Stability of Linearly Switched Demand Dynamics in Supply Chain Based on Inventory

Yubao Wang\textsuperscript{1,2,3,*} and Weihua Wang\textsuperscript{1,2,3}

\textsuperscript{1}School of Economics and Finance, Xi’an Jiaotong University, Xi’an, China
\textsuperscript{2}School of Management, University of Liverpool, Liverpool, UK
\textsuperscript{3}International Business School, University of Xi’an Jiaotong-Liverpool, Suzhou, China

Email: wybao@xjtu.edu.cn

Abstract. The organized supply chain is one of the best competitive ways for the retailer. The dynamics and stability of a supply chain under inventory and price with the feature of piecewise linearity is analysed. This switched linear model composed of four subsystems for the first time. On the basis of linear switched model, some stabilities are derived and the global dynamic behavior is theoretically examined. It is illustrated that the combination of the stock-dependent demand function and inventory can bring about the stability result under, but also arise more complicated nonlinear fluctuations under \( b_1 \geq 1 \).

Keywords. Supply chain, Stock-dependent demand function, Linearly switched system, Stability.

1. Introduction

The supply chain is considered a complex system, and can present the complicated dynamic properties, such as bullwhip effect (BE), stability, chaos and inventory deficit (inventory drift). The problem in supply chain of the inventory has received considerable attention in models with stock-dependent demand [1]. In practice, the demand of some items such as deteriorating items may affected by the stock level, constant selling price, and replenishment strategy. Due to that the fresh products’ life cycle is very short and the high freshness requirements, including the low-temperature storage and cold-chain distribution are required, which are costly. The researchers think the high fluctuations of inventory and demand in supply chains come from the storage technological improvement and sustainability in a background of fierce market competition. These behaviours characterize the supply chain as a dynamical system. Based on the complexity and dynamic characteristics, an insignificant change may usher supply chains into a chaotic state. Thus, it is crucial to study the dynamics and intricate behaviours in a complex supply.

There are some factors that cause the dynamic complexity in supply chain, such as lead times, costs of the channels (including single sale, dual-channel sale, O2O, etc.), ordering rules, selling price, sales team’s creativity, trade credit, substitutable products, etc. The most critical factors are substitution products and seasonal effect. The ordering rule and information sharing play the important role in controlling the inventory and determining order quantities. Eliminating supply chain volatility is essential to prevent BE from expanding. Alony & Munoz (2007) review the various methods of modelling the dynamics of supply chains, and suggest a combined discrete event and continuous simulation modelling approach [2]. Omid et al. (2017) set a price-dependent stochastic demand function, studied the problem of collaborative prices and inventory based on price-dependent stochastic demand and quantity discount t [3]. Modak et al. (2018) established a dual-channel demand function on manufacturers’ online and retailer’s offline random demand, they also proposed a dual-channel supply chain model based on random demand for price and delivery lead time, and analysed the influence of
delivery lead time and customer channels preference on supply chain coordination [4]. The impact of fair preferences and concerns on supply chain decision-making has been studied too, but it is necessary that the researchers give the insight to the dynamical systems of supply chain. This paper investigates the dynamic stability of a single-channel (offline) supply chain composed of four subsystems instead of three subsystems (Wei et al., 2013) [5] under the stock-dependent demand function. The rest is organized as follows. Section 2 describes the supply chain with stock-dependent demand and develops a switched linear model with four subsystems under inventory and price. In section 3, the stability of the switched linear system is studied carefully. Conclusions are provided in section 4.

2. Model Description

The supply chain is a periodic single item system, which is consist with manufacturers and many retailers, and in which demand depends on the retailers’ inventory. Owing to the complexity of supply chains, the investigations into stock-dependent demand have focused on the discrete dynamics. In fact, these models are characteristic of the continuous-time ones. To simplify the analysis, the following assumptions and notations are used.

2.1. Assumptions

There are a manufacturer and a retailer in the market. The manufacturer has sufficient products to meet the retailer’s order. The retailer receives the goods ordered only in one lead time \( t = 0 \) from the manufacturer and the replenishment rate is infinite.

