Tiny Groups Tackle Byzantine Adversaries

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Abstract—A popular technique for tolerating malicious faults in open distributed systems is to establish small groups of machines, each of which has a non-faulty majority. These groups are used as building blocks to design attack-resistant systems. Despite over a decade of active research, prior constructions require group sizes of \( \Omega(\log n) \), where \( n \) is the number of machines in the system. This group size is important since communication and state costs scale polynomially with this parameter. Given the stubbornness of this \( \Omega(\log n) \) barrier, a natural question is whether better bounds are possible.

Here, we demonstrate how to reduce the group size exponentially to \( O(\log \log n) \) while maintaining strong security guarantees, despite (i) a dynamic setting where machines join and depart over time, and (ii) an attacker that controls a constant fraction of the total computational resources belonging to machines in the system. This reduction yields significant improvements in cost.

I. INTRODUCTION

Byzantine fault tolerance addresses the challenge of performing useful work when machines (nodes) in a system are malicious. Information routed through or stored at a faulty node can be discarded or corrupted, and tasks executed on such nodes may fail or output an erroneous value.

A popular technique for overcoming these challenges is to arrange nodes into sets called groups where each has a non-faulty majority. We can then ensure the following:

- Secure routing is possible. For groups \( G_1 \) and \( G_2 \) along a route, all members of \( G_1 \) transmit messages to all members of \( G_2 \). This all-to-all exchange, followed by majority filtering by each non-faulty node in \( G_2 \), guarantees correctness of communication between groups despite malicious nodes.
- Computation is performed by all members of a group via protocols for Byzantine agreement (BA) or more general secure multiparty computation to guarantee that tasks execute correctly. In this way, each group simulates a reliable processor upon which jobs can be run.

The use of groups provides a scalable approach to designing an attack-resistant distributed system, since they avoid the need to have all \( n \) nodes perform BA in concert, or communicate via a system-wide pairwise exchange of messages.

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1Such sets have appeared under different names in the literature, such as “swarms”, “clusters”, and “quorums”. Our choice of “groups” aligns with pioneering work in this area.

2Groups also improve robustness in other ways. Members may agree to ignore a node if it misbehaves too often, hence reducing spamming. Data may also be redundantly stored at multiple group members.

Designing attack-resistant systems using groups has been an active topic of research for over a decade, with many intriguing theoretical results. However, since groups are building blocks for the system, their size \( |G| \) impacts costs:

(i) Cost of Group Communication. Group members must often act in concert despite malicious faults; for example, executing distributed key generation or generating random numbers. Such protocols require messages be exchanged between all members; we label this as group communication and it has \( \Theta(|G|^2) \) message cost.

(ii) Cost of Secure Routing. Routing via all-to-all exchange between two groups incurs \( \Omega(|G|^2) \) message complexity. Given a route of length \( D \), communication between any two groups requires \( O(D|G|^2) \) messages.

(iii) Cost of State Maintenance. Each node \( w \) must maintain state on its neighbors; this includes both the members of all groups to which \( w \) belongs and the members of neighboring groups. This requires storing link information, as well as periodically testing links for liveness.

In each case above, reducing \( |G| \) would directly reduce cost. Unfortunately, in prior results, \( |G| = \Omega(\log n) \) is key to ensuring that all groups have a non-faulty majority with high probability (w.h.p.). Without this property, all previous group constructions succumb to adversarial attack. A natural question is: Are there new ideas that allow us to decrease \( |G| \) while maintaining strong security guarantees?

Our main result is a surprising tradeoff: group size can be reduced exponentially while still allowing routing in all but a vanishingly small portion (an \( o(1) \)-fraction) of the network. Our solution relies, in part, on proof-of-work (PoW) and demonstrates another application of this technique for combating malicious behavior in distributed systems.

3Improvements are possible, but they come with caveats. Results in [17, 43] lower the cost to \( O(D|G|) \) in expectation but require a non-trivial (expander-like) construction, and [49] further reduces this to \( O(D) \) in expectation but with a poly(|G|) message cost each time routing tables are updated which is expensive even with moderate churn.

4Constructions in [17, 47] have each node belonging to \( \eta \geq 1 \) groups for a state overhead of \( \Omega(|G|/\eta) \). Also, if each group links to \( \Delta \) neighboring groups, then \( O(|G|\Delta) \) links must be maintained; typically, \( \Delta = O(\log n) \).

5With probability at least \( 1 - 1/n^c \) for a tunable constant \( c \geq 1 \).
A. Defining \( \varepsilon \)-Robustness

Consider a system of \( n \) nodes where a \( \beta \)-fraction suffer malicious faults; such nodes may deviate arbitrarily from any prescribed protocol to derail operations in the network. The following defines our notion of \( \varepsilon \)-\textit{robustness:}

For a small \( \varepsilon > 0 \), at least \((1-\varepsilon)n\) groups have an non-faulty majority and can securely route messages to each other.

The parameter \( \beta < 1/2 \) is a constant while \( \varepsilon = o(1) \). We consider the following questions:

**Is this a useful concept?** Consider decentralized storage and retrieval of data. This definition guarantees all but an \( \varepsilon \)-fraction of data is reachable and maintained reliably. Example applications include distributed databases, name services, and content-sharing networks. Alternatively, consider \( n \) jobs in an open computing platform that are run on individual machines. This definition guarantees that all but an \( \varepsilon \)-fraction of those jobs can be correctly computed.

**Is satisfying this definition trivial?** Given \( \Theta(n) \) non-faulty nodes, this definition characterizes simulating \((1-\varepsilon)n\) reliable processors and the ability to route information between them. If we ignore the use of groups or, equivalently, consider groups each consisting of a single node, then we trivially have \((1-\beta)n\) reliable processors, but routing between them is challenging. For example, establishing links between all pairs of nodes will give secure routing, but this is hardly scalable.

**Do previous solutions solve this problem?** Prior results using groups are subsumed by this definition and they address the case of \( w.h.p. \) preserving a non-faulty majority in each group for \( 2 \leq n \leq 64 \). More recently, Guerraoui et al. [20] give similar guarantees when the system size can vary polynomially.

Several results have focused on reducing communication costs when the good majority of all groups is guaranteed (via an algorithm like the cuckoo rule) [43], [48]. Here too, group size impacts performance and \( |G| = 30 \) incurs significant latency in PlanetLab experiments [49]. Groups have also been used in conjunction with quoruming malicious nodes [26], [41]; however, full dynamism is not addressed.

**Attack-Resistance Without Groups.** Other distributed constructions exist that do not explicitly use groups [13], [16], [42]. However, the associated techniques retain some form of \( \Omega(\log n) \) redundancy with regards to data placement or route selection and, therefore, incur the typical poly(\( \log n \)) cost.

In [12], [25], [35], malicious faults are tolerated by routing along multiple diverse routes. However, it is unclear that these systems can provide theoretical guarantees on robustness.

**Byzantine resistance when \( O(n/\text{poly}(\log n)) \) nodes may depart and join per time step is examined in [2], [3].** In this challenging model, (roughly) \( O(\sqrt{n}) \) Byzantine nodes can be tolerated. Our result addresses more moderate churn while tolerating \( \Theta(n) \) Byzantine groups.

Central authorities (CAs) have been used in prior results [12], [40] to achieve robustness. While our results can be used in conjunction with a CA, it is not always plausible to assume such an authority is available and immune to attack. For this reason, our work does not depend on a CA.

**Computational Puzzles.** Proof-of-work (PoW) via computational puzzles has been used to mitigate the Sybil attack [14] whereby an adversary overwhelms a system with a large number of identifiers (IDs). We note that such PoW schemes have been proposed in decentralized settings; for examples, see [21], [29]. However, such PoW schemes only \( \text{limit the number of Sybil IDs} \) — typically commensurate with the amount of computational power available to the adversary — and the problem of tolerating these adversarial IDs must still be addressed by other means; for examples, see [44], [49].

A prominent example of how PoW can provide security is Bitcoin. However, note that analyses of Bitcoin and related systems typically assume a communication primitive that allows a node to disseminate a value to all other nodes within a known bounded constant amount of time despite an adversary [19], [30], [32]. In contrast, our results do not assume the existence of such a primitive.

B. Related Work

**Tolerating Byzantine Faults via Groups.** The use of groups for building attack-resistant distributed systems has received significant attention, and all address the case of \( \varepsilon = 1/\text{poly}(n) \). Early results obtain a poly(\( \log n \)) cost assuming constraints on the amount of system dynamism [7], [16], [17], [22], [36].

