Abstract

The vicious random walker problem on a one dimensional lattice is considered. Many walkers take simultaneous steps on the lattice and the configurations in which two of them arrive at the same site are prohibited. It is known that the probability distribution of \(N\) walkers after \(M\) steps can be written in a determinant form. Using an integration technique borrowed from the theory of random matrices, we show that arbitrary \(k\)-th order correlation functions of the walkers can be expressed as quaternion determinants whose elements are compactly expressed in terms of symmetric Hahn polynomials.
1 Introduction

The vicious walker problem first introduced by Fisher[1] and then developed by Forrester[2] (see also [3]) recently attracts much attention in mathematical physics. Fascinating connections to other research fields, such as Young tableaux in combinatorics[4], Kardar-Parisi-Zhang (KPZ) universality in the theory of growth process[5] and the theory of random matrices[6], have been revealed one after another. In the context of random matrix theory, the ensemble of vicious walkers in one dimension corresponds to a discretization of the Gaussian ensembles of random matrices. Therefore the theory of discretized random matrices is expected to shed light on all of the related problems. Indeed discretized random matrices in the guise of discrete Coulomb gases are central to the work of Johansson[4, 7, 8] in these directions.

Suppose that there are \( N \) walkers on a one dimensional lattice. In the lock step version of the model (as distinct from the random turns version considered in the recent work[9]), at each time step each walker moves to the left or right one lattice site with equal probability. Walkers are ”vicious” so that two or more walkers are prohibited to arrive at the same site simultaneously. The \( j \)-th walker starts at the position \( x = 2j - 2 \) and, after \( M \) steps, arrives at \( x = X_j \). The walker configurations form nonintersecting paths in the \( x-t \) plane. An example is given in Figure 1. Furthermore it has long been realized[10] that there is a one-to-one correspondence between such nonintersecting paths and random rhombus tilings of a hexagon, or suitable truncation thereof, involving three types of rhombi. This is also illustrated in Figure 1.

The number of lock step paths is known to be expressed as a binomial determinant[11, 12, 13]

\[
P_1(X_1, X_2, \cdots, X_N) = \det \left[ \left( \frac{M + X_j}{2} - l + 1 \right) \right]_{j,l=1,2,\cdots,N}.
\] (1.1)

The binomial determinant can be further rewritten as a product formula

\[
P_1(X_1, X_2, \cdots, X_N) = 2^{-N(N-1)/2} \prod_{j=1}^{N} \frac{(M + N - j)!}{(M + X_j)! (M - X_j)^2 + N - 1)!} \prod_{j \neq l} (X_j - X_l),
\] (1.2)

where \( X_1 < X_2 < \cdots < X_N \). Now we introduce new variables

\[
x_j = \frac{X_j - N + 1}{2}, \quad L = M + N - 1
\] (1.3)

in order to obtain a compact and symmetric expression

\[
P_1(x_1, x_2, \cdots, x_N) = C_{MN} \prod_{j=1}^{N} \sqrt{w(x_j)} \prod_{j \neq l} | x_j - x_l |.
\] (1.4)

Here

\[
C_{MN} = \prod_{j=1}^{N} (M + N - j)!
\] (1.5)
Figure 1: Nonintersecting paths representing a vicious walker configuration with \( N = 3 \) walkers and \( M = 4 \) steps. In the top diagram these paths are drawn in the \( x-t \) plane. In the bottom diagram superimposed on the paths is the equivalent rhombi tiling, involving left sloping rhombi (step to the left), right sloping rhombi (step to the right) and vertical rhombi.

and

\[
w(x) = \frac{1}{\left[ \left( \frac{L}{2} + x \right)! \left( \frac{L}{2} - x \right)! \right]^x}. \tag{1.6}
\]

In some applications of the vicious walker problem one imposes the additional constraint that each walker returns to the initial position after \( 2M \) steps. One example is the rhombus tiling problem of Figure 1 with the region to be tiled extended to be symmetrical about \( t = M \) and thus made into a hexagon\[^8\]. In such cases the number of paths is given by the square of \( P_1(x_1, x_2, \cdots, x_N) \):

\[
P_2(x_1, x_2, \cdots, x_N) = C^2_{MN} \prod_{j=1}^{N} w(x_j) \prod_{j>l}^{N} |x_j - x_l|^2. \tag{1.7}
\]

As will be revised below, the function \( w(x) \) is a special case of the Hahn weight function from the theory of discrete orthogonal polynomials. The probability density (1.7) with the Hahn weight is intimately related to Hahn polynomials and so has been termed the Hahn ensemble\[^7, 8\].

