BLOWUP RESULTS AND CONCENTRATION IN FOCUSING SCHRÖDINGER-HARTREE EQUATION

YINGYING XIE, JIAN SU AND LIQUAN MEI∗

School of Mathematics and Statistics
Jiaotong University
Xi’an, Shanxi 710049, China

(Communicated by Juncheng Wei)

Abstract. This paper is concerned with the Cauchy problem of the Schrödinger-Hartree equation. Applying the profile decomposition of bounded sequence in $H^1(\mathbb{R}^N) \cap H^{S_c}(\mathbb{R}^N)$ and corresponding variational structure, a refined Gagliardo-Nirenberg inequality is established and the sharp constant for this inequality is deduced. Secondly, via construction and analysis of some invariant manifolds, we derive a different criterion of global existence and blowup results. Under the discussion of Bootstrap argument, we additionally obtain other sufficient condition for global existence. Finally, A compactness result is applied to show that the blowup solutions with bounded $H^{S_c}$ norm definitely have concentration properties related to a fixed $H^{S_c}$ norm of certain standing waves.

1. Introduction. In the study of modern mathematical physics, the mathematical basis of quantization theory is a crucial subject. Nonlinear wave system with nonlinear Schrödinger equation as its core is an important support of quantization theory and a basic mathematical model in quantum mechanics. When the potential field depends on the wave function, the nonlinear Schrödinger equation is derived in the general form of

$$iu_t + \Delta u + f(u) = 0, \quad (t, x) \in [0, T) \times \mathbb{R}^N, \ 0 < T \leq +\infty. \quad (\text{NLSE})$$

If the nonlinearity $f(u)$ is given by $|u|^{q-1} u \ (1 < q < \frac{N+2}{N-2})$, NLSE is called the classical nonlinear Schrödinger equation, which can describe the propagation of optical pulses in dispersive and nonlinear media, self-trapping in nonlinear optics [12] and Langmuir wave in plasma physics [25]. In recent decades, a series of important advances have been made in classical nonlinear Schrödinger equation, such as the local well-posedness for Cauchy problem [10], the existence and asymptotic behavior of global solutions [10, 19, 21], the existence and stability of standing waves [2, 20] as well as blowup of solutions in finite time [11, 17, 24]. If the nonlinearity term $f(u)$ is given by $(\frac{1}{|x|^\gamma} * |u|^2)u \ (0 < \gamma < N)$, NLSE models the time evolution of nonrelativistic bosonic system in the mean field approximation of multi-body interaction [8],

2010 Mathematics Subject Classification. Primary: 35Q55; Secondary: 74H35.

Key words and phrases. Schrödinger-Hartree equation, profile decomposition, Gagliardo-Nirenberg inequality, bootstrap argument, concentration properties.

The work is partially supported by Science Challenge Project, No. TZ2016002.

* Corresponding author: Liquan Mei.
which is called the nonlinear Hartree equation and its research for Cauchy problem can be found in [1,3,15,22].

In this paper, we consider a more generalized nonlinear Schrödinger equation, that is the focusing Schrödinger-Hartree equation

\[
\begin{align*}
    iu_t + \Delta u + (K_\alpha * |u|^p)|u|^{p-2} u &= 0, \\
    u(0, x) = u_0, \quad (t, x) \in [0, T) \times \mathbb{R}^N.
\end{align*}
\]

(1)

Here \( u = u(t, x) : [0, T) \times \mathbb{R}^N \to \mathbb{C} \ (0 < T \leq +\infty) \) is a unknown smooth function, \( i \) is the imaginary unit, \( \triangle \) is the Laplace operator on \( \mathbb{R}^N \) and \( p \) is a real constant satisfying \( 2 < p \leq \frac{N+\alpha}{N-2} \) \((0 < \alpha < N)\). Moreover, \( K_\alpha : \mathbb{R}^N \to \mathbb{R} \) is the Riesz potential denoted by

\[
K_\alpha(x) = \frac{\Gamma(\frac{N-\alpha}{2})}{\Gamma(\frac{\alpha}{2}) 2^{\alpha} \pi^{\frac{N}{2}} |x|^{N-\alpha}},
\]

where \( \Gamma \) represents the Gamma function.

Eq. (1) is also called the nonlinear Choquard or Choquard-Pekar equation. When \( p = 2 \), Eq. (1) equates to the nonlinear Hartree equation. When \( p = 2, N = 3 \) and \( \alpha = 2 \), the equation was first proposed by Pekar to describe the quantum mechanics of a stationary polaron [18]. Later, Choquard [14] used Eq. (1) to depict a trapped electron in an approximation to Hartree-Fock theory of one-component plasma.

There have been some studies on Eq. (1) itself. When \( \alpha = 2, 2 < p \leq \frac{N+\alpha}{N-2} \) \((0 < \alpha < N)\), the researchers in [16] established the existence, regularity and positivity of ground states, and proved in addition the existence of solutions was sharp, namely that if \( p \leq 1 + \frac{\alpha}{N} \) or \( p \geq \frac{N+\alpha}{N-2} \), Eq. (1) did not exist any nontrivial sufficient regular variational solution. Particularly, the orbital stability of ground states applied variational method was investigated in [23]. When \( 1 + \frac{\alpha}{N} < p < 1 + \frac{2+\alpha}{N} \) with \( 0 < \alpha < N \), the authors in [6] considered the local and global well-posedness, the \( H^2 \) regularity, as addendum of the mass concentration of blowup results in the the case of \( p = 1 + \frac{2+\alpha}{N} \) for Eq. (1) with a harmonic potential \(|x|^2 u\). Based on the profile decomposition of bounded sequences in \( H^s \) and variational methods, Feng and Zhang [7] studied the orbital stability of standing waves for the fractional Schrödinger-Hartree equation with a \( L^2 - critical \) nonlinearity \(|u|^{\frac{4}{N}} u\).

In this paper, we expect to find some conditions of the global existence and blowup solutions by means of the energy-mass control for Eq. (1), under the assumption of \( 1 + \frac{2+\alpha}{N} < p < \frac{N+\alpha}{N-2} \) \((0 < \alpha < N)\), which can be considered as \( H^1 - subcritical \) case. Furthermore, the dynamical behavior of blowup solutions should be investigated, especially the concentration properties. Precisely speaking, using the profile decomposition of bounded sequence in \( \dot{H}^1(\mathbb{R}^N) \cap H^{S_c}(\mathbb{R}^N) \) and corresponding variational structure, a refined Gagliardo-Nirenberg inequality is established and the sharp constant for this inequality is deduced. A compactness result is applied to show that the blowup solutions with bounded \( \dot{H}^{S_c} \) norm definitely have concentration properties related to a fixed \( \dot{H}^{S_c} \) norm of certain standing waves. Moreover, via construction and analysis of some invariant manifolds, we derive a different criterion of global existence and blowup for Eq. (1), which can be
extended to a sharp criterion. Under the discussion of Bootstrap argument, we additionally obtain the other sufficient condition for global existence.

