Virtual reshaping and invisibility in obstacle scattering

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Abstract

We consider reshaping an obstacle virtually by using transformation optics in acoustic and electromagnetic scattering. Among the general virtual reshaping results, the virtual minification and virtual magnification in particular are studied. Stability estimates are derived for scattering amplitude in terms of the diameter of a small obstacle, which implies that the limiting case for minification corresponds to a perfect cloaking, i.e., the obstacle is invisible to detection.

(Some figures in this article are in colour only in the electronic version)

1. Introduction

Since the pioneering work on transformation optics and cloaking [9, 10, 13, 24], there has been an avalanche of studies on designs of various striking cloaking devices, e.g., invisibility cloaking devices [6, 12], field rotators [1], concentrators [17], electromagnetic wormholes [4, 5], superscatterers [25], etc. We refer to a recent survey paper [7] for a comprehensive review and related literature. The crucial observation is that certain PDEs governing the wave phenomena are form invariant under transformations, e.g., the Hemholtz equation for acoustic scattering and Maxwell’s equations for electromagnetic (EM) scattering. Hence, one could form new acoustic or EM material parameters (in the physical space) by pushing forward the old ones (in the virtual space) via a mapping \( F \). Such materials/media are called transformation media [24]. It turns out that the wave solutions in the virtual space with the old material parameters and in the physical space with the new material parameters are also related to the push-forward \( F \). Those key ingredients pave the way for the design of optical devices with customized effects on wave propagation.

In this paper, we shall be concerned with cloaking devices for acoustic and electromagnetic obstacle scattering. As is known, there are two types of scatterers which are under a wide study for acoustic and EM scattering, namely, the penetrable medium and the impenetrable obstacle. For a medium, the acoustic or EM wave can penetrate inside, and basically the medium accounts for the coefficients in the governing PDEs. Whereas for an obstacle,
the acoustic or EM wave cannot penetrate inside and only exists in the exterior of the object, and the obstacle is related to the domain of definitions for the governing PDEs. The cloakings for acoustic or EM media have been extensively studied in transformation optics in the existing literature and the theory has been well established, we again refer to the review paper [7] for related discussion. For our current study, the cloakings for obstacles are considered and it is shown that the domain of definitions for certain PDEs can also be pushed forward under transformations. Using the transformation optics, one can push forward an obstacle in the virtual space to form a different obstacle in the physical space, and the ambient space around the virtual obstacle is then pushed forward to a cloaking medium around the physical obstacle. With a suitable push-forward $F$, it is shown that the scattering amplitude in the physical space coincides with that in the virtual space. That is, if one intends to recover the physical obstacle after being cloaked by the corresponding scattering measurements, then the reconstruction will give the image of the obstacle in the virtual space, but not the physical one, namely, the physical obstacle is virtually reshaped with the cloaking. Principally, it has been shown that one can achieve any desired virtual reshaping effect provided an appropriate transformation $F$ can be found between the virtual space and the physical space.

Particularly, we consider virtually magnifying and minifying an obstacle. By magnification, we mean that the size of the virtual obstacle is larger than that of the underlying physical one. That is, under acoustic and EM wave detection, the cloaking makes the obstacle look bigger than its original size. Whereas by minification, we actually mean virtually shrinking the obstacle, that is, the size of the virtual obstacle is smaller than that of the physical one. In the limiting case of minification, the virtual obstacle collapses to a single point, and this formally corresponds to a perfect cloaking, namely, the physical obstacle becomes invisible to detection. We note that in this case, the push-forward $F$ blows up a single point in the virtual space to a ‘hole’ (which is actually the physical obstacle) in the physical space. Hence, the map $F$ is intrinsically singular, and the obtained transformation medium is inevitably singular. Correspondingly, the transformed PDEs in the physical space are no longer uniformly elliptic which also becomes singular. Therefore, in order to rigorously justify the perfect cloaking, we need to deal with the singular PDEs. Basically, one would encounter the same problems in treating perfect cloakings for acoustic or EM medium and several approaches are proposed to deal with such singularities. For perfect cloaking of conductivity equation, which can be considered as optics at zero frequency, the invisibility is mathematically justified in [10] by using the removability of point singularities for harmonic functions, whereas an alternative treating is provided in [12], where near invisibility is introduced from a regularization viewpoint and the invisibility is rigorously justified based on certain stability estimates for the conductivity equation with small inclusions. For the finite frequency cases, a novel notion of finite energy solutions is introduced in [8] and the invisibility cloaking of acoustic and EM medium is then justified directly. For the perfect cloaking of obstacles considered in the present paper, we shall follow the approach in [12] to mathematically justify the invisibility by taking the limit of near invisibility. To that end, we derive certain stability estimates for scattering amplitudes in terms of the diameter of a small obstacle in both acoustic and EM scattering. Those stability estimates are then used to show that the limiting process of minification cloaking corresponds to a process of near-invisibility cloaking, which in turn implies the desired invisibility result of the perfect cloaking. For practical considerations, all our reshaping studies are conducted within multiple scattering, that is, there is more than one obstacle component included.

Finally, we would like to mention some unique determination results in inverse obstacle scattering, where one utilizes acoustic or EM scattering measurements to identify an unknown/inaccessible obstacle. The uniqueness/identifiability results correspond to
circumstances under which one cannot virtually reshape an obstacle. In the case that
the obstacle is situated in a homogeneous background medium, the uniqueness theory for
inverse obstacle scattering is relatively well established, and we refer to [16] for a survey and
relevant literature. Whereas in [11, 15, 22], the recovery of an obstacle included in certain
inhomogeneous (isotropic) medium is considered. It is shown in [11] and [22] that if the
isotropic medium is known \textit{a priori}, then the included obstacle is uniquely determined by
the associated scattering amplitude. Under the assumption that the isotropic medium and the
included obstacle have only planar contacts, it is proved in [15] that one can recover both the
medium and the obstacle by the associated scattering amplitude. The argument in [15] also
implies that an obstacle surrounded by an isotropic medium cannot produce the same scattering
amplitude as another pure obstacle. This result essentially indicates that transformation media
for virtually reshaping an obstacle must be anisotropic.

The rest of the paper is organized as follows. In section 2, we consider the reshaping
for acoustic scattering, where virtual minification and magnification are first considered
consecutively, and then we present a general reshaping result. Similar study has been
conducted for reshaping a EM perfectly conducting obstacle in section 3.

