Fundamental domains in $\text{PSL}(2, \mathbb{R})$ for Fuchsian groups

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Abstract

In this paper, we provide a necessary and sufficient condition for a set in $\text{PSL}(2, \mathbb{R})$ or in $T^1 \mathbb{H}^2$ to be a fundamental domain of a given Fuchsian group via its respective fundamental domain in the hyperbolic plane $\mathbb{H}^2$.

Keywords: Fundamental domain; Hyperbolic plane; $\text{PSL}(2, \mathbb{R})$; Fuchsian group

1 Introduction

Fundamental domains arise naturally in the study of group actions on topological spaces. The concept fundamental domain is used to describe a set in a topological space under a group action of which the images under the action tessellate the whose space. The term fundamental domain is well-known in the model of the hyperbolic plane $\mathbb{H}^2$ for the action of Fuchsian groups via Möbius transformations. If there exists a point in $\mathbb{H}^2$ that is not a fixed point for all elements different from
the unity in a Fuchsian group $\Gamma$ then there always exist a convex and connected fundamental domain for $\Gamma$ named Dirichlet domain (see [1, 3]). Other examples of fundamental domains are Ford domains [4]. The Poincaré’s polygon theorem [2, 3, 4] provides a fundamental domain which is a polygon for the Fuchsian group generated by the side-pairing transformations. In this case, if the polygon has finite edges (and hence it is relatively compact), the Fuchsian group is finitely generated and the space of $\Gamma$-orbits denoted by $\Gamma \backslash \mathbb{H}^2$ is compact. Fundamental domains have several applications for the study of $\Gamma \backslash \mathbb{H}^2$. If the action of $\Gamma$ has no fixed points, the quotient space $\Gamma \backslash \mathbb{H}^2$ has a Riemann surface structure that is a closed Riemann surface of genus at least 2 and has the hyperbolic plane $\mathbb{H}^2$ as the universal covering. Furthermore, it is well-known that any compact orientable surface with constant negative curvature is isometric to a factor $\Gamma \backslash \mathbb{H}^2$. If the Fuchsian group has a finite-area fundamental domain then all the fundamental domains have finite area and have the same area. This area is defined for the measure of the quotient space $\Gamma \backslash \mathbb{H}^2$. In addition, the space $\Gamma \backslash \mathbb{H}^2$ is compact if and only if every fundamental domain in $\mathbb{H}^2$ for $\Gamma$ is relatively compact [3].

There exists a bijection $\Theta : T^1 \mathbb{H}^2 \to \text{PSL}(2, \mathbb{R})$. The natural Riemannian metric on $\text{PSL}(2, \mathbb{R})$ induces a left-invariant metric function (a metric in usual sense). The topology induced from this metric is the same as the quotient topology induced from the one in $\text{SL}(2, \mathbb{R})$. The Sasaki metric on the unit tangent bundle $T^1 \mathbb{H}^2$ with respect to the hyperbolic metric on $\mathbb{H}^2$ makes $\Theta$ an isometry. This induces an isometry from $T^1 (\Gamma \backslash \mathbb{H}^2)$ to the collection of right cosets $\Gamma g$ of $\Gamma$ in $\text{PSL}(2, \mathbb{R})$ denoted by $\Gamma \backslash \text{PSL}(2, \mathbb{R})$ that is also obtained from a left action of Fuchsian group $\Gamma$ on $\text{PSL}(2, \mathbb{R})$. Furthermore, there is an action of $\Gamma$ on the unit tangent bundle $T^1 \mathbb{H}^2$ by the differential of elements in $\Gamma$ and this arises fundamental domains for $\Gamma$ in $T^1 \mathbb{H}^2$ also. However, up to now there have not any results about fundamental domains for $\Gamma$ in $\text{PSL}(2, \mathbb{R})$ or in $T^1 \mathbb{H}^2$. The aim of this paper is to study fundamental domains in $\text{PSL}(2, \mathbb{R})$ and in $T^1 \mathbb{H}^2$ for Fuchsian groups. A necessary and sufficient condition for a set in $\text{PSL}(2, \mathbb{R})$ or in $T^1 \mathbb{H}^2$ to be a
fundamental domain via its respective fundamental domain in $\mathbb{H}^2$ is provided.

The paper is organized as follows. In the next section we present action of Fuchsian groups on the hyperbolic plane $\mathbb{H}^2$, the unit tangent bundle $T^1\mathbb{H}^2$ and $\text{PSL}(2, \mathbb{R})$ and some basic examples of fundamental domain in $\mathbb{H}^2$. Main results are stated and proved in Section 3.

