Abstract

Hu’s metrization theorem for bornological universes is shown to hold in ZF and it is adapted to a quasi-metrization theorem for bornologies in bitopological spaces. The problem of uniform quasi-metrization of quasi-metric bornological universes is investigated. Several consequences for natural bornologies in generalized topological spaces in the sense of Delfs and Knebusch are deduced. Some statements concerning (uniform)-(quasi)-metrization of bornologies are shown to be relatively independent of ZF.
1 Introduction

A bitopological space is a triple \((X, \tau_1, \tau_2)\) where \(X\) is a set and \(\tau_1, \tau_2\) are topologies in \(X\). A quasi-pseudometric in a set \(X\) is a function \(d : X \times X \to [0; +\infty)\) such that, for all \(x, y, z \in X\), \(d(x, y) \leq d(x, z) + d(z, y)\) and \(d(x, x) = 0\). A quasi-pseudometric \(d\) in \(X\) is called a quasi-metric if, for all \(x, y \in X\), the condition \(d(x, y) = 0\) implies \(x = y\) (cf. [Kel], [FL]).

Let \(d\) be a quasi-pseudometric in \(X\). The conjugate of \(d\) is the quasi-pseudometric \(d^{-1}\) defined by \(d^{-1}(x, y) = d(y, x)\) for \(x, y \in X\). The \(d\)-ball with centre \(x \in X\) and radius \(r \in (0; +\infty)\) is the set \(B_d(x, r) = \{y \in X : d(x, y) < r\}\). The collection \(\tau(d) = \{V \subseteq X : \forall x \in V \exists n \in \omega B_d(x, \frac{1}{2n}) \subseteq V\}\) is the topology in \(X\) induced by \(d\). The triple \((X, \tau(d), \tau(d^{-1}))\) is the bitopological space associated with \(d\).

Definition 1.1. A bitopological space \((X, \tau_1, \tau_2)\) is (quasi)-metrizable if there exists a (quasi)-metric \(d\) in \(X\) such that \(\tau_1 = \tau(d)\) and \(\tau_2 = \tau(d^{-1})\) (cf. pp. 74–75 of [Kel]).

One can find a considerable number of quasi-metrization theorems in [Am] and in other sources (cf. [FL]).

We recall that, according to [Al]–[Hu], a boundedness in a set \(X\) is a (non-void) ideal of subsets of \(X\). A boundedness \(B\) in \(X\) is called a bornology in \(X\) if every singleton of \(X\) is a member of \(B\) (cf. 1.1.1 in [H-N]).

Definition 1.2 (cf. Definition 4.1 of [Hu]). If \(B\) is a boundedness in \(X\), then a collection \(A\) is called a base for \(B\) if \(A \subseteq B\) and every set of \(B\) is a subset of a member of \(A\). A second-countable boundedness is a boundedness which has a countable base.

Definition 1.3. Let \((X, \tau_1, \tau_2)\) be a bitopological space. A boundedness \(B\) in \(X\) will be called \((\tau_1, \tau_2)\)-proper if, for each \(A \in B\), there exists \(B \in B\) such that \(\text{cl}_{\tau_2} A \subseteq \text{int}_{\tau_1}(B)\). If \(\tau = \tau_1 = \tau_2\) and the boundedness \(B\) is \((\tau, \tau)\)-proper, we will say that \(B\) is \(\tau\)-proper.

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Let us notice that if \((X, \tau)\) is a topological space, then a boundedness \(\mathcal{B}\) in \(X\) is \(\tau\)-proper if and only if the universe \(((X, \tau), \mathcal{B})\) is proper in the sense of Definition 3.4 of [Hu].

**Definition 1.4.** (i) We say that a bornological biuniverse is an ordered pair \(((X, \tau_1, \tau_2), \mathcal{B})\) where \((X, \tau_1, \tau_2)\) is a bitopological space and \(\mathcal{B}\) is a bornology in \(X\).

(ii) A bornological universe is an ordered pair \(((X, \tau), \mathcal{B})\) where \((X, \tau)\) is a topological space and \(\mathcal{B}\) is a bornology in \(X\) (cf. Definition 1.2 of [Hu]).

**Definition 1.5.** Let \(d\) be a quasi-metric in \(X\) and let \(A\) be a subset of \(X\). Then:

(i) \(A\) is called \(d\)-bounded if there exist \(x \in X\) and \(r \in (0; +\infty)\) such that \(A \subseteq B_d(x, r)\);

(ii) if \(A\) is not \(d\)-bounded, we say that \(A\) is \(d\)-unbounded;

(iii) \(B_d(X)\) is the collection of all \(d\)-bounded subsets of \(X\).

For a quasi-metric \(d\) in \(X\), a set \(A \subseteq X\) can be simultaneously \(d\)-bounded and \(d^{-1}\)-unbounded.

**Example 1.6.** For \(x, y \in \omega\), let \(d(x, y) = 0\) if \(x = y\) and \(d(x, y) = 2^x\) if \(x \neq y\). Then \(\omega = B_d(0, 2)\), so \(\omega\) is \(d\)-bounded. However, for arbitrary \(x \in \omega\) and \(r \in (0; +\infty)\), if \(y \in \omega\) is such that \(2^y > r\), then \(y \notin B_d^{-1}(x, r)\). Therefore, \(\omega\) is \(d^{-1}\)-unbounded.