We are interested in the number of paths under the condition that \( k \) walkers take fixed positions \( x_1, x_2, \cdots, x_k \) after \( M \) steps \((k \leq N)\). This number is given by the correlation functions

\[
I_k^{(\beta)}(x_1, x_2, \cdots, x_k) = \frac{1}{(N-k)!} \sum_{x_{k+1}=-\infty}^{\infty} \sum_{x_{k+2}=-\infty}^{\infty} \cdots \sum_{x_N=-\infty}^{\infty} P_\beta(x_1, x_2, \cdots, x_N) \tag{1.8}
\]

in both the cases \( \beta = 1 \) and \( \beta = 2 \).
In the case with returning walkers \((\beta = 2)\), the evaluation of the correlation functions is relatively easy. Introducing monic orthogonal polynomials
\[ C_j(x) = x^j + \cdots \]
with orthogonality relations
\[ \sum_{x=-\infty}^{\infty} w(x) C_j(x) C_l(x) = \delta_{jl} h_j, \]
we can readily obtain
\[ I_k^{(2)}(x_1, x_2, \cdots, x_k) = C_{MN}^2 \prod_{j=0}^{N-1} h_j \det[K(x_j, x_l)]_{j,l=1,2,\cdots,k} \]
with
\[ K(x, y) = \sqrt{w(x)w(y)} \sum_{j=0}^{N-1} \frac{1}{h_j} C_j(x) C_j(y) \]
\[ = \sqrt{w(x)w(y)} \frac{1}{h_{N-1}} \frac{C_N(x)C_{N-1}(y) - C_{N-1}(x)C_N(y)}{x - y}. \]

We call \( C_j(x) \) symmetric Hahn polynomials and define them in terms of the Hahn polynomials in §2.

The main purpose of this paper is to derive an analogous formula for the correlation functions with no returning constraint \((\beta = 1)\). For that purpose, we again make use of the above symmetric Hahn polynomials to rewrite the correlation functions in the form of a determinant although now with quaternion elements. The Hahn polynomials and their symmetrization are introduced in §2. In §3, we introduce quaternion determinant formulas for the correlation functions \( I_k^{(1)}(x_1, \cdots, x_k) \). In §4, the continuous limit to the Gaussian ensembles is discussed.

## 2 Symmetric Hahn Polynomials

The Hahn polynomials \( Q_n(x) \) are orthogonal polynomials with respect to the weight function \[ w_H(x) = \frac{(a + x)! (L + b - x)!}{a! x! b! (L - x)!} \]
on a discrete measure \((x\) is restricted to be an integer). When the coefficient of the highest order term is fixed as
\[ Q_n(x) = \frac{(n + a + b + 1)_n}{(a + 1)_n (-L)_n} x^n + \cdots, \]
we have the orthogonality relation
\[ \sum_{x=0}^{L} w_H(x) Q_m(x) Q_n(x) = H_n \delta_{mn}, \]
where
\[ H_n = \frac{(-1)^n n!(b + 1)_n (n + a + b + 1)_{L+1}}{L! (2n + a + b + 1) (-L)_n (a + 1)_n}. \]
Here \((a)_n = \Gamma(a + n)/\Gamma(a)\). Moreover there is a dual orthogonality relation (completeness relation)

\[
\sum_{n=0}^{L} \frac{1}{H_n} Q_n(x)Q_n(y) = \frac{\delta_{xy}}{w_H(x)},
\]

which yields the orthogonality relation of the dual Hahn polynomials.

The weight \(L\) is the special case \(a = b = -L - 1\) of (2.1) with shift of coordinate \(x \mapsto x + L/2\). The monic orthogonal polynomials with respect to \((1.6)\), which we have denoted \(C_n(x)\) and termed the symmetric Hahn polynomials, are therefore given in terms of the Hahn polynomials \(Q_n(x)\) by

\[
C_n(x) = \left. \frac{(a+1)n(-L)_n}{(n+a+b+1)_n} Q_n \left( x + \frac{L}{2} \right) \right|_{a=b=-L-1}.
\]

and satisfy the orthogonality relation

\[
\sum_{x=-\infty}^{\infty} w(x) C_j(x) C_l(x) = \delta_{jl} h_j
\]

with

\[
h_j = \frac{j!(2L - 2j + 1)!(2L - 2j)!}{(2L - j + 1)!![(L - j)!!]^4}.
\]

The corresponding dual orthogonality relation reads

\[
\sum_{n=0}^{L} \frac{1}{h_n} C_n(x)C_n(y) = \frac{\delta_{xy}}{w(x)}.
\]

Since the weight function \(w(x)\) defined in \((1.6)\) is symmetric \((w(-x) = w(x))\), \(C_n(x)\) are even or odd polynomials corresponding to the parity of \(n\) \((C_n(-x) = (-1)^n C_n(x))\).