\section{Preliminaries}

In this paper we introduce the necessary background material which is well-known in \[2, 3, 4\]. The unity of arbitrary group is always denoted by $e$.

\subsection{Fundamental domains}

Let $X$ be a non-empty set and let $G$ be a group. Let $\rho : G \times X \to X$ be a (left) group action, that is, $\rho(e, x) = x$ and $\rho(g_1, \rho(g_2, x)) = \rho(g_1 g_2, x)$ for all $x \in X$ and $g_1, g_2 \in G$. For a subset $A \subset X$, denote $\rho(g, A) = \{\rho(g, x), x \in A\}$.

\textbf{Definition 2.1.} Let $G$ be a group and let $X$ be a topological space. Suppose that $\rho : G \times X \to X$ is an action. A non-empty open set $F \subset X$ is said to be a \textit{fundamental domain} for $G$, if

(a) $\bigcup_{g \in G} \rho(g, \overline{F}) = X$, and

(b) $\rho(g, F) \cap F = \emptyset$ for all $g \in G \setminus \{e\}$.

Here $e$ is the unity of $G$ and $\overline{F}$ denotes the closure of $F$ in $X$.

Due to the fact that $G$ is a group, condition (b) is equivalent to

$$\rho(g_1, F) \cap \rho(g_2, F) = \emptyset \quad \text{for all} \quad g_1, g_2 \in G, \quad g_1 \neq g_2.$$ 

We will introduce some examples in the next subsection.
2.2 $\mathbb{H}^2$ and $\text{PSL}(2, \mathbb{R})$

The hyperbolic plane is the upper half plane $\mathbb{H}^2 = \{(x, y) \in \mathbb{R}^2 : y > 0\}$, endowed with the hyperbolic $(g_z)_{z \in \mathbb{H}^2}$, where $g_z(\xi, \eta) = \frac{\xi \eta_1 + \xi_2 \eta_2}{(\text{Im} \ z)^2}$ for $\xi = (\xi_1, \xi_2), \eta = (\eta_1, \eta_2) \in T_{z} \mathbb{H}^2 \cong \mathbb{C}$. The group of Möbius transformations $\text{Möb}(\mathbb{H}^2) = \{ z \mapsto \frac{az + b}{cz + d} : a, b, c, d \in \mathbb{R}, ad - bc = 1 \}$ can be identified with the projective group $\text{PSL}(2, \mathbb{R}) = \text{SL}(2, \mathbb{R})/\{ \pm E_2 \}$ by means of the isomorphism

$$\Phi \left( \pm \begin{pmatrix} a & b \\ c & d \end{pmatrix} \right) = z \mapsto \frac{az + b}{cz + d},$$

where $\text{SL}(2, \mathbb{R})$ is the group of all real $2 \times 2$ matrices with unity determinant, and $E_2$ denotes the unit matrix.

Let $\Gamma$ be a Fuchsian group, which is a discrete subgroup in $\text{PSL}(2, \mathbb{R})$. We consider the action $\rho : \Gamma \times \mathbb{H}^2 \to \mathbb{H}^2$, $\rho(\gamma, z) = \Phi(\gamma)(z)$ for $(\gamma, z) \in \Gamma \times \mathbb{H}^2$. The action is called free if $\Phi(\gamma)(z) = z$ for some $z \in \mathbb{H}^2$ then $\gamma = e$. In this case, there always exist fundamental domains:

**Proposition 2.1.** Let $\Gamma \subset \text{PSL}(2, \mathbb{R})$ be a Fuchsian group and take $z_0 \in \mathbb{H}^2$ such that $z_0 \not= \Phi(\gamma)(z_0)$ holds for all $\gamma \in \Gamma \setminus \{ e \}$. Then the Dirichlet region

$$D_{z_0}(\Gamma) = \left\{ z \in \mathbb{H}^2 : d_{\mathbb{H}^2}(z, z_0) < d_{\mathbb{H}^2}(z, T(z_0)) \text{ for all } T = \Phi(\gamma), \gamma \in \Gamma \setminus \{ e \} \right\}$$

is a fundamental domain for $\Gamma$ which contains $z_0$.

See [1, Lemma 11.5] for a proof. Note that such a $z_0$ does exist if the action of $\Gamma$ on $\mathbb{H}^2$ is free.

