Monotone 3-Sat-4 is \( \mathcal{NP} \)-complete.

Andreas Darmann, Janosch Döcker

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Abstract

Monotone 3-Sat-4 is a variant of the satisfiability problem for boolean formulae in conjunctive normal form. In this variant, each clause contains exactly three literals—either all or none of them are positive, i.e., no clause contains both a positive and a negative literal—and every variable appears at most four times in the formula. Moreover, every clause consists of three distinct literals. We show that Monotone 3-Sat-4 is \( \mathcal{NP} \)-complete.

1 Introduction

The satisfiability problem for boolean formulae in conjunctive normal form—or one of its many variants—is frequently used in order to show that some decision problem is \( \mathcal{NP} \)-hard; for an introduction in the theory of \( \mathcal{NP} \)-completeness we refer to Garey and Johnson [GJ79]. Here, the motivation for looking into monotone variants of this problem is a conjecture attributed to Sarah Eisenstat in the scribe notes [DKY14] of an MIT lecture. The conjecture states that Monotone 3-Sat-5 is \( \mathcal{NP} \)-hard.

The notation \( r \)-Sat-\( s \) denotes the variant of the satisfiability problem where every clause contains exactly \( r \) distinct variables and each variable appears in at most \( s \) clauses. When we use \((p, q)\) instead of \( r \) this means that every clause contains either \( p \) or \( q \) distinct variables. We write clauses as subsets of a finite set \( V \) of variables, emphasizing that all variables need to be different in the variants of the satisfiability problem we consider in this paper. A \( k \)-clause contains exactly \( k \) distinct variables and a clause is called monotone if either all contained literals are positive or all of them are negative, respectively. A mixed clause is a clause which is not monotone, i.e., it contains at least one positive and at least one negative literal. Let \( C \) be a \( k \)-clause. The notation \( \text{Var}(C) \) means that we remove negations if there are any, i.e., we map \( C \) to the monotone \( k \)-clause containing the same variables in their unnegated form.

Monotone \( r \)-Sat-\( s \) is the restriction of \( r \)-Sat-\( s \) such that all clauses are monotone. It is known that the monotone satisfiability problem for boolean formulae in conjunctive normal form is \( \mathcal{NP} \)-hard [Gol78] and remains hard even if every clause contains exactly three distinct variables (see [Li97]).

In this paper, we prove the conjecture mentioned above and show that even Monotone 3-Sat-4 remains hard. The latter problem is a restriction of 3-Sat-4 which was proven to be \( \mathcal{NP} \)-hard by Tovey [Tov84]. Tovey also showed

\footnote{Algorithmic Lower Bounds: Fun with Hardness Proofs (Fall '14), Prof. Erik Demaine, Teaching assistants: Sarah Eisenstat, Jayson Lynch}
that 3-Sat-3 is trivial, i.e., instances of this problem are always satisfiable. Consequently, Monotone 3-Sat-3 is trivial as well.

2 Hardness of Monotone 3-Sat-s for $s \geq 4$

Let $I := (V, C)$ be any 3-Sat-4 instance (for the proof that 3-Sat-4 is $\mathcal{NP}$-complete see the work by Tovey [Tov84]). Applying Gold’s [Gol78, p. 314f] replacement rule to each mixed clause yields an equisatisfiable Monotone (2, 3)-Sat-4 instance $I' := (V', C')$: Consider any mixed clause $C = C^+ \cup C^-$, where $C^+$ contains the positive literals of $C$ and $C^-$ the negative literals, respectively. Then, creating a new variable $u$ and replacing $C$ with the two clauses $C^+ \cup \{u\}$ and $C^- \cup \{\bar{u}\}$ yields an equisatisfiable instance with one mixed clause less. Since $|C^+| + |C^-| = |C| = 3$, one of the introduced clauses has size 2 and the other one has size 3. The replacement does not change the number of appearances of any variable $v \in V$ and the created variable appears exactly twice in $I'$. Thus, we have shown:

Lemma 1. Monotone (2, 3)-Sat-4 is $\mathcal{NP}$-complete.

