SPECIAL UNIPOTENT REPRESENTATIONS OF REAL CLASSICAL GROUPS: COUNTING AND REDUCTION

DAN BARBASCH, JIA-JUN MA, BINYONG SUN, AND CHEN-BO ZHU

Abstract. Let $G$ be a real reductive group in Harish-Chandra’s class. We derive some consequences of theory of coherent continuation representations to the counting of irreducible representations of $G$ with a given infinitesimal character and a given bound of the complex associated variety. When $G$ is a real classical group (including the real metaplectic group), we investigate the set of special unipotent representations of $G$ attached to $\mathcal{O}$, in the sense of Arthur and Barbasch-Vogan. Here $\mathcal{O}$ is a nilpotent adjoint orbit in the Langlands dual of $G$ (or the metaplectic dual of $G$ when $G$ is a real metaplectic group). We give a precise count for the number of special unipotent representations of $G$ attached to $\mathcal{O}$. We also reduce the problem of constructing special unipotent representations attached to $\mathcal{O}$ to the case when $\mathcal{O}$ is analytically even (equivalently for a real classical group, has good parity in the sense of Mœglin). The paper is the first in a series of two papers on the classification of special unipotent representations of real classical groups.

Contents

1. Introduction 1
2. The main results 4
3. Generalities on coherent families of highest weight modules 16
4. Generalities on coherent families of Casselman-Wallach representations 28
5. Counting irreducible representations 37
6. Separating good parity and bad parity for coherent continuation representations 39
7. Special unipotent representations in type $A$ 49
8. Special unipotent representations in type $BCD$: counting 52
9. Special unipotent representations in type $BCD$: reduction to good parity 60
10. Combinatorics of painted bipartitions 63
Acknowledgements 73
References 74

1. Introduction

1.1. Background and goals. In [Art84, Art89], Arthur introduced certain families of representations of a reductive algebraic group $G$ over $\mathbb{R}$ or $\mathbb{C}$, in connection with his conjecture on square-integrable automorphic forms. Arthur’s representations, the special unipotent representations in the title of this paper, were made precise in the work of Barbasch-Vogan [BV85] (for groups over $\mathbb{C}$; the same works for groups over $\mathbb{R}$, see [ABV91, Chapter 27]). They are defined in the language of primitive ideals and are attached to a nilpotent adjoint orbit $\mathcal{O}$ in the Langlands dual of $G$.

Apart from their clear interest for the theory of automorphic forms [Art84, Art89, Art13], special unipotent representations belong to a fundamental class of unitary representations which are associated to nilpotent coadjoint orbits in the Kirillov philosophy (the orbit method; see [Kir62, Kos70, Vog87b]). These are known informally as unipotent representations, which are

2020 Mathematics Subject Classification. 22E46, 22E47.

Key words and phrases. Classical group, special unipotent representation, coherent continuation, Weyl group representation, primitive ideal, cell, infinitesimal character, associated variety.
expected to play a central role in the classification problem of the unitary dual of a real reductive group and therefore have been a subject of great interest. See [Vog87a, Vog87b, Vog91]. The aforementioned work of Barbasch-Vogan [BV85] was also motivated by this classification problem.

In the context alluded to earlier, a long standing problem in representation theory, known as the Arthur-Barbasch-Vogan conjecture ([Art89, Section 4], [ABV91, Introduction]), is to show that every special unipotent representation is unitary. In a series of two papers, the authors will construct and classify special unipotent representations of real classical groups (including real metaplectic groups), and will prove the Arthur-Barbasch-Vogan conjecture for these groups as a consequence of the classification. For quasi-split classical groups, the unitarity of special unipotent representations is independently established in [AARM24, ARM], from the perspective of the endoscopic classification of representations [Art13, Mok15].

The current paper is the first in the series, which has the following three goals.

(a) The first is to derive a consequence of the theory of coherent continuation representations to the counting, in the form of an inequality, of irreducible representations of $G$ with a given infinitesimal character and a given bound of the complex associated variety. Largely this is achieved by building on existing results in Kazhdan-Lusztig theory.

(b) The second is to prove that the inequality just alluded to is in fact an equality under certain technical hypothesis on $G$, which holds for a real classical group and yields precise and explicit counting of special unipotent representations attached to $\mathcal{O}$. This involves some new technical tools on $\tau$-invariants for a general $G$, as well as elaborate combinatorial constructions/arguments in the case of a real classical group $G$.

(c) The third is to reduce the problem, in the case of a real classical group $G$, of constructing special unipotent representations attached to $\mathcal{O}$ to the case when $\mathcal{O}$ is analytically even (Definition 2.4), or equivalently, has good parity in the sense of Mœglin [Mœg11]. This may be viewed as reduction to integral infinitesimal character which holds for the more general setting as in [ABV91]. Therefore the core of the classification problem of special unipotent representations reduces to the case when $\mathcal{O}$ has good parity. We remark that results of similar nature were proved by Mœglin and Renard in [MR19].

1.2. Approach. We will work in the category of Casselman-Wallach representations [Wal92, Chapter 11]. The main tool for the counting of irreducible representations in general and special unipotent representations in particular is the coherent continuation representation (the idea first appeared in [BV83b]). This is a representation of a certain integral Weyl group, and it can be compared, through a certain injective map of $\mathcal{K}$ groups (due to Casian), with an analogous coherent continuation representation for the category of highest weight modules. The latter has been intensively studied in Kazhdan-Lusztig theory and is amenable to detailed analysis through the theory of primitive ideals (as in the work of Joseph and Barbasch-Vogan), as well as the theory of cells and special representations (in the sense of Lusztig). In particular one may derive precise information on what representations of the integral Weyl group may contribute to the counting, in terms of the Springer correspondence. (This was done in [BV85, Section 5] for complex semisimple groups.) In effectively carrying out the mentioned steps, we build on earlier ideas of several authors including Joseph [Jos80a, Jos80b, Jos85], Vogan [Vog81], Barbasch-Vogan [BV82, BV83a, BV85], and Casian [Cas86]. Weaving things together, we arrive at a counting inequality on irreducible representations of $G$ with a given infinitesimal character and a given bound of the complex associated variety. The inequality is valid for any real reductive group $G$ in Harish-Chandra’s class.

For the precise counting of irreducible representations, a key technical issue is a certain relationship (expected by Vogan) between cell representations in the Casselman-Wallach setting (Harish-Chandra cells) and the highest weight module setting (double cells). (This was known to hold for certain $G$, notably for unitary groups [BV83b] and for complex semisimple groups [BV85, Section 3].) We resolve this issue assuming a certain equality of $\tau$-invariants and (a weak
form of) Vogan duality, both of which hold true for classical groups (including the real metaplectic group). Together with an explicit formula of the coherent continuation representation (due to Barbasch-Vogan) and explicit branching rules of Weyl group representations (via the Littlewood-Richardson rule), this will finally yield the counting of special unipotent representations of real classical groups, explicitly described in terms of certain combinatorial constructs called painted bipartitions. It is worthwhile to note that in explicating the counting formula, we need to sum over several Weyl group representations in a certain Lusztig left cell, which we prove to have the same contribution by capitalizing on the induction mechanism provided by a certain notion of descent for painted bipartitions (defined in this article).

We now outline our approach to the third goal. The case of type $A$ groups is relatively easy and so we focus on groups of type $B$, $C$, and $D$. The basic idea is to appeal to the theory of endoscopy and its relation to the Kazhdan-Lusztig-Vogan algorithm for nonintegral infinitesimal character (see [ABV91, Chapter 15]). Specifically when $\mathcal{O}$ is not of good parity, we introduce an $\mathcal{O}$-relevant parabolic subgroup $P$, whose Levi component is of the form $G'_b \times G_g$, where $G'_b$ is a general linear group, and $G_g$ is of the same classical type as $G$. In their respective Langlands duals, we have nilpotent adjoint orbits $\mathcal{O}'_b$ and $\mathcal{O}_g$ (determined by $\mathcal{O}$). We will show that the normalized smooth parabolic induction from $P$ to $G$ yields a bijective map from a pair of special unipotent representations of $G'_b$ (attached to $\mathcal{O}'_b$) and $G_g$ (attached to $\mathcal{O}_g$) to special unipotent representations of $G$ (attached to $\mathcal{O}$). Firstly, there is a classical group $G_b$ with the following properties:

- $G_b \times G_g$ is an endoscopy group of $G$ (except for the real metaplectic group, in which case there is a metaplectic analog);
- $G'_b$ is naturally isomorphic to the Levi component of an $\mathcal{O}_b$-relevant parabolic subgroup $P_b$ of $G_b$.

Here $\mathcal{O}_b$ is a nilpotent adjoint orbit in the Langlands dual of $G_b$ (determined by $\mathcal{O}$). General considerations as well as detailed information about coherent continuation representations allow us to separate bad parity and good parity, namely we may determine special unipotent representations of $G$ in terms of those of $G_b$ and $G_g$. The normalized smooth parabolic induction from $P_b$ to $G_b$ yields a bijection from special unipotent representations of $G'_b$ (attached to $\mathcal{O}'_b$) to special unipotent representations of $G_b$ (attached to $\mathcal{O}_b$). On the other hand, the Kazhdan-Lusztig-Vogan algorithm [Vog83, ABV91, RT03] implies that irreducibility is preserved in the endoscopy setting. This will finally imply the bijectivity of the induction map from $P$ to $G$.

1.3. Organization. In Section 2, we state our main results firstly on the counting of irreducible representations in our general setup, and secondly on the explicit counting of special unipotent representations of real classical groups. For the latter, the more involved cases of groups of type $B$, $C$ and $D$, the main results are in Theorem 2.20 (reduction to good parity) and Theorem 2.27 (counting in the case of good parity). In Section 3, we review some generalities on coherent continuation representations for highest weight modules, to be precise for the category $\text{Rep}(\mathfrak{g}, \mathfrak{b})$ (see Section 3.2 for the notation), as well as the full subcategory of $\text{Rep}(\mathfrak{g}, \mathfrak{b})$ defined by a prescribed support condition. Following [FJMN21], we introduce a new notion of the $\tau$-invariant of a Weyl group representation, and examine its role in a certain duality notion of double cells (Section 3.8). In Section 4, we examine some analogous results on coherent continuation representations for Casselman-Wallach representations, and their interplay with the corresponding results in the highest weight module setting. As mentioned in Section 1.2, we also establish a certain relationship between Harish-Chandra cells and double cells assuming some equality of $\tau$-invariants and a weak form of the Vogan duality (Section 4.5). In Section 5, we prove the first part of our main results on the counting of irreducible representations with a given infinitesimal character and a given bound of the complex associated variety. In Section 6, we separate good parity and bad parity for coherent continuation representations, as a preparation for the reduction step in Section 9. Section 7 and Section 8 are devoted to proof of the second part of our main results, which explicitly counts special unipotent representations, for groups of type $A$, and groups of type $B$, $C$ and $D$, respectively. In Section 9, we carry out the reduction step to the case of good parity and prove Theorem 2.20. In Section 10, we develop combinatorics of
painted bipartitions which we need for the proof of Theorem 2.27 in Section 8. This includes a notion of descent for painted bipartitions, helpful in explicating combinatorially the counting formula (mentioned earlier), and critical in matching combinatorial parameters of the current paper with special unipotent representations to be constructed in [BMSZ21] (the second paper in the series) by the method of theta lifting ([How79, How89]).

2. The main results

2.1. Lie algebra notation. Let \( \mathfrak{g} \) be a reductive complex Lie algebra. Its universal enveloping algebra is denoted by \( \mathcal{U}(\mathfrak{g}) \), and the center of \( \mathcal{U}(\mathfrak{g}) \) is denoted by \( \mathcal{Z}(\mathfrak{g}) \). Let \( ^\ast \mathfrak{h} \) denote the universal Cartan subalgebra of \( \mathfrak{g} \) (also called the abstract Cartan subalgebra in [Vog82]). Recall that for every Borel subalgebra \( \mathfrak{b} \) of \( \mathfrak{g} \), there is an identification \( ^\ast \mathfrak{h} = \mathfrak{b}/[\mathfrak{b}, \mathfrak{b}] \). Let \( W \subset \text{GL}(^\ast \mathfrak{h}) \) denote the Weyl group. In general we denote by superscript \( * \) the linear dual of a vector space. By the Harish-Chandra isomorphism, there is a 1-1 correspondence between \( W \)-orbits of \( \nu \in ^\ast \mathfrak{h}^\ast \) and algebraic characters \( \chi_\nu : \mathcal{Z}(\mathfrak{g}) \to \mathbb{C} \). We say that an ideal of \( \mathcal{U}(\mathfrak{g}) \) has infinitesimal character \( \nu \) if it contains the kernel of \( \chi_\nu \). By a result of Duflo ([Dix96], [Bor76, Section 3]), there is a unique maximal ideal of \( \mathcal{U}(\mathfrak{g}) \) that has infinitesimal character \( \nu \). Write \( I_\nu \) for this maximal ideal.

(We remark that every maximal ideal of \( \mathcal{U}(\mathfrak{g}) \) is primitive.)

Let \( \text{Nil}(\mathfrak{g}) \) (resp. \( \text{Nil}(\mathfrak{g}^\ast) \)) be the set of nilpotent elements in \( [\mathfrak{g}, \mathfrak{g}] \) (resp. \( [\mathfrak{g}, \mathfrak{g}]^\ast \)). Denote by \( \text{Ad}(\mathfrak{g}) \) the inner automorphism group of \( \mathfrak{g} \), and put
\[
\text{Nil}(\mathfrak{g}) := \text{Ad}(\mathfrak{g}) \backslash \text{Nil}(\mathfrak{g}), \quad \text{Nil}(\mathfrak{g}^\ast) := \text{Ad}(\mathfrak{g}) \backslash \text{Nil}(\mathfrak{g}^\ast),
\]
the set of \( \text{Ad}(\mathfrak{g}) \)-orbits in \( \text{Nil}(\mathfrak{g}) \) and \( \text{Nil}(\mathfrak{g}^\ast) \) (which are finite). The Killing form on \( [\mathfrak{g}, \mathfrak{g}] \) yields an identification \( \text{Nil}(\mathfrak{g}) = \text{Nil}(\mathfrak{g}^\ast) \). Since \( \mathfrak{g} \) is the direct sum of its center with \( [\mathfrak{g}, \mathfrak{g}] \), \( [\mathfrak{g}, \mathfrak{g}]^\ast \) is viewed as a subspace of \( \mathfrak{g}^\ast \) in the obvious way.

2.2. Counting irreducible representations. Let \( G \) be a real reductive group in Harish-Chandra’s class ([Kna02]) whose complexified Lie algebra equals \( \mathfrak{g} \). In the rest of this paper, unless mentioned otherwise, we use the corresponding lowercase Gothic letter to denote the complexified Lie algebra of a Lie group. Let \( \text{Rep}(G) \) be the category of Casselman-Wallach representations of \( G \), whose Grothendieck group (with \( \mathbb{C} \)-coefficients) is denoted by \( K(G) \). Let \( \text{Irr}(G) \) be the set of isomorphism classes of irreducible Casselman-Wallach representations of \( G \), which forms a basis of \( K(G) \).

Let \( \nu \in ^\ast \mathfrak{h}^\ast \) and let \( S \) be an \( \text{Ad}(\mathfrak{g}) \)-stable Zariski closed subset of \( \text{Nil}(\mathfrak{g}^\ast) \). Let \( \text{Irr}_{\nu,S}(G) \) denote the subset of \( \text{Irr}(G) \) consisting of representations that have infinitesimal character \( \nu \) and whose complex associated variety is contained in \( S \). Our aim is to count the set \( \text{Irr}_{\nu,S}(G) \).

To do this, we form the subset \( \Lambda \) of \( ^\ast \mathfrak{h}^\ast \), which is the \( \nu \)-translate of the root lattice. Let \( W(\Lambda) \) be the integral Weyl group of \( \nu \), which equals the stabilizer of \( \Lambda \) in \( W \). We consider the space \( \text{Coh}_\Lambda(K(G)) \) of \( K(G) \)-valued coherent families on \( \Lambda \), which is a finite-dimensional representation of \( W(\Lambda) \). This is called the coherent continuation representation. See Section 3.1.

Attached to \( S \), we will define a certain subset \( \text{Irr}_{S}(W(\Lambda)) \) of \( \text{Irr}(W(\Lambda)) \) by using the Springer correspondence (see Definition 3.31).

Let \( \text{Sp}_{2n}(\mathbb{R}) \) \( (n \in \mathbb{N} := \{0, 1, 2, \ldots \}) \) denote the metaplectic double cover of the symplectic group \( \text{Sp}_{2n}(\mathbb{R}) \) (it does not split for \( n > 0 \)), to be called a real metaplectic group. Recall that \( G \) is is said to be linear if it has a faithful finite-dimensional representation.

**Theorem 2.1.** We have the inequality
\[
\sharp(\text{Irr}_{\nu,S}(G)) \leq \sum_{\sigma \in \text{Irr}_{S}(W(\Lambda))} [1_{W_\nu} : \sigma] \cdot [\sigma : \text{Coh}_\Lambda(K(G))],
\]
where \( 1_{W_\nu} \) denotes the trivial representation of the stabilizer \( W_\nu \) of \( \nu \) in \( W \). The equality holds if the Coxeter group \( W(\Lambda) \) has no simple factor of type \( F_4 \), \( E_6 \), \( E_7 \), or \( E_8 \), and \( G \) is linear or isomorphic to a real metaplectic group.

Here and henceforth, \([ : ]\) indicates the multiplicity of the first (irreducible) representation in the second one, and \( \sharp \) indicates the cardinality of a finite set.
For any (two-sided) ideal $I$ of $\mathcal{U}(\mathfrak{g})$, let $\text{Irr}(G)$ denote the subset of $\text{Irr}(G)$ consisting of representations that are annihilated by $I$. Note that for the maximal ideal $I_\nu$, 
$$\text{Irr}_{I_\nu}(G) = \text{Irr}_{I_\nu}(G),$$
where $O_{\nu}$ is the associated variety of $I_\nu$. In this case, work of Lusztig [Lus84] and Barbasch-Vogan [BV85] imply that
$$[1_{I_\nu} : \sigma] \leq 1$$
for all $\sigma \in \text{Irr}_{O_{\nu}}(W(A))$, and the set
$$\{ \sigma \in \text{Irr}_{O_{\nu}}(W(A)) \mid [1_{I_\nu} : \sigma] \neq 0 \}$$
can be explicitly described by the Lusztig left cell $L_{\rho_\nu}$ (see (3.25) for its definition). Therefore Theorem 2.1 has the following consequence.

**Corollary 2.2.** We have the inequality
$$\sharp(\text{Irr}_{I_\nu}(G)) \leq \sum_{\sigma \in \text{Irr}_{L_{\rho_\nu}}(V(A))} \vert \sigma : \text{Coh}_{A}(K(G)) \vert.$$

The equality holds if the Coxeter group $W(A)$ has no simple factor of type $F_4, E_6, E_7, \text{ or } E_8$, and $G$ is linear or isomorphic to a real metaplectic group.

Theorem 2.1 and Corollary 2.2 will be proved in Section 5. The equalities in Theorem 2.1 and Corollary 2.2 will be proved more generally for any $G$ satisfying Conjecture 4.14.

### 2.3. Special unipotent representations of real classical groups.

Suppose that $\ast$ is one of the 10 labels, $G$ is a classical Lie group of type $\ast$, and $\tilde{G}$ is a complex Lie group, as in the following table ($n, p, q \in \mathbb{N}$).

| Label $\ast$ | Classical Lie Group $G$ | Langlands (or Metaplectic) Dual Group $\tilde{G}$ |
|--------------|------------------------|----------------------------------|
| $A^\mathbb{R}$ | $\text{GL}_n(\mathbb{R})$ | $\text{GL}_n(\mathbb{C})$ |
| $A^\mathbb{H}$ | $\text{GL}_2(\mathbb{H})$ $(n \text{ even})$ | $\text{GL}_n(\mathbb{C})$ |
| $A$ | $U(p, q)$ | $\text{GL}_{p+q}(\mathbb{C})$ |
| $\tilde{A}$ | $\tilde{U}(p, q)$ | $\text{GL}_{p+q}(\mathbb{C})/\{\pm 1_{p+q}\}$ $(1_{p+q} \text{ is the identity matrix})$ |
| $B$ | $\text{SO}(p, q)$ $(p + q \text{ odd})$ | $\text{Sp}_{p+q-1}(\mathbb{C})$ |
| $D$ | $\text{SO}(p, q)$ $(p + q \text{ even})$ | $\text{SO}_{p+q}(\mathbb{C})$ |
| $C$ | $\text{Sp}_{2n}(\mathbb{R})$ | $\text{SO}_{2n+1}(\mathbb{C})$ |
| $\tilde{C}$ | $\text{Sp}_{2n}(\mathbb{R})$ | $\text{Sp}_{2n}(\mathbb{C})$ |
| $D^\ast$ | $\text{SO}^\ast(2n)$ | $\text{SO}_{2n}(\mathbb{C})$ |
| $C^\ast$ | $\text{Sp}^\ast(\frac{p+q}{2}, \frac{p+q}{2})$ $(p, q \text{ even})$ | $\text{SO}_{p+q+1}(\mathbb{C})$ |

In this table $\tilde{U}(p, q)$ is the (linear) double cover of $U(p, q)$ defined by a square root of the determinant character. In the last column, $\tilde{G}$ is the the Langlands dual of the complexification of $G$, except for $G = \text{Sp}_{2n}(\mathbb{R})$ or $U(0, 0)$. In the case of $G = \text{Sp}_{2n}(\mathbb{R})$ we replace the Langlands dual by $\text{Sp}_{2n}(\mathbb{C})$, call it the “metaplectic” dual [Wei18, BMSZ23], and still write $\tilde{G}$ (or $\tilde{G}_{\text{mp}}$ with the subscript mp for emphasis).

Write $\mathfrak{g}$ for the Lie algebra of $\tilde{G}$. Let $\mathfrak{a} \mathfrak{h}$ denote the universal Cartan subalgebra of $\mathfrak{g}$. In all cases we identify $\mathfrak{a} \mathfrak{h}$ with $\mathfrak{a} \mathfrak{h}$ in the standard way, except for $\tilde{C}$ where the identification is via the half of the trace form on the Lie algebra $\mathfrak{g} = \mathfrak{sp}_{2n}(\mathbb{C})$.

Let $\tilde{O} \subseteq \tilde{g}$ be a nilpotent orbit, namely an Ad($\mathfrak{g}$)-orbit in $\text{Nil}(\mathfrak{g})$. Write $\mathfrak{d}_{\tilde{O}}$ for the Young diagram attached to $\tilde{O}$ ([CM93]). It determines the nilpotent orbit $\tilde{O}$, unless $\mathfrak{g} = \mathfrak{o}_{4k}(\mathbb{C})$ ($k \in \mathbb{N}^+ := \{1, 2, 3, \ldots \}$) and all the row lengths are even. (In the latter case, $\tilde{O}$ is called very even, and there are precisely two nilpotent orbits with the same Young diagram as that of $\tilde{O}$.) When there is no confusion, we will not distinguish between $\tilde{O}$ and $\mathfrak{d}_{\tilde{O}}$. 


Let \( \{\hat{e}, \hat{h}, \hat{f}\} \) be an \( \mathfrak{sl}_2 \)-triple in \( \mathfrak{g} \) representing \( \hat{O} \), namely \( \hat{e} \in \hat{O} \). The element \( \hat{h}/2 \) is semisimple, and its \( \text{Ad}(\mathfrak{g}) \)-orbit is uniquely determined by \( \hat{O} \). Using the identification
\[
\text{Ad}(\mathfrak{g}) \setminus \{\text{semisimple element in} \ \mathfrak{g}\} = W^{\mathfrak{a} \mathfrak{h}} = W^{\mathfrak{a} \hat{h}^*},
\]
we pick an element \( \lambda_{\hat{O}} \in \mathfrak{h}^* \) that represents the same element in \( W^{\mathfrak{a} \hat{h}^*} \) as \( \hat{h}/2 \). As in [BV85, Section 5], \( \lambda_{\hat{O}} \) determines an infinitesimal character (namely an algebraic character of \( \mathbb{Z}(\mathfrak{g}) \)), also called the infinitesimal character associated to \( \hat{O} \). Write \( I_{\hat{O}} := I_{\lambda_{\hat{O}}} := I_{\hat{O} \lambda_{\hat{O}}} \) for the maximal ideal of \( \mathcal{U}(\mathfrak{g}) \), only dependent on \( \hat{O} \). (The subscript \( * \) is one of the labels in the table.)

Let
\[
\hat{O} := d_{BV}(\hat{O}) \in \overline{\text{Nil}(\mathfrak{g}^*)}
\]
denote the Barbasch-Vogan dual of \( \hat{O} \), namely the unique Zariski open \( \text{Ad}(\mathfrak{g}) \)-orbit in the associated variety of \( I_{\hat{O}} \). See [BV85, Appendix], and [BMSZ23, Section 1] for the case of \( * = \hat{C} \). (This duality notion is a recasting in terms of representation theory of a duality notion defined combinatorially by Spaltenstein in [Spa82].)

Following Barbasch-Vogan [BV85], define the set of the special unipotent representations of \( G \) attached to \( \hat{O} \) by
\[
\text{Unip}_{\hat{O}}(G) := \text{Unip}_{\lambda_{\hat{O}}}(G)
\]
\[
:= \left\{ \begin{array}{ll}
\left\{ \pi \in \text{Irr}_{I_{\hat{O}}}(G) \mid \pi \text{ is genuine} \right\}, & \text{if } * \in \{\hat{A}, \hat{C}\}; \\
\text{Irr}_{I_{\hat{O}}}(G), & \text{otherwise}.
\end{array} \right.
\]

Here “genuine” means that the central subgroup \( \{\pm 1\} \) of \( G \), which is the kernel of the covering homomorphism \( \hat{U}(p, q) \to \text{U}(p, q) \) or \( \hat{S}\text{P}_{2n}(\mathbb{R}) \to \text{S}\text{P}_{2n}(\mathbb{R}) \), acts on \( \pi \) through the nontrivial character. As we noted earlier, \( \text{Irr}_{I_{\hat{O}}}(G) = \pi_{\lambda_{\hat{O}}} \mathcal{O}(G) \).

**Remark 2.3.** In representation theory literature, the set \( \text{Unip}_{\hat{O}}(G) \) is also called the weak Arthur/ABV packet attached to \( \hat{O} \), which is the union of Arthur/ABV packets ranging over unipotent Arthur parameters whose restriction to \( \text{SL}_2(\mathbb{C}) \) is determined by \( \hat{O} \) via the Jacobson-Morozov theorem. See [ABV91, Corollary 27.13].

The main goal of this paper and the second paper in the series [BMSZ21] is to count the set \( \text{Unip}_{\hat{O}}(G) \) and to construct all the representations in \( \text{Unip}_{\hat{O}}(G) \), both explicitly.

Motivated by [ABV91], we introduce, for a nilpotent orbit \( \hat{O} \subseteq \mathfrak{g} \), a key property to be called analytically even. For classical groups, this notion coincides with Mœglin’s notion of good parity [Mœg11]. The notion will be used throughout the article and will play an important role in the classification of special unipotent representations (for unitary groups and for groups of type \( BCD \)).

Let \( \{\hat{e}_{pn}, \hat{h}_{pn}, \hat{f}_{pn}\} \) be an \( \mathfrak{sl}_2 \)-triple in \( \mathfrak{g} \) such that \( \hat{e}_{pn} \) belongs to the principal nilpotent orbit. Then the element \( \exp(\pi \sqrt{-1} \hat{h}_{pn}) \in \hat{G} \) is a central element fixed by all holomorphic automorphisms of \( \hat{G} \), and is independent of the choice of the triple \( \{\hat{e}_{pn}, \hat{h}_{pn}, \hat{f}_{pn}\} \). Here \( \exp : \mathfrak{g} \to \hat{G} \) denotes the exponential map, and \( \pi \) is the usual circumference ratio.

**Definition 2.4.** (a) A semisimple element \( a \in \mathfrak{g} \) is said to be analytically integral if
\[
\exp(2\pi \sqrt{-1} a) = \exp(\pi \sqrt{-1} \hat{h}_{pn})
\]
(b) A nilpotent orbit \( \hat{O} \subseteq \mathfrak{g} \) is said to be analytically even if \( \hat{h}/2 \in \mathfrak{g} \) is analytically integral. Here and as before, \( \{\hat{e}, \hat{h}, \hat{f}\} \) is an \( \mathfrak{sl}_2 \)-triple in \( \mathfrak{g} \) representing \( \hat{O} \), namely \( \hat{e} \in \hat{O} \).

**Remarks 2.5.** (i) Definition 2.4 obviously generalizes to all connected complex reductive groups. (ii) Part (b) of the definition is independent of the choice of \( \{\hat{e}, \hat{h}, \hat{f}\} \). (iii) Every analytically even nilpotent orbit is even, and when the complex group \( \hat{G} \) is an adjoint group, every even nilpotent orbit is also analytically even.

We also define a notion of good parity and bad parity (à la Mœglin [Mœg11, page 356]). Note that if \( * \in \{\hat{A}^n, \hat{A}^\mathbb{R}, \hat{C}, \hat{C}, \hat{D}^*\} \), then \( n \) equals the rank of \( \mathfrak{g} \). Throughout the article, we will let \( n \) denote the rank of \( \mathfrak{g} \), in all cases.
Definition 2.6. We call an integer to have good parity (with respect to $\star$ and $n = \text{rank } g$) if it has the same parity as

$$
\begin{aligned}
    n, & \quad \text{if } \star \in \{ A^R, A^H, A \}; \\
    1 + n, & \quad \text{if } \star = \tilde{A}; \\
    1, & \quad \text{if } \star \in \{ C, C^*, D, D^* \}; \\
    0, & \quad \text{if } \star \in \{ B, \tilde{C} \}.
\end{aligned}
$$

Otherwise, we say the integer has bad parity.

The following lemma is easily checked.

Lemma 2.7. (a) Suppose that $\star \neq \tilde{A}$. Then a nilpotent orbit $\tilde{O} \subseteq \tilde{g}$ is analytically even if and only if all row lengths of its Young diagram have good parity.

(b) Suppose that $\star = \tilde{A}$. Then a nilpotent orbit $\tilde{O} \subseteq \tilde{g}$ is analytically even if and only if all row lengths of its Young diagram have the same parity (or equivalently, $\tilde{O}$ is even).

2.4. General linear groups. All special unipotent representations of $\text{GL}_n(\mathbb{R})$ and $\text{GL}_2(\mathbb{H})$ are obtained via normalized smooth parabolic induction from quadratic characters (see [Vog86, Page 450]). We will review their classifications in the framework of this article.

For a Young diagram $\tilde{i}$, write $r_1(i) \geq r_2(i) \geq r_3(i) \geq \cdots$ for its row lengths, and similarly, write $c_1(i) \geq c_2(i) \geq c_3(i) \geq \cdots$ for its column lengths. Denote by $|\tilde{i}| := \sum_{i=1}^{\infty} r_i(i)$ the total size of $\tilde{i}$.

For any Young diagram $\tilde{i}$, we introduce the set $\text{Box}(\tilde{i})$ of boxes of $\tilde{i}$ as follows:

$$(2.3) \quad \text{Box}(\tilde{i}) := \{ (i, j) \in \mathbb{N}^+ \times \mathbb{N}^+ \mid j \leq r_i(i) \}.$$ 

A subset of $\mathbb{N}^+ \times \mathbb{N}^+$ of the form (2.3) is also said to constitute the Young diagram $\tilde{i}$.

We also introduce five symbols $\bullet$, $s$, $r$, $c$ and $d$, and make the following definition.

Definition 2.8. A painting on a Young diagram $\tilde{i}$ is an assignment (we place a symbol in each box) $P : \text{Box}(\tilde{i}) \to \{ \bullet, s, r, c, d \}$ with the following properties:

1. If we remove the boxes painted with $\{s, r, c, d\}$, $\{r, c, d\}$, $\{c, d\}$, or $\{d\}$, the remainder still constitutes a Young diagram;
2. every row of $\tilde{i}$ has at most one box painted with $s$, and has at most one box painted with $r$;
3. every column of $\tilde{i}$ has at most one box painted with $c$, and has at most one box painted with $d$.

A painted Young diagram is a pair $(\tilde{i}, P)$ consisting of a Young diagram $\tilde{i}$ and a painting $P$ on $\tilde{i}$.

Remark 2.9. (i) The first requirement for a painted Young diagram says that the symbols $\bullet$, $s$, $r$, $c$ and $d$ should be painted in the order as indicated so that the painted boxes with any earlier painted symbols constitute a Young diagram throughout.

(ii) The specific requirements of painted Young diagrams are motivated by the branching rules of Weyl group representations (via the Littlewood-Richardson rule), and will be used to count special unipotent representations in Sections 7 and 8.

Example 2.10. The following represents a painted Young diagram.

```
\[ \begin{array}{cccc}
\bullet & \bullet & \bullet & \bullet \\
\bullet & r & d & \\
s & r & & \\
d & d & & \\
\end{array} \] 
```
Each of the followings does not represent a painted Young diagram.

\[
\begin{array}{ccc}
d & c \\
c & d \\
c & r \\
\end{array}
\]

Definition 2.11. Suppose that \( * \in \{ A^\mathbb{R}, A^\mathbb{H} \} \). A painting \( \mathcal{P} \) on a Young diagram \( \nu \) has type \( * \) if
1. the symbols of \( \mathcal{P} \) are in
\[
\{ \bullet, c, d \}, \quad \text{if } * = A^\mathbb{R}; \\
\{ \bullet \}, \quad \text{if } * = A^\mathbb{H};
\]
2. every column of \( \nu \) has an even number of boxes painted with \( \bullet \).

Denote by \( \text{PAP}_*(\nu) \) the set of paintings on \( \nu^t \) that has type \( * \), where \( \nu^t \) is the transpose of \( \nu \).

The middle letter A in PAP refers to the common A in \( \{ A^\mathbb{R}, A^\mathbb{H} \} \).

It is easy to check that if \( * = A^\mathbb{R} \), then
\[
\sharp(\text{PAP}_*(\check{\mathcal{O}})) = \prod_{i \in \mathbb{N}^+} (1 + \text{the number of rows of length } i \text{ in } \check{\mathcal{O}}),
\]
and if \( * = A^\mathbb{H} \), then
\[
\sharp(\text{PAP}_*(\check{\mathcal{O}})) = \begin{cases} 
1, & \text{if all row lengths of } \check{\mathcal{O}} \text{ are even;} \\
0, & \text{otherwise.}
\end{cases}
\]

Here and henceforth \( \sharp \) indicates the cardinality of a finite set.

If \( * = A^\mathbb{R} \), for every \( \mathcal{P} \in \text{PAP}_*(\check{\mathcal{O}}) \) we attach a representation \( \pi_{\mathcal{P}} \) of \( G \) as in what follows. Let \( P_R \) be the standard parabolic subgroup of \( G \) with Levi component
\[
\text{GL}_{r_1(\check{\mathcal{O}})}(\mathbb{R}) \times \text{GL}_{r_2(\check{\mathcal{O}})}(\mathbb{R}) \times \cdots \times \text{GL}_{r_k(\check{\mathcal{O}})}(\mathbb{R}),
\]
where \( k \) is the number of nonempty rows of \( \check{\mathcal{O}} \). On each factor \( \text{GL}_{r_j(\check{\mathcal{O}})}(\mathbb{R}) \), put the trivial character or the sign character according to whether the \( j \)-th column of \( \mathcal{P} \) ends in \( \{ \bullet, c \} \) or \( \{ d \} \). This yields a character of \( P_R \) and let \( \pi_{\mathcal{P}} \) be the resulting normalized induced representation of \( G \).

Similarly, if \( * = A^\mathbb{H} \), for every \( \mathcal{P} \in \text{PAP}_*(\check{\mathcal{O}}) \) we attach a representation \( \pi_{\mathcal{P}} \) of \( G \) as in what follows. Let \( P_H \) be the standard parabolic subgroup of \( G \) with Levi component
\[
\text{GL}_{r_1(\check{\mathcal{O}})/2}(\mathbb{H}) \times \text{GL}_{r_2(\check{\mathcal{O}})/2}(\mathbb{H}) \times \cdots \times \text{GL}_{r_k(\check{\mathcal{O}})/2}(\mathbb{H}),
\]
where \( k \) is the number of nonempty rows of \( \check{\mathcal{O}} \). Put the trivial character on each factor \( \text{GL}_{r_j(\check{\mathcal{O}})/2}(\mathbb{H}) \). This yields a character of \( P_H \) and let \( \pi_{\mathcal{P}} \) be the resulting normalized induced representation of \( G \).

Then in both cases \( \pi_{\mathcal{P}} \) is irreducible and belongs to \( \text{Unip}_\check{\mathcal{O}}(G) \) (see [Vog86, Theorem 3.8] and [ABV91, Example 27.5]). We summarize the classifications by the following theorem. A proof using the methods of this paper is in Section 7.2.

Theorem 2.12 (cf. [Vog86]). Suppose that \( * \in \{ A^\mathbb{R}, A^\mathbb{H} \} \). Then the map
\[
\begin{array}{ccc}
\text{PAP}_*(\check{\mathcal{O}}) & \rightarrow & \text{Unip}_\check{\mathcal{O}}(G), \\
\mathcal{P} & \mapsto & \pi_{\mathcal{P}}
\end{array}
\]
is bijective.

2.5. Unitary groups. Similar to Definition 2.11, we make the following definition.

Definition 2.13. Suppose that \( * \in \{ A, \tilde{A} \} \). A painting \( \mathcal{P} \) on a Young diagram \( \nu \) has type \( * \) if
1. the symbols of \( \mathcal{P} \) are in \( \{ \bullet, s, r \} \), and
2. every row of \( \nu \) has an even number of boxes painted with \( \bullet \).

Denote by \( \text{PAP}_*(\nu) \) the set of paintings on \( \nu^t \) that has type \( * \), where \( \nu^t \) is the transpose of \( \nu \).
Now suppose that $\iota$ is a Young diagram and $P$ is a painting on $\iota$ that has type $A$ or $\tilde{A}$. Define the signature of $P$ to be the pair
\begin{equation}
(\iota P, qP) := \left(\frac{\sharp(P^{-1}(\bullet))}{2} + \sharp(P^{-1}(r)), \frac{\sharp(P^{-1}(\bullet))}{2} + \sharp(P^{-1}(s))\right).
\end{equation}

**Example 2.14.** Suppose that
\[
\tilde{O} = \begin{array}{ccc}
\bullet & & \\
& & \\
& & \\
\end{array}
\quad \text{and} \quad P = \begin{array}{ccc}
\bullet & & \\
& & \\
& & \\
\end{array}
\in \text{PAP}_A(\tilde{O}).
\]

Then $(\iota P, qP) = (6, 5)$.

Given two Young diagrams $\iota$ and $\jmath$, write $\iota \cup \jmath$ for the Young diagram whose multiset of nonzero row lengths equals the union of those of $\iota$ and $\jmath$. Also write $2\iota = \iota \cup \iota$. Similarly, we write $\iota \cup \jmath$ for the Young diagram whose multiset of nonzero column lengths equals the union of those of $\iota$ and $\jmath$.

For unitary groups, we have the following counting result.

**Theorem 2.15.** Suppose that $\star \in \{A, \tilde{A}\}$. If there is a Young diagram decomposition
\begin{equation}
\tilde{O} = \tilde{O}_b \cup 2\tilde{O}_b',
\end{equation}

such that all nonzero row lengths of $\tilde{O}_b$ have good parity (namely the parity of $p + q$ when $\star = A$ and the parity of $p + q + 1$ when $\star = \tilde{A}$) and all nonzero row lengths of $\tilde{O}_b'$ have bad parity, then
\[
\sharp(\text{Unip}_{\tilde{O}}(G)) = \sharp\{ P \in \text{PAP}_\star(\tilde{O}_b) \mid (\iota P + |\tilde{O}_b'|, qP + |\tilde{O}_b'|) = (p, q)\}.
\]

Otherwise $\sharp(\text{Unip}_{\tilde{O}}(G)) = 0$.

In particular, when $\star = \tilde{A}$ and $p + q$ is odd, the set $\text{Unip}_{\tilde{O}}(\tilde{U}(p, q))$ is empty.

Suppose that $\star \in \{A, \tilde{A}\}$ so that $G = U(p, q)$ or $\tilde{U}(p, q)$. By Theorem 2.15, the set $\text{Unip}_{\tilde{O}}(G)$ is empty unless there is a decomposition as in (2.6) such that
\begin{equation}
p, q \geq |\tilde{O}_b'|.
\end{equation}

Assume this holds, and define two groups $G_b' := \text{GL}_b(C)$ and
\[
G_b := \begin{cases}
U(p - |\tilde{O}_b'|, q - |\tilde{O}_b'|), & \text{if } \star = A; \\
\tilde{U}(p - |\tilde{O}_b'|, q - |\tilde{O}_b'|), & \text{if } \star = \tilde{A}.
\end{cases}
\]

Then up to conjugation there is a unique real parabolic subgroup $P$ of $G$ whose Levi quotient (namely its quotient by the unipotent radical) is naturally isomorphic to $G_b' \times G_b$. Note that $\tilde{O}_b$ has good parity with respect to $\star$ and $|\tilde{O}_b|$.

Let $\pi_{\tilde{O}_b'}$ denote the unique element in $\text{Unip}_{\tilde{O}_b'}(G_b')$ (see Section 2.9 for a review on complex classical groups). Then for every $\pi_b \in \text{Unip}_{\tilde{O}_b}'(G_b')$, the normalized smooth parabolic induced representation (from $P$ to $G$) $\pi_{\tilde{O}_b'} \times \pi_b$ is irreducible by [Mat96, Theorem 3.2.2], and is an element of $\text{Unip}_{\tilde{O}_b}(G)$ (cf. [MR19, Theorem 5.3]). (This also follows from the calculation of the wavefront cycle of the induced representation ([Bar00, Corollary 5.0.10].))

**Theorem 2.16.** With the assumptions in (2.6) and (2.7), and notation as above, the parabolic induction map
\begin{equation}
\begin{array}{c}
\text{Unip}_{\tilde{O}_b}(G_b) \\
\pi_b
\end{array} \rightarrow \begin{array}{c}
\text{Unip}_{\tilde{O}_b}(G) \\
\pi_{\tilde{O}_b'} \times \pi_b
\end{array}
\end{equation}

is bijective.

Theorem 2.15 and Theorem 2.16 will be proved in Section 7.3.
2.6. Orthogonal and symplectic groups: relevant parabolic subgroups. In this subsection and the next two subsections, we assume that \( \star \in \{ B, C, \tilde{C}, C^*, D, D^* \} \). Then there are Young diagram decompositions
\[
\mathbf{d}_O = \mathbf{d}_b \sqcup \mathbf{d}_g \quad \text{and} \quad \mathbf{d}_b = 2 \mathbf{d}_b'
\]
such that \( \mathbf{d}_b \) has bad parity in the sense that all its nonzero row lengths have bad parity, and \( \mathbf{d}_g \) has good parity in the sense that all its nonzero row lengths have good parity. Put
\[
n_b := |\mathbf{d}_b|.
\]

Write \( V \) for the standard module of \( \mathfrak{g} \), which is either a complex symmetric bilinear space or a complex symplectic space. Denote by \( \mathcal{G}_C(V) \) the identity connected component of the isometry group of \( V \), which is a complexification of \( G \). Recall that \( \mathcal{O} \subseteq \mathfrak{g}^* \) is the Barbasch-Vogan dual of \( \mathcal{O} \). Note that when \( \star \in \{ D, D^* \} \), \( \mathcal{O} \) is very even (namely all its row lengths are even) if and only if \( \mathcal{O} \) is very even.

**Lemma 2.17.** Up to conjugation by \( \mathcal{G}_C(V) \), there is a unique totally isotropic subspace \( X_{\mathcal{O}} \) of \( V \) of dimension \( n_b \) that satisfies the following condition:
\[
\mathcal{O} = \text{Ind}^\mathcal{G}(\mathfrak{g}(X_{\mathcal{O}}) \times \mathfrak{g}(V_{\mathcal{O}})) \mathcal{O}' \times \mathcal{O}_g \quad \text{(parabolic induction of nilpotent orbit)}.
\]
Here \( V_{\mathcal{O}} \) is a non-degenerate subspace of \( V \) of dimension \( \dim V - 2n_b \) that is perpendicular to \( X_{\mathcal{O}} \), \( \mathfrak{g}(V_{\mathcal{O}}) \) denotes the Lie algebra of the isometry group of \( V_{\mathcal{O}} \), \( \mathfrak{g}(X_{\mathcal{O}}) \) denotes the general linear Lie algebra, \( \mathfrak{g}(X_{\mathcal{O}}) \times \mathfrak{g}(V_{\mathcal{O}}) \) is viewed as a Levi subalgebra of \( \mathfrak{g} \) as usual, \( \mathcal{O}_b \subseteq (\mathfrak{g}(X_{\mathcal{O}}))^* \) is the nilpotent orbit whose Young diagram is the transpose of \( \mathbf{d}_b' \), and \( \mathcal{O}_g \subseteq (\mathfrak{g}(V_{\mathcal{O}}))^* \) is the Barbasch-Vogan dual of the nilpotent orbit corresponding to the Young diagram \( \mathbf{d}_g \).

**Proof.** Excluding the case when \( \star \in \{ D, D^* \} \), \( V \neq 0 \), and \( \mathcal{O} \) is very even, there is a unique totally isotropic subspace \( X_{\mathcal{O}} \) of \( V \) of dimension \( n_b \), up to conjugation by \( \mathcal{G}_C(V) \). The condition in (2.11) is automatic by [BV85, Proposition A2, c)]. When \( \star \in \{ D, D^* \} \), \( V \neq 0 \), and \( \mathcal{O} \) is very even, there are exactly two totally isotropic subspaces of \( V \) of dimension \( n_b \), up to conjugation by \( \mathcal{G}_C(V) \). Denote by \( \mathcal{O}' \subseteq \mathfrak{g}^* \) the other nilpotent orbit having the same Young diagram as that of \( \mathcal{O} \). The two totally isotropic subspaces of \( V \) of dimension \( n_b \) are distinguished by the requirement in (2.11), i.e., relating \( \mathcal{O} \) with \( X_{\mathcal{O}} \), and \( \mathcal{O}' \) with \( X_{\mathcal{O}'} \), again by [BV85, Proposition A2, c)]. \( \square \)

**Definition 2.18.** (a) A parabolic subalgebra \( \mathfrak{p} \) of \( \mathfrak{g} \) is said to be \( \mathcal{O} \)-relevant if it is the stabilizer of a totally isotropic subspace \( X_{\mathcal{O}} \) of \( V \) in Lemma 2.17.
(b) A parabolic subgroup of \( G \) is said to be \( \mathcal{O} \)-relevant if its complexified Lie algebra is \( \mathcal{O} \)-relevant.
(c) The orbit \( \mathcal{O} \) is said to be \( G \)-relevant, if there is a parabolic subgroup of \( G \) which is \( \mathcal{O} \)-relevant.

We summarize by the following proposition, which is clear.

