Passivity-Based Generalization of Primal-Dual Dynamics for Non-Strictly Convex Cost Functions

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Abstract

In this paper, we revisit primal-dual dynamics for convex optimization and present a generalization of the dynamics based on the concept of passivity. It is then proved that supplying a stable zero to one of the integrators in the dynamics allows one to eliminate the assumption of strict convexity on the cost function based on the passivity paradigm together with the invariance principle for Carathéodory systems. We then show that the present algorithm is also a generalization of existing augmented Lagrangian-based primal-dual dynamics, and discuss the benefit of the present generalization in terms of noise reduction and convergence speed.

Key words: Primal-dual dynamics; Convex optimization; Passivity; Distributed optimization; Invariance principle for Carathéodory systems.

1 Introduction

Stimulated by strong needs for solving a large-scale optimization problem over a spatially distributed network, primal-dual dynamics [1], a continuous-time algorithm to solve convex optimization, has attracted attention again in recent years due to its decomposable nature under separability of cost and constraint functions [2]. The continuous-time algorithm mitigates the computational efforts, furthermore, it does not require network components to install any optimization solver differently from the other distributed optimization algorithms [3]. The continuous-time algorithm mitigates the computational efforts, furthermore, it does not require network components to install any optimization solver differently from the other distributed optimization algorithms [3]. Besides, it is pointed out in [4,5] that the impact of disturbances and noises added in the optimization process is analyzed from the control engineering point of view, which is important in the applications to online and/or distributed optimization.

The primal-dual dynamics is known to be closely related to so-called passivity [6,7], and it has been revealed that the algorithm is interpreted as a passivity-preserving interconnection of passive systems [8,9,10,11,12,13]. The passivity-based perspective brings several advantages. For example, the design flexibility inherent in passivity-based design allows one to stably interconnect other passive components such as physical dynamics [8,9,10] and communication delays with appropriate passivation techniques [10,11]. Robustness against the aforementioned disturbances may also be analyzed based on the celebrated passivity theorem [10]. In addition, the authors in [12,13] point out that the design flexibility brought by passivity contributes to accelerating the convergence speed and/or enhancing robustness. On the other hand, all of the above papers require strict convexity of the cost function, which may limit applications of the solutions.

Relatively few publications have addressed relaxation of the strict convexity assumption based on so-called augmented Lagrangian. Richert and Cortés [14] present a generalization of the primal-dual dynamics and prove asymptotic optimality for linear programming problems. Cherukuri et al. [15] also present an augmented Lagrangian-based solution to general convex optimization under strict convexification of the constraint function. Zhang et al. [16] present a solution relying on a projection operator to convex constrained sets although
it requires subprocesses to solve optimization to compute the projection.

In this paper, we revisit the paradigm of [12,13]. We start with hypothesizing that supplying stable zeros to transfer functions, namely leading the phase, in the primal-dual dynamics is a key to remove the strict convexity assumption through a toy linear programming problem. We then present a passivity-based generalization of the primal-dual dynamics so that zeros are added to the intended transfer functions. The above hypothesis is then shown to be valid, namely asymptotic optimality is proved for general convex cost functions under existence of the zeros, based on the passivity paradigm. It is further demonstrated that the present algorithm is also a generalization of the existing augmented Lagrangian-based algorithm [14,15], and the benefit of the generalization is exemplified through simulation.

The major contributions of this paper are summarized as below:

(i) the passivity-based approaches [8,9,10,11,12,13] are extended to general convex optimization with non-strictly convex cost function, and

(ii) a generalization of the augmented Lagrangian-based algorithm [14,15] is presented, and the design flexibility inherent in passivity-based design is shown to contribute to noise/disturbance reduction and convergence acceleration.

Additional contributions are as follows:

(iii) the passivity-based generalized primal-dual dynamics with general inequality constraints are presented in this paper for the first time, and

(iv) strict convexification of the constraint function required in [15] may spoil separability, whereas the present approach does not require such reformulation and accordingly broadens the class of problems solvable in a distributed fashion.

2 Preliminaries

This section is intended to present terminologies, associated results and notations used in this paper.

Let us first introduce the notion of passivity [6,7].

**Definition 1** Consider a system \( \Sigma \), described by a state model with state \( x \in \mathbb{R}^n \), input \( u \in \mathbb{R}^N \) and output \( y \in \mathbb{R}^N \). The system \( \Sigma \) is said to be passive if there exists a positive semi-definite function \( S : \mathbb{R}^n \to \mathbb{R}_{\geq 0} := [0, \infty) \), called storage function, such that

\[
\mathcal{L}_\Sigma S(x) \leq y^T u
\]

holds for all states \( x \in \mathbb{R}^n \) and all inputs \( u \in \mathbb{R}^N \), where the symbol \( \mathcal{L}_\Sigma \) represents Lie derivative along \( \Sigma \).

We next introduce convex functions defined below.

**Definition 2** A function \( f : \mathbb{R}^n \to \mathbb{R} \) is said to be convex if the following inequality holds for all \( x, y \in \mathbb{R}^n \).

\[
f(x) - f(y) \leq (\nabla f(x))^T (x - y) \tag{1}
\]

From (1), we immediately have so-called monotone condition:

\[
(\nabla f(x) - \nabla f(y))^T (x - y) \geq 0, \; \forall x, y \in \mathbb{R}^n. \tag{2}
\]

If \( f \) is strictly convex, the inequality (2) strictly holds as long as \( x \neq y \) [11].

Let us next introduce so-called KKT condition for the optimization problem:

\[
\begin{aligned}
\text{minimize} & \quad f(x) \\
\text{subject to} & \quad g(x) \leq 0, \; Ax - b = 0,
\end{aligned} \tag{3}
\]

where \( x \) is the decision variable, \( f : \mathbb{R}^n \to \mathbb{R} \) is the cost function, \( g : \mathbb{R}^n \to \mathbb{R}^m \) is the inequality constraint function, \( A \in \mathbb{R}^{r \times n} \) and \( b \in \mathbb{R}^r \) are the constant matrix and vector for equality constraint, respectively. Denote the \( l \)-th element of the function \( g \) as \( g_l(1 \leq l \leq m) : \mathbb{R}^n \to \mathbb{R} \). The KKT condition is given as below [2].

\[
\begin{aligned}
\nabla f(x^*) + \nabla g(x^*) \lambda^* + A^T \mu^* &= 0, \tag{4a} \\
Ax^* - b &= 0, \tag{4b} \\
\lambda^* &\geq 0, \quad g(x^*) \leq 0, \quad \lambda^* \circ g(x^*) = 0, \tag{4c}
\end{aligned}
\]

where the symbol \( \circ \) describes the Hadamard product. The set of the KKT solutions is now defined as

\[
\chi^* := \{ (x^*, \mu^*, \lambda^*) \in \mathbb{R}^n \times \mathbb{R}^r \times \mathbb{R}^m_{\geq 0} \mid (4) \text{ holds} \}.
\]

3 Generalized primal-dual dynamics and passivity

Throughout this paper, we consider the optimization problem (3) satisfying the following assumption.