The next step is to replace the clauses of size 2. Li [Li97, p. 295] observed that a clause $\{x, y\}$ is satisfiable if and only if $\{x, y, u\}, \{x, y, v\}, \{x, y, w\}$ are satisfiable and $\{\bar{x}, \bar{y}\}$ is satisfiable if and only if $\{\bar{x}, \bar{y}, \bar{u}\}, \{\bar{x}, \bar{y}, \bar{v}\}, \{\bar{x}, \bar{y}, \bar{w}\}, \{u, v, w\}$ are satisfiable, where $u, v$ and $w$ are distinct new variables. Note that this replacement rule increases the number of appearances of the variables $x$ and $y$. We show that this can be avoided by defining a suitable replacement rule which only creates new variables with at most five appearances.

In the following we define multiple rules $R_i$ that replace a monotone 2-clause $C$ in a collection $K$ of clauses by monotone 3-clauses $C_1, C_2, \ldots, C_j$ so that $C$ is satisfiable if and only if $C_1, C_2, \ldots, C_j$ are satisfiable and

$$\left(\bigcup_{k=1}^j \text{Var}(C_k)\right) \cap \left(\bigcup_{C \in K} \text{Var}(C)\right) \subseteq \text{Var}(C),$$

i.e., with the exception of the two variables appearing in $C$ all other variables appearing in $C_k$, $1 \leq k \leq j$, are new variables. The rules are of the form

$$R^C_k := \begin{cases} \{x, y\} \equiv C_1, C_2, \ldots, C_j, \\ \{\bar{x}, \bar{y}\} \equiv C_1', C_2', \ldots, C_j', \end{cases},$$

where $K$ is a collection of clauses, i.e., the context in which the rule is applied. In the following we omit the $K$ in the rule definitions to increase readability. Note that applying such a rule changes the context for further applications of the same or different rules. The notation $\{x, y\} \equiv C_1, C_2, \ldots, C_j$, means that the clause $\{x, y\}$ is satisfiable if and only if the clauses $C_1, C_2, \ldots, C_j$ are satisfiable; the other case is defined in the same way. We write $R_i(C)$ to denote
a rule application respecting the properties mentioned above: If \( C \) consists of two positive literals, then we replace \( C \) according to the top case of the rule; and if \( C \) consists of two negative literals we replace \( C \) according to the bottom case. We use the notation \( \Delta^v_R, \Delta^w_R, \) and \( \Delta^\text{new}_R \) to denote the maximum number by which an application of rule \( R_i \) to a clause \( \{ x, y \} \) or \( \{ \bar{x}, \bar{y} \} \) increases the appearances of \( x, y \), and the new variables, respectively.

**Replacement rule** \( R_1 \) Let \( R_1 \) denote Li’s replacement rule, which looks in our notation as follows:

\[
R_1 := \begin{cases} 
\{ x, y \} \equiv \{ x, y, u \}, \{ x, y, v \}, \{ x, y, w \}, \{ \bar{u}, \bar{v}, \bar{w} \} \\
\{ \bar{x}, \bar{y} \} \equiv \{ \bar{x}, \bar{y}, \bar{u} \}, \{ \bar{x}, \bar{y}, \bar{v} \}, \{ \bar{x}, \bar{y}, \bar{w} \}, \{ u, v, w \}.
\end{cases}
\]

We have

\[
\Delta^v_{R_1} = \Delta^w_{R_1} = \Delta^\text{new}_{R_1} = 2.
\]

**Replacement rule** \( R_2 \) As an intermediate step we define a second replacement rule:

\[
R_2 := \begin{cases} 
\{ x, y \} \equiv \{ x, y, u \}, \{ x, y, v \}, R_1(\{ \bar{u}, \bar{v} \}) \\
\{ \bar{x}, \bar{y} \} \equiv \{ \bar{x}, \bar{y}, \bar{u} \}, \{ \bar{x}, \bar{y}, \bar{v} \}, R_1(\{ u, v \}).
\end{cases}
\]

