Iterated Monodromy Groups of Exponential Maps

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Abstract. This paper introduces iterated monodromy groups for transcendental functions and discusses them in the simplest setting, for post-singularly finite exponential functions. These groups are self-similar groups in a natural way, based on an explicit construction in terms of kneading sequences. We investigate the group theoretic properties of these groups, and show in particular that they are amenable, but they are not elementary subexponentially amenable.

1. Introduction

In the iteration theory of rational maps, iterated monodromy groups are self-similar groups associated to post-singularly finite dynamical systems. These groups encode the Julia set of a rational function from the point of view of symbolic dynamics \[ \text{Nek05}. \] Conversely, many classical examples of self-similar groups with exotic geometric properties, such as the Fabrykowski-Gupta \[ \text{FG91} \] and the Basilica group \[ \text{GZ02} \], arise in a natural way as iterated monodromy groups of rational maps.

Much of the study of symbolic dynamics of quadratic polynomials has been done in terms of dynamic rays, as well as in terms of kneading sequences \[ \text{BSc02, MT88, Thu09} \], before Iterated Monodromy Groups were introduced as a new and powerful tool \[ \text{Nek05, BN06} \]. The relationships between these groups and kneading sequences were developed in \[ \text{BN08} \].

This paper is a first in a series of papers that study of iterated monodromy groups of entire functions. Here we focus on a particularly fundamental class of functions, the exponential family, motivated by the well known strong analogy between the combinatorics of quadratic polynomials and exponential maps (see e.g. \[ \text{BDH} + 00 \]). Like polynomials, exponential maps have so far only been studied in terms of rays and kneading sequences (see e.g. \[ \text{SZ03} \]) resulting in a complete classification in \[ \text{LSV08} \], based on \[ \text{HSS09} \].

In this paper, we introduce iterated monodromy groups for exponential maps and compare them to self-similar groups defined just in terms of formal kneading sequences. For an exponential map \( f \), we show that the iterated monodromy action of \( f \) is conjugate to the self-similar group action defined by the kneading sequence of

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For all kneading sequences, we show that the obtained group is a left-orderable amenable group that is residually solvable, but not residually finite.

We give a short background in holomorphic dynamics in section 2, with a special focus on the exponential family. Next in section 3 we provide the algebraic and graph theoretic background to define the iterated monodromy group of a post-singularly finite entire function. We give an explicit description of the iterated monodromy group in terms of kneading automata in section 4, see Theorem 4.6. The structure of the orbital Schreier graphs is investigated in section 5, where we show in Theorem 5.5 that every component of the (reduced) orbital Schreier graph is a tree with countably many ends. This result together with the work in [Rei20] is then used in section 6, where we collect group theoretic properties of the iterated monodromy groups of exponential functions, in particular amenability (see Theorem 6.5).

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2. Dynamics of Exponential Maps

2.1. General entire dynamics. We give a very short introduction into transcendental dynamics relevant to our needs, see [Sch10] for a survey. We start with the definition of a post-singularly finite entire function.

**Definition 2.1.** Let $f: \mathbb{C} \to \mathbb{C}$ be an entire function. A critical value is the image of a critical point, i.e. $f(c)$ where $f'(c) = 0$. An asymptotic value is a limit $\lim_{t \to \infty} f(\gamma(t))$ where $\gamma: [0, \infty) \to \mathbb{C}$ is a path with $\lim_{t \to \infty} \gamma(t) = \infty$. The set of singular values is defined as $S(f) = \{\text{critical and asymptotic values}\}$ and the set of post-singular values is $P(f) = \bigcup_{n \geq 0} f^n(S(f))$. The map $f$ is called post-singularly finite if $P(f)$ is finite.

The following lemma is the basis of our consideration:

**Lemma 2.2 (Sch10, Theorem 1.13).** Let $f$ be an entire function. Then $f$ restricts to an unbranched covering from $\mathbb{C} \setminus f^{-1}(S(f))$ to $\mathbb{C} \setminus S(f)$. △

In fact, an alternative definition of $S(f)$ is that $S(f)$ is the smallest closed subset $S$ such that $f$ restricts to an unbranched covering over $\mathbb{C} \setminus S$. As $P(f)$ is a closed and contains $S(f)$, we see that that $f$ also restricts to an unbranched covering from $\mathbb{C} \setminus f^{-1}(P(f))$ to $\mathbb{C} \setminus P(f)$. As $P(f)$ is forward invariant, we have that $P(f) \subset f^{-1}(P(f)) \subset f^{-2}(P(f)) \subset \ldots$ is an increasing chain of closed subsets. From this we can show by induction that $f^n$ restricts to an unbranched covering from $\mathbb{C} \setminus f^{-n}(P(f))$ to $\mathbb{C} \setminus P(f)$, using the fact that compositions of coverings of manifolds are again coverings.

The escaping set $I(f)$ is the set of points which escape to infinity under the iteration of $f$, i.e.

$I(f) = \{z: \lim_{n \to \infty} f^n(z) = \infty\}.$
Definition 2.3. A dynamic ray is a maximal injective curve $\gamma: (0, \infty) \to I(f)$ with $\gamma(t) \to \infty$ as $t \to \infty$. We say that $\gamma$ lands at $a$ if $\gamma(t) \to a$ for $t \to 0$.

We should note that this definition is not the precise standard definition given in [Sch10], however, it is appropriate in the study of post-singularly finite exponential maps as done in [LSV08]. We will only use dynamic rays for exponential maps, so this is not an issue for us.

2.2. Combinatorics of exponential maps. The exponential family is the family of functions $E_\lambda(z) = \lambda \exp(z)$ for $\lambda \in \mathbb{C}^* := \mathbb{C} \setminus \{0\}$. The only singular value of $\lambda \exp(z)$ is 0. It is the limiting value along the negative real axis. It is also an omitted value, so for the exponential family, Lemma 2.2 specialized to the well-known fact that every function in the exponential family is a covering from $\mathbb{C}$ to $\mathbb{C}^*$.

This is in fact a universal covering, and the group of deck transformations are given by translations of $2\pi i$. In the following, we will often consider collections which form a free orbit under translations with multiplies of $2\pi i$. A prime example is the set of preimages $E^{-1}_\lambda(z)$ of any point $z \in \mathbb{C}^*$. As $\text{S}(E_\lambda(z)) = \{0\}$, we have $\text{P}(E_\lambda(z)) = \{E^n_\lambda(0): n \geq 0\} = \{0, \lambda, E_\lambda(\lambda), \ldots\}$.

