SIMPLICITY OF THE AUTOMORPHISM GROUPS OF ORDER
AND TOURNAMENT EXPANSIONS OF HOMOGENEOUS
STRUCTURES

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Abstract. We define the notions of a free fusion of structures and a weakly
stationary independence relation. We apply these notions to prove simplicity
for the automorphism groups of order and tournament expansions of homoge-
neous structures like the bounded Urysohn space, the random graph, and the
random poset.

1. Introduction

This paper contributes to the study of the automorphism groups of countable
structures. Such groups are natural examples of separable and completely metris-
able topological groups. The richness of their topological properties have recently
brought to light a crucial interplay between Fraïssé amalgamation theory and other
area of mathematics like topological dynamics, Ramsey theory, and ergodic theory.
(See [1] and [4].)

The program of understanding the normal subgroup structure of these groups
dates back at least to the ’50s, when Higman [3] proved that Aut(\(\mathbb{Q}, <\)), the group
of order-preserving permutations of the rational numbers, has very few normal
subgroups.¹

In recent years, Macpherson and Tent [7] proved simplicity for a large collection
of groups that arise in a similar fashion as automorphism groups of homogeneous
structures. Their methods encompass a number of examples that had been con-
sidered before by various authors: the random graph [11], the \(K_n\)-free graphs and
random tournaments [8], and many others. However, as the authors of [7] pointed
out, their framework does not apply to ordered or even partially ordered structures,
in particular not to the random posets whose automorphism group was proved to
be simple in [2].

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¹In fact, the only nontrivial normal subgroups of Aut(\(\mathbb{Q}, <\)) are the obvious ones: the one consisting
of those automorphisms that fix point-wise some interval \((a, \infty)\), the one consisting of those
automorphisms that fix point-wise some interval \((-\infty, b)\), and their intersection.
A few years later, Tent and Ziegler [9] introduced the notion of a stationary independence relation and investigated automorphism groups of structures allowing for such a relation. Their approach is very general: apart from recovering the cases from [7] it applies to the random poset and many homogeneous metric structures like the Urysohn space and its variations. However, ordered homogeneous structures like the ordered random graph and tournaments do not carry such a stationary independence relation.

In this paper we weaken the notion of a stationary independence relation from [9] to study the automorphism groups of many order and tournament expansions of structures arising naturally in Fraïssé amalgamation theory. We believe that such weakly stationary independence relations will also be useful in other expansions of homogeneous structures.

Before stating our main theorem, we introduce some terminology.

Definition 1.1. Let $L_i, i = 1, 2$, be disjoint relational languages and let $M_i, i = 1, 2$, countable homogeneous $L_i$-structures on the same universe $M$. We call an $L^* = L_1 \cup L_2$ structure $M^*$ on $M$ the free fusion of $M_1$ and $M_2$ if $M^* \upharpoonright L_i = M_i$, $i = 1, 2$, and

(*) for every non-algebraic $L_i$-type $p_i$ for $i = 1, 2$, their union $p_1 \cup p_2$ is realized in $M^*$.

For any $L^*$-type $p$ and $L \subset L^*$, we write $p_L$ for its restriction to $L$.

We are particularly interested in the following special cases:

I. **Order expansion** Let $L_1 = L$ be a relational language and $M = M_1$ be a homogeneous $L$-structure on a set $M$. Let $L_2 = \{<\}$ and $M_2 \cong \mathbb{Q}$ be a dense linear ordering on $M$. In this case we denote the free fusion of $M_1$ and $M_2$ by $M_<$ and call it an *order expansion* of $M$. Thus, an $L^*$-structure $M_<$ is an order expansion of $M$ if $<$ is a total order on $M$, $M_< \upharpoonright L_1 = M$, and $M_<$ satisfies the following property:

(*) For every non-algebraic 1-type $p_L$ over a finite set $A$, and every interval $(a, b) \subseteq M$, there is a realization of $p$ in $(a, b)$.

II. **Tournament expansion** Let $L_1$ be a relational language and $M = M_1$ be a homogeneous 1-structure on a set $M$. Let $L_2 = \{\to\}$ and $M_2$ be a random tournament on $M$. In this case we denote the free fusion of $M_1$ and $M_2$ by $M_\to$, and call it a *tournament expansion* of $M$. Thus, an $L^*$-structure $M_\to$ is a tournament expansion of $M$ if $\to$ is a tournament on $M$, $M_\to \upharpoonright L_1 = M$, and $M_\to$ satisfies the following property:

(*) For every non-algebraic 1-type $p_L$ over a finite set $X$, and two disjoint finite subsets $A, B \subseteq M$, there is a realization $x$ of $p$ such that $x \to a$ for all $a \in A$ and $b \to x$ for all $b \in B$.
Remark 1.2. Note that if $M_i, i = 1, 2$, is the Fraïssé limit of some $L_i$-class $C_i, i = 1, 2$, then a structure $M^*$ is the free fusion of $M_1$ and $M_2$ if and only if $M^*$ is the Fraïssé limit of the $L^*$-class $C^*$ where an $L^*$-structure $A^*$ is in $C^*$ if and only if $A \upharpoonright L_i \in C_i, i = 1, 2$.

