AUTOMORPHISMS OF GRAPHS CORRESPONDING TO CONJUGACY CLASSES OF FINITE-RANK SELF-ADJOINT OPERATORS

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Abstract. We consider the graph whose vertex set is a conjugacy class $\mathcal{C}$ consisting of finite-rank self-adjoint operators on a complex Hilbert space $H$. The dimension of $H$ is assumed to be not less than 3. In the case when operators from $\mathcal{C}$ have two eigenvalues only, we obtain the Grassmann graph formed by $k$-dimensional subspaces of $H$, where $k$ is the smallest dimension of eigenspaces. Classical Chow’s theorem describes automorphisms of this graph for $k > 1$. Under the assumption that operators from $\mathcal{C}$ have more than two eigenvalues we show that every automorphism of the graph is induced by a unitary or anti-unitary operator up to a permutation of eigenspaces with the same dimensions. In contrast to this result, Chow’s theorem states that there are graph automorphisms induced by semilinear automorphisms not preserving orthogonality if $\mathcal{C}$ is formed by operators with precisely two eigenvalues.

1. Introduction

The Grassmann graph is the simple graph whose vertices are $m$-dimensional subspaces of a vector space $V$ (over a field) and two such subspaces are adjacent (connected by an edge) if their intersection is $(m-1)$-dimensional. Chow’s theorem describes automorphisms of this graph under the assumption that $1 < m < \dim V - 1$ (in the two remaining cases the Grassmann graph is a complete graph, and thus every bijective transformation of the vertex set is a graph automorphism). The theorem states that every automorphism is induced by a semilinear automorphism of $V$ or, only when $\dim V = 2m$, by a semilinear isomorphism of $V$ to the dual vector space $V^*$. The original version of Chow’s theorem concerns Grassmann graphs of finite-dimensional vector spaces, but the same holds for the infinite-dimensional case (see, for example, Chapter 2).

The Grassmannian of $m$-dimensional subspaces of a complex Hilbert space $H$ can be naturally identified with the conjugacy class of rank-$m$ projections. Note that projections can be characterised as self-adjoint idempotents in the algebra of bounded operators and play an important role in operator theory and mathematical foundations of quantum mechanics. By Gleason’s theorem, rank-one projections correspond to pure states of quantum mechanical systems. Classical Wigner’s theorem describes symmetries of the space of pure states, i.e. the conjugacy class of rank-one projections. Molnár extended this result on other conjugacy classes of finite-rank projections.

Two $m$-dimensional subspaces of $H$ are adjacent if and only if the difference of the corresponding projections is an operator of rank 2, i.e. of the smallest possible rank (the rank of the difference of two finite-rank self-adjoint operators from the same conjugacy class cannot be equal to 1). Chow’s theorem reformulated in terms of...
projections was successfully applied to prove some Wigner-type theorems [2, 8, 7, 11] (see [3] for a detailed description of the topic).

In [9], the authors extend the adjacency relation, described above, from conjugacy classes of projections to conjugacy classes of finite-rank self-adjoint operators. By the spectral theorem (its finite-dimensional version) every finite-rank self-adjoint operator on $H$ is a real linear combination of finite-rank projections whose images are mutually orthogonal (these images are the eigenspaces of the operator). A conjugacy class of such operators is completely determined by the spectrum and the dimensions of the eigenspaces. If the spectrum consists of $k$ eigenvalues, then the associated graph has $\binom{k}{2}$ different types of adjacency corresponding to pairs of eigenvalues; in particular, we obtain a Grassmann graph if there are two eigenvalues only. Let us have a look at the case of more than two eigenvalues. Some combinatorial properties of the graph (for example, connectedness and cliques) are investigated in [9], the main result concerns graph automorphisms under the assumption that the dimensions of all eigenspaces are greater than 1 (in Section 4 we explain why reasonings from [9] do not work for the general case). In the present paper, we determine graph automorphisms without any assumption. Using a new approach based on Johnson graphs we show that every automorphism is induced by a unitary or anti-unitary operator up to a permutation of eigenspaces with the same dimensions. For Grassmann graphs (the case of two eigenvalues) the above statement fails, as there are graph automorphisms induced by semilinear automorphisms of $H$ that do not preserve orthogonality.

2. Result

Let $H$ be a complex Hilbert space of dimension not less than 3. The projection on a closed subspace $X \subset H$ will be denoted by $P_X$. Two operators $A$ and $B$ on $H$ are (unitary) conjugate if there is a unitary operator $U$ on $H$ such that $B = UAU^*$. A conjugacy class of finite-rank self-adjoint operators on $H$ is completely determined by the spectrum $\sigma = \{\lambda_i\}_{i \in I}$ of operators from this class and the family $d = \{n_i\}_{i \in I}$, where $n_i$ is the dimension of eigenspaces corresponding to the eigenvalue $\lambda_i$. This conjugacy class will be denoted by $\mathcal{G}(\sigma, d)$. Note that the index set $I$ is finite, each $\lambda_i$ is real and at most one of $n_i$ is infinite. If $n_i$ is infinite, then $\lambda_i = 0$, i.e. the corresponding eigenspaces are the kernels of operators from $\mathcal{G}(\sigma, d)$. For every $A \in \mathcal{G}(\sigma, d)$ we have

$$A = \sum_{i \in I} a_i P_{X_i},$$

where $\{X_i\}_{i \in I}$ is a collection of mutually orthogonal subspaces whose sum coincides with $H$ and $\dim X_i = n_i$ for all $i \in I$.

Operators $A, B \in \mathcal{G}(\sigma, d)$ are said to be adjacent if the following conditions are satisfied:

(A1) the rank of $B - A$ is equal to 2;

(A2) $\text{Im}(B - A)$ and $\text{Ker}(B - A)$ are invariant to both $A$ and $B$.

