ON BEST LINEAR UNBIASED ESTIMATION AND PREDICTION UNDER A CONSTRAINED LINEAR RANDOM-EFFECTS MODEL

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Abstract. This paper is concerned with solving some fundamental estimation, prediction, and inference problems on a linear random-effects model with its parameter vector satisfying certain exact linear restrictions. Our work includes deriving analytical formulas for calculating the best linear unbiased predictors (BLUPS) and the best linear unbiased estimators (BLUEs) of all unknown parameters in the model by way of solving certain constrained quadratic matrix optimization problems, characterizing various mathematical and statistical properties of the predictors and estimators, establishing various fundamental rank and inertia formulas associated with the covariance matrices of predictors and estimators, and presenting necessary and sufficient conditions for several equalities and inequalities of covariance matrices of the predictors and estimators to hold.

1. Introduction. Regression analysis is described as certain procedure for finding the mathematical function which best describes the relationship between a dependent variable and one or more independent variables. Linear statistical models are a class of well-known and widely-used parametric regression models in the realm of statistics and applications, which include many concrete forms and various extensions according to given data structures. There are many types of linear regression models, differing in the types of response variable that can be examined and in the strength of parametric assumptions made. Hence they are still topics for research and discussion in statistics and applications. In this paper, we reconsider linear random-effects model (LREM), which is such an extension of linear regression models that allow unknown parameters in the models to be certain random variables. On the other hand, they can also be viewed as special cases of hierarchical linear models or multilevel models. This kind of models has been applied to account for hierarchical data as they provide regression equations for each hierarchical level. It is interesting and important both in their own right and as a starting point for the development of more complicated classes of linear statistical models.

In this paper, we consider the following most commonly used LREM:

\[
y = X\beta + \epsilon, \quad \beta = A\alpha + \gamma, \quad (1.1)
\]

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where \( \mathbf{y} \in \mathbb{R}^{n \times 1} \) is an observed random vector, \( \mathbf{X} \in \mathbb{R}^{n \times p} \) is a known matrix of arbitrary rank, \( \mathbf{\beta} \in \mathbb{R}^{p \times 1} \) is an unknown random parameter vector, \( \mathbf{A} \in \mathbb{R}^{p \times k} \) is a known matrix of arbitrary rank, \( \mathbf{\alpha} \in \mathbb{R}^{k \times 1} \) is a fixed but unknown vector, \( \mathbf{\gamma} \in \mathbb{R}^{p \times 1} \) is an unknown random vector, and \( \mathbf{\varepsilon} \in \mathbb{R}^{n \times 1} \) is a random error vector.

The research in this paper is motivated by a series of previous and recent contributions concerning estimation and inference issues under linear random-effects models subject to restrictions. Here we assume that \( \mathbf{\alpha} \) in (1.1) satisfies the following linear matrix restriction

\[
\mathbf{C} \mathbf{\alpha} = \mathbf{d},
\]

where \( \mathbf{C} \in \mathbb{R}^{q \times k} \) is a known matrix of arbitrary rank, and \( \mathbf{d} \in \mathbb{R}^{q \times 1} \) is a known vector, where the linear restriction equation is consistent. In statistical practice, the matrix equation in (1.2) is an integral part of the regression model, and of course it should be used in the statistical inference of the unknown parameters in (1.1).

Recall that a conventional algebraic technique in the examination of parametric regression models with restrictions to (1.1) and (1.2) are observed and satisfy the following model equations

\[
\mathbf{y}_* = \mathbf{X}_* \mathbf{\beta} + \mathbf{\varepsilon}_* = \mathbf{X}_* \mathbf{A} \mathbf{\alpha} + \mathbf{X}_* \mathbf{\gamma} + \mathbf{\varepsilon}_*, \quad \mathbf{C} \mathbf{\alpha} = \mathbf{d},
\]

which can also be classified as a special case of the constrained linear mixed-effects models (CLMEMs). Assume now that future observations of the response variables under (1.1) and (1.2) are observed and satisfy (1.3), (1.4) in combined forms of vectors.

On the other hand, we can assemble the two regression equations in (1.4) and (1.5) to obtain the following combined vector representation

\[
\mathbf{\tilde{y}} = \mathbf{Z} \mathbf{\alpha} + \mathbf{\tilde{X}} \mathbf{\gamma} + \mathbf{\tilde{\varepsilon}}, \quad \mathbf{\tilde{y}} = \begin{bmatrix} \mathbf{y} \\ \mathbf{d} \end{bmatrix}, \quad \mathbf{Z} = \begin{bmatrix} \mathbf{X} \mathbf{A} \\ \mathbf{C} \end{bmatrix}, \quad \mathbf{\tilde{X}} = \begin{bmatrix} \mathbf{X} \\ \mathbf{0} \end{bmatrix}, \quad \mathbf{\tilde{\varepsilon}} = \begin{bmatrix} \mathbf{\varepsilon} \\ \mathbf{0} \end{bmatrix}.
\]

In this situation, we call (1.5) the original model, and (1.6) an augmented model of (1.5), respectively. In order to make statistical inference under (1.1)–(1.6), assume that the expectation vector and the covariance matrix of the combined random vector in (1.6) satisfy

\[
\begin{bmatrix} \mathbf{\gamma} \\ \mathbf{\varepsilon} \\ \mathbf{\varepsilon}_* \end{bmatrix} = \mathbf{0}, \quad \text{Cov} \begin{bmatrix} \mathbf{\gamma} \\ \mathbf{\varepsilon} \\ \mathbf{\varepsilon}_* \end{bmatrix} = \begin{bmatrix} \mathbf{V}_{11} & \mathbf{V}_{12} & \mathbf{V}_{13} \\ \mathbf{V}_{21} & \mathbf{V}_{22} & \mathbf{V}_{23} \\ \mathbf{V}_{31} & \mathbf{V}_{32} & \mathbf{V}_{33} \end{bmatrix}, \quad \mathbf{V}_{ij} \triangleq \mathbf{V},
\]

where \( \mathbf{V} \in \mathbb{R}^{(p+n+n_*) \times (p+n+n_*)} \) is a known non-negative definite (nd) matrix of arbitrary rank. Under (1.7), the covariance matrix of the combined random vector
\( \hat{y} \) in (1.5) is given by

\[
\text{Cov}(\hat{y}) = \text{Cov}(\hat{X}\gamma + \varepsilon) = \Gamma V\Gamma', \quad \Gamma = \begin{bmatrix} X & I_n & 0 \\ 0 & 0 & 0 \end{bmatrix}.
\]  

(1.8)

This \( \text{Cov}(\hat{y}) \) is a known matrix under the assumptions in (1.1)–(1.7), and will occur in the statistical inferences of (1.5) and (1.6).

