Quantum Phases of a Vortex String

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We argue that the world-sheet dynamics of magnetic $k$-strings in the Higgs phase of the mass-deformed $\mathcal{N} = 4$ theory, is controlled by a bosonic $O(3)$ sigma model with anisotropy and a topological $\theta$ term. The theory interpolates between a massless $O(2)$ symmetric regime, a massive $O(3)$ symmetric phase and another massive phase with a spontaneously broken $\mathbb{Z}_2$ symmetry. The first two phases are separated by a Kosterlitz-Thouless transition. When $\theta = \pi$, the $O(3)$ symmetric phase flows to an interacting fixed point; sigma model kinks and their dyonic partners become degenerate, mirroring the behaviour of monopoles in the parent gauge theory. This leads to the identification of the kinks with monopoles confined on the string.

Introduction. – The formation and dynamics of colour flux tubes is of fundamental importance to the physics of gauge theories that exhibit confinement. The flux tubes or “QCD strings” have their own intrinsic dynamics and degrees of freedom. This raises an intriguing question, namely, what is the relation between the world-sheet degrees of freedom of the confining strings and the underlying gauge theory physics. This is a difficult problem in general, but can become tractable if the gauge theory has global symmetries that yield light, internal modes on the world-sheet with non-trivial dynamics. In certain examples with such internal symmetries and adequate amounts of supersymmetry, the connection between the supersymmetric dynamics of magnetic flux tubes or vortices, and their parent four dimensional field theories can be demonstrated beautifully [1]. In this letter we reveal a similar connection for a purely bosonic sigma model on a flux tube that resides in an $\mathcal{N} = 1$ supersymmetric gauge theory. The novel feature of our example is, in the absence of supersymmetry, the appearance of a rich quantum phase structure as a function of its parameters, some features of which reflect the 4D physics.

The gauge theory in question is a mass deformation of $\mathcal{N} = 4$ SUSY Yang-Mills, the so-called $\mathcal{N} = 1^*$ theory realized in its Higgs phase, and the vortex strings in this vacuum are the magnetic versions of confining $k$-strings. The $\mathcal{N} = 4$ SUSY gauge theory can be viewed as the theory of an $\mathcal{N} = 1$ SUSY vector multiplet $\mathcal{W}^{a}$, and three adjoint chiral multiplets $\Phi_{a}$, ($a = 1, 2, 3$). Suitable deformations of this theory can lead to rich infrared physics. One such deformation is the $\mathcal{N} = 1^*$ theory, corresponding to non-zero masses for the three chiral multiplets. The resulting $\mathcal{N} = 1$ theory, with gauge group $SU(N)$, has a large number of vacuum states, and classically these are in one-to-one correspondence with the partitions of $N$ into integers [2]. Of particular interest are vacua with a mass gap. Interestingly, every possible massive phase, including Higgs and confining, of an $SU(N)$ gauge theory with adjoint matter, is realized by one of the vacua. The $\mathcal{N} = 1^*$ theory has a classical superpotential,

$$W = \text{Tr} \left( \Phi_{1}[\Phi_{3}, \Phi_{3}] + \frac{3}{2} \sum_{a=1}^{3} m_{a} \Phi_{a}^{2} \right),$$

where the cubic term is the superpotential of the conformal $\mathcal{N} = 4$ theory. Classical ground states are determined by the F-term equations $\partial W / \partial \Phi_{a} = 0$, which are solved by $N$-dimensional representations of the $SU(2)$ algebra. The irreducible representation yields the “Higgs vacuum”

$$\Phi_{1,2} = i \sqrt{m_{1,2}} [J_{1,2}]_{N \times N}, \quad \Phi_{3} = i \sqrt{m_{3}} [J_{3}]_{N \times N},$$

where $J_{a}$ are the $N$-dimensional generators of $SU(2)$. The gauge symmetry is completely broken in this vacuum and magnetic degrees of freedom are confined. Amongst the many other massive vacua, there is also one with $\Phi_{a} = 0$, where the gauge group is completely unbroken and the quantum dynamics confines electric degrees of freedom. The different massive phases of the theory are exchanged and permuted by the action of the $SL(2, \mathbb{Z})$, Montonen-Olive duality group of the parent $\mathcal{N} = 4$ theory. In particular, the electric-magnetic duality or S-duality: $g_{YM} \rightarrow 1 / g_{YM}$, swaps the Higgs and confining phases. Consequently, for a fixed $N$, the physics of the confining vacuum at strong coupling $g_{YM} \gg 1$ is well described by the Higgs vacuum at weak coupling, $g_{YM} \ll 1$.

