Nucleon Mass Corrections to the $p \leftrightarrow n$ Rates During Big Bang Nucleosynthesis

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Abstract

The thermal rates for converting neutrons to protons, and vice versa, are calculated, including corrections of order 1 MeV divided by a nucleon mass. The results imply that the primodial helium abundance predicted for big bang nucleosynthesis has been systematically underestimated by about $\Delta Y_4 = 0.0012$, i.e., $\Delta Y_4/Y_4 \approx .005$. 
1 Introduction

The purpose of this paper is to evaluate nucleon mass corrections to the rate of weak transitions that interconvert neutrons and protons during the early stages of big bang nucleosynthesis. In the usual calculation of these rates the nucleon mass is ignored; i.e. one includes all energies and momenta in the MeV range, specifically, ratios of the electron mass, $m_e$, the temperature, $T$, and the neutron-proton mass difference, $Q = m_n - m_p = 1.2933$ MeV; but factors such as $Q/m_N$, i.e., a low energy scale divided by a nucleon mass, are ignored. These factors are individually of order a tenth of a percent, but it will be shown that together they cause roughly a 0.5% increase in the helium abundance predicted by big bang nucleosynthesis calculations. Such a systematic correction is significant in that it is comparable to the largest uncertainty in the standard hot big bang calculation - that due to uncertainty in the neutron halflife. Further, the increase in the predicted helium abundance translates into a tighter constraint on the density of baryons as well as a strengthening of particle constraints based on big bang nucleosynthesis - such as the limit on the number of neutrino species that may be in thermal equilibrium in the early Universe.

As an example of the sort of effect that is usually ignored, consider the neutron to proton abundance ratio in thermal equilibrium. Including the first correction in an expansion in inverse powers of $m_N$ this ratio is

$$\frac{x_n}{x_p} \approx e^{-Q/T} \left( \frac{m_n}{m_p} \right)^{3/2} \approx e^{-Q/T} (1 + 1.5 \frac{Q}{m_N}) = 1.00207e^{-Q/T}. \quad (1)$$

Usually one includes just the ‘Boltzman factor’ and ignores the correction, which is small, 0.2%. Thus, even if freeze out of the weak reactions occurred at the same time, one might expect the neutron abundance to be slightly higher if nucleon mass corrections were included.

Of course, it is essential that the neutron fraction drops out of thermal equilibrium as the weak reactions become slow compared to the expansion rate of the Universe, and so one does not calculate the neutron abundance by equilibrium arguments in a numerical calculation. Instead one evaluates the rates for $p \leftrightarrow n$ conversions and tracks carefully the
maintenance of approximate equilibrium at high temperatures and the failure to maintain equilibrium at low temperatures. It is, therefore, necessary to evaluate the change in the rates - not just the equilibrium neutron fraction. There are many $1/m_N$ corrections to the $p \leftrightarrow n$ rates and it is the purpose of this paper to enumerate and evaluate them in a systematic fashion.

The spirit of this paper is similar to those which evaluated the electromagnetic radiative, thermal, and coulomb corrections to the $p \leftrightarrow n$ processes. In both cases, the corrections are a few percent at most. To achieve a satisfactory level of accuracy, one part in a thousand, it is necessary to evaluate only the first correction, but not effects of order $1/m_N^2$ or, in the electromagnetic case, of order $\alpha^2$. Nor is it necessary to consider terms of order $\alpha/m_N$.

Another similarity in the two problems concerns the normalization of the corrections. The nucleosynthesis numerical codes typically normalize the weak rates to the experimental value of the neutron mean lifetime, $\tau_n = 889.1 \pm 2.1$ sec. Thus, when evaluating a purported correction to the rates one must also evaluate the same sort of corrections for neutron decay, and adjust the corrections appropriate for BBN accordingly. So, for example, the largest term in the order $\alpha$ radiative correction to the weak rates is a constant which also shows up in neutron decay. Similarly, a good part of the coulomb correction to the weak rates also cancels. Thus, the early numerical code of Wagoner contained a simple coulomb correction and no radiative correction, but although individual reactions have corrections of $\sim 5\%$, the net effect of a more detailed treatment results in less than a 1% correction to Wagoner’s results. In contrast, the $1/m_N$ corrections are of order 1% to the reaction rates, but the comparable correction to neutron decay is smaller due to kinematic thresholds. As a result, nearly the whole of the effects discussed here survive to affect the helium abundance.

With these thoughts in mind the rest of the paper is ordered as follows. In section 2, the main results are presented - the corrections to the $p \leftrightarrow n$ rates to first order in $1/m_N$. In section 3, similar effects are considered for neutron decay. Section 4 combines the results from the previous two sections to arrive at an expected change in the helium abundance. Section 5 contains a discussion of the significance of the results.
First, however, it may be useful to the reader to clarify some of the notation used later. Except where the neutron or proton mass is explicitly indicated by $m_n$ or $m_p$, the nucleon mass is given as $m_N$. In the formulae for cross-sections, rates, etc., $m_N$ refers to the initial nucleon mass, but to the extent that the formulae are only accurate to $1/m_N$ it makes no difference which nucleon mass is actually used. Also, $E_1$ and $k_1$ are the energy and momentum of the initial lepton in the rest frame of the fluid. Unless specifically noted, the energy $E_3$ denotes the quantity $E_1 + dQ$, where $dQ = \pm Q$. This is only equal to the outgoing lepton energy in the infinite mass limit, $m_N \to \infty$. $k_3$ is the corresponding momentum, $k_3 = (E_3^2 + m_3^2)^{1/2}$. During nucleosynthesis the temperature describing the neutrino distribution, $T_\nu$, is not equal to the temperature of the rest of the plasma (including the nucleons), denoted by $T_\gamma$. When the temperature $T_3$ is used, it refers to the temperature which describes the outgoing lepton.

2 Corrections to Scattering Processes

There are six processes that contribute to $p \leftrightarrow n$ conversion in the early Universe; neutron decay $n \rightarrow \bar{\nu}_e e^- p$, inverse neutron decay $e^- \nu_e p \rightarrow n$, and four scattering processes, $\nu_e n \rightarrow e^- p$, $e^- p \rightarrow \nu_e n$, $\nu_e p \rightarrow e^+ n$, and $e^+ n \rightarrow \bar{\nu}_e p$. The most critical time is when these reactions are ‘freezing out’, i.e., when they are just failing to maintain thermal equilibrium. This occurs at a temperature $T_F \approx 0.8$ MeV. At that time the scattering processes dominate over neutron decay and inverse decay by a factor of about 1000. The reactions which convert neutrons to protons are some 6 times greater than the inverse reactions, due to the nuclear mass difference affecting phase space. The reactions involving antileptons are nearly equal in importance to those involving leptons. To achieve an accuracy of 0.1% it therefore seems sufficient to consider just corrections to the scattering rates; however, because of the role played by neutron decay in normalizing the weak rates those corrections must also be examined. Accordingly, this section presents corrections to scattering and the next examines neutron decay.
The rate for two body scattering reactions in a medium may be written in the form

\[ \Gamma(12 \rightarrow 34) = \left( \prod_i \int \frac{d^3k_i}{(2\pi)^32E_i} \right) (2\pi)^4 \delta^4(\sum_i p_i) |M|^2 n_1 n_2 (1 - n_3)(1 - n_4), \]  

where \( p_i \) is the four momentum, \( k_i \) is the three momentum, \( E_i \) is the energy of each particle, and for the problem at hand all particles obey Fermi statistics. The occupation numbers \( n_i \) take thermal equilibrium values only for those species actually in equilibrium. In this notation the squared matrix element, \( |M|^2 \), has been summed over all spin degrees of freedom and it is assumed that the \( n_i \) do not depend on spin. The presence or absence of right-handed neutrinos is irrelevant for the evaluation of the scattering rates. For the reactions of interest, let particles 2 and 4 be the in and out nucleons, respectively, and let particles 1 and 3 be the leptons or anti-leptons.

### 2.1 The infinite mass limit

Before going into the details of nucleon mass corrections it is appropriate to evaluate the reaction rates in the limit of infinite nucleon mass. In this limit the energies of the in and out leptons are related by \( E_1 + dQ = E_3 \), where \( dQ = \pm Q \) depending on whether a neutron or proton is the initial nucleon. Further, neither lepton occupation number will depend upon the scattering angles. The rate can then be rewritten in the familiar form

\[ \Gamma(12 \rightarrow 34) = \int \frac{d^3k_1}{(2\pi)^3} \frac{d^3k_2}{(2\pi)^3} \sigma_{\text{rel}} n_1 n_2 (1 - n_3)(1 - n_4), \]  

where \( \sigma \) is the cross-section for the reaction summed over both initial and final spins and \( v_{\text{rel}} \) is the relative velocity of the two initial particles, which for infinite mass nucleons may be taken to be just the initial lepton velocity, \( v_1 \). It is useful to concentrate on the rate per initial state in the absence of blocking, \( \gamma \equiv \sigma_{\text{rel}} v_1 \). For infinite mass nucleons, this quantity becomes

\[ \sigma v_{\text{rel}} \to f_\alpha(\sigma_0 v_1) = f_\alpha \gamma_0 \frac{2G_f^2 \cos^2 \theta_c (1 + 3c_s^2)}{\pi} E_3 k_3 = f_\alpha A E_3 k_3, \]  

where \( \sigma_0 \) is the cross-section for a lepton of energy \( E_1 \) incident on an infinitely heavy nucleon, \( G_f = 1.1164 \times 10^{-5} \text{ GeV}^{-2} \) is Fermi’s constant, \( \theta_c \) is the Cabibbo angle, with
\[ \cos \theta_c = 0.975, \quad \text{and} \quad c_a \equiv \frac{g_A}{g_V} = 1.257 \]

is the ratio of the axial vector to vector coupling of the nucleon for charged currents. The last relation in Eq. 4 serves as a definition of the constant \( A \), a factor which will be common to all the weak reactions. The coulomb and radiative corrections to the rates are embodied in an electromagnetic correction factor \( f_\alpha \); however, as explained in the introduction, all corrections to the weak rates are small and may be treated independently. It is therefore acceptable to ignore \( f_\alpha \) except when worrying about the overall normalization of the rates, and so the factor \( f_\alpha \) will be dropped.