The condition (A1) implies (A2) if $|I| = 2$ [8, Example 2]. If $|I| \geq 3$, then there are pairs $A, B \in \mathcal{G}(\sigma, d)$ satisfying (A1) and such that (A2) fails [9, Examples 4, 5].

There is a geometric interpretation of operator adjacency. Let $X, Y \subset H$ be closed subspaces of the same finite dimension or the same finite codimension. We say that $X$ and $Y$ are adjacent if $X \cap Y$ is a hyperplane in both $X$ and $Y$, or, equivalently, $X^\perp, Y^\perp$ are hyperplanes in $X + Y$. Observe that $X, Y$ are adjacent if and only if $X^\perp, Y^\perp$ are adjacent. In the case when $I = \{1, 2\}$, operators $A, B \in \mathcal{G}(\sigma, d)$ are adjacent ($A - B$ is of rank 2) if and only if the eigenspaces of $A, B$ corresponding to a certain $\lambda_i$ are adjacent; the latter immediately implies that the eigenspaces corresponding to $\lambda_3 - \lambda_1$ also are adjacent. Consider the general case. Let $X_i$ and $Y_i$
be the eigenspaces of $A$ and $B$ (respectively) corresponding to $a_i$. The conditions (A1) and (A2) hold if and only if there are distinct $i, j \in I$ such that the following assertions are fulfilled:

- $X_i$ and $X_j$ are adjacent to $Y_i$ and $Y_j$, respectively;
- $X_t = Y_t$ for all $t \in I \setminus \{i, j\}$.

In this case, we say that the operators $A, B$ are $(i, j)$-adjacent.

Let $\Gamma(\sigma, d)$ be the simple graph whose vertex set is $G(\sigma, d)$ and two operators are connected by an edge in this graph if they are adjacent. This graph is connected [9, Theorem 2]. If $\sigma' = \{a'_i\}_{i \in I}$ is a collection of mutually distinct real numbers such that $a'_i = 0$ if $n_i$ is infinite, then the map

$$\sum_{i \in I} a_i P_{X_i} \rightarrow \sum_{i \in I} a'_i P_{X_i}$$

is an isomorphism between $\Gamma(\sigma, d)$ and $\Gamma(\sigma', d)$.

Note that when $I = \{1, 2\}$ and $n_1$ is finite, $\Gamma(\sigma, d)$ is isomorphic to the Grassmann graph of $n_1$-dimensional subspaces of $H$. If $n_2$ is finite, then $\Gamma(\sigma, d)$ also is isomorphic to the Grassmann graph formed by $n_2$-dimensional subspaces of $H$.

Let $U$ be a unitary or anti-unitary operator on $H$. Consider the bijective transformation sending every $A \in G(\sigma, d)$ to $UAU^*$. If $A = \sum_{i \in I} a_i P_{X_i}$, then

$$UAU^* = \sum_{i \in I} a_i P_{U(X_i)}$$

which shows that this transformation is an automorphism of the graph $\Gamma(\sigma, d)$.

Let $S(d)$ be the group of all permutations $\delta$ on $I$ satisfying $n_i = n_{\delta(i)}$. For every $\delta \in S(d)$ and $A = \sum_{i \in I} a_i P_{X_i} \in G(\sigma, d)$ the operator

$$\delta(A) = \sum_{i \in I} a_{\delta(i)} P_{X_{\delta(i)}}$$

also belongs to $G(\sigma, d)$. The transformation sending every $A \in G(\sigma, d)$ to $\delta(A)$ is an automorphism of $\Gamma(\sigma, d)$.

**Theorem 1.** If $|I| \geq 3$ and $f$ is an automorphism of $\Gamma(\sigma, d)$, then there is a unitary or anti-unitary operator $U$ on $H$ and a permutation $\delta \in S(d)$ such that

$$f(A) = U\delta(A)U^*$$

for all $A \in G(\sigma, d)$.

**Remark 1.** Suppose that $I = \{1, 2\}$ and $n_1 \leq n_2$. Then $\Gamma(\sigma, d)$ is isomorphic to the Grassmann graph of $n_1$-dimensional subspaces. Let $S$ be a semilinear automorphism of $H$, i.e.

$$S(x + y) = S(x) + S(y)$$

for all $x, y \in H$ and there is an automorphism $\alpha$ of the field $\mathbb{C}$ such that

$$S(ax) = \alpha(a) S(x)$$

for all $a \in \mathbb{C}$ and $x \in H$. The bijective transformation sending every $a_1 P_{X_1} + a_2 P_{X_2}$ from $G(\sigma, d)$ to

$$a_1 P_{S(X_1)} + a_2 P_{S(X_1)^\perp}$$

is a graph automorphism. If $n_1 = n_2$, then the same holds for the transformation sending every $a_1 P_{X_1} + a_2 P_{X_2} \in G(\sigma, d)$ to

$$a_1 P_{S(X_1)^\perp} + a_2 P_{S(X_1)}$$

1If $H$ is finite-dimensional, the orthocomplementation map provides an isomorphism between the Grassmann graphs formed by $m$-dimensional and $(\dim H - m)$-dimensional subspaces of $H$. 
(the transposition of the eigenspaces). By Chow’s theorem \[1\], there are no other graph automorphisms if \( n_1 > 1 \). In the case when \( n_1 = 1 \), any two operators from the conjugacy class are adjacent and every bijective transformation is a graph automorphism.

**Remark 2.** In \[5\], Theorem \[1\] is proved under the additional assumption that \( n_i > 1 \) for all \( i \in I \). It is required due to applied arguments. In Section 4, we explain why these arguments fail in the general case.

