LANCZOS POTENTIALS AND CURVATURE-FREE CONNECTIONS ALIGNED TO A GEODESIC SHEAR-FREE EXPANDING NULL CONGRUENCE

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Abstract

By the method of $\rho$-integration we obtain all Lanczos potentials $L_{ABCA'}$ of the Weyl spinor that, in a certain sense, are aligned to a geodesic shear-free expanding null congruence. We also obtain all spinors $H_{ABA'B'} = Q_{AB}o_{A'}o_{B'}$, $Q_{AB} = Q_{(AB)}$ satisfying $\nabla_{(A'B'}H_{B)A'B'} = L_{ABCA'}$. We go on to prove that $H_{ABA'B'}$ can be chosen so that $\Gamma_{ABCA'} = \nabla_{(A'B'}H_{B)C'A'B'}$ defines a metric asymmetric curvature-free connection such that $L_{ABCA'} = \Gamma_{(ABC)A'}$ is a Lanczos potential that is aligned to the geodesic shear-free expanding congruence. These results are a generalization to a large class of algebraically special spacetimes (including all vacuum ones for which the principal null direction is expanding) of the curvature-free connection of the Kerr spacetime found by Bergqvist and Ludvigsen, which was used in a construction of quasi-local momentum. In conclusion we give a corresponding definition of quasi-local momentum in this more general class of spacetimes and examine some of its properties in the special case of a Kerr-Schild spacetime.

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1 Preliminaries

1.1 Introduction and conventions

The purpose of this paper is to determine Lanczos potentials for the Weyl spinor $\Psi_{ABCD}$ and their $H$-potentials that have a particularly simple algebraic structure, in a class of spacetimes admitting a geodesic shear-free expanding null congruence (including all vacuum ones), and to use these potentials to construct curvature-free asymmetric connections. Such a construction has already been performed in the Kerr spacetime [11], where the curvature-free connection was used to construct a quasi-local momentum for the Kerr spacetime. In this section we will give the preliminary results that we need, concerning Lanczos potentials, $H$-potentials and $\rho$-integration. In the final part of this section we give an outline of the remainder of the paper.

We will use spacetime definitions and conventions from [19]. In particular this means that the metric $g_{ab}$ is assumed to have signature $(+−−−)$. We will use spinors for our calculations, but as all results are local in nature there is no need to postulate the existence of a global spinor structure on spacetime. Penrose’s abstract index notation [19] will be used throughout this paper; Latin letters will denote tensor indices, primed and unprimed capital Latin letters will denote spinor indices. However, on differential forms (completely antisymmetric tensors) occurring under an integral sign the indices will be suppressed and the differential form will be written as a bold-faced letter. All spinor dyads $(o^A, o^A')$ will be assumed to be normalized, i.e., $o_A o^A = 1$. $\nabla_{AA'}$ denotes the Levi-Civita connection, i.e., the uniquely defined metric and torsion-free (symmetric) connection on spacetime.

1.2 Lanczos potentials and $H$-potentials

It is well-known [18], [8], [15] that there always exists a completely symmetric spinor $L_{ABCA'} = L_{(ABC)A'}$ such that

$$\Psi_{ABCD} = 2\nabla_{(A'A}L_{BCD)A'}$$

(1)

where $\Psi_{ABCD}$ is the Weyl spinor. This equation is called the Weyl-Lanczos equation and $L_{ABCA'}$ is called a Lanczos potential of $\Psi_{ABCD}$. In fact [15], given any symmetric spinor $W_{ABCD}$ it can be shown that it has a Lanczos potential $L_{ABCA'}$.

It is important to note that a Lanczos potential is far from unique. It is shown in [15] that given any symmetric spinors $W_{ABCD}$, $\zeta_{BC}$ there exists a Lanczos potential of $W_{ABCD}$ (unique up to its values on a spacelike past-compact hypersurface) such that

$$\nabla^{A'A}L_{ABCA'} = \zeta_{BC}.$$
For a recent, very simple proof of this fact, see [4].

The spinor $\zeta_{BC}$ is called the differential gauge of $L_{ABCA'}$ and when $\zeta_{BC} = 0$, i.e.,

$$\nabla^{AA'}L_{ABCA'} = 0$$

$L_{ABCA'}$ is said to be in Lanczos differential gauge. Then the Weyl-Lanczos equation can be written

$$\Psi_{ABCD} = 2\nabla^{AA'}L_{BCDA'}.$$  

However, in this paper we will not impose the Lanczos differential gauge condition. Instead we prefer $F_{BC}$ to remain arbitrary and indeed the Lanczos potentials that we find will only satisfy Lanczos differential gauge in very special circumstances.

We now take this one step further and ask: Given a symmetric spinor $L_{ABCA'}$, does there exist a spinor $H_{ABA'B'}$ such that

$$L_{ABCA'} = \nabla^{(A'B'}H_{BC)A'B'}$$  \hspace{1cm} (2)

where $H_{ABA'B'}$ is completely symmetric, i.e., $H_{ABA'B'} = H_{(AB)(A'B')}$. In the case when $L_{ABCA'}$ is a Lanczos potential of the Weyl spinor, $H_{ABA'B'}$ would then be a gravitational analogue of the flat space Hertz potential in electromagnetic theory.

Illge gives a partial answer to this question. He shows [15] that if such a potential exists it has to satisfy a restrictive condition that is algebraic in the $H$-potential. This rules out the existence of such a Hertz-type potential in general. However, in Einstein spacetimes the $H$-potential vanishes from this condition and it turns out to be possible to prove the existence of a completely symmetric $H$-potential for an arbitrary symmetric $L_{ABCA'}$ in these spacetimes [6].

We remark that in an $H$-space [14] in ‘complex general relativity’, it is always possible to find a very simple Lanczos potential of the Weyl spinor, that in turn has a very simple $H$-potential; however a general result of the nature of the one in [4] does not exist, as far as we know, for these spaces.

If we remove the requirement of symmetry over the unprimed indices of $H_{ABA'B'}$, it follows from [13] that such a potential exists in all spacetimes, but in this paper we will only consider completely symmetric $H$-potentials so this result is of limited interest to us.

For a lot of our calculations in this paper we will use the GHP-formalism. For a normalized spinor dyad $(o^A, l^A)$ it is conventional to define the dyad components of the Lanczos potential, the so-called Lanczos scalars, as

\begin{align*}
L_0 &= L_{ABCA'}o^Ao^Bo^Co^A' \quad L_4 = L_{ABCA'}o^Ao^Bo^Co^A' \\
L_1 &= L_{ABCA'}o^Ao^B_iC^o^A' \quad L_5 = L_{ABCA'}o^Ao^B_iC^o^A' \\
L_2 &= L_{ABCA'}o^A_iB^iC^o^A' \quad L_6 = L_{ABCA'}o^A_iB^iC^o^A' \\
L_3 &= L_{ABCA'}l^Al^Bl^Cl^A' \quad L_7 = L_{ABCA'}l^Al^Bl^Cl^A' . \end{align*}  \hspace{1cm} (3)
The Weyl-Lanczos equation can then be translated into GHP-formalism:

\[
\frac{1}{2} \Psi_0 = \partial L_0 - \bar{ \mathbf{P}} L_4 - \bar{ \gamma} L_0 + 3 \sigma L_1 + \bar{ \rho} L_4 - 3 \kappa L_5
\]

\[
2 \Psi_1 = 3 \partial L_1 - 3 \bar{ \mathbf{P}} L_5 - \bar{ \gamma}' L_4 + \bar{ \mathbf{P}}' L_0 - (\bar{ \rho}' - 3 \bar{ \rho}) L_0 - 3 (\bar{ \gamma}' - \bar{ \gamma}) L_1
\]

\[
+ 6 \sigma L_2 - (3 \bar{ \gamma}' - \bar{ \gamma}) L_4 - 3 (\bar{ \rho} - \bar{ \rho}) L_5 - 6 \kappa L_6
\]

\[
\Psi_2 = \partial L_2 - \bar{ \mathbf{P}} L_6 - \bar{ \gamma}' L_5 + \bar{ \mathbf{P}}' L_1 + \kappa' L_0 - (\bar{ \rho}' - 2 \bar{ \rho}) L_1 - (\bar{ \gamma}' - 2 \bar{ \gamma}) L_2
\]

\[
+ \sigma L_3 - \bar{ \gamma}' L_4 - (2 \bar{ \gamma}' - \bar{ \gamma}) L_5 - (2 \bar{ \rho} - \bar{ \rho}) L_6 - \kappa L_7
\]

\[
2 \Psi_3 = \partial L_3 - \bar{ \mathbf{P}} L_7 - 3 \partial L_6 + 3 \bar{ \mathbf{P}}' L_2 + 6 \kappa' L_1 - 3 (\bar{ \rho}' - \rho') L_2 - (\bar{ \gamma}' - 3 \bar{ \gamma}) L_3
\]

\[
- 6 \sigma L_5 - 3 (\bar{ \gamma}' - \bar{ \gamma}) L_6 - (3 \bar{ \rho} - \bar{ \rho}) L_7
\]

\[
\frac{1}{2} \Psi_4 = \bar{ \mathbf{P}}' L_3 - \bar{ \gamma}' L_7 + 3 \kappa' L_2 - \bar{ \rho}' L_3 - 3 \sigma' L_6 + \bar{ \tau} L_7
\]

(4)

These equations will be used to integrate the Weyl-Lanczos equation for a large class of algebraically special spacetimes in the following sections.

We define the dyad components of $H_{ABA'B'}$ as

\[
H_{00'} = H_{ABA'B'} o^{A'o^B'o^{A'}o'B'}
\]

\[
H_{01'} = H_{ABA'B'} o^{A'o^B'o^{A'}l'l'B'}
\]

\[
H_{11'} = H_{ABA'B'} o^{A'l^B'o^{A'}l'l'B'}
\]

\[
H_{20'} = H_{ABA'B'} o^{A'o^B'o^{A'}l'l'B'}
\]

\[
H_{21'} = H_{ABA'B'} o^{A'o^B'o^{A'}l'l'B'}
\]

\[
H_{22'} = H_{ABA'B'} o^{A'o^B'o^{A'}l'l'B'}
\]

(5)

Then (2) becomes, in GHP-formalism:

\[
L_0 = \partial H_{00'} - \bar{ \mathbf{P}} H_{01'} - \bar{ \tau} H_{00'} + 2 \bar{ \rho} H_{01'} + \bar{ \kappa} H_{02'} + 2 \sigma H_{10'} - 2 \kappa H_{11'}
\]

\[
3 L_1 = \bar{ \mathbf{P}}' H_{00'} - \bar{ \mathbf{P}} H_{01'} + 2 \partial H_{01'} - 2 \partial H_{11'}
\]

\[
+ 2 (\bar{ \rho'} - \bar{ \rho}) H_{00'} + 2 (\bar{ \tau} - \bar{ \tau}') H_{01'} - \bar{ \sigma} H_{02'} + 2 (\bar{ \tau} - \bar{ \tau}') H_{10'}
\]

\[
- 2 (\rho - \rho') H_{11'} - 2 \kappa H_{12'} + 2 \sigma H_{20'} + 2 \kappa H_{21'}
\]

\[
3 L_2 = 2 \bar{ \mathbf{P}}' H_{10'} + 2 \partial H_{11'} + \partial H_{20'} - \bar{ \mathbf{P}} H_{21'}
\]

\[
+ 2 \kappa' H_{00'} - 2 \partial H_{01'} + 2 (\bar{ \rho'} - \bar{ \rho}) H_{10'} + 2 (\bar{ \tau} - \bar{ \tau}') H_{11'}
\]

\[
- 2 \sigma H_{12'} - (2 \bar{ \tau} - \bar{ \tau}') H_{20'} - 2 (\rho - \rho') H_{21'} - \kappa H_{22'}
\]

\[
3 L_3 = \bar{ \mathbf{P}}' H_{20'} - 2 \partial H_{21'} + 2 \kappa' H_{10'} + 2 \kappa H_{11'} - \bar{ \partial} H_{20'} + 2 \partial H_{21'} - \bar{ \sigma} H_{22'}
\]

\[
L_4 = \partial H_{01'} - \bar{ \mathbf{P}} H_{02'} + \bar{ \sigma} H_{00'} - 2 \partial' H_{01'} + \bar{ \rho} H_{01'} + 2 \sigma H_{10'} - 2 \kappa H_{11'}
\]

\[
3 L_5 = \bar{ \mathbf{P}}' H_{01'} - \bar{ \gamma} H_{02'} + 2 \partial H_{11'} - 2 \bar{ \mathbf{P}} H_{12'}
\]

\[
+ 2 \kappa' H_{00'} + 2 (\rho - \rho') H_{01'} + (\bar{ \tau} - \bar{ \tau}') H_{02'} + 2 \sigma' H_{10'}
\]

\[
+ 2 (\bar{ \tau} - \bar{ \tau}') H_{11'} - 2 (\rho - \rho') H_{12'} + 2 \sigma H_{20'} - 2 \kappa H_{21'}
\]

\[
3 L_6 = 2 \bar{ \mathbf{P}}' H_{11'} - 2 \partial' H_{12'} - \bar{ \partial} H_{21'} - \bar{ \mathbf{P}} H_{22'}
\]

\[
+ 2 \kappa' H_{01'} - 2 \partial H_{02'} + 2 \kappa' H_{01'} + 2 (\rho - \rho') H_{11'}
\]

\[
+ 2 (\bar{ \tau} - \bar{ \tau}') H_{12'} + \bar{ \gamma} H_{20'} + 2 (\bar{ \tau} - \bar{ \tau}') H_{21'} - (\rho - \rho') H_{22'}
\]

\[
L_7 = \bar{ \mathbf{P}}' H_{21'} - \bar{ \gamma} H_{22'} + 2 \kappa' H_{11'} - 2 \partial' H_{12'} + 2 \kappa' H_{20'} - 2 \partial' H_{21'} + \bar{ \tau} H_{22'}
\]

(6)
and we will also integrate these equations for the Lanczos potentials obtained from the GHP Weyl-Lanczos equations.

