PROPERTIES OF CLIFFORD LEGENDRE POLYNOMIALS

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Abstract. Clifford-Legendre and Clifford-Gegenbauer polynomials are eigenfunctions of certain differential operators acting on functions defined on $m$-dimensional euclidean space $\mathbb{R}^m$ and taking values in the associated Clifford algebra $\mathbb{R}_m$. New recurrence and Bonnet type formulae for these polynomials are proved, as their Fourier transforms are computed. Explicit representations in terms of spherical monogenics and Jacobi polynomials are given, with consequences including the interlacing of the zeros. In the case $m = 2$ we describe a degeneracy between the even- and odd-indexed polynomials.

1. Introduction

In one dimension, the Legendre polynomials $\{P_n\}_{n=0}^\infty$ have been long been used as spectral elements in schemes for solving differential equations, and are used in the numerical computation of prolate spheroidal wave functions. It’s important in this context that the Legendre polynomials satisfy certain recurrence relations, and in particular the Bonnet formula which expresses the product $xP_n(x)$ as a linear combination of $P_{n+1}(x)$ and $P_{n-1}(x)$ - see e.g. [4,13]. This property allows for the reduction of the computation of prolate spheroidal wavefunctions to the computation of the eigenvectors and eigenvalues of a tri-diagonal matrix.

Clifford algebras are generalizations to higher dimensions of the normed division algebras of real and complex numbers and the quaternions; they are normed algebras where the multiplication is non-commutative, but still associative [12]. With a view to constructing higher dimensional prolate spheroidal wavefunctions, here we investigate Clifford algebra-valued polynomials on the $m$-dimensional euclidean space $\mathbb{R}^m$ known as Clifford-Gegenbauer polynomials and the special case of Clifford-Legendre polynomials. These polynomials take values in the $2^m$-dimensional Clifford algebra $\mathbb{R}_m$. Orthogonal polynomials in Clifford analysis were introduced in [7] and their properties and applications were demonstrated in [5,8].

The key result of the paper is the Bonnet type formula for the Clifford-Legendre polynomials. We give explicit representations of these polynomials as products of Jacobi polynomials in the radial direction and spherical monogenics. We compute the Fourier transform of the restriction of Clifford-Legendre polynomials to the unit ball in $\mathbb{R}^m$ and demonstrate that despite the absence of a higher-dimensional Sturm-Liouville theory, the zeros of the Clifford-Legendre polynomials are (radially) interlaced.

In the second section, we give background related to Clifford algebra and Clifford analysis. In section 3, we define the Clifford-Gegenbauer polynomials and investigate some of their properties. In Section 4, we compute the Fourier transform of the restrictions of the Clifford-Legendre Polynomials to balls and use this to provide a suitable normalization. Plots of some normalized Clifford-Legendre Polynomials are

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then provided. In section 5, we investigate some properties of the Clifford-Legendre differential equation and will prove that the radial part of the Clifford-Legendre polynomials are Jacobi polynomials and that their zero sets are interlaced. Finally, in the last section we obtain Bonnet type formula for Clifford-Legendre polynomials.

2. Background

Let $\mathbb{R}^m$ be $m$-dimensional euclidean space and let $\{e_1, e_2, \ldots, e_m\}$ be an orthonormal basis for $\mathbb{R}^m$. We endow these vectors with the multiplicative properties
\[
e_j e_j = \begin{cases} 1, & j = 1, \ldots, m, \\ -1, & j = 1, \ldots, m. \end{cases}
\]
For any subset $A = \{j_1, j_2, \ldots, j_k\} \subseteq \{1, \ldots, m\}$, we consider the formal product $e_A = e_{j_1} e_{j_2} \ldots e_{j_k}$. Moreover for the empty set $\emptyset$ one puts $e_\emptyset = 1$ (the identity element). The Clifford algebra $\mathbb{R}_m$ is then the $2^m$-dimensional algebra
\[\mathbb{R}_m = \left\{ \sum_{A \subseteq M} \lambda_A e_A : \lambda_A \in \mathbb{R} \right\} \]
Every element $\lambda = \sum_{\lambda \subseteq M} \lambda_A e_A \in \mathbb{R}_m$ may be decomposed as $\lambda = \sum_{k=0}^m [\lambda]_k$, where $[\lambda]_k = \sum_{|A|=k} \lambda_A e_A$ is the so-called $k$-vector part of $\lambda (k = 0, 1, \ldots, m)$.

Denoting by $\mathbb{R}_m^k$ the subspace of all $k$-vectors in $\mathbb{R}_m$, i.e., the image of $\mathbb{R}_m$ under the projection operator $[\cdot]_k$, one has the multi-vector structure decomposition $\mathbb{R}_m = \mathbb{R}_m^0 \oplus \mathbb{R}_m^1 \oplus \cdots \oplus \mathbb{R}_m^m$, leading to the identification of $\mathbb{R}$ with the subspace of real scalars $\mathbb{R}_m^0$ and of $\mathbb{R}_m^m$ with the subspace of real Clifford vectors $\mathbb{R}_m^1$. The latter identification is achieved by identifying the point $(x_1, \ldots, x_m) \in \mathbb{R}^m$ with the Clifford number $x = \sum_{j=1}^m e_j x_j \in \mathbb{R}_m^1$. The Clifford number $e_M = e_1 e_2 \cdots e_m$ is called the pseudoscalar; depending on the dimension $m$, the pseudoscalar commutes or anti-commutes with the $k$-vectors and squares to $\pm 1$. The Hermitian conjugation is the real linear mapping $\lambda \mapsto \bar{\lambda}$ of $\mathbb{R}_m$ to itself satisfying
\[
\bar{\lambda \mu} = \bar{\mu} \bar{\lambda}, \quad \bar{\lambda_A e_A} = \bar{\lambda} \bar{e_A}, \quad \bar{\lambda} \in \mathbb{R}, \\
\bar{e_j} = -e_j, \quad j = 1, \ldots, m.
\]
The Hermitian conjugation leads to a Hermitian inner product and its associated norm on $\mathbb{R}_m$ given respectively by
\[
\langle \lambda, \mu \rangle = [\bar{\lambda} \mu]_0 \quad \text{and} \quad |\lambda|^2 = [\bar{\lambda} \lambda]_0 = \sum_A |\lambda_A|^2.
\]
The product of two vectors splits up into a scalar part and a 2-vector, also called a bivector:
\[
x y = -\langle x, y \rangle + x \wedge y
\]
where $\langle x, y \rangle = -\sum_{j=1}^m x_j y_j \in \mathbb{R}_m^0$, and $x \wedge y = \sum_{i=1}^m \sum_{j=i+1}^m e_i e_j (x_j y_i - x_i y_j) \in \mathbb{R}_m^2$. Note that the square of a vector variable $x$ is scalar-valued and equals the norm squared up to minus sign:
\[
x^2 = -\langle x, x \rangle = -|x|^2.
\]
Clifford analysis offers a function theory which is a higher-dimensional analogue of the theory of holomorphic functions of one complex variable. The functions considered are defined in the Euclidean space $\mathbb{R}^m$ and take their values in the Clifford algebra $\mathbb{R}_m$.

The central notion in Clifford analysis is monogenicity, which is a multidimensional counterpart of holomorphy in the complex plane.

Let $\Omega \subset \mathbb{R}^m$, $f : \Omega \to \mathbb{R}^m$ and $n$ a non-negative integer. We say $f \in C^n(\Omega, \mathbb{R}^m)$ if $f$ and all its partial derivatives of order less than or equal to $n$ are continuous.

**Definition 2.1.** Let $\Omega \subset \mathbb{R}^m$. A function $f \in C^1(\Omega, \mathbb{R}^m)$ is said to be left monogenic in that region if

$$\partial_x f = 0.$$ 

Here $\partial_x$ is the Dirac operator in $\mathbb{R}^m$, i.e.,

$$\partial_x = \sum_{j=1}^m e_j \partial_{x_j},$$

where $\partial_{x_j}$ is the partial differential operator $\frac{\partial}{\partial x_j}$. The Euler operator is defined on $C^1(\Omega, \mathbb{R}^m)$ by

$$E = \sum_{j=1}^m x_j \partial_{x_j}.$$ 

If $k$ is a non-negative integer and $f \in C^1(\mathbb{R}^m \setminus \{0\}, \mathbb{R}^m)$ is homogeneous of degree $k$ (i.e., $f(\lambda x) = \lambda^k f(x)$ for all $\lambda > 0$ and $x \in \mathbb{R}^m$) then $Ef = kf$. The Laplace operator is factorized by the Dirac operator as follows:

$$\Delta_m = -\partial_x^2.$$ (2.1)

The notion of right monogenicity is defined in a similar way by letting the Dirac operator act from the right. It is easily seen that if a Clifford algebra-valued function $f$ is left monogenic, its Hermitian conjugate $\bar{f}$ is right monogenic.

**Theorem 2.2.** (*Clifford-Stokes theorem*) Let $\Omega \subset \mathbb{R}^m$, $f, g \in C^1(\Omega)$ Assume that $C$ is a compact orientable $m$-dimensional manifold with boundary $\partial C$. Then for each $C \subset \Omega$, one has

$$\int_{\partial C} f(x)n(x) g(x) d\sigma(x) = \int_C \left[ (f(x) \partial_x g(x) + f(x)(\partial_x g(x))) \right] dx.$$ 

where $n(x)$ is the outward-pointing unit normal vector on $\partial C$.

