction model \( 2 \).

Due to the long-range nature of the interaction, our action \( 1 \) is non-local so that approximation strategies based on the local potential approximation \( 19 \) cannot be used. However, we can make our action \( 1 \) local by decoupling the interaction by means of a real Hubbard-Stratonovich field \( \psi(x) \) conjugate to the composite field \( \varphi^2(x) \). In momentum space the decoupled action is then

\[
S[\varphi, \psi] = \frac{1}{2} k \left[ G_0^{-1}(k) \varphi_{-k} \varphi_{k} + V_0^{-1}(k) \psi_{-k} \psi_{k} \right] + \frac{i}{2} k \int k_1 \int k_3 \delta_{k_1 + k_2 + k_3,0} \psi_{k_1} \varphi_{k_2} \varphi_{k_3},
\]

with \( G_0^{-1}(k) = r_0 + c_0 k^2 \) and \( V_0(k) \) given in Eq. \( 2 \). We have introduced the notation \( k \int d^D k/(2\pi)^D \) and \( \delta_{k,0} = (2\pi)^D \delta(k) \). Because the fields \( \varphi(x) \) and \( \psi(x) \) are real, their Fourier components satisfy \( \varphi_{-k} = \varphi^*_k \) and \( \psi_{-q} = \psi^*_q \). Note however, that the conjugate field \( \psi \) does not represent a relevant physical excitation but rather a mediator of the long range interaction.

Following the usual procedure we now replace the propagator of the order parameter field by the cutoff-dependent deformation \( G_0(k) \rightarrow G_{0,\Lambda}(k) \) such that for \( k \lesssim \Lambda \) the deformed propagator is small while for \( k \gtrsim \Lambda \) we recover the bare propagator. Since we are eventually interested in the behaviour close to the Wilson-Fisher fixed point the cutoff is thereby introduced only in the order parameter field. It is then straightforward \( 21 \) to write down formally exact FRG flow equations for the irreducible vertices of the model \( 3 \) describing the evolution of these vertices when the cutoff parameter \( \Lambda \) is reduced. Of particular interest are the flow equations of the self-energy \( \Sigma_\Lambda(k) \) of the order-parameter field \( \varphi \) and the self-energy \( \Pi_\Lambda(k) \) of the conjugate field \( \psi \), which are shown graphically in Fig. \( 4 \). Obviously, these flow equations depend on various higher order irreducible vertices with three and four external legs. For our purpose it is fortunately sufficient to retain only the mixed three-legged vertex \( \Gamma_{\Lambda}^{\varphi \varphi \varphi}(k_1; k_2, k_3) \) which is the only higher-order vertex with a non-zero initial value \( \Gamma_{\Lambda_0}^{\varphi \varphi \varphi}(k_1; k_2, k_3) = i \). The exact flow equation for this vertex is shown graphically in Fig. \( 4 \) (c).

Neglecting all other vertices on the right-hand sides of the flow equations in Fig. \( 4 \) we obtain a closed system of integro-differential equations for the three functions \( \Sigma_\Lambda(k), \Pi_\Lambda(k) \) and \( \Gamma_{\Lambda}^{\varphi \varphi \varphi}(k_1; k_2, k_3) \). To further reduce the complexity of the problem, we neglect the momentum-dependence of the three-legged vertex, setting \( \Gamma_{\Lambda}^{\varphi \varphi \varphi}(k_1; k_2, k_3) \approx i \gamma_\Lambda \). As discussed below, this approximation is controlled as long as the renormalized interaction is long-range, and we then obtain the closed system of integro-differential equations,

\[
\partial_\Lambda \Sigma_\Lambda(k) = \gamma_\Lambda^2 \int_q \hat{G}_\Lambda(q) V_\Lambda(q + k),
\]

\[
\partial_\Lambda \Pi_\Lambda(k) = \gamma_\Lambda^2 \int_q \hat{G}_\Lambda(q) G_\Lambda(q + k),
\]

\[
\partial_\Lambda \gamma_\Lambda = -2 \gamma_\Lambda \int_q \hat{G}_\Lambda(q) G_\Lambda(q) V_\Lambda(q).
\]

