Four-dimensional $N = 1 \mathbb{Z}_N \times \mathbb{Z}_M$ Orientifolds

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Abstract

We calculate the tadpole equations and their solutions for a class of four-dimensional orientifolds with orbifold group $\mathbb{Z}_N \times \mathbb{Z}_M$, and we present the massless bosonic spectra of these models. Surprisingly we find no consistent solutions for the models with $\mathbb{Z}_2 \times \mathbb{Z}_4$ and $\mathbb{Z}_4 \times \mathbb{Z}_4$ orbifold groups.

PACS: 11.25.Mj, 11.25.-w
Keywords: Superstrings; Open strings; Orientifold

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1 Introduction

In the search for string compactifications to four dimensions breaking supersymmetry to $N = 1$, orbifolds have played an important role, as they yield solvable models \[1\]. In the past attention focused on heterotic orbifolds \[2\]. With the advent of D-branes, (see \[3\] and references therein), a different class of theories came under investigation, where the orbifold group includes a world sheet symmetry: orientifolds \[4, 5\]. These are models in which, beside unoriented closed strings, open strings ending on various types of D-branes occur.

The simplest example of such a model is the type I string theory. This theory can be viewed as an orientifold of the type IIB string theory. Type IIB strings are symmetric under an exchange of their left and right moving sectors, and this symmetry can be divided out. The first effect of this is a truncation of the closed string spectrum to those states invariant under the symmetry; this truncated theory, however, is inconsistent due to non-vanishing tadpoles. The situation is cured by adding open strings, ending on space-filling nine-branes, and supporting an $SO(32)$ gauge group.

In \[6\] evidence was presented for a duality relating this type I theory to heterotic strings with the same gauge group. Studying type I orbifolds therefore provides a means for investigating non-perturbative aspects of the heterotic orbifolds considered previously. So far most attention has been paid to six-dimensional compactifications, where orientifolds typically describe a phase involving more than one tensor multiplet, a situation that is out of reach of perturbative heterotic theory.

In these lower dimensional orientifold models, obtained by dividing out symmetries from toroidally compactified type I theories, apart from nine-branes also five-branes have to be introduced. It was found by Gimon and Polchinski \[5\] that in these models further consistency conditions have to be met, due to the interplay between open strings ending on nine-branes and five-branes. Their work on the $Z_2 \times K3$ orientifold was subsequently extended to other $K3$ orientifolds in \[7\] and \[8\], and to a four-dimensional model with orbifold group $Z_2 \times Z_2$ in \[9\]. In the latter case three differently oriented sets of five-branes appear, with more stringent restrictions on the representation of the group elements in the open string sectors, arising from the mixing between different sectors. Further four-dimensional orientifolds were presented in \[11, 12\].

Here we extend the collection of four-dimensional orientifolds with $N = 1$ supersymmetry to models with spacetime orbifold groups of the form $Z_N \times Z_M$. Apart from the spacetime orbifold group elements, which are of the form $\alpha^n \beta^m$, with $\alpha (\beta)$ the generator of $Z_N (Z_M)$, the orientifold group involves the elements...
\[ \Omega \alpha^n \beta^m, \text{ where } \Omega \text{ is the world sheet orientation reversal. This introduces orientifold fixed planes, necessitating the addition of D-branes to compensate their RR-charges. We find that the algebraic consistency conditions that have to be imposed are, in some cases, too restrictive to have a solution.} \]

The plan of the paper is as follows. We first recall the general construction of orientifold models. Then we give representations of the action of the orientifold group elements on the open string Chan-Paton factors, consistent with tadpole cancellation and further algebraic consistency conditions of the type presented first in [5]. This generalises similar solutions in [7, 8, 9]. In the next subsection the closed string untwisted and twisted states are calculated. Finally we use the solutions of the Chan-Paton representations to obtain the open string spectra. The results of the tadpole computations are presented in an appendix.

2 Four-dimensional \( \mathbb{Z}_N \times \mathbb{Z}_M \)-orientifolds

2.1 Orientifolds

Let us briefly recall the general features of orientifolds [3]. One starts with a type IIB compactification on an orbifold. This model has a symmetry \( \Omega \), exchanging left and right movers on the world sheet. Dividing out this symmetry produces the orientifold. The closed string spectrum consists of those states invariant under \( \Omega \). In addition one has to include open strings ending on Dirichlet branes; these D-branes are necessary to cancel the charge of the orientifold planes, the fixed planes under the orientifold group elements.

The number of D-branes, as well as the action of the group elements on them, can be determined by a calculation of the amplitudes for the exchange of massless antisymmetric tensor particles from the RR-sector, which couple to the charges of orientifold planes and D-branes. The total contribution for each species of these particles should vanish in the exchange between the boundary states consisting of both orientifold planes and D-branes, since this amplitude is proportional to the square of the total charge.

The action of the group elements on the D-branes is encoded in the transformation of the Chan-Paton matrices \( \lambda_{ij} \) associated to open strings ending on D-branes. Under a symmetry \( \alpha \), these Chan-Paton matrices are conjugated by a unitary matrix \( \gamma_\alpha \),

\[ \lambda \rightarrow \gamma_\alpha \lambda \gamma_\alpha^{-1} \quad (2.1) \]

These \( \gamma_\alpha \) should form a projective representation of the group. The cancellation of the charges, or tadpole conditions, places restrictions on the traces of (products of) the \( \gamma_\alpha \). Furthermore there are restrictions originating from the fact that
quantities associated to symmetries are conserved in string interactions. The prototype of these constraints is the fact that in the presence of nine-branes with $\gamma_\Omega$ symmetric, five-branes have antisymmetric $\gamma_\Omega$; this is a consequence of the fact that both sectors are related via 59-strings [4].