*Proof.* See the proof at [9] □

**Definition 2.3.** A left monogenic homogeneous polynomial $Y_k$ of degree $k$ ($k \geq 0$) on $\mathbb{R}^m$ is called a left solid inner spherical monogenic of order $k$. The set of all left solid inner spherical monogenics of order $k$ will be denoted by $M^+_l(k)$. It can be shown [9] that the dimension of $M^+_l(k)$ is given by

$$\dim M^+_l(k) = \frac{(m+k-2)!}{(m-2)!k!} = \binom{m+k-2}{k} = d_k,$$
We may choose an orthonormal basis for each \( M_t^+(k) \), \((k \geq 0)\) i.e., a collection \( \{Y_k^j\}_{j=1}^{d_k} \) which spans \( M_t^+(k) \) and for which
\[
\int_{S^{m-1}} Y_k^j(\theta)Y_k^{j'}(\theta)d\theta = \delta_{jj'}.
\]

**Remark 2.4.** By direct calculation we see that
\[
e_j x = -2x_j - xe_j,
\]
\[
\partial_x E = \partial_x + E\partial_x.
\]

**Lemma 2.5.** If \( Y_k(x) \in M_t^+(k) \), and \( P_m \) is a polynomial of degree \( m \), of a single variable, then
\[
\partial_x[P_m(|x|^2)Y_k(x)] = Q_m(|x|^2)Y_k(x),
\]
\[
\partial_x[P_m(|x|^2)Y_k(x)] = xQ_{m-1}(|x|^2)Y_k(x)
\]
where \( Q_m \) and \( Q_{m-1} \) are polynomials of degree \( m \) and \( m-1 \), of a single variable, respectively.

**Remark 2.6.** It’s important to note that although the polynomial \( P_m \) in Lemma 2.5 has degree \( m \), the polynomial \( P_m(|x|^2) \) has degree \( 2m \).

**Remark 2.7.** If \( f : \mathbb{R}^m \to \mathbb{R}_m \) we define \( Qf \) by \( Qf(x) = xf(x) \). Then we have that
\[
\partial_x Q = -mI - Q\partial_x - 2E,
\]
\[
\partial_x^2 Q = Q\partial_x^2 - 2\partial_x.
\]

**Remark 2.8.** If \( \{A_j\}_{j=1}^{\infty} \), \( \{B_j\}_{j=1}^{\infty} \) are sequences of real numbers for which \( B_j = A_{j+1} - A_j \) and \( d = B_{j+1} - B_j \), is constant, then the general term of \( \{A_n\} \) is given by
\[
A_n = A_1 + (n-1)B_1 + \frac{(n-1)(n-2)}{2}d.
\]

We will need also the following lemma which is easy to obtain by direct calculation.

**Lemma 2.9.** For \( Y_k \in M_t^+(k) \) and \( s \in \mathbb{N} \) the following fundamental formulas hold:
\[
\partial_x[x^sY_k(x)] = \begin{cases} -sx^{s-1}Y_k(x) & \text{for } s \text{ even}, \\ -(s+2k+m-1)x^{s-1}Y_k(x) & \text{for } s \text{ odd}. \end{cases} \tag{2.2}
\]
and for \( s \geq 2 \),
\[
\Delta_m[x^sY_k(x)] = \begin{cases} -s(s+2k+m-2)x^{s-2}Y_k(x) & \text{for } s \text{ even}, \\ -(s+2k+m-2)(s-1)x^{s-2}Y_k(x) & \text{for } s \text{ odd}. \end{cases} \tag{2.3}
\]

3. **Clifford-Gegenbauer and Clifford-Legendre Polynomials**

For \( r > 0 \), let \( B(r) \) be the ball of radius \( r \) and centre 0 in \( \mathbb{R}^m \), i.e.,
\[
B(r) = \{x \in \mathbb{R}^m : |x| \leq r\}.
\]
The class of \( k \)-times continuously differentiable functions \( f : B(r) \to \mathbb{R}_m \) is denoted \( C^k(B(r), \mathbb{R}_m) \).

**Definition 3.1.** Given \( \alpha > -1 \), let \( D_\alpha \) be the differential operator defined on \( C^1(B(1), \mathbb{R}_m) \) by
\[
D_\alpha f(x) = (1 + x^2)^{-\alpha}\partial_x((1 + x^2)^{\alpha+1}f(x)). \tag{3.1}
\]
**Definition 3.2.** Let $\alpha > -1$ and let $Y_k^i \in M_+^i(k)$ be fixed where $i \in \{1, 2, 3, \ldots, d_k\}$ and $m, n \in \mathbb{N}$. Then the **Clifford-Gegenbauer polynomial**, $C_{n,m}^\alpha(Y_k^i)(x)$, is defined by

$$C_{n,m}^\alpha(Y_k^i)(x) = D_\alpha D_{\alpha+1} \cdots D_{\alpha+n-1}Y_k^i(x). \quad (3.2)$$

The following description of the Clifford-Gegenbauer polynomials is a generalization of classical Rodrigues’ formula for Gegenbauer polynomials on the line.

**Theorem 3.3.** *(Rodrigues’ Formula)* The Clifford Gegenbauer polynomials $C_{n,m}^\alpha(Y_k^i)(x)$ are also determined by

$$C_{n,m}^\alpha(Y_k^i)(x) = (1 + x^2)^{-\alpha} \partial_x^n((1 + x^2)^{\alpha+n}Y_k^i(x)). \quad (3.3)$$

**Proof.** See the proof at [9]. \qed

As a consequence of the Rodrigues’ formula *(3.3)*, we have that the Clifford-Gegenbauer polynomials are eigenfunctions of a differential operator as below.

**Theorem 3.4.** *(Differential equation for the Gegenbauer polynomials)* For all $n, k \in \mathbb{N}$ and $\alpha > -1$, the Clifford-Gegenbauer polynomial $C_{n,m}^\alpha(Y_k^i)(x)$ is an eigenfunction of the differential operator $D_\alpha \partial_x$ with real eigenvalue $C(\alpha, n, m, k)$, i.e.,

$$D_\alpha \partial_x(C_{n,m}^\alpha(Y_k^i)(x)) = C(\alpha, n, m, k)C_{n,m}^\alpha(Y_k^i)(x), \quad (3.4)$$

where

$$C(\alpha, n, m, k) = \begin{cases} n(2\alpha + n + m + 2k) & \text{if } n \text{ is even} \\ (2\alpha + n + 1)(n + m + 2k - 1) & \text{if } n \text{ is odd}. \end{cases}$$

**Proof.** For the proof, see [9] \qed

**Definition 3.5.** The Clifford-Legendre polynomial $C_{n,m}^0(Y_k^i)(x)$ is the special case of Clifford-Gegenbauer polynomial $C_{n,m}^\alpha(Y_k^i)(x)$ that arises when $\alpha = 0$. From Theorem 3.3 we have

$$C_{n,m}^0(Y_k^i)(x) = \partial_x^n((1 - |x|^2)^nY_k^i(x)).$$

**Lemma 3.6.** Let $C_{n,m}^0(Y_k^i)(x)$ be a Clifford-Legendre polynomial and $E$ be the Euler operator. Then

$$E[C_{n,m}^0(Y_k^i)(x)] = (n + k)C_{n,m}^0(Y_k^i)(x) - 2n\partial_x(C_{n,m}^0(Y_k^i)(x)).$$

**Proof.** An application of the binomial theorem gives

$$E[C_{n,m}^0(Y_k^i)(x)] = E[\partial_x^n((1 - |x|^2)^nY_k^i(x))] = \sum_{r=0}^{n} \binom{n}{r} (-1)^r E[\partial_x^{n-r}[|x|^{2r}Y_k^i(x)]]. \quad (3.5)$$

Note that if $g$ is homogeneous of degree $k$ (i.e., $g(\lambda x) = \lambda^kg(x)$ for $\lambda > 0$) then $E(g(x)) = kg(x)$. By Lemma 2.9 we have

$$\partial_x[|x|^{2r}Y_k^i(x)] = 2r(|x|^{2r-2}Y_k^i(x))$$

and

$$\partial_x^2[|x|^{2r}Y_k^i(x)] = -2r(m + 2r + 2k - 2)|x|^{2r-2}(Y_k^i(x)).$$
We conclude that \( \partial_x [|x|^{2r}Y_k^i(x)] \) is homogeneous of degree \( 2r + k - 1 \) and \( \partial_x^2 [|x|^{2r}Y_k^i(x)] \) is homogeneous of degree \( 2r + k - 2 \). An inductive argument gives us that \( \partial_x^n [|x|^{2r}Y_k^i(x)] \) is homogeneous of degree \( 2r - n + k \) so that

\[
E\partial_x^n [|x|^{2r}Y_k^i(x)] = (2r - n + k)\partial_x^n [|x|^{2r}Y_k^i(x)].
\]  

(3.6)

Applying (3.6) to (3.5) yields

\[
E\partial_x^n [C_{n,m}^0(Y_k^i(x))] = \sum_{r=0}^{n} \binom{n}{r} (-1)^r (2r + k - n)\partial_x^n [|x|^{2r}Y_k^i(x)]
\]

\[
= \partial_x^n \left[ \sum_{r=0}^{n} \binom{n}{r} (-1)^r 2r |x|^{2r}Y_k^i(x) \right]
\]

\[
+ (k - n)\partial_x^n \left[ \sum_{r=0}^{n} \binom{n}{r} (-1)^r |x|^{2r}Y_k^i(x) \right].
\]

(3.7)

However, note that differentiating the equation \((1 - t^2)^n = \sum_{r=0}^{n} \binom{n}{r} (-1)^r t^{2r}\) with respect to \( t \) gives \(-2nt(1 - t^2)^{n-1} = \sum_{r=0}^{n} \binom{n}{r} (-1)^r 2rt^{2r-1}\) and applying this to (3.7) yields

\[
E[C_{n,m}^0(Y_k^i(x))] = \partial_x^n [-2n|x|^2(1 - |x|^2)^{n-1}Y_k^i(x)] + (k - n)\partial_x^n [(1 - |x|^2)^nY_k^i(x)]
\]

\[
= \partial_x^n [2n(1 - |x|^2 - 1)(1 - |x|^2)^{n-1}Y_k^i(x)] + (k - n)C_{n,m}^0(Y_k^i(x))
\]

\[
= (n + k)(C_{n,m}^0(Y_k^i(x))) - 2n\partial_x(C_{n,m}^0(Y_k^i(x))).
\]

\[\square\]