Observe that

\[
\{ x, y \} \text{ is satisfiable } \iff \{ x, y, u \}, \{ x, y, v \}, R_1(\{ \bar{u}, \bar{v} \}) \text{ are satisfiable}
\]

and

\[
\{ \bar{x}, \bar{y} \} \text{ is satisfiable } \iff \{ \bar{x}, \bar{y}, \bar{u} \}, \{ \bar{x}, \bar{y}, \bar{v} \}, R_1(\{ u, v \}) \text{ are satisfiable},
\]

where \( u \) and \( v \) are distinct new variables. We have

\[
\Delta^v_{R_2} = \Delta^w_{R_2} = 1 \text{ and } \Delta^\text{new}_{R_2} = \max(2 + \Delta^v_{R_1}, 2 + \Delta^w_{R_1}, \Delta^\text{new}_{R_1}) = 4.
\]

**Replacement rule** \( R_3 \) Using the preceding rule—and implicitly also Li’s rule—we can define a replacement rule with the desired properties:

\[
R_3 := \begin{cases} 
\{ x, y \} \equiv \{ x, y, u \}, R_2(\{ \bar{u}, \bar{v} \}), R_2(\{ \bar{u}, \bar{w} \}), R_2(\{ v, w \}) \\
\{ \bar{x}, \bar{y} \} \equiv \{ \bar{x}, \bar{y}, \bar{u} \}, R_2(\{ u, v \}), R_2(\{ u, w \}), R_2(\{ \bar{v}, \bar{w} \}).
\end{cases}
\]

Observe that

\[
\{ x, y \} \text{ is satisfiable } \iff \{ x, y, u \}, \{ \bar{u} \} \text{ are satisfiable}
\]

\[
\iff \{ x, y, u \}, R_2(\{ \bar{u}, \bar{v} \}), R_2(\{ \bar{u}, \bar{w} \}), R_2(\{ v, w \}) \text{ are satisfiable}
\]

and

\[
\{ \bar{x}, \bar{y} \} \text{ is satisfiable } \iff \{ \bar{x}, \bar{y}, \bar{u} \}, R_2(\{ u, v \}), R_2(\{ u, w \}), R_2(\{ \bar{v}, \bar{w} \}) \text{ are satisfiable},
\]

\[
3
\]
where $u$, $v$ and $w$ are distinct new variables. We have
\[
\Delta_{R_3} = \Delta_{R_3}^v = 0 \text{ and } \Delta_{R_3}^{uw} = \max(3 + 2\Delta_{R_2}^u, 2 + 2\Delta_{R_2}^v, 2 + 2\Delta_{R_2}^w, \Delta_{R_2}^{uw}) = 5.
\]

An application of Rule $R_3$ replaces one clause with 19 new clauses using 18 new variables and reduces the number of 2-clauses by one. Actually, 17 clauses and 16 variables suffice, since we could have used $R_4(C)$ instead of $R_3(C)$ for $C \in \{\{v, w\}, \{\bar{v}, \bar{w}\}\}$ in the definition of $R_3$. The reason for not doing so is that $R_3$ and the calculation of $\Delta_{R_3}^{uw}$ appear a little simpler the way it is now. The number of necessary applications of $R_3$ is exactly the number of 2-clauses (of a monotone instance, of course). Since applying $R_3$ only introduces variables appearing at most five times and leaves the number of appearances of all other variables unchanged, we have proven:

**Theorem 1.** **Monotone 3-Sat-5** is $\mathcal{NP}$-complete.

Now, we show that Monotone 3-Sat-4 is $\mathcal{NP}$-complete. Again, we start with an instance of Monotone (2, 3)-Sat-4 and the goal is to get rid of the clauses of size 2 while preserving equisatisfiability. In order to achieve that, we present a finite collection of monotone 3-clauses $C_z$ such that no variable appears more than four times and a designated variable $z$ appears exactly three times, and show that this collection is satisfiable if and only if $z$ is set to true. If there is a clause of the form $\{x, y\}$ in the instance, we replace this clause with $\{\bar{x}, \bar{y}, z\}$ and add $C_z$ to the instance. The result is an equisatisfiable Monotone (2, 3)-Sat-4 instance with one negative 2-clause less. Of course, all variables appearing in $C_z$ are newly created. By negating every variable appearance in $C_z$, we can force $z$ to be set to false. Therefore, we can get rid of clauses of the form $\{x, y\}$ analogously. The collection $C_z$ is given by the following 25 clauses.