In this subsection, $f$ will always denote a post-singularly finite function in the exponential family. In this setting, 0 is a strictly preperiodic point, as it is an omitted value and has finite forward orbit. We denote the preperiod of 0 as $k$ and the period of 0 as $p$, so $\text{P}(f) = \{0, f(0), \ldots f^{k+p-1}(0)\}$ with $f^{k+p}(0) = f^k(0)$.

The dynamics of post-singularly finite exponential maps can be studied via dynamical rays, as seen in the following theorem:

Theorem 2.4 ([SZ03]). Let $f(z) = \lambda \exp(z)$ be a post-singularly finite function in the exponential family. Then there is a dynamic ray landing at 0 which is preperiodic.

We collect some facts about dynamic rays of exponential maps that are all discussed in [SZ03, LSV08].

Fact 2.5. (1) Two different dynamic rays do not intersect, but they might land at the same point.

(2) The preimage of a dynamic ray is a family of dynamic rays forming a free orbit under translations with multiplies of $2\pi i$.

(3) If $\gamma$ lands at $a$, then for every $b \in f^{-1}(a)$ there is a unique preimage component of $\gamma$ landing at $b$.

(4) If $\gamma$ lands at 0, then all preimage components separate the plane, the connected components of $\mathbb{C} \setminus f^{-1}(a)$ also form a free orbit under translations with multiplies of $2\pi i$.

Definition 2.6. A ray spider is a family $\mathbb{S} = (\gamma_a)_{a \in \text{P}(f)}$ such that $\gamma_a$ is a dynamic ray landing at $a$ for each $a \in \text{P}(f)$.

Remark 2.7. In this definition, we do not require any invariance properties.

Our notion of a ray spider is a special case of the general notion of spiders given in [SZ03]. By Theorem 2.4, there exists a ray spider: if $\gamma$ is a dynamic ray landing at 0, then $\gamma_{f^i(0)} = f^i(\gamma)$, $0 \leq i < k+p$ is a ray spider. This spider is not necessarily
forward invariant, as it might happen that $f^k(\gamma) \neq f^{k+p}(\gamma)$ (the period of the rays may be a multiple of the period of the landing point). This is not an issue in our construction as we will consider the family of pullbacks of a given spider.

**Definition 2.8.** Let $S = (\gamma_a)_{a \in P(f)}$ be a ray spider. The pullback of $S$ is the ray spider $(\tilde{\gamma}_a)$ where $\tilde{\gamma}_a$ is the unique preimage of $\gamma_{f(a)}$ landing at $a$.

The dynamical partition associated to $S$ is the partition of $\mathbb{C} \setminus f^{-1}(\gamma_0)$ into its connected components. We denote the connected component of 0 by $U_0 = \mathbb{U}_0 + 2\pi in$. Note that the dynamical partition only depends on the ray landing at 0.

The kneading sequence of $f$ is the sequence $(k_n)_{n \in \mathbb{N}}$ so that $f^n(0) \in U_{k_n}$. The kneading sequence is in fact independent of $S$, see [LSV08] for a more detailed discussion.

**Example 2.9.** Let $k \in \mathbb{Z} \setminus 0$, and consider $f(z) = 2k\pi i \exp(z)$. For this map, 0 is mapped to $2k\pi i$, which is a fixed point of $f$. Hence $f$ is post-singularly finite with $P(f) = \{0, 2k\pi i\}$. Let $\gamma_0$ be a dynamic ray landing at 0, and let $U$ be the associated dynamical partition. Then $0 \in U_0$ by definition of $U_0$ and $2k\pi i \in U_k = U_0 + 2\pi in$, so the kneading sequence of $f$ is $0k$.

### 3. Iterated Monodromy Groups

#### 3.1. The dynamical preimage tree $\mathcal{T}$

Let $f$ be a post-singularly finite entire function and $t \in \mathbb{C} \setminus P(f)$.

**Definition 3.1.** Choose a base point $t \in \mathbb{C} \setminus P(f)$. Let $L_n := f^{-n}(t)$ be the preimage of $t$ under the $n$-th iterate of $f$.

The dynamical preimage tree $\mathcal{T}$ is a rooted tree with vertex set $\bigsqcup_{n \geq 0} L_n$ (where $\bigsqcup$ denotes disjoint union) and edges $w \to f(w)$ for $w \in L_{n+1}, f(w) \in L_n$. Its root is $t$.

The dynamical preimage tree is always a regular rooted tree, i.e. all vertices have the same number of children. For polynomials, this number is the degree of the polynomial. For transcendental entire functions, every vertex has countably infinite many children. We will show in subsection 5.3 that for postsingularly finite exponential maps, the dynamical preimage tree has an extra regularity based on the periodicity of the exponential map.
3.2. Iterated Monodromy Action. Each level of \( \mathcal{T} \) is the preimage of \( t \) under a covering map, namely \( f^n : \mathbb{C} \setminus f^{-n}(\mathbb{P}(f)) \to \mathbb{C} \setminus \mathbb{P}(f) \). Hence \( \pi_1(\mathbb{C} \setminus \mathbb{P}(f), t) \) acts on \( L_n \) via path lifting: if \( \gamma : [0, 1] \to \mathbb{C} \setminus \mathbb{P}(f) \) is a loop based on \( t \) and \( v \in L_n \) is a \( n \)-th preimage of \( t \), then there is a unique lift \( \gamma^n \) making the following diagram commute:

\[
\gamma^n \quad \downarrow f^n \\
(\mathbb{C} \setminus f^{-n}(\mathbb{P}(f)), v) \quad \rightarrow \quad (\mathbb{C} \setminus \mathbb{P}(f), t)
\]

So \( \gamma^n(0) = v \), and \( \gamma^n(1) \in L_n \) might be another \( n \)-th preimage. We define \( [\gamma](v) := \gamma^n(1) \). Using the homotopy lifting properties of coverings, we can see that this defines an action of \( \pi_1(\mathbb{C} \setminus \mathbb{P}(f), t) \) on \( L_n \). If \( w \in L_{n+1} \) is a child of \( v \), then the following diagram commutes (by uniqueness of lifts):

\[
\gamma^n \quad \downarrow f^n \\
(\mathbb{C} \setminus f^{-n-1}(\mathbb{P}(f)), w) \quad \rightarrow \quad (\mathbb{C} \setminus \mathbb{P}(f), t)
\]

By commutativity of the diagram \( f(\gamma^n(1)) = \gamma^n(1) \) so \( [\gamma](w) \) is also a child of \( [\gamma](v) \). This means that actions on the levels are compatible and give rise to an action of \( \pi_1(\mathbb{C} \setminus \mathbb{P}(f)) \) on \( \mathcal{T} \). This is the *iterated monodromy action*.