One then determines the structure of the Chan-Paton matrices invariant under the action of the symmetry group. This yields the gauge group carried by the open string vectors and the representations of the open string matter.

### 2.2 The orientifold groups

The models we calculate are orientifolds with four-dimensional $N = 1$ supersymmetry. They arise from type IIB string theory compactified on a six-dimensional torus modded out by a symmetry group $\mathbb{Z}_N \times \mathbb{Z}_M$, as well as the orientation reversal symmetry $\Omega$. The group is abelian and its elements are therefore of the form $\Omega^n \beta^m$, $\alpha^n \beta^m$, with $\alpha, \beta$ generators of the two cyclic groups.

Parametrising the torus by the three complex numbers $z_1, z_2$ and $z_3$, we choose the action of the groups $\mathbb{Z}_N$ and $\mathbb{Z}_M$ as

\[
(z_1, z_2, z_3) \rightarrow (e^{\frac{2\pi i n}{N}} z_1, e^{-\frac{2\pi i n}{N}} z_2, z_3), \quad \text{resp.} \quad (z_1, z_2, z_3) \rightarrow (z_1, e^{\frac{2\pi i m}{M}} z_2, e^{-\frac{2\pi i m}{M}} z_3).
\]

(2.2)

For this to be a symmetry, we choose the three two-tori to be either the $SU(2) \times SU(2)$ or the $SU(3)$ root lattice. These are then invariant under the groups $\mathbb{Z}_2, \mathbb{Z}_4$ resp. $\mathbb{Z}_2, \mathbb{Z}_3, \mathbb{Z}_6$. (The model with $N = M = 3$ was already considered in [12].) The action on the spinors is defined by $e^{\frac{2\pi i n}{N} (J_{45} - J_{67})}$, $e^{\frac{2\pi i m}{M} (J_{67} - J_{89})}$. From this we can determine that the unbroken supercharges are given by spinors $(s_0, s_1, s_2, s_3, s_4)$, with $s_3 = s_4 = s_5$. Since they are also GSO-projected (chiral) this leaves us with four components, or one spinor in four dimensions. Due to the $\Omega$ projection left and rightmovers are identified, so we have $N = 1$ in $d = 4$.

Since in every model $\Omega$ is one of the group elements, they will all contain nine-branes. Five-branes also occur for those models with additional elements of order two.

### 2.3 Solutions to the tadpole equations

The tadpole equations are obtained from the calculation of closed string massless RR-sector exchange between boundary states consisting of all orientifold planes and D-branes. This calculation is most easily done by viewing the closed string tree-level exchange diagrams as open string one loop diagrams. The different boundary conditions for orientifold planes and D-branes give rise to cylinder, Möbius strip and Klein bottle diagrams, from which the part associated to the
required exchange can be easily extracted. The details and results of these calculations are deferred to the appendix.

Here we turn to the solutions of the tadpole equations. These can be obtained by generalising those found in [7, 9]. We will start with the simplest case of $Z_3 \times Z_3$, where only nine-branes are present. The trace of the unit matrix is 32, as will be the case in all models. The number of dynamical nine-branes, however, is smaller, as these 32 include the various images under the symmetries. The form of the tadpoles in this case is given in eqs. (A.9, A.10). The equations are solved by gamma-matrices

$$
\gamma_{\frac{1}{3},0} = \text{diag}(e^{\frac{2\pi i}{3}}, e^{-\frac{2\pi i}{3}}, 1, 1) \otimes I_8, \quad \gamma_{0,\frac{1}{3}} = \text{diag}(e^{-\frac{2\pi i}{3}}, 1, e^{\frac{2\pi i}{3}}, 1, e^{-\frac{2\pi i}{3}}, 1, e^{\frac{2\pi i}{3}}, 1) \otimes I_4,
$$

(2.3)

(where the indices indicate the elements of the two orbifold groups) together with the representation

$$
\gamma_\Omega = 
\begin{pmatrix}
0 & 1 & 0 & 0 \\
1 & 0 & 0 & 0 \\
0 & 0 & 0 & 1 \\
0 & 0 & 1 & 0 \\
\end{pmatrix} \otimes I_8,
$$

(2.4)

which swaps conjugate numbers in both matrices. The general $n,m$ matrices are simply obtained by defining

$$
\gamma_{n,m} = \gamma_{\frac{1}{3},0}^{n} \gamma_{0,\frac{1}{3}}^{m}.
$$

The next simplest models are $Z_3 \times Z_6$ and $Z_2 \times Z_3$. In this case we have, in addition to the nine-branes, 32 five-branes stretched along the 0...5 dimensions. The tadpoles that should vanish in this case are given in eqs. (A.11-A.14). For the nine-branes we find

$$
\gamma_{0,\frac{1}{5}} = \text{diag}(i, -i) \otimes \text{diag}(e^{\frac{2\pi i}{5}}, e^{\frac{2\pi i}{5}}, 1, 1) \otimes I_4, \quad \gamma_{\frac{1}{5},0} = I_4 \otimes \text{diag}(e^{-\frac{2\pi i}{5}}, 1, e^{\frac{2\pi i}{5}}, 1) \otimes I_2,
$$