1. $\{u, w, z\}$
2. $\{u, v, z\}$
3. $\{\bar{w}, \bar{v}, \bar{g}\}$
4. $\{w, \bar{v}, h\}$
5. $\{w, \bar{v}, \bar{r}\}$
6. $\{g, h, i\}$
7. $\{\bar{m}, \bar{n}, \bar{g}\}$
8. $\{\bar{m}, \bar{n}, \bar{h}\}$
9. $\{\bar{m}, \bar{n}, \bar{i}\}$
10. $\{m, a, b\}$
11. $\{n, a, b\}$
12. $\{\bar{u}, \bar{a}, \bar{r}\}$
13. $\{\bar{u}, \bar{b}, \bar{r}\}$
14. $\{r, z, f\}$
15. $\{\bar{d}, \bar{e}, \bar{a}\}$
16. $\{\bar{d}, \bar{e}, \bar{b}\}$
17. $\{p, q, d\}$
18. $\{p, q, e\}$
19. $\{\bar{f}, \bar{p}, \bar{c}\}$
20. $\{\bar{f}, \bar{q}, \bar{e}\}$
21. $\{r, c, j\}$
22. $\{j, \bar{p}, \bar{k}\}$
23. $\{j, \bar{q}, \bar{k}\}$
24. $\{k, c, \ell\}$
25. $\{\ell, \bar{j}, \bar{f}\}$

Assume that the above collection of clauses is satisfiable by a truth assignment in which $z$ is set false.

First, we show that this implies that $u$ has to be set true. If $u$ is set false, then the first two clauses imply that both $w$ and $v$ need to be set true. Clauses 3, 4, 5 thus yield that all three of $g, h, i$ have to be set false, in contradiction with clause 6. Thus, $u$ has to be set true.

By clause 6 at least one of $g, h, i$ has to be set true. Thus, clauses 7, 8, 9 imply that at least one of $m, n$ has to be set false. As a consequence, clauses 10, 11 yield that at least one of $a, b$ needs to be set true. In turn, by clauses 12, 13 this means that $r$ has to be set false (recall that $u$ is set true). Since both $r, z$ are set false, $f$ must be set true due to clause 14. By the fact that at least
one of $a, b$ is true, clauses 15, 16 imply that at least one of $d, e$ is set false. In turn, by the next two clauses this means that at least one of $p, q$ must be set true. In addition, recalling that $f$ is set true, clauses 19, 20 imply that $c$ has to be set false. Also recalling that $r$ is set false, this means that $j$ has to be set true due to clause 21. Now, clauses 22, 23 imply—since at least one of $p, q$ is true—that $k$ has to be set false. Hence, as a consequence of clause 24 and the fact that both $k, c$ are set false, $\ell$ has to be set true. That is, all of $\ell, j, f$ are set true, in contradiction with clause 25. Therewith, there in no satisfying truth assignment for the above formula in which $z$ is set false.

On the other hand, it is not hard to verify that the formula is satisfiable; e.g., setting all variables of the set $\{z, g, a, r, e, p, k\}$ true and the remaining ones false yields a satisfying truth assignment.

Finally, note that $z$ occurs exactly 3 times, while none of the other variables is contained in more than four clauses. Thus, we have shown:

**Theorem 2.** Monotone 3-Sat-4 is $\mathcal{NP}$-complete.

### 3 Conclusion

We have proven that Monotone 3-Sat-4 is $\mathcal{NP}$-complete. The correctness of the conjecture mentioned in the introduction stating that Monotone 3-Sat-5 is $\mathcal{NP}$-hard follows immediately from this result. Nonetheless, we also provided a proof of the conjecture since the proof is interesting in itself.

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### References

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