Entanglement-Assisted Quantum Quasi-Cyclic Low-Density Parity-Check Codes

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We investigate the construction of quantum low-density parity-check (LDPC) codes from classical quasi-cyclic (QC) LDPC codes with girth greater than or equal to 6. We have shown that the classical codes in the generalized Calderbank-Shor-Steane (CSS) construction do not need to satisfy the dual-containing property as long as pre-shared entanglement is available to both sender and receiver. We can use this to avoid the many 4-cycles which typically arise in dual-containing LDPC codes. The advantage of such quantum codes comes from the use of efficient decoding algorithms such as sum-product algorithm (SPA). It is well known that in the SPA, cycles of length 4 make successive decoding iterations highly correlated and hence limit the decoding performance. We show the principle of constructing quantum QC-LDPC codes which require only small amounts of initial shared entanglement.

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I. INTRODUCTION

Low-density parity-check (LDPC) codes were first proposed by Gallager [1] in the early 1960s, and were rediscovered [2,3,4] in the 90s. It has been shown that these codes can achieve a remarkable performance that is very close to the Shannon limit. Sometimes, they perform even better [5] than their main competitors, the Turbo codes. These two families of codes are called modern codes.

A $(J, L)$-regular LDPC code is defined to be the null space of a binary parity check matrix $H$ with the following properties: (1) each column consists of $J$ “ones” (each column has weight $J$); (2) each row consists of $L$ “ones” (each row has weight $L$); (3) both $J$ and $L$ are small compared to the length of the code $n$ and the number of rows in $H$. Several methods of constructing good families of regular LDPC codes have been proposed [1,6,7]. However, probably the easiest method is based on circulant permutation matrices [7], which was inspired by Gallager’s original LDPC construction.

We define a cycle of a linear code to be of length $2s$ if there is an ordered list of $2s$ matrix elements of $H$ such that: (1) all $2s$ elements are equal to 1; (2) successive elements in the list are obtained by alternatingly changing the row or column only (i.e., two consecutive elements will have either the same row and different columns, or the same column and different rows); (3) the positions of all the $2s$ matrix elements are distinct, except the first and last ones. We call the cycle of the shortest length the girth of the code.

There are various methods for decoding classical LDPC codes [6]. Among them, sum-product algorithm (SPA) decoding [8] provides the best trade-off between error-correction performance and decoding complexity. It has been shown that the performance of SPA decoding very much depends on the cycles of shortest length [8] — in particular, cycles of length 4. These shortest cycles make successive decoding iterations highly correlated, and severely limit the decoding performance. Therefore, to use SPA decoding, it is important to design codes without short cycles, especially cycles of length 4.

Because classical LDPC codes have such good performance — approaching the channel capacity in the limit of large block size — there has been considerable interest in finding quantum versions of these codes. However, quantum low-density parity-check codes [9,10,11,12] are far less studied than their classical counterparts. The main obstacle comes from the dual-containing constraint of the classical codes that are used to construct the corresponding quantum codes. While this constraint was not too difficult to satisfy for relatively small codes, it is a substantial barrier to the use of highly efficient LDPC codes. The second obstacle comes from the bad performance of the iterative decoding algorithm. Though the SPA decoding can be directly used to decode the quantum errors, its performance is severely limited by the many 4-cycles, which are usually the by-product of the dual-containing property, in the standard quantum LDPC codes [10].

In this paper we will show that, by using the entanglement-assisted formalism [13,14], these two obstacles of standard quantum LDPC codes can be overcome. By allowing the use of pre-shared entanglement between senders and receivers, the dual-containing constraint can be removed. Constructing quantum LDPC codes from classical LDPC codes becomes transparent. That is, arbitrary classical binary or quaternary codes can be used to construct quantum codes via the CSS or generalized CSS constructions [12]. Moreover, we can easily construct quantum LDPC codes from classical LDPC codes with girth at least 6. We make use of classical quasi-cyclic LDPC codes in our construction, and show that the resulting entanglement-assisted quantum
LDPC codes have good performance; we compare them to examples of standard LDPC codes already proposed in the literature with similar net rates, and show that the new quasicyclic codes have lower block-error rates.

This paper is organized as follows. We discuss properties of binary circulant matrices, and give a brief introduction to classical QC-LDPC codes in section II. We also prove a few interesting lemmas regarding classical QC-LDPC codes in this section. In section III we discuss the principle of constructing quantum QC-LDPC codes from classical QC-LDPC codes, such that the resulting quantum QC-LDPC codes require only a small amount of initial pre-shared entanglement. We also provide two examples of such constructions. In section IV we compare the performance of the quantum QC-LDPC codes illustrated in section III with some previously proposed quantum LDPC codes. Finally, in section V we conclude.