**Assumption 3** The functions \( f, g_l \) \((l = 1, \ldots, m)\) are convex, continuously differentiable, and their gradients \( \nabla f, \nabla g_l \) \((l = 1, \ldots, m)\) are locally Lipschitz. The feasible set of (3) is nonempty, and the function \( f \) has a minimum value in the feasible set.
It is well-known, under Assumption 3, that $x^*$ is an optimal solution to (3) if and only if there exist $(\mu^*, \lambda^*) \in \mathbb{R}^r \times \mathbb{R}^m_{\geq 0}$ such that $(x^*, \mu^*, \lambda^*) \in \chi^*$ [2]. Remark that $x^*$ is not always unique due to the lack of strict convexity of the cost function $f$.

### 3.1 Primal-dual gradient dynamics

In this subsection, we first deal with the following primal-dual gradient dynamics [1,17] as a solution to (3).

$$
\begin{align}
\dot{x} &= -\nabla f(x) - \nabla g(x)\lambda - A^\top \mu, \quad (5a) \\
\dot{\mu} &= Ax - b, \quad (5b) \\
\dot{\lambda} &= [g(x)]^+_\lambda, \lambda(0) \geq 0, \quad (5c)
\end{align}
$$

where $x \in \mathbb{R}^n$, $\mu \in \mathbb{R}^r$, and $\lambda \in \mathbb{R}^m_{\geq 0}$ are variables corresponding to the primal variable, the dual variable for the equality constraint and the dual variable for the inequality constraint, respectively. The operator $[\cdot]_\lambda^+$ in (5c) is defined as

$$[\sigma]_\lambda^+ := \begin{cases} 0 & \text{if } \varepsilon = 0 \text{ and } \sigma < 0, \\ \sigma & \text{otherwise}, \end{cases}$$

for scalars $\varepsilon, \sigma \in \mathbb{R}$. For vectors $\varepsilon, \sigma \in \mathbb{R}^m$, $[\sigma]_\varepsilon^+$ denotes the vector whose $i$-th component is $[\sigma_i]_{\varepsilon_i}^+$, $i = 1, \ldots, N$. For convenience, the mode satisfying the upper condition in (6) is called mode 1, and the other is mode 2. The block diagram of (5) is then illustrated in Fig. 1, where

$$\psi := A^\top \mu, \eta := \nabla g(x)\lambda, \ u := -\eta - \psi,$$

and the notation $(\cdot)^+$ means to keep output signal non-negative, defined as below. For a transfer function $\frac{1}{\alpha s + \beta}$, the system

$$y(s) = \left(\frac{1}{\alpha s + \beta}\right)^+ z(s) \tag{7}$$

means that $y(s) = \frac{1}{\alpha s + \beta} z(s)$ under the constraint of $y \geq 0$. In other words, if $\alpha \neq 0$, (7) means

$$\dot{y} = \left[-\frac{\beta}{\alpha} y + \frac{1}{\alpha} z\right]_y^+,$$

with non-negative initial value $y(0) \geq 0$. If $\alpha = 0$ and $\beta > 0$, then (7) means that

$$y = \frac{1}{\beta} \max\{0, z\}.$$

The primal-dual gradient dynamics (5) is known to satisfy the following facts concerning passivity and convergence [11,12], where we take the notations

$$\bar{x} := x - x^*, \quad \bar{u} := u - \nabla f(x^*),$$

$$\bar{\psi} := \psi - A^\top \mu^*, \quad \bar{\eta} := \eta - \nabla g(x^*)\lambda^*,$$

for a fixed $(x^*, \mu^*, \lambda^*) \in \chi^*$.

**Fact 4** Suppose that Assumption 3 holds. Then, the system (5) satisfies the following properties regardless of the selection of $(x^*, \mu^*, \lambda^*) \in \chi^*$.

- The system (5a) is passive from $\bar{u}$ to $\bar{x}$,
- the system (5b) is passive from $\bar{x}$ to $\bar{\psi}$,
- the system (5c) is passive from $\bar{x}$ to $\bar{\eta}$,
- $(x^*, \mu^*, \lambda^*)$ is a stable equilibrium of the system (5) in the sense of Lyapunov, and
- the trajectories of $(x, \mu, \lambda)$ generated by (5) approaches one of the constants included in $\chi^*$ as the time goes to infinity, if the cost function $f$ is strictly convex.

It is to be noted that only the last item requires strict convexity of the cost function. Indeed, the dynamics without this additional assumption does not ensure asymptotic optimality as exemplified in the following trivial example.

Let us consider the problem (3) with $x \in \mathbb{R}$, $f(x) = 0 \forall x$, $A = 1$ and $b = 0$ and without inequality constraints, which satisfies Assumption 3. It is also trivially confirmed that the (unique) optimal solution is $x^* = 0$. The primal-dual dynamics (5) for the problem is then given as

$$\dot{x} = -\mu, \quad \dot{\mu} = x. \tag{8}$$

The dynamics (8) is a feedback interconnection of two single integrators whose open-loop transfer function has the phase equal to $-180$deg over the whole frequency domain and hence the phase margin is (deg). Accordingly, the dynamics does not drive $x$ to $x^* = 0$. 

![Fig. 1. Block diagram of the primal-dual gradient dynamics (5). The system enclosed by the solid line is passive from $\bar{u}$ to $\bar{x}$. The system enclosed by the dashed line is passive from $\bar{x}$ to $\bar{\psi}$. The system enclosed by the dashed-dotted line is passive from $\bar{x}$ to $\bar{\eta}$.](Image 313x611 to 540x720)
The above toy problem also provides informative knowledge in terms of overcoming the drawback of the primal-dual dynamics. We know that asymptotic stability of \( x = 0 \) for (8) is ensured by just adding a compensator leading the phase. Inspired by the fact, we present a generalization of the primal-dual dynamics in the next subsection so that the phase lead compensation can be added in this specific example.

