ELECTROWEAK BARYON NUMBER VIOLATION

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Electroweak baryon number violation may play a crucial role for the creation of the matter-antimatter asymmetry in the early universe. In this talk, we review the basic mechanism, which relies on the behavior of chiral fermions in nontrivial Yang-Mills gauge field backgrounds.

1 Introduction

The conditions for baryogenesis in the early universe are well-known (cf. Refs. [1, 2]):
1. C and CP violation,
2. thermal nonequilibrium,
3. violation of baryon number conservation.

How realistic are these requirements? Well, noninvariance under the charge conjugation transformation (C) and the combined charge conjugation and parity reflection transformation (CP) have been observed in the laboratory. Also, thermal nonequilibrium can perhaps be expected for certain (brief) epochs in the history of the early universe, as described by the Hot Big Bang Model. But no experiment has ever seen baryon number violation, i.e., \( \Delta B \equiv B(t_{\text{out}}) - B(t_{\text{in}}) \neq 0 \).

Strictly speaking, we know of only one physical theory that is expected to display baryon number violation: the electroweak Standard Model. The problem is, however, that the relevant processes of the Standard Model are only known at relatively low scattering energies,

\[
E_{\text{center–of–mass}} \ll E_{\text{Sphaleron}} \approx M_W / \alpha \approx 10^4 \text{ GeV} ,
\]

and, worse, their cross-sections are negligible,

\[
\sigma_{\Delta B \neq 0} |_{\text{low–energy}} \propto \exp[-4\pi \sin^2 \theta_w / \alpha] \approx 0 ,
\]

with \( \theta_w \) the weak mixing angle (\( \sin^2 \theta_w \approx 1/4 \)) and \( \alpha \) the fine-structure constant (\( \alpha \approx 1/137 \)). Similarly, baryon-number-violating transition rates are negligible at low temperatures, \( T \ll T_c \), where \( T_c \approx 10^2 \text{ GeV} \) sets the scale of the electroweak phase transition.

Clearly, we should study electroweak baryon number violation for the conditions relevant to the early universe—that is, for high temperatures,

\[
T \gtrsim 10^2 \text{ GeV} .
\]

The problem is difficult but well-posed, at least within the context of the Standard Model.

In this contribution, we focus on the microscopic process of electroweak baryon number violation. This means that we must really deal with the fermions [3–9].

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Figure 1: Potential energy over a slice of configuration space, parameterized by the Chern–Simons number.

2 First steps

Consider chiral \( SU(2) \) Yang–Mills–Higgs theory with vanishing Yukawa couplings. Actually, forget about the Higgs field, which should be reasonable for temperatures above the electroweak phase transition. Natural units with \( c = h = k = 1 \) are used throughout.

Now recall the existence of the well-known triangle anomaly in the AAA-diagram, which occurs provided the VVV-diagram is anomaly-free [3]. Here, V and A indicate vector and axialvector vertices, respectively. (Note the obvious: the triangle anomaly is calculated with Feynman diagrams. In other words, the calculation is perturbative, with the interactions “turned off” in the asymptotic regions of spacetime; cf. Sec. 2 of Ref. [10]. The importance of this remark will become clear later on.)

The gauge vertices of the electroweak Standard Model are V–A and the corresponding current must be nonanomalous (gauge invariance is needed for unitarity). Instead, the fermion number current is anomalous [4] and the baryon (\( B \)) and lepton (\( L \)) number, for \( N_{\text{fam}} = 3 \) families of quarks and leptons, change as follows:

\[
\Delta(B - L) = 0 , \quad \Delta(B + L) = 2 N_{\text{fam}} \times \Delta N_{\text{CS}} .
\]

In the second equation, we have on the left-hand side the difference of certain fermion charges between the times \( t_{\text{in}} \) and \( t_{\text{out}} \), and on the right-hand side a characteristic of the gauge field background, also between \( t_{\text{in}} \) and \( t_{\text{out}} \). Specifically, this gauge field characteristic is

\[
\Delta N_{\text{CS}} \equiv N_{\text{CS}}(t_{\text{out}}) - N_{\text{CS}}(t_{\text{in}}) ,
\]

where the Chern–Simons number \( N_{\text{CS}}(t) \) is a particular functional of the \( SU(2) \) gauge field in the temporal gauge (\( A_0 = 0 \)) at time \( t \),

\[
N_{\text{CS}}(t) = N_{\text{CS}}[\vec{A}(\vec{x},t)] .
\]