Full dynamism was achieved by Awerbuch and Scheideler in a series of breakthrough results [8]–[10]; they propose a cuckoo rule that w.h.p. preserves a non-faulty majority in all groups over \( n^{\Theta(1)} \) joins/departures when the system size remains \( \Theta(n) \). More recently, Guerraoui et al. [20] give similar guarantees when the system size can vary polynomially.

Simulations of the cuckoo rule are conducted in [45]. The trade-off between group size and security is examined, and findings suggest that \( |G| \) must be fairly large. For \( n = 8,192 \) (the largest size examined) and \( \beta \approx 0.002 \), \( |G| = 64 \) preserves a non-faulty majority in each group for \( 10^5 \) joins/departures; \( \beta \approx 0.07 \) is possible with suggested improvements in [45].

C. Our Model and Preliminaries

A node is \textit{good} if it obeys the protocol; otherwise, the node is Byzantine or \textit{bad}.

**System Size.** We assume that \( n \) nodes are always present even under churn; that is, when a node leaves, another is assumed to join; this is a popular model considered in much of the previous literature for on tolerating \( \Theta(n) \) Byzantine faults; for example, [7]–[10], [13], [16], [17], [22], [36]. Additionally, our results hold when the system size is \( \Theta(n) \) — that is, the size changes by a constant factor — but we omit these details in this extended abstract.

\footnote{While this may not be sufficient for general computation, it is valuable for tasks where an \( o(1) \) error rate or bias can be tolerated; for example, obtaining statistics on a group of machines to measure network performance.}

\footnote{High churn without Byzantine fault tolerance is considered in [1], [4], [5].}
The Adversary. At most $\beta n$ nodes are bad and used by an adversary to attack the system, where $\beta < 1/2$ is a positive constant; that is, there exist $\Theta(n)$ Byzantine nodes. This is a challenging model since a single adversary allows the bad nodes to perfectly collude and coordinate their attacks. The adversary also knows the network topology and all message contents sent between nodes; however, the adversary does not know the random bits generated locally by a good node.

Since the adversary controls at most $\beta n$ nodes, it wields proportional computational power. This has implications for the number of IDs that the adversary can generate within a bounded amount of time; the details are deferred to Section IV.

We assume that each node (good or bad) does not differ significantly in terms of computing power. This is a common starting assumption when using PoW to design attack-resistant, open systems (for example, see [21], [29], [39]). Moreover, computational disparities are typically bounded by a small to moderate constant (see [39]); therefore, we may account for these IDs are u.a.r. from the ID space $[0, 1)$.

Groups. We assume each group has size $\Theta(\log \log n)$. Each node $w$ has its own group $G_w$ and $w$ is referred to as the leader. A group $G$ is good if (i) $d_1 \log \log n \leq |G| \leq d_2 \log \log n$ for sufficiently large positive constants $d_1 < d_2$, and (ii) the number of bad nodes in $G$ is at most $(1 + \delta)\beta |G|$ for some tunably small constant $\delta > 0$ depending only on $n$; otherwise, the group is bad. Note that groups are not necessarily disjoint; in addition to being the leader of $G_w$, node $w$ may belong to other groups. Group construction is described in Section III-A.

Input Graph. Our result builds off an input graph $H$ on $N$ nodes that satisfies the following:\footnote{For simplicity in our proofs, we let $\beta$ be small since we do not try to optimize this quantity. However, larger values of $\beta$ are likely possible.} Assuming there is no adversary and that IDs are uniformly at random (u.a.r.) in $[0, 1)$, the following properties hold for $H$ with probability at least $1 − N^{-c}$ for a tunable constant $c > 1$:

- **P1 - Search Functionality.** There exists a search (or routing) algorithm that, for any key value in $[0, 1)$, returns the node responsible for the corresponding resource (i.e., data item, computational job, network printer, etc.). A search requires traversing $D = O(\log N)$ nodes.

- **P2 - Load Balancing.** A randomly chosen node is responsible for at most a $(1 + \delta)/N$-fraction of the key values (and the corresponding resources) for an arbitrarily small $\delta > 0$ depending on sufficiently large $N$.

- **P3 - Linking Rules.** Each node $w$ links to nodes in a neighbor set $L_w$ and the rules for forming $L_w$ are specified by a protocol known to all nodes: $|L_w| = O(\log^2 n)$ for some constant $\gamma > 0$. Any node may determine the elements in $L_w$ by performing searches; in particular, finding the nearest-clockwise node of a point in $[0, 1)$.

There are also $O(\poly(\log n))$ nodes whose IDs dictate that $w$ is one of their neighbors (typically, this is $O(|L_w|)$ in expectation; see our full version [23]). Again, any node may verify this by performing searches. The number of links on which a node is incident is the degree of $w$, and every node has the same degree asymptotically.

- **P4 - Congestion Bound.** The congestion is $C = \polylog(N)/N$ where congestion is the maximum probability (over all nodes) that a node is traversed in a search initiated at a randomly chosen node for a randomly chosen point in $[0, 1)$.

We emphasize that $H$ is not assumed to be secure against bad nodes. Rather, any such $H$ provides a viable topology that, using our result, can be made attack-resistant.

Note that many constructions for $H$ exist such as Chord [46], the distance-halving construction [37], Viceroy [31], Chord++ [6], D2B [18], FISSIONE [28], and Tapestry [50].

Node IDs and PoW in Our System. Each node owns an ID which, for simplicity, is a value in $[0, 1)$. Note that adequate precision is obtained using $O(\log n)$ bits. Important properties that our system guarantees are:

- IDs expire after a period of time that can be set by the system designers.

- A claim to own an ID can be verified by any good node.

- The adversary possesses (roughly) $\beta n$ IDs at most, and these IDs are u.a.r. from the ID space $[0, 1)$.

We show how to enforce these properties via a PoW scheme whereby a node must solve a computational puzzle in order to obtain an ID. Given space constraints and that the bulk of our results are proved without the need to reference these details, we delay their discussion until Section IV.

Finally, we make the random oracle assumption [11]: there exist hash functions, $h$, such that $h(x)$ is uniformly distributed over $h$’s range, when any $x$ in the domain of $h$ is input to $h$ for the first time. We assume that both the input and output domains are the real numbers between 0 and 1. In practice, $h$ may be a cryptographic hash function, such as SHA-2 [38], with inputs and outputs of sufficiently large bit lengths.

The following well-known concentration results are used.

**Theorem 1.** (Chernoff Bounds [34]) Let $X_1, \ldots, X_N$ be independent indicator random variables such that $\Pr(X_i) = p$ and let $X = \sum_{i=1}^N X_i$. For any $\delta$, where $0 < \delta < 1$, the following holds:

\[ \Pr(X > (1 + \delta) E[X]) \leq e^{-\delta^2 E[X]/3} \]
\[ \Pr(X < (1 - \delta) E[X]) \leq e^{-\delta^2 E[X]/2} \]

**Theorem 2.** (Method of Bounded Differences [13]) Let $f$ be a function of the variables $X_1, \ldots, X_N$ such that for any $b, b'$ it holds that $|f(X_1, \ldots, X_i = b, \ldots, X_N) - f(X_1, \ldots, X_i = b', \ldots, X_N)| \leq c_i$ for $i = 1, \ldots, N$. Then, the following holds:

\[ \Pr(f > E[f] + \epsilon) \leq e^{-\epsilon^2/(2 \sum c_i^2)} \]
\[ \Pr(f < E[f] - \epsilon) \leq e^{-\epsilon^2/(2 \sum c_i^2)} \]
D. Overview and Our Main Result

As discussed above, reducing group size is desirable, but gives rise to the possibility of bad groups. In Section II, we demonstrate how to achieve $1/\text{poly}(\log n)$-robustness with groups of size $\Theta(\log \log n)$ when there is no churn. This argument leverages the bound on congestion given by the input graph, along with a careful tallying of the fraction of ID space that cannot be securely searched.

This result is applied in Section III where we show that $O(1/\text{poly}(\log \log n))$-robustness can be maintained with churn. A key component of our construction is the use of two graphs (composed of groups) that, when used in tandem, limit the number of bad groups that can be formed.

Finally, in Section IV we describe how PoW is used to provide the guarantees on node IDs discussed in Subsection I-C. The main challenge is defending against an adversary that wishes to store a large number of IDs for use in a massive future attack (i.e., a pre-computation attack).

Our main result is the following:

**Theorem 3.** Assume an input graph $H$ that satisfies $P1 - P4$. If the adversary has at most $\beta n$ computational power, then our construction using $|G| = O(\log \log n)$ provides the following guarantees w.h.p. over a polynomial number of join and departure events:

- all but an $O(1/\text{poly}(\log n))$-fraction of groups are good,
- all but an $O(1/\text{poly}(\log n))$-fraction of nodes can successfully search for all but an $O(1/\text{poly}(\log n))$-fraction of the resources.

