In this paper, we analyze magnon and heat currents induced by magnetic field and temperature differences in insulating ferromagnetic junctions as depicted in Fig. 1. We calculate all transport coefficients for magnon transport and establish Onsager relations between them. We predict some universal thermomagnetic relations for magnon transport in ferromagnetic junctions and estab-
lish a magnon analog of the Wiedemann-Franz (WF) law,15 which is known to govern charge transport at low temperatures.18,19 With respect to the SSE or SPE, we demonstrate that the magnon Seebeck and Peltier coefficients behave universally in the low temperature limit. We also show that these features are extremely robust with respect to multi-magnon effects. However, we find that magnon-magnon interactions can give rise to deviations from the Onsager reciprocity relation between the magnon Seebeck and Peltier coefficients. Such a mechanism may offer an explanation for recent experiments by Dejene et al.20 where deviations from the Onsager relation between the spin Seebeck and Peltier coefficients were observed.

The paper is organized as follows. In Sec. II we introduce the model system for a ferromagnetic junction. In Sec. III we describe the magnon transport in harmonic approximation driven by gradients of magnetic fields or heat and derive all Onsager transport coefficients perturbatively. There we also find the WF law and the universal behavior of the Seebeck and Peltier coefficients. In Sec. IV we discuss multi-magnon effects such as three- and four magnon terms. In Sec. V we give some concrete estimates for YIG systems. Finally, we summarize and give some conclusions in Sec. VI. The technical details are deferred to App. A-E.

II. SYSTEM

We consider a magnetic junction formed by two ferromagnetic insulators (FI), as illustrated in Fig. 1. The temperature of the left (right) FI is $T_{l(r)}$ and the cross-sectional area of the junction interface is $A$. Due to a finite overlap of the wave functions there exists in general a finite exchange interaction between the spins located at the boundaries between the two FIs. Thus, only the boundary spins, denoted as $S_{l}$ and $S_{r}$, in the left and right FI, respectively (see Fig. 1), are relevant for magnon transport across the junction interface. The exchange interaction between the two FIs may be described by the Hamiltonian $\mathcal{H}_{\text{ex}} = -J_{\text{ex}} \sum \langle \tau \rangle \Gamma_{\tau} \cdot \Gamma_{\tau}$, where $J_{\text{ex}} > 0$ is the exchange interaction, weakly coupling the two FIs. Assuming magnetic order along the magnetic field, defining the $z$-direction, we perform a Holstein-Primakoff expansion to leading order, $S_{l(r)} = \sqrt{2S} a_{l(r)} + \mathcal{O}(S^{-1/2})$ and $S_{l(r)} = S - a_{l(r)}^\dagger a_{l(r)}$, where $[a_{l}, a_{l}^\dagger] = \delta_{l,r}$, we obtain

$$\mathcal{H}_{\text{ex}} = -J_{\text{ex}} S \sum_{k_{l, k_{r}}} \sum_{k_{l, k_{r}}} a_{\Gamma_{l}, k} a_{\Gamma_{r}, k}^\dagger + \text{H.c.}, \quad (1)$$

where $k = (k_{x}, k_{y}, k_{z})$, $\mathbf{k}' = (k_{x}', k_{y}, k_{z})$, $k_{\perp} = (0, k_{y}, k_{z})$, and the bosonic operator $a_{\Gamma_{l}, k}^\dagger$ ($a_{\Gamma_{r}, k}$) creates (annihilates) a boundary magnon at the right/left FI. We note that the $k_{z}$-momentum of magnons is not conserved at the sharp junction interface, whereas the perpendicular momentum $k_{\perp}$ is conserved. The tunneling Hamiltonian $\mathcal{H}_{\text{ex}}$ thus gives the time-evolution of the magnon number operators in both FIs and generates the magnon and heat currents. In obtaining Eq. (1), we have assumed large spins $S \gg 1$, and hence the $\mathcal{O}(S^{0})$ term in Eq. (1) indeed becomes negligible.

Assuming cubic lattices, each of the three-dimensional FIs can be described by a Heisenberg spin Hamiltonian in the presence of a Zeeman term. The time reversal symmetry is broken by the assumed ferromagnetic order and by the magnetic field. Within the long wave-length approximation and in the continuum limit, the magnon dispersion relation in each FI reads

$$\omega_{k}^{l(r)} = 2JSa_{l(r)}^2k^2 + g\mu_{B}B_{l(r)}, \quad (2)$$

where $J > 0$ is the isotropic exchange interaction between the nearest neighbor spins in each FI, $a$ denotes the lattice constant, $k = | \mathbf{k} |$ is the wave vector modulus, and $B_{l(r)}$ is the magnetic field for magnons in the left (right) FI along the $z$-axis. We assume that $J_{\text{ex}} \ll J$ and treat $\mathcal{H}_{\text{ex}}$ perturbatively. We remark that in addition to the tunneling Hamiltonian given by Eq. (1), other $\mathcal{O}(J_{\text{ex}}S)$-terms actually arise from $\mathcal{H}_{\text{ex}}$ due to the ferromagnetic order and act as effective magnetic fields for the boundary magnons. Including such effects, an effective magnetic field in Eq. (2) is introduced. Thus, even in the absence of an external magnetic field, $B_{l(r)} > 0$ for the boundary magnons.

