On the existence of a conformal and an absolutely continuous invariant measure for transcendental entire maps

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Abstract

We identify a class of hyperbolic transcendental entire maps and we prove that some of its elements generate a class of potentials for which exhibit a conformal and invariant probability Gibbs measure. The methods and techniques from the thermodynamic formalism can be extended to this class of potentials. To complement this study we highlight that the dynamics of such a map on some subset of the Julia set is conjugated to the shift map over a code space with countable alphabet and the euclidean metric on the complex plane induces a metric on the symbolic space which is not compatible with the shift standard metric. From this fact, we provide a general description of the thermodynamic formalism from symbolic dynamic outlook, by studying the shift map acting on a non-compact and invariant subset of the full shift space with a countably infinite alphabet and a class of weakly Hölder continuous potentials, to prove the existence of a conformal and absolutely continuous invariant probability measure.

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1 Introduction

Great deal of attention has been paid to the study of the thermodynamic formalism of certain transcendental maps. In particular, the ergodic theory of the exponential family $E_\lambda(z) = \lambda \exp(z)$ has been described in great detail for a large class of parameters, see [McM87, St91a, Ka99, UZ03, UZ04, IS06, Ba07, BK07, CS07, MU10] and references therein.

When we deal with transcendental entire function, some difficulties arise in the study of the thermodynamic formalism, on the one hand the Julia set is never compact, so problems of convergence appear and the standard arguments such as Schauder-Tychonoff Fixed Point Theorem can not be applied. On the other hand,
the dynamics restricted to the Julia set is not Markov, so unfortunately the symbolic spaces we need to model Julia sets of these maps fall out of the framework developed by Sarig in [Sa99] and Mauldin-Urbański [MU01], who developed the thermodynamic formalism for topologically mixing Markov shift spaces with infinitely many symbols.

In the present work, we highlight a class of transcendental entire maps (See Definition in §1.1) that includes the exponential family, and we prove that there is a hyperbolic transcendental entire map, which generates a large class of potentials for which the existence of a conformal measure is guaranteed. The main novelty in this paper is the identification of a class of potentials which is different from the ones studied earlier in the literature by, for example, Mayer and Urbański [MU10] in the study of the thermodynamic formalism (see Theorem 1).

To develop a thermodynamic formalism of these maps with these associated potentials (see Theorem 2) the techniques used in [MU10] can be applied without making many changes, however it is worth addressing an alternative approach taking advantage of the properties of their symbolic representation of these maps acting on invariant subsets of the Julia sets, whose code spaces have a natural topology that is inherited from the Euclidean topology. So, the second part of this work is addressed to obtain results from a symbolic dynamics setting, by means of an approximation argument, considering restrictions of the map to subsets of the Julia set that can be encoded by a full shift on $2N + 1$ symbols, with $N$ as large as desired and since that space is compact and Markov the usual results hold and a conformal measure $\nu_N$ exists. So, the next step is to show that the sequence $\{\nu_N\}_N$ is tight and so this sequence has an accumulation point $\nu$. So the measure $\nu$ would be a conformal measure for the original map. All of these results are proved for the shift map acting on some non-compact and invariant subset of the full shift space with a countably infinite number of symbols, equipped with a metric that is not necessarily equivalent to the natural shift metric, therefore for a weakly Hölder continuous potentials the existence of a conformal and absolutely continuous invariant probability measure are proved. To do this, we impose some additional conditions on the symbolic space and potentials that are derived from the inherent properties of the transcendental dynamics. See Theorem 3, Theorem 4, Theorem 5.

The paper is organized as follows. In §1.1 we define the class of transcendental entire maps and a class of potentials to state and prove the Theorem 1. After gather several properties of the dynamics in Lemma 1 and properties of potentials in Proposition 1 the proof of Theorem 1 follows from techniques in [MU10]. After giving several metric conditions on the symbolic space, some preliminary Lemmas of the shift map and potentials in §3 we prove the respective Theorem 2, Theorem 3 and Theorem 4.

1.1 Hyperbolic transcendental entire maps

Given a transcendental entire function $f : \mathbb{C} \to \mathbb{C}$, the Fatou set $F(f)$ is the subset of $\mathbb{C}$ where the iterates $f^n$ of $f$ form a normal family and its complement is namely called the Julia set, which is denoted by $J(f)$. 
Denote by $\text{Sing}(f^{-1})$ the set of finite singularities of the inverse function $f^{-1}$, which is the set of critical values (images of critical points) and asymptotic values of $f$ together with their finite limit points. The post-singular set $PS(f)$ of $f$ is defined as,

$$PS(f) := \bigcup_{n=0}^{\infty} f^n(\text{Sing}(f^{-1})),$$

and $\rho_f := \limsup_{z \to \infty} \frac{\log \log |f(z)|}{\log |z|}$ is namely called the order of $f$.

Let $\mathcal{F}$ denote the class of transcendental entire functions $f$ satisfying the following properties

1. It is of finite order;

2. satisfies the rapid derivative growth condition: There are $\alpha_2 > 0$ $\alpha_1 > \alpha_2$ and $\kappa > 0$ such that for every $z \in J(f) \setminus f^{-1}(\infty)$ we have

$$|f'(z)| \geq \kappa^{-1}|z|^{\alpha_1}|f(z)|^{\alpha_2};$$

3. it is of Disjoint type, that is, the set $\text{Sing}(f^{-1})$ is contained in a compact subset of the immediate basin $B = B(z_0)$ of an attracting fixed point $z_0 \in \mathbb{C}$. (See [BJR12])

Note that each $f \in \mathcal{F}$ belongs to the Eremenko-Lyubich class

$$\mathcal{B} := \{ f : \mathbb{C} \to \mathbb{C} : \text{Sing}(f^{-1}) \text{ is bounded} \}.$$ 

It was proved in [Er92] that for $f \in \mathcal{B}$ all the Fatou components of $f$ are simply connected. Hence the immediate basin $B$ is simply connected. Moreover each $f \in \mathcal{F}$ is hyperbolic in the sense that the closure of $PS(f)$ is disjoint from the Julia set and $PS(f)$ is compact. We have that $f$ has no wandering and Baker domains, so $B$ is the only Fatou component of $f$, see [Er89][Er92][GK86].

Examples in the class $\mathcal{F}$ include the family $\lambda \exp(z)$ for $\lambda \in (0, 1/e)$, the family of maps $\lambda \sin(z)$ for $\lambda \in (0, 1)$, and $\lambda g(z)$, where $\lambda \in \mathbb{C}\{0\}$ and $g$ is an arbitrary map of finite order such that $\text{Sing}(g^{-1})$ is bounded and $|\lambda|$ is enough small, other examples are the expanding entire maps $\sum_{j=0}^{p+q} a_j e^{(j-p)z}$, $p, q > 0$, $a_j \in \mathbb{C}$, studied early in [CS07].

**Potentials**

Fix $f \in \mathcal{F}$. Since the immediate attraction basin $B = B(z_0)$ of an attracting fixed point $z_0$ is simply connected, there exists a bounded simply connected domain $D \subset \mathbb{C}$, such that its closure $\overline{D} \subset B$, and boundary $\partial D$ is an analytic Jordan curve. Moreover, $\text{Sing}(f^{-1}) \subset D$ and $f(D) \subset D$, for more details see [BK07] Lemma 3.1. Following [BK07], the pre-images of $\mathbb{C}\setminus \overline{D}$ by $f$ consists of countably many unbounded connected components called tracts of $f$. We denote the collection of all these tracts by $\mathcal{R}$. 
Since the closure of each tract is simply connected, there exists an open simple arc \( \alpha : (0, \infty) \to \mathbb{C}\setminus \mathcal{T} \), which is disjoint from the union of the closures of all tracts and such that \( \alpha(t) \) tends to a point of \( \partial D \) as \( t \) tends to \( 0^+ \), and \( \alpha(t) \) tends to \( \infty \) as \( t \) tends to \( +\infty \). We use this curve to define the fundamental domains on each tract as follows: since for every \( T \in \mathcal{B} \) the map \( f|_T \) is a cover of \( \mathbb{C}\setminus \mathcal{T} \), we have \( T \setminus f^{-1}(\alpha) \) is the union of infinitely many disjoint simply connected domains \( S \) such that the function

\[
f|_S : S \to \mathbb{C}\setminus (\mathcal{T} \cup \alpha)
\]
is bijective. Given \( T \in \mathcal{B} \), we denote by \( S_T \) the collection of connected components of \( T \setminus f^{-1}(\alpha) \). The elements of

\[
S := \bigcup_{T \in \mathcal{B}} S_T
\]
are called fundamental domains.

For each \( S \in \mathcal{S} \), we have that the restriction \( f|_S \) is univalent, so we denote its inverse branch by \( g_S := (f|_S)^{-1} : \mathbb{C}\setminus (\mathcal{T} \cup \alpha) \to S \). For \( n \geq 1 \) and each \( j \in \{0, 1, \cdots, n\} \) denote by \( S_j \) an element of \( \mathcal{S} \) and put \( g_{S_0S_1 \cdots S_n} = g_{S_0} \circ \cdots \circ g_{S_n} \). Then,

\[
g_{S_0 \cdots S_n}(\mathbb{C}\setminus (\mathcal{T} \cup \alpha)) = \{ z \in \mathbb{C} : f^j(z) \in S_j, \text{ for every } j = 0, \cdots, n \}.
\]

For each sequence \( S = (S_0S_1 \cdots) \in \mathcal{S}^\infty \), let \( K_S := \bigcap_{n=0}^{\infty} g_{S_0S_1 \cdots S_n}(\mathbb{C}\setminus (\mathcal{T} \cup \alpha)) \). Then, the Julia set of \( f \) is given by the disjoint union of \( K_S \), that is

\[
J(f) = \bigcup_{S \in \mathcal{S}^\infty} K_S.
\]

Since \( f \) has finite order and of disjoint-type, following [RRRS11], the Julia set \( J(f) \) is a Cantor bouquet, that is a union of uncountably many pairwise disjoint curves tending to infinity (hair) and each curve is attached to the unique point accessible from the immediate basin \( B \), called endpoint of the hair. More precisely, either \( K_S \) is empty or there is a homeomorphism \( h_S : [0, +\infty) \to K_S \) such that \( \lim_{t \to +\infty} h_S(t) = \infty \), and such that for every \( t > 0 \) we have \( \lim_{n \to +\infty} f^n(h_S(t)) = \infty \). In the latter case \( z_S := h_S(0) \) is the only point of \( K_S \) accessible from the immediate basin \( B \). See also [Ba07], which generalizes previous results for the exponential map having an attracting fixed point of [DG87].

Let \( \rho_f \) be the order of \( f \) and \( \alpha_1, \alpha_2 > 0 \) be the corresponding constants of the rapid derivative growth condition of \( f \). Fix \( \tau \in (0, \alpha_2) \) and let \( \gamma : \mathbb{C}\setminus \{0\} \to \mathbb{R} \cup \{\infty\} \) defined by \( \gamma(z) = \frac{1}{|z|^\tau} \). Let \( \theta \) be the Riemannian metric on \( \mathbb{C}\setminus \{0\} \) defined by

\[
d\theta(z) = \gamma(z)|dz|,
\]

If \( U \) is simply connected domain in the Riemann sphere \( \overline{\mathbb{C}} \), we say that a point \( z \in \partial U \) is accessible from \( U \) if there exists a curve \( v : [0, \infty) \to U \) such that \( \lim_{t \to +\infty} v(t) = z \).
and we derive $f$ with respect to $\theta$ instead of the Euclidean metric. So, for each $z \in \mathbb{C}\setminus\{0\}$ we have

$$|f'(z)|_\theta = |f'\circ f(z)|_{\gamma(z)} = |f'(z)|_{|f(z)|^\tau}.$$  \hspace{1cm} (3)

Denote by $\mathcal{C}$ the set of functions $\psi$ from $\bigcup_{S \in \mathcal{S}} S$ to $\mathbb{R}^+$ that are bounded from above and are constant on each element of $\mathcal{S}$.

$$\mathcal{C} := \left\{ \psi : \bigcup_{S \in \mathcal{S}} S \to \mathbb{R}^+ : \psi \text{ is bounded from above and constant over each } S \in \mathcal{S} \right\}.$$  \hspace{1cm} (4)

Let us define the following class of potentials for $f$:

$$\mathcal{P}_f = \left\{ \varphi_{\psi,t}(z) = \log \psi(z) - t \log |f'(z)|_\theta, \psi \in \mathcal{C}, t > \frac{\rho_f}{\alpha_1 + \alpha_2} \right\}.$$  \hspace{1cm} (5)

Observe that this class contains potentials $-t \log |f'|_\theta$, which from (3) are cohomologous to $-t \log |f'\circ f|_{\gamma(z)}$.

For each $f \in \mathcal{F}$ we denote by $\mathcal{T}_f$ the class of tame potentials stated in [MU10], that is

$$\mathcal{T}_f := \left\{ \varphi = h - t \log |f'|_\theta; \ h \text{ is bounded weakly Hölder function, } t > \frac{\rho'}{\alpha_1 + \alpha_2} \right\}.$$  \hspace{1cm} (6)

Although the class $\mathcal{F}$ does not include most of the functions considered in [MU10], our first result highlights that the class of potentials $\mathcal{P}_f$ determined by $f \in \mathcal{F}$ intersects the class of potentials considered in [MU10], which the difference is non-empty. So the first result is the following

**Theorem 1.** There exists $f \in \mathcal{F}$ such that $\mathcal{P}_f \cap \mathcal{T}_f \neq \emptyset$ and $\mathcal{P}_f \setminus \mathcal{T}_f \neq \emptyset$.