The generalized order-up-to policy is used by the retailer. The formula is

\[ O(t) = R(t) + \alpha(S(t) - S(t-1)). \]

Where \( O(t) \), \( R(t) \) is the retailer’s order quantities and the customer demand rate at the time \( t \), respectively; \( S(t), S(t-1) \) is the order-up-to policy quantity at the time \( t \) and \( (t-1) \). \( \alpha \) is a constant.

The retailer decides its stock and forecasts the customer demand rate at time \( t \).

The customer demand can be fulfilled by the retailer in the remaining time of each period.

2.2. Model Construction

Since the system is continuous, the differential equation can be used to characterize the dynamics of inventory and order. The retailer receives goods from the manufacturer at the time \( t \).

\[ R(t) = O(t) \]

The variable of inventory is closely related to the amount received from the manufacturer and sold to the customers. The differential equation of inventory level is obtained as

\[ \frac{d(I(t))}{dt} = R(t) - D(t) \]  

i.e.,

\[ \frac{d(I(t))}{dt} = O(t) - D(t) \]

The retailer forecasts the demand at time \( t \) as \( F(t) \), which is,

\[ \frac{d(F(t))}{dt} = \theta(D(t) - F(t)) \]

where \( \theta \) stands for the smoothing coefficient. As the \( \theta \) increases, the change in forecasted demand is bigger than the change of actual demand. Let \( 0 < \theta < 1 \) stabilize the forecasting process. For the retailer, the order is the key decision-making variable with the manufacturer. Order-up-to policies have been extensively considered. However, order-up-to policies are not good policies for improving supply chain dynamics, reducing order and inventory fluctuations (such as BE).

In order to compare the generalized and classic order-up-to policy, the classic order-up-to policy should be introduced firstly, which is described by

\[ O(t) = S(t) - I(t) \]

where \( I(t) \) is the inventory level.

The generalized order-up-to policy the retailer uses is represented as
\[ O(t) = \max\{F(t) + \mu D(t) - I(t)\}, 0\}, \] (6)

where \( D(t) \) is the target inventory level and \( \mu \) is a replenishment parameter \((0 < \mu)\) to restore the inventory level to a desired position. The target inventory level is set by \( D(t) = \eta F(t) \), where \( 0 < \eta \). The function \( \max\{} \) in (6) is used to avoid negative orders. If the positive constraint in the ordering rules is considered, we can find that the order-up-to policy is a special case of the generalized ordering policy, when the case is \( \mu = 1 \). Then the formula (6) becomes \( O(t) = S(t) - I(t) \) and \( S(t) = (1 + \eta)F(t) \). In order to discover the relationship between stabilities and cost, let \( IC(n) \) stand for the average inventory cost for \( n \) periods. \( C_h, C_p \) stand for the shortage cost per unit per period, respectively.

From the formulas (1) - (6), \( I(t) - D(t) \) is the inventory level at the end of period \( t \). Then, \( IC(n) \) for \( n \) periods is formulated by

\[ IC(n) = \frac{1}{T} \int_0^T \left| I(t) - D(t) \right| dt \] (7)

where

\[ C_{hp} = \begin{cases} C_h, & \text{if } I(t) - D(t) \geq 0, \\ C_p, & \text{if } I(t) - D(t) < 0. \end{cases} \]

It is necessary to determine the demand function to explore the nonlinear dynamics. To much research have been made in order to more realistic situations in inventory management and explain the dynamics of the stock-dependent demand since the economic order quantity (EOQ) model is put forward by Harris (1913). Yang (2014) developed an inventory model under a power, nonlinear stock-demand function and holding cost rate [1]. Piecewise linear function and power-form function are adopted to characterize the relationship between inventory and demand in this paper, which is described by

\[ D(t) = \begin{cases} a + bI(t), I(t) \geq 0, \\ a, & \text{otherwise} \end{cases} \] (8)

where \( a \) and \( b \) are nonnegative constants. The dynamic equations are given by

\[ \dot{I}(t) = O(t) - D(t) \]
\[ \dot{F}(t) = \theta(D(t) - F(t)), 0 < \theta < 1 \]
\[ O(t) = \max\{F(t) + \mu(\eta F(t) - I(t)), 0\}, 0 < \mu, 0 < \eta \]
\[ D(t) = a + bI(t), 0 < a, 0 < b \]

