Maintaining Entanglement for Three Qubit Gate System via Circulant Symmetry

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Abstract

Based on the circulant symmetry we give a proposal about the physical realization of the quantum Fourier transform gate. This symmetry allows us to construct a set of eigenvectors independently on the magnitude of physical parameters characterizing our system and as a result, the entanglement will be protected. The implementation of the present gate requires an adiabatic transition from each spin product state to Fourier modes. The fidelity was numerically calculated and the results show important values. Finally, we describe that we can accelerate the gate by using the counter-driving field.

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1 Introduction

In 1980 Paul Benioff proposed a quantum mechanical model of the Turing machine [1], which opened a new field of research called quantum computers (QCs). In 1982 Richard Feynman showed that QCs can be used to simulate complex systems (living cell, city traffic, human brain, universe ···) [2]. Four decades later, Seth Lloyd showed that an array of nuclear magnetic resonance spins are the basic units of a QC [3]. QC is a device that harnesses quantum phenomena to process information in a way that maintain quantum coherence [4]. Consequently, QC can solve the most hard computational problems that today’s most powerful supercomputers cannot solve, and never will [5]. As an example, Shor’s quantum algorithm (SQA) for factoring large numbers [6] is the most seminal motivation behind the development of QCs [5]. It is well-known in quantum computing that the physical implementation of SQA requires a gate of particular importance called quantum Fourier transform (QFT) [7]. QFT is the quantum implementation of the discrete Fourier transform over the amplitudes of a wavefunction and acts on a vector \( |v\rangle \in \mathbb{C}^N \) as

\[
\text{QFT}: \quad |x\rangle \mapsto |y\rangle = \frac{1}{\sqrt{N}} \sum_{j=0}^{N-1} e^{2\pi i x j/N} |j\rangle
\]

QFT is widely used as a key subroutine in several factoring algorithms like for instance quantum amplitude estimation [8] and quantum counting [9].

Concerning the physical implementation of logical QFT, the circulant Hamiltonians are addressed [10], which due to the fact that the eigenspectrum of circulant matrix is spanned by the Fourier modes [11, 12]. Recently, Ivanov and Vitanov [13] have constructed a Hamiltonian based on two spins emerged in a magnetic field, generating a Rabi oscillations, and by adjusting the coupling strength of the spin-spin interaction a circulant symmetry was obtained. Consequently, they showed that the eigenvectors do not depend on the magnitude of the physical parameters, which entails the protection of entanglement and then the obtained system can be used as a logical QFT. Two schemes for implementing quantum phase gates via adiabatic passage and phase control of the driving fields were proposed by Wu et al. [14]. They studied the experimental feasibility, gate fidelity and decoherence effect for both schemes. Moreover, the shortcuts to adiabaticity have become instrumental in preparing and driving internal and motional states in atomic, molecular and solid-state physics, a complete review can found be in [15].

Motivated by the results developed in [13], we study a Hamiltonian describing three spins in a magnetic field, coupled via linear and non-linear interactions. The last coupling generally arises when the interaction medium is non-linear [16–18] and remains an important ingredient to generate a circulant Hamiltonian. To build a logical QFT, we propose two choices by adjusting the physical parameters, the eigenvectors of our circulant Hamiltonian do not depend on the parameters, which protects entanglement during the gate implementation as long as the circulant symmetry is maintained. To breaks the circulant symmetry during the transition, we involve an energy offset Hamiltonian

\[
H_0(t) = \sum_{j=1}^{3} \Delta_j(t) \sigma_j^z.
\]

By adjusting the physical parameters and the detuning \( \Delta_j(t) \), we realize the circulant symmetry at end of transition \( t_f \). By sinusoidally modulating in time the physical parameters, we show that it is possible to adiabatically obtain from any initial state a superposition of the quantum Fourier modes with high fidelity. As the adiabatic evolution is robust but still limited by the non-adiabatic transition, we introduce an counter-driving Hamiltonian \( H_{CD}(t) \) [19] to suppress these transitions. Under suitable conditions of the physical parameters, we determine the eigenvectors associated to the total
Hamiltonian. These allow us to combine our gate with the short-cut to adiabacity scheme in order to accelerate the gate. To physically implement the gate, we suggest an extension of the proposal described in [13].

The outlines of our paper are summarized as follows. In Sec. 2, we propose an Hamiltonian describing three qubit involving different interaction in addition to Rabi oscillations and show how to obtain the circulant symmetry. The adiabatic transition technique is performed to end up with the Fourier modes in Sec. 3. The physical implementation of the obtained QFT gate will be discussed in Sec. 4. By sinusoidally modulating the parameters, the gate and creation of entangled state fidelities together with the non-degeneracy of frequencies will be numerically analyzed in Sec. 5. We discuss the shortcut to adiabacity scheme combined with our gate in Sec. 6. Finally, we close by concluding our work.

2 Theoretical model

To achieve our task we consider three spins emerged in a magnetic field forming a three qubit gate system as presented in Figure 1.

It can be described by a Hamiltonian involving two types of interaction, such as

$$H = J_1(\sigma_1^+ e^{-i\phi_{12}} + \sigma_1^- e^{i\phi_{12}})(\sigma_2^+ e^{-i\phi_{21}} + \sigma_2^- e^{i\phi_{21}}) + J_2(\sigma_2^+ e^{-i\phi_{23}} + \sigma_2^- e^{i\phi_{23}})(\sigma_3^+ e^{-i\phi_{32}} + \sigma_3^- e^{i\phi_{32}}) + J_3(\sigma_1^+ e^{-i\phi_{13}} + \sigma_1^- e^{i\phi_{13}})(\sigma_3^+ e^{-i\phi_{31}} + \sigma_3^- e^{i\phi_{31}})$$

$$+ \Omega_1(\sigma_1^+ e^{i\theta_1} + \sigma_1^- e^{-i\theta_1}) + \Omega_2(\sigma_2^+ e^{i\theta_2} + \sigma_2^- e^{-i\theta_2}) + \Omega_3(\sigma_3^+ e^{i\theta_3} + \sigma_3^- e^{-i\theta_3})$$

where $\sigma_j^+ = |\uparrow_j\rangle \langle \downarrow_j|$ and $\sigma_j^- = |\downarrow_j\rangle \langle \uparrow_j|$ stand for spin flip operators, $|\uparrow_j\rangle$ and $|\downarrow_j\rangle$ being the qubit states of the $j$th spin, with $j = 1, 2, 3$. The first term describes the interaction between spins 1 and 2 with phases $\phi_{12}$ and $\phi_{21}$. The second term for the interaction between spins 2 and 3 with phases $\phi_{23}$ and $\phi_{32}$. The third one shows the coupling between spins 1 and 3 with phases $\phi_{13}$ and $\phi_{31}$. The terms including the Rabi frequencies $\Omega_j$ are the single-qubit transitions with phases $\theta_j$. The last term describes the coupling between the three spins with phases $\phi_j$ [18]. Note in passing that our Hamiltonian can be seen as a generalization to three qubits used by Ivanov and Vitanov [13].
It is convenient for our task to consider the matrix form of the Hamiltonian (1) and then in the basis $B_c = \{\uparrow\uparrow\downarrow, \uparrow\downarrow\downarrow, \downarrow\downarrow\downarrow, \uparrow\downarrow\uparrow, \downarrow\uparrow\downarrow, \uparrow\uparrow\uparrow\}$ we have

$$H = \begin{pmatrix}
0 & \Omega_2 e^{i\theta_2} & \Omega_1 e^{i\theta_1} & \Omega_3 e^{i\theta_3} & J_1 e^{-i\xi_1} & J_1 e^{-i\xi_2} & J_1 e^{-i\xi_3} & J_1 e^{-i\xi_4} \\
\Omega_2 e^{-i\theta_2} & 0 & J_2 e^{-i\xi_4} & \Omega_3 e^{i\theta_3} & J_1 e^{-i\xi_6} & J_1 e^{-i\xi_5} & \Omega_1 e^{i\theta_1} & \Omega_3 e^{i\theta_3} \\
\Omega_3 e^{-i\theta_3} & J_2 e^{-i\xi_4} & 0 & \Omega_1 e^{i\theta_1} & J_1 e^{-i\xi_6} & \Omega_1 e^{i\theta_1} & J_1 e^{-i\xi_2} & \Omega_1 e^{i\theta_1} \\
J_2 e^{i\xi_1} & \Omega_2 e^{-i\theta_2} & \Omega_3 e^{-i\theta_3} & 0 & J_1 e^{-i\xi_10} & J_1 e^{-i\xi_12} & \Omega_1 e^{i\theta_1} & \Omega_3 e^{i\theta_3} \\
\Omega_1 e^{-i\theta_1} & J_3 e^{i\xi_5} & J_1 e^{i\xi_6} & \Omega_3 e^{-i\theta_3} & 0 & J_1 e^{i\xi_10} & \Omega_3 e^{i\theta_3} & \Omega_3 e^{i\theta_3} \\
J_3 e^{i\xi_2} & \Omega_1 e^{i\theta_1} & J_1 e^{i\xi_6} & \Omega_1 e^{i\theta_1} & J_1 e^{-i\xi_10} & J_1 e^{-i\xi_12} & \Omega_1 e^{i\theta_1} & \Omega_3 e^{i\theta_3} \\
J_1 e^{i\xi_3} & J_1 e^{i\xi_8} & \Omega_1 e^{-i\theta_1} & J_3 e^{i\xi_5} & \Omega_2 e^{-i\theta_2} & J_2 e^{i\xi_4} & 0 & \Omega_3 e^{i\theta_3} \\
J_1 e^{i\xi_7} & J_1 e^{i\xi_3} & J_3 e^{i\xi_2} & \Omega_1 e^{-i\theta_1} & J_2 e^{i\xi_1} & \Omega_2 e^{-i\theta_2} & \Omega_3 e^{-i\theta_3} & 0 
\end{pmatrix}$$

