Vector potential normal form classification for completely integrable solenoidal nilpotent singularities

Majid Gazor∗, Fahimeh Mokhtari

Department of Mathematical Sciences, Isfahan University of Technology
Isfahan 84156-83111, Iran

Jan A. Sanders

Department of Mathematics, Faculty of Sciences, Vrije Universiteit,
Amsterdam 1081 HV, The Netherlands

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Abstract

We introduce a \(\mathfrak{sl}_2\)-invariant family of nonlinear vector fields with a non-semisimple triple zero singularity. In this paper we are concerned with characterization and normal form classification of these vector fields. We show that the family constitutes a Lie algebra structure and each vector field from this family is solenoidal, completely integrable and rotational. All such vector fields share a common quadratic invariant. We provide a Poisson structure for the Lie algebra from which the second invariant for each vector field can be readily derived. We show that each vector field from this family can be uniquely characterized by two alternative representations; one uses a vector potential while the other uses two functionally independent Clebsch potentials. Our normal form results are designed to preserve these structures and representations. The results are implemented in Maple in order to compute vector potential and the Clebsch potential normal forms of a given vector field from this family. Some practical normal form coefficient formulas for degrees of up to four are presented.

Keywords: Normal form classification; Triple zero singularity; \(\mathfrak{sl}_2\)-Lie algebra representation; Clebsch potentials; Completely integrable vector field.

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1 Introduction

We are concerned with nonlinear normal form classification of a \(\mathfrak{sl}_2\)-Lie algebra generated family of vector fields with a non-semisimple nilpotent linear part, i.e., \(N := x\frac{\partial}{\partial y} + 2y\frac{\partial}{\partial z}\). Jacobson-Morozov theorem provides the other two generators, that they form a triple along with \(N\), for a \(\mathfrak{sl}_2\)-Lie algebra. Consider \(N\) as a differential operator. Then by the adjoint action of the \(\mathfrak{sl}_2\)-Lie algebra on (nonlinear) vector fields, we introduce a \(\mathfrak{sl}_2\)-invariant family of vector fields. We show that the set of all such nonlinear vector fields whose linear part is a multiple scalar of \(N\) constitutes a Lie algebra, where we denote it by \(\mathcal{B}\).

∗ Corresponding author. Phone: (98-31) 33913634; Fax: (98-31) 33912602; Email: mgazor@cc.iut.ac.ir.
An important goal in our normal form results is to detect, compute and preserve possible symmetries and geometrical features of a vector field’s flow. Hence, several geometric properties for our introduced \( \mathfrak{sl}_2 \)-invariant family are carefully studied in this paper. A natural dynamics analysis of such vector fields must respect these geometrical features which have applications in different applied disciplines. However, the classical normal form computations typically destroy certain symmetries of the truncated normal form system. These may include the system’s properties such as volume-preserving, Clebsch potentials, vector potentials, etc. Thus in either of these cases, the dynamics analysis of the truncated normal form is not appropriate. Hence it is important to use permissible transformations which preserve the system’s symmetry; also see \([14, 16, 19]\). One of our most important claimed contributions here is that our (truncated) normal form results preserve all the different representations (described below) in the paper such as vector potential, Clebsch potentials, and the volume-preserving property.

Solenoidal vector fields appear in disciplines such as magnetic fields and fluid mechanics; e.g., see \([21, 23–25]\). Computing the invariants of such vector fields is an important goal in this paper. Any solenoidal vector field, say \( v \), takes a vector potential representation, that is, there exists a vector field, say \( w \), whose curl is \( v \), i.e., \( \nabla \times w = v \). We prove that all vector fields in \( \mathcal{B} \) are solenoidal and provide the method and formulas for deriving their vector potential normal forms. We further introduce a Poisson algebra that is Lie-isomorphic to \( \mathcal{B} \) through a Lie isomorphism \( \psi \). Another most important claimed contributions in this paper is that the Lie isomorphism \( \psi \) associates a first integral \( \psi(v) \) to each vector field \( v \) from \( \mathcal{B} \). We further show that the quadratic polynomial \( \Delta := xz - y^2 \) is a second first integral for all vector fields in \( \mathcal{B} \).

Analytic normalization of an analytic vector field has close relations with complete integrability of the vector field; e.g., see \([34, 42, 46]\). We recall that two first integrals for \( v \) in \( \mathcal{B} \) are called functionally independent when their gradients have a rank of 2 for almost everywhere; e.g., see \([42, \text{page 3553}] \) and \([34]\). The level curves of these invariants provide a comprehensive understanding about the orbits in the state space associated with the vector field’s flow. Each vector field \( v \) from the \( \mathfrak{sl}_2 \)-invariant Lie algebra \( \mathcal{B} \) is a completely integrable solenoidal vector field; i.e., we show that the invariants \( \Delta \) and \( \psi(v) \) for each \( v \in \mathcal{B} \) are functionally independent. There is another alternative representation for completely integrable solenoidal vector fields, that is given by the two of the vector field’s functionally independent invariants. Indeed, we prove that each vector field \( v \) in \( \mathcal{B} \) equals the exterior product of the gradients of \( \Delta = xz - y^2 \) and \( \psi(v) \); the latter is obtained through the Lie isomorphism \( \psi \) between \( \mathcal{B} \) and our introduced Poisson algebra. The first integrals in the exterior product are referred as Clebsch potentials or Euler potentials of the vector field \( v \); e.g., see \([23, 24]\). We refer to \( \Delta \) by the primary Clebsch potential and \( \psi(v) \) as the secondary Clebsch potential for \( v \). We further conclude that these families of triple zero singularities are rotational vector field, that is, their curl is non-zero. This implies that these are not gradient vector fields.

Finally, we prove that \( \mathcal{B} \) is the set of all multiple scalars of solenoidal vector fields such as \( v \), that is given by
\[
v := -x \frac{\partial}{\partial y} - 2y \frac{\partial}{\partial z} + v(x, y, z),
\]
where \( v : \mathbb{R}^3 \to \mathbb{R}^3 \) denotes a vector field without constant and linear parts,
\[
\text{div}(v) = 0, \quad \text{and} \quad v(\Delta) = 0 \quad \text{where} \quad \Delta = xz - y^2.
\]

Note that for our convenience we interchange the uses of notations and terminologies such as vector fields, differential systems and differential operators. We also remark that \( x \) and \( \Delta \) are the two generators of the invariant algebra for the linear vector field \( N \). We refer to the vector field \( v \) in equation (1.1)-(1.2)
by a completely integrable system since it has two functionally independent invariants $\Delta$ and $\psi(v)$. The Poisson structure and the Lie isomorphism $\psi$ provide a practical method for deriving the second first integral within the invariant algebra of vector fields given by $(1.1)-(1.2)$ and their normal forms.

Normal form classification of nilpotent singularities has been a challenging task. Even in the two-dimensional case, there have been numerous important contributions in various types and approaches; e.g., see [1–3, 5, 10, 13, 18, 20, 22, 36–39, 41, 43, 45]. There have only been a few contributions in three-dimensional state space cases; see [11, 44] where hypernormalization is performed up to degree three; also see [15–17] and [27–31]. In this paper we provide a complete normal form classification for all vector fields $v$ in equations $(1.1)-(1.2)$, that is, the set of all completely integrable solenoidal nilpotent singularities where $\Delta$ is one of their invariants and a multiple scalar of $N$ is their linear part. These vector fields and their normal forms are uniquely characterized by their secondary Clebsch potential. Indeed, the primary Clebsch potential $\Delta$ is always preserved throughout the normalization steps while the normalizing transformations naturally reflect the normal form changes into the secondary Clebsch potential. In Theorem 5.1, we prove that a vector field given by $(1.1)-(1.2)$ can be either linearized or uniquely transformed into the formal normal form vector field

$$w := -x \frac{\partial}{\partial y} - 2y \frac{\partial}{\partial z} + z^p(z \frac{\partial}{\partial y} + 2y \frac{\partial}{\partial x}) + \sum_{k=p+1}^{\infty} \sum_{i=0}^{\lfloor \frac{k-p}{2} \rfloor} b_{i,k}z^i(xz - y^2)^{k-2i+p}(z \frac{\partial}{\partial y} + 2y \frac{\partial}{\partial x}),$$

(1.3)

where $b_{i,k} \in \mathbb{R}$ and $p$ is a natural number. In addition, the secondary Clebsch potential normal form is given by

$$I(x, y, z) = x + \frac{1}{p+1} z^{p+1} + \sum_{k=p+1}^{\infty} \sum_{i=0}^{\lfloor \frac{k-p}{2} \rfloor} \frac{b_{i,k}}{i+1} z^{i+1}(xz - y^2)^{k-2i+p}.$$  

(1.4)

The normal form invariant $(1.4)$ can sometimes be used for further reduction of the normal form vector fields; e.g., see section 5.

Now we describe the organization of the rest of this paper. We introduce a family of $\mathfrak{sl}_2$-invariant irreducible vector spaces of vector fields in section 2. We further prove that this family constitutes a Lie algebra and derive their associated structure constants. Section 3 is devoted for introducing a Poisson algebra and prove that it is Lie isomorphic to $\mathcal{B}$. Next, we discuss the geometrical properties of $\mathcal{B}$ family in section 4. In particular, we show that our $\mathfrak{sl}_2$-invariant introduced family of vector fields are fully characterized by equations $(1.1)-(1.2)$. Two further representations for each such vector field are presented in this section by using their Clebsch potentials and vector potentials. Section 5 is dedicated to study the normal form classification for vector fields $(1.1)-(1.2)$. Some practical formulas for normal form coefficients of up to degree three for a given triple zero singularity $(1.1)-(1.2)$ is presented. Finally, we introduce two more $\mathfrak{sl}_2$-invariant families of vector fields in Appendix A. These along with the family given by equations $(1.1)-(1.2)$ would amount to three family types of $\mathfrak{sl}_2$-invariant vector fields so that each three dimensional vector field can be uniquely Taylor expanded in terms of vector polynomial generators from these three types.

### 2 Algebraic structures

Let

$$N := x \frac{\partial}{\partial y} + 2y \frac{\partial}{\partial z}, \quad M := z \frac{\partial}{\partial y} + 2y \frac{\partial}{\partial x}, \quad \text{and} \quad H := [M, N] = MN - NM = 2z \frac{\partial}{\partial z} - 2x \frac{\partial}{\partial x}.$$  

(2.1)

The triple $\{M, N, H\}$ generates a $\mathfrak{sl}_2$ Lie algebra, i.e.,

$$[M, N] = H, \quad [H, M] = 2M, \quad [H, N] = -2N.$$
We denote $N^n f v = (N^n f)v$ for the iterative action of $N$ as a differential operator on the scalar function $f$ that is also multiplied with $v$. Further for a vector field $v$, $N^n v$ is inductively defined by

$$N^n v := \text{ad}_N v, \quad \text{and} \quad N^n v := \text{ad}_N N^{n-1} v \quad \text{for} \ n > 1.$$ 

We often put the underline only on the last element on which the operator acts, i.e., $N^n fgh := hN^n fg$ and $NfM := Nfg$. Note that $N$ as an operator distinguishes vector fields from scalar functions: the operator $N$ merely acts on scalar functions as a differential operator while it acts as a Lie operator on vector fields. By [6, Proposition 2], $\mathbb{R}[[z, \Delta]]$ is the invariant ring for $M$. For a homogeneous scalar polynomial function $f : \mathbb{R}^3[x, y, z] \to \mathbb{R}$ in $\ker M = \langle \Delta, z \rangle$, there exists an integer $\omega_f \in \mathbb{Z}$ so that

$$Hf = \omega_f f,$$

where $\omega_f$ and $f$ are called the eigenvalue and eigenfunction of the differential operator $H$, respectively. The algebra of first integrals for $M$ is the same as $\ker M = \langle \Delta, z \rangle$; see [7, 28, Chapter 2] and [33, Chapter 9] for more details on the representation of $\mathfrak{sl}_2$ and normal form theory. In this section we use the $\mathfrak{sl}_2$-triple (2.1) to generate a family of irreducible $\mathfrak{sl}_2$-invariant vector spaces. The set of all such invariant vector spaces constitutes a Lie algebra. This consists of all completely integrable and solenoidal vector fields of triple zero singularities, that is, their linear part is a scalar multiple of $N$ in (2.1). Vector fields from this family can be considered as alternatives in three dimensional state space for completely integrable Hamiltonian systems which always require even dimensions in state space; also see [14, page 2812].

**Notation 2.1.**

- The following notations frequently appear in this paper.

  $\sigma_1 := \sigma_1(s_1, s_2, k_1, k_2) = s_1 + s_2 + k_1 + k_2, \quad \sigma_2 := \sigma_2(q_1, q_2, i_1, i_2) = q_1 + q_2 - i_1 - i_2. \quad (2.2)$

  - We denote $e_1, e_2$ and $e_3$ for the standard basis of $\mathbb{R}^3$ and $\kappa_{l,i} := \frac{n!}{(i-l)!}$.
  - We use the Pochhammer $k$-symbol notation for any $a, b \in \mathbb{R}, k \in \mathbb{N}$ as $(a)_b^k := \prod_{j=0}^{k-1}(a + jb)$.
  - Throughout this paper we frequently use some constants or variables with negative powers in the denominator (or numerator) of a fraction. The reader should merely treat this as a formal misuse of notation to shorten the formulas.