The selection rule [4] shows that the fermion number \( B + L \) changes as long as the Chern–Simons number of the gauge field changes. But there is an energy barrier for transitions between gauge field vacua with different Chern–Simons number (Fig. 1). The top of this energy barrier corresponds to the Sphaleron configuration, which has \( N_{\text{CS}} = 1/2 \mod 1 \) (see Sec. 5 of Ref. [11]).

In a seminal paper [5], ’t Hooft calculated the amplitude for tunneling through the barrier. For this, he used the so-called BPST instanton, which is a finite-action solution of the imaginary-time Yang–Mills theory, i.e., the theory in Euclidean spacetime \( (M, g) = (\mathbb{R}^4, \delta_{\mu\nu}) \). The tunneling process has then

\[
\Delta N_{\text{CS}}[A_{\text{finite Euclidean action}}] = Q[A_{\text{finite Euclidean action}}] \in \mathbb{Z} ,
\]

where the topological charge \( Q \) corresponds to the winding number of a particular map

\[
S^3|_{|x|=\infty} \rightarrow SU(2) \sim S^3 .
\]
It is important to understand this last statement. The decisive observation is that any gauge field with finite Euclidean action becomes pure gauge towards infinity \(|x|^2 \equiv x_1^2 + x_2^2 + x_3^2 + x_4^2 \rightarrow \infty\), for \(x_\mu \in \mathbb{R}\)). Towards infinity, the Yang–Mills gauge field \(A_\mu(x)\) can then be written as \(-\partial_\mu g g^{-1}\), with \(g(\hat{x}) \in SU(2)\) for \(\hat{x}_\mu \equiv x_\mu / |x| \in S^3\). Hence, the gauge field at infinity is characterized by \(g(\hat{x})\), which corresponds to the map \(\mathbb{S}^3\). [Note that the \(SU(2)\) manifold has the topology of the three-dimensional sphere \(S^3\), because any group element \(g \in SU(2)\) can be written as \(g = \vec{n} \cdot i\vec{\sigma} + n_4 \mathbb{I}\), with \(|\vec{n}|^2 + n_4^2 = 1\).] The topological charge \(Q\), now, measures how many times \(g(\hat{x})\) wraps around \(SU(2)\) as \(\hat{x}\) ranges over the 3-sphere at infinity. This explains why the gauge-invariant topological charge \(Q\) of Eq. \(\mathbb{S}^3\) is an integer. Reference \([4]\), incidentally, gives the selection rule \(\mathbb{S}^3\) in the form \(\Delta(B + L) = 2N_{\text{fam}}\), at least for configurations with topological charge \(Q = 1\). For later use, we prefer to write the relation \(\mathbb{S}^3\) in terms of \(\Delta N_{\text{CS}}\).

The property \(\mathbb{S}^3\) holds only for transitions from vacuum to vacuum, as far as the Yang–Mills gauge field is concerned. Practically, this means that the result can only be relevant for processes \(\mathbb{S}^3\) at very low energies or temperatures. As mentioned above, the cross-section is then effectively zero by the tunneling factor \(\mathbb{S}^3\), but, at least, \(\Delta(B + L)\) is an integer.

3 Crucial question

For real-time processes at high energies, e.g. in Minkowski spacetime \((M, g) = (\mathbb{R}^4, \eta_{\mu\nu})\), the topological charge \(Q\) is, in general, a noninteger. The reason is that the energy density of a physical Yang–Mills gauge field (with a conserved nonzero total energy) is never exactly zero outside a bounded spacetime region; cf. Ref. \([12]\). The implication is, of course, that the expression for \(\Delta(B + L)\) can no longer just have 2\(N_{\text{fam}}\)\(Q\) on the right-hand side, as might be expected from the triangle anomaly [compare with Eqs. \(\mathbb{S}^3\) and \(\mathbb{S}^3\) above].

