Energy and Randić energy of special graphs

Jahfar T K * and Chithra A V †

Department of Mathematics, National Institute of Technology, Calicut, Kerala, India-673601

Abstract

In this paper, we determine the Randić energy of the $m$-splitting graph, the $m$-shadow graph and the $m$-duplicate graph of a given graph, $m$ being an arbitrary integer. Our results allow the construction of an infinite sequence of graphs having the same Randić energy. Further, we determine some graph invariants like the degree Kirchhoff index, the Kemeny’s constant and the number of spanning trees of some special graphs. From our results, we indicate how to obtain infinitely many pairs of equienergetic graphs, Randić equienergetic graphs and also, infinite families of integral graphs.

AMS classification: 05C50

Keywords: $m$-splitting graph, $m$-shadow graph, $m$-duplicate graph, energy, Randić energy, equienergetic graphs, integral graphs.

1 Introduction

In this paper, we consider simple connected graphs. Let $G = (V, E)$ be a simple graph of order $p$ and size $q$ with vertex set $V(G) = \{v_1, v_2, ..., v_p\}$ and edge set $E(G) = \{e_1, e_2, ..., e_q\}$. The degree of a vertex $v_i$ in $G$ is the number of edges incident to it and is denoted by $d_i = d_G(v_i)$. The adjacency matrix $A(G) = [a_{ij}]$ of the graph $G$ is a square symmetric matrix of order $p$ whose $(i, j)^{th}$ entry is defined by

$$a_{i,j} = \begin{cases} 1, & \text{if } v_i \text{ and } v_j \text{ are adjacent,} \\ 0, & \text{otherwise.} \end{cases}$$

The eigenvalues $\lambda_1, \lambda_2, ..., \lambda_p$ of the graph $G$ are defined as the eigenvalues of its adjacency matrix $A(G)$. If $\lambda_1, \lambda_2, ..., \lambda_t$ are the distinct eigenvalues of $G$, the spectrum of $G$ can be written as

$$Spec(G) = \begin{pmatrix} \lambda_1 & \lambda_2 & ... & \lambda_t \\ m_1 & m_2 & ... & m_t \end{pmatrix},$$

where $m_j$ indicates the algebraic multiplicity of the eigenvalue $\lambda_j$, $1 \leq j \leq t$ of $G$. The energy [11] of the graph $G$ is defined as

\*jahfartk@gmail.com
\†chithra@nitc.ac.in
\[ \varepsilon(G) = \sum_{i=1}^{p} |\lambda_i|. \] More results on graph energy are reported in \cite{3,11}. The Randić matrix \( R(G) = [R_{i,j}] \) of a graph \( G \) is a square matrix of order \( p \) whose \((i,j)\)th entry is
\[
R_{i,j} = \begin{cases} 
1/\sqrt{d_id_j}, & \text{if } v_i \text{ and } v_j \text{ are adjacent}, \\
0, & \text{otherwise}.
\end{cases}
\]

The eigenvalues of \( R(G) \) are called Randić eigenvalues of \( G \) and it is denoted by \( \rho_i, 1 \leq i \leq p \). If \( \rho_1, \rho_2, \ldots, \rho_s \) are the distinct Randić eigenvalues of \( G \), then the Randić spectrum of \( G \) can be written as \( RS(G) = \left( \begin{array}{c} \rho_1 \\
\rho_2 \\
\vdots \\
\rho_s
\end{array} \right) \). In analogy to this, two graphs \( G_1 \) and \( G_2 \) of same order are said to be Randić equienergetic if \( \varepsilon(G_1) = \varepsilon(G_2) \) \cite{13}. The normalized Laplacian matrix \( L(G) = (L_{ij}) \) is the square matrix of order \( p \) whose \((i,j)\)th entry is defined as
\[
L_{ij} = \begin{cases} 
1, & \text{if } v_i = v_j \text{ and } d_i \neq 0, \\
-1/\sqrt{d_id_j}, & \text{if } v_i \text{ and } v_j \text{ are adjacent in } G, \\
0, & \text{otherwise}.
\end{cases}
\]

For a graph \( G \) without isolated vertices, the normalized Laplacian matrix can be written as
\[
L(G) = I_n - D^{-\frac{1}{2}}A(G)D^{-\frac{1}{2}}.
\]
The eigenvalues of the matrix \( L(G) \) are called the normalized Laplacian eigenvalues of \( G \) and it is denoted by \( 0 = \hat{\mu}_1(G) \leq \hat{\mu}_2(G) \leq \cdots \leq \hat{\mu}_p(G) \). Let \( G \) be a graph without isolated vertices. Then its normalized Laplacian matrix \( L(G) \) and Randić matrix \( R(G) \) are related by \( L(G) = I - R(G) \). The normalized Laplacian eigenvalues \( \hat{\mu}_i(G) \) and Randić eigenvalues \( \rho_i(G) \) are related by \( \hat{\mu}_i(G) = 1 - \rho_i(G) \), for \( i = 1, 2, \ldots, p \). The degree Kirchhoff index of connected graph \( G \) is defined in \cite{3} as
\[
Kf^*(G) = \sum_{i<j} d_id_j r_{i,j}^*
\]
where \( r_{i,j}^* \) denotes the resistance distance \cite{15} between vertices \( v_i \) and \( v_j \) in a graph \( G \). In \cite{6}, the authors proved that
\[
Kf^*(G) = 2q \sum_{i=2}^{p} \frac{1}{\hat{\mu}_i(G)}.
\]
In general, the computation of the degree Kirchhoff index of a graph is a difficult thing. Here we obtained the formula for finding the degree Kirchhoff index of some families of
graphs. The Kemeny’s constant $K(G)$ of a connected graph $G$ [5] is defined in terms of normalized Laplacian as

$$K(G) = \sum_{i=2}^{p} \frac{1}{\hat{\mu}_i(G)}.$$ 

The various applications of the Kemeny’s constant to perturbed Markov chains, random walks on directed graphs are studied in [13]. The number of spanning trees (distinct spanning subgraphs of $G$ that are trees) of $G$ [8] can be expressed in terms of the normalized Laplacian eigenvalues as

$$t(G) = \prod_{i=1}^{p} d_i \prod_{i=2}^{p} \hat{\mu}_i(G) \sum_{i=1}^{p} d_i.$$ 

We use the notations $K_n, C_n$ and $K_{1,n-1}$ throughout this paper to denote the complete graph, the cycle and the star graph on $n$ vertices respectively. Let $J_m$ be the $m \times m$ matrix of all ones and $I_m$ be the identity matrix of order $m$.