The main purpose of this paper is to make a new theoretical analysis on (1.1) under the assumptions in (1.2)–(1.8). In order to obtain some general statistical inference results, we don’t assume any probability distribution assumptions for the random vectors in the models. In addition, we give no further restrictions to the patterns of the submatrices \( V_{ij} \) in (1.7) although they are usually taken as certain prescribed forms for a given linear random-effects model in the statistical literature. In other words, if \( V \) is assumed to be unknown or is given with some parametric forms, such as,

\[
V = \text{diag}(V_{11}, V_{22}, V_{33}), \quad V = \text{diag}(\sigma^2 I_p, \tau^2 I_n, \rho^2 I_n), \quad V = \sigma^2 I_p + \tau^2 I_n + \rho^2 I_n^*,
\]

etc, where \( \sigma^2, \tau^2, \) and \( \rho^2 \) are arbitrary positive scaling factors, people can first substitute various estimators of \( V \), or \( \sigma^2, \tau^2, \) and \( \rho^2 \) from the observed data in (1.6) into the given models and then make further statistical inference to (1.1)–(1.6). Note further from (1.5), (1.6), and (1.7) that \( \hat{y} \) and \( y^* \) are correlated. Hence, it is possible to establish certain predictors of \( y^*, X\beta, \) and \( \varepsilon^* \) in (1.4) from the original observation vector \( \hat{y} \) in (1.5) under the assumptions in (1.1)–(1.8).

It can be seen that statistical inference problems under the assumptions in (1.1)–(1.8) become complicated in comparison with those under a simple linear regression situation. In fact, there was much discussion on LREMs and their many areas of applications in the statistical literature, see e.g., [1, 23] among others. In recent years, people have approached LREMs by means of various novel matrix analysis techniques, and have established some explicit procedure of deriving closed-form formulas for calculating the best linear unbiased predictors (BLUPs) and the best linear unbiased estimators (BLUEs) of all unknown parameters in given LREMs, see e.g., [2, 3, 5–8, 11, 18, 21, 25, 33, 37, 39, 43] under various linear statistical model assumptions. Observe that there are future observations, unknown fixed-effects, random-effects, and error-term vectors in (1.1) and (1.4). Thus it is necessary to propose and establish certain joint linear estimators and predictors of these unknown terms. In fact, much previous and recent work on the problems of joint estimations and predictions of all unknown parameters in a given model were proposed and studied in the statistical literature; see e.g., [4, 9–11, 13–17, 30, 31, 33, 36–39, 43] among others. In view of these facts, we construct the following two general vectors of parametric functions involving the fixed-effects, random-effects, and error terms

\[
h = S\alpha + T\gamma + K\varepsilon + K^*\varepsilon^*,
\]  

(1.9)

where \( S \in \mathbb{R}^{s \times k}, T \in \mathbb{R}^{s \times p}, K \in \mathbb{R}^{s \times n}, \) and \( K^* \in \mathbb{R}^{s \times n^*} \) are given matrices of arbitrary ranks. Note that (1.9) includes all the unknown parameter vectors \( \alpha, \gamma, \varepsilon, \) and \( \varepsilon^* \) in (1.1) and (1.4) as its special cases. Hence the combined form can help establish a unified estimation and prediction theory under the assumptions in (1.1)–(1.8).

In this situation, we see that

\[
E(h) = Sh, \quad \text{Cov}(h) = \begin{bmatrix} \text{Cov}(\hat{y}) & \text{Cov}(\hat{y}, h) \\ \text{Cov}(h, \hat{y}) & \text{Cov}(h) \end{bmatrix} = \begin{bmatrix} \Gamma V\Gamma' & \Gamma V R \Gamma' \\ R \Gamma V' & R \Gamma V R' \end{bmatrix}
\]  

(1.10)
hold, where \( R = \{ T, K, K_s \} \).

As some novel contribution in this respect, we present in this paper a review of some recent developments concerning predictions and estimations of all unknown parameter vectors in the contexts of (1.1)–(1.10). Our work includes to solve the following three problems:

(I) Establish the fundamental matrix equation and analytical expression of the
BLUP of \( h \) in (1.9).

(II) Discuss algebraic and statistical properties of the BLUP and establish additive
decompositions of the BLUP.

(III) Derive formulas related to the covariance matrices of the BLUPs of \( h \) and its
components in (1.9).

We shall deal with many tedious algebraic operations for the given matrices and
vectors when solving these problems because there are 14 given matrices in (1.1)–
(1.7) and (1.9). Hence we need to prepare a variety of matrix analysis tools in the
next section.