III. ONSAGER COEFFICIENTS

The magnetic field and temperature differences defined by $\Delta B \equiv B_{r} - B_{l}$ and $\Delta T \equiv T_{r} - T_{l}$, respectively, generate the magnon and heat currents,18,24 $I_{m}$ and $I_{Q}$, where

$$I_{m} = -i(J_{\text{ex}}S/\hbar) \sum_{k_{l}, k_{r}} g\mu_{B}a_{\Gamma_{l}, k} a_{\Gamma_{r}, k'}^\dagger + \text{H.c.} \quad \text{and} \quad I_{Q} = -i(J_{\text{ex}}S/\hbar) \sum_{k, k'} \omega_{k}^{l(r)} a_{\Gamma_{l}, k} a_{\Gamma_{r}, k'}^\dagger + \text{H.c.}$$

Within the linear response regime, each Onsager coefficient $L_{ij}$ ($i, j = 1, 2$) and the matrix $\hat{L}$ are defined by

$$\begin{pmatrix} I_{m} \\ I_{Q} \end{pmatrix} = \begin{pmatrix} L_{11} & L_{12} \\ L_{21} & L_{22} \end{pmatrix} \begin{pmatrix} -\Delta B \\ \Delta T \end{pmatrix} \equiv \hat{L} \begin{pmatrix} -\Delta B \\ \Delta T \end{pmatrix}. \quad (3)$$

A straightforward perturbative calculation in $J_{\text{ex}}$ based on the Schwinger-Keldysh formalism and up to $\mathcal{O}(J_{\text{ex}}^2)$ gives each coefficient
\[
L^{11} = \frac{(g\mu_B)^2A}{2\hbar} \left( \frac{J_{ex}}{4\pi Ja} \right)^2 \sum_{n=1}^{\infty} [-\text{Ei}(-ne^2)e^{-nb}],
\]
\[
L^{12} = \frac{g\mu_B k_B A}{2\hbar} \left( \frac{J_{ex}}{4\pi Ja} \right)^2 \left[ \sum_{n=1}^{\infty} (1/n + b)[-\text{Ei}(-ne^2)e^{-nb} + Li_1(e^{-b})] \right],
\]
\[
L^{21} = \frac{g\mu_B k_B TA}{2\hbar} \left( \frac{J_{ex}}{4\pi Ja} \right)^2 \left[ \sum_{n=1}^{\infty} (1/n + b)[-\text{Ei}(-ne^2)e^{-nb} + Li_1(e^{-b})] \right],
\]
\[
L^{22} = \frac{k_B^2 TA}{2\hbar} \left( \frac{J_{ex}}{4\pi Ja} \right)^2 \left[ 3Li_2(e^{-b}) + 26Li_1(e^{-b}) + \sum_{n=1}^{\infty} \left( \frac{2}{n^2} + \frac{2b}{n} + b^2 \right)[-\text{Ei}(-ne^2)e^{-nb}] \right],
\]

where
\[
b = \frac{g\mu_B B}{k_BT}.
\]

We have denoted \( B_i \equiv B \) and \( T_i \equiv T \) for simplicity, and a phenomenological magnon lifetime \( \tau \) in \( \epsilon \equiv \hbar \beta/(2\tau) \ll 1 \), where \( \beta \equiv (k_BT)^{-1} \), has been introduced and is mainly due to nonmagnetic impurity scatterings. Interestingly, the WF law holds in the same way where the role of the charge \( e \) is played by \( g\mu_B \). The magnetic Lorenz number becomes independent of any material parameters except the \( g \)-factor which is material specific. Interestingly, the WF law holds in the same way for magnons, which are bosonic excitations, as for electrons which are fermions. The linear-in-\( T \) behavior can be traced back to the Onsager relation Eq. (6), and is mainly due to nonmagnetic impurity scatterings.

The coefficient \( L^{11} \) is identified with the magnetic magnon conductance \( G \), and, in analogy to charge transport, \( L^{12}, L^{21}, L^{22} \) we obtain the thermomagnetic ratio \( K/G \), characterizing magnon and heat transport. Its behavior is plotted in Fig. 2. At low temperatures, \( \Pi \) as function of \( b \) where \( \epsilon = 10^{-10} \), at low temperatures \( b = O(10) \), the ratio reaches the constant '1' and the WF law for magnon transport \[ Eq. (7) \] is realized.

\[
\Pi = TS.
\]

At low temperatures, \( \epsilon/(2\tau) \ll k_BT \ll g\mu_B B \), the coefficients reduce to
\[
S \equiv \frac{B}{T}, \quad \Pi \equiv B.
\]

This is a remarkable result: The magnon Seebeck and Peltier coefficients become universal at low temperatures, i.e., they are completely independent of any material parameters (including the \( g \)-factor) and are solely determined by the applied magnetic field and temperature.

Similarly, in analogy to charge transport in metals, we refer to \( S \equiv L^{12}/L^{11} \) as magnon Seebeck coefficient (i.e., thermomagnetic power), and \( \Pi \equiv L^{21}/L^{11} \) as magnon Peltier coefficient. The Onsager relation Eq. (6) provides the Thomson relation (i.e., Kelvin-Onsager relation)
\[
\Pi = TS.
\]

![Fig. 2](Image)

FIG. 2: (Color online) Plot of the ratio \( (g\mu_B/k_B)^2[K/(GT)] \) as function of \( b \) where \( \epsilon = 10^{-10} \). At low temperatures \( b = O(10) \), the ratio reaches the constant '1' and the WF law for magnon transport \[ Eq. (7) \] is realized.
and the thermal magnon conductance behave in terms of \( \epsilon \) as

\[
L^J \sim \ln \epsilon, \quad K \sim \ln \epsilon. \quad (11)
\]

Thus, all coefficients show a weak logarithmic dependence on \( \tau \), i.e., \( L^J, K \sim \ln \tau \). In addition, Eq. (11) implies that both currents arise from terms of order \( O(J^2_{\text{ex}}) \). Therefore, even when an electric field is applied to the interface, the resulting Aharonov-Casher phase cannot play any significant role in the transport of such noncondensed magnons. Moreover, even when a magnetic field difference \( \Delta B \neq 0 \) is generated, the noncondensed magnon current becomes essentially a dc one. This is in sharp contrast to the condensed magnon current \( J^2 \) which arises from the \( O(J^2_{\text{ex}}) \)-term.

