Existence and data-dependence theorems for fractional impulsive integro-differential system

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Abstract
In this article we have considered a fractional order impulsive integro-differential equation (IDE) in Caputo’s sense for the unique solution and data dependence results. We take help of the Banach fixed point theory and basic literature of fractional calculus. The results are examined with the help of an expressive numerical example for an application of the results.

MSC: Primary 26A33; secondary 34A12; 34K40; 47H08
Keywords: Fractional impulsive IDE; Banach fixed point theorem; Stability analysis; Uniqueness of solution

1 Introduction
Modeling with the help of fractional order integral and differential operators is very common in the community of engineers and scientists due to the applications. Recently, experts of theory and numeric methods have given interesting tools for the study of fractional order models. In the theoretical aspects of the models, fixed point theorems play a vital role. We suggest the readers for more detail about the fractional calculus and its application to the work in [1–7]. Among the fractional operators, the Caputo–Fabrizio [8–10] and the Atangana–Baleanu fractional differential operators with nonsingular kernel [4–7,11–15] are recently well studied operators.

Recently, some researchers have focused on the different types of FDEs with impulses for the existence of solutions (EUS). Here, we highlight some of them. Sousa et al. [16] considered the investigation of existence results and Ulam-stability by the help of fixed point approach of an impulsive system. Xu and Liu [17] studied the boundedness criteria for delay impulsive system and provided an application. Zhang and Xiong [18] used some properties of the Mittag-Leffler function with one and two parameters for the existence and stability results. Zhao et al. [19] evaluated fractional order impulsive systems with Dirichlet boundaries by the help of Morse theory for the EUS. Heidarkhani et al. [20] investigated multiple solutions with the help of three critical points approach.

For the application of the IDEs with impulses, we recommend the readers the recent work [21–23]. Keeping the importance of the study, we are considering the following im-
pulsive IDE for the existence, stability, and numerical solution:

\[
\begin{cases}
\varepsilon^c_{\alpha} D^\vartheta x(t) = \Psi(t, x(t)) + \int_a^t K_0(t, s, x(s)) \, ds, \\
x(t) = \frac{1}{T(\vartheta)} \int_{t_k}^t (t - s)^{\vartheta - 1} G_k(s, x(s)) \, ds, \quad t \in (t_k, \delta_k], \\
\beta_1 x(0) + \beta_2 x(b) = \kappa(0),
\end{cases}
\] (1.1)

where \(\Psi : [a, b] \times \mathbb{R} \rightarrow \mathbb{R}, K_0 : [a, b] \times [a, b] \times \mathbb{R} \rightarrow \mathbb{R}, \Psi : [a, b] \rightarrow \mathbb{R}\) are continuous functions in the arguments with \(\Psi(t, x(t))|_{t=0} = 0\). The \(\varepsilon^c_{\alpha} D^\vartheta\) is Caputo’s differential operator of order \(\vartheta \in (0, 1]\). We consider the split of the interval \([a, b]\) with respect to \(t_k, \delta_k\) such that \(a < t_k < \delta_k < b\) for \(k = 1, 2, 3, \ldots, m\) and assume \(\delta_{m+1} = b\). We consider Banach space \(C^1([0, b], \mathbb{R}^n)\) of all the continuous functions with norm \(\|x\|_{\infty} = \sup_{t \in [0, b]} \|x(t)\|\), where \(\|\cdot\|\) is a complete norm in \(\mathbb{R}^n\), where \(C^1([0, b])(M) = \{x \in C^1([0, b], \mathbb{R}^n) : \|x\|_{\infty} \leq M \text{ for all } M > 0\}\).

**Definition 1.1** Fractional order integral of \(\zeta : (0, +\infty) \rightarrow \mathbb{R}\) for order \(\kappa > 0\) is

\[
I^\kappa_0 \zeta(t) = \frac{1}{\Gamma(\kappa)} \int_0^t (t - s)^{\kappa - 1} \zeta(s) \, ds,
\]

such that the integral is defined on \((0, +\infty)\), where

\[
\Gamma(\kappa) = \int_0^{+\infty} e^{-s}s^{\kappa-1} \, ds.
\]

**Definition 1.2** For a fractional order \(\kappa > 0\), Caputo’s derivative for \(\zeta(t) : (0, +\infty) \rightarrow \mathbb{R}\) is given by

\[
D^\kappa \zeta(t) = \frac{1}{\Gamma(\kappa - k)} \int_0^t (t - s)^{\kappa - k - 1} (\zeta^{(k)}(s)) \, ds
\]

for \(k = [\kappa] + 1\), where \([\kappa]\) is used for the integer part of \(\kappa\).

