Compactness results for the Kähler-Ricci flow

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Abstract

We consider the Kähler-Ricci flow $\frac{\partial}{\partial t} g_{i\bar{j}} = g_{i\bar{j}} - R_{i\bar{j}}$ on a compact Kähler manifold $M$ with $c_1(M) > 0$, of complex dimension $k$. We prove the $\epsilon$-regularity lemma for the Kähler-Ricci flow, based on Moser's iteration. Assume that the Ricci curvature and $\int_M |Rm|^k dV$ are uniformly bounded along the flow. Using the $\epsilon$-regularity lemma we derive the compactness result for the Kähler-Ricci flow. Under our assumptions, if $k \geq 3$ in addition, using the compactness result we show that $|Rm| \leq C$ holds uniformly along the flow. This means the flow does not develop any singularities at infinity. We use some ideas of Tian from [28] to prove the smoothing property in that case.

1 Introduction

There has been a lot of interest in compactness theorems for Riemannian manifolds under different geometric assumptions (see e.g. [1], [3], [27], [29], [7]). For example, in the first three references the authors showed independently that if $(M_i, g_i)$ is a sequence of Einstein manifolds of real dimension $n$ ($n = 2k$), such that

(i) $\text{diam}(M_i, g_i) \leq C$,

(ii) $\text{Vol}_{g_i}(M_i) \geq \delta$ and

(iii) $\int_{M_i} |Rm|^k dV_{g_i} \leq C$,

for uniform constants $C, \delta$, then there is a subsequence of $(M_i, g_i)$ converging to a Kähler Einstein orbifold $(M_\infty, g_\infty)$, with finitely many isolated singularities. Let us denote by $\mathcal{K}_+(C, \delta, n)$ and $\mathcal{K}_-(C, \delta, n)$ the sets of all Kähler Einstein manifolds $(M, g)$ satisfying conditions (i), (ii), (iii), with $\text{Ric}(g) = w g$ and $\text{Ric}(g) = -w g$, respectively. In [28] it was proved that both, $\mathcal{K}_+$ and $\mathcal{K}_-$ are compact for $n \geq 3$, that is, $(M_\infty, g_\infty)$ is a smooth Kähler manifold.
and the convergence is smooth everywhere. The main tool in this smoothing property is Kohn’s estimate for $\bar{\partial}$-operators on strictly pseudoconvex CR-manifolds.

Especially after the breakthrough Perelman (22) made in using the Ricci flow to complete Hamilton’s program in proving the geometrization conjecture, there has been an increasing interest in studying the Kähler Ricci flow as well. Let $(M,g)$ be a Kähler manifold with $c_1(M) > 0$. Then we can assume $c_1(M) = [\text{Ric}(g)]$ is given by the Kähler form $w_g$ and the Kähler Ricci flow is a solution to

$$\frac{\partial}{\partial t} g_{ij} = g_{ij} - R_{ij} = \partial_i \bar{\partial}_j u,$$

$$g_{ij}(0) = g_{ij}. \tag{1}$$

Notice that the stationary solution to (1) is a Kähler Einstein metric in $K_+$. The generalization of Kähler Einstein metrics are Kähler Ricci solitons.

**Definition 1.** We will say that a solution $g(t)$ to (1) is a Kähler Ricci soliton if it moves by one parameter family of automorphisms $\phi(t)$, or equivalently satisfies the equation

$$R_{ij} + \rho g_{ij} = \mathcal{L}_X (g),$$

where $X$ is a holomorphic vector field induced by automorphisms $\phi(t)$. We say $g(t)$ is a gradient Kähler Ricci soliton if $X = \nabla f$ for a smooth function $f$ that satisfies $f_{ij} = 0$.

In [7] we showed the analogous compactness result for the set of gradient Kähler Ricci solitons satisfying similar geometric conditions to those in (i), (ii) and (iii). The next interesting problem arises in understanding the parabolic version of compactness results, that is, the compactness results regarding the solutions to (1). This is tightly related to questions of converging Kähler Ricci flows and understanding the possible limits.

In [4] Cao proved that a solution to (1) exists forever, that is, a singularity does not occur at finite time. It is very likely to happen that the flow develops singularities at infinity. A lot of progress has been made in studying the limits of the Kähler Ricci flow. It is especially important to know when we can expect to get Kähler Einstein metrics in a limit. Among first works on that topic is the work by Chen and Tian ([8], [9]) where they proved that if $M$ admits a Kähler Einstein metric with positive scalar curvature and if an initial metric has a nonnegative bisectional curvature that is positive at least at one point, then the Kähler Ricci flow converges exponentially fast to the Kähler Einstein metric with constant bisectional curvature. After the
work of Perelman \([22]\) appeared, in \([6]\) it was proved that if the bisectional curvature \(R_{i\bar{j}j}\) is positive at time \(t = 0\) then the curvature operator stays uniformly bounded along the flow. In \([31]\) it has been proved that if (in any complex dimension) \(M\) is a compact Kähler manifold which admits a Kähler-Ricci soliton \((g_{RS}, X)\) defined by a vector field \(X\), then any solution \(g(\cdot, t)\) of (1) converges to \(g_{RS}\), if the initial Kähler metric \(g_0\) is invariant under \(K_X\), an 1-parameter subgroup of a group of automorphisms of \(M\) generated by \(X\).

In the recent preprint \([23]\), the authors have proved the convergence of the Kähler-Ricci flow to a Kähler-Einstein metrics under certain assumptions. By the result in \([24]\), if the curvature is uniformly bounded along the flow then it sequentially converges to Kähler-Ricci solitons. In \([25]\) we have proved that if \(|\text{Ric}(g(t))| \leq C\) along the flow, for a uniform constant \(C\), then for any sequence \(t_i \to \infty\) there is a subsequence so that \((M, g(t_i + t)) \to (M_\infty, g_\infty(t))\), where \(M_\infty\) is smooth outside a closed singular set \(S\) which is of codimension at least 4 and the convergence is smooth outside \(S\).

Due to Perelman (see also \([26]\)) we have the following uniform estimates for the Kähler Ricci flow: there are uniform constants \(C\) and \(\kappa\) such that for all \(t\),

1. \(|u(t)|_{C^1} \leq C\),
2. \(\text{diam}(M, g(t)) \leq C\),
3. \(|R(g(t))| \leq C\),
4. \((M, g(t))\) is \(\kappa\)-noncollapsed.

This together with the uniform lower bound on Ricci curvatures along the flow gives a uniform upper bound on the Sobolev constant, that is, there is a uniform constant \(C_S\) so that for any \(v \in C^1_0(M)\) we have that

\[
\left( \int_M v^{\frac{4n}{2n-2}} dV_{g(t)} \right)^{\frac{2n}{2n-2}} \leq C_S \int_M |\nabla v|^2 dV_{g(t)},
\]

for all times \(t \geq 0\). This enables us to work with integral estimates. By the recent results in \([34]\) and \([35]\), that have been obtained independently, we have that due to Perelman’s uniform upper bound on the scalar curvature, (2) holds uniformly along the flow without any geometric assumptions on the curvatures.

In this paper we first show the following compactness result for the Kähler-Ricci flow.
Theorem 2. Let \( g(t) \) be the Kähler Ricci flow on a compact, Kähler manifold \( M \), with \( c_1(M) > 0 \), with Ricci curvatures uniformly bounded and with
\[
\int_M |\text{Rm}(g(t))|^{n/2} dV_{g(t)} \leq C \text{ along the flow.}
\]
Then for every sequence \( t_i \to \infty \) there is a subsequence so that \( (M, g(t_i + t)) \) converges to \( (M_\infty, g_\infty(t)) \), where

(a) \( M_\infty \) is an orbifold with finitely many isolated singular points, \( \{p_1, \ldots, p_N\} \), and the convergence is smooth outside those singular points.

(b) The limit metric \( g_\infty \) is a Kähler Ricci soliton in an orbifold sense, that is, satisfies the Kähler Ricci soliton equation,
\[
(g_\infty)_{ij} - R_{ij}(g_\infty) = \partial_i \partial_j f_\infty,
\]
\[
\partial_i \partial_j f_\infty = 0,
\]
off the singular points. Moreover, for every singular point \( p_j \), there is a neighbourhood in \( M_\infty \) which lifts to an open set \( D_j \subset \mathbb{C}^{n/2} \), and the lifting of an orbifold metric \( g_\infty \) satisfies equivalent equations to (3) in \( D_j \).

