A Generalized Modification of the Kumaraswamy Distribution for Modeling and Analyzing Real-Life Data

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Abstract In this paper, a generalized modification of the Kumaraswamy distribution is proposed, and its distributional and characterizing properties are studied. This distribution is closed under scaling and exponentiation, and has some well-known distributions as special cases, such as the generalized uniform, triangular, beta, power function, Minimax, and some other Kumaraswamy related distributions. Moment generating function, Lorenz and Bonferroni curves, with its moments consisting of the mean, variance, moments about the origin, harmonic, incomplete, probability weighted, L, and trimmed L moments, are derived. The maximum likelihood estimation method is used for estimating its parameters and applied to six different simulated data sets of this distribution, in order to check the performance of the estimation method through the estimated parameters mean squares errors computed from the different simulated sample sizes. Finally, four real-life data sets are used to illustrate the usefulness and the flexibility of this distribution in application to real-life data.

Keywords Generalized Modification of the Kumaraswamy Distribution, Maximum Likelihood Estimator, Moments, Simulation Study, Applications.

AMS 2010 subject classifications 60E05, 62E15, 65C05.

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1. Introduction

There are a lot of popular statistical distributions that used for modeling and analyzing real data, and so did their generalization by mixing these distributions with each other. In particular, the generalization of some well-known statistical distributions provides flexible and effective modeling in the case when the standard distribution cannot statistically fit the data. Even these generalization distributions may have some limitations in modeling and analyzing some real data, especially, non-standard and abnormal data. New modifications need to be considered and studied in order to use the modeling and analysis such as non-standard data real-life data.

The Kumaraswamy distribution was first introduced in 1980, by Kumaraswamy [49] as a probability density function for double-bounded random processes, then later so many researchers continued further studied about the distribution. Garg [27], Nadarajah [59], Jones [45], Mitnik [56], Gholizadeh et al [29], and Mitnik [57], developed further theoretical research on the distribution. In particular, Gholizadeh et al [29] considered classical and Bayesian point and interval estimators for the shape parameter of the Kumaraswamy distribution using Monte-Carlo simulation. Hussain [36] used the maximum likelihood estimation and Bayesian estimation methods to estimate the parameters of the

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Kumaraswamy distribution using both simple random sampling and ranked set sampling techniques, and used a simulation study to compare the resultant estimators in terms of their biases and mean square errors.

Tables 1a and 1b show lists of distributions that are modified and/or generalized version of the Kumaraswamy distribution. All researchers showed in Tables 1a and 1b studied some of its structural, mathematical, statistical properties of their distribution, as well as some proposed parameters estimates methods. In particular, de Pascoa et al [21] introduced the Kumaraswamy generalized gamma distribution, as a capable distribution of modeling bathtub-shaped hazard rate functions, study some of its properties, adopted the method of maximum likelihood and a Bayesian procedure to estimate its parameters, and illustrated its usefulness on two real data.

Cordeiro and de Castro [14] introduced the Kumaraswamy generalized distribution, as a distribution that has flexible different shapes for the hazard function to be used in modeling survival data. Lemonte et al [50] proposed the exponentiated Kumaraswamy distribution as a generalization of the Kumaraswamy distribution and studied some of its properties, then showed that it can be used effectively in analyzing lifetime data. Bourguignon et al [10] introduced the Kumaraswamy Pareto distribution, studied some of its structural properties, estimating its parameters using the method of maximum likelihood, and used a real data set to compare this distribution with other well-known distributions. Çkmakyapan and Kadılar [11] studied some mathematical properties of Kumaraswamy Lindley distribution, and parameter estimation using the method of maximum likelihood and concluded that this distribution is a useful tool to analyze customer lifetime duration in marketing research. Sharma and Chakrabarty [74] introduced the size-biased form of Kumaraswamy distribution and studied some of its distributional and characterizing properties, and used the maximum likelihood and the matching quantiles methods to estimate its parameters in order to fit four simulated data sets, as well as, a given real life data set, representing the measurements of tensile strength of 30 polyester fibers.

| Year | Name                                                        | Reference                        |
|------|--------------------------------------------------------------|----------------------------------|
| 1997 | The Marshall-Okin Kumaraswamy distribution                  | Marshall et al [33]              |
| 2011 | The Kumaraswamy generalized distribution                    | Cordeiro and de Castro [14]      |
|      | The Kumaraswamy generalized gamma distribution              | De Pascoa et al [21]             |
| 2012 | The Kumaraswamy normal distribution                         | Correa et al [19]                |
|      | The Kumaraswamy log-logistic distribution                   | De Santana et al [22]            |
|      | The Kumaraswamy Gumbel distribution                         | Cordeiro et al [15]              |
|      | The Kumaraswamy modified Weibull distribution               | Cordeiro et al [16]              |
|      | The Kumaraswamy half-normal generalized distribution        | Cordeiro et al [17]              |
|      | The Kumaraswamy inverse Weibull distribution                | Shahbaz et al [71]               |
| 2013 | The exponentiated Kumaraswamy distribution                  | Lemonte et al [50]               |
|      | The Kumaraswamy Pareto distribution                         | Bourguignon et al [10]           |
|      | The Kumaraswamy generalized Pareto distribution             | Nadarajah and Eljabri [60]       |
|      | The Kumaraswamy-generalized exponentiated Pareto distribution | Shams [72]                       |
|      | The Kumaraswamy-generalized Lomax distribution              | Shams [73]                       |
|      | Kumaraswamy-power series distribution.                      | Bidram and Nekoukhou [8]         |
|      | The Kumaraswamy GP Distribution                             | Nadarajah and Eljabri [59]       |
|      | The Kumaraswamy Burr XII distribution                       | Paranalba et al [65]             |
|      | Kumaraswamy linear exponential distribution                | Elbatal [23]                     |
From Tables 1a and 1b above, we can see that through the large number of amendments on or modifications of the Kumaraswamy distribution that have been studied over the past twenty years, we can know the importance of this distribution on the one hand, and on the other hand, the multiplicity of
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generalization and modifications of this distribution, as these were among the most important reasons
that prompted us to do this study.

In this paper, a generalized modification of the Kumaraswamy distribution is proposed in Section 2,
and its properties consisting of boundaries, limits, mode, quantities, reliability and hazard functions,
and Renyi entropy, are studied in Section 3. In Section 4, we considered distributions related to this
distribution as special cases. In Section 5, order statistics distribution is derived. In Section 6, its
mean deviations, moment generating function, and with its moments consisting of the mean, variance,
moments about the origin, harmonic, incomplete, probability weighted, L, and the trimmed L moments
and its Lorenz and Bonferroni curves are obtained. The maximum likelihood estimation method for
estimating its parameters was used in Section 7 and its results were applied in Section 8 to six different
models simulated data sets of this distribution, in order to check the performance of the estimation
method through the estimated parameters mean squares errors computed from the different simulated
sample sizes. Finally, in Section 9, four real-life data sets are used in order to show the usefulness and
the flexibility of this distribution in application to real-life data.

2. The Kumaraswamy Distribution and its Generalized Modification of

Definition 1 (Kumaraswamy, [49]):
The rv X having a probability density function (pdf), f is given by;

\[
f(x) = \begin{cases} 
abx^{a-1}(1-x^a)^{b-1}, & 0 < x < 1 \\
0, & \text{otherwise}
\end{cases}
\] (1)

where a and b are non-negative numbers, is said to have the Kumaraswamy distribution with parameters
a and b.

We note that the domain of the function f given by (1) is [0,1], and the parameters a and b are shape
parameters. A generalized modification of (1) is given below;

Let \(0 < a, b, c, \alpha, \beta < \infty\), such that \(\alpha < \beta\), and define the function f by:

\[
f(x; a, b, c, \alpha, \beta) = \begin{cases} 
\frac{bc}{a} (\frac{\beta - \alpha}{\beta - a})^c (\frac{x}{a})^{b-1} \left[\frac{\beta - (\frac{x}{a})^b}{\beta - a}\right]^{c-1}, & aa^{\frac{1}{\beta}} < x < a^{\beta^{\frac{1}{\alpha}}}
\\
0, & \text{otherwise}
\end{cases}
\] (2)

Let us write \(f(x)\) instead of \(f(x; a, b, c, \alpha, \beta)\) for simplicity.

Now, we have the following proposition

Proposition 1:
The function \(f\) defined by (2) is a pdf and its cumulative distribution function (CDF) \(F\) given by;

\[
F_X(x; a, b, c, \alpha, \beta) = \begin{cases} 
0, & x \leq aa^{\frac{1}{\beta}}
\\
1 - \left[\frac{\beta - (\frac{x}{a})^b}{\beta - a}\right]^c, & aa^{\frac{1}{\beta}} < x < a^{\beta^{\frac{1}{\alpha}}}
\\
1, & x \geq a^{\beta^{\frac{1}{\alpha}}}
\end{cases}
\] (3)

Proof:

Since \(0 < a, b, c, \alpha, \beta < \infty, \alpha < \beta,\) and \(aa^{\frac{1}{\beta}} < x < a^{\beta^{\frac{1}{\alpha}}}\), then \(\alpha < \left(\frac{x}{a}\right)^b < \beta,\) hence \(\beta - \left(\frac{x}{a}\right)^b > 0,\) implying that \(f\) given in (2) is non negative. Now; let

\[
\left(\frac{x}{a}\right)^b = \beta w
\] (4)
then $x = a \beta^{1/b} w^{1/b}$, and $dx = \frac{a \beta}{b} \beta^{1-1/b} w^{1-1/b} dw$, therefore;

$$\int_{-\infty}^{+\infty} f(x)dx = \int_{a a^{1/b}}^{a a^{1/b}} \frac{b c}{a} (\beta - \alpha)^{-c} \left(\frac{x}{a}\right)^{b-1} \left[\beta - \left(\frac{x}{a}\right)^b\right]^{c-1} dx$$

$$= c \beta^c (\beta - \alpha)^{-c} \int_{a}^{1} (1 - w)^{c-1} dw = -\beta^c (\beta - \alpha)^{-c} (1 - w)^c |_{1/b}^{1} = 1.$$

It follows that, for any $x$ such that, $aa^{1/b} < x < a \beta^{1/b}$,

$$F(x) = \int_{-\infty}^{x} f(y)dy = -\beta^c (\beta - \alpha)^{-c} (1 - w)^c |_{\alpha}^{1/b} = 1 - \left[\frac{\beta - \left(\frac{x}{a}\right)^b}{\beta - \alpha}\right]^c$$

Note that for, $aa^{1/b} < x < a \beta^{1/b}$, we have that: $\frac{d}{dx}F_X(x; a, b, c, \alpha, \beta) = f(x; a, b, c, \alpha, \beta)$.