Now we present some technical results which play a central role in this paper.

**Lemma 2.2.** Let $f$ be a homogeneous scalar polynomial function. Then,

$$N^n f M = N^n f M - nN^{n-1}fH - n(n-1)N^{n-2}fN, \quad \text{for any} \ n \in \mathbb{N}. \quad (2.3)$$

**Proof.** The proof is by induction. For $n = 1$, we have

$$NfM = N(f)M + f[N, M] = N(f)M - fH = NfM - fH.$$ 

By the induction hypothesis we have

$$N^{n+1} f M = [N, N^n f M - nN^{n-1}fH - n(n-1)N^{n-2}fN]$$

$$= N^n f M + N^n f NM - nN^n f H - nN^{n-1}fN - n(n-1)N^{n-1}fN$$

$$- n(n-1)N^{n-2}fNN - (N^n f MN - nN^{n-1}fHN - n(n-1)N^{n-2}fNN)$$

$$= N^n f M + N^n f [N, M] - nN^n f H - nN^{n-1}f[N, H] - n(n-1)N^{n-1}fN$$

$$= N^{n+1} f M - (n + 1)N^n f H - n(n+1)N^{n-1}fN.$$ 

This proves the statement; also see [7, Proposition 1].
There is an important corollary to this lemma.

**Corollary 2.3.** For any homogeneous function \( f \in \ker M \), we have
\[
N^n f M = \frac{2(\omega_f-n+2)\kappa_f N^{n+1} f}{(\omega_f+2)\kappa_f} \frac{\partial}{\partial z} + (\omega_f-n+2)\kappa_f \frac{\partial}{\partial y} - \frac{\partial}{\partial x} - 2(\omega_f+2)\kappa_f \frac{\partial}{\partial y} N^n - \frac{\partial}{\partial x} N^n. \tag{2.4}
\]

**Lemma 2.4.** For each \( l \in \mathbb{N}_0 \), the following equalities hold:
\[
\frac{\partial}{\partial x} N^l = l(l-1)N^{l-2} \frac{\partial}{\partial z} + lN^l \frac{\partial}{\partial y} + N^l \frac{\partial}{\partial x}, \quad \frac{\partial}{\partial y} N^l = 2lN^l \frac{\partial}{\partial z} + N^l \frac{\partial}{\partial y}. \tag{2.5}
\]

**Proof.** The proof is by induction on \( l \). For instance, by the induction hypothesis we have
\[
\frac{\partial}{\partial y} N^l = 2lN^{l-1} \frac{\partial}{\partial z} + N^l \frac{\partial}{\partial y} = 2(l+1)N^l \frac{\partial}{\partial z} + N^{l+1} \frac{\partial}{\partial y}.
\]
This proves the second equality. \( \square \)

**Lemma 2.5.** Let \( q = 2s + r \), where \( r = 0 \) or \( r = 1 \) and \( s \in \mathbb{N}_0 \). Then,
\[
N^q \frac{\partial}{\partial x} = \sum_{n=0}^{s} \eta_n^{q,i} x^{s-n} y^r z^{i-s-r-n} \Delta^n, \quad N^q \frac{\partial}{\partial y} = \sum_{n=0}^{s} \zeta_n^{q,i} x^{s-n} y^{r+1} z^{i-s-r-n} \Delta^n, \tag{2.6}
\]
where
\[
\eta_n^{q,i} := \frac{(-1)^n(s)_{n-i}(i)_{s-n} p^{s+n+r} (2i-1)_{s-n} q^{s+n+r}}{n!(2i-1)^{n-i}_{2}}, \quad \zeta_n^{q,i} := \frac{q^i (2s+r)^{2s+n}}{(s-n)(2n+r)!(i-n-s-r)!} \tag{2.7}
\]
for \( i \geq s + n + r \), while \( \eta_n^{0,i} := \zeta_n^{0,i} := 0 \) for \( i < q + n - s \). In particular, \( N^q \frac{\partial}{\partial z} = 0 \) when \( s < q - i \).

**Proof.** The proof is straightforward by an induction on \( q \). \( \square \)

**Lemma 2.5** enables us to formulate \( N^q \frac{\partial}{\partial x} N^q \frac{\partial}{\partial y} \) in terms of \( x, y, z \) and \( \Delta \).

**Proposition 2.6.** Let \( q_1 = 2s_1 + r_1 \) and \( q_2 = 2s_2 + r_2 \), where \( r_1, r_2 \in \{0, 1\} \) and \( s_1, s_2 \in \mathbb{N}_0 \). Then,
\[
N^{q_1} \frac{\partial}{\partial x} N^{q_2} \frac{\partial}{\partial y} = \sum_{n=0}^{\min\{s_1 + s_2 - \sigma_2, s_1 + s_2\}} \eta_{n, i, j}^{q_1, q_2} x^{s_1 + s_2 - \sigma_2, i, j} \Delta^n, \tag{2.8}
\]
and
\[
N^{q_1} \frac{\partial}{\partial x} N^{q_2} \frac{\partial}{\partial y} = \sum_{n=0}^{\min\{s_1 + s_2 - \sigma_2, s_1 + s_2\}} \eta_{n, i, j}^{q_1, q_2} x^{s_1 + s_2 - \sigma_2, i, j} \Delta^n \tag{2.9}
\]
hold where
\[
\eta_{n, i, j}^{q_1, q_2} := \sum_{r=0}^{n} \zeta_{n, i, j}^{q_1, q_2} \frac{\partial}{\partial x} N^r, \quad \eta_{n, i, j}^{q_1, q_2} := \sum_{r=0}^{n} \eta_{n, i, j}^{q_1, q_2} \frac{\partial}{\partial x} N^r \tag{2.10}
\]

Further, \( N^{q_1} \frac{\partial}{\partial x} N^{q_2} \frac{\partial}{\partial y} = 0 \) when either \( s_1 < q_1 - i \) or \( s_2 < q_2 - j \). In particular, \( N^{q_1} \frac{\partial}{\partial x} N^{q_2} \frac{\partial}{\partial y} = 0 \) for \( s_1 + s_2 < \sigma_2(q_1, q_2, i, j) = q_1 + q_2 - i - j \).

**Proof.** Given Lemma 2.5, the proof is trivial. \( \square \)

The following theorem provides an alternative formula for the expansion of \( N^n \frac{\partial}{\partial x} N^n \frac{\partial}{\partial y} \).
Theorem 2.7. Let $q_1 = 2s_1 + r_1$, $q_2 = 2s_2 + r_2$ and $\sigma_2(q_1, q_2, i, j) = q_1 + q_2 - i - j$. Then,

$$N^{q_1}_{s_1}{z^q}^{q_2}_{s_2} = \sum_{p = \max\{\sigma_2, 0\}} C^{q_1, q_2}_{p, i, j} N^{2p + |r_2 - r_1|}{z^p}{s_1 + s_2 - p + \frac{|r_1 + r_2|}{2}},$$

(2.9)

where $C^{q_1, q_2}_{p, i, j}$ is given by

$$\sum_{r = 0}^{s_1 + s_2 + \frac{|r_1 + r_2|}{2} - p} \left( \frac{\eta_{n,i,j}^{q_1, q_2} - \eta_{n-1,i,j}^{q_1, q_2}}{\eta_0} \right)^{s_1 + s_2 - p - r} \left( \frac{\eta_{n-1,i,j}^{q_1, q_2}}{\eta_0} \right)^{s_1 + s_2 + r - p} \left( \frac{\eta_{n-1,i,j}^{q_1, q_2}}{\eta_0} \right)^{2p + |r_2 - r_1|} \left( \frac{(p + 1)(s_1 + s_2 - p - r - |r_1 + r_2|)}{\eta_0^{s_1 + s_2 - p + r - |r_1 + r_2|}} \right)^{s_1 + s_2 - p + r + \frac{|r_1 + r_2|}{2}} \left( \frac{\eta_{n-1,i,j}^{q_1, q_2}}{\eta_0} \right)^{s_1 + s_2 + r + p - r - \frac{|r_1 + r_2|}{2}} \left( \frac{\eta_{n,i,j}^{q_1, q_2}}{\eta_0} \right)^{2p + |r_2 - r_1|} \left( \frac{(p + 1)(s_1 + s_2 - p + r + |r_1 + r_2|)}{\eta_0^{s_1 + s_2 - p + r + |r_1 + r_2|}} \right)^{s_1 + s_2 + p + r + \frac{|r_1 + r_2|}{2}}.$$

(2.10)

Proof. A polynomial expansion for the left hand side in (2.9) is derived in equation (2.7) while by using the first equation in (2.6), the right hand side is given by

$$\sum_{p = \max\{\sigma_2, 0\}}^{s_1 + s_2 + \frac{|r_1 + r_2|}{2}} C^{q_1, q_2}_{p, i, j} \sum_{n = 0}^{\eta_{n,i,j}^{q_1, q_2}} N^n_{s_1 + s_2 - p + n + \frac{|r_1 + r_2|}{2}}.$$

Given Lemma 2.5, we remark that $N^{2p + |r_2 - r_1|}{z^{2p + |r_2 - r_1|}} = 0$ for $p < \sigma_2$.

When $r_1 = r_2 = 0$, equation (2.9) is equivalent with the following polynomial equation

$$\sum_{n = 0}^{\eta_{n,i,j}^{q_1, q_2}} N^n_{s_1 + s_2 - n + \frac{|r_1 + r_2|}{2}} \sum_{p = s_1 + s_2 - n}^{\eta_{n,i,j}^{q_1, q_2}} C^{q_1, q_2}_{p, i, j} N^p_{s_1 + s_2 - p + \frac{|r_1 + r_2|}{2}}.$$

Hence for each $i, j, s_1, s_2$, and $0 \leq n \leq \min\{s_1 + s_2 - \sigma_2, s_1 + s_2\}$, we have

$$\tilde{\eta}_{n,i,j}^{q_1, q_2} = \sum_{k = 0}^{n} \eta_{n-k,i,j}^{q_1, q_2 - 2k, i-j-2k} C^{q_1, q_2}_{s_1 + s_2 - k, i,j}.$$

These introduce a family of upper triangular linear matrix equations. The determinant of the coefficient matrix is given by $\prod_{n = 0}^{\min\{s_1 + s_2 - \sigma_2, s_1 + s_2\}} \eta_0^{q_1 + q_2 - 2n, i-j-2n} \neq 0$. These together with the first equation in (2.6) give rise to

$$C^{q_1, q_2}_{p, i, j} = \sum_{r = 0}^{s_1 + s_2 - p} \left( \frac{\eta_{n,i,j}^{q_1, q_2} - \eta_{n-1,i,j}^{q_1, q_2}}{\eta_0} \right)^{s_1 + s_2 - p - r} \left( \frac{\eta_{n-1,i,j}^{q_1, q_2}}{\eta_0} \right)^{s_1 + s_2 + r - p} \left( \frac{\eta_{n-1,i,j}^{q_1, q_2}}{\eta_0} \right)^{2p + |r_2 - r_1|} \left( \frac{(p + 1)(s_1 + s_2 - p - r - |r_1 + r_2|)}{\eta_0^{s_1 + s_2 - p + r - |r_1 + r_2|}} \right)^{s_1 + s_2 - p + r + \frac{|r_1 + r_2|}{2}} \left( \frac{\eta_{n,i,j}^{q_1, q_2}}{\eta_0} \right)^{s_1 + s_2 + r + p - r - \frac{|r_1 + r_2|}{2}} \left( \frac{\eta_{n-1,i,j}^{q_1, q_2}}{\eta_0} \right)^{2p + |r_2 - r_1|} \left( \frac{(p + 1)(s_1 + s_2 - p + r + |r_1 + r_2|)}{\eta_0^{s_1 + s_2 - p + r + |r_1 + r_2|}} \right)^{s_1 + s_2 + p + r + \frac{|r_1 + r_2|}{2}}.$$

Now let $r_1 = r_2 = 1$. By substituting $y^2 = xz - \Delta$ into equation (2.7), $N^{q_1,i,j}z^{q_2,i,j}$ is given by

$$\sum_{n = 1}^{\min\{s_1 + s_2 - \sigma_2, s_1 + s_2\}} \left( \tilde{\eta}_{n,i,j}^{q_1, q_2} - \tilde{\eta}_{n-1,i,j}^{q_1, q_2} \right) x^{s_1 + s_2 - n + 1} x^{i+j-s_1-s_2-n-1} \Delta^n + \tilde{\eta}_{0,i,j}^{q_1, q_2} x^{s_1 + s_2 + 1} x^{i+j-s_1-s_2} \Delta^{s_1 + s_2 + 1}.$$

Hence the family of linear equations and its solutions are derived by

$$\tilde{\eta}_{n,i,j}^{q_1, q_2} - \tilde{\eta}_{n-1,i,j}^{q_1, q_2} = \sum_{k = 0}^{n} \tilde{\eta}_{n-k,i,j}^{q_1, q_2 - 2k, i-j-2k} C^{q_1, q_2}_{s_1 + s_2 - k, i,j},$$

and

$$C^{q_1, q_2}_{n,i,j} := \sum_{r = 0}^{s_1 + s_2 + 1 - n} \left( \tilde{\eta}_{n,i,j}^{q_1, q_2} - \tilde{\eta}_{n-1,i,j}^{q_1, q_2} \right) \left( n+1 \right)^{s_1 + s_2 - n - r + 1} \left( n-\sigma_2-1 \right)^{s_1 + s_2 - n - r + 1} \left( s_1 + s_2 - n - r + 1 \right)^{s_1 + s_2 - n - r + 1} \left( s_1 + s_2 - n - r + 1 \right)^{s_1 + s_2 - n - r + 1} \left( s_1 + s_2 - n - r + 1 \right)^{s_1 + s_2 - n - r + 1},$$

respectively.