That is, we achieve $O(1/\text{poly}(\log \log n))$-robustness.

To illustrate our cost improvements, we also establish:

**Corollary 1.** Using any of the constructions in [18], [37], or [37] as an input graph $H$, our result gives the following bounds on cost:

- group communication incurs $O(\text{poly}(\log \log n))$ messages,
- secure routing incurs $O(D \text{poly}(\log \log n))$ messages,
- expected $O(\text{poly}(\log \log n))$ state maintenance.

Note that these are substantial improvements over the costs described in Section I, particularly with respect to group computation and state maintenance.

**Can we do better?** We offer some intuition for why significantly improving on our result seems unlikely. With $|G| = \Theta(\log \log n)$, the probability of a bad group is roughly $1/\text{poly}(\log \log n)$. All constructions with $o(\log n)$ degree require $D = \Omega(\log n/\text{poly}(\log \log n))$ hops to perform a search. Thus, the probability of avoiding bad groups along the search path is (roughly) at most $\sum_{i=1}^{D} 1/\text{poly}(\log \log n)$ by a union bound, and this can be less than 1; we show this rigorously.

Now, consider a smaller group of size, say, $\Theta(\log \log \log n)$. The probability of a bad group is (roughly) $1/\text{poly}(\log \log \log n)$ and over $D$ hops, a union bound no longer bounds the probability of a failed search by less than 1.

In this sense, our choice of $|G|$ appears to be pushing the limits of what is possible when using groups to design attack-resistant systems.

II. The Static Case

We first prove results for the static case since this is a useful step in addressing the dynamic case.

A. The Group Graph

Given our input graph $H$, our approach involves the creation of a group graph $G$ which can be viewed as replacing each vertex $w$ in $H$ with group $G_w$.

Other aspects are analogous to $H$. The neighbor set $L_w$ consists of groups as elements. Edges are directed from group $G_w$ to group $G_v$, denoted by $(G_w, G_v)$, and signifies that $G_v \in L_w$. A search in $G$ proceeds over these edges as it would in $H$, except that groups are being traversed instead of individual nodes. The edge directionality indicates the way in which a search traversing $G_w$ proceeds; other communication may occur in both directions. The edge $(G_w, G_v)$ is realized in the network by all-to-all links between (at least) the good members in $G_w$ and $G_v$; see Figure I.

For ease of presentation, we often speak of groups being uniformly distributed in the ID space; by this, we mean the leaders of the groups are uniformly distributed. Congestion is similarly defined for a group graph: the probability that a random lookup traverses a group (that is, traverses the leader and, by extension, at least the good members of its group).

We will refer to blue and red groups where the former corresponds to good groups with their neighbors correctly established, while red groups correspond to bad groups or those groups that have an incorrect neighbor set. The utility of this coloring scheme will become clearer when we address churn in Section III-B.

In proving $G$ is $\varepsilon$-robust, we consider steps $S1-S3$ that capture the impact of the adversary:

- **S1.** $G$ inherits the properties of the input graph $H$; each group $G_w$ has $|L_w|$ neighbors, $O(\text{poly}(\log n))$ degree, and $G$ has congestion $C = O(\log^c n/n)$ for a constant $c \geq 0$.
- **S2.** Each group is red independently with probability $p_f \leq 1/\log^k n$ for a tunable constant $k > 0$.
- **S3.** The adversary adds or deletes edges between red groups only.

**Overview of Analysis.** We clarify a few points before presenting our arguments in the next section. A search in $G = (V, E)$ is said to fail if it traverses any red group. Pessimistically, we assume the result of a failed search is dictated by the adversary; otherwise, the search will succeed.

The value of $p_f$ in $S2$ corresponds to the probability that a group is bad or does not have the correct neighbor set. To provide intuition, note that if we select $\Theta(\log \log n)$ nodes u.a.r., then the probability that more than a $1/3$-fraction are bad is $O(1/\text{poly}(\log n))$ by a Chernoff bound. A similar bound can be derived on the probability of incorrectly setting up neighbors. By setting the group size, we can set $k$ to as large a constant as we please. However, keeping $p_f$ upper bounded by $1/\log^k n$ with churn is non-trivial, and this is argued later in Section III-B.

Edges in $G$ are directed from a group to its neighbors and a search traverses these edges as it proceeds. In our analysis,
special attention is paid to those edges \((G, G')\) where \(G\) is blue and \(G'\) is red; we call \(G'\) a \textit{border group}. Informally, border groups can be viewed as sitting at the boundary point to the “community” of red groups and a search that encounters a border group fails.

Since the adversary controls all red groups, it is free to insert or delete edges between red groups; hence the motivation for \(S3\). However, edges involving at least one blue group are not modified. This corresponds to the fact that the adversary cannot modify the blue group’s notion of who its neighbors are (since this is kept consistent by the good nodes who are in the majority), although the red group may certainly ignore or corrupt incoming messages from that blue group.

B. Analysis

In \(G\), a search starting at any group \(G_i\) and terminating at the first red group is a \textit{route}. Starting at any group \(G_i\), the union of all routes induces a \textit{search tree} with \(G_i\) at the root. There is one search tree per group.

For a group \(G_v\), we define \textit{responsibility} of \(G_v\) to be the sum over all \(n\) search trees, \(\mathcal{T}\), of the probability that a search using \(\mathcal{T}\) for a random point in \([0, 1]\) will traverse \(G_v\) (by this, we mean at least the good group members partake in the search); denote this by \(\rho(v)\). We are interested in the aggregate responsibility of all red groups. Note that this is equivalent to examining the aggregate responsibility of all border groups since a search must traverse a border group before traversing any other red group.

**Lemma 1.** With high probability \(\rho(v) = O(\log^c n)\) for each border group \(G_v\).

**Proof.** By \(S1\), w.h.p. the search in a random tree for a random point will traverse \(G_v\) with probability \(C = O(\log^c n/n)\).

For a fixed search tree \(\mathcal{T}\), we say that a search for a key value \textit{fails} if the appropriate route in \(\mathcal{T}\) traverses a red group. Let \(X\) be a random variable that is the sum over all search trees, \(\mathcal{T}\), of the probability that a search in \(\mathcal{T}\) for a random key value fails. The randomness of \(X\) depends on which groups are red.

**Lemma 2.** \(E(X) = O(p_f n \log^c n)\).

**Proof.** For some fixed group \(G_v\) in some fixed search tree, let \(X_{v}\) be a random variable that equals \(\rho(v)\) if \(G_v\) is a border group, and 0 otherwise. Note that \(X = \sum_{v} X_{v}\). By linearity of expectation, w.h.p. \(E(\sum_{v} X_{v}) = \sum_{v} E(X_{v}) = O(p_f n \log^c n)\) by Lemma 1.

**Lemma 3.** \(Pr(X \geq (1 + \epsilon) p_f n \log^c n) \leq e^{-\Omega(\epsilon^2 p_f^2 n / \log n)}\) where \(\epsilon > 0\) is an arbitrarily small constant depending only on \(n\).

**Proof.** For some fixed group \(G_v\) in some fixed tree, let \(X_{v}\) be a random variable that equals \(\rho(v)\) if \(G_v\) is a border group, and 0 otherwise. We will bound \(\sum_{v} X_{v}\), which is always at least as large as \(X\). Let \(f(X_1, ... , X_{n}) = \sum_{v} X_{v}\). By Lemma 4 for any fixed \(X_i\), \(|f(\ldots , X_i = x , \ldots ) - f(\ldots , X_i = x' , \ldots )| = O(\log^c n)\). Thus, we can apply Theorem 2 with \(c' = O(\log^c n)\) for all \(1 \leq i \leq n\). We have that:

\[
Pr(|X - E(X)| \geq \lambda) \leq e^{-\lambda^2/(d n \log^c n)}
\]

for some constant \(d > 0\). Setting \(\lambda = c p_f n \geq cn/\log^k n\):

\[
Pr(|X - E(X)| \geq \lambda) \leq e^{-\lambda^2 n/(2 \log^{2(c+k)} n)}
\]

By Lemma 2 \(E(X) = O(p_f n \log^c n)\) gives the result.

**Lemma 4.** With probability at least \(1 - e^{-\Omega(p_f^2 n / \log^{2(c+k)} n)}\) any search from a random group to a random point in \([0, 1]\) succeeds with probability \(1 - O(1/\log^{k-c} n)\) where \(k \geq c + 1\).

