Tilings of Sphere by Congruent Pentagons I: Edge Combinations $a^2b^2c$ and $a^3bc$

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Abstract

We develop the basic tools for classifying edge-to-edge tilings of sphere by congruent pentagons. Then we prove such tilings for edge combination $a^2b^2c$ are three families of pentagonal subdivisions of the platonic solids, with 12, 24 and 60 tiles. We also prove that such tilings for edge combination $a^3bc$ are two unique double pentagonal subdivisions, with 48 and 120 tiles.

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1 Introduction

Mathematicians have studied tilings for more than 100 years. A lot is known about tilings of the plane or the Euclidean space. However, results about tilings of the sphere are relatively rare. A major achievement in this regard is the complete classification of edge-to-edge tilings of the sphere by congruent triangles. The classification was started by Sommerville [4] in 1923 and completed by Ueno and Agaoka [5] in 2002. For tilings of the sphere by congruent pentagons, we know the classification for the minimal case of 12 tiles [1, 3].

Spherical tilings are relatively easier to study than planar tilings, simply because the former involve only finitely many tiles. The classifications in...
not only give the complete list of tiles, but also the ways the tiles are fit together. It is not surprising that such kind of classifications for the planer tilings are only possible under various symmetry conditions, because the quotients of the plane by the symmetries often become compact.

Like the earlier works, we restrict ourselves to edge-to-edge tilings of the sphere by congruent polygons, such that all vertices have degree $\geq 3$. These are mild and natural assumptions that simplify the discussion. The polygon in such a tiling must be triangle, quadrilateral, or pentagon (see [6], for example). We believe that pentagonal tilings should be relatively easier to study than quadrilateral ones because 5 is an “extreme” among 3, 4, 5. Indeed, many of the various restrictions on pentagonal tilings in Section 2 have no similar counterparts for quadrilateral tilings. In fact, a preliminary exploration on quadrilateral tilings [6] showed the difficulty of the problem.

In Lemma 9 we know that the lengths of five edges of the pentagon in our tiling may have five possible combinations: $a^2b^2c, a^3bc, a^3b^2, a^4b, a^5$. Here $a^2b^2c$ means the five edge lengths are $a, a, b, b, c$, with $a, b, c$, distinct. The following is the classification for this edge combination.

**Theorem.** Edge-to-edge tilings of the sphere by congruent pentagons with edge combination $a^2b^2c$ ($a, b, c$ distinct) are the following.

1. Pentagonal subdivision of tetrahedron, with 12 tiles.
2. Pentagonal subdivision of octahedron (or cube), with 24 tiles.
3. Pentagonal subdivision of icosahedron (or dodecahedron), with 60 tiles.

Pentagonal subdivision is introduced in Section 3.1. The operation can be applied to any tiling on oriented surface, and dual tilings give the same pentagonal subdivision tiling. Therefore the five platonic solids give three pentagonal subdivisions. Note that we already proved in [1, 3] that edge-to-edge tilings of sphere by (the minimal number of) 12 congruent pentagons is the deformed dodecahedron. This is exactly the pentagonal subdivision of tetrahedron in the theorem. We also note that each tiling in the theorem allows two free parameters.

The following is the classification when the five edges have lengths $a, a, a, b, c$.

**Theorem.** Edge-to-edge tilings of the sphere by congruent pentagons with edge combination $a^3bc$ ($a, b, c$ distinct) are the following.

1. Double pentagonal subdivision of octahedron (or cube), with 48 tiles.
2. Double pentagonal subdivision of icosahedron (or dodecahedron), with 120 tiles.

Double pentagonal subdivision is introduced in Section 3.2. Unlike pentagonal subdivision, each tiling in the theorem allows only one specific pentagon, and we provide the exact values in Section 3.2. In fact, we may also get a tiling (with 24 tiles) by applying the double pentagonal subdivision to tetrahedron. However, the specific pentagon has \( b = c \) and therefore the tiling has edge combination \( a^3b^2 \). Moreover, the tiling is a degenerate case \( (b = c \text{ in } a^2b^2c) \) of the pentagonal subdivision of octahedron (or cube). Therefore the tiling is not listed in the theorem.

Our next paper [7] classifies tilings with edge combination \( a^3b^2 \), and is much more complicated. The third in our series [2] classifies tilings with edge combination \( a^5 \), i.e., by congruent equilateral pentagons. This requires completely different technique, because we can no longer rely on edge length information. The remaining case is the edge combination \( a^4b \), which we call almost equilateral. The case is much more challenging and will be the subject of another series.

This paper is organized as follows. Section 2 is the basic facts for tilings of the sphere by congruent pentagons. Section 3 introduces pentagonal and double pentagonal subdivisions. We also calculate the specific pentagons used in the double pentagonal subdivision tilings. Sections 4 and 5 prove the two classification theorems.

We would like to thank Ka-yue Cheuk and Ho-man Cheung. Some of their initial work on the pentagonal subdivision are included in this and the next paper.

2 Basic Facts

2.1 Vertex

Consider an edge-to-edge tiling of the sphere by pentagons, such that all vertices have degree \( \geq 3 \). Let \( v, e, f \) be the numbers of vertices, edges, and
tiles. Let \( v_k \) be the number of vertices of degree \( k \). We have

\[
2 = v - e + f,
\]

\[
2e = 5f = \sum_{k=3}^{\infty} kv_k = 3v_3 + 4v_4 + 5v_5 + \cdots,
\]

\[
v = \sum_{k=3}^{\infty} v_k = v_3 + v_4 + v_5 + \cdots.
\]

Then it is easy to derive \( 2v = 3f + 4 \) and

\[
\frac{f}{2} - 6 = \sum_{k \geq 4} (k - 3)v_k = v_4 + 2v_5 + 3v_6 + \cdots, \tag{2.1}
\]

\[
v_3 = 20 + \sum_{k \geq 4} (3k - 10)v_k = 20 + 2v_4 + 5v_5 + 8v_6 + \cdots. \tag{2.2}
\]

By (2.1), \( f \) is an even integer \( \geq 12 \). Since tilings by 12 congruent pentagons have been classified by [1, 3], we may assume \( f > 12 \). We also note that by (2.1), \( f = 14 \) implies \( v_4 = 1 \) and \( v_i = 0 \) for \( i > 4 \). By [8, Theorem 1], this is impossible. Therefore we will always assume that \( f \) is an even integer \( \geq 16 \).

The equality (2.2) shows that most vertices have degree 3. We call vertices of degree \( > 3 \) high degree vertices.

**Lemma 1.** Any pentagonal tiling of the sphere has a tile, such that four vertices have degree 3 and the fifth vertex has degree 3, 4 or 5.

We call three types of tiles in the lemma 3\(^3\)-tile, 3\(^4\)4-tile, and 3\(^4\)5-tile. The neighborhood of this special tile is given by the first three of Figure 1. In case of 3\(^4\)4-tile or 3\(^4\)5-tile, we denote the fifth vertex \( H \) (of degree 4 or 5) by dot. The fourth of Figure 1 is shared by the three neighborhoods, and we call it the partial neighborhood. We will always label the tiles in the partial neighborhood as in the picture.

**Proof.** If a pentagonal tiling of the sphere has no tile described in the lemma, then any tile either has at least one vertex of degree \( \geq 6 \), or has at least two vertices of degree 4 or 5. Since a degree \( k \) vertex is shared by at most \( k \) tiles, the number of tiles of first kind is \( \leq \sum_{k \geq 6} kv_k \), and the number of tiles of the second kind is \( \leq \frac{1}{2}(4v_4 + 5v_5) \). Therefore we have

\[
f \leq 2v_4 + \frac{5}{2}v_5 + \sum_{k \geq 6} kv_k.
\]

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Figure 1: Neighborhood and partial neighborhood of a special tile.

On the other hand, by (2.1), we have

$$f - \left(2v_4 + \frac{5}{2}v_5 + \sum_{k \geq 6} kv_k\right) = 12 + \frac{1}{2}v_5 + \sum_{k \geq 6} (k - 6)v_k > 0.$$ 

We get a contradiction. \(\square\)

Lemma 2. If a pentagonal tiling of the sphere has no $3^5$-tile, then $f \geq 24$. Moreover, if $f = 24$, then each tile is a $3^44$-tile.

Proof. If there is no $3^5$-tile, then any tile has at least one vertex of degree $\geq 4$. This implies $f \leq \sum_{k \geq 4} kv_k$. By (2.1), we get

$$f = 2f - f \geq 24 + \sum_{k \geq 4} 4(k - 3)v_k - \sum_{k \geq 4} kv_k$$

$$= 24 + \sum_{k \geq 4} 3(k - 4)v_k \geq 24.$$ 

Moreover, the equality happens if and only if $v_i = 0$ for $i > 4$ and $f = 4v_4$. Since there is no $3^5$-tile, this means that each tile is a $3^44$-tile. \(\square\)

Lemma 3. If a pentagonal tiling of the sphere has no $3^5$-tile and $3^44$-tile, then $f \geq 60$. Moreover, if $f = 60$, then each tile is a $3^45$-tile.

Proof. If there is no $3^5$-tile and $3^44$-tile, then any tile either has at least two vertices of degree 4, or has at least one vertex of degree $\geq 5$. This implies $f \leq \frac{1}{2}4v_4 + \sum_{k \geq 5} kv_k$. By (2.1), we get

$$f = 5f - 4f \geq 60 + \sum_{k \geq 4} 5(k - 3)v_k - 8v_4 - \sum_{k \geq 5} 4kv_k$$

$$= 60 + 2v_4 + \sum_{k \geq 5} 5(k - 6)v_k \geq 60.$$ 

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Moreover, the equality happens if and only if \( v_4 = v_6 = v_7 = \cdots = 0 \) and \( f = 5v_5 \). Since there is no 3\(^5\)-tile and 3\(^4\)4-tile, this means that each tile is a 3\(^4\)5-tile.

2.2 Angle

The most basic property about angles is that the sum of all angles (angle sum) at a vertex is \( 2\pi \). Another basic property is the sum of all angles in pentagon.

**Lemma 4.** If all tiles in a tiling of sphere by \( f \) pentagons have the same five angles \( \alpha, \beta, \gamma, \delta, \epsilon \), then

\[
\alpha + \beta + \gamma + \delta + \epsilon = 3\pi + \frac{4}{f}\pi.
\]

**Proof.** Since the angle sum at each vertex is \( 2\pi \), the total sum of all angles is \( 2\pi v \). Moreover, the sum of five angles in each tile is \( \Sigma = \alpha + \beta + \gamma + \delta + \epsilon \), the same for all the tiles. Therefore the total sum of all angles is also \( \Sigma f \). We get \( 2\pi v = \Sigma f \), and by \( 3f = 2v - 4 \), we further get

\[
\Sigma = 2\pi \frac{v}{f} = 3\pi + \frac{4}{f}\pi. \quad \square
\]

The lemma does not require that the angles are arranged in the same way in all tiles, and does not require that the edges are straight (i.e., great arcs). However, if we additionally know that all edges are straight, then all tiles have the same area \( \Sigma - 3\pi \), and the equality in the lemma follows from the fact that the total area \((\Sigma - 3\pi)f\) is the area \( 4\pi \) of the sphere.

The angles in the lemma refer to the values, and some angles among the five may have the same value. For example, if the five values are \( \alpha, \alpha, \alpha, \beta, \beta \), with \( \alpha \neq \beta \) (different values), then we say the pentagon has angle combination \( \alpha^3\beta^2 \). The following is about the distribution of angle values.