(2)

where we have involved the new angles $\xi_1 = \phi_{23} + \phi_{32}, \xi_2 = \phi_{13} + \phi_{31}, \xi_3 = \phi_{12} + \phi_{21}, \xi_4 = \phi_{23} - \phi_{32}, \xi_5 = \phi_{13} - \phi_{31}, \xi_6 = \phi_{12} - \phi_{21}, \xi_7 = \phi_1 + \phi_2 + \phi_3, \xi_8 = \phi_1 + \phi_2 - \phi_3, \xi_9 = \phi_1 - \phi_2 + \phi_3, \xi_{10} = \phi_1 - \phi_2 - \phi_3.$

In what follows, we are going to find conditions on the involved parameters to end up with the Hamiltonian (2) as a circulant matrix [11, 12]. The benefit of circulant matrix is that its eigenvectors are the vectors of columns of the discrete quantum Fourier transform and therefore they do not depend on the elements of the circulant matrix. The eigenvectors of our circulant matrix can be mapped in the spin basis $B_c$ as

$$|\psi_0\rangle = \frac{1}{\sqrt{2}} \left( |\uparrow\downarrow\downarrow\rangle + |\downarrow\uparrow\downarrow\rangle + |\uparrow\downarrow\uparrow\rangle + |\downarrow\uparrow\uparrow\rangle + |\uparrow\uparrow\downarrow\rangle + |\uparrow\uparrow\uparrow\rangle \right)$$

(3)

$$|\psi_1\rangle = \frac{1}{\sqrt{2}} \left( |\uparrow\downarrow\downarrow\rangle + \omega |\downarrow\uparrow\downarrow\rangle + i \omega |\uparrow\downarrow\uparrow\rangle - |\downarrow\uparrow\downarrow\rangle - \omega |\uparrow\downarrow\uparrow\rangle - i \omega |\uparrow\uparrow\uparrow\rangle \right)$$

(4)

$$|\psi_2\rangle = \frac{1}{\sqrt{2}} \left( |\uparrow\downarrow\downarrow\rangle + i |\downarrow\uparrow\downarrow\rangle - |\downarrow\uparrow\downarrow\rangle - i |\downarrow\uparrow\uparrow\rangle + |\downarrow\uparrow\uparrow\rangle + i |\uparrow\uparrow\downarrow\rangle - |\uparrow\uparrow\downarrow\rangle - i \omega |\uparrow\uparrow\uparrow\rangle \right)$$

(5)

$$|\psi_3\rangle = \frac{1}{\sqrt{2}} \left( |\uparrow\downarrow\downarrow\rangle + \omega |\downarrow\uparrow\downarrow\rangle - i |\downarrow\uparrow\downarrow\rangle + \omega |\downarrow\uparrow\downarrow\rangle - |\downarrow\uparrow\downarrow\rangle - i \omega |\uparrow\uparrow\downarrow\rangle + i |\uparrow\uparrow\downarrow\rangle - \omega |\uparrow\uparrow\uparrow\rangle \right)$$

(6)

$$|\psi_4\rangle = \frac{1}{\sqrt{2}} \left( |\uparrow\downarrow\downarrow\rangle - |\downarrow\uparrow\downarrow\rangle + |\downarrow\uparrow\downarrow\rangle - |\downarrow\downarrow\downarrow\rangle + |\uparrow\uparrow\uparrow\rangle + |\uparrow\downarrow\downarrow\rangle - |\downarrow\downarrow\downarrow\rangle \right)$$

(7)

$$|\psi_5\rangle = \frac{1}{\sqrt{2}} \left( |\uparrow\downarrow\downarrow\rangle - \omega |\downarrow\uparrow\downarrow\rangle + i |\downarrow\uparrow\downarrow\rangle - i \omega |\downarrow\uparrow\downarrow\rangle - |\downarrow\uparrow\downarrow\rangle + \omega |\uparrow\uparrow\downarrow\rangle - i |\uparrow\downarrow\downarrow\rangle + i \omega |\uparrow\uparrow\downarrow\rangle \right)$$

(8)

$$|\psi_6\rangle = \frac{1}{\sqrt{2}} \left( |\uparrow\downarrow\downarrow\rangle - i |\downarrow\uparrow\downarrow\rangle - |\downarrow\uparrow\downarrow\rangle + i |\downarrow\uparrow\downarrow\rangle + |\downarrow\uparrow\downarrow\rangle - i |\uparrow\uparrow\downarrow\rangle - i \omega |\uparrow\downarrow\downarrow\rangle + i |\uparrow\uparrow\downarrow\rangle \right)$$

(9)

$$|\psi_7\rangle = \frac{1}{\sqrt{2}} \left( |\uparrow\downarrow\downarrow\rangle - i \omega |\downarrow\uparrow\downarrow\rangle - i \downarrow\downarrow\downarrow\rangle - \omega |\downarrow\downarrow\downarrow\rangle - |\downarrow\downarrow\downarrow\rangle + i \omega |\uparrow\downarrow\downarrow\rangle + i |\uparrow\downarrow\downarrow\rangle + \omega |\uparrow\downarrow\downarrow\rangle \right)$$

(10)

with the phase factor $\omega = \exp(i \frac{\pi}{4})$. At this stage we show the possibilities giving rise to the wondered circulant symmetries. More precisely, by requiring suitable conditions fulfilled by the physical parameters, we end up with two cases. Indeed, the first configuration

$$J_1 = \Omega_2$$

(11)

$$J = J_2 = J_3 = \Omega_3$$

(12)

$$\Omega_1 = 0$$

(13)

$$\theta_2 = \theta_3 = \phi_{32} = \phi_3 = \phi_{21} = -\phi_{31} = \varphi$$

(14)

$$\theta_1 = \phi_{23} = \phi_2 = \phi_1 = \phi_{12} = \phi_{13} = 0$$

(15)
can be injected into (2) to get the first circulant Hamiltonian

\[
H_{\text{cir}}^{(1)} = \begin{pmatrix}
0 & J e^{i\varphi} & J_1 e^{i\varphi} & J e^{-i\varphi} & 0 & J e^{i\varphi} & J_1 e^{-i\varphi} & J e^{-i\varphi} \\
J e^{-i\varphi} & 0 & J_1 e^{i\varphi} & J e^{i\varphi} & 0 & J e^{-i\varphi} & J_1 e^{i\varphi} & J e^{i\varphi} \\
J_1 e^{-i\varphi} & J e^{-i\varphi} & 0 & J e^{i\varphi} & J_1 e^{i\varphi} & 0 & J e^{-i\varphi} & J_1 e^{i\varphi} \\
J e^{i\varphi} & J_1 e^{-i\varphi} & J e^{i\varphi} & 0 & J e^{-i\varphi} & J_1 e^{i\varphi} & 0 & J e^{i\varphi} \\
0 & J e^{i\varphi} & J_1 e^{-i\varphi} & J e^{i\varphi} & 0 & J e^{-i\varphi} & J_1 e^{i\varphi} & J e^{i\varphi} \\
J e^{-i\varphi} & 0 & J_1 e^{i\varphi} & J e^{i\varphi} & 0 & J e^{-i\varphi} & J_1 e^{i\varphi} & J e^{i\varphi} \\
J_1 e^{i\varphi} & J e^{-i\varphi} & 0 & J e^{i\varphi} & J_1 e^{i\varphi} & 0 & J e^{-i\varphi} & J_1 e^{i\varphi} \\
J e^{i\varphi} & J_1 e^{i\varphi} & J e^{-i\varphi} & 0 & J e^{i\varphi} & J_1 e^{i\varphi} & 0 & J e^{i\varphi}
\end{pmatrix}.
\]

(16)

As for the second configuration

\[
J_1 = \Omega_2 \\
J = J_2 = J_3 = \Omega_3 \\
\theta_2 = \theta_3 = \phi_{32} = \phi_3 = \phi_{21} = -\phi_{31} = \varphi \\
\theta_1 = \phi_{23} = \phi_2 = \phi_1 = \phi_{12} = \phi_{13} = 0
\]

