GLOBAL ANALYTIC SOLUTIONS OF THE SEMICONDUCTOR BOLTZMANN-DIRAC-BENNEY EQUATION WITH RELAXATION TIME APPROXIMATION

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ABSTRACT. The semiconductor Boltzmann-Dirac-Benney equation
\[ \partial_t f + \nabla (p) \cdot \nabla f - \nabla \rho f (x,t) \cdot \nabla p f = \frac{F_x (p) - f}{\tau}, \quad x \in \mathbb{R}^d, \quad p \in B, \quad t > 0 \]
is a model for ultracold atoms trapped in an optical lattice. The global existence of a solution is shown for small \( \tau > 0 \) assuming that the initial data are analytic and sufficiently close to the Fermi-Dirac distribution \( F_\lambda \). This system contains an interaction potential \( \rho f \) being significantly more singular than the Coulomb potential, which causes major structural difficulties in the analysis.

The key technique is based on the ideas of Mouhot and Villani by using Gevrey-type norms which vary over time. The global existence result for small initial data is also generalized to
\[ \partial_t f + L f = Q(f), \]
where \( L \) is a generator of a \( C^0 \)-group with \( \| e^{tL} \| \leq C e^{\omega t} \) for all \( t \in \mathbb{R} \) and \( \omega > 0 \) and, where further additional analytic properties of \( L \) and \( Q \) are assumed.

1. Introduction. The semiconductor Boltzmann-Dirac-Benney equation is a model describing ultracold atoms in an optical lattice. An optical lattice is a spatially periodic structure that is formed by interfering optical laser beams. The interference produces an optical standing wave that may trap neutral atoms [4]. The underlying experiment has been proved to be a powerful tool to study physical phenomena that occur in solid state materials. Simply speaking, a solid crystal consists of ions and electrons. Because of the mass difference, the electrons in average move much faster than the ions in a semi-classical picture. Therefore, from a modeling point of view, one may assume that the positions of the ions are fixed and form a regular periodic structure. However, comparing the theory to the experiment, one faces certain difficulties as impurities lead to defects in the periodic structure.

The experiment of ultracold atoms in an optical lattice can be considered as a physical toy-model for solid state materials. The ultracold atoms represent the electrons and the optical lattice mimics the periodic structure of the ions. The
advantage of the optical lattice is the absence of impurities. Thus, one expects a better accordance of the experiment with the theory. Moreover, the dynamics of the ultracold atoms, i.e. at a temperature of magnitude of some nanokelvin, can be followed on the time scale of milliseconds. This facilitates the study physical phenomena in an optical lattice being difficult to observe in solid crystals. Furthermore, they are promising candidates to realize quantum information processors [17] and extremely precise atomic clocks [2].

The main difference consists of the use of uncharged atoms, whereas electrons are negatively charged. Ultracold fermions may be described with a Fermi-Hubbard model with a Hamiltonian that is a result of the lattice potential created by interfering laser beams and short-ranged collisions [12]. They assume that the ultracold atoms interact only with their nearest neighbors. For more details see [20].

In this article we are focusing on a semi-classical picture which is able to model qualitatively the observed cloud shapes [25]. The effective dynamics are modeled by a Boltzmann transport equation describing the macroscopic particle density \( \rho \) and the microscopic particle density \( f \). The interaction potential is given by \( V \). The interaction potential is given by \( V \). The macroscopic particle density \( \rho \) models the strength of the on-site interaction between spin-up and spin-down components \( [25] \). The band energy \( \varepsilon(p) \) is given by the periodic dispersion relation

\[
\varepsilon(p) = -2\varepsilon_0 \sum_{i=1}^{d} \cos(2\pi p_i), \quad p \in \mathbb{T}^d.
\]

The constant \( \varepsilon_0 \) is a measure for the tunneling rate of a particle from one lattice site to a neighboring one. This dispersion relation also occurs as an approximation for the lowest energy band in semiconductors (see [1]). Let \( \rho_f := \int_{\mathbb{T}^d} f dp \) be the macroscopic particle density. The interaction potential is given by \( V_f = -U \rho_f \), where \( U > 0 \) models the strength of the on-site interaction between spin-up and spin-down components \( [25] \).

Finally, the semiconductor Boltzmann-Dirac-Benney equation is given by

\[
\partial_t f + \nabla \varepsilon(p) \cdot \nabla_x f - U \nabla_x \rho_f \cdot \nabla f = Q(f),
\]

where \( Q(f) \) is a collision operator. There are several choices for the collision operator. The natural choice of the collision operator is a two particle collision operator neglecting the three or more particle scattering

\[
Q_{ee}(g)(p) := \sum_{G \in 2\pi\mathbb{Z}^d} \int_{\mathbb{T}^d} \int_{\mathbb{T}^d} Z(p) \left( g(p)g(p')g(p'')(1 - \eta g(p''))(1 - \eta g(p''')) \right.

\[
- g(p'')g(p''')(1 - \eta g(p))(1 - \eta g(p')) \bigg) \frac{d\mathcal{H}^{d-1}_{p''} \varepsilon_{tot}(p)}{|\nabla p''| \varepsilon_{tot}(p)} dp'.
\]

for some \( \eta \geq 0 \), where \( p = (p, p', p'', p''') \) and \( \mathcal{H}^{d-1}_{p''} \) denotes the \( d-1 \) dimensional Hausdorff measure w.r.t. \( p'' \). The function \( Z(p) \) models the probability of a scattering event from state \( (p, p_1) \) to the state \( (p_2, p_3) \). Moreover, the total change of momentum and energy are denoted by

\[
p_{tot}(p) := p + p' - p'' - p''' \quad \text{and} \quad \varepsilon_{tot}(p) = \varepsilon(p) + \varepsilon(p') - \varepsilon(p'') - \varepsilon(p'''),
\]

respectively. The sum over \( G \) runs over all reciprocal lattice vectors \( G \in 2\pi\mathbb{Z}^d \). Note that in fact only finite summands contribute to the sum since \( p_{tot} \) is bounded. This
scattering operator is also well-known as the electron-electron scattering operator [3].

Comparing the semiconductor Boltzmann-Dirac-Benney equation to the semiconductor Boltzmann equation with Coulomb interaction, there are two major differences. First, the band energy $\epsilon$ is a bounded function in contrast to the parabolic band approximation $\epsilon(p) = \frac{1}{2}|p|^2$, which is usually assumed [18]. Second, the potential $V_f$ is proportional to the macroscopic particle density $\rho_f = \int T d\mathbf{f} dp$. In semiconductor physics, the interaction potential $\Phi_f$ between the electrons is often modeled by the Coulomb potential [18]. Hence, $\Phi_f$ is determined self-consistently from the Poisson equation $-\Delta \Phi_f = \rho_f$ and therefore much more regular than $V_f$.

**Fermi-Dirac distribution.** Due to the complexity of the two particle scattering operator, the analysis of (1) with $Q = Q_{ee}$ is very difficult. Therefore, we search for a less complicated physical approximation of $Q_{ee}$. In [18], Jüngel proves in Proposition 4.6 that the zero set of $Q_{ee}$ consists of Fermi-Dirac distribution functions, i.e. it holds formally that $Q_{ee}(g) = 0$ if and only if there exists a $\lambda = (\lambda_0, \lambda_1) \in \mathbb{R}^2$ with

$$g(p) = F_\lambda(p) := \frac{1}{\eta + e^{-\lambda_0 - \lambda_1 \epsilon(p)}}.$$ 

Hence, $F_\lambda$ annihilates the collision operator and can be seen as an equilibrium distribution. For $\eta = 1$, we obtain the Fermi-Dirac distribution, while for $\eta = 0$, $F_\lambda$ equals the Maxwell-Boltzmann distribution. The parameter $\lambda_0, \lambda_1$ are sometimes called entropy parameters, where physically $-\lambda_1$ equals the inverse temperature and $-\lambda_0/\lambda_1$ the chemical potential.

Note that we have assumed a bounded band energy. This implies that the equilibrium $F_\lambda$ is integrable w.r.t. $p$ even if $\lambda_1 > 0$, which means that the absolute temperature may be negative. In fact, negative absolute temperature can be realized in experiments with ultracold atoms [24]. Negative temperatures occur in equilibrated (quantum) systems that are characterized by an inverted population of energy states. The thermodynamical implications of negative temperatures are discussed in [23].

**Relaxation time approximation.** The idea of the relaxation time approximation is to assume that the collision operator drives the solution into the equilibrium. We define

$$Q(g)(p) := \frac{F_\lambda(p) - g(p)}{\tau}$$

for some $\lambda \in \mathbb{R}^2$, $\tau > 0$ and $g = g(p)$ being a heuristic approximation of $Q_{ee}$ [1]. The parameter $\tau$ is called the relaxation time and represents the average time between two scattering events. Since $F_\lambda$ is a fixed function, the relaxation time approximation collision operator neither conserves the local particle nor the local energy. The simplest version of the relaxation time approximation is to assume that $\lambda_1$ vanishes. Then, $F_{\lambda_0,0}$ equals a constant $\rho \in [0, 1/\eta]$.

**Known results.** In a previous paper [6], the semiconductor Boltzmann-Dirac-Benney equation is investigated with a BGK-type collision operator