**Lemma 5.** If an angle appears at every degree 3 vertex in a tiling of sphere by pentagons with the same angle combination, then the angle appears at least 2 times in the pentagon.

**Proof.** If an angle \( \theta \) appears only once in the pentagon, then the total number of times \( \theta \) appears in the whole tiling is \( f \), and the total number of non-\( \theta \)
angles is $4f$. If we also know that $\theta$ appears at every degree 3 vertex, then $f \geq v_3$, and non-$\theta$ angles appear $\leq 2v_3$ times at degree 3 vertices. Moreover, non-$\theta$ angles appear $\leq \sum_{k \geq 4} kv_k$ times at high degree vertices. Therefore

$$4v_3 \leq 4f \leq 2v_3 + \sum_{k \geq 4} kv_k.$$  

Then by (2.2), we have

$$0 \geq 4v_3 - \left(2v_3 + \sum_{k \geq 4} kv_k\right) = \sum_{k \geq 4} (2(3k - 10) - k)v_k = \sum_{k \geq 4} 5(k - 4)v_k.$$  

This implies $v_i = 0$ for $i > 4$ and $f = v_3 = 2v_4$, and contradicts (2.1). \hfill \square

Unlike Lemma 4, which is explicitly about the values of angles, Lemma 5 only counts the number of angles. The key in counting is to distinguish angles. We may use the value as the criterion for two angles to be the “same”. We may also use the edge lengths bounding angles as the criterion. The observation will be used in the proof of Lemma 9. The observation also applies to the subsequent Lemmas 6, 7, 8.

**Lemma 6.** If an angle appears at least twice at every degree 3 vertex in a tiling of sphere by pentagons with the same angle combination, then the angle appears at least 3 times in the pentagon.

**Proof.** If an angle $\theta$ appears only once in the pentagon, then by Lemma 5, it cannot appear at every degree 3 vertex. If it appears twice in the pentagon, then the total number of $\theta$ in the whole tiling is $2f$, and the total number of non-$\theta$ angles is $3f$. If we also know that $\theta$ appears at least twice at every degree 3 vertex, then $2f \geq 2v_3$, and non-$\theta$ angles appear $\leq v_3$ times at degree 3 vertices. Moreover, non-$\theta$ angles appear $\leq \sum_{k \geq 4} kv_k$ times at high degree vertices. Therefore

$$3v_3 \leq 3f \leq v_3 + \sum_{k \geq 4} kv_k.$$  

This leads to the same contradiction as in the proof of Lemma 5. \hfill \square

The proof of Lemma 6 can be easily modified to get the following.

**Lemma 7.** If two angles together appear at least twice at every degree 3 vertex in a tiling of sphere by pentagons with the same angle combination, then the two angles together appear at least 3 times in the pentagon.
The following is about angles not appearing at degree 3 vertices.

**Lemma 8.** Suppose an angle $\theta$ does not appear at degree 3 vertices in a tiling of sphere by pentagons with the same angle combination.

1. There can be at most one such angle $\theta$.
2. The angle $\theta$ appears only once in the pentagon.
3. $2v_4 + v_5 \geq 12$.
4. One of $\alpha \theta^3$, $\theta^4$, $\theta^5$ is a vertex, where $\alpha \neq \theta$.

The first statement implies that the angle $\alpha$ in the fourth statement must appear at a degree 3 vertex.

**Proof.** Suppose two angles $\theta_1$ and $\theta_2$ do not appear at degree 3 vertices. Then the total number of times these two angles appear is at least $2f$, and is at most the total number $\sum_{k \geq 4} kv_k$ of angles at high degree vertices. Therefore we have $2f \leq \sum_{k \geq 4} kv_k$. On the other hand, by (2.1), we have

$$2f - \sum_{k \geq 4} kv_k = 24 + \sum_{k \geq 4} 3(k - 4)v_k > 0.$$ 

The contradiction proves the first statement.

The argument above also applies to the case $\theta_1 = \theta_2$, which means the same angle appearing at least twice in the pentagon. This proves the second statement.

The first two statements imply that $\theta$ appears exactly $f$ times. Since this should be no more than the total number $\sum_{k \geq 4} kv_k$ of angles at high degree vertices, by (2.1), we have

$$0 \geq f - \sum_{k \geq 4} kv_k = 12 - 2v_4 - v_5 + \sum_{k \geq 6} (k - 6)v_k.$$ 

This implies the third statement.

For the last statement, we assume that $\alpha \theta^3, \theta^4, \theta^5$ are not vertices. This means that $\theta$ appears at most twice at any degree 4 vertex, and at most four times at any degree 5 vertex. Since $\theta$ also does not appear at degree 3 vertices, the total number of times $\theta$ appears is $\leq 2v_4 + 4v_5 + \sum_{k \geq 6} kv_k$. 

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However, the number of times $\theta$ appears should also be $f$. Therefore $f \leq 2v_4 + 4v_5 + \sum_{k \geq 6} kv_k$. On the other hand, by (2.1), we have

$$f - \left(2v_4 + 4v_5 + \sum_{k \geq 6} kv_k\right) = 12 + \sum_{k \geq 6} (k - 6)v_k > 0.$$  

We get a contradiction.

2.3 Edge

Two pentagons have the same edge combination if the five edge lengths are equal. For example, the first two pentagons in Figure 2 have the same edge combination $a^2b^2c$. If we further have the same edge length arrangement in the two pentagons, then the two pentagons are edge congruent. For example, the first two pentagons in Figure 2 have the respective edge length arrangements $a, b, a, b, c$ and $a, a, b, b, c$, and are therefore not edge congruent.

**Lemma 9.** In an edge-to-edge tiling of the sphere by edge congruent pentagons, the edge lengths of the pentagon is arranged in one of the six ways in Figure 2, with distinct edge lengths $a, b, c$. Moreover, the first of Figure 2 cannot be a $3^5$-tile, and, in case the pentagon is a $3^4$- or $3^45$-tile, the fifth vertex of degree 4 or 5 is opposite to the $c$-edge.

Proof. The lemma is an extension of [3, Proposition 7]. We follow the argument in the earlier paper. By purely numerical consideration, there are seven possible edge combinations ($a, b, c, d, e$ are distinct)

$$abcde, a^2bcd, a^2b^2c, a^3bc, a^3b^2, a^4b, a^5.$$

For $abcde$, without loss of generality, we may assume that the edges are arranged as in the first of Figure 3. By Lemma 1, we may further assume
that the vertex shared by $c, d$ has degree 3. Let $x$ be the third edge at the
vertex. Then $x, c$ are adjacent in a tile, and $x, d$ are adjacent in another tile.
Since there is no edge in the pentagon that is adjacent to both $c$ and $d$, we
get a contradiction.

![Figure 3: Not suitable for tiling.](image)

The combination $a^2bcd$ has two possible arrangements, given (without loss
of generality) by the second (adjacent $a$) and third (separated $a$) of Figure
3. By Lemma 1 we may further assume a degree 3 vertex with the third
degree $x$. In the second of Figure 3 the edge $x$ is adjacent to both $c$ and $d$, a
contradiction. In the third of Figure 3 the edge $x$ is adjacent to both $a$ and
d, again a contradiction.

The combination $a^2b^2c$ has three possible arrangements, given by the first
and second of Figure 2 and fourth of Figure 3. In the fourth of Figure 3,
we may assume (by Lemma 1) a degree 3 vertex with the third edge $x$. The
degree $x$ is adjacent to both $a$ and $c$, a contradiction.

The combination $a^3bc$ has two possible arrangements, given by the third
of Figure 2 ($b, c$ adjacent) and the left of Figure 4 ($b, c$ separated). In case
$b, c$ are separated, the pentagon has one $a^2$-angle that we denote by $\alpha$, two
$ab$-angles, and two $ac$-angles. Since there are no $b^2$-angle, $c^2$-angle, $bc$-angle,
any degree 3 vertex must be one of three on the left of Figure 4. This implies
that the $a^2$-angle $\alpha$ appears at every degree 3 vertex. By Lemma 5 (and the
remark after the proof of the proposition), the pentagon should have at least
two $a^2$-angles, a contradiction.

![Figure 4: Not suitable for tiling.](image)

The combination $a^3b^2$ has two possible arrangements, given by the fourth
of Figure 2 ($b$ adjacent) and the right of Figure 4 ($b$ separated). In case
the two \( b \)-edges are separated, the pentagon has one \( a^2 \)-angle \( \alpha \) and four \( ab \)-angles. Since there is no \( b^2 \)-angle, any degree 3 vertex must be one of two on the right of Figure \([4]\). This implies that the \( a^2 \)-angle \( \alpha \) appears at every degree 3 vertex. By Lemma \([5]\) the pentagon should have at least two \( a^2 \)-angles, a contradiction.

Finally, the first of Figure \([2]\) has three \( ab \)-vertices (shared by \( a \)-edge and \( b \)-edge). If any such \( ab \)-vertex has degree 3, then the third edge at the vertex is adjacent to both \( a \) and \( b \). The only such edge in the pentagon is \( c \).

If two \( ab \)-vertices of degree 3 are adjacent, then we have two \( c \)-edges at the two vertices, and these two \( c \)-edges belong to the same pentagon, a contradiction. Therefore we cannot have adjacent \( ab \)-vertices of degree 3. The observation implies the final statement of the lemma. \( \Box \)

### 2.4 Vertex Configuration

In this paper, we study tilings by pentagons with edge combinations \( a^2b^2c \) and \( a^3bc \). By Lemma \([9]\) there are three possible ways of arranging the edges. Then we denote the angles as in Figure \([5]\).

![Figure 5: Edges and angles for \( a^2b^2c \) and \( a^3bc \), with \( a, b, c \) distinct.](image)

Lemma \([9]\) only assumes that the edge lengths \( a, b, c \) are distinct. As remarked after Lemma \([5]\) this may cause some ambiguity about angles. One source of ambiguity is that angles with distinct notations may have equal value. In this case, we may further use the edge lengths to distinguish the angles. The simplest case is the second of Figure \([5]\) where the angles \( \alpha, \beta, \gamma, \delta, \epsilon \) can be unambiguously described as \( ab \)-angle, \( a^2 \)-angle, \( b^2 \)-angle, \( ac \)-angle, \( bc \)-angle. In the first of Figure \([5]\) there is no ambiguity for the \( bc \)-angle \( \delta \) and \( ac \)-angle \( \epsilon \). However, \( \alpha, \beta, \gamma \) are all \( ab \)-angles, and can be distinguished only by referring to the pentagon (not adjacent to \( \delta, \epsilon \), adjacent to \( \delta \), and adjacent to \( \epsilon \)). Similarly, in the third of Figure \([5]\) \( \alpha, \beta, \gamma \) are \( bc \)-angle, \( ab \)-angle, \( ac \)-angle. Then \( \delta \) is the \( a^2 \)-angle adjacent to \( \beta \), and \( \epsilon \) is the \( a^2 \)-angle adjacent to \( \gamma \).
We introduce a notation for angle and edge configurations at vertices. First we note that angles are bounded by edges of certain lengths. For example, for \( \alpha \), we indicate the bounding edges by denoting \( |\alpha| \) and \( \alpha | \) for the first and second of Figure 5, and \( |\alpha| \) (or \( \alpha | \)) for the third of Figure 5. Then we may denote the vertices in Figure 6 by \( |\beta| |\gamma| \), \( |\alpha| |\delta| \), and \( |\alpha| |\beta| \). Here \( \cdots \) consists of the additional angles at the vertex, and is called the remainder. We may also denote \( |\alpha| |\delta| \cdots \) by \( \delta |\alpha| \), \( |\alpha| \delta \cdots \), \( \alpha |\delta| \cdots \), \( \delta |\alpha| \cdots \), etc. More briefly, we may omit the edge lengths and denote \( |\alpha| |\delta| \cdots \) by \( \alpha |\delta| \cdots \), \( |\alpha| \delta \cdots \), \( \alpha |\delta| \cdots \), \( \delta |\alpha| \cdots \). Without indicating the edge lengths, the notation no longer implies order or adjacency, unless we explicitly say consecutive. For example, \( \alpha \delta \cdots \) simply means that vertex has at least one \( \alpha \) and at least one \( \delta \), and consecutive \( \alpha \delta \) means \( \alpha \) and \( \delta \) are adjacent at a vertex.