For a sequence \( (M, g(t_i + t)) \) converging to \( (M_\infty, g_\infty) \) in the sense described by the previous theorem in (a) and (b), we will say it converges to a Kähler Ricci soliton in an orbifold sense. We will use Theorem 2 to prove the following smoothing theorem that shows the flow does not develop any singularities at infinity in the case \( k > 3 \).

Theorem 3. Let \( g(t) \) be a Kähler Ricci flow on a compact, Kähler manifold \( M \) of complex dimension \( k \) (\( n = 2k \)), with \( c_1(M) > 0 \), Ricci curvatures uniformly bounded and with
\[
\int_M |\text{Rm}(g(t))|^{n/2} dV_{g(t)} \leq C \text{ along the flow.}
\]
Then if \( k \geq 3 \), the curvature operator is uniformly bounded along the flow.

The proof of the later theorem relies on Kohn’s estimate for \( \bar{\partial} \)-operator that is proved in [28] and Perelman’s pseudolocality theorem (22). The estimate from [28] works only if \( n \geq 6 \). We expect that in the case of surfaces \( (n = 4) \) the curvature stays uniformly bounded along the flow as well. It is known by classification theory of complex surfaces that \( \mathbb{CP}^{2q} \mathbb{CP}^2 \) \((0 \leq q \leq 8)\) and \( \mathbb{CP}^1 \times \mathbb{CP}^1 \) are the only compact 4-manifolds which admit complex structures with positive first Chern class. By the result of Tian in [27] it is known that each of the surfaces \( \mathbb{CP}^1 \times \mathbb{CP}^1 \) and \( \mathbb{CP}^{2q} \mathbb{CP}^2 \) admits a Kähler Einstein metric for \( q = 0 \) and \( 3 \leq q \leq 8 \). Cao ([10]) and Koiso ([21]) independently showed that \( \mathbb{CP}^{2q} \mathbb{CP}^2 \) admits a nontrivial Kähler Ricci soliton metric. Wang and Zhu ([32]) showed the same result later for \( \mathbb{CP}^{2q} \mathbb{CP}^2 \). Up to the invariance of the initial metric under \( K_X \), by [31], we have that for
any Kähler Ricci flow on a compact Kähler surface $M$ with $c_1(M) > 0$, the curvature operator stays uniformly bounded along the flow. Moreover the flow always converges.

The organization of the paper is as follows. In section 2 we will show the $\epsilon$-regularity for the Kähler-Ricci flow. The proof will be based on standard Moser’s iteration argument. In section 3 we will prove the compactness result for the Kähler-Ricci flow, Theorem 2, using the $\epsilon$-regularity lemma established in the previous section and some ideas of [1], [3] and [27]. In section 4 we will show that some of the results from [28] regarding the convergence of plurianticanonical divisors apply also to our case. In section 5 we will prove Theorem 3 that is, the smoothing property of the compactness result in the case $k \geq 3$, which yields the uniform curvature bound along the flow. In other words, we will show we have the smooth convergence in Theorem 2.

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2 An $\epsilon$-regularity lemma for the Kähler-Ricci flow

In this section we will establish the $\epsilon$-regularity lemma that is a parabolic analogue of the $\epsilon$-regularity lemma in the elliptic setting, that appeared independently in [1], [3] and [27]. It will allow us obtain pointwise curvature bounds away from finitely many curvature concentration points. More precisely, the obtained quadratic curvature decay will be useful in understanding the orbifold structure of a limit of the Kähler Ricci flow when $t \to \infty$. For the proof of the $\epsilon$-regularity lemma we will need Moser’s weak maximum principle argument in a form proved my Dai, Wei and Ye in [11]. Along these lines, the $\epsilon$-regularity lemma for the mean curvature flow has been established in [14].

**Theorem 4** (Moser’s weak maximum principle). Let $f, b$ be smooth nonnegative functions on $N \times [0, T]$ which satisfy the following inequality

$$\frac{d}{dt} f \leq \Delta f + bf,$$

on $N \times [0, T]$. If $b$ satisfies

$$\sup_{t \in [0, T]} (\int_N b^{q/2})^{2/q} \leq \beta,$$

then
for some \( q > n \). Let \( C_S = \max_{[0,T]} C_S(g(t)) \) and \( l = \max_{[0,t]} \|\frac{dg}{dt}\|_{C^0} \). Then given any \( p_0 \) there exists a constant \( C = C(n, q, p_0, C_S, l, T, R) \) such that for \( x \in M \times (0, T] \)

\[
|f(x, t)| \leq C t^{-\frac{n+2}{2p_0}} (\int_0^t \int_{B_R} f(t, s)^{1/p_0})^{1/p_0},
\]

where \( B_R \) is a geodesic ball of radius \( R \), defined in terms of metric \( g(0) \).

Denote by \( f = |Rm| \). By the evolution equation of the curvature operator we have that \( f \) satisfies

\[
\frac{d}{dt} f \leq \Delta f + Cf^2,
\]

for some uniform constant \( C \). Assume the following condition, that we will refer to as to a euclidean volume growth.

\[
\text{Vol}_t B_t(p, r) \leq Cr^n, \quad \text{for all } t \geq 0 \ p \in M \text{ and } 0 \leq r \leq \text{diam}(M, g(t)).
\]

(5)

Assuming the euclidean volume growth condition, we have the following proposition.

**Proposition 5.** There are \( C = C(n, C_S) \), \( \epsilon = \epsilon(n, C_S) \) and \( 0 < \xi_1, \xi_2 < 1 \), such that if for some \( t > 0 \) and \( \delta \leq r^2 \) (we may assume \( \delta \) is comparable with \( r^2 \)),

\[
\sup_{s \in [0, \delta]} \int_{B_t(p, 3r)} f^{n/2} dV_{t+s} < \epsilon,
\]

(6)

then

\[
\sup_{B_t(p, 4r) \times [\xi_2 \delta, \delta]} |f| \leq C \left( \frac{1}{r^2} + \frac{1}{\delta} \right) \sup_{s \in [0, \delta]} \left( \int_{B_t(p, 2r)} f^{n/2} \right)^{2/n} dV_{t+s},
\]

(7)

where \( p \) is any point in \( M \) and our estimates are independent of the initial time \( t \).

By the independent results of Zhang ([35]) and Ye ([34]), there are positive constants \( A \) and \( B \), depending only on the geometric quantities of \( g(0) \), so that for all \( t \in [0, \infty) \) and all \( u \in W^{1,2}(M) \),

\[
\left( \int_M |u|^{\frac{2n}{n-2}} dV_t \right)^{\frac{n-2}{n}} \leq A \int_M (|\nabla u|^2 + \frac{R}{4} u^2) dV_t + B \int_M u^2 dV_t.
\]

Due to Perelman, \( |R| \leq C \) along the flow and the above inequality becomes

\[
\left( \int_M |u|^{\frac{2n}{n-2}} dV_t \right)^{\frac{n-2}{n}} \leq A \int_M |\nabla u|^2 + B_1 \int_M u^2 dV_t,
\]

(8)
for a different constant $B_1$.

We will first give a proof of Proposition 5 under the assumption that $|\text{Ric}| \leq C$ for all $t \geq 0$. Later we will show how we can replace the Ricci curvature condition with the euclidean volume growth condition.

Let $\eta$ be a cut off function, so that $\eta = 1$ in $B_t(p, r)$ and $\eta = 0$ outside the ball $B_t(p, 2r)$ and $|\nabla \eta|^2(t) \leq \frac{C}{r^2}$. Since,

$$\frac{d}{dt}|\nabla \eta|^2 = |\nabla \eta|^2 - \text{Ric}(\nabla \eta, \nabla \eta),$$

and therefore $|\frac{d}{dt} \ln |\nabla \eta|^2| \leq C$, it easily follows that,

$$\sup_{s \in [t-r^2, t+r^2]} |\nabla \eta|^2(s) \leq \frac{\tilde{C}}{r^2},$$

for a uniform constant $\tilde{C}$.

