The space of associated metrics on a symplectic manifold

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Abstract

In this work the spaces of Riemannian metrics on a closed manifold $M$ are studied. On the space $\mathcal{M}$ of all Riemannian metrics on $M$ the various weak Riemannian structures are defined and the corresponding connections are studied. The space $\mathcal{A}M$ of associated metrics on a symplectic manifold $M, \omega$ is considered in more detail. A natural parametrization of the space $\mathcal{A}M$ is defined. It is shown, that $\mathcal{A}M$ is a complex manifold. A curvature of the space $\mathcal{A}M$ and quotient space $\mathcal{A}M/\mathcal{D}_\omega$ is found. The spaces $\mathcal{A}M$ in cases when $M$ is a two-dimensional sphere and two-dimensional torus are considered as application of general results. The critical metrics of the functional of the scalar curvature on $\mathcal{A}M$ are considered. The finite dimensionality of the space of associated metrics of a constant scalar curvature with Hermitian Ricci tensor is shown.

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§1. Preliminaries.

1.1. The topology in spaces of tensor fields. Let $M$ be a smooth (i.e. $C^\infty$), closed, orientable $n$-manifold, $TM$ is its tangent bundle, $T^p_q M$ the bundle of $(p,q)$-tensors and $S_2 M$ the bundle of symmetric 2-forms on $M$.

The symbol $\Gamma(T^p_q M)$ will denote space of all smooth $(p,q)$-tensor fields on $M$. In particular: $\Gamma(TM)$ is the space of all smooth vector fields on $M$,

$S_2 = \Gamma(S_2 M)$ is the space of all smooth symmetric 2-forms on $M$.

Define topology in these spaces. For this purpose we fix some smooth Riemannian metric $g$ on $M$ and denote its covariant derivative as $\nabla$. In local coordinates $x^1, \ldots, x^n$ on $M$ let: $g_{ij}(x)$ are components of the metric tensor $g$, $g^{ij}(x)$ are components of inverse matrix to $g_{ij}(x)$ and $\mu_g = (\det g_{ij})^{1/2} \, dx^1 \wedge \cdots \wedge dx^n$ is Riemannian volume form.

The metric $g$ defines an inner product on the space $\Gamma(T^p_q M)$ of tensor fields of any type $(p,q)$. If $T_1$ and $T_2$ are $(p,q)$-tensor fields, then their inner product is set by the formula:

$$ (T_1, T_2)_g = \int_M g^{i_1 k_1} \cdots g^{i_q k_q} g_{j_1 i_1} \cdots g_{j_p i_p} \, T_1^{j_1 \cdots j_p} T_2^{k_1 \cdots k_q} \, d\mu_g. \tag{1.1} $$

The metric $g$ allows to define a stronger inner product on the space $\Gamma(T^p_q M)$ of $(p,q)$-tensor fields. Let $s$ is an integer non-negative number and $T_1$ and $T_2$ are $(p,q)$-tensor fields. Then

$$ (T_1, T_2)^s_g = \sum_{i=0}^s (\nabla^i T_1, \nabla^i T_2)_g, \tag{1.2} $$

where $\nabla^i = \nabla \circ \cdots \circ \nabla$ is $i$-th degree of the covariant derivative and $(\nabla^i T_1, \nabla^i T_2)$ is the inner product (1.1).

Denote as $H^s(T^p_q M)$ the completion of space $\Gamma(T^p_q M)$ with respect to the topology in $\Gamma(T^p_q M)$ given by the inner product (1.2). The space $H^s(T^p_q M)$ is called a space of $(p,q)$-tensor fields Sobolev class $H^s$. It is a Hilbert space. Denote corresponding norm as $\| \cdot \|_s$.

The fundamental property of $H^s$ spaces is the Sobolev embedding theorem which states: If $s \geq \frac{n}{2} + 1 + k$, then $H^s \subset C^k$, and the inclusion is a continuous linear mapping \[15\]. If $s \geq \frac{n}{2} + 1 + k$, then every tensor field $T$ of class $H^s$ is of a differentiable class $C^k$. The further restrictions on $s$ connect with necessity to ensure an appropriate class of a smoothness of tensor fields from $H^s(T^p_q M)$.

Define topology in the space $\Gamma(T^p_q M)$ of smooth $(p,q)$-tensor fields on $M$ by a set of norms $\{ \| \cdot \|_s, \, s \geq 0 \}$. Then $\Gamma(T^p_q M)$ is the Frechet space.

In further it is supposed, that the space $\Gamma(TM)$ of smooth vector fields on $M$ and the space $S_2 = \Gamma(S_2 M)$ of smooth symmetric 2-forms on $M$ are endowed just this topology and, thus, are infinite dimensional Frechet spaces.

The symbol $S_2^s = H^s(S_2 M)$ will denote the Hilbert space of symmetric 2-forms of a class $H^s, \, s > \frac{n}{2} + 2$.

**Remark.** One can show, that the topology on spaces $\Gamma(T^p_q M)$ defined above does not depend on a choice of the metric $g$ on $M$. 

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1.2. Differential operators $\alpha_g$ and $\delta_g$ and the decomposition of Berger-Ebin.

Let $g$ is some smooth Riemannian structure on manifold $M$ and $\nabla$ is its covariant derivative of Riemannian connection. Let $\nabla_i$ is a covariant derivative along the vector field $\frac{\partial}{\partial x^i}$ in local coordinates $x^1, \ldots, x^n$ on $M$.

The symbol $g^{-1}$ will denote operation of lifting of the first index of tensor. If $a$ is a symmetric 2-form on $M$, then $A = g^{-1}a = g^{ik}a_{kj}$ is an endomorphism of the tangent bundle $TM$.

The trace (with respect to $g$) of symmetric 2-form $a$ is defined by the formula $\text{tr}_g a = \text{tr} A = g^{ij}a_{ij}$.

The covariant divergence $\delta_g a$ of the symmetric 2-form $a$ on $M$ is a vector field on $M$ defined by equality $(\delta_g a)^i = -\nabla_j a^{ij}$, where $a^{ij} = g^{ik}g^{jl}a_{kl}$.

Thus, the covariant divergence is a differential operator of the 1-st order, $\delta_g : S_2 \longrightarrow \Gamma(TM)$, $(\delta_g a)^i = -\nabla_j a^{ij}$.

Enter one more differential operator $\alpha_g : \Gamma(TM) \longrightarrow S_2$, $\alpha_g(X) = \frac{1}{2}L_Xg$,

where $L_Xg$ is Lie derivative along the vector field $X$ on $M$, $L_Xg = \nabla_iX_j + \nabla_jX_i$.

The spaces $\Gamma(TM)$ of vector fields on $M$ and $S_2$ of the symmetric 2-forms have natural inner products:

$$ (X,Y)_g = \int_M g(X,Y)d\mu_g, \quad X,Y \in \Gamma(TM), $$

$$ (a,b)_g = \int_M g(a,b)d\mu_g = \int_M g^{ik}g^{jl}a_{ij}b_{kl}d\mu_g, \quad a,b \in S_2. $$

where $\mu_g = (\det g_{ij})^{1/2} dx^1 \wedge \cdots \wedge dx^n$ is Riemannian volume form of the metric $g$.

It just follows from the Stokes theorem, that operator $\alpha_g$ is the adjoint of $\delta_g$ [7]: for any $X \in \Gamma(TM)$ and $a \in S_2$,

$$(\alpha_g(X), a) = (X, \delta_g a).$$

Since $\alpha_g$ is operator with an injective symbol, there is following orthogonal Berger-Ebin decomposition [5] of the space $S_2$:

$$ S_2 = S_2^0 \oplus \alpha_g(\Gamma(TM)),$$

Where $S_2^0 = \ker \delta_g = \{a \in S_2; \delta_g a = 0\}$ is the space of divergence-free symmetric 2-forms. Then each 2-form $a \in S_2$ has unique representation:

$$ a = a^0 + L_Xg,$$

where $\delta_g a^0 = 0$. The components $a^0$ and $L_Xg$ are orthogonal and they are defined in a unique way.
There is one more (pointwise) orthogonal decomposition of the space $S_2$:

$$S_2 = S_2^T \oplus S_2^C,$$

where $S_2^T = \{ a \in S_2; \; \text{tr}_g a = 0 \}$ is the space of traceless symmetric 2-forms and $S_2^C = \{ a \in S_2; \; a = \sigma g, \; \sigma \in C^\infty(M, \mathbb{R}) \}$. Each 2-form is represented in a unique way as:

$$a = \left( a - \frac{1}{n} (\text{tr} a) g \right) + \frac{1}{n} (\text{tr} a) g.$$

1.3. The space of Riemannian metrics. Let $\mathcal{M}$ is the space of Riemannian structures on a manifold $M$. The space $\mathcal{M}$ is an open convex cone in the Frechet space $S_2$ (the space $S_2$ consists of all smooth symmetric 2-forms on $M$ and $\mathcal{M}$ of all positive defined symmetric 2-forms). So the space $\mathcal{M}$ is the Frechet manifold and for any $g \in \mathcal{M}$, the tangent space $T_g \mathcal{M}$ is naturally identified with the space $S_2$.

The manifold $\mathcal{M}$ has a canonical weak Riemannian structure. Namely, if $a, b \in T_g \mathcal{M} = S_2$ are two smooth symmetric 2-forms on $M$, which represent elements of tangent space $T_g \mathcal{M}$, then their inner product is defined by the formula:

$$(a, b)_g = \int_M \text{tr}(g^{-1} a g^{-1} b) d\mu_g = \int_M g^{ik} g^{jl} a_{ij} b_{kl} d\mu_g.$$ (1.4)

This structure on $\mathcal{M}$ is called weak because the inner product in the tangent space $T_g \mathcal{M} = S_2$ defines weaker topology, than the topology of the Frechet space. Detail research of the manifold $\mathcal{M}$ with a canonical Riemannian structure (1.4) is in work of D. Ebin [17].

The curvature and geodesics of the space $\mathcal{M}$ were studied in the work [24] and [28]. Other weak Riemannian structures on $\mathcal{M}$ have been considered in the work of author [59].

Recall main facts about the space $\mathcal{M}$ obtained in the works [17], [24] and [28].

Let elements $a, b, c \in T_g \mathcal{M} = S_2$ represent constant (parallel in $S_2$) vector fields on $\mathcal{M}$ and $A, B, C$ are their endomorphisms, $A = g^{-1} a$, $B = g^{-1} b$, $C = g^{-1} c$. The covariant derivative $\nabla^0$ of Riemannian connection on $\mathcal{M}$ of corresponding weak Riemannian structure (1.4) is obtained in [17]:

$$\nabla^0_a b = -\frac{1}{2} (aB + bA) + \frac{1}{4} (\text{tr}(A)b + \text{tr}(B)a - \text{tr}(AB)g),$$ (1.5)

where $aB = a_{ik} b^k_j$.

The curvature tensor has been found in works [24] and [28] (see also §2 of this work):

$$R^0(a, b) = -\frac{1}{4} g [[A, B], C] - \frac{1}{16} \text{tr}(C) (\text{tr}(A)b - \text{tr}(B)a) + \frac{n}{16} \left( \text{tr}(AC) b^T - \text{tr}(BC) a^T \right),$$ (1.6)

where $[A, B] = AB - BA$ and $a^T = a - \frac{1}{n} \text{tr}(A) g$ is traceless part of tensor $a$.

Geodesics on $\mathcal{M}$ have been found in works [24] and [28]. At first, we will show them in the form of [28].
Theorem 1.1 Let $g_0 \in \mathcal{M}$ and $a \in T_{g_0} \mathcal{M}$. Then geodesic on $\mathcal{M}$, going out from $g_0$ in direction $a$ is curve

$$G(t) = g_0 e^{\alpha(t)I + \beta(t)A^T},$$

where $I$ is identity endomorphism, $A^T$ is the traceless part of an endomorphism $A = g_0^{-1}a$ and $\alpha(t), \beta(t)$ are smooth functions on $M$, which depend on $t$ and look like:

$$\alpha(t) = \frac{2}{n} \ln \left( 1 + \frac{t}{4} \text{tr} A \right)^2 + \frac{n}{16} \text{tr}((A^T)^2)t^2,$$

$$\beta(t) = \begin{cases} \frac{t}{1 + \frac{t}{4} \text{tr} A}, & \text{if } \text{tr}((A^T)^2) = 0 \\ \frac{4}{\sqrt{ntr((A^T)^2)}} \arctan \left( \frac{\sqrt{ntr((A^T)^2)}}{4 + t \text{tr} A} \right), & \text{if } \text{tr}((A^T)^2) \neq 0, \end{cases}$$

where $\arctan$ takes values in $\left(-\frac{\pi}{2}, \frac{\pi}{2}\right)$, in those points of the manifold, where $\text{tr}(A) \geq 0$, and in points $x \in M$, where $\text{tr}(A) < 0$ we suppose:

$$\arctan \left( \frac{\sqrt{ntr((A^T)^2)}}{4 + t \text{tr} A} \right) = \begin{cases} \arctan \text{ in } [0, \frac{\pi}{2}) \text{ for } t \in [0, -\frac{4}{\text{tr} A}], \\ \frac{\pi}{2} \text{ for } t = -\frac{4}{\text{tr} A}, \\ \arctan \text{ in } (\frac{\pi}{2}, \pi) \text{ for } t \in (-\frac{4}{\text{tr} A}, \infty). \end{cases}$$

Let $N^a = \{x \in M; A^T(x) = 0\}$, and if $N^a \neq \emptyset$, then let $t^a = \inf\{\text{tr}A(x) \mid x \in N^a\}$. Then geodesic $g(t)$ is defined for $t \in [0, \infty)$, if $N^a = \emptyset$ or if $t^a \geq 0$, and geodesic is defined for $t \in \left[0, -\frac{4}{t^a}\right)$ if $t^a < 0$.

1.4. Decomposition of space $\mathcal{M}$. Let $\text{Vol}(M) \subset \Gamma(\Lambda^n M)$ is the space of smooth volume forms on $M$, i.e. space of smooth nondegenerate $n$-forms on $M$, which give the same orientation as initial orientation on $M$. The natural projection $\nu : \mathcal{M} \rightarrow \text{Vol}(M)$ is defined it takes each metric $g \in \mathcal{M}$ to Riemannian volume form $\mu_g = (\det g)^{1/2} dx^1 \wedge \cdots \wedge dx^n$. Fiber of bundle $\nu$ over $\mu \in \text{Vol}(M)$ is the space $\mathcal{M}_\mu$ of metrics with the same Riemannian volume form $\mu$.

The bundle $\nu : \mathcal{M} \rightarrow \text{Vol}(M)$ is trivial, the volume form $\mu \in \text{Vol}(M)$ defines decomposition of $\mathcal{M}$ in direct product:

$$\iota_\mu : \text{Vol}(M) \times \mathcal{M}_\mu \rightarrow \mathcal{M}, \quad (\nu, h) \rightarrow \left( \frac{\nu}{\mu} \right)^{2/n} h,$$

$$\varphi_\mu : \mathcal{M} \rightarrow \text{Vol}(M) \times \mathcal{M}_\mu, \quad g \rightarrow \left( \mu_g, \left( \frac{\mu}{\mu_g} \right)^{2/n} g \right),$$

where the positive function $\frac{\nu}{\mu}$ is defined by equality $\nu = \frac{\nu}{\mu} \mu$.

The metric $g \in \mathcal{M}$ defines section

$$S_g : \text{Vol}(M) \rightarrow \mathcal{M}, \quad \nu \rightarrow \left( \frac{\nu}{\mu_g} \right)^{2/n} g.$$
Using decomposition $\mathcal{M} = \text{Vol}(M) \times \mathcal{M}_\mu$, in the work [24] the following expressions of geodesics on space $\mathcal{M}$ are obtained.

**Theorem 1.2** [24] The geodesics in $\mathcal{M} = \text{Vol}(M) \times \mathcal{M}_\mu$ with initial position $(\mu, g)$ and initial velocity $(\beta, b) \in \Gamma(L^n M) \times S^2_2$ is

$$g_t = \left( q(t)^2 + r^2 t^2 \right)^{2/n} g \exp \left( \frac{1}{r} \arctan \left( \frac{rt}{q} \right) B \right),$$

where $B = g^{-1} b$, $\exp$ is the exponential mapping, $q(t) = 1 + \frac{1}{2} \beta t$, $r = \frac{1}{4} \sqrt{n \text{tr}(B^2)}$. (If $r = 0$, replace the exponential term by 1.) The change in the volume form is given by the formula

$$\mu(g_t) = \left( q(t)^2 + r^2 t^2 \right) \mu.$$

**Remark.** All facts explained above are true for the space $\mathcal{M}^s$ of Riemannian metrics on $M$ of Sobolev class $H^s$, $s > \frac{n}{2} + 2$.

1.5. **Action of group of diffeomorphisms on the space of metrics.**

Let $\mathcal{D}^{s+1}$ be the group of diffeomorphisms of the manifold $M$ of Sobolev class $H^{s+1}$, $s > \frac{n}{2} + 2$. Then group $\mathcal{D}^{s+1}$ acts on the smooth Hilbert manifold $\mathcal{M}^s$ in the following way:

$$A : \mathcal{M}^s \times \mathcal{D}^{s+1} \rightarrow \mathcal{M}^s, \quad A(g, \eta) = \eta^* g,$$

$$\eta^* g(x)(X, Y) = g(\eta(x))(d\eta(X), d\eta(Y)),$$

for any vector fields $X, Y$ on $M$ and any $x \in M$.

The group $\mathcal{D}^{s+1}$ is the smooth Hilbertian manifold, but it is not a Lie group, because group operations are only continuous. The action $\mathcal{D}^{s+1}$ on $\mathcal{M}^s$ is also only continuous. Nevertheless D.Ebin has shown [17], that for any metric $g \in \mathcal{M}$ and its group of isometries $I(g)$, orbit $g\mathcal{D}^{s+1}$ of action $A$ is a smooth closed submanifold, which is diffeomorphic to quotient space $\mathcal{D}^{s+1}/I(g)$. Differential of mapping

$$A_g : \mathcal{D}^{s+1} \rightarrow \mathcal{M}^s, \quad A_g(\eta) = \eta^* g,$$

is Lie derivative:

$$dA_g : \Gamma(TM) \rightarrow S^2_2, \quad dA_g(X) = L_X g = 2\alpha_g(X).$$

Moreover, there is following

**Theorem 1.3.** (Slice theorem, [17]). Let $s > \frac{n}{2} + 2$. For each metric $g \in \mathcal{M}^s$ there exists a smooth submanifold $S^s_g \subset \mathcal{M}^s$ containing $g$, such that

1) If $\eta \in I(g)$, then $\eta^* (S^s_g) = S^s_g$,
2) If $\eta \in \mathcal{D}^{s+1}$ and $\eta^* (S^s_g) \cap S^s_g \neq \emptyset$, then $\eta \in I(g)$,
3) There exists a local cross section $\chi : \mathcal{D}^{s+1}/I(g) \rightarrow \mathcal{D}^{s+1}$ defined in a neighbourhood $U^{s+1}_g$ of identity coset $[e] \in \mathcal{D}^{s+1}/I(g)$, such that mapping

$$F : S^s_g \times U^{s+1}_g \rightarrow \mathcal{M}^s, \quad F(h, u) = (\chi(u))^* h,$$
is a homeomorphism onto a neighbourhood $V^s$ of element $g \in M^s$.

Notice, that the map $F: S_g \times U \to M$ of the slice theorem is ILH-smooth, as for any $s \geq 2n + 5$ is hold [33]:

$$S_g^s = S_g^{2n+5} \cap M^s, \quad U^{s+1} = U^{2n+5} \cap (D^{s+1}/I(g)),$$

$$V^s = V^{2n+5} \cap M^s, \quad \chi^{s+1} = \chi^{2n+5} U^{s+1}$$

and for any $k \geq 0$ mappings

$$F^{s+k}: S_g^{s+k} \times U^{s+1+k} \to V^s,$$

$$p^{s+k} \times q^{s+k}: V^{s+k} \to S_g^s \times U^{s+1}$$

are $C^k$ differentiable.

The quotient space $M/D$ is not a manifold. Actually, $D$ does not act freely. Elements $g \in M$ have isotropy groups $I(g)$ depending on $g \in M$. It is known [17], that the set of metrics $g \in M$ with trivial group of isometries is an open dense set $M^*$ in $M$. Group $D$ acts on the space $M^*$ freely. On the slice theorem, we obtain, that the quotient space $M^*/D$ is an ILH-smooth manifold.

Since the tangent space to an orbit $gD$ is identified with space

$$\alpha_g (\Gamma(TM)) = \{ h \in S_2; \ h = L_X g, \ X \in \Gamma(TM) \},$$

then (Berger-Ebin decomposition):

$$T_g M = S_2 = S_2^2 \oplus T_g (gD).$$

We obtain, that the tangent space $T_{[g]}(M^*/D)$ is identified with space

$$S_2^0 = \{ h \in S_2; \ \delta_g h = 0 \}$$

of divergenceless 2-forms $h$.

1.6. ILH-manifolds.

**Definition 1.1.** Topological vector space $E$ is called ILH-space, if $E$ is an inverse limit of Hilbert spaces $\{ E^s; \ s = 1, 2, \ldots \}$, such that $E^l \subset E^s$, if $l \geq s$, and this inclusion is a bounded linear operator.

We will denote $E = \lim_{\leftarrow} E^s$.

**Definition 1.2.** Topological space $X$ is called $C^k$-ILH-manifold, which are modeled on ILH-space $E$, if

a) $X$ is an inverse limit of $C^k$-smooth Hilbert manifolds $\{ X^s \}$, modeled on $\{ E^s \}$, and if $l \geq s$, $X^l \subset X^s$;

b) For any point $x \in X$ there are open neighbourhoods $U^s(x)$ of point $x$ in $X^s$ and homeomorphisms $\psi^s$ of neighbourhoods $U^s(x)$ on open subsets $V^s(x) \subset E^s$, which define $C^k$-coordinates on $X^s$ in a neighbourhood of the point $x$, such that $U^l(x) \subset U^s(x)$ at $l \geq s$ and $\psi^{s+l}(y) = \psi^s(y)$ for any point $y \in U^{s+l}(x)$.
Definition 1.3. Let $X, Y$ are $C^k$-ILH-manifolds. The mapping $\varphi : X \to Y$ is called $C^k$-ILH-differentiable, if $\varphi$ is an inverse limit of $C^k$-differentiable mappings of Hilbert manifolds $X^l$ and $Y^s$, i.e. if for any $s$ there is a number $l(s)$ and $C^k$-differentiable mapping $\varphi^s : X^l(s) \to Y^s$, such that $\varphi^s(x) = \varphi^{s+1}(x)$, $\forall x \in X^{l(s+1)}$ and $\varphi = \lim_{l \to \infty} \varphi^s$.

1.7. Some final notation.

End$(TM)$ is the vector bundle of endomorphisms $K : TM \to TM$ of the tangent bundle. Fiber over a point $x \in M$ consists of all endomorphisms of the tangent space $T_x M$. Endomorphism $K : TM \to TM$ will be also called an operator, acting on the tangent bundle.

$A(M)$ is bundle over $M$ whose fiber $A_x(M)$ over the point $x \in M$ consists of automorphisms $J_x$ of the tangent space $T_x M$, such that: $J_x^2 = -I_x$, where $I_x$ is identity automorphism of the space $T_x M$. It is supposed, that dimension $n$ of manifold $M$ is even, $n = 2m$.

Definition 1.4. An almost complex structure (a.c.s.) on $M$ is the smooth section $J$ of bundle $A(M)$.

Thus, the almost complex structure on $M$ is a smooth automorphism $J : TM \to TM$, such that $J^2 = -I$, where $I$ is identity automorphism.

$\mathcal{A} = \Gamma(A(M))$ is space of all smooth almost complex structures on $M$. This is the space of smooth sections of bundle $A(M)$, therefore $\mathcal{A}$ is infinite-dimensional smooth ILH-manifold \cite{[1]}. 


§2. Natural weak Riemannian structures on space of the Riemannian metrics.

As above, there is a canonical weak Riemannian structure on the space \( \mathcal{M} \) of all Riemannian metrics on manifold \( M \) (1.4). In this paragraph we shall consider a series of other natural weak Riemannian structures on \( \mathcal{M} \) and we shall obtain their formulas for covariant derivative, curvature tensor, sectional curvatures and geodesic.

2.1. A flat structure. Let \( g_0 \) is fixed Riemannian metric on \( M \). The formula

\[
(a, b)_0^\alpha = \int_M \text{tr}(g_0^{-1}ag_0^{-1}b)d\mu_0 + \alpha \int_M \text{tr}(g_0^{-1}a)\text{tr}(g_0^{-1}b)d\mu_0,
\]

where \( a, b \in S_2 \), \( \alpha \in \mathbb{R} \) is some number and \( d\mu_0 = d\mu(g_0) \) is the Riemannian volume form, defines the symmetric form in a vector space \( S_2 \). It is positive defined for \( \alpha > -\frac{1}{n} \) and nondegenerate for \( \alpha \neq -\frac{1}{n} \). Therefore for \( \alpha > -\frac{1}{n} \) we obtain a flat weak Riemannian structure on \( S_2 \).

The covariant derivative \( d \) is a usual directional derivative in a vector space \( S_2 \): if \( b = b(g) \) is vector field on \( M \), \( d_a b = \frac{d}{dt}|_{t=0} b(g + ta) \). The curvature tensor is equal to zero. Geodesics are: \( g_t = g + ta \).

2.2. A conformally flat structure. Consider the following weak Riemannian structure on \( M \):

\[
\langle a, b \rangle_g = \int_M \text{tr}(g_0^{-1}ag_0^{-1}b)d\mu(g),
\]

where \( a, b \in S_2 \), \( g_0 \) is fixed metric on \( M \). In contrast to the previous case, the volume form \( \mu(g) \) depends on \( g \in \mathcal{M} \). It differs from \( \mu(g_0) \) on smooth positive function: \( \mu(g) = \rho(g)\mu(g_0) \). Function \( \rho(g) \) we shall call a denseness of the Riemannian metric \( g \) with respect to \( g_0 \).

Write weak Riemannian structure (2.2) as

\[
\langle a, b \rangle_g = \int_M \text{tr}(g_0^{-1}ag_0^{-1}b)\rho(g)d\mu(g_0),
\]

To find a covariant derivative \( D \) of metric (2.2) on \( \mathcal{M} \) it is used usual ”six-term formula” \([29]\). As volume form \( \mu(g_0) \) is constant in the integral (2.3) (i.e. does not depend on \( g \in \mathcal{M} \)) and application of six-term formula has no differentiation on \( x \in M \), then it is enough to calculate a covariant derivative \( D_x \), curvature tensor \( K_x \) and geodesics for the Riemannian metric

\[
\langle a, b \rangle_{g,x} = \text{tr}(g_0^{-1}ag_0^{-1}b)(x)\rho(g)(x)
\]

on the space \( \mathcal{M}_x \) of inner products on tangent space \( T_x M \) at each point \( x \in M \).

We will use following notation:

\[
A = g^{-1}a, \ G = g^{-1}g_0, \ A_0 = g_0^{-1}a, \ G_0 = g_0^{-1}g.
\]
Theorem 2.1. Weak Riemannian structure (2.2) on $\mathcal{M}$ has the following characteristics:

1) Covariant derivative

$$D_{\alpha}b = d_{\alpha}b + \frac{1}{4}(\text{tr}(A)b + \text{tr}(B)a) - \frac{1}{4}\text{tr}(A_0B_0)g_0G, \quad (2.6)$$

2) Curvature tensor

$$K(a, b)c =
= -\frac{1}{16}((4\text{tr}(AC) + \text{tr}(A)\text{tr}(C))b - (4\text{tr}(BC) - \text{tr}(B)\text{tr}(C))a) - \frac{1}{16}\text{tr}(A_0C_0)g_0(4B + \text{tr}(B))G + \frac{1}{16}\text{tr}(B_0C_0)g_0(4A + \text{tr}(A))G + \frac{1}{16}\text{tr}(G^2)(\text{tr}(A_0C_0)b - \text{tr}(B_0C_0)a).$$

3) The sectional curvature $K_\sigma$ in a plane section $\sigma$, given by orthonormal vectors $a, b \in T_g\mathcal{M}$ is equal to zero, if $a, b$ are scalar tensors ($a = \alpha g$, $b = \beta g$), and in case, when $a, b$ are traceless ($\text{tr}A = 0$, $\text{tr}B = 0$), it is expressed by the formula

$$K_\sigma = \int_M \left(-\frac{1}{2}\text{tr}(AB)\text{tr}(A_0B_0) + \frac{1}{4}(\text{tr}(A^2)\text{tr}(B_0^2) + \text{tr}(B^2)\text{tr}(A_0^2)) - \frac{1}{16}\text{tr}(G^2)(\text{tr}(A_0^2)\text{tr}(B_0^2) - (\text{tr}(A_0B_0))^2) \right) d\mu(g).$$

4) Geodesics $g_t$ on $\mathcal{M}$ are solutions of the following differential equation of the second order,

$$\frac{d}{dt} \left(\rho(g) \frac{dg}{dt}\right) = kg_0G,$$

where $k = k(x)$ is positive function on $\mathcal{M}$, which does not depend on $t$ and is equal to one eighth length of an initial velocity of geodesics $g_t(x)$ on $\mathcal{M}_x$, $x \in \mathcal{M}$. 

Proof. We will make computations on $\mathcal{M}_x$, $x \in \mathcal{M}$. The metric (2.4) on $\mathcal{M}_x$ is conformally equivalent to the metric $(a, b)_x = \text{tr}(g_0^{-1}ag_0^{-1}b)(x)$. Therefore to find a derivative $D_x$ we can apply usual formula (see for example [29]):

$$D_{\alpha}b = d_{\alpha}b + \frac{1}{2}(a(\psi) + b(\psi)a - (a, b)_x d\psi),$$

where $\psi = \ln(\rho(g))$, $d\psi$ is gradient of function $\psi$. A curvature tensor is found in the same way. Directional derivative $a(\psi)$ and gradient $d\psi$ of function $\psi$ are simply found: if $g_t = g + ta$, then from equality $\mu(g_t) = \rho(g_t)\mu$ is obtained

$$a(\rho) = \left.\frac{d}{dt}\right|_{t=0} \rho(g_t) = \frac{1}{2}\text{tr}(g^{-1}a)\rho(g) = \frac{1}{2}\text{tr}(A)\rho(g).$$

For $\psi = \ln(\rho)$ is obtained,

$$a(\psi) = \frac{1}{2}\text{tr}(g^{-1}a) = \frac{1}{2}\langle g_0g^{-1}a_x, g_0 \rangle.$$
Therefore

\[ a(\psi) = \frac{1}{2} \text{tr}(A), \quad d\psi = \frac{1}{2} g_0 G, \quad (2.7) \]

Now expression for a covariant derivative is:

\[ D_a b = d_a b + \frac{1}{4} (\text{tr}(A)b + \text{tr}(B)a) - \frac{1}{4} \text{tr}(A_0 B_0)g_0 G, \]

To find a curvature tensor \( K_x \) of conformally equivalent metric it is needed a Hessian tensor \( H_{\psi}(a) = d_a(d\psi) \) and Hessian \( h_{\psi}(a, b) = \langle d_a(d\psi), b \rangle_{0,x} \). They are easily calculated,

\[
d_a(d\psi) = \frac{d}{dt} \Bigg|_{t=0} \left( \frac{1}{2} g_0 g^{-1}_t g_0 \right) = -\frac{1}{2} g_0 g^{-1} a g^{-1} g_0 = -\frac{1}{2} g_0 A G,
\]

\[
h_{\psi}(a, b) = -\frac{1}{2} \text{tr} \left( g_0^{-1} g_0 g^{-1} a g^{-1} g_0 g^{-1} b \right) = -\frac{1}{2} \text{tr} \left( g^{-1} a g^{-1} b \right) = -\frac{1}{2} \text{tr}(A B).
\]

Put the given expressions in the formula for a curvature tensor of conformally equivalent metric [29], we obtain a curvature tensor \( K_x \) on \( M_x \) and, therefore, on \( M \).

Consider the equation of geodesics

\[ D_a a = a' + \frac{1}{2} \text{tr}(A)a - \frac{1}{2} \text{tr}(A_0^2)d\psi = 0. \quad (2.8) \]

If \( g_t \) is geodesic, then velocity \( a = g'_t \) has a constant length: \( \langle a, a \rangle_{g,x} = \text{tr}(A_0^2)\rho(g)(x) = c(x) \).