**Notation.** Throughout this paper, we use the following notation. \( 2^* = \frac{2N}{N-2} \), \( S_c = \frac{N}{2} - \frac{2+\alpha}{2p-2} \), \( P_c = \frac{2N}{N-2S_c} = \frac{2N(p-1)}{2+\alpha} \). \( H^1(\mathbb{R}^N) = W^{1,2}(\mathbb{R}^N) \) denotes the standard Sobolev space. For real constant \( s \), we define the pseudo-differential operator \((-\triangle)^s\) as \( \mathcal{F}[(-\triangle)^s u](\xi) \equiv |\xi|^{2s}\mathcal{F}[u](\xi) \), which directly used to define the homogeneous Sobolev space \( H^s(\mathbb{R}^N) := \left\{ u \in S'(\mathbb{R}^N) : \int_{\mathbb{R}^N} |\xi|^{2s}|\mathcal{F}[u](\xi)|^2d\xi < +\infty \right\} \) with its norm \( \|u\|_{H^s(\mathbb{R}^N)} = \|(-\triangle)^{\frac{s}{2}} u\|_{L^2(\mathbb{R}^N)} \), where \( \mathcal{F}[u] \) represents the Fourier transform of \( u \) on \( \mathbb{R}^N \), and \( S' \) stands for the space of tempered distributions. Without loss of the simplicity, \( C > 0 \) will represent different positive constants when it does not create any confusion. We abbreviate \( L^q(\mathbb{R}^N) \), \( H^1(\mathbb{R}^N) \), \( H^s(\mathbb{R}^N) \) and \( \int_{\mathbb{R}^N} dx \) as \( L^q \), \( H^1 \), \( \dot{H}^s \) and \( \int dx \). Moreover, we define \( \Sigma := \left\{ u \in H^1 | \int_{\mathbb{R}^N} |x|^2|u|^2dx < +\infty \right\} \).

### 2. Preliminaries.

#### 2.1. Some known results.

In this subsection, we will provide some relevant pre-liminaries. First of all, by the similar discussion as that in [6], we derive the following local well-posedness result of Cauchy problem (1).

**Lemma 2.1.** ([4]) Suppose that \( 2 \leq p < \frac{N+\alpha}{N-2} \). Then there exists a time \( T = T(\|u_0\|_{H^1}) \) such that Eq. (1) has a unique solution \( u(t,x) \) in \( C([0,T);H^1) \) for any initial data \( u_0 \in H^1 \). \( T \) satisfies the property: either \( T = +\infty \) or else \( T < +\infty \) and \( \lim_{t \to T^-} \|\nabla u(t,x)\|_{L^2}^2 = +\infty \). In addition, for all \( t \in [0,T) \), the following conservation laws of \( u(t,x) \) hold.

- **(i) Conservation of mass:**
  \[
  M(u) = \int |u(t,x)|^2dx = M(u_0).
  \]

- **(ii) Conservation of mass:**
  \[
  E(u) = \int |\nabla u(t,x)|^2dx - \frac{1}{p} \int (K_\alpha * |u(t,x)|^p)|u(t,x)|^pdx = E(u_0).
  \]

Weinstein [24] provided the existence of blowup solutions, which supplemented the result [11] due to convexity method. If \( u \in \Sigma \), then \( \|u\|_{L^2}^2 \leq \frac{C}{N} \|xu\|_{L^2}^2 \|\nabla u\|_{L^2}^2 \).

That is, to prove the blowup of solutions is to prove the variance
\[
J(t) = \int |x|^2|u(t,x)|^2dx
\]
vanishes in a finite time \( T > 0 \), which can be replaced by showing that \( J(t) \) is concave.

**Lemma 2.2.** Suppose that \( 2 \leq p < \frac{N+\alpha}{N-2} \). Let \( u_0 \in \Sigma \) and \( u(t,x) \) be a solution of the Eq. (1). Then the virial identity is given by
\[
J''(t) = 8 \int |\nabla u(t,x)|^2dx - \frac{4Np - 4N - 4\alpha}{p} \int (K_\alpha * |u(t,x)|^p)|u(t,x)|^pdx.
\]

Next, we recall the following Hardy-Littlewood-Sobolev inequality.

**Lemma 2.3.** ([13]) Let \( 0 < \lambda < N \) and assume that \( f \in L^r \), \( g \in L^s \) with \( 1 < r, s < \infty \) and
\[
\frac{\lambda}{N} + \frac{1}{r} + \frac{1}{s} = 2,
\]
Let Lemma 2.4. Cauchy problem (1), which is a bridge for constructions of the global existence and blowup to the of bounded sequence in \(\dot{\mathcal{H}}\) refined Gagliardo-Nirenberg inequality based on the profile decomposition principle have \(\dot{\mathcal{H}}\) A refined Gagliardo-Nirenberg inequality will be a challenge due to the strong singularity of (\(\dot{\mathcal{H}}\)) conjecture the results of Lemma 2.1 also stand for \(N\) then \[ \int \int \]

5004 YINGYING XIE, JIAN SU AND LIQUAN MEI

is given by

Then the best constant \(C_\alpha > 0\) of the Gagliardo-Nirenberg inequality

\[
\int (K_\alpha * |u|^p)|u|^p dx \leq C_\alpha \left( \int |\nabla u|^2 dx \right)^{\frac{2p-Np-N+\alpha}{2}} \left( \int |u|^2 dx \right)^{\frac{2p-Np-N+\alpha}{2}}
\]

is given by

\[
C_\alpha = \frac{2p}{2p-Np+N+\alpha} \left( \frac{Np-N-\alpha}{2p-Np+N+\alpha} \right)^{\frac{2p-Np-N+\alpha}{2}} \|Q\|_{L^2}^{-2}\|Q\|_{L^2}^{-2p}.
\]

2.2. A refined Gagliardo-Nirenberg inequality. In this subsection, we obtain a refined Gagliardo-Nirenberg inequality based on the profile decomposition principle of bounded sequence in \(\dot{\mathcal{H}} \cap \dot{\mathcal{H}}^S\), which is given in [27]. It is remarkable that we have \(\dot{\mathcal{H}} \hookrightarrow L^{\frac{2p}{N-2}}\) and \(\dot{\mathcal{H}}^S \hookrightarrow L^p\) by Sobolev imbeddings.

Lemma 2.5. For a bounded sequence \(\{u_n\}_{n=1}^\infty\) in \(\dot{\mathcal{H}} \cap \dot{\mathcal{H}}^S\), there exists a subsequence of \(\{u_n\}_{n=1}^\infty\) (still written by itself), a sequence of functions \(\{U^i\}_{i=1}^\infty \subset \dot{\mathcal{H}} \cap \dot{\mathcal{H}}^S\) and for any \(i \geq 1\), a family of sequences \(\{x_n^i\}_{i=1}^\infty \subset \mathbb{R}^N\) such that:

(i) There stands the following pairwise orthogonality as \(n \to +\infty\):

\[
\forall i \neq j, |x_n^i - x_n^j| \to +\infty.
\]

(ii) For every \(m \geq 1\), one has

\[
u_n(x) = \sum_{i=1}^m U^i(x - x_n^i) + r_n^m(x)
\]

with \(\limsup_{n \to \infty} \|r_n^m\|_{L^q} \to 0\) as \(m \to \infty\) for any \(q \in (P_c, 2^*)\). Furthermore, one has

\[
\|\nabla u_n\|_{L^2}^2 = \sum_{i=1}^m \|\nabla U^i\|_{L^2}^2 + \|\nabla r_n^m\|_{L^2}^2 + o_n(1)
\]

and

\[
u_n^2 = \sum_{i=1}^m \|U^i\|_{\dot{\mathcal{H}}^S}^2 + \|r_n^m\|_{\dot{\mathcal{H}}^S}^2 + o_n(1),
\]

where \(o_n(1) \to 0\) as \(n \to +\infty\).
When \( 1 + \frac{2 + \alpha}{N} < p < \frac{N + \alpha}{N - 2} \), we have \( P_c = \frac{2N}{N - 2} < \frac{2N}{N + \alpha} < 2^* = \frac{2N}{N - 2} \). For \( \frac{2N}{N - 2} = \frac{\theta}{p^*} + 1 - \frac{\theta}{2p^*} \), where \( \theta = \frac{2p(N + \alpha)}{2p + (N + \alpha)} = \frac{N - 2}{p} \), applying the Hardy-Littlewood-Sobolev inequality, interpolation inequality and Sobolev imbeddings, we can deduce the following Gagliardo-Nirenberg inequality in \( \dot{H}^1 \cap \dot{H}^S \):

\[
\int (K_\alpha * |u|^p) |u|^p dx \leq C \|u\|_{\dot{H}^{2N/\alpha}}^{2p} \leq C \|u\|_{L^{2\theta}}^{2p(1-\theta)} \leq C_\alpha \|u\|_{L^{2\theta}}^{2p-2} \|\nabla u\|_{L^2}^2.
\]

Drawing lessons from Weinstein [24], the best constant \( C_\alpha^* \) can be derived by studying the existence of the minimizer of the functional structure

\[
J_\alpha(u) = \frac{\|u\|_{L^{2\theta}}^{2p-2} \|\nabla u\|_{L^2}^2}{\int (K_\alpha * |u|^p) |u|^p dx}.
\]

More precisely, we have the following proposition.

