Quasi-particle structures around a pair of half-quantum vortices in chiral $p$-wave super conductors

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Abstract. Quasi-particle structures around a pair of half-quantum vortices (HQVs) and two singly quantized vortices are investigated, using the Bogoliubov-de Gennes equation. For this purpose, a new numerical method, which incorporates the two phase singularity points, is developed, using elliptic coordinates and the (modified) Mathieu functions. We investigate the local density of states around two singly quantized vortices and a pair of HQVs in chiral $p$-wave superconductors. As a result, we found that there is a difference between them: Quasi-particle bound states interfere with each other for two singly quantized vortices case, and do not for a pair of HQVs case.

1. Introduction

Triplet superconductors including Sr$_2$RuO$_4$ are interesting [1,2], because Cooper pairs have a degree of freedom of spin states. Recently a half height magnetization steps were observed in a micrometer-sized annular shaped Sr$_2$RuO$_4$ due to this degree of freedom of spins [3].

H.-Y. Kee et al. suggested that a pair of half-quantum vortices (HQVs) with a $d$-soliton exist in Sr$_2$RuO$_4$, where the pairing symmetry is assumed to be chiral $p$-wave ($p_x \pm ip_y$) [4]. The $d$-soliton is a domain wall between two regions where $d$-vectors are anti-parallel. Since $d$-vectors rotate by $\pi$ at both ends of the $d$-soliton and phase must be rotated by $\pi$, a pair of HQVs exists at both ends of the $d$-soliton. Although pairs of HQVs have not been discovered directly, previously mentioned half height magnetization experiment might be related to HQVs.

Previously, using the finite elements method, Suematsu et al. have solved the Bogoliubov-de Gennes (BdG) equation and obtained the local density of states (LDOS) around singly quantized vortices in nano-structured superconductor [5].

In contrast to the previous method, we focus on two vortices state. So, we have developed a new numerical method to solve the BdG equation for two vortices state in order to analyze quasi-particle
excitations around a pair of HQVs at both ends of the $d$-soliton, using the Mathieu functions and elliptic coordinates \cite{6}. Using this method, we have obtained the LDOS around two singly quantized vortices in $s$-wave superconductors and found that quasi-particle bound states of two vortices interfere with each other \cite{7}. In this study, we investigate the LDOS around a pair of HQVs and two singly quantized vortices in chiral $p$-wave superconductors and compared them.

2. Method

First, we consider an elliptic disk with two foci located at $\left(\pm h_y, 0 \right)$, and put two singly quantized vortices or a pair of HQVs at the foci in $p$-wave superconductors. Also, we assume that wave functions of quasi-particles vanish at the edge of the elliptic disk. In $p$-wave superconductors, we assume that all $d$-vectors do not rotate and are in the same direction for two singly quantized vortices case and that all $d$-vectors exist in $xy$-plane for a pair of HQVs case.

We use a differential form of the BdG equation. For a pair of HQVs case, because the equations for the spin-up and spin-down electrons decouple, the BdG equations for each spin become as,

$$\begin{align*}
-\frac{h^2}{2m} \nabla^2 - \mu &+ \frac{1}{2hk_F} \left[ \frac{1}{2} \left( i \partial_x + \partial_y \right) \Delta^s + \Delta^s \left( i \partial_x + \partial_y \right) \right] u^s = Eu^s, \\
-\frac{h^2}{2m} \nabla^2 - \mu &+ \frac{1}{2hk_F} \left[ \frac{1}{2} \left( -i \partial_x + \partial_y \right) \Delta^s + \Delta^s \left( -i \partial_x + \partial_y \right) \right] v^s = Ev^s,
\end{align*}$$

where $u$ and $v$ are wave functions for electron and hole components of quasi-particles, and $s = \uparrow\uparrow$ or $\downarrow\downarrow$ represents spin of quasi-particles. Here $\mu$ is the chemical potential. $k_F$ is the Fermi momentum. For a pair of HQVs case, we assume the order parameter as

$$\begin{align*}
\Delta^\uparrow\downarrow \left( x, y \right) &= \Delta_0 \exp \left[ i \tan^{-1} \left( \frac{y}{x - h_y} \right) \right] \tanh \frac{\sqrt{(x - h_y)^2 + y^2}}{\xi}, \\
\Delta^\downarrow\uparrow \left( x, y \right) &= \Delta_0 \exp \left[ i \tan^{-1} \left( \frac{y}{x + h_y} \right) \right] \tanh \frac{\sqrt{(x + h_y)^2 + y^2}}{\xi},
\end{align*}$$

where $\xi$ is the coherence length. Equations (3) and (4) express that spin-up and spin-down sectors of quasi-particles have a vortex at the one end of the $d$-soliton and no vortex at the other end of that.