1.3 Some spacetimes admitting a geodesic shear-free expanding null congruence

In [2] the GHP-equations for the spin coefficients and curvature components were $\rho$-integrated. In this section we will simply quote the results. We assume that spacetime admits a geodesic, shear-free null congruence $l^a = o^A \sigma_A$ and that its Ricci spinor satisfies the condition

$$\Phi_{ABA'B'} o^A o^B = 0.$$ (7)

For various technical reasons we also restrict the scalar curvature to be constant and the null congruence to be expanding. Any spacetime that satisfies all these conditions will be said to be of class $G$. Take $o^A$ as the first spinor of a spinor dyad. In GHP-formalism the above conditions are equivalent to

$$\Phi_{00} = \Phi_{01} = \Phi_{02} = 0, \quad \kappa = \sigma = 0, \quad \rho \neq 0, \quad \Lambda = \text{constant}$$ (8)

By the Goldberg-Sachs theorem we obtain

$$\Psi_0 = \Psi_1 = 0,$$ (9)

and so the spacetime is algebraically special.

We can use a null rotation about $o^A$ to achieve $\tau = 0$, and the Ricci equations [19] then imply that also

$$\tau' = \sigma' = 0.$$ (10)

Whenever a dyad is chosen in this way for an arbitrary spacetime of class $G$ it will be said to be in standard form.

We introduce Held’s [14] modified operators which can be written

$$\hat{\partial} = \frac{1}{\rho} \partial, \quad \hat{\partial}' = \frac{1}{\rho'} \partial', \quad \hat{P}' = \hat{P}' + \frac{p}{2\rho} (\Psi_2 + 2\Lambda) + \frac{q}{2\rho} (\overline{\Psi}_2 + 2\Lambda)$$ (11)

in this dyad. Note that our definition of $\hat{P}'$ is slightly modified from Held’s (by the inclusion of $\Lambda$ in the non-vacuum case). The purpose of using Held’s modified operators is simply to reduce the length of calculations; in particular the new operators have the nice properties

$$[\hat{P}, \hat{\partial}] = [\hat{P}, \hat{\partial}] = 0$$ (12)

and

$$[\hat{P}, \hat{P}']\eta = \left[ -\frac{1}{2\rho}(\Psi_2 + 2\Lambda) - \frac{1}{2\rho}(\overline{\Psi}_2 + 2\Lambda) \right]\hat{P}\eta$$ (13)
so that, in particular, if $\eta^o$ satisfies $\tilde{\eta}^o = 0$ (a degree sign will throughout the paper, be used to denote a quantity that is killed by $\tilde{P}$) then

$$\tilde{P}\tilde{\eta}^o = \left[\tilde{P}, \tilde{\eta}^o\right] = 0$$

and the same result is true if $\tilde{\eta}^o$ is replaced with $\tilde{\rho}^o$ or $\tilde{P}^o$.

We will now give the results of the integration. More details can be found in [3].

First of all, the GHP-operators acting on $\rho$ are

$$\begin{align*}
\tilde{P}\rho &= \rho^2 \\
\tilde{\rho} &= 0 \\
\tilde{\rho}^o &= \rho^2 \tilde{\rho}^o \\
\tilde{\Phi}^o &= \rho^2 \tilde{\Phi}^o + \frac{1}{\rho} \Lambda 
\end{align*}$$

where $\Omega^o = \frac{1}{\rho} - \frac{1}{\rho}$ is the twist of the congruence. From these we obtain the useful relations

$$\begin{align*}
\tilde{P}\Omega^o &= 0 \\
\tilde{P}^o &= \tilde{\rho}^o - \rho^o \\
\tilde{\rho}^o &= 2\Omega^o \tilde{\rho}^o + \Psi^o - \Psi^2
\end{align*}$$

(15)

The curvature scalars and the spin coefficients are

$$\begin{align*}
\rho' &= \tilde{\rho}\rho^o - \frac{1}{2}(\rho^2 + \tilde{\rho}\rho)\Psi^2 - \rho^2 \tilde{\Phi}^o + \frac{1}{\rho} \Lambda \\
\kappa' &= \kappa^o - \rho\Psi_3^o - \frac{1}{2} \rho^2 \tilde{\Phi}^2 - \frac{1}{2} \rho^2 \tilde{\Phi}^o - \rho^4 \tilde{\Phi}^2_{11} - \rho^4 \tilde{\Phi}^o_{11} - \rho^3 \tilde{\Phi}^2_{11} \tilde{\Omega}^o \\
\Psi_2 &= \rho^3 \tilde{\Phi}_3^o + 2\rho^3 \tilde{\Phi}_{11} \\
\Psi_3 &= \rho^2 \tilde{\Phi}_3^o + 2\rho^2 \Phi_3^o + 2\rho^2 \tilde{\Phi}_2^o + \rho^2 \tilde{\Phi}_1^o + \rho^3 \tilde{\Phi}_1^o + 3\rho \tilde{\Phi}_{11} \tilde{\Omega}^o \\
\Psi_4 &= \rho^3 \tilde{\Phi}_3^o + \rho^2 \tilde{\Phi}_3^o + 2\rho^2 \tilde{\Phi}_2^o + \rho^2 \tilde{\Phi}_1^o + \rho^3 \tilde{\Phi}_{11} \tilde{\Omega}^o \\
\Phi_1 &= \rho^2 \tilde{\Phi}_{11} \\
\Phi_2 &= \rho^2 \tilde{\Phi}_{21} + \rho^2 \tilde{\Phi}_{11} \tilde{\Omega}^o \\
\Phi_3 &= \rho^2 \tilde{\Phi}_{22} + \rho^2 \tilde{\Phi}_{21} + \rho^3 \tilde{\Phi}_{11} \tilde{\Omega}^o \\
\Phi_4 &= \rho^3 \tilde{\Phi}_{11} \tilde{\Omega}^o + 3\tilde{\Phi}_{11} \tilde{\Omega}^o + 3\tilde{\Phi}_{11} \tilde{\Omega}^o
\end{align*}$$

(14)
In Section 2 we \( \rho \)-integrate the Weyl-Lanczos equations and obtain their general solution in the case when \( L_{ABC \cdot A'} = M_{ABC \cdot A'} \), for spacetimes of class \( G \) where \( l^a = o^A o^{A'} \) is the geodesic shear-free expanding null-congruence.

In Section 3 we consider the equation

\[
L_{ABC \cdot A'} = \nabla_{(A'} H_{BC)A'B'}. 
\]
where $L_{ABCA'}$ is found in Section 2 and $H_{ABA'B'}$ is completely symmetric; we use the results and techniques from Section 2 to find its general solution for the case $H_{ABA'B'} = Q_{AB}o_{A'}o_{B'}$. In particular it is shown that such an $H$-potential always exists, providing the function of integration $L_{7}^{2}$ from Section 2 vanishes, which is a permissible choice.

Section 4 concerns itself with metric connections $\hat{\nabla}_{AA'}$ defined by

$$
\hat{\nabla}_{AA'} \xi^{B} = \nabla_{AA'} \xi^{B} + 2\Gamma^{B}_{\ CAA'} \xi^{C} \tag{19}
$$

where

$$
\Gamma_{ABCA'} = L_{ABCA'} + \varepsilon_{AC} \chi_{BA'} + \varepsilon_{BC} \chi_{AA'}
$$

and $L_{ABCA'}$ is symmetric over its unprimed indices. We remark that a spacetime equipped with such a connection is called a Riemann-Cartan spacetime. It has been shown [11] that in the Kerr spacetime a particular choice of such a connection, due to the fact that it has vanishing curvature, can be used to define quasi-local momentum. This particular choice of $\Gamma_{ABCA'}$ can also be written

$$
\Gamma_{ABCA'} = \nabla_{(A'B'}H_{B)CA'B'},
$$

where $H_{ABA'B'} = Q_{AB}o_{A'}o_{B'}$ for some spinor $Q_{AB} = Q_{(AB)}$.

It was subsequently shown [9] for this choice of $\Gamma_{ABCA'}$, that the symmetric part $L_{ABCA'}$ is actually a Lanczos potential of the Weyl spinor in the Kerr spacetime. It is therefore of interest to see if the Lanczos- and $H$-potentials found in Section 3 and 4 can be used to define a connection that has vanishing curvature for these more general spacetimes.

We show that any connection $\hat{\nabla}_{AA'}$ defined by (19) from a Lanczos potential of the type investigated in Section 2, has vanishing Weyl curvature, i.e., $\hat{\Psi}_{ABCD} = 0$. We also show that we can accomplish $\hat{\Sigma}_{AB} = 0$ if and only if the Lanczos potential we start from possesses an $H$-potential of the type investigated in Section 3. We go on to prove that in spacetimes where $\Lambda = 0$ or $\hat{\partial} \Omega^{o} = 0$ we can also eliminate $\hat{\Lambda}$ by choosing the functions of integration $L_{6}^{2} = -\Lambda$ and $H_{22}^{o} = -\hat{\partial} L_{5}^{2} - \Omega^{o} \Lambda$.

When we look at the Ricci spinor $\hat{\Phi}_{ABA'B'}$ it is shown that three of its components always vanish, and providing $\Lambda = 0$ the remaining six components can be eliminated by fixing another function of integration $H_{42}^{o} = 3\Omega^{o} L_{2}^{2}$ and demanding that the three remaining functions of integration $L_{4}^{2}$, $L_{3}^{2}$ and $H_{22}^{o}$ are solutions of a coupled system of third order equations involving only the differential operator $\hat{\partial}$, and a first order non-linear equation involving the operators $\hat{\partial}$ and $\hat{\partial}'$ only. We go on to prove that all these conditions can be simultaneously satisfied and hence, providing $\Lambda = 0$, a completely curvature-free connection can always be constructed in this manner.

In Section 5 we examine the Bergqvist-Ludvigsen construction of quasi-local momentum in class $\mathcal{G}$ spacetimes with vanishing Ricci scalar, and in greater detail in the special case of Kerr-Schild spacetimes belonging to this class.
Section 6 discusses possible ways of continuing this work, and also contains a few concluding remarks.

2 All Lanczos potentials of the Weyl spinor that are aligned to $o^{A'}$

In this section we will find all Lanczos potentials of $\Psi_{ABCD}$ in spacetimes of class $G$, that have the algebraic structure $L_{ABCA'} = M_{ABCO}o^{A'}$ with $o^A$ as in the previous section. Such a Lanczos potential will be said to be aligned to $o^{A'}$.

Thus, we assume once again that we have a spacetime of class $G$ with a spinor dyad in standard form. That $L_{ABCA'}$ is aligned to $o^{A'}$ amounts to choosing the Lanczos scalars

$$L_0 = L_1 = L_2 = L_3 = 0$$

The existence of such Lanczos potentials in these spacetimes has already been shown by Torres del Castillo [21], [22]. He actually proves existence in the slightly more general class of spacetimes that does not require $\Lambda$ to be a constant and also allows $\rho = 0$. However, his approach differs significantly from ours and he is therefore unable to find all Lanczos potentials of this type.

The Weyl-Lanczos equations in GHP-formalism then become

$$0 = -\bar{\rho} L_4 + \rho L_4$$
$$0 = -3\bar{\rho} L_5 - \rho \tilde{\partial}' L_4 - 3(\rho - \bar{\rho}) L_5$$
$$\Psi_2 = -\bar{\rho} L_6 - \rho \tilde{\partial}' L_5 - (2\rho - \bar{\rho}) L_6$$
$$2\Psi_3 = -\bar{\rho} L_7 - 3\rho \tilde{\partial}' L_6 - (3\rho - \bar{\rho}) L_7$$
$$\frac{1}{2} \Psi_4 = -\rho \tilde{\partial}' L_7.$$  

The first equation can immediately be $\rho$-integrated:

$$0 = \frac{1}{\rho} \bar{\rho} L_4 - L_4 = \bar{\rho} L_4 = \bar{\rho} L_4.$$  

so that

$$L_4 = \bar{\rho} L_4.$$  

Then

$$\tilde{\partial}' L_4 = \bar{\rho} \tilde{\partial}' L_4$$

which substituted into the second equation gives

$$0 = \frac{\rho}{\bar{\rho}} \bar{\rho} L_5 + \left(\frac{\rho^2}{\bar{\rho}} - \rho\right) L_5 + \frac{1}{3} \rho^2 \tilde{\partial}' L_4 = \bar{\rho} \left(\frac{\rho}{\bar{\rho}} L_5 + \frac{1}{3} \rho \bar{\rho} \tilde{\partial}' L_4\right)$$

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Thus,
\[ L_5 = \frac{\bar{\rho}}{\rho} L_5^o - \frac{1}{3} \rho \bar{\rho} \bar{\rho}^r L_4^o \] (23)

Substituting this into the third equation and \( \rho \)-integrating in the same way gives, using the expression for \( \Phi_2 \), an expression for \( L_6 \)
\[ L_6 = \frac{\bar{\rho}}{\rho^2} L_6^o - \frac{\bar{\rho}}{\rho} \bar{\rho}^r L_5^o + \frac{1}{6} \bar{\rho} (\bar{\rho}^r L_4^o + 3 L_6^o \bar{\rho}^r \Omega^o) - \frac{1}{4} \rho^2 \Phi_2^o - \frac{1}{12} \rho \rho \Phi_2^o - \frac{1}{2} \rho^2 \bar{\rho} \Phi_1^o. \] (24)

We can also \( \rho \)-integrate the fourth equation to get an expression for \( L_7 \)
\[ L_7 = \frac{\bar{\rho}}{\rho^3} L_7^o - \frac{3}{2} \rho \bar{\rho} \bar{\rho}^r \bar{L}_6^o + \frac{3}{2} \rho (\bar{\rho}^r L_5^o + 2 L_6^o \bar{\rho}^r \Omega^o) - \frac{1}{2} \rho \Phi_3^o \]
\[ - \frac{1}{6} \rho (\bar{\rho}^r L_4^o + 3 L_6^o \bar{\rho}^r \Omega^o + 9 \bar{\rho}^r \Omega^o \bar{\rho} L_5^o + \Phi_3^o) - \frac{1}{4} \rho^2 \bar{\rho} \bar{\rho}^r \Phi_2^o - \frac{1}{2} \rho \rho \bar{\rho} \Phi_1^o \]
\[ - \frac{1}{2} \rho^3 \Phi_2^o \rho \bar{\rho} \bar{\rho}^r \rho \Phi_1^o - \frac{1}{7} \rho^3 \rho \bar{\rho} \Phi_1^o \bar{\rho} \rho \Phi_1^o. \] (25)

