A NONLINEAR FRACTIONAL REACTION-DIFFUSION SYSTEM APPLIED TO IMAGE DENOISING AND DECOMPOSITION

ABDELGHAFOUR ATLAS
Ecole Nationale des Sciences Appliquées de Marrakech, Université Cadi Ayyad
B.P. 575 Avenue Abdelkrim Al Khattabi Marrakech, Morocco

MOSTAPA BENDAHMANE
Institut de Mathématiques de Bordeaux and INRIA-Carmen Bordeaux Sud-Ouest
Université de Bordeaux, 33076 Bordeaux Cedex, France

FAHAD KARAMI∗, DRISS MESKINE AND OMR OUBBIH
Ecole Supérieure de Technologie d’Essaouira, Université Cadi Ayyad
B.P. 383 Essaouira El Jadida, Essaouira, Morocco

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Abstract. This paper is devoted to the mathematical and numerical study of a new proposed model based on a fractional diffusion equation coupled with a nonlinear regularization of the Total Variation operator. This model is primarily intended to introduce a weak norm in the fidelity term, where this norm is considered more appropriate for capturing very oscillatory characteristics interpreted as a texture. Furthermore, our proposed model profits from the benefits of a variable exponent used to distinguish the features of the image. By using Faedo-Galerkin method, we prove the well-posedness (existence and uniqueness) of the weak solution for the proposed model. Based on the alternating direction implicit method of Peaceman-Rachford and the approximations of the Grünwald-Letnikov operators, we develop the numerical discretization of our fractional diffusion equation. Experimental results claim that our model provides high-quality results in cartoon-texture-edges decomposition and image denoising. In particular, our model can successfully reduce the staircase phenomenon during the image denoising. Furthermore, small details, texture and fine structures still maintained in the restored image. Finally, we compare our numerical results with the existing models in the literature.

1. Introduction. Over the last decade, image processing has seen several important developments. Denoising and texture extraction are the two most interesting titles that have made a significant contribution to this discipline, which has raised much interest from researchers. Our main goal is to achieve both noise reduction and preservation of important image characteristics such as texture extraction and reduction of the staircase effect. It is important to look for the restored image \( u(i) \) as

\[
f(i) = u(i) + \eta(i),
\]

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\]

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∗ Corresponding author: Fahd Karami.
where \( f(i) \) is the noisy image corrupted by the noise perturbation \( \eta(i) \) at a pixel \( i \) which often considered the stationary Gaussian with zero mean and variance \( \sigma^2 \). By our proposed model, we do not only recover a restored image \( u \), but we also decompose \( u \) into the cartoon and the texture parts, which will be denoted by \( u \) and \( f - u \), respectively. The challenge is to recover an image corrupted by the noise while preserving edges, boundaries and textures. Many current methods are available to treat this problem using a variety of mathematical domains such as partial differential equations [1, 4, 35], variational models [26, 36] and wavelet thresholding [10].

However, nonlocal method have proven to be very powerful in image processing (see for e.g. [9, 41]). In this method, the main idea is the use of the similarities between neighboring or overlapping patches in an image. For typical applications of this new method, we refer the readers to [17, 22, 23, 26, 28, 31].

Nonlinear diffusion is a widely used method in image processing, which has become an active destination for several important applications in the literature [1, 3, 4, 35, 36]. Note that the basic idea of the nonlinear method is to reduce the noise without smoothing out fine structures and significant parts of the image content, precisely edges or other small details that are very important for the representation of the image.

Image decomposition and denoising problem are naturally related to the solution \( u \) of the following minimization

\[
\min_u \int_\Omega |Du|dx + \lambda \|f - u\|_*, \tag{2}
\]

where \( \Omega \subset \mathbb{R}^N \) (\( N \) equal to 2 or 3 in practical situations) is an open bounded domain, \( \lambda > 0 \) is a weight parameter, \( \|f - u\|_* \) is the fidelity term and \( \int_\Omega |Du|dx \) is a regularizing term, stands for the total variation. Observe that in the case when \( u \) is smooth, then \( \int_\Omega |Du|dx = \int_\Omega |\nabla u|dx \). Note that a different choice of the norm \( \|\cdot\|_* \) of fidelity terms corresponds to different models in the literature. The most basic choice for \( \|\cdot\|_* \) is the \( L^2 \)-norm, initiated by the seminal work of Rudin-Osher-Fatemi in [36]. In this case, the problem (2) is formally equivalent to the following nonlinear equation

\[
f - u = \frac{1}{2\lambda} \text{div} \left( \frac{\nabla u}{|\nabla u|} \right). \tag{3}
\]

This model performs very well for image denoising while preserving edges. However, the smaller details, texture, and fine structures are destroyed if the parameter \( \lambda \) is too small. This is why in Meyer [32] the image denoising can be extended as image decomposition for texture extraction. In this problem, the author assumes that a given image \( f \) is the sum of two components \( u \) and \( v \), where \( u \) is a representation of \( f \) and \( v \) is an oscillatory component composed of noise and texture. Meyer observed that if \( f \) is a characteristic function, which is small relative to a certain norm than of \( L^2 \)-norm. Then, the \((BV, L^2)\) model does not provide the expected decomposition of \( f \), because it treated such a function \( v \) as oscillations like noise (see for e.g. Rudin-Osher-Fatemi [36]). To overcome this inaccuracy, Meyer proposed a weak norm in the fidelity term, more suitable for dealing with oscillatory patterns.
composed of noise or texture. It is defined as follows

$$\min_u \int_\Omega |\nabla u| \, dx + \lambda \|f - u\|_X,$$

where the space $X$ is the Banach space consisting the distributions $(f - u)$ with

$$X = (W^{1,1}(\Omega))^* = \{ \text{div} \, \tilde{g} : \tilde{g} \in L^\infty(\Omega) \} := G,$$

or

$$X = \{ \text{div} \, \tilde{g} : \tilde{g} \in \text{BMO}(\Omega) \} := F,$$

or

$$X = \{ \triangle g : g \in \text{Zygmund} \} := E.$$

Meyer’s $(BV, X)$ model handles the oscillatory patterns and the texture very well. However, this convex model cannot be easily computed, due to the form of the $X-$norm of $(f - u)$. Inspired by the proposals of Meyer in [32], a rich literature of models have been proposed and analyzed both theoretically and computationally. Next, Osher-Vese [39] proposed the first practical approach to approximate the Meyer’s $(BV,G)$ model and make it computationally amenable. In [34], Osher et al proposed another approach which consists of considering minimizers

$$\min_u \int_\Omega |\nabla u| \, dx + \lambda \|f - u\|^{2}_{\mathcal{H}^{-1}(\Omega)},$$

where the semi-norm $\| \cdot \|^{2}_{\mathcal{H}^{-1}(\Omega)}$ is defined by: $\| \cdot \|^{2}_{\mathcal{H}^{-1}(\Omega)} = \int_\Omega |\nabla \Delta^{-1}(\cdot)|^2 \, dx$. After, Aujol-Aubert [7, 8] addressed the original Meyer’s problem $(BV,G)$ and proposed an alternate method to minimize

$$\min_u \int_\Omega |\nabla u| \, dx + \lambda \|f - u - v\|_{L^2(\Omega)},$$

s.t. $\|v\|_G \leq \mu$.

Moreover, in [30] Lieu and Vese generalized (8) by proposing a decomposition model of negative Hilbert-Sobolev space, formulated by the following minimization

$$\min_u \int_\Omega |\nabla u| \, dx + \lambda \|u - f\|_{H^{-s}(\Omega)}, \quad \text{for} \quad s \geq 0.$$

Recently, Giga et al [21] proposed to study this previous minimization in terms of the variation flow. The authors presented a consistent approach to the construction of the minimizing total variation sequence in the $H^{-s}(\Omega)$ space with $s \in (0, 1]$. More precisely, the authors considered the following $(2s + 2)-$order parabolic differential equation

$$\frac{\partial u}{\partial t} = (-\Delta)^s \left( \text{div} \left( \frac{\nabla u}{|\nabla u|} \right) \right),$$

with periodic boundary conditions and initial data in $H^{-s}(\Omega)$. Among other models related to the image decomposition, we refer the reader to [19, 20, 27, 29], where the authors are progressively approaching the functional spaces: $(BV,F)$ and $(BV,E)$ proposed by Meyer in [32].
Now, following the ideas of Osher-Solé-Vese [34] and Giga et al. [21], the corresponding Euler-Lagrange equation to the minimization (10) is formally given as follows

$$2\lambda(f - u) = (-\Delta)^s \left( \text{div} \left( \frac{\nabla u}{|\nabla u|} \right) \right).$$

(12)

We notice that to the right side of (12) is not well defined at the locations where $|\nabla u| = 0$. It is well known that the TV operator presents a mathematical difficulty related to defining a notion of solution. Some appropriate sophisticated definitions can be founding in [5]. Moreover, numerical simulations induce drawback phenomena known as the staircase effect. To avoid this, we can introduce some perturbation $\epsilon$ in dominator and replace $|\nabla u|$ by $\sqrt{|\nabla u|^2 + \epsilon}$. In [3], Afraites et al. propose a nonlinear regularization of a TV operator that is considered more appropriate to reduce the staircase effect. Our Figure 1 illustrates the comparison between these two relaxations. We observe that the relaxation proposed by [3] is more appropriate to converge on the TV operator when the gradient is large. In this paper, we propose a new nonlinear regularization of the TV operator. It profits from the benefits of a variable exponent depending on the space variable, thus leading to a generalization of the proposed one in [3]. Based on the variable exponent proposed, we can adaptively command the diffusion mode in accordance with the image features and sufficiently preserve small details. Because of the controllability of the diffusion mode, our new nonlinear regularization of the TV will not create new features and will avoid the staircase effect. Namely, at the edges where the gradient is large, the proposed variable exponent quickly approaches to 0 and our nonlinear regularization proposed of the TV behaves like the standard TV operator and guarantees rapid convergence. In this case, we propose an Orlicz-operator as a
nonlinear regularization term and we generalize the equation (12). Thus, we arrive at the following energy

$$2\lambda (f - u) = (-\Delta)^s \left( \text{div} \left( \frac{A_{\mu(x)}(x, |\nabla u|)}{|\nabla u|} \nabla u \right) \right).$$

(13)

Here, the function $A_{\mu(\cdot)} : \Omega \times \mathbb{R} \to \mathbb{R}$ is chosen to monotonically close to 1 when $t$ tends to infinity (for more details consult (17)-(20)). Note that when $A_{\mu(\cdot)} \equiv 1$ we get the equation in (12). The proposed operator in (13) is designed to be more convex in regions of the moderate gradient (e.g., away from edges) and behave like the standard TV operator near discontinuities thus preserve the edges and suppress the staircase effect. Moreover, we note that the above equations are of $(2s+2)$-order which require more mathematical regularity. Solving (13) leads to a highly nonlinear PDE and, therefore its numerical solution is a non-trivial task. To overcome this difficulty, we follow the decomposition suggested in [3, 25], introducing a splitting into the coupled system

\begin{align*}
\begin{cases}
- \text{div} \left( \frac{A_{\mu(x)}(x, |\nabla u|)}{|\nabla u|} \nabla u \right) + 2\lambda w &= 0, \\
(-\Delta)^s w + (f - u) &= 0.
\end{cases}
\end{align*}