![Diagram of adjacent angle deduction](image)

Figure 6: Adjacent angle deduction.

If the first vertex in Figure 6 is part of a tiling by the first of Figure 5, then we may determine the arrangements for the three tiles and get vertices \( |\alpha| \cdots \), \( |\alpha| \delta \cdots \) just off \( |\beta| |\gamma| \cdots \). The vertices are obtained by the following process. First we denote angles \( \alpha, \delta \) adjacent to \( \beta \) (in the pentagon) by \( |\beta| |\delta| \) (or \( |\beta| \alpha | \delta | \)), and denote angles \( \alpha, \epsilon \) adjacent to \( \gamma \) by \( |\gamma| |\epsilon| \alpha | \). We may then combine these into the adjacent angle deduction

\[
|\beta| |\gamma| \rightarrow |\delta| |\alpha| |\delta| \epsilon | \alpha |.
\]

On the right, we can clearly see the vertices \( |\alpha| \cdots \) and \( |\alpha| \delta \cdots \). Similarly, if the second and third of Figure 6 are parts of tilings by the second and third of Figure 5, and we get adjacent angle deductions

\[
|\alpha| \epsilon |\delta| \rightarrow |\beta| |\gamma| |\delta| \epsilon | \beta |, \quad |\alpha| |\delta| \beta \rightarrow |\beta| |\gamma| |\beta| |\alpha| |\delta| |\beta |.
\]

The adjacent angle deduction is an efficient notation that deduce new vertices from existing ones without drawing pictures. But we need to be aware of the possible ambiguities for the angles, because the configuration at a vertex alone may not be enough to uniquely identify some angles. The
readers are advised to double check adjacent angle deductions by drawing pictures.

An adjacent angle deduction may have several outcomes, due to possibly different ways of arranging the tiles in compatible way. For example, consecutive $\gamma \delta$ for the third of Figure 5 has two possible adjacent angle deductions

$$\gamma \delta = l\gamma l \delta l \rightarrow l\alpha \epsilon l\beta \gamma l, \ l\alpha \epsilon l\gamma \beta l.$$ 

We denote all the possible outcomes by $l\alpha \epsilon l(\beta \gamma)l$, where $(\beta \gamma)$ can be either $\beta \gamma$ or $\gamma \beta$. 

The adjacent angle deduction is symmetric with respect to flipping (corresponding to the change of vertex orientation). For example, the adjacent angle deduction above for $\gamma \delta$ is also

$$l\delta l \gamma l \rightarrow l\gamma l \beta l \alpha l, \ l\beta l \gamma l \alpha l.$$ 

If all the angles at a vertex are included, then the adjacent angle deduction is circular. For example, a vertex $\gamma^2 \delta$ for the third of Figure 5 is $l\gamma l \gamma l \delta l$, and is also $l\delta l \gamma l \gamma$ or $l\gamma l \delta l \gamma$. Correspondingly, we have the same adjacent angle deductions

$$l\gamma l \gamma l \delta l \rightarrow l\epsilon l \alpha l(\beta \gamma)l, \ l\delta l \gamma l \gamma l \rightarrow l(\beta \gamma)l \alpha l \epsilon l, \ l\gamma l \delta l \gamma l \rightarrow l\alpha l(\beta \gamma)l \epsilon l.$$ 

As an application of the technique of adjacent angle deduction, let us consider a vertex $\gamma^k$ in a tiling by the second of Figure 5. The vertex is $k$ copies of $l\gamma l$, and adjacent angle deduction at the vertex is

$$l\gamma l \gamma l \cdots l\gamma l \gamma l \rightarrow l(\alpha \epsilon)l(\alpha \epsilon)l \cdots l(\alpha \epsilon)l(\alpha \epsilon)l.$$ 

Since the deduction is circular, we find that on the right, the number of $\alpha l \alpha$ is the same as the number of $l \epsilon l$. In particular, if $\alpha l \alpha$ appears on the right, then $l \epsilon l$ also appears.

\section{Subdivision Tiling}

\subsection{Pentagonal Subdivision}

Given any tiling of an oriented surface, we add two vertices to each edge. If an edge belongs to a tile, then we can use the orientation to label the
two vertices as the first and the second, with respect to the tile (and the orientation). See the labels for the five edges of the pentagonal tile on the left of Figure 7. Note that each edge is shared by two tiles, and the labels from the viewpoints of two tiles are different. See the edge shared by the pentagon and the triangle below it.

We further add one vertex at the center of each tile, and connect the center vertex to the first vertex of each boundary edge of the tile. This divides an $m$-gon tile into $m$ pentagons. See the right of Figure 7. The process turns any tiling into a pentagonal tiling. We call the process *pentagonal subdivision*.

Each tile in the pentagonal subdivision has three edge vertices, one old vertex (from the original tiling), and one center vertex. The edge vertices have degree 3. The degree of the old vertex remains the same. The degree of the center vertex is the “degree” (number of edges) of the original tile. It is easy to see that the pentagonal subdivisions of a tiling and its dual tiling are combinatorially the same, with old vertices and center vertices exchanged.

Suppose we start with a regular tiling, and the distance from the new edge vertices to the nearby original vertices are the same. Then the pentagonal subdivision is a tiling by congruent pentagons. See Figure 8. In fact, we no longer require the original edges to be straight. This means that we only require $\alpha + \delta + \epsilon = 2\pi$ instead of $\alpha + \delta = \epsilon = \pi$. Moreover, if the original regular tiling is made up of $m$-gons and each vertex has degree $n$, then $\beta = \frac{2\pi}{m}$ and $\gamma = \frac{2\pi}{n}$.

The regular tilings of the sphere are the five platonic solids. Since dual tilings give the same pentagonal subdivision, the pentagonal subdivision gives three tilings of the sphere by congruent pentagons.

1. Pentagonal subdivision of tetrahedron: $f = 12$. 
2. Pentagonal subdivision of cube or octahedron: \( f = 24 \).

3. Pentagonal subdivision of dodecahedron or icosahedron: \( f = 60 \).

The first tiling is the deformed dodecahedron \( T_5 \) in [3]. Further by [1], this is the only edge-to-edge tiling of the sphere by 12 congruent pentagons. The second tiling is the first of Figure 9.

The three tilings are pentagonal subdivisions of three regular triangular tilings (corresponding to \( m = 3 \) and \( n = 3, 4, 5 \)): tetrahedron, octahedron, icosahedron. We draw each tile by the dotted (spherical) regular triangle in the second of Figure 9 with angle \( \frac{2\pi}{n} = \frac{2\pi}{3}, \frac{\pi}{2}, \frac{2\pi}{5} \) at the vertex \( V \). Let \( C \) be the center of triangle, and let \( M \) be the middle point of an edge of triangle. The pentagonal subdivision is completely determined by a point \( P \) in the following way. The point \( P \) determines another point \( Q \), by requiring that \( M \) is the middle point of the great arc \( PQ \). Moreover, rotating \( P \) around \( C \) by \( \frac{2\pi}{3} \) gives \( P' \). Then \( C, P, Q, V, P' \) are the vertices of a tile in the pentagonal subdivision. Rotating this tile around \( C \) by \( \frac{2\pi}{3} \) and \( \frac{4\pi}{3} \) give two
more tiles, and the three tiles together form the pentagonal subdivision of one regular triangular tile of the original platonic solid. The whole pentagonal subdivision tiling is obtained by replacing each triangle in the platonic solid by the second of Figure 9. The description gives a one-to-one correspondence between the tiling and the location of $P$, subject to the only condition that the three pentagons in the second of Figure 9 do not overlap. In particular, the pentagonal subdivision tiling allows two free parameters. The study of the 2-dimensional moduli of pentagonal subdivision tiling will be the subject of another paper.

### 3.2 Double Pentagonal Subdivision

Given any tiling of a surface, we have the dual tiling. The original tiling and the dual tiling give a combined tiling by quadrilaterals. If the surface is also oriented, then we may use the orientation to further cut each quadrilateral into two halves in a compatible way. The process turns any tiling into a pentagonal tiling. We call the process *double pentagonal subdivision*.

![Figure 10: Double pentagonal subdivision.](image)

Similar to pentagonal subdivision, we hope that the double pentagonal subdivision of a platonic solid can be a tiling of the sphere by congruent pentagons. Since the dual tiling gives the same double pentagonal subdivision, we only need to consider the double pentagonal subdivision of one triangle tile in regular tetrahedron, octahedron, icosahedron (each vertex has degree $n = 3, 4, 5$). This is given by the first of Figure 11. By the angle sums at the vertices, we get the angles of the pentagonal tile in the second of Figure 11.

$$
\alpha = \frac{1}{2}\pi, \quad \beta = \left(1 - \frac{1}{n}\right)\pi, \quad \gamma = \delta = \frac{2}{3}\pi, \quad \epsilon = \frac{2}{n}\pi.
$$

The number of tiles are respectively $f = 24, 48, 120$.

Unlike pentagonal subdivision, all angles of the pentagon in a double pentagonal subdivision tiling are fixed. Note that a general pentagon has
7 degrees of freedom. On the other hand, the specific values of 5 angles and 3 equal edges amount to 7 equations. Therefore we expect the double pentagonal subdivision tilings to be isolated examples. This can be argued as follows. We connect three edges of length $a$ together at angles $\delta = \frac{2}{3}\pi$ and $\epsilon = \frac{2}{n}\pi$. Then at the two ends, we produce the $b$-edge and $c$-edge at angles $\beta = (1 - \frac{1}{n})\pi$ and $\gamma = \frac{2}{3}\pi$ with the existing $a$-edges. They meet and create a pentagon $P(a)$ that is determined by $a$. In fact, $b, c, \alpha$ are also determined by $a$. The area of $P(a)$ is

$$A(a) = \alpha + \beta + \gamma + \delta + \epsilon - 3\pi = \alpha + \left(\frac{1}{n} - \frac{2}{3}\right)\pi,$$

and is a strictly increasing function of $a$. When $a$ is very small, the pentagon is very small, and therefore $\alpha(a) = \left(\frac{2}{3} - \frac{1}{n}\right)\pi + A(a)$ is slightly bigger than $\left(\frac{2}{3} - \frac{1}{n}\right)\pi$. When $a$ grows, $A(a)$ will reach $\left(\frac{1}{n} - \frac{1}{6}\right)\pi$, and the angle $\alpha$ will reach the desired value $\frac{1}{3}\pi$, and we get the desired pentagon.