Let multiply the equation (2.8) on \( \rho(g) \),

\[
a' \rho + \frac{1}{2} \text{tr}(A)a \rho - \frac{1}{2} \text{tr}(A_0^2)\rho d\psi = 0.
\]

Then

\[
\frac{d}{dt}(a \rho) = \frac{1}{2} \langle a, a \rangle_{g,x} \, d\psi = k(x) g_0 G.
\]

The theorem is proved.

**Remark 1.** Setting formulas for a covariant derivative, curvature tensor and geodesics on \( M \) give us the same characteristic of Riemannian manifold \( M_x \) with the metric

\[
\langle a, b \rangle_{g,x} = \text{tr}(g_0^{-1} a g_0^{-1} b)(x) \sqrt{\det g_{ij}(x)}.
\]

**Remark 2.** For more general Riemannian structure on \( M_x \)

\[
\langle a, b \rangle^\alpha_{g,x} = \text{tr}(g_0^{-1} a g_0^{-1} b)\rho(g)(x) + \alpha \text{tr}(g_0^{-1} a)\text{tr}(g_0^{-1} b)\rho(g)(x)
\]

expressions for a gradient and Hessian are the following:

\[
d\psi = \frac{1}{2} g_0 G - \frac{\alpha}{2(1 + \alpha n)} \text{tr}(G)g_0,
\]

\[
H_{\psi}(a) = -\frac{1}{2} g_0 A G + \frac{\alpha}{2(1 + \alpha n)} \text{tr}(AG)g_0.
\]
\[ h_\psi(a, b) = -\frac{1}{2} \text{tr}(AB). \]

Using these formulas we can easily obtain a covariant derivative and curvature tensor on \( M \) of a weak Riemannian structure

\[ \langle a, b \rangle^\alpha_g = \int_M \text{tr}(A_0 B_0) d\mu(g) + \alpha \int_M \text{tr}(A_0) \text{tr}(B_0) d\mu(g). \]

### 2.3. A homogeneous structure.

The inner product in a point \( g \in M \) is defined by the formula:

\[ \langle a, b \rangle_g = \int_M \text{tr}(AB) d\mu(g_0). \] (2.9)

where \( a, b \in T_g M, g_0 \) is fixed metric on \( M, A = g^{-1}a \).

**Theorem 2.2.** Weak Riemannian structure (2.9) has the following geometric characteristics:

1) Covariant derivative,

\[ \nabla_a b = d_a b - \frac{1}{2} (AB + bA), \]

2) Curvature tensor,

\[ R(a, b)c = -\frac{1}{4} g [[A, B], C], \]

3) Sectional curvature \( K_\sigma \) in a plane section \( \sigma \), given by orthonormal pair \( a, b \in T_g M; \)

\[ K_\sigma = \frac{1}{4} \int_M \text{tr} \left( [A, B]^2 \right) d\mu(g_0), \]

4) Geodesics, going out from a point \( g \in M \) in direction \( a \in T_g M \) look like \( g_t = g e^{tA} \).

**Proof.** As the volume form \( \mu(g_0) \) is constant (does not depend from \( g \in M \)), then computations can be done with integrand expression from (2.9), i.e. on the finite-dimensional manifold \( M \) with the metric \( (a, b)_{g,x} = \text{tr}(AB)(x) \)

Let \( a, b \in S_{2,x}(M) \). We will think, that they define parallel vector fields on \( M \subset S_{2,x}(M) \), then \( d_a b = 0 \) and \( [a, b] = d_a b - d_b a = 0 \). For \( g \in M \) we have \( A = g^{-1}a \) and \( B = g^{-1}b \). Let \( g_t = g + tc \) is curve on \( M \), going out in direction \( c \in S_{2,x}(M) \). Then,

\[ d_c A = \left. \frac{d}{dt} \right|_{t=0} A = \left. \frac{d}{dt} \right|_{t=0} g^{-1}a = -g^{-1}cg^{-1}a = -CA. \]

To find a covariant derivative \( \nabla_a b \) we apply six-term formula

\[ 2(\nabla_a b, c) = a(b, c) + b(c, a) - c(b, a) + (c, [a, b]) + (b, [c, a]) - (a, [b, c]) = \]

\[ = d_a (\text{tr}BC) + d_b (\text{tr}CA) - d_c (\text{tr}BA) = -\text{tr}(ABC) - \text{tr}(BAC) - \text{tr}(BCA). \]
Therefore for a constant vector fields $a$ and $b$ on $\mathcal{M}_x$ we have:

$$\nabla_a b = -\frac{1}{2}(aB + bA) = \Gamma_g(a, b).$$

Calculate a curvature tensor $R(a, b)c = \nabla_a \nabla_b c - \nabla_b \nabla_a c$.

$$\nabla_a \nabla_b c = d_a(\nabla_b c) + \Gamma_g(a, \nabla_b c) = \left. \frac{d}{dt} \right|_{t=0} \left( -\frac{1}{2}(bg_t^{-1}c + cgt^{-1}b) \right) - \frac{1}{2}(ag^{-1}\nabla_b c + \nabla_b c A) =$$

$$= -\frac{1}{2}(-bg^{-1}ag^{-1}c - cgt^{-1}b) + \frac{1}{4}(a(BC + CB) + (bCA + cBA)) =$$

$$= \frac{1}{2}(bAC + cAB) + \frac{1}{4}(aBC + aCB + bCA + CBA).$$

Similarly,

$$\nabla_b \nabla_a c = \frac{1}{2}(aBC + cBA) + \frac{1}{4}(bAC + bCA + aCB + cAB).$$

From this we obtain,

$$R(a, b)c = \nabla_a \nabla_b c - \nabla_b \nabla_a c = -\frac{1}{4}((aB - bA)C - c(AB - BA)) =$$

$$= -\frac{1}{4}g((AB - BA)C - C(AB - BA)) = -\frac{1}{4}g[[A, B], C].$$

Find geodesics of this metric. Let $g_t$ is geodesic and $a = g'_t$. Then $\nabla_a a = a' - aA = 0$. Write the equation $a' - aA = 0$ as $g_t^{-1}a' - A^2 = 0$. Then the last equation is equivalent to the following: $A' = 0$, where $A = g_t^{-1}a$. Solution of the last one is a constant operator: $A(t) = A$. Then $a(t) = g_tA$, or $g'_t = g_tA$. Therefore $g_t = g e^{tA}$.

**Corollary.** Submanifold $\mathcal{M}_\mu$ of metrics $g$ with the same volume form $\mu$ and submanifold $\mathcal{P}g$ of pointwise conformally equivalent to $g \in \mathcal{M}$ metrics are totally geodesic in $\mathcal{M}$ with respect to weak Riemannian structure (2.9).

**Proof.** If the initial velocity $a$ of geodesic $g_t$ touches a submanifold $\mathcal{M}_\mu$, $\text{tr} A = 0$. Therefore for geodesic $g_t = ge^{tA}$ we have $\mu(g_t) = \mu(g) \det^{1/2}(e^{tA}) = \mu(g)$, therefore, $g_t \in \mathcal{M}_\mu$. Similarly, if $a$ touches submanifold $\mathcal{P}g$, then $a = \alpha g$, where $\alpha$ is function on $M$. Then $A = \alpha I$ and $e^{tA} = e^{\alpha t} I$.

**Remark 1.** Just the same results are obtained for more general weak Riemannian structure

$$(a, b)_{g, \alpha} = \int_M \text{tr}(AB) d\mu(g_0) + \alpha \int_M \text{tr}(A)\text{tr}(B) d\mu(g_0).$$

**Proof.** It is enough to calculate a covariant derivative. On the six-term formula we have:

$$2(\nabla_a b, c) = a(b, c)_{g, \alpha} + b(c, a)_{g, \alpha} - c(b, a)_{g, \alpha} =$$
\[ d_a (\text{tr}(BC) + \alpha \text{tr}(B) \text{tr}(C)) + d_b (\text{tr}(CA) \alpha \text{tr}(C) \text{tr}(A)) - d_c (\text{tr}(BA) \alpha \text{tr}(B) \text{tr}(A)) = \]
\[ = -\text{tr}(ABC) - \text{tr}(BAC) - \alpha \text{tr}(AB) \text{tr}(C) - \alpha \text{tr}(B) \text{tr}(AC) - \]
\[ -\text{tr}(BCA) - \text{tr}(CBA) - \alpha \text{tr}(BC) \text{tr}(A) - \alpha \text{tr}(C) \text{tr}(BA) + \]
\[ + \text{tr}(CBA) + \text{tr}(BCA) + \alpha \text{tr}(CB) \text{tr}(A) + \alpha \text{tr}(B) \text{tr}(CA) = \]
\[ = \text{tr}((AB + BA)C) - \alpha (\text{tr}(AB + BA) \text{tr}(C)) = -(aB + bA, c)_{g, \alpha}. \]

Sectional curvature
\[
K_{\sigma, \alpha} = (R(a, b, a))_{g, \alpha} = -\frac{1}{4} \text{tr}([A, B] A) - \frac{1}{4} \text{tr}([A, B]) \text{tr}(A) =
\]
\[ = -\frac{1}{4} \text{tr}([A, B] A) = -\frac{1}{4} \text{tr}([A, B][B, A]) = \frac{1}{4} \text{tr}([A, B]^2). \]

**Remark 2.** Weak Riemannian structure (2.9) is most convenient for study of the manifold \( \mathcal{M} \). It has simple formulas for a covariant derivative, curvature and geodesics, the submanifolds \( \mathcal{M}_\mu \) and \( \mathcal{P}_g \) are totally geodesic in \( \mathcal{M} \). But the structure (2.9) is non-invariant with respect to action of group of diffeomorphisms \( D(M) \).

**2.4. General canonical structure.** We shall consider a weak Riemannian structure, which is more general, than canonical,
\[
(a, b)_{g}^{\alpha} = \int_{M} \text{tr}(AB) d\mu(g) + \alpha \int_{M} \text{tr}(A) \text{tr}(B) d\mu(g). \tag{2.10}
\]

If \( \alpha \neq -1 \), then given structure is called as Devitt metric, it arises at Hamilton description of a general relativity theory \[48\].

**Theorem 2.3.** Weak Riemannian structure (2.10) has the following geometric characteristics:

1) Covariant derivative
\[
\nabla_a b = d_a b - \frac{1}{2} (aB + bA) + \frac{1}{4} (\text{tr}(A)b + \text{tr}(B)a) - \frac{1}{4(1 + \alpha n)} (\text{tr}(AB) + \alpha \text{tr}(A) \text{tr}(B)) g,
\]

2) Curvature tensor,
\[
R^a(a, b)c = -\frac{1}{4} g[[A, B], C] - \frac{1}{16} \text{tr}(C)(\text{tr}(A)b - \text{tr}(B)a) +
\]
\[ + \frac{n}{16(1 + \alpha n)} ((a, c)^{\alpha}_x b^T - (b, c)^{\alpha}_x a^T),
\]

where \( (a, c)^{\alpha}_x = \text{tr}(AC) + \alpha \text{tr}(A) \text{tr}(C) \), \( b^T = b - \frac{1}{n} \text{tr}(B) g \) is a traceless part of a tensor \( b \),

3) Sectional curvature \( R^a_{\sigma} \) in a plane section \( \sigma \), given by orthonormal pair \( a, b \in T_g \mathcal{M} \):
\[
K^a_{\sigma} = \frac{1}{4} \int_{M} \text{tr}([A, B])^2 d\mu(g) - \frac{n}{16(1 + \alpha n)} \int_{M} \left( (a, a)^{\alpha}_x (b, b)^{\alpha}_x - ((a, b)^{\alpha}_x)^2 \right) d\mu(g) +
\]

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+ \frac{1}{16} \int_M \left( \text{tr}(A^2)\text{tr}(B) + \text{tr}(B^2)\text{tr}(A) - 2\text{tr}(A)\text{tr}(B)\text{tr}(AB) \right) d\mu(g); 

4) The geodesics in $\mathcal{M} = \text{Vol}(M) \times M_\mu$ with initial position $(\mu, g)$ and initial velocity $(\beta, b) \in \Gamma(A^n M) \times S^2_\text{T}$ is

$$g_t = \left( q(t)^2 + r^2 t^2 \right)^{2/\alpha} g \exp \left( \frac{1}{r} \arctan \left( \frac{rt}{q} \right) B \right),$$

where $B = g^{-1}b$, exp is exponential mapping, $q(t) = 1 + \frac{1}{2} \beta t$, $r = \frac{1}{4} \sqrt{\text{tr}(B^2)}$. If $r = 0$, replace the exponential term by 1.

Proof. Write weak Riemannian structure (2.10) as

$$(a, b)_g^\alpha = \int_M (\text{tr}(AB) + \alpha \text{tr}(A)\text{tr}(B)) \rho(g) d\mu(g_0).$$

Integrand expression defines Riemannian metric on $\mathcal{M}_x$, which is conformally equivalent to metric $(a, b)_x^\alpha = \text{tr}(AB) + \alpha \text{tr}(A)\text{tr}(B)$, considered in item 2.3. Therefore covariant derivative and a curvature tensor are easily obtained from a covariant derivative $\nabla$ and a curvature tensor $R$ with using of equalities:

$$\psi = \ln \rho(g), \quad d\psi = \frac{1}{2(1 + \alpha n)} g, \quad H_\psi(a) = 0.$$ 

Set view of geodesics. Suppose, that $g_t$ is geodesic and $a(t) = \frac{d}{dt} g_t$. Then from the equation $\nabla^a_a = 0$ we obtain,

$$a' = aA + \frac{1}{2} \text{tr}(A)a - \frac{1}{4(1 + \alpha n)} \left( \text{tr}(A^2) - \alpha \text{tr}^2(A) \right) g = 0.$$

It is easy to see, that for $A(t) = g_t^{-1}a$ the derivative $A'$ is found from the formula $A' = g_t^{-1} (a' - aA)$. Therefore equation of geodesics is written as:

$$A' + \frac{1}{2} \text{tr}(A)A - \frac{1}{4(1 + \alpha n)} \left( \text{tr}(A^2) - \alpha \text{tr}^2(A) \right) = 0.$$ 

Enter the following value: $f = \text{tr}(A)$, $E = A - \frac{I}{n} I$, $v = \text{tr}(E^2)$, where $I$ is identity operator. Then $A = E + \frac{I}{n} I$ and $\text{tr}(A^2) = v + \frac{I^2}{n}$. Substituting it in the equation of geodesics, we obtain the pair of the equations,

$$f' + \frac{1}{4} f^2 - \frac{1}{4(1 + \alpha n)} v = 0,$$

$$E' + \frac{1}{2} fE = 0.$$ 

Now use decomposition of space $\mathcal{M}$ in direct product $\mathcal{M} = \text{Vol}(M) \times M_\mu$. Then $g_t$ is represented as a pair $g_t = (\mu_t, h_t)$. Let $\mu_t = \rho(t)\mu$. We have $g_t = \rho^{2/n}(t)h_t$ and
\(a(t) = \frac{2}{n} p' g_t + g_t h_t^{-1} h_t'.\) Therefore \(A(t) = \frac{2}{n} \rho' I + h_t^{-1} h_t'.\) It follows that \(f = \frac{2}{n} \rho'\) and \(E = h_t^{-1} h_t'.\) Let \(u = \rho',\) then previous sistem of equations rewrited as:

\[ u' + \frac{1}{2} u^2 - \frac{1}{8(1 + \alpha n)} v = 0, \quad E' = -uE.\]

As \(v' = (\text{tr}(E^2))' = 2\text{tr}(EE') = 2\text{tr}(-uE^2) = -2uv,\) then by differentiation of the first equation, we obtain, \(u'' + uu' = -\frac{v'}{4(1 + \alpha n)} = -2u(u' + \frac{1}{2} u^2),\) i.e.,

\[ u'' + 3uu' + u^3 = 0.\]

This equation, and the second one \(E' = -uE,\) have coincided with the similar equations in work \([24]\) for a determination of the geodesics of canonical metric. Therefore geodesics \(g_t\) of our Riemannian structure \((a,b)_{a}^g\) coincide with geodesics of canonical Riemannian structure on \(M,\) found in work \([24]\).

2.5. Non-Riemannian connection. In this section we consider connection on \(M,\) which takes an intermediate position between Riemannian connection \(\nabla\) of the section 2.3 and canonical connection \(\nabla^0.\) It is defined by the following covariant derivative:

\[ \nabla_a b = d_a b - \frac{1}{2}(ab + ba) + \frac{1}{8} \text{tr}(A)b + \text{tr}(B)a.\]

It is easy to see, that the bilinear form

\[ Q(a, b) = \int_M \text{tr}(A)\text{tr}(B)d\mu(g) \]

is invariant with respect to \(\nabla: AQ(b,c) = Q(\nabla_a b, c) + Q(b, \nabla_a c).\)

**Theorem 2.4.** 1) Curvature tensor of connection \(\nabla\) expressed by the formula

\[ R(a, b)c = -\frac{1}{4}q[[A, B], C] - \frac{1}{64} (\text{tr}(A)b - \text{tr}(B)a) \text{tr}(C).\]

2) Geodesics on \(M,\) going out from a point \(g \in M\) in direction \(A_0 = \frac{42}{n} I + B, \) \(\text{tr}(B) = 0\) look like:

\[ g_t = \begin{cases} 
(\beta t + 1)^{\frac{4}{n}} g \exp(\frac{\beta}{n} \ln(\beta t + 1)B), & \beta \neq 0 \\
g \exp(Bt), & \beta = 0
\end{cases} \]

**Proof.** Curvature tensor is calculated immediately. Consider the equation of geodesics \(\nabla_a a = a' - aA + \frac{1}{4} \text{tr}(A)a = 0.\) Applying an operator \(A(t) = g_t^{-1} a(t),\) we reduce it to a view

\[ A' + \frac{1}{4} \text{tr}(A)A = 0.\]

Let \(f = \text{tr}(A)\) and \(A = \frac{4}{n} I + E\) is decomposition \(A\) on a scalar and traceless parts. The equation for \(A\) falls to two ones:

\[ f' + \frac{1}{4} f^2 = 0, \quad E' + \frac{1}{4} fE = 0.\]
Solution of the first equation, with the initial condition \( f(0) = \beta \), is found immediately: \( f = 4\beta(\beta t + 4)^{-1} \). Before to decide the second equation, we decompose \( g_t \) on two components according to the decomposition \( \mathcal{M} = \text{Vol}(M) \times \mathcal{M}_\mu \). Then \( g_t = (\mu_t, h_t) \). Let \( \mu_t = \rho(t)\mu \). We have \( g_t = \rho^{2/n}(t)h_t \) and \( a(t) = \frac{2}{n} \rho g_t + g_t h_t^{-1} h_t' \). Therefore \( A(t) = \frac{2}{n} \rho I + h_t^{-1} h_t' \). Consequently \( f = 2\rho' \rho \) and \( E = h_t^{-1} h_t' \). From the first equation, taking into account the initial condition \( \rho(0) = 1 \), we find: \( \rho = \frac{1}{16} (\beta t + 4)^2 \). Solution of the second equation \( E' + \frac{1}{4} f E = 0 \) is easily found now: \( E(t) = \frac{E(0)}{\sqrt{\rho}} = \frac{4B}{\beta t + 4} \). Let \( g_0 \) is initial point of geodesic \( g_t \).

It is easy to see, that \( E(t) = h_t^{-1} h_t' = (\ln(g_0^{-1} h_t))' \). From the equation \( (\ln(g_0^{-1} h_t))' = \frac{4B}{\beta t + 4} \) is obtained \( \ln(g_0^{-1} h_t) = \frac{4\ln(\beta t + 4)B}{\beta} + c \). Since when \( t = 0 \) it is fulfilled \( h_0 = g_0, c = -\frac{4\ln 4}{\beta} B \).

Therefore
\[
g_0^{-1} h_t = e^c \exp\left(\frac{4\ln(\beta t + 4)}{\beta} B \right) = \exp\left(\frac{4}{\beta} \ln\left(\frac{\beta t + 4}{4}\right) B \right), \quad \beta \neq 0,
\]
\[
g_0^{-1} h_t = \exp(Bt), \quad \beta = 0.
\]

As \( g_t = \rho^{2/n}(t)h_t \), from the last expression, after replacement \( \beta \) on \( 4\beta \), we just obtain the view of geodesic \( g_t \), which is pointed in the theorem.

**Remark.** We have considered basic cases of natural weak Riemannian structures on \( \mathcal{M} \). Other variants also can be studied, it is possible to find corresponding covariant derivatives and curvature tensor. For example, if
\[
(a, b)_g = \int_M \text{tr}(A_0 B_0) d\mu(g_0) + \alpha \int_M \text{tr}(A)\text{tr}(B) D\mu(g),
\]
then the covariant derivative has view:
\[
\nabla_a b = d_a b - \frac{1}{2} (aB + bA) - \frac{\alpha}{1 + \alpha \rho \text{tr}(G^2)} \left( \text{tr}(AB) - \frac{1}{4} \text{tr}(A)\text{tr}(B) \right) g_0 g^{-1} g_0,
\]
where the function \( \rho \) is found from a condition \( \mu(g) = \rho(g) \mu(g_0) \). The remaining characteristics are found similarly.
§3. The space of associated Riemannian metrics on a symplectic manifold.

In this paragraph we shall suppose, that manifold \(M\) is symplectic. It means, that on \(M\) the closed nondegenerate 2-form \(\omega\) of class \(C^\infty\) is given. The manifold \(M\) has an even dimension, \(\dim M = 2n\).

3.1. The spaces of associated metrics and almost complex structures. On a symplectic manifold it is natural to consider the metrics, which are compatible with the symplectic form \(\omega\). It seams, that such metrics are well connected with almost complex structures (further a.c.s.) on \(M\).

If the a.c.s. \(J\) is given on \(M\), then it is always convenient to have also the Riemannian metric \(g\). It is natural to demand from \(g\) to be Hermitian with respect to \(J\): \(g(JX, JY) = g(X, Y)\). As known \([33]\), there exists Hermitian metric \(g\), for every a.c.s. \(J\), but it is not unique.

The task is to connect only one Hermitian metric with every a.c.s. \(J\). In case of Riemannian surfaces the complex structure defines a class of the conformally equivalent metrics, the choice of metric in this class is defined by the demand of a constancy of curvature. If \(n > 1\) then the almost complex structure \(J\) does not define a class of the conformally equivalent metrics. However there is means of unique choice of Hermitian metric \(g\) on a symplectic manifold. Let’s give the necessary definitions.

**Definition 3.1.** An almost complex structure on a manifold \(M\) is the endomorphism of a tangent bundle \(J : TM \to TM\), such that: \(J^2 = -I\), where \(I\) is identity automorphism.

**Definition 3.2.** An almost complex structure \(J\) on \(M\) is called positive associated with the symplectic form \(\omega\), if for any vector fields \(X, Y\) on \(M\) the following conditions are hold:
1) \(\omega(JX, JY) = \omega(X, Y)\),
2) \(\omega(X, JX) > 0\), if \(X \neq 0\).

**Definition 3.3.** Each positive associated a.c.s. \(J\) defines the Riemannian metric \(g\) on \(M\) by equality
\[
g(X, Y) = \omega(X, JY),
\]
which is also called associated.

The associated metric \(g\) has the following properties:
1) \(g(JX, JY) = g(X, Y)\),
2) \(g(JX, Y) = \omega(X, Y)\).

**Remark.** Sometimes positive associated almost complex structure \(J\) is called calibrating 2-form \(\omega\) (an exterior 2-form \(\omega\) on \(X\) is called \(J\)-calibrated if \(\omega(JX, JY) = \omega(X, Y)\) and \(\omega(X, JX) > 0\) for \(X \neq 0\) \([31]\)).

Almost complex structure \(J\), satisfying to the condition of positiveness \(\omega(X, JX) > 0\), if \(X \neq 0\) is also called tamed to form \(\omega\) (we say that an exterior 2-form \(\omega\) on \(M\) tames an almost-complex structure \(J\) if \(\omega(X, JX) > 0\) for \(X \neq 0\) \([31]\)).
Our terminology is offered by D.Blair \[8\], \[12\] and it seems more natural. It also corresponds to the terminology used in case of contact manifolds.

Let $\mathcal{A}$ is the space of all smooth almost complex structures on $M$. It is the space of smooth sections of the bundle $A(M)$ over $M$, fiber of which is $A_x(M)$ over the point $x \in M$ consists of automorphisms $J_x$ of tangent space $T_xM$, such that: $J_x^2 = -I_x$, where $I_x$ is identity automorphism of space $T_xM$. As the space $\mathcal{A} = \Gamma(A(M))$ is space of sections it is infinite-dimensional, smooth ILH-manifold \[1\].

In this paragraph we shall also consider the following spaces:

$\mathcal{A}_\omega$ is the space of all smooth positive associated almost complex structures on a symplectic manifold $M^{2n}, \omega$;

$\mathcal{AM}$ is the space of all smooth associated metrics on a symplectic manifold $M^{2n}, \omega$.

It is clear, that $\mathcal{AM}$ is the space of all smooth almost Kählerian metrics on a symplectic manifold, which fundamental form coinciding with $\omega$.

These spaces $\mathcal{A}_\omega$ and $\mathcal{AM}$ are spaces of smooth sections of corresponding bundles over $M$. Therefore \[1\] they are infinite-dimensional, smooth ILH-manifolds. The corresponding series of smooth Hilbert manifolds form spaces $\mathcal{A}_s^\omega$ and $\mathcal{AM}_s^\omega$ of positive associated almost complex structures and, respectively, associated metrics of Sobolev class $H^s, s > n + 1$. In this paragraph we shall not use the spaces $\mathcal{A}_s^\omega$ and $\mathcal{AM}_s^\omega$, we only remark, that for these spaces are fulfilled all facts, which are obtained for spaces $\mathcal{A}_\omega$ and $\mathcal{AM}$.

Let $J \in \mathcal{A}$ is an almost complex structure on $M$. Find tangent space $T_JA$. For this purpose there is enough to differentiate the condition $J^2 = -1$. Let $J_t$ is differentiable set of almost complex structures and $K = \frac{d}{dt}\big|_{t=0} J_t$ tangent element to the curve $J_t$ on $\mathcal{A}$. Differentiating a condition $J_t^2 = -1$, we obtain,

$$JK + KJ = 0.$$ 

Let $\text{End}_J(TM)$ is the space of smooth endomorphisms $K : TM \to TM$, anticommutating with $J$. Thus,

$$T_J\mathcal{A} = \text{End}_J(TM).$$

Now we find a tangent space $T_J\mathcal{A}_\omega$ to a manifold of positive associated almost complex structures. Let $J \in \mathcal{A}_\omega$ is positive associated a.c.s. And $g$ is its associated metric on $M$. It is enough to differentiate two conditions:

$$J_t^2 = -1, \quad \omega(J_tX, J_tY) = \omega(X, Y).$$

Let $P = \frac{d}{dt}\big|_{t=0} J_t$ is tangent element to a curve $J_t$ on $\mathcal{A}_\omega$. Differentiating the written out conditions, we obtain

$$JP + PJ = 0, \quad \omega(PX, JY) + \omega(JX, PY) = 0.$$ 

The second condition is written more convenient via the associated metric:

$$g(PX, Y) - g(X, PY) = 0$$
This is condition of symmetry of an endomorphism \( P \).

Thus, the tangent space \( T_J\mathcal{A}_\omega \) consists of symmetrical endomorphisms \( P \), anticommutating with \( J \)

\[
T_J\mathcal{A}_\omega = \{ P \in \text{End}(TM); \quad PJ = -JP, \quad g(PX,Y) = g(X, PY) \}.
\]

Let \( \text{End}_{SJ}(TM) \) is the space of smooth symmetrical endomorphisms \( P : TM \to TM \), anticommutating with \( J \). We obtain, that

\[
T_J\mathcal{A}_\omega = \text{End}_{SJ}(TM).
\]

Similarly one can show, that the tangent space \( T_g\mathcal{AM} \) to manifold \( \mathcal{AM} \) at a point \( g \) consists of the anti-Hermitian symmetric 2-forms on \( M \),

\[
T_g\mathcal{AM} = \{ h \in S_2; \quad h(JX, JY) = -h(X, Y), \quad \forall X, Y \in \Gamma(TM) \}.
\]

The condition to be anti-Hermitian is \( h_{ik}J^k_j = h_{kj}J^k_i \) in local coordinates. From property of \( h \) to be anti-Hermitian we obtain, in particular, that \( \text{tr}h = 0 \).

Denote spaces of anti-Hermitian and Hermitian symmetric 2-forms on \( M \) as \( S_{2A} \) and \( S_{2H} \) respectively. The natural(pointwise) decomposition takes place

\[
S_2 = S_{2A} \oplus S_{2H}, \quad (3.2)
\]

\[
h(X, Y) = \frac{1}{2} (h(X, Y) - h(JX, JY)) + \frac{1}{2} ((h(X, Y) + h(JX, JY)),
\]

It is orthogonal with respect to inner product (1.4) in \( S_2 \). Then,

\[
T_g\mathcal{AM} = S_{2A}.
\]

The correspondence between positive associated almost complex structures and associated metrics is defined by the diffeomorphism

\[
G : \mathcal{A}_\omega \longrightarrow \mathcal{AM},
\]

\[
J \longrightarrow G(J) = g, \quad g(X, Y) = \omega(X, JY). \quad (3.3)
\]

In coordinates, \( g_{ij} = (G(J))_{ij} = \omega_{ik}J^k_j \).

Inverse diffeomorphism:

\[
J : \mathcal{AM}^* \longrightarrow \mathcal{A}_\omega^*, \quad g \longrightarrow J, \quad J^i_j = \omega^{ik}g_{kj}.
\]

Differentiating the relation \( g_t(X, Y) = \omega(X, J_t Y) \), we find the differential of the diffeomorphism \( G \):

\[
dG : T_J\mathcal{A}_\omega \longrightarrow T_g\mathcal{AM}, \quad P \longrightarrow h = \omega P,
\]

\[
h(X, Y) = \omega(X, PY) = g(X, PJY). \quad (3.4)
\]

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There is also the following relation

\[ h(JX,Y) = h(X,JY) = -g(X,PY). \]  

(3.5)

### 3.2. A parametrization of the spaces \( \mathcal{A}_\omega \) and \( \mathcal{AM} \)

As it already was marked, the spaces \( \mathcal{A}, \mathcal{A}_\omega \) and \( \mathcal{AM} \) are smooth ILH-manifolds. Therefore one can enter local maps on them by the usual way \([1]\). We shall show, that these spaces allow a more natural parametrization via Cayley transformation.

Let \( J_0 \) is some fixed almost complex structure. As above, the tangent space \( T_{J_0}A \) consists of endomorphisms \( K : TM \rightarrow TM \), anticommutating with a.c.s. \( J_0, KJ_0 = -J_0K \).

Therefore, for exponential mapping \( e^K \) we have \( J_0e^K = e^{-K}J_0 \). It just follows from here, that

\[ J = J_0e^K \]

is an almost complex structure. The last relation gives a parametrization of the space \( \mathcal{A} \) in a neighbourhood of the element \( J_0 \) by endomorphisms \( K \), anticommutating with \( J_0 \):

\[ E : \text{End}_{J_0}(TM) \rightarrow \mathcal{A}, \quad K \mapsto J = J_0e^K. \]

Sometimes in the theory of matrices instead of a transcendental dependence \( w = e^{iz} \) the rational one: \( w = \frac{1+iz}{1-iz}, \ z = i\frac{1-w}{1+w} \) is used. Apply this transformation to an operator \( K \), possessing a property \( KJ_0 = -J_0K \), we obtain:

\[ J = J_0(1 + K)(1 - K)^{-1}. \]

It is easy to see, that

\[ J_0(1 + K)(1 - K)^{-1} = (1 - K)(1 + K)^{-1}J_0. \]

Therefore, \( J \) is an almost complex structure. At the definition of such almost complex structure the nondegeneracy of an operator \( 1 - K \) was assumed. It is enough for this purpose, that at each point \( x \in M \) the operator \( K(x) \) does not have eigenvalues equal to unit. It is clear, that the set of such endomorphisms is an open set in the space \( \text{End}_{J_0}(TM) \) of all endomorphisms \( K : TM \rightarrow TM \), anticommutating with \( J_0 \). Denote this set as \( \mathcal{V}(J_0) \),

\[ \mathcal{V}(J_0) = \{ K \in \text{End}(TM); \ KJ_0 = -J_0K, \ 1 - K \text{ is converted} \}. \]

**Proposition 3.1.** Relations

\[ J = J_0(1 + K)(1 - K)^{-1}, \]  

(3.6)

\[ K = (1 - J_0)^{-1}(1 + J_0), \]  

(3.7)

state the one-to-one correspondence between the set of endomorphisms \( K : TM \rightarrow TM \), anticommutating with a.c.s. \( J_0 \), such that \( 1 - K \) is converted and the set of almost complex structures \( J \) on \( M \), for which an endomorphism \( 1 - J_0 \) is converted.
**Proof.** Let $1 - K$ is converted. Then for a.c.s. $J = J_0(1 + K)(1 - K)^{-1}$ we have $JJ_0 = J_0(1 + K)(1 - K)^{-1}J_0,$

$$1 - JJ_0 = -J_0J_0 - J_0(1 + K)(1 - K)^{-1}J_0 = -J_0(1 - K + (1 + K))(1 - K)^{-1}J_0 =$$

$$= -2J_0(1 - K)^{-1}J_0$$

is converted operator.