**Proof of Proposition 5.** Since the equations (4), (6) and (7) are scale invariant, we may assume the radius of the ball is $r = 1$. In other words, by rescaling our metric by factor $r^{-2}$, that is $\bar{g}(t) = r^{-2}g(r^2t)$, it is enough to show that there are uniform constants $C, \xi_1$ and $\xi_2$ so that

$$\sup_{B_t(p, \xi_1) \times [\xi_2 br^{-2}, \delta r^{-2}]} |f|(x, t + s) \leq C \sup_{s \in [0, \delta r^{-2}]} (\int_{B_t(p, 2)} f^{\frac{n}{2}}dV_{t+s})^{2/n},$$

for some uniform constant $C$, if $\sup_{s \in [0, \delta r^{-2}]} \int_{B_t(p, 3)} f^{n/2}dV_{t+s} < \epsilon$ (where $f(t) = |\text{Rm}|(\bar{g}(t))$). Multiply (4) by $\eta^2 f$, where $\eta$ is a cut off function with compact support in $B_t(p, 3)$, which is equal to 1 on $B_t(p, 2)$ and such that $|\nabla \eta| \leq C$ ($|\nabla \eta|$ is computed in norm $\bar{g}(t)$). Furthermore, since $|\text{Ric}|(\bar{g}(t)) \leq Cr^2$, for all $t$, we have

$$\frac{d}{ds}|\nabla \eta|^2(t + s) \leq Cr^2|\nabla \eta|^2(t + s),$$

and $|\nabla \eta|(t + s) \leq \tilde{C}$ for $s \in [0, \delta r^{-2}]$. After multiplying (4) by $\eta^2 f$ and integrating by parts we get

$$\int \eta^2 |\nabla f|^2 - 2 \int \eta f |\nabla \eta| |\nabla f| + \frac{1}{2} \frac{d}{ds} \int f^2 \eta^2 \leq C \int \eta^2 f^2 + C \int \eta^2 f^3. \quad (10)$$

By Hölder, Sobolev and interpolation inequalities we have

$$\int \eta f |\nabla f| |\nabla \eta| \leq \frac{1}{2} \int \eta^2 |\nabla f|^2 + C \int |\nabla \eta|^2 f^2, \quad (11)$$
\begin{align*}
\int \eta^2 f^3 &= \int (\eta f^2) \cdot f \leq \int_{B_t(p,3)} f^{n/2} / \sqrt{n} \left( \int (\eta f)^{2n/3} \right)^{n-2} / n \\
&\leq \epsilon C_S \left( \int |\nabla f|^2 \eta^2 + \int |\nabla \eta|^2 f^2 \right). \quad (12)
\end{align*}

Combining (10), (11) and (12) and choosing \( \epsilon \) small so that we can absorb \( \epsilon C_S \left( \int |\nabla f|^2 \eta^2 \right) \) into the left hand side of (10) yield

\[
\frac{d}{ds} \int \eta^2 f^2 + C \int |\nabla f|^2 \eta^2 \leq C \int f^2 (\eta^2 + |\nabla \eta|^2). \quad (13)
\]

This implies

\[
\int_0^{\delta r^{-2}} \int_{B_t(p,2)} |\nabla f|^2 \leq C \left( \int_0^{\delta r^{-2}} \int_{B_t(p,3)} f^2 dV_t + \int_{B_t(p,3)} f^2 dV_t \right) \leq \tilde{C},
\quad (14)
\]

for a uniform constant \( \tilde{C} \), since

\[
\int_M f^2 dV_t \leq \left( \int_M f^{4/3} dV_t \right)^{4/3} \cdot (\text{Vol}_t(M))^{(n-4)/n} \leq C,
\]

for a uniform constant \( C \). We also have the evolution equation for \( |\nabla f|^2 \),

\[
\frac{d}{ds} |\nabla f|^2 \leq \Delta |\nabla f|^2 - 2 |\nabla^2 f|^2 + C |\nabla f|^2 f. \quad (16)
\]

Let \( \eta \) be a cut off function with compact support in \( B_t(p, 2) \) such that it is identically 1 on \( B_t(p, 1) \). Multiply (16) by \( \eta^2 \) and integrate it over \( M \).

\[
\frac{d}{ds} \int \eta^2 |\nabla f|^2 \leq C \int \eta^2 |\nabla f|^2 - 2 \int \eta |\nabla |\nabla f|| \nabla f| \nabla \eta - \int \eta^2 |\nabla^2 f|^2 + C \int \eta^2 |\nabla f|^2 f. \quad (17)
\]

Similarly as before we can estimate

\[
2 \int \eta |\nabla |\nabla f|| \nabla f| \leq \frac{1}{3} \int \eta^2 |\nabla |\nabla f||^2 + C \int \eta^2 |\nabla f|^2 |\nabla f|^2, \quad (18)
\]

\[
\int \eta^2 |\nabla f|^2 f \leq \left( \int f^{n/2} / \sqrt{n} \left( \int (|\nabla f| \eta)^{2n/3} \right)^{n-2} / n \right) \leq \epsilon C_S \left( \int |\nabla |\nabla f||^2 \eta^2 + \int |\nabla f|^2 |\nabla \eta|^2 \right). \quad (19)
\]
Combining (17), (18) and (19), choosing \( \epsilon \) small enough and using that 
\[ |\nabla |\nabla f|| \leq |\nabla^2 f| \]
yield
\[
\frac{d}{ds} \int |\nabla f|^2 \eta^2 \leq C \int |\nabla f|^2 (\eta^2 + |\nabla \eta|^2) \leq C \int_{B_t(p,2)} |\nabla f|^2. \tag{20}
\]

As in [11] choose a cut off function \( \psi(s) \) such that \( \psi(s) = 0 \) for \( 0 \leq t \leq \delta r^{-2}/4, \) \( \frac{1}{\delta r^{-2}}(t-\delta r^{-2}/4) \) for \( t \in [\delta r^{-2}/4, \delta r^{-2}/2] \) and \( 1 \) for \( t \in [\delta r^{-2}/2, \delta r^{-2}] \).

Multiplying (20) by \( \psi(s) \) we obtain
\[
\frac{d}{ds} \left( \psi \int |\nabla f|^2 \eta^2 \right) \leq (C \psi + \psi') \int |\nabla f|^2 \eta^2 + C \psi \int f^2 |\nabla \eta|^2.
\]

Integrating in \( s \) we obtain
\[
\int \eta^2 |\nabla f|^2 (t+s) \leq C \int_0^{\delta r^{-2}} \int_{B_t(p,2)} |\nabla f|^2 dV_{t+s} ds + C \int_0^{\delta r^{-2}} \int_{B_t(p,3)} f^2 dV_{t+s} ds \leq \tilde{C},
\]
for a uniform constant \( \tilde{C} \), independent of \( t \) and \( s \in [\delta r^{-2}/2, \delta r^{-2}] \). By (14) we get
\[
\sup_{[\delta r^{-2}/2, \delta r^{-2}]} \left| \int_{B_t(p,1)} |\nabla f|^2 \right| \leq C \int_0^{\delta r^{-2}} \int_{B_t(p,3)} f^2 dV_{t+s} ds + \int_{B_t(p,3)} f^2 dV_t,
\]
for a uniform constant \( C \). By Sobolev inequality we get
\[
\sup_{[\delta r^{-2}/2, \delta r^{-2}]} \left( \int_{B_t(p,1)} f^{2n/(n-2)} dV_{t+s} \right)^{(n-2)/n} \leq \tilde{C}_1 \int_0^{\delta r^{-2}} \int_{B_t(p,3)} f^2 dV_{t+s} ds + \int_{B_t(p,3)} f^2 dV_t.
\]

(21)

In other words, the \( L^{2n/(n-2)} \)-norm of \( |f| \) can be estimated in terms of the \( L^2 \)-norm of \( |f| \). Similarly, by multiplying (11) by \( \eta^2 f^{n+1/(n-2)} \) and proceeding as above, we have
\[
\sup_{s \in [\delta r^{-2}/(1+1/2), \delta r^{-2}]} \left( \int_{B_t(p,1/3)} f^{2(n/(n-2))^2} dV_{t+s} \right)^{(n-2)/n} \leq \tilde{C}_1 \int_0^{\delta r^{-2}} \int_{B_t(p,1)} f^{2n/(n-2)} dV_{t+s} ds + \int_{B_t(p,1)} f^{2n/(n-2)} dV_{t+s} \leq C.
\]
Continuing the above process \( l \) times, so that \((\frac{n}{n-2})^l \geq n\), using (15) we obtain
\[
\sup_{s \in [\delta r^{-2}(\frac{\sum l=0^{2-j}}{2}), \delta r^{-2}]} \left( \int_{B_t(p,1/3')} f^{2(n/(n-2))} dV_t+s \right)((n-2)/n)^l \leq C(l) \sup_{s \in [0, \delta r^{-2}]} \left( \int_{B_t(p,1)} f^2 dV_s \right)^{1/2}.
\]
(22)

By Theorem 4, its proof and by the estimate (22) we get
\[
\sup_{B_t(p,1/(2\cdot 3')) \times [\delta r^{-2}(\frac{\sum l=0^{2-j}}{2}), \delta r^{-2}]} \left| f(x,t) \right| \leq C(l) \sup_{s \in [0, \delta r^{-2}]} \left( \int_{B_t(p,1)} f^2 dV_s \right)^{1/2} \leq \tilde{C}(l) \sup_{s \in [0, \delta r^{-2}]} \left( \int_{B_t(p,1)} f^{\frac{2}{n}} dV_s \right)^{\frac{n}{2}}.
\]

If we rescale metric \( \tilde{g} \) back to our original metric \( g(t) \) we get (7) by putting \( \xi_1 := \frac{1}{2^{3r}} \) and \( \xi_2 := \left( \frac{\sum l=0^{2-j}}{2} \right) \).