**Symbolic representation**

Let $\Sigma := \{s = (s_0s_1 \ldots) : s_j \in \mathbb{Z}, \text{ for all } j \geq 0\}$ be the full shift space, and the shift metric is defined as follows, for for some $\theta \in (0,1)$,

$$d(\underline{s}, \underline{t}) = \theta^\inf\{k : s_k \neq t_k\} \wedge \{\infty\}.$$  \hspace{1cm} (7)

For every $n \geq 1$, we denote a finite word $s_0 \cdots s_{n-1}$ in $\mathbb{Z}^n$ simply by $s^*$, so we follows the following notation for cylinders

$$[s^*] = \{\underline{w} \in \Sigma : w_i = s_i, 0 \leq i \leq n-1\}$$

and for $s \in \mathbb{Z}$, we simply denote $[s] = \{\underline{w} \in \Sigma : w_0 = s\}$. Let $\sigma : \Sigma \to \Sigma$ be the left-sided shift map, given by $\sigma(s_0s_1 \cdots) = (s_1s_2 \cdots)$. 
Observe that by definition the set $S$ given in (1) is countably infinite, so we identify $S$ with $\mathbb{Z}$. Put

$$X := \{ S \in \mathbb{Z}^n : K_S \neq \emptyset \} \subseteq \Sigma_Z.$$  
(6)

Let

$$Z = \bigcup_{S \in X} K_S.$$

From (2), we have for each $S \in \mathbb{Z}^n$, $f(K_S) = K_{\sigma(S)}$, then the function $f$ on the Julia set $J(f)$ is semi-conjugate to $\sigma$ on $X$, however $f$ on the set

$$\mathcal{EP} := \{ z_S = h_S(0) : S \in X \},$$

is conjugate to $\sigma|X$. Hence the set $X$ is completely $\sigma$-invariant.

The set $\mathcal{EP}$ defined above, is the set of endpoints of hairs $K_S$ and it satisfies the following properties, it is the set of accessible points from the immediate attraction basin $B$. It is totally disconnected, however $\mathcal{EP} \cup \{ \infty \}$ is connected, see [BJR12]. Moreover following [Ba08], the Hausdorff dimension of this set is equal to two, generalizing previous results of Karpiński [Ka99] for the exponential map $f_{\lambda}(z) = \lambda e^z$ with parameters $\lambda \in (0, 1/e)$. This exponential map is probably the best known example in the family $\mathcal{F}$, its Julia set is a Cantor bouquet and the set of endpoints is modelled by the symbolic space of all allowable sequences, see [DK84] and [DG87].

In the following, we state some properties concerning to the dynamics $(X, \sigma|X)$, endowed with a metric inherited from the euclidean metric on $J(f)$. It does not necessarily generate the topology induced by the cylinder sets.

Let $H : X \times [0, +\infty) \to J(f)$, and $H|_{X \times \{0\}} : X \times \{0\} \to \mathcal{EP}$ defined by $H(s, 0) = h_s(0)$, we have that $H$ induces a metric $\rho$ on $X$

$$\rho(s, w) := |h_s(0) - h_w(0)|.$$

The shift map $\sigma : X \to X$ is continuous respect to $\rho$.

Given $s \in \Sigma_Z$ and $w^* \in \mathbb{Z}^n$ let us write $w^*s = (w_0 \cdots w_{n-1}s_0s_1 \cdots)$. For a set $A \subseteq \Sigma_Z$ write $w^*A = \{ w^s : s \in A \}$. For $s \in X$ and $\delta > 0$ we define the following sets with respect to the metric $\rho$.

$$B(s, \delta) := \{ w \in X : \rho(s, w) < \delta \} = \{ w \in X : |h_s(0) - h_w(0)| < \delta \},$$

$$\overline{B}(0, \delta) := \{ s \in X : \rho(s, 0) \leq \delta \}.$$

$$\mathcal{B}_0(s, \delta) := \{ w \in X : \rho(s, w) < \delta & b_0 = a_0 \}.$$

For every $n \geq 1$ and $s \in X$ define

$$\mathcal{B}_n(s, \delta) := \{ w \in X : \sigma^j(w) \in \mathcal{B}_0(\sigma^j(s), \delta), \text{ for all } j = 0, 1, \cdots, n \}.$$

The set $X$ endowed with the metric $\rho$ is non-compact, however it can be approximated by a increasing sequence of compact and invariant subsets. Indeed, for all $N \geq 1$, define
\[ \Sigma_N := \{ \underline{s} = (s_0 s_1 \cdots) \in X : \text{for } j \geq 0, s_j \in \{-N, \cdots, N\} \}, \]

so, the following holds

**Lemma 1.**
1. For all \( N \geq 1 \), \( \Sigma_N \subset X \), \( \Sigma_N \) is compact with respect to \( \rho \) and invariant by \( \sigma \). Moreover, for each compact subset \( \Lambda \) of \( X \) with respect to the metric \( \rho \), so that \( \sigma(\Lambda) \subset \Lambda \), we have, there exists \( N_0 \geq 1 \), such that \( \Lambda \subset \Sigma_{N_0} \).

2. There exists \( \delta_0 \) such that the following condition holds

There exist \( C > 0 \) and \( \lambda > 1 \) such that for every \( n \in \mathbb{N} \) and \( \underline{s}, \underline{t} \in X \) and \( u^* \in \mathbb{Z}^n \), if \( \rho(\underline{s}, \underline{t}) < \delta_0 \) then we have

\[ \rho(u^* \underline{s}, u^* \underline{t}) \leq C \lambda^{-n} \rho(\underline{s}, \underline{t}). \]

3. For every \( R > 0 \) there exists \( n \geq 1 \) such that for every \( \underline{s} \in B(0, R) \), we have \( \sigma^n(B(\underline{s}, \delta_0)) \supset B(0, R) \). Thus \((X, \sigma)\) is topologically mixing.

4. The set \( \bigcup_{N \geq 1} \Sigma_N \) is dense in \( X \).

For the next theorem, we remark about the definition of a conformal measure stated in §3. In §4 we give the definition of a Gibbs measure of potentials appropriately adapted for the transcendental functions.

**Theorem 2.** Let \( f \in \mathcal{F} \). Then for every potential \( \phi \in \mathcal{P}_f \) we have the following properties.

1. There exists a unique \( e^{P(\phi)-\phi} \)-conformal measure \( \nu_\phi \) of \( f \).

2. There exists a unique probability Gibbs state \( \mu_\phi \). That is, \( \mu_\phi \) is \( f \)-invariant and equivalent to \( \nu_\phi \). Moreover, both measures are ergodic and supported on the radial Julia set \( J_r(f) \), where

\[ J_r(f) = \{ z \in J(f) : \lim_{n \to \infty} f^n(z) < \infty \}. \]

3. For each \( w \in J(f) \), we have \( P(\phi) = \lim_{n \to \infty} \frac{1}{n} \log \sum_{z \in f^{-n}(w)} \exp \left( \sum_{j=0}^{n-1} \phi \circ f^j(z) \right) \).

**2 Proof of results**

**2.1 Proof of Theorem 1**

Consider the exponential family \( \{ f_\lambda(z) = \lambda e^z, \lambda \in (0, 1/e) \} \). Each \( f_\lambda \) belongs to \( \mathcal{F} \), because has order equal to 1, satisfies the rapid derivative growth condition with \( \alpha_1 = 0 \) and \( \alpha_2 = 1 \), and since 0 is the only singular value of \( f_\lambda \), so this set is hyperbolic. Moreover the potentials \(-t \log |z| = -t \log |f_\lambda'(z)| + \log \gamma_1 - \log \gamma_1 \circ f_\lambda \), where \( \gamma_1 = |z|^{-t} \), are tame potentials and also belong to the class \( \mathcal{P}_{f_\lambda} \).
On the other hand, let \( \mathbb{D} \) be the open unit disk in \( \mathbb{C} \), then

\[
 f_\lambda \left( \left\{ z : \text{Re } z < \ln \left( \frac{1}{\lambda} \right) \right\} \right) = \mathbb{D} \setminus \{0\},
\]

and since \( 1 < \ln \left( \frac{4}{3} \right) \) we have \( f_\lambda(\mathbb{D}) \subset \mathbb{D} \). Moreover since the immediate basin \( B \) of the attracting fixed point is the only Fatou component of \( f_\lambda \) we have \( \overline{\mathbb{D}} \subset B \).

Since \( f_\lambda^{-1}(\mathbb{C} \setminus \overline{\mathbb{D}}) = \left\{ z : \text{Re } z > \ln \left( \frac{4}{3} \right) \right\} \), the only tract of \( f_\lambda \) is the half plane \( T = \left\{ z : \text{Re } z > \ln \left( \frac{4}{3} \right) \right\} \). Let us consider the ray \( \alpha : (0, \infty) \to \mathbb{C} \setminus \mathbb{D} \) defined by \( \alpha(t) = -(1 + t) \), then

\[
 f_\lambda^{-1}(\alpha(0, \infty)) = \bigcup_{k \in \mathbb{Z}} \left\{ x + (2k - 1)\pi i : x > \ln \left( \frac{1}{\lambda} \right) \right\},
\]

and for each \( k \in \mathbb{Z} \), put \( S_k := \left\{ z : \text{Re } z > \ln \left( \frac{4}{3} \right), (2k - 1)\pi < \text{Im } z < (2k + 1)\pi \right\} \). Then \( T \setminus f_\lambda^{-1}(\alpha(t)) \) is the disjoint union of the fundamental domains \( S_k \).

Following [ISC0], let \( c : J(f_\lambda) \to \mathbb{R}^+ \) be a function such that for each \( k \in \mathbb{Z} \), is constant on \( J(f_\lambda) \cap (S_{-k} \cup S_k) \) and we denote by \( c_k \) its value on this set. Furthermore we assume that the sequence \( (c_k)_{k \in \mathbb{Z}} \) of positive numbers satisfies

\[
 \lim_{k \to \infty} \frac{\log c_k}{\log k} = -\infty. \quad (7)
\]

Define \( \varphi(z) := \log \left( c(z) |z|^{-t} \right) \), where \( t > 0 \), \( c(z) = c_k \) if \( z \in S_{-k} \cup S_k \) and the sequence \( (c_k)_{k \in \mathbb{Z}} \) satisfies [1]. Observe that any potential as above \( \varphi(z) = \log(c(z) |z|^{-t}) \) satisfies \( \lim_{k \to +\infty} c_k \to 0 \), so, \( \varphi \) is not a tame potential however this potential belongs to the class \( \mathcal{P}_{f_\lambda} \) because the function \( c \) is bounded on each \( S_k \) and \( |E_\lambda'(z)|_\theta = |z| \).

### 2.2 Proof Lemma [1]

1. By Denjoy-Carleman-Ahlfors theorem implies that the transcendental entire functions of finite order have only a finite number of tracts. We will assume for simplicity that for \( f \in \mathcal{F} \) there is only one tract \( T \), and there is no complication in the generalization to a finite number of tracts.

Let \( N \geq 1 \) and denote by \( S_N \) the union of \( 2N + 1 \) fundamental domains in \( T \), that is

\[
 S_N = \bigcup_{k=-N}^N S_k
\]

and define

\[
 K_N := \{ z \in S_N : \text{for every } j \geq 0, f^j(z) \in S_N \}.
\]

Let \( (s^m_{\lambda})_{m \geq 1} \) be a sequence in \( \Sigma_N \). Taking a subsequence if it is necessary, one can assume that for some \( R \) large enough, the subsequence \( (s^m_{\lambda})_{m \geq 1} \) is contained in \( \Sigma_N \cap B(0, R) \). So for every \( m \geq 1 \) there is there is an endpoint \( z_m = h_m(0) \in K_N \cap B(z_0, R) \). Since \( K_N \cap B(z_0, R) \) is bounded and \( K_N \) is closed in \( J(f) \) we have there is a subsequence converging to some point \( z \in K_N \). Let \( \lambda \) be the itinerary associated to \( z \), then \( \lambda \in \Sigma_N^+ \) and \( g(\lambda^m_{z}, \lambda) = |z_m^\lambda - z| \to 0, j \to \infty \).
On the other hand, let \( \Lambda \) be a compact subset of \( X \) with respect to the metric \( \rho \) with \( \sigma(\Lambda) \subset \Lambda \). Let \( s \in \Lambda \) and let \( z \in H(\Lambda) \) with itinerary \( s \), then since the compact subset \( H(\Lambda) \) intersects only a finite numbers of tracts (see [BKO07] Lemma 3.2), there exists \( N_0 \geq 1 \) such that for every \( j \geq 0 \) we have \( |s_j| \leq N_0 \). Therefore \( \Lambda \subset \Sigma_{N_0} \).

2. Follows from the derivative grown condition and the uniformly expanding property of \( f \), see [MU10] Proposition 4.4.

3. Following [MU10] Lemma 4.2, we have for each \( R > 0 \) there exists \( n \geq 1 \) such that for every \( \sigma \in B(z_0, R) \), \( f^n(B(\sigma, \delta_0)) \supset B(z_0, R) \), then the property follows. Let \( s, \ell \in X \) and \( U = B(s, \delta_1), V = B(\ell, \delta_2) \), for \( \delta_1, \delta_2 > 0 \). By expanding property in Part 2, follows there is \( n_1 > 0 \) such \( \sigma^{n_1}(U) \supset B(\sigma^{n_1} s, \delta_0) \). Let \( R \) be large enough such that \( \sigma^{n_1} s, \ell \in B(0, R) \) and there is \( N_1 \) such that

\[
\sigma^{N_1}(B(\sigma^{n_1} s, \delta_0)) \supset B(0, R).
\]

So, \( \sigma^{N_1+n_1}(U) \supset \sigma^{N_1} B(\sigma^{n_1} s, \delta_0) \supset B(0, R) \). Taking \( m > N_1+n_1 \), we have for every \( k \geq m \), \( \sigma^k(U) \cap V \neq \emptyset \).