### 2 Integral form

This section is reserved for the integral form of fractional order impulsive system (1.1)

**Theorem 2.1** For \(\vartheta \in (0, 1]\) and \(\mathbb{H}(t, x(t)) \in C[a, b]\) such that \(x(t)\) is a solution of

\[
\begin{cases}
\varepsilon^c_{\alpha} D^\vartheta x(t) = \mathbb{H}(t, x(t)), \\
x(t) = \frac{1}{T(\vartheta)} \int_{t_k}^t (t - s)^{\vartheta - 1} G_k(s, x(s)) \, ds, \quad t \in (t_k, \delta_k], \\
\beta_1 x(0) + \beta_2 x(b) = \kappa(0),
\end{cases}
\] (2.1)
provided that

\[
x(t) = \begin{cases} 
\frac{x(0)}{\beta_1} - \frac{\beta_2}{\beta_1 \Gamma(\theta)} \int_{t_k}^{t} (\delta_k - s)^{\theta-1} G_k(s, x(s)) \, ds + \int_{t_k}^{b} (b-s)^{\theta-1} \mathbb{H}(s, x(s)) \, ds \\
+ \frac{1}{\Gamma(\theta)} \int_{0}^{t} (t-s)^{\theta-1} \mathbb{H}(s, x(s)) \, ds, \\
\text{for } t \in [0, t_1], \\
\frac{1}{\Gamma(\theta)} \int_{t_k}^{t} (t-s)^{\theta-1} G_k(s, x(s)) \, ds, \\
\text{for } t \in [t_k, \delta_k], \\
\vdots \\
\frac{1}{\Gamma(\theta)} \int_{t_k}^{t} (t-s)^{\theta-1} G_k(s, x(s)) \, ds + \frac{1}{\Gamma(\theta)} \int_{t_k}^{t} (t-s)^{\theta-1} \mathbb{H}(s, x(s)) \, ds, \\
\text{for } t \in [\delta_k, t_{k+1}].
\end{cases}
\]  

(2.2)

Proof. We divide the proof in parts as follows.

Case-I For \( t \in (0, t_1] \), applying the integral operator \( \mathcal{I}^\theta \) on (2.1), we have

\[
x(t) = x(0) + \frac{1}{\Gamma(\theta)} \int_{0}^{t} (t-s)^{\theta-1} \mathbb{H}(s, x(s)) \, ds.
\]  

(2.3)

Case-II For \( t \in (\delta_k, t_{k+1}] \), applying the integral operator \( \mathcal{I}^\theta \) on (2.1), we have

\[
x(t) = x(\delta_k) + \frac{1}{\Gamma(\theta)} \int_{t_k}^{t} (t-s)^{\theta-1} \mathbb{H}(s, x(s)) \, ds,
\]  

(2.4)

where by the help of the impulsive relation

\[
x(t) = \frac{1}{\Gamma(\theta)} \int_{t_k}^{t} (t-s)^{\theta-1} G_k(s, x(s)) \, ds,
\]  

(2.5)

we get

\[
x(\delta_k) = \frac{1}{\Gamma(\theta)} \int_{t_k}^{\delta_k} (\delta_k - s)^{\theta-1} G_k(s, x(s)) \, ds.
\]  

(2.6)

Thus, (2.4) implies

\[
x(t) = \frac{1}{\Gamma(\theta)} \int_{t_k}^{\delta_k} (\delta_k - s)^{\theta-1} G_k(s, x(s)) \, ds + \frac{1}{\Gamma(\theta)} \int_{t_k}^{t} (t-s)^{\theta-1} \mathbb{H}(s, x(s)) \, ds.
\]  

(2.7)

Now, using the condition \( \beta_1 x(0) + \beta_2 x(t) = \kappa(0) \), we have

\[
x(0) = \frac{\kappa(0)}{\beta_1} - \frac{\beta_2}{\beta_1 \Gamma(\theta)} \left[ \int_{t_k}^{\delta_k} (\delta_k - s)^{\theta-1} G_k(s, x(s)) \, ds + \int_{t_k}^{b} (b-s)^{\theta-1} \mathbb{H}(s, x(s)) \, ds \right].
\]  

(2.8)

Thus, by the help of (2.3) and (2.8), for \( t \in [0, t_1] \), we have

\[
x(t) = \frac{\kappa(0)}{\beta_1} - \frac{\beta_2}{\beta_1 \Gamma(\theta)} \left[ \int_{t_k}^{\delta_k} (\delta_k - s)^{\theta-1} G_k(s, x(s)) \, ds + \int_{t_k}^{b} (b-s)^{\theta-1} \mathbb{H}(s, x(s)) \, ds \right]
\]

\[+ \frac{1}{\Gamma(\theta)} \int_{0}^{t} (t-s)^{\theta-1} \mathbb{H}(s, x(s)) \, ds.
\]  

(2.9)
Case-III For $t \in (t_k, \delta_k]$, we have

$$x(t) = \frac{1}{\Gamma(\theta)} \int_{t_k}^{t} (t-s)^{\theta-1} G_k(s,x(s)) \, ds. \quad (2.10)$$

This completes the proof. 