**Remark 5** The primal dual dynamics (5) is known to provide a distributed algorithm if (5) is separable [2]. In addition, the distributed optimization problem

\[
\begin{align*}
\text{minimize} & \sum_{i=1}^{N} f_i(x) \\
\text{subject to} & g_i(x) \leq 0, \ A_i x - b_i = 0, \forall i = 1, \ldots, N,
\end{align*}
\]

with private costs \( f_i \) \( (i = 1, \ldots, N) \) and private constraints \( g_i(x) \leq 0, \ A_i x - b_i = 0, \forall i = 1, \ldots, N \) for agents \( i = 1, \ldots, N \) connected by an undirected graph with graph Laplacian \( L \) can be equivalently transformed into

\[
\begin{align*}
\text{minimize} & \sum_{i=1}^{N} f_i(x_i) + \frac{1}{2} x^\top (L \otimes I_n) x \\
\text{subject to} & g_i(x_i) \leq 0, \ A_i x_i - b_i = 0, \forall i = 1, \ldots, N, \\
& (L \otimes I_n) x = 0,
\end{align*}
\]

where the symbol \( \otimes \) describes the Kronecker product. The primal dual dynamics (5) for the new problem provides a distributed algorithm based on the PI consensus algorithm [11]. These distribution of primal-dual gradient dynamics (5) is due to the diagonal structure of the integrator matrices in Fig. 1.

### 3.2 Generalized primal-dual dynamics

In this subsection, we generalize the primal-dual dynamics mainly to ensure asymptotic optimality even in the absence of strict convexity assumption. For notational simplicity, we define the signals \( v, h \) and \( w \) as

\[
\begin{align*}
v &= -\nabla f(x) - \nabla g(x) \lambda - A^\top \mu, \quad (9a) \\
h &= Ax - b, \quad (9b) \\
w &= g(x). \quad (9c)
\end{align*}
\]

Let us present the generalized primal-dual dynamics. The basic design policy is to allow one to add the phase lead compensators to the open-loop transfer functions, while preserving passivity of the subsystems colored by gray in Fig. 1, formulated as

\[
\begin{align*}
x(s) &= M(s) v(s), \quad (10a) \\
\mu(s) &= H(s) h(s), \quad (10b) \\
\lambda(s) &= G^+(s) w(s), \quad (10c)
\end{align*}
\]

where \( M(s) \) and \( H(s) \) are the transfer function matrices, \( G^+(s) \) is the transfer function matrix with the operator in (7). Although we might be able to take a more general unstructured form, we restrict the matrices \( M(s), H(s) \) and \( G^+(s) \) to the diagonal structure as:

\[
\begin{align*}
M(s) &= \text{diag} \left( M_1(s), \ldots, M_n(s) \right), \quad (11a) \\
H(s) &= \text{diag} \left( H_1(s), \ldots, H_r(s) \right), \quad (11b) \\
G^+(s) &= \text{diag} \left( G^+_1(s), \ldots, G^+_m(s) \right), \quad (11c)
\end{align*}
\]

with

\[
\begin{align*}
M_i(s) &= \frac{c_i}{s} + \sum_{k=2}^{n_i} \frac{c_{ik}}{s + a_{ik}} + d_i, \quad (12a) \\
H_j(s) &= \frac{\bar{c}_{ij}}{s} + \sum_{q=2}^{r_j} \frac{\bar{c}_{jq}}{s + \bar{a}_{jq}} + \bar{d}_j, \quad (12b) \\
G^+_i(s) &= \left( \frac{\bar{c}_{ii}}{s} \right)^+ + \sum_{p=2}^{m_i} \left( \frac{\bar{c}_{ip}}{s + \bar{a}_{ip}} \right)^+ + \left( \bar{d}_i \right)^+, \quad (12c)
\end{align*}
\]

where

\[
\begin{align*}
a_{in_i} > \cdots > a_{i2} > 0, \ c_{ik} > 0 \ (k = 1, \ldots, n_i), \ d_i \geq 0, \\
\bar{a}_{jr_j} > \cdots > \bar{a}_{j2} > 0, \ \bar{c}_{jq} > 0 \ (q = 1, \ldots, r_j), \ \bar{d}_j \geq 0, \\
\bar{a}_{jm_i} > \cdots > \bar{a}_{i2} > 0, \ \bar{c}_{ip} > 0 \ (p = 1, \ldots, m_i), \ \bar{d}_i \geq 0.
\end{align*}
\]

It is easy to confirm that (12) allows one to add the phase lead compensator to the integrators in the primal dual dynamics (5). We also immediately see that \( M(s) \) and \( H(s) \) are passive since \( M_i(s) \) and \( H_j(s) \) are defined by a parallel connection of passive systems, which is known to preserve passivity. More precise descriptions on the issue together with passivity of \( G^+(s) \) will be presented in the next subsection.

The block diagram of the algorithm (10)–(12) is then illustrated in Fig. 2, where \( \psi, \eta \) and \( u \) are defined in the same way as Subsection 3.1.

**Remark 6** At this moment, we focus on the diagonal structure of \( M(s), H(s) \) and \( G^+(s) \) in (11) for convenience of the subsequent technical discussions, but a more general form will be presented in the end of the next section. However, the diagonal structure is itself of particular importance since it trivially preserves the distributed nature of the primal-dual dynamics pointed out in Remark 5.

### 3.3 Passivity analysis

In this subsection, we confirm that the subsystems colored by light gray in Fig. 2 ensure passivity.
Lemma 8 Suppose that Assumption 3 holds. Then, the system given by (10b), (11b) and (12b) is passive from \( \tilde{x} \) to \( u \) where

\[
\begin{align*}
\dot{\zeta}_{j1} &= \tilde{e}_{j1} h_j, \\
\dot{\zeta}_{jq} &= -\tilde{a}_{jq} \zeta_{jq} + \tilde{c}_{jq} h_j, \quad q = 2, \ldots, r_j, \\
\mu_j &= 1_m^T \lambda_j + \tilde{d}_j h_j,
\end{align*}
\]

where \( \zeta_{jq} \) is the \( q \)-th element of the state \( \zeta_j \in \mathbb{R}^{r_j} \).

**Proof.** See Appendix A.3. □

Let us first consider the primal dynamics (10a). Now, a state space representation of \( M_i(s) \) in (12a) is given as

\[
\begin{align*}
\dot{\xi}_{11} &= c_{11} v_i, \\
\dot{\xi}_{ik} &= -a_{ik} \xi_{ik} + c_{ik} v_i, \quad k = 2, \ldots, n_i, \\
x_i &= 1_{n_i}^T \xi_i + d_i v_i,
\end{align*}
\]

where \( \xi_{ik} \) is the \( k \)-th element of the state \( \xi_i \in \mathbb{R}^{n_i} \) and \( 1_{n_i} \) is the \( n_i \)-dimensional all-ones vector. Then, we obtain the following lemma.