For $g = \left\{ \pm \begin{pmatrix} a & b \\ c & d \end{pmatrix} \right\} \in \text{PSL}(2, \mathbb{R})$, the trace of $g$ is defined by

$$\text{tr}(g) = |a + d|.$$ 

Element $g$ is called hyperbolic if $|\text{tr}(g)| > 2$, elliptic if $|\text{tr}(g)| < 2$ and parabolic if $|\text{tr}(g)| = 2$. Recall that the action of $\Gamma$ on $\mathbb{H}^2$ is free if and only if $\Gamma$ does not contain elliptic elements.
For any $g \in \text{PSL}(2, \mathbb{R})$, the cyclic group $\langle g \rangle = \{g^n : n \in \mathbb{Z}\} \subset \text{PSL}(2, \mathbb{R})$ is a Fuchsian group. We will consider fundamental domains for $\langle g \rangle$ with special classes of $g$. For $t \in \mathbb{R}$, denote

$$A_t = \begin{pmatrix} e^{t/2} & 0 \\ 0 & e^{-t/2} \end{pmatrix}, B_t = \begin{pmatrix} 1 & t \\ 0 & 1 \end{pmatrix}, D_t = \begin{pmatrix} \cos \frac{t}{2} & \sin \frac{t}{2} \\ -\sin \frac{t}{2} & \cos \frac{t}{2} \end{pmatrix} \in \text{SL}(2, \mathbb{R})$$

and respectively

$$a_t = [A_t], b_t = [B_t], d_t = [D_t] \in \text{PSL}(2, \mathbb{R}).$$

**Proposition 2.2.** (a) For any $t > 0$, the set

$$F_t = \{z = x + iy \in \mathbb{H}^2 : 1 < y < e^t\}$$

is a fundamental domain in $\mathbb{H}^2$ for the Fuchsian group $\langle a_t \rangle$.

(b) For any $t > 0$, the set

$$E_t = \{z = x + iy \in \mathbb{H}^2 : 0 < x < t\}$$

is a fundamental domain in $\mathbb{H}^2$ for the Fuchsian group $\langle b_t \rangle$.

**Proof:** (a) Obviously $F_t$ is open and

$$\overline{F_t} = \{z = x + iy \in \mathbb{H}^2 : 1 \leq y \leq e^t\}.$$ 

We have $\Phi(a_{jt}) = e^{jt} \text{id}$, with $\text{id} : \mathbb{H}^2 \to \mathbb{H}^2$ denoting the identity. Here

$$\Phi(a_{jt})(\overline{F_t}) = e^{jt} \text{id}(\{z = x+iy \in \mathbb{H}^2 : 1 \leq y \leq e^t\}) = \{z \in \mathbb{H}^2 : e^{jt} \leq y \leq e^{(j+1)t}\},$$

so that $\bigcup_{j \in \mathbb{Z}} \Phi(a_{jt})(\overline{F_t}) = \mathbb{H}^2$ and $\Phi(a_{jt})(F_t) \cap \Phi(a_{kt})(F_t) = \emptyset$ for $j \neq k$.

(b) It is proved analogously to (a). \qed

The collection of right co-sets $\Gamma g$ of $\Gamma$ in $\text{PSL}(2, \mathbb{R})$ denoted by $\Gamma \setminus \text{PSL}(2, \mathbb{R})$ can be also obtained by $\Gamma$-orbits of the left action

$$\varrho : \Gamma \times \text{PSL}(2, \mathbb{R}) \to \text{PSL}(2, \mathbb{R}), \varrho(\gamma, g) = \gamma g \text{ for } \gamma \in \Gamma, g \in \text{PSL}(2, \mathbb{R}). \quad (2.1)$$

This leads to the concept fundamental domain in $\text{PSL}(2, \mathbb{R})$. 5
Remark 2.1. If $F \subset \text{PSL}(2, \mathbb{R})$ is a fundamental domain for $\Gamma$ and $\gamma \in \Gamma \setminus \{e\}$, then $\gamma F$ is a fundamental domain disjoint from $F$. For, it is obvious that $\gamma F$ is open since $F$ open and $\gamma F = \gamma F$ in $\text{PSL}(2, \mathbb{R})$. Therefore

$$\bigcup_{\gamma' \in \Gamma} \gamma' F = \bigcup_{\gamma' \in \Gamma} (\gamma' \gamma)(F) = \bigcup_{\eta \in \Gamma} \eta F = \text{PSL}(2, \mathbb{R}),$$

and for $\gamma' \in \Gamma \setminus \{e\}$,

$$(\gamma F) \cap \gamma' F = \gamma F \cap (\gamma' \gamma)F = \emptyset$$

by (a) in Definition 2.1, due to $\gamma \neq \gamma' \gamma$. ♦

2.3 $T^1\mathbb{H}^2$

The unit tangent bundle of $\mathbb{H}^2$ is defined by

$$T^1\mathbb{H}^2 = \{(z, \xi) : z \in \mathbb{H}^2, \xi \in T_z\mathbb{H}^2, \|\xi\|_z = g_z(\xi, \xi)^{1/2} = 1\}.$$  \hspace{1cm} (2.2)