**Definition 3.2.** Let \( f \) be a post-singularly finite entire function, \( t \in \mathbb{C} \setminus \mathbb{P}(f) \). Let \( \phi : \pi_1(\mathbb{C} \setminus \mathbb{P}(f), t) \to \text{Aut}(\mathcal{T}) \) be the group homomorphism induced by the iterated monodromy action. The iterated monodromy group of \( f \) with base point \( t \) is the image of \( \phi \). By the first factor theorem we have

\[
\text{IMG}(f) \cong \frac{\pi_1(\mathbb{C} \setminus \mathbb{P}(f), t)}{\ker \phi}
\]

This definition depends a priori on the base point \( t \in \mathbb{C} \setminus \mathbb{P}(f) \). For a different base point \( t' \), every path from \( t \) to \( t' \) gives rise to an isomorphism of preimage trees over \( t \) and over \( t' \), so we can identify the groups up to inner automorphisms. See [Nek05] Proposition 5.1.2 for a detailed discussion in the rational case.

3.3. \( \mathbb{Z} \)-regular rooted trees. We use the following definition of rooted trees:

**Definition 3.3.** A *rooted tree* is a tuple \( T = (V, E, r) \) such that \( (V, E) \) forms a tree (with vertex set \( V \) and edge set \( E \)) and \( r \in V \), which we call the *root* of \( T \). We endow \( T \) with the unique orientation so that all vertices are reachable from the root, i.e. for every vertex \( v \), there is a directed path from the root to \( v \).

If \( (v, w) \) is a directed edge for this orientation, we say that \( w \) is a *child* of \( v \) and \( v \) is the *parent* of \( w \). If \( v \) has no children, we call it a *leaf*.

If \( w \) is reachable from \( v \), we say that \( w \) is a descendant of \( v \) and \( v \) is an ancestor of \( w \). We denote by \( T_v \) the rooted tree which is the induced subgraph on the set of
descendants of \( v \) together with \( v \) as the new root. An end of a rooted tree \( T \) is a sequence \( v_n \), so that \( v_0 \) is the root of \( T \) and \( v_{n+1} \) is a child of \( v_n \). We denote by \( \partial T \) the set of ends of \( T \).

We will mainly consider countable infinite trees without leaves.

In fact, \( \partial T \) can be defined without fixing a root of \( T \), one way is by considering equivalence classes of geodesic rays, where two geodesic rays are equivalent if they have a common tail. Given a root \( r \) and a geodesic ray \( \gamma \), there is always a unique geodesic ray starting at \( r \) equivalent to \( \gamma \). Also, \( \partial T \) is a totally disconnected Hausdorff space with clopen subset \( \partial T_v \subset \partial T \). The topology is also independent of the root. If \( T \) is a locally finite tree without leaves, then \( \partial T \) is compact.

**Definition 3.4.** A \( \mathbb{Z} \)-regular rooted tree \( T \) is a tuple \((V, E, r, \eta)\), where \((V, E, r)\) is a rooted tree and \( \eta \) is a right \( \mathbb{Z} \)-action \( \eta: V \setminus \{r\} \times \mathbb{Z} \rightarrow V \setminus \{r\} \) such that for all vertices \( v \in V \), the set of its children forms a free orbit under the action.

Note that this implies that the root is fixed by the action, as it is the only vertex without a parent. Also the tree has no leaves, as the empty set is not a free orbit under a \( \mathbb{Z} \)-action.

An isomorphism between \( \mathbb{Z} \)-regular rooted trees is a tree isomorphism which preserves the root and commutes with the additional right \( \mathbb{Z} \)-actions. We denote by \( \text{Aut}_\mathbb{Z}(T) \) the group of automorphisms of \( T \) as a \( \mathbb{Z} \)-regular rooted tree. Every element of \( \text{Aut}_\mathbb{Z}(T) \) preserves the root of \( T \) and acts by a translation on the first level. We denote by \( \rho: \text{Aut}_\mathbb{Z}(T) \rightarrow \mathbb{Z} \) the group homomorphism given by the first level action. The kernel of \( \rho \) is the stabilizer of the first level, as every element of \( \text{Aut}_\mathbb{Z}(T) \) acts by translation, this is also the stabilizer of any vertex on the first level. For a vertex \( v \in V \) and a subgroup \( G \subset \text{Aut}_\mathbb{Z}(T) \) we denote the stabilizer of \( v \) in \( G \) by \( \text{Stab}_G(v) \). We denote the stabilizer of the first level as \( \text{Stab}_G \).

Note that \( \text{Aut}_\mathbb{Z}(T) \) also acts on \( \partial T \). This action is an fact faithful, as every vertex is part of a sequence defining an end.

**Example 3.5.** The standard \( \mathbb{Z} \)-regular tree has as vertex set \( \mathbb{Z}^* \), the set of finite words in \( \mathbb{Z} \). Its root is the empty word \( \emptyset \). Its edges are all pairs of the form \((v, vn)\) for \( v \in \mathbb{Z}^*, n \in \mathbb{Z} \) (here \( vn \) denotes the word \( v \) concatenated with the letter \( n \)). So for each vertex \( v \), the set of its children are all words obtained by concatenating one letter to it. Also, the set of ends can be identified with the set of right-infinite words, which we denote by \( \mathbb{Z}^\omega \).

The right action is given by

\[
\eta.vn, m) = v(n + m).
\]

So the action is by translation on the last letter. By abuse of notation, we will denote the standard \( \mathbb{Z} \)-regular tree also by \( \mathbb{Z}^* \).

The subgroups of \( \text{Aut}_\mathbb{Z}(\mathbb{Z}^*) \) were studied in [OS10] under the name of \( \mathbb{Z}C \)-groups. Note that if \( T \) is a \( \mathbb{Z} \)-regular rooted tree and \( v \) is a vertex of \( T \), then \( T_v \) is also a \( \mathbb{Z} \)-regular rooted tree. However, in general we have no canonical choice of an isomorphism between \( T \) and \( T_v \). This is different for the standard \( \mathbb{Z} \)-regular tree:

**Definition 3.6.** For \( g \in \text{Aut}_\mathbb{Z}(\mathbb{Z}^*) \), \( v \in \mathbb{Z}^* \) let \( g_{|v} \) denote the unique element in \( \text{Aut}_\mathbb{Z}(\mathbb{Z}^*) \) such that \( g(vw) = g(v)g_{|v}(w) \). We say that \( g_{|v} \) is the section of \( g \) at \( v \).
We will use the following set of easily verifiable cocycle equations:

\[(g|_w)v = g|_{v w}\]
\[(g h)|_v = g|_{h(v)} h|_v\]