4. Let \( s = (s_0 s_1 \cdots) \in X \) and \( \epsilon > 0 \). Then, there exists \( n_1 > 0 \) such that

\[
B(\sigma^{n_1} s, \delta_0) \subset \sigma^{n_1} B(s, \epsilon).
\]

It is enough to take \( n_1 > 0 \) such that \( C \lambda^{-n_1} \delta_0 < \epsilon \). So, we have

\[
s_0 s_1 \cdots s_{n_1-1} B(\sigma^{n_1} s, \delta_0) \subset B(s, C \lambda^{-n_1} \delta_0) \subset B(s, \epsilon).
\]

Hence, \( B(\sigma^{n_1} s, \delta_0) \subset \sigma^{n_1} B(s, \epsilon) \). Since \( X = \bigcup_{R > 0} B(0, R) \), then for some \( R > 0 \) we have \( \sigma^{n_1} s \in B(0, R) \). Moreover from Part 3, there is \( n_2 > 0 \) such that \( B(0, R) \subset \sigma^{n_2} B(\sigma^{n_1} s, \delta_0) \). Therefore, for \( n = n_1+n_2 \) we follows that \( B(0, R) \subset \sigma^n B(s, \epsilon) \).

Hence, the set \( \sigma^n B(s, \epsilon) \) contains the sequence \( 0 = 00 \cdots \). Let \( w^* = w_0 \cdots w_{n-1} \in \mathbb{Z}^n \) such that \( w^* B(0, R) \subset \sigma^n B(s, \epsilon) \). Then, \( w_0 \cdots w_{n-1} 0 \in B(s, \epsilon) \). Then, taking \( N := \max\{|w_0|, \cdots, |w_{n-1}|\} \) we conclude \( w_0 w_1 \cdots w_n 0 \in \Sigma_N \). See also Lemma 3 and Lemma 2.

Let \( CB(J(f), \mathbb{R}) \) be denote the Banach space of bounded continuous functions on \( J(f) \). For each potential \( \varphi \in \mathscr{T}_f \), the transfer operator associated to \( \varphi \) and denoted by \( \mathcal{L}_\varphi \) acts continuously on \( CB(J(f), \mathbb{R}) \). So for each \( \psi \in CB(J(f), \mathbb{R}) \),

\[
\mathcal{L}_\varphi \psi(z) = \sum_{f(w) = z} \psi(w) \exp(\varphi(w)).
\]

Following [MU10], the thermodynamic formalism was stated for a large class of hyperbolic meromorphic functions \( f : \mathbb{C} \rightarrow \overline{\mathbb{C}} \) of finite order \( \rho' \) satisfying a rapid growth condition and for a class of tame potentials. To prove Theorem 2 one could apply without making too many changes the approach used in [MU10] to the family of transcendental entire maps and potentials under study. For instance, the Proposition 1 seems to works without any modifications for potentials \( \phi_c \) with \( c \) being only bounded from above in [MU10]. Otherwise one can obtain this result
as a consequence of the Theorems\[3,4\] and\[5\] realizing that the class of potentials hyperbolic entire maps and potentials studied fit in the hypothesis of the results developed in the next section for a symbolic version.

**Proposition 1.** Given \( f \in \mathcal{F} \), we have each potential \( \varphi_{c,t} \in \mathcal{P}_f \) satisfies the following properties.

1. \( \sup_{w \in X} \mathcal{L}_{\varphi_{c,t} \circ H}(1)(w) < \infty \)

2. \( \lim_{R \to \infty} \sum_{s \in \mathbb{Z}} \exp \left( \sup_{w \in [s] \cap (X \setminus B(q,R))} \varphi_{c,t} \circ H(w) \right) = 0. \)

3. \( \lim_{z_w \to \infty} \mathcal{L}_{\varphi_{c,t} \circ H} \mathbb{1}(w) = 0. \)

**Proof.** Let \( \varphi_{c,t} = \log c - t \log |f'|_{\theta} \) be a potential in \( \mathcal{P}_f \) and \( \psi \in \text{CB}(J(f), \mathbb{R}) \),

\[
\mathcal{L}_{\varphi_{c,t} \circ H} \psi \circ H(w) = \sum_{\sigma(\alpha) = w} \psi(z_x) c(z_x) |f'(z_x)|^{-\alpha t} = \sum_{\sigma(\alpha) = w} \psi(z_x) c(z_x) |f'(z_x)|^{-\alpha t} |z_x|^{-\alpha t} |f(z_x)|^{\alpha t} = |z_x|^{\alpha t} \sum_{\sigma(\alpha) = w} \psi(z_x) c(z_x) |f'(z_x)|^{-\alpha t} |z_x|^{-\alpha t}.
\]

Since \( f \) satisfies the derivative growth condition, we have

\[
\mathcal{L}_{\varphi_{c,t} \circ H}(1)(w) \leq \kappa t |z_w|^{\alpha t} \sum_{\sigma(\alpha) = w} c(z_x) |z_x|^{-\alpha t} |f(z_x)|^{-\alpha t} |z_x|^{-\alpha t} = \kappa t |z_w|^{\alpha t} \sum_{\sigma(\alpha) = w} c(z_x) |z_x|^{-\alpha t} |z_x|^{-\alpha t} \leq \frac{\kappa t}{|z_w|^{\alpha t} |z_x|^{\alpha t}} \sum_{\sigma(\alpha) = w} c(z_x) |z_x|^{-\alpha t} |z_x|^{-(\alpha t + \alpha)})
\]

Since \( f \) is a transcendental entire function of finite order \( \rho \) and \( t > \rho \) \( (\tau + \alpha) \), then the Borel-Picard Theorem (see \[MU10\] Theorem 3.5) states that the series has the exponent of convergence equal to \( \rho \). So the last sum is finite. Following \[MU10\] Proposition 3.6, there exists \( M_t > 0 \) such for all \( w \in X \) we have

\[
\mathcal{L}_{\varphi_{c,t} \circ H}(1)(w) \leq \frac{M_t}{|z_w|^{\alpha t} |z_x|^{\alpha t}} \sum_{\sigma(\alpha) = w} c(z_x) |z_x|^{-\alpha t} |z_x|^{-(\alpha t + \alpha)}.
\]

So, the equation (8) implies \( \lim_{R \to \infty} \sum_{s \in \mathbb{Z}} \exp \left( \sup_{w \in [s] \cap (X \setminus B(q,R))} \varphi_{c,t} \circ H(w) \right) = 0 \) and

\( \lim_{z_w \to \infty} \mathcal{L}_{\varphi_{c,t} \circ H} \mathbb{1}(w) = 0. \)

\( \square \)
3 Symbolic dynamic outlook

In this section we give a very general approach that will complement the results in previous section. To do that, we give certain conditions to guarantee what is desired from a symbolic perspective.

Let

$$\Sigma^+ := \{ \mathbf{a} = (a_0 a_1 \ldots) : a_j \in \mathbb{Z}, \text{ for all } j \geq 0 \} = \mathbb{Z}^\mathbb{N},$$

be the full shift space on a countably infinite number of symbols endowed with the product topology, and let \( \sigma : \Sigma^+ \to \Sigma^+ \) be the shift transformation defined by \( \sigma(a_0 a_1 \cdots) = (a_1 a_2 \cdots). \) For every \( N \geq 1 \) let

$$\Sigma^+_N := \{ \mathbf{a} = (a_0 a_1 \cdots) \in \Sigma^+ : a_j \in \{-N, \cdots, N\} \}.$$  \( \text{(9)} \)

The purpose of this section is to develop a thermodynamic formalism for the dynamics of \( \sigma \) on some subset \( X \) of \( \Sigma^+ \) which is completely invariant, that is, satisfies

$$\sigma(X) = X = \sigma^{-1}(X).$$

Thus the restricted transformation \( \sigma|_X \) and all of its iterates are well defined. We endow \( X \) with some metric \( \rho \) which is not necessarily compatible with the product metric on \( \Sigma^+ \), in the sense that it is not assumed that the metric \( \rho \) on \( X \) can be extended to the full shift in the way that \( \sigma \) is still continuous.

We do not assume that \( X \) is compact, however, we assume that,

for each \( N \geq 1 \), the set \( X \) contains \( \Sigma^+_N \), and that \( \Sigma^+_N \) is a compact with respect to \( \rho \). Moreover we assume that for each subset \( \Lambda \subset X \), that is compact with respect to \( \rho \) such that \( \sigma(\Lambda) \subset \Lambda \) there exists \( N \geq 1 \) such that \( \Lambda \subset \Sigma^+_N \).

In Section 3.1 we give conditions on \((X, \rho)\) that implies existence of a conformal measure and an absolutely continuous invariant measure for a certain class of weakly Hölder continuous potentials, see Theorem 3, Theorem 4 and Theorem 5.

3.1 Conditions on the space \((X, \rho)\) and the main results

Given \( \mathbf{a} = (a_0 a_1 \ldots) \in \Sigma^+ \) and given a word of finite length \( b^* = b_0 \cdots b_{n-1} \in \mathbb{Z}^n \) let us write

$$b^* \mathbf{a} = (b_0 \cdots b_{n-1} a_0 a_1 \cdots).$$

Given a set \( A \subset \Sigma^+ \) let us write \( b^* A = \{ b^* \mathbf{a} : \mathbf{a} \in A \} \). For \( \mathbf{a} \in X \) and \( \delta > 0 \) we define

$$B(\mathbf{a}, \delta) := \{ \mathbf{b} \in X : \rho(\mathbf{a}, \mathbf{b}) < \delta \},$$

$$B_0(\mathbf{a}, \delta) := \{ \mathbf{b} \in X : \rho(\mathbf{a}, \mathbf{b}) < \delta \text{ and } b_0 = a_0 \}.$$  \( \text{Let us denote} \)

$$B(0, R) = \{ \mathbf{b} \in X : \rho(0, \mathbf{b}) \leq R \}.$$  \( \text{By} \ \mathbb{B}_n(\mathbf{a}, \delta) \ \text{we denote the} \ (n, \delta) \text{-ball at} \ \mathbf{a} \ \text{in} \ X, \ \text{namely} \)

$$\mathbb{B}_n(\mathbf{a}, \delta) := \{ \mathbf{b} \in X : \sigma^j(\mathbf{b}) \in \mathbb{B}_0(\sigma^j(\mathbf{a}), \delta), \ \text{for all} \ j = 0, 1, \cdots, n \}.$$
We assume that there exists $\delta_0 > 0$ such that the metric $\rho$ satisfies the exponential shrinking of preimages on balls (Exp) and connection on balls (Con) given below.

(Exp) There exist $C_{ES} > 0$ and $\lambda_{ES} > 1$ such that, for all $n \in \mathbb{N}$, $\underline{a}, \underline{b} \in X$ and $c^* \in \mathbb{Z}^n$, if $\rho(\underline{a}, \underline{b}) < \delta_0$ then we have

$$\rho(c^* \underline{a}, c^* \underline{b}) \leq C_{ES} \lambda_{ES}^{-n} \rho(\underline{a}, \underline{b}).$$

(Con) For every $R > 0$ there exists $n \geq 1$ such that for all $\underline{a} \in B(0, R)$ we have

$$\sigma^n B(\underline{a}, \delta_0) \supset B(0, R).$$

So the plan here is as follows.

**Weakly Hölder continuous potentials**

Given $\delta_1 > 0$ and $\alpha \in (0, 1]$. We say that a function $\phi : X \to \mathbb{R}$ is uniformly $\delta_1$-locally $\alpha$-Hölder if there exists $L \geq 0$ such that for all $\underline{a}, \underline{b}, \underline{c} \in X$ satisfying $\underline{a}, \underline{b} \in B_0(\underline{c}, \delta)$ we have

$$|\phi(\underline{a}) - \phi(\underline{b})| \leq L(\rho(\underline{a}, \underline{b}))^\alpha.$$

Given $\delta \in (0, \delta_0]$, $n \geq 0$ we define the $(n, \delta)$-variation of the potential $\phi : X \to \mathbb{R}$ (See [Sa99]) by

$$\text{Var}_n(\phi) := \sup_{\underline{a} \in X} \sup_{\underline{b}, \underline{c} \in B_n(\underline{a}, \delta)} |\phi(\underline{b}) - \phi(\underline{c})|.$$ 

Given $r \in (0, 1)$ and $\delta \in (0, \delta_0]$, we say that a potential $\phi : X \to \mathbb{R}$ is $(\delta, r)$-weakly Hölder continuous if there exist $C > 0$ such that for every $n \geq 0$

$$\text{Var}_n(\phi) \leq Cr^n.$$

We denote by $\text{BV}_r(X, \mathbb{R})$ the space of all $(\delta, r)$-weakly Hölder continuous. We work with fixed $\alpha \in (0, 1)$ and $\delta \in (0, \delta_0]$ and we prove that every uniformly $\delta$-locally $\alpha$-Hölder continuous potential is $(\delta, r)$-weakly Hölder continuous with $r = \lambda_{ES}^{-\alpha}$ (Lemma [3]). We assume further that the following condition (Delta) is satisfied.