### 3. Proof of Theorem \[1\]

Throughout this section we assume that \( |I| \geq 3 \).

#### 3.1. One technical result.

For every integer \( m \) satisfying \( 0 < m < \dim H \) we denote by \( G_m(H) \) the Grassmannian formed by \( m \)-dimensional subspaces of \( H \).

**Proposition 1.** Let \( m \) and \( l \) be integer satisfying \( 0 < m < l < \dim H \). If \( g \) and \( h \) are bijective transformations of \( G_m(H) \) and \( G_l(H) \) (respectively) such that for any \( X \in \mathcal{G}_m(H) \) and \( Y \in \mathcal{G}_l(H) \) we have

\[
X \subset Y \iff g(X) \subset h(Y),
\]

then \( g \) and \( h \) are induced by the same semilinear automorphism of \( H \), i.e. there is a semilinear automorphism \( S \) of \( H \) such that

\[
g(X) = S(X) \quad \text{and} \quad h(Y) = S(Y)
\]

for all \( X \in \mathcal{G}_m(H) \) and \( Y \in \mathcal{G}_l(H) \).

**Proof.** In \[5\] Proposition 3.4], the statement is proved for finite-dimensional vector spaces, but the same reasonings work in the general case. \( \square \)

#### 3.2. \((i,j)\)-connected components.

Let \( i \) and \( j \) be distinct indices from \( I \). We say that operators \( A, B \in \mathcal{G}(\sigma, d) \) are \((i,j)\)-connected if there is a sequence of operators

\[
A = C_0, C_1, \ldots, C_m = B,
\]

where \( C_{t-1}, C_t \) are \((i,j)\)-adjacent for all \( t \in \{1, \ldots, m\} \). Two operators from \( \mathcal{G}(\sigma, d) \) are \((i,j)\)-connected if and only if for every \( p \in I \setminus \{i,j\} \) they have the same eigenspace corresponding to \( a_p \) \[9\] Lemma 1]. An \((i,j)\)-connected component of \( \Gamma(\sigma, d) \) is a subset of \( \mathcal{G}(\sigma, d) \) maximal with respect to the property that any two operators are \((i,j)\)-connected.

At least one of \( n_i, n_j \) is finite. Suppose that \( n_i \) is finite and consider the pair \((\sigma, d)_{-i,+j}\) obtained from \((\sigma, d)\) as follows: \( n_j \) is replaced by \( n_j + n_i \) and \( a_i, n_i \) are removed respectively from \( \sigma \) and \( d \). This pair defines the conjugacy class of finite-rank self-adjoint operators \( \mathcal{G}((\sigma, d)_{-i,+j}) \) whose spectrum is \( \sigma \setminus \{a_i\} \), the dimension of eigenspaces corresponding to \( a_j \) is \( n_i + n_j \) and the dimensions of eigenspaces corresponding to the remaining \( a_i \) are \( n_t \).

**Remark 3.** If \( n_i \) is infinite, then \( a_i = 0 \) and, consequently, \( a_j \neq 0 \). Since eigenspaces corresponding to non-zero eigenvalues of finite-rank operators cannot be infinite-dimensional, we do not obtain a conjugacy class of finite-rank self-adjoint operators in this case.

For every operator \( T \in \mathcal{G}((\sigma, d)_{-i,+j}) \) we denote by \( \mathcal{G}(T) \) the set of all operators \( A \in \mathcal{G}(\sigma, d) \) such that

\[
A = T + (a_i - a_j)P_X,
\]

where \( X \) is an \( n_i \)-dimensional subspace in the eigenspace of \( T \) corresponding to \( a_j \).

In other words, \( \mathcal{G}(T) \) stands for all \( A \in \mathcal{G}(\sigma, d) \) satisfying the following conditions:

- for every \( t \in I \setminus \{i,j\} \) the eigenspaces of \( A \) and \( T \) corresponding to \( a_t \) are coincident;
For any mutually distinct $a_j$ the eigenspace of $T$ corresponding to $a_j$ is the orthogonal sum of the eigenspaces of $A$ corresponding to $a_i$ and $a_j$.

Note that for every $A \in \mathcal{G}(\sigma, d)$ there is a unique $T \in \mathcal{G}(\{\sigma, d\}_{i, j})$ such that $A \in \mathcal{G}(T)$.

Now, assume that $n_j$ is also finite and consider the operator $Q \in \mathcal{G}(\{\sigma, d\}_{i, j})$ such that for every $t \in I \setminus \{i, j\}$ the eigenspaces of $Q$ and $T$ corresponding to $a_t$ are coincident and the eigenspace of $Q$ corresponding to $a_i$ coincides with the eigenspace of $T$ corresponding to $a_j$. Then $\mathcal{G}(T) = \mathcal{G}(Q)$.

By [9] Lemma 2, $\{\mathcal{G}(T) : T \in \mathcal{G}(\{\sigma, d\}_{i, j})\}$ is the family of all $(i, j)$-connected components of $\mathcal{G}(\sigma, d)$.

**Remark 4.** The restriction of the graph $\Gamma(\sigma, d)$ to every $(i, j)$-connected component is isomorphic to the Grassmann graph formed by $n_i$-dimensional subspaces of $H'$, where $H'$ is a complex Hilbert space of dimension $n_i + n_j$.

**Lemma 1.** Operators $T, Q \in \mathcal{G}(\{\sigma, d\}_{i, j})$ are adjacent if and only if there are adjacent operators $A \in \mathcal{G}(T)$ and $B \in \mathcal{G}(Q)$.