The dynamical system can be expressed as

\[ \dot{I}(t) = \max\{F(t) + \mu(\eta F(t) - I(t)), 0\} - D(t) \]
\[ \dot{F}(t) = \theta(D(t) - F(t)) \] (10)

3. Stability Analysis of the Switched Demand System
A switched system is a dynamical system that consists a few subsystems and a logical rule that coordinates switching between these subsystems. Switched linear systems are an important class of hybrid dynamic systems which consists of a family of LTI system and a switching law specifying the switching between them. Our model is piecewise linear which enables us to characterize the dynamic system. From (8) and (10), the system switches among the following four subsystems:

3.1. Switched Systems and Switched Rules
(1) The retailer has no inventory \((I(t) \leq 0)\) and no order \((F(t) + \mu(\eta F(t) - I(t)) \leq 0)\), i.e.

\[ \dot{I}(t) = -a \]
\[ \dot{F}(t) = \theta(a - F(t)) \] (11)
(2) The retailer has no inventory \((I(t) \leq 0)\) but has an order \(F(t) + \mu(\eta F(t) - I(t)) > 0\), i.e.,
\[
I(t) = (1 + \mu \eta)F(t) - \mu I(t) - a
\]
\[
F(t) = \theta(a - F(t))
\]
(12)

(3) The retailer has an inventory \((I(t) > 0)\) but no order \((F(t) + \mu(\eta F(t) - I(t)) \leq 0)\), i.e.,
\[
\dot{I}(t) = -(a + bI(t))
\]
\[
\dot{F}(t) = \theta(a + bI(t) - F(t))
\]
(13)

(4) The retailer has an inventory \((I(t) > 0)\) and an order \((F(t) + \mu(\eta F(t) - I(t)) > 0)\), i.e.,
\[
\dot{I}(t) = -(\mu + b)I(t) + (1 + \mu \eta)F(t) - a
\]
\[
\dot{F}(t) = \theta bI(t) - \theta F(t) + \theta a
\]
(14)

There are two switched rules: one is \(I(t) \leq 0\) or \(I(t) > 0\), the other is \(F(t) + \mu(\eta F(t) - I(t)) \leq 0\) or \(F(t) + \mu(\eta F(t) - I(t)) > 0\), i.e., the switched signal function is obtained as
\[
\sigma(t) = \begin{cases} 
1, & \text{if } I(t) \leq 0 \wedge (1 + \mu \eta)F(t) - \mu I(t) < 0 \\
2, & \text{if } I(t) \leq 0 \wedge (1 + \mu \eta)F(t) - \mu I(t) \geq 0 \\
3, & \text{if } I(t) > 0 \wedge (1 + \mu \eta)F(t) - \mu I(t) \leq 0 \\
4, & \text{if } I(t) \geq 0 \wedge (1 + \mu \eta)F(t) - \mu I(t) > 0 
\end{cases}
\]
(15)

where the symbol \(\wedge\) stands for “and”. Let \(x(t) = (I(t), F(t))^T\) be a state vector, the switched linear model is represented as
\[
\dot{x}(t) = A_{\sigma(t)}x(t) + B
\]
(16)

where
\[
A_1 = \begin{pmatrix} 0 & 0 \\ 0 & -\theta \end{pmatrix}, A_2 = \begin{pmatrix} -\mu & 1 + \mu \eta \\ -\theta & 0 \end{pmatrix}, A_3 = \begin{pmatrix} -b & 0 \\ \theta b & -\theta \end{pmatrix}, A_4 = \begin{pmatrix} -(\mu + b) & 1 + \mu \eta \\ b \theta & -\theta \end{pmatrix}, \text{ and } B = \begin{pmatrix} -a \\ \theta a \end{pmatrix}.
\]

The subsystem \(i\) is described by \(\dot{x}(t) = A_{\sigma(t)}x(t) + B\), \(i = \sigma(t) \in \{1, 2, 3, 4\}\).

