On the Interquark Potential calculation from Dirac Brackets

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Abstract

We obtain the binding energy of an infinitely heavy quark-antiquark pair from Dirac brackets by computing the expectation value of the pure QCD Hamiltonian. This procedure exploits the rich structure of the dressing around static fermions. Some subtle points related to exhibiting explicitly the interquark energy are considered.

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I. INTRODUCTION

It is well known that to gain a deeper insight into gauge theories a proper study of the concepts of screening and confinement is of considerable importance. In this connection the binding energy of an infinitely heavy quark-antiquark pair represents a fundamental concept in QCD which is expected to play an important role in the understanding of quark confinement. In this respect we recall that asymptotic freedom allows one to use perturbation theory for the description of high energy phenomena. But within this framework, we cannot explain low energy phenomena such as the permanent confinement of quarks and gluons. The reason is that the infrared divergences and gauge dependence make bound-state equations very hard to approximate. We further note that the choice of the gauge has a strong
influence on the propagator. For instance, the studies of Caracciolo et al. [1] were crucial in the investigation of the ambiguity in the definition of the gluon propagator.

In view of the above-mentioned difficulties much attention has been recently devoted to formulations of QCD in which gauge-invariant variables are explicitly constructed. By doing so it has been possible to obtain a more direct physical insight into the description of charged fields [2,3]. As a result of this development the matter fields (quarks) are now dressed by a cloud of gauge fields. Using this approach, the pure QCD correction to the Coulomb potential in 3+1 dimensions has also been derived more recently by Lavelle et al. [4]. In particular, it was showed that such correction contains a dominant anti-screening contribution and another one which corresponds to screening by physical gluons.

On the other hand, it is well known that gauge theories fall into the class of constrained systems, where a systematic procedure for quantizing such systems has been given by Dirac [5]. The point we wish to emphasize, however, is that because of the structure of the constraints the resulting equal-time commutation relations involve in general the coupling constant. This raises the question of how to recover from them the interquark potential between charged fields following the conventional path via the expectation value of the QCD Hamiltonian. This problem is addressed in the present letter.

**II. INTERQUARK POTENTIAL**

Before computing explicitly the interquark potential, we shall first present the Hamiltonian analysis for the Yang-Mills field coupled to an external source \( J^0 \). We start with the Lagrangian

\[
\mathcal{L} = -\frac{1}{4} Tr (F_{\mu\nu} F^{\mu\nu}) - A^a_0 J^0 = -\frac{1}{4} F^{a\mu\nu} F_{a\mu\nu} - A^a_0 J^0. \tag{1}
\]

Here \( A_\mu(x) = A^a_\mu(x) T^a \), where \( T^a \) is a hermitian representation of the semi-simple and compact gauge group; and \( F^{a\mu\nu} = \partial_\mu A^a_\nu - \partial_\nu A^a_\mu + gf^{abc} A^b_\mu A^c_\nu \), with \( f^{abc} \) the structure constants of the gauge group. The Dirac procedure [5] as applied to (1) is straightforward. The
canonical momenta are $\Pi^{a\mu} = -F^{a0\mu}$, which results in the usual primary constraints $\Pi^0_0 = 0$, and $\Pi^{ai} = F^{ai0}$. The canonical Hamiltonian following from the above Lagrangian is:

$$H_c = \int d^3x \left( -\frac{1}{2} \Pi^a_i \Pi_{ai} + \Pi^a_i \partial^i A^{a0} + \frac{1}{4} F^a_{ij} F^{aij} - gf^{abc} \Pi^{ai} A^b_l A^c_i + A^a_0 J^0 \right).$$  \hspace{1cm} (2)

The persistence of the primary constraints leads to the following secondary constraints

$$\Omega^{a(1)}(x) = \partial_i \Pi^{ai} + gf^{abc} A^b_i \Pi^c_i - J^0 \approx 0. \hspace{1cm} (3)$$