II. PRELIMINARY

A. Properties of binary circulant matrices

We begin with a well-known proposition for binary circulant matrices.

Proposition II.1 The set of binary circulant matrices of size $r \times r$ forms a ring isomorphic to the ring of polynomials of degree less than $r$: $\mathbb{F}_2[X]/(X^r - 1)$.

Let $M$ be an $r \times r$ circulant matrix over $\mathbb{F}_2$. We can uniquely associate with $M$ a polynomial $M(X)$ with coefficients given by entries of the first row of $M$. If $c = (c_0, c_1, \ldots, c_{r-1})$ is the first row of the circulant matrix $M$, then

$$M(X) = c_0 + c_1X + c_2X^2 + \cdots + c_{r-1}X^{r-1}. \quad (1)$$

Adding or multiplying two circulant matrices is equivalent to adding or multiplying their associated polynomials modulo $X^r - 1$. We now give some useful properties of these matrices and polynomials.

The first lemma is a well-known result in the theory of cyclic codes [13].

Lemma II.2 Let $M(X)$ be the polynomial associated with the $r \times r$ binary circulant matrix $M$. If $\text{gcd}(M(X), X^r - 1) = K(X)$, and the degree of $K(X)$ is $k$, then the rank of $M$ is $r - k$.

In the following, we will discuss some particular cases of the circulant matrix $M$ that will play important roles in the later section.

Theorem II.3 Let $r = pq$, where $p, q > 1$. Let $c = (c_0, c_1, \cdots, c_{r-1})$ be the first row of an $r \times r$ circulant matrix $M$.

1. If $c_{(k+p)i \mod r} = 1$, for some $k$ and $i = 0, 1, \cdots, (q - 1)$, and 0 otherwise, then $\text{rank}(M) = p$.

Proof

1. If $c_{(k+p)i \mod r} = 1$, for some $k$ and $i = 0, 1, \cdots, (q - 1)$, and 0 otherwise, we have

$$M(X) = X^k \left(1 + X^p + X^{2p} + \cdots + X^{(q-1)p}\right)$$

$$= X^k \left(\frac{X^r - 1}{X^p - 1}\right).$$

Since $\text{gcd}(X^k, X^p - 1) = 1$, the following holds

$$K(X) = \text{gcd}(M(X), (X^r - 1))$$

$$= 1 + X^p + X^{2p} + \cdots + X^{(q-1)p}.$$

Then the degree of $K(X)$ is $p - 1$. Therefore, by lemma II.2 the rank of $M$ is $p$.

2. In this case, the polynomial is

$$M(X) = X^k \left(1 + X + X^2 + \cdots + X^{(p-1)}\right)$$

$$= X^k \left(\frac{X^r - 1}{X^r - 1(1 + X^p + X^{2p} + \cdots + X^{(q-1)p})}\right)$$

and

$$K(X) = 1 + X + X^2 + \cdots + X^{(p-1)}.$$

Then the degree of $K(X)$ is $p - 1$. Therefore, by lemma II.2 the rank of $M$ is $r - p + 1$.

We also have the following corollary.

Corollary II.4 Suppose $r = pq$, and $p, q > 1$. If the weight of each row is $p$, and $K(X) \neq 1$, then the rank $\kappa$ of matrix $M$ is upper-bounded by $r - p + 1$.

Proof Since the weight of $c$ is $p$, the lowest possible degree of $M(X)$ that divides $X^r - 1$ is $p - 1$, wherein

$$M(X) = 1 + X + X^2 + \cdots + X^{(p-1)} \quad (2)$$

Then item 2 of theorem II.3 confirms the rank $\kappa$ is at most $r - p + 1$.

B. Classical quasi-cyclic LDPC codes

Definition II.5 A binary linear code $C(H)$ of length $n = r \cdot L$ is called quasi-cyclic (QC) with period $L$ if any codeword which is cyclically right-shifted by $L$ positions is again a codeword. Such a code can be represented by a parity-check matrix $H$ consisting of $r \times r$ blocks (by properly rearranging the coordinates of the code), each of which is an (in general different) $r \times r$ circulant matrix.
By the isomorphism mentioned in Prop. [11.1] we can associate with each quasi-cyclic parity-check matrix $H \in \mathbb{F}_2^{r \times L}$ a $J \times L$ polynomial parity-check matrix $H(X) = [h_{j,l}(X)]_{j \in [J], l \in [L]}$ where $h_{j,l}(X)$ is the polynomial, as defined in [1], representing the $r \times r$ circulant submatrix of $H$, and the notation $[J] := \{1, 2, \cdots, J\}$.

