Abstract
We develop a fully stochastic theory for age-structured populations via quantum field theoretical methods. The operator formalism of Doi is first developed, whereby birth and death events are represented by creation and annihilation operators, and the complete probabilistic representation of the age-chart of a population is represented by states in a suitable Hilbert space. We then use this formalism to rederive several results in companion paper [1], including an equation describing the moments of the age-distribution, and the distribution of the population size. The functional representation of coherent states used by Peliti to analyze discrete Fock space is then adapted to incorporate continuous age parameters, and a path integral formulation constructed. We apply these formalisms to a range of birth-death processes and show that although many of the results from Doi-Peliti formalism can be derived in a purely probabilistic way, the efficient formalism offered by second quantization methods provides a powerful technique that can manage algebraically complex birth death processes in a compact manner.

Keywords
Doi-Peliti Quantization · Age-Structure · Birth-Death Process

1 Introduction

Birth-death processes are continuous time stochastic processes used to model populations with variable size, such as queueing systems, epidemics, predator-prey dynamics, and fluctuating populations, for example. The archetype model is Markovian, with dynamics that only depend upon the current population size. However, many models of interest do not satisfy such constraints and alternative methods are required. One such class of problems involve age-structured populations. The classical approach to analyzing age-structure in a population is the McKendrick-von Foerster equation [5, 12, 15], along with subsequent generalisations [9–11, 20]. However, although these classical deterministic approaches successfully model the variation in age-structure through time, the fluctuations in population size are not captured, and more general methods are required, as we have demonstrated for simple processes in [8], and for more complex birth-death processes in the companion paper to this article [1]. A full review of various approaches to age-structured modelling can also be found in [1].

Quantum field theory is typically used to address sub-atomic particle interactions, and this theoretical framework is well equipped to deal with analytical difficulties that arise when changes in particle number result in changes in dimensionality of phase space. Doi [2, 3] noticed the similarity with population dynamics; individuals can be born or die, just as particles can divide or combine. He tailored field theory methods accordingly and applied them to chemical reaction kinetics using perturbative techniques. Peliti [17] took discrete probability distributions, such as the number of individuals in a population, and adapted quantum techniques applicable to Fock spaces to construct a path integral formulation for a range of Markovian birth-death processes. In particular, a coherent state resolution of the identity was utilized to perform the time slicing required in a path integral construction, and a functional state representation utilized. The Doi methods are well suited to continuous variables (such as molecular positions and
momenta), whereas the Peliti methods are well suited to discrete systems. The models we consider have elements of both; the age dependence of the population is continuous, and the number of individuals is discrete. We will adapt both the Doi and Peliti techniques to the models we consider.

This paper is organised as follows. In the next section we outline how the Doi formalism can be applied to the simple age-structured birth death process developed in [8]. In Section 3 we show how this structure can be used to derive PDEs describing the time evolution of the population-size-age-structure probability distribution. We also provide derivations of age-structured moments of the population. In Section 4 we use perturbative analysis to investigate the generating function for the population-size distribution, showing that Feynman perturbation diagrams correspond to physical realisations of the branching processes. In Section 5 we develop a suitable formulation of coherent states that enables a path integral representation of the population-size generating function. In Sections 6-8 we highlight the use of these methods with more complex birth-death processes, including degeneracies arising from binary fission, spatial dependencies occurring in Brownian trees, and the multi-species nature of sexual reproduction. Conclusions complete the paper. This is a companion paper to [1], where probabilistic derivations of our equations, and more detailed methods of solution, can be found.

2 Operator Formalism

Here, we introduce a Doi-Peliti operator formalism tailored to age-dependent birth-death processes. To avoid confusion, we largely adapt the notation in [2, 3], where possible. We first need to describe the underlying microscopic model. We assume that we have a population of individuals, such that individuals of age \( q \) die at age-dependent death rate \( \mu(q) \). We also assume that individuals produce offspring at age-dependent birth rate \( \beta(q) \). This is a budding or simple mode of birth [8] (such as found in yeast, for example), where the parent does not die during birth. We later consider a fission mode of birth, where the parent “dies” immediately after giving birth to two identical offspring. Cell division is a possible realisation of fission, which is treated in more detail in [1] and in section 6 below. We will also make use of the event rate, defined as \( \gamma(q) = \beta(q) + \mu(q) \).

We first define state vectors \( |q\rangle \equiv |q_1, q_2, \ldots, q_n\rangle \) to represent a set of \( n \) individuals with ages \( q_1, q_2, \ldots, q_n \). Unlike our previous analyses in [1] and [8], we use \( q \) to denote age in order to make the connection with field theory methods more explicit. In this representation individuals are indistinguishable and the order of the components \( q_i \) is immaterial. We use \( |\phi\rangle \) to represent the ‘vacuum’ state of an empty population. The notation \( |\phi\rangle \) is used rather than the usual \( |0\rangle \) to distinguish the vacuum state from a population of one newborn individual of age zero.

Next, we introduce annihilation and creation operators \( \psi_q \) and \( \psi_q^\dagger \) which satisfy standard commutation relations:

\[
[\psi_q, \psi_p^\dagger] = \delta(q - p), \quad [\psi_q, \psi_p] = [\psi_q^\dagger, \psi_p^\dagger] = 0. \tag{1}
\]

States can be constructed using creation operators: \( |q_1, q_2, \ldots, q_n\rangle = \psi_{q_1}^\dagger \psi_{q_2}^\dagger \ldots \psi_{q_n}^\dagger |\phi\rangle \). From these states and commutation relations, we obtain the normalization

\[
\langle q_m | p_n \rangle = \delta_{mn} \sum_{\pi \in S_n} \prod_{i=1}^n \delta(q_i - p_{\pi(i)}), \tag{2}
\]

where \( (\phi|\phi) \equiv 1 \) and \( S_n \) is the symmetry group of permutations on \( n \) symbols. The annihilation operators are assumed to kill the vacuum state: \( \psi_q |\phi\rangle = 0 \). Note that although we will use this formalism to model (positive) ages, we place no restrictions on the states \( |q_1, q_2, \ldots, q_n\rangle \) which can contain negative entries. It is relatively straightforward to use Eq. 2 to verify the following resolution of the identity operator:

\[
I = \sum_{m=0}^{\infty} \int \frac{dm}{m!} \langle q_m | q_m \rangle. \tag{3}
\]

Next, we define the function \( \rho(t; q_m; t) \) to represent the probability density for a population of \( n \) individuals such that if we randomly label them \( 1, 2, \ldots, n \), they have ages \( q_1, q_2, \ldots, q_n \). More details of this representation can be found in [1, 8]. Then \( \rho(t; q_m; t) \equiv f_{\mu}(q_m; t)/n! \) where \( f_{\mu} \) is the distribution used in Doi [2, 3]. The function \( \rho(t; q_m; t) \) has the advantage of being interpretable as a probability distribution over \( \mathbb{R}^n \). On the other hand, although \( f_{\mu} \) is not a probability distribution over this domain, it can be interpreted as a probability distribution for ordered ages over the analytically more difficult region.
defined by \( \{ q_n : q_1 \leq \ldots \leq q_n \} \). We adopt the \( f_n \) representation, which tends to result in algebraically simpler expressions, although \( p_n \) will be referred to in comparison to other work \([1, 8]\). We thus represent the distribution \( f_n \) with the following superposition of states:

\[
|f(t)\rangle = \sum_{n=0}^{\infty} \int_{\mathbb{R}^n} \frac{dq_n}{n!} f_n(q_n; t) |q_n\rangle.
\]  

(4)

The evolution of the state can be described as follows:

\[
\frac{\partial}{\partial t} |f(t)\rangle = \zeta |f(t)\rangle,
\]

(5)

where \( \zeta = \zeta_0 + \zeta_b + \zeta_d \) is an operator that can be decomposed into three parts. The term \( \zeta_0 \) describes the increase of all age variables in time, the term \( \zeta_b \) represents the increase in population size due to birth and \( \zeta_d \) represents the decrease in population size due to death. Then we have, following the Doi formalism \([2, 3]\), the following expressions:

\[
\zeta_0 = \int dq \, \psi_q^\dagger \frac{\partial}{\partial q} \psi_q, \quad \zeta_b = \int dq \, \beta(q)(\psi_q^\dagger \psi_q - \psi_q^\dagger \psi_q^\dagger \psi_q), \quad \zeta_d = \int dq \, \mu(q)(\psi_q^\dagger \psi_q - \psi_q^\dagger \psi_q).
\]

(6)

Thus for example, the second term in \( \zeta_b \) contains creation operator \( \psi_q^\dagger \), representing the birth of a new individual of age zero, and the annihilation and creation operators \( \psi_q \) and \( \psi_q^\dagger \) are a bookkeeping measure that preserves the parental individual of age \( q \). These states and operators are given in the Schrödinger representation where the operators are constant and the states vary in time, with Eq. 5 having a formal solution of the form

\[
|f(t)\rangle = e^{-\zeta t} |f(0)\rangle.
\]

(7)

We complete this section by introducing coherent states. If we take any complex function \( u(q) \) of the real value \( q \), with conjugate \( \bar{u}(q) \), we can construct the the following state superposition:

\[
|u\rangle = e^{\int dq \, u(q) \psi_q^\dagger} |\phi\rangle = \sum_{n=0}^{\infty} \frac{1}{n!} \int dq_n \, u(q_1) \ldots u(q_n) \psi_{q_1}^\dagger \ldots \psi_{q_n}^\dagger |\phi\rangle.
\]

(8)

These coherent states satisfy the following eigenstate property:

\[
\psi_q^\dagger |u\rangle = \sum_{n=0}^{\infty} \frac{1}{n!} \int dq_n \, u(q_1) \ldots u(q_n) \psi_q^\dagger \psi_{q_1}^\dagger \ldots \psi_{q_n}^\dagger |\phi\rangle = \sum_{n=0}^{\infty} \frac{1}{n!} \int dq_n \, u(q_1) \ldots u(q_n) \delta(p - q) \prod_{i=1}^{n} \psi_{q_i}^\dagger |\phi\rangle = u(p) |u\rangle.
\]

(9)

Although the creation operators do not have eigenstates, we find that they do have the functional derivative representation

\[
\psi_q^\dagger |u\rangle = \psi_q^\dagger e^{\int dq \, u(q) \psi_q^\dagger} |\phi\rangle = \frac{\partial}{\partial u(p)} e^{\int dq \, u(q) \psi_q^\dagger} |\phi\rangle = \frac{\partial}{\partial u(p)} |u\rangle.
\]

(11)

Coherent states also satisfy the following normalisation property,

\[
\langle u | u \rangle = \langle \phi | e^{\int dq \, \bar{u}(q) e^{\int dq \, u(q) \psi_q^\dagger}} | \phi \rangle = \langle \phi | e^{\int dq \, \bar{u}(q) e^{\int dq \, u(q) \psi_q^\dagger}} | \phi \rangle = e^{\int dq \, \bar{u}(q) u(q)}
\]

(12)

where we have used the Baker-Campbell-Hausdorff theorem to commute operators \([13, 18]\). We use this frequently throughout and quote the formulation we use for convenience. For operators \( X \) and \( Y \) we have \( e^{X} \cdot Y = e^{ad_X(Y)} \cdot e^{X} \), where we have the adjoint action \( ad_X(Y) = [X,Y] \).

