A NOTE ON WHITEHEAD’S QUADRATIC FUNCTOR

B. MIRZAI, F. Y. MOKARI, AND D. C. ORDINOLA

Abstract. For an abelian group $A$, we give a precise homological description of the kernel of the natural map $\Gamma(A) \to A \otimes \mathbb{Z} A$, $\gamma(a) \mapsto a \otimes a$, where $\Gamma$ is whitehead’s quadratic functor from the category of abelian groups to itself.

Introduction

Whitehead’s quadratic functor is an important functor, which first appeared in the context of algebraic topology. This is a functor from the category of abelian groups to itself and usually is denoted by $\Gamma$. Most of important aspects of this functor is known and its has been generalized in various ways.

For an abelian group $A$, we give a precise homological description of the kernel of the natural map

$$\Gamma(A) \to A \otimes \mathbb{Z} A, \quad \gamma(a) \mapsto a \otimes a$$

which it is known to be 2-torsion. The cokernel of this map is isomorphism to $H_2(A, \mathbb{Z})$, the second integral homology group of $A$.

In this short article we give a precise homological description of the kernel of the above map. As our main result we prove that we have the exact sequence

$$0 \to H_1(\Sigma_2, \text{Tor}^\mathbb{Z}_1(2^\infty A, 2^\infty A)) \to \Gamma(A) \to A \otimes \mathbb{Z} A \to H_2(A, \mathbb{Z}) \to 0,$$

where $2^\infty A$ is the 2-power torsion subgroup of $A$, $\Sigma_2 := \{\text{id}, \sigma^z\}$ the symmetric group with two elements and $\sigma^z$ being the involution on $\text{Tor}^\mathbb{Z}_1(2^\infty A, 2^\infty A)$ induced by the involution $A \times A \to A \times A$, $(a, b) \mapsto (b, a)$.

If $A \to B$ is a homomorphism of abelian groups, by $B/A$ we mean coker$(A \to B)$. For a group $A$, $nA$ is the subgroup of $n$-torsion elements of $A$. For prime $p$, $p^\infty A$ is the $p$-power torsion subgroup of $A$.

1. Whitehead’s quadratic functor

A function $\psi : A \to B$ of (additive) abelian groups is called a quadratic map if

1. for any $a \in A$, $\psi(a) = \psi(-a)$,
(2) the function \( A \times A \to B \) with \( (a, b) \mapsto \psi(a + b) - \psi(a) - \psi(b) \) is bilinear.

For any abelian group \( A \), there is a universal quadratic map

\[ \gamma : A \to \Gamma(A) \]

such that for any quadratic map \( \psi : A \to B \), there is a unique group homomorphism \( \Psi : \Gamma(A) \to B \) such that \( \Psi \circ \gamma = \psi \). It is easy to see that \( \Gamma \) is a functor from the category of abelian groups to itself.

The functions \( \phi : A \to A/2 \) and \( \psi : A \to A \otimes \mathbb{Z} A \), given by \( \phi(a) = \bar{a} \) and \( \psi(a) = a \otimes a \) respectively, are quadratic maps. Thus we get the canonical homomorphisms

\[ \Phi : \Gamma(A) \to A/2, \gamma(a) \mapsto a \quad \text{and} \quad \Psi : \Gamma(A) \to A \otimes \mathbb{Z} A, \gamma(a) \mapsto a \otimes a. \]

Clearly \( \Phi \) is surjective and \( \text{coker}(\Psi) = A \wedge A \cong H_2(A, \mathbb{Z}) \). Furthermore we have the bilinear pairing

\[ \left\langle , \right\rangle : A \otimes \mathbb{Z} A \to \Gamma(A), \ [a, b] := \gamma(a + b) - \gamma(a) - \gamma(b). \]

It is easy to see that for any \( a, b, c \in A \), \( [a, b] = [b, a] \), \( \Phi[a, b] = 0 \), \( \Psi[a, b] = a \otimes b + b \otimes a \) and \( [a + b, c] = [a, c] + [b, c] \). Using (1) and this last equation, for any \( a, b, c \in A \), we obtain

(a) \( \gamma(a) = \gamma(-a) \),

(b) \( \gamma(a + b + c) - \gamma(a + b) - \gamma(a + c) - \gamma(b + c) + \gamma(a) + \gamma(b) + \gamma(c) = 0. \)

Using these properties we can construct \( \Gamma(A) \). Let \( A \) be the free abelian group generated by the symbols \( w(a), a \in A \). Set \( \Gamma(A) := A/\mathcal{R} \), where \( \mathcal{R} \) denotes the relations (a) and (b) with \( w \) replaced by \( \gamma \).