(Delta) If $\phi$ is $(\delta', r)$-weakly Hölder continuous then $\phi$ is $(\delta, r)$-weakly Hölder continuous, where $\delta' = \min\{\delta, \delta/C_{ES}\}$. 

Since this condition could be a bit difficult to verify directly, we show that the topological condition ((Path)) implies (Delta) (Proposition [3]).

(Path) There exists $\ell \geq 1$ such that for all $\underline{a} \in X$ and for all $\underline{b}, \underline{c} \in B_0(\underline{a}, \delta)$, there exists a sequence

$$\underline{b} = \underline{a}_0, \underline{a}_1, \cdots , \underline{a}_\ell = \underline{c}$$

such that for all $j \geq 0$ we have $\rho(\underline{a}_j, \underline{a}_{j+1}) < \delta'$. 
The transfer operator and the pressure

We say that a potential $\phi$ is summable if it satisfies

$$\sup_{a \in X} \left\{ \sum_{b: \sigma(b) = a} \exp(\phi(b)) \right\} < \infty.$$

(10)

We say that $\phi$ is bounded on balls if for all $R > 0$ we have

$$\sup_{c \in B(0,R)} |\phi(c)| < \infty.$$

(11)

If we assume that the potential $\phi$ satisfies (10) and (11) then for any real bounded continuous function $g$ on $X$, the transfer operator

$$L^\phi(g)(a) := \sum_{b: \sigma(b) = a} e^{\phi(b)} g(b),$$

is well-defined. We denote the space of real bounded continuous functions on $X$ by $\text{CB}_b(X, \mathbb{R})$. For every $n \geq 1$ and $a \in X$ we put

$$S_n(a) = \sum_{k=1}^{n-1} \phi \circ \sigma^k(a).$$

So, for every $n \geq 1$ and $a \in X$ we have

$$L^\phi_n(g)(a) = \sum_{b: \sigma^n(b) = a} e^{S_n(a)} g(b).$$

Since our space $X$ is not compact the topological pressure is not defined in the standard way, so we define the pressure $P(\phi)$ of a potential weakly Hölder continuous $\phi$ with respect to $\sigma$ as the supremum over all the topological pressures of $\phi$ restrict to $\Sigma_N^+$, that is

$$P(\phi) := \sup_{N \geq 1} P_{\text{top}}(\phi|_{\Sigma_N^+}),$$

(12)

where $P_{\text{top}}(\phi|_{\Sigma_N^+})$ is the topological pressure of $\phi$ over $\Sigma_N^+$, that is defined for each $N \geq 1$

$$P_{\text{top}}(\phi|_{\Sigma_N^+}) := \lim_{n \to \infty} \frac{1}{n} \log \sum_{c^* \in \mathbb{Z}^n, c^* \in \Sigma_N^+} e^{S_n(c^*)}.$$

To guarantee the existence of a conformal measure we have to impose certain conditions on $X$, $\rho$, and the potential $\phi$. Namely, notice that due to the lack of compactness of $X$ the Schauder-Tychonoff Fixed Point Theorem cannot be applied.

For any $a^* \in \mathbb{Z}^n$ the cylinder $[a^*]$ is defined by

$$[a^*] := \{ b \in X : b_i = a_i, 0 \leq i \leq n - 1 \}.$$

Let us assume that the following conditions are satisfied.
1. The transformation $\sigma : X \to X$ can be continuously extended to $\overline{\sigma} : \overline{X}^\rho \to \overline{X}^\rho$, where $\overline{X}^\rho$ is the completion of $X$ with respect to the metric $\rho$.

2. For any $R > 0$, the set $B(0, R)$ is compact in $\overline{X}^\rho$.

3. For every $R > 0$ and for every $b \in \overline{X}^\rho \setminus X$ there exists $N \geq 1$ such that for $n \geq N$, we have $\sigma^n(b) \in \overline{X}^\rho \setminus B(0, R)$.

4. Given $R > 0$ and $k \in \mathbb{Z}$, let
   \[ [k, R] := \{a \in [k] \cap X : \rho(0, a) > R\}. \]
   Assume that every potential $\phi \in \text{BV}_r(X, \mathbb{R})$, satisfies
   \[
   \sum_{k \in \mathbb{Z}} e^{\sup_{[k]}} < \infty, \tag{13}
   \]
   \[
   \lim_{R \to \infty} \sum_{k \in \mathbb{Z}} e^{\sup_{[k, R]}} = 0. \tag{14}
   
5. Every potential $\phi \in \text{BV}_r(X, \mathbb{R})$ has a continuous extension to $\overline{X}^\rho$.

If it does not lead to misunderstanding we will frequently denote $\overline{\sigma}$ simply by $\sigma$.

Set
\[ X_{\text{rad}} := \{a \in \overline{X}^\rho : \omega_\sigma(a) \cap X \neq \emptyset\}. \]

Let us state the main results of this paper. The proofs can be found in Section 3.4.

**Theorem 3.** Let $\sigma : X \to X$ be the shift map and let $\phi \in \text{BV}_r(X, \mathbb{R})$ satisfying the conditions (i) (ii), (iii), (iv) and (v). Then there exists a measure $\nu$ that is $e^{P(\phi) - \phi}$-conformal and satisfies $\nu(X_{\text{rad}}) = 1$.

We obtain the measure $\nu$ as the weak limit of a tight sequence of measures $\{\nu_N\}_{N \in \mathbb{N}}$, where every measure $\nu_N$ is conformal with respect to $\sigma|_{\Sigma^+_N}$. We recall the definition of tightness in Section 3.4.

**Theorem 4.** Let $\sigma : X \to X$ be the shift map and let $\phi \in \text{BV}_r(X, \mathbb{R})$ satisfying the conditions (i) (ii), (iii), (iv) and (v). Then there exists an invariant measure $\mu$ that is absolutely continuous with respect to $\nu$ and is a Gibbs measure for $\phi$.

Moreover we say that a potential $\phi \in \text{BV}_r(X, \mathbb{R})$ is rapidly decreasing if
\[
\lim_{R \to \infty} \sup_{a \in \overline{X}^\rho \setminus B(0, R)} \mathcal{L}_0 \mathbb{I}(a) = 0. \tag{15}
\]

Under this additional assumption we are able to prove the following result.

**Theorem 5.** Let $\phi$ be a potential satisfying (10), (11) and (15). Then for each $a \in \overline{X}^\rho$ we have that the limit $\lim_{n \to \infty} \frac{1}{n} \log \mathcal{L}_0^n \mathbb{I}(a)$ exists and is equal to the pressure $P(\phi)$. This limit is independent of $a$. 
Notice that from (8) each potential \( \phi_{c,t} \in \mathcal{P} \) satisfies the condition of summability given in (10). The condition of bounded on ball given in (11) follows from the rapid growth condition. Moreover, from equation (8) implies that the condition (13) is satisfied. The condition (14) follows since for \( R > 0 \) and \( U_R := \{ z \in \mathbb{C} : |z| > R \} \), we have that for every \( \phi_{c,t} \in \mathcal{P} \) and for every \( w \in J(f) \),

\[
\lim_{R \to \infty} \sum_{S \in S} \exp \left( \sup_{z \in S \cap U_R} \phi_{c,t}(z) \right) = 0.
\]

and

\[
\lim_{w \to \infty} \mathcal{L}_{\phi_{c,t}} 1(w) = 0.
\]

Thus Theorem 2 can follow as a consequence of the Theorem 3, Theorem 4 and Theorem 5.

3.2 Some preliminary Lemmas

Throughout all this section we let \( X \) be a subset of \( \Sigma^+ \) such that \( \sigma(X) = X \) and \( \sigma^{-1}(X) = X \). Furthermore we assume \( X \) is endowed with a metric \( \rho \) satisfying (Exp) with constants \( \delta_0 > 0, C_{ES} > 0 \) and \( \lambda_{ES} > 1 \).

The following properties follow immediately from the definitions, and will be used several times. For all \( \delta > 0, \underline{a} \in X, n, m \geq 0 \) and \( c^* \in \mathbb{Z}^n \) we have

\[
\sigma^n(B_{m+n}(\underline{a}, \delta)) \subseteq B_m(\sigma^n \underline{a}, \delta),
\]

and

\[
B_{m+n}(c^* \underline{a}, \delta) \subseteq c^* B_m(\underline{a}, \delta).
\]

Then the following lemmas hold.

**Lemma 2.** Fix \( n_0 \geq 0 \) such that \( C_{ES}^{-1} \lambda_{ES}^{-n_0} \leq \min \{ 1, 1/C_{ES} \} \). Then for all \( \delta \in (0, \delta_0], n, m \geq 0, c^* \in \mathbb{Z}^n \) and \( \underline{a} \in X \), we have

1. \( B_{m+n}(c^* \underline{a}, \delta) \subseteq c^* B_m(\underline{a}, \delta) \subseteq B_m(c^* \underline{a}, C_{ES}^{-1} \lambda_{ES}^{-n} \delta) \).

   In particular taking \( n = n_0 \), we have

   \[
   B_{m+n_0}(c^* \underline{a}, \delta) \subseteq B_m(c^* \underline{a}, \min \{ \delta, \delta/C_{ES} \}).
   \]

2. \( B_{m+n}(c^* \underline{a}, \min \{ \delta, \delta/C_{ES} \}) \subseteq c^* B_m(\underline{a}, \min \{ \delta, \delta/C_{ES} \}) \subseteq B_{m+n}(c^* \underline{a}, \delta) \).

**Proof.** 1. The first inclusion is (17). The second inclusion follows since for \( \underline{b} \in c^* B_m(\underline{a}, \delta) \) there is \( \underline{b}' \in B_m(\underline{a}, \delta) \) such that \( \underline{b} = c^* \underline{b}' \) and for each \( j = 0, \ldots, m \) we have

   \[
   \rho(\sigma^j \underline{a}, \sigma^j \underline{b}') < \delta \text{ and } \underline{a}_j = \underline{b}_j.
   \]

   Then by (Exp) implies that for such \( j \) we have

   \[
   \rho(\sigma^j (c^* \underline{a}), \sigma^j (c^j \underline{b}')) < C_{ES}^{-1} \lambda_{ES}^{-n} \delta.
   \]
2. Set $\delta' = \min\{\delta, \delta/C_{ES}\}$. The first inclusion is (17) with $\delta$ replaced by $\delta'$. To prove the second inclusion let $b \in c^*B_m(\underline{a}, \delta')$. Then there exists $b' \in B_m(a, \delta)$ such that $b = c^*b'$, hence for all $j = 0, 1, \cdots, m$ we have

$$\varrho(\sigma^j b', \sigma^j a) < \delta' \text{ and } b'_j = a_j.$$  

Moreover by (Exp) for each $j = 0, 1, \cdots, n$ we have

$$\varrho(\sigma^j (c^* b'), \sigma^j (c^* a)) \leq C_{ES} \lambda_{ES}^{j-n} \varrho(b', a) < C_{ES} \delta' \leq \delta.$$

Lemma 3. For all $\underline{a} \in X$ and $r > 0$ there exists $n > 0$ such that

$$B(\sigma^n \underline{a}, \delta_0) \subseteq \sigma^n B(\underline{a}, r).$$

Proof. Take $n > 0$ such that $C_{ES} \lambda_{ES}^{-n} \delta_0 < r$. Thus by (Exp) we have

$$a_0 a_1 \cdots a_{n-1} B(\sigma^n \underline{a}, \delta_0) \subseteq B(\underline{a}, C_{ES} \lambda_{ES}^{-n} \delta_0) \subseteq B(\underline{a}, r).$$

And so we have

$$B(\sigma^n \underline{a}, \delta_0) \subseteq \sigma^n B(\underline{a}, r).$$

Lemma 4. If the metric $\varrho$ satisfies (Con), then the set $\bigcup_{N \geq 1} \Sigma^+_{N}$ is dense in $X$.

Proof. Let $\underline{a} \in X$ and $r > 0$. Then by Lemma 3 there exists $n_1 > 0$ such that

$$B(\sigma^{n_1} \underline{a}, \delta_0) \subseteq \sigma^{n_1} B(\underline{a}, r).$$  \hfill (18)

Since $X = \bigcup_{R > 0} B(\underline{0}, R)$, then for some $R > 0$ we have $\sigma^{n_1} \underline{a} \in B(\underline{0}, R)$. Moreover by (Con) there exists $n_2 > 0$ such that

$$B(\underline{0}, R) \subseteq \sigma^{n_2} B(\sigma^{n_1} \underline{a}, \delta_0).$$  \hfill (19)

Therefore from (18) and (19) we have for $n = n_1 + n_2$ it follows $B(\underline{0}, R) \subseteq \sigma^n B(\underline{a}, r)$. Hence $\sigma^n B(\underline{a}, r)$ contains the sequence $\underline{0} = 00 \cdots$. Let $c^* \in \mathbb{Z}^n$ such that $c^* B(\underline{0}, R) \subseteq B(\underline{a}, r)$, then $c_0 \cdots c_{n-1} \underline{0} \in B(\underline{a}, r)$. So, taking $N := \max\{|c_0|, \cdots, |c_{n-1}|\}$ we conclude $c_0 c_1 \cdots c_n \underline{0} \in \Sigma^+_{N}$.