These Lanczos scalars will give a Lanczos potential if and only if the fifth equation of (21) is satisfied. By substituting the above expression for \( L_7 \) into this equation, and using the formula
\[ \bar{\rho} = \frac{\rho}{1 + \rho \Omega^o} \]
we find that the fifth equation of (21) is satisfied if and only if
\[ 0 = \bar{\rho}^r L_7^o - 3 \rho (\bar{\rho}^r L_6^o + L_7^o \bar{\rho}^r \Omega^o) + \frac{3}{2} \rho^2 (\bar{\rho}^r L_5^o + 2 L_6^o \bar{\rho}^r \Omega^o + 6 \bar{\rho}^r \Omega^o \bar{\rho} L_5^o + \frac{1}{3} \Phi_3^o) \]
\[ - \frac{1}{6} \rho^3 (\bar{\rho}^r L_4^o + 3 L_6^o \bar{\rho}^r \Omega^o + 12 \bar{\rho}^r \Omega^o \bar{\rho} L_5^o + 18 \bar{\rho}^r \Omega^o \bar{\rho}^2 L_5^o + 18 L_6^o (\bar{\rho}^r \Omega^o)^2 \]
\[ + \bar{\rho} \Phi_3^o - 3 \Omega^o \Phi_3^o \] (26)

By repeatedly applying \( \bar{\Phi} \) to the RHS of the above expression, and dividing by \( \rho^2 \), it is easy to show that equation (26) is satisfied if and only if each coefficient vanishes. Thus, the above Lanczos scalars will yield a Lanczos potential of \( \Psi_{ABCD} \) if and only if the functions of integration satisfy
\[ 0 = \bar{\rho}^r L_5^o \]
\[ 0 = \bar{\rho}^r L_6^o + L_7^o \bar{\rho}^r \Omega^o \]
\[ 0 = \bar{\rho}^r L_5^o + 2 L_6^o \bar{\rho}^r \Omega^o + 6 \bar{\rho}^r \Omega^o \bar{\rho} L_5^o + \frac{1}{3} \Phi_3^o \]
\[ 0 = \bar{\rho}^r L_4^o + 3 L_6^o \bar{\rho}^r \Omega^o + 12 \bar{\rho}^r \Omega^o \bar{\rho} L_5^o + 18 \bar{\rho}^r \Omega^o \bar{\rho}^2 L_5^o + 18 L_6^o (\bar{\rho}^r \Omega^o)^2 \]
\[ + \bar{\rho} \Phi_3^o - 3 \Omega^o \Phi_3^o \] (27)
Since \( \mathcal{L} \mathcal{H} = 0 \) it follows that the first of the above equations can locally be solved for \( L_4 \). Once we have done that, the second equation can be solved for \( L_6 \). Similarly, the third and fourth equation can be solved for \( L_5 \) and \( L_7 \) respectively, irrespective of the values of \( \Omega \), \( \Psi_3 \) and \( \Psi_4 \). Hence, we have proved the following theorem:

**Theorem 2.1** For any spacetime of class \( \mathcal{G} \) with spinor dyad in standard form, all Lanczos potentials of the Weyl spinor that are aligned to \( o^A' \) are given by

\[
\begin{align*}
L_4 &= \bar{\rho} L_4^o \\
L_5 &= \frac{\bar{\rho}}{\rho} L_5^o - \frac{1}{3} \bar{\rho} \partial' L_4^o \\
L_6 &= \frac{\bar{\rho}}{\rho^3} L_6^o - \frac{\bar{\rho}}{\rho} \partial' L_5^o + \frac{1}{6} \bar{\rho} \partial' L_4^o + 3 L_5^o \partial' \Omega^o - \frac{1}{4} \rho^2 \Psi_2^o - \frac{1}{12} \rho \bar{\rho} \Psi_2^o - \frac{1}{2} \rho^2 \bar{\rho} \Phi_1^o \\
L_7 &= \frac{\bar{\rho}}{\rho^3} L_7^o - \frac{3 \bar{\rho}}{\rho^2} \partial' L_6^o + \frac{3 \bar{\rho}}{2 \rho} \partial' L_5^o + 2 L_6^o \partial' \Omega^o - \frac{1}{2} \rho \bar{\rho} \Psi_3^o \\
&\quad - \frac{1}{2} \rho \bar{\rho} \bar{\rho} \Phi_1^o - \frac{1}{4} \rho^2 \partial' \Omega_3^o - \frac{1}{2} \rho \bar{\rho} \bar{\rho} \Phi_1^o - \frac{1}{2} \rho \bar{\rho} \bar{\rho} \Phi_2^o - \frac{1}{2} \rho^2 \bar{\rho} \Phi_3^o \\
&= \frac{1}{4} \rho^2 \partial' \Omega_3^o - \frac{1}{2} \rho \bar{\rho} \bar{\rho} \Phi_1^o - \frac{1}{2} \rho \bar{\rho} \bar{\rho} \Phi_2^o - \frac{1}{2} \rho \bar{\rho} \bar{\rho} \Phi_3^o.
\end{align*}
\]

where the functions \( L_4^o, L_5^o, L_6^o \) and \( L_7^o \) are subject to the conditions (27). In particular, there always exists a local Lanczos potential that is aligned to \( o^A' \).

For future reference, we note that a particular solution of the first two equations (27) is \( L_5^o = 0, L_6^o = -\Lambda \).

### 3 All \( H \)-potentials of Lanczos potentials of the Weyl spinor that are aligned to \( o^A' \)

We will say that a completely symmetric spinor \( H_{ABA'B'} \) is aligned to \( o^A' \) if it has the algebraic structure \( H_{ABA'B'} = Q_{A oA oB} o_{B'} \). In this section we will find all such spinors \( H_{ABA'B'} \) that are solutions of the equation

\[
L_{ABC A'} = \nabla_{(A} B' H_{BC)} A'B'
\] (29)

where \( L_{ABC A'} \) is a Lanczos potential of the Weyl spinor, i.e.,

\[
\Psi_{ABCD} = 2 \nabla_{(A} B'C'D) A',
\]

in spacetimes of class \( \mathcal{G} \) with spinor dyad in standard form.

First we note that if \( H_{ABA'B'} \) is aligned to \( o^A' \) and satisfies (29) then

\[
L_{ABC A'} o_{A'} = o^A' \nabla_{(A} B' H_{BC)} A'B' = -Q_{(BC oA'} o_{B')} \nabla_{A) B'oA'} = \kappa Q_{(BC A)} - \sigma Q_{(BC oA)} = 0
\]
so that $L_{ABC'A'}$ has the algebraic structure $L_{ABC'A'} = M_{ABC}a_{A'}$ for some symmetric spinor $M_{ABC}$ and is therefore itself aligned to $o^A$. Hence, it suffices to solve equation (29) for the Lanczos potentials found in the previous section. We remark that since the spacetimes we are considering are not necessarily Einstein, and since we are only considering $H$-potentials that are aligned to $o^A$, their existence is not guaranteed by the results in [1].

If $H_{ABA'B'}$ is aligned to $o^A$, it follows that only the components $H_{02'}$, $H_{12'}$, and $H_{22'}$ are non-zero and from the above calculation we see that four out of the eight GHP-equations are identically satisfied. The remaining four become, using Held’s operators

\[
L_4 = -\frac{1}{2} H_{02'} + \rho H_{02'}
\]

\[
3L_5 = -2\rho H_{12'} - \rho \delta^j H_{02'} - 2(\rho - \rho) H_{12'}
\]

\[
3L_6 = -\rho H_{22'} - 2\rho \delta^j H_{12'} - (2\rho - \rho) H_{22'}
\]

\[
L_7 = -\rho \delta^j H_{22'}
\]

(30)

The first three of these equations can now be $\rho$-integrated in the same way as in the previous section and after some calculations we obtain

\[
H_{02'} = \frac{\bar{\rho}}{\rho} L_5^{(2)} + \rho H_{02'}^{(2)}
\]

\[
H_{12'} = \frac{3}{2} \frac{\bar{\rho}}{\rho^2} L_6^{(3)} + \frac{\bar{\rho}}{\rho} H_{12'}^{(3)} - \frac{1}{2} \bar{\rho}(\delta^j H_{02'}^{(3)} - L_5^{(3)} \delta^j \Omega^c)
\]

\[
H_{22'} = \frac{3}{2} \frac{\bar{\rho}}{\rho^3} L_6^{(4)} + \frac{\bar{\rho}}{\rho^2} H_{22'}^{(4)} - \frac{1}{2} \frac{\bar{\rho}}{\rho} \left[4 \delta^j H_{12'}^{(4)} + \delta^j \delta^2 L_4^{(4)} - 9 \delta^j \delta^2 \Omega^c\right] + \frac{1}{4} \rho \Psi_2^c
\]

\[+ \frac{1}{2} \bar{\rho}(\delta^2 H_{02'}^{(4)} + 2 H_{12'}^{(4)} \delta^2 \Omega^c - L_5^{(4)} \delta^2 \Omega^c \delta^j L_4^{(4)} + \frac{1}{2} \Psi_2^c)
\]

\[+ \frac{1}{2} \rho \bar{\rho} \phi_1^c
\]

(31)

These $H$-scalars now give an $H$-potential of a Lanczos potential of the Weyl spinor if and only if the last equation of (30) is satisfied. By substituting the above expressions for $L_7$ and $H_{22'}$ into this equation we find that it is satisfied if and only if

\[
0 = L_7^{(4)} + \rho^2 \left(\delta^j H_{22'}^{(4)} + \frac{3}{2} \delta^j L_5^{(4)} - 6 L_5^{(2)} \delta^j \Omega^c\right)
\]

\[+ 2 \rho^4 \left(\delta^2 H_{12'}^{(4)} + H_{22'}^{(4)} \delta^2 \Omega^c + \frac{1}{3} \delta^2 L_4^{(4)} + 2 L_5^{(2)} \delta^2 \Omega^c - \frac{3}{2} \delta^j \Omega^c \delta^j L_5^{(4)} + \frac{1}{3} \Psi_3^c\right)
\]

\[+ \frac{1}{2} \rho^4 \left(\delta^3 H_{02'}^{(4)} + 2 H_{12'}^{(4)} \delta^2 \Omega^c + 6 \delta^j \Omega^c \delta^j H_{12'}^{(4)} - L_5^{(2)} \delta^3 \Omega^c - 2 \delta^2 \Omega^c \delta^j L_4^{(4)}
\]

\[+ 9 L_5^{(2)} (\delta^j \Omega^c)^2 + \frac{1}{2} \delta \Psi_2^c - \Omega^c \Psi_3^c - \Phi_{21}^c\right)
\]

(32)
By repeatedly taking $\mathbf{P}$ of the above equation and dividing by $\rho^2$ we obtain the following necessary and sufficient conditions for $H_{AB}^*_{AB'}$, to be an $H$-potential of a Lanczos potential of the Weyl spinor.

\begin{align*}
0 &= L_0^o \\
0 &= \tilde{\partial}^' H_{22}' + \frac{3}{2} \tilde{\partial}^' L_5 - 6L_0^o \tilde{\partial}^' \Omega^o \\
0 &= \tilde{\partial}^2 H_{12}' + H_{22}^o \tilde{\partial}^' \Omega^o + \frac{1}{3} \tilde{\partial}^3 L_4 - 2L_0^o \tilde{\partial}^2 \Omega^o - \frac{3}{2} \tilde{\partial}^3 \Omega^o \tilde{\partial}^' L_5 + \frac{1}{3} \Psi_3^o \\
0 &= \tilde{\partial}^3 H_{02}' + 2H_{12}^o \tilde{\partial}^2 \Omega^o + 6\tilde{\partial}^3 \Omega^o \tilde{\partial}^' H_{12}' - L_0^o \tilde{\partial}^3 \Omega^o - 2\tilde{\partial}^2 \Omega^o \tilde{\partial}^' L_4^o \\
&= -9L_0^o (\tilde{\partial}^' \Omega^o)^2 + \frac{1}{2} \tilde{\partial}^' \Psi_2^o - \Omega^o \Psi_3^o - \Phi_2^1,
\end{align*}

(33)

Now, because $[\mathbf{P},\tilde{\partial}^'] = 0$ the second of these equations must have a local solution, $H_{22}'$. By substituting this solution into the third equation, a local solution, $H_{12}^o$, of this equation must exist, and similarly the fourth equation must have a local solution $H_{02}'$. Thus, a Lanczos potential of the Weyl spinor has an $H$-potential that is aligned to $\sigma^A$ if and only if $L_0^o = 0$.

Summing up, we have proved the following result:

**Theorem 3.1** For any spacetime of class $G$ with spinor dyad in standard form, all $H$-potentials that are aligned to $\sigma^A$, of Lanczos potentials of the Weyl spinor, are given by

\begin{align*}
H_{02} &= \frac{\tilde{\partial}}{\rho} L_0^o + \tilde{\rho} H_{02}' \\
H_{12} &= \frac{3}{2} \frac{\tilde{\partial}}{\rho^2} L_5^o + \frac{\tilde{\rho}}{\rho} H_{12}^o - \frac{1}{2} \tilde{\rho} (\tilde{\partial}^' H_{02}' - L_0^o \tilde{\partial}^' \Omega^o) \\
H_{22} &= \frac{3}{\rho^3} L_0^o + \frac{\tilde{\rho}}{\rho^2} H_{22}^o - \frac{1}{2} \rho (\tilde{\partial}^2 H_{12}' + \tilde{\partial}^3 L_4 + 9L_0^o \tilde{\partial}^2 \Omega^o) + \frac{1}{4} \rho \Psi_2^o \\
&\quad + \frac{1}{2} \rho \tilde{\rho} (\tilde{\partial}^2 H_{02}' + 2H_{12}^o \tilde{\partial}^' \Omega^o - L_0^o \tilde{\partial}^2 \Omega^o - \tilde{\partial}^' \Omega^o \tilde{\partial}^' L_4 + \frac{1}{2} \Psi_2^o) \\
&\quad + \frac{1}{2} \rho \tilde{\rho} \Phi_{11}^1,
\end{align*}

(34)

The functions of integration $L_4^o, L_5^o, L_0^o, H_{02}', H_{12}'$ and $H_{22}'$ are subject to the conditions

\begin{align*}
0 &= \tilde{\partial}^2 L_6^o \\
0 &= \tilde{\partial}^3 L_5^o + 2L_0^o \tilde{\partial}^2 \Omega^o + 6\tilde{\partial}^3 \Omega^o \tilde{\partial}^' L_6^o + \frac{1}{3} \Psi_4^o \\
0 &= \tilde{\partial}^4 L_4^o + 3L_0^o \tilde{\partial}^3 \Omega^o + 12\tilde{\partial}^2 \Omega^o \tilde{\partial}^' L_4^o + 18\tilde{\partial}^3 \Omega^o \tilde{\partial}^2 L_6^o + 18L_0^o (\tilde{\partial}^' \Omega^o)^2 \\
&\quad + \tilde{\partial}^3 \Psi_3^o - 3\Omega^o \Psi_4^o
\end{align*}

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\[ 0 = \frac{\partial'}{\partial} H^2 + \frac{3}{2} \partial'^2 L^2 - 6 L^2 \partial' \Omega^2 \]

\[ 0 = \frac{\partial'}{\partial} H_1 + H_2 \partial' \Omega^2 + \frac{1}{3} L_3 \partial' L^2 - 2 L_3 \partial'^2 \Omega^2 - \frac{3}{2} \partial' \Omega^2 L^2 + \frac{1}{3} \Psi^2 \]

\[ 0 = \frac{\partial'}{\partial} H_0 + 2 H_1 \partial' \Omega^2 + 9 \partial' \Omega^2 \partial' H_1 - L_3 \partial'^3 \Omega^2 - 2 \partial'^3 \Omega^2 \partial' L^2 \]

\[-9 L_3 (\partial' \Omega^2)^2 + \frac{1}{2} \partial' \Psi^2 - \Omega^2 \Psi^2 - \Phi^2 \]

(35)

and in particular, there always exists a local H-potential that is aligned to \( o^A' \).

