J.L. Ramírez Alfonsín and M. Skałba

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Primes in numerical semigroups

J.L. Ramírez Alfonsín*, a and M. Skałba b

a UMI2924 - Jean-Christophe Yoccoz, CNRS-IMPA, Brazil and Univ. Montpellier, CNRS, Montpellier, France
b Institute of Mathematics, University of Warsaw, Banacha 2, 02-097 Warszawa, Poland
E-mails: jorge.ramirez-alfonsin@umontpellier.fr (J.L. Ramírez Alfonsín), skalba@mimuw.edu.pl (M. Skałba)

Abstract. Let 0 < a < b be two relatively prime integers and let \langle a, b \rangle be the numerical semigroup generated by a and b with Frobenius number \( g(a, b) = ab - a - b \). In this note, we prove that there exists a prime number \( p \in \langle a, b \rangle \) with \( p < g(a, b) \) when the product \( ab \) is sufficiently large. Two related conjectures are posed and discussed as well.

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Let 0 < a < b be two relatively prime integers. Let \( S = \langle a, b \rangle = \{ n \mid n = ax + by, x, y \in \mathbb{Z}, x, y \geq 0 \} \) be the numerical semigroup generated by a and b. A well-known result due to Sylvester [5] states that the largest integer not belonging to S, denoted by \( g(a, b) \), is given by \( ab - a - b \). \( g(a, b) \) is called the Frobenius number (we refer the reader to [3] for an extensive literature on the Frobenius number).

We clearly have that any prime \( p \) larger than \( g(a, b) \) belongs to \( \langle a, b \rangle \). A less obvious and more intriguing question is whether there is a prime \( p \leq g(a, b) \) belonging to \( \langle a, b \rangle \).

In this note, we show that there always exists a prime \( p \in \langle a, b \rangle, p < g(a, b) \) when the product \( ab \) is sufficiently large. The latter is a straightforward consequence of the below Theorem.

Let 0 < u < v be integers. We define

\[ \pi_S[u,v] = |\{ p \text{ prime} \mid p \in S, u \leq p \leq v \}|. \]

For short, we may write \( \pi_S \) instead of \( \pi_S[0, g(a, b)] \).

**Theorem 1.** Let 3 ≤ a < b be two relatively prime integers and let \( S = \langle a, b \rangle \) be the numerical semigroup generated by a and b. Then, for any fixed \( \varepsilon > 0 \) there exists \( C(\varepsilon) > 0 \) such that

\[ \pi_S > C(\varepsilon) \frac{g(a, b)}{\log(g(a, b))^{2+\varepsilon}}. \]
for \(ab\) sufficiently large.

Let us quickly introduce some notation and recall some facts needed for the proof of Theorem 1.  
Let \(S = \langle a, b \rangle\) and let \(0 < u < v\) be integers. We define

\[n_S[u, v] = |\{n \in \mathbb{N} | u \leq n \leq v, n \in S\}| \]

and

\[n_S^c[u, v] = |\{n \in \mathbb{N} | u \leq n \leq v, n \not\in S\}|.\]

For short, we may write \(n_S\) instead of \(n_S[0, g(a, b)]\) and \(n_S^c\) instead of \(n_S^c[0, g(a, b)]\). The set of elements in \(n_S^c = \mathbb{N} \setminus S\) are usually called the gaps of \(S\).

It is known [3] that \(S\) is always symmetric, that is, for any integer \(0 \leq s \leq g(a, b)\)

\[s \in S \quad \text{if and only if} \quad g(a, b) - s \not\in S.\]

It follows that

\[n_S = \frac{g(a, b) + 1}{2}.\]

We may now prove Theorem 1.

**Proof of Theorem 1.** Let \(\varepsilon > 0\) be fixed. We distinguish two cases.

**Case 1.** Suppose that \(a > (\log(ab))^{1+\varepsilon}\). Let us take \(c = ab/(\log(ab))^{1+\varepsilon}\). It is known [1] that if \(k \in [0, \ldots, g(a, b)]\) then

\[n_S[0, k] = \sum_{i=0}^{\left\lfloor \frac{k}{c} \right\rfloor} \left\lfloor \frac{k - ib}{a} \right\rfloor + 1.\]

In our case, we obtain that

\[n_S[0, c] \leq \left\lfloor \frac{c}{a} \right\rfloor + \left\lfloor \frac{c}{b} \right\rfloor + \left\lfloor \frac{c-b}{a} \right\rfloor + 1 \leq \left\lfloor \frac{c}{a} \right\rfloor + \left\lfloor \frac{c}{b} \right\rfloor + \left\lfloor \frac{c}{a} \right\rfloor + 1\]

\[\leq \frac{c}{a} + \frac{c}{b} + \frac{c^2}{ab} + 1 = \frac{bc + ac + c^2 + ab}{ab} < \frac{2c^2 + c^2 + 2c^2}{ab} = \frac{4c^2}{ab} = \frac{4ab}{(\log(ab))^{2+2\varepsilon}}\]

where the last inequality holds since \(c > b > a\).

Due to the symmetry of \(S\), we have

\[n_S^c[g(a, b) - c, g(a, b)] = n_S[0, c] < \frac{4ab}{(\log(ab))^{2+2\varepsilon}}. \tag{1}\]

Let \(\pi(x)\) be the number of primes integers less or equals to \(x\). We have

\[\pi(g(a, b)) - \pi(g(a, b) - c) \gg \frac{c}{\log(ab)} = \frac{ab}{(\log(ab))^{2+2\varepsilon}} \tag{2}\]

when \(ab\) is large enough. The latter follows from Prime Number Theorem for short intervals (when \(c = ab/(\log(ab))^{1+\varepsilon}\) is large enough in comparison to \(g(a, b) = ab - a - b\)).