$$Q_{BGK}(f) = \rho_f(1 - \eta \rho_f)\left(F_f - f\right),$$  

where $\tau > 0$ is the relaxation time and $F_f$ is determined by

$$F_f(x, p, t) = \frac{1}{\eta + e^{-\lambda_0(x,t) - \lambda_1(x,t)\epsilon(p)}} , \quad x \in \mathbb{R}^d, \quad p \in \mathbb{T}^d, \quad t > 0,$$
where $(\tilde{\lambda}_0, \tilde{\lambda}_1)$ are the Lagrange multipliers resulting from the local mass and energy conservation constraints, i.e.
\[
\int_{\mathbb{T}^d} (F_f - f)dp = 0, \quad \int_{\mathbb{T}^d} (F_f - f)\varepsilon dp = 0.
\]
In [6], it is shown that (1) with $Q = Q_{BGK}$ is ill-posed in the following sense.
Let $k \in \mathbb{N}$, $\theta > 0$ and $\gamma > 0$, $U \neq 0$. There exist $\lambda \in \mathbb{R}^2$ and a time $\tau > 0$ such that there exist solutions $f_\delta : \mathbb{R}^d \times \mathbb{T}^d \times [0, \tau] \to [1, \eta^{-1}]$ of (1) with $Q = Q_{BGK}$ such that
\[
\lim_{\delta \to 0} \|f_\delta(\cdot, t) - \mathcal{F}_{\lambda}\|_{L^1(B_\theta(x,p))} = \infty \quad \text{for all } x \in \mathbb{R}^d, p \in \mathbb{T}^d, t \in (0, \tau).
\]
A sufficient condition for the critical $\tilde{\lambda}$ is given in [6] by
\[
1 < U\lambda_1 \int_{\mathbb{T}^d} \mathcal{F}_{\lambda}(p)(1 - \eta\mathcal{F}_{\lambda}(p))dp.
\]
This result reflects the theory of the Vlasov-Dirac-Benney equation with is the counterpart of the semiconductor Boltzmann-Dirac-Benney equation for free particle without collisions, i.e. with $\epsilon(p) = \frac{1}{2}|p|^2$ and $Q(f) = 0$.
The Vlasov-Dirac-Benney equation is therefore given by
\[
\partial_t f + p \cdot \nabla_x f - \nabla \rho_f(x,t) \cdot \nabla_p f = 0 \quad (3)
\]
for $x \in \mathbb{R}^d, p \in \mathbb{R}^d$ and $t > 0$. In spatial dimension one, this equation can be used to describe the density of a fusion plasma in a strong magnetic field in direction of the field [11].
The Vlasov-Dirac-Benney equation is a limit of a scaled non-linear Schrödinger equation [10]. Comparing the standard Vlasov-Poisson equation, we see that the interaction potential $\Phi_f := -\frac{1}{|x|} * \rho_f$ is long ranged by means of that the support of the kernel $1/|x|$ is the whole space. The interaction potential of the Vlasov-Dirac-Benney equation can be rewritten using the $\delta$ distribution as $V_f := -U\delta_0 * \rho_f$. Therefore $V_f$ is called a short ranged Dirac potential, which motivated the “Dirac” in the name of the Vlasov-Dirac-Benney equation [8]. The name Benney is due to its relation to the Benney equation in dimension one (for details see [8]). Moreover, the Vlasov-Dirac-Benney equation can also be derived by a quasi-neutral limit of the Vlasov-Poisson equation [15].
The Vlasov-Dirac-Benney equation first appeared in [16], where only local in time solvability was shown for analytic initial data in spatial dimension one. In [8], Bardos and Besse show that this system is not locally weakly $(H^m - H^1)$ well-posed in the sense of Hadamard. Moreover, the Vlasov-Dirac-Benney equation is actually ill-posed in $d = 3$, requiring that the spatial domain is restricted to the 3-dimensional torus $\mathbb{T}^3$ [14]: the flow of solutions does not belong to $C^\alpha(H^{s,m}(\mathbb{R}^3 \times \mathbb{T}^3), L^2(\mathbb{R}^3 \times \mathbb{T}^3))$ for any $s \geq 0, \alpha \in (0, 1]$ and $m \in \mathbb{N}_0$. Here, $H^{s,m}(\mathbb{R}^3 \times \mathbb{T}^3)$ denotes the weighted Sobolev space of order $s$ with weight $(x,u) \mapsto \langle u \rangle^m := (1 + |u|^2)^{m/2}$.
More precisely, [14] provides a stationary solution $\mu = \mu(u)$ of (3) and a family of solutions $(f_\varepsilon)_{\varepsilon > 0}$, times $t_\varepsilon = O(\varepsilon \log \varepsilon)$ and $(x_0, u_0) \in \mathbb{T}^3 \times \mathbb{R}^3$ such that
\[
\lim_{\varepsilon \to 0} \frac{\|f_\varepsilon - \mu\|_{L^2([0,t_\varepsilon] \times B_\varepsilon(x_0) \times B_\varepsilon(u_0))}}{\|(u)^m(f_\varepsilon|_{\varepsilon = 0} - \mu)\|_{H^\alpha(\mathbb{T}^3 \times \mathbb{R}^3)}} = \infty, \quad (4)
\]
where $B_\varepsilon(x_0)$ denotes the ball with radius $\varepsilon$ centered at $x_0$. 
These results show the main difference between the well-posed Vlasov-Poisson equation and the Vlasov-Dirac-Benney equation.

In [15], Han-Kwan and Rousset consider the quasi-neutral limit of the Vlasov-Poisson equation. By proving uniform estimates on the solution of the scaled Vlasov-Poisson equation they show that the scaled solution converges to a unique local solution $f \in C([0,T], H^{2m-1,2r}(\mathbb{R}^3 \times T^3))$ of the Vlasov-Dirac-Benney equation. For this, they require that the initial data $f_0 \in H^{2m,2r}(\mathbb{R}^3 \times T^3)$ satisfies the Penrose stability condition

$$\inf_{x \in \mathbb{T}^d} \inf_{(\gamma,\tau,\eta) \in (0,\infty) \times \mathbb{R}^d \setminus \{0\}} \left| 1 - \int_0^\infty e^{-|\gamma+\tau|^s} \frac{i\eta}{1 + |\eta|^2} \cdot (F_0 \nabla_v f)(x,\eta) ds \right| > 0,$$

where $F_0$ denotes the Fourier Transform in $v$.

Note that the Vlasov-Dirac-Benney equation embeds into a larger class of ill-posed equation: Han-Kwan and Nguyen write Eq. (3) as a particular case of

$$\partial_t f + Lf = Q(f,f), \quad x \in \mathbb{T}^d, z \in \Omega$$

in which $L$ (resp. $Q$) is a a linear (resp. bilinear) integro-differential operator in $(x,z)$ and $\Omega$ is a open subset of $\mathbb{R}^k$ [14]. They also state a version of (4) for the generalized setting by using the techniques of [21].

1.1. Main results. The semiconductor Boltzmann-Dirac-Benney equation and the Vlasov-Dirac-Benney equation have only been treated locally so far. A global existence result is still missing. The aim of this article is to show that the semiconductor Boltzmann-Dirac-Benney equation admits global solutions if we use the relaxation time approximation, namely

$$\partial_t f + \nabla \varepsilon(p) \cdot \nabla_x f - U \nabla_x \rho_f \cdot \nabla_x f = \frac{\mathcal{F}_\lambda(p) - f}{\tau}.$$  \hspace{1cm} (5)

For this we require analytic initial data being close to the Fermi-Dirac distribution

$$\mathcal{F}_\lambda(p) = \frac{1}{\eta + e^{-\lambda_0 - \lambda_1 \varepsilon(p)}}, \quad p \in \mathbb{T}^d.$$

This is due to the singular short ranged potential.

**Theorem 1.** Let $\lambda \in \mathbb{R}^2_+, U > 0$ and $k \in \mathbb{N}$. Then there exist $\tau_0, \varepsilon, \nu > 0$ such that if $f_0 \in S(\mathbb{R}^d \times \mathbb{T}^d)$ satisfies

$$\sum_{\alpha,\beta \in \mathbb{N}^d_0} \frac{\rho^{(\alpha+\beta)}}{\alpha! \beta!} \| \partial_x^\alpha \partial_p^\beta (f_0 - \mathcal{F}_\lambda) \|_{L^2(\mathbb{R}^d \times \mathbb{T}^d)} \leq \varepsilon$$  \hspace{1cm} (6)

then for all $\tau \in (0, \tau_0)$, (5) has a unique global analytic solution with $f|_{t=0} = f_0$ satisfying

$$\| f(t) - \mathcal{F}_\lambda \|_{H^s_x L^p_t} \leq C e^{-\left(\frac{\lambda - \lambda_1}{2m}\right)t} \quad \text{for all } t \geq 0$$

for some $C > 0$. Moreover, for all $f_0, \tilde{f}_0 \in S(\mathbb{R}^d \times \mathbb{T}^d)$ satisfying (6), we have

$$\| f(t) - \tilde{f}(t) \|_{H^s_x L^p_t} \leq C e^{-\left(\frac{\lambda - \lambda_1}{2m}\right)t} \sum_{\alpha,\beta \in \mathbb{N}^d_0} \frac{\rho^{(\alpha+\beta)}}{\alpha! \beta!} \| \partial_x^\alpha \partial_p^\beta (f_0 - \tilde{f}_0) \|_{L^2(\mathbb{R}^d \times \mathbb{T}^d)} \quad \text{for all } t \geq 0$$

for some $C > 0$, where $f, \tilde{f}$ are the solution of (5) with $f(0) = f_0$ and $\tilde{f}(0) = \tilde{f}_0$, respectively.

We can also improve this result and obtain a better estimate for the solution $f$. For this, however, we require different spaces.
There exists an $F$-\null

\begin{definition}
Let $S(\mathbb{R}^d \times T^d)$ and $C_b^\infty(\mathbb{R}^d \times T^d)$ be the Schwartz space and the space of bounded smooth functions, respectively.

- For $\lambda \in \mathbb{R}^d_+: = \{x, y) \in \mathbb{R}^2 : y \geq 0\}$, let $k \in \mathbb{N}$, $k > \frac{d}{2}$. We define $Y := S(\mathbb{R}^d \times T^d)$ and $X := H^k_x L^2_p(\mathbb{R}^d \times T^d)$ equipped with the scalar product

$$\langle f, g \rangle_X := \sum_{|\alpha| \leq k} \langle \partial^\alpha_x f, \partial^\alpha_x g \rangle_0,$$

where

$$\langle f, g \rangle_0 := \int_{\mathbb{R}^d} \int_{T^d} f(x, p)g(x, p) \frac{dpdx}{\mathcal{F}_\lambda(p)(1 - \eta\mathcal{F}_\lambda(p))} + U\lambda_1 \int_{\mathbb{R}^d} \rho_f \rho_g \, dpdx$$

and $\rho_f(x) := \int_{T^d} f(x, p) \, dp$.

- For $\lambda_1 = 0$, we can alternatively define $Y := C_b^\infty(\mathbb{R}^d \times T^d)$ and $X := C_b^0(\mathbb{R}^d \times T^d)$.

\end{definition}

\begin{definition}
Let $\nu \in [0, \infty)$. We define

$$\|\phi\|_{Y^\nu_t} := \sum_{|\alpha + \beta| \leq 1, \alpha, \beta \in \mathbb{N}_0} \|e^{tL}\partial^\alpha_x \partial^\beta_p e^{-tL} \phi\|_X$$

for $\phi \in Y$ and $t \in \mathbb{R}$, where $e^{tL}$ is generated by

$$Lf(x, p) := \nabla \varepsilon(p) \cdot \nabla_x f(x, p) - U\nabla_x \int_{\mathbb{R}^d} f(x, p') \, dp' \cdot \nabla_p \mathcal{F}_\lambda(p).$$

We show in Lemma 18 that this is well-defined.

\begin{theorem}
Let $\lambda \in \mathbb{R}^d_+, U > 0$. Then there exist $\tau_0, \varepsilon, \nu_0 > 0$ such that if

$$\|f_0 - \mathcal{F}_\lambda\|_{Y^\nu_t} \leq \varepsilon\nu$$

for some $\nu \leq \nu_0$, then (7) has a unique global analytic solution $f$ with $f|_{t=0} = f_0$ for all $\tau \in (0, \tau_0)$. The solution satisfies

$$\|f(t) - \mathcal{F}_\lambda\|_{Y^\nu_{\exp(-\frac{t}{\tau})}} \leq 2\varepsilon e^{-\left(\frac{\tau}{\tau_0}\right)} t$$

for all $t \geq 0$.

Moreover, for all $f_0, \tilde{F}_0 \in Y$ satisfying (7), we have

$$\|f(t) - \tilde{f}(t)\|_{Y^\nu_{\exp(-\frac{t}{\tau})}} \leq 2e^{-\left(\frac{\tau}{\tau_0}\right)} \|f_0 - \tilde{F}_0\|_{Y^\nu_0}$$

for all $t \geq 0$ where $f, \tilde{f}$ are the solution of (5) with $f(0) = f_0$ and $\tilde{f}(0) = \tilde{F}_0$, respectively.

As in [14], we can generalize these results to a more abstract setting. Let $X$ be a Banach space and $Y \subset X$ be dense. Moreover, let $A = (A_1, \ldots, A_n) : D(A) \subset X \to X^n$ be a linear operator with $Y \subset D(A)$ such that $A(Y) \subset Y^n$ and $[A_i, A_j] = 0$ for all $i, j = 1, \ldots, n$. For $x_0 \in D(A)$, we consider the non-linear Cauchy-problem

$$\partial_t x = F(x), \quad \text{with} \quad x(0) = x_0,$$

where $F : D(A) \to X$ satisfies the following conditions:

- $\partial \bar{x} \in D(A)$ with $F(\bar{x}) = 0$
- $F$ is Gâteaux differentiable at $\bar{x}$ and $Lu := DF(\bar{x})u$ fulfills (H2a)
- $L : D(L) \subset X \to X$ is a generator of a $C_0$ group $(e^{tL})_{t \in \mathbb{R}}$ with $Y \subset D(L)$ and $L(Y) \subset Y$ as well as

$$\|e^{tL}\|_X \leq C_L e^{\omega t} \quad \text{for all} \quad t \in \mathbb{R}$$

and some $C_L \geq 1$ and $\omega > 0$. 