2. Notation and some preliminary results. We start with an outline of nota-
tion and some preliminary results adopted in the sequel. We denote by \( \mathbb{R}^{m \times n} \) the
set of all \( m \times n \) real matrices, by \( A' \) the transpose of \( A \), \( r(A) \) the rank of \( A \), i.e.,
the maximum order of the invertible submatrix of \( A \); by \( \mathcal{R}(A) = \{ Ax : x \in \mathbb{R}^{n \times 1}\} \)
the range of a matrix \( A \in \mathbb{R}^{m \times n} \); by \( I_m \) the identity matrix of order \( m \). Both
\( i_+(A) \) and \( i_-(A) \), called the partial inertia of \( A = A' \in \mathbb{R}^{m \times m} \), are defined to be
the number of the positive and negative eigenvalues of \( A \) counted with multi-
licities, respectively. For brief, we use \( i_+(A) \) to denote both numbers. The
four notations \( A \succ 0, A \succcurlyeq 0, A \prec 0, \) and \( A \preccurlyeq 0 \) indicate that \( A \) is a symmetric
positive definite, non negative definite(nde), negative definite, non positive definite,
respectively. Two symmetric matrices \( A \) and \( B \) of the same size are said to satisfy
the inequalities \( A \succ B, A \succcurlyeq B, A \prec B, \) and \( A \preccurlyeq B \) in the Löwner partial ordering
if \( A - B \) is positive definite, positive semi-definite, negative definite, and negative
semi-definite, respectively. The Moore–Penrose generalized inverse of a matrix \( M \),
denoted by \( M^\dagger \), is defined to be the unique matrix \( G \) satisfying the four Penrose
equations MGM = M, GMG = G, (MG)' = MG, and (GM)' = GM. In what
follows, we denote by \( P_M = MM^\dagger, M^\perp = E_M = I_m - MM^\dagger, \( and \) F_M = I_n - M^\dagger M \),
the three orthogonal projectors induced from \( M \), respectively. Further information
about the orthogonal projectors \( P_M, E_M, \) and \( F_M \) with their applications in the
linear statistical models can be found, e.g., in [20,24,27,32].

We next introduce the matrix analysis methods that we shall use in the sequel.
Recall that block matrix and the rank of matrix are two basic conceptual objects
in linear algebra and matrix theory. On the other hand, they have been taken as
two useful analysis tools for dealing with various basic and advanced problems in
theoretical and computational mathematics because they provide us the capacity of
constructing and analyzing various simple and complicated matrix expressions and
equalities in a clear and concise way.

Lemma 2.1. Let \( A, B \in \mathbb{R}^{m \times n} \), or \( A = A', B = B' \in \mathbb{R}^{m \times m} \). Then the following
results hold.

(a) \( A = B \) if and only if \( r(A - B) = 0 \).

(b) \( A - B \) is nonsingular if and only if \( r(A - B) = m \).

(c) \( A \succ B (A \prec B) \) if and only if \( i_+(A - B) = m (i_-(A - B) = m) \).
Lemma 2.2 ([22]). Let $A \in \mathbb{R}^{m \times n}$, $B \in \mathbb{R}^{m \times k}$, and $C \in \mathbb{R}^{l \times n}$. Then

\[
\begin{align*}
    r[A, B] &= r(A) + r(E_A B) = r(B) + r(E_B A), \\
    r\begin{bmatrix} A \\ C \end{bmatrix} &= r(A) + r(CF_A) = r(C) + r(AF_C), \\
    r\begin{bmatrix} A' \\ B' \\ 0 \end{bmatrix} &= r[A, B] + r(B).
\end{align*}
\]  

In addition, the following results hold.

(a) $r[A, B] = r(A) \Leftrightarrow \mathcal{R}(B) \subseteq \mathcal{R}(A) \Leftrightarrow AA'B = B \Leftrightarrow E_A B = 0$.

(b) $r\begin{bmatrix} A \\ C \end{bmatrix} = r(A) \Leftrightarrow \mathcal{R}(C') \subseteq \mathcal{R}(A') \Leftrightarrow CA'A = C \Leftrightarrow CF_A = 0$.

(c) $r[A, B] = r(A) + r(B) \Leftrightarrow \mathcal{R}(A) \cap \mathcal{R}(B) = \{0\} \Leftrightarrow \mathcal{R}((E_B A)') = \mathcal{R}(B') \Leftrightarrow \mathcal{R}(AF_C) = \mathcal{R}(C) \Leftrightarrow \mathcal{R}(AF_C) = \mathcal{R}(A)$.

(d) $r\begin{bmatrix} A' \\ B' \\ C' \end{bmatrix} = r(A) + r(C) \Leftrightarrow \mathcal{R}(A') \cap \mathcal{R}(C') = \{0\} \Leftrightarrow \mathcal{R}(CF_A) = \mathcal{R}(C) \Leftrightarrow \mathcal{R}(AF_C) = \mathcal{R}(A)$.

Lemma 2.3 ([34]). If $\mathcal{R}(A_1') \subseteq \mathcal{R}(B_1')$, $\mathcal{R}(A_2) \subseteq \mathcal{R}(B_1)$, $\mathcal{R}(A_3') \subseteq \mathcal{R}(B_2')$, and $\mathcal{R}(A_3) \subseteq \mathcal{R}(B_2)$, then

\[
\begin{align*}
    r(A_1 B_1^\dagger A_2) &= r\begin{bmatrix} B_1 \\ A_1 \end{bmatrix} - r(B_1), \\
    r(A_1 B_1^\dagger A_2 B_3^\dagger A_3) &= r\begin{bmatrix} 0 & B_2 & A_3 \\ B_1 & A_2 & 0 \\ A_1 & 0 & 0 \end{bmatrix} - r(B_1) - r(B_2).
\end{align*}
\]

The fundamental inertia formulas in the following lemma are well known or follow directly from the definition of the inertia of symmetric matrix.

Lemma 2.4 ([35]). Let $A = A' \in \mathbb{R}^{m \times m}$, $B \in \mathbb{R}^{m \times n}$, $D = D' \in \mathbb{R}^{n \times n}$, and assume that $P \in \mathbb{R}^{m \times m}$ is nonsingular. Then

\[
\begin{align*}
    r(A) &= i_+(A) + i_-(A), \\
    i_{\pm}(PAP') &= i_{\pm}(A) \quad \text{(Sylvester’s law of inertia)}, \\
    i_{\pm}(A^\dagger) &= i_{\pm}(A), \quad i_{\pm}(-A) = i_{\mp}(A), \\
    i_{\pm}\begin{bmatrix} A \\ 0 \end{bmatrix} &= i_{\pm}(A) + i_{\pm}(D), \\
    i_{\pm}\begin{bmatrix} 0 \\ B' \\ 0 \end{bmatrix} &= i_{\pm}\begin{bmatrix} 0 \\ B' \\ 0 \end{bmatrix} = r(B), \\
    i_{\pm}\begin{bmatrix} A \\ B' \\ 0 \end{bmatrix} &= r(B) + i_{\pm}(E_B A E_B), \\
    i_{\pm}\begin{bmatrix} A \\ B' \\ 0 \end{bmatrix} &= r[A, B], \quad i_{\pm}\begin{bmatrix} A \\ B' \\ 0 \end{bmatrix} = r(B), \quad \text{if } A \succeq 0, \\
    i_{\pm}\begin{bmatrix} A \\ B' \\ D \end{bmatrix} &= i_{\pm}(A) + i_{\pm}\begin{bmatrix} 0 \\ E_A B \\ B'E_A D - B'A^\dagger B \end{bmatrix}.
\end{align*}
\]
In particular,
\[
    i_\pm \begin{bmatrix} A & B \\ B' & D \end{bmatrix} = i_\pm (A) + i_\pm (D - B'A) \quad \text{if } \mathcal{R}(B) \subseteq \mathcal{R}(A).
\]

The following lemma is well known in matrix theory and applications.