Thus, a structure $M^*$ is an order expansion of a Fraïssé limit $M$ if and only if $M^*$ is the Fraïssé limit of the $L^*$-class $C^*$ where an $L^*$-structure $A^*$ is in $C^*$ if and only if $A \upharpoonright L \in C$. Equivalently, $C^*$ consists of all $A \in C$ expanded by all possible orderings. Similarly for the tournament expansion of a Fraïssé limit.

Our main theorem can now be stated as follows:

**Theorem 1.3.** Assume that $M$ is one of the following:

1. the Fraïssé limit of a free, transitive and nontrivial amalgamation class;
2. the bounded countable Urysohn space; or
3. the random poset.

If $M^*$ is an order expansion of $M$, then $G := \text{Aut}(M^*)$ is simple. The same holds if $M^*$ is a tournament expansion of (1) or (2).

Apart from the ordered bounded Urysohn spaces, Theorem 1.3 implies simplicity of the automorphism groups of various countable structures including the ordered random graph, the ordered random hypergraph, the ordered $K_n$-free graphs and their hypergraph analogues.

**Remark 1.4.**

1. To transfer the result from the countable bounded Urysohn space to the complete (and hence uncountable) one, one uses the exact same arguments as in [9] and [10].
2. In the same way as in [9], we can also conclude that for the ordered Urysohn space $U_\prec$, the quotient of $\text{Aut}(U_\prec)$ modulo the normal subgroup of automorphisms of bounded displacement is a simple group.

Theorem 1.3 is proved in two main steps: we first define a notion of moving maximally adapted to free fusion structures and prove:

**Theorem 1.5.** Let $M^*$ be the free fusion of a homogeneous $L_1$-structure $M_1$ carrying a stationary independence relation with an $L_2$-structure $M_2$. If $g \in G$ moves maximally, then any element of $G$ is the product of at most eight conjugates of $g$ and $g^{-1}$.

We will see below that Theorem 1.3 applies to any strictly increasing automorphism of $(\mathbb{Q}, \prec)$, see Remark 2.13 and to any automorphism $g$ of the random tournament such that $a \to g(a)$ for all $a$. Thus we obtain as a corollary:

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2We call an amalgamation class nontrivial, if its limit is not an indiscernible set and transitive if the automorphism group of its Fraïssé limit is transitive.
Corollary 1.6. If \( g \) is a strictly increasing automorphism of \((\mathbb{Q}, <)\) or an automorphism of the random tournament such that \( a \to g(a) \) for all \( a \), then any element of \( G \) is the product of at most eight conjugates of \( g \) and \( g^{-1} \).

In Section 4 we prove simplicity of \( G \) for the ordered random posets and complete the proof of Theorem 1.3 by proving:

Proposition 1.7. If \( M^* \) is an order or tournament expansion of a structure \( M \) as in Theorem 1.3 (1) or (2) and \( h \in G = \text{Aut}(M^*) \), then there is some \( g \in \langle h \rangle^G \) that moves maximally.

Clearly, Proposition 1.7 and Theorem 1.5 imply Theorem 1.3 for the cases (1) and (2).

2. Background and definitions

First we recall the definition of a stationary independence relation due to Tent and Ziegler.

Definition 2.1. [9, Definition 2.1] Let \( M \) be a countable structure with universe \( M \) and let \( \downarrow \) be a ternary relation between finite subset of \( M \). We say that \( \downarrow \) is a stationary independence relation on \( M \) if for all finite sets \( A, B, C, D \subseteq M \) the following hold:

(i) (Invariance) Whether \( A \) and \( B \) are independent over \( C \) depends only on the type of \( ABC \).
(ii) (Monotonicity) \( A \downarrow B CD \) implies that \( A \downarrow B C \) and \( A \downarrow BC D \).
(iii) (Transitivity) \( A \downarrow B C \) and \( A \downarrow BC D \) implies \( A \downarrow B D \).
(iv) (Symmetry) \( A \downarrow B C \) if and only if \( C \downarrow B A \).
(v) (Existence) If \( p \) is a type over \( B \) and \( C \) is a finite set, there is some \( a \) realizing \( p \) such that \( a \downarrow_B C \).
(vi) (Stationarity) If the tuples \( x \) and \( y \) have the same type over \( B \) and are both independent from \( C \) over \( B \), then \( x \) and \( y \) have the same type over \( BC \).

Remark 2.2. [9] Lemma 2.4] Note that by Transitivity and Monotonicity we have

\[
x_1x_2 \downarrow_{A;B} y_1y_2 \quad \text{if and only if} \quad \begin{bmatrix} x_1 \downarrow_{A;B} y_1 \quad \text{and} \quad x_2 \downarrow_{A_{x_1;B}y_1} y_2 \end{bmatrix}
\]

where \( A \downarrow_{C;D} B \) denotes the conjunction of \( AC \downarrow_D B \) and \( A \downarrow_{C;BD} \).