**Proposition 1.** Let \( 1 + \frac{2 + \alpha}{N} < p < \frac{N + \alpha}{N - 2} \). Then the best constant \( C_\alpha^* \) is given by

\[
C_\alpha^* = p \|W\|_{L^{2-2p}}^{2-2p},
\]

where \( W \) is the ground state of the following elliptic equation

\[
-\Delta u + (-\Delta)^S u - (K_\alpha * |u|^p)|u|^{p-2}u = 0.
\]

**Proof.** Considering the following variational structure

\[
J := \inf \{J_\alpha(v), v \in \dot{H}^1 \cap \dot{H}^S, v \neq 0 \}.
\]

Obviously, \( J_\alpha(v) \) satisfies the scaling properties. Let \( u = \mu v(\lambda x) \), \( \mu, \lambda > 0 \), which implies

\[
\|u\|_{\dot{H}^{2\theta}}^2 = \mu^2 \lambda^{2s-N} \|v\|_{\dot{H}^S}^2, \quad \|\nabla u\|_{L^2}^2 = \mu^2 \lambda^{2s-N} \|\nabla v\|_{L^2}^2,
\]

\[
\int (K_\alpha * |u|^p) |u|^p dx = \mu^{2p} \lambda^{\alpha-N} \int (K_\alpha * |v|^p) |v|^p dx.
\]

Let

\[
\mu = \frac{\|v\|_{\dot{H}^S}^{2s-N}}{\|\nabla v\|_{L^2}^{2s-N}}, \quad \lambda = \left( \frac{\|v\|_{\dot{H}^S}}{\|\nabla v\|_{L^2}} \right)^{\frac{1}{s-N}}.
\]

We have \( \|u\|_{\dot{H}^S} = \|\nabla u\|_{L^2} = 1 \). Therefore, choose a minimizing sequence \( \{u_n\}_{n=1}^\infty \subset \dot{H}^1 \cap \dot{H}^S \) of variational structure (11) such that

\[
\|u_n\|_{\dot{H}^S} = \|\nabla u_n\|_{L^2} = 1,
\]

\[
J_\alpha(u_n) = \frac{1}{\int (K_\alpha * |u_n|^p) |u_n|^p dx} \to J, \text{ as } n \to \infty.
\]

Obviously, it follows (14) that \( \{u_n\}_{n=1}^\infty \) is bounded in \( \dot{H}^1 \cap \dot{H}^S \). Applying Lemma 2.5, \( u_n \) can be decomposed by

\[
u_n(x) = \sum_{i=1}^m U^i(x - x_n^i) + r_n^m(x)
\]

\[
= \sum_{i=1}^m U^i + r_n^m.
\]
Moreover, we have
\[
\sum_{i=1}^{m} ||U_i^j||_{H^s_x}^2 \leq ||u_n||_{H^{s_c}}^2 = 1, \quad \sum_{i=1}^{m} ||\nabla U_i^j||_{L^2}^2 \leq ||\nabla u_n||_{L^2}^2 = 1. \tag{17}
\]

On the other hand, we can deduce that
\[
\int \left( K_\alpha \ast \left| \sum_{i=1}^{m} U_i^j \right|^p \right) \left| \sum_{i=1}^{m} U_i^j \right|^p \, dx = \sum_{i=1}^{m} \int (K_\alpha \ast |U_i^j|^p) |U_i^j|^p \, dx + o_n(1), \tag{18}
\]
where \( o_n(1) \to 0 \) as \( n \to \infty \). Indeed, using the elementary inequality
\[
\left| \sum_{i=1}^{m} A_i^{q} - \sum_{i=1}^{m} |A_i|^q \right| \leq C \sum_{i \neq j} |A_i||A_j|^{q-1},
\]
we have
\[
\begin{align*}
\left| \sum_{i=1}^{m} U_i(x - x_n^i) \right|^p \left| \sum_{i=1}^{m} U_i(y - x_n^i) \right|^p \, dxdy \\
- \sum_{i=1}^{m} \int \int \frac{|U_i(x - x_n^i)|^p |U_i(y - x_n^i)|^p}{|x - y|^{N-\alpha}} \, dxdy \\
\leq C \left( \sum_{i \neq j} \int \int \frac{|U_i(x - x_n^i)||U_j(x - x_n^j)|^{p-1} |U_k(x - x_n^k)|^p}{|x - y|^{N-\alpha}} \, dxdy \\
+ \sum_{i \neq j} \int \int \frac{|U_i(y - x_n^i)||U_j(y - x_n^j)|^{p-1} |U_k(y - x_n^k)|^p}{|x - y|^{N-\alpha}} \, dxdy \\
+ \sum_{i \neq j} \int \int \frac{|U_i(x - x_n^i)||U_j(y - x_n^j)|^p}{|x - y|^{N-\alpha}} \, dxdy \right).
\end{align*}
\]

By the pairwise orthogonality of the family \( \{x_n^i\}_{i=1}^{\infty} \) and Hardy-Littlewood-Sobolev inequality, we have
\[
\begin{align*}
\sum_{i \neq j} \int \int \frac{|U_i(x - x_n^i)||U_j(x - x_n^j)|^{p-1} |U_k(x - x_n^k)|^p}{|x - y|^{N-\alpha}} \, dxdy \\
\leq \sum_{i \neq j} \left\| |U_n^i||U_j^j|^{p-1} \right\|_{L^{\frac{2N}{N-\alpha}}} \left\| \sum_{k=1}^{m} U_k^k \right\|_{L^{\frac{2N}{N-\alpha}}} \to 0, \text{ as } n \to \infty.
\end{align*}
\]

Similarly, the second term on the right side can be estimated. Finally,
\[
\sum_{i \neq j} \int \int \frac{|U_i(x - x_n^i)|^p |U_j(y - x_n^j)|^p}{|x - y|^{N-\alpha}} \, dxdy \\
= \sum_{i \neq j} \int \int \frac{|U_i(x)|^p |U_j(y)|^p}{|x - y - x_n^i + x_n^j|^{N-\alpha}} \, dxdy \\
\leq \sum_{i \neq j} \frac{C}{|x_n^i - x_n^j|^{N-\alpha}} \left\| U_i \right\|_{L^p}^p \left\| U_j \right\|_{L^p}^p \\
\to 0, \text{ as } n \to \infty.
\]
Thus, we have (18). Consequently, we deduce that as $n \to \infty$ and $m \to \infty$,

$$
\sum_{i=1}^{m} \int (K_{\alpha} * |U_{n}^{i}|^p |U_{n}^{i}|^{p} dx \to \frac{1}{J}.
$$

(19)

Next, we show that the existence of minimizer for variational structure (11). From the definition of $J$, for each $1 \leq i \leq m$, $U^{i}$ satisfies

$$
\int \left( K_{\alpha} * |U^{i}|^p \right) |U^{i}|^p dx \leq \frac{1}{J} \| U^{i} \|_{H^{s}}^{2p-2} \| \nabla U^{i} \|_{L^2}. \tag{20}
$$

