Critical solutions of nonminimally coupled scalar field theory and first-order thermodynamics of gravity

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Analytical solutions of nonminimally coupled scalar field cosmology corresponding to critical scalar field values constitute a potential challenge to the recent first-order thermodynamics of scalar-tensor gravity (a formalism picturing general relativity as the zero-temperature equilibrium state for modified gravity). The critical solutions are unstable with respect to homogeneous perturbations, hence unphysical.

I. INTRODUCTION

The idea that the Einstein equations may not be fundamental and that gravity could, after all, be an emergent phenomenon, has been contemplated for a long time [1–10]. Probably the most significant step in the emergent gravity program occurred when Jacobson derived the Einstein equations purely with thermodynamical considerations [11]. A decade later, modified gravity was brought into this picture when the field equations of (quadratic) metric gravity were derived with a similar thermodynamical procedure (here $R$ denotes the Ricci scalar of the spacetime metric $g_{ab}$ and $f(R)$ is a nonlinear function of $R$). A key idea was that general relativity (GR) constitutes a state of thermal equilibrium while modified gravity is an excited state [12]. In this picture, the approach of modified gravity to GR would be a sort of relaxation to thermal equilibrium. The role of a “viscosity of gravity” in this process was suggested in [12] and clarified in later work [13].

In spite of many years of research on spacetime thermodynamics, little progress has really been made with respect to the original works [11, 12]. In particular, no equation describing the approach to the GR equilibrium state of modified gravity has been found and the order parameter (the “temperature of gravity”) has not been identified. This state of ignorance about these key ingredients is very disappointing, especially in view of the fact that modified gravity is extremely popular nowadays and is the subject of intense research. The main motivation for studying alternative theories of gravity comes from cosmology: the standard Λ-Cold Dark Matter model of the universe based on Einstein theory invokes a completely ad hoc dark energy to explain the present acceleration of the universe discovered in 1998 with Type Ia supernovae (see [14] for a review). Many researchers dissatisfied with the idea of dark energy have resorted to modified gravity at large scales instead [15, 16]. While this approach has its problems, it has also been shown as a proof of principle that it can explain the cosmic acceleration without introducing dark energy. The class of $f(R)$ theories of gravity is particularly popular for this purpose (see the reviews [17, 18]).

There are other independent and sound motivations to study deviations from GR. As soon as corrections are introduced, gravity deviates from GR and exhibits higher order equations of motion or extra degrees of freedom. Likewise, the low-energy limit of the simplest string theory, the bosonic string, is an $\omega = -1$ Brans-Dicke theory [20, 21] (where $\omega$ is the parameter of the theory called “Brans-Dicke coupling” [22]). Returning to cosmology, but in the early (instead of late) universe realm, Starobinski inflation [23] seems to be the inflationary scenario currently favoured by observations [24] and is based on the Lagrangian density $R + \alpha R^2$ motivated by quantum corrections to the Einstein-Hilbert Lagrangian $R$.

From the point of view of emergent gravity it is intuitive that, as soon as extra degrees of freedom are added to the usual massless spin two modes of GR and are excited, one deviates from GR in what could be called a thermally excited state, and that GR represents a state of “thermal equilibrium” at lower “temperature of gravity”. The problem is that this “temperature of gravity” and the “approach to equilibrium” are unknown.

A recent alternative approach [25, 26] to the general picture of modified gravity as an excited state of GR is completely different from Jacobson’s thermodynamics of spacetime. It is minimalistic in its assumptions and, contrary to spacetime thermodynamics, does not need to assume results from quantum field theory in curved spacetime (the Unruh temperature of uniformly accelerated observers) or horizon thermodynamics (the Bekenstein-Hawking entropy). The key idea is writing the field equations of modified gravity in the form of effective Einstein equations with a right-hand side formed from all geometric terms different from the Einstein tensor (plus “real” matter, if present). This is possible when the gradient of the gravitational scalar field in the theory is timelike. It is a fact that, for the classes of theories examined (“first generation” scalar-tensor gravity and viable Horndeski