Now \( \gamma : A \to \Gamma(A) \) is given by \( a \mapsto \bar{w}(a) \).

Using this properties one can show that for any nonnegative integer \( n \), we have

\[ \gamma(na) = n^2 \gamma(a). \]

It is known that the sequence

\[ A \otimes \mathbb{Z} A \xrightarrow{[\ , \ ]} \Gamma(A) \xrightarrow{\Phi} A/2 \to 0 \]

is exact and the kernel of \([\ , \ ]\) is generated by the elements of the form \( a \otimes b - b \otimes a, a, b \in A \). Therefore we have the exact sequence

(1.1) \[ 0 \to H_0(\Omega_2, A \otimes \mathbb{Z} A) \xrightarrow{\gamma} \Gamma(A) \xrightarrow{\Phi} A/2 \to 0, \]

where \( \Omega_2 := \{ \text{id}, \omega \} \) and \( \omega \) is the involution \( \omega(a \otimes b) = b \otimes a \) on \( A \otimes \mathbb{Z} A \).

It is easy to see that the composition

\[ A \otimes \mathbb{Z} A \xrightarrow{[\ , \ ]} \Gamma(A) \xrightarrow{\Psi} A \otimes \mathbb{Z} A \]
Whitehead’s quadratic functor takes $a \otimes b$ to $a \otimes b + b \otimes a$. Moreover the composition
\[
\Gamma(A) \xrightarrow{\Psi} A \otimes \mathbb{Z} A \xrightarrow{\delta} \Gamma(A)
\]
coincide with multiplication by 2. Thus $\ker(\Psi)$ is 2-torsion.

To give a homological description of the kernel of $\psi$, we will need the following fact.

**Proposition 1.1.** For any abelian group $A$, $\Gamma(A) \simeq H_4(K(A, 2), \mathbb{Z})$, where $K(A, 2)$ is the Eilenberg-Maclane space of type $(A, 2)$.

*Proof.* See [3, Theorem 21.1] \[\square\]

2. **Tor-functor and third homology of abelian groups**

Let $A$ and $B$ be abelian groups. For any positive integer $n$ there is a natural homomorphism
\[
\tau_n : nA \otimes \mathbb{Z} nB \to n\text{Tor}_1^\mathbb{Z}(A, B).
\]
We denote the image of $a \otimes b$, under $\tau_n$ by $\tau_n(a, b)$.

For any pair of integers $s$ and $n$ such that $n = sm$, the maps $\tau_n$ are related by the commutative diagrams
\[
\begin{array}{c}
\begin{array}{ccc}
& & nA \otimes \mathbb{Z} nB \\
& p_m \otimes \text{id} & \downarrow \tau_n \\
\tau_s & \downarrow & n\text{Tor}_1^\mathbb{Z}(A, B) \\
& \text{id} \otimes i_m & \\
sA \otimes \mathbb{Z} sB & \tau_s & nA \otimes \mathbb{Z} nB,
\end{array}
\end{array}
\]
\[
\begin{array}{c}
\begin{array}{ccc}
& & nA \otimes \mathbb{Z} nB \\
& \text{id} \otimes p_m & \text{id} \otimes \text{id} \\
\tau_s & \downarrow & n\text{Tor}_1^\mathbb{Z}(A, B) \\
& \text{id} \otimes i_m & \\
sA \otimes \mathbb{Z} sB & \tau_s & nA \otimes \mathbb{Z} nB,
\end{array}
\end{array}
\]
in which $i_m : sA \to nA$ and $p_m : nA \to sA$ are the inclusion and the map induced by multiplication by $m$ respectively. The commutativity of these diagrams expresses the relations
\[
\tau_n(a, b) = \tau_s(ma, b), \quad \text{for} \quad a \in nA \text{ and } b \in sB,
\]
and
\[
\tau_n(a', b') = \tau_s(a', mb'), \quad \text{for} \quad a' \in sA \text{ and } b' \in nB.
\]
The following proposition is well-known [1, Proposition 3.5].
Proposition 2.1. The induced map \( \tau : \lim_I nA \otimes nB \to \text{Tor}^Z_1(A, B) \), where \( I \) is the inductive system of objects \( nA \otimes \mathbb{Z} nB \) determined by the above diagrams for varying \( n \), is an isomorphism.

