Weighted Simplicial Complexes and Weighted Analytic Torsions

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Abstract

A weighted simplicial complex is a simplicial complex with values (called weights) on the vertices. In this paper, we consider weighted simplicial complexes with \( \mathbb{R}^2 \)-valued weights. We study the weighted homology and the weighted analytic torsion for such weighted simplicial complexes.

1 Introduction

During the 20-th century, the theory of R-torsion and analytic torsion of Riemannian manifolds was developed by K. Reidemeister [14], John Milnor [11], D. B. Ray and I. Singer [13], Werner Müller [12], J. Cheeger [1], etc. Recently, Alexander Grigor’yan, Yong Lin and Shing-Tung Yau [4] applied the R-torsion and the analytic torsion theory to digraphs and path complexes (cf. [5, 6, 7, 8, 9]). A discrete version of the R-torsion and the analytic torsion theory on digraphs has been given in [4].

Simplicial complexes are useful combinatorial models in algebraic topology. In recent decades, (abstract) simplicial complexes have been found to have various applications in data sciences. Let \( V \) be a finite set with a total order \( \prec \). An (abstract) simplicial complex \( K \) on \( V \) is a subset of the power set \( 2^V \) such that for any \( \sigma \in K \) and any non-empty subset \( \tau \subseteq \sigma \), we have that \( \tau \in K \). An element \( \sigma \in K \) is called a simplex, and an element \( v \in V \) is called a vertex. For any simplex \( \sigma \in K \), we can always write \( \sigma \) as \( \{v_0, v_1, \ldots, v_n\} \) for some non-negative integer \( n \), where \( v_0, v_1, \ldots, v_n \in V \) and \( v_0 \prec v_1 \prec \ldots \prec v_n \). We write \( n = \text{dim } \sigma \) and call it the dimension of \( \sigma \). Letting \( \sigma \) run over all the simplices in \( K \), the maximum of \( \text{dim } \sigma \) is denoted as \( \text{dim } K \) and is called the dimension of \( K \).

A weighted simplicial complex is a simplicial complex with some functions that assign values to the vertices (or to the simplices). In 1990, Robert J. MacG. Dawson [2] studied the (weighted) homology for weighted simplicial complexes. In recent years, the weighted homology of weighted simplicial complexes have been further explored in [15, 16]. Some other topological features for weighted simplicial complexes, such as the fundamental groups, the discrete Morse functions, the Hodge-Laplace operators, etc. have been studied in [15, 19-20]. In 2020, Zhenyu Meng, D Vijay Anand, Yupeng Lu, Jie Wu and Kelin Xia [10] have found amazing applications of the weighted homology for weighted simplicial complexes in biological data analysis, with significant effects.

In this paper, we consider the weighted simplicial complexes \((K, f, g)\) where both \( f \) and \( g \) are real functions on the set \( V \) of the vertices. We use \( f \) to twist the boundary operators and use \( g \) to give a symmetric and semi-positive definite quadratic form on the chain complexes. We study the weighted homology equipped with the symmetric and semi-positive definite quadratic forms. Then we study the weighted analytic torsions for weighted simplicial complexes.

For any real number \( r \), we let \( \epsilon(r) = 1 \) if \( r \neq 0 \) and let \( \epsilon(r) = 0 \) if \( r = 0 \). Then for any function \( f \) on \( V \), \( \epsilon(f) \) is a function on \( V \) given by \( \epsilon(f)(v) = \epsilon(f(v)) \) for any \( v \in V \). For any real-valued function \( f \) on \( V \), let \( K_f^\gamma \) be the largest sub-simplicial complex of \( K \) such that \( f \) is non-vanishing on all of its vertices. We will prove the following theorem.

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Theorem 1.1 (Main Result). Let \( K \) be a simplicial complex with the set \( V \) of its vertices. Let \((f, g)\) be a \( \mathbb{R}^2 \)-valued function on \( V \). Let \( T(K, f, g) \) be the \((f, g)\)-weighted analytic torsion of \( K \). Then

(i) for any non-vanishing real function \( h \) on \( V \), we have

\[
T(K, \epsilon(g)fh, \epsilon(f)gh) = T(K, \epsilon(g)f, \epsilon(f)g);
\]

(ii) for any non-zero real constant \( c \) we have

\[
T(K, \epsilon(g)f, c\epsilon(g)g) = |c|^{s(K, \epsilon(f))(\epsilon(g))} T(K, \epsilon(g)f, \epsilon(f)g);
\]

and

\[
T(K, c\epsilon(g)f, \epsilon(f)g) = |c|^{-s(K, \epsilon(f))(\epsilon(g))} T(K, \epsilon(g)f, \epsilon(f)g).
\]

Here

\[
s(K, \epsilon(f)(\epsilon(g))) = \sum_{n \geq 0} (-1)^{n} \dim \partial_n C_n(K, \epsilon(f)(\epsilon(g)); \mathbb{R}).
\]

As a by-product, we prove in Theorem 3.1 that for any weighted simplicial complex \((K, f, g)\) and any \( n \geq 0 \), there is a symmetric and semi-positive definite quadratic form \( \langle \cdot, \cdot \rangle_g \) on the \( n \)-th weighted homology \( H_n(K, f, g; \mathbb{R}) \). In addition, if \( g \) is non-vanishing, then the quadratic form \( \langle \cdot, \cdot \rangle \) is an inner product. We also prove in Theorem 3.9 and Corollary 3.10 that for any non-vanishing real function \( h \) on \( V \), there is a linear isometry from \( H_n(K, fh, gh; \mathbb{R}) \) to \( H_n(K, f, g; \mathbb{R}) \). Thus the \((f, g)\)-weighted homology \( H_n(K, f, g; \mathbb{R}) \) only depends on the ratio function \( f/g \) for any non-vanishing functions \( f \) and \( g \). In particular, if \( f = g \), then there is a linear isometry from the weighted homology \( H_n(K, f, f; \mathbb{R}) \) to the usual (unweighted) homology \( H_n(K; \mathbb{R}) \) (cf. [16] Section 5).

The remaining part of this paper is organized as follows. In Section 2, we give some preliminaries in linear algebra. In Section 3, we study the weighted homology of weighted simplicial complexes. We prove Theorem 3.9 and Theorem 3.1. In Section 4, we study the weighted analytic torsions for weighted simplicial complexes and prove Theorem 1.1.

2 Preliminaries

Let \( W \) be a (finite dimensional) real vector space. Let \( \langle \cdot, \cdot \rangle \) be a symmetric semi-positive definite quadratic form on \( W \). Consider the sub-space

\[
N = \{ v \in W \mid \langle v, v \rangle = 0 \} = \{ v \in W \mid \langle v, u \rangle = 0 \text{ for any } u \in W \}.
\]

Note that the second equality is obtained by using the Cauchy-Schwarz inequality. We call \( N \) the null sub-space for the quadratic form \( \langle \cdot, \cdot \rangle \). The next lemma proves that the quotient space of \( W \) by \( N \) has an inherited inner product.

**Lemma 2.1.** The quotient space \( W/N \) has an inner product \( \langle \cdot, \cdot \rangle \) which is inherited from the quadratic form \( \langle \cdot, \cdot \rangle \) on \( W \).

**Proof.** For any \( v_1 + N, v_2 + N \in W/N \), we define

\[
\langle v_1 + N, v_2 + N \rangle = \langle v_1, v_2 \rangle.
\]

(2.1)

To prove (2.1) is well-defined, we let \( v'_1 = v_1 + n_1 \) and \( v'_2 = v_2 + n_2 \) where \( n_1, n_2 \in N \). By the Cauchy-Schwarz inequality,

\[
|\langle v_1, n_2 \rangle|^2 \leq \langle v_1, v_1 \rangle \langle n_2, n_2 \rangle.
\]

(2.2)
Since \( n_2 \in N \), \( \langle n_2, n_2 \rangle = 0 \). Hence (2.2) implies \( \langle v_1, n_2 \rangle = 0 \). Similarly, we have \( \langle n_1, v_2 \rangle = 0 \) and \( \langle n_1, n_2 \rangle = 0 \). Therefore, we have (2.1), and the inherited quadratic form \((\quad,\quad)\) is well-defined on \( W/N \).