**Definition 2:**
The rv $X$ is said to have a generalized modification of the Kumaraswamy distribution (GMKD) with parameters $a, b, c$, and $\beta$, written as $X \sim GMKD(a, b, c, \alpha, \beta)$, if its pdf is given by (2), or equivalently, its CDF is given by (3).

Figure 1 shows some plots of the pdf of the GMKD for some of its parameter’s values, inducting that this distribution has a lot of various different flexible shapes.
We first note that, \( f(x; 1, b, c, 1, 1) \) is the pdf of Kumaraswamy distribution with parameters \( b \) and \( c \), given by (1). Furthermore, we have the following represents the method or the transformation that can be used for modifying the Kumaraswamy pdf given in (1) in order to get what we have called the GMKD.

**Proposition 2:**

Let the rv \( Y \) be a rv having the Kumaraswamy distribution with parameters \( b \) and \( c \), and the rv \( X = a\left[\alpha + (\beta - \alpha)Y^b\right]^\frac{1}{b} \), where \( 0 < a, \alpha, \beta < \infty \), such that \( \alpha < \beta \), then \( X \sim GMKD(a, b, c, \alpha, \beta) \).

**Proof:**

First, it is easy to find that the CDF of \( Y \) is given, for \( 0 < y < 1 \), by:

\[
F_Y(y) = 1 - \left[ 1 - y^b \right]^c
\]

Since \( 0 < a, b, c, \alpha, \beta < \infty \) and \( \alpha < \beta \), then \( 0 < y < 1 \) is equivalent to \( a\alpha^\frac{1}{b} < x < a\beta^\frac{1}{b} \), therefore; for \( a\alpha^\frac{1}{b} < x < a\beta^\frac{1}{b} \), we have that:

\[
F_X(x) = P(X \leq x)
= P\left(a\left[\alpha + (\beta - \alpha)Y^b\right]^\frac{1}{b} \leq x\right)
= P\left(Y \leq \left[\frac{\left(\frac{x}{a}\right)^b - a}{\beta - \alpha}\right]^\frac{1}{b}\right)
= F_Y\left(\left[\frac{\left(\frac{x}{a}\right)^b - a}{\beta - \alpha}\right]^\frac{1}{b}\right)
= 1 - \left[ 1 - \left(\frac{\left(\frac{x}{a}\right)^b - a}{\beta - \alpha}\right)^b\right]^c
= 1 - \left[ \frac{\beta - \left(\frac{x}{a}\right)^b}{\beta - \alpha} \right]^c.
\]

**3. Some Properties of the GMKD**

**3.1. Boundaries and Some Limits of the pdf**

Let us study the behavior of the pdf of the GMKD\((a, b, c, \alpha, \beta)\). At the boundary’s points, we have from (2) that:

\[
f\left(aa^\frac{1}{b}\right) = \frac{bc}{a} \frac{a^{1-\frac{1}{b}}}{\alpha (\beta - \alpha)}
\]

and:

\[
f\left(a\beta^\frac{1}{b}\right) = 0,
\]

Therefore:

\[
\lim_{a \to 0^+} f\left(aa^\frac{1}{b}\right) = 0,
\]

and

\[
\lim_{b \to 1} f\left(aa^\frac{1}{b}\right) = \frac{c}{a(\beta - \alpha)}
\]
3.2. The Mode:

We have for $a a^{\frac{1}{b}} < x < a b^{\frac{1}{b}}$ that:

$$\frac{\partial}{\partial x} f(x) = \frac{b c}{a} (\beta - a)^{-c} \left(\frac{x}{a}\right)^{b-2} \left[\beta - \left(\frac{x}{a}\right)^b\right]^{c-2} \left\{\left(\frac{b-1}{a}\right) \beta + \left(1 - \frac{b c}{a}\right) \left(\frac{x}{a}\right)^b\right\}$$

Hence, if $bc \neq 1$, provided that $\frac{a}{\beta} < \frac{1-b}{1-bc} < 1$, which is satisfied if $b > 1$ and $c > 1$, then $\frac{\partial}{\partial x} f(x) = 0$ has a root given by:

$$x_0 = a \left[\left(\frac{1-b}{1-bc}\right) \beta\right]^{\frac{1}{b}}$$

And that:

$$\frac{\partial^2}{\partial x^2} f(x_0) = \frac{c b^{c-1} (1-bc)}{a^3 (\beta - a)^c} \beta^{2b-3} + c-2 \left(\frac{1-b}{1-bc}\right)^{\frac{2b-3}{b}} \left(\frac{1-c}{1-bc}\right)^{c-2}$$

Therefore, if $b > 1$ and $c > 1$, then $bc > 1$, $0 < \frac{1-b}{1-bc} < 1$ and $0 < \frac{1-c}{1-bc} < 1$, hence $\frac{\partial^2}{\partial x^2} f(x_0)$ is negative, implying that a single-mode exists. Note that the condition $\frac{a}{\beta} < \frac{1-b}{1-bc} < 1$ implies that $aa^{\frac{1}{b}} < x_0 < a b^{\frac{1}{b}}$.

Therefore, we have proved the following.

3.3. Proposition 3:

If $b > 1$ and $c > 1$, then a single-mode exists for the GMKD$(a, b, c, \alpha, \beta)$.

3.4. Quantile Function

Let $0 < p < 1$, then the quantile function of the rv $X \sim GMKD(a, b, c, \alpha, \beta)$, $Q$, defined by:

$$Q(u) = \inf\{x \in \mathbb{R}; p \leq F(x)\}$$

can be found using (3), to be;

$$Q(u) = a \left[\beta - (\beta - a) (1-u)^{\frac{1}{\beta}}\right]^{\frac{1}{\beta}}$$

In particular, the median of $X$, $Med (X)$; is given by;

$$Med (X) = a \left[\beta - \left(\beta - a\right)\right]^{\frac{1}{\beta}}$$

3.5. Reliability Function

The survival function of $X \sim GMKD(a, b, c, \alpha, \beta)$, using (3), is given by;

$$\bar{F}(x) = 1 - F(x) = \begin{cases} 
1, & x \leq aa^{\frac{1}{b}} \\
\left[\frac{\beta - (\frac{x}{a})^{\frac{1}{b}}}{\beta - a}\right]^c, & aa^{\frac{1}{b}} < x < a b^{\frac{1}{b}} \\
0, & x \geq a b^{\frac{1}{b}}
\end{cases}$$
3.6. Hazard Function

The hazard function of the rv \(X \sim GMKD(a,b,c,\alpha,\beta)\), \(h(x)\), using (2) and (6), is given for \(a^{\frac{1}{\beta}} < x < a^{\frac{1}{\beta}}\), by;

\[
h(x) = \frac{f(x)}{1-F(x)} = \frac{bc}{a} \left( \frac{x}{a} \right)^{b-1}\frac{1}{\beta - \left( \frac{x}{a} \right)^{\beta}}\]

If \(b \geq 1\), then \(h(x)\) is increasing; and if \(b < 1\), then \(h(x)\) is increasing for \(x > a[(1-b)\beta]^{\frac{1}{\beta}}\), =0 for \(x = a[(1-b)\beta]^{\frac{1}{\beta}}\), and decreasing for \(x < a[(1-b)\beta]^{\frac{1}{\beta}}\).

3.7. Renyi Entropy

Let us compute the Renyi entropy as a measure of variation of the uncertainty of the rv \(X \sim GMKD(a,b,c,\alpha,\beta)\). For \(\theta > 0\) such that \(\theta \neq 1\), we have for the rv \(X \sim GMKD(a,b,c,\alpha,\beta)\) that;

\[
I_X(\theta) = \frac{1}{1-\theta} \log \int_{-\infty}^{+\infty} [f(x)]^\theta dx
\]

\[
I_X(\theta) = \frac{1}{1-\theta} \log \left[ \frac{bc}{a} \left( \frac{x}{a} \right)^{b-1} \int_{a^{\frac{1}{\beta}}}^{a^{\frac{1}{\beta}}} \left( \frac{\beta - \left( \frac{x}{a} \right)^{\beta}}{\beta - a} \right)^{(c-1)\theta} dx \right]
\]

Using the transformation given by (4), we have that;

\[
I_X(\theta) = \log(a) - \log(b) + \frac{\theta}{1-\theta} \log(c) + \frac{1}{1-\theta} \left[ \left( \frac{1}{b} \right)\theta + \frac{1}{b - 1} \right] \log(\beta) - c\theta \log(\beta - a) +
\]

\[
\log \left[ B^* \left( \left( 1 - \frac{1}{b} \right)(\theta - 1) - 1, (c - 1)\theta - 1; \frac{\alpha}{\beta} \right) \right]
\]

where \(B^*(a,b;z)\) is defined by;

\[
B^*(a,b;z) = \int_z^1 x^{a-1}(1-x)^{b-1} dx
\]

Or equivalently;

\[
B^*(a,b;z) = B(a,b) - B_z(a,b)
\]

where \(B\) and \(B_z\) are, respectively, the beta and the incomplete beta functions, Abramowitz and Stegun ([3], p. 258), defined by;

\[
B(a,b) = \int_0^1 x^{a-1}(1-x)^{b-1} dx
\]

and

\[
B_z(a,b) = \int_0^x x^{a-1}(1-x)^{b-1} dx
\]