**Proposition 3.7.** If \( f : \mathbb{R}^m \rightarrow \mathbb{R} \) has continuous partial derivatives up to order \( n \geq 0 \), then

\[
\partial_x^l[(1 - |x|^2)f] = A_l\partial_x^{l-2}f + B_lE\partial_x^{l-2}f + C_lx\partial_x^{l-1}f + (1 - |x|^2)\partial_x^lf
\]

(3.8)

for \( 0 \leq l \leq n \), where

\[
A_l = l^2 + l(m - 2) + \frac{(1 - m)(1 - (-1)^l)}{2}; \quad B_l = 2l - 1 + (-1)^l; \quad C_l = (-1)^l - 1.
\]

**Proof.** We first prove (by induction on \( m, n \)) that there exist constants \( A_l, B_l, C_l \) such that (3.8) holds for \( 0 \leq l \leq n \). Note that (3.8) is satisfied for \( l = 0 \) with \( A_0 = B_0 = C_0 = 0 \). By direct calculation, we find

\[
\partial_x[(1 - |x|^2)f(x)] = -2xf(x) + (1 - |x|^2)\partial_xf(x)
\]

so that (3.8) is satisfied for \( l = 1 \) with \( A_1 = B_1 = 0, C_1 = -2 \). Suppose there are constants \( A_k, B_k, C_k \) for which (3.8) holds for \( 2 \leq k \leq n - 1 \). Then

\[
\partial_x^{k+1}[(1 - |x|^2)f(x)]
\]

(3.9)

\[
= \partial_x \left[ A_k\partial_x^{k-2}f(x) + B_kE\partial_x^{k-2}f(x) + C_kx\partial_x^{k-1}f(x) + (1 - |x|^2)\partial_x^kf(x) \right]
\]

\[
= A_k\partial_x^{k-1}f(x) + B_kE\partial_x^{k-2}f(x) + C_kx\partial_x^{k-1}f(x) + \partial_x[(1 - |x|^2)\partial_x^kf(x)]
\]

\[
+ (-C_k - 2)x\partial_x^{k-1}f(x) + (1 - |x|^2)\partial_x^kf(x)
\]

(3.10)
so that (3.8) is verified \( l = k + 1 \). We conclude that there are constants \( A_k, B_k, C_k \)
for which (3.8) is satisfied for \( 0 \leq k \leq n - 1 \). Comparing (3.9) with (3.8), we find the
recurrence relations

\[
\begin{align*}
A_{k+1} &= A_k + B_k - mC_k \\
B_{k+1} &= B_k - 2C_k \\
C_{k+1} &= -C_k - 2
\end{align*}
\]  

Equation (3.13) with initial condition \( C_0 = 0 \) has solution \( C_k = -1 + (-1)^k \). Substituting this into
(3.12) and applying the initial condition \( B_0 = 0 \) gives \( B_k = 2k - 1 + (-1)^k \). Finally, substituting \( C_k = -1 + (-1)^k \) and \( B_k = 2k - 1 + (-1)^k \) into
(3.11) and applying the initial condition \( A_0 = 0 \) gives

\[
A_k = k^2 + (m - 2)k + \frac{(1 - m)(1 - (-1)^k)}{2}
\]

We now consider recurrence formulae for the Clifford-Legendre and Clifford-Gegenbauer polynomials.

**Theorem 3.8.** The Clifford-Legendre polynomials \( \{C_{n,m}^0(Y_k^i)(x)\}_{n=0}^\infty \) satisfy the following recurrence formula;

\[
\partial_x[C_{n+1,m}^0(Y_k^i)(x)] = \alpha_{n,k,m}[C_{n,m}^0(Y_k^i)(x)] + \beta_{n,k}\partial_x[C_{n-1,m}^0(Y_k^i)(x)] - C_{n+1}x\partial_x[C_{n,m}^0(Y_k^i)(x)]
\]

where

\[
\begin{align*}
\alpha_{n,k,m} &= [A_{n+1} + (n + k + 1)B_{n+1} - (m + 2n + 2k)C_{n+1} + C(0, n, m, k)] \\
\beta_{n,k} &= 2n(2C_{n+1} - B_{n+1})
\end{align*}
\]

\( A_n, B_n, C_n \) are as in the statement of Proposition 3.7 and \( C(0, n, m, k) \) is the given
eigenvalue in Theorem 3.4.

**Proof.** An application of Proposition 3.7 gives

\[
\partial_x[C_{n+1,m}^0(Y_k^i)(x)] = \partial_x\partial_x^n(1 - |x|^2)^m(1 - |x|^2)^nY_k(0)
\]

\[
\begin{align*}
&= \partial_x[A_{n+1}\partial_x^n(1 - |x|^2)^mY_k(x) + B_{n+1}E\partial_x^{n-1}(1 - |x|^2)^nY_k(x)] \\
&+ C_{n+1}x\partial_x^n[(1 - |x|^2)^mY_k(x)] + (1 - |x|^2)\partial_x^n[(1 - |x|^2)^nY_k(x)] \\
&= C_{n+1}C_{n,m}^0(Y_k)(x) + B_{n+1}\partial_xE\partial_x^{n-1}(1 - |x|^2)^nY_k(x) \\
&+ \partial_xQ\partial_x^n[(1 - |x|^2)^nY_k(x)] + \partial_x[(1 - |x|^2)\partial_xC_{n,m}^0(Y_k)(x)]
\end{align*}
\]  

Remarks 2.4 and 2.7 can be applied to the second and third terms on the right hand
side of (3.14) to obtain

\[
\begin{align*}
\partial_x[C_{n+1,m}^0(Y_k^i)(x)] &= A_{n+1}C_{n,m}^0(Y_k)(x) + B_{n+1}(I + E)C_{n,m}^0(Y_k)(x) \\
&+ C_{n+1}\partial_xQC_{n,m}^0(Y_k)(x) + LC_{n,m}^0(Y_k)(x) \\
&= A_{n+1}C_{n,m}^0(Y_k)(x) + B_{n+1}(I + E)C_{n,m}^0(Y_k)(x) \\
&+ C_{n+1}(-mI - Q\partial_x - 2E)C_{n,m}^0(Y_k)(x) + LC_{n,m}^0(Y_k)(x)
\end{align*}
\]  

(3.15)
where $\mathcal{L}$ is the differential operator of Theorem 3.4. An application of Theorem 3.4 and Lemma 3.6 to (3.15) yields
\[
\partial_x[C_{n+1,m}^0(Y_k^i)(x)] = A_{n+1}C_{n,m}^0(Y_k^i)(x) + B_{n+1}C_{n,m}^0(Y_k^i)(x) + (n + k)C_{n,m}^0(Y_k^i)(x) - 2n\partial_x C_{n,m}^0(Y_k^i)(x)
\]
\[
+ C_{n+1}[-mC_{n,m}^0(Y_k^i)(x) - x\partial_x C_{n,m}^0(Y_k^i)(x) - 2[(n + k)C_{n,m}^0(Y_k^i)(x) - 2n\partial_x C_{n,m}^0(Y_k^i)(x)]
\]
\[
+ \mathcal{L}C_{n,m}^0(Y_k^i)(x)
\]
\[
= [A_{n+1} + (n + k + 1)B_{n+1} - (m + 2(n + k))(n, m, k)]C_{n,m}^0(Y_k^i)(x)
\]
\[
+ [4nC_{n+1} - 2nB_{n+1}]\partial_x C_{n,m}^0(Y_k^i)(x) - C_{n+1}x\partial_x C_{n,m}^0(Y_k^i)(x)
\]
\[
= [\alpha_{n,k,m}I + \beta_{n,k}\partial_x - C_{n+1}Q\partial_x]C_{n,m}^0(Y_k^i)(x)
\]
where $\alpha_{n,k,m}$ and $\beta_{n,k}$ are as in the statement of the Theorem. \(\square\)

The following differential recurrence formula is valid for the Clifford-Gegenbauer polynomials $C_{n,m}^0(Y_k^i)(x)$.