The rest of the paper is organized as follows. In Section 2, we give a list of some previously known results which are useful for further reference in this paper. In Section 3, Randić energy of the $m$-splitting graph, the $m$-shadow graph and the $m$-duplicate graphs are obtained. In Section 4, our results allow the construction of an infinitely many integral and Randić integral graphs. Also, our results show how to construct equienergetic and Randić equienergetic graphs. In Section 5, we discuss the graph invariants like the degree Kirchhoff index, the Kemeny’s constant and the number of spanning trees of resulting graphs from various graph operations.

2 Preliminaries

In this section, we recall the concepts of the $m$-splitting graph, the $m$-shadow graph and the $m$-duplicate graph of a graph and list some results that will be used in the subsequent sections.

**Definition 2.1.** [9] The Kronecker product of two graphs $G_1$ and $G_2$ is the graph $G_1 \times G_2$ with vertex set $V(G_1) \times V(G_2)$ and the vertices $(x_1, x_2)$ and $(y_1, y_2)$ are adjacent if and only if $(x_1, y_1)$ and $(x_2, y_2)$ are edges in $G_1$ and $G_2$ respectively.

**Definition 2.2.** [9] Let $A, B \in \mathbb{R}^{n \times n}$, then the Kronecker product of $A$ and $B$ is defined as follows

$$A \otimes B = \begin{bmatrix} a_{11}B & a_{12}B & a_{13}B & \ldots & a_{1n}B \\ a_{21}B & a_{22}B & a_{23}B & \ldots & a_{2n}B \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ a_{m1}B & a_{m2}B & a_{m3}B & \ldots & a_{mn}B \end{bmatrix}.$$ 

**Proposition 2.1.** [9] Let $A, B \in \mathbb{R}^{m \times n}$. Let $\lambda$ be an eigenvalue of matrix $A$ with corresponding eigenvector $x$ and $\mu$ be an eigenvalue of matrix $B$ with corresponding eigenvector $y$, then $\lambda \mu$ is an eigenvalue of $A \otimes B$ with corresponding eigenvector $x \otimes y$. 

3
Definition 2.3. Let $G$ be a simple $(p,q)$ graph. Then the $m$-splitting graph of a graph $G$, $\text{Spl}_m(G)$ is obtained by adding to each vertex $v$ of $G$ new $m$ vertices say, $v_1,v_2,...,v_m$ such that $v_i, 1 \leq i \leq m$ is adjacent to each vertex that is adjacent to $v$ in $G$. The adjacency matrix of the $m$-Splitting graph of the graph $G$ is

$$A(\text{Spl}_m(G)) = \begin{bmatrix} A(G) & A(G) & A(G) & \ldots & A(G) \\ A(G) & O & O & \ldots & O \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ A(G) & O & O & \ldots & O \end{bmatrix}_{(m+1)p}.$$ 

If $m = 1$, the $m$-Splitting graph of the graph $G$ is known as splitting graph of $G$ [21], denoted by $\text{Spl}(G)$. The number of vertices and the number of edges in $\text{Spl}_m(G)$ are $(m + 1)p$ and $(m + 1)q$ respectively.

Proposition 2.2. The energy of the $m$-splitting graph of $G$ is $\varepsilon(\text{Spl}_m(G)) = \sqrt{1 + 4m\varepsilon(G)}$.

Definition 2.4. Let $G$ be a simple $(p,q)$ graph. Then the $m$-shadow graph $D_m(G)$ of a connected graph $G$ is constructed by taking $m$ copies of $G$ say, $G_1,G_2,...,G_m$ then join each vertex $u$ in $G_1$ to the neighbors of the corresponding vertex $v$ in $G_j, 1 \leq i \leq m, 1 \leq j \leq m$. The adjacency matrix of the $m$-shadow graph of $G$ is

$$A(D_m(G)) = \begin{bmatrix} A(G) & A(G) & A(G) & \ldots & A(G) \\ A(G) & A(G) & A(G) & \ldots & A(G) \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ A(G) & A(G) & A(G) & \ldots & A(G) \end{bmatrix}_{mp}.$$ 

If $m = 2$, the $m$-shadow graph of $G$ is known as shadow graph of $G$ [16]. The number of vertices and the number of edges in $D_m(G)$ are $pm$ and $m^2q$ respectively.

Proposition 2.3. The energy of the $m$-shadow graph of $G$ is $\varepsilon(D_m(G)) = m\varepsilon(G)$.

Definition 2.5. Let $G = (V,E)$ be a simple $(p,q)$ graph with vertex set $V$ and edge set $E$. Let $V'$ be a set such that $V \cap V' = \emptyset$, $|V| = |V'|$ and $f : V \rightarrow V'$ be bijective (for a $\in V$ we write $f(\alpha)$ as $\alpha'$ for convenience). A duplicate graph of $G$ is $D(G) = (V_1,E_1)$, where the vertex set $V_1 = V \cup V'$ and the edge set $E_1$ of $D(G)$ is defined as, the edge $ab$ is in $E$ if and only if both $ab'$ and $a'b$ are in $E_1$. In general the $m$-duplicate graph of the graph $G$, $D^m(G)$ is defined as $D^m(G) = D^{m-1}(D(G))$. The number of vertices and the number of edges in the $m$-duplicate graph of the graph are $2^mp$ and $2^mq$ respectively. With suitable labeling of the vertices, the adjacency matrix of $D(G)$ is

$$A(D(G)) = \begin{bmatrix} O_{p \times p} & A(G) \\ A(G) & O_{p \times p} \end{bmatrix}.$$ 

Proposition 2.4. The energy of the duplicate graph of $G$ is $\varepsilon(D(G)) = 2\varepsilon(G)$.