Vortices in $\mathcal{N} = 1^*$ theory. – The Higgs vacuum at weak coupling admits classical vortex solutions [3, 4]. These solutions carry a discrete magnetic flux, taking values in $\pi_{1}(SU(N) / \mathbb{Z}_{N}) \simeq \mathbb{Z}_{N}$. Solutions carrying $k$ units of $\mathbb{Z}_{N}$ flux, the so-called magnetic $k$-strings (S-dual to confining $k$-strings) were studied extensively in [4], specifically when the mass deformation parameters were equal: $m_{a} = m$. In this situation, the gauge theory has a global $O(3)$ symmetry, under which $\Phi_{a}$ transform as a triplet, which is broken by the VEVs [2] in the Higgs phase. However, a combination of global colour rotations and the broken $O(3)$ symmetry, leaves the Higgs VEVs invariant and a “colour-flavour locked” symmetry, $O(3)_{C+F}$ remains.

The configurations of interest have the adjoint scalars $\Phi_{a}$ approaching their VEVs in the Higgs vacuum asymptotically, along with a phase winding around the vortex. This phase rotation corresponds to a gauge transformation (at infinity) which is single-valued in $SU(N) / \mathbb{Z}_{N}$. 

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and is generated by
\[ Y_k = \frac{i}{k} \, \text{Diag}(k, \ldots, k, -(N-k), \ldots, -(N-k)), \] (2)
so that the resulting chromomagnetic flux is proportional to \( Y_k \) and \( \exp(i \, A_{\mu}^k |_{-\infty}) = \exp(2\pi i k/N) \). The flux picks out a specific direction in the colour-flavour space and the associated string is truly non-Abelian, as we illustrate with the example for \( SU(2) \) gauge group. For the \( SU(2) \) gauge theory, the \( \mathbb{Z}_2 \) vortex ansatz \([3]\) in singular gauge reads,
\[ \Phi_1 = im \, \psi(r) \, \tau_1; \quad \Phi_2 = im \, \psi(r) \, \tau_2; \quad \Phi_3 = im \kappa(r) \, \tau_3, \]
\[ \mathcal{A} = f(r) \, Y_1 \, \hat{\phi}, \quad Y_1 = \tau_3. \] (3)
Here \( \tau_a \) are generators of \( SU(2) \), and \( \psi, \kappa, f \) are the vortex profile functions which can be solved for numerically. Crucially, the solution preserves a \( U(1) \) subgroup of the \( O(3) \) gauge group, corresponding to rotations in the \( 1-2 \) plane.

A generic colour-flavour rotation will change the internal orientation of the non-Abelian flux and generate a family of such solutions. There is therefore an \( S^2 \simeq SO(3)/U(1) \) moduli space of solutions. This picture and the solutions can be generalized to the case of a general \( k \)-string for \( SU(N) \) gauge group \([3]\) and in all cases there is an \( S^2 \) moduli space of solutions.

**Vortex Sigma model.**—By making the internal zero modes depend slowly on the vortex world-sheet coordinates, and plugging in the associated ansatz into the gauge theory action, it is possible to systematically derive the effective 1 + 1-dimensional sigma model on the world-sheet \([2,4]\). The vortex solutions break all four supercharges of the theory, and therefore we do not expect fermionic internal zero modes (apart from the four generated by the broken supercharges); an explicit search supports this. We thus obtain a bosonic sigma model with an \( S^2 \simeq \mathbb{CP}^1 \) target space, which we may conveniently view as an \( O(3) \) nonlinear sigma model:
\[ \mathcal{L}_{\sigma} = \frac{1}{2g^2} (\partial_\alpha \vec{n})^2 + \frac{\theta}{4e} \, e^{\alpha \beta} \vec{n} \cdot \left( \partial_\alpha \vec{n} \times \partial_\beta \vec{n} \right); \quad \vec{n} \cdot \vec{n} = 1; \quad \theta = k(N-k) \theta_{YM}. \] (4)
Here \( \vec{n} \equiv (n_1, n_2, n_3) \) is an \( O(3) \) unit vector parametrizing the internal orientation of the non-Abelian flux. \( \theta_{YM} \) is the vacuum angle of the 4D gauge theory, and the bare coupling \( g_{\sigma} = g_{YM}/C_{k,N} \ll 1 \) with \( C_{k,N} \) a numerically determined function of \( k \) and \( N \).

