Remarks on the existence of bilaterally symmetric extremal Kähler metrics on $\mathbb{CP}^2\#2\overline{\mathbb{CP}^2}$

He, Weiyong
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The study of extremal Kähler metric is initiated by the seminal works of Calabi [4], [5]. Let $(M, [\omega])$ be a compact Kähler manifold with fixed Kähler class $[\omega]$. For any Kähler metrics $g$ in the fixed Kähler class $[\omega]$, the Calabi energy $C(g)$ is defined as

$$C(g) = \int_M s^2 d\mu,$$

where $s$ is the scalar curvature of $g$. The extremal Kähler metric is the critical point of the Calabi energy. The Euler-Lagrange equation is

$$\bar{\partial} \nabla_1^0 s = 0.$$ 

In other words, $\Xi = \nabla_1^0 s$ is a holomorphic vector field (we call it extremal vector field from now on). From PDE point of view, the existence of the extremal metric is to solve a 6th order nonlinear elliptic equation. According to Chen [6] (c.f. Donaldson [9] for algebraic case), there is a priori greatest lower bound for the Calabi energy in any fixed Kähler class. This a priori lower bound can be computed explicitly as

$$A([\omega]) = \left( c_1 \cdot [\omega] \right)^2 \frac{1}{[\omega]^2} - \frac{1}{32\pi^2} \mathcal{F}(\Xi, [\omega]),$$

where $\mathcal{F}(\Xi, [\omega])$ is the Futaki invariant of class $[\omega]$. Note that the extremal vector field $\Xi$ is determined [10] up to conjugation without the assumption of the existence of an extremal metric.

By E. Calabi [5], extremal Kähler metrics minimizes the Calabi energy locally. By X.X. Chen [6] and S.K. Donaldson [9], we know

$$A([\omega]) \leq \frac{1}{32\pi^2} \min_{g \in [\omega]} C(g),$$

where the equality holds when there is an extremal Kähler metric in $[\omega]$.

In an amazingly beautiful work, Chen-LeBrun-Weber [8] proved the existence of bilaterally symmetric extremal Kähler metrics on $\mathbb{CP}^2\#2\overline{\mathbb{CP}^2}$ by global
deformation method. More strikingly, it contains an extremal class where the extremal metric is conformal to an Einstein metric with positive scalar curvature. \( \mathbb{CP}^2 \# 2 \mathbb{CP}^2 \) can be also described as \( \mathbb{CP}^1 \times \mathbb{CP}^1 \) blowing up at one point. We use \( F_1, F_2 \) to denote the Poincaré dual of two factors \( \mathbb{CP}^1 \times \mathbb{CP}^1 \), and \( E \) denotes the exceptional divisor. The term "bilaterally symmetric" is introduced in [8] to describe the Kähler class which are invariant under the interchange \( F_1 \leftrightarrow F_2 \). The "bilaterally symmetric" class can be described by \( \omega_x = (1 + x)(F_1 + F_2) - xE \) for \( 0 < x < \infty \). Let \( f(x) = \mathcal{A}(\omega_x) \), and it is shown that \( f(x) < 9 \) (c.f. [8]). Set \( L \) to be the smallest number of \( f^{-1}(8) \), Chen-LeBrun-Weber [8] proved the following theorem regarding the existence of extremal Kähler metrics

**Theorem A** [8] For any \( x \in (0, L) \), let \( \omega_x = (1 + x)(F_1 + F_2) - xE \) denote the Kähler class of on \( M = \mathbb{CP}^2 \# 2 \mathbb{CP}^2 \), then there is an extremal metric in \( \omega_x \) for any \( x \in (0, L) \).

Their method is through large scale deformation. The existence of extremal Kähler metrics is promised by the results of Arezzo-Pacard-Singer [2] when \( x \) is small enough (also for \( x \) big enough). According to LeBrun-Simanca [11], the set which admits extremal Kähler metric is open. Following the work of Chen-Weber [7] on moduli space of extremal Kähler metrics in complex surface, a sequence of bilaterally symmetric extremal metrics will converge to an extremal metric with finite orbifold points. However the orbifold singularities can only arise as a very specific mechanism of curvature concentration for critical metrics [1], [12], [7]. The key idea of Chen-LeBrun-Weber [8] is thorough careful analysis of the bubble formation and they conclude that, for bubble to arise, the original Kähler class must admit some Lagrange cycle with negative self-intersection number. And they show that when \( f(x) < 8 \), there is no such Lagrange cycle. It follows that the orbifold singularities will never occur.