In Ref. [15], recurrence relations for the Hahn polynomials are presented. In the symmetric limit \(a = b = -L - 1\), we find

\[
C_{n+1}(x) = xC_n(x) - \omega_n C_{n-1}(x)
\]

and

\[
-x^2 \left[ C_n(x) - C_n(x-1) \right] = \alpha_n C_{n+1}(x) + \beta_n C_n(x) + \gamma_n C_{n-1}(x),
\]

where

\[
\omega_n = \frac{n(2L - n + 2)(L - n + 1)^2}{4(2L - 2n + 3)(2L - 2n + 1)}.
\]

\[
\alpha_n = -n, \quad \beta_n = -\frac{1}{2} n(2L - n + 1)
\]

and

\[
\gamma_n = -\frac{n(2L - n + 1)(2L - n + 2)(L - n + 1)^2}{4(2L - 2n + 3)(2L - 2n + 1)}.
\]
3 Correlation Functions

In the theory of random real symmetric matrices, it is known that the correlation functions among the eigenvalues are expressed as quaternion determinants \[16, 17, 18\]. Although in most cases the integration method has been applied to continuous eigenvalue distributions, several authors have made attempts to evaluate the correlation functions on discrete measures \[19, 20, 21, 22\]. Here we can employ a similar procedure to rewrite the multiple sum (1.8) in a quaternion determinant form.

3.1 Quaternion Determinant

As the first step let us introduce the quaternion determinant. A quaternion is defined as a linear combination of four basic units \(\{1, e_1, e_2, e_3\}\):

\[
q = q_0 + q \cdot e = q_0 + q_1e_1 + q_2e_2 + q_3e_3, \quad (3.1)
\]

where \(q_0, q_1, q_2\) and \(q_3\) are real or complex numbers. The first part \(q_1\) is called the scalar part of \(q\). The multiplication laws of the four basic units are given by

\[
1 \cdot 1 = 1, \quad 1 \cdot e_j = e_j \cdot 1 = e_j, \quad j = 1, 2, 3; \quad (3.2)
\]

\[
e_1^2 = e_2^2 = e_3^2 = e_1e_2e_3 = -1. \quad (3.3)
\]

Note that the quaternion multiplication is associative but in general not commutative. The dual \(\hat{q}\) of a quaternion \(q\) is defined as

\[
\hat{q} = q_0 - q \cdot e. \quad (3.4)
\]

A matrix \(Q\) with quaternion elements \(q_{jl}\) has a dual matrix \(\hat{Q} = [\hat{q}_{lj}]\). The quaternion units are represented as \(2 \times 2\) matrices

\[
1 \rightarrow \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}, \quad e_1 \rightarrow \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix},
\]

\[
e_2 \rightarrow \begin{bmatrix} 0 & -i \\ i & 0 \end{bmatrix}, \quad e_3 \rightarrow \begin{bmatrix} i & 0 \\ 0 & -i \end{bmatrix}, \quad (3.5)
\]

so that any \(2 \times 2\) matrix with complex elements can be identified as a \(2 \times 2\) representation of a quaternion.

We define a quaternion determinant \(T\text{det}\) of a self dual \(Q\) (i.e., \(Q = \hat{Q}\)) as

\[
T\text{det} \, Q = \sum_P (-1)^{N-l} \prod_{a}^{l} (q_{ab}q_{bc} \cdots q_{da})_{0}. \quad (3.6)
\]

Here \(P\) denotes any permutation of the indices \((1, 2, \cdots, N)\) consisting of \(l\) exclusive cycles of the form \((a \rightarrow b \rightarrow c \rightarrow \cdots \rightarrow d \rightarrow a)\) and \((-1)^{N-l}\) is the parity of \(P\). The subscript \(0\) has a meaning that we take the scalar part of the product over each cycle.
3.2 Skew Orthogonal Polynomials