To prove \((\quad,\quad)\) is an inner product on \( W/N \), we need to verify that it is strictly positive-definite. Suppose \( \langle v + N, v + N \rangle = 0 \). Then \( \langle v, v \rangle = 0 \), which implies \( v \in N \). Therefore, \((\quad,\quad)\) is strictly positive-definite on \( W/N \), thus it is an inner product. \(\blacksquare\)

As a generalization of Lemma 2.1 the next lemma shows that any quotient space of \( W \) inherits a quadratic form.

**Lemma 2.2.** For any sub-space \( U \) of \( W \), the quotient space \( W/U \) inherits a symmetric semi-positive definite quadratic form \((\quad,\quad)\) from \( W \). Moreover, \((\quad,\quad)\) is an inner product on \( W/U \) if and only if \( N \subseteq U \).

**Proof.** Let \( U \) be a sub-space of \( W \). Restricted to \( U \), the quadratic form \((\quad,\quad)\) on \( W \) gives a quadratic form \((\quad,\quad)\) on \( U \), which is still symmetric and semi-positive definite. By Lemma 2.1 \( U/(U \cap N) \) inherits an inner product \((\quad,\quad)\). Note that as vector spaces, \( U \cong U/(U \cap N) \oplus (U \cap N) \). (2.3)

We let \( \iota \) be the isomorphism from the right-hand side of (2.3) to the left-hand side of (2.3) and let \( U_1 \) be the image \( \iota(U/(U \cap N)) \). Then

\[ U = U_1 \oplus (U \cap N). \] (2.4)

We point out that the isomorphism \( \iota \) may not be unique hence the sub-space \( U_1 \) may not be unique as well. Suppose both \( \iota \) and \( U_1 \) are fixed. For any \( u \in U \), with respect to (2.4) we can write \( u = u_1 + n \) uniquely where \( u_1 \in U_1 \) and \( n \in U \cap N \). Then for any \( u, u' \in U \) we have

\[ \langle u, u' \rangle = \langle u_1, u'_1 \rangle. \] (2.5)

The left-hand side of (2.5) is a semi-positive definite quadratic form and the right-hand-side of (2.5) is an inner product. By a similar argument of the Gram-Schmidt process, we can extend \( U_1 \) to be a sub-space \( W_1 \) of \( W \) such that

\[ W = W_1 \oplus N \] (2.6)

as vector spaces where \( W_1 \) inherits an inner product \((\quad,\quad)\). We consider the orthogonal complement \( \perp_{W_1} U_1 \) of \( U_1 \) in \( W_1 \). Then

\[ W_1 = (\perp_{W_1} U_1) \oplus U_1 \] (2.7)

as Euclidean spaces. It follows from (2.6) and (2.7) that for any \( v \in W \), we can write

\[ v = v_0 + v_2 + n \] (2.8)

where \( v_0 \in U_1 \), \( v_2 \in \perp_{W_1} U_1 \) and \( n \in N \). Moreover, once \( W_1 \) is fixed, the expression (2.8) is unique. With the help of (2.8) we define the quadratic form on \( W/U \) by setting

\[ \langle v + U, v' + U \rangle = \langle v_2, v'_2 \rangle \] (2.9)

for any \( v, v' \in W \). Note that the right-hand side of (2.9) does not depend on the representatives \( v \) of \( v + U \) and \( v' \) of \( v' + U \). Hence (2.9) gives a well-defined quadratic form on \( W/U \), which is obviously symmetric and semi-positive definite.

We point out that \( W_1 \), the extension of \( U_1 \), may not be unique. Nevertheless, for any two such extensions \( W_1 \) and \( W'_1 \) of \( U_1 \), if we write the corresponding decompositions in
as \( v = v_0 + v_2 + n \) and \( v = \bar{v}_0 + \bar{v}_2 + \bar{n} \) respectively, then for any \( v \in W, v_0 = \bar{v}_0; \) and for any \( v, v' \in V, \)

\[
\langle v_2, v'_2 \rangle = \langle \bar{v}_2, \bar{v}'_2 \rangle.
\]

(2.10)

It follows from (2.10) that the inherited quadratic form on \( W/U \) given by (2.9) does not depend on the extension \( W_1 \). We obtain the first assertion.

By the above argument, it follows from (2.4) and (2.6) that

\[
W/U \cong (\perp_{W_1} U_1) \oplus (N/(U \cap N)).
\]

We see that \( (\ , \ ) \) is an inner product on \( W/U \) if and only if it is strictly positive definite on \( W/U \), which happens if and only if

\[
W/U \cong \perp_{W_1} U_1.
\]

(2.11)

Therefore, (2.11) holds if and only if

\[
N/(U \cap N) = 0,
\]

which happens if and only if \( N \subseteq U \). We obtain the second assertion.

With the help of Lemma (2.2), we have the next lemma for linear maps between vector spaces with quadratic forms.

**Lemma 2.3.** Let \( W \) and \( W' \) be vector spaces with (symmetric and semi-positive definite) quadratic forms \( (\ , \ ) \) and \( (\ , \)' \) respectively. Let \( \varphi : W \to W' \) be a linear isomorphism such that for any \( a, b \in W, (\varphi(a), \varphi(b))' = (a, b) \). Let \( U \subset W \) and \( U' \subset W' \) be subspaces such that \( \varphi(U) = U' \). Then \( \varphi \) induces a linear isomorphism

\[
\varphi_* : W/U \to W'/U'
\]

such that the induced symmetric and semi-positive definite quadratic forms on \( W/U \) and \( W'/U' \) are preserved.

**Proof.** It is clear that \( \varphi \) is a linear isomorphism. By Lemma (2.2), there are induced symmetric and semi-positive definite quadratic forms \( (\ , \) \) on \( W/U \) and \( (\ , \)' \) on \( W'/U' \). Moreover, \( \varphi \) preserves the decompositions (2.6) and (2.7). We verify that \( \varphi \) preserves the quadratic forms by

\[
\langle v + U, w + U \rangle = \langle v_2, w_2 \rangle = \langle \varphi(v_2), \varphi(w_2) \rangle' = \langle \varphi(v + U), \varphi(w + U) \rangle'.
\]

Here \( v + U \) and \( w + U \) are any two elements in \( W/U \), and \( v_2 \) and \( w_2 \) are in \( \perp_{W_1} U_1 \) given by (2.8). We obtain the lemma.

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3 Weighted Simplicial Complexes and Weighted Homology

Let \( K \) be a simplicial complex. Let \( V \) be the set of the vertices of \( K \). Let \( f \) and \( g \) be two real-valued functions on \( V \). Then

\[
(f, g) : V \to \mathbb{R}^2
\]
is a vector-valued function assigning a point in the plane to each vertex in $V$. We call both $f$ and $g$ a \textbf{vertex-weight} on $K$ and call the ordered triple $(K,f,g)$ a \textbf{vertex-weighted simplicial complex}.

Let $n \geq 0$. For any $n$-simplex $\sigma = \{v_0,v_1,\ldots,v_n\}$ in $K$, we define the induced weights of $f$ and $g$ on $\sigma$ respectively as

$$f(\sigma) = \prod_{i=0}^{n} f(v_i), \quad g(\sigma) = \prod_{i=0}^{n} g(v_i).$$

Let $C_n(K;\mathbb{R})$ be the vector space consisting of all the formal linear combinations of the $n$-simplices in $K$. The $n$-th \textbf{$f$-weighted boundary operator} is a linear map

$$\partial_n^f: C_n(K;\mathbb{R}) \longrightarrow C_{n-1}(K;\mathbb{R})$$

defined by

$$\partial_n^f\{v_0,v_1,\ldots,v_n\} = \sum_{i=0}^{n} (-1)^{i} f(v_i)\{v_0,\ldots,\hat{v}_i,\ldots,v_n\} \quad (3.1)$$

for any $n$-simplex $\{v_0,v_1,\ldots,v_n\}$ of $K$. Particularly, for any $n$-simplex $\sigma$ in $K$ with $f(\sigma) \neq 0$, $\partial_n f(\sigma) f(\sigma)$ can be alternatively expressed as

$$\partial_n^f \sigma = \sum_{i=0}^{n} (-1)^{i} \frac{f(\sigma)}{f(d_i,\sigma)} d_i \sigma \quad (3.2)$$

where for each $0 \leq i \leq n$, $d_i \sigma$ is the $i$-th $(n-1)$-face of $\sigma$ obtained by removing the $i$-th vertex of $\sigma$. Note that $f(\sigma) \neq 0$ implies $f(d_i,\sigma) \neq 0$ for each $0 \leq i \leq n$. Hence (3.2) makes sense. It can be verified that

$$\partial_n^f \partial_n^{f+1} = 0$$

for any $n \geq 0$. Thus

$$\{C_n(K;\mathbb{R}), \partial_n^f\}_{n \geq 0} \quad (3.3)$$

is a chain complex.