(14)

Moreover, the solution $(u, w)$ of system (14) is a stable state of the following evolutionary fractional reaction-diffusion system

\begin{align*}
\begin{cases}
\frac{\partial u}{\partial t} - \text{div} \left( \frac{A_{\mu(x)}(x, |\nabla u|)}{|\nabla u|} \nabla u \right) + 2\lambda w &= 0, & \text{in } Q := \Omega \times (0, T), \\
\frac{\partial w}{\partial t} + (-\Delta)^s w + (f - u) &= 0, & \text{in } Q := \Omega \times (0, T), \\
A_{\mu(x)}(x, |\nabla u|) \frac{\nabla u}{|\nabla u|} \cdot \vec{n} &= 0, & \text{in } \Sigma := \partial \Omega \times (0, T), \\
K_s w &= 0, & \text{in } \Sigma_s := \mathbb{R}^N / \Omega \times (0, T), \\
w(0, x) = f, & w(0, x) = 0, & \text{in } \Omega,
\end{cases}
\end{align*}

(15)

where the operator $K_s w$ denotes the nonlocal normal derivative or the fractional Neumann boundary condition formally defined, for smooth functions, by

$$K_s w(x) := \frac{C_{N,s}}{2} \int_{\Omega} \frac{w(x) - w(y)}{|x - y|^{N+2s}} \, dy, \quad x \in \mathbb{R}^N / \Omega.$$  

(16)

Indeed, we refer to [14] for several comments, justifications, and reasons to consider such operators, and for this reason, we shall skip these motivations. Furthermore, $A_{\mu(\cdot)}$ is a function defined by

$$A_{\mu(x)}(x, t) = \begin{cases}
G(x, t), & \text{if } t \leq \delta, \\
\frac{t}{\delta \log^\mu(x)(1 + t)} G(x, \delta), & \text{if } t > \delta,
\end{cases} \quad \text{for } x \in \Omega.$$  

(17)

The parameters $\delta, \lambda$ are nonnegative and $\mu : \Omega \to [0, \infty)$ is a continuous function in $\Omega$, such that

$$0 < \min_{x \in \Omega} \mu(x) = \mu_0 \leq \mu(x) \leq \mu_1 = \max_{x \in \Omega} \mu(x).$$  

(18)
Next, \( t \to G(\cdot, t) \in [0, +\infty] \) is an increasing convex function for all \( t \geq 0 \) and satisfying
\[
0 < G_0 \leq G(x, \delta) \in L^\infty(\Omega) \quad \text{and} \quad \lim_{t \to 0} \frac{G(x, t)}{t} = 0 \quad \text{for a.e.} \ x \in \Omega. \quad (19)
\]
A typical example of such function \( G \) is \( G(x, t) = t^2 \exp(t) \) for a.e. \( x \in \Omega \). We observe that if \( \mu(\cdot) \) is close to 0, we get
\[
\log \frac{\mu(u)}{|\nabla u|} (1 + |\nabla u|) |\nabla u| \sim |\nabla u| \text{ in } |\nabla u| > \delta, \quad (20)
\]
and when \( |\nabla u| \leq \delta \), \( \frac{A_{\mu(\cdot)}(x, |\nabla u|)}{|\nabla u|} \nabla u \) behaves like \( \exp(|\nabla u|) \nabla u \). Now we define the fractional Laplacian (see [14] for more details)
\[
(-\Delta)^s u(x) := \frac{C_{N,s}}{2} \text{P.V.} \int_{\mathbb{R}^N} \left( \frac{u(x) - u(y)}{|x - y|^{N+2s}} \right) dy, \quad \text{for } x \in \Omega, \ s \in (0, 1), \quad (21)
\]
where \( C_{N,s} \) is a positive normalizing constant, which is given by
\[
C_{N,s} := 2^{2s} \pi^{-\frac{N}{2}} \frac{\Gamma(\frac{N+2s}{2})}{\Gamma(-s)}. \quad (22)
\]
Herein, \( \Gamma \) denotes the Gamma function. We underline that the definition (21) remains valid for functions defined in some open domain \( \Omega \subset \mathbb{R}^N \). In this case, we denote by
\[
\left( (-\Delta)^s u, v \right) := \frac{C_{N,s}}{2} \int_{\Omega} \int_{\Omega} \frac{(u(x) - u(y))(v(x) - v(y))}{|x - y|^{N+2s}} dy dx, \quad (23)
\]
the duality pairing corresponding to the fractional Laplacian. From the point of view of numerical analysis, the numerical approximation of the fractional derivatives presents fundamental difficulties because some good properties of the classical approximation operators are lost. Inspired by the papers [13, 33], our fractional-order equation in our system (15) is discretized by using of the ADI-method of Peaceman-Rachford. This method is based on the Crank-Nicholson scheme and the approach Grünwald-Letnikov operators difference.

The paper is organized as follows: in Section 2, we set out some basic definitions of the inhomogeneous Orlicz-Sobolev and the fractional-Sobolev spaces, as well as our main result of this work. We show in Section 3 the well-posedness (existence and uniqueness) of the proposed weak solutions to our system (15). Finally, Section 4 is devoted to the numerical experiments using the proposed numerical scheme of our system.

2. Existence of weak solution.

2.1. Preliminaries. Let \( \Omega \) be a bounded open subset of \( \mathbb{R}^N \), \( T > 0 \) and set \( Q = \Omega \times ]0, T[. \) Let \( \mathcal{N} : \mathbb{R}^+ \to \mathbb{R}^+ \) be an N-function, i.e. \( \mathcal{N} \) is continuous, convex, with \( \mathcal{N}(t) > 0 \) for \( t > 0 \), \( \frac{\mathcal{N}(t)}{t} \to 0 \) as \( t \to 0 \) and \( \frac{\mathcal{N}(t)}{t} \to \infty \) as \( t \to \infty \). Equivalently, \( \mathcal{N} \) admits the representation: \( \mathcal{N}(t) = \int_0^t a(\tau) \ d\tau \) where the function \( a : \mathbb{R}^+ \to \mathbb{R}^+ \) is non-decreasing, right continuous, with \( a(0) = 0 \), \( a(t) > 0 \) for \( t > 0 \) and \( a(t) \to \infty \) as \( t \to \infty \). The N-function \( \mathcal{N} \) conjugate to \( \mathcal{N} \) is defined by \( \mathcal{N}(t) = \int_0^t \pi(\tau) \ d\tau \), where
the function $\tilde{a}: \mathbb{R}^+ \to \mathbb{R}^+$ is given by $\tilde{a}(t) = \sup\{s:a(s) \leq t\}$ (see [2, 16] for more details). The N-function $\mathcal{N}$ is said to satisfy the $\Delta_2$ condition if, for some $k > 0$:
\begin{equation}
\mathcal{N}(2t) \leq k \mathcal{N}(t) \quad \text{for all } t \geq 0.
\end{equation}
Note that, when this inequality holds only for $t \geq t_0 > 0$, $\mathcal{N}$ is said to satisfy the $\Delta_2$ condition near infinity. Next, we define the Orlicz class $E_\mathcal{N}(\Omega)$ by
\begin{equation}
E_\mathcal{N}(\Omega) = \left\{ u \in \Omega, \int_\Omega \mathcal{N}\left(\frac{|u(x)|}{\lambda}\right) dx < \infty \right\}.
\end{equation}
The Orlicz space (also called the generalized Orlicz spaces) $L_\mathcal{N}(\Omega)$ is the vector space generalized by $E_\mathcal{N}(\Omega)$, that is, $L_\mathcal{N}(\Omega)$ is the smallest linear space containing the set $E_\mathcal{N}(\Omega)$, equivalently
\begin{equation}
L_\mathcal{N}(\Omega) = \left\{ u : \text{there exists } \lambda > 0 \text{ such that}, \int_\Omega \mathcal{N}\left(\frac{|u(x)|}{\lambda}\right) dx < \infty \right\},
\end{equation}
which is equipped with the (Luxemburg) norm
\begin{equation}
\|u\|_{L_\mathcal{N}(\Omega)} = \inf\left\{ \lambda > 0, \int_\Omega \mathcal{N}\left(\frac{|u(x)|}{\lambda}\right) dx \leq 1 \right\}.
\end{equation}
Note that, if the set is clear from the context we abbreviate: $\|u\|_{L_\mathcal{N}(\Omega)}$ by $\|u\|_{\mathcal{N},\Omega}$.

The space $L_\mathcal{N}(\Omega)$ is reflexive if and only if $\mathcal{N}$ and $\overline{\mathcal{N}}$ satisfy the $\Delta_2$ condition (near infinity only if $\Omega$ has finite measure).

We now turn to the Orlicz-Sobolev spaces. $W^1L_\mathcal{N}(\Omega)$ (resp., $W^1E_\mathcal{N}(\Omega)$) is the space of all functions $u$ such that $u$ and its distributional derivatives up to order 1 lie $L_\mathcal{N}(\Omega)$ (resp., $E_\mathcal{N}(\Omega)$). This is a Banach space under the norm
\begin{equation}
\|u\|_{1,\mathcal{N},\Omega} = \sum_{|\alpha| \leq 1} \|D^\alpha u\|_{\mathcal{N},\Omega},
\end{equation}
where $\alpha = (\alpha_1, \alpha_2, \cdots, \alpha_n)$, $\alpha_j \geq 0$, $j = 1, \cdots, n$ with $|\alpha| = \alpha_1 + \alpha_2 + \cdots + \alpha_n$ and $D^\alpha u$ denote the distributional derivatives of multi-index $\alpha$.

Thus $W^1L_\mathcal{N}(\Omega)$ and $W^1E_\mathcal{N}(\Omega)$ can be identified with subspaces of the product of $(N + 1)$ copies of $L_\mathcal{N}(\Omega)$. Denoting this product by $\prod L_\mathcal{N}$, we will use the weak topologies $\sigma\left(\prod L_\mathcal{N}, \prod E_\mathcal{N}\right)$ and $\sigma\left(\prod L_\mathcal{N}, \prod \overline{L_\mathcal{N}}\right)$. The space $W^1_0E_\mathcal{N}(\Omega)$ is defined as the (norm) closure of the Schwartz space $\mathcal{D}(\Omega)$ in $W^1E_\mathcal{N}(\Omega)$ and the space $W^{1,\mathcal{N}}(\Omega)$ as the $\sigma\left(\prod L_\mathcal{N}, \prod E_\mathcal{N}\right)$ closure of $\mathcal{D}(\Omega)$ in $W^1L_\mathcal{N}(\Omega)$.