(H2b) For \(0 \neq \alpha \in \mathbb{N}_0^n\), we define \(L_0 := L\) and 
\[L_{\alpha + \bar{e}_i} := [L_\alpha, A_i], \quad \text{where} \quad \bar{e}_i = (0, \ldots, 1_i, \ldots, 0) \quad \text{for} \quad i = 1, \ldots, n.\]

There exist \(C \geq 0\) for \(i = 1, \ldots, n\) and \(r \in [0, \infty)\) with \(Cr < \omega/(nC_\ell^2)\) such that 
\[
\|L_\alpha y\|_X \leq C\alpha! r^{\|\alpha\|} \sum_{i=1}^n \|A_i y\|_X
\]
for all \(\alpha \in \mathbb{N}_0^n\) and all \(y \in Y\).

(H3) Define \(Q(y) := F(y) - F(\bar{x}) - DF(\bar{x})y\).

(H3a) We assume that 
\[
\|A^\alpha Q(y)\|_X \leq \sum_{\gamma \leq \alpha} \left(\frac{\alpha}{\gamma}\right) \|A^{\alpha - \gamma} y\|_X \sum_{|\beta| \leq 1} M_{|\beta|} \|A^{\gamma + \beta} y\|_X
\]
for all \(\alpha \in \mathbb{N}_0^n, y \in Y\) and some \(M_{\beta} \geq 0\).

(H3b) We have the following Lipschitz estimate 
\[
\|A^\alpha (Q(y) - Q(x))\|_X 
\leq \sup_{z, z' \in [x, y]} \sum_{\gamma \leq \alpha} \left(\frac{\alpha}{\gamma}\right) \left(\|A^{\alpha - \gamma} z\|_X \sum_{|\beta| \leq 1} M'_{|\beta|} \|A^{\gamma + \beta} (y - x)\|_X \right) + \|A^{\alpha - \gamma} (y - x)\|_X \sum_{|\beta| \leq 1} M'_{|\beta|} \|A^{\gamma + \beta} z'\|_X
\]
for all \(\alpha \in \mathbb{N}_0^n, y \in Y\) and some \(M'_{\beta} \geq 0\), where \([x, y] := \{sx + (1 - s)y : s \in [0, 1]\}\).

We now generalize Definition 2 for these properties.

**Definition 3.** Let \(\nu \in [0, \infty)\). We define 
\[
\|y\|_{X_{\nu}^\alpha} := \sum_{\alpha \in \mathbb{N}_0^n} \frac{\nu^{\|\alpha\|}}{\alpha!} \|A^\alpha y\|_X
\]
for \(y \in Y\).

**Theorem 3.** Assume that (H1)-(H3) hold. Then for every positive \(\nu_0 < \frac{1}{\nu}(1 - \sqrt{nC_\ell^2 / \omega})\), there exists an \(\varepsilon > 0\) such that if 
\[
\|x_0 - \mathcal{Y}\|_{X_{\nu}^\alpha} \leq \varepsilon \nu
\]
for some \(\nu \leq \nu_0\), then (8) has a strong solution \(x\) with \(x|_{t=0} = x_0\) satisfying 
\[
\|x(t) - \mathcal{Y}\|_{X_{\nu}^\alpha - \omega t} \leq 2C_L e^{-\omega t} \nu \quad \text{for all} \quad t \geq 0.
\]

Moreover, for all \(x_0, y_0 \in Y\) fulfilling (9), we have 
\[
\|x(t) - y(t)\|_{X_{\nu}^\alpha - \omega t} \leq 2C_L e^{-\omega t} \|x_0 - y_0\|_{X_{\nu}^\alpha} \quad \text{for all} \quad t \geq 0,
\]
where \(x, y\) are the solution of (8) with \(x(0) = x_0\) and \(y(0) = y_0\), respectively.

**Remark 4.** The operator \(L_\alpha\) is well-defined, because 
\[
[L_{\alpha + \bar{e}_i}, A_j] = ([L_\alpha, A_i], A_j) 
= -[A_j, L_\alpha] - [A_i, A_j] L_\alpha = [[L_\alpha, A_j], A_i] = [L_{\alpha + \bar{e}_i}, A_i]
\]
for $i, j = 1, \ldots, n$ according to the Jacoby identity and the the assumption $[A_i, A_j] = 0$.

**Example 1.** Let $\tilde{Q} : D(A) \times D(A) \to X$ be bilinear fulfilling

$$\left\| \tilde{Q}(x, y) \right\|_X \leq C_Q \sum_{i=1}^n (\|A_ix\|_X \|y\|_X + \|x\|_X \|A_iy\|_X)$$

(10)
as well as

$$A_i\tilde{Q}(x, y) = \tilde{Q}(A_ix, y) + \tilde{Q}(x, A_iy) \quad \text{for all } x, y \in Y, \; i = 1, \ldots, N$$

(11)
and some $C_Q$. Then it holds

$$\left\| A^\alpha \tilde{Q}(x, x) \right\|_X \leq 2C_Q \sum_{i=0}^n \sum_{\beta \leq \alpha} \left( \begin{array}{c} \alpha \\ \beta \end{array} \right) \|A^\alpha_{\beta\gamma}x\|_X \|A_{\beta\gamma}^\alpha x\|_X$$

and

$$\left\| A^\alpha(\tilde{Q}(x, x) - Q(y, y)) \right\|_X \leq 2C_Q \sup_{z \in \{x, y\}} \sum_{i=1}^n \sum_{\gamma \leq \alpha} \left( \begin{array}{c} \alpha \\ \gamma \end{array} \right) \left( \|A^\alpha_{\beta\gamma}z\|_X \|A_{\beta\gamma}^\alpha (y - x)\|_X + \|A^\alpha_{\beta\gamma} (y - x)\|_X \|A_{\beta\gamma}^\alpha z\|_X \right)$$

for all $\alpha, \beta \in \mathbb{N}_0^n$, $x, y \in Y$. In particular, $Q(y) := \tilde{Q}(y, y)$ satisfies the assumption of (H3) with $M_0 = \tilde{M}_0 = 0$, $M_1 = \tilde{M}_1 = 2C_Q$.

**Proof.** According to the Leibniz formula, it holds

$$\left\| A^\alpha \tilde{Q}(x, y) \right\|_X \leq \sum_{\beta \leq \alpha} \left( \begin{array}{c} \alpha \\ \beta \end{array} \right) \|Q(A^\beta x, A^{\alpha - \beta} y)\|_X$$

$$\leq C_Q \sum_{i=0}^n \sum_{\beta \leq \alpha} \left( \begin{array}{c} \alpha \\ \beta \end{array} \right) \left( \|A^{\beta + \hat{e}_i} x\|_X \|A^{\alpha - \beta} y\|_X + \|A^\beta x\|_X \|A^{\alpha - \beta + \hat{e}_i} y\|_X \right).$$

(12)

This implies the first assertion setting $y = x$. Since $\tilde{Q}$ is bilinear, we have

$$\left\| A^\alpha(\tilde{Q}(x_1, y_1) - \tilde{Q}(x_2, y_2)) \right\|_X \leq \left\| A^\alpha \tilde{Q}(x_1 - x_2, y_1) \right\|_X + \left\| A^\alpha Q(x_2, y_2 - y_1) \right\|_X.$$  This implies directly the second assertion using (12). \hfill $\Box$

2. **Preliminary commutator estimates for $L$.**

**Lemma 5.** Let $\alpha \in \mathbb{N}_0^n$. Then

$$[L, A^\alpha] = \sum_{0 \neq \gamma \leq \alpha} \left( \begin{array}{c} \alpha \\ \gamma \end{array} \right)(-1)^{\gamma_\alpha - 1} L_\gamma A^{\alpha - \gamma}. $$
Lemma 6. Let \( C, r \) be as in (H2b). Then for \( \nu < 1/r \) it holds

\[
\sum_{\alpha \leq N} \frac{\nu^{|\alpha|}}{\alpha!} \| [L, A^\alpha] y \|_X \leq \frac{nC\nu r}{(1 - \nu r)^n} \sum_{\alpha \geq N} \frac{\nu^{|\alpha|}}{\alpha!} \sum_{i=1}^n \| A^{\alpha+i} y \|_X
\]

for \( y \in Y \) and \( N \in \mathbb{N}_0^n \).

Proof. Let \( \| \cdot \|'_X := C \sum_{i=1}^n \| A^{\alpha_i} \|_X \). Using Lemma 5 and the hypothesis (1.1), we have

\[
\| [L, A^\alpha] x \|_X \leq \sum_{0 \leq \gamma \leq \alpha} \frac{\alpha}{\gamma} \| L_\gamma A^{\alpha-\gamma} x \|_X \leq \sum_{0 \leq \gamma \leq \alpha} \frac{\alpha^{|\gamma|}}{(\alpha - \gamma)!} \| A^{\alpha-\gamma} x \|'_X = \alpha! \sum_{\gamma \leq \alpha} \frac{\nu^{|\alpha-\gamma|}}{\gamma!} \| A^\gamma x \|'_X.
\]

Define \( \delta = \nu r \). Then for \( N \in \mathbb{N}_0^n \) and \( i = 1, \ldots, n \), it holds

\[
\sum_{\alpha \leq N} \frac{\nu^{|\alpha|}}{\alpha!} \| [L, A^\alpha] x \|_X \leq \sum_{\alpha \leq N} \sum_{\gamma \leq \alpha} \frac{\nu^{|\alpha-\gamma|}}{\gamma!} \| A^\gamma x \|'_X \leq \sum_{\alpha \leq N} \sum_{\gamma \leq \alpha} \frac{\nu^{|\alpha-\gamma|}}{\gamma!} \| A^\gamma x \|'_X \leq n\delta \sum_{\alpha \geq N} \frac{\nu^{|\alpha|}}{\gamma!} \| A^\gamma x \|'_X
\]
using the Cauchy-product for finite sums. Thus, we obtain the assertion by estimating $\sum_{\alpha \leq N} \delta^{[\alpha]} \leq \frac{1}{(1-\sigma)^n}$. \hfill \Box

3. Time depending collisions. Instead of the norm $\|\cdot\|_X$ and the r.h.s. $Q$, we can also use a time depending norm $\|\cdot\|_{X_t}$ on $Y$ and a time depending collision operator $Q_t$, respectively. Then we need the following assumptions.

Let $L$ be a generator of a strong continuous group $e^{tL}$ on $X$. There exists $C, r \geq 0$ such that

$$\|e^{tL}L_\alpha y\|_{X_t} \leq C\alpha! r^{[\alpha]} \sum_{i=1}^n \|e^{tL}A_i y\|_{X_t}$$  \hspace{1cm} (H2')

for all $\alpha \in \mathbb{N}_0^n$ and all $y \in Y$, where $L_0 = L$ and $L_{\alpha+\epsilon} := [L_\alpha, A_\epsilon]$.

