Sharp and Simple Bounds for the Raw Moments of the Binomial and Poisson Distributions

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Abstract

We prove the inequality $E\left[\left(\frac{X}{\mu}\right)^k\right] \leq \left(\frac{k}{\mu} \log(1+\frac{k}{\mu})\right)^k \leq \exp\left(\frac{k^2}{2\mu}\right)$ for sub-Poissonian random variables $X$, such as Binomially or Poisson distributed variables, with mean $\mu$. The asymptotic behaviour $E[\left(\frac{X}{\mu}\right)^k] = 1 + O(k^2/\mu)$ matches a lower bound of $1 + \Omega(k^2/\mu)$ for small $k^2/\mu$. This improves over previous uniform raw moment bounds by a factor exponential in $k$.

1 Introduction

Suppose we sample an urn of $n$ balls, each coloured red with probability $p$ and otherwise blue. What is the probability that a sample of $k$ balls, with replacement, from this urn consists of only red balls? Such questions are of interest to sample-efficient statistics and the derandomisation of algorithms.

If $R \sim \text{Binomial}(n, p)$ denotes the number of red balls in the urn, the probability of drawing a single red ball from the urn is $R/n$. Thus, the probability that a sample of $k$ balls from the urn is all red is given by $(R/n)^k$, or $P = E[\left(\frac{R}{n}\right)^k]$ when the probability is taken over both sample phases. Whenever the urn is large ($n$ is large), $R/n$ concentrates around $p$, so sampling from the urn is equivalent to sampling from the original distribution and $P \approx p^k$. Indeed, from Jensen’s inequality, we can see that $p^k$ is always a lower bound: $P = E[\left(R/n\right)^k] \geq E[(R/n)^k] = p^k$. Previous authors have shown a nearly matching upper bound of $C^kp^k$ in the range $k/(np) = O(1)$ for some constant $C > 1$. (See eq. (1) below for details.) In this note, we improve the upper bound to $P \leq p^k(1 + k/(2np))^k$, which shows that when $k = o(\sqrt{np})$, the factor $C^k$ can be replaced by just $1 + o(1)$.

1.1 Related work

One direct approach to computing the Binomial moments expands them using the Stirling numbers of the second kind: $E[X^k] = \sum_{i=0}^{k} \binom{k}{i} n^i p^i$, where $n^i = n(n-1)\cdots(n-i+1)$. This equality can be derived as a sum of the much easier to compute “factorial moments”, $E[X^k] = n^k p^k$. See Knoblauch (2008) for details. Taking the leading two terms of the sum, one finds that $E[X^k] = (np)^k \left(1 + \binom{k}{2} \frac{1-p}{np} + O(1/n^2)\right)$ as $n \to \infty$. However, this
approach does not work when \( k \) is not constant with respect to \( n \). Similarly, for the Poisson distribution, the moments can be expressed as the so-called Bell (or Touchard) polynomials in \( \mu \): 

\[
E[X^k] = \sum_{i=0}^{k} \{k\}_i \mu^i.
\]

This sum gives a simple lower bound 

\[
E[X^k] \geq \{k\}_k \mu^k + \{k\}_{k-1} \mu^{k-1} = \mu^k \left(1 + \frac{k(k-1)}{2\mu^2}\right),
\]

matching our upper bound asymptotically when \( k = O(\sqrt{\mu}) \). However, as in the Binomial case, the sum does not easily yield a uniform bound. We give the details of both lower bounds in Section 2.2.

A different approach uses the powerful results on moments of independent random variables by Latała (1997) and Pinelis (1995). In the case of Binomial and Poisson random variables, they yield:

\[
\left( c \frac{k/\mu}{\log(1 + k/\mu)} \right)^k \leq E[(X/\mu)^k] \leq \left( C \frac{k/\mu}{\log(1 + k/\mu)} \right)^k
\]

for some universal constants \( c < 1 < C \). The bound is tight up to the factor \( (C/c)^k \), which is negligible when the overall growth is \( O(k^k) \). However, when \( k/\mu \to 0 \), we expect the upper bound to be 1, and so the factor \( C^k \) in the upper bound can be overwhelmingly large.