The \( O(3) \) sigma model has instantons and the associated topological \( \theta \)-term is fed in from the the gauge theory vacuum angle through the simple but non-trivial relation above. This relation has been obtained explicitly by using both semiclassical methods and the large-N gravity dual picture of \( \mathcal{N} = 1^* \) theory \([4]\). The sigma model description of the dynamics is valid on length scales larger than the vortex thickness which can be estimated to be \( \Lambda^{-1} \sim (m_\sqrt{\mathcal{N}})^{-1} \). (This follows from the connection between \( \mathcal{N} = 1^* \) vortices and noncommutative instantons on \( S^2 \times \mathbb{R}^2 \).)

It is well known that the \( O(3) \) model is asymptotically free, and for generic \( \theta \), it has a mass gap. The dynamical scale of the theory is \( \Lambda_{\sigma} \sim \Lambda \exp(-2\pi / g^2_{\sigma}) \), and its spectrum consists of a single massive triplet. When \( \theta = \pi \) however, the theory is known to flow to a \( c = 1 \) conformal fixed point described by the \( SU(2) \) Wess-Zumino-Witten model at level one \([4]\).

We now wish to consider what happens to the sigma model when we move away from the \( O(3) \)-symmetric limit. In the four dimensional gauge theory it is natural to consider the case where \( m_1 = m_2 = m \) whilst \( m_3 \neq m \). Then the \( O(3) \) global symmetry is explicitly broken to \( O(2) \), corresponding to rotations in the \( \Phi_1-\Phi_2 \) plane. When \( m_3 \ll m \), the theory can be viewed as softly broken \( \mathcal{N} = 2^* \) gauge theory and in the opposite regime \( m_3 \gg m \) it flows toward an \( \mathcal{N} = 1 \) superconformal field theory with two adjoint chiral multiplets and a quartic superpotential.

From the point of view of the \( k \)-string sigma model, it makes sense to consider only small deviations from the \( O(3) \) symmetric situation, so that it is still meaningful to regard the internal orientation as a quasi-modulus. To this end we introduce the deviation \( \Delta \equiv m_3^2 - m^2 \). As long as \( |\Delta/m^2| \ll 1 \), the \( O(3) \) breaking will manifest itself as a deformation of the sigma model above. We can then explicitly compute this deformation potential at linear order in \( \Delta \). At this lowest order, it is consistent to take the unmodified vortex profiles (e.g. \([3]\) for \( SU(2) \)), perform a generic colour-flavor rotation and substitute into the 4D action with \( m_3 \neq m \) to obtain the effective deformed sigma model in 2D, and we find
\[ \mathcal{L}_{\sigma} \rightarrow \mathcal{L}_{\sigma} - A_{k,N} \Delta (n_3)^2, \] (5)
where \( A_{k,N} > 0 \), is a constant that can only be determined numerically for each \( k \) and \( N \). At higher order in \( \Delta \), the vortex solution itself will be modified and the potential will be complicated. However, the lowest order contribution is already interesting. The key point here is that, depending on the sign (and magnitude) of \( \Delta \), the sigma model is in one of three possible phases. Let us discuss these in succession. Classically, when \( \Delta < 0 \), we expect that \( n_3 = \pm 1 \) are the vacua, while for \( \Delta > 0 \), the equator of the target sphere becomes the vacuum manifold. This picture is confirmed in Fig.1, by computing numerically, the tensions of the exact vortex solutions oriented along two different directions.

\( \Delta > \Delta_{\text{c}}^2 \): When \( \Delta \) is much larger than the dynamical scale \( \Lambda_{\sigma}^2 \) of the undeformed \( O(3) \) theory (still ensuring \( \Delta/m^2 \ll 1 \) or \( m_3 \gg m \)), the effective coupling \( g_{\sigma}(|\Delta|) \) is weak and the classical potential forces \( n_3 \) to vanish, keeping the orientation in the 1-2 plane. The resulting \( O(2) \) symmetry is not broken due to Coleman’s theorem, but there is a massless free boson which is the angular degree of freedom. This model contains vortex-instantons
which are suppressed and dilute for large $\Delta$, but as $\Delta$ is decreased, and the effective coupling $g_\sigma$ increases, they become important. The “vortices inside the vortex” come in two varieties. One that circulates around the equator at infinity and moves off at its core to the north pole, while the second kind moves to the south pole at the core. The topological charges of these two types of vortex instantons are $\pm 1/2$ and so are the merons of the $O(3)$ model. As $\Delta$ is decreased so that $\Delta \sim \Lambda^2_\sigma$, there will be a critical value of the effective coupling $g_\sigma(|\Delta|)$, at which the vortices will condense, following a Kosterlitz-Thouless transition. Thus, a mass gap is generated by meron condensation and the theory enters the massive $O(3)$ symmetric regime. At $\theta = \pi$, this mechanism fails and the model remains massless due to a cancellation between merons of positive and negative topological, meron charge $z$. The Coulomb gas of the two kinds of vortices can be mapped to a sine-Gordon model for general $\theta$, with the action $\mathcal{L} \sim g_\sigma^2 (\partial_\phi)^2/2 - 2\zeta \cos \theta \cos(\phi)$ ($\zeta$ is the vortex fugacity). At $\theta = \pi$ this theory is massless.