**Corollary 2.1** By replacing $H(t,x(t))$ with $\Psi(t,x(t)) + \int_{a}^{t} K_0(t,s,x(s)) \, ds$, while keeping the conditions and order of derivative the same as in the theorem above, we get the following solution for fractional impulsive system (1.1):

$$x(t) = \begin{cases} \frac{x(0)}{\Gamma(\theta)} - \frac{\beta}{\Gamma(\theta) \Gamma(\gamma)} \int_{t_k}^{t} (t-k-s)^{\theta-1} G_k(s,x(s)) \, ds + \int_{t_k}^{a} (b-s)^{\theta-1} (\Psi(s,x(s)) \\
+ \int_{a}^{t} K_0(s,z,x(z)) \, dz) \, ds + \frac{1}{\Gamma(\theta)} \int_{0}^{t} (t-s)^{\theta-1} (\Psi(s,x(s)) \\
+ \int_{a}^{t} K_0(s,z,x(z)) \, dz) \, ds, \quad \text{for } t \in [0,t_1], \\
\vdots \\
\frac{1}{\Gamma(\theta)} \int_{t_k}^{t} (t-s)^{\theta-1} G_k(s,x(s)) \, ds + \frac{1}{\Gamma(\theta)} \int_{t_k}^{t} (t-s)^{\theta-1} (\Psi(s,x(s)) \\
+ \int_{a}^{t} K_0(s,z,x(z)) \, dz) \, ds, \quad \text{for } t \in [\delta_k, t_{k+1}]. \end{cases} \quad (2.11)$$

### 3 Theorems for EUS

For the main results of this manuscript, we convert the suggested problem (1.1) into a fixed point problem. For this, we introduce the following operator:

$$F_x(t) = \begin{cases} \frac{x(0)}{\Gamma(\theta)} - \frac{\beta}{\Gamma(\theta) \Gamma(\gamma)} \int_{t_k}^{t} (t-k-s)^{\theta-1} G_k(s,x(s)) \, ds + \int_{t_k}^{a} (b-s)^{\theta-1} (\Psi(s,x(s)) \\
+ \int_{a}^{t} K_0(s,z,x(z)) \, dz) \, ds + \frac{1}{\Gamma(\theta)} \int_{0}^{t} (t-s)^{\theta-1} (\Psi(s,x(s)) \\
+ \int_{a}^{t} K_0(s,z,x(z)) \, dz) \, ds, \quad \text{for } t \in [0,t_1], \\
\vdots \\
\frac{1}{\Gamma(\theta)} \int_{t_k}^{t} (t-s)^{\theta-1} G_k(s,x(s)) \, ds + \frac{1}{\Gamma(\theta)} \int_{t_k}^{t} (t-s)^{\theta-1} (\Psi(s,x(s)) \\
+ \int_{a}^{t} K_0(s,z,x(z)) \, dz) \, ds, \quad \text{for } t \in [\delta_k, t_{k+1}]. \end{cases} \quad (3.1)$$

The following assumptions are assumed for the proof of our main results:

$(A_1)$ $\Psi(t,x(t)) : [0,b] \times \mathbb{R} \to \mathbb{R}$, $K_0(t,s,x(s)) : [0,b] \times [0,b] \times \mathbb{R} \to \mathbb{R}$ are continuous functions, and there exist constants $K_1, K_2, K_3, K_4, K_5, K_6 \in \mathbb{R}$ such that

1. $|\Psi(t,x(t)) - \Psi(t,y(t))| \leq K_1 |x(t) - y(t)|$ for all $t \in [0,b]$ and $x, y \in \mathbb{R}$;
2. $|\Psi(t,x(t))| \leq K_2 + K_3 |x(t)|$;
3. $|K_0(t,s,x(t)) - K_0(t,s,y(t))| \leq K_4 |x(t) - y(t)|$ for all $t, s \in [0,b]$ and $x, y \in \mathbb{R}$;
4. $|K_0(t,s,x(t))| \leq K_5 + K_6|x(t)|$.

(A2) $G_k: I_k \times \mathbb{R} \rightarrow \mathbb{R}$ for $I_k = [\delta_k, \delta_k]$, for $k = 1, 2, \ldots, p$ are continuous and there exist positive constants $\eta_k$, $\eta_k$ such that

1. $|G_k(t,x(t)) - G_k(t,y(t))| \leq \eta_k|x - y|$ for all $x, y \in \mathbb{R}$, $t \in I_k$, $k = 1, 2, \ldots, p$;

2. $|G_k(t,x(t))| \leq \eta_k$ for all $t \in I_k$ and $x(t) \in \mathbb{R}$.