\[ f(R) = R + \alpha R^2 \]
Theories), this effective stress-energy tensor assumes the form of a dissipative fluid with a spacelike heat flux and shear and bulk viscosity. The next step consists of applying to this dissipative fluid Eckart’s first-order thermodynamics. The latter is well-known to be non-causal and to suffer from instabilities but nevertheless is still the most widely used model of dissipative fluid in relativity. We regard first-order thermodynamics of scalar-tensor gravity as a first step to be eventually replaced by a more realistic model. In spite of its crudeness, the first-order thermodynamics of scalar-tensor gravity has identified clearly a notion of “temperature of gravity” and has provided an equation describing the approach to the GR equilibrium state, or departures from it. Generic predictions of the formalism are: a) GR is the zero-temperature state of equilibrium; b) near spacetime singularities or near singularities of the effective gravitational coupling, gravity is “hot” in the sense that it departs from GR, with “temperature” diverging at these singularities; c) the expansion of spacetime generally “cools” gravity bringing it closer to the GR equilibrium state; d) theories in which the gravitational scalar field is non-dynamical (e.g., cuscuton gravity) are also states of equilibrium at zero temperature. This fact is explained by the fact that no extra degree of freedom with respect to GR is excited.

The first-order thermodynamics of scalar-tensor gravity introduced in for “first generation” scalar-tensor theories has been generalized to viable Horndeski gravity and then applied to scalar-tensor cosmology. Although the formalism is intriguing in many respects and explains certain features of scalar-tensor gravities, or of particular solutions of these theories, it is crucial to attempt to falsify the main ideas and predictions and look for places where the formalism could fail. In this regard, the inspection of particular theories or of particular analytical solutions could uncover corners where the formalism breaks down, eventually highlighting its limits of validity or leading to its rejection altogether. Certain critical solutions of nonminimally coupled scalar field cosmology constitute a potential challenge to the first-order thermodynamics because they are associated with an effective gravitational coupling that diverges identically through the entire cosmological dynamics and with an ill-defined effective temperature of gravity. Although it has been shown that the universe cannot pass through such singular points dynamically, strictly speaking these special solutions evade this theoretical result because they are already at infinite . We would like to study the stability of these solutions. If stable, one should worry about them because the effective temperature is undefined as a result of the product of the divergent with a vanishing quantity, and the first order thermodynamical formalism could potentially break down.

In the next section we make these arguments explicit, introducing the field equations of nonminimally coupled scalar field cosmology and the key equations of the first-order thermodynamics of scalar-tensor gravity, as well as the critical solutions corresponding to infinite . The stability of these solutions with respect to homogeneous perturbations is studied in Section while contains our conclusions and a discussion.

We adopt the notation of Ref., using units in which the speed of light is unity and the metric signature is . is Newton’s constant, and is the potential of the gravitational scalar field.

II. EQUATIONS AND UNPERTURBED SOLUTIONS

The action for gravity with a nonminimally coupled scalar field is

\[
S = \int d^4x \sqrt{-g} \left[ \frac{1}{\kappa} - \xi \phi^2 \right] R - \frac{1}{2} \nabla^a \phi \nabla_c \phi - V(\phi) \right],
\]

where is the Ricci scalar of the spacetime metric with determinant . The vacuum field equations are

\[
G_{ab} = \frac{\kappa}{1 - \xi \phi^2} \left[ \nabla_a \phi \nabla_b \phi - \frac{1}{2} g_{ab} \nabla^c \phi \nabla_c \phi - \frac{V}{2} g_{ab} \right.
\]

\[
+ \xi (g_{ab} \Box - \nabla_a \nabla_b) \left( \phi^2 \right),
\]

\[
\Box \phi - \frac{dV}{d\phi} - \xi R \phi = 0,
\]