To show how we can remove the hypothesis on the Ricci curvature bound, note that the main tool in the proof of Proposition 5 is the Sobolev inequality (2) that we have due to [34] and [35]. We have used the Ricci curvature bound to control the norms \( \sup_{s \in [-r^2,r^2]} \left| \nabla \eta \right|_{t+s} \), where \( \eta \) is the cut off function with a support in \( B_t(p, r) \), independently of \( t \) and \( s \in [-r^2, r^2] \). We can overcome that difficulty by choosing the cut off function differently, if we assume the euclidean volume growth condition. The difference is that we will let the cut off function evolve in time as the metric evolves, unlike in the situation above where the cut off function was time independent.

**Corollary 6.** Proposition 5 still holds if we replace the condition on a uniform bound of Ricci curvatures along the flow with a weaker condition (5).

**Proof.** Let \( \eta_0 \) be a cut off function as in Proposition 5 so that \( \left| \nabla \eta_0 \right|^2 \leq \frac{C}{r^2} \), where the norm of \( \nabla \eta_0 \) is taken with respect to metric \( g(t) \). Let \( \eta \) evolve \( \eta(s) \) for \( s \in [0, \delta] \), in a ball \( B_t(p, 2r) \) as follows, (for our purposes we can assume \( \delta \) being comparable with \( r^2 \))
\[
\frac{\partial}{\partial s} \eta = \Delta \eta, \\
\eta(x, 0) = \eta_0(x), \\
\eta(x, s) = 0, \text{ for } (x, s) \in \partial B_t(p, 2r) \times [0, \delta],
\]
(23)
where $\Delta$ is taken with respect to a changing metric $g(t+s)$. The evolution equation for $\nabla \eta$ is

$$
\frac{\partial}{\partial s} |\nabla \eta|^2 = \Delta |\nabla \eta|^2 - |\nabla \nabla \eta|^2 - |\bar{\nabla} \eta|^2,
$$

where $|\nabla \eta|^2$ is taken with respect to $g(t+s)$. By the maximum principle it follows

$$
|\nabla \eta|^2(\cdot, s) \leq \max_{\bar{B}(p,2r) \times [0,\delta]} |\nabla \eta|^2 \leq \frac{C}{r^2},
$$

(24)

\[ \eta(x,s) \geq 0, \quad \text{for all } s \in [0,\delta], \]

$$
\max_{B_t(p,2r) \times [0,\delta]} |\eta(x,s)| \leq 1.
$$

By the results in [34] and [35] we have Sobolev inequality (8) holding uniformly along the flow. In the following claim we will show $\eta$ has a uniform lower bound and therefore we can apply almost the same proof of Proposition 5 to get the result.

**Claim 7.** There exist uniform constants $\delta_1, \delta_2 > 0$, independent of $t$ and $r > 0$, so that $\eta(x,t+s) \geq \delta_1 > 0$, for all $x \in B_t(p,\delta_2 r)$ and all $s \in [0,\delta]$.

**Proof.** All we need to show is that $\eta(p,t+s) \geq \alpha$ for all $s \in [0,\delta]$ and a uniform constant $\alpha$, independent of $t, p \in M$ and $r > 0$. This together with (24) will yield the result. If we integrate (23) over $B_t(p,2r)$ we get

$$
\frac{d}{ds} \int_{B_t(p,2r)} \eta(x,t+s) \, dV_{t+s} = \int_{B_t(p,2r)} \eta(x,t+s)(k - R) \, dV_{t+s},
$$

which, since $\eta(\cdot, t+s) \geq 0$ implies

$$
\left| \frac{d}{ds} \int_{B_t(p,2r)} \eta(x,t+s) \, dV_{t+s} \right| \leq C \int_{B_t(p,2r)} \eta(x,t+s) \, dV_{t+s},
$$

and therefore, since $\eta(x,t) = 1$ for $x \in B_t(p,r)$ and $\text{Vol}_t B_t(p,r) \geq \kappa r^n$,

$$
\int_{B_t(p,2r)} \eta(x,t+s) \, dV_{t+s} \geq e^{-Cs} \int_{B_t(p,2r)} \eta(x,t) \, dV_t \geq C_1 \int_{B_t(p,r)} \eta(x,t) \, dV_t \geq C_1 \kappa r^n = Ar^n,
$$

for all $s \in [0,\delta], p \in M$ and $r > 0$. By condition (5) this implies

$$
\frac{1}{\text{Vol}_{t+s}(B_t(p,2r))} \int_{B_t(p,2r)} \eta(x,t+s) \, dV_{t+s} \geq A_1,
$$

(25)
for a uniform constant $A_1$, independent of $r > 0$, $p \in M$ and $s \in [0, \delta]$. Fix $s \in [0, \delta]$. By (24), there exists at least one $q_1 \in B_t(p, 2r)$ so that $\eta(q_1, t + s) \geq A_1$. By (24) there exists a uniform constant $\delta > 0$ so that $\eta(x, t + s) \geq A_1/2$ for all $x \in B_t(q_1, \delta r)$. If $p \in B_t(q_1, \delta r)$ we are done. If not, let $r_1 = \text{dist}_{t+s}(q_1, p)/4$ and consider the ball $B_t(p, 2r_1)$. Radius $r_1$ will play the role of $r$ above. Since $q_1$ is not in $B_t(p, 2r_1)$, by the same arguments as before we can find $q_2 \in B_t(p, 2r_1)$ so that $\eta(x, t + s) \geq A_1/2$ for all $x \in B_t(q_2, \delta r_1)$. This way we will construct a sequence $\{q_j\}$ such that $q_j \to p$ as $j \to \infty$ and $\eta(q_j, t + s) \geq A_1/2$. By the continuity of $\eta$, $\eta(p, t + s) \geq A_1/2 =: \alpha$. Since $s$ was an arbitrary number in $[0, \delta]$, we are done.

As we remarked earlier, having the uniform Sobolev inequality (8) and Claim 7 finishes the proof of Corollary 6.

3 Convergence to a Kähler-Ricci soliton orbifold with finitely many singular points

In this section we will prove the compactness result for the Kähler-Ricci flow, which is dimension independent.

Proof of Theorem 2. Take $\epsilon << \epsilon_0$ small, where $\epsilon_0$ is taken from Proposition 5. Define

\[ D_t^r = \{x \in M \mid \int_{B_t(p, 2r)} |\text{Rm}|^{n/2} dV_t < \epsilon\}, \]
\[ L_t^r = \{x \in M \mid \int_{B_t(x, 2r)} |\text{Rm}|^{n/2} dV_t > \epsilon\}. \]

By the covering argument, similarly as in [7] we can get there is a uniform upper bound on a number $N$ of points, $\{x_{ij}^r \}$, that are the centres of balls of radius $2r$ covering sets $L_t^r$, so that the corresponding concentric balls of radius $r$ are disjoint. Notice that $N$ is independent of both $i$ and $r > 0$. To see that, assume

\[ \int_{B_t(x_{ij}^r, 2r)} |\text{Rm}|^{n/2} dV_t \geq \epsilon. \]