Conversely, suppose, that $1 - JJ_0$ is converted, then it follows from $1 - J_0J_0 = -J(J + J_0),$ that the operator $J + J_0$ is converted. From the relation (3.6), we obtain

$$J - J_0 = (J + J_0)K, \quad J(1 + J_0J) = J(1 - J_0K)K, \quad K = (1 - J_0J)^{-1}(1 + J_0).$$

Moreover

$$K = (J + J_0)^{-1}(J - J_0).$$

Then $1 - K = 1 + (J + J_0)^{-1}(J - J_0) = (J + J_0)^{-1}(J + J_0 + J - J_0) = (J + J_0)^{-1}2J$ is converted operator.

**Remark.** The algebraic more understandable relation follows from the relation (3.6)

$$J = (1 - K)J_0(1 - K)^{-1}, \quad (3.8)$$

The set

$$U(J_0) = \{J \in \mathcal{A}; \, 1 - JJ_0 \text{ isomorphism} TM\}$$

is an open set in the space $\mathcal{A}$. Therefore map

$$\Phi : U(J_0) \longrightarrow V(J_0), \quad J \mapsto K,$$

$$K = (1 - J_0J)^{-1}(1 + J_0), \quad (3.9)$$

gives local coordinates in a neighbourhood of the element $J_0$. If $K = \Phi(J)$, it is obvious that

$$J = J_0(1 + K)(1 - K)^{-1}.$$

The ”change-of-coordinates formulas“ are easily obtained from (3.9). If $J \in U(J_0) \cap U(J_1)$ and $K = (1 - J_0J)^{-1}(1 + J_0), \quad P = (1 - J_1J)^{-1}(1 + J_1)$, then

$$P = (1 - (1 - K)(1 + K)^{-1}J_0J_1)^{-1}(1 + (1 - K)(1 + K)^{-1}J_0J_1). \quad (3.10)$$

Parametrize the space $\mathcal{A}_\omega$ of positive associated almost complex structures. The element $J \in \mathcal{A}_\omega$ has two properties:

1) $\omega(JX, JY) = \omega(X, Y),$

2) $\omega(X, JX) > 0$, if $X \neq 0.$
It was shown earlier, that the first property insures symmetry of the tangent element \( K \in T_J \mathcal{A}_\omega \). The second property of a positiveness marks out an open set in the space \( \mathcal{A} \) of all almost complex structures. Enter notation for this set:

\[
U = \{ J \in \mathcal{A}; \ \omega(X, JX) > 0, \text{ if } X \neq 0 \}.
\]

Further, the simple analysis shows, that if \( J \) and \( J_0 \) are positive almost complex structures, they are close, in the sense that the both belong to coordinate neighbourhood \( U(J_0) = \{ J \in \mathcal{A}; \ 1 - JJ_0 \text{ is isomorphism } TM \} \) entered above.

**Proposition 3.2.** Let \( J_0 \) is positive associated almost complex structure. An almost complex structure \( J \) is positive if and only if the following operators are positive

\[
-J_0 J, \quad J^T J_0, \quad -JJ_0, \quad J_0 J^T,
\]

where the transposition is taken with respect to the metric \( g_0 \), associated with \( J_0 \).

**Proof.** For associated a.c.s. \( J_0 \) we have \( g_0(J_0 X, J_0 Y) = g_0(X, Y) \) and \( g_0(X, J_0 Y) = -\omega(X, Y) \). Let \( J \in U \) is positive a.c.s. Then the operator \( S = -J_0 J \) is positive if and only if \( J \) is positive: \( g_0(X, SX) = -g_0(X, J_0 JX) = \omega(X, JX) > 0 \). For an operator \( -JJ_0 \) we represent an arbitrary vector \( X \) as \( X = J_0 Y \), then \( g_0(X, -JJ_0 X) = -g_0(J_0 Y, J_0 J_0 J_0 Y) = g_0(J_0 Y, JY) = \omega(Y, JY) > 0 \). The transposed operators \( J^T J_0, \ J_0 J^T \) are also positive.

**Corollary 1.** Let \( J_0 \) is positive associated almost complex structure. An almost complex structure \( J \) is positive iff it can be presented as

\[
J = J_0 S,
\]

where the operator \( S \) is positive with respect to the metric \( g_0 \), associated with \( J_0 \).

**Corollary 2.** Let \( J_0 \) is positive associated almost complex structure. If \( J \) is any other positive a.c.s., then the operator \( 1 - JJ_0 \) is converted.

**Corollary 3.** Positive almost complex structure \( J \) can be represented as \( J = J_0 (1 + K)(1 - K)^{-1} \), where an operator \( (1 + K)(1 - K)^{-1} \) is positive defined with respect to the metric \( g_0 \), associated with \( J_0 \).

**Proof.** Operator \( 1 - JJ_0 \) is converted because it is a sum of two positive operators: identity 1 and \( -JJ_0 \). As \( 1 - JJ_0 \) is converted, the almost complex structure \( J \) can be represented as \( J = J_0 (1 + K)(1 - K)^{-1} \), where the operator \( S = (1 + K)(1 - K)^{-1} \) is positively defined with respect to the metric \( g_0 \), associated with \( J_0 \).

The elements of the space \( \mathcal{A}_\omega \) are characterized as follows.

**Proposition 3.3.** Let \( J_0 \) is positive associated almost complex structure and \( g_0 \) is corresponding to \( J_0 \) associated metric. The almost complex structure \( J \) is positive associated iff it is represented as \( J = J_0 (1 + P)(1 - P)^{-1} \), where the endomorphism \( P : TM \to TM \) has properties:
1) \( PJ_0 = -J_0 P \),
2) \( P \) is symmetric with respect to \( g_0 \),
3) \( 1 - P^2 \) is positive with respect to \( g_0 \).

**Proof.** Let \( J \in \mathcal{A}_\omega \). It is easily checked, that the operator \( S = -J_0 J \) is symmetric with respect to \( g_0 \):

\[
g_0(X, -J_0 JY) = \omega(X, JY) = -\omega(JX, Y) = \omega(JX, J_0 JY) = g_0(JX, J_0 Y) = g_0(-J_0 JX, Y).
\]

Therefore, \( P = -(1 + S)^{-1}(1 - S) \) is also symmetric with respect to \( g_0 \). Moreover, the operator \( S \) is positive, we shall express \( S \) through \( P \), \( S = (1 + P)(1 - P)^{-1} = (1 - P^2)((1 - P)^{-1})^2 \). Then \( 1 - P^2 = S(1 - P)^2 > 0 \).

Conversely, let operator \( P \) is symmetric with respect to \( g_0 \), and \( 1 - P^2 \) is positive. Then \( 1 - P \) is converted and the operator \( S = (1 + P)(1 - P)^{-1} \) is symmetric. As \( PJ_0 = -J_0 P \), it is easy to see, that \( J = J_0 S \) is a.c.s. and

\[
\omega(JX, Y) = \omega(J_0 SX, Y) = -\omega(SX, J_0 Y) = -g_0(SX, J_0 Y) = -g_0(X, SY) =
\]

\[
= g_0(J_0 J_0 X, SY) = -g_0(J_0 J_0 SY) = \omega(X, -J_0 SY) = -\omega(X, JY).
\]

It follows from here, that \( \omega(JX, JY) = \omega(X, Y) \). The second property, \( \omega(X, JX) > 0 \), follows from a positiveness of \( 1 - P^2 \). In fact, \( S = (1 - P^2)((1 - P)^{-1})^2 > 0 \), therefore

\[
\omega(X, JX) = \omega(X, J_0 SX) = g_0(X, SX) > 0, \quad \text{if} \ X \neq 0.
\]

**Remark.** If \( J = J_0 (1 + P)(1 - P)^{-1} \) is positive associated a.c.s. and \( g \) is associated metric, corresponding to it, then \( J_0 = J(1 - P)(1 + P)^{-1} \) and it is easily checked, that the endomorphism \( P \) has properties:

1) \( J P = -P J \),
2) \( P \) is symmetric with respect to \( g \),
3) \( 1 - P^2 \) is positive with respect to \( g \).

Recall, that the tangent space \( T_J \mathcal{A}_\omega \) at a point \( J_0 \) coincides with the space \( \text{End}_{S,J_0}(TM) \) of smooth symmetrical endomorphisms \( P : TM \to TM \), anticommutating with \( J_0 \). The condition of positiveness \( 1 - P^2 > 0 \) marks out an open set in this space. Denote it as \( \mathcal{P}_{J_0} \):

\[
\mathcal{P}_{J_0} = \{ P \in \text{End}_{S,J_0}(TM) : 1 - P^2 > 0 \}.
\]

It follows from the proposition 3.3 that map

\[
\Psi : \mathcal{P}_{J_0} \to \mathcal{A}_\omega, \quad P \mapsto J = J_0 (1 + P)(1 - P)^{-1} \quad (3.11)
\]

gives a global parametrization of the space \( \mathcal{A}_\omega \) of positive associated almost complex structures.

**Proposition 3.4.** Relation \( J = J_0 e^P \) defines the one-to-one correspondence between the space \( \text{End}_{S,J_0}(TM) \) of symmetrical endomorphisms \( P : TM \to TM \), anticommutating with \( J_0 \) and the space \( \mathcal{A}_\omega \) of positive associated almost complex structures.
Proof. As $P$ anticommutes with $J_0$, $J = J_0 e^P$ is a.c.s. It follows from symmetry of $P$, that $e^P$ is symmetrical and positive, therefore $J = J_0 e^P$ is the positive associated almost complex structure. Conversely, if $J \in \mathcal{A}_\omega$, $J = J_0 (1 + P)(1 - P)^{-1}$ and the operator $(1 + P)(1 - P)^{-1}$ is positive and symmetrical. Therefore the logarithm is unique defined to it: a symmetrical operator $Q$, such that $e^Q = J_0 (1 + P)(1 - P)^{-1}$.

It follows from a proposition 3.4, that one more parametrization of space $\mathcal{A}_\omega$ is given by map

$$E_S : \text{End}_{S, J_0}(TM) \longrightarrow \mathcal{A}_\omega, \quad P \mapsto J = J_0 e^P.$$

Consider the matter of difference of the positive associated almost complex structures. Let $J_0$ is some positive associated almost complex structure, and $g_0$ is corresponding metric. The space $\mathcal{A}$ of all almost complex structures is parametrized by endomorphisms $K : TM \rightarrow TM$, anticommuting with $J_0$. As $J_0^T = -J_0$, it follows from equality $KJ_0 = -J_0 K$, that

$$J_0 K^T = -K^T J_0.$$

Therefore operator $K$ is decomposed in a sum $K = P + L$ of symmetrical $K$ and skew-symmetric $L$ endomorphisms, each of which also anticommutes with $J_0$,

$$\text{End}_{J_0}(TM) = \text{End}_{S, J_0}(TM) \oplus \text{End}_{A, J_0}(TM).$$

At an exponential parametrization of space $\mathcal{A}$

$$E : \text{End}_{J_0}(TM) \longrightarrow \mathcal{A}, \quad K \mapsto J = J_0 e^K,$$

the subspace $\text{End}_{S, J_0}(TM)$ of symmetrical endomorphisms parametrizes associated almost complex structures, and subspace $\text{End}_{A, J_0}(TM)$ of antisymmetric endomorphisms is used for a parametrization of the other, which are not associated almost complex structures.

Thus, the submanifold, which is transverse to $\mathcal{A}_\omega$ is parametrized by map

$$E_A : \text{End}_{A, J_0}(TM) \longrightarrow \mathcal{A}, \quad L \mapsto J = J_0 e^L.$$

As the endomorphism $L$ is skew-symmetric, then $e^L$ is orthogonal transformation, anticommutating with $J_0$.

Therefore, the submanifold, which is transverse to $\mathcal{A}_\omega$ in a neighbourhood of the element $J_0$ forms an orthogonal almost complex structures $J$ of the view $J = J_0 O$, where $O$ is orthogonal transformation, anticommuting with $J_0$.

Finally, give a parametrization of the space of the associated metrics. As we know, there is a natural diffeomorphism between positive associated almost complex structures and associated metrics:

$$G : \mathcal{A}_\omega \longrightarrow \mathcal{AM},
\quad J \longrightarrow G(J) = g, \quad g(X, Y) = \omega(X, JY).$$

As $J = J_0 (1 + P)(1 - P)^{-1}$,

$$g(X, Y) = \omega(X, JY) = \omega(X, J_0 (1 + P)(1 - P)^{-1} Y) =$$
We obtain a global parametrization of the space \( \mathcal{AM} \) of the associated metrics:

\[
\Psi_{AM}: \mathcal{P}_{J_0} \rightarrow \mathcal{AM}, \quad P \rightarrow g = g_0(1 + P)(1 - P)^{-1},
\]

\[
g(X,Y) = g_0(X, (1 + P)(1 - P)^{-1}Y).
\]

Other parametrization of the space \( \mathcal{AM} \) is given by map

\[
E_{AM}: \text{End}_{S,J_0}(TM) \rightarrow \mathcal{AM}, \quad P \mapsto g = g_0 e^P,
\]

\[
g(X,Y) = g_0(X, e^P Y).
\]

Find an expression of differential of the mapping \( \Psi_{AM} \). As \( \mathcal{P}_{J_0} \) is domain in the space \( \text{End}_{S,J_0}(TM) \), then \( T_P \mathcal{P}_{J_0} = \text{End}_{S,J_0}(TM) \). Therefore, differential \( d \Psi_{AM} \) at a point \( P \) is mapping of the following spaces:

\[
d\Psi_{AM}: \text{End}_{S,J_0}(TM) \rightarrow T_P \mathcal{AM}.
\]

For the fixed element \( P \in \mathcal{P}_{J_0} \) and any symmetric operator \( A \), anticommutating with \( J_0 \), we consider the line \( P_t = P + tA \) on domain \( \mathcal{P}_{J_0} \). Then

\[
g_t = g_0(1 + P_t)(1 - P_t)^{-1}, \quad J_t = J_0(1 + P_t)(1 - P_t)^{-1}.
\]

Let \( h_A = \frac{d}{dt} \bigg|_{t=0} g_t \) and \( K_A = \frac{d}{dt} \bigg|_{t=0} J_t \). To find these values the following obvious equality is used:

\[
\frac{d}{dt} \bigg|_{t=0} (1 - P_t)^{-1} = (1 - P)^{-1} A(1 - P)^{-1}.
\]

Then

\[
d\Psi_{AM}(P; A) = h_A = g_0 \left( A(1 - P)^{-1} + (1 + P)(1 - P)^{-1} A(1 - P)^{-1} \right). \quad (3.15)
\]

The expression obtained above can be transformed by three means.

1) We take into account, that \( g = g_0(1 + P)(1 - P)^{-1} \), then

\[
h_A = g_0 \left( A(1 - P)^{-1} + g A(1 - P)^{-1} \right). \quad (3.16)
\]

2) Transform (3.15) as follows:

\[
g_0 \left( A(1 - P)^{-1} + (1 + P)(1 - P)^{-1} A(1 - P)^{-1} \right) = g_0 \left( 1 + (1 + P)(1 - P)^{-1} \right) A(1 - P)^{-1} =
\]

\[
g_0 \left( 1 - P + 1 + P \right)(1 - P)^{-1} A(1 - P)^{-1} = 2g_0(1 - P)^{-1} A(1 - P)^{-1}.
\]

Then

\[
h_A = 2g_0(1 - P)^{-1} A(1 - P)^{-1}. \quad (3.17)
\]
3) Instead of \( g_0 \) it is more convenient to have \( g \),

\[
g_0 \left( A(1 - P)^{-1} + (1 + P)(1 - P)^{-1}A(1 - P)^{-1} \right) = g_0 \left( 1 + (1 + P)(1 - P)^{-1} \right) A(1 - P)^{-1} =
\]

\[
= g_0(1 + P)(1 - P)^{-1} \left( (1 - P)(1 + P)^{-1} + 1 \right) A(1 - P)^{-1} =
\]

\[
= g_0(1 + P)(1 - P)^{-1}(1 - P + 1 + P)(1 + P)^{-1}A(1 - P)^{-1} =
\]

\[
= 2g(1 + P)^{-1}A(1 - P)^{-1}.
\]

Thus,

\[
h_A = 2g(1 + P)^{-1}A(1 - P)^{-1}. \tag{3.18}
\]

Write an operator \((1 + P)^{-1}\) as \((1 - P)(1 - P)^{-1}(1 + P)^{-1} = (1 - P)(1 - P^2)^{-1}\), then

\[
h_A = 2g(1 - P)(1 - P^2)^{-1}A(1 - P)^{-1}. \tag{3.19}
\]

We shall accept the last formula as the basic expression of differential

\[
d \Psi_{AM} : \text{End}_{S,h_0}(TM) \longrightarrow T_g \mathcal{A}M,
\]

\[
d \Psi_{AM}(A) = h_A = 2g(1 - P)(1 - P^2)^{-1}A(1 - P)^{-1}.
\]

It is easily found the inverse mapping:

\[
d \Psi_{AM}^{-1}(h) = A = \frac{1}{2}(1 - P)^{-1}(1 - P^2)g^{-1}h(1 - P). \tag{3.20}
\]

In a case of an almost complex structure \( J_t = J_0(1 + P_t)(1 - P_t)^{-1} \) is similarly obtained for an operator \( K_A = \left. \frac{d}{dt} \right|_{t=0} J_t \):

\[
K_A = J_0 \ A(1 - P)^{-1} + J \ A(1 - P)^{-1}, \tag{3.21}
\]

\[
K_A = 2J_0(1 - P)^{-1}A(1 - P)^{-1}, \tag{3.22}
\]

\[
K_A = 2J(1 + P)^{-1}A(1 - P)^{-1}, \tag{3.23}
\]

\[
K_A = 2J(1 - P)(1 - P^2)^{-1}A(1 - P)^{-1}. \tag{3.24}
\]

### 3.3. A complex structure of the space \( \mathcal{A}M \).

The space \( \mathcal{A}M \) has a natural almost complex structure, which is constructed as follows.

The tangent space \( T_g \mathcal{A}M \) at \( g \in \mathcal{A}M \) consists of all symmetric \( J \)-anti-Hermitian 2-forms \( h \) on \( M \), where \( J \) is almost complex structure corresponding to the metric \( g \). As the form \( h \) is anti-Hermitian, i.e. \( h(JX, JY) = -h(X, Y) \), the 2-form \( hJ \), defined by equality \( (hJ)(X, Y) = h(X, JY) \), is also symmetric and anti-Hermitian. Therefore on each tangent space \( T_g \mathcal{A}M \) the operator acts

\[
J_g : T_g \mathcal{A}M \longrightarrow T_g \mathcal{A}M, \quad J_g(h) = hJ. \tag{3.25}
\]

It is obvious, that \( J_g^2 = -1 \). Therefore, on the manifold \( \mathcal{A}M \) the almost complex structure \( J \) is defined.
On the other hand, the model space $\text{End}_{S, J_0}(TM)$ of a global parametrization

$$\Psi_{AM} : \mathcal{P}_{J_0} \to AM, \quad P \to g = g_0(1 + P)(1 - P)^{-1},$$

has a complex structure:

$$\text{End}_{S, J_0}(TM) \to \text{End}_{S, J_0}(TM), \quad A \mapsto AJ_0. \quad (3.26)$$

Therefore one can think, that the space $AM$ is infinite-dimensional complex manifold.

**Theorem 3.1.** The almost complex structure $J$ on the manifold $AM$ is integrable. The corresponding complex structure also coincides with a complex structure on $AM$, obtained by a parametrization $\Psi_{AM} : \mathcal{P}_{J_0} \to AM$.

**Proof.** $d\Psi_{AM}(A) = h_A = 2g(1 - P)(1 - P^2)^{-1}A(1 - P)^{-1},$

$$A \mapsto h_A \mapsto h_AJ = 2g(1 - P)(1 - P^2)^{-1}A(1 - P)^{-1}(1 - P)J_0(1 - P)^{-1} =$$

$$= 2g(1 - P)(1 - P^2)^{-1}AJ_0(1 - P)^{-1} \mapsto AJ_0.$$

Remark, that the weak Riemannian structure on $AM$ is Hermitian with respect to the complex structure $J$. Indeed, if $a, b \in T_g AM$ are arbitrary tangent elements, then operators corresponding to them, $A = g^{-1}a$, $B = g^{-1}b$ anticommute with $J$ and we obtain

$$(J(a), J(b))_g = (aJ, bJ)_g = \int_M \text{tr}(AJBJ)d\mu = \int_M \text{tr}(AB)d\mu = (a, b)_g.$$

Define the fundamental form of the Hermitian weak Riemannian structure $(a, b)_g$ on $AM$,

$$\Omega_g(a, b) = (aJ, bJ)_g = \int_M \text{tr}(AJB)d\mu. \quad (3.27)$$

It is obvious, that it is the nondegenerate skew-symmetric 2-form on $AM$.

**Theorem 3.2.** The fundamental form $\Omega_g$ of the Hermitian weak Riemannian structure $(a, b)_g$ on $AM$ is closed.

**Proof.** Show, that the exterior differential of the form $\Omega_g$ is equal to zero at an arbitrary point $g_0 \in AM$, $d\Omega_{g_0} = 0$. For this purpose we use coordinates on $AM$ with origin at a point $g_0$: $g = g_0(1 + P)(1 - P)^{-1}$. The value of an operator $P = 0$ corresponds to the point $g_0$. Therefore it is enough to show, that $d\Omega_P = 0$ at $P = 0$. We use the standard formula for an external product for an external product:

$$d\Omega(A_0, A_1, A_2) = A_0\Omega(A_1, A_2) - A_1\Omega(A_0, A_2) + A_2\Omega(A_0, A_1) -$$

$$-\Omega([A_0, A_1], A_2) + \Omega([A_0, A_2], A_1) + \Omega([A_1, A_2], A_0).$$

Let $A_0, A_1, A_2$ are constant vector fields of operators on the space $\text{End}_{S, J_0}(TM)$. Then all Lie brackets are equal to zero. Let’s show, that the other addends $A_i\Omega(A_j, A_k)$ are equal to zero too.

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The field \( h_A \) on \( \mathcal{AM} \) corresponds to operator \( A \in \text{End}_{S,J_0}(TM) \) under the following formula:
\[
A \mapsto h_A = 2g(1 + P)^{-1}A(1 - P)^{-1}.
\]
Then \( J(h_A) = h_{AJ_0} \). Therefore
\[
g^{-1}J(h_A) = g^{-1}h_{AJ_0} = 2(1 + P)^{-1}AJ_0(1 - P)^{-1} = H_{AJ_0}.
\]
We obtain an expression of \( \Omega \) in the coordinate map:
\[
\Omega_P(A, B) = \Omega_g(h_A, h_B) = (h_AJ, h_B)_g = (h_{AJ_0}, h_B)_g =
\]
\[
= 4 \int_M \text{tr}(H_{AJ_0}H_B)d\mu = 4 \int_M \text{tr}((1 + P)^{-1}AJ_0(1 - P)^{-1}(1 + P)^{-1}B(1 - P)^{-1})d\mu =
\]
\[
= 4 \int_M \text{tr}((1 - P^2)^{-1}AJ_0(1 - P^2)^{-1}B)d\mu.
\]
Let in this formula \( A = A_1, B = A_2 \) are constant operators (i.e. do not depend on \( P \)). On a linear property of an integral and trace, it is enough to differentiate the expression \( (1 - P^2)^{-1}A_1J_0(1 - P^2)^{-1}A_2 \) on \( P \) to find derivative \( A_0\Omega(A_1, A_2) \). One can think, that \( P_t = tA_0 \). As \( \frac{d}{dt}|_{t=0}(1 - P^2)^{-1} = \frac{d}{dt}|_{t=0}(1 - t^2A_0^2)^{-1} = 0 \), \( A_0\Omega(A_1, A_2) = \frac{d}{dt}|_{t=0}\Omega P_t(A_1, A_2) = 0 \). The theorem is proved.

**Corollary.** Manifold \( \mathcal{AM} \) is Kähler.

### 3.4. Local expressions. The Beltrami equation.

Let \( J_0 \) is positive associated almost complex structure and \( g_0 \) is corresponding associated metric. The almost complex structure \( J_0 \) defines decomposition of the complexification \( TM^C \) of the tangent bundle \( TM \),
\[
TM^C = T^{10}(J_0) \oplus T^{01}(J_0),
\]
on subbundles \( T^{10}(J_0) \) and \( T^{01}(J_0) \), on which the complexified operator \( J_0 \) acts as multiplication on \( i \) and \( -i \) respectively.

Let \( \partial_1, \ldots, \partial_n \) is local basis of sections of the bundle \( T^{10}(J_0) \), \( dz^1, \ldots, dz^n \) is dual basis of the bundle \( T^{*10}(J_0) \) and \( \overline{\partial}_1, \ldots, \overline{\partial}_n \) is complex conjugate basis of sections of the bundle \( T^{01}(J_0) \), \( d\overline{z}^1, \ldots, d\overline{z}^n \) is dual basis of \( T^{*01}(J_0) \).

As \( g_0 \) is \( J_0 \)-Hermitian metric, it follows, that
\[
g_{\alpha\beta} = g_0(\partial_\alpha, \partial_\beta) = 0, \quad g_{\overline{\alpha}\overline{\beta}} = g_0(\overline{\partial}_\alpha, \overline{\partial}_\beta) = 0, \quad \alpha, \beta = 1, \ldots, n.
\]
Let
\[
g_{\alpha\overline{\beta}} = g_0(\partial_\alpha, \overline{\partial}_\beta), \quad g_{\overline{\alpha}\beta} = g_0(\overline{\partial}_\alpha, \partial_\beta), \quad \alpha, \beta = 1, \ldots, n.
\]
These coefficients have properties
\[
g_{\alpha\overline{\beta}} = g_{\overline{\alpha}\beta}, \quad g_{\alpha\overline{\beta}} = \overline{g_{\overline{\alpha}\beta}},
\]
which are implied from symmetry and hermiticity of the metric \( g_0 \).
The metric $g_0$ is expressed as follows:

$$g_0 = 2g_{\alpha\beta} \, dz^\alpha d\overline{z}^\beta.$$ 

Let now $J$ is another positive associated almost complex structure. Then $J = J_0(1 + P)(1 - P)^{-1}$, where $P : TM \to TM$ is a symmetric endomorphism, anticommutating with $J_0$, satisfying to the condition of positiveness $1 - P^2 > 0$.

As

$$J_0(P(\partial_\alpha)) = -PJ_0(\partial_\alpha) = -P(i\partial_\alpha) = -iP(\partial_\alpha),$$

that $P(\partial_\alpha)$ is a local section of the bundle $T^{01}(J_0)$, therefore

$$P(\partial_\alpha) = \overline{P}^\beta_{\alpha} \bar{\partial}_\beta, \quad \alpha, \beta = 1, \ldots, n \quad (3.28)$$

where $P^\beta_{\alpha}$ is matrix of complex-valued functions. Thus, the operator $P$ in the complex basis is locally given by a matrix of view:

$$P = \begin{pmatrix} 0 & P^\beta_{\alpha} \\ P^\beta_{\alpha} & 0 \end{pmatrix}, \quad P^\beta_{\alpha} = \overline{P}^\beta_{\alpha}. \quad (3.29)$$

The condition of symmetry of an operator $P$ is expressed by the relation:

$$P_{\alpha\beta} = P_{\beta\alpha}, \quad \text{where} \quad P_{\alpha\beta} = g_{\alpha\tau} P^\tau_{\beta}. \quad (3.30)$$

Invariant form of an operator $P$:

$$P = P^\beta_{\alpha} \bar{\partial}_\beta \otimes dz^\alpha + P^\beta_{\alpha} \partial_\beta \otimes d\overline{z}^\alpha \quad (3.31).$$

Positive associated almost complex structure $J = J_0(1 + P)(1 - P)^{-1}$ defines another decomposition of the complexification $TM^C$,

$$TM^C = T^{10}(J) \oplus T^{01}(J).$$

Dependence of this expansion upon $J$ becomes clear if we use an operator $P$.

**Proposition 3.5.** If $J = J_0(1 + P)(1 - P)^{-1}$ then

$$T^{10}(J) = (1 - P)(T^{10}(J_0)), \quad T^{01}(J) = (1 - P)(T^{01}(J_0)). \quad (3.32)$$

**Vector fields**

$$\partial_\alpha(J) = \partial_\alpha - P^\beta_{\alpha} \bar{\partial}_\beta, \quad \overline{\partial}_\alpha(J) = \overline{\partial}_\alpha - P^\beta_{\alpha} \partial_\beta \quad (3.33)$$

form local basis of sections of bundles $T^{10}(J)$ and $T^{01}(J)$ respectively.

**Proof.** It is enough to prove, that the vector fields $\partial_\alpha(J) = \partial_\alpha - P^\beta_{\alpha} \bar{\partial}_\beta$ form the basis of the bundle $T^{10}(J)$. From a nondegeneracy of an endomorphism $1 - JJ_0$ the nondegeneracy of an operator $1 - P$, which is equal to $-2(1 - JJ_0)^{-1}JJ_0$, follows. Therefore, vector fields
\[ \partial_\alpha(J) = (1 - P)(\partial_\alpha) \] are linearly independent. Let’s show, that \[ J(\partial_\alpha(J)) = i\partial_\alpha(J), \]

\[ J(\partial_\alpha(J)) = J(1 - P)\partial_\alpha = J_0(1 + P)(1 - P)^{-1}(1 - P)\partial_\alpha = \]

\[ = J_0(1 + P)\partial_\alpha = (1 - P)J_0\partial_\alpha = i(1 - P)\partial_\alpha = i\partial_\alpha(J). \]

**Corollary.** The dual bases of the forms \( dz_1(J), \ldots, dz_n(J) \) and \( d\bar{z}_1(J), \ldots, d\bar{z}_n(J) \) for bases of vector fields \( \partial_1(J), \ldots, \partial_n(J) \) and \( \partial_1(J), \ldots, \partial_n(J) \) have the following expressions:

\[ dz^\alpha(J) = D^\alpha_\gamma (dz^\gamma + P^\alpha_\beta d\bar{z}^\beta), \quad (3.34) \]

\[ d\bar{z}^\alpha(J) = D^\alpha_\gamma (d\bar{z}^\gamma + P^\alpha_\beta dz^\beta), \quad (3.35) \]

where \( D^\alpha_\gamma \) is matrix of an operator \( D = (1 - P^2)^{-1} : T^{10}(J_0) \rightarrow T^{01}(J_0). \)

It follows from the proposition 3.5 and formula (3.31), that the operator \( P \), giving an almost complex structure \( J \) is many-dimensional generalization of Beltrami coefficient in the Beltrami equation \( \frac{\partial f}{\partial \bar{z}} - \mu \frac{\partial F}{\partial z} = 0. \)

Indeed, geometrical sense (see for example [16]) of Beltrami coefficient is that \( \mu \) is tensor field on a Riemann surface \( M \) of view: \( \mu = \mu_\partial \frac{\partial}{\partial \bar{z}} \otimes d\bar{z}. \) The invariant form (3.31) of the operator \( P \) has the same sense. Therefore many-dimensional generalization of the Beltrami equation has the following view:

\[ \bar{\partial}_\alpha(J)(f) = \frac{\partial f}{\partial \bar{z}^\alpha} - P^\alpha_\beta \frac{\partial f}{\partial z^\beta} = 0, \quad \alpha = 1, \ldots, n, \quad (3.36) \]

where \( f \) is function on \( M. \)

Each positive associated a.c.s. \( J \) defines associated Hermitian metric \( g \) by equality \( g(X,Y) = \omega(X, JY). \) Recall the expression of the metric \( g \) via the Riemannian metric \( g_0 \) and operator \( P: \)

\[ g(X,Y) = g_0((1 + P)X, (1 - P)^{-1}Y) = g_0((1 + P)X, (1 + P)DY), \]

where \( D = (1 - P^2)^{-1}. \)

Assume, for a simplicity that a.c.s. \( J_0 \) is integrable and \( z_1, \ldots, z_n \) are corresponding complex local coordinates on \( M. \) Let in these coordinates the Hermitian form corresponding to a Riemannian structure \( g_0 \) has an aspect \( g_0 = 2g_{a\overline{\beta}} d z^a d \overline{z}^\beta. \) Then from (3.34) and (3.35) is obtained the following local expression for the Hermitian form of the associated metric \( g: \)

\[ g = 2g_{a\overline{\beta}} D^\alpha_\gamma (dz^\alpha + P^\alpha_\beta d\bar{z}^\beta) (d\bar{z}^\gamma + P^\gamma_\nu dz^\nu). \quad (3.37) \]
§4. Decomposition of the space of Riemannian metrics on a symplectic manifold.