**Proof.** The first part of Lemma 4 in [9].

**Remark 5.** If $T, Q \in \mathcal{G}(\{\sigma, d\}_{i, j})$ are $(t, s)$-adjacent for some $t, s \in I \setminus \{i, j\}$, then any adjacent $A \in \mathcal{G}(T)$ and $B \in \mathcal{G}(Q)$ also are $(t, s)$-adjacent. In the case when $T, Q \in \mathcal{G}(\{\sigma, d\}_{i, j})$ are $(t, j)$-adjacent for a certain $t \in I \setminus \{i, j\}$, there are $(t, j)$-adjacent $A \in \mathcal{G}(T)$ and $B \in \mathcal{G}(Q)$ as well as $(t, i)$-adjacent $A' \in \mathcal{G}(T)$ and $B' \in \mathcal{G}(Q)$.

### 3.3. Relations to automorphisms of the Johnson graph.

Let $f$ be an automorphism of the graph $\Gamma(\sigma, d)$.

**Lemma 2.** For any distinct $i, j \in I$ there are distinct $i', j' \in I$ such that $f$ sends every $(i, j)$-connected component to an $(i', j')$-connected component and $f^{-1}$ sends every $(i', j')$-connected component to an $(i, j)$-connected component. Furthermore, we have $n_{i'}, n_{j'} > 1$ if and only if $n_i, n_j > 1$.

**Proof.** Lemma 9 in [9].

By Lemma 2, $f$ induces a permutation $\tau$ on the set of all 2-element subsets of the index set $I$. Recall that the Johnson graph $J(I, 2)$ is the simple graph whose vertex set is formed by all 2-element subsets of $I$ and two distinct subsets are adjacent vertices in this graph if their intersection is non-empty.

**Lemma 3.** The transformation $\tau$ is an automorphism of $J(I, 2)$.

The proof is based on the following technical lemma.

**Lemma 4.** The following assertions are fulfilled:

1. Suppose that $|I| \geq 4$ and for some mutually distinct $i, j, i', j' \in I$ there are operators $A, B, C \in \mathcal{G}(\sigma, d)$ such that $C$ is $(i, j)$-adjacent to $A$ and $(i', j')$-adjacent to $B$. Then there is an operator $C' \in \mathcal{G}(\sigma, d)$ which is $(i', j')$-adjacent to $A$ and $(i, j)$-adjacent to $B$.

2. For any mutually distinct $i, j, t \in I$ there are operators $A, B, C \in \mathcal{G}(\sigma, d)$ such that $C$ is $(i, j)$-adjacent to $A$ and $(j, t)$-adjacent to $B$ and there is no operator from $\mathcal{G}(\sigma, d)$ which is $(j, t)$-adjacent to $A$ and $(i, j)$-adjacent to $B$.

**Proof.** (1). For every $s \in I \setminus \{i, j, i', j'\}$ the eigenspaces of $A, B, C$ corresponding to $a_s$ are coincident. For $s \in \{i, j, i', j'\}$ we denote by $X_s$ and $Y_s$ the eigenspaces of $A$ and $B$, respectively. The eigenspaces of $C$ corresponding to $a_s$, $s \in \{i, j\}$ and
s ∈ \{i', j'\} coincide with Y_s and X_s (respectively). This means that X_s, Y_s are adjacent for all s ∈ \{i, j, i', j'\}; moreover

\[ X_i + X_j = Y_i + Y_j \quad \text{and} \quad X_{i'} + X_{j'} = Y_{i'} + Y_{j'} . \]

Consider the operators C' ∈ \(G(\sigma, d)\) whose eigenspace corresponding to \(a_s, s ∈ I \setminus \{i, j, i', j'\}\) coincides with \(X_s = Y_s\) and the eigenspaces corresponding to \(a_s, s ∈ \{i, j\}\) and \(s ∈ \{i', j'\}\) coincide with \(X_s\) and \(Y_s\) (respectively). This operator is as required.

(2). Suppose that \(A, B, C\) are operators from \(G(\sigma, d)\) such that \(C\) is \((i, j)\)-adjacent to \(A\) and \((j, t)\)-adjacent to \(B\). If \(s ∈ I \setminus \{i, j, t\}\), then the eigenspaces of \(A, B, C\) corresponding to \(a_s\) are coincident. For \(s ∈ \{i, j, t\}\) we denote by \(X_s\) and \(Y_s\) the eigenspaces of \(A\) and \(B\), respectively. The eigenspaces of \(C\) corresponding to \(a_i\) and \(a_j\) coincide with \(Y_i\) and \(X_t\) (respectively) and, consequently, these subspaces are orthogonal. If \(C' ∈ G(\sigma, d)\) is \((j, t)\)-adjacent to \(A\) and \((i, j)\)-adjacent to \(B\), then the eigenspaces of \(C'\) corresponding to \(a_i\) and \(a_j\) are \(X_i\) and \(Y_t\) (respectively) which implies that \(X_i, Y_t\) are orthogonal.

Now, for any operator \(A ∈ G(\sigma, d)\) whose eigenspace corresponding to \(a_s, s ∈ I\) is denoted by \(X_s\) we construct \(B, C ∈ G(\sigma, d)\) satisfying the required conditions.

Let us take any 1-dimensional subspace \(P ∈ X_i\). Notice that it is the orthogonal complement of \(X_i\) in \(X_i + P\). Next, take a hyperplane \(Y_t ⊂ X_i + P\) distinct from \(X_i\) and denote by \(Q\) the orthogonal complement of \(Y_t\) in \(X_i + P\) (it is 1-dimensional). It is clear that \(X_i, Y_t\) are adjacent and \(P ≠ Q\), i.e. \(Q\) is not orthogonal to \(X_i\). The subspace \(Y_t\) is orthogonal to \(X_t\) (since \(X_t\) and \(P ⊂ X_j\) both are orthogonal to \(X_i\) and \(Y_t ⊂ X_i + P\)).