By the $\kappa$-noncollapsing that was proved by Perelman (see [26]), there is a uniform constant $\kappa > 0$ so that

\[ \text{Vol}_t B_t(p, r) \geq \kappa r^n, \quad \text{for all } t > 0 \text{ and all } r > 0 \text{ } p \in M. \] (26)
Since $\text{Ric} \geq -C$, by Bishop-Gromov comparison principle, for every $r > \delta$ and every $p \in M$,\[
\frac{\text{Vol}_t B_t(p, r)}{V_C(r)} \leq \frac{\text{Vol}_t B_t(p, \delta)}{V_C(\delta)},\]
where $V_C(r)$ is the volume of a ball of radius $r$ in a space form of constant sectional curvature $-C$. If we let $\delta \to 0$, the right hand side of the previous inequality converges to a constant equal to the volume of a unit ball in $\mathbb{R}^n$. Call it $w_n$. Then,
\[
\frac{\text{Vol}_t B_t(p, r)}{r^n} \leq w_n \frac{V_C(r)}{r^n} \leq C_1,
\]
for a uniform constant $C_1$ and $r \leq r_0$. Notice that (26) and (27) imply there is a uniform upper bound $m$ on the number of disjoint balls of radius $r$ contained in a ball of radius $2r$. Then,
\[
N \epsilon \leq \sum_j \int_{B_t (x_i, 2r)} |Rm|^{n/2} dV_t \leq m \int_M |Rm|^{n/2} dV_t \leq Cm = \tilde{C},
\]
which yields
\[
N \leq \frac{\tilde{C}}{\epsilon},
\]
(28)
In [25] we have proved that given the Kähler-Ricci flow with uniformly bounded Ricci curvatures, then for every sequence $t_i \to \infty$ there exists a subsequence such that $(M, g(t_i + t)) \to (Y, \bar{g}(t))$. The convergence is smooth outside a singular set $S$, which is closed and at least of codimension four. In [26] we showed that $\bar{g}(t)$ solves the Kähler-Ricci soliton equation off $S$.

**Proposition 8.** The closed set $S$ consists of finitely many points.

**Proof.** The proof goes by contradiction. Assume the proposition is false. Since $S$ is a closed subset of $Y$, for every $r$ we can find a finite cover of $S$ with balls of radius $2r$ so that the corresponding concentric balls of radius $r$ stay disjoint. Choose $r > 0$ small so that the number $L$ of above balls of radius $2r$ covering $S$ is bigger than $1000[\frac{\epsilon}{\tilde{C}}]$, where the constants are taken from (28). We can always do that if $S$ is not just a set of finitely many isolated points. Denote by $\bar{p}_j$, for $1 \leq j \leq L$ the centers of those balls. Let $S(r) = \bigcup_{j=1}^L B_{2r}(\bar{p}_j)$. From [25] there are points $p_j^i \in M$ and diffeomorphisms $\phi_i$ from $Y \setminus S(r)$ into $M$, containing $M \setminus S_i(3r)$, where
$S_i(3r) = \bigcup_{j=1}^{l_i} B_{t_i}(p^j_{i}, 3r)$, such that $\phi^*_ig_i$ converges smoothly to $\bar{g}$. That means the curvatures $|Rm|(\cdot, t_i)$ are uniformly bounded on $M \setminus S_i(3r)$. We may assume the balls $B_{t_i}(p^j_{i}, r/2)$ are disjoint for $i$ big enough. Since the reason for the formation of the singular set $S$ is the curvature blow up, we can find points $q^j_{i} \in B_{t_i}(p^j_{i}, 3r)$ so that $\max_{B_{t_i}(q^j_{i}, 4r)} |Rm|(x, t_i + s) = |Rm|(q^j_{i}, t_i) := Q^j_{i}$ (at least for $i$ big enough). If that maximum is attained at some time $s \neq 0$, we can just replace $t_i$ by $t_i + s$ and continue the consideration. We may assume $Q^j_{i} \to \infty$ as $i \to \infty$ for every $j$, since otherwise $S_i(3r) \cap B_{t_i}(q^j_{i}, 4r)$ would at the same time give rise to a part of the singular set $S$ and also converge to a smooth part of $Y$ (due to the uniform curvature bounds), which is not possible.

**Claim 9.** There exists a uniform constant $C_1$, so that for every $r > 0$, every sequence of points $q_i \in M$ and a sequence of times $t_i \to \infty$, with the property that $\max_{B_{t_i}(q_i, r)} |Rm|(x, t_i + s) = |Rm|(q_i, t_i)$, we have that for every $0 < \rho \leq r$ there exists an $i_0$ with the property

$$
\int_{B_{t_i}(q_i, \rho)} |Rm|^{\frac{n}{2}}(t_i + s) \, dV_{t_i + s} \leq C_1 \int_{B_{t_i}(q_i, \rho)} |Rm|^{\frac{n}{2}} \, dV_{t_i} ,
$$

for all $s \in [-\rho^2, \rho^2]$ and all $i \geq i_0$.

**Proof.** Assume the claim were not true. Then for every $j$ there would exist an $0 < \rho_j \leq r_j$, so that for every $i$ there would exist $k_{ij} > i$ with the property,

$$
\int_{B_{k_{ij}}(q_{k_{ij}}, \rho_j)} |Rm|^{\frac{n}{2}} \, dV_{k_{ij}} \leq \frac{1}{j} \int_{B_{k_{ij}}(q_{k_{ij}}, \rho_j)} |Rm|^{\frac{n}{2}}(t_{k_{ij}} + s_{k_{ij}}) \, dV_{t_{k_{ij}} + s_{k_{ij}}} ,
$$

for some $s_{k_{ij}} \in [-r_j^2, r_j^2]$. Since $Q_i \to \infty$, we may choose a subsequence $k_{ij}$ so that $\rho_j \geq (Q_{k_{ij}})^{-1}$. If we define a sequence of rescaled metrics $\tilde{g}_{k_{ij}}(\tau) = Q_{k_{ij}}g(t_{k_{ij}} + \tau(Q_{k_{ij}})^{-1})$, (30) can be rewritten as

$$
\int_{B_{\tilde{g}_{k_{ij}}(0)}(q_{k_{ij}}, r_j(Q_{k_{ij}})^{-1/2})} |Rm|^{\frac{n}{2}} \, dV_{\tilde{g}_{k_{ij}}(0)} \leq \frac{C}{j} ,
$$

since $\int_M |Rm|^{\frac{n}{2}} \, dV \leq C$, uniformly along the flow. The pointed sequence of solutions $(B_{\tilde{g}_{k_{ij}}}(\tau)q_{k_{ij}}, r_j \sqrt{Q_{k_{ij}}}), \tilde{g}_{k_{ij}}(\tau), q_{k_{ij}})$ converge to a complete solution $(X, \tilde{g}_\infty(\tau), q_\infty)$ with $|Rm|\tilde{g}_\infty(q_\infty, 0) = 1$. Therefore taking the limit as $i, j \to \infty$ in (31) yields a contradiction since

$$
0 \neq \int_{X} |Rm|^{\frac{n}{2}} \, dV_{\tilde{g}_\infty} \leq \liminf_{i,j \to \infty} \int_{B_{\tilde{g}_{k_{ij}}(0)}(q_{k_{ij}}, r_j \sqrt{Q_{k_{ij}}})} |Rm|^{\frac{n}{2}} \, dV_{\tilde{g}_{k_{ij}}(0)} = 0 .
$$
Take $C_1$ as in the Claim 9. Take $\epsilon > 0$ small so that $C_1\epsilon < \epsilon_0$, where $\epsilon_0$ is taken from Proposition 5. If $r > 0$ and a sequence $\{q^i_j\}$ (for $1 \leq j \leq L$) are as we have constructed in the paragraph just prior to Claim 9 (depending on the chosen $\epsilon$), then we have the following:

**Claim 10.** There exists $i_0$ so that

$$\int_{B_{r_i}(q^i_j, r)} |\text{Rm}|^{\frac{n}{2}} dV_{t_i} > \epsilon, \quad \text{for } i \geq i_0,$$

for all $1 \leq j \leq L$.

**Proof.** Assume the claim is not true. Then for some $j \in \{1, \ldots, L\}$, by Claim 9, we have

$$\int_{B_{r_i}(q^i_j, r)} |\text{Rm}|^{\frac{n}{2}} dV_{t_i+s} \leq C_1\epsilon < \epsilon_0,$$

for all $s \in [-r^2, r^2]$. By Proposition 5,

$$|\text{Rm}|(q^i_j, t_i) \leq \frac{C}{r^2}, \quad \text{for } i \geq i_0,$$

which contradicts $Q^i_j \to \infty$ as $i \to \infty$.