For heavy nucleons, low baryon density and low lepton asymmetry, it is appropriate to approximate \( n_2 \) by a Boltzmann distribution and ignore \( n_4 \) entirely. Integrating over nucleon momentum and lepton direction, converting the lepton momentum integral to one over energy, and using thermal distributions for the leptons one gets the rate,

\[
\Gamma(12 \rightarrow 34) = \int dE_1 \frac{dT}{dE_1} = \frac{N_2}{4\pi^2} \int dE_1 \frac{E_1 k_1 \gamma_0}{(1 + e^{E_1/T_1})(1 + e^{-E_3/T_3})} = \frac{A N_2}{4\pi^2} \int dE_1 \frac{E_1 k_1 E_3 k_3}{(1 + e^{E_1/T_1})(1 + e^{-E_3/T_3})},
\]

where \( N_2 \) is the spatial density of initial nucleons, and \( T_i \) is the temperature describing lepton \( i \). Eq. 5 defines the differential interaction rate \( \frac{dT}{dE_1} \), which is plotted in Fig. 1 for each of the four scattering processes. The plots were generated using temperatures \( T_\gamma = 0.8 \text{ MeV} \), and \( T_\nu = 0.07926 \text{ MeV} \). This is near the conventional “freezeout temperature”, \( i.e. \) that temperature where the equilibrium abundances are equal to the final values, as if the interactions were very rapid and then turned off abruptly. The freezeout point is high enough that electron annihilation has caused the photon temperature to increase by only a small amount over the neutrino temperature. Note that the \( n \rightarrow p \) rates in Fig. 1 are some 6 times greater than the \( p \rightarrow n \) rates, as required to maintain equilibrium at this temperature.

To give a better feel for the important points in determining the neutron fraction, Fig. 2 shows the integrated scattering rates, \( \Gamma \), for the four scattering processes as a function of the photon temperature, \( T_\gamma \). The expansion rate, \( H(T_\gamma) \), is also shown; along with the
free neutron decay rate. The $p \to n$ reactions freezeout first, and become increasingly unimportant at lower temperatures. The $n \to p$ scattering rates freeze out later. They are more important than free neutron decay down to a temperature $\sim 0.2$ MeV, but what really counts is the comparison to $H$. After $T_\gamma \approx 0.5$ MeV the most significant comparison to $H$ is free neutron decay just at the time the “deuterium bottleneck” breaks, at which time $T_\gamma \approx 0.07$ MeV. Corrections to the scattering rates for $T_\gamma \lesssim 0.5$ MeV are not very important.

Apart from electromagnetic corrections, Eq. 5 is the reaction rate used in nucleosynthesis calculations. There are several points where infinite mass nucleons were used. Merely writing the reaction in the form of a cross-section required that the final state occupation numbers did not depend on the scattering angles, and this depends on the assumption that no recoil energy goes to the nucleon. The vector and axial vector cross-section, Eq. 4, has corrections of order $1/m_N$. In addition, the vector and axial vector Lagrangian must be corrected for nucleon structure effects, such as momentum dependent form factors, or new terms in the effective low energy effective Lagrangian, such as ‘weak magnetism’. The relative velocity, $v_{rel}$, must be corrected for the nucleon velocity. One must average over the Boltzmann distribution, $n_2$, to produce a ‘thermal averaged’ cross-section $\times$ relative velocity $\times$ blocking effects due to the Fermi statistics. Since the nucleon velocity $v_{nuc}$ is of order $(T_\gamma/m_N)^{1/2}$ one must expand in the nuclear velocity to second order to get corrections to first order in the nucleon mass. One implication of this is that there may be correlations between corrections that are first order in $\vec{v}_{nuc}$. Although first order terms vanish when angle averaged their correlations may not, and can therefore contribute at first order in $1/m_N$.

As presented here, these calculations are done by evaluating corrections to the rates, $\gamma = \sigma v_{rel}$, and the blocking factors $f_b = 1 - n_3$, as a function of the initial lepton energy. After taking appropriate angular combinations, the corrections are integrated over lepton energy to produce corrections to the conversion rates per nucleon, which may be used in the rate equations to solve for the neutron abundance as a function of time.
The most obvious corrections to consider are $1/m_N$ corrections to the cross-section, which are combined with the zeroth order, or infinite nucleon mass, values for $v_{rel}$ and $f_b$. The $1/m_N$ corrections to $\sigma$ will be calculated in section 2.2. When the nucleon velocity is taken into account there will be corrections to $\sigma$ due to the altered lepton energy, as well as corrections intrinsic to $v_{rel}$. These are evaluated in section 2.3 along with the corrections to the blocking factors. These corrections and their correlations will be expressed as effective corrections to the rate $\gamma_0$, which can be multiplied by the zeroth order $v_{rel}$ and $f_b$.

As a preliminary to this, consider the differential cross-section to order $1/m_N$,

$$4\pi \frac{d\sigma}{d\Omega} = \sigma_0 + \sigma_1 + \sigma_\alpha \cos \alpha.$$  \hspace{1cm} (6)

The zeroth order total cross-section, $\sigma_0$, is given by Eq. 4. The $1/m_N$ correction to the total cross-section is $\sigma_1$. The relevant angular dependence of the differential cross-section is given by $\sigma_\alpha$, which is to be multiplied by $\cos \alpha$ with $\alpha$ being the center of mass scattering angle. Terms higher order in $\cos \alpha$ are suppressed by two powers of $m_N$ (or powers of $G_f$) and may be dropped. If there were no corrections to the blocking factors for the final state leptons $\sigma_\alpha$ would integrate to zero when averaged over scattering angle; however, there are corrections to the blocking factors. Since these are suppressed by factors of $m_N$ one need only keep the zeroth order term of $\sigma_\alpha$,

$$\sigma_\alpha = \sigma_0 \left( \frac{1 - c^2_a}{1 + 3c^2_a} \right) \frac{k_1 k_3}{E_1 E_3}.  \hspace{1cm} (7)$$

Discussion of the corrections to the rates due to $\sigma_\alpha$ is postponed till later, after evaluating the corrections to the lepton blocking factors in section 2.3.

### 2.2 Corrections to the cross-section

There are two important corrections in $\sigma_1$, one that arises from including the weak magnetism term in the interaction, and one that arises from modifications to the final state phase space due to the recoil of the nucleon. They may be treated independently to first order in $m_N$. 


The effective low energy weak Lagrangian is,

$$\mathcal{L}_w = \frac{G_f}{\sqrt{2}} J^\mu_{lep} J_{\text{had}, \mu},$$

where the leptonic current has the usual $V - A$ structure and the hadronic weak current is given by

$$J^\mu_{\text{had}} = \cos \theta c \bar{\psi}_N \left( \gamma^\mu \left( 1 - c_a \gamma_5 \right) + \frac{f_2}{m_N} \sigma^{\mu\nu} q_{\nu} + f_{ps} \gamma_5 q^\mu \right) \psi_N,$$

where $f_2 = 1.81$ is the anomalous weak charged current magnetic moment of the nucleon, $f_{ps}$ is the pseudoscalar coupling to the nucleon, and $q$ is the momentum transfer to the nucleon. At higher energies, one would treat the couplings $g_A$, $g_V$, $f_2$, and $f_{ps}$ as form factors with corrections of order $q^2/M_i^2$, where the $M_i$ differ for the different interactions and are experimentally determined to be in the range 500 – 1000 MeV. Thus, the form factor corrections may reasonably be assumed to be higher order than the $1/m_N$ corrections considered in this paper.