(A3) For $\mathcal{M} = \|x(t)\|$, $x_1 = \frac{\int_{I_k} (\delta_k - s)^{\alpha-1} G_k(s,x(s)) \, ds}{\Gamma(\alpha)}$,

\[
\begin{align*}
\|x(t)\| & = \frac{\int_{I_k} (\delta_k - s)^{\alpha-1} G_k(s,x(s)) \, ds}{\Gamma(\alpha)} \\
& + \frac{1}{\Gamma(\alpha)} \int_{I_k} (b-s)^{\alpha-1} \left( \Psi(s,x(s)) + \int_a^s K_2(s,z,x(z)) \, dz \right) \, ds \\
& + \frac{1}{\Gamma(\alpha)} \int_{I_k} (t-s)^{\alpha-1} \left( \Psi(s,x(s)) + \int_a^s K_3(s,z,x(z)) \, dz \right) \, ds \\
& \leq \frac{\int_{I_k} (\delta_k - s)^{\alpha-1} \|G_k(s,x(s))\| \, ds}{\Gamma(\alpha)} \\
& + \frac{1}{\Gamma(\alpha)} \int_{I_k} (b-s)^{\alpha-1} \left( K_2 + K_3 \|x(t)\| \right) \, ds \\
& + \frac{1}{\Gamma(\alpha+1)} \int_{I_k} (t-s)^{\alpha-1} \left( K_2 + K_3 \|x(t)\| \right) \, ds \\
& \leq \frac{\int_{I_k} (\delta_k - s)^{\alpha-1} \|G_k(s,x(s))\| \, ds}{\Gamma(\alpha)} \\
& + \frac{1}{\Gamma(\alpha+1)} \int_{I_k} (b-s)^{\alpha} \left( K_2 + K_3 \|x(t)\| \right) \, ds \\
& + \frac{1}{\Gamma(\alpha+1)} \int_{I_k} (t-s)^{\alpha} \left( K_2 + K_3 \|x(t)\| \right) \, ds \\
& \leq \frac{\int_{I_k} (\delta_k - s)^{\alpha-1} \|G_k(s,x(s))\| \, ds}{\Gamma(\alpha)} \\
& + \frac{1}{\Gamma(\alpha+1)} \int_{I_k} (b-s)^{\alpha} \left( K_2 + K_3 \|x(t)\| + (K_5 + K_6 \|M\|) \right) \\
& + \frac{1}{\Gamma(\alpha+1)} \int_{I_k} (t-s)^{\alpha} \left( K_2 + K_3 \|x(t)\| + (K_5 + K_6 \|M\|) \right)
\end{align*}
\]

\[ \leq \frac{\int_{I_k} (\delta_k - s)^{\alpha-1} \|G_k(s,x(s))\| \, ds}{\Gamma(\alpha)} + \frac{1}{\Gamma(\alpha+1)} (b-t)^{\alpha} \left( K_2 + K_3 \|x(t)\| + (K_5 + K_6 \|M\|) \right) \\
+ \frac{1}{\Gamma(\alpha+1)} (t-s)^{\alpha} \left( K_2 + K_3 \|x(t)\| + (K_5 + K_6 \|M\|) \right) \]

\[ \leq \frac{\int_{I_k} (\delta_k - s)^{\alpha-1} \|G_k(s,x(s))\| \, ds}{\Gamma(\alpha)} + \frac{1}{\Gamma(\alpha+1)} (b-t)^{\alpha} \left( K_2 + K_3 \|x(t)\| + (K_5 + K_6 \|M\|) \right) \\
+ \frac{1}{\Gamma(\alpha+1)} (t-s)^{\alpha} \left( K_2 + K_3 \|x(t)\| + (K_5 + K_6 \|M\|) \right) \]

\[ \leq \frac{\int_{I_k} (\delta_k - s)^{\alpha-1} \|G_k(s,x(s))\| \, ds}{\Gamma(\alpha)} + \frac{1}{\Gamma(\alpha+1)} (b-t)^{\alpha} \left( K_2 + K_3 \|x(t)\| + (K_5 + K_6 \|M\|) \right) \\
+ \frac{1}{\Gamma(\alpha+1)} (t-s)^{\alpha} \left( K_2 + K_3 \|x(t)\| + (K_5 + K_6 \|M\|) \right) \]

\[ \leq \frac{\int_{I_k} (\delta_k - s)^{\alpha-1} \|G_k(s,x(s))\| \, ds}{\Gamma(\alpha)} + \frac{1}{\Gamma(\alpha+1)} (b-t)^{\alpha} \left( K_2 + K_3 \|x(t)\| + (K_5 + K_6 \|M\|) \right) \\
+ \frac{1}{\Gamma(\alpha+1)} (t-s)^{\alpha} \left( K_2 + K_3 \|x(t)\| + (K_5 + K_6 \|M\|) \right) \]

\[ \leq \frac{\int_{I_k} (\delta_k - s)^{\alpha-1} \|G_k(s,x(s))\| \, ds}{\Gamma(\alpha)} + \frac{1}{\Gamma(\alpha+1)} (b-t)^{\alpha} \left( K_2 + K_3 \|x(t)\| + (K_5 + K_6 \|M\|) \right) \\
+ \frac{1}{\Gamma(\alpha+1)} (t-s)^{\alpha} \left( K_2 + K_3 \|x(t)\| + (K_5 + K_6 \|M\|) \right) \]

\[ \leq \frac{\int_{I_k} (\delta_k - s)^{\alpha-1} \|G_k(s,x(s))\| \, ds}{\Gamma(\alpha)} + \frac{1}{\Gamma(\alpha+1)} (b-t)^{\alpha} \left( K_2 + K_3 \|x(t)\| + (K_5 + K_6 \|M\|) \right) \\
+ \frac{1}{\Gamma(\alpha+1)} (t-s)^{\alpha} \left( K_2 + K_3 \|x(t)\| + (K_5 + K_6 \|M\|) \right) \]