**Proof.** Fix any \(\epsilon > 0\). By Lemma 3 we have \(Pr(X \geq (1 + \epsilon) p_f n \log^c n) \leq e^{-\Omega(\epsilon^2 p_f^2 n / \log^{2(c+k)} n)}\). We can consider \(X\) to be the total space over all search trees that can not be reached because of the red groups. Hence, if we pick a tree uniformly at random from which to start a search, and search for a random point, then the probability of failure is exactly \(X/n = O(1/\log^{k-c} n)\) by \(S2\).
II. THE DYNAMIC CASE

We now consider the case where nodes can join and depart. We still make the assumptions about IDs described in Section I-C. Time is divided into disjoint consecutive windows of $T$ steps called epochs indexed by $j \geq 1$; we discuss how $T$ is set in Subsection IV-A. In any epoch $j$, there are:

- two old group graphs $G_{j-1}$ and $G_{j-1}^{-1}$, each with $n$ nodes.
- two new group graphs $G_j^1$ and $G_j^2$, each with $\leq n$ nodes.

We emphasize that the use of two group graphs per epoch is critical. A naive approach is to use a single group graph in the current epoch in order to build a new group graph in the next epoch. However, this approach will fail because errors from bad groups will accumulate over time and we give some intuition for why.

Informally, in epoch $j$, we have a process where (1) bad groups build new bad groups, and (2) good groups build bad groups with some failure probability $p_j^f > 0$ that depends on the current number of bad groups. Therefore, in the next epoch $j + 1$, the population of bad groups has increased and so has $p_j^f$. Left unchecked, this increasing error probability will surpass the desired value of $1/\log^6 n$.

By using two group graphs, we can upper bound $p_j^f$ by this value. This is critical to invoking our result for the static case (in particular, recall S2).

The new group graphs are built using the old group graphs over the $n$ deletions and additions that occur in the current epoch $j$; we describe this in Subsection III-A. By the end of epoch $j$, the old group graphs $G_{j-1}$ and $G_{j-1}^{-1}$ are no longer needed, and the new ones $G_j^1$ and $G_j^2$ are complete.

A. Building New Group Graphs

We describe how the new group graphs $G_j^1$ and $G_j^2$ are created. Then, in Section III-B we prove that w.h.p. this construction preserves $\varepsilon$-robustness.

Preliminaries. Assume the system is in epoch $j$. Each good node $v$ already in the system maintains the same ID in $G_j^1$ and $G_j^2$.

Any node $v$ (already participating in the system or a newcomer) that wishes to participate in the next epoch $j + 1$ must begin generating an ID by the halfway point of the current epoch $j$. Generating this ID requires an expenditure of computational power as described in Section IV.

Recall from Section I-C that IDs expire after a tunable period of time. Upon creation, the new ID will be active throughout epoch $j + 1$ allowing $v$ to initiate searches via $G_v$ and for $v$ to be added to other groups. When $v$’s ID expires, its group $G_v$ (this includes $v$) should remain in both old graphs for an additional $T$ steps. During these steps, $G_v$ will forward communications, but $v$ cannot initiate searches using $G_v$, nor can $v$ be added to new groups; we say that $v$’s ID is passive.

Nodes are assumed to know when the system came online (i.e. step 0)\footnote{This is a fixed parameter included as part of the application, along with $T$, the hash functions, and various constants.}. Since $T$ is set when the system is designed, any node that wishes to join knows when the current epoch ends and the next one begins. Some synchronization between devices is implicit. In practice, this is rarely a problem given the near-ubiquitous Internet access (see the Network Time Protocol [33]) available to users.

A new node with a random ID is bootstrapped into the new group graph by a bootstrapping group denoted by $G_w$. Throughout, we assume that a joining node knows a good bootstrapping group; we discuss this further in the Appendix of our full version [23].

Making a Group-Membership Request. In epoch $j$, group graphs $G_j^1$ and $G_j^2$ are being built using secure routing provided by $G_{j-1}^1$ and $G_{j-1}^{-1}$. A node $w$ uses the same ID in $G_j^1$ and $G_j^2$.

Consider how $G_w$ is added to $G_j^1$. The $i$th member of $G_w$ is the successor node to $h_1(w, i)$ — denoted by $\text{suc}(h_1(w, i))$ — in the old group graphs for $i = 1, ..., d_2 \ln \ln n$ where $h_1$ is a secure hash function and $d_2$ is defined with respect to group size in Section I-C. That is, in both $G_{j-1}^1$ and $G_{j-1}^{-1}$, a search for each successor of $h_1(w, i)$ is performed; this is executed by the bootstrapping group and $\text{suc}(h_1(w, i))$ is solicited for membership in $G_w$. Note that if different nodes are returned by the two searches, the node with the ID closest clockwise to $h_1(w, i)$ is selected since this aligns with the linking rules.

During epoch $j$, all IDs in $G_{j-1}^1$ and $G_{j-1}^{-1}$ are active (and will remain in a passive state over the next epoch) and so can be used as members for new groups in $G_j^1$.

A similar process occurs to form $G_w$ in $G_j^2$, except that a different secure hash function $h_2$ is used. Thus, the membership of $G_w$ is likely different in each group graph.

Making a Neighbor Request. If $w$ and $u$ are neighbors in the input graph, then $G_w$ and $G_u$ should be neighbors in the group graph (recall that this entails all-to-all links between the members of both groups). By property P3 of the input graph $H$, each node $u \in L_w$ is dictated by $w$’s ID. On behalf of $w$, $G_w$ performs a search to locate each such neighbor $u$ in both old group graphs (again, favoring the closest-clockwise result). In this way, $G_w$ allows $u$ (and $G_u$) to learn about $w$ and agree to set up a link in the respective group graph.

Verifying Requests. The adversary may attempt to have many good nodes join as neighbors or members of a bad group. This attack is problematic since good nodes have their resources consumed by maintaining too many neighbors or joining too many groups; this increases the state cost (see Section I). To prevent this attack, any such request must be verified:

Verifying a Group-Membership Request. When node $u$ in $G_j^{i-1}$ is asked to become a member of group $G_u$ in $G_j^j$, node $u$ must verify that this request aligns with the linking rules; recall that this is assumed possible by property P3 of the input graph.

To do this verification, $u$ first checks in $G_j^{i-1}$ whether $h_1(w, i)$ for the appropriate $i$ (that accompanies the request) has a value in $[u - (c' \ln n)/n, u]$ for a constant $c' > 0$ sufficiently large (we relax notation such that $u$ refers to the

\footnote{As with much of the literature, we do not address concurrency. Since a join or departure requires updating only $\text{poly}(\log n)$ links in a group graph, we assume that there is sufficient time between events to do so.}
name of the node as well as its ID value). If \( h_1(w, i) \) does not fall within this distance, then \( u \) rejects the request; otherwise, the same check is performed in \( G'_2 \). See the proof in the Appendix of our full version \([23]\) for why this step in the verification is useful in giving low expected state cost.

If the request has not been rejected, \( u \) performs a search on \( h_1(w, i) \) in both group graphs \( G^{-1}_1 \) and \( G^{-1}_2 \). If either returns \( u \), then the request is considered verified and \( u \) becomes a neighbor of \( w \); otherwise, the request is rejected.

Note that \( u \) may erroneously reject a membership request; this is addressed in Lemma \([10]\). Conversely, \( u \) may erroneously accept a membership request, and the expected state cost is addressed in Lemma \([10]\).

Verifying a Neighbor Request. A node \( u \) that is asked to become a neighbor of node \( w \) (and thus establish links between the members of \( G_u \) and \( G_w \)) must verify the request. Similar to a membership request, \( u \) can determine via a search in \( G^{-1}_1 \) and \( G^{-1}_2 \) whether \( u \) should indeed be a neighbor of \( w \).

If either search returns \( u \), then the request is verified and \( u \) becomes a neighbor of \( w \); otherwise, the request is rejected.

Note that \( u \) may erroneously reject a neighbor request; this is addressed in Lemma \([8]\). Also, \( u \) may erroneously accept a request; the expected state cost is addressed in Lemma \([10]\).

Performing a Search. Throughout epoch \( j \), each new node \( w \) performs secure searches only using the old group graphs \( G^{-1}_1 \) and \( G^{-1}_2 \). This is done by forwarding the request to \( G_{new} \) and forwarding from the search on that position. Since \( G_{new} \) was active when \( w \) joined, then \( G_{new} \) should remain – even if in a passive state – to facilitate searches for another \( T \) steps.

Over the duration of epoch \( j \), note that \( G_w \) may not be able to reliably perform searches in the new group graphs \( G'_1 \) and \( G'_2 \) since they are still under construction. For example, \( w \) might be the first node to join \( G'_1 \) and \( G'_2 \).