The full squared matrix element for scattering with the current in Eq. 9 is given in Appendix A, but here only the relevant terms are kept. The pseudoscalar coupling is usually approximated by the pion pole term. At low momentum transfer this leads to a suppression of the amplitude by a factor of $\sim g_{\pi NN} m_e q^2 / (m_\pi^2 f_\pi)$, where $g_{\pi NN}$ is the $\pi$-nucleon coupling and $f_\pi$ is the pion decay constant. Since this is small it is dropped from further discussion. Weak magnetism is generated by the $f_2$ term. There is an explicit factor of $1/m_N$ in the coupling, so one may ignore the square of the weak magnetism term, but there may be interference between weak magnetism and the vector and axial vector interactions. The interference with the vector interaction vanishes at order $1/m_N$, which leaves just a correction proportional to $c_a f_2$,

$$\gamma_{wm} = \sigma_{wm} v_1 = \gamma_0 \frac{c_a f_2}{1 + 3 c_a^2} \frac{2(E_3 k_1^2 + E_1 k_3^2)}{E_1 E_3 m_N},$$

where the zeroth order $v_{rel}$ is acceptable since there is already one power of $1/m_N$ in the correction $\sigma_{wm}$.

Next, consider the $1/m_N$ corrections to the usual $V$ plus $A$ interactions. These will be referred to collectively as the ‘recoil’ correction, since a major component of the correction
may be understood as a reduction in the phase space for the outgoing lepton due to the
energy carried off by the nucleon. The correction is calculated in the frame of the target
nucleon by 1) expressing the differential cross-section in terms of the invariants \( s, t, \) and the
particle masses, 2) expressing \( s \) and \( t \) in terms of the incident lepton energy, 3) integrating
over phase space, and 4) extracting all terms to the required power of \( 1/m_N. \) One must
keep the full expression for \( s, s = m_2^2 + 2E_1m_2 + m_1^2, \) since the leading part of \( s \) cancels in
some parts of the calculation. The invariant \( t \) may be written as \( t = t_0 + \delta t \cos \alpha, \) where to
first order in \( 1/m_N \)

\[
t_0 = -2E_1E_3 + m_1^2 + m_3^2 + \frac{2k_1^2E_3 + E_1(k_1^2 + k_3^2)}{m_N},
\]

\[
\delta t = 2k_1k_3 - \frac{k_1(2E_1k_3^2 + E_3(k_1^2 + k_3^2))}{k_3m_N}.
\]

Integrating over \( dt, \) keeping just the term proportional to \( m_N, \) and applying the appropriate
normalization yields

\[
\gamma_{rec} = \frac{1}{f_2} \gamma_{wm} - \gamma_0 \frac{(2E_1k_3^2 + E_3(k_1^2 + k_3^2))}{2k_3^2m_N} + \gamma_0 \frac{1}{1 + 3c_a^2} \frac{(m_1^2 - m_3^2 - Q^2)}{2E_3m_N}
\]

\[
+ \gamma_0 \frac{c_a^2}{1 + 3c_a^2} \frac{(6E_1^2E_3 - 6E_1E_3^2 - 3E_1k_1^2 - 4E_3k_1^2 - E_1k_3^2)}{2E_1E_3m_N},
\]

(12)

Note that the interference between the A and V currents has exactly the same structure as
that between the axial vector and weak magnetism interactions.

### 2.3 Thermal averages of \( \sigma v_{rel} \) and \( f_b \)

For the remaining corrections one must perform averages over scattering angle and/or ther-
mal averages over the nucleon momentum. The strategy presented here is to evaluate these
corrections separately for the lepton blocking factor \( f_b \), and for the product \( \sigma v_{rel}. \) Each is
developed as a power series in the cosines of the scattering angle \( \alpha \) and of the incident angle
of the initial lepton momentum relative to the nucleon momentum, labeled by \( \theta. \) It is only
necessary to include terms up to \( \cos^2 \theta, \) since each factor of \( \cos \theta \) comes accompanied by the
nucleon velocity, which is of order \( \sqrt{T_{\gamma}/m_N}. \) Further, terms first order in \( \cos \theta \) and \( \cos \alpha \)
integrate to zero and may be dropped, although only after the two series are multiplied together to pick up the angular correlations between the corrections to $\sigma_{v_{\text{rel}}}$ and $f_b$.

The thermal averaged $\sigma_{v_{\text{rel}}}$ for a lepton of energy $E_1$ is given by

$$\langle \sigma_{v_{\text{rel}}} \rangle = \frac{\int k_2^2 dk_2 d(\cos \theta)}{4\pi^2} \sigma_{v_{\text{rel}} n_2}. \tag{13}$$

Eq. 4 can be used for $\sigma_{v_{\text{rel}}}$ with just two changes. First, one must use the lepton energy in the nucleon rest frame,

$$E_1' = E_1 \gamma (1 - v_1 v_{\text{nuc}} \cos \theta), \tag{14}$$

where here $\gamma = 1/(1 - v_{\text{nuc}}^2)^{(1/2)}$ is the relativistic $\gamma$ factor for the initial nucleon. Second, $\sigma_{v_{\text{rel}}}$ must be multiplied by a factor

$$v'_{\text{rel}} = 1 - v_1 v_{\text{nuc}} \cos \theta, \tag{15}$$

to account for the change in lepton flux seen in the nucleon rest frame. The thermal average is then done by expanding in powers of $v_{\text{nuc}}$ and replacing $v_{\text{nuc}}^2$ by its thermal average, $v_{\text{nuc}}^2 \rightarrow 3T\gamma/m_N$. This procedure is totally equivalent to the more standard practice of using the Lorentz invariant cross-section with $E_1'$ and using the Lorentz invariant flux factor

$$v_{\text{rel}} = \left( (\vec{v}_1 - \vec{v}_{\text{nuc}})^2 - (\vec{v}_1 \times \vec{v}_{\text{nuc}})^2 \right)^{1/2}. \tag{16}$$

However, a difficulty arises in the use of Eq. 14. When $v_{\text{rel}}$ is expanded in terms of $v_{\text{nuc}}$, terms of order $v_{\text{nuc}}/v_1$ are generated, but there is a region of phase space where the lepton velocity is small compared to $v_{\text{nuc}}$, and this expansion is not valid. Using the rest frame $\sigma_{v_{\text{rel}}}$ and Eq. 17 avoids this problem for the incident lepton velocity.

The result of performing the thermal average is an effective correction to $\sigma_{v_{\text{rel}}}$ for incident lepton energy $E_1$,

$$\gamma_{th,0} = \gamma_0 \frac{T}{m_N} \left( \frac{3E_1^2 + 2k_1^2}{2E_1E_3} + \frac{3k_1^2 E_1 + 3E_1^2 E_3 + 2k_1^2 E_3}{2E_1 k_3^2} - \frac{k_1^2 E_3^2}{2k_3^2} \right). \tag{17}$$

The last term in Eq. 17 presents a problem akin to that just discussed concerning $v_{\text{rel}}$; namely, when the reaction energy is near threshold the final state lepton velocity will be
small if that lepton is massive, i.e. it is an electron or positron. This is not a problem for the reaction $\nu_e n \rightarrow e^- p$ since the positive $Q$ value always keeps the electron energy well above threshold, but it is a problem for $\nu_e p \rightarrow e^+ n$. It is also not a difficulty for reactions with final state neutrinos since then $k_3 = E_3$.

The anomalous powers of $k_3$ are symptomatic of a deeper problem with the thermal averaging. The averaging procedure adopted here is only valid when the change in outgoing lepton momentum due to nuclear mass effects is small compared to its value when the nucleon mass is taken to infinity. This is not true near threshold\cite{11,12}, where $k_3 \rightarrow 0$. As an example, consider an incident lepton whose energy is the threshold energy for a nucleon at rest. Then, for those nucleons moving with $\cos \theta < 0$ the effective reaction energy is above threshold. Thus, after thermal averaging, the threshold should no longer be sharp. Fortunately, the reaction rates are not dominated by the behavior near threshold, since phase space vanishes there. The error introduced by the adopted procedure seems to be acceptably small, as will be discussed later.

Now consider the lepton blocking factor $f_b$. Assuming that the leptons are in thermal equilibrium (see Dodelson and Turner\cite{13} for a discussion of this point) the blocking factor is

$$f_b = 1 - n(E'_3) = \frac{1}{1 + e^{-E'_3/T_3}},$$

where $E'_3 \neq E_3$ is the true energy of the outgoing lepton. The factor $f_b$ depends only on the energy of the outgoing lepton. Unfortunately, $E'_3$ is a function of both the scattering angles and the relative motion of the initial lepton and nucleon, so an integration over all of phase space is unavoidable.