We may call \(B^*(a,b;z)\) given by (9), “the upper beta function at \(z\)."
4. Distributions Related to GMKD

4.1. Special Cases of GMKD

4.1.1. GMKD(1, a, b; 0; 1) is the Kumaraswamy distribution, Kumaraswamy [49], with pdf;
\[
f(x) = ab \beta^{a-1} (x^a (1-x^a))^{b-1}, \quad 0 < x < 1
\]

4.1.2. GMKD(1, 1, 1; 0; 1) is the standard uniform distribution with pdf;
\[
f(x) = 1, \quad 0 < x < 1
\]

4.1.3. GMKD(1, 2, 1; 0; 1) is the triangular distribution with pdf;
\[
f(x) = 2x, \quad 0 < x < 1
\]

4.1.4. GMKD(1, b; 1; 0; 1) is the beta distribution with parameters 1 and b, with pdf;
\[
f(x) = b x^{b-1}, \quad 0 < x < 1
\]

4.1.5. GMKD(1, c; 0; 1) is the beta distribution with parameters c and 1, with pdf;
\[
f(x) = c \left[1 - x\right]^{c-1}, \quad 0 < x < 1
\]

4.1.6. GMKD(1, δ, α, β) is the power function distribution with pdf;
\[
f(x) = \frac{δ}{(β - α)} \left[\frac{β - x}{β - α}\right]^{δ-1}, \quad a < x < β
\]

4.1.7. GMKD(λ, aθ, b, 0; 1) is the Kumaraswamy power function distribution, Abdul-Moniem [2], with pdf;
\[
f(x) = \frac{abθ}{λ} \left(\frac{x}{λ}\right)^{aθ-1} \left[1 - \left(\frac{x}{λ}\right)^{aθ}\right]^{b-1}, \quad 0 < x < λ
\]

4.1.8. GMKD(1, β, γ; 0; 1) is the Minimax distribution, McDonald [54], with pdf;
\[
f(x) = β \gamma x^{β-1} \left[1 - x^{β}\right]^{γ-1}, \quad 0 < x < 1
\]

4.1.9. GMKD(ε, -a, θ; 0; 1) is the exponentiated Pareto distribution, Gupta et al [33], with pdf;
\[
f(x) = θaε^a x^{-(a+1)} \left[1 - \left(\frac{x}{c}\right)^{-a}\right]^{θ-1}, \quad 0 < c < x
\]

4.1.10. GMKD(α, β, 1; 0; 1) is the generalized uniform distribution, Tiwari et al [76], with pdf;
\[
f(x) = \frac{β}{α} \left(\frac{x}{α}\right)^{β-1}, \quad 0 < x < α
\]

4.2. Exponentiation Property

Proposition 4: GMKD is closed under exponentiation.
Let the rv \( X \sim GMKD(a, b, c, α, β) \) and the rv \( Y = X^k \), where \( 0 < k < \infty \), then \( Y \sim GMKD(a^k, \frac{b}{k}, c, α, β) \).

Proof:
\[
F_Y(y) = P(Y \leq y) = P(X^k \leq y) = P \left( X \leq y^\frac{1}{k} \right) = F_X \left( y^\frac{1}{k}; a, b, c, α, β \right)
\]
Therefore;
\[
F_Y(y) = 1 - \left[ \frac{\beta - \left( \frac{y^k}{a} \right)^b}{\beta - a} \right]^c = 1 - \left[ \frac{\beta - \left( \frac{y^k}{a} \right)^b}{\beta - a} \right]^c = F_X\left(\frac{y}{a^k}, \frac{b}{k}, c, a, \beta\right)
\]

4.3. Scaling Property

**Proposition 5:** GMKD is closed under scaling.
Let the rv \( X \sim GMKD(a, b, c, \alpha, \beta) \) and the rv \( Y = kX \), where \( 0 < k < \infty \), then \( Y \sim GMKD(ka, b, c, \alpha, \beta) \).

**Proof:**

\[
F_Y(y) = P(Y \leq y) = P(kX \leq y) = P\left(\frac{X \leq \frac{y}{k}}{k} = F_X\left(\frac{y}{ka}; a, b, c, \alpha, \beta\right)\right.
\]
\[
= 1 - \left[ \frac{\beta - \left( \frac{y^k}{ka} \right)^b}{\beta - a} \right]^c = F_X(y; ka, b, c, \alpha, \beta)
\]

4.4. Related Distributions of GMKD

**Lemma 1:**
Let the rv \( X \sim GMKD(1, b, 1, 0, 1) \) then;

1. The rv \( Y \) defined by \( Y = \theta - b^2 \log(X) \) has an exponential distribution with parameters \( \theta \) and \( b \), Johnson et al \[43\] p. 494, with CDF given by;
\[
F_Y(y) = 1 - e^{-(\frac{y}{b})}, \quad x > \theta
\]

2. The rv \( Y \) defined by \( Y = \mu - \beta \log\left[-b \log(\beta \log(X^b))\right] \) has a Gumbel (generalized extreme value type-I) distribution with parameters \( \mu \) and \( \beta \), Forbes et al \[26\] p. 98 , with CDF given by;
\[
F_Y(y) = e^{-e^{-(\frac{y-\mu}{\beta})}}
\]

3. The rv \( Y \) defined by \( Y = a + \beta \log\left(\frac{X^b}{1-X^b}\right) \) has a logistic distribution with parameters \( a \) and \( \beta \), Johnson et al \[44\] p. 115, with CDF given by;
\[
F_Y(y) = \left[1 + e^{-(\frac{y-a}{\beta})}\right]^{-1}
\]

4. The rv \( Y \) defined by \( Y = \frac{k}{X} \), where \( k \) is a positive constant, has a Pareto distribution with parameters \( k \) and \( b \), Johnson et al \[43\] p. 574, with CDF given by;
\[
F_Y(y) = 1 - \left(\frac{k}{y}\right)^b
\]

5. The rv \( Y \) defined by \( Y = \xi + a \left[-\log(X^b)\right]^\beta \) has a Weibull distribution with parameters \( \xi, a, \) and \( \theta \), Johnson et al \[43\] p. 629, with its CDF given by;
\[
F_Y(y) = 1 - e^{-(\frac{y-\xi}{a})^\theta}
\]
Proof: 
(1.1) Consider the distribution of the rv \( Y = \frac{1}{X} \);
\[
F_Y(y) = P(Y \leq y) = P\left( \frac{1}{X} \leq y \right) \\
= P\left( \frac{1}{y} \leq X \right) \\
= 1 - P\left( X < \frac{1}{y} \right) \\
= 1 - F_X\left( \frac{1}{y}; \beta, 1, 0, 1 \right) \\
= 1 - \left[ 1 - \left( 1 - \left( \frac{1}{y} \right)^\beta \right) \right] \\
= 1 - (y)^{-\beta}
\]
Therefore, the rv \( Y \) has a Pareto distribution with parameters \( \gamma \) and \( 0 \).
Proof of (1.2) through (1.5) can be shown on the same lines as the proof of (1.1).

Lemma 2: 
(2.1) Let the rv \( X \sim \text{GMKD}(1, 1, 1, 0, 1) \) then the rv \( Y \) defined by \( Y = \left( \frac{1}{X} \right)^{\frac{1}{\gamma}} e^{-\frac{y}{\beta}} \) has a log-logistic distribution with parameters \( \delta \) and \( \gamma \), Johnson et al. [42] p. 151, with CDF given by;
\[
F_Y(y) = 1 - \left[ 1 + y^\delta e^\gamma \right]^{-1}, \quad y \geq 0
\]
(2.2) Let the rv \( X \sim \text{GMKD}(a, b, c, \alpha, \beta) \) then the rv \( Y \) defined by \( Y = \beta - \left( \frac{x}{a} \right)^b \) has the generalized uniform distribution, Tiwari et al. [76], with CDF given by;
\[
F_Y(y) = \frac{y^c}{\beta - a}, \quad 0 \leq y \leq \beta - a
\]
(2.3) Let the rv \( X \sim \text{GMKD}(a, b, c, \alpha, \beta) \) then the rv \( Y \) defined by \( Y = (\beta - a)^{-1} \left[ \beta - \left( \frac{x}{a} \right)^b \right] \) has the beta distribution with parameters 1 and \( c \), with CDF given by;
\[
F_Y(y) = y^c, \quad 0 \leq y \leq 1
\]
(2.4) Let the rv \( U \) has the standard uniform distribution, \( U(0,1) \), then the rv \( X \) defined by \( X = a \left[ \beta - (\beta - a)(1 - u)^{\frac{1}{\beta}} \right]^{\frac{1}{\gamma}} \), then \( X \sim \text{GMKD}(a, b, c, \alpha, \beta) \).
Proof: 
On the same lines as the proof of Lemma 1.