We say that \( g \in \text{Aut}_Z(\mathbb{Z}^*) \) is of finite activity on level \( n \) if the set \( \{ v \in \mathbb{Z}^n : g|_v \neq 1 \} \) is finite. We define \( \text{aut}_Z(\mathbb{Z}^*) \) as the group of automorphisms which have finite activity on every level. We will many work with subgroups of \( \text{aut}_Z(\mathbb{Z}^*) \). As we work with an infinite alphabet, we have to take care for the wreath recursion. The wreath recursion for \( \text{aut}_Z(\mathbb{Z}^*) \) is

\[
\text{aut}_Z(\mathbb{Z}) \cong \bigoplus_{x \in \mathbb{Z}} \text{aut}_Z(\mathbb{Z}) \rtimes \mathbb{Z}
\]

\[
g \mapsto (x \mapsto g|_{x}, \rho(g))
\]

We say a subgroup \( G \subset \text{Aut}_Z(\mathbb{Z}^*) \) is self-similar if \( g|_v \in G \) for all \( g \in G \) and \( v \in \mathbb{Z}^* \). A subgroup \( G \subset \text{Aut}_Z(\mathbb{Z}^*) \) is self-replicating if for all \( v \in \mathbb{Z}^* \) and \( g \in G \) there exists an \( h \in G \) with \( h|_v = g \). It is easy to see that is is enough to check this on the first level.

**Lemma 3.7.** Let \( f \) be a post-singularly finite exponential function, \( t \in \mathbb{C}\setminus \mathbb{P}(f) \). Then the dynamical preimage tree of \( f \) with base point \( t \) is a \( \mathbb{Z} \)-regular tree and \( \text{IMG}(f) \) is a subgroup of \( \text{Aut}_Z(\mathcal{T}) \).

**Proof.** The \( \mathbb{Z} \)-regular structure is given by translation by multiples of \( 2\pi i \). As two complex numbers have the same value under the exponential map if and only if they differ by a multiple of \( 2\pi i \), it is clear that this really defines a \( \mathbb{Z} \)-regular structure. Also, if \( w \) is an \( n \)-th preimage of \( t \), and \( \gamma \) is a loop on \( t \), for the lift \( \gamma^w \), the \( 2\pi i \) translate of \( \gamma^w \) is also a lift of \( \gamma \) by the \( 2\pi i \) periodicity of \( f^n \). This shows that the iterated monodromy action commutes which the \( \mathbb{Z} \) action given by the \( \mathbb{Z} \)-regular structure, so \( \text{IMG}(f) \subset \text{Aut}_Z(\mathcal{T}) \).

\[ \square \]

4. **Combinatorial description**

4.1. **Automata.**

**Definition 4.1.** An automaton \( A \) is a map \( \tau : Q \times X \to X \times Q \). We call \( Q \) the *state set* and \( X \) the *alphabet*. We will write the components of \( \tau(a, x) \) often as \( (a(x), a|x) \). Here \( a(x) \in X \) is called the image of \( x \) under \( a \), and \( a|x \) is the restriction of \( a \) at \( x \).

A *group automaton* is an automaton such that for all \( a \in Q \), the map \( x \mapsto a(x) \) is a bijection on \( Q \). If the alphabet is \( \mathbb{Z} \), that automaton is a \( \mathbb{Z} \)-automaton if for all \( a \in Q \), the map \( x \mapsto a(x) \) is a translation on \( \mathbb{Z} \), i.e. equal to the map \( x \mapsto x + n \) for some \( n \in \mathbb{Z} \).

We will only consider automata which have a distinguished identity state \( 1 \), i.e. a state such that \( \tau(1, x) = (x, 1) \) for all \( x \in X \). We can draw automata using Moore diagram. As vertices we take the state set \( Q \), and if \( \tau(a, x) = (y, b) \), we draw an edge from \( a \) to \( b \) labeled \( x|y \). Here is an example of a Moore diagram, of the so-called binary adding machine.

\[
1 \rightarrow 0 \leftrightarrow \alpha \quad 0 \rightarrow 1\]

\[
1 \leftrightarrow 0, 1, 0, 1
\]
Definition 4.2. Let $A$ be an automaton given by $\tau: Q \times X \to X \times Q$. We extend $\tau$ to a map $Q \times X^* \to X^* \times Q$ recursively via

$$\tau(a, xv) = (a(x)a|x(v), a|x(v))$$

If $A$ is a group automaton, then for each $a \in A$, the extended map $X^* \to X^*$ induces a tree automorphism of the regular $X$-tree. If $A$ is a $\mathbb{Z}$-automaton, it is a automorphism preserving the regular $\mathbb{Z}$-tree structure.

4.2. Kneading automata.

Definition 4.3. Given two words $x_1 \ldots x_k, y_1 \ldots y_p \in \mathbb{Z}^*$ with $x_k \neq y_p$ the automaton $K(x_1 \ldots x_k, y_1 \ldots y_p)$ has alphabet $\mathbb{Z}$ and states $a_1 \ldots a_k, b_1 \ldots b_p$ (and the identity state $1$) and the following transition function:

$$\tau(a_1, z) = (z + 1, 1)$$
$$\tau(a_i+1, x_i) = (x_i, a_i)$$
$$\tau(b_1, x_k) = (x_k, a_k)$$
$$\tau(b_1, y_p) = (y_p, b_p)$$
$$\tau(b_i+1, y_i) = (y_i, b_i)$$
$$\tau(q, z) = (z, 1)$$ for all other cases.

We note that $K(x_1 \ldots x_k, y_1 \ldots y_p)$ is a $\mathbb{Z}$-automaton, indeed $a_1$ acts on $\mathbb{Z}$ by the translation by one, and all other states act on $\mathbb{Z}$ as the identity. Figure 2 shows a reduced Moore diagram of $K(x_1 \ldots x_k, y_1 \ldots y_p)$, where labels with only one letter $z$ are abbreviations for the label $z|z$ and all trivial arrows ending in the identity state have been omitted.

Example 4.4. The automaton $K(0, k)$ with $k \in \mathbb{Z} \setminus \{0\}$ has the following (non-reduced) Moore diagram:

Here $n$ stands for any element of $\mathbb{Z}$, and $z$ for any element of $\mathbb{Z} \setminus \{0, k\}$.