The relevant corrections to the blocking factor are derived in Appendix B and given in Eq. B.6. There are four corrections, organized by factors of $\cos \alpha$ and $\cos \theta$, that constitute the blocking factor up to order $1/m_N$. The corrections are normalized by $f_{b,0}$, the zeroth order term, so that the full blocking factor is

$$f_b = f_{b,0}(1 + f_{b,1} + f_{b,\theta} + f_{b,\alpha} + f_{b,\alpha\theta}).$$

$f_{b,1}$ is first order in $1/m_N$ and should be combined only with the zeroth order part of $\sigma v_{rel}$.
to produce an effective correction to the cross-section

$$\gamma_{b} = \gamma_{0} f_{b,1},$$

(20)

which generates a correction to the rate when integrated over incident lepton energy. The next term $f_{b,\theta}$ is of order $1/\sqrt{m_{N}}$ and proportional to $\cos \theta$. When combined with the $\cos \theta$ correction to $\sigma_{\nu_{e}l_{\nu}}$ due to thermal averaging, an effective $1/m_{N}$ correction is produced

$$\gamma_{th,\theta} = \gamma_{0} \frac{k_{1}^{2}(E_{1}^{2} + k_{2}^{2}) + E_{3}k_{3}^{2}}{E_{1}E_{3}k_{3}m_{N}(1 + e^{E_{3}/T_{3}})} T_{3}.$$  

(21)

Finally there are two pieces that are proportional to $\cos \alpha$, which must be combined with the $\cos \alpha$ dependent part of the differential cross-section, $\sigma_{\alpha}$. The first piece, $f_{b,\alpha\theta}$ is proportional to $\cos \theta/\sqrt{m_{N}}$ and must be combined with only the $\cos \theta$ part of $\sigma_{\alpha \nu_{e}l_{\nu}}$ to yield a correction

$$\gamma_{\alpha,\theta} = \gamma_{0} \frac{c_{a}^{2} - 1}{1 + 3c_{a}^{2}} \frac{2E_{3}k_{1}^{2} + E_{1}k_{3}^{2}}{3E_{1}E_{3}m_{N}(1 + e^{E_{3}/T_{3}})} T_{3}.$$  

(22)

The second piece, $f_{b,\alpha}$ is angle independent but of order $1/m_{N}$ and is combined only with the leading piece of $\sigma_{\alpha}$ to yield a second correction

$$\gamma_{\alpha} = \gamma_{0} \frac{c_{a}^{2} - 1}{1 + 3c_{a}^{2}} \frac{k_{1}k_{3}}{3E_{1}E_{3}m_{N}} \left( \frac{k_{1}^{2}k_{3}^{2} - (E_{3}k_{1}^{2} + 2E_{1}k_{3}^{2})T_{3}}{k_{1}k_{3}T_{3}T_{3}(1 + e^{E_{3}/T_{3}})} - \frac{(e^{E_{3}/T_{3}} - 1)k_{1}k_{3}T_{3}}{T_{3}^{2}(1 + e^{E_{3}/T_{3}})^{2}} \right).$$  

(23)

In the following section, these two terms are combined to form a single correction, $\gamma_{\alpha,\text{tot}}$.

2.4 Results for the corrected rates

In the previous two sections six corrections to the weak $p \leftrightarrow n$ rates that are formerly of order $1/m_{N}$ were identified: $\gamma_{wm}$, $\gamma_{rec}$, $\gamma_{th,0}$, $\gamma_{b}$, $\gamma_{th,\theta}$, and $\gamma_{\alpha,\text{tot}}$; which should be combined with $f_{b,0}$ and integrated over $E_{1}$ to produce corrections to the rates. Fig. 3 shows a plot of $\gamma_{i}/\gamma_{0}$ for each of the six corrections to the reaction $\nu_{e}n \rightarrow e^{-}p$, at $T_{\gamma} = 0.8$ MeV.

First, consider the three small terms $\gamma_{th,\theta}$, $\gamma_{b}$, and $\gamma_{\alpha,\text{tot}}$, which have all been exagerated by a factor of 100 in the figure. Clearly, these three terms are much smaller than the other three. The main reason for this is easy to understand. Due to the $E^{2}$ dependence of the cross-section and powers of $E$ in phase space, the rates are dominated by leptons with $E \sim 5T$. At this point the blocking factors are small, and corrections to them are even
smaller. This can be seen explicitly in Appendix B, where it is shown that each correction to $f_b$ carries at least one extra factor of $(1 + e^{E_b/T})^{-1}$. In addition, the proliferation of terms in the expansions leading to $\gamma_{wm}$, $\gamma_{\text{rec}}$, and $\gamma_{\text{th},0}$ is greater than for the terms associated with the blocking factors. A third factor suppresses $\gamma_{\alpha,tot}$, namely that it is proportional to $1 - c_a^2$, which is numerically about a tenth of $1 + 3c_a^2$ which comes into the thermal corrections. For all these reasons, the three small corrections are dropped from most of the discussion that follows.

Now turn to the three larger corrections, beginning with that for weak magnetism. Fig. 4 shows $\gamma_{wm}$ weighted by phase space considerations to produce a differential interaction rate per baryon, $d\Gamma_{wm}/dE_1$, that can be found by substituting $\gamma_{wm}$ for $\gamma_0$ in Eq. 5. The scale for this graph should be compared to Fig. 1. The corrections to each reaction are of order 1% at $T_\gamma = 0.8$ MeV, but apart from small contributions near thresholds one can see that there is an almost exact cancellation between the lepton reactions ($\nu_e n \rightarrow e^- p$ and $e^- p \rightarrow \nu_e n$) and the anti-lepton interactions ($e^+ n \rightarrow \overline{\nu}_e p$ and $\overline{\nu}_e p \rightarrow e^+ n$). This is due to an effective change in sign for the value of $c_a$ when considering leptonic and antileptonic scattering, i.e., the anti-leptonic current is right handed. Thus, although the corrections are large for each of the individual reactions, the net effect on nucleosynthesis due to weak magnetism is fairly small. It is not, however, totally negligible. There are differences in the phase space details for the different channels, and the neutrino temperature is in fact less than the electron temperature. As a result, when the photon temperature is $0.5 < T_\gamma < 2$ MeV the $e^+ n \rightarrow \overline{\nu}_e p$ channel is slightly more important than the $\nu_e n \rightarrow e^- p$ channel, and weak magnetism causes a small decrease in $\Gamma_{n\rightarrow p}$. This, in turn, causes a slight increase in $x_n$. At cooler temperatures, the electron density drops and the $e^+ n \rightarrow \overline{\nu}_e p$ channel becomes insignificant, but by then the $\nu_e n \rightarrow e^- p$ channel is also small and the weak magnetism corrections are not important then.

Next, consider Fig. 5 which shows the correction to the differential reaction rate due to recoil effects, $\gamma_{\text{rec}}$. Here the sign of the effect is the same for all reactions. The final state phase space for the outgoing lepton is reduced and this causes a reduction in the
cross-sections at $T = 0.8$ MeV of about 1%. The magnitude of the reduction increases with temperature. This can be seen by examining the $\gamma_{rec}$ curve in Fig. 3 where the fractional increase in the recoil effect is seen to increase approximately linearly with energy. When weighted by a thermal distribution the fractional change in rate will increase with $T$.

Even though all the reactions are affected in a similar way that does not imply that there will be no effect on nucleosynthesis. Since all the rates are reduced, freezeout of the neutron-proton ratio will take place a little earlier, when the neutron abundance is higher. As a result there will be more helium. Further, the rates are not reduced in proportion to the zeroth order rates, so there may be a shift in $x_n/x_p$ even at high temperatures, when the rates are fast. These effects will be discussed further in section 4.

The third important correction is that due to thermal averaging over the nucleon momentum, illustrated in Fig. 6. Here again all reactions are affected in a similar way, only now the rates are slightly increased. The increase is due to the fact that the average collision energy is slightly enhanced by the nucleon motion, and since the cross-sections increase with energy, the rates increase due to this effect. Comparison of Fig. 5 and Fig. 6 shows that the thermal averaging effect is about 1/3 the effect due to recoil, so that the net effect of the two processes is to decrease the reaction rates.

The total reduction in rate arises from integrating over initial lepton energies. Fig. 7 shows the fractional change in rate, $\delta = \Delta \Gamma / \Gamma$ for the four scattering reactions as a function of $T_{\gamma}$. The curves include all six terms shown in Fig. 3. The reduction increases nearly linearly with temperature, although there are deviations at low temperatures. The linear increase is a consequence of the fact that of the three small parameters, $m_e/m_N$, $Q/m_N$, and $\sim 5T/m_N$, the latter is by far the largest. The coefficient, 5, reflects the increase of cross-section and phase space with initial lepton energy.

Another feature of Fig. 7 is that at high temperatures the corrections to $\Gamma_{n\rightarrow p}$ are either less positive or more negative than the corresponding corrections to $\Gamma_{p\rightarrow n}$. This is not unexpected since the order $1/m_N$ correction to the equilibrium abundance of neutrons should result in a 0.2% increase in $x_n/x_p$, and this must be reflected by a change in the
rates which maintain equilibrium. At low temperatures two things happen. First, the neutrino and photon temperatures are no longer equal, so equilibrium arguments no longer apply. Second, the difficulties with the threshold behavior in the $\nu_e p \rightarrow e^+ n$ channel become apparent. Fortunately, the threshold behavior does not become a problem until $T_\gamma \lesssim 0.5$ MeV and by that time the absolute rate of the $\nu_e p \rightarrow e^+ n$ reactions is so small (see Fig. 2) that the error to the correction to $x_n$ is insignificant.

3 Corrections to Neutron Decay

As mentioned in the introduction, the $p \leftrightarrow n$ rates used in big bang nucleosynthesis calculations are not usually calculated from first principles, but are normalized to the experimental lifetime for neutron decay. Originally this had the advantage of partially accounting for some of the effects left out of the calculation, such as the coulomb and radiative corrections. In the present case, this convention requires us to calculate the recoil corrections for neutron decay, since those corrections are, in effect, already included in the numerical BBN codes.