Next we need to define skew orthogonal polynomials in order to give the quaternion
determinant expressions for the correlation functions. The Schmidt’s orthogonalization
procedure enables us to construct monic polynomials $R_n(x)$ of degree $n$ which satisfy

$$\langle R_{2m}(x), R_{2n+1}(y) \rangle = -\langle R_{2n+1}(x), R_{2m}(y) \rangle = r_m \delta_{mn},$$

$$\langle R_{2m}(x), R_{2n}(y) \rangle = 0, \quad \langle R_{2m+1}(x), R_{2n+1}(y) \rangle = 0,$$

where

$$\langle f(x), g(y) \rangle = \frac{1}{2} \sum_{y=-\infty}^{\infty} \left[ \sum_{x=-\infty}^{y-1} \sqrt{w(x)} f(x) - \sum_{x=y+1}^{\infty} \sqrt{w(x)} f(x) \right] \sqrt{w(y)} g(y).$$

When the weight function $w(x)$ is given by eq. (1.6), the skew orthogonal polynomials
$R_n(x)$ are compactly expressed in terms of the symmetric Hahn polynomials. To show
this we utilize a method analogous to that employed by Nagao and Wadati[23] for the
continuous analogue of (3.8). Let us first note the identity

$$\langle f(x), g(y+1) - g(y) \rangle = -\langle f(x), \frac{\sqrt{w(y)} - \sqrt{w(y-1)}}{\sqrt{w(y)}} g(y) \rangle$$

$$- \frac{1}{2} \sum_{x=-\infty}^{\infty} \sqrt{w(x)} f(x) \left[ \sqrt{w(x)} g(x+1) + \sqrt{w(x-1)} g(x) \right]$$

and make substitutions

$$f(x) = R_n(x),$$

$$g(x) = \left[ \frac{L}{2} - x + 1 \right] \left[ \frac{L}{2} + x \right] C_l(x-1).$$

Noting

$$\frac{\sqrt{w(x)} - \sqrt{w(x-1)}}{\sqrt{w(x)}} = \frac{1 - 2x}{L/2 - x + 1}$$

and

$$\sqrt{w(x)} g(x+1) + \sqrt{w(x-1)} g(x)$$

$$= \sqrt{w(x)} \left[ \frac{L}{2} - x \right] \left[ \frac{L}{2} + x + 1 \right] C_l(x) + \sqrt{w(x)} \left[ \frac{L}{2} + x \right]^2 C_l(x-1)$$

$$= \sqrt{w(x)} [L - 1 + \alpha_l] C_{l+1}(x) + \sqrt{w(x)} \left[ \frac{L(L+1)}{2} + \beta_l \right] C_l(x)$$

$$+ \sqrt{w(x)} [(L-1) \omega_l + \gamma_l] C_{l-1}(x),$$

(3.12)
we obtain
\[
\langle R_n(x), \left[ \frac{L}{2} - y \right] \left[ \frac{L}{2} + y + 1 \right] C_l(y) \rangle - \langle R_n(x), \left[ \frac{L}{2} + y \right]^2 C_l(y - 1) \rangle = -\frac{1}{2} \sum_{x=-\infty}^{\infty} w(x) R_n(x)(L - l - 1)C_{l+1}(x) \quad \text{and} \quad -\frac{1}{4} \sum_{x=-\infty}^{\infty} w(x) R_n(x)(L - l + 1)(L - l)C_{l+1}(x)
\]
\[
+ \frac{1}{8} \sum_{x=-\infty}^{\infty} w(x) R_n(x) \frac{l(2L - l + 2)(L - l + 1)^2(L - l + 2)}{(2L - 2l + 3)(2L - 2l + 1)} C_{l-2}(x).
\] (3.13)

Putting the expansion
\[
R_{2m}(x) = \sum_{j=0}^{m} \alpha_{2m,2j} C_{2j}(x),
\]
\[
R_{2m+1}(x) = \sum_{j=0}^{m} \alpha_{2m+1,2j+1} C_{2j+1}(x)
\] (3.14)

\((\alpha_{jj} = 1)\) into (3.13) and using eqs. (2.7) and (3.7), we arrive at
\[
R_{2m}(x) = C_{2m}(x),
\]
\[
R_{2m+1}(x) = C_{2m+1}(x) - \frac{L - 2m}{L - 2m + 1} \frac{h_{2m}}{h_{2m-1}} C_{2m-1}(x)
\] (3.15)

and
\[
r_m = \frac{1}{4}(L - 2m)h_{2m},
\] (3.16)

where \(h_n\) is given by eq. (2.8).