Next we calculate the exact size of the pentagons in the double pentagonal subdivision tilings. For a triangle face in the tetrahedron, octahedron, icosahedron, we form the triangle in the first of Figure 12 by connecting the center point $\delta$ to the middle point $\alpha$ of the edge and the vertex $\epsilon$ (the points are the center and the middle by symmetry). We know the angles of the triangle, and denote the edges by $x, y, z$. The edges are determined by the angles

$$\cos x = \cot \frac{1}{3}\pi \cot \frac{1}{n}\pi, \quad \cos y = \frac{\cos \frac{1}{n}\pi}{\sin \frac{1}{3}\pi}, \quad \cos z = \frac{\cos \frac{1}{n}\pi}{\sin \frac{1}{3}\pi}.$$

The first of Figure 11 shows that the $x$-edge connects the two edges of three connected $a$-edges at alternating angles $\delta$ and $\epsilon$. See the second of
Figure 12: Calculate pentagon in double pentagonal subdivision tiling.

Figure 12. By [1, Lemma 3], we have

\[
\cos x = \cos^3 a + \sin a (\sin a \cos a \cos \epsilon + \cos a \sin a \cos \delta)
- \sin a \sin a (\sin \epsilon \sin \delta + \cos a \cos \epsilon \cos \delta)
= \cos^3 a (1 - \cos \delta)(1 - \cos \epsilon) + \cos^2 a \sin \delta \sin \epsilon
+ \cos a (\cos \delta + \cos \epsilon - \cos \delta \cos \epsilon) - \sin \delta \sin \epsilon.
\]

For \(f = 24\), this means

\[
\cos x = \frac{1}{3} = \frac{9}{4} \cos^3 a + \frac{3}{4} \cos^2 a - \frac{5}{4} \cos a - \frac{3}{4}.
\]

The cubic equation has only one real solution (the approximate value means \(0.1486\pi \leq a < 0.1487\pi\))

\[
\cos a = \frac{2}{9} \sqrt[3]{19 + 3\sqrt{33}} + \frac{8}{9\sqrt[3]{19 + 3\sqrt{33}}} - \frac{1}{9}, \quad a \approx 0.1486\pi.
\]

The first of Figure 11 also gives the triangle in the third of Figure 12. Then we have the cosine law

\[
\cos y = \cos a \cos b + \sin a \sin b \cos \beta.
\]

This can be regarded as a quadratic equation for \(\cos b\) or \(\sin b\). Then we can get a precise formula for \(\cos b\) and \(\sin b\) in square and cubic roots, and we get \(b \approx 0.2056\pi\). Since \(\beta = \gamma = \delta = \epsilon\) for \(f = 24\), the pentagon is symmetric, and we have \(b = c\).

For \(f = 48\), we get the cubic equation for \(\cos a\)

\[
\cos x = \frac{1}{\sqrt{3}} = \frac{3}{2} \cos^3 a + \frac{\sqrt{3}}{2} \cos^2 a - \frac{1}{2} \cos a - \frac{\sqrt{3}}{2}.
\]
The cubic equation has only one real solution
\[
\cos a = \frac{1}{9} \sqrt[3]{186\sqrt{3} + 54\sqrt{35}} + \frac{4}{3\sqrt[3]{186\sqrt{3} + 54\sqrt{35}}} - \frac{1}{9}\sqrt{3}, \quad a \approx 0.1278\pi.
\]
We can also get the precise formula for \( \cos b, \sin b, \cos c, \sin c \), and then get 
\( b \approx 0.0840\pi, \ c \approx 0.1627\pi \).

For \( f = 120 \), we get the cubic equation for \( \cos a \)
\[
\cos x = \frac{\sqrt{5} + 1}{\sqrt{6(5 - \sqrt{5})}} = \frac{1}{8} \left( 3(5 - \sqrt{5})\cos^3 a + \sqrt{6(5 + \sqrt{5})}\cos^2 a \\
+ (-7 + 3\sqrt{5})\cos a - \sqrt{6(5 + \sqrt{5})} \right)
\]
The cubic equation has only one real solution. We have precise formula for \( \cos a, \cos b, \cos c \), and then get \( a \approx 0.0960\pi, b \approx 0.0238\pi, c \approx 0.1081\pi \).

### 4 Tiling for Edge Combination \( a^2b^2c \)

By Lemma 9, for an edge-to-edge tiling of the sphere by congruent pentagons
with edge combination \( a^2b^2c \), the pentagon is the first or the second of Figure 5. Moreover, in case the pentagon is a special tile in Lemma 1, the first must be a \( 3^44 \)- or \( 3^45 \)-tile, and the vertex \( H \) of degree 4 or 5 is the vertex where \( \alpha \) is. Therefore our classification can be divided into four cases in Figure 13, with the dot vertex being the vertex \( H \) of the special tile. The case that there is a \( 3^5 \)-tile is part of the second pentagon, with the dot vertex having degree 3. The detailed argument is in Section 4.2. This means that there is no \( 3^5 \)-tile in Sections 4.1, 4.3, 4.4, 4.5. In particular, the dot vertex always has degree 4 or 5 in these sections, and by Lemma 2 we may assume \( f \geq 24 \) in these sections.

![Figure 13: Cases for edge combination \( a^2b^2c \).](image)
Our strategy is to first tile the partial neighborhood in the fourth of Figure 1. Then we study the configuration around $H$, and further tile beyond the partial neighborhood if necessary. We will always start by assuming that the center tile $P_i$ is given by the pentagon in Figure 13.

To facilitate discussion, we denote by $P_i$ the tile labeled $i$, by $E_{ij}$ the edge shared by $P_i, P_j$, and by $V_{ijk}$ the vertex shared by $P_i, P_j, P_k$. We denote by $A_{i,jk}$ the angle of $P_i$ at $V_{ijk}$, and by $\theta_i$ the angle $\theta$ in $P_i$. When we say a tile is determined, we mean that we know all the edges and angles of the tile.

4.1 Case 1 $(a^2b^2c)$

Let the first of Figure 13 be the center tile $P_1$ in the partial neighborhood in the first of Figure 14. Since $E_{23}$ and $E_{56}$ are adjacent to both $a$ and $b$, we get $E_{23} = E_{56} = c$. This determines (all edges and angles of) $P_2, P_3, P_5, P_6$. Then we know three edges of $P_4$, which further determines $P_4$.

The angle sums at $\beta\delta\epsilon, \gamma\delta\epsilon$ and the angle sum for pentagon (Lemma 4) imply $\beta = \gamma$ and $\alpha + \beta = (1 + \frac{4}{7})\pi$. By the edge length consideration, we have $H = \alpha^2 \beta \gamma, \alpha \beta^2 \gamma, \alpha \beta \gamma^2, \alpha \beta \gamma \delta \epsilon$. By $2\alpha + \beta + \gamma = 2(\alpha + \beta) > 2\pi$ and $\alpha + \beta + \gamma + \delta + \epsilon = (3 + \frac{4}{7})\pi > 2\pi$, we get $H \neq \alpha^2 \beta \gamma, \alpha \beta \gamma \delta \epsilon$.

Note that the partial neighborhood tiling is symmetric with respect to the exchange $a \leftrightarrow b, \beta \leftrightarrow \gamma, \delta \leftrightarrow \epsilon$. Therefore up to the symmetry, we may assume $H = \alpha \beta^2 \gamma$. By the extra angle sum at $H$, we get $\alpha = (\frac{1}{2} + \frac{6}{7})\pi, \beta = \gamma = (\frac{1}{2} - \frac{2}{7})\pi, \delta + \epsilon = (\frac{3}{2} + \frac{2}{7})\pi$.

We have adjacent angle deduction at $H = |\alpha|\beta|\beta|\gamma$.

$$|\beta|\gamma| \rightarrow |\alpha|\delta|\alpha|\epsilon|.$$ 

This gives a vertex $\alpha|\delta|\cdots = |\alpha|\delta|\cdots$. Since an angle bounded by $c$-edge is $\delta$ or $\epsilon$, the vertex is $|\alpha|\delta|\delta|\cdots$ or $|\alpha|\delta|\epsilon|\cdots$. By $\alpha + \delta + \epsilon = (2 + \frac{8}{7})\pi > 2\pi$, $|\alpha|\delta|\delta|\cdots$ is not a vertex. Therefore the vertex is $|\alpha|\delta|\delta|\cdots$, and we have adjacent angle deduction $|\delta|\delta| \rightarrow |\beta|\epsilon|\beta|$ at the vertex. Therefore $\epsilon|\epsilon|\cdots = |\epsilon|\cdots$ is a vertex. Since there is no $a^2$-angle, we get $|\epsilon|\cdots = \theta|\epsilon|\rho\cdots$ for at least two angles $\theta, \rho$.

Since $|\alpha|\delta|\delta|\cdots$ is a vertex, the angle sum at the vertex implies $\delta \leq \frac{1}{2}(2\pi - \alpha) = (\frac{3}{2} - \frac{4}{7})\pi$. Then $\epsilon = (\frac{3}{2} + \frac{5}{7})\pi - 2 \delta \geq (\frac{4}{2} + \frac{5}{7})\pi > \alpha > \beta = \gamma$. This implies that the minimal value among the five angles is either $\beta$ or $\delta$. Therefore the angle sum at $\theta|\epsilon|\rho\cdots$ is $\geq 2\beta + 2\epsilon$ or $2\delta + 2\epsilon$. By $\beta + \epsilon \geq (\frac{1}{2} - \frac{4}{7})\pi + (\frac{4}{2} + \frac{5}{7})\pi > \pi$ and $\delta + \epsilon = (\frac{3}{2} + \frac{2}{7})\pi > \pi$, we get a contradiction.
4.2 Case 2 \((a^2b^2c)\)

Let the second of Figure 13 be the center tile \(P_1\) in the partial neighborhood in the second and third of Figure 14. The two pictures show two possible ways of arranging edges and angles of \(P_4\). In the first picture, we may determine \(P_3, P_5\), and then get \(b^2\)-angle \(\gamma_2\) and \(a^2\)-angle \(\beta_6\). In the second picture, we may get \(b^2\)-angle \(\gamma_3\) and \(a^2\)-angle \(\beta_5\). Then \(E_{23} = a\) or \(c\), and either way gives \(\gamma_2\). Similarly, we get \(\beta_6\).

In the second of Figure 14, the angle sums at \(\alpha\delta\epsilon, \beta\gamma^3, \gamma^3\) and the angle sum for pentagon imply \(f = 12\). By [1, 3], the whole tiling is the deformed dodecahedron, which is the pentagonal subdivision of regular tetrahedron.

We note that by the edge length consideration, the tile \(P_1\) in the third of Figure 14 cannot be a \(3^5\)-tile. Therefore the case of edge combination \(a^2b^2c\) and there is a \(3^5\)-tile is included in the second of Figure 14. This means that we may assume there is no \(3^5\)-tile in the subsequent discussion for edge length combination \(a^2b^2c\).

In the third of Figure 14 by the edge length consideration, the vertex \(H = \alpha^2\beta\gamma, \alpha^2\beta\gamma^2, \alpha^2\beta^2\gamma, \alpha\beta\gamma\delta\epsilon\). By Lemma 4, \(\alpha\beta\gamma\delta\epsilon\) is not a vertex. Moreover, the first three possibilities imply \(2\alpha + \beta + \gamma < 2\pi\). Combined with the angle sums at \(\beta\delta^2, \gamma\epsilon^2\), we get

\[
\alpha + \beta + \gamma + \delta + \epsilon = \frac{1}{2}((2\alpha + \beta + \gamma) + (\beta + 2\delta) + (\gamma + 2\epsilon)) < 3\pi,
\]

again contradicting Lemma 4.