□

Lemma 2.8. The formal power series ring $\mathbb{R}[[z, \Delta]]$ is the ring of invariants for $M$. 
Proof. Obviously, \( \mathbb{R}[[z, \Delta]] \) is an invariant ring for \( M \). We prove that this is actually the ring of invariants for \( M \) using generating functions; see [28, Lemma 4.7.9] and [6, 9, 33]. The generating function for \( \mathbb{R}[[z, \Delta]] \) is given by

\[
\frac{1}{(1 - u^2)(1 - t^2)}.
\]

(2.11)

Here, \( z \) has an eigenvalue 2 while the eigenvalue for \( \Delta \) is 0. Hence, differentiating with respect to \( u \) at \( u = 1 \) leads to

\[
\frac{(1-u^2)(1-t^2)+2u^2t(1-t^2)}{(1-u^2)(1-t^2)^2}|_{u=1} = \frac{(1-u^2)(1-t^2)+2(1-t^2)}{(1-u^2)(1-t^2)^2} = \frac{1}{(1-t)^3}.
\]

The latter is the generating function for all formal power series associated with three variables. The proof is complete by a differential form version of [28, Lemma 4.7.9].

\[\square\]

**Theorem 2.9.** Let \( V = \text{span}\{N^n z^i \frac{\partial}{\partial x^i}, N^n z^i \frac{\partial}{\partial y^i}, N^n z^i \frac{\partial}{\partial z^i} \mid n, i, k \in \mathbb{N}_0 \} \) and

\[\mathcal{K} = \text{span}\{z^i \Delta k \frac{\partial}{\partial x^i}, z^i \Delta k \frac{\partial}{\partial y^i}, z^i \Delta k \frac{\partial}{\partial z^i} \mid i, k \in \mathbb{N}_0 \},\]

where \( \mathbb{N}_0 \) denotes nonnegative integers. Then, \( \mathcal{K} = \ker \text{ad}_M \) and \( V \) is the set of all three dimensional formal vector fields.

Proof. We first claim that the homogeneous polynomial set

\[\{N^n z^{i+1} \Delta k \frac{\partial}{\partial x^i}, N^n z^{i+1} \Delta k \frac{\partial}{\partial y^i}, N^n z^{i+1} \Delta k \frac{\partial}{\partial z^i} \mid n, i, k \in \mathbb{N}_0 \}\]

is a linearly independent set. Since \([N, f X] = NF X + f [N, X] \) and \([N, M] = [N, E] = 0 \), we imply that the set of polynomials \( N^n z^i \Delta k M \) and \( N^n z^i \Delta k E \) for different \( n, i, k \) are linearly independent. On the other hand due to \( \text{ad}_N \frac{3 \partial}{\partial x} = -\text{ad}_N \frac{2 \partial}{\partial y} = -2 \text{ad}_N \frac{\partial}{\partial z} = 0 \),

\[
N^n z^{i+1} \Delta k \frac{\partial}{\partial x} = N^n z^{i+1} \Delta k \frac{\partial}{\partial y} - N^n z^{i+1} \Delta k \frac{\partial}{\partial z} - 2N^n z^{i+1} \Delta k \frac{\partial}{\partial z}.
\]

Then, the proof of our claim is complete by the linear independency of polynomials \( N^n z^{i+1}, 2yN^n z^i \) and \( xN^n z^i \).

Since \( M \) commutes with \( E, \frac{\partial}{\partial x^i}, \frac{\partial}{\partial y^i}, \frac{\partial}{\partial z^i} \), and \( M z^i = 0 \), \( \mathcal{K} \) is a subspace for \( \ker \text{ad}_M \). By equation (2.11), the generating function for \( \mathbb{R}[[z, \Delta]] \) is \( \frac{1}{(1-u^2)(1-t^2)} \) and thus, the generating function for \( \ker \text{ad}_M \) is given by \( \frac{1}{(1-u^2)(1-t^2)} \). This concludes that \( \mathcal{K} = \ker \text{ad}_M \). Furthermore, \( V \) is the space of all three dimensional vector fields with three variables and its generating function is given by \( \frac{1}{(1-t)^3} \). \[\square\]

We define

\[
B^l_{i,k} := \frac{N^{l+1} z^{i+1} \Delta k M}{K_l+1,2i+2}, \quad \text{for} \quad -1 \leq l \leq 2i + 1, i, k \in \mathbb{N}_0 = \mathbb{N} \cup \{0\}.
\]

(2.12)

By taking \( f := z^i \Delta k \) in equation (2.11), \( \omega_f = 2i \) and

\[
B^l_{i,k} = \frac{2(2i-l)(i+1)N^{l+1} z^{i+1} \Delta k \frac{\partial}{\partial x} + (i-l)N^{l+1} z^{i+1} \Delta k \frac{\partial}{\partial y} - (l+1)N^{l+1} z^{i+1} \Delta k \frac{\partial}{\partial z}}{(i+1)K_l+1,2i+2}.
\]

(2.13)

Hence, \( B^0_{i,0} := -N \) and \( B^1_{i,0} := -M \). Now we introduce \( \mathcal{B} \) as the vector space spanned by all nonlinear vector fields from this nilpotent linear part \( B^1_{i,0} = N, \ i.e., \)

\[
\mathcal{B} := \text{span} \left\{ B^l_{0,0} + \sum b^l_{i,k} B^l_{i,k} \mid -1 \leq l \leq 2i + 1, i \in \mathbb{N}, k \in \mathbb{N}_0, b^l_{i,k} \in \mathbb{R} \right\}.
\]

(2.14)

Note that two more families of vector fields associated with Theorem 2.9 are defined in Appendix A.
Theorem 2.10. Denote \( \sigma_1(s_1, s_2, k_1, k_2) \) and \( \sigma_2(q_1, q_2, i_1, i_2) \) for the indices defined in equations (2.2). The vector space \( \mathcal{B} \) is a Lie algebra with structure constants given by

\[
[B_{i_1,k_1}^{q_1}, B_{i_2,k_2}^{q_2}] = \sum_{j=\max\{\sigma_2-1, 1\}}^{s_1+s_2+|\frac{s_1+s_2}{2}|} a_{q_1,q_2}^{i_1,i_2,j} B_{j+i_1,i_2}^{2j+|r_2-r_1|} \left( 2j+|r_2-r_1|, \sigma_1-j+|\frac{s_1+s_2}{2}| \right),
\]  

(2.15)

where

\[
a_{q_1,q_2}^{i_1,i_2,j} = \sum_{p=1}^{3} \frac{\left( p,q_1,q_2 \right)_{\frac{i_1+k_1}{2}} \left( p,q_1,q_2 \right)_{\frac{i_2+k_2}{2}} C_{j,i_2}^{q_1+3-p,q_2-3+p} C_{j,i_1}^{q_1+3-p,q_2-3+p}}{(2j+|r_2-r_1|+1)(2j+|r_2-r_1|+1)}.
\]

(2.16)

The constants \( C_{i_1,i_2}^{q_1,q_2} \) follow equation (2.10) and

\[
l_{p,q_1,q_2}^{i_1,i_2} \quad \text{and} \quad p_{q_1,q_2}^{i_1,i_2} \quad \text{are defined by equations (2.17).}
\]

(2.17)

Proof. By equations (2.13), (2.9), and Lemma 2.4, the third component of \( B_{i_1,k_1}^{q_1} B_{i_2,k_2}^{q_2} \) is given by

\[
\sum_{p=1}^{3} l_{p,q_1,q_2}^{i_1,i_2} N_{q_1+3-p,q_2-3+p} \Delta_{-k_1-k_2} = \sum_{j=\max\{\sigma_2-1, 0\}}^{s_1+2-s_2+|\frac{s_1+s_2}{2}|} l_{p,q_1,q_2}^{i_1,i_2,j} B_{j+i_1,i_2}^{2j+|r_2-r_1|} \left( 2j+|r_2-r_1|, \sigma_1-j+|\frac{s_1+s_2}{2}| \right) \cdot e_3,
\]

where \( l_{p,q_1,q_2}^{i_1,i_2} \) for \( p = 1, 2, 3 \) are defined by equations (2.17). Now using the latter and equation (2.13), the third component of \( [B_{i_1,k_1}^{q_1}, B_{i_2,k_2}^{q_2}] = B_{i_1,k_1}^{q_1} B_{i_2,k_2}^{q_2} - B_{i_2,k_2}^{q_1} B_{i_1,k_1}^{q_2} \) is given by

\[
\sum_{j=\max\{\sigma_2-1, 1\}}^{s_1+s_2+|\frac{s_1+s_2}{2}|} \sum_{p=1}^{3} l_{p,q_1,q_2}^{i_1,i_2,j} N_{q_1+3-p,q_2-3+p} \Delta_{-k_1-k_2} = \sum_{j=\max\{\sigma_2-1, 1\}}^{s_1+2-s_2+|\frac{s_1+s_2}{2}|} p_{q_1,q_2}^{i_1,i_2,j} B_{j+i_1,i_2}^{2j+|r_2-r_1|} \left( 2j+|r_2-r_1|, \sigma_1-j+|\frac{s_1+s_2}{2}| \right) \cdot e_3,
\]

when \( 2j + |r_2 - r_1| + 1 \neq 0 \). The second component of \( B_{i_1,k_1}^{q_1} B_{j,k_2}^{q_2} \) is given by

\[
\sum_{j=\max\{\sigma_2-1, -1\}}^{s_1+2-s_2+|\frac{s_1+s_2}{2}|} \sum_{p=1}^{3} p_{p,q_1,q_2}^{i_1,i_2,j} \Delta_{k_1+k_2} N_{q_1+3-p,q_2-2+p} \Delta_{-k_1-k_2} = \sum_{j=\max\{\sigma_2-1, 1\}}^{s_1+2-s_2+|\frac{s_1+s_2}{2}|} p_{p,q_1,q_2}^{i_1,i_2,j} B_{j+i_1,i_2}^{2j+|r_2-r_1|} \left( 2j+|r_2-r_1|, \sigma_1-j+|\frac{s_1+s_2}{2}| \right) \cdot e_2,
\]

where the constants \( l_{p,q_1,q_2}^{i_1,i_2} \) are defined by equations (2.17). Again through the equation (2.13), the second component of \( [B_{i_2,k_2}^{q_1}, B_{i_1,k_1}^{q_2}] \) is given by

\[
\sum_{j=\max\{\sigma_2-1, 1\}}^{s_1+s_2+|\frac{s_1+s_2}{2}|} \sum_{p=1}^{3} p_{p,q_1,q_2}^{i_1,i_2,j} C_{j+k_1,j+k_2}^{q_1+3-p,q_2-2+p} \Delta_{j+k_1+k_2} = \sum_{j=\max\{\sigma_2-1, 1\}}^{s_1+2-s_2+|\frac{s_1+s_2}{2}|} p_{p,q_1,q_2}^{i_1,i_2,j} B_{j+i_1,i_2}^{2j+|r_2-r_1|} \left( 2j+|r_2-r_1|, \sigma_1-j+|\frac{s_1+s_2}{2}| \right) \cdot e_2.
\]

(2.18)

On the other hand

\[
a_{q_1,q_2}^{i_1,i_2,j} = \sum_{p=1}^{3} p_{p,q_1,q_2}^{i_1,i_2,j} C_{j+k_1,j+k_2}^{q_1+3-p,q_2-2+p} \Delta_{j+k_1+k_2} = \sum_{p=1}^{3} p_{p,q_1,q_2}^{i_1,i_2,j} B_{j+i_1,i_2}^{2j+|r_2-r_1|} \left( 2j+|r_2-r_1|, \sigma_1-j+|\frac{s_1+s_2}{2}| \right),
\]

when \( 2j + 1 + |r_2 - r_1| \neq 0 \).
We remark that the third component of \( B_{-s_2-1,s_1+1}^{-1} \) is always zero and this corresponds to the condition \( 2j + 1 + |r_2 - r_1| = 0 \). Indeed, this condition occurs when \( j = -1 \) and

\[
(q_1 := 2s_1 \text{ and } q_2 := 2s_2 + 1) \quad \text{or} \quad (q_1 := 2s_1 + 1 \text{ and } q_2 := 2s_2).
\]