It is easy to check that there are no further constraints in the theory, and that the above constraints are first class. The corresponding total (first class) Hamiltonian that generates the time evolution of the dynamical variables is given by

$$H = H_c + \int d^3x \left( c^a_0(x) \Pi^0_0(x) + c^a_1(x) \Omega^{a(1)}(x) \right),$$  \hspace{1cm} (4)

where $c^a_0$ and $c^a_1$ are arbitrary functions. Since $\Pi^0_0 \approx 0$ for all time and $A^a_0(x) = [A^a_0(x), H] = c^a_0(x)$, which is completely arbitrary, we discard $A^a_0(x)$ and $\Pi^0_0(x)$ because they add nothing to the description of the system. The Hamiltonian then takes the form

$$H = \int d^3x \left( -\frac{1}{2} \Pi^a_i \Pi_{ai} + \Pi^a_i \partial^i A^{a0} + c^a(x) \left( \partial^i \Pi^a_i + gf^{abc} A^b_i \Pi^c_i - J^0 \right) \right),$$  \hspace{1cm} (5)

where $c^a(x) = c^a_1(x) - A^a_0(x)$.

Therefore, we have the first class constraints $\Omega^{a(1)}(x)$, which appear at the secondary level. Now the presence of the arbitrary quantities $c^a(x)$ are undesirable since we have no way of giving them a meaning in a quantum theory. To circumvent this trouble, we introduce a supplementary condition on the vector potential such that the full set of constraints becomes second class. A particularly appealing and useful choice is given by

$$\Omega^{(2)}_a(x) = \frac{1}{\lambda} d\lambda (x - \xi)^k A^{(a)}_k (\xi + \lambda (x - \xi)) \approx 0,$$  \hspace{1cm} (6)

where $0 \leq \lambda \leq 1$ is the parameter describing the spacelike straight path $x^k = \xi^k + \lambda (x - \xi)^k$, on a fixed time slice. This supplementary condition is the non-Abelian generalization of the gauge condition discussed in [8], which leads to the Poincaré gauge [8]. For simplicity we restrict our considerations to $\xi^k = 0$. As a consequence, (3) becomes
\[
\Omega^{(2)}_0 (x) = \int_0^1 d\lambda x^k A_k^a (\lambda x) \approx 0.
\] (7)

Now we come to the calculation of the Dirac brackets. By following the Dirac’s procedure one arrives at

\[
\left\{ A_i^a (x) , A_j^b (y) \right\}^* = 0 = \left\{ \Pi_i^a (x) , \Pi_j^b (y) \right\}^* .
\] (8)

\[
\left\{ A_i^a (x) , \Pi^{bj} (y) \right\}^* = \delta^{ab} \delta_i^j \delta^{(3)} (x - y) - \int_0^1 d\lambda \left( \delta^{ab} \frac{\partial}{\partial x^i} - gf^{abc} A_c^i (x) \right) x^j \delta^{(3)} (\lambda x - y) .
\] (9)

Note the presence of the last term on the right-hand side which depends on \( g \). In passing we note that similar Dirac brackets were obtained independently in reference [9].

We are now equipped to compute the interaction energy between pointlike sources in pure QCD, where a fermion is localized at the origin \( 0 \) and an antifermion at \( y \). In order to accomplish this purpose, we will calculate the expectation value of the energy operator \( H \) in the physical state \( |\Omega\rangle \), which we will denote by \( \langle H \rangle_\Omega \). From our above discussion, we see that \( \langle H \rangle_\Omega \) reads

\[
\langle H \rangle_\Omega = \langle \Omega | \int d^3x \left( -\frac{1}{2} \Pi_i^a \Pi^a_i + \frac{1}{4} F_{ij} F^{ij} \right) |\Omega\rangle .
\] (10)

Since the fermions are taken to be infinitely massive (static), this can be further simplified as

\[
\langle H \rangle_\Omega = \langle \Omega | \int d^3x \left( -\frac{1}{2} \Pi_i^a (x) \Pi^{ia} (x) \right) |\Omega\rangle .
\] (11)