4.3 Case 3 \((a^2b^2c)\)

Let the third of Figure 13 be the center tile \(P_1\) in the partial neighborhood in Figure 15. We consider two possible arrangements of \(P_5\). The first picture
shows one arrangement, and the second and third show the other arrangement. We label the three partial neighborhood tilings as Cases 3.1, 3.2, 3.3.

We also recall that, after Case 2, the degree of $H$ is 4 or 5.

In the first picture, we get $b_2$-angle $\gamma_4$ and $a_2$-angle $\beta_6$. Then $E_{34} = a$ or $c$, and either way gives $\gamma_3$ and $E_{23} = b$. This determines $P_2$.

In the second and third pictures, we may determine $P_4$, $P_6$. Then the two pictures show two possible arrangements of $P_3$. In the second picture, we get $a_2$-angle $\beta_2$. In the third picture, we may determine $P_2$.

![Figure 15: Case 3 for $a^2b^2c$.](image)

**Case 3.1**

The angle sums at $\alpha^2\gamma, \gamma\epsilon^2, \beta\delta^2$ and the angle sum for pentagon imply

\[
\alpha = \epsilon = \pi - \frac{1}{2} \gamma, \quad \beta = \frac{8}{7} \pi, \quad \delta = (1 - \frac{4}{7}) \pi.
\]

The angle $A_{6,12} = \alpha$ or $\delta$ since it is adjacent to $\beta_6$. By the edge length consideration and the fact that $\alpha^2\gamma, \beta\delta^2$ are vertices, we get $H = \alpha^2\beta^2, \alpha^2\beta^3, \alpha\beta^2\delta\epsilon$.

Suppose $H = \alpha\beta^2\delta\epsilon$. The extra angle sum at $H$ implies

\[
\alpha = \epsilon = (\frac{1}{2} - \frac{6}{7}) \pi, \quad \beta = \frac{8}{7} \pi, \quad \gamma = (1 + \frac{12}{7}) \pi, \quad \delta = (1 - \frac{4}{7}) \pi.
\]

By the edge length consideration, we have $H = \alpha\alpha\alpha\cdots$, and adjacent angle deduction $\alpha\alpha\alpha \rightarrow \beta\gamma\gamma\delta\delta$. Therefore $\gamma\gamma\gamma\cdots$ is a vertex, contradicting $\gamma > \pi$.

Suppose $H = \alpha^2\beta^2, \alpha^2\beta^3$. The extra angle sum at $H$ implies

\[
H = \alpha^2\beta^2: \alpha = \epsilon = (1 - \frac{6}{7}) \pi, \quad \beta = \frac{8}{7} \pi, \quad \gamma = \frac{16}{7} \pi, \quad \delta = (1 - \frac{4}{7}) \pi;
\]

\[
H = \alpha^2\beta^3: \alpha = \epsilon = (1 - \frac{12}{7}) \pi, \quad \beta = \frac{8}{7} \pi, \quad \gamma = \frac{24}{7} \pi, \quad \delta = (1 - \frac{4}{7}) \pi.
\]

By the edge length consideration, we have $H = \alpha\alpha\alpha\cdots$, and adjacent angle deduction $\alpha\alpha\alpha \rightarrow \beta\gamma\gamma\beta\beta$. Therefore $\gamma\gamma\gamma\cdots$ is a vertex. If the remainder of $\gamma\gamma\gamma\cdots$ has one $\alpha$ or $\epsilon$, then by the edge length consideration,
the remainder has another $\alpha$ or $\epsilon$. By $\gamma + \alpha > \pi$ and $\gamma + \epsilon > \pi$, we get a contradiction. Therefore there is no $\alpha$ and $\epsilon$ in the remainder. Then by the edge length consideration, the remainder also has no $\beta$, and we get $I\gamma I\gamma I\cdots = I\gamma I\gamma I\cdots I\gamma I = \gamma^k$. We have adjacent angle deduction

$$I\gamma I\gamma I\cdots I\gamma I \rightarrow I(\alpha\epsilon)I(\alpha\epsilon)I\cdots I(\alpha\epsilon)I(\alpha\epsilon)I.$$

Recall that the initial pair $|\alpha|\alpha|$ is obtained from adjacent angle deduction on $|\alpha|\alpha|$. Therefore we have $\alpha|\alpha|$ on the right side of the adjacent angle deduction above. This implies that $\epsilon|\epsilon|$ also appears. See the example at the end of Section 2.4. Therefore we get $f \geq 24$ in case $H = \alpha^2\beta^2$ and $f \geq 36$ in case $H = \alpha^2\beta^3$. This implies $4\epsilon > 2\pi$ in both cases, and we get a contradiction.

**Case 3.2**

The angle sums at $\alpha^2\beta, \alpha\delta\epsilon, \gamma^3$ and the angle sum for pentagon imply

$$\alpha = (\frac{5}{6} - \frac{2}{7})\pi, \beta = (\frac{1}{3} + \frac{4}{7})\pi, \gamma = \frac{2}{3}\pi, \delta + \epsilon = (\frac{7}{6} + \frac{2}{7})\pi.$$

The angle $A_{2,16} = \alpha$ or $\delta$. By the edge length consideration and the fact that $\alpha^2\beta, \alpha\delta\epsilon$ are vertices, we get $H = \beta^2\delta^2, \beta^3\delta^2$. The extra angle sum at $H$ implies

$$H = \beta^2\delta^2: \delta = (\frac{5}{3} - \frac{4}{7})\pi, \epsilon = (\frac{1}{2} + \frac{6}{7})\pi;$$

$$H = \beta^3\delta^2: \delta = (\frac{1}{2} - \frac{9}{7})\pi, \epsilon = (\frac{2}{3} + \frac{8}{7})\pi.$$

Consider the vertex $|\epsilon|\epsilon|\cdots$ shared by $P_3, P_4$ in the second of Figure 15. By the edge length consideration, the vertex is $\theta|\epsilon|\epsilon|\cdots$, with $\theta, \rho = \delta$ or $\epsilon$. By $\delta + \epsilon > \pi$ and $2\epsilon > \pi$, we get a contradiction.

**Case 3.3**

The angle sums at $\alpha\delta\epsilon, \gamma^3$ and the angle sum for pentagon imply

$$\alpha + \delta + \epsilon = 2\pi, \beta = (\frac{1}{3} + \frac{4}{7})\pi, \gamma = \frac{2}{3}\pi.$$

By the edge length consideration, we have $H = \alpha^2\beta^3, \beta^3\delta^2, \beta^4, \beta^5$.  

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Suppose \( H = \alpha^2 \beta^3 \). The extra angle sum at \( H \) implies
\[
\alpha = \left( \frac{1}{2} - \frac{6}{7} \right) \pi, \ \beta = \left( \frac{1}{3} + \frac{4}{7} \right) \pi, \ \gamma = \frac{2}{3} \pi, \ \delta + \epsilon = \left( \frac{3}{2} + \frac{6}{7} \right) \pi.
\]

By the third of Figure 15, we have adjacent angle deduction \( \beta \alpha \beta \alpha \beta \rightarrow \alpha \delta \beta \gamma \beta \alpha \delta \) at \( H \). This gives a vertex \( \beta \alpha \beta \cdots = \beta \beta \delta \beta \cdots \) or \( \beta \beta \delta \beta \cdots \).

The vertex \( \beta \beta \delta \beta \cdots \) implies \( \delta \leq \pi - \frac{1}{2} \beta = \left( \frac{5}{6} - \frac{2}{7} \right) \pi, \epsilon = \left( \frac{3}{2} + \frac{8}{7} \right) \pi - \delta \geq \left( \frac{2}{3} + \frac{8}{7} \right) \pi, \) and adjacent angle deduction \( \delta \delta \rightarrow \beta \epsilon \beta \). Then we have a vertex \( \epsilon \epsilon \cdots \), with the remainder having at least one \( \gamma \) or at least two from \( \alpha, \epsilon \), by the edge length consideration. By \( \epsilon \geq \left( \frac{2}{3} + \frac{8}{7} \right) \pi \) and \( f \geq 24 \), this implies that the angle sum at \( \epsilon \epsilon \cdots \) is always > 2\( \pi \), a contradiction.

The remainder of the vertex \( \beta \alpha \beta \cdots \) has value \( \left( \frac{1}{6} - \frac{10}{7} \right) \pi < \alpha, \beta, \gamma \), and therefore has only \( \delta, \epsilon \). By the edge length consideration, the remainder has at least one \( \delta \) and at least one \( \epsilon \). This contradicts \( \delta + \epsilon > \left( \frac{1}{6} - \frac{10}{7} \right) \pi \).

Suppose \( H = \beta^3 \delta^2 \). The extra angle sum at \( H \) implies
\[
\alpha + \epsilon = \left( \frac{3}{2} + \frac{6}{7} \right) \pi, \ \beta = \left( \frac{1}{3} + \frac{4}{7} \right) \pi, \ \gamma = \frac{2}{3} \pi, \ \delta = \left( \frac{1}{2} - \frac{6}{7} \right) \pi.
\]

By the third of Figure 15, we have adjacent angle deduction \( \beta \beta \alpha \beta \beta \alpha \beta \rightarrow \alpha \delta \beta \epsilon \beta \alpha \delta \beta \alpha \delta \) at \( H \). This gives vertices \( \beta \beta \beta \alpha \beta \) and \( \epsilon \epsilon \beta \beta \alpha \beta \). By the edge length consideration, we have \( \beta \beta \beta \alpha \beta \cdots = \beta \beta \beta \alpha \beta \cdots, \beta \beta \beta \alpha \beta \cdots, \beta \beta \beta \alpha \beta \cdots \). We also know that the remainder of \( \epsilon \epsilon \cdots \) has at least one \( \gamma \) or at least two from \( \alpha, \epsilon \).

The angle sum at \( \beta \beta \beta \alpha \beta \cdots \) or \( \beta \beta \beta \alpha \beta \cdots \) gives an upper bound for \( \alpha \) that further implies (using \( \alpha + \epsilon \)) \( \epsilon > \frac{1}{2} \pi \). Then \( \alpha + 2 \epsilon = (\alpha + \epsilon) + \epsilon > 2 \pi, 2 \gamma + 2 \epsilon > 2 \pi, 4 \epsilon > 2 \pi \), so that \( \epsilon \epsilon \cdots \) is not a vertex, a contradiction.

The remainder of \( \beta \beta \beta \alpha \beta \cdots \) has value \( \left( \frac{1}{6} - \frac{10}{7} \right) \pi < \beta, \gamma, \delta \), and therefore has only \( \alpha, \epsilon \). By the edge length consideration, the remainder has at least one \( \alpha \) and at least one \( \epsilon \). This contradicts \( \alpha + \epsilon > \left( \frac{1}{6} - \frac{10}{7} \right) \pi \).

We will discuss the remaining cases \( H = \beta^4, \beta^5 \) in Section 4.5.