**Proposition 2.19.** (a) Up to conjugation by \( G \), there is at most one parabolic subgroup of \( G \) that is \( \mathcal{O} \)-relevant.
(b) If \( \star = D^* \), \( V \neq 0 \), and \( \mathcal{O} \) is very even, then there are precisely two orbits in \( \text{Nil}(\mathfrak{g}) \) that has the same Young diagram as that of \( \mathcal{O} \). Between these two orbits, exactly one is \( G \)-relevant. Excluding this special case, \( \mathcal{O} \) is \( G \)-relevant if and only if
\[
\begin{align*}
\text{either } \star & \in \{ B, D, C^* \} \text{ and } p, q \geq n_b, \quad \text{or } \star \in \{ C, \tilde{C}, D^* \}. \\
\end{align*}
\]

When (2.12) holds, we put
\[
G_g := \begin{cases} 
\text{SO}(p - n_b, q - n_b), & \text{if } \star \in \{ B, D \}; \\
\text{SO}^*(2n - 2n_b), & \text{if } \star = D^*; \\
\text{Sp}_{2n-2n_b}(\mathbb{R}), & \text{if } \star = C; \\
\tilde{\text{Sp}}_{2n-2n_b}(\mathbb{R}), & \text{if } \star = \tilde{C}; \\
\text{Sp}(\frac{2n - 2n_b}{2}, \frac{2n - 2n_b}{2}), & \text{if } \star = C^*.
\end{cases}
\]
Then the Levi quotient of every $\mathcal{O}$-relevant parabolic subgroup of $G$ is naturally isomorphic to $G'_b \times G_g$ (or $(G'_b \times G_g)/\{\pm 1\}$ when $\star = \tilde{\mathcal{O}}$), where

$$G'_b := \begin{cases} \text{GL}_{m_b}(\mathbb{R}), & \text{if } \star \in \{B, C, D\}; \\ \text{GL}_{m_b}(\mathbb{R}), & \text{if } \star = \tilde{C}; \\ \text{GL}_{m_b}(\mathbb{H}), & \text{if } \star \in \{C^*, D^*\}. \end{cases}$$

Here $\text{GL}_{m_b}(\mathbb{R})$ is the double cover of $\text{GL}_{m_b}(\mathbb{R})$ that fits the following Cartesian diagram of Lie groups:

$$\begin{array}{ccc} \text{GL}_{m_b}(\mathbb{R}) & \longrightarrow & \text{GL}_{m_b}(\mathbb{R}) \\ \downarrow & & \downarrow \text{sign of det}(g) \\ \{\pm 1, \pm \sqrt{-1}\} & \xrightarrow{x \mapsto x^2} & \{\pm 1\}. \end{array}$$

Unless otherwise mentioned, we will use the corresponding lower case Gothic letter to denote the complexified Lie algebra of a Lie group. For example, $\mathfrak{g}'_b$ is the complexified Lie algebra of $G'_b$. Let $\mathfrak{g}'_b$ denote the Langlands dual of $\mathfrak{g}_b$, and let $\mathcal{O}'_b \subseteq \mathfrak{Nil}(\mathfrak{g}'_b)$ denote the nilpotent orbit with Young diagram $d'_b$. Likewise let $\mathfrak{g}'_g$ denote the Langlands dual (or the metaplectic Langlands dual when $\star = \tilde{C}$) of $\mathfrak{g}_g$, and let $\mathcal{O}'_g \subseteq \mathfrak{Nil}(\mathfrak{g}'_g)$ denote the (unique) nilpotent orbit with Young diagram $d'_g$.

### 2.7. Orthogonal and symplectic groups: reduction to good parity.

Define

$$\text{Unip}_{\mathcal{O}_b}(\text{GL}_{m_b}(\mathbb{R})) := \{\pi \in \text{Irr}_{\mathfrak{g}'_b}(\text{GL}_{m_b}(\mathbb{R})) \mid \pi \text{ is genuine}\}.$$ 

Here and as before, “genuine” means that the central subgroup $\{\pm 1\}$ acts through the nontrivial character. Then we have a bijective map

$$\text{Unip}_{\mathcal{O}_b}(\text{GL}_{m_b}(\mathbb{R})) \to \text{Unip}_{\mathcal{O}_b}(\text{GL}_{m_b}(\mathbb{R})), \quad \pi \mapsto \pi \otimes \tilde{\chi}_{m_b},$$

where $\tilde{\chi}_{m_b}$ is the character given by the left vertical arrow of (2.15).

In Section 9, we will prove the following theorem.

**Theorem 2.20.** If $G$ has an $\mathcal{O}$-relevant parabolic subgroup $P$, then the normalized smooth parabolic induction from $P$ to $G$ yields a bijection

$$\text{Unip}_{\mathcal{O}_b}(G'_b) \times \text{Unip}_{\mathcal{O}_g}(G_g) \longrightarrow \text{Unip}_{\mathcal{O}}(G), \quad (\pi'_b, \pi'_g) \mapsto \pi'_b \otimes \pi'_g.$$

Otherwise,

$$\text{Unip}_{\mathcal{O}}(G) = \emptyset.$$

By Theorem 2.20, we have the more specific results on counting as follows.

(a) Assume that $\star \in \{B, D\}$ so that $G = \text{SO}(p, q)$. Then

$$\sharp(\text{Unip}_{\mathcal{O}}(G)) = \begin{cases} \sharp(\text{Unip}_{\mathcal{O}_b}(G'_b)) \times \sharp(\text{Unip}_{\mathcal{O}_g}(\text{GL}_{m_b}(\mathbb{R}))), & \text{if } p, q \geq m_b; \\ 0, & \text{otherwise}. \end{cases}$$

(b) Assume that $\star = C^*$ so that $G = \text{Sp}(\frac{q}{2}, \frac{q}{2})$. Then

$$\sharp(\text{Unip}_{\mathcal{O}}(G)) = \begin{cases} \sharp(\text{Unip}_{\mathcal{O}_g}(G'_g)), & \text{if } p, q \geq m_b; \\ 0, & \text{otherwise}. \end{cases}$$

(c) Assume that $\star \in \{C, \tilde{C}\}$ so that $G = \text{Sp}_{2n}(\mathbb{R})$ or $\tilde{\text{Sp}}_{2n}(\mathbb{R})$. Then

$$\sharp(\text{Unip}_{\mathcal{O}}(G)) = \sharp(\text{Unip}_{\mathcal{O}_g}(G'_g)) \times \sharp(\text{Unip}_{\mathcal{O}_b}(\text{GL}_{m_b}(\mathbb{R}))).$$

(d) Assume that $\star = D^*$ so that $G = \text{SO}^*(2n)$. Then

$$\sharp(\text{Unip}_{\mathcal{O}}(G)) = \begin{cases} 0, & \text{if } \mathcal{O} \text{ is not } G\text{-relevant}; \\ \sharp(\text{Unip}_{\mathcal{O}_g}(G'_g)), & \text{otherwise}. \end{cases}$$
2.8. Orthogonal and symplectic groups: the case of good parity. We now assume that \( \tilde{O} \) has good parity, namely \( \tilde{O} = \tilde{O}_g \). By Theorem 2.20, the counting problem in general is reduced to this case.

**Definition 2.21.** A \( \star \)-pair is a pair \((i, i + 1)\) of consecutive positive integers such that

\[
\begin{cases} 
  i & \text{is odd, if } \star \in \{C, \tilde{C}, C^*\}; \\
  i & \text{is even, if } \star \in \{B, D, D^*\}.
\end{cases}
\]

A \( \star \)-pair \((i, i + 1)\) is said to be

- vacant in \( \tilde{O} \), if \( r_i(\tilde{O}) = r_{i+1}(\tilde{O}) = 0 \);
- balanced in \( \tilde{O} \), if \( r_i(\tilde{O}) = r_{i+1}(\tilde{O}) > 0 \);
- tailed in \( \tilde{O} \), if \( r_i(\tilde{O}) - r_{i+1}(\tilde{O}) \) is positive and odd;
- primitive in \( \tilde{O} \), if \( r_i(\tilde{O}) - r_{i+1}(\tilde{O}) \) is positive and even.

Denote \( PP_\star(\tilde{O}) \) the set of all \( \star \)-pairs that are primitive in \( \tilde{O} \).

**Remark 2.22.** When \( \star \neq \tilde{C} \), the power set of \( PP_\star(\tilde{O}) \) gives another description of Lusztig’s canonical quotient attached to \( \tilde{O} \). The set \( PP_\star(\tilde{O}) \) appears implicitly in [Son01, Section 5].

We give algorithms of how to explicitly count special unipotent representations for each type. First we attach to \( \tilde{O} \) a pair of Young diagrams

\[
(t_\tilde{O}, J_\tilde{O}) := (t_\star(\tilde{O}), J_\star(\tilde{O})),
\]

as follows. (This is to be viewed as a Weyl group representation. See Section 8.3.)

Recall for a Young diagram \( \gamma \), the row and column lengths are written as

\[
r_1(\gamma) \geq r_2(\gamma) \geq r_3(\gamma) \geq \cdots \\ c_1(\gamma) \geq c_2(\gamma) \geq c_3(\gamma) \geq \cdots.
\]

**The case when \( \star = B \).** In this case, the nilpotent orbit \( \tilde{O} \) is of type \( C \), and has even rows only. Define

\[
(c_i(t_\tilde{O}), c_{i+1}(J_\tilde{O})) = \left( \frac{r_{2i}(\tilde{O})}{2}, \frac{r_{2i+1}(\tilde{O})}{2} \right).
\]

(In words: The largest row of size \( r_1(\tilde{O}) \) contributes a column of size \( \frac{r_1(\tilde{O})}{2} \) to \( J_\tilde{O} \). Pair up the remaining rows, adding a row of size 0 if there are an even number of nonzero rows. Each pair of rows \( (r_{2i}(\tilde{O}), r_{2i+1}(\tilde{O})) \) contributes a column of size \( \frac{r_{2i}(\tilde{O})}{2} \) to \( t_\tilde{O} \) and \( \frac{r_{2i+1}(\tilde{O})}{2} \) to \( J_\tilde{O} \).)

**The case when \( \star = \tilde{C} \).** In this case, the nilpotent orbit \( \tilde{O} \) is of type \( C \), and has even rows only. Define, for all \( i \geq 1 \),

\[
(c_i(t_\tilde{O}), c_{i+1}(J_\tilde{O})) = \left( \frac{r_{2i-1}(\tilde{O})}{2}, \frac{r_{2i}(\tilde{O})}{2} \right).
\]

(In words: Pair up rows, adding a row of size 0 if there are an odd number of nonzero rows. Each pair of rows \( (r_{2i-1}(\tilde{O}), r_{2i}(\tilde{O})) \) contributes a column of size \( \frac{r_{2i-1}(\tilde{O})}{2} \) to \( t_\tilde{O} \) and \( \frac{r_{2i}(\tilde{O})}{2} \) to \( J_\tilde{O} \).)

**The case when \( \star = \{C, C^*\} \).** In this case, the nilpotent orbit \( \tilde{O} \) is of type \( B \), and has odd rows only. Define, for all \( i \geq 1 \),

\[
(c_i(t_\tilde{O}), c_{i+1}(J_\tilde{O})) = \begin{cases} 
  (0, 0), & \text{if } (2i - 1, 2i) \text{ is vacant in } \tilde{O}; \\
  \left( \frac{r_{2i-1}(\tilde{O}) - 1}{2}, 0 \right), & \text{if } (2i - 1, 2i) \text{ is tailed in } \tilde{O}; \\
  \left( \frac{r_{2i-1}(\tilde{O}) - 1}{2}, \frac{r_{2i}(\tilde{O}) + 1}{2} \right), & \text{otherwise}.
\end{cases}
\]

(In words: The number of nonzero rows is odd, say \( 2k + 1 \). Pair them up, by adding a row of size zero. For each pair of rows \( (r_{2i-1}(\tilde{O}), r_{2i}(\tilde{O})) \), add a column of size \( \frac{r_{2i-1}(\tilde{O}) - 1}{2} \) to \( J_\tilde{O} \) and \( \frac{r_{2i}(\tilde{O}) + 1}{2} \) to \( t_\tilde{O} \). For the last pair \( (r_{2k+1}(\tilde{O}), 0) \), only add a column of size \( \frac{r_{2k+1}(\tilde{O}) - 1}{2} \) to \( J_\tilde{O} \).)
The case when $\star \in \{D, D^*\}$. In this case, the nilpotent orbit $\check{O}$ is of type $D$, and has odd rows only. Define

$$c_1(t_{\check{O}}) = \begin{cases} 0, & \text{if } r_1(\check{O}) = 0; \\ \frac{r_1(\check{O})+1}{2}, & \text{if } r_1(\check{O}) > 0, \end{cases}$$

and for all $i \geq 1$,

$$(c_i(j_{\check{O}}), c_{i+1}(t_{\check{O}})) = \begin{cases} (0, 0), & \text{if } (2i, 2i+1) \text{ is vacant in } \check{O}; \\ \left(\frac{r_2(\check{O})-1}{2}, 0\right), & \text{if } (2i, 2i+1) \text{ is tailed in } \check{O}; \\ \left(\frac{r_2(\check{O})-1}{2}, \frac{r_{2i+1}(\check{O})+1}{2}\right), & \text{otherwise}. \end{cases}$$

(In words: The number of nonzero rows is even, say $2k$. The largest row of size $r_1(\check{O})$ contributes a column of size $\frac{r_1(\check{O})+1}{2}$ to $t_{\check{O}}$. Pair up the remaining rows, by adding a row of size 0. For each pair of rows $(r_{2i}(\check{O}), r_{2i+1}(\check{O}))$, add a column of size $\frac{r_{2i}(\check{O})-1}{2}$ to $j_{\check{O}}$ and $\frac{r_{2i+1}(\check{O})+1}{2}$ to $t_{\check{O}}$. For the last pair $(r_{2k}(\check{O}), 0)$, only add a column of size $\frac{r_{2k}(\check{O})-1}{2}$ to $j_{\check{O}}$.)

Example 2.23. Suppose that $\star = C$, and $\check{O}$ is the following Young diagram which has good parity.

```
+---+---+
|   |   |
|   |   |
+---+---+
```

Then $PP_\star(\check{O}) = \{(1, 2), (5, 6)\}$

and

$$(t_{\check{O}}, j_{\check{O}}) = \begin{array}{c|c}
\begin{array}{|c|c|c|}
\hline
\bullet & \bullet & \bullet \\
\hline
\end{array} & \begin{array}{|c|c|}
\hline
\bullet & \bullet \\
\hline
\end{array} \\
\hline
\end{array}.$$
(3) the symbols of $Q$ are in

$$\left\{ \{\bullet, s, r, d\}, \text{ if } \gamma = B^+ \text{ or } B^-; \right.$$ 

$$\{\bullet, s\}, \text{ if } \gamma = C; \right.$$ 

$$\{\bullet\}, \text{ if } \gamma = D; \right.$$ 

$$\{\bullet, r, d\}, \text{ if } \gamma = \tilde{C}; \right.$$ 

$$\{\bullet, s, r\}, \text{ if } \gamma = C^*; \right.$$ 

$$\{\bullet, r\}, \text{ if } \gamma = D^*. \right.$$ 

Remark 2.25. In the above definition, the pairs $(i, j)$ parametrize irreducible representations of the classical Weyl group $W(B) = W(C)$ (of type $B$ or $C$). We give some informal explanation on how painted bipartitions arise. For each conjugacy class of Cartan subgroups $H$ of $G$, there is a subgroup $W(H) = H_\gamma \times \cdots \times W_\gamma \times W_\gamma \times W_d$ in $W(\gamma)$ (roughly the integral Weyl group of the real classical group labeled by $\gamma$), a 1-dimensional character $\chi_H$ of $W(H)$, and the painted bipartitions will count the multiplicities of a certain $W(\gamma)$-representation in $\text{Ind}_{W(H)}^{W(\gamma)} \chi_H$. There is another subgroup $W_s$ containing $H_\gamma$ and $W_\gamma$, $W_r$, $W_c$, $W_d$ are all Weyl groups of classical type. They depend on the type of the group being considered, and the Cartan subgroup $H$. The induction from $H_\gamma$ to $W_s$ (of a certain quadratic character) is encoded by the painting with the symbol $\bullet$. The character $\chi_H$ is $\text{sgn}$ on $W_s$, $W_r$, and trivial on $W_c$, $W_d$. The painting with the symbols $s, r, c, d$ reflects (use of) the Littlewood-Richardson rule (more precisely the Pieri rule). See Section 8.4 for details.

For any painted bipartition $\tau$ as in Definition 2.24, we write

$$\nu_\tau := i, \ P_\tau := P, \ j_\tau := j, \ Q_\tau := Q, \ \tau_\tau := \gamma, \ |\tau| := |i| + |j|,$$

and

$$\tau_\tau := \left\{ \begin{array}{ll}
B, \text{ if } \gamma = B^+ \text{ or } B^-; \\
\gamma, \text{ otherwise.}
\end{array} \right.$$ 

We further define a pair $\text{Sign}(\tau) := (p_\tau, q_\tau)$ of natural numbers given by the following recipe:

(a) If $\tau_\tau \in \{B, D, C^*\}$, then $(p_\tau, q_\tau)$ is given by counting the various symbols appearing in $(i, P)$, $(j, Q)$ and $\{\alpha\}$:

$$\left\{ \begin{array}{l}
p_\tau := (#\bullet) + 2(#r) + (#c) + (#d) + (#B^+);
q_\tau := (#\bullet) + 2(#s) + (#c) + (#d) + (#B^-).
\end{array} \right.$$ 

(2.17)

Here

$$\#\bullet := \#(P^{-1}(\bullet)) + \#(Q^{-1}(\bullet)),$$

the total number of boxes painted with $\bullet$ in $P$ and $Q$, and the other terms are defined in the obvious way.

(b) If $\tau_\tau \in \{C, \tilde{C}, D^*\}$, then $p_\tau := q_\tau := |\tau|$.

We also define a classical group

$$G_\tau := \left\{ \begin{array}{l}
\text{SO}(p_\tau, q_\tau), \text{ if } \tau_\tau = B \text{ or } D; \\
\text{Sp}_{2|\tau}(\mathbb{R}), \text{ if } \tau_\tau = C; \\
\tilde{\text{Sp}}_{2|\tau}(\mathbb{R}), \text{ if } \tau_\tau = \tilde{C}; \\
\text{Sp}(\frac{2|\tau}{2}, \mathbb{R}), \text{ if } \tau_\tau = C^*; \\
\text{SO}^*(2|\tau), \text{ if } \tau_\tau = D^*.
\end{array} \right.$$ 

Define

$$\text{PBP}_s(\hat{O}) := \{ \tau \text{ is a painted bipartition } | \tau_\tau = \bullet, \text{ and } (i_\tau, j_\tau) = (i_{\hat{O}}, j_{\hat{O}}) \} ,$$

and

$$\text{PBP}_G(\hat{O}) := \{ \tau \in \text{PBP}_s(\hat{O}) \mid G_\tau = G \} .$$

Here “$G_\tau = G$” amounts to saying that $(p_\tau, q_\tau) = (p, q)$ if $\tau_\tau \in \{B, D, C^*\}$. 
Example 2.26. Suppose that \( \ast = B \) and
\[
\hat{O} = \begin{array}{cccc}
\ast & \ast & \ast & \ast \\
\ast & \ast & \ast & \ast \\
\ast & \ast & \ast & \ast \\
\ast & \ast & \ast & \ast \\
\end{array}
\].

Then
\[
\tau := \begin{array}{ccc}
\ast & \ast & \ast \\
\ast & \ast & \ast \\
\end{array} \times \begin{array}{ccc}
\ast & \ast & \ast \\
\ast & \ast & \ast \\
\end{array} \times B^{+} \in \text{PBP}_{\ast}(\hat{O}),
\]
and
\[
G_{\tau} = \text{SO}(10,9).
\]

We now state our final result on the explicit counting of special unipotent representations. It will be proved in Section 8.4.

Theorem 2.27. Assume that \( \ast \in \{B, C, D, \hat{C}, \hat{C}^{\ast}, D^{\ast}\} \), and \( \hat{O} \) has good parity. Then
\[
\sharp(\text{Unip}_{\hat{O}}(G)) = \begin{cases} 
\sharp(\text{PBP}_{\ast}(\hat{O})), & \text{if } \ast \in \{C^{\ast}, D^{\ast}\}; \\
2^{\sharp(\text{PP}_{\ast}(\hat{O}))} \cdot \sharp(\text{PBP}_{G}(\hat{O})), & \text{if } \ast \in \{B, C, D, \hat{C}\}.
\end{cases}
\]

In [BMSZ21], the authors will construct all representations in \( \text{Unip}_{\hat{O}}(G) \) by the method of theta lifting, when \( \hat{O} \) has good parity. See [BMSZ21, Section 3].

We shall illustrate the contents of Theorem 2.27 for the case \( \ast = C \). The nilpotent orbit \( \hat{O} \) is of type \( B \) and all rows have odd sizes. The number of nonzero rows is odd, say \( 2k + 1 \). Label them as
\[
2r_{1} + 1 \geq 2r_{2} + 1 \geq \cdots \geq 2r_{2k+1} + 1 > 0.
\]

Associate a pair of Young diagrams \((i_{\hat{O}}, j_{\hat{O}})\) with columns
\[
(r_{2} + 1, r_{4} + 1, \ldots, r_{2k} + 1) \times (r_{1}, r_{3}, \ldots, r_{2k-1}, r_{2k+1}).
\]

We will actually associate a larger set of such pairs of Young diagrams to \( \hat{O} \) (see Section 8.3 for details). Line up the numbers \( r_{1}, r_{2}, \ldots, r_{2k+1} \) as follows:
\[
(r_{1}, r_{2}) \cdots (r_{2k-1}, r_{2k})(r_{2k+1}).
\]

The set \( \text{PP}_{\ast}(\hat{O}) \) consists of pairs \((2i - 1, 2i)\) such that \( r_{2i-1} > r_{2i} \) (where \( 1 \leq i \leq k \)). For each subset \( \varphi \subseteq \text{PP}_{\ast}(\hat{O}) \), form a new pair of Young diagrams by replacing the pair of numbers \((r_{2i} + 1, r_{2i-1})\) in \((i_{\hat{O}}, j_{\hat{O}})\) by \((r_{2i-1} + 1, r_{2i})\), where \((2i - 1, 2i) \in \varphi \). In this way we obtain \( 2^{\sharp(\text{PP}_{\ast}(\hat{O}))} \) number of pairs of Young diagrams associated to \( \hat{O} \); they parametrize the so-called Lusztig left cell associated to \( \hat{O} \) (see Proposition 8.3).

For a pair of Young diagrams \((i, j)\) as in the above, we associate the set of painted bipartitions \( \tau = (i, P) \times (j, Q) \times C \), by the painting rule specified in Definition 2.24 for \( \gamma = C \). It turns out that the set of painted bipartitions associated to each of the \( 2^{\sharp(\text{PP}_{\ast}(\hat{O}))} \) pairs of Young diagrams shares the same cardinality (see Section 10.1). Theorem 2.27 says that the total number of painted bipartitions described above gives the number of special unipotent representations associated to \( \hat{O} \), for \( G = \text{Sp}_{2n}(\mathbb{R}) \).

Example 2.28. Suppose that \( \ast = C \), \( G = \text{Sp}_{4}(\mathbb{R}) \), and
\[
\hat{O} = \begin{array}{ccc}
\ast & \ast & \ast \\
\end{array}.
\]

Then \( \text{PP}_{\ast}(\hat{O}) = \{(1,2)\} \), and the set \( \text{PBP}_{G}(\hat{O}) \) has 4 elements in total:
\[
\begin{array}{cccc}
\ast \times \ast \times C, & \ast \times \ast \times C, & \ast \times \ast \times C, & \ast \times \ast \times C.
\end{array}
\]

Thus there are precisely 8 special unipotent representations of \( G \) attached to \( \hat{O} \).
2.9. The case of complex classical groups. Special unipotent representations of complex classical groups are all well-understood ([BV85], [Bar89]) and known to be unitarizable ([Bar89]). We briefly review their counting and constructions in what follows. As the methods of this paper and [BMSZ21] work for complex classical groups as well, we will present the results in the complex case parallel to those of this paper and [BMSZ21]. For this subsection, we introduce five more labels $A^C, B^C, D^C, C^C,$ and $\overline{C}^C,$ and let $\ast$ be one of them. Let $G$ be a complex classical group of type $\ast,$ namely

$$G = \text{GL}_n(\mathbb{C}), \quad \text{SO}_{2n+1}(\mathbb{C}), \quad \text{SO}_{2n}(\mathbb{C}), \quad \text{Sp}_{2n}(\mathbb{C}), \quad \text{or} \quad \text{Sp}_{2n}(\mathbb{C}) \quad (n \in \mathbb{N}),$$

respectively. Let $\mathfrak{g}_0$ denote the Lie algebra of $G,$ which is a complex Lie algebra. The Langlands dual (or metaplectic Langlands dual when $\ast = \overline{C}^C$) $\mathfrak{g}_0$ of $\mathfrak{g}_0$ is respectively defined to be

$$\mathfrak{gl}_n(\mathbb{C}), \quad \mathfrak{sp}_{2n}(\mathbb{C}), \quad \mathfrak{o}_{2n}(\mathbb{C}), \quad \mathfrak{o}_{2n+1}(\mathbb{C}), \quad \text{or} \quad \mathfrak{sp}_{2n}(\mathbb{C}).$$

Let $\tilde{O} \in \tilde{\mathfrak{nil}}(\mathfrak{g}_0).$ In the real case we have a maximal ideal $I_{\tilde{O}} := I_{\ast, \tilde{O}}$ of $\mathcal{U}(\mathfrak{g}_0).$

Write $\mathfrak{g}_0$ for the complex Lie algebra equipped with a conjugate linear isomorphism $\sim : \mathfrak{g}_0 \to \mathfrak{g}_0.$ The latter induces a conjugate linear isomorphism $\sim : \mathcal{U}(\mathfrak{g}_0) \to \mathcal{U}(\mathfrak{g}_0).$ Note that $\mathfrak{g}_0 \times \mathfrak{g}_0$ equals the complexified Lie algebra $\mathfrak{g}$ of $G.$ Define the set of special unipotent representations of $G$ attached to $\tilde{O}$ by

$$\text{Unip}_{\tilde{O}}(G) := \text{Unip}_{\ast, \tilde{O}}(G) := \{ \pi \in \text{Irr}(G) \mid \pi \text{ is annihilated by } I_{\tilde{O}} \otimes \mathcal{U}(\mathfrak{g}_0) + \mathcal{U}(\mathfrak{g}_0) \otimes \mathcal{T}_{\mathfrak{g}} \}. $$

If $\ast = A^C$ so that $G = \text{GL}_n(\mathbb{C}),$ then $\text{Unip}_{\tilde{O}}(G)$ is a singleton whose unique element is given by the normalized smooth parabolic induction $\text{Ind}_{P}^G 1_{P},$ where $P$ is the standard parabolic subgroup whose Levi component equals

$$\text{GL}_{r_1(\tilde{O})}(\mathbb{C}) \times \text{GL}_{r_2(\tilde{O})}(\mathbb{C}) \times \cdots \times \text{GL}_{r_{\epsilon_1(\tilde{O})}(\tilde{O})}(\mathbb{C}),$$

and $1_{P}$ denotes the trivial representation of $P.$

Now suppose that $\ast \in \{ B^C, D^C, C^C, \overline{C}^C \}.$ Write

$$d_{\tilde{O}} = d_{b} \bigcup d_{g} \quad \text{and} \quad d_{b} = 2d_{b}',$n

as in (2.9), and put $n_b := |d_{b}'|$ as before. Let $p_{0}$ be a parabolic subalgebra of $\mathfrak{g}_0$ that is $\tilde{O}$-relevant (defined in Section 2.6). Let $P$ be the parabolic subgroup of $G$ with Lie algebra $p_0.$ Then the Levi quotient of $P$ is naturally isomorphic to $G_{b}' \times G_{g},$ where $G_{b}' := \text{GL}_{n_{b}}(\mathbb{C})$ and

$$G_{g} := \begin{cases} 
\text{SO}_{2n-2n_{b}+1}(\mathbb{C}), & \text{if } \ast = B^C; \\
\text{SO}_{2n-2n_{b}}(\mathbb{C}), & \text{if } \ast = D^C; \\
\text{Sp}_{2n-2n_{b}}(\mathbb{C}), & \text{if } \ast \in \{ C^C, \overline{C}^C \}.
\end{cases}$$

Define the set $\text{PP}_{\ast}(\tilde{O}_{b})$ as in the real case. Then by the work of Barbasch-Vogan ([BV85, Corollary 5.29], integral case) and Barbasch ([Bar89], general case; see also [MR17, Theorem 6.12 and Theorem 10.1]), we have that

$$\sharp(\text{Unip}_{\tilde{O}}(G)) = \sharp(\text{Unip}_{\tilde{O}_{b}}(G_{b}')) \times \sharp(\text{Unip}_{\tilde{O}_{g}}(G_{g})) = 2^\sharp(\text{PP}_{\ast}(\tilde{O}_{g})).$$

As in the real case, every representation in $\text{Unip}_{\tilde{O}}(G)$ is obtained through irreducible parabolic induction from $P$ to $G$ via those of $\text{Unip}_{\tilde{O}_{b}}(G_{b}'), \text{Unip}_{\tilde{O}_{g}}(G_{g})$ (cf. Theorem 2.20), and every representation in $\text{Unip}_{\tilde{O}_{b}}(G_{b}')$ is obtained via iterated theta lifting (see [Bar17, Theorem 3.5.1], [Møg17] and [BMSZ21]). The method of [BMSZ21] gives an alternative proof that all representations in $\text{Unip}_{\tilde{O}}(G)$ are unitarizable.

3. Generalities on coherent families of highest weight modules

In this section, we recall some generalities on coherent continuation representations for highest weight modules, as a preparation for the study of coherent continuation representations in the Casselman-Wallach setting. (The theory of primitive ideals, developed by Joseph and Barbasch-Vogan among others, plays a key role.) Using $\tau$-invariant, we formalize a certain duality notion
of double cells (Proposition 3.38), useful in understanding their counterparts in the Casselman-Wallach setting.

We retain the notation of Section 2.1. Write
\[ \Delta^+ \subseteq \Delta \subseteq a\mathfrak{h}^* \quad \text{and} \quad \Delta^+ \subseteq \Delta \subseteq a\mathfrak{h} \]
for the positive root system, the root system, the positive coroot system, and the coroot system, respectively, for the reductive complex Lie algebra \( \mathfrak{g} \). Write \( Q_\mathfrak{g} \) and \( Q^\mathfrak{g} \) for the root lattice and the weight group of \( \mathfrak{g} \), respectively. Namely \( Q_\mathfrak{g} \) is the subgroup of \( a\mathfrak{h}^* \) spanned by \( \Delta \), and
\[ Q^\mathfrak{g} := \{ \nu \in a\mathfrak{h}^* \mid \langle \nu, \alpha \rangle \in \mathbb{Z} \quad \text{for all} \ \alpha \in \Delta \}. \]

3.1. Coherent continuation representations. Throughout this article, the coefficient ring of all Grothendieck groups will be \( \mathbb{C} \). When no confusion is possible, for every object \( O \) in an abelian category, we use the same symbol to indicate the Grothendieck group element represented by the object \( O \).

Let \( Q \) be a \( W \)-stable subgroup of \( Q^\mathfrak{g} \) containing \( Q_\mathfrak{g} \):
\[ Q_\mathfrak{g} \subseteq Q \subseteq Q^\mathfrak{g}. \]
Denote by \( \text{Rep}(\mathfrak{g}, Q) \) the category of all finite-dimensional representations of \( \mathfrak{g} \) whose weights (viewed as elements of \( Q^\mathfrak{g} \subseteq a\mathfrak{h}^* \)) are contained in \( Q \). Write \( \mathcal{R}(\mathfrak{g}, Q) \) for the Grothendieck group of this category, which is a commutative \( \mathbb{C} \)-algebra with multiplication the tensor product of representations: \( X \cdot Y := X \otimes Y \), where \( X, Y \in \mathcal{R}(\mathfrak{g}, Q) \).

Let \( \Lambda \subseteq a\mathfrak{h}^* \) be a \( Q \)-coset. Let \( W_\Lambda \) denote the stabilizer of \( \Lambda \) in \( W \). Specifically, if \( \Lambda = \lambda + Q \) for a \( \lambda \in a\mathfrak{h}^* \), then
\[ W_\Lambda := \{ w \in W \mid w\lambda - \lambda \in Q \}. \]

Definition 3.1. Given an \( \mathcal{R}(\mathfrak{g}, Q) \)-module \( K \), a \( K \)-valued \( \Lambda \)-family is an assignment of subspaces \( \{ K_\nu \}_{\nu \in \Lambda} \) of \( K \) such that
\[ \begin{align*}
\bullet & \ K_{\nu w} = K_\nu \quad \text{for all} \ w \in W_\Lambda \text{ and} \ \nu \in \Lambda; \\
\bullet & \ \text{for all representations} \ F \ \text{in} \ \text{Rep}(\mathfrak{g}, Q) \ \text{and all} \ \nu \in \Lambda, \\
F \cdot K_\nu & \subseteq \sum_{\mu \ \text{is a weight of} \ F} K_{\nu + \mu}. 
\end{align*} \]

Definition 3.2 ([Jan79], [Sch77]). Let \( K \) be an \( \mathcal{R}(\mathfrak{g}, Q) \)-module, and \( \{ K_\nu \}_{\nu \in \Lambda} \) a \( K \)-valued \( \Lambda \)-family. A \( \mathcal{K} \)-valued coherent family on \( \Lambda \) based on \( \{ K_\nu \}_{\nu \in \Lambda} \) is a map
\[ \Psi : \Lambda \rightarrow \mathcal{K} \]
satisfying the following two conditions:
\[ \begin{align*}
\bullet & \ \text{for all} \ \nu \in \Lambda, \ \Psi(\nu) \in K_\nu; \\
\bullet & \ \text{for all representations} \ F \ \text{in} \ \text{Rep}(\mathfrak{g}, Q) \ \text{and all} \ \nu \in \Lambda, \\
F \cdot (\Psi(\nu)) & = \sum_{\mu} \Psi(\nu + \mu), 
\end{align*} \]
where \( \mu \) runs over all weights of \( F \), counted with multiplicities.

When specifying a \( \mathcal{K} \)-valued coherent family on \( \Lambda \) based on \( \{ K_\nu \}_{\nu \in \Lambda} \), we will often explicitly describe \( K \) as a Grothendieck group, while the \( \mathcal{R}(\mathfrak{g}, Q) \)-module structure and the \( \mathcal{K} \)-valued \( \Lambda \)-family \( \{ K_\nu \}_{\nu \in \Lambda} \) are the ones which are clear from the context. (Often the \( \mathcal{K} \)-valued \( \Lambda \)-family is specified by the infinitesimal character). When the \( \mathcal{K} \)-valued \( \Lambda \)-family \( \{ K_\nu \}_{\nu \in \Lambda} \) is specified or clear from the context, we will just call it a \( \mathcal{K} \)-valued coherent family on \( \Lambda \).

Given a \( \mathcal{K} \)-valued \( \Lambda \)-family \( \{ K_\nu \}_{\nu \in \Lambda} \), let \( \text{Coh}_\Lambda(\mathcal{K}) \) denote the vector space of all \( \mathcal{K} \)-valued coherent families on \( \Lambda \) based on \( \{ K_\nu \}_{\nu \in \Lambda} \). It is a representation of \( W_\Lambda \) under the action
\[ (3.1) \quad (w \cdot \Psi)(\nu) = \Psi(w^{-1} \nu), \quad \text{for all} \ w \in W_\Lambda, \ \Psi \in \text{Coh}_\Lambda(\mathcal{K}), \ \nu \in \Lambda. \]
This is called a coherent continuation representation (based on \( \{ K_\nu(\mathfrak{g}, \mathfrak{b}) \}_{\nu \in \Lambda} \).
The assignment $\mathcal{K} \mapsto \text{Coh}_\Lambda(\mathcal{K})$ is functorial in the following sense: suppose that $\mathcal{K}'$ is another $\mathcal{R}(g, Q)$-module with a $\mathcal{K}'$-valued $\Lambda$-family $\{\mathcal{K}_\nu\}_{\nu \in \Lambda}$, and $\eta : \mathcal{K} \rightarrow \mathcal{K}'$ is an $\mathcal{R}(g, Q)$-homomorphism such that $\eta(\mathcal{K}_\nu) \subseteq \mathcal{K}'_\nu$ for all $\nu \in \Lambda$, then
\[
(3.2) \quad \eta_* : \text{Coh}_\Lambda(\mathcal{K}) \rightarrow \text{Coh}_\Lambda(\mathcal{K}'), \quad \Psi \mapsto \eta \circ \Psi
\]
is a well-defined $W_\Lambda$-equivariant linear map.

### 3.2. Coherent continuation representation for highest weight modules.

Coherent families for highest weight modules were introduced by Jantzen [Jan79]. We refer the reader to [Hum08, Section 7] for basic facts on coherent families in this setting.

Let $b$ be a Borel subalgebra of $g$, and let $\text{Rep}(g, b)$ denote the category of finitely generated $g$-modules that are unions of finite-dimensional $b$-submodules. For each $\nu \in {}^b h^*$, let $\text{Rep}_\nu(g, b)$ denote the full subcategory of $\text{Rep}(g, b)$ consisting of modules that have generalized infinitesimal character $\nu$ (by definition, a $g$-module has generalized infinitesimal character $\nu$ if every vector in it is annihilated by $(\ker(\chi_\nu))^k$ for some $k \in \mathbb{N}^+$).

**Definition 3.3.** Write $\mathcal{K}(g, b)$ for the Grothendieck group of $\text{Rep}(g, b)$. It is an $\mathcal{R}(g, Q)$-module under the tensor product. Similarly define
\[
\mathcal{K}_\nu(g, b) := \text{the Grothendieck group of } \text{Rep}_\nu(g, b),
\]
which is a subspace of $\mathcal{K}(g, b)$. Denote by $\text{Coh}_\Lambda(\mathcal{K}(g, b))$ the coherent continuation representation based on $\{\mathcal{K}_\nu(g, b)\}_{\nu \in \Lambda}$, as in (3.1).

Write $\rho \in {}^b h^*$ for the half sum of positive roots. For each $\nu \in {}^b h^*$, define the Verma module
\[
M(\nu) := M(g, b, \nu) := U(g) \otimes_{U(b) C_{\nu-\rho}} C_{\nu-\rho},
\]
where $C_{\nu-\rho}$ is the one-dimensional $^b h$-module corresponding to the character $\nu - \rho \in {}^b h^*$, and every $^b h$-module is viewed as a $b$-module via the canonical map $b \rightarrow {}^b h$. Write $L(\nu) := L(g, b, \nu)$ for the unique irreducible quotient of $M(\nu)$.

An element $\nu \in {}^b h^*$ is called dominant if
\[
(3.3) \quad \langle \nu, \tilde{\alpha} \rangle \not\in \mathbb{N}^+ \quad \text{for all } \tilde{\alpha} \in \tilde{\Delta}^+,
\]
and regular if
\[
(3.4) \quad \langle \nu, \tilde{\alpha} \rangle \neq 0 \quad \text{for all } \tilde{\alpha} \in \tilde{\Delta}.
\]

For every $w \in W$, define a map
\[
\Psi_w : \Lambda \rightarrow \mathcal{K}(g, b), \quad \nu \mapsto M(w\nu).
\]
Then $\Psi_w \in \text{Coh}_\Lambda(\mathcal{K}(g, b))$, and
\[
(3.5) \quad \{\Psi_w\}_{w \in W} \text{ is a basis of } \text{Coh}_\Lambda(\mathcal{K}(g, b)).
\]
Similarly there is a unique coherent family $\overline{\Psi}_w \in \text{Coh}_\Lambda(\mathcal{K}(g, b))$ such that
\[
(3.6) \quad \overline{\Psi}_w(\nu) = L(w\nu) \quad \text{for all regular dominant element } \nu \in \Lambda.
\]
Then $\{\overline{\Psi}_w \mid w \in W\}$ is also a basis of the coherent continuation representation $\text{Coh}_\Lambda(\mathcal{K}(g, b))$.

Let $W$ act on $\mathcal{K}(g, b)$ as $\mathcal{R}(g, Q)$-module automorphisms by
\[
(3.7) \quad w \cdot (M(\nu)) = M(w\nu) \quad \text{for all } \nu \in {}^b h^*.
\]
By the functority (3.2), this yields an action of $W$ on $\text{Coh}_\Lambda(\mathcal{K}(g, b))$ as automorphisms of $W_\Lambda$-representations. The resulting action of $W \times W_\Lambda$ on $\text{Coh}_\Lambda(\mathcal{K}(g, b))$ is explicitly given by
\[
(3.8) \quad \Psi_w = \Psi_{w_1} \Psi_{w_2}^{-1} \quad \text{for all } w_1, w_2 \in W, w_2 \in W_\Lambda, w \in W.
\]
Let $\Lambda^0 \subseteq {}^b h^*$ denote the $Q^0$-coset containing $\Lambda$, and let $\Lambda_\tilde{g} \subseteq \Lambda$ be a $Q_{\tilde{g}}$-coset. Put
\[
(3.9) \quad \Delta(\Lambda) := \{\alpha \in \Delta \mid \langle \tilde{\alpha}, \nu \rangle \in \mathbb{Z} \text{ for some (and all) } \nu \in \Lambda\}.
\]
Here and henceforth, $\tilde{\alpha} \in {}^b h$ denotes the coroot corresponding to $\alpha$. This is a root system with the corresponding coroots
\[
(3.10) \quad \tilde{\Delta}(\Lambda) := \{\tilde{\alpha} \in \tilde{\Delta} \mid \langle \tilde{\alpha}, \nu \rangle \in \mathbb{Z} \text{ for some (and all) } \nu \in \Lambda\}.
\]
Let \( W(\Lambda) \subseteq W \) denote the Weyl group of the root system \( \Delta(\Lambda) \), to be called the integral Weyl group (attached to \( \Lambda \)). Then \( W(\Lambda_g) = W(\Lambda) = W(\Lambda^g) = W(\Lambda_g) \subseteq W(\Lambda) \subseteq W(\Lambda^g) \).

To understand the coherent continuation representation \( \text{Coh}_\Lambda(K(g, b)) \), it suffices to consider the case when \( \Lambda = \Lambda_g \) by the following lemma.

**Lemma 3.4.** The restriction map \( \text{Coh}_{\Lambda_g}(K(g, b)) \rightarrow \text{Coh}_\Lambda(K(g, b)) \) is a \( W \times W(\Lambda) \)-equivariant linear isomorphism, and the restriction map \( \text{Coh}_\Lambda(K(g, b)) \rightarrow \text{Coh}_{\Lambda_g}(K(g, b)) \) is a \( W \times W(\Lambda) \)-equivariant linear isomorphism.

**Proof.** This follows from (3.4). Essentially a coherent family on \( \Lambda \) extends uniquely to one on \( \Lambda_g \) and a coherent family on \( \Lambda_g \) extends uniquely to one on \( \Lambda \), for any \( \Lambda_g \subseteq \Lambda \subseteq \Lambda^g \). \( \square \)

### 3.3. Jantzen matrix.

Define the Jantzen matrix \( \{ a_\Lambda(w_1, w_2) \}_{w_1, w_2 \in W} \), where \( a_\Lambda(w_1, w_2) \in \mathbb{Z} \) is specified by the following equation in \( \text{Coh}_\Lambda(K(g, b)) \):

\[
\Psi_{w_1} = \sum_{w_2 \in W} a_\Lambda(w_1, w_2)\Psi_{w_2},
\]

for each \( w_1 \in W \).

**Lemma 3.5 (Bernstein-Gelfand-Gelfand).** Let \( w_1, w_2 \in W \). If \( w_1 W(\Lambda) \neq w_2 W(\Lambda) \), then \( a_\Lambda(w_1, w_2) = 0 \).

**Proof.** This is a consequence of the theorem of Bernstein-Gelfand-Gelfand on the composition factors of a Verma module. See [Hum08, Corollary 5.2]. \( \square \)

Put \( \Delta^+(\Lambda) := \Delta(\Lambda) \cap \Delta^+ \)
and

\[
D(\Lambda) := \{ w' \in W \mid w'(\Delta^+(\Lambda)) \subseteq \Delta^+ \}.
\]

Then the group multiplication yields a bijective map \( D(\Lambda) \times W(\Lambda) \rightarrow W \).

**Lemma 3.6 (Joseph).** For all \( w' \in D(\Lambda) \) and \( w_1, w_2 \in W(\Lambda) \),

\[
(3.9) \quad a_\Lambda(w'w_1, w'w_2) = a_\Lambda(w_1, w_2).
\]

**Proof.** This is shown by relating highest weight modules with principal series representations of complex semisimple groups. See [Jos79, Theorem 4.12 and Theorem 5.4]. \( \square \)

**Remark 3.7.** Equation (3.9) is also a direct consequence of a theorem of Soergel (see [Soe90, Section 2.5, Theorem 11]). The matrix \( \{ a_\Lambda(w_1, w_2) \}_{w_1, w_2 \in W(\Lambda)} \) only depends on \( W(\Lambda) \) as a Coxeter group. More precisely, suppose that \( (g', \Lambda', W(\Lambda')) \) is a triple of the same type as \( (g, \Lambda, W(\Lambda)) \), and \( \eta : W(\Lambda) \rightarrow W(\Lambda') \) is a group isomorphism that restricts to a bijection between the sets of simple reflections, then

\[
a_\Lambda(w_1, w_2) = a_{\Lambda'}(\eta(w_1), \eta(w_2))
\]

for all \( w_1, w_2 \in W(\Lambda) \).
Recall that a polynomial function on $a\mathfrak{h}^*$ or $a\mathfrak{h}$ is said to be $W(\Lambda)$-harmonic if it is annihilated by all the $W(\Lambda)$-invariant constant coefficient differential operators without constant term.

For every $w \in W$, define a polynomial function $\tilde{p}_{\Lambda,w}$ on $a\mathfrak{h} \times a\mathfrak{h}^*$ by

$$
(3.10) \quad \tilde{p}_{\Lambda,w}(x, \nu) := \sum_{w_1 \in W} a_{\Lambda}(w, w_1) : (x, w_1 \nu)^{ma_{\Lambda,w}}, \quad \text{for all } x \in a\mathfrak{h}, \nu \in a\mathfrak{h}^*,
$$

where $m_{\Lambda,w}$ is the smallest non-negative integer (which always exists) that makes the right-hand side of (3.10) a nonzero polynomial function.

**Lemma 3.8** (Joseph [Jos84, Section 5.1]). Let $w \in W$. There is a $W(\Lambda)$-harmonic polynomial function $p'_{\Lambda,w}$ on $a\mathfrak{h}$ and a $W(\Lambda)$-harmonic polynomial function $p_{\Lambda,w}$ on $a\mathfrak{h}^*$ such that

$$
\tilde{p}_{\Lambda,w}(x, \nu) = p'_{\Lambda,w}(x) \cdot p_{\Lambda,w}(\nu), \quad \text{for all } x \in a\mathfrak{h}, \nu \in a\mathfrak{h}^*.
$$

The two harmonic polynomial functions are nonzero, homogeneous of degree $m_{\Lambda,w}$ and are uniquely determined up to scalar multiplication.