**Theorem 3.9.** The Clifford-Gegenbauer polynomials $\{C_{n,m}^0(Y_k^i)\}_{n,k=0}^\infty$ satisfy
\[
\partial_x[C_{n+1,m}^0(Y_k^i)(x)] = 4(n + \alpha + 1)(n + \alpha + k + \frac{m}{2})C_{n,m}^0(Y_k^i)(x) + 2\alpha\frac{C_{n-1,m}^0(Y_k^i)(x)}{(1 - |x|^2)}
\]
\[
- 4(n + \alpha + 1)(n + \alpha)\partial_x[C_{n-1,m}^0(Y_k^i)(x)].
\]

**Proof.** Note that
\[
\partial_x[(1 - |x|^2)^{n+\alpha}Y_k^i(x)] = -2(n + \alpha + 1)(1 - |x|^2)^{n+\alpha-1}
\]
\[
\times [2(n + \alpha)|x|^2 - (2k + m)(1 - |x|^2)]Y_k^i(x)
\]
and consequently
\[
\partial_x[C_{n+1,m}^0(Y_k^i)(x)] = \partial_x[(1 - |x|^2)^{-\alpha}\partial_x^{n-1}[-2(n + \alpha + 1)(1 - |x|^2)^{n+\alpha-1}
\]
\[
\times [2(n + \alpha)|x|^2 - (2k + m)(1 - |x|^2)]Y_k^i(x)]
\]

With using $|x|^2 = -(1 - |x|^2)^2$, we have
\[
\partial_x[C_{n+1,m}^0(Y_k^i)(x)] = 2m(n + 1 + \alpha)\partial_x[(1 - |x|^2)^{-\alpha}\partial_x^{n-1}[(1 - |x|^2)^{n+\alpha}Y_k^i(x)]
\]
\[
+ 4(1 + n + \alpha)((n + \alpha)\partial_x[(1 - |x|^2)^{-\alpha}\partial_x^{n-1}[(1 - |x|^2)^{n+\alpha}Y_k^i(x)]
\]
\[
- (n + \alpha)\partial_x[(1 - |x|^2)^{-\alpha}\partial_x^{n-1}[(1 - |x|^2)^{(n-1)+\alpha}Y_k^i(x)]
\]
\[
+ k\partial_x[(1 - |x|^2)^{-\alpha}\partial_x^{n-1}[(1 - |x|^2)^{n+\alpha}Y_k^i(x)])
\]
\[
= [2m(n + 1 + \alpha) + 4(n + 1 + \alpha)(n + \alpha) + 4k(n + 1 + \alpha)]
\]
\[
\times [(1 - |x|^2)^{-\alpha}\partial_x^{n-1}[(1 - |x|^2)^{n+\alpha}Y_k^i(x)]
\]
\[
+ 2\alpha(2x)(1 - |x|^2)^{-\alpha-1}\partial_x^{n-1}[(1 - |x|^2)^{n+\alpha}Y_k^i(x)]
\]
\[
- 4(n + 1 + \alpha)(n + \alpha)\partial_x[(1 - |x|^2)^{-\alpha}\partial_x^{n-1}[(1 - |x|^2)^{(n-1)+\alpha}Y_k^i(x)]].
\]
By the definition 3.5 we have that
\[
\partial_x[C_{n+1,m}^\alpha(Y_k^\nu)(x)] = [2m(n + 1 + \alpha) + 4(n + 1 + \alpha)(n + \alpha) + 4k(n + 1 + \alpha)](C_{n,m}^\alpha(Y_k^\nu)(x)) + 2\alpha x(1 - |x|^2)^{-\alpha-1}\partial_x^{-1}[(1 - |x|^2)^{(n-1)+\alpha}Y_k^\nu(x))] - 4(n + 1 + \alpha)(n + \alpha)\partial_xC_{n-1,m}^\alpha(Y_k^\nu)(x) = 4(n + 1 + \alpha)(n + \alpha + k + \frac{m}{2})[C_{n,m}^\alpha(Y_k^\nu)(x)] + 2\alpha(1 - |x|^2)^{-1}C_{n-1,m}^\alpha(Y_k^\nu)(x)] - 4(n + 1 + \alpha)(n + \alpha)\partial_xC_{n-1,m}^\alpha(Y_k^\nu)(x)]
\]
which is the desired recurrence.

Putting \(\alpha = 0\) in Theorem 3.9 gives the following recurrence relation for Clifford-Legendre polynomials.

**Corollary 3.10.** For \(n \geq 1\), \(k \geq 0\) and \(1 \leq i \leq d_k\), the Clifford-Legendre polynomials \(C_{n,m}^0(Y_k^\nu)(x)\) satisfy the recurrence relation

\[
\partial_xC_{n+1,m}^0(Y_k^\nu)(x) = 4(n + 1)\{(n + k + \frac{m}{2})C_{n,m}^0(Y_k^\nu)(x) - n\partial_xC_{n-1,m}^0(Y_k^\nu)(x)]\}.
\]

**Remark 3.11.** When \(n\) is odd, the recurrence formulas of Theorem 3.8 and Corollary 3.10 for the Clifford-Legendre polynomials are identical. Since \(B_n = 2n - 1 + (-1)^n\), if \(n\) is odd, \(B_{n+1} = 2(n + 1)\). So the coefficients \(\beta_{n,k}\) from Proposition 3.8 become \(\beta_{n,k} = -4(n + 1)\) and the coefficients \(\alpha_{n,k}\) become \(\alpha_{n,k} = 4(n + 1)(n + k + \frac{m}{2})\).

The next result gives an explicit representation for the Clifford-Legendre polynomials for the even and odd cases separately.

**Theorem 3.12.** Let \(N, k \geq 0\) and \(1 \leq i \leq d_k\). Then we have

\[
C_{2N+1,m}^0(Y_k^\nu)(x) = -\frac{2^{2N+1}(2N + 1)!}{N!} \sum_{l=0}^{N} \binom{N}{l} \frac{\Gamma(l + k + \frac{m}{2} + N + 1)}{\Gamma(l + k + \frac{m}{2} + 1)} (-1)^l |x|^{2l}Y_k^\nu(x),
\]

\[
C_{2N,m}^0(Y_k^\nu)(x) = \frac{2^{2N}(2N)!}{N!} \sum_{l=0}^{N} \binom{N}{l} \frac{\Gamma(l + k + \frac{m}{2} + N)}{\Gamma(l + k + \frac{m}{2} + 1)} (-1)^l |x|^{2l}Y_k^\nu(x).
\]

**Proof.** When \(n = 2N + 1\) is odd, repeated application of Lemma 2.9 and the binomial theorem give

\[
C_{2N+1,m}^0(Y_k^\nu)(x) = 2^{2N+1} \Gamma(1 - |x|^2)2N+1Y_k^\nu(x)]
\]

\[
= \sum_{j=0}^{2N+1} \binom{2N+1}{j} (-1)^j |x|^{2j}Y_k^\nu(x)
\]

\[
= -\sum_{j=N+1}^{2N+1} \binom{2N+1}{j} (-1)^j \Gamma(j + k + \frac{m}{2} - 1) \frac{\Gamma(j + k + \frac{m}{2} + N + 1)}{\Gamma(j + k + \frac{m}{2} + N + 1)} \frac{|x|^{2j-2N+1}Y_k^\nu(x)}{2j}
\]

\[
= -\sum_{l=0}^{N} \binom{2N+1}{N+l} (-1)^l 2^{2N} \frac{\Gamma(m/2 + k + l + N + 1)}{\Gamma(m/2 + k + l + 1)} 2(l + 1)|x|^{2l}Y_k^\nu(x)
\]

\[
= -\frac{2^{2N+1}(2N + 1)!}{N!} \sum_{l=0}^{N} \binom{N}{l} \frac{\Gamma(l + k + \frac{m}{2} + N + 1)}{\Gamma(l + k + \frac{m}{2} + 1)} (-1)^l |x|^{2l}Y_k^\nu(x).
\]
Similarly, when \( n = 2N \) is even,

\[
C^{0}_{2N,m}(Y^{i}_{k})(x) = \partial^{2N}_{x}[(1 - |x|^{2})^{2N}Y^{i}_{k}(x)]
\]

\[
= \sum_{j=0}^{2N} \binom{2N}{j} (-1)^{j} \partial^{2N}_{x} |x|^{2j}Y^{i}_{k}(x)
\]

\[
= \sum_{j=N}^{2N} \binom{2N}{j} (-1)^{j+N} 2^{N} \frac{(j)!}{(j-N)!} \frac{\Gamma(j+k+m)}{\Gamma(j+k+m-N)}|x|^{2j-2N}Y^{i}_{k}(x)
\]

\[
= \sum_{l=0}^{N} \binom{2N}{N+l} (-1)^{l} 2^{N} \frac{(N+l)!}{(l)!} \frac{\Gamma(m+k+l+N)}{\Gamma(m+k+l)}|x|^{2l}Y^{i}_{k}(x)
\]

\[
= \frac{2^{2N}(2N)!}{N!} \sum_{l=0}^{N} \binom{N}{l} (-1)^{l} \frac{\Gamma(m+k+l+N)}{\Gamma(m+k+l)}|x|^{2l}Y^{i}_{k}(x).
\]

\( \square \)

**Corollary 3.13.** If \( C^{0}_{2N,m}(Y^{i}_{k})(x) \), and \( C^{0}_{2N+1,m}(Y^{i}_{k})(x) \) are Clifford-Legendre polynomials, then there exist polynomials \( P_{N,k} \), and \( Q_{N,k} \) of degree \( m \) such that

\[
C^{0}_{2N,m}(Y^{i}_{k})(x) = P_{N,k,m}(|x|^{2})Y^{i}_{k}(x),
\]

\[
C^{0}_{2N+1,m}(Y^{i}_{k})(x) = Q_{N,k,m}(|x|^{2})aY^{i}_{k}(x).
\]

4. **Normalisation of the Clifford-Legendre Polynomials**

In this section we compute the Fourier transforms of the Clifford-Legendre and their \( L^{2} \)-norms as a consequence. Plots of the polynomials are provided, and a curious degeneracy observed in the case \( m = 2 \).

We consider the Clifford algebra-valued inner product of the functions \( f, g : \mathbb{R}^{m} \rightarrow \mathbb{R}_{m} \) by

\[
\langle f, g \rangle = \int_{\mathbb{R}^{m}} f(x)g(x) \, dx,
\]

where \( dx \) is Lebesgue measure on \( \mathbb{R}^{m} \). The associated norm \( \| \cdot \|_{2} \) is given by

\[
\|f\|_{2}^{2} = \langle f, f \rangle = \left( \int_{\mathbb{R}^{m}} |f(x)|^{2} \, dx \right)^{1/2}.
\]

The right Clifford-module of Clifford algebra-valued measurable functions on \( \mathbb{R}^{m} \) for which \( \|f\|_{2} < \infty \) is a right Hilbert Clifford-module which we denote by \( L^{2}(\mathbb{R}^{m}, \mathbb{R}_{m}) \).