The \textbf{$g$-weighted quadratic form} on $C_n(K;\mathbb{R})$ is a symmetric and semi-positive definite quadratic form

$$\langle \ , \ \rangle_g : C_n(K;\mathbb{R}) \times C_n(K;\mathbb{R}) \longrightarrow \mathbb{R} \quad (3.4)$$

given by

$$\langle \sigma, \tau \rangle_g = g(\sigma)g(\tau)\delta(\sigma,\tau) = \prod_{i=0}^{n} g(v_i)g(u_i)\delta(v_i, u_i)$$

for any $n$-simplices $\sigma = \{v_0,v_1,\ldots,v_n\}$ and $\tau = \{u_0,u_1,\ldots,u_n\}$ in $K$. Here for any vertices $v$ and $u$, we use the notation $\delta(v,u) = 0$ if $v \neq u$ and $\delta(v,u) = 1$ if $v = u$; and for any simplices $\sigma$ and $\tau$, we use the notation $\delta(\sigma,\tau) = 0$ if $\sigma \neq \tau$ and $\delta(\sigma,\tau) = 1$ if $\sigma = \tau$.

We notice that for each $n \geq 0$, (3.4) gives a $g$-weighted quadratic form on the vector space in the chain complex (3.3). We use the following notation

$$C_*(K,f,g;\mathbb{R}) = \{C_n(K;\mathbb{R}), \partial_n^f, \langle \ , \ \rangle_g\}_{n \geq 0} \quad (3.5)$$
to denote the chain complex \((3.3)\) with the \(f\)-weighted boundary operators \(\partial^f_i\) and the \(g\)-weighted quadratic forms \((\ ,\ )_g\). The \(n\)-th \((f,g)\)-weighted homology of \(\mathcal{K}\) (with coefficients in real numbers) is defined as the \(n\)-th homology group of the chain complex \(C_n(\mathcal{K}, f, g; \mathbb{R})\), i.e. the quotient space

\[
H_n(\mathcal{K}, f, g; \mathbb{R}) = \text{Ker}(\partial^f_i)/\text{Im}(\partial^f_{i+1}).
\]

The next theorem says that for each \(n \geq 0\), the quadratic form \((\ ,\ )_g\) on \(C_n(\mathcal{K}; \mathbb{R})\) induces a quadratic form, which is still denoted as \((\ ,\ )_g\), on the \(n\)-th \((f,g)\)-weighted homology group \(H_n(\mathcal{K}, f, g; \mathbb{R})\).

**Theorem 3.1.** For any vertex-weighted simplicial complex \((\mathcal{K}, f, g)\) and any \(n \geq 0\), \(H_n(\mathcal{K}, f, g; \mathbb{R})\) is a vector space with a symmetric and semi-positive definite quadratic form \((\ ,\ )_g\).

**Proof.** In Lemma 2.2, we let the space \(W\) be \(\text{Ker}\partial^f_i\) and let the sub-space \(U\) of \(W\) be \(\text{Im}\partial^f_{i+1}\). As sub-spaces of \(C_n(\mathcal{K}; \mathbb{R})\), both \(W\) and \(U\) have an inherited symmetric and semi-positive definite quadratic forms \((\ ,\ )_g\). Thus by Lemma 2.2 the quotient space \(W/U\), which by definition is the \((f,g)\)-weighted homology \(H_n(\mathcal{K}, f, g; \mathbb{R})\), has an induced symmetric and semi-positive definite quadratic form \((\ ,\ )_g\).

For each \(n \geq 0\), the null sub-space of \(C_n(\mathcal{K}; \mathbb{R})\) with respect to \((\ ,\ )_g\) is given by

\[
N_n(\mathcal{K}, g; \mathbb{R}) = \{ b \in C_n(\mathcal{K}; \mathbb{R}) \mid (b, b)_g = 0 \}
= \{ b \in C_n(\mathcal{K}; \mathbb{R}) \mid (b, a)_g = 0 \text{ for any } a \in C_n(\mathcal{K}; \mathbb{R}) \}.
\]

We denote \(N_*(\mathcal{K}, g; \mathbb{R})\) for the graded vector space \(\{N_n(\mathcal{K}, g; \mathbb{R}) \mid n = 0, 1, 2, \ldots\}\). By applying the second assertion in Lemma 2.2 to the proof of Theorem 3.1, the next corollary follows immediately.

**Corollary 3.2.** For any vertex-weighted simplicial complex \((\mathcal{K}, f, g)\) and any \(n \geq 0\), \((\ ,\ )_g\) is an inner product on \(H_n(\mathcal{K}, f, g; \mathbb{R})\) if and only if \(N_n(\mathcal{K}, g; \mathbb{R})\) is a sub-space of \(\partial^f_{i+1}C_n(\mathcal{K}; \mathbb{R})\).

For each vertex \(v \in V\), we let \(\text{star}_\mathcal{K}(v)\) be the open star of \(v\) in \(\mathcal{K}\) given by the set of the simplices containing \(v\) as a vertex:

\[
\text{star}_\mathcal{K}(v) = \{ \sigma \in \mathcal{K} \mid v \in \sigma \}.
\]

We have the next lemma.

**Lemma 3.3.** Let \((\mathcal{K}, f, g)\) be a vertex-weighted simplicial complex. Then for any simplex \(\sigma \in \mathcal{K}\),

\[
g(\sigma) = 0 \iff \sigma \in \bigcup_{v \in V, g(v) \geq 0} \text{star}_\mathcal{K}(v).
\]

**Proof.** Let \(\sigma \in \mathcal{K}\). We write \(\sigma\) in the form \(\{v_0, v_1, \ldots, v_n\}\) for some \(n \geq 0\). Since \(f\) is a vertex-weight, it follows from the definition of \(f(\sigma)\) that \(f(\sigma) = 0\) if and only if \(f(v_i) = 0\) for some \(0 \leq i \leq n\). This happens if and only if \(\sigma\) is in the open star of \(v_i\) for some \(v_i \in V\) with \(f(v_i) = 0\). Thus we obtain \((3.6)\). Consequently, by the definition of \(N_n(\mathcal{K}, g; \mathbb{R})\), it follows from \((3.6)\) that for each \(n \geq 0\), \(N_n(\mathcal{K}, g; \mathbb{R})\) has a basis consisting of all the \(n\)-simplices in \((3.7)\). We obtain the lemma. \(\square\)
The following lemma shows that if we take the quotient space of \( N_n(K, g; \mathbb{R}) \) in \( C_n(K; \mathbb{R}) \), then we will obtain the chain complex of the maximal sub-simplicial complex \( \mathcal{K}_g^\ast \) of \( K \) where \( g \) is non-vanishing.