2. Virtual reshaping for acoustic scattering

2.1. The Helmholtz equation

Let $M$ be an open subset of $\mathbb{R}^3$ with the Lipschitz continuous boundary $\partial M$ and connected
complement $M^+ := \mathbb{R}^3 \setminus M$. Let $(M^+, g)$ be a Riemannian manifold such that $g$ is Euclidean
outside of a sufficiently large ball $B_R$ containing $M$. Here and in the following, $B_R$ shall
denote an Euclidean ball centered at origin and of radius $R$. In wave scattering, $M$ denotes
an impenetrable obstacle, and the Riemannian metric $g$ corresponds to the surrounding
medium with the Euclidean metric $g_0 := \delta_{ij}$ representing the vacuum. In acoustic scattering,
$\sigma = (\sigma^{ij})_{i,j=1}^3$ with $\sigma^{ij} := \sqrt{|g|} g^{ij}$ is the anisotropic acoustic density and $\sqrt{|g|} = |\sigma|$
is the bulk modulus, where $(g^{ij})_{i,j=1}^3$ is the matrix inverse of the matrix $(g_{ij})_{i,j=1}^3$, and
$|g| = \det g, |\sigma| = \det \sigma$. Formally, we have the following one-to-one correspondence between
a material parameter tensor and a Riemannian metric

$$\sigma^{ij} = |g|^{1/2} g^{ij} \quad \text{or} \quad g^{ij} = |\sigma|^{-1} \sigma^{ij}. \quad (2.1)$$

We consider the scattering for a time-harmonic plane incident wave $u^I = \exp\{ik \cdot \theta\}, \theta \in S^2$
due to the obstacle $M$ together with the surrounding medium $(M^+, g)$. The total wave field
is governed by the Helmholtz equation

$$\Delta_{x} u + k^2 u = 0 \quad \text{in} \ M^+, \quad (2.2)$$

$$u|_{\partial M} = 0, \quad (2.3)$$

where the Laplace–Beltrami operator associated with $g$ is given in local coordinates by

$$\Delta_{x} u = \frac{1}{\sqrt{|g|}} \sum_{i,j=1}^3 \frac{\partial}{\partial x_i} \left( \sqrt{|g|} g^{ij} \frac{\partial u}{\partial x_j} \right).$$

The homogeneous Dirichlet boundary condition (2.3) means that the wave pressure vanishes
on the boundary of the obstacle. $M$ is usually referred to as a sound-soft obstacle. The
scattered wave field is as usual assumed to satisfy the Sommerfeld radiation condition.
Taking advantage of the one-to-one correspondence (2.1) between (positive definite) acoustic
densities $\sigma$ and Riemannian metrics $g$, we proceed to mention a few facts about the form
invariance of the Helmholtz equation under transformations. For a smooth diffeomorphism
\( F : \Omega_1 \to \Omega_2 \), the metric \( g(x) \) transforms as a covariant symmetric two-tensor,
\[
\tilde{g}_{ij}(y) := (F \circ g)_{ij}(y) = \sum_{j,m=1}^{3} \frac{\partial x^i}{\partial y^j} \frac{\partial x^m}{\partial y^l} g_{lm} \bigg|_{x=F^{-1}(y)},
\] (2.4)
and then, for \( u = \tilde{u} \circ F \), we have
\[
(\Delta_x + k^2)u = 0 \iff (\tilde{\Delta}_y + \tilde{k}^2)\tilde{u} = 0.
\] (2.5)
Alternatively, using (2.1), one could work with the Helmholtz equation of the following form:
\[
\sum_{i,j=1}^{3} \left[ \partial_i (\sigma^{ij} \partial_j u) + k^2 |\sigma| u \right] = 0,
\] (2.6)
and then, for \( u = \tilde{u} \circ F \), we have
\[
\sum_{i,j=1}^{3} \left[ \partial_i (\tilde{\sigma}^{ij} \partial_j \tilde{u}) + k^2 |\tilde{\sigma}| \tilde{u} \right] = 0.
\] (2.7)
Here \( \tilde{\sigma} \) is the push-forward of \( \sigma \) which, by using (2.1) and (2.4), can be readily shown to be given by
\[
\tilde{\sigma} = F_* \sigma := \frac{(DF)^T \cdot \sigma \cdot (DF)}{|\det DF|} \circ F^{-1},
\] (2.8)
where \( DF \) denotes the (matrix) differential of \( F \) and \( (DF)^T \) its transpose.

Throughout, we shall work with \( \sigma \in L^\infty(M^+)^{3 \times 3} \) and \( F \) is orientation preserving, invertible with both \( F \) and \( F^{-1} \) (uniformly) Lipschitz continuous over \( M^+ \). So, it is appropriate to work with the following Sobolev space for the scattering solution to (2.2)–(2.3):
\[
H_{\text{loc}}^1(M^+) = \{ u \in \mathcal{D}'(M^+); u \in H^1(M^+ \cap B_R) \text{ for each finite } R \text{ with } M \subseteq B_R \}.
\]
The system (2.2)–(2.3) or (2.6) and (2.3) is well posed and has a unique solution, \( u \in H_{\text{loc}}^1(M^+) \) (see [19]). Noting that the corresponding metric outside a ball \( B_R \supset M \) is Euclidean, we know that \( u \) is smooth outside \( B_R \). Furthermore, the solution \( u(x, \theta, k) \) admits asymptotically as \( |x| \to +\infty \) the development (see [3])
\[
u(x, \theta, k) = e^{ikx \cdot \theta} + \frac{e^{ik|x|}}{|x|} A(\theta', \theta, k) + \mathcal{O} \left( \frac{1}{|x|^2} \right),
\] (2.9)
where \( \theta' = x/|x| \in S^2 \). The analytic function \( A(\theta', \theta, k) \) is known as the scattering amplitude or far-field pattern. According to the celebrated Rellich’s theorem, there is a one-to-one correspondence between the scattering amplitude \( A(\theta', \theta, k) \) and the wave solution \( u(x, \theta, k) \).

Throughout, we consider the scattering amplitude for the virtual reshaping effects.

We shall denote by \( M \oplus (M^+, g) \) a cloaking device with an obstacle \( M \) and the corresponding cloaking medium \( (M^+, g) \). The metric \( g \) is always assumed to be Euclidean outside a sufficiently large ball containing \( M \), namely, the cloaking medium is compactly supported. If we know the support of the cloaking medium, say \( M \setminus \tilde{M} \), we also write \( M \oplus (M^+, \tilde{M}, g) \) to denote the cloaking device.

**Definition 2.1.** We say that \( M \oplus (M^+, g) \) (virtually) reshapes the obstacle \( M \) to another obstacle \( \tilde{M} \), if the scattering amplitudes coincide for \( M \oplus (M^+, g) \) and \( \tilde{M} \), i.e.
\[
A(\theta', \theta, k; M \oplus (M^+, g)) = A(\theta', \theta, k; \tilde{M}).
\]

We would like to remark that according to the correspondence (2.1), the cloaking device in definition (2.1) can also be written as \( M \oplus (M^+, \sigma) \), where \( \sigma \) is the (anisotropic) acoustic density for the cloaking medium.
2.2. Virtual minification by cloaking

We first consider the reshaping effects for a special class of obstacles, which are star shaped and referred to as $l^p$-ball-shaped obstacles in the following. They are domains in $\mathbb{R}^3$ of the form

$$\{ x \in \mathbb{R}^3; \|x\|_p = r \},$$

where $p \in [1, +\infty]$, $r > 0$ is a constant and for $x = (x_1, x_2, x_3)$

$$\|x\|_p = \left( \sum_{i=1}^{3} |x_i|^p \right)^{1/p}.$$

Obviously, $\| \cdot \|_2 = | \cdot |$ and an $l^2$-ball is exactly an Euclidean ball. For $l^p$-ball-shaped obstacles, we can give the transformation rule explicitly and correspondingly, the cloaking material parameters for those obstacles can be derived explicitly. Henceforth, we write $B_{R,p}$ to denote an $l^p$-ball of radius $R$ and centered at origin, whereas as prescribed earlier, we write $B_R$.