**Lemma 2.5** ([26]). The linear matrix equation \( AX = B \) is consistent if and only if \( r(A, B) = r(A) \), or equivalently, \( AA'B = B \). In this case, the general solution of the equation can be written in the following parametric form \( X = A'B + (I - A'A)U \), where \( U \) is an arbitrary matrix.

It is well known that statistical theory and optimization theory have been closely linked in many ways. In particular, statisticians search best estimations and predictions of unknown parameters in parametric regression models by means of various mathematical optimization tools. In order to directly solve the classic matrix minimization problem in (3.1), we shall use the following remarkable result concerning analytical solutions of a constrained quadratic matrix-valued function minimization problem.

**Lemma 2.6** ([30]). Let \( A \in \mathbb{R}^{p \times q} \), \( B \in \mathbb{R}^{n \times q} \), \( C \in \mathbb{R}^{m \times p} \), and \( D \in \mathbb{R}^{n \times m} \) be given, \( M \in \mathbb{R}^{m \times m} \) be nnd matrix, and assume that the matrix equation \( GA = B \) is solvable for the variable matrix \( G \in \mathbb{R}^{n \times p} \). Then the following constrained quadratic matrix-valued function optimization problem in the Löwner partial ordering
\[
    f(G) = (GC + D)M(GC + D)' = \min \quad \text{s.t. } GA = B
\]
is always solvable for \( G \). In this situation, the variable matrix satisfying (2.15) is determined by the matrix equation
\[
    G_0 \begin{bmatrix} A & CMC'A^\perp \end{bmatrix} = \begin{bmatrix} B, -DMC'A^\perp \end{bmatrix}.
\]
Correspondingly, the analytical expressions of \( G_0 \) and \( f(G_0) \) as well as an additive decomposition of \( f(G) \) are given by
\[
    G_0 = \arg\min_{GA = B} f(G) = \begin{bmatrix} B, -DMC'A^\perp \end{bmatrix} \begin{bmatrix} A, CMC'A^\perp \end{bmatrix}' + U \begin{bmatrix} A, CMC' \end{bmatrix}^\perp',
\]
\[
    f(G_0) = \min_{GA = B} f(G) = WMW' - WMC' (A^\perp CMC'A^\perp)' CMW',
\]
\[
    f(G) = f(G_0) + (GC + D)MC' (A^\perp CMC'A^\perp)' CM (GC + D)',
\]
where \( W = BA'C + D \) and \( U \in \mathbb{R}^{n \times p} \) is arbitrary.

3. Fundamental theory of BLUPs and BLUEs under a CLREM. In this section, we establish a fundamental framework on BLUPs and BLUEs under the assumptions in (1.1)–(1.10) by means of the conventional statistical techniques in the mainstream of regression theory. We begin with the classic conceptional definition of consistency of (1.5), which was first introduced and studied in [28, 29].

**Definition 3.1.** Model (1.5) is said to be consistent if and only if the vector inclusion \( \hat{y} \in \mathcal{R}[Z, \Gamma V'T] \) holds with probability 1.

In the following, we make statistical inference of the CLREM in (1.5) under the assumption that the model is consistent. The definition of predictability and the BLUP of \( h \) in (1.9) is given below.
This page contains mathematical definitions and theorems related to the BLUP (Best Linear Unbiased Predictor) in the context of constrained linear random-effects models. The text is discussing the properties and equations related to the BLUP, including the matrix equation and predictions under certain conditions.

**Definition 3.2.** Let $h$ be as given in (1.9). The linear statistic $G\hat{y}$, where $G \in \mathbb{R}^{n \times (n+q)}$, is said to have the same expectation with $h$, or $h$ is said to be predictable by $\hat{y}$ in (1.5), if and only if $E(G\hat{y} - h) = 0$ holds.

**Definition 3.3.** Let $h$ be as given in (1.9) and assume that $h$ is predictable under (1.5). If there exists a matrix $G$ such that

\[ \text{Cov}(G\hat{y} - h) = \min \text{ subject to } E(G\hat{y} - h) = 0 \]

holds in the Löwner partial ordering, then the linear statistic $G\hat{y}$ is defined to be the BLUP of $h$ in (1.9), and is denoted by

\[ G\hat{y} = \text{BLUP}(h) = \text{BLUP}(S\alpha + T\gamma + K\varepsilon + K_*\varepsilon_*) . \]

In particular, if $h = S\alpha$, then the linear statistic $G\hat{y}$ satisfying (3.1) is the BLUE of $S\alpha$ under (1.5), and is denoted by

\[ G\hat{y} = \text{BLUE}(S\alpha) . \]

In what follows, we present an algebraic process for establishing and solving the BLUP’s equation of the parameter space in (1.9). Note from (1.5) and (1.9) that $G$ holds for some $G$. It is obvious that $E(h - \hat{y}) = 0$ holds if and only if $E(G\hat{y} - h) = 0$ holds for some $G$. Then the expectation and covariance of $G\hat{y} - h$ can be expressed as

\[ E(G\hat{y} - h) = E((GZ - S)\alpha) = (GZ - S)\alpha , \]

\[ \text{Cov}(G\hat{y} - h) = (G\tilde{X} - T)\gamma + (G\tilde{I} - K)\varepsilon - K_*\varepsilon_* , \]

\[ \tilde{I} = \begin{bmatrix} I_n & 0 \end{bmatrix} . \]

Hence, we convert the constrained covariance matrix minimization problem in (3.1) to a quadratic matrix-valued function minimization problem of $f(G)$ subject to the consistent linear matrix equation $GZ = S$. We have the following result on the predictability of $h$ in (1.9).