The coherent states defined here generalize states of the form \( |z\rangle \) used by Doi \([2, 3]\), which are defined by setting the function \( u(q) \equiv z \) to be constant. Doi noticed the form \( |z\rangle \) allows many summary statistics of interest to be simply expressed. The more general coherent state \( |u\rangle \equiv |u(q)\rangle \) we have introduced will later be used in the path integral formulation of our problem.
3 Feature Extraction

We now use the formalism outlined above to derive equations for the distribution \( f_n(q_n; t) \) and obtain the associated moments.

3.1 Complete Population-Size-Age-Structure Distribution

Here, we derive a hierarchy of equations that describe the distribution \( f_n(q_n; t) \). If we condition upon an initial distribution \( f_n(q_n; 0) \), \( f_n(q_n; t) \) is the projection of the state \( |f(t)\rangle \) onto the fundamental state \( |q_n\rangle \):

\[
f_n(q_n; t) = \langle q_n | f(t) \rangle = \langle q_n | e^{-\mathcal{L}t} | f(0) \rangle.
\]

Since both \( \langle q_n | \) and \( | f(0) \rangle \) are constant in time, we can differentiate Eq. 13 with respect to time to find

\[
\frac{\partial f_n(q_n; t)}{\partial t} = -\langle q_n | \zeta | f(t) \rangle = -\langle q_n | \zeta_0 | f(t) \rangle - \langle q_n | \zeta_b | f(t) \rangle - \langle q_n | \zeta_d | f(t) \rangle.
\]

Next we use the commutation relations to calculate the left action of the operators \( \zeta_0, \zeta_b \) and \( \zeta_d \) upon \( | q_n \rangle \). The first term gives

\[
\langle q_n | \zeta_0 | f(t) \rangle = \int dp \langle \phi | \psi_{q_1} \psi_{q_2} \ldots \psi_{q_n} \psi_{\beta} \frac{\partial}{\partial p} \psi_{\beta} | f(t) \rangle = \int dp \langle \phi | \sum_{i=1}^n \delta(p-q_i) \prod_{j \neq i} \psi_{q_j} \frac{\partial}{\partial p} \psi_{\beta} | f(t) \rangle
\]

\[
= \sum_{i=1}^n \frac{\partial}{\partial q_i} \langle \phi \prod_{j \neq i} \psi_{q_j} | f(t) \rangle = \sum_{i=1}^n \frac{\partial}{\partial q_i} \langle q_n | f(t) \rangle = \sum_{i=1}^n \frac{\partial}{\partial q_i} f_n(q_n; t).
\]

For the birth term, a similar derivation yields

\[
\langle q_n | \zeta_b | f(t) \rangle = \int dp \beta(p) \langle \phi | \psi_{q_1} \psi_{q_2} \ldots \psi_{q_n} (\psi_{\beta}^\dagger \psi_{p} - \psi_{p}^\dagger \psi_{\beta} | f(t) \rangle
\]

\[
= \sum_{i=1}^n \beta(q_i) \left[ f_n(q_n; t) - \sum_{j \neq i} \delta(q_j) f_{n-1}(q_n^{(-j)}; t) \right].
\]

where \( q_n^{(\neg j)} = [q_1, \ldots, q_{i-1}, q_{i+1}, \ldots, q_n] \) represents the age-chart with all \( n \) ages except the \( j \)th one, which is omitted. Finally, the death term yields

\[
\langle q_n | \zeta_d | f(t) \rangle = \sum_{i=1}^n \mu(q_i) f_n(q_n; t) - \int dp \mu(p) f_{n+1}(q_n, p; t).
\]

Upon combining these results for any strictly positive age chart \( q_n \) we lose the delta functions in Eq. 16 and obtain the following set of hierarchical equations

\[
\frac{\partial f_n(q_n; t)}{\partial t} + \sum_{i=1}^n \frac{\partial f_n(q_n; t)}{\partial q_i} = f_n(q_n; t) \sum_{i=1}^n \left[ \mu(q_i) + \beta(q_i) \right] + \int dp \mu(p) f_{n+1}(q_n, p; t).
\]

These equations are identical to those found in [8], under the equivalence \( \rho_n \equiv f_n/n! \). Furthermore, integrating Eq. 14 with respect to the variable \( q_j \) over a small interval containing the boundary \( q_j = 0 \) captures a delta function from Eq. 16 and recovers the boundary condition found in [8]:

\[
f_n(q_n^{(-j)}, 0; t) = f_{n-1}(q_n^{(-j)}; t) \sum_{i \neq j} \beta(q_i).
\]

In Eqs. 18 and 19, we have thus derived a complete hierarchy describing the evolution of the function \( f_n(q_n; t) \), and thus, a complete stochastic description of the population size and age structure of the entire population. These equations mirror the BBKGY hierarchies seen in gas kinetics [16, 21]. Further details concerning these equations, including methods of solution, can be found in [1, 8].
3.2 Age-Structured Moments

We now consider the moments of the age-structure. Now, the density $X^{(n)}(\mathbf{q}_n; t)$ for $n$ individuals with age-chart vector $\mathbf{q}_n$ is given by the following expectation of the number operator $\psi_{q_1}^\dagger \cdots \psi_{q_n}^\dagger \psi_{q_n} \cdots \psi_{q_1}$ (see Doi [2] for more details):

$$X^{(n)}(\mathbf{q}_n; t) = \langle 1| \psi_{q_1}^\dagger \cdots \psi_{q_n}^\dagger \psi_{q_n} \cdots \psi_{q_1}| f(t) \rangle = \langle 1| \psi_{q_1} \cdots \psi_{q_n} e^{-\zeta t} | f(0) \rangle .$$

Note that in this expression $\langle 1 \rangle$ represents a coherent state with function $u(q) \equiv 1$ (rather than a state representing a single individual of age 1). We have used the fact that $\langle 1 \rangle$ is a left eigenstate of $\psi_{q_1}^\dagger$, with eigenvalue 1, as seen in Eq. 10. We now derive an analytic equation for $X^{(1)}(q) \equiv X(q)$ in detail. Differentiating Eq. 20 with respect to time, we find

$$\frac{\partial X}{\partial t} = - \langle 1| \psi_q \zeta | f(t) \rangle = - \langle 1| \psi_q (\zeta_0 + \zeta_\beta + \zeta_d) | f(t) \rangle ,$$

which yields three terms on the right-hand side that can be written in the forms

$$\langle 1| \psi_q \zeta_0 | f(t) \rangle = \langle 1| \psi_q \int dp \psi_{q, \mu}^\dagger \frac{\partial}{\partial p} \psi_{p} | f(t) \rangle = \int dp \langle 1| \delta(q - p) \frac{\partial}{\partial p} \psi_{p} | f(t) \rangle + \int dp \langle 1| \psi_{p}^\dagger \psi_{p} \frac{\partial}{\partial p} \psi_{p} | f(t) \rangle$$

$$= \frac{\partial}{\partial q} \langle 1| \psi_q | f(t) \rangle + \int dp \frac{\partial}{\partial p} \langle 1| \psi_q \psi_{p} | f(t) \rangle = \frac{\partial X(q)}{\partial q} + \int dp \frac{\partial}{\partial p} X^{(2)}(q, p),$$

$$\langle 1| \psi_q \zeta_\beta | f(t) \rangle = \langle 1| \psi_q \int dp \beta(p) \left[ \psi_{p}^\dagger \psi_{p} - \psi_{p}^\dagger \psi_{p}^\dagger \psi_{p} \right] | f(t) \rangle$$

$$= \beta(q) X(q) + \int dp \beta(p) \langle 1| \psi_q \psi_{p} | f(t) \rangle - \beta(q) \int dr \beta(p) \langle 1| \psi_{r} \psi_{p} | f(t) \rangle = - \beta(q) \int dp \beta(p) X(p),$$

and

$$\langle 1| \psi_q \zeta_d | f(t) \rangle = \langle 1| \psi_q \int dp \mu(p) \left[ \psi_{p}^\dagger \psi_{p} - \psi_{p} \right] | f(t) \rangle$$

$$= \mu(q) X(q) + \int dp \mu(p) \langle 1| \psi_q \psi_{p} | f(t) \rangle - \int dp \mu(p) \langle 1| \psi_{q} \psi_{p} | f(t) \rangle = \mu(q) X(q).$$

Next we assume $X^{(2)}(q, p)$ will be vanishingly small for extreme $p$, and the integral-differential term above vanishes, leaving the PDE

$$\frac{\partial X}{\partial t} + \frac{\partial X}{\partial q} + \mu(q) X = \delta(q) \int dp \beta(p) X(p).$$

For any $q > 0$ we lose the delta function and recover the McKendrick-von Foerster equation, as expected [8]. If we integrate $q$ across a vanishing small interval containing $q = 0$ we also recover the McKendrick-von Foerster boundary condition.

Higher order correlations $X^{(n)}(\mathbf{q}_n; t)$ obey the following equation, which can be derived in much the same way as Eq. 25; by differentiating Eq. 20 with respect to time and calculating the resulting matrix elements, the details of which are left to the reader:

$$\frac{\partial X^{(n)}}{\partial t} + \sum_{i=1}^{n} \frac{\partial X^{(n)}}{\partial q_i} + X^{(n)} \sum_{i=1}^{n} \mu(q_i) = \sum_{i=1}^{n} \delta(q_i) \left[ \int dp \beta(p) X^{(n)}(\mathbf{q}_n^{(i-1)}, p) + X^{(n-1)}(\mathbf{q}_n^{(i-1)}) \sum_{j \neq i} \beta(q_j) \right].$$

These novel equations are equivalent to those in [1], giving a natural way to describe the stochastic fluctuations in age-structured populations that the mean-field McKendrick-von Foerster equation fails to capture. A probabilistic derivation and solutions to Eq. 26 can be found in [1], along with a derivation of the variance of the size of sub-populations of individuals with ages belonging to any interval $[a, b]$. 