Write the scattering rate for one of the channels as

$$\Gamma_{sc} = \Gamma_{sc,0}(1 + \delta_{sc}),$$

(24)

where $\Gamma_{sc,0}$ is the zeroth order scattering rate, and $\delta_{sc}$ is the $1/m_N$ term normalized to $\Gamma_{sc,0}$. Similarly, the neutron decay rate may be written as

$$\Gamma_n = \Gamma_{n,0}(1 + \delta_n),$$

(25)

where the decay rate is approximated by the sum of zeroth and first order terms in an expansion in $1/m_N$. The zeroth order scattering and neutron decay rates are related, schematically, $\Gamma_{sc,0} = B\Gamma_{n,0}$, where $B$ is some function of temperature and the particle masses. Since the nucleosynthesis codes are normalized to the experimental decay rate, but include no recoil corrections they effectively use a scattering rate $\Gamma'_{sc} = B\Gamma_n$. The correction to the current calculations may then be estimated

$$\Gamma_{sc} = B\Gamma_{n,0}(1 + \delta_{sc})$$

$$\approx \Gamma'_{sc}(1 + \delta_{sc} - \delta_n),$$

(26)
In the last section the various $\delta_{sc}$ were calculated, implicitly; in this section the corresponding $\delta_n$ is evaluated.

For laboratory neutron decay it is only necessary to evaluate the recoil corrections - there are no thermal averages, nor any blocking factors. Although weak magnetism affects the angular correlations of the decay products, its effects drop out of the total decay rate at first order because the interference term with the axial current is zero when integrated over leptonic phase space. This can be used as a check of the calculation.

The decay to three bodies can be put into a form similar to that for scattering processes,

$$\Gamma_{n \rightarrow e^- e^- p} = \int dE_1 \frac{k_2^2}{2\pi^2} \gamma_{n \rightarrow e^- e^- p},$$

where $\gamma_{n \rightarrow e^- e^- p}$ is identical in form to that for the $n \rightarrow p$ cross-sections but with $s$ evaluated for an ‘initial’ lepton energy equal to minus the energy of the corresponding lepton in the decay. It is then straightforward to use $\gamma_{n \rightarrow e^- e^- p} = \gamma_0 + \gamma_{rec} + \gamma_{wm}$. Graphs of the corresponding differential decay spectra and corrections are shown in Fig. 8. One can see that weak magnetism contributes to the asymmetry but not to the total decay rate; however, the recoil correction does reduce the decay rate, by an amount

$$\delta_n \approx -0.00201.$$  

Noting that we have not included Coulomb and radiative corrections, the value for the zeroth order neutron halflife is $\tau_{n,0} = 1/\Gamma_{n \rightarrow e^- e^- p} = 964.70$ sec, while the halflife including recoil is $\tau_{n,rec} = 966.66$. In the next section, where the rate equations are solved for $x_n$, it will be advisable to account for as much of the Coulomb and radiative corrections as possible so as to isolate the corrections due to nucleon mass effects. To do this it should be adequate to adjust both the neutron decay rate and the scattering cross-sections, by a constant factor. This can be done easily by increasing the effective Fermi constant by $(966.66/889.1)^{1/2} = 1.0427$.

Wilkinson has performed a comprehensive examination of the corrections to neutron decay. In an effort to obtain a reliable accuracy at the level of one part in $10^4$, he evaluated all effects that would plausibly contribute at a level $10^{-5}$. These include recoil, weak
magnetism, radiative, and coulomb corrections to second order as well as other small corrections, e.g. due to the finite size of the nucleons. Specifically, his Table 4 includes a recoil correction of $\delta_n = +0.0017$. This result differs from Eq. 28 in magnitude and sign (!), but the difference is due solely to different definitions of what is meant by the recoil correction.

Wilkinson writes the decay rate as

$$\Gamma_n = B' \int_1^{E_0} dE_e E_e k_e (E_0 - E_e)^2 (1 + R(E_e, E_0, m_N)),$$

(29)

where $B'$ is a constant, and $E_0 = Q - (Q^2 - m_e^2)/(2m_N)$ is the electron endpoint energy, including recoil effects. He then identifies the recoil correction as

$$\Delta \Gamma_{n, rec} = B \int_1^{E_0} dE_e E_e k_e (E_0 - E_e)^2 R(E_e, E_0, m_N),$$

(30)

but this does not include the correction to the integral due to the change in the electron endpoint energy from $Q$ to $E_0$,

$$\Delta \Gamma'_{n, rec} = B \int_1^{E_0} dE_e E_e k_e (E_0 - E_e)^2 - \int_1^{Q} dE_e E_e k_e (Q - E_e)^2.$$ (31)

The change due to the endpoint of integration is small since the integrand vanishes there in any event, but the decrease in the integrand by $\approx E_e^2 k_e (Q^2 - m_e^2)/m_N$ is significant. Wilkinson includes this term in his definition of the zeroth order phase space integral, whereas in the current paper $\Delta \Gamma'_{n, rec}$ is included as part of the recoil correction. The current nucleosynthesis codes assume that the change in lepton energy is $E_3 - E_1 = Q$, which is the zeroth order value for the endpoint conventions used in this paper. Even though Wilkinson puts $\Delta \Gamma'_{n, rec}$ into the zeroth order phase space integral, it is still present in his full phase space factor, correct to second order in $1/m_N$. Therefore, results of neutron decay based on Wilkinson’s work should be valid.

The recoil correction to neutron decay should not be applied to neutron decay in the early Universe, since the rate used in the code is the experimentally determined value. There is, however, a small thermal correction to neutron decay due to the thermal averaged time dilation factor. The neutron decay rate should be divided by a factor of $(1 + 1.5T/m_N)$. Since neutron decay is more important at late times when $T \approx 0.1$ MeV this correction, although technically of order $1/m_N$, is numerically quite small.
4 Estimate of the change in $Y_4$

All the pieces are now in place to estimate the change in $Y_4$. No effort will be made in this paper to incorporate the modified $p \leftrightarrow n$ rates in a full nucleosynthesis code. Rather it should be sufficient to examine the evolution of the neutron fraction down to $T \approx 0.07$ MeV with and without the nucleon mass corrections. The increase in $Y_4$ due to these corrections is given by twice the increase in the neutron fraction, $\Delta Y_4 = 2\Delta x_n$.

To perform the evolution, a simplified numerical model of the early Universe was constructed. One sector included neutrons, protons, electrons, and photons in thermal equilibrium at a temperature $T_\gamma$. The other contained three neutrino species in equilibrium at a temperature $T_\nu$. Account was taken of $e^+e^-$ annihilation for keeping track of the energy density and the expansion rate of the Universe, so that in general $T_\gamma \neq T_\nu$. The effect of different temperatures was included in the rate calculations.

The zeroth order scattering rates, Eq. [\ref{eq:sc}], and the corrections Eqs. [\ref{eq:sc1}, \ref{eq:sc2}, \ref{eq:sc3}, \ref{eq:sc4}, \ref{eq:sc5}], [\ref{eq:sc6}, \ref{eq:sc7}]{\ref{eq:sc8} were calculated on a logarithmic temperature grid and interpolating functions were created that reproduced the numerical integration (at new points) to better than a part in $10^4$ over the temperature range, $50 \text{ keV} < T_\gamma < 10 \text{ MeV}$. This was done for each of the four channels. The experimental rate for neutron decay was modified by the thermal lorentz dilation factor. The rates for $e^-\nu_ep \rightarrow n$ were inferred using $\Gamma_{n \rightarrow \nu_ep \rightarrow n}$ and the known equilibrium neutron fraction, under the assumption that $T_\gamma = T_\nu$. Since this channel is numerically unimportant the error introduced by this procedure is not important. The reaction rates are then

\begin{align}
\Gamma_{sc} &= \Gamma_{sc,0}(1 + \delta_{sc} - \delta_n) \\
\Gamma_{n \rightarrow \nu_ep \rightarrow n} &= \frac{T_\gamma}{m_n}(1 - 1.5 \frac{T_\gamma}{m_n}) \\
\Gamma_{e^-\nu_ep \rightarrow n} &= \Gamma_{n \rightarrow \nu_ep \rightarrow n}e^{-Q/T} \tag{32}
\end{align}

These rates were used to solve for $x_n$ by

\begin{align}
\frac{dx_n}{dT_\nu} = -\frac{1}{HT_\nu} \frac{dx_n}{dt} \tag{33}
\end{align}
with
\[
\frac{dx_n}{dt} = \Gamma_{p \to n}(1 - x_n) - \Gamma_{n \to p}x_n, \quad (34)
\]
where \(\Gamma_{p \to n}\) and \(\Gamma_{n \to p}\) are sums over the appropriate reaction rates. \(H\) is the expansion rate given by
\[
H^2 = \frac{8\pi G_N}{3} \sum_i \rho_i(m_i, T_i) \quad (35)
\]
where \(G_N\) is Newton’s constant and \(\rho_i\) is the density in species \(i\) calculated for the appropriate mass and temperature. The photon and neutrino temperatures were derived assuming adiabatic expansion and totally decoupled neutrinos.