In [17], authors remarked that, the $m$-duplicate graph of $G$, $D^m(G) = G \times K_2 \times K_2... \times K_2$ ($K_2$ repeats $m$-times) . The energy of the $m$- duplicate graph of $G$ is $\varepsilon(D^m(G)) = \varepsilon(G) \varepsilon(K_2) \ldots \varepsilon(K_2) = 2^m\varepsilon(G)$ [3].
3 Randić energy of the $m$-splitting, the $m$-shadow and the $m$-duplicate graphs

In this section, we present the Randić energy of the $m$-splitting graph, the $m$-shadow graph and the $m$-duplicate graphs of $G$. Also, we obtain some new families of Randić equienergetic graphs. In addition, our results show how to construct infinitely many families of integral graphs.

**Theorem 3.1.** Let $G$ be a simple $(p,q)$ graph without isolated vertices. Then the Randić energy of the $m$-splitting graph of $G$ is $\varepsilon_R(Spl_m(G)) = \frac{2m+1}{m+1}\varepsilon_R(G)$.

**Proof.** The Randić matrix of $m$- splitting graph of $G$ is $R(Spl_m(G))$

\[
\begin{pmatrix}
((m+1)D)^{-\frac{1}{2}} & O & \ldots & O \\
O & D^{-\frac{1}{2}} & \ldots & O \\
\vdots & \vdots & \ddots & \vdots \\
O & O & \ldots & D^{-\frac{1}{2}}
\end{pmatrix}
\begin{pmatrix}
A(G) & A(G) & A(G) & \ldots & A(G) \\
A(G) & O & O & \ldots & O \\
A(G) & O & \ldots & O \\
\vdots & \vdots & \ddots & \vdots \\
O & O & \ldots & D^{-\frac{1}{2}}
\end{pmatrix}
\begin{pmatrix}
((m+1)D)^{-\frac{1}{2}} & O & \ldots & O \\
O & D^{-\frac{1}{2}} & \ldots & O \\
\vdots & \vdots & \ddots & \vdots \\
O & O & \ldots & D^{-\frac{1}{2}}
\end{pmatrix}
\]

\[
= \frac{1}{\sqrt{m+1}} D^{-\frac{1}{2}} A(G) D^{-\frac{1}{2}}, \quad \text{where} \quad B = \begin{pmatrix}
\frac{1}{\sqrt{m+1}} & 1 & 1 & \ldots & 1 \\
1 & 0 & 0 & \ldots & 0 \\
\vdots & \vdots & \vdots & \ddots & \vdots \\
1 & 0 & 0 & \ldots & 0
\end{pmatrix}^{(m+1)}.
\]

The eigenvalues of $B$ are $\sqrt{m+1}, -\frac{m}{\sqrt{m+1}}$ and 0, and 0 has multiplicity $m - 1$. So spectrum of $B$ is

\[
\text{Spec}(B) = \left(0, \sqrt{m+1}, -\frac{m}{\sqrt{m+1}}\right).
\]

Thus the Randić spectrum of the $m$-splitting graph is,

\[
RS(Spl_m(G)) = \begin{pmatrix}
0 & \rho_1 & \rho_2 & \ldots & \rho_p \\
p(m-1) & 1 & 1 & \ldots & 1 \\
\vdots & \vdots & \vdots & \ddots & \vdots \\
1 & 0 & 0 & \ldots & 0
\end{pmatrix}^{(m+1)}.
\]

Hence the Randić energy of the $m$-splitting graph of $G$ is $\varepsilon_R(Spl_m(G)) = \frac{2m+1}{m+1}\varepsilon_R(G)$.

If $m = 1$ in Theorem 3.1, we get the Randić energy of splitting graph of $G$ is $\varepsilon_R(Spl(G)) = \frac{3}{2}\varepsilon_R(G)$.

**Corollary 3.2.** Let $G_1$ and $G_2$ be Randić equienergetic graphs. Then $Spl_m(G_1)$ and $Spl_m(G_2)$ are Randić equienergetic.
In [19], Rojo et al. have obtained the construction of bipartite graphs having the same Randić energy. We indicate how to obtain infinitely many pairs of graphs (other than bipartite graphs) having the same Randić energy. The following theorem gives some information how to construct a new family of graphs having the same Randić energy as that of $G$.

**Theorem 3.3.** Let $G$ be a simple $(p, q)$ graph without isolated vertices. Then Randić energy of the $m$-shadow graph of $G$, $m > 1$ is $\varepsilon_R(D_m(G)) = \varepsilon_R(G)$.

**Proof.** The Randić matrix of the $m$-shadow graph of $G$ is $R(D_m(G))$

$$
\begin{pmatrix}
(mD)^{-\frac{1}{2}} & O & O & \ldots & O \\
O & (mD)^{-\frac{1}{2}} & O & \ldots & O \\
\vdots & \vdots & \ddots & \vdots & \vdots \\
O & O & O & \ldots & (mD)^{-\frac{1}{2}}
\end{pmatrix}
\times
\begin{pmatrix}
A(G) & A(G) & A(G) & \ldots & A(G) \\
A(G) & A(G) & A(G) & \ldots & A(G) \\
\vdots & \vdots & \ddots & \vdots & \vdots \\
A(G) & A(G) & A(G) & \ldots & A(G)
\end{pmatrix}
\times
\begin{pmatrix}
(mD)^{-\frac{1}{2}} & 0 & O & \ldots & O \\
O & (mD)^{-\frac{1}{2}} & O & \ldots & O \\
\vdots & \vdots & \ddots & \vdots & \vdots \\
O & O & O & \ldots & (mD)^{-\frac{1}{2}}
\end{pmatrix}

\begin{pmatrix}
1 & 1 & 1 & \ldots & 1 \\
1 & 1 & 1 & \ldots & 1 \\
\vdots & \vdots & \ddots & \vdots & \vdots \\
1 & 1 & 1 & \ldots & 1
\end{pmatrix}
\otimes
\frac{1}{m}D^{-\frac{1}{2}}A(G)D^{-\frac{1}{2}}

= J_m \otimes \frac{1}{m}D^{-\frac{1}{2}}A(G)D^{-\frac{1}{2}}.

