Mathematical Model of Easter Island Society Collapse

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Abstract

In this paper we consider a mathematical model for the evolution and collapse of the Easter Island society, starting from the fifth century until the last period of the society collapse (fifteen century). Based on historical reports, the available primary sources consisted almost exclusively on the trees. We describe the inhabitants and the resources as an isolated system and both considered as dynamic variables. A mathematical analysis about why the structure of the Easter Island community collapse is performed. In particular, we analyze the critical values of the fundamental parameters driving the interaction humans-environment and consequently leading to the collapse. The technological parameter, quantifying the exploitation of the resources, is calculated and applied to the case of other extinguished civilization (Copán Maya) confirming, with a sufficiently precise estimation, the consistency of the adopted model.

Key words: Social System, Evolution, Ecology

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1 Introduction

Easter Island history is a very famous example of an evolved human society that collapsed for over exploiting its fundamental resources [123] that in this case were essentially in palm trees. It were covering the island[4] when, few dozens of individuals, first landed around 400 A.D. Its advanced culture was developed in a period of one thousand years approximately. Its ceremonial rituals and associated construction were demanding more and more natural resources especially palm trees. The over exploitation of this kind of tree, very necessary as a primary resource (tools construction, cooking, erosion barrier, etc.) was related with the collapse. In this paper, a mathematical
model concerning growing and collapse of this society is presented. Different
to usual like Lotka-Volterra models [5, 6] where the carrying capacity variation
becomes from external natural forces, in this work it is directly connected
with the population dynamics. Namely, population and carrying capacity are
interacting dynamics variables, so generalizing the Leslie model prey-predator.

The general mathematical treatment of a model describing such a complex
society is a very hard task and probably not unique. Our aim is to settle
the most simple model describing with acceptable precision the evolution of East-
erner society. With the idea of writing a model that could be generalized to
a more complex system, we first divide the elements into two categories: the
resource quantity $R_i$ with $i = 1, 2 \cdots k$ and the inhabitants numbers (species)
$N_i$ with $i = 1, 2 \cdots m$. With the concept of resources we are meaning resources
in a very large sense, it could be oil, trees, food and so on. The several kind
of resources are described by the index $i$. In similar way, with the concept
of inhabitants, we are meaning different species of animals or internal sub-
division of human being in country or town or even tribes. Leaving the idea
of a constant quantity of resources that leads to the logistic equation for the
number on inhabitants [7], the aim of this paper is to include in the dynamical
description of the time evolution of the system the resources too which can
not be considered as constant. A generalization of the logistic equation to an
arbitrary number of homogeneous species interacting among the individuals
(with non constant resources) can be written as

$$\frac{d}{dt}N_i = r_i N_i \left( 1 - \frac{N_i}{N_{ci} (R_1 \cdots R_k)} \right) - \sum_{j=1,i\neq j}^{m} \chi_{ij} N_j N_i.$$  \hspace{1cm} (1)

Where $r_i$ is the usual growing rate for species $i$. In the denominator it appears
the carrying capacity of the system with respect to the number of inhabitants
$N_{ci} (R_1 \cdots R_k)$. Beside the dependence on the resources $R_i$ we could have also
a dependence on an other species that would be then a "resource" for some
other species. This fact is expressed even by the quantities $\chi_{ij}$ that in general
are not symmetric expression ($\chi_{ij} \neq \chi_{ji}$) since that the prey is a resource for
the predator and not the inverse. Similarly, for the resources we have:

$$\frac{d}{dt}R_i = r'_i R_i \left( 1 - \frac{R_i}{R_{ci}} \right) - \sum_{j=1}^{m} \alpha_{ij} N_j R_i.$$  \hspace{1cm} (2)

It is clear that set of equations (2) could be formally included into Eq. (1)
redefining the quantities $\alpha_{ij}$. Nevertheless, we shall keep this distinction for
the sake of clarity especially referred to resources, such as trees, oil or oxygen,
where the carrying capacity is not determined by other species and can be
considered a constant ($R_{ci}$). Also the meaning of the parameters $r'_i$ is more
or less the same of the analogous parameters $r_i$. It is suitable to define $r'_i$ as

\[2\]
renewability ratios”, since describe the capacity of the resources to renew itself and clearly are depending on the kind of resource. For example, the renewability of the oil is clearly zero since the period of time to get oil from a natural process is of the order of geological time-scale processes. In general all parameters of Eqs. (1) and (2) are time dependent, including stochasticity. We can assume reasonably slowly time-varying for the ancient societies so that we can consider it as constant[5], particularly the α’s. Anyway the set of α\textsubscript{ij} is worthy of a more detailed discussion. We can call this set of parameters technological parameters in the sense that they carry the information about the capacity to exploit the resources of the habitat. We shall see in the next section that the technological parameters combined with the renewability ratios will be the key point to decide whether, or not, a society is destined to collapse.