The standard tensorial multi-dimensional Fourier transform given by:

\[
\mathcal{F}f(\xi) = \int_{\mathbb{R}^{m}} \exp(-2\pi i(x, \xi))f(x) \, dx
\]

(4.1)

whenever \( f \in L^{1}(\mathbb{R}^{m}, \mathbb{R}_{m}) \). As is shown in [9], the Fourier transform extends to a unitary mapping on \( L^{2}(\mathbb{R}^{m}, \mathbb{R}_{m}) \).
Theorem 4.1. (Plancherel theorem) For all \( f, g \in L^2(\mathbb{R}^m, \mathbb{R}_m) \) the Parseval formula holds:
\[
\langle f, g \rangle = \langle \mathcal{F}f, \mathcal{F}g \rangle.
\]
In particular, for each \( f \in L^2(\mathbb{R}^m, \mathbb{R}_m) \) one has:
\[
\|f\|_2 = \|\mathcal{F}f\|_2.
\]

By Theorem 2.2, it is possible to prove the following orthogonality property of homogeneous monogenic polynomials.

Lemma 4.2. Let \( Y_k \in M_l^+(k) \) and \( Y_{k'} \in M_{l'}^+(k') \). Then
\[
\int_{S^{m-1}} Y_k(\theta)\bar{Y}_{k'}(\theta)d\theta = 0.
\]

The orthogonality of the Clifford-Legendre polynomials is proved in \cite{9}.

Lemma 4.3. The Clifford-Legendre polynomials
\[
\{C_{n,m}^{0}(Y_k)(x) : n \geq 0, k \geq 0, 1 \leq i \leq d_k\}
\]
form an orthogonal basis for the functions in \( L^2(B(1), \mathbb{R}_m) \).

The following well-known result appears as Lemma 9.10.2 in \cite{2}.

Lemma 4.4. Let \( \xi, \theta \in S^{m-1}, r > 0 \) and \( Y_k \in M_l^+(k) \). Then
\[
\int_{S^{m-1}} e^{-2\pi ir(\xi,\theta)}Y_k(\theta)d\sigma(\theta) = \frac{2\pi(-i)^k}{r^{\frac{m}{2}}-1}J_{k+\frac{m}{2}-1}(2\pi r)Y_k(\hat{\xi}),
\]
where \( J_{k+\frac{m}{2}-1} \) is a Bessel function of the first kind.

Lemma 4.5. If \( f \in C^n(B(1), \mathbb{R}^m) \) \( (n \geq 1) \) and \( 0 \leq k \leq n \), then
\[
\partial_{x}^k((1 - |x|^2)^n f(x)) = (1 - |x|^2)^{n-k}f_k(x)
\]
with \( f_k \in C^{n-k}(B(1), \mathbb{R}^m) \).

Proof. The proof is by induction on \( k \). Equation (4.2) is clearly true when \( k = 0 \). Suppose (4.2) holds for \( k = l \) \( (0 \leq l \leq n - 1) \), i.e.,
\[
\partial_{x}^{l}((1 - |x|^2)^{n} f(x)) = (1 - |x|^2)^{n-l}f_l(x)
\]
with \( f_l \in C^{n-l}(B(1), \mathbb{R}^m) \). Then,
\[
\partial_{x}^{l+1}((1 - |x|^2)^{n} f(x)) = \partial_{x}^{l}((1 - |x|^2)^{n-l}f_l(x))
\]
\[
= \sum_{j=1}^{m} e_j[(n-l)(1 - |x|^2)^{n-l-1}(-2x_j)f_l(x) + (1 - |x|^2)^{n-l}\partial_{x_j}f_l(x)]
\]
\[
= -2(n-l)x(1 - |x|^2)^{n-l-1}f_l(x) + (1 - |x|^2)^{n-l}\partial_{x}f_l(x)
\]
\[
= (1 - |x|^2)^{n-l-1}\partial_{x}f_{l+1}(x),
\]
where \( f_{l+1}(x) = -2(n-l)x f_{l}(x) + (1 - |x|^2)\partial_{x}f_{l}(x) \).

Theorem 4.6. The Fourier transform of the restriction of the Clifford-Legendre polynomial \( C_{n,m}^{0}(Y_k)(x) \) to the unit ball \( B(1) \), is given by
\[
\mathcal{F}(C_{n,m}^{0}(Y_k))(\xi) = (-1)^k \xi^{n+k}2^n n! \xi^{n} \left(\frac{J_{k+\frac{m}{2}+n}(2\pi|\xi|)}{|\xi|^{\frac{m}{2}+n+k}}\right)Y_k(\hat{\xi}).
\]
Proof. We apply the Rodrigues’ formula for the Clifford-Legendre polynomials and the Clifford-Stokes theorem to find
\[
\mathcal{F}(C_{n,m}^0(Y_k^i))(\xi) = \int_{B(1)} e^{-2\pi i \langle x, \xi \rangle} \partial_x^n[(1 - |x|^2)^n Y_k^i(x)] dx
\]
\[
= \int_{S^{m-1}} e^{-2\pi i \langle x, \xi \rangle} \partial_x^{n-1}[(1 - |x|^2)^n Y_k^i(x)] dx
\]
\[
- \int_{B(1)} (e^{-2\pi i \langle x, \xi \rangle} \partial_x^n[(1 - |x|^2)^n Y_k^i(x)]) dx.
\]
By Lemma 4.5, the restriction of \( \partial_x^{n-1}[(1 - |x|^2)^n Y_k^i(x)] \) to the unit sphere \( S^{m-1} \) is zero, so that
\[
\mathcal{F}(C_{n,m}^0(Y_k^i))(\xi) = (2\pi i \xi) \int_{B(1)} e^{-2\pi i \langle x, \xi \rangle} \partial_x^{n-1}[(1 - |x|^2)^n Y_k^i(x)] dx.
\]
By applying Theorem 2.2 repeatedly, we find
\[
\mathcal{F}(C_{n,m}^0(Y_k^i))(\xi) = (2\pi i \xi)^n \int_{B(1)} e^{-2\pi i \langle x, \xi \rangle} [(1 - |x|^2)^n Y_k^i(x)] dx
\]
\[
= (2\pi i \xi)^n \int_0^1 r^{m-1+k} (1 - r^2)^n \int_{S^{m-1}} e^{-2\pi i r \omega \xi} Y_k^i(\omega) d\omega dr
\]
\[
= (2\pi i \xi)^n (2\pi)(-i)^k \frac{Y_k^i(\xi)}{|\xi|^\frac{m}{2}-1} \int_0^1 r^{m+k} (1 - r^2)^n J_{k+\frac{m}{2}-1}(2\pi r |\xi|) dr,
\]
where we have used the homogeneity of \( Y_k^i \) and Lemma 4.4 in the last step. The last integral can be computed from (12, 6.567 1) to yield the result. \( \square \)

Corollary 4.7. The \( L^2 \)-norm of the restriction of the Clifford-Legendre polynomial \( C_{n,m}^0(Y_k^i) \) to the unit ball \( B(1) \) is given by
\[
||C_{n,m}^0(Y_k^i)(x)||_2^2 = \frac{2^{2n}(n!)^2}{2k + 2n + m}.
\]

Proof. We apply the Theorem 4.1 and the assumption that \( Y_k^i \) is \( L^2(S^{m-1}) \)-normalized to find
\[
||C_{n,m}^0(Y_k^i)(x)||^2 = ||\mathcal{F}(C_{n,m}^0(Y_k^i))||^2
\]
\[
= 2^{2n}(n!)^2 \int_{\mathbb{R}^m} \left| \frac{Y_k^i(\xi)J_{k+\frac{m}{2}+n}(2\pi |\xi|)}{|\xi|^\frac{m}{2}+k} \right|^2 d\xi
\]
\[
= 2^{2n}(n!)^2 \int_0^\infty \left( \int_{S^{m-1}} |Y_k^i(\omega)|^2 d\omega \right) |J_{k+\frac{m}{2}+n}(2\pi r)|^2 r^{-1} dr
\]
\[
= \frac{2^{2n}(n!)^2}{2k + 2n + m}
\]
where the last integral has computed from (12, 6.5742). \( \square \)
We therefore define the normalised Clifford-Legendre polynomials $C_{n,m}^0(Y_k)$ by
\[ C_{n,m}^0(Y_k) = \frac{\sqrt{2k+2n+m}}{2^{n+m}n!} C_{n,m}^0(Y_k). \] (4.4)

According to the Definition 2.3, in dimension $m = 2$ we have
\[ \dim M_1^+(k) = \frac{(m+k-2)!}{(m-2)!k!} = 1. \]

Consequently, when $m = 2$ the function
\[ Y_k(r \cos \theta, r \sin \theta) = \frac{r^k}{\sqrt{2\pi}} [e_1 \cos k\theta - e_2 \sin k\theta] \] (4.5)

itself forms an orthonormal basis for $M_1^+(k)$ and in this case the Clifford-Legendre polynomials (described explicitly in Theorem 3.12) take the form
\[ C_{2N,2}^0(Y_k)(x) = F_{N,k}^1 e_1 + F_{N,k}^2 e_2 \]
\[ C_{2N+1,2}^0(Y_k)(x) = G_{N,k}^1 + G_{N,k}^2 e_1 e_2 \]

where $F_{N,k}^1, F_{N,k}^2, G_{N,k}^1, G_{N,k}^2$ are real-valued functions defined on the unit ball $B(1)$. In Figures 4-4 below, these functions are plotted for various values of $N$ and $k$.