Note that whenever a new node is bootstrapped into (or departs) \( G'_1 \) and \( G'_2 \), group \( G_w \) may need to update its neighbor links in \( G'_1 \) and \( G'_2 \) and this is done via searches in the old group graphs \( G^{-1}_1 \) and \( G^{-1}_2 \); in expectation, this will occur \( O(L_w) \) times given that IDs are u.a.r. in \([0, 1]\).

Once epoch \( j + 1 \) starts, the new group graphs \( G'_1 \) and \( G'_2 \) are to be used. At this point, \( G_w \) will initiate any search using its own links in these graphs (rather than relying on \( G_{new} \) which may no longer be present in the system).

B. Analysis

In this section, we prove that old group graphs satisfying \( S_1, S_2 \) and \( S_3 \) can be used to construct new group graphs that preserve \( S_1, S_2 \), and \( S_3 \). Due to space constraints, our proofs are provided in the Appendix of our full version \([23]\).

Properties P1-P4 of input graph \( H \) are critical to our arguments. However, a prerequisite to these properties is that all IDs are selected uniformly at random (see Section \([1C]\) which is untrue if the adversary chooses to add only some of its bad nodes; for example, maybe only bad nodes with IDs in \([0, \frac{1}{2}]\) are added by the adversary. Intuitively, this should not interfere with any of the properties.

In the following, we may consider \( H' \) to be a modified input graph which uses the same construction as \( H \), but is subject to an adversary that only includes a subset of its nodes (from a larger set of nodes with u.a.r. IDs).

Lemma 5. Consider a graph \( H' \) where the nodes are formed from two sets:

- \( N_1 \) consists of at least \((1 - \beta)n\) nodes with IDs selected u.a.r. from \([0, 1]\).
- \( N_2 \) is an arbitrary subset of at most \(\beta n\) nodes with IDs selected u.a.r. from \([0, 1]\).

W.h.p., under the same construction as the input graph \( H \), graph \( H' \) has properties P1 - P4.

Throughout, the above result is assumed — that properties P1-P4 continue to hold if the adversary includes only a subset of its IDs — even if we do not always make it explicit (for example, P1 is used throughout, P2 is used in Lemma \([6]\), P3 in Lemma \([7]\) and P4 in Lemma \([10]\).

As described above, for a node \( u \), there are searches on random key values (via hashing under the random oracle assumption) in order to find members for group \( G_u \). But if that key value maps to a bad node, then this results in a bad member added to the group. We can bound the probability of this event:

Lemma 6. W.h.p. a random key value in an old group graph maps to a bad node with probability at most \((1 + \delta')\beta\) for an arbitrarily small constant \(\delta' > 0\) depending only on sufficiently large \(n\).

In the following, let \( q_f = O(1/\log^{k-c} n) \) be the probability that a search for a random key in an old group graph \( G^{-1}_1 \) fails; this is dictated by Lemma \([4]\). Recall that a group \( G \) is good if \( d_1 \ln \ln n \leq |G| \leq d_2 \ln \ln n \) members, for constants \( d_1 < d_2 \), and at most \((1 + \delta)\beta |G| \) members are bad.

Lemma 7. A new group is bad with probability at most \(O(q_f^2 d_2 \log n + 1/\log^{d'} n) \) for a tunable constant \(d' > 0\) depending on \(d_2\).

Proof. For a new node \( w \), there are two ways in which \( G_w \) may end up bad when it issues its member requests. First, a search for a group member may fail; that is, the search encounters a bad group anywhere along the route. Given a point \( h_1(w, i) \), the probability that both searches in the old group graphs fail is at most \( q_f^2 \). By a union bound, the probability of such a dual failure occurring over \( d_2 \ln \ln n \) searches is \( O(q_f^2 d_2 \log \log n) \).

Second, the search succeeds but returns \( \text{suc}(h(w, i)) \) where \( \text{suc}(h(w, i)) \) is a bad node (even though its group has a good majority). Since \( h(w, i) \) is a random point (under the random oracle assumption), this event occurs with probability at most \((1 + \delta')\beta \) by Lemma \([6]\) for an arbitrarily small constant \(\delta' > 0\) given sufficiently large \(n\). Over \( d_2 \ln \ln n \) searches, the expected number of such events is at most \((1 + \delta')\beta d_2 \ln \ln n \). The probability of exceeding this expectation (and having too many bad nodes be added to the group) by more than a small constant factor is \(O(1/\log^{d'} n) \) by a Chernoff bound where the constant \(d' > 0\) is tunable depending only on sufficiently large \(d_2\).
Finally, the node being asked to join may reject the request. This occurs if both searches used to verify the request fail. By a union bound, this happens with probability at most $O(q_f^2 d_2 \log \log n)$.

A group $G_w$ should link to all groups with leaders in the neighbor set $L_w$. Recall that $|L_w| = O(\log^\gamma n)$ for some constant $\gamma > 0$. If $G_w$ (1) links to any group whose leader is not in $L_w$, or (2) fails to link to any group whose leader is in $L_w$, then $G_w$ is said to be confused. We now bound the probability of a confused group being created.

**Lemma 8.** Each group in a new group graph is confused independently with probability at most $O(q_f^2 \log^\gamma n)$.

**Proof.** Since the bootstrapping group is good, the only way in which a group $G_w$ is confused about a member of $L_w$ is if (1) the two searches for a neighbor point in the old group graphs both fail or (2) the corresponding target group erroneously rejects the request.

Applying Lemma 4, case (1) occurs with probability at most $q_f^2$ per element of $L_w$. Over $O(\log^\gamma n)$ potential neighbors, a union bound limits the probability of this occurring over all elements of $L_w$ by $O(q_f^2 \log^\gamma n)$. Regarding case (2), the target group will perform two searches per request and only reject if both fail; therefore, we get the same bound as in case (1).

We now prove that w.h.p. each new group graph is $\varepsilon$-robust.

**Lemma 9.** Assume the old group graphs are $\varepsilon$-robust and that the adversary adds at most $(1 + \varepsilon)\beta n$ nodes with u.a.r. IDs to a new group graph for an arbitrarily small constant $\varepsilon > 0$ depending only on sufficiently large $n$. Then, w.h.p., each new group graph is $\varepsilon$-robust.

**Proof.** To prove this result, we demonstrate equivalence between the construction of a new group graph and steps S1, S2, and S3 in Section III-A. Recall the terminology in Section III-A and designate all bad groups and confused groups as red, and all other groups as blue.

**Equivalence to S1.** By assumption, the adversary has at most $(1 + \varepsilon)\beta n$ u.a.r. IDs. The good nodes also have u.a.r. IDs. Using all of these IDs would give congestion $C$ corresponding to the input graph $H$ and thus step S1 would be satisfied. However, the adversary may choose to employ only a subset of its IDs — how does this affect the congestion? By Lemma 3, the resulting congestion is $O(C)$ and we have equivalence with S1 up to a constant factor (which does not affect the argument in Lemma 4).

**Equivalence to S2.** Satisfying S2 requires enforcing that for each construction of a new group graph, the probability of a red group (the group is bad or confused) is at most $p_f \leq 1/\log^k n$ for a tunable constant $k > 0$. By Lemmas 7 and 8, each group is red independently with probability at most:

\[
\begin{align*}
\leq & \quad O(q_f^2 \log^\gamma n) + O(q_f^2 d_2 \log \log n + 1/\log^{d'} n) \\
\leq & \quad O\left(\frac{\log \log n}{\log^{2(k-\varepsilon) - \gamma} n} + \frac{1}{\log^{d'} n}\right) \leq \frac{1}{\log^n n}
\end{align*}
\]

The last line follows by setting $d_2$ to be sufficiently large such that $d'$ exceeds $k$; note $d_2$ is fixed at the beginning and never needs to be changed throughout the lifetime of the network. Then, setting $k > 2c + \gamma$ to be a sufficiently large constant yields the necessary inequality with $p_f$.

**Equivalence to S3.** The incorrect link structure of confused groups corresponds to the adversary’s ability to add or delete edges between red groups (i.e., step S3).

Finally, Lemma 4 applies to the new group graph and completes the proof.

We prove bounds on the state a good node maintains due to (1) membership in groups, and (2) being a neighbor of a group. This is done by analyzing the verification process described in Section III-A; see [23] for the proof.

**Lemma 10.** In expectation, each good node $w$ in a group graph is a member of $O(\log \log n)$ groups and maintains state on $O(|L_w|)$ groups that are either neighbors or have $w$ as a neighbor.