Hence, the constant \( a_{q_1,q_2}^{-1} \) are derived through equation (2.18). Now the proof is complete by derivation of the formula for the first component of \( [B_{q_1}^{i_1,k_1}, B_{q_2}^{i_2,k_2}] \)

\[
\sum_{j = \max \{ s_2 - 1, -1 \}}^{s_1 + s_2 + [\frac{q_1 + q_2}{2}] - 1} \sum_{p=1}^{3} \frac{\rho_{q_1,q_2}^{i_1,i_2} C_{j,i+1,j+1}^{i_1,i_1+1} + 1}{(2j + 1 - 2s_2 + |r_2 - r_1|)(2j + 1 - s_2 + |r_2 - r_1|)^{\frac{1}{2} + 1}} \cdot e_1,
\]

where

\[
\rho_{q_1,q_2}^{i_1,i_2} := \frac{-(q_2 + 1)^2(2i_1 + q_1 + 1)(2i_2 - q_1 + 1)}{(1 + 1)(q_2 + 2, i_2 + 1) + 2}, \quad \rho_{q_1,q_2}^{i_1,i_2} := \frac{-(q_2 + 1)(2i_2 + q_2 + 1)(i_1 + 1)}{2q_2 + 2q_2 + 2q_2 + 1 + 2}, \quad \rho_{q_1,q_2}^{i_1,i_2} := \frac{(q_1 + 1)(2i_2 + q_2 + 1)(i_1 + 1)}{q_2 + 2q_2 + 2q_2 + 2q_2 + 1 + 2},
\]

and the equality

\[
\rho_{q_1,q_2}^{i_1,i_2} = \frac{\sum_{p=1}^{3} \rho_{q_1,q_2}^{i_1,i_2} C_{j,i+1,j+1}^{i_1,i_1+1} + 1}{(2j + 1 - 2s_2 + |r_2 - r_1|)(2j + 1 - s_2 + |r_2 - r_1|)^{\frac{1}{2} + 1}}.
\]

Now we illustrate the structure constants for a few examples.

**Example 2.11.** The following examples are computed by using a Maple program:

\[
[B_{8,3,5}^2, B_{5,2}^2] = \begin{pmatrix} 1152 & 112 \\ 785213 & 503217 \end{pmatrix}, \quad [B_{6,1,4}^7, B_{6,1,4}^1] = \begin{pmatrix} 512 & 429429 \\ 637 & 31603 \end{pmatrix}, \quad [B_{5,6,7}^7, B_{7,8}^5] = \begin{pmatrix} 224 & 27440 \\ 347633 & 6605027 \end{pmatrix}.
\]

Now we remark that

\[
B_{i,k}^l = \Delta^k B_{i,0}^l
\]

for all nonnegative integers \( l, i, k \); this is due to the equality \( N(\Delta) = 0 \). Further recall that \( \Delta \) is invariant under the \( \mathfrak{sl}_2 \)-action. Yet the following demonstrates the complexity of the structure constants:

\[
[B_{5,0,3}^0, B_{4,1}^4] = \begin{pmatrix} 256 & 512 \\ 297297 & 42471 \end{pmatrix}, \quad [B_{3,1}^1, B_{3,3}^3] = \begin{pmatrix} 512 & 416 \\ 3927 & 3927 \end{pmatrix}, \quad [B_{2,4}^5, B_{7,5,2}^3] = \begin{pmatrix} 1312 & 143 \\ 1881 & 24 \end{pmatrix}.
\]

Similar to equations (3.8a)-(3.8h)], we further present some Lie brackets that they are particularly useful for our normal form results:

\[
[B_{0,0,0}^0, B_{i,k}^l] = (l - i)B_{i,k}^l, \quad [B_{1,0,0}^1, B_{i,k}^l] = (l - 2i + 1)B_{i,k}^{l+1}, \quad [B_{0,0}^{-1,0}, B_{i,k}^l] = (l + 1)B_{i,k}^{-l-1},
\]

\[
[B_{p,0}^{-1,0}, B_{i,k}^l] = \sum_{j = \max \{ q - 2i - p, 0 \}}^{1 + [\frac{q - 2i - p}{2}] - 1} a_{j,p}^{l} B_{2j + q + 2i + p + |r - 1|, s - 1 + k + j + [\frac{q + 2i + p}{2}]}.
\]
3 Poisson algebra structure

We consider the ring of formal power series $\mathbb{R}[[x, y, z]]$ and define a Poisson bracket on the ring’s variables by

$$\{x, y\} = x, \quad \{x, z\} = 2y, \quad \{y, z\} = z. \quad (3.1)$$

Since $f$ and $g$ from $\mathbb{R}[[x, y, z]]$ have each a unique representation as formal power series in $x$, $y$, and $z$, the Leibniz rule and bilinearity of the Poisson bracket are sufficient to uniquely determine Poisson structure for all elements in $\mathbb{R}[[x, y, z]]$. In particular, the Poisson bracket is independent of the splitting functions in $\mathbb{R}[[x, y, z]]$, i.e.,

$$\{f, g(hk)\} = (gh)k. \quad (3.2)$$

Indeed, by the Leibniz rule we have

$$\{f, (gh)k\} = \{f, g\}hk + gk\{f, h\} + gh\{f, k\} - kg\{f, h\} - kh\{f, g\} = 0.$$

Hence, the structure constants associated with monomials are given by

$$\{x^iy^jz^k, x^my^nz^p\} = (in + jp - kn - jm)x^{i+m}y^{n+j-1}z^{k+p} + 2(ip - km)x^{i+m-1}y^{n+j}z^{k+p-1}, \quad (3.3)$$

for arbitrary nonnegative integers $m, n, p, i, j, k$. Now define

$$b_{i,k}^l := -\frac{\text{ad}_{x}^{l+1, i+1} \Delta^k}{(i + 1) \kappa_{i+1, 2i+2}}$$

for $-1 \leq l \leq i + 1$, and $i, k \in \mathbb{N}_0$, \quad (3.4)

where $\text{ad}_x f := \{x, f\}$ and $\text{ad}_x^{n} f := \{x, \text{ad}_x^{n-1} f\}$ for $f \in \mathbb{R}[[x, y, z]]$ and $n > 1$.

**Corollary 3.1.** The following formulas provide two alternative polynomial expansions for each $b_{i,k}^l$ in terms of $x, y, z$ and $\Delta$:

$$b_{s+1, i, k}^l = \sum_{j=0}^{s} \frac{-\eta_{j}^{2s+1, i+1} x^{s-j} y z^{s-j} \Delta^{k+j}}{(i+1) \kappa_{s+1, 2s+2}}, \quad b_{s+1, i, k}^l = \sum_{j=0}^{s} \frac{-\zeta_{j}^{2s+1, i+1} x^{s-j} y z^{s-j} \Delta^{k+j}}{(i+1) \kappa_{s+1, 2s+2}} \quad (3.5)$$

where

$$\eta_{j}^{2s+1, i+1} := \frac{(-1)^{(s)j}(s+1)(i+1)^{s+j+1}(2i+1)^{s+2+j}}{(j+1)! (2j+2)!}, \quad \zeta_{j}^{2s+1, i+1} := \frac{(i+1)(2s+1)!2^{2j}}{(s-j)! (2j)! (i-j-s)!}. \quad (3.6)$$

**Proof.** Due to the previous lemma, the actions of $\text{ad}_x$ and $\text{ad}_N$ on $z^l$ are identical. Hence, our claim readily follows from Lemma 2.5. \qed

Now we define a vector space $\mathcal{B}$ as

$$\mathcal{B} := \text{span}\{b_{0,0}^l + \sum_{i,k} \beta_{i,k}^l b_{i,k}^l | -1 \leq l \leq 2i + 1, i, k \in \mathbb{N}_0, \beta_{i,k}^l \in \mathbb{R}\}. \quad (3.6)$$

The following two lemmas show that $\mathcal{B}$ is a Poisson algebra and it is Lie-isomorphic to $\mathcal{B}$. 

**Lemma 3.2.** The space \( \mathcal{B} \) is invariant under the Poisson bracket and the linear map

\[
\Psi : \mathcal{B} \to \mathcal{B}, \quad \text{defined by} \quad \Psi(b_{i,k}^l) = B_{i,k}^l,
\]

is a Lie isomorphism.

**Proof.** By the Leibniz rule we have

\[
\text{ad}_{z^i}^{n+1} = z \text{ad}_{z^i}^n + n \text{ad}_{z^i} \text{ad}_{z^i}^{n-1}(x)z^i + \frac{n(n-1)}{2!} \text{ad}_{z^i}^2 \text{ad}_{z^i}^{n-2}. = z \text{ad}_{z^i}^n + 2ny \text{ad}_{z^i}^{n-1}(x)z^i + n(n-1)x \text{ad}_{z^i}^{n-2}.
\]

Due to equation (3.7), we have \( \Psi(x) = N, \Psi(y) = \frac{H}{2}, \) and \( \Psi(z) = -M. \) Further,

\[
\text{ad}_{z^i}^{n+1} \Delta^k = \text{ad}_{z^i}^{n+1} \Delta^k.
\]

The actions of \( \text{ad}_{z^i}^n \) on \( z^i \Delta^k \) and \( \text{ad}_N \) on \( z^i \Delta^k \) are identified through \( \Psi. \) Then, the proof follows an induction on \( n, \) structure constants (3.31) and those governing the \( \mathfrak{sl}_2 \) triple \( M, N, \) and \( H. \)

Now we present a ring structure constants for \( \mathcal{B} \) so that \( \mathcal{B} \) is a Poisson algebra.

**Lemma 3.3.** The space \( \mathcal{B} \) is a Poisson algebra. In particular, let \( q_1 = 2s_1 + r_1 \) and \( q_2 = 2s_2 + r_2. \) Then, the ring structure constants are given by

\[
\mathfrak{b}^{q_1-1}_{t_1-1,k_1} \mathfrak{b}^{q_2-1}_{t_2-1,k_2} = \frac{1}{s_1s_2s_3s_4s_5s_6s_7} \sum_{p=\max\{\sigma,0\}}^{s_1+s_2+1} \frac{C_{p+i,j}^{q_1,q_2} \Delta_{p+i,j}^{s_1+p+1}}{(2p-s_2+r_2-r_1+1)(3p-s_2+r_2-r_1+1)}.\]

**Proof.** The proof directly follows from (3.4) and the formulas given in Theorem 2.7. Indeed, we have

\[
\mathfrak{b}^{q_1-1}_{t_1-1,k_1} \mathfrak{b}^{q_2-1}_{t_2-1,k_2} = \frac{N^{q_1+1}N^{q_2+1} \Delta^{k_1+k_2}}{s_1s_2s_3s_4s_5s_6s_7} \sum_{p=\max\{\sigma,0\}}^{s_1+s_2+1} \frac{C_{p+i,j}^{q_1,q_2} \Delta_{p+i,j}^{s_1+p+1}}{(2p-s_2+r_2-r_1+1)(3p-s_2+r_2-r_1+1)}.\]

The following theorem presents a property that is similar to the Hamiltonian cases, i.e., the rate of change of functions along with vector fields from \( \mathcal{B} \) can be computed by the Poisson bracket.

**Theorem 3.4.** For each \( l, i, k, \) we have

\[
B_{i,k}^l = \sum_{j=1}^{3} \{ x_j, \Psi^{-1}(B_{i,k}^l) \} e_j, \quad \text{where} \quad x_1 := x, x_2 := y, x_3 := z.
\]

This representation for \( B_{i,k}^l \) indicates that the family of vector fields from \( \mathcal{B} \) and their associated dynamics are uniquely determined by their secondary Clebsch potentials. Furthermore, the change rate of any formal power series in \( (x, y, z) \), say \( F : \mathbb{R}^3 \to \mathbb{R}, \) along a vector field \( \nu \) from \( \mathcal{B} \) is given by

\[
\frac{dF}{dt} := \{ F, \Psi^{-1}(\nu) \}.
\]

**Proof.** From equations (2.13), (3.7) and (3.4), we have

\[
B_{i,k}^l = (l-2i-1)B_{i,k}^{l+1} \frac{\partial}{\partial x} + (l-i)B_{i,k}^l \frac{\partial}{\partial y} + (l+1)B_{i,k}^{l-1} \frac{\partial}{\partial z}.
\]

Then, the proof follows the Lie isomorphism (3.7), the formulas (2.13), \( \Psi(x) = N, \Psi(y) = \frac{H}{2}, \Psi(z) = -M, \)

\[
B_{0,0}^l = -N \quad \text{and} \quad B_{0,0}^l = -M. \]The vector field representation \( \nu \) from \( \mathcal{B} \) in Poisson bracket form (3.9) directly follows from equation (3.8), the linearity and the continuity (in filtration topology) of the Lie isomorphism \( \psi, \) the continuity and bilinearity of the Poisson bracket, and finally, the chain and Leibniz rules.
4 Geometrical features of integrable solenoidal vector fields

Definition 4.1. A vector field \( v \) is called solenoidal (nondissipative, incompressible, or volume-preserving) when \( \text{div}(v(x)) = 0 \), and otherwise the vector field \( v \) is called generalized dissipative, i.e., \( \text{div}(v(x)) \neq 0 \). When \( v(x) = \nabla f(x) \) for a scalar function \( f(x) \), the vector field \( v \) is called a gradient or a globally potential vector field. Examples of this are the gravitational potential, a mechanical potential energy, and the electric potential energy. The vector field \( v(x) \) is said to be nonpotential (non-gradient) when there exists at least a point \( x \in \mathbb{R}^3 \) such that \( \text{curl}(v(x)) \neq 0 \); e.g., see [40] page 1.