Inspired by the idea of [8], we extend their result to show that the existence of bilaterally symmetric extremal Kähler metrics on \( \mathbb{CP}^2 \# 2 \mathbb{CP}^2 \) for any \( x \in (0, \infty) \) in this short note. The readers are enthusiastically referred to [8] for the historic background of this problem as well as an excellent list of references. Following the scheme in ([8]), we show that

**Theorem 0.1.** For any \( x \in (0, \infty) \), let \( \omega_x = (1 + x)(F_1 + F_2) - xE \) denote the Kähler class of on \( M = \mathbb{CP}^2 \# 2 \mathbb{CP}^2 \), then there is an extremal metric in \( \omega_x \) for any \( x \in (0, \infty) \).

We keep the notations of [8]. Our observation is that, without assuming \( \mathcal{A}(\omega_x) < 8 \), the proposition ([8], Proposition 26) still holds.

**Proposition 0.2.** Let \( g_i \) be a sequence of unit-volume bilaterally symmetric extremal Kähler metrics on \( (M, J) = \mathbb{CP}^2 \# 2 \mathbb{CP}^2 \) such that the corresponding Kähler class

\[
[\omega_i] = \frac{(1 + x_i)(F_1 + F_2) - x_i E}{\sqrt{1 + 2x_i + x_i^2/2}}
\]
satisfy $A \leq x_i \leq B$, where $A < B$ are any two fixed positive number. Then there is a subsequence $g_i$ of metrics and a sequence of diffeomorphisms $\Psi_j : M \to M$ such that $\Psi_j^* g_i$ converges in the smooth topology to an extremal Kähler metric on the smooth 4-manifold $M$ compatible with some complex structure $\tilde{J} = \lim_{j \to \infty} \Psi_j^* J$.

Recall for a compact smooth 4-manifold $(M, g)$ the Gauss-Bonnet formula says
\[
\frac{1}{8\pi^2} \int_M \left( |W_+|^2 + |W_-|^2 + \frac{s^2}{24} - \frac{|\text{Ric}_0|^2}{2} \right) d\mu = \chi(M)
\]
and the signature formula reads
\[
\frac{1}{12\pi^2} \int_M (|W_+|^2 - |W_-|^2) d\mu = \tau(M).
\]
If $(X, g_{\infty})$ is any ALE 4-manifold with finite group $\Gamma \subset SO(4)$ at infinity, then the Gauss-Bonnet formula becomes
\[
\frac{1}{8\pi^2} \int_X \left( |W_+|^2 + |W_-|^2 + \frac{s^2}{24} - \frac{|\text{Ric}_0|^2}{2} \right) d\mu_{g_{\infty}} = \chi(X) - \frac{1}{|\Gamma|}
\]
and the signature formula becomes
\[
\frac{1}{12\pi^2} \int_X (|W_+|^2 - |W_-|^2) d\mu_{g_{\infty}} = \tau(X) + \eta(S^3/\Gamma),
\]
where $\chi(X)$ is the Euler characteristic of non-compact manifold $X$ and $\eta(S^3/\Gamma)$ is called $\eta$ invariant. When $(X, g_{\infty})$ is scalar flat Kähler, the formulas simplify to
\[
\frac{1}{8\pi^2} \int_X \left( |W_-|^2 - \frac{|\text{Ric}_0|^2}{2} \right) d\mu_{g_{\infty}} = \chi(X) - \frac{1}{|\Gamma|}
\]
and
\[
-\frac{1}{12\pi^2} \int_X |W_-|^2 d\mu_{g_{\infty}} = \tau(X) + \eta(S^3/\Gamma).
\]

Our first observation is that the lemmas ([8], Lemma 21 and Lemma 22) hold without the assumption on $\mathcal{A}(\omega)$.