It follows from (17) that there exists $i_{0} \geq 1$ such that $\| U^{i_{0}} \|_{H^{s}} \geq 1$, which implies $\| U^{i_{0}} \|_{H^{s}} = 1$. In other words, there exists only one term $U^{i_{0}} \neq 0$ such that

$$
\| U^{i_{0}} \|_{H^{s}} = \| \nabla U^{i_{0}} \|_{L^2} = 1, \quad \int \left( K_{\alpha} * |U^{i_{0}}|^p \right) |U^{i_{0}}|^p dx = \frac{1}{J}. \tag{22}
$$

Thus, $U^{i_{0}}$ is a minimizer of $J_{\alpha}$. Based on the standard variational principle, the minimizer satisfies

$$
\frac{dJ_{\alpha}(U^{i_{0}} + \tau u)}{d\tau} \bigg|_{\tau=0} = 0, \text{ for all } u \in C^{\infty}_{c}. \tag{23}
$$

After some direct computations, one has

$$
\frac{d}{d\tau} \| U^{i_{0}} + \tau u \|^{2p-2}_{H^{s}} \bigg|_{\tau=0} = (2p - 2) \text{Re} \int (-\Delta)^{S_{\alpha}} U^{i_{0}} \bar{u} dx,
$$

and

$$
\frac{d}{d\tau} \| \nabla (U^{i_{0}} + \tau u) \|^{2}_{L^2} \bigg|_{\tau=0} = 2 \text{Re} \int \Delta U^{i_{0}} \bar{u} dx
$$

and

$$
\frac{d}{d\tau} \int (K_{\alpha} * |U^{i_{0}} + \tau u|^p) |U^{i_{0}} + \tau u|^p dx \bigg|_{\tau=0} = 2p \text{Re} \int (K_{\alpha} * |U^{i_{0}}|^p) |U^{i_{0}}|^{p-2} U^{i_{0}} \bar{u} dx.
$$

Then we deduce easily that $U^{j_{0}}$ is a solution of the following elliptic equation

$$
-\Delta u + (p - 1)(-\Delta)^{S_{\alpha}} u - pJ(K_{\alpha} * |u|^p)|u|^{p-2} u = 0.
$$

Setting $U^{j_{0}} = \mu W(\lambda x)$, where

$$
\mu = (p - 1)^{\frac{N-2S_{\alpha}}{4-2S_{\alpha}}} \left( \frac{1}{pJ} \right)^{\frac{2p-2}{2-2S_{\alpha}}} , \quad \lambda = (p - 1)^{\frac{3}{2-2S_{\alpha}}},
$$

Then, $W$ satisfies Eq. (13) and the best constant $C_{\alpha}$ is derived. \qed
Theorem 2.6. Let $1 + \frac{2 + \alpha}{N} < p < \frac{N + \alpha}{N - 2}$. Suppose that $u_0 \in \dot{H}^1 \cap \dot{H}^{S_c}$ and $u(t, x)$ is the corresponding solution to Eq. (1) such that $\sup_{t \in [0, T]} \|u\|_{H^{S_c}} < \|W\|_{H^{S_c}}$, where $W$ is the solution of the elliptic equation (13). Then $u(t, x)$ exists globally.

Proof. By the conservation law of energy and inequality (11), we immediately have

$$E(u) = \int |\nabla u|^2 dx - \frac{1}{p} \int (K_\alpha \ast |u|^p) |u|^p dx \geq \|\nabla u\|_{L^2}^2 \left(1 - \left(\|W\|_{H^{S_c}}\right)^{2p-2}\right).$$

Together with the assumption $\sup_{t \in [0, T]} \|u\|_{H^{S_c}} < \|W\|_{H^{S_c}}$, we deduce that $\|\nabla u(t, x)\|_{L^2}^2$ is bounded for all time. In consideration of the local well-posedness results, we derive that the solution $u(t, x)$ exists globally.

3. Thresholds of global existence and blowup.

3.1. A criterion based on energy-mass control. In this subsection, we will derive a new criterion of the global existence and blowup based on energy-mass control and some invariant manifolds generated by the Cauchy problem (1). For convenience, we define

$$G := \left\{ u \in H^1 : \Phi(\|u\|_{H^1}^2) < E(u) + M(u) < \frac{p - 1}{p} \left(\frac{1}{C_\alpha}\right)^{\frac{1}{p-1}}, \|u\|_{H^1}^2 < \left(\frac{1}{C_\alpha}\right)^{\frac{1}{p-1}} \right\},$$

$$B := \left\{ u \in H^1 : \Phi(\|u\|_{H^1}^2) < E(u) + M(u) < \frac{Np - N - \alpha - 2}{Np - N - \alpha} \left(\frac{2p}{Np - N - \alpha}\right)^{\frac{1}{p-1}} \left(\frac{1}{C_\alpha}\right)^{\frac{1}{p-1}}, \|u\|_{H^1}^2 > \left(\frac{2p}{Np - N - \alpha}\right)^{\frac{1}{p-1}} \left(\frac{1}{C_\alpha}\right)^{\frac{1}{p-1}} \right\}.$$

Here $\Phi(z) := z - \frac{C_\alpha}{p} z^p$, $z > 0$. Moreover, we denote $\overline{G} := G \cup \{0\}$.

Proposition 2. Let $1 + \frac{2 + \alpha}{N} < p < \frac{N + \alpha}{N - 2}$ and $0 < \alpha < N$. Then $G$ and $B$ are invariant evolution flows generated by the Cauchy problem (1). Precisely speaking, if the initial data $u_0 \in \overline{G}$ (resp. $B$), then the solution $u(t, x)$ of the Cauchy problem (1) still satisfies $u(t, x) \in \overline{G}$ (resp. $B$) for all $t \in [0, T)$.

Proof: Assume that the initial data $u_0 \in \overline{G}$ and $u(t, x)$ is the solution of Cauchy problem (1). By local well-posedness results, one has

$$E(u(t)) + M(u(t)) < \frac{p - 1}{p} \left(\frac{1}{C_\alpha}\right)^{\frac{1}{p-1}}, \quad t \in [0, T).$$

To testify $u(t) \in \overline{G}$, it suffices to prove

$$\|u(t)\|_{H^1}^2 < \left(\frac{1}{C_\alpha}\right)^{\frac{1}{p-1}}, \quad t \in [0, T).$$  \hfill (24)

If not, the inequality $\|u(t)\|_{H^1}^2 \geq \left(\frac{1}{C_\alpha}\right)^{\frac{1}{p-1}}$ holds. By continuity of $\|u(t)\|_{H^1}^2$, there exists a $t_0 \in [0, T)$ such that

$$\|u(t_0)\|_{H^1}^2 = \left(\frac{1}{C_\alpha}\right)^{\frac{1}{p-1}}.$$
On the other hand, we can deduce
\[ E(u(t_0)) + M(u(t_0)) \]
\[ = \int \left( |\nabla u(t_0)|^2 + |u(t_0)|^2 \right) dx - \frac{1}{p} \int (K_\alpha * |u(t_0)|^p) |u(t_0)|^p dx \]
\[ \geq \| \nabla u(t_0) \|_{L^2}^2 + \| u(t_0) \|_{L^2}^2 - \frac{C_\alpha}{p} \| \nabla u(t_0) \|_{L^2}^{N_p-N-\alpha} \| u(t_0) \|_{L^2}^{2p-Np+N+\alpha} \]
\[ > \| \nabla u(t_0) \|_{L^2}^2 + \| u(t_0) \|_{L^2}^2 - \frac{C_\alpha}{p} (\| \nabla u(t_0) \|_{L^2}^2 + \| u(t_0) \|_{L^2}^2)^p \]
\[ = \| u(t_0) \|_{H^1}^2 - \frac{C_\alpha}{p} \| u(t_0) \|_{H^1}^{2p} \]
\[ = F(\| u(t_0) \|_{H^1}^2) = \frac{p-1}{p} \left( \frac{1}{C_\alpha} \right)^{\frac{1}{\frac{2}{p}}} \]

which contradicts \( E(u(t_0)) + M(u(t_0)) < \frac{p-1}{p} \left( \frac{1}{C_\alpha} \right)^{\frac{1}{\frac{2}{p}}} \). Then, (24) stands. In other words, \( \overline{G} \) is an invariant set under the flows generated by the Cauchy problem (1).