We also note that the Lanczos scalars of the Lanczos potentials obtained in this theorem, are given by (28) with \( L^2 = 0 \) and for future reference, we also note that a simple particular solution of the first equation is \( L^2 = -\Lambda \).

4 Lanczos potentials and curvature-free connections

4.1 Riemann-Cartan equations

It is well-known [19] that given any spinor \( \Gamma_{ABCA'} = \Gamma_{(AB)CA'} \) we can define a metric connection \( \hat{\nabla}_{AA'} \) by the equation

\[ \hat{\nabla}_{AA'} \xi^B = \nabla_{AA'} \xi^B + 2 \Gamma_{C' AA'} \xi^C \]

(36)

and providing \( \Gamma_{ABCA'} \neq 0 \) the connection \( \hat{\nabla}_{AA'} \) will have non-zero torsion. The curvature of such a connection can be described by its curvature spinors \( \hat{\Psi}_{ABCD}, \hat{\Phi}_{ABA'B'} = \hat{\Phi}_{(AB)(A'B')}, \hat{\Sigma}_{AB} = \hat{\Sigma}_{(AB)} \) and \( \Lambda \), through the formula [1], [3]

\[ \hat{R}_{abcd} = \varepsilon_{A'B'C'D'} \left[ \hat{\Psi}_{ABCD} + 2 (\varepsilon_{B(C\hat{\Sigma}_{CD})A} + \varepsilon_{A(C\hat{\Sigma}_{CD})B}) + \hat{\Phi}_{ABCD} \right] \left[ \varepsilon_{A'B'C'D'} \right] + c.c \]

(37)

where \( c.c \) stands for the complex conjugate of the entire expression.

Note that if the torsion is non-zero then \( \hat{\Phi}_{ABA'B'} \) and \( \Lambda \) are in general complex quantities and \( \hat{\Sigma}_{AB} \) is in general non-zero.

The curvature spinors of \( \hat{\nabla}_{AA'} \) are related to the curvature spinors of \( \nabla_{AA'} \), [1], [3]

\[ \hat{\Psi}_{ABCD} = \Psi_{ABCD} - 2 \nabla_{(A} E_{E' BCD)E'} - 4 \Gamma_{E(AB)E'} E_{E' CD)}E' + 1 \sum_{E'} \sum_{E''} \Gamma_{E' E''} \Gamma_{E''} \]

\[ \hat{\Lambda} = \Lambda - \frac{1}{3} \nabla_{E' E''} \Gamma_{E' E''} - \frac{1}{3} \Gamma_{EFGE} \Gamma_{E' E''} + \frac{1}{3} \Gamma_{EFGE} \Gamma_{E' E''} \]

\[ \hat{\Sigma}_{AB} = \frac{1}{4} \nabla_{E' E''} \Gamma_{E(AB)E'} - \frac{1}{4} \nabla_{(A} E_{E' B)E'} - \frac{1}{2} \Gamma_{E(AB)E'} \Gamma_{E' E''} \]

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Now, $\Gamma_{ABCA'}$ can be decomposed into a symmetric $(3,1)$-spinor $L_{ABCA'}$ and a complex covector $\chi_{AA'}$ according to

$$\Gamma_{ABCA'} = L_{ABCA'} + \varepsilon_{AC} \chi_{BA'} + \varepsilon_{BC} \chi_{AA'}$$

(39)

where $L_{ABCA'} = \Gamma_{(ABC)A'}$ and $\chi_{AA'} = \frac{1}{3} \Gamma_{ABB'}A'$. It can then be shown [1] that the curvature spinors of $\hat{\nabla}_{AA'}$ can be expressed as

$$\hat{\Psi}_{ABCD} = \Psi_{ABCD} - 2\nabla_{(A} \epsilon_{E'F'} L_{BCD)E'} - 8\chi_{(A} \epsilon_{E'F'} L_{BCD)B'} + 4L_{(AB \epsilon_{E'} L_{CD)F'} + 4\epsilon_{EF'F''} \chi_{EE'}$$

$$\hat{\Lambda} = \Lambda - \nabla_{E'F'} \chi_{EE'} - \frac{1}{3} \epsilon_{EFF'} \epsilon_{E'F'} + 4\chi_{EF'} \chi_{EE'}$$

$$\hat{\Sigma}_{AB} = \frac{1}{4} \nabla_{E'F'} \epsilon_{E'F'} + \nabla_{(A} \epsilon_{B')}E' + 3L_{AB \epsilon_{E'} E'}$$

$$\hat{\Phi}_{ABA'B'} = \hat{\Phi}_{ABA'B'} - 2\nabla_{(A} \epsilon_{E'F'} \epsilon_{B'|E')} + 2\nabla_{(A} \epsilon_{A} \epsilon_{B')} + 2\nabla_{(A} \epsilon_{B')} + 2\nabla_{(A} \epsilon_{B')} + 8\epsilon_{E'F'} \epsilon_{E'}$$

$$+ 16\epsilon_{E'F'} \epsilon_{E'}$$

(40)

We note that the corresponding equation in both [1] and [3] unfortunately contains a misprint in the coefficient of the last term. These equations will be used to find connections on the spacetimes studied in the previous sections, that are curvature-free and for which $L_{ABCA'}$ is a Lanczos potential of the Weyl spinor that is aligned to $oA'$.}

### 4.2 Kerr-Schild spacetimes, Lanczos potentials, curvature-free connections and quasi-local momentum

In [11] Bergvist and Ludvigsen study the Kerr spacetime. It is known to be a special case of a Kerr-Schild spacetime, i.e., its metric can be written

$$g_{ab} = \eta_{ab} + 2f l_{a} l_{b}$$

(41)

where $\eta_{ab}$ is a flat metric, $l^{a} = o^{A} o^{A'}$ is a null vector that, in the Kerr case, is geodesic and shear-free and $f$ is a real function that can be written

$$f = \frac{\rho + \bar{\rho}}{4\rho} \Psi_{2}$$

(42)

in the Kerr case. If we put

$$H_{ABA'B'} = f o_{A} o_{B} o_{A'} o_{B'} = f l_{a} l_{b}$$

(43)
it was shown in \[11\] that the spinor

\[
\Gamma_{ABCA^\prime} = \nabla_{(A} B^\prime H_{B)CA^\prime B^\prime}
\]
defines a metric connection with non-zero torsion, but vanishing curvature, i.e., \( \hat{R}_{abcd} = 0 \). In \[9\] it was subsequently shown that the spinor

\[
L_{ABCA^\prime} = \Gamma_{(ABC)A^\prime} = \nabla_{(A} B^\prime H_{BC)A^\prime B^\prime}
\]
is a Lanczos potential of the Weyl spinor that is aligned to \( o^{A^\prime} \).

These results were generalized in \[13\] and \[3\]. The final result is that in any Kerr-Schild spacetime where \( r^a = o^a o^{A^\prime} \) is geodesic and shear-free, the above construction yields a metric, asymmetric, curvature-free connection \( \hat{\nabla}_{AA^\prime} \) with the property that \( L_{ABCA^\prime} = \Gamma_{(ABC)A^\prime} \) is a Lanczos potential of the Weyl spinor that is aligned to \( o^{A^\prime} \).

In \[10\], \[11\] Bergqvist and Ludvigsen used the curvature-free connection \( \hat{\nabla}_{AA^\prime} \) described previously, to define quasi-local momentum in the Kerr spacetime. In this section we will review this construction.

That \( \hat{\nabla}_{AA^\prime} \) is curvature-free means that it is integrable, i.e., parallel propagation is path independent. From this fact we can easily prove that the spinor fields that satisfy the equation

\[
\hat{\nabla}_{AA^\prime} \xi_B = 0 \tag{44}
\]
form a 2-dimensional vector space over the complex numbers. We will call this vector space of spinor fields \( S \) (with indices according to the abstract index notation \[19\] when appropriate). For a spinor field \( \xi_A \in S_A \) we define the spinor

\[
\varphi_{AB} = \xi_{(A} \nabla_{B)} C^C \bar{\xi}_{C^\prime} - \bar{\xi}_{C^\prime} \nabla_{(A} C^C \xi_{B)} \tag{45}
\]
and the (antisymmetric) 2-form

\[
F_{ab} = i (\varepsilon_{AB} \varphi_{A^\prime B^\prime} - \varepsilon_{A^\prime B^\prime} \varphi_{AB}) \tag{46}
\]
Bergqvist and Ludvigsen prove that \( F_{ab} \) is actually a closed 2-form, i.e., \( \nabla_{[a} F_{b]c} = 0 \). Given a spacelike 2-surface \( \Sigma \) they then define the quasi-local momentum \( P_{AA^\prime}(\Sigma) \) as a 1-form on the hermitian part of \( S^A \otimes \bar{S}^{A^\prime} \), by the equation

\[
P_{AA^\prime}(\Sigma) \xi_A \bar{\xi}_{A^\prime} = \frac{1}{8 \pi} \int_{\Sigma} F \tag{47}
\]
This defines the action of \( P_{AA^\prime}(\Sigma) \) on null vector fields in the hermitian part of \( S^A \otimes \bar{S}^{A^\prime} \) and by linearity its action is then defined on all of the hermitian part of \( S^A \otimes \bar{S}^{A^\prime} \). We note that this definition is genuinely quasi-local as we have made no reference to the asymptotic properties of the Kerr spacetime. \( P_{AA^\prime}(\Sigma) \) can also be shown to, in a certain sense, agree with the Bondi momentum when \( \Sigma \) is a cross section of future null infinity.
4.3 Connections and Lanczos potentials in class $\mathcal{G}$ spacetimes that are aligned to $o^A$

4.3.1 Connections for which $\hat{\Psi}_{ABCD} = 0$, $\hat{\Sigma}_{AB} = 0$

We will now give a similar construction using the Lanczos potentials and $H$-potentials that were found in the previous sections, as in the Kerr-Schild case. Thus, suppose once again that we have an arbitrary class $\mathcal{G}$ spacetime with spinor dyad in standard form.

If we choose $H_{ABA'B'}$ to be aligned to $o^A'$ then, as is already shown, $L_{ABCA'}$ will automatically be aligned to $o^A'$. In a similar way, it is easy to show that $\chi_{AA'} = \lambda_A o_A'$ for some spinor $\lambda_A$. It automatically follows that all the product terms in the first three equations of (40) vanish. Moreover, if we choose $H_{ABA'B'}$ as in Theorem 3.1, so that $L_{ABCA'}$ is a Lanczos potential of the Weyl spinor, it is easily seen that $\Psi_{ABCD} = 0$. Hence, we immediately get the result.

**Proposition 4.1** Let $H_{ABA'B'}$ be as in Theorem 3.1. Then the spinor

$$\Gamma_{ABCA'} = \nabla_{\langle A B'} H_{B|C'A'B'}$$

defines a metric connection $\hat{\nabla}_{AA'}$ through the equation

$$\hat{\nabla}_{AA'} \xi^B = \nabla_{AA'} \xi^B + 2 \Gamma_{C'B'A'} \xi^C$$

that is $\hat{\Psi}$-flat, i.e., $\hat{\Psi}_{ABCD} = 0$.

We will next choose a particular class of $H$-potentials that will ensure that the curvature spinor $\hat{\Sigma}_{AB}$ vanishes. We will do this in two steps. First we will $\rho$-integrate the GHP-version of the corresponding equations from (40) to get $\chi_{AA'}$. Then we note that from the definition of $\chi_{AA'}$ we have

$$\chi_{AA'} = \frac{1}{3} \Gamma_{AB}^{B'} A'A' = -\frac{1}{6} \nabla^{B'B'} H_{ABA'B'},$$

(48)

so we then substitute our expressions for the various quantities into the GHP-version of this equation to get the possible choices for $H_{ABA'B'}$.