A third option is to use a Rosenthal bound, such as the following by Berend and Tassa (2010), (see also Johnson et al., 1985):

\[
E[X^k] \leq B_k \max\{\mu, \mu^k\}.
\]

Here, \( B_k \) is the \( k \)th Bell number, which Berend and Tassa show satisfies the uniform bound \( B_k < \left( \frac{0.792k}{\log(k+1)} \right)^k \). For large \( k \), a precise asymptotic bound, \( B_k^{1/k} = \frac{k}{e \log k} (1 + o(1)) \), is given by (e.g. de Bruijn, 1981; Ibragimov and Sharakhmetov, 1998). Unfortunately, the Rosenthal bound is incomparable to the other bounds in this paper when \( \mu < 1 \), as it grows with \( \mu \) rather than \( \mu^k \). However, for \( \mu \geq 1 \) and integral, we show a matching asymptotic lower bound in the second half of Section 2.2. That indicates that the upper bound of this paper could be improved by a factor \( e^{-k} \) for large \( k \).

Finally, Ostrovsky and Sirota (2017) give another asymptotically sharp bound in a recent preprint. Using a technique based on moment generating functions, similar to this paper, they bound the Bell polynomial, which as discussed above, is equivalent to bounding the moments of a Poisson random variable. The bound holds when \( k \geq 2\mu \):

\[
E[(X/\mu)^k]^{1/k} \leq \frac{k/\mu}{e \log(k/\mu)} \left( 1 + C(\mu)^{\log\log(k/\mu)} \right) \frac{\log\log(k/\mu)}{\log(\mu)}
\]

if \( k \geq 2\mu \),

where \( C(\mu) > 0 \) is some “constant” depending only on \( \mu \). In the range \( k < 2\mu \), Ostrovsky and Sirota only gives the bound \( E[(X/\mu)^k] \leq 8.9758^k \), so similarly to the other bounds presented, it loses an exponential factor in \( k \) compared to Theorem 1 below, for smaller \( k \).

## 2 Bounds

The theorem considers “sub-Poissonian” random variables, which are variables \( X \), satisfying the requirement 

\[
E[\exp(tX)] \leq \exp(\mu(e^t - 1)).
\]

Such sub-Poissonian include many simple distributions, such as the Poisson or Binomial distribution. We give more examples in Section 3.
**Theorem 1.** Let $X$ be a non-negative random variable with mean $\mu > 0$ and moment-generating function $E[\exp(tX)]$ bounded by $\exp(\mu(e^t - 1))$ for all $t > 0$. Then for all $k > 0$ and any $\alpha > 0$:

$$E[(X/\mu)^k] \leq \left(\frac{k/\mu}{e^{1-\alpha \log(1+\alpha k/\mu)}}\right)^k.$$  

The theorem has a free parameter, $\alpha$, which is optimally set such that $1 + \alpha k/\mu = e^{W(k/\mu)}$, where $W$ is the Lambert-W function, which is defined by $W(x)e^{W(x)} = x$.\footnote{The Lambert-W function has multiple branches. We always refer to the main one (sometimes called the 0th), in which $W(x)$ and $x$ are both positive.}

In practice the following two corollaries may be easier to work with.

**Corollary 1.**

$$E[(X/\mu)^k] \leq \left(\frac{k/\mu}{\log(1+k/\mu)}\right)^k \leq \left(1 + \frac{k}{2\mu}\right)^k \leq \exp\left(\frac{k^2}{2\mu}\right).$$

**Proof.** For the first inequality, set $\alpha = 1$ in Theorem 1. The second bound, we use a standard logarithmic inequality, $\log(1+x) \leq 1 + x/2$ (see e.g. Topsøe, 2007, eq. 6). The last bound is the standard $1 + x \leq \exp(x)$.

In the range $k = O(\sqrt{\mu})$ we show a matching lower bound of $1+\Omega(k^2/\mu)$ in Section 2.2, eq. (9).

**Corollary 2.** Let $x = k/\mu$, then

$$E[(X/\mu)^k]^{1/k} \leq \frac{x e^{1/\log(e+x)}}{e \log(1 + x/\log(e + x))} = \frac{x}{e \log x} \left(1 + O\left(\frac{\log \log x}{\log x}\right)\right) \quad \text{as } x \to \infty.$$ \hspace{1cm} (4)

**Proof.** Take $\alpha = 1/\log(e+x)$. For $x > 0$ we have $\log(e+x) > 0$ and so $\alpha > 0$ as required by Theorem 1.

Corollary 2 matches our lower bound in eq. (10), as well as Ostrovsky and Sirota in eq. (3), but without the restriction on the range of $k/\mu$.