IV. MULTI-MAGNON EFFECTS

So far we have considered the transport of essentially noninteracting magnons. Now we take multimagron effects into account. Two kinds of effects are considered below; the first one corresponds to a three-magnon splitting\(^{22}\), which arises due to higher order terms in the \( 1/S \)-expansion of the Holstein-Primakoff transformation\(^{22}\), while the second one appears because of magnon-magnon interactions\(^{21,22,24}\) and is due to the anisotropy\(^{26}\) of the exchange interaction between neighboring spins in each FI.

We begin by considering higher order terms of the Holstein-Primakoff transformation\(^{22}\),

\[
S^+_i = \sqrt{2S}[1 - a_i^\dagger a_i/(4S)]a_i + O(S^{-3/2}),
\]

and thereby include the three-magnon splittings, \( a_i^\dagger a_j a_k a_l \), into \( \mathcal{H}_{\text{ex}} \). One can expect that each coefficient becomes smaller because such terms correspond to the replacement of the operator \( a_i \) with \( [1 - a_i^\dagger a_i/(4S)]a_i^\dagger \). A straightforward calculation\(^{26}\) based again on the Schwinger-Keldysh\(^{27,28}\) formalism gives the simple result for the modified matrix \( \hat{L}_2 \) up to \( O(J^2_{\text{ex}}) \)

\[
\hat{L}_2 = \left[ 1 - \frac{1}{4\pi^{3/2}S} \left( \frac{k_B T}{2JS} \right)^{3/2} \text{Li}_{3/2}(e^{-b}) \right] \hat{L}. \quad (12)
\]

Eq. (12) shows that, although each coefficient becomes smaller due to the three-magnon splittings, the modified matrix \( \hat{L}_2 \) is still characterized by the noninteracting one \( \hat{L} \) [Eq. (4)]; \( \hat{L}_2 \propto \hat{L} \). Consequently, the Onsager and the Thomson relations, Eqs. (9) and (10), remain satisfied. In addition, the magnon WF law [Eq. (7)] and the Seebeck and Peltier coefficients, Eqs. (16), at low temperature remain valid. These thermomagnetic properties are therefore robust against the three-magnon splittings.

One should note\(^{22}\) that these properties actually hold in any order of the \( 1/S \)-expansion of the Holstein-Primakoff transformation.

An anisotropic\(^{21,22,24}\) exchange interaction between nearest-neighbor spins in each FI gives rise to magnon-magnon interactions of the type\(^{26}\)

\[
\mathcal{H}_m = -J_m \sum_{\langle ij \rangle} a_i^\dagger a_i^\dagger a_i a_j. \quad \text{The symbol } \langle ij \rangle \text{ indicates summation over nearest-neighbor spins in each FI. The magnitude and the sign of the interaction } J_m \text{ depends on the anisotropy } \mathcal{B} \text{ of the exchange interaction. We assume here a small anisotropy } |J_m| \ll J. \quad \text{A straightforward but tedious calculation}\(^{26}\) gives us the modified matrix up to } O(J^2_{\text{ex}}J_m)
\]

\[
\hat{L}_3 = \left( L^{11} + \delta L^{11} L^{12} \right), \quad (13)
\]

where

\[
\delta L^{11} = \frac{J^2_{\text{ex}}J_m \sqrt{k_B T} \tau A}{16 \sqrt{2S \pi^{5/2} S^2}} (g \mu_B)^2 \text{Li}_{3/2}(e^{-b}) \times \ln(\sqrt{2JS} \beta a/\epsilon), \quad (14a)
\]

\[
\delta L^{21} = \frac{J^2_{\text{ex}}J_m \sqrt{k_B T} \tau A}{16 \sqrt{2S \pi^{5/2} S^2}} g \mu_B^2 \text{Li}_{3/2}(e^{-b}) \times \left[ JS(a \Lambda)^2 - e^2/(2\beta) + [JS(a \Lambda)^2 + g \mu_B B] \times \ln(\sqrt{2JS} \beta a/\epsilon) \right], \quad (14b)
\]