Let $M = B_{R_1,p}$ with $R_1 > 0$. Let $R_0, R_2$ be such that $0 < R_0 < R_1 < R_2$. We define the map, $F: B_{R_0,p} \mapsto B_{R_1,p}$ by

$$x := F(y) = \begin{cases} y, & \text{for } \|y\|_p \geq R_2, \\ \left( \frac{R_1 - R_0}{R_2 - R_0} R_2 + \frac{R_1 - R_0}{R_2 - R_1} \|y\|_p \right)^{\frac{1}{p}}, & \text{for } R_0 < \|y\|_p < R_2. \end{cases}$$

(2.10)

It is noted that $F$ is (uniformly) Lipschitz continuous over $B_{R_0,p}^+$ and maps $B_{R_1,p} \setminus B_{R_0,p}$ to $B_{R_2,p} \setminus B_{R_1,p}$. For $R_1 < \|x\|_p < R_2$, let

$$g_1(x) = (F_\ast g_0)(x).$$

(2.11)

Set

$$g(x) = \begin{cases} g_1(x), & \text{for } R_1 < \|x\|_p < R_2, \\ g_0(x), & \text{for } \|x\|_p \geq R_2. \end{cases}$$

(2.12)

We have

**Proposition 2.2.** The cloaking device $B_{R_0,p} \bigoplus (B_{R_1,p}^+, g)$ with $g$ defined in (2.12), reshapes $B_{R_1,p}$ virtually to $B_{R_2,p}$. That is, the physical obstacle $B_{R_1,p}$ with the cloaking material $(B_{R_0,p} \setminus B_{R_1,p}, g_1)$ is virtually minified to the obstacle $B_{R_2,p}$ with a minification ratio $\kappa := R_0/R_1$ (see figure 1 for a schematic illustration).

**Proof.** Let $u(y) \in H_{\text{loc}}^1(B_{R_0,p}^+)$ be the unique solution to the Helmholtz equation (2.2)–(2.3) corresponding to the obstacle $B_{R_0,p}$. Whereas, we let $v(x) \in H_{\text{loc}}^1(B_{R_1,p}^+)$ be the unique
solution to the Helmholtz equation (2.2)–(2.3) corresponding to \( B_{R_1,p} \oplus (B_{R_1,p}', g) \). Define \( \tilde{u}(x), x \in B_{R_1,p}' \), to be such that \( u = \tilde{u} \circ F \), i.e. \( \tilde{u} = F_*u = (F^{-1})^*u \). It is clear that \( \tilde{u} \in \mathcal{H}^{2\lambda}_{0,0}(B_{R_1,p}) \) since \( F \) is bijective and both \( F \) and \( F^{-1} \) are (uniformly) Lipschitz continuous. Moreover, noting \( F(\partial B_{R_0,p}) = \partial B_{R_1,p} \), we know \( \tilde{u}|_{\partial B_{R_1,p}} = 0 \).

By the invariance of the Helmholtz equation under transformation, it is readily seen that 
\[
A \left( \theta', \theta, k; B_{R_1,p} \oplus (B_{R_1,p}', g) \right) = A(\theta', \theta, k; B_{R_1,p}). \tag{2.13}
\]
For an Euclidean ball \( B_{R_0} \subset \mathbb{R}^3 \), by separation of variables, we have
\[
A(\theta', \theta, k; B_{R_0}) = \frac{i}{k} \sum_{n=0}^{\infty} (2n + 1) \frac{j_n(kR_0)}{h_n^{(1)}(kR_0)} P_n(\cos \psi), \tag{2.14}
\]
where \( j_n(t) \) and \( h_n^{(1)}(t) \) are respectively, the \( n \)th order spherical Bessel function and spherical Hankel function of first kind, \( P_n(t) \) is the Legendre polynomial, and \( \psi = \zeta(\theta, \theta') \). Using the asymptotic properties
\[
j_n(t) = O(t^n), \quad h_n^{(1)}(t) = O(t^{-n-1}), \quad n = 0, 1, \ldots, \quad \text{as} \quad t \to +0,
\]
it is straightforward to show
\[
A(\theta', \theta, k; B_{R_0}) = O(R_0) \quad \text{as} \quad R_0 \to +0. \tag{2.15}
\]
Now we consider the limiting case for minification, namely \( k \to +0 \) or equivalently \( R_0 \to +0 \). By (2.13) and (2.15),
\[
A \left( \theta', \theta, k; B_{R_1} \oplus (B_{R_1}', g) \right) = A(\theta', \theta, k; B_{R_0}) = O(R_0) \tag{2.16}
\]
as \( R_0 \to +0 \). That is

**Proposition 2.3.** The limit for minification of an Euclidean ball \( B_{R_0} \) in proposition 2.2 gives a perfect cloaking, namely, it makes the obstacle invisible to detection.

In the limiting case with \( k = 0 \), the transformation in (2.10) becomes
\[
x = H(y) := \begin{cases} y, & \text{for} \quad \|y\|_p \geq R_2, \\ \left( R_1 + \frac{R_2 - R_1}{R_2}, \frac{y}{\|y\|_p} \right), & \text{for} \quad 0 < \|y\|_p < R_2, \end{cases} \tag{2.17}
\]
which maps \( \mathbb{R}^3 \setminus \{0\} \) to \( \mathbb{R}^3 \setminus B_{R_1,p} \), i.e., it blows up the single point \( \{0\} \) to \( B_{R_1,p} \). It is remarked that the map \( H \) in (2.17) with \( p = 2 \) is exactly that used in [9, 10] for perfect cloaking of the conductivity equation, and in [24] for perfect cloaking of electromagnetic material tensors. Next, we take the case with \( p = 2 \) as an example for a simple analysis of the perfect cloaking medium. The corresponding metric \((H_0g_0)(x)\) in \( B_{R_1} \setminus \overline{B}_{R_1} \) is singular near the cloaking interface, namely \( \partial B_{R_1} \). In fact, considering in the standard spherical coordinates on \( B_{R_1} \setminus \{0\} \), \((r, \phi, \theta) \mapsto (r \sin \theta \cos \phi, r \sin \theta \sin \phi, r \cos \theta) \) \( \in \mathbb{R}^3 \) and by (2.4), it can be easily calculated that
\[
\tilde{g} := H_0g_0 = \begin{pmatrix} \lambda^2 & 0 & 0 \\ 0 & \lambda^2(r - R_1)^2 & 0 \\ 0 & 0 & \lambda^2(r - R_1)^2 \sin^2 \theta \end{pmatrix},
\]
where \( \lambda = R_2/(R_2 - R_1) \). That is, \( \tilde{g} \) has one eigenvalue bounded from below (with eigenvector corresponding to the radial direction) and two eigenvalues of order \((r - R_1)^2\) approaching zero as \( r \to +R_1 \). Hence, if the perfect cloaking is analyzed directly, one needs to deal with

\[ \lambda^{1/2} = O(\epsilon) \]
the degenerated elliptic equation near the cloaking interface. So, a suitable choice of the class of 
weak solutions to the singular equation must be purposely introduced, as the finite energy solutions considered in [8] for invisibility cloaking devices of acoustic and electromagnetic media. Clearly, our earlier analysis on the perfect cloaking of an Euclidean ball avoids the singular equation by taking the limit. This is similar to [12] for the analysis of perfect cloaking of conductivities in electrical impedance tomography by regularization. Here we would like to point out that there is no theoretical result available showing that the limit of the regularized solutions obtained by the approach of the current paper by sending \( k \to 0 \) is the finite energy solutions in the sense of [8]. A further study in this aspect may provide more insights into the invisibility cloaking.