### 3.3 The Case \(N\) Even

Let us first consider the case \(N\) is even \((N = 2\nu)\). Independent of the functional form of \(w(x)\), the number of paths has a quaternion determinant expression \([17]\)
\[
P_1(x_1, x_2, \cdots, x_N) = C_{MN} 2^\nu \prod_{n=0}^{\nu-1} r_n \mathrm{Tdet} [f(x_j, x_l)]_{j,l=1,2,\cdots,2\nu},
\] (3.17)

where the quaternion elements are represented as
\[
f(x, y) = \begin{bmatrix}
S(x, y) & I(x, y) \\
D(x, y) & S(y, x)
\end{bmatrix}
\] (3.18)

with
\[
S(x, y) = \sqrt{w(y)} \sum_{n=0}^{(N/2)-1} \frac{1}{r_n} \Phi_{2n}(x) R_{2n+1}(y) - \Phi_{2n+1}(x) R_{2n}(y),
\]
\[
D(x, y) = \sqrt{w(y)} \sum_{n=0}^{(N/2)-1} \frac{1}{r_n} [R_{2n}(x) R_{2n+1}(y) - R_{2n+1}(x) R_{2n}(y)],
\]
\[
I(x, y) = -\sum_{n=0}^{(N/2)-1} \frac{1}{r_n} [\Phi_{2n}(x) \Phi_{2n+1}(y) - \Phi_{2n+1}(x) \Phi_{2n}(y)] - \epsilon(x - y)
\] (3.19)
and
\[ \Phi_n(x) = \sum_{y=-\infty}^{\infty} \epsilon(x - y) \sqrt{w(y)} R_n(y). \] (3.20)

Here \( \epsilon(x) \) is defined as
\[ \epsilon(x) = \begin{cases} 
1/2, & x > 0, \\
0, & x = 0, \\
-1/2, & x < 0.
\end{cases} \] (3.21)

Using the skew orthogonality relation (3.7), we can prove the following. Let us define
\[ Q_n(x_1, x_2, \ldots, x_n) = \text{Tdet}[f(x_j, x_l)]_{j,l=1,2,\ldots,n}. \] (3.22)

Then
\[ \sum_{x_n = -\infty}^{\infty} Q_n(x_1, x_2, \ldots, x_n) = (N - n + 1) Q_{n-1}(x_1, x_2, \ldots, x_{n-1}). \] (3.23)

Recursive use of (3.23) on (3.22) leads to
\[ I^{(1)}_{k}(x_1, x_2, \ldots, x_k) = C_{MN} 2^{\nu} \left( \prod_{n=0}^{\nu-1} r_n \right) \text{Tdet}[f(x_j, x_l)]_{j,l=1,2,\ldots,k}. \] (3.24)

Thus the correlation functions are expressed in a quaternion determinant form.

In the continuous case, it is known [24] that for \( w(x) \) a classical weight the summations (3.19) can be simplified. Similar simplifications can be undertaken here. First we note the function \( \Phi_n(x) \) can be rewritten as
\[ \Phi_n(x) = \frac{1}{2} \sum_{y=-\infty}^{\infty} \sum_{z'=-\infty}^{\infty} \sum_{z=-\infty}^{\infty} [\delta_{yz'z'} - \delta_{xz'z}'] \sqrt{w(y)} R_n(y). \] (3.25)

We put the dual orthogonality relation (2.9) and an expansion
\[ C_{2m}(x) = \sum_{j=0}^{m} \beta_{2m} 2_j R_{2j}(x), \]
\[ C_{2m+1}(x) = \sum_{j=0}^{m} \beta_{2m+1} 2_{j+1} R_{2j+1}(x) \] (3.26)
in the above equation to find
\[ \Phi_{2n}(x) = \sqrt{w(x)} \sum_{p=n}^{[\nu-1]/2} \frac{C_{2p+1}(x)}{h_{2p+1}} \beta_{2p+1} 2_{n+1} r_n, \]
\[ \Phi_{2n+1}(x) = -\sqrt{w(x)} \sum_{p=n}^{[\nu]/2} \frac{C_{2p}(x)}{h_{2p}} \beta_{2p} 2_{n} r_n. \] (3.27)

Here \([x]\) is the largest integer not exceeding \( x \). Substituting (3.27) into (3.19) and using (3.15) yields
\[ S(x, y) = \sqrt{w(x)w(y)} \sum_{n=0}^{N-1} \frac{1}{h_n} C_n(x) C_n(y) \]
Making use of

\[ \Phi_{N-2}(x) = \sqrt{w(x)} \left( \sum_{p=N/2}^{[L-1]/2} \beta_{2p+1} \frac{C_{2p+1}(x)}{h_{2p+1}} \right) \]

and \( R_{N-2}(x) = C_{N-2}(x) \), we can easily find

\[ S(x, y) = \sqrt{w(x)w(y)} \sum_{n=0}^{N-2} \frac{1}{h_n} \frac{C_n(x)}{C_n(y)} \]

Note that the first term on the right hand side is precisely \([1.1]\) with \( N \mapsto N - 1 \). This gives a compact expression of \( S(x, y) \) in terms of the symmetric Hahn polynomials.