2 Easter island collapse model

The particular history of Easter Island society presents several advantages for modelling its evolution [9]. In fact it can be with very good approximation considered a closed system. The peculiar style of life and culture allows us to consider a basic model where trees are essentially the only kind of resources. Many of activities of the ancient inhabitants involved the trees, from building and transport the enormous Moai, to build boats for fishing, etc. In fact the cold water was not adapt to the fish life and the impervious shape of the coast made difficult fishing. Finally from historical reports it can be inferred that the inhabitants did not change the way to exploit their main resource, even very near to exhaust it so that we can consider the technological parameter as constant. Considering Eqs. (1) and (2) for one inhabitant species and one kind of resource, we obtain:

\[
\frac{d}{dt} N = rN \left[1 - \frac{N}{N_c(R)}\right], \tag{3}
\]

\[
\frac{d}{dt} R = r'R \left[1 - \frac{R}{R_c}\right] - \alpha NR, \tag{4}
\]

where we introduce the notation: α\textsubscript{11} ≡ α. The unknown function \(N_c(R)\) has to satisfy few properties. For a quantity of primary unlimited resource, \(R \to \infty\), even \(N_c(R) \to \infty\) that means that the population can grow unlimited too. In the opposite case \(R \to 0\) clearly also the population must vanish, \(N_c(R) \to 0\), and finally when the resource is constant we are back to the ordinary equation (logistic) so that \(N_c(R) = \text{const.}\) It is clear that the choice of this relation is quite arbitrary but following the simplicity criteria we can select \(N_c(R) = \beta R\), where \(\beta\) is a positive parameter. This choice formalizes
the intuition that the maximum number of individuals tolerated by a niche is proportional to the quantity of resources. We note that in (4) the interacting term depend on the variable \( R \). Namely, for \( R = 0 \) no variation of resource exist \( \left( \frac{d}{dt} R = 0 \right) \) corresponding to a biological criterion and different from this one of reference [10]. A more sophisticate model should include also fishing as resource and consider for \( N_c(R) \) an expression such as \( N_c(R) = \beta R + R_f \) where \( R_f \) is the fishing carrying capacity. This resource was limited near the coast and could not be fully exploited without boats, so that, we are going to neglect this resource. We can rewrite Eqs. (3) and (4) as:

\[
\frac{d}{dt} N = rN \left[ 1 - \frac{N}{\beta R} \right], \tag{5}
\]

\[
\frac{d}{dt} R = r' R \left[ 1 - \frac{R}{R_c} - \alpha E N \right], \text{ where } \alpha_E = \frac{\alpha}{r'}. \tag{6}
\]

The dimensionless parameter \( \alpha_E \) is the ratio between the technological parameter \( \alpha \), representing the capability of to exploit the resources, and \( r' \) the renewability parameter representing the capability of the resources to regenerate. We will call \( \alpha_E \) deforestation parameter since it gives a measure of the rapidity with which the resources are going to exhaust and then a measure of the reversibility or irreversibility of the collapse. Using the historical data we can have an estimation of the parameters. At the origin \( (t = 0) \) we can assume that the trees were covering the entire island surfaces of 160 km\(^2\). When the first humans arrived to the island, around the 400 A.D., their number was of the order of few dozens of individuals and it grew until to reach the maximum \( N_M \sim 10000 \) around the 1300 A.D.