Choosing a sharp momentum cutoff, the propagator and single-scale propagator of the order-parameter field are

\[
G_\Lambda(k) = \frac{\Theta(k - \Lambda)}{r_0 + c_0 k^2 + \Sigma_\Lambda(k)},
\]

\[
\hat{G}_\Lambda(k) = -\frac{\delta(k - \Lambda)}{r_0 + c_0 k^2 + \Sigma_\Lambda(k)},
\]

while the scale-dependent effective interaction is

\[
V_\Lambda(k) = \frac{1}{m_0 + b_0 k^2 + \Pi_\Lambda(k)}.
\]

Eqs. \( 4 \) \( 6 \) form a closed system of integro-differential equations for the two self-energies \( \Sigma_\Lambda(k) \) and \( \Pi_\Lambda(k) \) and

![Graphical representations of exact FRG flow equations: (a) Self-energy \( \Sigma_\Lambda(k) \) of the order-parameter field \( \varphi \); (b) self-energy \( \Pi_\Lambda(k) \) of the conjugate field \( \psi \); (c) mixed three-legged vertex \( \Gamma_{\Lambda}^{\varphi \varphi \varphi}(k_1; k_2, k_3) \). Here the solid lines represent the cutoff-dependent \( \varphi \)-propagator \( G_\Lambda(k) \), while the wiggly lines represent the \( \psi \)-propagator \( V_\Lambda(k) \). Slashed lines represent the corresponding single-scale propagators of the order parameter field.](image)
the vertex $\gamma_\Lambda$. Although these equations may in principle be solved numerically without further approximations, to make progress analytically we expand the self-energies for small momenta up to order $k^2$, $\Omega(k) = r - r_0 + (c - c_0)k^2 + O(k^4)$, $\Pi(k) = m - m_0 + a|k| + (b - b_0)k^2 + O(k^4)$. Note that for sharp momentum cutoff the expansion of $\Pi(k)$ has a non-analytic term proportional to $|k|$. Substituting the expansions (10) and (11) into our flow equations (4, 5, 6) it is straightforward to derive RG flow equations for the six couplings $r$, $c$, $m$, $a$, $b$ and $\gamma$. In order to find the scaling solution corresponding to the Wilson-Fisher fixed point it is convenient to introduce the dimensionless rescaled couplings $r = r/(c\Lambda^2)$, $a = a\Lambda/m$, and $b = b\Lambda^2/m$, which are considered as functions of the logarithmic flow parameter $t = \ln(t)$. This choice furthermore reveals that the two parameters $m$ and $\gamma$ only appear in the combination $m^2/\gamma^2$ in the resulting flow equations. It is thus natural to reduce the number of relevant parameters by one by introducing the rescaled version of the momentum independent part of the interaction as $g = \Omega D \Lambda^{-1} \gamma^3/(2\pi D \Lambda^2)$, where $\Omega_D$ is the surface area of the $D$-dimensional unit sphere. We obtain

\[ \partial_t r = (2 - \eta r) - \frac{g}{1 + r_L} s, \]
\[ \partial_t g = (4 - 2\eta) - \frac{g^2}{1 + r_L} \left[ \frac{1}{2} + \frac{2}{s} \right], \]
\[ \partial_t a = - \frac{1 + \frac{g}{2(1 + r_L)^2}}{2 + \frac{g}{(1 + r_L)^2}} a - \frac{g}{2} \frac{\Omega_D - 1}{\Omega_D} \left( \frac{D - 1}{D + 1} - r \right), \]
\[ \partial_t b = - \frac{1 + \frac{g}{2(1 + r_L)^2}}{2 + \frac{g}{(1 + r_L)^2}} b + \frac{g}{2} \frac{4 - D}{D + 1} - r. \]

where the scale-dependent coupling $s = 1 + a + b$ is large if the interaction is long-range and reduces to a number of order unity for short-range interaction. The flowing anomalous dimension $\eta = -(\lambda \partial \Lambda^2 c_\Lambda)/c_\Lambda$ is given by

\[ \eta = \frac{g}{(1 + r_L)s^2} \left[ \frac{(a_0 + b_0)^2}{Ds} - \frac{D - 1}{2D} a - b \right]. \]

In the rest of this work, we carefully analyse the RG flow encoded in Eqs. (12, 13).