4 Perturbation Expansion and Feynman Diagrams

A natural quantity to consider is the distribution of the population size. As described in Doi [2, 3], \( \rho_n(t) = \frac{1}{n!} \int dq_n f_n(q_n, t) \) is the probability of population size \( n \) at time \( t \), and the generating function \( F(z, t) = \sum_{n=0}^{\infty} \rho_n(t) z^n \) can be expressed as

\[
F(z, t) = \langle z | f(t) \rangle = \langle z | e^{-\zeta t} | f(0) \rangle.
\] (27)

To evaluate the right-hand side of this expression we need to determine the action of \( \zeta \) on \( \langle z \rangle \). This is not straightforward and we turn to perturbative methods [2, 3]. Since the approach is standard, we only highlight salient features of the derivation. More details on perturbative expansions can be found in standard quantum field theory texts such as [13, 14, 18].

The representation used so far has been the Schrödinger representation, where the states are time dependent and the operators are constant in time. The Heisenberg representation shifts the time dependence to the operators, having constant states. The Interaction representation sits between the two, using the non-interaction part of the Hamiltonian to shift the representation. In our situation, the birth and death events represent the interactions, and the non-interactive part of our ‘Hamiltonian’ \( \zeta \) is described by the operator \( \z_0 \). If \( |s\rangle \) and \( \mathcal{O} \) denote generic states and operators we have the transformations

\[
\mathcal{O}_I = e^{\zeta_0 t} \mathcal{O} e^{-\zeta_0 t} \equiv \mathcal{O}(t), \quad |s\rangle_I = e^{\zeta_0 t} |s\rangle,
\] (28)

where the subscript \( I \) labels the interaction representation. We also use time dependence of \( \mathcal{O}(t) \) to indicate an operator in the interaction representation.

Note that the rightmost operators of Eq. 6 are all annihilation operators, meaning that operators \( \z_0 \) and \( \z \) kill the vacuum state \( |\phi\rangle \) and so \( e^{-\z_0 t} |\phi\rangle = e^{\zeta_0 t} |\phi\rangle = |\phi\rangle \). That is, the vacuum state in the Schrödinger, Interaction and Heisenberg representations are identical and invariant in time, and so is well defined when simply written as \( |\phi\rangle \).

We write Eq. 27 as

\[
F(z, t) = \langle z | e^{-\z_0 t} e^{\z_0 t} e^{-\z t} | f(0) \rangle = \langle z | e^{\zeta_0 t} e^{-\zeta_0 t} e^{-\z_0 t} e^{\z_0 t} | f(0) \rangle = \langle z | T e^{-\int_0^t \z_0(s) + \z(s) ds} | f(0) \rangle,
\] (29)

where \( T \) denotes the Dyson time ordering operator (see [2, 3, 13, 14, 18]). The derivation proves to be simpler to handle if we move from the ‘position’ representation of age to the Fourier-transformed ‘momentum’ representation. More specifically, we introduce the operators

\[
a_k = \frac{1}{\sqrt{2\pi}} \int dq \psi_q e^{-ikq}, \quad a_k^\dagger = \frac{1}{\sqrt{2\pi}} \int dq \psi_q^* e^{ikq},
\] (30)

which satisfy commutation relations

\[
[a_k, a_{\ell}^\dagger] = \delta(k - \ell), \quad [a_k, a_{\ell}] = [a_k^\dagger, a_{\ell}^\dagger] = 0.
\] (31)

It is straightforward to verify that the operators \( \zeta_0, \zeta_b \) and \( \zeta_d \) can be written as

\[
\zeta_0 = i \int dk ka_k^\dagger a_k,
\]

\[
\zeta_b = \frac{1}{\sqrt{2\pi}} \int dk dl \tilde{\beta}(k) a_{k+l}^\dagger a_l - \frac{1}{2\pi} \int dk dl dr \tilde{\beta}(k) a_{k+l}^\dagger a_{k+r}^\dagger a_l a_{r},
\]

\[
\zeta_d = \frac{1}{\sqrt{2\pi}} \int dk dl \tilde{\mu}(k) a_{k+l}^\dagger a_l - \int dk \tilde{\mu}(k) a_{-k}.
\]

The terms \( \tilde{\beta} \) and \( \tilde{\mu} \) denote Fourier transforms of the birth and death rates. For simple models, such as those containing constant birth and death rates, these are divergent terms. However, we can reasonably define \( \beta(p) \) and \( \mu(p) \) to tend to zero for sufficiently large age \( p \) and be Fourier invertible. This assumption will have little effect on the dynamics and provide convergent terms.
Next, we expand the generating function. Note that with Eq. 30 we can construct coherent state expansion \( \langle z | = \langle z | e^{-\zeta_0 t} = \langle \phi | e^{-\zeta_0 t} e^{\sqrt{2\pi}a_0} e^{-\zeta_0 t} = \langle \phi | e^{\sqrt{2\pi}a_0(0)}. \) Then, writing
\[
|f(0)\rangle = \sum_{m=0}^{\infty} \int \frac{dk_m}{m!} \hat{f}_m(k_m; 0) \prod_j \hat{a}_{kj}^0(0) |\phi\rangle,
\]
where we have used the Fourier inversion of Eq. 30 to represent \( |f(0)\rangle \) in ‘momentum’ space, we find that Eq. 29 can be written as,
\[
F(z,t) = \langle \phi | e^{\sqrt{2\pi}a_0(0)} T e^{-\int_0^t [\zeta(s)+\zeta_0(s)] ds} \sum_{m=0}^{\infty} \int \frac{dk_m}{m!} \hat{f}_m(k_m; 0) \prod_j \hat{a}_{kj}^0(0) |\phi\rangle.
\]
Then expanding both exponentials, we obtain
\[
F(z,t) = \sum_{n,r,m=0}^{\infty} z^n \frac{(2\pi)^{n/2}}{m!n!} \int ds_r \int dk_m f_m(k_m; 0) \langle \phi | T a_0(t)^n \prod_{i=1}^{r} [-\zeta_0(s_i) + \zeta_d(s_i)] \prod_j \hat{a}_{kj}^0(0) |\phi\rangle,
\]
where we have used \( a_0(t) = a_0(0) \). This follows from the Baker-Campbell-Hausdorff formula, \( a_0(t) = e^{\int_0^t a_0 \hat{a}_0} e^{-\zeta_0 t} = e^{\int_0^t ad\zeta_0} a_0 = a_0 = a_0(0) \), where we have used the adjoint action \( ad\zeta_0(a_0) \equiv [\zeta_0, a_0] = 0 \) from Eqs. 31 and 32. We have also removed a \( r! \) denominator in the sum by restricting the \( s_r \) integral to the region where \( 0 \leq s_1 \leq \ldots \leq s_r \leq t \). We can then use Eqs. 33 and 34 to expand the term \( \prod_{i=1}^{r} [-\zeta_0(s_i) + \zeta_d(s_i)] \) in creation and annihilation operators and use Wick’s theorem to obtain a product of terms of the form \( \langle \phi | T a_0(t) a_0^0(s) |\phi\rangle \). For \( t > s \) this is equivalent to \( \langle \phi | e^{\int_0^t a_k e^{-\zeta_0 t} e^{\zeta_0 s} a_0^0 e^{-\zeta_0 s}} |\phi\rangle = \langle \phi | a_k a_0^0 e^{-\hat{a}_0(t-s)} |\phi\rangle \) where we have used the Baker-Campbell-Hausdorff theorem, along with the relation \( [\zeta_0, a_0^0] = il a_0^0 \). For \( t < s \) the order of \( a_k \) and \( a_0^0 \) is reversed and the term is killed. Thus we find:
\[
\langle \phi | T a_k(t) a_0^0(s) |\phi\rangle = \begin{cases} 
\delta(k-l)e^{-ik(t-s)}, & t > s, \\
0, & t < s.
\end{cases}
\]
Each term in the sum in Eq. 35 can be represented by a Feynman diagram such as that in Fig. 1. These are constructed as follows. We start with \( m \) lines at time 0, corresponding to \( m \) founder individuals, and end with \( n \) lines at time \( t \), corresponding to \( n \) surviving individuals. In between, we have three types of event that are associated with terms arising from \( \prod_{i=1}^{r} [-\zeta_0(s_i) + \zeta_d(s_i)] \) when substituting with Eqs.
33 and 34. We can have birth events, such as at time \( s_1 \) in Fig. 1, where we pick up a term \( \hat{\beta}(l_3) \) (from the second term in Eq. 33), death events, such as at time \( s_2 \), where we pick up a term \( \hat{\mu}(l_4) \) (from the second term in Eq. 34), and null events, such as at time \( s_3 \), where we pick up a term \( -\hat{\beta}(l_6) - \hat{\mu}(l_6) = -\hat{\gamma}(l_6) \) (from the first terms in Eqs. 33 and 34).

