GENERIC REPRESENTATIONS IN $L$-PACKETS

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Abstract. We give the details of the construction of a map to restate a conjectural expression about adjoint group action on generic representations in $L$-packets. We give an application of the construction to give another proof of the classification of the Knapp-Stein $R$-group associated to a unitary unramified character of a torus. Finally we prove the conjecture for unramified $L$-packets.

1. Introduction

Let $G$ be a quasi-split connected reductive group defined over a local field $k$ of characteristic zero and let $Z$ be the center of $G$. Let $B$ be a $k$-Borel subgroup of $G$ and let $T$ be a maximal $k$-torus in $B$. Let $U$ be the unipotent radical of $B$. A character $\psi : U(k) \to \mathbb{C}^\times$ is called generic if the stabilizer of $\psi$ in $T(k)$ is exactly the center $Z(k)$. An irreducible admissible representation $\pi$ of $G$ is called generic ($\psi$-generic) if there exists a generic character $\psi$ of $U(k)$ such that $\text{Hom}_{G(k)}(\pi, \text{Ind}_{U(k)}^G(\psi)) \neq 0$.

The conjectural local Langlands program partitions the irreducible admissible representations of $G$ into finite sets known as $L$-packets. Each $L$-packet is expected to be parametrized by an arithmetic object called the Langlands parameter, which is an admissible homomorphism from the Weil-Deligne group $W'_k$ of $k$ to the $L$-group $L_G$ of $G$. See [Bor79a] for the definitions and statements.

To each Langlands parameter $\varphi$, one can associate a finite group $S_{\varphi}$ (see [Art06 Section 1, eq. (1.1)]). It is expected that the associated $L$-packet $\Pi_\varphi$ is parametrized by the irreducible representations $\hat{S}_{\varphi}$ of $S_{\varphi}$ [Art06 Section 1]. The parametrization will depend on the choice of a Whittaker datum for $G$, which is a $G(k)$-conjugacy class of pairs $(B, \psi)$, where $\psi$ is a generic character of $U(k)$. When $\Pi_\varphi$ is generic, i.e., it has a generic representation, the $\psi$-generic representation in $\Pi_\varphi$ is then required to correspond to the trivial representation of $S_{\varphi}$. The parametrization is also expected to satisfy certain conjectural endoscopic character identity [Kal13].

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When $\varphi$ is a tempered parameter, i.e., a parameter whose image projects onto a relatively compact subset of the complex dual $\hat{G}$ of $G$, Shahidi’s tempered $L$-packet conjecture [Sha90, §9] states that $\Pi_{\varphi}$ must be generic.

Let $\Gamma_k$ be the absolute Galois group of $k$ and write $H^1(k, -)$ for $H^1(\Gamma_k, -)$. In Section 3, we construct a map $\gamma_{\varphi} : R_{\varphi} := \pi_0(Z_G(\text{Im}(\varphi))) \to H^1(k, X(Z))$, where $X(Z)$ is the character lattice of $Z$ and $\varphi$ is any Langlands parameter. Using Tate duality, we get the dual map $\hat{\gamma}_{\varphi} : H^1(k, Z) \to \hat{R}_{\varphi}$, where $\hat{R}_{\varphi}$ is the set of irreducible representations of $R_{\varphi}$. Let $p : t \in T \to \hat{t} \in T_{\text{ad}} := T/Z$ be the adjoint morphism. The finite abelian group $T_{\text{ad}}(k)/p(T(k)) \to H^1(k, Z)$ acts simply transitively on the set of $T(k)$-orbits of generic characters [DR10, §3]. The map $\zeta_{\varphi} := \hat{\gamma}_{\varphi}|T_{\text{ad}}(k)/p(T(k))$ factors through $\hat{S}_{\varphi}$ (see [GGP12, Sec. 9(4)], also [Kal13, Sec. 3]).

Now fix a parametrization $\rho \in \hat{S}_{\varphi} \mapsto \pi_{\rho} \in \Pi_{\varphi}$ by making the choice of a Whittaker datum. The following is a version of the conjecture in [GGP12, Sec. 9(3)] for generic $L$-packets.

**Conjecture.** A representation $\pi_{\rho} \in \Pi_{\varphi}$ is $\psi$-generic iff $\pi_t \cdot \rho$ is $t \cdot \psi$ generic for all $t \in T_{\text{ad}}(k)$, where $t \cdot \psi$ is $\rho \otimes \zeta_{\varphi}(t)$.

The map $\hat{\gamma}_{\varphi}$ was constructed in [Kuo10] in a very special case ($G$ split semisimple and $\varphi$ is the parameter associated to a unitary character of $T(k)$). For depth zero supercuspidal $L$-packets, the conjecture follows from [DR10]. When $G$ is semisimple and split and the $L$-packet is formed by the constituents of a unitary principal series, the conjecture follows from [Kuo02]. In [Kal13], Kaletha gives a proof of the above conjecture for classical groups using very general arguments.

Now let $G$ be unramified and let $\varphi$ be the parameter associated to a unitary unramified character $\lambda$ of $T(k)$. The construction of the map $\gamma_{\lambda}$ allows one to obtain a nice description of the group $R_{\varphi}$ as a certain subgroup of an extended affine Weyl group (Proposition 10). Using this, in Theorem 11, we obtain in a conceptual and uniform way, the classification of the Knapp-Stein $R$-group associated to $\lambda$. This kind of classification was obtained by Keys [Key82, §3] in a case by case manner. For split groups, using different methods, another way of getting the classification obtained by Keys was recently given by Kamran and Plymen [KP13]. Our situation is more general and we also describe the isomorphism, which has a simple description.