The integration was started at \(T_\nu = 10\) MeV. For the zeroth order case the initial neutron to proton ratio was set to \((x_n/x_p)_0 = e^{-Q/T}\), but for the calculation with \(1/m_N\) corrections the initial value was set to \((x_n/x_p)_1 = e^{-Q/T}(1 + 1.5Q/m_N)\). In fact, the end results are essentially independent of initial conditions since the reaction rates are so fast that dynamic equilibrium is quickly achieved.

Fig. 9 shows the resulting \(x_n\). The equilibrium values \(x_{n,eq}\) are also shown to illustrate the freezeout of the \(p \leftrightarrow n\) scattering reactions, followed by the slower neutron decay. The breaking of the deuterium bottleneck is defined, in an \textit{ad hoc} way, to occur when \(x_n = 0.12\). This happens at \(T_d = 0.071\) MeV.

The zeroth order and corrected results for \(x_n\) are so close that the difference cannot be shown in Fig. 9. To bring out the correction, \(x_{n,1} - x_{n,0}\) is plotted as the solid curve in Fig. 10. The maximum correction occurs around freezeout, but is diminished by neutron decay until the deuteron bottleneck breaks and the remaining neutrons are cooked into \(^4\)He. The correction to \(x_n\) at this point is \(\Delta x_n(T_d) \approx 0.0006\) yielding a correction to the helium abundance of \(\Delta Y_4 \approx 0.0012\).

It is interesting that at high temperatures the reaction rates for the model with nucleon mass corrections do not appear to reproduce the equilibrium neutron fraction, shown as the dotted curve in Fig. 10. The difference can be understood as being due to corrections that are second order in \(1/m_N\) - both in the equilibrium abundance and in the rates.

A test of this can be done by forming a residual which should vanish through first order
in $1/m_N$ when $T_\gamma = T_\nu$,

$$\delta_2 = 1 - \frac{\Gamma_{p\to n}}{\Gamma_{n\to p}} \left( e^{-Q/T} (1 + 1.5Q/m_N) \right) \sim \mathcal{O}\left( \frac{1}{m_N^2} \right).$$  \hspace{1cm} (36)

A graph of $\delta_2$ is shown as the solid curve in Fig. 11. At high temperatures $\delta_2$ is increasing because the second order corrections are increasing. At $T = 10$ MeV one finds $\delta_2 \approx 10^{-4}$ which accounts for most of the difference between $\Delta$ and $\Delta_{eq}$ in Fig. 10. At lower temperatures, $T_\gamma \sim 1$ MeV, there is no problem with the corrected rates producing corrected equilibrium fractions, rather one only needs to ascertain that $\delta_2$ is much less than the individual first order corrections $\delta_{sc,i}$. Indeed, the residual is much smaller than the individual corrections (typically a few percent) for $0.5 \text{ MeV} < T < 10 \text{ MeV}$.

At very low temperatures $\delta_2$ again becomes significant. The problem goes back to the poor threshold behavior of the $\nu_{e}p \to e^+n$ reaction. This was checked by arbitrarily taking $m_e = 0$, which should alleviate the threshold problems, and increasing $m_N$. In that case, $\delta_2$ scaled as $1/m_N^2$ across the full temperature range $0.1 \text{ MeV} < T < 10 \text{ MeV}$.

Fig. 11 also shows several other examples of $\delta_2$ with different terms included in the rates. The solid curve at the bottom shows $\delta_2$ in the limit of infinite mass nucleons, and $m_e = 0$. The $10^{-7}$ level of the result reflects the accuracy of the numerical integration. The dotted curves show $\delta_2$ for $m_e = 0$ in the cases where $\Gamma$ includes a) recoil, b) thermal averaging, c) recoil and thermal averaging, and d) recoil, thermal averaging, it and the small blocking corrections. For both cases c) and d), $\delta_2$ is smaller than in the previous case as more of the terms necessary to achieve thermal equilibrium are included. The $10^{-4}$ magnitude for case d) is indicative of the $1/m_N^2$ nature of $\delta_2$. Note that it is not necessary to include the weak magnetism corrections in this analysis, since one can consistently imagine another world where $f_2 = 0$, and $\delta_2$ should still vanish to second order.

The conclusion of these investigations is that the numerical accuracy of the approximations and numerical integrations is adequate for temperatures below $\sim 3 \text{ MeV}$. The major weakness is the poor threshold behavior, which induces errors of order the correction in the $\nu_{e}p \to e^+n$ channel for $T_\gamma \lesssim 0.5 \text{ MeV}$. Since this channel is not so important then, the numerical accuracy of the corrections presented here are estimated to be about 10%
(1σ equivalent). There are also errors at higher temperatures since the corrections are only first order in $T/m_N$, but these errors are dynamically erased by the fast reaction rates that persist down to freezeout.

It would be useful to have a simple approximation for the $1/m_N$ corrections, since encoding the full expression into a nucleosynthesis code and performing the phase space integrals at each step would be a time consuming exercise. An approximation, linear in $T_\gamma$ was developed,

\[
\begin{align*}
\delta_{n\rightarrow p} &= -0.00185 - 0.01032 \frac{T_\gamma}{\text{MeV}} \\
\delta_{p\rightarrow n} &= +0.00136 - 0.01067 \frac{T_\gamma}{\text{MeV}},
\end{align*}
\]

which represents averages for the two channels that enter into the forward or back reactions. As such these may be readily applied to the polynomial formulae used in Wagoner’s code to approximate the $n \rightarrow p$ and $p \rightarrow n$ reaction rates. Before doing this one must separate out those pieces due to neutron decay and inverse decay and treat them on a separate footing, as in Eq. [32]. The approximations in Eq. [37] do not include the correction to the neutron lifetime, so this must be added in separately.

The result of carrying out this procedure for the simplified model of the early Universe used in this paper is shown as the dashed curve in Fig. [10]. The solution for $x_n$ matches that derived from direct integration of the rates to better than 10% for temperatures less than 2 MeV. This is comparable to the estimated uncertainty in the calculation of the rates due to the improper treatment of the threshold effects. The parameters in Eq. [37] were chosen by fitting the $p \rightarrow n$ reactions in the temperature range $0.7 < T_\gamma < 2$ MeV, and the $n \rightarrow p$ reactions in the range $0.3 < T_\gamma < 2$ MeV. These ranges cover freezeout for the different channels and avoid, for the most part, sensitivity to the threshold behavior of the rates.

Finally, to isolate the effects of weak magnetism, $x_n$ was calculated with a set of rates where $f_2$ was set to zero. The resulting increase in $x_n$ was 0.00045 instead of 0.0006; i.e., about 1/4th of the net increase in $Y_4$ can be attributed to weak magnetism. The bulk of this contribution comes at $0.5 < T_\gamma < 2$ MeV where the $e^+n \rightarrow \overline{\nu}_e p$ channel is slightly
more important than the $\nu_e n \rightarrow e^- p$ channel because of kinematics and also because $T_\gamma$ is slightly greater than $T_\nu$.

It doesn’t really make sense to perform a similar calculation to try and isolate the recoil vs. the thermal averaging corrections. The point of the analysis of the residual $\delta_2$ is that both are necessary to achieve a sensible thermodynamic result if one were to take $T_\gamma = T_\nu$. Even so, including just recoil corrections to the rates, leads to a change in $x_n$ of just 0.0002. This is somewhat surprising since the recoil corrections were larger than and of the opposite sign to the thermal averaging corrections. Based on this, one might have expected the recoil corrections to give a correction to $x_n$ of order $\sim 0.0008$, which would be partly compensated by the thermal averaging corrections. This is not the case. An explanation can be found in the details of Fig. 1 and Fig. 5, where the corrections can be seen to be not simply proportional to the zeroth order rates.

5 Discussion

The main point of this paper is that the primordial helium abundance predicted by big bang nucleosynthesis calculations should be increased by $\Delta Y_4 \approx 0.0012$. It is difficult to attach a firm level of uncertainty to this number, but the results displayed in Fig. 11 and the accompanying text suggest that an uncertainty of 10% should be inferred.

This is a significant correction, but does not dramatically alter the conclusions that may be drawn from studies of BBN. Consider the changes implied for the baryon density of the Universe as inferred from nucleosynthesis calculations. Walker and Kernan have recently analyzed the uncertainties in the big bang helium calculation, but they do not include the corrections due to nucleon mass effects. Adapting their result to include the results presented here, the primordial helium abundance for the standard cosmology with three neutrino species is

$$Y_4 = 0.2410 + 0.0117 \ln\left(\frac{\eta_{10}}{3}\right) \pm 0.0017 \pm 0.0002,$$

(38)

where $\eta_{10} = 10^{10} n_B / n_\gamma$ parameterizes the baryon density. Without the $1/m_N$ corrections, the first coefficient would be 0.2398, instead of 0.2410. In Eq. 38, the first uncertainty
represents a 2\(\sigma\) error due to uncertainties in the nuclear reaction network, of which “80-90%” is due to uncertainty in measurements of the neutron decay rate. The second uncertainty allows for some of the smaller corrections to the weak interaction rates, for example, the deviation of the neutrino spectrum from thermal equilibrium. The nucleon mass corrections, approximately 0.0012 in \(Y_4\), are equivalent to about a 1.5\(\sigma\) shift in the neutron decay rate, and are much larger than any other known uncertainty in calculating the weak rates.