4.5. Generate GMKD random variates 
Using result Lemma 2(4), we can generate \( \text{GMKD}(a, b, c, \alpha, \beta) \) random variates as follows;
1. Generate \( u \sim U(0,1) \).
2. Set \( x = a \left[ \beta - (\beta - a)(1 - u)^{\frac{1}{\beta}} \right]^{\frac{1}{\gamma}} \).
5. Order Statistics

Let \( X_1, X_2, \ldots, X_n \) be a random sample of size \( n \) from \( GMKD(a, b, c, \alpha, \beta) \), and let \( X_{1:n}, X_{2:n}, \ldots, X_{n:n} \) be their order statistics, then for \( i = 1, 2, 3, \ldots, n \), the pdf of \( i \)-th order statistics \( X_{i:n} \), is given for by:

\[
f_{i:n}(x; a, b, c, \alpha, \beta) = \begin{cases} \frac{n!}{(i-1)!(n-i)!} f(x) [F(x)]^{i-1} [1 - F(x)]^{n-i}, & \text{if } aa^{\frac{1}{\beta}} < x < a^{\beta}b \\ 0 & \text{otherwise,} \end{cases}
\]

Hence, for \( aa^{\frac{1}{\beta}} < x < a^{\beta}b \), we have from (2) and (3) that;

\[
f_{i:n}(x; a, b, c, \alpha, \beta) = \frac{n!}{(i-1)!(n-i)!} \frac{bc}{a(\beta-a)} \left( \frac{x}{a} \right)^{b-1} \left[ 1 - \left( \frac{\beta - \left( \frac{x}{a} \right)^b}{\beta - \alpha} \right)^{i-1} \left( \frac{\beta - \left( \frac{x}{a} \right)^b}{\beta - a} \right)^{n-i} \right] f(x)
\]

Since \( i = 1, 2, 3, \ldots, n \), we have that;

\[
\left[ 1 - \left( \frac{\beta - \left( \frac{x}{a} \right)^b}{\beta - \alpha} \right)^{i-1} \right] = \sum_{j=0}^{i-1} \binom{i-1}{j} (-1)^j \left( \frac{\beta - \left( \frac{x}{a} \right)^b}{\beta - \alpha} \right)^j
\]

Hence, using (13), we can write (12) as;

\[
f_{i:n}(x; a, b, c, \alpha, \beta) = \frac{n!}{(i-1)!(n-i)!} \frac{bc}{a(\beta-a)} \left( \frac{x}{a} \right)^{b-1} \sum_{j=0}^{i-1} \binom{i-1}{j} (-1)^j \left( \frac{\beta - \left( \frac{x}{a} \right)^b}{\beta - \alpha} \right)^{c(j+1)+c-1}
\]

Let for \( 0 \leq j \leq i \leq n \);

\[A(i, j; n) = (-1)^j \frac{n!}{(n-i)!(i-j)!j!} \]

And using the fact that the pdf of the rv \( X, f \), given by (2), satisfies that;

\[
\frac{a(\beta-a)}{bc} f(x; a, b, c, \alpha, \beta) = \left( \frac{x}{a} \right)^{b-1} \left[ \frac{\beta - \left( \frac{x}{a} \right)^b}{\beta - \alpha} \right]^{c-1}
\]

Then the pdf of the rv \( X_{i:n}, f_{i:n} \), given by (12), can be written as;

\[
f_{i:n}(x; a, b, c, \alpha, \beta) = \sum_{j=0}^{i-1} \frac{A(i, j; n)}{(n-i+j+1)} f(x; a, b, (n-i+j+1)c, \alpha, \beta)
\]

6. Moments

6.1. Moments about the Origin

Let \( k = 1, 2, 3, \ldots \), then the moment of the rv \( X \sim GMKD(a, b, c, \alpha, \beta) \), of order \( k \) about zero is given by;

\[
E(X^k) = \int_{aa^{\frac{1}{\beta}}}^{a^{\beta}b} x^k \frac{bc}{a(\beta-a)} \left( \frac{x}{a} \right)^{b-1} \left[ \frac{\beta - \left( \frac{x}{a} \right)^b}{\beta - \alpha} \right]^{c-1} dx
\]
Using the transformation given by (4), we have that:

\[ E(X^k) = c a^k \beta^\frac{k}{\beta - \alpha} \int_{\frac{b}{\beta}}^{1} w^k (1 - w)^{c-1} \, dw \]  

\[ = c a^k \beta^\frac{k}{\beta - \alpha} \int_{\frac{b}{\beta}}^{1} w^k \left(1 + \frac{c}{\alpha} \right) B^* \left(\frac{k}{b} + 1, c; \frac{a}{\beta} \right), \]

where \( B^*(; ; z) \) is “the upper beta function at \( z \)” given by (9).

Now expand \((1 - w)^{c-1}\) in the integral of (15), using the binomial series expansion, Abramowitz and Stegun [3] p.14, to get that;

\[ E(X^k) = c a^k \beta^\frac{k}{\beta - \alpha} \sum_{i=0}^{\infty} \binom{c - 1}{i} \left(\frac{-1}{i + 1}\right)^i \left[ 1 - \left(\frac{a}{\beta}\right)^{\frac{k}{b} + i + 1} \right] \]

Therefore, an interesting relation can be seen from (16) and (17), for the upper beta function, is given by;

\[ B^* \left(\frac{k}{b} + 1, c; \frac{a}{\beta} \right) = \sum_{i=0}^{\infty} \binom{c - 1}{i} \left(\frac{-1}{i + 1}\right)^i \left[ 1 - \left(\frac{a}{\beta}\right)^{\frac{k}{b} + i + 1} \right] \]

or equivalently, in a general form, given in the following proposition;

**Proposition 6:**

\[ B^*(x, c; y) = \sum_{i=0}^{\infty} \binom{c - 1}{i} \left(\frac{-1}{i + 1}\right)^i \left[ 1 - y^{x+i} \right] \]

### 6.2. Mean and Variance

Using (16), the mean of \( X \sim GMKD(a, b, c, \alpha, \beta) \) is given by;

\[ E(X) = c a^2 \beta^\frac{2}{\beta - \alpha} \left[ \frac{B^* \left(\frac{1}{b} + 1, c; \frac{a}{\beta} \right)}{\Gamma \left(\frac{1}{b} + c + 1\right)} \right] \]

And hence the variance;

\[ Var(X) = c a^2 \beta^\frac{2}{\beta - \alpha} \left[ \frac{B^* \left(\frac{2}{b} + 1, c; \frac{a}{\beta} \right)}{\Gamma \left(\frac{2}{b} + c + 1\right)} - c \left(\frac{\beta}{\beta - \alpha}\right)^c \left[ \frac{B^* \left(\frac{1}{b} + 1, c; \frac{a}{\beta} \right)}{\Gamma \left(\frac{1}{b} + c + 1\right)} \right]^2 \right] \]

In particular, if \( \alpha = 0 \) and \( \beta = 1 \), then

\[ E(X) = c a \frac{\Gamma \left(\frac{1}{b} + 1\right) \Gamma(c)}{\Gamma \left(\frac{1}{b} + c + 1\right)} \]

\[ Var(X) = c a^2 \left[ \frac{\Gamma \left(\frac{2}{b} + 1\right) \Gamma(c)}{\Gamma \left(\frac{2}{b} + c + 1\right)} - c \left(\frac{\Gamma \left(\frac{1}{b} + 1\right) \Gamma(c)}{\Gamma \left(\frac{1}{b} + c + 1\right)} \right)^2 \right] \]

and;

\[ E(X^k) = c a^k \frac{\Gamma \left(\frac{k}{b} + 1\right) \Gamma(c)}{\Gamma \left(\frac{k}{b} + c + 1\right)} \]
6.3. The Moment Generating Function

Similarly, the moment generating function of the rv \( X \sim GMKD(a, b, c, \alpha, \beta) \), \( M_X(t) \), can be found to be:

\[
M_X(t) = E\left(e^{tX}\right) = c \left(\frac{\beta}{\beta - \alpha}\right)^c \sum_{i=0}^{\infty} \frac{(at)^i}{i!} B^i\left(\frac{i}{b} + 1, c; \frac{\alpha}{\beta}\right)
\]

6.4. Harmonic Mean

The harmonic mean of \( X \sim GMKD(a, b, c, \alpha, \beta) \), on the same lines as that of the moment of \( X \), is given by:

\[
E\left(\frac{1}{X}\right) = c \left(\frac{\beta}{\beta - \alpha}\right)^c B^\left(1 - \frac{1}{b}, c; \frac{\alpha}{\beta}\right)
\]

6.5. Incomplete Moments

The k-th incomplete moment of \( X \sim GMKD(a, b, c, \alpha, \beta) \), \( I(z, k) \), is defined by:

\[
I(z, k) = \int_{-\infty}^{\infty} x^k f(x) \, dx
\]

\[
= \int_{a}^{\infty} x^k \frac{bc}{a} \left(\frac{\beta - \alpha}{\alpha}\right)^{-c} \left(\frac{x}{a}\right)^{b-1} \left[\frac{x}{a} - \left(\frac{\beta}{\alpha}\right)^c\right] \, dx
\]

\[
= c a^k \beta^k \left(\frac{\beta}{\beta - \alpha}\right)^c \left[\frac{1}{\alpha^k} \left(\frac{1}{b} + 1, c; \frac{\alpha}{\beta}\right) - \frac{1}{\beta} \left(\frac{1}{b} + 1, c; \frac{\alpha}{\beta}\right)\right]
\]

6.6. Mean Deviations

The mean deviation of \( X \) about its mean \( \mu = E(X) \), \( MD(\mu) \), is given by:

\[
MD(\mu) = E|X - \mu|
\]

Which can be found, Cordeiro et al [15], to be:

\[
MD(\mu) = 2[\mu F(\mu) - I(\mu - 1)]
\]

Hence, using (3), (18) and (19), for the rev \( X \sim GMKD(a, b, c, \alpha, \beta) \), we have that:

\[
MD(\mu) = 2c a \beta^c \left(\frac{\beta}{\beta - \alpha}\right)^c \left\{B^\left(\frac{1}{b} + 1, c; \frac{\mu}{b}\right)\left(\frac{1}{b} + 1, c; \frac{\alpha}{\beta}\right) - B^\left(\frac{1}{b} + 1, c; \frac{\alpha}{\beta}\right)\left(\frac{\beta}{\alpha}\right)^c\right\}
\]

Similarly, the mean deviation of \( X \) about its median \( m \), \( MD(m) \), is given by:

\[
MD(m) = \mu - 2I(m, 1)
\]

\[
MD(m) = c a \beta^c \left(\frac{\beta}{\beta - \alpha}\right)^c \left[\frac{1}{b} + 1, c; \frac{\beta}{\beta - \alpha}\right] + B^\left(\frac{1}{b} + 1, c; \frac{\beta}{\beta - \alpha}\right) - 2B^\left(\frac{1}{b} + 1, c; \frac{\beta}{\beta - \alpha}\right)\left(\frac{1}{b} + 1, c\right)
\]
6.7. Probability Weighted Moments