Let $M^m$ be a smooth oriented manifold and $\text{Vol}(M) \subset \Gamma(\Lambda^m M)$ is the space of smooth volume forms on $M$, i.e. the space of smooth nondegenerate $m$-forms on $M$, defined orientation which coincides with the original one on $M$. The natural projection is defined

$$vol : \mathcal{M} \longrightarrow \text{Vol}(M), \quad g \longmapsto \mu_g = \sqrt{\det g} dx^1 \wedge \ldots \wedge dx^m.$$

A fiber of the bundle $vol$ over $\mu \in \text{Vol}(M)$ is the space $M_{\mu}$ of metrics with the same Riemannian volume form $\mu$.

The fixing of a volume form $\mu$ defines decomposition of the space $\mathcal{M}$ in the direct product:

$$\varphi_{\mu} : \mathcal{M} \longrightarrow \text{Vol}(M) \times M_{\mu}, \quad g \longmapsto (\mu_g, \rho^{-\frac{2}{m}} g),$$

where $\rho$ is density of the volume form $\mu_g$ with respect to $\mu$, i.e. function on $M$ is defined from equality $\mu_g = \rho \mu$.

The inverse mapping:

$$\iota_{\mu} : \text{Vol}(M) \times M_{\mu} \longrightarrow \mathcal{M}, \quad (\nu, h) \longmapsto \rho^\frac{2}{m} h, \quad \nu = \rho \mu.$$

At such decomposition, the space $\text{Vol}(M)$ of volume forms corresponds to the space of the metrics $C_g$, which are conformally equivalent to the fixed metric $g \in M_{\mu}$:

$$\text{Vol}(M) \times \{g\} \longrightarrow C_g, \quad \nu = \rho \mu_g, \quad \nu \longmapsto \rho^\frac{2}{m} g.$$

Thus, the space of all Riemannian metrics on a manifold $M$ is decomposed in a direct product of the space of metrics with a fixed Riemannian volume form and the space of pointwise conformally equivalent metrics. The similar construction is possible in case of symplectic and contact manifolds.

Let $M^{2n}, \omega$ is symplectic manifold.

We remind, that the almost complex structure $J$ on $M$ is called positive associated to the symplectic form $\omega$, if for any vector fields $X, Y$ on $M$,

1) $\omega(JX, JY) = \omega(X, Y),$

2) $\omega(X, JX) > 0$, if $X \neq 0$.

Every such a.c.s. $J$ defines the Riemannian metric $g$ on $M$ by equality:

$$g(X, Y) = \omega(X, JY),$$

which is also called associated.

Let $\mathcal{AM}_\omega$ is the space of all smooth associated metrics on $M$.

The symplectic form $\omega$ on a manifold $M$ defines a well known projection of the space $\mathcal{M}$ on $\mathcal{AM}_\omega$ as follows:

Let $g' \in \mathcal{M}$ is any metric. There is a unique skew-symmetric automorphism $A$ of a tangent bundle $TM$, such that:

$$\omega(X, Y) = g'(AX, Y), \quad A^T = -A.$$
Apply the polar decomposition to the endomorphism $A$

$$A = JH,$$

where $J$ is orthogonal operator and $H$ is positive symmetrical one. The endomorphism $-A^2 = A^T A = AA^T$ is positive defined and symmetrical with respect to $g'$. Then $H = (-A^2)^{\frac{1}{2}}$ is positive square root from $-A^2$. As it is known, it commutes with operators $A$ and $J$. Suppose $J = A(-A^2)^{-\frac{1}{2}}$. It is easily checked, that $J$ is an almost complex structure:

$$J^2 = A(-A^2)^{-\frac{1}{2}} A(-A^2)^{-\frac{1}{2}} = A^2(-A^2)^{-1} = -I.$$

The formula

$$g(X, Y) = \omega(X, JY)$$

defines the Riemannian metric on $M$. Indeed,

$$g(X, Y) = \omega(X, JY) = g'(AX, JY) = g'(AX, A(-A^2)^{-\frac{1}{2}} Y) =
= g'(X, (-A^2)(-A^2)^{-\frac{1}{2}} Y) = g'(X, (-A^2)^{\frac{1}{2}} Y).$$

As the operator $(-A^2)^{\frac{1}{2}}$ is positive and symmetrical, then $g$ is the Riemannian metric. Positivity of an almost complex structure $J$ also follows from here: $\omega(X, JX) = g(X, X) > 0$.

The metric $g$ is $J$-Hermitian:

$$g(JX, JY) = \omega(JX, -Y) = -g'(AJX, Y) = -g'(AA(-A^2)^{-\frac{1}{2}} X, Y) =
= g'(-A^2)^{\frac{1}{2}} X, Y) = g'(X, (-A^2)^{\frac{1}{2}} Y) = g(X, Y).$$

The symplectic form $\omega$ is also $J$-invariant:

$$\omega(JX, JY) = g(JX, Y) = -g(X, JY) = -\omega(X, J^2 Y) = \omega(X, Y).$$

Therefore, $J$ is a positive associated almost complex structure, and metric $g(X, Y) = \omega(X, JY)$ is associated metric, which corresponds to the structure $J$.

We have got a required projection

$$p_\omega : \mathcal{M} \longrightarrow \mathcal{AM}_\omega, \quad g' \mapsto g, \quad g(X, Y) = g'(X, (-A^2)^{\frac{1}{2}} Y). \quad (4.1)$$

As the operator $J$ on construction is orthogonal with respect to $g'$, the metric $g'$ is also Hermitian with respect to the a.c.s. $J$. One can show, that the fiber $p_\omega^{-1}(g)$ consists of all $J$-Hermitian metrics. Let $\mathcal{M}_J$ denotes the set of all $J$-Hermitian Riemannian metrics on $M$.

**Lemma 4.1.** For any associated metric $g \in \mathcal{AM}_\omega$ and its corresponding a.c.s. $J$, the inverse image of the element $g$ at a projection $p_\omega$ coincides with a set $\mathcal{M}_J$ of all $J$-Hermitian Riemannian metrics on $M$:

$$p_\omega^{-1}(g) = \mathcal{M}_J.$$
Proof. We already remarked, that any metric $g'$, which is projected in $g$ is $J$-Hermitian, therefore $p_\omega^{-1}(g) \subset \mathcal{M}_J$. Show the converse. Let $g' \in \mathcal{M}_J$ is $J$-Hermitian metric. As $g$ and $g'$ are $J$-Hermitian metrics, there is a symmetric positive operator $B$ commuting with $J$ and such that $g'(X,Y) = g(X,BY) = g(BX,Y)$.

$$\omega(X,Y) = g(JX,Y) = g'(JX,B^{-1}Y) = g'(B^{-1}JX,Y).$$

Therefore $A = B^{-1}J$, $-A^2 = -B^{-1}JB^{-1}J = -B^{-2}J^2 = B^{-2}$. As $B^{-1}$ is symmetric, $(-A^2)^{-\frac{1}{2}} = (B^{-2})^{-\frac{1}{2}} = B$. Then the almost complex structure corresponding to the metric $g'$ coincides with original: $(-A^2)^{-\frac{1}{2}}A = BB^{-1}J = J$. Therefore $p_\omega(g') = g$.

Let $g_0 \in \mathcal{AM}_\omega$ is some fixed associated metric and $J_0$ is almost complex structure, corresponding to it. Any other associated metric $g$, $J$ can be represented as

$$g(X,Y) = g_0(X,(1+P)(1-P)^{-1}Y), \quad J = J_0(1+P)(1-P)^{-1},$$

where the operator $P$, anticommutating with an a.c.s. $J_0$, is symmetric with respect to $g_0$ and $1-P^2$ is positive with respect to $g_0$.

There is a natural question: Are other metrics of fibers $\mathcal{M}_{J_0}$ and $\mathcal{M}_J$ connected by similar relations? The answer is given by the following

Lemma 4.2. Let $g_0 \in \mathcal{M}_{J_0}$ is an arbitrary $J_0$-Hermitian metric and $J = J_0(1+P)(1-P)^{-1}$. Then $g'$, defined by equality:

$$g'(X,Y) = \frac{1}{2} \left( g_0 \left( (1+P)(1-P)^{-1}X,Y \right) + g_0' \left( X, (1+P)(1-P)^{-1}Y \right) \right) \quad (4.2)$$

is $J$-Hermitian metric. The fundamental form $\omega'(X,Y) = g'(JX,Y)$ of the metric $g'$ is expressed via $\omega$ as follows:

$$\omega'(X,Y) = \frac{1}{2} \left( \omega(X,Y) + \omega \left( (1+P)(1-P)^{-1}X, (1+P)(1-P)^{-1}Y \right) \right). \quad (4.3)$$

Proof. Symmetry of $g'(X,Y)$ follows from the definition. The positive definiteness of $g'(X,Y)$ just follows from the positive definiteness of operator $(1+P)(1-P)^{-1}$ with respect to $g_0$. $J$-hermiticity of metric $g'(X,Y)$ at once follows from $J_0$-hermiticity of $g_0'$ and the following property

$$J = J_0(1+P)(1-P)^{-1} = (1-P)(1+P)^{-1}J_0.$$ 

The fundamental form $\omega'(X,Y) = g'(JX,Y)$ of the metric $g'$ is found from elementary calculation.

It follows from this lemma, that for the associated metrics $g_0$, $J_0$ and $g$, $J$ the fibers $\mathcal{M}_{J_0}$ and $\mathcal{M}_J$ are naturally diffeomorphic. Indeed, formulas

$$g_0(X,Y) = g(X,(1-P)(1+P)^{-1}Y), \quad J_0 = J(1-P)(1+P)^{-1},$$

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allow to define the inverse mapping of fibers $\mathcal{M}_J \to \mathcal{M}_{J_0}$:

$$g'_0(X,Y) = \frac{1}{2} \left( g' \left( (1 - P)(1 + P)^{-1}X,Y \right) + g' \left( X,(1 - P)(1 + P)^{-1}Y \right) \right). \tag{4.4}$$

**Theorem 4.1.** The space $\mathcal{M}$ is a smooth trivial bundle over $\mathcal{AM}_\omega$. A fiber over the element $(g, J) \in \mathcal{AM}_\omega$ is the space $\mathcal{M}_J$ of all $J$-Hermitian Riemannian metrics on $M$.

**Proof.** Let $(g_0, J_0) \in \mathcal{AM}_\omega$ is associated structure. The space $\mathcal{AM}_\omega$ is parametrized by domain

$$\mathcal{P}_{J_0} = \{ P \in \text{End}_{S,J_0}(TM) : 1 - P^2 > 0 \}$$

of the space $\text{End}_{S,J_0}(TM)$ of smooth symmetrical endomorphisms $P : TM \to TM$ anticommutating with $J_0$.

The fiber $\mathcal{M}_{J_0}$ of fiber bundle $p_\omega$ over the point $g_0$ consists of all $J_0$-Hermitian metrics, it is an open set in the linear Frechet space of smooth symmetric $J_0$-Hermitian forms on $M$.

One can take the following ILH-smooth mapping as a required map:

$$\Psi_M : \mathcal{P}_{J_0} \times \mathcal{M}_{J_0} \longrightarrow \mathcal{M}, \quad (P, g'_0) \longrightarrow g',$$

where $P \in \mathcal{P}_{J_0}$, $g'_0 \in \mathcal{M}_{J_0}$ and the metric $g'$ is defined by equality (4.2). It follows from the lemma 2, that $\mathcal{M}_{J_0}$ is diffeomorphic mapped on fiber $\mathcal{M}_J$ of fiber bundle $p_\omega$ over the point $g(X,Y) = g_0((1 + P)X, (1 - P)^{-1}Y)$.

The inverse mapping for $\Psi_M$ is defined by the correspondence (4.4): $\mathcal{M}_J \to \mathcal{M}_{J_0}$, $g' \longrightarrow g'_0$ together with the projection $p_\omega : \mathcal{M} \longrightarrow \mathcal{AM}_\omega$. The theorem is proved.

**Conclusion.** The space of all Riemannian metrics on a symplectic manifold $M$ is decomposed in a direct product of the space of associated metrics and the space of $J$-conformally equivalent metrics.

Finally, we remark, that one more projection $q : \mathcal{M} \longrightarrow \Lambda^2_0(M)$ of the space of metrics $\mathcal{M}$ on the space $\Lambda^2_0(M)$ of all smooth nondegenerate skew-symmetric 2-forms can be defined. If $g' \in \mathcal{M}$ and $J = p_\omega(g')$ is corresponding a.c.s., then

$$q(g') = \omega', \quad \omega'(X,Y) = g'(JX,Y).$$

The inverse image $q^{-1}(\omega')$ of the element $\omega' \in \Lambda^2_0(M)$ is a set $\mathcal{AM}_{\omega'}$ of Riemannian metrics, associated with the nondegenerate 2-form $\omega'$ on $M$.

At mapping $q$ the fiber $\mathcal{M}_J$ of projection $p_\omega$ passes in the space of the nondegenerate exterior 2-forms, which are invariant with respect to a.c.s. $J$, i.e. such forms $\omega'$, that $\omega'(JX, JY) = \omega'(X,Y)$. 

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§5. A curvature of the space of associated metrics.

In this paragraph we shall find geodesics and sectional curvatures of the space $\mathcal{AM}$ of almost Kähler metrics on a symplectic manifold $M^{2n}, \omega$.

It is well known [33], that the Riemannian volume form $\mu_g$ of a metric $g \in \mathcal{AM}$ is expressed via the form $\omega$: $\mu_g = \frac{1}{n!} \omega^n$. Therefore, the space $\mathcal{AM}$ is into manifold $M_\mu$ of metrics with the same volume form $\mu = \frac{1}{n!} \omega^n$.

The manifold $M_\mu$, is a smooth ILH-submanifold in $\mathcal{M}$ [17] and inherits a weak Riemannian structure, which is expressed as follows. If $a, b \in T_g M_\mu$ are two smooth symmetric 2-forms on $M$, representing elements of tangent space $T_g M_\mu$, then their inner product is defined by the formula:

$$(a, b)_g = \int_M \text{tr} A B \, d\mu,$$

(5.1)

where $A = g^{-1} a = g^{ik} a_{kj}$. Recall, that the tangent space $T_g M_\mu$ consists of traceless symmetric 2-forms,

$T_g M_\mu = S^T_2 = \{ h \in S_2 : \text{tr}_g h = 0 \}.$

As in (5.1) the volume form does not depend on $g$, $\mu_g = \mu$, it will be more convenient to use another weak Riemannian structure, defined by the formula (2.9) from §2, on all the space $\mathcal{M}$. In this case submanifold $M_\mu \subset \mathcal{M}$ is (see §2) totally geodesic in $\mathcal{M}$. Therefore, from the theorem 2.2 we obtain the following characteristics of the space $M_\mu$:

1) A covariant derivative:

$$\nabla_a b = d_a b - \frac{1}{2} (a B + b A),$$

2) A tensor of curvature:

$$R(a, b)c = -\frac{1}{4} g \left[ [A, B], C \right],$$

3) A sectional curvature $K_\sigma$ in plane section $\sigma$, given by orthonormal pair $a, b \in T_g M_\mu$:

$$K_\sigma = \frac{1}{4} \int_M \text{tr} ([A, B]^2) \, d\mu,$$

4) Geodesics, going out from a point $g \in M_\mu$ in direction $a \in T_g M_\mu$ look like $g_t = g e^{tA}$.

**Proposition 5.1.** The manifold $\mathcal{AM}$ is totally geodesic submanifold in the space $M_\mu$ of all Riemannian structures on $M$ with the same volume form $\mu = \frac{1}{n!} \omega^n$.

**Proof.** Geodesics on $M_\mu$ look like $g_t = ge^{tA}$, where $A = g^{-1} a, a \in T_g M_\mu$. Show, that if $a \in T_g \mathcal{AM}$, then geodesic $g_t$ is on $\mathcal{AM}$. If $a \in T_g \mathcal{AM}$, then the operator $A = g^{-1} a$ is symmetric and anticommutes with $J$, $JA = -AJ$. Therefore, $Je^{tA} = e^{-tA} J$. Show, that $g_t = ge^{tA}$ is a family of associated metrics, corresponding to a family of positive
associated almost complex structures \( J_t = J e^{tA} \). For this purpose it is enough to show, that \( g_t(X,J_tY) = -\omega(X,Y) \), but it is obvious:

\[
g_tJ_t = ge^{tA}J e^{tA} = gJ e^{-tA}e^{tA} = gJ = -\omega.
\]

The proposition is proved.

Recall, that

\[
\text{End}_{SJ}(TM) = \{ P \in \text{End}(TM); \quad PJ = -JP, \quad g(PX,Y) = g(X,PY) \}
\]

is the space of smooth symmetrical endomorphisms \( P : TM \to TM \), anticommutating with \( J \).

**Corollary.** Map

\[
E_{AM} : \text{End}_{SJ_0}(TM) \to AM, \quad P \mapsto g = g_0e^P, \quad g(X,Y) = g_0(X,e^PY)
\]

gives normal coordinates on the space \( AM \) in a neighbourhood of the element \( g_0 \).

The tangent space \( T_gAM \) at a point \( g \in AM \) consists of all symmetric anti-Hermitian 2-forms \( h \) on \( M \). As the form \( h \) is anti-Hermitian, i.e. \( h(JX,JY) = -h(X,Y) \), the 2-form \( hJ \), defined by equality \( (hJ)(X,Y) = h(X,JY) \), is also symmetric and anti-Hermitian. Therefore, on the space \( T_gAM \) the operator

\[
J : T_gAM \to T_gAM, \quad J(h) = hJ.
\]

is defined. Obviously, that \( J^2 = -1 \). Therefore, on the space \( AM \) the almost complex structure \( J \) is defined. It is immediately checked, that the inner product \( (5.1) \) is Hermitian with respect to the a.c.s. \( J \) on \( AM \) (see also §3).

It follows from the property of the submanifold \( AM \) to be totally geodesic in \( M_\mu \), that all properties of a curvature pointed above for the space \( M_\mu \) hold for the space \( AM \).

**Theorem 5.2.** The space \( AM \) has the following geometric characteristics.

1) A tensor of curvature is following

\[
R(a,b)c = -\frac{1}{4}g [[A,B],C], \quad (5.2)
\]

where \( a, b, c \in T_gAM, \quad A = g^{-1}a \).

2) A sectional curvature \( K_\sigma \) in plane section \( \sigma \), given by orthonormal pair \( a, b \in T_gAM \) is expressed by the formula

\[
K_\sigma = \frac{1}{4} \int_M \text{tr} ([A,B]^2) \ d\mu. \quad (5.3)
\]

In particular, a holomorphic sectional curvature has a view:

\[
K(a),Ja = -\int_M \text{tr}(A^4)d\mu. \quad (5.4)
\]

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3) Geodesics, going out from a point \( g \in \mathcal{AM} \) in direction \( a \in T_g \mathcal{AM} \) are given by the following way: \( g_t = g e^{tA} \).

Fix a positive associated a.c.s. \( J_0 \) and corresponding associated metric \( g_0 \). Let \( \mathcal{P}_{J_0} \) is domain in the space \( \text{End}_{S_{J_0}}(TM) \), consisting of endomorphisms \( P : TM \to TM \), for which the operator \( 1 - P^2 \) is positively defined with respect to \( g_0 \).

Consider a global parametrization of the space \( \mathcal{AM} \) of associated metrics:

\[
\Psi_{\mathcal{AM}} : \mathcal{P}_{J_0} \longrightarrow \mathcal{AM}, \quad P \to g = g_0(1 + P)(1 - P)^{-1},
\]

\[
g(X,Y) = g_0(X,(1 + P)(1 - P)^{-1}Y).
\]

Find expression of metric and curvature of the space \( \mathcal{AM} \) in these coordinates. The differential of mapping \( \Psi_{\mathcal{AM}} \) acts as follows:

\[
d \Psi_{\mathcal{AM}} : \text{End}_{S_{J_0}}(TM) \longrightarrow T_g \mathcal{AM},
\]

\[
d \Psi_{\mathcal{AM}}(A) = h_A = 2g(1 - P)(1 - P^2)^{-1}A(1 - P)^{-1}.
\]

The inverse mapping:

\[
d \Psi_{\mathcal{AM}}^{-1}(h) = A = \frac{1}{2}(1 - P)^{-1}(1 - P^2)g^{-1}h(1 - P).
\]

It follows from these formulas and the theorem 5.2

**Theorem 5.3.** The space \( \mathcal{AM} \) has the following geometric characteristics.

1) An inner product is given by the formula:

\[
(A, B)_P = 4 \int_M \text{tr} \left( (1 - P^2)^{-1}A(1 - P^2)^{-1}B \right) d\mu,
\]

where \( A, B \in T_P \mathcal{P}_{J_0} = \text{End}_{S_{J_0}}(TM) \).

2) A covariant derivative of vector fields given by (constant) operators \( A \) and \( B \):

\[
\nabla_A B = AP(1 - P^2)^{-1}B + B(1 - P^2)^{-1}A.
\]

3) A curvature tensor has a view:

\[
R(A, B)C = -(1 - P^2) \left[ [(1 - P^2)^{-1}A, (1 - P^2)^{-1}B], (1 - P^2)^{-1}C \right],
\]

where \( A, B, C \in T_P \mathcal{P}_{J_0} = \text{End}_{S_{J_0}}(TM) \).

4) A sectional curvature \( K_\sigma \) in a plane section \( \sigma \), given by orthonormal pair \( A, B \) is found by the formula:

\[
K_\sigma = \int_M \text{tr} \left( [(1 - P^2)^{-1}A, (1 - P^2)^{-1}A]^2 \right) d\mu.
\]

In particular, a holomorphic sectional curvature has a view:

\[
K(A, AJ_0) = -\int_M \text{tr} \left( (1 - P^2)^{-1}A \right)^4 d\mu.
\]
5) Geodesics, going out from a point $g_0$ in directions $A \in \text{End}_{S,j_0}(TM)$ are curves $P(t)$ on domain $\mathcal{P}_{j_0}$ of a view:

$$P(t) = \tanh(tA) = \left(e^{tA} + e^{-tA}\right)^{-1} \left(e^{tA} - e^{-tA}\right).$$

**Proof.**

1) The inner product.

$$(A, B)_g = (h_A, h_B)_g = \int_M \text{tr} \left(g^{-1}h_Ag^{-1}h_B\right) d\mu =$$

$$= 4 \int_M \text{tr} \left(\left(1 - P\right)(1 - P^2)^{-1}A(1 - P^2)^{-1}B(1 - P)^{-1}\right) d\mu =$$

$$= 4 \int_M \text{tr} \left(\left(1 - P^2\right)^{-1}A(1 - P^2)^{-1}B\right) d\mu.$$

2) The covariant derivative is immediately calculated under the six-term formula. In view of a constancy of operators $A, B, C$ it is:

$$(\nabla_A B, C)_g = \frac{1}{2} \left(A(B, C)_g + B(A, C)_g - C(A, B)_g\right).$$

In calculations we use a linearity of an integral and trace, and also the formula

$$\left((1 - P^2)^{-1}\right)' = (1 - P^2)^{-1}(AP + PA)(1 - P^2)^{-1},$$

where $P_t = P + tA$ is variation of an operator $P$ in direction $A$.

3) The curvature tensor. Let $h_A, h_B, h_C \in T_g \mathcal{A} \mathcal{M}$, then

$$R(h_A, h_B)h_C = -\frac{1}{4} g \left[[H_A, H_B], H_C\right],$$

where $H_A = g^{-1}h_A = 2(1 - P)(1 - P^2)^{-1}A(1 - P)^{-1}$. Substituting this expression, we obtain:

$$R(h_A, h_B)h_C = -2g(1 - P) \left[[\left(1 - P^2\right)^{-1}A, 1 - P^2)^{-1}B\right], (1 - P^2)^{-1}C\right) (1 - P)^{-1}.$$ Therefore

$$R(A, B)C = d \Psi^{-1}_{AM}(A) R(h_A, h_B)h_C = -(1 - P^2) \left[[\left(1 - P^2\right)^{-1}A, (1 - P^2)^{-1}B\right], (1 - P^2)^{-1}C\right].$$

4) The sectional curvature is calculated similarly via using of expression $d \Psi_{AM}(A) h_A = 2g(1 - P)(1 - P^2)^{-1}A(1 - P)^{-1}$.

5) Let $A \in \text{End}_{S,j_0}(TM)$ and $h = d \Psi_{AM}(A) = 2g_0A$. Then geodesic going out from a point $g_0$ in direction $h \in \mathcal{A} \mathcal{M}_{g_0}$ is a curve $g_t = g_0e^{tA}$. Let $P(t)$ is corresponding curve on the domain $\mathcal{P}_{j_0}$. Then $g_t = g_0e^{tA} = g_0(1 + P_t)(1 - P_t)^{-1}$. From the last formula $P(t)$ we express:

$$P(t) = \left(e^{tA} + e^{-tA}\right)^{-1} \left(e^{tA} - e^{-tA}\right) = \tanh(tA).$$
§6. Orthogonal decompositions of the space of symmetric tensors on an almost Kähler manifold.

We consider a manifold $M$ with the closed nondegenerate 2-form $\omega$ of class $C^\infty$, $\dim M = m = 2n$.

Let $\mathcal{M}$ is the space of all Riemannian metrics on $M$. The group $\mathcal{D}$ of smooth diffeomorphisms of manifold $M$ acts naturally on the space $\mathcal{M}$:

$$A : \mathcal{M} \times \mathcal{D} \longrightarrow \mathcal{M}, \quad A(g, \eta) = \eta^* g,$$

where the metric $\eta^* g$ is defined by equality:

$$\eta^* g(x)(X, Y) = g(\eta(x)) (d\eta(X), d\eta(Y)),$$

for any vector fields $X, Y$ on $M$ and any $x \in M$. It is known \[17\], that for any metric $g \in \mathcal{M}$, orbit $g \mathcal{D}$ of action $A$ is a smooth closed submanifold. The tangent space $T_g(g \mathcal{D})$ to an orbit consists of symmetric 2-forms of the view $h = L_X g = \nabla_i X_j + \nabla_j X_i$ and, therefore, coincides with an image of a differential operator

$$\alpha_g : \Gamma(TM) \longrightarrow S_2, \quad \alpha_g(X) = L_X g,$$

where $L_X g$ is Lie derivative along a vector field $X$ on $M$,

$$T_g(g \mathcal{D}) = \alpha_g(\Gamma(TM)).$$

Adjoint operator for $\alpha_g$ is the covariant divergence, $\alpha^*_g = 2\delta_g$, where

$$\delta_g : S_2 \longrightarrow \Gamma(TM), \quad (\delta_g a)^i = -\nabla_j a^{ij}.$$

The following orthogonal Berger-Ebin decomposition \[3\] of the space $S_2 = T_g \mathcal{M}$ hold:

$$S_2 = S_2^0 \oplus \alpha_g(\Gamma(TM)), \quad (6.1)$$

where $S_2^0 = \ker \delta_g = \{ a \in S_2 ; \delta_g a = 0 \}$ is the space of divergence-free symmetric 2-forms. Accordingly, each 2-form $a \in S_2$ is represented in an unique way as:

$$a = a^0 + L_X g,$$

where $\delta_g a^0 = 0$. The components $a^0$ and $L_X g$ are orthogonal and defined by an unique way.

Let $\mathcal{A} \mathcal{M}$ is the space of associated metrics on a symplectic manifold $(M, \omega)$. Under the action of all group $\mathcal{D}$ of smooth diffeomorphisms, the space $\mathcal{A} \mathcal{M}$ is not invariant. It is easy to see, that the group $\mathcal{D}_\omega$ of smooth symplectic diffeomorphisms of a manifold $M$, i.e. such diffeomorphisms $\eta : M \rightarrow M$, that save the symplectic form $\eta^* \omega = \omega$, acts on the space $\mathcal{A} \mathcal{M}$.
The group $\mathcal{D}_\omega$ acts also on the space $\mathcal{A}_\omega$ of positive associated almost complex structures: if $\eta \in \mathcal{D}_\omega$ and $J \in \mathcal{A}_\omega$, then $\eta^* J = (d\eta)^{-1} \circ (J \circ \eta) \circ d\eta$. The equivariance of a diffeomorphism $G : \mathcal{A}_\omega \rightarrow \mathcal{AM}$ (see §3) is easily checked.

In this paragraph we shall state orthogonal decompositions of the spaces $S_2$ and $S_{2A}$, with respect to action of group $\mathcal{D}_\omega$.

Recall, that the space $S_{2A}$ of anti-Hermitian symmetric 2-forms is tangent to a manifold $\mathcal{AM}$, on $M$,

$$T_g \mathcal{AM} = S_{2A} = \{ h \in S_2; \; h(JX, JY) = -h(X, Y), \; \forall X, Y \in \Gamma(TM) \},$$

where $J$ is an almost complex structure, corresponding to the metric $g$.

As all considered spaces are ILH-manifolds, then decompositions have to be obtained for finite class of smoothness too.

Let $\mathcal{M}^s$ is the space of all metrics on $M$ of Sobolev class $H^s$, $s \geq \frac{1}{2} \dim M + 2$ and $\mathcal{AM}^s$ is the space of associated metrics of class $H^s$. For $g \in \mathcal{M}^s$ the tangent space $T_g \mathcal{M}^s$ to the manifold $\mathcal{M}^s$ is the space $S_2^s$ of symmetric 2-forms of class $H^s$ on $M$, and the tangent space $T_g \mathcal{AM}^s$ coincides with the space $S_{2A}^s$ of anti-Hermitian symmetric 2-forms on $M$.

Let $\mathcal{D}_\omega$ and $\mathcal{D}_\omega^*$ are groups of smooth and, of class $H^s$ symplectic diffeomorphisms of manifold $M$ respectively. The group $\mathcal{D}_\omega$ is ILH-Lie group with a Lie algebra $T_e \mathcal{D}_\omega$, consisting of smooth locally Hamilton vector fields $X$ on $M$. For each $s$ the group $\mathcal{D}_\omega^s$ is a continuous group and smooth Hilbert manifold. The tangent space at unit $T_e \mathcal{D}_\omega^s$ consists of locally Hamilton vector fields $X$ on $M$ of class $H^s$.

At first we consider an ILH-Lie group $\mathcal{G}$ of exact symplectic diffeomorphisms. Its Lie algebra is the algebra $\mathcal{H}$ of Hamilton vector fields $X_F$ on $M$ [12]. The arbitrary Hamilton vector field $X_F$ can be presented as $X_F = J \text{grad} F$, where $F$ is some function on $M$, called a Hamiltonian of the field $X_F$ and $J$ is almost complex structure corresponding to the metric $g$. Therefore orthogonal complement $\mathcal{H}^\perp$ to $\mathcal{H}$ in the space $\Gamma(TM)$ of all vector fields consists of vector fields $V$ on $M$, satisfying to the condition $\text{div} J V = 0$.

Fix the Riemannian metric $g \in \mathcal{AM}^s$ and consider its orbit $g \mathcal{G}$. The tangent space $T_g (g \mathcal{G})$ consists of 2-forms of the view $h = L_{X_F} g$. In this connexion we consider a differential operator, acting on functions:

$$D_g : H^{s+2}(M, \mathbb{R}) \rightarrow S_2^s, \quad D_g(F) = L_{X_F} g.$$

Let $\text{Im} \; D_g$ is image of an operator $D$ and $\text{Ker} \; D_g^s$ is kernel of a adjoint operator $D_g^* : S_2^s \rightarrow H^{s-2}(M, \mathbb{R})$.

Recall, that the Riemannian metric $g$ defines an inner product (1.4) in the space $S_2^s$, with respect to which the adjoint operator is defined.

**Theorem 6.1.** The space $S_2^s$ is decomposed in a direct sum of orthogonal subspaces

$$S_2^s = \text{Ker} \; D_g^* \oplus \text{Im} \; D_g. \quad (6.2)$$

Accordingly with it each symmetric 2-form $h$ of class $H^s$ is represented as

$$h = h^* + L_{X_F} g, \quad (6.3)$$
where \( X = X_F \) is the Hamilton vector field of class \( H^{s+1} \), and \( h^* \) satisfies to a condition \( \text{div}(J\delta h^*) = 0 \).