Finally in Theorem 12, we prove the conjecture for unramified $L$-packets (see Sec. 5). We do not assume the packet to be tempered.

## 2. Preliminaries

### 2.1. Group Cohomology.

For details about this subsection, see [Ser97] Ch. 5.
Let $\Gamma$ be a topological group and let
\begin{equation}
1 \to A \to B \to C \to 1
\end{equation}
be a short exact sequence of $\Gamma$-groups. Assume that $A$ is central subgroup of $B$. Then $C$ acts on $B$ by inner automorphisms and it acts trivially on $A$. Let $\gamma : \Gamma \to C$ be a co-cycle in $C$, i.e., it satisfies the relation $\gamma(ab) = \gamma(a)\gamma(b)$ for all $a, b \in \Gamma$. By twisting the short exact sequence in (2.1) by $\gamma$, we get another short exact sequence
\begin{equation}
1 \to A \to \gamma B \to \gamma C \to 1
\end{equation}
From this we get a long exact cohomology sequence
\begin{equation}
\begin{array}{c}
1 \to H^0(\Gamma, A) \to H^0(\Gamma, \gamma B) \to H^0(\Gamma, \gamma C) \to H^1(\Gamma, A) \to \\
H^1(\Gamma, \gamma B) \to H^1(\Gamma, \gamma C)
\end{array}
\end{equation}

2.2. Affine roots and affine transformations.

2.2.1. The group $\Omega$. Let $\Psi = (X, R, \Delta, \breve{X}, \breve{R}, \breve{\Delta})$ be a based root datum in the sense of [Spr79, 1.9]. So $X$ and $\breve{X}$ are free abelian groups in duality by a pairing $X \times \breve{X} \to \mathbb{Z}$, $R$ is a root system in the vector space $Q \otimes \mathbb{R}$, where $Q$ is the root lattice, $\breve{R}$ is the set of co-roots, $\Delta \subset R$ is a basis and $\breve{\Delta}$ is the dual basis. Let $W = W(\Psi)$ be the Weyl group. The set $\Delta$ determines an alcove $C$ in $V := X \otimes \mathbb{R}$ in the following way. Let $\Delta = \{\breve{\alpha}_1, \ldots, \breve{\alpha}_l\}$ and let $\breve{\beta} = \sum_{i=1}^l n_i \breve{\alpha}_i$ be the highest co-root. Then $C$ is the alcove in $V$ defined by $C = \{x \in V : \breve{\alpha}_0(x) \geq 0, \ldots, \breve{\alpha}_l(x) \geq 0\}$, where $\breve{\alpha}_0 = 1 - \breve{\beta}$. Let $\breve{W} = W \times X$ and $\breve{W}^\circ = W \times Q$. Let $\Omega$ be the stabilizer of $C$ in $\breve{W}$. Then $\breve{W} = \Omega \times \breve{W}^\circ$.

Now assume that $\Psi$ is semisimple [Spr79, 1.1] and $R$ is an irreducible root system in $V$. Let $c_0$ be the weighted barycenter of $C$, characterized by the equations $\breve{\alpha}_i(c_0) = 1/h$ for $i = 0, \ldots, l$, where $h$ is the Coxeter number. For any $w \in W$, let $\breve{w}$ be the affine map $x \in V \mapsto w(x - c_0) + c_0$. It is the unique affine map fixing $c_0$ with tangent part $w$. The following lemmas follow from [AYY13, Lemma 6.2].

**Lemma 1.** For any $w \in W$, the following are equivalent:
\begin{align*}
(1) & \quad \breve{w} \in \breve{W}.
(2) & \quad \breve{w} \in \Omega.
\end{align*}

**Lemma 2.** There is an isomorphism $\iota : \Omega \to X/Q$ defined by any of the following ways
\begin{align*}
(1) & \quad \iota(\breve{w}) = (w^{-1} - 1)c_0 + Q.
(2) & \quad \text{The natural projection } \breve{W} \to \breve{W}/\breve{W}^\circ = X/Q \text{ restricted to } \Omega.
\end{align*}
2.2.2. Based root datum. Let $\Psi = (X, R, \Delta, \hat{X}, \hat{R}, \hat{\Delta})$ be a reduced based root datum. Let $\theta$ be a finite group acting on $\Psi$. In [Yu], Jiu-Kang Yu defines the following 6-touple $\Psi = (X, R, \Delta, \hat{X}, \hat{R}, \hat{\Delta})$:

\[
\begin{align*}
X &= X_\theta / \text{torsion}, \\
\hat{X} &= \hat{X}^\theta, \\
R &= \{a : a \in R\}, \quad \text{where } a := a|_X \\
\hat{R} &= \{\hat{a} : \alpha \in R\}, \\
\Delta &= \{a : a \in \Delta\}, \\
\hat{\Delta} &= \{\hat{\alpha} : \alpha \in \Delta\}.
\end{align*}
\]

The explanation for the defining formulas is as follows. We first note that $X$ and $\hat{X}$ are free abelian groups, dual to each other under the canonical pairing $(x, y) \mapsto <x, y>$, for $x \in X$, $y \in \hat{X} \subset \hat{X}$, where $x$ is any preimage of $x$ in $X$. Define $\hat{\alpha}$ for $\alpha \in R$ as follows:

\[
(2.2) \quad \hat{\alpha} = \begin{cases} 
\sum_{a \in R : \alpha(a) = 0} \hat{a}, & \text{if } 2\alpha \notin R \\
2 \sum_{a \in R : \alpha(a) = 0} \hat{a}, & \text{if } 2\alpha \in R
\end{cases}
\]

In [Yu], Jiu-Kang Yu proves the following.