(2.5)

so that the $\gamma_{0,\frac{1}{5}}$ and $\gamma_{\frac{1}{5},0}$ are -1 and +1 times the matrices of the previous case. For $\gamma_{0,\frac{1}{5}}$ we can take $\gamma_{0,\frac{1}{5}}$, and $\gamma_{\frac{1}{5},0} = -\gamma_{\frac{1}{5},0}$. Further we can again take the matrix $\gamma_{\Omega,5}$ to exchange all conjugate entries, i.e.

$$
\gamma_{\Omega,9} = 
\begin{pmatrix}
0 & 1 \\
1 & 0 \\
\end{pmatrix} \otimes 
\begin{pmatrix}
0 & 1 & 0 & 0 \\
1 & 0 & 0 & 0 \\
0 & 0 & 0 & 1 \\
0 & 0 & 1 & 0 \\
\end{pmatrix} \otimes I_4.
$$

(2.6)

For $\gamma_{\Omega,5}$ we can take $\gamma_{\Omega,5} \gamma_{0,\frac{1}{5}}$. This is antisymmetric, as it should be.

For the $Z_2 \times Z_3$ model’s solution we can use the same solution, discarding those elements not belonging to the $Z_2$ subgroup of $Z_6$. 

5
We now arrive at the somewhat more complicated models involving three different sets of five-branes: 32 5_1-branes along the 4 − 5 plane, 32 5_2-branes along the 6 − 7 plane, and 32 5_3-branes along the 8 − 9 plane. The basic example of this kind, with orbifold group \( \mathbb{Z}_2 \times \mathbb{Z}_2 \), was explored in [9]. The algebraic consistency conditions relating the Chan-Paton representation of the orientifold group are more involved, creating \( \gamma \)'s in the different sectors (nine-branes and different five-branes) that do not commute. Here we will employ the solution of [4] to construct the generalisation to the models under consideration.

The tadpole equations are obtained by setting the expressions (A.3 – A.8) to zero. For the \( \mathbb{Z}_6 \times \mathbb{Z}_6 \) model, we find for the nine-branes

\[
\begin{align*}
\gamma_{0,1,9} &= i\sigma_2 \otimes I_2 \otimes \text{diag}(e^{\frac{2\pi i}{3}}, e^{-\frac{2\pi i}{3}}, 1, 1), \\
\gamma_{1,1,9} &= -\sigma_3 \otimes i\sigma_2 \otimes \text{diag}(e^{\frac{2\pi i}{3}}, 1, e^{-\frac{2\pi i}{3}}, 1), \\
\gamma_{2,0,9} &= \gamma_{1,1,9}^{-1} \gamma_{0,1,9}^{-1},
\end{align*}
\]

with the appropriate identity factors. This choice gives the answers of [4] when the third power is taken. The five-brane matrices can be constructed similarly:

\[
\begin{align*}
\gamma_{0,1,5_1} &= i\sigma_2 \otimes I_2 \otimes \text{diag}(e^{\frac{2\pi i}{3}}, e^{-\frac{2\pi i}{3}}, 1, 1), \\
\gamma_{1,1,5_1} &= \sigma_3 \otimes I_2 \otimes \text{diag}(e^{\frac{2\pi i}{3}}, 1, e^{-\frac{2\pi i}{3}}, 1), \\
\gamma_{2,0,5_1} &= \gamma_{1,1,5_1}^{-1} \gamma_{0,1,5_1}^{-1},
\end{align*}
\]

\[
\begin{align*}
\gamma_{0,1,5_2} &= -\sigma_1 \otimes I_2 \otimes \text{diag}(e^{\frac{2\pi i}{3}}, e^{-\frac{2\pi i}{3}}, 1, 1), \\
\gamma_{1,1,5_2} &= i\sigma_2 \otimes I_2 \otimes \text{diag}(e^{\frac{2\pi i}{3}}, 1, e^{-\frac{2\pi i}{3}}, 1), \\
\gamma_{2,0,5_2} &= -\gamma_{1,1,5_2}^{-1} \gamma_{0,1,5_2}^{-1},
\end{align*}
\]

\[
\begin{align*}
\gamma_{0,1,5_3} &= \sigma_3 \otimes I_2 \otimes \text{diag}(e^{\frac{2\pi i}{3}}, e^{-\frac{2\pi i}{3}}, 1, 1), \\
\gamma_{1,1,5_3} &= -\sigma_1 \otimes I_2 \otimes \text{diag}(e^{\frac{2\pi i}{3}}, 1, e^{-\frac{2\pi i}{3}}, 1), \\
\gamma_{2,0,5_3} &= \gamma_{1,1,5_3}^{-1} \gamma_{0,1,5_3}^{-1},
\end{align*}
\]

while the representation for \( \Omega \) is \( I \) times the \( \mathbb{Z}_3 \times \mathbb{Z}_3 \) structure for nine-branes and \( I \otimes i\sigma_2 \) times this structure for all the five-branes. Again we can find the \( \mathbb{Z}_2 \times \mathbb{Z}_6 \) (and the \( \mathbb{Z}_2 \times \mathbb{Z}_2 \)) model by restriction from these matrices.