\[ \leq \frac{\int_{I_k} (\delta_k - s)^{\alpha-1} \|G_k(s,x(s))\| \, ds}{\Gamma(\alpha)} + \frac{1}{\Gamma(\alpha+1)} (b-t)^{\alpha} \left( K_2 + K_3 \|x(t)\| + (K_5 + K_6 \|M\|) \right) \\
+ \frac{1}{\Gamma(\alpha+1)} (t-s)^{\alpha} \left( K_2 + K_3 \|x(t)\| + (K_5 + K_6 \|M\|) \right) \]

\[ \leq \frac{\int_{I_k} (\delta_k - s)^{\alpha-1} \|G_k(s,x(s))\| \, ds}{\Gamma(\alpha)} + \frac{1}{\Gamma(\alpha+1)} (b-t)^{\alpha} \left( K_2 + K_3 \|x(t)\| + (K_5 + K_6 \|M\|) \right) \\
+ \frac{1}{\Gamma(\alpha+1)} (t-s)^{\alpha} \left( K_2 + K_3 \|x(t)\| + (K_5 + K_6 \|M\|) \right) \]

\[ \leq \frac{\int_{I_k} (\delta_k - s)^{\alpha-1} \|G_k(s,x(s))\| \, ds}{\Gamma(\alpha)} + \frac{1}{\Gamma(\alpha+1)} (b-t)^{\alpha} \left( K_2 + K_3 \|x(t)\| + (K_5 + K_6 \|M\|) \right) \\
+ \frac{1}{\Gamma(\alpha+1)} (t-s)^{\alpha} \left( K_2 + K_3 \|x(t)\| + (K_5 + K_6 \|M\|) \right) \]

\[ \leq \frac{\int_{I_k} (\delta_k - s)^{\alpha-1} \|G_k(s,x(s))\| \, ds}{\Gamma(\alpha)} + \frac{1}{\Gamma(\alpha+1)} (b-t)^{\alpha} \left( K_2 + K_3 \|x(t)\| + (K_5 + K_6 \|M\|) \right) \\
+ \frac{1}{\Gamma(\alpha+1)} (t-s)^{\alpha} \left( K_2 + K_3 \|x(t)\| + (K_5 + K_6 \|M\|) \right) \]
By the help of (3.2)–(3.3), we have

$$\frac{\kappa(0)}{\beta_1} + \frac{\beta_2 L_2}{\beta_1 \Gamma(\vartheta + 1)} (\delta_k - t_k)^\vartheta + \left( \frac{\beta_2}{\beta_1 \Gamma(\vartheta + 1)} (b - t_k)^\vartheta + \frac{1}{\Gamma(\vartheta + 1)} (t - s)^\vartheta \right)(K_2 + K_3 M + K_5 + K_6 M)$$

$$\leq \frac{\kappa(0)}{\beta_1} + \frac{b^\vartheta \beta_2 L_2}{\beta_1 \Gamma(\vartheta + 1)} + \left( \frac{b^\vartheta \beta_2}{\beta_1 \Gamma(\vartheta + 1)} + \frac{b^\vartheta}{\Gamma(\vartheta + 1)} \right)(K_2 + K_3 M + K_5 + K_6 M). \quad (3.2)$$

Now, for the $t \in (\delta_k, t_{k+1}]$, we have

$$\| T x(t) \| = \left\| \frac{1}{\Gamma(\vartheta)} \int_{t_k}^{b} (\delta_k - s)^{\vartheta-1} G_k(s, x(s)) \, ds \right\|
+ \frac{1}{\Gamma(\vartheta)} \int_{t_k}^{t} (t - s)^{\vartheta-1} \left( \psi(s, x(s)) + \int_{a}^{s} K_0(s, z, x(z)) \, dz \right) \, ds$$

$$\leq \frac{1}{\Gamma(\vartheta)} \int_{t_k}^{b} (\delta_k - s)^{\vartheta-1} L_2 \, ds
+ \frac{1}{\Gamma(\vartheta)} \int_{t_k}^{t} (t - s)^{\vartheta-1} \left( K_2 + K_3 |x(t)| + \int_{a}^{s} (K_5 + K_6 |x(t)|) \, dz \right) \, ds$$

$$\leq \frac{L_2}{\Gamma(\vartheta + 1)} (\delta_k - t_k)^\vartheta + \frac{1}{\Gamma(\vartheta + 1)} (t - t_k)^\vartheta (K_2 + K_3 M + (K_5 + K_6 M))$$

$$\leq \frac{1}{\Gamma(\vartheta + 1)} L_2 b^\vartheta + \frac{1}{\Gamma(\vartheta + 1)} b^\vartheta (K_2 + K_3 M + (K_5 + K_6 M)). \quad (3.3)$$