The probability weighted moments of order s and r of \(X \sim GMKD(a,b,c,\alpha,\beta)\), \(\rho_{s,r}\), is given by:

\[
\rho_{s,r} = E(X^s[f(X)]^r)
\]

Using the transformation given in (4), we have that:

\[
\rho_{s,r} = a^{s-r+1}b^{r-1}c^r \frac{\beta^{\frac{s}{b}+r\left(\frac{c}{b}-1\right)+1}}{(\beta-a)^{r}} B^*(\frac{s}{b}+1,r\left(\frac{c}{b}-1\right)+1;\frac{a}{\beta})
\]

6.8. Moments of Order Statistics

Let \(X_1,X_2,\ldots,X_n\) be a random sample of size n from \(GMKD(a,b,c,\alpha,\beta)\), and let \(X_{1:n},X_{2:n},\ldots,X_{n:n}\) be their order statistics, then for \(i = 1, 2, 3,\ldots, n\), we have that:

\[
E(X_{i:n}^m) = \int_{\alpha}^{1} x^m f_{i:n}(x; a, b, c, \alpha) dx
\]

Hence, using (14) we have that:

\[
E(X_{i:n}^m) = \sum_{j=0}^{i-1} \frac{A(i,j;n)}{(n-i+j+1)} \int_{\alpha}^{1} x^m f(x; a, b, (n-i+j+1)c, \alpha, \beta) dx
\]

Therefore, using (16), we have that:

\[
E(X_{i:n}^m) = ca^m b^m \sum_{j=0}^{i-1} \frac{A(i,j;n)B^*(\frac{m}{b}+1,(n-i+j+1)c;\frac{a}{\beta})}{\left(\frac{\beta}{\beta-a}\right)^{(n-i+j+1)c}}
\]

(20)

6.9. L-Moments

Let \(r = 1, 2, 3,\ldots\) then the r-th L-moment \(\gamma_r\) of \(X \sim GMKD(a,b,c,\alpha,\beta)\), is given by:

\[
\gamma_r = \frac{1}{r} \sum_{i=0}^{r-1} (-1)^i \binom{r-1}{i} E(X_{r-i:r})
\]

which is, with the use of (20), can be found to be:

\[
\gamma_r = \frac{ca^m b^m}{r} \sum_{i=0}^{r-1} (-1)^i \binom{r-1}{i} \sum_{j=0}^{i-1} A(r-i,j;r) B^*(\frac{1}{b}+1,(i+j+1)c;\frac{a}{\beta}) \left(\frac{\beta}{\beta-a}\right)^{(i+j+1)c}
\]

In particular, the first L-moments are given by:

\[
\gamma_1 = ca^m b^m \left(\frac{\beta}{\beta-a}\right) c^* B^*(\frac{1}{b}+1,c;\frac{a}{\beta})
\]

And:

\[
\gamma_2 = ca^m b^m \left(\frac{\beta}{\beta-a}\right) c^* \left[1 - 2 \left(\frac{\beta}{\beta-a}\right)^c\right] B^*(\frac{1}{b}+1,2c;\frac{a}{\beta})
\]
Hence, the L-coefficient of variation \( \tau \), Bilkova\[9\], defined by;
\[
\tau = \frac{\gamma_2}{\gamma_1}
\]
is given by;
\[
\tau = \left[ 1 - 2\left( \frac{\beta}{\beta - \alpha} \right)^c \right] \frac{B^*\left( \frac{1}{b} + 1, 2c; \frac{a}{\beta} \right)}{B^*\left( \frac{1}{b} + 1, c; \frac{a}{\beta} \right)}
\]

6.10. The L and the Trimmed L-Moments
Let \( r = 1, 2, 3, \ldots \) then the \( r \)-th trimmed L-moments (TL-moments) of the rv \( X \sim GMKD(a, b, c, \alpha, \beta) \), \( \gamma_{r}^{(s,t)} \), is given by;
\[
\gamma_{r}^{(s,t)} = \frac{1}{r} \sum_{k=0}^{r-1} (-1)^k \binom{r-1}{k} E(X_{r+s-k:r+s+t})
\]
which is, with the use of (20), can be found to be;
\[
\gamma_{r}^{(s,t)} = \frac{ca\beta^c}{r} \sum_{k=0}^{r-1} (-1)^k \binom{r-1}{k} \sum_{j=0}^{r+s-k-1} A(r+s-k,j;r+s+t) B^*\left( \frac{1}{b} + 1, (t+k+j+1)c; \frac{\alpha}{\beta} \right) \left( \frac{\beta}{\beta - \alpha} \right)^{(t+k+j+1)c}
\]
Hence, the \( r \)-th L-moments of the rv \( X \), \( \gamma_r \), can be found, since the L-moments is a special case of the TL-moments, namely, when \( s = t = 0 \), that is \( \gamma_r = \gamma_{r}^{(0,0)} \).

6.11. Lorenz and Bonferroni Curves
For \( 0 < \pi < 1 \), the Lorenz curve, \( L(\pi) \), and Bonferroni curves, \( B(\pi) \), for the rv \( X \sim GMKD(a, b, c, \alpha, \beta) \), are given by;
\[
L(\pi) = \frac{I(Q(\pi), 1)}{\mu}
\]
and
\[
B(\pi) = \frac{I(Q(\pi), 1)}{\pi \mu}
\]
where \( Q(\pi) \) is the quantile function of the rv \( X \) at \( \pi \), and \( I(z, k) \) is the incomplete moment of the rv \( X \) given by (19). Therefore, using (16) and (19), we have that;
\[
L(\pi) = \frac{B_{\frac{1}{\beta}}(\frac{\pi}{a})^\beta \left( \frac{1}{b} + 1, c \right) - B_{\frac{1}{\beta}} \left( \frac{1}{b} + 1, c \right)}{B^*\left( \frac{1}{b} + 1, c; \frac{a}{\beta} \right)}
\]
And similarly, that;
\[
B(\pi) = \frac{B_{\frac{1}{\beta}}(\frac{\pi}{a})^\beta \left( \frac{1}{b} + 1, c \right) - B_{\frac{1}{\beta}} \left( \frac{1}{b} + 1, c \right)}{\pi B^*\left( \frac{1}{b} + 1, c; \frac{a}{\beta} \right)}
\]
7. Parameters Estimation of the GMKD

We will use the maximum likelihood estimation (MLE) method for estimating the parameters of the GMKD. Let \( x_1, x_2, \ldots, x_n \) be a random sample from \( \text{GMKD}(a, b, c, \beta) \), as given by (2), then the likelihood function \( L = L(a, b, c, \beta; x_1, x_2, \ldots, x_n) \) can be written as,

\[
L = \prod_{i=1}^{n} \left\{ \frac{bc}{a} (\beta - \alpha)^{-c} \left( \frac{x_i}{a} \right)^{b-1} \left[ -\left( \frac{x_i}{a} \right)^{b} \right]^{c-1} \right\} \\
= \left[ \frac{bc}{a^b(\beta - \alpha)^c} \right] \prod_{i=1}^{n} \left\{ \left( \frac{x_i}{a} \right)^{b-1} \left[ -\left( \frac{x_i}{a} \right)^{b} \right]^{c-1} \right\}
\]

Hence;

\[
\log L = n \log (b) + n \log (c) - nb \log (a) - nc \log (\beta - \alpha) + (b - 1) \sum_{i=1}^{n} \log (x_i) + (c - 1) \sum_{i=1}^{n} \log \left[ -\left( \frac{x_i}{a} \right)^{b} \right]
\]

Therefore;

\[
\frac{\partial}{\partial a} \log L = -\frac{n b}{a} + b \frac{c}{a} \sum_{i=1}^{n} \frac{x_i}{a} - \frac{(c - 1) b}{\beta - \left( \frac{x_i}{a} \right)^{b}} \tag{21}
\]

Since, \( \frac{\partial^2}{\partial a^2} \log L \) can be shown to be not in a simple form, therefore a local maximum of \( L \) at \( a \) has to be explicitly examined. Now;

\[
\frac{\partial}{\partial b} \log L = \frac{n}{b} - n \log (a) + \sum_{i=1}^{n} \log (x_i) - (c - 1) \sum_{i=1}^{n} \frac{x_i}{a} \log \left( \frac{x_i}{a} \right) \tag{22}
\]

Hence also, \( \frac{\partial^2}{\partial b^2} \log L \) can be shown to be not in a simple form, therefore a local maximum of \( L \) at \( b \) has to be explicitly examined. Now;

\[
\frac{\partial}{\partial c} \log L = \frac{n}{c} - n \log (\beta - \alpha) + \sum_{i=1}^{n} \log \left[ -\left( \frac{x_i}{a} \right)^b \right] \tag{23}
\]

Hence;

\[
\frac{\partial^2}{\partial c^2} \log L = -\frac{n}{c^2}
\]

Therefore, \( \frac{\partial^2}{\partial c^2} \log L < 0 \), which indicates that \( L \) has a local maximum at \( c \). Similarly,

\[
\frac{\partial}{\partial \beta} \log L = -\frac{nc}{\beta - \alpha} + (c - 1) \sum_{i=1}^{n} \frac{1}{\beta - \left( \frac{x_i}{a} \right)^{b}} \tag{24}
\]

Hence;

\[
\frac{\partial^2}{\partial \beta^2} \log L = \frac{nc}{(\beta - \alpha)^2} - (c - 1) \sum_{i=1}^{n} \frac{1}{\beta - \left( \frac{x_i}{a} \right)^b} \frac{1}{(\beta - \left( \frac{x_i}{a} \right)^b)^2}
\]