**Proof.** This follows from [Jos80b, Lemma 2.3 (i), Lemma 2.5] and [BV83a, Theorem 2.6 (b)]. □

### 3.4. Left, right, and double cells

We define a basal vector space to be a complex vector space $V$ equipped with a basis $B \subseteq V$, and call elements of $B$ the basal elements in $V$. A subspace of a basal space $V$ is called a basal subspace if it is spanned by a set of basal elements of $V$. Every basal subspace is obviously a basal space.

As an example, if $K$ is the Grothendieck group of an abelian category in which all objects have finite length, then $K$ is a basal vector space with the irreducible objects as the basal elements.

**Definition 3.9.** Let $E$ be a finite group. A basal representation of $E$ is a basal vector space carrying a representation of $E$. A basal subrepresentation of a basal representation $V$ is a subrepresentation of $V$ that is simultaneously a basal subspace.

Let $V$ be a basal representation of a finite group $E$, with basal elements $B \subseteq V$. For each subset $S \subseteq B$, write $\langle S \rangle$ for the smallest basal subrepresentation of $V$ containing $S$. For each $\phi \in B$, write $\langle \phi \rangle := \langle \{\phi\} \rangle$ for simplicity. We define an equivalence relation $\approx$ on $B$ by

$$
\phi_1 \approx \phi_2 \quad \text{if and only if} \quad \langle \phi_1 \rangle = \langle \phi_2 \rangle \quad (\phi_1, \phi_2 \in B).
$$

An equivalence class of the relation $\approx$ on the set $B$ is called a cell in $V$.

**Definition 3.10.** Let $C$ be a cell in $V$ and put $\overline{C} := \langle C \rangle \cap B$. Define the cell representation attached to $C$ by

$$
V(C) := \overline{C}/(\overline{C} \setminus C).
$$

In the notation of Definition 3.10, $V(C)$ is a representation of $E$, and the cosets $\{\phi + (\overline{C} \setminus C)\}_{\phi \in C}$ form a basis of $V(C)$.

We view $\text{Coh}_\Lambda(K(\mathfrak{g}, \mathfrak{b}))$ as a basal representation of $W \times W_\Lambda$ with the basal elements $\{\overline{W}_w | \ w \in W\}$. Write

$$
(3.11) \quad \text{Coh}^L_\Lambda(K(\mathfrak{g}, \mathfrak{b})) := \text{Coh}_\Lambda(K(\mathfrak{g}, \mathfrak{b})),
$$

to be viewed as a basal representation of $W \times W(\Lambda)$. Likewise, write

$$
\text{Coh}^R_\Lambda(K(\mathfrak{g}, \mathfrak{b})) := \text{Coh}_\Lambda(K(\mathfrak{g}, \mathfrak{b})),
$$

to be viewed as a basal representation of $W$, and write

$$
\text{Coh}^R_\Lambda(K(\mathfrak{g}, \mathfrak{b})) := \text{Coh}_\Lambda(K(\mathfrak{g}, \mathfrak{b})),
$$

to be viewed as a basal representation of $W(\Lambda)$ (as a subgroup of $W_\Lambda$).

**Definition 3.11.**

- Cells in $\text{Coh}^L_\Lambda(K(\mathfrak{g}, \mathfrak{b}))$ are called left cells.
- Cells in $\text{Coh}^R_\Lambda(K(\mathfrak{g}, \mathfrak{b}))$ are called right cells.
- Cells in $\text{Coh}^{LR}_\Lambda(K(\mathfrak{g}, \mathfrak{b}))$ are called double cells.
For every set $S$ of basal elements in $\text{Coh}_\Lambda(K(g,b))$, write $\langle S \rangle_L$ for the smallest basal subrepresentation of $\text{Coh}_\Lambda^L(K(g,b))$ containing $S$. Similarly write $\langle S \rangle_R$ and $\langle S \rangle_{LR}$.

It is clear that every left (right) cell is contained in a (unique) double cell.

3.5. Left cells and classification of primitive ideals. Let $\text{Ann}(M) \subseteq U(g)$ denote the annihilator ideal of a $U(g)$-module $M$. By a result of Duflo [Duf77, Theorem 1], each primitive ideal in the universal enveloping algebra $U(g)$ has the form $\text{Ann} \overline{\Psi}_w(\nu)$ for a $w \in W$ and a dominant $\nu \in \mathfrak{h}^*$.

For each $w \in W$, let $p_{\Lambda,w}$ be the $W(\Lambda)$-harmonic polynomial function as in Lemma 3.8.

Lemma 3.12. Let $w_1, w_2 \in W$. The following conditions are equivalent.

(i) The basal elements $\overline{\Psi}_{w_1}$ and $\overline{\Psi}_{w_2}$ lie in a common left cell.

(ii) For some regular dominant $\nu \in \Lambda$, $\text{Ann} \overline{\Psi}_{w_1}(\nu) = \text{Ann} \overline{\Psi}_{w_2}(\nu)$.

(iii) For every regular dominant $\nu \in \Lambda$, $\text{Ann} \overline{\Psi}_{w_1}(\nu) = \text{Ann} \overline{\Psi}_{w_2}(\nu)$.

(iv) The equality $\mathbb{C} \cdot p_{\Lambda,w_1} = \mathbb{C} \cdot p_{\Lambda,w_2}$ holds.

Proof. The equivalence of (i) and (ii) is the equivalence of $(a)$ and $(b)$ in [BV83a, Proposition 2.9].

The equivalence of (ii) and (iii) follows from the translation principle, see [Vog78, Lemma 2.7].

The equivalence of (ii) and (iv) is [Jos80b, Theorem 5.1 and 5.5] when $w_1 W(\Lambda) = w_2 W(\Lambda)$.

The general case can be deduced by relating the problem to the setting of complex semisimple groups using [Jos79, Theorem 4.12] and [Duf77, Proposition 3.7]. See Lemma 3.6. \hfill $\Box$

Using Lemma 3.12, we attach a polynomial function $p_{\mathcal{L}}$ on $\mathfrak{h}^*$ to every left cell $\mathcal{L}$ in $\text{Coh}_\Lambda(K(g,b))$, which is uniquely determined up to scalar multiplication.

Lemma 3.13. For every $w \in W$ and every simple reflection $s \in W(\Lambda)$, the following statements are equivalent.

(i) $s \cdot p_{\Lambda,w} = -p_{\Lambda,w}$.

(ii) $s \cdot \overline{\Psi}_w = -\overline{\Psi}_w$.

(iii) $w(\alpha_s) \in \Delta^+$, where $\alpha_s \in \Delta^+(\Lambda)$ is the simple root corresponding to $s$.

Proof. The equivalence between (i) and (ii) is due to the fact that $p_{\Lambda,w}$ is essentially the Bernstein degree polynomial, see [Jos80b, Theorem 4.10] and [Vog78, p85].

The equivalence between (ii) and (iii) is due to Jantzen, see [BV83a, Proposition 2.20]. \hfill $\Box$

Definition 3.14. The $\tau$-invariant of a basal element $\Psi := \overline{\Psi}_w \in \text{Coh}_\Lambda(K(g,b))$ ($w \in W$) is the set of simple reflections satisfying the equivalent conditions in Lemma 3.13:

$$\tau_\Psi := \{ s \in W(\Lambda) \mid s \cdot p_{\Lambda,w} = -p_{\Lambda,w} \}.$$  

This only depends on the left cell containing $\Psi$, and thus the $\tau$-invariant $\tau_\mathcal{L}$ of a left cell $\mathcal{L}$ is well defined.

For each $\nu \in \mathfrak{h}^*$, write $W_{\nu}$ for the stabilizer of $\nu$ in $W$. Note that $W_{\nu} \subseteq W(\Lambda)$, for any $\nu \in \Lambda$. The following is a reformulation of Jantzen’s result on the translation to a wall, see [Hum08, Theorem 7.9] and also [Vog81, Corollary 7.3.22].

Lemma 3.15 (Jantzen). Let $w \in W$ and let $\nu \in \Lambda$ be a dominant element. Then

$$\overline{\Psi}_w(\nu) = \begin{cases} L(w\nu), & \text{if } \tau_\Lambda,w \cap W_{\nu} = \emptyset; \\ 0, & \text{otherwise}. \end{cases}$$

Lemma 3.15 and the translation principle (see [Vog79b, Lemma 2.7]) imply the following classification of primitive ideals at a general infinitesimal character.

Proposition 3.16 (Joseph). Let $\nu \in \Lambda$ be a dominant element. Then the map

$$\{ w \in W \mid \tau_{\Lambda,w} \cap W_{\nu} = \emptyset \} \rightarrow \{ \text{primitive ideal of } U(g) \text{ of infinitesimal character } \nu \}$$

is surjective. Furthermore for all $w_1, w_2$ in the domain of this map,

$$\text{Ann}(L(w_1\nu)) = \text{Ann}(L(w_2\nu)) \quad \text{if and only if} \quad \mathbb{C} \cdot p_{\Lambda,w_1} = \mathbb{C} \cdot p_{\Lambda,w_2}.$$
By the work of Macdonald, Lusztig, and Spaltenstein ([Car93, Chapter 11]), the ideal \( \operatorname{Ann}(\Psi(\nu)) \) only depends on the left cell \( L \) that \( \Psi \) belongs to. The map

\[
(3.12) \quad \frac{L \to \operatorname{Ann}(\Psi(\nu))}{\{ \text{left cell } L \text{ in } \operatorname{Coh}_A(K(\mathfrak{g}, \mathfrak{b})) \mid \tau_L \cap W_\nu = \emptyset \}}
\]

is a well-defined bijection where \( \Psi \) is an arbitrary element in \( L \). In particular when \( \nu \in \Lambda \) is regular dominant, for each basal element \( \Psi \) in \( \operatorname{Coh}_A(K(\mathfrak{g}, \mathfrak{b})) \), the left cell corresponding to the primitive ideal \( \operatorname{Ann}(\Psi(\nu)) \) is just the left cell containing \( \Psi \).

The corollary justifies the following definition.

**Definition 3.18.** For every left cell \( L \) in \( \operatorname{Coh}_A(K(\mathfrak{g}, \mathfrak{b})) \) and every dominant element \( \nu \in \Lambda \), define

\[
(3.13) \quad I_{\nu, L} := \operatorname{Ann}(\Psi(\nu)), \quad \text{for (any) } \Psi \in L.
\]

Conversely, for every primitive ideal \( I \) of infinitesimal character \( \nu \in \Lambda \), write \( L_{\nu, I} \) for the left cell that corresponds to it under the bijection (3.12).

We record the following lemma for later use.

**Lemma 3.19.** Let \( \nu \in \Lambda \) be a dominant element. Then for all primitive ideals \( I_1 \) and \( I_2 \) of \( \mathcal{U}(\mathfrak{g}) \) of infinitesimal character \( \nu \),

\[
I_1 \subseteq I_2 \quad \text{if and only if} \quad \langle L_{\nu, I_1} \rangle_L \supseteq \langle L_{\nu, I_2} \rangle_L.
\]

**Proof.** This follows from Corollary 3.17 and [BV83a, Proposition 2.9]. \( \square \)

### 3.6. Double cells and special representations.

For every \( \sigma \in \operatorname{Irr}(W) \), its fake degree is defined to be

\[
(3.14) \quad a(\sigma) := \min\{k \in \mathbb{N} \mid \sigma \text{ occurs in the } k\text{-th symmetric power } S^k(\mathfrak{a}\mathfrak{h})\}.
\]

This is well-defined since every \( \sigma \in \operatorname{Irr}(W) \) occurs in the symmetric algebra \( S(\mathfrak{a}\mathfrak{h}) \). The representation \( \sigma \) is said to be univalent if it occurs in \( S^{a(\sigma)}(\mathfrak{a}\mathfrak{h}) \) with multiplicity one, where \( \mathfrak{a}\mathfrak{h} := \operatorname{Span}(\Delta) \) denotes the span of the coroots.

Recall Lusztig’s notion of a special representation of a Weyl group ([Lus79]). An irreducible representation of \( W \) is said to be Springer if it corresponds to the trivial local system on a nilpotent orbit in \( \mathfrak{g}^* \) via the Springer correspondence [Spr78]. Note that every special irreducible representation is Springer, and every Springer representation is univalent [BM81].

We have a decomposition

\[
(3.15) \quad \mathfrak{a}\mathfrak{h} = (\Delta(\Lambda))^\perp \oplus \operatorname{Span}(\Delta(\Lambda))
\]

where

\[
(\Delta(\Lambda))^\perp := \{ x \in \mathfrak{a}\mathfrak{h} \mid \langle x, \alpha \rangle = 0 \text{ for all } \alpha \in \Delta(\Lambda) \}.
\]

For every univalent irreducible representation \( \sigma_0 \) of \( W(\Lambda) \), whenever it is convenient, we view it as a subrepresentation of \( S^{(\sigma_0)}(\mathfrak{a}\mathfrak{h}) \) via the inclusions

\[
\sigma_0 = \mathbb{C} \otimes \sigma_0 \subseteq S^0((\Delta(\Lambda))^\perp) \otimes S^{(\sigma_0)}(\operatorname{Span}(\Delta(\Lambda))) \subseteq S^{(\sigma_0)}(\mathfrak{a}\mathfrak{h})
\]

By the work of Macdonald, Lusztig, and Spaltenstein ([Car93, Chapter 11]), the \( W \) subrepresentation of \( S^{(\sigma_0)}(\mathfrak{a}\mathfrak{h}) \) generated by \( \sigma_0 \) is irreducible and univalent, with the same fake degree as that of \( \sigma_0 \). This irreducible representation of \( W \) is called the \( j \)-induction of \( \sigma_0 \), to be denoted by \( j_W^W(\Lambda) \sigma_0 \).

If \( \sigma_0 \) is special, then the \( j \)-induction \( j_W^W(\Lambda) \sigma_0 \) is Springer. Write

\[
(3.15) \quad O_{\sigma_0} \in \overline{\operatorname{Nil}(\mathfrak{g}^*)} \quad \text{for the nilpotent orbit corresponding to } j_W^W(\Lambda) \sigma_0,
\]

via the Springer correspondence. Then

\[
\dim O_{\sigma_0} = 2 \cdot (\#(\Delta^+) - a(\sigma_0)).
\]

See [Hot84, Theorem 3].
Definition 3.20. (a) Define a preorder \( \preceq \) on \( \text{Irr}(W(\Lambda)) \) by
\[
\sigma_1 \preceq \sigma_2 \quad \text{if and only if} \quad \langle D_1 \rangle_{LR} \supseteq \langle D_2 \rangle_{LR}
\]
for some double cells \( D_i \) in \( \text{Coh}_{\Lambda}(K(\mathfrak{g}, \mathfrak{h})) \) such that \( \sigma_i \) occurs in \( \text{Coh}_{\Lambda}^{LR}(K(\mathfrak{g}, \mathfrak{b}))(D_i) \) \( (i = 1, 2) \).
(b) Define an equivalence relation \( \approx \) on \( \text{Irr}(W(\Lambda)) \) by
\[
\sigma_1 \approx \sigma_2 \quad \text{if and only if} \quad \sigma_1 \preceq \sigma_2 \quad \text{and} \quad \sigma_2 \preceq \sigma_1.
\]
An equivalence class of this equivalence relation is called a double cell in \( \text{Irr}(W(\Lambda)) \).

We remark that the preorder \( \preceq \) only depends on \( W(\Lambda) \) as a Coxeter group (see [BV83a, Proposition 2.28]) and the notion of double cells in Definition 3.20 (b) coincides with that of Lusztig in [Lus82] (also [Car93, Section 13.2]).

A basic property of this preorder is the following result.

Proposition 3.21 ([BV83a, Proposition 2.25]). For all \( \sigma_1, \sigma_2 \in \text{Irr}(W(\Lambda)) \), \( \sigma_1 \preceq \sigma_2 \) if and only if \( \sigma_2 \otimes \text{sgn} \preceq \sigma_1 \otimes \text{sgn} \).

Here and henceforth \( \text{sgn} \) denotes the sign character of an appropriate Weyl group.

Suppose that \( D \) is a double cell in \( \text{Coh}_{\Lambda}(K(\mathfrak{g}, \mathfrak{b})) \). Pick a \( w \in W \) such that \( \overline{w} \in D \), and put
\[
\langle D \rangle := m_{\Lambda, w}, \quad \text{(as in (3.10))}
\]
Note that \( m_D \) is independent of the choice of \( w \) (see [BV83a, Corollary 2.15 and 2.16]). For every \( \Psi = \sum_{w \in W} a_w \Psi_w \in \langle D \rangle_{LR}, \) define a polynomial function \( \tilde{\rho}_\Psi \) on \( ^a\mathfrak{h} \times ^a\mathfrak{h}^* \) by
\[
\tilde{\rho}_\Psi(x, \nu) := \sum_{w \in W} a_w \cdot (x, w\nu)^m_D, \quad \text{for all} \quad x \in ^a\mathfrak{h}, \nu \in ^a\mathfrak{h}^*.
\]

Then the linear map
\[
\langle D \rangle_{LR} \to S(^a\mathfrak{h}^*) \otimes S(^a\mathfrak{h}), \quad \Psi \mapsto \tilde{\rho}_\Psi \quad \text{(S indicates the symmetric algebra)}
\]
is \( W \times W(\Lambda) \)-equivariant and descends to a linear map
\[
\text{Coh}_{\Lambda}^{LR}(K(\mathfrak{g}, \mathfrak{b}))(D) \to S(^a\mathfrak{h}^*) \otimes S(^a\mathfrak{h}).
\]
See [Jos80b, § 3] and [BV83a, Corollary 2.15].

Let \( \text{Irr}^{sp}(W(\Lambda)) \) denote the subset of \( \text{Irr}(W(\Lambda)) \) consisting of the special irreducible representations. We recollect some results due to Joseph and Barbasch-Vogan in the following proposition. See [Jos80b, Section 5] and [BV83a, Section 2].

Proposition 3.22. Let \( D \) be a double cell in \( \text{Coh}_{\Lambda}(K(\mathfrak{g}, \mathfrak{b})) \).

(a) The image of (3.19) is isomorphic to \( \left( J_W^{W(\Lambda)}(\sigma_D) \right) \otimes \sigma_D \) for a unique \( \sigma_D \) in \( \text{Irr}^{sp}(W(\Lambda)) \).
Moreover, \( a(\sigma_D) = m_D \) and for every \( \sigma' \in \text{Irr}(W(\Lambda)) \) that occurs in \( \langle D \rangle_{LR} \), either \( \sigma' = \sigma_D \) or \( a(\sigma') > m_D \).
(b) The representation \( \sigma_D \) is the unique element of \( \text{Irr}^{sp}(W(\Lambda)) \) that occurs in the double cell representation \( \text{Coh}_{\Lambda}^{LR}(K(\mathfrak{g}, \mathfrak{b}))(D) \). Moreover, the map
\[
\{ \text{double cell in} \text{Coh}_{\Lambda}(K(\mathfrak{g}, \mathfrak{b})) \} \to \text{Irr}^{sp}(W(\Lambda)), \quad \sigma_D \mapsto \sigma_D
\]
is bijective.
(c) As a representation of \( W \times W(\Lambda) \),
\[
\text{Coh}_{\Lambda}^{LR}(K(\mathfrak{g}, \mathfrak{b}))(D) \cong \bigoplus_{\sigma \in \text{Irr}(W(\Lambda)), \sigma \preceq \sigma_D} \left( \text{Ind}_W^{W(\Lambda)}(\sigma) \right) \otimes \sigma.
\]

Definition 3.23. For a double cell \( D \) in \( \text{Coh}_{\Lambda}(K(\mathfrak{g}, \mathfrak{b})) \), call \( \sigma_D \) as in Proposition 3.22 the special irreducible representation attached to \( D \).

Recall the definition of the polynomial function \( p(L) \) after Lemma 3.12, where \( L \) is a left cell in \( \text{Coh}_{\Lambda}(K(\mathfrak{g}, \mathfrak{b})) \).
Definition 3.25. Define the Goldie rank representation of a primitive ideal $I$ of infinitesimal character $\nu$, where $\nu \in \Lambda$ is dominant, to be
\begin{equation}
\sigma_{\nu,I} := \sigma_{D_{\nu,I}} \in \text{Irr}^p(W(\Lambda)).
\end{equation}
Here $D_{\nu,I}$ is the double cell containing the left cell $L_{\nu,I}$.

For a primitive ideal $I$ of infinitesimal character $\nu$, we may thus attach the nilpotent orbit $O_{\sigma_{\nu,I}} \in \text{Nil}(\mathfrak{g}^*)$, via the Goldie rank representation $\sigma_{\nu,I}$ as in (3.15).

The following result of Joseph determines the associated variety of a primitive ideal.

Proposition 3.26 ([Jos85, Theorem 3.10]). Let $\nu \in \Lambda$ be a dominant element and let $I$ be a primitive ideal of $\mathcal{U}(\mathfrak{g})$ of infinitesimal character $\nu$. Then the associated variety $AV(I)$ equals the Zariski closure $\overline{O_{\sigma_{\nu,I}}} \subseteq \mathfrak{g}^*$ of the Zariski closed subset $O_{\sigma_{\nu,I}}$ in $\mathfrak{g}^*$.

Definition 3.27. Fix $S \subseteq \text{Nil}(\mathfrak{g}^*)$, an $\text{Ad}(\mathfrak{g})$-stable Zariski closed subset. Let $\text{Rep}_S(\mathfrak{g}, \mathfrak{b})$ be the full subcategory of $\text{Rep}(\mathfrak{g}, \mathfrak{b})$ consisting of modules $M$ such that
\[ AV(\text{Ann}(M)) \subseteq S. \]
Denote by $K_S(\mathfrak{g}, \mathfrak{b})$ its Grothendieck group.

As before, we have a coherent continuation representation $\text{Coh}_\Lambda(K_S(\mathfrak{g}, \mathfrak{b}))$ of $W_\Lambda$, which is a subrepresentation of $\text{Coh}_\Lambda(K(\mathfrak{g}, \mathfrak{b}))$.

For every $w \in W$, put
\[ O_{\Lambda,w} := O_{\sigma_D}, \]
where $D$ is the double cell in $\text{Coh}_\Lambda(K(\mathfrak{g}, \mathfrak{b}))$ containing $\overline{\Psi}_w$.

Lemma 3.28. Let $w \in W$. The coherent family $\overline{\Psi}_w$ belongs to $\text{Coh}_\Lambda(K_S(\mathfrak{g}, \mathfrak{b}))$ if and only if $O_{\Lambda,w} \subseteq S$.

Proof. This follows from Lemma 3.15, Proposition 3.16, Proposition 3.26 and [Vog81, Proposition 7.3.11] (for the translation outside the dominant cone).

Lemma 3.29. Let $w_1, w_2 \in W$. If $\langle \overline{\Psi}_{w_1} \rangle_L \supseteq \langle \overline{\Psi}_{w_2} \rangle_L$, then $O_{\Lambda,w_1} \supseteq O_{\Lambda,w_2}$.

Proof. Pick a regular dominant element $\nu \in \Lambda$. Put $I_i := \text{Ann}(\overline{\Psi}_{w_i}(\nu))$ ($i = 1, 2$). Lemma 3.19 implies that $I_1 \subseteq I_2$ and hence
\[ AV(I_1) \supseteq AV(I_2). \]
See [BK76, Korollar 3.6]. On the other hand, Proposition 3.26 implies that $AV(I_i) = O_{\Lambda,w_i}$ ($i = 1, 2$). Hence the lemma follows.

Lemma 3.30. The space $\text{Coh}_\Lambda(K_S(\mathfrak{g}, \mathfrak{b}))$ is a $W \times W_\Lambda$-submodule of $\text{Coh}_\Lambda(K(\mathfrak{g}, \mathfrak{b}))$ spanned by the basal elements
\begin{equation}
\{ \overline{\Psi}_w \mid w \in W, O_{\Lambda,w} \subseteq S \}.
\end{equation}

Proof. Lemma 3.28 implies that $\text{Coh}_\Lambda(K_S(\mathfrak{g}, \mathfrak{b}))$ is spanned by the set (3.21). Since the space $\text{Coh}_\Lambda(K_S(\mathfrak{g}, \mathfrak{b}))$ is clearly $W_\Lambda$-stable, Lemma 3.29 implies that it is also $W$-stable. This proves the lemma.
Definition 3.31. Define

\[ \text{Irr}^\text{sp}_S(W(\Lambda)) := \{ \sigma_0 \in \text{Irr}^\text{sp}(W(\Lambda)) \mid O_{\sigma_0} \subseteq S \} \]

and

\[ \text{Irr}_S(W(\Lambda)) := \{ \sigma \in \text{Irr}(W(\Lambda)) \mid \text{there is an } \sigma_0 \in \text{Irr}^\text{sp}_S(W(\Lambda)) \text{ such that } \sigma \cong \sigma_0 \} . \]

By Lemma 3.30, \( \text{Coh}_A(K_S(g, b)) \) is a basal subrepresentation of \( \text{Coh}_A^L(K(g, b)) \).

Proposition 3.32. As a representation of \( W \times W(\Lambda) \),

\[ \text{Coh}_A(K_S(g, b)) \cong \bigoplus_{\sigma \in \text{Irr}_S(W(\Lambda))} \left( \text{Ind}^W_{W(\Lambda)} \sigma \right) \otimes \sigma. \]

Proof. Write \( B_0 \) for the set of basal elements in \( \text{Coh}_A(K_S(g, b)) \). Then it is elementary to see that we may choose a filtration

\[ B_0 \supset B_1 \supset \cdots \supset B_k = \emptyset \quad (k \in \mathbb{N}) \]

such that

- \( \langle B_i \rangle_{LR} \) is spanned by \( B_i \) \((i = 0, 1, 2, \ldots, k)\),
- \( D_i := B_i \setminus B_{i+1} \) is a double cell in \( \text{Coh}_A(K(g, b)) \) for all \( i = 0, 1, \ldots, k - 1 \), and
- for all \( i_1, i_2 \in \{0, 1, \ldots, k - 1\} \), \( \langle D_{i_1} \rangle_{LR} \subseteq \langle D_{i_2} \rangle_{LR} \) implies that \( i_1 \geq i_2 \).

Then Proposition 3.22 and Lemma 3.28 imply that

\[ \{ \sigma_{D_0}, \sigma_{D_1}, \ldots, \sigma_{D_{k-1}} \} = \text{Irr}^\text{sp}_S(W(\Lambda)). \]

Thus we have that

\[
\begin{align*}
\text{Coh}_A(K_S(g, b)) & \cong \bigoplus_{i=0}^{k-1} \langle B_i \rangle_{LR} / \langle B_{i+1} \rangle_{LR} \\
& \cong \bigoplus_{i=0}^{k-1} \text{Coh}_A(K(g, b))(D_i) \\
& \cong \bigoplus_{\sigma \in \text{Irr}_S(W(\Lambda))} \left( \text{Ind}^W_{W(\Lambda)} \sigma \right) \otimes \sigma \quad \text{(by Proposition 3.22 (c))}. 
\end{align*}
\]

Let \( \nu \in \Lambda \) be a dominant element. Recall that for a primitive ideal \( I \) of infinitesimal character \( \nu \), we have the corresponding left cell \( L_{\nu, I} \) (Section 3.5). In the rest of the section, we focus on left cells which come from maximal ideals.

Recall from the introductory section the maximal ideal \( I_{\nu} \) with the infinitesimal character \( \nu \). Write

\[ (3.22) \quad O_{\nu} \in \text{Nil}(g^*) \]

for the nilpotent orbit whose Zariski closure \( O_{\nu} \subseteq g^* \) equals the associated variety of \( I_{\nu} \). Note that in the notation of Proposition 3.26, \( O_{\nu} = O_{\sigma_{\nu, I_{\nu}}} \).

Recall also the \( J \)-induction defined in [Lus82, § 11] (see also [Lus84, (4.1.7)]).

Proposition 3.33 (Barbasch-Vogan). Let \( \nu \in \Lambda \) be a dominant element. As \( W \)-representations,

\[ \text{Coh}_A^K(g, b)(L_{\nu, I_{\nu}}) \cong \text{Ind}^W_{W(\Lambda)} \left( \left( J^{W(\Lambda)}_{W_{\nu}} \text{sgn} \right) \otimes \text{sgn} \right) \]

and as \( W(\Lambda) \)-representations,

\[ (3.23) \quad \left( J^{W(\Lambda)}_{W_{\nu}} \text{sgn} \right) \otimes \text{sgn} \cong \bigoplus_{\sigma \in \text{Irr}_{W_{\nu}}(W(\Lambda)), |1_{W_{\nu}} \cdot \sigma| \neq 0} \sigma. \]

Moreover,

\[ \sigma_{\nu, I_{\nu}} \cong \left( J^{W(\Lambda)}_{W_{\nu}} \text{sgn} \right) \otimes \text{sgn}, \]

and \( \sigma_{\nu, I_{\nu}} \) is the unique special irreducible representation of \( W(\Lambda) \) that occurs in (3.23).
Proof. In view of Proposition 3.22 (b) and Lemma 3.29, this follows by the same line of proof as [BV85, Corollary 5.30]. □

The isomorphism (3.23) and explicit formulas of the J-induction (see [Lus84, §4.4-4.13]) imply that

\[ [1_{W_{\nu}} : \sigma] \leq 1 \]

for all dominant \( \nu \in \Lambda \) and all \( \sigma \in \text{Irr}_{W_{\nu}}(W(\Lambda)) \).

**Definition 3.34.** We call the set

\[ L_{C_{\nu}} := \left\{ \sigma \in \text{Irr}(W(\Lambda)) \mid \sigma \text{ occurs in } \left( J_{W_{\nu}}^{W(\Lambda)} \text{sgn} \right) \otimes \text{sgn} \right\} \]

the Lusztig left cell attached to \( \nu \in \Lambda \) (which is not necessarily dominant).

**Lemma 3.35.** Let \( \nu \in \Lambda \) be a dominant element and let \( I \) be a primitive ideal of \( U(g) \) of infinitesimal character \( \nu \). Then \( I = I_{\nu} \) if and only if \( \sigma_{\nu,I} \sim \left( J_{W_{\nu}}^{W(\Lambda)} \text{sgn} \right) \otimes \text{sgn} \).

**Proof.** The “only if” part follows from Proposition 3.33. For the proof of the “if” part, note that \( I \subseteq I_{\nu} \). Then \( \sigma_{\nu,I} \sim \sigma_{\nu,I_{\nu}} \) implies that \( I \) and \( I_{\nu} \) have the same associated variety by Proposition 3.26. Thus \( I = I_{\nu} \) by the maximality of \( I_{\nu} \) and the proof is complete. □

3.8. \( \tau \)-invariants and duals of double cells. Let \( \Pi(\Lambda) \) denote the set of simple reflections in \( W(\Lambda) \). For each subset \( S \subseteq \Pi(\Lambda) \), define an element

\[ e_S := \frac{1}{\sharp(W_{\Pi(\Lambda) \setminus S}) \cdot \sharp(W_S)} \sum_{g \in W_{\Pi(\Lambda) \setminus S}} \sum_{h \in W_S} \text{sgn}(h) gh \]

in the group algebra \( \mathbb{C}[W(\Lambda)] \), where \( W_S \) and \( W_{\Pi(\Lambda) \setminus S} \) are the subgroups of \( W(\Lambda) \) generated by \( S \) and \( \Pi(\Lambda) \setminus S \) respectively.

Following [FJMN21], for every finite-dimensional representation \( \sigma \) of \( W(\Lambda) \), define its \( \tau \)-invariant to be the set

\[ \tau_{\sigma} := \{ S \subseteq \Pi(\Lambda) \mid \text{the } e_S \text{ action on } \sigma \text{ is nonzero} \}. \]

For every double cell \( D \) in \( \text{Coh}_{LR}(K(g,b)) \), we define its \( \tau \)-invariant to be the set

\[ \tau_D := \{ \tau_{\Psi} \mid \Psi \in D \}. \]

Recall that we have a representation \( \sigma_D \in \text{Irr}^{sp}(W(\Lambda)) \) attached to \( D \) (Definition 3.23).

**Lemma 3.36.** For every double cell \( D \) in \( \text{Coh}_{LR}(K(g,b)) \), \( \tau_D = \tau_{\sigma_D} \).

**Proof.** Let \( M \) denote the cell representation \( \text{Coh}_{LR}(K(g,b))(D) \) to be viewed as a \( W(\Lambda) \) representation. By [FJMN21, Theorem 2.10], we know that \( \tau_D = \tau_M \). Since \( \sigma_D \) occurs in \( M \), we have that

\[ \tau_{\sigma_D} \subseteq \tau_M = \tau_D. \]

Now it remains to show that \( \tau_D \subseteq \tau_{\sigma_D} \). Let \( \Psi \in D \) and put \( S := \tau_{\Psi} \). Write

\[ e_S = Q(S') \cdot R(S), \]

where \( S' := \Pi(\Lambda) \setminus S \),

\[ R(S) := \frac{1}{\sharp(W_S)} \sum_{h \in W_S} \text{sgn}(h) h \]

and

\[ Q(S') := \frac{1}{\sharp(W_{S'})} \sum_{g \in W_{S'}} g. \]

For each \( \Phi \in D \), write \([\Phi] \in M \) for the class represented by \( \Phi \). Then

\[ R(S) : [\Psi] = [\Psi], \]
and [FJMN21, Proof of Proposition 2.6] implies that
\[ Q(S') \cdot [\Psi] = [\Psi] + \sum_{\Psi' \in D, \tau_{\Psi'} \not\subseteq S} c_{\Psi'}[\Psi'], \]
where \( c_{\Psi'} \in \mathbb{C} \). Thus
\[ (3.26) \quad e_S \cdot [\Psi] = [\Psi] + \sum_{\Psi' \in D, \tau_{\Psi'} \not\subseteq S} c_{\Psi'}[\Psi']. \]

Through Proposition 3.22, we view \( \sigma_D \) as a subspace of \( S^{m_D}(\alpha_{\mathfrak{h}}) \). The same proposition also implies that there is a \( W(\Lambda) \)-homomorphism
\[ \phi : M \rightarrow \sigma_D \]
such that \( \phi([\Phi]) \) is a nonzero scalar multiple of \( p_{\mathcal{L}_\Phi} \) for all \( \Phi \in D \), where \( \mathcal{L}_\Phi \) is the left cell containing \( \Phi \). Applying \( \phi \) to the quality (3.26), we obtain that
\[ e_S \cdot p_{\mathcal{L}_\Phi} = p_{\mathcal{L}_\Phi} + \sum_{\Psi' \in D, \tau_{\Psi'} \not\subseteq S} c'_{\Psi'}p_{\mathcal{L}_{\Psi'}}, \]
where \( c'_{\Psi'} \in \mathbb{C} \). This is nonzero by Lemma 3.12 and Proposition 3.24, and by noting that \( \mathcal{L}_\Phi \neq \mathcal{L}_{\Psi'} \) for all \( \Psi' \in D \) with \( \tau_{\Psi'} \not\subseteq S \). Therefore \( S \in \tau_{\sigma_D} \) and the proof is complete. \( \square \)

**Proposition 3.37.** Let \( \sigma_1, \sigma_2 \in \text{Irr}^{sp}(W(\Lambda)) \). If \( \tau_{\sigma_1} = \tau_{\sigma_2} \), then \( \sigma_1 = \sigma_2 \).

**Proof.** Clearly it suffices to consider the case when the root system \( \Delta(\Lambda) \) is irreducible. The lemma follows from Lemma 3.36 and the results of [FJMN21] (Theorem 2.12 for classical groups and the discussion in § 6 for exceptional groups). \( \square \)

Let \((\mathfrak{g}, \mathfrak{b}, \Lambda)\) be a triple with the same property as the triple \((\mathfrak{g}, \mathfrak{b}, \Lambda)\). Let \( \bar{W} \) denote the Weyl group of \( \bar{\mathfrak{g}} \). As before we have the basal space \( \text{Coh}_\Lambda(\mathcal{K}(\bar{\mathfrak{g}}, \mathfrak{b})) \) carrying a representation of \( \bar{W} \times \bar{W}(\Lambda) \), where \( \bar{W}(\Lambda) \subseteq \bar{W} \) is the integral Weyl group attached to \( \Lambda \).

Suppose we are given an identification \( W(\Lambda) = \bar{W}(\Lambda) \) of Coxeter groups, namely a group isomorphism that sends simple reflections to simple reflections. We say that two subsets \( \tau_1 \) and \( \tau_2 \) of the power set of \( \Pi(\Lambda) \) are dual to each other if
\[ \tau_2 = \{ \Pi(\Lambda) \setminus S \mid S \in \tau_1 \}. \]

**Proposition 3.38.** We are given an identification \( W(\Lambda) = \bar{W}(\Lambda) \) of Coxeter groups. For every double cell \( D \) in \( \text{Coh}_\Lambda(\mathcal{K}(\bar{\mathfrak{g}}, \mathfrak{b})) \), there is a unique double cell \( \bar{D} \) in \( \text{Coh}_\Lambda(\mathcal{K}(\bar{\mathfrak{g}}, \mathfrak{b})) \) whose \( \tau \)-invariant is dual to that of \( D \). Moreover, the map
\[ \{ \sigma \in \text{Irr}(W(\Lambda)) \mid \sigma \approx \sigma_D \} \rightarrow \{ \bar{\sigma} \in \text{Irr}(\bar{W}(\Lambda)) \mid \bar{\sigma} \approx \sigma_{\bar{D}} \} \]
is well-defined and bijective.

**Proof.** Let \( \mathcal{C} := \{ w \in W(\Lambda) \mid \bar{\Psi}_w \in D \} \). Then \( \mathcal{C} \) is a double cell of \( W(\Lambda) \) in the sense of Kazhdan-Lusztig ([KL79]). By [BV83a, Corollary 2.23], \( D \) has the form
\[ \{ \bar{\Psi}_{w'} \mid w' \in D(\Lambda), w \in \mathcal{C} \} \quad \text{(see (3.8) for the definition of } D(\Lambda)). \]
Let \( w_0 \) be the element in \( W(\Lambda) \) such that \( w_0(\Delta^+(\Lambda)) \cap \Delta^+(\Lambda) = \emptyset \). Similar to \( D(\Lambda) \), we have a subgroup \( \bar{D}(\Lambda) \) of \( \bar{W} \).

By [BV83a, Corollary 2.24],
\[ \bar{D} := \{ \bar{\Psi}_{w'w_0w} \mid w' \in \bar{D}(\Lambda), w \in \mathcal{C} \} \]
is a double cell in \( \text{Coh}_\Lambda(\mathcal{K}(\bar{\mathfrak{g}}, \mathfrak{b}, \Lambda)) \), where \( \bar{\Psi}_{w'w_0w} \) is a basal element of \( \text{Coh}_\Lambda(\mathcal{K}(\bar{\mathfrak{g}}, \mathfrak{b})) \) defined as in (3.5). By Lemma 3.13, we have that
\[ \tau_{\bar{\Psi}_{w'w}} = \tau_{\bar{\Psi}_w} = \Pi(\Lambda) \setminus \tau_{\bar{\Psi}_w}, \quad \tau_{\bar{\Psi}_{w'w_0}} = \Pi(\Lambda) \setminus \tau_{\bar{\Psi}_{w'w_0}} \]
for each \( w' \in D(\Lambda), w'' \in \bar{D}(\Lambda) \) and \( w \in W(\Lambda) \). This implies that \( \tau_D \) is dual to \( \tau_{\bar{D}} \).

The uniqueness of \( \bar{D} \) follows from Proposition 3.37 and the bijection between double cells and special representations as in Proposition 3.22 (b). See also [FJMN21, § 6].
The assertion on the operation of tensoring with sgn is in [BV83a, Proposition 2.25]. □

The double cell $D$ in Proposition 3.38 will be referred as the dual (in $\text{Coh}_A(K(\mathfrak{g}, \mathfrak{h}))$) of $D$.

4. Generalities on coherent families of Casselman-Wallach representations

Coherent families of group representations were introduced by Schmid [Sch77]. See also [Zuc77] and [SV80]. We refer the reader to [Vog81, Chapter 7] as a general reference for coherent families in this setting.

Let $G$ be a real reductive group in Harish-Chandra’s class (which may be linear or non-linear). The following assumptions hold for the real Lie groups under consideration.

The main purpose of this section is to establish good control of $\text{Coh}_A(K(G))$ of $W_\Lambda$.

For every $\nu \in \Lambda$, we have the evaluation map at $\nu$:

$$\text{ev}_\nu : \text{Coh}_A(K(G)) \to K_\nu(G).$$

Theorem 4.1 (Schmid, Zuckerman). The map $\text{ev}_\nu$ is surjective for every $\nu \in \Lambda$, and it is bijective when $\nu$ is regular.

Proof. The surjectivity is due to Schmid and Zuckerman, see [Vog81, Theorem 7.2.7]. The injectivity (for $\nu$ regular) is due to Schmid, see [Vog81, Proposition 7.2.23]. □

We proceed to describe the parameter set for the coherent continuation representation.

Suppose that $H$ is a Cartan subgroup of $G$. Recall that by our convention, its complexified Lie algebra is denoted by $\mathfrak{h}$. As usual, denote by $\Delta_0 \subseteq \mathfrak{h}^*$ the root system of $\mathfrak{g}$ with respect to $\mathfrak{h}$ and $W_\mathfrak{h}$ the corresponding Weyl group. Write $t$ for the complexified Lie algebra of the unique maximal compact subgroup of $H$. A root $\alpha \in \Delta_0$ is called real if $\alpha|_t = 0$, and imaginary if $\alpha \in t$.

An imaginary root $\alpha \in \Delta_0$ is said to be compact if the root spaces $\mathfrak{g}_\alpha$ and $\mathfrak{g}_{-\alpha}$ are contained in the complexified Lie algebra of a common compact subgroup of $G$. 


Note that every Casselman-Wallach representation of $H$ is finite dimensional. Since $G$ is in Harish-Chandra's class, for every $\Gamma \in \operatorname{Irr}(H)$, there is a unique element $d\Gamma \in \mathfrak{h}^*$ such that the differential of $\Gamma$ is isomorphic to a direct sum of one-dimensional representations of $\mathfrak{h}$ attached to $d\Gamma$.

For every Borel subalgebra $\mathfrak{b}$ of $\mathfrak{g}$ containing $\mathfrak{h}$, write

$$
\xi_\mathfrak{b} : a\mathfrak{h} \rightarrow \mathfrak{h}
$$

for the linear isomorphism attached to $\mathfrak{b}$, namely the inverse of the composition of

$$
\mathfrak{h} \subseteq \mathfrak{b} \rightarrow \mathfrak{b}/[\mathfrak{b}, \mathfrak{b}] = a\mathfrak{h}.
$$

The transpose inverse of the map (4.1) is still denoted by $\xi_\mathfrak{b} : a\mathfrak{h}^* \rightarrow \mathfrak{h}^*$. Write

$$
W(a\mathfrak{h}^*, \mathfrak{h}^*) := \{\xi_\mathfrak{b} : a\mathfrak{h}^* \rightarrow \mathfrak{h}^* \mid \mathfrak{b} \text{ is a Borel subalgebra of } \mathfrak{g} \text{ containing } \mathfrak{h}\}.
$$

Recall that $\Delta^+ \subseteq a\mathfrak{h}^*$ denotes the set of positive roots. For every element $\xi \in W(a\mathfrak{h}^*, \mathfrak{h}^*)$, put

$$
(4.2) \quad \delta(\xi) := \frac{1}{2} \sum_{\alpha \text{ is an imaginary root in } \xi \Delta^+} \alpha - \sum_{\beta \text{ is a compact imaginary root in } \xi \Delta^+} \beta \in \mathfrak{h}^*.
$$

**Definition 4.2.** Write $\mathcal{P}_\Lambda(G)$ for the set of all triples $\gamma = (H, \xi, \Gamma)$, where $H$ is a Cartan subgroup of $G$, $\xi \in W(a\mathfrak{h}^*, \mathfrak{h}^*)$, and

$$
\Gamma : \Lambda \rightarrow \operatorname{Irr}(H), \quad \nu \mapsto \Gamma_\nu
$$

is a map with the following properties:

- $\Gamma_{\nu+\beta} = \Gamma_\nu \otimes \xi(\beta)$ for all $\beta \in Q$ and $\nu \in \Lambda$;
- $d\Gamma_\nu = \xi(\nu) + \delta(\xi)$ for all $\nu \in \Lambda$.

Here $\xi(\beta)$ is naturally viewed as a character of $H$ by using the homomorphism $\iota : H \rightarrow H_\mathbb{C}$, and $H_\mathbb{C}$ is the Cartan subgroup of $G_\mathbb{C}$ containing $\iota(H)$.

The group $G$ acts on $\mathcal{P}_\Lambda(G)$ in the the standard way, and we define the set of parameters for $\operatorname{Coh}_\Lambda(K(G))$ to be

$$
(4.3) \quad \mathcal{P}_\Lambda(G) := G \setminus \mathcal{P}_\Lambda(G).
$$

For each $\gamma \in \mathcal{P}_\Lambda(G)$ that is represented by $\gamma = (H, \xi, \Gamma)$, by [Vog81, Theorem 8.2.1], we have two $K(G)$-valued coherent families $\Psi_\gamma$ and $\overline{\Psi}_\gamma$ on $\Lambda$ such that

$$
(4.4) \quad \Psi_\gamma(\nu) = X(\Gamma_\nu, \xi(\nu)) \quad \text{and} \quad \overline{\Psi}_\gamma(\nu) = \overline{X}(\Gamma_\nu, \xi(\nu))
$$

for all regular dominant element $\nu \in \Lambda$. Here $X(\Gamma_\nu, \xi(\nu))$ is the standard representation defined in [Vog81, Notational Convention 6.6.3] and $\overline{X}(\Gamma_\nu, \xi(\nu))$ is its unique irreducible subrepresentation (see [Vog81, Theorem 6.5.12]).

By Langlands classification, $\{\overline{\Psi}_\gamma\}_{\gamma \in \mathcal{P}_\Lambda(G)}$ is a basis of $\operatorname{Coh}_\Lambda(K(G))$ ([Vog81, Theorem 6.6.14]), and we view $\operatorname{Coh}_\Lambda(K(G))$ as a basal representation of $W_\Lambda$ with this basis. The family $\{\Psi_\gamma\}_{\gamma \in \mathcal{P}_\Lambda(G)}$ is also a basis of $\operatorname{Coh}_\Lambda(K(G))$ ([Vog81, Proposition 6.6.7]).

**Remark 4.3.** For every dominant element $\nu \in \Lambda$, the set

$$
\{\gamma \in \mathcal{P}_\Lambda(G) \mid \overline{\Psi}_\gamma(\nu) \neq 0\}
$$

is identified with the set of final characters for $G$ with infinitesimal character $\nu$, by sending the class of $\gamma = (H, \xi, \Gamma) \in \mathcal{P}_\Lambda(G)$ to the $G$-conjugacy class of

$$
(H, \text{the set of imaginary roots in } \xi(\Delta^+), \Gamma_\nu).
$$

See [ABV91, Chapter 11].