**Theorem 3.** [Jiu-Kang Yu] The 6-touple $\Psi = (X, R, \Delta, \hat{X}, \hat{R}, \hat{\Delta})$, with the canonical pairing between $X$ and $\hat{X}$ and the correspondence $R \to \hat{R}$, $\alpha \mapsto \hat{\alpha}$, is a based root datum. Moreover, the homomorphism $W(\Psi) \to GL(\hat{X})$, $w \mapsto w|_{\hat{X}}$ is injective and the image is $W(\Psi)$.

The above Theorem for simply connected groups is proved in [Rec10, Sec. 3.3].

3. A CONSTRUCTION AND A CONJECTURE

3.1. Construction. Let $G$ be a quasi-split group defined over a local field $k$ of characteristic zero. Let $T$ be a maximal $k$-torus of $G$ which is contained in a $k$-Borel subgroup $B$. Let $\hat{G}_{sc}$ be the simply connected cover of the derived group $\hat{G}_{der}$ of $\hat{G}$, where $\hat{G}$ is the complex dual of $G$. Let $\hat{T} \subset \hat{G}$ be the torus dual to $T$ and $\hat{T}_{sc}$ be the pull back of $(\hat{T} \cap \hat{G}_{der})^0$ via $\hat{G}_{sc} \to \hat{G}_{der}$. Let $X = X(T)$ (resp. $\hat{X} = \hat{X}(T)$) denote the group of characters (resp. co-characters) of $T$. Let $Z$ be the center of $G$ and let $\hat{\mathfrak{z}}$ be the Lie algebra of the center $\hat{Z}$ of $\hat{G}$. Then $\hat{G} := \hat{G}_{sc} \times \hat{\mathfrak{z}}$ is the topological universal cover of $\hat{G}$. We have a short exact sequence

\[
(3.1) \quad 1 \to \pi_1(\hat{G}) \to \hat{G} \to \hat{G} \to 1,
\]
where \( \pi_1(\hat{G}) \) is the topological fundamental group of \( \hat{G} \). Let \( Q \) denote the root lattice. Then from [Spr79 2.15],
\[
X(Z) \cong X/Q. 
\]
(3.2)

The algebraic fundamental group of \( \hat{G} \) is \( \hat{X}(\hat{T})/\hat{X}(\hat{T}_{sc}) = X/Q \). Since \( \hat{G} \) is a complex algebraic group, its algebraic fundamental group is the same as its topological fundamental group (see [BGA14]). Therefore
\[
X/Q \cong \pi_1(\hat{G}). 
\]
(3.3)

Let \( W_k \) (resp. \( \Gamma_k \)) denote the Weil group (resp. absolute Galois group) of \( k \). Define \( W'_k := W_k \) if \( k \) is archimedean and \( W'_k := W_k \times \text{SL}(2, \mathbb{C}) \) if \( k \) is non-archimedean. \( W'_k \) is called the Weil-Deligne group of \( k \). Let \( \varphi : W'_k \to \mathbb{L}G \) be a Langlands parameter (see [Bor79b, Sec. 8.2]). View \( \varphi \) as an admissible homomorphism. Then \( \varphi \) determines a co-cycle \( \phi_{|W_k} : W_k \to \mathbb{L}G \to \hat{G} \). We can twist the exact sequence (3.1) by the co-cycle \( \phi \) (see Section 2.1). Then using the isomorphism \( X(Z) \cong \pi_1(\hat{G}) \), we get
\[
\hat{\gamma} : H^0(W_k, \phi \hat{G}) \to H^1(W_k, X(Z)). 
\]
Since \( H^0(W_k, \phi \hat{G}) \supset Z_{\hat{G}}(\text{Im}(\varphi)) \), by restriction this induces
\[
\hat{\gamma}' : Z_{\hat{G}}(\text{Im}(\varphi)) \to H^1(W_k, X(Z)). 
\]
Since this map is continuous and \( H^1(W_k, X(Z)) \) is discrete, \( \ker(\hat{\gamma}') \supset (Z_{\hat{G}}(\text{Im}(\varphi)))^\circ \). Thus we get a map
\[
\gamma_{\varphi} : R \varphi := \pi_0(Z_{\hat{G}}(\text{Im}(\varphi))) \to H^1(W_k, X(Z)). 
\]
(3.4)

Since \( R \varphi \) is finite, \( \gamma_{\varphi} \) induces
\[
\gamma_{\varphi}^0 : R \varphi \to H^1(W_k, X(Z))^{\text{tor}}. 
\]

By [Kar11 Theorem 4.1.3 (ii)], we have a functorial isomorphism
\[
H^1(W_k, X(Z))^{\text{tor}} = H^1(k, X(Z)). 
\]
Here we are abbreviating \( H^1(\Gamma_k, -) \) by the notation \( H^1(k, -) \). We thus get a map
\[
\gamma_{\varphi} : R \varphi \to H^1(k, X(Z)). 
\]
(3.5)

By Tate Duality ([Mil06 Corr. 2.4]), we have an isomorphism
\[
H^1(k, X(Z)) \cong \text{Hom}(H^1(k, Z), \mathbb{C}^\times). 
\]
Using the isomorphism (3.6) in (3.5), we get a map
\[
\hat{\gamma}_{\varphi} : H^1(k, Z) \to \widehat{R \varphi}, 
\]
(3.7)
where \( \hat{R}_\varphi \) is the set of irreducible representations of \( R_\varphi \). Since \( H^1(k, X(Z)) \) is abelian, the image of \( \hat{\varphi} \) lies in the group of one dimensional representations of \( R_\varphi \).