For all $\underline{a}, \underline{b} \in X$ and $k \geq 1$, we define

$$\varrho_k(\underline{a}, \underline{b}) := \max_{0 \leq j \leq k} \{\varrho(\sigma^j \underline{a}, \sigma^j \underline{b})\}.$$  

Lemma 5. Let $\delta \in (0, \delta_0]$ and $\delta' = \min\{\delta, \delta/C_{ES}\}$. Assume that $\varrho$ satisfies (Path). Then for all $k \geq 1$, $\underline{a} \in X$ and $\underline{a}, \underline{b} \in B_k(\underline{d}, \delta)$ we have that there exists a sequence $\underline{a} = \underline{a}_0, \underline{a}_1, \cdots, \underline{a}_k = \underline{b}$ such that for every $j \in \{0, \cdots, k-1\}$,

$$\varrho_k(\underline{a}_j, \underline{a}_{j+1}) < \delta'.$$
Proof. We will proceed by induction in \( k \). By assumption (Path), the desired assertion is satisfied for \( k = 1 \). Suppose that this is true for \( k > 1 \). For \( a, b \in B_{k+1}(d, \delta) \) we have \( \sigma(a), \sigma(b) \in B_k(\sigma(d), \delta) \), thus there exists a sequence \( \sigma(a) = a_0, \cdots, a_k = \sigma(b) \) such that for every \( j \in \{0, \cdots, \ell^k - 1\} \) we have
\[
\rho_k(a_j, a_{j+1}) < \delta'.
\]
Then, let \( c^* \) the first word from \( d \) such that by item 1 of Lemma 2 \( c^*B_0(\sigma(d), \delta') \subset B_0(c^* \sigma(d), \delta) \) and for all \( j = 0, 1, \cdots, \ell^k - 1 \),
\[
\rho(c^* a_j, c^* a_{j+1}) < \delta.
\]
Then for every \( j = 0, \cdots, \ell^k - 1 \) there exists a sequence \( c^* a_j = d_j^0, d_j^1, \cdots, d_j^\ell = c^* a_{j+1} \), such that for every \( 0 \leq r \leq \ell - 1 \) we have \( \rho(d_j^r, d_j^{r+1}) < \delta' \). So there is a path \( a = d_0^0, \cdots, d_j^r, d_0^1, \cdots, d_1^1, \cdots, d_{\ell^k-1}^1, \cdots, d_\ell^{\ell-1} = b \), such that for all \( 0 \leq j \leq \ell^k - 1 \) and \( 0 \leq r \leq \ell - 1 \) we have
\[
\rho_{k+1}(d_j^r, d_j^{r+1}) < \delta'.
\]

\[\square\]

### 3.3 Potentials

For the rest of this paper we fix \( \alpha \in (0, 1] \) and \( \delta \in (0, \delta_0) \). Given \( \delta_1 > 0 \). We say that a function \( \phi \in C(X, \mathbb{R}) \) is uniformly \( \delta_1 \)-locally \( \alpha \)-H"older (ULH) if, there exists \( L \geq 0 \) such that.

For all \( a, b, c \in X \), satisfying \( a, b \in B_0(c, \delta) \) we have
\[
|\phi(a) - \phi(b)| \leq L(\rho(a, b))^\alpha.
\]
We say that a function \( \phi \) is locally \( \alpha \)-H"older if there is \( \delta_1 > 0 \) such that \( \phi \) is \( \delta_1 \)-locally \( \alpha \)-H"older.

We denote by \( H_{\alpha, \delta} \) the vector space of all bounded uniformly \( \delta \)-locally \( \alpha \)-H"older potentials. We endow \( H_{\alpha, \delta} \) with the norm \( \| \cdot \|_{\alpha, \delta} \) defined as follows. First, we define the \( \alpha, \delta \)-variation of \( \phi \) by
\[
v_{\alpha, \delta}(\phi) := \inf \{ L \geq 0 : \text{ for all } a, b \in X, \text{ if } \rho(a, b) < \delta \text{ then } |\phi(a) - \phi(b)| \leq L(\rho(a, b))^\alpha \},
\]
and \( \| \phi \|_{\infty} = \sup \{|\phi(a)| : a \in X\} \). Then the norm \( \| \cdot \|_{\alpha, \delta} \) on \( H_{\alpha, \delta} \) is defined by
\[
\| \phi \|_{\alpha, \delta} := v_{\alpha, \delta}(\phi) + \| \phi \|_{\infty}.
\]
The vector space \( H_{\alpha, \delta} \) endowed with the norm \( \| \cdot \|_{\alpha, \delta} \) is a Banach space.
For every potential \( \phi \in C(X, \mathbb{R}) \) and for every integer \( n \geq 0 \) we recall that the \textit{n-variation} of \( \phi \) is defined by

\[
\text{Var}_n(\phi) := \sup_{\substack{a \in X \\ b \in \mathbb{B}_n(a, \delta)}} \sup \ \ |\phi(b) - \phi(a)|. \tag{20}
\]

We say \( \phi \in C(X, \mathbb{R}) \) is \textit{weakly Hölder continuous} if it satisfies the following condition.

There exist \( C > 0 \) and \( 0 < r < 1 \) such that, for all \( n \geq 0 \) we have

\[
\text{Var}_n(\phi) \leq Cr^n.
\]

We denote by \( \text{BV}(X, \mathbb{R}) \) the space of all weakly Hölder continuous potentials. We put

\[
\text{BV}_r(X, \mathbb{R}) := \{ \phi \in C(X, \mathbb{R}) : \text{Var}_n(\phi) \leq Cr^n,
\text{ for all } n \geq 0, \text{ and for some } C \geq 0 \},
\]
and for \( \phi \in \text{BV}_r(X, \mathbb{R}) \), let

\[
w_r(\phi) := \sup \left\{ \frac{\text{Var}_n(\phi)}{r^n} : n \geq 1 \right\},
\]
and

\[
\|\phi\|_r = \|\phi\|_\infty + w_r(\phi).
\]

**Proposition 2.** The function \( \| \cdot \|_r \) is a norm on \( \text{BV}_r(X, \mathbb{R}) \) that makes it a Banach space.

\[
\begin{proof}
\text{Follows easily that } \| \cdot \|_r \text{ is a norm in } \text{BV}(X, \mathbb{R}). \text{ To prove that } \text{BV}(X, \mathbb{R}) \text{ is a Banach space with the norm } \| \cdot \|_r, \text{ let } \{ \phi(n) \}_{n \geq 1} \text{ be a Cauchy sequence in } \text{BV}(X, \mathbb{R}). \text{ So for every } \epsilon > 0, \text{ there exists } N \in \mathbb{N} \text{ such that for all } m, n \geq N
\|\phi(n) - \phi(m)\|_\infty + w_r(\phi(n) - \phi(m)) < \epsilon. \text{ Since } \|\phi\|_\infty \leq \|\phi\|_r \text{ and } w_r(.) \leq \|\phi\|_r, \text{ then } \{ \phi(n) \}_{n \geq 1} \text{ is a Cauchy sequence with respect to the norm } \| \cdot \|_\infty (\text{this implies there exists } \phi \in C(X, \mathbb{R}) \text{ such that } \phi(n) \to \phi, n \to \infty) \text{ and the sequence } \{ w_r(\phi(n)) \}_{n \geq 1} \text{ is also a Cauchy sequence.}

\text{Observe first that } \phi \in \text{BV}(X, \mathbb{R}). \text{ Let } \epsilon > 0 \text{ then there exists } N \text{ such that for all } m \geq N \|\phi(m)(b) - \phi(b)\| < \epsilon, \text{ for all } b \in X. \text{ Then for all } b, c \in \mathbb{B}_n(a, \delta) \text{ with } a \in X \text{ we have}
\]

\[
|\phi(b) - \phi(c)| \leq |\phi(b) - \phi(m)(b)| + |\phi(m)(b) - \phi(m)(c)| + |\phi(m)(c) - \phi(c)|
\leq \frac{\epsilon}{2} + C_{(m)v} r^m_{(m)v} + \frac{\epsilon}{2} = C_{(m)v} r^m_{(m)v} + \epsilon
\]

Let \( C_{(N)v} = \max\{ C_{(m)v} : m \geq N \} \) and \( r^N_{(N)v} = \max\{ r^m_{(m)v} : m \geq N \} \), then in particular we have

\[
|\phi(b) - \phi(c)| \leq C_{(N)v} r^N_{(N)v} + \epsilon.
\]

Therefore

\[
V_n(\phi) \leq C_{(N)v} r^N_{(N)v}, \text{ for all } n \geq 1.
\]
\end{proof}
\]
Lemma 6. Every uniformly $\delta$-locally $\alpha$-Hölder continuous potential is weakly Hölder continuous with constant $r = \lambda_{ES}^{-\alpha}$.

Proof. By part 1 of Lemma 2 with $c^*a$ replaced by $a$ and with $m = 0$, we obtain

$$\mathbb{B}_n(a, \delta) \subseteq \mathbb{B}_0(a, C_{ES}\lambda_{ES}^{-n}\delta).$$

So, if $\phi$ is uniformly $\delta$-locally $\alpha$-Hölder, then we have that there exists $L \geq 0$ such that for all $n \geq 0$, $a \in X$ and $b, c \in \mathbb{B}_n(a, \delta)$ we have

$$|\phi(b) - \phi(c)| \leq L\rho(b, c) \leq L(C_{ES}\lambda_{ES}^{-n})^\alpha \delta^\alpha = LC_{ES}^\alpha \delta^\alpha (\lambda_{ES})^n.$$

For every $n \geq 1$ put

$$S_n\phi = \sum_{k=0}^{n-1} \phi \circ \sigma^k.$$

If $n = 0$, then put $\phi_0 \equiv 0$.

Lemma 7. Let $\phi \in \text{BV}_r(X, \mathbb{R})$, and assume that (Delta) holds. Then for all $n \geq 0$ we have the following assertions.

1. For all $m \geq 0$ we have

$$V_{m+n}(S_n\phi) \leq \left( w_r(\phi) \frac{r^n}{1-r} \right) r^m.$$

2. For every $c^* \in \mathbb{Z}^n$ and for all $a \in X$, let

$$\phi_{n,c^*}(a) := S_n\phi(c^*a),$$

then $\phi_{n,c^*} \in \text{BV}_r(X, \mathbb{R})$.

Proof. 1. Let $d \in X$ and $m \geq 0$. By property [16] we have that for each $j = 0,1,\ldots,n-1$,

$$\sigma^j\mathbb{B}_{m+n}(d, \delta) \subseteq \mathbb{B}_{m+n-j}(\sigma^j d, \delta).$$

Then, for each $a, b \in \mathbb{B}_{m+n}(d, \delta)$ we have

$$|S_n\phi(a) - S_n\phi(b)| \leq \sum_{j=0}^{n-1} |\phi(\sigma^j a) - \phi(\sigma^j b)| \leq \sum_{j=0}^{n-1} V_{m+n-j}(\phi) \leq w_r(\phi) \sum_{j=0}^{n-1} r^{(m+n-j)} \leq w_r(\phi) r^{m+n} \frac{1}{1-r} = \left( w_r(\phi) \frac{r^n}{1-r} \right) r^m. \quad (21)$$
2. Let \( n \geq 0, \ c^* \in \mathbb{Z}^n \) and \( m \geq 0 \). We have to prove that \( \phi_{n,c^*} \) is \((\delta, r)\)-weakly Hölder, however by condition (Delta) is enough to show that \( \phi_{n,c^*} \) is \((\delta', r)\)-weakly Hölder. In fact, let \( \mathbf{d} \in X \) and \( \mathbf{a}, \mathbf{b} \in \mathbb{B}_m (\mathbf{d} , \delta') \). By part 2 of Lemma 2 we have

\[
c^* \mathbb{B}_m (\mathbf{d} , \delta') \subseteq \mathbb{B}_{m+n}(c^* \mathbf{d} , \delta).
\]

Hence, using the estimation (21) of part 1 of this lemma we get, for \( c^* \mathbf{a}, c^* \mathbf{b} \in \mathbb{B}_{m+n}(c^* \mathbf{d} , \delta) \)

\[
|S_n \phi(c^* \mathbf{a}) - S_n \phi(c^* \mathbf{b})| \leq \left( w_r (\phi) \frac{r^n}{1 - r} \right) r^m.
\]

It is the same to have

\[
|\phi_{n,c^*}(\mathbf{a}) - \phi_{n,c^*}(\mathbf{b})| \leq \left( w_r (\phi) \frac{r^n}{1 - r} \right) r^m.
\]

Thus it follows that the function \( \phi_{n,c^*} \) is \((\delta, r)\)-weakly Hölder continuous.

\( \square \)

**Lemma 8.** Let \( \phi \in \mathbb{B}_r (X, \mathbb{R}) \). Then for all \( n \geq 1, \ c^* \in \mathbb{Z}^n, \ m \geq 1, \ \mathbf{d} \in X \) and \( \mathbf{a}, \mathbf{b} \in \mathbb{B}_m (\mathbf{d} , \delta) \), if we put \( K := w_r (\phi_{n,c^*}) \), then

\[
\left| e^{S_n \phi(c^* \mathbf{a})} - e^{S_n \phi(c^* \mathbf{b})} \right| \leq e^K \left( e^{S_n \phi(c^* \mathbf{b})} \right).
\]

**Proof.** Let \( m \geq 1, \ \mathbf{d} \in X \) and \( \mathbf{a}, \mathbf{b} \in \mathbb{B}_m (\mathbf{d} , \delta) \). By item 2 of Lemma 7 we have for \( n \geq 1 \) and \( c^* \in \mathbb{Z}^n \) the function \( \phi_{n,c^*} \in \mathbb{B}_r (X, \mathbb{R}) \). Thus

\[
|S_n \phi(c^* \mathbf{a}) - S_n \phi(c^* \mathbf{b})| \leq w_r (\phi_{n,c^*}) r^m \leq w_r (\phi_{n,c^*}) = K.
\]

Then

\[
e^{S_n \phi(c^* \mathbf{a})} - S_n \phi(c^* \mathbf{b}) - 1 \leq e^K - 1,
\]

and

\[
1 - e^{-(S_n \phi(c^* \mathbf{b}) - S_n \phi(c^* \mathbf{a}))} \leq 1 - e^{-K} \leq e^K - 1.
\]

Therefore

\[
|e^{S_n \phi(c^* \mathbf{a})} - S_n \phi(c^* \mathbf{b}) - 1| \leq e^K - 1.
\]

\( \square \)

In the following proposition we prove that the condition (Delta) follows immediately as consequence of the assumption (Path).