**Lemma 3.4.** The vector $h$ in (1.9) is predictable by $\hat{y}$ in (1.5), i.e., $E(G\hat{y} - h) = 0$ holds for some $G$, if and only if

\[ \mathcal{R}(S') \subseteq \mathcal{R}(Z') . \]

**Proof.** It is obvious that $E(G\hat{y} - h) = 0 \iff GZ\alpha - S\alpha = 0$ for all $\alpha \iff GZ = S$. By Lemma 2.5, the matrix equation is consistent if and only if (3.7) holds.

Our main results on the matrix equation and explicit formulas of the BLUP of $h$ in (1.9), as well as some fundamental properties of the BLUP are presented below.

**Theorem 3.5.** Assume that $h$ in (1.9) is predictable by $\hat{y}$ in (1.5), namely, (3.7) holds, and let $\Gamma$, $R$, and $\tilde{I}$ be as given in (1.8), (1.10), and (3.4), respectively. Then

\[ \text{Cov}(G_0\hat{y} - h) = \min \text{ s.t. } E(G_0\hat{y} - h) = 0 \]

\[ \iff G_0[Z, \text{Cov}(\hat{y})Z^\perp] = [S, \text{Cov}(h, \hat{y})Z^\perp] . \]

The matrix equation in (3.8) is consistent, i.e.,

\[ [S, \text{Cov}(h, \hat{y})Z^\perp][Z, \text{Cov}(\hat{y})Z^\perp]^\perp[Z, \text{Cov}(\hat{y})Z^\perp] = [S, \text{Cov}(h, \hat{y})Z^\perp] \]

(3.9)
holds under (3.7), and the general solution of the matrix equation and the corresponding BLUP(h) can be written as
\[
\text{BLUP}(h) = G_0\hat{y},
\]
where the general solution \( G_0 \) of the matrix equation in (3.8) is
\[
G_0 = \arg\min_{G \in S} f(G) = [S, \text{Cov}(h, \hat{y})Z^+] \{ [Z, \text{Cov}(\hat{y})Z^+]^T + U[Z, \text{Cov}(\hat{y})Z^+] \}
\]
where \( U \in \mathbb{R}^{s\times(n+q)} \) is arbitrary. The corresponding \( f(G_0) \) under (3.6), namely, the covariance matrix of the difference of BLUP(h) - h is
\[
f(G_0) = \text{Cov}(\text{BLUP}(h) - h) = ([S, RVT'Z^+] \{ [Z, \Gamma VT'Z^+] | T - R \} V \times ([S, RVT'Z^+] \{ [Z, \Gamma VT'Z^+] | T - R \})^T.
\]
The difference of the covariance matrices of h and BLUP(h) is
\[
\text{Cov}(h) - \text{Cov}(\text{BLUP}(h)) = RVR' - ([S, RVT'Z^+] \{ [Z, \Gamma VT'Z^+] \} \Gamma VT')
\]
\[
\times ([S, RVT'Z^+] \{ [Z, \Gamma VT'Z^+] \} \Gamma V').
\]
In addition, the following results hold.
(a) \([27, \text{p. 123}] \) if and only if \( r[Z, \Gamma V'] = n + q \).
(b) \( G_0 \) is unique if and only if \( r[Z, \Gamma V] = n + q \).
(c) BLUP(h) is unique with probability 1 if and only if (1.5) is consistent.
(d) The covariance matrix of BLUP(h) is
\[
\text{Cov}(\text{BLUP}(h)) = ([S, RVT'Z^+] \{ [Z, \Gamma VT'Z^+] \} \Gamma VT')
\]
\[
\times ([S, RVT'Z^+] \{ [Z, \Gamma VT'Z^+] \} \Gamma V')^T.
\]
Proof. Under (3.6), we see from Lemma 2.5 that the first part of (3.8) is equivalent to finding a solution \( G_0 \) of the consistent matrix equation \( GZ = S \) such that
\[
f(G) \succeq f(G_0) \quad \text{for all } GZ = S
\]
holds in the Löwner partial ordering. Furthermore from Lemma 2.6, there always exists a solution \( G_0 \) of \( GZ = S \) such that (3.17) holds, and the \( G_0 \) is determined by the matrix equation \( G_0[Z, \Gamma VT'Z^+] = [S, RVT'Z^+] \), establishing the matrix equation in (3.8). Solve the equation by Lemma 2.5 to give (3.11). Also from (3.6),
\[
f(G_0) = \text{Cov}(G_0\hat{y} - h) = \text{Cov}(h - G_0\hat{y}) = (G_0\Gamma - R)\Gamma V(G_0\Gamma - R)',
\]
Corollary 3.6. Let $h$ be as given in (1.9). Then the following results hold.

(a) If $h$ is predictable by $\tilde{y}$ in (1.5), then $Ph$ is predictable by $\tilde{y}$ in (1.5) as well for any matrix $P \in \mathbb{R}^{p \times s}$, and $\text{BLUP}(Ph) = P\text{BLUP}(h)$ holds.

(b) If $h$ in (1.9) is predictable by $\tilde{y}$ in (1.5), then $S\alpha$ is estimable by $\tilde{y}$ in (1.5) as well, and the $\text{BLUP}$ of $h$ can be decomposed as the sum

$$\text{BLUP}(h) = \text{BLUE}(S\alpha) + \text{BLUP}(T\gamma) + \text{BLUP}(K\varepsilon) + \text{BLUP}(K,\varepsilon_*),$$

(3.18)

and the following formulas for covariance matrices hold

$$\text{Cov}\{\text{BLUP}(S\alpha), \text{BLUP}(T\gamma + K\varepsilon + K_*\varepsilon_*)\} = 0,$$

(3.19)

$$\text{Cov}(\text{BLUP}(h)) = \text{Cov}(\text{BLUE}(S\alpha)) + \text{Cov}(\text{BLUP}(T\gamma + K\varepsilon + K_*\varepsilon_*)).$$

(3.20)