### 3.2. Statement of a conjecture.

Let \( U \) be the unipotent radical of \( B \) and let \( p : G \to G_{\text{ad}} := G/Z \) be the adjoint morphism. We denote by the same symbol, the induced map \( p : T \to T_{\text{ad}} := T/Z \).

**Definition 4.** A character \( \psi : U(k) \to \mathbb{C}^\times \) is generic if its stabilizer in \( T_{\text{ad}}(k) \) is trivial.

The group \( T_{\text{ad}}(k) \) acts simply transitively on the set of generic characters of \( U(k) \). Hence the finite abelian group \( T_{\text{ad}}(k)/p(T(k)) \) acts simply transitively on the set of \( T(k) \)-orbits of generic characters.

**Definition 5.** The pure inner forms of \( G \) are the groups \( G' \) over \( k \) which are obtained by inner twisting by elements in the pointed set \( H^1(k, G) \).

All pure inner forms have the same center \( Z \) over \( k \). Let \( G' \) be a pure inner form of \( G \). Denote the maximal torus of \( G' \) (resp. \( G'_{\text{ad}} \)) corresponding to \( T \) (resp. \( T_{\text{ad}} \)) by \( T' \) (resp. \( T'_{\text{ad}} \)). We will denote the adjoint morphism for all inner forms by the same symbol \( p \).

We have a canonical inclusion \( T'_{\text{ad}}(k)/p(T'(k)) \hookrightarrow H^1(k, Z) \) and a canonical isomorphism \( T'_{\text{ad}}(k)/p(T(k)) \cong G'_{\text{ad}}(k)/p(G'(k)) \) (Lemma 5.1 [DR10]). Equation (3.7) thus induces

\[ \zeta'_\varphi : G'_{\text{ad}}(k)/p(G'(k)) \to \hat{R}_\varphi. \]

Let \( \hat{\Pi}_\varphi \) denote the Vogan \( L \)-packet associate to \( \varphi \). It is the union of the standard \( L \)-packets associated to \( \varphi \) of \( G \) and all its pure inner forms. By standard, we mean \( L \)-packets as defined in [Bor79a]. Let \( \rho \in \hat{R}_\varphi \mapsto \pi_\rho \in \hat{\Pi}_\varphi \) be the parametrization defined after the choice of a Whittaker datum. Assume that this parametrization is compatible with Deligne’s normalization of the local Artin map (see [GGP12, Sec. 3]). Let \( \Pi'_\varphi \) be the standard \( L \)-packet of \( G' \) contained in \( \hat{\Pi}_\varphi \). The following is a conjecture in [GGP12, Sec. 9 (3)].

**Conjecture 6.** For \( g \in G'_{\text{ad}}(k) \), \( \pi_\rho \circ \text{Ad}(g) = \pi_{g\cdot \rho} \), where \( g \cdot \rho = \rho \otimes \zeta'_{\varphi}(g) \) and \( \pi_\rho \in \Pi'_\varphi \). Thus \( \pi_\rho \) is \( \psi \)-generic iff \( \pi_{g\cdot \rho} \) is \( g \cdot \psi \) generic.

We have a natural inclusion \( \pi_0(\hat{Z}^{\Gamma_k}) \subset \hat{R}_\varphi \). Let \( \tau \in \hat{R}_\varphi \). In [GGP12, Sec. 9(4)], it is explained that the pure inner form of \( G \) which acts on the representation corresponding to the parameter \( (\varphi, \tau) \) is determined by the character \( \tau | \pi_0(\hat{Z}^{\Gamma_k}) \). Thus the standard \( L \)-packet \( \Pi_\varphi \subset \hat{\Pi}_\varphi \) of \( G \) is parametrized by \( \tau \in \hat{R}_\varphi \) whose restriction to \( \pi_0(\hat{Z}^{\Gamma_k}) \) is trivial. In other words, the standard \( L \)-packet is parametrized by the irreducible representations \( \hat{S}_\varphi \hookrightarrow \hat{R}_\varphi \) of the group \( S_\varphi := \pi_0(Z_G(\text{Im}(\varphi))/\hat{Z}^{\Gamma_k}) \). The map \( \zeta_\varphi : G_{\text{ad}}(k)/p(G(k)) \to \hat{R}_\varphi \) must factor through \( \hat{S}_\varphi \). Conjecture (8) for standard generic \( L \)-packets can be stated as:
Conjecture 6’. \( \pi_\rho \in \Pi_\varphi \) is \( \psi \)-generic iff \( \pi_{g \cdot \rho} \) is \( g \cdot \psi \) generic, where \( \rho \in \widehat{R}_\varphi \), \( g \in G_{\text{ad}}(k) \), and where \( g \cdot \rho = \rho \otimes \zeta_\varphi(g) \).

Remark 7. In [Kal13, Sec. 3], Kaletha constructs a map \( \zeta_\varphi : G_{\text{ad}}(k)/p(G(k)) \to \widehat{S}_\varphi \). In [Kal13, Sec. 1, eq. (1.1)], he states the above conjecture in a more precise manner by comparing the parametrization of a tempered \( L \)-packet for different choices of Whittaker data. He also points out that the action of \( g \in G_{\text{ad}}(k) \), should send \( \rho \in \widehat{S}_\varphi \) to \( \rho \otimes \zeta_\varphi(g) \) or \( \rho \otimes \zeta_\varphi^{-1}(g) \) depending on which of the two possible normalizations of the local Artin map one chooses. The normalization in Conjecture 6 uses Deligne’s normalization [GGP12, Sec. 3].