The coherent continuation representation $\operatorname{Coh}_\Lambda(K(G))$ is independent of $Q$ in the sense of the following lemma.

**Lemma 4.4.** For a $Q_\mathfrak{g}$-coset $\Lambda_\mathfrak{g}$ in $\Lambda$, form the coherent continuation representation $\operatorname{Coh}_\Lambda_\mathfrak{g}(K(G))$ by using the adjoint representation $G \rightarrow \operatorname{Ad}(\mathfrak{g})$. Then the restriction yields a linear isomorphism

$$
\operatorname{Coh}_\Lambda(K(G)) \cong \operatorname{Coh}_\Lambda_\mathfrak{g}(K(G)).
$$
Proof. This is a consequence of the fact that the restriction yields a bijective map
\[ \mathcal{P}_\Lambda(G) \to \mathcal{P}_\Lambda^b(G). \]

The cross action of \( W_\Lambda \) on the set \( \mathcal{P}_\Lambda(G) \) is defined by ([Vog82, Definition 4.2]):
\[ w \times (H, \xi, \Gamma) = (H, \xi w^{-1}, (\nu \mapsto \Gamma_v \otimes (\xi w^{-1} \nu + \delta(\xi w^{-1}) - \xi \nu - \delta(\xi))). \]
This commutes with the action of \( G \) and thus descends to an action on \( \mathcal{P}_\Lambda(G) \):
\[ (W_\Lambda \times \mathcal{P}_\Lambda(G)) \to \mathcal{P}_\Lambda(G), \quad (w, \gamma) \mapsto w \times \gamma. \]

Denote by \( W_{b,t} \) the stabilizer of the space \( t \) in the Weyl group \( W_h \), whose detailed structure is analyzed in [Vog82, Section 3]. Write \( \Delta_{b,im} \) for the set of imaginary roots in \( \Delta_h \), which is a root system. The corresponding Weyl group is denoted by \( W(\Delta_{b,im}) \), which is identified with a normal subgroup of \( W_{b,t} \). Then there is a unique quadratic character
\[ \text{sgn}_{im} : W_{b,t} \to \mathbb{C}^\times \]
such that
- its restriction to \( W(\Delta_{b,im}) \) equals the sign character, and
- its restriction to \( W_{b,t,\Delta_{b,im}}^+ \) is trivial for some (and hence all) positive system \( \Delta_{b,im}^+ \) of \( \Delta_{b,im} \), where \( W_{b,t,\Delta_{b,im}}^+ \) denotes the stabilizer of the set \( \Delta_{b,im}^+ \) in \( W_{b,t} \).

Note that with the notation as above, we have that
\[ W_{b,t} = W_{b,t,\Delta_{b,im}}^+ \times W(\Delta_{b,im}). \]

Since \( G \) is in Harish-Chandra’s class, the real Weyl group
\[ W_H := (\text{the normalizer of } H \text{ in } G)/H \]
is identified with a subgroup of \( W_{b,t} \). Choose a representative \( \gamma = (H, \xi, \Gamma) \) for an element \( \gamma \in \mathcal{P}_\Lambda(G) \). Write \( W_{\gamma} \) for the stabilizer of \( \gamma \) in \( W_\Lambda \) under the cross action. Then for any \( w \in W_{\gamma} \), \( \xi w \xi^{-1} \in W_H \subseteq W_{b,t} \), and we have a quadratic character
\[ \text{sgn}_\gamma : W_{\gamma} \to \mathbb{C}^\times, \quad w \mapsto \text{sgn}_{im}(\xi w \xi^{-1}). \]
This quadratic character is independent of the representative \( \gamma \).

The coherent continuation representation may be computed by using the basis of standard modules \( \{ \Psi_\gamma \}_{\gamma \in \mathcal{P}_\Lambda(G)} \) (see [Vog82, Section 14]). The following result is due to Barbasch-Vogan, in a suitably modified form from [BV83b, Proposition 2.4]. As its proof follows the same line as that of [BV83b, Proposition 2.4], we just state the precise result.

**Theorem 4.5 (cf. [BV83b, Proposition 2.4]).** As a representation of \( W_\Lambda \),
\[ \text{Coh}_\Lambda(\mathcal{K}(G)) \cong \bigoplus_{\gamma} \text{Ind}_{W_\gamma}^{W_{b,t}} \text{sgn}_\gamma, \]
where \( \gamma \) runs over a representative set of the \( W_\Lambda \)-orbits in \( \mathcal{P}_\Lambda(G) \) under the cross action.

### 4.2. Some properties of the coherent continuation representation \( \text{Coh}_\Lambda(\mathcal{K}(G)) \)
We present some properties of the coherent continuation representation as a basal representation.

**Lemma 4.6.** Let \( \Psi \) be a basal element in \( \text{Coh}_\Lambda(\mathcal{K}(G)) \). Then there is a unique left cell \( \mathcal{L}_\Psi \) in \( \text{Coh}_\Lambda(\mathcal{K}(g, b)) \) such that \( \text{Ann}(\Psi(\nu)) = I_{\nu, \mathcal{L}_\Psi} \) for every dominant element \( \nu \in \Lambda \).

**Proof.** Since a coherent family restricted to the dominant cone can be constructed via the translation functor, the proposition follows from [Vog79b, Lemma 2.7 and Lemma 2.8].

For every basal element \( \Psi \in \text{Coh}_\Lambda(\mathcal{K}(G)) \), define its \( \tau \)-invariant to be
\[ \tau_\Psi := \tau_{\mathcal{L}_\Psi}, \]
where \( \mathcal{L}_\Psi \) is as in Lemma 4.6. A basic fact about \( \tau_\Psi \) is the following: For each simple reflection \( s \) in \( W(\Lambda) \), \( s \in \tau_\Psi \) if and only if \( s \cdot \Psi = -\Psi \). See [Vog79a, Theorem 2.4] and [Vog81, Corollary 7.3.9].
\textbf{Lemma 4.7 ([Vog81, Corollary 7.3.23]).} Let \( \nu \in \Lambda \) be a dominant element. Then for every basal element \( \Psi \in \text{Coh}_\Lambda(K(G)) \), \( \Psi(\nu) \) is
\[
\begin{cases}
\text{irreducible}, & \text{if } \tau_\Psi \cap W_\nu = \emptyset; \\
0, & \text{otherwise}.
\end{cases}
\]

Let \( \mathcal{D} \) be a double cell in \( \text{Coh}_\Lambda(K(\mathfrak{g}, \mathfrak{b})) \). Denote by
\[
\text{Coh}_{\mathcal{D}}(K(G))
\]
the basal subspace of \( \text{Coh}_\Lambda(K(G)) \) spanned by all the basal elements \( \Psi \) such that \( \mathcal{L}_\Psi \subseteq \langle \mathcal{D} \rangle_{LR} \). Recall that \( \langle \mathcal{D} \rangle_{LR} \) is the smallest basal subrepresentation of \( \text{Coh}_\Lambda^{LR}(K(\mathfrak{g}, \mathfrak{b})) \) containing \( \mathcal{D} \).

The following lemma should be known in the expert community and it follows from the properties of tensoring with finite dimensional representations and the containment of annihilators. We include a proof for the sake of completeness.

\textbf{Lemma 4.8.} For every double cell \( \mathcal{D} \) in \( \text{Coh}_\Lambda(K(\mathfrak{g}, \mathfrak{b})) \), the space \( \text{Coh}_{\mathcal{D}}(K(G)) \) is a basal \( W(\Lambda) \)-subrepresentation of \( \text{Coh}_\Lambda(K(G)) \).

\textit{Proof.} By passing to a suitable covering group, we assume without loss of generality that the derived group of \( G_C \) is simply connected.

Let \( \Psi \) be a basal element in \( \text{Coh}_{\mathcal{D}}(K(G)) \), and let \( \alpha \) be a simple root in \( \Delta(\Lambda) \). We need to show that \( s_\alpha \cdot \Psi \in \text{Coh}_{\mathcal{D}}(K(G)) \), where \( s_\alpha \in W(\Lambda) \) is the simple reflection associated to \( \alpha \).

Our assumption of simply connectedness implies that there is a regular dominant element \( \mu \in \Lambda \) such that \( \langle \mu, \check{\alpha} \rangle \in 2\mathbb{Z} \). Put
\[
\nu := \mu - \frac{\langle \mu, \check{\alpha} \rangle}{2} \alpha \in \Lambda,
\]
which is a dominant element such that \( W_\nu = \{1, s_\alpha\} \).

If \( \Psi(\nu) = 0 \), then \( s_\alpha \cdot \Psi = -\Psi \in \text{Coh}_{\mathcal{D}}(K(G)) \). Now we assume that \( \Psi(\nu) \neq 0 \). Then \( \Psi(\nu) \) is irreducible.

Let \( F \) denote the irreducible representation of \( G \) with extremal weight \( \frac{1}{2} \langle \mu, \check{\alpha} \rangle \alpha \) that occurs in the symmetric power \( S(\mathfrak{g}) \). Then there are two nonzero \( G \)-homomorphisms
\[
\Psi(\nu) \xrightarrow{\iota_1} \Psi(\nu) \otimes F \xrightarrow{\iota_2} \Psi(\mu)
\]
such that \( \iota_2 \circ \iota_1 = 0 \). Moreover, \( \Psi(\mu) \) occurs in the composition series of \( \Psi(\nu) \otimes F \) with multiplicity two, and
\[
(s_\alpha \cdot \Psi)(\mu) = \text{Pr}_\mu(\Psi(\nu) \otimes F) - \Psi(\mu) \in K(G),
\]
where \( \text{Pr}_\mu \) denotes the projection map
\[
\text{Pr}_\mu : K(G) = \bigoplus_{\lambda \in W(\Lambda)_{\nu}^*} K_{\lambda}(G) \to K_{\mu}(G).
\]
See [Vog81, Theorem 7.3.16 and Corollary 7.3.18].

By (4.9), we know that \( s_\alpha \cdot \Psi \) is a positive integral linear combination of basal elements of \( \text{Coh}_\Lambda(K(G)) \). Let \( \Psi' \) be a basal element that occurs in the positive integral combination. Then
\[
\text{Ann}(\Psi'(\mu)) \supseteq \text{Ann}(\Psi(\nu) \otimes F)).
\]

Pick an element \( w \in W \) such that \( \text{Ann}(\Psi(\mu)) = \text{Ann}(\overline{\Psi}_w(\mu)) \), where \( \overline{\Psi}_w \) is defined as in (3.5). The translation principle [Vog79b, Lemma 2.7] implies that \( \text{Ann}(\Psi(\nu)) = \text{Ann}(\overline{\Psi}_w(\nu)) \). In particular, \( \overline{\Psi}_w(\nu) \) is nonzero and hence irreducible.

As in the group case, \( \overline{\Psi}_w(\mu) \) occurs as a subrepresentation of \( \overline{\Psi}_w(\nu) \otimes F \) (and also as a quotient representation), and occurs in the composition series of \( \overline{\Psi}_w(\nu) \otimes F \) with multiplicity two. Moreover,
\[
(s_\alpha \cdot \overline{\Psi}_w)(\mu) = \text{Pr}_\mu(\overline{\Psi}_w(\nu) \otimes F) - \overline{\Psi}_w(\mu) \in K(\mathfrak{g}, \mathfrak{b}),
\]
where \( \text{Pr}_\mu \) denotes the projection map analogous to (4.10).
We have an ideal $I \subseteq \mathcal{U}(g)$ that is a finite product of primitive ideals of $\mathcal{U}(g)$ with infinitesimal character distinct from $\mu$, and a sequence $w_1, w_2, \ldots, w_{k-1}, w_k = w$ in $W$ ($k \in \mathbb{N}^+$) such that

$$\operatorname{Ann}(\overline{\Psi}(\nu) \otimes F) \supseteq \operatorname{Ann}(\overline{\Psi}(\mu)) \cdot \operatorname{Ann}(\overline{\Psi}(\mu)) \cdot \operatorname{Ann}(\overline{\Psi}(\mu)) \cdot \cdots \cdot \operatorname{Ann}(\overline{\Psi}(\mu)) \cdot I,$$

and

$$\tag{4.12} (s_\alpha \cdot \overline{\Psi}) = \overline{\Psi}_1 + \overline{\Psi}_2 + \cdots + \overline{\Psi}_k \in \mathcal{K}(g, b).$$

Then

$$\operatorname{Ann}(\overline{\Psi}(\mu)) \supseteq \operatorname{Ann}(\overline{\Psi}(\mu)) \cdot \operatorname{Ann}(\overline{\Psi}(\mu)) \cdot \operatorname{Ann}(\overline{\Psi}(\mu)) \cdot \cdots \cdot \operatorname{Ann}(\overline{\Psi}(\mu)) \cdot I.$$ 

Since the primitive ideal $\operatorname{Ann}(\overline{\Psi}(\mu))$ is prime, the above implies that $\operatorname{Ann}(\overline{\Psi}(\mu)) \supseteq \operatorname{Ann}(\overline{\Psi}(\mu))$ for some $i \in \{1, 2, \ldots, k\}$. Hence $\mathcal{L}_\Psi \subseteq (\overline{\Psi}_i) \mathcal{L}$ by Lemma 3.19. On the other hand, $\overline{\Psi}_i$ is in $(\overline{\Psi})_R$ by (4.12). We conclude that $\overline{\Psi} \in \mathcal{Coh}_{\Lambda,D}(\mathcal{K}(G))$. The proof is complete. \qed

4.3. The coherent continuation representation $\mathcal{Coh}_\Lambda(\mathcal{K}(g, H, b'))$. Let $H$ be a Cartan subgroup of $G$. Since a Cartan subgroup can be disconnected, we need a mild generalization of the category of $b$-modules considered earlier. Let $b'$ be a Borel subalgebra of $g$ containing the complexified Lie algebra $h$ of $H$. (In the sequel, we will need to use several Cartan subgroups. We shall reserve the notation $b$ for a fixed Borel subalgebra of $g$ as in Section 3.2.) Recall that a $(g, H)$-module is defined to be a $g$-module $V$ together with a locally-finite representation of $H$ on it such that

- $h \cdot (X \cdot (h^{-1} \cdot u)) = (\operatorname{Ad}_h(X)) \cdot u$, for all $h \in H, X \in \mathcal{U}(g), u \in V$ (Ad stands for the Adjoint representation);
- the differential of the representation of $H$ and the restriction of the representation of $g$ yields the same representation of $h$ on $V$.

Let $\mathcal{R}_\Lambda(g, H, b')$ denote the category of finitely generated $(g, H)$-modules that are unions of finite-dimensional $b'$-submodules. As before, we form the Grothendieck group $\mathcal{K}(g, H, b')$, and the coherent continuation representation $\mathcal{Coh}_\Lambda(\mathcal{K}(g, H, b'))$ of $\mathcal{W}(\Lambda)$.

Write $\mathcal{R}_\Lambda(g, H, b')$ for the set of all pairs $(\nu, \Gamma)$ where $w \in W$, and

$$\Gamma : \Lambda \to \operatorname{Irr}(H), \quad \nu \mapsto \Gamma_\nu,$$

is a map such that

- $\Gamma_{\nu + \beta} = \Gamma_\nu \otimes \xi_\nu(\beta)$ for all $\beta \in Q$ and $\nu \in \Lambda$;
- $d\Gamma_\nu = \xi_\nu(wv - \rho)$ for all $\nu \in \Lambda$.

Here $\xi_\nu$ is defined in (4.1), $\xi_\nu(\beta)$ is viewed as a character of $H$ by pulling-back through the homomorphism $\iota : H \to H_C$, and $H_C$ is the Cartan subgroup of $G_C$ containing $\iota(H)$.

For each $\sigma \in \operatorname{Irr}(H)$, put

$$M(\sigma) := \mathcal{U}(g) \otimes_{\mathcal{U}(b')} \sigma,$$

which is a module in $\mathcal{R}(g, H, b')$, where the $\mathcal{U}(g)$-action is given by the left multiplication, and the $H$-action is given by

$$h \cdot (X \otimes u) := \operatorname{Ad}_h(X) \otimes h \cdot u, \quad h \in H, X \in \mathcal{U}(g), u \in \sigma.$$

As before, the $(g, H)$-module $M(\sigma)$ has a unique irreducible quotient, to be denoted by $L(\sigma)$.

For each $(w, \Gamma) \in \mathcal{R}_\Lambda(g, H, b')$, we have two $\mathcal{K}(g, H, b')$-valued coherent families $\Psi_{w, \Gamma}$ and $\overline{\Psi}_{w, \Gamma}$ on $\Lambda$ such that

$$\tag{4.13} \Psi_{w, \Gamma}(\nu) = M(\Gamma_\nu) \quad \text{for all } \nu \in \Lambda,$$

and

$$\tag{4.14} \overline{\Psi}_{w, \Gamma}(\nu) = L(\Gamma_\nu) \quad \text{for all regular dominant element } \nu \in \Lambda.$$

As before, both $\{\Psi_{w, \Gamma} \mid (w, \Gamma) \in \mathcal{R}_\Lambda(g, H, b')\}$ and $\{\overline{\Psi}_{w, \Gamma} \mid (w, \Gamma) \in \mathcal{R}_\Lambda(g, H, b')\}$ are bases of $\mathcal{Coh}_\Lambda(\mathcal{K}(g, H, b'))$.

We view $\mathcal{Coh}_\Lambda(\mathcal{K}(g, H, b'))$ as a basis representation of $\mathcal{W}(\Lambda)$ by using the second basis.

The analog of Lemma 4.6 for $\mathcal{Coh}_\Lambda(\mathcal{K}(g, H, b'))$ still holds (with the same proof), and we define similarly the left cell $\mathcal{L}_\Psi$ in $\mathcal{Coh}_\Lambda(\mathcal{K}(g, b))$ for every basal element $\Psi(= \overline{\Psi}_{w, \Gamma}) \in \mathcal{Coh}_\Lambda(\mathcal{K}(g, H, b'))$. 

Let $\mathcal{D}$ be a double cell in $\text{Coh}_\Lambda(\mathcal{K}(\mathfrak{g}, \mathfrak{b}))$ as before. Define

$$\text{Coh}_{\Lambda, \mathcal{D}}(\mathcal{K}(\mathfrak{g}, H, b'))$$

to be the basal subspace of $\text{Coh}_\Lambda(\mathcal{K}(\mathfrak{g}, H, b'))$ spanned by all the basal elements $\Psi$ such that $L_\Psi \subseteq \langle \mathcal{D} \rangle_{LR}$. By the same argument as that of Lemma 4.8, the space $\text{Coh}_{\Lambda, \mathcal{D}}(\mathcal{K}(\mathfrak{g}, H, b'))$ is a basal $W(\Lambda)$-subrepresentation of $\text{Coh}_\Lambda(\mathcal{K}(\mathfrak{g}, H, b'))$.

**Lemma 4.9.** The representation $\text{Coh}_{\Lambda, \mathcal{D}}(\mathcal{K}(\mathfrak{g}, H, b'))$ of $W(\Lambda)$ is isomorphic to a subrepresentation of $(\langle \mathcal{D} \rangle_{LR})^k$, for some $k \in \mathbb{N}$.

**Proof.** Let $Q$ act on the set $\text{Irr}(H)$ by

$$\beta \cdot \sigma = \sigma \otimes \xi(\beta), \quad \beta \in Q, \quad \sigma \in \text{Irr}(H),$$

where $\xi(\beta)$ is as in (4.1). For each $Q$-orbit $\Omega \subseteq \text{Irr}(H)$, let $\text{Rep}_\Omega(\mathfrak{g}, H, b')$ be the full subcategory of $\text{Rep}(\mathfrak{g}, H, b')$ whose objects are modules $V$ such that every irreducible subquotient of $V|_H$ (viewed as a representation of $H$) belongs to $\Omega$. Write $K_{\Omega, g}(H, b')$ for the Grothendieck group of the category $\text{Rep}_\Omega(\mathfrak{g}, H, b')$. Then $\text{Coh}_\Lambda(K_{\Omega, g}(H, b'))$ is a basal $W(\Lambda)$-representation. As $W(\Lambda)$-representations,

$$\text{Coh}_{\Lambda, \mathcal{D}}(\mathcal{K}(\mathfrak{g}, H, b')) = \bigoplus_{i=1}^k \text{Coh}_{\Lambda, \mathcal{D}}(\mathcal{K}_{\Omega_i}(\mathfrak{g}, H, b')),$$

where $\Omega_1, \Omega_2, \cdots, \Omega_k$ ($k \in \mathbb{N}$) are certain $Q$-orbits in $\text{Irr}(H)$, and

$$\text{Coh}_{\Lambda, \mathcal{D}}(\mathcal{K}_{\Omega_i}(\mathfrak{g}, H, b')) := \text{Coh}_\Lambda(K_{\Omega_i}(\mathfrak{g}, H, b')) \cap \text{Coh}_{\Lambda, \mathcal{D}}(\mathcal{K}(\mathfrak{g}, H, b')).$$

Define a linear isomorphism

$$T_{b, b'} : \mathcal{K}(\mathfrak{g}, \mathfrak{b}) \to \mathcal{K}(\mathfrak{g}, \mathfrak{b}')$$

such that $T_{b, b'}(M(\mathfrak{g}, \mathfrak{b}, \nu)) = M(\mathfrak{g}, \mathfrak{b}', \nu)$ for all $\nu \in \mathfrak{a}^\ast$. The map $T_{b, b'}$ is $W$-equivariant and induces an isomorphism

$$\text{Coh}_\Lambda(\mathcal{K}(\mathfrak{g}, \mathfrak{b})) \to \text{Coh}_\Lambda(\mathcal{K}(\mathfrak{g}, \mathfrak{b}')).$$

By using this isomorphism we assume without loss of generality that $\mathfrak{b} = \mathfrak{b}'$.

Denote by $\mathcal{F}$ the forgetful functor from $\text{Rep}(\mathfrak{g}, H, b')$ to $\text{Rep}(\mathfrak{g}, b')$. Since $\Psi \mapsto \mathcal{F} \circ \Psi$ gives an injective $W(\Lambda)$-homomorphism from $\text{Coh}_{\Lambda, \mathcal{D}}(\mathcal{K}_{\Omega_i}(\mathfrak{g}, H, b'))$ to $(\langle \mathcal{D} \rangle_{LR})^k$, the lemma follows.

### 4.4. An embedding of coherent continuation representations.

As before $\mathcal{D}$ is a double cell in $\text{Coh}_\Lambda(\mathcal{K}(\mathfrak{g}, \mathfrak{b}))$. The purpose of this subsection is to prove the following proposition.

**Proposition 4.10.** The representation $\text{Coh}_{\Lambda, \mathcal{D}}(\mathcal{K}(G))$ of $W(\Lambda)$ is isomorphic to a subrepresentation of $(\langle \mathcal{D} \rangle_{LR})^k$, for some $k \in \mathbb{N}$.

Let $\{H_1, H_2, \cdots, H_r\}$ ($r \in \mathbb{N}^+$) be a set of representatives of the conjugacy classes of Cartan subgroups of $G$. For each $i = 1, 2, \cdots, r$, fix a Borel subalgebra $\mathfrak{b}_i$ of $\mathfrak{g}$ that contains the complexified Lie algebra of $H_i$.

**Lemma 4.11.** There is an injective $\mathcal{R}(\mathfrak{g}, Q)$-module homomorphism

$$\gamma_G : \mathcal{K}(G) \to \bigoplus_{i=1}^r \mathcal{K}(\mathfrak{g}, H_i, \mathfrak{b}_i)$$

such that

$$\gamma_G(\mathcal{K}_I(G)) \subseteq \bigoplus_{i=1}^r \mathcal{K}_I(\mathfrak{g}, H_i, \mathfrak{b}_i)$$

for every ideal $I$ of $\mathcal{U}(\mathfrak{g})$. Here $\mathcal{K}_I(G)$ is the Grothendieck group of $\text{Rep}_I(G)$, $\text{Rep}_I(G)$ is the full subcategory of $\text{Rep}(G)$ consisting of the representations that are annihilated by $I$, and $\mathcal{K}_I(\mathfrak{g}, H_i, \mathfrak{b}_i)$ is the similarly defined subspace of $\mathcal{K}(\mathfrak{g}, H_i, \mathfrak{b}_i)$. 


Proof. This follows from the work of Casian ([Cas86]). Since the proposition is not explicitly stated in [Cas86], we briefly recall the argument of Casian for the convenience of the reader.

Let $n_i$ denote the nilpotent radical of $[g, g] \cap b_i$ ($i = 1, 2, \cdots, r$). For every $q \in \mathbb{Z}$, let $\gamma^q_{n_i}$ denote the $q$-th right derived functor of the following left exact functor from the category of $g$-modules to itself:

$$V \rightarrow \{ u \in V \mid n_i^k \cdot u = 0 \text{ for some } k \in \mathbb{N}^+ \}.$$ 

Fix a Cartan involution $\theta$ of $G$ and write $K$ for its fixed point group (which is a maximal compact subgroup of $G$). Without loss of generality we assume that all $H_i$'s are $\theta$-stable.

For every Casselman-Wallach representation $V$ of $G$, write $V_{[K]}$ for the space of $K$-finite vectors in $V$, which is a $(g, K)$-module of finite length. Then $\gamma^q_{n_i}(V_{[K]})$ is naturally a representation in $\text{Rep}(g, H_i, b_i)$ ([Cas86, Corollary 4.9]).

We define a linear map

$$\gamma_G : \mathcal{K}(G) \rightarrow \bigoplus_{i=1}^{r} \mathcal{K}(g, H_i, b_i)$$

given by

$$\gamma_G(V) = \left\{ \sum_{q \in \mathbb{Z}} (-1)^q \gamma^q_{n_i}(V_{[K]}) \right\}_{i=1, 2, \ldots, r}$$

for every Casselman-Wallach representation $V$ of $G$. A form of the Osborne conjecture (see [Cas86, Theorem 3.1]) and [Cas86, Corollary 4.9]) implies that the map $\gamma_G$ is well-defined and injective.

Proposition 4.11 of [Cas86] implies that the functor $\gamma^q_{n_i}$ commutes with tensor product with finite-dimensional representations. Thus $\gamma_G$ is an $\mathcal{R}(g, Q)$-homomorphism. Finally, [Cas86, Corollary 4.15] implies that $\gamma_G$ satisfies the property in (4.15).

Proof of Proposition 4.10. Lemma 4.11 implies that the representation $\text{Coh}_{\Lambda, D}(\mathcal{K}(G))$ of $W(\Lambda)$ is isomorphic to a subrepresentation of $\bigoplus_{i=1}^{r} \text{Coh}_{\Lambda, D}(\mathcal{K}(g, H_i, b_i))$. Together with Lemma 4.9, this implies Proposition 4.10.

4.5. Harish-Chandra cells and double cells. We now discuss the consequences of Proposition 4.10. We view $\text{Coh}_\Lambda(\mathcal{K}(G))$ as a basal representation of $W(\Lambda)$. A cell in $\text{Coh}_\Lambda(\mathcal{K}(G))$ will be called a Harish-Chandra cell.

We fix a Harish-Chandra cell $C$.

Lemma 4.12. There is a unique double cell $D$ in $\text{Coh}_\Lambda(\mathcal{K}(g, b))$ such that $L_\Psi \subseteq D$ for every basal element $\Psi$ in $C$. Here $L_\Psi$ is the left cell defined in Lemma 4.6.

Proof. Suppose $\Psi_1$ and $\Psi_2$ are two basal elements in $C$. For $i = 1, 2$, let $D_{\Psi_i}$ be the double cell containing $L_{\Psi_i}$. By Lemma 4.8, we have $\langle D_{\Psi_1} \rangle_{LR} \subset \langle D_{\Psi_2} \rangle_{LR}$ since $\Psi_1 \in \langle \Psi_2 \rangle$. Since the role of $\Psi_1$ and $\Psi_2$ is symmetric, we also have $\langle D_{\Psi_2} \rangle_{LR} \subset \langle D_{\Psi_1} \rangle_{LR}$. Hence $\langle D_{\Psi_1} \rangle_{LR} = \langle D_{\Psi_2} \rangle_{LR}$. Together with Proposition 3.22 (a) and (b), it implies that $D_{\Psi_1} = D_{\Psi_2}$. This proves the existence and uniqueness of the double cell.

Definition 4.13. We call the double cell $D$ as in Lemma 4.12 the double cell attached to $C$ and $\sigma_C := \sigma_D$ the special irreducible representation of $W(\Lambda)$ attached to $C$.

The following conjecture is widely anticipated, although to our knowledge no proof has appeared in the literature. See [Vog82, page 1055].

Conjecture 4.14. Suppose that $\sigma \in \text{Irr}(W(\Lambda))$ occurs in $\text{Coh}_\Lambda(\mathcal{K}(G))(C)$. Then $\sigma \approx \sigma_C$.

The conjecture holds for complex semisimple groups (see [BV85, Section 3]). It also holds for unitary groups by [Bož02, Theorem 5]. We note McGovern’s observation ([McG98, Page 213]) which amounts to the assertion in Conjecture 4.14 for reductive linear groups in Harish-Chandra’s class. It appears to us that the argument is inadequate as presented. In what follows we will give a proof assuming a certain equality of $\tau$-invariants (Hypothesis 4.15) and a weak
form of Vogan duality (Hypothesis 4.22), both of which hold true for classical groups (including the real metaplectic group).

Define the $\tau$-invariant of $C$ to be the set

$$\tau_C := \{ \tau_{\Psi} : \Psi \in C \}.$$  

By Lemma 4.12, we have $\tau_C \subseteq \tau_D$, where $D$ is the double cell attached to $C$.

To continue our discussion, we make the following hypothesis.

**Hypothesis 4.15.** With the notation as above, $\tau_C = \tau_D$.

**Remark 4.16.** In view of Lemma 3.36, Hypothesis 4.15 will follow from [Vog82, Corollary 14.11] which asserts that the special representation $\sigma_D$ occurs in the cell representation of $C$. However, [Vog82, Corollary 14.11] implicitly assumes that the Gelfand-Kirillov dimension of $\Psi'$ (namely the Gelfand-Kirillov dimension of $\Psi'(\nu)$, where $\nu$ is an arbitrary regular dominant element in $\Lambda$) is strictly less than that of $\Psi$ for any two basal elements $\Psi'$ and $\Psi$ in $\text{Coh}_\Lambda(K(G))$ such that $\langle \Psi' \rangle \nsubseteq \langle \Psi \rangle$. We thank David Vogan for confirming that this has not been established in the literature.

**Proposition 4.17.** Suppose that the Coxeter group $W(\Lambda)$ has no simple factor of type $F_4$, $E_6$, $E_7$, or $E_8$. Then Hypothesis 4.15 holds.

We proceed to prove Proposition 4.17 by reducing it to a question on double cells, by the use of the so-called edge transport theorems (see [GJ19]).

Recall the integral root system $\Delta(\Lambda)$ from (3.7).

Let $B_\Lambda(G)$ be the set of basal elements in $\text{Coh}_\Lambda(K(G))$. For two simple roots $\alpha, \beta \in \Delta(\Lambda)$ spanning a root system of type $A_2$, $B_2$ (or $C_2$), or $G_2$, define a map

$$T_{\alpha,\beta} : B_\Lambda(G) \to \{ \text{subset of } B_\Lambda(G) \}$$

such that, for each $\Psi \in B_\Lambda(G)$,

$$T_{\alpha,\beta}(\Psi) = \begin{cases} 
\{ \Psi' \in B_\Lambda(G) : s_{\alpha} \in \tau_{\Psi'}, s_{\beta} \notin \tau_{\Psi'} \text{, and} \\
\text{the coefficient of } \Psi' \text{ in } s_{\alpha} \cdot \Psi \text{ is non-zero} \}, & \text{if } s_{\alpha} \notin \tau_{\Psi} \text{ and } s_{\beta} \in \tau_{\Psi}； \\
\emptyset, & \text{otherwise.}
\end{cases}$$

Here and as usual, $s_\alpha$ denotes the simple reflection associated to $\alpha$.

Let $L\Lambda$ denote the set of left cells in $\text{Coh}_\Lambda(K(\mathfrak{g}, \mathfrak{b}))$. For each $w \in W$, write $L_w \in L\Lambda$ for the left cell containing the basal element $\Psi_w$. The map $T_{\alpha,\beta}$ can also be defined for the category of highest weight modules (see [Vog79a, Theorem 3.2]) and descends to a map on the set of left cells (see [Vog79a, Corollary 3.6 and 3.9]). More precisely, we have a map

$$T_{\alpha,\beta}^L : L\Lambda \to \{ \text{subset of } L\Lambda \}$$

such that, for each $w \in W(\Lambda)$,

$$T_{\alpha,\beta}^L(L_w) = \begin{cases} 
\{ L_{w'} : w' \in W(\Lambda), s_{\alpha} \in \tau_{L_{w'}}, s_{\beta} \notin \tau_{L_{w'}}, \text{ and} \\
\text{the coefficient of } \Psi_{w'} \text{ in } s_{\alpha} \cdot \Psi_w \text{ is non-zero} \}, & \text{if } s_{\alpha} \notin \tau_{L_w} \text{ and } s_{\beta} \in \tau_{L_w}； \\
\emptyset, & \text{otherwise.}
\end{cases}$$

When four simple roots $\alpha_1, \alpha_2, \alpha_3, \alpha_4$ in $\Delta(\Lambda)$ span a root system of type $D_4$, one can define $D_4$-maps

$$T_{\alpha_i} : B_\Lambda(G) \to \{ \text{subset of } B_\Lambda(G) \}$$

and

$$T_{\alpha_i}^L : L\Lambda \to \{ \text{subset of } L\Lambda \} \quad (i = 1, 2, 3, 4)$$

as in [GV92, Theorem 2.15].

Let $(T, T^L)$ be a pair $(T_{\alpha,\beta}, T_{\alpha,\beta}^L)$ or $(T_{\alpha_i}, T_{\alpha_i}^L)$ as above. Then

- for each basal element $\Psi \in B_\Lambda(G)$, every element in $T(\Psi)$ is in the same Harish-Chandra cell as $\Psi$;
- for each left cell $L \in L\Lambda$, every left cell in $T^L(L)$ is contained in the same double cell as $L$;

...
Remark 4.19. For the root systems of type $F_4, E_6, E_7,$ or $E_8$, one expects similar results on double cells by including some other maps defined in a similar fashion as $T^{L}_{\alpha, \beta}$ and $T^{L}_{\alpha_i}$ as above. Let

\[ \{ \mathcal{L} \} := \bigcup_{k=1}^{\infty} \mathcal{L}_k. \]

**Proposition 4.18.** Suppose that the Coxeter group $W(\Lambda)$ has no simple factor of type $F_4, E_6, E_7,$ or $E_8$. Then for each $\mathcal{L} \in \mathcal{L}_\Lambda$, $[\mathcal{L}]$ is the set of all left cells in the double cell containing $\mathcal{L}$.

**Proof.** The proposition is easily reduced to the cases when the root system $\Delta(\Lambda)$ is simple.

The proposition is a classical result of Knuth, Joseph, Vogan, and Kazhdan-Lusztig, when $\Delta(\Lambda)$ is of type $A$ (see [KL79, p. 177] and [BV82, p. 172]), and a result of Devra Garfinkle Johnson when $\Delta(\Lambda)$ is of type $B$ or $C$ (see [Gar93, Theorem 3.2.2]), or $D$ (cf. [JG91, MP23]). (We thank Devra Garfinkle Johnson for confirming this result when $\Delta(\Lambda)$ is of type $D$.)

Note that the root system of type $D_2$ (resp. $D_3$) is isomorphic to the root system of type $A_1 \oplus A_1$ (resp. $A_3$).

Suppose the root system $\Delta(\Lambda)$ is of type $G_2$. Let $\{ \alpha, \beta \}$ be the set of simple roots. There are three double cells in total. Among the three double cells, two contain a single left cell in each, and the other is the union of two left cells. The $\tau$-invariant of the latter is $\{ \{ s_\alpha \}, \{ s_\beta \} \}$. See [Gar93, p. 412]. Clearly it suffices to consider the double cell with two left cells, and since they are interchanged by $T^{L}_{\alpha, \beta}$ (or $T^{L}_{\beta, \alpha}$), the proposition clearly holds.

Now Proposition 4.18 implies that Hypothesis 4.15 holds via (4.16) as well as the two properties just above it. This concludes the proof of Proposition 4.17.

**Remark 4.20.** We say that $(\hat{G}, \hat{G}_C, \hat{\iota} : G \to G_{C, \Lambda, C})$ is a weak Vogan dual of $(G, G_C, \iota : G \to G_C, \Lambda, C)$ if there is given an identification $W(\Lambda) = \hat{W}(\hat{\Lambda})$ of Coxeter groups such that

\[ Coh_{\Lambda}(\hat{K}(\hat{G}))(C^*) \cong (Coh_{\Lambda}(K(G))(C)) \otimes \sgn \]

as representations of $W(\Lambda) = \hat{W}(\hat{\Lambda})$.

**Lemma 4.21.** Suppose that $(\hat{G}, \hat{G}_C, \hat{\iota} : \hat{G} \to \hat{G}_{C, \Lambda, C^*})$ is a weak Vogan dual of $(G, G_C, \iota : G \to G_{C, \Lambda, C})$. Then the $\tau$-invariants $\tau_{C^*}$ and $\tau_C$ are dual to each other.

**Proof.** Write $M := Coh_{\Lambda}(K(G))(C)$, to be viewed as a representation of $W(\Lambda)$. Likewise write $M^* := Coh_{\Lambda}(K(\hat{G}))(C^*)$, to be viewed as a representation of $\hat{W}(\hat{\Lambda})$. Then by [FJMN21, Theorem 2.10],

\[ \tau_M = \tau_C, \quad \tau_{M^*} = \tau_{C^*}, \]

and $\tau_M$ is dual to $\tau_{M^*}$. Therefore the lemma follows. 

\[ \square \]
Hypothesis 4.22. The 5-tuple $(G, G_C, i : G \to G_C, \Lambda, \mathcal{C})$ has a weak Vogan dual.

Theorem 4.23. Suppose that the Coxeter group $W(\Lambda)$ has no simple factor of type $F_4$, $E_6$, $E_7$, or $E_8$. Assume that Hypothesis 4.22 holds, then for every $\sigma \in \operatorname{Irr}(W(\Lambda))$ occurring in $\operatorname{Coh}_\Lambda(\mathcal{K}(G))(\mathcal{C})$,

$$\sigma \approx \sigma_C.$$

Proof. Let $(\hat{G}, \hat{G}_C, \hat{i} : \hat{G} \to \hat{G}_C, \hat{\Lambda}, \hat{\mathcal{C}})$ be a weak Vogan dual of $(G, G_C, i : G \to G_C, \Lambda, \mathcal{C})$. Similar to $\sigma_C$, we have a special representation $\sigma_{C^*} \in \operatorname{Irr}^{\text{sp}}(\hat{\mathcal{C}})$ associated to the Harish-Chandra cell $\hat{\mathcal{C}}$.

Suppose that $\sigma \in \operatorname{Irr}(W(\Lambda))$ occurs in the cell representation $\operatorname{Coh}_\Lambda(\mathcal{K}(\hat{G}))(\hat{\mathcal{C}})$. Then Proposition 4.10 implies that

$$(4.17)\quad \sigma_{C^*} \leq \sigma_{LR}.$$ 

Since $\sigma \otimes \text{sgn}$ occurs in the cell representation $\operatorname{Coh}_\Lambda(\mathcal{K}(\hat{G}))(\hat{\mathcal{C}})$, Proposition 4.10 also implies that

$$\sigma_{C^*} \leq \sigma \otimes \text{sgn}.$$ 

Proposition 3.21 then implies that

$$(4.18)\quad \sigma \leq \sigma_{C^*} \otimes \text{sgn}.$$ 

Let $\mathcal{D}$ (resp. $\mathcal{D}^*$) be the double cell attached to $\mathcal{C}$ (resp. $\mathcal{C}^*$). Proposition 4.17 implies that

$$\tau_{\mathcal{C}} = \tau_{\mathcal{D}} \quad \text{and} \quad \tau_{\mathcal{C}^*} = \tau_{\mathcal{D}^*}.$$ 

Since $\tau_{\mathcal{C}^*}$ is dual to $\tau_{\mathcal{C}}$ by Lemma 4.21, we know that $\tau_{\mathcal{D}^*}$ is dual to $\tau_{\mathcal{D}}$. Proposition 3.38 then implies that $\mathcal{D}^* = \mathcal{D}$ and so

$$\sigma_{C^*} \otimes \text{sgn} = \sigma_{\mathcal{D}^*} \otimes \text{sgn} = \sigma_{\mathcal{D}} \otimes \text{sgn} \approx \sigma_{\mathcal{D}} = \sigma_C.$$ 

Together with (4.17) and (4.18), we finally conclude that $\sigma \approx \sigma_C$ and the theorem is proved. □

In the above proof we have used the fact that $\sigma_{\mathcal{D}} \otimes \text{sgn} \approx \sigma_{\mathcal{D}}$. We remark that $\sigma_{\mathcal{D}} \otimes \text{sgn} = \sigma_{\mathcal{D}}$ in most cases including the cases when $W(\Lambda)$ is of classical type (see [Lus84, Chapter 4]).

Since weak Vogan duals exist for linear groups ([Vog82, Corollary 14.9]) and the real metaplectic group ([RT00], [RT03, Theorem 5.2]), we obtain the following corollary.

Corollary 4.24. Suppose that the Coxeter group $W(\Lambda)$ has no simple factor of type $F_4$, $E_6$, $E_7$, or $E_8$, and $G$ is linear or isomorphic to a real metaplectic group. Then for every $\sigma \in \operatorname{Irr}(W(\Lambda))$ occurring in $\operatorname{Coh}_\Lambda(\mathcal{K}(G))(\mathcal{C})$,

$$\sigma \approx \sigma_C.$$ 

5. Counting irreducible representations

The purpose of this section is to prove two counting results on irreducible representations, valid for any $G$ satisfying Conjecture 4.14. Special versions of the counting results are stated as Theorem 2.1 and Corollary 2.2 in Section 2.2, in view of Corollary 4.24.

For a basal element $\Psi \in \operatorname{Coh}_\Lambda(\mathcal{K}(G))$, let

$$\mathcal{O}_\Psi := \mathcal{O}_{\sigma_\Psi} \in \overline{\Nil(g^*)}$$

be the nilpotent orbit corresponding to $j_W^W(\Lambda) \tau_\Psi$ (as in (3.15)). Here $\sigma_{\Psi}$ is the special irreducible representation of $W(\Lambda)$ attached to the Harish-Chandra cell containing $\Psi$, as in Definition 4.13. By Proposition 3.26 and the translation principle [Vog79b, Lemma 2.7],

$$\text{AV}(\text{Ann}(\Psi(\nu))) = \overline{\mathcal{O}_\Psi} \quad (\text{the Zariski closure of } \mathcal{O}_\Psi)$$

for all dominant $\nu \in \Lambda$ such that $\Psi(\nu) \neq 0$. Using the properties of translation outside the dominant cone as in [Vog81, Proposition 7.3.11], we conclude that

$$\Psi \in \operatorname{Coh}_\Lambda(\mathcal{K}_{\sigma_\Psi}(G)).$$
Recall that $S$ is a fixed $\text{Ad}(\mathfrak{g})$-stable Zariski closed subset of $\text{Nil}(\mathfrak{g}^*)$. Recall also that $\text{Irr}_{\nu,S}(G)$ is the set of isomorphism classes of irreducible representations in $\text{Rep}_{\nu,S}(G)$, where $\nu \in ^0\mathfrak{h}^*$. 

**Proposition 5.1** (Vogan). For every $\nu \in \Lambda$, the evaluation map
\[ \text{ev}_\nu : \text{Coh}_\Lambda(K_S(G)) \rightarrow K_{\nu,S}(G), \quad \Psi \mapsto \Psi(\nu) \]
descends to a linear isomorphism
\[ \text{Coh}_\Lambda(K_S(G))_{W_\nu} \xrightarrow{\sim} K_{\nu,S}(G), \]
where the subscript group indicates the coinvariant space. Consequently,
\[ \sharp(\text{Irr}_{\nu,S}(G)) = \dim [1_{W_\nu} : \text{Coh}_\Lambda(K_S(G))]. \]

**Proof.** Without loss of generality we assume that $\nu$ is dominant. Put
\[ B_S := \{ \Psi \mid \Psi \text{ is a basal element of } \text{Coh}_\Lambda(K(G)) \text{ such that } O_\Psi \subseteq S \}, \]
which is a basis of $\text{Coh}_\Lambda(K_S(G))$ by Lemma 4.6.

It follows that
\[
\text{ker}(\text{ev}_\nu) = \text{Span} \{ \Psi \in B_S \mid \Psi(\nu) = 0 \} = \text{Span} \{ \Psi - s \cdot \Psi \mid \Psi \in B_S, s \text{ is a simple reflection in } W_\nu \} \quad \text{(by Lemma 4.7)}
\]
\[ \subseteq \text{Span} \{ \Psi - w \cdot \Psi \mid \Psi \in \text{Coh}_\Lambda(K_S(G)), w \in W_\nu \} \subseteq \text{ker}(\text{ev}_\nu). \]

Therefore
\[ \text{ker}(\text{ev}_\nu) = \text{Span} \{ \Psi - w \cdot \Psi \mid \Phi \in \text{Coh}_\Lambda(K_S(G)), w \in W_\nu \}. \]

Together with Theorem 4.1, this implies the first part. The second part is immediate since
\[ \sharp(\text{Irr}_{\nu,S}(G)) = \dim K_{\nu,S}(G). \]

**Proposition 5.2.** For all $\sigma \in \text{Irr}(W(\Lambda)) \setminus \text{Irr}_S(W(\Lambda))$,
\[ [\sigma : \text{Coh}_\Lambda(K_S(G))] = 0. \]

**Proof.** This is implied by Proposition 4.10 and Proposition 3.32. □

Proposition 5.1 and Proposition 5.2 imply that for every $\nu \in \Lambda$,
\[ \sharp(\text{Irr}_{\nu,S}(G)) = \sum_{\sigma \in \text{Irr}_S(W(\Lambda))} [1_{W_\nu} : \sigma] \cdot [\sigma : \text{Coh}_\Lambda(K_S(G))]. \tag{5.1} \]

**Theorem 5.3.** For all $\nu \in \Lambda$,
\[ \sharp(\text{Irr}_{\nu,S}(G)) \leq \sum_{\sigma \in \text{Irr}_S(W(\Lambda))} [1_{W_\nu} : \sigma] \cdot [\sigma : \text{Coh}_\Lambda(K(G))], \]
and the equality holds if Conjecture 4.14 holds for $G$.