with \( a \Lambda = \sqrt{5/(JS \beta)} \). These results imply the violation of the Onsager relation in Eq. (10) and of the Thomson relation in Eq. (9) due to magnon-magnon interactions. Notice that \( \delta L^{11} = O(J^2_{\text{ex}}J_m) \) and \( \delta L^{21} = O(J^2_{\text{ex}}J_m) \), while \( L^{11} = O(J^2_{\text{ex}}J^0_m) \) and \( L^{21} = O(J^2_{\text{ex}}J^0_m) \). We recall that the Onsager reciprocal relation\(^{12,35,36}\) could in principle be already violated in the noninteracting case since the time reversal asymmetry is broken by the ferromagnetic order and the magnetic field right from the outset. Still, we have microscopically found that the relation remains satisfied even in the presence of the three-magnon and higher order splitting terms of the Holstein-Primakoff transformation\(^{21}\). However, the anisotropy induced magnon-magnon interaction \( J_m \) provides a ‘nonlinearity’ \( \delta L^{21} = O(J^2_{\text{ex}}J_m) \) in terms of the perturbative terms \( J_{\text{ex}} \) and \( J_m \), and consequently the matrix \( \hat{L}_3 \) cannot be reduced to the form \( \hat{L} = O(J^2_{\text{ex}}J^0_m) \). Using a mean field argument or a more rigorous microscopic calculation\(^{26}\), one can show that the magnon-magnon interaction acts as an effective magnetic field and that the total magnetic field difference \( \Delta B_{\text{tot}} \) may be written as \( \Delta B_{\text{tot}} = (1 + b_m) \Delta B \) with \( b_m = O(J_m) \). The term \( b_m \) gives \( \delta L^{11} = O(J_m) \) and \( \delta L^{21} = O(J_m) \). The Onsager relation is thus violated due to the nonlinearity caused by the anisotropy induced magnon-magnon interaction. The magnitude of the effective magnetic field difference can be estimated by \( b_m \sim \delta L^{21}/L^{21} \).

These multi-magnon contributions, \( \delta L^{11} \) and \( \delta L^{21} \), generally affect also the thermomagnetic properties and can lead to deviations from our previous results. However, at low temperatures, where the WF law, Eq. (7), and the universality of Seebeck and Peltier coefficients hold, these deviations become negligible\(^{26}\) because \( |\delta L^{11}/L^{11}| \ll 1 \) for typical parameter values (see below). So far we have assumed bulk FIs (see Fig. 1) where magnetic dipole-dipole interactions are negligible\(^{35}\).
Such dipolar effects, however, become important in thin films, resulting in a modified dispersion for magnons.\(^{25}\) Still, the WF law remains valid in this case too,\(^{31}\) underlining the universality of this law.

V. ESTIMATES FOR EXPERIMENTS

The magnon currents can be experimentally measured by using, for instance, the method proposed in Refs. \(^{[21,38]}\). Since the magnons, being moving magnetic dipoles with magnetic moment \(g \mu_B \mathbf{e}_z\), produce electric fields, magnon currents can be detected by measuring the resulting voltage drop perpendicular to the current direction and magnetic field. For an estimate, we assume the following experimental parameter values.\(^{4,13,29,39–41}\): \(J = 100 \text{meV}\), \(J_{\text{ex}} = 10 \text{meV}\), \(J_m^1 = 1 \text{meV}\), \(a = 1 \text{Å}\), \(A = 3 \text{cm}^2\), \(g = 2\), \(\tau = 100 \text{ns}\), \(\Delta T = 0.5 \text{K}\), and \(B = 50 \text{mT}\) (5T) and \(T = 300 \text{K}\) (0.7K) for the high (low)\(^{30}\) temperature regime. Using a similar set-up as in Ref. \(^{[21]}\), we find that the resulting voltage drop is in the nV (\(\mu\text{V}\)) range for high (low) temperatures. Although small, such values are within experimental reach and are actually about \(10^6\) (\(10^3\)) times larger than the one (\(~\mu\text{V}\)) predicted for currents in condensed magnon systems.\(^{22}\) Alternatively, attaching a metal (e.g. Pt) to the FIs and using the inverse spin Hall effect to convert magnon currents into electric currents,\(^{10}\) the magnon currents could also be detected\(^{8,11,16}\) by measuring the resulting Hall voltage in the metal. Finally, we mention that the temperature difference \(\Delta T\) can be experimentally produced by applying microwaves of different frequencies to each FI or by local laser heating\(^{39,43}\).

VI. SUMMARY

We have studied the thermomagnetic transport behavior of a ferromagnetic insulating junction and determined the Onsager coefficients in linear response regime. We found that at low temperatures the magnon transport obeys an analog of the Wiedemann-Franz law where the ratio of heat to magnon conductance is linear in temperature. Like its electronic counterpart the WF law found here is universal and does not depend on material parameters except the \(g\)-factor. Quite remarkably, it exhibits the same linear-in-\(T\) behavior at low temperatures as the one for electronic transport in spite of the fact that the quantum-statistical properties of bosons and fermions are fundamentally different, in particular in the low temperature regime where quantum effects dominate.

The temperature scale, however, for electrons is given by the Fermi temperature \((\sim 10^4 \text{ K for normal metals})\), while it is the magnetic field for magnons (a few Kelvins). Obviously, these two scales are very different, and that is why the electronic counterpart of the WF law is valid typically at much higher temperatures than the magnonic one, but in both cases the WF law applies when the system temperature is low relative to its respective temperature scale. Moreover, we showed that the magnon Seebeck and Peltier coefficients become universal at low temperatures.

As an outlook we mention that it would be interesting to explore the regime beyond the weak junction coupling studied here and see if the WF law can be extended to such a regime as well.

Finally, it would be interesting to test our predictions experimentally in candidate systems like insulating ferromagnets such a YIG material in the bulk or thin film limit.