4. Description of \( R \)-group

Let the notations be as in Section 3. Assume that \( G \) is unramified, i.e., it is quasi-split and split over an unramified extension of \( k \). We also assume \( k \) to be non-archimedean. Let \( I \) be the inertia subgroup of \( W_k \) and let \( \sigma \) be the Frobenious element in \( W_k/I \). Throughout this section, we will abbreviate \( H^1(W_k/I, -) \) by the notation \( H^1(\sigma, -) \).

4.1. Case of an unramified parameter. Let \( \bar{s} \in \hat{T} \) and let \( \varphi \) be the Langlands parameter determined by the map \( \sigma \mapsto \bar{s} \). Let \( s \) be a lift of \( \bar{s} \) in \( \hat{T}_{\text{sc}} \times \hat{z} \).

Let \( H^1(\sigma, \hat{G})_{\text{ss}} \subset H^1(\sigma, \hat{G}) \) denote the \( \sigma \)-conjugacy classes of the semisimple elements of \( \hat{G} \), where \( \hat{G} = \hat{G}_{\text{sc}} \times \hat{z} \) as in Section 3. Denote by \([t]\), the class of \( t \in \hat{G}_{\text{ss}} \) in \( H^1(\sigma, \hat{G})_{\text{ss}} \). Let \( A \) denote the co-invariant of \( A \) with respect to \( \sigma \). We have \( H^1(\sigma, A) \cong A \). Let \( x \) denote the image of \( x \in A \) in \( A \). Then there is an action of \( H^1(\sigma, A) \) on \( H^1(\sigma, \hat{G})_{\text{ss}} \) given by
\[
x \cdot [t] := [xt]
\]
for \( x \in A, t \in \hat{G}_{\text{ss}} \).

Denote by \( A_{\varphi} \) the stabilizer of \([s]\) in \( A \).

Lemma 8. The map \( \gamma_\varphi \) in equation (3.5) induces an isomorphism \( R_\varphi \cong A_{\varphi} \).

Proof. We have
\[
R_\varphi \cong \ker(H^1(\sigma, A) \to H^1(\sigma, \hat{G}))
\begin{align*}
&= \{ x \in A | g^{-1}x(\sigma g)s^{-1} = 1 \text{ for some } g \in \hat{G} \} \\
&= \{ x \in A | x \cdot [s] = [s] \} \\
&= A_{\varphi}.
\end{align*}
\]

Remark. The above Lemma is also proved in [Yu].
4.2. **Action of $\Omega$.** Let $\Psi = (X, R, \Delta, \hat{X}, \hat{R}, \hat{\Delta})$ be the based root datum of $(G, B, T)$. So $X$ (resp. $\hat{X}$) is the group of characters (resp. co-characters) of $T$, $R$ (resp. $\hat{R}$) is the set of roots (resp. co-roots) of $T$ in the Lie algebra of $G$ and $\Delta$ (resp. $\hat{\Delta}$) is a basis in $R$ (resp. $\hat{R}$) determined by $B$. Let $\Psi = (X, R, \Delta, \hat{X}, \hat{R}, \hat{\Delta})$ be the based root datum obtained from $\Psi = (X, R, \Delta, \hat{X}, \hat{R}, \hat{\Delta})$ by the construction given in 2.2.2. Let $Q$ be the lattice generated by $R$. Let $C$ be the alcove in $V := X \otimes \mathbb{R}$ determined by $\Delta$. Let $W = W(\Psi)$ be the Weyl group of of the based root datum $\Psi$. By Theorem 3, it is the relative Weyl group of $G$. Let $\Omega \sim X/Q$ be the stabilizer in $W \ltimes X$ of $C$ (see Section 2.2.1).

By [Bor79a, Lemma 6.5] (or more directly by [Mis15, Prop. 11]), we have

\[
\hat{T}_\sigma/W \cong (\hat{G} \rtimes \sigma)_{ss}/\text{Int}(\hat{G}),
\]

where $(\hat{G} \rtimes \sigma)_{ss}$ is the set of semisimple elements in $\hat{G} \rtimes \sigma$ and $\text{Int}(\hat{G})$ denotes the group of inner automorphisms of $\hat{G}$.

Let $\hat{T}_{\text{cpt}}$ be the maximal compact subtorus in $\hat{T}$. Write $\hat{T} = X \otimes \mathbb{C}^\times$. Under this identification, $\hat{T}_{\text{cpt}} = X \otimes (\mathbb{R}/\mathbb{Z}) \cong X \otimes \mathbb{R}/X$. Let $\hat{G}_{\text{cpt}}$ be the set of those semi-simple elements of $\hat{G}$ which lie in some maximal compact subtorus of $\hat{G}$. The isomorphism in (4.1) induces an isomorphism

\[
\hat{G}_{\text{cpt}} \rtimes \sigma/\text{Int}(\hat{G}) \cong (\hat{T}_{\text{cpt}})_\sigma/W \\
\cong X \otimes \mathbb{R}/W \times X \\
= X \otimes \mathbb{R}/((W \times Q) \times \Omega) \\
\leftarrow \longrightarrow \overline{C}/\Omega,
\]

where $\overline{C}$ is the closure of the alcove $C$ determined by $\Delta$.