**Proof.** The first assertion is immediate from (5.1). As a representation of $W(\Lambda)$,
\[ \text{Coh}_\Lambda(K(G)) \cong \bigoplus_{C \text{ is a cell in } \text{Coh}_\Lambda(K(G))} \text{Coh}_\Lambda(K(G))(C). \]

Lemma 4.12 implies that
\[ \text{Coh}_\Lambda(K_S(G)) \cong \bigoplus_{C \text{ is a cell in } \text{Coh}_\Lambda(K(G)), \sigma_C \in \text{Irr}_S^\nu(W(\Lambda))} \text{Coh}_\Lambda(K(G))(C), \]
where $\sigma_C$ is the special irreducible representation attached to $C$, as in Definition 4.13. Under the assumption that Conjecture 4.14 holds for $G$,
\[ [\sigma : \text{Coh}_\Lambda(K_S(G))] = [\sigma : \text{Coh}_\Lambda(K(G))] \quad \text{for all } \sigma \in \text{Irr}_S(W(\Lambda)). \]

Together with (5.1), this implies the second assertion. □
Recall that $O_\nu$ is the nilpotent orbit whose Zariski closure $\overline{O_\nu} \subseteq g^*$ equals the associated variety of the maximal ideal $I_\nu$. Recall also the Lusztig left cell $C_{\nu,0}$ from (3.25).

**Corollary 5.4.** For all $\nu \in \Lambda$, 
\[
\sharp(\text{Irr}_{\nu,\overline{O_\nu}}(G)) \leq \sum_{\sigma \in C_{\nu,0}} |\sigma : \text{Coh}_\Lambda(K(G))|, 
\]
and the equality holds if Conjecture 4.14 holds for $G$.

**Proof.** In view of (3.24) and Proposition 3.33, this follows from Theorem 5.3. \qed

We prove the following proposition for later use.

**Proposition 5.5.** Let $\Psi$ be a basal element of $\text{Coh}_\Lambda(K(G))$, and let $\nu \in \Lambda$ be a dominant element. Then $\Psi(\nu) \in \text{Irr}_{\nu,\overline{O_\nu}}(G)$ if and only if $\sigma_\Psi \cong (j^{W(\Lambda)}_{W_{\nu}} \text{sgn}) \otimes \text{sgn}$ and no element of $\tau_\Psi$ fixes $\nu$.

**Proof.** First suppose that $\sigma_\Psi \cong (j^{W(\Lambda)}_{W_{\nu}} \text{sgn}) \otimes \text{sgn}$ and there is no element of $\tau_\Psi$ that fixes $\nu$. Then Lemma 4.7 implies that $\Psi(\nu) \in \text{Irr}(G)$ and Lemma 4.12 implies that
\[
\sigma_{\nu,\text{Ann}(\Psi(\nu))} \cong (j^{W(\Lambda)}_{W_{\nu}} \text{sgn}) \otimes \text{sgn}. 
\]
Thus $\text{Ann}(\Psi(\nu)) = I_{\nu}$ by Lemma 3.35, and hence $\Psi(\nu) \in \text{Irr}_{\nu,\overline{O_\nu}}(G)$.

Now suppose that $\Psi(\nu) \in \text{Irr}_{\nu,\overline{O_\nu}}(G)$. Lemma 4.7 implies that no element of $\tau_\Psi$ fixes $\nu$. As $\text{Ann}(\Psi(\nu)) = I_{\nu}$, Lemma 3.35 and Lemma 4.12 imply that $\sigma_\Psi \cong (j^{W(\Lambda)}_{W_{\nu}} \text{sgn}) \otimes \text{sgn}$. This completes the proof of the proposition. \qed

6. Separating Good Parity and Bad Parity for Coherent Continuation Representations

From now on, $\star$ will be one of the 10 labels, and $G$ will be a classical group of type $\star$, as in Sections 2.3-2.8. We have the complex Lie group
\[
G_\mathbb{C} := \begin{cases}
\text{GL}_n(\mathbb{C}), & \text{if } \star \in \{A^R, A^\mathbb{R}\};
\text{GL}_{p+q}(\mathbb{C}), & \text{if } \star \in \{A, A^\mathbb{R}\};
\text{SO}_{p+q}(\mathbb{C}), & \text{if } \star \in \{B, D\};
\text{SO}_{2n}(\mathbb{C}), & \text{if } \star = D^*; \\
\text{Sp}_{2n}(\mathbb{C}), & \text{if } \star \in \{C, C^\mathbb{R}\};
\text{Sp}_{p+q}(\mathbb{C}), & \text{if } \star = C^*,
\end{cases}
\]
and $\iota : G \rightarrow G_\mathbb{C}$ is the usual complexification homomorphism except when $\star \in \{A^\mathbb{R}, C\}$ where it factors through the cover homomorphism (see Section 4 for the general setup).

The main purpose of this section is to separate good parity and bad parity for coherent continuation representations. (The broad ideas are from [ABV91].) We give explicit details in order to bring out the key role of standard representations. A reader who is familiar with [ABV91] may wish to skip this section.) We will first describe structure data of the classical group $G$ (such as the universal Cartan subalgebra and the analytic weight lattice) in terms of the standard representation of $G$. In the Langlands dual Lie algebra $\tilde{g}$, we define the good parity and bad parity parts, also in terms of the standard representation of $\tilde{g}$. We then proceed to separate good parity and bad parity in all structure data necessary for the description of coherent continuation representations. This is not difficult, but requires care for the sake of clarity.

We will freely use the notation of Sections 2.3-2.8. One additional convention is as follows.

If $\star \in \{A^\mathbb{R}, C\}$, $\mathcal{K}_{\star}(G)$ will denote the Grothendieck group of the category of genuine Casselman-Wallach representations of $G$. Otherwise, put $\mathcal{K}(G) := \mathcal{K}(G)$. Similar notations such as $\mathcal{P}_{\star}(G)$, $\mathcal{K}_{\star}(G)$ and $\text{Irr}_{\nu}^\prime(G)$ will be used without further explanation.
6.1. **Standard representations of classical groups.** For (mostly) efficiency reasons, we would like to lump certain cases (such as $B$ and $D$) together when discussing standard representations. Partition the set of labels into subsets

$$\{\{A^R\}, \{A^H\}, \{A, \tilde{A}\}, \{B, D\}, \{C, \tilde{C}\}, \{C^*\}, \{D^*\}\},$$

and denote by $[\star]$ the equivalence class containing $\star$.

Given a label $\star$, we define the notion of a $[\star]$-structure, which consists of a finite dimensional complex vector space $V$, together with some additional data. The group $G(V)$ will be defined as the (real) subgroup of $GL(V)$ fixing the $[\star]$-structure. Its Zariski closure in $GL(V)$ will be denoted by $G_\star(V)$. For cases $A$ and $\tilde{C}$ there will be a cover, and our classical group (specified by $\star$ and $V$) will be denoted by $G_\star(V)$.

**The case when $\star \in \{A^R, A^H\}$.** In this case, a $[\star]$-structure consists of a finite dimensional complex vector space $V$, and a conjugate linear automorphism $j : V \to V$ such that

$$j^2 = \begin{cases} 
1, & \text{if } \star = A^R; \\
-1, & \text{if } \star = A^H.
\end{cases}$$

The group $G(V)$ is $GL_n(\mathbb{R})$ when $j^2 = 1$, and $GL_n(\mathbb{H})$ when $j^2 = -1$. Here $n = \dim V$.

**The case when $\star \in \{A, \tilde{A}\}$.** In this case, a $[\star]$-structure consists of a finite dimensional complex vector space $V$, and a non-degenerate Hermitian form $\langle , \rangle : V \times V \to \mathbb{C}$ (which is by convention linear on the first variable and conjugate linear on the second variable). The Hermitian form has signature $(p, q)$ so $G(V)$ is $U(p, q)$ with $p + q = \dim V$.

**The case when $\star \in \{C, \tilde{C}, C^*\}$.** In this case, a $[\star]$-structure consists of a finite dimensional complex vector space $V$, a symplectic form $\langle , \rangle : V \times V \to \mathbb{C}$ together with a conjugate linear automorphism $j : V \to V$ such that

$$j^2 = \begin{cases} 
1, & \text{if } \star \in \{C, \tilde{C}\}; \\
-1, & \text{if } \star = C^*.
\end{cases}$$

and for all $u, v \in V$,

$$\langle j(u), j(v) \rangle = \overline{\langle u, v \rangle}, \quad \text{ (bar indicates the complex conjugate)}.$$

When $j^2 = 1$, the group $G(V)$ is $Sp_{2n}(\mathbb{R})$ ($n = \frac{\dim V}{2}$). When $j^2 = -1$, there are several $G_\star(V)$ conjugacy classes of $j$'s and $G(V)$ is $Sp(\frac{p}{2}, \frac{q}{2})$ with $p, q$ even and $p + q = \dim V$.

**The case when $\star \in \{B, D, D^*\}$.** In this case, a $[\star]$-structure consists of a finite dimensional complex vector space $V$, a non-degenerate symmetric bilinear form $\langle , \rangle : V \times V \to \mathbb{C}$, a conjugate linear automorphism $j : V \to V$, and $\omega \in \wedge^{\dim V} V$ (to be called the orientation), subject to the following five conditions:

- (6.2) $j^2 = \begin{cases} 
1, & \text{if } \star \in \{B, D\}; \\
-1, & \text{if } \star = D^*.
\end{cases}$

- For all $u, v \in V$,

  $$\langle j(u), j(v) \rangle = \overline{\langle u, v \rangle}.$$

- The subgroup of $GL(V)$ stabilizing $\langle , \rangle$, to be denoted by $G(\langle , \rangle)$, is a real orthogonal group $O(p, q)$ (with $p + q = \dim V$) when $\star \in \{B, D\}$, and the quaternionic orthogonal group $SO^*(2n)$ (with $n = \frac{\dim V}{4}$) when $\star = D^*$. The data $\omega$ satisfies the additional conditions below.

- With respect to the symmetric bilinear form $\langle , \rangle : \wedge^{\dim V} V \times \wedge^{\dim V} V \to \mathbb{C}$ given by

  $$\langle u_1 \wedge u_2 \wedge \cdots \wedge u_{\dim V}, v_1 \wedge v_2 \wedge \cdots \wedge v_{\dim V} \rangle = \det \left( [\langle u_i, v_j \rangle]_{i,j=1,2,\ldots,\dim V} \right),$$

  we have

  (6.4) $\langle \omega, \omega \rangle = 1$. 


• if $\star = D^*$, then
\begin{equation}
\omega = \sqrt{-1} u_1 \wedge j(u_1) \wedge \sqrt{-1} u_2 \wedge j(u_2) \wedge \cdots \wedge \sqrt{-1} u_{\dim V} \wedge j(u_{\dim V}),
\end{equation}
for some vectors $u_1, u_2, \ldots, u_{\dim V} \in V$;

• If $\star \in \{B, D\}$ and $\dim V = 0$, then
\begin{equation}
\omega = 1 \quad \text{(as an element of $\wedge^{\dim V} V = \mathbb{C}$)}.
\end{equation}

The group $G(V)$ is a real special orthogonal group or a quaternionic orthogonal group fixing a preferred orientation.

Remarks 6.1. (a) In the case of $\star \in \{C, \widetilde{C}\}$, the existence of the symplectic form implies that $\dim V$ is even. In the case of $\star \in \{A^\mathbb{H}, C^\mathbb{R}, D^*\}$, the existence of $j$ also implies that $\dim V$ is even.

(b) We comment on the three conditions on the orientation $\omega$ in the case of $\star \in \{B, D, D^*\}$. Note that there are precisely two elements $\omega \in \wedge^{\dim V} V$ (negative of each other) satisfying $\langle \omega, \omega \rangle = 1$, the condition in (6.4). When $\star \in \{B, D\}$ and $\dim V \neq 0$, these two elements are conjugate under $G((\cdot, \cdot), j)$. However, when $\star \in \{B, D\}$ and $\dim V = 0$, or $\star = D^*$, the group $G((\cdot, \cdot), j)$ is connected and acts trivially on $\wedge^{\dim V} V$, and the aforementioned two elements are not conjugate under $G((\cdot, \cdot), j)$. The two conditions (6.5) and (6.6) thus ensure that the orientation $\omega$, as part of the $\star$-structure on $V$, is unique up to conjugation by $G((\cdot, \cdot), j)$ in all cases.

(c) When the group $G(V)$ is a real special orthogonal group of type $D$, or a quaternionic orthogonal group, we will need to use the orientation $\omega$ to identify the universal Cartan subalgebra with $\mathbb{C}^n$, where $n$ is the rank of the group. See (6.8).

For a $\star$-structure, by abuse of terminology we will call the underlying vector space $V$ a $\star$-space. We will also write $(\cdot, \cdot)_V := (\cdot, \cdot)$, $j_V := j$, and $\omega_V := \omega$ (when the additional structures are present). Given two $\star$-spaces $V_1$ and $V_2$, a linear isomorphism $V_1 \to V_2$ is said to be an isomorphism of $\star$-spaces if it preserves the $\star$-structure. The product $V_1 \times V_2$ is a $\star$-space in the obvious way.

For a $\star$-space $V$, put
\[
G_\star(V) := \begin{cases}
\text{the det}^{\frac{1}{2}}\text{-double cover of } G(V), & \text{if } \star = \widetilde{A}; \\
\text{the metaplectic double cover of } G(V), & \text{if } \star = \widetilde{C}; \\
G(V), & \text{otherwise}.
\end{cases}
\]

The group $G_\star(V)$ only depends on the isomorphism class of $V$ as a $\star$-space.

Now we assume that $G$ is identified with $G_\star(V)$. We call $V$ the standard representation of $G$. Recall than $n$ is the rank of $g$ in all cases. We fix a flag
\begin{equation}
\{0\} = V_0 \subseteq V_1 \subseteq \ldots \subseteq V_n
\end{equation}
in $V$ such that

- $\dim V_i = i$ for all $i = 1, 2, \ldots, n$;
- if $\star \in \{B, D, C, \widetilde{C}, C^\mathbb{R}, D^*\}$, then $V_n$ is totally isotropic (with respect to the bilinear form $(\cdot, \cdot)_V : V \times V \to \mathbb{C}$);
- if $\star \in \{D, D^*\}$, then $V_n$ is $\omega_V$-compatible in the sense that
\[
\omega_V = u_1 \wedge u_2 \wedge \cdots \wedge u_{2n}
\]
for some elements $u_1, u_2, \ldots, u_{2n} \in V_{2n}$ such that $u_1, u_2, \ldots, u_n \in V_n$ and
\[
\langle u_i, u_j \rangle = \begin{cases}
1, & \text{if } i + j = 2n + 1; \\
0, & \text{if } i + j \neq 2n + 1,
\end{cases}
\]
for all $i, j = 1, 2, \ldots, 2n$.

Remarks 6.2. (a) When $\star \in \{D, D^*\}$ and $\dim V \neq 0$ so that $G_\mathbb{C}$ is a non-trivial complex even special orthogonal group, up to conjugation by $G_\mathbb{C}$ there are precisely two $n$-dimensional totally isotropic subspaces of $V$. Among these two subspaces, exactly one is $\omega_V$-compatible.
(b) In all cases, up to conjugation by $G_C$ there exists a unique flag as in (6.7) satisfying the aforementioned three conditions. The stabilizer of the flag (6.7) in $G_C$ is a Borel subgroup of $G_C$, and every Borel subgroup of $G_C$ is uniquely of this form.

(c) When $*=D^*$ and $n$ is even, the condition (6.5) ensures that all $j_V$-stable totally isotropic subspace of $V$ of dimension $n$ are $\omega_V$-compatible.

The stabilizer of the flag (6.7) in $g$ is a Borel subalgebra of $g$. Using this Borel subalgebra, we get an identification

$$a_h = \prod_{i=1}^{n} \mathfrak{gl}(V_i/V_{i-1}) = \mathbb{C}^n.$$  

This identification is independent of the choice of the flag (6.7). As in (the beginning of) Section 4, we assume that $Q=Q_1$ (the analytic weight lattice). Note that

$$Q = \mathbb{Z}^n \subseteq \mathbb{C}^n = (\mathbb{C}^n)^*,$$

and the positive roots are

$$\Delta^+ = \begin{cases} 
\{e_i - e_j \mid 1 \leq i < j \leq n\}, & \text{if } * \in \{A^R, A^H, A, \tilde{A}\}; \\
\{e_i \pm e_j \mid 1 \leq i < j \leq n\}, & \text{if } * \in \{D, D^*\}; \\
\{e_i \pm e_j \mid 1 \leq i < j \leq n\} \cup \{e_i \mid 1 \leq i \leq n\}, & \text{if } * = B; \\
\{e_i \pm e_j \mid 1 \leq i < j \leq n\} \cup \{\pm e_i \mid 1 \leq i \leq n\}, & \text{if } * \in \{C, \tilde{C}, C^*\}.
\end{cases}$$

Here $e_1, e_2, \ldots, e_n$ is the standard basis of $\mathbb{C}^n$. The Weyl group

$$W = \begin{cases} 
S_n, & \text{if } * \in \{A^R, A^H, A, \tilde{A}\}; \\
W_n, & \text{if } * \in \{B, C, \tilde{C}, C^*\}; \\
W'_n, & \text{if } * \in \{D, D^*\},
\end{cases}$$

where $S_n \subseteq \text{GL}_n(\mathbb{Z})$ is the group of the permutation matrices, $W_n \subseteq \text{GL}_n(\mathbb{Z})$ is the subgroup generated by $S_n$ and all the diagonal matrices with diagonal entries $\pm 1$, and $W'_n \subseteq \text{GL}_n(\mathbb{Z})$ is the subgroup generated by $S_n$ and all the diagonal matrices with diagonal entries $\pm 1$ and determinant 1.

6.2. Good parity and bad parity in the Langlands dual. Recall that an integer has good parity (which depends on $*$ and $n = \text{rank} \, \mathfrak{g}$) if it has the same parity as

$$\begin{cases} 
n, & \text{if } * \in \{A^R, A^H, A\}; \\
1 + n, & \text{if } * = \tilde{A}; \\
1, & \text{if } * \in \{C, C^*, D, D^*\}; \\
0, & \text{if } * \in \{B, \tilde{C}\}.
\end{cases}$$

Otherwise it has bad parity.

Let $\hat{V}$ denote the standard representation of $\mathfrak{g}$ so that $\delta$ is identified with a Lie subalgebra of $\mathfrak{gl}(\hat{V})$. When $* \in \{A^R, A^H, A, \tilde{A}\}$, $\mathfrak{g} = \mathfrak{gl}(\hat{V})$. When $* \in \{B, \tilde{C}\}$, the space $\hat{V}$ is equipped with a $\mathfrak{g}$-invariant symplectic form $\langle \, , \rangle_{\hat{V}} : \hat{V} \times \hat{V} \to \mathbb{C}$. When $* \in \{C, C^*, D, D^*\}$, the space $\hat{V}$ is equipped with a $\mathfrak{g}$-invariant non-degenerate symmetric bilinear form $\langle \, , \rangle_{\hat{V}} : \hat{V} \times \hat{V} \to \mathbb{C}$ together with an element $\omega_{\hat{V}} \in \wedge^2 \hat{V}$ with $\langle \omega_{\hat{V}}, \omega_{\hat{V}} \rangle_{\hat{V}} = 1$ (see (6.4)).

As in (6.8) we also have that $a_h = \mathbb{C}^n$. Thus we have an identification $a_h = a_h^*$ that identifies $\mathfrak{g}$ as the Langlands dual (or the metaplectic Langlands dual) of $\mathfrak{g}$.

Recall the semisimple element $\lambda_0^\circ \in \mathfrak{g}$ that equals half of the neutral element in an $\mathfrak{sl}_2$-triple attached to $\hat{O}$. Its conjugacy class is uniquely determined by $\hat{O}$ under $\text{Ad}(\mathfrak{g})$. Fix a Lie algebra homomorphism

$$\phi : \mathfrak{sl}_2(\mathbb{C}) \to \mathfrak{g}$$

such that

$$\phi \left( \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix} \right) \in \hat{O} \quad \text{and} \quad \phi \left( \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix} \right) = 2\lambda_0^\circ.$$
We view $\check{V}$ as an $\mathfrak{sl}_2(\mathbb{C})$-module via $\phi$. Put

$$\check{V}_b := \text{sum of irreducible } \mathfrak{sl}_2(\mathbb{C})\text{-submodules of } \check{V} \text{ with bad parity dimension.}$$

Define $\tilde{V}_g$ similarly by adding the $\mathfrak{sl}_2(\mathbb{C})$-submodules of $\check{V}$ with good parity dimension. Then

(6.12) $$\check{V} = \check{V}_b \times \tilde{V}_g.$$ 

Write $\hat{g}_b$ for the Lie subalgebra of $\hat{g}$ consisting of the elements that annihilate $\check{V}_g$ and stabilize $\check{V}_b$. Define $\tilde{g}_g$ similarly so that $\hat{g}_b \times \tilde{g}_g$ is the stabilizer of the decomposition (6.12) in $\hat{g}$. We have natural isomorphisms

$$\langle \underbrace{\mathfrak{g}_b, \tilde{g}_g}_\text{if } \ast \in \{ A, C, C^\ast, D, D^\ast \} \rangle \cong \begin{cases} \langle \mathfrak{gl}_{n_b}(\mathbb{C}), \mathfrak{gl}_{n_g}(\mathbb{C}) \rangle, & \text{if } \ast \in \{ A^R, A^\ast, A, \tilde{A} \}; \\
\langle \mathfrak{sp}_{2n_b}(\mathbb{C}), \mathfrak{sp}_{2n_g}(\mathbb{C}) \rangle, & \text{if } \ast \in \{ B, \tilde{C} \}; \\
\langle \mathfrak{o}_{2n_b}(\mathbb{C}), \mathfrak{o}_{2n_g+1}(\mathbb{C}) \rangle, & \text{if } \ast \in \{ C, C^\ast \}; \\
\langle \mathfrak{o}_{2n_b}(\mathbb{C}), \mathfrak{o}_{2n_g}(\mathbb{C}) \rangle, & \text{if } \ast \in \{ D, D^\ast \}, \end{cases}$$

where $n_b$ and $n_g$ are ranks of $\hat{g}_b$ and $\tilde{g}_g$ respectively so that $n_b + n_g = n$. Note that when $\ast \in \{ B, C, \tilde{C}, C^\ast, D, D^\ast \}$, $n_b$ agrees with the one defined in (2.10).

When $\ast \in \{ B, C, \tilde{C}, C^\ast, D, D^\ast \}$, (6.12) is an orthogonal decomposition, and the form $\langle , \rangle_\check{V}$ on $\check{V}$ restricts to bilinear forms on $\check{V}_b$ and $\tilde{V}_g$, to be respectively denoted by $\langle , \rangle_{\check{V}_b}$ and $\langle , \rangle_{\tilde{V}_g}$. When $\ast \in \{ C, C^\ast, D, D^\ast \}$, we fix an element $\omega_{\check{V}_b} \in \wedge^{\dim \check{V}_b} \check{V}_b$ and an element $\omega_{\tilde{V}_g} \in \wedge^{\dim \tilde{V}_g} \tilde{V}_g$ such that

(6.13) $$\langle \omega_{\check{V}_b}, \omega_{\tilde{V}_g} \rangle_{\check{V}_b} = \langle \omega_{\check{V}_b}, \omega_{\tilde{V}_g} \rangle_{\tilde{V}_g} = 1 \quad \text{and} \quad \omega_{\check{V}_b} \wedge \omega_{\tilde{V}_g} = \omega_{\check{V}}.$$ 

Then as in (6.8), in all cases we have identifications

$$\begin{array}{c}
a_{\hat{g}_b} = \mathbb{C}^{n_b}, \\
\ast = \tilde{A} \quad \text{and} \quad p + q \text{ is odd} ; \\
a_{\tilde{g}_g} = \mathbb{C}^{n_g} \quad \text{and} \quad \ast = a_{\hat{g}_b} \times a_{\tilde{g}_g} = \mathbb{C}^n,
\end{array}$$

where $a_{\hat{g}_b}$ and $a_{\tilde{g}_g}$ are the universal Cartan subalgebras of $\hat{g}_b$ and $\tilde{g}_g$, respectively.

The homomorphism $\phi$ in (6.11) has image in (6.12), so induces Lie algebra homomorphisms

$$\phi_\check{g} : \mathfrak{sl}_2(\mathbb{C}) \to \tilde{g}_g \quad \text{and} \quad \phi_\hat{g} : \mathfrak{sl}_2(\mathbb{C}) \to \hat{g}_b.$$

By using these two homomorphisms, we obtain $\check{O}_g \in \text{Nil}(\tilde{g}_g)$, $\check{\lambda}_{\check{O}_g} \in \tilde{g}_g$, $\hat{O}_b \in \text{Nil}(\hat{g}_b)$, and $\lambda_{\hat{O}_b} \in \hat{g}_b$ as in (6.11). Then

$$\mathbf{d}_{\check{O}} = \mathbf{d}_{\check{O}_g} \mathbf{d}_{\hat{O}_b}.$$ 

Put

(6.14) $$\ast_g := \begin{cases} A, & \text{if } \ast = \tilde{A} \text{ and } p + q \text{ is odd;} \\
\ast, & \text{otherwise}, \end{cases}$$

and

(6.15) $$\ast_b := \begin{cases} \tilde{A}, & \text{if } \ast = A \text{ and } p + q \text{ is odd;} \\
D, & \text{if } \ast = C; \\
D^\ast, & \text{if } \ast = C^\ast; \\
\ast, & \text{otherwise}. \end{cases}$$

The orbit $\check{O}_g$ has good parity (i.e. all its nonzero row lengths have good parity) with respect to $\ast_g$ and $n_g$, while the orbit $\hat{O}_b$ has bad parity with respect to $\ast_b$ and $n_b$. 

Put

$$A_b := \begin{cases} \mathbb{Z}^{n_b}, & \text{if } \ast \in \{ A^R, A \} \text{ and } n \text{ is even, or } \ast \in \{ A^\ast, B, \tilde{C} \}; \\
\left\{ \frac{1}{2}, \frac{3}{2}, \ldots, \frac{1}{2} \right\} + \mathbb{Z}^{n_b}, & \text{otherwise}, \end{cases}$$
which is the $\mathbb{Z}^{n_b}$-coset in $\mathbb{C}^{n_b} = \mathfrak{a}^*\mathfrak{h}$ containing some (equiv. all) elements in $\mathfrak{a}^*\mathfrak{h}$ that represent the $\text{Ad}(\mathfrak{g})$-conjugacy class of $\Lambda_{\mathfrak{h}_b}$. Likewise put

$$
\Lambda_{g} := \begin{cases} 
(\frac{1}{2}, \frac{1}{2}, \ldots, \frac{1}{2}) + \mathbb{Z}^{n_g}, & \text{if } \star \in \{A^{\mathbb{R}}, A\} \text{ and } n \text{ is even, or } \star \in \{A^{\mathbb{H}}, B, C\}; \\
\mathbb{Z}^{n_g}, & \text{otherwise},
\end{cases}
$$

(6.16)

which is the $\mathbb{Z}^{n_g}$-coset in $\mathbb{C}^{n_g} = \mathfrak{a}^*\mathfrak{h}_g$ containing some (equiv. all) elements in $\mathfrak{a}^*\mathfrak{h}_g$ that represent the $\text{Ad}(\mathfrak{g}_g)$-conjugacy class of $\Lambda_{\mathfrak{h}_g}$. Finally put

$$
\Lambda := \Lambda_b \times \Lambda_g \subseteq \mathfrak{a}^*\mathfrak{h}_b \times \mathfrak{a}^*\mathfrak{h}_g = \mathbb{C}^{n_b} \times \mathbb{C}^{n_g} = \mathbb{C}^{n} = \mathfrak{a}^*\mathfrak{h} = \mathfrak{a}^*.
$$

### 6.3. Separating the universal Cartan subalgebra.

When $\star \in \{B, C, \tilde{C}, C^*, D, D^*\}$, we have defined in Section 2.6 the notion of an $\mathcal{O}$-relevant parabolic subgroup of $G$ (Definition 2.18). We extend this notion to the other cases: If $\star \in \{A, \tilde{A}\}$, then a parabolic subgroup of $G$ is said to be $\mathcal{O}$-relevant if it is the stabilizer group of a $\mathfrak{y}_p$-stable subspace of $V$ of dimension $n_b$. If $\star \in \{A, \tilde{A}\}$, then a parabolic subgroup of $G$ is said to be $\mathcal{O}$-relevant if it is the stabilizer group of a totally isotropic subspace of $V$ of dimension $\frac{2n}{2}$ (in particular, the existence of such a parabolic subgroup implies that $n_b$ is even).

Recall the notions in Section 4.1, particularly Definition 4.2 of the data $\gamma = (H, \xi, \Gamma)$ parametrizing irreducible Casselman-Wallach representations. Recall also, Section 2.3, $d_{\mathcal{O}}$ denotes the Young diagram attached to $\mathfrak{g}$.

Through a direct verification, we record the following lemma without proof.

**Lemma 6.3.** For every $\gamma = (H, \xi, \Gamma) \in \mathcal{P}_\Lambda^\prime(G)$, $H$ is contained in a parabolic subgroup of $G$ that is $\mathcal{O}^\prime$-relevant for some $\mathcal{O}^\prime \in \mathcal{N}(\mathfrak{g})$ with $d_{\mathcal{O}^\prime} = d_{\mathcal{O}}$.

**Remark 6.4.** The $\mathcal{O}^\prime$ in the above lemma is nothing but $\mathcal{O}$ (which we fix), except in the case when $\star \in \{D, D^*\}$, $V \neq 0$, and $\mathcal{O}$ is very even. See Section 2.6 for more discussions on the notion of relevant parabolic subgroups.

In the rest of this paper we assume that $G$ has a parabolic subgroup that is $\mathcal{O}^\prime$-relevant for some $\mathcal{O}^\prime \in \mathcal{N}(\mathfrak{g})$ with $d_{\mathcal{O}^\prime} = d_{\mathcal{O}}$. Otherwise Lemma 6.3 and the surjectivity of the evaluation map in Theorem 4.1 imply that $\text{Unip}_{\mathcal{O}^\prime}(G)$ is empty. The assumption is equivalent to saying that

$$
n_b \text{ is even if } \star = \tilde{A},
$$

and

$$
p, q \geq \begin{cases} 
\frac{m_b}{2}, & \text{if } \star \in \{A, \tilde{A}\}; \\
\frac{n_b}{2}, & \text{if } \star \in \{B, C^*, D\}.
\end{cases}
$$

(6.17)

(6.18)

See Proposition 2.19. In particular, $\star_g = \star$ in all cases.

Define two classical groups

$$
(G_b, G_g) = \begin{cases} 
(GL_{n_b}(\mathbb{R}), GL_{n_g}(\mathbb{R})), & \text{if } \star = A^{\mathbb{R}}; \\
(GL_{n_b}(\mathbb{H}), GL_{n_g}(\mathbb{H})), & \text{if } \star = A^{\mathbb{H}}; \\
(U(\frac{n_b}{2}, \frac{n_g}{2}), U(p, q)), & \text{if } \star = A \text{ and } p + q \text{ is even}; \\
(U(\frac{n_b}{2}, \frac{n_g}{2}), \tilde{U}(p, q)), & \text{if } \star = A \text{ and } p + q \text{ is odd}; \\
(\tilde{U}(\frac{n_b}{2}, \frac{n_g}{2}), \tilde{U}(p, q)), & \text{if } \star = \tilde{A}; \\
(SO(n_b, n_b + 1), SO(p, q)), & \text{if } \star = B; \\
(SO(n_b, n_b), Sp_{2n_b}(\mathbb{R})), & \text{if } \star = C; \\
(SO^*(2n_b), Sp(\frac{n_b}{2}, \frac{n_g}{2})), & \text{if } \star = C^*; \\
(Sp_{2n_b}(\mathbb{R}), Sp_{2n_g}(\mathbb{R})), & \text{if } \star = \tilde{C}; \\
(SO(n_b, n_b), SO(p, q)), & \text{if } \star = D; \\
(SO^*(2n_b), SO^*(2n_g)), & \text{if } \star = D^*.
\end{cases}
$$

(6.19)
where
\[
(p_g, q_g) = \begin{cases} 
(p - \frac{n_b}{2}, q - \frac{n_b}{2}), & \text{if } * \in \{A, A^\vee\}; \\
(p - n_b, q - n_b), & \text{if } * \in \{B, C^*, D\}.
\end{cases}
\]

Then \(G_b\) has type \(*_b\) and \(G_g\) has type \(*\). See (6.14) and (6.15). We remark that \(G_b \times G_g\) is an endoscopic group of \(G\) ([ABV91, Chapter 21]), when \(* \neq \tilde{C}\).

**Definition 6.5.** Define \(V_b\) to be the \([*_b]\)-space such that \(G_b\) is identified with \(G_{*_b}(V_b)\). Likewise define \(V_g\) to be the \([*]\)-space such that \(G_g\) is identified with \(G_*(V_g)\).

As in (6.1), we have the complex groups \(G_b, C\) and \(G_g, C\), and the complexification homomorphisms \(i_b : G_b \to G_b, C\) and the complexification homomorphisms \(i_g : G_g \to G_g, C\).

Write \(\mathfrak{h}_b\) and \(\mathfrak{h}_g\) for the universal Cartan subalgebras for \(g_b\) and \(g_g\), respectively. Then we have identifications
\[
\mathfrak{h}_b = \mathfrak{c} \subseteq \mathfrak{c}_g = \mathfrak{h}_g.
\]

Using these identifications, we view \(\hat{g}_b\) as the Langlands dual (or the metaplectic Langlands dual when \(* = \tilde{C}\)) of \(g_b\), and view \(\hat{g}_g\) as the Langlands dual (or the metaplectic Langlands dual when \(* = \tilde{C}\)) of \(g_g\).

We also have identifications
\[
\mathfrak{h}_b^* \times \mathfrak{h}_g^* = \mathfrak{c}^{*b} \times \mathfrak{c}^{*g} = \mathfrak{c}^* = \mathfrak{h}^*.
\]

Under this identification,
\[
\Delta_b^+ = \Delta_b \cap \Delta^+ \quad \text{and} \quad \Delta_g^+ = \Delta_g \cap \Delta^+,
\]
where \(\Delta_b^+ \subseteq \Delta_b \subseteq \mathfrak{h}_b^*\) are respectively the positive root system and the root system of \(\mathfrak{g}_b\), and likewise \(\Delta_g^+ \subseteq \Delta_g \subseteq \mathfrak{h}_g^*\) are respectively the positive root system and the root system of \(\mathfrak{g}_g\).

Similar to (6.9), we have the analytic weight lattices
\[
Q_b = \mathbb{Z}^{n_b} \subseteq \mathfrak{c}^{n_b} = (\mathfrak{c}^{*b})^* = \mathfrak{h}_b^*;
\]
and
\[
Q_g = \mathbb{Z}^{n_g} \subseteq \mathfrak{c}^{n_g} = (\mathfrak{c}^{*g})^* = \mathfrak{h}_g^*.
\]

Then \(\Lambda_b\) is the unique \(Q_{n_b}\)-coset in \(\mathfrak{h}_b^*\) that contains some (equiv. all) elements in \(\mathfrak{h}_b^*\) that represents the \(\text{Ad}(\hat{g}_b)\)-conjugacy class of \(\lambda_{\hat{g}_b}^0\). Likewise, \(\Lambda_g\) is the unique \(Q_{n_g}\)-coset in \(\mathfrak{h}_g^*\) that contains some (equiv. all) elements in \(\mathfrak{h}_g^*\) that represents the \(\text{Ad}(\hat{g}_g)\)-conjugacy class of \(\lambda_{\hat{g}_g}^0\).

Write \(W_b \subseteq \text{GL}(\mathfrak{h}_b)\) for the Weyl group for \(g_b\) so that
\[
W_b = \begin{cases} 
S_{n_b}, & \text{if } * \in \{A^R, A^H, A, A\}; \\
W_{n_b}, & \text{if } * \in \{B, C, C\}; \\
W_{n_b}', & \text{if } * \in \{C, C^*, D, D^*\},
\end{cases}
\]
and write \(W_g \subseteq \text{GL}(\mathfrak{h}_g)\) for the Weyl group for \(g_g\) so that
\[
W_g = \begin{cases} 
S_{n_g}, & \text{if } * \in \{A^R, A^H, A, A\}; \\
W_{n_g}, & \text{if } * \in \{B, C, C, C^*\}; \\
W_{n_g}', & \text{if } * \in \{D, D^*\}.
\end{cases}
\]

Let \(W_g' \subseteq \text{GL}(\mathfrak{h}_g)\) denote the integral Weyl group of \(\Lambda_g\), namely the Weyl group of the root system
\[
\{ \alpha \in \Delta_g : \langle \alpha, \nu \rangle \in \mathbb{Z} \text{ for some (and all) } \nu \in \Lambda_g \}.
\]

Then
\[
W_g' = \begin{cases} 
W_{n_g}', & \text{if } * = \tilde{C}; \\
W_g, & \text{otherwise}.
\end{cases}
\]

Note that
\[
W(A) = W_b \times W_g' \subseteq W_b \times W_g \subseteq W_A.
\]
Let \( \lambda_{\mathfrak{O}_b} \in \Lambda_b \) be the unique dominant element that represents the \( \text{Ad}(\mathfrak{g}_b) \)-conjugacy class of \( \lambda^0_{\mathfrak{O}_b} \). Also take a dominant element \( \lambda_{\mathfrak{O}_g} \in \Lambda_g \) that represents the \( \text{Ad}(\mathfrak{g}_g) \)-conjugacy class of \( \lambda^0_{\mathfrak{O}_g} \), which is unique unless \( \star = \mathring{C} \). Put \( \lambda_{\mathfrak{O}} := (\lambda_{\mathfrak{O}_b}, \lambda_{\mathfrak{O}_g}) \in \Lambda \).

**Remark 6.8.** The element \( \lambda_{\mathfrak{O}} \in \mathfrak{a}^* \) is dominant and represents the conjugacy class of \( \lambda^0_{\mathfrak{O}} \).

**Proof.** Note that for all \( \alpha \in \Delta \),

\[
\langle \lambda_{\mathfrak{O}}, \alpha \rangle \in \mathbb{Z} \quad \text{implies} \quad \alpha \in \Delta_b \cup \Delta_g.
\]

This implies that \( \lambda_{\mathfrak{O}} \) is dominant.

It is easy to see that there exists an element \( \lambda'_{\mathfrak{O}_b} \in \mathfrak{a}^*_b \) that represents the conjugacy class of \( \lambda^0_{\mathfrak{O}_b} \), and an element \( \lambda'_{\mathfrak{O}_g} \in \mathfrak{a}^*_g \) that represents the conjugacy class of \( \lambda^0_{\mathfrak{O}_g} \) such that \( \lambda'_{\mathfrak{O}} := (\lambda'_{\mathfrak{O}_b}, \lambda'_{\mathfrak{O}_g}) \) represents the conjugacy class of \( \lambda^0_{\mathfrak{O}} \). Since \( \lambda_{\mathfrak{O}_b} \) is \( W_b \)-conjugate to \( \lambda'_{\mathfrak{O}_b} \), and \( \lambda_{\mathfrak{O}_g} \) is \( W_g \)-conjugate to \( \lambda'_{\mathfrak{O}_g} \), \( \lambda_{\mathfrak{O}} \) is \( W \)-conjugate to \( \lambda'_{\mathfrak{O}} \). Thus \( \lambda_{\mathfrak{O}} \in \mathfrak{a}^* \) also represents the conjugacy class of \( \lambda^0_{\mathfrak{O}} \). \( \Box \)

**Definition 6.7.** We say that a \([*]\)-space \( V_0 \) is split if

- \( \star \in \{ A^R, A^H, C, \mathring{C} \} \), or
- \( \star \in \{ A, \mathring{A} \} \) and there exists a totally isotropic subspace of \( V_0 \) of dimension \( \dim V_0/2 \), or
- \( \star \in \{ B, D, C^*, D^* \} \) and there exists a \( j_{V_0} \)-stable totally isotropic subspace of \( V_0 \) of dimension \( \dim V_0/2 \).

**Remark 6.8.** In the above definition, \( j_{V_0} \) is part of the structure data of the standard representation \( V_0 \), which fixes a real or quaternionic structure. See Section 6.1.

Up to isomorphism a split \([*]\)-space is determined by its dimension.

Recall from Definition 6.5 the spaces \( V_b \) and \( V_g \). We define a modification of \( V_b \) as follows.

**Definition 6.9.** Denote by \( V^g_b \) the split \([*]\)-space with

\[
\dim V^g_b = \begin{cases} 
\dim V_b - 1, & \text{if } \star = B; \\
\dim V_b, & \text{otherwise}.
\end{cases}
\]

For simplicity, write \( G^g_b := G_*(V^g_b) \). Note that

\[
(6.23) \quad V \cong V^g_b \times V_g
\]

as \([*]\)-spaces.

### 6.4. Matching relevant Cartan subgroups

By a \( \star \)-representation of a Lie group \( E \), we mean a \([*]\)-space \( V_1 \) together with a Lie group homomorphism \( E \to G_*(V_1) \). An isomorphism of \( \star \)-representation is defined to be a \([*]\)-space isomorphism \( f : V_1 \to V_2 \) of \( \star \)-representations that makes the diagram

\[
\begin{array}{ccc}
E & \xrightarrow{\text{the isomorphism induced by } f} & G_*(V_2). \\
\downarrow & & \downarrow \\
G_*(V_1) & \xrightarrow{\text{the isomorphism induced by } f} & G_*(V_2).
\end{array}
\]

commute. The product \( V_1 \times V_2 \) is obviously a \( \star \)-representation of \( E \) whenever \( V_1 \) and \( V_2 \) are \( \star \)-representations of \( E \).

**Definition 6.10.** We say that a Cartan subgroup \( H_b \) of \( G_b \) is relevant if

- \( \star \in \{ A^R, A^H \} \), or
- \( \star \in \{ A, \mathring{A} \} \) and \( H_b \) stabilizes a totally isotropic subspace of \( V_b \) of dimension \( \frac{m_b}{2} \), or
- \( \star \in \{ B, C, \mathring{C}, D, C^*, D^* \} \) and \( H_b \) stabilizes a \( j_{V_b} \)-stable totally isotropic subspace of \( V_b \) of dimension \( n_b \).
For a relevant Cartan subgroup $H_b$ of $G_b$ as in the above definition, $V_b$ is naturally a $*$-representation of $H_b$.

The following lemma can be verified directly. Recall $V_b^g$ from Definition 6.9.

**Lemma 6.11.** There exits a unique (up to isomorphism) $*$-representation of $H_b$ on the $[*]$-space $V_b^g$ with the following property.

(a) When $*$ = $B$, $V_b \cong V_1 \times V_b^g$ as $*$-representations of $H_b$ for a one-dimensional $*$-representation $V_1$ of $H_b$ with trivial action.

(b) When $* \in \{A, A^*, C, C^*, B\}$, $V_b \cong V_b^g$ as representations of $H_b$.

(c) When $* \in \{A^R, A^H, C, D, D^*\}$, $V_b \cong V_b^g$ as $*$-representations of $H_b$.

We view $V_b^g$ as a $*$-representation of $H_b$ as in Lemma 6.11.

Let $H_g$ be a Cartan subgroup of $G_g$ so that $V_g$ is naturally an $H_g$-representation. Then $V_b^g \times V_g$ is naturally a $*$-representation of $H_b \times H_g$. The isomorphism (6.23) yields a Lie group homomorphism

$$H_b \times H_g \to G.$$  

Write $H$ for the image of the homomorphism (6.24), and let $\zeta : H_b \times H_g \to H$ denote the homomorphism induced by (6.24). Note that $H$ is a Cartan subgroup of $G$. We call the pair $(H, \zeta)$ a matching of $H_b \times H_g$, which is uniquely defined up to conjugation by $G$.

When $* = A$ and $p + q$ is odd, $H_b \cong (\mathbb{C}^\times)^{\frac{n_b}{2}} \times \{\pm 1\}$. In all cases, we put

$$H'_b := \begin{cases} 
\text{the identity connected component of } H_b, & \text{if } * = A \text{ and } p + q \text{ is odd;} \\
H_b, & \text{otherwise.}
\end{cases}$$

Then the homomorphism $\zeta : H'_b \times H_g \to H$ is a double cover when $* \in \{A, C\}$, and is an isomorphism in all other cases.

**6.5. Matching the parameters.** Given a parameter $\gamma_b \in \mathcal{P}'_{A_b}(G_b)$ that is represented by $(H_b, \xi_b, \Gamma_b)$, and a parameter $\gamma_g \in \mathcal{P}'_{A_g}(G_g)$ that is represented by $(H_g, \xi_g, \Gamma_g)$, we define a parameter $\varphi(\gamma_b, \gamma_g) \in \mathcal{P}'_{A}(G)$ as in what follows. Note that Lemma 6.3 implies that $H_b$ is relevant. Take a matching $(H, \zeta)$ of $H_b \times H_g$. Let $\xi$ be the composition of

$$\alpha_b^{*} = a_{b_b}^{*} \times a_{b_g}^{*} \xi_b \xi_g \rightarrow b_b^{*} \times b_g^{*}$$

the transpose inverse of the complexified differential of $\xi$, $b^{*}$.

Let $\Gamma : \Lambda \to \text{Irr}'(G)$ be the map

$$\nu = (\nu_b, \nu_g) \mapsto \Gamma_{b,\nu_b} \otimes \Gamma_{g,\nu_g},$$

where $\Gamma_{b,\nu_b} \otimes \Gamma_{g,\nu_g}$ (which is originally defined as an irreducible representation of $H_b \times H_g$) is viewed as an irreducible representation of $H$ via the descent through the homomorphism $\xi : H'_b \times H_g \to H$.

**Lemma 6.12.** The triple $(H, \xi, \Gamma)$ defined above is an element of $\mathcal{P}'_{A}(G)$.

**Proof.** Note that every imaginary root of $G$ with respect to $H$ is either an imaginary root of $G_b$ with respect to $H_b$ or an imaginary root of $G_g$ with respect to $H_g$ (here $\mathfrak{h}$ is identified with $\mathfrak{h}_b \times \mathfrak{h}_g$ via $\xi$). This implies that $\delta(\xi) = \delta(\xi_b) + \delta(\xi_g)$, and the lemma then easily follows. \(\square\)

Now we define $\varphi(\gamma_b, \gamma_g) \in \mathcal{P}'_{A}(G)$ to be the $G$-orbit of the triple $(H, \xi, \Gamma)$ defined above. This is independent of the choices of the representatives $(H_b, \xi_b, \Gamma_b)$, $(H_g, \xi_g, \Gamma_g)$, and the matching $(H, \zeta)$. It is easy to see that the map

$$(6.25) \quad \varphi : \mathcal{P}'_{A_b}(G_b) \times \mathcal{P}'_{A_g}(G_g) \to \mathcal{P}'_{A}(G)$$

is $W_b \times W_g$-equivariant under the cross actions.

**Proposition 6.13.** If either $* = C^*$ and $n_b > 0$, or $* = D^*$ and $n_b, n_g > 0$ (so $W_b \times W_g$ is of index 2 in $W_A$), then the map (6.25) is injective and

$$\mathcal{P}'_{A}(G) = \varphi(\mathcal{P}'_{A_b}(G_b) \times \mathcal{P}'_{A_g}(G_g)) \sqcup w \times \left( \varphi(\mathcal{P}'_{A_b}(G_b) \times \mathcal{P}'_{A_g}(G_g)) \right)$$
for every \( w \in W_A \setminus (W_b \times W_g) \), where “\( w \times \)” indicates the cross action of \( w \). The map (6.25) is bijective in all other cases.