By similar discussion as above, we can prove that \( B \) is an invariant set under the flows generated by the Cauchy problem (1).

Next, we state the main result of this subsection.

**Theorem 3.1.** Let \( 1 + \frac{2+\alpha}{N} < p < \frac{N+\alpha}{N} \) and \( 0 < \alpha < N \). Assume that \( u(t,x) \) is the solution of the Cauchy problem (1) corresponding to the initial data \( u_0 \). Then we have

(i) if \( u_0 \in \overline{G} \), then the solution \( u(t,x) \) exists globally in time;

(ii) if \( u_0 \in B \cap \Sigma \), then the solution \( u(t,x) \) blows up in a finite time.

**Proof.** (i) Suppose \( u_0 \in \overline{G} \), it follows from Proposition 2 that \( u(t,x) \in \overline{G} \). Hence,

\[ \| u \|_{H^1}^2 < \left( \frac{1}{C_\alpha} \right)^{\frac{1}{\frac{2}{p}}} \]

and we directly obtain \( u(t,x) \) is bounded in \( H^1 \).

(ii) Suppose \( u_0 \in B \), it follows from Proposition 2 that \( u(t,x) \in B \). Then we can estimate the virial identity

\[ \frac{d^2}{dt^2} \int |x|^2 |u|^2 dx = 8 \int |\nabla u|^2 dx - \frac{4Np - 4N - 4\alpha}{p} \int (K_\alpha * |u|^p) |u|^p dx \]
\[ = 4(Np - N - \alpha)(E(u) + M(u)) - 4(Np - N - \alpha - 2) \int |\nabla u|^2 dx \]
\[ - 4(Np - N - \alpha) \int |u|^2 dx \]
\[ < 4(Np - N - \alpha) \left( E(u_0) + M(u_0) - \frac{Np - N - \alpha - 2}{Np - N - \alpha} \| u \|_{H^1}^2 \right) \]
\[ \leq -C < 0. \]

Then applying the technique of Glassy [11], we deduce that the solution \( u(t,x) \) blows up in a finite time. This completes the proof. \( \Box \)
Remark 2. It is obvious to check that 
\[
\left(\frac{1}{c_\alpha}\right)^{\frac{1}{\alpha}} > \frac{Np-N-\alpha-2}{Np-N-\alpha} \left(\frac{2p}{Np-N-\alpha}\right)^{\frac{1}{p-1}}
\]
for \(1 + \frac{2+\alpha}{N} < p < \frac{N+2}{2}\) with \(0 < \alpha < N\). If we reduce the energy-mass to \(\frac{Np-N-\alpha-2}{Np-N-\alpha} \left(\frac{2p}{Np-N-\alpha}\right)^{\frac{1}{p-1}} \left(\frac{1}{c_\alpha}\right)^{\frac{1}{\alpha}}\) in \(\mathcal{G}\), Theorem 3.1 turns a sharp threshold of global existence and blowup for Cauchy problem (1).

3.2. A criterion based on bootstrap argument. Now we are set out to investigate the bootstrap argument of Cauchy problem (1). Let us start with the following proposition.

Proposition 3. Let \(1 + \frac{2+\alpha}{N} < p < \frac{N+2}{2}\) and \(Q\) be the unique solution of the nonlinear elliptic equation (13). If the initial data \(u_0\) satisfying \(\int |\nabla u_0|^2 dx < \left(\frac{p}{C_p}\right)^{\frac{2}{Np-N-\alpha}} \left(\frac{1}{M(u_0)}\right)^{-\frac{Np-N-\alpha-2}{Np-N-\alpha}}\), then \(E(u) > 0\).

Proof. Suppose \(u \in H^1(\{0\})\), one has
\[
E(u) = \int |\nabla u|^2 dx - \frac{1}{p} \int (K_\alpha * |u|^p) |u|^p dx
\geq \int |\nabla u|^2 dx - \frac{C_\alpha}{p} \left(\int |\nabla u|^2 dx\right)^{\frac{Np-N-\alpha}{2}} \left(\int |u|^2 dx\right)^{\frac{2p-Np+N+\alpha}{2}},
\]
the result is proved by conservation laws.

Lemma 3.2. (bootstrap argument) Let \(\varphi = \varphi(t)\) be a nonnegative continuous function on \([0,T]\) satisfying
\[
\varphi(t) \leq \epsilon_1 + \epsilon_2 \varphi(t)^\eta, \quad t \in [0,T],
\]
where \(\epsilon_1, \epsilon_2 > 0\) and \(\eta > 1\) are constants such that
\[
\epsilon_1 < \frac{\eta-1}{\eta} \left(\frac{1}{\epsilon_2}\right)^{\frac{1}{\eta-1}}, \quad \varphi(0) \leq \left(\frac{1}{\eta \epsilon_2}\right)^{\frac{1}{\eta-1}}. \tag{25}
\]
Then, one has
\[
\varphi(t) \leq \frac{\eta}{\eta-1} \epsilon_1, \quad t \in [0,T].
\]

Proof. Construct auxiliary function
\[
Z(\varphi) = \varphi - b \varphi^\eta, \quad \varphi \geq 0.
\]
It is obvious to check that \(\varphi_0 = \left(\frac{1}{\eta \epsilon_2}\right)^{\frac{1}{\eta-1}}\) is the maximum point of \(Z(\varphi)\) with maximum \(Z_{\text{max}} = Z(\varphi_0) = \frac{\eta-1}{\eta} \left(\frac{1}{\epsilon_2}\right)^{\frac{1}{\eta-1}} > \epsilon_1\), which implies \(\varphi_0 > \frac{\eta}{\eta-1} \epsilon_1 \geq \varphi(0)\).

On the other hand, \(\varphi(t) \leq \epsilon_1 + \epsilon_2 \varphi(t)^\eta\) holds for all \(t \in [0,T]\), thus \(Z(\varphi(t)) = \varphi(t) - b \varphi(t)^\eta \leq \epsilon_1\). It suffices to prove that \(\varphi(t) < \varphi_0\) for all \(t \in [0,T]\). Indeed, if there exists \(t_1 \in (0,T)\) such that \(\varphi(t_1) \geq \varphi_0 > \varphi(0)\), then by the continuity of \(Z(\varphi)\) and \(\varphi(t)\), we know that there exists some \(t_2 \in (0,T)\) satisfies \(\varphi(t_2) = \varphi_0\), so \(\varphi(t_2)\) is the maximum point of \(Z(\varphi)\), which contradicts with \(Z(\varphi(t)) \leq \epsilon_1\). This completes the proof.

From the above lemma, we can construct a new invariant flow concerned the global existence of solutions. For convenience, we write \(k = Np - N - \alpha\), and then \(2 < k < 2p\) under the condition of \(1 + \frac{2+\alpha}{N} < p < \frac{N+2}{2}\) with \(N \geq 3\).
**Theorem 3.3.** Let $1 + \frac{2+\alpha}{N} < p < \frac{N+\alpha}{N-2}$ and $N \geq 3$. Denote

$$K := \left\{ u \in H^1 : \Phi(\|u\|_{H^1}) < E(u) + M(u) < \frac{2p-2}{2p-k} \|Q\|_{L^2}^2, \|u\|_{H^1}^2 < \frac{2p-2}{2p-k} \left( \frac{k}{k-2} \right)^{\frac{k-2}{2p-k}} \|Q\|_{L^2}^2 \right\}.$$  

Then $K$ is invariant under the flows generated by the Cauchy problem (1). Consequently, if $u(t, x)$ is the solution corresponding to the initial data $u_0 \in K$, then $u(t, x)$ globally exists in $H^1$.