Annihilation and creation operators are then placed at the end and beginning of each segment. For example, this gives five horizontal segments in Fig. 1, and five terms in the Wick product. We then find that the term from Eq. 35 corresponding to this Feynman diagram is:

\[
\frac{z^2(2\pi)^{\frac{d}{2}}}{2!^d} \int_{0<s_1<s_2<s_3<\cdots} ds_{3,d} \int_{0<s_1<s_2<s_3<\cdots} dl_{k_1} \langle \phi|T\alpha_0(t)a_{k_1}^\dagger(0)|\phi \rangle \\
\times \frac{1}{2\pi} \int dl_1 dl_3 dl_4 dk_2 \langle \phi|T\alpha_{-l_4}(s_2)a_{l_4+1}(s_1)|\phi \rangle \langle \phi|T\alpha_{l_4}(s_1)a_{l_4}^\dagger(0)|\phi \rangle \hat{\mu}(l_4)\hat{\beta}(l_3). \\
\times \frac{1}{\sqrt{2\pi}} \int dl_5 dl_6 dl_7 \langle \phi|T\alpha_0(t)a_{l_5+1}(s_3)|\phi \rangle \langle \phi|T\alpha_{l_5}(s_3)a_{l_5}^\dagger(s_1)|\phi \rangle \hat{\gamma}(l_6) \cdot \hat{f}_2(k_2;0)
\]

\[
= \frac{z^2}{2!^d} \int_{0<s_1<s_2<s_3<\cdots} ds_{3,d} \cdot \int dp_2 \hat{f}_2(p_2;0) \cdot 1 \cdot \hat{\gamma}(s_3-s_1) \cdot \hat{\beta}(p_2+s_1) \cdot \hat{\mu}(p_2+s_2). \tag{37}
\]

Note that we have split the integral on the left-hand side into three terms corresponding to the three individuals that are represented in the Feynman diagram. Substitution using Eq. 36 then allows reverse Fourier transforms to give the final integral of birth and death rates weighted by the original distribution of ages given on the right-hand side of Eq. 37. The final integral has intuitive appeal. The birth term \( \hat{\beta}(p_2+s_1) \) is associated with the birth event occurring at time \( s_1 \), the parent having an initial age of \( p_2 \). This individual also dies at age \( p_2+s_2 \), which is the event corresponding to term \( \hat{\mu}(p_2+s_2) \). The null term \( -\hat{\gamma}(s_3-s_1) \) is associated with the newborn individual, at a time \( s_3-s_1 \) after their birth. We then integrate this structure over the unknown times \( s_1, s_2 \) and \( s_3 \), weighted by the initial distribution of ages \( f_2(p_2;0) \). We now utilise three observations to show that we get the simplified form of Eq. 37 in general.

For our first observation; (i) note that we can associate variables to individuals represented in the Feynman diagram. In Fig. 1, for example, the variable \( k_1 \) can be associated to the individual represented by the upper line, the variables \( k_2, l_1, l_3, l_4, s_1 \) and \( s_2 \) can be associated to the individual represented by the middle line, and the variables \( l_2, l_5, l_6 \) and \( s_3 \) associated to the lower. More generally, a death event at time \( s \) results in two terms of the form \( \hat{\mu}(l_j) \) and \( a_{-l_j}(s) \). The two variables implicated, \( s \) and \( l_j \), can be associated with the dying individual. A birth event at time \( s \) results in three terms of the form \( \hat{\beta}(l_j), a_{l_j}(s) \) and \( a_{l_j}^\dagger(s) \). The three variables \( s, l_j, l_j \) can again be associated with a single individual.

The second and third observations we require are; (ii) the integrals always reduce to inverse Fourier transforms which resolve to a product of birth, death and null terms, such as that given in Eq. 37, and (iii) that the null terms corresponding to each individual can be absorbed into a propagator, and all null events dropped from the Feynman diagram. We next establish the veracity of these two observations in more detail.

The terms associated with any single individual (that is, the terms associated with a horizontal line in the Feynman diagram) take the general form given in Fig. 2. In this formulation, the individual survives across a time interval \([l_0, l_1] \subset [0, t]\). The time \( l_0 \) can correspond to the initial time \( (l_0 = 0) \) for a founder individual, or can correspond to the moment of birth \((l_0 > 0) \) of the individual. Similarly, the time \( l_1 \) can correspond to the final time point \((l_1 = t) \) of a surviving individual, or can correspond to the time \((l_1 < t) \) of the individuals death.

Consider the term \( \hat{\eta}_0(k) \). If \( l_0 > 0 \) we have a birth event and set \( \hat{\eta}_0(k) = 1 \) because the birth rate function \( \hat{\beta} \) is associated with the parental individual (such as at time \( s_1 \) in Fig. 1). If \( l_0 = 0 \) we have a founder individual and \( \hat{\eta}_0(k) \) represents an initial distribution term \( \hat{f}(k) \), where \( k \) is one of the components of \( k \) (such as \( k_2 \) at time \( 0 \) in Fig. 1).

Now, consider \( \hat{\eta}_{n+1}(r) \). If \( t_1 < l_3 \), we can choose a null event with \( \hat{\eta}_{n+1}(r) \equiv \mu(r) \) (such as at time \( t_1 = s_2 \) in Fig. 1). If \( t_1 = l_3 \), we can have a termination term, in which case \( \hat{\eta}_{n+1}(r) \equiv 1 \) and no \( r \) integral is required.

The internal terms \( \hat{\eta}_j(k_j) \) denote either birth terms \( \hat{\beta}(l_j) \) for which the individual is a parent, or null terms \(-\hat{\gamma}(l_j)\).
Next, we use Eqs. 33, 34 and 36 on Eq. 35 to combine terms corresponding to each individual into inverse Fourier transforms, such as the example in Eq. 37, by evaluating an integral of the following form (integration over the \( s \) variables is ignored for the moment):

\[
(-1)^N \int dr^l d l^k \delta(r - (l'^n + l_n)) \prod_{j=2}^n \delta(l_j - (l'_{j-1} + l_{j-1})) \delta(l_1 - k) 
\]

\[
e^{-ir(t_1-t_0)} \sum_{j=2}^n \delta(j(s_j-s_{j-1})-il_1) \hat{\eta}_{n+1}(r) \prod_{j=1}^n \hat{\eta}_j(l'_j) \hat{\eta}_0(k),
\]

(38)

where \( N \) represents the number of null events, which by Eqs 34 and 35 will have negative signs associated with them.

Then integrating with respect to the \( l \) and \( k \) variables, the delta functions send \( k \to l_1, l_1 \to l_2 - l'_1, l_2 \to l_3 - l'_2 \) and so on, along with \( l_n \to r - l'_n \). Equivalently, \( k \to -(r + l'_1 + \ldots + l^n_n), l_1 \to -(r + l'_1 + \ldots + l^n_n), l_j \to -(r + l'_1 + \ldots + l^n_n) \). The exponent in Eq. 38 then becomes \( il'_1 (s_1 - t_0) + il'_2 (s_2 - t_0) + \ldots + il'_n (s_n - t_0) + ir(t_1 - t_0) \). The integral then reduces to the following form:

\[
(-1)^N \int dr^l \hat{\eta}_{n+1}(r) e^{ir(t_1-t_0)} \prod_{j=1}^n \hat{\eta}_j(l'_j) e^{il'_j (s_j-t_0)} \hat{\eta}_0(-(r + l'_1 + \ldots + l'^n_n)).
\]

(39)

We now consider various cases. Let \( t_0 \) denote a birth event. Then \( \hat{\eta}_0(k) \equiv 1 \) and the integrals separate, resulting in a product of inverse Fourier transforms, which simplify to

\[
(-1)^N \eta_{n+1}(t_1 - t_0) \prod_{j=1}^n \eta_j(s_j - t_0).
\]

(40)

In the case that \( \hat{\eta}_0 \equiv \hat{f} \) is the initial age distribution at time \( t_0 = 0 \), the integrals do not directly separate and instead we end up with a time shifted product of inverse Fourier transforms. If \( p \) is the initial age of the individual in question, and \( \mathbf{p} \) is the initial age-chart of the population, we get a contribution of the form

\[
(-1)^N \eta_{n+1}(p + t) \prod_{j=1}^n \eta_j(p + s_j) f(p; 0).
\]

(41)

If the event at time \( t_1 \) is a termination event, then \( r = 0 \), there is no \( r \) integral, and the \( \eta_{n+1} \) term can be ignored in both Eqs. 40 and 41. We then multiply these terms across the individuals in the Feynman diagram, leaving an integral across internal times \( s \) and initial ages \( \mathbf{p} \) such as that given in the right-hand side of Eq. 37. This establishes observation (ii).

To establish observation (iii), we combine the null terms into a single propagator as follows. Note that for any Feynman diagram devoid of null terms, we can place any number of null terms on each individuals horizontal timeline. These can be collected together as follows.

In the case that the individual has a positive birth time, we see from Eq. 40 that the initial distribution \( f \) plays no role. If we have \( N \) null events, then performing the \( s \) integral we have thus far ignored, we collect a factor of the form:
Then summing over all the possible values of $N$ we obtain the propagator $U(0, t_1 - t_0)$ where

$$U(q, q') = \exp \left\{ - \int_q^{q'} ds \gamma(s) \right\}. \quad (43)$$

This is the well known survival function, describing the probability that no birth or death event has occurred to the individual between the ages of $q$ and $q'$. In the case where we have a founder individual with a birth time before $t_0 = 0$, the initial distribution is involved in Eq. 41. However, we can still collect the null terms into a factor $U(q, q')$, where $q$ and $q'$ are the initial and final ages of the individual. We thus find that we can drop all the null events from the Feynman diagrams and add a propagator for each individual, and observation (iii) is established. The Feynman diagrams then simply become realizations of birth-death processes, with a term of the type

$$\sum_{s \leq r_1 \leq \cdots \leq r_N \leq t} \frac{(-1)^N}{N!} \int_{t_0}^{t_1} ds \gamma(s - t_0) \cdot \prod_{i=1}^{N} \gamma(s_i - t_0) = \frac{(-1)^N}{N!} \left\{ \int_{t_0}^{t_1} ds \gamma(s - t_0) \right\}^N. \quad (42)$$

5 Coherent States and Path Integral Formulation

The path integral formulation of quantum mechanics works well in part because the fundamental position and momentum states are eigenstates of terms in Hamiltonians corresponding to many fields of interest. These fundamental states are then used to construct resolutions of the identity, which are applied between time slices across the time period of interest, resulting in path integrals [4, 19]. This technique will not work for the systems we consider. Specifically, we have fundamental states $\{|q\rangle\}$ that represent the age-charts of populations of size $n$. However, the ‘Hamiltonians’ $\zeta$ we consider are functions of creation and annihilation operators, and the states $|p\rangle$ are not eigenstates for these operators. Creation operators increase the minimum occupation number for any state superposition indicating that an eigenstate will not exist. However, eigenstates exist for annihilation operators, the coherent states we have seen in Eq. 10. To construct a path integral, we thus need a resolution of the identity in terms of coherent states. There are two possible approaches. One generalises that of Peliti [17] and is the approach we take, as detailed in Appendix A. The other approach adapts techniques more commonly found in quantum field theoretic applications, as detailed in Appendix B, along with an explanation why two path integral formulations are possible.