For the models involving \( \mathbb{Z}_4 \) on the other hand, it turns out we cannot satisfy the consistency conditions. Let us take \( \mathbb{Z}_4 \times \mathbb{Z}_4 \) as an example, and concentrate on the gammas in the nine-brane sector. We may again choose \( \gamma_{\Omega} \) to be the
identity matrix. This then determines $\gamma_{\frac{1}{2},0}$, $\gamma_{0,\frac{1}{2}}$ and $\gamma_{\frac{1}{2},\frac{1}{2}}$ to be antisymmetric matrices, as in the $\mathbb{Z}_2 \times \mathbb{Z}_2$ model of [9]. To see this, one uses that the tadpole conditions require e.g. $\gamma_{\Omega,\frac{1}{2},\frac{1}{2}}$ to be symmetric in the $5_3$ sector. From consideration of the process where two $9_{5_3}$ states go into a $99$ or a $5_35_3$ state, one obtains that $\gamma_{\Omega,\frac{1}{2},0}$ should have the opposite symmetry property on the nine-branes. Since we chose $\gamma_\Omega = I$ in this sector, we find that also $\gamma_{0,\frac{1}{2}}$ should be antisymmetric in the $9$ sector. All matrices are required to form a projective representation of the orbifold group, so that we have

$$\gamma_{\frac{1}{2},0} = aI, \quad \gamma_{0,\frac{1}{2}} = bI, \quad \text{and} \quad c\gamma_{\frac{1}{2},0} \gamma_{0,\frac{1}{2}} = \gamma_{0,\frac{1}{2}} \gamma_{\frac{1}{2},0}.$$  \hspace{1cm} (2.11)

Here $a$, $b$, and $c$ are arbitrary phases. Since $\gamma_{\frac{1}{2},0}^2 \gamma_{0,\frac{1}{2}}^2$ should equal, up to a phase, $\gamma_{\frac{1}{2},\frac{1}{2}}$, it is antisymmetric. From this and the third identity in (2.11), we find that $c^4 = -1$. But as $\gamma_{0,\frac{1}{2}} \gamma_{\frac{1}{2},0} \gamma_{0,\frac{1}{2}} = c\gamma_{\frac{1}{2},0}$ (again the third equation in (2.11)), we also have, by conjugating this three times more, that $\gamma_{0,\frac{1}{2}} \gamma_{\frac{1}{2},0} \gamma_{0,\frac{1}{2}} = c^4 \gamma_{\frac{1}{2},0}$. However, if we now use the second identity in (2.11), the phase $b$ drops out and we obtain that $c^4 = 1$. The same contradiction can be derived for the $\mathbb{Z}_2 \times \mathbb{Z}_4$ case. So we have found the surprising result that the consistency conditions do not admit a solution to these models, which at first sight seem to be as reasonable as the others.

This concludes the solutions to the tadpole equations.

### 2.4 The massless closed string spectra

The spectrum of the models consists of closed and open string states. The closed string states are those type IIB orbifold states that are invariant under $\Omega$. The states group together in multiplets of the $N = 1$ supersymmetry.

The closed string spectrum in the untwisted sector is built up from the following massless states:

| Sector | state | $(z, w)$ | helicity |
|--------|-------|----------|----------|
| NS | $\psi^\mu|0>$ | $1$ | $\pm 1$ |
| | $\psi^1|0>$ | $e^{\pm 2\pi iz}$ | $2 \times 0$ |
| | $\psi^2|0>$ | $e^{\pm 2\pi i(w-\bar{z})}$ | $2 \times 0$ |
| | $\psi^3|0>$ | $e^{\pm 2\pi iw}$ | $2 \times 0$ |
| R | $s_1 = s_2 = s_3 = s_4 >$ | $1$ | $\pm \frac{1}{2}$ |
| | $s_1 = s_2 = -s_3 = -s_4 >$ | $e^{\pm 2\pi iz}$ | $\pm \frac{1}{2}$ |
| | $s_1 = -s_2 = s_3 = -s_4 >$ | $e^{\pm 2\pi i(z-w)}$ | $\pm \frac{1}{2}$ |
| | $s_1 = -s_2 = -s_4 = s_4 >$ | $e^{\pm 2\pi iw}$ | $\pm \frac{1}{2}$ |
The three internal tori are labelled by the numbers 1, 2, 3. The \((z, w)\) column lists the behaviour of the states under the element \(z = \frac{n}{N}\) of \(\mathbb{Z}_N\) and \(w = \frac{m}{M}\) of \(\mathbb{Z}_M\). In the final column the helicities of the states are listed. Only states surviving the GSO projection are given. The bosonic closed string states are those combinations invariant under the orbifold group, and left-right symmetric in the NSNS sector, antisymmetric in the RR sector. For the \(\mathbb{Z}_3 \times \mathbb{Z}_3, \mathbb{Z}_3 \times \mathbb{Z}_6\) and \(\mathbb{Z}_6 \times \mathbb{Z}_6\) models this gives the massless bosonic spectrum

\[
\begin{align*}
\text{NSNS} & : \quad \pm 2 + 4 \times 0 \\
\text{RR} & : \quad 4 \times 0,
\end{align*}
\]

making up the gravity multiplet and four chiral multiplets; for \(\mathbb{Z}_2 \times \mathbb{Z}_3\) and \(\mathbb{Z}_2 \times \mathbb{Z}_6\) we find

\[
\begin{align*}
\text{NSNS} & : \quad \pm 2 + 6 \times 0 \\
\text{RR} & : \quad 4 \times 0,
\end{align*}
\]

which gives five chiral multiplets.

Next we determine the massless twisted closed string sectors. To this end we determine the twisted contributions to the cohomology of the orbifold model, which give the twisted RR ground states of the IIB orbifold compactification \([10]\). Of these states only half survive the orientifold projection \(\Omega\), and these sit together with an equal number of NSNS states in chiral supermultiplets. We carry out explicitly only the calculation of the \(\mathbb{Z}_3 \times \mathbb{Z}_3\) and \(\mathbb{Z}_2 \times \mathbb{Z}_3\) models, and just give the results of the others.