**Lemma 7** Suppose that Assumption 3 holds. Then, the system given by (10a), (11a) and (12a) is passive from \( \tilde{u} \) to \( \tilde{x} \) for the following storage function:

\[
S := \sum_{i=1}^n S_i, \quad S_i := \frac{1}{2 \epsilon_{11}} |\xi_{11} - x_i^*|^2 + \sum_{k=2}^{n_i} \frac{1}{2 \epsilon_{kk}} |\xi_{ik}|^2,
\]

where \( x_i^* \) is the \( i \)-th element of \( x^* \).

**Proof.** See Appendix A.1. □

We next treat the dynamics (10b). A state space representation of \( H_j(s) \) in (12b) is given as

\[
\begin{align*}
\dot{\zeta}_{j1} &= \tilde{e}_{j1} h_j, \\
\dot{\zeta}_{jq} &= -\tilde{a}_{jq} \zeta_{jq} + \tilde{c}_{jq} h_j, \quad q = 2, \ldots, r_j, \\
\mu_j &= 1_m^T \lambda_j + \tilde{d}_j h_j,
\end{align*}
\]

where \( \zeta_{jq} \) is the \( q \)-th element of the state \( \zeta_j \in \mathbb{R}^{r_j} \).

**Lemma 8** Suppose that Assumption 3 holds. Then, the system given by (10b), (11b) and (12b) is passive from \( \tilde{x} \) to \( \tilde{\psi} \) for the following storage function:

\[
W := \sum_{j=1}^r W_j, \quad W_j := \frac{1}{2 \epsilon_{11}} |\zeta_{j1} - \mu_j^*|^2 + \sum_{q=2}^{r_j} \frac{1}{2 \epsilon_{qq}} |\zeta_{jq}|^2,
\]

where \( \mu_j^* \) is the \( j \)-th element of \( \mu^* \).

**Proof.** See Appendix A.2. □

Let us finally consider the dynamics (10c). The system \( G_i^+(s) \) is formulated as follows with state \( \rho_i \in \mathbb{R}^{m_i} \).

\[
\begin{align*}
\dot{\rho}_{11} &= [\tilde{e}_{11} w_i]_{11}^+, \\
\dot{\rho}_p &= [-\tilde{a}_{lp} \rho_p + \tilde{c}_{lp} w_i]_{1_p}^+, \quad p = 2, \ldots, m_i, \\
\lambda_i &= 1_{m_i}^T \rho_i + \tilde{d}_i \max\{0, w_i\},
\end{align*}
\]

with an initial state \( \rho_i(0) \geq 0 \), where \( \rho_p \) is the \( p \)-th element of \( \rho_i \).

**Lemma 9** Suppose that Assumption 3 holds. Then, the system given by (10c), (11c) and (12c) is passive from \( \tilde{x} \) to \( \tilde{\eta} \) for the following storage function:

\[
U := \sum_{l=1}^m U_l, \quad U_l := \frac{1}{2 \epsilon_{11}} |\rho_{l1} - \lambda^*_l|^2 + \sum_{p=2}^{m_i} \frac{1}{2 \epsilon_{pp}} |\rho_{lp}|^2,
\]

where \( \lambda^*_l \) is the \( l \)-th element of \( \lambda^* \).

**Proof.** See Appendix A.3. □

Lemmas 7–9 mean that the dynamics (10)–(12) is regarded as a passivity-preserving interconnection of passive systems. Accordingly, we immediately have the following result [6].

**Lemma 10** Define the function \( V := S + W + U \). If Assumption 3 is satisfied, the Lie derivative along with the system (10) of \( V \), denoted by \( \mathcal{L}_A V \), satisfies \( \mathcal{L}_A V \leq 0 \) under (10)–(12).
PROOF. From (A.3), (A.4) and (A.11), we obtain
\[ L_A V = L_p S + L_k W + L_j U \]
\[ \leq - \sum_{i=1}^{n} \left( d_i v_i^2 + \sum_{k=2}^{n_i} \delta_{ik} \xi_k^2 \right) + \ddot{x}^T \ddot{u} \]
\[ - \sum_{j=1}^{r} \left( \ddot{d}_j h_j^2 + \sum_{q=2}^{r_j} \delta_{jq} \eta_j^2 \right) + \ddot{x}^T \ddot{\psi} \]
\[ - \sum_{l=1}^{m} \tilde{d}_l \max\{0, w_l\}^2 + \sum_{p=2}^{m_l} \tilde{\delta}_{lp} \rho_p^2 \right) + \ddot{x}^T \ddot{\eta} \]
\[ \leq \ddot{x}^T (\ddot{u} + \ddot{\psi} + \ddot{\eta}) = 0, \quad (16) \]
since \( \ddot{u} + \ddot{\psi} + \ddot{\eta} = 0 \) is ensured by (4a). \( \square \)

4 Convergence analysis

In this section, we analyze convergence of the trajectories of \( (x, \mu, \lambda) \) generated by (10)–(12) to the set \( \chi^* \). In the sequel, we use the notations \( \ddot{u} := \sum_{i=1}^{n_i} n_i, \ddot{r} := \sum_{j=1}^{r_j} r_j \) and \( \ddot{m} := \sum_{l=1}^{m} m_l \). Also, we define the notations of the state variables as
\[ \xi := [\xi_1^T \ldots \xi_n^T]^T \in \mathbb{R}^\ddot{n}, \]
\[ \zeta := [\zeta_1^T \ldots \zeta_r^T]^T \in \mathbb{R}^{\ddot{r}}, \]
\[ \rho := [\rho_1^T \ldots \rho_m^T]^T \in \mathbb{R}^{\ddot{m}}. \]

The proof relies on the invariance principle for Carathéodory systems [17,18]. It is thus first proved that the system (10)–(12) satisfies the assumptions required by the principle.

Lemma 11 Under Assumption 3, the system (10)–(12) with state \((\xi, \zeta, \rho)\) satisfies the following properties.

(i) There exists a compact and invariant subset \( S \subset \mathbb{R}^n \times \mathbb{R}^r \times \mathbb{R}^m \).
(ii) For each point \((\xi_0, \zeta_0, \rho_0) \in S\), there exists a unique solution of (10)–(12) starting at \((\xi_0, \zeta_0, \rho_0)\).
(iii) The omega-limit set of the unique solution of (10)–(12) is invariant.
(iv) The Lie derivative of the continuous and differentiable function \( V \) along (10) satisfies \( L_A V(\xi, \zeta, \rho) \leq 0 \) for all \((\xi, \zeta, \rho) \in S\).