Let \(Y_t\) be an \(n_t\)-dimensional subspace containing \(Q\) and adjacent to \(X_t\). Note that \(Q\) is not contained in \(X_t\) (\(Q\) is not orthogonal to \(X_i\)) which implies that \(Y_t = (X_i ∩ Y_t) + Q\). Thus, \(Y_t\) is orthogonal to \(Y_t\) (since \(X_t\) and \(Q\) are orthogonal to \(Y_t\)). We have

\[ Y_t + Y_t ⊂ X_i + X_j + X_t \]

and write \(Y_t\) for the orthogonal complement of \(Y_t\) in \(X_i + X_j + X_t\). Denote by \(B\) the operator from \(G(\sigma, d)\) whose eigenspaces corresponding to \(a_s, s ∈ I \setminus \{i, j, t\}\) and \(s ∈ \{i, j, t\}\) coincide with \(X_s\) and \(Y_t\), respectively.

We have \(Y_t ⊂ X_i + X_j\), so denote by \(Z_j\) the orthogonal complement of \(Y_t\) in \(X_i + X_j\). The subspaces \(X_i, Y_i\) are adjacent and, consequently, \(X_j, Z_j\) are adjacent (as the orthogonal complements of \(X_i\) and \(Y_i\) in \(X_i + X_j\)). The inclusion \(Z_j ⊂ X_i + X_j\) implies that \(Z_j\) is orthogonal to \(X_t\). Consider the operator \(C ∈ G(\sigma, d)\) defined as follows:

- the eigenspace of \(C\) corresponding to \(a_s, s ∈ I \setminus \{i, j, t\}\) coincides with \(X_s\);
- the eigenspaces of \(C\) corresponding to \(a_i, a_j, a_t\) coincide with \(Y_i, Z_j, X_t\) (respectively).

This operator is \((i, j)\)-adjacent to \(A\). Note that

\[ Z_j + X_t = Y_j + Y_t . \]

Since \(X_t, Y_t\) are adjacent, \(Z_j, Y_j\) are adjacent as the orthogonal complements of \(X_t\) and \(Y_t\) in \((1)\). Therefore, \(C\) is \((j, t)\)-adjacent to \(B\).

Recall that \(Q ⊂ Y_t\) is not orthogonal to \(X_i\), i.e. \(X_i\) and \(Y_t\) are not orthogonal. This means that there is no operator in \(G(\sigma, d)\) which is \((j, t)\)-adjacent to \(A\) and \((i, j)\)-adjacent to \(B\). □

**Proof of Lemma 3.** If \(|I| = 3\), then the intersection of any two distinct 2-element subsets of \(I\) is non-empty and the statement is trivial.
Let $|I| \geq 4$ and let $i, j, t \in I$ be mutually distinct indices. Suppose that
\[
\tau(\{i, j\}) = \{i', j'\} \quad \text{and} \quad \tau(\{i, t\}) = \{s', t'\}.
\]
By the statement (2) from Lemma 4 there are $A, B, C \in \mathcal{G}(\sigma, d)$ such that $C$ is $(i, j)$-adjacent to $A$ and $(j, t)$-adjacent to $B$ and there is no operator from $\mathcal{G}(\sigma, d)$ which is $(j, t)$-adjacent to $A$ and $(i, j)$-adjacent to $B$. The operator $f(C)$ is $(i', j')$-adjacent to $f(A)$ and $(s', t')$-adjacent to $f(B)$. If $i', j', s', t'$ are mutually distinct, then the statement (1) from Lemma 4 implies the existence of $C' \in \mathcal{G}(\sigma, d)$ which is $(s', t')$-adjacent to $f(A)$ and $(i', j')$-adjacent to $f(B)$. Then $f^{-1}(C')$ is $(j, t)$-adjacent to $A$ and $(i, j)$-adjacent to $B$ which is impossible. Therefore,
\[
\tau(\{i, j\}) \cap \tau(\{i, t\}) \neq \emptyset.
\]
Applying the same arguments to $f^{-1}$ and $\tau^{-1}$ we establish that $\tau$ is an automorphism of $J(I, 2)$. \hfill \Box

If $|I| \neq 4$, then every automorphism of $J(I, 2)$ is induced by a permutation on the set $I$. In the case when $|I| = 4$, an automorphism of $J(I, 2)$ is induced by a permutation on $I$ or it is the composition of an automorphism induced by a permutation and the automorphism sending every 2-element subset $J \subset I$ to the complement $I \setminus J$. Therefore, one of the following possibilities is realised:
- there is a permutation $\delta$ on the set $I$ such that $f$ sends $(i, j)$-adjacent operators to $(\delta(i), \delta(j))$-adjacent operators;
- $|I| = 4$ and there is a permutation $\delta$ on $I$ such that $f$ sends $(i, j)$-adjacent operators to $(i', j')$-adjacent operators, where $\{i', j'\} = I \setminus \{\delta(i), \delta(j)\}$.
In the second case, $\tau$ transfers the collection of 2-element subsets of $I$ containing a certain $i \in I$ to the collection of 2-element subsets contained in a certain 3-element subset of $I$.