For the orientifold with orbifold group \(\mathbb{Z}_3 \times \mathbb{Z}_3\) we have the group generators \(\alpha\), acting on the complex torus coordinates with charges \((\frac{1}{3}, \frac{2}{3}, 0)\), and \(\beta\) : \((0, \frac{1}{3}, \frac{2}{3})\). First we have states twisted by \(\alpha\), whose fixed point set is 9 tori; of the torus cohomology only half is invariant under \(\beta\); in total we end up with one chiral multiplet for each such torus; there are 6 order three elements like \(\alpha\), so this gives 54 multiplets. Then we have states twisted by \(\alpha^2 \beta\), which have 27 fixed points; there are two such twisted sectors, giving in total 27 chiral multiplets. Adding these up, we arrive at 81 twisted sector chiral multiplets. For \(\mathbb{Z}_2 \times \mathbb{Z}_3\), with generators \(\alpha = (\frac{1}{2}, \frac{1}{2}, 0), \beta = (0, \frac{1}{3}, \frac{2}{3})\), we have the following states: in the \(\alpha\) twisted sector we have 16 fixed tori; of these, four are singlets under the \(\mathbb{Z}_3\) group, the other 12 sit in four triplets. The singlets contribute only half a torus cohomology invariant under \(\beta\); for the triplets one can construct linear combinations of 0–, 1– and 2–forms that are invariant under \(\beta\), giving a whole torus cohomology per triplet. This sector therefore produces 12 chiral multiplets. The sector twisted by \(\beta\) has 9 fixed tori, 3 singlets and three doublets of \(\mathbb{Z}_2\).
making up 9 chiral multiplets, and the same for \( \beta^2 \). Finally, the sectors twisted by \( \alpha \beta, \alpha \beta^2 \) have 12 fixed points; together they produce 12 chiral multiplets. This adds up to 42 chiral multiplets for the \( \mathbb{Z}_2 \times \mathbb{Z}_3 \) model.

The calculation for the other models is similar, but somewhat more work. The results are

| model       | number of chiral multiplets |
|-------------|-----------------------------|
| \( \mathbb{Z}_3 \times \mathbb{Z}_3 \) | 81                          |
| \( \mathbb{Z}_3 \times \mathbb{Z}_2 \) | 42                          |
| \( \mathbb{Z}_2 \times \mathbb{Z}_6 \) | 50                          |
| \( \mathbb{Z}_3 \times \mathbb{Z}_6 \) | 71                          |
| \( \mathbb{Z}_6 \times \mathbb{Z}_6 \) | 81                          |

### 2.5 The massless open string spectrum

We will now determine the massless states arising in the open string sector. Different states occur depending on whether the string ends on nine-branes (99-sector) or 5i-branes (5i5i-sector), which we will call the unmixed sectors, or whether the string has its two ends on two different types of branes, the mixed sectors (5i9, 5i5j-sectors). In the unmixed sectors, the massless states in the bosonic (NS) sector are obtained by acting with one oscillator on the ground state, which includes a Chan-Paton factor \( \lambda_{ij} \) labelling which branes the string ends on; the mixed sector NS ground states are massless themselves. The form of the Chan-Paton factors is then obtained by demanding the states to be invariant under the orientifold group.

We will only analyse the situation where there are no Wilson lines on the branes, and with all five-branes of a given kind located on one fixed point. This will give the maximum gauge symmetry. Furthermore we will so obtain a T self dual configuration, so that nine- and five-branes will produce the same gauge groups and matter content. All \( \lambda \) will be 32 by 32 hermitian matrices.

The constraints on the \( \lambda_{ij} \) are as follows. In the 99-sector we have four distinct sets of states, depending on whether the oscillator has an index in spacetime (the gauge bosons) or in one of the three internal tori (two scalars each). On all these oscillators \( \Omega \) acts as \(-1\), the orbifold group element \((\frac{n}{N}, \frac{m}{M})\) distinguishes the different categories. Demanding the total state to be invariant we have
The 55-sectors give the same constraints, apart from an extra minus in the action of $\Omega$ on the directions orthogonal to the five-brane. In the mixed sectors, 95$_i$ and 5,5$_j$, the NS ground state is a massless spinor in the four directions with ND boundary conditions. Considering for example the 5$_1$9 sector, these will be the 6789 directions. There will be two states, labelled by their spin in the two tori, which are equal due to GSO projection. There is no constraint on the Chan-Paton matrix from the orientation reversal: this relates the 5$_1$9 state to a 95$_1$ state. We only have

| state | \( \frac{n}{N}, \frac{m}{M} \) | $\lambda = \gamma_{n,m,9}\gamma_{n,m,9}^{-1}$ | $\lambda = -\gamma_{\Omega,9}\lambda^T\gamma_{\Omega,9}^{-1}$ |
|-------|--------------------------|------------------|------------------|
| $\psi^\mu|0, ij > \lambda_{ij}$ | $\lambda = \gamma_{n,m,9}\lambda_{n,m,9}^{-1}$ | $\lambda = -\gamma_{\Omega,9}\lambda^T\gamma_{\Omega,9}^{-1}$ |
| $\psi^{1\pm}|0, ij > \lambda_{ij}$ | $\lambda = e^{\pm \frac{2\pi n}{N}}\gamma_{n,m,9}\lambda_{n,m,9}^{-1}$ | $\lambda = -\gamma_{\Omega,9}\lambda^T\gamma_{\Omega,9}^{-1}$ |
| $\psi^{2\pm}|0, ij > \lambda_{ij}$ | $\lambda = e^{\pm \frac{2\pi m}{M}}\gamma_{n,m,9}\lambda_{n,m,9}^{-1}$ | $\lambda = -\gamma_{\Omega,9}\lambda^T\gamma_{\Omega,9}^{-1}$ |
| $\psi^{3\pm}|0, ij > \lambda_{ij}$ | $\lambda = e^{\pm \frac{2\pi m}{M}}\gamma_{n,m,9}\lambda_{n,m,9}^{-1}$ | $\lambda = -\gamma_{\Omega,9}\lambda^T\gamma_{\Omega,9}^{-1}$ |

and similar states for the other sectors.