Let $\hat{\mathfrak{g}}_{\text{cpt}} := X/Q \otimes \mathbb{R}$. It is the Lie algebra of the maximal compact subtorus of $\hat{Z}$. Let $\hat{G}_{\text{cpt}} = \hat{G}_{\text{sc}} \times \hat{\mathfrak{g}}_{\text{cpt}}$. Then

\[
\hat{G}_{\text{cpt}} \rtimes \sigma/\text{Int}(\hat{G}) \cong \hat{T}_{\text{cpt}}/W \\
\cong ((X_{\text{sc}} \otimes (\mathbb{R}/\mathbb{Z})) \times (X/Q \otimes \mathbb{R}))/W \\
\cong X \otimes R/(Q \rtimes W) \\
\text{since } X_{\text{sc}} = Q \\
\leftarrow \longrightarrow \overline{C}.
\]

We have $A \cong (X/Q)_{\sigma} \rightarrow X/Q \cong \Omega$. In Lemma 9 below, we will show that the action of $A$ on $\hat{G}_{\text{cpt}} \rtimes \sigma/\text{Int}(\hat{G}) \subset (\hat{G} \rtimes \sigma)_{ss}/\text{Int}(\hat{G})$ is compatible with the action of $\Omega$ on $\overline{C}$. Now $G$ is isogenous to $Z^\circ \times (G_{\text{sc}})_{\text{der}}$, where $(G_{\text{sc}})_{\text{der}}$ is the simply connected cover of the derived group of $G$ and $Z^\circ$ is the identity component of the center of $G$. Since any simply connected semisimple group is the direct product of
almost simple groups, it suffices to prove the compatibility in the case when $G$ is almost simple.

Let $\phi \in A$ and let $a$ be a lift of $\phi$ in $A$. Let $c_0$ be the weighted barycenter of $C$ and let $a \mapsto \tilde{\omega}_a$ under the surjection $A \twoheadrightarrow \Omega$, where $\omega_a \in W$ and $\tilde{\omega}_a$ is the affine transformation $x \in X \otimes \mathbb{R} \mapsto \omega_a(x - c_0) + c_0$ (see Section 2.2.1).

Let $[s] \mapsto x_s$ under the bijection $\hat{G}^\text{cpt} \times \sigma/\text{Int}(\hat{G}) \leftrightarrow \hat{C}$, where $[s]$ denotes the class of $s \in \hat{G}^\text{cpt}$. Without loss of generality, we can assume that $s \in \hat{T}^\text{cpt}$.

**Lemma 9.** We have $\tilde{\omega}_a \cdot x_s = x_{[as]}$.

**Proof.** Let $\tilde{W}^\sigma = W \ltimes \mathbb{Q}$. We have

\[
\tilde{\omega}_a \cdot x_s = \omega_a(x_s - c_0) + c_0 = \omega_a \cdot x_s + (1 - \omega_a)c_0 = \omega_a(x_s) + (\omega_a^{-1} - 1)c_0.
\]

By Lemma 2 $\tilde{\omega}_a \mapsto (\omega_a^{-1} - 1)c_0 + \mathbb{Q}$ under the isomorphism $\Omega \cong X/\mathbb{Q}$. Using this we get that $x_{[a]} \equiv (\omega_a^{-1} - 1)c_0 \mod \tilde{W}^\sigma$. Thus

\[
\tilde{\omega}_a \cdot x_s \equiv x_s + x_{[a]} \mod \tilde{W}^\sigma = x_{[as]} \mod \tilde{W}^\sigma.
\]

Since $\tilde{\omega}_a \cdot x_s \in \hat{C}$ and $x_{[as]} \in \hat{C}$, we conclude that

$\tilde{\omega}_a \cdot x_s = x_{[as]}$.

\[\square\]

### 4.3. Tempered parameter

Let $\lambda$ be a unitary unramified character of $T(k)$. Let $\lambda \mapsto [\tilde{s}]$ under the bijection

\[\text{Hom}(T(k), S^1)/W \cong \hat{G}^\text{cpt} \times \sigma/\text{Int}(\hat{G}),\]

where $\tilde{s}$ can be chosen to be in $\tilde{T}$. Here $S^1$ denotes the unit circle in $\mathbb{C}$. Let $\varphi$ be the Langlands parameter determined by the map $\sigma \mapsto \tilde{s}$. Let $s$ be a lift of $\tilde{s}$ in $\tilde{T}_{sc} \times \tilde{\varnothing}$.

Let $\Omega_{\varphi}$ be the stabilizer of $x_s \in \hat{C}$ in $\Omega$. We have

**Proposition 10.** $S_{\varphi} \cong \Omega_{\varphi}$.

**Proof.** By [Key57], Lem. 2.5(iii)], $\pi_0(\tilde{T}^\sigma) = \pi_0(\tilde{T}^\sigma)$. But $\pi_0(\tilde{T}^\sigma) \cong (X_{\sigma})^{\text{tor}}$. Since $S_{\varphi} \cong R_{\varphi}/\pi_0(\tilde{T}^\sigma)$, by Lemma 8 we get that $S_{\varphi} \cong \mathbb{A}_{\varphi}/(X_{\sigma})^{\text{tor}} = \Omega_{\varphi}$. Lemma 9 shows that $S_{\varphi}$ and $\Omega_{\varphi}$ have compatible actions on $\hat{G}^\text{cpt} \times \sigma/\text{Int}(\hat{G})$ and $\hat{C}$ respectively. \[\square\]