3.4. The case $|I| \neq 4$. Let $i \in I$. We say that two operators $A, B \in \mathcal{G}(\sigma, d)$ are $\overline{\tau}$-connected if there is a sequence
\[
A = A_0, A_1, \ldots, A_t = B
\]
such that for every $s \in \{1, \ldots, t\}$ the operators $A_{s-1}, A_s$ are $(i_s, j_s)$-adjacent and $i \notin \{i_s, j_s\}$.
Let $\mathcal{G}(i)$ be the set of all eigenspaces of operators from $\mathcal{G}(\sigma, d)$ corresponding to $a_i$. If $n_i$ is finite, then $\mathcal{G}(i)$ is the Grassmannian of $n_i$-dimensional subspaces. In the case when $n_i$ is infinite, $\mathcal{G}(i)$ is formed by all closed subspaces of codimension
\[
n^i = \sum_{j \notin \{i\}} n_j.
\]
For every $S \in \mathcal{G}(i)$ we denote by $[S]_i$, the set of all operators from $\mathcal{G}(\sigma, d)$ whose eigenspaces corresponding to $a_i$ coincide with $S$. Any two operators from $[S]_i$ are $\overline{\tau}$-connected. Conversely, if $\mathcal{X}$ is a subset of $\mathcal{G}(\sigma, d)$, where any two elements are $\overline{\tau}$-connected, then all operators from $\mathcal{X}$ have the same eigenspace corresponding to $a_i$, i.e. $\mathcal{X}$ is contained in a certain $[S]_i$. Therefore,
\[
\{[S]_i : S \in \mathcal{G}(i)\}
\]
can be characterised as the family of all subsets of $\mathcal{G}(\sigma, d)$ maximal with respect to the property that any two elements are $\overline{\tau}$-connected.
Suppose that $\tau$ (the automorphism of $J(I, 2)$ associated to $f$) is induced by a permutation $\delta$ on $I$ (this holds if $|I| \neq 4$). The automorphism $f$ sends $\overline{\tau}$-connected operators to $\overline{\delta}(\tau)$-connected operators. Therefore, for every $S \in \mathcal{G}(i)$ there is $S' \in \mathcal{G}(\delta(i))$ such that
\[
f([S]_i) = [S']_{\delta(i)};
in other words, there is a map

\[ f_i : G(i) \to G(\delta(i)) \]

satisfying

\[ f([S]_i) = [f_i(S)]_{\delta(i)} \]

for every \( S \in G(i) \). The map \( f_i \) is bijective (since \( f \) is bijective).

**Lemma 5.** The map \( f_i \) is adjacency preserving in both directions.

**Proof.** The statement is a consequence of the following fact: \( S, T \in G(i) \) are adjacent if and only if for every \( t \in I \setminus \{i\} \) there are \((i, t)\)-adjacent operators \( A \in [S]_i \) and \( B \in [T]_t \).

**Lemma 6.** The permutation \( \delta \) belongs to \( S(d) \).

To prove Lemma 5 we need to shed some light on isomorphisms between Grassmann graphs (see [3, Chapter 2] for the details).

**Remark 6.** The Grassmann graphs formed by \( m \)-dimensional and \( m' \)-dimensional subspaces of \( H \) are isomorphic if and only if \( m = m' \) or \( H \) is finite-dimensional and \( \dim H = m + m' \). In the case when \( H \) is infinite-dimensional, the Grassmann graph formed by closed subspaces of codimension \( m \) is isomorphic to the Grassmann graph of \( m \)-dimensional subspaces (the orthocomplementation map) and, consequently, it is isomorphic to the Grassmann graph of \( m' \)-dimensional subspaces if and only if \( m = m' \).

**Proof of Lemma 6.** We need to show that \( n_i = n_{\delta(i)} \) for every \( i \in I \).

If \( n_i \) and \( n_{\delta(i)} \) both are finite, then \( f_i \) is an isomorphism between the Grassmann graphs of \( n_i \)-dimensional and \( n_{\delta(i)} \)-dimensional subspaces of \( H \) (Lemma 5). By Remark 6 \( n_i = n_{\delta(i)} \) or \( H \) is finite-dimensional and \( \dim H = n_i + n_{\delta(i)} \). In the second case, we have \( |I| = 2 \) which is impossible.

Suppose that one of \( n_i, n_{\delta(i)} \), say \( n_i \), is infinite. If \( \delta(i) \neq i \), then \( n_{\delta(i)} \) is finite and \( f_i \) is an isomorphism between the Grassmann graphs formed by closed subspaces of codimension \( n' \) and \( n_{\delta(i)} \)-dimensional subspaces of \( H \) (Lemma 5). By Remark 6 we have \( n' = n_{\delta(i)} \) which means that \( |I| = 2 \), a contradiction.

By Lemma 6 we can assume that \( f \) preserves each type of adjacency (otherwise, we replace \( f \) by the automorphism \( \delta^{-1}f \)). Then each \( f_i \) is a bijective transformation of \( G(i) \) and

\[ f \left( \sum_{i \in I} a_i P_{X_i} \right) = \sum_{i \in I} a_i P_{f_i(X_i)}, \]

if \( X_i \in G(i) \) and \( H \) is the orthogonal sum of all \( X_i \). Therefore, \( X \in G(i) \) and \( Y \in G(j) \) are orthogonal if and only if \( f_i(X) \) and \( f_j(Y) \) are orthogonal.

Suppose that \( \dim H = n \) is finite. Since \( |I| \geq 3 \), for any distinct \( i, j \in I \) there is \( t \in I \setminus \{i, j\} \). Consider the bijective transformation \( h \) of \( G_{n-n_t}(H) \) which sends every \((n-n_t)\)-dimensional subspace \( Z \) to \( f_t(Z^\perp)^\perp \). For \( X \in G_{n}(H) \) and \( Z \in G_{n-n_t}(H) \) we have \( X \subset Z \) if and only if \( X \) and \( Z^\perp \) are orthogonal. The latter holds if and only if \( f_i(X) \) and \( f_t(Z^\perp) \) are orthogonal or, equivalently, \( f_t(X) \) is contained in \( f_t(Z^\perp)^\perp \). Therefore,

\[ X \subset Z \iff f_i(X) \subset h(Z). \]

By Proposition 1 \( f_i \) and \( h \) are induced by the same semilinear automorphism of \( H \). Similarly, we establish that \( f_j \) and \( h \) are induced by the same semilinear automorphism of \( H \). So, all \( f_i \) are induced by the same semilinear automorphism \( U \) of \( H \). Since \( U \) sends orthogonal vectors to orthogonal vectors, it is a scalar
multiple of a unitary or anti-unitary operators [8 Proposition 4.2]. For every non-zero scalar \( a \) and every subspace \( X \subset H \) we have \( aU(X) = U(X) \), i.e. we can assume that \( U \) is unitary or anti-unitary. This gives the claim.