where is the Ricci tensor, is the Einstein tensor, and is the curved spacetime of d’Alembertian. In the spatially flat Friedmann-Lemaître-Robertson-Walker (FLRW) geometry

\[
ds^2 = -dt^2 + a^2(t) \left( dx^2 + dy^2 + dz^2 \right),
\]

the field equations assume the form

\[
6 \left[ 1 - \xi (1 - 6\xi) \kappa \phi^2 \right] \left( \dot{H} + 2H^2 \right) - \kappa (6\xi - 1) \dot{\phi}^2
\]

\[-4\kappa V + 6\kappa \xi \phi V' = 0,
\]

\[
\frac{\kappa}{2} \dot{\phi}^2 + 6\kappa \phi \dot{\phi}^2 - 3H^2 \left( 1 - \kappa \phi^2 \right) + \kappa V = 0,
\]

\[
\dot{\phi} + 3H \phi + \xi R \phi + V = 0,
\]
where an overdot denotes differentiation with respect to the comoving time $t$ and $H \equiv \dot{a}/a$ is the Hubble function.

When $\phi \neq 0$, two of the three equations are independent and one can derive one of them from the other two. Since the cosmic scale factor $a(t)$ of a spatially flat universe only enters the equations of motion in the combination $H \equiv \dot{a}/a$, the phase space consists of the three dimensions $(\phi, \dot{\phi}, H)$. However, Eq. (2.6) is a first order constraint ("Hamiltonian" or "scalar" constraint) akin to an energy conservation equation in point particle mechanics, that forces the orbits of the solutions to move on a two-dimensional subset of this three-dimensional phase space. This feature is made evident by the fact that Eq. (2.10) yields $\phi$ for any given pair of values of $\phi$ and $H$:

$$
\dot{\phi}(\phi, H) = -6\xi H\phi \pm \left[36\xi^2 H^2 \phi^2 + \frac{6H^2}{\kappa} \left(1 - \kappa \phi^2\right)\right]^{-1/2} - 2V(\phi). \quad (2.8)
$$

The double sign in front of the square root in Eq. (2.8) describes the fact that the "energy" submanifold is usually comprised of two sheets which join at the points of the phase space where the argument of this square root vanishes (if the latter is negative there is a region of the phase space forbidden to the orbit of the solutions). This structure of the $(H, \phi, \dot{\phi})$ phase space is discussed at length in [54] for nonminimally coupled scalar field cosmology and in [56] for more general scalar-tensor cosmology.

The nonminimal coupling of the scalar field introduces, in principle, the possibility of a negative effective gravitational coupling [57–61]

$$
G_{\text{eff}} = \frac{G}{1 - \kappa \xi \phi^2}. \quad (2.9)
$$

Although there are rather compelling reasons to select conformal coupling $\xi = 1/6$ [62, 70], we discuss general, but positive, values of the nonminimal coupling constant $\xi$. When $\xi > 0$ there are two critical values of the nonminimally coupled scalar $\phi$,

$$
\pm \phi_c \equiv 1 \pm \frac{1}{\sqrt{\kappa \xi}}. \quad (2.10)
$$

By defining

$$
\psi \equiv 1 - \kappa \xi \phi^2, \quad (2.11)
$$

the action [24, 12] is rewritten as the more familiar scalar-tensor action

$$
S = \frac{1}{2\kappa} \int d^4 x \sqrt{-g} \left[\psi R - \frac{\omega(\psi)}{\psi} \nabla^\mu \psi \nabla_\mu \psi - U(\psi)\right], \quad (2.12)
$$

where $U(\psi) = V(\phi(\psi))$.