Remark 2.3. Recall from [9] that if \( M \) is the limit of a Fraïssé class of structures with free amalgamation, then \( M \) admits a stationary independence relation: define

\[3\] It was noted by several people that Transitivity follows from the other axioms. We include it here for convenience.
$A \downarrow_B C$ if and only if $ABC$ is isomorphic to the free amalgam of $A$ and $C$ over $B$, i.e. if and only if for every $n$-ary relation $R$ in $L$, if $d_1, \ldots, d_n$ is an $n$-tuple in $A \cup B \cup C$ with some $d_i \in A \setminus B$ and $d_j \in C \setminus B$, then $R(d_1, \ldots, d_n)$ does not hold. (See [9, Example 2.2].)

For the stationary independence relation on the (bounded) Urysohn space and other metric spaces, we put $A \downarrow_C B$ if and only if for all $a \in A, b \in B$ there is some $c \in C$ such that $d(a, b) = d(a, c) + d(c, b)$, and $A \downarrow B$ if and only if for all $a \in A, b \in B$ the distance $d(a, c)$ is maximal, see [9].

In the same vein, the random poset carries a natural stationary independence relation, namely $A \downarrow_C B$ if and only if $A \cap B \subset C$ and for all $a \in A \setminus C, b \in B \setminus C$ such that $a <_{po} b$ or $b <_{po} a$ there is some $c \in C$ such that $a <_{po} c <_{po} b$ or $b <_{po} c <_{po} a$, where we write $<_{po}$ for the partial order of the random poset (see [5] 4.2.1). Note that we have $A \downarrow B$ if and only if no element of $a$ is comparable in the partial order to any element of $B$, i.e. if for all $a \in A, b \in B$ we have $a \not<_{po} b$ and $b \not<_{po} a$.

**Definition 2.4.** Let $M^*$ be the free fusion of countable $L_i$-structures $M_i, i = 1, 2$, with universe $M$ and let $\downarrow$ be a ternary relation between finite subset of $M$. We say that $\downarrow$ is a weakly stationary independence relation on $M^*$ if it satisfies (Invariance), (Monotonicity), (Symmetry), (Existence) and

$(v')$ (Weak Stationarity) If $x$ and $y$ have the same $L^*$-type over $B$ and are both independent from $C$ over $B$, then $x$ and $y$ have the same $L_1$-type over $BC$. Thus, if furthermore $tp_{L_2}(x/BC) = tp_{L_2}(y/BC)$, then $x$ and $y$ have the same $L^*$-type over $BC$.

We first note the following:

**Proposition 2.5.** Let $M^*$ be the free fusion of a homogeneous $L_1$-structure $M_1$ carrying a stationary independence relation $\downarrow$ with an $L_2$-structure $M_2$. Then on $M^*$ the relation $\downarrow$ is a weakly stationary independence relation.

**Proof.** All properties except (Existence) follow immediately. To see that (Existence) holds for $L^*$-types, let $p$ be an $L^*$-type over a finite set $B$ and let $C$ be a finite set. By (Existence) for $L_1$-types there is a realization $a$ of $p_{L_1}$ with $a \downarrow_C B$. By $(*)$, there is a realization $b$ of $tp_{L_1}(a/BC)$ realizing $p_{L_2}$. Then $b$ realizes $p$ and $b \downarrow_B C$. \hfill $\square$

We first note the following adaptation from [9]:

**Lemma 2.6.** If $M^*$ is the free fusion of a homogeneous $L_1$-structure $M_1$ carrying a stationary independence relation $\downarrow$ with some $L_2$-structure $M_2$, then $G = Aut(M^*)$ has a dense conjugacy class.

**Proof.** Clearly, $G$ contains a dense conjugacy class if and only if for any finite tuples $\bar{x}, \bar{y}, \bar{a}, \bar{b}$ with $tp(\bar{x}) = tp(\bar{y})$ and $tp(\bar{a}) = tp(\bar{b})$ there are tuples $\bar{x}', \bar{y}'$ such that
Definition 2.7. Let $M$ be an $L$-structure, $L_2 \subseteq L$, and let $G = \text{Aut}(M)$. We say that $g \in G$ is $L_2$-homogeneous if for all $x \in M$, $A \subseteq M$ and $a \in \text{Fix}(A)$ we have

$$\text{tp}_{L_2}(g(x)/xA) = \text{tp}_{L_2}(g^a(x)/xA).$$

Remark 2.8. Note that if $L_2$ contains only binary relations, then $g \in G$ is $L_2$-homogeneous if and only if for all $x \in M$ and $h \in G$ we have

$$\text{tp}_{L_2}(g(x)/x) = \text{tp}_{L_2}(g^h(x)/x).$$

Example 2.9. Let $M^*$ be the free fusion of a homogeneous $L_1$-structure $M_1$ carrying a stationary independence relation with an $L_2$-structure $M_2$ and let $G = \text{Aut}(M^*)$.