**FIGURE 1.** Graph of $e_1$ part of normalized Clifford-Legendre polynomial $C_{0,2}^0(Y_1)$.

**FIGURE 2.** Graph of $e_2$ part of normalized Clifford-Legendre polynomial $C_{0,2}^0(Y_1)$.

In dimension $m = 2$, it can be easily seen from (4.5) that $xY_k(x) = e_1 Y_{k+1}(x)$. As a consequence, we have the following degeneracy between even and odd Clifford-Legendre polynomials.
Theorem 4.8. In dimension $m = 2$, the normalised Clifford-Legendre polynomials satisfy

$$\overline{C}_{2N+1,2}^{0}(Y_k)(x) = -e_1 \overline{C}_{2N,2}^{0}(Y_{k+1})(x).$$

Proof. Putting $m = 2$ in the explicit representations of Theorem 3.12 and applying the normalization (4.4) gives

$$\overline{C}_{2N+1,2}^{0}(Y_k)(x) = -\sqrt{4N + 2k + 4} \frac{N!}{N!} \sum_{l=0}^{N} \binom{N}{l} \binom{l + k + N + 1}{N} (-1)^l |x|^{2l} x Y_k(x).$$

However, $x Y_k = e_1 Y_{k+1}$, so

$$\overline{C}_{2N+1,2}^{0}(Y_k)(x) = -e_1 \sqrt{4N + 2k + 4} \frac{N!}{N!} \sum_{l=0}^{N} \binom{N}{l} \binom{l + k + N + 1}{N} (-1)^l |x|^{2l} Y_{k+1}(x)$$

$$= -e_1 \overline{C}_{2N,2}^{0}(Y_{k+1})(x).$$

□

5. Connections Between Clifford Legendre Polynomials and Jacobi Polynomials

In this section, we prove that the radial part of the Clifford-Legendre polynomials are shifted re-scaled Jacobi polynomials. This observation provides an explanation of the observed interlacing of the zeros of the Clifford-Legendre polynomials.
Let \( m \geq 2 \) be arbitrary and \( n = 2N \) be even. We can write
\[
C_{2N,m}^0(Y_k)(x) = P_{N,k,m}(|x|^2)Y_k(x)
\]
and we call the polynomials \( P_{N,k,m}(|x|^2) \) the radial part of the Clifford-Legendre polynomial \( C_{2N,m}^0(Y_k) \). Let \( \mathcal{L} = -(\Delta + 2x\partial_x) \) be the differential operator that appears in Theorem 3.4. Then we have
\[
\mathcal{L}(C_{2N,m}^0(Y_k^i)(x)) = C(0, 2N, m, k)C_{2N,m}^0(Y_k^i)(x).
\]
We aim to determine a differential operator \( T_0 \) for which \( P_{N,k,m} \) is an eigenfunction. We have
\[
\mathcal{L}(C_{2N,m}^0(Y_k^i)(x)) = \left[ \partial_x(1 - |x|^2) C_{2N,m}^0(Y_k^i)(x) \right] = C(0, 2N, m, k)C_{2N,m}^0(Y_k^i)(x).
\]
On the other hand, since \( Y_k^i \) is left monogenic,
\[
\mathcal{L}(C_{2N,m}^0(Y_k^i)(x)) = \partial_x \left[ (1 - |x|^2)^2 + 2(1 - |x|^2)|x|^2 \partial Y_k^i(x) \right]
\]
However, since \( e_j x = -x e_j - 2x_j \) and \( EY_k^i = kY_k^i \), we have
\[
\mathcal{L}(C_{2N,m}^0(Y_k^i)(x)) = 2 \left[ -2x^2 P_{N,k,m}^0(1 - |x|^2)Y_k^i(x) - \frac{\partial}{\partial x_j} \frac{\partial}{\partial x_j} \frac{\partial}{\partial x_j} \frac{\partial}{\partial x_j} \right]
\]
We conclude from (5.2) that the functions \( P_{N,k,m} \) satisfy
\[
t(1 - t) P_{N,k,m}^0(t) + \left( \frac{m}{2} + k \right) - \left( 1 + \frac{m}{2} + k \right) t \right] P_{N,k,m}^0(t) = -\frac{C(0, 2N, m, k)}{4} P_{N,k,m}(t),
\]
(5.3)
i.e., $$T_0 P_{N,k,m} = -\frac{C(0, 2N, m, k)}{4} P_{N,k,m}$$ where $$T_0$$ is the differential operator

$$T_0 = t(1-t) \frac{d^2}{dt^2} + \left[ \left( \frac{m}{2} + k \right) - \left( \frac{1 + m}{2} + k \right) t \right] \frac{d}{dt}$$

where $$t \in (0, 1)$$. On putting $$t = \frac{s + 1}{2}$$ ($$s \in (-1, 1)$$) and $$R_{N,k,m}(s) = P_{N,k,m}(t)$$, equation (5.3) becomes

$$S_0 R_{N,k,m} = -\frac{C(0, 2N, m, k)}{4} R_{N,k,m}$$

where $$S_0$$ is the differential operator

$$S_0 = (1 - s^2) \frac{d^2}{ds^2} + \left[ \left( k + \frac{m}{2} - 1 \right) - \left( \frac{m}{2} + k + 1 \right) s \right] \frac{d}{ds}.$$  \hspace{1cm} (5.4)

The Jacobi polynomials $$P_{n}^{(\alpha, \beta)}(x)$$ ($$n \geq 0$$, $$\alpha, \beta > -1$$, $$x \in [-1, 1]$$) satisfy the differential equation

$$(1 - x^2)y''(x) + [\beta - \alpha - (\alpha + \beta + 2)x]y'(x) = -n(n + \alpha + \beta + 1)y(x).$$  \hspace{1cm} (5.5)

Since the eigenvalues of the Jacobi differential equation operator are non-degenerate we have

$$P_{N,k,m}(|x|^2) = c_{N,k,m} P_{N}^{(0, k + \frac{m}{2} - 1)}(2|x|^2 - 1)$$  \hspace{1cm} (5.6)

for real constants $$c_{N,k,m}$$. We aim to find $$c_{N,k,m}$$ when $$C_{2N,m}^0(Y_k^i)(x)$$ is normalized. In this case, the homogeneity of $$Y_k^i$$ and its normalization on the unit sphere give

$$1 = \int_{B(1)} |C_{2N,m}^0(Y_k^i)(x)|^2 dx = \int_{B(1)} |P_{N,k,m}(|x|^2 Y_k^i(x)|^2 dx$$

$$= \int_{B(1)} |c_{N,k,m} P_{N}^{(0, k + \frac{m}{2} - 1)}(2|x|^2 - 1) Y_k^i(x)|^2 dx$$

$$= c_{N,k,m}^2 \int_{S^{n-1}} \int_0^1 |P_{N}^{(0, k + \frac{m}{2} - 1)}(2r^2 - 1)|^2 r^{2k} |Y_k^i(\theta)|^2 r^{m-1} dr d\theta$$

$$= \frac{c_{N,k,m}^2}{2^{k + \frac{m}{2} + 1}} \int_{-1}^1 (s + 1)^{k + \frac{m}{2} - 1} |P_{N}^{(0, k + \frac{m}{2} - 1)}(s)|^2 ds. \hspace{1cm} (5.7)$$

But from page 983 we see that the final integral in (5.7) equals $$\frac{2^{k + \frac{m}{2}}}{k + \frac{m}{2} + 2N}$$ and we conclude that $$c_{N,k,m} = \pm \sqrt{2(k + \frac{m}{2} + 2N)}$$. The odd case may be treated similarly. These calculations are summarised below.

**Theorem 5.1.** Let $$C_{2N,m}^0(Y_k^i)(x)$$ and $$C_{2N+1,m}^0(Y_k^i)(x)$$ be normalized Clifford-Legendre polynomials. Then the radial part of these functions are shifted, scaled and renormalised Jacobi polynomials, i.e.,

$$C_{2N,m}^0(Y_k^i)(x) = \pm \sqrt{2(k + \frac{m}{2} + 2N)} P_{N}^{(0, k + \frac{m}{2} - 1)}(2|x|^2 - 1) Y_k^i(x)$$

$$C_{2N+1,m}^0(Y_k^i)(x) = \pm \sqrt{2(k + \frac{m}{2} + 1 + 2N)} P_{N}^{(0, k + \frac{m}{2} + 1)}(2|x|^2 - 1) x Y_k^i(x).$$
Remark 5.2. As we have seen, the radial part of the Clifford-Legendre polynomial $C_{n,m}^0(Y_i^k)(x)$ is a Jacobi polynomials of degree $\left\lfloor \frac{n}{2} \right\rfloor$. Since (e.g., see [2,11]) the degree $n$ Jacobi polynomials have exactly $n$ simple zeros on $[-1,1]$, and we conclude that each Clifford-Legendre polynomial $C_{n,m}^0(Y_i^k)(x)$ will be zero on precisely $n$ distinct spheres of radius $r < 1$ centred at the origin. By appealing to the Sturm-Liouville theory associated with the Jacobi polynomials (e.g., see [1,6]) the radii of the spheres on which the even polynomials $\{C_{2N,m}^0(Y_i^k)\}_{N=0}^\infty$ interlace, as do the radii of the spheres on which the odd polynomials $\{C_{2N+1,m}^0(Y_i^k)\}_{N=0}^\infty$ vanish. What’s not clear is that the even and odd polynomial zero sets interlace, i.e., that the radii of the spheres on which the polynomials $\{C_{n,m}^0(Y_i^k)\}_{n=0}^\infty$ are eigenfunctions of different differential operators.

The following result on the interlacing of the zeros of Jacobi polynomials comes from [10].

Theorem 5.3. Let $n \in \mathbb{N}$, $\alpha \geq -1$, $\beta \geq -1$ and let

$$-1 < x_1 < x_2 < \cdots < x_n < 1,$$
$$-1 < t_1 < t_2 < \cdots < t_n < 1,$$
$$-1 < y_1 < y_2 < \cdots < y_n < 1,$$

where $0 < t < 2$. Then

$$-1 < x_1 < t_1 < y_1 < x_2 < t_2 < y_2 < \cdots < x_n < t_n < y_n < 1.$$

Proof. For the proof see [10]. \(\square\)

Theorem 5.4. The radii of the spheres forming the zero sets of the Clifford-Legendre polynomials $C_{n,m}^0(Y_i^k)(x)$ are interlaced.