Let \( \sigma_0 : A \otimes B \to B \otimes A \) and \( \sigma_1 : \text{Tor}^Z_1(A, B) \to \text{Tor}^Z_1(B, A) \) be induced by interchanging the groups \( A \) and \( B \). It is well known that the diagram

\[
\begin{array}{ccc}
A \otimes \mathbb{Z} nB & \xrightarrow{\sigma_0} & nB \otimes \mathbb{Z} nA \\
\downarrow{\tau_n} & & \downarrow{\nu'_n} \\
n\text{Tor}^Z_1(A, B) & \xrightarrow{-\sigma_1} & n\text{Tor}^Z_1(B, A)
\end{array}
\]

commutes. By passing to the inductive limit, the same is true for the diagram

\[
\begin{array}{ccc}
\lim_I (nA \otimes \mathbb{Z} nB) & \xrightarrow{\sigma_0} & \lim_I (nB \otimes \mathbb{Z} nA) \\
\downarrow{\tau} & & \downarrow{\nu'} \\
\text{Tor}^Z_1(A, B) & \xrightarrow{-\sigma_1} & \text{Tor}^Z_1(B, A).
\end{array}
\]

It is useful to observe that the map \( \sigma_1 : \text{Tor}^Z_1(A, B) \to \text{Tor}^Z_1(B, A) \) is indeed induced by the involution \( A \otimes \mathbb{Z} B \to B \otimes \mathbb{Z} A \) given by \( a \otimes b \mapsto -b \otimes a \) and therefore \( -\sigma_1 \) is induced by the involution \( a \otimes b \mapsto b \otimes a \).

Let \( \Sigma_2 \) be the symmetric group of order 2. For an abelian group \( A \), \( \Sigma_2 \) acts on \( A \otimes \mathbb{Z} A \) and \( \text{Tor}^Z_1(A, A) \), through \( \sigma_0 \) and \( \sigma_1 \). Let us denote the symmetric group by \( \Sigma_2^c \), rather than simply by \( \Sigma_2 \), when it acts on \( \text{Tor}^Z_1(A, A) \) as

\[
(\sigma^c, x) \mapsto -\sigma_1(x).
\]

We need the following well-known lemma on the third homology of abelian groups [5, Lemma 5.5], [1, Section 6].

Proposition 2.2. For any abelian group \( A \) we have the exact sequence

\[
0 \to \bigwedge^3_A \mathbb{Z} A \to H_3(A, \mathbb{Z}) \to \text{Tor}^Z_1(A, A)^{\Sigma_2^c} \to 0,
\]

where the right side homomorphism is obtained from the composition

\[
H_3(A, \mathbb{Z}) \xrightarrow{\Delta_A} H_3(A \times A, \mathbb{Z}) \to \text{Tor}^Z_1(A, A),
\]

\( \Delta_A \) being the diagonal map \( A \to A \times A, a \mapsto (a, a) \).
3. The kernel of $Ψ : Γ(A) → A ⊗ A$

We study the kernel of $Ψ : Γ(A) → A ⊗ Z A$. If $Θ = [ , ] : A ⊗ Z A → Γ(A)$, then from the commutative diagram

$$
\begin{array}{cccccc}
0 & → & \ker(Θ) & → & A ⊗ Z A & → & \im(Θ) & → & 0 \\
& & ↓ & & ↓Θ & & ↓γ & & \\
0 & → & \ker(Ψ) & → & Γ(A) & → & A ⊗ Z A & → & 0
\end{array}
$$

and exact sequence (1.1) we obtain the exact sequence

$$\ker(Ψ) → A/2 → (A ⊗ Z A)_{Ω_2} → H_2(A, Z) → 0,$$

where $(A ⊗ Z A)_{Ω_2} = (A ⊗ Z A)/(a ⊗ b + b ⊗ a | a, b ∈ A)$ and $δ(\bar{a}) = \bar{a} ⊗ \bar{a}$. But the sequence

$$0 → A/2 → (A ⊗ Z A)_{Ω_2} → H_2(A, Z) → 0$$

is always exact. Thus the map $\ker(Ψ) → A/2$ is trivial, which shows that

$$\ker(Γ(A) → A ⊗ Z A) ⊆ \im(A ⊗ Z A → Γ(A)).$$

We give a precise description of the kernel of $Ψ$.