3.2. Stability Analysis

The switched system is subject to the original one in a period \(\sigma(0 < \sigma < 1)\) as it crosses over the boundaries, i.e., \(I(t) = 0\) or \((1 + \mu \eta)F(t) - \mu I(t) = 0\). Then, the system changes according to (15). There are the following results about the stability of the subsystems.

Lemma 1. \(I(t)\) approaches \(-\infty\) and \(F(t)\) approaches 0 where \(I_0 = I(0) \leq 0\), \(F_0 = F(0) \leq 0\) and \((1 + \mu \eta)F_0 - \mu I_0 \leq 0\) as the time \(t\) tends to \(+\infty\) under these conditions of \(a > 0, \eta > 0\) and \(0 < \theta < 1\) in the subsystems (I) \((\sigma(t) = 1)\). This subsystem crosses over this line: \((1 + \mu \eta)F - \mu I = 0\) \((I < 0)\). The solution is \(x(t) = (-at + I_0, a + (F_0 - a)e^{-\theta t})^T\).

Lemma 2. \((I(t), F(t))\) approach \(M_2(\alpha \eta, a)\) where \(I_0 = I(0) \leq 0\) and \((1 + \mu \eta)F_0 - \mu I_0 \leq 0\) as the time \(t\) tends to \(+\infty\) under these conditions of \(a > 0, b > 0, \eta > 0, \mu > 0\) and \(0 < \theta < 1\) in the subsystems (II) \((\sigma(t) = 2)\). This subsystem crosses over this line: \(I = 0\) and \((1 + \mu \eta)F - \mu I > 0\).

The equilibrium of the subsystem (II) is \(M_2(\alpha \eta, a)\) and the two eigenvalues about the characteristic equation \(A_2\) are \(-\mu(0)\) and \(-\theta(0)\). Hence, the equilibrium of the system (II) \(M_2(\alpha \eta, a)\) is globally asymptotically stable.

Lemma 3. \((I(t), F(t))\) approach \(M_3(-a/b, 0)\) where \(I_0 = I(0) > 0\) and \((1 + \mu \eta)F_0 - \mu I_0 < 0\) as the time \(t\) tends to \(+\infty\) under these conditions of \(a > 0, b > 0, \eta > 0, \mu > 0\) and \(0 < \theta < 1\) in the
subsystems (III) \((\sigma(t) = 3)\). This subsystem may cross over these lines: \((1 + \mu \eta)F - \mu I = 0\) \((I < 0)\) or 
\(I = 0\) \(((1 + \mu \eta)F - \mu I > 0)\).

The equilibrium of the subsystem (III) is \(M_s(\frac{-a}{b}, 0)\) and the two eigenvalues about the characteristic equation \(A_s\) are \(-b < 0\) and \(-\theta < 0\). Hence, the equilibrium of the subsystem (III) \(M_s(\frac{-a}{b}, 0)\) is globally asymptotically stable.

Lemma 4. \((I(t), F(t))\) approach \(M_s(\frac{\alpha_0}{1-b\eta}, \frac{a}{1-b\eta})\) where \(I_0 = I(0)(\geq 0)\) and \((1 + \mu \eta)F_0 - \mu I_0 \geq 0\) as the time \(t\) tends to \(+\infty\) with these conditions: \(a > 0\), \(b > 0\), \(\eta > 0\), \(\mu > 0\), \(0 < \theta < 1\) and \(b\eta < 1\) in the subsystems (IV) \((\sigma(t) = 4)\).

The equilibrium of the subsystem (IV) is \(M_s(\frac{\alpha_0}{1-b\eta}, \frac{a}{1-b\eta})\) and the two eigenvalues about the characteristic equation \(A_s\) are \((-b + \mu + \theta + \sqrt{(b + \mu + \theta)^2 - 4\mu \theta(1-b\eta)})\) \(-b + \mu + \theta - \sqrt{(b + \mu + \theta)^2 - 4\mu \theta(1-b\eta)})/2\) \((< 0)\) and \((< 0)\). It is globally stable when \(b\eta < 1\) in the switched system (16) with these conditions: \(a > 0\), \(b > 0\), \(\eta > 0\), \(\mu > 0\) and \(0 < \theta < 1\).