The eigenvalues of $J_m$ are the simple eigenvalue $m$ and 0, and 0 has multiplicity $m - 1$. Therefore, the Randić spectrum of the $m$-shadow graph is

$$
RS(D_m(G)) = \begin{pmatrix}
0 & \rho_1 & \rho_2 & \cdots & \rho_p \\
\rho_1(m-1) & 1 & 1 & \cdots & 1
\end{pmatrix}.
$$

Hence $\varepsilon_R(D_m(G)) = \varepsilon_R(G)$. \hfill \qed

The following Proposition helps us to construct infinite sequence of Randić integral graphs.

**Proposition 3.1.** Let $G$ be a simple $(p, q)$ graph and $m \geq 2$ an integer. Then $G$ is Randić integral if and only if the $m$-shadow graph of $G$ is Randić integral.

For example, $D_m(C_4)$ and $D_m(K_{1,4})$ are Randić integral for every $m$.

The following theorem gives a relation between the Randić energy of the $m$-duplicate graph of the graph and Randić energy of the original graph.

**Theorem 3.4.** Let $G$ be a simple $(p, q)$ graph and $D^m(G)$ be the $m$-duplicate of graph $G$. Then $\varepsilon_R(D^m(G)) = 2^m\varepsilon_R(G)$. 

---

6
Proof. The Randić matrix of $D^m(G)$ is

$$
R(D^m(G)) = \begin{bmatrix}
O & O & O & \ldots & O & R(G) \\
O & O & O & \ldots & R(G) & O \\
\vdots & \vdots & \vdots & \ldots & \vdots & \vdots \\
R(G) & O & O & \ldots & O & O \\
\end{bmatrix}^{2^m} = \begin{bmatrix}
0 & 0 & 0 & \ldots & 0 & 1 \\
0 & 0 & 0 & \ldots & 1 & 0 \\
\vdots & \vdots & \vdots & \ldots & \vdots & \vdots \\
1 & 0 & 0 & \ldots & 0 & 0 \\
\end{bmatrix} \otimes R(G).
$$

Then the Randić spectrum of $D^m(G)$ is

$$
RS(D^m(G)) = \begin{pmatrix}
-\rho_1 & -\rho_2 & \ldots & -\rho_p & \rho_1 & \rho_2 & \ldots & \rho_p \\
2^{m-1} & 2^{m-1} & \ldots & 2^{m-1} & 2^{m-1} & 2^{m-1} & \ldots & 2^{m-1} \\
\end{pmatrix}.
$$

Hence Randić energy of $D^m(G)$ is $\varepsilon_R(D^m(G)) = 2^m \varepsilon_R(G)$. □

Remark 3.1. The graphs $D^m(G)$ and $D_{2m}(G)$ are non-cospectral equienergetic but not Randić equienergetic.

Proposition 3.2. Let $G$ be a simple $(p, q)$ graph and $m \geq 1$. Then $G$ is Randić integral if and only if the $m$-duplicate graph of $G$ is Randić integral.

4 Energy and Randić energy of some non-regular graphs

In this section, we define some new operations on a graph $G$ and calculate the energy and Randić energy of the resultant graphs. Moreover, our results allow the construction of new pairs of equienergetic and Randić equienergetic graphs.

Operation 4.1. Let $G$ be a simple $(p, q)$ graph and $D_m(G), m > 3$ be the the $m$-shadow graph of $G$ and $G_1, G_2, \ldots, G_m$ are the $m$ copies of $G$ in $D_m(G)$. The graph $H^m_i(G)$ is defined by $H^m_i(G) = D_m(G) - E(G_i) - E(G_j), \text{ for a pair } i \neq j, 1 \leq i, j \leq m$.

The number of vertices and the number of edges in $H^m_i(G)$ are $pm$ and $(m^2 - 2)q$ respectively.

We can easily compute the energy of $H^m_i(G)$ in terms of energy of $G$.

Theorem 4.1. The energy of the graph $H^m_i(G)$ is $\varepsilon(H^m_i(G)) = \left[1 + \sqrt{m^2 + 2m - 7}\right] \varepsilon(G)$. 

7
Proof. With the suitable labeling of the vertices, the adjacency matrix of $H_1^m(G)$ is

$$A(H_1^m(G)) = \begin{bmatrix}
O & A(G) & A(G) & \ldots & A(G) & A(G) \\
A(G) & A(G) & A(G) & \ldots & A(G) & A(G) \\
\vdots & \vdots & \vdots & \ldots & \vdots & \vdots \\
A(G) & A(G) & A(G) & \ldots & A(G) & A(G) \\
A(G) & A(G) & A(G) & \ldots & A(G) & 0
\end{bmatrix}_{pm}$$

$$= \begin{bmatrix}
0 & 1 & 1 & \ldots & 1 & 1 \\
1 & 1 & 1 & \ldots & 1 & 1 \\
\vdots & \vdots & \vdots & \ldots & \vdots & \vdots \\
1 & 1 & 1 & \ldots & 1 & 0
\end{bmatrix}_m \otimes A(G) = V_1 \otimes A(G),$$

where $V_1 = \begin{bmatrix}
0 & 1 & 1 & \ldots & 1 & 1 \\
1 & 1 & 1 & \ldots & 1 & 1 \\
\vdots & \vdots & \vdots & \ldots & \vdots & \vdots \\
1 & 1 & 1 & \ldots & 1 & 0
\end{bmatrix}_m$

The simple eigenvalues of $V_1$ are $\frac{m-1+\sqrt{m^2+2m-7}}{2}, \frac{m-1-\sqrt{m^2+2m-7}}{2}, -1, 0$ and $1$ have multiplicity $m-3$. Thus spectrum of $H_1^m(G)$ is

$$Spec(H_1^m(G)) = \left(\frac{m-1+\sqrt{m^2+2m-7}}{2}\lambda_i, \frac{m-1-\sqrt{m^2+2m-7}}{2}\lambda_i, -\lambda_i, 0, 1 \right), 1 \leq i \leq p.$$ 

Hence $\varepsilon(H_1^m(G)) = \left[1 + \sqrt{m^2+2m-7}\right] \varepsilon(G).$ \hfill $\Box$

**Corollary 4.2.** Let $G_1$ and $G_2$ be equienergetic graphs. Then $H_1^m(G_1)$ and $H_1^m(G_2)$ are equienergetic for all $m > 3$.