When $G$ is almost simple and simply connected, the non-trivial $\Omega$ are given by the table below (see [Kan01, Sec. 9-4] and [Ree10, Table-1]).
|        | \( \Omega \)          |
|--------|------------------------|
| \( A_n \) | \( \mathbb{Z}/(n+1)\mathbb{Z} \) |
| \( B_n \) | \( \mathbb{Z}/2\mathbb{Z} \) |
| \( C_n \) | \( \mathbb{Z}/2\mathbb{Z} \) |
| \( D_n \) (n even) | \( \mathbb{Z}/2\mathbb{Z} \times \mathbb{Z}/2\mathbb{Z} \) |
| \( D_n \) (n odd) | \( \mathbb{Z}/2\mathbb{Z} \) |
| \( E_6 \) | \( \mathbb{Z}/3\mathbb{Z} \) |
| \( E_7 \) | \( \mathbb{Z}/2\mathbb{Z} \) |
| \( ^2A_{2n-1} \) (n ≥ 3) | \( \mathbb{Z}/2\mathbb{Z} \) |
| \( ^2D_{n+1} \) (n ≥ 2) | \( \mathbb{Z}/2\mathbb{Z} \) |

**Table 1.**

Let \( R_\lambda \) be the Knapp-Stein \( R \)-group associated to \( \lambda \) (see [Key87] §2 for definition). By [Key87] Prop. 2.6, \( R_\lambda \cong S_\varphi \). Using Proposition 10, we obtain \( \Omega_\varphi \cong R_\lambda \). In fact, the isomorphism is given by the restriction of the natural projection \( W \ltimes X \to W \) to \( \Omega_\varphi \). We get

**Theorem 11.** Let \( G \) be an almost simple, simply connected, unramified group defined over a non-archimedean local field \( k \). The non-trivial \( R_\lambda \) that can appear are precisely the subgroups of \( \Omega \) in table 1.

(see also [KP13]).

This gives the classification obtained by Keys in [Key82] §3 in the case of unramified groups.

5. **Unramified \( L \)-packet**

Let the notations be as in Section 3. Assume further that \( G \) is unramified and that \( k \) is non-archimedean. As in Section 4, let \( I \) be the inertia subgroup of \( W_k \) and let \( \varphi \) be the Frobenius element in \( W_k/I \).

An unramified \( L \)-packet consists of those irreducible subquotients of an unramified principal series representation of \( G(k) \) which have a non-zero vector fixed by some hyperspecial subgroup of \( G(k) \). Unramified \( L \)-packets are in bijective correspondence with \( (\hat{G} \times \sigma)_{ss}/\text{Int}(\hat{G}) \). Let \( \varphi \) be a Langlands parameter determined by the \( \sigma \)-conjugacy class of a semi-simple element and let \( \Pi_\varphi \) be the associated unramified \( L \)-packet. The \( L \)-packet \( \Pi_\varphi \) is parametrized by \( \hat{S}_\varphi \), where \( S_\varphi := \pi_0(Z_G(\text{Im}(\varphi)))/\hat{Z}\Gamma_\varphi \) as in Section 3 after making the choice of a hyperspecial point. We denote the parametrization by \( \rho \in \hat{S}_\varphi \mapsto \pi_\rho \in \Pi_\varphi \).

Let \( K \) be a compact subgroup of \( G(k) \). Denote by \([K] \), the \( G(k) \)-conjugacy class of \( K \). If \( \pi \) is a representation of \( G(k) \), we denote by \( \pi^K \) the \( K \)-fixed points of the space realizing \( \pi \). By the notation \( \pi^{[K]} \neq 0 \), we mean that \( \pi \) has a non-zero vector fixed by some (therefore any) conjugate of \( K \).
The conjugacy classes of hyperspecial subgroups of \( G(k) \) form a single orbit under \( T_{ad}(k) \). The author, in his Ph.D. thesis [Mis13, Theorem 2.2.1] (also [Mis12, Thm. 1]) constructs a map \( T_{ad}(k)/p(T(k)) \to \hat{S}_\varphi \). For the action of \( T_{ad}(k) \) on \( \hat{S}_\varphi \) given by this map, he shows that \( \pi_{t,\rho}^{[K]} \neq 0 \iff \pi_{\rho}^{[K]} \neq 0 \) for all \( t \in T_{ad}(k) \), \( \rho \in \hat{S}_\varphi \), where \( K \) is a hyperspecial subgroup of \( G(k) \). Using this result, we have

**Theorem 12.** Let \( \Pi_\varphi \) be an unramified \( L \)-packet associated to a Langlands parameter \( \varphi \). Then \( \pi_\rho \in \Pi_\varphi \) is \( \psi \)-generic iff \( \pi_{t,\rho} \) is \( t \cdot \psi \) generic for all \( t \in T_{ad}(k) \).

**Proof.** Given \( \pi_\rho \in \Pi_\varphi \), let \( K \) be a hyperspecial subgroup such that \( \pi_{\rho}^{[K]} \neq 0 \). We can write \( K \) as the stabilizer \( G(k)_{x} \) of some hyperspecial point \( x \) in the Bruhat-Tits building of \( G(k) \). Without loss of generality we can assume \( x \) to lie in the apartment associated to \( T \). We have that \( (\text{Ind}_{G(k)}^{U(k)} \psi)^{G(k)}_{x} \neq 0 \) iff there exists \( g \in G \) such that \( \psi |_{g^{-1}G(k)_{x} \cap U(k)} \equiv 1 \). Without loss of generality, we can assume that \( g = 1 \). Let \( t \in T_{ad}(k) \).

\[
\text{Hom}_{G(k)}(\pi_\rho, (\text{Ind}_{U(k)}^{G(k)} \psi)) \neq 0 \iff (\text{Ind}_{U(k)}^{G(k)} \psi)^{G(k)}_{x} \neq 0
\]

\[
\iff \psi |_{G(k)_{x} \cap U(k)} \equiv 1
\]

\[
\iff t \cdot \psi |_{G(k)_{x} \cap U(k)} \equiv 1
\]

\[
\iff (\text{Ind}_{U(k)}^{G(k)} t \cdot \psi)^{t \cdot [G(k)_{x}]} \neq 0
\]

\[
\iff \text{Hom}_{G(k)}(\pi_{t,\rho}, (\text{Ind}_{U(k)}^{G(k)} t \cdot \psi)) \neq 0
\]

\( \square \)

**Remark 13.** Note that we do not assume \( \varphi \) to be tempered. However, if the associated \( L \)-packet is not generic, then the above statement could be vacuous.