It is difficult to determine the primordial abundance of \(^4\)He through direct observation due to a) the inert nature of neutral helium, b) chemical pollution through stellar burning, and c) the high accuracy of the measurement required - better than 1% is desired. Walker, et al. suggest a primordial abundance in the range 0.22 < \(Y_4\) < 0.24. The limits are suggestive of 95% confidence levels, but there is no statistically rigorous upper bound to the helium abundance. For the sake of argument then, take 0.24 as the upper limit and allow the uncertainty due to the neutron lifetime in Eq. 38 to be favorable at the 2\(\sigma\) level, i.e., allow the two uncertainties to add −0.0017 to the helium abundance. These constraints require \(\eta_{10} < 3.18\). Without the 0.0012 correction due to nucleon masses the corresponding number is \(\eta_{10} < 3.53\). These numbers should be compared with the constraints derived from comparisons of observations of \(D, \ ^3\)He, and \(^7\)Li and BBN calculations, 2.8 < \(\eta_{10}\) < 4.0.

Taken at face value, a substantial portion (but not all) of the allowed parameter space for \(\eta_{10}\) is eliminated by the nucleon mass correction; however, one should always keep in mind the difficulties of helium observations. If the upper limit were \(Y_4\) < 0.245 there would be no significant constraint from the consideration of \(^4\)He. On the other hand, the discussion in the previous paragraph was based on two separate favorable assumptions, both at the 2\(\sigma\) level: a) allowing \(Y_4\) = 0.24, and b) taking the neutron lifetime to be near the lower end of the allowed range. Dropping either of these assumptions from the favorable to the neutral category eliminates any allowed values of \(\eta_{10}\).

Another use of the primordial helium abundance is to constrain the energy density at

\[1\text{ In }0.0017\text{ this }\pm\text{ was derived using a 3.5 sec }1\sigma\text{ uncertainty in the neutron lifetime}\text{, where the current particle data book uncertainty is }2.1\text{ sec, so the uncertainty due to the reaction network should become }\sim\pm0.0014\text{. The corresponding decrease in the maximum allowed value of }\eta_{10}\text{ would decrease from 3.18 to 3.10.}\]
the time of nucleosynthesis. This is often parameterized by the number of neutrino species, \( \Delta Y_4 = 0.012(N_\nu - 3) \). The \( 1/m_N \) corrections are equivalent to 0.1 neutrino species. Again, belief in constraints placed on particle physics models depends upon one’s faith in the helium observations.

Over the years, there have been several papers written which treat recoil corrections in \( n \rightarrow p \) processes. In addition to the Wilkinson paper on neutron decay, Fayans\(^{19}\) and Vogel\(^{18}\) have studied recoil and weak magnetism corrections to the \( \nu_e p \rightarrow e^+ n \) reaction in the context of laboratory neutrino oscillation experiments. The results here are in agreement with Fayans for both recoil and weak magnetism. There is also agreement with Vogel concerning weak magnetism. It is more difficult to compare to Vogel’s results for the recoil correction, since he gives the correction in terms of the final state lepton energy, whereas the results in this paper express the corrections in terms of the initial lepton energy. I know of no paper which deals with the thermal corrections or the corrections to the blocking factors that are relevant for the big bang nucleosynthesis scenario.

Acknowledgements While this work was underway I became aware of the work by Kernan\(^{16}\) and Walker\(^{17}\), who were also beginning to look into the question of recoil corrections. I am indebted to them for sharing the details of their previous work and thoughts about the issues presented here. I would also like to thank S.M. Barr, P. Vogel, J. Engel and E.W. Kolb for useful discussions. This work was partially supported by DOE grant DE-AC02-78ER05007, and by the University of Delaware Research Foundation.

Appendix A: The squared matrix element

For completeness, here is the spin summed squared matrix element for the Lagrangian in Eq. 8 and Eq. 9, to leading order in \( G_f \). The terms are grouped by coupling constant and expressed in terms of relativistic invariants. The invariant \( u \) has been eliminated in favor of \( s \) and \( t \) and particle masses. The particle identifications are; 1: incoming lepton, 2: incoming baryon, 3: outgoing lepton, 4: outgoing baryon. In this expression the vector and axial couplings are given explicitly as \( g_A \) and \( g_V \), instead of specifying the ratio \( c_a = g_A/g_V \).
as in the text.

\[ M^2 = G_f^2 (4f_{ps}^2 (t - (m_2 - m_4)^2) + (m_1^2 + m_3^2) t - (m_1^2 - m_3^2)^2) + \]
\[ 8g_A^2 (2m_1^2 m_3^2 + 2m_2^2 m_4^2 + 2s^2 + ((m_2 + m_4)^2 - t)(m_1^2 + m_3^2 - t) \]
\[ - 2s(m_1^2 + m_2^2 + m_3^2 + m_4^2 - t)) + 8g_A^2 (2m_1^2 m_3^2 + 2m_2^2 m_4^2 + 2s^2 + ((m_2 - m_4)^2 - t)(m_1^2 + m_3^2) - t) \]
\[ - 2s(m_1^2 + m_2^2 + m_3^2 + m_4^2 - t)) + \frac{8f_{2gV}^{m_N}}{(m_2 - m_4)(m_1^2 - m_2)(m_3^2 - t) - (m_1^2 - m_3^2)s} + \]
\[ m_4(-m_1^2 + t)(m_2^2 - m_3^2 + t) + m_2(-m_3^2 + t)(m_1^2 + m_3^2 + t)) + 16g_A g_V ((m_1^2 - m_3^2)(m_2^2 - m_4^2) + (m_1^2 + m_2^2 + m_3^2 + m_4^2 - 2s - t)(m_2 + m_4) \]
\[ + 16g_A f_{ps} ((m_2 - m_4)(m_1^2 m_3^2 - m_2^2 m_4^2) + (-m_1^2 + m_3^2)(t - m_2 + m_4)) \]
\[ + m_4^2 m_2^2 (m_1^2 - m_3^2 - t) - m_2 m_4^2 (m_1^2 - m_3^2 + t)) + \frac{f_{2g}^2}{m_N^2} (2m_1^2 (m_1^2 - m_3^2 - t) + 2m_4^2 (-m_1^2 + m_3^2 - t) - (m_1^2 + m_3^2)^2 t + (m_1^2 + m_3^2) t^2 \]
\[ + m_4^2 ((m_1^2 + m_3^2)^2 - 4m_1^2 - 3m_3^2 t + m_1^2 t + 2t^2) + m_2^2 ((m_1^2 + m_3^2)^2 - 4m_1^2 - 3m_1^2 t + m_3^2 t + 2t^2 \]
\[ - 2m_2 m_4^2 ((m_1^2 - m_3^2)^2 + (m_1^2 + m_3^2 - 2t)) + 4s((m_1^2 - m_3^2)(m_1^2 - m_2^2) + (m_1^2 + m_2^2 + m_3^2 + m_4^2 - t)) - 4s^2 t)) \]

(A.1)

When evaluating the weak magnetism coupling numerically, \( m_N \) was set equal to the initial hadron mass, \( m_2 \).

**Appendix B: Corrections due to the final state occupation number**

In this appendix, the corrections to the final state blocking factor, \( f_b \) are derived. The corrections can be put into four categories based on their angular dependences, which also determines how they will be combined with various corrections to \( \sigma v_{rel} \). There are
corrections that are independent of both $\cos \alpha$ and $\cos \theta$, corrections that are linear in $\cos \alpha$ or in $\cos \theta$, and terms that are proportional to $\cos \alpha \cos \theta$. Let us refer to these four terms as $f_{b,1}$, $f_{b,\alpha}$, $f_{b,\theta}$, and $f_{b,\alpha \theta}$. There are also terms that are proportional to $\cos^2 \theta$, but these are already of order $1/m_N$ and so may be averaged over $\theta$ immediately, i.e. $\cos^2 \theta \to 1/3$, and included in $f_{b,1}$ or $f_{b,\alpha}$.

Denote the true value of the outgoing lepton energy by $E'_3$, its value in the infinite nucleon mass case by $E_3$, and the difference by $\epsilon$; i.e. $E'_3 = E_3 + \epsilon$. The blocking factor can then be written as

$$f_b = \frac{1}{1 + e^{E_3/T_3} e^{\epsilon/T_3}}$$

$$= \frac{1}{1 + a \left( 1 + \frac{a e}{1 + a T_3} + \frac{a(a-1) e^2}{2(1+a)^2 T_3^2} \cdots \right)}$$

$$\approx f_{b,0}(1 + f_{b,\theta} + f_{b,\alpha \theta} + f_{b,1} + f_{b,\alpha})$$

(B.1)

where $a = e^{E_3/T_3}$, and the last equation defines the normalization to the corrections. In Eq. (B.1) the blocking factor has been expanded to second order in $\epsilon$ in recognition of the fact that the energy correction will have terms of order $v_{nuc} \sim (T/m_N)^{1/2}$ which need to be included to second order.