Moreover, we assume that

$$\|e^{tL}A^\alpha Q_t(y)\|_{X_t} \leq e^{-\omega t} \sum_{\gamma \leq \alpha} \left( \begin{array}{c} \alpha \\ \gamma \end{array} \right) \|e^{tL}A^{\alpha-\gamma} y\|_{X_t} \sum_{|\beta| \leq 1} M_{|\beta|} \|e^{tL}A^{\gamma+\beta} y\|_{X_t}$$  \hspace{1cm} (H3a')

holds for all $t > 0$, $\alpha \in \mathbb{N}_0^n$, $y \in Y$ and some $M_{\beta} \geq 0$ and some $\omega > Cr$.

$$\|e^{tL}A^\alpha (Q_t(y) - Q_t(x))\|_X \leq e^{-\omega t} \sup_{z, \tilde{z} \in [x, y]} \sum_{\gamma \leq \alpha} \left( \begin{array}{c} \alpha \\ \gamma \end{array} \right) \left( \|A^{\alpha-\gamma} z\|_{X_t} \sum_{|\beta| \leq 1} M_{|\beta|} \|e^{tL}A^{\gamma+\beta} (y - x)\|_{X_t} \right.$$\hspace{1cm} (H3b')

$$\left. + \|A^{\alpha-\gamma} (y-x)\|_{X_t} \sum_{|\beta| \leq 1} M_{|\beta|} \|e^{tL}A^{\gamma+\beta} z\|_{X_t} \right)$$

for all $t > 0$, $\alpha \in \mathbb{N}_0^n$, $y \in Y$ and some $M_{\beta} \geq 0$, where $[x, y] := \{sx + (1-s)y : s \in [0, 1]\}$. Moreover, we need an estimate on the time derivative of the norm, i.e.,

$$\partial_t \|x(t)\|_{X_t} \leq \|\partial_t x(t)\|_{X_t}$$  \hspace{1cm} (H4')

for all $x \in C^1([0, \infty), X)$.

Lemma 7. For $\|\cdot\|_{X_t} := \|\cdot\|_X$ the modified hypothesis (H2')-(H4') are a consequence of the original ones (H2)-(H3) since $\|e^{tL}\|_{L(X)} \leq C_1 e^{\omega t}$ for $t \in \mathbb{R}$. Note that, we have to multiply the constant $C$ from (H2b) by $C_1^2$ to obtain the constant of (H2').

With the same arguments as in the proof of Lemma 6, we can prove its corresponding version:

Lemma 8. Let $C, r$ be as in (H2'). Then for $\nu < 1/r$, it holds

$$\sum_{\alpha \leq N} \frac{\nu^{[\alpha]}}{\alpha!} \|e^{tL}[L, A^\alpha] y\|_{X_t} \leq \frac{nC^2\nu^r}{(1-\nu r)^n} \sum_{\alpha \leq N} \frac{\nu^{[\alpha]}}{\alpha!} \sum_{i=1}^n \|e^{tL}A^{\alpha+\epsilon_i} y\|_{X_t}$$

for $y \in Y$ and $N \in \mathbb{N}_0^n$.

4. Transformed equation. As in the previous section, we may assume that $Q = Q_t$ depends directly on time and that we have a time depending norm such that (H2')-(H4') are fulfilled.
Lemma 10. Let $t \in \mathbb{R}$ and $y \in Y$, we define
\[ A_L := e^{tL}A e^{-tL} \] and $Q_{tL}(y) := e^{tL}Q_t(e^{-tL}y)$.
Thus, if $u$ is a solution of
\[ \partial_t u = Q_{tL}(u) \quad \text{with} \quad u(0) = u_0 := x_0 - \bar{x}, \] then $x(t) := \bar{x} + e^{-tL}u(t)$ solves (8) with $x(0) = x_0$.

The main strategy in this paper is to solve (13) by using the following time depended analytic semi-norms, which are a generalization of the norms found in [22].

Definition 5.
\[ \|y\|_{X^T_r} := \sum_{\alpha \in \mathbb{N}_0^N} \frac{\nu(|\alpha|)}{\alpha!} A^\alpha_t y \]
for $y \in Y$ and $t \in \mathbb{R}$.

Lemma 9. Let $y \in Y$ and $t \in \mathbb{R}$, $\nu \geq 0$. Then
\[ \|Q_{tL}(y)\|_{X^T_r} \leq e^{-\omega t} \|y\|_{X^T_r} \sum_{|\beta| \leq 1} M_{|\beta|} \|A^\beta_{tL} y\|_{X^T_r}. \]

Proof. We start making use of (H3a') and the multinomial formula to see
\[
\|Q_{tL}(y)\|_{X^T_r} = \sum_{\alpha \in \mathbb{N}_0^N} \nu(\alpha) \|e^{tL}A^\alpha e^{-tL}y\|_{X^T_r},
\]
\[
\leq e^{-\omega t} \sum_{\alpha \in \mathbb{N}_0^N} \nu(\alpha) \|e^{tL}A^\alpha e^{-tL}y\|_{X^T_r} \sum_{\alpha \in \mathbb{N}_0^N} \nu(\alpha) \sum_{|\beta| \leq 1} M_{|\beta|} \|A^\beta_{tL} e^{-tL}y\|_{X^T_r},
\]
\[
= e^{-\omega t} \|y\|_{X^T_r} \sum_{|\beta| \leq 1} M_{|\beta|} \|A^\beta_{tL} y\|_{X^T_r}. \]

Likewise, we can show the following Lipschitz estimate using (H3b') instead of (H3a').

Lemma 10. Let $y_1, y_2 \in Y$ and $t \in \mathbb{R}$, $\nu \geq 0$. Then
\[ \|Q_{tL}(y_2) - Q_{tL}(y_2)\|_{X^T_r} \leq e^{-\omega t} \sum_{|\beta| \leq 1} M_{|\beta|} \left( \|y_2\|_{X^T_r} \|A^\beta_{tL}(y_2 - y_1)\|_{X^T_r} + \|y_2 - y_1\|_{X^T_r} \|A^\beta_{tL} y_1\|_{X^T_r} \right). \]

Proposition 11. Let $\nu_0 < 1/r$ and
\[ \mu \geq \mu_0 := \frac{nCr}{(1 - \nu_0 r)^n}, \]
where $C$ is given by (H2'). We define $\nu(t) = \nu_0 \exp(-\mu t)$. Then
\[ \|u(t)\|_{X^T_L} + \sum_{t} \int_s^t (\mu - \mu_0) \nu(\tau) \|A^\alpha_{L\tau} u(\tau)\|_{X^T_L} d\tau \]
\[ \leq \|u(s)\|_{X^T_L} + \sum_{t} \int_s^t \|\partial_L u(\tau)\|_{X^T_L} d\tau \]
for $t > s \geq 0$ and $u \in C^0([0, \infty), X)$ such that $u(t) \in Y$ for all $t \geq 0$ and $t \mapsto A^\alpha_{tL} u(t) \in C^1([0, \infty), X)$ for all $\alpha \in \mathbb{N}_0^N$. 

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λ entails using $y$ for $L$ since $(0 \leq t < \infty)$, the dominated convergence theorem implies $\left| \int_0^T P_{u,N}(\nu(t), \nu(s)) ds \right| \leq \sum_{i=1}^n \left| \int_0^T e^{Lt}[L, A^\alpha]e^{-Lt}u(t) \right|_{X^\alpha}$ as $N \to \infty$. Let $\alpha \in \mathbb{N}_0^\alpha$ and $0 < s < t$. Since $\partial_t \cdot \|X\| \leq \|\partial_t \cdot \|X\|$, we have

$$\left| \int_0^T P_{u,N}(\nu(t), \nu(s)) ds \right| \leq \sum_{i=1}^n \left| \int_0^T e^{Lt}[L, A^\alpha]e^{-Lt}u(t) \right|_{X^\alpha} (t - s)$$

using

$$[\partial_t, A^\alpha]y = e^{Lt}A^\alpha[\partial_t, e^{-Lt}]y + [\partial_t, e^{Lt}]A^\alpha e^{-Lt}y$$

$$= e^{Lt}LA^\alpha e^{-Lt}y + e^{Lt}A^\alpha Le^{-Lt}y$$

$$= e^{Lt}[L, A^\alpha]e^{-Lt}y$$

for $y \in Y$. This implies

$$|P_{u,N}(\nu(t), \nu(s))| \leq \sum_{i=1}^n \left| \int_0^T e^{Lt}[L, A^\alpha]e^{-Lt}u(t) \right|_{X^\alpha} (t - s).$$

The estimate

$$\frac{\nu(t) - \nu(s)}{\alpha!} \geq \sum_{i=1}^n \left| \frac{\nu(t) - \nu(s)}{\alpha - \hat{\epsilon}}! \right| (\nu(t) - \nu(s))$$

entails

$$|P_{u,N}(\nu(t), \nu(s))| \leq \sum_{i=1}^n \left| \int_0^T e^{Lt}[L, A^\alpha]e^{-Lt}u(t) \right|_{X^\alpha} (t - s)$$

$$\leq \sum_{i=1}^n \left| \int_0^T e^{Lt}[L, A^\alpha]e^{-Lt}u(t) \right|_{X^\alpha} (t - s).$$

Thus, $P_{u,N}(\nu(t), \nu(s))$ is Lipschitz continuous w.r.t. $t$ and belongs to $W^{1,\infty}((0, T))$ with

$$\frac{d}{dt}P_{u,N}(\nu(t), \nu(s)) \leq P_{\partial u,N}(\nu(t), \nu(t)) + R_N(\nu(t), t) + \dot{\nu}(t)Q_N(\nu(t), t),$$

since $P_{u,N}, P_{\partial u,N}$ and $Q_N$ are continuous. By $P_{\partial u,N}(\nu(t), \tau) \leq \|\partial_t u(t)\|_{X^\alpha(t)} \in L^1(0, T)$, the dominated convergence theorem implies

$$\int_0^T P_{\partial u,N}(\nu(t), \tau) d\tau \to \int_0^T \|\partial_t u(t)\|_{X^\alpha(t)} d\tau$$
as \( N \to \infty \). According to the monotone convergence theorem we have

\[
\int_s^t \nu(\tau)Q_N(\nu(\tau), \tau)\,d\tau \to \sum_{i=1}^n \int_s^t \nu(\tau) \left\| A_{tL}^{\tilde{\varepsilon}} u(\tau) \right\|_{X_{tL}^r} \,d\tau.
\]

Then Lemma 8 yields

\[
R_N(\nu(t), t) \leq \frac{nC\nu(t)r}{(1 - \nu(t)r)^n} \sum_{\alpha \geq N} \frac{\nu(t)^{[\alpha]}}{\alpha!} \sum_{i=1}^n \left\| e^{tL} A^{\alpha + \varepsilon} e^{-tL} u(t) \right\|_X
\]

\[
\leq \mu_0 \nu(t)Q_N(\nu(t), t)
\]

for \( N \in \mathbb{N}_0 \) recalling \( \mu_0 := \frac{nC}{(1 - \nu_0 r)^n} \) and \( \nu(t) \leq \nu_0 \). Finally, we obtain

\[
\|u(t)\|_{X_t^r(\nu)} + \sum_{i=1}^n \int_s^t \left( (\mu - \mu_0) \nu(\tau) \left\| A_{tL}^{\tilde{\varepsilon}} u(\tau) \right\|_{X_t^r(\nu)} - \|\partial_t u(\tau)\|_{X_{tL}^r(\nu)} \right) \,d\tau
\]

\[
\leq \limsup_{N \to \infty} P_{u, N}(\nu(t), t)
\]

\[
- \int_s^t (\hat{\nu}(\tau)Q_N(\nu(\tau), \tau) + R_N(\nu(\tau), \tau) + P_{\partial_t u, N}(\nu(\tau), \tau)) \,d\tau
\]

\[
\leq \|u(s)\|_{X_s^r(\nu)}.
\]

This finishes the proof using \( \hat{\nu} = -\mu \nu \).

\[\square\]