Theorem 4.2. For every \( v \in \mathcal{B} \), \( v \) is solenoidal.

Proof. By the Leibniz rule and \( B_{i,k}^l = \Delta^k B_{i,0}^l \), we observe that

\[
\nabla \cdot B_{i,k}^l = \nabla \cdot \Delta^k B_{i,0}^l = \Delta^k \nabla \cdot B_{i,0}^l + \nabla(\Delta^k) \cdot B_{i,0}^l. \tag{4.1}
\]

We claim that \( \nabla \cdot B_{i,0}^l = 0 \) and \( \nabla(\Delta^k) \cdot B_{i,0}^l = 0 \). Using Lemma 2.4, the partial derivatives of \( N^l_{s,i+1} \) are given by

\[
\frac{\partial}{\partial x} N^l_{s,i+1} = (l+1)(l+2)(i+1)N^l_{s,i}, \quad \frac{\partial}{\partial y} N^l_{s,i+1} = 2(l+1)(i+1)N^l_{s,i}, \quad \frac{\partial}{\partial z} N^l_{s,i+1} = (i+1)N^l_{s,i}. \tag{4.2}
\]

Equation (2.13) and Lemma 2.4 give rise to

\[
\nabla \cdot B_{i,0}^l = (l+1)(i+1) \left( \frac{(l+2)(2i-l+1)}{k_{i+2,2i+2}} + \frac{2(i-l)}{k_{i+1,2i+2}} - \frac{1}{k_{i,2i+2}} \right) N^l_{s,i} = 0. \tag{4.3}
\]

On the other hand for \( l = 2s \), equation (2.13) gives rise to

\[
\nabla(\Delta^k) \cdot B_{i,0}^l = \frac{k(2i-s-1)\Delta^{k-1} N^{2s+2}_{s+1,i+1}}{(i+1)k_{s+1,2i+2}} - \frac{2k(i-s)\Delta^{k-1} N^{2s+2}_{s+1,i+1}}{(i+1)k_{s+1,2i+2}} - \frac{k(s+1)\Delta^{k-1} N^{2s+2}_{s+1,i+1}}{(i+1)k_{s+1,2i+2}} = 2k! (2s+1)! \sum_{j=0}^{s} \frac{(s+1)(s-j-s+1)!}{2^j (2i-s+1)! (2j+1)!} = 0.
\]

The last equality is derived from Lemma 2.3. Hence, \( \nabla(\Delta^k) \cdot B_{i,0}^l = 0 \) due to

\[
(s+1)(i-j-s+1) - j(i-2s) - (i-s+1)(s-j+1) = 0. \tag{4.4}
\]

When \( l = 2s + 1 \), \( \nabla(\Delta^k) \cdot B_{i,0}^l \) is given by

\[
\frac{k(2i-2s)\Delta^{k-1} N^{2s+3}_{s+1,i+1}}{(i+1)k_{s+1,2i+2}} - \frac{2k(i-2s-1)\Delta^{k-1} N^{2s+3}_{s+1,i+1}}{(i+1)k_{s+1,2i+2}} - \frac{k(2s+2)\Delta^{k-1} N^{2s+3}_{s+1,i+1}}{(i+1)k_{s+1,2i+2}} = \sum_{j=0}^{s} \frac{k(s+1)(2i-2s-1)\Delta^{k-1}}{(2i+2)!(i-2s-1)!} \frac{(i-2s-1)!}{(s+j!)}. \tag{4.5}
\]

Hence \( \nabla(\Delta^k) \cdot B_{i,0}^l = 0 \) due to the equation

\[
\frac{(i-2s-1)!}{(s+j!)} = 0. \tag{4.6}
\]

Equations (4.3), (4.4), (4.4) and (4.1) conclude the proof.

Theorem 4.3. Let \( i \) and \( k \) be arbitrary nonnegative integers and \(-1 \leq l \leq i + 1\).
• Polynomials \( b_{i,0}^l \) and \( \Delta \) are two first integrals for \( B_{i,k}^l \), i.e.,

\[
B_{i,k}^l(b_{i,0}^l) = 0 \quad \text{and} \quad B_{i,k}^l(\Delta) = 0.
\]

Indeed for every \( v \in \mathcal{B} \), \( \Psi^{-1}(v) \in \mathcal{B} \) and \( \Delta \) are two first integrals for \( v \).

• A Clebsch potential representation for \( B_{i,k}^l \) is given by

\[
B_{i,k}^l = \Delta^k(\nabla b_{i,0}^l \times \nabla \Delta).
\]

Equation (4.6) provides an alternative representation for each vector field \( v \in \mathcal{B} \) by using the primary and secondary Clebsch potentials \( \Delta \) and \( \Psi^{-1}(v) \in \mathcal{B} \).

• The polynomial functions \( b_{i,0}^l \) and \( \Delta \) are two functionally independent first integrals for \( B_{i,0}^l \).

• The ring of invariants for \( B_{i,k}^{-1} \), \( B_{i,k}^l \), and \( B_{i,k}^{2l+1} \) includes \( \langle \Delta, z \rangle, \langle \Delta, y \rangle \) and \( \langle \Delta, x \rangle \), respectively.

**Proof.** By equation (5.4), Leibniz rule and \( Nz = \{x, z\} = 2y \), we have \( b_{i,0}^l := -\frac{N_{i+1, i+1}^{l+1, l+1}}{(i+1)!} \). Hence, equation (2.13) and Lemma 2.4 imply

\[
B_{i,k}^l(b_{i,0}^l) = \frac{(2i + 1 - l)!(lN_{i+2, i+1}^{l+1, l+1}z_i + 2(i - l)N_{i+1, i}^{l+1, l+1}z_i - (2i + 2 - l)N_{i+1, i}^{l+1, l+1}z_i)}{-\Delta^{-k}(l + 1)^{-1}(i + 1)(2i + 2)!}.
\]

Now we claim that

\[
lN_{i+2, i+1}^{l+1, l+1}z_i + 2(i - l)N_{i+1, i}^{l+1, l+1}z_i - (2i + 2 - l)N_{i+1, i}^{l+1, l+1}z_i = 0.
\]

Equality (4.7) is trivial for the case \( l = 0 \). Let \( l \neq 0 \), \( l := 2s + r \), \( r = 0 \) and 1. By equation (2.7), we have

\[
B_{i,k}^l(b_{i,0}^l) = \sum_{n=0}^{l} f_r(n)x^{l-n}z^{2i-l-n}y^{2n+1},
\]

where \( f_r(n) := \sum_{p=0}^{n} F_r(n, p) \) for all \( 0 \leq n \leq l \),

\[
F_0(n, p) := \frac{(2s)!d_1(i+1)!d_2(i+p-n-s-1)!^{-1}}{(2p)!d_1(s-p)!d_2(s-p-n+1)!d_2(2n-2p)!} + \frac{(2i-s)!d_2(i-p-s-1)!^{-1}}{(2n-2p+1)!d_1(s-p-n)!d_2(s-p-n+1)!d_2(2n-2p+1)!}
\]

and \( F_1(n, p) \) is given by

\[
\frac{2^{2n+1}d_1(i+1)!d_2(i+p-n-s-1)!^{-1}}{(s-n+p)!d_2(2p)!d_1(s-n+p-1)!d_2(2n-2p)!} + \frac{2^{i-2s-1}d_2(i-p-s-1)!^{-1}}{(s+1-p)!d_2(s+1-n+p+1)!d_2(2n-2p+1)!}.
\]

Now we follow Zeilberger’s algorithm [32] Chapter 6) to prove \( f_r(n) = 0 \). Let

\[
G_0(n, p) := \frac{p(2s-i)(2i+2s^2+5p-2is-2i-2n-5)}{(2n-2p+3)(n-p+1)!d_2(2p)!d_1(s-p+1)!d_2(2n-2p+1)!d_2(2n-2p+1)!d_1(s+p-n+1)!d_2(s+p-n+1)!i^{i+p-n-s-1}}.
\]

Then,

\[
-2(2n+3)F_0(n+1, p) + (n-i-1)F_0(n, p) = G_0(n, p + 1) - G_0(n, p).
\]

Next, we add both sides of the equality (4.8) over \( p \) for all \( 0 \leq p \leq n - 1 \). Hence \( G_0(n, n) \) is given by

\[
(n-i-1)f_0(n) - (4n+6)f_0(n+1) + (4n+6)(F_0(n+1, n) + F_0(n+1, n+1)) = (n-i-1)F_0(n, n).
\]
On the other hand \( G_0(n, n) = 2(2n + 3) \left( F_0(n + 1, n) + F_0(n + 1, n + 1) \right) - (n - i - 1)F_0(n, n) \) and
\[
-2(2n+3)f_0(n+1) + (n - i - 1)f_0(n) = 0.
\]

Thereby, \( f_0(n) = \frac{f_0(0)}{2^n} \prod_{j=0}^{n-1} \frac{j-i-1}{2j+3} \); also see \([32, \text{ page } 103]\). Since
\[
f_0(0) = F_0(0, p) = \frac{1}{(s-1)!s(i-s)(i-s-1)!} = 0,
\]
\( f_0(n) = 0 \) for any \( n \). Now let \( l = 2s + 1 \), i.e., \( r := 1 \). Let
\[
G_1(n, p) = \frac{p^{2n+2}2l(i+1)!/(2s+1)!/(2s+2)!/(2s+3)!}{(i-p-s)(2n-2p)(s+1-p)(n-p+1)(i-2s-1)} \frac{1}{(s-1)!(i-s)!}\]
and thus,
\[
(-3 - 2n)F_1(n + 1, p) + 2(n - i - 1)F_1(n, p) = G_1(n, p + 1) - G_1(n, p).
\]

Hence,
\[
-(2n+3)f_1(n+1) + 2(n - i - 1)f_1(n) = 0,
\]
\( f_1(0) = F_1(0, 0) = 0 \) and finally, \( f_1(n) = 0 \). Hence, \( B^l_{i,k}(b^l_{i,0}) = 0 \) and \( b^l_{i,0} \) is an invariant function for \( B^l_{i,k} \).

Equation (4.11) and Theorem 4.2 give rise to \( B^l_{i,k}(\Delta) = \nabla(\Delta) \cdot B^l_{i,k} = 0 \). Hence, \( \Delta \) is also a first integral for \( B^l_{i,k} \).

By Lemma 2.4
\[
\nabla b^l_{i,0} = \frac{1}{\kappa_{i+1,2i+2}} \left( l(l+1)N^{l-1}z^{i}2(l+1)N^{l+1}z^{i} \right).
\]

Since \( B^l_{i,k} \) is tangent to the level surfaces of \( b^l_{i,0} \) and \( \Delta \) for any \( (x, y, z) \in \mathbb{R}^3 \), there exists a function \( S^l_{i,k}(x, y, z) \) such that
\[
B^l_{i,k} = S^l_{i,k} \nabla b^l_{i,0} \times \nabla \Delta.
\]

Hence,
\[
(S^l_{i,0} \nabla b^l_{i,0} \times \nabla \Delta) \cdot e_3 - B^l_{i,0} \cdot e_3 = -S^l_{i,0} \frac{\log N^{l-1}z^{i} + zN^{l+1}z^{i}}{2(l+1)\kappa_{i+1,2i+2}} = 0.
\]

Therefore by equation (2.5),
\[
\sum_{n=0}^{s} \zeta^{2s+1, i} x^{s-n} y^{2n} z^{i-s+1} = 0.
\]

Thus, \( S^l_{i,0} = 1 \) due to
\[
\frac{\kappa^{2s+1, i}}{\kappa_{i+1,2i+2}} - \frac{\zeta^{2s+1, i}}{2(l+1)\kappa_{i+1,2i+2}} = 0.
\]

for all \( 0 \leq n \leq s \), and \( \zeta^{2s-1, i} = 0 \). The condition \( S^l_{i,0} = 1 \) implies \( S^l_{i,k} = \Delta^{k} \).

