Locating chromatic number of direct product of some graphs

M. Vivekanandan\textsuperscript{1}* and R. Srinivasan\textsuperscript{2}

Abstract
The locating chromatic number of \( G \), denoted \( \chi_{Lc}(G) \) is the least \( r \) in such a way that \( G \) requires a locating colouring with \( r \) colours. In this paper, we determine the values of the locating chromatic number of direct product graphs among the graphs, complete graph \((K_4)\), path graph \((P_5)\).

Keywords
Chromatic Number, Locating colouring, Locating Chromatic number, Direct product.

AMS Subject Classification
05C90.

\textsuperscript{1}Department of Mathematics, Parisutham institute of technology and Science, Thanjavur-613006, Tamil Nadu, India.
\textsuperscript{2}Department of Ancient Science, Faculty of Science, Tamil University, Thanjavur-613006, Tamil Nadu, India.
*Corresponding author: \textsuperscript{1}vivekmaths85@gmail.com;
Article History: Received 01 February 2020; Accepted 29 March 2020

Contents
1 Introduction ................................................. 363
2 The Locating Chromatic number of product of two Path graphs ...................... 364
3 The locating Chromatic number of the product of Complete graph and Path graph . 364
4 Conclusion .................................................. 365
References ..................................................... 365

1. Introduction

Let \( G = (V,E) \) be a linear graph which consists a set of two objects called vertices and edges denoted as \( V = \{v_1,v_2,\ldots,v_n\} \) and \( E = \{e_1,e_2,\ldots,e_n\} \) in such a way that every edge \( e_k, 1 \leq k \leq n \), is associated with an unordered pair of vertices, say \((v_i,v_j)\). In this paper we only deal the graphs which are without self-loops and parallel edges. Painting every vertices in the graph using some colours provided different colours to be used to the vertices which are adjacent, this process is called proper colouring and also referred as simply colouring of a graph. A graph is called properly coloured graph if all the vertices of a graph is painted a colour according to the proper colouring. In so many different ways a graph can be properly coloured, in which the minimum number of colours used to colour all the vertices of a graph using proper colouring is called chromatic number. If \( r \) different colours required for its proper colouring, and not less, is called \( r \) chromatic graph and \( r \) is the chromatic number of the graph denoted by \( \chi(G) \). Let \( v_1,v_2 \) be any two distinct vertices of a graph \( G \), then the distance between \( v_1 \) & \( v_2 \), denoted as \( d(v_1,v_2) \), is the length of the smallest path between them, and \( v_1 \) be any vertex of \( G \), \( P \) be the subset of the vertex set of \( G \), then the distance between \( v_1 \) and \( P \) is given by \( d(v_1,P) = \min \{d(v_1,v_2) \mid v_2 \in P\} \).

Definition 1.1 ([3]). Let \( c \) be a proper \( r \)-coloring of a connected graph \( G \) and \( \Pi = \{P_1,P_2,\ldots,P_r\} \) be an ordered partition of \( V(G) \) into the resulting classes. For a vertex \( v_1 \) of \( G \), the colour code of \( v_1 \) with respect to \( \Pi \) is defined to be the ordered \( r \) tuple

\[
c_{\Pi}(v_1) = (d(v_1,P_1),d(v_1,P_2),\ldots,d(v_1,P_r))
\]

If different colour codes have assigned to different vertices of \( G \), then \( c \) is called a locating colouring of \( G \). The least number of colours required for locating colouring is called locating chromatic number of \( G \), denoted as \( \chi_{Lc}(G) \).