The effective temperature $T$ of scalar-tensor gravity in the context of Eckart’s first-order thermodynamics for the effective fluid equivalent of the Brans-Dicke-like scalar $\phi$ was derived in [25, 26]. Its product with the effective thermal conductivity $K$ is [25, 26]

$$
K_T = \frac{\sqrt{-\nabla^\mu \psi \nabla_\mu \psi}}{\kappa \psi} = \frac{2\kappa |\phi| \sqrt{-\nabla^\mu \phi \nabla_\mu \phi}}{1 - \kappa \xi \phi^2}. \quad (2.13)
$$

This formula is derived in the general theory. Technically, in unperturbed FLRW cosmology, the spatial heat flux and shear viscosity are absent in order to preserve the spatial homogeneity and isotropy (but Eq. (2.13) still makes sense, as discussed in [49]), but the viscous pressure and bulk viscosity remain and are given by [25, 26, 30, 49]

$$
P_{\text{viscous}} = -\zeta \Theta, \quad \zeta = -\frac{K_T}{3} \quad (2.14)
$$

according to Eckart’s constitutive laws, where $\Theta = 3H$ is the expansion scalar and $\zeta$ is the effective bulk viscosity coefficient [25, 26, 30, 49]. As for $K_T$, the critical solutions imply an ill-defined bulk viscosity coefficient $\zeta$. However, as soon as the critical FLRW universes are perturbed (as we do here), the effective temperature and bulk and shear viscosity coefficients are well-defined again.

Here we examine a few analytical solutions of FLRW cosmology sourced by a nonminimally coupled scalar field that describe spatially flat FLRW universes with constant Ricci scalar

$$
R = 6 \left(\dot{H} + 2H^2\right) = 6C \quad (2.15)
$$

reported in [54, 55]. They are given by

$$
\phi = \pm \phi_c \equiv \frac{1}{\sqrt{\kappa \xi}} \quad (2.16)
$$

and

$$
H_c(t) = \sqrt{\frac{C}{2}} \tanh \left(\sqrt{2C} t\right) \quad \text{for} \ C > 0, \quad (2.17)
$$

$$
H_c(t) = \frac{1}{2t} \quad \text{for} \ C = 0, \quad (2.18)
$$

$$
H_c(t) = -\sqrt{\frac{|C|}{2}} \tan \left(\sqrt{2|C|} t\right) \quad \text{for} \ C < 0, \quad (2.19)
$$

corresponding to the scale factors

$$
a_c(t) = a_0 \sqrt{\cosh \left(\sqrt{2C} t\right)}, \quad (2.20)
$$

$$
a_c(t) = a_0 \sqrt{t}, \quad (2.21)
$$

$$
a_c(t) = a_0 \sqrt{\cos \left(\sqrt{2C} t\right)}, \quad (2.22)
$$

respectively, where $a_0$ is a constant.
There are also critical de Sitter spaces\footnote{We refer to both exponentially expanding or contracting spatially flat FLRW universes (in comoving time) as “de Sitter spaces”, although this terminology is normally restricted to expanding spaces.} with constant Hubble function \[54\]

\[\phi = \pm \phi_c, \quad H_c = \pm \sqrt{\frac{C}{2}} \quad \text{for} \quad C > 0, \quad (2.23)\]

or \(a(t) = a_0 e^{H_c t}\). de Sitter universes with constant scalar field are the only fixed points of the dynamical system \[25, 26\] in phase space. For \(C = 0\), these de Sitter universes degenerate into Minkowski spacetime.

All these solutions satisfy the field equations \[25, 26\] provided that

\[V_c \equiv V(\pm \phi_c) = 0, \quad (2.24)\]

\[V_c' \equiv \frac{dV}{d\phi} \bigg|_{\pm \phi_c} = \mp 6\xi C \phi_c. \quad (2.25)\]

The critical values \(\pm \phi_c\) of the scalar field are precisely those for which the effective gravitational coupling \(G_{\text{eff}}\) diverges. On the one hand, \(\phi = \text{const.}\) implies \(\mathcal{K}T = 0\); on the other hand, \(\phi = \pm \phi_c\) implies an infinite \(G_{\text{eff}} = G \left[ 1 - (\phi/\phi_c)^2 \right]^{-\frac{2}{3}}\). As a result, the effective temperature \[25, 26\] of the effective dissipative fluid is ill-defined. This feature is a puzzle for the first-order thermodynamics of scalar-tensor gravity. Its resolution comes from the realization that these analytical solutions of nonminimally coupled scalar field cosmology are unstable with respect to homogeneous perturbations. Therefore, they are not physically relevant since they will be destroyed by arbitrarily small perturbations and are not expected to occur in nature.