**Proof.** In view of Lemma 6.3, this follows by a direct verification. \( \square \)

Define a linear map
\[
(6.26) \quad \varphi : \text{Coh}_{\Lambda_b}(\mathcal{K}'(G_b)) \otimes \text{Coh}_{\Lambda_g}(\mathcal{K}'(G_g)) \to \text{Coh}_{\Lambda}(\mathcal{K}'(G))
\]
such that
\[
\varphi(\Psi_{\gamma_b} \otimes \Psi_{\gamma_g}) = \Psi_{\varphi(\gamma_b, \gamma_g)}
\]
for all \( \gamma_b \in \mathcal{P}'_{\Lambda_b}(G_b) \) and \( \gamma_g \in \mathcal{P}'_{\Lambda_g}(G_g) \).

**Proposition 6.14.** The linear map (6.26) is \( W_b \times W_g \)-equivariant and injective. Moreover
\[
\varphi(\overline{\Psi}_{\gamma_b} \otimes \overline{\Psi}_{\gamma_g}) = \overline{\Psi}_{\varphi(\gamma_b, \gamma_g)}
\]
for all \( \gamma_b \in \mathcal{P}'_{\Lambda_b}(G_b) \) and \( \gamma_g \in \mathcal{P}'_{\Lambda_g}(G_g) \).

**Proof.** This follows from the Kazhdan-Lusztig-Vogan algorithm (see [Vog83, Theorems 1.6 and 1.12] and [ABV91, Chapter 15]). The case of the odd orthogonal group is treated in [GI19, Section 3] and the case of the real metaplectic group is treated in [RT03, Theorem 5.2]. \( \square \)

By Proposition 6.13 and Proposition 6.14, we have the following corollary.

**Corollary 6.15.** If \(* \in \{ C^*, D^* \} \), then
\[
\text{Coh}_{\Lambda}(\mathcal{K}'(G)) \cong \text{Ind}_{W_b \times W_g}^{W_A}(\text{Coh}_{\Lambda_b}(\mathcal{K}'(G_b)) \otimes \text{Coh}_{\Lambda_g}(\mathcal{K}'(G_g)))
\]
as representations of \( W_A \). In all the other cases,
\[
\text{Coh}_{\Lambda}(\mathcal{K}'(G)) \cong \text{Coh}_{\Lambda_b}(\mathcal{K}'(G_b)) \otimes \text{Coh}_{\Lambda_g}(\mathcal{K}'(G_g))
\]
as representations of \( W_b \times W_g \).

The following proposition is a reformulation of results in [Mat04, Lemma 4.1.3] and [GI19, Lemma 3.3].

**Proposition 6.16.** Let \( \nu = (\nu_b, \nu_g) \in \Lambda_b \times \Lambda_g = \Lambda \). Then there is a unique linear map \( \varphi_{\nu} : \mathcal{K}'_{\nu_b}(G_b) \otimes \mathcal{K}'_{\nu_g}(G_g) \to \mathcal{K}'_{\nu}(G) \) that makes the diagram
\[
\begin{array}{ccc}
\text{Coh}_{\Lambda_b}(\mathcal{K}'(G_b)) & \otimes & \text{Coh}_{\Lambda_g}(\mathcal{K}'(G_g)) \\
\downarrow_{\text{ev}_{\nu_b} \otimes \text{ev}_{\nu_g}} & & \downarrow_{\text{ev}_{\nu}} \\
\mathcal{K}'_{\nu_b}(G_b) & \otimes & \mathcal{K}'_{\nu_g}(G_g) \\
\varphi_{\nu} & & \varphi_{\nu}
\end{array}
\]
commutes, where \( \text{ev} \) indicates the evaluation maps. Moreover, \( \varphi_{\nu} \) is injective, and
\[
\varphi_{\nu}(\text{Irr}'_{\nu_b}(G_b) \times \text{Irr}'_{\nu_g}(G_g)) \subseteq \text{Irr}'_{\nu}(G).
\]

**Proof.** The first assertion follows directly from Proposition 5.1, since the map \( \varphi = W(\Lambda) = W_b \times W_g \)-equivariant. Since \( \varphi \) is injective, Proposition 5.1 also implies that \( \varphi_{\nu} \) is injective.

For the proof of the second assertion, we assume without loss of generality that \( \nu \) is dominant. Let \( \pi_b \in \text{Irr}'_{\nu_b}(G_b) \) and \( \pi_g \in \text{Irr}'_{\nu_g}(G_g) \). Pick basal elements
\[
\Psi_b \in \text{Coh}_{\Lambda_b}(\mathcal{K}'(G_b)) \quad \text{and} \quad \Psi_g \in \text{Coh}_{\Lambda_g}(\mathcal{K}'(G_g))
\]
such that \( \Psi_b(\nu_b) = \pi_b \) and \( \Psi_g(\nu_g) = \pi_g \). Then
\[
\varphi_{\nu}(\pi_b \otimes \pi_g) = \text{ev}_{\nu} \varphi(\Psi_b \otimes \Psi_g).
\]
Proposition 6.14 implies that \( \varphi(\Psi_b \otimes \Psi_g) \) is a basal element of \( \text{Coh}_{\Lambda}(\mathcal{K}'(G)) \). Thus by Lemma 4.7, \( \text{ev}_{\nu}(\varphi(\Psi_b \otimes \Psi_g)) \) is either irreducible or zero. It is clearly nonzero since \( \varphi_{\nu} \) is injective. This proves the last assertion of the proposition. \( \square \)
7. Special unipotent representations in type $A$

We begin the explicit counting of special unipotent representations for a real classical group $G$.

Recall that $\mathcal{K}'(G)$ is the Grothendieck group of the category of genuine Casselman-Wallach representations of $G$ if $\ast \in \{	ilde{A}, \tilde{C}\}$, and $\mathcal{K}'(G) := \mathcal{K}(G)$ otherwise.

Recall we have the Lusztig left cell $^Lc_\lambda$ attached to $\lambda \in \Lambda$ (Definition 3.34). To simplify the notation as well as the terminology, we will also write $^Lc_\lambda := ^Lc_{\lambda_{\tilde{O}}}$, and call it the Lusztig left cell attached to $\tilde{O}$. To reiterate, we repeat its definition: $(\lambda_{\tilde{O}} \in \Lambda)$

$$^Lc_{\lambda_{\tilde{O}}} := \left\{ \sigma \in \text{Irr}(W(\Lambda)) \mid \sigma \text{ occurs in } \left( j_{W_{\lambda_{\tilde{O}}}}^{W(\Lambda)}(\text{sgn}) \otimes \text{sgn} \right) \right\}. $$

As an obvious variant of Corollary 5.4 for $\nu = \lambda_{\tilde{O}}$, we have that

$$\sharp(\text{Unip}_{\lambda_{\tilde{O}}}(G)) = \sum_{\sigma \in ^Lc_{\lambda_{\tilde{O}}}} |\sigma : \text{Coh}_\Lambda(\mathcal{K}'(G))|.$$  

(The equality holds because of Corollary 4.24.) Here $\Lambda \subseteq \mathfrak{a}^*$ is defined in (6.16).

7.1. Some Weyl group representations. The group $S_n$ is identified with the permutation group of the set $\{1, 2, \ldots, n\}$, and $W_n$ is identified with $S_n \ltimes \{\pm 1\}^n$.

Define a quadratic character

$$\varepsilon : W_n \to \{\pm 1\}, \quad (s, (x_1, x_2, \ldots, x_n)) \mapsto x_1 x_2 \cdots x_n.$$  

Then $W_n$ is the kernel of this character. As always, sgn denotes the sign character (of an appropriate Weyl group). Since $S_n$ is a quotient of $W_n$, we may inflate the sign character of $S_n$ to obtain a character of $W_n$, to be denoted by sgn:

$$\overline{\text{sgn}} : \ W_n \rightarrow S_n \xrightarrow{\text{sgn}} \{\pm 1\}.$$  

We have

$$\varepsilon = \text{sgn} \otimes \overline{\text{sgn}}.$$  

We also have the natural embedding

$$W_n \hookrightarrow S_{2n}$$  

via the homomorphism determined by

$$S_n \ni (i, i + 1) \mapsto (2i - 1, 2i + 1)(2i, 2i + 2), \quad (1 \leq i \leq n - 1),$$

and

$$\{\pm 1\}^n \ni (1, \cdots, 1, \underbrace{-1}_{j\text{-th term}}, 1, \cdots, 1) \mapsto (2j - 1, 2j), \quad (1 \leq j \leq n).$$

Here $(i, i + 1), (2i - 1, 2i + 1)$, etc., indicate the transpositions in the permutation groups. Note that $\varepsilon$ is also the restriction of sgn of $S_{2n}$ to $W_n$.

Let $\mathcal{YD}_n$ be the set of Young diagrams of total size $n$. Suppose that $\ast \in \{A^\mathbb{R}, A^\mathbb{H}, \tilde{A}\}$ in the rest of this section. We identify $\mathcal{YD}_n$ with the set $\overline{\text{Nil}}(\mathfrak{g}) = \mathfrak{gl}_n(\mathbb{C}) \setminus \overline{\text{Nil}}(\mathfrak{gl}_n(\mathbb{C}))$ of complex nilpotent orbits and also with the set $\text{Irr}(S_n)$ via the Springer correspondence (see [Car93, 11.4]). More specifically, for $\mathcal{O}' \in \overline{\text{Nil}}(\mathfrak{g}) = \mathcal{YD}_n$, the Springer correspondence is given by Macdonald’s construction for $S_n$ via $j$-induction:

$$\text{Springer}(\mathcal{O}') = \prod_{i \in \mathbb{N}^+} S_{c_i^{\mathbb{N}}(\mathcal{O}_n)} \times \prod_{i \in \mathbb{N}^+} S_{c_i^{\mathbb{N}}(\mathcal{O}_n)}.$$  

Recall that $W(\Lambda) = S_{n_b} \times S_{n_g}$. It is easy to verify that

$$W_{\lambda_{\tilde{O}}} = \prod_{i \in \mathbb{N}^+} S_{c_i^{\mathbb{N}}(\mathcal{O}_n)} \times \prod_{i \in \mathbb{N}^+} S_{c_i^{\mathbb{N}}(\mathcal{O}_n)}.$$  

$$^Lc_{\lambda_{\tilde{O}}} = \left\{ \tau_{\lambda_{\tilde{O}}} \right\}, \quad \text{where } \tau_{\lambda_{\tilde{O}}} := \left( j_{W_{\lambda_{\tilde{O}}}}^{W(\Lambda)}(\text{sgn}) \otimes \text{sgn} \right) = \text{Springer}(\mathcal{O}_b^\mathbb{R}) \otimes \text{Springer}(\mathcal{O}_g^\mathbb{R}).$$
Here and as before, a superscript “t” indicates the transpose of a Young diagram. Also note that the last tensor represents the external tensor product of representations of $S_{n_b}$ and $S_{n_g}$.

The following Propositions 7.1-7.3 follow from Theorem 4.5 by direct computation. We omit the details.

**Proposition 7.1.** Suppose that $\ast = A^R$. For each $l \in \mathbb{N}$, put
\[
C_l := \bigoplus_{t, c, d \in \mathbb{N}} \text{Ind}_{W_t \times S_c \times S_d}^{S_l} \varepsilon \otimes 1 \otimes 1.
\]
Then
\[
\text{Coh}_{\Lambda_b}(K'_b(G_b)) \cong C_{n_b} \quad \text{and} \quad \text{Coh}_{\Lambda_g}(K'_g(G_g)) \cong C_{n_g}.
\]

**Proposition 7.2.** Suppose that $\ast = A^H$. For each even number $l \in \mathbb{N}$, put
\[
C_l := \text{Ind}_{S_l}^{W_l} \varepsilon.
\]
Then
\[
\text{Coh}_{\Lambda_b}(K'_b(G_b)) \cong C_{n_b} \quad \text{and} \quad \text{Coh}_{\Lambda_g}(K'_g(G_g)) \cong C_{n_g}.
\]

**Proposition 7.3.** Suppose that $\ast \in \{A, \tilde{A}\}$. Then
\[
\text{Coh}_{\Lambda_b}(K'_b(G_b)) \cong \text{Ind}_{W_{n_b}}^{S_{n_b}} 1
\]
and
\[
\text{Coh}_{\Lambda_g}(K'_g(G_g)) \cong \bigoplus_{t, s, r \in \mathbb{N}} \text{Ind}_{W_t \times S_s \times S_r}^{S_{n_g}} 1 \otimes \text{sgn} \otimes \text{sgn}.
\]

7.2. **Special unipotent representations of $GL_n(\mathbb{R})$ and $GL_n(\mathbb{H})$.** Recall from Definition 2.11 and Definition 2.13 the set $\text{PAP}_\ast(\tilde{O})$, which is the set of paintings on $\tilde{O}$ that has type $\ast \in \{A^R, A^H, A, \tilde{A}\}$. Recall that if $\ast = A^R$, then
\[
\text{(7.8)} \quad \#(\text{PAP}_\ast(\tilde{O})) = \prod_{i \in \mathbb{N}^+} \left( \# \{ i \in \mathbb{N}^+ \mid r_i(\tilde{O}) = r \} + 1 \right).
\]
Also note that if $\ast = A^H$, then
\[
\text{(7.9)} \quad \#(\text{PAP}_\ast(\tilde{O})) = \begin{cases} 1, & \text{if } \tilde{O} = \tilde{O}_g; \\ 0, & \text{otherwise}. \end{cases}
\]

**Proposition 7.4.** Suppose that $\ast \in \{A^R, A^H\}$. Then
\[
[\tau_{\lambda_{\tilde{O}}} : \text{Coh}_{\lambda}(K'(G))] = \#(\text{PAP}_\ast(\tilde{O}_g)) \times \#(\text{PAP}_\ast(\tilde{O}_b)) = \#(\text{PAP}_\ast(\tilde{O})).
\]

**Proof.** In view of Proposition 7.1 and the description of the left cell representation $\tau_{\lambda_{\tilde{O}}}$ in (7.7), the first equality follows from Pieri’s rule ([GW09, Corollary 9.2.4]) and the following branching formula (see [BV83b, Lemma 4.1 (b)]):
\[
\text{(7.10)} \quad \text{Ind}_{W_t}^{S_{2t}} \varepsilon = \bigoplus_{\sigma \in \text{YD}_{2t}} \sigma \quad (t \in \mathbb{N}).
\]
The last equality follows from (7.8) and (7.9). \qed

We are in the setting of Theorem 2.12. Recall the map
\[
\text{PAP}_\ast(\tilde{O}) \to \text{Unip}_{\tilde{O}}(G),
\]
where $\pi_P$ is defined in Section 2.4. It is proved in [Vog86, Theorem 3.8] that the above map is injective. Then the map is bijective by (7.2) and Proposition 7.4. This proves Theorem 2.12.
7.3. Special unipotent representations of unitary groups. In this subsection, we suppose \( \star \in \{ A, \bar{A} \} \) so that

\[
G = \begin{cases} 
U(p, q), & \text{if } \star = A; \\
\bar{U}(p, q), & \text{if } \star = \bar{A}.
\end{cases}
\]

For \( \mathcal{P} \in \text{PAP}_n(\bar{O}) \), we have defined its signature \((pp, qp)\) in (2.5). Recall that \( \mathcal{O} := d_{BV}(\bar{O}) = \bar{O}' \subseteq \text{Nil}(g^*) \). Let \( \overline{\text{Nil}}_G(\mathcal{O}) \) denote the set of \( G \)-orbits in \((\sqrt{-1}g_0^*) \cap \mathcal{O})\), where \( g_0 \) denotes the Lie algebra of \( G \) which equals \( u(p, q) \).

We first consider the case when \( \bar{O} = \bar{O}_g \). In this good parity setting, we will state a counting result on \( \text{Unip}_\bar{O}(G) \). The elements in \( \text{Unip}_\bar{O}(G) \) can be constructed by cohomological induction explicitly and they are irreducible and unitary. See [BV83b, Theorem 4.2], and also [Mat96, for the construction of all elements of \( \text{Unip}_\bar{O}(G) \), [Tra04, Section 2] and [MR19, Section 4]. We refer the reader to [BMSZ21, Section 12] for the construction of all elements of \( \text{Unip}_\bar{O}(G) \) by the method of theta lifting.

**Theorem 7.5** (cf. [BV83b, Theorem 4.2] and [Tra04, Theorem 2.1]). Suppose that \( \bar{O} = \bar{O}_g \). Then

\[
\sharp(\text{Unip}_\bar{O}(G)) = \sharp \{ \mathcal{P} \in \text{PAP}_n(\bar{O}) \mid (pp, qp) = (p, q) \} = \sharp(\overline{\text{Nil}}_G(\mathcal{O})).
\]

Moreover, for every \( \pi \in \text{Unip}_\bar{O}(G) \), its wavefront set \( WF(\pi) \) is the closure in \( \sqrt{-1}g_0^* \) of a unique orbit \( \mathcal{O}_\pi \in \overline{\text{Nil}}_G(\mathcal{O}) \), and the map

\[
\text{Unip}_\bar{O}(G) \rightarrow \overline{\text{Nil}}_G(\mathcal{O}), \quad \pi \mapsto \mathcal{O}_\pi
\]

is bijective.

**Proof.** In view of the counting formula in (7.2), the first equality in (7.11) follows from Proposition 7.3, (7.7), (7.10), and Pieri’s rule ([GW09, Corollary 9.2.4]). The second equality in (7.11) will follow directly from the bijectivity of (7.12).

The assignment of wavefront set yields a bijection

\[
\{ \text{cell in the basal representation } \text{Coh}_A(K'_\mathcal{O}(G))/\text{Coh}_A(K'_\mathcal{O}(G)) \} \rightarrow \overline{\text{Nil}}_G(\mathcal{O}),
\]

and every cell representation in \( \text{Coh}_A(K'_\mathcal{O}(G))/\text{Coh}_A(K'_\mathcal{O}(G)) \) is irreducible and isomorphic to \( \tau_{\lambda_\mathcal{O}} \). See [BV83b, Theorem 4.2] and [Bož02, Theorem 5] (we have used [SV00, Theorem 1.4] to rephrase the result in terms of real nilpotent orbits).

Note that (3.24) implies that

\[
[W_{\lambda_\mathcal{O}} : \tau_{\lambda_\mathcal{O}}] = 1.
\]

Recall from Lemma 6.6 that \( \lambda_\mathcal{O} \in ^*\mathfrak{a}_\mathcal{O}^* \) is dominant. As in the proof of Theorem 5.3, for each cell \( \mathcal{C} \) in \( \text{Coh}_A(K'_\mathcal{O}(G)) \) that is not a cell in \( \text{Coh}_A(K'_\mathcal{O}(G)) \), there is a unique element \( \Psi_\mathcal{C} \in \mathcal{C} \) such that \( \Psi_\mathcal{C}(\lambda_\mathcal{O}) \neq 0 \). Then

\[
\text{Unip}_\bar{O}(G) = \{ \Psi_\mathcal{C}(\lambda_\mathcal{O}) \} \mathcal{C} \text{ is a cell in } \text{Coh}_A(K'_\mathcal{O}(G)) \text{ that is not a cell in } \text{Coh}_A(K'_\mathcal{O}(G)).
\]

This implies the bijectivity assertion of the theorem. \( \square \)

**Lemma 7.6.** If \( \sharp(\text{Unip}_\bar{O}(G)) \neq 0 \), then \( \bar{O}_b = 2\bar{O}'_b \) for some Young diagram \( \bar{O}'_b \).

**Proof.** Suppose that \( \sharp(\text{Unip}_\bar{O}(G)) \neq 0 \). Then

\[
[\tau_{\lambda_\mathcal{O}_b} : \text{Coh}_{A_b}(K'(G_b))] \neq 0.
\]

Recall from Proposition 7.3 that

\[
\text{Coh}_{A_b}(K'(G_b)) \cong \text{Ind}_{W_t}^{S_2t} 1, \quad (t = \frac{m_b}{2}).
\]

Thus (7.13) is the same as saying that

\[
[\text{Springer}(\bar{O}'_b) : \text{Ind}_{W_t}^{S_2t} 1] \neq 0.
\]
Similar to (7.10), we have that
\[
\text{Ind}_{\bar{W}_t}^{\mathbb{S}_t} 1 = \bigoplus_{\sigma \in \mathbb{Y}_2 t} \sigma \quad (t \in \mathbb{N}).
\]
This implies the lemma. \(\square\)

Now we assume that \(\bar{O}_b = 2\bar{O}_b'\) for some Young diagram \(\bar{O}_b'\).

The group \(G\) has an \(\bar{O}\)-relevant parabolic subgroup \(P\) whose Levi quotient is naturally isomorphic to \(G'_b \times G_k\), where \(G'_b := \text{GL}\left(\frac{k}{2}\right)\). Let \(\pi'_b\) denote the unique element in \(\text{Unip}_{\bar{O}_b'}(\text{GL}\left(\frac{k}{2}\right))\).

Then for every \(\pi_{\bar{g}} \in \text{Unip}_{\bar{O}}(G_{\bar{g}})\), the normalized smooth parabolic induction \(\pi_{\bar{g}} \rtimes \pi_{\bar{g}}\) is irreducible by [Mat96, Theorem 3.2.2] and is an element of \(\text{Unip}_P(G)\) (cf. [MR19, Theorem 5.3]).

**Theorem 7.7.** The equality
\[
\sharp(\text{Unip}_P(G)) = \sharp(\text{Unip}_{\bar{O}(G)}(G_{\bar{g}}))
\]
holds, and the map
\[
\pi_{\bar{g}} : \text{Unip}_{\bar{O}}(G_{\bar{g}}) \to \text{Unip}_P(G),
\]
is bijective.

**Proof.** By the counting formula in (7.2),
\[
\sharp(\text{Unip}_P(G)) = [\tau_{\lambda_P} : \text{Coh}_A(K'(G))].
\]
Similarly,
\[
\sharp(\text{Unip}_{\bar{O}}(G_{\bar{g}})) = [\tau_{\lambda_{\bar{O}}} : \text{Coh}_{A_b}(K'(G_{\bar{g}}))].
\]
The proof of Lemma 7.6 shows that
\[
[\tau_{\lambda_{\bar{O}}} : \text{Coh}_{A_b}(K'(G_{\bar{g}}))] = 1.
\]
The above three equalities clearly imply (7.15).

By the calculation of the wavefront set of the induced representations ([Bar00, Corollary 5.0.10]), Theorem 7.5 implies that all the representations \(\pi_{\bar{g}'} \rtimes \pi_{\bar{g}}\), where \(\pi_{\bar{g}}\) varies in \(\text{Unip}_{\bar{O}}(G_{\bar{g}})\), have pairwise distinct wavefront sets. Thus the map (7.16) is injective. Hence it is bijection by the counting assertion (7.15). \(\square\)

Theorems 2.15 and 2.16 then follow from Theorems 7.5 and 7.7.

**Remark 7.8.** When \(\bar{O} \neq \bar{O}_k\), the wavefront set \(\text{WF}(\pi)\), where \(\pi \in \text{Unip}_P(G)\), may not be the closure of a single orbit in \(\text{Nil}_G(O)\).

8. **Special unipotent representations in type BCD: counting**

In this section and the next one, we assume that \(\star \in \{B, C, \bar{C}, C^*, D, D^*\}\). We will give the detailed description of the coherent continuation representations and the Lusztig left cells, which yield the explicit counting of special unipotent representations via the Littlewood-Richardson rule.

8.1. **The coherent continuation representations.** Let
\[
H_t := \mathcal{W}_t \times \{\pm 1\}^t, \quad (t \in \mathbb{N}),
\]
to be viewed as a subgroup in \(\mathcal{W}_{2t}\) such that
- the first factor \(\mathcal{W}_t\) sits in \(S_{2t} \subseteq \mathcal{W}_{2t}\) as in (7.5) and (7.6),
- the element \((1, \cdots, 1, \underbrace{-1}_{i\text{-th term}}, 1, \cdots, 1) \in \{\pm 1\}^t\) acts on \(\mathbb{C}^{2t}\) by
  \[
  (x_1, x_2, \cdots, x_{2t}) \mapsto (x_1, \cdots, x_{2i-2}, -x_{2i}, -x_{2i-1}, x_{2i+1}, \cdots, x_{2t}).
  \]
Note that $H_t$ is also a subgroup of $W_{2t}$. Define a quadratic character

$$\eta : H_t = W_t \times \{ \pm 1 \} \rightarrow \{ \pm 1 \},$$

$$(g, (a_1, a_2, \cdots, a_t)) \mapsto a_1a_2\cdots a_t.$$  

Recall from Section 6.3 the classical groups $G_b$ and $G_g$, the $\mathbb{Z}^n$-coset $\Lambda = \Lambda_b \times \Lambda_g$, the Weyl group $W_b$ of $g_b$ and the Weyl group $W_g$ of $g_g$.

The following Propositions 8.1 and 8.2 also follow from Theorem 4.5 by direct computation. We omit the details. cf. [McG98, Applications] and [AdC09]. The following subsection gives an example of type $B$ of the structure data relevant for the computation.

**Proposition 8.1.** As representations of $W_b$,

$$\text{Coh}_{W_b}(\mathcal{K}'(G_b)) \cong \begin{cases} \bigoplus_{0 \leq p_b - (2t + a + 2r) \leq 1, 0 \leq q_b - (2t + a + 2s) \leq 1} \text{Ind}_{H_t \times W_2 \times W_2} \eta \otimes 1 \otimes \text{sgn} \otimes \text{sgn}, & \text{if } * = B; \\
\bigoplus_{2t + a + c + d = n_b} \text{Ind}_{W_2 \times W_2} \eta \otimes 1 \otimes \eta \otimes 1 \otimes \text{sgn} \otimes \text{sgn} \otimes \text{sgn}, & \text{if } * = C; \\
\bigoplus_{2t + a + a' = n_g} \text{Ind}_{W_2} \eta \otimes \eta \otimes 1, & \text{if } * = C'; \\
\bigoplus_{(t + r, t + s) = (\frac{p_b}{2}, \frac{q_b}{2})} \text{Ind}_{W_2 \times W_2} \eta \otimes \eta \otimes 1 \otimes 1, & \text{if } * = C^*; \\
\bigoplus_{2t + c + d + 2r = p_g} \text{Ind}_{W_2 \times W_2} \eta \otimes 1 \otimes 1 \otimes 1, & \text{if } * = D; \\
\bigoplus_{2t + a = n_g} \text{Ind}_{W_2} \eta \otimes \text{sgn}, & \text{if } * = D^*, \\
\end{cases}$$

where in the case when $* \in \{ C, D \}$, the right-hand side space is viewed as a representation of $W_b = W_{2b}$, by restriction.

Recall the quadratic character $\text{sgn}$ of $W_n$ in (7.4).

**Proposition 8.2.** As representations of $W_g$,

$$\text{Coh}_{W_g}(\mathcal{K}'(G_g)) \cong \begin{cases} \bigoplus_{0 \leq p_g - (2t + a + 2r) \leq 1, 0 \leq q_g - (2t + a + 2s) \leq 1} \text{Ind}_{H_t \times W_2 \times W_2} \eta \otimes 1 \otimes \text{sgn} \otimes \text{sgn}, & \text{if } * = B; \\
\bigoplus_{2t + a + c + d = n_g} \text{Ind}_{W_2 \times W_2} \eta \otimes 1 \otimes \eta \otimes 1 \otimes \text{sgn} \otimes \text{sgn} \otimes \text{sgn}, & \text{if } * = C; \\
\bigoplus_{2t + a + a' = n_g} \text{Ind}_{W_2} \eta \otimes \eta \otimes 1, & \text{if } * = C'; \\
\bigoplus_{(t + r, t + s) = (\frac{p_b}{2}, \frac{q_b}{2})} \text{Ind}_{W_2 \times W_2} \eta \otimes \eta \otimes 1 \otimes 1 \otimes 1, & \text{if } * = C^*; \\
\bigoplus_{2t + c + d + 2r = p_g} \text{Ind}_{W_2 \times W_2} \eta \otimes 1 \otimes 1 \otimes 1, & \text{if } * = D; \\
\bigoplus_{2t + a = n_g} \text{Ind}_{W_2} \eta \otimes \text{sgn}, & \text{if } * = D^*, \\
\end{cases}$$

where in the case when $* = D$, the right-hand side space is viewed as a representation of $W_g = W_{2g}$ by restriction.

**8.2. An example: type $B$.** In this subsection, we give the relevant structure data leading to the results of Proposition 8.1 and Proposition 8.2 when $G = SO(p, q)$ with $p + q = 2n + 1$.

We retain the notation of Section 4 and Section 6. The set of the conjugate classes of Cartan subgroups is parameterized by

$$(8.3) \quad \{ (t, a) \in \mathbb{N} \times \mathbb{N} \mid p - 2t - a \geq 0, q - 2t - a \geq 0 \}. $$

Fixing an element $(t, a)$ in (8.3), a representative of the conjugate class is given by a Cartan subgroup $H_{t,a}$ isomorphic to

$$(\mathbb{C}^\times)^l \times (\mathbb{R}^\times)^a \times SO(2)^{n-2t-a}$$

with the set of real roots

$$\{ \pm (e_{2i-1} - e_{2i}) : 1 \leq i \leq t \} \cup \{ \pm e_i : 2t + 1 \leq i \leq 2t + a \}$$

and the set of imaginary roots

$$\{ \pm (e_{2i-1} + e_{2i}) : 1 \leq i \leq t \} \cup \{ \pm e_i \pm e_j : 2t + a + 1 \leq i < j \leq n \} \cup \{ e_k : 2t + a + 1 \leq k \leq n \}. $$

For a root $\alpha$, write $s_\alpha$ the corresponding reflection. Then the real Weyl group $W_{H_{t,a}}$ is given by

$$W_{H_{t,a}} = S_t \ltimes (W(A_1)^l \times W(A_1)^l) \times W_a \times W_s \times W_s,$$

where
\[ W(A_1) \text{ is the Weyl group of type } A_1; \]
\[ r = [(p - 2t - a)/2] \text{, and } s = [(q - 2t - a)/2], \text{ (note that } r + s = n - 2t - a); \]
\[ S_i \text{ is generated by } s_{e_{2i-1} - e_{2i+1}}, s_{e_{2i} - e_{2i+2}} \text{ for } 1 \leq i < t; \]
\[ \text{the first } W(A_1)^t \text{ is generated by } s_{e_{2i-1} - e_{2i}} \text{ for } 1 \leq i \leq t; \]
\[ \text{the second } W(A_1)^t \text{ is generated by } s_{e_{2i-1} + e_{2i}} \text{ for } 1 \leq i \leq t; \]
\[ W_a, W_r \text{ and } W_s \text{ are identified respectively with the subgroups of } W_n \text{ acting on the } \]
subspaces spanned by \{ e_i : 2t + 1 \leq i \leq 2t + a \}, \{ e_i : 2t + a + 1 \leq i \leq 2t + a + r \}, \text{ and } \{ e_i : 2t + a + r + 1 \leq i \leq n \}.\]

Note that the imaginary roots have the same span as
\[ \{ e_1 + e_2, e_3 + e_4, \ldots, e_{2t-1} + e_{2t}, e_{2t+a+1}, e_{2t+a+2}, \ldots, e_n \}. \]

The quadratic character \( \text{sgn}_\text{im} \), when restricted to \( W_{H_{t,a}} \), is

- the trivial character on \( S_t \), the first \( W(A_1)^t \) factor and \( W_a \),
- the sign character on the second \( W(A_1)^t \) factor, and \( W_s \times W_r \).

We identify \( H_t = W_t \times \{ \pm 1 \}^t \) with \( S_t \times (W(A_1)^t \times W(A_1)^t) \), viewed as a subgroup of \( W_{2t} \) as in (8.1).

Let \( P_{t,a} \) denote the set of parameters in \( P_{\Lambda}(G) \) which are represented by \( (H, \xi, \Gamma) \in P_{\Lambda}(G) \) with \( H \) conjugate to \( H_{t,a} \). (Recall \( \Lambda = \Lambda_0 \times \Lambda_g \) is the \( \mathbb{Z}^n \)-coset defined in (6.16).) Under the cross action, the \( W_\Lambda \)-orbits of \( P_{t,a} \) is parameterized by a pair of numbers \( (c, d) \) in
\[ \{ (c, d) \in \mathbb{N} \times \mathbb{N} \mid n_b - c - d \text{ is a non-negative even integer and } c + d \leq a \}. \]

Write \( t_1 = (n_b - c - d)/2 \) and \( t_2 = t - t_1 \). Set
\[ \nu := (0, \ldots, 0, \frac{1}{2}, \ldots, \frac{1}{2}) \in \Lambda_b \times \Lambda_g = \Lambda \]

Then there is a unique \( W_\Lambda \)-orbit \( P_{t,a,c,d} \) in \( P_{\Lambda}(G) \) represented by an element \( \gamma_{t,a,c,d} = (H_{t,a}, \xi, \Gamma) \in P_{\Lambda}(G) \) satisfying the following conditions:

- the group \( H_{t,a} \) has a decomposition \( H_{t,a} = H_b \times H_g \), with
  \[ H_b = (\mathbb{C}^\times)^{t_1} \times (\mathbb{R}^\times)^{c+d}, \quad H_g = (\mathbb{C}^\times)^{t_2} \times (\mathbb{R}^\times)^{a-c-d} \times \text{SO}(2)^{n-2t-a}; \]
- \( \xi^{(\alpha_b h_b)} = h_b \) and \( \xi^{(\alpha_g h_g)} = h_g \),
  where \( h_b \) and \( h_g \) are respectively the complexified Lie algebras of \( H_b \) and \( H_g \);
- \( \Gamma_{\nu_1^{(\pm 1)^{c+d}}} = \underbrace{1 \otimes \cdots \otimes 1}_{c} \otimes \text{sgn} \otimes \cdots \otimes \text{sgn} \), where \( \{ \pm 1 \}^{c+d} \) is viewed as a subgroup of \( H_{t,a} \) via the inclusions
  \[ \{ \pm 1 \}^{c+d} \subseteq (\mathbb{R}^\times)^{c+d} \subseteq H_b = (\mathbb{C}^\times)^{t_1} \times (\mathbb{R}^\times)^{c+d} \subseteq H_{t,a}; \]
- \( \Gamma_{\nu_1^{(\pm 1)^{a-c-d}}} = (\mathbb{R}^\times)^{a-c-d} \otimes (\mathbb{R}^\times)^{a-c-d} \otimes \text{SO}(2)^{n-2t-a} \subseteq H_{t,a}. \]

Recall that the real Weyl group \( W_{H_{t,a}} \) may be identified as a subgroup of the abstract Weyl group via \( \xi \). Then the cross stabilizer \( \gamma_{t,a,c,d} \) of \( \gamma_{t,a,c,d} \) is the following subgroup of \( W_\Lambda \):
\[ (H_{t_1} \times W_c \times W_d) \times (H_{t_2} \times S_{a-c-d} \times W_s \times W_r) \subseteq W_b \times W_g, \]
where
- \( H_{t_1} \times H_{t_2} \) is the natural subgroup of \( H_{t,a} \subseteq W_{2t}, \)
- \( W_c \times W_d \times S_{a-c-d} \) is the natural subgroup of \( W_c \times W_d \times W_{a-c-d} \subseteq W_a. \)

With the above structure data one may deduce the two formulas on the coherent continuation representations stated in Section 8.1.
8.3. The Lusztig left cells. To ease the notation, for every sequence $a_1 \geq a_2 \geq \cdots \geq a_k \geq 0$ ($k \geq 0$) of integers, we let $[a_1, a_2, \cdots, a_k]_{\text{col}}$ denote the Young diagram whose $i$-th column has length $a_i$ if $1 \leq i \leq k$ and length 0 otherwise. Likewise, we let $[a_1, a_2, \cdots, a_k]_{\text{row}}$ denote the Young diagram whose $i$-th row has length $a_i$ if $1 \leq i \leq k$ and length 0 otherwise.

As usual, we identify $\text{Irr}(W_t)$ ($t \in \mathbb{N}$) with the set of bipartitions $\tau = (\tau_L, \tau_R)$ of total size $t$ ([Car93, Section 11.4]). Here and henceforth, a bipartition means a pair of Young diagrams, and the total size refers to $|\tau_L| + |\tau_R|$. We also let $(\tau_L, \tau_R)I \in \text{Irr}(W_t')$ denote the irreducible representation given by

- the restriction of $(\tau_L, \tau_R) \in \text{Irr}(W_t)$ if $\tau_L \neq \tau_R$, and
- the induced representation $\text{Ind}_L^W \tau_L$ if $\tau_L = \tau_R$.

Take an element $w \in W_t$ such that $wW_t'$ generates the group $W_t/W_t'$. Define $(\tau_L, \tau_R)II \in \text{Irr}(W_t')$ to be the twist of $(\tau_L, \tau_R)I$ by the conjugation by $w$, namely there is a linear isomorphism $\kappa : (\tau_L, \tau_R)I \rightarrow (\tau_L, \tau_R)II$ such that

$$\kappa(g \cdot u) = (wgw^{-1}) \cdot (\kappa(u)), \quad \text{for all } g \in W_t, \ u \in (\tau_L, \tau_R)I.$$

Note that

$$(\tau_L, \tau_R) = (\tau_R, \tau_L)I \quad \text{and} \quad (\tau_L, \tau_R)II = (\tau_R, \tau_L)II$$

in all cases, and $(\tau_L, \tau_R)I = (\tau_L, \tau_R)II$ when $\tau_L \neq \tau_R$.

In the rest of this subsection, we describe the Lusztig left cell $^L \mathcal{Q}_{\lambda_O} (\subseteq \text{Irr}(W(\Lambda)))$ attached to $\lambda_O$ (Definition 3.34). Here the integral Weyl group $W(\Lambda) = W_b \times W_g'$, and is described in Section 6.3.

Recall that if $* \in \{C, C^*, D, D^*\}$, then $W_b = W_{b_0}$. In this case, we say that

$$(8.4) \quad \hat{\mathcal{O}}_b \text{ has type I, if the number of negative entries of } \lambda_{\hat{\mathcal{O}}_b} \text{ has the same parity as } \frac{n_b}{2};$$

otherwise we say that $\hat{\mathcal{O}}_b$ has type II.

Recall also that if $* = \tilde{C}$, then $W_g' = W_{g_0}$. In this case, we say that

$$(8.5) \quad \hat{\mathcal{O}}_g \text{ has type I if } \frac{n_g}{2} \text{ is even, and type II if } \frac{n_g}{2} \text{ is odd.}$$

Define two Young diagrams

$$(8.6) \quad \tau_{L,b} := \begin{cases} \frac{1}{2}(r_1(\hat{\mathcal{O}}_b) + 1), \frac{1}{2}(r_2(\hat{\mathcal{O}}_b') + 1), \cdots, \frac{1}{2}(r_c(\hat{\mathcal{O}}_b'') + 1)_{\text{col}}, & \text{if } * \in \{B, \tilde{C}\}; \\ \frac{1}{2}(r_1(\hat{\mathcal{O}}_b''), \frac{1}{2}r_2(\hat{\mathcal{O}}_b'), \cdots, \frac{1}{2}r_c(\hat{\mathcal{O}}_b''))_{\text{col}}, & \text{if } * \in \{C, C^*, D, D^*\}; \end{cases}$$

and

$$(8.7) \quad \tau_{R,b} := \begin{cases} \frac{1}{2}(r_1(\hat{\mathcal{O}}_b') - 1), \frac{1}{2}(r_2(\hat{\mathcal{O}}_b') - 1), \cdots, \frac{1}{2}(r_c(\hat{\mathcal{O}}_b''') - 1)_{\text{col}}, & \text{if } * \in \{B, \tilde{C}\}; \\ \frac{1}{2}(r_1(\hat{\mathcal{O}}_b'''), \frac{1}{2}r_2(\hat{\mathcal{O}}_b''), \cdots, \frac{1}{2}r_c(\hat{\mathcal{O}}_b''))_{\text{col}}, & \text{if } * \in \{C, C^*, D, D^*\}; \end{cases}$$

where $c := c_r(\hat{\mathcal{O}}_b)$.

Define an irreducible representation $\tau_b \in \text{Irr}(W_b)$ attached to $\hat{\mathcal{O}}_b$ by

$$(8.8) \quad \tau_b := \begin{cases} (\tau_{L,b}, \tau_{R,b}), & \text{if } * \in \{B, \tilde{C}\}; \\ (\tau_{L,b}, \tau_{R,b})I, & \text{if } * \in \{C, C^*, D, D^*\} \text{ and } \hat{\mathcal{O}}_b \text{ has type I}; \\ (\tau_{L,b}, \tau_{R,b})II, & \text{if } * \in \{C, C^*, D, D^*\} \text{ and } \hat{\mathcal{O}}_b \text{ has type II.} \end{cases}$$

Recall the set $\text{PP}_*(\hat{\mathcal{O}}_g)$ from Definition 2.21. Put

$$A(\hat{\mathcal{O}}) := A(\hat{\mathcal{O}}_g) := \text{the power set of } \text{PP}_*(\hat{\mathcal{O}}_g),$$
which is identified with the free $\mathbb{F}_2$-vector space with free basis $\text{PP}_*(\tilde{\mathcal{O}}_g)$. Here $\mathbb{F}_2 := \mathbb{Z}/2\mathbb{Z}$ is the field with two elements only. Note that $\{\emptyset, \text{PP}_*(\tilde{\mathcal{O}}_g)\}$ is a subgroup of $A(\tilde{\mathcal{O}})$. Define

$$ \tilde{A}(\tilde{\mathcal{O}}) := \tilde{A}(\tilde{\mathcal{O}}_g) := \begin{cases} A(\tilde{\mathcal{O}}_g)/\{\emptyset, \text{PP}_*(\tilde{\mathcal{O}}_g)\}, & \text{if } \ast = \tilde{C}; \\ A(\tilde{\mathcal{O}}_g), & \text{otherwise}. \end{cases} $$

Generalizing (2.16), for each $\varphi \in A(\tilde{\mathcal{O}})$, we define a pair of Young diagrams

$$ (t_\varphi, j_\varphi) := (t_\ast(\tilde{\mathcal{O}}, \varphi), j_\ast(\tilde{\mathcal{O}}, \varphi)) $$

as in what follows.

If $\ast = B$, then

$$ c_1(j_\varphi) = \frac{r_1(\tilde{\mathcal{O}}_g)}{2}, $$

and for all $i \geq 1$,

$$ (c_i(t_\varphi), c_{i+1}(j_\varphi)) = \begin{cases} \left( \frac{r_{2i+1}(\tilde{\mathcal{O}}_g)}{2}, \frac{r_{2i}(\tilde{\mathcal{O}}_g)}{2} \right), & \text{if } (2i, 2i + 1) \in \varphi; \\ \left( \frac{r_{2i}(\tilde{\mathcal{O}}_g)}{2}, \frac{r_{2i+1}(\tilde{\mathcal{O}}_g)}{2} \right), & \text{otherwise}. \end{cases} $$

If $\ast = \tilde{C}$, then for all $i \geq 1$,

$$ (c_i(t_\varphi), c_i(j_\varphi)) = \begin{cases} \left( \frac{r_{2i}(\tilde{\mathcal{O}}_g)}{2}, \frac{r_{2i-1}(\tilde{\mathcal{O}}_g)}{2} \right), & \text{if } (2i - 1, 2i) \in \varphi; \\ \left( \frac{r_{2i-1}(\tilde{\mathcal{O}}_g)}{2}, \frac{r_{2i}(\tilde{\mathcal{O}}_g)}{2} \right), & \text{otherwise}. \end{cases} $$

If $\ast \in \{C, C^*\}$, then for all $i \geq 1$,

$$ (c_i(j_\varphi), c_i(t_\varphi)) = \begin{cases} \left( \frac{r_{2i}(\tilde{\mathcal{O}}_g) - 1}{2}, \frac{r_{2i-1}(\tilde{\mathcal{O}}_g) + 1}{2} \right), & \text{if } (2i - 1, 2i) \in \varphi; \\ (0, 0), & \text{if } (2i - 1, 2i) \text{ is vacant in } \tilde{\mathcal{O}}_g; \\ \left( \frac{r_{2i-1}(\tilde{\mathcal{O}}_g) - 1}{2}, \frac{r_{2i}(\tilde{\mathcal{O}}_g) + 1}{2} \right), & \text{if } (2i - 1, 2i) \text{ is tailed in } \tilde{\mathcal{O}}_g; \end{cases} $$

If $\ast \in \{D, D^*\}$, then

$$ c_1(t_\varphi) = \begin{cases} 0, & \text{if } r_1(\tilde{\mathcal{O}}_g) = 0; \\ \frac{r_1(\tilde{\mathcal{O}}_g)}{2} + 1, & \text{if } r_1(\tilde{\mathcal{O}}_g) > 0, \end{cases} $$

and for all $i \geq 1$,

$$ (c_i(j_\varphi), c_{i+1}(t_\varphi)) = \begin{cases} \left( \frac{r_{2i+1}(\tilde{\mathcal{O}}_g) - 1}{2}, \frac{r_{2i}(\tilde{\mathcal{O}}_g) + 1}{2} \right), & \text{if } (2i, 2i + 1) \in \varphi; \\ (0, 0), & \text{if } (2i, 2i + 1) \text{ is vacant in } \tilde{\mathcal{O}}_g; \\ \left( \frac{r_{2i}(\tilde{\mathcal{O}}_g) - 1}{2}, \frac{r_{2i+1}(\tilde{\mathcal{O}}_g) + 1}{2} \right), & \text{if } (2i, 2i + 1) \text{ is tailed in } \tilde{\mathcal{O}}_g; \end{cases} $$

In the notation above, we have

$$ (t_{\varphi}, j_{\varphi}) = (t_{\ast}(\tilde{\mathcal{O}}, \emptyset), j_{\ast}(\tilde{\mathcal{O}}, \emptyset)). $$

Here $(t_{\tilde{\mathcal{O}}}, j_{\tilde{\mathcal{O}}})$ is the pair of Young diagrams attached to $\tilde{\mathcal{O}}$, as in (2.16).

We define an element $\tau_\varphi \in \text{Irr}(W'_g)$ by

$$ \tau_\varphi := \begin{cases} (t_\varphi, j_\varphi), & \text{if } \ast \in \{B, C, C^*\}; \\ (t_\varphi, j_\varphi)|_{II}, & \text{if } \ast \in \{D, D^*\}; \\ (t_\varphi, j_\varphi)|_{I}, & \text{if } \ast = \tilde{C} \text{ and } \tilde{\mathcal{O}}_g \text{ has type I}; \\ (t_\varphi, j_\varphi)|_{II}, & \text{if } \ast = \tilde{C} \text{ and } \tilde{\mathcal{O}}_g \text{ has type II}. \end{cases} $$

Note that if $\ast = \tilde{C}$, then $\tau_\varphi = \tau_{\varphi^c}$, where $\varphi^c$ is the complement of $\varphi$ in $\text{PP}_*(\tilde{\mathcal{O}}_g)$. Therefore in all cases, $\tau_\varphi \in \text{Irr}(W'_g)$ is defined for every $\varphi \in A(\tilde{\mathcal{O}})$. 

