One more proof of the Erdős–Turán inequality, and an error estimate in Wigner’s law.

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Erdős and Turán [3] have proved the following inequality, which is a quantitative form of Weyl’s equidistribution criterion.

Proposition 1 (Erdős – Turán). Let υ be a probability measure on the unit circle $\mathbb{T} = \mathbb{R}/2\pi \mathbb{Z}$. Then, for any $n_0 \geq 1$ and any arc $A \subset \mathbb{T}$,

$$\left| \nu(A) - \frac{\text{mes} A}{2\pi} \right| \leq K_1 \left\{ \frac{1}{n_0} + \sum_{n=1}^{n_0} \left| \hat{\nu}(n) \right| \right\},$$

where

$$\hat{\nu}(n) = \int_{\mathbb{T}} \exp(-in\theta) d\nu(\theta),$$

and $K_1 > 0$ is a universal constant.

A number of proofs have appeared since then, an especially elegant one given by Ganelius [5]. In most of the proofs, the indicator of $A$ is approximated by its convolution with an appropriate (Fejér-type) kernel. We shall present another proof, based on the arguments developed by Chebyshev, Markov, and Stieltjes to prove the Central Limit Theorem (see Akhiezer [1, Ch. 3]). In this approach, the indicator of $A$ is approximated from above and from below by certain interpolation polynomials. The argument does not use the group structure on $\mathbb{T}$, and thus works in a more general setting.

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In Section 1, we formulate a slightly different proposition and show that it implies Proposition 1. In Section 2 we reproduce the part of the arguments of Chebyshev, Markov, and Stieljes that we need for the sequel. For the convenience of the reader, we try to keep the exposition self-contained. In Section 3 we apply the construction of Section 2 to prove the Erdős–Turán inequality. In Section 4 we formulate another inequality that can be proved using the same construction. As an application to random matrices, we use an inequality from [4] and deduce a form of Wigner’s law with a reasonable error estimate.

1 Introduction

Let the measure $\sigma_1$ on $\mathbb{R}$ be defined by

$$d\sigma_1(x) = \frac{1}{\pi}(1 - x^2)^{-1/2} \, dx.$$ 

Let $T_n(\cos \theta) = \cos n\theta$ be the Chebyshev polynomials of the first kind; these are orthogonal with respect to $\sigma_1$. We shall prove the Erdős–Turán inequality in the following form:

**Proposition 2.** Let $\mu$ be a probability measure on $\mathbb{R}$. Then, for any $n_0 \geq 1$ and any $x_0 \in \mathbb{R}$,

$$\left| \mu[x_0, +\infty) - \sigma_1[x_0, +\infty) \right| \leq K_2 \left\{ \frac{1}{n_0} + \sum_{n=1}^{n_0} \frac{1}{n} \left| \int_{\mathbb{R}} T_n(x) \, d\mu(x) \right| \right\}. \quad (2)$$

Proposition 2 implies Proposition 1. Let $\nu$ be a measure on $\mathbb{T}$, and let $A \subset \mathbb{T}$ be an arc. Rotate $\mathbb{T}$ (together with $\nu$ and $A$) moving the center of $A$ to 0; this does not change the right-hand side of (1).

Denote $\nu_1(B) = \nu(B) + \nu(-B)$; $\nu_1$ is a measure on $[0, \pi]$. The change of variables $x = \cos \theta$ pushes it forward to $\mu_1$ on $[-1, 1]$. Now apply Proposition 2 to $\mu_1$, observing that

$$\int_{-1}^{1} T_n(x) \, d\mu_1(x) = \Re \hat{\nu}(n).$$

\[\square\]

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1 We do not assume that $\text{supp } \mu \subset [-1, 1]$
2 The Chebyshev–Markov–Stieltjes construction

Let $\sigma$ be a probability measure on $\mathbb{R}$ (with finite moments); let $S_0, S_1, \ldots$ be the orthogonal polynomials with respect to $\sigma$. For a probability measure $\mu$ on $\mathbb{R}$, denote

$$\varepsilon_n = \varepsilon_n(\mu) = \int_{\mathbb{R}} S_n(x) d\mu(x), \quad n = 1, 2, 3, \ldots$$

We shall estimate the distance between $\mu$ and $\sigma$ in terms of the numbers $\varepsilon_n$.