Generally, there are two ways of constructing $(J, L)$-regular QC-LDPC by using circulant matrices [10]:

Definition II.6 We say that a QC-LDPC code is Type-I if it is given by a polynomial parity-check matrix $H(X)$ such that all entries are non-zero monomials. We say that a QC-LDPC code is Type-II if it is given by a polynomial parity-check matrix $H(X)$ with either binomials, monomials, or zero.

1. Type-I QC-LDPC

To give an example, let $r = 16$, $J = 3$, and $L = 8$. The following polynomial parity check matrix

$$H(X) = \begin{bmatrix} X & X & X & X & X & X & X & X \\ X^2 & X^5 & X^3 & X^2 & X^5 & X^3 & X^5 & X^2 \\ X^2 & X^3 & X^4 & X^5 & X^6 & X^7 & X^8 & X^9 \end{bmatrix}$$ \hspace{1cm} (3)$$

allows a Type-I $(3, 8)$-regular QC-LDPC code of length $n = 16 \cdot 8 = 128$. Later on, we will also express $H(X)$ by its exponent matrix $H_E$. For example, the exponent matrix of $H(X)$ is

$$H_E = \begin{bmatrix} 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 \\ 2 & 5 & 3 & 5 & 2 & 5 & 3 & 5 \\ 2 & 3 & 4 & 5 & 6 & 7 & 8 & 9 \end{bmatrix}.$$ \hspace{1cm} (4)$$

The difference of two arbitrary rows of the exponent matrix $H_E$ is defined as

$$d_{ij} = c_i - c_j = ((c_{i,k} - c_{j,k}) \mod r)_{k \in [L]},$$ \hspace{1cm} (5)$$

where $c_i$ is the $i$-th row of $H_E$ and $r$ is the size of the circulant matrix. We then have

$$d_{21} = (1, 4, 2, 4, 1, 4, 2, 4)$$
$$d_{31} = (1, 2, 3, 4, 5, 6, 7, 8)$$
$$d_{32} = (0, 14, 1, 0, 4, 2, 5, 4).$$

We call an integer sequence $d = (d_0, d_1, \cdots, d_{L-1})$ multiplicity even if each entry appears an even number of times. For example, $d_{21}$ is multiplicity even, but $d_{32}$ is not, since only 0 and 4 appear an even number of times. We call $d$ multiplicity free if no entry is repeated; for example, $d_{31}$.

A simple necessary condition for Type-I $(J, L)$-regular QC-LDPC codes to give girth $g \geq 6$ is given in [2]. However, a stronger result (both sufficient and necessary condition) is shown in [9]. We state these theorems from [9] without proofs.

Theorem II.7 A Type-I QC-LDPC code $C(H_E)$ is dual-containing if and only if $c_i - c_j$ is multiplicity even for all $i$ and $j$, where $c_i$ is the $i$-th row of the exponent matrix $H_E$.

Theorem II.8 There is no dual-containing Type-I QC-LDPC having girth $g \geq 6$.

Theorem II.9 A necessary and sufficient condition for a Type-I QC-LDPC code $C(H_E)$ to have girth $g \geq 6$ is $c_i - c_j$ to be multiplicity free for all $i$ and $j$.

2. Type-II QC-LDPC

Take $r = 16$, $J = 3$, and $L = 4$. The following is an example of a Type-II $(3, 4)$-regular QC-LDPC code:

$$H(X) = \begin{bmatrix} X + X^4 & 0 & X^7 + X^{10} & 0 \\ X^5 & X^6 & X^{11} & X^{12} \\ 0 & X^2 + X^9 & 0 & X^7 + X^{13} \end{bmatrix}.$$ \hspace{1cm} (6)$$

The exponent matrix of $H(X)$ is

$$H_E = \begin{bmatrix} (1, 4) & \infty & (7, 10) & \infty \\ 5 & 6 & 11 & 12 \\ \infty & (2, 9) & \infty & (7, 13) \end{bmatrix}.$$ \hspace{1cm} (7)$$

Here we denote $X^\infty = 0$.