**Definition 6.** Let

\[
C_L^r([0, \infty); Y) := \{ u : [0, \infty) \to Y \text{ s.t. } t \mapsto A_{tL}^{\tilde{\varepsilon}} u(t) \in C^r([0, \infty), X), \ \alpha \in \mathbb{N}_0^n \}
\]

for \( i \in \mathbb{N}_0 \). We define \( \Phi : C_L^0([0, \infty); Y) \to C_L^0([0, \infty); Y) \) by

\[
\Phi(u)(t) := u(0) + \int_0^t Q_{tL}(u(\tau))\,d\tau.
\]

Let \( \nu_0 < 1/r \) and

\[
\mu \geq \mu_0 := \frac{nCr}{(1 - \nu_0 r)^n}
\]

with \( C \) as in (H2'). We define \( \nu(t) = \nu_0 \exp(-\mu t) \) and

\[
\|u\|_{\nu_0, \mu} := \sup_{t \geq 0} \left( \|u(t)\|_{X_t^r(\nu_0)} + (\mu - \mu_0) \sum_{i=1}^n \int_0^t \nu(\tau) \left\| A_{tL}^{\tilde{\varepsilon}} u(\tau) \right\|_{X_t^r(\nu_0)} \,d\tau \right)
\]

for \( u \in C_L^0([0, \infty); Y); \)

**Lemma 12.** Let \( \nu_0 < 1/r \) and assume that \( \omega > \mu_0 \). Then

\[
\|\Phi(u)\|_{\nu_0, \omega} \leq \|u(0)\|_{X_0^0(\nu_0)} + \max \left\{ \frac{M_0}{\omega}, \frac{M_1}{\nu_0(\omega - \mu_0)} \right\} \|u\|^2_{\nu_0, \omega}.
\]

**Proof.** Applying Proposition 11 to \( \Phi(u) \), we obtain

\[
\|\Phi(u)\|_{\nu_0, \omega} = \sup_{t \geq 0} \left( \|\Phi(u)(t)\|_{X_t^r(\nu_0)} + (\omega - \mu_0) \sum_{i=1}^n \int_0^t \nu(\tau) \left\| A_{tL}^{\tilde{\varepsilon}} \Phi(u)(\tau) \right\|_{X_t^r(\nu_0)} \,d\tau \right)
\]

\[
\leq \|\Phi(u)(0)\|_{X_0^r(\nu_0)} + \int_0^\infty \|\partial_t \Phi(u)(\tau)\|_{X_t^r(\nu_0)} \,d\tau
\]

\[
= \|u(0)\|_{X_0^r(\nu_0)} + \int_0^\infty \|Q_{tL}(u(\tau))\|_{X_t^r(\nu_0)} \,d\tau.
\]
Thus,

\[ \| \Phi(u) \|_{\nu_0, \omega} \leq \| u(0) \|_{X_0^{(0)}} + \int_0^\infty e^{-\omega \tau} \| u(\tau) \|_{X_{\nu_0}^\rho} \sum_{|\beta| \leq 1} M|\beta| \left\| A_{L, \tau}^\beta u(\tau) \right\|_{X_{\nu_0}^\rho} \, d\tau \]

\[ \leq \| u(0) \|_{X_0^{(0)}} + \| u \|_{\nu_0, \omega} \sum_{|\beta| \leq 1} \int_0^\infty e^{-\omega \tau} M|\beta| \left\| A_{L, \tau}^\beta u(\tau) \right\|_{X_{\nu_0}^\rho} \, d\tau. \]

For \( \beta = 0 \), we estimate

\[ \int_0^\infty e^{-\omega \tau} M_0 \left\| A_{L, \tau}^\beta u(\tau) \right\|_{X_{\nu_0}^\rho} \, d\tau = \int_0^\infty e^{-\omega \tau} d\tau \sup_{0 \leq \tau < \infty} M_0 \| u(\tau) \|_{X_{\nu_0}^\rho} \leq \frac{M_0}{\omega} \| u \|_{\nu_0, \omega}. \]

In the remaining cases where \( |\beta| = 1 \), we have

\[ \int_0^\infty e^{-\omega \tau} M_1 \left\| A_{L, \tau}^\beta u(\tau) \right\|_{X_{\nu_0}^\rho} \, d\tau \leq \frac{M_1}{\nu_0(\omega - \mu_0)} \int_0^\infty \nu_0 e^{-\omega \tau} \left\| A_{L, \tau}^\beta u(\tau) \right\|_{X_{\nu_0}^\rho} \, d\tau \]

\[ \leq \frac{M_1}{\nu_0(\omega - \mu_0)} \| u \|_{\nu_0, \omega}. \]

Finally, we conclude with

\[ \| \Phi(u) \|_{\nu_0, \omega} \leq \| u(0) \|_{X_0^{(0)}} + \max \left\{ \frac{M_0}{\omega}, \frac{M_1}{\nu_0(\omega - \mu_0)} \right\} \| u \|_{\nu_0, \omega}^2 \]

the assertion. \( \square \)

**Lemma 13.** Assuming the same hypothesis as in the previous lemma, let \( u, v \in C_L^0([0, \infty), Y) \) such that \( R = \max \{ \| u \|_{\nu_0, \omega}, \| v \|_{\nu_0, \omega} \} \). Then

\[ \| \Phi(u) - \Phi(v) \|_{\nu_0, \omega} \leq \| u(0) - v(0) \|_{X_0^{(0)}} + 2R \max \left\{ \frac{M_0}{\omega}, \frac{M_1}{\nu_0(\omega - \mu_0)} \right\} \| u - v \|_{\nu_0, \omega} \]

**Proof.** Let \( u, v \in C_L^0([0, \infty); Y) \) such that \( \| u \|_{\nu, \mu}, \| v \|_{\nu, \mu} \leq R \). We have

\[ \| \Phi(u) - \Phi(v) \|_{\nu_0, \omega} = \sup_{t \geq 0} \left( \| \Phi(u)(t) - \Phi(v)(t) \|_{X_{\nu_0}^\rho(t)} \right. \]

\[ + \left. (\omega - \mu_0) \sum_{n=1}^\infty \int_0^t \nu(\tau) \left\| A_{L, \tau}^n (\Phi(u)(\tau) - \Phi(v)(\tau)) \right\|_{X_{\nu_0}^\rho(\tau)} \, d\tau \right) \]

\[ \leq \| u(0) - v(0) \|_{X_0^{(0)}} + \int_0^\infty \| Q_{L, \tau}(u(\tau)) - Q_{L, \tau}(v(\tau)) \|_{X_{\nu_0}^\rho(\tau)} \, d\tau \]

For the next step, we have to use the condition (H3b) and proceed similarly as in the proof of Lemma 12. Note that for \( \xi \in [v, u] := \{ t \mapsto s(t)v(t) + (1 - s(t))u(t) : s(t) \in [0, 1] \} \), we observe that \( \| \xi \|_{\nu_0, \omega} \leq R \). Thus,

\[ \| \Phi(u) - \Phi(v) \|_{\nu_0, \omega} \leq \| u(0) - v(0) \|_{X_0^{(0)}} + \int_0^\infty e^{-\omega \tau} \| u(\tau) - v(\tau) \|_{X_0^{(0)}} \sum_{|\beta| \leq 1} M|\beta| \left\| A_{L, \tau}^\beta \xi(\tau) \right\|_{X_{\nu_0}^\rho} \, d\tau \]

\[ + \sup_{\xi \in [v, u]} \int_0^\infty e^{-\omega \tau} \| \xi(\tau) \|_{X_{\nu_0}^\rho} \sum_{|\beta| \leq 1} M|\beta| \left\| A_{L, \tau}^\beta (u(\tau) - v(\tau)) \right\|_{X_{\nu_0}^\rho} \, d\tau \]

\[ \leq 2R \max \left\{ \frac{M_0}{\omega}, \frac{M_1}{\nu_0(\omega - \mu_0)} \right\} \| u - v \|_{\nu_0, \omega}. \]

This terminates the proof. \( \square \)
Likewise,

**Theorem 15.** Let \( \nu \) denote the subspace of \( C^0([0, \infty), Y) \) such that \( \|u\|_{\nu, \mu} < \infty \) for all \( u \in Z \). Note that \( Z \) endowed with \( \|\cdot\|_{\nu, \omega} \) is a Banach space. For \( R > 0 \), we define \( Z_R := \{u \in Z : \|u\|_{\nu, \omega} \leq R \text{ and } u(0) := u_0\} \).

**Proposition 14.** Let \( \nu_0 < \frac{1}{2}(1 - \sqrt{\frac{nCr}{2}}) \), and let \( R > 0 \) satisfy

\[
C_0 := 1 - R \max \left\{ \frac{M_0}{\omega}, \frac{M_1}{\nu_0(\omega - \mu_0)} \right\} > 0, \quad C_1 := 1 - 2R \max \left\{ \frac{M_0'}{\omega}, \frac{M_1'}{\nu_0(\omega - \mu_0)} \right\} > 0,
\]

where \( \mu_0 = \frac{nCr}{(1 - \nu_0r)2} > \omega \). Then for all \( u_0 \in Y \) with

\[
\|u_0\|_{X_0^{\nu_0}} \leq C_0R,
\]

the equation (13) has a unique solution \( u \) in \( Z_R \) satisfying \( u|_{t=0} = u_0 \). Moreover, let \( u_0, w_0 \) satisfy (14) and let \( u, w \) be the solution of (13) with \( u(0) = u_0 \) and \( w(0) = w_0 \), respectively. Then

\[
C_1\|u - w\|_{\nu_0, \omega} \leq \|u_0 - w_0\|_{X_0^{\nu_0}}.
\]

**Proof.** We combine the last two lemmata with the Banach fixed-point theorem to see that \( \Phi : Z_R \to Z_R \) is a contraction and admits a unique fixed point \( u \). By the definition of \( Z \) we easily see that \( u \) is differentiable with w.r.t. \( t \) in \( X \) such that \( u \) is a strong solution of (13).

**Theorem 15.** Let \( \omega, C, r \) be as in (H2’),(H3a’) and (H3b’). Then for every positive \( \nu_0 < \frac{1}{2}(1 - \sqrt{\frac{nCr}{2}}) \), there exists an \( \varepsilon > 0 \) such that if

\[
\|u_0\|_{X_0^{\nu_0}} \leq \varepsilon
\]

for some \( \nu \leq \nu_0 \), then (13) has a strong solution \( u \) with \( u|_{t=0} = u_0 \), with

\[
\|u(t)\|_{X_{\nu_0}^{\nu_t}} \leq 2\varepsilon \nu \quad \text{for all } t \geq 0.
\]

Moreover, for all \( u_0, w_0 \in Y \) satisfying (15), we have

\[
\|u(t) - w(t)\|_{X_{\nu_0}^{\nu_t}} \leq 2\|u_0 - w_0\|_{X_0^{\nu_0}} \quad \text{for all } t \geq 0,
\]

where \( u, w \) are the solution of (13) with \( u(0) = u_0 \) and \( w(0) = w_0 \), respectively.