Recall the Lusztig left cell \( ^L \mathcal{S}_O \) attached to \( \mathcal{O} \), which is the set of all \( \sigma \in \text{Irr}(W(\Lambda)) \) that occurs in the multiplicity free representation

\[
\left( J_{W(\Lambda)}^W \text{ sgn} \right) \otimes \text{ sgn}.
\]

**Proposition 8.3** (cf. Barbasch-Vogan [BV85, Proposition 5.28]). The map

\[
\bar{\Lambda}(\mathcal{O}) \to \ ^L \mathcal{S}_O,
\]

\( \bar{\delta} \mapsto \tau_\bar{b} \otimes \tau_{\bar{\delta}} \)

is well-defined and bijective, and

\[
\tau_\mathcal{O} := \tau_\bar{b} \otimes \tau_{\bar{\delta}}
\]

is the unique special representation in \( ^L \mathcal{S}_O \). Moreover,

(8.11) \( \bar{J}_W^W(\tau_\mathcal{O}) \) corresponds to \( d_{BV}(\mathcal{O}) \) under the Springer correspondence,

and

(8.12) \( d_{BV}(\mathcal{O}) = (\mathcal{O}_b^\dagger \sqcup (\mathcal{O}_b^\prime)^\dagger \sqcup d_{BV}(\mathcal{O}_2) \)

as Young diagrams.

**Proof.** When \( \mathcal{O} = \mathcal{O}_g \) and \( \ast \neq \tilde{C} \), the proposition is proved in [BV85, Proposition 5.28]. When \( \ast \neq \tilde{C} \), (8.11) and (8.12) are proved in [BV85, Proposition A2]. In general, the proposition follows from Lusztig’s formula of \( J \)-induction in [Lus84, §4.4-4.6], as well as the explicit form of the Springer correspondence [Sho79]. \( \square \)

Recall that \( W'_n = W_{n_g} \) when \( \ast \in \{ \tilde{C}, D, D^* \} \). Since the representation theory of \( W_n \) is more elementary than that of \( W'_n \), we prefer to work with \( W_n \) instead of \( W'_n \) in some situations. For this reason, we also define for all cases

(8.13) \( \tilde{\tau}_\varphi = (\iota_\varphi, J_\varphi) \in \text{Irr}(W_{n_g}), \quad \varphi \in A(\mathcal{O}). \)

For later use, we record the following lemma, which follows immediately from our explicit descriptions of \( \tau_\bar{b} \) and \( \tau_{\bar{\delta}} \).

**Lemma 8.4.** Let \( \varphi \in A(\mathcal{O}). \) If \( \ast = \tilde{C} \), then

\[
\text{Ind}^{W_{n_g} \times W_{n_g}}_{W_{n_b} \times W_{n_b}} \tau_\bar{b} \otimes \tau_{\bar{\delta}} \cong \begin{cases} 
\tau_\bar{b} \otimes \tau_{\bar{\delta}}, & \text{if } PP_\ast(\mathcal{O}_g) = \emptyset; \\
(\tau_\bar{b} \otimes \tau_{\bar{\delta}}) \oplus (\tau_\bar{b} \otimes \tau_{\bar{\varphi}}), & \text{otherwise.}
\end{cases}
\]

If \( \ast \in \{ D, D^* \} \), then

\[
\text{Ind}^{W_{n_g} \times W_{n_g}}_{W_{n_b} \times W_{n_b}} \tau_\bar{b} \otimes \tau_{\bar{\delta}} \cong \begin{cases} 
\tilde{\tau}_b, & \text{if } n_g = 0; \\
(\tilde{\tau}_b \otimes \tau_{\bar{\varphi}}) \oplus (\tilde{\tau}_b \otimes \tau_{\bar{\varphi}}), & \text{otherwise.}
\end{cases}
\]

Here \( \tilde{\tau}_b = \text{Ind}^{W_{n_b}}_{W_{n_b}} \tau_\bar{b} \) and \( \tilde{\tau}_{\bar{\varphi}} := \tilde{\tau}_{\bar{\varphi}} \otimes \varepsilon \) (recall the quadratic character \( \varepsilon \) from (7.3)).

### 8.4. From coherent continuation representation to counting

We have defined in (2.18) the set \( \text{PBP}_\ast(\mathcal{O}) \) when \( \mathcal{O} \) has good parity. Similarly, we make the following definition in the bad parity case.

**Definition 8.5.** Let \( \text{PBP}_\ast(\mathcal{O}_b) \) be the set of all triples \( \tau = (i, \mathcal{P}) \times (j, \mathcal{Q}) \times \ast \) where \( (i, \mathcal{P}) \) and \( (j, \mathcal{Q}) \) are painted Young diagrams such that

- \( (i, j) = (\tau_{L_b}, \tau_{R_b}) \) (see (8.6) and (8.7));
- the symbols of \( \mathcal{P} \) are in

\[
\left\{ \bullet, c, d \right\}, \quad \text{if } \ast \in \{ B, \tilde{C} \};
\]

\[
\left\{ \ast, d \right\}, \quad \text{if } \ast \in \{ C, D \};
\]

\[
\left\{ \bullet \right\}, \quad \text{if } \ast \in \{ C^*, D^* \};
\]
\* the symbols of $\mathcal{Q}$ are in
\[
\begin{cases}
\{ \bullet \}, & \text{if } \star \in \{ B, \tilde{C}, C^*, D^* \}; \\
\{ \bullet, c \}, & \text{if } \star \in \{ C, D \}.
\end{cases}
\]

We introduce some additional notation. For each $\mathcal{O}$ with good parity, and $\varphi \subseteq \text{PP}(\mathcal{O})$, let
\[
\text{(8.14)} \quad \text{PBP}_*(\mathcal{O}, \varphi) := \{ \tau \text{ is a painted bipartition } | \star = \star, (i_\tau, j_\tau) = (i_\varphi, j_\varphi) \}
\]
and
\[
\text{(8.15)} \quad \text{PBP}_G(\mathcal{O}, \varphi) := \{ \tau \text{ is a painted bipartition } | G_\tau = G, (i_\tau, j_\tau) = (i_\varphi, j_\varphi) \}.
\]
Note that in the notation above, we have $\text{PBP}_*(\mathcal{O}) = \text{PBP}_*(\mathcal{O}, \emptyset)$ and $\text{PBP}_G(\mathcal{O}) = \text{PBP}_G(\mathcal{O}, \emptyset)$.

Put
\[
\widehat{\text{PBP}}_*(\mathcal{O}) := \bigsqcup_{\varphi \subseteq \text{PP}(\mathcal{O})} \text{PBP}_*(\mathcal{O}, \varphi)
\]
and
\[
\widehat{\text{PBP}}_G(\mathcal{O}) := \bigsqcup_{\varphi \subseteq \text{PP}(\mathcal{O})} \text{PBP}_G(\mathcal{O}, \varphi).
\]

Recall the notion of $\mathcal{O}$ being $G$-relevant (Section 2.6). Note that if $\star = D^*$, then in view of the Remarks 6.2 (c), $\mathcal{O}$ is not $G$-relevant if and only if $\mathcal{O}$ has bad parity and type II (see (8.4)).

**Proposition 8.6.** If $\star = D^*$ and $\mathcal{O}$ is not $G$-relevant, then
\[
\sum_{\varphi \in \Lambda(\mathcal{O})} [\eta_b \otimes \tau_\varphi : \text{Coh}_\Lambda(\mathcal{K}'(G))] = 0.
\]

In all other cases,
\[
\sum_{\varphi \in \Lambda(\mathcal{O})} [\eta_b \otimes \tau_\varphi : \text{Coh}_\Lambda(\mathcal{K}'(G))] = \sharp(\text{PBP}_*(\mathcal{O}_b)) \cdot \sharp(\widehat{\text{PBP}}_G(\mathcal{O}_g)).
\]

**Proof.** We use the following formulas ([McG98, p220 (6)]) to compute the multiplicities:
\[
\text{Ind}_{H_t}^{W_{2t}} \eta \approx \bigoplus_{\sigma \in \text{Irr}(S_t)} (\sigma, \sigma),
\]
\[
\text{Ind}_{H_t}^{W_{2t}} \eta \cong \bigoplus_{\sigma \in \text{Irr}(S_t)} (\sigma, \sigma)_I,
\]
\[
\text{Ind}_{S_t}^{W_{s+r}} \text{sgn} \cong \bigoplus_{s+r=t} \text{Ind}_{W_{s+r}}^{W_s} \text{sgn} \otimes \text{sgn} \cong \bigoplus_{s+r=t} ([s]_{\text{col}}, [r]_{\text{col}}),
\]
\[
\text{Ind}_{S_t}^{W_{s+r}} 1 \cong \bigoplus_{c+d=t} \text{Ind}_{W_{s+r}}^{W_{c+d}} 1 \otimes \varepsilon \cong \bigoplus_{c+d=t} ([c]_{\text{row}}, [d]_{\text{row}}),
\]
where $t \in \mathbb{N}$.

We skip the details when $\star \in \{ B, \tilde{C}, C, D, C^* \}$, and present the computation for $\star = D^*$, which is the most complicated case (in certain aspect). Suppose that $\star = D^*$ so that $G = \text{SO}^*(2n)$. If $\mathcal{O}$ has bad parity, then
\[
\sum_{\varphi \in \Lambda(\mathcal{O})} [\eta_b \otimes \tau_\varphi : \text{Coh}_\Lambda(\mathcal{K}'(G))] = \begin{cases}
[\eta_b : \text{Coh}_\Lambda(\mathcal{K}'(G))] \\
[\eta_b : \text{Ind}_{H_t}^{W_{2t}} \eta] \\
1 = \sharp(\text{PBP}_*(\mathcal{O}_b)) \cdot \sharp(\widehat{\text{PBP}}_G(\mathcal{O}_g)), & \text{if } \mathcal{O} \text{ is } G\text{-relevant}; \\
0, & \text{if } \mathcal{O} \text{ is not } G\text{-relevant}.
\end{cases}
\]
Now we assume that \( \mathcal{O} \) does not have bad parity so that \( n_\mathcal{O} > 0 \). Put

\[
W'_{n_b,n_g} := W'_{n} \cap (W_{mb} \times W_{ng}) \supseteq W'_{mb} \times W'_{ng} = W_b \times W_g.
\]

Recall that

\[
\tilde{\tau}_b := \text{Ind}_{W'_{mb}}^{W_{mb}} \tau_b = (\mathcal{O}_b', \mathcal{O}_b') \in \text{Irr}(W_{mb})
\]
and

\[
\tilde{\tau}_\varphi := (\tau_\varphi, J_\varphi) \in \text{Irr}(W_{ng}) \quad \text{for all } \varphi \subseteq \text{PP}(\mathcal{O}_g).
\]

Note that \( \tau_\varphi \neq J_\varphi \) since \( c_1(\tau_\varphi) > c_1(J_\varphi) \), which implies that

\[
\text{Ind}_{W'_{mb} \times W'_{ng}} \tau_b \otimes \tau_\varphi \cong (\tilde{\tau}_b \otimes \tilde{\tau}_\varphi)|_{W'_{mb} \times W'_{ng}}.
\]

For ease of notation, write \( W'' := W'_{mb} \). Put

\[
C_b := \text{Ind}_{W''}^{W_{mb}} \eta \quad \text{and} \quad C_\varphi := \bigoplus_{2t+a=n_g} \text{Ind}_{W''}^{W_{ng}} \eta \otimes \text{sgn},
\]
where \( C_\varphi \) is viewed as a representation of \( W''_{mb} \) by restriction. For every finite group \( E \) and any two finite-dimensional representations \( V_1 \) and \( V_2 \) of \( E \), put

\[
[V_1,V_2]_E := \dim \text{Hom}_E(V_1,V_2).
\]

For each \( \varphi \in A(\mathcal{O}_g) \), we have that

\[
[\tau_b \otimes \tau_\varphi : \text{Coh}_A(\mathcal{K}'(G))] = [\tau_b \otimes \tau_\varphi : \text{Ind}_{W''_{mb} \times W''_{ng}} \otimes C_\varphi, \text{Ind}_{W''_{mb} \times W''_{ng}} \otimes C_\varphi]|_{W''_{mb} \times W''_{ng}}
\]

\[
= [\text{Ind}_{W''_{mb} \times W''_{ng}} \otimes \tau_b, \otimes \tau_\varphi : \text{Ind}_{W''_{mb} \times W''_{ng}} \otimes C_\varphi]|_{W''_{mb} \times W''_{ng}}
\]

\[
= [\tilde{\tau}_b \otimes \tilde{\tau}_\varphi : \text{Ind}_{W''_{mb} \times W''_{ng}} \otimes C_\varphi]|_{W''_{mb} \times W''_{ng}}
\]

\[
= [\tilde{\tau}_b : \text{Ind}_{W''_{mb}} \otimes C_\varphi]|_{W'_{mb}}
\]

\[
= (\text{PB}^*(\mathcal{O}_b)) : (\text{PB}^*_G(\mathcal{O}_g, \varphi)).
\]

The last equality follows from induction in stages, the branching rules in (8.16), specifically \( \text{Ind}_{W''_{mb}} \eta \cong \bigoplus_{\sigma \in \text{Irr}(\mathcal{S})}(\sigma, \sigma) \) and \( \text{Ind}_{W''_{mb}} \text{sgn} \cong \bigoplus_{s+t=r}([s]_{\text{col}}, [r]_{\text{col}}) \), and the Pieri rule ([GW09, Corollary 9.2.4]) (which is a special case of the Littlewood-Richard rule). For the case at hand, note that a painted bipartition \( \tau = (\tau, P) \times (\eta, Q) \times D^* \) in \( \text{PB}^*_G(\mathcal{O}_g, \varphi) \) has symbols \( \bullet, s \) in \( P \) and symbols \( \bullet, r \) in \( Q \). In applying the branching rules, the \( \sigma \) in the summation \( \bigoplus_{\sigma \in \text{Irr}(\mathcal{S})}(\sigma, \sigma) \) corresponds to filling an identical set of boxes in \( P \) and \( Q \) with the symbol \( \bullet \), and the \( [s]_{\text{col}} \) (resp. \( [r]_{\text{col}} \)) in the summation \( \bigoplus_{s+t=r}([s]_{\text{col}}, [r]_{\text{col}}) \) corresponds to filling a column consisting of symbols \( s \) in \( P \) (resp. a column consisting of symbols \( r \) in \( Q \)), with each possible way of constructing a bipartition \( \tau \) in \( \text{PB}^*_G(\mathcal{O}_g, \varphi) \) contributing 1 to the multiplicity \( [\tilde{\tau}_\varphi : \text{Ind}_{W''_{ng}} \otimes C_\varphi]|_{W'_{ng}} \). Likewise each possible way of constructing a bipartition in \( \text{PB}^*(\mathcal{O}_b) \) contributes 1 to the multiplicity \( [\tilde{\tau}_b : \text{Ind}_{W''_{mb}} \otimes C_\varphi]|_{W'_{mb}} \).

Now the equality (7.2), Proposition 8.3 and Proposition 8.6 imply the following corollary.

**Corollary 8.7.** The following equality holds:

\[
\sharp(\text{Unip}_G(\mathcal{O})) = \begin{cases} 0, & \text{if } \ast = D^* \text{ and } \mathcal{O} \text{ is not } G\text{-relevant;} \\ \sharp(\text{PB}^*(\mathcal{O}_b)) \cdot \sharp(\text{PB}^*_G(\mathcal{O}_g)), & \text{otherwise.} \end{cases}
\]
When \( \hat{O} \) has good parity, we will see from Proposition 10.1 and Proposition 10.2 that

\[
\sharp(\PBP_G(\hat{O})) = \begin{cases} 
\sharp(\PBP_G(\hat{O})), & \text{if } \ast \in \{C^*, D^*\}; \\
\sharp\sharp(\PBP_G(\hat{O})), & \text{if } \ast \in \{B, C, D, \tilde{C}\}. 
\end{cases}
\]

Corollary 8.7 will thus imply Theorem 2.27.

9. Special unipotent representations in type \( BCD \): Reduction to good parity

The goal of this section is to prove Theorem 2.20. Corollary 8.7 implies that \( \Unip_\hat{O}(G) \) is empty if \( \hat{O} \) is not \( G \)-relevant (this notion is defined in Section 2.6). Thus we further assume that \( \hat{O} \) is \( G \)-relevant. With our earlier assumptions (6.17) and (6.18), this is equivalent to saying that \( \hat{O} \) has type I when \( \ast = D^* \) and \( \hat{O} \) has bad parity.

If \( \ast = C^* \), or \( \ast = D^* \) and \( \hat{O} \) does not have bad parity, by possibly changing \( \omega_{\hat{V}_b} \) and \( \omega_{\hat{V}_g} \) defined in (6.13) to their negatives, we assume without loss of generality that \( \hat{O} \subset \) has type I.

9.1. Separating bad parity and good parity for special unipotent representations.

By Proposition 6.16, we have an injective linear map

\[
\varphi_{\lambda_{\hat{O}}} : \mathcal{K}_{\lambda_{\hat{O}}}'(G_h) \otimes \mathcal{K}_{\lambda_{\hat{O}}}'(G_g) \to \mathcal{K}_{\lambda_{\hat{O}}}'(G)
\]

and an injective map

\[(9.1) \quad \varphi_{\lambda_{\hat{O}}} : \Irr_{\lambda_{\hat{O}}}'(G_h) \times \Irr_{\lambda_{\hat{O}}}'(G_g) \to \Irr_{\lambda_{\hat{O}}}'(G).
\]

**Proposition 9.1.** The map (9.1) restricts to a bijective map

\[
\varphi_{\lambda_{\hat{O}}} : \Unip_{\hat{O}}(G_h) \times \Unip_{\hat{O}}(G_g) \to \Unip_{\hat{O}}(G).
\]

**Proof.** Let \( \pi_b \in \Irr_{\lambda_{\hat{O}}}'(G_h) \) and \( \pi_g \in \Irr_{\lambda_{\hat{O}}}'(G_g) \). Pick a basal element \( \Psi_b \) of \( \Coh_{\lambda_{\hat{O}}}(\mathcal{K}'(G_h)) \) such that \( \Psi_b(\lambda_{\hat{O}}) = \pi_b \). Likewise pick a basal element \( \Psi_g \) of \( \Coh_{\lambda_{\hat{O}}}(\mathcal{K}'(G_g)) \) such that \( \Psi_g(\lambda_{\hat{O}}) = \pi_g \). Write \( \Psi := \varphi(\Psi_b \otimes \Psi_g) \). Denote by \( C \) the Harish-Chandra cell in \( \Coh_{\lambda_{\hat{O}}}(\mathcal{K}'(G)) \) containing \( \Psi \), and similarly define the Harish-Chandra cells \( C_b \ni \Psi_b \) and \( C_g \ni \Psi_g \).

Put \( \pi := \varphi_{\lambda_{\hat{O}}}^{-1}(\pi_b, \pi_g) = \Psi(\lambda_{\hat{O}}) \). Recall the integral Weyl group \( W(\Lambda) = W_b \times W_g' \). Then

\[
\pi \in \Unip_{\hat{O}}(G) \\
\Leftrightarrow \sigma_{C} \cong \begin{pmatrix} W(\Lambda) \\ \text{sgn} \end{pmatrix} \otimes \text{sgn} \quad \text{(by Proposition 5.5)} \\
\Leftrightarrow \sigma_{C_b} \cong \begin{pmatrix} W_b \\ \text{sgn} \end{pmatrix} \otimes \text{sgn} \quad \text{and} \quad \sigma_{C_g} \cong \begin{pmatrix} W_g' \\ \text{sgn} \end{pmatrix} \otimes \text{sgn} \\
\Leftrightarrow \pi_b \in \Unip_{\hat{O}}(G_h) \quad \text{and} \quad \pi_g \in \Unip_{\hat{O}}(G_g) \quad \text{(by Proposition 5.5)}.
\]

Here \( W_{b,\lambda_{\hat{O}}b} \) denotes the stabilizer of \( \lambda_{\hat{O}} \) in \( W_b \), and likewise \( W_{g,\lambda_{\hat{O}}g}' \) denotes the stabilizer of \( \lambda_{\hat{O}} \) in \( W_g' \).

In particular, we have proved that

\[
\varphi_{\lambda_{\hat{O}}}(\Unip_{\hat{O}}(G_h) \times \Unip_{\hat{O}}(G_g)) \subseteq \Unip_{\hat{O}}(G).
\]

On the other hand, Corollary 8.7 implies that

\[
\sharp(\Unip_{\hat{O}}(G_h) \times \Unip_{\hat{O}}(G_g)) = \sharp\Unip_{\hat{O}}(G).
\]

This proves the proposition since the map (9.1) is injective. \( \square \)
9.2. The case of bad parity. Recall from (2.14) the group

\[ G'_{b} := \begin{cases} 
GL_{m_{b}}(\mathbb{R}), & \text{if } \star \in \{ B, C, D \}; \\
\hat{GL}_{m_{b}}(\mathbb{R}), & \text{if } \star = \tilde{C}; \\
GL_{2b}(\mathbb{H}), & \text{if } \star \in \{ C^{*}, D^{*} \}.
\end{cases} \]

Lemma 9.2. The equalities

\[ \sharp(PBP^*(\hat{O}_{b})) = \sharp(PAP_{\star}(\hat{O}_{b})) = \sharp(\text{Unip}_{\hat{O}_{b}}(G'_{b})) \]

hold, where

\[ \star' := \begin{cases} 
\mathbb{R}, & \text{if } \star \in \{ B, C, \tilde{C}, \tilde{D} \}; \\
\mathbb{H}, & \text{if } \star \in \{ C^{*}, D^{*} \}.
\end{cases} \]

Proof. Suppose that \( \star \in \{ C^{*}, D^{*} \} \). Then

\[ \sharp(PBP^*(\hat{O}_{b})) = \sharp(PAP_{\star}(\hat{O}_{b})) = 1. \]

Suppose that \( \star \in \{ B, C, \tilde{C}, D \} \). It is easy to see that we have a bijection

\[ PBP^*(\hat{O}_{b}) \rightarrow PAP_{\star}(\hat{O}_{b}), \]

where \( P' \) is defined by the condition that

\[ P(c_{j}(\tau_{L,b}), j) = d \iff P'(r_{j}(\hat{O}_{b}), j) = d, \quad \text{for all } j = 1, 2, \ldots, c_{1}(\hat{O}_{b}). \]

The last equality is in Theorem 2.12 (which is proved in Section 7.2).

Let \( P_{b} \) be a parabolic subgroup of \( G_{b} \) that is \( \hat{O}_{b}, \text{-relevant} \) as defined in Section 2.6. Then \( G'_{b} \)

is naturally isomorphic to the Levi quotient of \( P_{b} \).

Proposition 9.3. For every \( \pi' \in \text{Unip}_{\hat{O}_{b}}(G'_{b}) \), the normalized induced representation \( \text{Ind}_{P_{b}}^{G_{b}} \pi' \)

is irreducible and belongs to \( \text{Unip}_{\hat{O}_{b}}(G_{b}) \). Moreover, the map

\[ \text{Unip}_{\hat{O}_{b}}(G'_{b}) \rightarrow \text{Unip}_{\hat{O}_{b}}(G_{b}), \]

\[ \pi' \rightarrow \text{Ind}_{P_{b}}^{G_{b}} \pi' \]

is bijective.

Proof. By the construction of \( \pi' \) (see Section 2.4) and a result of Barbasch [Bar00, Corollary 5.0.10], the wavefront cycle of \( \text{Ind}_{P_{b}}^{G_{b}} \pi' \) is a single orbit \( \mathcal{O} \) with multiplicity one and its complex-

ification is \( d_{BV}(\hat{O}_{b}) \). Note that every irreducible summand of \( \text{Ind}_{P_{b}}^{G_{b}} \pi' \) belongs to \( \text{Unip}_{\hat{O}_{b}}(G_{b}) \).

Hence \( \text{Ind}_{P_{b}}^{G_{b}} \pi' \) has to be irreducible.

We now suppose that \( \star = D \) so that \( G_{b} = SO(n_{b}, n_{b}) \). We will prove the injectivity of the map (9.3). Fix a split Cartan subgroup

\[ H = (\mathbb{R}^{\times})^{n_{b}} \subseteq G'_{b} \subseteq G_{b} \]

and write \( H = MA \) where \( M = \{ \pm 1 \}^{n_{b}} \) is the compact part of \( H \) and \( A = (\mathbb{R}^{\times})^{n_{b}} \) is the split part of \( H \). We identify a character of \( A \) with a vector in \( C^{n_{b}} \) as usual. Let

\[ K := \{(g_{1}, g_{2}) \in O(n_{b}) \times O(n_{b}) \mid \det(g_{1}) \cdot \det(g_{2}) = 1\} \]

be a maximal compact subgroup of \( G_{b} \), and let \( B \) be a Borel subgroup of \( G_{b} \) containing \( H \) and the unipotent radical of \( P_{b} \).

For each integer \( l \) such that \( 0 \leq l \leq n_{b} \), put

\[ \delta_{l} = 1_{\pm}^{\otimes} \otimes \cdots \otimes 1_{\pm}^{\otimes} \otimes \text{sgn}_{\pm}^{\otimes} \otimes \cdots \otimes \text{sgn}_{\pm}^{\otimes} \in \text{Irr}(M), \]

where \( 1_{\pm} \) denotes the trivial character of \( \{ \pm 1 \} \) and \( \text{sgn}_{\pm} \) denotes the non-trivial character of \( \{ \pm 1 \} \). It is a fine \( M \)-type ([Vog81, Definition 4.3.8]). Let \( \lambda_{l} \) be the restriction to \( K \) of the
irreducible \( \Omega(n_h) \times \Omega(n_h) \)-representation \( \Lambda^{n_h-1} \mathbb{C}^{n_h} \otimes \Lambda^0 \mathbb{C}^{n_h} \). Then \( \lambda_i \) is irreducible and is a fine \( K \)-type, see [BGG75, §6].

Vogan proves that there is a well-defined injective map

\[
\mathcal{X} : \quad W_H / \text{Irr}(H) \to \text{Irr}(G_b),
\]

where \( W_H \) denotes the real Weyl group of \( G_b \) with respect to \( H \), \( l_\chi \in \{0, 1, \ldots, n_h\} \) is the integer such that \( \chi|_{M} \) is conjugate to \( \delta_\chi \) by \( W_H \), and \((\text{Ind}_{B}^{G_b} \chi)(\lambda_i)\) denotes the unique irreducible subquotient in the normalized induced representation \( \text{Ind}_{B}^{G_b} \chi \) containing the \( K \)-type \( \lambda_i \). See [Vog81, Theorem 4.4.8].

As in (2.4), every representation in \( \text{Unip}_{\mathcal{O}_b}(G'_b) \) is a normalized smooth parabolic induction

\[
\pi' = 1_{n_1} \times \cdots \times 1_{n_r} \times \text{sgn}_{n_{r+1}} \times \cdots \times \text{sgn}_{n_k},
\]

where \( k \geq r \geq 0 \), \( n_1, n_2, \ldots, n_k \) are positive integers with \( n_1 + n_2 + \cdots + n_k = n_b \), \( 1_i \) and \( \text{sgn}_i \) respectively denote the trivial character and the sign character of \( \text{GL}_i(\mathbb{R}) \) \( (i \in \mathbb{N}^+) \).

Write

\[
\delta_{n'} := \delta_{n_1+n_2+\cdots+n_k} \in \text{Irr}(M),
\]

\[
\nu_{n'} := (\frac{n_1-1}{2}, \frac{n_2-3}{2}, \ldots, \frac{1-n_1}{2}, \frac{n_2-1}{2}, \frac{n_3-3}{2}, \ldots, \frac{1-n_k}{2}) \in \text{Irr}(A).
\]

Clearly the map \( \mathcal{Q} : \text{Unip}_{\mathcal{O}_b}(G'_b) \to W_H / \text{Irr}(H) \),

\[
\pi' \mapsto \quad \text{the \, \text{W}_H\text{-orbit of } \delta_{n'} \otimes \nu_{n'}}
\]

is well-defined and injective.

Note that the map (9.3) equals the composition \( \mathcal{X} \circ \mathcal{Q} \), and hence it is also injective. This proves the injectivity of (9.3) in the case when \( * = D \). The same proof works in the case when \( * \in \{B, C, \tilde{C}\} \) and we omit the details. When \( * \in \{C^*, D^*\} \), the map (9.3) has to be injective since its domain is a singleton. Thus (9.3) is injective in all cases.

The bijection of (9.3) follows from the injectivity and the counting inequalities below:

\[
|\text{PAP}_{\nu}(\tilde{\mathcal{O}}'_b)| = \left| \text{Unip}_{\mathcal{O}'_b}(G'_b) \right| \leq \left| \text{Unip}_{\mathcal{O}_b}(G_b) \right| \leq \left| \text{PBP}^*{\nu}(\tilde{\mathcal{O}}_b) \right| = |\text{PAP}_{\nu}(\tilde{\mathcal{O}}'_b)|.
\]

Here \( \nu' \in \{ A^R, \mathcal{A}^\mathbb{R} \} \) is as in (9.2), the first inequality follows from the injectivity of (9.3), the second inequality follows from Corollary 8.7, and the last equality is in Lemma 9.2.

9.3. Coherent continuation representations and parabolic induction. The normalized smooth parabolic induction from \( P_b \) to \( G_b \) yields a linear map

\[
\text{Ind} : \mathcal{K}'(G'_b) \to \mathcal{K}'(G_b).
\]

This induces a linear map

\[
\text{Ind} : \text{Coh}_{\lambda,b}(\mathcal{K}'(G'_b)) \to \text{Coh}_{\lambda,b}(\mathcal{K}'(G_b)),
\]

\[
\Psi'_b \mapsto \quad (\nu \mapsto \text{Ind}(\Psi'_b(\nu))).
\]

Let \( P \) be an \( \tilde{\mathcal{O}} \)-relevant parabolic subgroup of \( G \) as in Theorem 2.20. Then \( G'_b \times G_b \) (or its quotient by \( \{ \pm 1 \} \) when \( * = \tilde{C} \) is naturally isomorphic to the Levi quotient of \( P \). The normalized smooth parabolic induction using \( P \) yields a linear map

\[
\text{Ind} : \mathcal{K}'(G'_b) \otimes \mathcal{K}'(G_b) \to \mathcal{K}'(G),
\]

which further induces a linear map (see [Vog81, Corollary 7.2.10])

\[
\text{Ind} : \text{Coh}_{\lambda,b}(\mathcal{K}'(G'_b)) \otimes \text{Coh}_{\lambda,b}(\mathcal{K}'(G_b)) \to \text{Coh}_{\lambda}(\mathcal{K}'(G)),
\]

\[
\Psi'_b \otimes \Psi'_b \mapsto \quad ((\nu_b, \nu_b) \mapsto \text{Ind}(\Psi'_b(\nu_b) \otimes \Psi'_b(\nu_b))).
\]

The following result follows from Langlands’ construction of standard modules. See [Vog81, Lemma 6.6.12 and Theorem 6.6.15].
Lemma 9.4. The diagram
\[
\begin{array}{ccc}
\text{Coh}_{A_b}(\mathcal{K}'(G_b')) & \otimes & \text{Coh}_{A_b}(\mathcal{K}'(G_g)) \\
\text{Ind} \otimes \text{Id} & \text{Ind} & \varphi \\
\text{Coh}_{A_b}(\mathcal{K}'(G_b)) & \otimes & \text{Coh}_{A_b}(\mathcal{K}'(G_g')) & \rightarrow & \text{Coh}_{A}(\mathcal{K}'(G))
\end{array}
\]
commutes.

Proposition 9.5. For all \( \pi' \in \text{Unip}_{\mathcal{O'}}(G'_b) \) and \( \pi_g \in \text{Unip}_{\mathcal{O}_g}(G_g) \), the normalized smooth parabolic induction \( \text{Ind}_{\mathcal{O}}^G(\pi' \otimes \pi_g) \) is irreducible and belongs to \( \text{Unip}_{\mathcal{O}}(G) \).

Proof. By using the evaluation maps, the commutative diagram (9.4) descends to a commutative diagram
\[
\begin{array}{ccc}
\mathcal{K}'_{\lambda, \mathcal{O}}(G'_b) & \otimes & \mathcal{K}'_{\lambda, \mathcal{O}_g}(G_g) \\
\text{Ind} \otimes \text{Id} & \text{Ind} & \varphi_{\lambda, \mathcal{O}} \\
\mathcal{K}'_{\lambda, \mathcal{O}}(G'_b) & \otimes & \mathcal{K}'_{\lambda, \mathcal{O}_g}(G_g) & \rightarrow & \mathcal{K}'_{\lambda}(G).
\end{array}
\]
Here \( \varphi_{\lambda, \mathcal{O}} \) is as in Proposition 6.16. Thus \( \text{Ind}_{\mathcal{O}}^G(\pi' \otimes \pi_g) \) is irreducible by Propositions 9.3 and 6.16. As before, [Bar00, Corollary 5.0.10] implies that \( \text{Ind}_{\mathcal{O}}^G(\pi' \otimes \pi_g) \in \text{Unip}_{\mathcal{O}}(G) \).

By Proposition 9.5, we have a well-defined map
\[
\text{Ind} : \text{Unip}_{\mathcal{O'}}(G'_b) \times \text{Unip}_{\mathcal{O}_g}(G_g) \rightarrow \text{Unip}_{\mathcal{O}}(G).
\]
In view of the commutative diagram (9.5), Propositions 6.16 and 9.3 imply that the map (9.6) is injective. On the other hand, Propositions 9.1 and 9.3 imply that the domain and codomain of the map (9.6) have the same cardinality. Thus the map (9.6) is bijective. This completes the proof of Theorem 2.20.

10. Combinatorics of painted bipartitions

Throughout this section, we assume that \( * \in \{B, C, \vec{C}, C^*, D, D^*\} \), and \( \mathcal{O} = \mathcal{O}_g \), namely \( \mathcal{O} \) has good parity.

Recall the set \( \text{PP}_*(\mathcal{O}) \) of primitive \( * \)-pairs in \( \mathcal{O} \) (Definition 2.21). For each subset \( \varphi \) of \( \text{PP}_*(\mathcal{O}) \), we have defined a pair \( (i_{\varphi}, j_{\varphi}) \) of Young diagrams in Section 8.3. Our main object of interest is the following set of disjoint union:
\[
\widehat{\text{PBP}}_G(\mathcal{O}) := \bigsqcup_{\varphi \in \text{PP}(\mathcal{O})} \text{PBP}_G(\mathcal{O}, \varphi),
\]
where \( \text{PBP}_G(\mathcal{O}, \varphi) \) is defined in (8.15).

10.1. Main combinatorial results. We say that the orbit \( \mathcal{O} \) is quasi-distinguished if there is no \( * \)-pair that is balanced in \( \mathcal{O} \) (see Definition 2.21).

For the quaternionic groups, we have the following non-existence result on painted bipartitions.

Proposition 10.1. Suppose that \( * \in \{C^*, D^*\} \). If \( \text{PBP}_G(\mathcal{O}, \varphi) \) is nonempty for some \( \varphi \subseteq \text{PP}_*(\mathcal{O}) \), then \( \mathcal{O} \) is quasi-distinguished and \( \varphi = \emptyset \). Consequently,
\[
\sharp(\text{PBP}_G(\mathcal{O})) = \#(\text{PBP}_G(\mathcal{O})).
\]

Proof. Suppose that \( \tau = (i_{\varphi}, P) \times (j_{\varphi}, Q) \times * \in \text{PBP}_G(\mathcal{O}, \varphi) \). Assume by contradiction that \( \varphi \neq \emptyset \).

First assume that \( * = C^* \). Pick an element \( (2i - 1, 2i) \in \varphi \). Then we have that
\[
c_i(i_{\varphi}) = \frac{1}{2}(r_{2i} - 1) + 1 > \frac{1}{2}(r_{2i} - 1) = c_i(j_{\varphi}).
\]
By the requirements of a painted bipartition, we also have that
\[ c_i(t_p) \begin{cases} \upvarnothing \{ j \in \mathbb{N}^+ \mid (i, j) \in \text{Box}(t_p), P(i, j) = \bullet \} \\ \upvarnothing \{ j \in \mathbb{N}^+ \mid (i, j) \in \text{Box}(j_p), Q(i, j) = \bullet \} \end{cases} \leq c_i(j_p). \]
This contradicts (10.1) and therefore \( \varnothing = \emptyset \).
Now assume that \( \star = D^\ast \). Pick an element \((2i, 2i + 1) \in \varnothing \). Then we have that
\[ (10.2) \quad c_{i+1}(t_p) = \frac{1}{2}(r_{2i}(\mathcal{O}) + 1) > \frac{1}{2}(r_{2i+1}(\mathcal{O}) - 1) = c_i(j_p). \]
By the requirements of a painted bipartition, we also have that
\[ c_{i+1}(t_p) \leq \upvarnothing \{ j \in \mathbb{N}^+ \mid (i, j) \in \text{Box}(t_p), P(i, j) = \bullet \} \leq c_i(j_p). \]
This contradicts (10.2) and therefore \( \varnothing = \emptyset \).

It is also straightforward to show that \( \mathcal{O} \) is quasi-distinguished from the requirements of a painted bipartition. We omit the details. \( \square \)

In view of Proposition 10.1, we assume in the rest of this section that \( \mathcal{O} \) is quasi-distinguished when \( \star \in \{ C^\ast, D^\ast \} \).

In the rest of the section, we will define a notion of descent for painted bipartitions (in \( \text{PPB}_G(\mathcal{O}, \varnothing) \)) which will be used as an induction mechanism and we will deduce certain counting formulas by a generating function approach. The following proposition will then be an immediate consequence of a certain independence property of the generating functions in Proposition 10.11 and Proposition 10.12.

**Proposition 10.2.** Suppose that \( \star \in \{ B, C, \tilde{C}, D \} \). Then
\[ \sharp(\text{PPB}_G(\mathcal{O}, \varnothing)) = \sharp(\text{PPB}_G(\mathcal{O}, \emptyset)), \quad \text{for all } \varnothing \subseteq \text{PP}_\star(\mathcal{O}). \]

Consequently,
\[ \sharp(\text{PPB}_G(\mathcal{O})) = 2^\sharp(\text{PP}_\star(\mathcal{O})) \cdot \sharp(\text{PPB}_G(\mathcal{O})). \]

10.2. **Type C and \( \tilde{C} \): shape shifting.** In this subsection, we assume that \( \star \in \{ C, \tilde{C} \} \). We will define a shape shifting operation on painted bipartitions which will be used in defining the notion of descent.

Recall that \( \mathcal{O} \) is an orbit with good parity. In this subsection we assume that \((1, 2) \in \text{PP}_\star(\mathcal{O})\), namely the pair \((1, 2)\) is primitive in \( \mathcal{O} \). Let \( \varnothing \) be a subset of \( \text{PP}_\star(\mathcal{O}) \) such that \((1, 2) \notin \varnothing \). Put \( \varnothing_\tau := \varnothing \cup \{ (1, 2) \} \subseteq \text{PP}_\star(\mathcal{O}) \).

Note that
\[ (c_1(t_{p_\tau}), c_1(j_{p_\tau})) = \begin{cases} (c_1(j_\varnothing) + 1, c_1(t_\varnothing) - 1) & \text{when } \star = C; \\ (c_1(j_\varnothing), c_1(t_\varnothing)) & \text{when } \star = \tilde{C}, \end{cases} \]
and
\[ (c_i(t_{p_\tau}), c_i(j_{p_\tau})) = (c_i(t_\varnothing), c_i(j_\varnothing)), \quad \text{for } i = 2, 3, 4, \ldots. \]

For \( \tau := (t_\varnothing, P_\varnothing) \times (j_\varnothing, Q_\varnothing) \times \alpha \in \text{PPB}_G(\mathcal{O}, \varnothing) \), we will define an element
\[ \tau_\tau := (t_{p_\tau}, P_{p_\tau}) \times (j_{p_\tau}, Q_{p_\tau}) \times \alpha \in \text{PPB}_G(\mathcal{O}, \varnothing_\tau) \]
by the following recipe.

**The case when \( \star = C \).** Note that \( c_1(t_{p_\tau}) > c_1(t_\varnothing) \geq 1 \) since \((1, 2) \in \text{PP}_\star(\mathcal{O})\). For all \((i, j) \in \text{Box}(t_{p_\tau}) \), we define \( P_{p_\tau}(i, j) \) case by case as in what follows.

(a) Suppose that \( P_{\tau}(c_1(t_\varnothing), 1) \neq \bullet \).
If $c_1(t_p) \geq 2$ and $P_{\tau}(c_1(t_p) - 1, 1) = c$, we define

$$P_{\tau}(i, j) := \begin{cases} r, & \text{if } j = 1 \text{ and } c_1(t_p) - 1 \leq i \leq c_1(t_{\bar{\tau}}) - 2; \\ c, & \text{if } (i, j) = (c_1(t_{\bar{\tau}}) - 1, 1); \\ d, & \text{if } (i, j) = (c_1(t_{\bar{\tau}}), 1); \\ P_{\tau}(i, j), & \text{otherwise}. \end{cases}$$

Otherwise, we define

$$P_{\tau}(i, j) := \begin{cases} r, & \text{if } j = 1 \text{ and } c_1(t_p) \leq i \leq c_1(t_{\bar{\tau}}) - 1; \\ c, & \text{if } (i, j) = (c_2(t_p), 2); \\ d, & \text{if } (i, j) = (c_1(t_{\bar{\tau}}), 1); \\ P_{\tau}(i, j), & \text{otherwise}, \end{cases}$$

(b) Suppose that $P_{\tau}(c_1(t_p), 1) = \bullet$.

- If $c_2(t_p) = c_1(t_p)$ and $P_{\tau}(c_1(t_p), 2) = r$, we define

$$P_{\tau}(i, j) := \begin{cases} r, & \text{if } j = 1 \text{ and } c_1(t_p) \leq i \leq c_1(t_{\bar{\tau}}) - 1; \\ c, & \text{if } (i, j) = (c_1(t_{\bar{\tau}}) - 1, 1); \\ d, & \text{if } (i, j) = (c_1(t_{\bar{\tau}}), 1); \\ P_{\tau}(i, j), & \text{otherwise}. \end{cases}$$

Otherwise, we define

$$P_{\tau}(i, j) := \begin{cases} r, & \text{if } j = 1 \text{ and } c_1(t_p) \leq i \leq c_1(t_{\bar{\tau}}) - 2; \\ c, & \text{if } (i, j) = (c_1(t_{\bar{\tau}}) - 1, 1); \\ d, & \text{if } (i, j) = (c_1(t_{\bar{\tau}}), 1); \\ P_{\tau}(i, j), & \text{otherwise}. \end{cases}$$

Note that $\text{Box}(j_{\bar{\tau}}) \subseteq \text{Box}(j_p)$. For all $(i, j) \in \text{Box}(j_{\bar{\tau}})$, we define

$$Q_{\tau}(i, j) := Q_{\tau}(i, j).$$

**The case when $* = \bar{\tau}$.** For all $(i, j) \in \text{Box}(j_{\bar{\tau}})$, define

$$P_{\tau}(i, j) := \begin{cases} s, & \text{if } j = 1 \text{ and } Q_{\tau}(i, j) = r; \\ c, & \text{if } j = 1 \text{ and } Q_{\tau}(i, j) = d; \\ P_{\tau}(i, j), & \text{otherwise}. \end{cases}$$

For all $(i, j) \in \text{Box}(j_{\bar{\tau}})$, define

$$Q_{\tau}(i, j) := \begin{cases} r, & \text{if } j = 1 \text{ and } P_{\tau}(i, j) = s; \\ d, & \text{if } j = 1 \text{ and } P_{\tau}(i, j) = c; \\ Q_{\tau}(i, j), & \text{otherwise}. \end{cases}$$

In both cases, it is routine to check that $\tau_{\bar{\tau}}$ is a painted bipartition and is an element of $\text{PBP}_G(\bar{\Theta}, \varphi_{\bar{\tau}})$.

**Lemma 10.3.** The map

$$T_{\bar{\tau}}: \text{PBP}_G(\bar{\Theta}, \varphi) \to \text{PBP}_G(\bar{\Theta}, \varphi_{\bar{\tau}}), \quad \tau \mapsto \tau_{\bar{\tau}}$$

defined above is bijective.

**Proof.** Let $\tau' = (t_{\bar{\tau}}, P_{\bar{\tau}}, (j_{\bar{\tau}}, Q_{\bar{\tau}})) \times \alpha \in \text{PBP}_G(\bar{\Theta}, \varphi_{\bar{\tau}})$. We will define a painted bipartition $\tau \in \text{PBP}_G(\bar{\Theta}, \varphi)$ by the following recipe.

**The case when $* = \bar{\tau}$.** Note that

$$c_1(t_{\bar{\tau}}) = c_1(j_p) + 1 > c_1(t_p) = c_1(j_{\bar{\tau}}) + 1.$$
- If $c_1(j_{\tau'}) \geq 1$ and $P_{\tau'}(c_1(j_{\tau'}), 1) = r$, for all $(i, j) \in \text{Box}(j_{\tau'})$ we define

$$P_{\tau}(i, j) := \begin{cases} c, & \text{if } (i, j) = (c_1(j_{\tau'}), 1); \\ d, & \text{if } (i, j) = (c_1(j_{\tau'}), 1); \\ P_{\tau'}(i, j), & \text{otherwise},\end{cases}$$

and for all $(i, j) \in \text{Box}(j_{\tau'})$ we define

$$Q_{\tau}(i, j) := \begin{cases} s, & \text{if } j = 1 \text{ and } c_1(j_{\tau'}) \leq i \leq c_1(j_{\tau'}); \\ Q_{\tau'}(i, j), & \text{otherwise}.\end{cases}$$

- Otherwise, for all $(i, j) \in \text{Box}(j_{\tau'})$ we define

$$P_{\tau}(i, j) := \begin{cases} \bullet, & \text{if } (i, j) = (c_1(j_{\tau'}), 1); \\ P_{\tau'}(i, j), & \text{otherwise},\end{cases}$$

and for all $(i, j) \in \text{Box}(j_{\tau'})$ we define

$$Q_{\tau}(i, j) := \begin{cases} \bullet, & \text{if } (i, j) = (c_1(j_{\tau'}), 1); \\ s, & \text{if } j = 1 \text{ and } c_1(j_{\tau'}) + 1 \leq i \leq c_1(j_{\tau'}); \\ Q_{\tau'}(i, j), & \text{otherwise}.\end{cases}$$

(b) Suppose that $P_{\tau'}(c_1(j_{\tau'}), 1) = r$.