The correction to the outgoing lepton energy, $\epsilon$, can be derived by a series of Lorentz transformations. Start by choosing a coordinate system where the nucleon moves along the x-axis, then a) boost by $\beta$ to the rest frame of the nucleon, b) rotate by $\theta_1$ so that the lepton lies along the x-axis, and c) boost by $\beta_{cm}$ to the center of mass frame. After scattering, the inverse Lorentz transformation is applied and the final state lepton energy is determined as a function of the scattering angles and initial parameters. The result of this procedure is

$$E'_3 = (1 - \beta \beta_{cm} \cos \theta_1) \gamma \gamma_{cm} E_{3,cm}$$

$$+ \cos \alpha (-\beta_{cm} + \beta \cos \theta_1) \gamma \gamma_{cm} k_{3,cm} - \beta \gamma \cos \psi \sin \alpha \sin \theta_1 k_{3,cm}$$

(B.2)

where $E_{3,cm} = (s + m_3^2 - m_4^2)/(2\sqrt{s})$ is the final lepton energy in the center of mass frame for the collision and $s$ is the usual relativistic invariant. Using $s = m_1^2 + m_2^2 + 2E_{1,cm}m_2$, where $E_{1,cm} = E_1 \gamma (1 - \beta v_1 \cos \theta)$ is the initial lepton energy in the center of mass frame,
produces a result in terms of the initial energies and momenta in the fluid rest frame. The
other quantities used here are, $k_{3,\text{cm}}$ - the three momentum corresponding to $E_{3,\text{cm}}$, $\psi$ - the
azimuthal scattering angle, and $\gamma$ and $\gamma_{\text{cm}}$ - the relativistic $\gamma$ factors corresponding to the
two boosts. To make contact with notation in the rest of the paper, $v_{\text{nuc}} \equiv \beta$.

To leading order in $m_N$, $\beta_{\text{cm}} = -k_1/m_2$ and $\gamma_{\text{cm}} = 1$. Further, $\beta \sim 1/\sqrt{m_N}$, so all
terms of order $\beta^3$, or $\beta \beta_{\text{cm}}$ may be dropped. This allows Eq. [B.2] to be reduced to

$$E'_3 = \gamma E_{3,\text{cm}} + \cos \alpha (\beta_{\text{cm}} + \beta \cos \theta_1) k_{3,\text{cm}} - \beta \cos \psi \sin \alpha \sin \theta_1 k_{3,\text{cm}}. \quad (B.3)$$

Next, extract the leading piece from $\gamma E_{3,\text{cm}}$,

$$\gamma E_{3,\text{cm}} = \gamma (E_{1,\text{cm}} + Q - \frac{k_1^2 + k_3^2}{m_2})$$

$$= E_3 + E_1 (\beta^2 - \beta v_1 \cos \theta) - \frac{k_1^2 + k_3^2}{m_2}, \quad (B.4)$$

where after extracting a $1/m_N$ term the initial lepton momenta, $k_1$ and $k_3$, may be used
without further correction. Eliminating $E_3$ from Eq. [B.3] and Eq. [B.4] gives,

$$\epsilon = (E_1 (\beta^2 - \beta v_1 \cos \theta) - \frac{k_1^2 + k_3^2}{m_2})$$

$$+ \cos \alpha (\beta_{\text{cm}} + \beta \cos \theta_1) k_{3,\text{cm}} - \beta \cos \psi \sin \alpha \sin \theta_1 k_{3,\text{cm}}$$

$$\equiv \epsilon_1 + \epsilon_2 + \epsilon_3, \quad (B.5)$$

where the $\epsilon_i$ are defined respectively by the previous line of Eq. [B.3].

To determine all relevant contributions to the $f_b$ one must include both first and second
order terms in $\epsilon$. The $\epsilon_3$ term contributes only to $f_{b,1}$ after being squared and angle averaged.
Since such a term is second order in $\beta$, one may use $\theta_1 = \theta$ and $k_{3,\text{cm}} = k_3$. The other two
terms are more complicated. There are contributions to both $f_{b,\alpha}$ and $f_{b,\alpha \theta}$ from $\epsilon_2$ and
from the product $\epsilon_1 \epsilon_2$. In the linear contribution from $\epsilon_2$ one must keep $k_{3,\text{cm}}$ and $\cos \theta_1$ to
sufficient accuracy; $k_{3,\text{cm}} \approx k_3 - (\beta \cos \theta E_3 k_1)/k_3$ and $\cos \theta_1 \approx \cos \theta - \beta (1 - \cos^2 \theta) E_1/k_1$.
For all the other terms it is sufficient to take $k_{3,\text{cm}} = k_3$ and $\theta_1 = \theta$. The $\epsilon_1^2$ and $\epsilon_2^2$ terms
contribute to $f_{b,1}$, whereas the linear term in $\epsilon_1$ contributes to both $f_{b,1}$ and to $f_{b,\theta}$.

The zeroth order blocking factor and four corrections are then,
\[ f_{b_0} = \frac{1}{1 + e^{E_3/T_3}} \]
\[ f_{b_1} = \frac{-k_1^2 - k_3^2 + 3(E_1 + E_3)T_\gamma}{2m_N T_3 (1 + e^{E_3/T_3})} + \frac{(k_1^2 + k_3^2)(1 - e^{E_3/T_3})T_\gamma}{2m_N T_3^2 (1 + e^{E_3/T_3})^2} \]
\[ f_{b,\theta} = \frac{(3T_\gamma)^{1/2} \cos \theta k_1}{m_N^{1/2} T_3 (1 + e^{E_3/T_3})} \]
\[ f_{b,\alpha} = \cos \alpha \left( \frac{k_1^2 k_3^2 - (E_3 k_1^2 + 2E_1 k_3^2)T_\gamma}{k_1 k_3 m_N T_3 (1 + e^{E_3/T_3})} - \frac{(e^{E_3/T_3} - 1)k_1 k_3 T_\gamma}{m_N T_3^2 (1 + e^{E_3/T_3})^2} \right) \]
\[ f_{b,\alpha\theta} = \frac{(3T_\gamma)^{1/2} \cos \alpha \cos \theta k_3}{m_N^{1/2} T_3 (1 + e^{E_3/T_3})}. \] (B.6)
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Figure 1: The zeroth order differential rates $\frac{d\Gamma}{dE_1}$ for the $p \leftrightarrow n$ reactions, see Eq. 5. Results are shown for $T_\gamma = 0.8$ MeV, and $T_\nu = 0.7926$.

Figure 2: The zeroth order scattering rates for $p \leftrightarrow n$. The neutron decay rate and the expansion rate, $H$, are shown as bold solid lines.

Figure 3: The six corrections $\gamma_i$ normalized to $\gamma_0$ for the reaction $\nu_e n \to e^- p$, as a function of initial neutrino energy.
Figure 4: The weak magnetism correction for the four scattering reactions. Since the $c_a f_2$ interference changes sign for reactions with antileptons rather than leptons, the sum of the corrections is near zero. $T_\gamma = 0.8$ MeV.

Figure 5: The same as Fig. 4, but for the recoil corrections, $\gamma_{rec}$. Here the corrections are all negative, since the phase space for the outgoing lepton is reduced due to the recoil of the nucleon.

Figure 6: The same as Fig. 4, but for $\gamma_{th}$, the correction due to thermal averaging over the initial nucleon distribution.

Figure 7: The sum of the $1/m_N$ corrections to the $p \leftrightarrow n$ scattering rates, expressed as a percentage of the zeroth order rates.

Figure 8: The neutron decay spectrum and the $1/m_N$ corrections.

Figure 9: The fraction of baryons in neutrons, $x_n = n_n/n_B$, as a function of the photon temperature, $T_\gamma$. The deuterium bottleneck is defined by where $x_n = 0.12$. The equilibrium abundance is shown as a dotted line.

Figure 10: The change in $x_n$ due to the inclusion of nucleon mass corrections. The solid curve shows the result using the full formulae for the corrections. The dashed curve, $\Delta_{lin}$, shows the result using the linear approximation to the correction from Eq. 37. The correction to the equilibrium abundance is shown as a dotted line.

Figure 11: The residual, $\delta_2$ (see text), as a function of temperature. The ‘dips’ occur when $\delta_2$ goes through zero as two terms cancel.
\[ \frac{d\Gamma_0}{dE_1} \text{ (MeV-sec)}^{-1} \]

- \( \nu n \rightarrow e^- p \)
- \( \nu^+ p \rightarrow e^+ n \)
- \( e^+ n \rightarrow \nu^+ p \)
- \( e^- p \rightarrow \nu n \)
$$\nu n \rightarrow e^- p$$
$$\bar{\nu} p \rightarrow e^+ n$$
$$e^+ n \rightarrow \bar{\nu}^+ p$$
$$e^- p \rightarrow \nu n$$
$T_D$ (defined by $x_n = 0.12$)
\[ \Delta = \chi_n - \chi_{n,0} \]

\[ T_D \]

\[ \Delta_{lin} \]

\[ \Delta_{eq} \]

\[ T_\gamma (\text{MeV}) \]
\[ \Gamma_i (\text{sec}^{-1}) \]

\[ E_e (\text{MeV}) \]