$\Delta < 0$ and $|\Delta| > \Lambda^2_\sigma$: When $m_3 < m$, the parameter $\Delta$ is negative and the sigma model potential has two discrete minima at $n_3 = \pm 1$ corresponding to the north and south poles respectively. These two degenerate vacua, are clearly the ground states as long as $|\Delta| \gg \Lambda^2_\sigma$ and the sigma model coupling is weak. A choice of vacuum spontaneously breaks the $\mathbb{Z}_2$ symmetry under $n_3 \rightarrow -n_3$. In this semiclassical regime the spectrum consists of massive perturbative excitations and kinks that interpolate between the two vacua. Since the model still has a $U(1)$ symmetry generated by rotations in the 1-2 plane, the kink (and anti-kink) solutions have a one-parameter degeneracy corresponding to this internal rotation angle. One can then have solutions where this internal collective coordinate is time-dependent and kinks rotate around the $z$-axis. Bohr-Sommerfeld quantization of the semiclassical solution implies that the associated conserved charge is quantized and the sigma model kinks are “dyonic” $z$. The semiclassical kinks (anti-kinks) can be labelled by the topological kink number $T = \pm 1$ (-1), and the global $U(1)$ charge $S$. The mass of the $(S,T)$ kink is $\tilde{M}^2_{S,T} = A_{k,N} \Delta \left[ \frac{T^2}{4g_\sigma^2} + (S + \frac{\theta}{2\pi} T)^2 \right]$. (6)

The formula incorporates a 2D version of the Witten effect, whereby a non-zero vacuum angle induces a $U(1)$ charge, $T \frac{\partial}{\partial T}$, for the kink. As $\theta$ is smoothly varied from zero to $2\pi$, the semiclassical kink spectrum undergoes a rearrangement. At $\theta = \pi$ there is a level crossing, and the $(S,+1)$ and $(-S -1,+1)$ states (and their charge conjugates) become degenerate.

The north and south pole vacua with $n_3 = \pm 1$ correspond to two different orientations of the non-Abelian magnetic flux. Taking the flux in the $n_3 = +1$ vacuum to be proportional to the matrix $Y_k$, we can obtain the the $k$-string flux in the second vacuum by performing a colour-flavour rotation in the 1-3 plane.

$$e^{i\pi J_2} Y_k e^{-i\pi J_2} = -Y_{N-k}. \quad (7)$$

Thus, in going from the north to the south pole, we do not change the $N$-ality of the string, but we interpolate between a $k$-string and an anti-$(N-k)$-string. The interpolating kink carries zero $N$-ality and is akin to a baryon vertex or a “gluelump” (Fig. 2).

FIG. 1: Tensions of the $n_3 = \pm 1$ vortex (round markers) and of the “equatorial” vortex (square markers) for different values of $m_3/m$ and $N_c = 2$.

FIG. 2: The kink interpolating between the two sigma model vacua pictured as a “gluelump”, and the level crossing as a function of $\theta$.

As $|\Delta|$ is decreased and approaches the dynamical scale $\Lambda^2_\sigma$, the two classical vacua above mix quantum mechanically, resulting in a single global ground state, and a second local minimum. The kinks and anti-kinks interpolate between these local minima, and are actually doublets of $SO(3) \simeq SU(2)$ $\frac{3}{2}$, and form a stable bound state transforming as a massive triplet. Thus we expect that the sigma model must undergo a phase transition in between the two massive regimes discussed above for $\Delta < 0$. One can verify the presence of such a phase transition easily, for the anisotropic $O(n)$ model in the large-$n$ limit. In addition, in the $O(3)$ model, at the special value of $\theta = \pi$, for small enough $|\Delta|$, we expect two degenerate vacua, with deconfined kinks and anti-kinks. At this point the model is massless and the exact S-matrix between the $SU(2)$ doublets is also known.