The difference of two arbitrary rows of $H_E$ is defined similarly to (6) with the following additional rules: (i) if for some entry $c_{i,k}$ is $\infty$, then the difference of $c_{i,k}$ and any other arbitrary term is again $\infty$; (ii) if the entries $c_{i,k}$ and $c_{j,k}$ are both binomial, then the difference of $c_{i,k}$ and $c_{j,k}$ contains four terms. In this example, we have

$$d_{21} = ((4, 1), \infty, (4, 1), \infty)$$
$$d_{31} = (\infty, \infty, \infty, \infty)$$
$$d_{32} = (\infty, (12, 3), \infty, (11, 1))$$
$$d_{11} = ((0, 3, 13, 0), \infty, (0, 3, 13, 0), \infty)$$
$$d_{22} = (0, 0, 0, 0)$$
$$d_{33} = (\infty, (0, 9, 7, 0), \infty, (0, 10, 6, 0)).$$

The definition of multiplicity even and multiplicity free is the same, except that we do not take $\infty$ into account. For example, $d_{32}$ is multiplicity free, since there is no pair with the same entry except $\infty$. Unlike Type-I QC-LDPC codes, whose $d_{ii}$ is always the zero vector, $d_{ii}$ of Type-II QC-LDPC codes can have non-zero entries. Therefore it is possible to have cycles of length 4 in a single layer if $d_{ii}$ is not multiplicity free. Each layer is a set of rows of size $r$ in the original parity check matrix $H$ that corresponds to one row of $H_E$. For example, $d_{31}$ is multiplicity even, therefore the first layer of this Type-II regular QC-LDPC parity check matrix contains 4-cycles.

In the following, we will generalize theorems II.7, II.8, II.9 from the previous section to include the Type-II QC-LDPC case.
Theorem II.10 \( C(H_E) \) is a dual-containing Type-II regular QC-LDPC code if and only if \( c_i - c_j \) is multiplicity even for all \( i \) and \( j \).

**Proof** Let \( H(X) = [h_{j,i}(X)]_{j \in [J], i \in [L]} \) be the polynomial parity check matrix associated with a Type-II \((J, L)\)-regular QC-LDPC parity check matrix \( H \). Denote the transpose of \( H(X) \) by \( H(X)^T = [h_{j,i}^T(X)]_{i \in [L], j \in [J]} \), and we have

\[
\hat{h}_{i,j}(X) = \begin{cases} 
0 & \text{if } h_{j,i}(X) = 0 \\
X^{r-k} & \text{if } h_{j,i}(X) = X^k \\
X^{r-k_1} + X^{r-k_2} & \text{if } h_{j,i}(X) = X^{k_1} + X^{k_2} 
\end{cases}.
\]

(8)

Let \( \hat{H}(X) = H(X)H(X)^T \), and let the \((i, j)\)-th component of \( \hat{H}(X) \) be \( \hat{h}_{i,j}(X) \). Then

\[
\hat{h}_{i,j}(X) = \sum_{l \in [L]} h_{i,l}(X)h_{j,l}(X).
\]

(9)

The condition that \( d_{ij} \) is multiplicity even implies that \( \hat{h}_{i,j}(X) = 0 \) modulo \( X^t - 1 \), and vice versa. \( \square \)

**Theorem II.11** A necessary and sufficient condition for a Type-II regular QC-LDPC code \( C(H_E) \) to have girth \( g \geq 6 \) is that \( c_i - c_j \) be multiplicity free for all \( i \) and \( j \).

**Proof** The condition that \( c_i - c_j \) is multiplicity free for all \( i \) and \( j \) guarantees that there is no 4-cycle between layer \( i \) and layer \( j \), and visa versa. \( \square \)

**Theorem II.12** There is no dual-containing QC-LDPC having girth \( g \geq 6 \).

**Proof** This proof follows directly from theorem II.10 and theorem II.11. If the Type-II regular QC-LDPC code is dual-containing, then by theorem II.10 \( c_i - c_j \) must be multiplicity even for all \( i \) and \( j \). However, theorem II.11 says that this QC-LDPC must contain cycles of length 4. \( \square \)

III. CONSTRUCTION OF QUANTUM QC-LDPC CODES FROM CLASSICAL QC-LDPC CODES

It has been shown that any classical linear binary or quaternary code can be used to construct a corresponding entanglement-assisted quantum error-correcting code [13, 14].

**Theorem III.1** Let \( C(H) \) be a binary classical \([n, k, d]\) code with parity check matrix \( H \). We can obtain a corresponding \([n, 2k - n + c, d, c]\) EAQECC, where \( c = \text{rank}(HH^T) \) is the number of ebits needed.