Let us also mention here that, as was first established by Dirac [10], the physical states \( |\Omega\rangle \) correspond to the gauge invariant ones. It is helpful to recall at this stage that in the Abelian case \( |\Omega\rangle \) may be written as [11]

\[
|\Omega\rangle \equiv \overline{\Psi} (y) \Psi (0) \rangle = \overline{\psi} (y) \exp \left( ig \int_0^y dz^i A_i (z) \right) \psi (0) |0\rangle ,
\] (12)

where \( |0\rangle \) is the physical vacuum state and the line integral appearing in the above expression is along a spacelike path starting at \( 0 \) and ending at \( y \), on a fixed time slice. It should be
clear from this discussion that the strings between fermions have been introduced in order to have a gauge-invariant function $|\Omega\rangle$. According to this viewpoint the fermion fields are now dressed by a cloud of gauge fields.

The expression (12) may be extended on account of the fact that we have non-Abelian fields. Accordingly, we can write a state which has a fermion at 0 and an antifermion at $y$ as

$$|\Omega\rangle = \bar{\psi}(y) U(y, 0) \psi(0) \langle 0\rangle,$$  

(13)

where

$$U(y, 0) \equiv P \exp \left( ig \int_0^y dz^i A^a_i(z) T^a \right).$$  

(14)

As before, the line integral is along a spacelike path on a fixed time slice, $P$ is the path-ordering prescription and $|0\rangle$ is the physical vacuum state.

This last point gives us an opportunity to compare our work with the standard Wilson loop procedure \[11\], to make sure that the known results are recovered from the general expression (13) in the weak coupling limit. In effect, due to asymptotic freedom, the short distance behavior of the interquark potential is determined by perturbation theory. According to this, at weak coupling, one can expand

$$U(y, 0) \equiv P \exp \left( ig \int_0^y dz^i A^a_i(z) T^a \right) = P \left( 1 + ig \int_0^y dz^i A^a_i(z) T^a + ... \right).$$  

(15)

This implies that, at lowest order in $g$, the non-Abelian generalization of the dressing framework is the same as in the Abelian theory. Thus, at short distances, one should expect to obtain the known Coulomb potential in addition to a correction of order $g^4$, as we will now show.

With this in view, our next task is the computation of the expectation value of $H$ in the physical state $|\Omega\rangle$ given by the expression (14). From the above Hamiltonian analysis one distinguishes here an Abelian part (proportional to $C_F$) and a non-Abelian part (proportional to the combination $C_F C_A$). We first consider the Abelian part which is identical to the $QED$ case. To do this, we shall begin by observing that
\( \Pi_1^a (x) |\Omega \rangle = \psi (y) U(y, 0) \psi (0) \Pi_1^a (x) |0 \rangle + g T^a \int_0^y \, dz \delta (x - z) |\Omega \rangle. \)  \hfill (16)

Using this in (14) we then evaluate the interaction energy in the presence of the static charges

\[ V^g = \frac{1}{2} g^2 tr T^a T^a \int_0^y \, dz \int_0^y \, dz' \delta (z - z'), \]  \hfill (17)

remembering that the integrals over \( z^i \) and \( z'^i \) are zero except on the contour of integration. Writing the purely group theoretic factor \( tr T^a T^a = C_F \), the expression (17) leads to

\[ V^g (L) = \frac{1}{2} g^2 C_F kL, \]  \hfill (18)