Let $x_1 < x_2 < \cdots < x_{n0}$ be the zeros of $S_{n0}$. Construct the polynomials $P, Q$ of degree $\leq 2n_0 - 2$, so that

$$P(x_k) = \begin{cases} 0, & 1 \leq k < k_0 \\ 1, & k_0 \leq k \leq n_0 \end{cases}; \quad P'(x_k) = 0 \quad \text{for} \quad k \neq k_0;$$

$$Q(x_k) = \begin{cases} 0, & 1 \leq k \leq k_0 \\ 1, & k_0 < k \leq n_0 \end{cases}; \quad Q'(x_k) = 0 \quad \text{for} \quad k \neq k_0 .$$

Lemma 3 (Chebyshev–Markov–Stieltjes).

$$P \geq 1_{[x_{k0}, +\infty)} \geq 1_{(x_{k0}, +\infty)} \geq Q .$$

Proof. Let us prove for example the first inequality. The derivative $P'$ of $P$ vanishes at $x_k$, $k \neq k_0$, and also at intermediate points $x_k < y_k < x_{k+1}$, $k \neq k_0, n_0$. The degree of $P'$ is at most $2n_0 - 3$, hence it has no more zeroes. Now, $P(x_{k0}) > P(x_{k0-1})$; hence $P$ is increasing on $(x_{k0-1}, y_{k0+1})$. Therefore $P'$ is decreasing on $(y_{k0+1}, y_{k0+2})$, increasing on $(x_{k0+2}, y_{k0+3})$, et cet. Thus $P(x) \geq 1$ for $x \geq x_{k0}$. Similarly, $P(x) \geq 0$ for $x < x_{k0}$.

Let $P = \sum_{n=0}^{n0} p_n S_n$, $Q = \sum_{n=0}^{n0} q_n S_n$. Then

$$\mu(x_{k0}, +\infty) \leq \int_{\mathbb{R}} P(x) d\mu(x) = p_0 + \sum_{n=1}^{2n_0-2} \varepsilon_n p_n$$

$$= q_0 + (p_0 - q_0) + \sum_{n=1}^{2n_0-2} \varepsilon_n p_n$$

$$\leq \sigma(x_{k0}, +\infty) + (p_0 - q_0) + \sum_{n=1}^{2n_0-2} |\varepsilon_n| p_n .$$
Similarly,

\[ \mu(x_{k_0}, +\infty) \geq \sigma(x_{k_0}, +\infty) - (p_0 - q_0) - \sum_{n=1}^{2n_0-2} |\varepsilon_n||q_n|. \]

Therefore

\[ |\mu(x_{k_0}, +\infty) - \sigma(x_{k_0}, +\infty)| \leq (p_0 - q_0) + \sum_{n=1}^{2n_0-2} |\varepsilon_n| \max(|p_n|, |q_n|). \] (3)

Thus we need to estimate \(p_0 - q_0, |p_n|, |q_n|\). This can be done using the following observation (which we have also used in [S].) Let \(R\) be the Lagrange interpolation polynomial of degree \(n_0 - 1\), defined by

\[ R(x_k) = \delta_{kk_0}, \ k = 1, 2, \ldots, n_0 . \]

Equivalently,

\[ R(x) = \frac{S_{n_0}(x)}{S_{n_0}'(x_{k_0})(x - x_{k_0})}. \] (4)

**Lemma 4.** \(P - Q = R^2\).

**Proof.** The polynomial \(P - Q\) has multiple zeroes at \(x_k, k \neq k_0\). Therefore \(R^2 \mid (P - Q)\). Also, \(\deg R^2 = 2n_0 - 2 \geq \deg(P - Q)\), and

\[ R^2(x_{k_0}) = 1 = P(x_{k_0}) - Q(x_{k_0}) . \]

Thus

\[ p_0 - q_0 = \int_{\mathbb{R}} R^2(x) d\sigma(x) \] (5)

and

\[ |p_n| = \left| \int_{\mathbb{R}} P(x)S_n(x) d\sigma(x) \right| \]

\[ \leq \left| \int_{x_{k_0}}^{\infty} S_n(x) d\sigma(x) \right| + \left| \int_{\mathbb{R}} (P(x) - 1_{[x_{k_0}, +\infty)}(x))S_n(x) d\sigma(x) \right| \]

\[ \leq \left| \int_{x_{k_0}}^{\infty} S_n(x) d\sigma(x) \right| + \int_{\mathbb{R}} R^2(x)|S_n(x)|d\sigma(x) . \] (6)
Similarly,
\[ |q_n| \leq | \int_{x_{k_0}}^{\infty} S_n(x) d\sigma(x) | + \int_{\mathbb{R}} R^2(x) |S_n(x)| d\sigma(x) . \]