1. If $M_2$ is the trivial structure, any $g \in G$ is $L_2$-homogeneous.
2. If $M_2$ is an order expansion of $M_1$, then $g \in G$ is $\prec$-homogeneous if and only if $g$ is strictly increasing or strictly decreasing.
3. If $M_2$ is a tournament expansion of $M_1$, then $g \in G$ is $\rightarrow$-homogeneous if and only if $a \rightarrow g(a)$ for all $a \in M$ or $g(a) \rightarrow a$ for all $a \in M$.
4. If $M_2$ is the random graph, then $g \in G$ is $E$-homogeneous if and only if $E(a,g(a))$ for all $a \in M$ or $\neg E(a,g(a))$ for all $a \in M$.

Note that the free fusion of two structures both having a stationary independence relation has again a stationary independence relation. Therefore, while all our methods will obviously transfer, we do not consider expansions by random graphs in this paper.

Definition 2.10. Let $M^*$ be the free fusion of a homogeneous $L_1$-structure $M_1$ carrying a stationary independence relation with an $L_2$-structure $M_2$ and let $G = \text{Aut}(M^*)$. We say that $g \in G$ moves maximally if

1. $g$ is $L_2$-homogeneous; and
2. every type over a finite set $X$ has a realization $x$ such that

$$x \Downarrow_{X:g(X)} g(x).$$

When $x$ is a realization as in (1), we say that $x$ is moved maximally by $g$.

Remark 2.11. Note that by Remark 2.7 for an automorphism of $M^*$ to move maximally it suffices that every 1-type over a finite set has a realization that is moved maximally.

We will frequently use the following refinement of the maximal moving condition:
Proposition 2.12. If \( g \in G \) is moving maximally, then for every \( n \)-type \( p \) over a finite set \( X \) and \( L_2 \)-type \( q_{L_2} \) over \( A \) such that \( q_{L_2} \) implies \( p_{L_2} \) there is some \( y \) realizing \( p \cup q_{L_2} \) which is moved maximally by \( g \).

Proof. Suppose that \( g \) moves maximally and consider a type \( p \) over a finite set \( X \) and an \( L_2 \)-type \( q_{L_2} \) over \( A \subset M^n \) such that \( q_{L_2} \) implies \( p_{L_2} \). Let \( y \) be a realization of \( p_{L_1} \) such that \( y \upharpoonright_X A \). By \( (*) \) we can choose a realisation \( z \) of \( tp_{L_1}(y/XA) \cup q_{L_2} \).

Thus \( z \upharpoonright_X A \). Since \( g \) moves maximally, there is a realization \( c \) of \( tp(z/XA) \) such that \( c \upharpoonright_{XA} g(XA) g(c) \). Furthermore, since \( c \upharpoonright_X A \) \( g(XA) g(c) \) and \( c \upharpoonright_X A \), we get \( c \downarrow_X g(X) g(c) \) by Transitivity. Similarly, \( g(c) \downarrow_{g(X)} g(A) \) and \( cXA \downarrow_{g(XA)} g(c) \) imply \( cX \downarrow_{g(X)} g(c) \) and hence we see that \( c \downarrow_{X;g(X)} g(c) \).

Thus \( c \) realizes \( p \cup q_{L_2} \) and is moved maximally by \( g \) over \( X \). \( \square \)

Remark 2.13. (1) Note that if \( g \) moves maximally, then clearly so do \( g^{-1} \) and all conjugates of \( g \).

(2) An indiscernible set \( M \) carries a stationary independence relation by setting \( A \) to be independent from \( B \) over \( C \) if \( A \cap B \subset C \). Any permutation of \( M \) with infinite support moves maximally with respect to this stationary independence relation (see [9]). Then \( M_{<} \) is isomorphic \((\mathbb{Q},<)\), and we conclude that given any strictly increasing automorphism \( g \) of \((\mathbb{Q},<)\), any \( h \in \text{Aut}(\mathbb{Q}) \) can be written as a product of at most eight conjugates of \( g \) and \( g^{-1} \).

Similarly, in this case the tournament expansion \( M_{<} \) is just the countable random tournament and we conclude that for any automorphism \( g \) such that \( a \rightarrow g(a) \) any \( h \in G \) can be written as a product of at most eight conjugates of \( g \) and \( g^{-1} \).

3. PROOF OF THE MAIN RESULT

In this section we prove Theorem 1.5 using the general strategy of [6] and [9]. Let \( M^* \) be as in the hypothesis of Theorem 1.5 and \( G = \text{Aut}(M^*) \). For \( A \subseteq M \), let \( \text{Fix}(A) \) denote the pointwise stabiliser of \( A \). Note that as usual for \( A, B \subseteq M \), we write \( AB \) for their union \( A \cup B \).

Lemma 3.1. (1) If \( A \downarrow_B C \) and \( D \) is arbitrary, then there is some \( D' \) such that \( tp(D'/BC) = tp(D/BC) \) and \( A \downarrow_B CD' \).

(2) Let \( A \downarrow_B C \) and \( g_1, \ldots, g_n \in G \). Then there is \( e \in \text{Fix}(BC) \) with \( A \downarrow_B C g_1^e(C) \cdots g_n^e(C) \).