The direct product is also referred in many terminologies such as tensor product, cardinal product, weak direct product, relational product, Kronecker product, or conjunction. Direct product was introduced by Whitehead and Russel [1]. Direct product satisfy commutative and associative properties [2]

Definition 1.2 ([2]). Let \( G = (V_1,E_1) \) and \( H = (V_2,E_2) \) be any two simple graphs with \( V_1 = \{v_1,v_2,\ldots,v_t\} \) and \( V_2 = \{v'_1,v'_2,\ldots,v'_j\} \). Then the direct product of the graphs \( G \) and
Let $H$ be the resulting graph of the product of complete graph with $x$ vertices denoted as $K_x$ and the path graph with $y$ vertices denoted as $P_y$. Vertices of $H$ can be represented in $x \times y$ matrix, that is, $H$ contains $x$ number of rows and $y$ number of columns. Clearly the complete graph with $x$ vertices ($K_x$) is isomorphic to the induced subgraph on the vertices of every column and by the same way the path graph with $y$ vertices ($P_y$) is isomorphic to the induced

\[ \chi_{LC}(H_1 \times H_2) \leq \chi_{LC}(H_1) \chi_{LC}(H_2). \]

**Proof.** Let $H_1$ be any connected graph with $x_1$ locating colouring with the colour class $C_1, C_2, \ldots, C_{x_1}$, that is, $x_1 = \chi_{LC}(H_1)$ and let $H_2$ be any connected graph with $x_2$ locating colouring with the colour class $C'_1, C'_2, \ldots, C'_{k_2}$, that is, $x_2 = \chi_{LC}(H_2)$. For every $i \in [x_1]$ and every $j \in [x_2]$, $C_i \times C'_j$ is an independent set in $H_1 \times H_2$. Therefore $\{C_i \times C'_j, i \in [x_1], j \in [x_2]\}$ of vertices of $H_1 \times H_2$ will be in the colour classes of proper colouring of $H_1 \times H_2$. To verify that the above is a locating colouring, let us consider any two distinct vertices say $(a_1, b_1)$ and $(a_2, b_2)$ in the colour class $C_i \times C'_j$ provided that $a_1 \neq a_2$. Also $d(b_1 \times C'_j) = d(b_2 \times C'_j) = 0$, then there exists $l$ such that $d((a_1 \times C_l) \neq d((a_2 \times C_l)).$ Therefore

\[ d((a_1, b_1)C_i \times C'_j) = d((a_1, C_i) + d(b_1, C'_j) = d((a_1, C_i) + 0 \neq d((a_2, C_i) + d((a_2, b_2), C_i \times C'_j). \]

It is very clearly indicating that the above mentioned colouring is a locating colouring.

Suppose $H_1$ and $H_2$ are complete graph with two vertices (i.e. $K_2$), then we have

\[ \chi_{LC}(K_2 \times K_2) = \chi_{LC}(C_4) = 4 = \chi_{LC}(K_2) \chi_{LC}(K_2) \]

Hence the above mentioned inequality holds good. 

**Theorem 2.2.** Let $x$ and $y$ be any arbitrary integers and $y$ exceeds 2, $x$ exceeds $y$ then the locating chromatic number of the product of two path graphs with $x$ and $y$ vertices respectively is four, that is, $\chi_{LC}(P_x \times P_y) = 4$ provided $y \geq 2$ \& $x \geq y$.

**Proof.** Let $P_x$ and $P_y$ be any two path graphs with $x$ and $y$ vertices provided $y \geq 2$ \& $x \geq y$. Clearly there is an induced cycle $C_4$ available with 3 colours in every proper 3 colouring of the product of two path graphs with $x$ and $y$ vertices. Here there are two vertices on this cycle with the same colour. Hence $\chi_{LC}(P_x \times P_y) \geq 4$. For every $i \in [x]$ and $j \in [y]$, let $u_{i,j}$ be the vertex in the mesh of $P_x \times P_y$ as $i$th row and $j$th column. Now $P_x \times P_y$ is the product of two graphs with the proper 2 colouring with the colour set $\{c_1, c_2\}$ denoting as $C$. Let us define the colouring class $C'$ as follows

\[ C'(u_{1,1}) = c_3, C'(u_{1,y}) = c_4 \text{ and } C'(u_{i,j}) = C(u_{i,j}), \]

then we have

\[ d(u_{i,j}, u_{1,1}) = i + j - 2 \]

\[ d(u_{i,j}, u_{1,n}) = n + i - (j + 1). \]

From the above it is clear that, distinct vertices have distinct colour codes, hence it completes the proof.
subgraph on the vertices of every row. Let \( u_{i,j} \) be the vertex of the resulting graph of the product of complete graph with \( x \) vertices and the path graph with \( y \) vertices which is in the \( i \)th row and \( j \)th column of the \( x \) by \( y \) matrix where \( i \leq x \) and \( j \leq y \). Therefore which is clearly indicates that \((i, j)\) is the colour of \( u_{i,j} \).