The question, then, is what does appear on the right-hand side,

\[
\Delta(B + L) \propto \text{which gauge field characteristic } ??
\]

As will be shown in Section \(\mathbb{S}^3\), the answer is fundamentally different for dissipative or nondissipative Yang–Mills gauge field solutions. Here, a gauge field is called dissipative if its energy density approaches zero uniformly as \(t \rightarrow \pm \infty\).

At this point, let us introduce some further terminology \(\mathbb{S}^3\). A spherically symmetric gauge field solution is called strongly dissipative, if both the (3+1)-dimensional and (1+1)-dimensional energy densities approach zero uniformly for large times \((t \rightarrow \pm \infty)\), and weakly dissipative, if the (3+1)-dimensional energy density dissipates with time but not the (1+1)-dimensional energy density. [Note that the (1+1)-dimensional energy density divided by a factor \(4\pi r^2\) corresponds to a spherically symmetric energy density in \(3 + 1\) dimensions.]

4 Spectral flow

It suffices for our calculations to consider \(SU(2)\) Yang–Mills theory with a single isodoublet of left-handed Weyl fermions. [A fully consistent \(SU(2)\) Yang–Mills theory requires an even number of chiral isodoublets \(\mathbb{S}^3\), which is the case for the electroweak Standard Model, with 3 isodoublets of left-handed quarks and 1 isodoublet of left-handed leptons per family. The fermion number \(B + L\) of the electroweak Standard Model follows then from the fermion number of our simplified model by multiplication with a factor of \((3 \times 1/3 + 1 \times 1) \times N_{\text{fam}} = 2N_{\text{fam}}\).]

Start from the eigenvalue equation of the corresponding time-dependent Dirac Hamiltonian,

\[
H(\vec{x}, t) \Psi(\vec{x}, t) = E(t) \Psi(\vec{x}, t).
\]

Then, the resulting spectral flow \(\mathcal{F}\) is related to the fermion number violation we are after. The definition of spectral flow is as follows: \(\mathcal{F}[t_{\text{out}}, t_{\text{in}}]\) is the number of eigenvalues of the
operator considered (here, the Dirac Hamiltonian) that cross zero from below minus the number of eigenvalues that cross zero from above, for the time interval \([t_{\text{in}}, t_{\text{out}}]\) with \(t_{\text{in}} < t_{\text{out}}\). See Fig. 2 for a sketch and Ref. [6] for references to the mathematical literature.

5 Old and new results on spectral flow

For \(SU(2)\) Yang–Mills theory with a single isodoublet of chiral fermions, the “crucial question” of Eq (9) can be rephrased as follows: precisely which gauge fields lead to nontrivial spectral flow of the Dirac eigenvalues?

The answer is known [6–8] for the case of strongly dissipative \(SU(2)\) gauge fields:

\[
\mathcal{F}[t_{\text{out}}, t_{\text{in}}] = \Delta N_{\text{winding}}[t_{\text{out}}, t_{\text{in}}] \\
\equiv N_{\text{CS}}[\vec{A}_{\text{associated vacuum}}(\vec{x}, +\infty)] - N_{\text{CS}}[\vec{A}_{\text{associated vacuum}}(\vec{x}, -\infty)],
\]

provided the time interval considered, \(\Delta t \equiv t_{\text{out}} - t_{\text{in}}\), is sufficiently large. Here, the “associated vacuum” at \(t = +\infty\) is the (zero-energy) vacuum configuration which the (finite-energy) gauge field would approach starting from \(t = t_{\text{out}}\) and similarly for the “associated vacuum” at \(t = -\infty\), starting from \(t = t_{\text{in}}\) but in the reversed direction. The right-hand side of Eq. (11) is then the difference of two integers, even though the relevant topological charge \(Q\) of the gauge field may be a noninteger.

For strongly dissipative \(SU(2)\) gauge fields in the electroweak Standard Model, the spectral flow result (11) reproduces the selection rule (4) with \(\Delta N_{\text{CS}}\) replaced by \(\Delta N_{\text{winding}}\). Note that \(\Delta(B - L)\) vanishes, because the left-handed quark and lepton isodoublets behave in the same way, namely with identical spectral flow as given by Eq. (11).