In order to achieve the similar invisibility result for a general \( l^p \)-ball-shaped obstacles, we need to derive stability estimates similar to (2.16) for generally shaped obstacles with small diameters. This is given by lemma 2.4, proved using the boundary integral representation rather than separation of variables, and the obstacles could be generally star shaped. On the other hand, from a practical viewpoint, we consider the scattering with multiple scattering components and only some of the components are cloaked. We shall show that the virtual reshaping takes effect only for those cloaked components and the other uncloaked components remain unaffected. Particularly, those perfectly cloaked components will be invisible, even though there is scattering interaction between the obstacle components. We are now in a position to present the key lemma. In the sequel, we let \( \tilde{\mathbb{B}} \) be a simply connected set in \( \mathbb{R}^3 \) whose boundary is star shaped with respect to the origin of the form \( \partial \tilde{\mathbb{B}} = \partial r_0(\theta) \), where \( \theta \in S^2, r_0(\theta) \in C^2(S^2) \) and \( \delta > 0 \). Let \( \partial \tilde{\Omega}_0 \) be the domain, \( \{ x \in \mathbb{R}^3; |x| < r_0(\theta) \} \).

**Lemma 2.4.** Suppose \( M_1 \cap \partial \tilde{\Omega}_0 = \emptyset \), then we have

\[
A(\theta', \theta, k; M_1 \cup \tilde{\mathbb{B}}) = A(\theta', \theta, k; M_1) + \mathcal{O}(\delta) \quad \text{as} \quad \delta \to +0.
\]

**Proof.** Let \( \Phi(x, y) = e^{ik|x-y|/(4\pi|x-y|)} \) be the fundamental solution to the Helmholtz operator \((\Delta + k^2)\). We know that \( u(x; k, \theta) \in C^2(\mathbb{R}^3 \setminus \overline{M_1 \cup \tilde{\mathbb{B}}}) \cap C(\mathbb{R}^3 \setminus (M_1 \cup \tilde{\mathbb{B}})) \) and can be represented in the form (see [3])

\[
u(x; M_1 \cup \tilde{\mathbb{B}}) = e^{ikx \cdot \theta} + \int_{\partial M_1} \left\{ \frac{\partial \Phi(x, y)}{\partial v(y)} - i\Phi(x, y) \right\} \phi_1(y) \, ds(y)

+ \int_{\partial \tilde{\mathbb{B}}} \left\{ \frac{\partial \Phi(x, y)}{\partial v(y)} - i\eta \Phi(x, y) \right\} \phi_2(y) \, ds(y),
\]

where \( \phi_1 \in C(\partial M_1) \) and \( \phi_2 \in C(\partial \tilde{\mathbb{B}}) \) are density functions, and \( \eta \neq 0 \) is a real coupling parameter. The densities \( \phi_1 \) and \( \phi_2 \) are unique solutions to the following integral equation (see [3]):

\[
\psi(x) + 2 \int_{\partial M_1} \left\{ \frac{\partial \Phi(x, y)}{\partial v(y)} - i\Phi(x, y) \right\} \phi_1(y) \, ds(y)

+ 2 \int_{\partial \tilde{\mathbb{B}}} \left\{ \frac{\partial \Phi(x, y)}{\partial v(y)} - i\eta \Phi(x, y) \right\} \phi_2(y) \, ds(y) = -2e^{ikx \cdot \theta},
\]

for \( x \in \partial M_1 \cup \partial \tilde{\mathbb{B}} \), where \( \psi(x) := \phi_1(x) \) for \( x \in \partial M_1 \) and \( \psi(x) := \phi_2(x) \) for \( x \in \partial \tilde{\mathbb{B}} \). We introduce the integral operators

\[
(S \psi_1)(x) = 2 \int_{\partial M_1} \Phi(x, y) \psi_1(y) \, ds(y), \quad (K \psi_1)(x) = 2 \int_{\partial M_1} \frac{\partial \Phi(x, y)}{\partial v(y)} \psi_1(y) \, ds(y)
\]

\[
(S_2 \psi_2)(x) = 2 \int_{\partial \tilde{\mathbb{B}}} \Phi(x, y) \psi_2(y) \, ds(y), \quad (K_2 \psi_2)(x) = 2 \int_{\partial \tilde{\mathbb{B}}} \frac{\partial \Phi(x, y)}{\partial v(y)} \psi_2(y) \, ds(y),
\]
and set
\[ h_1(x) := -2e^{ikx}, \quad x \in \partial M_1; \quad h_2(x) := -2e^{ikx}, \quad x \in \partial B. \]

Then equation (2.20) can be rewritten as
\[
\begin{align*}
[\varphi_1 + K_1\varphi_1 - iS_1\varphi_1 + K_2\varphi_2 - i\eta S_2\varphi_2](x) &= h_1(x), \quad x \in \partial M_1, \\
[\varphi_2 + K_2\varphi_2 - i\eta S_2\varphi_2 + K_1\varphi_1 - iS_1\varphi_1](x) &= h_2(x), \quad x \in \partial B.
\end{align*}
\]

(2.21) \quad (2.22)

It is remarked that the integral operators involved in equations (2.21) and (2.22) with weakly singular integral kernels have to be understood in the sense of Cauchy principle values and we refer to [3] and [19] for related mapping properties. Clearly, \( \varphi_1 \) and \( \varphi_2 \) are functions dependent on \( \delta \). We next study their asymptotic behaviors as \( \delta \to +0 \). To this end, we fix \( \delta > 0 \) but being sufficiently small and take \( \eta = \delta^{-1} \).