We can now prove Theorem 2 and Corollary 1.

**Proof.** Lemma 9 guarantees w.h.p. that in the new group graphs, all but a $1/\poly(\log n)$-fraction of groups are good, and all but a $1/\poly(\log n)$-fraction of nodes can search for all but a $1/\poly(\log n)$-fraction of the resources.

Given that groups have size $O(\log \log n)$, it follows that group communication incurs $O((\log \log n)^2)$ messages. Recall that secure routing proceeds via all-to-all communication between members of groups and that searches have maximum length $D$ (P1 in Section I-C). Thus, the message complexity is $O(D(\log \log n)^2)$.

To bound the expected state cost, we invoke Lemma 10. Each good node $w$ belongs to $O(\log \log n)$ groups in expectation. This implies $O((\log \log n)^2)$ expected state cost to keep track of the members of these groups.

In terms of links to and from other groups, $w$ maintains state on $O(|L_w|) = O(\poly(\log n))$ groups in expectation. However, the constructions for $H$ defined in [17], [18], or [31] provide the properties P1-P4, but with a better bound of $O(1)$ expected degree. Using any of these constructions, the state cost incurred by these neighboring groups is $O(\log \log n)$ in expectation. Thus, the total expected state cost is $O((\log \log n)^2) + O(\log \log n) = O((\log \log n)^2)$.

**IV. COMPUTATIONAL PUZZLES**

Up to this point, we have assumed that the adversary can inject into each new group graph at most $(1 + \varepsilon)\beta n$ bad nodes with u.a.r. IDs, and that these IDs can be verified and forced to expire after a period of time (Section I-C). Given space constraints, we limit our discussion to the main ideas of how to use computational puzzles to guarantee these properties.

**A. Generating an ID**

All nodes are assumed to know two secure hash functions, $f$ and $g$, with range and domain $[0, 1)$ and that both hash functions satisfy the random oracle assumption.
In the current epoch $i$, node $w$ is assumed to possess a “globally-known” random string $r_{i-1}$ of $\ell \ln n$ bits. By “globally-known”, we mean known to all good nodes except the $1/\poly(\ln n)$-fraction from our earlier analysis. We motivate $r_{i-1}$ and describe how it is generated in Subsection IV-B.

Starting at step $T/2$ in the current epoch, each good node begins generating a new ID for use in the next epoch (see Subsection IV-B) as described below.

**Description of ID Generation.** To generate an ID, a good node $w$ selects a value $\sigma_w$ of $\ell \ln n$ random bits (matching the length of $r_{i-1}$). Then, $w$ XORs these two strings to get $\sigma_w \oplus r_{i-1}$, and checks if $g(\sigma_w \oplus r_{i-1}) \leq \tau$; if so, then $f(g(\sigma_w \oplus r_{i-1}))$ is a valid ID. We assume the value $\tau$ is set small enough such that w.h.p. a node requires $(1+\epsilon)T/2$ steps to find a $\sigma_w$ that satisfies this inequality, where $\epsilon > 0$ is a tunable (small) positive constant and $T > 0$ is a parameter set when the system is initialized.

The value of $T$ can be large to amortize the cost of forcing nodes to depart (and possibly rejoin) over a long period of time; preferably $T > n$ since new group graphs are being built over $T$ steps. Given an application domain, designers may estimate the rate of churn for their application and set a (loose) upper bound on $n$, then they can set $T$ accordingly.

**Why Use Two Hash Functions?** Consider using a single secure hash function $f$ to assign IDs; that is, if $g(x) < \tau$, then $x$ is a valid ID. Then, for example, the adversary may restrict itself to small inputs $x$ in order to confine its solutions to yielding small IDs. In other words, the IDs obtained by the adversary will not be u.a.r. from $[0, 1]$. This can be solved via composing two secure hash functions, $f$ and $g$, as described above. See [23] for the proof of the following:

**Lemma 11.** W.h.p., the adversary generates at most $(1+\epsilon)\beta n$ IDs over $(1+\epsilon)(T/2)$ steps and these IDs are u.a.r. in $[0, 1]$.

To simplify our analysis, we have omitted (as Lemma 11 implies) that the adversary can generate up to $(3(1+\epsilon)\beta n$ IDs for use in the next epoch. To counter this, we can revise the adversary’s power from $\beta$ to $\beta/3$ and results of previous sections hold. Note that all such IDs will be invalidated when the next random string is created.

**ID Verification.** Upon receiving a message from some node $w$, a good node $u$ verifies $w$’s ID. This could be done naively by having $w$ send $\sigma_w$ to $u$ who checks that $g(\sigma_w \oplus r_{i-1}) \leq \tau$ and that $f(g(\sigma_w \oplus r_{i-1}))$ evaluates correctly to the claimed ID (note that $u$ already has $r_i$ since it is globally-known). Unfortunately, this allows $u$ to steal $\sigma_w$ if $u$ is bad.

To avoid this issue, we assume a zero-knowledge scheme for revealing the pre-image of the hashing; such a scheme is provided for the SHA family [24]. This allows $w$ to prove the validity of $\sigma_w$ without revealing it.

If $w$’s ID cannot be verified, then $u$ simply ignores $w$ going forward. Note that $w$’s current ID will not be valid in the next epoch since it is signed by the older string $r_{i-1}$ (rather than the next globally-known random string $r_i$); that is, $w$’s ID will have expired. Nodes with IDs that are not verified are effectively removed from the system; they may consort with bad nodes, but they have no interactions with good nodes.

**B. Generating Global Random Strings**

Imagine if no random string was used in the creation of IDs described above in Subsection IV-B. The adversary would know the format of the ID-generation puzzles, and so could spend time computing a large number of IDs, and then use these IDs all at once to overwhelm the system at some future point. This is a pre-computation attack.

Signing IDs with a random string prevents such an attack as it is impossible for the adversary to know far in advance how to generate IDs. We provide a scheme where random strings are generated and propagated in the system to be used in ID generation. Due to space constraints, this content is provided in the Appendix of our full version [23]. Our overall result is:

**Lemma 12.** W.h.p., the protocol for propagating strings (i) guarantees that, for each good node $w$, its string used for generating an ID is known to is each good node, (ii) the number of strings stored by each node is $O(\ln n)$, and (iii) has message complexity $O(n \ln T)$.

Note that, averaged over the epoch, this message cost is low.

**V. Conclusion and Future Work**

We showed that groups of size $O(\log \log n)$ can be used to tolerate a powerful Byzantine adversary. Our result partly relies on PoW to limit the number of IDs the adversary controls. While it is interesting that this technique can be leveraged to reduce group size, an open question is whether the computational costs for the PoW component can be reduced. Might there be a way to avoid the constant solving of puzzles? Is there an approach that would only utilize puzzle solving when malicious IDs are present?

**References**

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Appendix

VI. Proofs for Section III-B

Lemma 5 Consider a graph $H'$ where the nodes are formed from two sets:
- $N_1$ consists of at least $(1 - \beta)n$ nodes with IDs selected u.a.r. from $[0, 1]$.
- $N_2$ is an arbitrary subset of at most $\beta n$ nodes with IDs selected u.a.r. from $[0, 1]$.

W.h.p., under the same construction as the input graph $H$, graph $H'$ has properties P1 - P4.

Proof. View the ID space as a unit ring, and place on it the nodes from $N_1$ and $N_2$. Let the total number of nodes be $m$ where $m \geq (1 - \beta)n$.

Moving clockwise from any node, consider a contiguous interval of length $(\lambda \ln m)/m$ where $\lambda > 0$ is any constant. Since IDs in $N_2$ are selected from a larger set of IDs u.a.r. in $[0, 1]$, intuitively the adversary’s choice of $N_2$ cannot significantly change the density/sparseness of nodes on the ring. By Chernoff bounds, regardless of how $N_2$ is selected, and for any $\lambda > 0$, the following holds:
- with probability at least $1 - (m - \lambda)/(m + \lambda)$, every interval contains at least $(\lambda/2) \ln m$ nodes, and
- with probability at least $1 - m^{-\lambda/12}$, every interval contains at most $(3\lambda/2) \ln m$ nodes.

A placement of $m$ nodes on the ring that satisfies these two properties is a $\lambda$-well-spread placement. Observe that no matter how the adversary chooses $N_2$, w.h.p. the adversary’s influence on the distribution of IDs is characterized by some $\lambda$-well-spread placement. In other words, we can ignore the adversary since the issue now reduces to: what is the probability of a $\lambda$-well-spread placement that degrades a property in $H'$?