### 4.4 Case 4 \((a^2b^2c)\)

Let the fourth of Figure 13 be the center tile \( P_1 \) in the partial neighborhood in Figure 16. We consider two possible arrangements of \( P_6 \). The first and second pictures show one arrangement. The third picture shows the other arrangement. We label the three partial neighborhood tilings as Cases 4.1, 4.2, 4.3. Again, the degree of \( H \) is 4 or 5.
In the first and second pictures, we may determine $P_5$ and $b^2$-angle $\gamma_4$. Then the two pictures show two possible arrangements of $P_4$. In the first picture, we may determine $P_3$ and $a^2$-angle $\beta_2$. In the second picture, we get $a^2$-angle $\beta_3$. Then $E_{56} = b$ or $c$, and either way gives $\beta_2$. In the third picture, we get $b^2$-angle $\gamma_5$. Then $E_{34} = a$ or $c$, and either way gives $\gamma_4$. This further determines $P_3$ and gives $a^2$-angle $\beta_2$.

Figure 16: Case 4 for $a^2b^2c$.

In Case 4.1, the angle sums at $\alpha\delta\epsilon, \beta^3, \gamma^3$ and the angle sum for pentagon imply $f = 12$. After Case 2, the case can be dismissed.

In Case 4.2, since $\alpha\delta\epsilon$ is a vertex, the remainder of $H = \beta\delta\epsilon \cdots$ has no $\alpha$. Then by the edge length consideration, we have $H = \beta\delta^2\epsilon^2$. Then angle sums at $\alpha^2\beta, \gamma^3$ and $H$ imply

$$\alpha + \beta + \gamma + \delta + \epsilon = \frac{1}{2}(2\alpha + \beta) + \frac{1}{2}3\gamma + \frac{1}{2}(\beta + 2\delta + 2\epsilon) = \frac{8}{3}\pi,$$

contradicting the angle sum for pentagon.

In Case 4.3, the angle sums at $\alpha^2\gamma, \beta^3, \gamma\epsilon^2$ and the angle sum for pentagon imply

$$\alpha = \epsilon = \pi - \frac{1}{2}\gamma, \beta = \frac{2}{3}\pi, \delta = \left(\frac{1}{3} + \frac{4}{\gamma}\right)\pi.$$
$\beta \gamma \delta \epsilon$, we know $\beta \gamma \delta \epsilon \cdots$ is a vertex. For $H = \alpha^2 \beta^2 \gamma^2$, this contradicts $\gamma > \pi$. For $H = \alpha^2 \beta^2$, the remainder of $\beta \gamma \delta \epsilon \cdots$ has value $(\frac{2}{3} - \frac{16}{27})\pi < \alpha, \beta, \gamma, 2\delta, \epsilon$. Therefore the remainder is a single $\delta$. However, $\gamma^2 \delta$ is not a vertex by edge length consideration. We get a contradiction.

We will discuss the remaining case $H = \delta^4$ in Section 4.5.

4.5 Pentagonal Subdivision

After Sections 4.1, 4.2, 4.3, 4.4, the only remaining cases for the edge combination $a^2 b^2 c$ are the following.

Case 3.3, $H = \beta^4$: The extra angle sum at $H$ gives
$$\alpha + \delta + \epsilon = 2\pi, \beta = \frac{1}{2}\pi, \gamma = \frac{2}{3}\pi, f = 24.$$

Case 3.3, $H = \beta^5$: The extra angle sum at $H$ gives
$$\alpha + \delta + \epsilon = 2\pi, \beta = \frac{2}{5}\pi, \gamma = \frac{2}{3}\pi, f = 60.$$

Case 4.3, $H = \delta^4$: The extra angle sum at $H$ gives
$$\alpha = \epsilon = \pi - \frac{1}{2}\gamma, \beta = \frac{2}{3}\pi, \delta = \frac{1}{4}\pi, f = 24.$$

The partial neighborhood tiling for Case 3.3 is the third of Figure 15 and for Case 4.3 is the third of Figure 16. Moreover, we know there is no $3^3$-tile after Case 2.

By Lemma 2 for $f = 24$, we know each tile is a $3^4$-tile. In Case 4.3, $H = \delta^4$, the four tiles $P_1, P_2, P_3, P_4$ around a vertex $\delta^4$ must be the first of Figure 17. Since every tile is a $3^4$-tile, all vertices of the four tiles except the central $\delta^4$ have degree 3. In particular, we have tiles $P_5, P_6, P_7$ as indicated, with their edges (those not shared with $P_1, P_2, P_3$) and angles to be determined. Up to the symmetry of vertical flipping, we may assume that the edges of $P_5$ are given as indicated. This determines one $b$-edge and one $c$-edge of $P_6$, and therefore determines all edges of $P_6$. Then we find that $P_7$ has 3 $b$-edges, a contradiction. This proves that Case 4.3, $H = \delta^4$, allows no tiling.

Next consider Case 3.3, $H = \beta^4$. Again every tile is a $3^4$-tile. Consider the four tiles around $\beta^4$. First assume that adjacent tiles always have opposite orientations. Then we get the second of Figure 17. Since every tile is a $3^4$-tile, the vertex indicated by the dot has degree 3. This implies that there
is a $c^2$-angle, a contradiction. Therefore there are two adjacent tiles around $\beta^4$ with the same orientation, say $P_1, P_2$ in the third of Figure 17. We may assume that the edges and angles of $P_1, P_2$ are given as indicated. Since every tile is a $3^44$-tile, we also have tiles $P_3, P_4, \ldots, P_8$, with their edges and angles to be determined.

The edges $E_{15} = c$ and $E_{25} = b$ determine $P_5$. The edges $E_{16} = b$ and $E_{56} = a$ determine $P_6$. Since $P_7$ already has two $b$-edges, and $E_{28} = c$, we have $E_{78} = a$. This determines $P_7, P_8$. Then $E_{23} = a$ and $E_{38} = b$ determine $P_3$. This shows that, by starting with $P_1, P_2$, we can derive $P_3$. By repeating with $P_2, P_3$ in place of $P_1, P_2$, we can further derive $P_4$. We have shown that the neighborhood of $\beta^4$ is uniquely given by four tiles with the same orientation.

It remains to show that the vertex $\beta^2\cdots$ shared by $P_5, P_6$ is $\beta^4$. If the vertex has degree 3, then by the edge length consideration, it is $\beta^3$, contradicting the fact that $\beta^4$ is already a vertex. Since every tile is a $3^44$-tile, the vertex $\beta^2\cdots$ has degree 4, and the vertex $\alpha\epsilon\cdots$ shared by $P_5, P_7$ has degree 3. Therefore we have a tile $P_9$ outside $P_5, P_7$. Then $E_{59} = a$ and $E_{79} = c$ determine $P_9$. In particular, we know $A_{9,56} = \beta$. Since $V_{569}$ has degree 4, by the edge length consideration, we get $V_{569} = \beta^4$.

By starting at a vertex $V_{1234} = \beta^4$, we find another vertex $V_{569} = \beta^4$. We may repeat the argument that started from $V_{1234} = \beta^4$ by restarting from $V_{569} = \beta^4$. After repeating six times, we get the pentagonal subdivision of cube (or octahedron).

The final remaining case is Case 3.3, $H = \beta^5$. After finishing Case 3.3, $H = \beta^4$, and Case 4.3, $H = \delta^4$, we may further assume that there is no $3^44$-tile. By Lemma 3 for $f = 60$, we know each tile is a $3^45$-tile. Among the five
tiles around $\beta^5$, there must be two adjacent tiles with the same orientation. Then the argument given by the third of Figure 17 can be carried out, because the argument never uses $P_4$. The argument shows that all five tiles have the same orientation, and also derives another degree 5 vertex $V_{569} = \beta^3 \cdots$, with three tiles $P_5, P_6, P_9$ having the same orientation.

By the edge length consideration, we have $V_{569} = \beta^5, \alpha^2 \beta^3, \beta^3 \delta^2$. The later two cases are given by Figure 18 with $P_1, P_2$ (these are $P_6, P_9$ in the third of Figure 17) having the same orientation. In both pictures, we can determine $P_3, P_4$, and then determine $P_5$. Since every tile is a $3^45$-tile, the vertex $e^2 \cdots$ shared by $P_1, P_3$ and the vertex $\alpha \beta \cdots$ shared by $P_2, P_4$ have degree 3. By the edge length consideration, the vertices are $\gamma e^2, \alpha^2 \beta$. Moreover, $P_1, P_3, P_5$ also share a vertex $\beta \delta^2$. We also have vertices $\alpha \delta \epsilon, \gamma^3$ in the third of Figure 18. The angle sums at the five degree 3 vertices imply all the angles are $\frac{2}{3} \pi$, contradicting $\beta = \frac{2}{5} \pi$. Therefore $V_{569} = \beta^3 \cdots$ is $\beta^5$, and the process of deriving new $V_{569} = \beta^5$ from existing $V_{123} = \beta^5$ can be repeated. After repeating the process twelve times, we get the pentagonal subdivision of dodecahedron (or icosahedron).

Figure 18: Case 3.3, $H = \beta^5$: $\beta^3 \cdots$ must be $\beta^5$.

5 Tiling for Edge Combination $a^3bc$

By Lemma 9, for an edge-to-edge tiling of the sphere by congruent pentagons with edge combination $a^3bc$, the pentagon is the third of Figure 5. Again we start with the neighborhood of a special tiling in Lemma 1. In case the tile is $3^44$- or $3^45$-tile, we use the dot to represent the vertex $H$ of degree 4 or 5. The classification is then divided into three cases in Figure 19. We will include the discussion of the neighborhood of $3^5$-tile in the first case. This means we will assume there is no $3^5$-tile in the second and third cases.
5.1 Case 1(\(a^3bc\))

Let the first of Figure 19 be the center tile \(P_1\) in the partial neighborhood in Figure 20. We consider whether \(E_{23} = a\) or \(E_{56} = a\). If both are not \(a\)-edges, then we get the first picture. If one is \(a\)-edge and the other is not, by the symmetry of exchanging \(b\) and \(c\), we may assume that \(E_{23} = b\) and \(E_{56} = a\). This is the second picture. If both are \(a\)-edges, then we get the third picture. We label the three partial neighborhood tilings as Cases 1.1, 1.2, 1.3.

In Case 1.1, since \(V_{123} = \alpha \beta \gamma\) is a vertex, the vertex \(V_{126} = \alpha \beta \gamma \cdots\) has empty remainder, so that \(P_1\) is a 3\(5\)-tile. We also note that, by the edge length consideration, the tile \(P_1\) cannot be a 3\(5\)-tile in Cases 1.2 and 1.3. Therefore Case 1.1 is equal to the existence of 3\(5\)-tile for \(a^3bc\). This means that, after Case 1.1, we may assume there is no 3\(5\)-tile. By Lemma 1, this implies \(f \geq 24\).

Case 1.1

No matter how we arrange angles of \(P_4\), we always get vertices \(\delta^2 \epsilon\) and \(\delta \epsilon^2\). The angle sums at \(\alpha \beta \gamma, \delta^2 \epsilon, \delta \epsilon^2\) and the angle sum for pentagon imply \(f = 12\). By Lemma 1, the tiling is the deformed dodecahedron, and cannot have edge combination \(a^3bc\).
Case 1.2

By the edge length consideration, we have $H = \alpha^3 \beta \gamma, \alpha^2 \gamma^2 \ldots$. Since $\alpha \beta \gamma$ is a vertex, we have $H \neq \alpha^3 \beta \gamma$. We will further show that $\alpha + \gamma > \pi$, so that $H \neq \alpha^2 \gamma^2 \ldots$.