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Appendix A: MAGNON AND HEAT CURRENTS

Assume cubic lattices, each three-dimensional FI is described by a Heisenberg spin Hamiltonian in the presence of a Zeeman term,

\[
\mathcal{H}_{\text{FI}} = -\sum_{\langle ij \rangle} \mathbf{S}_i \cdot \mathbf{J} \cdot \mathbf{S}_j - g \mu_B \mathbf{B}_{\text{(r)}} \cdot \sum_i \mathbf{S}_i, \tag{A1}
\]

where \(\mathbf{J}\) denotes a diagonal \(3 \times 3\)-matrix with \(\text{diag}(\mathbf{J}) = \mathbf{J}\{1,1,\eta\}\). The exchange interaction between neighboring spins in the ferromagnetic insulator is \(J > 0\), \(\eta\) denotes the anisotropy of the spin Hamiltonian, and \(B_{\text{(r)}} = B_{\text{(r)}} \mathbf{e}_z\) is an applied magnetic field to the left (right) FI (\(\mathbf{e}_z\) denotes the unit vector along the \(z\)-axis). The symbol \(\langle ij \rangle\) indicates summation over neighboring spins in each FI and \(\mathbf{S}_i\) denotes the spin of length \(S\) at lattice site \(i\). Within our microscopic calculation,\(^{22}\) we find that in the continuum limit, the magnon-magnon interaction \(\mathcal{H}_m\) could arise from the \(\eta \neq 1\) anisotropic spin Hamiltonian \(\mathcal{H}_{\text{FI}}\) as the \(O(1-\eta)\) term

\[
\mathcal{H}_m = -J_m a^3 \int \mathbf{d} \mathbf{r} a^\dagger(\mathbf{r}) a^\dagger(\mathbf{r}') a(\mathbf{r}) a(\mathbf{r}'), \tag{A2}
\]

where \(J_m \equiv -J(1-\eta) = O(S^0)\), the lattice constant \(a\), and \([a(\mathbf{r}), a^\dagger(\mathbf{r}')] = \delta(\mathbf{r} - \mathbf{r}')\). Therefore the magnon-magnon interaction does not influence the magnon transport between \(\eta = 1\) isotropic FIs in any significant manner,\(^{21,22}\) and we can neglect them in the isotropic case. Assuming the isotropic FI, within the long wavelength approximation, the magnon dispersion in each FI is given by Eq. \(\text{(2)}\) in the main text, i.e., \(\omega_k^{(r)} = 2JSu^2k^2 + g \mu_B B_{\text{(r)}}\).
We then consider a magnetic junction formed by two ferromagnetic insulators, as illustrated in Fig. 1. The temperature of the left (right) FI is $T_{l(r)}$ and the cross-section area of the junction interface is $A$. Due to a finite overlap of the wave functions, there exists in general a finite exchange interaction between the spins located at the boundaries between the two FIs. Thus, only the boundary spins, denoted as $S_{l}$ and $S_{r}$, in the left and right FI, respectively (see Fig. 1), are relevant for the boundary spins, denoted as $\Gamma_l$. Thus, only the finite overlap of the wave functions, there exists in general a finite exchange interaction between the two FIs. Assuming magnetic order along the magnetic field, defining the z-direction, we perform a Holstein-Primakoff expansion to leading order, $S_{l,r}^z = \sqrt{2S_{l,r}} \frac{1}{2} + O(S^{-1/2})$ and $S_{l,r}^z = S - a_{l,r}^\dagger a_{l,r}$, where $[a_i, a_i^\dagger] = \delta_{l,r}$, we obtain Eq. (1) in the main text.

The tunneling Hamiltonian $H_{\text{ex}}$ gives the time-evolution of the magnon number operators in the FIs and generates the magneton and heat current operators $L_{m}$ and $L_{Q}$ by $H_{\text{ex}}$. $L_{m} = -i(J_{\text{ex}}S/h) \sum_k \sum_{k'} g_{\mu} \Gamma_l(k_\mu \Gamma_r(k'_{\mu}) + \text{H.c.}$ and $L_{Q} = -i(J_{\text{ex}}S/h) \sum_k \sum_{k'} \sum_{\nu} \omega_{\nu} \Gamma_l(k_{\nu} \Gamma_r(k'_{\nu}) + \text{H.c.}$ Using the Schwinger-Keldysh formalism and treating $H_{\text{ex}}$ perturbatively ($J_{\text{ex}} \ll J$), they can be evaluated up to $O(J_{\text{ex}}^2)$,

$$\langle L_{m} \rangle = -\frac{2}{\hbar} \left( \frac{J_{\text{ex}}S}{L} \right)^2 \sum_k g_{\mu} \sum_{k'_{\nu}} \left[ n(\omega_{\nu}^k) - n(\omega_{\nu}^{k'}) \right] \frac{\hbar/(2\tau)}{[2JSa^2(k_x^2 - k_{x'}^2)]^2 + [\hbar/(2\tau)]^2},$$

$$\langle L_{Q} \rangle = -\frac{2}{\hbar} \left( \frac{J_{\text{ex}}S}{L} \right)^2 \sum_k \sum_{k'_{\nu}} \omega_{\nu} \left[ n(\omega_{\nu}^k) - n(\omega_{\nu}^{k'}) \right] \frac{\hbar/(2\tau)}{[2JSa^2(k_x^2 - k_{x'}^2)]^2 + [\hbar/(2\tau)]^2}.$$  

For large $L$, we can replace the sums by integrals. The summation over $k_{\nu}'$ in Eqs. (A5a) and (A5b) thus becomes

$$\sum_{k_{\nu}'} \left[ n(\omega_{\nu}^k) - n(\omega_{\nu}^{k'}) \right] \frac{\hbar/(2\tau)}{[2JSa^2(k_x^2 - k_{x'}^2)]^2 + [\hbar/(2\tau)]^2} \approx \frac{L}{2\pi} \int \frac{\beta}{2JSa^2} dx' \left[ \frac{\epsilon}{(x'^2 - x^2)^2 + \epsilon^2} \right]^2,$$

where $x'^2 \equiv 2JSa^2 \beta x_x$, $x'^2 \equiv 2JSa^2 \beta x', \epsilon \equiv \hbar \beta/(2\tau)$. In the last equation [Eq. (A6b)], we have assumed that $\tau$ is large such that $\epsilon \ll 1$.