For $g \in \text{PSL}(2, \mathbb{R})$ we consider the derivative operator

$$\mathcal{D}g : T^1\mathbb{H}^2 \to T^1\mathbb{H}^2$$

defined as

$$\mathcal{D}g(z, \xi) = (T(z), T'(z)\xi),$$

where $T = \Phi(g)$. Then $\mathcal{D}$ is well-defined. Explicitly, if $g = \left\{ \pm \begin{pmatrix} a & b \\ c & d \end{pmatrix} \right\}$, then

$$T(z) = \frac{az + b}{cz + d} \text{ and } ad - bc = 1,$$

whence

$$\mathcal{D}g(z, \xi) = \left( \frac{az + b}{cz + d}, \frac{\xi}{(cz + d)^2} \right).$$  \hspace{1cm} (2.3)

Let $\Gamma \subset \text{PSL}(2, \mathbb{R})$ be a subgroup. Consider the group action

$$\kappa : \Gamma \times T^1\mathbb{H}^2 \to T^1\mathbb{H}^2, \ k(\gamma, (z, \xi)) = \mathcal{D}g(z, \xi) \text{ for } \gamma \in \Gamma, (z, \xi) \in T^1\mathbb{H}^2.$$  

If $\Gamma = \text{PSL}(2, \mathbb{R})$ then the action is simply transitive (see [1] Lemma 9.2), that is, for given $(z, \xi), (w, \eta) \in T^1\mathbb{H}^2$, there exists a unique $g \in \text{PSL}(2, \mathbb{R})$ such that $\kappa(g, (z, \xi)) = \mathcal{D}g(z, \xi) = (w, \eta)$. In particular, we have the following result:
Lemma 2.1. For each \((z, \xi) \in T^1 \mathbb{H}^2\), there is a unique \(g \in \text{PSL}(2, \mathbb{R})\) such that 
\[Dg(i, i) = (z, \xi).\]

Note that if \((z, \xi) \in T^1 \mathbb{H}^2\) then \(g = \{ \pm \begin{pmatrix} a & b \\ c & d \end{pmatrix} \}\) is defined by 
\[
\begin{align*}
\frac{ac + bd}{c^2 + d^2} &= \text{Re}\ z, \\
\frac{1}{c^2 + d^2} &= \text{Im}\ z, \\
\frac{2cd}{(c^2 + d^2)^2} &= \text{Re}\ \xi, \\
\frac{d^2 - c^2}{(c^2 + d^2)^2} &= \text{Im}\ \xi.
\end{align*}
\]

(2.4)

We will use these relations afterwards.

3 Main results

This section deals with the relation of fundamental domains for a Fuchsian group in \(\mathbb{H}^2\), \(\text{PSL}(2, \mathbb{R})\) and in \(T^1 \mathbb{H}^2\).

The main result of this paper is the following:

Theorem 3.1. Let \(\Gamma \subset \text{PSL}(2, \mathbb{R})\) be a Fuchsian group. For \(F \subset \mathbb{H}^2\), denote 
\[\mathcal{F} = \{ g = b_x a_{in, y} d_\theta : x + iy \in F, \theta \in [0, 2\pi) \} \subset \text{PSL}(2, \mathbb{R})\].

Then \(F\) is a fundamental domain for \(\Gamma\) in \(\mathbb{H}^2\) if and only if \(\mathcal{F}\) is a fundamental domain for \(\Gamma\) in \(\text{PSL}(2, \mathbb{R})\).

Remark 3.1. Recall that if \(\Gamma\) contains no elliptic elements then there always exist fundamental domains in \(\mathbb{H}^2\) for \(\Gamma\) and hence fundamental domains in \(\text{PSL}(2, \mathbb{R})\) do always exist. The collection of \(\Gamma\)-orbits of the action \(\rho\) (see (2.1)) denoted by \(\Gamma \backslash \text{PSL}(2, \mathbb{R}) = \{ \Gamma g, g \in \text{PSL}(2, \mathbb{R}) \}\) is compact if and only if the quotient space \(\Gamma \backslash \mathbb{H}^2\) is compact if and only if there is a relatively compact fundamental domain (in \(\mathbb{H}^2\) or in \(\text{PSL}(2, \mathbb{R})\)) for \(\Gamma\). In this case all fundamental domains of \(\Gamma\) is relatively compacts. For proofs of statements in \(\mathbb{H}^2\), see [3, Chapter 3].