Returning to the simple \(SU(2)\) model with a single left-handed isodoublet, consider next the spherically symmetric gauge field solutions of Lüscher and Schechter (LS), which describe collapsing and re-expanding shells of energy [14]. For three particular cases of these analytic solutions (which are, in fact, “weakly dissipative”), the change of winding number and the spectral flow have been calculated explicitly [9],

\[
\begin{align*}
\text{LS case 1 (low energy)} : & \quad \Delta N_{\text{winding}} = 0 \quad \text{and} \quad \mathcal{F} = 0, \\
\text{LS case 2 (moderate energy)} : & \quad \Delta N_{\text{winding}} = 1 \quad \text{and} \quad \mathcal{F} = 1, \\
\text{LS case 3 (high energy)} : & \quad \Delta N_{\text{winding}} = 1 \quad \text{and} \quad \mathcal{F} = -1.
\end{align*}
\]

Apparently, the spectral flow need not equal the change of winding number, at least for high enough energies with respect to a Sphaleron-like barrier of the potential energy. In other words, the previous result (11) does not hold in general.
The correct relation for the spectral flow in a generic spherically symmetric SU(2) gauge field background follows from the existence of another gauge field characteristic, \( \Delta \text{twist} \), so that

\[
\mathcal{F} = \Delta N_{\text{winding}} + \Delta \text{N}_{\text{twist}} \tag{13}
\]

Here, \( N_{\text{twist}}(t) \) is an integer number which can be calculated directly from the SU(2) gauge field configuration at time \( t \). For the special cases considered in Eq. (12), the relation (13) is verified with

\[
\begin{align*}
\Delta \text{N}_{\text{twist}} &= 0, & \text{for LS case 1 and 2}, \\
\Delta \text{N}_{\text{twist}} &= -2, & \text{for LS case 3}.
\end{align*}
\]

(14)

It should be emphasized that the new selection rule (13) has two integers on the right-hand side, whereas the topological charge \( Q \) may be a noninteger. Indeed, the LS cases 2 and 3 have \( Q = 0.70 \) and \( Q = 0.13 \), respectively.

For weakly dissipative or nondissipative SU(2) Yang–Mills gauge fields in the electroweak Standard Model, one has thus

\[
\begin{align*}
\Delta (B - L) &= 0, \\
\Delta (B + L) &= 2 N_{\text{fam}} \times (\Delta N_{\text{winding}} + \text{extra terms})
\end{align*}
\]

with \( \Delta N_{\text{winding}} \) as defined by the right-hand side of Eq. (11). According to Eq. (13), there is a single “extra term” for the case of spherically symmetric fields, namely \( \Delta \text{N}_{\text{twist}} \). But, in general, the “extra terms” of Eq. (13) are not known. Note also that the issue of gauge invariance deserves particular care.

6 Summary

To our knowledge, there is only one established theory in elementary particle physics for which baryon number violation can be expected to occur, namely the electroweak Standard Model. (Evaporating black holes may or may not violate baryon number conservation. Our current understanding does not allow for definitive statements about the ultimate fate of black holes; cf. Ref. [15].)

Most discussions of electroweak baryogenesis (cf. Ref. [16]) have been based on the selection rule (4), which holds in particular for the tunneling process at low energies [4, 5]. As remarked in the second paragraph of Section 2, this relation has first been derived from perturbation theory [3, 10], with the interactions in the asymptotic regions of spacetime “turned-off.” It is then not altogether surprising that we have found relation (4) to be invalid for high-energy gauge field backgrounds which are weakly dissipative or nondissipative [9]. Of course, precisely these fields are relevant to the physics of the early universe.

At this moment, we have only a partial result for the correct selection rule, namely Eqs. (13) and (15) for the case of spherically symmetric Yang–Mills gauge fields. To generalize this result to arbitrary Yang–Mills gauge fields will be difficult, but is absolutely necessary for a serious discussion of electroweak baryon number violation in the early universe.

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