Now we can use the representations of the $\gamma$'s we found to determine the representations of these states. The results are collected in the table.
This concludes the calculation of the spectra.

The notation $1^2$ is short for $1, 1$, etc.

The subscript indicates $U(1)$ charge; these are antisymmetric tensors of $U(2)$: singlets under $SU(2)$, but charge 2 under $U(1)$.
3 Discussion

We have calculated the spectra of certain four-dimensional orientifold models with \( N = 1 \) supersymmetry. The spacetime symmetry divided out was of the form \( \mathbb{Z}_N \times \mathbb{Z}_M \), with the first factor acting on the 4567 directions, and the second on the 6789 directions. We found solutions to the tadpole equations in most cases, and used these to compute the open string massless spectrum. We also computed the closed string twisted sectors in these cases. Unexpectedly, two of the models under consideration, \( \mathbb{Z}_2 \times \mathbb{Z}_4 \) and \( \mathbb{Z}_4 \times \mathbb{Z}_4 \), turned out not to allow a solution to the tadpole equations that satisfied the consistency conditions.

The orientifold models are expected to have dual heterotic orbifolds. This follows from the ten-dimensional heterotic–type I duality [6]. In fact, such duals were constructed for the \( N = M = 3 \) case in [12, 13]. It would be interesting to see whether a heterotic approach to the two inconsistent models may shed some light on the reason for their inconsistency.

We have not exhausted all possibilities for orientifolds with \( \mathbb{Z}_N \times \mathbb{Z}_M \) orbifold group. It is for instance possible to introduce discrete torsion, or to have models where the groups act differently on the tori.

Acknowledgements

The author thanks Erik Verlinde for advice and discussions. This work is supported by FOM.

A The tadpole calculation

We briefly discuss the calculation of the RR-tadpoles due to the orientifolds and branes. Demanding them to vanish gives equations for the unitary matrices \( \gamma \) representing the action of the orientifold group on the Chan-Paton factors.

The RR-charges can most easily be calculated by considering the exchange of RR closed strings between the various orientifold planes and D-branes. Schematically the amplitude for such a process is

\[
\mathcal{A} = \int dl \sum <\text{boundary}_1|e^{-2\pi l(p^2+m^2)}|\text{boundary}_2>,
\]

Here \( l \) denotes the proper time along the tube, the sum is over all RR-states propagating in the loop, and the boundaries are crosscaps for orientifold planes, and normal boundaries for D-branes. In the limit \( l \to \infty \) the massless exchange diverges; the above diagram is then proportional to \( \int dl Q_1 Q_2 \), the RR-charges of
the boundaries. If we take both boundaries to be the combined state of all orientifold planes and D-branes, this is proportional to the total charge squared, which should vanish since the space transverse to the branes and planes is compact.

The various closed string exchange diagrams can be reinterpreted as open string loops. Two crosscaps give a Klein Bottle (KB) diagram, one crosscap and a boundary a Möbius strip (MS), and two boundaries a cylinder (C). The RR-channel states then correspond to the open string traces with \((-1)^F\) for the cylinder, the R-sector for the MS and R with \((-1)^F\) and NS for the KB. (Note, however, that due to the zero-modes the traces with \((-1)^F\) in the R-sector vanish). A consistency check is that if we add up all diagrams (per twisted sector) we should get a square, since then we are computing the total (orientifold and brane) charge squared.

We will use the following notation. The orientifold we are interested in has orbifold group \(\mathbb{Z}_N \times \mathbb{Z}_M\), which acts on the three complex coordinates of the six-torus as

\[
(z_1, z_2, z_3) \to (e^{\frac{2\pi im}{N}} z_1, e^{-\frac{2\pi in}{N}} z_2, z_3), \quad \text{resp.} \quad (z_1, z_2, z_3) \to (z_1, e^{\frac{2\pi im}{M}} z_2, e^{-\frac{2\pi im}{M}} z_3).
\]

(A.1)

The branes involved are nine-branes, and possibly five-branes stretched along four-dimensional spacetime plus one of the three internal tori; they are referred to as \(5_1, 5_2\), and \(5_3\). The \(\gamma\)'s carry indices \(n, m\) and possibly \(\Omega\) to denote the group element, and a 9 or 5, to indicate in which sector they act.

One has to compute the various loop amplitudes, which are traces with insertion of the different orbifold group elements, and a factor \(\Omega\) for MS and KB. The traces include a trace over the Chan-Paton indices when the diagram has a boundary on the different D-branes (i.e. for the C and MS diagrams). The results can be conveniently expressed in terms of Jacobi's \(\vartheta\)-functions (for their definition, see e.g. [7]). From supersymmetry one has that the sum of NS and R-sectors, with and without \((-1)^F\), vanishes for each diagram. In the calculation, this is a consequence of the various identities for sums of products of \(\vartheta\)-functions.