(17-20)
in addition to \(\Omega_1 \neq 0\), we obtain the second circulant Hamiltonian

\[
H_{\text{cir}}^{(2)} = \begin{pmatrix}
0 & J e^{i\varphi} & J_1 e^{i\varphi} & J e^{-i\varphi} & \Omega_1 & J e^{i\varphi} & J_1 e^{-i\varphi} & J e^{-i\varphi} \\
J e^{-i\varphi} & 0 & J_1 e^{i\varphi} & J e^{i\varphi} & \Omega_1 & J e^{-i\varphi} & J_1 e^{i\varphi} & J e^{i\varphi} \\
J_1 e^{-i\varphi} & J e^{-i\varphi} & 0 & J e^{i\varphi} & J_1 e^{i\varphi} & \Omega_1 & J e^{-i\varphi} & J_1 e^{i\varphi} \\
J e^{i\varphi} & J_1 e^{-i\varphi} & J e^{i\varphi} & 0 & J e^{-i\varphi} & J_1 e^{i\varphi} & 0 & J e^{i\varphi} \\
\Omega_1 & J e^{i\varphi} & J_1 e^{-i\varphi} & J e^{i\varphi} & 0 & J e^{-i\varphi} & J_1 e^{i\varphi} & J e^{i\varphi} \\
J e^{-i\varphi} & \Omega_1 & J e^{i\varphi} & J_1 e^{-i\varphi} & J e^{i\varphi} & 0 & J e^{-i\varphi} & J_1 e^{i\varphi} \\
J_1 e^{i\varphi} & J e^{-i\varphi} & \Omega_1 & J e^{i\varphi} & J_1 e^{i\varphi} & 0 & J e^{-i\varphi} & J_1 e^{i\varphi} \\
J e^{i\varphi} & J_1 e^{i\varphi} & J e^{-i\varphi} & \Omega_1 & J e^{i\varphi} & J_1 e^{i\varphi} & 0 & J e^{i\varphi}
\end{pmatrix}.
\]

(21)

In the forthcoming analysis, we focus only on one of the above Hamiltonians let say for instance \(H_{\text{cir}}^{(1)}\) and investigate its basic features.

### 3 Adiabatic transition to Fourier modes

The adiabatic transition to Fourier modes can be achieved by using some controls. For this, we discuss two of them, which are the energy offset and Rabi frequencies.

#### 3.1 Controlling by energy offset

To realize an adiabatic evolution to the circulant Hamiltonian states (Fourier modes), we add an energy offset \(H_0(t)\)

\[
H_0(t) = \Delta_1(t)\sigma_1^z + \Delta_2(t)\sigma_2^z + \Delta_3(t)\sigma_3^z
\]

(22)

where the time-dependent detuning \(\Delta_j(t)\) of \(j^{th}\) spin are necessary to control the adiabatic transition from computational spin states to quantum Fourier states (3-10). Consequently, we have now the
Hamiltonian

\[ H(t) = H_0(t) + H_{\text{eff}}^{(1)}(t). \] (23)

We recall that in the adiabatic limit, the system remains in the same eigenstate of \( H(t) \) (23) in all time \cite{13}. By choosing a particular time dependence of couplings and detunings, the eigenstates of \( H(t) \) will be those of \( H_0(t) \) at \( t_i \) and the dynamics drives them to the Fourier modes at \( t_f \). Therefore, the adiabatic evolution maps each computational spin state to a Fourier mode, generating the QFT in a single interaction step. The adiabatic evolution demands that the non-degeneracy between the eigen-frequencies of \( H(t) \) is larger at any time than the non-adiabatic coupling between each pair of the eigenstates \( |\lambda_{\pm}\rangle, |\delta_{\pm}\rangle, |\mu_{\pm}\rangle \), and \( |\gamma_{\pm}\rangle \) of \( H(t) \). Otherwise, we have

\[
\begin{align*}
|\mu_{\pm}(t) - \lambda_{\pm}(t)| & \gg |(\partial_t \mu_{\pm}(t)|\lambda_{\pm}(t))| \\
|\lambda_{+}(t) - \lambda_{-}(t)| & \gg |(\partial_t \lambda_{+}(t)|\lambda_{-}(t))| \\
|\lambda_{+}(t) - \lambda_{-}(t)| & \gg |(\partial_t \lambda_{+}(t)|\lambda_{-}(t))| \\
|\delta_{+}(t) - \delta_{-}(t)| & \gg |(\partial_t \delta_{+}(t)|\delta_{-}(t))| \\
|\delta_{+}(t) - \delta_{-}(t)| & \gg |(\partial_t \delta_{+}(t)|\delta_{-}(t))| \\
|\gamma_{+}(t) - \gamma_{-}(t)| & \gg |(\partial_t \gamma_{+}(t)|\gamma_{-}(t))|. 
\end{align*}
\] (24-29)

To simplify our problem let us choose \( \varphi = \frac{\pi}{2} \) and then we show that the eigenvalues \( \lambda_{\pm}, \delta_{\pm}, \mu_{\pm}, \gamma_{\pm} \) of the Hamiltonian \( H(t) \) (23) are given by (A.1-A.4) in Appendix A. Now, it is worthy to mention that the circulant symmetry is broken. We suppose that the system is initially prepared in the computational product states \( |\psi_{s_1s_2s_3}\rangle = |s_1s_2s_3\rangle \ (s_j = \downarrow_j, \uparrow_j) \), which are eigenstates of the Hamiltonian \( H_0(t) \). As a result, the initial parameters should verify

\[ \Delta_{1,2,3}(t_i) \gg J(t_i), J_1(t_i) \] (30)

and then \( H(t) \) goes to \( H_0(t) \). Consequently the eigenvalues become

\[
\begin{align*}
\lambda_{\pm}(t_i) &= \pm [\Delta_1(t_i) + \Delta_2(t_i) + \Delta_3(t_i)] \\
\delta_{\pm}(t_i) &= \pm [\Delta_1(t_i) + \Delta_2(t_i) - \Delta_3(t_i)] \\
\mu_{\pm}(t_i) &= \pm [\Delta_1(t_i) - \Delta_2(t_i) + \Delta_3(t_i)] \\
\gamma_{\pm}(t_i) &= \pm [\Delta_1(t_i) - \Delta_2(t_i) - \Delta_3(t_i)]
\end{align*}
\] (31-34)

as well as the eigenvectors of \( H(t) \) coincide with the computational spin states, i.e. \( |\psi(t_i)\rangle = |s_1s_2s_3\rangle \), which are

\[
\begin{align*}
|\lambda_{+}\rangle &= |\downarrow\downarrow\downarrow\rangle, \quad |\lambda_{-}\rangle = |\uparrow\uparrow\uparrow\rangle \\
|\delta_{+}\rangle &= |\downarrow\downarrow\uparrow\rangle, \quad |\delta_{-}\rangle = |\uparrow\uparrow\downarrow\rangle \\
|\mu_{+}\rangle &= |\downarrow\uparrow\downarrow\rangle, \quad |\mu_{-}\rangle = |\uparrow\downarrow\uparrow\rangle \\
|\gamma_{+}\rangle &= |\uparrow\uparrow\uparrow\rangle, \quad |\gamma_{-}\rangle = |\uparrow\downarrow\downarrow\rangle.
\end{align*}
\] (35-38)

To prevent the degeneracy, it is necessary to have the condition \( \Delta_1(t_i) \neq \Delta_j(t_i) \) with \( i, j = 1, 2, 3 \) and we can have the equidistant eigen-frequencies by requiring \( \Delta_1(t_i) = 2\Delta_2(t_i) = 4\Delta_3(t_i) \). Furthermore,
to obtain the Fourier modes at final time \( t_f \) of transition, the couplings parameters together with detunings should verify the condition

\[ \Delta_{1,2,3}(t_f) \ll J(t_f), J_1(t_f). \]  

(39)

As a consequence, the total Hamiltonian evolves to circulant one, i.e. \( H(t) \to H_{\text{cir}}^{(1)}(t) \), and the corresponding eigenspectrum becomes that of \( H_{\text{cir}}^{(1)}(t) \), such as

\[
\begin{align*}
|\lambda_+\rangle &= |\psi_0\rangle, \\
|\lambda_-\rangle &= |\psi_7\rangle \\
|\delta_+\rangle &= |\psi_1\rangle, \\
|\delta_-\rangle &= |\psi_6\rangle \\
|\mu_+\rangle &= |\psi_2\rangle, \\
|\mu_-\rangle &= |\psi_5\rangle \\
|\gamma_+\rangle &= |\psi_3\rangle, \\
|\gamma_-\rangle &= |\psi_4\rangle.
\end{align*}
\]

(40)

(41)

(42)

(43)

Additionally, the realization of the QFT relies on the adiabatic following of each of the instantaneous eigenvectors

\[
\begin{align*}
|↓↓↓\rangle &\rightarrow e^{i\alpha_1} |\psi_0\rangle \\
|↓↓↑\rangle &\rightarrow e^{i(\alpha_2 - \frac{\pi}{2})} |\psi_1\rangle \\
|↓↑↓\rangle &\rightarrow e^{i\alpha_3} |\psi_2\rangle, \\
|↓↑↑\rangle &\rightarrow e^{i\alpha_4} |\psi_3\rangle \\
|↑↓↓\rangle &\rightarrow e^{-i\alpha_1} |\psi_4\rangle \\
|↑↓↑\rangle &\rightarrow e^{-i\alpha_2} |\psi_5\rangle \\
|↑↑↓\rangle &\rightarrow e^{-i\alpha_3} |\psi_6\rangle \\
|↑↑↑\rangle &\rightarrow e^{-i\alpha_4} |\psi_7\rangle.
\end{align*}
\]