The instability of anisotropic Bianchi universes as \(\phi \to \pm \phi_c\) was already established by Starobinski in \[51\] (and recovered in \[52\]): \(\phi\) can never pass dynamically through the critical values \(\pm \phi_c\) where \(G_{\text{eff}}\) diverges, since the shear \(\sigma_{ab}\) and the Kretschmann scalar \(R_{abcd} R^{abcd}\) diverge there, for both classical and semiclassical \(\phi\) \[51\]. However, for the exact solutions \((\phi, H) = (\pm \phi_c, H_c(t))\) the situation is somehow different. The singularity \(G_{\text{eff}} = \infty\) is not approached dynamically but this quantity is identically infinite. The effective temperature

\[\mathcal{K}T = \frac{2\xi |\phi| \sqrt{-V' \phi V'' \phi}}{1 - \kappa \xi \phi^2} \sim \infty \cdot 0 \quad (2.26)\]

is ill-defined. Normally, a constant \(\phi\) reduces the theory to GR and \(\mathcal{K}T\) to zero, but singularities are “hot” in the sense that scalar-tensor gravity deviates from GR in their proximity \[23, 26\]. Moreover, in the context of Eckart’s effective thermodynamics, “singularity” should be intended either as a spacetime singularity or as a singularity of the effective gravitational coupling, as discussed in \[25, 26\]. Starobinski’s result suggests that the fine-tuned solutions \(\phi = \pm \phi_c\), \(H = H_c(t)\) are dynamically unstable with respect to anisotropic perturbations; here we examine the stability of these critical solutions with respect to homogeneous perturbations.

### III. Perturbing the Critical Solutions

Before discussing the stability of the critical solutions \[21, 24\] of nonminimally coupled scalar field cosmology with respect to homogeneous perturbations, we note that established formalisms for more general inhomogeneous perturbations, such as the gauge-invariant formalism of Bardeen-Ellis-Bruni \[71, 72\] in Hwang’s version for modified gravity \[72, 82\], are not applicable where the effective gravitational coupling diverges. However, homogeneous perturbations make all the critical solutions unstable, which suffices to establish their instability.

The homogeneous perturbations are described by

\[\phi(t) = \pm \phi_c + \delta \phi(t), \quad (3.1)\]

\[H(t) = H_c(t) + \delta H(t). \quad (3.2)\]

It is difficult to make sense of a negative effective gravitational coupling \[51, 61\]; moreover, the scalar field \(\phi\) cannot cross dynamically the barriers \(\phi = \pm \phi_c\) separating regions with positive effective gravitational coupling \[24\] from regions with negative \(G_{\text{eff}} \[51, 52\]. Therefore, we impose that this coupling is always positive. This requirement means that perturbations \(\delta \phi(t)\) of critical solutions with \(\phi = \pm \phi_c\) must have \(\delta \phi \leq 0\) to keep \(\phi \leq \phi_c\), while perturbations \(\delta \phi(t)\) of critical solutions with \(\phi = - \phi_c\) have \(\delta \phi \geq 0\) to maintain \(|\phi| \leq \phi_c\). As a consequence of this requirement, the effective temperature \[2.13\] of the \(\phi\)-fluid is positive-definite. In short,

\[1 - \kappa \xi \phi^2 = 1 - \kappa \xi (\pm \phi_c + \delta \phi)^2 \simeq \mp 2 \phi_c \delta \phi \geq 0. \quad (3.3)\]