Hence, first we wish to solve the equations

$$0 = \hat{\Sigma}_{AB} = \frac{1}{4} \nabla^{EE'} L_{ABEE'} + \nabla_{\langle A E'} \chi_{B)E'},$$

(49)

We note that since by assumption $\chi_{AA'} = \lambda_A o_A'$ it has only two non-vanishing components, namely

$$\chi_{01'} = \chi_{AA'} o^A l^A'$$

$$\chi_{11'} = \chi_{AA'} l^A A'$$
Then the GHP-version of (49) becomes

\[
0 = -\hat{\Phi} \chi_{01'} + \hat{\rho} \chi_{01'} + \frac{1}{4}(\hat{\Phi} L_5 - \rho \hat{\delta}' L_4 - (3\rho + \hat{\rho}) L_5)
\]

(50)

\[
0 = -\hat{\Phi} \chi_{11'} - \rho \hat{\delta}' \chi_{01'} - (\rho - \hat{\rho}) \chi_{11'} + \frac{1}{2}(\hat{\Phi} L_6 - \rho \hat{\delta}' L_5 - (2\rho + \hat{\rho}) L_6)
\]

(51)

\[
0 = -\rho \hat{\delta}' \chi_{11'} + \frac{1}{4}(\hat{\Phi} L_7 - \rho \hat{\delta}' L_6 - (\rho + \hat{\rho}) L_7)
\]

(52)

By using the Weyl-Lanczos equations we can eliminate \(L_i', i = 4, 5, 6\) from the above equations and by substituting the expressions from Section 3 for the Lanczos scalars it is possible to \(\rho\)-integrate the first two of these equations,

\[
\chi_{01'} = \frac{\hat{\delta}}{\rho} L_5^0 + \hat{\rho} \chi_{01'}^0
\]

\[
\chi_{11'} = 2 \frac{\hat{\rho}}{\rho^2} L_6^0 + \frac{\hat{\delta}}{\rho} \chi_{11'}^0 - \hat{\rho}(\hat{\delta}' \chi_{01'}^0 - L_5^0 \hat{\delta}' \Omega^\circ) + \frac{1}{12} \hat{\rho} \hat{\rho} \Psi_2^0.
\]

(53)

We now need to substitute this into the third equation, but before we do that we will temporarily drop the assumption that \(\chi_{AA'} = \lambda_A \Omega_A\) so that we allow for a non-zero \(L_2^0\). Then the third equation becomes

\[
0 = L_7^0 + \rho^2(\hat{\delta}' \chi_{311'} + \frac{1}{2} \hat{\delta}' \hat{\delta}^2 L_5^0 - 3L_5^0 \hat{\delta}' \Omega^\circ)
\]

\[-\rho^3(\hat{\delta}' \chi_{011'} + \chi_{111'} \hat{\delta}' \Omega^\circ - L_5^0 \hat{\delta}' \hat{\delta}' \Omega^\circ - \hat{\delta}' \Omega^\circ \hat{\delta}' L_3^0 + \frac{1}{6} \Omega^\circ)
\]

(54)

By identifying coefficients in the same way as in the previous sections we obtain the conditions

\[
0 = L_7^0
\]

\[
0 = \hat{\delta}' \chi_{311'}^0 + \frac{1}{2} \hat{\delta}' \hat{\delta}^2 L_5^0 - 3L_5^0 \hat{\delta}' \Omega^\circ
\]

\[
0 = \hat{\delta}' \chi_{011'}^0 + \chi_{111'} \hat{\delta}' \Omega^\circ - L_5^0 \hat{\delta}' \hat{\delta}' \Omega^\circ - \hat{\delta}' \Omega^\circ \hat{\delta}' L_3^0 + \frac{1}{6} \Omega^\circ
\]

(55)

By the commutator \([\hat{\Phi}, \hat{\delta}] = 0\) it follows that we can solve the second of these equations for \(\chi_{111'}^0\), substitute the result into the third equation and solve it for \(\chi_{011'}^0\). Hence, it follows that we can choose \(\chi_{AA'}\) so that \(\Sigma_{AB} = 0\) if and only if \(L_7 = 0\). Recall from the previous section that our Lanczos potential \(L_{ABCA'}\) possessed an \(H\)-potential if and only if \(L_7 = 0\) so the Lanczos potentials that allow us to obtain a connection of the above type, with \(\Sigma_{AB} = 0\) are precisely the Lanczos potentials that possess an \(H\)-potential that is aligned to \(\omega^{\chi^1}\). However, it remains to be seen whether the \(H\)-potential can be chosen so that \(\chi_{AA'} = -\frac{1}{6} \nabla^{BB'} H_{ABA'B'}\), i.e., so that

\[
\Gamma_{ABCA'} = \nabla(A_B H_{(B)})_{CA'B'}
\]

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This will be the topic of our next investigation.

The GHP-version of the equation $\chi_{AA'} = -\frac{1}{6}\nabla^B B' H_{ABA'B'}$, as two of the four equations are identically satisfied, is

\begin{align*}
6\chi_{01'} &= -b H_{02'} + \rho \tilde{\partial}' H_{02'} + (2\rho + \tilde{\rho}) H_{12'} \quad (56) \\
6\chi_{11'} &= -b H_{22'} + \rho \tilde{\partial}' H_{12'} + (\rho + \tilde{\rho}) H_{22'} \\
\end{align*}

We can use the equations (30) to eliminate the quantities $\tilde{\partial}' H_{02'}$ and $\tilde{\partial}' H_{12'}$ from these equations and we can substitute the expressions for the Lanczos-, $H$- and $\chi$-scalars obtained previously, into these equations. Then they become, after some simplification

\begin{align*}
0 &= 2\chi^{\circ}_{01'} - H^{\circ}_{12'} - \frac{1}{3} \tilde{\partial}' L_4^{\circ} \\
0 &= 2\chi^{\circ}_{11'} - H^{\circ}_{22'} - \frac{1}{2} \tilde{\partial}' L_5^{\circ} - \rho(2\tilde{\partial}' \chi^{\circ}_{01'} - \tilde{\partial}' H_{12'} - \frac{1}{3} \tilde{\partial}' L_4^{\circ})
\end{align*}

We see that if the first of these conditions is satisfied, then the expression within parenthesis in the second is identically zero. Hence, the conditions simplify to

\begin{align*}
\chi^{\circ}_{01'} &= \frac{1}{2} H^{\circ}_{12'} + \frac{1}{6} \tilde{\partial}' L_4^{\circ} \\
\chi^{\circ}_{11'} &= \frac{1}{2} H^{\circ}_{22'} + \frac{1}{4} \tilde{\partial}' L_5^{\circ}
\end{align*}

(58)

We have chosen the $H$-scalars to satisfy (33). Thus, we need to check that the $\chi$-scalars defined by (58) satisfy (59). We obtain, according to (33)

\begin{align*}
0 &= \frac{1}{2} \left( \tilde{\partial}' H^{\circ}_{22'} + \frac{3}{2} \tilde{\partial}' L_5^{\circ} - 6 L_6^{\circ} \tilde{\partial}' \Omega^{\circ} \right) \\
&= \tilde{\partial}' \left( \frac{1}{2} H^{\circ}_{22'} + \frac{1}{4} \tilde{\partial}' L_5^{\circ} \right) + \frac{1}{2} \tilde{\partial}' L_5^{\circ} - 3 L_6^{\circ} \tilde{\partial}' \Omega^{\circ} \\
&= \tilde{\partial}' \chi_{11'} + \frac{1}{2} \tilde{\partial}' L_5^{\circ} - 3 L_6^{\circ} \tilde{\partial}' \Omega^{\circ}
\end{align*}

which is precisely (59). For $\chi^{\circ}_{01'}$ we obtain

\begin{align*}
0 &= \frac{1}{2} \left( \tilde{\partial}' H^{\circ}_{12'} + H^{\circ}_{22'} \tilde{\partial}' \Omega^{\circ} - 2 L_5^{\circ} \tilde{\partial}' \Omega^{\circ} - \frac{3}{2} \tilde{\partial}' \Omega^{\circ} \tilde{\partial}' L_5^{\circ} + \frac{1}{3} \tilde{\partial}' L_4^{\circ} + \frac{1}{3} \Psi_3^{\circ} \right) \\
&= \tilde{\partial}' \left( \frac{1}{2} H^{\circ}_{12'} + \frac{1}{6} \tilde{\partial}' L_4^{\circ} \right) + \left( \frac{1}{2} H^{\circ}_{22'} + \frac{1}{4} \tilde{\partial}' L_5^{\circ} \right) \tilde{\partial}' \Omega^{\circ} - L_6^{\circ} \tilde{\partial}' \Omega^{\circ} \\
&= \tilde{\partial}' \chi_{01'} + \chi^{\circ}_{11'} \tilde{\partial}' \Omega^{\circ} - L_5^{\circ} \tilde{\partial}' \Omega^{\circ} - \tilde{\partial}' \Omega^{\circ} \tilde{\partial}' L_5^{\circ} + \frac{1}{6} \Psi_3^{\circ}
\end{align*}

which is also condition (55).

The following result can now easily be proved:
Theorem 4.2 In spacetimes of class $\mathcal{G}$, a spinor $\Gamma_{ABC} = \Gamma_{(AB)CA'}$ that is aligned to $o^A$ (i.e., $\Gamma_{ABC} = N_{ABCOA'}$ with $N_{ABC} = N_{(AB)C}$) whose symmetric part $L_{ABC} = \Gamma_{(ABC)A'}$ is a Lanczos potential of the Weyl spinor, defines a connection $\hat{\nabla}_{AA'}$ for which $\hat{\Psi}_{ABCD} = 0$ and $\hat{\Sigma}_{AB} = 0$ if and only if it can be written

$$\Gamma_{ABC} = \nabla_{(A'B'}H_{B)CA'B'}$$

for some spinor $H_{ABA'B'} = Q_{ABOAOB'}$ with $Q_{AB} = Q_{(AB)}$. With a dyad in standard form, the Lanczos- and $\hat{H}$-scalars for these spinors are given by (28), (34) and (35) with $L_{\odot 7} = 0$. The $\chi$-scalars are given by (53) and (58).

Proof: Suppose the $\chi$- and $\hat{H}$-scalars are related as in (58), i.e., $\chi_{AA'} = -\frac{1}{6} \nabla_{EE'} H_{ABA'B'}$. Then the above calculations prove that the conditions (55) and (33) are equivalent. Since (55) is equivalent to the vanishing of $\hat{\Sigma}_{AB}$ and since (33) is equivalent to $H_{ABA'B'}$ being an $H$-potential of a Lanczos potential of the Weyl spinor, the theorem follows.

We also note that, in particular, it follows that such spinors $\Gamma_{ABC}A'$ and $H_{ABA'B'}$ exist in every spacetime of class $\mathcal{G}$. We remark that this partial result was proved in [1] using a particular construction of Lanczos potentials by Torres del Castillo. [21], [22].

4.3.2 Connections for which $\hat{\Lambda} = 0$

We will now check whether our choice of $H$-potential also allows us to put $\hat{\Lambda} = 0$. According to (40) the condition for this is

$$0 = \Lambda - \nabla_{EE'} \chi_{EE'} = \Lambda - \frac{i}{2} \chi_{11'} + \rho \tilde{\partial} \chi_{01'} + (\rho + \bar{\rho}) \chi_{11'}.$$

By using the expression (53) for the $\chi$-scalars, we arrive at the condition

$$0 = L_{6}^0 + \Lambda + \rho(2\chi_{11'} + \tilde{\partial}^2 L_{5}^0 + \Omega^\circ \Lambda)$$

By identifying coefficients in the usual way we obtain the result that $\hat{\Lambda} = 0$ if and only if

$$0 = L_{6}^0 + \Lambda$$

$$0 = 2\chi_{11'} + \tilde{\partial}^2 L_{5}^0 + \Omega^\circ \Lambda$$

(59)

As remarked above, the first of these conditions satisfies the $L_{\odot 5}$-equation of (27) identically, as we have already chosen $L_{\odot 7} = 0$ in order to get $\hat{\Sigma}_{AB} = 0$ and in order to obtain an $H$-potential of $L_{ABC}A'$ and we assumed that $\Lambda$ is constant.

We now check the second condition by substituting it into (53)

$$0 = 2\tilde{\partial} \chi_{11'} + \tilde{\partial}^2 L_{5}^0 - 6L_{5}^0 \tilde{\partial} \Omega^\circ = \tilde{\partial}^2 (-\tilde{\partial} L_{5}^0 - \Omega^\circ \Lambda) + \tilde{\partial}^2 L_{5}^0 + 6\Lambda \tilde{\partial} \Omega^\circ = 5\Lambda \tilde{\partial} \Omega^\circ.$$
Hence, it is satisfied if and only if at least one of the conditions $\Lambda = 0$ and $\tilde{\Omega} = 0$ is satisfied. The second of these conditions is easily seen to be equivalent to the perhaps more familiar looking GHP-condition $\tilde{\rho} = 0$ which is satisfied, e.g., if $\rho = \bar{\rho}$.

If we now define $H$-scalars according to (58) it is clear that the conditions (13), for $H_{ABA'B'}$ to be an $H$-potential of $L_{ABCA'}$, are also identically satisfied if and only if $\Lambda = 0$ or $\tilde{\Omega} = 0$.