(44)

(45)

(46)

(47)

(48)

(49)

(50)

(51)

where \( \alpha_j \) are the global adiabatic phases appearing due to the adiabatic evolution \([13, 15, 20, 21]\)

\[
\begin{align*}
\alpha_1 &= \int_{t_i}^{t_f} \lambda_+(t) \, dt, \\
\alpha_2 &= \int_{t_i}^{t_f} \delta_+(t) \, dt, \\
\alpha_3 &= \int_{t_i}^{t_f} \mu_+(t) \, dt, \\
\alpha_4 &= \int_{t_i}^{t_f} \gamma_+(t) \, dt
\end{align*}
\]

(52)

and we have

\[
\begin{align*}
\lambda_-(t) &= -\lambda_+(t), \\
\delta_-(t) &= -\delta_+(t), \\
\mu_-(t) &= -\mu_+(t), \\
\gamma_-(t) &= -\gamma_+(t).
\end{align*}
\]

(53)

However, after a specific tuning of the detuning \( \Delta_j(t) \), \( \alpha_j \) reduce to

\[
\begin{align*}
\alpha_1 &= 2p\pi, \\
\alpha_2 &= 2m\pi, \\
\alpha_3 &= 2n\pi, \\
\alpha_4 &= 2k\pi
\end{align*}
\]

(54)

with \( p, m, n, k \) are four integer numbers. This choice leads to realize the following unitary quantum gate

\[
G = \frac{1}{2\sqrt{2}} \begin{pmatrix}
1 & -i & 1 & 1 & 1 & 1 & 1 \\
1 & -i\omega & i\omega & -1 & -\omega & -i & -i\omega \\
1 & 1 & -1 & -i & 1 & i & -1 \\
1 & \omega & -i & \omega & -1 & -i\omega & i & -\omega \\
1 & i & 1 & -1 & 1 & -1 & 1 & -1 \\
1 & i\omega & i & -i\omega & -1 & \omega & -i & \omega \\
1 & -1 & -1 & i & 1 & -i & -1 & i \\
1 & -\omega & -i & -\omega & -1 & i\omega & i & \omega
\end{pmatrix}.
\]

(55)
As a result one can show that up to an additional phase $-\frac{\pi}{2}$ the determinant of $G$ is equal one, i.e. $\det(G) = 1$, which is necessary for adiabatic evolution. Thereby, it is worthy to note that the gate $G$ is a QFT one.

3.2 Controlling by Rabi frequencies

Now we discuss how to control the Rabi frequencies $\Omega_2(t)$ and $\Omega_3(t)$ by considering the Hamiltonian $H^{(1)}(t)$ (B.2) in Appendix B without using the energy offset $H_0(t)$ (22). To achieve this goal, we drive $\Omega_2(t)$ and $\Omega_3(t)$ in a way that they becomes equal to the couplings $J_1$ and $J$, respectively. We summarize in what follows the process of control in three steps.

- **Initially**: Let us assume that $\Omega_2(t_i) \gg J_1$ and $\Omega_3(t_i) \gg J$, then the eigenvectors of $H^{(1)}(t)$ (B.2) are the rotating computational spin states $|\psi(t_i)\rangle = |s'_1 s'_2 s'_3 \rangle$ ($s'_j = \pm j$) such as
  \begin{align*}
  |\pm_1\rangle &= \frac{1}{\sqrt{2}} (|\downarrow_1\rangle \pm |\uparrow_1\rangle) \\
  |\pm_2\rangle &= \frac{1}{\sqrt{2}} (e^{i\varphi} |\downarrow_2\rangle \pm |\uparrow_2\rangle) \\
  |\pm_3\rangle &= \frac{1}{\sqrt{2}} (|\downarrow_3\rangle \pm |\uparrow_3\rangle)
  \end{align*}

- **Transition**: The adiabatic transition from the initial eigenvectors to Fourier modes is given by the mappings
  \begin{align*}
  |--\rangle &\rightarrow e^{-i\beta_0} |\psi_0\rangle \\
  |--\rangle &\rightarrow e^{-i\beta_1} |\psi_1\rangle \\
  |--\rangle &\rightarrow e^{-i\beta_2} |\psi_2\rangle \\
  |--\rangle &\rightarrow e^{-i\beta_3} |\psi_3\rangle \\
  |+ -\rangle &\rightarrow e^{-i\beta_4} |\psi_4\rangle \\
  |+ -\rangle &\rightarrow e^{-i\beta_5} |\psi_5\rangle \\
  |+ -\rangle &\rightarrow e^{-i\beta_6} |\psi_6\rangle \\
  |+ +\rangle &\rightarrow e^{-i\beta_7} |\psi_7\rangle
  \end{align*}

where the adiabatic phases $\beta_i$ read as
  \begin{equation}
  \beta_i = \int_{t_i}^{t_f} \Lambda_i(t) \, dt, \quad i = 1, \ldots, 7
  \end{equation}

and the eigenfrequencies $\Lambda_i(t)$ (B.3-B.10) of $H^{(1)}(t)$ (B.2) are showed in Appendix B.

- **Finally**: We adiabatically decrease $\Omega_2(t)$ together with $\Omega_3(t)$ to end up with
  \begin{align*}
  \Omega_2(t_f) &= J_1, \quad \Omega_3(t_f) = J.
  \end{align*}

As a result, the circulant symmetry will be established and the Fourier modes will be derived as well.
At this stage, we mention that in the case of the gate realization based on the energy offset, the exact eigenvectors of the Hamiltonian (23) can not exactly be obtained. However, under the consideration made here the derivation of eigenvectors can be achieved, see Appendix B. Thus, to accelerate the gate we combine the gate scheme with the short-cut to adiabaticity.

4 Gate implementation

To give a physical implementation of our system we generalize the process proposed by Ivanov and Vitanov [13] to three qubit case. Indeed, to realize the circulant Hamiltonian, we proceed with trapped ions [22–24]. A crystal with $3N$ ions with mass $M$ is considered, the trap axis are $z$ and $x$ with the frequencies $\Omega_z$ and $\Omega_x$, respectively. Each ion of the crystal is a qubit described by two typical levels $|\uparrow\rangle, |\downarrow\rangle$ with an energy gap $\omega_0$. Moreover, the virtual excitations generated between two coupled ions undergo a small radial vibrations around the equilibrium positions of the ions. Then as an illustration, we presents our physical implementation depicted in Figure 2.

![Figure 2](image)

**Figure 2** — (color online) A proposal of the physical implementation of the interaction between three coupled trapped ions, with laser frequencies $\Omega_{j,Lr} = \omega_0 - \nu - \Delta_j(t)$ and $\Omega_{j,Lb} = \omega_0 + \nu - \Delta_j(t)$ that generate a spin dependent force at the frequency $\nu$.

As for our Hamiltonian (1), the implementation of kinetic terms together with bilinear couplings are perfectly discussed in [13]. Regarding the implementation of the trilinear coupling we suggest that it can be seen as a coupling between two coupled ions with an extra third ion [18]. To be clear, the small radial vibrations around the equilibrium positions between two coupled ions are displayed by a set of collective vibrational modes with the Hamiltonians [16,25]

$$H_{ph1} = \sum_n \Omega_n a_n^\dagger a_n, \quad H_{ph2} = \sum_n \Omega_n \hat{b}_n^\dagger \hat{b}_n, \quad H_{ph3} = \sum_n \Omega_n c_n^\dagger c_n$$  \hspace{1cm} (60)

whereas the internal energy is

$$H_q = \frac{1}{2} \sum_j \omega_0 \sigma_j^z$$  \hspace{1cm} (61)

and then the free Hamiltonian becomes

$$H_0 = H_q + \sum_j H_{phj}$$  \hspace{1cm} (62)
As claimed above, generating the trilinear coupling will be leaned on the interaction between two coupled ions with a third one. This can be achieved, for instance, by using two pairs of noncopropagating laser beams along the radial directions having the frequencies

\[ \Omega_{j,L_r} = \omega_0 - \nu - \Delta_j(t), \quad \Omega_{j,L_b} = \omega_0 + \nu - \Delta_j(t) \]

that generate a spin dependent force at frequency \( \nu \).