We have, then, the following path integral resolution of the identity,

$$I = \int \mathcal{D}u \mathcal{D}v \, e^{-i \int dq u(q) v(q) |i\rangle \langle v|}, \quad (45)$$

where the integrals over $u$ and $v$ are over real functions such that

$$\int \mathcal{D}u \mathcal{D}v \equiv \lim_{Q \to \infty} \prod_{Q} \int_{-Q}^{Q} \frac{d[u(q)] d[v(q)]}{2\pi}. \quad (46)$$

We next construct a path integral representation of the amplitude between two coherent states, using the resolution of the identity from Eq. 45 at the time slices. To do this we first obtain matrix elements for
the evolution operators $\zeta_0$, $\zeta_b$ and $\zeta_d$ between coherent states $\langle u|$ and $|iv\rangle$. We find, using the eigenvalue properties of Eq. 10, that

$$\langle u|\zeta_0|iv\rangle = \langle u|iv\rangle i \int dz u \frac{\partial v}{\partial q},$$  \hspace{1cm} (47)

$$\langle u|\zeta_b|iv\rangle = \langle u|iv\rangle i(1 - u(0)) \int dz \beta uv,$$  \hspace{1cm} (48)

$$\langle u|\zeta_d|iv\rangle = \langle u|iv\rangle i \int dz \mu v(u - 1).$$  \hspace{1cm} (49)

For small time interval $\epsilon$, and using the normalisation property of Eq. 12, we find

$$\langle u|e^{-\zeta}|iv\rangle = \exp \left\{ -i \int dz \left[ \frac{uv}{\epsilon} + u \frac{\partial v}{\partial q} + \left( 1 - u(0) \right) \beta uv + \mu v(u - 1) \right] \right\}. \hspace{1cm} (50)$$

Taking a product of such time slices over a time interval $[0, T]$, we obtain the following path integral formulation:

$$\langle u_T|e^{-\zeta T}|iv_0\rangle = \int_{z=v_0}^{z=iT} \mathcal{D}u \mathcal{D}v \exp \left\{ -i \int dz dt \left[ u \left( \frac{\partial v}{\partial t} + \frac{\partial v}{\partial q} + \gamma v \right) \right] \right.$$  

$$\left. + i \int dz dt \left[ \beta uv u(0, t) + \mu v \right] + i \int dz dt v_T \right\},$$  \hspace{1cm} (51)

where $u(q, t)$ and $v(q, t)$ are now real functions of age and time. We can now use this construction to investigate specific quantities such as the generating function $F(z, T)$, $F(z, T) = \langle z|e^{-\zeta T}|f(0)\rangle = \int \mathcal{D}u \mathcal{D}v \mathcal{D}u_0 \mathcal{D}v_0 \langle z|iv_T\rangle \langle u_T|e^{-\zeta T}|iv_0\rangle \langle u_0|f(0)\rangle e^{-i \int dz [u v_0 + u_T v_T]},$  \hspace{1cm} (52)

where we have used resolutions of the identity at time points 0 and $T$. Using Eqs 8 and 12, we obtain the following identities

$$\langle z|iv_T\rangle = e^{iz} \int dz v_T, \hspace{1cm} \langle u_0|f(0)\rangle = \sum_m \int \frac{dq}{m!} \prod_{i=1}^{m} u_0(q_i) f_m(q_m; 0).$$  \hspace{1cm} (53)

Upon substitution into Eq. 52, we find

$$F(z, T) = \sum_m \int \frac{dq}{m!} f_m(q_m; 0) \int \mathcal{D}u \mathcal{D}v \prod_{i=1}^{m} u_0(q_i) \exp \left\{ -i \int dz dt \left[ u \left( \frac{\partial v}{\partial t} + \frac{\partial v}{\partial q} + \gamma v \right) \right] \right.$$  

$$\exp \left\{ i \int dz dt v \left( \beta u u(0) + \mu v \right) \right\} \exp \left\{ i \int dz \left[ z v_T - u_0 v_0 \right] \right\}.$$  \hspace{1cm} (54)

We now consider two cases; a pure death process, for which the path integral can be calculated exactly, and a birth-death process, which we calculate by expansion methods.

A pure death process is defined by $\beta(q) = 0$. The functional integration over $u(q, t)$ forces the constraint $\frac{\partial}{\partial t} + \frac{\partial}{\partial q} + \mu v = 0$, which we recognise as the McKendrick-von Foerster equation. Since there is no restriction on the sign of $q$, we obtain a solution purely in terms of the boundary $v_0(q)$,

$$v_t(q) = v_0(q - t) \exp \left\{ - \int_0^t \mu(q - t + s) ds \right\}.$$  \hspace{1cm} (55)

After substitution and integration of the $\mu v$ term with respect to time,
and ages at death, are indicated over time interval $[0, T]$ for the five individuals implicated.

\[
F(z,T) = \sum_{m} \int \frac{d\mathbf{q}_{m}}{m!} f_{m} (\mathbf{q}_{m};0) \int \mathcal{D}u_{0} \mathcal{D}v_{0} \prod_{i=1}^{m} u_{0}(q_{i}) \cdot \\
\exp \left\{ -i \int dq v_{0} \left[ u_{0} - z e^{-\int_{0}^{T} \mu(q+s)ds} - (1 - e^{-\int_{0}^{T} \mu(q+s)ds}) \right] \right\} \cdot \\
= \sum_{m} \int \frac{d\mathbf{q}_{m}}{m!} f_{m} (\mathbf{q}_{m};0) \prod_{i=1}^{m} \left[ ze^{-\int_{0}^{T} \mu(q+s)ds} + (1 - e^{-\int_{0}^{T} \mu(q+s)ds}) \right]. \tag{56}
\]

For the last step, the integral over $v_{0}$ forces the constraint $u_{0} = ze^{-\int_{0}^{T} \mu(q+s)ds} + (1 - e^{-\int_{0}^{T} \mu(q+s)ds})$ which is then substituted for $u_{0}$, leaving us with a formula that can be derived by probability arguments; the term $\exp \left\{ -\int_{0}^{T} \mu(q+s)ds \right\}$ is precisely the survival term found in [8], and we collect one such term per surviving individual for each power of $z$ as expected.

In the general case that includes both birth and death, we expand $\langle z|iv_{T} \rangle = e^{iz \int dq v_{T}}$ in Eq. 54 to obtain

\[
F(z,T) = \sum_{m,n} \frac{z^{n}}{n!} \int \frac{d\mathbf{p}_{m}}{m!} d\mathbf{q}_{m} f_{m} (\mathbf{p}_{m};0) \int \mathcal{D}u \mathcal{D}v \prod_{i=1}^{m} u_{0}(p_{i}) \prod_{j=1}^{n} iv_{T}(q_{j}) \cdot \\
\exp \left\{ -i \int dp dt u \left( \frac{\partial v}{\partial t} + \frac{\partial u}{\partial t} + \gamma v \right) - i \int dp u_{0} v_{0} \right\} \exp \left\{ i \int dp dt v (\beta u(0) + \mu) \right\}. \tag{57}
\]

The first exponential in this functional integral is quadratic and can be integrated exactly. The second exponential must be evaluated by expansion. The two terms in the argument $v (\beta u(0) + \mu)$ correspond to birth and death and expansion results in associated Feynman diagrams, each representing a birth-death process with corresponding terms we now describe. Details of the derivation relating Eq. 57 to the terms we now assign to the Feynman diagram can be found in Appendix C.

We have $m$ lines starting at time zero, terminating in $n$ lines at time $T$, with birth and death events inbetween (see Fig. 3, for example). This represents a process starting and ending with $m$ and $n$ individuals, respectively. The $m$ lines are assigned an initial age vector $\mathbf{p}_{m}$. Each line arising from a birth at time $t > 0$ is assigned an initial age $p_{\text{initial}} = 0$. Each line has a final age $p_{\text{final}} = p_{i} + t$ and initial age $p_{\text{initial}} = p_{i}$, where $t$ is the time duration along the line. These ages are functions of $\mathbf{p}_{m}$ and $\mathbf{t}_{k}$, where $k$ is the number of birth-death events with $0 < t_{1} < \ldots < t_{k} < T$. Each birth is assigned a term $\beta(p_{\text{initial}})$ and each death a term $\mu(p_{\text{final}})$. Each edge is assigned the propagator $U(p_{\text{initial}}, p_{\text{final}}) = e^{-\int_{p_{\text{initial}}}^{p_{\text{final}}} \gamma(x)dx}$. 
We then integrate the product of these terms and initial distribution $\frac{1}{m!} f_m(p_m; 0)$ over the initial ages and time variables. The integral for Fig. 3, for example, then reduces to

$$\frac{e^3}{3!} \int dp_2 \int dt_5 \frac{f_2(p_2; 0)}{2!} \cdot \beta_2(p_1 + t_1) \beta_2(p_2 + t_3) \delta_2(t_5 - t_1) \cdot \mu_3(p_1 + t_2) \mu_3(p_2 + t_4) \cdot U(p_1, p_1 + t_2)U(p_2, p_2 + t_4)U(0, T - t_1)U(0, T - t_3)U(0, T - t_5).$$

Note that the propagators correspond to survival probabilities for the five individuals over their corresponding lifetimes in Fig. 3. For example, the founder individual with initial age $p_1$ dies at age $p_1 + t_2$ and is associated with propagator $U(p_1, p_1 + t_2).$ This formulation precisely matches that derived earlier by perturbative expansion, and mirrors the formulation obtained using probabilistic arguments in [8].

We now turn to some model systems to highlight the various methods introduced.

6 Fission Processes

We now describe some of the issues arising when utilizing these methods on binary fission, where two individuals simultaneously arise at the moment the parental individual terminates, resulting in a pair of twins with identical ages, such as in cell division, for example. This is a degenerate process because some ages are duplicated (whilst both twins are alive) and some are not (when one of a pair of twins dies). We shall consider the mean-field behaviour and the full probability density for the population-size-age-chart.

The first thing to note is that such microscopic degeneracy is automatically handled in this formalism. For example, if we apply the number operator $\psi^\dagger \psi$ to the degenerate state $|p, p, q\rangle,$ where one age $p$ is duplicated, we obtain the density $\langle 1|\psi^\dagger \psi \psi |p, p, q\rangle = 2\delta(r - p) + \delta(r - q)$ and find the duplicated ages are correctly accounted for.

For the fission process, the operators $\zeta_0$ and $\zeta_d$ are identical to those in previous sections, but the birth operator becomes $\zeta_0 = \int dp \beta(p)|\psi^\dagger \psi \psi - \psi^\dagger \psi^\dagger \psi |p\rangle,$ where the two $\psi^\dagger$ operators in the latter term account for the birth of twins, and $\psi^\dagger$ represents termination of the parent.