It should be also noted that, using eq. (3.27), we can derive

\[ \epsilon(y - x) = \sum_{n=0}^{[L-1]/2} \frac{1}{r_n} [\Phi_{2n}(x)\Phi_{2n+1}(y) - \Phi_{2n+1}(x)\Phi_{2n}(y)], \]

which yields

\[ I(x, y) = \sum_{n=N/2}^{[L-1]/2} \frac{1}{r_n} [\Phi_{2n}(x)\Phi_{2n+1}(y) - \Phi_{2n+1}(x)\Phi_{2n}(y)]. \]

To conclude this subsection, let us remark on the "partition function"

\[ I_0^{(1)} = C_M N 2^{\nu} \left( \prod_{n=0}^{\nu-1} r_n \right) \]

\[ = \frac{(N/2)-1}{2} \frac{(2n)![(2L - 4n + 1)(2L - 4n)!]}{2[(2L - 2n + 1)!(L - 2n)]^2}. \]

It can be readily rewritten as

\[ I_0^{(1)} = \prod_{l=0}^{N-1} \frac{1}{(2M + 2l + 1)!} \left( \frac{(2n)![(2M + 4n + 3)(2M + 4n + 2)(2M + 2n + 1)!]}{2[(M + 2n + 1)!]^2} \right), \]

so that

\[ \frac{I_0^{(1)}|_{N \to N+2}}{I_0^{(1)}|_{N \to N}} = \frac{N!(2M + N + 1)!}{(M + N + 1)!(M + N)!} = \prod_{1 \leq i \leq j \leq M} N + i + j + 1. \]
Therefore, noting
\[
I^{(1)}_0\big|_{N\to 2} = \frac{1}{2} \frac{(2M + 2)!}{[(M + 1)!]^2} = \prod_{1 \leq i \leq j \leq M} \frac{i + j + 1}{i + j - 1},
\]  
we obtain
\[
I^{(1)}_0 = \prod_{1 \leq i \leq j \leq M} \frac{N + i + j - 1}{i + j - 1}.
\]  
This expression is already known. It was conjectured by Essam and Guttmann [1] and subsequently proved in Ref. [12] using the correspondence between the paths of vicious walkers and Young tableaux. In the context of Young tableaux, this enumeration problem is reduced to a special case of the result known as Bender-Knuth conjecture.

### 3.4 The Case N Odd

In order to deal with the case \(N\) odd \((N = 2\nu + 1)\), we introduce the polynomials \(R^{\text{odd}}_n(x)\),
\[
R^{\text{odd}}_n(x) = R_n(x) - \frac{s_n}{s_{N-1}} R_{N-1}(x), \quad n = 0, 1, 2, \ldots, N - 2,
\]  
where
\[
s_n = \sum_{x = -\infty}^{\infty} \sqrt{w(x)} R_n(x).
\]  
Then we know from [25] that we can express the number of paths in a quaternion determinant form
\[
P_1(x_1, x_2, \ldots, x_N) = C_{MN} s_{N-1} 2^\nu (\prod_{n=0}^{\nu-1} r_n) \text{det}[f^{\text{odd}}(x_j, x_l)]_{j,l=1,2,\ldots,2\nu+1},
\]  
where the \(2 \times 2\) representations of the quaternion elements are given by
\[
f^{\text{odd}}(x, y) = \begin{bmatrix} S^{\text{odd}}(x, y) & I^{\text{odd}}(x, y) \\ D^{\text{odd}}(x, y) & S^{\text{odd}}(y, x) \end{bmatrix}
\]  
with
\[
S^{\text{odd}}(x, y) = \sqrt{w(y)} \sum_{n=0}^{(N-3)/2} \frac{1}{r_n} [\Phi^{\text{odd}}_{2n}(x) R^{\text{odd}}_{2n+1}(y) - \Phi^{\text{odd}}_{2n+1}(x) R^{\text{odd}}_{2n}(y)] + \frac{1}{s_{N-1}} \sqrt{w(y)} R_{N-1}(y),
\]
\[
D^{\text{odd}}(x, y) = \sqrt{w(x)w(y)} \sum_{n=0}^{(N-3)/2} \frac{1}{r_n} [R^{\text{odd}}_{2n}(x) R^{\text{odd}}_{2n+1}(y) - R^{\text{odd}}_{2n+1}(x) R^{\text{odd}}_{2n}(y)],
\]
\[
I^{\text{odd}}(x, y) = -\sum_{n=0}^{(N-3)/2} \frac{1}{r_n} [\Phi^{\text{odd}}_{2n}(x) \Phi^{\text{odd}}_{2n+1}(y) - \Phi^{\text{odd}}_{2n+1}(x) \Phi^{\text{odd}}_{2n}(y)] + \frac{1}{s_{N-1}} [\Phi_{N-1}(x) - \Phi_{N-1}(y)] - \epsilon(x - y)
\]  
\]
\[ \Phi_n(x) = \sum_{y=-\infty}^{\infty} \epsilon(x - y) \sqrt{w(y)} R_n(y). \]  