Substituting (59) into the equations (34) and (35) and using that $\Lambda\tilde{\Omega} = 0$ proves the following result:

**Lemma 4.3** Given a spacetime of class $\mathcal{G}$, there exists a spinor $H_{ABA'B'} = Q_{AB\theta A'B'}$, $Q_{AB} = Q_{(AB)}$ such that the spinor

$$\Gamma_{ABCA'} = \nabla_{(A} B'_{H]CA'B'}$$

defines a metric, asymmetric connection for which

$$\Psi_{ABCD} = 0, \quad \Sigma_{AB} = 0, \quad \hat{\Lambda} = 0$$

if and only if $\Lambda = 0$ or $\tilde{\Omega} = 0$ ($\Leftrightarrow \tilde{\rho} = 0$). All such spinors $H_{ABA'B'}$ are given by

\[
\begin{align*}
H_{02'} &= \frac{\bar{\rho}}{\rho} L^2 + \bar{\rho} H^2_{02'} \\
H_{12'} &= \frac{3}{2} \frac{\bar{\rho}}{\rho^2} L^2 + \frac{\bar{\rho}}{\rho} H^2_{12'} - \frac{1}{2} \left( \tilde{\rho} \tilde{H}^0_{02'} - L^2_4 \tilde{\rho} \tilde{\Omega} \right) \\
H_{22'} &= -3 \frac{\bar{\rho}}{\rho^3} \Lambda - \frac{\bar{\rho}}{\rho^2} \left( \frac{3}{2} \tilde{\rho} L^2_3 + \tilde{\Omega}^2 \Lambda \right) - \frac{1}{2} \left( \tilde{\rho} \tilde{H}^0_{12'} - L^2_4 \tilde{\rho} \tilde{\Omega} \right) + \frac{1}{2} \rho \Omega \Psi_2 + \frac{1}{2} \tilde{\rho} \tilde{H}^0_{12'} + 2 H^2_{12'} \tilde{\rho} \tilde{\Omega} \tilde{\Omega} - 2 L^2_4 \tilde{\rho} \tilde{\Omega} - \tilde{\rho} \tilde{\Omega} \tilde{\Omega} L^2_4 + \frac{1}{2} \Omega \Psi_2 \\
&= \frac{1}{2} \tilde{\rho} \tilde{\rho} \tilde{H}^0_{12'} + \frac{1}{2} \rho \Omega \Psi_2 + \frac{1}{2} \tilde{\rho} \tilde{H}^0_{12'} + 2 H^2_{12'} \tilde{\rho} \tilde{\Omega} \tilde{\Omega} - 2 L^2_4 \tilde{\rho} \tilde{\Omega} - \tilde{\rho} \tilde{\Omega} \tilde{\Omega} L^2_4 + \frac{1}{2} \Omega \Psi_2 \\
&= \frac{1}{2} \frac{\bar{\rho}}{\rho^3} \tilde{\rho} \tilde{H}^0_{12'} + \frac{1}{2} \tilde{\rho} \tilde{H}^0_{12'} + 2 H^2_{12'} \tilde{\rho} \tilde{\Omega} \tilde{\Omega} - 2 L^2_4 \tilde{\rho} \tilde{\Omega} - \tilde{\rho} \tilde{\Omega} \tilde{\Omega} L^2_4 + \frac{1}{2} \Omega \Psi_2 \tag{60}
\end{align*}
\]

where $L^2_4, L^2_5, H^2_{02'}$ and $H^2_{12'}$ are subject to the conditions

\[
\begin{align*}
0 &= \tilde{\rho} L^2_5 + \frac{1}{3} \Psi^2_4 \\
0 &= \tilde{\rho}^2 L^2_4 + 3 L^2_5 \tilde{\rho} \tilde{\Omega} + 12 \tilde{\rho} \tilde{\Omega} \tilde{\rho} L^2_5 + 18 \tilde{\rho} \tilde{\Omega} \tilde{\rho} L^2_5 + 3 \tilde{\rho} \tilde{\Omega} \tilde{\rho} - 3 \tilde{\rho}^2 \tilde{\Omega} \tilde{\rho} \\
0 &= \tilde{\rho}^2 H^0_{12'} + \frac{1}{3} \tilde{\rho} \tilde{\rho} L^2_4 - 2 L^2_5 \tilde{\rho} \tilde{\Omega} - 3 \tilde{\rho} \tilde{\Omega} \tilde{\rho} L^2_5 + \frac{1}{3} \Omega \Psi^2_4 \\
0 &= \tilde{\rho}^3 H^0_{02'} + 2 H^0_{12'} \tilde{\rho} \tilde{\Omega} + 6 \tilde{\rho} \tilde{\Omega} \tilde{\rho} H^0_{12'} - L^2_4 \tilde{\rho} \tilde{\Omega} - 2 \tilde{\rho} \tilde{\Omega} \tilde{\Omega} L^2_4 \\
&= -9 L^2_5 (\tilde{\rho} \tilde{\Omega})^2 + \frac{1}{2} \tilde{\rho} \tilde{\rho} \tilde{\rho} - \tilde{\rho} \tilde{\rho} \tilde{\rho} - \frac{1}{2} \tilde{\rho} \tilde{\rho} \tilde{\rho} \sum_{21} \tag{61}
\end{align*}
\]
4.3.3 The Ricci spinor of $\hat{\nabla}_{AA'}$

In this section we will consider the Ricci spinor of $\hat{\nabla}_{AA'}$. We will therefore assume that $L_{ABCA'}$ and $\chi_{AA'}$ are both aligned to $o^{A'}$ and have been chosen to give $\tilde{\Psi}_{ABCD} = 0, \Sigma_{AB} = 0, \Lambda = 0$ in a spacetime of class $\mathcal{G}$ with dyad in standard form. Thus, in particular we assume that $\Lambda \tilde{\Omega}^5 = 0, L^5_6 = -\Lambda$. Put

$$M_{ABC} = L_{ABCA'}^{A'} = L_7o_0B_0C - 3L_6o_0(A_0B_1C) + 3L_5o_0(A_1B_1C) - L_4A_1B_1C$$

$$\lambda_A = \chi_{AA'}^{A'} = \chi_{11'}o_A - \chi_{01'}o_{A'}$$  \hspace{1cm} (62)

Then the complex conjugate of the fourth equation of (60) becomes

$$\tilde{\Phi}_{ABA'B'} = \Phi_{ABA'B'} - 2M_{ABE} \nabla_{(A'}^{(A'}o_{B')} - 2o_{(A'} \nabla_{B')}^{E}M_{ABE}$$

$$+ 4\lambda_{(A} \nabla_{B)(A'}^{(A'}o_{B')} + 4o_{(A'} \nabla_{B')}^{(A} \lambda_{B)} + 4M_{A}^{EF}M_{BEF}o_{A}o_{B'}$$

$$+ 8M_{ABE}^{(A} \lambda^{(A}Eo_{A}o_{B')} + 16\lambda_{A}^{(A} \lambda_{B}o_{A}o_{B')}$$  \hspace{1cm} (63)

Since $\hat{\Phi}_{ABA'B'}$ is in general non-hermitian it has 9 complex components defined according to the usual convention [19].

Since

$$o^{A'}o_{B'}\nabla_{A'A'}^{A'A'}o_{B'} = \hat{\sigma}o_A - \bar{\kappa}_A = 0,$$

it follows from (63) that $\hat{\tilde{\Phi}}_{ABA'B'}o^{A'}o^{B'} = 0$ so that

$$\hat{\Phi}_{00'} = \hat{\Phi}_{01'} = \hat{\Phi}_{02'} = 0.$$

The ‘next’ three components become, in GHP-formalism using Held’s modified operators

$$\hat{\Phi}_{10'} = \hat{\Phi}_{L_5} - (3\rho - \bar{\rho})L_5 - \rho \hat{\tilde{\sigma}}L_4 + 2\hat{\Phi}11' + 2\rho \hat{\Phi}01'$$

$$\hat{\Phi}_{11'} = \hat{\Phi}_{L_6} - (2\rho - \bar{\rho})L_6 - \rho \hat{\tilde{\sigma}}L_5 + \hat{\Phi}11' + (\rho + \bar{\rho})\chi_{11'} + \rho \hat{\tilde{\sigma}}\chi_{01'}$$

$$\hat{\Phi}_{12'} = \hat{\Phi}_{L_7} - (\rho - \bar{\rho})L_7 - \rho \hat{\tilde{\sigma}}L_6 + 2\rho \hat{\tilde{\sigma}}\chi_{11'}$$  \hspace{1cm} (64)

We use (21) and (32) to eliminate the terms containing $\hat{\tilde{\sigma}}$ and use our expressions for the curvature components, Lanczos scalars and $\chi$-scalars to obtain the following result from the first two equations of (64)

$$\hat{\Phi}_{10'} = 0, \hat{\Phi}_{11'} = 0$$

if and only if

$$\chi_{01'}^\circ = \frac{1}{6} \hat{\tilde{\sigma}} L_4^\circ + \frac{3}{2} \hat{\Omega}^5 L_5^\circ$$

$$\chi_{11'}^\circ = \frac{1}{2} \hat{\tilde{\sigma}} L_5^\circ - 3\hat{\Omega} L^\circ A + \rho(\hat{\tilde{\sigma}} \chi_{01'}^\circ - \frac{1}{6} \hat{\tilde{\sigma}}^2 L_4^\circ - \frac{3}{2} L_5^\circ \hat{\tilde{\sigma}} \hat{\Omega}^5 - \frac{3}{2} \hat{\Omega}^5 \hat{\tilde{\sigma}} L_5^\circ).$$  \hspace{1cm} (65)
respectively. We see that the expression within parenthesis vanishes identically if \( \Phi'_{10} = 0 \) so that we obtain

\[
\chi'_{11} = -\frac{1}{2} \partial' L_5^2 - 3\Omega^2 \Lambda.
\]

However, from (69) we have that

\[
\chi'_{11} = -\frac{1}{2} \partial' L_5^2 - \frac{1}{2} \Omega^2 \Lambda,
\]

so we obtain a necessary condition \( \Omega^2 \Lambda = 0 \).

Assuming the first two equations of (64) hold, the third is easily seen to be equivalent to

\[
0 = \partial' L_4^2 + 9\Omega^2 \partial' L_5^2 + 9\partial' \Omega^2 \partial' L_5^2 + 3L_5^2 \partial' \Omega^2 + \Psi_3^2,
\]

using our expressions for \( \Phi_{21}, \Psi_3 \) and \( L_7 \). This proves that

\[
\Phi_{10}' = \Phi_{11}' = \Phi_{12}' = 0
\]

if and only if our class \( G \) spacetime with dyad in standard form is such that \( \Lambda = 0 \) or \( \Omega^2 = 0 \) and in addition

\[
\chi_{50}' = \frac{1}{6} \partial' L_4^2 + \frac{3}{2} \Omega^2 L_5^2
\]

\[
\chi_{11}' = -\frac{1}{2} \partial' L_5^2
\]

\[
0 = \partial' L_4^2 + 9\Omega^2 \partial' L_5^2 + 9\partial' \Omega^2 \partial' L_5^2 + 3L_5^2 \partial' \Omega^2 + \Psi_3^2
\]

(67)

It remains to check that these choices satisfy the conditions from the previous chapters:

\[
0 = \partial' L_5^2 + \frac{1}{3} \Psi_4^4
\]

\[
0 = \partial' L_4^2 + 3L_5^2 \partial' \Omega^2 + 12\partial' \Omega^2 \partial' L_5^2 + 18\partial' \Omega^2 \partial' L_5^2 + \partial' \Psi_3^2 - 3\Omega^2 \Psi_4^2
\]

\[
0 = \partial' \chi_{50}' + \chi_{11} \partial' \Omega^2 - L_5^2 \partial' \Omega^2 - \partial' \Omega^2 \partial' L_5^2 + \frac{1}{6} \Psi_3^2
\]

(68)

We will now show that the equations (67) and the first equation of (68) implies the last two equations of (68). First it is easily verified that the second equation of (68) can be rewritten

\[
0 = \partial' (\partial' L_4^2 + 9\Omega^2 \partial' L_5^2 + 9\partial' \Omega^2 \partial' L_5^2 + 3L_5^2 \partial' \Omega^2 + \Psi_3^2) - 9\Omega^2 (\partial' L_5^2 + \frac{1}{3} \Psi_4^4)
\]

so it is indeed identically satisfied. Substituting the first two equations of (67) into the third equation of (68) it becomes, after simplification \( \frac{1}{6} \) times the third equation of (67) so it is also identically satisfied.
According to [58], any $H$-potential of the spinor $\Gamma_{ABCA'}$ must satisfy

$$H_{12}^o = 2\chi_{01'} - \frac{1}{3} \ddot{\theta}' L_4^o = 3\Omega^5 L_5^o$$
$$H_{22}^o = 2\chi_{11'} - \frac{1}{2} \ddot{\theta}' L_5^o = -\frac{3}{2} \ddot{\theta}' L_5^o$$

and in addition $H_{12}^o$ must satisfy

$$0 = \dddot{\theta}^2 H_{12}^o - 3 \dddot{\theta}' \Omega^5 \ddot{\theta}' L_5^o + \frac{1}{3} \dddot{\theta}^2 L_4^o - 2L_5^o \dddot{\theta}' \Omega^5 + \frac{1}{3} \Psi_3^o$$
$$= \frac{1}{3} (\dddot{\theta} L_4^o + 9\Omega^5 \dddot{\theta}' L_5^o + 9 \dddot{\theta}' \Omega^5 \dddot{\theta}' L_5^o + 3L_5^o \dddot{\theta}' \Omega^5 + \Psi_3^o)$$

which is identically satisfied. This proves the following result,

**Lemma 4.4** Given a spacetime of class $\mathcal{G}$ with dyad in standard form, there exists a spinor $H_{ABAB'}Q_{ABO'O'B'}, Q_{AB} = Q_{(AB)}$ such that the spinor

$$\Gamma_{ABCA'} = \nabla_{(A} H_{B)CA'B'}$$

defines a metric, asymmetric connection for which all curvature quantities vanish except $\Phi_{22'}$, $\Phi_{21'}$ and $\Phi_{22'}$ if and only if $\Lambda = 0$ or $\Omega^o = 0$ ($\Leftrightarrow \rho = \bar{\rho}$). All such spinors $H_{ABAB'}$ are given by

$$H_{02'} = \frac{\bar{\rho}}{\rho} L_4 + \bar{\rho} H_{02'}$$
$$H_{12} = \frac{3}{2} \frac{\bar{\rho}}{\rho} L_5^o + 3 \bar{\rho} \Omega^5 L_5^o - \frac{1}{2} \bar{\rho} (\ddot{\theta}' H_{02'} - L_4^o \dddot{\theta}' \Omega^5)$$
$$H_{22'} = -3 \frac{\bar{\rho}}{\rho} \Lambda - \frac{3}{2} \frac{\bar{\rho}}{\rho} \ddot{\theta}' L_5^o - \frac{1}{2} \bar{\rho} (\ddot{\theta}^2 L_4^o + 12 \Omega^5 \dddot{\theta}' L_5^o + 3L_5^o \dddot{\theta}' \Omega^5) + \frac{1}{4} \rho \Psi_2^o$$
$$+ \frac{1}{2} \bar{\rho} (\ddot{\theta}^2 H_{02'} + 6L_5^o \dddot{\theta}' \Omega^5 - L_4^o \dddot{\theta}' \Omega^5 - \dddot{\theta}' \Omega^5 \dddot{\theta}' L_4^o + \frac{1}{2} \Psi_2^o)$$
$$+ \frac{1}{2} \rho \bar{\rho} \Phi_1^o$$

where $L_4^o, L_5^o$ and $H_{02'}$ are subject to the conditions

$$0 = \dddot{\theta}^2 L_4^o + \frac{1}{3} \Psi_3^o$$
$$0 = \dddot{\theta}^2 L_5^o + 3L_5^o \dddot{\theta}' \Omega^5 + 9 \dddot{\theta}' \Omega^5 \dddot{\theta}' L_5^o + 9 \dddot{\theta}' \Omega^5 \dddot{\theta}' L_5^o + \Psi_3^o$$
$$0 = \dddot{\theta}^2 H_{02'} + 6L_5^o \dddot{\theta}' \Omega^5 + 18 \dddot{\theta}' \Omega^5 \dddot{\theta}' L_5^o - L_4^o \dddot{\theta}' \Omega^5 - 2 \dddot{\theta}' \Omega^5 \dddot{\theta}' L_4^o$$
$$+ 9L_5^o (\dddot{\theta}' \Omega^5)^2 + \frac{1}{2} \dddot{\theta}' \Psi_2^o - \Omega^5 \Psi_3^o - \Phi_21^o$$