(c) If $\alpha$ in (1.9) is estimable under (1.5), then the $h$ in (1.9) is predictable by $\tilde{y}$ in (1.5). In this case,

$$\text{BLUP} \begin{bmatrix} \alpha \\ \gamma \\ \varepsilon \\ \varepsilon_* \end{bmatrix} = \begin{bmatrix} \text{BLUE}(\alpha) \\ \text{BLUP}(\gamma) \\ \text{BLUP}(\varepsilon) \\ \text{BLUP}(\varepsilon_*) \end{bmatrix},$$

(3.21)

$$\text{BLUP}(h) = \text{SBLE}(\alpha) + \text{TBLUP}(\gamma) + \text{KBLUP}(\varepsilon) + K_*,\text{BLUP}(\varepsilon_*).$$

(3.22)

Proof. The predictability of $Ph$ follows from $\mathcal{R}(Z') \supseteq \mathcal{R}(S') \supseteq \mathcal{R}(SP^T)$. Also from (3.11),

$$\text{BLUP}(Ph) = (SP, \text{PRV}\Gamma'Z^\perp \parallel Z, \text{GVT}'Z^\perp \parallel \Gamma^\perp)\tilde{y}$$

$$= P ([S, \text{RV}\Gamma'Z^\perp \parallel Z, \text{GVT}'Z^\perp \parallel \Gamma^\perp] + U_1[Z, \text{GVT}'Z^\perp \parallel \Gamma^\perp])\tilde{y}$$

$$= P\text{BLUP}(h),$$

where $U = PU_1$, as required for (a).

Note that $[S, \text{RV}\Gamma'Z^\perp \parallel Z, \text{GVT}'Z^\perp \parallel \Gamma^\perp]$ in (3.11) can be decomposed as

$$[S, \text{RV}\Gamma'Z^\perp \parallel Z, \text{GVT}'Z^\perp \parallel \Gamma^\perp]$$

$$= [S, 0 \parallel Z, \text{GVT}'Z^\perp \parallel \Gamma^\perp] + [0, T, 0 \parallel \text{VT}'Z^\perp \parallel Z, \text{GVT}'Z^\perp \parallel \Gamma^\perp]$$

$$+ [0, 0, K, 0 \parallel \text{VT}'Z^\perp \parallel Z, \text{GVT}'Z^\perp \parallel \Gamma^\perp]$$

$$+ [0, 0, 0, K_* \parallel \text{VT}'Z^\perp \parallel Z, \text{GVT}'Z^\perp \parallel \Gamma^\perp],$$
Corollary 3.7. Let 

\[ \text{BLUE}(S\alpha), \text{BLUE}(T\gamma + K\varepsilon + K_s\varepsilon_s) \] 

establishing (3.18). We also obtain from (3.11) the covariance matrix between 

\[ \text{BLUE}(S\alpha), \text{BLUE}(T\gamma + K\varepsilon + K_s\varepsilon_s) \] 

as follows 

\[
\text{Cov}\{ \text{BLUE}(S\alpha), \text{BLUE}(T\gamma + K\varepsilon + K_s\varepsilon_s) \} = \begin{bmatrix} 0 \\ S \end{bmatrix} 0 || Z, \Gamma\Gamma'Z^\perp | | \Gamma\Gamma'([0, RV\Gamma'Z^\perp] [Z, \Gamma\Gamma'Z^\perp])'.
\] (3.23)

Applying (2.5) to (3.23) and simplifying, we obtain 

\[
r(Cov\{ \text{BLUE}(S\alpha), \text{BLUE}(T\gamma + K\varepsilon + K_s\varepsilon_s) \})
\]

\[
= r([S, 0 || Z, \Gamma\Gamma'Z^\perp])\Gamma\Gamma'([0, RV\Gamma'Z^\perp] [Z, \Gamma\Gamma'Z^\perp])
\]

\[
= r\begin{bmatrix} 0 \\ Z, \Gamma\Gamma'Z^\perp \\ [S, 0] \\ [Z, 0] \\ [Z, 0] \\ [S, 0] \\ \end{bmatrix} + 2r [Z, \Gamma\Gamma']
\]

\[
= r[Z, \Gamma\Gamma'] + 2r [Z, \Gamma\Gamma'] = 0,
\]

which implies that 

\[ \text{Cov}\{ \text{BLUE}(S\alpha), \text{BLUE}(T\gamma + K\varepsilon + K_s\varepsilon_s) \} \]

is a zero matrix, establishing (3.19). Eq. (3.20) follows from (3.18) and (3.19). Eqs. (3.21) and (3.22) follow from (a) and (3.18).

Corollary 3.7. Let \( V \) and \( \Gamma \) be as given in (1.7) and (1.8), respectively, and assume that 

\[
h_1 = S_1\alpha + T_1\gamma + K_1\varepsilon + K_s\varepsilon_s, \quad h_2 = S_2\alpha + T_2\gamma + K_2\varepsilon + K_s\varepsilon_s
\]

are predictable under (1.5), where \( S_1, S_2 \in \mathbb{R}^{s \times k}, T_1, T_2 \in \mathbb{R}^{s \times p}, K_1, K_2 \in \mathbb{R}^{s \times n}, \) and \( K_s, K_s \in \mathbb{R}^{s \times n} \) are known matrices, and denote \( R_i = [T_i, K_i, K_s], i = 1, 2. \) Then the following results hold.

(a) The sum \( h_1 + h_2 \) is predictable under (1.5), and its BLUP satisfies 

\[
\text{BLUP}(h_1 + h_2) = \text{BLUP}(h_1) + \text{BLUP}(h_2).
\] (3.24)

(b) \( \text{BLUP}(h_1) = \text{BLUP}(h_2) \iff S_1 = S_2 \) and \( \mathcal{R}(\Gamma\Gamma'R_i - \Gamma\Gamma'R') \subseteq \mathcal{R}(\Gamma) \).