First of all, we note that the above system of flow equations has a fixed point with one relevant direction and finite $\eta$ which we identify with the Wilson-Fisher fixed point. In three dimensions, the numerical values of our rescaled couplings at the fixed point are $r_\infty = -0.170$, $g_\infty = 0.269$, $a_\infty = -0.115$, $b_\infty = 0.065$, and $\eta_\infty = -0.00957$. At first sight, it seems that our truncation is not satisfactory, as it yields a negative anomalous dimension at the fixed point. Moreover, keeping in mind that the dimensionless coupling $b$ can be identified with the square of the range of the interaction in units of the ultraviolet cutoff, we see that at the fixed point the interaction is short range, corresponding to $b_\infty \ll 1$. It is therefore not surprising that our truncation strategy, which relies on the long-range nature of the interaction, breaks down as soon as the flowing coupling $b$ ceases to be large compared with unity. On the other hand, for $b_\infty \gg 1$ our truncation is controlled by the small parameter $1/b_\infty$ and is expected to be quantitatively accurate in this regime. In fact, by perturbatively calculating the modification of the flow equations (12, 13) due to the higher-order vertices shown in Fig. 1 and the momentum-dependent part of the three-point vertex $\Gamma^{\pi \pi \pi}_k$ (Fig. 1), we find that all corrections involve at least an additional factor of $1/s = 1/(1 + a_L + b_\infty)$, which is small if $b_\infty \gg 1$. It is then natural to stop the RG flow at some finite scale $l$, where $l$ is still reasonably small. If the Ginzburg scale [21, 23, 25] (which can be identified...
with the upper limit of the momentum range where the order-parameter correlation function at the critical point scales as $k^{-2+\eta}$ is larger than $\Delta_0 e^{-l_i}$, the RG trajectory at $l = l_*$ already "feels" the Wilson-Fisher fixed point so that we expect that the flowing $\eta_l$ at scale $l = l_*$ will be a reasonable approximation for the critical exponent $\eta$.

To investigate whether such a scale $l_*$ really exists, we solve the flow equations \(12\text{–}16\) numerically. For $a_0 = 0$ and given initial values $\eta_0$ and $b_0$, the initial $r_0$ is thereby fine-tuned such that for $l \to \infty$ the RG trajectory flows into the fixed point. The flow of the couplings $r_1$, $g_l$, $a_l$, $b_l$, $s_l$, and $\eta_l$ with initial condition $\eta_0 = 1$, and $b_0 = 10^{10}$ in $D = 3$ is shown in Fig. 2. The crucial observation is now that the flowing $\eta_l$ exhibits a local maximum $\eta_*$ at a finite scale $l = l_*$. At this scale the dimensionless parameter $1/s_*$, which controls the accuracy of our truncation is still small but rapidly approaches a number of order unity for $l \gtrsim l_*$. Moreover, from Fig. 2 we see that close to the scale $l_*$ the couplings $r_1$, $g_l$, and $a_l$ exhibit a local extremum before monotonously approaching their fixed point values. It is therefore reasonable to assume that the scale $\Delta_0 e^{-l_i}$ defines the boundary of the Ginzburg regime and estimate $\eta \approx \eta_*$. As the control parameter $1/s_*$ is roughly 0.1 at this point, we expect that in this way we can obtain $\eta$ with an accuracy $\Delta\eta/\eta$ of about 10%. We checked that these features are robust with respect to variations of the initial conditions by changing the initial range of the interaction, parametrized by $b_0$, for fixed bare interaction $g_0$. While the resulting $l_*$ grows with increasing $b_0$, we observe a convergence of the corresponding maximum $\eta_*$. In Fig. 3 we present our results for $\eta_*$ as a function of the initial value $b_0$ for $g_0$ in the range between $10^{-2}$ and $10^2$. The value of $\eta_*$ obtained in this way converges for $b_0 \to \infty$ to $\eta_c = 0.03651$. Amazingly, this is only one percent larger than currently most accurate results \[14\text{–}16\], although a priori we would have expected agreement only at the 10% level. While we cannot exclude the possibility that this agreement is accidental, we believe that it is due a cancellation of the corrections of order in $1/s_l^2$ to the FRG flow equations in $D = 3$. This point certainly deserves further attention.