We say that $u_n$ converges to $u$ for the modular convergence in $W^1L_\mathcal{N}(\Omega)$ if for some $\lambda > 0$, $\int_\Omega \mathcal{N}\left(|D^\alpha u_n - D^\alpha u|/\lambda\right) dx \to 0$ for all $|\alpha| \leq 1$. This implies that $(u_n)$ converges to $u$ in $W^1L_\mathcal{N}(\Omega)$ for the weak topology $\sigma\left(\prod L_\mathcal{N}, \prod \overline{L_\mathcal{N}}\right)$. Note that, if $u_n \to u$ in $L_\mathcal{N}$ for the modular convergence and $v_n \to v$ in $L_\mathcal{N}$ for the modular convergence, we have
\begin{equation}
\int_\Omega u_n v_n dx \to \int_\Omega uv dx \quad \text{as } n \to \infty.
\end{equation}
Indeed, let $\lambda > 0$ and $\mu > 0$ such that
\begin{equation}
\int_\Omega \mathcal{N}\left(\frac{|u_n - u|}{\lambda}\right) dx \to 0 \quad \text{and} \quad \int_\Omega \overline{\mathcal{N}}\left(\frac{|v_n - v|}{\mu}\right) dx \to 0,
\end{equation}
respectively, and then
and, since \( u_n v_n - u v = (u_n - u)(v_n - v) + u_n v + uv_n - 2uv \) and by using Young's inequality we arrive to

\[
\frac{1}{\lambda \mu} \left| \int_\Omega \left( u_n v_n - u v \right) \, dx \right| \leq \int_\Omega \mathcal{N} \left( \frac{|u_n - u|}{\lambda} \right) \, dx + \int_\Omega \mathcal{N} \left( \frac{|v_n - v|}{\mu} \right) \, dx
\]

\[
+ \frac{1}{\lambda \mu} \left| \int_\Omega \left( u_n v + uv_n - 2uv \right) \, dx \right|
\]

therefore, by letting \( n \to \infty \) in the last side, we get the results.

Let \( W^{-1}L_\mathcal{N}(\Omega) \) (resp., \( W^{-1}E_\mathcal{N}(\Omega) \)) denote the space of distributions on \( \Omega \) which can be written as sums of derivatives of order \( \leq 1 \) of functions in \( L_\mathcal{N} \) (resp., \( E_\mathcal{N} \)). It is a Banach space under the usual quotient norm.

For each \( \alpha \in \mathbb{N}^N \), let denote by \( D_\alpha^u \) the distributional derivatives on \( Q \) of order 1 with respect to the variable \( x \in \mathbb{R}^N \). The inhomogeneous Orlicz-Sobolev spaces are defined as follows

\[
W^{1,\sigma}L_\mathcal{N}(Q) = \left\{ u \in L_\mathcal{N}(Q) : D_\alpha^u \in L_\mathcal{N}(Q), \forall |\alpha| \leq 1 \right\},
\]

\[
W^{1,\sigma}E_\mathcal{N}(Q) = \left\{ u \in E_\mathcal{N}(Q) : D_\alpha^u \in E_\mathcal{N}(Q), \forall |\alpha| \leq 1 \right\}.
\]

The last space is a subspace of the first one, and both are Banach spaces under the norm

\[
\| u \|_{1,\sigma,\mathcal{N},\Omega} = \sum_{|\alpha| \leq 1} \| D_\alpha^u \|_{\mathcal{N},\Omega}.
\]

We can show that they form a complementary system when \( \Omega \) satisfies the segment property (we refer to [38] Definition 4.1.15 for more details and the enforce therein). The spaces are considered as subspaces of the product space \( \prod L_\mathcal{N}(Q) \) which has \( (N + 1) \) copies. We shall also considered the weak topologies \( \sigma \left( \prod L_\mathcal{N}, \prod E_\mathcal{N} \right) \) and \( \mathcal{N} \left( \prod L_\mathcal{N}, \prod E_\mathcal{N} \right) \). The space \( W^{1,\sigma}L_\mathcal{N}(Q) \) is defined as the (norm) closure in \( W^{1,\sigma}L_\mathcal{N}(Q) \) of \( \mathcal{D}(Q) \). As follows in [24], we can see that when \( \Omega \) has the segment property then each element \( u \) of the closure of \( \mathcal{D}(Q) \) with respect of the weak * topology \( \sigma \left( \prod L_\mathcal{N}, \prod E_\mathcal{N} \right) \) is limit, in \( W^{1,\sigma}L_\mathcal{N}(Q) \), of some subsequence \( (u_n) \in \mathcal{D}(Q) \) the modular convergence; i.e., the exists \( \lambda > 0 \) such that for all \( |\alpha| \leq 1 \), we have

\[
\int_Q \mathcal{N} \left( \frac{|D_\alpha^u u_n - D_\alpha^u u|}{\lambda} \right) \, dt \to 0 \quad \text{as} \quad n \to \infty.
\]

This implies that \( (u_n) \) converges to \( u \) in \( W^{1,\sigma}L_\mathcal{N}(Q) \) for the weak topology \( \sigma \left( \prod L_\mathcal{N}, \prod E_\mathcal{N} \right) \). Consequently

\[
\mathcal{D}(Q)^{\sigma(\prod L_\mathcal{N}, \prod E_\mathcal{N})} = \mathcal{D}(Q)^{\sigma(\prod L_\mathcal{N}, \prod E_\mathcal{N})}.
\]

We will denote this space by \( W^{1,\sigma}_0L_\mathcal{N}(Q) \). Furthermore,

\[
W^{1,\sigma}_0E_\mathcal{N}(Q) = W^{1,\sigma}_0L_\mathcal{N}(Q) \cap \prod E_\mathcal{N}.
\]

Now, we make a reminder of fractional Sobolev Spaces \( W^{s,p}(\mathbb{R}^N) \) [12], also called as Aronszajn, Glagliardo or Slobodeckij spaces introduced in [6, 18, 37]. For any
We denote by $W^{s,p}(\Omega)$, the fractional Sobolev space which is an intermediary Banach space between $L^p(\Omega)$ and $W^{1,p}(\Omega)$, endowed with the norm $\|u\|_{W^{s,p}(\Omega)} := \|u\|_{L^p(\Omega)} + \|u\|_{L^p(\Omega)}$. Functions in the space $W^{s,p}(\Omega)$ can be defined in the whole space $\Omega$. The main result in this work is the following theorem.

\begin{align}
L_{\mu(t)} := t \log^{c_{t}}(1 + t).
\end{align}

The main result in this work is the following theorem.

**Theorem 2.1.** Assume that $s \in (0, 1)$, $u_0 \in L^2(\Omega)$ and $f \in L^2(\Omega)$, then for every $\lambda > 0$, there exists an unique solution

\begin{align}
(u, w) \in W^{1,s} L_\mathcal{M}(Q) \times L^2(0, T; W^{s,2}(\Omega)) \cap \left[ L^\infty (0, T; L^2(\Omega)) \right]^2,
\end{align}

satisfying

\begin{align}
-\left\langle \frac{\partial \phi}{\partial t}, u \right\rangle_Q + \int_0^T \int_\Omega L_{\mu(t)}(x, \nabla u) \nabla \phi dxdt + \int_{\Omega} w\phi dx \bigg|_0^T \\
+ 2\lambda \int_0^T \int_\Omega w\phi dxdt = 0,
\end{align}

\begin{align}
-\left\langle \frac{\partial \varphi}{\partial t}, w \right\rangle_Q + \frac{C_{N,s}}{2} \int_0^T \int_\Omega \int_\Omega \frac{(w(x, t) - w(y, t))(\varphi(x, t) - \varphi(y, t))}{|x - y|^{N+2s}} dx dy dt \\
+ \int_{\Omega} w\varphi dx \bigg|_0^T + \int_0^T \int_\Omega (f - u)\varphi dxdt = 0,
\end{align}

for every $\tau \in (0, T)$ and every test-functions

\begin{align}
\phi \in W^{1,s} L_\mathcal{M}(Q) \cap L^2(Q), \quad \frac{\partial \phi}{\partial t} \in W^{-1,s} L_{\text{loc}}(Q) + L^2(Q),
\end{align}

\begin{align}
\varphi \in L^2(0, T; W^{s,2}(\Omega)), \quad \frac{\partial \varphi}{\partial t} \in L^2(0, T; (W^{s,2}(\Omega))^\prime).
\end{align}
Note that our approximate solution satisfies the following weak formulation

\[
\begin{align*}
\text{Proof.} & \quad \text{By Faedo-Galerkin nonlinear method, we prove our main result. The primary idea is to split the proof into the following four parts} \\
3. & \quad \text{Well-posedness of weak solution.} \quad \text{We choose a sequence } \{v_1, v_2, \ldots\} \text{ in } \mathbb{D}(\Omega) \text{ and } \Xi_m = \text{Span}\{v_1, v_2, \ldots, v_m\} \text{ such that } \cup_{m=1}^{\infty} \Xi_m \text{ is dense in } W^{s,2}(\Omega) \text{ (for } s \text{ large enough) and } W^{s,2}(\Omega) \text{ is continuously embedded in } C^1(\Omega). \text{ Next, we define the following Faedo-Galerkin solutions to (15)} \\
& \quad u^m(x, t) = \sum_{k=1}^{m} e_k^m(t) \vartheta_k(x), \quad w^m(x, t) = \sum_{k=1}^{m} h_k^m(t) \vartheta_k(x), \quad (45)
\end{align*}
\]

where \(e_k^m, h_k^m : [0, T) \to \mathbb{R}\) are supposed to be measurable bounded functions. For \(k \in \{1, 2, \ldots, m\}\), the coefficients \(e_k^m\) and \(h_k^m\) are obtained from the following system of ordinary differential equations

\[
\begin{align*}
&\int_{\Omega} u^m \vartheta_k dx + \int_{\Omega} L_{\mu}(x, \nabla u^m) \nabla \vartheta_k dx + 2\lambda \int_{\Omega} w^m \vartheta_k dx = 0, \quad (46) \\
&\int_{\Omega} w^m \vartheta_k dx + \frac{C_{N,s}}{2} \int_{\Omega} \int_{\Omega} \frac{(w^m(x, t) - u^m(y, t))(\vartheta_k(x) - \vartheta_k(y))}{|x-y|^{N+2s}} dxdy \\
&\quad + \int_{\Omega} (f - u^m) \vartheta_k dx = 0, \quad (47)
\end{align*}
\]

with \(u^m(\cdot, 0) = \varphi_0 \in \cup_{m=1}^{\infty} \Xi_m, \; w^m \to \varphi_0 \text{ strongly in } L^2(\Omega), \text{ as } m \to \infty \) and \(w^m(\cdot, 0) = 0\). Now, using the same steps given in [3] and applying the standard theory of ordinary differential equations in [11], we obtain easily the existence of a local solution to equations (46)-(47) over interval \([0, T)\) independent of \(m\). To obtain the limit solutions (as \(m \to \infty\)) of (46)-(47) and proving the existence of \(u\) and \(w\), we need the following a priori estimates lemma.