Besides, using the weak coupling assumption, one can legitimately apply the optical rotating-wave approximation and as a result, the interacting part of our system reads now as

\[
H_1 = \sum_j \Delta_j \sigma_j^z + \Omega_x e^{i k z_1} \cos(\nu t) \left( \sigma_1^+ e^{i \phi_1} + \sigma_1^- e^{-i \phi_1} \right) + \Omega_x e^{i k z_2} \cos(\nu t) \left( \sigma_2^+ e^{i \phi_2} + \sigma_2^- e^{-i \phi_2} \right) \\
+ \Omega_x e^{i k z_3} \cos(\nu t) \left( \sigma_3^+ e^{i \phi_3} + \sigma_3^- e^{-i \phi_3} \right) + \sum_j \Omega_j \left( \sigma_j^+ e^{i \theta_j} + \sigma_j^- e^{-i \theta_j} \right) \\
+ \Omega_x e^{i k z_3} \sin(\nu t) \left( \sigma_1^+ e^{i \varphi_1} + \sigma_1^- e^{-i \varphi_1} \right) \left( \sigma_2^+ e^{i \varphi_2} + \sigma_2^- e^{-i \varphi_2} \right)
\]

where \( \Omega_x, \Omega_z, \Omega_j, \Omega_\alpha \) are the Rabi frequencies, \( \phi_j, \theta_j \) are the laser phases, \( \varphi_{1,2} \) are the phases resulted from trilinear interaction and the spatial arguments

\[
k z_1 = \sum_n \eta_{1,n} (\hat{a}_n^\dagger e^{i \Omega_n t} + \hat{a}_n e^{-i \Omega_n t})
\]

\[
k z_2 = \sum_n \eta_{2,n} (\hat{b}_n^\dagger e^{i \Omega_n t} + \hat{b}_n e^{-i \Omega_n t})
\]

\[
k z_3 = \sum_n \eta_{3,n} (\hat{c}_n^\dagger e^{i \Omega_n t} + \hat{c}_n e^{-i \Omega_n t})
\]

involving the Lamb-Dicke parameters

\[
\eta_{j,n} = b_{j,n} k \sqrt{\hbar / 2M \Omega_n}
\]

with \( b_{j,n} \) are the normal mode transformation matrix for the \( j \)th ion. Since the dimensionless parameter \( \eta_{j,n} \) is small, hence, we can make the Lamb-Dicke approximation, \( \Delta k \langle \hat{x}_1 \rangle \ll 1, \Delta k \langle \hat{x}_2 \rangle \ll 1, \Delta k \langle \hat{x}_3 \rangle \ll 1 \), to end up with

\[
H_1 = \sum_j \Delta_j \sigma_j^z + \sum_n J_{1,n} \cos(\nu t) \left( \sigma_1^+ e^{i \phi_1} + \sigma_1^- e^{-i \phi_1} \right) \left( \hat{a}_n^\dagger e^{i \Omega_n t} + \hat{a}_n e^{-i \Omega_n t} \right) \\
+ \sum_j \Omega_j \left( \sigma_j^+ e^{i \theta_j} + \sigma_j^- e^{-i \theta_j} \right) + \sum_n J_{2,n} \cos(\nu t) \left( \sigma_2^+ e^{i \phi_2} + \sigma_2^- e^{-i \phi_2} \right) \left( \hat{b}_n^\dagger e^{i \Omega_n t} + \hat{b}_n e^{-i \Omega_n t} \right) \\
+ \sum_n J_{3,n} \cos(\nu t) \left( \sigma_3^+ e^{i \phi_3} + \sigma_3^- e^{-i \phi_3} \right) \left( \hat{c}_n^\dagger e^{i \Omega_n t} + \hat{c}_n e^{-i \Omega_n t} \right) \\
+ \sum_n h_n \sin(\nu t) \left( \sigma_1^+ e^{i \varphi_1} + \sigma_1^- e^{-i \varphi_1} \right) \left( \sigma_2^+ e^{i \varphi_2} + \sigma_2^- e^{-i \varphi_2} \right) \left( \hat{c}_n^\dagger e^{i \Omega_n t} + \hat{c}_n e^{-i \Omega_n t} \right)
\]

where \( J_{1,n} = \eta_{1,n} \Omega_x, \quad J_{2,n} = \eta_{2,n} \Omega_x \) and \( J_{3,n} = \eta_{3,n} \Omega_z \) are the spin-phonon coupling and \( h_n = \eta_{3,n} \Omega_\alpha \) is the trilinear coupling. Moreover, during a slow dynamics the beatnote frequency \( \nu \) isn’t resonant with any radial vibration mode, i.e. \( |\Omega_n - \nu| \gg J_{j,n}, h_n \). Additionally, the phonons are virtually excited,
then they should be eliminated from the dynamics, and as consequence, the spin states in different sites become coupled to each other. For different three sites $j^{th}$, $p^{th}$ and $q^{th}$, we have

$$H_1 = \Delta_j \sigma_j^z + \Delta_p \sigma_p^z + \Delta_q \sigma_q^z + \Omega_j \left( \sigma_j^+ e^{i\theta_j} + \sigma_j^- e^{-i\theta_j} \right) + \Omega_p \left( \sigma_p^+ e^{i\theta_p} + \sigma_p^- e^{-i\theta_p} \right)$$
$$+ \Omega_q \left( \sigma_q^+ e^{i\theta_q} + \sigma_q^- e^{-i\theta_q} \right) + J_1 \left( \sigma_j^+ e^{i\phi_j} + \sigma_j^- e^{-i\phi_j} \right) \left( \sigma_p^+ e^{i\phi_p} + \sigma_p^- e^{-i\phi_p} \right)$$
$$+ J_2 \left( \sigma_p^+ e^{i\phi_p} + \sigma_p^- e^{-i\phi_p} \right) \left( \sigma_q^+ e^{i\phi_q} + \sigma_q^- e^{-i\phi_q} \right) + J_3 \left( \sigma_j^+ e^{i\phi_j} + \sigma_j^- e^{-i\phi_j} \right) \left( \sigma_q^+ e^{i\phi_q} + \sigma_q^- e^{-i\phi_q} \right)$$

(70)

where the couplings between two ions are given by

$$J_1 = \sum_n J_{j,n} J_{p,n} \frac{1}{\nu^2 - \Omega_n^2}, \quad J_2 = \sum_n J_{p,n} J_{q,n} \frac{1}{\nu^2 - \Omega_n^2}, \quad J_3 = \sum_n J_{j,n} J_{q,n} \frac{1}{\nu^2 - \Omega_n^2}$$

(71)

and that of the trilinear coupling between three ions

$$J = \sum_n J_{j,n} J_{p,n} J_{q,n} \frac{1}{\nu^2 - \Omega_n^2}$$

(72)

At this level, it is clearly seen that one can realize the circulant Hamiltonian $H_{	ext{cir}}^{(1)}(t)$ (16) by adjusting the coupling parameters.

5 Numerical analysis

In what follows, we choose the following time modulation of the couplings $J_1(t)$, $J(t)$ and the detunings $\Delta_j(t)$ for the gate implementation

$$J_1(t) = J_{01} \sin^2(\omega' t)$$

(73)

$$J(t) = J_0 \sin^2(\omega' t)$$

(74)

$$\Delta_j(t) = \Delta_j \cos^2(\omega' t)$$

(75)

where the characteristic parameter $\omega'$ controls the adiabaticity of the transition and the interaction time $t$ varies as $t \in [0, t_{\text{max}}]$ with $t_{\text{max}} = \frac{\pi}{2 \omega'}$. This time dependence guarantees both conditions

$$\Delta_{1,2,3}(0) \gg J(0), J_1(0), \quad \Delta_{1,2,3}(t_{\text{max}}) \ll J(t_{\text{max}}), J_1(t_{\text{max}}).$$

(76)

The adiabatic transition to Fourier modes can be carried without using the detuning $\Delta_j$. In fact, we can simply vary the Rabi frequencies to finally get the Fourier modes, such as

$$J(t) = J_0 \sin^2(\omega' t)$$

(77)

$$J_1(t) = J_{01} \sin^2(\omega' t)$$

(78)

$$\Omega_2(t) = J_0 + \Upsilon_0 \cos^2(\omega' t)$$

(79)

$$\Omega_3(t) = J_0 + \Upsilon_0' \cos^2(\omega' t)$$

(80)

with $\Upsilon_0$ and $\Upsilon_0'$ are the adding amplitude for the control of the adiabaticity of transition in the second and the third qubit, respectively.
5.1 Eigenfrequencies

We numerically show in Figure 3, the eigenfrequencies $\lambda_{\pm}(t), \delta_{\pm}(t), \mu_{\pm}(t), \gamma_{\pm}(t)$ of the Hamiltonian $H(t)$ (23) versus time under suitable choices of the coupling parameters and detunings. As expected, all eigenfrequencies are separated from each others, which entails in its turn the suppression of any transition to a superposition of eigenstates. The degeneracy of the energies should be avoided during the simulated time, and this is due to the fact that, degeneracy entails the prevention of the gate implementation at hand. The gap between the eigenvalues decrease during the simulate time, to avoid their degeneracy during the evolution, we have introduced the detuning frequencies $\Delta_j$. Additionally, we mention that the amplitude coupling $J_0$ and $J_{01}$ are important to prevent the degeneracy at final time $t_{max}$.

![Figure 3](image)

Figure 3 – (color online) Eigenfrequencies of the Hamiltonian $H(t)$ (23) as a function of the time. The parameters are set to $J_0/2\pi = 1$ kHz, $J_{01}/2\pi = 2$ kHz, $\Delta_1/2\pi = 120$ kHz, $\Delta_2/2\pi = 60$ kHz, $\Delta_3/2\pi = 30$ kHz, $\varphi = \pi/2$, $\omega'/2\pi = 0.15$ kHz.