**Proof** See [17]. \( \square \)

**Remark** An \([n, k', d; c]\) EAQECC encodes \( k' \) logic qubits into \( n \) physical qubits with the help of \( c \) ebits \((c \) copies of maximally entangled states). We define the net rate of such an EAQECC to be \( \frac{r}{c} \). The definition of net rate only takes account of the effective qubits that are sent through a channel if we trade the stronger resource of pure identity qubit channel with the weaker resource of pure entanglement. When \( c = 0 \), this quantity is equal to the “rate” of a standard QECC. Another way to think of this is that the net rate is the rate we can achieve if we “borrow” \( c \) ebits in order to send the codeword, then “pay them back” by using \( c \) communication qubits to establish a new \( c \) ebits. Because this catalytic mode makes no net consumption of ebits, it is quite reasonable to compare the net rate of an EAQECC to the rate of a standard QECC.

In the following, we will consider conditions that will give us \((J, L)\)-regular QC-LDPC codes \( C(H) \) with girth \( g \geq 6 \) and with the rank of \( HH^T \) as small as possible. Let \( \hat{H}(X) = H(X)H(X)^T \) be the polynomial representation of \( HH^T \). In general, \( \hat{H}(X) \) represents a square symmetric matrix \( \hat{H} \) with size \( Jr \times Jr \) that contains \( J^2 \) circulant \( r \times r \) matrices represented by \( \hat{h}_{i,j}(X) \) as defined in (9). Next, we provide two examples to illustrate two different ways of minimizing the rank of the square symmetric matrix represented by \( \hat{H}(X) \). This would minimize the number of ebits when we use the classical code \( C(H) \) to construct the EAQECC.

The first method is to make the matrix \( \hat{H} = HH^T \) become a circulant matrix with a small rank. This can be achieved by properly choosing \( H(X) \) such that the elements \( \hat{h}_{i,j}(X) \) in \( \hat{H}(X) \) satisfying:

\[
\hat{h}_{i,j}(X) = \hat{h}_{i+1,j+1}(X),
\]

(10)

for \( i, j = 0, 1, \ldots, J-2 \). First notice that each polynomial matrix \( \hat{h}_{i,j}(X) \) in \( \hat{H}(X) \) is a circulant matrix of size \( r \times r \), and the polynomial matrix \( \hat{H}(X) \) contains \( J^2 \) such circulant matrices. Since condition (10) guarantees \( \hat{H} \) is itself circulant, \( \hat{H}(X) \) can be represented by some polynomial \( g(X) \) in \( \mathbb{F}_2[X]/(X^{4r} - 1) \). The rank \( \kappa \) of \( \hat{H} \) can then be read off by lemma II.2 If \( \text{gcd}(g(X), X^{4r} - 1) = K(X) \), and the degree of \( K(X) = k \), then \( \kappa = Jr - k \).

Let’s look at an example of this type using a classical Type-I QC-LDPC code. Consider \( r = 16, J = 3, L = 8 \), and the following polynomial parity check matrix \( H(X) \):

\[
H(X) = \begin{bmatrix} 
X & X & X & X & X & X & X & X \\
X & X^2 & X^3 & X^4 & X^5 & X^6 & X^7 & X^8 \\
X & X^3 & X^5 & X^7 & X^9 & X^{11} & X^{13} & X^{15} 
\end{bmatrix}.
\]

(11)

Simple calculation shows that

\[
\hat{h}_{i,j}(X) = \begin{cases} 
0, & i = j, \\
\sum_{k=0}^{7} X^k, & i = j + 1 \\
\sum_{k=0}^{7} X^{2k}, & i = j + 2 
\end{cases}
\]

(12)
Since (12) satisfies (10), $\hat{H}(X)$ represents a circulant matrix, and the polynomial associated with $\hat{H}$ is

$$g(X) = X^{16} \left( \sum_{k=0}^{7} X^k \right) + X^{32} \left( \sum_{k=0}^{7} X^{2k} \right).$$

The degree of $\gcd(g(X), X^{48} - 1) = 30$, therefore by lemma II.2, the number of obits that was needed to construct the corresponding quantum code is only 18. Actually, (11) gives us a $[128, 58; 6; 18]$ EAQECC, and we will refer to this example as “Ex1” later in section IV. The net rate of this code is $(k - c)/n = 40/128$.