Proof. First note that by Theorem 5.3 with $t = 1$, the zero sets of $P_n^{(0,k+\frac{m}{2}-1)}(t)$, $P_n^{(0,k+\frac{m}{2})}(t)$ and $P_n^{(0,k+\frac{m}{2}-1)}(t)$ on $[-1,1]$ are interlaced. By Theorem 5.1, the radii of the zero sets of $C_{2n,m}^0(Y_i^k)$, $C_{2n+1,m}^0(Y_i^k)$ and $C_{2n+2,m}^0(Y_i^k)$ are interlaced. \(\square\)

Figure 5. Classic Jacobi polynomials on the real line
6. Bonnet Type Formula for the Clifford-Legendre Polynomials

In this final section, we will prove the Bonnet type formula which expresses \( xC^0_{n,m}(Y^i_k)(x) \) as a linear combination of \( C^0_{n-1,m}(Y^i_k)(x) \) and \( C^0_{n+1,m}(Y^i_k)(x) \). This is the main motivation for this work, as it allows for the efficient computation of Clifford-prolate functions in higher dimensions [3].

**Lemma 6.1.** The even Clifford-Legendre polynomials \( C^0_{2n,m}(Y^i_k)(x) \) defined on \( \mathbb{R}^m \) satisfy

\[
\int_{B(1)} C^0_{2n'}(Y^i_k')(x) x C^0_{2n}(Y^i_k')(x) \, dx = 0,
\]

for all \( n, n', k, k', i, i' \).

**Proof.** We use the decomposition [5,1] of the Clifford-Legendre polynomials as products of their radial and tangential parts to obtain

\[
\int_{B(1)} C^0_{2n'}(Y^i_k')(x) x C^0_{2n}(Y^i_k')(x) \, dx = \int_{B(1)} P_{n',k',m}(|x|^2) Y^i_k'(x) x P_{n,k,m}(|x|^2) Y^i_k(x) \, dx
\]

\[
= \int_0^1 P_{n',k',m}(r^2) P_{n,k,m}(r^2) r^{k+k'+2} \, dr \left( \int_{S^{m-1}} Y^i_k'(\theta) \partial Y^i_k(\theta) \, d\theta \right) = 0,
\]

because of the Clifford-Stokes theorem [2,2] and the monogenicity of \( Y^i_k \) and \( Y^i_k' \).

**Lemma 6.2.** For the Clifford-Legendre polynomials \( C^0_{n,m}(Y^i_k) \) satisfy the integration by parts formulae

\[
\int_{B(1)} \partial_x^{2n}[xC^0_{2n'+1,m}(Y^i_k)(x)] [(1 - |x|^2)2nY^i_k(x)] \, dx = - \int_{B(1)} C^0_{2n'+1,m}(Y^i_k)(x) x C^0_{2n,m}(Y^i_k)(x) \, dx
\]

\[
\int_{B(1)} \partial_x^{2n+1}[xC^0_{2n',m}(Y^i_k)(x)] [(1 - |x|^2)2n+1Y^i_k(x)] \, dx = - \int_{B(1)} C^0_{2n',m}(Y^i_k)(x) x C^0_{2n+1,m}(Y^i_k)(x) \, dx.
\]

**Proof.** We will prove only the first of these formulae as the proof of the second is similar. Note that an application of the Rodrigues’ formula (3.3) and the Clifford-Stokes theorem gives

\[
\int_{B(1)} C^0_{2n'+1,m}(Y^i_k)(x) x C^0_{2n,m}(Y^i_k)(x) \, dx = - \int_{B(1)} x C^0_{2n'+1,m}(Y^i_k)(x) \partial_x^{2n}[1 - |x|^2]2nY^i_k(x) \, dx
\]

\[
= - \int_{S^{m-1}} x C^0_{2n'+1,m}(Y^i_k)(x) \partial_x^{2n-1}[1 - |x|^2]2nY^i_k(x) \, d\sigma(x)
\]

\[
+ \int_{B(1)} (x C^0_{2n'+1,m}(Y^i_k)(x)) \partial_x \partial_x^{2n-1}[1 - |x|^2]2nY^i_k(x) \, dx
\]

\[
= - \int_{B(1)} \partial_x [xC^0_{2n'+1,m}(Y^i_k)(x)] \partial_x^{2n-1}[1 - |x|^2]2nY^i_k(x) \, dx \tag{6.1}
\]

since the integrand of the integral over the unit sphere vanishes. The result follows by repeated application of (6.1). \( \square \)
In the next result, we show that \( xc_{n,m}^0(Y_k^i)(x) \) can be written as a linear combination \( C_{n_1,m}^0(Y_{k_1}^i)(x) \) and \( C_{n_1+1,m}^0(Y_{k_1}^i)(x) \).

**Lemma 6.3.** There exist real constants \( \{a_{n,k,i}; n \geq 0, k \geq 0, 1 \leq i \leq d_k\} \) and \( \{b_{n,k,i}; n \geq 0, k \geq 0, 1 \leq i \leq d_k\} \) such that
\[
x C_{n,m}^0(Y_k^i)(x) = C_{n_1,m}^0(Y_{k_1}^i)(x)a_{n,k,i} + C_{n_1+1,m}^0(Y_{k_1}^i)(x)b_{n,k,i}.
\]

**Proof.** Since \( \{C_{n,m}^0(Y_k^i); n \geq 0, k \geq 0, 1 \leq i \leq d_k\} \) forms an orthonormal basis for \( L^2(B(1, \mathbb{R}) \) there are Clifford constants \( \{a_{n,k,i}\} \) and \( \{b_{n,k,i}\} \) such that
\[
x C_{2n,m}^0(Y_{k}^i)(x) = \sum_{n'} \sum_{i'} \sum_{k'} C_{2n_1+1,m}^0(Y_{k_1}^{i'})(x) a_{n',k',i'}
+ \sum_{n'} \sum_{i'} \sum_{k'} C_{2n_1,m}^0(Y_{k_1}^{i'})(x) b_{n',k',i'}.
\]

We multiply both sides of (6.2) by \( C_{2M,m}^0(Y_{k''}^{i''})(x) \) from left side and integrate over \( B(1) \) and apply Lemma 6.1 to find
\[
0 = \int_{B(1)} C_{2M,m}^0(Y_{k''}^{i''})(x) x C_{2n,m}^0(Y_k^i)(x) dx
= \sum_{n',i',k'} \left( \int_{B(1)} C_{2M,m}^0(Y_{k''}^{i''})(x) C_{2n_1+1,m}^0(Y_{k_1}^{i'})(x) dx \right) a_{n',k',i'}
+ \sum_{n',i',k'} \left( \int_{B(1)} C_{2M,m}^0(Y_{k''}^{i''})(x) C_{2n_1,m}^0(Y_{k_1}^{i'})(x) dx \right) b_{n',k',i'}
= \int_{B(1)} |C_{2M,m}^0(Y_{k''}^{i''})(x)|^2 b_{M,k'',i''}
\]
where in the last step we have used the orthogonality of the Clifford-Legendre polynomials on \( B(1) \). We conclude that \( b_{M,k'',i''} = 0 \) and therefore (6.2) simplifies to
\[
x C_{2n,m}^0(Y_k^i)(x) = \sum_{n'} \sum_{i'} \sum_{k'} C_{2n_1+1,m}^0(Y_{k_1}^{i'})(x) a_{n',k',i'}.
\]

We recall the radial decompositions
\[
C_{2n,m}^0(Y_k^i)(x) = P_{n,k,m}(|x|^2)Y_k^i(x);
C_{2n+1,m}^0(Y_k^i)(x) = x Q_{n,k,m}(|x|^2)Y_k^i(x)
\]
and apply them to (6.3) to obtain
\[
P_{n,k,m}(r^2)r^kY_{k'}^i(\omega) = \sum_{n',i',k'} Q_{n',k',m}(r^2)r^kY_{k'}^i(\omega) a_{n',k',i'}
\]
with \( r > 0 \) and \( \omega \in S^{m-1} \). Multiplying both sides of this equation by \( Y_{k''}^{i''}(\omega) \) and integrating over \( S^{m-1} \) yields
\[
P_{n,k,m}(r^2)r^k \int_{S^{m-1}} Y_{k''}^{i''}(\omega)Y_{k'}^i(\omega) d\omega = \sum_{n',i',k'} Q_{n',k',m}(r^2)r^k \int_{S^{m-1}} Y_{k''}^{i''}(\omega)Y_{k'}^i(\omega) d\omega a_{n',k',i'}
\]
and the orthogonality of \( \{Y_k^i; k \geq 0, 1 \leq i \leq d_k\} \) on \( S^{m-1} \) gives
\[
P_{n,k,m}(r^2)r^k \delta_{i',i} = \sum_{n',i',k'} Q_{n',k',m}(r^2)r^k \delta_{i',i} a_{n',k',i'} = Q_{n',k',m}(r^2)r^k a_{n',k',i'}
\]
We conclude that \(a_{n',k',i} = 0\) unless \(i = i' = i''\) and \(k = k' = k''\). As a consequence, (6.4) simplifies to
\[
P_{n,k,m}(t) = \sum_{n'=0}^{\infty} Q_{n',k,m}(t) a_{n',k,i}.
\] (6.5)