**Lemma 3.4.** Let \( (K, f, g) \) be a vertex-weighted simplicial complex. Then the sub-simplicial complex

\[
\mathcal{K}_g^\ast = \{ \sigma \in K \mid g(\sigma) \neq 0 \}
\]

is the maximal sub-simplicial complex of \( K \) such that \( g \) is non-vanishing on all of its vertices. Moreover, for each \( n \geq 0 \), with respect to the inner products \( \langle \cdot , \cdot \rangle_g \), there is a canonical linear isometry

\[
\iota_n : C_n(\mathcal{K}_g^\ast; \mathbb{R}) \longrightarrow C_n(K; \mathbb{R})/N_n(K, g; \mathbb{R}).
\] (3.8)

**Proof.** Let \( (K, f, g) \) be a vertex-weighted simplicial complex. It follows from Lemma 3.3 that the maximal sub-simplicial complex \( \mathcal{K}_g^\ast \) such that \( g \) is non-vanishing on all of its vertices is the complement of the union \( (3.7) \) of open stars

\[
\mathcal{K}_g^\ast = K \setminus \left( \bigcup_{v \in V , \atop g(v) = 0} \text{star}_K(v) \right).
\]

Consequently, for each \( n \geq 0 \), we have a linear isomorphism \( \iota_n \) in (3.8) sending an \( n \)-simplex \( \sigma \in \mathcal{K}_g^\ast \) to the co-set \( \sigma + N_n(K, g; \mathbb{R}) \). By Lemma 2.7, for each \( n \geq 0 \), the quotient space \( C_n(K; \mathbb{R})/N_n(K, g; \mathbb{R}) \) has an induced inner product \( \langle \cdot , \cdot \rangle_g \). Since \( \mathcal{K}_g^\ast \) is a sub-simplicial complex of \( K \), \( C_n(\mathcal{K}_g^\ast; \mathbb{R}) \) is a sub-space of \( C_n(K; \mathbb{R}) \) hence it inherits a quadratic form \( \langle \cdot , \cdot \rangle_g \). Note that \( \langle \cdot , \cdot \rangle_g \) is strictly positive-definite on \( C_n(\mathcal{K}_g^\ast; \mathbb{R}) \) hence it is an inner product on \( C_n(\mathcal{K}_g^\ast; \mathbb{R}) \). For any simplices \( \sigma \) and \( \tau \) in \( \mathcal{K}_g^\ast \), it can be verified that \( \iota_n \) preserves the inner products:

\[
\langle \iota_n(\sigma), \iota_n(\tau) \rangle_g = \langle \sigma + N_n(K, g; \mathbb{R}), \tau + N_n(K, g; \mathbb{R}) \rangle_g = \langle \sigma + \mu, \tau + \nu \rangle_g = \langle \sigma, \tau \rangle_g + \langle \mu, \nu \rangle_g + \langle \mu, \nu \rangle_g = \langle \sigma, \tau \rangle_g.
\]

Here \( \mu \) and \( \nu \) are arbitrary elements in \( N_n(K, g; \mathbb{R}) \). Thus \( \iota_n \) is a linear isometry. \( \square \)

By Lemma 3.4, for each \( n \geq 0 \) we can define the \( f \)-weighted reduced boundary operator

\[
\bar{\partial}_n^f : C_n(K; \mathbb{R})/N_n(K, g; \mathbb{R}) \longrightarrow C_n(K; \mathbb{R})/N_n(K, g; \mathbb{R})
\] (3.9)

as

\[
\bar{\partial}_n^f = \iota_n \circ (\partial_n^f |_{\mathcal{K}_g^\ast}) \circ \iota_n^{-1}.
\] (3.10)

Here

\[
\partial_n^f |_{\mathcal{K}_g^\ast} : C_n(\mathcal{K}_g^\ast; \mathbb{R}) \longrightarrow C_{n-1}(\mathcal{K}_g^\ast; \mathbb{R})
\] (3.11)

is the \( n \)-th \( f \)-weighted boundary operator of \( \mathcal{K}_g^\ast \), which is obtained by restricting the \( f \)-weighted boundary operator \( \partial_n^f \) of \( C_n(K; \mathbb{R}) \) to the sub-chain complex \( C_n(\mathcal{K}_g^\ast; \mathbb{R}) \). Similar with (3.5), we use the following notations

\[
C_\ast(\mathcal{K}_g^\ast, f, g; \mathbb{R}) = \{ C_n(\mathcal{K}_g^\ast; \mathbb{R}), \partial_n^f, \langle \cdot , \cdot \rangle_g \}_{n \geq 0}
\]

and

\[
\tilde{C}_\ast(\mathcal{K}, f, g; \mathbb{R}) = \{ C_n(\mathcal{K}; \mathbb{R})/N_n(\mathcal{K}, g; \mathbb{R}), \bar{\partial}_n^f, \langle \cdot , \cdot \rangle_g \}_{n \geq 0}
\]

to denote the chain complexes with the \( f \)-weighted boundary operators \( \partial_n^f \) (or \( \bar{\partial}_n^f \)) and the \( g \)-weighted inner products \( \langle \cdot , \cdot \rangle_g \). We have the next proposition.
Proposition 3.5. Let \((K, f, g)\) be a vertex-weighted simplicial complex. Then we have an isomorphism of chain complexes

\[
\iota_\ast : \tilde{C}_\ast(K, f, g; \mathbb{R}) \rightarrow C_\ast(K^\times, f, g; \mathbb{R})
\]

where for each \(n \geq 0\), \(\iota_n\) is a linear isometry with respect to the corresponding inner products \(\langle \cdot, \cdot \rangle_g\).

Proof. The proof follows by the definition of \(\iota_\ast\) in Lemma 3.4. \(\qed\)

By taking the homology of the chain complexes, the isomorphism \(\iota_\ast\) induces an isomorphism of homology groups. The next corollary follows from Proposition 3.5 immediately.

Corollary 3.6. Let \((K, f, g)\) be a vertex-weighted simplicial complex. Then for each \(n \geq 0\), the weighted homology \(H_n(K^\times, f, g; \mathbb{R})\) is isometrically isomorphic to the homology \(H_n(\tilde{C}_\ast(K, f, g; \mathbb{R}))\) of the chain complex \(\tilde{C}_\ast(K, f, g; \mathbb{R})\) with respect to the inner products \(\langle \cdot, \cdot \rangle_g\). \(\qed\)

In the remaining part of this section, we study the effects of \(f\) and \(g\) on the weighted homology \(H_n(K, f, g; \mathbb{R})\). We prove the next lemma by using an argument similar with [17, Lemma 3.1].

Lemma 3.7. Let \(K\) be a simplicial complex with the set \(V\) of its vertices. Let \(f\) and \(g\) be any two real-valued functions on \(V\). Let \(h\) a non-vanishing real-valued function on \(V\). Then there is a canonical chain isomorphism

\[
\varphi : C_\ast(K, f, g; \mathbb{R}) \rightarrow C_\ast(K, fh, gh; \mathbb{R})
\]

given by

\[
\varphi\left(\sum x_{v_0,v_1,\ldots,v_n}\{v_0,v_1,\ldots,v_n\}\right) = \sum \frac{x_{v_0,v_1,\ldots,v_n}}{h(v_0)h(v_1)\cdots h(v_n)}\{v_0,v_1,\ldots,v_n\}
\]

such that

\[
\langle \varphi(a), \varphi(b) \rangle_{gh} = \langle a, b \rangle_g
\]

for any \(n\)-chains \(a\) and \(b\) in \(C_\ast(K, f, g; \mathbb{R})\).

Proof. Let \(n \geq 0\). We denote an \(n\)-simplex of \(K\) as \(\{v_0, v_1, \ldots, v_n\}\) where the vertices \(v_i\) are distinct for \(i = 0, 1, \ldots, n\) and \(v_i \prec v_j\) for \(0 \leq i < j \leq n\) with respect to the total order \(\prec\) on \(V\).

Firstly, we take

\[
\sum x_{v_0,v_1,\ldots,v_n}\{v_0,v_1,\ldots,v_n\}
\]

to be any \(n\)-chain in \(C_n(K; \mathbb{R})\). Then we have

\[
\partial^n_{fh} \circ \varphi\left(\sum x_{v_0,v_1,\ldots,v_n}\{v_0,v_1,\ldots,v_n\}\right)
= \partial^n_{fh}\left(\sum \frac{x_{v_0,v_1,\ldots,v_n}}{h(v_0)h(v_1)\cdots h(v_n)}\{v_0,v_1,\ldots,v_n\}\right)
= \sum \frac{x_{v_0,v_1,\ldots,v_n}}{h(v_0)h(v_1)\cdots h(v_n)} \sum_{i=1}^{n} (-1)^i f(v_i)h(v_i)\{v_0,\ldots,\hat{v}_i,\ldots,v_n\}
= \sum x_{v_0,v_1,\ldots,v_n} \sum_{i=0}^{n} (-1)^i f(v_i) \frac{\{v_0,\ldots,\hat{v}_i,\ldots,v_n\}}{h(v_0)\cdots h(v_i)\cdots h(v_n)}
= \varphi \circ \partial^n_{fh}\left(\sum x_{v_0,v_1,\ldots,v_n}\{v_0,v_1,\ldots,v_n\}\right).
\]
Thus $\varphi$ is a chain map.