The previous discussion and Claim 10 imply that there is an $i_0$ so that for every $\bar{p}_j$, where $1 \leq j \leq L$, we can find a sequence of points $\{q^i_j\}_{i \geq i_0} \in M$ so that

$$\int_{B_{r_i}(q^i_j, 2r)} |\text{Rm}|^{\frac{n}{2}} dV_{t_i} > \epsilon, \quad \text{(32)}$$

for all $i \geq i_0$. By our careful choices of $r$ and the covering of $S$, (by increasing the constant 1000 in the number of balls of radius $2r$ covering $S$, if necessary), for every fixed $i \geq i_0$, the number of points $q^i_j \in L^r_i$ with the property that the balls $B_{r_i}(q^i_j, r)$ are disjoint is at least $\left\lceil \frac{\epsilon}{r^2} \right\rceil + 2$, which contradicts (28).

This finishes the proof of Proposition 8. □

Combining Proposition 8 and the results from (25), so far we have proved the following: if $g(t)$ is the Kähler-Ricci flow, with $|\text{Ric}|(\cdot, t) \leq C$ and $\int_M |\text{Rm}|^{\frac{n}{2}} dV_t \leq C$, uniformly along the flow, then for every sequence of times $t_i \to \infty$ there exists a subsequence so that $(M, g(t_i + t))$ converges to $(M_\infty, g_\infty(t))$ in the following sense. There are finitely many points $p_1, \ldots, p_N \in M_\infty$...
$M_\infty$ and \{\(p_j^i\)\} \(\in M\), for \(1 \leq j \leq N\), so that for every \(r > 0\) there are diffeomorphisms \(\phi_i\) from \(M_\infty \cup \bigcup_{j=1}^{N} B_\infty(p_j, r)\) into \(M\), with the image of \(\phi_i\) containing \(M \cup \bigcup_{i=1}^{N} B_{t_i}(p_j^i, 2r)\), and \(\phi_i^* g(t_i + t)\) smoothly converging to the Kähler-Ricci soliton \(g_\infty(t)\), outside the singular points. This in particular means \(M_\infty\) is smooth outside finitely many points. We would like to understand the structure of those singular points.

By Fatou’s lemma,

$$\int_{M_\infty} |Rm|^{\frac{n}{2}} dV_\infty \leq C < \infty.$$

By the continuity of volume under the condition of the lower bound on Ricci curvatures of a sequence of manifolds (see [10]), we have

$$\lim_{i \to \infty} Vol_{t_i}(B_{t_i}(x_i, r)) = Vol_\infty(B_\infty(x_\infty, r)),$$

for every sequence \(x_i \in M\) and \(r > 0\) such that a sequence of balls \(B_{t_i}(x_i, r)\) converges in Gromov-Hausdorff topology, as \(i \to \infty\), to a ball \(B_\infty(x_\infty, r)\). This together with (27) yield

$$Vol_\infty(B_\infty(x, r)) \leq 2C_1 r^n, \quad \text{for all } x \in M_\infty.$$

Let \(\epsilon << \epsilon_0\). The previous estimate implies there is an \(r_0\) so that for \(r \leq r_0\), for every \(x \in M_\infty\),

$$\int_{B_\infty(x, r)} |Rm|^{\frac{n}{2}} dV_{g_\infty(t)} < \epsilon,$$

for all \(t \in [-2r^2, 2r^2]\). Fix some \(r > 0\) and denote by \(D_r^i = M \setminus \bigcup_{j=1}^{N} B_{t_i}(p_j^i, r)\).

By the definition of convergence, there exists an \(i_0\) so that for \(i \geq i_0\), all \(x \in D_r^i\) and \(t \in [-r^2, r^2]\),

$$\int_{B_{t_i}(x, r)} |Rm|^{\frac{n}{2}} dV_{t_i + t} < 2\epsilon < \epsilon_0.$$

By Proposition 5 we have

$$\sup_{[t_i - \delta r^2, t_i + \delta r^2]} |Rm| \leq \frac{C}{r^2},$$

for uniform constants \(C\) and \(\delta\). By Shi’s estimates,

$$\sup_{[t_i - \delta_1 r^2, t_i + \delta_1 r^2]} |D^k Rm| \leq C(k, n, r),$$

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where we can assume without no loss of generality that \( \delta_1 = 1 \). We can extract a subsequence, so that \( (D_2^{2r}, g(t_{i} + t)) \) converges to a smooth solution to the Ricci flow, \( (D_2^{2r}, g_{\infty}(t)) \), for \( t \in [-r^2, r^2] \). As in \[26\] and \[25\], we can show \( g_{\infty}(t) \) satisfies the Kähler Ricci soliton equation,

\[
\text{Ric}(g_{\infty}) + \nabla \bar{\nabla} f_{\infty} - g_{\infty} = 0, \\
\nabla \nabla f_{\infty} = 0.
\]  

(33)

We now choose a sequence \( \{r_l\} \to 0 \) with \( r_{l+1} < r_l/2 \) and perform the above construction for every \( l \). If we set \( D_i(r_l) = \{ x \in M | x \in D_i(r_{i,j}) \text{ for some } j \leq l \} \) then we have

\[ D_i(r_l) \subset D_i(r_{l+1}) \subset \cdots \subset M. \]

For each fixed \( r_l \), by the same arguments as above, each sequence \( \{D_i(r_l), g(t_{i} + t)\} \) for \( t \in [-r_l^2, r_l^2/2] \), has a smoothly convergent subsequence to a smooth limit \( D(r_l) \) with a metric \( g_{\infty}^l \), satisfying the Kähler Ricci soliton condition. We can now set \( D = \bigcup_{l=1}^{\infty} D(r_l) \) with the induced metric \( g_{\infty} \) that coincides with \( g^l \) on \( D(r_l) \) and which is smooth on \( D \).

Following section 5 in \[1\] we can show there are finitely many points \( \{x_i\} \) so that \( M_\infty = D \cup \{x_i\} \) is a complete length space with a length function \( g_{\infty}(0) \), which restricts to the Kähler Ricci soliton on \( D \). By (33), since \( |\text{Ric}(g_{\infty}(t))| \leq C \), we get,

\[
\sup_{M_\infty \setminus \{x_1, \ldots, x_N\}} |D^2 f_{\infty}| \leq \tilde{C}.
\]

Similarly as in \[26\] we get \( |\nabla f_{\infty}|, |f_{\infty}| \) are uniformly bounded on \( M_\infty \setminus \{x_1, \ldots, x_N\} \).

We will include below the computation from \[7\], that holds for all \( x \in M_\infty \setminus \{p_1, \ldots, p_N\} \). Denote \( \text{Rm}(g_{\infty}) \) shortly by \( \text{Rm} \). By Bochner-Weitzenböck formulas we have

\[
\Delta |\text{Rm}|^2 = -2\langle \Delta \text{Rm}, \text{Rm} \rangle + 2|\nabla \text{Rm}|^2 - \langle Q(\text{Rm}), \text{Rm} \rangle,
\]

(34)

where \( Q(\text{Rm}) \) is quadratic in \( \text{Rm} \). The Laplacian of the curvature tensor in the Kähler case reduces to

\[
\Delta R_{ijkl} = \nabla_i \nabla_j R_{jk} + \nabla_j \nabla_k R_{il} + S_{ijkl},
\]

where \( S(\text{Rm}) \) is quadratic in \( \text{Rm} \). Since our metric \( g_{\infty} \) is the soliton metric \( g_{\infty} \), satisfying (33) outside \( \{p_1, \ldots, p_N\} \), by commuting the covariant
derivatives, we get

\[ \Delta R_{ijkl} = (f_\infty)_{jkl} + (f_\infty)_{i\bar{k}\bar{l}} + S_{ijkl} \]

\[ = (f_\infty)_{j\bar{k}l} + \nabla_i(R_{j\bar{k}m}(f_\infty))_m + \nabla_j(R_{i\bar{k}m}(f_\infty))_m + S_{ijkl} \]

\[ = \nabla_i(R_{j\bar{k}m}(f_\infty))_m + \nabla_j(R_{i\bar{k}m}(f_\infty))_m + S_{ijkl} \]

\[ = \nabla R_m \ast \nabla (f_\infty), \quad (35) \]

where we have effectively used the fact that \((f_\infty)_{ij} = (f_\infty)_{\bar{i}\bar{j}} = 0\) and \(A \ast B\) denotes any tensor product of two tensors \(A\) and \(B\) when we do not need precise expressions. By using that \(|f_\infty|_{C^1(M_\infty \setminus \{p_1, \ldots, p_N\})} \leq C\), and identities (34) and (35) we get,

\[ \Delta |Rm|^2 \geq -C|\nabla Rm||Rm| + 2|\nabla Rm|^2 - C|Rm|^3. \]

By interpolation inequality we have

\[ \Delta |Rm|^2 \geq (2 - \theta)|\nabla Rm|^2 - C(\theta)|Rm|^2 - C|Rm|^3 \]

\[ \geq (2 - \theta)|\nabla |Rm||^2 - C(\theta)|Rm|^2 - C|Rm|^3, \]

for some small \(\theta\). Also,

\[ \Delta |Rm|^2 = 2\Delta |Rm||Rm| + 2|\nabla |Rm||^2, \]

and therefore,

\[ \Delta |Rm||Rm| \geq -\theta/2|\nabla |Rm||^2 - C(\theta)|Rm|^2 - C|Rm|^3. \quad (36) \]

By the same arguments as in [7] (see sections 5 and 6), we can show

(a) \(M_\infty = D \cup \{p_1, \ldots, p_N\}\) is a complete orbifold with isolated singularities \(\{p_1, \ldots, p_N\}\).