after subtracting the term \( \langle H \rangle_0 = \langle 0 | H | 0 \rangle \), where \( |y| \equiv L \) and \( k = \delta^{(2)} (0) \). This calculation shows that special care is required in order to clarify the appearance of this peculiar result. It may be remarked, however, that the origin of the divergence is quite clear, so that it is possible to extract the Coulomb potential from the infinite contribution. Notice that the origin of the divergent factor \( k \) is due to the fact that the thickness of the string is nonvanishing only on the contour of integration. It is worth stressing at this stage that a more careful examination of the term \( \frac{g^2}{2} \int d^3 x \left( \int_0^y \, dz \delta^{(3)} (x - z) \right)^2 \) reproduces exactly the expected Coulomb interaction between charges after subtracting the self-energy term, as was discussed in [6]. Having made this observation, we write immediately the standard result for the potential to order \( g^2 \), that is,

\[ V^g (L) = - \frac{1}{4\pi} g^2 C_F \frac{1}{L}. \]  \hfill (19)

Let us also mention here that if we had considered a modified form for the supplementary condition (6), which is equivalent to the Coulomb gauge [6], the result for the potential would have been the same.

We now turn our attention to the non-Abelian part. From the expressions (9) and (11), we see that the \( g^4 \) contribution may be written in the form

\[ V^{g^4} = \int d^3 x \langle 0 | (I^\lambda)^2 | 0 \rangle, \]  \hfill (20)
where

$$I^i = g^2 f^{bac} T^b \int d^3 z \int \frac{1}{|z|} d\lambda A^c_k (z) z^i \delta(x - \lambda z).$$  \hfill (21)$$

The expression (21) may then be further manipulated as described in [3]. Thus, if we use spherical coordinates, we find that

$$I^i = g^2 f^{bac} T^b |z|^2 \int \frac{1}{|z|} d\lambda A^c_k (z) \sum_{lm} Y^*_{lm} (\theta', \varphi') Y_{lm} (\theta, \varphi).$$  \hfill (22)$$

Now, by employing (22) we can reduce (20) to

$$V_{g^4} = \frac{1}{2} g^4 tr \int d^3 x \langle 0 | \left( f^{bac} T^b \frac{z^i}{|z|} \int \frac{1}{|z|} d\lambda A^c_k (z) \sum_{lm} Y^*_{lm} (\theta', \varphi') Y_{lm} (\theta, \varphi) \right)^2 | 0 \rangle,$$  \hfill (23)$$

which, by introducing the integration variable \( r = \frac{x}{z} \) and using usual properties for the spherical harmonics, may be rewritten as

$$V_{g^4} (L) = \frac{1}{2} g^4 C_A C_F \left( -\frac{1}{L} \right) \int d^3 z \int d^3 z' D_{ij} (z, z').$$  \hfill (24)$$

Here \( D_{ij} (z, z') \) is the propagator, which is diagonal in colour and taken in an arbitrary gauge. Thus, in order to carry out this calculation, we choose, for example, \( D_{ij} (z, z') \) in the Feynman gauge. Hence expression (24) reduces to

$$V_{g^4} (L) = -\frac{1}{8 \pi^2} g^4 C_A C_F \left( -\frac{1}{L} \right) \int d^3 z \int d^3 z' \frac{1}{(z - z')^2}.$$  \hfill (25)$$

This allows us to derive the \( g^4 \) contribution

$$V_{g^4} (L) = -g^4 \frac{1}{4 \pi^2} C_A C_F \frac{1}{L} \log (\mu L),$$  \hfill (26)$$

where \( \mu \) is a cutoff. By putting together Eqs. (19) and (26), we obtain for the total interquark potential

$$V (L) = -g^2 C_F \frac{1}{4 \pi L} \left( 1 + \frac{g^2}{\pi} C_A \log (\mu L) \right).$$  \hfill (27)$$

In this way one obtains the known heavy interquark potential at lowest order in \( g \) [4]. However, the central difference between the above analysis and that of Ref. [4] rests in the
fact that the potential (27) is directly recovered from the constraints structure of the theory we have discussed. In this context, the present investigation complements the discussion done in Ref. [4], as well as it reveals the general viability of our analysis. Thus it seems a challenging job to extend the scope of applicability of the above analysis. We expect to report on progress along these lines soon.

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