Proof. The proof of [9] 3.1 works here as well since it does not use Stationarity. \( \square \)

Proposition 3.2. Consider \( g_1, \ldots, g_4 \in G \) and finite sets \( X_0, \ldots, X_4 \) such that \( g_i(X_{i-1}) = X_i \). Then for \( i = 1, \ldots, 4 \) there are \( a_i \in \text{Fix}(X_{i-1}X_i) \) and extensions \( X_i \subset Y_i \) such that
(1) \(g^{a_i}(Y_{i-1}) = Y_i\);
(2) \(Y_0 \perp_{Y_1} Y_2\) and \(Y_2 \perp_{Y_3} Y_4\).

Proof. The corresponding proof in [9] does not use Stationarity and hence works here without any modification. □

Proposition 3.3. Let \(g_1, \ldots, g_4 \in G\) move maximally and assume that \(g_2 = g_3^{-1}\). Let \(Y_0, \ldots, Y_4\) be finite sets such that \(g_i(Y_{i-1}) = Y_i\) for \(i = 1, \ldots, 4\). Assume also that \(Y_0 \perp_{Y_1} Y_2\) and \(Y_2 \perp_{Y_3} Y_4\). Let \(x_0\) and \(x_4\) be two tuples such that \(g_4g_3g_2g_1\) maps \(tp(x_0/Y_0)\) to \(tp(x_4/Y_4)\). Then for \(i = 1, \ldots, 4\), there are \(a_i \in \text{Fix}(Y_{i-1}Y_i)\) such that

\[g_4^{a_4} \cdots g_1^{a_1}(x_0) = x_4.\]

For the proof we need two lemmas:

Lemma 3.4. Let \(g \in G\) move maximally, let \(X, Y, C\) be finite sets such that \(g(X) = Y\) and \(X \perp_Y C\) and let \(x\) be a tuple. Then there is some \(a \in \text{Fix}(XY)\) such that

\[g^a(x) \perp_Y C.\]

Proof. The proof of [9. 3.5] goes through. □

Lemma 3.5. Let \(g \in G\) move maximally and let \(X, Y\) be finite sets with \(g(X) = Y\). Suppose we have tuples \(x = (x_0, \ldots, x_n), y = (x_0', \ldots, x_n')\) with \(g(tp(x/X)) = tp(y/Y)\) and \(tp_{L_2}(y_i/x_i) = tp_{L_2}(g(x_i)/x_i)\) for \(i = 0, \ldots, n\). and such that

\[x \perp_{X;Y} y.\]

Then there is some \(a \in \text{Fix}(XY)\) such that \(g^a(x) = y\).

Proof. Note that by Remark 2.22 it suffices to prove this for singletons \(x\) and \(y\) and use induction. By Proposition 2.12 we can choose a realisation \(x'\) of \(tp(x/X)\) which is moved maximally by \(g\) and such that \(tp_{L_2}(x'/XY) = tp_{L_2}(x/XY)\). Since \(x' \perp_X Y\), we have \(tp(x'/XY) = tp(x/XY)\) by Proposition 2.5. Choose \(a_1 \in \text{Fix}(XY)\) with \(a_1(x) = x'\). Then \(g^{a_1}\) moves \(x\) maximally over \(X\). Setting \(y' = g^{a_1}(x)\) we have \(tp(y'/XY) = tp(y/XY)\) and thus \(tp_{L_2}(y'/xXY) = tp_{L_2}(y'/xXY)\) as \(g\) is \(L_2\)-homogeneous. By Proposition 2.5 we conclude that \(tp(y'/xXY) = tp(y/xXY)\). Choose \(a_2 \in \text{Fix}(xXY)\) with \(a_2(y) = a_2(y')\). Then \(g^{a_1a_2}(x) = y\). □

Proof of Proposition 3.3. Two applications of Lemma 3.4 yield \(a_0 \in \text{Fix}(Y_0Y_1)\) and \(a_4 \in \text{Fix}(Y_3Y_4)\) such that for

\[x_1 = g_1^{a_1}(x_0)\]
\[x_3 = (g_4^{-1})^{a_4}(x_4)\]

we have

\[x_1 \perp_{Y_1} Y_2\] and \(Y_2 \perp_{Y_3} x_3.\]
Choose a realisation $x_2$ of the type $g_2(tp(x_1/Y_1)) = g_3^{-1}(tp(x_3/Y_3))$ (over $Y_2$) such that $x_2 \downarrow Y_2 x_1 Y_3 x_3$.

Since $g_2 = g_3^{-1}$ is $L_2$-homogeneous, we have $tp_{L_2}(x_2 x_1) = tp_{L_2}(x_2 x_3)$ and hence Lemma 3.5 yields $a_2 \in \text{Fix}(Y_1 Y_2)$ and $a_3 \in \text{Fix}(Y_2 Y_3)$ such that $g_2^{a_2}(x_1) = x_2 = (g_3^{-1})^{a_3}(x_3)$.

**Proposition 3.6.** Let $g_1, \ldots, g_4 \in G$ move maximally, $g_2 = g_3^{-1}$. Then, for any open set $U \subseteq G^4$, there is some open set $W \subseteq G$ such that the image $\phi(U)$ under the map $\phi: G^4 \to G: (h_1, \ldots, h_4) \mapsto g_4^{h_4} g_3^{h_3} g_2^{h_2} g_1^{h_1}$ is dense in $W$.