5.2 Gate fidelity

The gate fidelity is a tool to compare how close two gates, or more generally operations, are to each other [26]. In other word, it expresses the probability that one state will pass a test to identify as the other one. We recall that fidelities higher than 99.99% for a single-qubit gate and 99.9 % for an entangling gate in a two-ion crystal have been developed in [27–29]. Generally, for theoretical density matrix $\rho_0$ and reconstructed density matrix $\rho$ it is defined by

$$F(\rho_0, \rho) = \left( \frac{\text{Tr}\sqrt{\rho_0 \rho \sqrt{\rho_0}}}{} \right)^2.$$  (81)

By applying the Uhlmann theorem [30], (81) can take a simple form

$$F(\rho_0, \rho) = |\langle \psi_0 | \psi \rangle|^2.$$  (82)

with $\psi_0$ and $\psi$ are theoretical and reconstructed purified state vectors. As for our system, we have [13]

$$F_{\text{Gate}}(t) = \frac{1}{16} \left| \sum_{s_1,s_2,s_3} \langle s_1 s_2 s_3 | G^+ G'(t) | s_1 s_2 s_3 \rangle \right|^2.$$  (83)
where $s_j = \uparrow_j, \downarrow_j$, $\mathcal{G}$ is the three-qubit QFT (55) and $\mathcal{G}'(t)$ is the real transform. In Figure 4, we present the gate fidelity versus the evolution time by choosing the detunings $\Delta_{1,2,3}$ such that the adiabatic phases are given in (54). The unitary propagator $\mathcal{G}'(t)$ converges to $\mathcal{G}$ as time progresses. We notice that for a nonlinear coupling $J_0 = 1$ kHz and $t = 0.4875$ ms the gate reaches a high fidelity (96%).

Figure 4 - (color online) Gate fidelity calculated from the numerical simulation with the Hamiltonian $H(t)$ (23). The parameters are set to $J_0/2\pi = 1$ kHz, $\Delta_1/2\pi = 20$ kHz, $\Delta_2/2\pi = 10$ kHz, $\Delta_3/2\pi = 6$ kHz, $\varphi = \frac{\pi}{2}$ and $\omega' = 0.505$ kHz.

By using the Hamiltonian (B.2) together with the quantum Fourier states (3-10), one can end up with the fidelity of the adiabatic transitions between the rotating computational spin states $|s'_1 s'_2 s'_3\rangle (s'_j = \pm j)$

$$F_{ad}(t) = \frac{1}{16} |\sum_{i=0}^{7} \langle \psi_i | \Lambda_i \rangle|^2$$

and more explicitly we have

$$F_{ad}(t) = \frac{1}{16} |\langle \psi_0 | \Lambda_0 \rangle + \langle \psi_1 | \Lambda_1 \rangle + \langle \psi_2 | \Lambda_2 \rangle + \langle \psi_3 | \Lambda_3 \rangle + \langle \psi_4 | \Lambda_4 \rangle + \langle \psi_5 | \Lambda_5 \rangle + \langle \psi_6 | \Lambda_6 \rangle + \langle \psi_7 | \Lambda_7 \rangle|^2.$$ (85)

Figure 5 - (color online) Fidelity of adiabatic transition with $J_0/2\pi = 2.1$ kHz, $\Upsilon_0/2\pi = 1.9$ kHz, $J_{01}/2\pi = 2.4$ kHz, $\Upsilon_0'/2\pi = 2$ kHz, $\varphi = \frac{\pi}{4}$ and $\omega'/2\pi = 0.3$ kHz.

Figure 5 presents the good fidelity of the adiabatic transition within a shorter interaction time $t_{max} = 0.835$ ms. It is clearly seen that our results show the possibility to obtain high fidelity (71%).
5.3 Creation of entangled states

To create entangled states one has to suitably prepare the initial state in a superposition of spin states, which is due to the fact that the action of the QFT on the computational basis creates a superposition, but they are not entangled. For concreteness, we assume that the system is initially prepared in the following state

\[ |\psi(0)\rangle = \frac{1}{2} (|\downarrow_1\downarrow_2\downarrow_3\rangle + e^{i\alpha_1} |\downarrow_1\uparrow_2\downarrow_3\rangle + e^{i\alpha_2} |\downarrow_2\downarrow_3\rangle + e^{i\alpha_3} |\downarrow_2\uparrow_3\rangle + e^{i\alpha_4} |\uparrow_2\downarrow_3\rangle) \]  

(86)

Performing our three qubits gates, we obtain the entangled state

\[ |\psi(0)\rangle \rightarrow |\psi(t_f)\rangle = \frac{1}{2} (|\psi_0\rangle + |\psi_1\rangle + |\psi_2\rangle + |\psi_3\rangle) \]  

(87)

Let us emphasis that by rotating the initially prepared state such as

\[ |\psi(0)\rangle = \frac{1}{2} \left( |\uparrow_1\downarrow_2\downarrow_3\rangle - e^{-i\beta_1} |\downarrow_1\uparrow_2\downarrow_3\rangle - e^{-i\beta_2} |\downarrow_2\downarrow_3\rangle + e^{-i\beta_3} |\downarrow_2\uparrow_3\rangle \right) \]  

(88)

we end up with the same transformed entangled state (87). Thereby, the fidelity of the creation of the entangled state is defined by

\[ F(t) = \frac{1}{2} |\langle \psi(t_f) | (e^{-i\beta_0} |\Lambda_0(t)\rangle + e^{-i\beta_1} |\Lambda_1(t)\rangle + e^{-i\beta_2} |\Lambda_2(t)\rangle + e^{-i\beta_3} |\Lambda_3(t)\rangle)|^2. \]  

(89)

By adjusting the parameters \( \omega' \), \( J_{01} \) and \( J_0 \) one can reach high fidelity of the creation of entangled states as presented in Figure 6-A, Figure 6-B and Figure 6-C.

![Figure 6](image-url) - (color online) Fidelity of Entangled state calculated from the numerical simulation of the Hamiltonian (B.2). (A): \( \Upsilon_0 = 1.8 \text{ kHz}, \Upsilon_0' = 1.7 \text{ kHz}, J_{01} = 2.1 \text{ kHz}, J_0 = 2.3 \text{ kHz} \) and the gate time \( t = 0.31 \text{ ms} \). (B): The same values with \( \omega'/2\pi = 0.5 \text{ kHz} \) and vary the coupling strength \( J_{01} \). (C): The same values in (A) with \( \omega'/2\pi = 0.605 \text{ kHz} \) and vary the coupling strength \( J_0 \).

6 Short-cut to adiabaticity

Now we add an auxiliary interaction \( H_{CD}(t) \) (counter-driving-field) [15,31] to the reference Hamiltonian \( H^{(1)}(t) \) (B.2) in order to suppress the non-adiabatic transitions and reduce the gate time. As a result, the Hamiltonian will take the form

\[ H_T(t) = H^{(1)}(t) + H_{CD}(t) \]  

(90)
such that the interaction is

\[ H_{\text{CD}}(t) = i\hbar \sum_{i=0}^{7} \langle \Lambda_i(t) \rangle \langle \Lambda_i(t) \rangle \]  \tag{91} 

and the time-dependent eigenvectors \(|\Lambda_i(t)\rangle\) of \(H^{(1)}(t)\) are given in (B.11-B.18). After some algebra, we obtain

\[ H_{\text{CD}}(t) = -\partial_t \kappa(t) \begin{pmatrix} 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{pmatrix} \]  \tag{92} 

Using the time-dependent coupling parameters (77-80) and the eigenvectors (B.11-B.18), we explicitly determine the counter-driving-field

\[ \partial_t \kappa(t) = \frac{1}{2} \left( \frac{\omega' J_{01}(J_{01} + \Upsilon_0) \sin(2\omega't)}{J_{01}^2 \sin^2(\omega't) + [\Upsilon_0 \sin^2(\omega't) - (J_{01} + \Upsilon_0)]^2} + \frac{\omega' J_0(J_0 + \Upsilon'_0) \sin(2\omega't)}{J_0^2 \sin^2(\omega't) + [\Upsilon'_0 \sin^2(\omega't) - (J_0 + \Upsilon'_0)]^2} \right) \]  \tag{93} 

In Figure 7 we show the shape of the counter-driving field (93) as a function of the time and by varying \(\omega'\), \(J_{01}\) and \(J_0\). The counter-driving term should be zero at \(t = 0\), because the system starts in the rotation computational spin states. We mention also, that at \(t_{\text{max}}\) the system end up with the Fourier modes.

**Figure 7** – (color online) Behavior of countrer-driving field with \(\Upsilon_0/2\pi = 0.5\) kHz and \(\Upsilon'_0/2\pi = 2\) kHz. The other parameters are \(J_{01} = 1.5\) kHz, \(J_0 = 1\) kHz (blue line), \(J_{01} = 1.9\) kHz, \(J_0 = 1.3\) kHz (cyan line), \(J_{01} = 2\) kHz, \(J_0 = 1.7\) kHz (red line).
7 Conclusion

We generalized the work done by Ivanov and Vitanov [13] dealing with two-qubit quantum gate and entanglement protected by circulant symmetry to a system of three-qubit quantum gates. In fact, we have constructed a discrete system based on three qubits emerged in a magnetic field. A special symmetry called circulant is obtained only by adjusting the Rabi frequencies and the coupling parameters characterizing our system. We have showed that our eigenvectors do not depend on the magnitude of the physical parameters, which entails the protection of entanglement. These eigenvectors lead to obtain the quantum Fourier transform (QFT) modes, which imply the realization of QFT gate. To discuss the implementation of the gate, the eigenfrequencies should be non degenerate. To this aim we have added an Hamiltonian $H_0(t)$, that breaks the circulant symmetry and favors the adiabatic transition process.