The perturbations in Eqs. \[3.1\], \[3.2\] are substituted in Eqs. \[2.6\] - \[2.7\]. Keeping only first order terms in \(\delta \phi\) and \(\delta H\), the field equations \[2.25\] - \[2.27\] become

\[\pm 6 \kappa \xi \phi_c H_c (\delta \phi + \delta H) + H_c (6 \kappa \xi \phi_c^2 - 1) \delta H + \kappa V_c' \delta \phi = 0, \quad (3.4)\]

\[\delta \ddot{\phi} + 3 H_c \delta \dot{\phi} + \xi R \delta \phi \pm \xi \phi_c \delta R + V_c'' \delta \phi = 0, \quad (3.5)\]

with \(\delta R = 6 (\delta \dot{H} + 4 H_c \delta H)\) from Eq. \[2.13\]. Using \[8.2\] and \[2.10\], Eq. \[8.1\] becomes

\[\delta \ddot{\phi} + \left[ H_c - \frac{C}{H_c} \right] \delta \phi = 0 \quad (3.6)\]
when \( H_c \neq 0 \) (The special case \( H_c = 0 \) is treated in sec. [III D]). The solution of this homogeneous ODE is given by

\[
\delta \phi(t) = \exp \left\{ - \int dt \left[ H_e(t) - \frac{C}{H_e(t)} \right] \right\}.
\]

The integral is computed using \((2.15)\), obtaining

\[
\delta \phi(t) = \delta \phi_0 \, a(t) \, H_c.
\]

where now the sign of the (small) constant \( \delta \phi_0 \) must be such that the effective gravitational coupling \((2.9)\) remains positive.

One can then use Eq. \((3.6)\) to remove derivatives of \( \delta \phi \) from the first two terms of \((3.5)\):

\[
\delta H + 3H_c \delta \dot{\phi} = \left[ -H_c + \frac{C}{H_c} \right] \delta \phi + \left[ -\dot{H}_c - \frac{C \dot{H}_c}{H_c^2} \right] \delta \phi
\]

\[
+ 3H_c \left[ -H_c + \frac{C}{H_c} \right] \delta \phi = 2C \delta \phi + 6H_c \delta \phi
\]

and rewrite Eq. \((3.5)\) as

\[
\delta H + 4H_c \delta H = \frac{3}{6 \delta \phi_c} (6\delta \phi \ddot{\phi} + 2V_{\phi}) \delta \phi \equiv -D_{\pm} \delta \phi,
\]

where the source term has been moved to the right-hand side. The integrating factor associated with this inhomogenous ODE is

\[
\exp \left\{ 4 \int dt \, H_c \right\} = \exp \left\{ 4 \int dt \left( \frac{\dot{\phi}}{\delta \phi} - \frac{H_c}{H_c} \right) \right\}
\]

\[
\sim \left( \frac{\delta \phi}{H_c} \right)^4,
\]

where Eqs. \((3.6)\) and \((2.15)\) have been used to write

\[
\delta \dot{\phi} = \left[ -H_c + 2H_c + \frac{\dot{H}_c}{H_c} \right] \delta \phi = \left( H_c + \frac{\dot{H}_c}{H_c} \right) \delta \phi
\]

leading to

\[
H_c = \frac{\delta \dot{\phi}}{\delta \phi} - \frac{\dot{H}_c}{H_c}
\]

if \( H_c \) does not vanish identically. The general solution computed using Eq. \((3.10)\) reads

\[
\delta H = \delta \phi_0 \left( -\frac{D_{\pm}}{5} a(t) \right) + \delta H_0 \left( \frac{1}{a(t)^2} \right),
\]

where \( \delta H_0 \) is a (small) integration constant for the homogeneous solution of Eq. \((3.6)\) while the first term of \((3.11)\) is a particular solution of Eq. \((3.6)\). The solutions \((3.7)\) and \((3.11)\) depend on only two free initial conditions \( \delta \phi_0 \) and \( \delta H_0 \), which is consistent with the presence of the phase space constraint \((2.8)\) and the resulting dimensionality of the phase space.