Since \( B^l_{i,k} \neq 0 \) for almost everywhere, the last two claims are immediately concluded from the first and second claim, and Lemma 4.3. \( \square \)
Define the grading function
\[ \delta(B_{i,k}^l) := i + 2k; \] (4.11)
that is, the standard degree of homogeneous vector fields minus one. This makes the space \( (\mathcal{B}, [-,-]) \) a graded Lie algebra. Hence, for \( N \in \mathbb{N}_0 \) the vector space
\[ \mathcal{B}_N := \text{span}\{B_{N-2k,k}^l : l = -1, \ldots, 2(N - 2k) + 1, k = 0, \ldots, \lfloor \frac{N}{2} \rfloor \}, \]
consists of all \( \delta \)-homogenous vector fields of grade \( N \). For \( v \in \mathcal{B} \), we define (also see [12, Definition 3.2])
\[ \text{Terms}(v) := \bigcup_{p=1}^3 \text{Terms}(v \cdot e_p), \]
while
\[ \text{Terms}(v \cdot e_p) := \{ \text{All monomials contributing in Taylor expansion } v \cdot e_p \}. \]
For an instance we have \( \text{Terms}(2x^2 + 3xy + 5) = \{1, x^2, xy\} \). Thereby, for any two \( \delta \)-grade homogeneous vector fields \( v_1 \) and \( v_2 \), where \( \delta(v_1) \neq \delta(v_2) \),
\[ \text{Terms}(v_1 \cdot e_p) \cap \text{Terms}(v_2 \cdot e_p) = \emptyset, \text{ for any } p = 1, 2, 3. \]
When \( i, k \in \mathbb{N}_0, -1 \leq l \leq 2i + 1, \) and \( N := i + 2k \), we define a condition for a nonnegative integer \( m \) by
\[ 0 \leq m \leq \min \{ \lfloor \frac{N}{2} \rfloor, \lfloor \frac{2N-2k-l+1}{2} \rfloor, \lfloor \frac{2k+l+1}{2} \rfloor \} \] (4.12)
and next, a set \( P_{i,k}^l \) by
\[ P_{i,k}^l := \{(m_1, m_2, m_3) : m_1 = l + 2(k - m_3), m_2 = N - 2m_3, \text{the condition } (4.12) \text{ holds for } m_3 \}. \]

**Lemma 4.4.** Let \( B_{i,k}^l \in \mathcal{B}, (m_1, m_2, m_3) \in P_{i,k}^l, \text{ and } p \in \{1, 2, 3\} \). Then,
1. \( \text{Terms}(B_{m_2,m_3}^{m_1} \cdot e_p) \neq \emptyset \) when \( l \neq (3 - p)i + 2 - p \). Otherwise, \( \text{Terms}(B_{m_2,m_3}^{m_1} \cdot e_p) = \emptyset \), i.e., for \( (p,l) = (1, 2i + 1), (p,l) = (2, i) \) and \( (p,l) = (3, -1) \).
2. When \( \text{Terms}(B_{i,k}^l \cdot e_p) \neq \emptyset, \)
   \[ \text{Terms}(B_{m_2,m_3}^{m_1} \cdot e_p) \subseteq \text{Terms}(B_{i,k}^l \cdot e_p) \text{ and } P_{i,k}^l = P_{m_2,m_3}^{m_1}. \]
3. Let \( \text{Terms}(B_{i,k}^l) = \text{Terms}(B_{i',k'}^{l'}). \) Then, natural numbers \( l - l' \) and \( i - i' \) are even. Furthermore, the inequalities \( k \neq k', l \neq l' \) and \( i \neq i' \) are equivalent. In particular, \( k = k' \) implies \( (l, i, k) = (l', i', k') \).
4. The set
   \[ \{ \text{Terms}(B_{N,0}^{N-1} \cdot e_p), \text{Terms}(B_{N,0}^{N} \cdot e_p), \text{Terms}(B_{N,0}^{2N+1} \cdot e_p), \text{Terms}(B_{N,0}^{N}) \text{ for } 0 \leq l \leq 2N, l \neq N \} \]
   is a partition for
   \[ \bigcup \{ \text{Terms}(v_N \cdot e_p), v_N \in \mathcal{B} \}. \]
5. \( \text{Terms}(b_{i,k}^l) = \text{Terms}(b_{m_2,m_3}^{m_1}) \neq \emptyset. \)
6. For any \( 0 \neq v \in \mathcal{B} \) and nonnegative integer \( N \), there exists a unique polynomial vector \( v^j_{N-2k,0} \in \mathbb{R}\{B^j_{N-2k,0}\} \) for each \(-1 \leq j \leq 2N - 4k + 1\) and \(0 \leq k \leq \min\{\lfloor \frac{2N-j+1}{4} \rfloor, \lfloor \frac{N}{2} \rfloor\} \) so that
\[
v = \sum_{N=0}^{\infty} \sum_{j=-1}^{2N+1} \sum_{k=0}^{\min\{\lfloor \frac{2N-j+1}{4} \rfloor, \lfloor \frac{N}{2} \rfloor\}} v^j_{N-2k,0} \Delta^k. \tag{4.13}
\]
Furthermore, \( v = 0 \) if and only if \( v^j_{N-2k,0} = 0 \) for all \( j, k, N \).

Proof. The claim in part \( 1 \) directly follows from equation (2.13). For claim \( 2 \), let \( B^{l', k'}_{r', l'} \) be the vector field in \( \mathcal{B} \) with \((l, i, k) \neq (l', i', k')\). Here we only consider the case \( p = 3 \). Using the polynomial expansion for \( \Delta^k, \Delta^{k'} \), Lemma 2.5 and definition (2.12), the monomials appearing in \( B^{l}_{i, k} \) and \( B^{l', k'}_{r', l'} \) follow
\[
P(x, y, z) = x^s n_1 + y^{2n_1 + r + 2(l - p_1)} z^{n_2 - s - r + p_1}, Q(x, y, z) = x'^{s'} n_2 + y^{2n_2 + r' + 2(l' - p_2)} z'^{n_2 - s' - r' + p_2},
\]
for some \( n_1, n_2, p_1, p_2 \), where \( l = 2s + r \) and \( l' = 2s' + r' \). Let \( P = Q \). Then,
\[
s - s' = n_1 - n_2 + p_2 - p_1, r - r' = n_2 - n_1 - k + p_1 - p_2, i - i' = n_1 - n_2 + s - s' + r - r' + p_2 - p_1. \tag{4.14}
\]
By substituting the first equation in (4.14) into the third one, we have
\[
i - i' = l - l'. \tag{4.15}\]
Since the vector fields \( B^{l}_{i, k} \) and \( B^{l', k'}_{r', l'} \) have the same \( \delta \)-grade,
\[
i - i' = 2(k' - k). \tag{4.16}\]
Therefore, \( i - i' = l - l' = 2(k' - k) \) and \( i + 2k = i' + 2k' \). Hence, the inequalities \( i' \geq 0 \) and \(-1 \leq l' \leq 2i' + 1 \) are equivalent to the inequalities (4.12) on \( m := k' \). The equality \( P^l_{i,k} = P^{m_1}_{m_2, m_3} \) is due to the definition.

Part \( 3 \) follows equations (4.16) and (4.15). The claim \( 4 \) is a direct corollary of the claim \( 2 \). The proof of part \( 3 \) is similar to claims \( 2 \) and \( 4 \) and the claim is consistent with equation (4.6).

Finally for the claim \( 5 \), consider \( v := \sum_{N=0}^{\infty} w^N \) for \( w^N \in \mathcal{B} \). Then by claim \( 5 \), for any \( N \) we have
\[
w^N := \sum_{j=-1}^{2N+1} v^j_{N-2k,0} w^j_N e_p \in \text{span} \text{Terms}(B^j_{N,0} \cdot e_p).
\]
Now we Taylor-expand \( w^j_N \) in terms of \( \Delta \), that is,
\[
w^j_N = \sum_{k=0}^{\min\{\lfloor \frac{2N-j+1}{4} \rfloor, \lfloor \frac{N}{2} \rfloor\}} v^j_{N-2k,0} \Delta^k, \quad \text{where} \quad v^j_{N-2k,0} \cdot e_p \in \text{span} \text{Terms}(B^j_{N-2k,0} \cdot e_p). \tag{4.17}
\]
Hence, \( v^j_{N-2k,0} \) does not share any monomial vector field with any \( B \)-terms except \( B^j_{N-2k,0} \). This is because of the claim in part \( 2 \) and that we have already excluded the powers of \( \Delta \) in (4.17). Therefore, the expansion of \( v^j_{N-2k,0} \) in terms of the \( B \)-term generators of \( \mathcal{B} \) only includes \( B^j_{N-2k,0} \).

Remark 4.5. Given Lemma 4.4, the same statements trivially hold for other \( \mathfrak{so}_2 \)-generated vector fields from spaces \( \mathcal{A} \) and \( \mathcal{C} \) defined in Appendix A. In particular, for any \( v \in \mathcal{C} \) and \( w \in \mathcal{A} \), there exist uniquely determined constants \( c^j_{N,k} \) and \( a^j_{N,k} \) so that
\[
v = \sum_{N=0}^{\infty} \sum_{j=-1}^{2N+1} \sum_{k=0}^{\min\{\lfloor \frac{2N-j+1}{4} \rfloor, \lfloor \frac{N}{2} \rfloor\}} c^j_{N,k} C^j_{N-2k,0} \Delta^k, \tag{4.18}
\]
\[
w = \sum_{N=0}^{\infty} \sum_{j=-2}^{2N+2} \sum_{k=0}^{\min\{\lfloor \frac{2N-j+2}{4} \rfloor, \lfloor \frac{N}{2} \rfloor\}} a^j_{N,k} A^j_{N-2k,0} \Delta^k. \tag{4.19}
\]
The following theorem provides a concrete characterization for vector fields in $\mathcal{B}$.

**Theorem 4.6.** Let $v$ be a three dimensional vector field. The conditions $\text{div}(v) = 0$ and $v(\Delta) = 0$ hold if and only if $v \in \mathcal{B}$.

**Proof.** Given Theorem 4.2 and Theorem 4.3 (part 4.3), we only need to prove the only if part. Let $v = u + w_C + w_A$, where $u \in \mathcal{B}$, $w_A \in \mathcal{A}$ and $w_C \in \mathcal{C}$. Hence, $\nabla \cdot (w_C + w_A) = 0$ and $w_C(\Delta) + w_A(\Delta)$. By equations (4.18) and (4.19),

$$w_C + w_A = \sum_{N=0}^{\infty} \left( \sum_{j=0}^{2N} \sum_{k=0}^{\min\{2N+1-j,|N|\}} C_{j}^{N-k,0} \Delta^k + \sum_{j=-2}^{2N+2} \sum_{k=0}^{\min\{2N+2-j,|N+2|\}} A_{j}^{N-2k,0} \Delta^k \right),$$

where $w_{C,N-2k,0} \in \mathbb{R}\{C_{j}^{N-k,0}\}$ and $w_{A,N-2k,0} \in \mathbb{R}\{A_{j}^{N-k,0}\}$. Since

$$\text{Terms}(\mathbb{R}\{C_{j}^{N-k,0}\} + \mathbb{R}\{A_{j}^{N-k,0}\}) \cdot e_p \cap \text{Terms}(\mathbb{R}\{C_{j}^{N-k,0}\} + \mathbb{R}\{A_{j}^{N-k,0}\}) \cdot e_p = \emptyset$$

for all $p = 1, 2, 3$ when $(j_1, k_1) \neq (j_2, k_2)$,

$$\text{div } w_{C,N-2k,0} + \text{div } w_{A,N-2k,0} = 0 \quad \text{and} \quad w_{C,N-2k,0}(\Delta) + w_{A,N-2k,0}(\Delta) = 0.$$

By Theorem A.1, $\text{div } w_{A,N-2k,0} = 0$ and so $\text{div } w_{C,N-2k,0} = 0$ for all $j, k, N$. By Lemma 2.4, equation (A.5), and item 6 in Lemma 4.4, the condition $\text{div } w_{C,N-2k,0} = 0$ implies that $w_{C,N-2k,0} = 0$. Thereby, $w_{A,N-2k,0}(\Delta) = 0$. Now Theorem A.1 concludes the proof through the equation $w_{A,N-2k,0} = 0$ for all $j, k, N$. Indeed, $w_A = w_C = 0$ and $v = u \in \mathcal{B}$.  \hfill \qed

**Theorem 4.7.** The following hold.

1. The set $\{\text{Terms}(b_{i}^{N,0}) : 1 \leq l \leq 2N + 1\}$ forms a partition for $\mathcal{B}_N$.

2. For $p = 1, 2, 3$, and $(x_1, x_2, x_3) := (x, y, z)$, $\{x_p, \text{Terms}(b_{i}^{l,0})\} \subseteq \text{Terms}(b_{i}^{l+2-p})$. When $(p, l) \neq (1, i + 1)$, $(p, l) \neq (2, i)$ and $(p, l) \neq (3, -1)$, $\text{Terms}(x_p, \text{Terms}(b_{i}^{l,0})) = \text{Terms}(b_{i}^{l+2-p})$.