Or equivalently;

\[
\frac{\partial^2}{\partial \beta^2} \log L = (a_1 - a_2) c + a_1
\]
where;

\[
a_1 = \frac{n}{(\beta - \alpha)^2}
\]

and

\[
a_2 = \sum_{i=1}^{n} \left[ \frac{1}{\beta - \left( \frac{x_i}{a} \right)^b} \right]^2
\]

Therefore, if \( a_1 > a_2 \) and \( c < a_2 \), then \( \frac{\partial^2}{\partial \beta^2} \log L < 0 \), or if \( a_1 < a_2 \) and \( c > a_2 \), then \( \frac{\partial^2}{\partial \beta^2} \log L < 0 \), and therefore, a local maximum of \( L \) at \( \beta \) has to be explicitly examined. Finally,

\[
\frac{\partial}{\partial \alpha} \log L = \frac{nc}{\beta - \alpha}
\]

and that;

\[
\frac{\partial^2}{\partial \alpha^2} \log L = \frac{nc}{(\beta - \alpha)^2}
\]

implying that \( \frac{\partial}{\partial \alpha} \log L > 0 \) and that \( \frac{\partial^2}{\partial \alpha^2} \log L > 0 \), then an alternative way to find the MLE of \( \alpha \) has to be considered. Since \( \alpha a^{\frac{1}{b}} < x < a \beta^{\frac{1}{b}} \), then the MLE of \( \alpha a^{\frac{1}{b}} \) and \( \alpha \beta^{\frac{1}{b}} \) are; respectively, \( x_{1:n} \) and \( x_{n:n} \), and hence;

\[
\frac{\alpha a^{\frac{1}{b}}}{\alpha \beta^{\frac{1}{b}}} = \frac{x_{1:n}}{x_{n:n}}
\]

Or equivalently;

\[
\frac{\alpha}{\beta} = \left( \frac{x_{1:n}}{x_{n:n}} \right)^{\frac{1}{b}}
\]

And hence;

\[
\hat{\alpha} = \left( \frac{x_{1:n}}{x_{n:n}} \right)^{\frac{1}{b}} \hat{\beta}
\]

Now, letting \( \frac{\partial}{\partial \alpha} \log L = 0 \), we have from (21) that;

\[
(c - 1) \sum_{i=1}^{n} \left[ \frac{x_i}{\beta - \left( \frac{x_i}{a} \right)^b} \right] - n = 0
\]

And letting \( \frac{\partial}{\partial \beta} \log L = 0 \), then from (22) we have that;

\[
\frac{n}{b} - n \log(a) + \sum_{i=1}^{n} \log(x_i) - (c - 1) \sum_{i=1}^{n} \left( \frac{x_i}{\beta} \right)^b \log \left( \frac{x_i}{a} \right) = 0
\]

Similarly, letting \( \frac{\partial}{\partial \beta} \log L = 0 \), then from (23) we have that;

\[
\frac{n}{\beta} - n \log(\beta - \alpha) + \sum_{i=1}^{n} \log \left[ \beta - \left( \frac{x_i}{a} \right)^b \right] = 0
\]
Hence, from (25) we have that;

\[ c = \frac{n}{n \log (\beta - \alpha) - \sum_{i=1}^{n} \log \left[ \beta - \left( \frac{x_i}{\alpha} \right)^\beta \right]} \]

And finally, letting \( \frac{\partial}{\partial \beta} \log L = 0 \), then from (24), we have that;

\[ (c - 1) \sum_{i=1}^{n} \frac{1}{\beta - \left( \frac{x_i}{\alpha} \right)^\beta} - \frac{nc}{\beta - \alpha} = 0 \]

Then the MLE of the parameters \( a, b, c, \alpha \) and \( \beta \), can be found by solving the following equations;

\[ A_1(\hat{a}) = 0 \]
\[ A_2(\hat{b}) = 0 \]
\[ \hat{c} = \frac{n}{n \log \left( \frac{\hat{\beta}}{\hat{\alpha}} \right) - \sum_{i=1}^{n} \log \left[ \frac{\hat{\beta}}{\hat{\alpha}} - \left( \frac{x_i}{\hat{\alpha}} \right)^\hat{b} \right]} \rightarrow \hat{\alpha} = \left( \frac{x_{1:n}}{x_{n:n}} \right)^\frac{\hat{b}}{\hat{\beta}} \]

and,

\[ A_3(\hat{\beta}) = 0 \]

using any numerical procedure, say, Newton Rapson method, where;

\[ A_1(\hat{a}) = (\hat{c} - 1) \sum_{i=1}^{n} \frac{\left( \frac{x_i}{\alpha} \right)^\hat{b}}{\frac{\hat{\beta}}{\hat{\alpha}} - \left( \frac{x_i}{\alpha} \right)^\hat{b}} - n \]

\[ A_2(\hat{b}) = \frac{n}{\hat{b}} - n \log(\hat{a}) + \sum_{i=1}^{n} \log(x_i) - (\hat{c} - 1) \sum_{i=1}^{n} \frac{\left( \frac{x_i}{\alpha} \right)^\hat{b} \log \left( \frac{x_i}{\alpha} \right)}{\frac{\hat{\beta}}{\hat{\alpha}} - \left( \frac{x_i}{\alpha} \right)^\hat{b}} \]

\[ A_3(\hat{\beta}) = (\hat{c} - 1) \sum_{i=1}^{n} \frac{1}{\frac{\hat{\beta}}{\hat{\alpha}} - \left( \frac{x_i}{\alpha} \right)^\hat{b}} - \frac{n\hat{c}}{\frac{\hat{\beta}}{\hat{\alpha}} - \hat{\alpha}} \]

8. A Simulation Study

Using the results given in Section 7 and the Absoft Pro Fortran compiler for computing, different GMKD models data sets were simulated, in order to check the performance of the MLEs of the parameters of each model through their mean squares errors (MSE) computed from different simulated sample sizes. The steps are given below;

1. Six different GMKD models are considered, that have different pdf's shapes and variable ranges.
2. Six sample sizes, namely; 15, 30, 50, 100, 200, and 300 are used.
3. For each sample size, 5,000 random variates are generated from each of the given GMKD models.
4. For each sample size and for each GMKD model, the parameters are estimated using the MLE method given in Section 9.

5. The means, standard deviation (SD), bias, and MSE for each of the parameters are computed for each random sample for each sample size of the given GMKD models.

Table 2 shows the actual and the MLEs parameters values of the different simulated GMKD data sets, and Figure 2 shows their corresponding pdf’s plots, while Tables 3a and 3b present the bias of the parameters of the different simulated GMKD data sets for each sample size, while Tables 4a and 4b present the MSE of the parameters of the different simulated GMKD data sets for each sample size, while.

| Data Set | Value | Parameters | Variable Range |
|----------|-------|------------|----------------|
|          |       | a | b | c | α | β | Minimum | Maximum |
| 1        | Actual | 1.3 | 1.3 | 2 | 0.2 | 1 | 0.377 | 1.3 |
|          | MLE    | 1.280123 | 1.333333 | 2.02672 | 0.198141 | 1.012141 | 0.38 | 1.292 |
| 2        | Actual | 1.95 | 2.27 | 2.5 | 0.1 | 3 | 0.71 | 3.164 |
|          | MLE    | 1.93213 | 2.26713 | 2.38939 | 0.1041 | 2.9668 | 0.71 | 3.135 |
| 3        | Actual | 2.0 | 0.8 | 2.75 | 0 | 2 | 0.0 | 4.757 |
|          | MLE    | 2.15889 | 0.837778 | 2.66798 | 0.002103 | 1.88474 | 0.001 | 4.611 |
| 4        | Actual | 2 | 3.5 | 1.25 | 0.015 | 2.5 | 0.602 | 2.599 |
|          | MLE    | 1.98913 | 3.49911 | 1.25318 | 0.015111 | 2.511031 | 0.6 | 2.588 |
| 5        | Actual | 2.3 | 1.65 | 1.35 | 0 | 0.9 | 0 | 2.158 |
|          | MLE    | 2.29789 | 1.65356 | 1.34917 | 0.001041 | 0.89256 | 0.033 | 2.145 |
| 6        | Actual | 2.5 | 0.15 | 0.5 | 0 | 1 | 0 | 2.5 |
|          | MLE    | 2.51023 | 0.151713 | 0.497749 | 0.000675 | 0.999351 | 0 | 2.5 |

Figure 2. Plots of the Actual and Simulated GMKD pdf’s
Table 3a: The Bias of the Parameters of the Simulated GMKD Data Sets for each Sample Size n.

| n   | Actual Value | Bias          |       |       |       |       |       |       |       |       |       |       |       |       |       |       |       |       |       |       |       |       |       |       |       |       |       |       |       |       |       |       |       |       |       |       |       |       |       |       |       |       |       |       |       |       |       |       |       |       |       |       |       |       |       |       |       |       |       |       |       |       |       |       |       |       |       |       |       |       |       |       |       |       |       |       |       |       |       |       |       |       |       |       |       |       |       |       |       |       |       |       |       |       |       |       |       |       |       |       |       |       |       |       |       |       |       |       |       |       |       |       |       |       |       |       |       |       |       |       |       |       |       |       |       |       |       |       |       |       |       |       |       |       |       |       |       |       |       |       |       |       |       |       |       |       |       |       |       |       |       |       |       |       |       |       |       |       |       |       |       |       |       |       |       |       |       |       |       |       |       |       |       |       |       |       |       |       |       |       |       |       |       |       |       |       |       |       |       |       |       |       |       |       |       |       |       |       |       |       |       |       |       |       |       |       |       |       |       |       |       |       |       |       |       |       |       |       |       |       |       |       |       |       |       |       |       |       |       |       |       |       |       |       |       |       |       |       |       |       |       |       |       |       |       |       |       |       |       |       |       |       |       |       |       |       |       |       |       |       |       |       |       |       |       |       |       |       |       |       |       |       |       |       |       |       |       |       |       |       |       |       |       |       |       |       |       |       |       |       |       |       |       |       |       |       |       |       |       |       |       |       |       |       |       |       |       |       |       |       |       |       |       |       |       |       |       |       |       |       |       |       |       |       |       |       |       |       |       |       |       |       |       |       |       |       |       |       |       |       |       |       |       |       |       |       |       |       |       |       |       |       |       |       |       |       |       |       |       |       |       |       |       |       |       |       |       |       |       |       |       |       |       |       |       |       |       |       |       |       |       |       |       |       |       |       |       |       |       |       |       |       |       |       |       |       |       |       |       |       |       |       |       |       |       |       |       |       |       |       |       |       |       |       |       |       |       |       |       |       |       |       |       |       |       |       |       |       |       |       |       |       |       |       |       |       |       |       |       |       |       |       |       |       |       |       |       |       |       |       |       |       |       |       |       |       |       |       |       |       |       |       |       |       |       |       |       |       |       |       |       |       |       |       |       |       |       |       |       |       |       |       |       |       |       |       |       |       |       |       |       |       |       |       |       |       |       |       |       |       |       |       |       |       |       |       |       |       |       |       |       |       |       |       |       |       |       |       |       |       |       |       |       |       |       |       |       |       |       |       |       |       |       |       |       |       |       |       |       |       |       |       |       |       |       |       |       |       |�, June 2020