In the sequel, without loss of generality, we may assume that \( \text{dist}(\partial M_1, \partial B_0) > c_0 > 0 \), otherwise one can shrink \( B_0 \) to \( 1/2B_0 \). By straightforward calculations, it can be easily shown that
\[
\|K_2 - i\eta S_2\|_{C(\partial B) \to C(\partial B)} = O(\delta), \quad \|K_1 - iS_1\|_{C(\partial M_1) \to C(\partial B)} = O(1). \tag{2.23}
\]

Next, for \( x \in \partial B_0 \), we define
\[
(K_0\phi)(x) = 2\int_{\partial B_0} \frac{\partial \Phi_0(x, y)}{\partial \nu(y)} \phi(y) \, ds(y), \quad (S_0\phi)(x) = 2\int_{\partial B_0} \Phi_0(x, y) \phi(y) \, ds(y),
\]
where \( \phi \in C(\partial B_0) \) and \( \Phi_0(x, y) = 1/(4\pi|x - y|) \) is the fundamental solution to the Laplace operator. It is known that both \( S_0 \) and \( K_0 \) are compact operators in \( C(\partial B_0) \) (see [2]). By changing the integration to the boundary of the reference obstacle \( \partial B_0 \), we have
\[
\begin{align*}
(S_2\phi)(x) &= 2\int_{\partial B} \frac{e^{ik|x-y|}}{|x-y|} \phi(y) \, ds(y) = 2\delta \int_{\partial B_0} \frac{e^{ik|x-y'|}}{|x'-y'|} \phi(\delta y') \, ds(y'), \\
(K_2\phi)(x) &= 2\int_{\partial B} \frac{\partial}{\partial x} \left( \frac{e^{ik|x-y|}}{|x-y|} \right) \partial y \phi(y) \, ds(y) = 2 \int_{\partial B_0} \frac{e^{ik|x-y'|}}{|x'-y'|} \partial y \phi(\delta y') \, ds(y'),
\end{align*}
\]
where \( \phi \in C(\partial B), \ x, \ y \in \partial B \) and \( x' := x/\delta, \ y' = y/\delta \in \partial B_0 \). Then by using the power series expansion of the exponential function \( e^{ik|x-y|} \), we have by direct calculations
\[
\|1/\delta S_2 - S_0\|_{C(\partial B_0) \to C(\delta B_0)} = O(\delta), \quad \|K_2 - K_0\|_{C(\partial B_0) \to C(\delta B_0)} = O(\delta^2). \tag{2.24}
\]

By changing the integration to \( \partial B_0 \) and using the results in (2.24), we have from (2.22) that
\[
\varphi_2(\delta x) = [I + K_0 - iS_0 + O(\delta)]^{-1}[h_2(\delta x) - (K_1 - iS_1)\varphi_1(\delta x)], \quad x \in \partial B_0. \tag{2.25}
\]

It is noted here that \( (I + K_0 - iS_0) \) is bounded invertible (see [2]). Then, plugging (2.25) into (2.21) and using the relations in (2.23), we further have
\[
\varphi_1 = [I + K_1 - iS_1 + O(\delta)]^{-1}(h_1 + O(\delta)), \tag{2.26}
\]
which, by noting \( I + K_1 - iS_1 \) is invertible (see [3]), gives
\[
\varphi_1 = \hat{\varphi}_1 + O(\delta), \tag{2.27}
\]
Using the estimates in (2.27) together with (2.25) implies that
\[ \varphi_2 = O(1). \] (2.28)

Finally, by (2.19) we know
\[
A(\theta', \theta, k; M_1 \cup B) = \frac{1}{4\pi} \int_{\partial M_1} \left( \hat{\varphi}_1(y) \right) ds(y) + \frac{1}{4\pi} \int_{\partial B} \left( \hat{\varphi}_1(y) \right) ds(y).
\] (2.29)

Using the estimates in (2.27) and (2.28) to (2.29) and changing the integration over \( \partial B \) to \( \partial B_0 \), we have
\[
A(\theta', \theta, k; M_1 \cup B) = A(\theta', \theta, k; M_1) + O(\delta),
\] where we have made use of the fact that
\[
A(\theta', \theta, k; M_1) = \frac{1}{4\pi} \int_{\partial M_1} \hat{\varphi}_1(y) \left( \frac{\partial e^{-ik\theta' \cdot y}}{\partial v(y)} \right) ds(y).
\]

The proof is completed. \( \square \)

Remark 2.5. If \( \partial M_1 \cup \partial B \) is only Lipschitz continuous (whence \( r_0(\theta) \in C^{0,1}(\mathbb{S}^2) \)), one can make use of the mapping properties of relevant boundary layer potential operators presented in [19] and derive similar estimate.

Proposition 2.6. Suppose that \( M_1 \cap B_{R_1, p} = \emptyset \). The cloaking device \( \left( (M_1 \cup B_{R_1, p}) \bigoplus ((M_1 \cup B_{R_1, p})^+, \hat{g}) \right) \), where \( \hat{g}(x) = g_1(x) \) in (2.11) for \( R_1 < \|x\|_p < R_2 \) and \( g_0(x) \) for \( x \in (M_1 \cup B_{R_1, p})^+ \), reshapes the obstacle \( M_1 \cup B_{R_1, p} \) to \( M_1 \cup B_{R_0, p} \). Furthermore, the limiting case with \( R_0 = 0 \) corresponds to the perfect cloaking of \( B_{R_0, p} \), namely
\[
A \left( \theta', \theta, k; (M_1 \cup B_{R_0, p}) \bigoplus ((M_1 \cup B_{R_0, p})^+, \hat{g}) \right) = A(\theta', \theta, k; M_1) \) \quad (2.30)
\]

Proof. Let \( F \) be the transformation in (2.10) and let \( \hat{F} \) be the restriction of \( F \) over \( (M_1 \cup B_{R_0})^+ \). Clearly, \( \hat{g} = \hat{F}_* g_0 \). By a similar argument as the proof of proposition 2.2, it is easily seen that \( M_1 \cup B_{R_0, p} \) is virtually reshaped to \( M_1 \cup B_{R_0, p} \) by the cloaking of \( \hat{g} \), i.e.,
\[
A \left( \theta', \theta, k; (M_1 \cup B_{R_0, p}) \bigoplus ((M_1 \cup B_{R_0, p})^+, \hat{g}) \right) = A(\theta', \theta, k; M_1 \cup B_{R_0, p} \).
\]

Next, by lemma 2.4,
\[
A(\theta', \theta, k; M_1 \cup B_{R_0, p}) = A(\theta', \theta, k; M_1) + O(R_0) \quad \text{as} \quad R_0 \to +0, \quad (2.31)
\] and hence the limiting case with \( R_0 = 0 \) yields an ideal cloaking of \( B_{R_0, p} \). \( \square \)

Remark 2.7. Clearly, proposition 2.6 implies a same invisibility result for perfectly cloaking an \( l^p \)-ball as that in proposition 2.3 for perfectly cloaking an Euclidean ball.
2.3. Virtual magnification by cloaking

Let \( 0 < R_0 < R_1 < R_2 \), and let \( B_{R_0,p} \) be the obstacle which we intend to virtually magnify to \( B_{R_1,p} \) by using a cloaking for \( B_{R_0,p} \) supported in \( B_{R_2,p} \) \( \setminus B_{R_0,p} \) (see figure 2). We define \( \tau = R_1 / R_0 \) to be the magnification ratio.