**Theorem 4.3.** The Randić energy of the graph $H_1^m(G), m > 3$ is $\varepsilon_R(H_1^m(G)) = \varepsilon_R(G) + 2e_{R(G)}.$

Proof. The Randić matrix of $H_1^m(G)$ is $R(H_1^m(G))$

$$= \begin{bmatrix}
(m-1)D^{-\frac{1}{2}} & O & \ldots & O \\
O & (mD)^{-\frac{1}{2}} & \ldots & O \\
\vdots & \vdots & \ddots & \vdots \\
O & O & \ldots & (m-1)D^{-\frac{1}{2}}
\end{bmatrix}_{pm} \begin{bmatrix}
O & A(G) & A(G) & \ldots & A(G) \\
A(G) & A(G) & A(G) & \ldots & A(G) \\
\vdots & \vdots & \vdots & \ldots & \vdots \\
A(G) & A(G) & A(G) & \ldots & 0
\end{bmatrix}_{pm} \begin{bmatrix}
(m-1)D^{-\frac{1}{2}} & O & \ldots & O \\
O & (mD)^{-\frac{1}{2}} & \ldots & O \\
\vdots & \vdots & \ddots & \vdots \\
O & O & \ldots & (m-1)D^{-\frac{1}{2}}
\end{bmatrix}_{pm}$$

where $D$ and $A(G)$ denote the diagonal and adjacency matrices of $G$, respectively.
The following theorem gives a relation between the energy of the original graph.

**Theorem 4.5.** The energy of the graph \( H_2^m(G) \) is 

\[
\varepsilon(H_2^m(G)) = \left[ m - 2 + 2\sqrt{m - 1} \right] \varepsilon(G), \quad m \geq 2.
\]
Proof. With the suitable labeling of the vertices, the adjacency matrix of $H_2^m(G)$ is

\[
A(H_2^m(G)) = \left[\begin{array}{cccccc}
A(G) & A(G) & A(G) & \cdots & A(G) & A(G) \\
A(G) & A(G) & O & \cdots & O & O \\
A(G) & O & A(G) & \cdots & O & O \\
\vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\
A(G) & O & O & \cdots & A(G) & O \\
A(G) & O & O & \cdots & O & A(G)
\end{array}\right]_{pn}
\]

where $W_1 = \left[\begin{array}{cccc}
\sqrt{m-1} & 1 \\
1 & 1 \\
\vdots & \vdots \\
1 & 1 \\
1 & 0 \\
1 & 0 \\
1 & 0
\end{array}\right]_m$. Let $X = \left[\begin{array}{c}
\sqrt{m-1} \\
1 \\
1 \\
\vdots \\
1 \\
1 \\
1
\end{array}\right]_{m \times 1}$, then $W_1X = (1 + \sqrt{m-1})X$ and let $Y = \left[\begin{array}{c}
-\sqrt{m-1} \\
1 \\
1 \\
\vdots \\
1 \\
1
\end{array}\right]_{m \times 1}$, then $W_1Y = (1 - \sqrt{m-1})Y$. Let $E_j = \left[\begin{array}{c}
0 \\
\vdots \\
0 \\
-f_j
\end{array}\right]_{m \times 1}$, $1 \leq j \leq m-2$, where $f_j$ is the column vector having $j^{th}$ entry one, all other entries zeros. Then $W_1E_j = E_j$. So the simple eigenvalues of $W_1$ are $1 + \sqrt{m-1}, 1 - \sqrt{m-1}$, and $1$ has multiplicity $m-2$. Thus spectrum of $H_2^m(G)$ is

\[
\text{Spec}(H_2^m(G)) = \left\{(1 + \sqrt{m-1})\lambda_i, (1 - \sqrt{m-1})\lambda_i, \lambda_i \mid 1 \leq i \leq p, m \leq 2 \right\}
\]

Hence we get, $\varepsilon(H_2^m(G)) = \left[m - 2 + 2\sqrt{m-1}\right] \varepsilon(G)$. \(\square\)

**Corollary 4.6.** Let $G$ be an integral graph and $m - 1$ a perfect square. Then $H_2^m(G)$ is integral.

**Corollary 4.7.** Let $G_1$ and $G_2$ be equienergetic graphs. Then $H_2^m(G_1)$ and $H_2^m(G_2)$ are equienergetic for all $m > 1$.

**Remark 4.1.** If $m = 2$, the graph $H_2^m(G)$ coincide with the shadow graph $D_2(G)$.  

10
Theorem 4.8. The Randić energy of the graph \( H_2^m(G), m > 3 \) is \( \varepsilon_R(H_2^m(G)) = \varepsilon_R(G) + \frac{(m+1)(m-2)\varepsilon_R(G)}{2m} \).

Proof. The Randić matrix of \( H_2^m(G) \) is \( R(H_2^m(G)) \)

\[
\begin{pmatrix}
\frac{1}{m} & \frac{1}{\sqrt{2m}} & \frac{1}{\sqrt{2m}} & \cdots & \frac{1}{\sqrt{2m}} \\
\frac{1}{\sqrt{2m}} & 0 & 0 & \cdots & 0 \\
\vdots & \vdots & \vdots & \ddots & \vdots \\
\frac{1}{\sqrt{2m}} & 0 & 0 & \cdots & \frac{1}{2}
\end{pmatrix}_m
\otimes D^{-\frac{1}{2}}A(G)D^{-\frac{1}{2}}
\]

\[
= W_2 \otimes D^{-\frac{1}{2}}A(G)D^{-\frac{1}{2}}, \text{ where } W_2 =
\begin{pmatrix}
\sqrt{\frac{m}{2}} & \frac{1}{\sqrt{2m}} & \frac{1}{\sqrt{2m}} & \cdots & \frac{1}{\sqrt{2m}} \\
1 & 0 & 0 & \cdots & 0 \\
\vdots & \vdots & \vdots & \ddots & \vdots \\
1 & 0 & 0 & \cdots & \frac{1}{2}
\end{pmatrix}_m
\]