Let \( i \) be an index from \( I \) such that \( n_i \) is infinite. Then \( n' = \sum_{j \in I \setminus \{i\}} n_j \) is finite. As above, we consider the bijective transformation \( h \) of \( \mathcal{G}_{n'}(H) \) sending every \( n' \)-dimensional subspace \( Z \) to \( f_i(Z^\perp) \) and show that for all \( j \in I \setminus \{i\} \) the transformations \( f_j \) are induced by the same semilinear automorphism \( U \) of \( H \). We can assume that \( U \) is unitary or anti-unitary (because it preserves orthogonality).

For every \( X \in \mathcal{G}(i) \) there are mutually orthogonal subspaces \( X_j \in \mathcal{G}(j) \), \( j \in I \setminus \{i\} \) whose sum is the orthogonal complement of \( X \). Then \( f_i(X) \) is the orthogonal complement of

\[
\sum_{j \in I \setminus \{i\}} f_j(X_j) = U \left( \sum_{j \in I \setminus \{i\}} X_j \right) = U(X^\perp) = U(X)^\perp
\]

which implies that \( f_k(X) = U(X) \), i.e. \( f_k \) also is induced by \( U \).

3.5. The case \(|I| = 4\). Suppose that \( I = \{1, 2, 3, 4\} \) and \( n_1 \geq n_2 \geq n_3 \geq n_4 \). Then one of the following possibilities is realised:

1. \( n_1 \geq n_2 > 1 \);
2. \( n_1 > 1 \) and \( n_2 = n_3 = n_4 = 1 \);
3. \( n_1 = n_2 = n_3 = n_4 = 1 \).

The case (1). The automorphism \( f \) transfers \((1,2)\)-connected components to \((i,j)\)-connected components for some \( i, j \in I \). Since \( n_1, n_2 \) both are greater than 1, Lemma 2 shows that

\[
n_1 = n_i, n_2 = n_j \quad \text{or} \quad n_1 = n_j, n_2 = n_i.
\]

Without loss of generality we assume that the first possibility is realised. Then the permutation \((1, i)(2, j)\) belongs to \( S(d) \) and the automorphism \((1, i)(2, j)f \) preserves the \((1,2)\)-adjacency. We can replace \( f \) by \((1, i)(2, j)f \), which means that we can assume that \( f \) preserves the \((1,2)\)-adjacency. There is a one-to-one correspondence between \((1,2)\)-connected components and operators from the conjugacy class \( \mathcal{G}((\sigma, d)-2,+1) \). In our case, \((\sigma, d)-2,+1 = (\sigma', d')\), where

\[
\sigma' = \{a_1, a_3, a_4\}, \quad d' = \{n_1 + n_2, n_3, n_4\}
\]

and the associated set of indices is \( \{1, 3, 4\} \). The automorphism \( f \) induces a bijective transformation \( f' \) of \( \mathcal{G}(\sigma', d') \). Lemma 4 shows that \( f' \) is an automorphism of the graph \( \Gamma(\sigma', d') \). By Subsection 3.4, for every \( A \in \mathcal{G}(\sigma', d') \) we have

\[
f'(A) = U \delta'(A) U^*,
\]

where \( U \) is a unitary or anti-unitary operator and \( \delta' \in S(d') \) is a permutation on \( \{1, 3, 4\} \). Since \( n_1 + n_2 > n_3 \geq n_4 \), we get \( \delta'(1) = 1 \). Therefore, \( \delta' \) is identity or the transposition \((3, 4)\). In the second case, we have \( n_3 = n_4 \) and \((3, 4) \in S(d) \). The automorphism \((3, 4)f \) of \( \Gamma(\sigma, d) \) induces the automorphism \((3, 4)f' \) of \( \Gamma(\sigma', d') \). The latter preserves each type of adjacency. So, we can assume that \( \delta' \) is identity and \( f' \) preserves each type of adjacency. This immediately implies that \( f \) preserves the \((3,4)\)-adjacency. Now, we show that \( f \) preserves the \((1,3)\)-adjacency and \((2,3)\)-adjacency or interchanges them.

Suppose that \( A, B \in \mathcal{G}(\sigma, d) \) are \((1,3)\)-adjacent. Consider \( A', B' \in \mathcal{G}(\sigma', d') \) such that \( A \in \mathcal{G}(A') \) and \( B \in \mathcal{G}(B') \) (see Subsection 3.2). Then \( A' \) and \( B' \) also are \((1,3)\)-adjacent. The same holds for \( f'(A') \) and \( f'(B') \) (since \( f' \) preserves each type of adjacency). We have

\[
f(A) \in \mathcal{G}(f'(A')), \quad f(B) \in \mathcal{G}(f'(B'))
\]
and \( f'(A'), f'(B') \) are \((1,3)\)-adjacent. By Remark 5, \( f(A), f(B) \) are \((1,3)\)-adjacent or \((2,3)\)-adjacent. We apply the same arguments to a pair of \((2,3)\)-adjacent operators from \( \mathcal{G}(\sigma, d) \) and get the claim.