Remark The parity check matrix $H$ of (11) gives a $[128, 84, 6]$ classical code. The rate of the QC-LDPC code is actually slightly higher than $(L - J)/L$, since $H$ usually contains linearly dependent rows. For example, each layer of $H$ contains the all-one vector; therefore, we can find at least $(J - 1)$ linearly dependent rows in $H$. The second method is to minimize the rank of each circulant matrix inside $\hat{H}$, that is, to minimize the rank of the circulant matrix represented by $\hat{h}_{i,j}(X), \forall i, j$. Let the rank of $\hat{H}$ be $\kappa$. Then

$$\kappa \leq \sum_{i=1}^{J} \max_{j \in [J]} \kappa_{i,j}. \tag{13}$$

This upper bound is not tight in general, e.g., when $L$ is odd in the Type-I $(J, L)$-regular QC-LDPC codes, the bound of (13) gives $Jr$, which is equal to the number of rows of $\hat{H}$. This is because $\kappa_{i,i} = r$ for every $i$. However, with certain restrictions (e.g., even $L$), we can obtain a reasonable upper bound for $\kappa$.

**Theorem III.2** Given a $(J, L)$-regular QC-LDPC code with polynomial parity check matrix $H(X)$ such that the exponent matrix of $\hat{H}(X) = H(X)H(X)^T$ is multiplicity free, if whenever $\hat{h}_{i,j}(X) \neq 0$, $\gcd(\hat{h}_{i,j}(X), X^r - 1) \neq 1$, then the rank $\kappa$ is upper bounded by $J(r - L + 1)$.

**Proof** Let $\hat{h}_{i,j}$ be the circulant matrix associated with the polynomial $\hat{h}_{i,j}(X)$. Since the exponent matrix of $\hat{H}(X)$ is multiplicity free, then the weight of each row vector of $\hat{h}_{i,j}$ is $L$ whenever $\hat{h}_{i,j}(X) \neq 0$. By corollary II.4

$$\kappa_{i,j} \leq r - L + 1. \tag{14}$$

Define the entanglement consumption rate to be $\kappa/n$. Since the rank decides the number of EPR pairs that are required by the corresponding entanglement-assisted quantum code, we want $\kappa/n$ to be as small as possible. It is easy to see that this bound becomes tighter when we pick $L$ much larger than $J$:

$$\frac{\kappa}{n} \leq J\frac{(r - L + 1)}{n} \leq J\frac{r}{Lr} = \frac{J}{L}. \tag{15}$$

In the following, we present an example showing that the restriction in theorem III.2 is achievable. This example comes from a classical Type-II QC-LDPC code. Consider $r = 16$, $J = 3$, $L = 8$, and the following polynomial parity check matrix $H(X)$:

$$H(X) = \begin{bmatrix} X + X^2 & 0 & X + X^4 & 0 & X + X^6 & 0 & X + X^6 & 0 \\ X^5 & X^5 & X^6 & X^6 & X^7 & X^7 & X^8 & X^8 \\ 0 & X + X^2 & 0 & X + X^4 & 0 & X + X^6 & 0 & X + X^8 \end{bmatrix}. \tag{16}$$

Simple calculation shows that

$$\hat{h}_{i,j}(X) = \begin{cases} 0, & (i, j) = (2, 2), (1, 3), \text{or (3, 1)} \\ \sum_{k=0}^{7} X^{1+2k}, & (i, j) = (1, 1), (3, 3) \\ \sum_{k=0}^{7} X^k, & (i, j) = (2, 1), (2, 3) \end{cases} \tag{17}$$

In this example, (16) satisfies the statement given in theorem II.2. Therefore, the rank of $H(X)$ is upper bounded by 27. The polynomial parity check matrix in (15) gives a $[128, 58; 6; 18]$ quantum QC-LDPC code, and we will refer to this example as “Ex2” in section IV. It also has net rate $(k - c)/n = 40/128$, just like Ex1.

Remark Though no general guidelines of using these two methods to construct the desired polynomial parity check matrix $H(X)$ are given, examples can be obtained with a simple search even when the code length is very long. This is because if we want to construct classical QC-LDPC codes with long length, we will increase the parameter $r$ rather than $L$ and $J$. Choosing larger $r$ increases the distance property of the QC-LDPC codes. Therefore, in general, $L$ and $J$ are not very big, which makes the search not difficult to perform.

**IV. PERFORMANCE**

In this section, we compare the performance of the quantum LDPC codes given in Sec. III to simulation results for two constructions currently available in the dual-containing quantum LDPC codes literature. The
criterion for comparison between these quantum LDPC codes is the net rate (this is equal to the definition of “rate” for standard QECCs). If the quantum codes are constructed by the CSS construction from classical QC-LDPC codes with same parameters \(J\) and \(L\), the corresponding standard QECCs and EAQECCs have almost same rate, which equals \(\frac{L-2J}{L}\). Slight differences in net rates are possible because of the possibility of different numbers of linearly dependent rows in two parity check matrices of classical QC-LDPC codes.