(b) A limit metric \(g_\infty\) on \(D\) can be extended to an orbifold metric on \(M_\infty\) (denote this extension by \(g_\infty\) as well). More precisely, in an orbifold lifting around singular points, in an appropriate gauge, the Kähler Ricci soliton metric \(g_\infty\) can be smoothly extended over the origin in a ball in \(C^n/2\).

This finishes the proof of Theorem 2. \(\Box\)
4 Convergence of plurianticanonical divisors

We will prove Theorem 3 by contradiction. Assume there is a sequence \( t_i \to \infty \) such that \( Q_i := \max_M |Rm|(\cdot, t_i) \to \infty \) as \( i \to \infty \). Considering the sequence of solutions \((M, g(t_i + t))\) and using the ideas of Tian, we will show that a subsequence \((M, g(t_i))\) smoothly converges to a smooth Kähler manifold \((M_{\infty}, g_{\infty})\), which will contradict the fact \( Q_i \to \infty \). By Theorem 2, we may assume \((M, g(t_i + t))\) converges to a Kähler Ricci soliton \((M_{\infty}, g_{\infty})\) in an orbifold sense, with finitely many orbifold points \( p_1, p_2, \ldots, p_l \).

A line bundle \( E \) on \( M_{\infty} \) is a line bundle on the regular part \( M_{1\infty}^{0} \) such that for each local lifting of a singular point \( x \), \( \pi_{x} : \tilde{B}_{x} \to M_{\infty} \), the pullback \( \pi_{x}^{\ast} E \) on \( \tilde{B}_{x} \backslash \pi_{x}^{-1}(x) \) can be extended to the whole \( \tilde{B}_{x} \). We will consider plurianticanonical line bundles \( K_{\infty}^{-m} \) for \( m \in \mathbb{N} \). A global section of \( K_{\infty}^{-m} \) is an element in \( H^{0}(M_{\infty}^{0}, K_{\infty}^{-m}) \), which can be extended across the singular points in the above sense. Therefore, \( H^{0}(M_{\infty}, K_{\infty}^{-m}) \) can be understood as a linear space of all global sections of \( K_{\infty}^{-m} \) and \( g_{\infty} \) induces a hermitian orbifold metric on \( K_{\infty}^{-m} \).

The proof of Theorem 3 will follow from a sequence of lemmas and claims. The following Lemma is taken from [28] and [27].

Lemma 11.

(i) Let \( S^{i} \) be a global holomorphic section of \( H^{0}(M, K_{M}^{-m}) \), \( \int_{M} ||S^{i}||_{g(t_i)} \ dV_{t_i} = 1 \), where \( ||\cdot||_{g(t_i)} \) is the hermitian metric of \( K_{M}^{-m} \) induced by \( g(t_i) \). Then there is a subsequence \( \{S^{i}\} \) converging to a global holomorphic section \( S^{\infty} \) in \( H^{0}(M_{\infty}, K_{\infty}^{-m}) \).

(ii) Any section \( S \) in \( H^{0}(M_{\infty}, K_{\infty}^{-m}) \) is the limit of some sequence \( \{S^{i}\} \) where \( S^{i} \in H^{0}(M, K_{M}^{-m}) \).

In particular, (i) and (ii) imply the dimension of \( H^{0}(M, K_{M}^{-m}) \) is the same as that of \( H^{0}(M_{\infty}, K_{\infty}^{-m}) \).

Sections \( S^{i} \) and \( S^{\infty} \) are not on the same Kähler manifolds. From the definition of the convergence, for every compact set \( K \subset M_{\infty} \) there are diffeomorphisms \( \phi_{i} \) from compact subsets \( K_{i} \subset M_{\infty} \) so that \( (\phi_{i}^{-1})^{\ast} g(t_i) \to g_{\infty} \) and \( \phi_{i}^{\ast} \circ J_{i} \circ (\phi_{i}^{-1})^{\ast} \to J_{\infty} \), as \( i \to \infty \). The convergence of sections in the previous lemma means that the sections \( \phi_{i}^{\ast}(S^{i}) \) converge to a section \( S^{\infty} \) of \( K_{\infty}^{-m} \) in the \( C^{\infty} \)-topology.

Proof of Lemma 11. Let \( S^{i} \) be as in the statement of the lemma. If \( \Delta_{i} \) is the laplacian and \( ||\cdot||_{i} \) the inner product with respect to metric \( g(t_i) \), by a
direct computation, we have
\[
\Delta_i(||S^i||^2_i)(x) = ||D_i S^i||^2_i(x) - \text{Ric}(S^i, S^i)(x),
\]
where \(D_i\) is the covariant derivative with respect to \(g(t_i)\). Since \text{Ric} is uniformly bounded along the flow,
\[
\Delta_i(||S^i||^2_i)(x) \geq -C||S^i||^2_i(x),
\]
(37)

Since we have \(\int_M ||S^i||_i \ dV_{t_i} = 1\) and Sobolev inequality (2) which holds for all \(t_i \geq 0\), with the uniform upper bound on the Sobolev constant, applying Moser’s iteration to (37), there is a uniform constant \(C = C(n)\) such that
\[
\sup_M (||S^i||^2_i(x)) \leq C, \quad \text{for all } i.
\]
(38)

Once we have the estimate (38) we can proceed as in the proof of Lemma 2.1 in [28]. We need to show that for every integer \(j > 0\), the \(j\)-th covariant derivatives of \(\phi_i S^i\) (\(\phi_i\) is a diffeomorphism that comes from a definition of convergence) are bounded in every compact set \(K \subset M_\infty \{p_1, \ldots, p_l\}\). Depending only on \(K\) there is an \(r > 0\) so that for every \(x \in K\) the geodesic ball \(B_r(x, g(t_i))\) is uniformly biholomorphic to an open subset of \(\mathbb{C}^k\). On each \(B_r(x, g(t_i))\), the section \(S^i\) is represented by a holomorphic function. We can use well-known Cauchy integral formula to get uniform bounds on \(j\)-th covariant derivatives of \(S^i\).

Absolutely the same proof as that of Lemma 2.2 works also in our case to prove that any section \(S\) in \(H^0(M_\infty, K^{-m}_{M_\infty})\) is the limit of some sequence \(\{S^i\}\) with \(S^i \in H^0(M, K^{-m}_M)\). This in particular implies that the dimension of \(H^0(M_\infty, K^{-m}_{M_\infty})\) is the same as that of \(H^0(M, K^{-m}_M)\). \(\square\)