Remark 4.5. We see that every non-trivial state has exactly one edge ending in it, so for every non-trivial state there is a unique left-infinite path ending in it. This implies that $K(x_1 \ldots x_k, y_1 \ldots y_p)$ is a bounded activity automaton in the sense of [Sid04]: For any length $m$, there are $n + k$ paths of length $m$ ending in a non-trivial state in the Moore diagram, so for any $q$, the set $\{v \in \mathbb{Z}^m: q|v \neq 1\}$ has cardinality bounded by $n + k$.

We denote by $K(x_1 \ldots x_k, y_1 \ldots y_p)$ the group of automorphisms of $\mathbb{Z}^*$ generated by $K(x_1 \ldots x_k, y_1 \ldots y_p)$. 
THEOREM 4.6. Let $f$ be a post-singularly finite exponential function with kneading sequence $x_1 \ldots x_k y_1 \ldots y_p \in \mathbb{Z}^2$. Then the iterated monodromy action of $f$ is conjugate to the action of $K(x_1 \ldots x_k, y_1 \ldots y_p)$ on $\mathbb{Z}^+$. 

In particular, for functions of the form $2\pi ik \exp(z)$ with $k \in \mathbb{Z} \setminus \{0\}$, the iterated monodromy action is conjugate to the action of the automata group $K(0, k)$ discussed in Example 4.1.

PROOF. We choose a ray spider $S_0$ for $f$ and consider the sequence $S_n$, where $S_{n+1}$ is the pullback of $S_n$. We denote by $\gamma_{z,n}$ the ray in $S_n$ landing at $z$, also let $U_{*,n}$ be the dynamical partition induced by $S_n$. Choose a base point $t \in \mathbb{C} \setminus \bigcup_{z \in P(f)} \gamma_{z,0}$. We recursively define an isomorphism between the dynamical preimage tree $T$ and the standard $\mathbb{Z}$-tree $Z^*$. We send the root $t$ to the empty word $\emptyset$. Suppose we already defined the bijection on $L_n \subset T$, and let $w \in L_n$ be mapped to $v \in \mathbb{Z}^n$. Then for the dynamical partition $U_{*,n}$, there is exactly one child of $w$ in each component. We send the child lying in $U_{m,n}$ to $vm$. 

By construction, this defines an isomorphism of $\mathbb{Z}$-trees. The complement of each ray spider is a simply connected domain. For two points $w_1, w_2 \in \mathbb{C} \setminus \bigcup_{z \in P(f)} \gamma_{z,n}$, let $g_n(w_1, w_2)$ be a path from $w_1$ to $w_2$ crossing no ray of $S_n$, and let $g_{z,n}(w_1, w_2)$ be a path from $w_1$ to $w_2$ crossing only the ray of $\gamma_{z,n}$ once in a positive sense (so that $g_n(w_1, w_2)$ composed with $g_{z,n}(w_1, w_2)$ has winding number $1$ around $z$) and no other ray of $S_n$. The homotopy classes of $g_n$ are well defined in the fundamental groupoid $\Pi_1(\mathbb{C} \setminus P(f))$. Let us investigate the lifting behavior of these homotopy classes: let $w, w' \in \mathbb{C} \setminus \bigcup_{z \in P(f)} \gamma_{z,n}$ and let $v \in f^{-1}(w)$. 

Let $g^v_n(w, w')$ (or $g^v_{z,n}(w, w')$) denote the lift of $g_n(w, w')$ (respectively $g_{z,n}(w, w')$). Then $g^v_n(w, w')$ is a path in $\mathbb{C}$ meeting no preimage of $\gamma_{z,n}$ for $z \in P(f)$. Let $v'$ be the preimage of $w'$ in the same component of $U_{*,n}$ as $v$. Then $g^v_n(w, w')$ must be homotopic to $g_{n+1}(v, v')$. Similarly, $g^v_{0,n}(w, w)$ is a path which doesn’t cross any ray of $S_{n+1}$, and as $g_{0,n}(w, w)$ has winding number $1$ around $0$, the lift $g^v_{0,n}(w, w)$ must end in $v+2\pi i$. Hence $g^v_{0,n}(w, w') \equiv g_{n+1}(v, v'+1)$ and by composition $g^v_{0,n}(w, w') \equiv g_{n+1}(v, v'+1)$. 

Now $\pi_1(\mathbb{C} \setminus P(f), t)$ is freely generated by $(g_{z,0}(t), z \in P(f))$. Numerate $P(f)$ by $z_1 = 0, z_{i+1} = f(z_i), 1 \leq i \leq n + k - 1$. We claim that the group homomorphism...
given by
\begin{align}
  g_{z,0}(t,t) &\mapsto a_i, 1 \leq i \leq k \\
  g_{z,0}(t,t) &\mapsto b_{i-k}k + 1 \leq i \leq n + k
\end{align}

conjugates the iterated monodromy action of \( f \) to the action of \( \mathcal{K}(x_1 \ldots x_k, y_1 \ldots y_p) \).

This follows from the pullback behavior.

\[\square\]

5. Schreier Graphs

For this section, we fix \( x_1 \ldots x_k, y_1 \ldots y_p \) with \( x_k \neq y_p \). We will give a combinatorial description of the action of \( \mathcal{K}(x_1 \ldots x_k, y_1 \ldots y_p) \) on the standard \( \mathbb{Z} \)-tree \( \mathbb{Z}^* \).

We will work in this section with the generating set \( S := \{a_1, \ldots, a_k, b_1, \ldots, b_p\} \) of \( \mathcal{K}(x_1 \ldots x_k, y_1 \ldots y_p) \).

**Definition 5.1.** Let \( n \in \mathbb{N} \). The \( n \)-th level Schreier graph has vertex set \( \mathbb{Z}^n \) and edges \( v \rightarrow s(v) \) for \( v \in \mathbb{Z}^n, s \in S \). The orbital Schreier graph \( \Gamma_\infty \) has the ends of the standard \( \mathbb{Z} \)-tree as vertex set (which can be identified with \( \mathbb{Z}^\infty \)) and also has edges \( v \rightarrow s(v) \) for \( v \in \mathbb{Z}^\infty, s \in S \).

The reduced Schreier graph \( \overline{\Gamma}_n \) and reduced orbital Schreier graph \( \overline{\Gamma}_\infty \) are obtained by deleting all loops of \( \Gamma_n \) respectively \( \Gamma_\infty \).