Let \( K : \mathbb{R}^3 \setminus B_{R_1,p} \mapsto \mathbb{R}^3 \setminus B_{R_0,p} \) be defined by

\[
x := K(y) = \begin{cases} y, & \text{for } \|y\|_p \geq R_2, \\ \left( \frac{R_0 - R_1}{R_2 - R_1} R_2 + \frac{R_0 - R_1}{R_2 - R_1} \|y\|_p \right) \frac{y}{\|y\|_p}, & \text{for } R_0 < \|y\|_p < R_2. \end{cases}
\]

It is verified directly that \( K \) maps the \( l^p \)-annulus \( R_1 \leq \|y\|_p \leq R_2 \) to the \( l^p \)-annulus \( R_0 \leq \|y\|_p \leq R_2 \). Moreover, \( K \) is bijective and both \( K \) and \( K^{-1} \) are (uniformly) Lipschitz continuous. Let

\[
\hat{g} = K^* g_0
\]

be the metric in \( B_{R_0,p}^* \). Clearly, \( \hat{g} \) is Euclidean outside \( B_{R_2,p} \).

**Proposition 2.8.** The cloaking device \( B_{R_0,p} \oplus (B_{R_1,p} \setminus B_{R_0,p}, \hat{g}) \) with \( \hat{g} \) defined in (2.32), reshapes \( B_{R_0,p} \) virtually to \( B_{R_1,p} \). That is, the physical obstacle \( B_{R_0,p} \) with the cloaking material \( (B_{R_1,p} \setminus B_{R_0,p}, \hat{g}) \) is virtually magnified to the obstacle \( B_{R_1,p} \) with a magnification ratio \( \tau = R_1 / R_0 \).

**Proof.** Let \( u \in H^1_{\text{loc}}(B_{R_0,p}^*) \) be the solution to the Helmholtz equation (2.2)–(2.3) associated with the obstacle \( B_{R_0,p} \). Define \( \tilde{u} = K_* u \in H^1_{\text{loc}}(B_{R_0,p}^*) \). Again, by the invariance of the Helmholtz equation under transformation together with the fact that \( \tilde{u}|_{\partial B_{R_0,p}} = u|_{\partial B_{R_0,p}} = 0 \), we see that \( \tilde{u} \) is the scattering solution corresponds to \( B_{R_0,p} \oplus (B_{R_1,p}^*, \hat{g}) \). That is,

\[
A \left( \partial', \theta, k; B_{R_0,p} \oplus (B_{R_1,p}^*, \hat{g}) \right) = A(\theta', \theta, k; B_{R_1,p}).
\]

In proposition 2.2, we use \( F \) to compress the vacuum to achieve a transformation-based minification device, whereas in proposition 2.8, we use \( K \) to loosen up the vacuum to achieve a transformation-based magnification device. Note that \( R_2 > R_1 \), the cloaking device is of size larger than the virtual obstacle image, though the virtual obstacle could be of size arbitrarily close to the cloaking device. Hence, the cloaking in proposition 2.8 is not of magnification in
the real sense. However, our magnification result is still of particular practical interests, e.g.,
if one is only interested in recovering an obstacle without knowing a priori that it is cloaked,
then the scattering reconstruction will give a virtually magnified obstacle. On the other hand,
we would like to mention that in [20, 21, 23], it is demonstrated that a coated cylindrical core
can be extended beyond the cloaking shell into the matrix, where the cloaking material must
be negative refractive indexed, namely, the corresponding metric $g$ has negative eigenvalues.
A general strategy is presented in [14] on how to devise a negative refractive indexed (NRI)
cloaking by using the transformation optics. There, the transformation $F$ is neither injective
nor orientation preserving, which maps a right-handed medium to left-handed medium. Based
on NRI cloaking, it is shown in [18, 25] that one can virtually reshape a cylindrical perfect
conductor of size bigger than the cloaking device. However, all the aforementioned results
are essentially based on exerting transformation directly to the analytical solutions, which is
not of the main theme of the present paper.

2.4. Virtually reshaping acoustic obstacles by cloaking

Our discussion so far has been mainly concerned with the minification and magnification of
obstacles by cloaking. Clearly, we may consider virtually reshaping an obstacle arbitrarily
provided a suitable transform can be found with which we can make essential use of the
transformation invariance of the Helmholtz equation. Let $M$ be an obstacle with $m$ pairwise
disjoint simply connected components $M_l, l = 1, 2, \ldots, m$, i.e., $M = \bigcup_{l=1}^{m} M_l$. Let $M_l \subset M_l'$ and $M_l' \cap M_{l'}' = \emptyset$, for $l, l' = 1, 2, \ldots, m$ and $l \neq l'$. Set $M' = \bigcup_{l=1}^{m} M_l'$. Let $\tilde{M} = \bigcup_{l=1}^{m} \tilde{M}$. Suppose there exist $F_l: \mathbb{R}^3 \setminus \tilde{M} \mapsto \mathbb{R}^3 \setminus M_l, \quad l = 1, 2, \ldots, m$
such that $F_l$ is orientation preserving and invertible with $F_l$ and $F_l^{-1}$ Lipschitz continuous,
and $F_l = \text{id}$ outside $M_l'$. Set $g_l = (F_l)^* g_0$ and let $M_l \bigoplus (M'_l \setminus M_l, g_l)$ be a cloaking device for
$M_l$.

**Theorem 2.9.** The cloaking device $M'$ virtually reshapes the obstacle $M$ to $\tilde{M}$. That is

$$A(\theta', \theta, k; M \bigoplus (M'_l \setminus M_l, g')) = A(\theta', \theta, k; \tilde{M}),$$

where $g'$ is $g_l$ in $M'_l \setminus M_l$.

The proof is already clear from our earlier discussion on minification and magnification. We
have several important consequences of the theorem.

**Remark 2.10.** Suppose that some of the components of $M$ are uncloaked, say $M_l$ for $1 \leq l \leq m' < m$, and this corresponds to taking $M_l = M_l' = \tilde{M}$ and $F_l = \text{id}$ for $l = 1, 2, \ldots, m'$.

**Remark 2.11.** If for some $M_l$ being star shaped w.r.t. certain point, and the transformation $F_l^{-1}$ shrinks $M_l$ only in the radial direction to $\tilde{M}_l$, then the case with $M_l$ degenerated to a single point corresponds to an ideal cloaking for $M_l$. By using a similar argument as that for proposition 2.6 together with the estimate in lemma 2.4, one has that $M_l$ is invisible to
detection. In fact, by repeating the argument, the same conclusion holds when there are more
than one obstacle component, which is perfectly cloaked.

It is noted that in remark 2.11, the perfectly cloaked obstacle components are required to
be star shaped, and this is because we need to make use of the estimate in lemma 2.4 to achieve
invisibility. In order to show the prefect cloakings of more generally shaped obstacles, one
may need different thoughts.
In the rest of this section, we shall indicate that all our previous results on virtual reshaping in space dimension three can be straightforwardly extended to the two-dimensional case. In fact, for the two-dimensional scattering problem, the (positive definite) acoustic density \( \sigma \in L^\infty(M^*)^{2 \times 2} \) also transforms according to (2.8). Therefore, the reshaping result presented in theorem 2.9 is still valid in \( \mathbb{R}^2 \). In order to achieve invisibility for perfect cloaking of star-shaped obstacles in \( \mathbb{R}^2 \), one needs to show a similar estimate to lemma 2.4. Indeed, replacing \( \Phi(x, y) \) by the first kind Hankel function \( \frac{1}{2} H_0^{(1)}(k|x - y|) \) of order zero in the proof of lemma 2.4 and using the corresponding mapping properties of the integral operators involved (see [3]), one can obtain by similar arguments the following estimate to the scattering problem in \( \mathbb{R}^2 \) (see (2.18) for comparison):

\[
A(\theta', \theta, k; M_1 \cup \mathbb{B}) = A(\theta', \theta, k; M_1) + O(|\log \delta|^{-1}) \quad \text{as} \quad \delta \to +0. \tag{2.33}
\]

Obviously, with (2.33) one can show that the perfect cloaking of a star-shaped obstacles in \( \mathbb{R}^2 \) makes it invisible to detection.