Proof. (3.24) follows from Corollary 3.6(a) and (3.18). By Theorem 3.5, the two equations for the coefficient matrices of \( \text{BLUP}(h_1) = G_1\hat{y} \) and \( \text{BLUP}(h_2) = G_2\hat{y} \) are given by 

\[
G_1[Z, \Gamma\Gamma'Z^\perp] = [S_1, R_1\Gamma'Z^\perp], \quad G_2[Z, \Gamma\Gamma'Z^\perp] = [S_2, R_2\Gamma'Z^\perp].
\]
This pair of matrix equations have a common solution if and only if

\[
\begin{bmatrix}
Z & \Gamma V' Z^\perp \\
S_1 & R_1 V' Z^\perp
\end{bmatrix}
\begin{bmatrix}
Z & \Gamma V' Z^\perp \\
S_2 & R_2 V' Z^\perp
\end{bmatrix}
= \begin{bmatrix}
r[Z, \Gamma V' Z^\perp, Z, \Gamma V' Z^\perp]
\end{bmatrix},
\tag{3.25}
\]

where

\[
\begin{bmatrix}
Z & \Gamma V' Z^\perp \\
S_1 & R_1 V' Z^\perp
\end{bmatrix}
\begin{bmatrix}
Z & \Gamma V' Z^\perp \\
S_2 & R_2 V' Z^\perp
\end{bmatrix}
= \begin{bmatrix}
r\begin{bmatrix}
Z & \Gamma V' Z^\perp \\
S_2 & R_2 V' Z^\perp
\end{bmatrix} + r[S_2 - S_1, (R_2 V' - R_1 V') Z^\perp]
\end{bmatrix},
\]

and \( r[S_2 - S_1, (R_2 V' - R_1 V') Z^\perp] = 0 \). Hence, (3.25) is equivalent to

\[
\begin{bmatrix}
Z & \Gamma V' Z^\perp \\
S_1 & R_1 V' Z^\perp
\end{bmatrix}
\begin{bmatrix}
0 & 0
\end{bmatrix}
= 0,
\]

which is further equivalent to (b) by Lemma 2.2 (b).

From (3.18), we directly obtain two fundamental decomposition identities for \( y \) and \( y^* \) in (1.3) and (1.4) as follows.

**Corollary 3.8.** The vector \( y \) and \( \hat{y} \) in (1.3) and (1.5) are always predictable by \( \hat{y} \) in (1.5), while \( y^* \) and \( \tilde{y} \) in (1.4) and (1.6) are predictable by \( \hat{y} \) in (1.5) if and only if \( \mathcal{R}((X, A)') \subseteq \mathcal{R}(Z') \). In this case, the following two additive decomposition identities hold

\[
y = \text{BLUP}(X\beta) + \text{BLUP}(\varepsilon)
= \text{BLUE}(XA\alpha) + \text{BLUP}(X\gamma) + \text{BLUP}(\varepsilon),
\tag{3.26}
\]

\[
\text{BLUP}(y_*) = \text{BLUP}(X_*\beta) + \text{BLUP}(\varepsilon_*)
= \text{BLUE}(X_*A\alpha) + \text{BLUP}(X_*\gamma) + \text{BLUP}(\varepsilon_*),
\tag{3.27}
\]

\[
\hat{y} = \text{BLUE}(Z\alpha) + \text{BLUP}(\bar{X}\gamma) + \text{BLUP}(\bar{\varepsilon}),
\tag{3.28}
\]

\[
\text{BLUP}(\tilde{y}) = \begin{bmatrix}
\hat{y} \\
\text{BLUP}(y_*)
\end{bmatrix}.
\tag{3.29}
\]

The unconditional additive decomposition equalities in (3.26) and (3.27) can be viewed as two fundamental facts and formulas in the statistical inferences of linear regression models, especially, we take them as orthodox optimization criteria in the comparison of other predictors/estimators of unknown parameters in CLREMs. Moreover, we are able to obtain the formulas in the preceding theorems and corollaries, various properties of the BLUPs under (1.5). For example, it is easy to see from (3.12) that \( \text{Cov}(\text{BLUP}(h) - h) \) plays a key role in defining the BLUP of \( h \) in (1.9). Thus we need to know more features of \( f(G_0) = \text{Cov}(\text{BLUP}(h) - h) \) from mathematical point of view. In recent years, the matrix rank/inertia methodology have been introduced into the analysis of linear regression models; see e.g., [5–8, 12–14, 16, 19, 40–42]. They proved many closed-form formulas for calculating ranks/inertias of covariance matrices of projectors/estimators under linear regression models and gave many valuable applications of the formulas in characterising performance of projectors/estimators. We next present a group of explicit formulas for calculating the rank and inertia of the difference \( \text{Cov}(\text{BLUP}(h) - h) - H \), where \( H \) is a symmetric matrix.
Theorem 3.9. Let \( h \) be as given in (1.9), and assume that \( h \) is predictable by \( \hat{\mathbf{y}} \) in (1.3). Also, let \( H \in \mathbb{R}^{s \times s} \) be a symmetric matrix, and denote

\[
M = \begin{bmatrix}
\text{Cov}(\hat{\mathbf{y}}) & \text{Cov}(\hat{\mathbf{y}}, h) & Z \\
\text{Cov}(h, \hat{\mathbf{y}}) & \text{Cov}(h) - H S & \Gamma V' \\
Z' & S' & 0
\end{bmatrix} = \begin{bmatrix}
\Gamma V' & \Gamma V R' & Z \\
\Gamma V & R V' & R V R' - H S \\
Z' & S' & 0
\end{bmatrix}.
\]

Then the following inertia and rank formulas hold.

\[
r\{\text{Cov}(\text{BLUP}(h) - h) - H\} = r(M) - r[Z, \Gamma V] - r(Z), \quad (3.30)
\]

\[
i_+ \{\text{Cov}(\text{BLUP}(h) - h) - H\} = i_+(M) - r[Z, \Gamma V], \quad (3.31)
\]

\[
i_- \{\text{Cov}(\text{BLUP}(h) - h) - H\} = i_-(M) - r(Z) \quad (3.32)
\]

In consequence, the following results hold.