The authors in [10] proposed four dual-containing constructions of quantum LDPC codes. Among these constructions, we use one construction in particular (Ref. [10] called it construction “B”) as a benchmark, since it performed the best of their constructions, especially in the low quantum rate (less than 0.5) and medium code length (less than 10000 qubits) case. Comparison of the performance of all these 4 constructions is illustrated in Fig. 17 on page 35 of [10]. The construction is as follows: take an \(n/2 \times n/2\) cyclic matrix \(C\) with row weight \(L/2\), and define

\[
H_0 = [C, C^T].
\]

We then delete some rows from \(H_0\) to obtain a matrix \(H\) with \(m\) rows. It is easy to verify that \(H\) is dual-containing. Therefore, by the CSS construction [18, 19], we can obtain standard quantum LDPC codes of length \(n\). The advantage of this construction is that the choice of \(n, m,\) and \(L\) is completely flexible; however, the column weight \(J\) is not fixed. We picked \(n = 128, m = 48,\) and \(L = 8\), and called this [[128,32]] quantum LDPC code “Ex-MacKay.” It has rate \(k/n = 32/128\), lower than the net rate of Ex1 and Ex2.

The second example is a standard quantum LDPC code that was constructed from classical QC-LDPC codes [9]. This construction is the first example of standard quantum LDPC codes with no 4-cycles.

**Theorem IV.1** Let \(P\) be an integer which is greater than 2 and \(\sigma\) an element of \(\mathbb{Z}_P^* := \{z : z^{-1}\text{ exists}\}\) with \(\text{ord}(\sigma) \neq |\mathbb{Z}_P^*|\), where \(\text{ord}(\sigma) := \min\{m > 0 | \sigma^m = 1\}\) and \(|X|\) means the cardinality of a set \(X\). If we pick any \(\tau \in \mathbb{Z}_P^* = \{1, \sigma, \sigma^2, \cdots \}\), define

\[
c_{j,l} := \begin{cases} \sigma^{-j-l} & 0 \leq l < L/2 \\ -\sigma^{-j-l+1} & L/2 \leq l < L \end{cases},
\]

\[
d_{k,l} := \begin{cases} \sigma^{-k+l} & 0 \leq l < L/2 \\ -\sigma^{-k+l} & L/2 \leq l < L \end{cases},
\]

and define the exponent matrix \(H_C\) and \(H_D\) as

\[
H_C = [c_{j,l}]_{j \in [J], l \in [L]}, \quad H_D = [d_{k,l}]_{k \in [K], l \in [L]},
\]

where \(L/2 = \text{ord}(\sigma)\) and \(1 \leq J, K \leq L/2\), then \(H_C\) and \(H_D\) can be used to construct quantum QC-LDPC codes with girth at least 6.

Here, we pick the set of parameters \((J, L, P, \sigma, \tau)\) to be \((3, 8, 15, 2, 3)\), to get a code with similar block size and rate to the other examples. The exponent matrices \(H_C\) and \(H_D\) described in theorem IV.1 are

\[
H_C = \begin{bmatrix} 1 & 2 & 4 & 8 & 6 & 12 & 9 & 3 \\ 8 & 1 & 2 & 4 & 12 & 9 & 3 & 6 \end{bmatrix}, \quad (17)
\]

\[
H_D = \begin{bmatrix} 9 & 3 & 6 & 12 & 14 & 11 & 7 & 14 \\ 12 & 9 & 3 & 6 & 13 & 11 & 7 & 14 \end{bmatrix}, \quad (18)
\]

and by the CSS construction, it will give a \([120,38,4]\) quantum QC-LDPC code. We will call this code “Ex-H1”. It has rate \(k/n = 38/120\), slightly higher (just over 1%) than Ex1 and Ex2.

![FIG. 1: (Color online). Performance of quantum LDPC codes with SPA decoding, and 100-iteration](image)

In the simulation, the channel is assumed to be the depolarizing channel, which creates \(X\) errors, \(Y\) errors, and \(Z\) errors with equal probability \(f_m\). Since all these quantum LDPC codes are CSS-type quantum codes, \(Z\) errors and \(X\) errors can be decoded and corrected separately by the sum-product algorithm using a standard classical correction algorithm [10]. Because the depolarizing channel can be thought of as producing a separate list of \(X\) errors and \(Z\) errors (with a \(Y\) error being both an \(X\) and a \(Z\)), doing two separate “classical” correction steps gives an accurate simulation of quantum error correction. For an EAQECC we must include Bob’s half of the shared ebits in the error correction step; these bits, not having passed through the channel, are of course error-free.