For completeness, we report the effective temperatures and bulk viscosity coefficients for the perturbed FLRW critical solutions. We have \(^3\)

\[
K T = \frac{2 \xi |\phi| \sqrt{-V_c \nabla \cdot \nabla \phi}}{1 - \kappa \xi \phi^2} = \frac{1}{\kappa} \left| \frac{\delta \phi}{\delta \phi} \right|
\]

leading to first order, where in the last line we used the inequality \((3.6)\). Using Eq. \((3.10)\), this yields

\[
K T = -3\zeta = \frac{1}{\kappa} \left| \frac{C}{H_c} - H_c \right|,
\]

which is independent of the perturbation to this order, while the shear viscosity coefficient in the general theory is \( \eta = 3\zeta/2 \). \(^{23, 26}\)

We now study the stability of the specific critical solutions \((2.16)-(2.23)\).

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\(^3\) This expression is only valid for attractive gravity and the absolute value making \(K T\) positive-definite comes from the restriction \((3.6)\) on the sign of \(\delta \phi\).
B. Critical solution II \((C > 0)\)

Combining Eqs. (3.7) and (3.11) with the solutions (2.17) and (2.20) yields

\[
\delta H = \delta \phi_0 \left( -\frac{D_+}{5} a_0 \cosh^{1/2} \left[ \sqrt{2C} t \right] \right) + \delta H_0 \left( \frac{1}{a_0} \text{sech} \left[ \sqrt{2C} t \right] \right),
\]

(3.16)

\[
\delta \phi = \sqrt{\frac{C}{2}} \delta \phi_0 a_0 \frac{\sinh \left[ \sqrt{2C} t \right]}{\cosh^{1/2} \left[ \sqrt{2C} t \right]}.
\]

(3.17)

As shown in Fig. 2 both the particular part of \(\delta H\) and \(\delta \phi\) lead to divergences for \(t \to \infty\). Therefore, the solution for \(C > 0\) is unstable.

The associated effective temperature and bulk viscosity are

\[
\kappa T = -3\zeta = \frac{\sqrt{2C}}{\kappa} \left[ 1 + \cosh^2 \left( \sqrt{2C} t \right) \right].
\]

In the late-time limit \(t \to +\infty\),

\[
\kappa T \to \frac{1}{\kappa} \sqrt{\frac{C}{2}}.
\]

It seems that, \(\kappa T\) decreases asymptotically towards a constant at \(t \to \infty\), which is consistent with the idea that expansion (\(C > 0\) corresponds to an expanding universe) "cools" gravity \[25, 26\], but \(\kappa T\) stops before reaching the GR state of equilibrium. All this is made irrelevant by the fact that the solution is unstable. The same value of \(\kappa T\) is obtained at all times for the critical de Sitter space in Sec. [III]D.
C. Critical solution III \((C < 0)\)

Substituting now the solutions \((2.19)-(2.22)\) in Eqs. \((3.7)-(3.11)\) leads to the perturbations

\[
\delta H = \delta \phi_0 \left( -\frac{D_{\pm a_0}}{5} \cos^{1/2} \left[ \sqrt{2} C t \right] \right) + \delta H_0 \left( \frac{1}{a_0^4} \sec^{2} \left[ \sqrt{2} C t \right] \right),
\]

\[
\delta \phi = -\sqrt{\frac{C}{2}} \delta \phi_0 a_0 \sin \left[ \sqrt{2} C t \right].
\]

Figure 4 shows how the homogenous part of the solution for \(\delta H\) and \(\delta \phi\) diverges for \(t \to \pi/(2\sqrt{2}|C|)\) and the solution is unstable.