Secondly, since $h$ is non-vanishing on $V$, it follows from (3.13) directly that $\varphi$ is a linear isomorphism from $C_n(\mathcal{K}; \mathbb{R})$ to itself.

Thirdly, let

$$\sum x_{\{v_0, v_1, \ldots, v_n\}} \{v_0, v_1, \ldots, v_n\}$$

and

$$\sum y_{\{u_0, u_1, \ldots, u_n\}} \{u_0, u_1, \ldots, u_n\}$$

be any two of the $n$-chains in $C_n(\mathcal{K}; \mathbb{R})$. Then

$$\langle \varphi(\sum x_{\{v_0, v_1, \ldots, v_n\}} \{v_0, v_1, \ldots, v_n\}), \varphi(\sum y_{\{u_0, u_1, \ldots, u_n\}} \{u_0, u_1, \ldots, u_n\}) \rangle_{gh}$$

= $$\sum \sum \frac{x_{\{v_0, v_1, \ldots, v_n\}}}{h(v_0)h(v_1) \cdots h(v_n)} \cdot \frac{y_{\{u_0, u_1, \ldots, u_n\}}}{h(u_0)h(u_1) \cdots h(u_n)} \langle \{v_0, v_1, \ldots, v_n\}, \{u_0, u_1, \ldots, u_n\} \rangle_{gh}$$

= $$\sum \sum \frac{x_{\{v_0, v_1, \ldots, v_n\}}}{h(v_0)h(v_1) \cdots h(v_n)} \cdot \frac{y_{\{u_0, u_1, \ldots, u_n\}}}{h(u_0)h(u_1) \cdots h(u_n)} \prod_{i=0}^{n} g(v_i)g(u_i)\delta(v_i, u_i)$$

= $$\langle \sum x_{\{v_0, v_1, \ldots, v_n\}} \{v_0, v_1, \ldots, v_n\}, \sum y_{\{u_0, u_1, \ldots, u_n\}} \{u_0, u_1, \ldots, u_n\} \rangle_{g}.$$ 

Thus $\varphi$ preserves the quadratic forms with respect to $\langle \ , \ \rangle_{gh}$ and $\langle \ , \ \rangle_{g}$.

Summarizing all the above three steps, we obtain the lemma.

The next corollary follows directly by applying Lemma 3.7 to the sub-simplicial complex $\mathcal{K}_g^\ast$.

**Corollary 3.8.** Let $\mathcal{K}$ be a simplicial complex with the set $V$ of its vertices. Let $f$ and $g$ be any two real-valued functions on $V$. Let $h$ a non-vanishing real-valued function on $V$. Then there is a canonical chain isomorphism

$$\varphi : C_\ast(\mathcal{K}_g^\ast, f, g; \mathbb{R}) \to C_\ast(\mathcal{K}_g^\ast, fh, gh; \mathbb{R})$$

(3.14)

given by (3.15) which is an isometry with respect to the inner products $\langle \ , \ \rangle_g$ and $\langle \ , \ \rangle_{gh}$ respectively.

**Proof.** Note that $g$ is non-vanishing on the set of the vertices of $\mathcal{K}_g^\ast$. Thus $\langle \ , \ \rangle_g$ is an inner product on $C_n(\mathcal{K}_g^\ast; \mathbb{R})$ for each $n \geq 0$. Moreover, since $h$ is non-vanishing, $\langle \ , \ \rangle_{gh}$ is an inner product on $C_n(\mathcal{K}_g^\ast; \mathbb{R})$ as well for each $n \geq 0$.

With the help of Lemma 3.7 we have the next theorem.

**Theorem 3.9.** Let $(\mathcal{K}, f, g)$ be a vertex-weighted simplicial complex where $f$ and $g$ are any real-valued functions on $V$. Let $h$ be a non-vanishing real function on $V$. Then for each $n \geq 0$, we have a linear isomorphism

$$\varphi_\ast : H_n(\mathcal{K}, f, g; \mathbb{R}) \to H_n(\mathcal{K}, fh, gh; \mathbb{R})$$

(3.15)

which preserves the quadratic forms with respect to $\langle \ , \ \rangle_g$ and $\langle \ , \ \rangle_{gh}$. Here $\varphi_\ast$ is induced by the chain isomorphism $\varphi$ in Lemma 3.7.

**Proof.** Since $\varphi$ in Lemma 3.7 is a chain isomorphism that preserves the quadratic forms, it follows with the help of Lemma 2.20 that $\varphi_\ast$ induces a linear isomorphism $\varphi_\ast$ from the $n$-th $(f, g)$-weighted homology group $H_n(\mathcal{K}, f, g; \mathbb{R})$ to the $n$-th $(fh, gh)$-weighted homology group $H_n(\mathcal{K}, fh, gh; \mathbb{R})$ that preserves the quadratic forms on the quotient spaces. The theorem follows.
Remark 1: Particularly, if both $f$ and $g$ are non-vanishing on $V$, then Theorem 3.9 implies that the $(f, g)$-weighted homology $H_*(\mathcal{K}, f, g; \mathbb{R})$ (as graded vector space with quadratic forms $\langle \ , \rangle_g$) only depends on the ratio function $f/g$ (or $g/f$ equivalently).

Similar with Theorem 3.9, the next corollary follows from Corollary 3.8.

**Corollary 3.10.** Let $(\mathcal{K}, f, g)$ be a vertex-weighted simplicial complex. Then for each $n \geq 0$ and each non-vanishing function $h$ on the set of the vertices, we have a linear isometry

$$\varphi_* : H_n(\mathcal{K}_f^\times, f, gh; \mathbb{R}) \longrightarrow H_n(\mathcal{K}_g^\times, fh, gh; \mathbb{R}).$$

(3.16)

In particular, taking $f = g$ and $h = 1/f$, we have for each $n \geq 0$,

$$H_n(\mathcal{K}_f^\times, f, f; \mathbb{R}) \cong H_n(\mathcal{K}_{(f)}^\times, \mathbb{R})$$

as Euclidean spaces.

\[ \square \]

## 4 Weighted Analytic Torsions for Weighted Simplicial Complexes

Throughout this section, we let $(\mathcal{K}, f, g)$ be a vertex-weighted simplicial complex where $\mathcal{K}$ is a simplicial complex and $f$ and $g$ are arbitrary real-valued functions on the set $V$ of the vertices of $\mathcal{K}$.

For each $n \geq 0$, we let $\langle \ , \rangle_g$ be the inner product on $C_n(\mathcal{K}; \mathbb{R})/N_n(\mathcal{K}, g; \mathbb{R})$ induced by $g$. Let

$$(\delta_n^f)_g^* : C_{n-1}(\mathcal{K}; \mathbb{R})/N_n(\mathcal{K}, g; \mathbb{R}) \longrightarrow C_n(\mathcal{K}; \mathbb{R})/N_n(\mathcal{K}, g; \mathbb{R})$$

be the adjoint linear map of $\delta_n^f$ (which is given in (3.9) and (3.10)) with respect to the inner product $\langle \ , \rangle_g$ such that

$$\langle \delta_n^f(a), b \rangle_g = \langle a, (\delta_n^f)_g^*(b) \rangle_g$$

for any $n$-chains $a$ and $b$ in $\tilde{C}_n(\mathcal{K}, f, g; \mathbb{R})$. The $n$-th $(f, g)$-weighted (reduced) Hodge-Laplace operator (on the quotient chain complex) is the linear map

$$\tilde{\Delta}_n^{f,g} : \tilde{C}_n(\mathcal{K}; \mathbb{R}) \longrightarrow \tilde{C}_n(\mathcal{K}; \mathbb{R})$$

given by

$$\tilde{\Delta}_n^{f,g}(c) = (\delta_n^f)_g^* \delta_n^f(c) + (\delta_{n+1}^f)_g^* \delta_{n+1}^f(c)$$

for any $n$-chains $c$ in $\tilde{C}_n(\mathcal{K}, f, g; \mathbb{R})$.