3 Proof of Proposition 2

We apply the framework of Section 2 to \( \sigma = \sigma_1, S_n = T_n \). Let \( x_{k_0} = \cos \theta_0, 0 \leq \theta_0 \leq \pi/2 \). Then
\[ T_n'(\cos \theta_0) \cdot - \sin \theta_0 = n_0 \sin n \theta_0 , \]
and hence
\[ |T_n'(x_0)| = \frac{n_0}{|\sin \theta_0|} = \frac{n_0}{\sqrt{1 - x_{k_0}^2}} . \]

Thus, according to (5),
\[ p_0 - q_0 = \int_{\mathbb{R}} \frac{T_n'(x_0)^2}{T_n'(x_0)^2(x - x_0)^2} d\sigma_1(x) = \frac{\sin^2 \theta_0}{4 \pi n_0^2} \int_0^\pi \frac{\cos^2 n_0 \theta}{\sin^2 \frac{\theta + \theta_0}{2} \sin^2 \frac{\theta - \theta_0}{2}} d\theta . \]

Now,
\[ \int_0^{\theta_0/2} \leq \int_0^{\theta_0/2} C_1 d\theta / \theta_0^4 \leq C_1 / \theta_0^3 \leq C_2 n_0 / \theta_0^2 , \]
\[ \int_{\theta_0/2}^{\theta_0 - \pi / (3 n_0)} \leq C_3 \int_{\theta_0/2}^{\theta_0 - \pi / (3 n_0)} \frac{d\theta}{\theta_0^2 (\theta - \theta_0)^2} \leq \frac{C_4 n_0}{\theta_0^5} , \]
and similarly
\[ \int_{\theta_0 + \pi / (3 n_0)}^{\pi} \leq C_5 n_0 / \theta_0^2 . \]

Finally,
\[ |T_n'(\cos \theta)| = n_0 \left| \frac{\sin n_0 \theta}{\sin \theta} \right| \geq n_0 / (C_6 \theta_0) \geq |T_n'(\cos \theta_0)| / C_7 \]
for \( \theta - \theta_0 \leq \pi / (3 n_0) \), hence
\[ \int_{\theta_0 - \pi / (3 n_0)}^{\theta_0 + \pi / (3 n_0)} \frac{T_n'(\cos \theta)^2 d\theta}{T_n'(\cos \theta_0)^2 (\cos \theta - \cos \theta_0)^2} \leq C_8 / n_0 . \]
Therefore
\[ p_0 - q_0 \leq C/n_0. \] (7)

Next,
\[
\int_{x_{k_0}}^{\infty} T_n(x) d\sigma_1(x) = \int_0^{\theta_0} \cos n\theta \frac{d\theta}{\pi} = \frac{\sin n\theta_0}{n\pi};
\]
(8)
\[
\int_{\mathbb{R}} R^2(x)|T_n(x)|d\sigma_1(x) = \int_0^{\pi} \frac{\cos^2 n_0\theta}{\sin^2 \theta_0} (\cos \theta - \cos \theta_0)^2 \left| \cos n\theta \right| \frac{d\theta}{\pi} \leq C_1 \theta_0^3 \int_0^{\pi} \frac{\cos^2 n_0\theta}{\sin^2 \theta_0} \frac{d\theta}{\sin^2 \theta_0};
\]

Now,
\[
\int_0^{\theta_0/2} \leq C_2/\theta_0 \leq C_3 n_0/\theta_0^2;
\]
\[
\int_{\theta_0/2}^{\theta_0 - \pi/(3n_0)} \leq C_4 \int_{\theta_0/2}^{\theta_0 - \pi/(3n_0)} \frac{d\theta}{\theta_0^2 (\theta - \theta_0)^2} \leq C_5 n_0/\theta_0^2,
\]
and similarly
\[
\int_{\theta_0 + \pi/(3n_0)}^{\theta_0 - \pi/(3n_0)} \leq C_6 n_0/\theta_0^2;
\]
\[
\int_{\theta_0 - \pi/(3n_0)}^{\theta_0 + \pi/(3n_0)} \leq (C_7/n_0)(n_0^2/\theta_0^2) = C_7 n_0/\theta_0^2.
\]

Therefore
\[
\int_{\mathbb{R}} R^2(x)|T_n(x)|d\sigma_1(x) \leq C_8/n_0.
\] (9)
Combining (6), (8) and (9), we deduce:
\[ |p_n| \leq C/n. \] (10)
Similarly, \[ |q_n| \leq C/n. \]