We see that both currents [Eqs. (A5a) and (A5b)] are characterized by the difference of Bose-distribution functions $n(\omega)$. After integration over $x'$, Eq. (A6b) reduces to $n(\omega_{\nu}^k) - n(\omega_{\nu}^{k'})$. Within the linear response regime, this difference becomes

$$n(\omega_{\nu}^k) - n(\omega_{\nu}^{k'}) \approx \frac{\beta g_{\mu} \Gamma_{l}(\epsilon_{\nu}^k - 1)}{\epsilon_{\nu}^{k'}} \Delta B, \quad \text{for } \Delta T = 0,$$

$$n(\omega_{\nu}^k) - n(\omega_{\nu}^{k'}) \approx \frac{-\beta g_{\mu} \Gamma_{l}(\epsilon_{\nu}^{k'} - 1)}{\epsilon_{\nu}^{k'}} \Delta T, \quad \text{for } \Delta B = 0.$$

We have introduced the magnetic field and temperature differences

$$\Delta B \equiv B_l - B_t, \quad \Delta T \equiv T_r - T_l,$$

and have denoted $B_t \equiv B$ and $T_t \equiv T$ for convenience.
The difference of the Bose-distribution functions has been expanded in powers of $\Delta T \ll T$ and $\Delta B \ll B$. Eq. (A7) yields

$$\frac{[n(\omega_k^+)] - n(\omega_k^-)]}{[n(\omega_k^+) - n(\omega_k^-)]} \bigg|_{\Delta T = 0} / \Delta T = -\frac{\omega_k^+}{g\mu_B T}. \tag{A9}$$

Within linear response, we can also approximate the magnetic field terms in all Green functions in Eq. (A4) by $B$. Finally, one can see that Eq. (A9) is responsible for the Onsager reciprocal relation, given in Eq. (6), and the Onsager relation, Eq. (6), holds accordingly. This remains in place even when three-magnon splittings [Eq. (A3)] and any higher order terms of the Holstein-Primakoff expansion are taken into account [see Eqs. (E1) and (E2) as an example]: in any order of the expansion, the currents are characterized by the difference of the Bose-distribution functions [Eq. (A7) and (A9)]. Consequently, the Onsager relation [Eqs. (A9) and (6)] holds.

This is changed by the anisotropy induced magnon-magnon interaction and the Onsager relation becomes violated, as we shall see below.

Appendix B: ONSAGER COEFFICIENTS

Within the linear response regime [Eq. (A7)], Eqs. (A5a) and (A5b) provide the Onsager coefficients $L^{ij}$ in Eqs. (1a)-(1d) of the main text. The coefficients, $L^{21}$ and $L^{12}$, are seen to satisfy the Onsager relation Eq. (9). The coefficient $L^{11}$ is identified with the magnetic magnon conductance $G$, and the thermal magnon conductance $K$ is defined by $K \equiv L^{21} - L^{21}L^{12}/L^{11}$. In analogy to charge transport, this definition follows from the condition that the magnon current $\langle I_m \rangle$ induced by the applied thermal difference $\Delta T$ be zero. This gives rise to an induced magnetic field difference $\Delta B_{ind} = \Delta T L^{12}/L^{11}$, and thus to the thermal magnon current $\langle I_Q \rangle = K \Delta T$ (in the absence of an applied field gradient, i.e., $\Delta B = 0$).

Eqs. (A5a) and (A5b) determine the general thermomagnetic ratio $K/G$ for magnon and heat transport, see Fig. 2 in the main text. At low temperatures, $1 \ll b \equiv g\mu_B B/(k_B T)$, the ratio reduces to Eq. (7) in the main text (see also Fig. 2). We note that the temperature should be still such that $\epsilon \ll 1$ remains satisfied, i.e., $h/(2\tau) \ll k_B T \ll g\mu_B B$. In the opposite limit, $\epsilon \gg 1$, both currents, Eqs. (A5a) and (A5b), are seen to be exponentially vanishing as $T \to 0$.

The Onsager relation [Eq. (6)] provides the Thomson relation (i.e., the Kelvin-Onsager relation) in Eq. (6) of the main text. At low temperatures, $h/(2\tau) \ll k_B T \ll g\mu_B B$, the magnon Seebeck and Peltier coefficients reduce to Eq. (10) in the main text (see Fig. 3), and, quite strikingly, become universal.

We remark that at low temperatures, each Onsager coefficient $L^{ij}$ and the thermal magnon conductance behave in terms of $\epsilon$ as in Eq. (11) in the main text. Thus, all coefficients show a weak logarithmic dependence on $r$, i.e., $L^{ij} \sim \ln r$.

![FIG. 3: (Color online) Plot of magnon Seebeck coefficient $S$ as function of $b$ where $\epsilon = 10^{-10}$. At low temperatures $b = O(10^2)$, the rescaled coefficient $g\mu_B S/(k_B b)$ approaches unity asymptotically and realizes the universal relation [Eq. (10)].](image)

Appendix C: YIG THIN FILM

So far we have assumed that both FIs are of bulk shape such that surface effects due to magnetic dipole-dipole interactions are negligible. However, it is interesting to consider also a thin film geometry since such systems are of great experimental interest, in particular for YIG films.