**Proof.** For function \( F \in H^{s+1}(M, \mathbb{R}) \) we have \( J(\text{grad} F) = X_F \). Therefore operator \( D_g \) is a composition of three operators, \( D_g = \alpha_g \circ J \circ \text{grad} \) (recall, that \( \alpha_g(X) = L_X g \)). The operators \( \text{grad} \) and \( \alpha_g \) have injective symbols [3], consequently, \( D_g \) has an injective symbol too. From [3] (theorem 6.1), we obtain decomposition (6.2). Find a adjoint operator, \( D^*_g = (\alpha_g \circ J \circ \text{grad})^* = \text{grad}^* \circ J^* \circ (\alpha_g)^* = - \text{div} \circ (\text{grad}^*) \circ (2\delta_g) = 2 \text{div} \circ J \circ \delta_g \). Therefore \( h^* \in \text{Ker} \ D^*_g \) satisfies to the condition \( \text{div} \ (J\delta g h^*) = 0 \).

**Remark.** The space of 2-forms \( h^* \) of class \( H^s \), satisfying to the condition,

\[
\text{div}(J\delta h^*) = 0,
\]

we shall designate by the symbol \( S^{s*}_2 \). Sometimes we shall write the decomposition (6.2) as

\[
S^s_2 = S^{s*}_2 \oplus \alpha_g(\mathcal{H}^{s+1}),
\]

(6.2)

where \( \mathcal{H}^{s+1} \) is the space of Hamilton vector fields on \( M \) of class \( H^{s+1} \).

Consider the space \( \mathcal{A}\mathcal{M}^s \) of associated metrics. As the group \( \mathcal{G}^{s+1} \) acts on \( \mathcal{A}\mathcal{M}^s \), the tangent space \( T_g \mathcal{A}\mathcal{M}^s = S^s_{2A} \) contains the tangent space to an orbit \( T_g(g\mathcal{G}^{s+1}) \). As \( T_g(g\mathcal{G}^{s+1}) = \alpha_g(T_e\mathcal{G}^{s+1}) = \alpha_g(\mathcal{H}^{s+1}) \),

\[
\alpha_g(\mathcal{H}^{s+1}) \subset S^s_{2A}.
\]

Therefore we obtain decomposition of the space \( S^s_{2A} \) of symmetric anti-Hermitian 2-forms.

**Corollary 1.** The space \( S^s_{2A} \) is decomposed in an orthogonal direct sum

\[
S^s_{2A} = S^{s*}_{2A} \oplus \alpha_g(\mathcal{H}^{s+1}), \quad h = h^* + L_X g,
\]

where \( X = X_F \) is the Hamilton vector field of class \( H^{s+1} \), and \( h^* \) is anti-Hermitian and satisfies to the condition \( \text{div}(J\delta h^*) = 0 \).

The space \( S^{s*}_2 \) can be decomposed on orthogonal subspaces according to Berger-Ebin decomposition

\[
S^s_2 = S^{0s}_2 \oplus \alpha_g(\Gamma^{s+1}(TM)).
\]

As in our case \( \text{Im} \ D_g = \alpha_g(\mathcal{H}) \subset \Gamma(TM) \), the space \( S^{0s}_2 \) is in \( \text{Ker} \ D^*_g \). Therefore the space \( \text{Ker} \ D^*_g \) can be decomposed on orthogonal subspaces according to Berger-Ebin decomposition:

\[
\text{Ker} \ D^*_g = S^{0s}_2 \oplus B^s,
\]

where

\[
B^s = \text{Ker} \ D^*_g \cap \alpha_g(\Gamma^{s+1}(TM)) = \{ h \in S^{s*}_2; \ h = L_Y g, \ \text{div} J\delta g(h) = 0 \}.
\]

**Theorem 6.2.** The space \( S^s_2 \) is decomposed in a direct sum of orthogonal subspaces

\[
S^{s*}_2 = S^{0s}_2 \oplus B^s \oplus \alpha_g(\mathcal{H}^{s+1}).
\]

(6.4)
According to this, each symmetric 2-form $h$ of class $H^s$ is represented in an unique way as

$$h = h^0 + L_Y g + L_X g,$$

where $X = X_F$ is Hamilton vector field of class $H^{s+1}$, $h^0$ has the property $\delta_g(h^0) = 0$, and vector field $Y$ of class $H^s$ is such that $\text{div} J\delta_g Y = 0$.

**Remark.** For the vector field $\delta_g L_Y g$ there is a following expression from [19]:

$$\delta_g L_Y g = \Delta Y - \text{grad}(\text{div} Y) - 2\text{Ric}(g)Y,$$

where $\text{Ric}(g)$ is Ricci tensor and $(\Delta Y)_e = (dd^* + d^*d)(Y_e)$, $Y_e$ is 1-form obtained from a vector field $Y$ by omitting of an index.

State variant of the last decomposition, when instead of the space $\mathcal{H}$ of Hamilton vector fields on $M$ the space $T_e\mathcal{D}_\omega$ of locally Hamilton vector fields is taken. It is known, that $T_e\mathcal{D}_{s+1}^\omega / \mathcal{H}^{s+1}$ is isomorphic to the first cohomology group $H^1(M, \mathbb{R})$ of manifold $M$. One can think, that $H^1(M, \mathbb{R})$ is represented by vector fields of class $C^\infty$ and $T_e\mathcal{D}_{s+1}^\omega = \mathcal{H}^{s+1} \oplus H^1(M, \mathbb{R})$ is direct sum.

The operator $\alpha_g : \Gamma^{s+1}(TM) \longrightarrow S_2^s, \alpha_g(X) = L_X g$, maps the finite-dimensional space $H^1(M, \mathbb{R})$ on the closed finite-dimensional subspace $\alpha_g(H^1(M, \mathbb{R})) \subset S_2^s$. We form a direct sum

$$\text{Im } D_g \oplus \alpha_g(H^1(M, \mathbb{R})) = \alpha_g(T_e\mathcal{D}_{s+1}^\omega).$$

Consider the orthogonal complement

$$S_2^{s\omega} = \alpha_g(T_e\mathcal{D}_{s+1}^\omega)_{\perp}$$

in the space $S_2^s$, then

$$S_2^s = S_2^{s\omega} \oplus \alpha_g(T_e\mathcal{D}_{s+1}^\omega).$$

Let $(J\delta_g h)_e$ is 1-form obtained from a vector field $J\delta_g h$ by omitting of an index via the metric tensor $g$. Then the belonging of $h$ to the space $\text{Ker } D_g$ is expressed by condition of co-closed: $d^*(J\delta_g h)_e = 0$, and the space $S_2^{s\omega}$ consists of such $h$, that 1-form $(J\delta_g h)_e$ is $d^*$-exact, where $d^*$ is codifferential.

**Theorem 6.3.** The space $S_2^s$ is decomposed in a direct sum of orthogonal subspaces

$$S_2^s = S_2^{s\omega} \oplus \alpha_g(T_e\mathcal{D}_{s+1}^\omega).$$

(6.5)

Each form $h \in S_2^s$ is represented in an unique way as $h = h^\omega + L_X g$, where $X$ is locally Hamilton vector field of class $H^s$, and $h^\omega$ is such that the 1-form $(J\delta_g h)_e$ is $d^*$-exact.

Consider the space $\mathcal{AM}^s$ of associated metrics. As the group $\mathcal{D}_{s+1}^\omega$ acts on $\mathcal{AM}^s$, the tangent space $T_\gamma \mathcal{AM}^s = S_2^s$ contains the space $\alpha_g(T_e\mathcal{D}_{s+1}^\omega)$. It is tangent to an orbit $g\mathcal{D}_{s+1}^\omega$ of action of group $g\mathcal{D}_{s+1}^\omega$ on $\mathcal{AM}^s$.

From the theorem 6.3 is obtained

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\textbf{Corollary 2.} The space $S^s_{2A}$ is decomposed in an orthogonal direct sum

$$S^s_{2A} = S^0_{2A} \oplus \alpha_g(T_eD^s_{\omega}),$$

where $S^0_{2A} = S^0_{2A} \cap S^s_{2A}$ consists of anti-Hermitian symmetric 2-forms $h$ of class $H^s$ such, that the form $(J\delta g) h$ is $d^*$-exact.

As $S^0_{2A}$ contains the space $S^0_{2}$ of divergence-free forms, $S^0_{2A}$ is decomposed further in correspondence with Berger-Ebin decomposition:

$$S^0_{2A} = S^0_{2} \oplus \tilde{B}^s.$$

Thus,

$$\tilde{B}^s = S^0_{2} \cap \alpha_g(\Gamma^{s+1}(TM)) \quad \text{and} \quad B^s \cong \tilde{B}^s \oplus \alpha_g(H^1(M, \mathbb{R})).$$

\textbf{Theorem 6.4.} The space $S^s_{2A}$ is decomposed in the orthogonal direct sum

$$S^s_{2A} = S^0_{2A} \oplus \tilde{B}^s \oplus \alpha_g(T_eD^s_{\omega}).$$

According to this each tensor field $h \in S^s_{2A}$ is represented in an unique way as

$$h = h^0 + L_Y g + L_X g,$$

where $X$ is locally Hamilton vector field of class $H^s$, $h^0$ has the property $\delta g(h^0) = 0$, and the vector field $Y$ of class $H^s$ is such, that the 1-form $\gamma(Y) = (J(\Delta Y - \text{grad} \text{ div} Y - 2\text{Ric}(g))Y)_z$ is $d^*$-exact.

\textbf{Proof.} It is only required to prove the property of the field $Y$. Let $Y$ is such vector field, that $L_Y g \in B^s$. It follows from an orthogonality of the spaces $B^s$ and $\alpha_g(T_eD^s_{\omega})$, that for any $X \in T_eD^s_{\omega}$ we have $(L_Y g, L_X g)_g = 0$ (here $(..)_g$ is inner product in the space $S_2$). On the other hand, $(L_Y g, L_X g) = (L_Y g, \alpha_g(X)) = (\alpha_g^*(L_Y g), X) = 2(\delta g(L_Y g), X)$. Therefore vector field $\delta g(L_Y g)$ is orthogonal to $T_eD^s_{\omega}$. It is equivalent to $d^*$-exactness of the 1-form $(J\delta g L_Y g)_z$. Now we use the formula from [19]:

$$\delta g L_Y g = \Delta Y - \text{grad}(\text{div} Y) - 2\text{Ric}(g) Y.$$

\textbf{Remark.} In decomposition (6.8) vector fields $X$ and $Y$ are defined on $h$ up to a Killing vector field.

Consider decomposition (6.8) of anti-Hermitian 2-forms.

\textbf{Lemma 6.1.} If $h \in S^s_{2A}$, then in decomposition (6.8) the second component $L_Y g$ is unique defined on $h^0$.

\textbf{Proof.} We can think, that $h = h^0 + L_Y g \in S^0_{2A}$. Assume, that there are two forms $h_1, h_2 \in S^s_{2A}$, such, that $h_1 \neq h_2$ and $h_1 = h^0 + L_Y g$, $h_2 = h^0 + L_Y g$. Then $L_{Y_1 - Y_2} g = h_1 - h_2 \in S^s_{2A} \subset T_g \mathcal{A} \mathcal{M}^s = S^s_{2A}$. On the other hand, $L_{Y_1 - Y_2} g \in \alpha_g(T_eD^s_{\omega})$. It follows from the
M. Gromov’s maximality theorem. Indeed, let $\mathcal{D}(\mathcal{AM}^s)$ is group of all $H^s$ - diffeomorphisms of a manifold $M$, saving the space $\mathcal{AM}^s$. This is the closed subgroup in the group $\mathcal{D}^{s+1}$ of diffeomorphisms saving the volume form $\mu = \frac{1}{n!} \omega^n$ and $\mathcal{D}^{s+1}_\omega \subset \mathcal{D}(\mathcal{AM}^s) \subset \mathcal{D}^{s+1}_\omega$. Under the Gromov’s theorem the group $\mathcal{D}(\mathcal{AM}^s)$ coincides either with $\mathcal{D}^{s+1}_\omega$, or with $\mathcal{D}^{s+1}$. If we shall assume, that $\mathcal{D}(\mathcal{AM}^s) = \mathcal{D}^{s+1}_\omega$, for any $Y \in T_e \mathcal{AM}^s$, i.e. for any divergence-free field $Y$, 2-form $L_Y g$ is anti-Hermitian. It is obviously impossible. Therefore $\mathcal{D}(\mathcal{AM}^s) = \mathcal{D}^{s+1}_\omega$. Therefore, it follows from $L_{Y_1 - Y_2} g \in T_g \mathcal{AM}^s$ that $L_{Y_1 - Y_2} g \in \alpha_g(T_e \mathcal{D}^{s+1}_\omega)$. The simultaneous inclusion $L_{Y_1 - Y_2} g \in S^{2s}_{2A}$ is possible only at $L_{Y_1 - Y_2} = 0$, i.e. at $L_{Y_1} = L_{Y_2}$.

In decomposition $S^{2s}_{2A} = S^{2s}_{2A} \oplus \alpha_g(T_e \mathcal{D}^{s+1}_\omega)$ of the space $S^{2s}_{2A}$ of anti-Hermitian forms the first component $S^{2s}_{2A}$ can be decomposed further in correspondence with Berger-Ebin decomposition:

$$S^{2s}_{2A} = A S^{0s} \oplus C^s, \quad h = h^0 + L_Y g. \quad (6.9)$$

Thus $C^s \subset \widetilde{B}^s$ and the second component $L_Y g$ is unique defined by $h_0$.

**Corollary 3.** In the decomposition (6.9) the space $S^{2s}_{2A}$ is isomorphic to the closed subspace $A S^{0s}$ in the space $S^{2s}_2 = \text{Ker} \delta_g$. The isomorphism is stated by projection $p(h) = p(h^0 + L_Y g) = h^0$.

Describe the space $A S^{0s}$, i.e. describe those divergence-free elements $h^0 \in S^{0s}_2$, which are components of decomposition of anti-Hermitian forms $h$.

Let $S^{0s}_{2H}$ is the space of Hermitian symmetric 2-forms of class $H^s$ on $M$. Natural orthogonal decomposition takes place

$$S^s_2 = S^{s}_{2A} \oplus S^{s}_{2H},$$

$$h(X, Y) = \frac{1}{2} (h(X, Y) - h(JX, JY)) + \frac{1}{2} ((h(X, Y) + h(JX, JY))).$$

**Theorem 6.5.** The element $h^0 \in S^{0s}_2$ is component of decomposition (6.8) of element $h \in S^{s}_{2A}$ iff $h^0$ is orthogonal in $S^{0s}_2$ to the subspace $S^{0s}_{2H} = S^{0s}_2 \cap S^{s}_{2H}$ of the Hermitian divergence-free forms.

**Proof.** Let $h = h^0 + L_Y g$ is anti-Hermitian form. As $h \perp S^{0s}_{2H}$ and $L_Y g \perp S^{0s}_2$, $h^0 = h - L_Y g$ is orthogonal to intersection $S^{0s}_{2H} = S^{s}_{2H} \cap S^{0s}_2$. Conversely, suppose, that $h^0 \perp S^{0s}_{2H} = S^{s}_{2H} \cap S^{0s}_2$. Then $h^0 \in (S^{0s}_{2H})^\perp = (S^{s}_{2H} \cap S^{0s}_2)^\perp = (S^{s}_{2H})^\perp \cup (S^{0s}_2)^\perp = S^{s}_{2A} \cup \alpha_g(\Gamma^{s+1}(TM))$. Therefore $h^0$ is a linear combination of an element $L_V g, V \in \Gamma^{s+1}(TM)$ and anti-Hermitian form $h$, $h^0 = h - L_Y g$. We have obtained, that $h = h^0 + L_Y g$ is the anti-Hermitian form, at which $h^0$ is a divergence-free component.

**Corollary 4.**

$$A S^{0s}_2 = (S^{0s}_{2H})^\perp,$$

where the orthogonal complement is taken in the space $S^{0s}_2$ of divergence-free forms.
Corollary 5. Nonzero Hermitian form $h^0$ can not be a divergence-free component of decomposition (6.7) of element $h \in S^{s}_{2A} = T_{g} \mathcal{A} \mathcal{M}^{s}$.

Decompose component $h^0$ on Hermitian and anti-Hermitian parts,

$$h^0 = (h^0)_H + (h^0)_A.$$  

We have got, that the component $(h^0)_A$ of the nonzero form $h^0$ is not equal to zero too. At the same Hermitian part $(h^0)_H$, the second component $(h^0)_A$ is defined up to the anti-Hermitian divergence-free form $h^0_A \in S^{0s}_{2A} = S^s_2 \cap S^{0s}_2$. Therefore if $(h^0)_H \neq 0$, $\delta_g((h^0)_H) \neq 0$, and so $\delta_g((h^0)_A) \neq 0$.

Consider Hermitian part $(h^0)_H$ of a divergence-free component $h^0$ of the forms $h \in S^{s}_{2A}$.

Define mapping

$$J : \Gamma(T^0_2 M) \longrightarrow \Gamma(T^0_2 M),$$

which takes the 2-form $a(X,Y)$ on $M$ in the 2-form $J(a)(X,Y) = a(X,JY)$. The mapping $J$ commutes with an operator of taking of Hermitian part $a_H$ of the form $a$:

$$J(a_H) = (Ja)_H.$$  

Note an obvious isomorphism

$$J : S^s_{2H} \oplus S^s_{2A} \longrightarrow H^s(\Lambda^2_H(M)) \oplus S^s_{2A},$$

where $\Lambda^2_H(M)$ is bundle of Hermitian skew-symmetric 2-forms on $M$. If $\varphi \in \Lambda^2_H(M)$, it is easy to see, that $J\varphi(X,Y) = \varphi(X,JY)$ is Hermitian symmetric 2-form.

**Theorem 6.6.** The space $S^s_{2A}$ is isomorphic to a direct sum of orthogonal subspaces

$$S^s_{2A} \cong A S^0_{2} \oplus \alpha_g(T \mathcal{D}^{s+1}_\omega). \quad (6.10)$$

Each tensor field $h \in S^s_{2A}$ is unique represented as

$$h = h^0 + L_Y g + L_X g, \quad (6.11)$$

where $X$ is locally Hamilton vector field of class $H^s$, the component $L_Y g$ is unique defined on $h_0$, a vector field $Y$ and 2-form $h^0$ have properties:

1) $\delta_g h^0 = 0$,
2) $h^0_H = -J((L_Y \omega)_H)$,
3) $\gamma(Y) = (J(\Delta Y - \text{grad div} Y - 2Ric(g)Y)) \ast$ is $d^\ast$-exact 1-form.

**Proof.** From the lemma 1 is obtained, that $L_Y g$ is unique defined by $h^0$. The properties 1 and 3 follow from the theorem 6.4. Let $Y$ is vector field from decomposition (6.11) and $\overline{h} = L_Y g$. Applying Lie derivative $L_Y$ to the left and right parts of equality

$$g(JV,W) - g(V,JW) = \omega(V,W) + \omega(JV,JW)$$

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and taking into account Leibniz rule, we obtain

\[ h(JV, W) - h(V, JW) = L_Y \omega(V, W) + L_Y \omega(JV, JW) = 2(L_Y \omega)_H(V, W). \tag{6.12} \]

Substituting instead of \( W \) a vector field \( JW \), we have

\[ 2(h)_H = h(JV, JW) + h(V, W) = L_Y \omega(V, JW) - L_Y \omega(JV, W) = 2J(L_Y \omega)_H(V, W). \]

Thus, \((h)_H = J(L_Y \omega)_H\). As the form \( h = h^0 + L_Y g + L_X g = h^0 + \overline{h} + L_X g \) belongs to the space \( S^0_{2A} = T^*_g \mathcal{AM}^s \) and \( L_X g \in T^*_g \mathcal{AM}^s \), \((h)_H = 0 \) and \((L_X g)_H = 0 \). Therefore \((h^0 + \overline{h})_H = 0 \), so \((h^0)_H = -h)_H. \] Finally we have \((h^0)_H = -J(L_Y \omega)_H\).

**Remark.** We have obtained, that for any vector field \( Y \) on \( M \) the equalities take place:

\[ L_Y \omega = -J(L_Y g) - gL_Y J, \quad J(L_Y \omega)_H = (L_Y g)_H. \]

**Proposition 6.7.** The space \( AS^0_{2s} \) consists of symmetric 2-forms \( h^0 \in S^0_{2s} \), having a property,

\[ (h^0)_H = J(d\beta)_H, \]

where \( \beta \) is 1-form on \( M \).

**Proof.** As \( L_Y \omega \) is exact form, in one direction the statement follows from the theorem 6.6. Conversely, let \( h^0 \) is such that \( \delta_g h^0 = 0 \) and \((h^0)_H = J(d\beta)_H\) for some exact 2-form \( d\beta \). Find a vector field \( Y \), possessing properties 2,3 of theorems 6.6. For this purpose we shall find a vector field \( Z \) from equality \( i_Z \omega = -\beta \). From here \( L_Z \omega = -d\beta \). Decomposition (6.11) implies, that \( L_Z g \) is orthogonal \( S^0_{2s} \), therefore under the theorem 6.4 \( L_Z g \) is decomposed as \( L_Z g = L_Y g + L_X g \), where \( Y \) has the necessary property 3. As \( X \in T_e \mathcal{D}_{2s}^s \), \( L_X \omega = 0 \). As a result \( L_Y \omega = L_Z \omega = -d\beta \) and \((h^0)_H = J(L_Y \omega)_H\).

**Remark.** One can point decomposition of the space \( S_2 \), dual to the decomposition (6.2) in some sense. Namely, any symmetric form \( h \in S_2 \) is decomposed in a sum of orthogonal addends

\[ h = h^1 + L_X g, \tag{6.13} \]

where \( X \) is exact divergence-free vector field on \( M \), and \( h^1 \) has the property, that a vector field \( J\delta_g h^1 \) is locally Hamilton. If one requires of a field \( X \) to be only divergence-free in (6.13), then vector field \( J\delta_g h^1 \) should be Hamilton.

Recall, that exact divergence-free is such vector field \( X \), for which the 1-form \( X_\sharp = g^{-1}X \) is \( d^\ast \) - exact.

The decomposition (6.13) turns out the same as (6.2), via using of a differential operator \( R : \Gamma(\Lambda^2\mathcal{T}M) \rightarrow S_2 \), which is defined by equality \( R(\varphi) = \alpha_g(d^\ast \varphi)^\sharp \), where the symbol \( ^\sharp \) designates operation of a raising of an index.
§7. A curvature of a quotient space \( \mathcal{AM}/\mathcal{D}_\omega \).

Let \( \mathcal{AM} \) is the space of all smooth associated metrics on a symplectic manifold \( M, \omega \). The group \( \mathcal{D}_\omega \) of symplectic diffeomorphisms of a manifold \( M \) acts natural on this space \( \mathcal{AM} \). Let \( \mathcal{D}_{\omega_0} \) is connected component of the unit of group \( \mathcal{D}_\omega \). Lie algebra of the group \( \mathcal{D}_\omega \) consists of locally Hamilton vector fields on \( M \).

Suppose that the symplectic structure \( \omega \) allows integrable associated almost complex structures. In this case commutator \( \mathcal{G} = [\mathcal{D}_{\omega_0}, \mathcal{D}_{\omega_0}] \) is \([50], [3]\) the connected closed ILH-Lie subgroup of the group \( \mathcal{D}_{\omega_0} \) with a Lie algebra \( \mathcal{H} \), consisting of Hamilton vector fields on \( M \). At that \( \mathcal{D}_{\omega_0}/\mathcal{G} \cong H^1(M, \mathbb{R})/\Gamma \), where \( \Gamma \) is some discrete subgroup of the first group of cohomologies \( H^1(M, \mathbb{R}) \).

Consider the matter of a curvature of the space \( \mathcal{AM}/\mathcal{G} \) of classes of equivalent associated metrics. The quotient space \( \mathcal{AM}/\mathcal{G} \) is not a manifold, in general, as it has singularities corresponding to such of metrics \( g \), that there isometry group \( I(g) \) has nontrivial intersection with the group \( \mathcal{G} \). Calculate a curvature \( \mathcal{AM}/\mathcal{G} \) in the regular points \([g] = g\mathcal{G} \), i.e. in such that \( I(g) \cap \mathcal{G} = e \). The set of such classes \([g] \) is open in \( \mathcal{AM}/\mathcal{G} \). Indeed, it contains metrics with discrete group \( I(g) \). The set of metrics with discrete (and even with trivial) isometry group is open and is everywhere dense in the space of metrics \( \mathcal{AM} \) \([17]\).

The weak Riemannian structure on \( \mathcal{AM} \) is invariant \([3]\) with respect to action of the group \( \mathcal{D}_\omega \). Therefore on a regular part of the space \( \mathcal{AM}/\mathcal{G} \) a weak Riemannian structure such, that the projection \( p : \mathcal{AM} \rightarrow \mathcal{AM}/\mathcal{G} \) is a Riemannian submersion, is natural defined \([43]\). The vertical subbundle \( V(\mathcal{AM}) \) consists of subspaces, which are tangents to orbits \( g\mathcal{G} \). Recall, that \( V_g = T_g(g\mathcal{G}) \) consists of 2-forms of the view \( a = L_X g \), where \( L_X \) is the Lie derivate along a Hamilton vector field \( X \).

It is shown in §6, that the horizontal subspace \( H_g = V_g^\perp \) consists of the anti-Hermitian symmetric 2-forms \( a \) such, that \( \text{div} J_\delta g a = 0 \), where \( \delta_g \) is covariant divergence: \( (\delta_g a)^i = -\nabla_k a^{kj} \). Recall that for an operator \( \delta_g : S_2 \rightarrow \Gamma(TM) \) the adjoint operator \( \delta^*_g : \Gamma(TM) \rightarrow S_2 \) is expressed via a Lie derivate: \( \delta^*_g = \frac{1}{2} \alpha_g(X) = \frac{1}{2} L_X g \).

Let \( \overline{\pi}, \overline{b} \in T_g(\mathcal{AM}/\mathcal{G}) \). As \( p : \mathcal{AM} \rightarrow \mathcal{AM}/\mathcal{G} \) is Riemannian submersion, the sectional curvature \( \overline{K}(\overline{\pi}, \overline{b}) \) of the space \( \mathcal{AM}/\mathcal{G} \) is under the formula \([13], [7]\):

\[
\overline{K}(\overline{\pi}, \overline{b}) = K(a, b) + \frac{3}{4} \frac{[[a, b]^V, [a, b]^V]_g}{\|a \wedge b\|^2},
\]

(7.1)

where \( a, b \in T_g \mathcal{AM} \) are horizontal lifts of vectors \( \pi, b \), \( K(a, b) \) is sectional curvature of the space \( \mathcal{AM} \), \( [a, b]^V \) is vertical part of a Lie commutator of horizontal vector fields on \( \mathcal{AM} \), which continue \( a, b \in T_g \mathcal{AM} \).

For an evaluation \( [a, b]^V \) we will need two differential operators:

1) Elliptic operator of the 4-th order:

\[
E_g : C^\infty(M, \mathbb{R}) \rightarrow C^\infty(M, \mathbb{R}), \quad E_g(f) = \text{div} J_\delta g \alpha_g J_\text{grad} f,
\]

(7.2)

It is obvious, that its kernel consists of a constant functions;
2) Differential operator of the 1-st order \( \Box \):

\[
\Box : S_2 \rightarrow \Gamma(S_2 M \otimes TM), \quad (\Box a)^k_{ij} = a^k_{ij} = \frac{1}{2}(\nabla_i a^k_j + \nabla_j a^k_i - \nabla^k a_{ij}), \quad a \in S_2.
\]

The geometrical sense of an operator \( \Box \) is, that it is a differential of mapping \( \Gamma : M \rightarrow RConn \), which takes a Riemannian metric \( g \in M \) to Riemannian connection without a torsion \( \Gamma(g) \). If \( g_t \) is curve on the space of metrics with tangent vector \( a = \frac{d}{dt}|_{t=0} g_t \) and \( \Gamma^k_{ij}(g_t) \) are Christoffel symbols of a Riemannian connection of the metric \( g_t \),

\[
(\Box a)^k_{ij} = \frac{d}{dt}|_{t=0} \Gamma^k_{ij}(g_t).
\]

Define a contraction \( \Box a \) and \( \delta g a \) with the symmetric 2-form \( b \):

\[
(\Box a, b)^k = a^k_{ij} b^{ij}, \quad (\delta g a, b)^k = b^k_i (\delta g a)^i.
\]  

(7.3)

Introduce the notation:

\[
\{a, b\} = (\delta g b, a) + (\Box b, a) - (\delta g a, b) - (\Box a, b).
\]  

(7.4)

**Theorem 7.1.** If \( a, b \) are horizontal vector fields on the space \( \mathcal{AM} \), then

\[
[a, b]^V = -\alpha_g J \text{grad } E_g^{-1} \text{div} (\{a, b\}).
\]  

(7.5)

**Theorem 7.2.** Let \( [g] \) is regular point of the quotient space \( \mathcal{AM}/G \) and \( g \in \mathcal{AM} \) is any metric from the class \( [g] \). The sectional curvature \( K(a, b) \) of the space \( \mathcal{AM}/G \) in a plane section, defined by pair of vectors \( \bar{a}, \bar{b} \in T_{[g]}(\mathcal{AM}/G) \), is expressed by the formula

\[
\overline{K}(\bar{a}, \bar{b}) = K(a, b) + \frac{3}{4 ||a \wedge b||^2} \int_M (\text{div} J\{a, b\}) \left( E_g^{-1}(\text{div} J\{a, b\}) \right) d\mu_g.
\]  

(7.6)

where \( a, b \in T_g \mathcal{AM} \) are horizontal lifts of vectors \( \bar{a}, \bar{b} \) and \( K(a, b) \) is sectional curvature of the space \( \mathcal{AM} \).

**Proof of the theorem 7.1.** The vertical subspace \( V_g \) consists of 2-forms of a view \( h = L_X g \), where \( X \) is Hamiltonian vector field, \( X = J \text{grad } f \). Thus, \( h = \alpha_g J \text{grad } f \). For such forms \( h \) the equality \( \text{div} J \delta_g h = \text{div} J \delta_g \alpha_g J \text{grad } f = E_g(f) \), where \( E_g \) is operator defined earlier by the formula (7.2), is held. Therefore orthogonal projection of the tangent space \( T_g \mathcal{AM} \) on vertical \( V_g \) can be given as follows:

\[
h^V = \alpha_g J \text{grad } E_g^{-1} \text{div} J \delta_g(h).
\]  

(7.7)

Find a projection of Lie commutator \( [a, b] \) of horizontal vector fields \( a, b \) on \( \mathcal{AM} \). As \( \mathcal{AM} \subset M \subset S_2 \), \( [a, b]_g = db(a) - da(b) \), where \( db(a) \) is usual derivative of a field \( b \) in direction \( a \) in the vector space \( S_2 \).
At first we shall obtain $\text{div} J \delta_g (db(a))$ in a considered point $g$. Take a curve $g_t$ on $\mathcal{M}$, going out in direction $a$, $\frac{d}{dt} \big|_{t=0} g_t = a$. Vector field $b$ is horizontal, $\text{div}_t J_t \delta_{g_t} (b(g_t)) = 0$, therefore

$$\left. \frac{d}{dt} \right|_{t=0} \text{div}_t J_t \delta_{g_t} (b(g_t)) = 0. \tag{7.8}$$

Calculate derivatives of operators $\text{div}_t$, $J_t$, $\delta_{g_t}$,

$$\left. \frac{d}{dt} \right|_{t=0} \text{div}_t X = \left. \frac{d}{dt} \right|_{t=0} (\nabla_k (t) X^k) = \left. \frac{d}{dt} \right|_{t=0} \left( \frac{\partial X^k}{\partial x^k} + \Gamma^k_{ki} X^i \right) = a^k_{ki} X^i = 0,$$

where $a^k_{ij} = (\Box a)^k_{ij} = \frac{1}{2} \left( \nabla_i a^k_j + \nabla_j a^k_i - \nabla^k a_{ij} \right)$. Taking into account, that $\text{tr} g^{-1} a = a^k_k = 0$, we obtain, that $a^k_{ki} = \frac{1}{2} \left( \nabla_i a^k_j + \nabla_k a^k_i - \nabla^k a_{ik} \right) = 0$.