To study the mean field behaviour, we define the mean density $X(p) = \langle 1|\psi^\dagger \psi \psi p e^{-\zeta_0} |f(0)\rangle$ as before and differentiate with respect to time. The derivation is largely the same as that for Eq. 25, resulting in a McKendrick-von Foerster-like equation,

$$\frac{\partial X}{\partial t} + \frac{\partial X}{\partial p} = -[\beta(p) + \mu(p)] X.$$  \hspace{1cm} (59)

Note that the difference between this microscopic model and that encapsulated in Eq. 25 is that both birth from, and death to, an individual results in their termination. This is reflected in the right-hand side of Eq. 59. The boundary condition is also modified to account for the duplicated offspring:

$$X(0) = 2 \int dp \beta(p) X(p).$$  \hspace{1cm} (60)

If we let $f_n(p_n; t)$ denote the age-chart distribution as before, the age-duplication results in degenerate equations. Specifically, we can differentiate $f_n(p_n; t) = \langle p_n | e^{-\zeta_0} | f(0)\rangle$ in the same way as the derivation of Eqs. 18 and 19, which results in the following equation and boundary condition:

$$\frac{\partial f_n}{\partial t} + \sum_{i=1}^n \frac{\partial f_n}{\partial p_i} = -f_n \sum_{i=1}^n [\beta(p_i) + \mu(p_i)] + \int dq \mu(q) f_{n+1}(p_n; q; t),$$

$$f_n(p_{n-1}; 0, t) = \sum_{i=1}^{n-1} \delta(p_i) \int dq \beta(q) f_{n-1}(p_{n-1}; q, t).$$  \hspace{1cm} (61)

The delta function in the boundary condition arises because of the duplication of a new born individual of age zero. This is difficult to deal with analytically and was handled in [1] by splitting the population into pairs of twins, with identical ages, and individuals with unique ages. We now describe how to implement such a formalism by adopting a multi-species Doi-Peliti paradigm.

We treat the two classes (individuals and twins) as separate species and have two pairs of creation and annihilation operators; $\psi^\dagger, \psi$ and $\chi^\dagger, \chi,$ respectively. These obey the usual commutation relations (Eq.
1) and the two classes of operators also commute with each other (e.g., \([\psi, \chi] = 0\)). We then represent \(m\) individuals with unique ages by age-chart \(p_m\) and \(n\) sets of twins by age-chart \(q_n\), resulting in a general state of the form

\[
|p_m; q_n\rangle \equiv \psi_p^\dagger \cdots \psi_{p_m}^\dagger \chi_q^\dagger \cdots \chi_{q_n}^\dagger |\phi\rangle. \tag{62}
\]

If \(\frac{1}{m!n!} f_{m,n}(p_m; q_n; t)\) is the associated probability density we have the state representation

\[
|f(t)\rangle = \sum_{m,n=0}^{\infty} \int \frac{dp_m}{m!} \frac{dq_n}{n!} f_{m,n}(p_m; q_n; t) \, |p_m; q_n\rangle. \tag{63}
\]

The evolution operators take on the following form:

\[
\zeta_0 = \int dp\, \psi_p^\dagger \frac{\partial}{\partial p} \psi_p + \int dp\, \chi_p^\dagger \frac{\partial}{\partial p} \chi_p,
\]

\[
\zeta_6 = \int dp\, \beta(p) \left[ \psi_p^\dagger \psi_p - \chi_p^\dagger \chi_p \right] + 2 \int dp\, \beta(p) \left[ \chi_p^\dagger \chi_p - \chi_p^\dagger \psi_p^\dagger \psi_p \right],
\]

\[
\zeta_d = \int dp\, \mu(p) \left[ \psi_p^\dagger \psi_p - \psi_p^\dagger \psi_p \right] + 2 \int dp\, \mu(p) \left[ \chi_p^\dagger \chi_p - \chi_p^\dagger \psi_p^\dagger \psi_p \right]. \tag{64}
\]

These operators reflect the microscopic fission process, generalizing the operators in Eq. 6 that represent the simpler, non-fission budding birth-death process. For example, the last term \(\chi_p^\dagger \psi_p^\dagger \psi_p\) in \(\zeta_6\) represents the event that one individual from a pair of twins divides into a newborn pair of twins; the operator \(\chi_p\) represents the annihilation of the pair of twins of age \(p\), the term \(\chi_p^\dagger\) represents the creation of newborn twins of age zero, and the creation operator \(\psi_p^\dagger\) represents the single remaining individual of age \(p\). The coefficient 2 reflects the fact that each of the individuals in the initial pair of age \(p\) can be annihilated.

We differentiate \(f_{m,n}(p_m; q_n; t) = \langle p_m; q_n | e^{-q f(t)} | 0 \rangle\) with respect to time and derive bulk equations in the same manner as before to yield

\[
\frac{\partial}{\partial t} f_{m,n} + \sum_{i=1}^{m} \frac{\partial}{\partial p_i} f_{m,n} + \sum_{j=1}^{n} \frac{\partial}{\partial q_j} f_{m,n} = -f_{m,n} \left[ \sum_{i=1}^{m} (\beta(p_i) + \mu(p_i)) + 2 \sum_{j=1}^{n} (\beta(q_j) + \mu(q_j)) \right]
\]

\[
+ \int dr \, \mu(r) f_{m+1,n}(p_m, r; q_n) + 2 \sum_{i=1}^{m} \mu(p_i) f_{m-1,n+1}(p_m^{(-i)}, q_n, p_i), \tag{65}
\]

\[
f_{m+1,n}(p_m, 0; q_n) = 0, \tag{66}
\]

\[
f_{m,n+1}(p_m; q_n, 0) = \int dr \, \beta(r) f_{m,n-1}(p_m, r; q_n) + 2 \sum_{j=1}^{m} \beta(p_j) f_{m-1,n+1}(p_m^{(-j)}, q_n, p_j). \tag{67}
\]

These results agree with the equations obtained via a probabilistic derivation in [1]. We can also obtain the mean field equations given in [1] by defining densities \(A(p) = \langle 1 | \psi_p^\dagger \psi_p | f(t) \rangle\) and \(B(p) = \langle 1 | \chi_p^\dagger \chi_p | f(t) \rangle\), where we define coherent state:

\[
|1\rangle \equiv e^{\int dp \psi_p^\dagger \psi_p + \int dq \chi_q^\dagger \chi_q} |\phi\rangle. \tag{68}
\]

Differentiating \(A(p) = \langle 1 | \psi_p e^{-q f(t)} | 0 \rangle\) and \(B(p) = \langle 1 | \chi_p e^{-q f(t)} | 0 \rangle\) with respect to time, using the commutator relations and eigenstate properties of |1\rangle, we find

\[
\langle 1 | \psi_r \cdot \partial_r | f(t) \rangle = \frac{\partial A}{\partial r}, \quad \langle 1 | \chi_r \cdot \partial_r | f(t) \rangle = \frac{\partial B}{\partial r}, \tag{69}
\]

\[
\langle 1 | \psi_r | f(t) \rangle = \mu(r)(A - 2B), \quad \langle 1 | \chi_r | f(t) \rangle = 2\mu(r)B, \tag{70}
\]

\[
\langle 1 | \psi_r \cdot \partial_r | f(t) \rangle = \beta(r)(A - 2B), \quad \langle 1 | \chi_r \cdot \partial_r | f(t) \rangle = 2\beta(r)B - \delta(r) \int dp \beta(p)(A + 2B), \tag{71}
\]

which result in the following equations and boundary conditions:
the correlation densities, population-size variance and population-size-age-position density. The latter is
case considered in [20], we can use the Doi formalism to consider other features of interest, such as
Eq. 81 is equivalent to the space-age models to be found in Webb [20]. However, unlike the mean-field
\[ \mu \quad \text{and} \quad \phi \]
Brownian motion, and its position \( q \) in independent newborn Brownian paths at a rate \( \beta \) can be incorporated into an age-structured Doi-Peliti formalism. In this process, Brownian paths spawn
We now introduce age-dependent Brownian trees, to highlight how other variables, such as position,
\[ 7 \]
Brownian Trees

We now introduce age-dependent Brownian trees, to highlight how other variables, such as position,
Eq. 65. This example also serves to show that different Doi-Peliti models can reveal different levels of
detail from the same underlying stochastic process.

These equations agree with the system in [1], although unlike [1], the derivation here does not require
Eq. 72
\[ \frac{\partial A}{\partial t} + \frac{\partial A}{\partial r} = - (\beta(r) + \mu(r))(A - 2B), \]
\[ \frac{\partial B}{\partial t} + \frac{\partial B}{\partial r} = - 2(\beta(r) + \mu(r))B, \]
\[ A(0) = 0, \]
\[ B(0) = \int dp (A + 2B). \]

7 Brownian Trees

We now introduce age-dependent Brownian trees, to highlight how other variables, such as position, can be incorporated into an age-structured Doi-Peliti formalism. In this process, Brownian paths spawn
independent newborn Brownian paths at a rate \( \beta(p,q) \) that depends upon the age \( p \) of the parental
Brownian motion, and its position \( q \). The new Brownian path starts from the position of the parental
Brownian path at the moment of birth. Each path also dies at a rate \( \mu(p,q) \). Creation and annihilation
operators \( \phi_{pq} \) and \( \phi_{pq}^{\dagger} \) satisfy the commutation relation \( [\phi_{pq}, \phi_{pq}^{\dagger}] = \delta(p - p')\delta(q - q') \). If \( \frac{1}{m} f_m(p_m, q_m) \)
represents the age-position probability distribution, we have the following state representation:
\[ |f(t)\rangle = \sum_{m=0}^{\infty} \int \frac{dp_m dq_m}{m!} f_m(p_m; q_m; t) |p_m; q_m\rangle, \]
defined in terms of fundamental state
\[ |p_m; q_m\rangle \equiv \psi_{p_1,q_1}^{\dagger} \cdots \psi_{p_m,q_m}^{\dagger} |\phi\rangle. \]