Let \( X^* = \begin{pmatrix} \sqrt{\frac{m}{2}} \\ 1 \\ \vdots \\ 1 \end{pmatrix}_{m \times 1} \), then \( W_2X^* = 1.X^* \) and let \( Y^* = \begin{pmatrix} -(m-1) \frac{\sqrt{\frac{m}{2}}}{m} \\ 1 \\ \vdots \\ 1 \end{pmatrix}_{m \times 1} \), then

\[
W_2Y^* = \frac{-(m-2)}{2m} Y^*. \quad \text{Let } E_j \text{ be as in Theorem 4.5, then } W_2E_j = \frac{1}{2}E_j. \text{ So the simple eigenvalues of } W_2 \text{ are } 1, \frac{-(m-2)}{2m}, \text{ and } \frac{1}{2} \text{ has multiplicity } m-2. \text{ Thus Randić spectrum of } H_2^m(G) \text{ is}
\]

\[
RS(H_2^m(G)) = \begin{pmatrix} \rho_i & \frac{-(m-2)}{2m} \rho_i \\ \frac{1}{2} \rho_i & m-2 \end{pmatrix}, 1 \leq i \leq p.
\]

Hence \( \varepsilon_R(H_2^m(G)) = \varepsilon_R(G) + \frac{(m+1)(m-2)\varepsilon_R(G)}{2m}. \)

Corollary 4.9. Let \( G_1 \) and \( G_2 \) be Randić equienergetic graphs. Then \( H_2^m(G_1) \) and \( H_2^m(G_2) \) are Randić equienergetic for all \( m > 3 \).

Operation 4.3. Let \( G \) be a simple \((p,q)\) graph with vertex set \( V(G) = \{v_1, v_2, \ldots, v_p\} \) and \( G_1, G_2, \ldots, G_{m-1} \) are the \( m-1 \) copies of \( G \). Define a graph \( H_3^m(G), m > 1 \) with vertex set \( V(H_3^m(G)) = V(G) \cup \bigcup_{i=1}^{m-1} V(G_i) \) and edge set \( E(H_3^m(G)) \) consisting only of those edges joining \( i^{th} \) vertex of \( G_j, 1 \leq j \leq m-1, \) to the neighbors of \( v_i \) in \( G, 1 \leq i \leq p \) and then removing edges of \( G \).

The number of vertices and the number of edges in \( H_3^m(G) \) are \( pm \) and \( 3(m-1)q \) respectively.

Theorem 4.10. The energy of the graph \( H_3^m(G) \) is \( \varepsilon(H_3^m(G)) = \left[ m-2 + \sqrt{4m-3} \right] \varepsilon(G). \)
Proof. The adjacency matrix of $H_3^m(G)$ is

$$A(H_3^m(G)) = \begin{bmatrix} O & A(G) & A(G) & \ldots & A(G) & A(G) \\ A(G) & A(G) & O & \ldots & O & O \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ A(G) & O & O & \ldots & O & A(G) \end{bmatrix}_{pm}$$

$$= \begin{bmatrix} 0 & 1 & 1 & \ldots & 1 & 1 \\ 1 & 1 & 0 & \ldots & 0 & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ 1 & 0 & 0 & \ldots & 0 & 1 \end{bmatrix}_m \otimes A(G)$$

$$= Z_1 \otimes A(G), \text{ where } Z_1 = \begin{bmatrix} 0 & 1 & 1 & \ldots & 1 & 1 \\ 1 & 1 & 0 & \ldots & 0 & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ 1 & 0 & 0 & \ldots & 0 & 1 \end{bmatrix}.$$ 

Let $P = \begin{bmatrix} -1+\sqrt{4m-3} \\ 2 \\ 1 \\ \vdots \\ 1 \end{bmatrix}_{m \times 1}$, then $Z_1 P = \left(\frac{1+\sqrt{4m-3}}{2}\right)P$ and let $Q = \begin{bmatrix} -1-\sqrt{4m-3} \\ 2 \\ 1 \\ \vdots \\ 1 \end{bmatrix}_{m \times 1}$, then $Z_1 Q = \left(\frac{1-\sqrt{4m-3}}{2}\right)Q$. Let $E_j$ be as in Theorem 4.5, then $Z_1 E_j = E_j$. So the simple eigenvalues of $Z_1$ are $\frac{1+\sqrt{4m-3}}{2}, \frac{1-\sqrt{4m-3}}{2}$, and 1 has multiplicity $m-2$. Thus spectrum of $H_3^m(G)$ is

$$\text{Spec}(H_3^m(G)) = \left(\frac{1+\sqrt{4m-3}}{2}\lambda_i, \frac{1-\sqrt{4m-3}}{2}\lambda_i, \lambda_i \right)_{1 \leq i \leq p}.$$ 

Hence we have

$$\varepsilon(H_3^m(G)) = \left[ m - 2 + \sqrt{4m-3} \right] \varepsilon(G).$$

Corollary 4.11. Let $G$ be an integral graph. Then $H_3^m(G)$ is an integral graph if $4m-3$ is a perfect square.

For example, $H_3^3(K_2)$, $H_3^5(K_2)$, $H_3^{13}(K_2)$, $H_3^{21}(K_2)$ etc.

Corollary 4.12. Let $G_1$ and $G_2$ be equienergetic graphs. Then $H_3^m(G_1)$ and $H_3^m(G_2)$ are equienergetic for all $m > 1$.

Theorem 4.13. The Randić energy of graph $H_3^m(G), m > 2$ is $\varepsilon_R(H_3^m(G)) = \varepsilon_R(G) + \frac{(m-1)\varepsilon_R(G)}{2}$.