Let $A = g^{-1} a$. Then, it is known ($\S 4$), that

$$\left. \frac{d}{dt} \right|_{t=0} J_t = J A.$$

Further, taking into account that

$$\left. \frac{d}{dt} \right|_{t=0} g^k_{ti} = -a^k_{ti} \quad \text{and} \quad \left. \frac{d}{dt} \right|_{t=0} (\nabla_k b_{is}) = -a^q_{ki} b_{qs} - a^q_{ks} b_{iq},$$

we obtain:

$$\left. \frac{d}{dt} \right|_{t=0} \delta_{g_t} (b) = \left. \frac{d}{dt} \right|_{t=0} (-g^{ki} g^{si} \nabla_k b_{ls}) =

= a^l_{ki} g^{si} \nabla_k b_{ls} + g^{kl} a^s_{ki} \nabla_k b_{ls} + g^{kl} g^{si} a^q_{ks} b_{qs} + g^{kl} a^q_{ks} b_{ls} =

= a^l_{ki} \nabla_k b^i_l - (A(\delta g)b)^i_l - (B(\delta g)a)^i_l + g^{si} a^q_{ks} b^k_q.$$

Using these expressions of derivatives, equality (7.8) can be written as

$$\text{div} J \left( (A(\delta g)b)^i_l + a^l_{ki} \nabla_k b^i_l (A(\delta g)b)^i_l (B(\delta g)a)^i_l + g^{si} a^q_{ks} b^k_q \right) + \text{div} J \delta_g (db(a)) = 0.$$

From here we obtain, that

$$\text{div} J \delta_g (db(a)) = -\nabla_p \left( J^p_q (a^l_{ki} \nabla_k b^i_l) - b^i_l (\delta g a)^k + \frac{1}{2} \nabla^i (a_{ki}) b^{kl} \right) .$$

div$J\delta_g(da(b))$ is similarly calculated. Using operator $\Box$, we see, that

$$\text{div} J \delta_g [a, b]_g = -\text{div} J (\delta g (da(a) + (\Box b, a) - (\delta g a, b) - (\Box a, b)).$$

This equality and relation (7.7) imply the formula (7.5).

**Proof of the theorem 7.2** follows at once from the theorem 7.1 and formulas (7.1), (7.5). Indeed, using conjugacy of operators $\delta_g^* = \alpha_g$, $J^* = -J$, grad$^* = -\text{div}$, one can just see, that

$$\langle [a, b]_g, [a, b]_g \rangle_g = (\alpha_g J \text{ grad } E^{-1}_g \text{div} J \{a, b\}, \alpha_g J \text{ grad } E^{-1}_g \text{div} J \{a, b\}) =

= \int_M \left( \text{div} J \{a, b\} \right) \left( E^{-1}_g \left( \text{div} J \{a, b\} \right) \right) d\mu_g.$$
8. The Spaces of the associated metrics on sphere and torus.

8.1. Associated metrics on a sphere $S^2$.

In this section we shall find sectional curvatures of the space $\mathcal{AM}$ of associated metrics on a sphere $S^2$ with a natural symplectic structure. In particular, it will be shown, that the sectional curvature of the space $\mathcal{AM}$ is negative.

8.1.1. A view of associated metrics. Consider unit sphere $S^2$ in $\mathbb{R}^3$, with spherical coordinates

\[
\begin{align*}
x &= \cos \varphi \cos \theta \\
y &= \sin \varphi \cos \theta \\
z &= \sin \theta
\end{align*}
\]

$\varphi \in (0, 2\pi)$, $\theta \in (-\pi/2, \pi/2)$.

Then the canonical metric $g_0$ on $S^2$ and volume form look like:

\[
g_0 = \cos^2 \theta d\varphi^2 + d\theta^2, \quad \mu = \omega = \cos \theta d\varphi \wedge d\theta, \quad J_0 = \begin{pmatrix} 0 & -\cos^{-1} \theta \\ \cos \theta & 0 \end{pmatrix}.
\]

Let $z = \sin \theta$, then

\[
g_0 = (1 - z^2)d\varphi^2 + \frac{1}{1 - z^2}dz^2, \quad \mu = \omega = d\varphi \wedge dz, \quad J_0 = \begin{pmatrix} 0 & -(1 - z^2)^{-1} \\ (1 - z^2)^{-1} & 0 \end{pmatrix}.
\]

Choose an orthonormalized frame

\[
e_1 = \frac{1}{\sqrt{1 - z^2}} \frac{\partial}{\partial \varphi}, \quad e_2 = \sqrt{1 - z^2} \frac{\partial}{\partial z}
\]

and dual coframe

\[
e^*_1 = \sqrt{1 - z^2} \, d\varphi, \quad e^*_2 = \frac{1}{\sqrt{1 - z^2}} \, dz.
\]

Then $g_0 = (e^*_1)^2 + (e^*_2)^2, \quad J_0 = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}$.

In a corresponding complex frame

\[
\partial = \frac{1}{2}(e_1 - ie_2), \quad \overline{\partial} = \frac{1}{2}(e_1 + ie_2),
\]

\[
d\bar{w} = e^*_1 + ie^*_1, \quad d\bar{w} = e^*_1 - ie^*_1
\]

matrixes of tensors $g_0 = dw \, d\overline{w}$ and $J_0$ look like:

\[
g_0 = \frac{1}{2} \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \quad J_0 = \begin{pmatrix} i & 0 \\ 0 & -i \end{pmatrix}.
\]

The endomorphism $P$ in this case is defined by one complex-valued function $p(z, \varphi)$ on a coordinate map $U = (-1, 1) \times (0, 2\pi)$. We obtain the following values,

\[
P = \begin{pmatrix} 0 & \overline{p} \\ p & 0 \end{pmatrix}, \quad P^2 = p\overline{p} \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}, \quad 1 - P^2 = (1 - |p|^2) \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} > 0, \quad \text{if } |p| < 1,
\]

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\[ D = (1 - P^2)^{-1} = \frac{1}{1 - |p|^2} \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}, \quad (1 + P)^2 = \begin{pmatrix} 1 + \overline{p} \overline{\alpha} & 2\overline{p} \\ 2p & 1 + p\overline{\alpha} \end{pmatrix}. \]

Associated metric and a.c.s. \( J \), corresponding to an operator \( P \) look like:

\[ g(P) = D^{-1}(1 + P)g_0(1 + P) = \frac{1}{2(1 - p\overline{\alpha})} \begin{pmatrix} 2p & 1 + p\overline{\alpha} \\ 2p & 1 + p\overline{\alpha} \end{pmatrix}, \quad (8.1) \]

\[ J = J_0D(1 + P)^2 = \frac{i}{1 - p\overline{\alpha}} \begin{pmatrix} 1 + p\overline{\alpha} & 2\overline{p} \\ 2p & 1 + p\overline{\alpha} \end{pmatrix}. \quad (8.2) \]

\[ \partial(J) = \partial - p\overline{\partial}, \quad \overline{\partial}(J) = \overline{\partial} - \overline{\partial}\partial. \]

The anti-Hermitian symmetric 2-form \( a \) is also set by one complex-valued function:

\[ a = \begin{pmatrix} \alpha & 0 \\ 0 & \overline{\alpha} \end{pmatrix}, \quad A = g_0^{-1}a = 2 \begin{pmatrix} 0 & \overline{\alpha} \\ \alpha & 0 \end{pmatrix}. \]

Let

\[ \Psi_{AM} : \mathcal{P}_{J_0} \rightarrow \mathcal{AM}, \quad P \rightarrow g = g_0(1 + P)(1 - P)^{-1}, \]

\[ g(X,Y) = g_0(X, (1 + P)(1 - P)^{-1}Y). \]

is global parametrization of the space \( \mathcal{AM} \) of associated metrics and

\[ d \Psi_{AM} : \text{End}_{S,J_0}(TM) \rightarrow T_g\mathcal{AM}, \]

\[ d\Psi_{AM}(A) = h_A = 2g_0(1 - P)^{-1}A(1 - P)^{-1}. \]

is differential of mapping \( \Psi_{AM} \).

Then the element \( h \in T_g\mathcal{AM} \) has the following expression via function \( p \) of an operator \( P \) and function \( \alpha \) of an operator \( A \):

\[ A = \begin{pmatrix} 0 & \overline{\alpha} \\ \alpha & 0 \end{pmatrix} \mapsto h_A = 2g_0(1 - P)^{-1}A(1 - P)^{-1} = \frac{1}{(1 - |p|^2)^2} \begin{pmatrix} \alpha + p^2\overline{\alpha} & p\overline{\alpha} + \overline{p}\alpha \\ p\overline{\alpha} + \overline{p}\alpha & \overline{\alpha} + \overline{p}^2\alpha \end{pmatrix}. \quad (8.3) \]

Operator \( H_A = g^{-1}h_A \):

\[ H_A = \frac{2}{(1 - |p|^2)^2} \begin{pmatrix} p\overline{\alpha} - \overline{p}\alpha & \overline{\alpha} - \overline{p}^2\alpha \\ \alpha - p^2\overline{\alpha} & p\alpha - p\overline{\alpha} \end{pmatrix}. \quad (8.4) \]

Write out expressions of associated metric \( g(P) \), a.c.s. \( J \) and 2-form \( h = 2g_0P \) in real basis \( \frac{\partial}{\partial z}, \frac{\partial}{\partial \varphi} \). We shall write the function \( p(z, \varphi) \) as \( p = r e^{i\psi} = r(\cos \psi + i \sin \psi) \), where \( r = r(z, \varphi) \), \( \psi = \psi(z, \varphi) \) are functions on domain \( U \subset S^2 \).

\[ g(P) = \frac{1}{1 - r^2} \begin{pmatrix} (1 - z^2)(1 + r^2 + 2r \cos \psi) & -2r \sin \psi \\ -2r \sin \psi & \frac{1 + r^2 - 2r \cos \psi}{1 - z^2} \end{pmatrix}, \quad (8.5) \]

\[ J(P) = \frac{1}{1 - r^2} \begin{pmatrix} 2r \sin \psi & \frac{1 + r^2 - 2r \cos \psi}{1 - z^2} \\ \frac{1 + r^2 - 2r \cos \psi}{1 - z^2} & -2r \sin \psi \end{pmatrix}, \quad (8.6) \]
If \( \alpha(z, \varphi) = u(z, \varphi) + iv(z, \varphi) \),
\[
a = g_0 A = \begin{pmatrix} (1 - z^2)u & -v \\ -v & -\frac{u}{1 - z^2} \end{pmatrix},
\]
(8.7)

8.1.2. A curvature of the space \( \mathcal{AM}(S^2) \). The sectional curvature \( K(a, b) \) in the plane section, given by the elements \( a, b \in T_g \mathcal{AM}(S^2) \) find by the formula:
\[
K(a, b) = \frac{1}{4\|a \wedge b\|^2} \int_M \text{tr}([A, B]^2) d\mu,
\]
where \( M = S^2 \), \( A = g^{-1} a \), \( \mu = \omega = d\varphi \wedge dz \).

Find expression of a sectional curvature in a coordinate map \( \Psi_{\mathcal{AM}} \). Let \( g_0 \) is canonical metric, \( g = g_0(1 + P)(1 - P)^{-1} \) is any associated metric and \( J \) is complex structure, corresponding to it.

Take an arbitrary elements \( A, B \in \text{End}_{S, g_0}(TM) \). In a complex frame \( \partial_1, \bar{\partial}_1 \) they are set by matrixes of a view
\[
A = \begin{pmatrix} 0 & \alpha \\ \bar{\alpha} & 0 \end{pmatrix}, \quad B = \begin{pmatrix} 0 & \beta \\ \bar{\beta} & 0 \end{pmatrix},
\]
where \( \alpha, \beta \) are complex functions of variables \( z, \varphi \).

Tangent elements \( h_A, h_B \) of the space \( T_g \mathcal{AM}(S^2) \) correspond to the operators \( A \) and \( B \):
\[
h_A = 2g(1 - P)(1 - P^2)^{-1} A(1 - P)^{-1}, \quad h_B = 2g(1 - P)(1 - P^2)^{-1} B(1 - P)^{-1}
\]
and operators \( H_A = g^{-1}h_A, \ H_B = g^{-1}h_B \):
\[
H_A = 2(1 - P)(1 - P^2)^{-1} A(1 - P)^{-1}, \quad H_B = 2(1 - P)(1 - P^2)^{-1} B(1 - P)^{-1}.
\]

Operator bracket has a view:
\[
[H_A, H_B] = 4(1 - P)[(1 - P^2)^{-1} A, 1 - P^2)^{-1} B](1 - P)^{-1}.
\]

In a two-dimensional case we have, \( (1 - P^2)^{-1} = \frac{1}{1 - |p|^2} \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \) , therefore operators \( A \) and \( B \) commute with \( (1 - P^2)^{-1} \).

Begin to calculate a sectional curvature on the plane \( \sigma \), generated by the elements \( h_A, h_B \in T_g \mathcal{AM}(S^2) \).
\[
\|h_A\|^2 = (h_A, h_A)_g = 4 \int_M \text{tr}((1 - P^2)^{-1} A(1 - P^2)^{-1} A) d\mu = 4 \int_M \frac{\text{tr}(A^2)}{(1 - |p|^2)^2} d\mu.
\]
Therefore
\[
\|h_A \wedge h_B\|^2 = \|h_A\|^2 \|h_B\|^2 - (h_A, h_B)_g^2 =
\]
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Similarly,
\[\int_M \text{tr}([H_A, H_B]) d\mu = 16 \int_M \frac{\text{tr}(AB)}{(1-|p|^2)^2} d\mu.\]

Therefore, the formula for calculation of a sectional curvature becomes:
\[K_\sigma = \frac{1}{4} \frac{\int_M \text{tr}(AB)^2 d\mu}{\int_M \frac{\text{tr}(A)}{(1-|p|^2)^2} d\mu} \left( \int_M \frac{\text{tr}(B)}{(1-|p|^2)^2} d\mu - \left( \int_M \frac{\text{tr}(AB)}{(1-|p|^2)^2} d\mu \right)^2 \right). \quad (8.9)\]

As the operators \(A\) and \(B\) are set by functions \(\alpha\) and \(\beta\), then values \(\text{tr}(AB)^2\), \(\text{tr}(A^2)\), \(\text{tr}(B^2)\), \(\text{tr}(AB)\) in this formula can also be expressed through \(\alpha\) and \(\beta\).

\[A = \begin{pmatrix} 0 & \bar{\alpha} \\ \alpha & 0 \end{pmatrix}, \quad A^2 = \begin{pmatrix} \alpha\bar{\alpha} & 0 \\ 0 & \alpha \bar{\alpha} \end{pmatrix} = |\alpha|^2 E, \quad \text{tr}A^2 = 2|\alpha|^2.\]

\[AB = \begin{pmatrix} 0 & \bar{\alpha} \\ \alpha & 0 \end{pmatrix} \begin{pmatrix} 0 & \beta \\ \beta & 0 \end{pmatrix} = \begin{pmatrix} \alpha\beta & 0 \\ 0 & \alpha \bar{\beta} \end{pmatrix}, \quad \text{tr}AB = \bar{\alpha}\beta + \alpha \bar{\beta} = 2\text{Re}(\bar{\alpha}\beta).\]

\[[A, B] = AB - BA = \begin{pmatrix} \alpha\beta - \alpha\bar{\beta} \\ 0 \end{pmatrix} = 2i\text{Im}(\alpha\bar{\beta}) \begin{pmatrix} -1 & 0 \\ 0 & 1 \end{pmatrix}, \quad \text{tr}[A, B]^2 = -4\left(\text{Im}(\alpha\bar{\beta})\right)^2.\]

**Theorem 8.1.** Let \(A, B \in \text{End}_{\mathcal{S}, h_0}(TM)\) are operators given by functions \(\alpha\) and \(\beta\) and \(h_A, h_B\) are elements of the tangent space \(T_g\mathcal{AM}(S^2)\), corresponding to them. Then the sectional curvature \(K_\sigma\) of the space \(\mathcal{AM}(S^2)\) in the plane section \(\sigma\), generated by the elements \(h_A, h_B \in T_g\mathcal{AM}(S^2)\) is expressed by the formula
\[K_\sigma = -\frac{1}{2} \frac{\int_M \left(\text{Im}(\alpha\bar{\beta})\right)^2 d\mu}{\int_M \frac{|\alpha|^2}{(1-|p|^2)^2} d\mu} \left( \int_M \frac{|\beta|^2}{(1-|p|^2)^2} d\mu - \left( \int_M \frac{\text{Re}(\bar{\alpha}\beta)}{(1-|p|^2)^2} d\mu \right)^2 \right). \quad (8.10)\]

In particular:
1) If functions \(\alpha\) and \(\beta\) are simultaneously either real, or pure imaginary, then \(K_\sigma = 0\).
2) If one of functions \(\alpha\) and \(\beta\) is real, and other is pure imaginary, then
\[K_\sigma = -\frac{1}{2} \frac{\int_M \left(\text{Im}(\alpha\bar{\beta})\right)^2 d\mu}{\int_M \frac{|\alpha|^2}{(1-|p|^2)^2} d\mu} \left( \int_M \frac{|\beta|^2}{(1-|p|^2)^2} d\mu \right). \quad (8.11)\]

3) The holomorphic sectional curvature is limited from above by negative constant:
\[K(h_A, Jh_A) \leq -\frac{1}{8\pi}. \quad (8.12)\]
Proof. If functions $\alpha$ and $\beta$ are simultaneously either real, or pure imaginary, then 
$$\text{tr}[A, B]^2 = -8 \left(\text{Im}(\alpha \overline{\beta})\right)^2 = 0.$$ 
In case, when one of functions $\alpha$ and $\beta$ is real, and other is pure imaginary, 
$$\text{tr}AB = 2\text{Re}(\overline{\alpha} \beta) = 0 \text{ and } \left(\text{Im}(\alpha \overline{\beta})\right)^2 = \left(\text{Im}(\alpha \beta)\right)^2.$$

Prove the last statement. In a complex frame, when one gives elements $A \in \text{End}_{S,J_0}(T^*M)$ by complex functions, operator of a complex structure on $\text{End}_{S,J_0}(T^*M)$

$$J : \text{End}_{S,J_0}(T^*M) \longrightarrow \text{End}_{S,J_0}(T^*M), \quad A \mapsto AJ_0$$

acts as multiplication of function $\alpha$ on the number $i$. Therefore if $B = JA$, then $\beta = i\alpha$
and $\text{tr}[A, B]^2 = -8 \left(\text{Im}(\alpha i\overline{\alpha})\right)^2 = -8 \left(|\alpha|^2\right)^2, \text{tr}AB = 2\text{Re}(\overline{\alpha} i\alpha) = 0$. Therefore the formula (8.10) becomes

$$K(h_A, Jh_A) = -\frac{1}{2} \left(\frac{\int_M \frac{|\alpha|^4}{(1-|\alpha|^2)^2} \, d\mu}{\int_M \frac{|\alpha|^2}{(1-|\alpha|^2)^2} \, d\mu}\right)^2.$$  \hspace{1cm} (8.12) 

Now we use a Cauchy-Bunyakovskii inequality $\left(\int_M f \, d\mu\right)^2 \leq \int_M f^2 d\mu \int_M d\mu$ for function $f = \frac{|\alpha|^2}{(1-|\alpha|^2)^2}$. As $\int_M d\mu = 4\pi$ for a two-dimensional unit sphere $M = S^2$, then inequality

$$\frac{1}{4\pi} \leq \frac{\int_M f^2 d\mu}{(\int_M f \, d\mu)^2}.$$ 

implies the required evaluation $K(h_A, Jh_A) \leq -\frac{1}{8\pi}$.

Find sectional curvatures without using of a parametrization of the space $\mathcal{A}M$. Let $g \in \mathcal{A}M$ is any associated metric and $J$ is complex structure, corresponding to it. The volume form $\mu(g)$ coincides with the symplectic form $\omega = d\varphi \wedge dz$.

Choose on a coordinate map $(\varphi, z)$ on a sphere $S^2$ a field of orthonormalized (with respect to $g$) frames $e_1, e_2$. Let $\partial_1 = \frac{1}{2}(e_1 - ie_2), \partial_2 = \frac{1}{2}(e_1 + ie_2)$ is corresponding field of complex frames.

We already marked, that in a complex frame the element $a \in T_g\mathcal{A}M$ has a matrix

$$a = \begin{pmatrix} \alpha & 0 \\ 0 & \overline{\alpha} \end{pmatrix},$$

where $\alpha(z, \varphi)$ is complex function on a map $U \subset S^2$. When one gives elements $a \in T_g\mathcal{A}M(S^2)$ by complex functions $\alpha(z, \varphi)$ on $U \subset S^2$, the operator of an almost complex structure

$$J : T_g\mathcal{A}M(S^2) \longrightarrow T_g\mathcal{A}M(S^2)$$

acts as multiplication of function $\alpha(z, \varphi)$ on $i$.

The functions $e^{i(kz+l\varphi)}, k \in \pi \mathbb{Z}, l \in \mathbb{Z}$, form full orthogonal system of functions on domain $U = \{ (\varphi, z); \varphi \in (0, 2\pi), z \in (-1, 1) \}$. They define complex basis in $T_g\mathcal{A}M$. The real basis of the space $T_g\mathcal{A}M$ corresponds to functions

$$\cos(kz + l\varphi), \sin(kz + l\varphi), i \cos(kz + l\varphi), i \sin(kz + l\varphi), k \in \pi \mathbb{Z}, l \in \mathbb{Z}. \hspace{1cm} (8.13)$$

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Remark. Coordinate map \((U; \varphi, z)\) has singularities in two points of a sphere \(z = 1\) and \(z = -1\). The 2-form \(a\), defined by function \(\alpha(z, \varphi)\) on \(U\) can have singularities at \(z = \pm 1\) and \(\varphi = 2\pi\). At calculation of sectional curvatures the integration on a sphere is used. As the singularities of limited functions on a sphere concentrated on set of zero measure, do not influence on values of integrals, then the functions of the system (8.13) can be used for calculation of sectional curvatures of the space \(\text{AM}(S^2)\).

Theorem 8.2. If functions \(\alpha(z, \varphi), \beta(z, \varphi)\) from system (8.13), which gives \(a, b \in T_g\text{AM}\), are simultaneously real, or pure imaginary, then \(K(a, b) = 0\). If the form \(a\) is given by real function, and form \(b\) is given by pure imaginary function (8.13), then

\[
K(a, b) = \begin{cases} 
-\frac{1}{8\pi}, & \text{if } |\alpha| \neq |\beta|, \\
-\frac{1}{16\pi}, & \text{if } |\alpha| = |\beta| \neq 1, \\
-\frac{1}{8\pi}, & \text{if } |\alpha| = |\beta| = 1.
\end{cases}
\]

In particular, for any basis function \(\alpha\) the holomorphic sectional curvature \(K(a, ja)\) takes values

\[
K(a, ja) = \begin{cases} 
-\frac{3}{16\pi}, & |\alpha| \neq 1, \\
-\frac{1}{8\pi}, & |\alpha| = 1.
\end{cases}
\]

Proof. For operators \(A = g^{-1}a, B = g^{-1}b\) corresponding to the forms \(a, b\) we have:

\[
A = 2 \begin{pmatrix} 0 & \bar{\alpha} \\ \alpha & 0 \end{pmatrix}, \quad B = 2 \begin{pmatrix} 0 & \bar{\beta} \\ \beta & 0 \end{pmatrix}, \quad [A, B] = 8i \begin{pmatrix} \text{Im}(\bar{\alpha}\beta) & 0 \\ 0 & -\text{Im}(\bar{\alpha}\beta) \end{pmatrix}.
\]

Thus, \([A, B] = 0\) if and only if \(\text{Im}(\bar{\alpha}\beta) = 0\). In particular, so will be, if \(\alpha\) and \(\beta\) are simultaneously real, or pure imaginary. The first statement of the theorem follows from here. Further, in the second case we have \(\text{tr}(AB) = 8\text{Re}(\bar{\alpha}\beta) = 0\). From the formulas

\[
\int_M \text{tr}([A, B]^2) dz d\varphi = -128 \int_M (\text{Im}(\bar{\alpha}\beta))^2 dz d\varphi,
\]

\[
\|a \wedge b\|^2 = \int_M \text{tr}(A^2) d\mu \int_M \text{tr}(B^2) d\mu \left(\int_M \text{tr}(AB) d\mu\right)^2 = 64 \int_M |\alpha|^2 dz d\varphi \int_M |\beta|^2 dz d\varphi,
\]

we obtain expression of a sectional curvature:

\[
K_\sigma = -\frac{1}{2} \frac{\int_M (\text{Im}(\alpha\beta))^2 d\mu}{\int_M |\alpha|^2 d\mu \int_M |\beta|^2 d\mu}.
\]

If

\[
a = \cos(kz + l\varphi), \text{ or } \sin(kz + l\varphi),
\]

\[
\beta = i \cos(pz + q\varphi), \text{ or } i \sin(pz + q\varphi),
\]

then the direct calculations give values, indicated in the theorem. Indeed, we consider all essentially various cases.
1) Let, for example,
\[ \alpha = \cos(kz + l\varphi), \quad \beta = i\cos(pz + q\varphi), \quad (k, l) \neq (0, 0), \quad (p, q) \neq (0, 0), \quad (k, l) \neq (p, q). \]

Designate \( \zeta = kz + l\varphi, \quad \psi = pz + q\varphi. \) Then \( |\alpha|^2 = \cos^2 \zeta, \quad |\beta|^2 = \cos^2 \psi \) and \( (\text{Im}(\overline{\alpha}\beta))^2 = (\cos \zeta \cos \psi)^2 = \frac{1}{4} (\cos(\zeta + \psi) + \cos(\zeta - \psi))^2. \)

\[
\int_0^{2\pi} \int_{-1}^1 (\text{Im}(\overline{\alpha}\beta))^2 \, d\varphi \, dz =
\]

\[
= \frac{1}{4} \int_0^{2\pi} d\varphi \int_{-1}^1 (\cos^2(\zeta + \psi) + 2 \cos(\zeta + \psi) \cos(\zeta - \psi) + \cos^2(\zeta - \psi)) \, dz = \frac{1}{4} (2\pi + 2\pi) = \pi.
\]

Then \( |\alpha|^2 = \cos^2 \zeta, \quad |\beta|^2 = \cos^2 \psi. \)

\[
\|a \wedge b\|^2 = \int_U \cos^2 \zeta \, dz \, d\varphi \int_U \cos^2 \psi \, dz \, d\varphi = 4\pi^2.
\]

Therefore
\[
K(a, b) = -\frac{1}{\frac{1}{2} \cdot \frac{\pi}{4\pi^2}} = -\frac{1}{8\pi}.
\]

2) Let \( \alpha = 1, \quad \beta = i\cos(pz + q\varphi), \quad (p, q) \neq (0, 0). \) Designate \( \psi = pz + q\varphi. \) Then \( (\text{Im}(\overline{\alpha}\beta))^2 = (\cos \psi)^2 \) and \( |\alpha|^2 = 1, \quad |\beta|^2 = \cos^2 \psi. \)

\[
\int_0^{2\pi} \int_{-1}^1 (\text{Im}(\overline{\alpha}\beta))^2 \, d\varphi \, dz = \int_0^{2\pi} d\varphi \int_{-1}^1 \cos^2(\psi) \, dz = 2\pi.
\]

Therefore
\[
K(a, b) = -\frac{1}{\frac{2\pi}{(4\pi)(2\pi)}} = -\frac{1}{8\pi}.
\]

3) Now let (holomorphic sectional curvature),
\[ \alpha = \cos(kz + l\varphi), \quad \beta = i\cos(kz + l\varphi), \quad (k, l) \neq (0, 0). \]

Designate \( \zeta = kz + l\varphi. \) Then \( (\text{Im}(\overline{\alpha}\beta))^2 = (\cos^2 \zeta)^2 = \frac{1}{4} (1 + \cos(2\zeta))^2 \) and \( |\alpha|^2 = \cos^2 \zeta, \quad |\beta|^2 = \cos^2 \zeta. \)

\[
\int_0^{2\pi} \int_{-1}^1 (\text{Im}(\overline{\alpha}\beta))^2 \, d\varphi \, dz = \frac{1}{4} \int_0^{2\pi} d\varphi \int_{-1}^1 (1 + 2 \cos(2\zeta) + \cos^2(2\zeta)) \, dz = \frac{1}{4} (4\pi + 2\pi) = \frac{3\pi}{2}.
\]

Then \( \|a \wedge b\|^2 = \int_U \cos^2 \zeta \, dz \, d\varphi \int_U \cos^2 \zeta \, dz \, d\varphi = (2\pi)(2\pi). \)

Therefore
\[
K(a, ia) = -\frac{1}{\frac{3\pi}{2\cdot(2\pi)(2\pi)}} = -\frac{3}{16\pi}.
\]
4) The last case (holomorphic sectional curvature): \( \alpha = \pm 1, \ \beta = \pm i \). Then \( \text{Im}(\alpha \beta) \) \( = 1, \ |\alpha|^2 = 1, \ |\beta|^2 = 1 \). Therefore

\[
\int_0^{2\pi} \int_{-1}^1 (\text{Im}(\alpha \beta))^2 \, d\varphi \, dz = 4\pi, \quad \|a \wedge b\|^2 = \int_U dzd\varphi \int_U zd\varphi = (4\pi)(4\pi).
\]

\[
K(1,i) = -\frac{1}{2} \frac{4\pi}{(4\pi)(4\pi)} = -\frac{1}{8\pi}.
\]

The theorem is proved.

Finally we shall write out the equation \( \text{div} J\delta_g a = 0 \), when \( g = g_0 \). The covariant divergence of the form \( a \) is a vector field

\[
\delta_g a = -2 \left( (1 - z^2) \frac{\partial u}{\partial z} - \frac{\partial v}{\partial \varphi} - 2zu \right) \frac{\partial}{\partial z} + 2 \left( \frac{\partial v}{\partial z} + \frac{1}{1 - z^2} \frac{\partial u}{\partial \varphi} - \frac{2z}{1 - z^2} v \right) \frac{\partial}{\partial \varphi}.
\]

Calculating value \( \text{div} J\delta_g a \) we obtain the equation, which picks out horizontal directions \( h \):

\[
-\frac{1}{2} \text{div} J\delta_g a =
\]

\[
(1 - z^2) \frac{\partial^2 v}{\partial z^2} - \frac{1}{1 - z^2} \frac{\partial^2 v}{\partial \varphi^2} - 4z \frac{1}{1 - z^2} \frac{\partial v}{\partial z} - 2v + 2 \frac{\partial^2 u}{\partial z \partial \varphi} = 2z \frac{\partial u}{1 - z^2} \frac{\partial \varphi}{\partial \varphi} = 0. \quad (8.14)
\]
8.2. Associated metrics on a torus $T^2$.

In this section we shall find sectional curvatures of the space $\mathcal{AM}$ of associated metrics on a torus $T^2$ with a natural symplectic structure. In particular, values of sectional curvatures in regular points of the quotient space $\mathcal{AM}(T^2)/\mathcal{G}$, where $\mathcal{G}$ is the group of symplectic diffeomorphisms of a torus, which Lie algebra is algebra of Hamilton vector fields on $T^2$, will be found.

8.2.1. A view of associated metrics. Consider a torus $T^2 = \mathbb{R}^2/2\pi\mathbb{Z}^2$ with coordinates $(x, y) \bmod 2\pi$, the flat metric $g_0 = dx^2 + dy^2$ and complex structure $J_0$: $z = x + iy$. A volume form: $\mu = dx \wedge dy$. In complex basis

$$\frac{\partial}{\partial z} = \frac{1}{2} \left( \frac{\partial}{\partial x} - i \frac{\partial}{\partial y} \right), \quad \frac{\partial}{\partial \overline{z}} = \frac{1}{2} \left( \frac{\partial}{\partial x} + i \frac{\partial}{\partial y} \right).$$

matrixes of tensor $g_0 = dz \, d\overline{z}$ and operator of a complex structure $J_0$ look like the same ones in case of sphere. The endomorphism $P$ is defined by one complex-valued function $p(x, y)$ on a torus $T^2$. Associated metric and a.c.s. $J$, corresponding to an operator, $P$ have the same form, as well as in case of a sphere $S^2$.

$$\partial(J) = \frac{\partial}{\partial z} - \overline{P} \frac{\partial}{\partial \overline{z}}, \quad \overline{\partial}(J) = \frac{\partial}{\partial \overline{z}} - P \frac{\partial}{\partial z}.$$ 

Write out expressions of associated metric $g(P)$, a.c.s. $J$ and 2-form $h = 2g_0 P$ in real basis $\frac{\partial}{\partial x}, \frac{\partial}{\partial y}$. Write function $p(x, y)$ as $p = r \, e^{i\psi} = r(\cos \psi + i \sin \psi)$, where $r = r(x, y)$, $\psi = \psi(x, y)$ is function on a torus $T^2$.

$$g(P) = \frac{1}{1 - r^2} \begin{pmatrix} 1 + r^2 + 2r \cos \psi & -2r \sin \psi \\ -2r \sin \psi & 1 + r^2 - 2r \cos \psi \end{pmatrix},$$

$$J(P) = \frac{1}{1 - r^2} \begin{pmatrix} 2r \sin \psi & -(1 + r^2 - 2r \cos \psi) \\ 1 + r^2 + 2r \cos \psi & -2r \sin \psi \end{pmatrix}.$$ 

8.2.2. A curvature of the space $\mathcal{AM}(T^2)$. The expression of sectional curvature in a coordinate map $\Psi_{\mathcal{AM}}$ is the same, as in case of a sphere. Let $g_0$ is canonical metric, $g = g_0(1 + P)(1 - P)^{-1}$ is any associated metric and $J$ is complex structure, corresponding to it.