**Proof.** Using Proposition 3.2 and Lemma 3.5, the proposition follows exactly as in [9, Prop. 2.13]. □

**Theorem 3.7.** If $g \in G$ moves maximally, then any element of $G$ is the product of at most eight conjugates of $g$ and $g^{-1}$.

**Proof.** We can use Proposition 3.6 and follow the proof in [9, Thm 2.7]. □

4. Obtaining automorphisms that move maximally

In this section we prove Proposition 1.7 assuming that $M$ is either the random poset, the Fraïssé limit of a nontrivial free amalgamation class or the bounded (countable) Urysohn space and that $M_<$ is an order expansion of $M$. The arguments for the tournament expansion $M_\to$ are very similar and we will give some details in Proposition 4.10 below.

To prove Proposition 1.7 for $M_<$, we construct a increasing automorphism $g \in \langle h \rangle^G$ moving maximally starting from any $h \in G$. This is done in three steps.

1. construct a fixed point free $h_1 = [h, f_1] \in \langle h \rangle^G$;
2. construct a strictly increasing $h_2 = [h_1, f_2] \in \langle h_1 \rangle^G$;
3. construct a strictly increasing $h_3 = [h_2, f_3] \in \langle h_1 \rangle^G$ moving maximally.

We take care of each of these steps in the next lemmas.

**Lemma 4.1.** (cp. [17, 3.4(ii)]) No element of $G \setminus \{\text{id}\}$ fixes any interval pointwise.

**Proof.** Suppose otherwise. Choose $h \in G \setminus \{1\}$ and $a, c, d \in M$ such that $h$ fixes $[c, d]$ pointwise and $a \neq h(a)$.

If $M_<$ is the Fraïssé limit of a free amalgamation class, then since $M$ is not an indiscernible set, by [17, Cor. 2.10] any non-algebraic $L$-type has a realization which is moved by $h$. By homogeneity, there exists a finite set $Y \subseteq M$ such that $\text{tp}(x_1/Y_1) = \text{tp}(x_3/Y_3)$ (over $Y_2$) such that $x_2 \downarrow Y_2 x_1 Y_3 x_3$.

Note that steps (2) and (3) could have been done in a single step. We separate them for the sake of readability.
\[\text{tp}_L(a/Y) \neq \text{tp}_L(h(a)/Y)\] as otherwise we could build an automorphism by back-and forth fixing all elements in \(M \setminus \{a, h(a)\}\) and taking \(a\) to \(h(a)\). By property (\(*\)), \(\text{tp}(Y/ah(a))\) is realized by some finite set \(B \subseteq (c, d)\). Since \(h\) fixes \(B\) pointwise, we see that \(\text{tp}(h(a)/B) = \text{tp}(a/B)\), a contradiction.

If \(M_\prec\) is the ordered bounded countable Urysohn space, then using (\(*\)) we can pick some \(x \in (c, d)\) with \(d(a, x) = 1\) and \(d(h(a), x) = 1/2\), contradicting the fact that \(h\) is an isometry fixing \(x\).

If \(M_\prec\) is the ordered random poset, assume \(a \not\leq_{\text{po}} h(a)\). Using (\(*\)) we can pick some \(x \in (c, d)\) with \(x <_{\text{po}} a\) and \(x \not<_{\text{po}} h(a)\). Then \(\text{tp}(h(a)/x) \neq \text{tp}(a/x)\), contradicting the assumption that \(h\) fixes \(x\).

\textbf{Corollary 4.2.} A nontrivial element of \(G\) does not fix the set of realizations \(D = p(M_\prec)\) of any non-algebraic \(L^*\)-type \(p\) over a finite set \(A\).

\textit{Proof.} We can assume that \(p = q \cup \{a < x < b\}\) for some \(a, b \in M \cup \{-\infty, \infty\}\) where \(q\) is a complete \(L\)-type over \(A\). Let \(D' = q(M)\), so \(D = D' \cap (a, b)\). By condition (\(*\)) \(D' \cap (a, b)\) is dense in \((a, b)\), so if \(h\) fixes \(D\) pointwise, then \(h\) is the identity on \((a, b)\) and hence \(h = \text{id}\) by Lemma 4.1.

\textbf{Lemma 4.3.} There is some \(g = [h, f] \in \langle h \rangle^G\) which is fixed point free.

\textit{Proof.} Using Corollary 4.2 this follows exactly as in Lemma 2.11 of [7]. Note that we can use condition (\(*\)) to ensure that \(f\) preserves the ordering.

To deal with order expansions, we need the following terminology:

\textbf{Definition 4.4.} An element \(g \in G\) is \textit{unboundedly increasing} (resp. \textit{decreasing}) if it is increasing (resp. decreasing) and for every \(a < b\) in \(M_\prec\) there is \(m \in \mathbb{N}\) such that \(g^m(a) > b\) (resp. \(g^m(b) < a\)).

The following observation will be helpful:

\textbf{Lemma 4.5.} If \(M_\prec\) is an ordered homogeneous structure and \(g \in G\) is unboundedly increasing, then we can identify \(M_\prec\) with the rationals \(\mathbb{Q}\) in such a way that \(g(x) = x + 1\) for all \(x \in M_\prec\).