3.2. A priori estimates. We have the following lemma.

**Lemma 3.1.** Let \((u^m, w^m)\) be a solution of the problem (46)-(47). Then, there exists a constant \(C > 0\) independent of \(m\) such that

\[
\begin{align*}
&\|u^m\|_{L^\infty(0, T; L^2(\Omega))}^2 + \frac{G_0}{\delta \log^{1/(1+\delta)}} \int_0^T \int_{\Omega} |\nabla u^m| \log^{1/(1+\delta)} (1 + |\nabla u^m|) dx dt \leq C, \quad (48) \\
&\|w^m\|_{L^\infty(0, T; L^2(\Omega))}^2 + \lambda C_{N,s} \int_0^T \int_{\Omega} \int_{\Omega} \frac{|u^m(x, t) - u^m(y, t)|^2}{|x-y|^{N+2s}} dxdy dt \leq C.
\end{align*}
\]

**Proof.** Note that our approximate solution satisfies the following weak formulation

\[
\begin{align*}
&\int_{\Omega} u^m \varphi^m dx + \int_{\Omega} L_{\mu}(x, \nabla u^m) \nabla \varphi^m dx + 2\lambda \int_{\Omega} w^m \varphi^m dx = 0, \quad (49)
\end{align*}
\]
Using the initial condition $u_0(x, y) = u(x, y)$ for all $\varphi \in W^{1, \infty}(Q)$ and $\varphi_w \in L^2\left(0, T; W^{s, 2}(\Omega)\right)$. Now, let $\tau \in [0, T]$, taking $u^{m}(x, \varphi_w(x))$ as a test function in (49) (respectively in (50)), we obtain

$$
\frac{1}{2} \int_{\Omega} u^{m}(\tau) dx + \frac{C_N}{2} \int_{\Omega} \int_{\Omega} \frac{|u^{m}(x, t) - u^{m}(y, t)|^2}{|x - y|^{N + 2s}} dxdy
$$

$$
+ \int_{\Omega} (f - u^{m}) \varphi_w dx = 0,
$$

(50)

for all $\varphi_w \in W^{1, \infty}(Q)$ and $\varphi_w \in L^2\left(0, T; W^{s, 2}(\Omega)\right)$. Now, let $\tau \in [0, T]$, taking $u^{m}(x, \varphi_w(x))$ as a test function in (49) (respectively in (50)), we obtain

$$
\frac{1}{2} \int_{\Omega} u^{m}(\tau) dx + \frac{C_N}{2} \int_{\Omega} \int_{\Omega} \frac{|u^{m}(x, t) - u^{m}(y, t)|^2}{|x - y|^{N + 2s}} dxdydt
$$

$$
+ \int_{\Omega} (f - u^{m}) u^{m} dx dt = \frac{1}{2} \int_{\Omega} u^{m}(x, 0)^2 dx.
$$

(52)

Multiplying (52) by $2\lambda$ and adding the previous equality, we get

$$
\frac{1}{2} \int_{\Omega} u^{m}(\tau) dx + \lambda \int_{\Omega} u^{m}(\tau) dx + \int_{0}^{\tau} \int_{\Omega} \mathcal{L}_{\mu}(x, \nabla u^{m}) \nabla u^{m} dx dt
$$

$$
+ \lambda C_N \int_{0}^{\tau} \int_{\Omega} \frac{|u^{m}(x, t) - u^{m}(y, t)|^2}{|x - y|^{N + 2s}} dxdydt + 2\lambda \int_{0}^{\tau} \int_{\Omega} f u^{m} dx dt
$$

$$
= \frac{1}{2} \int_{\Omega} u^{m}(x, 0)^2 dx + \lambda \int_{\Omega} u^{m}(x, 0)^2.
$$

(53)

Using the initial condition $u^{m}(x, 0) = u_0^{m}$ and $u^{m}(x, 0) = 0$, then the equation (53) becomes

$$
\frac{1}{2} \int_{\Omega} u^{m}(\tau) dx + \lambda \int_{\Omega} u^{m}(\tau) dx + \int_{0}^{\tau} \int_{\Omega} \mathcal{L}_{\mu}(x, \nabla u^{m}) \nabla u^{m} dx dt
$$

$$
+ \lambda C_N \int_{0}^{\tau} \int_{\Omega} \frac{|u^{m}(x, t) - u^{m}(y, t)|^2}{|x - y|^{N + 2s}} dxdydt + 2\lambda \int_{0}^{\tau} \int_{\Omega} f u^{m} dx dt
$$

$$
= \frac{1}{2} \int_{\Omega} (u_0^{m})^2 dx.
$$

(54)

On the one hand, applying Young’s inequality, we arrive to

$$
\left| \int_{0}^{\tau} \int_{\Omega} f u^{m} dx dt \right| \leq \frac{1}{2} \int_{0}^{\tau} \int_{\Omega} f^2 dx dt + \frac{1}{2} \int_{0}^{\tau} \int_{\Omega} (u^{m})^2 dx dt,
$$

$$
\leq \frac{T}{2} \int_{\Omega} f^2 dx + \frac{1}{2} \int_{\Omega} (u^{m})^2 dx dt.
$$

(55)
On the other hand, thanks to the definition of $\mathcal{L}_{m(t)}$ (consult (39) and (17)–(20) for more details) we have
\[
\int_0^\tau \int_\Omega |\mathcal{L}_{m(t)}(x, \nabla u^m) \nabla u^m| \, dx \, dt = \int_0^\tau \int_{|\nabla u^m| \leq \delta} G(x, |\nabla u^m|) \, dx \, dt
\]
\[
+ \int_0^\tau \int_{|\nabla u^m| > \delta} \frac{G(x, \delta)}{\delta \log^{\mu^m(x)}(1 + \delta)} |\nabla u^m| \log^{\mu^m(x)}(1 + |\nabla u^m|) \, dx \, dt,
\]
\[
\geq \int_0^\tau \int_{|\nabla u^m| \leq \delta} \frac{G(x, \delta)}{\delta \log^{\mu^m(x)}(1 + \delta)} |\nabla u^m| \log^{\mu^m(x)}(1 + |\nabla u^m|) \, dx \, dt
\]
\[
+ \int_0^\tau \int_{|\nabla u^m| > \delta} \frac{G(x, \delta)}{\delta \log^{\mu^m(x)}(1 + \delta)} |\nabla u^m| \log^{\mu^m(x)}(1 + |\nabla u^m|) \, dx \, dt.
\]
Next, we use (55) and (56), to deduce from (54)
\[
\frac{1}{2} \int_\Omega u^m(\tau)^2 \, dx + \lambda \int_\Omega w^m(\tau)^2 \, dx
\]
\[
+ \lambda C_{N,s} \int_0^\tau \int_\Omega \int_\Omega \frac{|w^m(x, t) - w^m(y, t)|^2}{|x - y|^{N+2s}} \, dx \, dy \, dt
\]
\[
+ \frac{G_0}{\delta \log^{\mu^m(x)}(1 + \delta)} \int_0^\tau \int_\Omega |\nabla u^m| \log^{\mu^m(x)}(1 + |\nabla u^m|) \, dx \, dt,
\]
\[
\leq T \int_\Omega G(x, \delta) \, dx + \frac{1}{2} \int_\Omega (u_0^m)^2 \, dx + \lambda T \int_\Omega f^2 \, dx + \lambda \int_0^\tau \Theta_m(t) \, dt.
\]
Now, setting $\Theta_m(\tau) = \int_\Omega u^m(\tau)^2 \, dx$, we deduce from (57)
\[
0 \leq \Theta_m(\tau) \leq T \int_\Omega G(x, \delta) \, dx + \frac{1}{2} \int_\Omega (u_0^m)^2 \, dx + \lambda T \int_\Omega f^2 \, dx + \lambda \int_0^\tau \Theta_m(t) \, dt.
\]
Using Gronwall inequality, we get
\[
0 \leq \Theta_m(\tau) \leq \left[ T \int_\Omega G(x, \delta) \, dx + \frac{1}{2} \int_\Omega (u_0^m)^2 \, dx + \lambda T \int_\Omega f^2 \, dx + \right] \exp(\lambda \tau), \quad \forall \tau \in [0, T].
\]
Since $u_0^m \to u_0$ strongly in $L^2(\Omega)$, as $m \to \infty$, this implies that
\[
\int_\Omega u^m(\tau)^2 \, dx \leq C,
\]
for some constant $C > 0$ depending only on $T$ and $\lambda$. Consequently, we obtain
\[
\max_{0 \leq \tau < T} \int_\Omega u^m(\tau)^2 \, dx \leq C,
\]
and
\[
\int_0^T \int_\Omega \frac{|w^m(x, t) - w^m(y, t)|^2}{|x - y|^{N+2s}} \, dx \, dy \, dt \leq C. \tag{62}
\]

Then, we deduce that \(w^m\) is bounded in \(L^\infty(0,T;L^2(\Omega)) \cap L^2(0,T;W^{s,2}(\Omega))\).