Take the arbitrary elements $A, B \in \text{End}_{S,J_0}(TM)$. In a complex frame $\partial_1, \overline{\partial}_1$ they are set by matrixes of the view

$$A = \begin{pmatrix} 0 & \overline{\alpha} \\ \alpha & 0 \end{pmatrix}, \quad B = \begin{pmatrix} 0 & \overline{\beta} \\ \beta & 0 \end{pmatrix},$$

where $\alpha, \beta$ are complex functions of variables $x, y$.

Theorem 8.3. Let $A, B \in \text{End}_{S,J_0}(TM)$ are operators, given by functions $\alpha$ and $\beta$ and $h_A, h_B$ are the elements of tangent space $T_g \mathcal{AM}(T^2)$, corresponding to them. Then
the sectional curvature $K_\sigma$ of the space $\mathcal{AM}(S^2)$ in the plane section $\sigma$, generated by the elements $h_A, h_B \in T_g\mathcal{AM}(T^2)$, is expressed by the formula

$$K_\sigma = -\frac{1}{2} \int_M \frac{(\Im(\alpha\beta))^2}{(1-|p|^2)^4} d\mu \int_M \frac{|\alpha|^2}{(1-|p|^2)^2} d\mu - \left( \int_M \frac{\Re(\alpha\beta)}{(1-|p|^2)^2} d\mu \right)^2,$$

where $M = T^2$. In particular:

1) If functions $\alpha$ and $\beta$ are simultaneously either real, or pure imaginary, then $K_\sigma = 0$.
2) If one of functions $\alpha$ and $\beta$ is real, and other pure imaginary, then

$$K_\sigma = -\frac{1}{2} \int_M \frac{(\Im(\alpha\beta))^2}{(1-|p|^2)^4} d\mu \int_M \frac{|\alpha|^2}{(1-|p|^2)^2} d\mu \int_M \frac{|\beta|^2}{(1-|p|^2)^2} d\mu - \left( \int_M \frac{\Re(\alpha\beta)}{(1-|p|^2)^2} d\mu \right)^2.$$

3) The holomorphic sectional curvature is limited from above by negative constant:

$$K(h_A, Jh_A) \leq -\frac{1}{8\pi^2}.$$

**Proof.** The reason of appearance of number $\pi^2$, instead of $\pi$, is that the area of a sphere is equal to $4\pi$, and the area of our torus is equal to $4\pi^2$.

Find sectional curvatures without using of a parametrization of the space $\mathcal{AM}$. Let $g \in \mathcal{AM}$ is any associated metric and $J$ is complex structure, corresponding to it. The volume form $\mu(g)$ coincides with the symplectic form $\omega = dx \wedge dy$.

Choose a field of orthonormalized (with respect to $g$) frames on torus $T^2 e_1, e_2$. Let $\partial_1 = \frac{1}{2}(e_1 - ie_2), \overline{\partial}_1 = \frac{1}{2}(e_1 + ie_2)$ is field of corresponding complex frames.

We already marked, that in a complex frame the element $a \in T_g\mathcal{AM}$ has a matrix

$$a = \begin{pmatrix} \alpha & 0 \\ 0 & \overline{\alpha} \end{pmatrix},$$

where $\alpha(x,y)$ is complex $2\pi$ – periodic function of variables $x, y$. When one gives elements $a \in T_g\mathcal{AM}(T^2)$ by complex functions $\alpha(x,y)$, operator of an almost complex structure

$$J : T_g\mathcal{AM}(T^2) \longrightarrow T_g\mathcal{AM}(T^2)$$

acts as multiplication of function $\alpha(x,y)$ on $i$.

The full orthogonal system of functions on a torus $T^2$ is formed by functions

$$p_{kl} = e^{i(kx+ly)}, \quad k, l \in \mathbb{Z}, \quad (x, y) \in T^2.$$

One can find a curvature $R = \frac{1}{\det g(p)} R_{1212}$ of associated metric $g(p_{kl})$.

The direct count in a system of analytical calculations MapleV gives for a set of the metrics $g_t = g(tp_{kl})$ the following expression of a Gaussian curvature:

$$R = -\frac{t}{1-t^2} \left( (k^2 - l^2) \cos(kx + ly) - 2kl \sin(kx + ly) \right).$$

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The functions $e^{i(kx+ly)}$ define the complex basis in $T_g\mathcal{AM}$. The real basis of the space $T_g\mathcal{AM}$ corresponds to functions

$$\cos(kx+ly), \sin(kx+ly), i\cos(kx+ly), i\sin(kx+ly), \quad k, l \in \mathbb{Z}. \quad (8.15)$$

**Theorem 8.4.** If functions $\alpha(x,y), \beta(x,y)$ from system (8.15), which give 2-forms $a,b \in T_g\mathcal{AM}$ are simultaneously either real, or pure imaginary, then $K(a,b) = 0$. If the form $a$ is set by real function, and form $b$ is pure imaginary function (8.15), then

$$K(a,b) = \begin{cases} 
-\frac{1}{8\pi^2}, & \text{if } |\alpha| \neq |\beta|, \\
-\frac{1}{16\pi^2}, & \text{if } |\alpha| = |\beta| \neq 1, \\
-\frac{1}{8\pi^2}, & \text{if } |\alpha| = |\beta| = 1.
\end{cases}$$

In particular, for any basis function $\alpha$ the holomorphic sectional curvature $K(a,Ja)$ takes values

$$K(a,Ja) = \begin{cases} 
-\frac{3}{16\pi^2}, & |\alpha| \neq 1, \\
-\frac{1}{8\pi^2}, & |\alpha| = 1.
\end{cases}$$

In case, when $\alpha(x,y) = e^{i(kx+ly)}, \beta(x,y) = e^{i(px+qy)},$

$$K(a,b) = \begin{cases} 
-\frac{1}{16\pi^2}, & \beta \neq \pm i\alpha, \\
-\frac{1}{8\pi^2}, & \beta = \pm i\alpha.
\end{cases}$$

**Proof.** Let $\alpha = e^{i\varphi}, \beta = E^{i\psi}$, where $\varphi, \psi$ are functions on $T^2$ of view $\varphi = kx + ly, \psi = px + qy$, we assume, that $\varphi \neq \psi$. In this case we have the following expressions:

$$|\alpha|^2 = 1, \quad |\beta|^2 = 1,$$

$$(\text{Im}(\alpha\beta))^2 = (\text{Im}(e^{i(\psi-\varphi)}))^2 = \sin^2(\psi - \varphi) = \frac{1 - \sin 2(\psi - \varphi)}{2}.$$ 

$$\int_0^{2\pi} \int_0^{2\pi} (\text{Im}(\pi\beta))^2 \, dx \, dy = \int_0^{2\pi} \int_0^{2\pi} \frac{1 - \sin 2(\psi - \varphi)}{2} \, dx \, dy = \begin{cases} 
2\pi^2, & \text{if } \psi - \varphi \neq \pm \pi/2, \\
4\pi^2, & \text{if } \psi - \varphi = \pm \pi/2.
\end{cases}$$

$$\|a \land b\|^2 = \int_0^{2\pi} \int_0^{2\pi} dx \, dy \int_0^{2\pi} \int_0^{2\pi} dx \, dy - \left(\int_0^{2\pi} \int_0^{2\pi} \cos(\psi - \varphi) \, dx \, dy\right)^2 = (4\pi^2)(4\pi^2).$$

Therefore

$$K(a, b) = \begin{cases} 
-\frac{1}{2} \frac{2\pi^2}{(4\pi^2)} = -\frac{1}{16\pi^2}, & \text{if } \beta \neq \pm i\alpha, \\
-\frac{1}{2} \frac{4\pi^2}{(4\pi^2)} = -\frac{1}{8\pi^2}, & \text{if } \beta = \pm i\alpha.
\end{cases}$$

The theorem is completely proved.

**8.2.3. A curvature of the space $\mathcal{AM}/\mathcal{G}$.** Let $\overline{a}, \overline{b} \in T_{\overline{g}}(\mathcal{AM}/\mathcal{G})$. As $p : \mathcal{AM} \rightarrow \mathcal{AM}/\mathcal{G}$ is Riemannian submersion, the sectional curvature $K(\overline{a}, \overline{b})$ of the space $\mathcal{AM}/\mathcal{G}$ is under the formula [33], [7]:

$$\overline{K}(\overline{a}, \overline{b}) = K(a,b) + \frac{3}{4} \left(\frac{[a,b]^V}{\|a \land b\|^2}\right)_g,$$ 

(8.16)
where \( a, b \in T_g\mathcal{AM} \) are horizontal lifts of vectors \( \bar{a}, \bar{b} \), \( K(a, b) \) is sectional curvature of the space \( \mathcal{AM} \), \([a, b]^{V}\) is vertical part of Lie commutator of horizontal vector fields on \( \mathcal{AM} \), which continue \( a, b \in T_g\mathcal{AM} \).

Consider the flat metric \( g_0 = dz \, d\bar{z} \) on \( T^2 \). Let symmetric 2-form \( h \) in real basis \( dx, dy \) has a view

\[
h = u \, dx^2 - 2v \, dxdy - u \, dy^2, \quad (8.17)
\]

Corresponding quadratic differential \( h = pdz \, d\bar{z} + \bar{p} \, dz \, d\bar{z} \) is set by function \( p = 1/2(u + iv) \).

The covariant divergence \( \delta_g h = -\nabla^i h_{ij} \) represents a vector field

\[
\delta_g h = -2 \left( \frac{\partial u}{\partial x} - \frac{\partial v}{\partial y} \right) \frac{\partial}{\partial x} + 2 \left( \frac{\partial v}{\partial x} + \frac{\partial u}{\partial y} \right) \frac{\partial}{\partial y}, \quad (8.18)
\]

Mark incidentally, that the equality \( \delta_g h = 0 \) means, that \( h = pdz \, d\bar{z} \) is holomorphic quadratic differential. Find \( \text{div} J\delta_g h \).

\[
J\delta_g h = -2 \left( \frac{\partial v}{\partial x} + \frac{\partial u}{\partial y} \right) \frac{\partial}{\partial x} - 2 \left( \frac{\partial u}{\partial x} - \frac{\partial v}{\partial y} \right) \frac{\partial}{\partial y} - \frac{1}{2} \text{div} J\delta_g h = \frac{\partial^2 v}{\partial x^2} - \frac{\partial^2 v}{\partial y^2} + 2 \frac{\partial^2 u}{\partial x \partial y}. \quad (8.19)
\]

In the complex form:

\[
\text{div} J\delta_g h = -8 \text{Im} \left( \frac{\partial^2 p}{\partial \bar{z}^2} \right). \quad (8.20)
\]

The horizontal space \( S^*_z \mathcal{A} \), orthogonal to an orbit \( g_0\mathcal{G} \subset \mathcal{AM} \) consists of quadratic differentials \( a = pdz^2 \), satisfying to the equation,

\[
\frac{1}{4} \text{Im} \left( \frac{\partial^2 p}{\partial \bar{z}^2} \right) = \frac{\partial^2 v}{\partial x^2} - \frac{\partial^2 v}{\partial y^2} + 2 \frac{\partial^2 u}{\partial x \partial y} = 0. \quad (8.21)
\]

The basis of solutions of this equation is given by complex functions \( p(x, y) \) of a view

\[
p(x, y) = (f_1(x) + f_2(y)) + i \left( f_3(x + y) + f_4(x - y) \right). \]

Basis of real functions \( p = u(x, y) \), defining horizontal forms, i.e., satisfying to the equation,

\[
\frac{\partial^2 u}{\partial x \partial y} = 0,
\]

form the functions of a view

\[
\cos kx, \ \sin kx, \ \cos ly, \ \sin ly, \quad k, l \in \mathbb{Z}. \quad (8.22)
\]

Basis of pure imaginary functions \( p = iv(x, y) \), defining horizontal forms, i.e., satisfying to the equation,

\[
\frac{\partial^2 v}{\partial x^2} - \frac{\partial^2 v}{\partial y^2} = 0,
\]
is formed by functions of a view

\[ i \cos p(x + y), \ i \sin p(x + y), \ i \cos q(x - y), \ i \sin q(x - y), \ p, q \in \mathbb{Z}. \] (8.23)

Find a sectional curvature of the quotient space \( \mathcal{AM}/G \) in case, when the horizontal forms \( a, b \), given by functions of view (8.22) and (8.23), are taken as the elements \( a, b \in T_{[g]} \mathcal{AM}/G \).

**Theorem 8.5.** Sectional curvature \( \mathcal{K}(\pi, \overline{b}) \) of the space \( \mathcal{AM}/G \) at a point \( [g_0] \) in direction of the horizontal forms \( a, b \in T_{g_0} \mathcal{AM} \) takes the following values:

1) If \( \alpha \) is \( \cos kx \) or \( \sin kx \), and \( \beta \) is \( \cos ly \) or \( \sin ly \), then

\[ \mathcal{K}(\pi, \overline{b}) = \frac{3}{8\pi^2} \frac{k^2l^2}{(k^2 + l^2)^2}, \]

in remaining cases of real functions, \( \mathcal{K}(\pi, \overline{b}) = 0 \).

2) If \( \alpha \) is \( i \cos p(x + y) \) or \( i \sin p(x + y) \), and \( \beta \) is \( i \cos q(x - y) \) or \( i \sin q(x - y) \), then

\[ \mathcal{K}(\pi, \overline{b}) = \frac{3}{32\pi^2} \frac{p^2q^2}{(p^2 + q^2)^2}, \]

in remaining cases of pure imaginary functions, \( \mathcal{K}(\pi, \overline{b}) = 0 \).

3) If \( \alpha \) is any real function from (8.22) and \( \beta \) is imaginary function of a view (8.23), then

\[ \mathcal{K}(\pi, \overline{b}) = \begin{cases} \frac{1}{4\pi^2}, & \text{if } |\alpha| \neq 1, |\beta| \neq 1, \\ \frac{1}{8\pi^2}, & \text{if } |\alpha| = 1 \text{ or } |\beta| = 1. \end{cases} \]

**Proof.** We shall use the formula (7.6),

\[ \mathcal{K}(\pi, \overline{b}) = K(a, b) + \frac{3}{4|a \wedge b|^2} \int_M (\text{div} J\{a, b\}) (E^{-1}_g(\text{div} J\{a, b\})) d\mu_g, \]

where:

\( E_g(f) = \text{div} J \delta_g \alpha_g J \text{grad} f, \)

\( \{a, b\} = A(\delta_g b) + (\square (b), a) - B(\delta_g a)(\square (a), b), \)

\( A(\delta_g b) = a^i_k (\delta_g b)^i, \ (\delta_g b)^i = -\nabla_j b^i_j, \)

\( (\square (b), a) = b^i_j a^i_j, \)

\( (\square (b))^k_{ij} = b^k_{ij} = \frac{1}{2} (\nabla_i b^k_j + \nabla_j b^k_i - \nabla^k b_{ij}), \ b \in S^2. \)

In our case of a two-dimensional torus \( T^2 \) and flat metric \( g = dx^2 + dy^2 \), the operator \( E_g \) has a view:

\[ E_g f = \frac{1}{2} \left( \frac{\partial^4 f}{\partial x^4} + 2 \frac{\partial^4 f}{\partial x^2 \partial y^2} + \frac{\partial^4 f}{\partial y^4} \right) = \frac{1}{2} \Delta^2 f. \]
It is convenient now to set the symmetric 2-forms $a, b$ in real base $dx, dy$. The 2-form $a$, corresponding to complex function $\alpha = 1/2(u + iv)$ has a matrix $a = \begin{pmatrix} u & -v \\ -v & -u \end{pmatrix}$ in real basis. Calculate $\delta_g a, \Box (a)$ for the following basic horizontal forms.

**Type 1.** $a = \begin{pmatrix} \cos kx & 0 \\ 0 & -\cos kx \end{pmatrix}$.

Covariant divergence: $\delta_g a = (k \sin kx, 0)$.

Tensor $a_{ij}^k$:

\[
\begin{array}{c}
\begin{array}{c}
a_{11}^1 = \frac{1}{2} k \sin kx, \\
a_{12}^1 = 0, \\
a_{22}^1 = -\frac{1}{2} k \sin kx, \\
a_{11}^2 = 0, \\
a_{12}^2 = \frac{1}{2} k \sin kx, \\
a_{22}^2 = 0.
\end{array}
\end{array}
\]

**Type 2.** $a = \begin{pmatrix} \cos ly & 0 \\ 0 & -\cos ly \end{pmatrix}$.

Covariant divergence: $\delta_g a = -(0, l \sin ly)$.

Tensor $a_{ij}^k$:

\[
\begin{array}{c}
\begin{array}{c}
a_{11}^1 = 0, \\
a_{12}^1 = -\frac{1}{2} l \sin ly, \\
a_{22}^1 = 0, \\
a_{11}^2 = \frac{1}{2} l \sin ly, \\
a_{12}^2 = 0, \\
a_{22}^2 = \frac{1}{2} l \sin ly.
\end{array}
\end{array}
\]

**Type 3.** $a = \begin{pmatrix} 0 & -\cos p(x + y) \\ -\cos p(x + y) & 0 \end{pmatrix}$.

Covariant divergence: $\delta_g a = -(p \sin p(x + y), p \sin p(x + y))$.

Tensor $a_{ij}^k$:

\[
\begin{array}{c}
\begin{array}{c}
a_{11}^1 = 0, \\
a_{12}^1 = 0, \\
a_{22}^1 = p \sin p(x + y), \\
a_{11}^2 = p \sin p(x + y), \\
a_{12}^2 = 0, \\
a_{22}^2 = 0.
\end{array}
\end{array}
\]

**Type 4.** $a = \begin{pmatrix} 0 & -\cos q(x - y) \\ -\cos q(x - y) & 0 \end{pmatrix}$.

Covariant divergence: $\delta_g a = -(q \sin q(x - y), q \sin q(x - y))$.

Tensor $a_{ij}^k$:

\[
\begin{array}{c}
\begin{array}{c}
a_{11}^1 = 0, \\
a_{12}^1 = 0, \\
a_{22}^1 = -q \sin q(x - y), \\
a_{11}^2 = q \sin q(x - y), \\
a_{12}^2 = 0, \\
a_{22}^2 = 0.
\end{array}
\end{array}
\]

**Remark.** In case, when for definition of the form $a$ the function sin is taken, the values $\delta_g a$ and $\Box (a)$ are similarly found with elementary replacement of functions sin and cos.

Begin calculation of a sectional curvature $K(a, b)$ of the space $\mathcal{AM}/G$.

**First part.** Let

\[
a = \begin{pmatrix} \cos kx & 0 \\ 0 & -\cos kx \end{pmatrix}, \quad b = \begin{pmatrix} \cos ly & 0 \\ 0 & -\cos ly \end{pmatrix}, \quad k \neq 0, \ l \neq 0.
\]

In this case: $K(a, b) = 0,$

\[
\Box (b), a) = 0, \quad \Box (a), b) = 0,
\]

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\[ A(\delta_g b) = (0, l \cos kx \sin ly), \quad B(\delta_g a) = (k \sin kx \cos ly, 0), \]
\[ \{a, b\} = (-k \sin kx \cos ly, l \cos kx \sin ly), \]
\[ J\{a, b\} = -(l \cos kx \sin ly, k \sin kx \cos ly), \]
\[ \text{div} J\{a, b\} = 2kl \sin kx \sin ly = kl(\cos(kx - ly) - \cos(kx + ly)), \]
\[ E_g^{-1}(\text{div} J\{a, b\}) = 2kl \frac{1}{(k^2 + l^2)^2} (\cos(kx - ly) - \cos(kx + ly)), \]
\[ \int_{T^2} (\text{div} J\{a, b\}) E_g^{-1}(\text{div} J\{a, b\}) \, dx \, dy = \frac{2k^2l^2}{(k^2 + l^2)^2} (2\pi^2 + 2\pi^2), \]
\[ \|a\|^2 = \int_{T^2} \text{tr}(A^2) \, dx \, dy = 2 \int_{T^2} \cos^2 kx \, dx \, dy = 4\pi^2 = \|b\|, \]
\[ (a, b) = 2 \int_{T^2} \cos kx \cos ly \, dx \, dy = 0, \]

Therefore
\[ \mathbf{K}(\pi, \bar{b}) = \frac{3}{8\pi^2} \frac{k^2l^2}{(k^2 + l^2)^2}. \]

If
\[ a = \begin{pmatrix} \cos kx & 0 \\ 0 & -\cos kx \end{pmatrix}, \quad b = \begin{pmatrix} \sin ly & 0 \\ 0 & -\sin ly \end{pmatrix}, \quad k \neq 0, \ l \neq 0, \]
then similarly,
\[ \mathbf{K}(\pi, \bar{b}) = \frac{3}{8\pi^2} \frac{k^2l^2}{(k^2 + l^2)^2}. \]

Let now
\[ a = \begin{pmatrix} \cos kx & 0 \\ 0 & -\cos kx \end{pmatrix}, \quad b = \begin{pmatrix} \cos lx & 0 \\ 0 & -\cos lx \end{pmatrix}. \]
In this case: \( K(a, b) = 0, \)
\[ (\square (b), a) = 0, \quad (\square (a), b) = 0, \]
\[ A(\delta_g b) = (l \sin lx \cos kx, 0), \quad B(\delta_g a) = (k \sin kx \cos lx, 0), \]
\[ \{a, b\} = (l \sin lx \cos kx - k \sin kx \cos lx, 0), \]
\[ \text{div} J\{a, b\} = 0. \]

Therefore \( \mathbf{K}(\pi, \bar{b}) = 0. \)

If
\[ a = \begin{pmatrix} \cos kx & 0 \\ 0 & -\cos kx \end{pmatrix}, \quad b = \begin{pmatrix} \sin lx & 0 \\ 0 & -\sin lx \end{pmatrix}, \]
then similarly, \( \mathbf{K}(\pi, \bar{b}) = 0. \)

Second part. Let
\[ a = \begin{pmatrix} 0 & -\cos p(x + y) \\ -\cos p(x + y) & 0 \end{pmatrix}, \quad b = \begin{pmatrix} 0 & -\cos q(x - y) \\ -\cos q(x - y) & 0 \end{pmatrix}, \quad p^2 + q^2 \neq 0. \]
In this case: \( K(a, b) = 0, \)
\[
\begin{align*}
&\begin{array}{ccc}
\left\langle \mathbf{a}, \mathbf{b} \right\rangle = 0, & \left\langle \mathbf{a}, \mathbf{b} \right\rangle = 0,
\end{array} \\
&\begin{array}{ccc}
A(\delta g) = (q \cos p(x + y) \sin q(x - y), -q \cos p(x + y) \sin q(x - y)),
\end{array} \\
&\begin{array}{ccc}
B(\delta g) = (p \sin p(x + y) \cos q(x - y), p \sin p(x + y) \cos q(x - y)),
\end{array} \\
&\begin{array}{ccc}
\{a, b\} = (q \cos p(x + y) \sin q(x - y) - p \sin p(x + y) \cos q(x - y), \\
- q \cos p(x + y) \sin q(x - y) - p \sin p(x + y) \cos q(x - y)),
\end{array} \\
&\begin{array}{ccc}
J\{a, b\} = (q \cos p(x + y) \sin q(x - y) + p \sin p(x + y) \cos q(x - y),
\end{array} \\
&\begin{array}{ccc}
q \cos p(x + y) \sin q(x - y) - p \sin p(x + y) \cos q(x - y)),
\end{array} \\
&\begin{array}{ccc}
\text{div} J\{a, b\} = -2pq \sin p(x + y) \sin q(x - y) =
\end{array} \\
&\begin{array}{ccc}
= pq \cos((p - q)x + (p + q)y) - \cos((p + q)x + (p - q)y)),
\end{array} \\
&\begin{array}{ccc}
E^{-1}_y(\text{div} J\{a, b\}) =
\end{array} \\
&\begin{array}{ccc}
= \frac{-2pq}{(p^2 + q^2)(p^2 + q^2)} \cos((p - q)x + (p + q)y) \cos((p + q)x + (p - q)y)),
\end{array} \\
&\begin{array}{ccc}
\int_{T^2} (\text{div} J\{a, b\}) E^{-1}_y(\text{div} J\{a, b\}) \, dx \, dy = \frac{2p^2q^2}{(2p^2 + 2q^2)(2\pi^2 + 2\pi^2)},
\end{array} \\
&\begin{array}{ccc}
\|a\|^2 = 4\pi^2 = \|b\|^2, \quad (a, b)_g = 0, \quad K(a, b) = 0.
\end{array}
\]

Therefore
\[
\overline{K}(\mathbf{a}, \mathbf{b}) = \frac{3}{32\pi^2} \frac{p^2q^2}{(p^2 + q^2)^2}.
\]

If
\[
a = \begin{pmatrix}
0 & -\cos p(x + y) \\
-\cos p(x + y) & 0
\end{pmatrix}, \quad b = \begin{pmatrix}
0 & -\sin q(x - y) \\
-\sin q(x - y) & 0
\end{pmatrix}, \quad p^2 + q^2 \neq 0.
\]

then similarly
\[
\overline{K}(\mathbf{a}, \mathbf{b}) = \frac{3}{32\pi^2} \frac{p^2q^2}{(p^2 + q^2)^2}.
\]

Let now,
\[
a = \begin{pmatrix}
0 & -\cos p(x + y) \\
-\cos p(x + y) & 0
\end{pmatrix}, \quad b = \begin{pmatrix}
0 & -\cos q(x + y) \\
-\cos q(x + y) & 0
\end{pmatrix}, \quad p \neq q.
\]

In this case: \( K(a, b) = 0, \) \( \left\langle \mathbf{a}, \mathbf{b} \right\rangle = 0, \) \( \left\langle \mathbf{a}, \mathbf{b} \right\rangle = 0, \) \( \text{div} J\{a, b\} = 0 \) and \( \overline{K}(\mathbf{a}, \mathbf{b}) \). If
\[
a = \begin{pmatrix}
0 & -\cos p(x + y) \\
-\cos p(x + y) & 0
\end{pmatrix}, \quad b = \begin{pmatrix}
0 & -\sin q(x + y) \\
-\sin q(x + y) & 0
\end{pmatrix}, \quad p \neq q.
\]

then also \( \overline{K}(\mathbf{a}, \mathbf{b}) \).
**Third part.** Let

\[
a = \begin{pmatrix} -\cos kx & 0 \\ 0 & -\cos kx \end{pmatrix}, \quad b = \begin{pmatrix} 0 & -\cos p(x + y) \\ -\cos p(x + y) & 0 \end{pmatrix}, \quad p^2 + q^2 \neq 0.
\]

In this case: \( K(a, b) = -\frac{1}{8\pi^2}, \)

\[
\begin{align*}
\square (b, a) &= (-p \cos kx \sin p(x + y), p \cos kx \sin p(x + y)), \\
\square (a, b) &= (0, -k \sin kx \cos p(x + y)), \\
A(\delta_g b) &= (-p \cos kx \sin p(x + y), p \cos kx \sin p(x + y)), \\
B(\delta_g a) &= (0, -k \sin kx \cos p(x + y)), \\
\{a, b\} &= 2 (-p \cos kx \sin p(x + y), p \cos kx \sin p(x + y)) + 2 (0, k \sin kx \cos p(x + y)), \\
J\{a, b\} &= -2p (\cos kx \sin p(x + y), k \cos kx \sin p(x + y)) - 2k (0, \sin kx \cos p(x + y)), \\
\text{div} J\{a, b\} &= 4kp \sin kx \sin p(x + y) - 2(k^2 + 2p^2) \cos kx \cos p(x + y) = \\
&= (2kp - k^2 - 2p^2) \cos((k - p)x - py) - (k^2 + 2kp + 2p^2) \cos((k + p)x + py), \\
E^{-1}_g (\text{div} J\{a, b\}) &= 2 \frac{2kp - k^2 - 2p^2}{((k - p)^2 + p^2)^2} \cos((k-p)x- py) - 2 \frac{k^2 + 2kp + 2p^2}{((k + p)^2 + p^2)^2} \cos((k + p)x + py), \\
\int_{T^2} (\text{div} J\{a, b\}) ~ E^{-1}_g (\text{div} J\{a, b\}) \ dx \ dy = \\
&= \left( \frac{2(2kp - k^2 - 2p^2)^2}{((k - p)^2 + p^2)^2} + 2 \frac{(k^2 + 2kp + 2p^2)^2}{((k + p)^2 + p^2)^2} \right) 2\pi^2 = 8\pi^2, \\
\|a\|^2 &= 4\pi^2 = \|b\|^2, \quad (a, b)_g = 0,
\end{align*}
\]

Therefore

\[
K(\pi, \bar{b}) = 2 \frac{2kp - k^2 - 2p^2}{((k - p)^2 + p^2)^2} + 2 \frac{k^2 + 2kp + 2p^2}{((k + p)^2 + p^2)^2} = \frac{1}{8\pi^2}.
\]

The same value of a sectional curvature turns out, when the 2-form \(a\) is given by functions \(\cos kx, \cos p(x + y), \sin ly\) or when the 2-form \(b\) is given by functions \(\cos p(x + y), \cos q(x - y), \sin q(x - y)\). If one of functions has a view \(\pm 1\) or \(\pm i\), then \(\{a, b\} = 0\). Therefore \(K(\pi, \bar{b}) = K(a, b) = -\frac{1}{8\pi^2}\). The theorem is proved.
§9. Critical associated metrics on a symplectic manifold.

Consider a functional on the space of Riemannian metrics $\mathcal{M} \rightarrow \mathbb{R}$, $R(g) = \int_M r(g) \, d\mu(g)$, where $r(g)$ is scalar curvature of the metric $g \in \mathcal{M}$. Functional $R$ is invariant with respect to an action of group of diffeomorphisms $D$ on $\mathcal{M}$. The gradient of a functional $R$ is simple found \[7\]:

$$dR(g; h) = \int_M \left( dr(g; h) + r(g) \frac{1}{2} tr_g h \right) \, d\mu(g) =$$

$$= \int_M \left( \triangle (tr_g h) + \delta_g \delta_g h g(h, Ric(g)) + r(g) \frac{1}{2} tr_g h \right) \, d\mu(g) =$$

(under the Stokes formula, taking into account $\partial M = \emptyset$)

$$= \int_M g \left( \frac{1}{2} r(g) g - Ric(g), h \right) \, d\mu(g) = \left( \frac{1}{2} r(g) g - Ric(g), h \right)_g,$$

where $Ric(g)$ is Ricci tensor, $g(h, Ric(g))$ is pointwise inner product of tensor fields on $M$, $g(h, Ric(g)) = tr (g^{-1} h g^{-1} Ric(g))$, $tr_g h = tr g^{-1} h$. Thus,

$$\text{grad } R = \frac{1}{2} r(g) g - Ric(g).$$

Recall, that a metric $g$ on $M$ is called Einsteinian, if its Ricci tensor is proportional to a metric tensor $g$,

$$Ric(g) = \lambda g, \quad \lambda \in \mathbb{R}.$$  

Einsteinian metrics are critical for the functional $R$ on the manifold $\mathcal{M}_1$ of metrics with the same volume (equal to unit). Recall, that

$$T_g \mathcal{M}_1 = \{ h \in S_2; \int_M tr_g h \, d\mu(g) = 0 \}.$$  

The orthogonal complement to $T_g \mathcal{M}_1$ in the space $T_g \mathcal{M} = S_2$ consists of 2-forms proportional to $g$: $h = cg$, $c \in \mathbb{R}$. The metric $g \in \mathcal{M}_1$ is critical for the functional $R$ on $\mathcal{M}_1$ if and only if $\text{grad } R(g)$ is orthogonal to $T_g \mathcal{M}_1$, i.e. if for some number $c \in \mathbb{R}$,

$$\frac{1}{2} r(g) g - Ric(g) = cg.$$  

is held. We just obtain from here, that $r(g) = \text{const}$ and $Ric(g) = \lambda g$, $\lambda \in \mathbb{R}$.

Consider the functional $R$ on the manifold $\mathcal{AM}$ of associated metrics on a symplectic manifold $M, \omega$. In this case a set of critical metrics is much wider. D.Blair has shown \[9\],
that a metric \( g \in \mathcal{AM} \) is critical for the functional \( R \) on \( \mathcal{AM} \) if and only if Ricci tensor \( Ric(g) \) is Hermitian with respect to almost complex structure \( J \), corresponding to \( g \).