5 Smoothness property in complex dimensions \(\geq 3\)

Given a complex manifold \(X\) with strongly pseudoconvex boundary \(Y\), we define \(\mathcal{B}^{p,q}(Y)\) to be the space of smooth sections of \(\Lambda^p(X) \cap \Lambda^q(T^*_X Y \otimes \mathbb{C})\). The \(\bar{\partial}\) operator of \(X\) induces \(\bar{\partial}_b : \mathcal{B}^{p,q}(Y) \rightarrow \mathcal{B}^{p,q+1}(Y)\). Let \(\bar{\partial}_b^k\) be the adjoint operator of \(\bar{\partial}_b\) on \(Y\). Since \(\bar{\partial}_b^0 = 0\), we have the boundary complex
\[
0 \rightarrow \mathcal{B}^{p,0} \xrightarrow{\partial_b} \mathcal{B}^{p,1} \rightarrow \cdots \xrightarrow{\partial_b} \mathcal{B}^{p,n-1} \rightarrow 0.
\]
The cohomology of this boundary complex is called Kohn-Rossi cohomology and is denoted by \(H^{p,q}(\mathcal{B})\).
By Theorem 2 we have that a sequence \((M, g(t))\) converges to a Kähler Ricci soliton in an orbifold sense. In other words, there are points \(p_{1i}, \ldots, p_{Ni}\) in \((M, g(t))\) and \(p^{1, \infty}, \ldots, p^{N, \infty}\) in \(M_{\infty}\) so that: for every \(r > 0\) there are diffeomorphisms \(\phi_i\) from compact sets \(M_{\infty}\ \{(B_r(p_{1i}) \cup \cdots \cup B_r(p_{Ni}))\} \) into \(M\), containing \(M \backslash B_{2r}(p_{1i}, g(t_i)) \cup \cdots \cup B_{2r}(p_{Ni}, g(t_i))\) so that \(\phi_i^* g_i\) and \(\phi_i^* \circ J_i \circ (\phi_i^{-1})^*\) converge to \(g_\infty\) and \(J_\infty\), respectively. We would like to understand the holomorphic structure of \(B_r(p_i)\), \(1 \leq l \leq N\), for sufficiently small \(r\) and big \(i\). The same problem arose in [28] where the sequence of Kähler-Einstein manifolds converges to a Kähler-Einstein orbifold with isolated singularities. The main tool there was Kohn’s estimate for \(\square_b\)-operators that works only when \(k \geq 3\). Let \(S_{\infty} := \partial B_r(p_\infty)\) (where \(p_\infty\) is one of the points \(\{p_{1, \infty}, \ldots, p_{N, \infty}\}\)) be the level surface of the distance function \(\rho_\infty(\cdot, p_\infty)\). The Levi form on \(S_{\infty}\) is \((L_1, L_2) := 2(\partial \bar{\partial} \rho_\infty(\cdot, p_\infty), L_1 \wedge L_2)\).

It is positive definite for \(r\) small since \(\rho(\cdot, p_\infty)\) is convex near \(p_\infty\). Let \(S_{\infty}\) be the level surface \(\{x \in (M, g(t))| \rho_\infty(p_{\infty}, \phi_i^{-1}(x)) = r\}\), which is also a smooth, pseudoconvex manifold. Denote by \(\hat{S}_i\) the universal covering of \(S_{\infty}\), it is diffeomorphic to \(S^{n-1}\). By using the result in [33] that \(H^{0,1}(B(S^{n-1})) = 0\) for \(n \geq 6\) (or \(k \geq 3\)), Tian [28] showed the following Kohn’s estimate for \(\partial_b\)-operator,

\[
C||u||_2^2 \leq ||\partial_b u||_2^2 + ||\bar{\partial}_b u||_2^2,
\]

for any \(u \in B^{0,1}(\hat{S}_i)\) and a uniform constant \(C\). If \(\lambda\) is the smallest eigenvalue of \(\square_b = \partial_b \bar{\partial}_b + \bar{\partial}_b \partial_b\), the following estimate is equivalent to \(\lambda \geq c > 0\). Tian used this to show there exist embeddings \(k_{ij} : \hat{S}_j \rightarrow \mathbb{C}^k\) so that \(k_{ij}(\hat{H}_j)\) converge to \(S^{n-1}\) as submanifolds of \(\mathbb{C}^k\), in sufficiently nice topology. Following the same proof as that of Tian in [28] we get the following proposition.

**Proposition 12.** Let \((M, g(t))\) be the Kähler Ricci flow and \(t_i \rightarrow \infty\) as at the beginning of section 4. Then either a sequence \(\{(M, g(t_i))\}\) converges to a Kähler-Ricci soliton in \(C^0\)-topology, or there is a smooth Kähler-Ricci soliton \((M_\infty, g_\infty)\) so that a sequence \(\{(M, g(t_i))\}\) converges to \((M_\infty, g_\infty)\) in the \(C^0\)-topology, outside finitely many points.

The proof of the above proposition uses results from section 4 more precisely Lemma 11 to get the orthonormal bases \(\{S_j\}^{N_0}_j\) of \(H^0(M, K_{M}^{-m})\), with respect to metric \(g(t_i)\), that converges to the basis \(\{S_0^\infty, \ldots, S_{N_0}^\infty\}\) of \(H^0(M_{\infty}, g_\infty)\) (defining the Kodaira’s embedding of \(M_{\infty}\) into \(\mathbb{CP}^{N_0}\)). In particular, for \(j\) sufficiently large, these \(\{S_j\}\) give embeddings of \(M\) into \(\mathbb{CP}^{N_0}\). The proof of Proposition 12 can be found in section 3 of [28].
Proof of Theorem \[3. \] The proof is by contradiction. Assume there is a sequence \( t_i \to \infty \) so that \( \max_M |\mathrm{Rm}|(\cdot, t_i) \to \infty \) as \( i \to \infty \). Take \( \eta > 0 \) small. Since \( |\mathrm{Ric}| \leq C \) along the flow, for every \( i \), the metrics \( g(t_i + t) \) are uniformly equivalent for \( t \in [-\eta, \eta] \), with constants independent of \( i \), that is,

\[
c g(t_i + t) \leq g(t_i) \leq C g(t_i + t), \quad \text{for all } i \text{ and all } t \in [-\eta, \eta].
\]

This means that by Proposition \[12. \] either a sequence of solutions \( \{(M, g(t_i + t))\} \) converges to a Kähler-Ricci soliton solution in \( C^p \)-topology, or there is a smooth Kähler-Ricci soliton \( (M_\infty, g_\infty(t)) \) so that a sequence \( \{(M, g(t_i + t))\} \) converges to \( (M_\infty, g_\infty(t)) \) in the \( C^p \)-topology, outside finitely many points, for every \( t \in [-\eta, \eta] \). This yields each \( B_r(p_\infty, g_\infty(t)) \), for \( t \in [-\eta, \eta] \), with small \( r \), is a smooth ball in \( \mathbb{C}^k \). Therefore, for \( i \) large enough and \( t \in [-\eta, \eta] \), \( B_r(p_\infty, g(t_i + t)) \) are smooth balls in \( \mathbb{C}^k \) as well.

To finish the proof of Theorem \[3. \] we will need one of the achievements of Perelman, known as the pseudolocality theorem for the Ricci flow (\[22. \]). It says that for every \( \alpha > 0 \) there exists \( \delta > 0 \), \( \epsilon > 0 \) with the following property. Suppose we have a smooth solution to the Ricci flow \( (g_{ij})_t = -2R_{ij} \) for \( 0 \leq t \leq (\epsilon r_0)^2 \) and assume that at \( t = 0 \) we have \( R(x) \geq -r_0^{-2} \) and \( \mathrm{Vol}(\partial \Omega)^n \geq (1 - \delta)c_n \mathrm{Vol}(\Omega)^{n-1} \) for any \( x \) and \( \Omega \subset B(x_0, r_0) \), where \( c_n \) is the euclidean isoperimetric constant. Then we have an estimate \( |\mathrm{Rm}|(x,t) \leq \alpha t^{-1} + (\epsilon r_0)^{-2} \), whenever \( 0 < t \leq (\epsilon r_0)^2 \), \( d(x,t) = \mathrm{dist}_t(x, x_0) \leq \epsilon r_0 \).

We will apply the pseudolocality theorem to each of the solutions \( \tilde{g}_i(t) = g(t_i - \eta + t) \), for \( i \geq i_0 \). Fix \( \alpha > 0 \) and choose \( \epsilon, \delta > 0 \) as in Perelman’s result. Since for \( r_0 = r \) the conditions of the pseudolocality theorem are satisfied, \( |\mathrm{Rm}|(\tilde{g}_i)(x,t) \leq \alpha t^{-1} + (\epsilon r)^{-2} \), whenever \( 0 < t < (\epsilon r)^2 \) and \( \mathrm{dist}_{\tilde{g}_i(t)}(x, p_i(t)) \leq \epsilon r \). If \( (\epsilon r)^2 < \eta \), apply Perelman’s result to \( \tilde{g}_i(\cdot, t) \) starting at \( t = (\epsilon r)^2 / 2 \), since it also satisfies the assumptions of his theorem. This will help us extend our uniform in \( i \) estimates on \( |\mathrm{Rm}|(\tilde{g}_i(t)) \) past time \( (\epsilon r)^2 \). Repeat this procedure until we hit \( t = \eta \). To summarize, we get that

\[
\sup_M |\mathrm{Rm}|(\cdot, t_i) \leq C(n, \epsilon, \alpha, \eta, r), \quad \text{for all } i \geq i_0,
\]

and we get a contradiction with \( \max_M |\mathrm{Rm}|(\cdot, t_i) \xrightarrow{i \to \infty} \infty \). \[\square\]

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