We show now that the WF law also holds for a thin film, where now the length of the junction in $x$-direction $L$ (see Fig. 1) is short compared to all other dimensions. Due to the dipole-dipole interaction, the magnon dispersion relation in each FI changes and becomes $\omega_k^{(r)} = D(k^2 - k^2_m)^2 + g\mu_B B_{(r)}$, where $k_m \sim 10^4$ cm for e.g. YIG thin films. The parameter $D$ is due to the long-range dipole-dipole interaction as well as the exchange interaction between the nearest-neighbor spins. The main contribution comes from $k \approx k_m$ where $\omega_k^{(r)} \approx 4Dk^2_m(k - k_m)^2 + g\mu_B B_{(r)}$ for $k \approx k_m$. As we have seen in the main text, the WF law is realized in strong magnetic fields $B_{(r)} \sim$ a few T (leading to a large energy gap). Repeating the same perturbative calculation as before, but under the restriction that only the lowest magnon subband in $x$-direction ($k_x = k'_x = 0$) needs
to be taken into account due to finite-size quantization (which actually simplifies the calculation), the Onsager coefficients in thin films become

\[
\begin{align*}
L^{11} &= \frac{\tau A}{4\pi^2 D k_m^2} \left( \frac{J_{ex} S a}{L} \right)^2 (g\mu_B)^2 L_{i0}(e^{-b}), \\
L^{12} &= \frac{\tau A}{4\pi^2 D k_m^2} \left( \frac{J_{ex} S a}{L} \right)^2 g\mu_B k_B \left[ L_{i1}(e^{-b}) + b L_{01}(e^{-b}) \right], \\
L^{21} &= \frac{\tau A}{4\pi^2 D k_m^2} \left( \frac{J_{ex} S a}{L} \right)^2 g\mu_B k_B T \left[ L_{i1}(e^{-b}) + b L_{01}(e^{-b}) \right], \\
L^{22} &= \frac{\tau A}{4\pi^2 D k_m^2} \left( \frac{J_{ex} S a}{L} \right)^2 k_B^2 T \left[ 2 L_{i2}(e^{-b}) + b L_{01}(e^{-b}) + b^2 L_{00}(e^{-b}) \right].
\end{align*}
\]

The Onsager relation, \( L^{21} = T \cdot L^{12} \), again holds.

In particular, the ratio \( K / G \) reduces to \(^3\2\)

\[
\frac{K}{G} \equiv \left( \frac{k_B}{g\mu_B} \right)^2 T, \quad \text{when} \quad k_B T \ll g\mu_B B. \tag{C2}
\]

Again, the ratio is linear in temperature, and, quite remarkably, again with a universal prefactor (in particular independent of the dipole-dipole interaction \( D \)). Thus, we see that the WF law holds also for thin films and in the presence of dipole-dipole interactions. Thus, we conclude that the linear-in-\( T \) behavior is extremely robust against microscopic details.

We note that each Onsager coefficient itself [Eqs. \( \text{(A1a)-} \text{(A1d)} \)] drastically changes in terms of temperature dependence as well as magnon lifetime \( \text{etc} \). as compared to the bulk results (see main text). However, these differences all cancel out in the ratio \( K / G \) and the linear-in-temperature behavior remains.

Finally, at low temperatures \( k_B T \ll g\mu_B B \), the magnon Seebeck and Peltier coefficients for thin films are reduced to \( S \equiv B/T \) and \( \Pi \equiv B \) as for the bulk case given in the main text.

**Appendix D: MAGNON-MAGNON INTERACTIONS**

Assuming an anisotropic\(^2\1,2,2,34\) exchange interaction among the nearest neighboring spins in each FI [Eq. \( \text{(A1)} \)], magnon-magnon interactions in Eq. \( \text{(A2)} \) may arise from such an anisotropic Heisenberg spin Hamiltonian. The sign and the magnitude of the magnon-magnon interaction \( J^{(r)}_{m} \) depends on the anisotropy in the left (right) FI and it is assumed to be small \( | J^{(r)}_{m} | \ll J \) to be treated perturbatively; in Eqs. \( \text{(A1a)} \) and \( \text{(A1b)} \), they are set \( J^{(r)}_{m} = J^{(r)}_{m} \equiv J_{m} \) for simplicity.

Using the Schwinger-Keldysh formalism\(^27,28\), we find up to \( \mathcal{O}(J^2_{m} J_{m}) \)

\[
\langle I_m \rangle = \frac{(J_{ex} S)^2 J_m^2 a^2}{2\pi^2 I^2} \left( \frac{k_B T}{2 JS} \right)^{3/2} \text{Li}_{3/2}(e^{-b}) \sum_{k, k'} \int d\omega g\mu_B \times \text{Re}(G^a_{l,k,\omega} G^a_{l,k',\omega} G^b_{l,k,\omega} G^b_{l,k',\omega} \quad \text{D1})
\]

where \( G^{a(\ell>/)} \) is the bosonic advanced (retarded/lesser/greater) Green function. Thus the magnon current \( \langle I_m \rangle = \mathcal{O}(J^2_{m}) \) arises as the nonlinearity in terms of the perturbative terms \( \langle J^{(r)}_{m} \rangle \) and \( J_{ex} \). After some straightforward manipulations [Eq. \( \text{A1a} \)], Eq. \( \text{D1} \) is reduced to the form

\[
\langle I_m \rangle = C_1 (J^{(r)}_{m} \mathcal{N}^{\ell} + J^{(r)}_{m} \mathcal{N}^{r}) \tag{D1}
\]