On the other hand, for each $n \geq 0$, we also use $\langle \ , \rangle_g$ to denote the inner product on $C_n(\mathcal{K}_g^\times; \mathbb{R})$ with respect to $g$. Let

$$(\delta_n^f |_{\mathcal{K}_g^\times})_g^* : C_{n-1}(\mathcal{K}_g^\times; \mathbb{R}) \longrightarrow C_n(\mathcal{K}_g^\times; \mathbb{R})$$

be the adjoint linear map of $\delta_n^f |_{\mathcal{K}_g^\times}$ (which is given in (3.11)) with respect to the inner products $\langle \ , \rangle_g$ such that

$$\langle \delta_n^f(a), b \rangle_g = \langle a, (\delta_n^f |_{\mathcal{K}_g^\times})_g^*(b) \rangle_g$$
for any $n$-chains $a$ and $b$ in $C_*(K^X_\phi, f, g; \mathbb{R})$. The $n$-th $(f, g)$-weighted Hodge-Laplace operator of $K^X_\phi$ is the linear map

$$\Delta_n^{f, g} : C_n(K^X_\phi; \mathbb{R}) \rightarrow C_n(K^X_\phi; \mathbb{R})$$

given by

$$\Delta_n^{f, g}(c) = (\partial_n^f |_{K^X_\phi})_* \partial_n^f(c) + (\partial_n^g |_{K^X_\phi})_* \partial_n^g(c)$$

for any $n$-chains $c$ in $C_*(K^X_\phi, f, g; \mathbb{R})$.

The reduced Hodge-Laplace operator $\tilde{\Delta}_n^{f, g}$ and the Hodge Laplace operator $\Delta_n^{f, g}$ can be identified via the isomorphism $\iota_n$. In fact, it follows from Proposition 3.5 and [17, Lemma 2.2] that for each $n \geq 0$,

$$\Delta_n^{f, g} = \iota_n \circ \tilde{\Delta}_n^{f, g} \circ (\iota_n)^{-1}.$$  

Since $\iota_n$ is an isometry, it follows that counted with multiplicities, the multi-set of the eigenvalues of $\Delta_n^{f, g}$ and the multi-set of the eigenvalues of $\tilde{\Delta}_n^{f, g}$ are equal. For simplicity, we do not distinguish $\Delta_n^{f, g}$ and $\tilde{\Delta}_n^{f, g}$. We define the $n$-th $(f, g)$-weighted Hodge-Laplace operator of the vertex-weighted simplicial complex $(K, f, g)$ as $\Delta_n^{f, g}$.

We observe that $\Delta_n^{f, g}$ is symmetric and semi-positive definite. Hence its matrix representatives are diagonalizable and its eigenvalues are non-negative. We use $\lambda$ to denote any eigenvalue of $\Delta_n^{f, g}$. The zeta function $\zeta_n(s)$ is defined by

$$\zeta_n(s) = \sum_{\lambda > 0} \frac{1}{\lambda^s}.$$  

The $(f, g)$-weighted analytic torsion of $(K, f, g)$ is given by

$$\log T(K, f, g) = \frac{1}{2} \sum_{n=0}^{N} (-1)^n \zeta'(0), \quad (4.1)$$

where $N$ is the dimension of $K$. The next proposition follows from Lemma 3.7 immediately.

**Proposition 4.1.** For any simplicial complex $K$, any real-valued weights $f$ and $g$ on the set $V$, and any non-vanishing real function $h$ on $V$, we have

$$T(K, hf, hg) = T(K, f, g).$$

**Proof.** By Lemma 3.7 and [17, Lemma 2.2], we have

$$\Delta_n^{f, g} = \varphi^{-1} \circ \Delta_n^{hf, hg} \circ \varphi$$

which implies that all the eigenvalues (counted with multiplicities) of $\Delta_n^{f, g}$ for $K^{hf, hg}$ are equal. By the definition (4.1) of the weighted analytic torsions, the proposition follows.

We fix $f$ and multiply $g$ by a non-zero scalar $c$. By applying [4, Corollary 3.8] to the $R$-torsion, we have

**Proposition 4.2.** Let $K$ be a simplicial complex of dimension $N$ with the set of its vertices $V$. Let $f$ and $g$ be two real-valued weights on $V$ such that for any $v \in V$, $g(v) \neq 0$ implies $f(v) \neq 0$. Let $c \neq 0$ be a real constant. Then

$$T(K, f, cg) = t(K, c, \epsilon(g)) T(K, f, g) \quad (4.2)$$

where $t(K, c, \epsilon(g)) = |c|^{s(K, \epsilon(g))}$ and

$$s(K, \epsilon(g)) = \sum_{n=0}^{N} (-1)^n \dim \partial_n C_n(K^X_{\epsilon(g)}; \mathbb{R}).$$
Proof. Since for any \( v \in V, g(v) \neq 0 \) implies \( f(v) \neq 0 \), we have that both \( f \) and \( g \) are non-vanishing on the set of the vertices of \( K^g_\ast \). Hence the ratio function \( g/f \) is well-defined on \( K^g_\ast \). Consequently, with the help of Theorem 3.9, \( H_n(K^g_\ast, f, g; \mathbb{R}) \) is linearly isometric to \( H_n(K^R_\ast, 1, g/f; \mathbb{R}) \), which is linearly isomorphic to \( H_n(K^R_\ast; \mathbb{R}) \). For any \( n \geq 0 \) and any \( a, b \in C_n(K^R_\ast; \mathbb{R}) \), we have
\[
\langle a, b \rangle_{cg} = c^{d(n+1)} \langle a, b \rangle_g.
\]
Thus by [4, Corollary 3.8], we have \([4],[12]\) where
\[
t(K, c, \epsilon(g)) = \prod_{n=0}^{N} \langle c \rangle^{(-1)^n(n+1)} \left( \dim C_n(K^R_\ast; \mathbb{R}) - \dim H_n(K^R_\ast; f, g; \mathbb{R}) \right)
\]
\[
= \langle c \rangle^{\sum_{n=0}^{N}(-1)^n(n+1)} \left( \dim C_n(K^R_\ast; \mathbb{R}) - \dim H_n(K^R_\ast; \mathbb{R}) \right)
\]
\[
= \langle c \rangle^{\sum_{n=0}^{N}(-1)^n n} \left( \dim C_n(K^R_\ast; \mathbb{R}) - \dim H_n(K^R_\ast; \mathbb{R}) \right).
\]
(4.3)
The last equality of (4.3) follows from the formulas for Euler characteristic number
\[
\chi(K^R_\ast) = \sum_{n=0}^{N} (-1)^n \dim C_n(K^R_\ast; \mathbb{R}) = \sum_{n=0}^{N} (-1)^n \dim H_n(K^R_\ast; \mathbb{R}).
\]
Therefore, with the help of [17, Lemma 2.1], the right-hand side of the last equality in (4.3) equals to \( |c|^{s(K,c(\epsilon))} \).

We fix \( g \) and multiply \( f \) by a non-zero scalar \( c \). By a straight-forward calculation of the analytic torsion, we have

**Proposition 4.3.** Let \( K \) be a simplicial complex of dimension \( N \) with the set of its vertices \( V \). Let \( f \) and \( g \) be a real-valued weights on \( V \) such that for any \( v \in V, g(v) \neq 0 \) implies \( f(v) \neq 0 \). Let \( c \neq 0 \) be a real constant. Then
\[
T(K, cf, g) = t(K, c, \epsilon(g))^{-1} T(K, f, g).
\]
(4.4)

**Proof.** The proof is an analogue of the proof for [17] Proposition 3.7. Let \( n \geq 0 \). It follows directly that
\[
\partial_n^{cf} = c\partial_n^f, \quad (\partial_{n+1}^{cf} \mid K^g_\ast)^* = c(\partial_{n+1}^f \mid K^g_\ast)^*.
\]
Thus we have
\[
\Delta_{n}^{cf,g} = c^2 \Delta_{n}^{f,g}.
\]
Consequently, with the help of [4] (3.21),
\[
\log T(K, cf, g) = \frac{1}{2} \sum_{n=0}^{N} (-1)^n n \left( -\sum_{\lambda_i > 0} \log(\lambda_i) \right)
\]
\[
- \frac{1}{2} \sum_{n=0}^{N} \langle c \rangle^{(-1)^n n (2 \log |c|) \left( \dim C_n(K^R_\ast; \mathbb{R}) - \dim H_n(K^R_\ast; \mathbb{R}) \right)}
\]
\[
= \frac{1}{2} \sum_{n=0}^{N} \langle c \rangle^{(-1)^n n} - \left( \sum_{\lambda_i > 0} \frac{1}{\lambda_i^2} \right) - (\log |c|) s(K, \epsilon(g))
\]
\[
= \log T(K, f, g) - (\log |c|) s(K, \epsilon(g)).
\]
Taking the exponential map on both sides of the equations, we have \([4],[3]\).
Finally, sumarizing Proposition 4.2 and Proposition 4.3 we obtain the proof of Theorem 1.1.