(3.43)

The skew orthogonality relation among \( R_n(x) \) enables us to carry out the multiple integration one by one as before and arrive at

\[ I^{(1)}_k(x_1, x_2, \cdots, x_k) = C_{MN} s_{N-1} 2^{-n} (\prod_{n=0}^{\nu-1} r_n) \text{det}[f^{\text{odd}}(x_j, x_l)]_{j,l=1,2,\cdots,k}. \]  

(3.44)

This is the quaternion determinant formula for the correlation functions in the case \( N \) odd.

Inserting (3.38) into (3.42), we can rewrite \( S^{\text{odd}}(x, y) \) as

\[ S^{\text{odd}}(x, y) = S(x, y)|_{N \to N-1} + \frac{\sqrt{w(y)}}{s_{N-1}} R_{N-1}(y) \]

\[ + \frac{\sqrt{w(y)}}{s_{N-1}} \left( \sum_{n=0}^{(N-3)/2} \frac{s_{2n}}{r_n} \right) [\Phi_{2n+1}(x) R_{N-1}(y) - \Phi_{N-1}(x) R_{2n+1}(y)]. \]  

(3.45)

If we note from eqs. (3.19) and (3.30) that

\[ \sqrt{w(y)} \sum_{n=0}^{(N-3)/2} \frac{s_{2n}}{r_n} R_{2n+1}(y) = 2 \lim_{x \to \infty} S(x, y)|_{N \to N-1} \]

\[ = \sqrt{w(y)} \frac{s_{N-3}}{r_{(N-3)/2}} C_{N-2}(y), \]  

(3.46)

this can be further simplified as

\[ S^{\text{odd}}(x, y) = S(x, y)|_{N \to N-1} + \frac{\sqrt{w(y)}}{s_{N-1}} C_{N-1}(y) \]

\[ + \sqrt{w(y)} \frac{s_{N-3}}{s_{N-1} r_{(N-3)/2}} [\varphi_{N-2}(x) C_{N-1}(y) - \varphi_{N-1}(x) C_{N-2}(y)], \]  

(3.47)

where

\[ \varphi_n(x) = \sum_{y=-\infty}^{\infty} \epsilon(x - y) \sqrt{w(y)} C_n(y). \]  

(3.48)

We remark that \( s_n \) as defined by (3.39) can be written in a closed form. For this purpose we require the generating functions for the dual Hahn polynomials in Ref. [14]. One of them can be written in terms of the Hahn polynomials as

\[ (1 - t)^{L-n} {2F}_1(-n, -n - b; a + 1; t) = \sum_{x=0}^{L} \frac{(-L)_x}{x!} Q_n(x) t^x. \]  

(3.49)

Putting \( t = -1 \) and \( a = b = -L - 1 \), we deduce

\[ s_n = \begin{cases} 2L-n \frac{n!(L-(n/2))!(2L-2n+1)!}{(n/2)!(2L-n+1)!(L-n)!2}, & n \text{ even,} \\ 0, & n \text{ odd.} \end{cases} \]  

(3.50)
As an application, since the ”partition function” \( I_{0}^{(1)} \) can be rewritten as

\[
I_{0}^{(1)} = C_{MN} s_{N-1} 2^{\nu} \left( \prod_{n=0}^{\nu-1} r_{n} \right)
\]

\[
= s_{N-1} M! \prod_{i=0}^{N-2} \frac{1}{(2M + 2i + 3)!} \prod_{n=0}^{(N-3)/2} \frac{(2n)!(2M + 4n + 5)!(2M + 4n + 4)!(2M + 2n + 3)!}{2[(M + 2n + 2)!]^{2}},
\]

we can utilize the summation formula (3.50) to derive

\[
\left. \frac{I_{0}^{(1)}}{N \to N+2} \right| \frac{I_{0}^{(1)}}{N \to N} = \frac{N!(2M + N + 1)!}{(M + N + 1)!(M + N)!} = \prod_{1 \leq i \leq j \leq M} \frac{N + i + j + 1}{N + i + j + 1},
\]

which leads to

\[
I_{0}^{(1)} = \prod_{1 \leq i \leq j \leq M} \frac{N + i + j + 1}{i + j + 1}.
\]

This is again a known result [12], related to the Bender-Knuth conjecture in the theory of Young tableaux.