PROOF. The item (iv) was already proved in Lemma 10. The item (i) also immediately holds since the function \( V \) is radially unbounded. Since (10)–(12) is a projected dynamics, the item (ii) can be proved by following the same procedure as Lemma 4.3 in [17]. Additionally, the assumption (iii) can be also proved in the same way as Lemma 4.4 in [17] and Lemma 4.1 in [7]. \( \square \)

We are now ready to show the main result of this paper.

Theorem 12 Consider the system (10)–(12). Assume that the transfer function \( M_i(s) \) has one or more stable zeros, for all \( i \). If Assumption 3 holds, then \((x, \mu, \lambda)\) approaches one of the constants included in \( \chi^* \) as the time goes to infinity.

PROOF. From Lemma 11, the invariance principle for Carathéodory systems [17,18] is applied to the system (10)–(12), and hence any solution of (10)–(12) starting at \( S \) converges to the largest invariant set in \( \text{cl}\{(\xi, \zeta, \rho) \in S | L_A V(\xi, \zeta, \rho) = 0\} \). From (16), \( L_A V = 0 \) implies that
\[ d_i v_i^2 + \sum_{k=2}^{n_i} \delta_{ik} \xi_k^2 \equiv 0 \forall i = 1, \ldots, n, \quad (17) \]
\[ \ddot{d}_j h_j^2 + \sum_{q=2}^{r_j} \delta_{jq} \eta_j^2 \equiv 0 \forall j = 1, \ldots, r, \quad (18) \]
\[ \tilde{d}_l \max\{0, w_l\}^2 + \sum_{p=2}^{m_l} \tilde{\delta}_{lp} \rho_p^2 \equiv 0 \forall l = 1, \ldots, m. \quad (19) \]

In the sequel, we consider the system trajectories identically satisfying (17)–(19). First, we focus on (17) and the dynamics (13). Since \( M_i(s) \) has one or more zeros, \( d_i > 0 \) or \( n_i \geq 2 \) must be satisfied. If \( d_i > 0 \), we have \( v_i = 0 \) and, otherwise, we have \( \xi_{ik} = 0 \) and \( \eta_{jk} = 0 \) for all \( k = 2, \ldots, n_i \). In the latter case, substituting \( \xi_{ik} = 0 \) and \( \eta_{jk} = 0 \) into (13b) yields \( v_i = 0 \). We thus conclude that \( v_i = 0 \) holds for all \( i \), and hence the state trajectories must identically satisfy
\[ \nabla f(x) + \nabla g(x)\lambda + A^T \mu = 0. \quad (20) \]

We also see from (13a) and (13c) that \( \xi_{i1} = 0 \) and \( x_i \) must be constant for all \( i \).

Next, we focus on (18) and the dynamics (14). Since \( x \) is constant as shown above, \( h = Ax - b \) must be also constant. Now, if \( h_j \neq 0 \) for some \( j \), \( \zeta_j \) must diverge from (14a), which contradicts boundedness of \( \zeta_j \). We thus conclude that the trajectories meet \( \zeta_j \equiv 0 \). In other words, the following equation identically holds.
\[ Ax - b = 0 \quad (21) \]

Accordingly, \( \zeta_j \) is constant. We also have \( \xi_{jq} \equiv 0 \forall q = 2, \ldots, r_j \) from (18). In summary, we conclude from (14c) that the trajectories satisfying (17)–(19) meet \( \mu_j \equiv \bar{c}_{j1}\zeta_j \) and it is identically constant.
Let us next consider about (19) and the dynamics (15). Since $x$ is constant as shown above, $w = g(x)$ is also constant. Now, let us focus on (15a). If mode 1 is active, then $\dot{\rho}_1 = 0$ holds. Otherwise, $\dot{\rho}_1 = w_l$ holds, i.e., $\rho_1$ is constant. Then, $\dot{\rho}_1 = 0$ holds since $\dot{\rho}_1 \neq 0$ contradicts the boundedness of $\rho_1$. Thus, we have $\dot{\rho}_1 = 0$ for all $l=1,\ldots,m$. This means that

$$\rho_1 \geq 0, \ w_l \leq 0, \ \rho_1 w_l = 0, \ \forall l=1,\ldots,m$$

(22)
is identically satisfied. From (19), we obtain

$$\dot{d}_l \max\{0,w_l\} \equiv 0, \ \rho_p = 0, \ \forall p=2,\ldots,m.$$  

From (15c), we also see that $\lambda \equiv \hat{c}_1 \rho_1$ and it is identically constant. Moreover, (22) means that the trajectories identically satisfy

$$\lambda \geq 0, \ g(x) \leq 0, \ \lambda \circ g(x) = 0.$$  

(23)

In summary, we conclude that the state trajectories identically satisfying (17)–(19) provide a constant $(x, \mu, \lambda)$ satisfying (20), (21) and (23). In other words, $(x, \mu, \lambda) \in \chi^*$ holds in the positively invariant set in $\mathcal{L}_AV \equiv 0$. This completes the proof.

Remark that the assumption on the zeros of $M_i(s)$ validates the hypothesis extracted from the toy problem in the end of the previous section that leading the phase is the key to relax the strict convexity assumption.

Similar generalizations of the primal-dual dynamics are presented [12,13]. In these publications, $M_i(s)$ is assumed to be proper, positive real and having a pole at the origin, which is almost compatible with ours, and asymptotic optimality is proved under strict convexity of the cost function. The primary contribution of this paper relative to [12,13] is to show that the strict convexity assumption can be relaxed to convexity under the additional condition on the zeros of $M_i(s)$. Besides, we can add two additional contributions as below.

First, our algorithm can treat a general convex constraint function $g$, while [12] and [13] deal with the problems without inequality constraints and with linear inequality constraints, respectively. Namely, we immediately have a fully generalized result of [12,13] as follows.

**Corollary 13** Consider the system (10)–(12). Suppose that Assumption 3 holds. If the cost function $f$ is strictly convex, $(x, \mu, \lambda)$ approaches one of the constants included in $\chi^*$ as the time goes to infinity.

**PROOF.** Noticing (A.2) and (16), $\mathcal{L}_AV \equiv 0$ implies

$$(x - x^*)^T(\nabla f(x) - \nabla f(x^*)) \equiv 0.$$  

(24)

Due to strict convexity of $f$, (24) is equivalent to $x \equiv x^*$. Also, $x$ is constant because $x^*$ is the unique solution to (3). From these results, we can prove that $\mu$ and $\lambda$ are constant in the same way as Theorem 12. In addition, we obtain that $x$ satisfies (21) and (23). Since $x$, $\mu$ and $\lambda$ are constant, $v = -\nabla f(x) - \nabla g(x) \lambda - A^T \mu$ is also constant. Then, we have $v \equiv 0$ since $\xi_i = v_i \neq 0$ contradicts the boundness of $\xi$. Thus, (20) holds. As a result, $(x, \mu, \lambda)$ is the constant satisfying KKT conditions when $\mathcal{L}_AV \equiv 0$. This completes the proof.