**Theorem 3.1.** Let \( x \) and \( y \) be the positive integers provided \( x \geq 3 \) and \( y \geq 2 \) and \( H \) be the resulting graph of the product of complete graph with \( x \) vertices and the path graph with \( y \) vertices and there is a locating \((x + 1)\) colouring then \( X \) be its colouring matrix. Then every two successive columns of \( X \) has contrasting lost colours.

**Proof.** Let us consider the number of vertices of a complete graph be \( 3 \) i.e. \( x = 3 \) and \( X \) be the matrix of the resulting graph of the product of complete graph with \( x \) vertices and the path graph with \( y \) vertices \( (K_x \times P_y) \). We need to prove that every two successive columns of \( X \) has contrasting lost colours.

By the method of contradiction, there are two colours which are successive say \( X_j \) and \( X_{j+1} \) of \( X \) with the same lost colour, say \( c_4 \)

Let us assume that \( X_j = [c_1 \ c_2 \ c_3]^T \), obviously \( X_{j+1} \) is the alternate of \( X_j \) then \( X_{j+1} = [c_3 \ c_1 \ c_2]^T \) or otherwise \( X_{j+1} = [c_2 \ c_3 \ c_1]^T \). If \( j = 1 \) and \( X_j \) will be the first column and in row \( i \) in which the colour \( c_4 \) can be used, then the distance between the vertices \( u_{i,1} \) and \( u_{i,1+1} \) are equal to the colour class \( c_4 \). Which is contradiction to the fact that different colours to be used for same distance vertices. For \( j+1 = y \), the above discussed procedure can be followed, not only for that, the same argument can be followed pertaining the colour \( c_4 \) have the index more than \( j+1 \) or otherwise all the indices smaller than \( j \)

Therefore assume that there are two indices \( j_1 \) and \( j_2 \) whereas \( j_1 < j, j < j+1 \) & \( j+1 < j_2 \) in such a way that the colour \( c_4 \) appears in both the columns \( j_1 < j_2 \). If \( j-j_1 = j_2-j \), then the distance between the columns \( j_1 \) and \( j_2 \) is equal to the colour code \( c_4 \). Therefore at the minimum, two vertices of the same distances have the same colour code. Therefore WLG assume that \( j-j_1 < j_2-j \), and the colour code \( c_4 \) available in \( I \) row of the column \( X_{j_1} \) Now the distance between the vertices \( u_{3,j} \) and \( u_{1,j+1} \) are equal and having the colour code \( c_4 \), which is the contradiction to the fact that the same distance having vertices assigning different colour codes.

For \( x \geq 4 \), then the same kind of approach can be viewed, we can find two vertices with the same distance having the same colour code. Which is contradiction to the fact that the same distance having vertices assigning different colour codes.

Therefore our assumption is wrong. Hence every two successive columns of \( X \) has contrasting lost colours.

**Theorem 3.2.** Let \( x \geq 3 \) and \( y \geq 2 \) then \( \chi_{LC}(K_x \times P_y) = x + 1 \) provided \( x \geq y - 1 \).

**Proof.** Let \( H \) be the resulting graph of the product of the graphs complete graph with \( x \) vertices and path graph with \( y \) vertices. By Proposition (2), we have \( \chi_{LC}(H) = x + 1 \), let us assigning \( x + 1 \) colouring to \( H \). Let \( [c_{x+1} \ c_1 \ c_2 \ldots \ c_{x-2} \ c_{x+2}]^T \) be the first column vector of the colouring matrix \( X \) and the other columns are \([c_1 \ c_2 \ldots c_{x-1} \ c_1]^T\) and \([c_x \ c_1 \ c_2 \ldots c_{x-1}]^T\) then the distinct vertices of \( H \) have distinct colour codes, that is, no two distinct vertices with the same distance and same colour code, clearly it is locating colouring of \( H \).