3. Virtual reshaping for electromagnetic scattering

3.1. The Maxwell’s equations

We define Maxwell’s equations for the scatterer \( M \bigoplus (M^*, g) \) as that introduced in section 2.1. Using the metric \( g \), we define a (positive definite) electric permittivity tensor \( \varepsilon \) and a magnetic permeability tensor \( \mu \) by

\[
\varepsilon_{ij} = \mu_{ij} = |g|^{1/2} g^{ij} \quad \text{on} \quad M^* . \tag{3.1}
\]

It is clear that \( \varepsilon = (\varepsilon_{ij})_{i,j=1}^3 \) and \( \mu = (\mu_{ij})_{i,j=1}^3 \) are invariantly defined and transform as a product of a \( (+) \) density and a contravariant symmetric two-tensor with the same rule as that for acoustic density \( \sigma \) in (2.8). We consider the scattering due to the scatterer \( M \bigoplus (M^*, g) \) corresponding to some incident wave field. The resulting total electric and magnetic fields, \( E \) and \( H \) in \( M^* \), are defined as differential 1-forms, given in some local coordinates by

\[
E = E_j \, dx^j \quad \text{and} \quad H = H_j \, dx^j .
\]

Here and in the following, we use Einstein’s summation convention, summing over indices appearing both as sub- and super-indices in formulae. Then \( (E, H) \) satisfies Maxwell’s equations on \( (M^*, g) \) at frequency \( k \)

\[
dE = i k \, *_g H, \quad dH = -i k \, *_g E , \tag{3.2}
\]

where \( *_g \) denotes the Hodge operator on 1-forms given by

\[
*_g (E_j \, dx^j) = \frac{1}{2} |g|^{1/2} g^{ij} E_i s_{lpq} \, dx^p \wedge dx^q = \frac{1}{2} \varepsilon^{ij} E_j s_{lpq} \, dx^p \wedge dx^q ,
\]

with \( s_{lpq} \) denoting the Levi-Civita permutation symbol, and \( s_{lpq} = 1 \) (resp. \( s_{lpq} = -1 \)) if \((l, p, q) \) is an even (resp. odd) permutation of \((1, 2, 3) \) and zero otherwise. By introducing, for \( H = H_j \, dx^j \), the notation

\[
(*_g H)^j = s_{lpq} \frac{\partial}{\partial x^p} H_q,
\]

the exterior derivative may then be written as

\[
d(H_q \, dx^q) = \frac{\partial H_q}{\partial x^p} \, dx^p \wedge dx^q = \frac{1}{2} (\text{curl} \, H)^j s_{lpq} \, dx^p \wedge dx^q .
\]

Hence, in a fix coordinate, the Maxwell’s equations (3.2) can be written as

\[
(*_g E)^j = i k \mu^{jl} H_j , \quad (*_g H)^j = -i k \varepsilon^{ij} E_j . \tag{3.3}
\]
Without loss of generality, we take the incident fields to be the normalized time-harmonic electromagnetic plane waves,
\[ E^i(x) := \frac{i}{k} \nabla \times p e^{ikx \cdot \theta}, \quad H^i(x) := \nabla p e^{ikx \cdot \theta}, \]
where \( p \in \mathbb{R}^3 \) is a polarization. As usual, the radiation fields are assumed to satisfy the Silver–Müller radiation condition. To complete the description, we further assume that the obstacle \( M \) is perfectly conducting, and we have the following two types of boundary conditions on \( \partial M \): the perfect electric conductor (PEC) boundary condition
\[ \nu \times E|_{\partial M} = 0, \]
or the perfect magnetic conductor (PMC) boundary condition
\[ \nu \times H|_{\partial M} = 0, \]
where \( \nu \) is the Euclidean normal vector of \( \partial M \).

We shall work with \( \varepsilon, \mu \in (L^\infty(M^*))^{3 \times 3} \). It is convenient to introduce the following Sobolev spaces:
\[ H(\nabla \times; \Omega) = \{ u \in L^2(\Omega); \nabla u \in L^2(\Omega) \}, \]
\[ H_{loc}(\nabla \times; M^+) = \{ u \in \mathcal{D}'(M^+); u \in H(\nabla \times; M^+ \cap B_\rho) \text{ for each finite } \rho \text{ with } M \subset B_\rho \}. \]

Then it is known that there exists a unique solution \((E, H) \in H_{loc}(\nabla \times; M^+) \oplus H_{loc}(\nabla \times; M^+)\) to the EM scattering problem. Moreover, the solution \( E(x, k, p, \theta) \) admits asymptotically as \( |x| \to +\infty \) the development (see [3])
\[ E(x, k, p, \theta) = E^i(x) + \frac{e^{ik|x|}}{|x|} E_\infty(\theta', k, p, \theta) + O\left( \frac{1}{|x|^2} \right), \quad (3.4) \]
where \( \theta' = x/|x| \in S^2 \). The analytic function \( E_\infty(\theta', k, p, \theta) \) is known as the electric far-field pattern. Similar to definition 2.1, we introduce

**Definition 3.1.** We say that \( M \bigoplus(M^+, g) \) (virtually) reshapes the obstacle \( M \) to another obstacle \( \tilde{M} \), if the electric far-field patterns coincide for \( M \bigoplus(M^+, g) \) and \( \tilde{M} \), i.e.
\[ E_\infty(\theta', k, p, \theta; M \bigoplus(M^+, g)) = E_\infty(\theta', k, p, \theta; \tilde{M}). \]

We would also like to remark that the cloaking device in definition 3.1 can also be written as \( M \bigoplus(M^*, \varepsilon, \mu) \) according to the correspondence \((3.1)\), where \( \varepsilon \) and \( \mu \) are respectively, electric permittivity and magnetic permeability for the cloaking medium.