Subsequently, instead of adding an energy offset we have shown that it is possible to control the transition by using only the Rabi frequencies. By using the second route, the gate scheme together with the short-cut to adiabaticity have been discussed and as a result we have found the suppression of the non adiabatic transition and accelerating the gate. In addition, in the framework of the trapped ions, we have suggested a possible physical implementation of the constructed Hamiltonian. By assuming a particular sinusoidal modulation, several fidelities are discussed, and the results show the possibility to achieve high fidelities only by adjusting the physical parameters. The physical realization of a three qubit QFT is the key subroutine in several quantum algorithms, then it turns out that our present three qubits gate can significantly reduce the number of several gates in a quantum algorithm.

Appendix A  Eigenfrequencies of $H(t)$ with energy offset

We show that the eigenfrequencies of the Hamiltonian $H(t)$ (23) can be written as

$$\lambda_{\pm} = \pm \sqrt{\frac{A}{4} - S + \frac{1}{2} \sqrt{-4S^2 - 2p + \frac{q}{S}}}$$  \hspace{1cm} (A.1)

$$\delta_{\pm} = \pm \sqrt{\frac{A}{4} - S - \frac{1}{2} \sqrt{-4S^2 - 2p + \frac{q}{S}}}$$  \hspace{1cm} (A.2)

$$\mu_{\pm} = \pm \sqrt{\frac{A}{4} + S + \frac{1}{2} \sqrt{-4S^2 - 2p - \frac{q}{S}}}$$  \hspace{1cm} (A.3)

$$\gamma_{\pm} = \pm \sqrt{\frac{A}{4} + S - \frac{1}{2} \sqrt{-4S^2 - 2p - \frac{q}{S}}}$$  \hspace{1cm} (A.4)

where we have set

$$p = \frac{1}{8} (8B - 3)$$  \hspace{1cm} (A.5)

$$q = \frac{1}{8} (-1 + 4B + 8C)$$  \hspace{1cm} (A.6)

$$S = \frac{1}{2} \sqrt{\frac{1}{3} \left| -2p + \left( Q + \frac{\Delta_0}{Q} \right) \right|}$$  \hspace{1cm} (A.7)
\[ \Delta_0 = B^2 + 3C + 12D \]
\[ Q = \sqrt{\frac{1}{2} \Delta + \sqrt{\Delta^2 - 4\Delta_0^2}} \]
\[ \Delta = 2B^3 + 9BC + 27D + 27C^2 - 72BD \]

and the involved quantities are

\[ A = -16J^2 - 4\Delta_1^2 - 4\Delta_2^2 - 4\Delta_3^2 - 8J_1^2 \]
\[ B = 32J^2\Delta_1^2 + 32J^2\Delta_2^2 + 48J^2\Delta_3^2 + 128J^2J_1^2 + 6\Delta_1^4 + 4\Delta_1^2\Delta_2^2 + 4\Delta_1^2\Delta_3^2 + 16\Delta_1^2J_1^2 + 6\Delta_2^4 + 4\Delta_2^2\Delta_3^2 + 24\Delta_2^2J_1^2 + 6\Delta_3^4 + 8\Delta_3^2J_1^2 + 16J_1^4 \]
\[ C = -16J^2\Delta_1^4 - 32J^2\Delta_1^2\Delta_2^2 - 128J^2\Delta_1^2J_1^2 - 16J^2\Delta_2^4 - 128J^2\Delta_2^2J_1^2 - 48J^2\Delta_3^4 - 256J^2J_1^4 - 4\Delta_1^6 + 4\Delta_1^4\Delta_2^2 + 4\Delta_1^4\Delta_3^2 - 8\Delta_1^3J_1^2 + 4\Delta_1^2\Delta_2^4 - 40\Delta_1^2\Delta_2^2\Delta_3^2 + 4\Delta_1^2\Delta_3^4 - 32\Delta_1^2\Delta_3^2J_1^2 - 4\Delta_2^6 + 4\Delta_2^4\Delta_3^2 - 24\Delta_2^4J_1^2 + 4\Delta_2^2\Delta_3^4 + 16\Delta_2^2\Delta_3^2J_1^2 - 32\Delta_2^2J_1^4 + 4\Delta_3^6 + 8\Delta_3^4J_1^2 - 32\Delta_3^2J_1^4 \]
\[ D = 4\Delta_1^4\Delta_2^4\Delta_3^2 + 4\Delta_1^2\Delta_2^2\Delta_3^2 + 4\Delta_1^2\Delta_2^2\Delta_3^4 - 4\Delta_1^6\Delta_2^2 - 4\Delta_1^4\Delta_2^4 + 6\Delta_1^4\Delta_3^4 - 4\Delta_1^2\Delta_3^6 - 4\Delta_2^4\Delta_3^2 + 6\Delta_2^4\Delta_3^4 - 4\Delta_2^2\Delta_3^6 + 16\Delta_2^4J_1^4 - 8\Delta_3^6J_1^2 + 16\Delta_3^4J_1^4 + 16J^2\Delta_3^6 + 8\Delta_2^2J_1^4 + 16\Delta_2^2\Delta_3^4 + 24\Delta_2^2\Delta_3^2J_1^2 + 128J^2\Delta_3^4J_1^2 + 256J^2\Delta_3^2J_1^4 + 16J^2\Delta_1^4\Delta_3^2 + 32J^2\Delta_1^2\Delta_3^4 + 16J^2\Delta_2^4\Delta_3^2 - 32J^2\Delta_2^2\Delta_3^4 + 128J^2\Delta_2^2\Delta_3^2J_1^2 + 128J^2\Delta_1^2\Delta_3^2J_1^2 + 128J^2\Delta_1^2\Delta_3^2J_1^2 + \Delta_1^8 + \Delta_2^8 + \Delta_3^8 \]

**Appendix B  Energy spectrum of** \(H^{(1)}(t)\)

The Hamiltonian \(H^{(1)}(t)\) with \(\Omega_2(t)\) (i.e. \(J_1 = \Omega_2(t)\) and \(J = \Omega_3(t)\) are not always respected) takes the form

\[ H^{(1)}(t) = J_1(\sigma_1^+ + \sigma_1^-)(\sigma_2^+ e^{-i\varphi} + \sigma_2^- e^{i\varphi}) + \Omega_3(t)(\sigma_2^+ + \sigma_2^-)(\sigma_3^+ e^{-i\varphi} + \sigma_3^- e^{i\varphi}) \]
\[ + \Omega_3(t)(\sigma_1^+ + \sigma_1^-)(\sigma_3^+ e^{i\varphi} + \sigma_3^- e^{-i\varphi}) + \Omega_2(t)(\sigma_1^+ e^{i\varphi} + \sigma_1^- e^{-i\varphi}) \]
\[ + \Omega_3(t)(\sigma_3^+ e^{i\varphi} + \sigma_3^- e^{-i\varphi}) + J(\sigma_1^+ + \sigma_1^-)(\sigma_2^+ + \sigma_2^-)(\sigma_3^+ e^{-i\varphi} + \sigma_3^- e^{i\varphi}) \]

and in matrix we have

\[ H^{(1)}(t) = \]
\[
\begin{pmatrix}
0 & \Omega_3(t)e^{i\varphi} & \Omega_2(t)e^{i\varphi} & \Omega_2(t)e^{-i\varphi} & 0 & \Omega_3(t)e^{i\varphi} & J_1e^{-i\varphi} & J_1e^{i\varphi} \\
\Omega_3(t)e^{-i\varphi} & 0 & \Omega_2(t)e^{i\varphi} & \Omega_2(t)e^{-i\varphi} & 0 & \Omega_3(t)e^{i\varphi} & 0 & \Omega_3(t)e^{i\varphi} \\
\Omega_2(t)e^{-i\varphi} & \Omega_3(t)e^{i\varphi} & 0 & \Omega_3(t)e^{i\varphi} & J_1e^{i\varphi} & J_1e^{i\varphi} & 0 & \Omega_3(t)e^{i\varphi} \\
\Omega_3(t)e^{i\varphi} & \Omega_2(t)e^{-i\varphi} & \Omega_3(t)e^{-i\varphi} & 0 & J_1e^{i\varphi} & J_1e^{i\varphi} & \Omega_3(t)e^{-i\varphi} & 0 \\
0 & \Omega_3(t)e^{i\varphi} & J_1e^{-i\varphi} & J_1e^{-i\varphi} & 0 & \Omega_3(t)e^{i\varphi} & \Omega_2(t)e^{i\varphi} & \Omega_3(t)e^{i\varphi} \\
\Omega_3(t)e^{-i\varphi} & 0 & \Omega_3(t)e^{i\varphi} & \Omega_2(t)e^{-i\varphi} & \Omega_3(t)e^{-i\varphi} & 0 & \Omega_3(t)e^{-i\varphi} & 0 \\
J_1e^{i\varphi} & J_1e^{i\varphi} & \Omega_3(t)e^{i\varphi} & \Omega_2(t)e^{i\varphi} & \Omega_3(t)e^{i\varphi} & 0 & \Omega_3(t)e^{i\varphi} & 0 \\
J_1e^{-i\varphi} & J_1e^{-i\varphi} & 0 & \Omega_3(t)e^{-i\varphi} & \Omega_2(t)e^{-i\varphi} & \Omega_3(t)e^{-i\varphi} & 0 & \Omega_3(t)e^{-i\varphi} & 0
\end{pmatrix}
\]
For simplicity we choose $\varphi = \frac{\pi}{4}$ and show theta the eigenfrequencies time-dependent of the Hamiltonian $H^{(1)}(t)$ take the form