Let \( w_m \in \mathbb{Z}^m \) be the reverse of the length \( m \) prefix of \( x_1 \ldots x_ky_1 \ldots y_p \). In the Moore diagram in Figure 2 we see that \( w_m \) is the concatenation of the labels of the unique path \( p \) of length \( m \) ending in \( a_1 \). Let \( c_m \) be the starting state of \( p \) (so \( c_m = a_m \) for \( m \leq k \), and \( c_m = b_{m'} \) for appropriate \( m' \) otherwise). Then \( c_m | w_m = a_1 \) and \( s|v \neq a_1 \) for all other pairs of a state \( s \) and \( v \in \mathbb{Z}^m \). As \( a_1 \) is the only state which acts non-trivially on the first level, we have
\[
c_m | w_m(i) = i + 1 \\
s|v(i) = i \text{ for other pairs.}
\]

Since additionally \( a_1 \) only restricts to the identity state, we also have that if \( s(v) = w \) with \( v \neq w \in \mathbb{Z}^m \) for some state \( s \), then \( s(vi) = wi \) for all \( i \in \mathbb{Z} \). In fact \( v \) and \( w \) must differ in exactly one position.

This discussion can be summarized in the following lemma:

**Lemma 5.2.** The Schreier graph \( \overline{\Gamma}_{m+1} \) can be obtained from \( \overline{\Gamma}_m \) in the following way: take as vertex set \( vx \) where \( v \in \mathbb{Z}^m, x \in \mathbb{Z} \). For edges we have the following two construction rules:
\begin{itemize}
  \item \( v \rightarrow v' \) edge in \( \overline{\Gamma}_m \)
  \item \( vi \rightarrow v'i \) is an edge in \( \overline{\Gamma}_{m+1} \) for all \( i \in \mathbb{Z} \).
\end{itemize}

See Figure 3 for a visualization of the construction rules.

**Example 5.3.** We can use this construction to produce the first few \( \overline{\Gamma}_m \) for the group \( \mathcal{K}(0,1) \). As in Example 1.4, we name the generators \( a \) and \( b \) instead of \( a_1 \) and \( b_1 \). Note that \( a \) acts by translation on the first level, and \( b \) acts trivially on the first level, so \( \overline{\Gamma}_1 \) is just a bi-infinite line. To use the construction rule, we note that \( w_1 = 0 \), so we obtain \( \overline{\Gamma}_2 \) as a comb in Figure 4. The loops at 1, 0 and 1, 1 are of course not present in the reduced Schreier graph, but we did include them here for they are the loops which “split up” in the further generations: as \( b \) restricts to \( a \)
at 1, 0, we obtain $\Gamma_3$ by connecting $\mathbb{Z}$ many copies of $\Gamma_2$ by an bi-infinite line going through the copies of 1, 0.

With this inductive description we can prove the following:
Lemma 5.4. For all \( m \in \mathbb{N} \), the reduced Schreier graph \( \Gamma_m \) is a tree with countably (or finitely) many ends.

Proof. We do induction over \( m \). For \( m = 1 \), the Schreier graph \( \Gamma_1 \) is a bi-infinite line, so it is in particular a tree with finitely many ends. Now by Lemma 5.2, \( \Gamma_{m+1} \) is the union of countably many copies of \( \Gamma_m \) and a bi-infinite line intersecting each copy in one point. So it is again a tree. We claim that have the following inductive description of the space of ends:

\[
\partial \Gamma_{m+1} \cong \mathbb{Z} \times \partial \Gamma_m \cup \{-\infty, +\infty\}
\]

Here the right hand space is a compactification of \( \mathbb{Z} \times \partial \Gamma_m \), where \(-\infty\) has the open sets \( U_{<n} := \{ z \in \mathbb{Z} : z < n \} \times \partial \Gamma_m \cup \{-\infty\} \) as neighborhood basis, and similarly \(+\infty\) has the open sets \( U_{>n} := \{ z \in \mathbb{Z} : z > n \} \times \partial \Gamma_m \cup \{+\infty\} \) as neighborhood basis. The identification in \( \Gamma_1 \) works as follows: we take \( w_m \) as our root of \( \Gamma_m \) and \( w_m0 \) as the root of \( \partial \Gamma_{m+1} \). Then we have the following identifications:

1. We send \(-\infty\) to the end \( (w_m(-i))_{i \in \mathbb{N}} \), i.e. we walk the bi-infinite line in the negative direction.
2. We send \(+\infty\) to the end \( (w_m(+i))_{i \in \mathbb{N}} \), i.e. we walk the bi-infinite line in the positive direction.
3. Given a pair \((z,v) \in \mathbb{Z} \times \partial \Gamma_m \), we identify it with the end which is given by the concatenation of the path from \( w_m0 \) to \( w_mz \) together with the sequence \( v \) truncated to \( m \). This means that first walk to the root of the copy of \( \Gamma_m \) labeled by \( z \), and then go the end defined by the sequence \( v \) in this copy.

Using Lemma 5.2, it is easy to check that this indeed defines a homeomorphism as given in \( \Gamma_1 \). Now is a \( \mathbb{Z} \times \partial \Gamma_m \cup \{-\infty, +\infty\} \) is countable union of countable set, so \( \partial \Gamma_{m+1} \) is countable.

Let us fix some notation related to the orbital Schreier graph \( \Gamma_\omega \). For \( u \in \mathbb{Z}^\omega \), let \( T_m(u) \) be the induced subgraph of \( \Gamma_\omega \) on the set \( \{ u' \in \mathbb{Z}^\omega : u_i = u'_i \text{ for all } i > m \} \). We denote the union \( \bigcup_{m \in \mathbb{N}} T_m(u) \) by \( T(u) \).

Theorem 5.5. The connected component of \( u \) in \( \Gamma_\omega \) is \( T(u) \). It is a tree with countably many ends.

Proof. The projection to the prefix of length \( m \) is a bijection from the vertex set of \( T_m(u) \) to \( \mathbb{Z}^m \). It gives rise to graph isomorphism from \( T_m(u) \) to \( \Gamma_m \), as the generating set acts by changing at most one letter at once. So \( T(u) \) is an increasing union of trees, hence it is also a tree.

Each end of \( T(u) \) either stays in some \( T_m(u) \) or leaves all \( T_m(u) \). The first kind is a countable union of countable sets, hence we only need to consider ends leaving all \( T_m(u) \). Let \( E_m \) be the set of edges in \( T(u) \) leaving \( T_m(u) \). We have a map \( E_m \to E_{m-1} \) which sends an edge \( e \) leaving \( T_m(u) \) to the unique edge leaving \( T_{m-1}(u) \) on the geodesic from \( u \) to \( e \). It is possible that an edge is send to itself, if it leaves multiple subtrees at once. Now the set of ends leaving all \( T_m(u) \) is isomorphic to \( \lim E_m \). Now the sets \( E_m \) have uniform bounded cardinality. This can be seen as follows: Let \( w \) be the \( m \)-suffix of \( u \). Then an edge in \( E_m \) corresponds to a pair \( v \in \mathbb{Z}^m, q \in S \cup S^{-1} \) with \( qv(w) \neq w \), in particular the restriction \( q|v \) is not trivial. But \( K(x_1, \ldots, x_k, y_1, \ldots, y_p) \) is a bounded activity automaton, so the number of pairs \( \{v, q\} \in \mathbb{Z}^m \times (S \cup S^{-1}) \) with \( q|v \neq 1 \) is uniformly bounded, and so
are the sets $E_m$. Hence the inverse limit has finite cardinality, so in total we have countably many ends.