3. $\ker \text{ad}_x = \mathbb{R}\{[x]\}$, $\ker \text{ad}_y = \mathbb{R}\{[y, x]\}$, and $\ker \text{ad}_z = \mathbb{R}\{[z]\}$.

**Proof.** The first item follows items 3 and 4 in Theorem 4.4 and Lemma 3.2. The second and third claim follows from the structure constants and Lemma 3.2.  \hfill \qed

An alternative representation for vector fields in $\mathcal{B}$ are based on the vector potential. Each solenoidal vector field $v$ has always a vector potential that is unique modulo gradient vector fields. Vector potential frequently appears in the classical and quantum mechanics, e.g., see [25]. Vector potential is called magnetic vector potential in electrodynamics while the curl of the magnetic vector potential is called magnetic field; see [4].

**Theorem 4.8.** There exists a vector potential $\phi_{i,k}^l$ such that $B_{i,k}^l = \text{curl}(\phi_{i,k}^l)$, where

$$\phi_{i,k}^l := b_{i,k}^l \nabla \Delta = \frac{\Delta^{N+1,N+1} \cdot \Delta^2}{N+1,2N+2} (z, -2y, x).$$

**Proof.** From equation (4.6) and the equality $\nabla f \times \nabla g = \nabla \times f \nabla g$ for all scalar functions $f$ and $g$, we have

$$B_{i,k}^l = \nabla \times (b_{i,k}^l \nabla \Delta).$$

Thereby, $b_{i,k}^l \nabla \Delta$ is a vector potential for $B_{i,k}^l$.  \hfill \qed
**Remark 4.9.** An alternative vector potential for solenoidal vector fields is available through the computational approach on [26, page 21]. Indeed, there exists a vector potential $\Phi_{i,k}^l$ such that $B_{i,k}^l = \text{curl}(\Phi_{i,k}^l)$, where

$$
\Phi_{i,k}^l = \left(\frac{(l-i)z^{N+1}l^{i+l}-(l+i)y^{N+l^i+1}}{(l+i)(2k+i+3)\Delta^k}, \frac{(2i+1-l)z^{N+2}l^{i+1}+(l+1)x^{N+l^i+1}}{(l+i)(2k+i+3)\Delta^k}, \frac{(l-2i-1)y^{N+2}l^{i+1}+(i-l)x^{N+l^i+1}}{(l+i)(2k+i+3)\Delta^k}\right). \quad (4.21)
$$

In particular, $\Phi_{0,0}^1 = (-\Delta, 0, 0)$. The proof here follows [26, page 21]. Indeed, define

$$P(X) := \int_0^1 t B_{i,k}^l(tX) \, dt = \frac{1}{(2k+i+3)} B_{i,k}^l, \quad \text{where} \quad X := (x, y, z).$$

Then, a vector potential for $B_{1,0}^1$ is given by $\Phi_{1,0}^1$, i.e.,

$$\phi_{1,0}^1 = \left(\frac{y^2}{4}, -\frac{xyz}{2}, \frac{xy^2}{4}\right) \quad \text{and} \quad \Phi_{1,0}^1 = \left(\frac{xyz^2}{3}, -\frac{y^2}{2}, \frac{x^2}{2}\right), \quad \Phi_{1,0}^1 = \left(\frac{y^2}{4}, -\frac{xyz}{2}, \frac{xy^2}{4}\right).$$

Nonlinear vector fields from $B$ are rotational vector fields; i.e., they have a nonzero curl.

**Theorem 4.10.** All vector fields from $B$ have a non-zero curl. In particular, the null space of curl operator on the formal sum of vector field types (2.12) is given by $\mathbb{R}B_{0,0}^0$.

**Proof.** Note that $\text{curl}(B_{0,0}^0) = 0$. Let $v \in B$ and $\text{curl}(v) = 0$. By equation (4.13), Lemma 4.4, linearity and continuity in filtration topology of curl operator,

$$v = \sum_{N=0}^{\infty} \sum_{j=-1}^{2N+1} \sum_{k=0}^{\min\{\frac{2N-j+1}{2}, \frac{N}{2}\}} v_{N-2k,0}^j \Delta^k, \quad \text{where} \quad v_{N-2k,0}^j \in \mathbb{R}\{B_{N-2k,0}^j\},$$

and $\text{curl}(v_{N-2k,0}^j \Delta^k) = 0$ for all $j, l, N$. Now let $\text{curl}(B_{i,k}^l) = 0$. Hence, all three components of $\text{curl}(B_{i,k}^l)$ are zero. The first component of $\text{curl}(B_{i,k}^l)$ is given by $-\frac{\partial}{\partial y} (l+1)\Delta^k \frac{N-l^i+1}{(i+1)N-l^i+1} - \frac{\partial}{\partial z} (l+1)\Delta^k \frac{N-l^i+1}{(i+1)N-l^i+1}$ and by Lemma 2.4 we have

$$\nabla \times B_{i,k}^l \cdot e_1 = -\frac{2l(l+1)\Delta^k \frac{N-l^i+1}{(i+1)N-l^i+1}}{\kappa_{i+1,2l+2}^2} + 2k(l+1)\Delta^k \frac{N-l^i+1}{(i+1)N-l^i+1} - \frac{(l-i)\Delta^k \frac{N-l^i+1}{(i+1)N-l^i+1}}{\kappa_{i+1,2l+2}^2} - \frac{k(i-l)\Delta^k \frac{N-l^i+1}{(i+1)N-l^i+1}}{\kappa_{i+1,2l+2}^2} = 0. \quad (4.22)$$

Since Terms($\Delta^k \frac{N-l^i+1}{(i+1)N-l^i+1}$), Terms($\Delta^k \frac{N-l^i+1}{(i+1)N-l^i+1}$), Terms($\Delta^k \frac{N-l^i+1}{(i+1)N-l^i+1}$), and Terms($\Delta^k \frac{N-l^i+1}{(i+1)N-l^i+1}$) are pairwise disjoint sets of monomial terms, equation (4.22) holds if and only if $l(l+1) = k(l+1) = (i-l) = k(i-l) = 0$. The later is equivalent with $i = k = l = 0$. This completes the proof. \qed

**Example 4.11.** Let

$$v := A_{0,1}^0 - 3C_{2,0}^2 = -3xy^2 \frac{\partial}{\partial x} - 3xyz \frac{\partial}{\partial y} - 3y^2z \frac{\partial}{\partial z}.$$ 

Then, $v(\Delta) = 0$ while $\text{div}(v) = -3xz - 6y^2 \neq 0$. 
Consider the vector field
\[ A_{i,0}^{-2} := z^{i+1} \frac{\partial}{\partial x}, \quad \text{for any } i \in \mathbb{N}_0. \]

This family has two first integrals of \( y \) and \( z \) while \( A_{i,0}^{-2} \) is also solenoidal for all \( i \). These vector fields do not generate a Lie algebra with the nilpotent linear part \( B_{0,0}^1 \), indeed,
\[ [A_{i,0}^{-2}, A_{i,0}^{-2}] = 0, \quad [B_{0,0}^1, A_{i,0}^{-2}] = -2(i + 1)A_{i,0}^{-1}. \]

This indicates that the family of vector fields in \( \mathcal{B} \) does not represent the set of all solenoidal vector fields with two independent first integrals.

## 5 Normal form classification

**Theorem 5.1.** The vector field (1.1)-(1.2) is either linearizable in the first level normalization step or there exists a natural number \( p \) so that the first level normal form of the vector field (1.1)-(1.2) is given by

\[ w := -x \frac{\partial}{\partial y} - 2y \frac{\partial}{\partial z} + z^p(z \frac{\partial}{\partial y} + 2y \frac{\partial}{\partial x}) + \sum_{k=p+1}^{\infty} \sum_{i=0}^{\lfloor \frac{k+p}{2} \rfloor} b_{i,k} z^i(xz - y^2)^{k-2i+p}(z \frac{\partial}{\partial y} + 2y \frac{\partial}{\partial x}), \quad (5.1) \]

where \( b_{i,k} \in \mathbb{R} \). Furthermore, the normal form vector field (5.1) is always an affine in the infinite level normal form. In addition, the infinite level secondary Clebsch potential normal form is given by

\[ I(x, y, z) = x + \frac{1}{p+1} z^{p+1} + \sum_{k=p+1}^{\infty} \sum_{i=0}^{\lfloor \frac{k+p}{2} \rfloor} \frac{b_{i,k}}{i+1} z^{i+1}(xz - y^2)^{k-2i+p}. \quad (5.2) \]

Here, the quadratic polynomial \( \Delta = xz - y^2 \) stands for the primary Clebsch potential.

**Proof.** The normal form
\[ w := B_{0,0}^1 + \sum_{i,k \in \mathbb{N}_0} b_{i,k} B_{i,k}^{-1}, \quad (5.3) \]

is readily available given the \( \mathfrak{sl}_2 \)-style normal form and the fact that \( \ker(\text{ad}_M) = \text{span}\{B_{-1}^{-1}\} \); for more information see [2,3]. Let \( r := \min\{i \mid b_{i,0} \neq 0\} \) and \( r > 0 \). Define a new grading function by \( \delta(B_{i,k}^l) := lr + 2i + k \). Then, \( \mathcal{B} \) is a \( \delta \)-graded Lie algebra and
\[ \mathcal{B}_p := B_{0,0}^1 + b_{p,0} B_{p,0}^{-1} \in \mathcal{B}_p. \]

Linear invertible transformations can be used to rescale the coefficient \( b_{p,0} \) into \( b_{p,0} := 1 \). Following [2,3] we define
\[ \Gamma := \text{ad}(B_{0,0}^{-1}) \circ \text{ad}(\mathcal{B}_p). \quad (5.4) \]

By the structure constants and equation (2.19),
\[ \Gamma(B_{i,k}^q) = (q - 2i - 1)(q + 2)B_{i,k}^q + b_{p,0} \sum_{j=\max\{q-2-i-p,-1\}}^{s-1+[\frac{q+i}{2}]} \frac{a_{j,p,i} B_{2j-q+1+i+p}^{-1}(s-1+k-j)_{i+1}}{(2j + |r - 1| + 1)^{-1}}. \]
Hence for any \( i \) and \( k \) \((q = 2i + 1)\), there is a possibility of a vector polynomial in kernel \( \Gamma \). This is due to a similar argument used by [2,3]. On the other hand \( B_{r,k}^{2i+1} = -x^{i+1}\Delta^k \), \( \Psi((b_{0,0}^{i+1}\Delta^k)) = B_{r,k}^{2i+1} \), and \( B_{r,k}^{r+1}\Delta^k := \Psi \left( (\psi^{-1}(B_r))^{i+1}\Delta^k \right) = \Psi((-x - z^{r+1})^{i+1}\Delta^k) \in \ker \Gamma \).

These polynomial vectors are extended to a symmetry for the normal form vector field (5.1), through
\[
\Psi \left( (\psi^{-1}(w))^{i+1}\Delta^k \right).
\]

This proves that there is no possibility of any hypernormalization beyond the \( \mathfrak{sl}_2 \)-style normalized vector field (5.1).

\[ \square \]

**Corollary 5.2.** The following presents five alternative representations for the normal form (5.1):

1. A formal sum of B-terms:
\[
w := B_{0,0}^1 + B_{p,0}^{-1} + \sum_{k=p+1}^{\infty} \sum_{i=0}^{\lfloor \frac{k-1}{2} \rfloor} b_{i,k} B_{i,k+p-2i}^{-1}.
\]

2. The secondary invariant:
\[
w := \Psi \left( -x - z^p - \sum_{k=p+1}^{\infty} \sum_{i=0}^{\lfloor \frac{k-1}{2} \rfloor} b_{i,k} z^{i+1}(xz - y^2)^{k+p-2i} \right).
\]

Here, \( \Psi \) is the Lie isomorphism given by equation (3.7).

3. Vector potential:
\[
w := \nabla (xz - y^2) \times \nabla \left( x + z^{p+1} + \sum_{k=p+1}^{\infty} \sum_{i=0}^{\lfloor \frac{k-1}{2} \rfloor} \frac{\Delta_{i+1}^k}{i+1} \right)\left( z, -2y, x \right).
\]

4. Functionally independent Clebsch potentials:
\[
w := \nabla (xz - y^2) \times \nabla \left( x + z^{p+1} + \sum_{k=p+1}^{\infty} \sum_{i=0}^{\lfloor \frac{k-1}{2} \rfloor} \frac{\Delta_{i+1}^k}{i+1} \right)\left( z, -2y, x \right).
\]

5. Poisson bracket:
\[
w = \sum_{p=1}^{3} \{x_p, I(x, y, z)\} \cdot e_p
\]

where \( I(x, y, z) \) is the invariant given in equation (5.2). Furthermore, \( \{I(x, y, z), \Delta\} = 0 \).