We argue by contradiction as follows. Recall the guarantee that the input graph $H$ has some property $P$ with probability at least $1 - 1/m^d$ for a tunable constant $c > 0$ (Section III-A). Now assume there exists a set $S$ of $\lambda$-well-spread placements that violates this property $P$ for $H'$, and that these occur with aggregate probability $1/m^d > 1/m^c$ for some positive constant $d$. But this yields a contradiction since, with probability at least $1/m^d > 1/m^c$, placing $m$ nodes u.a.r. on the ring would yield a placement from $S$ for the input graph $H$ and, therefore, violate the guarantee of property $P$ for $H$.

Lemma 6 W.h.p. a random key value in an old group graph maps to a bad node with probability at most $(1 + \delta^2)\beta$ for an arbitrarily small constant $\delta > 0$ depending only on sufficiently large $n$.

Proof. By property P2 of the input graph $H$, w.h.p. a randomly chosen node in $H$ is responsible for at most a $(1 + \delta)/n$-fraction of the key values for an arbitrarily small $\delta > 0$ depending on sufficiently large $n$ (and, by Lemma 5 this holds even if the adversary does not add all of its bad IDs). Since the IDs of the adversary are u.a.r., the $\beta n$ bad nodes are responsible for at most a $(1 + \delta)\beta$-fraction of the key values.

Lemma 10 In expectation, each good node $w$ in a group graph is a member of $O(\log \log n)$ groups and maintains state on $O(|L_w|)$ groups that are either neighbors or have $w$ as a neighbor.

Proof. We perform our analysis with respect to a good node $w$. First, we analyze the state cost incurred by $w$ due to group-membership requests. Second, we analyze the state cost incurred by $w$ due to (i) neighbors $w$ links to, and (ii) those neighbor requests $w$ receives.

State Cost of Group-Membership Requests. In the absence of adversarial interference, the expected number of group-member requests that $w$ receives is $O(\log \log n)$ given that such requests are distributed among all nodes u.a.r. and each group requires $O(\log \log n)$ members.

Now, we consider the impact of the adversary on this state cost. Consider node $w$ receives a request to join some group $G_w$ in the first group graph. As described in Section III-A, node $w$ first checks that $h_1(w, i)$ for the appropriate $i$ lies in the interval $\mathcal{I} = [u - (\ell \ln n)/n, u]$, where we relax notation and use $u$ to also denote the ID of node $u$. We reiterate a well-known argument: since IDs are u.a.r., the probability that two nodes are separated by more than a $(\ell \ln n)/n$ distance is at most $(1 - (\ell \ln n)/n)^n \leq 1/n^\Omega(\ell)$. Therefore, if $h_1(w, i)$ lies outside of this interval, then w.h.p. $w$ is not the successor of $h_1(w, i)$ and cannot be a valid neighbor. In this case, the request is safely rejected.

If $h_1(w, i)$ does lie within the interval, then $w$ performs a search on $h_1(w, i)$. If this returns $u$, then $u$ accepts the request. Given the $\varepsilon$-robustness guarantee in old group graphs, with probability at least $1 - O(1/\log^{k_c} n)$, this acceptance is correct (recall Lemma 4). In other words, the probability of accepting an erroneous neighbor request is at most $1/\text{poly}(\log n)$ where we can tune this polynomial.

How many neighbor requests with an $h_1(w, i) \in \mathcal{I}$ does $w$ receive? Given that $h_1(w, i)$ is u.a.r. (given the random oracle assumption) the probability that $h_1(w, i)$ maps to this interval is $(\ell \ln n)/n$. By property P3, $i = O(\log^2 n)$ for some constant $\gamma > 0$, and so over at most $n$ nodes that might own valid IDs in $\mathcal{I}$, there are $O(\log^2 n)$ such requests received by $w$; this holds w.h.p. by a standard Chernoff bound.

It follows that $w$ erroneously accepts $O(\log^2 n)/\text{poly}(\log n) = O(1)$ malicious requests in expectation as long as $k$ is sufficiently large with respect to $c$.

State Cost of $w$’s Neighbors. Recall by property P3 of the input graph $H$, that $w$ links to $O(|L_w|)$ other nodes. Therefore, in each group graph, $w$ links to $O(|L_w|)$ groups as neighbors.

State Cost of Neighbor Requests. Via Lemma 5 properties P1 and P3 guarantees that $w$ can determine independently whether it should indeed be a neighbor of some node $u$, and there are at most poly$(\log n)$ such nodes $u$. Using the old group graphs, wherein $\varepsilon$-robustness is guaranteed w.h.p., $w$ initiates a search to check that it should indeed be a neighbor. With a tunable probability at least $1 - O(1/\log^{k_c} n)$ node $w$ can detect if the request is erroneous. Therefore, in expectation, the number
of erroneous acceptances is at most $o(|L_u|)$ so long as our constant $c$ is sufficiently large (recall Lemma 4).

VII. GENERATING GLOBAL RANDOM STRINGS

**Lemma 11** W.h.p., the adversary generates at most $(1+\epsilon)\beta n$ IDs over $(1+\epsilon)(T/2)$ steps where these IDs are u.a.r. in $[0,1)$.

**Proof.** Since the adversary has $\beta n$ computational power to expend over this epoch, w.h.p. it can generate at most $(1+\epsilon)\beta n$ solutions $\sigma_j$ such that $g(\sigma_j \oplus r_{i-1}) \leq \tau$ within $(1+\epsilon)(T/2)$ steps where the constant $c > 0$ can be made arbitrarily small depending only on sufficiently large $n$. By the random oracle assumption, applying $f$ to these solutions yields at most $(1+\epsilon)\beta n$ IDs u.a.r. from $[0,1)$.

**Generating Global Random Strings.** During epoch $i$, each node $w$ forms a solution set of the $d_0 \ln n$ smallest random strings $R_i^w$ for some constant $d_0 \geq 1$. Over epoch $i$, all good nodes generate random strings and the smallest are collected independently by each node $w$ to create $R_i^w$. To generate a string in epoch $i$, a node $w$ uses a string $r_{i-1}$ — the globally-known string from the previous epoch — and an individually generated random string, $s_u$, to compute the output $t_u = h(s_u \oplus r_{i-1})$.

**Bins and Counters.** To facilitate our discussion of how to propagate strings and ease our subsequent analysis, we describe a system of bins and counters maintained by each good node $w$. The bins $B_j$ correspond to intervals in the ID space where $B_j = \{1/2^j, 1/2^{j-1}\}$ for $j = 1, 2, ..., b\ln(nT)$ where $b \geq 1$ is a sufficiently large constant. Since $T$ is known and there are standard techniques for obtaining a constant-factor approximation to $\ln n$, calculating $\ln(nT) = \ln(n) + \ln(T)$ to within a constant factor is possible.

Each bin $B_j$ has an associated counter $C_j$. Consider that $w$ receives a string $s_u$ with corresponding output $t_u$ that falls within the interval defined by $B_j$; we say that $B_j$ contains $s_u$. If $t_u$ is smaller than the other values $w$ has seen so far contained in $B_j$, and $C_j \leq c_0 \ln n$ for some sufficiently large constant $c_0 \geq 1$, then $w$ increments $C_j$ and forwards $s_u$ onto its neighbors. After $C_j = c_0 \ln n$, no value landing within $B_j$ is ever forwarded.

The intuition is that, if $c_0 \ln n$ strings are found with "record-breaking" outputs in $B_j$, then w.h.p. smaller strings exist with outputs belonging to $B_{j+1}$. In other words, those strings corresponding to $B_j$ will not be candidates a globally-known string and so they can be ignored.

**Protocol for Propagating Strings.** The propagation of strings is broken into *phases* which make up the first half of an epoch. We describe the protocol for good nodes (although bad nodes can deviate arbitrarily).

Phase 1 executes over steps $1$ to $T/2 - 2d' \ln n$ for a constant $d' > 1$ of the current epoch $i$. Over this time, each node $w$

\[13\] A standard technique for estimating $\ln n$ to within a constant factor is as follows. For nodes with u.a.r. IDs, the distance $d(u,v)$ between any two nodes $u$ and $v$ satisfies $\ln(1/\delta) \leq d(u,v) \leq \ln(1/\delta')$ with high probability. Therefore, with high probability, $\ln(1/\delta) = \Theta(\ln n)$ and this holds even when an adversary decides to omit its nodes in the ID space (see Chapter 4 in [7]),

generates random strings with associated outputs. After Phase 1 ends, nodes no longer generate new random strings.

Phase 2 begins at step $T/2 - 2d' \ln n + 1$ and runs for $d' \ln n$ steps. Each node $w$ (using its group $G_w$) selects the string $s_u^w$ with the smallest output $t_u^w$ that was generated in Phase 1, and then sends $s_u^w$ its neighbors. Node $w$ updates the corresponding bin and counter, as described earlier.