From the amplitude, one then extracts the divergent behaviour in the long tube limit. This limit coincides with the limit of the modular parameter of the open string diagram going to zero. The asymptotics can be calculated using the transformation properties of the theta-functions under modular transformations [7].

As an example we give the amplitude for the cylinder diagram. Here different cases can be distinguished: either boundary of the cylinder can lie on a nine-brane or on a five-brane. To be specific let us take both boundaries on a nine-brane. In the sector where we include the group element labelled by \((n, m)\) (so that this
twisted RR field strength propagates in the loop in the closed string tree channel perspective) we should calculate the trace

\[
\text{Tr} \left( \frac{1}{2NM} \alpha^n \beta^m \left( 1 + (-1)^F \right) e^{-2\pi t L} \right)
\]

(A.2)

The trace is over both NS and R sector states, the latter contributing with a minus sign. \(\alpha\) and \(\beta\) are the generators of the orbifold group; the factor in front is the division by the order of the orientifold group.

The total amplitude in this case is given by the expression

\[
\mathcal{A}_{99}^C(n, m) = \frac{v_4}{16NM} (\text{Tr} \gamma_{n,m,9})^2 \left( \frac{1}{8 \sin \frac{\pi n}{N} \sin \frac{\pi m}{M} \sin \pi \left( \frac{m}{M} - \frac{n}{N} \right)} \right)^2
\]

\[
\int \frac{dt}{t^3} \sin \pi \left( \frac{n}{N} - \frac{m}{M} \right) f_1(t)^{-3} \vartheta_1 \left( \frac{n}{N} | t \right)^{-1} \vartheta_1 \left( \frac{m}{M} | t \right)^{-1} \vartheta_1 \left( \frac{n}{N} - \frac{m}{M} | t \right)^{-1} \times
\]

\[
\left( \vartheta_3(0 | t) \vartheta_3 \left( \frac{n}{N} | t \right) \vartheta_3 \left( \frac{m}{M} | t \right) - \vartheta_4(0 | t) \vartheta_4 \left( \frac{n}{N} | t \right) \vartheta_4 \left( \frac{m}{M} | t \right) - \vartheta_2(0 | t) \vartheta_2 \left( \frac{n}{N} | t \right) \vartheta_2 \left( \frac{m}{M} | t \right) \right).
\]

The factor \(v_4\) in front is the regularised dimension of the non-compact directions, in string units; it arises from the integration over the non-compact momenta, which is part of the trace. The gamma-matrices reflect the action of the group element on the nine-brane Chan-Paton indices. The subsequent goniometric factor comes from the action of the group element on the compact space [7], and is equal to one over the number of fixed points. Under the integral sign we find the oscillator contributions. The inverse theta-functions represent the bosonic contributions, which appear in the denominator. Their definitions include goniometric functions that have to be cancelled, hence the three sines. The arguments of the theta-functions show the action of the group element on the associated complex coordinates. The function \(f_1\) is associated to oscillators in the uncompact directions; its definition can also be found in [7]. The expression in brackets represents the fermionic oscillators, appearing in the numerator. The first term comes from the NS-sector, the second one from the NS-sector with the inclusion of \((-1)^F\) in the trace, while the third one is the R-sector (with an additional minus sign because of the fermion loop). The total expression vanishes; the contribution of the RR exchange only is given by the second term.

The massless RR exchange is obtained by taking the limit of infinite tree channel parameter \(l\), which in the case of the cylinder is related to \(t\) as \(t = \frac{1}{2\pi l}\). This will be proportional to the nine-brane charge w.r.t. the twisted RR-potential squared.
The limiting behaviour can be obtained using the transformation properties of the Jacobi functions under modular transformations.

Similar computations were carried out for the other sectors \((5,9,5i,5j)\), as well as for the MS and KB amplitudes. For the latter two, the element \(\Omega\) is to be added in the trace; the diagrams involving the group element \(\alpha^n\beta^m\) are associated to exchange of the \(2n,2m\)-twisted RR-potential, between orientifold-plane and D-brane (in the case of the MS), or between two orientifold-planes (KB). Hence, different diagrams contribute to the various channels depending on whether \(N, M\) are odd or even. These diagrams give contributions involving the charge of the orientifold-planes.

In the end, the total charge of planes and branes with respect to all different twisted RR-potentials should vanish. This gives a constraint on the traces of the different \(\gamma\)-matrices. Let us start with the case where \(N, M\) are both even. In the odd-twisted sectors, i.e. \(n\) or \(m\) is odd, we only get a contribution from the cylinders, proportional to:

\[
\frac{1}{\sin \pi \frac{n}{N} \sin \pi \frac{m}{M} \sin \pi \left| \frac{n}{N} - \frac{m}{M} \right|} \times \\
\left[ \text{Tr} \gamma_{n,m,9} + 4 \sin \pi \frac{m}{M} \sin \pi \left( \frac{n}{N} - \frac{m}{M} \right) \text{Tr} \gamma_{n,m,51} \\
- 4 \sin \pi \frac{n}{N} \sin \pi \frac{m}{M} \text{Tr} \gamma_{n,m,52} - 4 \sin \pi \frac{n}{N} \sin \pi \left( \frac{n}{N} - \frac{m}{M} \right) \text{Tr} \gamma_{n,m,53} \right]^2
\]