Now, for $t \in (t_k, \delta_{k+1})$, where $k = 1, 2, 3, \ldots, p$ and $x(t) \in \mathbb{B}$, we have

$$\| T x(t) \| = \left\| \frac{1}{\Gamma(\vartheta)} \int_{t_k}^{t} (t - s)^{\vartheta-1} G_k(s, x(s)) \, ds \right\|
\leq \frac{1}{\Gamma(\vartheta)} \int_{t_k}^{t} (t - s)^{\vartheta-1} \| G_k(s, x(s)) \| \, ds$$

$$\leq \frac{L_2}{\Gamma(\vartheta + 1)} (t - t_k)^\vartheta
\leq \frac{L_2}{\Gamma(\vartheta + 1)} b^\vartheta. \quad (3.4)$$

By the help of (3.2)–(3.3), we have $\| T x \| \leq \chi$. This implies that $T : \mathbb{B} \rightarrow \mathbb{B}$.

**Step 2.** Now, we show that $T$ is a strict contraction. For this, we assume $x(t), y(t) \in \mathbb{R}$.

And consider the following three cases.

Case I. For $t \in [0, t_1]$, we have

$$\| T x(t) - T y(t) \|$$

$$= \left\| \frac{\kappa(0)}{\beta_1} - \frac{\beta_2}{\beta_1} \Gamma(\vartheta) \int_{t_k}^{b} (\delta_k - s)^{\vartheta-1} G_k(s, x(s)) \, ds \right\|
+ \frac{1}{\Gamma(\vartheta)} \int_{t_k}^{b} (b - s)^{\vartheta-1} \left( \psi(s, x(s)) + \int_{a}^{s} K_0(s, z, x(z)) \, dz \right) \, dz$$
\[+ \frac{1}{\Gamma(\theta)} \int_{t_0}^{t} (t - s)^{\theta - 1} \left( \Psi(s, x(s)) + \int_{a}^{s} \kappa_0(s, z, x(z)) \, dz \right) \, ds\]

\[= \left( \frac{x(0)}{\beta_1} - \frac{\beta_2}{\beta_1} \left[ \frac{1}{\Gamma(\theta)} \int_{t_0}^{b} (\delta_k - s)^{\theta - 1} \mathcal{G}_k(s, y(s_k)) \, ds \right] \right)\]

\[+ \frac{1}{\Gamma(\theta)} \int_{t_0}^{b} (b - s)^{\theta - 1} \left( \Psi(s, y(s)) + \int_{a}^{s} \kappa_0(s, z, y(z)) \, dz \right) \, ds\]

\[+ \frac{1}{\Gamma(\theta)} \int_{t_0}^{t} (t - s)^{\theta - 1} \left( \Psi(s, y(s)) + \int_{a}^{s} \kappa_0(s, z, y(z)) \, dz \right) \, ds\]

\[\leq \beta_2 \left[ \frac{1}{\Gamma(\theta)} \int_{t_0}^{\delta_k} (\delta_k - s)^{\theta - 1} \mathbb{I}_q \|x - y\| \, ds \right] \]

\[+ \frac{1}{\Gamma(\theta)} \int_{t_0}^{b} (b - s)^{\theta - 1} \left( \mathbb{K}_4 \|x - y\| + \int_{a}^{s} \mathbb{K}_4 \|x - y\| \, dz \right) \, ds\]

\[+ \frac{1}{\Gamma(\theta)} \int_{t_0}^{t} (t - s)^{\theta - 1} \left( \mathbb{K}_4 \|x - y\| + \int_{a}^{s} \mathbb{K}_4 \|x - y\| \, dz \right) \, ds\]

\[\leq \frac{\beta_2}{\beta_1} \left[ \frac{b^\theta}{\Gamma(\theta + 1)} \mathbb{I}_q + \frac{b^\theta}{\Gamma(\theta + 1)} (\mathbb{K}_1 + b \mathbb{K}_4) \right] \|x - y\|. \quad (3.5)\]

For \( t \in (\delta_k, t_{k+1}] \), we have

\[\|T x(t) - T y(t)\|\]

\[= \left( \frac{1}{\Gamma(\theta)} \int_{t_0}^{\delta_k} (\delta_k - s)^{\theta - 1} \mathcal{G}_k(s, x(s_k)) \, ds + \frac{1}{\Gamma(\theta)} \int_{t_k}^{t} (t - s)^{\theta - 1} \left( \Psi(s, x(s)) \right. \right.\]

\[+ \int_{a}^{s} \kappa_0(s, z, x(z)) \, dz) \, ds\]

\[= \left. \frac{1}{\Gamma(\theta)} \int_{t_0}^{\delta_k} (\delta_k - s)^{\theta - 1} \mathcal{G}_k(s, x(s_k)) \, ds \right. \]

\[- \frac{1}{\Gamma(\theta)} \int_{t_0}^{t} (t - s)^{\theta - 1} \left( \Psi(s, y(s)) - \int_{a}^{s} \kappa_0(s, z, y(z)) \, dz \right) \, ds\]

\[\leq \frac{1}{\Gamma(\theta)} \int_{t_0}^{\delta_k} (\delta_k - s)^{\theta - 1} \left| \mathcal{G}_k(s, x(s_k)) - \mathcal{G}_k(s, y(s_k)) \right| \, ds\]

\[+ \frac{1}{\Gamma(\theta)} \int_{t_0}^{t} (t - s)^{\theta - 1} \left( \|\Psi(s, x(s)) - \Psi(s, y(s))\| \right.\]
Theorem 4.1 Assume that here, we present data-dependence of the solution of impulsivesystem (1.1). We follow the results studied given in [12, 24].