**Theorem 3.1.** For any abelian group $A$, we have the exact sequence

$$0 → H_1(Σ^e_2, \Tor^Z_1(2∞ A, 2∞ A)) → Γ(A) → A ⊗ Z A → H_2(A, Z) → 0.$$

**Proof.** If $A ⊢ B ⊔ C$ is an extension of abelian groups, then standard classifying space theory gives a (homotopy theoretic) fibration of Eilenberg-MacLane spaces $K(A, 1) → K(B, 1) → K(C, 1)$. From this we obtain the fibration [4, Lemma 3.4.2]

$$K(B, 1) → K(C, 1) → K(A, 2).$$

For the group $A$, the morphism of extensions

$$\begin{array}{ccc}
A & ← & i_1 \rightarrow A × A \xrightarrow{p_2} A \\
& ↓ & ↓\mu & ↓ \\
A & ← & = \rightarrow A \rightarrow \{1\},
\end{array}$$

where $i_1(a) = (a, 1)$, $p_2(a, b) = b$ and $μ(a, b) = ab$, induces the morphism of fibrations

$$\begin{array}{ccc}
K(A × A, 1) & → & K(A, 1) \xrightarrow{\Psi} K(A, 2) \\
& ↓ & ↓ & ↓ \\
K(A, 1) & → & K(\{1\}, 1) \rightarrow K(A, 2).
\end{array}$$
By analysing the Serre spectral sequences associated to this morphism of fibrations, we obtain the exact sequence

$$0 \to \ker(\Psi) \to H_4(K(A, 2)) \xrightarrow{\Psi} A \otimes_{\mathbb{Z}} A \to H_2(A) \to 0,$$

where

$$\ker(\Psi) \cong H_3(A, \mathbb{Z})/\mu_*(A \otimes_{\mathbb{Z}} H_2(A, \mathbb{Z}) \oplus \text{Tor}_1^Z(A, A)).$$

By Proposition 2.2 we have the exact sequence

$$0 \to \Lambda^3_{\mathbb{Z}} A \to H_3(A, \mathbb{Z}) \to \text{Tor}_1^Z(A, A) \to 0.$$

Clearly $\mu_*(A \otimes_{\mathbb{Z}} H_2(A, \mathbb{Z})) \subseteq \Lambda^3_{\mathbb{Z}} A$. Therefore

$$\ker(\Psi) \cong \text{Tor}_1^Z(A, A)/\Delta_{\mathbb{Z}} \circ \mu_*(\text{Tor}_1^Z(A, A)).$$

We prove that the map $\Delta \circ \mu : A \times A \to A \times A$, which is given by $(a, b) \mapsto (ab, ab)$, induces the map

$$\text{id} + \sigma^\epsilon : \text{Tor}_1^Z(A, A) \to \text{Tor}_1^Z(A, A).$$

By studying the map $(\Delta \circ \mu)_* : H_2(A \times A) \to H_2(A \times A)$ using the fact that $A \otimes A \simeq H_2(A \times A)/(H_2(A) \oplus H_2(A))$ (the Künneth Formula), one sees that $\Delta \circ \mu$ induces the map

$$A \otimes A \to A \otimes A, \quad a \otimes b \mapsto a \otimes b - b \otimes a,$$