In order to prove Theorem 3.1 we need the following factorization which is called NAK decomposition (so-called Iwasawa decomposition).
Lemma 3.1. If \( G = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \text{SL}(2, \mathbb{R}) \) then \( G = B_x A_{\ln y} d_\theta \) with
\[
x = \frac{ac + bd}{c^2 + d^2}, \quad y = \frac{1}{c^2 + d^2}, \quad \theta = -2 \arg(d + ic).
\]

Lemma 3.2. (a) If \( g = [G] \in \text{PSL}(2, \mathbb{R}) \) for \( G = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \text{SL}(2, \mathbb{R}) \) then
\[
g = B_x A_{\ln y} d_\theta \text{ with } x = \frac{ac + bd}{c^2 + d^2}, \quad y = \frac{1}{c^2 + d^2}, \quad \theta = -2 \arg(d + ic).
\]

(b) \( \text{PSL}(2, \mathbb{R}) = \{ b_x a_{\ln y} d_\theta : x + iy \in \mathbb{H}^2, \theta \in [0, 2\pi) \} \)
\[
= \{ b_x a_{\ln y} d_\theta : x + iy \in \mathbb{H}^2, \theta \in \mathbb{R} \}.
\]

Proof: (a) This follows directly from Lemma 3.1. (b) According to (a), every element \( g = \{ \pm \begin{pmatrix} a & b \\ c & d \end{pmatrix} \} \in \text{PSL}(2, \mathbb{R}) \) has the decomposition \( g = b_x a_{\ln y} d_\theta \) for \( x + iy \in \mathbb{H}^2 \) and \( \theta = -2 \arg(d + ic) \in (-2\pi, 0] \). It remains to verify that we can find some \( \theta' \in [0, 2\pi) \) such that \( d_{\theta'} = d_\theta \) and as a consequence, \( g = b_x a_{\ln y} d_{\theta'} \). Indeed, the matrix \( D_\theta \) changes by an overall sign if \( \theta \) changes by \( 2\pi \) and so does the matrix \( G = B_x A_{\ln y} D_\theta \). Therefore we can find a unique \( k \in \mathbb{Z} \) such that \( \theta' := 2k\pi + \theta \in [0, 2\pi) \) to have \( d_{\theta'} = d_\theta \). This implies the first equality in (b). The latter follows from \( d_{\theta + 2k\pi} = d_\theta \) for all \( \theta \in [0, 2\pi) \) and \( k \in \mathbb{Z} \). □

Proof of Theorem 3.1. First, denote
\[
\hat{F} = \{ G = B_x A_{\ln y} D_\theta \in \text{SL}(2, \mathbb{R}) : x + iy \in F, \theta \in [0, 2\pi) \}.
\]

It is easy to see that \( \hat{F} \) is open in \( \text{SL}(2, \mathbb{R}) \) and since the projection \( \pi : \text{SL}(2, \mathbb{R}) \to \text{PSL}(2, \mathbb{R}) \) is an open map and \( \pi(\hat{F}) = \mathcal{F} \), it follows that \( \mathcal{F} \) is open in \( \text{PSL}(2, \mathbb{R}) \) (note that \( d_{\theta + 2k\pi} = d_\theta \) for \( \theta \in [0, 2\pi) \) and \( k \in \mathbb{Z} \)). To establish part (a) in the definition, we first claim that the closure of \( \mathcal{F} \) in \( \text{PSL}(2, \mathbb{R}) \) is
\[
\overline{\mathcal{F}} = \{ g = b_x a_{\ln y} d_\theta \in \text{PSL}(2, \mathbb{R}) : x + iy \in \overline{F}, \theta \in [0, 2\pi) \}.
\]
Indeed, it suffices to check that

$$\overline{F} = \{ G = B_x A_{\ln y} D_\theta : x + iy \in \overline{F}, \theta \in [0, 2\pi) \}, \quad (3.3)$$

where $\overline{F}$ denotes the closure of $F$ in $\mathbb{H}^2$ and $\overline{\hat{F}}$ denotes the closure of $\hat{F}$ in $\text{SL}(2, \mathbb{R})$. The set in the right-hand side of (3.3) is denoted by $\text{cl}(\hat{F})$. For every $G \in \text{cl}\hat{F}$, we show that $G_n \to G$ for some sequence $(G_n)_n \subset \hat{F}$. Writing $G = B_x A_{\ln y} D_\theta$, we have $x + iy \in \overline{F}$ by the definition of $\text{cl}(\hat{F})$. Let

$$(x_n + iy_n)_n \subset F$$

be such that $x_n + iy_n \to x + iy$ in $\mathbb{H}^2$ as $n \to \infty$. Then $x_n \to x$ as well as $y_n \to y$ in $\mathbb{R}$. Taking $G_n = B_{x_n} A_{\ln y_n} D_\theta \in \text{cl}(\hat{F})$, we obtain $G_n \to G$ in $\text{SL}(2, \mathbb{R})$ after a short check.