\[
\| x(t) - \tilde{x}(t) \| = \left[ \frac{1}{\Gamma'(\theta + 1)} b^\beta \kappa_s + \frac{1}{\Gamma'(\theta + 1)} b^\beta (\kappa_1 + b\kappa_2) \right] \| x - y \|. \tag{3.6}
\]

Now, for \( t \in (t_k, \delta_k] \), where \( k = 1, 2, 3, \ldots, p \) and \( x(t) \in \mathbb{B} \), we have

\[
\| T x(t) - T y(t) \| = \left\| \frac{1}{\Gamma'(\theta)} \int_{t_k}^t (t-s)^{\theta-1} G_k(s,x(s)) \| ds - \frac{1}{\Gamma'(\theta)} \int_{t_k}^t (t-s)^{\theta-1} G_k(s,y(s)) \| ds \right\|
\leq \frac{\kappa_s}{\Gamma'(\theta + 1)} (t - t_k)^{\theta} \| x - y \|
\leq \frac{\kappa_s}{\Gamma'(\theta + 1)} b^\beta \| x - y \|. \tag{3.7}
\]

Thus, with the help of (3.5)–(3.7), we have that the operator \( T \) is a contraction, and by the Banach fixed point theorem \( T \) has a unique fixed point. This further implies that fractional impulsive system (1.1) has a unique solution, which accomplishes the proof. \( \square \)

4 Data dependence

Here, we present data-dependence of the solution of impulsivesystem (1.1). We follow the results studied given in [12, 24].

Theorem 4.1 Assume that \( (A_1) \) to \( A_3 \) are satisfied. Then, for \( x(t), \tilde{x}(t) \) satisfying (2.11), we have \( \| x(t) - \tilde{x}(t) \| < \zeta \).

Proof With the help of Theorem 2.1, we have

\[
x(t) - \tilde{x}(t) = \frac{\kappa_s}{p_1} - \frac{\rho_1}{\rho_2} \int_{t_k}^t (t-s)^{\theta-1} G_k(s,x(s)) ds + \int_{t_k}^t (b - s)^{\theta-1} (\Psi (s,x(s)) + f_a K_0(s,z,x(z)) dz) ds
\]

\[
+ \frac{1}{\Gamma'(\theta)} \int_{t_k}^t (t-s)^{\theta-1} (\Psi (s,x(s)) + f_a K_0(s,z,x(z)) dz) ds
\]

\[
- \frac{\kappa_s}{p_1} - \frac{\rho_1}{\rho_2} \int_{t_k}^t (t-s)^{\theta-1} G_k(s,\tilde{x}(s)) ds + \int_{t_k}^t (b - s)^{\theta-1} (\Psi (s,\tilde{x}(s)) + f_a K_0(s,z,\tilde{x}(z)) dz) ds
\]

\[
- \frac{1}{\Gamma'(\theta)} \int_{t_k}^t (t-s)^{\theta-1} (\Psi (s,\tilde{x}(s)) + f_a K_0(s,z,\tilde{x}(z)) dz) ds,
\]

for \( t \in [0, t_k) \),

\[
\frac{1}{\Gamma'(\theta)} \int_{t_k}^t (b - s)^{\theta-1} G_k(s,x(s)) ds + \frac{1}{\Gamma'(\theta)} \int_{t_k}^t (t-s)^{\theta-1} (\Psi (s,x(s)) + f_a K_0(s,z,x(z)) dz) ds
\]

\[
+ \int_{t_k}^t (b - s)^{\theta-1} (\Psi (s,\tilde{x}(s)) + f_a K_0(s,z,\tilde{x}(z)) dz) ds
\]

\[
- \frac{1}{\Gamma'(\theta)} \int_{t_k}^t (t-s)^{\theta-1} G_k(s,\tilde{x}(s)) ds - \frac{1}{\Gamma'(\theta)} \int_{t_k}^t (b - s)^{\theta-1} G_k(s,\tilde{x}(s)) ds + \frac{1}{\Gamma'(\theta)} \int_{t_k}^t (t-s)^{\theta-1} G_k(s,\tilde{x}(s)) ds,
\]

for \( t \in [t_k, t_{k+1}) \),

\[
\frac{1}{\Gamma'(\theta)} \int_{t_k}^t (t-s)^{\theta-1} G_k(s,x(s)) ds - \frac{1}{\Gamma'(\theta)} \int_{t_k}^t (t-s)^{\theta-1} G_k(s,\tilde{x}(s)) ds,
\]

for \( t \in [t_k, \delta_k] \).
Case 1. Then, for \( t \in [0, t_1] \), we have

\[
\|x(t) - \overline{x}(t)\|_\infty \leq \frac{|\kappa(0) - \overline{\kappa}(0)|}{\beta_1} + \frac{\beta_2}{\beta_1 \Gamma(\theta)} \left[ \int_{t_k}^{t} (\delta_k - s)^{\theta - 1} \|G_k(s, x(s)) - G_k(s, \overline{x}(s))\| ds \\
+ \int_{t_k}^{t} (t - s)^{\theta - 1} \left( \Psi(s, x(s)) - \Psi(s, \overline{x}(s)) \right) ds \right] \]