(71)
The remaining components of equation (63) can now be written in GHP-formalism, using Held’s modified operators, as

\[ \Phi_{20'} = -2\tilde{\Phi}' L_4 + \left[ 2\rho' + \frac{3\Psi_2}{\rho} - \frac{\Psi_2}{\rho} + \frac{4\Lambda}{\rho} \right] L_4 + 2\tilde{\rho}\tilde{\partial}L_5 + 4\tilde{\rho}\tilde{\partial}\chi_{01'} \]
\[ + 8 \left( L_5 L_4 - L_5^2 + L_4^2 \chi_{11'} - L_5^2 \chi_{01'} + 2\chi_{11'}^2 \right) \]
\[ \Phi_{21'} = \Phi_{12} - 2\tilde{\Phi}' L_5 + \left[ 4\rho' + \frac{\Psi_2}{\rho} - \frac{\Psi_2}{\rho} + \frac{4\Lambda}{\rho} \right] L_5 + 2\tilde{\rho}\tilde{\partial}L_6 - 2\kappa' L_4 \]
\[ + 2\bar{\Phi}' \chi_{01'} - \frac{2\rho'}{\rho} \bar{\Phi}' \chi_{01'} + 2\tilde{\rho}\tilde{\partial}\chi_{11'} \]
\[ + 4 \left( L_4 L_7 - L_5 L_6 + 2L_5^2 \chi_{11'} - 2L_5 \chi_{01'} + 4\chi_{01'} \chi_{11'} \right) \]
\[ \Phi_{22'} = \Phi_{22} - 2\tilde{\Phi}' L_6 + \left[ 6\rho' + \frac{\Psi_2}{\rho} - \frac{\Psi_2}{\rho} + \frac{4\Lambda}{\rho} \right] L_6 + 2\tilde{\rho}\tilde{\partial}L_7 - 4\kappa' L_5 \]
\[ + 4\bar{\Phi}' \chi_{11'} + \left[ \frac{2\Psi_2}{\rho} + \frac{2\Psi_2}{\rho} + \frac{8\Lambda}{\rho} \right] \chi_{11'} + 4\kappa' \chi_{01'} \]
\[ + 8 \left( L_5 L_7 - L_6^2 + L_6^2 \chi_{11'} - L_7 \chi_{01'} + 2\chi_{11'}^2 \right) \]  \hspace{1cm} (72)

where we have used that \( \Omega^o \Lambda = 0 \) and hence that \( \frac{\Lambda}{\rho} = \frac{\Lambda}{\rho} \). At a first glance it seems unlikely that these equations can be solved since they are highly nonlinear, but we shall see that the situation is manageable. We will start by looking at the non-twisting case, i.e., \( \Omega^o = 0 \) (it is not necessary to make this separation into two cases \( \Omega^o = 0 \) and \( \Omega^o \neq 0 \), but it simplifies the calculations greatly). By substituting our previous equations into the first equation of (72), we find that \( \Phi_{20'} = 0 \) if and only if

\[ 0 = \Lambda^o L_4^o \]
\[ 0 = 3\partial L_5^o - \tilde{\Phi}' L_4^o - 6L_5^o \tilde{\partial}^o L_3^o + 6L_5^o \tilde{\partial} L_4^o \] \hspace{1cm} (73)

Continuing with the second equation of (72) we obtain that \( \Phi_{20'} = \Phi_{21'} = 0 \) if and only if

\[ 0 = \Lambda^o L_4^o \]
\[ 0 = \Lambda^o L_5^o \]
\[ 0 = 3\partial L_5^o - \tilde{\Phi}' L_4^o - 6L_5^o \tilde{\partial}^o L_3^o + 6L_5^o \tilde{\partial} L_4^o \] \hspace{1cm} (74)

After a very long calculation the last equation of (72) gives us that \( \Phi_{20'} = \Phi_{21'} = \Phi_{22'} = 0 \), and hence that \( R_{abcd} = 0 \) if and only if \( \Lambda = 0 \) and in addition

\[ 3\partial L_5^o - \tilde{\Phi}' L_4^o - 6L_5^o \tilde{\partial}^o L_3^o + 6L_5^o \tilde{\partial} L_4^o = 0 \]  \hspace{1cm} (75)

along with all other conditions derived previously. Before we look at the possibility of satisfying all the conditions we have obtained, we will also look at the
non-twisting case $\Omega^o \neq 0$. Then, by our previous conditions we already have $\Lambda = 0$. If we substitute our previous equations into the three equations (72), a very long calculation indeed reveals that $\Phi_{0'} = \Phi_{21'} = \Phi_{22'} = 0$, and hence that $\tilde{R}_{abcd} = 0$ if and only if

$$3\partial L^5_5 - \tilde{P}' L^5_5 - 6L^5_5 \tilde{\sigma}' L^5_5 + 6L^5_5 \tilde{\sigma}' L^5_4 + 18\Omega^o L^5_5 = 0$$

along with the previously derived conditions.

This proves the following result

**Theorem 4.5** In a spacetime of class $C$ with dyad in standard form a necessary condition for $L_{ABCA'}$ and $\chi_{AA'}$ to define a completely curvature-free connection is that $\Lambda = 0$. All such connections are given by (28) and (53) where the functions of integration satisfy the conditions

$$\chi_{01'} = -\frac{1}{6} \tilde{\sigma}' L^5_5 + \frac{3}{2} \Omega^o L^5_5$$
$$\chi_{11'} = -\frac{1}{2} \tilde{\sigma}' L^5_5$$
$$0 = L^5_5$$
$$0 = L^5_5$$
$$0 = \tilde{\sigma}' L^5_5 + \frac{1}{3} \Psi^5_4$$
$$0 = \tilde{\sigma}' L^5_4 + 9\Omega^o \tilde{\sigma}' L^5_5 + 9\tilde{\sigma}' \tilde{\sigma}' L^5_5 + 3L^5_5 \tilde{\sigma}' \Omega^o + \Psi^5_3$$
$$0 = 3\tilde{\sigma}' L^5_5 - \tilde{P}' L^5_5 - 6L^5_5 \tilde{\sigma}' L^5_5 + 6L^5_5 \tilde{\sigma}' L^5_4 + 18\Omega^o L^5_5$$

All $H$-potentials satisfying $\nabla_{(AB'} H_{B'A'B''} = \Gamma_{ABCA'}$ are given by equation (77) subject to the condition

$$0 = \tilde{\sigma}' H^o_{20'} + 6L^5_5 \Omega^o \tilde{\sigma}' \Omega^o + 18\Omega^o \tilde{\sigma}' \Omega^o \tilde{\sigma}' L^5_5 - L^5_4 \tilde{\sigma}' \Omega^o - 2\tilde{\sigma}' \Omega^o \tilde{\sigma}' L^5_4$$
$$+ 9L^5_5 (\tilde{\sigma}' \Omega^o)^2 + \frac{1}{2} \tilde{\sigma}' \Psi^5_2 - \Omega^o \Psi^5_3 - \Phi^5_{21}$$

Note that at this moment we have not yet proved that all these conditions can be simultaneously satisfied.

### 4.3.4 The existence of completely curvature-free connections

In this section we will show that $\Lambda = 0$ is also a sufficient condition for the existence of a curvature-free connection of the type discussed previously. As seen in the previous theorem we need to find a solution to the equations

$$0 = \tilde{\sigma}' L^5_5 + \frac{1}{3} \Psi^5_4$$
$$0 = \tilde{\sigma}' L^5_4 + 9\Omega^o \tilde{\sigma}' L^5_5 + 9\tilde{\sigma}' \tilde{\sigma}' L^5_5 + 3L^5_5 \tilde{\sigma}' \Omega^o + \Psi^5_3$$
$$0 = 3\tilde{\sigma}' L^5_5 - \tilde{P}' L^5_5 - 6L^5_5 \tilde{\sigma}' L^5_5 + 6L^5_5 \tilde{\sigma}' L^5_4 + 18\Omega^o L^5_5$$

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We observe that the first equation can be written
\[
0 = \partial^3 L^5_3 + \frac{1}{3} \Psi^5_4 = \partial^j (\partial^2 L^5_5 - \frac{1}{3} \kappa^\circ)
\]
Thus, the first equation is satisfied, e.g., if
\[
\partial^2 L^5_5 = \frac{1}{3} \kappa^\circ.
\] (80)

We observe that via the commutators it is easy to show that there actually exists functions \(L^5_5\) that satisfy this equation. Then the second equation of (79) can be rewritten
\[
0 = \partial^j (\partial^2 L^5_4 + 3 \partial^2 \partial^j \Omega^\circ + 6 \Omega^\circ \partial^j L^5_5 - \rho^\circ)
\]
so it in turn is satisfied if, e.g.,
\[
\partial^2 L^5_4 = \rho^\circ - 3 \partial^2 \partial^j \Omega - 6 \Omega^\circ \partial^j L^5_5.
\] (81)

We note that \(L^5_4\) satisfies this equation if and only if it also satisfies the condition
\[
\partial^j L^5_4 = -3 \Omega^5 L^5_5 + \alpha^\circ
\] (82)
for some function \(\alpha^\circ\) that satisfies
\[
\partial^j \alpha^\circ = \rho^\circ - 3 \Omega^\circ \partial^j L^5_5.
\] (83)

Applying the \([\tilde{P}', \tilde{\partial}']\)-commutator to \(L^5_4\) then gives us the following necessary and sufficient condition for the existence of a solution \(L^5_4\):
\[
\tilde{P}' \alpha^\circ = 3 \partial \partial' L^5_5.
\]
Applying the same commutator to \(\alpha^\circ\) we find that it is identically satisfied and hence there exists a function \(\alpha^\circ\) that satisfies both of the above conditions. It follows that the conditions for \(L^5_4\) also satisfies the commutators identically, and therefore there actually exists solutions of (79).

Thus, our final result is

**Theorem 4.6** In a spacetime of class \(G\) with dyad in standard form there exists a Lanczos potential of the Weyl spinor \(L_{ABC'A'}\) and a covector \(\chi_{AA'}\), both aligned to \(\alpha^\circ\) such that the resulting connection \(\nabla_{AA'}\) is completely curvature-free (i.e., \(\hat{R}_{abcd} = 0\)) if and only if \(\Lambda = 0\).

A possible choice of \(L_{ABC'A'}\) and \(\chi_{AA'}\) is given by
\[
L_4 = \bar{\rho} L^5_4
\]
\[
L_5 = \frac{\bar{\rho} L^5_5}{\rho} - \frac{1}{3} \partial \partial' L^5_4
\]
\[ L_6 = -\frac{\ddot{\rho}}{\rho} \dot{\theta} L_5^0 + \frac{1}{6} \ddot{\rho} (\rho^o - 6 \Omega^o \dot{\theta} L_5^0) - \frac{1}{4} \rho^2 \Psi_2^0 - \frac{1}{12} \rho \dot{\rho} \Psi_2^0 - \frac{1}{2} \rho^2 \ddot{\rho} \dot{\Phi}_1^0 \]
\[ L_7 = \frac{1}{2} \kappa^o - \frac{1}{2} \dot{\rho} \Psi_3^0 - \frac{1}{4} \rho^2 \dot{\theta} \dot{\Psi}_2^0 - \frac{1}{7} \dot{\rho} \Phi_{21} - \frac{1}{4} \rho^3 \dot{\theta} \dot{\Psi}_2^0 \Omega^o - \frac{1}{2} \rho^2 \dot{\rho} \dot{\Phi}_{11}^0 \]
\[ \chi_{01'} = \frac{\ddot{\rho}}{\rho} L_5^0 + \rho \left( \frac{1}{6} \dot{\theta} L_5^0 + \frac{3}{2} \Omega^o L_5^0 \right) \]
\[ \chi_{11'} = \frac{1}{2} \dot{\theta} L_5^0 - \frac{1}{6} \ddot{\rho} \rho^o + \frac{1}{12} \rho \ddot{\rho} \Psi_2^0 \]

where
\[ \dot{\theta}^2 L_5^0 = \frac{1}{3} \kappa^o \]
\[ \dot{\theta}^2 L_4^0 = \rho^o - 3 L_5^0 \dot{\theta} \Omega^o - 6 \Omega^o \dot{\theta} L_5^0 \]
\[ \dot{L}_4' = 3 \ddot{\theta} L_5^0 - 6 L_4' \dot{\theta} L_5^0 + 6 L_5^0 \dot{\theta} L_4^0 + 18 \Omega^o L_5^2 \]

and in particular, there always exists functions \( L_4^0, L_5^0 \) satisfying these conditions.

All \( H \)-potentials of these connections that are aligned to \( \sigma^{A'} \) are given by
\[ H_{02'} = \frac{\ddot{\rho}}{\rho} L_4^0 + \rho H_{02'} \]
\[ H_{12'} = \frac{3 \rho}{2 \rho^2} \dot{\theta} L_5^0 + \frac{3 \dot{\rho}}{\rho} L_5^0 - \frac{1}{2} \dot{\rho} \left( \dot{\theta} H_{02'} - L_4' \dot{\theta} \Omega^o \right) \]
\[ H_{22'} = \frac{3 \rho}{2 \rho^2} \dot{\theta} L_5^0 - \frac{3 \dot{\rho}}{\rho} L_5^0 - \frac{1}{2} \rho \rho^o + \frac{1}{2} \rho \ddot{\rho} \Psi_2^0 + \frac{1}{2} \rho \dot{\rho} \dot{\Phi}_{11}^0 \]

where \( H_{02'} \) satisfies
\[ 0 = \ddot{\theta}^3 H_{02'} + 6 \dot{L}_5^0 \Omega^o \dot{\theta}^2 \Omega^o + 18 \Omega^o \dot{\theta} \Omega^o \dot{\theta} L_5^0 - L_4^0 \dot{\theta} \dot{\theta} \dot{\theta} L_5^0 - 2 \dot{\theta}^2 \Omega^o \dot{\theta} L_4^0 \]
\[ + 6 L_5^0 (\dot{\theta}^o)^2 + \frac{1}{2} \dot{\theta}^o \Psi_2^0 - \Omega^o \Psi_3^0 - \Phi_{21}^0 \]

and in particular, such a function \( H_{02'}^0 \) exists.