(a) \( \text{Cov}(\text{BLUP}(h) - h) \succ H \iff i_+(M) = r[Z, \Gamma V] + s \).
(b) \( \text{Cov}(\text{BLUP}(h) - h) \prec H \iff i_-(M) = r[Z, \Gamma V] + s \).
(c) \( \text{Cov}(\text{BLUP}(h) - h) \succcurlyeq H \iff i_+(M) = r[Z] \).
(d) \( \text{Cov}(\text{BLUP}(h) - h) \preccurlyeq H \iff i_-(M) = r[Z] \).
(e) \( \text{Cov}(\text{BLUP}(h) - h) = H \iff r(M) = r[Z, \Gamma V] + r(Z) \).

Proof. Note from (3.12) that

\[
H - \text{Cov}(\text{BLUP}(h) - h) = H - \left( [S, \Gamma V] \Gamma V \Gamma [S, \Gamma V] ^\dagger \Gamma - R \right) V \left( [S, \Gamma V] \Gamma V \Gamma [S, \Gamma V] ^\dagger \Gamma - R \right) ^\dagger.
\]

Applying (2.14) to this expression and simplifying, we obtain

\[
i_\pm (H - \text{Cov}(\text{BLUP}(h) - h)) = i_\pm \left( H - \left( [S, \Gamma V] \Gamma V \Gamma [S, \Gamma V] ^\dagger \Gamma - R \right) V \left( [S, \Gamma V] \Gamma V \Gamma [S, \Gamma V] ^\dagger \Gamma - R \right) ^\dagger \right)
\]

\[
= i_\pm \left( [S, \Gamma V] \Gamma V \Gamma [S, \Gamma V] ^\dagger \Gamma - R \right) V \left( [S, \Gamma V] \Gamma V \Gamma [S, \Gamma V] ^\dagger \Gamma - R \right) ^\dagger - i_\pm (V)
\]

\[
= i_\pm \left( \begin{bmatrix} V & -V R' \end{bmatrix} \right)
\]

\[
+ \begin{bmatrix} 0 & 0 \\ 0 & [S, \Gamma V] \Gamma V \Gamma [S, \Gamma V] ^\dagger \end{bmatrix} \left( \begin{bmatrix} [Z, \Gamma V] \Gamma V \Gamma [S, \Gamma V] ^\dagger \end{bmatrix} \right)^\dagger \begin{bmatrix} 0 & 0 \\ 0 & [S, \Gamma V] \Gamma V \Gamma [S, \Gamma V] ^\dagger \end{bmatrix} + i_\pm (V)
\]

\[
= i_\pm \left( \begin{bmatrix} 0 & -Z & -\Gamma V \Gamma Z' & \Gamma V & 0 \\ -Z' & 0 & 0 & 0 & S' \\ -Z' \Gamma V & 0 & 0 & 0 & Z' \Gamma V R' \\ \Gamma V' & 0 & 0 & V & -V R' \\ 0 & S & \Gamma V' \Gamma Z' & -R V & H \end{bmatrix} - i_\pm (V) \right)
\]
Corollary 3.10. Let \( h \) be as given in (1.9), and assume that \( h \) is predictable by \( \tilde{y} \) in (1.5). Then

\[
 r\{\text{Cov}(\text{BLUP}(h) - h)\} = r\left[ \text{Cov}\left[ \begin{array}{c} \tilde{y} \\ h \end{array} \right], \begin{bmatrix} Z \\ S \end{bmatrix} \right] - r[\text{Cov}(\tilde{y}), Z]. \tag{3.33}
\]

In consequence, the following results hold.

(a) \( \text{Cov}(\text{BLUP}(h) - h) \succ 0 \leftrightarrow r\left[ \text{Cov}\left[ \begin{array}{c} \tilde{y} \\ h \end{array} \right], \begin{bmatrix} Z \\ S \end{bmatrix} \right] = r[\text{Cov}(\tilde{y}), Z] + s. \)

(b) \( \text{Cov}(\text{BLUP}(h) - h) = 0 \leftrightarrow \text{BLUP}(h) = h \) holds with probability 1 \( \Leftrightarrow \)

\[
r\left[ \text{Cov}\left[ \begin{array}{c} \tilde{y} \\ h \end{array} \right], \begin{bmatrix} Z \\ S \end{bmatrix} \right] = r[\text{Cov}(\tilde{y}), Z].
\]
Corollary 3.11. Assume that $h$ in (1.9) is predictable by $\hat{y}$ in (1.5), and let $G$ satisfy $\mathbb{E}(G\hat{y} - h) = 0$. Then

$$r\{\text{Cov}(G\hat{y} - h) - \text{Cov}(\text{BLUP}(h) - h)\} = r\left[\frac{GFV\Gamma' - RV\Gamma'}{Z'}\right] - r(Z). \quad (3.34)$$

Proof. Notice that $(Z\Gamma V\Gamma'Z\perp)\dagger \succeq 0$ and

$$\mathcal{R}\left((GFV\Gamma'Z\perp - RV\Gamma'Z\perp)\right) \subseteq \mathcal{R}(Z\Gamma V\Gamma'Z\perp) = \mathcal{R}(Z\Gamma V) .$$

Then, we obtain from (2.2) and (3.13) that

$$r(\text{Cov}(G\hat{y} - h) - \text{Cov}(\text{BLUP}(h) - h))$$

$$= r((GFV\Gamma'Z\perp - RV\Gamma'Z\perp)(Z\perp\Gamma V\Gamma'Z\perp)\dagger)$$

$$= r(GFV\Gamma'Z\perp - RV\Gamma'Z\perp)$$

$$= r\left[\frac{GFV\Gamma' - RV\Gamma'}{Z'}\right] - r(Z),$$

as required for (3.34).

4. Conclusions. We have presented a group of theoretical results and facts on matrix equations and formulas for computing the BLUPs of joint vectors of all unknown parameters in the context of (1.1)–(1.10) by employing a mixed regression framework, including a set of analytical formulas for calculating ranks/inertias of covariance matrices of the BLUPs, and have used the formulas to describe some fundamental equalities and inequalities for the covariance matrices of the BLUPs. This research shows that fundamental problems on features and performances of BLUPs/BLUEs can be appropriately formulated and solved by means of various matrix analysis tools. It is believed that more intriguing and sophisticated formulas, equalities, and inequalities associated with various types of predictors and estimators can be derived under statistical models, which, we believe, will provide more theoretical support to regression analysis.

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