Proof. The Randić matrix of $H_3^m(G)$ is $R(H_3^m(G))$

$$\begin{bmatrix} (m-1)D + \frac{1}{2} & O & O & \ldots & O \\ O & (2D)^{\frac{1}{2}} & O & \ldots & O \\ \vdots & \vdots & \ddots & \ddots & \vdots \\ O & O & O & \ldots & (2D)^{\frac{1}{2}} \end{bmatrix}_{pm} \cdot \begin{bmatrix} O & A(G) & A(G) & \ldots & A(G) \\ A(G) & A(G) & O & \ldots & O \\ \vdots & \vdots & \ddots & \ddots & \vdots \\ A(G) & O & O & \ldots & A(G) \end{bmatrix}_{pm} \cdot \begin{bmatrix} (m-1)D + \frac{1}{2} & O & O & \ldots & O \\ O & (2D)^{\frac{1}{2}} & O & \ldots & O \\ \vdots & \vdots & \ddots & \ddots & \vdots \\ O & O & O & \ldots & (2D)^{\frac{1}{2}} \end{bmatrix}_{pm}$$

12
Theorem 5.1. Let results for $D$ number of spanning trees of $\text{Spl}$. In this section, we compute the degree Kirchhoff index, the Kemeny’s constant and the

5 Applications

Let $P^* = \begin{bmatrix} \sqrt{m+1} & 1 & \ldots & 1 \\ 1 & 1 & \ldots & 1 \\ \vdots & \vdots & \ddots & \vdots \\ 1 & 1 & \ldots & 1 \end{bmatrix}_{m \times 1}$ then $Z_2 P^* = 1.P^*$ and let $Q^* = \begin{bmatrix} -(m-1) \sqrt{ \frac{2}{m-1}} \\ 1 \\ \vdots \\ 1 \end{bmatrix}_{m \times 1}$ then $Z_2 Q^* = -\frac{1}{2} Q^*$. Let $E_j$ be as in Theorem 4.13 then $Z_2 E_j = \frac{1}{2} E_j$. So the simple eigenvalues of $Z_2$ are $1$, $-\frac{1}{2}$, and $\frac{1}{2}$ has multiplicity $m-2$. Thus Randić spectrum of $H_3^m(G)$ is

$$\text{RS}(H_3^m(G)) = \left( \rho_i, \frac{1}{2} \rho_i, \frac{1}{2} \rho_i \right)_{1 \leq i \leq m-2}$$

Hence $\varepsilon_R(H_3^m(G)) = \varepsilon_R(G) + \frac{(m-1) \varepsilon_R(G)}{2}$. □

Corollary 4.14. Let $G_1$ and $G_2$ be Randić equienergetic graphs. Then $H_3^m(G_1)$ and $H_3^m(G_2)$ are Randić equienergetic for all $m > 1$.

5 Applications

In this section, we compute the degree Kirchhoff index, the Kemeny’s constant and the number of spanning trees of $\text{Spl}_m(G)$ in terms of original graph. Analogous to this, results for $D_m(G), D^m(G), H_1^m(G), H_2^m(G)$ and $H_3^m(G)$ are included in Appendix.

Theorem 5.1. Let $G$ be a simple connected $(p, q)$ graph with Randić spectrum $\{\rho_1, \rho_2, \ldots, \rho_p\}$. Then

$$K(\text{Spl}_m(G)) = p(m-1) + K(G) + \sum_{i=2}^{p} \frac{m+1}{1+m(1+\rho_i(G))}.$$ 

Theorem 5.2. Let $G$ be a simple connected $(p, q)$ graph with Randić spectrum $\{\rho_1, \rho_2, \ldots, \rho_p\}$. Then

$$K f^*(\text{Spl}_m(G)) = 2(m+1)q \left[ p(m-1) + \sum_{i=2}^{p} \frac{m+1}{1+m(1+\rho_i(G))} \right] + (m+1)K f^*(G).$$
From the following theorem, we obtain the number of spanning trees of graphs in terms of Randić eigenvalues.

**Theorem 5.3.** Let $G$ be a connected simple $(p, q)$ graph with Randić spectrum $\{\rho_1, \rho_2, ..., \rho_p\}$. Then

$$t(Spl_m(G)) = \frac{(m + 1)^p (\prod_{i=1}^p d_i) m^m t(G) \prod_{i=1}^p \left(1 + \frac{m \rho_i(G)}{m+1}\right)}{2m + 1}.$$

**Conclusion** In this paper, we compute the energy and Randić energy of some specific graphs which are obtained by some graph operations on $G$. Also, our results show how to construct some new class of graphs having the same Randić energy as that of $G$. In addition, some new family of equienergetic, Randić equienergetic, integral and Randić integral graphs are obtained. Moreover, we discuss some graph invariants like the degree Kirchhoff index, the Kemeny’s constant and the number of spanning trees of graph $Spl_m(G)$.

6 Appendix

Let $G$ be a connected graph, then $D_m(G)$, $H_1^m(G)$, $H_2^m(G)$ and $H_3^m(G)$ are connected. Here we discuss the degree Kirchhoff index, the Kemeny’s constant and the number of spanning trees of graphs $D_m(G)$, $H_1^m(G)$, $H_2^m(G)$ and $H_3^m(G)$.

**Theorem 6.1.** Let $G$ be a simple connected $(p, q)$ graph with Randić spectrum $\{\rho_1, \rho_2, ..., \rho_p\}$. Then

1. $K(D_m(G)) = p(m - 1) + K(G)$.
2. $K(H_1^m(G)) = m - 3 + K(G) + \sum_{i=2}^{p} \frac{m-1}{m-1+\rho_i(G)} + \sum_{i=2}^{p} \frac{m(m-1)}{m^2-m+(m-2)\rho_i(G)}$.
3. $K(H_2^m(G)) = m - 3 + K(G) + \sum_{i=2}^{p} \frac{2m}{2m+(m-2)\rho_i(G)} + \sum_{i=2}^{p} \frac{2}{2-\rho_i(G)}$.
4. $K(H_3^m(G)) = 6(m-1)q \left[ K(G) + \sum_{i=2}^{p} \frac{2}{2+\rho_i(G)} + \sum_{i=2}^{p} \frac{2}{2-\rho_i(G)} \right]$.

**Theorem 6.2.** Let $G$ be a simple connected $(p, q)$ graph with Randić spectrum $\{\rho_1, \rho_2, ..., \rho_p\}$. Then

1. $Kf^*(D_m(G)) = 2m^2 q (p(m - 1)) + m^2 Kf^*(G)$.
2. $Kf^*(H_1^m(G)) = 2(m^2 - 2)q \left[ m - 3 + \sum_{i=2}^{p} \frac{m-1}{m-1+\rho_i(G)} + \sum_{i=2}^{p} \frac{m(m-1)}{m^2-m+(m-2)\rho_i(G)} \right] + (m^2 - 2) Kf^*(G)$. 