Consider the edge $E_{45}$. If $E_{45} = a$, then $V_{134}$ and $V_{145}$ are combinations of $\delta, \epsilon$. If the combinations are different, then the angle sums at the two vertices imply $\delta = \epsilon = \frac{2}{7} \pi$. Combined with the angle sum at $\alpha \beta \gamma$ and the angle sum for pentagon, we get $f = 12$, contradicting $f \geq 24$. The requirement of the same combination of $\delta, \epsilon$ implies $A_{4,13} = \epsilon$, $A_{4,15} = \delta$, $A_{5,14} = \delta$. This further determines $P_4, P_5$, and we get the first of Figure 21. If $E_{45} = b$ or $c$, then we may determine $P_4, P_5$ and get the second and third of Figure 21. The angle sums at the four degree 3 vertices and the angle sum for pentagon imply

$$E_{45} = a: \alpha + \gamma = (1 + \frac{4}{7})\pi, \beta = \delta = (1 - \frac{4}{7})\pi, \epsilon = \frac{5}{7} \pi;$$
$$E_{45} = b: \alpha + \gamma = (\frac{3}{7} - \frac{2}{7})\pi, \beta = (\frac{1}{7} + \frac{2}{7})\pi, \delta = (1 - \frac{4}{7})\pi, \epsilon = \frac{5}{7} \pi;$$
$$E_{45} = c: \alpha = \beta = (\frac{1}{2} + \frac{2}{7})\pi, \gamma = \epsilon = (1 - \frac{4}{7})\pi, \delta = \frac{8}{7} \pi.$$

In all cases, we have $\alpha + \gamma > \pi$.

![Figure 21: Case 1.2 for $a^3bc$.](image)

Case 1.3

By the edge length consideration, we have $H = \alpha^4, \alpha^3 \beta \gamma$. In the third of Figure 20, we also note that, since $b$ and $c$ are adjacent, one of $E_{34}$ and $E_{45}$ is $a$. By the symmetry of exchanging $b$ and $c$, we may further assume that $E_{45} = a$. Then we consider three possibilities for $E_{34}$.

The case $E_{34} = a$ is illustrated by Figure 22. The vertices $V_{134}$ and $V_{145}$ are combinations of $\delta, \epsilon$. If the combinations are different, then the
angle sums at the two vertices imply $\delta = \epsilon = \frac{2}{3}\pi$. The angle sums at $V_{123}$ and $V_{156}$ further imply $\beta = \gamma = \frac{2}{3}\pi$. By [3, Lemma 21], we get $b = c$, a contradiction. Therefore $V_{134}$ and $V_{145}$ are the same combinations of $\delta, \epsilon$.

This implies $A_{4,13} = \delta, A_{4,15} = \epsilon, A_{3,14} = A_{5,14}$. Then we determine $P_4$ and get two pictures according to $A_{3,14} = A_{5,14} = \delta$ or $\epsilon$.

Figure 22: Case 1.3 for $a^3bc, E_{34} = a, E_{45} = a$.

In the first of Figure 22, the angle sums at $\beta, \gamma$ imply $\delta = \epsilon = \frac{2}{3}\pi$. The angle sum for pentagon implies

$$\alpha = \frac{4}{7}\pi + \frac{1}{2}\epsilon, \beta = \gamma = \delta = \pi - \frac{1}{2}\epsilon.$$ 

By $3\alpha + \beta + \gamma > 2\pi$, we get $H \neq \alpha^3\beta\gamma$. On the other hand, if $H = \alpha^4$, then the extra angle sum at $H$ implies

$$\alpha = \frac{1}{2}\pi, \beta = \gamma = \delta = (\frac{1}{2} + \frac{4}{7})\pi, \epsilon = (1 - \frac{8}{7})\pi.$$ 

By the edge length consideration, the remainder of the vertex $\beta\beta\cdots$ shared by $P_3, P_4$ has two angles from $\alpha, \beta$. This implies that the angle sum at the vertex is $> 2\pi$, a contradiction. By exchanging $b \leftrightarrow c, \beta \leftrightarrow \gamma, \delta \leftrightarrow \epsilon$, the same argument shows that the second of Figure 22 also leads to contradiction.

The case $E_{34} = b$ is illustrated by Figure 23. This determines $P_3, P_4$. Then two possible ways of arranging $P_5$ give two pictures.

In the first of Figure 23, the angle sums at $\beta\delta, \beta^2\epsilon, \gamma^2\delta, \delta^2\epsilon$ imply $\beta = \gamma = \delta = \epsilon$. By [3, Lemma 21], this implies $b = c$, a contradiction. In the second of Figure 23, if $H = \alpha^3\beta\gamma$, then the angle sums at $\beta^2\epsilon, \gamma^2\delta, \delta^3, H$ and the angle sum for pentagon imply

$$\alpha = (\frac{1}{4} + \frac{1}{7})\pi, \beta = (\frac{7}{12} - \frac{3}{7})\pi, \delta = \gamma = \frac{2}{3}\pi, \epsilon = (\frac{5}{6} + \frac{6}{7})\pi.$$ 

Then the remainder of the vertex $\beta\beta\cdots$ shared by $P_2, P_3$ has value $(\frac{1}{3} - \frac{12}{7})\pi < \beta, \gamma, \delta, \epsilon$. Since $\alpha$ is a $bc$-angle, we get a contradiction.
We will discuss the remaining case $H = \alpha^4$ (for second of Figure 23) in Section 5.4.

The case $E_{34} = c$ is illustrated by Figure 24. Then two possible ways of arranging $P_5$ give two pictures.

In the first of Figure 24 if $H = \alpha^3\beta\gamma$, then the angle sums at $\beta^2\delta, \gamma^2\epsilon, \delta\epsilon^2$, $H$ and the angle sum for pentagon imply

$$\alpha = \left(\frac{1}{4} + \frac{1}{f}\right)\pi, \beta = \epsilon = \left(\frac{1}{2} - \frac{4}{f}\right)\pi, \gamma = \left(\frac{3}{4} + \frac{3}{f}\right)\pi, \delta = (1 + \frac{12}{f})\pi.$$  

The vertex $\gamma\delta\cdots$ shared by $P_4, P_5$ is $\alpha\gamma\delta\cdots$ or $\beta\gamma\delta\cdots$. By $2\gamma + \delta > \alpha + \gamma + \delta > 2\pi$, we get a contradiction. Therefore $H = \alpha^4$, and the angle sums at $\beta^2\delta, \gamma^2\epsilon, \delta\epsilon^2, H$ and the angle sum for pentagon imply

$$\alpha = \frac{1}{2}\pi, \beta = \epsilon = (1 - \frac{8}{f})\pi, \gamma = (\frac{1}{2} + \frac{4}{f})\pi, \delta = \frac{16}{f}\pi.$$  

By $\alpha + \beta + \gamma > 2\pi$, a vertex $\alpha\cdots$ is not $\alpha\beta\gamma\cdots$. Then by the edge length consideration, we have $\alpha\cdots = \alpha\alpha\alpha\cdots, \alpha\alpha\alpha\cdots$. Moreover, the remainder of $\alpha\alpha\alpha\cdots$ has at least two angles from $\alpha, \gamma$, and the remainder of $\alpha\alpha\alpha\cdots$ has at least...
has at least two angles from $\alpha, \beta$. By $\alpha = \frac{1}{4}\pi$, $\alpha + \beta > \pi$, $\alpha + \gamma > \pi$, we conclude that $\alpha \cdots = \alpha^4$. This gives $P_7, P_8$ and determine all the edges and angles of the two tiles. Note that $f = 24$ implies $\beta = \gamma = \delta = \epsilon$. By [3, Lemma 21], this implies $b = c$, a contradiction. Therefore $f > 24$, and the remainder of the vertex $|\beta|\beta|\cdots$ shared by $P_4, P_7$ has value $\delta < \alpha, \beta, \gamma, \epsilon$. This implies the vertex $|\beta|\beta|\cdots$ is $\beta^2\delta$. Now we have a vertex $\gamma^2\delta\cdots$ shared by $P_4, P_5, P_8$, with $\theta$ adjacent to $\delta$. Then $\theta = \beta$ or $\epsilon$, and $2\gamma + \delta + \theta > 2\pi$, a contradiction.

In the second of Figure 24 the angle sums at $\beta^2\epsilon, \gamma^2\epsilon, \delta^2\epsilon$ and the angle sum for pentagon imply

$$\alpha = \frac{1}{7}\pi + \frac{1}{2}\epsilon, \beta = \gamma = \delta = \pi - \frac{1}{2}\epsilon.$$  

By $3\alpha + \beta + \gamma > 2\pi$, we have $H \neq \alpha^3\beta\gamma$. Therefore $H = \alpha^4$, and the extra angle sum at $H$ implies

$$\alpha = \frac{1}{2}\pi, \beta = \gamma = \delta = (\frac{1}{2} + \frac{4}{7})\pi, \epsilon = (1 - \frac{8}{7})\pi.$$  

By the edge length consideration, the vertex $|\beta|\delta|\cdots$ shared by $P_4, P_5$ is $|\alpha|\beta|\delta|\cdots$ or $|\beta|\beta|\delta|\cdots$. The remainder of $|\alpha|\beta|\delta|\cdots$ has value $\left(\frac{1}{2} - \frac{2}{7}\right)\pi$, which is nonzero and strictly less than all the angles. We get a contradiction. The remainder of $\beta^2\delta\cdots$, has value $\left(\frac{1}{2} - \frac{12}{7}\right)\pi$, which is strictly less than all the angles. Therefore the remainder is zero. This implies $f = 24$ and $\beta = \gamma = \delta = \epsilon$, which by [3, Lemma 21] further implies $b = c$, again a contradiction.

5.2 Case 2($a^3bc$)

Let the second of Figure 19 be the center tile $P_1$ in the partial neighborhood in the first of Figure 25. The $b$-edge of $P_2$ and the $c$-edge of $P_3$ imply $E_{23} = a$. This determines $P_2, P_3$. In particular, $\alpha \beta \gamma$ is a vertex. Then $H = |\alpha|\beta|\cdots$ has no $\gamma$. Therefore $H = |\alpha|\alpha|\beta|\cdots = \alpha^2\beta^2, \alpha^2\beta^2\delta, \alpha^2\beta^2\epsilon$. The angle sum at $H$ implies $\alpha + \beta \leq \pi$. Then the angle sum at $\alpha \beta \gamma$ implies $\gamma \geq \pi$. Therefore $\gamma^2\cdots$ is not a vertex. On the other hand, we have adjacent angle deduction $|\alpha|\alpha|\rightarrow|\beta\gamma|\gamma\beta|$. Therefore $\gamma^2\cdots$ is a vertex, a contradiction.

5.3 Case 3($a^3bc$)

Let the third of Figure 19 be the center tile $P_1$ in the partial neighborhood in the second of Figure 25. The $b$-edge of $P_3$ and the $c$-edge of $P_4$ imply
Figure 25: Cases 2 and 3 for $a^3bc$.