In the case where both \(n\) and \(m\) are even, there are also contributions of the \(MS\) and twisted and untwisted \(KB\) amplitudes. We find a contribution proportional to

\[
\frac{1}{\sin 2\pi \frac{n}{N} \sin 2\pi \frac{m}{M} \sin 2\pi \left( \frac{n}{N} - \frac{m}{M} \right)} \times \\
\left[ \text{Tr} \gamma_{2n,2m,9} + 4 \sin 2\pi \frac{m}{M} \sin 2\pi \left( \frac{n}{N} - \frac{m}{M} \right) \text{Tr} \gamma_{2n,2m,51} \\
- 4 \sin 2\pi \frac{n}{N} \sin 2\pi \frac{m}{M} \text{Tr} \gamma_{2n,2m,52} - 4 \sin 2\pi \frac{n}{N} \sin 2\pi \left( \frac{n}{N} - \frac{m}{M} \right) \text{Tr} \gamma_{2n,2m,53} \\
- 32 \left( \cos 2\pi \frac{n}{N} \cos^2 \pi \frac{m}{M} - \sin^2 \pi \frac{m}{M} + \frac{1}{2} \sin 2\pi \frac{n}{N} \sin 2\pi \frac{m}{M} \right) \right]^2.
\]

For this we used that

\[
\text{Tr} \left( \gamma_{m,n,\Omega} \gamma_{m,n,\Omega}^T \right) = \pm \text{Tr} \gamma_{2m,2n}.
\]

When \(n = 0\) we have to include momentum and winding states along the \(4-5\)-torus; the result then is

\[
\frac{1}{\sin^2 \pi \frac{m}{M}} \left[ \text{Tr} \gamma_{0,m,9} - 4 \sin^2 \pi \frac{m}{M} \text{Tr} \gamma_{0,m,51} \right]^2
\]
and
\[
[\text{Tr}\gamma_{0,m,5_2} - \text{Tr}\gamma_{0,m,5_3}]^2. \tag{A.6}
\]
for odd \(m\), while for even \(m\) we find
\[
\frac{1}{\sin^2 2\pi \frac{m}{M}} \left[\text{Tr}\gamma_{0,2m,9} - 4 \sin^2 2\pi \frac{m}{M} \text{Tr}\gamma_{0,2m,5_1} - 32 \cos 2\pi \frac{m}{M}\right]^2, \tag{A.7}
\]
and
\[
[\text{Tr}\gamma_{0,2m,5_2} - \text{Tr}\gamma_{0,2m,5_3}]^2. \tag{A.8}
\]
Similar expressions are found for the \(m = 0\) and \(m = n\) sectors. Finally, the expression for \(m = n = 0\) guarantees that there are 32 branes of each kind.

The result when \(M, N\) are both odd (so we have no five-branes, just nine-branes) is
\[
\frac{1}{\sin 2\pi \frac{n}{N} \sin 2\pi \frac{m}{N} \sin 2\pi \left(\frac{n}{N} - \frac{m}{M}\right)} \left[\text{Tr}\gamma_{2n,2m,9} - 32 \cos \pi \frac{n}{N} \cos \pi \frac{m}{M} \cos \pi \left(\frac{n}{N} - \frac{m}{M}\right)\right]^2. \tag{A.9}
\]
Again for \(n = 0\) we obtain
\[
\frac{1}{\sin^2 2\pi \frac{m}{M}} \left[\text{Tr}\gamma_{0,2m,9} - 4 \sin^2 2\pi \frac{m}{M} \text{Tr}\gamma_{0,2m,5_1} - 32 \cos^2 \pi \frac{m}{M}\right]^2, \tag{A.10}
\]
while the \(n = m = 0\) result tells us that there are 32 nine-branes and no five-branes.

Finally, when \(N\) is odd, \(M\) even, we have, for odd \(m\)
\[
\frac{1}{\sin \pi \frac{n}{N} \sin \pi \frac{m}{M} \sin \pi \left|\frac{n}{N} - \frac{m}{M}\right|} \left[\text{Tr}\gamma_{n,m,9} + 4 \sin \pi \frac{m}{M} \sin \pi \left(\frac{n}{N} - \frac{m}{M}\right) \text{Tr}\gamma_{n,m,5_1}\right]^2, \tag{A.11}
\]
and for even \(m\)
\[
\frac{1}{\sin 2\pi \frac{n}{N} \sin 2\pi \frac{m}{M} \sin 2\pi \left(\frac{n}{N} - \frac{m}{M}\right)} \times
\left[\text{Tr}\gamma_{2n,2m,9} + 4 \sin 2\pi \frac{m}{M} \sin 2\pi \left(\frac{n}{N} - \frac{m}{M}\right) \text{Tr}\gamma_{2n,2m,5_1}
- 32 \cos \pi \frac{n}{N} \cos \pi \frac{m}{M} \cos \pi \left(\frac{n}{N} - \frac{m}{M}\right) \cos \pi \frac{n}{N} \sin \pi \frac{m}{M} \sin \pi \left(\frac{n}{N} - \frac{m}{M}\right)\right]^2. \tag{A.12}
\]
If \(n = 0\) the answer is that of the even \(N, M\) situation, eqs. \((A.3) - (A.8)\), while for \(m = 0\) we find
\[
\frac{1}{\sin^2 2\pi \frac{n}{N}} \left[\text{Tr}\gamma_{2n,0,9} - 32 \cos^2 \pi \frac{n}{N}\right]^2 \tag{A.13}
\]
and
\[
[\text{Tr}\gamma_{2n,0,5_1} - 8]^2. \tag{A.14}
\]
Finally we again should have 32 nine- and 51-branes for the untwisted channel cancellation.

Requiring all these expressions to vanish, one obtains the tadpole equations. The solutions to these equations are presented in section 2.3.

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