Again, \(\mathcal{K}T\) and \(\zeta\) are computed, giving

\[
\mathcal{K}T = -3\zeta = \frac{\sqrt{2|C|}}{\kappa} \left| 3 \cos^2 \left( \frac{\sqrt{2|C|}}{2} t \right) - 1 \right| \sin \left( \frac{\sqrt{2|C|}}{2} t \right).
\]

and we have

\[
\mathcal{K}T \to +\infty \ \text{as} \ t \to 0,
\]

\[
\mathcal{K}T \to +\infty \ \text{as} \ t \to \pm \frac{\pi}{2\sqrt{2}|C|},
\]

\[
\mathcal{K}T \to 0, \ \text{at} \ t = \pm \frac{\pi}{6\sqrt{2|C|}}.
\]

D. Critical de Sitter spaces \((C > 0)\)

For the special de Sitter spaces obtained for \(C > 0\) and given by Eq. \((2.23)\), the perturbations are

\[
\delta H = \delta \phi_0 \left[ -\frac{D_{\pm a_0}}{5} \exp \left( \pm \sqrt{\frac{C}{2}} t \right) \right] + \delta H_0 \left[ \frac{1}{a_0^4} \exp \left( \mp 4\sqrt{\frac{C}{2}} t \right) \right],
\]

\[
\delta \phi = \pm \sqrt{\frac{C}{2}} \delta \phi_0 a_0 \exp \left( \pm \sqrt{\frac{C}{2}} t \right),
\]

where the new \(\pm\) sign represents the two possible signs for \(H_c\) and is independent of the sign appearing in \(D_{\pm}\).

These perturbations are illustrated in Figs. 4 and 5. In each case, the exponential behavior is different for the homogeneous and particular components of the solution for \(\delta H\). Since there is always one diverging exponential for \(t \to \infty\), both solutions are unstable.

\(\mathcal{K}T\) and the bulk viscosity coefficient are time-independent,

\[
\mathcal{K}T = -3\zeta = \frac{1}{\kappa} \sqrt{\frac{C}{2}}
\]

These de Sitter spaces with constant effective temperature would be interpreted as metastable states, similarly to other special solutions discussed in [5].

If \(C = 0\), these de Sitter universes degenerate into a Minkowski spacetime for which Eq. \((2.23)\) (with \(V_c = 0,\)
The critical scalar field solutions correspond to ill-defined temperature and bulk viscosity coefficients in the first-order thermodynamics of scalar-tensor gravity recently developed and one would like to understand their role. We have studied their stability with respect to homogeneous perturbations. Perturbed solutions of different type have different $\mathcal{K}T$ close to the critical solutions. More precisely, $\mathcal{K}T$ is independent of $\delta \phi_0$ and $\delta H_0$, hence it is valid for arbitrarily small perturbations at any given time. While $\mathcal{K}T$ is time-independent for the critical de Sitter solutions, it varies for the other solutions. This means that, approaching the $\phi = \pm \phi_c$ states from different directions in phase space, one obtains different values of $\mathcal{K}T$, which is consistent with the fact that this quantity is undetermined at the critical scalar field value.

In order to keep the effective gravitational coupling $G_{\text{eff}}$ positive near the critical field values $\pm \phi_c$, one must impose the condition $\mp 2 \phi_c \delta \phi \geq 0$ on the scalar field perturbations. The analysis of the previous section established that the critical solutions are unstable.

Although we have reported effective temperature and bulk viscosity for the critical solutions (2.16)-(2.23), their physical meaning is very questionable or irrelevant because all these FLRW solutions are unstable and are destroyed already by homogeneous perturbations. Therefore, these critical solutions would not be realized in nature and they are of no real concern for the first order thermodynamics of scalar-tensor gravity. This remark is particularly important for de Sitter solutions with constant $\mathcal{K}T$, $\zeta$, and $\eta$ and for the late-time limit of the solution for $R > 0$, which converges to a state where these quantities are also constant. If stable, these analytical solutions of nonminimally coupled scalar field cosmology would correspond to new states of equilibrium far away from GR, but they are unstable instead.

Having addressed this potential challenge, the first-order thermodynamical formalism will be developed further in future work.

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