**Proof of Theorem 1.1.** Given a weighted simplicial complex \((K, f, g)\), We consider the new weighted simplicial complex \((K, f', g')\) where \(f' = \epsilon(g)f\) and \(g' = \epsilon(f)g\). Then we have

\[
\epsilon(f') = \epsilon(g') = \epsilon(f)\epsilon(g).
\]

Applying Proposition 4.1 to \((K, f'g')\), we obtain Theorem 1.1 (i). Applying Proposition 4.2 and Proposition 4.3 to \((K, f', g')\), we obtain Theorem 1.1 (ii).

For any chain complex \(\{C_n\}_{n=0}^{N}\) where each \(C_n\) has an inner product, the analytic torsion can be obtained by calculating the R-torsion (cf. [4, Section 3]). A complete construction of the R-torsion has been given in [4, Section 3] and reviewed in [17, Section 2] hence it will not be restated here. Particularly, our weighted analytic torsions for weighted simplicial complexes can be obtained by calculating the corresponding R-torsions. In the remaining part of this paper, we give some examples of weighted analytic torsions for weighted simplicial complexes by calculating the R-torsions. The calculations here are similar with [17, Section 4].

**Example 4.4.** Let \(V = \{v_0, v_1, v_2, v_3\}\). Let \(f\) and \(g\) be two real-valued non-vanishing functions on \(V\) such that (i). \(f\) is non-vanishing on \(V\); (ii). \(g(v_0), g(v_1), g(v_2) \neq 0\) and \(g(v_3) = 0\). In the following, we use \(v_0\) to denote a 0-simplex \(\{v_0\}\), use \(v_0v_1\) to denote a 1-simplex \(\{v_0, v_1\}\), etc.

(1). Let

\[
K = \{v_0, v_1, v_2, v_0v_1, v_1v_2, v_0v_2, v_0v_1v_2, v_0v_3, v_1v_3, v_2v_3\}
\]

Then

\[
K_{\epsilon(f), \epsilon(g)} = K_\epsilon = \{v_0, v_1, v_2, v_0v_1, v_1v_2, v_0v_2, v_0v_1v_2\}
\]

is a solid triangle. Similar with the calculation in [4, Example 3.10] and [17, Example 4.2], we have

\[
T(K, f, g) = T(K_\epsilon, f, g) = \sqrt{\sum_{i=0}^{2} \frac{f(v_i)^2}{g(v_i)^2}}
\]

(2). Let

\[
\mathcal{L} = \{v_0, v_1, v_2, v_0v_1, v_1v_2, v_0v_2, v_0v_3, v_1v_3, v_2v_3\}.
\]

Then

\[
\mathcal{L}_{\epsilon(f), \epsilon(g)} = \mathcal{L}_\epsilon = \{v_0, v_1, v_2, v_0v_1, v_1v_2, v_0v_2\}.
\]

We choose the following \((\cdot, \cdot)\)\_\(g\)-orthonormal bases

\[
\omega_0 = \left\{ \frac{v_0}{g(v_0)}, \frac{v_1}{g(v_1)}, \frac{v_2}{g(v_2)} \right\}
\]

and

\[
\omega_1 = \left\{ \frac{v_0v_1}{g(v_0)g(v_1)}, \frac{v_1v_2}{g(v_1)g(v_2)}, \frac{v_0v_2}{g(v_0)g(v_2)} \right\}
\]
in $C_1(L_g^\times;\mathbb{R})$. Choose also the basis

$$b_0 = \left\{ \frac{f(v_0)v_1 - f(v_1)v_0}{g(v_0)g(v_1)}, \frac{f(v_1)v_2 - f(v_2)v_1}{g(v_1)g(v_2)} \right\}$$

in $\partial_1^0C_1(L_g^\times;\mathbb{R})$. Then its lift is

$$\tilde{b}_1 = \left\{ \frac{v_0v_1}{g(v_0)g(v_1)}, \frac{v_1v_2}{g(v_1)g(v_2)} \right\}.$$ 

Taking the orthogonal complement of $\partial_1^0C_1(L_g^\times;\mathbb{R})$ in $\text{Ker}\partial_0^1$ with respect to $\langle \cdot, \cdot \rangle_g$, we have

$$h_0 = \left\{ \frac{1}{\sum_{i=0}^2 (f(v_i))^2} \left( \frac{f(v_0)}{g(v_0)}^2 v_0 + \frac{f(v_1)}{g(v_1)}^2 v_1 + \frac{f(v_2)}{g(v_2)}^2 v_2 \right) \right\}.$$ 

Moreover, note that since $C_2(L_g^\times;\mathbb{R}) = 0$, we have $\partial_2 C_2(L_g^\times;\mathbb{R}) = 0$, which implies $b_1 = 0$. Hence

$$h_1 = \left\{ \frac{f(v_0)v_1v_2 - f(v_1)v_0v_2 + f(v_2)v_0v_1}{g(v_0)g(v_1)g(v_2)} \right\}.$$ 

Thus

$$[h_0, h_0, \tilde{b}_0/\omega_0] = \left[ \frac{f(v_1)}{g(v_1)} \right] \left[ \sum_{i=0}^2 \frac{f(v_i)^2}{g(v_i)^2} \right]$$

and

$$[h_1, h_1, \tilde{b}_1/\omega_1] = \left[ \frac{f(v_1)}{g(v_1)} \right].$$

Therefore,

$$T(L, f, g) = T(L_g^\times, f, g) = \left[ \sum_{i=0}^2 \frac{f(v_i)^2}{g(v_i)^2} \right]^{1/2}.$$

**Example 4.5.** Let $V = \{v_0, v_1, v_2, v_3, v_4\}$. Suppose both $f$ and $g$ are non-vanishing functions on $V$.

(1) Let

$$K = \{v_0, v_1, v_2, v_3, v_4, v_0v_1, v_1v_2, v_2v_3, v_3v_4\}.$$ 

Then similar with the calculation in [4 Example 3.9] and [17 Example 4.1], we have

$$T(K, f, g) = \left[ \frac{1}{\sum_{i=0}^3 (f(v_i))^2} \left( \sum_{k=0}^3 \frac{f(v_k)}{g(v_k)} \prod_{0 \leq j < k} \frac{f(v_{j+1})}{g(v_{j+1})} \prod_{k \leq j \leq 2} \frac{f(v_j)}{g(v_j)} \right) \right]^{1/2}.$$ 

(2) Let

$$L = \{0, 1, 2, 3, 01, 12, 23, 02, 03, 13\}.$$ 

We choose the following $\langle \cdot, \cdot \rangle_g$-orthonormal bases

$$\omega_0 = \left\{ \frac{v_0}{g(v_0)}, \frac{v_1}{g(v_1)}, \frac{v_2}{g(v_2)}, \frac{v_3}{g(v_3)} \right\}.$$
We choose the \( \langle \ , \rangle _{g} \)-orthonormal bases

\[
\omega_0 = \left\{ \frac{v_i}{g(v_i)} \mid 0 \leq i \leq 4 \right\}
\]

in \( C_0(\mathcal{L}; \mathbb{R}) \) and

\[
\omega_1 = \left\{ \frac{v_0 v_1}{g(v_0)g(v_1)} , \frac{v_1 v_2}{g(v_1)g(v_2)} , \frac{v_2 v_3}{g(v_2)g(v_3)} , \frac{v_0 v_2}{g(v_0)g(v_2)} , \frac{v_0 v_3}{g(v_0)g(v_3)} , \frac{v_1 v_3}{g(v_1)g(v_3)} \right\}
\]

in \( C_1(\mathcal{L}; \mathbb{R}) \). Choose also the basis

\[
b_0 = \left\{ \frac{f(v_0)v_1 - f(v_1)v_0}{g(v_0)g(v_1)} , \frac{f(v_1)v_2 - f(v_2)v_1}{g(v_1)g(v_2)} , \frac{f(v_2)v_3 - f(v_3)v_2}{g(v_2)g(v_3)} \right\}
\]

in \( \partial_{1}^{g} C_1(\mathcal{L}; \mathbb{R}) \) and set its lift

\[
\tilde{b}_1 = \left\{ \frac{\sqrt{3} v_0 v_1}{g(v_0)g(v_1)} , \frac{\sqrt{3} v_1 v_2}{g(v_1)g(v_2)} , \frac{\sqrt{3} v_2 v_3}{g(v_2)g(v_3)} \right\}.
\]