**Proof.** Suppose $u_0 \in K$, by Lemma 2.1 and Proposition 1, we have

$$\int |\nabla u|^2 dx = E(u) + \frac{1}{p} \int (K_\alpha \ast |u|^p) |u|^p dx \leq E(u) + C_\alpha \left( \frac{1}{p} \right) \left( \int |\nabla u|^2 dx \right)^{\frac{N-\alpha}{2}} \left( \int |u|^2 dx \right)^{\frac{2p-N+\alpha}{2}} \leq (\alpha) / \left( \frac{k-2}{k-2} \right) \left( \frac{2p-2}{2p-k} \right)^{\frac{k-2}{2p-k}} (2p-2-M(u_0))^{\frac{k-2}{2p-k}}$$

Choosing $\varphi(t) = \int |\nabla u(t)|^2 dx$, $\epsilon_1 = E(u_0)$, $\epsilon_2 = C_\alpha \frac{p}{p} M(u_0)^{\frac{k-2}{2}}$ and $\eta = \frac{k}{2}$, by Lemma 3.2, it suffices to prove that the relationship in (25) holds. Based on the Young’s inequality, we estimate

$$E(u_0) + M(u_0) < \frac{2p-2}{2p-k} \|Q\|_{L^2}^2$$

$$= \left( \frac{2p-2}{k} \left( \frac{2p}{kC_\alpha} M(u_0) \frac{k-2p}{k} \right)^{\frac{k-2}{2p-k}} \right)^{\frac{k-2}{2p-k}} \frac{2p-2-k}{2p-k} M(u_0)$$

$$\leq \frac{k-2}{k} \left( \frac{2p}{kC_\alpha} M(u_0) \frac{k-2p}{k} \right)^{\frac{k-2}{2p-k}} + M(u_0)$$

$$= \left( 1 - \frac{1}{\epsilon_1} \right) \left( \frac{1}{(\epsilon_1)_{1/1}} + M(u_0), \right)$$

$$\|u_0\|_{H^1}^2 = \int |\nabla u_0|^2 dx + M(u_0)$$

$$< \frac{2p-2}{2p-k} \left( \frac{k}{k-2} \right)^{\frac{k-2}{2p-k}} \|Q\|_{L^2}^2$$

$$= \left( \frac{2p-2}{k} \left( \frac{2p}{kC_\alpha} M(u_0) \frac{k-2p}{k} \right)^{\frac{k-2}{2p-k}} \right)^{\frac{k-2}{2p-k}} \frac{2p-2-k}{2p-k} M(u_0)$$

$$\leq \left( \frac{2p}{kC_\alpha} M(u) \frac{k-2p}{k} \right)^{\frac{k-2}{2p-k}} + M(u_0)$$

$$= \frac{1}{(\epsilon_1)_{1/1}} + M(u_0),$$

then, we directly have

$$\|u\|_{H^1}^2 = \int |\nabla u|^2 dx + M(u) \leq \frac{\eta}{\eta-1} E(u) + M(u) \leq \frac{2p-2}{2p-k} \left( \frac{k}{k-2} \right)^{\frac{k-2}{2p-k}} \|Q\|_{L^2}^2$$
which implies $K$ is invariant under the flows generated by the Cauchy problem (1) and $u(t, x)$ globally exists in $H^1$.

So far, we have derived some sufficient conditions for global existence and blowup of solutions. However, their relationship has not yet been established. Next, we will intend to study more details of these results. For conciseness, we rewrite

$$G(k) := \frac{(2p - 2)k}{(2p)^{\frac{2}{p+1}}} \frac{1}{(2p - k) \frac{k-2}{p-2}} \|Q\|_{L^2}^2, \quad g(k) := \frac{k}{(2p - k) \frac{k-2}{p-2}} \|Q\|_{L^2}^2;$$

$$B(k) := \frac{2p - 2}{2p - k} \|Q\|_{L^2}^2, \quad b(k) := \frac{2p - 2}{2p - k} \left( \frac{k}{k - 2} \right) \frac{k-2}{p-2} \|Q\|_{L^2}^2;$$

$$H(k) := \frac{2p - 2}{2p - k} \|Q\|_{L^2}^2, \quad h(k) := \frac{2p - 2}{2p - k} \left( \frac{k}{k - 2} \right) \frac{k-2}{p-2} \|Q\|_{L^2}^2.$$

Consequently, the comparison of $G(k), B(k)$ and $H(k)$ is shown in Fig. 1 (see Appendix). Furthermore, the relationship among $g(k), b(k)$ and $h(k)$ is exhibited in Fig. 2 (see Appendix). Thus, we derive $H(k) \geq G(k) \geq B(k), b(k) \geq b(k) \geq g(k)$ and the following results.

**Proposition 4.** If $1 + \frac{2+\alpha}{N} < p < \frac{N+\alpha}{N-2}$ and $N \geq 3$, then we have $\overline{C} \subset \mathcal{K} \cup \{0\}$.

**Remark 3.** Consider $\Omega := K \cap B$. Thus

$$\Omega := \left\{ u \in H^1 : \Phi(\|u\|_{H^1}^2) < E(u) + M(u) < \frac{k - 2}{k + 2} \frac{1}{2p - k} \frac{k-2}{p-2} \|Q\|_{L^2}^2, \quad \left( \frac{k}{2p - k} \right) \frac{k-2}{p-2} \|Q\|_{L^2}^2 < \|u\|_{H^1}^2 \leq \frac{2p - 2}{2p - k} \left( \frac{k}{k - 2} \right) \frac{k-2}{p-2} \|Q\|_{L^2}^2 \right\}.$$ 

But from Theorem 3.1 and Theorem 3.3, we observe if the initial data $u_0 \in \Omega \cap \Sigma$, then the solution $u(t, x)$ globally exists and also blows up in a finite time, that is a contradiction. So we declare that $\Omega \cap \Sigma = \emptyset$.

4. **Concentration of blowup results.** In this section, applying the similar techniques with that in [5, 26], we investigate the concentration of blowup solutions for Eq. (1). Precisely speaking, using the profile decomposition of bounded sequence in $\dot{H}^1(\mathbb{R}^N) \cap \dot{H}^{S_c}(\mathbb{R}^N)$, we obtain a compactness result, which will be used to prove the blowup solutions with bounded $\dot{H}^{S_c}$ norm definitely have concentration properties related to a fixed $\dot{H}^{S_c}$ norm of certain standing waves.