Hence \( \chi_{LC}(H) = x + 1 \) or \( \chi_{LC}(H) = x + 2 \). We are in a position to prove that if \( \chi_{LC}(H) = x + 1 \) then \( x \geq y - 1 \).

Let us assume that \( \chi_{LC}(H) = x + 1 \) and \( X \) be the corresponding matrix of the locating \( x + 1 \) colouring of \( H \). Clearly it indicates that \( X \) contains \( x \) rows and one colour will be missing in every column. By Theorem 3.1, every two successive columns of \( X \) has contrasting lost colours and also all the columns of \( X \) have at the minimum of one full colour, which shows that \( H \) has \( x + 1 \) columns at the maximum. More precisely, assume \( x \geq y - 1 \) and proving \( \chi_{LC}(H) = x + 1 \) for \( x \in \{3, 4\} \) then \( E_1 \) be the matrix of \((K_3 \times P_4)\) and \((K_4 \times P_5)\), then

First column of \( E_1 = [c_1 \ c_2 \ c_3]^T \)
Second column of \( E_1 = [c_4 \ c_1 \ c_2]^T \)
Third column of \( E_1 = [c_2 \ c_4 \ c_3]^T \)
Fourth column of \( E_1 = [c_3 \ c_1 \ c_4]^T \).

And \( E_2 \) be the matrix of \((K_4 \times P_5)\), then
First column of \( E_2 = [c_1 \ c_2 \ c_3 \ c_4]^T \)
Second column of \( E_2 = [c_5 \ c_3 \ c_1 \ c_2]^T \)
Third column of \( E_2 = [c_1 \ c_3 \ c_2 \ c_4]^T \)
Fourth column of \( E_2 = [c_5 \ c_3 \ c_4 \ c_1]^T \)

Fifth column of \( E_2 = [c_4 \ c_5 \ c_3 \ c_2]^T \).

From the above classifications of \( E_1 \) and \( E_2 \), we came to know that distinct columns have distinct missing colours and therefore distinct colour codes are assigned to the vertices which are in same colour. There are absolutely \( x + 1 \) colourful vertices. Therefore this is locating colouring. Hence \( x \) belonging to either three or four and by the condition \( y \leq x + 1 \). Hence \( \chi_{LC}(K_x \times P_y) = x + 1 \).

**4. Conclusion**

In this paper, we studied colouring of the graph and analyze the locating chromatic number of graphs. In particularly studied the locating chromatic number of the direct product of path graphs and direct product of complete graph with path graph. More precisely, the above said techniques were analyzed through matrix method.

**References**

[1] A.N. Whitehead and B. Russel, *Principia Mathematica*, Volume 2, Cambridge University Press, Cambridge, (1912).
[2] W. Imrich and S. Klavzar, *Product graphs: Structure and Recognition*, John-Wiley & Sons, New York, USA (2000)

[3] G. Chartrand, D. Erwin, M.A. Henning, P.J. Slater, and P. Zhang, The locating chromatic number of a graph, *Bull. Inst. Combin. Appl.*, 36(2002), 89–101.

[4] A. Behtoei and B. Omoomi, On the locating chromatic number of Kneser graphs, To appear in *Discrete Appl. Math.*, 2020.

[5] Asmiati, H. Assiyatun, and E.T. Baskoro, Locating Chromatic number of amalgamation of stars, *ITB J. Sci.*, 43A(2001) 1–8.

[6] G. Chartrand, D. Erwin, M.A. Henning, P.J. Slater, and P. Zhang, Graphs of order $n$ with locating chromatic number $n - 1$. *Discrete Math.*, 269(13)(2003), 65–79.

[7] G. Chartrand, V. Saenpholphat, and P. Zhang, Resolving edge colourings in graphs, *Ars Combin.*, 74(2005), 33–47.

[8] Narsingh Deo, *Graph Theory with Applications to Engineering and Computer Science*, Prentice Hall Inc., India, 2000.

*********
ISSN(P):2319 – 3786
Malaya Journal of Matematik
ISSN(O):2321 – 5666
*********