Associating Brownian motion with simple diffusion, we have an evolution operator \( \zeta_p + \zeta_q + \zeta_k + \zeta_d \)
with age, diffusion, birth and death components
\[ \zeta_p = \int dp dq \psi_{pq}^{\dagger} \frac{\partial}{\partial p} \psi_{pq}, \]
\[ \zeta_q = \int dp dq D \psi_{pq}^{\dagger} \frac{\partial^2}{\partial q^2} \psi_{pq}, \]
\[ \zeta_k = \int dp dq \beta(p,q) \left[ \psi_{pq}^{\dagger} \psi_{pq} - \psi_{pq}^{\dagger} \psi_{pq}^{\dagger} \psi_{pq} \right], \]
\[ \zeta_d = \int dp dq \mu(p,q) \left[ \psi_{pq}^{\dagger} \psi_{pq} - \psi_{pq} \psi_{pq}^{\dagger} \right]. \]

If we define coherent state \( |1\rangle = e^{\int dp dq \psi_{pq}^{\dagger} |\phi\rangle} \) the mean-field age-position density is given by
\[ X(p,q) = \langle 1 | \psi_{pq}^{\dagger} \psi_{pq} | f(t) \rangle = \langle 1 | \psi_{pq} e^{-\zeta_d} | f(0) \rangle. \]
Differentiating \( X(p,q) \) with respect to time in much the same way as above results in the following system:
\[ \frac{\partial X}{\partial t} + \frac{\partial X}{\partial p} = D \frac{\partial^2 X}{\partial q^2} - \mu(p,q)X, \]
\[ X(0,q) = \int dp \beta(p,q) X(p,q). \]

Now if \( \beta(p,q) = \beta(p) \) and \( \mu(p,q) = \mu(p) \) are just age-dependent processes, we obtain a separable
solution of the form \( X(p,q) = P(p)Q(q) \) where \( P(p) \) is a solution to the age-structure McKendrick-von
Foerster equation and \( Q(q) \) is a solution to the diffusion heat equation. More general processes \( \beta(p,q) \)
and \( \mu(p,q) \) will likely lead to more interesting (but less tractable) processes. The formulation given in
Eq. 81 is equivalent to the space-age models to be found in Webb [20]. However, unlike the mean-field
case considered in [20], we can use the Doi formalism to consider other features of interest, such as
the correlation densities, population-size variance and population-size-age-position density. The latter is
derived via probabilistic methods in [1].
8 Sexual Reproduction

Our final example is of a multi-species nature, where we have two types of individuals, male and female. This is also our first example of a true interaction model, where two individuals come together to produce offspring. Here, we define male and female creation and annihilation operators \( \psi_p, \psi_q \) and \( \chi_p, \chi_q \), respectively. These follow the usual commutation relations, with operators \( \psi \) and \( \chi \) commuting. Then we adopt Eq. 63 to describes a general state for our system, where \( m \) and \( n \) index the number of males and females in the population. We then have an evolution operator of the form \( \zeta = \zeta_0 + \zeta_b + \zeta_d \), where

\[
\zeta_0 = \int dp \psi_p^\dagger \frac{\partial}{\partial p} \psi_p + \int dp \chi_p^\dagger \frac{\partial}{\partial p} \chi_p, \\
\zeta_b = \int dp dq \beta(p, q) \left[ \chi_p^\dagger \chi_q^\dagger \psi_p \chi_q - \lambda_m \chi_p^\dagger \chi_q^\dagger \psi_0 \chi_q - \lambda_f \chi_p^\dagger \chi_q^\dagger \chi_p \psi_q \right], \\
\zeta_d = \int dp \mu_m(p) \left[ \psi_p^\dagger \psi_p - \psi_p \right] + \int dp \mu_f(p) \left[ \chi_p^\dagger \chi_p - \chi_p \right].
\]

The term \( \beta(p, q) \) is the intrinsic rate that males of age \( p \) and females of age \( q \) produce offspring, with males and females arising with probabilities \( \lambda_m \) and \( \lambda_f \), respectively (\( \lambda_m + \lambda_f = 1 \)). The death rates for males and females are \( \mu_m(p) \) and \( \mu_f(p) \), respectively.

The pairwise interaction results in coupled mean field equations. Specifically, if we have \( X(p) = \langle 1|\psi_p^\dagger \psi_p e^{-c t}|f(0)\rangle \) and \( Y(p) = \langle 1|\chi_p^\dagger \chi_p e^{-c t}|f(0)\rangle \), we can differentiate \( X(p) \) and \( Y(p) \) with respect to time and use the commutation relations to produce the following equations:

\[
\frac{\partial X}{\partial t} + \frac{\partial X}{\partial p} = -\mu_m(p)X, \quad \frac{\partial Y}{\partial t} + \frac{\partial Y}{\partial p} = -\mu_f(p)Y,
\]

\[
X(0) = \lambda_m \int dp \beta(p, q)Z(p, q), \quad Y(0) = \lambda_f \int dp \beta(p, q)Z(p, q),
\]

where \( Z(p, q) = \langle 1|\chi_p^\dagger \chi_q^\dagger \psi_p \chi_q e^{-c t}|f(0)\rangle \). Although the main equations are simply McKendrick-von Foerster equations, the boundary conditions involve the correlation function \( Z(p, q) \). We can similarly obtain a PDE for \( Z(p, q) \), however, this will involve boundary conditions that require higher order correlations and a hierarchy of equations will result. The population dynamics of sexual reproduction is therefore non-trivial even in the mean-field setting.

9 Conclusions

In this work we have used age-structured population modelling to highlight Doi-Peliti methods of population inference. In particular, we have used a general formulation of coherent states that enables both continuous variables, as analysed by Doi \[2, 3\] and discrete variables, as analysed by Peliti \[17\] to be analysed in a single framework. We note that to do so involves abandoning the elegant functional representation used by Peliti \[17\], where states \( |q_n\rangle \equiv z^n \) representing population-size \( n \) are employed. When we have more general states such as \( |q_n\rangle \) to be analysed, it is not clear how one would extend such a functional representation, and more classical path integral techniques have to be employed.

A limitation of the Doi-Peliti approach would appear to be non-linear effects. For example, in \[1, 8\] it is possible to analyse birth and death rates, \( \beta_n(a) \) and \( \mu_n(a) \), as functions of both age \( a \) and population-size \( n \), although the resultant equations are difficult to solve. The second quantization approaches we have introduced model interactions at the local level; adapting these techniques to birth and death rates that are functions of population-size will likely involve a model where all individuals are simultaneously interacting, an approach somewhat beyond the scale of the present study.

However, the framework we have introduced provides a compact machinery set that can efficiently deal with many age-structured models that become complicated when dealt with in a probabilistic manner \[1, 8\]. This machinery can be naturally developed in both perturbative and (two) path integral formulations. We have seen that the perturbative and path integral expansions result in summations over diagrams of birth-death processes. The results we have provided all have probabilistic derivations.

Going forward, the real challenge afforded by this framework is therefore to find alternative expansion or path integral techniques that will provide distinct methods of inference that are not obvious when taking more conventional probabilistic approaches.
Appendix A: Resolution of the Identity

We outline why Eq. 45 is equivalent to the identity operator. More information about using coherent states and resolutions of the identity can be found in [6]. The path integral in Eq. 45 is understood in the following sense:

\[ I = \lim_{P \to -\infty} \prod_{i=-P/\epsilon}^{P/\epsilon} \frac{\epsilon}{2\pi} \int d[u(p_i)] \ d[v(p_i)] e^{-i\epsilon \sum_{i=-P/\epsilon}^{P/\epsilon} u(p_i)v(p_i)} |iv\rangle \langle u|, \tag{87} \]

where age has been discretized over the interval [−P, P] such that \( p_i = i\epsilon \).

To interpret the coherent states \(|u\rangle\) and \(|iv\rangle\) we need to use a discretized form of fundamental states \(|p_n\rangle\). To do this we use an occupation number state representation. For example, suppose we have a coarse grain resolution of age so that there are five possible values, \( p_{-2}, p_{-1}, p_0, p_1 \) and \( p_2 \). Suppose, furthermore, that we have a discretized state \(|p_4\rangle = |p_0, p_{-2}, p_0, p_1\rangle\) comprised of the ages of four individuals. Then we write \(|p_4\rangle \equiv |1, 0, 2, 1, 0\rangle_0\), where the subscript 0 indicates an occupation number state representation. For example, two of the four individuals have middle age \( p_0 \), and so the middle occupation number is 2 in the occupation state. Then we expand and descretize a general coherent state \(|g\rangle\) as:

\[ |g\rangle = e^{\int dp\ g(p)\psi(p)\phi} = e^{\sum g(p_i)\psi(p_i)|\phi\rangle} = \sum_{n=0}^{\infty} \frac{\epsilon^n}{n!} \sum_{\sum_i x_i = n} \left( \frac{n}{x!} \right)^P \prod_{i=-P/\epsilon}^{P/\epsilon} g(p_i)^{x_i} |x\rangle_0, \tag{88} \]

where we use the notation \( \left( \frac{n}{x!} \right)^P \equiv \frac{n!}{x!^{P/\epsilon}} \cdots \frac{n!}{x!^{P/\epsilon}} \). Then if we similarly define \( x! \equiv x_{-P/\epsilon}! \cdots x_{P/\epsilon}! \) and use Eq. 88 to obtain a discretized version of \(|iv\rangle \langle u|\), Eq. 87 can be written as

\[ I = \lim_{P \to -\infty} \sum_{m,n=0}^{\infty} \epsilon^{m+n} \sum_{\{x_i\}} \frac{1}{x! y!} \left( \frac{n}{x!} \right)^P \prod_{i=-P/\epsilon}^{P/\epsilon} \left[ \int \frac{\epsilon}{2\pi} d[u(p_i)] \ d[v(p_i)] e^{-i\epsilon u(p_i)v(p_i)} u(p_i)^x_i (iv(p))^y_i \right] |y\rangle_0 \langle x|_0. \tag{89} \]

Now, integration by parts establishes the following identity [7, 17]:

\[ \int \frac{du \ dv}{2\pi} e^{-iuv} u^m (iv)^n = \int du \ u^m \left( -\frac{\partial}{\partial u} \right)^n \delta(u) = m! \delta_{mn}, \tag{90} \]

which can be used to simplify Eq. 89, giving

\[ I = \lim_{P \to -\infty} \sum_{m=0}^{\infty} \epsilon^m \sum_{\{x_i\}} \frac{1}{x!} |x\rangle_0 \langle x|_0. \tag{91} \]