**Proposition 3.** Let \( \delta' = \min\{\delta, \delta/C_{ES}\} \). If the metric \( g \) satisfies the condition (Path) then \( \phi \) satisfies the condition (Delta), it means if \( \phi \) is \((\delta', r)\)-weakly Hölder then \( \phi \) is \((\delta, r)\)-weakly Hölder.

**Proof.** Let \( \phi \) be a \((\delta', r)\)-weakly Hölder potential. Let \( \mathbf{d} \in X, \ m \geq 0, \ a, b \in \mathbb{B}_m (\mathbf{d}, \delta) \). We have by part 1 of Lemma 2 that there exists \( n_0 \) such that for all \( n \geq 0 \)

\[
\mathbb{B}_{n+n_0}(\mathbf{d} , \delta) \subseteq \mathbb{B}_n (\mathbf{d} , \delta').
\]

If $m \geq n_0$ then $B_m(d, \delta) \subseteq B_{m-n_0}(d, \delta')$. Thus we have

$$|\phi(a) - \phi(b)| \leq w_r(\ell)^{m-n_0} = (w_r(\ell)^{r-n_0})^{r^m}.$$ 

If $1 < m < n_0$, then by condition (Path) we have that Lemma 4 is satisfied. So, for all $a, b \in B_m(d, \delta)$, there is a sequence $a = c_0, \cdots, c_m = b$ such that for all $j = 0, \cdots, \ell^m - 1$ we have $c_m(c_j, c_{j+1}) \leq \delta'$. Hence

$$|\phi(a) - \phi(b)| \leq \sum_{j=0}^{\ell^m-1} |\phi(c_j) - \phi(c_{j+1})| \leq (\ell^m w_r(\ell))^{r_m}.$$ 

\[\square\]

### Topological Pressure

From now on we assume that the metric $\varrho$ given in the introduction satisfies (Con) and that any potential $\phi$ is weakly Hölder continuous potential.

Let $\Lambda$ be a compact subset of $X$ such that $\sigma(\Lambda) = \Lambda$, let $\phi \in BV_r(X, \mathbb{R})$ and $a \in X$. Set

$$Z_n(\Lambda, \phi, a) := \sum_{c^* \in \mathbb{Z}^r, c^* a \in \Lambda} e^{S_n \phi(c^* a)}.$$ 

The standard topological pressure of $\phi$ on $\Lambda$ is given by

$$P_{\text{top}}(\phi|\Lambda) = \lim_{n \to \infty} \frac{1}{n} \log (Z_n(\Lambda, \phi, a)),$$  \hspace{1cm} (22)

see [PU10]). Since $X$ is not compact, we define the pressure $P(\phi)$ of a potential $\phi \in BV_r(X, \mathbb{R})$ on $X$ with respect to $\sigma$ as follows

$$P(\phi) := \sup_{N \geq 1} P_{\text{top}}(\phi|\Sigma_N^+),$$  \hspace{1cm} (23)

where for each $N \geq 1$

$$P_{\text{top}}(\phi|\Sigma_N^+) := \lim_{n \to \infty} \frac{1}{n} \log (Z_n(\Sigma_N^+, \phi, a)).$$

**Proposition 4.** The following assertion holds

$$\sup \{P_{\text{top}}(\phi|\Lambda) : \Lambda \subseteq X \text{ is compact, } \sigma(\Lambda) = \Lambda\} = \sup_{N \geq 1} P_{\text{top}}(\phi|\Sigma_N^+).$$

**Proof.** Let $\Lambda$ be a compact subset of $X$ such that $\sigma(\Lambda) = \Lambda$. Then there exists $N \geq 1$ such that $\Lambda \subset \Sigma_N^+$ (see hypothesis of $X$ in the introduction). Hence for every $a \in \Lambda$ we have

$$Z_n(\Lambda, \phi, a) \leq Z_n(\Sigma_N^+, \phi, a).$$

Therefore $P_{\text{top}}(\phi|\Lambda) \leq \sup_{N \geq 1} P_{\text{top}}(\phi|\Sigma_N^+)$. Since it is satisfied for every compact and $\sigma$-invariant set $\Lambda$, we have

$$P(\phi) = \sup \{P_{\text{top}}(\phi|\Lambda) : \Lambda \subseteq X \text{ is compact, } \sigma(\Lambda) = \Lambda\}.$$

\[\square\]
For the rest of this section we assume that each $\phi \in \text{BV}_r(X, \mathbb{R})$ satisfies the following conditions

- **Summable** condition, that is
  \[
  \sup_{a \in X} \left\{ \sum_{b : \sigma(b) = a} \exp(\phi(b)) \right\} < \infty.
  \]

- **Bounded on balls**. That is, if for all $R > 0$ we have
  \[
  \sup_{c \in B(0, R)} |\phi(c)| < \infty.
  \]

Thus, for every $g \in \text{CB}_b(X, \mathbb{R})$ we have that the transfer operator $\mathcal{L}_\phi$ associated to the potential $\phi$ which is given by

\[
\mathcal{L}_\phi(g)(a) = \sum_{b : \sigma(b) = a} e^{\phi(b)} g(b),
\]

is well defined on $\text{CB}_b(X, \mathbb{R})$, In particular when $g$ is the function identically equal to 1, we have

\[
\mathcal{L}_\phi(1)(a) = \sum_{b : \sigma(b) = a} e^{\phi(b)}.
\]

Notice that for every $n \geq 1$ and $a \in X$ the iterates of $\mathcal{L}_\phi$ are given by

\[
\mathcal{L}_\phi^n(g)(a) = \sum_{c^* \in \mathbb{Z}^n} e^{S_n \phi(c^*)} g(c^* a).
\]

**Lemma 9.** Let $\phi \in \text{BV}_r(X, \mathbb{R})$. For every $n \geq 0$ and $m \geq 0$ there exists a constant $K > 0$ such that for all $d \in X$ and $a, b \in \mathbb{B}_m(d, \delta)$ we have

\[
|\mathcal{L}_\phi^n 1(b) - \mathcal{L}_\phi^n 1(a)| \leq e^K \mathcal{L}_\phi^n 1(a).
\]

**Proof.** Let $n \geq 0$, $c^* \in \mathbb{Z}^n$ and let $K = w_r(\phi_{n, c^*})$ be the constant in Lemma \[\square\]

Then for $a, b \in \mathbb{B}_m(d, \delta)$ we have

\[
|e^{S_n \phi(c^* b)} - e^{S_n \phi(c^* a)}| \leq e^K \left( e^{S_n \phi(c^* b)} \right).
\]

Therefore,

\[
|\mathcal{L}_\phi^n 1(b) - \mathcal{L}_\phi^n 1(a)| \leq \sum_{c^* \in \mathbb{Z}^n} e^{S_n \phi(c^* b)} - \sum_{c^* \in \mathbb{Z}^n} e^{S_n \phi(c^* a)}
\]

\[
\leq \sum_{c^* \in \mathbb{Z}^n} \left| e^{S_n \phi(c^* b)} - e^{S_n \phi(c^* a)} \right|
\]

\[
\leq e^K \sum_{c^* \in \mathbb{Z}^n} e^{S_n \phi(c^* a)} \leq e^K \mathcal{L}_\phi^n 1(a).
\]

\[\square\]
Lemma 10. For every potential $\phi \in BV_r(X, \mathbb{R})$ that is summable and bounded on balls, and for every $R > 0$ there exists $M_{\phi, R} > 0$ such that for every $n \geq 0$ and for all $a, b \in B(0, R)$ we have

$$L^n(a) \leq M_{\phi, R} \cdot L^n(b).$$

Proof. Let $R > 0$. From (Con) we have there exists $n_0 = n_0(R) \geq 1$ such that for all $a, b \in B(0, R)$ there exists $c \in BV(a, \delta_0)$ such that $\sigma^{n_0}(c) = b$. Therefore, we have

$$L^{n+n_0}(a) = \sum_{d \in \sigma^{n+n_0}(d) = b} e^{\phi_n + n_0(d)} = \sum_{d \in \sigma^{n+n_0}(d) = b} e^{\phi_n(d)} \cdot e^{\phi_{n_0}(\sigma^n(d))}$$

$$= \sum_{d \in \sigma^n(c) = b} \sum_{d \in \sigma^n(d) = c} e^{\phi_n(d)} \cdot e^{\phi_{n_0}(\sigma^n(d))}$$

$$\geq \sum_{d \in \sigma^n(c) = b} e^{\phi_n(d)} \cdot e^{\phi_{n_0}(\sigma^n(c))} = e^{\phi_{n_0}(\sigma^n(c))} L^n(c).$$

Observe that

$$L^{n+n_0}(b) = \sum_{d \in \sigma^{n+n_0}(d) = b} e^{\phi_n + n_0(d)} = \sum_{d \in \sigma^{n+n_0}(d) = b} e^{\phi_n(d)} \cdot e^{\phi_{n_0}(\sigma^n(d))}$$

$$= \sum_{d \in \sigma^n(c) = b} \sum_{d \in \sigma^n(d) = c} e^{\phi_n(d)} \cdot e^{\phi_{n_0}(\sigma^n(d))}$$

$$\leq \sum_{d \in \sigma^n(c) = b} \left( e^{\phi_n(c)} \cdot \sum_{d \in \sigma^n(d) = c} e^{\phi_{n_0}(\sigma^n(d))} \right)$$

$$\leq \|L^n\|_{\infty} L^n(b).$$

By Lemma, we have, for every $n \geq 0$, $L^n(a) \leq (1 + e^K) L^n_\phi(c).$

Therefore,

$$L^n(a) \leq (1 + e^K)e^{-\phi_{n_0}(c)} L^{n+n_0}_\phi(b)$$

$$\leq (1 + e^K)e^{-\phi_{n_0}(c)} \cdot L_\phi(b) \cdot L^{n+n_0}_\phi(b)$$

$$\leq (1 + e^K)M_{\phi, R} \cdot L_\phi(b) \cdot L^{n+n_0}_\phi(b),$$

where $M_{\phi, R} = \sup_{c \in B(0, R+\delta_0)} e^{-\phi_{n_0}(c)}$, which is bounded by hypothesis.

Corollary 1. Let $\phi \in BV_r(X, \mathbb{R})$, then we have that

$$\lim_{n \to \infty} \frac{1}{n} \log L^n_\phi(a)$$

is independent of $a \in X$. 
Proof. Let \( a, b \in X \), and set \( R = \max\{\varrho(0, a), \varrho(0, b)\} + 1 \). So, let \( M = M_{\varrho, R} \) as in Lemma 10. It is such that for all \( n \geq 0 \) we have

\[
\mathcal{L}_\varrho^n I(a) \leq M \mathcal{L}_\varrho^n I(b).
\]

Hence

\[
\limsup_{n \to \infty} \frac{1}{n} \log \mathcal{L}_\varrho^n I(a) \leq \limsup_{n \to \infty} \frac{1}{n} \log \mathcal{L}_\varrho^n I(b).
\]

Thus the corollary follows immediately.

We will show that the sequence \( \frac{1}{n} \log \mathcal{L}_\varrho^n I(a) \) with \( a \in X \) actually converges and its limit is precisely \( P(\varphi) \).

3.4 The existence of conformal measures

Throughout this section we consider the following notation. For any \( a^* = a_0 \cdots a_{n-1} \in \mathbb{Z}^n \), the cylinder \([a^*]\) is defined by

\[
[a^*] := \{ b \in X : b_i = a_i, 0 \leq i \leq n-1 \}.
\]

For topologically mixing Markov shift spaces with infinitely many symbols, the existence of conformal measures is not very immediate, see for instance [MU01]. Due our space \( X \) is a bit more complicated due the lack of compactness and since fall out of the framework developed by [MU01], we have to impose on \( X \) further assumptions to state the existence of conformal measures.

- Assume that the transformation \( \sigma : X \to X \) can be continuously extended to \( \overline{\sigma} : \overline{X}^0 \to \overline{X}^0 \), where \( \overline{X}^0 \) is the completion of \( X \) with respect to the metric \( \varrho \).

- For all \( R > 0 \), we denote by \( \overline{B}(0, R) \) the closed ball in \( \overline{X}^0 \) centered in \( 0 \) and radius \( R \). Namely

\[
\overline{B}(0, R) := \{ b \in \overline{X}^0 : \varrho(0, b) \leq R \},
\]

and assume that this ball \( \overline{B}(0, R) \) is compact in \( \overline{X}^0 \).

- We also assume that for every \( R > 0 \) and for every \( b \in \overline{X}^0 \setminus X \) there exists \( N \geq 1 \) such that for \( n \geq N \), we have \( \overline{\sigma}^n(b) \in \overline{X}^0 \setminus \overline{B}(0, R) \). If it does not lead to misunderstanding we will frequently denote \( \overline{\sigma} \) simply by \( \sigma \).