14
3. \( K^*(H_m^3(G)) = 2(3m-2)q \left[ m - 3 + \sum_{i=2}^{p} \frac{2m}{2m+(m-2)\rho_i(G)} + \sum_{i=2}^{p} \frac{2}{2-\rho_i(G)} \right] + (3m-2)K^*(G). \)

4. \( K^*(H_m^3(G)) = 6(m-1)q \left[ \sum_{i=2}^{p} \frac{2}{2+\rho_i(G)} + \sum_{i=2}^{p} \frac{2}{2-\rho_i(G)} \right] + 3(m-1)K^*(G). \)

From the following theorem, we obtain the number of spanning trees of graphs in terms of Randić eigenvalues.

**Theorem 6.3.** Let \( G \) be a connected simple \((p, q)\) graph with Randić spectrum \( \{\rho_1, \rho_2, ..., \rho_p\} \). Then

1. \( t(D_m(G)) = \frac{m^p}{m^2} \prod_{i=1}^{p} d_i^{m-1}t(G). \)

2. \( t(H_1^m(G)) = \frac{m^{(m-2)p}(m-1)^2}{m^2} \prod_{i=1}^{p} d_i^{m-1}t(G) \prod_{i=1}^{p} \left( 1 + \frac{\rho_i(G)}{m-1} \right) \prod_{i=1}^{p} \left( 1 + \frac{2(m-2)\rho_i(G)}{m(m-1)} \right). \)

3. \( t(H_2^m(G)) = \frac{2^{(m-1)p}m^p}{3m-2} \prod_{i=1}^{p} d_i^{m-1}t(G) \prod_{i=1}^{p} \left( 1 + \frac{\rho_i(G)}{2m} \right) \prod_{i=1}^{p} \left( 1 + \frac{\rho_i(G)}{2} \right). \)

4. \( t(H_3^m(G)) = \frac{2^{(m-1)p}(m-1)^p}{3m-3} \prod_{i=1}^{p} d_i^{m-1}t(G) \prod_{i=1}^{p} \left( 1 - \frac{\rho_i(G)}{2} \right) \prod_{i=1}^{p} \left( 1 + \frac{\rho_i(G)}{2} \right). \)

**References**

[1] Saeid Alikhani and Nima Ghanbari. Randić energy of specific graphs. *Applied Mathematics and Computation*, 269:722–730, 2015.

[2] Ilkay Altındag. Some statistical results on Randić energy of graphs. *MATCH Commun. Math. Comput. Chem*, 79:331–339, 2018.

[3] R Balakrishnan. The energy of a graph. *Linear Algebra and its Applications*, 387:287–295, 2004.

[4] S Burcu Bozkurt, A Dilek Güngör, Ivan Gutman, and A Sinan Cevik. Randić matrix and Randić energy. *MATCH Commun. Math. Comput. Chem*, 64:239–250, 2010.

[5] Steve Butler. Algebraic aspects of the normalized Laplacian. In *Recent Trends in Combinatorics*, pages 295–315. Springer, 2016.

[6] Haiyan Chen and Fuji Zhang. Resistance distance and the normalized Laplacian spectrum. *Discrete Applied Mathematics*, 155(5):654–661, 2007.

[7] Zheng-Qing Chu, Saima Nazeer, Tariq Javed Zia, Imran Ahmed, and Sana Shahid. Some new results on various graph energies of the splitting graph. *Journal of Chemistry*, 2019, 2019.

[8] Fan R.K Chung. Spectral graph theory. CBMS regional conference series in mathematics. *American Mathematical Society*, 92, 1997.
[9] Dragoš M Cvetković, Michael Doob, and Horst Sachs. Spectra of graphs: Theory and applications. *Academic Press, New York*, 10, 1980.

[10] Kinkar Ch Das, Shaowei Sun, and Ivan Gutman. Normalized Laplacian eigenvalues and Randić energy of graphs. *MATCH Commun. Math. Comput. Chem*, 77:45–59, 2017.

[11] Ivan Gutman. The energy of a graph. *Ber. Math.-Statist. Sekt. Forsch. Graz*, (100-105):Ber. No. 103, 22, 1978.

[12] Ivan Gutman, Boris Furtula, and Ş Burcu Bozkurt. On Randić energy. *Linear Algebra and its Applications*, 442:50–57, 2014.

[13] Jeffrey J Hunter. The role of Kemeny’s constant in properties of Markov chains. *Communications in Statistics-Theor. and Methods*, 43(7):1309–1321, 2014.

[14] G Indulal and A Vijayakumar. On a pair of equienergetic graphs. *MATCH Commun. Math. Comput. Chem*, 55(1):83–90, 2006.

[15] Douglas J Klein and Milan Randić. Resistance distance. *Journal of mathematical chemistry*, 12(1):81–95, 1993.

[16] Emanuele Munarini, Claudio Perelli Cippo, Andrea Scagliola, and Norma Zagaglia Salvi. Double graphs. *Discrete mathematics*, 308(2-3):242–254, 2008.

[17] H.P Patil and V Raja. On tensor product of graphs, girth and triangles. *Iranian Journal of Mathematical Sciences and infomatics*, 10(1):139–147, 2015.

[18] Harishchandra S Ramane, Hanumappa B Walikar, Siddani Bhaskara Rao, B Devadas Acharya, I Gutman, PR Hampiholi, and Sudhir R Jog. Equienergetic graphs. *Kragujevac Journal of Mathematics*, (26):5–13, 2004.

[19] Oscar Rojo and Luis Medina. Construction of bipartite graphs having the same Randić energy. *Match-communications in mathematical and in computer chemistry*, 68(3):805–814, 2012.

[20] E Sampathkumar. On duplicate graphs. *J. Indian Math. Soc*, 37:285–293, 1973.

[21] E Sampathkumar and H.B Walikar. On splitting graph of a graph, j. *Karnatak Univ. Sci.*, 25(13):13–16, 1980.

[22] Samir K Vaidya and Kalpesh M Popat. Energy of m-splitting and m-shadow graphs. *Far East Journal of Mathematical Sciences*, 102:1571–1578, 2017.