Similarly, for \( t \in (\delta_k, t_{k+1}] \), we have

\[
\|x(t) - \overline{x}(t)\|_\infty \leq \frac{|\kappa(0) - \overline{\kappa}(0)|}{\beta_1} + \frac{\beta_2}{\beta_1 \Gamma(\theta)} \left[ \int_{t_k}^{t} (\delta_k - s)^{\theta - 1} \|G_k(s, x(s)) - G_k(s, \overline{x}(s))\| ds \\
+ \int_{t_k}^{t} (t - s)^{\theta - 1} \left( \Psi(s, x(s)) - \Psi(s, \overline{x}(s)) \right) ds \right] \]

(4.2)
\begin{equation}
\left\| \mathcal{K}_0(s, z, x(z)) - \mathcal{K}_0(s, z, x(z)) \right\| ds \leq \frac{1}{\Gamma(\theta + 1)} b^\theta k_x \| x - \bar{x} \| + \frac{1}{\Gamma(\theta + 1)} b^\theta (K_1 \| x - \bar{x} \| + b K_4 \| x - \bar{x} \|) \leq \frac{b^\theta}{\Gamma(\theta + 1)} \left[ K_x + (K_1 + b K_4) \right] \| x - \bar{x} \|. \tag{4.3}
\end{equation}

Finally, for \( t \in [t_k, \delta_k] \), we have
\begin{align*}
\left\| x(t) - \bar{x}(t) \right\|_\infty &= \left\| \frac{1}{\Gamma(\theta)} \int_{t_k}^t (t - s)^{\theta-1} \mathcal{G}_k(s, x(s)) ds - \frac{1}{\Gamma(\theta)} \int_{t_k}^t (t - s)^{\theta-1} \mathcal{G}_k(s, \bar{x}(s)) ds \right\| \\
&\leq \frac{1}{\Gamma(\theta)} \int_{t_k}^t (t - s)^{\theta-1} \left\| \mathcal{G}_k(s, x(s)) - \mathcal{G}_k(s, \bar{x}(s)) \right\| ds \\
&\leq \frac{1}{\Gamma(\theta + 1)} b^\theta k_x \| x - \bar{x} \|. \tag{4.4}
\end{align*}

Thus, by the help of (4.2)-(4.4), \( \| x - \bar{x}(t) \| \leq \zeta \). \hfill \square

5 Application

In order to give verification of the existence and data dependence theorems, here we give the following illustrative model.

Example 5.1 Assume that \( \theta = 0.5, \beta_1 = 5, \beta_2 = 2, b = 0.5, I = [0, 1] \) and
\begin{align*}
\begin{cases}
\mathcal{C}D^\theta_0 x(t) = \frac{1}{100} \cos(t) + \int_0^t \sin(x(t)) dt, & t \in [0, 1], t \neq \frac{\pi}{2}, \\
x(t) = \frac{\sin(t)}{t}, & t \in (t_k, \delta_k), \\
\beta_1 x(0) + \beta_2 x(b) = \frac{1}{100}.
\end{cases} \tag{5.1}
\end{align*}

Then from (5.1) one can easily evaluate from \( M < \frac{1}{2}, b = 1, a = 0, K_1 = 1/100, K_5 = K_2 = 0, K_3 = 1/100 = K_6, K_x = L_x = \frac{1}{7}, \chi < 1 \). Thus, by the help of Theorem 3.1, system (5.1) has a unique solution \([0, 1]\).

6 Conclusion

In this article we have considered a fractional order impulsive IDE (1.1) for the existence of unique solution, data dependence, and stability results. We have used basic results from the fixed point theory and literature for fractional order calculus. The results are examined with the help of an expressive numerical example.

Acknowledgements
The authors are very obliged to the editor and referees for their supportive input and useful feedback.

Funding
This research was funded by the Dean of Scientific Research at Princess Nourah bint Abdulrahman University through the Fast-track Research Funding Program.

Availability of data and materials
Not applicable.
Competing interests
The authors have no competing interests regarding the publication of this article.

Authors’ contributions
All the authors have equal contributions in this article. All authors read and approved the final manuscript.

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Publisher’s Note
Springer Nature remains neutral with regard to jurisdictional claims in published maps and institutional affiliations.

Received: 29 April 2020 Accepted: 7 July 2020 Published online: 03 September 2020

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