**Proposition 5.** Let $\{u_n\}_{n=1}^\infty$ be a bounded sequence of $\dot{H}^1 \cap \dot{H}^{S_c}$ such that

$$\lim_{n \to \infty} \int (K_{\alpha} * |u_n|^p) |u_n|^p dx \geq \gamma_1 > 0, \quad \lim_{n \to \infty} \|\nabla u_n\|_{L^2}^2 \leq \gamma_2.$$

Then, there exists $\{x_n\}_{n=1}^\infty$ such that, up to a subsequence,

$$u_n(x + x_n) \rightharpoonup U \text{ weakly in } \dot{H}^1 \cap \dot{H}^{S_c}$$

with

$$\|U\|_{\dot{H}^{S_c}}^{2p-2} \geq \frac{\gamma_1}{p^2} \|W\|_{\dot{H}^{S_c}}^{2p-2},$$

where $W$ solves the nonlinear elliptic equation (13).
Proof. Using Proposition 2.5 to the bounded sequence \( \{u_n\}_{n=1}^{\infty} \), we derive the profile decomposition \( u_n(x) = \sum_{i=1}^{m} U^i(x - x_n^i) + r^m_n(x) \) such that (9) and (10) hold. On the other hand,
\[
\lim_{n \to \infty} \int \left( K_\alpha \left[ \sum_{i=1}^{m} U^i(x - x_n^i) \right] \right)^p \left[ \sum_{i=1}^{m} U^i(x - x_n^i) \right]^p dx = \sum_{i=1}^{m} \int (K_\alpha \ast |U^i|^p) |U^i|^p dx.
\]
Thus, we have
\[
\gamma_1 \leq \lim \sup_{n \to \infty} \sum_{i=1}^{\infty} \int (K_\alpha \ast |U^i_n|^p) |U^i_n|^p dx.
\]
Moreover, in consideration of the Gagliardo-Nirenberg inequality (11), we derive that
\[
\sum_{i=1}^{\infty} \int (K_\alpha \ast |U^i_n|^p) |U^i_n|^p dx
\leq \sum_{i=1}^{\infty} C_n \| \nabla U^i \|^2_{L^2} \| U^i \|^2_{H^s_c}
\leq C_n \sup \{ \| U^i \|^2_{H^s_c} \mid i \geq 1 \} \sum_{i=1}^{\infty} \| \nabla U^i \|^2_{L^2}
\leq \gamma_2 C_n \sup \{ \| U^i \|^2_{H^s_c} \mid i \geq 1 \},
\]
which suggests
\[
\sup \{ \| U^i \|^2_{H^s_c} \mid i \geq 1 \} \leq \frac{\gamma_1}{\gamma_2 C_n} = \frac{\gamma_1}{\gamma_2} \| W \|^2_{H^s_c}.
\]
By the convergence of \( \sum_{i=1}^{\infty} \| U^i \|^2_{H^s_c} \), we infer that there exists some \( i_0 \geq 1 \) such that \( \sup \{ \| U^i \|^2_{H^s_c} \mid i \geq 1 \} = \| U^{i_0} \|^2_{H^s_c} \), which gives
\[
\| U^{i_0} \|^2_{H^s_c} \geq \frac{\gamma_1}{\gamma_2} \| W \|^2_{H^s_c}.
\]
Now, we consider
\[
u_n(x + x_n^i) = U^{i_0}(x) + \sum_{i \neq i_0} U^i(x + x_n^i - x_n^i) + u^m_n(x + x_n^i).
\]
According to (7) and (8), we have
\[
U^i(x + x_n^i - x_n^i) \to 0 \quad \text{weakly in } \dot{H}^1 \cap \dot{H}^{s_c} \quad \text{for every } i \neq i_0,
\]
\[
u_n^m(x + x_n^i) \to 0 \quad \text{weakly in } \dot{H}^1 \cap \dot{H}^{s_c} \quad \text{for } m \to \infty.
\]
Consequently, one has
\[
u_n(x + x_n^i) \to U^{i_0} \quad \text{weakly in } \dot{H}^1 \cap \dot{H}^{s_c}.
\]
This completes the proof of Proposition 5.

The foremost result obtained in this section is as follows.

**Theorem 4.1.** Let \( u_0 \in \dot{H}^1 \cap \dot{H}^{s_c} \). Suppose the solution \( u(t, x) \) of Eq. (1) corresponding the initial data \( u_0 \) blows up in finite time \( T > 0 \) satisfying
\[
\sup_{t \in [0, T]} \| u(t, x) \|_{H^{s_c}} < +\infty.
\]

\[\tag{32}\]
Assume that $\theta(t) > 0$ such that
\[
\lim_{t \to T^-} \theta(t) \|\nabla u(t, x)\|_{L^2}^{\frac{1}{2s_c}} = +\infty.
\]
(33) Then, there exists $x(t) \in \mathbb{R}^N$ which suggests
\[
\liminf_{t \to T^-} \int_{|x-x(t)| \leq \theta(t)} |(-\Delta)^{\frac{s_c}{2}} u(t, x)|^2 dx \geq \|W\|_{H_{sc}}^2,
\]
(34) where $W$ solves the nonlinear elliptic equation (13).

Proof. Choose $\{t_n\}_{n=1}^\infty$ be an arbitrary sequence satisfying $t_n \to T$ as $n \to \infty$. Take
\[
\chi_n = \left(\frac{\|\nabla W\|_2}{\|\nabla u(t_n)\|_2}\right)^{\frac{1}{s_c}}, \quad v_n = \chi_n^{\frac{N-2s_c}{2}} u(t_n, \chi_n x).
\]
By means of assumption (32), we derive
\[
\|v_n\|_{H_{sc}} = \|u(t_n)\|_{H_{sc}} < +\infty, \quad \|\nabla v_n\|_{L^2} = \chi_n^{1-s_c} \|\nabla u(t_n)\|_{L^2} = \|\nabla W\|_{L^2}
\]
and
\[
E(v_n) = \int |\nabla v_n|^2 dx - \frac{1}{p} \left(\int (K_n * |v_n|^p) |v_n|^p dx \right) = \chi_n^{2-2s_c} \int |\nabla u(t_n)|^2 dx - \frac{1}{p} \chi_n^{2-2s_c} \int (K_n * |u(t_n)|^p) |u(t_n)|^p dx = \chi_n^{2-2s_c} E(u(t_n)).
\]
From the local well-posedness established in Proposition 2.1, we derive $\chi_n \to 0$ and consequently $E(v_n) \to 0$ as $n \to \infty$, which suggests that $\int (K_n * |v_n|^p) |v_n|^p dx \to p \|\nabla W\|_{L^2}^2$.

Now, we assign $\gamma_1 = p \|\nabla W\|_{L^2}^2$ and $\gamma_2 = \|\nabla W\|_{L^2}^2$. Applying Proposition 5 to the bounded sequence $\{v_n\}_{n=1}^\infty$, we have a family $\{x_n\}_{n=1}^\infty$, and a profile $V$ with
\[
\|V\|_{H_{sc}}^{2p-2} \geq \frac{\gamma_1}{\gamma_2} \|W\|_{H_{sc}}^{2p-2} = \|W\|_{H_{sc}}^{2p-2}.
\]
(35) such that, up to a subsequence,
\[
\chi_n^{\frac{N-2s_c}{2}} u(t_n, \chi_n (x_0 + x_0)) \to V \quad \text{weakly in} \quad \dot{H}^1 \cap \dot{H}^{s_c}.
\]
Thus, it follows from the definition of $\dot{H}^s$ that
\[
\chi_n^{\frac{N-2s_c}{2}} (-\Delta)^{\frac{s_c}{2}} u(t_n, \chi_n(x_0 + x_0)) \to (-\Delta)^{\frac{s_c}{2}} V \quad \text{weakly in} \quad L^2.
\]
Consequently, for every positive constant $C$,
\[
\int_{|x| \leq C} \|(-\Delta)^{\frac{s_c}{2}} V\|^2 dx \leq \liminf_{n \to \infty} \int_{|x| \leq C} \chi_n^{\frac{N-2s_c}{2}} (-\Delta)^{\frac{s_c}{2}} u(t_n, \chi_n(x_0 + x_0)) dx = \liminf_{n \to \infty} \int_{|x-x_0| \leq \chi_n C} \|(-\Delta)^{\frac{s_c}{2}} u(t_n, x)\|^2 dx.
\]
Assumption (33) suggests that $\frac{\theta(t_n)}{\chi_n} \to +\infty$ as $n \to \infty$, which directly shows
\[
\int_{|x| \leq C} \|(-\Delta)^{\frac{s_c}{2}} V\|^2 dx \leq \liminf_{n \to \infty} \sup_{y \in \mathbb{R}^N} \int_{|x-y| \leq \theta(t_n)} \|(-\Delta)^{\frac{s_c}{2}} u(t_n, x)\|^2 dx.
\]
Due to the arbitrariness of $\{t_n\}_{n=1}^\infty$, we deduce that
\[
\int_{|x| \leq C} \|(-\Delta)^{\frac{s_c}{2}} V\|^2 dx \leq \liminf_{t \to T^-} \sup_{y \in \mathbb{R}^N} \int_{|x-y| \leq \theta(t)} \|(-\Delta)^{\frac{s_c}{2}} u(t, x)\|^2 dx.
\]
For each fixed $t \in [0, T)$, the mapping $y \mapsto \int_{|x-y| \leq \theta(t)} |(-\Delta)^{\frac{s}{2}} u(t, x)|^2 dx$ is continuous, and tends to zero as $y \to +\infty$. Thus, there exists $x(t) \in \mathbb{R}^N$ such that