Each neighbor $u$ verifies $s_u^w$. Using $t_u^w$, node $u$ decides whether to forward $s_u^w$ to its own neighbors (except for $w$) and, if so, updates the corresponding bin and counter; otherwise, $u$ ignores this value. At the end of Phase 2, each node $w$ selects the string with the smallest output it has seen so far; this is denoted by $s_u^{w*}$. Phase 3 starts at step $T/2 - d' \ln n + 1$ and runs for the final $d' \ln n$ steps. Over these steps, nodes no longer generate new strings, although they will still propagate them according to the above rules.

At the end of the phase, each node $w$ creates its solution set $R_i^w$ in the following way. Node $w$ finds the largest $j$ for which $B_j$ contains at least one element. Then, $w$ takes the union of subsequent bins for decreasing $j$ until there are $d_0 \ln n$ elements; the collection of these elements form $R_i^w$.

This concludes the propagation protocol. We note that immediately (at step $T/2 + 1$) node $w$ will start generating a new ID signed with the string $s_u^{w*}$ chosen in Phase 2.

**Discussion.** The adversary may prevent good nodes from agreeing on the same solution set. As mentioned in Subsection [IV-A] a 1/poly$(\log n)$-fraction may be unable to partake in the propagation process even with our secure routing, and their loss is already incorporated into our analysis in Subsection [III-B]. Therefore, we address the giant component of $(1 - 1/poly(\log n))n$ good nodes that can reach each other (and this set of nodes is implied by the terminology “good nodes”).

The critical source of disagreement between good nodes is that the adversary may delay releasing a string $s'$ (or multiple strings) with a small output. For example, if this occurs right before the end of Phase 2, then only a subset of good nodes receive $s'$ and their respective solution sets differ from the other good nodes. In this way, two good nodes $u$ and $w$ can choose different strings $s_u^{w*}$ and $s_u^{w*}$.

We sketch how this disagreement is handled, but first we address the simpler case where there is no adversarial interference.

**With No Adversary:** Note that the propagation of a string in the giant component requires at most $d' \ln n$ steps. Therefore, since all nodes send their string at the beginning of Phase 1, then by the end of Phase 2, all nodes accept the same set of strings and agree on the minimum string.

Furthermore, in Phase 3, nothing will occur (since no strings are released late) and so any nodes $w$ and $u$ are guaranteed w.h.p. to have $R_i^w = R_i^u$. What are the outputs corresponding to these solution sets? There are $\Theta(n)$ nodes computing for $\Theta(T)$ steps, so the smallest output in a set $R_i^w$ is $\Theta(1/nT)$ and w.h.p. no larger than $O(\ln n/\sqrt{T})$. 
With an Adversary: The adversary can propagate a string $s'$ with a small output late in Phase 2. If $w$ receives $s'$ while $u$ does not, then $R_u^w \neq R_v^w$. We argue that (1) the size of each solution set remains bounded by $\Theta(\ln n)$, and (2) that the string $s_{w_1}^u$ used by each good node $w$ belongs to every other good nodes’ solution set; these two properties enable efficient and correct verification (described below).

How many solutions $s'$ could $w$ receive and add to $R_u^w$? As noted above, this solution set will hold outputs of value $\Theta(\ln n)$. Since the adversary has bounded computational power of $\beta n$, w.h.p. there cannot be more than $d^w \ln n$ solutions with output value $\Theta(\frac{\ln n}{d^w})$ for some constant $d^w > 0$. This is true even if the adversary computes over the entire epoch. We set the constant $c_0$ used in the bin counters such that $c_0 \geq d^w$ in order to make sure that no smallest values are omitted.

Now assume that $w$ selects $s_{w_1}^u$, but that $s_{w_1}^u$ is not present in good node $u$’s solution set $R_u^w$. We will derive a contradiction. If $s_{w_1}^u$ originated from a good node, then $w$ received $s_{w_1}^u$ by the end of Phase 3 since $2d^w \ln n$ steps are more than sufficient for the propagation of a string in the giant component. Else, $s_{w_1}^u$ originated from the adversary. Since $s_{w_1}^u$ was held by $w$ by the end of Phase 2, the addition $d^w \ln n$ steps in Phase 3 would have allowed $s_{w_1}^u$ to reach $u$ and be added to $R_u^w$. In either case, this yields the contradiction.

Finally, what is the message complexity of the propagation protocol? Recall that for each bin, the associated counter restricts to $\Theta(\ln n)$ the number of times a node forwards a string to its neighbors. Given that there are $\Theta(\ln(nT))$ bins, the total number of times a node can forward a string is $\Theta(\ln(nT))$. The number of messages sent between any pair of neighboring groups is $O(\Theta(2^w) = O((\log \log n)^2))$ and the degree in the group graph is $O(\log (\log n))$. Therefore, the total message complexity over $O(n)$ nodes is $O(\Theta(\ln n/nT))$ where $O$ accounts for $\log \log n$ terms.

The above discussion supports the following:

**Lemma 12** With high probability, the protocol for propagating strings (i) guarantees that, for each good node $w$ in the component, $s_{w_1}^u$ is contained within the solution set of every good node in the component, (ii) $|R_u^w| = \Theta(\ln n)$, and (iii) has message complexity $O(\ln n/nT)$.

**Verifying IDs.** For simplicity, our discussion of ID verification in Subsection 3.4 assumed that a single $r_{i-1}$ was agreed upon. However, not much changes when using solution sets.

To generate an ID for use in epoch $i + 1$, node $w$ uses $s_{w_1}^u$ to sign its ID. By the above discussion, we are guaranteed w.h.p. that $s_{w_1}^u$ belongs to the solution set of each good node. Therefore, a good node $u$ that wishes to verify $w$’s new ID checks whether this ID was signed by any of the strings in $R_u^w$; this requires checking only $\Theta(\ln n)$ elements by the above discussion.

In our discussion of the properties of the input graph in Section 5 we made the following statement in P3: “... there are $O(\log N)$ nodes whose IDs dictate $w \in L_u$”. We discuss this here further using Chord as an example; however, the same property holds for the other input graphs we specify in Section 3.

We consider the version of Chord where IDs are in $[0, 1)$. Node $u$ has neighbors (in its “Distance Table”) that are found by taking points $2^{-i}$ and linking to the successor of each such point, for $i = 1, \ldots, O(\log (N))$. Therefore, if $u$ is claiming $w$ as a neighbor, then $w$ can examine the index $i$ and immediately determine if $u + 2^{-i}$ is “close enough” to $w$. For example, if $u + 2^{-i} = 0.5$, but $w = 0.9$, then clearly the successor $u + 2^{-i}$ will not be $w$ and the request is erroneous.

How close is “close enough” such that $w$ must perform a search? Since IDs are u.a.r. in $[0, 1]$, it is easy to see that w.h.p. the largest interval between any two nodes is $\Theta(\log N)$. Therefore, if $u + 2^{-i}$ is outside of the subinterval $[w - \Theta(\log N), w]$, then $w$ can ignore the request. Otherwise, it must perform a search to see if the successor of $u + 2^{-i}$ is indeed $w$.

How many nodes have a neighbor link that falls into this interval $[w - \Theta(\log N), w]$ and cannot be rejected out of hand? Again, a standard argument can be made that $O(\log N)$ nodes fall within any interval of this size, and this is tight w.h.p. by a Chernoff bound; a request from each such node must be checked via a search. Since the degree of each node has degree $\Theta(\log N)$, this means that only $O(\log(\log n))$ nodes can make neighbor requests as claimed.

**IX. Bootstrapping Groups**

A standard assumption in the literature is that a node knows how to contact another node already in the system in order to be bootstrapped. In the absence of an adversary, this seems plausible and the assumption holds true in practice.

The bootstrapping issue is less clear in a Byzantine setting and we consider it an open challenge, although not within the scope of our work here. We can imagine that a node might know (i.e. have IP addresses and port numbers) for an entire group which can then act as $G_{boot}$. However, it is unclear how this information would be provided.

If the information is advertised on a server, then this becomes a point of attack. Alternatively, if this information is hard-coded into the application that is downloaded onto the node, then we must rely on someone to do the hard-coding.

Or perhaps another distributed system is in place to facilitate bootstrapping akin to Vuze for BitTorrent, but then that system must also be Byzantine fault tolerant.

Note that these issues arise whether all groups are good, or “almost all” groups are good. We conjecture that a solution is possible via PoW — having a group be able to offer a bootstrapping service only if it solves a puzzle — but we leave this issue to future work.