**Proof.** First, we recall \( \mu_0 := \frac{nCr}{(1 - \nu_0r)^2} \). We choose \( R' > 0 \) such that

\[
R' < \min \left\{ \frac{\omega}{M_0}, \frac{\omega - \mu_0}{M_1} \right\}, \quad R' < \frac{1}{4} \min \left\{ \frac{\omega}{M_0'}, \frac{\omega - \mu_0}{M_1'} \right\}.
\]

With this, we define \( R := R'\nu \) and \( \varepsilon := 1 - R' \max \left\{ \frac{M_0\nu_0}{\omega}, \frac{M_1}{\omega - \mu_0} \right\} \). Thus,

\[
C_0 := 1 - R \max \left\{ \frac{M_0}{\omega}, \frac{M_1}{\nu(\omega - \mu_0)} \right\} \geq \varepsilon > 0.
\]

Likewise,

\[
C_1 := 1 - 2R \max \left\{ \frac{M_0'}{\omega}, \frac{M_1'}{\nu(\omega - \mu_0)} \right\} \geq 1 - 2R' \max \left\{ \frac{M_0\nu_0}{\omega}, \frac{M_1}{(\omega - \mu_0)} \right\} \geq \frac{1}{2} > 0.
\]

Finally, we obtain the assertion by applying the theorem. Note that the estimate

\[
\|u(t)\|_{X_{\nu_0}^{\nu_t}} \leq 2\varepsilon \nu \quad \text{for all } t \geq 0
\]

is a direct consequence of

\[
\|u(t) - w(t)\|_{X_{\nu_0}^{\nu_t}} \leq 2\|u_0 - w_0\|_{X_0^{\nu_0}} \quad \text{for all } t \geq 0.
\]
for \( w(t) = w_0 = 0 \) and \( \| u_0 \|_{X_0^\nu} \leq \varepsilon \nu. \)

**Remark 16.** By shrinking \( R' > 0 \) such that
\[
R' < \min \left\{ \frac{\omega}{M_\nu \mu_0}, \frac{\omega - \mu_0}{M_\nu} \right\}, \quad R' \leq \frac{\gamma}{2} \min \left\{ \frac{\omega}{M_\nu \mu_0}, \frac{\omega - \mu_0}{M_\nu} \right\}
\]
is satisfied for a fixed positive \( \gamma \in (0, 1) \). We can show similarly as in the previous proof that
\[
\| u(t) - w(t) \|_{X_0^\nu} \leq \frac{1}{1 - \gamma} \| u_0 - w_0 \|_{X_0^\nu} \quad \text{for all } t \geq 0
\]
if \( \varepsilon := 1 - R' \max \left\{ \frac{M_{\nu 0}}{\omega}, \frac{M_{\nu}}{\omega - \mu_0} \right\} \).

**Proof of Theorem 3.** According to Lemma 7, Theorem 3 is a direct consequence of Theorem 15 for \( u_0 := x_0 - \bar{x} \) and \( x(t) := \bar{x} + e^{-tL}u(t) \):
\[
\| x(t) - \bar{x} \|_{X_0^\nu} = \sum_{\alpha \in \mathbb{N}_0^n} \frac{(\nu e^{-\omega t})|\alpha|}{\alpha!} \| A^\alpha (x(t) - \bar{x}) \|_X
\]
\[
\leq C_L e^{-\omega t} \sum_{\alpha \in \mathbb{N}_0^n} \frac{(\nu e^{-\omega t})|\alpha|}{\alpha!} \| e^{tL} A^\alpha e^{-tL} e^{tL} (x(t) - \bar{x}) \|_X
\]
\[
= C_L e^{-\omega t} \| e^{tL} (x(t) - \bar{x}) \|_{X_0^\nu} = C_L e^{-\omega t} \| u(t) \|_{X_0^\nu} \leq 2\varepsilon \nu C_L e^{-\omega t}
\]
for all \( t \geq 0 \). Likewise, we have
\[
\| x(t) - y(t) \|_{X_0^\nu} \leq 2C_L e^{-\omega t} \| x_0 - y_0 \|_{X_0^\nu}
\]
for every \( t \geq 0 \). \( \square \)

5. **The model case.** In this section, we consider the model equation (5) with \( \lambda = (\lambda_0, \lambda_1) \in \mathbb{R}^2, \lambda_1 \geq 0 \). The substitution
\[
g(x, p, t) := e^{\frac{t}{\lambda}} (f(x, p, t) - F_\lambda(p))
\]
leads to the system
\[
\begin{aligned}
\partial_t g + \nabla \varepsilon(p) \cdot \nabla_x g - U \nabla_x \int_{\mathbb{T}^d} gd' \cdot \nabla_p F_\lambda(p) &= U e^{\frac{t}{\lambda}} \nabla_x \int_{\mathbb{T}^d} gd' \cdot \nabla_p g, \\
g|_{t=0} &= g_0
\end{aligned}
\]
for \( g_0 := f_0 - F_\lambda \). Defining
\[
L f(x, p) := \nabla \varepsilon(p) \cdot \nabla_x f(x, p) - U \nabla_x \int_{\mathbb{T}^d} f(x, p') dp' \cdot \nabla_p F_\lambda(p)
\]
and
\[
Q_l(f)(x, p) := U e^{\frac{t}{\lambda}} \nabla_x \int_{\mathbb{T}^d} f(x, p') dp' \cdot \nabla_p f(x, p),
\]
we can rewrite (16) to
\[
\partial_t g + L g = Q_l(g). \quad (17)
\]
The idea is now to apply the general result, which requires the hypothesis (H2')-(H4').

**Definition 8.** Let \( \mathcal{S}(\mathbb{R}^d \times \mathbb{T}^d) \) and \( C^\infty_c(\mathbb{R}^d \times \mathbb{T}^d) \) be the Schwartz space and the space of bounded smooth functions, respectively. Let \( \lambda = (\lambda_0, \lambda_1) \in \mathbb{R}^2_+ := \{(x, y) \in \mathbb{R}^2 : y \geq 0\} \) and \( F_\lambda(p) = 1/(\eta + e^{-\lambda_0 - \lambda_1 \varepsilon(p)}) \) be the Fermi-Dirac distribution function.
Lemma 17. There exists a $C_\lambda > 0$ such that
\[
\|\rho_h g\|_X \leq C_\lambda \|h\|_X \|g\|_X
\]
for $\rho_h(x) := \int_{T^d} h(x,p)dp$. For $X = C^0_b(\mathbb{R}^d \times T^d)$, we can easily see that $C_\lambda = 1$ using $|T^d| = 1$. In the other case, the assertion is a consequence of the algebra properties of $H^k$ for $k \geq \frac{d}{2}$.

Lemma 18. $L : D(L) \subset X \to X$ is a generator of a $C_0$ contraction group $(e^{tL})_{t \in \mathbb{R}}$ with $L(Y) \subset Y \subset D(L)$.

The following proof is similar as the proof of Theorem 3.1 of [8].

Proof. The assertion is clear for $\lambda_1 = 0$ and $X = C^0_b(\mathbb{R}^d \times T^d)$ since then $(e^{tL})$ is a transport contraction group generated by $L = \nabla \varphi(p) \cdot \nabla \varphi$. Now, let $\lambda \in \mathbb{R}^*_+$ and $X = H^k_\beta L^2_p(\mathbb{R}^d \times T^d)$.

For $h \in Y := \mathcal{S}(\mathbb{R}^d \times T^d)$ with $\|h\|_X < \infty$ we have
\[
\langle Lh, h \rangle_X = \sum_{|\alpha| \leq k} \langle \partial_x^\alpha Lh, \partial_x^\alpha h \rangle_0
\]
\[
= \sum_{|\alpha| \leq k} \langle L\partial_x^\alpha h, \partial_x^\alpha h \rangle_0
\]
since we can easily show that $[L, \partial_{x_i}] = 0$ for $i = 1, \ldots, d$. Then abbreviating $g := \partial_x^\alpha h$, we have
\[
\langle Lg, g \rangle_0 = \int_{\mathbb{R}^d} \int_{T^d} (\nabla \varphi(p) \cdot \nabla \varphi g(x, p) - U \nabla \varphi \rho_g(x) \cdot \nabla \varphi \rho_g(p)) g(x, p) \frac{dpdx}{\mathcal{F}_\lambda(1 - \eta \mathcal{F}_\lambda)}
\]
\[
+ U\lambda_1 \int_{\mathbb{R}^d} \int_{T^d} \left( \nabla \varphi(p) \cdot \nabla \varphi g(x, p) - U \nabla \varphi \rho_g(x) \cdot \nabla \varphi \rho_g(p) \right) d\rho_g(x) dx
\]
\[
= \int_{\mathbb{R}^d} \int_{T^d} \frac{\nabla \varphi(p) \cdot \nabla \varphi g(x, p)^2}{2\mathcal{F}_\lambda(1 - \eta \mathcal{F}_\lambda)} dxdp
\]
\[
- \lambda_1 U \int_{\mathbb{R}^d} \nabla \varphi \rho_g(x) \cdot \int_{T^d} \nabla \varphi(p) g(x, p) d\rho_g(x)
\]
\[
- U \lambda_1 \int_{\mathbb{R}^d} \int_{T^d} \nabla \varphi(p) \cdot \nabla \varphi g(x, p) d\rho_g(x) dx
\]
\[
- U \int_{\mathbb{R}^d} \int_{T^d} \nabla \varphi \rho_g(x) \cdot \nabla \varphi \rho_g(p) d\rho_g(x) dx =: I_1 + I_2 + I_3 + I_4
\]
By the Gauss law, we see that \( I_1 = I_4 = 0 \). Moreover, \( I_2 = -I_3 \) implying that \( \langle Lg, g \rangle_0 = 0 \) and hence \( \langle Lh, h \rangle_X = 0 \). Thus, \( L \) is the closure of an anti-symmetric operator such that
\[
\| (\sigma + L)h \|_X \| h \|_X \geq |\langle (\sigma + L)h, h \rangle_X | = |\sigma| \| h \|_X
\]
for \( \sigma \in \mathbb{C} \) with \( \Re \sigma \neq 0 \). Next, as in [8], we want to show that \( L \) is indeed anti-adjoint. For this, we need show for \( \sigma \in \mathbb{R} \setminus \{0\} \) that \( (\sigma + L) \) is surjective onto \( X \) according to (cf. Theorem V-3.16 or Problem V-3.31 in [19]). Let \( h \in Y \). We have to find a solution to the equation
\[
\sigma f + Lf = h.
\]
Applying the Fourier transform w.r.t. \( x \) to (18), we obtain
\[
\sigma \hat{f}(\xi, p) + \nabla \rho \varepsilon(p) \cdot i\xi \hat{f}(\xi, p) - U i\xi \hat{\rho}_f(\xi) \cdot \nabla \rho \mathcal{F}_\lambda(p) = \hat{h}(\xi, p),
\]
where \( \hat{\rho}_f := \int_{\mathbb{T}^d} \hat{f} dp \) implying
\[
\hat{f} = \frac{1}{\sigma + \nabla \rho \varepsilon(p)} \left( \hat{h} + iU \xi \hat{\rho}_f \cdot \nabla \rho \mathcal{F}_\lambda \right).
\]
An integration of this equality leads to \( \hat{\rho}_f = \hat{\varrho} \) with
\[
\left( 1 - U \int_{\mathbb{T}^d} \frac{i\xi \cdot \nabla \rho \mathcal{F}_\lambda(p)}{\sigma + \nabla \rho \varepsilon(p) \cdot i\xi} dp \right) \hat{\rho}(\xi) = \int_{\mathbb{T}^d} \frac{\hat{h}(\xi, p)}{\sigma + \nabla \rho \varepsilon(p) \cdot i\xi} dp.
\]
Since \( \varepsilon(-p) = \varepsilon(p) \) implies \( \mathcal{F}_\lambda(-p) = \mathcal{F}_\lambda(p) \) and \( \nabla \varepsilon(-p) = -\nabla \varepsilon(p) \), we have
\[
\int_{\mathbb{T}^d} \frac{i\xi \cdot \nabla \rho \mathcal{F}_\lambda(p)}{\sigma + \nabla \rho \varepsilon(p) \cdot i\xi} dp = \lambda_1 \int_{\mathbb{T}^d} \frac{i\xi \cdot \nabla \rho \varepsilon(p)}{\sigma + \nabla \rho \varepsilon(p) \cdot i\xi} \mathcal{F}_\lambda(p)(1 - \eta \mathcal{F}_\lambda(p)) dp
\]
\[
= \lambda_1 \int_{\mathbb{T}^d} \frac{\sigma i\xi \cdot \nabla \rho \varepsilon(p)}{\sigma^2 + |\nabla \rho \varepsilon(p) \cdot \xi|^2} \mathcal{F}_\lambda(p)(1 - \eta \mathcal{F}_\lambda(p)) dp
\]
\[
+ \lambda_1 \int_{\mathbb{T}^d} \frac{|\xi \cdot \nabla \rho \varepsilon(p)|^2}{\sigma^2 + |\nabla \rho \varepsilon(p) \cdot \xi|^2} \mathcal{F}_\lambda(p)(1 - \eta \mathcal{F}_\lambda(p)) dp
\]
\[
= \lambda_1 \int_{\mathbb{T}^d} \frac{|\xi \cdot \nabla \rho \varepsilon(p)|^2}{\sigma^2 + |\nabla \rho \varepsilon(p) \cdot \xi|^2} \mathcal{F}_\lambda(p)(1 - \eta \mathcal{F}_\lambda(p)) dp \geq 0.
\]
Thus, we can define \( \hat{\varrho} \) by (19) and obtain
\[
|\hat{\varrho}(\xi)| \leq \left| \int_{\mathbb{T}^d} \frac{\hat{h}(\xi, p)}{\sigma + \nabla \rho \varepsilon(p) \cdot i\xi} dp \right|.
\]
We set \( \hat{f} = \frac{1}{\sigma + \nabla \varepsilon(p) \cdot i\xi} \hat{h} \) and have \( \hat{\rho}_f = \hat{\varrho} \). Therefore, we can easily see that there exists a constant \( C_\sigma > 0 \) independent of \( h \) such that
\[
\langle f, f \rangle_0 \leq C_\sigma^2 \langle h, h \rangle_0.
\]
Repeating this argument for \( \partial_x^\alpha h \) instead of \( h \) and using that \( \partial_x^\alpha \) commutes with \( L \), we see that
\[
\| f \|_X \leq C_\sigma \| h \|_X,
\]
which entails that \( f \in X \) which implies that \( f \in D(L) \). Finally, since \( \mathcal{S}(\mathbb{R}^d \times \mathbb{T}^d) \) is dense in \( X \) and \( L \) is a closed operator, we have that \( \sigma + L \) is bijective from \( D(L) \) onto \( X \). Thus, \( L \) is anti-adjoint and fulfills
\[
\| (\sigma + L)^{-1} \|_{L(X)} \leq \frac{1}{|\Re \sigma|} \text{ for } \sigma \in \mathbb{C} \setminus i\mathbb{R}.
\]
At this point, we have showed the hypothesis of the Hille-Yosida Theorem (see Corollary 3.7 of Chapter II in [13]) for the generation of a contraction group, which implies the assertion. \(\square\)