where \( C_1 \) is a \( \Delta B \)- and \( \Delta T \)-independent constant, and \( \mathcal{N}^{\ell(r)} \) defined by

\[
\mathcal{N}^{\ell} \equiv \frac{\sqrt{\pi}}{4a^3} \left( \frac{k_B T}{2 JS} \right)^{3/2} \text{Li}_{3/2}(e^{-b}), \tag{D3a}
\]

\[
\mathcal{N}^{r} \equiv \mathcal{N}^{\ell} + \mathcal{O}(\Delta B) + \mathcal{O}(\Delta T), \tag{D3b}
\]

is obtained by integrating the Bose-distribution function \( n(\omega_{k}^{(r)}) \) over the wavevector. Eq. \( \text{D1} \) indicates that the perturbation of the magnon-magnon interaction \( J^{(r)}_{m} \) work as the magnetic field difference \( \Delta B \) expansion. A mean field analysis actually shows that the magnon-magnon interaction works as an effective magnetic field. The \( C_1 \) term arises from \( G^a_{l,k,\omega} G^a_{l,k',\omega} G^b_{l,k,\omega} G^b_{l,k',\omega} \) in Eq. \( \text{D1} \) and the \( C_2 \) term from the rest. Since the difference of the Bose-distribution functions \( n(\omega_{k}^{(r)}) \) indeed gives the responses to \( \Delta B \) and \( \Delta T \) by itself [Eq. \( \text{A7} \)], the \( C_2 \) term becomes irrelevant in the linear response regime; the response to the temperature difference
The difference $\Delta B_{\text{tot}}$ may be written as $\Delta B_{\text{tot}} = (1 + b_m) \Delta B$, where $b_m = O(J_m N^\ell + J_m^r N^r)$ is the contribution of such magnon-magnon interactions. This contribution $b_m$ gives rise to the $C_1(\ell)$-term [Eqs. (13) and (D7)], and Eq. (13a)], and leads to $\delta L^{11} = O(J_m' N^\ell + J_m^r N^r)$ and $\delta L^{21} = O(J_m' N^r + J_m^r N^r)$. Thus, the Onsager relation is violated by the magnon-magnon interactions. The magnitude of the effective magnetic field difference can be estimated by $b_m \sim \Delta L^{21}/L^{21}$.

The contributions arising from magnon-magnon interactions, $\delta L^{11}$ and $\delta L^{21}$, generally affect also the thermomagnetic properties of magnon and heat transport. However, at low temperatures, $h/(2\tau) < k_B T < \mu_B B$, the ratios reduce to $\delta L^{11}/L^{11} \sim (3J_m\tau/2h)(k_B T/SJ)^{3/2}$ and similarly for $\delta L^{21}/L^{21}$. For typical parameter values (see main text) we find that e.g. $\delta L^{11}/L^{11} \sim 0.1$. Therefore, $\hat{L}_3 = L$ at low temperatures. Thus, the Onsager and the Thomson relations, Eqs. (6) and (9), the WF law for magnon transport, Eq. (7), and the thermomagnetic properties of magnon Seebeck and Peltier coefficients, Eq. (10), still hold at low temperatures.

**Appendix E: THREE-MAGNON SPLITTINGS**

Until now, we have essentially used the linearized Holstein-Primakoff transformation $S_i^+ = \sqrt{2S}[1 - a_i^\dagger a_i/(2S)]^{1/2}a_i + O(S)^{-1/2}$. Now, we consider the higher term $O(S)^{-1/2}$, $S_i^+ = \sqrt{2S}[1 - a_i^\dagger a_i/(4S)]a_i + O(S)^{-3/2}$, and include the three-magnon splittings, $a_i^\dagger a_i a_i/2S + \text{H.c.}$, into the Hamiltonian $H_{\text{ex}}$ to be treated perturbatively. We remark that when $S \rightarrow \infty$ (i.e. in the classical limit), the three-magnon splittings cease to work. In that sense, the three-magnon splitting can be regarded as a quantum effect.

Following the same procedure as before with the approximation Eq. (A7), the magnon current in linear response regime becomes in leading order

$$\langle I_m \rangle = \left[ 1 - \frac{1}{4\pi^3/2S} \frac{k_B T}{2JS} \frac{3/2}{\Delta B} \right] \frac{\text{Li}_{3/2}(e^{-b})}{\Delta T},$$

which is of order $O(J_m' e)$. The heat current can be evaluated in the same way and becomes up to $O(J_m' e)$

$$\langle I_Q \rangle = \left[ 1 - \frac{1}{4\pi^3/2S} \frac{k_B T}{2JS} \frac{3/2}{\Delta B} \right] \frac{\text{Li}_{3/2}(e^{-b})}{\Delta T},$$

Thus, the modified matrix $\hat{L}_2$ is given by [Eq. (3)]

$$\hat{L}_2 = \left[ 1 - \frac{1}{4\pi^3/2S} \frac{k_B T}{2JS} \frac{3/2}{\Delta B} \right] \text{Li}_{3/2}(e^{-b}) \hat{L}.$$
This means
\[
\dot{L}_2 \propto \dot{L} \quad \text{(E4a)} \\
\dot{L} \rightarrow \dot{L}, \quad \text{when } T \rightarrow 0. \quad \text{(E4b)}
\]

Therefore, the Onsager and the Thomson relations [Eqs. \(\text{[6]}\) and \(\text{[9]}\) hold. In addition, the WF law for magnon transport [Eq. \(\text{[7]}\) and the thermomagnetic properties [Eq. \(\text{[10]}\) at low temperatures remain valid even in the presence of three-magnon splittings.

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