## 5 Applications to quasi-local momentum

### 5.1 Quasi-local momentum in spacetimes of class \( G \)

Now that we have obtained curvature-free connections in the spacetimes of class \( G \), we will look at possible applications to physics. Thus, in this section we will
see how far the Bergqvist-Ludvigsen construction of quasi-local momentum can be taken in a general class $\mathcal{G}$ spacetime. In an analogous way as for the Kerr spacetime, let $\mathcal{S}_A$ denote the 2-dimensional complex vector space of spinor fields $\xi_A$ satisfying

$$\hat{\nabla}_{AA'}\xi_B = 0. \quad (88)$$

where $\hat{\nabla}_{AA'}$ is an arbitrary curvature-free connection given in Theorem 4.5. Put

$$\varphi_{AB} = \xi_A(\nabla_B)^C \xi_{C'} - \xi_C(\nabla_{A'}\xi_B) \quad (89)$$

and

$$F_{ab} = i(\varepsilon_{AB}\varphi_{A'B'} - \varepsilon_{A'B'}\varphi_{AB}). \quad (90)$$

Given a spacelike 2-surface $\Sigma$ we now define a 1-form $P_{AA'}$ on the hermitian part of $\mathcal{S}_A \otimes \mathcal{S}_{A'}$ by

$$P_{AA'}(\Sigma)\xi^A\xi^{A'} = \frac{1}{8\pi} \int_{\Sigma} F, \quad (91)$$

analogously to \[10\], \[11\].

Because $F_{ab}$ is a 2-form, $(dF)_{abc} = \nabla_{[a}F_{bc]}$ is a 3-form so its Hodge dual $(^*dF)_a$ is a 1-form which is much easier to calculate than $(dF)_{abc}$ and we have that

$$(^*dF)_a = \nabla_A B \varphi_{AB} + \nabla_{A'} B' \varphi_{A'B'}.$$  

By using \[88\] we obtain

$$\varphi_{AB} = 2(\bar{\Gamma}_{C'D'}(A\varepsilon_B)C - \Gamma_{C(AB)C'})\xi^C\xi^{C'}. \quad (92)$$

Decomposing $\Gamma_{ABCA'}$ yields

$$\varphi_{AB} = 2(3\bar{\chi}_C(A\varepsilon_B)C + \chi_{C'(A\varepsilon_B)C} - L_{ABCC'})\xi^C\xi^{C'}. \quad (93)$$

A very long spinor calculation involving both the equations \[88\] and \[89\] now reveals that

$$(^*dF)_a = -2\xi^B\xi^{B'}(\Phi_{ABA'B'} + 4(M_{ABC}o^C - \lambda_A o_B + 2o_A\lambda_B)$$

$$- \lambda_{A'} o_B + 2o_A\lambda_{B'}) - 36o_A o_{A'}\lambda_B\lambda_{B'}) = -\varepsilon^B_{B'}(\Phi_{ABA'B'} + \mathcal{F}_{ABA'B'} + \varepsilon_{A'B'}\mathcal{E}_{AB} + \varepsilon_{AB}\mathcal{E}_{A'B'}) \quad (94)$$

where $L_{ABCA'} = M_{ABCOA'}$ and $\chi_{AA'} = \lambda_{A0A'}$. Explicitly, the hermitian spinor $\mathcal{F}_{ABA'B'} = \mathcal{F}_{(AB)(A'B')}$ and the spinor $\mathcal{E}_{AB} = \mathcal{E}_{(AB)}$ are given by

$$\mathcal{F}_{ABA'B'} = 4(M_{ABC}o^C + o(A\lambda_B))(M_{A'B'C}o^{C'} + o(A'\lambda_{B'}))$$

$$- 36o_A o_{A'}\lambda_B\lambda_{B'})$$

$$\mathcal{E}_{AB} = 6o_A\lambda^A(M_{ABC}o^C - 2o(A\lambda_B)). \quad (95)$$
The components of $\mathcal{F}_{A\bar{A}'B'}$ and $\mathcal{E}_{A\bar{B}}$ in a spinor dyad $(\sigma^A, \iota^A)$ with $\iota^A$ arbitrary, are given by

\begin{align*}
\mathcal{E}_0 &= -6\chi_{01'}L_4 \\
\mathcal{E}_1 &= -6\chi_{01'}(\chi_{01'} - L_5) \\
\mathcal{E}_2 &= -6\chi_{01'}(L_6 - 2\chi_{11'}) \\
\mathcal{F}_{00'} &= 4L_4L_4 \\
\mathcal{F}_{10'} &= 2\overline{L_4}(2L_5 + \chi_{01'}) \\
\mathcal{F}_{20'} &= 4L_4(L_6 + \chi_{11'}) \\
\mathcal{F}_{11'} &= (2L_5 + \chi_{01'})(2\overline{L_5} + \chi_{01'}) - 9\chi_{01'}\overline{\chi_{01'}} \\
\mathcal{F}_{21'} &= 2(L_6 + \chi_{11'})(2\overline{L_5} + \chi_{01'}) - 18\chi_{11'}\overline{\chi_{01'}} \\
\mathcal{F}_{22'} &= 4(L_6 + \chi_{11'})(\overline{L_6} + \chi_{11'}) - 36\chi_{11'}\overline{\chi_{11'}}
\end{align*}

We remark that in an asymptotically flat spacetime an analogous construction can be performed. As our spin space $S$ we take the asymptotic spin space [20]. For $\xi^A$ asymptotically constant we define $\phi_{A\bar{B}}$ as in (89) and $\mathcal{F}_{ab}$ as in (90). Then $\mathcal{F}_{ab}$ is called the Nester-Witten 2-form, the resulting momentum $P_{AA'}(\Sigma_\infty)$ where $\Sigma_\infty$ is a spacelike cross-section of future null infinity, is called the Bondi momentum and the Hodge dual of the 1-form $-\xi^B\xi^{B'}(\mathcal{F}_{A\bar{A}'B'} + \varepsilon_{A'B'}\mathcal{E}_{A\bar{B}} + \varepsilon_{A\bar{B}}\mathcal{E}_{A'B'})$ is called the Sparling 3-form [20].

We first consider the case $\chi_{01'} = 0$, i.e., in components $L_4 = 0$, $L_5 = \chi_{01'}$ and $L_6 = 2\chi_{11'}$ (from (96)). In a spinor dyad in standard form, the functions of integration must satisfy $L_4^2 = 0$, $\chi_{01'}^2 = 0$, $L_5^2\overline{\delta}\Omega^2 = 0$ and also $\Psi_3^0 = 0$ according to the equations (28), (53) and (77). This implies that $\Psi_3^0 = 0$ so the spacetime has to be at least Petrov type III. Thus, the condition $M_{ABC} = 2\sigma_{(A}\lambda_{B)}$ places severe restrictions on a vacuum spacetime.

We also see that if $M_{ABC} = 2\sigma_{(A}\lambda_{B)}$, the only other possibility for $\mathcal{F}_{ab}$ to be closed is that $\chi_{01'} = 0$. In this case we also obtain $L_4 = 0$, $L_5 = 0$ and in addition

\[(L_6 + \chi_{11'})(\overline{L_6} + \chi_{11'}) = 9\chi_{11'}\overline{\chi_{11'}}.\]

Referring to (28) and (53) we find that the functions of integration must satisfy $L_4^2 = 0$, $L_5^2 = 0$ and $\chi_{01'} = 0$. These are also very restrictive conditions even though the last one is seen to be identically satisfied. From (77) we see that the vacuum spacetime must satisfy $\Psi_3^0 = 0$ and $\Psi_4^0 = 0$. 

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5.2 Kerr-Schild spacetimes of class $G$ with vanishing Ricci scalar

As an application of the results in the previous section we will now look at Kerr-Schild spacetimes of class $G$ with vanishing Ricci scalar. Following the conventions of Section 4.2 we obtain the Lanczos- and $\chi$-scalars

$$L_4 = 0, \quad L_5 = 0, \quad \chi_{01'} = 0$$

$$L_6 = -\frac{1}{6}(\mathbf{p}f + (2\rho - \bar{\rho})f)$$

$$L_7 = -\frac{1}{2}(\partial' f - \bar{\tau} f)$$

$$\chi_{11'} = -\frac{1}{12}(\mathbf{p}f - (\rho + \bar{\rho})f)$$

for arbitrary dyad spinor $\iota^A$, so we allow for the possibility of the dyad not being in standard form. Then we immediately obtain $E_{AB} = 0$ and in addition

$$\mathcal{F}_{00'} = 0, \quad \mathcal{F}_{10'} = 0, \quad \mathcal{F}_{20'} = 0$$

$$\mathcal{F}_{11'} = 0, \quad \mathcal{F}_{21'} = 0$$

$$\mathcal{F}_{22'} = \frac{f}{2}((\rho + \bar{\rho})\mathbf{p}f - (\rho^2 + \bar{\rho}^2)f).$$

However, it is easily shown that in these spacetimes

$$(\rho + \bar{\rho})\mathbf{p}f - (\rho^2 + \bar{\rho}^2)f = -2\Phi_{11}$$

by rewriting the relevant Newman-Penrose equations in [17]. Hence,

$$\mathcal{F}_{22'} = -f\Phi_{11}$$

and we can therefore write

$$\mathcal{F}_{ABA'B'} = -f\Phi_{11}o_Ao_BO_A'o_B'.$$  \hspace{1cm} (99)$$

We see that in particular the 2-form $F_{ab}$ is closed if and only if the Kerr-Schild spacetime is vacuum, similarly to the Bergqvist-Ludvigsen construction in the Kerr spacetime. Hence, if $\Sigma_1$ and $\Sigma_2$ are two spacelike hypersurfaces such that they together form the boundary of some 3-volume $V$, then $P_{AA'}(\Sigma_1) = P_{AA'}(\Sigma_2)$ according to Stokes' theorem, in the vacuum case.

6 Conclusions

In spacetimes of class $G$ with dyad in standard form we obtained, by the method of $\rho$-integration, all Lanczos potentials that are aligned to $o^A$, of the Weyl spinor and their $H$-potentials (also aligned to $o^A$). The resulting expressions for the
Lanczos scalars can be written as polynomials in $\rho$ and $\rho^{-1}$, divided by some power of the factor $(1 + \rho \Omega^2)$, by making use of the formula
\[
\bar{\rho} = \frac{\rho}{1 + \rho \Omega^2}.
\]
This is closely related to the peeling theorem in asymptotically flat spacetimes. We therefore expect it to be possible to extend the approach in this paper to such spacetimes and so it may be possible to integrate for Lanczos potentials and use them to construct curvature-free connections for (some) asymptotically flat spacetimes.

We remark that this paper can be viewed as an alternative existence proof for Lanczos potentials of the Weyl spinor and $H$-potentials of Lanczos potentials of the Weyl spinor, for spacetimes of class $\mathcal{G}$. We also remark that the existence proof for $H$-potentials of a general symmetric (3,1)-spinor in \cite{6} is valid only in Einstein spacetimes, whereas we have found $H$-potentials in the special case when $L_{ABCA'}$ is a Lanczos potential of the Weyl spinor that is aligned to the repeated principal spinor. A similar existence proof was obtained by Torres del Castillo \cite{21}, \cite{22} for a slightly more general class of spacetimes though, as mentioned above, he did not find all potentials of the type we have discussed. His approach was reminiscent of the $H$-space theory \cite{10}; it would be interesting to investigate which of the potentials found in this paper can be written in the form that he derived.

We also note that the condition that $L_{ABCA'}$ possesses an $H$-potential aligned to $o^{A'}$ is actually a necessary condition for $\Gamma_{ABCA'}$ to define a curvature-free connection in the case that we have studied (Theorem 4.2). This is an interesting result and it raises the question whether $H$-potentials of Lanczos potentials of the Weyl spinor offers possibilities for constructing curvature-free connections and quasi-local momentum in more general spacetimes. We also remark that hermitian $H$-potentials seem to play a role in the construction of angular momentum \cite{11}. It would therefore be of interest to investigate when hermitian $H$-potentials can be found.

It has been conjectured \cite{7} that the Lanczos potential is related to the NP spin coefficients. In \cite{3} Lanczos potentials for the Weyl spinor whose components can be directly equated to the NP spin coefficients of some normalized spinor dyad, were studied. It has been confirmed that such Lanczos potentials exist in many special classes of spacetimes namely, many stationary axially symmetric spacetimes and many cylindrically symmetric spacetimes \cite{14}, all conformally flat pure radiation spacetimes and all Kerr-Schild spacetimes where $L^a$ is geodesic and shear-free \cite{1}. Slight variations of the identification scheme also works for all type III, N and 0 spacetimes \cite{10}. If we, in a class $\mathcal{G}$ spacetime, choose a new normalized spinor dyad $(\xi_0^A, \xi_1^A)$ from the spinor fields in $S^A$, then the components of the spinor $\Gamma_{ABCA'}$ are precisely the NP spin coefficients of the dyad $(\xi_0^A, \xi_1^A)$. Hence, $L_{ABCA'} = \Gamma_{(ABC)A'}$ is a Lanczos potential of the Weyl spinor, whose components can be directly equated to the spin coefficients in the
manner described in [5].

An important application of these results is the construction of quasi-local momentum $P_{AA'}$ in spacetimes of class $G$ given in the previous section. The reason why we have not explored this application in greater detail is that in order to examine the properties of $P_{AA'}$, and also of the analogues of the Nester-Witten 2-form and the Sparling 3-form, in this more general class of spacetimes, we would need to impose extra restrictions on the global topology onto the class $G$. Since we feel this would obscure the results obtained so far, a detailed exploration of this application will be postponed to a future paper. Another development of the Bergqvist-Ludvigsen connection in the Kerr spacetime is Harnett’s [13] construction of twistors for the Kerr spacetime. Hopefully the results in this paper could be used to generalize this twistor construction to more general spacetimes.

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