Finding the equilibrium points of Eqs. (5) and (6) we obtain (see section III, for stability):

\[
N_0 = 0, \quad R_0 = R_c \quad \text{(unstable-saddle-point)} \tag{7}
\]

\[
N_e = \frac{\beta R_c}{1 + \alpha E \beta R_c}, \quad R_e = \frac{R_c}{1 + \alpha E \beta R_c} \quad \text{(stable)}. \tag{8}
\]

While the point \( (N_0, R_0) \) of Eq. (7) represents the trivial fact that in absence of human being the number of trees is constant (carrying capacity). Eq. (8) describes the fact that, due to the interaction humans-environment, the more interesting equilibrium point \( (N_e, R_e) \) does not coincide with \( (N_c, R_c) \) with \( N_c = \beta R_c \) since \( \alpha \neq 0 \). To study the stability of the point \( (N_e, R_e) \), we have linearize the system of Eqs. (5) and (6) around the equilibrium point. In fact, in the next section we shall show that it is a stable equilibrium point.
3 Equilibrium points and stability

Let us first cast Eqs. (5) and (6) in term of dimensionless quantities; setting $\nu(t) = N/N_c$, $\varrho(t) = R/R_c$ and $\tau = rt$, we have:

$$\frac{d}{d\tau} \nu = \nu \left[ 1 - \frac{\nu}{\varrho} \right]$$  \hspace{1cm} (9)

$$\frac{d}{d\tau} \varrho = \bar{\varrho} \left[ 1 - \varrho - \bar{\alpha} \nu \right], \quad \bar{\alpha} \equiv \alpha E N_c, \quad \bar{\varrho} \equiv \frac{r'}{r}.$$  \hspace{1cm} (10)

For sake of clarity we rewrite also the equilibrium point:

$$\nu_e = \frac{1}{1 + \bar{\alpha}}, \quad \varrho_e = \frac{1}{1 + \bar{\alpha}}$$  \hspace{1cm} (11)

with obvious meaning of the symbols. Perturbing the equilibrium point $\nu(\tau) = \nu_e [1 + \eta(\tau)]$ and $\varrho(\tau) = \varrho_e [1 + \varepsilon(\tau)]$ with $\eta(\tau)$ and $\varepsilon(\tau)$ infinitesimal functions, after straightforward algebra we obtain Eq. (8):

$$\frac{d}{d\tau} \eta = -\eta + \varepsilon$$  \hspace{1cm} (12)

$$\frac{d}{d\tau} \varepsilon = -\frac{\bar{\varrho}}{1 + \bar{\alpha}} [\bar{\alpha} \eta + \varepsilon]$$  \hspace{1cm} (13)

The eigenvalues of the system are:

$$\lambda_{1,2} = \frac{- (1 + \bar{\alpha} + \bar{\varrho}) \pm \sqrt{(1 + \bar{\alpha} + \bar{\varrho})^2 - 4\bar{\varrho}(1 + \bar{\alpha})^2}}{2(1 + \bar{\alpha})}$$  \hspace{1cm} (14)

Restricting ourself to the case of positive values of \(\bar{\alpha}\), Eq. (14) shows that both the eigenvalues always have a real negative part, so that the equilibrium point \((\nu_e, \varrho_e)\) is a stable equilibrium point. More in detail we have that for

$$\bar{\varrho} \leq \frac{1}{4}, \quad \bar{\alpha} \geq 0 \quad \text{and for} \quad \bar{\varrho} > \frac{1}{4}, \quad \bar{\alpha} \leq \frac{(\sqrt{\bar{\varrho}} - 1)^2}{2\sqrt{\bar{\varrho}} - 1}$$

the eigenvalues are real and negative so that the equilibrium point is reached in exponential damped way, otherwise the eigenvalues acquire an imaginary part and the system reach the equilibrium point via exponential damped oscillations.
Fig. 1. $N(t)/N_c$ versus time. In this numerical example $N(0)/N_c \sim 0.8$ and $N_{Max}/N_c \sim 3$

4 Collapse condition

Even if, mathematically speaking, the stable point $(N_e, R_e)$ is an acceptable result we have to take in account the biological constraints that allow to a specie to survive. A reasonable number of individuals is required for viability of a given species [11,12]. This is so because genetic diversity, social structures, encounters, etc., need a minimum numbers of individuals since under this critical numbers the species is not viable and collapse. It is worthy to stress that while the trees can reach an equilibrium point without the humans, Eq. (7), the opposite does not hold, as stated by Eq. (8).

Calling the minimum number of humans $N_{min}$ we can find a upper bound for the parameter $\alpha_E$ so that a civilization can survive. Imposing the condition that at the equilibrium point $N_e \geq N_{min}$ we obtain:

$$\alpha_E \geq \frac{1}{N_{min}} - \frac{1}{\beta R_c} \quad \text{(collapse condition).} \quad (15)$$

As further simplification of inequality (15), we assume that $N_{min} \ll \beta R_c$ and we find that $\alpha_E N_{min} \geq 1$ or $\alpha N_{min} \geq r'$. It is a natural condition since it tells that collapse exists when the production rate $r'$ is minor than the deforestation rate $\alpha N_{min}$.