### 2.1 The proof

Technically our bound is shown using the moment-generating function and some new sharp inequalities involving the Lambert-W function. We will use the following lemma:

**Lemma 1** (Hoorfar and Hassani, 2008). For all $y > 1/e$ and $x > -1/e$,

$$e^{W(x)} \leq \frac{x + y}{1 + \log y}. \hspace{1cm} (5)$$

We present an elementary proof of this fact for completeness:
Proof. Starting from \(1 + t \leq e^t\), substitute \(\log(y) - t\) for \(t\) to get \(1 + \log y - t \leq ye^{-t}\). Multiplying by \(e^t\) we get \(e^t(1 + \log y) \leq te^t + y\). Let \(t = W(x)\) s.t. \(te^t = x\). Rearranging, we get eq. (5). □

Taking \(y = e^{W(x)}\) in eq. (5) makes the two sides equal, so we can think of Lemma 1 as a way to turn a rough estimate into an upper bound.

We apply Lemma 1 to show a new bound on \(W(x)\) in a similar style. This lemma will be the main ingredient in proving Theorem 1.

**Lemma 2.** For all \(y > 1\) and \(x > 0\),

\[
\frac{1}{W(x)} + W(x) \leq \frac{y}{x} + \log \left( \frac{x}{\log y} \right),
\]

with equality if \(y = e^{W(x)}\).

**Proof.** The proof uses the identities \(W(x) = \log \left( \frac{x}{W(x)} \right)\) and \(\frac{1}{W(x)} = \frac{1}{x} \exp(W(x))\) which are simple rewritings of the definition \(W(x)e^{W(x)} = x\). The main idea is to introduce a new variable \(z > 0\), to be determined later, which allows us to control the effect of applying the logarithmic inequality \(\log x \geq 1 - 1/x\). We also use Lemma 1 which introduces another new variable \(y > 1\) to be determined.

We bound:

\[
\frac{1}{W(x)} + W(x) = \frac{1}{W(x)} + \log \left( \frac{x}{W(x)} \right)
\]

\[
= \frac{1}{W(x)} + \log \left( \frac{x}{W(x)} \right) - \log \left( \frac{x}{z} \right)
\]

\[
\leq \frac{1}{W(x)} + \log \left( \frac{x}{z} \right) - \left( 1 - \frac{z}{W(x)} \right)
\]

\[
= \frac{1 + z}{W(x)} - 1 + \log \left( \frac{x}{z} \right)
\]

\[
= e^{W(x)} \frac{1 + z}{x} - 1 + \log \left( \frac{x}{z} \right)
\]

\[
\leq \frac{x + y}{1 + \log(y)} \frac{1 + z}{x} - 1 + \log \left( \frac{x}{z} \right).
\]

\[
= \frac{y}{x} + \log \left( \frac{x}{\log y} \right).
\]

Here the last two steps come from the inequality eq. (5) in its general form, and the substitution \(z = \log y\). We can check that equality follows all the way through if we let \(y = e^{W(x)}\). □

We are now ready to prove the main theorem of the paper:

**Proof of Theorem 4.** Let \(m(t) = \text{E}[\exp(tX)]\) be the moment-generating function. We will bound the moments of \(X\) by

\[
\text{E}[X^k] \leq m(t) \left( \frac{k}{et} \right)^k,
\]  \hspace{1cm} (6)

4
which holds for all \( k \geq 0 \) and \( t > 0 \). This follows from the basic inequality \( 1 + z \leq e^z \), where we substitute \( tz/k - 1 \) for \( z \) to get \( tz/k \leq e^{tz/(k\mu)} \). Letting \( z = X \) and taking expectations, we get eq. (6).

We now define \( x = k/\mu \) and take \( t \) such that \( te^t = x \). In the notation of the Lambert-W function, this means \( t = W(x) \). We note that \( t > 0 \) whenever \( x > 0 \). We proceed to bound the moments of \( X/\mu \) using eq. (6):

\[
E[(X/\mu)^k] \leq m(t) \left( \frac{k}{e^t} \right)^k \mu^{-k}
\]

\[
\leq \exp(\mu(e^t - 1)) \left( \frac{k}{e^t \mu} \right)^k
\]

\[
= \exp(\mu(x/t - 1)) \left( \frac{e^t}{e} \right)^k
\]

\[
= \exp((k/x)(x/t - 1) + k(t - 1))
\]

\[
= \exp(kf(x))
\]

where we define \( f(x) := 1/t - 1/x + t - 1 \). Here eq. (7) came from the simple rewriting of the definition of \( t \), \( 1/t = e^t/x \)

