An eigenfunction method for particle acceleration at ultra-relativistic shocks

Axel W. Guthmann* , John G. Kirk*, Yves A. Gallant†,‡ and Abraham Achterberg‡

*Max-Planck-Institut für Kernphysik, Postfach 103980, D-69029 Heidelberg, Germany
† Astronomical Institute, Utrecht University, P.O. Box 80000, 3508 TA Utrecht, Netherlands
‡ Dublin Institute for Advanced Studies, 5 Merrion Square, Dublin 2, Ireland

Abstract. We adapt and modify the eigenfunction method of computing the power-law spectrum of particles accelerated at a relativistic shock front via the first-order Fermi process [6] to apply to shocks of arbitrarily high Lorentz factor. The power-law index of accelerated particles undergoing isotropic small-angle scattering at an ultra-relativistic, unmagnetized shock is found to be $s = 4.23 \pm 0.2$ (where $s = d \ln f / d \ln p$, with $f$ the Lorentz-invariant phase-space density and $p$ the momentum), in agreement with the results of Monte-Carlo simulations. We present results for shocks in plasmas with different equations of state and for Lorentz factors ranging from 5 to infinity.

THE METHOD

We study a stationary shock front in the $x−y$-plane. The accelerated particles are assumed to be test-particles without influence on the dynamics of the plasma or the jump conditions at the shock-front. The plasma flows along the $z$-axis, with constant velocities $u_−$ in the upstream ($z < 0$) region and $u_+$ downstream ($z > 0$), the velocities are related by the Rankine-Hugoniot jump conditions.

Test-particles are injected into the acceleration process and their interaction with the plasma flow is assumed to give rise to diffusion in the angle $\cos^{-1} \mu$ between a particle’s velocity and the shock normal. In the frame of the shock front this leads to a stationary transport equation valid for the local plasma rest frame and given in mixed coordinates as [6]

$$\Gamma(u + \mu) \frac{\partial f}{\partial z} = \frac{\partial}{\partial \mu} D_{\mu\nu}(1 - \mu^2) \frac{\partial f}{\partial \mu}$$

(1)

where the plasma speed $u$ is measured in units of the speed of light, $\Gamma = (1 - u^2)^{-1/2}$ is the Lorentz-factor, $f(p, \mu, z)$ is the (Lorentz invariant) phase-space density as

1) homepage: http://www.mpi-hd.mpg.de/theory/
a function of the particle momentum $p$, direction $\mu$ and position. $p$ and $\mu$ are measured in the local rest frame of the plasma, whereas $z$ is measured in the rest frame of the shock front.

Equation (1) is solved using the separation Ansatz [6]

$$f(p, u, \mu, x) = \sum_{i=-\infty}^{+\infty} g_i(p) Q_i(\mu, u) \exp \left( \frac{\Lambda_i z}{\Gamma} \right),$$

valid in each half-plane with $\Lambda_i$ and $Q_i$ the eigenvalues and eigenfunctions of the equation

$$\left\{ \frac{\partial}{\partial \mu} \left[ D_{\mu \mu} \frac{\partial}{\partial \mu} \right] - \Lambda_i(u + \mu) \right\} Q_i(\mu, u) = 0$$

The momentum distribution of particles with energy far above the injection energy range – those in which we are interested – takes the shape of a power-law $g_i(p) \propto p^{-s}$ with a power-law index $s$, since there is no preferred momentum scale in this range.

Matching the expansion (2) across the shock front according to Liouville’s Theorem and imposing physically realistic boundary conditions up and downstream leads to a nonlinear algebraic equation for the power law index $s$.

In [6] and [4] only the eigenfunctions with $i < 0$ were used and the method was applied to mildly relativistic shock speeds ($\Gamma \lesssim 5$). Here, we use the eigenfunctions with $i > 0$ and calculate them directly with a numerical scheme. In the limit $u_+ \rightarrow 1$ an analytic expression is available [7]. Four eigenfunctions ($i = 1, 3, 5, 7$) are shown in Fig. 1A as functions of the cosine $\mu_s = (\mu + u)/(1 + \mu u)$ of the angle between the particle direction and the shock normal, measured in the shock rest frame. For $i > 1$ they are oscillatory for $-1 < \mu_s < 0$ and for all $i > 0$ fall off monotonically in the range $0 < \mu_s < 1$.

RESULTS

The index $s$ of the momentum spectra of the accelerated particles in different cases are shown in Fig. 1B. The jump conditions investigated are those for a relativistic gas both up and downstream: $u_- u_+ = 1/3$ and for a strong shock in a medium with adiabatic index $4/3$ [5].

Also we investigate two different scattering operators, $D_{\mu \mu 1} = 1 - \mu^2$ (isotropic small/angle scattering) and $D_{\mu \mu 2} = (1 - \mu^2) \times (\mu^2 + 0.01)^{1/3}$ corresponding to scattering in weak Kolmogorov turbulence, together with a rough prescription for avoiding the lack of scattering at $\mu = 0$ [4]. For high upstream Lorentz-factors the power-law index settles at a value around 4.23 for all equations of state, which is reproduced in the limiting case $u_- \rightarrow 1$. The scattering operator has only a minor effect.
FIGURE 1. A) (left) The eigenfunctions $Q_i$ for $i = 1, 3, 5, 7$ for $\Gamma_- = 223$, as a function of the (cosine of the) angle between the particle speed and the shock normal, measured in the shock frame, for a relativistic gas. B) (right) The power-law index $s$ for relativistic gas with isotropic (solid, 2nd from top) and anisotropic (dashed-dotted, top) scattering operator and for a strong shock in a gas of adiabatic index $4/3$, with isotropic (dashed, 4th from top) and anisotropic (dotted, 3rd from top) scattering

SUMMARY

These results are in agreement with the asymptotic Monte-Carlo results of Gallant et al. [3] and those of Bednarz & Ostrowski [1] for $\Gamma_- \approx 200$. Anisotropic scattering, which has not been treated by Monte-Carlo simulations, leads to a slight steeping in the power-law spectrum, because fewer particles are able to cross the region $\mu \approx 0$ and return to the shock. From observations of GRB afterglows, Galama et al. [2] and Waxman [8] have found synchrotron spectral indices corresponding to $s \approx 4.25$, implying that the particles could indeed have been accelerated by the first order Fermi mechanism operating at an ultrarelativistic shock front.

This work was supported by the European Commission under the TMR programme, contract number ERBFMRX-CT98-0168

REFERENCES

1. Bednarz, J., Ostrowski, M., Phys. Rev. Lett. 80, 3911 (1998).
2. Galama, T. et al., Astrophysical Journal 500, L101 (1998).
3. Gallant, Y.A., Achterberg, A., Kirk, J.G., Astron. Astrophys. Suppl. Ser. 138, 549 (1999)
4. Heavens, A.F., Drury L.O’C., MNRAS 235, 997 (1988)
5. Kirk, J.G., Duffy, P., Journal of Physics G: Nucl. Part. Phys. 25, R163 (1999)
6. Kirk, J.G., Schneider, P., Astrophysical Journal 315, 425 (1987).
7. Kirk, J.G., Schneider, P., Astronomy & Astrophysics 225, 559 (1989)
8. Waxmann, E., Astrophysical Journal 485, L5 (1997).