Table 3b: The Bias of the Parameters of the Simulated GMKD Data Sets for each Sample Size n.

| n   | Actual Value | MSE          |       |       |       |       |       |       |       |       |       |       |       |       |       |       |       |       |       |       |       |       |       |       |       |       |       |       |       |       |       |       |       |       |       |       |       |       |       |       |       |       |       |       |       |       |       |       |       |       |       |       |       |       |       |       |       |       |       |       |       |       |       |       |       |       |       |       |       |       |       |       |       |       |       |       |       |       |       |       |       |       |       |       |       |       |       |       |       |       |       |       |       |       |       |       |       |       |       |       |       |       |       |       |       |       |       |       |       |       |       |       |       |       |       |       |       |       |       |       |       |       |       |       |       |       |       |       |       |       |       |       |       |       |       |       |       |       |       |       |       |       |       |       |       |       |       |       |       |       |       |       |       |       |       |       |       |       |       |       |       |       |       |       |       |       |       |       |       |       |       |       |       |       |       |       |       |       |       |       |       |       |       |       |       |       |       |       |       |       |       |�, June 2020

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Table 4a: The MSE of the Parameters of the Simulated GMKD Data Sets for each Sample Size $n$.  

| $n$ | Actual Value | MSE  |
|-----|--------------|------|
|     | $a$ | $b$ | $c$ | $\alpha$ | $\beta$ | $\delta$ | $\hat{b}$ | $\hat{c}$ | $\hat{d}$ | $\hat{\beta}$ |
| 15  | 1.3 | 1.3 | 2   | 0.2  | 1      | 0.5927  | 1.7371  | 2.9691  | 0.3803  | 1.8725  |
|     | 1.9 | 2.27| 2.5 | 0.1  | 3      | 2.0704  | 1.6583  | 3.0146  | 0.2945  | 0.9887  |
|     | 2   | 0.8 | 2.75| 0    | 2      | 0.9750  | 0.6742  | 3.2541  | 0.3980  | 1.0615  |
|     | 2   | 3.5 | 1.25| 0.015| 2.5    | 2.4903  | 1.1561  | 0.9519  | 0.3141  | 1.1174  |
|     | 2.3 | 1.65| 1.35| 0    | 0.9    | 1.0894  | 1.7070  | 1.0354  | 0.1967  | 1.0994  |
|     | 2.5 | 0.15| 0.5 | 0    | 1      | 1.3065  | 0.7012  | 1.0601  | 0.3277  | 1.0420  |
| 30  | 1.3 | 1.3 | 2   | 0.2  | 1      | 0.5789  | 1.3874  | 2.8764  | 0.2388  | 1.6527  |
|     | 1.9 | 2.27| 2.5 | 0.1  | 3      | 1.4840  | 1.5209  | 2.7058  | 0.2361  | 0.7022  |
|     | 2   | 0.8 | 2.75| 0    | 2      | 0.7488  | 0.5465  | 2.8353  | 0.2015  | 0.7482  |
|     | 2   | 3.5 | 1.25| 0.015| 2.5    | 2.4378  | 0.9590  | 0.7315  | 0.2884  | 0.9814  |
|     | 2.3 | 1.65| 1.35| 0    | 0.9    | 0.9746  | 1.6750  | 0.9971  | 0.1424  | 0.1058  |
|     | 2.5 | 0.15| 0.5 | 0    | 1      | 1.1496  | 0.5271  | 1.0367  | 0.1885  | 0.9625  |
| 50  | 1.3 | 1.3 | 2   | 0.2  | 1      | 0.5409  | 0.9923  | 2.5963  | 0.1884  | 0.8690  |
|     | 1.9 | 2.27| 2.5 | 0.1  | 3      | 1.4806  | 1.4228  | 2.6926  | 0.2551  | 0.6916  |
|     | 2   | 0.8 | 2.75| 0    | 2      | 0.6871  | 0.3610  | 2.6592  | 0.1720  | 0.6247  |
|     | 2.5 | 0.15| 0.5 | 0    | 1      | 1.1326  | 0.4787  | 0.7650  | 0.1711  | 0.9073  |


Table 4b: The MSE of the Parameters of the Simulated GMKD Data Sets for each Sample Size $n$.  

| $n$ | Actual Value | MSE  |
|-----|--------------|------|
|     | $a$ | $b$ | $c$ | $\alpha$ | $\beta$ | $\delta$ | $\hat{b}$ | $\hat{c}$ | $\hat{d}$ | $\hat{\beta}$ |
| 100 | 1.3 | 1.3 | 2   | 0.2  | 1      | 0.2661  | 0.1654  | 2.4133  | 0.1658  | 0.5843  |
|     | 1.9 | 2.27| 2.5 | 0.1  | 3      | 1.1363  | 1.0940  | 3.2125  | 0.2453  | 0.3996  |
|     | 2   | 0.8 | 2.75| 0    | 2      | 0.4507  | 0.1933  | 2.6239  | 0.1514  | 0.1601  |
|     | 2   | 3.5 | 1.25| 0.015| 2.5    | 2.1658  | 0.2596  | 0.1633  | 0.1886  | 0.7211  |
|     | 2.3 | 1.65| 1.35| 0    | 0.9    | 0.4748  | 0.3919  | 0.4946  | 0.1194  | 0.2988  |
|     | 2.5 | 0.15| 0.5 | 0    | 1      | 0.4727  | 0.2031  | 0.4871  | 0.1368  | 0.4452  |
| 200 | 1.3 | 1.3 | 2   | 0.2  | 1      | 0.2307  | 0.0728  | 2.0742  | 0.1318  | 0.3668  |
|     | 1.9 | 2.27| 2.5 | 0.1  | 3      | 1.1023  | 1.0374  | 2.1287  | 0.2103  | 0.1809  |
|     | 2   | 0.8 | 2.75| 0    | 2      | 0.2873  | 0.1468  | 2.1122  | 0.1197  | 0.0866  |
|     | 2.3 | 1.65| 1.35| 0    | 0.9    | 0.3564  | 0.3594  | 0.1107  | 0.0397  | 0.2369  |
|     | 2.5 | 0.15| 0.5 | 0    | 1      | 0.2310  | 0.0373  | 0.3756  | 0.0637  | 0.0890  |
| 500 | 1.3 | 1.3 | 2   | 0.2  | 1      | 0.1296  | 0.0680  | 2.0558  | 0.1303  | 0.2105  |
|     | 1.9 | 2.27| 2.5 | 0.1  | 3      | 1.0568  | 1.0234  | 2.0443  | 0.1810  | 0.1500  |
|     | 2   | 0.8 | 2.75| 0    | 2      | 0.1345  | 0.0454  | 2.0936  | 0.0729  | 0.0492  |
|     | 2   | 3.5 | 1.25| 0.015| 2.5    | 2.0158  | 0.1547  | 0.0577  | 0.0182  | 0.2845  |
|     | 2.3 | 1.65| 1.35| 0    | 0.9    | 0.1519  | 0.2200  | 0.0601  | 0.0373  | 0.0368  |
|     | 2.5 | 0.15| 0.5 | 0    | 1      | 0.2031  | 0.0157  | 0.0559  | 0.0430  | 0.0773  |
Figure 3 shows the behaviour of the plots of the MSE of each of the model estimated parameters for the different GMKD simulated data set, which are decreasing as the sample size increases.

9. Application of Fitting GMKD Model to Real-Life Data

We consider four real-life data sets in order to show the usefulness of the proposed estimation procedure to estimate and fit the GMKD model to these real-life data sets. The data sets are:

**Data Set 1**: Represents the first-semester mathematics course examination grades for the students of the Ahmed Bin Mohammed Military College for the academic semesters Spring 2011 till Fall 2018.

**Data Set 2**: Represents the second-semester mathematics course examination grades for the students of the Ahmed Bin Mohammed Military College for the academic semesters Spring 2011 till Fall 2018.

**Data Set 3**: Represents the first-semester introductory statistics course examination grades for the students of the Ahmed Bin Mohammed Military College for the academic semesters Spring 2011 till Spring 2018.

**Data Set 4**: Represents the second-semester introductory statistics course examination grades for the students of the Ahmed Bin Mohammed Military College for the academic semesters Spring 2011 till Spring 2018.