Finally, we use again (57) to obtain the assertion of Lemma 3.1. This completes the proof of Lemma 3.1. \(\square\)

Note that from Lemma 3.1, we have
\[
\int_0^T \int_{||\nabla u^m|>e^{-1}} |\nabla u^m| \log^\nu(1 + |\nabla u^m|) \, dx \, dt
\]
\[
\leq \int_0^T \int_{||\nabla u^m|>e^{-1}} |\nabla u^m| \log^\nu(x)(1 + |\nabla u^m|) \, dx \, dt, \tag{63}
\]
and
\[
\int_0^T \int_\Omega |\nabla u^m| \log^\nu(1 + |\nabla u^m|) \, dx \, dt
\]
\[
\leq \int_0^T \int_{||\nabla u^m|>e^{-1}} |\nabla u^m| \log^\nu(1 + |\nabla u^m|) \, dx \, dt
\]
\[
+ \int_0^T \int_{||\nabla u^m|<e^{-1}} |\nabla u^m| \log^\nu(1 + |\nabla u^m|) \, dx \, dt,
\]
\[
\leq (e - 1) \operatorname{meas}(Q) + C,
\]
for some constant \(C > 0\). Furthermore, we observe that
\[
\int_0^T \int_\Omega |\mathcal{L}_{\mu(x)}(x, \nabla u^m)| \, dx \, dt = \int_0^T \int_\Omega |\mathcal{A}_{\mu(x)}(x, \nabla u^m)| \, dx \, dt
\]
\[
\leq \frac{G_0}{\delta \log^{\nu_1}(1 + \delta)} \int_0^T \int_\Omega \log^{\nu}(1 + |\nabla u^m|) \chi_{[\delta<|\nabla u^m|\leq e^{-1}]} \, dx \, dt
\]
\[
+ \int_0^T \int_\Omega \frac{G_0}{\delta \log^{\nu_1}(1 + \delta)} \log^{\nu}(1 + |\nabla u^m|) \chi_{[|\nabla u^m|>e^{-1}]} \, dx \, dt
\]
\[
+ \int_0^T \int_\Omega G(x, \delta) \chi_{[|\nabla u^m|\leq \delta]} \, dx \, dt,
\]
\[
\leq C \int_0^T \int_\Omega (\log^{\nu_1}(1 + |\nabla u^m|) + 1) \, dx \, dt,
\]
where \(C = \max \left\{ \frac{G_0}{\delta \log^{\nu_1}(1 + \delta)}, G_0 \right\} > 0\). Thus, we get
\[
\int_0^\tau \int_\Omega \exp(\|L_{\mu(x)}(x, \nabla u^m)\|^{\frac{1}{\mu}}) \, dx \, dt \leq \int_0^\tau \int_\Omega \exp((\log \mu_1(1 + |\nabla u^m|) + 1)^{\frac{1}{\mu}}) \, dx \, dt,
\]
\[
\leq \int_0^\tau \int_\Omega \exp((2C)^{\frac{1}{\mu}}(\log(1 + |\nabla u^m|) + 1)) \, dx \, dt,
\]
\[
\leq C \int_0^\tau \int_\Omega (1 + |\nabla u^m|) \, dx \, dt \leq C,
\]
for some constant \( C > 0 \).

Now, to obtain the strong convergence of our Feado-Galerkin solution, we use the following compactness lemma

**Lemma 3.2.** For every \( m \in \mathbb{N} \), we have \( \frac{\partial u^m}{\partial t} \) is bounded in \( W^{-1,x} L_{\mathcal{M}}(Q) + L^2(Q) \) and \( \frac{\partial w^m}{\partial t} \) is bounded in \( L^2\left(0, T; (W^{s,2}(\Omega))'\right) \).

**Proof.** Combining (54) with the estimates (48) of Lemma 3.1, we have
\[
\lambda C_{N,s} \int_0^\tau \int_\Omega \int_\Omega \frac{|w^m(x, t) - w^m(y, t)|^2}{|x - y|^{N+2s}} \, dx \, dy \, dt + \int_0^\tau \int_\Omega L_{\mu(x)}(x, \nabla u^m) \nabla u^m \, dx \, dt \leq C,
\]
for some constant \( C > 0 \) not depending on \( m \). Now, we take \( \phi \in (E_M(Q))^N \) and \( \varphi \in (L^2(Q))^N \) satisfying \( \|\phi\|_{A, \Omega} = 1 \) and \( \|\varphi\|_{L^2(Q)} = 1 \) to deduce
\[
\int_0^\tau \int_\Omega \left( L_{\mu(x)}(\cdot, \nabla u^m) - L_{\mu(x)}(\cdot, \varphi) \right)(\nabla u^m - \varphi) \, dx \, dt \geq 0.
\]
This implies
\[
\int_0^\tau \int_\Omega L_{\mu(x)}(\cdot, \nabla u^m) \phi \, dx \, dt \leq \int_0^\tau \int_\Omega L_{\mu(x)}(\cdot, \nabla u^m) \nabla u^m \, dx \, dt + \int_0^\tau \int_\Omega L_{\mu(x)}(\cdot, \phi)(\nabla u^m - \varphi) \, dx \, dt.
\]
Using (67), we easily see that
\[
\left| \int_0^\tau \int_\Omega L_{\mu(x)}(\cdot, \nabla u^m) \phi \, dx \, dt \right| \leq C.
\]
Therefore, we have \( L_{\mu}(:, \nabla u^m) \) is bounded sequence in \( L_{\mathcal{M}}(Q) \) and so \( \frac{\partial u^m}{\partial t} \) is a bounded in \( W^{-1,x} L_{\mathcal{M}}(Q) + L^2(Q) \) and similarly one can prove that \( \frac{\partial w^m}{\partial t} \) is bounded in \( L^2\left(0, T; (W^{s,2}(\Omega))'\right) \), which complete the proof of Lemma.

### 3.3. Passage to the limit.

Thanks to Lemma 3.1, there exist a subsequences, still denoted by \((u^m, w^m)\), such that
\[
u^m \rightharpoonup u \text{ weakly in } \ W^{1,x} L_{\mathcal{M}}(Q) \text{ for } \sigma\left(\prod L_{\mathcal{M}}, \prod E_{\mathcal{M}}\right),
\]
\[
w^m \rightharpoonup w \text{ weakly in } L^2\left(0, T; W^{s,2}(\Omega)\right).
\]
Using Lemma 3.2, there exists χ ∈ \((L^\infty(\Omega))^N\) such that for a subsequence

\[
\mathcal{L}_{\mu(\cdot)}(\cdot, \nabla u^m) \rightharpoonup \chi \quad \text{weakly in} \quad L^\infty(Q) \quad \text{for} \quad \sigma \left( \prod L^\infty, \prod E_M \right),
\]

\[
\int_{\Omega} \frac{(u^m(x, t) - w^m(y, t))}{|x - y|^{N + 2s}} dy \quad \text{converges weakly to some} \quad V \quad \text{in} \quad L^2(Q).
\]

Consequently, as it is done by Elmahi and Meskine in [16] (see the proof of Theorem 2), we get

\[u^m \to u \quad \text{and} \quad w^m \to w \quad \text{strongly in} \quad L^1(Q).
\]

Letting \(m \to \infty\) in (49)-(50) and for all \((\phi, \varphi) \in C^\infty(0, \tau; D(\Omega)) \times C^\infty(0, \tau; \mathcal{D}(\Omega))\), we have

\[
- \int_0^\tau \int_{\Omega} u\phi_t \, dx \, dt + \int_0^\tau \int_{\Omega} \chi \nabla \phi dx \, dt + \int_{\Omega} u\phi dx \, \big|_0^\tau + 2\lambda \int_0^\tau \int_{\Omega} w\phi dx \, dt = 0,
\]

\[
- \int_0^\tau \int_{\Omega} w\varphi_t \, dx \, dt + \frac{C_{N,s}}{2} \int_0^\tau \int_{\Omega} \int_{\Omega} \frac{(w(x, t) - w(y, t))((\varphi(x) - \varphi(y))}{|x - y|^{N + 2s}} \, dx \, dy \, dt
\]

\[
+ \int_{\Omega} w\varphi dx \, \big|_0^\tau + \int_0^\tau \int_{\Omega} (f - u)\varphi dx \, dt = 0.
\]

Taking \((u^m - u^n)\chi_{(0, \tau)}\) (respectively \((w^m - w^n)\chi_{(0, \tau)}\)) as a test function in (49) (respectively in (50) ), we obtain

\[
\int_0^\tau \int_{\Omega} (u^m - u^n)_t (u^m - u^n) \, dx \, dt + \int_0^\tau \int_{\Omega} (w^m - w^n)_t (w^m - w^n) \, dx \, dt
\]

\[
+ \frac{C_{N,s}}{2} \int_0^\tau \int_{\Omega} \int_{\Omega} \frac{|(w^m(x, t) - w^m(x, t)) - (w^m(y, t) - w^m(y, t))}{|x - y|^{N + 2s}} \, dx \, dy \, dt
\]

\[
+ \int_0^\tau \int_{\Omega} \left( \mathcal{L}_{\mu(x)}(x, \nabla u^m) - \mathcal{L}_{\mu(x)}(x, \nabla u^n) \right) \nabla (u^m - u^n) \, dx \, dt = 0.
\]

Using the monotonicity of \(\mathcal{L}_{\mu(\cdot)}\), we obtain

\[
\int_{\Omega} |u^m(\tau) - u^n(\tau)|^2 \, dx + \int_{\Omega} |w^m(\tau) - w^n(\tau)|^2 \, dx \leq \int_{\Omega} |u^m(0) - u^n(0)|^2 \, dx
\]

\[
+ \int_{\Omega} |w^m(0) - w^n(0)|^2 \, dx.
\]

Consequently \(u^m\) and \(w^m\) are Cauchy sequences in \(C \left(0, T; L^2(\Omega)\right)\). Consequently

\[\langle u^m, w^m \rangle \to \langle u, w \rangle \quad \text{in} \quad C \left(0, T; L^2(\Omega)\right) \times C \left(0, T; L^2(\Omega)\right).
\]
Now, let us prove that $L_{\mu}(\cdot, \nabla u) = \chi$ a.e. in $Q$. Indeed, we proceed as [16], let $v^k \in \mathcal{D}(\Omega \times \mathbb{R})$ be a regularization the prolongation of $u$ such that
\begin{equation}
\frac{\partial v^k}{\partial t} \to \frac{\partial u}{\partial t} \text{ in } W^{-1,\infty}L_{\mathcal{M}}(Q) + L^2(Q).
\end{equation}
Moreover, we have $v^k \to u$ in $C\left(0, T; L^2(\Omega)\right)$ (consult Theorem 2 in [16]) and
\begin{equation}
\limsup_{m \to \infty} \limsup_{k \to \infty} \int_Q \frac{\partial u^m}{\partial t} (v^k - u^m) \, dx \, dt \leq 0.
\end{equation}
Collecting the previous convergences, letting $m \to \infty$ and $k \to \infty$ in (74) and using (80), we deduce
\begin{equation}
\limsup_{m \to \infty} \int_0^T \int_{\Omega} L_{\mu(x)}(x, \nabla u^m) \nabla (u^m - u) \, dx \, dt \leq 0.
\end{equation}

An application of the standard Minty Browder arguments, we get $L_{\mu(\cdot)}(\cdot, \nabla u) = \chi$ a.e. in $Q$. To finish the proof of (42) and (43), we let $\phi \in W^{1,\infty}L_{\mathcal{M}}(Q) \cap L^2(Q)$, and $\psi \in L^2\left(0, T; W^{s,2}(\Omega)\right)$ such that
\begin{equation}
\frac{\partial \phi}{\partial t} \in W^{-1,\infty}L_{\mathcal{M}}(Q) + L^2(Q) \quad \text{and} \quad \frac{\partial \psi}{\partial t} \in L^2\left(0, T; (W^{s,2}(\Omega))^{'\prime}\right).
\end{equation}
Observe that, there exists $(\phi^k, \psi^k) \in \mathcal{D}(\Omega \times \mathbb{R}) \times \mathcal{D}(\Omega \times \mathbb{R})$ such that
\begin{equation}
\phi^k \to \tilde{\phi} \quad \text{weakly in } W^{1,\infty}L_{\mathcal{M}}(\Omega \times \mathbb{R}) \quad \text{for } \sigma\left(\prod L_{\mathcal{M}}, \prod E_{\mathcal{M}}\right),
\end{equation}
\begin{equation}
\frac{\partial \phi^k}{\partial t} \to \frac{\partial \tilde{\phi}}{\partial t} \text{ in } W^{-1,\infty}L_{\mathcal{M}}(\Omega \times \mathbb{R}) + L^2(\Omega \times \mathbb{R}),
\end{equation}
\begin{equation}
\psi^k \to \tilde{\psi} \quad \text{weakly in } L^2\left(\mathbb{R}; W^{s,2}(\Omega)\right),
\end{equation}
\begin{equation}
\frac{\partial \psi^k}{\partial t} \to \frac{\partial \tilde{\psi}}{\partial t} \text{ weakly in } L^2\left(\mathbb{R}; (W^{s,2}(\Omega))^{'\prime}\right),
\end{equation}
where $\tilde{\phi}$ (respectively $\tilde{\psi}$) is a prolongation of $\phi$ (respectively $\psi$) in $\mathbb{R} \times \Omega$ (see [16] for the existence of $\tilde{\phi}$ and $\tilde{\psi}$). Letting $\phi^k$ (respectively $\psi^k$) as a test function in (74) (respectively (75)) and using the previous estimations, then we obtain (42) and (43).