In addition, [12,13] take the diagonal transfer function matrices $M(s)$ and $H(s)$ as in (11) and we can also generalize the structure based on the passivity paradigm. To this end, we first replace $M(s)$ and $H(s)$ in (11) by

$$M(s) = \text{diag} (M_1(s), \ldots, M_n(s)) + M'(s),$$

$$H(s) = \text{diag} (H_1(s), \ldots, H_n(s)) + H'(s),$$

(25a, 25b)

where $M'(s)$ and $H'(s)$ are possibly non-diagonal transfer function matrices assumed to be strictly positive real. The additional design flexibility associated with $M'(s)$ and $H'(s)$ may contribute to improvement of the performance. It is now not difficult to confirm that adding $M'(s)$ and $H'(s)$ does not affect all the signals at the stationary state. From passivity preservation w.r.t. parallel interconnections, both of $M(s)$ and $H(s)$ are preserved to be passive. Accordingly, we can immediately prove the following corollary.

**Corollary 14** Consider the system (10), (25), (11c) and (12). Assume that the transfer function $M_i(s)$ has one or more stable zeros, for all i. If Assumption 3 holds, then $(x, \mu, \lambda)$ approaches one of the constants included in $\chi^*$ as the time goes to infinity.

**PROOF.** Redefine the energy function $V$ by adding the storage functions of $M'(s)$ and $H'(s)$. It is then immediate to see from passivity preservation of $M(s)$ and $H(s)$ that the inequality (21) holds even after adding $M'(s)$ and $H'(s)$. The subsequent discussions are the same as Theorem 12. 

5 Relation to augmented Lagrangian method

In this section, we explore relations between the present algorithm (10) and augmented Lagrangian-based primal-dual dynamics in [14,15].

Let us first focus on [15], where the authors present the
the figure that (28) is equivalent to (10) with $M_l(s) = \frac{1}{s}$, $H_f(s) = \frac{1}{s}$ and $G_l^+(s) = \left(\frac{1}{s}\right)^+$. We immediately see from following dynamics to solve (27) is presented in [14].

$$\Phi \in \mathbb{R}^{m \times n}, \phi \in \mathbb{R}^m \text{ and } \theta \in \mathbb{R}^n. \text{ Then, the following dynamics to solve (27) is presented in [14].}$$

$$\begin{align*}
\dot{x} &= - (\nabla f(x) + \nabla g(x) \lambda + A^\top \mu), \quad (26a) \\
\mu &= \zeta + (A x - b), \quad (26b) \\
\zeta &= A x - b, \quad (26c) \\
\lambda &= \rho + \max\{0, g(x)\}, \quad (26d) \\
\dot{\rho} &= [g(x)]_\rho^+, \quad \rho(0) \geq 0. \quad (26e)
\end{align*}$$

It is not difficult to confirm that (26) is illustrated in the block in Fig. 3. Comparing Fig. 3 with Fig. 2, we immediately see that (26) is a special case of (10). Specifically, if we take $M_i(s) = \frac{1}{s}$, $H_f(s) = \frac{1}{s}$ and $G_l^+(s) = \left(\frac{1}{s}\right)^+ + (1)^+$. We thus conclude that the present dynamics is a generalization of (26).

Next, we investigate the relation to [14], where the authors address the following linear programming problem.

$$\begin{align*}
\text{minimize} \quad & \theta^\top x, \\
\text{subject to} \quad & \Phi x - \phi \leq 0,
\end{align*}$$

where $\Phi \in \mathbb{R}^{m \times n}$, $\phi \in \mathbb{R}^m$ and $\theta \in \mathbb{R}^n$. Then, the following dynamics to solve (27) is presented in [14].

$$\begin{align*}
x &= \xi - \theta - \Phi^\top \lambda, \quad (28a) \\
\dot{\xi} &= \theta - \Phi^\top \lambda, \quad (28b) \\
\dot{\lambda} &= [\Phi x - \phi]_\lambda^+, \quad \lambda(0) \geq 0, \quad (28c)
\end{align*}$$

which is illustrated in Fig. 4. We immediately see from the figure that (28) is equivalent to (10) with $M_i(s) = \frac{1}{s}$ and $G_l^+(s) = \left(\frac{1}{s}\right)^+$. It is thus concluded that (28) is also a special example of (10).

In the reminder of this section, we clarify benefits of the generalized algorithm (10) over [14,15]. While the advantage over [14] is obvious since our approach is not restricted to the linear programming, the result of [15]

As stated in [14,19], optimization algorithms may suffer from a variety of noises. For example, in online optimization, the cost and constraint functions may be defined by the real-time data including noises. In addition, distributed implementation of the algorithm may suffer from the noises at the communication channels. Regarding the noise reduction, it is to be noted that only one of the transfer functions $M_i(s), H_f(s)$ and $G_l^+(s)$ in [14,15] are strictly proper, which means that the gain decay of the open-loop systems over the high frequency domain is 20dB/dec. On the other hand, our approach allows one to choose strictly proper $M_i(s), H_f(s)$ and $G_l^+(s)$, which would achieve a better roll-off and hence better noise reduction. Besides, the generalization presented in this paper allows one to shape the open loop systems more flexibly, which would contribute to a better disturbance rejection and/or acceleration of convergence speed.

We exemplify the above hypothesis through simulation. Let us consider the linear programming problem (27) with

$$\Phi = \begin{bmatrix} -1 & 0 \\ 0 & -1 \\ 4 & 3 \\ 1 & 2 \end{bmatrix}, \quad \phi = \begin{bmatrix} 0 \\ 0 \\ 10 \\ 5 \end{bmatrix}, \quad \theta = \begin{bmatrix} -2 \\ -3 \end{bmatrix}$$

whose optimal solution is $x^* = [1 \ 2]^\top$. We prepare the following three dynamics to solve the problem.