Unfortunately, our collision term \(Q\) is very irregular. We cannot use the norm \(\|\cdot\|_X\) to show (H3a) and (H3b).

As we have seen in the proof of the general case, we work with time depending norm on the space \(Y\). Therefore, we can already use a time depending norm \(\|\cdot\|_{X_t}\) on the base space \(Y\).

**Definition 9.** Fix \(\delta > 0\) and let \(t \in \mathbb{R}\). We define
\[
\|f\|_{X_t} := e^{-\delta t} \sum_{|\alpha + \beta| \leq 1} \|e^{tL} \partial_\alpha^\beta \partial_p^\beta e^{-tL} f\|_X
\]
for \(f \in Y\).

For the proof of the hypothesis (H2'), we need the following lemma.

**Lemma 19.** There exist \(C > 0\) and \(r_0 > 0\) such that
\[
\|\partial_p^\beta \nabla_\rho \varepsilon(p) g\|_X + \|U \partial_p^\beta \nabla_\rho \mathcal{F}_\lambda(p) n_\rho\|_X \leq C \beta! r_0^{1/2} \|g\|_X
\]
for all \(\beta \in \mathbb{N}_0^d\) and \(g \in X\), where \(\rho_\gamma := \int_{T^d} gd\rho_\gamma\).

**Proof.** The proof is straight-forward using the analyticity of \(\varepsilon\) and \(\mathcal{F}_\lambda\). \(\square\)

**Lemma 20.** We have \([L, \partial_x] = 0\) and
\[
\|e^{tL} \tilde{L}_\alpha f\|_{X_t} \leq C \alpha! r^{1/2} \|e^{tL} \partial_x f\|_{X_t},
\]
for some \(C, r > 0\) and \(\tilde{L}_{\beta + \varepsilon} := [\tilde{L}_\beta, \partial_p], \tilde{L}_0 := L\).

**Proof.** The assertion \([L, \partial_x] = 0\) can be obtained by a straight-forward calculation. Then
\[
\partial_p\tilde{L}_\alpha f(x, p) = \partial_p \left( \nabla_\rho \varepsilon(p) \cdot \nabla_x f(x, p) - U \nabla_x \int_{T^d} f(x, p') dp' \cdot \nabla_\rho \mathcal{F}_\lambda(p) \right)
\]

\[
= \partial_p, \nabla_\rho \varepsilon(p) \cdot \nabla_x f(x, p) + \nabla_\rho \varepsilon(p) \cdot \nabla_x \partial_p f(x, p) - U \nabla_x \int_{T^d} f(x, p') dp' \cdot \nabla_\rho \partial_p \mathcal{F}_\lambda(p)
\]
and
\[
\tilde{L}_\alpha f(x, p) = \nabla_\rho \varepsilon(p) \cdot \nabla_x \partial_p f(x, p) - U \nabla_x \int_{T^d} \partial_p f(x, p') dp' \cdot \nabla_x \mathcal{F}_\lambda(p)
\]

imply that
\[
\tilde{L}_{\varepsilon_1} := [L, \partial_p] = -\partial_p, \nabla_\rho \varepsilon(p) \cdot \nabla_x + U \int_{T^d} \nabla_x f(x, p') dp' \cdot \nabla_\rho \partial_p \mathcal{F}_\lambda(p).
\]

We see that \(\tilde{L}_{\varepsilon_1}\) has a similar form to \(L\). Likewise to the calculation above, we obtain
\[
(-1)^{|\beta|} \tilde{L}_\beta f = \partial_p^\beta \nabla_\rho \varepsilon(p) \cdot \nabla_x f - U \int_{T^d} \nabla_x f dp' \cdot \nabla_\rho \partial_p^\beta \mathcal{F}_\lambda(p).
\]
According to Lemma 19, this implies that
\[ \| \tilde{L}_\beta f \|_X \leq C|\beta| \| \nabla_x f \|_X \leq C|\beta| \sum_{i=1}^d \| \partial_{x_i} f \|_X \]
for some \( C, r > 0 \). Furthermore, this implies that
\[
e^{\delta t} \| e^{t \tilde{L}} \tilde{L}_\beta f \|_{X_i} \leq \sum_{|\alpha| + |b| \leq 1} \| e^{t \tilde{L}} \partial_x^\alpha \partial_p^b \tilde{L}_\beta f \|_X \leq \sum_{|\alpha| + |b| \leq 1} \| \tilde{L}_\beta \partial_x^\alpha \partial_p^b f \|_X + \sum_{|b| = 1} \| \tilde{L}_\beta + f \|_X \leq C \sum_{|\alpha| = 1} \beta |\gamma| \sum_{|\alpha| + |b| \leq 1} \| \partial_x^{\alpha + \gamma} \partial_p^b f \|_X + \sum_{|b| = 1} (\beta + b)! r_0^{|\beta|+1} \| e^{t \tilde{L}} \partial_x^{\gamma} f \|_X \]
using that \( \| e^{t \tilde{L}} \| \leq 1 \) for all \( t \in \mathbb{R} \). Thus for every \( r_0 > r \) there exists a \( C_r > 0 \) such that
\[
e^{\delta t} \| e^{t \tilde{L}} \tilde{L}_\beta f \|_{X_i} \leq C_r \beta |\gamma| \sum_{|\alpha| = 1} \sum_{|\alpha| + |b| \leq 1} \| e^{t \tilde{L}} \partial_x^\alpha \partial_p^b \partial_x^{\gamma} f \|_X \leq C_r \beta |\gamma| \sum_{|\alpha| = 1} \sum_{|\alpha| + |b| \leq 1} \| e^{t \tilde{L}} \partial_x^\alpha \partial_p^b e^{-t \tilde{L}} e^{t \tilde{L}} \partial_x^{\gamma} f \|_X \]
\[ = C_r \beta |\gamma| \sum_{|\gamma| = 1} e^{\delta t} \| e^{t \tilde{L}} \partial_x^{\gamma} f \|_{X_i} \]
showing the assertion.

\[ \square \]

**Lemma 21.** Let \( t \in \mathbb{R} \) and \( f : \mathbb{R}^d \times \mathbb{T}^d \times \mathbb{R} \to \mathbb{R} \) be bounded and Lipschitz continuous in \( t \) and analytic on \( \mathbb{R}^d \times \mathbb{T}^d \). For \( \delta \geq C \) \( r > 0 \) with \( C, r > 0 \) given by Lemma 20, it holds
\[
\frac{d}{dt} \| f \|_X \leq \| \partial_t f \|_X .
\]

**Proof.** We can easily show that \( \frac{d}{dt} \| f \|_X \leq \| \partial_t f \|_X \) is satisfied. Then the lemma is a consequence of the following calculation
\[
\frac{d}{dt} \| f \|_{X_i} + \delta \| f \|_{X_i} = \frac{d}{dt} \left( e^{-\delta t} \sum_{|\alpha| + |b| \leq 1} \| e^{t \tilde{L}} \partial_x^\alpha \partial_p^b e^{-t \tilde{L}} f \|_X \right) + \delta \| f \|_X ,
\]
\[
\leq e^{-\delta t} \sum_{|\alpha| + |b| \leq 1} \| \partial_t \left( e^{t \tilde{L}} \partial_x^\alpha \partial_p^b e^{-t \tilde{L}} f \right) \|_X .
\]
Lemma 20. Let $|\alpha + \beta| \leq 1$

where, we have used that $\alpha = 0$ or $\beta = 0$ is fulfilled and $[L, \partial_x] = 0$ according to Lemma 20. Let $|\beta| = 1$. We apply again Lemma 20 and see

$$
\|e^{tL\tilde{L}_\beta} e^{-tL} f\|_X \leq C_L \|\tilde{L}_\beta e^{-tL} f\|_X
$$

Combining this with the estimate above, we obtain

$$
\frac{d}{dt} \|f\|_X, + \delta \|f\|_X, \leq C\delta \|f\|_X, + \|\partial_t f\|_X,
$$

This finishes the proof assuming that $\delta \geq C\epsilon$.