### 3.4. Uniqueness of solution.

The final step in the proof of the Theorem 2.1 is to show the uniqueness of the solution to the problem (15). Let us assume that the problem (15) admits two weak solutions $(u_1, w_1)$ and $(u_2, w_2)$ and taking $(u_1 - u_2, w_1 - w_2)$ as a test function in the definition of weak solutions, then we
have
\[\left\langle \left( u_1 - u_2, \frac{\partial (u_1 - u_2)}{\partial t} \right) \right\rangle_Q + 2\lambda \int_0^T \int_{\Omega} (u_1 - u_2)(u_1 - u_2)dxdt \]
\[+ \int_0^T \int_{\Omega} \left( \mathcal{L}_u(x, \nabla u_1) - \mathcal{L}_u(x, \nabla u_2) \right) \nabla (u_1 - u_2)dxdt = 0,\]
\[\left\langle w_1 - w_2, \frac{\partial (w_1 - w_2)}{\partial t} \right\rangle + \int_0^T \int_{\Omega} (w_1 - w_2)(w_1 - w_2)dxdt \]
\[+ C_{N,s} \int_0^T \int_{\Omega} \int_{\Omega} \frac{|w_1(x,t) - w_2(x,t) - w_1(y,t) + w_2(y,t)|^2}{|x-y|^{N+2s}} dydxdt = 0.\]
(84)

Using the monotonicity of the operator \(\mathcal{L}_{\mu(x)}\) and adding \(2\lambda \times (85)\) to (84), we obtain
\[\left\langle \left( u_1 - u_2, \frac{\partial (u_1 - u_2)}{\partial t} \right) \right\rangle_Q + 2\lambda \left\langle w_1 - w_2, \frac{\partial (w_1 - w_2)}{\partial t} \right\rangle \leq 0,\]
(85)
then, we have
\[\frac{1}{2} \int_{\Omega} (u_1 - u_2)^2 dx + \lambda \int_{\Omega} (w_1 - w_2)^2 dx \leq 0.\]
(86)
Consequently \(u_1(x,t) = u_2(x,t)\) and \(w_1(x,t) = w_2(x,t)\) a.e. in \(\Omega\). This completes the proof of the Theorem 2.1.

4. Algorithm and numerical results. This section is devoted to the numerical illustration of our new proposed model. We compare our numerical results with some existing models in the literature: the TV model (using \(L^2\)-norm from [36]) and the AAKM model (using \(H^{-1}\)-norm from [3]). To proceed with the numerical algorithm, we discretize our model (15) by using an explicit and implicit numerical scheme based on the finite difference method and the approximations of the fractional left and right derivatives of Riemann-Liouville.

For simplicity, we write our system (15) in the following form
\[
\begin{cases}
\frac{\partial u}{\partial t} - \text{div} \left( \mathcal{P}(x,t) \right) + 2\lambda w = 0, & \text{in } Q := \Omega \times (0, T),
\frac{\partial w}{\partial t} + (-\Delta)^s w - F = 0, & \text{in } Q := \Omega \times (0, T),
\frac{\partial u}{\partial n} = 0, & \text{in } \Sigma := \Gamma \times (0, T),
K_s w = 0, & \text{in } \Sigma_s := \mathbb{R}^N/\bar{\Omega} \times (0, T),
u(0) = f, \quad w(0) = 0, & \text{in } \Omega,
\end{cases}
\]
(88)
where, for all \(x \in \Omega\)
\[
\mathcal{P}(x,t) = \nabla u \exp(|\nabla u|) \chi[|\nabla u| \leq \delta] + \varrho(x) \nabla u \frac{\log(u(x)) (1 + |\nabla u|)}{|\nabla u|} \chi[|\nabla u| > \delta],
\]
\[
\varrho(x) = \frac{\delta \exp(\delta)}{\log^{\mu(x)} (1 + \delta)},
\]
\[
F = f - u.
\]
(89)
Recall that the function $f$ describes the noisy image and $u(\cdot, t)$ is the image with scale parameter $t$. Now, we propose the improved version of the variable exponent function, used to distinguish the features of the image by different values and to inhibit the influence of the noise

$$
\mu(x) = \frac{1}{1 + k \exp((|\nabla G_{\sigma_{\mu}} \ast u|^2)^k), \ k > 0, \ \sigma_{\mu} > 0,}
$$

(90)

where $k$ is constant, $\sigma_{\mu}$ is parameter depend in time (consult (107)) and $G_{\sigma_{\mu}} = 1/(4\pi \sigma_{\mu})^{N/2} \exp(-|x|^2/4\sigma_{\mu}^2)$ is the Gaussian kernel.

Next, we present the discretization of the proposed model. Assuming $\tau$ to be the time step size and $\ell$ the space grid size, we discretized time and space as follows

$$
t_n = n \tau, \quad n = 1, 2, \ldots,
$$

$$
x_i = i \ell, \quad i = 1, 2, \ldots, M,
$$

$$
y_j = j \ell, \quad j = 1, 2, \ldots, N,
$$

(91)

where $M \ell \times N \ell$ is the size of the original image. We denote $u_{n,i,j}^n$, $w_{n,i,j}^n$, $F_{n,i,j}^{n+1/2}$, $\mu_{n,i,j}$ and $\varrho_{n,i,j}$ the approximations of $u(x_i, y_j, t_n)$, $w(x_i, y_j, t_n)$, $F(x_i, y_j, t_{n+1/2})$, $\mu(x_i, y_j, t_n)$ and $\varrho(x_i, y_j, t_n)$ respectively. The discrete gradient is given by

$$
\nabla_{x}^+ u_{i,j}^n = \frac{u_{i+1,j}^n - u_{i,j}^n}{\ell}, \quad \nabla_{x}^- u_{i,j}^n = \frac{u_{i,j}^n - u_{i-1,j}^n}{\ell}
$$

$$
\nabla_{y}^+ u_{i,j}^n = \frac{u_{i,j+1}^n - u_{i,j}^n}{\ell}, \quad \nabla_{y}^- u_{i,j}^n = \frac{u_{i,j}^n - u_{i,j-1}^n}{\ell}
$$

(92)

We use the following norm to our numerical scheme

$$
|\nabla u_{i,j}^n| = \sqrt{(\nabla_{x}^+ u_{i,j}^n)^2 + (\nabla_{y}^+ u_{i,j}^n)^2}.
$$

(93)

We denote $\text{div}(P)_{i,j}^n$ the approximation of $\text{div}(P(x_i, y_j, t_n))$ defined by

$$
\text{div}(P)_{i,j}^n = \nabla_{x}^- \left(\nabla_{x}^+ u_{i,j}^n \exp(|\nabla u_{i,j}^n|) \chi_{\{\|\nabla u_{i,j}^n\| \leq \delta\}} + \varrho_{i,j} \nabla_{x}^+ u_{i,j}^n \log u_{i,j}^n (1 + |\nabla u_{i,j}^n|) \frac{1}{|\nabla u_{i,j}^n|} \chi_{\{\|\nabla u_{i,j}^n\| > \delta\}}\right)
$$

$$
+ \nabla_{y}^- \left(\nabla_{y}^+ u_{i,j}^n \exp(|\nabla u_{i,j}^n|) \chi_{\{\|\nabla u_{i,j}^n\| \leq \delta\}} + \varrho_{i,j} \nabla_{y}^+ u_{i,j}^n \log u_{i,j}^n (1 + |\nabla u_{i,j}^n|) \frac{1}{|\nabla u_{i,j}^n|} \chi_{\{\|\nabla u_{i,j}^n\| > \delta\}}\right).
$$

(94)

Using the finite difference method, the discrete explicit scheme of the first equation of (88), is given by

$$
\frac{u_{i,j}^{n+1} - u_{i,j}^n}{\tau} = \text{div}(P)_{i,j}^n - 2\lambda u_{i,j}^{n+1}.
$$

(95)

On the other hand, we define the fractional Laplacian in terms of approximations of the fractional left and right derivatives of Riemann-Liouville (see [13, 33]), as follows

$$
(-\Delta)^s w = c_s (D_{x}^{s,l} w + D_{x}^{s,r} w) + c_s (D_{y}^{s,l} w + D_{y}^{s,r} w),
$$

(96)

where $c_s = 1/2 \cos(\pi s/2)$ and $D_{x}^{s,l}$, $D_{x}^{s,r}$, $D_{y}^{s,l}$, $D_{y}^{s,r}$ are noted the left and right shifted-Grünwald difference operators, defined by

$$
D_{x}^{s,l} w |_{x_i} = \frac{1}{\ell^s} \sum_{k=0}^{i+1} g_k^{(s)} w |_{x_{i-k+1}} + O(\ell),
$$

(97)
We complete (95), (104) and (105) by the discrete version of boundary and initial and

\[ D_x^{s,r} w |_{x_i} = \frac{1}{\ell^s} \sum_{k=0}^{n-i+1} g_k^{(k)} w |_{x_{i+k-1}} + O(\ell), \]

such that \( g_k^{(k)} = (-1)^k \frac{\Gamma(s+1)}{\Gamma(s-k+1)\Gamma(k+1)} \Gamma(n) = n\Gamma(n-1) = (n-1)! \) and \( \lim_{\ell \to 0} O(\ell) \to 0. \)

For the second equation of problem (88), we develop the Crank-Nicolson difference scheme by using the formula given in (96)

\[ w^{n+1}_{i,j} - w^n_{i,j} = -\frac{c_s}{2} \left( (D_x^{s,l} w^n_{i,j}) + (D_y^{s,r} w^n_{i,j}) + (D_x^{s,l} w^n_{i,j}) + (D_y^{s,r} w^n_{i,j}) + D^{s+1}_{i,j} \right). \]