**Proof.** Follow equation (2.13), Lemma (3.2), item 2 in Theorem (4.3), and Theorem 3.4 respectively. \( \square \)

Now we consider further reduction of a normal form system given by
\[
w := -x \frac{\partial}{\partial y} - 2y \frac{\partial}{\partial z} + z^p \left( z \frac{\partial}{\partial y} + 2y \frac{\partial}{\partial z} \right) + \sum_{i=p+1}^{\infty} b_i z^i \left( z \frac{\partial}{\partial y} + 2y \frac{\partial}{\partial z} \right).
\]  

(5.5)

**Theorem 5.3.** There exists a near identity transformation so that the vector field (5.5) is transformed into the normal form vector field
\[
v := -2y \frac{\partial}{\partial z} + \left( -X + \sum_{i=p+1}^{\infty} b_i (i+2) \frac{Z^{i+1}}{i+1} \right) \frac{\partial}{\partial y}.
\]

(5.6)

Furthermore, \( X(t) := c \) is always constant. Hence, the normal form system (5.6) has a Hamiltonian
\[
H(Z, Y) := -\Delta(X, Y, Z) = Y^2 - cZ + \sum_{i=p+1}^{\infty} \frac{b_i}{i+1} Z^{i+2}.
\]

(5.7)

On the invariant manifold \( I(x, y, z) = 0 \), the normal form vector field takes a further hypernormalization given by
\[
v^{(\infty)} = -2y \frac{\partial}{\partial z} + \sum_{i=p}^{\infty} a_i z^{i+1} \frac{\partial}{\partial y}
\]

where \( a_i = 0 \) for \( i = (p+1)(m+1) \), \( m \in \mathbb{N}_0 \).
Proof. The key idea is to use the secondary Clebsch potential \[5.2\] as a near-identity transformation, i.e.,

\[(X, Y, Z) := I(x, y, z) = \left(x + \frac{1}{p+1}z^{p+1} + \sum_{i=p+1}^{\infty} b_i z^{i+1}, y, z\right).\]  

(5.8)

Hence, \(X(t)\) is constant. Then, the normal form vector field is given by

\[h := 2A_0^1 - cA_1^{-1} + \frac{(p+2)}{p+1} A_p^{-1} + \sum_{i=p+1}^{\infty} \frac{(i+2)b_i}{i+1} A_i^{-1},\]

in terms of notations used in \[2, 13\]. Hence, the second claim follows \[2, \text{Theorem 8.9}]. \qed

5.1 Truncated normal form coefficients

Consider a cubic-degree truncated triple zero vector field

\[v := -x\frac{\partial}{\partial y} - 2y\frac{\partial}{\partial z} + \sum_{i+j+k=2} x^i y^j z^k (a_{ijk}\frac{\partial}{\partial x} + b_{ijk}\frac{\partial}{\partial y} + c_{ijk}\frac{\partial}{\partial z}) \in \mathcal{B},\]

(5.9)

where \(a_{ijk}, b_{ijk}\) and \(c_{ijk} \in \mathbb{R}\). By Theorem 4.6, \(\text{div}(v) = 0\) and \(v(\Delta) = 0\). The first one implies

\[
\begin{align*}
  c_{002} &= -\frac{1}{2}(a_{101} + b_{011}), &  a_{200} &= -b_{110}, &  c_{020} &= 2b_{110}, &  c_{101} &= -(2a_{200} + b_{110}), &  a_{101} &= b_{011}, \\
  a_{020} &= 2b_{011}, &  c_{111} &= -a_{110}, &  a_{011} &= 2b_{002}, &  c_{110} &= 2b_{200}, &  c_{200} &= a_{002} = b_{011} = 0.
\end{align*}
\]

(5.10)

while \(v(\Delta) = 0\) concludes \(c_{300} = a_{003} = 0\) and

\[
\begin{align*}
  c_{111} &= -2(a_{210} + b_{120}), &  b_{012} &= -2c_{003}, &  b_{111} &= -2(a_{201} + c_{102}), &  a_{012} &= 2b_{003}, &  c_{210} &= 2b_{300}, &  a_{120} &= -c_{021}, \\
  a_{111} &= -2b_{021} - 2c_{102}, &  b_{210} &= -2a_{300}, &  a_{102} &= -b_{012} - 3c_{003}, &  c_{201} &= -3a_{300} - b_{210}, &  a_{030} &= 2b_{021}, \\
  c_{300} &= 2b_{120}, &  c_{012} &= -2(b_{021} + b_{102}), &  a_{01} &= -c_{102}, &  a_{210} &= -2(b_{201} + b_{120}), &  a_{021} &= -4a_{003}, &  c_{120} &= -4a_{300}.
\end{align*}
\]

Hence, the vector field \((5.9)\) can be written as

\[
v = B_{0,0}^1 + d_{1,0}^{-1}B_{1,0}^{-1} + d_{0,1}^0B_{0,1}^0 + d_{2,0}^1B_{2,0}^1 + d_{1,0}^0B_{1,0}^2 + d_{2,0}^0B_{2,0}^2 + d_{0,1}^0B_{0,1}^3 + d_{2,0}^1B_{2,0}^3 + d_{1,0}^0B_{1,0}^4 + d_{2,0}^1B_{2,0}^4 + d_{1,0}^0B_{1,0}^5 + d_{2,0}^0B_{2,0}^5,
\]

(5.11)

where 
\(d_{1,0}^{-1} = d_{0,1}^1 = \frac{1}{5}(4b_{021} - b_{201}), d_{1,0}^1 = (3b_{021} + 3b_{102})\), and 
\(d_{0,1}^0 = 2b_{021}, d_{2,0}^0 = -3c_{003}, d_{2,0}^0 = -(c_{021} + c_{102}), d_{1,0}^1 = a_{110}, d_{2,0}^1 = b_{003}, d_{2,0}^1 = -2b_{110}, d_{2,0}^0 = 3a_{300}, d_{1,0}^1 = \frac{1}{5}(b_{120} - 4b_{201}), d_{1,0}^1 = -b_{200}, d_{2,0}^1 = -(3b_{201} + 3b_{120}), d_{0,1}^0 = \frac{1}{5}(c_{021} - 4c_{102}).\)

Proposition 5.4. The quartic truncated normal form for equation \((5.11)\) is given by

\[w = -x\frac{\partial}{\partial y} - 2y\frac{\partial}{\partial z} + \left(2y((xz - y^2)(b_1^1 z + b_1^0) + z(b_0^3 z^2 + b_0^2 z + b_0^1))\right)\frac{\partial}{\partial x} + \left((xz - y^2)(b_1^1 z + b_1^0) + z^2(b_0^3 z^2 + b_0^2 z + b_0^1)\right)\frac{\partial}{\partial y},\]

whose first integrals are \(\Delta = xz - y^2\) and 
\[I(x, y, z) = x - b_1^0 y^2 + \frac{1}{2}(b_1^1 xz - b_1^1 y^2 + 4b_1^0 + 2b_0^3 z^2 + \frac{b_0^3 x^2}{2} + \frac{b_0^3 z^2}{3}).\]
where

\[
\begin{align*}
    b_0^1 &= b_{002}, \\
    b_0^2 &= b_{003} + b_{00210110} \frac{3b_{0111}^2}{4}, \\
    b_1^1 &= -\frac{a_{110}^3}{378} - \frac{b_{10}^2 b_{002}}{4} + \frac{(4c_{1021} + 2c_{0211}) b_{0111}^2}{15} + \frac{6 b_{101000} b_{002}}{21} + \frac{12 (b_{0100} + b_{1020}) b_{002}}{105} + \frac{8 b_{110000} + 12 (b_{0210} + b_{1020}) b_{01110}^2}{15} - \frac{4 b_{002000} b_{002}}{7} \\
    b_3^1 &= \frac{2b_{00210110}}{3} - \frac{6 b_{1010000} b_{0111}}{5} + \frac{4 (3 b_{0121} + 3 b_{0102}) b_{002}}{15} - \frac{6 b_{200000} b_{002}^2}{21} + \frac{b_{0222000} b_{01120}^2}{10} + 2 c_{0003} b_{0111}.
\end{align*}
\]

The vector potential normal form for vector field (5.11) is given by

\[
(x + b_0^1 \frac{\partial}{\partial z} + b_0^2 \Delta z + b_0^3 \frac{\partial^2}{\partial z^2} + b_0^4 \frac{\partial^3}{\partial z^3})(z, -2y, x).
\]

**Proof.** The normal form coefficients are derived using an implementation of the formulas in Maple. \(\square\)

### Appendix

Let

\[
\mathcal{A} := \text{span} \left\{ \sum d_{i,k}^l A_{i,k}^l \right\},
\]

and

\[
\mathcal{C} := \text{span} \left\{ \sum c_{i,k}^l C_{i,k}^l \right\},
\]

where

\[
A_{i,k}^l := \frac{N^{l+2} z^i \Delta^k \partial}{\kappa_{l+2,i+2}}, \quad \text{for} \quad -2 \leq l \leq 2i+2, i, k \in \mathbb{N}_0 := \mathbb{N} \cup \{0\},
\]

\[
C_{i,k}^l := \frac{N^{l+2} z^i \Delta^k E}{\kappa_{l,i}}, \quad \text{for} \quad 0 \leq l \leq 2i, i, k \in \mathbb{N}_0 := \mathbb{N} \cup \{0\},
\]

and \(E = x \frac{\partial}{\partial x} + y \frac{\partial}{\partial y} + z \frac{\partial}{\partial z}\). By equation (2.4), we have

\[
A_{i,k}^l := \frac{\Delta^k N^{l+2} z^{i+1}}{\kappa_{l+2,i+2}} \frac{\partial}{\partial x} - \frac{(l+2) \Delta^k N^{l+1} z^{i+1}}{(2l-i+1) \kappa_{l+1,i+2}} \frac{\partial}{\partial y} + \frac{(l+1)(l+2) \Delta^k N^{l+1} z^{i+1}}{(2l-i+1)(2l-i+2) \kappa_{l,i+2}} \frac{\partial}{\partial z},
\]

\[
C_{i,k}^l := \frac{z^i \Delta^k \partial}{\kappa_{l,i}} + \frac{y^i \Delta^k \partial}{\kappa_{l+1,i+2}} + \frac{z^i \Delta^k \partial}{\kappa_{l+1,i+2}}.
\]

**Theorem A.1.** For all \(-2 \leq l \leq 2i+2, i, k \in \mathbb{N}_0,\)

\[
\text{div}(A_{i,0}^l) = 0, \quad \nabla \Delta \cdot A_{i,k}^l \neq 0.
\]

**Proof.** By equation (4.2) and definition \(A_{i,k}^l\), we have

\[
\nabla \cdot A_{i,0}^l = (l+1)(l+2)(i+1) \left( \frac{1}{\kappa_{l+2,i+2}} - \frac{2}{\kappa_{l+1,i+2}} - \frac{1}{\kappa_{l,i+2}} \right) N^{l+2} z^i.
\]

Since the coefficient \(N^{l+2} z^i\) is zero, \(\text{div}(A_{i,k}^l) = 0\) for any \(i \in \mathbb{N}_0,\) and \(-2 \leq l \leq 2i+2.\)

When \(l\) is even, say \(l = 2s,\) by Lemma 2.3 we have

\[
\nabla \Delta \cdot A_{i,0}^l = \left( \frac{(2s+2)(2s+1)(2s+2)(2s)(2s-1)!}{(2s+2)!} \right) \sum_{n=0}^{s+1} \frac{4^n a^{n+1} y^{2n} z^{i-n+1}}{(s-n+1)(2n)(i-n)!} + \left( \frac{(2s+2)(2s)(2s-1)!}{(2s+2)!} \right) \sum_{n=0}^{s} \frac{4^n a^{n+1} y^{2n} z^{i-n+1}}{(s-n+1)(2n)(i-n-s+1)!} + \left( \frac{(2s+2)(2s)}{(2s+2)!} \right) \sum_{n=1}^{s+1} \frac{(2s+1)(2s+2)(2s+1)!}{(s-n+1)(2n+1)(i-n-s+1)!}.
\]
Since the coefficient of $x^{s+1}z^{i-s+1}$ is
\[
\frac{(i + 2)! (2i - 2s)! (2s + 2)!}{(2i + 2)! (s + 1)! (i - s + 1)!} \neq 0,
\]
\[\Delta \text{ is not a first integral for } A_{i,k}^l. \text{ The argument is similar for when } l \text{ is odd.} \]

**Theorem A.2.** Each three dimensional vector field $v$ can be uniquely expanded in terms of formal sums of polynomial generators $A_{i,k}^l$, $B_{i,k}^l$ and $C_{i,k}^l$ from $A$, $B$ and $C$, respectively.

**Proof.** This is a straightforward corollary of Theorem 2.9, Lemma 4.4 and Remark 4.5.

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