**Lemma 0.3.** $(X, g_{\infty})$ is the deepest bubble. Then $X$ is diffeomorphic to a region of $M$ which is invariant under $F_1 \leftrightarrow F_2$, and this $\mathbb{Z}_2$ action induces a holomorphic isometric involution of $(X, g_{\infty})$.

**Proof.** By the signature formula, we have that
\[
\int_M |W_-|^2 d\mu = -12\tau(M) + \int_M |W_+|^2 d\mu = 12\pi^2 + \int_M \frac{s^2}{24} d\mu
\]
for any Kähler metrics on $M = \mathbb{CP}^2\#2\overline{\mathbb{CP}^2}$. For any bilaterally symmetric Kähler class $[\omega]$ on $\mathbb{CP}^2\#2\overline{\mathbb{CP}^2}$, $A([\omega]) < 9$. Thus any bilaterally symmetric extremal Kähler metrics satisfy
\[ \int_M |W_-|^2 d\mu < 12\pi^2 + \frac{9}{24}32\pi^2 = 24\pi^2. \]

When $|\Gamma| \geq 2$, since $b_1(X) = b_3(X) = 0$ and $b_2(X) > 0$. Hence $\chi(X) \geq 2$, and the Gauss-Bonnet formula gives that
\[ \int_X |W_-|^2 d\mu_{g_\infty} \geq 8\pi^2(2 - 1/2) \geq 12\pi^2. \]

When $|\Gamma| = 1$, the signature formula gives that
\[ \int_X |W_-|^2 d\mu = 12\pi^2. \]

And then the same argument of ([8] Lemma 21) applies.

Lemma 0.4. Let $(X, g_\infty)$ be the deepest bubble. If $b_2(X) = 1$, then $X$ must be diffeomorphic to the line bundle of degree $-k$ over $\mathbb{CP}^3$ for $1 \leq k \leq 5$.

Proof. The proof follows ([8], Lemma 23). Since $X$ is diffeomorphic to the line bundle of degree $-k$ over $\mathbb{CP}^3$ for some $k > 0$. If $C$ denotes the homology class of the zero section, the Poincaré dual of $c_1$ is the rational homology class $\frac{k-2}{k}C$ and it follows that
\[ \int_X |\text{Ric}_0|^2 d\mu_{g_\infty} = -8\pi^2c_1^2 = 8\pi^2\left(\frac{k-2}{k}\right)^2. \]

Any bilaterally symmetric extremal Kähler metrics satisfy
\[ \int_M |\text{Ric}_0|^2 d\mu = \frac{1}{4} \int_M s^2 d\mu - 8\pi^2c_1^2(M) < 16\pi^2. \]

It follows that $k \leq 5$.

([8] Lemma 22) holds also.

Lemma 0.5. Let $(X, g_\infty)$ be the deepest bubble. If $b_2(X) = 2$, then $\Gamma \cong \mathbb{Z}_3$, and $X$ has intersection form
\[ \begin{pmatrix} -2 & 1 \\ 1 & -2 \end{pmatrix}. \]

Proof. Since $b_2(M_\infty) = 2$, the Gauss-Bonnet and signature formula give that
\[ \frac{1}{12\pi^2} \int_X |W_-|^2 d\mu_{g_\infty} = 2 - \eta(S^2/\Gamma) \]

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and
\[ \frac{1}{8\pi^2} \int_X \left( |W_\mu|^2 - \frac{|Ric_0|^2}{2} \right) d\mu_{g_{\infty}} = 3 - \frac{1}{|\Gamma|}. \]

It follows that
\[ \frac{3}{2} \eta(S^3/\Gamma) + \frac{1}{16\pi^2} \int_X |Ric_0|^2 d\mu_{g_{\infty}} = \frac{1}{|\Gamma|}, \quad (0.1) \]

And we know that
\[ \int_M |W_\mu|^2 d\mu = \frac{12\pi^2}{24} + \int_M \frac{s^2}{24} d\mu < 24\pi^2, \]

it follows that
\[ \frac{1}{16\pi^2} \int_X \frac{|Ric_0|^2}{2} < \frac{1}{|\Gamma|} \]

and
\[ \eta(S^3/\Gamma) > 0. \]