Based on the arguments above, we arrive at the phase diagram for the sigma model on the $k$-string in Fig. 3. It should be emphasized that above the K-T phase transition the L"uscher term $\frac{3}{2}$ associated to the (magnetic) confining string in $D = 4$ will jump from the value $(D - 2)/24$ to $(D - 1)/24$, due to the contribution of the extra $O(2)$ massless degree of freedom.

Kinks and Confined monopoles.– We will now attempt to establish a connection between the sigma model dynamics above and 4D gauge theory physics, by looking at the
FIG. 3: The phase plot of the $k$-string sigma model. The blue curve represents a K-T phase transition.

regime $\Delta < 0$ but with $m_3 \ll m$. We have seen that the physics of the sigma model can change dramatically at $\theta = \pi$. From (5), this corresponds to the gauge theory vacuum angle taking the values $\theta_{YM} = \pi/2(N - k)$. It is a priori not clear what is the significance of these values of $\theta_{YM}$. The answer to this question will also reveal the connection between the kinks above and confined monopoles, similar to the phenomena discovered in [10].

For $m_3 \ll m$, the $N = 1^*$ theory can be viewed as softly broken $N' = 2^*$ theory which is the theory of the $N = 2$ vector multiplet coupled to an adjoint hypermultiplet of mass $m$ [2]. The latter theory, with $m_3 = 0$, has a Coulomb branch moduli space of vacua where the $SU(N)$ gauge group is broken to $U(1)^{N-1}$. Singularities on the moduli space, where new light degrees of freedom appear and in particular, where the Seiberg-Witten curve undergoes maximal degeneration, descend to massive vacua of $N' = 1^*$ theory upon soft-breaking with $m_3 \neq 0$. (It should be emphasized that in this regime, the magnetic flux tubes are of the Abelian type and the sigma model description will be inapplicable). The Higgs vacuum singularity is located at $\Phi_3$, $\Phi_{1,2} = 0$. Denoting the diagonal elements of the $\Phi_3$ VEV at this point as $a = \{a_i\}$, $i = 1, \ldots, N$:

$$a = m \left[ \frac{(N-1)}{2}, \frac{(N-3)}{2}, \ldots, \frac{(N-3)}{2}, \frac{(N-1)}{2} \right].$$

(8)

At weak coupling $g_{YM} \ll 1$, in this vacuum, the $N' = 2^*$ theory has massless electric hypermultiplets, and a semiclassical spectrum consisting of a tower of BPS monopoles and dyons charged under the low-energy Abelian groups. The mass of a BPS state with magnetic and electric charges $(n_m, n_e)$ is given by

$$M = \sqrt{2} |a \cdot (n_m \tau + n_e) + m \sigma|, \quad \tau = \frac{\tilde{g}_{YM}}{2\pi} + \frac{Q_{YM}}{2\pi}.$$  

(9)

Crucially, the formula involves the charge $S$ for the global $U(1)$ symmetry of $N = 2^*$ theory which rotates $\Phi_1$ and $\Phi_2$ into each other. This descends to the global $U(1)$ symmetry visible in the vortex sigma model theory for $m_3 \ll m$. The BPS monopole in the $N' = 2^*$ theory, which carries the same magnetic charge as the sigma model kink, has (from (7))

$$n_m = Y_k - (-Y_{N-k}) = (1, \ldots, 1, 0, 0, \ldots, -1, -1),$$

and $n_e = 0$. The fermion zero modes of the monopole due to the matter multiplet $\Phi_{1,2}$ can be used to construct a multiplet with this magnetic charge and non-zero $S$ [11]. The mass of such a state, using (9), is $M_S = \sqrt{2m |k(N - k)\tau + S|}$.

It is clear from this formula that two such states with global charge $S$ and $-S - 1$, will become degenerate precisely when $k(N - k)\theta_{YM} = \pi$. But we also know that for $m_3 \neq 0$, in the vortex sigma model $\theta = k(N - k)\theta_{YM}$ and that the (dyonic) kinks with $U(1)$ charge $S$ and $-S - 1$ in the sigma model undergo a level crossing at $\theta = \pi$. Thus we are led to the conclusion that the BPS monopoles above are confined on the $k$-strings when a non-zero $m_3$ is introduced and then become identified with the kinks and their “dyonic” excitations.

Conclusions.— We have shown that the quantum dynamics of certain kinds of confining strings can exhibit novel phase transitions as parameters of the underlying gauge theory are varied. The Kosterlitz-Thouless transition from a massive to massless phase manifests itself as a jump in a physical observable, namely the Lüscher term. It would be interesting to find if such behaviour is generic to a wider class of field theories, and what it says about the underlying gauge theory.

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