Next, for any $g = b_x a_{\ln y} d_\theta \in \text{PSL}(2, \mathbb{R})$ we have $z := x + iy \in \Phi(\gamma)(\overline{F})$ for some $\gamma \in \Gamma$ as $F \subset \mathbb{H}^2$ is a fundamental domain for $\Gamma$. Take $\tilde{z} = \tilde{x} + i\tilde{y} = \Phi(\gamma^{-1})(z) \in \overline{F}$ and write $\gamma = [T]$ with $T = \left( \begin{array}{cc} t_{11} & t_{12} \\ t_{21} & t_{22} \end{array} \right) \in \text{SL}(2, \mathbb{R})$. Let

$$\tilde{\theta} = \theta + 2 \arg(t_{21}\tilde{z} + t_{22}) + 2k\pi \in [0, 2\pi)$$

for a unique $k \in \mathbb{Z}$ to obtain $h := b_{\tilde{x}} a_{\ln \tilde{y}} d_{\tilde{\theta}} \in \overline{F}$. Thus

$$x + iy = \frac{t_{11}\tilde{z} + t_{12}}{t_{21}\tilde{z} + t_{22}}$$

and

$$\theta = \tilde{\theta} - 2 \arg(t_{21}\tilde{z} + t_{22}) - 2k\pi$$

implies $g = \gamma h \in \gamma \overline{F}$ after a short calculation. This completes the proof for (a) in the definition.

For part (b), suppose on the contrary that there exists $g \in \mathcal{F} \cap \gamma \mathcal{F}$ for some $\gamma \in \Gamma \setminus \{ e \}$. Then $g = b_x a_{\ln y} d_\theta$ and $g = \gamma b_{x'} a_{\ln y'} d_{\theta'}$ for $x + iy \in \mathcal{F}$ and $x' + iy' \in \mathcal{F}$. A short calculation shows that $x + iy = \Phi(\gamma)(x' + iy') \in \mathcal{F} \cap \Phi(\gamma)(\mathcal{F})$ that contradicts to the fact that $F$ is a fundamental domain. Thus $\mathcal{F} \cap \gamma \mathcal{F} = \emptyset$ for all $\gamma \in \Gamma \setminus \{ e \}$.

Conversely, assume that $\mathcal{F}$ is a fundamental domain for $\Gamma$. Then $F \subset \mathbb{H}^2$ is open since $\mathcal{F}$ is open. For any $z = x + iy \in \mathbb{H}^2$, then $g = b_x a_{\ln y} \in \text{PSL}(2, \mathbb{R}) = \bigcup_{\gamma \in \Gamma} \gamma \overline{F}$ implies that $g = \gamma h$ for some $\gamma \in \Gamma$ and $h \in \overline{F} = \{ b_x a_{\ln y} d_\theta, x + iy \in \mathcal{F} \}$. Then $g \in \mathcal{F} \cap \gamma \mathcal{F}$ is open since $\mathcal{F}$ is open. For any $z = x + iy \in \mathbb{H}^2$, then $g = b_x a_{\ln y} \in \text{PSL}(2, \mathbb{R}) = \bigcup_{\gamma \in \Gamma} \gamma \overline{F}$ implies that $g = \gamma h$ for some $\gamma \in \Gamma$ and $h \in \overline{F} = \{ b_x a_{\ln y} d_\theta, x + iy \in \mathcal{F} \}$. Thus $\mathcal{F} \cap \gamma \mathcal{F} = \emptyset$ for all $\gamma \in \Gamma \setminus \{ e \}$. This completes the proof for (b).
Write $h = b_x a_{\ln y} d_{\theta}$. Then $\tilde{z} = \tilde{x} + i \tilde{y} \in \overline{F}$ and $z = \Phi(\gamma)(\tilde{z})$ yield $z \in \Phi(\gamma)(\overline{F})$. This proves (a) in Definition 2.1. Finally, assume that $z = x + iy \in F$ and $z = \Phi(\gamma)(z')$ for some $\gamma \in \Gamma \setminus \{e\}$ and $z' = x' + iy' \in F$. Then take $g = b_x a_{\ln y} d_{\pi}$ and $h = b_{x'} a_{\ln y'} d_{\theta}$ with $\theta = 2 \arg(h_{21}z + h_{22}) + 2k\pi$ for a unique $k \in \mathbb{Z}$ such that $\theta \in [0, 2\pi)$; here $h = \pi(H)$, $H = \left( \begin{array}{cc} h_{11} & h_{12} \\ h_{21} & h_{22} \end{array} \right)$. Then $g = \gamma h$ after a short calculation. This means that $F \cap \gamma F \neq \emptyset$ which is impossible since $F$ is a fundamental domain. \hfill \Box