So, \( \tau \) (the automorphism of \( J(I, 2) \) associated to \( f \)) preserves the collection of 2-element subsets of \( I \) containing 3. This guarantees that \( \tau \) is induced by a permutation on \( I \) (see the remark at the end of Subsection 3.3). Hence, we can apply the arguments from Subsection 3.4.

The case (3). Since \( n_1 = n_2 = n_3 = n_4 \), every permutation on \( I \) belongs to \( S(d) \). If \( f \) transfers the \((1,2)\)-adjacency to the \((i, j)\)-adjacency for some \( i, j \in I \), then the automorphism \((1, i)(2, j)f \) preserves the \((1,2)\)-adjacency. We assume that \( f \) preserves the \((1,2)\)-adjacency and repeat arguments used in the case (1).

The case (2). Since \( n_2 = n_3 = n_4 \), it suffices to show that \( f \) transfers the \((1,2)\)-adjacency to the \((1, i)\)-adjacency for a certain \( i \in \{2, 3, 4\} \). After that, we can repeat the above arguments.

Suppose to the contrary that \( f \) sends the \((1,2)\)-adjacency to the \((i, j)\)-adjacency for some \( i, j \in \{2, 3, 4\} \). Without loss of generality we can assume that \( (i, j) = (3, 4) \). Then, by Lemma 1, \( f \) induces an isomorphism \( f' \) of \( \Gamma((\sigma, d)_{-2,+1}) \) to \( \Gamma((\sigma, d)_{-4,+3}) \).

In the present case, \((\sigma, d)_{-2,+1} = (\sigma', d') \) and \((\sigma, d)_{-4,+3} = (\sigma'', d'') \), where
\[
\sigma' = \{a_1, a_3, a_4\}, \quad d' = \{n_1 + 1, 1, 1\}
\]
and
\[
\sigma'' = \{a_1, a_2, a_3\}, \quad d'' = \{n_1, 1, 2\}.
\]

Consider the inverse isomorphism \( f'^{-1} \). It sends the \((1,3)\)-adjacency of \( \Gamma(\sigma'', d'') \) to the \((t, s)\)-adjacency of \( \Gamma(\sigma', d') \) for some \( t, s \in \{1, 3, 4\} \). The dimensions of the eigenspaces of operators from \( \mathcal{G}(\sigma', d') \) corresponding to \( a_1 \) and \( a_2 \) both are greater than 1. Lemma 4 shows that the same holds for the dimensions of the eigenspaces of operators from \( \mathcal{G}(\sigma', d') \) corresponding to \( a_t \) and \( a_s \). The latter is impossible, since \( d' = \{n_1 + 1, 1, 1\} \); a contradiction.

4. Final remarks

Theorem 1 was proved in [9] under the assumption that \( n_i > 1 \) for all \( i \in I \). Now, we describe briefly the reasonings from [9] and explain why they cannot be exploited in the general case.

Some information concerning isomorphisms between Grassmann graphs can be found in Remark 5, but we need a bit more. Let \( X \) and \( X' \) be finite-dimensional subspaces of \( H \) (not necessarily of the same dimension). Consider the Grassmann graphs \( \Gamma \) and \( \Gamma' \) formed by \( m \)-dimensional subspaces of \( X \) and \( m' \)-dimensional subspaces of \( X' \), respectively. In the case when \( 1 < m < \dim X - 1 \), the graphs \( \Gamma \) and \( \Gamma' \) are isomorphic if and only if \( \dim X = \dim X' \) and \( m' \) is equal to \( m \) or \( \dim X - m \). However, for
\[
m \in \{1, \dim X - 1\} \quad \text{and} \quad m' \in \{1, \dim X' - 1\}
\]
the graphs are isomorphic even if \( \dim X \neq \dim X' \) (in this case, any two distinct vertices in each of these graphs are adjacent).

Assume for simplicity that all \( n_i \) are finite. Let \( f \) be an automorphism of \( \Gamma(\sigma, d) \). By Lemma 2, for any distinct \( i, j \in I \) there are distinct \( i', j' \in I \) such that \( f \) provides a one-to-one correspondence between \((i, j)\)-connected components and \((i', j')\)-connected components. Recall that the restriction of \( \Gamma(\sigma, d) \) to an \((i, j)\)-connected component is isomorphic to the Grassmann graph formed by \( n_{i'} \)-dimensional subspaces of a certain \((n_i + n_j)\)-dimensional subspace of \( H \). Therefore, \( f \) induces an isomorphism between this graph and the Grassmann graph of \( n_{i'} \)-dimensional subspaces of an \((n_{i'} + n_{j'})\)-dimensional subspace of \( H \).
If $n_i, n_j > 1$, then $\{n_i, n_j\} = \{n_i', n_j'\}$. This implies the existence of a permutation $\delta \in S(d)$ such that the automorphism $\delta f$ preserves the $(i, j)$-adjacency. The latter automorphism induces an automorphism of $\Gamma((\sigma, d)_{-i, +j})$. Applying this reduction recursively we obtain an automorphism of a Grassmann graph. This is one of the key methods used in [9].

If $n_i = 1$, then one of $n_i', n_j'$, say $n_i'$, also is equal to 1. We cannot assert that $n_j = n_j'$ in this case (see the above remark on isomorphisms of Grassmann graphs).

There is yet another reason. Suppose that $I = \{1, 2, 3\}$ and $n_1 = n_2 = n_3 = 1$. Then every permutation on $I$ belongs to $S(d)$ and we can assume that $f$ preserves each type of adjacency and, consequently, induces an automorphism of $\Gamma((\sigma, d)_{-i, +j})$ for any distinct $i, j \in I$. Each $\Gamma((\sigma, d)_{-i, +j})$ is isomorphic to the Grassmann graph formed by 1-dimensional subspaces of $\mathbb{C}^3$ and every bijective transformation of the vertex set is a graph automorphism.

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