- Case 1 ([14]) : $M_i(s) = \frac{s+1}{s}$, $M_l(s) = \frac{s^{+1}}{s}$, $G_l^+(s) = \left(\frac{1}{s}\right)^+$, \forall l
- Case 2 : $M_i(s) = \frac{s^{+1}}{s}$, $M_l(s) = \frac{s^{+1}}{s^{+25}}$, $G_l^+(s) = \left(\frac{1}{s}\right)^+$, \forall l
- Case 3 : $M_i(s) = \frac{s^{+1}}{s^{+10}}$, $M_l(s) = \frac{s^{+10}}{s^{+25}}$, $G_l^+(s) = \left(\frac{1}{s}\right)^+ + (\frac{1}{s^{+0.05}})^+$, \forall l
Note that all of them satisfy the assumptions in Theorem 12, and case 1 coincides with the algorithm in [14].

We run the algorithms with the above transfer functions while adding the zero mean Gaussian noises with frequency components greater than 10rad/s to ξ. The initial states are set as ξ(0) = 0, ρ(0) = 0 for all cases. We see from Fig. 5(a) that the algorithm of case 1, namely [14], is heavily affected by the noise. On the other hand, we also see that the effects are drastically reduced in Fig. 5(b) and (c), where the corresponding dynamics have strictly proper $M_i(s)$ and $G_i^+(s)$. Comparing (b) and (c), it is also confirmed that the convergence speed can be improved by appropriately supplying zeros to these transfer functions.

6 Conclusion

In this paper, we have presented a generalized primal-dual dynamics based on the concept of passivity. We have then proved asymptotic optimality for general convex optimization with a not necessarily strict convex cost function by supplying at least one stable zero to one of the integrators in the dynamics based on the passivity paradigm together with the invariance principle for Carathéodory systems. The present algorithm has also been shown to generalize existing augmented Lagrangian-based primal-dual dynamics. We have then demonstrated the benefit of the present generalization.

References

[1] K. Arrow, L. Hurwicz and H. Uzawa: Studies in Linear and Non-Linear Programming, Stanford University Press, 1958.
[2] S. Boyd and L. Vandenberghe: Convex Optimization, Cambridge University Press, 2004.
[3] S. Boyd, N. Parikh, E. Chu, B. Peleato and J. Eckstein: Distributed optimization and statistical learning via the alternating direction method and multipliers, Foundations and Trends in Machine Learning, vol. 3, no. 1, pp. 1–122, 2011.
[4] J. Wang and N. Elia: A control perspective for centralized and distributed convex optimization, Proceedings of 50th IEEE Conference on Decision and Control, pp. 557–561, 2011.
[5] S. Liu, Z. Qiu and L. Xie: Continuous-time distributed convex optimization with set constraints, IFAC Proceedings Volumes, vol. 47, no. 3, pp. 9762–9767, 2014.
[6] T. Hatanaka, N. Chopra, M. Fujita and M.W. Spong: Passivity-Based Control and Estimation in Networked Robotics, Springer-Verlag, 2015.
[7] H. Khalil: Nonlinear systems third edition, Prentice Hall, 2002.
[8] T. Stegink, C.D. Persis and A. van der Schaft: A unifying energy-based approach to stability of power grids with market dynamics, IEEE Transactions on Automatic Control, vol. 62, no. 6, pp. 2612–2622, 2017.
[9] T. Hatanaka, X. Zhang, W. Shi, M. Zhu and N. Li: An integrated design of optimization and physical dynamics for energy efficient buildings: a passivity approach, Proceedings of 1st IEEE Conference on Control Technology and Applications, pp. 1050–1057, 2017.
[10] T. Hatanaka, X. Zhang, W. Shi, M. Zhu and N. Li: Physics-integrated hierarchical/distributed HVAC optimization for multiple buildings with robustness against time delays, Proceedings of 56th IEEE Conference on Decision and Control, pp. 6573–6579, 2017.
[11] T. Hatanaka, N. Chopra, T. Ishizaki and N. Li: Passivity-based distributed optimization with communication delays using PI consensus algorithm, IEEE Transactions on Automatic Control, to appear, 2019.
[12] H. Yamamoto and K. Tsumura: Control of smart grids based on price mechanism and network structure, Mathematical Engineering Technical Reports, The University of Tokyo, METR 2012-11, 2012.
[13] J.T. Wen and M. Arcak: A unifying passivity framework for network flow control, IEEE Transactions on Automatic Control, vol. 49, no. 2, pp. 162–174, 2004.
[14] D. Richert and J. Cortés: Robust distributed linear programming, IEEE Transactions on Automatic Control, vol. 60, no. 10, pp. 2567–2582, 2015.
[15] A. Cherukuri, A.D. Domínguez-García and J. Cortés: Distributed coordination of power generators for a linearized optimal power flow problem, 2017 American Control Conference, pp. 3962–3967, 2017.
[16] H. Zhang, J. Wei, P. Yi and X. Hu: Projected primal-dual gradient flow of augmented Lagrangian with application to distributed maximization of the algebraic connectivity of a network, Automatica, vol. 98, pp. 34–41, to appear, 2018.
[17] A. Cherukuri, E. Mallada and J. Cortés: Asymptotic convergence of constrained primal-dual dynamics, Systems & Control Letters, vol. 87, pp. 10–15, 2016.
[18] A. Bacciotti and F. Ceragioli: Nonpathological Lyapunov functions and discontinuous Carathéodory systems, Automatica, vol. 42, no. 3, pp. 453–458, 2006.
[19] J.W. Simpson-Porco, B.K. Poolla, N. Monshizadeh and F. Dörfler: Input-output performance of linear-quadratic saddle-point algorithms with application to distributed resource allocation problems, arXiv, arXiv:1803.02182, 2018.

A Proof of lemmas

A.1 Proof of Lemma 7

The Lie derivative of $S_i$ along (10a), denoted by $\mathcal{L}_P S_i$, is given as

$$
\mathcal{L}_P S_i = (\xi_{i1} - x_{i1}^2) v_i + \sum_{k=2}^{n_i} \xi_{ik} v_i - \sum_{k=2}^{n_i} \delta_{ik} \xi_{ik}^2
$$

$$
= \left( \sum_{k=1}^{n_i} \xi_{ik} - x_{i1}^2 \right) v_i - \sum_{k=2}^{n_i} \delta_{ik} \xi_{ik}^2
$$

$$
= \bar{x}_i v_i - d_i v_i^2 - \sum_{k=2}^{n_i} \delta_{ik} \xi_{ik}^2,
$$
This completes the proof of Lemma 8. We thus obtain where \( \bar{\delta} \) holds because of (2). From (A.1) and (A.2), we have

\[
L_P S = \bar{x}^T v - \sum_{i=1}^{n} \left( d_i v_i^2 + \sum_{k=2}^{n_i} \delta_{ik} \xi_{ik}^2 \right). \tag{A.1}
\]

Due to (9a) and the definition of \( \bar{u} \), we have \( v = -(\nabla f(x) - \nabla f(x^*)) + \bar{u} \) and hence

\[
\bar{x}^T v = -\bar{x}^T (\nabla f(x) - \nabla f(x^*)) + \bar{x}^T \bar{u} \leq \bar{x}^T \bar{u}. \tag{A.2}
\]

holds because of (2). From (A.1) and (A.2), we have

\[
L_P S \leq \bar{x}^T \bar{u} - \sum_{i=1}^{n} \left( d_i v_i^2 + \sum_{k=2}^{n_i} \delta_{ik} \xi_{ik}^2 \right) \leq \bar{x}^T \bar{u}. \tag{A.3}
\]

This completes the proof of Lemma 7.