- Given \( R > 0 \) and \( k \) be an integer number. Let

\[
[k, R] := [k] \cap \overline{X}^0 \setminus \overline{B}(0, R) = \{ a \in [k] \cap \overline{X}^0 : \varrho(0, a) > R \}.
\]

And given a potential \( \varphi \in \text{BV}_r(X, \mathbb{R}) \), we assume that \( \varphi \) satisfies

\[
\sum_{k \in \mathbb{Z}} e^{\sup \varphi|_{[k]}} < \infty.
\]

\[
\sum_{k \in \mathbb{Z}} e^{\sup \varphi|_{[k, R]}} \to 0, \text{ when } R \to \infty.
\]
Finally we assume that $\phi$ has a continuous extension on $X^\Omega$.

Let us recall what a conformal measure means. Consider a measurable endomorphism $T : Y \to Y$ on a measurable space $(Y, \mathcal{F})$ and a measurable non-negative function $g$ on $Y$. A measure $m$ on $(Y, \mathcal{F})$ is called $g$-conformal for $T$ on $Y$ if for all measurable set $A$ which $T(A)$ is measurable and $T|_A$ is invertible we have

$$m(T(A)) = \int_A g \, dm.$$  

Observe that (24) implies that $m \circ T$ is absolutely continuous with respect to $m$ on the $\sigma$-algebra $\mathcal{F} \cap A$, for every set $A \in \mathcal{F}$ such that $T : A \to T(A)$ is a measurable isomorphism. In this case, the equality in (24) is equivalent to the fact that the corresponding Radon-Nikodym derivative $d(m \circ T)|_A dm|_A$ is equal to $g|_A$.

Let $C(X, \mathbb{R})^*$ be the dual space of $C(X, \mathbb{R})$, $\phi \in BV_r(X, \mathbb{R})$ and $L^*_\phi : C(X, \mathbb{R})^* \to C(X, \mathbb{R})^*$ the dual Ruelle transfer operator defined by

$$L^*_\phi(\nu)(g) = \nu(L\phi(g)).$$

For every $N \geq 1$ we have that $\Sigma^+_N$ is a compact metric space in $X$. So the Schauder-Tychonoff fixed point theorem guarantees that exists a fixed point $\nu_N$ (supported on $\Sigma^+_N$) of the map

$$\nu \to (L^*_\phi(\nu)(1))^{-1}L^*_\phi(\nu).$$

That is

$$(L^*_\phi(\nu_N)(1))^{-1}L^*_\phi(\nu_N) = \nu_N.$$  

Following the classical thermodynamic formalism for compact spaces, It is known that for each $N$ the fixed point $\nu_N$ is a $e^{P_{\text{top}}(\phi|_{\Sigma^+_N})-\phi|_{\Sigma^+_N}}$-conformal measure for $\sigma|_{\Sigma^+_N}$, see [PU10]. It means that for each $N$ we have

$$\nu_N(\sigma([k, R]_N)) = e^{P_{\text{top}}(\phi|_{\Sigma^+_N})} \int_{[k, R]_N} e^{-\phi|_{\Sigma^+_N}} d\nu_N(q),$$

where $[k, R]_N = \Sigma^+_N \cap [k, R]$ such that $\sigma|_{[k, R]_N}$ is invertible

**Remark 1.** Since for every $N \geq 1$ we have $P_{N+1}(\phi) \geq P_N(\phi)$, and $\lim_{N \to \infty} P_{\text{top}}(\phi|_{\Sigma^+_N}) = P(\phi)$, we have that $\lim_{N \to \infty} e^{P_{\text{top}}(\phi|_{\Sigma^+_N})}$ exists and is equal to $e^{P(\phi)}$. We will denote $e^{P(\phi)}$ by $\lambda$.

Since our goal in this section is to construct conformal measures without the condition of compactness, we first show that the family $\{\nu_N\}_{N \in \mathbb{N}}$ is tight and then to apply Prokhorov’s Theorem. We recall that tightness means that

for all $\varepsilon > 0$ there exists a compact set $K \subset X^\Omega$ such that for all $N$ we have $\nu_N(X^\Omega \setminus K) < \varepsilon$.

**Proposition 5.** The sequence of measures $\{\nu_N\}_{N \in \mathbb{N}}$ is tight.
Proof. Since for each $N$ the measure $\nu_N$ is $e^{P_{\text{top}}(\phi|_{\Sigma_N^+})-\phi|_{\Sigma_N^+}}$-conformal, we have

$$\nu_N(\sigma([k,R])) = e^{P_{\text{top}}(\phi|_{\Sigma_N^+})} \int_{[k,R]} e^{-\phi} d\nu_N(\mathfrak{a})$$

$$\geq e^{P_{\text{top}}(\phi|_{\Sigma_N^+})} \nu_N([k,R]) e^{-\sup \phi|_{[k,R]}}.$$ 

Hence

$$\nu_N([k,R]) \leq e^{-P_{\text{top}}(\phi|_{\Sigma_N^+})} \nu_N(\sigma([k,R])) e^{\sup \phi|_{[k,R]}} \leq e^{-P_{\text{top}}(\phi|_{\Sigma_N^+})} e^{\sup \phi|_{[k,R]}}.$$ 

Furthermore observe that

$$\nu_N(X \setminus B(0, R)) = \nu_N \left( \bigcup_{k \in \mathbb{Z}} [k,R] \right) \leq \sum_{k \in \mathbb{Z}} \nu_N([k,R]) \leq e^{-P_{\text{top}}(\phi|_{\Sigma_N^+})} \sum_{k \in \mathbb{Z}} e^{\sup \phi|_{[k,R]}}.$$ 

From [14] we have that the last term tends to zero when $R$ tends to infinity. Therefore from the compactness of the closed ball $\overline{B}(0, R)$ in $X^\sigma$ the tightness of $\{\nu_N\}_{N \geq 1}$ is proved.

The Prokhorov’s theorem allows us relate the tightness of measures to weak convergence in the space of probability measures.

**Theorem 6** (Prokhorov). If $P$ is a Polish space, that means a complete metrizable and separable space, and $\mathcal{M}(P)$ is the space of all Borel probability measures in $P$, then every tight family measures from $\mathcal{M}(P)$ is a pre-compact subset of $\mathcal{M}(P)$.

Therefore, for the sequence of measures $\{\nu_N\}_{N \geq 1}$ we have, there exists a subsequence $\{\nu_{N_i}\}_{i \geq 1}$ which converges in the weak topology to some probability measure $\nu$. It follows in particular that for every Borel set $A$ such that $\nu(A) = 0$ and for every bounded continuous function $g$ with bounded support we have

$$\lim_{i \to \infty} \int_A g d\nu_{N_i} = \int_A g d\nu.$$ 

(27)

Let

$$X_{\text{rad}} := \{ \mathfrak{a} \in X^\sigma : \omega_\sigma(\mathfrak{a}) \cap X \neq \emptyset \}.$$ 

For every $N \geq 1$, let $\overline{B}(0, N)$ be the closed ball in $X^\sigma$ and for $k$ an integer number let $|k|$ be the closure of $[k]$. Thus we consider a sequence of $A_N$, where

$$A_N := \overline{B}(0, N) \cap \bigcup_{k = -N}^N |k|.$$ 

For every $N \geq 1$ we have $A_N \subset A_{N+1}$ and $\bigcup_{N=1}^\infty A_N = X^\sigma$.

**Theorem 7.** The measure $\nu$ is $e^{P(\phi)-\phi}$-conformal and $\nu(X_{\text{rad}}) = 1$. 

Proof. Note that for each $N \geq 1$ the measure $\nu_N$ is $e^{P_{\text{top}}(\phi|_{\Sigma^+_N})}$-conformal for $\sigma|_{\Sigma^+_N}$ but not for $\sigma$ defined on $\overline{X^\phi}$. However, if $N$ large enough then for every $A \subset A_N$ such that $\sigma|_A$ is one-to-one, we have

$$\nu_N(\sigma(A)) = e^{P_{\text{top}}(\phi|_{\Sigma^+_N})} \int_A e^{-\phi} d\nu_N.$$  

To verify this, first we prove that

$$\sigma(A) \cap \Sigma^+_N = \sigma(A \cap \Sigma^+_N).$$  

Observing that $\sigma(A \cap \Sigma^+_N) \subseteq \sigma(A) \cap \Sigma^+_N \subseteq \sigma(A) \cap \Sigma^+_N$. To get the contrary inclusion, let $a \in \sigma(A) \cap \Sigma^+_N$, then there exists $b \in A$ such that $a = \sigma(b)$. We prove that $b \in \Sigma^+_N$. Let $c \in \mathbb{Z}$ a word such that $ca = b$. Then since $b \in A$ and $A \subset A_N$ we have that there exists $k \in \{-N, \ldots, N\}$ such that $ca \in [k]$, then we have

$$B_1(ca, \delta) \cap [k] \neq \emptyset.$$  

Thus by part 1 of Lemma 2 we have that $B_1(ca, \delta) \subseteq cB_0(a, \delta)$ and hence

$$cB_0(a, \delta) \cap [k] \neq \emptyset.$$  

Therefore $c \in \{-N, \ldots, N\}$ and so we obtain $b \in \Sigma^+_N$. Using (28) we can write

$$\nu_N(\sigma(A)) = e^{P_{\text{top}}(\phi|_{\Sigma^+_N})} \int_{A \cap \Sigma^+_N} e^{-\phi} d\nu_N = e^{P_{\text{top}}(\phi|_{\Sigma^+_N})} \int_A e^{-\phi} d\nu_N.$$  

And since the sequence $\{\nu_N\}$ converges weakly to $\nu$ we have that for every Borel set $A$ such that $\nu(\partial A) = 0$ it satisfies $\nu_N(A) \rightarrow \nu(A)$. In particular this holds for every bounded Borel $A$ such that $\nu(\partial A) = 0$ and $\nu(\partial \sigma(A)) = 0$. Then

$$\nu(\sigma(A)) = \lim_{i \to \infty} \nu_N_i(\sigma(A)) = \lim_{i \to \infty} e^{P_{\text{top}}(\phi|_{\Sigma^+_N})} \int_A e^{-\phi} d\nu_N_i = \lambda \int_A e^{-\phi} d\nu.$$  

Take an arbitrary bounded Borel set $A$ such that $\sigma|_A$ is injective. We have that $\text{Sing}(\sigma : \overline{X^\phi} \rightarrow \overline{X^\phi}) = \emptyset$ and $\nu(\overline{X^\phi}) = 1$. Then we obtain that $\nu$ is a conformal measure.

To prove that the measure $\nu$ is supported in $X_{\text{rad}}$, note that by the inequality (14), there exists $R > 0$ such that

$$\lambda^{-1} \sum_{k \in \mathbb{Z}} e^{\phi|_{[k,R]}} < 1/2.$$  

Let

$$\overline{X^\phi}(R, n) := \{a \in \overline{X^\phi} : (\sigma^k(a)) \in \overline{X^\phi} \setminus B(0, R) \text{ for } k = 0, \ldots, n - 1\}.$$
Then
\[ \nu(\overline{X}(R, n)) \geq \nu(\overline{X}(R, n + 1)) \cap [k]) \]
\[ = \lambda \int_{\overline{X}(R, n + 1) \cap [k]} e^{-\phi} d\nu \geq \lambda e^{\sup_{X \setminus B(0, R)}} e^{-\nu(\overline{X}(R, n + 1)) \cap [k]}. \]

Hence using (29), we get that
\[ \nu(\overline{X}(R, n)) \leq \lambda \sum_{k \in \mathbb{Z}} e^{\sup_{[k]} \nu(\overline{X}(R, n))}, \]
and then
\[ \nu(\overline{X}(R, n)) \leq (1/2)^n. \]

Thus \( \nu(X_{\text{rad}}) = 1. \)

**Corollary 2.** There exists a measure \( \nu \) which is \( e^{P(\phi) - \phi} \)-conformal for \( \sigma : X_{\text{rad}} \rightarrow X_{\text{rad}} \) and \( \nu(X_{\text{rad}}) = 1. \)

For the rest of this section we assume that \( \phi \in B_r(X, \mathbb{R}) \) is summable and bounded on balls see (10) and (11).