Table 5 shows some statistics for the actual grades data sets, Table 6a shows the actual and predicted first and second-semester Mathematics course examinations grades frequencies with model parameters estimates and the chi-squares goodness of fit for the proposed GMKD and the Kumaraswamy power function distribution (KPFD), which is close to the GMKD (see Section 4.1), and Table 6b shows the actual and predicted first and second-semester Introductory Statistics course examinations grades frequencies with model parameters estimates and the chi-squares goodness of fit for the proposed GMKD and the KPFD. The p-values of the Chi-Squares of for each grade four data set using the GMKD model inducting very good estimates statistically, as well as, better than all the KPFD models. These results can be seen visually also from Figure 4, illustrating the histograms and the fitted pdfs for each of the grade data sets.
Table 5: Some Statistics for Actual Grades Data Sets

| Statistics     | Examination | Math 1 | Math 2 | Stat 1 | Stat 2 |
|----------------|-------------|--------|--------|--------|--------|
| N              |             | 592    | 592    | 509    | 509    |
| Mean           |             | 9.032095 | 10.456081 | 10.368369 | 11.885069 |
| Std. Error of Mean |         | 0.204036 | 0.202118 | 0.208087 | 0.192046 |
| Median         |             | 8.5    | 10.5   | 10.5   | 12.5   |
| Mode           |             | 6.5*   | 9.5*   | 9.5*   | 13.5   |
| Variance       |             | 24.645330 | 24.184193 | 22.039726 | 18.772696 |
| Skewness       |             | 0.243412 | -0.004408 | 0.040214 | -0.211136 |
| Kurtosis       |             | -0.877241 | -0.932818 | -0.890339 | -0.832699 |
| Std. Error of Kurtosis |   | 0.200505 | 0.200505 | 0.216089 | 0.216089 |
| Minimum        |             | 0.0    | 0.0    | 0.0    | 1      |
| Maximum        |             | 20.0   | 20.0   | 20.0   | 20.0   |

* Multiple modes exist. The smallest value is shown

Figure 4. Histograms and the Fitted pdfs for the Grades Data Sets

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**Table 6a: Actual and Predicted Mathematics Exams 1 and 2 Grades Frequencies with Model Parameters Estimates and Goodness of Fit**

| Grades Range | Frequencies | Stat Exam 1 | Stat Exam 2 |
|--------------|-------------|-------------|-------------|
|              | Actual      | Predicted GMKB | Predicted KPFD | Actual      | Predicted GMKB | Predicted KPFD |
| 0.0 — 1.0    | 14          | 17          | 22          | 8          | 6          | 8          |
| 1.1 — 2.0    | 26          | 27          | 31          | 12         | 14         | 16         |
| 2.1 — 3.0    | 34          | 33          | 36          | 20         | 20         | 22         |
| 3.1 — 4.0    | 35          | 36          | 38          | 25         | 26         | 26         |
| 4.1 — 5.0    | 39          | 39          | 40          | 27         | 30         | 30         |
| 5.1 — 6.0    | 40          | 40          | 41          | 35         | 33         | 33         |
| 6.1 — 7.0    | 41          | 41          | 41          | 36         | 36         | 36         |
| 7.1 — 8.0    | 41          | 41          | 41          | 38         | 38         | 38         |
| 8.1 — 9.0    | 40          | 40          | 40          | 39         | 39         | 39         |
| 9.1 — 10.0   | 39          | 38          | 39          | 40         | 40         | 40         |
| 10.1 — 11.0  | 37          | 37          | 37          | 40         | 40         | 40         |
| 11.1 — 12.0  | 33          | 35          | 35          | 39         | 39         | 40         |
| 12.1 — 13.0  | 31          | 32          | 32          | 38         | 38         | 39         |
| 13.1 — 14.0  | 29          | 29          | 29          | 33         | 36         | 38         |
| 14.1 — 15.0  | 26          | 26          | 26          | 35         | 34         | 36         |
| 15.1 — 16.0  | 25          | 23          | 22          | 31         | 31         | 33         |
| 16.1 — 17.0  | 20          | 20          | 18          | 29         | 28         | 29         |
| 17.1 — 18.0  | 17          | 16          | 14          | 27         | 25         | 24         |
| 18.1 — 19.0  | 13          | 13          | 9           | 22         | 21         | 18         |
| 19.1 — 20.0  | 12          | 10          | 1           | 18         | 18         | 7          |
| Total        | 592         | 592         | 592         | 592        | 592        | 592        |

**Estimated Model Parameters**

|        | Stat Exam 1 | Stat Exam 2 |
|--------|-------------|-------------|
| a      | 21.111111   | 20          |
| b      | 1.449899    | 1.33345     |
| c      | 2.683171    | 1.911199    |
| α      | 0.010120    | α           |
| β      | 1.233333    | β           |

**Goodness of Fit**

|        | Stat Exam 1 | Stat Exam 2 |
|--------|-------------|-------------|
| $\chi^2$ | 1.43279416  | 28.032592   |
| df     | 14          | 15*         |
| p-value | 0.9999897   | p-value     |
|        | 0.0213659   | 0.09993212  |
|        |             | 0.17665     |

* The number of internals were adjusted in order to make the expected number of observations in each interval equal to or greater than 5, which is in turn effected the number of the degree of the freedom.
Table 6b: Actual and Predicted Statistics Exams 1 and 2 Grades Frequencies with Model Parameters

| Grades Range | Frequencies | Actual | Predicted | Predicted | Actual | Predicted | Predicted |
|--------------|-------------|--------|-----------|-----------|--------|-----------|-----------|
|              |             | Stat Exam 1 |             | Stat Exam 2 |             |
|              |             | GMKB     | KPFD       | GMKB     | KPFD       |
| 0.0 — 1.0    |             | 2        | 5          | 4         | 0       | 1          | 1         |
| 1.1 — 2.0    |             | 10       | 12         | 11        | 2       | 5          | 3         |
| 2.1 — 3.0    |             | 16       | 17         | 17        | 6       | 8          | 6         |
| 3.1 — 4.0    |             | 22       | 22         | 22        | 10      | 12         | 10        |
| 4.1 — 5.0    |             | 27       | 26         | 26        | 16      | 16         | 14        |
| 5.1 — 6.0    |             | 28       | 29         | 29        | 22      | 20         | 19        |
| 6.1 — 7.0    |             | 32       | 31         | 32        | 25      | 24         | 23        |
| 7.1 — 8.0    |             | 34       | 33         | 35        | 27      | 27         | 28        |
| 8.1 — 9.0    |             | 36       | 35         | 36        | 31      | 31         | 33        |
| 9.1 — 10.0   |             | 37       | 35         | 37        | 35      | 34         | 37        |
| 10.1 — 11.0  |             | 37       | 35         | 38        | 37      | 36         | 40        |
| 11.1 — 12.0  |             | 35       | 34         | 37        | 38      | 38         | 43        |
| 12.1 — 13.0  |             | 34       | 33         | 36        | 39      | 39         | 44        |
| 13.1 — 14.0  |             | 31       | 31         | 34        | 40      | 40         | 44        |
| 14.1 — 15.0  |             | 30       | 29         | 31        | 39      | 39         | 43        |
| 15.1 — 16.0  |             | 27       | 27         | 28        | 38      | 38         | 39        |
| 16.1 — 17.0  |             | 22       | 24         | 23        | 36      | 35         | 34        |
| 17.1 — 18.0  |             | 21       | 20         | 18        | 31      | 30         | 27        |
| 18.1 — 19.0  |             | 16       | 17         | 11        | 23      | 23         | 17        |
| 19.1 — 20.0  |             | 12       | 14         | 4         | 14      | 13         | 4         |
| Total        |             | 509      | 509        | 509       | 509     | 509        | 509       |

Estimated Model Parameters

|                  |        |        |        |        |        |        |
|                  | GMKB   | KPFD   | GMKB   | KPFD   | GMKB   | KPFD   |
| a                | 20.999999 | 20   | a      | 20.05333 | 20   |
| b                | 1.785556    | 1.799513 | b      | 2.172103   | 2.499990 |
| c                | 2.891111     | 1.997890   | c      | 1.771321    | 2.137350  |
| a                | 0.021012     | a        | 0.000111 | a        |
| β                | 1.388889     | β        | 1.02115 | β        |

Goodness of Fit

|                  |        |        |        |        |
|                  | χ²     |        | χ²     |        |
| df               | 14     | 14*    | df     | 13*    | 13*    |
| p-value           | 0.99856701 | 0.5148497 | p-value | 0.98507207 | 0.216531 |

* The number of internals were adjusted in order to make the expected number of observations in each interval equal to or greater than 5, which is in turn effected the number of the degree of the freedom.
10. Summary

A new generalized modification of the Kumaraswamy distribution is introduced, and its properties consisting of boundaries, limits, mode, quantities, reliability and hazard functions, and Renyi entropy, are studied, and some of its different various shapes are given to show its flexibility. This distribution is closed under scaling and exponentiation. Some of the well-known distributions, such as the generalized uniform, triangular, beta, power function, Minimax, and some other Kumaraswamy related distributions, are special cases of this distribution. Its order statistics, mean deviations, moment generating function, and Lorenz and Bonferroni curves, with its moments consisting of the mean, variance, moments about the origin, harmonic, incomplete, probability weighted, L, and the trimmed L moments are derived. We used the maximum likelihood estimation method for estimating its parameters, and are applied to six different models, having different pdf’s shapes, simulated data sets of this distribution, in order to check the performance of the estimation method through the estimated parameters mean squares errors computed from the different simulated sample sizes, which are shown to be decreasing as the sample size increases. Finally, four real-life data sets of students grades’ at Ahmed Bin Mohammed Military College, Doha-Qatar, the first two sets, representing the first and second semester mathematics course examination grades for the academic semesters Spring 2011 till Fall 2018, while the third and fourth sets representing the first and second semester introductory statistics course examination grades for the academic semesters Spring 2011 till Spring 2018, are used in order to show the usefulness and the flexibility of this distribution in application to real-life data sets, as well as our examinations grades data sets. We also used the Kumaraswamy power function distribution models, which is close to our proposed distribution. The results are very good, statistically, via the chi-squares goodness of fit tests, and visually, via the histograms and the fitted pdfs for each of our examinations grade data sets, and even better than all the Kumaraswamy power function distribution models.

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