Moreover, note that \( \tilde{b}_0 = 0 \). Taking the orthogonal complement of \( \partial_{1}^{g} C_1(\mathcal{L}; \mathbb{R}) \) in \( \text{Ker} \partial_{0}^{g} = C_0(\mathcal{L}; \mathbb{R}) \) with respect to \( \langle \ , \rangle _{g} \), we have

\[
h_0 = \left\{ \frac{\sum_{i=0}^{3} f(v_i)v_i}{\sqrt{\sum_{i=0}^{3} f(v_i)^2}} \right\}.
\]

Note that \( b_1 = 0 \). Thus

\[
h_1 = \left\{ \frac{v_0 v_2}{g(v_0)g(v_2)} , \frac{v_0 v_3}{g(v_0)g(v_3)} , \frac{v_1 v_3}{g(v_1)g(v_3)} \right\}.
\]

Thus

\[
[b_0, h_0, \tilde{b}_0/\omega_0] = \frac{1}{\sqrt{\sum_{i=0}^{3} f(v_i)^2}} \left| \sum_{k=0}^{3} \frac{f(v_k)}{g(v_k)} \prod_{0 \leq j \leq k-1} \frac{f(v_{j+1})}{g(v_{j+1})} \prod_{k \leq j \leq 2} \frac{f(v_j)}{g(v_j)} \right|
\]

and

\[
[b_1, h_1, \tilde{b}_1/\omega_1] = 1.
\]

Therefore,

\[
T(\mathcal{L}, f, g) = \frac{1}{\sqrt{\sum_{i=0}^{3} f(v_i)^2}} \left| \sum_{k=0}^{3} \frac{f(v_k)}{g(v_k)} \prod_{0 \leq j \leq k-1} \frac{f(v_{j+1})}{g(v_{j+1})} \prod_{k \leq j \leq 2} \frac{f(v_j)}{g(v_j)} \right|
\]

(3). Let

\[
\mathcal{M} = \{ v_0, v_1, v_2, v_3, v_4, v_0 v_1, v_0 v_2, v_0 v_3, v_0 v_4 \}.
\]

We choose the \( \langle \ , \rangle _{g} \)-orthonormal bases

\[
\omega_0 = \left\{ \frac{v_i}{g(v_i)} \mid 0 \leq i \leq 4 \right\}
\]

in \( C_0(\mathcal{L}; \mathbb{R}) \) and

\[
\omega_1 = \left\{ \frac{v_0 v_i}{g(v_0)g(v_i)} \mid 1 \leq i \leq 4 \right\}
\]

in \( C_1(\mathcal{M}; \mathbb{R}) \). Let

\[
b_0 = \left\{ \frac{f(v_0)v_i - f(v_i)v_0}{g(v_0)g(v_i)} \mid 1 \leq i \leq 4 \right\}.
\]
be a basis in $\partial_1^f (C_1(\mathcal{M}; \mathbb{R}))$. The lift of $b_0$ in $C_1(\mathcal{M}; \mathbb{R})$ is
\[ \tilde{b}_1 = \omega_1. \]

By taking the $\langle , \rangle_g$-orthogonal complement of $\partial_1^f (C_1(\mathcal{M}; \mathbb{R}))$ in $C_0(\mathcal{L}; \mathbb{R})$, we have
\[
\tilde{b}_0 = \frac{1}{\sqrt{\sum_{i=0}^{4} f(v_i)^2 g(v_i)^2}} \left( \sum_{i=0}^{4} \frac{f(v_i)}{g(v_i)} v_i \right).
\]

Moreover,
\[
\tilde{b}_0 = 0, \quad b_1 = 0, \quad h_1 = 0.
\]

It follows by a calculation similar with [4, Example 3.9] and [17, Example 4.1] that
\[
[b_0, h_0, \tilde{b}_0/\omega_0] = \frac{f(v_0)^3}{g(v_0)^3} \sqrt{\sum_{i=0}^{4} \frac{f(v_i)^2}{g(v_i)^2}}
\]

and
\[
[b_1, h_1, \tilde{b}_1/\omega_1] = 1.
\]

Consequently,
\[
T(\mathcal{M}, f, g) = \frac{f(v_0)^3}{g(v_0)^3} \sqrt{\sum_{i=0}^{4} \frac{f(v_i)^2}{g(v_i)^2}}.
\]

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References

[1] J. Cheeger, Analytic torsion and the heat equation, *Annals of Mathematics*, 109, 259-322, 1979.

[2] Robert. J. MacG. Dawson, Homology of weighted simplicial complexes, *Cahiers de Topologie et Géométrie Différentielle Catégoriques*, 31 no. 3, 229-243, 1990.

[3] David Fried, Analytic torsion and closed geodesics on hyperbolic manifolds, *Inventiones Mathematicae*, 84, 523-540, 1986.

[4] Alexander Grigor’yan, Yong Lin, Shing-Tung Yau, Torsion of digraphs and path complexes, arXiv: 2012.07302v1, 2020.

[5] Alexander Grigor’yan, Yong Lin, Yuri Muranov, Shing-Tung Yau, Homologies of path complexes and digraphs, arXiv: 1207.2834, 2013.

[6] Alexander Grigor’yan, Yong Lin, Yuri Muranov, Shing-Tung Yau, Homotopy theory for digraphs, *Pure and Applied Mathematics Quarterly*, 10 (4), 619-674, 2014.

[7] Alexander Grigor’yan, Yong Lin, Yuri Muranov, Shing-Tung Yau, Cohomology of digraphs and (undirected) graph, *Asian Journal of Mathematics*, 15 (5), 887-932, 2015.
[8] Alexander Grigor’yan, Yuri Muranov, Shing-Tung Yau, Homologies of digraphs and Künneth formulas, *Communications in Analysis and Geometry*, 25, 969-1018, 2017.

[9] Alexander Grigor’yan, Yuri Muranov, Shing-Tung Yau, Path complexes and their homologies, *Journal of Mathematical Sciences*, 248 (5), 564-599, 2020.

[10] Zhenyu Meng, D Vijay Anand, Yunpeng Lu, Jie Wu, Kelin Xia, Weighted persistent homology for biomolecular data analysis, *Scientific Reports*, 10, no. 2079, 2020.

[11] John Milnor, Whitehead torsion, *Bulletin of the American Mathematical Society*, 72, 358-426, 1966.

[12] Werner Müller, Analytic torsion and $R$-torsion of Riemannian manifolds, *Advance in Mathematics*, 28, 233-305, 1978.

[13] D. B. Ray, I. Singer, R-torsion and Laplacian on Riemannian manifolds, *Advance in Mathematics*, 7, 145-210, 1971.

[14] K. Reidemeister, Homotopieringe and Linsenräume, *Hamburger Abhaufl*, 11, 102-109, 1935.

[15] Shiquan Ren, Chengyuan Wu, Jie Wu, Weighted persistent homology, *Rocky Mountain Journal of Mathematics*, 48, no. 8, 2661-2687, 2018.

[16] Shiquan Ren, Chengyuan Wu, Jie Wu, Computational tools in weighted persistent homology, *Chinese Annals of Mathematics Series B*, 42, no. 2, 237-258, 2021.

[17] Shiquan Ren, Chong Wang, Weighted analytic torsions for weighted digraphs, arXiv: 2103.09552, 2021.

[18] Chengyuan Wu, Shiquan Ren, Jie Wu, Kelin Xia, Discrete Morse theory for weighted simplicial complexes, *Topology and Its Applications*, 270, no. 107038, 2020.

[19] Chengyuan Wu, Shiquan Ren, Jie Wu, Kelin Xia, Weighted (co)homology and weighted Laplacian, arXiv: 1804.06990.

[20] Chengyuan Wu, Shiquan Ren, Jie Wu, Kelin Xia, Weighted fundamental group, *Bulletin of the Malaysian Mathematical Sciences Society*, 43, 4065-4088, 2020.

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