This implies

\[ \left( 1 + \frac{\tau c_s}{2} D_x^{s,l} + \frac{\tau c_s}{2} D_x^{s,r} + \frac{\tau c_s}{2} D_y^{s,l} + \frac{\tau c_s}{2} D_y^{s,r} \right) w^{n+1}_{i,j} = \left( 1 - \frac{\tau c_s}{2} D_x^{s,l} - \frac{\tau c_s}{2} D_x^{s,r} - \frac{\tau c_s}{2} D_y^{s,l} - \frac{\tau c_s}{2} D_y^{s,r} \right) w^n_{i,j} + \tau F^{n+1/2}_{i,j}. \]

The formulation (100) reduces to

\[ \left( 1 + \frac{\tau c_s}{2} \delta^x + \frac{\tau c_s}{2} \delta^y \right) w^{n+1}_{i,j} = \left( 1 - \frac{\tau c_s}{2} \delta^x - \frac{\tau c_s}{2} \delta^y \right) w^n_{i,j} + \tau F^{n+1/2}_{i,j}, \]

where \( \delta^x = D_x^{s,l} + D_x^{s,r} \) and \( \delta^y = D_y^{s,l} + D_y^{s,r}. \)

Add into left and right of (101) the terms \( \frac{\tau^2 c_s^2}{4} \delta^x \delta^y w^{n+1}_{i,j} \) and \( \frac{\tau^2 c_s^2}{4} \delta^x \delta^y w^n_{i,j}, \) respectively. We deduce

\[ \left( 1 + \frac{\tau c_s}{2} \delta^x \right) \left( 1 + \frac{\tau c_s}{2} \delta^y \right) w^{n+1}_{i,j} = \left( 1 - \frac{\tau c_s}{2} \delta^x \right) \left( 1 - \frac{\tau c_s}{2} \delta^y \right) w^n_{i,j} + \tau F^{n+1/2}_{i,j}. \]

Observe that (102) can be solved by the alternating direction implicit method (ADI-Method) of Peaceman-Rachford. The formulation (102) can be split into the following three fractional time-steps (at time \( t_n \)):

1. First step, we solve the explicit equation in the x-direction (for each fixed \( x_i \))

\[ v^{n+1}_{i,j} = \left( 1 - \frac{\tau c_s}{2} \delta^y \right) w^n_{i,j}. \]

2. Second step, we solve the implicit equation in the y-direction (for each fixed \( y_j \))

\[ \left( 1 + \frac{\tau c_s}{2} \delta^y \right) w^{n+1}_{i,j} = \left( 1 - \frac{\tau c_s}{2} \delta^y \right) w^n_{i,j} + \frac{\tau}{2} F^{n+1/2}_{i,j}. \]

3. Final step, we solve the implicit equation in the x-direction (for each fixed \( x_i \))

\[ \left( 1 + \frac{\tau c_s}{2} \delta^x \right) w^{n+1}_{i,j} = 2w^n_{i,j} - v^n_{i,j}. \]

We complete (95), (104) and (105) by the discrete version of boundary and initial conditions

\[
\begin{align*}
  v^0_{i,j} &= f_{i,j}, \quad w^0_{i,j} = 0 \quad \forall \ 1 \leq i \leq M, \ 1 \leq j \leq N, \\
  u^0_{i,j} &= u_{i,j}, \quad u^0_{M,j} = u_{M-1,j}, \quad u^0_{i,j} = u_{i,1}, \quad u^0_{i,N} = u_{i,N-1}, \\
  w^0_{0,j} &= w_{0,j}, \quad u^0_{M-1,j} = u_{M-1,j}, \quad u^n_{i,j} = u^{n}_{i,1}, \quad u^n_{i,N} = u^{n}_{i,N-1}, \\
  v^{n}_{i,0} &= \left( 1 + \frac{\tau c_s}{2} \delta^y \right) w^n_{i,1}, \quad v^{n}_{i,N} = \left( 1 + \frac{\tau c_s}{2} \delta^y \right) w^n_{i,N},
\end{align*}
\]
\[ w_{0,j}^{n+1/2} = \left( 1 - \frac{\tau}{2} c_s \delta_{y}^{*} \right) w_{0,j}^{n+1} + \left( 1 + \frac{\tau}{2} c_s \delta_{y}^{*} \right) w_{0,j}^{n}, \]
\[ w_{M,j}^{n+1/2} = \left( 1 - \frac{\tau}{2} c_s \delta_{y}^{*} \right) w_{M,j}^{n+1} + \left( 1 + \frac{\tau}{2} c_s \delta_{y}^{*} \right) w_{M,j}^{n}. \]

Furthermore, we define the following approximations \( \mu_{i,j}^{n} \) and \( \varphi_{i,j}^{n} \) of \( \mu \) and \( \varphi \) (consult (89) and (90)):
\[ \mu_{i,j}^{n} = \frac{1}{1 + k \exp(\left| \nabla G_{\sigma^{n}_{\mu}} \ast u^{n}_{i,j} \right|^{2})} \quad \text{and} \quad \varphi_{i,j}^{n} = \frac{\delta \exp(\delta)}{\log \mu_{i,j}^{n} (1 + \delta)}. \] (106)

Herein, the discrete solution of \( \sigma_{\mu}^{n} \) is governed by
\[ \begin{cases} 
\sigma_{0}^{0} = \sigma_{\mu}, \\
\sigma_{\mu}^{n} = \sigma_{\mu}^{0} + n \left( \sigma_{\mu} - \sigma_{\mu} \right)/N,
\end{cases} \] (107)

where \( \sigma_{\mu} \) and \( \sigma_{\mu} \) are given positive constants.

To obtain our numerical results, we run our algorithm until we reach the best evaluation criterion through iteration (see (111) and its comments). For this, all algorithm parameters are heuristically chosen to make the algorithm works as well as possible. For our simulations, we choose \( \tau = 0.1, \ell = 1, k = 0.7 \) and \( \sigma_{\mu} = 0.5 \).

We show (in the first tests) the ability of our algorithm to denoise corrupted images with different noise levels (low, typical and high noises). Furthermore, we show a comparison results of our algorithm with some approaches that existed in the literature for image denoising, e.g. the TV model using \( L^{2} \)-norm (see [36]) and the AAKM model using \( H^{-1} \)-norm (see [3]). To have this we fix the parameter \( \sigma_{\mu} = \sigma \) (\( \sigma \) is noise level). In the test, we compare the residual part and the cartoon component of the restored images, obtained with different values of \( s \in [0, 1] \). For that, we fix the parameters \( \lambda = 10 \) and \( \sigma_{\mu} = 1 \). Through this experiment, we show the utility of introducing the weak norm of \( H^{-s} \) space and the capacity of the proposed approach to model the texture. Subsequently, we compare the restored images of our model (15) with the TV model from [36]. In this experiment, we mainly aim to reduce the staircase effect caused by the TV operator. In the last experiment, we compare the decomposition of the restored images into the cartoon part and texture component, obtained by our model and by some existing models in the literature, e.g. the TV model using \( L^{2} \)-norm from [36] and the AAKM model using \( H^{-1} \)-norm from [3].

Now, we present our numerical results obtained with the new proposed model and comparisons with models, e.g. the TV model using the \( L^{2} \)-norm from [36] and the AAKM model using \( H^{-1} \)-norm from [3].

| Noise level \( \sigma = 10 \) | Fig.3(A) Noisy | Fig.3(D) Restored | Fig.3(B) Noisy | Fig.3(E) Restored | Fig.3(C) Noisy | Fig.3(F) Restored |
|--------------------------|----------------|-------------------|----------------|-------------------|----------------|-------------------|
| SNR                     | 12.54          | 20.23             | 13.57          | 14.84             | 12.66          | 15.11             |
| PSNR                    | 28.14          | 35.82             | 27.22          | 29.16             | 28.17          | 30.60             |
| SSIM                    | 0.782          | 0.971             | 0.865          | 0.933             | 0.590          | 0.949             |

Table 1. The SNR and PSNR values of Noisy (\( \sigma = 10 \)) and Restored images of the first efficiency test.
Figure 2. Original images.

(a) (b) (c)

Noisy images $\sigma = 10$.

(d) (e) (f)

Restored images.

Figure 3. The efficiency test of our model to denoise corrupted images with $\sigma = 10$ (see Table 1).

For measuring the quality of these results, the signal-to-noise ratio (SNR) and peak signal-noise ratio (PSNR) tools are provided, which are defined as follows

\[
\text{PSNR} := 10 \log_{10} \left( \frac{M \times N}{\|u_0 - u\|_2} \times 255^2 \right),
\]
\[
\text{SNR} := 10 \log_{10} \left( \frac{\sigma_u}{\sigma_\eta} \right),
\]  

(108)
Figure 4. The efficiency test of our model to denoise corrupted images with $\sigma = 20$ (see Table 2).

| Noise level $\sigma = 20$ | Fig.4(A) | Fig.4(D) | Fig.4(B) Noisy | Fig.4(E) Noisy | Fig.4(C) Noisy | Fig.4(F) Noisy |
|--------------------------|----------|----------|----------------|----------------|----------------|----------------|
| SNR                      | 06.50    | 17.49    | 09.04          | 12.10          | 06.62          | 13.41          |
| PSNR                     | 22.10    | 33.09    | 22.02          | 25.38          | 22.13          | 28.90          |
| SSIM                     | 0.580    | 0.925    | 0.743          | 0.896          | 0.422          | 0.837          |

Table 2. The SNR and PSNR values of Noisy ($\sigma = 20$) and Restored images of the second efficiency test.

| Noise level $\sigma = 30$ | Fig.5(A) | Fig.5(D) | Fig.5(B) Noisy | Fig.5(E) Noisy | Fig.5(C) Noisy | Fig.5(F) Noisy |
|--------------------------|----------|----------|----------------|----------------|----------------|----------------|
| SNR                      | 03.01    | 16.12    | 05.51          | 10.07          | 03.06          | 12.44          |
| PSNR                     | 18.58    | 31.73    | 18.56          | 23.39          | 18.54          | 27.92          |
| SSIM                     | 0.501    | 0.884    | 0.650          | 0.885          | 0.325          | 0.780          |

Table 3. The SNR and PSNR values of Noisy ($\sigma = 30$) and Restored images of the third efficiency test.
where \( u_0, u \) and \( M \times N \) are the original image, the restored image and the size of the image, respectively. The functions \( \sigma_u \) and \( \sigma_n \) are the signal and noise standard deviations, respectively. We also use the structural similarity index (SSIM) [40] as a quantitative measure for image quality. Let us first define local similarity index computed on windows \( w_1 \) and \( w_2 \), as follows

\[
\text{ssim}(w_1, w_2) := \frac{(2\bar{w}_1\bar{w}_2 + C_1)(2\sigma_{w_1,w_2} + C_2)}{\bar{w}_1^2 + (\bar{w}_2)^2 + C_1 + \sigma_{w_1}^2 + \sigma_{w_2}^2 + C_2},
\]

(109)

where \( \bar{w}_1 \) and \( \bar{w}_2 \) are the average; \( \sigma_{w_1}^2 \) and \( \sigma_{w_2}^2 \) are the variance; \( \sigma_{w_1,w_2} \) is the covariance of \( w_1 \) and \( w_2 \); and \( C_1 \) and \( C_2 \) are two variables to stabilize the division with a low denominator. The overall SSIM is the mean of local similarity indices, i.e.,

\[
\text{SSIM} := \frac{1}{m} \sum_{i=1}^{m} \text{ssim}((w_1)_i, (w_2)_i),
\]

(110)

where \((w_1)_i, (w_2)_i\) are corresponding windows indexed by \( i \), and \( m \) is the number of windows, see [40] for more details. Here, we consider windows of size \( 8 \times 8 \).