More in general it can be showed, numerically, that considering the standard case with the starting population number $N(0) < N_c = \beta R_c$, we can have a solution that can exceed the value $\beta R_c$ (depending on initial conditions $N(0)$ and $\frac{dN}{dt}|_{t=0}$). It is worthy to stress that in the case of ordinary logistic map the population number never can exceed this limit value[7]. In the example of Fig. 1 the maximum of $N$ is reached at a value that is almost three times
\( \beta R_c \). Then, according to the region of the parameters that we are considering, the paths to the final equilibrium is exponentially fast, reaching eventually the point \( N_{\text{min}} \) and collapsing.

5 Deforestation rate estimation

As we saw in the previous section, the collapse condition (15) gives a sufficient condition on the deforestation rate per individual \( \alpha \). On other hand, the last period of tree extinction was governed essentially by the deforestation rate. In this way, we have the rate of tree extinction

\[
\frac{1}{R} \frac{dR}{dt} \sim -\alpha N. \tag{16}
\]

As discussed, the path to the equilibrium point is exponentially fast, so that a rough estimation of the left side of Eq. (16) is the time scale of the deforestation, \( \tau_F \), while the right side can be taken at the end of the collapse process (the equilibrium point):

\[
\frac{1}{\tau_F} \sim \alpha N_F, \tag{17}
\]

being \( N_F \) the final number of individuals. It can be deduced that the range of is \( \tau_F \sim 100 \) yrs. to \( \tau_F \sim 300 \) yrs. and \( N_F \sim 3000 \), the rate of deforestation (per individual) could be estimated as:

\[
\alpha \sim \frac{1}{\tau_F N_F} \text{ (yrs. individual)}^{-1} \tag{18}
\]

giving a range

\[
1.1 \times 10^{-6} < \alpha < 3.3 \times 10^{-6} \text{ (yrs. individual)}^{-1}. \tag{19}
\]

This estimation has validity in the case of exponential decay which is the our case, as it has been showed by the analysis performed in Sec. 3. Assuming the number of trees as proportional to the area \( A \) we can now estimate the rate-deforestation-area. The island has a surface at order of \( A \sim 160 \text{ km}^2 \) and initially it can be supposed that was covered of trees so that we can estimate:

\[
0.5 < \frac{dA}{dt} \sim \frac{A_0}{\tau_F} < 1.6 \text{ (km}^2/\text{yrs}). \tag{20}
\]
As comparison we can consider that in the last 500 years the deforestation of amazonian forest rate is 15 (km$^2$/yrs). Considering that in 500 years the deforestation technology became more and more efficient, especially in the last century, we can consider it as an upper limit, giving us an idea of the technological change. In the human history there are several examples of over exploiting the natural resources even if not so known as Easter island. In particular, the Copán Maya history has certain similarity with respect to the technology level and the over exploiting of the natural resources. In short, this ancient civilization reached almost 20,000 individuals and declined to 5000 individuals in the 9th century. Using the estimation of the technological parameter $\alpha$ obtained for eastern island civilization, we get a collapse time from Eq. (17) that is $60 < \tau < 180$ years. The collapse time based on historical reports is $\tau \sim 100$, showing that the adopted model is consistent with the available data. The estimation of the parameter is consistent with the idea that similar civilizations, in technological meaning, have similar capacity to exploit the natural resources.

6 Concluding remarks

A mathematical model considering the interaction among carrying capacity and population in an isolated system has been considered. The model takes into account the fact that a population can over exploit the carrying capacity without saturate, a fact of relevant importance on the path leading to a collapse of a society. Its application to the collapse of the Easter Island civilization has been presented. An estimation of the technological parameter $\alpha$ is obtained and applied to another ancient civilization, the Copán-Maya, with a reasonably precise expectation about their collapse time. All confirming the consistency of the adopted model. On the other hand, its relative reasonable prediction suggests a possible extension to more complex system. The effort to mathematical modelling of ancient civilizations could be important considering the actual human growing and resources exploitations. An adequate equilibrium between competition, demand and exploitation is the key of surviving.

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