The next result follows directly from Proposition 2.2 and Lemma 3.1.

**Corollary 3.1.** (a) For $t > 0$, the set

$$F_t = \{ g = b_x a_{\ln y} d_{\theta} \in \text{PSL}(2, \mathbb{R}) : x \in \mathbb{R}, 1 < y < e^t, \theta \in [0, 2\pi) \}$$

is a fundamental domain in PSL(2, $\mathbb{R}$) for the Fuchsian group $\langle a_t \rangle$.

(b) For $t > 0$, the set

$$\mathcal{E}_t = \{ g = b_x a_{\ln y} d_{\theta} \in \text{PSL}(2, \mathbb{R}) : 0 < x < t, y > 0, \theta \in [0, 2\pi) \}$$

is a fundamental domain in PSL(2, $\mathbb{R}$) for the Fuchsian group $\langle b_t \rangle$.

It is well-known that PSL(2, $\mathbb{Z}$) is a Fuchsian group and the set

$$F = \left\{ z \in \mathbb{H}_2 : |z| > 1, |\text{Re} z| < \frac{1}{2} \right\}$$

is a fundamental domain of PSL(2, $\mathbb{Z}$) in $\mathbb{H}_2$ (see [1, Proposition 9.18]). It follows from Lemma 3.2 and Theorem 3.1 that

**Corollary 3.2.** The set

$$\mathcal{F} = \left\{ g = [G] \in \text{PSL}(2, \mathbb{R}), G = \left( \begin{array}{cc} a & b \\ c & d \end{array} \right) \in \text{SL}(2, \mathbb{R}) : 2|ac + bd| < c^2 + d^2, (ac + bd)^2 + 1 > (c^2 + d^2)^2 \right\}$$

is a fundamental domain in PSL(2, $\mathbb{R}$) for PSL(2, $\mathbb{Z}$).  

10
The next result shows us how to find a fundamental domain for a cyclic group as we know a fundamental domain for the cyclic group generated by a conjugate element of its generator.

**Lemma 3.3.** Let \( g_1 \) and \( g_2 \) be conjugate in \( \text{PSL}(2, \mathbb{R}) \) and \( g_2 = h g_1 h^{-1} \) for \( h \in \text{PSL}(2, \mathbb{R}) \). Then if \( \mathcal{F}_1 \subset \text{PSL}(2, \mathbb{R}) \) is a fundamental domain for \( \langle g_1 \rangle \) then \( \mathcal{F}_2 = h \mathcal{F}_1 \) is a fundamental domain for \( \langle g_2 \rangle \).

**Proof.** Obviously \( \mathcal{F}_2 = h \mathcal{F}_1 \). Since \( \mathcal{F}_1 \) is a fundamental domain for \( \langle g_1 \rangle \), we have

\[
\bigcup_{j \in \mathbb{Z}} g_2^j \mathcal{F}_2 = \bigcup_{j \in \mathbb{Z}} h g_1^j h^{-1} h \mathcal{F}_1 = \bigcup_{j \in \mathbb{Z}} h g_1^j \mathcal{F}_1 = h \bigcup_{j \in \mathbb{Z}} g_1^j \mathcal{F}_1 = h \text{PSL}(2, \mathbb{R}) = \text{PSL}(2, \mathbb{R}),
\]

and if \( j \in \mathbb{Z}, g_2^j \neq e \), then \( g_1^j \neq e \) yields

\[
\mathcal{F}_2 \cap g_2^j \mathcal{F}_2 = h \mathcal{F}_1 \cap h g_1^j h^{-1} h \mathcal{F}_1 = h \mathcal{F}_1 \cap h g_1^j \mathcal{F}_1 = h (\mathcal{F}_1 \cap g_1^j \mathcal{F}_1) = \emptyset.
\]

Also both \( \mathcal{F}_1 \subset \text{PSL}(2, \mathbb{R}) \) and \( \mathcal{F}_2 \subset \text{PSL}(2, \mathbb{R}) \) are open.

\[\square\]

Recall that every hyperbolic (resp. parabolic) element is conjugate with \( a_t \) (resp. \( b_t \)) for some \( t \in \mathbb{R} \). Note that \( \langle a_t \rangle = \langle a_{-t} \rangle \) and \( \langle b_t \rangle = \langle b_{-t} \rangle \). The next result follows from the preceding lemma.