We say that \( \phi \) is rapidly decreasing if it satisfies.
\[ \lim_{R \to \infty} \sup_{a \in X_{\text{rad}} \setminus B(0, R)} (\mathcal{L}_\phi^\infty(I(a))) = 0. \]

We consider the normalized transfer operator \( \hat{\mathcal{L}}_\phi = e^{-P(\phi)} \mathcal{L}_\phi. \)

**Proposition 6.** Let \( \phi \) be a rapidly decreasing potential. The limit \( \lim_{n \to \infty} \frac{1}{n} \log \mathcal{L}_\phi^n I(a), \) with \( a \in X_{\text{rad}} \) exists, and
\[ P(\phi) = \lim_{n \to \infty} \frac{1}{n} \log \mathcal{L}_\phi^n I(a). \]

**Proof.** To prove this proposition is enough to prove that there exists \( L > 0 \) and, for all \( R > 0 \) there exists \( \ell_R > 0 \) such that for all \( n \geq 1 \) and all \( a \in B(0, R) \), we have
\[ \ell_R \leq \mathcal{L}_\phi^n I(a) \leq L. \]  

First, we prove the right hand side inequality. Since \( \phi \) is rapidly decreasing we have that there exists sufficiently large \( R_0 > 0 \) that
\[ \sup \left\{ \mathcal{L}_\phi I(a) : a \in X_{\text{rad}} \setminus B(0, R_0) \right\} \leq 1. \]

We will show by induction that for every \( n \geq 0, \)
\[ \| \mathcal{L}_\phi^n I \|_{\infty} \leq \frac{M_{\phi, R_0}}{\nu(B(0, R_0))} = L, \]  

where \( M_{\phi, R_0} \) is the constant coming from Lemma (11) with \( \mathcal{L}_\phi \) replaced by the operator \( \mathcal{L}_\phi. \)
To see this observe that for all $n \geq 0$ it follows
\[
\|\mathcal{L}_\phi^n 1\|_\infty \leq e^{-nP(\phi)} \|\mathcal{L}_\phi 1\|_\infty^n.
\] (33)

Thus for $n = 0$ is clear, since
\[
\|\mathcal{L}_\phi^0 1\|_\infty \leq 1 \leq M_{\phi,R_0} \leq \frac{M_{\phi,R_0}}{\nu(B(0,R_0))}.
\]

Suppose that
\[
\|\mathcal{L}_\phi^n 1\|_\infty \leq \frac{M_{\phi,R_0}}{\nu(B(0,R_0))}.
\]

We will now prove this inequality for $n + 1$. Using (15) and the fact that $\phi$ is summable we have, there exists $b \in X^\nu$ such that
\[
\|\mathcal{L}_\phi^{n+1} 1\|_\infty = \mathcal{L}_\phi^{n+1} 1(b).
\]

If $b \in X^\nu \setminus B(0,R_0)$, then by (31) and (32)
\[
\|\mathcal{L}_\phi^{n+1} 1\|_\infty = \mathcal{L}_\phi^{n+1} 1(b) = \mathcal{L}_\phi(\mathcal{L}_\phi^n 1)(b)
\leq e^{-P(\phi)} \sum_{c \in \sigma^{-1}(b)} e^{\phi(c)} \mathcal{L}_\phi^n 1(c)
\leq \|\mathcal{L}_\phi^n 1\|_\infty \mathcal{L}_\phi 1(b)
\leq \frac{M_{\phi,R_0}}{\nu(B(0,R_0))}.
\]

Otherwise, if $b \in B(0,R_0)$, then by Lemma 10 and the fact that for all $n \geq 0$, $\mathcal{L}_\phi^n 1 \nu = \nu$ we follow
\[
1 = \int d\nu = \int \mathcal{L}_\phi^{n+1} 1 d\nu
\geq \int_{B(0,R_0)} \mathcal{L}_\phi^{n+1} 1 d\nu \geq M_{\phi,R_0}^{-1} \nu(B(0,R_0)) \|\mathcal{L}_\phi^{n+1} 1\|_\infty.
\]

Thus the right hand inequality is proved.

Now we prove the other inequality in (30). Let $R_1 > R_0$ be such that $\nu(X^\nu \setminus B(0,R_1)) \leq 1/4L$. Let $R > R_1$ then we have
\[
1 = \int \mathcal{L}_\phi^n 1 d\nu \leq \int_{B(0,R)} \mathcal{L}_\phi^n 1 d\nu + \int_{X \setminus B(0,R)} \mathcal{L}_\phi^n 1 d\nu
\leq \int_{B(0,R)} \mathcal{L}_\phi^n 1 d\nu + L \nu(X^\nu \setminus B(0,R))
\leq \int_{B(0,R)} \mathcal{L}_\phi^n 1 d\nu + 1/4.
Hence for any \( n \geq 0 \) there exists \( a_n \in B(0, R_0) \) such that \( \mathcal{L}_\phi^n \mathds{1}(a_n) \geq 3/4 \). If \( a \in B(0, R) \) is any other point, then by the Lemma \[10\] we have

\[
3/4 \leq \mathcal{L}_\phi^n \mathds{1}(a_n) \leq M_{\phi,R} \mathcal{L}_\phi^n \mathds{1}(a).
\]

Thus, for each \( R \geq R_1 \) the inequality holds with \( l_R = 3M_{\phi,R}^{-1}/4 \), if \( R < R_1 \) then we just put \( l_R := l_{R_1} \). Therefore \( \lim_{n \to \infty} \frac{1}{n} \log \mathcal{L}_\phi^n \mathds{1}(a) \) exists and it is equal to \( P(\phi) \).

### 3.5 The existence of an invariant probability Gibbs measure

In this section we prove for weakly Hölder potentials the existence of an invariant measure \( \mu \) which is absolutely continuous with respect to the conformal measure \( \nu \) (given in Section \[3.4\]) and we also prove that \( \mu \) is a Gibbs measure. We give the definition of Gibbs measures of weakly Hölder potentials, with a definition appropriately adapted for the transcendental functions. A probability measure \( \eta \) on \( X \) and \( \mathcal{F} \) the Borel \( \sigma \)-algebra of sets is called a \textit{Gibbs measure} for a weakly Hölder potential \( \phi \) if there exist \( P \in \mathbb{R}, C \geq 1 \) such that for all \( a \in X \), there exists \( M = M(a) \) such that for all \( n \geq 1 \) and \( c^* \in \mathbb{Z}^n \) we have

\[
C^{-1} M(a) \leq \frac{\eta(c^*E_0(a, \delta))}{\exp(S_n \phi(c^*a) - nP)} \leq C. \tag{34}
\]

If additionally \( \nu \) is \( \sigma \)-invariant, we call \( \nu \) \textit{an invariant Gibbs measure}.

**Lemma 11.** Let \( \nu \) be the conformal measure from Theorem \[4\]. Then for every \( a \in X \) and for every \( r > 0 \), we have \( \nu(B(a, r)) > 0 \).

**Proof.** Let \( a \in X \) and \( r > 0 \). Since \( X = \bigcup_{R \geq 0} B(0, R) \) we have there exists \( R > 0 \) such that \( \nu(B(0, R)) > 0 \). By Lemma \[8\] there exists \( n_0 > 0 \) such that

\[
B(\sigma^{n_0} a, \delta_0) \subseteq \sigma^{n_0} (B(a, r)).
\]

Take \( R_1 > 0 \) such that \( R_1 \geq R \) and \( \sigma^{n_0} a \in B(0, R_1) \). By (Con) property we have there exists \( n_1 \) such that

\[
B(0, R_1) \subseteq \sigma^{n_1} (B(\sigma^{n_0} a, \delta_0)) \subseteq \sigma^{n_1 + n_0} (B(a, r)).
\]

Thus for \( n = n_0 + n_1 \) we have \( B(0, R) \subseteq \sigma^n (B(a, r)) \).

Since \( B(a, r) = \bigcup_{c^* \in \mathbb{Z}^n} B(a, r) \cap \{c^*\} \). Then

\[
\sigma^n (\bigcup_{c^* \in \mathbb{Z}^n} B(a, r) \cap \{c^*\}) \supset B(0, R).
\]

Then there exists \( c^* \in \mathbb{Z}^n \) such that

\[
\nu(\sigma^n (B(a, r) \cap \{c^*\})) > 0.
\]
Therefore, since $\sigma^n |_{B(\underline{a},r) \cap [c^*]}$ is one-to-one, we have

$$0 < \nu(\sigma^n(B(\underline{a}, r) \cap [c^*])) = \int_{B(\underline{a}, r) \cap [c^*]} e^{nP(\phi)-S_n \phi} \, d\nu$$

$$\leq e^{nP(\phi)} e^{-\inf_{|B(\underline{a}, r) \cap [c^*]} \nu(B(\underline{a}, r) \cap [c^*])}$$

$$\leq e^{nP(\phi)} e^{-S_n \phi(a)} C \nu(B(\underline{a}, r)),$$

where $C$ is the positive constant $\exp\left(\frac{w_\nu(\phi)}{1-r}\right)$ from Lemma 7. Thus

$$\nu(B(\underline{a}, r)) \geq \nu(\sigma^n B(\underline{a}, r) \cap [c^*]) e^{-nP(\phi) e^{S_n \phi(a)} C} > 0.$$

\[\square\]

**Proposition 7.** Let $\phi \in BV_r(X,\mathbb{R})$ satisfying the conditions (i) (ii), (iii), (iv) and (v). Then there exists an invariant measure $\mu$ that is absolutely continuous with respect to $\nu$ and it is a Gibbs measure for $\phi$.

**Proof.** Let $A \subset B(\underline{a}, \delta) \subset B(\underline{0}, R)$ be a Borel set. By the conformality of the measure $\nu$, we have that

$$\nu(A) = \int_{c^A} e^{nP(\phi)} e^{-S_n \phi} \, d\nu$$

and by Lemma 10 and again the conformality, we also have

$$\sum_{c^* \in \mathbb{Z}^n} e^{-nP(\phi)} e^{S_n \phi(c^*)} \geq M_{\phi,R}^{-1} e^{-nP(\phi)} \int_{B(\underline{0}, R)} L^n \mathbb{1} \, d\nu = M_{\phi,R}^{-1} \nu(B(\underline{0}, R)).$$

Then, by (21) and (35),

$$\nu(A) \geq e^{nP(\phi)} e^{-\inf(S_n \phi|_{e^{-n}(\underline{a}, \delta)})} \nu(c^* A) \geq e^{nP(\phi)} e^{-S_n \phi(c^*)} C^{-1} \nu(c^* A)$$

and

$$\nu(A) \leq e^{nP(\phi)} e^{-\inf(S_n \phi|_{e^n(\underline{a}, \delta)})} \nu(c^* A) \leq e^{nP(\phi)} e^{-S_n \phi(c^*)} C \nu(c^* A).$$

In particular

$$e^{-nP(\phi)} e^{S_n \phi(c^*)} \leq \frac{\nu(c^* B(\underline{a}, \delta))}{\nu(B(\underline{a}, \delta))}.$$

Then by (37), we have

$$\nu(\sigma^{-n}(A)) = \sum_{c^* \in \mathbb{Z}^n} \nu(c^* A) \leq \nu(A) C \sum_{c^* \in \mathbb{Z}^n} e^{-nP(\phi)} e^{S_n \phi(c^*)}$$

$$\leq \nu(A) C^2 / \nu(B(\underline{a}, \delta)),$$

and by (33) and (36)

$$\nu(\sigma^{-n}(A)) = \sum_{c^* \in \mathbb{Z}^n} \nu(c^* A) \geq \nu(A) C^{-1} \sum_{c^* \in \mathbb{Z}^n} e^{-nP(\phi)} e^{S_n \phi(c^*)}$$

$$\geq \nu(A) (CM_{\phi,R})^{-1} \nu(B(\underline{0}, R)).$$
Therefore,

$$\nu(A)(CM_{\phi,R})^{-1}\nu(B(\underline{\mathbf{a}}, R)) \leq \nu(\sigma^{-n}(A)) \leq \nu(A)C^2/\nu(B(\underline{\mathbf{a}}, \delta)) \quad (39)$$

In the Banach space of all bounded sequences of real numbers we consider a continuous linear functional $L$ called a Banach limit $L : \ell^\infty \to \mathbb{R}$ (see [BK85]) such that

- $L(\{a_n\}_{n \geq 1}) = L(\{a_{n+1}\}_{n \geq 1})$.
- $\liminf_{n \to \infty} a_n \leq L(\{a_n\}) \leq \limsup_{n \to \infty} a_n$
- If $\{a_n\}$ converges then $\lim_{n \to \infty} a_n = L(\{a_n\})$.

Then let $\nu$ be a measure defined by formula

$$\mu(A) = L(\nu \circ \sigma^{-n}(A))_{n=0}^\infty.$$

Since $L(\{a_n\}_{n \geq 1}) = L(\{a_{n+1}\}_{n \geq 1})$, the measure is invariant. Moreover, since $\liminf_{n \to \infty} a_n \leq L(\{a_n\}) \leq \limsup_{n \to \infty} a_n$, by (39) we get that for any Borel set $A \subset B(\underline{\mathbf{a}}, \delta) \subset B(\underline{\mathbf{a}}, R)$, we have

$$\nu(A)(CM_{\phi,R})^{-1}\nu(B(\underline{\mathbf{a}}, R)) \leq \mu(A) \leq \nu(A)C^2/\nu(B(\underline{\mathbf{a}}, \delta)) \quad (40)$$

Hence the measure $\mu$ is equivalent to the measure $\nu$.

**Proposition 8.** The measures $\nu$ and $\mu$ are Gibbs measure.

**Proof.** From (37) and (38) we have

$$C^{-1}\nu(B(\underline{\mathbf{a}}, \delta)) \leq \frac{\nu(c^{*B}(\underline{\mathbf{a}}, \delta))}{\exp(S_n\phi(c^{*\underline{\mathbf{a}}} - nP(\phi)))} \leq C. \quad (41)$$

Using Lemma [11] follows that for every $\underline{\mathbf{a}} \in X$, $\nu(B(\underline{\mathbf{a}}, \delta)) > 0$. Therefore we get that $\nu$ is a Gibbs measure with $P = P(\phi)$, $C = \exp(S_n\phi(c^{*\underline{\mathbf{a}}})$ and $M(a) = \nu(B(\underline{\mathbf{a}}, \delta))$.

We also have that from (40) and (41)

$$C^{-2}M_{\phi,R}^{-1}M(a)^2 \leq \frac{\mu(c^{*B}(\underline{\mathbf{a}}, \delta))}{\exp(S_n\phi(c^{*\underline{\mathbf{a}}} - nP(\phi)))} \leq C^3.$$

Therefore $\mu$ is a Gibbs measure.

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