□

Lemma 22. Recalling $Q_t(f)(x, p) := U e^{-\frac{1}{2} \nabla_x \int_{\mathbb{T}^d} f(x, p') dp' \cdot \nabla_p f(x, p)}$, we have

$$
\|e^{tL} \partial_x^\alpha \partial_p^\beta Q_t(f)\|_{X_t} \leq UC_L e^{(2\beta - \frac{1}{2})t} \sum_{(\alpha', \beta') \leq (\alpha, \beta)} \binom{\alpha}{\alpha'} \binom{\beta}{\beta'} \|e^{tL} \partial_x^{\alpha - \alpha'} \partial_p^{\beta - \beta'} f\|_{X_t} \times
$$

$$
\times \sum_{|a + b| = 1} \|e^{tL} \partial_x^{\alpha' + a} \partial_p^{\beta' + a} f\|_{X_t}.
$$

Proof. Let $\rho_f := \int_{\mathbb{T}^d} f dp$. We directly estimate using the Leibnitz rule that

$$
\|e^{tL} \partial_x^\alpha \partial_p^\beta \nabla_x \rho_f \cdot \nabla_p f\|_{X_t} \leq \sum_{\alpha' \leq \alpha} \binom{\alpha}{\alpha'} \|e^{tL} \left(\partial_x^{\alpha - \alpha'} \nabla_x \rho_f \cdot \partial_p^{\beta'} \nabla_p f\right)\|_{X_t}.
$$

Let $\alpha', \alpha'' \geq 0$ and $\beta' \geq 0$. By the definition of $\|\cdot\|_{X_t}$, we have

$$
e^{\delta t} \left| \|e^{tL} \left(\partial_x^{\alpha''} \nabla_x \rho_f \cdot \partial_p^{\beta'} \nabla_p f\right)\|_{X_t}\right|
$$

$$
\leq \sum_{|\gamma| = 1} \sum_{|a + b| \leq 1} \|e^{tL} \partial_x^a \partial_p^b \left(\partial_x^{\alpha'' + \gamma} \partial_p^{\beta'} \rho_f \partial_x^{\alpha'} \partial_p^{\beta' + \gamma} f\right)\|_X
$$

$$
\leq \sum_{|\gamma| = 1} \sum_{|a| \leq 1} \|e^{tL} \left(\partial_x^{\alpha'' + a + \gamma} \rho_f \partial_x^{\alpha'} \partial_p^{\beta' + \gamma} f\right)\|_X
$$

$$
+ \sum_{|\gamma| = 1} \sum_{|a + b| \leq 1} \|e^{tL} \left(\partial_x^{\alpha'' + \gamma} \rho_f \partial_x^{\alpha'} \partial_p^{\beta' + b + \gamma} f\right)\|_X.
$$
Now, we can use that $(e^{tL})$ is a strongly continuous contraction group with $\|e^{tL}\| \leq 1$ for all $t \in \mathbb{R}$ implying

$$\|e^{tL}(\rho_n g)\|_X \leq \|\rho_n g\|_X \leq C_X \|h\|_X \|g\|_X \leq C_X \|e^{tL}h\|_X \|e^{tL}g\|_X$$

for all $h, g \in X$ using Lemma 17. Thus,

$$\sum_{|\gamma|=1} \sum_{|a| \leq 1} \left\| e^{tL} \left( \partial_x^{\alpha''} + a + \gamma \rho_f \partial_x^{\beta''+\gamma} f \right) \right\|_X \leq C_X \sum_{|\gamma|=1} \sum_{|a| \leq 1} \left\| e^{tL} \partial_x^{\alpha''+a+\gamma} f \right\|_X \left\| e^{tL} \partial_x^{\beta''+\gamma} f \right\|_X$$

$$\leq C_X e^{2\delta t} \sum_{|\gamma|=1} \left\| e^{tL} \partial_x^{\alpha''+\gamma} f \right\|_X \left\| e^{tL} \partial_x^{\beta''+\gamma} f \right\|_X.$$

Likewise,

$$\sum_{|\gamma|=1} \sum_{|a+b| \leq 1} \left\| e^{tL} \left( \partial_x^{\alpha''+\gamma} \rho_f \partial_x^{\alpha'+a} \partial_p^{\beta''+b+\gamma} f \right) \right\|_X \leq C_X e^{2\delta t} \sum_{|\gamma|=1} \left\| e^{tL} \partial_x^{\alpha''+\gamma} f \right\|_X \left\| e^{tL} \partial_x^{\beta''+\gamma} f \right\|_X.$$

Combining both estimates ensures the assertion.

With the same arguments, we can easily show the following lemma concerning the desired Lipschitz estimate.

**Lemma 23.** Recalling $Q_t(f)(x,p) := U e^{-\frac{1}{2} \nabla_x \int_{\mathbb{T}^d} f(x,p') dp' \cdot \nabla_p f(x,p)},$ we have

$$\left\| e^{tL} \partial_x^{\alpha} \partial_p^{\beta} Q_t(f-g) \right\|_X \leq UC_X e^{(2\delta-\frac{1}{2})t} \sum_{(\alpha',\beta') \leq (\alpha,\beta)} \left( \frac{\alpha}{\alpha'} \right) \left( \frac{\beta}{\beta'} \right) \times$$

$$\times \left( \left\| e^{tL} \partial_x^{\alpha'-\alpha} \partial_p^{\beta'-\beta} (f-g) \right\|_X \sum_{|a+b|=1} \left\| e^{tL} \partial_x^{\alpha'+a} \partial_p^{\beta'+b} f \right\|_X \right) + \left\| e^{tL} \partial_x^{\alpha'-\alpha} \partial_p^{\beta'-\beta} (f-g) \right\|_X \sum_{|a+b|=1} \left\| e^{tL} \partial_x^{\alpha'+a} \partial_p^{\beta'+b} (f-g) \right\|_X \right). \quad (21)$$

**Theorem 24.** Let $C, r$ be as in Lemma 20 and $\delta = Cr$. Then for every positive $\nu_0 < \frac{1}{r},$ there exist $\epsilon > 0$ and $\tau_0 \in (0, 1/(2Cr))$ such that if

$$\|g_0\|_{X_t^0} \leq \epsilon \nu \quad (22)$$

for some $\nu \leq \nu_0,$ then (16) has a classical and analytic solution $g$ with $g|_{t=0} = g_0,$ with

$$\|g(t)\|_{X_t^{\nu \exp(-\frac{t}{\tau_0})}} \leq 2 \epsilon \nu \quad \text{for all } t \geq 0.$$

Moreover, for all $g_0, h_0 \in Y$ satisfying (22), we have

$$\|g(t) - h(t)\|_{X_t^{\nu \exp(-\frac{t}{\tau_0})}} \leq 2 \|g_0 - h_0\|_{X_0^0} \quad \text{for all } t \geq 0,$$

where $g, h$ are the solution of (16) with $g(0) = h_0$ and $g(0) = h_0$, respectively.

**Proof.** According to Lemma 20, (H2) is satisfied. Moreover, Lemma 21 yields (H4'). We set $M_0 := M_0' := 0$ and define $M_1 := M_1' := UC_X,$ where $C_X > 0$ is given by Lemma 17. Given $\nu_0 < 1/r,$ we choose $\omega_0 > \frac{2dCr}{(1 - \nu_0)^{2\delta}}$ and set $\tau_0 := \frac{1}{\omega_0 + 2\delta} < \frac{1}{2Cr}.$
By Lemmata 22 and 23, we obtain (H3a’) and (H3b’) with \( \tau \geq \tau_0 \) for \( \omega := \omega(\tau) := \frac{1}{\tau} - 2\delta \) for every \( \tau \leq \tau_0 \). Note that \( \omega \geq \frac{1}{\tau_0} - 2\delta = \omega_0 \). Thus,

\[
\nu_0 < \frac{1}{r} \left( 1 - \frac{2dCr}{\omega_0} \right) \leq \frac{1}{r} \left( 1 - \frac{2dCr}{\omega(\tau)} \right)
\]

(23)

for all \( \tau \leq \tau_0 \). Therefore, we can apply Theorem 15 and obtain the assertion using that \( \frac{1}{\tau} \geq \omega \). The solution is indeed classical, because \( g \) is differentiable in \( t \) and analytic in \( x \) and \( p \). One can moreover easily show by an bootstrap argument that \( g \) is also analytic in \( t \). Note that \( \varepsilon \) does not depend on \( \tau \) because \( \frac{1}{r} \left( 1 - \frac{2dCr}{\omega(\tau)} \right) \) is uniformly bounded from below by a constant greater than \( \nu_0 \) because of (23).

**Proof of Theorem 2.** For \( C, r > 0 \) as in Lemma 20, \( \nu_0 < \frac{1}{r} \), let \( \varepsilon > 0 \) and \( \tau_0 \in (0, 1/(2Cr)) \) be given by Theorem 24. For any \( \nu \leq \nu_0 \), assume that \( f_0 \) satisfies

\[ ||f_0 - F_\lambda||_{X_\nu^\varepsilon} \leq \varepsilon \nu. \]

Due to Theorem 24, (16) has a analytic solution \( g \) with \( g|_{t=0} = f_0 - F_\lambda \), with

\[ ||g(t)||_{X_\nu^{\varepsilon}(t, \nu)} \leq 2\varepsilon \nu \quad \text{for all } t \geq 0. \]

Then \( f(t) := e^{-\frac{t}{\nu}}g(t) + F_\lambda \) solve the original problem (5) and satisfies \( f(0) = f_0 \). Moreover, it holds

\[ ||f(t) - F_\lambda||_{X_\nu^{\varepsilon}(t, \nu)} = e^{-\frac{t}{\nu}}||g(t)||_{X_\nu^{\varepsilon}(t, \nu)} \leq 2\varepsilon \nu e^{-\frac{t}{\nu}} \]

for all \( t \geq 0 \). By Definition 9, we have that \( \| \cdot \|_{X_\nu^\varepsilon} = e^{-\delta t} \| \cdot \|_{Y_\nu} \) for \( t > 0 \) and especially \( \| \cdot \|_{X_\nu^\varepsilon} = \| \cdot \|_{Y_\nu^\varepsilon} \). Theorem 24 entails that \( \delta = Cr \leq 1/(2\tau_0) \leq 1/\tau_0 \). Thus,

\[ ||f(t) - F_\lambda||_{X_\nu^{\varepsilon}(t, \nu)} \leq e^{\delta t}||f(t) - F_\lambda||_{Y_\nu^{\varepsilon}(t, \nu)} \]

\[ \leq e^{\delta t}||f(t) - F_\lambda||_{Y_\nu^{\varepsilon}(t, \nu)} \leq 2\varepsilon \nu e^{-\left(\frac{t}{\nu} - \frac{\delta}{\varepsilon} \right)} \]

Likewise,

\[ ||f(t) - \tilde{f}(t)||_{X_\nu^{\varepsilon}(t, \nu)} \leq 2e^{-\left(\frac{1}{\nu} - \frac{\delta}{\varepsilon} \right) t}||f_0 - \tilde{f}_0||_{X_\nu^\varepsilon} \]

\[ = 2e^{-\left(\frac{1}{\nu} - \frac{\delta}{\varepsilon} \right) t}||f_0 - \tilde{f}_0||_{Y_\nu^\varepsilon} \quad \text{for all } t \geq 0 \]

where \( f, \tilde{f} \) are the solution of (5) with \( f(0) = f_0 \) and \( \tilde{f}(0) = \tilde{f}_0 \), respectively.

**Proof of Theorem 1.** Theorem 1 is actually a direct corollary of Theorem 2. We only need to apply the following two properties.

First, for \( \mu < \nu \) there exists a constant \( C_{\nu, \mu} > 0 \) such that

\[ ||h||_{Y_{\nu}^\varepsilon} \leq C_{\nu, \mu} \sum_{\alpha, \beta \in \mathbb{N}_0} \frac{\nu^{\alpha+\beta}}{\alpha! \beta!} ||\partial_\alpha x \partial_\beta_p h||_{X} \]

for all \( h \in Y \), which was proved in [6] Lemma 2.3 and originates from [22]. Second,

\[ ||h||_{X} \leq ||h||_{Y_{\nu}^\varepsilon} \leq ||h||_{Y_{\nu}^\varepsilon} \]

for all \( \nu \geq 0 \) and all \( h \in Y \).
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