We use elliptic coordinates $(\xi, \eta)$, which are given as

$$\begin{align*}
x &= h_y \cosh \xi \cos \eta, \\
y &= h_y \sinh \xi \sin \eta.
\end{align*}$$

For a pair of HQVs case, the BdG equation in the elliptic coordinates becomes

$$\begin{align*}
\left[ -\frac{h^2}{2m} \left( \cosh 2\xi - \cos 2\eta \right) \left( \partial_{\xi}^2 + \partial_{\eta}^2 \right) - \mu \right] u^s \left( \xi, \eta \right) + \frac{1}{hk_F} \frac{1}{h_y \sinh \left( \xi + i\eta \right)} \left[ \frac{1}{2} \left( i \partial_\xi + \partial_\eta \right) \Delta^s + \Delta^s \left( i \partial_\xi + \partial_\eta \right) \right] v^s \left( \xi, \eta \right) &= Eu^s \left( \xi, \eta \right), \\
\left[ -\frac{h^2}{2m} \left( \cosh 2\xi - \cos 2\eta \right) \left( \partial_{\xi}^2 + \partial_{\eta}^2 \right) - \mu \right] v^s \left( \xi, \eta \right) + \frac{1}{hk_F} \frac{1}{h_y \sinh \left( \xi - i\eta \right)} \left[ \frac{1}{2} \left( -i \partial_\xi + \partial_\eta \right) \Delta^s + \Delta^s \left( -i \partial_\xi + \partial_\eta \right) \right] u^s \left( \xi, \eta \right) &= Ev^s \left( \xi, \eta \right).
\end{align*}$$

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For the kinetic energy Hamiltonian in these equations, eigenfunctions are given by the Mathieu functions and the modified Mathieu functions. The series expansion of wave functions $u$ and $v$ using the Mathieu functions and the modified Mathieu functions are given as [6],

$$u^s(\xi, \eta) = \sum_{m=0, r=1}^{\infty} u^s_{m, r}(\xi, q_{mr}) ce_m(\eta, q_{mr}) + \sum_{m=1, r=1}^{\infty} u^s_{m, r}(\xi, q_{mr}) se_m(\eta, q_{mr}), \quad (9)$$

$$v^s(\xi, \eta) = \sum_{m=0, r=1}^{\infty} v^s_{m, r}(\xi, q_{mr}) ce_m(\eta, q_{mr}) + \sum_{m=1, r=1}^{\infty} v^s_{m, r}(\xi, q_{mr}) se_m(\eta, q_{mr}), \quad (10)$$

where $Ce_m(\xi, q_{mr})$ and $Se_m(\xi, q_{mr})$ are integer modified Mathieu functions, $ce_m(\xi, q_{mr})$ and $se_m(\xi, q_{mr})$ are integer Mathieu functions, and $q_{mr}$ and $\bar{q}_{mr}$ are characteristic numbers, which are determined by the boundary condition, $Ce_m(\xi_0, q_{mr}) = 0$ and $Se_m(\xi_0, \bar{q}_{mr}) = 0$, where $\xi_0$ is the value at the edge of the disk. We substitute these expansions into the BdG equation. We obtain eigenfunctions and eigenvalues, solve these equations and then calculate the LDOS and quasi-particle structures for a pair of HQVs.

3. Results and Discussion

We show numerical results of the LDOS around two singly quantized vortices and a pair of HQVs in chiral $p$-wave superconductors. We set $h_0/\xi_s = 1.0$, $k_F/\xi_s = 3.0$, where $\xi_s = E_F/k_F \Delta_0$ is the coherence length. We set order parameter as $\Delta_0/E_s = 0.2$ at zero temperature, the chemical potential as $\mu/E_c = 0.3$ where $E_c$ is the cut off energy, and temperature as $T/T_c = 0.01$, where $T_c$ is the critical temperature.

In figure 1, we show the quasi-particle bound states around two singly quantized vortices at $E/\Delta_0 = 0.02$ in chiral $p$-wave superconductor. We have found that quasi-particle bound states of two vortices interfere with each other, because the small LDOS peak exists between two vortex cores as that for $s$-wave superconductors case [7]. Suematsu et al. found these peaks in $s$-wave superconductors [5], too.

In figure 2, we show quasi-particle bound states around a pair of HQVs at $E/\Delta_0 = 0.02$ in chiral $p$-wave superconductor. Compared with figure 1, the LDOS peak between HQV cores is small. This can be explained as follows. Spin-up and spin-down electrons around a pair of HQVs exist independently as Ivanov pointed out [8]. Therefore, quasi-particle bound states of a pair of HQVs do not interfere with each other.
4. Conclusion

We have applied a new numerical method to solve the BdG equation for two singly quantized vortices and a pair of HQVs in chiral $p$-wave superconductors. And we have investigated the quasi-particle excitations. As a result, we have found that quasi-particle bound states of two vortices interfere with each other in chiral $p$-wave superconductors as in $s$-wave superconductors, but quasi-particle bound states do not interfere for a pair of HQVs case. How this interference is affected by the distance between vortices is a future problem.

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