\textit{Proof.} Pick some \(x_0 \in M_\prec\), identify the interval \([x_0, g(x_0)] \subseteq M_\prec\) in an order preserving way with \([0, 1] \subseteq \mathbb{Q}\) and extend.

Clearly, in the same way we can identify an unboundedly decreasing function on \(M_\prec\) with \(g(x) = x - 1\) for all \(x \in M_\prec\).

\textbf{Lemma 4.6.} If \(h \in G\) has no fixed point, then there is some \(g = [h, f] \in \langle h \rangle^G\) which is strictly increasing.

\textit{Proof.} Write \(M_\prec\) as an ordered union of intervals \(J_i, i \in I\), on which \(h\) is either strictly increasing or strictly decreasing. By further subdividing we may assume...
that on each interval \( J_i \) the automorphism \( h \) is either unboundedly increasing or unboundedly decreasing.

If \( h \) is unboundedly increasing on \( J_i \), then as in Lemma 4.5 we identify \( J_i \) with (a copy of) \( Q \) in such a way that we have \( h(x) = x + 1 \) for \( x \in J_i \). Similarly, if \( h \) is unboundedly decreasing on \( J_j \), we identify \( J_j \) with (a copy of) \( Q \) in such a way that we have \( h(x) = x - 1 \) for \( x \in J_j \).

Fix a positive \( \epsilon < \frac{1}{2} \). We construct an element \( f \in G \) by back and forth leaving each \( J_i \) invariant and such that if \( h \) is increasing on \( J_i \), then we choose \( f(x) \in \left[ \frac{x}{2}, \frac{x}{2} + \epsilon \right] \) for \( x \in J_i \), or, equivalently, \( f^{-1}(x) \in (2x - 2\epsilon, 2x] \). Similarly, if \( h \) is decreasing on \( J_i \), we choose \( f(x) \in [2x, 2x + \epsilon) \) for \( x \in J_i \), or, equivalently, \( f^{-1}(x) \in (\frac{x}{2} - \frac{\epsilon}{2}, \frac{x}{2}] \). Clearly, such an \( f \in G \) can be easily be constructed by back-and-forth thanks to condition (\ast). We then have \( h(f(x)) > f(h(x)) \), and so \([h, f](x) > x \) for any \( x \in M_< \). \( \square \)

**Lemma 4.7.** If \( M_< \) is as in Theorem 1 (1) or (2) and \( h \in G \) is strictly increasing, there is some \( g \in \langle h \rangle^G \) that moves maximally.

*Proof.* Since \( h \) is strictly increasing, we may identify \( M_< \) with an ordered sum of copies of \( Q \) in such a way that on each copy of \( Q \) we have \( h(x) = x + 1 \) for all \( x \in M_< \). Fix a positive \( \epsilon < \frac{1}{2} \).

**Case I:** If \( M_< \) is the ordered Fraïssé limit of a free amalgamation class, we define \( f \) by a back and forth construction like in [9] Lemma 5.1, with the additional requirement that on each copy of \( Q \) we have

\[
f(x) \in \left( \frac{x}{2} - \epsilon, \frac{x}{2} \right] \text{ for each } x \in M_<.
\]

Since this implies that \( f^{-1}(x) \in [2x, 2x + \epsilon) \), it follows that

\[
[h, f](x) > x + \delta
\]

with \( \delta = 2(\frac{1}{2} - \epsilon) > 0 \). Hence the commutator \([h, f]\) will again be strictly increasing.

So suppose that \( f' \) is already defined on a finite set \( A \) and let \( p \) be a type over a finite set \( X \). It suffices to show that \( f' \) has an extension \( f \) such that \([h, f]\) moves \( p \) maximally.

By possibly extending \( f' \) we can assume that \([h, f']\) is defined on \( X \) and that \( f'^{-1}h, f'(X) \subseteq A \). Now pick a realisation \( a \) of \( p \) independent from \( X' = U \cup h(X) \cup [h, f'](X) \) over \( X \) and such that \( h(a) \neq a \) which is possible by Corollary 4.2. Let \( B = h'(A) \) and pick a realisation \( b \) of \( f'(\text{tp}(a/A)) \) in such a way that \( b \perp_B h^{-1}(B) \) and \( b \in \left( \frac{a}{2} - \epsilon, \frac{a}{2} \right] \). Extend \( f' \) to \( Aa \) by setting \( f'(a) = b \). Next pick a realisation \( c \) of \( f'^{-1}(\text{tp}(h(b)/Bb)) \) such that \( c \) is independent from \( h(a)h(X) \) over \( Aa \), and \( c \in [2h(b), 2h(b) + \epsilon) \). Extend \( f' \) by setting \( f'(c) = h(b) \). Since weak stationary independence agrees with stationary independence on subsets of \( M_< \), the proof of Lemma 5.1 in [9] shows that \( a \perp_{X; h(X)} [h, f'](a) \).
**Case II:** Now suppose that $\mathcal{M}_<$ is the ordered bounded Urysohn space. If there is no $a \in \mathcal{M}_<$ with $d(a, h(a)) = 1$, then as in [10, Lemma 1.3] and using condition $(\ast)$ we construct some strictly increasing $h_1 \in \langle h \rangle^G$ as a product of conjugates of $h$ such that there is some $b \in \mathcal{M}_<$ with $d(b, h_1(b)) = 1$: let $0 < \epsilon < 1$ and $a \in \mathcal{M}_<$ such that $d(a, h(a)) = \epsilon$. Assume $h(a) > a$ (the other case being similar). Pick some $b \in (a, \infty)$ with $d(a, b) = 1$. Let $k > 1$ be such that $k \epsilon \geq 1$. Put $a_0 = a, a_k = b$ and, using $(\ast)$, pick $a_i \in (a_{i-1}, b)$ such that $d(a_{i-1}, a_i) = \epsilon, i = 1, \ldots, k$. Let $f_i \in G$ with $f_i(a_{i-1}, a_i) = (a_i, a_{i+1})$ and put $h_1 = h_{f_1} \cdots h_{f_k}$.