Indeed, any element \( h \in T_g \mathcal{AM} \) represents symmetric anti-Hermitian form, therefore \( \text{tr}_g h = 0 \) and then

\[
\begin{align*}
    dR(g; h) &= \int_M \left( -g(h, Ric(g)) + \frac{1}{2} \text{tr}_g h \right) \, d\mu(g) = \\
    &= -\int_M g(h, Ric(g)) \, d\mu(g) = 0.
\end{align*}
\]

From an arbitrary of the anti-Hermitian form \( h \) we obtain \( g(h, Ric(g)) = 0 \), and it follows from a pointwise orthogonality of the Hermitian and anti-Hermitian forms on \( M \), that the tensor \( Ric(g) \) is Hermitian.

**Conclusion.** In case of associated metrics analog of Einstein metrics is metrics with Hermitian Ricci tensor.

The important property of the space \( \mathcal{EM} \) of einsteinian metrics on \( M \) is the finite dimensionality of a moduli space \( \mathcal{EM}/\mathcal{D} \). Such question is natural for asking for the space of critical metrics of the functional \( R \) on the space of associated metrics.

In this paragraph we show a finite dimensionality of the space of classes of equivalent critical associated metrics of a constant scalar curvature. Consider the map

\[
ARic : \mathcal{AM} \longrightarrow S_{2A}, \quad g \longrightarrow Ric(g)_A, \quad (9.1)
\]

which takes each associated metric \( g \) to an anti-Hermitian part \( Ric(g)_A \) of Ricci tensor. Then the set of all critical metrics coincides with a set \( ARic^{-1}(0) \).

By symbol \( \mathcal{CAM}_c \) we shall designate a set of the critical associated metrics of constant scalar curvature, which is equal to \( c \). The space \( \mathcal{CAM}_c \) can be considered as a level set \( (ARic \times r)^{-1}(0, c) \) of mapping

\[
ARic \times r : \mathcal{AM} \longrightarrow S_{2A} \times C^\infty(M, R), \quad g \longrightarrow (Ric(g)_A, r(g)). \quad (9.2)
\]

This mapping extends to a smooth mapping

\[
ARic \times r : \mathcal{AM}^s \longrightarrow S_{2A}^{s-2} \times H^{s-2}(M, R)
\]

from Hilbert manifold \( \mathcal{AM}^s \) of associated metrics of the Sobolev class \( H^s \), \( s \geq 2n + 3 \), into the Hilbert space \( S_{2A}^{s-2} \times H^{s-2}(M, R) \) of forms and functions of class \( H^{s-2} \). It was proven in §3 that the manifold \( \mathcal{AM}^s \) is analytical. In further the analyticity of the continued mapping \( ARic \times r \) will be shown. Let \( \{ \mathcal{G}; \mathcal{G}^s, \ s \geq 2n + 5 \} \) is ILH-Lie group of exact symplectic diffeomorphisms \[42], \[50] and let \( I_\omega(g) = I(g) \cap \mathcal{G} \) is isometry group being also symplectic transformations.

The following theorem takes place, it just follows from the slice theorem, \[17], \[27], stated in general case for action of groups of diffeomorphisms on the space of metrics.
Slice theorem. Let $g \in \mathcal{AM}$. If $s \geq 2n + 5$, then there exists a submanifold $S^s_g$ in $\mathcal{AM}^s$ and a local section $\chi^{s+1} : I_\omega(g) \setminus G^{s+1} \rightarrow G^{s+1}$ defined on an open neighbourhood $U^{s+1}$ of a coset $[I_\omega(g)]$ which possesses the following properties:

1) If $\gamma \in I_\omega(g)$, then $\gamma^*(S^s_g) = S^s_g$. 
2) Let $\gamma \in G^{s+1}$. If $\gamma^*(S^s_g) \cap S^s_g \neq \emptyset$, then $\gamma \in I_\omega(g)$. 
3) The mapping $F : S^s_g \times U^{s+1} \rightarrow \mathcal{AM}^s$, $F^s_g(u) = \chi^{s+1}(u)^* g_1$, is homeomorphism on an open neighbourhood $V^s$ of the element $g$ from $\mathcal{AM}^s$. 

The general schema of construction of a slice $S^s_g$ is applied to our case too. Let $S^{s*}_{2A}$ is the space of anti-Hermitian symmetric 2-forms $h$ on $M$ of a class $H^s$, satisfying to the condition, 

$$\text{div} J\delta_g h = 0.$$ 

The slice $S^s_g$ is an image of a neighbourhood of zero $W^s \subset S^{s*}_{2A}$ at exponential mapping [17]. In case of the space of associated metrics the exponential map is set by usual exponential mapping (see §3):

$$\exp_g : S^{s*}_{2A} \rightarrow \mathcal{AM}^s, \quad \exp_g(h) = g e^H,$$

where $H = g^{-1} h$ and $e^H = 1 + H + \frac{1}{2} H^2 + \frac{1}{3!} H^3 + \ldots$. As the mapping $\exp_g$ is real-analytic, then the slice $S_g^s$ is a real-analytic submanifold in $\mathcal{AM}^s$. Note, that mapping $F$ of the slice theorem

$$F : S_g \times U \rightarrow \mathcal{AM}$$

is ILH-smooth, as for any $s \geq 2n + 5$:

$$S^s_g = S^{2n+5}_g \cap \mathcal{AM}^s, \quad U^{s+1} = U^{2n+6} \cap (I_\omega(g) \setminus G^{s+1}),$$

$$V^s = V^{2n+5} \cap \mathcal{AM}^s, \quad \chi^{s+1} = \chi^{2n+6} | U^{s+1}$$

is held, and for any $k \geq 0$ mappings

$$F^{s+1} : S^{s+k}_g \times U^{s+k+1} \rightarrow V^s,$$

$$p^{s+k} \times q^{s+k} : V^{s+k} \rightarrow S^s_g \times U^{s+1}$$

are $C^k$-differentiable.

The quotient space $S^s_g / I_\omega(g)$ describes a local structure of a quotient space $\mathcal{AM}^s / G^{s+1}$ in a neighbourhood of class $[g]$.

Let $g \in \mathcal{CAM}_c$ is critical metric of a constant scalar curvature, which is equal to $c$.

Definition 9.1. The set of critical associated metrics of a constant scalar curvature $c$, which are in slice $S_g \subset \mathcal{AM}$ at a point $g$, is called as a premoduli space of critical associated metrics of a constant scalar curvature in a neighbourhood $g \in \mathcal{AM}$.

Premoduli space will be denoted by a symbol $\mathcal{PM}(g)$,

$$\mathcal{PM}(g) = S_g \cap \mathcal{CAM}_c = \left( (\text{ARic} \times r)|_{S_g} \right)^{-1}(0, c).$$
The local moduli space is the quotient space $\mathcal{PM}(g)/I_\nu(g)$. It describes a local structure of the space $\mathcal{CAM}_c/\mathcal{G}$ in a neighbourhood of class $[g] = g\mathcal{G}$.

Let $\mathcal{PM}^s(g) = S^c_\mathcal{G} \cap \mathcal{CAM}^c_\mathcal{G}$ be premoduli space of critical metrics of a constant curvature of Sobolev class $H^s$, $s \geq 2n + 5$.

**Theorem 9.1.** Let $g \in \mathcal{CAM}_c$, then for any $s \geq 2n + 5$ there is a neighbourhood $W^s$ of the element $g$ in slice $S^c_\mathcal{G}$ such, that the space $\mathcal{PM}^s(g) \cap W^s$ is an analytical set of finite-dimensional real-analytic submanifold $Z \subset W^s$, whose tangent space $T_gZ$ has dimension independent on $s$ and consists of the anti-Hermitian $2$-forms of $C^\infty$-class.

For a proof we will need some facts about analytic mappings of Hilbert spaces. Recall, that the mapping $f$, defined on an open set $U$ of Hilbert space $E_1$ in a Hilbert space $E_2$ is called real-analytic, if in a neighbourhood of each point it is represented by a convergent power series.

Let $E_1$, $E_2$ are complex Hilbert spaces and $U \subset E_1$ is open set. The mapping $f : U \rightarrow E_2$ is called holomorphic, if $f$ is of $C^1$-class and in each point $x \in U$ the differential $df(x)$ commutes with complex structures $J_1$ and $J_2$ on $E_1$ and $E_2$.

We cite several known statements [25].

**Proposition 1.** Let $E_1$ and $E_2$ are complex Hilbert spaces and $U \subset E_1$ is open set. The holomorphic mapping $f : U \rightarrow E_2$ is real-analytic.

**Proposition 2.** Let $E_1$ and $E_2$ are real Hilbert spaces and $E^C_1$, $E^C_2$ are their complexifications. Let $U \subset E_1$ is open set and $f : U \rightarrow E_2$ is real-analytic mapping. Then there exists an open set $U^C \subset E^C_1$ including $U$ and such that $f$ extends to a holomorphic mapping $f^C : U^C \rightarrow E^C_2$.

**Proposition 3**[33]. Let $E_1$ and $E_2$ are real Hilbert spaces and $f$ is real-analytic mapping from $E_1$ in $E_2$, defined on an open neighbourhood of zero $0 \in E_2$. Suppose, that $f(0) = 0$ and image of a differential $df(0)$ is closed in $E_2$. Then there is an open neighbourhood $U$ of zero in $E_1$ such, that the set $f^{-1}(0) \cap U$ is a real-analytic set in a real-analytic submanifold $Z$ from $U$, the tangent space $T_0Z$ of which coincides with $\text{Ker}df(0)$.

**Lemma 9.1.** Let $E$, $F$ are vector bundles over a manifold $M$ and $E^C$, $F^C$ are their complexifications and $\Gamma^s(E)$, $\Gamma^s(F)$ are spaces of sections of the Sobolev class $H^s$. Let $U^s \subset \Gamma^s(E)$ is open set and $\Psi : U^s \rightarrow \Gamma^{s-k}(F)$, $s \geq \frac{n}{2} + k + 1$ is $C^\infty$-smooth mapping possessing a property: for any point $x \in M$ there is a neighbourhood $V_x \subset M$ of this point, such, that $\Psi$ defines mapping $\Psi|V_x : U^s \cap \Gamma^s(E|V_x) \rightarrow \Gamma^{s-k}(F|V_x)$ of a class $C^\infty$, which extends to holomorphic mapping

$$
\Psi^C|V_x : (U^s)^C \cap \Gamma^s(E^C|V_x) \rightarrow \Gamma^{s-k}(F^C|V_x),
$$

where $(U^s)^C$ is open set in $\Gamma^s(E^C)$, containing $U^s$. Then mapping $\Psi : U^s \rightarrow \Gamma^{s-k}(F)$ is real-analytic.

**Proof.** Let $J_E$ and $J_F$ are operators of complex structures in complexified fiber bundles $E^C$ and $F^C$. They define complex structures in Hilbert spaces of sections $\Gamma^s(E^C)$ and
\[ \Gamma^{s-k}(FC), \text{ which we shall designate by the same symbols } J_E \text{ and } J_F. \text{ Show, that the mapping } \Psi : (U^s)^C \rightarrow \Gamma^{s-k}(FC) \text{ is holomorphic, i.e. for any section } u \in (U^s)^C \text{ the equality } d\Psi(J_E(u)) = J_F(d\Psi(u)) \text{ is held. The operators } J_E \text{ and } J_F \text{ on the spaces } \Gamma^s(EC) \text{ and } \Gamma^s(FC) \text{ acts pointwise:} \\
\]

\[ J_E(u)(x) = J_E(u(x)), \quad J_F(v)(x) = J_F(v(x)), \]

therefore equality \( d\Psi(J_E(u)) = J_F(d\Psi(u)) \) needs to be only checked up at each point \( x \in M \):

\[ d\Psi(J_E(u))(x) = J_F(d\Psi(u))(x) \quad x \in M. \]

But the last is held on a condition in a neighbourhood \( V_x \) of each point \( x \in M \).

**Corollary 1.** If the mapping \( \Psi : U^s \rightarrow \Gamma^s(FC), \ s \geq \frac{n}{2} + 1, \) has a view

\[ \Psi(u) = \psi \circ u, \]

where \( \psi : E \rightarrow F \) is saving fibers map of class \( C^\infty \), defined on an open set in \( E \) and extendible to a mapping \( \psi^C : EC \rightarrow FC \), such that its restriction \( \psi^C \) on each fiber \( E^C_x \) is holomorphic, then \( \Psi \) is real-analytic mapping.

It follows from here, that the tensor operations (convolution, raising of an index etc.) determine analytical mappings of spaces of sections.

The partial derivative of a tensor field \( u \) with respect to coordinate \( x^i \) on base of \( M \) represents a linear operation in a neighbourhood of a point \( x \in M \). Therefore it extends to holomorphic mapping of complex fields \( u \) in a neighbourhood of a point \( x \).

**Corollary 2.** If the mapping \( \Psi : U^s \rightarrow \Gamma^{s-k}(F) \) in local coordinates in a neighbourhood of each point \( x \in M \) is analytically expressed through tensor operations and partial derivatives of sections \( u \in U^s \) to the order \( k \), then \( \Psi \) is real-analytic mapping.

In particular, mapping

\[ ARic \times r : \mathcal{M}^s \rightarrow S_{2A}^{s-2} \times H^{s-2}(M, R) \]

is real-analytic mapping at \( s \geq \frac{n}{2} + 3 \), since it is locally analytically expressed through the second partial derivatives of a metric tensor \( g \) and operations of convolution, raising of an index, taking of an anti-Hermitian part.

The slice \( S_g^s \) is a real-analytic submanifold in \( \mathcal{AM}^s \), as it is an image of a neighbourhood of zero of analytical mapping

\[ Exp_g : S_{2A}^{ss} \rightarrow \mathcal{AM}^s, \quad Exp_g(h) = g e^H, \]

where \( H = g^{-1} h \).

**Proof of the theorem 9.1.** Consider analytical mapping \( ARic \times r : \mathcal{AM}^s \rightarrow S_{2A}^{s-2} \times H^{s-2}(M, R) \) and take its restriction on an analytical submanifold \( S_g^s, g \in \mathcal{AM}, \)

\[ ARic \times r|_{S_g^s} : S_g^s \rightarrow S_{2A}^{s-2} \times H^{s-2}(M, R). \]
The premoduli space $\mathcal{PM}^s(g)$ is a level set,

$$\mathcal{PM}^s(g) = \left( ARic \times r|_S^g \right)^{-1} (0, c).$$

Therefore we can apply a proposition 3. It is necessary to show, that an image of a differential $d \left( ARic \times r|_S^g \right) (g)$ is closed. At first we find differential $d(ARic(g))$ of mapping

$$ARic : \mathcal{AM}^s \longrightarrow S\mathcal{f}^{s-2}_{2A}.$$

Let $h \in T_g \mathcal{AM}^s$ and let $g_t$ is curve which is going out from $g$ in direction $h$ and $J_t$ is its corresponding set of associated almost complex structures. The tangent vector $I = \frac{d}{dt}|_{t=0} J_t$ is expressed through $h$: $I = J \circ H$, where $H = g^{-1}h$. Let $X, Y \in \Gamma(TM)$. Differentiating of equality with respect to $t$

$$ARic(g_t)(X, Y) = \frac{1}{2} (Ric(g_t)(X, Y) - Ric(g_t)(J_tX, J_tY)),$$

we obtain,

$$(d(ARic)(g; h))(X, Y) = A(dRic(g; h)(X, Y)) -$$

$$-\frac{1}{2} (Ric(g)(IX, JY) + Ric(g)(JX, IY)) =$$

$$= \frac{1}{2} A \left( \Delta_L h - 2\delta_g^* \delta_g h \right) (X, Y) - \frac{1}{2} (Ric(g)(IX, JY) + Ric(g)(JX, IY)), $$

where $A \left( \Delta_L h - 2\delta_g^* \delta_g h \right)$ is anti-Hermitian part, $\delta_g$ is covariant divergence, $\delta_g^*$ is operator, which is adjoint to $\delta_g$, $\Delta_L h = -\nabla_k \nabla^k h_{ij} + R_{ikj} h^k_j + R_{ijk} h^k_i - 2R_{ijkl} h^{kl}$ is Lichnerowicz Laplacian, we take into account, that $h \in T_g \mathcal{AM}^s$ and, therefore, $tr_h h = 0$, the expression for $dRic(g)$ is obtained in work [2], see also [3].

Consider, that the point $g \in \mathcal{AM}$, in which the differential is calculated, is the critical metric. Then the Ricci tensor $Ric(g)$ is Hermitian, therefore:

$$d(ARic)(g; h)(X, Y) =$$

$$= \frac{1}{2} A \left( \Delta_L h - 2\delta_g^* \delta_g h \right) (X, Y) - \frac{1}{2} (Ric(g)(HX, Y) + Ric(g)(X, HY)).$$

or, omitting arguments $X, Y$,

$$d(ARic)(g; h) = \frac{1}{2} A \left( \Delta_L h - 2\delta_g^* \delta_g h \right) - \frac{1}{2} \left( H^T \circ Ric(g) + Ric(g) \circ H \right). \tag{9.4}$$

We have obtained, that $dARic(g)$ is a differential operator of the second order.

The differential of the mapping $r : \mathcal{AM}^s \longrightarrow H^{s-2}(M, R)$ is known [3] to be

$$dr(g, h) = \Delta \ tr_g h + \delta_g \delta_g h - g(h, Ric(g)).$$

Since $h$ is anti-Hermitian and $Ric(g)$ is Hermitian, it follows that $tr_g h = 0$ and $g(h, Ric(g)) = 0$. Therefore,

$$dr(g; h) = \delta_g \delta_g h. \tag{9.5}$$
We have obtained, that the differential \( d(ARic \times r) \) of mapping \( ARic \times r : \mathcal{AM}^s \rightarrow S^{s-2}_{2A} \times H^{s-2}(M, \mathbb{R}) \) at a critical point \( g \in \mathcal{AM} \) is a differential operator of the second order

\[
d(ARic \times r)(g) : S^{s}_{2A} \rightarrow S^{s-2}_{2A} \times H^{s-2}(M, \mathbb{R}),
\]

\[
h \mapsto \left( \frac{1}{2} A(\Delta_L h - 2\delta_g^* \delta_g h) - \frac{1}{2} (H^T \circ Ric(g) + Ric(g) \circ H), \delta_g \hat{\partial} h \right).
\]

Since the mapping \( ARic \) is restricted on the slice \( S^s_g \), we should to assume

\[
h \in T_g S^s_g = \{h \in S^s_{2A}; \text{ div } J \hat{\partial} h = 0\} = S^s_{2A},
\]

i.e. we should impose the additional condition: \( \delta_g J \hat{\partial} h = 0 \).

The following lemma is a basis for the proof of the theorem 9.1.

**Lemma 9.2.** For any associated metric \( g \in \mathcal{AM} \) the differential operator

\[
F_g : S_{2A} \rightarrow S_{2A} \times C^\infty(M, \mathbb{R}) \times C^\infty(M, \mathbb{R}),
\]

\[
h \mapsto \left( \frac{1}{2} A(\Delta_L h - 2\delta_g^* \delta_g h) - \frac{1}{2} (H^T \circ Ric(g) + Ric(g) \circ H), \delta_g \hat{\partial} h, \delta_g J \hat{\partial} h \right)
\]

has an injective symbol.

**Proof.** Recall the definition of a symbol of a differential operator \( D : \Gamma(E) \rightarrow \Gamma(E)(F) \) of the order \( k \), where \( E, F \) are vector bundles above \( M \), and \( \Gamma(E), \Gamma(F) \) are spaces of their sections [?]. Let \( x \in M \) is an arbitrary point. For any covector \( \xi \in T^*_x M \) there is a function \( f \) on \( M \) such, that \( f(x) = 0 \) and \( df(x) = \xi \). Let \( h \in \Gamma(E) \) is section, then expression

\[
\sigma_\xi(D) h_x = \frac{1}{k!} D (f^k h)(x)
\]

depends only on section \( h(x) \) of \( h \) at a point \( x \) and, thus defines linear mapping \( \sigma_\xi(D) : E_x \rightarrow F_x \) of fibers above a point \( x \) of bundles \( E \) and \( F \), which is named as a symbol of an operator \( D \).

The symbol \( \sigma(D) \) is called injective, if \( \sigma_\xi(D) : E_x \rightarrow F_x \) is injective for everyone \( x \in M \) and everyone nonzero \( \xi \in T^*_x M \). Let \( x \in M \) is any fixed point. Show an injectivity of a symbol \( \sigma_\xi(F_g) \) of an operator \( F_g \).

To the associated metric \( g \) on \( M \) there corresponds an associated almost complex structure \( J \), and it defines decomposition \( TM^C = T^{10} \oplus T^{01} \) of complexification \( TM^C \) of a tangent bundle \( TM \).

Let \( \partial_1, \ldots, \partial_n \) is basis of sections of a bundle \( T^{10} \) in a neighbourhood of a point \( x \) and \( \overline{\partial}_1, \ldots, \overline{\partial}_n \) is its corresponding basis of sections of \( T^{01} \). Choose dual basis \( dz^1, \ldots, dz^n, d\overline{z}^1, \ldots, d\overline{z}^n \) of bundle \( T^* M^C = T^{10} \oplus T^{01} \) (we pay attention, that \( dz^k \) is simple notation, complex coordinates \( z^1, \ldots, z^n \) on \( M \) in a neighbourhood of a point \( x \in M \) can be not defined).
If \( g_{αβ} = g(δ_α, δ_β) \), for our Hermitian form \( g \) on \( M \), continued on a complexification \( TM^C \) we have, \( g = 2g_{αβ}dz^αdz^β \) and \( g_{αβ} = \overline{g_{βα}} \). Note, that if \( \tilde{g} = g_{αβ}dz^α \otimes dz^β \), then \( g = 2\text{Re} \tilde{g} \). Take an arbitrary anti-Hermitian form \( h \). In a local coframe it has a view:

\[
h = h_{αβ}dz^αdz^β + \overline{h}_{αβ}d\overline{z}^αd\overline{z}^β,
\]

where \( h_{αβ} = h(δ_α, δ_β) \), \( h_{αβ} = h_{βα} \). Let \( \tilde{h} = h_{αβ}dz^α \otimes dz^β \), then \( h = 2\text{Re} \tilde{h} \).

Note obvious equalities:

\[
2\tilde{g} = g - i\omega, \quad 2\tilde{h} = h + i\text{Im}(2\tilde{h}), \quad \text{Im}(2\tilde{h})(u, v) = -h(u, Jv).
\]

Remark, that a symbol of an operator

\[
F_g(h) = \left( \frac{1}{2}A(\Delta h - 2\delta^*_gδ_gh) - \frac{1}{2}(H^T \circ Ric(g) + Ric(g) \circ H), \delta_gδ_gh, δ_gJδ_gh \right)
\]

coincides with a symbol of an operator

\[
h \rightarrow \left( \frac{1}{2}A(\overline{\Delta}h - 2\delta^*_gδ_gh), \delta_gδ_gh, δ_gJδ_gh \right), \quad (9.6)
\]

where \( \overline{\Delta} \) is Laplacian.

Let \( \nabla_α = \nabla_{δ_α} \) and \( \nabla_τ = \nabla_{\overline{δ}_α} \); then for the anti-Hermitian form \( h = h_{αβ}dz^αdz^β + \overline{h}_{αβ}d\overline{z}^αd\overline{z}^β \) we have:

\[
(\Delta h)_{αβ} = -\nabla_γ\nabla_αh_{βγ} + \nabla_\overline{γ}\nabla_\overline{α}h_{β\overline{γ}},
\]

\[
\delta_gh = - (\nabla_γh_{αγ}, \nabla_\overline{γ}h_{α\overline{γ}}),
\]

\[
Jδ_gh = -i(\nabla_γh_{αγ}, -\nabla_\overline{γ}h_{α\overline{γ}}),
\]

\[
δ_γδ_gh = \nabla^α\nabla_αh_{γγ} + \nabla^\overline{α}\nabla_\overline{α}h_{γ\overline{γ}},
\]

\[
δ_γJδ_gh = i(\nabla^α\nabla_αh_{γγ} - \nabla^\overline{α}\nabla_\overline{α}h_{γ\overline{γ}}),
\]

\[
A(δ^*_gδ_gh) = \frac{1}{2}\begin{pmatrix}
\nabla_α\nabla_γh_{βγ} + \nabla_β\nabla_γh_{αγ} & 0 \\
0 & \nabla_τ\nabla_\overline{τ}h_{β\overline{γ}} + \nabla_\overline{τ}\nabla_τh_{β\overline{γ}}
\end{pmatrix},
\]

Therefore for \( ξ = ξ_αdz^α + ξ_τd\overline{z}^τ \) we obtain,

\[
σ_ξ(\overline{\Delta})(h) = -ξ^γξ_γh_{αβ} - ξ^\overline{τ}ξ_\overline{τ}h_{αβ}.
\]

\[
σ_ξ(Aδ^*_gδ_g)(h) = \frac{1}{2}\begin{pmatrix}
ξ_αξ^γh_{βγ} + ξ_βξ^γh_{αγ} & 0 \\
0 & ξ_τξ^\overline{τ}h_{β\overline{γ}} + ξ_\overline{τ}ξ^\overline{τ}h_{β\overline{γ}}
\end{pmatrix},
\]

\[
σ_ξ(δ_gδ_gh) = ξ^αξ_εh_{αγ} + ξ^\overline{τ}ξ_εh_{α\overline{τ}} = h(ξ, ξ),
\]

\[
σ_ξ(δ_γJδ_gh) = i(ξ^αξ_εh_{αγ} - ξ^\overline{τ}ξ_εh_{α\overline{τ}}) = -2i\text{Im}h(ξ, ξ) = h(ξ, Jξ),
\]

We identify a covector \( ξ \in T_x^*M \) to a vector from \( T_xM \) with the help of metric tensor \( g \).
Our task is to show, that if \( \sigma_\xi(F_g)(h) = 0 \) for nonzero \( \xi \), then \( h = 0 \). Assume, that \( h \) satisfies to a condition \( \sigma_\xi(F_g)(h) = 0 \) for nonzero \( \xi \in T_x^*M \). In this case,

\[
h(\xi, \xi) = 0, \quad h(\xi, J\xi) = 0
\]

and for any vector \( u \in T_xM \) the equality is held

\[
\sigma_\xi A (\overline{\Delta h} - 2\delta_g^*\delta_g h) (u, u) = 0.
\]

Calculate the left part.

\[
\sigma_\xi A (\overline{\Delta h} - 2\delta_g^*\delta_g h) (u, u) = \]

\[
= -g(\xi, \xi)h(u, u) + \xi_\alpha u^\alpha h_{\beta\gamma} u^\beta \xi_f^\gamma + \xi_\beta u^\beta h_{\alpha\gamma} u^\alpha \xi_f^\gamma + \xi_\alpha u^\alpha \overline{h_{\beta\gamma} u^\beta \xi_f^\gamma} + \xi_\beta u^\beta \overline{h_{\alpha\gamma} u^\alpha \xi_f^\gamma} = \]

\[
= -g(\xi, \xi)h(u, u) + 2\overline{\xi}(\xi, u)\overline{h(u, \xi)} + 2\overline{g}(\xi, u)\overline{h(u, \xi)} = \]

\[
= -g(\xi, \xi)h(u, u) + 2\text{Re}\left(2\overline{g}(\xi, u)\overline{h(u, \xi)}\right) = \]

\[
= -g(\xi, \xi)h(u, u) + g(\xi, u)h(u, \xi) + \omega(\xi, u)\text{Im}(h(u, \xi)) = \]

\[
= -g(\xi, \xi)h(u, u) + g(\xi, u)h(u, \xi) - \omega(\xi, u)h(u, J\xi).
\]

Therefore for any vector \( u \in T_xM \), we obtain:

\[
-g(\xi, \xi)h(u, u) + g(\xi, u)h(u, \xi) - \omega(\xi, u)h(u, J\xi) = 0.
\]

Taking into account, that \( -\omega(\xi, u) = -g(J\xi, u) \), we rewrite the last equality as:

\[
-g(\xi, \xi)h(u, u) + g(\xi, u)h(u, \xi) - g(J\xi, u)h(u, J\xi) = 0.
\]  \( (9.7) \)

It is held for any \( u \in T_xM \). If the vector \( u \) is orthogonal to vectors \( \xi \) and \( J\xi \), two last addends vanish and we obtain,

\[
-g(\xi, \xi)h(u, u) = 0,
\]

so \( h(u, u) = 0 \) for any vector \( u \), orthogonal to two-dimensional subspace \( \mathbb{R}\{\xi, J\xi\} \) spanned by the vectors \( \xi \) and \( J\xi \). If \( u = \xi \) or \( u = J\xi \), the equality \( (9.7) \) is held for any form \( h \). However we have two equalities: \( h(\xi, \xi) = 0, h(\xi, J\xi) = 0 \). Taking into account, that \( h(J\xi, J\xi) = -h(\xi, \xi) \), we obtain, that the 2-form \( h \) vanishes on a two-dimensional subspace \( \mathbb{R}\{\xi, J\xi\} \). Thus, the 2-form \( h \) vanishes on all the space \( T_x^*M \). Therefore \( h = 0 \). The lemma is proved.

**Ending of proof of the theorem 9.1.** The differential operator

\[
F_g : S_{2A} \longrightarrow S \times C^\infty(M, \mathbb{R}) \times C^\infty(M, \mathbb{R}),
\]

\[
h \longrightarrow (d(ARic)(g; h), dr(g; h), \delta_g J\delta_g h)
\]

has an injective symbol. Therefore its kernel \( \text{Ker } F_g \subset S_{2A}^\circ \) is finite-dimensional and consists of forms \( h \) of class \( C^\infty \). It follows from an ellipticity of an operator \( F_g^* \circ F_g : S_{2A}^\circ \longrightarrow S_{2A}^\circ \).

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Besides the following Berger-Ebin decomposition in a direct sum of the closed orthogonal subspaces [5], takes place:

\[ S^{s-2}_{2A} \times H^{s-2}(M, \mathbb{R}) \times H^{s-2}(M, \mathbb{R}) = \text{Im}(F_g) \oplus \ker(F_g^*). \] (9.8)

Consider mapping

\[ ARic \times r : S^*_g \rightarrow S^{s-2}_{2A} \times H^{s-2}(M, \mathbb{R}). \]

The tangent space \( T_g S^*_g \) consists of 2-forms \( h \in S^{s}_{2A} \), satisfying to a condition: \( \text{div} J_{\delta g} h = 0 \). Since

\[ F_g(h) = (d(ARic \times r)(g; h), \text{div} J_{\delta g} h), \]

then the image of a differential \( d(ARic \times r)_g(T_g S^*_g) \) coincides with an image of a subspace \( \{ h \in S^s_{2A}; \text{div} J_{\delta g} h = 0 \} \) under the action of the operator \( F_g \). Therefore

\[ d(ARic \times r)_g(T_g S^*_g) = \text{Im}(F_g) \cap \left( S^{s-2}_{2A} \times H^{s-2}(M, \mathbb{R}) \times \{0\} \right). \]

So the image is closed as intersection of two closed subspaces. Designate \( F^{s-2} = d(ARic \times r)_g(T_g S^*_g) \). Let

\[ p : S^{s-2}_{2A} \times H^{s-2}(M, \mathbb{R}) \times \{0\} \rightarrow F^{s-2} \]

is orthogonal projection on the closed subspace (in correspondence with decomposition (9.8)), assume \( q = \text{id} - p \). The mapping \( p \circ (ARic \times r) : S^*_g \rightarrow F^{s-2} \) is analytical and its differential at a point \( g \) maps tangent space \( T_g S^*_g \) onto the whole space \( F^{s-2} \). By the implicit function theorem for analytical mapping of Hilbert manifolds there is a neighbourhood \( W^s \) of the element \( g \) in the slice \( S^*_g \) such, that the set \( Z = (P \circ (ARic \times r))^{-1}(0) \cap W^s \) represents a real-analytic submanifold in \( W^s \). Moreover the tangent space \( T_g Z \) coincides with the kernel \( \ker (d(ARic \times r)_g) = \ker F_g \) of an operator \( F_g \), which is finite-dimensional and consists of the forms \( h \) of class \( C^\infty \). Therefore \( Z \) is finite-dimensional analytical submanifold in \( S^*_g \). Now

\[ \mathcal{PM}^s(g) \cap Wof^s = (q \circ (ARic \times r)|Z)^{-1}(0) \]

is analytical set in \( Z \) as the inverse image of zero under the analytical mapping \( q \circ (ARic \times r) : Z \rightarrow (F)^\perp \). The theorem is proved.
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