We continue to bound \( f(x) \) using Lemma 2

\[
f(x) = \frac{1}{W(x)} + W(x) - 1 - \frac{1}{x}
\]

\[
\leq \frac{y}{x} + \log \left( \frac{x}{\log y} \right) - 1 - \frac{1}{x}
\]

\[
= \alpha - 1 + \log \left( \frac{x}{\log(1 + \alpha x)} \right),
\]

taking \( y = 1 + \alpha x \), which is greater than 1 when \( \alpha \) and \( x \) are both greater than 0.

Backing up, we have shown

\[
E[(X/\mu)^k] \leq \exp(kf(x)) \leq \left( \frac{x}{e^{1-\alpha \log(1 + \alpha x)}} \right)^k,
\]

which finishes the proof.

\[\square\]

2.2 Lower bound

As mentioned in the introduction, the expansion for the Poisson moments \( E[X^k] = \sum_{i=0}^{k} \binom{k}{i} \mu^i \) gives a simple lower bound by taking the two highest terms. We note that \( \binom{k}{k} = 1 \) and \( \binom{k}{k-1} = \binom{k}{2} \) to get

\[
E[X^k] \geq \mu^k \left( 1 + \frac{k(k-1)}{2\mu} \right),
\]

matching Theorem 1 asymptotically for \( k = O(\sqrt{\mu}) \).
The expansion for Binomial moments \( E[X^k] = \sum_{i=0}^{k} \binom{k}{i} n^i p^i \) yields a similar lower bound

\[
E[X^k] \geq np^k + \binom{k}{2} n^{k-1} p^{k-1} \\
= (np)^k \left( \frac{n^k}{n^k} \right) \left( 1 + \binom{k}{2} \frac{1}{(n-k+1)p} \right) \\
= (np)^k \left( \prod_{i=0}^{k-1} \frac{1}{n} \right) \left( 1 + \binom{k}{2} \frac{1}{(n-k+1)p} \right) \\
\geq (np)^k \left( 1 - \binom{k}{2} \frac{1}{n} \right) \left( 1 + \binom{k}{2} \frac{1}{np} \right) \\
= (np)^k \left( 1 + \binom{k}{2} \frac{1-p}{np} \left( 1 - \binom{k}{2} \frac{1}{n} \right) \right),
\]

which matches Theorem 1 for \( k = O(\sqrt{n}) \) and \( p \) not too close to 1.

We will investigate some more precise lower bounds as \( k/\mu \) gets large. As mentioned briefly in the introduction, there is a correspondence between the moments of a Poisson random variable and the Bell polynomials defined by \( B(k, \mu) = \sum_i \{i\}_k \mu^i \). In particular, \( E[X^k] = B(k, \mu) \), if \( \mu \) is the mean of the Poissonian random variable. The Bell polynomials are so named because \( B(k, 1) \) is the \( k \)th Bell number. By Dobinski’s formula \( B(k, 1) = \frac{1}{e} \sum_{i=0}^{\infty} \frac{i^k}{i!} \), the Bell numbers are generalised for real \( k \). We write these as \( B_x = B(x, 1) \).

We give a lower bound for \( E[(X/\mu)^k] \) by showing the following simple connection between the Bell polynomials and Bell numbers:

**Theorem 2.** Let \( k \) be a positive real number and \( \mu \geq 1 \) be an integer. Then

\[
B(k, \mu)/\mu^k \geq B_{k/\mu}^\mu.
\]

While the proof below assumes \( \mu \) is an integer, we will conjecture Theorem 2 to be true for any \( \mu \geq 1 \). Now by de Bruijn’s (1981) asymptotic expression for the Bell numbers:

\[
E[(X/\mu)^k]^{1/k} \geq B_{k/\mu}^{\mu/k} = \frac{k/\mu}{e \log(k/\mu)} \left( 1 + \Theta \left( \frac{\log \log(k/\mu)}{\log(k/\mu)} \right) \right)
\]

as \( k/\mu \to \infty \). (10)

matching our upper bound, eq. (4), the upper bound of Ostrovsky and Sirota, eq. (3), for large \( k \), as well as Latała’s uniform lower bound with a different constant.