$$\int_{|x-x(t)| \leq \theta(t)} |(-\Delta)^{\frac{s}{2}} u(t, x)|^2 dx = \sup_{y \in \mathbb{R}^N} \int_{|x-y| \leq \theta(t)} |(-\Delta)^{\frac{s}{2}} u(t, x)|^2 dx.$$ Combining with (35), one has

$$\liminf_{t \to T^-} \int_{|x-x(t)| \leq \theta(t)} |(-\Delta)^{\frac{s}{2}} u(t, x)|^2 dx \geq \|W\|^2_{\dot{H}^S},$$

which completes the proof.

5. **Concluding remarks.** In this paper, the thresholds of global existence and blowup as well as concentration properties for the nonlinear Schrödinger-Hartree equation are considered. With the aid of Bootstrap argument, we provide a new invariant evolution flows $K$ which involves the global existence of solutions. Further study shows $K \cap B \cap \Sigma$ is an empty set because global existence and blowup cannot appear at the same time. However, there still leaves a problem: is $K \cap B$ a empty set? It is a work that should be solved in the future. Moreover, by means of interpolation inequality and Sobolev imbeddings, we obtain a new Gagliardo-Nirenberg inequality in $\dot{H}^1 \cap \dot{H}^S$, which guarantees to derive the $\dot{H}^S$ concentration of blowup solutions. By similar discussion, we conjecture that blowup solutions with bounded $L^p_c$ norm definitely have concentration properties related to a fixed $L^p_c$ norm of certain standing waves.

**Appendix.** The comparison of $G(k)$, $B(k)$ and $H(k)$ is shown in Fig. 1. Furthermore, the relationship among $g(k)$, $b(k)$ and $h(k)$ is exhibited in Fig. 2.
REFERENCES

[1] T. Cazenave, *Semi-linear Schrödinger Equations*, Courant Lecture Notes in Mathematics, 10. New York University, Courant Institute of Mathematical Sciences, New York; American Mathematical Society, Providence, RI, 2003.

[2] T. Cazenave and P. L. Lions, Orbital stability of standing waves for some nonlinear Schrödinger equations, *Commun. Math. Phys.*, **85** (1982), 549–561.

[3] J. Chen and B. Guo, Strong instability of standing waves for a nonlocal Schrödinger equation, *Physica. D.*, **237** (2007), 142–148.

[4] B. Feng, Sharp threshold of global existence and instability of standing wave for the Schrödinger-Hartree equation with a harmonic potential, *Nonlinear. Anal. Real. World. Appl.*, **31** (2016), 132–145.

[5] B. Feng and Y. Cai, Concentration for blow-up solutions of the Davey-Stewartson system in $\mathbb{R}^3$, *Nonlinear. Anal. Real. World. Appl.*, **26** (2015), 330–342.

[6] B. H. Feng and X. Yuan, The global well-posedness and blow-up solutions for the generalized Choquard equation, *Evol. Equ. Control. Theory*, **4** (2015), 281–296.

[7] B. H. Feng and H. H. Zhang, Stability of standing waves for the fractional Schrödinger-Choiour equation, *Comput. Math. Appl.*, **75** (2018), 2499–2507.

[8] J. Fröhlich and E. Lenzmann, Mean-field limit of quantum bose gases and nonlinear Hartree equation, in *Seminaire EDP*, Ecole Polytechnique, (2004), 2003–2004.

[9] H. Genev and G. Venkov, Soliton and blow-up solutions to the time-dependent Schrödinger-Hartree equation, *Discrete Contin. Dyn. Syst. Ser. S.*, **5** (2012), 903–923.

[10] J. Ginibre and G. Velo, On a class of nonlinear Schrödinger equations. I. The Cauchy problem, general case, *J. Funct. Anal.*, **32** (1979), 1–32.

[11] R. T. Glassey, On the blowing up of solution to the Cauchy problem for nonlinear Schrödinger equations, *J. Math. Phys.*, **18** (1977), 1794–1797.

[12] P. L. Kelley, Self focusing of optical beams, *Phys. Rev. Lett.*, **15** (1965), 1005–1008.

[13] E. Lieb, *Analysis*, in *Graduate Studies in Mathematics*, American Mathematical Society, 2001.

[14] P. L. Lions, The Choquard equation and related questions, *Nonlinear. Anal.*, **4** (1980), 1063–1072.

[15] C. Miao, G. Xu and L. Zhao, On the blow up phenomenon for the $L^2$-critical focusing Hartree equation in $\mathbb{R}^3$, *Colloq. Math.*, **119** (2010), 23–50.
[16] V. Moroz and J. V. Schaftingen, Groundstates of nonlinear Choquard equations: Existence, qualitative properties and decay asymptotics, *J. Funct. Anal.*, **265** (2013), 153–184.

[17] T. Ogawa and Y. Tsutsumi, Blow-up of $H^1$ solution for the nonlinear Schrödinger equation, *J. Differ. Equations*, **92** (1991), 317–330.

[18] S. Pekar, *Untersuchung über Die Elektronentheorie Der Kristalle*, Akademie-Verlag, Berlin, 1954.

[19] W. A. Strauss, Nonlinear scattering theory at low energy, *J. Funct. Anal.*, **41** (1981), 110–133.

[20] W. A. Strauss, Existence of solitary waves in higher dimensions, *Commun. Math. Phys.*, **55** (1977), 149–162.

[21] Y. Tsutsumi, Scattering problem for nonlinear Schrödinger equation, *Nonlinear Ann. Inst. Henri. Poincaré Physique Théorique*, **43** (1985), 321–347.

[22] Y. J. Wang, Strong instability of standing waves for Hartree equation with harmonic potential, *Physica. D.*, **237** (2008), 998–1005.

[23] X. Wang, X. M. Sun and W. H. Lv, Orbital stability of generalized Choquard equation, *Bound. Value Probl.*, **2016** (2016), Paper No. 190, 8 pp.

[24] M. I. Weinstein, Nonlinear Schrödinger equations and sharp interpolations estimates, *Commun. Math. Phys.*, **87** (1982/83), 567–576.

[25] V. E. Zakharov, Collapse of Langmuir waves, *Sov. Phys. JETP*, **35** (1972), 909–912.

[26] S. H. Zhu, On the Davey-Stewartson system with competing nonlinearities, *J. Math. Phys.*, **57** (2016), 031501, 13pp.

[27] S. H. Zhu, *Dynamical Properties of Blow-Up Solutions to Nonlinear Schrödinger Equations*, Ph.D. thesis (in Chinese), Sichuan University, 2011.

Received for publication October 2019.

E-mail address: yingyingxie95@hotmail.com
E-mail address: jsu@xjtu.edu.cn
E-mail address: lqmei@mail.xjtu.edu.cn