6. Group theoretic properties

The groups $K(x_1 \ldots x_k, y_1 \ldots y_p)$ are examples of ZC-groups defined as $\text{OS10}$. In particular, they are left-orderable residually solvable groups.

In this section, we will always work with a fixed pair of sequences $x_1 \ldots x_k, y_1 \ldots y_p$ and we will just write $K$ instead of $K(x_1 \ldots x_k, y_1 \ldots y_p)$. We still use $S := \{a_1, \ldots, a_k, b_1, \ldots, b_p\}$ as our generating set.

**Lemma 6.1.** The abelianization of $K(x_1 \ldots x_k, y_1 \ldots y_p)$ is the free abelian group on $x_1, \ldots, x_k, y_1, \ldots, y_p$.

**Proof.** We have a family of group homomorphisms $\rho_n : \text{aut}_Z(Z^*) \to Z$ where $\rho_n : g \mapsto \sum_{v \in \mathbb{Z}^n} \rho(g|v)$.

Note that the sum is defined as $g|v$ is trivial for almost all $v$, so almost all summands are 0. By the cocycle equations 3.2 we see that $\rho_n$ is indeed a group homomorphism, and for all $g \in \text{aut}_Z(Z^*)$, we have $\rho_{n+1}(g) = \sum_{v \in \mathbb{Z}} \rho_n(g|v)$. The transition functions given in the Definition 4.3 translate to $\rho_n(a_1) = 1$, $\rho_n(s) = 0$ for all $s \in S \setminus \{a_0\}$, $\rho_{n+1}(a_{i+1}) = \rho_n(a_i)$, $\rho_{n+1}(b_1) = \rho_n(a_k) + \rho_n(b_p)$, $\rho_{n+1}(b_{j+1}) = \rho_n(a_j)$.

If we collect $\rho_0, \ldots, \rho_{k+p-1}$ to a group homomorphism $\overline{\rho} : \text{aut}_Z(Z^*) \to \mathbb{Z}^{k+p}$, we can show row by row that $(\overline{\rho}(a_1), \ldots, \overline{\rho}(a_k), \overline{\rho}(b_1), \ldots, \overline{\rho}(b_{k^p})) \in \mathbb{Z}^{(k+p) \times (k+p)}$ is the identity matrix. So $\overline{\rho}$ induces an isomorphism between the abelianization of $K$ and $\mathbb{Z}^{k+p}$.

**Lemma 6.2.** $K$ surjects onto the restricted wreath product $\mathbb{Z} \wr \mathbb{Z}$. In particular, $K$ is of exponential growth.

**Proof.** The action on $\mathbb{Z}^2$ gives a map to $\mathbb{Z} \wr \mathbb{Z}$. We see that $a_1$ and $a_2$ respectively $b_1$ if $k = 1$ are mapped to the standard generating set of $\mathbb{Z} \wr \mathbb{Z}$, so we have a surjection. As $\mathbb{Z} \wr \mathbb{Z}$ has exponential growth (see Par92, BT17 for a detailed discussion), $K$ also has exponential growth.

**Lemma 6.3.** The group $K(x_1 \ldots x_k, y_1 \ldots y_p)$ is level-transitive and self-replicating.

For the derived subgroup $K' \subset \text{Stab}_K$ we have the following: under the map $\text{Stab}_K \hookrightarrow \bigoplus_{x \in \mathbb{Z}} K$ induced by the wreath recursion, the image of $K'$ contains $\bigoplus_{x \in \mathbb{Z}} K'$ and the composition

\[
K' \hookrightarrow \text{Stab}_K \hookrightarrow \bigoplus_{x \in \mathbb{Z}} K \twoheadrightarrow K
\]

is surjective, where the last map is the projection map to any summand.
Proof. Note that \( a := a_1 \) acts just by translations on the first level, and every generator is the section of another generator. This already implies level-transitive and self-replicating. To show that the composition is surjective, it is easy to see that every generator of \( K \) is a section of a commutator of a generator and a sufficiently large power of \( a_1 \). So it is easy to see that \( K' \) surjects geometrically onto \( K \). As \( a_1 \) is just the first level shift, and \( K' \) is a normal subgroup of \( K \), to show that \( \bigoplus_{x \in \mathbb{Z}} K' \subset K' \), it is enough to show that \( K' \otimes 0 \subset K' \). Since \( K \) is self-replicating, it is enough to show that \( s, t \otimes 0 \subset K' \) for every commutator of two generators \( s, t \in S \). Now if \( c \) and \( d \) are the generators which have \( s \) and \( t \) as sections at \( z \) and \( w \), then a straightforward calculation shows \( [a^{-2} c a^{-w} d a^w] = [s, t] \).

Lemma 6.4. The groups \( K(x_1 \ldots x_k, y_1 \ldots y_p) \) are not residually finite.

Proof. By the previous lemma, \( \text{Stab}_K \) surjects onto \( K \), and since \( K \) is not abelian (it surjects onto an non-abelian group), neither is \( \text{Stab}_K \). Let \( x, y \in \text{Stab}_K \) be a non-commuting pair. Suppose \( K \) is residually finite, then there exists a group homomorphism \( \phi : K \to F \) to a finite group \( F \) such that \( \phi([x, y]) \) is non-trivial. But \( F \) is finite, so \( \phi(a_1) \) has finite order. So there is a \( n > 0 \) With \( \phi(a_1^{mn}) = 1 \) for all \( m \). Then \( \phi([x, y]) = \phi([a_1^{-mn} x a_1^{mn}, y]) \). Now under the wreath recursion, \( x \) and \( y \) have finite support in the direct sum \( \bigoplus_{x \in \mathbb{Z}} \text{aut}_\mathbb{Z}(\mathbb{Z}) \), so for \( m \) large enough, the support of \( a_1^{-mn} x a_1^{mn} \) and \( y \) will be disjoint, hence they commute. So \( \phi([x, y]) = \phi([a_1^{-mn} x a_1^{mn}, y]) \) is trivial, so we arrive at a contradiction.