$$\Lambda_0(t) = \sqrt{(\Omega_3 - J)^2 + J_1^2 + \Omega_2^2 + \sqrt{2(\Omega_3 - J)^2(J_1 - \Omega_2)^2}}$$ \hspace{1cm} (B.3)

$$\Lambda_1(t) = -\sqrt{(\Omega_3 - J)^2 + J_1^2 + \Omega_2^2 + \sqrt{2(\Omega_3 - J)^2(J_1 - \Omega_2)^2}}$$ \hspace{1cm} (B.4)

$$\Lambda_2(t) = \sqrt{(\Omega_3 - J)^2 + J_1^2 + \Omega_2^2 - \sqrt{2(\Omega_3 - J)^2(J_1 - \Omega_2)^2}}$$ \hspace{1cm} (B.5)

$$\Lambda_3(t) = -\sqrt{(\Omega_3 - J)^2 + J_1^2 + \Omega_2^2 - \sqrt{2(\Omega_3 - J)^2(J_1 - \Omega_2)^2}}$$ \hspace{1cm} (B.6)

$$\Lambda_4(t) = \sqrt{5\Omega_3^2 + 2J\Omega_3 + J^2 + J_1^2 + \Omega_2^2 + \sqrt{2\Omega_3^2(9\Omega_2^2 + 2J\Omega_2 + 9J_1^2) + 2J(J_1 + \Omega_2)^2(2\Omega_3 + J)}}$$ \hspace{1cm} (B.7)

$$\Lambda_5(t) = -\sqrt{5\Omega_3^2 + 2J\Omega_3 + J^2 + J_1^2 + \Omega_2^2 + \sqrt{2\Omega_3^2(9\Omega_2^2 + 2J\Omega_2 + 9J_1^2) + 2J(J_1 + \Omega_2)^2(2\Omega_3 + J)}}$$ \hspace{1cm} (B.8)

$$\Lambda_6(t) = \sqrt{5\Omega_3^2 + 2J\Omega_3 + J^2 + J_1^2 + \Omega_2^2 - \sqrt{2\Omega_3^2(9\Omega_2^2 + 2J\Omega_2 + 9J_1^2) + 2J(J_1 + \Omega_2)^2(2\Omega_3 + J)}}$$ \hspace{1cm} (B.9)

$$\Lambda_7(t) = -\sqrt{5\Omega_3^2 + 2J\Omega_3 + J^2 + J_1^2 + \Omega_2^2 - \sqrt{2\Omega_3^2(9\Omega_2^2 + 2J\Omega_2 + 9J_1^2) + 2J(J_1 + \Omega_2)^2(2\Omega_3 + J)}}$$ \hspace{1cm} (B.10)

and the associated eigenvectors are given by

$$|\Lambda_0(t)\rangle = \frac{1}{2\sqrt{2}} \left( e^{-i\alpha(t)|\downarrow\downarrow\downarrow\rangle} + e^{-i\alpha(t)|\downarrow\downarrow\uparrow\rangle} + |\downarrow\uparrow\downarrow\rangle + |\downarrow\uparrow\uparrow\rangle + e^{-i\alpha(t)|\uparrow\downarrow\downarrow\rangle} + e^{-i\alpha(t)|\uparrow\downarrow\uparrow\rangle} + |\uparrow\uparrow\downarrow\rangle + |\uparrow\uparrow\uparrow\rangle \right)$$ \hspace{1cm} (B.11)

$$|\Lambda_1(t)\rangle = \frac{1}{2\sqrt{2}} \left( e^{i\alpha(t)|\downarrow\downarrow\downarrow\rangle} + ie^{-i\alpha(t)|\downarrow\downarrow\uparrow\rangle} + |\downarrow\uparrow\downarrow\rangle + |\downarrow\uparrow\uparrow\rangle + e^{-i\alpha(t)|\uparrow\downarrow\downarrow\rangle} - e^{-i\alpha(t)|\uparrow\downarrow\uparrow\rangle} - |\uparrow\uparrow\downarrow\rangle - |\uparrow\uparrow\uparrow\rangle \right)$$ \hspace{1cm} (B.12)

$$|\Lambda_2(t)\rangle = \frac{1}{2\sqrt{2}} \left( e^{-i\alpha(t)|\downarrow\downarrow\downarrow\rangle} + i\omega e^{i\alpha(t)|\downarrow\downarrow\uparrow\rangle} + |\downarrow\uparrow\downarrow\rangle - |\downarrow\uparrow\uparrow\rangle + e^{-i\alpha(t)|\uparrow\downarrow\downarrow\rangle} + ie^{-i\alpha(t)|\uparrow\downarrow\uparrow\rangle} - |\uparrow\uparrow\downarrow\rangle - |\uparrow\uparrow\uparrow\rangle \right)$$ \hspace{1cm} (B.13)

$$|\Lambda_3(t)\rangle = \frac{1}{2\sqrt{2}} \left( e^{i\alpha(t)|\downarrow\downarrow\downarrow\rangle} + i\omega e^{i\alpha(t)|\downarrow\downarrow\uparrow\rangle} - |\downarrow\uparrow\downarrow\rangle + \omega |\downarrow\uparrow\uparrow\rangle - e^{i\alpha(t)|\uparrow\downarrow\downarrow\rangle} - i\omega e^{i\alpha(t)|\uparrow\downarrow\uparrow\rangle} + |\uparrow\uparrow\downarrow\rangle \right)$$ \hspace{1cm} (B.14)

$$|\Lambda_4(t)\rangle = \frac{1}{2\sqrt{2}} \left( e^{-i\alpha(t)|\downarrow\downarrow\downarrow\rangle} - e^{-i\alpha(t)|\downarrow\downarrow\uparrow\rangle} + |\downarrow\uparrow\downarrow\rangle - |\downarrow\uparrow\uparrow\rangle + e^{-i\alpha(t)|\uparrow\downarrow\downarrow\rangle} + e^{-i\alpha(t)|\uparrow\downarrow\uparrow\rangle} - |\uparrow\uparrow\downarrow\rangle + |\uparrow\uparrow\uparrow\rangle \right)$$ \hspace{1cm} (B.15)

$$|\Lambda_5(t)\rangle = \frac{1}{2\sqrt{2}} \left( e^{i\alpha(t)|\downarrow\downarrow\downarrow\rangle} - \omega e^{i\alpha(t)|\downarrow\downarrow\uparrow\rangle} - i\omega |\downarrow\uparrow\downarrow\rangle - e^{i\alpha(t)|\uparrow\downarrow\downarrow\rangle} + \omega e^{i\alpha(t)|\uparrow\downarrow\uparrow\rangle} + |\uparrow\uparrow\downarrow\rangle + i\omega |\uparrow\uparrow\uparrow\rangle \right)$$ \hspace{1cm} (B.16)

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\[ |A_6(t)\rangle = \frac{1}{2\sqrt{2}} \]
\[
\left( e^{-i\alpha(t)} |\downarrow\downarrow\rangle - ie^{-i\alpha(t)} |\downarrow\uparrow\rangle - |\uparrow\downarrow\rangle + e^{-i\alpha(t)} |\uparrow\uparrow\rangle - ie^{-i\alpha(t)} |\uparrow\uparrow\rangle - |\uparrow\uparrow\rangle + i |\uparrow\uparrow\rangle \right) \tag{B.17}
\]
\[ |A_7(t)\rangle = \frac{1}{2\sqrt{2}} \]
\[
\left( e^{i\alpha(t)} |\downarrow\downarrow\rangle - i\omega e^{i\alpha(t)} |\downarrow\uparrow\rangle - i |\downarrow\uparrow\rangle - \omega |\downarrow\uparrow\rangle - e^{i\alpha(t)} |\uparrow\downarrow\rangle + i\omega e^{i\alpha(t)} |\uparrow\uparrow\rangle + i |\uparrow\uparrow\rangle + \omega |\uparrow\uparrow\rangle \right) \tag{B.18}
\]
where we have defined
\[
\alpha(t) = \frac{\pi}{4} - \kappa(t) \tag{B.19}
\]
\[
\tan[\kappa(t)] = \frac{\Omega_2(t)}{2J_1} + \frac{\Omega_3(t)}{2J}. \tag{B.20}
\]

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