**Proposition 3.1.** Let \( g \in \text{PSL}(2, \mathbb{R}) \) be a hyperbolic element (resp. parabolic element). If \( h \in \text{PSL}(2, \mathbb{R}) \) and \( t \in \mathbb{R} \) are such that \( g = h^{-1} a_t h \) (resp. \( g = h^{-1} b_t h \)) then \( \mathcal{F} = h \mathcal{F}_{|t|} \) (resp. \( \mathcal{E} = h \mathcal{E}_{|t|} \)) is a fundamental domain for \( \Gamma = \langle g \rangle \), where \( \mathcal{F}_{|t|} \) (resp. \( \mathcal{E}_{|t|} \)) is a fundamental domain for \( \langle a_t \rangle \) (resp. \( \langle b_t \rangle \)) given by (3.4) (resp. (3.5)).

Next we define \( \Theta : T^1 \mathbb{H}^2 \to \text{PSL}(2, \mathbb{R}) \) by \( \Theta(z, \xi) = g \) for \( (z, \xi) \in T^1 \mathbb{H}^2 \), where \( g \in \text{PSL}(2, \mathbb{R}) \) satisfies \( \partial g(i, i) = (z, \xi) \). Then \( \Theta \) is well-defined and bijective owing to Lemma 2.1. Recall that there exist metrics on \( \text{PSL}(2, \mathbb{R}) \) and \( T^1 \mathbb{H}^2 \) such that \( \Theta \) is an isometry.
Lemma 3.4. Let $F \subset \mathbb{H}^2$ and denote $T^1 F = \{ (z, \xi) \in T^1 \mathbb{H}^2 : z \in F \}$. Then

$$\Theta(T^1 F) = \{ g \in \text{PSL}(2, \mathbb{R}) : g = b_x a_{\ln y} d_\theta : x + iy \in F, \theta \in [0, 2\pi) \}.$$ 

Proof: For any $g = b_x a_{\ln y} d_\theta \in \text{PSL}(2, \mathbb{R})$ with $x + iy \in F$, if $g = \{ \pm \begin{pmatrix} a & b \\ c & d \end{pmatrix} \}$ then we take $z = x + iy \in F$ and $\xi = \text{Re} \xi + i \text{Im} \xi$ satisfying

$$x = \frac{ac + bd}{c^2 + d^2}, \quad y = \frac{1}{c^2 + d^2}, \quad \text{Re} \xi = \frac{2cd}{(c^2 + d^2)^2}, \quad \text{Im} \xi = \frac{d^2 - c^2}{(c^2 + d^2)^2}.$$ 

Then $\|\xi\|_z = \frac{|\xi|}{y} = 1$ means that $(z, \xi) \in T^1 F$ and (2.4) shows $\Theta(z, \xi) = g$. On the other hand, for $(z, \xi) \in T^1 F$ and $\Theta(z, \xi) = g \in \text{PSL}(2, \mathbb{R})$. If $g = \{ \pm \begin{pmatrix} a & b \\ c & d \end{pmatrix} \}$ then $x = \frac{ac + bd}{c^2 + d^2}, y = \frac{1}{c^2 + d^2}$ by Lemma 3.2 (a). Once again (2.4) implies that $z = x + iy \in F$. This completes the proof. \(\Box\)

The relation of fundamental domains in $\mathbb{H}^2$ and in $T^1 \mathbb{H}^2$ is the following:

Theorem 3.2. Let $\Gamma$ be a Fuchsian group. A set $F \subset \mathbb{H}^2$ is a fundamental domain for $\Gamma$ if and only if $T^1 F \subset T^1 \mathbb{H}^2$ is a fundamental domain for $\Gamma$.

Proof: Let $F \subset \mathbb{H}^2$ and $\mathcal{F} = \{ g = b_x a_{\ln y} d_\theta, x + iy \in F, \theta \in [0, 2\pi) \} \subset \text{PSL}(2, \mathbb{R})$. Then $\Theta^{-1}(\mathcal{F}) = T^1 F$ by Lemma 3.4 and this follows from Theorem 3.1 and the fact that $\Theta$ is an isomorphism. \(\Box\)

See also [3, Exercise 24] for the sufficiency.

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