Finally, by combining equations (1) and (2), we obtain

\[\pi_S \geq \pi_S[g(a, b) - c, g(a, b)] \geq \pi(g(a, b)) - \pi(g(a, b) - c) - n_S^c[g(a, b) - c, g(a, b)]\]

\[\gg \frac{ab}{(\log(ab))^{2+2\varepsilon}} - \frac{4ab}{(\log(ab))^{2+2\varepsilon}} > 0\]

where the last inequality holds since \((\log(ab))^\varepsilon > 4\) for \(ab\) large enough for the fixed \(\varepsilon\). The above leads to the desired estimate of \(\pi_S\).
**Case 2.** Suppose that $3 \leq a \leq (\log(ab))^{1+\epsilon}$.

If $p \in [b, \ldots, g(a,b)]$ is a prime and $p \equiv b \pmod{a}$ then $p$ is clearly representable as $p = b + \frac{p-b}{a}$. By Siegel–Walfisz theorem $[2, 7]$, the number of such primes $p$, denoted by $N$, is

$$N = \frac{1}{\varphi(a)} \int_{b}^{g(a,b)} \frac{du}{\log u} + R$$

where $\varphi$ is the Euler totient function and $|R| < D''(\epsilon) \frac{g(a,b)}{(\log(g(a,b)))^{2+2\epsilon}}$ uniformly in $a$ and $g(a,b)$.

Since the function $1/\log u$ is decreasing on the interval $[b, g(a,b)]$ then

$$\int_{b}^{g(a,b)} \frac{du}{\log u} > (g(a,b) - b) \cdot \frac{1}{\log g(a,b)}$$

and therefore

$$N > \frac{1}{\varphi(a)} \cdot \frac{g(a,b) - b}{\log(g(a,b))} - D'(\epsilon) \frac{g(a,b)}{(\log(g(a,b)))^{2+2\epsilon}}.$$  \hfill (3)

Now, we have that

$$\frac{1}{\varphi(a)} \cdot \frac{g(a,b) - b}{\log(g(a,b))} = \frac{1}{\varphi(a)} \log(g(a,b))^{1+\epsilon} \left(1 - \frac{b}{g(a,b)}\right)$$

$$> \frac{1}{\log(ab)^{1+\epsilon}} \log(g(a,b))^{1+\epsilon} \left(1 - \frac{b}{g(a,b)}\right) \left(\text{since}\ (ab)^{1+\epsilon} \geq a > \varphi(a)\right)$$

$$> \left(\frac{\log(ab) - \log(3)}{\log(ab)}\right)^{1+\epsilon} \frac{1}{5} \cdot \frac{1}{F} > 0 \left(\text{since}\ g(a,b) > ab/3\ and\ \frac{b}{g(a,b)} \leq \frac{4}{5}\right)$$

for some absolute $F > 0$, uniformly for $ab \geq D''(\epsilon)$ with $a \geq 3$.

It yields to

$$\frac{1}{\varphi(a)} \cdot \frac{g(a,b) - b}{\log(g(a,b))} \geq F \cdot \frac{g(a,b)}{(\log(g(a,b)))^{2+2\epsilon}} \hfill (4)$$

and combining equations (3) and (4) we obtain

$$N > F \frac{g(a,b)}{(\log(g(a,b)))^{2+2\epsilon}}$$

for $ab$ large enough for the fixed $\epsilon$. The latter leads to the desired estimate of $\pi_S$ also in this case.

\[\Box\]

1. Concluding remarks

A number of computer experiments lead us to the following.

**Conjecture 2.** Let $2 \leq a < b$ be two relatively prime integers and let $S$ be the numerical semigroup generated by $a$ and $b$. Then,

$$\pi_S > 0.$$  

In analogy with the symmetry of $(a,b)$ mentioned above, our task of looking for primes in $(a,b)$ is related with the task of finding primes in $[g(a,b) - 1]/2, \ldots, g(a,b)]$. From this point of view, Conjecture 2 can be thought of as a counterpart of the famous Chebyshev theorem stating that there is always a prime in $[n, \ldots, 2n]$ for any $n \geq 2$, see [4, Chapter 3]. A way to attack Conjecture 2 could be by applying effective versions of Siegel–Walfisz theorem. For instance, one may try to use [6, Corollary 8.31] in order to get computable constants in our estimates. However, it is not an easy task to trace all constants appearing in the relevant estimates of $L(x, \chi)$ (but in principle possible). The remaining cases for small values $ab$ must to be treated by computer.
Conjecture 3. Let $2 \leq a < b$ be two relatively prime integers and let $S$ be the numerical semigroup generated by $a$ and $b$. Then,

$$\pi_S \sim \frac{\pi(g(a,b))}{2} \text{ for } a \to \infty.$$  

In the same spirit as the prime number theorem, this conjecture seems to be out of reach.

The famous Linnik's theorem asserts that there exist absolute constants $C$ and $L$ such that: for given relatively prime integers $a, b$ the least prime $p$ satisfying $p \equiv b \pmod{a}$ is less than $Ca^L$. It is conjectured that one can take $L = 2$, but the current record is only that $L \leq 5$ is allowed, see [8].

On the same flavor of Linnik's theorem that concerns the existence of primes of the form $ax+b$, Theorem 1 is concerning the existence of primes of the form $ax+by$ with $x, y \geq 1$ less than $ab$ for sufficiently large $ab$. This relation could shed light on in either direction.

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