We compare the performance of our examples in section [11] with these two dual-containing quantum LDPC codes in figure 1. The performances of Ex1 and Ex2 do not differ much. This is not surprising, since these two codes have similar parameters. The performance of Ex-MacKay is worse than Ex1 and Ex2, probably because there are so many 4-cycles in Ex-MacKay. These
cycles impair the performance of sum-product decoding algorithm. The entanglement-assisted quantum QC-LDPC codes also outperform the quantum QC-LDPC code of Ex-HI, probably because the classical QC-LDPC codes used to construct our examples have better distance properties than the classical QC-LDPC of ex-HI. This simulation result is also consistent with our result in [13]: better classical codes give better quantum codes.

Of course, these 4 simulation results are by no means an exhaustive study of all possible quantum LDPC codes. These four examples were chosen without optimizing for their error-correcting performance; in general, one expects generically similar results for two codes produced by the same construction, so these comparisons are likely to be typical. This does not, of course, argue that there are no standard quantum LDPC codes, or other entanglement-assisted LDPC codes for that matter, with performance superior to those in this paper. However, in general the results match our expectations: it is much easier to find classical codes with good performance (i.e., no 4-cycles and good distance properties), and from them construct quantum codes that use a relatively small amount of shared entanglement, than to satisfy the exact constraint of a dual-containing code; even more so than to simultaneously find a code that is dual-containing and contains no 4-cycles.

Moreover, since these codes use the CSS construction, it is not surprising that the iterative decoding algorithm works well: we are effectively doing two successive classical decoding steps, using classical codes that are known to be good, and reflecting the classical result that codes without 4-cycles tend to outperform codes with 4-cycles. Furthermore, Devetak’s proof of the quantum channel coding theorem shows that codes with a CSS-like structure are good enough to achieve capacity [20]. Therefore, it is quite possible that CSS-type quantum LDPC codes are sufficient to given performance as good as is practically possible. (Though we certainly do not rule out the possibility that studying general additive LDPC codes over GF(4) would reveal interesting properties—that is work for the future.)

V. CONCLUSIONS

There are two advantages of Type-II QC-LDPCs over Type-I QC-LDPCs. First, according to [16] certain configurations of Type-II QC-LDPC codes have larger minimum distance than Type-I QC-LDPC. Therefore, we can construct better quantum QC-LDPCs from classical Type-II QC-LDPC codes. Second, it seems likely that Type-II QC-LDPCs will have more flexibility in constructing quantum QC-LDPC codes with small amount of pre-shared entanglement, because of the ability to insert zero submatrices. However, further investigation of this issue is required.

By using the entanglement-assisted error correction formalism, it is possible to construct EAQECCs from any classical linear code, not just dual-containing codes. We have shown how to do this for two classes of quasi-cyclic LDPC codes (Type-I and Type-II), and proven a number of theorems that make it possible to bound how much entanglement is required to send a code block for codes of these types. Using these results, we have been able to easily construct examples of quantum QC-LDPC codes that require only a relatively small amount of initial shared entanglement, and that perform better (based on numerical simulations) than examples of previously constructed dual-containing quantum LDPC codes. Since in general the properties of quantum codes follows directly from the properties of the classical codes used to construct them, and the evidence of our examples suggests that the iterative decoders can also be made to work effectively on the quantum versions of these codes, this should make possible the construction of large-scale efficient quantum codes. These codes could be useful for quantum communications; conceivably they could also be used as building blocks for standard QECCs that might be of use in quantum computation, though that is still a subject for further research.

We are especially interested in developing a new quantum decoding algorithm in the future. Though the SPA decoding algorithm gives a reasonable trade-off between complexity and performance, it may not be the best choice for decoding quantum errors. One reason for concern is that SPA ignores the purely quantum phenomenon of degeneracy in the decoding process, which could possibly result in introducing more errors instead of correcting them. Though this issue can hopefully be fixed by adding simple heuristic methods on SPA, degeneracy has also been shown to lead to convergence problems for some codes [12], though we have not observed that effect on performance for the codes we present in this paper. If convergence problems prove to be common, we hope these can be overcome by a true quantum decoding algorithm.

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