### 3.2. Virtually reshaping electromagnetic obstacles by cloaking

We consider the virtual reshaping for electromagnetic obstacles by cloaking. Let \( M = \bigcup_{l=1}^m M_l, M^+ = \bigcup_{l=1}^m M^+_l, \tilde{M} = \bigcup_{l=1}^m \tilde{M}_l \) and \( F_l, l = 1, 2, \ldots, m \) be those introduced in section 2.4. Furthermore, we assume that \( F_l \) is normal preserving in the sense that
\[ \tilde{\nu}_l = v_l \circ F_l, \quad l = 1, 2, \ldots, m, \]
where \( \tilde{\nu}_l \) and \( v_l \) are, respectively, the Euclidean normals to \( \partial \tilde{M}_l \) and \( \partial M_l \). For example, if \( \tilde{M}_l \) and \( M_l \) are both star shaped w.r.t. the origin, say \( \partial \tilde{M}_l = \tilde{r}(\theta) \theta \) and \( \partial M_l = r(\theta) \theta \) with \( \tilde{r}/r = c \) being some constant, then \( F_l \) is normal preserving since one has
\[ \tilde{\nu}_l|_{r(\theta)\theta} = v_l|_{r(\theta)\theta} = \frac{r(\theta)\theta - \nabla r}{\sqrt{r^2 + |\nabla r|^2}}. \]
Particularly, if \( \tilde{M} \) is \( l^p \)-ball shaped, the transformation of the following form

\[
F(y) = (a + b\|y\|_p) \frac{y}{\|y\|_p},
\]

is normal preserving, which transforms an \( l^p \)-ball of radius \( \tilde{r} \) into another \( l^p \)-ball of radius \( r = a + b\tilde{r} \).

Concerning the virtual reshaping, we have

**Theorem 3.2.** The cloaking device \( M' \) virtually reshapes the obstacle \( M \) to \( \tilde{M} \). That is

\[
E_\infty(\theta', k, p, \theta; M \bigoplus (M' \setminus M, g')) = E_\infty(\theta', k, p, \theta; \tilde{M}),
\]

where \( g' \) is \( g \) in \( M' \setminus M \).

**Proof.** Let \( F : \mathbb{R}^3 \setminus \tilde{M} \mapsto \mathbb{R}^3 \setminus M \) be such that \( F|_{M' \setminus \tilde{M}} = F|_{M' \setminus \tilde{M}}, l = 1, 2, \ldots, m \) and \( F = id \) over \( \mathbb{R}^3 \setminus M' \). Let \((E, H) \in H_{\text{loc}}(\text{curl}; \tilde{M}^*) \oplus H_{\text{loc}}(\text{curl}; \tilde{M}^*) \) be the unique scattering solution corresponding to the perfect conducting obstacle \( \tilde{M} \). Define \( \hat{E} = F_{\text{curl}} E, \hat{H} = F_{\text{curl}} H \). Clearly, \((\hat{E}, \hat{H}) \in H_{\text{loc}}(\text{curl}; M^*) \oplus H_{\text{loc}}(\text{curl}; M^*) \) according to our requirements on the mappings \( F_l's, l = 1, 2, \ldots, m \). Moreover, noting \( F_l's, 1 \leq l \leq m \), are normal preserving, we know \( v \times \hat{E}|_{\tilde{M}} = 0 \) (resp. \( v \times \hat{H}|_{\tilde{M}} = 0 \)) if \( \tilde{M} \) is a perfectly electric conducting obstacle (resp. perfectly magnetic conducting obstacle). Hence, \((\hat{E}, \hat{H}) \) is the unique solution corresponding to the cloaking device \( M \bigoplus (M' \setminus M, g') \). Therefore, we have

\[
E_\infty(\theta', k, p, \theta; M \bigoplus (M' \setminus M, g')) = E_\infty(\theta', k, p, \theta; \tilde{M}).
\]

With theorem 3.2, all the virtual minification and magnification results for acoustic obstacle scattering can be straightforwardly extended to the electromagnetic obstacle scattering. In order to obtain similar invisibility results for a perfectly conducting obstacle when some of its star-shaped components are perfectly cloaked, we need a lemma similar to lemma 2.4 in the following for EM scattering.

**Lemma 3.3.** Let \( M_1, \mathbb{B}_0 \) and \( \mathbb{B} \) be the same as those in lemma 2.4, then we have

\[
E_\infty(\theta', k, p, \theta; M_1 \cup \mathbb{B}) = E_\infty(\theta', k, p, \theta; M_1) + O(\delta) \quad \text{as} \quad \delta \to +0. \tag{3.5}
\]

**Proof.** We first introduce the space \( T^{0,\alpha}(\partial M_1), 0 < \alpha \leq 1 \), consisting of the uniformly Hölder continuous tangential fields \( a \) equipped with the Hölder norm, and \( T^{0,\alpha}_a(\partial M_1) = \{ a \in T^{0,\alpha}(\partial M_1); \text{Div} a \in C^0(\partial M_1) \} \). Similarly, one can introduce \( T^{0,\alpha}_a(\partial M_1 \cup \partial \mathbb{B}) \). We know the solution \((E, H) \in C^{0,\alpha}(\mathbb{R}^3 \setminus (M_1 \cup \mathbb{B})) \oplus C^{0,\alpha}(\mathbb{R}^3 \setminus (M_1 \cup \mathbb{B})) \) can be expressed as (cf [3])

\[
E(x) = E'(x) + \text{curl} \int_{\partial M_1} a_1(y) \Phi(x, y) \, ds(y) + i \int_{\partial M_1} \nu(y) \times (S_0^a)(y) \Phi(x, y) \, ds(y)
+ \text{curl} \int_{\partial \mathbb{B}} a_2(y) \Phi(x, y) \, ds(y) + \text{in} \int_{\partial \mathbb{B}} \nu(y) \times (S_0^a)(y) \Phi(x, y) \, ds(y),
\]

and \( H(x) = \text{curl} E(x)/i k \), where \( \eta \neq 0 \) is a real coupling parameter. Here, \( S_0^a \) is the operator as defined in the proof of lemma 2.4 but with the integration domains changed according to the context and the densities \( a_1 \in T^{0,\alpha}_a(\partial M_1), a_2 \in T^{0,\alpha}_a(\partial \mathbb{B}) \) satisfy

\[
(a + M_1 a_1 + i \eta a_1 P S_0^a a_1 + M_2 a_2 + i \eta a_2 P S_0^a a_2)(x) = V(x), \quad x \in \partial M_1 \cup \partial \mathbb{B} \tag{3.6}
\]
where \(a(x) := a_1(x)\) for \(x \in \partial M_1\) and \(a(x) := a_2(x)\) for \(x \in \partial B\), and \(V(x) := -2v \times E'(x)\) for the PEC obstacle and \(V(x) := -2v \times H'(x)\) for the PMC obstacle. The operators involved in (3.6) are respectively given by

\[
\begin{align*}
(M_1a_1)(x) & := 2 \int_{\partial M_1} v(x) \times \nabla \{a_1(y)\Phi(x, y)\} \, ds(y), \\
(N_1b_1)(x) & := 2v(x) \times \nabla \nabla \int_{\partial M_1} v(y) \times b_1(y)\Phi(x, y) \, ds(y), \\
(M_2a_2)(x) & := 2 \int_{\partial B} v(x) \times \nabla \{a_2(y)\Phi(x, y)\} \, ds(y), \\
(N_2b_2)(x) & := 2v(x) \times \nabla \nabla \int_{\partial B} v(y) \times b_2(y)\Phi(x, y) \, ds(y), \\
Pc & := (v \times c) \times c.
\end{align*}
\]

We again refer to [2, 3] for relevant mapping properties of the above operators. Finally, a similar asymptotic analysis to that implemented in the proof of lemma 2.4, one can complete the proof.

Clearly, with lemma 3.3, we have the similar invisibility result for electromagnetic scattering as those remarked in remark 2.11 for acoustic scattering.

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