Thus to study the induced map on $\text{Tor}_1^Z(A, A)$ by $\Delta \circ \mu$ we should study the map induced on $\text{Tor}_1^Z(A, A)$ by the map

$$A \otimes A \to A \otimes A, \quad a \otimes b \mapsto a \otimes b + b \otimes a = (\text{id} + \iota)(a \otimes b),$$

where $\iota : A \otimes A \to A \otimes A$ is given by $a \otimes b \mapsto b \otimes a$. Let

$$0 \to F_1 \xrightarrow{\partial} F_0 \xrightarrow{\epsilon} A \to 0$$

be a free resolution of $A$. Then the sequence

$$0 \to F_1 \otimes F_1 \xrightarrow{\partial_2} F_0 \otimes F_1 \oplus F_1 \otimes F_0 \xrightarrow{\partial_1} F_0 \otimes F_0 \to 0$$

can be used to calculate $\text{Tor}_1^Z(A, A)$, where $\partial_2 = (\partial \otimes \text{id}_{F_1}, -\text{id}_{F_1} \otimes \partial)$, $\partial_1 = \text{id}_{F_0} \otimes \partial + \partial \otimes \text{id}_{F_0}$. The map $\text{id} + \iota : A \otimes A \to A \otimes A$ can be extended to the morphism of complexes

$$0 \to F_1 \otimes F_1 \xrightarrow{\partial_2} F_0 \otimes F_1 \oplus F_1 \otimes F_0 \xrightarrow{\partial_1} F_0 \otimes F_0 \to 0$$

$$0 \to F_1 \otimes F_1 \xrightarrow{\partial_2} F_0 \otimes F_1 \oplus F_1 \otimes F_0 \xrightarrow{\partial_1} F_0 \otimes F_0 \to 0,$$
where
\[
\begin{align*}
f_0(x \otimes y) &:= x \otimes y + y \otimes x, \\
f_1(x \otimes y, y' \otimes x') &:= (x \otimes y + x' \otimes y', y \otimes x + y' \otimes x'), \\
f_2(x \otimes y) &:= x \otimes y - y \otimes x.
\end{align*}
\]
Since
\[
f_1(x \otimes y, y' \otimes x') = (x \otimes y, y' \otimes x') + (x' \otimes y', y \otimes x),
\]
\(\Delta \circ \mu\) induces the map \(\text{id} + \sigma^e : \text{Tor}^Z_1(A, A) \to \text{Tor}^Z_1(A, A).\) Therefore
\[
\ker(\Psi) \simeq \text{Tor}^Z_1(A, A) / (\text{id} + \sigma^e)(\text{Tor}^Z_1(A, A)) = H_1(\Sigma_2, \text{Tor}^Z_1(A, A)).
\]
Finally since \(\text{Tor}^Z_1(A, A) = \text{Tor}^Z_1(A_T, A_T),\) \(A_T\) being the subgroup of torsion elements of \(A,\) and since for any torsion abelian group \(B,\)
\(B \simeq \bigoplus_{p \text{ prime}} p^{\infty} B,\) we have the isomorphism
\[
H_1(\Sigma_2, \text{Tor}^Z_1(A, A)) \simeq H_1(\Sigma_2, \text{Tor}^Z_1(2^{\infty} A, 2^{\infty} A)).
\]
This completes the proof of the theorem. \qed

**Corollary 3.2.** For any abelian group \(A,\) we have the exact sequence
\[
0 \to \lim_I H_1(\Sigma_2, 2^{\infty} A \otimes Z 2^{\infty} A) \to \Gamma(A) \xrightarrow{\Psi} A \otimes Z A \to H_2(A, Z) \to 0.
\]
In particular if \(2^{\infty} A\) is finite then we have the exact sequence
\[
0 \to H_1(\Sigma_2, 2^{\infty} A \otimes Z 2^{\infty} A) \to \Gamma(A) \xrightarrow{\Psi} A \otimes Z A \to H_2(A, Z) \to 0.
\]
**Proof.** This follows from Theorem 3.1 and Proposition 2.1. \qed

**References**

[1] Breen, L. On the functorial homology of abelian groups. Journal of Pure and Applied Algebra 142 (1999) 199–237.

[2] Brown, K. S. Cohomology of Groups. Graduate Texts in Mathematics, 87. Springer-Verlag, New York, 1994.

[3] Eilenberg, S., MacLane, S. On the groups \(H(\Pi, n),\) II: Methods of computation. Ann. of Math. 70 (1954), no. 1, 49–139.

[4] May, J. P., Ponto, K. More concise algebraic topology: Localization, completion, and model categories. Chicago Lectures in Mathematics. University of Chicago Press, Chicago, IL, 2012.

[5] Suslin, A. A. \(K_3\) of a field and the Bloch group. Proc. Steklov Inst. Math. 183 (1991), no. 4, 217–239.