**Proof of Theorem 2.** Let \( X, X_1, \ldots, X_\mu \) be i.i.d. Poisson variables with mean 1, then \( S = \sum_{i=1}^{\mu} X_i \) is Poisson with mean \( \mu \). We write \( ||X||_k = E[X^k]^{1/k} \). Then by the AG inequality:

\[
||S/\mu||_k = \left\| \frac{1}{\mu} \sum_{i=1}^{\mu} X_i \right\|_k \geq \left\| \left( \prod_{i=1}^{\mu} X_i \right)^{1/\mu} \right\|_k = \left\| \prod_{i=1}^{\mu} X_i \right\|_{k/\mu}^{1/\mu} = \left( \prod_{i=1}^{\mu} ||X_i||_{k/\mu} \right)^{1/\mu} = ||X||_{k/\mu}.
\]

(11)

Since \( X \) has mean 1 we have \( ||X||_{k/\mu} = B_{k/\mu}^{\mu/k} \), and as \( S \) has mean \( \mu \) we have \( ||S/\mu||_k = B(k, \mu)^{1/k}/\mu \). Thus, taking \( k \)th powers, eq. (11) is what we wanted to show. \( \square \)
For small $k/\mu$ this bound is less interesting since $B_x \to 0$ as $x \to 0$, rather than 1 as our upper bound. However, it is pretty tight, as we conjecture by the following matching upper bound in terms of the Bell numbers:

**Conjecture 1.** For all $k > 0$ and $\mu \geq 1$,

$$B_{k/\mu}^{1/(k/\mu)} \leq \frac{B(k, \mu)^{1/k}}{\mu} \leq B_{k/\mu+1}^{1/(k/\mu+1)}.$$  

Furthermore, for $0 < \mu \leq 1$, $\frac{B(k, \mu)^{1/k}}{\mu} \leq B_{k/\mu}^{1/(k/\mu)}$.

While the upper bound appears true numerically, it can’t follow from our moment-generating function bound eq. (8), since it drops below that for $k/\mu$ bigger than 40. The conjectured upper bound is even incomparable with our Theorem 1, since it is slightly above $\frac{k/\mu}{\log(1+k/\mu)}$ for very small $k/\mu$. The conjectured bound is weaker than eq. (2) by Berend and Tassa (2010) in the region $k < 2$ and $\mu < 1$, but for all other parameters, it is substantially tighter.

### 3 Sub-Poissonian Random Variables

We call a non-negative random variable $X$ sub-Poissonian if $E[X] = \mu$ and the moment-generating function, mgf., $E[\exp(tX)] \leq \exp(\mu(e^t - 1))$ for all $t > 0$. We will briefly show that this notion includes all sums of bounded random variables, such as the Binomial distribution.

If $X_1, \ldots, X_n$ are sub-Poissonian with mgf. $m_1(t), \ldots, m_n(t)$ and mean $\mu_1, \ldots, \mu_n$ respectively, then $\sum_i X_i$ is sub-Poissonian as well, since

$$E[\exp(t \sum_i X_i)] = \prod_i m_i(t) \leq \prod_i \exp(\mu_i(e^t - 1)) = \exp\left(\left(\sum_i \mu_i\right)(e^t - 1)\right).$$

Next, a random variable bounded in $[0, 1]$ with mean $\mu$ has mgf.

$$E[\exp(tX)] = 1 + \sum_{k=1}^{\infty} t^k \frac{E[X^k]}{k!} \leq 1 + \mu \sum_{k=1}^{\infty} t^k \frac{E[1^{k-1}]}{k!} = 1 + \mu(e^t - 1) \leq \exp(\mu(e^t - 1)).$$

Hence if $X = X_1 + \cdots + X_n$ where each $X_i \in [0, 1]$ we have $\mu = E[X] = \sum_i E[X_i]$ and by Theorem 1 that $E[(X/\mu)^k] \leq \frac{k/\mu}{\log(k/\mu+1)}$. In particular this captures sum of Bernoulli variables with distinct probabilities.

An example of a non-sub-Poissonian distribution is the geometric distribution with mean $\mu$. This has moment generating function $m(t) = \frac{1}{1-\mu(e^t - 1)}$, which is larger than $\exp(\mu(e^t - 1))$ for all $t > 0$. However, likely, similar methods to those in the proof of Theorem 1 will still apply to bound its moments.

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