$E_{34} = a$. This determines $P_3, P_4$, and further determines $P_2, P_5$. The angle sum at $\alpha \beta \gamma$ and the angle sum for pentagon imply

$$\alpha + \beta + \gamma = 2\pi, \quad \delta + \epsilon = (1 + \frac{4}{7})\pi.$$

We claim that the remainder of $H = \delta \epsilon \cdots$ has no $\beta \mid \beta$. If this is the case, then the angle sum at $H$ implies

$$\beta \leq \pi - \frac{1}{2}(\delta + \epsilon) = (\frac{1}{2} - \frac{2}{7})\pi, \quad \alpha + \gamma \geq 2\pi - \beta \geq (\frac{3}{2} + \frac{2}{7})\pi.$$

Moreover, we have adjacent angle deduction $|\beta \mid \beta| \rightarrow |\delta \alpha \mid \delta|$ at $H$. Therefore $|\alpha |\alpha | \cdots = \theta |\alpha |\alpha | \rho \cdots$, with $\theta, \rho = \alpha$ or $\gamma$, is a vertex. By $\alpha + \gamma > \pi$, one of $\theta, \rho$ is $\alpha$. Then $|\alpha |\alpha | \cdots = |\alpha |\alpha | \alpha | \cdots$, and the angle sum at the vertex implies $4\alpha \leq 2\pi$ or $3\alpha + \gamma \leq 2\pi$. Combined with $\alpha + \gamma \geq (\frac{3}{2} + \frac{2}{7})\pi$, we always get $\gamma > \pi$. On the other hand, we have adjacent angle deduction $|\alpha |\alpha | \rightarrow |\beta \gamma |\gamma \beta|$. Therefore $\gamma^2 \cdots$ is a vertex, and we get a contradiction.

This proves our claim that $H = \delta \epsilon \cdots$ has no $\beta \mid \beta$. By the same reason, $H$ has no $\gamma \mid \gamma$. Moreover, by $\delta + \epsilon > \pi$, we have $H \neq \delta^2 \epsilon^2 \cdots$. Since $H$ has degree 4 or 5, by the edge length consideration, we get $H = \delta^3 \epsilon, \delta \epsilon^3, \delta^4 \epsilon, \delta \epsilon^4$. We will continue studying the case in Section 5.4.

### 5.4 Double Pentagonal Subdivision

After Sections 5.1, 5.2, 5.3 the only remaining cases for the edge combination $a^3bc$ are the following.

Case 1.3, $H = a^4$, $E_{34} = b$, $E_{45} = a$: The extra angle sum at $H$ gives

$$\alpha = \frac{1}{2}\pi, \quad \beta = (\frac{5}{6} - \frac{4}{7})\pi, \quad \gamma = \delta = \frac{2}{3}\pi, \quad \epsilon = (\frac{1}{3} + \frac{8}{7})\pi.$$
Case 3, $H = \delta^3\epsilon, \delta\epsilon^3, \delta^4\epsilon, \delta\epsilon^4$: The extra angle sum at $H$ gives

$$\alpha + \beta + \gamma = 2\pi, \delta + \epsilon = (1 + \frac{4}{7})\pi.$$ 

The partial neighborhood tiling for Case 1.3 is the second of Figure 23, and for Case 3 is the second of Figure 25. Moreover, we know there is no $3^5$-tile, which by Lemma 2 implies $f \geq 24$.

For Case 1.3, $H = \alpha^4$, if $f = 24$, then $\beta = \gamma = \delta = \epsilon$. By [3, Lemma 21], this implies $b = c$, a contradiction. Therefore we have $f > 24$. If $\alpha^a\beta^b\gamma^c\delta^d\epsilon^e$ is a vertex, then we have

$$\frac{1}{2}a + \left(\frac{5}{6} - \frac{4}{7}\right)b + \frac{2}{3}(c + d) + \left(\frac{1}{3} + \frac{8}{7}\right)e = 2.$$ 

This implies

$$(3a + 5b + 4c + 4d + 2e - 12)f = 24(b - 2e).$$ 

By $f > 24$, we have $\beta > \frac{1}{3}\pi$. We also have $\epsilon > \frac{1}{3}\pi$ and specific values for $\alpha, \gamma, \delta$. This implies $a \leq 4, b \leq 5, c \leq 3, d \leq 3, e \leq 5$. We substitute the finitely many combinations of exponents satisfying the bounds into the equation above and solve for $f$. Those combinations yielding even $f > 24$ are given in Table 1. In the table, "$f = \text{all}$" means that the angle combinations can be vertices for any $f$. Further consideration by edge length shows that the vertices in the column "not vertex" are impossible.

| vertex | not vertex | $f$ | $\beta$ | $\epsilon$ |
|--------|------------|-----|---------|-----------|
| $\beta^2\epsilon, \gamma^2\delta, \delta^3, \alpha^4$ | $\gamma^2\delta^2, \gamma^3$ | all | - | - |
| $\alpha\beta^e, \epsilon^4$ | $\alpha^4\epsilon, \alpha^2\epsilon^2, \alpha\epsilon^2$ | 48 | $\frac{4}{7}\pi$ | $\frac{1}{3}\pi$ |
| $\delta\epsilon^3$ | $\gamma\epsilon^3$ | 72 | $\frac{2}{3}\pi$ | $\frac{1}{3}\pi$ |
| $\alpha\gamma\epsilon^2, \alpha\delta\epsilon^2$ | 96 | - | - | - |
| $\epsilon^5$ | $\beta\epsilon^3$ | 120 | $\frac{4}{7}\pi$ | $\frac{1}{3}\pi$ |
| $\alpha^4\epsilon$ | 192 | - | - | - |

Table 1: AVC for Case 1.3, $H = \alpha^4$: $\alpha = \frac{1}{2}\pi, \delta = \gamma = \frac{2}{3}\pi$.

The table tells us that the all the possible vertices (anglewise vertex combination) for $f = 48$ is

$$\text{AVC} = \{\alpha\beta^2, \beta^2\epsilon, \gamma^2\delta, \delta^3, \alpha^4, \epsilon^4\}.$$
We construct the tiling based on the AVC and the fact that there is a vertex $\delta^3$, such that the tiles around the vertex are arranged as $P_1, P_4, P_5$ in the second of Figure 23. Figure 26 illustrates such a vertex $\delta^3$, with three tiles $P_1, P_1', P_1''$ around the vertex as assumed. We will use $P_n', P_n''$ to denote the two rotations of $P_n$, and subsequent conclusions remain valid after rotations.

By the AVC, the vertex $\ldots \beta \epsilon \cdot \delta$ shared by $P_1, P_1'$ is $\beta^2 \epsilon$. This determines $P_2$ (and its two other rotations). The vertex $\ldots \alpha^2 \cdot \delta$ shared by $P_1, P_2$ is $\alpha^4$ (we will omit mentioning “by the AVC”). This determines $P_3, P_4$. We also know that $P_1', P_2, P_4'$ (and their rotations) share a vertex $\gamma^2 \delta$. The vertex $\ldots \gamma^2 \cdot \delta$ shared by $P_2, P_3$ is $\gamma^2 \delta$. This gives $P_5, \delta_5$. The vertex $\ldots \epsilon^2 \cdot \delta$ shared by $P_2, P_4'$ is $\epsilon^4$. This gives $\epsilon_5$. Then $\delta_5, \epsilon_5$ determine $P_5$. The vertex $\ldots \beta \epsilon \cdot \delta$ shared by $P_3, P_5$ is $\beta^2 \epsilon$. This determines $P_6$. The vertex $\ldots \delta^2 \cdot \epsilon$ shared by $P_3, P_6$ is $\delta^3$. This gives $P_7, \delta_7$. By the edge length consideration, the vertex $\ldots \beta \cdot \epsilon \cdot \delta$ shared by $P_3, P_4$ cannot be $\alpha \beta^2$ and is therefore $\beta^2 \epsilon$. This gives $\epsilon_7$. Then $\delta_7, \epsilon_7$ determine $P_7$. We find that $P_3, P_6, P_7$ around a vertex $\delta^3$ just like $P_1, P_1', P_1''$ around the starting vertex $\delta^3$. The argument for the tiling can therefore be repeated by starting from the new $\delta^3$.

![Figure 26: Construct double pentagonal subdivision.](image)

The tiles $P_1, P_2$ and their rotations form a local tiling that is exactly the same as the tiling of triangle in Figure 11. The repeated construction around the new $\delta^3$ creates another such tiling of triangle next to the existing tiling of triangle. Therefore further repetition gives the double pentagonal subdivision tiling.

The argument for $f = 48$ also applies to $f = 120$. By Table 1, the AVC
for $f = 120$ is $\{\beta^2 \epsilon, \gamma^2 \delta, \delta^3, \alpha^4, \epsilon^5\}$. The argument still starts with the same assumption on the neighborhood of one $\delta^3$ vertex. The only place $\epsilon^4$ is used for $f = 48$ is the vertex shared by $P_2, P_4, P'_5$. In fact, the only fact we use is that all angles at $\epsilon^2 \cdots = \epsilon^4$ are $\epsilon$. Therefore the argument still works, as all angles at $\epsilon^2 \cdots = \epsilon^5$ are $\epsilon$. At the end, we still get double pentagonal subdivision tiling.

For $f = 72$, the AVC is \(\{\beta^2 \epsilon, \gamma^2 \delta, \delta^3, \alpha^4, \delta^3\} \). The argument for $f = 48$ works until the vertex $\epsilon^2 \cdots$ shared by $P_2, P'_4$. By the AVC, we have $\epsilon^2 \cdots = \delta^3$. However, the circular adjacent angle deduction at $\delta^3$ shows that one of $\beta \gamma \cdots, \gamma \gamma \cdots, \gamma \epsilon \cdots$ should be a vertex. By the AVC, we find $\beta \gamma \cdots$ and $\gamma \epsilon \cdots$ are not vertices. Moreover, the only vertex $\gamma^2 \cdots$ in the AVC is $\gamma^2 \delta = \gamma \gamma \delta \neq \gamma \gamma \gamma \cdots$. The contradiction shows that $\epsilon^2 \cdots$ is not a vertex. Therefore there is no tiling for $f = 72$.

For general $f$ (i.e., $f \neq 48, 72, 120$), the AVC is \(\{\beta^2 \epsilon, \gamma^2 \delta, \delta^3, \alpha^4\} \). Again we repeat the argument for $f = 48$ until the vertex $\epsilon^2 \cdots$ shared by $P_2, P'_4$. Since $\epsilon^2 \cdots$ is not in the AVC, there is no tiling for general $f$.

This completes the discussion for Case 1.3, $H = \alpha^4$. The conclusion is the double pentagonal subdivisions for $f = 48$ and 120.

Now we turn to Case 3, with $H = \delta^3 \epsilon, \delta^3 \epsilon, \delta^4 \epsilon, \delta \epsilon^4$. In the second partial neighborhood tiling in Figure 25 if $H = \delta^3 \epsilon$, then $A_{6,12} = \delta$. This implies $A_{6,15} = \beta$ or $\epsilon$. By the edge length consideration, we get $A_{6,15} = \epsilon$. Therefore $\delta \epsilon^2$ is a vertex. Then the angle sums at $\delta \epsilon^2, \delta \epsilon^2$ and $\delta + \epsilon = (1 + \frac{4}{f})\pi$ imply $f = 20$, a contradiction. We get the same contradiction for $H \neq \delta \epsilon^3$.

The final case we need to consider is Case 3, $H = \delta^4 \epsilon, \delta \epsilon^4$. After finishing all the other cases for $a^3bc$, we may assume that there is no $3^3$-tile and no $3^4$-tile. By Lemma 3. This implies $f \geq 60$. On the other hand, for $H = \delta^4 \epsilon$, the argument for $H = \delta^3 \epsilon$ can be used to show that $\delta \epsilon^2$ is a vertex. Then the angle sums at $\delta^4 \epsilon, \delta \epsilon^2$ and $\delta + \epsilon = (1 + \frac{4}{f})\pi$ imply $f = 28$, contradicting $f = 60$. We get the same contradiction for $H = \delta \epsilon^3$.

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