Proof: According to the different regions of the initial values \((I_0, F_0)\) (or the switched point \((I_k, F_k)\), \(k = 1, 2, \ldots\)), we illustrate that the orbit of this system will approach to this equilibrium \(M_s(\frac{\alpha_0}{1-b\eta}, \frac{a}{1-b\eta})\).

It gets \(I(t) = -at + I_0\) and \(F(t) = a(F_0 - a)e^{\theta t} (t \geq 0)\) when the initial values \(I_0 < 0\), \(F_0 < 0\) and \(F_0 + \mu(\eta F_0 - I_0) < 0\).

From Lemma 1, it exists a time \(t_1(> 0)\) where \(I(t_1) < 0\) and \(F(t_1) + \mu(\eta F(t_1) - I(t_1)) = 0\). Then it enters the region: \(F + \mu(\eta F - I) > 0\), \(I < 0\) and the system satisfies the equation (12) (after the period \(\tau\)).

From Lemma 2, it exists a time \(t_2(> 0)\) where \(I(t_2) = 0\) and \(F(t_2) + \mu(\eta F(t_2) - I(t_2)) > 0\). Moreover, the system enters the region: \(F(t) + \mu(\eta F(t) - I(t)) > 0\), \(I > 0\) and the system satisfies the equation (14) (after the period \(\tau\)).

Given the initial values \((I_0, F_0)\) satisfy that \(I_0 > 0\) and \(F_0 + \mu(\eta F_0 - I_0) < 0\). From Lemma 3, it exists a time \(t_3(> 0)\) where \(I(t_3) = 0\) and \(F(t_3) + \mu(\eta F(t_3) - I(t_3)) < 0\). Then the system enters this region \(F < 0\), \(F + \mu(\eta F - I) < 0\) as above where the initial values \(I_0 < 0\), \(F_0 < 0\) and \(F_0 + \mu(\eta F_0 - I_0) < 0\). Otherwise, it exists a time \(t_4(> 0)\) where \(F(t_4) + \mu(\eta F(t_4) - I(t_4)) = 0\) and \(I(t_4) > 0\). Then the system enters this region \(I > 0\), \(F + \mu(\eta F - I) > 0\).

From Lemma 4, the switched system converges to \(M_s(\frac{\alpha_0}{1-b\eta}, \frac{a}{1-b\eta})\) included in the region \(F(t) + \mu(\eta F(t) - I(t)) > 0\), \(I > 0\) as the time \(t\) tends to \(+\infty\) \((b\eta < 1)\).

It is evident that the orbits of this linearly switched system in the different regions enter the region IV regardless of the numerical value of \(b\eta\). In this sense, the region IV is the attractive one. The switched system is stable as this region is stable according to \(b\eta < 1\). Figure 1 shows how the orbits of this switched system vary in the distinct initial values as \(b\eta < 1\).
Proposition: The linearly switched system may arise chaos or all of the variables may tend to as the time \( t \) increases under the condition of \( b\eta < 1 \).

**Figure 1.** Sketch for the orbits of the switched linear system in several initial values as \( b\eta < 1 \).

4. Conclusions
The dynamics between the inventory and the forecasting demand is discussed with the time \( t \) in the paper. There are two parameters \( b \) and \( \eta \) which are highly significant. If \( b\eta < 1 \), the inventory \( I(t) \) and the forecasting demand moves to \( \frac{a\eta}{1-b\eta} \) and \( \frac{a}{1-b\eta} \) as the time \( t \) increases, respectively. Here, the quantities of the initial inventory or the forecasting demand do not affect these results. However, the dynamics presents much more dynamical characteristics as \( b\eta \geq 1 \). It is affected not only by these parameters but the initial states. The inventory may move to \(+\infty\) , oscillate, and even produce chaos. This situation deserves to be investigated.

With the advancement of modern logistics technology and rapid growth in online sales, the multi-channel supply chain will create the diversified consumptions. Next, the dynamic dual-channel or multi-channel supply chain that depends on inventory demand for all parties to obtain maximum efficiency steadily will be studied.

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