Proof of Proposition 2. Substitute (7) and (10) into (3), taking
\[ m_0 = \lfloor n_0/2 \rfloor + 1 \]
instead of \( n_0 \). We deduce that (2) holds when \( x_0 = x_{k_0} \) is a non-negative zero of \( T_{m_0} \). By symmetry, a similar inequality holds for negative zeroes. For a general \( x_0 \in \mathbb{R} \), apply the inequality to the two zeroes of \( T_{m_0} \) that are adjacent to \( x_0 \) (one of them may formally be \( \pm \infty \)).
4 Another inequality, and an application to Wigner’s law

Let the measure $\sigma_2$ on $\mathbb{R}$ be defined by

$$d\sigma_2(x) = \frac{2}{\pi}(1 - x^2)^{1/2} \, dx.$$  

Let $U_n(\cos \theta) = \cos n\theta$ be the Chebyshev polynomials of the second kind; these are orthogonal with respect to $\sigma_2$.

**Proposition 5.** Let $\mu$ be a probability measure on $\mathbb{R}$. Then, for any $n_0 \geq 1$ and any $x_0 \in \mathbb{R}$,

$$|\mu[x_0, +\infty) - \sigma_2[x_0, +\infty)| \leq K_n \left\{ \frac{\rho(x_0; n_0)}{n_0} + \rho(x_0; n_0)^{1/2} \sum_{n=1}^{n_0} n^{-1} \left| \int_{\mathbb{R}} U_n(x) d\mu(x) \right| \right\},$$

where $\rho(x; n_0) = \max(1 - |x|, n_0^{-2})$.

Observe that $\rho \leq 1$. Similar inequalities with 1 instead of $\rho$ have been proved by Grabner [7] and Voit [9]. On the other hand, the dependence on $x$ in (11) is sharp, in the following sense: for any $x_0$, there exists a probability measure $\mu$ on $\mathbb{R}$ such that $\int_{\mathbb{R}} U_n(x) d\mu(x) = 0$ for $1 \leq n \leq n_0$, and

$$|\mu[x_0, +\infty) - \sigma_2[x_0, +\infty)| \geq C^{-1} \rho(x_0; n_0)/n_0,$$

where $C > 0$ is independent of $n_0$; cf. Akhiezer [1, Ch. 3].

The proof of Proposition 5 is parallel to that of Proposition 2; we apply the inequalities of Section 2 to the measure $\sigma_2$ and the polynomials $U_n$.

Grabner [7] and Voit [9] have applied their inequalities to estimate the cap discrepancy of a measure on the sphere. We present an application to random matrices.

Let $A$ be an $N \times N$ Hermitian random matrix, such that

1. $\{A_{uv} \mid 1 \leq u \leq v \leq N\}$ are independent,
2. $\mathbb{E}|A_{uv}|^{2k} \leq (Ck)^k$, $k = 1, 2, \ldots$;
3. the distribution of every $A_{uv}$ is symmetric, and $\mathbb{E}|A_{uv}|^2 = 1$ for $u \neq v$.  

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Let $\mu_A = N^{-1} \sum_{k=1}^{N} \delta_{\lambda_k(A)/(2\sqrt{N})}$ be the empirical measure of the eigenvalues of $A$ (which is a random measure). By [4, Theorem 1.5.3],

$$0 \leq \mathbb{E} \int_{\mathbb{R}} U_n(x) d\mu_A(x) \leq Cn/N , \quad 1 \leq n \leq N^{1/3}.$$ 

Applying Proposition 5, we deduce the following form of Wigner’s law:

**Proposition 6.** Under the assumptions 1.-3.,

$$\left| \mathbb{E} \# \left\{ k \mid \lambda_k > 2\sqrt{N}x_0 \right\} - N\sigma_2(x_0, +\infty) \right| \leq C \max \left( N^{2/3}(1 - |x_0|), 1 \right)$$

for any $x_0 \in \mathbb{R}$.

Better bounds are available for $x \in (-1 + \varepsilon, 1 - \varepsilon)$ (cf. Götzte and Tikhomirov [6], Erdős, Schlein, and Yau [2]). On the other hand, for $x$ very close to $\pm 1$, the right-hand side in our bound is of order $O(1)$, which is in some sense optimal.

**Remark 7.** A similar method allows to bound the variance of the number of eigenvalues on a half-line:

$$\mathbb{V} \# \left\{ k \mid \lambda_k > 2\sqrt{N}x_0 \right\} \leq C \max \left( N^{2/3}(1 - |x_0|), 1 \right)^{5/2};$$

therefore one can also bound the probability that $\# \left\{ k \mid \lambda_k > 2\sqrt{N}x_0 \right\}$ deviates from $N\sigma_2(x_0, +\infty)$.

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