Each occupancy vector \(|x\rangle_0\) with total occupation number \( m \) corresponds to \( \left( \frac{m}{x!} \right) \) possible states \(|p_m\rangle\). After taking the continuum limit we find

\[ I = \sum_{m=0}^{\infty} \int \frac{dp_m}{m!} |p_m\rangle \langle p_m|. \tag{92} \]

Thus we have recovered Eq. 3; a standard resolution of the identity, as required.
Appendix B: Alternative Path Integral Formulation

The fundamental property that enables the path integral formulation given in Eq. 51 is the resolution of the identity described in Appendix A. This relies on the fundamental integral representation of the Kronecker delta function given in Eq. 90, which was also the formulation used by Peliti. However, there also exists the following expression, which can be established by converting to polar coordinates and integrating the subsequent gamma distribution:

$$\int \frac{dz\,d\tau}{2\pi i} e^{-\tau z^m z^m} = m!\delta_{mn}. \quad (93)$$

In a manner largely identical to that in Appendix A, this can be used to construct a resolution of the identity more commonly seen in quantum mechanics:

$$I = \int \mathcal{D}f\mathcal{D}\bar{f} e^{-\int dp\,\bar{f}(p)f(p)} \langle f \rangle \langle f \rangle. \quad (94)$$

In much the same way as the derivation of Eqs. 47 - 49, we can use eigenstate properties to determine the action of operators $\zeta_0$, $\zeta_0$ and $\zeta_d$ upon coherent states $|f \rangle$ and $|g \rangle$, where $f$ and $g$ are possibly complex functions, to give the following:

$$\langle f|\zeta_0|g \rangle = \langle f|g \rangle \int dp\,\bar{f}(p) \frac{\partial g}{\partial p}(p), \quad (95)$$

$$\langle f|\zeta_0|g \rangle = \langle f|g \rangle (1 - \bar{f}(0)) \int dp\,\beta(p)\bar{f}(p)g(p), \quad (96)$$

$$\langle f|\zeta_d|g \rangle = \langle f|g \rangle \int dp\,\mu(p)g(p)(\bar{f}(p) - 1). \quad (97)$$

We then use these to construct the incremental term

$$\langle f|e^{-\zeta} |g \rangle = \exp\left\{-\epsilon \int dp\left[ -\frac{\bar{f}g}{\epsilon} + \bar{f}\frac{\partial g}{\partial p} + (1 - \bar{f}(0))\beta \bar{f}g + \mu g(\bar{f} - 1) \right]\right\}, \quad (98)$$

and thus arrive at following path integral formulation:

$$\langle f_T|e^{-\zeta T}|f_0 \rangle = \int_{f(p,T) = \bar{f}(p)} \mathcal{D}f\mathcal{D}\bar{f} \exp\left\{- \int dpdt \left[ \bar{f}(f_t + f_p) + \beta(1 - \bar{f}(0,t))f + \mu f(\bar{f} - 1) \right]\right\} \cdot \exp\left\{ \int dp\bar{f}(p,T)f(p,T) \right\}. \quad (99)$$

This provides an alternative path integral formulation to Eq. 51. We have not explored this representation any further.

Appendix C: Path Integral Calculation

We now provide an outline of the path integral calculation for Eq. 57. This is an adaptation of the path integral techniques found in [14]. If we expand the rightmost exponential in Eq. 57 containing a birth and death term, we obtain terms in the form of the following path integral, which we can write as a functional derivative of a generating functional

$$\int \mathcal{D}u\mathcal{D}v \prod_{i=1}^{m} u_i(p_i) \prod_{j=1}^{n} v_j(q_j) \exp \left\{ -i \int dpdt u \left( \frac{\partial v}{\partial t} + \frac{\partial v}{\partial p} + \gamma v \right) - i \int dp u \bar{v}_0 \right\} \cdot \left[ (-i)^m \partial^{m+n} j=0 \right]$$

where we have the generating functional
\[ I(j, k) = \int D\alpha D\beta \exp \left\{ -i \int dq dt \left( \frac{\partial \alpha}{\partial t} + \frac{\partial \beta}{\partial p} + \gamma \beta \right) - i \int dp u_0 v_0 + i \int dp dt [uj + vk] \right\}. \] (101)

Next we make the following ‘space-time’ transformation, where \((q, s) = (p - t, t)\) and define variables \(U(q, s) = u(p, t), V(q, s) = v(p, t), J(q, s) = j(p, t)\) and \(K(q, s) = k(p, t)\). We can thus recast the generating functional in the following form:

\[ I(j, k) = \int D\alpha D\beta \exp \left\{ -i \int dq ds U(q, s)D(q, s, s')V(q, s') - i \int dq U_0 V_0 + i \int dq ds [UJ + VK] \right\}, \] (102)

where the operator \(D(q, s, s') \equiv \delta(s - s') \left[ \frac{\partial}{\partial s} + \gamma(q + s) \right]\) has an inverse \(D^{-1}(q, s, s') = \theta(s - s')e^{-\int_0^{s'} \gamma(q + x)dx}\) in the sense that \(\int ds' D(q, s', s')D^{-1}(q, s', s'') = \int ds' D^{-1}(q, s', s'')D(q, s', s'') = \delta(s - s'')\).

We then complete the square by making the transformations

\[ U(q, s) = \hat{U}(q, s) + \int ds' K(q, s')D^{-1}(q, s', s), \] (103)

\[ V(q, s) = \hat{V}(q, s) + \int ds' D^{-1}(q, s, s')J(q, s'), \] (104)

which gives us

\[ I(j, k) = \exp \left\{ i \int dq ds ds' K(q, s)D^{-1}(q, s, s')J(q, s') \right\}. \] (105)

\[ \int D\hat{U} D\hat{V} \exp \left\{ -i \int dq ds ds' \hat{U}(q, s)D(q, s, s')\hat{V}(q, s') - i \int dq \hat{U}_0 \hat{V}_0 \right\}. \]

\[ \exp \left\{ -i \int dq \hat{V}_0 \int ds e^{-\int_0^s \gamma(q + x)dx} \right\}. \] (106)

Then to remove the last exponential factor we make the substitution

\[ \hat{U}_0(q) \rightarrow \hat{U}_0(q) - \int ds e^{-\int_0^s \gamma(q + x)dx}. \] (107)

Then substituting for \(j\) and \(k\) we find

\[ I(j, k) = I(0, 0) \exp \left\{ i \int dq ds dq' ds' \delta(q - q')k(q + s, s)D^{-1}(q, s, s')j(q' + s', s') \right\}, \]

\[ I(0, 0) = \int D\hat{U} D\hat{V} \exp \left\{ -i \int dq ds \hat{U} \left( \frac{\partial \hat{V}}{\partial s} - \gamma(q + s)\hat{V} \right) - i \int dq \hat{U}_0 \hat{V}_0 \right\}. \] (108)

The integral of \(I(0, 0)\) over \(\hat{U}\) forces the constraint \(\frac{\partial \hat{V}}{\partial s} - \gamma(q + s)\hat{V} = 0\), resulting in a term of the form \(\delta(\hat{V}(q, s))\), which integrates over \(\hat{V}\) to give \(I(0, 0) = 1\). We are finally in a position to use Wick’s theorem and find that Eq. 100 reduces to a product of terms of the form

\[ \frac{-i\delta^2 I(j, k)}{\partial j(p, t) \partial k(p', t')} = \delta((p - p') - (t - t')) \theta(t - t') \exp \left\{ - \int_p^{p'} \gamma(x)dx \right\}. \] (109)

The delta function ensures that the change \(p - p'\) in an individuals age is equal to the time passed, \(t - t'\). If we integrate over the \(p\) variables arising in the expansion of the rightmost exponential term in Eq. 57, we are then left with precisely the propagators that were assigned to the Feynman diagram.
References

1. Chou, T., Greenman, C.D.: A hierarchical kinetic theory of birth, death, and fission in age-structured interacting populations. J. Stat. Phys. XXX (2016)
2. Doi, M.: Second quantization representation for classical many-particle system. Journal of Physics A: Mathematical and General 9(9), 1465 (1976)
3. Doi, M.: Stochastic theory of diffusion-controlled reaction. Journal of Physics A: Mathematical and General 9(9), 1479 (1976)
4. Feynman, R.P., Hibbs, A.R.: Quantum mechanics and path integrals, vol. 2. McGraw-Hill New York (1965)
5. von Foerster, H.: Some remarks on changing populations in The Kinetics of Cell Proliferation. Springer (1959)
6. Fradkin, E.: Field theories of condensed matter systems. 82. Addison Wesley Publishing Company (1991)
7. Gardiner, C.W.: Handbook of stochastic methods. Springer (1985)
8. Greenman, C.D., Chou, T.: Kinetic theory of age-structured stochastic birth-death processes. Physical Review E 93, 012,112 (2015)
9. Gurtin, M.E., MacCamy, R.C.: Non-linear age-dependent population dynamics. Arch. Rational Mech. Anal pp. 281–300 (1974)
10. Gurtin, M.E., MacCamy, R.C.: Some simple models for nonlinear age-dependent population dynamics. Math. Biosci 43, 199–211 (1979)
11. Iannelli, M.: Mathematical theory of age-structured population dynamics. Applied Mathematics Monographs. Giardini editori e stampatori (1995)
12. Keyfitz, B.L., Keyfitz, N.: The McKendrick partial differential equation and its uses in epidemiology and population study. Mathl. Comput. Modelling 26, 1–9 (1997)
13. Maggiore, M.: A modern introduction to quantum field theory. Oxford University Press (2004)
14. Mandl, F., Shaw, G.: Quantum field theory. John Wiley & Sons (2010)
15. McKendrick, A.G.: Applications of mathematics to medical problems. Proc. Edinburgh Math. Soc. 44, 98–130 (1926)
16. McQuarrie, D.A.: Statistical Mechanics. University Science Books (2000)
17. Peliti, L.: Renormalisation of fluctuation effects in the A + A → A reaction. Journal of Physics A: Mathematical and General 19(6), L365 (1986)
18. Peskin, M.E., Schroeder, D.V.: An introduction to quantum field theory. Westview (1995)
19. Srednicki, M.: Quantum field theory. Cambridge University Press (2007)
20. Webb, G.F.: Population models structured by age, size, and spatial position. In: P. Magal, S. Ruan (eds.) Structured population models in biology and epidemiology, pp. 1–49. Springer, Berlin, Heidelberg (2008)
21. Zanette, D.H.: A BBGKY hierarchy for the extended kinetic theory. Physica A 162, 414–426 (1990)