**Remark 14.** Theorem 12 is a very special case of Conjecture 6 [Kal13, Thm. 3.3], Kaletha proves Conjecture 6 for tempered representations in the case when \( G \) is a quasi-split real \( K \)-group or a quasi-split \( p \)-adic classical group (in the sense of Arthur).

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**REFERENCES**

[Art06] J. Arthur. A note on \( L \)-packets. Pure Appl. Math. Q., 2(1, Special Issue: In honor of John H. Coates. Part 1):199–217, 2006.
[AYY13] J. An, J.-K. Yu, and J. Yu. On the dimension datum of a subgroup and its application to isospectral manifolds. *J. Differential Geom.*, 94(1):59–85, 2013.

[BGA14] M. Borovoi and C. D. González-Avilés. The algebraic fundamental group of a reductive group scheme over an arbitrary base scheme. *Cent. Eur. J. Math.*, 12(4):545–558, 2014.

[Bor79a] A. Borel. Automorphic L-functions. In *Automorphic forms, representations and L-functions (Proc. Sympos. Pure Math., Oregon State Univ., Corvallis, Ore., 1977)*, Part 2, Proc. Sympos. Pure Math., XXXIII, pages 27–61. Amer. Math. Soc., Providence, R.I., 1979.

[Bor79b] A. Borel. Automorphic L-functions. In *Automorphic forms, representations and L-functions (Proc. Sympos. Pure Math., Oregon State Univ., Corvallis, Ore., 1977)*, Part 2, Proc. Sympos. Pure Math., XXXIII, pages 27–61. Amer. Math. Soc., Providence, R.I., 1979.

[DR10] S. DeBacker and M. Reeder. On some generic very cuspidal representations. *Compos. Math.*, 146(4):1029–1055, 2010.

[GGP12] W.-T. Gan, B. H. Gross, and D. Prasad. Symplectic root numbers, central critical values, and restriction problems in the representation theory of classical groups. *Asterisque*, 2012.

[Kal13] T. Kaletha. Genericity and contragredience in the local Langlands correspondence. *Algebra Number Theory*, 7(10):2447–2474, 2013.

[Kan01] R. Kane. *Reflection groups and invariant theory*. CMS Books in Mathematics/Ouvrages de Mathématiques de la SMC, 5. Springer-Verlag, New York, 2001.

[Kar11] D. A. Karpuk. Weil-étale Cohomology over p-adic Fields. *ArXiv e-prints*, November 2011.

[Key82] D. Keys. Reducibility of unramified unitary principal series representations of p-adic groups and class-1 representations. *Math. Ann.*, 260(4):397–402, 1982.

[Key87] D. Keys. L-indistinguishability and R-groups for quasisplit groups: unitary groups in even dimension. *Ann. Sci. École Norm. Sup. (4)*, 20(1):31–64, 1987.

[KP13] T. Kamran and R. Plymen. K-theory and the connection index. *Bull. Lond. Math. Soc.*, 45(1):111–119, 2013.

[Kuo02] W. Kuo. Principal nilpotent orbits and reducible principal series. *Represent. Theory*, 6:127–159 (electronic), 2002.

[Kuo10] W. Kuo. The Langlands correspondence on the generic irreducible constituents of principal series. *Canad. J. Math.*, 62(1):94–108, 2010.

[Mil06] J. S. Milne. *Arithmetic duality theorems*. BookSurge, LLC, Charleston, SC, second edition, 2006.

[Mis12] M. Mishra. Structure of the Unramified L-packet. *ArXiv e-prints*, December 2012.

[Mis13] M. Mishra. *Structure of the unramified L-packet*. ProQuest LLC, Ann Arbor, MI, 2013. Thesis (Ph.D.)–Purdue University.

[Mis15] M. Mishra. Langlands parameters associated to special maximal parahoric spherical representations. *Proc. Amer. Math. Soc.*, 143(5):1933–1941, December 2015.

[Rec10] M. Reeder. Torsion automorphisms of simple Lie algebras. *Enseign. Math. (2)*, 56(1-2):3–47, 2010.

[Ser97] J.-P. Serre. *Galois cohomology*. Springer-Verlag, Berlin, 1997. Translated from the French by Patrick Ion and revised by the author.

[Sha90] F. Shahidi. A proof of Langlands’ conjecture on Plancherel measures; complementary series for p-adic groups. *Ann. of Math. (2)*, 132(2):273–330, 1990.
[Spr79] T. A. Springer. Reductive groups. In Automorphic forms, representations and $L$-functions (Proc. Sympos. Pure Math., Oregon State Univ., Corvallis, Ore., 1977), Part 1, Proc. Sympos. Pure Math., XXXIII, pages 3–27. Amer. Math. Soc., Providence, R.I., 1979.

[Yu] J.-K. Yu. A note on the relative root datum of quasi-split groups (preprint).

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