Theorem 6.5. The groups \( K(x_1 \ldots x_k, y_1 \ldots y_p) \) are amenable but not elementary subexponentially amenable.

Proof. We invoke Theorem B of [Rei20] to show that the groups \( K(x_1 \ldots x_k, y_1 \ldots y_p) \) are amenable. We already observed in Remark 4.5 that the groups are generated by bounded activity automata. Hence they are subgroups of \( \text{Aut}_{f.s.}^B(\mathbb{Z}^*; \mathbb{Z}) \). As the left action of \( \mathbb{Z} \) on itself is recurrent, by Theorem B of [Rei20] the group \( \text{Aut}_{f.s.}^B(\mathbb{Z}^*; \mathbb{Z}) \) is recurrent, and so are the subgroups \( K(x_1 \ldots x_k, y_1 \ldots y_p) \).

The groups have exponential growth by Lemma 6.2. Lemma 6.3 together with Corollary 3 of [Jus18] imply that the groups are not elementary subexponentially amenable.

We should note that [Jus18] only deals with finite alphabets. The proof can be easily modified to deal with subgroups of \( \text{aut}_\mathbb{Z}(\mathbb{Z}^*) \).
7. Outlook

This paper is the beginning of our study of iterated monodromy groups for entire transcendental maps and a stepping stone towards a more general discussion. The regularity of the monodromy of the exponential map simplifies the discussion and has consequences that are special to the exponential case. In particular, the left-order on the dynamical preimage tree heavily uses this regularity. For other entire transcendental functions, we should expect torsion elements in the monodromy group and torsion elements for some iterated monodromy groups of functions in that parameter space.

In an upcoming paper [Rei] we discuss the general structure of iterated monodromy groups of entire maps. In particular, we also apply the results of [Rei20] to show that the iterated monodromy groups of entire functions are amenable if and only their monodromy group is. For polynomials and the exponential family, the condition is trivially satisfied, as finite groups and abelian groups are amenable. However, there are entire maps with virtually free monodromy groups, so we have to impose this condition.

Moreover, we can also try to generalize from entire functions to meromorphic functions. Here a good starting family would be the functions of the form $M \circ \exp$ including tangent, where $M$ is a Mbius transform. We should think of this as the analogy to the family of bicritical rational maps, see also Appendix D of [Mi00]. In this case, we can also define iterated monodromy group for post-singularly finite maps and show that they are ZC-groups. So the class of ZC-groups, in particular subgroups of $\text{aut}_\mathbb{C}(\mathbb{C}^*)$ has many examples of self-similar groups coming from complex dynamics. This warrants a further general investigation of ZC-groups.

Outside of this family $M \circ \exp$, we should not expect to have the left-orderability of all IMGs in one parameter space, as it might be a special phenomenon due to the very rigid monodromy groups of exponential maps.

References

[BDH+00] Clara Bodelón, Robert L. Devaney, Michael Hayes, Gareth Roberts, Lisa R. Goldberg, and John H. Hubbard. Dynamical convergence of polynomials to the exponential. *J. Differ. Equations Appl.*, 6(3):275–307, 2000.

[BN06] Laurent Bartholdi and Volodymyr Nekrashevych. Thurston equivalence of topological polynomials. *Acta Math.*, 197(1):1–51, 2006.

[BN08] Laurent Bartholdi and Volodymyr Nekrashevych. Iterated monodromy groups of quadratic polynomials. I. *Groups Geom. Dyn.*, 2(3):309–336, 2008.

[BS02] Henk Bruin and Dierk Schleicher. Symbolic dynamics of quadratic polynomials. Technical Report 7, Institut Mittag-Leffler, 2001/2002.

[BT17] Michelle Bucher and Alexey Talmi. Minimal exponential growth rates of metabelian Baumslag-Solitar groups and lamplighter groups. *Groups Geom. Dyn.*, 11(1):189–209, 2017.

[FG91] Jacek Fabrykowski and Narain Gupta. On groups with sub-exponential growth functions. II. *J. Indian Math. Soc. (N.S.)*, 56(1-4):217–228, 1991.

[GŽ02] Rostislav I. Grigorchuk and Andrzej Żuk. On a torsion-free weakly branch group defined by a three state automaton. volume 12, pages 223–246. 2002. International Conference on Geometric and Combinatorial Methods in Group Theory and Semigroup Theory (Lincoln, NE, 2000).

[HSS09] John Hubbard, Dierk Schleicher, and Mitsuko Shishikura. Exponential Thurston maps and limits of quadratic differentials. *J. Amer. Math. Soc.*, 22(1):77–117, 2009.

[Jus18] Kate Juschenko. Non-elementary amenable subgroups of automata groups. *J. Topol. Anal.*, 10(1):35–45, 2018.
Bastian Laubner, Dierk Schleicher, and Vlad Vicol. A combinatorial classification of postsingularly finite complex exponential maps. *Discrete Contin. Dyn. Syst.*, 22(3):663–682, 2008.

John Milnor. On rational maps with two critical points. *Experiment. Math.*, 9(4):481–522, 2000.

John Milnor and William Thurston. On iterated maps of the interval. In *Dynamical systems*, pages 465–563. Springer, 1988.

Volodymyr Nekrashevych. *Self-similar groups*, volume 117 of *Mathematical Surveys and Monographs*. American Mathematical Society, Providence, RI, 2005.

AS Oliynyk and VI Sushchanskiı. The groups of ZC-automaton transformations. *Siberian mathematical journal*, 51(5):879–891, 2010.

Walter Parry. Growth series of some wreath products. *Trans. Amer. Math. Soc.*, 331(2):751–759, 1992.

Bernhard Reinke. Iterated monodromy groups of entire functions. In preparation.

Bernhard Reinke. Amenability of bounded automata groups on infinite alphabets. arXiv:2004.05029.

Dierk Schleicher. Dynamics of entire functions. In Graziano Gentili, Jacques Guenot, and Giorgio Patrizio, editors, *Holomorphic dynamical systems*, volume 1998 of *Lecture Notes in Math.*, pages 295–339. Springer, Berlin, 2010.

Said Sidki. Finite automata of polynomial growth do not generate a free group. *Geometriae Dedicata*, 108(1):193–204, 2004.

Dierk Schleicher and Johannes Zimmer. Periodic points and dynamic rays of exponential maps. *Ann. Acad. Sci. Fenn. Math.*, 28(2):327–354, 2003.

William P. Thurston. On the geometry and dynamics of iterated rational maps. In *Complex dynamics*, pages 3–137. A K Peters, Wellesley, MA, 2009. Edited by Dierk Schleicher and Nikita Selinger and with an appendix by Schleicher.

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