In the same way we can adapt [10, Lemma 2.4] to construct iterated commutators $[h, f]$ using $(\ast)$ to make sure that $f$ preserves the order and additionally satisfies

$$f(x) \in \left(\frac{x}{2} - \epsilon, \frac{x}{2}\right)$$

for each $x \in M$. Thus, after each step, the commutator will again be unboundedly increasing and we end up as in [10] Prop. 2.5 with an automorphism $g' \in \langle h \rangle^G$ which is unboundedly increasing and moves almost maximally, i.e. every type $p$ over a finite set $X$ has a realization $a$ such that $a \downarrow X \sim g'(a)$. An application of the previous argument as in [9] Lemma 5.3 then yields the required $g \in \langle h \rangle^G$ which is strictly increasing and moves maximally. This concludes the proof in the case of the ordered bounded Urysohn space and thus of Proposition [1.7] \hfill $\square$

Before turning to the ordered random poset, we first show the following:

**Lemma 4.8.** If $h \in G$ is strictly increasing, there is an unboundedly increasing $h' \in \langle h \rangle^G$.

**Proof.** Since $h$ is strictly increasing, we may identify $\mathcal{M}_<$ with a sum of copies of $Q$ such that on each copy we have $h(x) = x + 1$.

First assume that the sum of copies of $Q$ is infinite in both directions. Divide each copy of $Q$ into an ordered sum of two copies $Q_1 \cup Q_1$ (each $Q_1$ being again isomorphic to $Q$). Using assumption $(\ast)$, we define $f \in G$ by back-and-forth such that each half-copy of $Q$ is moved to the next one above: so $f(Q_1) = Q_2$ and $f(Q_2) = Q_1$ in the next copy of $Q$. If there is a first copy $Q_1 \cup Q_2$ we define $f$ in such a way that $f(Q_1) = Q_1 \cup Q_2$. And if there is a last copy $Q = Q_1 \cup Q_2$ we define $f$ in such a way that $f(Q_1 \cup Q_2) = Q_2$. Then $h' = h \cdot h'$ is unboundedly increasing. \hfill $\square$

To complete the proof of Theorem [1.3] in the case of order expansions, it is left to prove the following:

**Proposition 4.9.** If $\mathcal{M}^*$ is the ordered random poset, then $G$ is simple.

**Proof.** Let $h \in G$. By Lemmas [4.3, 4.6] and [4.8] we may assume that $h$ is unboundedly increasing in the sense of $\prec$. Now we can follow the steps of [2, Sec. 3] to
construct some $g \in \langle h \rangle^G$ which is unboundedly increasing in the sense of the partial order $<_\text{po}$. Using property (∗) we can make sure that at each step the result is again unboundedly increasing in the sense of the order $<$. It is easy to see using (∗) that any two elements of $G$ that are unboundedly increasing both in the sense of $<_\text{po}$ and in the sense of $<$ are conjugate. Adapting the proof of [2] Lemma 3.4] using (∗), any element $f$ of $G$ can be written as a product $f = g_1^{-1}g_2$ with $g_1,g_2$ unboundedly increasing in the sense of $<_\text{po}$ and $<$. Thus $G$ is simple also in this case.

**Proposition 4.10.** Assume that $M$ is the Fraïssé limit of a nontrivial free amalgamation class or the bounded (countable) Urysohn space and $M \rightarrow$ is a tournament expansion of $M$. For any $h \in G$, there is some $g \in \langle h \rangle^G$ moving maximally.

**Proof.** It is easy to see as in Corollary [4, 3.4(ii)] that a nontrivial element of $G$ does not fix pointwise the set of realizations of any nonalgebraic type over a finite set. Hence as in Lemma [4, (Case I) we can follow the construction of [9, 5.1] to construct an element $f \in G$ such that $g = [h,f]$ moves maximally in the sense of the stationary independence relation. By the axioms of the random tournament, we can construct $f$ in a way to ensure that $a \to [h,f](a)$ for all $a \in M$. Thus, $g = [h,f]$ moves maximally in the sense of a free fusion structure. \qed

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