4 Continuous Limit

In this last section we discuss the continuous limit to the Gaussian ensembles. Some aspects of this limit for the \( \beta = 2 \) Hahn ensemble [11] have been discussed in [8]. Let us first define

\[
\bar{w}(x) = \frac{\pi L}{2} \left( \frac{L!}{2L} \right)^{2} w \left( \frac{\sqrt{L}}{2} x \right).
\]

Then the Stirling’s asymptotic formula for the Gamma function yields

\[
w^{(G)}(x) = \lim_{L \to \infty} \bar{w}(x) = e^{-x^{2}}.
\]

Hence we can say that the vicious walker problem is reduced to the Gaussian ensembles of random matrices in this scaling limit, with (1.4) identical to the probability density for the Gaussian orthogonal ensemble.

We further define monic polynomials

\[
\bar{C}_{n}(x) = \left( \frac{2}{\sqrt{L}} \right)^{n} C_{n} \left( \frac{\sqrt{L}}{2} x \right)
\]

and

\[
\bar{C}_{n}^{(G)}(x) = \lim_{L \to \infty} \bar{C}_{n}(x).
\]

Then, from eq.(2.7), it can be readily seen that

\[
\frac{2}{\sqrt{L}} \sum_{x=\infty}^{\infty} \bar{w} \left( \frac{2}{\sqrt{L}} x \right) \bar{C}_{j} \left( \frac{2}{\sqrt{L}} x \right) \bar{C}_{l} \left( \frac{2}{\sqrt{L}} x \right) = \frac{\pi L}{2} \left( \frac{L!}{2L} \right)^{2} \left( \frac{2}{\sqrt{L}} \right)^{2j+1} h_{j}\delta_{jl}.
\]
In the asymptotic limit $L \to \infty$, the infinite sum in the above equation becomes a Riemann integral and results in
\[
\int_{-\infty}^{\infty} e^{-x^2} C_j^{(G)}(x) C_l^{(G)}(x) \, dx = \sqrt{\pi} \frac{j!}{2^j} \delta_{jl},
\]
which means
\[
C_n^{(G)}(x) = \frac{1}{2^n} H_n(x),
\]
where $H_n(x)$ is the Hermite polynomial. The corresponding skew orthogonal polynomials are introduced as
\[
\bar{R}_n(x) = \left(\frac{2}{\sqrt{L}}\right)^n R_n\left(\frac{\sqrt{L}}{2} x\right)
\]
and
\[
R_n^{(G)}(x) = \lim_{L \to \infty} \bar{R}_n(x).
\]
Then we can similarly find
\[
\langle R_{2m}^{(G)}(x), R_{2n+1}^{(G)}(y) \rangle_G = -\langle R_{2n+1}^{(G)}(x), R_{2m}^{(G)}(y) \rangle_G = \sqrt{\pi} \frac{(2m)!}{2^{2m}} \delta_{mn},
\]
\[
\langle R_{2m}^{(G)}(x), R_{2m}^{(G)}(y) \rangle_G = 0, \quad \langle R_{2m+1}^{(G)}(x), R_{2n+1}^{(G)}(y) \rangle_G = 0,
\]
where
\[
\langle f(x), g(y) \rangle_G = \frac{1}{2} \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} dy \left[ \int_y^\infty dx \ e^{-x^2/2} f(x) - \int_{-\infty}^y dx \ e^{-x^2/2} f(x) \right] e^{-y^2/2} g(y).
\]
The scaling limit of eq.(3.15)
\[
R_{2m}^{(G)}(x) = C_{2m}^{(G)}(x),
\]
\[
R_{2m+1}^{(G)}(x) = C_{2m+1}^{(G)}(x) - mC_{2m-1}^{(G)}(x)
\]
reproduces the known compact expression of $R_n^{(G)}(x)$ (see e.g. [18, 23, 24]).

The Gaussian ensembles have been thoroughly studied by many authors and asymptotic properties in the limit $N \to \infty$ are well known. In the vicious walker problem, Gaussian results are valid only when we first take the limit $M \to \infty$ in the above way and then analyze the large $N$ asymptotic behavior. Other scaling limits with respect to $M$ and $N$ are not so well understood and should be further investigated in future works.

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