**Stopping criterion.** Some standard stopping criteria are the relative error being small, the number of iterations being chosen with respect to the best SNR,
PSNR and SSIM values, or both, i.e.,
\[
\frac{\|u^{n+1} - u^n\|_{L^2(\Omega)}}{\|u^n\|_{L^2(\Omega)}} \leq Tol,
\]
with predefined tolerance value \(Tol\). In this paper, we choose to stop the global iteration when the relative error is smaller than \(1e^{-6}\). The better quality image will have higher values of SNR, PSNR and SSIM.

To set algorithm parameters (or to select a relevant training set), it is required to know the noise level appearing in the image, which permits directly to calculate the similarity indexes (SNR, PSNR and SSIM) of the original image. For practical situations where the noise level is unknown, one can apply soft-thresholding based on the statistical estimation of the original image. In this case, we refer the reader to [15], for more details, where the authors have proposed an abstract approach that sets up a problem of estimating a sequence in white Gaussian noise and relates this to a problem of optimal recovery in deterministic noise.

In order to show the accuracy of our model, we begin with validation tests, establishes of a restoration of corrupted images by different noise levels (low \(\sigma = 10\), typical \(\sigma = 20\) and high noises \(\sigma = 30\)). These results are provided in Figures 3, 4 and 5. From these results, we can clearly see that the proposed algorithm proves to be powerful to solve the problem of image denoising. Our algorithm is able to
remove the noise in different types of images performed with different noise levels (low, typical, and high noises).

Furthermore, Figures 6–8 illustrate the denoising results obtained by using our algorithm and some approaches that existed in the literature for image denoising, e.g. the TV model using $L^2$–norm (see [36]) and the AAKM model using $H^{-1}$–norm (see [3]). Here, the tests are performed on different types of images, e.g. the medical image (‘Lung’ image in Figure 6) and on the grayscale images (images: ‘Einstein’ in Figure 7 and ‘Lena’ in Figure 8). The test images are destroyed with an additive Gaussian white noise with a noise level of $\eta = 10$ and $\eta = 15$, respectively. From these experiments, we can see clearly that our algorithm performs better than other methods, especially, our model can preserve the edges, not create new features, and suppressed the staircase effect. Figures 6–8 also illustrate the values of the SSIM for each result associated with different methods. Comparing these details, we can be seen that the values of SSIM proven using our proposed model are the largest than the other models. We can conclude that our method reduces the staircase effect which is still remained in images restored by other methods. Then, the proposed system is more robust.

After that, we show the utility of introducing the weak $H^{-s}$–norm. For this, we make a comparison of cartoon component of $u$ and residual part of $u - f$ for
Figure 8. Image denoising performed on the grayscale image (‘Lena’ image). Noise removal results provided by our model and other denoising techniques.

Figure 9. This figure shows the ability to reduce the staircase effect between restored images with our proposed model and the TV model using the $L^2$-norm from [36]. Our model can successfully reduce the staircase phenomenon during the image denoising.
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\[
\begin{align*}
[H^{-1}\text{–norm}] & \quad \text{SNR} = 22.43 \\
[H^{-0.8}\text{–norm}] & \quad \text{SNR} = 38.93 \\
[H^{-0.5}\text{–norm}] & \quad \text{SNR} = 42.51 
\end{align*}
\]

Smooth images \( u \)

\[
\begin{align*}
[H^{-1}\text{–norm}] & \quad \text{SNR} = 25.36 \\
[H^{-0.3}\text{–norm}] & \quad \text{SNR} = 29.15 \\
[H^{-0.5}\text{–norm}] & \quad \text{SNR} = 34.75 
\end{align*}
\]

Smooth images \( u \)

Residual images \( f - u \)
Figure 10. The comparison results of different values of the fraction $s$ obtained by our proposed model. The first row contains smooth images $u$; the second row contains a residual part associated with $u - f$. 

\[ H^{-1}\text{-norm} \]
SNR = 21.70

\[ H^{0.4}\text{-norm} \]
SNR = 28.89

\[ H^{0.5}\text{-norm} \]
SNR = 34.67

Smooth images $u$

Residual images $f - u$
Figure 11. Comparisons results between the classical TV model ($L^2$-norm) from [36] and the AAKM model ($H^{-1}$-norm) from [3] with our proposed model ($H^{-0.5}$-norm).
Figure 12. Results on Scan’s image of line profile number 50. Red: original image; Green: cartoon part obtained by the TV model using the $L^2$-norm [36]; Violet: cartoon part obtained by the AAKM model using the $H^{-1}$-norm [3]; Blue: cartoon part obtained by our proposed model using the $H^{-0.5}$-norm.

Left: original image; Middle: edge detector; Right: contour lines, applied on the original image.

Left: cartoon part; Middle: edge detector; Right: contour lines, applied on the cartoon part obtained by the TV model using the $L^2$-norm from [36].
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Figure 13. Comparison results of edge detector, contour lines and texture applied on a cartoon image obtained by the classical TV model (using the $L^2$-norm) from [36] and AAKM model (using the $H^{-1}$-norm) from [3] with our proposed model (using the $H^{-0.5}$-norm).

Left: texture part obtained by the TV model using the $L^2$-norm from [36]; Middle: texture part obtained by the AAKM model using the $H^{-1}$-norm from [3]; Right: texture part obtained by our proposed model using the $H^{-0.5}$-norm.

Left: cartoon part; Middle: edge detector; Right: contour lines, applied on the cartoon part obtained by the AAKM model using the $H^{-1}$-norm from [3].

Left: cartoon part; Middle: edge detector; Right: contour lines, applied on the cartoon part obtained by our model using the $H^{-0.5}$-norm.

different values of $s \in [0, 1]$. We observe that the texture is better preserved for the values of $s$ close to $1/2$, (see Figure 10 for more details).

Subsequently, we compare the restored images from our model with the TV model from [36]. The staircase effect can be observed greatly on the zoom *Barbara’s hand*.
Figure 14. Results on Pepper’s image of line profile number 50. Red: original image; Green: cartoon part obtained by the TV model using the $L^2$-norm from [36]; Violet: cartoon part obtained by the AAKM model using the $H^{-1}$-norm from [3]; Blue: cartoon part obtained by our proposed model using the $H^{-0.5}$-norm.

Left: original image; Middle: edge detector; Right: contour lines, applied on the original image.

Left: cartoon part; Middle: edge detector; Right: contour lines, applied on the cartoon part obtained by the TV model using the $L^2$-norm from [36].
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Figure 15. Comparison results of edge detector, contour lines and texture applied on a cartoon image obtained by the classical TV model (the $L^2$-norm) from [36] and the AAKM model (the $H^{-1}$-norm) from [3] with our proposed model (the $H^{-0.5}$-norm).

Left: cartoon part; Middle: edge detector; Right: contour lines, applied on the cartoon part obtained by the AAKM model using the $H^{-1}$-norm from [3].

Left: cartoon part; Middle: edge detector; Right: contour lines, applied on the cartoon part obtained by our model using the $H^{-0.5}$-norm.

Left: texture part obtained by the TV model using the $L^2$-norm from [36]; Middle: texture part obtained by the AAKM model using the $H^{-1}$-norm from [3]; Right: texture part obtained by our proposed model using the $H^{-0.5}$-norm.

image of Figure 9a obtained by the TV model from [36]. On the other hand, the staircase has been successfully reduced in the image obtained by our model.

We conclude our simulations, by comparing the smooth $u$ and residual part of $u - f$ images between our model and some existing models in the literature; (the TV model [36] and the AAKM model [3]). In this experiment, it is clear that
the results provided in Figure 11 by our model are better than those provided by
the TV model $L^2$-norm and the local AAKM model $H^{-1}$-norm. To see more
difference between these models, we apply Canny’s edge detector and the contour
lines to the restored images. From Figures 13 and 15, we note that in the smooth
images, the edges are still kept in our model, while those of the AAKM model using
$H^{-1}$-norm are occasionally eliminated and those of the TV model using $L^2$-norm
are considerably eliminated. Moreover, Figures 12 and 14 represent the profile of
line number 50 of the restored images obtained by The TV model, The AAKM
model and the proposed one. The red curve precisely corresponds to the original
image, whereas the smooth images of the TV model, the AAKM model and the
proposed one, are respectively represented by green, violet and blue curves. The
lines of the smooth images associated with our proposed model are close to the lines
of the original image. While the lines of the smooth images associated with the TV
model and the AAKM model are far.

5. Conclusion. In this paper, we have proposed a new class of nonlinear fractional
reaction-diffusion systems for the reconstruction of textured images. Our
model combines a new nonlinear regularization of a TV operator with a fractional
Laplacian. Moreover, the proposed nonlinear regularization is based on a variable
exponent depending on the space variable. Using the proposed variable exponent,
we adaptively controlled the diffusion mode in accordance with the image features
and sufficiently preserve small details. Namely, at the edges where the gradient
is large, the proposed variable exponent quickly approaches to 0 and our nonlinear
regularization proposed of the TV behaves like the standard TV operator and
guarantees rapid convergence. Our model can preserve the edges, not create new
features and suppressed the staircase effect. On the other hand, the fractional
Laplacian derived from a weak norm of the dual of fractional-Sobolev space, which
is more appropriate for dealing with patterns and oscillations/textures. We men-
tion here that the dual of fractional-Sobolev space provided an exact approximation
of Meyer norm. We have shown the well-posedness (existence and uniqueness) of
the weak solution to our proposed model. To illustrate the efficiency of our model,
we have developed a numerical scheme based on the finite difference method and
the approximations of the fractional Laplacian. We have compared our numerical
results with some existing models in the literature.

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E-mail address: a.atlas@uca.ma
E-mail address: mostafa.bendahmane@u-bordeaux.fr
E-mail address: fa.karami@uca.ma
E-mail address: dr.meskine@uca.ac.ma
E-mail address: omar.oubbih@usmba.ac.ma