Since Lemma 0.3 shows that we still have a $\mathbb{Z}_2$ action which interchanges the two totally geodesic $\mathbb{C}P^1$s which generate $H^2(X, \mathbb{Z})$. The argument in (8) Lemma 22 applies and so the intersection form of $X$ must be given by
\[ \left( \begin{array}{cc} -k & 1 \\ 1 & -k \end{array} \right) \]

for some $k \geq 2$ and $\Gamma \cong \mathbb{Z}_{k^2-1}$. And at infinity the 3-manifold is a Lens space $L(k^2 - 1, k)$. In particular $\Gamma \neq \{1\}$. Since $|\Gamma| \neq 1$, by (0.1) we get that
\[ \eta(S^3/\Gamma) \leq \frac{1}{3}. \]

It means that
\[ 0 < \eta(S^3/\Gamma) \leq \frac{1}{3}. \]

For the Lens space $L(k^2 - 1, k) = S^3/\Gamma$, the $\eta$-invariant is given by [3],
\[ \eta(S^3/\Gamma) = \frac{1}{|\Gamma|} \sum_{i=1}^{k^2-2} \cot \frac{i\pi}{k^2 - 1} \cot \frac{ki\pi}{k^2 - 1} \]
\[ = \frac{1}{k^2 - 1} \left( \frac{2}{3} k^3 - 2k^2 + 2 \right). \quad (0.2) \]

It follows that $k = 2$ and $\Gamma \cong \mathbb{Z}_3$.

**Remark 0.6.** In this case, one can calculate the first Chern class in stead of the $\eta$-invariant as in Lemma 0.4. And the Poincaré dual of the first Chern class is the rational homology class
\[ \frac{k - 2}{k - 1} (E_1 + E_2), \]
where $E_1, E_2$ are two totally geodesic $\mathbb{CP}^1$ and they have intersection form

$$\begin{pmatrix} -k & 1 \\ 1 & -k \end{pmatrix}.$$ 

But the calculation of the eta-invariant will have independent interest for lens spaces. The formula is given by [3]. We carry out the example for lens spaces $L(k^2 - 1, k)$.

**Lemma 0.7.** Under the assumption of Proposition 0.2, for any $A, B$ fixed, $X$ cannot be as in Lemma 0.4 and Lemma 0.5.

**Proof.** The proof follows exactly ([8] Lemma 25). Since the limit metric $g_\infty$ on $X$ is by construction a pointed limit of larger and larger rescalings of the metrics $g_i$, the generators of $H_2(X, \mathbb{Z})$ must arise from smooth 2-sphere $S_i \subset M$ whose areas with respect to $g_i$ tend to zero as $i \to \infty$. When $b_2(X) = 1$, let $S_i$ be the smooth 2-sphere corresponding to the zero section $\mathbb{CP}^1$; when $b_2(X) = 2$, let $S_i$ be a 2-sphere corresponding to one of the two $\mathbb{CP}^1$ generators, and $\tilde{S}_i$ is the reflection under $F_1 \leftrightarrow F_2$. Take $\Sigma = [S_i] \in H^2(M, \mathbb{Z})$ when $b_2(X) = 1$, and $\Sigma = [S_i] + [\tilde{S}_i]$ when $b_2(X) = 2$. Since the homology class is $\mathbb{Z}_2$ invariant, we have

$$[S] = m(F_1 + F_2) + nE$$

for some integers $m$ and $n$ and the self-intersection condition gave that

$$2m^2 - n^2 = -k$$

for $k \leq 5$. Now any of unit-volume bilaterally symmetric Kähler classes $[\omega_i]$ is of the form

$$[\omega_i] = \frac{(1 + x_i)(F_1 + F_2) - x_i E}{\sqrt{1 + 2x_i + x_i^2/2}},$$

where $A \leq x_i \leq B$.