A.2 Proof of Lemma 8

We take the notation \( \hat{\lambda} := \lambda - \lambda^* \). The Lie derivative of \( U_t \) along (10c), denoted by \( L_t U_t \), is given by

\[
L_t U_t = -\frac{1}{\hat{c}_{t1}} \lambda^*_t [\hat{c}_{t1} w_t]^+_\rho_{t1} + \sum_{p=1}^{m_t} \frac{1}{\hat{c}_{tp}} \rho_p [-\hat{a}_{tp} \rho_p + \hat{c}_{tp} w_t]^+_\rho_{tp}, \tag{A.5}
\]

where \( \hat{a}_{t1} = 0 \). Now, the equation

\[
\rho_p [-\hat{a}_{tp} \rho_p + \hat{c}_{tp} w_t]^+_\rho_{tp} = \rho_p (-\hat{a}_{tp} \rho_p + \hat{c}_{tp} w_t). \tag{A.6}
\]

holds for any \( p = 1, \ldots, m_t \) since \( \rho_p = 0 \) must hold in the case of \([-\hat{a}_{tp} \rho_p + \hat{c}_{tp} w_t]^+_\rho_{tp} \neq -\hat{a}_{tp} \rho_p + \hat{c}_{tp} w_t \) from the definition of \([.]^+_\rho\). Suppose now that \([\hat{c}_{t1} w_t]^+_\rho_{t1} \neq \hat{c}_{t1} w_t \). Then, \([\hat{c}_{t1} w_t]^+_\rho_{t1} = 0 \) and \( \hat{c}_{t1} w_t < 0 \) must hold and hence

\[
0 = -\lambda^*_t [\hat{c}_{t1} w_t]^+_\rho_{t1} \leq -\lambda^*_t \hat{c}_{t1} w_t \tag{A.7}
\]

because of \( \lambda^*_t \geq 0 \). The inequality in (A.7) holds in the case of \([\hat{c}_{t1} w_t]^+_\rho_{t1} = \hat{c}_{t1} w_t \). Substituting (A.6) and (A.7) into (A.5) yields

\[
L_t U_t \leq \left( \rho_{t1} - \lambda^*_t \right) w_t + \sum_{p=2}^{m_t} \rho_p (-\hat{\delta}_p \rho_p + w_t), \tag{A.8}
\]

where \( \hat{\delta}_p := \hat{a}_{tp}/\hat{c}_{tp} > 0 \). (A.8) is further rewritten as

\[
L_t U_t \leq \left( \sum_{p=1}^{m_t} \rho_p - \lambda^*_t \right) w_t - \sum_{p=2}^{m_t} \hat{\delta}_p \rho_p^2 \leq \bar{\lambda}_t w_t - \hat{d}_t \max\{0, w_t\} w_t - \sum_{p=2}^{m_t} \hat{\delta}_p \rho_p^2,
\]

where \( \bar{\lambda}_t := \bar{\lambda}_t \hat{a}_{tp}/\hat{c}_{tp} > 0 \). (A.4) is further rewritten as

\[
L_t U_t \leq \left( \sum_{p=1}^{m_t} \rho_p - \lambda^*_t \right) w_t - \sum_{p=2}^{m_t} \hat{\delta}_p \rho_p^2
\]

This completes the proof of Lemma 8.
where $\tilde{\lambda}_l$ is the $l$-th element of $\tilde{\lambda}$.

Accordingly, $\mathcal{L}_I U$ satisfies

$$
\mathcal{L}_I U \leq \tilde{x}^\top g(x) - \sum_{l=1}^m \left( \hat{d}_l \max\{0, w_l\}^2 + \sum_{p=2}^{m_1} \hat{\delta}_{lp} \rho_p^2 \right).
$$

Here, we focus on $\tilde{\lambda}^\top g(x)$. Since $\lambda \geq 0$, $g(x^*) \leq 0$ and $\lambda^* \circ g(x^*) = 0$, we obtain

$$
\tilde{\lambda}^\top g(x) = \tilde{\lambda}^\top (g(x) - g(x^*)) + \lambda^\top g(x^*) - (\lambda^*)^\top g(x^*)
= \tilde{\lambda}^\top (g(x) - g(x^*)) + \lambda^\top g(x^*)
\leq \tilde{\lambda}^\top (g(x) - g(x^*)) = \sum_{l=1}^m \tilde{\lambda}_l (g_l(x) - g_l(x^*)).
$$

(A.9)

From convexity of $g_l$, we have $g_l(x) - g_l(x^*) \leq \bar{x}^\top \nabla g_l(x)$ and $g_l(x^*) - g_l(x) \leq -\bar{x}^\top \nabla g_l(x^*)$. Using these inequalities with $\lambda_l \geq 0$ and $\lambda^*_l \geq 0$, we have

$$
\tilde{\lambda}_l (g_l(x) - g_l(x^*))
= \lambda_l (g_l(x) - g_l(x^*)) + \lambda^*_l (g_l(x^*) - g_l(x))
\leq \lambda_l \bar{x}^\top \nabla g_l(x) - \lambda^*_l \bar{x}^\top \nabla g_l(x^*).
$$

(A.10)

From (A.9) and (A.10),

$$
\tilde{\lambda}^\top g(x) \leq \bar{x}^\top \left( \sum_{l=1}^m \lambda_l \nabla g_l(x) - \lambda^*_l \nabla g_l(x^*) \right) = \bar{x}^\top \eta
$$

holds. From these results, we obtain

$$
\mathcal{L}_I U \leq \bar{x}^\top \eta - \sum_{l=1}^m \left( \hat{d}_l \max\{0, w_l\}^2 + \sum_{p=2}^{m_1} \hat{\delta}_{lp} \rho_p^2 \right)
\leq \bar{x}^\top \eta.
$$

(A.11)

This completes the proof of Lemma 9.