- If $c_2(j_{\tau'}) = c_1(j_{\tau'}) + 1$ and $(P_{\tau'}(c_1(j_{\tau'}), 1), P_{\tau'}(c_2(j_{\tau'}), 2)) = (d, c)$, for all $(i, j) \in \text{Box}(j_{\tau'})$ we define

$$P_{\tau}(i, j) := \begin{cases} \bullet, & \text{if } (i, j) = (c_1(j_{\tau'}), 1); \\ r, & \text{if } (i, j) = (c_2(j_{\tau'}), 2); \\ P_{\tau'}(i, j), & \text{otherwise},\end{cases}$$

and for all $(i, j) \in \text{Box}(j_{\tau'})$ we define

$$Q_{\tau}(i, j) := \begin{cases} \bullet, & \text{if } (i, j) = (c_1(j_{\tau'}), 1); \\ s, & \text{if } j = 1 \text{ and } c_1(j_{\tau'}) + 1 \leq i \leq c_1(j_{\tau'}); \\ Q_{\tau'}(i, j), & \text{otherwise}.\end{cases}$$

- Otherwise, for all $(i, j) \in \text{Box}(j_{\tau'})$ we define

$$P_{\tau}(i, j) := \begin{cases} P_{\tau'}(c_1(j_{\tau'}), 1), & \text{if } (i, j) = (c_1(j_{\tau'}), 1); \\ P_{\tau'}(i, j), & \text{otherwise},\end{cases}$$

and for all $(i, j) \in \text{Box}(j_{\tau'})$ we define

$$Q_{\tau}(i, j) := \begin{cases} s, & \text{if } j = 1 \text{ and } c_1(j_{\tau'}) \leq i \leq c_1(j_{\tau'}); \\ Q_{\tau'}(i, j), & \text{otherwise}.\end{cases}$$

The case when $\ast = \tilde{\varphi}$. The definition of $T_{\tau', *}$ is given by the formulas (10.3) and (10.4) with the role of $\varphi$ and $\varphi_\tau$ switched.

In both cases, it is routine to check that

$$T_{\tau', *}: \text{PBP}_G(O, \varphi_\tau) \to \text{PBP}_G(O, \tilde{\varphi}), \quad \tau' \mapsto \tau$$

is well-defined and is the inverse of $T_{\tau', \tau'}$.  □
10.3. Naive descent of a painted bipartition. As before, let \( \star \in \{ B, C, D, \tilde{C}, C^*, D^* \} \) and let \( \check{O} \) be a Young diagram that has good parity.

Define a label
\[
\star' := \tilde{C}, D, C, B, D^*, \text{ or } C^*
\]
respectively if
\[
\star = B, C, D, \tilde{C}, C^*, \text{ or } D^*.
\]
We call \( \star' \) the Howe dual of \( \star \).

The dual descent of the Young diagram \( \check{O} \) is defined to be
\[
\check{O}' := \nabla(\check{O}) := \nabla_{\star'}(\check{O}) := \begin{cases} \square; & \text{if } \star \in \{ D, D^* \} \text{ and } |\check{O}| = 0; \\ \nabla_{\text{naive}}(\check{O}); & \text{otherwise}, \end{cases}
\]
where \( \square \) denotes the Young diagram that has total size 1, and \( \nabla_{\text{naive}}(\check{O}) \) denotes the Young diagram obtained from \( \check{O} \) by removing the first row. We also view \( \check{O}' \) as a nilpotent orbit in the Langlands dual (or metaplectic Langlands dual) of the complexified Lie algebra of a classical real Lie group of type \( \star' \).

For a Young diagram \( i \), its naive descent, which is denoted by \( \nabla_{\text{naive}}(i) \), is defined to be the Young diagram obtained from \( i \) by removing the first column. By convention, \( \nabla_{\text{naive}}(\emptyset) = \emptyset \).

We start with the notion of the naive descent of a painted bipartition. Let \( \tau = (i, \mathcal{P}) \times (j, \mathcal{Q}) \times \gamma \) be a painted bipartition such that \( \star_\tau = \star \). Put
\[
\gamma' = \begin{cases} B^+, & \text{if } \alpha = \tilde{C} \text{ and } c \text{ does not occur in the first column of } (i, \mathcal{P}); \\ B^-, & \text{if } \alpha = \tilde{C} \text{ and } c \text{ occurs in the first column of } (i, \mathcal{P}); \\ \star', & \text{if } \alpha \neq \tilde{C}. \end{cases}
\]

Lemma 10.4. If \( \star \in \{ B, C, C^* \} \), then there is a unique painted bipartition of the form \( \tau'_{\text{naive}} = (i', \mathcal{P}'_{\text{naive}}) \times (j', \mathcal{Q}'_{\text{naive}}) \times \gamma' \) with the following properties:

- \( (i', j') = (i, \nabla_{\text{naive}}(j)) \);
- for all \( (i, j) \in \text{Box}(i') \),
  \[
  \mathcal{P}'_{\text{naive}}(i, j) = \begin{cases} \bullet \text{ or } s, & \text{if } \mathcal{P}(i, j) \in \{ \bullet, s \}; \\ \mathcal{P}(i, j), & \text{if } \mathcal{P}(i, j) \notin \{ \bullet, s \}; \end{cases}
  \]
- for all \( (i, j) \in \text{Box}(j') \),
  \[
  \mathcal{Q}'_{\text{naive}}(i, j) = \begin{cases} \bullet \text{ or } s, & \text{if } \mathcal{Q}(i, j + 1) \in \{ \bullet, s \}; \\ \mathcal{Q}(i, j + 1), & \text{if } \mathcal{Q}(i, j + 1) \notin \{ \bullet, s \}. \end{cases}
  \]

Proof. First assume that the images of \( \mathcal{P} \) and \( \mathcal{Q} \) are both contained in \{\bullet, s\}. Then the image of \( \mathcal{P} \) is in fact contained in \{\bullet\}, and \( (i, j) \) is right interlaced in the sense that
\[
c_1(j) \geq c_1(i) \geq c_2(j) \geq c_2(i) \geq c_3(j) \geq c_3(i) \geq \cdots.
\]
Hence \( (i', j') := (i, \nabla(j)) \) is left interlaced in the sense that
\[
c_1(i') \geq c_1(j') \geq c_2(i') \geq c_2(j') \geq c_3(i') \geq c_3(j') \geq \cdots.
\]
Then it is clear that there is a unique painted bipartition of the form \( \tau'_{\text{naive}} = (i', \mathcal{P}'_{\text{naive}}) \times (j', \mathcal{Q}'_{\text{naive}}) \times \gamma' \) such that symbols of \( \mathcal{P}'_{\text{naive}} \) and \( \mathcal{Q}'_{\text{naive}} \) are both in \{\bullet, s\}. This proves the lemma in the special case when the images of \( \mathcal{P} \) and \( \mathcal{Q} \) are both contained in \{\bullet, s\}.

The proof of the lemma in the general case is easily reduced to this special case. \( \square \)

Lemma 10.5. If \( \star \in \{ \tilde{C}, D, D^* \} \), then there is a unique painted bipartition of the form \( \tau'_{\text{naive}} = (i', \mathcal{P}'_{\text{naive}}) \times (j', \mathcal{Q}'_{\text{naive}}) \times \gamma' \) with the following properties:

- \( (i', j') = (\nabla_{\text{naive}}(i), j) \);
• for all \((i, j) \in \text{Box}(\tau')\),
\[
P_{\text{naive}}'(i, j) = \begin{cases} 
\bullet \text{ or } s, & \text{if } P(i, j + 1) \in \{\bullet, s\}; \\
\mathcal{Q}(i, j), & \text{if } P(i, j + 1) \notin \{\bullet, s\};
\end{cases}
\]

• for all \((i, j) \in \text{Box}(\mathcal{Q})\),
\[
Q_{\text{naive}}'(i, j) = \begin{cases} 
\bullet \text{ or } s, & \text{if } Q(i, j) \in \{\bullet, s\}; \\
Q(i, j), & \text{if } Q(i, j) \notin \{\bullet, s\}.
\end{cases}
\]

Proof. The proof is similar to that of Lemma 10.4.

Example 10.6. Suppose that the nonzero row lengths of \(\mathcal{O}\) are 8, 6, 6, 6, 4, 4, 2. Then
\[
\tau = \begin{array}{cccc}
\bullet & \bullet & \bullet & c \\
\bullet & s & c & \\
s & c & \\
h & d & d & d
\end{array}
\times \begin{array}{cccc}
\bullet & \bullet & \bullet & c \\
\bullet & r & d & \\
r & d & d & d
\end{array} \in PBP_{Sp_{36}(\mathbb{R})}(\mathcal{O}).
\]

We have that
\[
\nabla_{\text{naive}}(\tau) = \begin{array}{cccc}
\bullet & c & \\
\bullet & c & d & d
\end{array} \times B^- \in PBP_{SO(14,15)}(\mathcal{O}).
\]

10.4. Descent of a painted bipartition. Given a subset \(\wp \subseteq PP_\bullet(\mathcal{O})\), the dual descent of \(\wp\) is defined to be the subset
\[
\wp' := \nabla(\wp) := \{(i, i + 1) \mid i \in \mathbb{N}^+, (i + 1, i + 2) \in \wp\} \subseteq PP_\wp(\mathcal{O}).
\]

Note that
\[
(t_\wp', j_\wp') = \begin{cases} 
(t_\wp, \nabla_{\text{naive}}(j_\wp)), & \text{if } \star = B, \text{ or } \star \in \{C, C^*\} \text{ and } (1, 2) \notin \wp; \\
(t_{j_{p}}, \nabla_{\text{naive}}(j_{p})), & \text{if } \star \in \{C, C^*\} \text{ and } (1, 2) \notin \wp; \\
\nabla_{\text{naive}}(t_{p}), j_{p})), & \text{if } \star \in \{D, D^*\}, \text{ or } \star = \tilde{C} \text{ and } (1, 2) \notin \wp; \\
\nabla_{\text{naive}}(t_{p}), j_{p})), & \text{if } \star = \tilde{C} \text{ and } (1, 2) \notin \wp,
\end{cases}
\]

where \(t_{wp} := \nabla(t_\wp, j_\wp)\), \(j_{wp} := j_\wp(\mathcal{O}, j_\wp)\), and \(\wp' = \wp \setminus \{(1, 2)\}\).

Recall from (8.14) the set
\[
\text{PBP}_\star(\mathcal{O}, \wp) := \{ \tau \text{ is a painted bipartition } | \, \star_\tau = \star, (t_\tau, j_\tau) = (t_\wp, j_\wp) \}.
\]

Suppose that
\[
\tau = (t_\wp, P) \times (j_\wp, Q) \times \gamma \in \text{PBP}_\star(\mathcal{O}, \wp).
\]

We will use the notion of Lemma 10.4 and Lemma 10.5, and define an element
\[
\tau' := (t_{wp}, P') \times (j_{wp}, Q') \times \gamma' \in \text{PBP}_\star(\mathcal{O}', \wp'),
\]
to be called the descent of \(\tau\).

The case when \(\star = B\).
(a) If
\[
\begin{cases}
\gamma = B^+; \\
(2, 3) \notin \wp; \\
r_2(\mathcal{O}) > 0; \\
Q(c_1(t_\wp), 1) \in \{r, d\},
\end{cases}
\]
we define
\[ P'(i, j) := \begin{cases} s, & \text{if } (i, j) = (c_1(\tau'), 1); \\ P'_{\text{naive}}(i, j), & \text{otherwise} \end{cases} \]
for all \((i, j) \in \text{Box}(\tau')\), and \(Q' := Q'_{\text{naive}}\).

(b) If
\[
\begin{cases} \gamma = B^+; \\ (2, 3) \in \wp; \\ Q(c_2(j_\wp), 1) \in \{r, d\}, \end{cases}
\]
we define \(P' := P'_{\text{naive}}\), and
\[ Q'(i, j) := \begin{cases} r, & \text{if } (i, j) = (c_1(\tau'), 1); \\ Q'_{\text{naive}}(i, j), & \text{otherwise} \end{cases} \]
for all \((i, j) \in \text{Box}(\tau')\).

(c) Otherwise, we define \(P' := P'_{\text{naive}}\) and \(Q' := Q'_{\text{naive}}\).

The case when \(\star = D\).

(a) If
\[
\begin{cases} \gamma = \hat{B}; \\ P(c_2(\tau), 2) = c; \\ P(i, 1) \in \{r, d\}, \end{cases}
\]
for all \(c_2(\tau) \leq i \leq c_1(\tau)\),
we define
\[ P'(i, j) := \begin{cases} r, & \text{if } (i, j) = (c_1(\tau'), 1); \\ P'_{\text{naive}}(i, j), & \text{otherwise} \end{cases} \]
for all \((i, j) \in \text{Box}(\tau')\), and \(Q' := Q'_{\text{naive}}\).

(b) If
\[
\begin{cases} (2, 3) \in \wp; \\ P(c_2(\tau) - 1, 1) \in \{r, c\}, \end{cases}
\]
we define
\[ P'(i, j) := \begin{cases} r, & \text{if } (i, j) = (c_1(\tau') - 1, 1); \\ P'(c_2(\tau) - 1, 1), & \text{if } (i, j) = (c_1(\tau'), 1); \\ P'_{\text{naive}}(i, j), & \text{otherwise} \end{cases} \]
for all \((i, j) \in \text{Box}(\tau')\), and \(Q' := Q'_{\text{naive}}\).

(c) Otherwise, we define \(P' := P'_{\text{naive}}\) and \(Q' := Q'_{\text{naive}}\).

The case when \(\star \in \{C, \tilde{C}, C^*, D^*\}\).

(a) If \((1, 2) \notin \wp\), we define
\[ \tau' := \nabla_{\text{naive}}(\tau). \]

(b) If \((1, 2) \in \wp\), then \(\star \in \{C, \tilde{C}\}\) (by Proposition 10.1) and we define
\[ \tau' := \nabla_{\text{naive}}(\tau_{\psi_1}), \]
where \(\psi_1 := \wp \setminus \{(1, 2)\}\), \(\tau_{\psi_1} := T_{\psi_1, \rho}^{-1}(\tau)\), and \(T_{\psi_1, \rho}\) is as in Lemma 10.3.

In all cases, it is routine to check that \(\tau'\) thus constructed is an element in \(\text{PBP}_{\star}(\hat{\wp}, \wp')\), which is called the descent of \(\tau\). To summarize, we have a map
\[
\nabla : \text{PBP}_{\star}(\hat{\wp}, \wp) \to \text{PBP}_{\star}(\hat{\wp'}, \wp').
\]
We call \(\nabla\) the descent map of painted bipartitions.
Example 10.7. Suppose that the nonzero row lengths of $\hat{\mathcal{O}}$ are 7, 7, 3. Then

$$\tau = \begin{bmatrix} \bullet & \bullet \\ \bullet & s \\ s & \bullet \\ r & c \end{bmatrix} \times D \in \text{PBP}_{SO(11,13)}(\hat{\mathcal{O}}).$$

We have that

$$\nabla_{\text{naive}}(\tau) = \begin{bmatrix} \bullet & \bullet & s \\ \bullet & \bullet \\ s & \bullet \\ c \end{bmatrix} \times C \quad \text{and} \quad \nabla(\tau) = \begin{bmatrix} \bullet & \bullet \end{bmatrix} \times \begin{bmatrix} \bullet & s \\ c \end{bmatrix} \times C,$$

both in $\text{PBP}_{Sp_{16}(\mathbb{R})}(\hat{\mathcal{O}})$.

10.5. Properties of the descent map. We first define the notion of the tail of a painted bipartition, when $\star \in \{B, D, C^\ast\}$.

Put

$$\tau_t := \begin{cases} D, & \text{if } \star \in \{B, D\}; \\ C^\ast, & \text{if } \star = C^\ast, \end{cases} \quad \text{and} \quad k := \begin{cases} \frac{r_1(\hat{\mathcal{O}}) - r_2(\hat{\mathcal{O}})}{2} + 1, & \text{if } \star \in \{B, D\}; \\ \frac{r_1(\hat{\mathcal{O}}) - r_2(\hat{\mathcal{O}}) - 1}{2}, & \text{if } \star = C^\ast. \end{cases}$$

Note that $k$ is a non-negative integer, and is positive when $\star \in \{B, D\}$.

From the pair $(\tau, \hat{\mathcal{O}})$, we define a Young diagram $\hat{\mathcal{O}}_t$ as follows.

- If $\star \in \{B, D\}$, then $\hat{\mathcal{O}}_t$ consists of two rows with lengths $2k - 1$ and 1.
- If $\star = C^\ast$, then $\hat{\mathcal{O}}_t$ consists of one row with length $2k + 1$.

Note that in all the three cases $\hat{\mathcal{O}}_t$ has good parity (with respect to $\tau_t$), $\text{PP}_{\tau_t}(\hat{\mathcal{O}}_t) = \emptyset$, and every element in $\text{PBP}_{\tau_t}(\hat{\mathcal{O}}_t)$ has the form

$$\begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_k \end{bmatrix} \times \emptyset \times D, \quad \begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_k \end{bmatrix} \times \emptyset \times D, \quad \text{or} \quad \emptyset \times \begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_k \end{bmatrix} \times C^\ast,$$

(10.7)

according to $\star = B, D$, or $C^\ast$, respectively. When $k = 0$ (and thus $\star = C^\ast$), the element in $\text{PBP}_{\tau_t}(\hat{\mathcal{O}}_t)$ is understood to be $\emptyset \times \emptyset \times C^\ast$.

When $\star \in \{B, D, C^\ast\}$, the tail $\tau_t \in \text{PBP}_{\tau_t}(\hat{\mathcal{O}}_t)$ of an element $\tau = (i, \mathcal{P}) \times (j, \mathcal{Q}) \times \gamma \in \text{PBP}_\ast(\hat{\mathcal{O}}, \mathcal{G})$ will be the painted bipartition in (10.7), specified by the multiset $\{x_1, x_2, \cdots, x_k\}$ case by case as follows.

**The case when $\star = B$.** In this case, $c_1(i) \leq c_1(j)$. The multiset $\{x_1, x_2, \cdots, x_k\}$ is the union of the multiset

$$\{Q(j, 1) \mid c_1(i) + 1 \leq j \leq c_1(j)\}$$

with the set

$$\left\{ \begin{array}{ll} \{ c \}, & \text{if } \alpha = B^+, \text{ and either } c_1(i) = 0 \text{ or } Q(c_1(i), 1) \in \{ \bullet, s \}; \\ \{ s \}, & \text{if } \alpha = B^-, \text{ and either } c_1(i) = 0 \text{ or } Q(c_1(i), 1) \in \{ \bullet, s \}; \\ \{ Q(c_1(i), 1) \}, & \text{otherwise.} \end{array} \right.$$ 

**The case when $\star = D$.** In this case, $c_1(i) > c_1(j)$ when $|\hat{\mathcal{O}}| > 0$. The multiset $\{x_1, x_2, \cdots, x_k\}$ is the union of the multiset

$$\{P(j, 1) \mid c_1(j) + 2 \leq j \leq c_1(i)\}$$
with the set
\[
\begin{cases}
\{ d \}, & \text{if } |\mathcal{O}|=0; \\
\{ c \}, & \text{if } r_2(\mathcal{O}) = r_3(\mathcal{O}) > 0, \quad \mathcal{P}(c_1(i), 1) \in \{ r, d \}, \text{ and } \\
\{ \mathcal{P}(c_1(j) + 1, 1) \}, & \text{otherwise}.
\end{cases}
\]

The case \( \ast = C^* \). In this case, \( \mathcal{O} \) is quasi-distinguished, \( \varphi = \emptyset \), and \( c_1(i) \leq c_1(j) \). The multiset \( \{ x_1, x_2, \ldots, x_k \} \) equals the multiset
\[
\{ \mathcal{Q}(j, 1) \mid c_1(i) + 1 \leq j \leq c_1(j) \}.
\]

We introduce one final notation when \( \ast \in \{ B, D \} \):
\[
x_\tau := \mathcal{P}_{\kappa}(k, 1),
\]
which is the symbol in the last box of the tail \( \tau \).

The following two propositions summarize key properties of the descent map. Both readily follow from the detailed description of the descent algorithm.

**Proposition 10.8.** Suppose that \( \ast \in \{ C, \tilde{C}, D^* \} \).
(a) If \( \ast = D^* \) or \( r_1(\mathcal{O}) > r_2(\mathcal{O}) \), then the descent map (10.6) is bijective.
(b) If \( \ast \in \{ C, \tilde{C} \} \) and \( r_1(\mathcal{O}) = r_2(\mathcal{O}) \), then the descent map (10.6) is injective with image
\[
\{ \tau' \in \text{PB}_\ast(\mathcal{O}', \varphi') \mid x_{\tau'} \neq s \}.
\]

Recall that \( \mathcal{O} = d_{BV}(\mathcal{O}) \) is the Barbasch-Vogan dual of \( \mathcal{O} \). Write
\[
\mathcal{O}'' := \nabla^2(\mathcal{O}) := \nabla(\mathcal{O}') \quad \text{and} \quad \varphi'' := \nabla^2(\varphi) := \nabla(\varphi').
\]

Then we have the double descent map
\[
\nabla^2 := \nabla \circ \nabla : \text{PB}_\ast(\mathcal{O}, \varphi) \to \text{PB}_\ast(\mathcal{O}'', \varphi '').
\]

**Proposition 10.9.** Suppose that \( \ast \in \{ B, D, C^* \} \) and \( (\ast, |\mathcal{O}|) \neq (D, 0) \). Consider the map
\[
\text{PB}_\ast(\mathcal{O}, \varphi) \to \text{PB}_\ast(\mathcal{O}'', \varphi') \times \text{PB}_\ast(\mathcal{O}_\tau), \quad \tau \mapsto (\nabla^2(\tau), \tau_\tau).
\]

(a) Suppose that \( \ast = C^* \) or \( r_2(\mathcal{O}) > r_3(\mathcal{O}) \). Then the map (10.8) is bijective, and for every \( \tau \in \text{PB}_\ast(\mathcal{O}) \),
\[
\text{Sign}(\tau) = (c_2(\mathcal{O}), c_2(\mathcal{O})) + \text{Sign}(\nabla^2(\tau)) + \text{Sign}(\tau_\tau).
\]

(b) Suppose that \( \ast \in \{ B, D \} \) and \( r_2(\mathcal{O}) = r_3(\mathcal{O}) > 0 \). Then the map (10.8) is injective with image
\[
\left\{(\tau'', \tau_0) \in \text{PB}_\ast(\mathcal{O}'', \varphi'') \times \text{PB}_D(\mathcal{O}_\tau) \mid \begin{array}{l}
either x_{\tau''} = d, \text{ or } \\
x_{\tau''} \in \{ r, c \} \text{ and } \mathcal{P}_{\tau_0}^{-1}(\{ s, c \}) \neq \emptyset \end{array}\right\},
\]
and for every \( \tau \in \text{PB}_\ast(\mathcal{O}, \varphi) \),
\[
\text{Sign}(\tau) = (c_2(\mathcal{O}) - 1, c_2(\mathcal{O}) - 1) + \text{Sign}(\nabla^2(\tau)) + \text{Sign}(\tau_\tau).
\]

The following corollary will be used in [BMSZ21].

**Corollary 10.10.** Suppose that \( \ast \in \{ B, D, C^* \} \). Denote
\[
\varepsilon_\tau := \begin{cases} 0, & \text{if } \ast \in \{ B, D \} \text{ and } x_\tau = d; \\
1, & \text{otherwise}.
\end{cases}
\]

Then the map
\[
\text{PB}_\ast(\mathcal{O}, \varphi) \to \text{PB}_\ast(\mathcal{O}'', \varphi') \times \mathbb{N} \times \mathbb{N} \times \mathbb{Z}/2\mathbb{Z}, \quad \tau \mapsto (\nabla(\tau), p_\tau, q_\tau, \varepsilon_\tau),
\]
is injective.
Proof. This is easy to verify in the special case when \( r_3(\bar{\Omega}) = 0 \) and

\[
(r_1(\bar{\Omega}), r_2(\bar{\Omega})) = \begin{cases} 
(2k - 2, 0), & \text{if } \star = B; \\
(2k - 1, 1), & \text{if } \star = D; \\
(2k - 1, 0), & \text{if } \star = C^*.
\end{cases}
\]

In general, it follows from the injectivity of \( \nabla \) in Proposition 10.8 and the signature formula in Proposition 10.9.

10.6. Generating functions. We introduce some generating functions to count painted bipartitions. When \( \star \in \{ B, D, C^* \} \), we define

\[
f_{\star, \bar{\Omega}, \bar{\varphi}}(p, q) := \sum_{\tau \in \text{PBP}_\star(\bar{\Omega}, \bar{\varphi})} p^{\tau_x} q^{\tau_y},
\]

where \( (p_\tau, q_\tau) \) is as in (2.17). This is an element in the polynomial ring \( \mathbb{Z}[p, q] \), where \( p \) and \( q \) are indeterminants. The coefficient of \( p^\tau q^\varphi \) in \( f_{\star, \bar{\Omega}, \bar{\varphi}}(p, q) \) equals the cardinality of \( \text{PBP}_{\star}(\bar{\Omega}, \bar{\varphi}) \) (note that \( G = \text{SO}(p, q) \) or \( \text{Sp}(\frac{k}{2}, \frac{k}{2}) \)).

When \( \star \in \{ B, D \} \), for each subset \( S \subseteq \{ c, d, r, s \} \), we also define

\[
\text{PBP}_\star^S(\bar{\Omega}, \bar{\varphi}) := \{ \tau \in \text{PBP}_\star(\bar{\Omega}, \bar{\varphi}) \mid x_\tau \in S \}
\]

and the corresponding generating function

\[
f_{\star, \bar{\Omega}, \bar{\varphi}}^S(p, q) := \sum_{\tau \in \text{PBP}_\star^S(\bar{\Omega}, \bar{\varphi})} p^{\tau_x} q^{\tau_y}.
\]

Note that

\[
f_{d, \bar{\Omega}, \bar{\varphi}}(1) = 1, \quad f_{c, r, \bar{\Omega}, \bar{\varphi}} := 0, \quad \text{and } f_{s, \bar{\Omega}, \bar{\varphi}} := 0.
\]

In the following we will compute \( f_{\star, \bar{\Omega}, \bar{\varphi}}^S(p, q) \) for every \( \star \in \{ B, D \} \), \( S \subseteq \{ c, d, r, s \} \), whose sum is the desired function \( f_{\star, \bar{\Omega}, \bar{\varphi}}^S \).

For every integer \( k \), we define

\[
\nu_k := \begin{cases} 
\sum_{i=0}^{k} p^{2i} q^{2(k-i)}, & \text{if } k \geq 0; \\
0, & \text{if } k < 0.
\end{cases}
\]

It is straightforward to check the following identities: for every \( k \in \mathbb{N} \),

\[
f_{\bar{B}, [2k]}(p, q) = \begin{cases} 
(p^2 q + pq^2)\nu_{k-1}, & \text{if } S = \{ d \}; \\
ppq + p^2 q^2\nu_{k-1}, & \text{if } S = \{ c, r \}; \\
q^{2k+1}, & \text{if } S = \{ s \},
\end{cases}
\]

and for every \( k \in \mathbb{N}^+ \),

\[
f_{\bar{D}, [2k-1]}(p, q) = \begin{cases} 
ppq\nu_{k-1} + p^2 q^2\nu_{k-2}, & \text{if } S = \{ d \}; \\
(p^2 + pq)\nu_{k-1}, & \text{if } S = \{ c, r \}; \\
q^{2k}, & \text{if } S = \{ s \}.
\end{cases}
\]

For every \( k \in \mathbb{N} \), we also define

\[
h_{k}^{d} := (p^2 q^2 + pq^3)\nu_{k-2},
\]

\[
h_{k}^{c, r} := ppq\nu_{k-1} + p^2 q^2\nu_{k-2},
\]

\[
h_{k}^{s} := q^{2k}.
\]

Proposition 10.11. Suppose that \( \star \in \{ B, D \} \) and \( r_2(\bar{\Omega}) > 0 \). Let \( k := \frac{r_1(\bar{\Omega}) - r_2(\bar{\Omega})}{2} + 1 \) and suppose that \( S = \{ d \}, \{ c, r \}, \) or \( \{ s \} \).

(a) If \( (2, 3) \in \text{PP}_\star(\bar{\Omega}) \), then

\[
f_{\star, \bar{\Omega}, \bar{\varphi}}^S(pq)^{r_1(\bar{\Omega})} f_{\bar{D}, [2k-1]}(p, q) \cdot f_{\star, \bar{\varphi}^1(\bar{\Omega}), \bar{\varphi}^2(\bar{\Omega})}.
\]
Proposition 10.12. Suppose that \(* \in \{ C, \tilde{C} \}\) and \(r_1(\mathcal{O}) > 0\).

(a) If \((1, 2) \in PP_\ast(\mathcal{O})\), then
\[
\#PP_\ast(\mathcal{O}, \varphi) = f_{\ast, \mathcal{V}(\mathcal{O}), \mathcal{V}(\varphi)}(1, 1) = \frac{(pq)^{r_1(\mathcal{O})-1}}{2^{r_2(\mathcal{O})+1}k} \cdot f_{\ast, \mathcal{V}(\mathcal{O}), \emptyset} + h^{\mathcal{O}}_k \cdot f^{(c,r)}_{\ast, \mathcal{V}(\mathcal{O}), \mathcal{V}(\varphi)}.
\]

(b) If \((1, 2) \notin PP_\ast(\mathcal{O})\), then
\[
\#PP_\ast(\mathcal{O}, \varphi) = \frac{f^{(c,r)}_{\ast, \mathcal{V}(\mathcal{O}), \mathcal{V}(\varphi)}(1, 1) + f^{(d)}_{\ast, \mathcal{V}(\mathcal{O}), \mathcal{V}(\varphi)}(1, 1)}{2}
\]

Proof. This follows from Proposition 10.8, after a routine calculation.

Proposition 10.13. Suppose that \(* = C^*\). Then
\[
f_{\ast, \mathcal{O}, \emptyset} = \begin{cases} \nu_{r_1(\mathcal{O})-1}, & \text{if } r_2(\mathcal{O}) = 0; \\ \frac{(pq)^{r_1(\mathcal{O})-1}}{2^{r_2(\mathcal{O})+1}k} \cdot f_{\ast, \mathcal{V}(\mathcal{O}), \emptyset}, & \text{if } r_2(\mathcal{O}) > 0. \end{cases}
\]

where \(k := \frac{r_1(\mathcal{O})-r_2(\mathcal{O})}{2} - 1\).

Proposition 10.14. Suppose that \(* = D^*\). Then
\[
\#PP_G(\mathcal{O}, \emptyset) = f_{\ast, \mathcal{V}(\mathcal{O}), \emptyset}(1, 1) = 1 + \nu_{r_1(\mathcal{O})-1}.
\]

The following result is clear by Proposition 10.13 and Proposition 10.14.

Proposition 10.15. Suppose that \(* \in \{ C^*, D^* \}\). Then the cardinality of \(PP_G(\mathcal{O})\) equals the number of \(G\)-orbits in \((\sqrt{-1}g_0^0) \cap \mathcal{O}\), where \(g_0^0\) denotes the Lie algebra of \(G\) and \(\mathcal{O} := d_{BV}(\mathcal{O})\).

Acknowledgements

The authors have learned much about coherent continuation representations from David Vogan, and are grateful to him for many helpful discussions/clarifications. The authors also thank Devra Garfinkle Johnson for fruitful discussions concerning Proposition 4.18 and Remark 4.19. The authors are further grateful to the referees for their helpful comments and detailed suggestions.

D. Barbasch is supported by NSF grant, Award Number 2000254. J.-J. Ma is supported by the National Natural Science Foundation of China (Grant No. 11701364 and Grant No. 11971305), the Fundamental Research Funds for the Central Universities (Grant No. 20720230022) and Xiamen University Malaysia Research Fund (Grant No. XMUMRF/2022-C9/IMAT/0019).

B. Sun is supported by National Key R & D Program of China (No. 2022YFA1005300 and 2020YFA0712600) and New Cornerstone Investigator Program. C.-B. Zhu is supported by MOE AcRF Tier 1 grants A-0004280-00-00 and A-8002490-00-00, and Provost’s Chair grant E-146-00-0018-01 in NUS.

C.-B. Zhu is grateful to Max Planck Institute for Mathematics in Bonn, for its warm hospitality and conducive work environment, where he spent the academic year 2022/2023 as a visiting scientist.
References

[AARM24] J. Adams, N. Arancibia Robert, and P. Mezo, *Equivalent definitions of Arthur packets for real classical groups*, Mem. Amer. Math. Soc. 300 (2024), no. 1503, vii+110 pp.

[ABV91] J. Adams, D. Barbasch, and D. A. Vogan, *The Langlands classification and irreducible characters for real reductive groups*, Progr. Math., vol. 104, Birkhäuser, 1991.

[AdC09] J. Adams and F. du Cloux, *Algorithms for representation theory of real reductive groups*, J. Inst. Math. Jussieu 8 (2009), no. 2, 209–259.

[ARM] N. Arancibia Robert and P. Mezo, *Equivalent definitions of Arthur packets for real unitary groups*, available at arXiv:2204.19715.

[Art84] J. Arthur, *On some problems suggested by the trace formula*, Lie group representations, II (College Park, Md.), Lecture Notes in Math., vol. 1041, 1984, pp. 1–49.

[Art89] ——, *Unipotent automorphic representations: conjectures, Orbites unipotentes et représentations, II*, Astérisque 171-172 (1989), 13–71.

[Art13] ——, *The Endoscopic Classification of Representations: Orthogonal and Symplectic Groups*, Amer. Math. Soc. Colloq. Publ., vol. 61, Amer. Math. Soc., Providence, RI (2013).

[Bar89] D. Barbasch, *The unitary dual for complex classical Lie groups*, Invent. Math. 96 (1989), no. 1, 103–176.

[Bar90] ——, *Orbital integrals of nilpotent orbits*, The mathematical legacy of Harish-Chandra, Proc. Sympos. Pure Math. 68 (2000), 97–110.

[Bar17] ——, *Unipotent representations and the dual pair correspondence*, J. Cogdell et al. (eds.), Representation Theory, Number Theory, and Invariant Theory, In Honor of Roger Howe. Progr. Math., vol. 323, Birkhäuser, 2017, pp. 47–85.

[BMS22] D. Barbasch, J.-J. Ma, B. Sun, and C.-B. Zhu, *Special unipotent representations of real classical groups: construction and unitarity (2021)*, available at arXiv:1712.05552.

[BMS23] ——, *On the notion of metaplectic Barbasch-Vogan duality*, Int. Math. Res. Not. IMRN 20 (2023), 17822–17852.

[BV82] D. Barbasch and D. A. Vogan, *Primitive ideals and orbital integrals in complex classical groups*, Math. Ann. 259 (1982), no. 2, 153–199.

[BV83a] ——, *Primitive ideals and orbital integrals in complex exceptional groups*, J. Algebra 80 (1983), no. 2, 350–382.

[BV83b] ——, *Weyl Group Representations and Nilpotent Orbits*, Representation Theory of Reductive Groups: Proc. Univ. Utah Conference (1982), 1983, pp. 21–33.

[BV85] ——, *Unipotent representations of complex semisimple groups*, Ann. of Math. 121 (1985), no. 1, 41–110.

[BGG75] I. N. Bernstein, I. M. Gel’fand, and S. I. Gel’fand, *Models of representations of compact Lie groups*, Funkcional. Anal. i Priložen. 9 (1975), no. 4, 61–62 (Russian).

[Bor79] A. Borel, *Automorphic L-functions*, Automorphic Forms, Representations and L-functions, Proc. Sympos. Pure Math., vol. 33, Part 2, 1979, pp. 27–61.

[Bor76] W. Borho, *Recent advances in enveloping algebras of semisimple Lie-algebras*, Séminaire Bourbaki, Exp. No. 489 (1976/77), 1–18.

[BK76] W. Borho and H. Kraft, *Über die Gelfand-Kirillov-Dimension*, Math. Ann. 220 (1976), no. 1, 1–24.

[BMP78] W. Borho and R. MacPherson, *Représentations des groupes de Weyl et homologie d’intersection pour les variétés nilpotentes*, C. R. Acad. Sci. Paris Sér. I Math. 292 (1981), no. 15, 707–710 (French, with English summary).

[Boš92] M. Božičević, *Double cells for unitary groups*, J. Algebra 254 (2002), no. 1, 115–124.

[Car93] R. W. Carter, *Finite groups of Lie type*, Wiley Classics Library, John Wiley & Sons, Ltd., Chichester, 1993.

[Cas86] L. G. Casian, *Primitive ideals and representations*, J. Algebra 101 (1986), no. 2, 497–515.

[CM93] D. H. Collingwood and W. M. McGovern, *Nilpotent orbits in semisimple Lie algebra: an introduction*, Van Nostrand Reinhold Co., 1993.

[Dix96] J. Dixmier, *Enveloping algebras*, Grad. Stud. Math., vol. 11, Amer. Math. Soc., 1996.

[Duf77] M. Duflo, *Sur la Classification des Ideaux Primitifs Dans L’algebre Enveloppante d’une Algebre de Lie Semi-Simple*, Ann. of Math. 105 (1977), no. 1, 107–120.

[FIJMN21] T. Folz-Donahue, S. G. Jackson, T. Milev, and A. G. Noël, *Sign signatures and characters of Weyl Groups*, Adv. in Appl. Math. 130 (2021), Paper No. 10225, 73.

[GI19] W. T. Gan and A. Ichino, *On the irreducibility of some induced representations of real reductive Lie groups*, Tunis. J. Math. 1 (2019), no. 1, 73–107.

[Gar93] D. Garfinkle, *On the classification of primitive ideals for complex classical Lie algebras. III*, Compos. Math. 88 (1993), no. 2, 187–234.

[GV92] D. Garfinkle and D. A. Vogan, *On the structure of Kazhdan-Lusztig cells for branched Dynkin diagrams*, J. Algebra 153 (1992), no. 1, 91–120.

[GJ19] D. Garfinkle Johnson, *Edge Transport from Parabolic Subgroups of Type D_4* (2019), available at arXiv:1907.09717.
[GW09] R. Goodman and N. R. Wallach, Symmetry, representations, and invariants, Grad. Texts in Math., vol. 255, Springer, Dordrecht, 2009.

[Hot84] R. Hotta, On Joseph’s construction of Weyl group representations, Tohoku Math. J. 36 (1984), 49–74.

[How79] R. Howe, θ-series and invariant theory, Automorphic Forms, Representations and L-functions, Proc. Sympos. Pure Math., vol. 33, Part 1, 1979, pp. 275–285.

[How89] ______, Transcending classical invariant theory, J. Amer. Math. Soc. 2 (1989), 535–552.

[Hum08] J. E. Humphreys, Representations of semisimple Lie algebras in the BGG category O, Grad. Stud. Math., vol. 94, Amer. Math. Soc., Providence, RI, 2008.

[Jan79] J. C. Jantzen, Moduln mit einem höchsten Gewicht, Lecture Notes in Math., vol. 750, Springer-Verlag, Berlin/Heidelberg/New York, 1979.

[Jos79] A. Joseph, Dixmier’s problem for Verma and principal series submodules, J. Lond. Math. Soc. (2) 20 (1979), no. 2, 193–204.

[Jos80a] ______, Goldie rank in the enveloping algebra of a semisimple Lie algebra. I, J. Algebra 65 (1980), no. 2, 269–283.

[Jos80b] ______, Goldie rank in the enveloping algebra of a semisimple Lie algebra. II, J. Algebra 65 (1980), no. 2, 284–306.

[Jos84] ______, On the variety of a highest weight module, J. Algebra 88 (1984), no. 1, 238–278.

[Jos85] ______, On the associated variety of a primitive ideal, J. Algebra 93 (1985), no. 2, 509–523.

[KL79] D. Kazhdan and G. Lusztig, Representations of Coxeter groups and Hecke algebras, Invent. Math. 53 (1979), no. 2, 165–181.

[Kir62] A. A. Kirillov, Unitary representations of nilpotent Lie groups, Uspehi Mat. Nauk 17 (1962), 57–110.

[Kna02] A. W. Knapp, Lie groups beyond an introduction, Progr. Math., vol. 200, Birkhäuser Boston, 2002.

[Kos70] B. Kostant, Quantization and unitary representations, Lectures in Modern Analysis and Applications III, Lecture Notes in Math., vol. 170, 1970, pp. 87–208.

[Lus79] G. Lusztig, A class of irreducible representations of a Weyl group, Nederl. Akad. Wetensch. Indag. Mat. 41 (1979), no. 3, 323–335.

[Lus82] G. Lusztig, A class of irreducible representations of a Weyl group. II, Nederl. Akad. Wetensch. Indag. Mat. 44 (1982), no. 2, 219–226.

[Lus84] ______, Characters of reductive groups over a finite field, Ann. of Math. Stud., vol. 107, Princeton University Press, Princeton, NJ, 1984.

[Mat96] H. Matumoto, On the representations of $U(m,n)$ unitarily induced from derived functor modules, Compos. Math. 100 (1996), no. 1, 1–39.

[Mat96] ______, On the representations of $Sp(p,q)$ and $SO^\ast(2n)$ unitarily induced from derived functor modules, Compos. Math. 140 (2004), no. 4, 1059–1096.

[McG98] W. M. McGovern, Cells of Harish-Chandra modules for real classical groups, Mem. Amer. Math. Soc. 140 (1999), no. 672, x+103 pp.

[MP23] W. M. McGovern and T. Pietraho, Representations of semisimple classical Lie algebras, IV (2023), available at arXiv:2309.02363.

[Mœg11] C. Mœglin, Multiplicité 1 dans les paquets d’Arthur aux places $p$-adiques, On certain L-functions, Clay Math. Proc., vol. 13, Amer. Math. Soc., Providence, RI, 2011, pp. 333–374 (French, with English summary).

[Mœg17] ______, Paquets d’Arthur Spéciaux Unipotents aux Places Archimédiennes et Correspondance de Howe, J. Cogdell et al. (eds.), Representation Theory, Number Theory, and Invariant Theory, In Honor of Roger Howe, Progr. Math., vol. 323, Birkhäuser, 2017, pp. 469–502.

[MR15] C. Mœglin and D. Renard, Paquets d’Arthur des groupes classiques complexes, Around Langlands correspondences, Contemp. Math., vol. 691, Amer. Math. Soc., Providence, RI, 2017, pp. 203–256 (French, with English and French summaries).

[MR19] ______, Sur les paquets d’Arthur des groupes unitaires et quelques conséquences pour les groupes classiques, Pacific J. Math. 299 (2019), no. 1, 53–88 (French, with English and French summaries).

[Mok15] C. P. Mok, Endoscopic classification of representations of quasi-split unitary groups, Mem. Amer. Math. Soc. 235 (2015), no. 1108, vi+248 pp.

[RT00] D. Renard and P. Trapa, Irreducible genuine characters of the metaplectic group: Kazhdan-Lusztig algorithm and Vogan duality, Represent. Theory 4 (2000), 245–295.

[RT03] ______, Irreducible characters of the metaplectic group. II. Functoriality, J. Reine Angew. Math. 557 (2003), 121–158.

[Sch77] W. Schmid, Two character identities for semisimple Lie groups, Non-commutative harmonic analysis (Actes Colloq., Marseille-Luminy, 1976). Lecture Notes in Math., Vol. 587, Springer, Berlin, 1977, pp. 196–225.

[SV00] W. Schmid and K. Vilonen, Characteristic cycles and wave front cycles of representations of reductive Lie groups, Ann. of Math. 151 (2000), no. 3, 1071–1118.

[Sho79] T. Shoji, On the Springer representations of the Weyl groups of classical algebraic groups, Comm. Algebra 7 (1979), 1713–1745.
[Soe90] W. Soergel, *Kategorie O, perverse Garben und Moduln über den Koinvarianten zur Weylgruppe*, J. Amer. Math. Soc. 3 (1990), no. 2, 421–445 (German, with English summary).

[Som01] E. Sommers, *Lusztig’s canonical quotient and generalized duality*, J. Algebra 243 (2001), no. 2, 790–812.

[Spa82] N. Spaltenstein, *Classes unipotentes et sous-groupes de Borel*, Lecture Notes in Math., vol. 946, Springer-Verlag, Berlin-New York, 1982.

[SV80] B. Speh and D. A. Vogan, *Reducibility of generalized principal series representations*, Acta Math. 145 (1980), no. 3-4, 227–299.

[Spr78] T. A. Springer, *A construction of representations of Weyl groups*, Invent. Math. 44 (1978), 279–293.

[Trapa] P. Trapa, *Annihilators and associated varieties of $A_q(\lambda)$ modules for $U(p,q)$*, Compos. Math. 129 (2001), no. 1, 1–45.

[Trapa04] , *Special unipotent representations and the Howe correspondence*, Univ. Aarhus Publ. Series 47 (2004), 210–230.

[Vog78] D. A. Vogan, *Gelfand-Kirillov dimension for Harish-Chandra modules*, Invent. Math. 48 (1978), no. 1, 75–98.

[Vog79a] , *A generalized $\tau$-invariant for the primitive spectrum of a semisimple Lie algebra*, Math. Ann. 242 (1979), no. 3, 209–224.

[Vog79b] , *Irreducible characters of semisimple Lie groups. I*, Duke Math. J. 46 (1979), no. 1, 61–108.

[Vog81] , *Representations of real reductive Lie groups*, Progr. Math., vol. 15, Birkhäuser, Boston, Mass., 1981.

[Vog82] , *Irreducible characters of semisimple Lie groups. IV. Character-multiplicity duality*, Duke Math. J. 49 (1982), no. 4, 943–1073.

[Vog83] , *Irreducible characters of semisimple Lie groups. III. Proof of Kazhdan-Lusztig conjecture in the integral case*, Invent. Math. 71 (1983), no. 2, 381–417.

[Vog86] , *The unitary dual of $GL(n)$ over an Archimedean field*, Invent. Math. 83 (1986), no. 3, 449–505.

[Vog87a] , *Representations of reductive Lie groups*, Proc. Intern. Congr. Math. (Berkeley, Calif., 1986), 1987, pp. 246–266.

[Vog87b] , *Unitary representations of reductive Lie groups*, Ann. of Math. Stud., vol. 118, Princeton University Press, 1987.

[Vog91] , *Associated varieties and unipotent representations*, Harmonic analysis on reductive groups, Proc. Conf., Brunswick/ME (USA) 1989, Progr. Math., vol. 101, 1991, pp. 315–388.

[Wal92] N. R. Wallach, *Real reductive groups II*, Academic Press Inc., 1992.

[Wei18] M. H. Weissman, *L-groups and parameters for covering groups*, Astérisque 398 (2018), 33–186.

[Zuc77] G. Zuckerman, *Tensor products of finite and infinite dimensional representations of semisimple Lie groups*, Ann. of Math. 106 (1977), 295–308.

**DEPARTMENT OF MATHEMATICS, 310 MALOTT HALL, CORNELL UNIVERSITY, ITHACA, NEW YORK 14853**

**Email address:** barbasch@math.cornell.edu

**SCHOOL OF MATHEMATICAL SCIENCES, XIAIEN UNIVERSITY, XIAMEN, 361005, CHINA**

**DEPARTMENT OF MATHEMATICS, XIAIEN UNIVERSITY MALAYSIA CAMPUS, SEPANG, SELANGOR DARUL EHSAN, 43900, MALAYSIA**

**Email address:** hoxide@xmu.edu.cn

**INSTITUTE FOR ADVANCED STUDY IN MATHEMATICS & NEW CORNERSTONE SCIENCE LABORATORY, ZHEJIANG UNIVERSITY, HANGZHOU, 310058, CHINA**

**Email address:** sunbinyong@zju.edu.cn

**DEPARTMENT OF MATHEMATICS, NATIONAL UNIVERSITY OF SINGAPORE, 10 LOWER KENT RIDGE ROAD, SINGAPORE 119076**

**Email address:** matzhucb@nus.edu.sg