Also we know that the area of $S_i$ measured by $g_i$ goes to zero when $i \to \infty$. By Wirtinger’s inequality we can get

$$|[\omega_i]| \leq 2 \text{ area}(S_i) \to 0.$$ 

It follows that

$$\frac{2m(1 + x_i) + nx_i}{\sqrt{1 + 2x_i + x_i^2/2}} \to 0.$$ 

Denote

$$\frac{2m(1 + x_i) + nx_i}{\sqrt{1 + 2x_i + x_i^2/2}} = \varepsilon_i,$$

we can get that

$$n = -2m \frac{1 + x_i}{x_i} + \varepsilon_i \frac{\sqrt{1 + 2x_i + x_i^2/2}}{x_i}.$$
Since \( \sqrt{1 + 2x_i + x_i^2/2} \) is uniformly bounded for \( x_i \in [A, B] \) and \( \epsilon_i \to 0 \) when \( i \to 0 \), then

\[
4m^2 \left( \frac{(1 + x_i)^2}{x_i^2} - \epsilon_i C(\epsilon, A, B) \right) \leq n^2 \leq 4m^2 \left( \frac{(1 + x_i)^2}{x_i^2} + \epsilon \right) + \epsilon_i C(\epsilon, A, B)
\]

where \( \epsilon \) is arbitrary small positive number and \( C(\epsilon) \) is independent of \( i \). We can take \( \epsilon = \frac{1}{100} \) and when \( i \) big enough, \( C(1/100, A, B)\epsilon_i < 1/100 \), then it gives that

\[
4m^2 \left( \frac{1 + x}{x^2} \right) - 2m^2 - \frac{m^2}{100} \leq k + 1/100.
\]

It follows that \( (2 - 1/100)m^2 < k + 1/100 \).

Since \( k \leq 5 \), it gives that \( m = 0, \pm 1 \). But \( m = 0 \) gives that \( n = 0 \), contradiction. If \( m = 1 \), then \( k = 2, n = -2 \). And \( m = -1 \), then \( k = 2, n = 2 \). For any cases,

\[
|\epsilon_i| = \left| \frac{2m(1 + x_i) + nx_i}{\sqrt{1 + 2x_i + x_i^2/2}} \right| = \frac{2}{\sqrt{1 + 2x_i + x_i^2/2}}
\]

is uniformly bounded for \( x_i \in [A, B] \). Contradiction. \( \square \)

Deepest bubbles can therefore never arise, Proposition 0.2 follows. By using the result ([8], Theorem 27), Proposition 0.3 implies that the existence of bilaterally symmetric extremal Kähler metrics in the bilaterally symmetric Kähler class for any \( x \in [A, B] \).

1 Appendix

Here we prove the identity in (0.2.)

\[
\sum_{i=1}^{k^2-2} \cot \frac{i\pi}{k^2-1} \cot \frac{ki\pi}{k^2-1} = \frac{2}{3}k^3 - 2k^2 + 2. \tag{1.1}
\]

and it follows that the eta-invariant for lens space \( L(k^2 - 1, k) \) is

\[
-\frac{1}{k^2-1} \left( \frac{2}{3}k^3 - 2k^2 + 2 \right).
\]

Lemma 1.1. \( k \in \mathbb{N} \),

\[
\sin (k+1)x = 2^k \prod_{i=0}^{k} \sin \left( x + \frac{i\pi}{k+1} \right). \tag{1.2}
\]
Proof.

\[ 2 \sin x = i(e^{-ix} - e^{ix}) = 1e^{-ix}(1 - e^{2ix}). \]

It follows that

\[ \prod_{i=0}^{k} 2^{k+1} \sin \left( x + \frac{i\pi}{k+1} \right) = \prod_{i=0}^{k} \left\{ ie^{-i(x + \frac{i\pi}{k+1})}(1 - e^{2i(x + \frac{i\pi}{k+1})}) \right\} \]

\[ = (i)^{k+1} e^{-(k+1)x - \frac{k\pi}{2} + 2i(k+1)x} \prod_{i=0}^{k} (e^{-2ix} - e^{ik\pi}) \]

\[ = 1(e^{-(k+1)x} - e^{(k+1)x}) \]

\[ = 2 \sin(k+1)x. \]

Lemma 1.2.

\[(k+1) \cot(k+1)x = \sum_{i=0}^{k} \cot \left( x + \frac{i\pi}{k+1} \right).\]