Let \(w_{k,m}(t) = 2t^{k+\frac{m}{2}}\). We multiply both sides by \(w_{k,m}(t)Q_{n'',k}(t)\) and integrate over \([0,1]\) to obtain
\[
\int_{0}^{1} Q_{n'',k,m}(t)P_{n,k,m}(t)w_{k,m}(t) dt = \sum_{n'=0}^{\infty} \left( \int_{0}^{1} Q_{n'',k,m}(t)Q_{n',k,m}(t)w_{k,m}(t) dt \right) a_{n,k,i} = \|Q_{n'',k,m}\|_{w_{k,m}} a_{n'',k,i}.
\] (6.6)

However, \(P_{N,k,m}(t) = c_{N,k,m}P_{N}^{(0,k+\frac{m}{2}-1)}(2t-1)\) and \(Q_{N,k,m}(t) = d_{N,k,m}P_{N}^{(0,k+\frac{m}{2}-1)}(2t-1)\), so the left hand side of (6.6) becomes
\[
\int_{0}^{1} Q_{n'',k,m}(t)P_{n,k,m}(t)w_{k,m}(t) dt = c_{n,k,m}d_{n,k,m} \int_{-1}^{1} P_{n}^{(0,k+\frac{m}{2}-1)}(s)P_{n}^{(0,k+\frac{m}{2}-1)}(s)(s+1)^{k+\frac{m}{2}} ds.
\] (6.7)

However, The Jacobi polynomial \(P_{n}^{(0,k+\frac{m}{2}-1)}\) has degree \(n''\) and is orthogonal to all polynomials of lower degree when the inner product on \([-1,1]\) is computed relative to the weight function \((s+1)^{k+\frac{m}{2}}\). We conclude from (6.7) that the left hand side of (6.6) is zero when \(n < n''\) and hence that \(a_{n'',k,i} = 0\) for \(n'' > n\). Hence, we have
\[
xC_{2n,m}^{0}(Y_{k}^{i})(x) = \sum_{n'=0}^{n} C_{2n'+1,m}^{0}(Y_{k}^{i})(x) a_{n',k,i}.
\] (6.8)

Multiplying both sides of (6.8) on the left by \(C_{2M+1,m}^{0}(Y_{k}^{i})(x)\) \((0 \leq M \leq n)\) and integrating over \(B(1)\) gives, with an application of the first part of Lemma 6.2,
\[
-\int_{B(1)} \partial_{x}^{2n}|x|^{2n}C_{2M+1,m}^{0}(Y_{k}^{i})(x) (1-|x|^{2})^{2n}Y_{k}^{i}(x) dx = \|C_{2M+1,m}^{0}(Y_{k}^{i})\|_{L^{2}(B(1))}^{2} a_{M,k,i}.
\] (6.9)

If \(F_{n}, G_{n}\) are polynomials of degree \(n\) and \(Y_{k}\) is a spherical monogenic of degree \(k\), a simple calculation shows that
\[
\partial_{x}(F_{n}(|x|^{2})Y_{k}(x)) = xH_{n-1}(|x|^{2})Y_{k}(x); \quad \partial_{x}(xG_{n}(|x|^{2})Y_{k}(x)) = I_{n}(|x|^{2})Y_{k}(x)
\]
where \(H_{n-1}\) and \(I_{n}\) are polynomials of degree \(n - 1\) and \(n\) respectively. We conclude that the left hand side of (6.9) is zero when \(2n > 2M + 2\) so that \(a_{M,k,i} = 0\) for \(M < n - 1\). We conclude from (6.8) that
\[
xC_{2n,m}^{0}(Y_{k}^{i})(x) = \sum_{n'=n-1}^{n} C_{2n'+1,m}^{0}(Y_{k}^{i})(x) a_{n',k,i}
\]
as required for the case of even Clifford-Legendre polynomials. A similar argument, using second part of Lemma 6.1, deals with the odd case. \(\square \)

**Theorem 6.4.** The Bonnet type Formula for the Clifford-Legendre polynomials is as follows:
We now equate coefficients of the powers of $|x|^2$, equation (6.10) may be written as
\[
x C_{2N+1,m}^0(Y_k^i)(x) = \alpha_{N,k} C_{2N+2,m}^0(Y_k^i)(x) + \beta_{N,k} C_{2N,m}^0(Y_k^i)(x),
\]
where
\[
\alpha_{N,k} = \frac{-1}{4\left(\frac{m}{2} + 2N + k + 1\right)}; \quad \beta_{N,k} = \frac{2(2N + 1)(\frac{m}{2} + N + k)}{(\frac{m}{2} + 2N + k + 1)}.
\]
(b) when $n$ is even,
\[
x C_{2N,m}^0(Y_k^i)(x) = \alpha'_{N,k} C_{2N+1,m}^0(Y_k^i)(x) + \beta'_{N,k} C_{2N-1,m}^0(Y_k^i)(x),
\]
where
\[
\alpha'_{N,k} = \frac{-\left(\frac{m}{2} + N + k\right)}{2(2N + 1)(\frac{m}{2} + 2N + k)}, \quad \beta'_{N,k} = \frac{4N^2}{(\frac{m}{2} + 2N + k)}.
\]

\textbf{Proof.} By Lemma 6.3, when $n = 2N + 1$ is odd, there are Clifford constants $\alpha_{N,k,i}$ and $\beta_{N,k,i}$ for which
\[
x C_{2N+1,m}^0(Y_k^i)(x) = C_{2N+2,m}^0(Y_k^i)(x)\alpha_{N,k,i} + C_{2N,m}^0(Y_k^i)(x)\beta_{N,k,i}. \quad (6.10)
\]
Now, by the explicit representation of Clifford-Legendre polynomials given in Theorem 3.12 equation (6.10) may be written as
\[
-\frac{2^{2N+1}(2N + 1)!}{N!} \sum_{j=0}^{N} \binom{N}{j} \frac{\Gamma(j + k + \frac{m}{2} + N + 1)}{\Gamma(j + k + \frac{m}{2} + 1)} (-1)^j |x|^{2j+2} \]
\[
= \left[\frac{2^{2N+2}(2N + 2)!}{(N + 1)!} \sum_{j=0}^{N+1} \binom{N+1}{j} \frac{\Gamma(j + k + \frac{m}{2} + N + 1)}{\Gamma(j + k + \frac{m}{2})} (-1)^j |x|^{2j}\right] \alpha_{N,k,i}
\]
\[
+ \left[\frac{2^{2N}(2N)!}{N!} \sum_{j=0}^{N} \binom{N}{j} \frac{\Gamma(j + k + \frac{m}{2} + N)}{\Gamma(j + k + \frac{m}{2})} (-1)^j |x|^{2j}\right] \beta_{N,k,i}.
\]
We now equate coefficients of the powers of $|x|^2$ on both sides of this equation. By equating coefficients of $|x|^{2N+2}$, we find
\[
-\frac{2^{2N+2}(2N + 2)!}{(N + 1)!} \frac{\Gamma(k + \frac{m}{2} + 2N + 2)}{\Gamma(N + k + \frac{m}{2} + 1)} (-1)^{N+1} \alpha_{N,k,i}
\]
\[
= \frac{2^{2N+1}(2N + 1)!}{N!} \frac{\Gamma(k + \frac{m}{2} + 2N + 1)}{\Gamma(N + k + \frac{m}{2} + 1)} (-1)^N \alpha_{N,k}
\]
from which we conclude that $\alpha_{N,k,i} = \alpha_{N,k} = \frac{-1}{4(\frac{m}{2} + 2N + k + 1)}$. Similarly, by equating the constant terms, we find that
\[
\left[\frac{2^{2N+2}(2N + 2)!}{(N + 1)!} \frac{\Gamma(k + \frac{m}{2} + N + 1)}{\Gamma(k + \frac{m}{2})}\right] \alpha_{N,k} + \left[\frac{2^{2N}(2N)!}{N!} \frac{\Gamma(k + \frac{m}{2} + N)}{\Gamma(k + \frac{m}{2})}\right] \beta_{N,k,i} = 0.
\]
By replacing $\alpha_{N,k}$ by its known value gives $\beta_{N,k,i} = \beta_{N,k} = \frac{2(2N+1)(\frac{m}{2}+N+k)}{(\frac{m}{2}+2N+k+1)}$. This completes the proof of part (a) of the Theorem.

Part (b) is proved in a similar manner. \qed

When applied to the construction of Clifford-prolate functions in higher dimensions, the Bonnet type formula is required for normalized Clifford-Legendre polynomials. In that context, the Bonnet type formula takes the form below.
Corollary 6.5. The Bonnet type formula for the normalized Clifford-Legendre polynomials is as follows:

(a) When $n = 2N$ is even,

$$x \overline{C}_{2N,m}^0(Y_k^j)(x) = A_{N,k,m} \overline{C}_{2N+1,m}^0(Y_k^j)(x) + B_{N,k,m} \overline{C}_{2N-1,m}^0(Y_k^j)(x)$$

where

$$A_{N,k,m} = -\left(\frac{m}{2} + N + k\right)\sqrt{m + 4N + 2k + 2}$$

$$B_{N,k,m} = \frac{N\sqrt{m + 4N + 2} - (m + 2 + k)\sqrt{m + 4N + 2k + 2}}{m + 2 + k + 2}$$

(b) When $n = 2N + 1$ is odd

$$x \overline{C}_{2N+1,m}^0(Y_k^i)(x) = A_{N,k,m}' \overline{C}_{2N+2,m}^0(Y_k^i)(x) + B_{N,k,m}' \overline{C}_{2N,m}^0(Y_k)(x)$$

where

$$A_{N,k,m}' = -\frac{(N + 1)\sqrt{m + 4N + 2k}}{m + 2N + k + 2}$$

$$B_{N,k,m}' = \frac{(m + 2N + k + 1)\sqrt{m + 4N + 2k + 2}}{m + 2N + k + 1}$$

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