Given the fact that our flow equations \[12\text{–}16\] are valid for arbitrary $D$, we may also use our method to calculate $\eta$ as a function of the dimensionality $D$ of the system. The result is shown in Fig. 4 together with a coinciding result obtained independently by Borel-resumming the $\epsilon$-expansion series \[10\] and conformal bootstrap calculations \[26\]. For $D \leq 3$ our value for $\eta$ agrees within the expected uncertainty of about 10% with the previous results, although the exact value is always slightly larger. In the opposite limit of small $\epsilon = 4 - D$ our method is only able to predict the order of magnitude of $\eta$. A natural explanation for this lack of quantitative accuracy for $\epsilon \ll 1$ is that in this case the Ginzburg momentum is exponentially small, $k_G \propto e^{-const/\epsilon}$ (see Refs. 21 and 22), so that the scale where the renormalized interaction of our model crosses over from long-range to short-range does not overlap with the Ginzburg regime.

In summary, we have developed a new method for calculating the critical exponent $\eta$ of the Ising universality class which uses the inverse range of the interaction of an effective Ginzburg-Landau-Wilson model as a small parameter to control the truncation of the vertex expansion of the FRG flow equations. Although the effective interaction becomes short-range as the RG is iterated, by stopping the RG flow at a finite scale where the range of

**FIG. 3.** Maximum $\eta_*$ of $\eta$ in $D = 3$ for different bare interactions $g_0$ as a function of the bare value $b_0$ of the dimensionless interaction-range parameter $b$. For $1/b_0 \to 0$ our results for $\eta_*$ converge to $\eta_c = 0.03651$. On the scale of the plot this cannot be distinguished from the results of Ref. \[14\text{–}16\]. The shaded region indicates the converged accuracy $\Delta\eta_c = \eta_c s_c$.

**FIG. 4.** Estimate for the critical exponent $\eta = \eta_c(D) \pm \Delta\eta_c(D)$ obtained with our method for dimensions $2 \leq D < 4$. The dashed line represents the coinciding result obtained independently by Borel-resumming the $\epsilon$-expansion series \[10\] and the conformal bootstrap calculations \[26\]. The latter intersects with out RG result close to $D = 3$. The result is shown in Fig. 4 together with a coinciding result obtained independently by Borel-resumming the $\epsilon$-expansion series \[10\] and conformal bootstrap calculations \[26\]. For $D \leq 3$ our value for $\eta$ agrees within the expected uncertainty of about 10% with the previous results, although the exact value is always slightly larger. In the opposite limit of small $\epsilon = 4 - D$ our method is only able to predict the order of magnitude of $\eta$. A natural explanation for this lack of quantitative accuracy for $\epsilon \ll 1$ is that in this case the Ginzburg momentum is exponentially small, $k_G \propto e^{-const/\epsilon}$ (see Refs. 21 and 22), so that the scale where the renormalized interaction of our model crosses over from long-range to short-range does not overlap with the Ginzburg regime.
the interaction is still large we were able to obtain a surprisingly accurate estimate of $\eta$ three dimensions. Note that a similar scheme to obtain the value of the critical exponent $\eta$ was already employed in the context of the $O(2)$ model in two dimensions [27, 28]. These works evaluate the corresponding anomalous dimension along a line of unstable pseudofixed points and find a local maximum of $\eta^{(2)}_{D=2} = 0.24$ close to the known value 0.25. The location of the maximum thereby coincides with a crossover from the ordered into the disordered phase, similar to our calculation where the extremum of $\eta$ is located at the scale where the interaction changes from long-range to short-range.

An implicit assumption underlying our method is that the Ginzburg regime extends to the scale where the RG flow is stopped to estimate $\eta$. This assumption seems to be valid in $2 \leq D \leq 3$, but does not hold for small $\epsilon = 4 - D$ where the Ginzburg scale is exponentially small. Our calculation can be systematically improved by taking the momentum-dependence of the three-point vertex and of higher order vertex corrections encoded in the different types of induced four-point vertices shown in Fig. [1] into account, which give rise to additional terms in the RG flow equations [12–16] involving higher powers of the small parameter $1/s_l$. Our method should also be useful to calculate $\eta$ for other universality classes, and can be extended to obtain the complete momentum-dependence of the self-energy and of the effective interaction.

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