Proof. Taking derivative on both sides of (1.2), it gives that

\[(k+1) \cos(k+1)x = 2^k \cos \left( x + \frac{j\pi}{k+1} \right) \prod_{i\neq j} \sin \left( x + \frac{i\pi}{k+1} \right).\]

Then divided by (1.1), we get

\[(k+1) \cot(k+1)x = \sum_{i=0}^{k} \cot \left( x + \frac{i\pi}{k+1} \right).\]

Lemma 1.3.

\[(k+1)^2 \cot^2(k+1)x + (k+1)k = \sum_{i=0}^{k} \cot^2 \left( x + \frac{i\pi}{k+1} \right).\]

Proof. In Lemma 1.2, taking derivative on both sides.

Lemma 1.4.

\[ \sum_{i=1}^{k} \cot^2 \frac{i\pi}{k+1} = \frac{k(k-1)}{3}. \]

Proof. In Lemma 1.3, by taking limit for \( x \to 0. \)

Now we can prove (1.1).
Proof. When $i$ is not the multiple of $k - 1$, then
\[
\cot \frac{i\pi}{k^2 - 1} \cot \frac{ki\pi}{k^2 - 1} = 1 + \cot \left( \frac{i\pi}{k^2 - 1} + \frac{ki\pi}{k^2 - 1} \right) \left( \cot \frac{i\pi}{k^2 - 1} + \cot \frac{ki\pi}{k^2 - 1} \right)
\]
\[
= 1 + \cot \frac{i\pi}{k-1} \left( \cot \frac{i\pi}{k^2 - 1} + \cot \frac{ki\pi}{k^2 - 1} \right).
\]
For each $j \in \{1, 2, \cdots, k-2\}$, we regroup the summation by if $i = j + (k-1)m$, where $0 \leq m \leq k$, it gives that
\[
\sum_{m=0}^{k} \cot \frac{i\pi}{k-1} \cot \frac{ki\pi}{k^2 - 1} = \cot \frac{j\pi}{k-1} \left( \sum_{m=0}^{k} \cot \frac{(j+(k-1)m)\pi}{k^2 - 1} \right)
\]
\[
= \cot \frac{j\pi}{k-1} \left( \sum_{m=0}^{k} \cot \left( \frac{j\pi}{k^2 - 1} + \frac{m\pi}{k+1} \right) \right)
\]
\[
= (k + 1) \cot^2 \frac{j\pi}{k-1},
\]
where we use Lemma 1.2. by taking $x = \frac{j\pi}{k^2 - 1}$. And similarly
\[
\sum_{m=0}^{k} \cot \frac{i\pi}{k-1} \cot \frac{ki\pi}{k^2 - 1} = \cot \frac{j\pi}{k-1} \left( - \sum_{m=0}^{k} \cot \left( - \frac{j\pi}{k^2 - 1} + \frac{m\pi}{k+1} \right) \right)
\]
\[
= (k + 1) \cot^2 \frac{j\pi}{k-1}.
\]
When $i = (k-1)j$, where $1 \leq j \leq k$, it gives that
\[
\sum_{j=1}^{k} \cot \frac{(k-1)j\pi}{k^2 - 1} \cot \frac{k(k-1)j\pi}{k^2 - 1} = - \sum_{j=1}^{k} \cot^2 \frac{j\pi}{k+1}.
\]
Sum all terms up, it gives that
\[
\sum_{i=1}^{k^2-2} \cot \frac{i\pi}{k^2 - 1} \cot \frac{ki\pi}{k^2 - 1} = \left( k^2 - 2 - k \right) + 2(k+1) \sum_{j=1}^{k^2-2} \frac{j\pi}{k-1} - \sum_{j=1}^{k^2-2} \frac{j\pi}{k+1}
\]
\[
= \left( k^2 - 2 - k \right) + 2(k+1) \frac{(k-2)(k-3)}{3} - \frac{k(k-1)}{3}
\]
\[
= \frac{2}{3} k^3 - 2k^2 + 2.
\]

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whe@math.wisc.edu

Department of Mathematics, University of Wisconsin-Madison, Madison, Wisconsin, 53706, USA

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