**Definition 1.7.** We say that a bornological biuniverse \(((X, \tau_1, \tau_2), \mathcal{B})\) is (quasi)-metrizable if there exists a (quasi)-metric \(d\) in \(X\) such that \(\tau_1 = \tau(d), \tau_2 = \tau(d^{-1})\) and \(\mathcal{B} = B_d(X)\).

It is obvious that if \(\tau\) is a topology in \(X\), then a bornological biuniverse \(((X, \tau, \tau), \mathcal{B})\) is metrizable if and only if the bornological universe \(((X, \tau), \mathcal{B})\) is metrizable in the sense of Definition 10.1 of [Hu]. Let us recall this definition.

**Definition 1.8.** Let \(((X, \tau), \mathcal{B})\) be a bornological universe. We say that:
(i) \(((X, \tau), \mathcal{B})\) is **metrizable (in the sense of Hu)** if there exists a metric \(d\) on \(X\) such that \(\tau = \tau(d)\) and \(\mathcal{B} = \{A \subseteq X : \text{diam}_d(A) < +\infty\}\) where \(\text{diam}_d(A) = \sup\{d(x, y) : x, y \in A\}\);

(ii) \(((X, \tau), \mathcal{B})\) is **quasi-metrizable** if there exists a quasi-metric \(d\) on \(X\) such that \(\tau = \tau(d)\) and, moreover, \(\mathcal{B}\) is the collection of all \(d\)-bounded sets.

We show in Section 4 that if a bornological biuniverse \(((X, \tau, \tau), \mathcal{B})\) is quasi-metrizable, then the bornological universe \(((X, \tau), \mathcal{B})\) is metrizable.

Of course, it is impossible to prove anything in mathematics without axioms. The basic set-theoretic system of axioms used in this paper is ZF (cf. [Ku1]-[Ku2]). If a relatively independent of ZF axiom \(A\) is added to ZF, we shall write \(\text{ZF} + A\) and clearly denote our theorems proved in \(\text{ZF} + A\) but not in \(\text{ZF}\). As far as set-theoretic axioms are concerned, we use standard notation from [Ku2] and [Her]. In particular, we denote \(\text{ZF} + \text{AC}\) by ZFC. If it is necessary, we use a modification of ZF signalled in [PW].

According to Theorem 1 of [Vr2] and Theorem 13.2 of [Hu], the following theorem can be called **Hu’s metrization theorem for bornological universes**:

**Theorem 1.9.** It holds true in ZFC that a bornological universe \(((X, \tau), \mathcal{B})\) is metrizable if and only if it is proper, while, simultaneously, \((X, \tau)\) is metrizable and \(\mathcal{B}\) has a countable base.

One of the main aims of our present work is to show that the proof to Hu’s metrization theorem in [Hu] highly involves the axiom CC of countable choice and to prove in ZF the following generalization of Theorem 1.9:

**Theorem 1.10.** It is true in ZF that a bornological biuniverse \(((X, \tau_1, \tau_2), \mathcal{B})\) is quasi-metrizable if and only if \(\mathcal{B}\) has a countable base and it is \((\tau_1, \tau_2)\)-proper, while the bitopological space \((X, \tau_1, \tau_2)\) is quasi-metrizable.

We deduce Theorem 1.9 from 1.10 and we prove a stronger theorem than 1.10 in Section 4 (Theorem 4.7). We also give some other applications of Theorem 1.10. Especially in Sections 2, 3 and 7, we give examples of unprovable in ZF results on bornological universes that were obtained by other authors probably either in ZFC or in naive preaxiomatic set theory. Section 5 contains a generalization of Theorem 13.5 of [Hu]. We pay a special attention to [GM] and, in Section 6, we modify the basic theorem of [GM] to get
necessary and sufficient conditions for a bornological quasi-metric universe to be uniformly quasi-metrizable (Theorem 6.5); furthermore, in Section 8, we modify a theorem about compact bornologies from [GM]. Finally, in Section 10, we offer relevant to bornologies concepts of quasi-metrizability for generalized topological spaces in the sense of Delfs and Knebusch (cf. [DK], [P1], [P2], [PW]) and give a number of illuminating examples. Section 9 concerns bornologies in generalized topological spaces and it is a preparation for Section 10. We close the paper with Section 11 where there are remarks about new topological categories.

We recommend [En] as a monograph on topology that we use. Our basic knowledge about category theory is taken from [AHS]. Models of set theory applied by us are described in [Her], [J1]-[J2] and [HR].

2 Countability

The axiom of countable choice is usually denoted by CC, ACC or CAC.

We shall use the following standard notions of finiteness and infinity:

**Definition 2.1.** A set $X$ is called:

(i) **finite** or **T-finite** (truly finite) if there exists $n \in \omega$ such that $X$ is equipollent with $n$;

(ii) **D-finite** or **Dedekind-finite** if no proper subset of $X$ is equipollent with $X$.

(iii) **infinite** or **T-infinite** if it is not finite, and **D-infinite** if it is not D-finite.

A set is T-finite if and only if it is finite in Tarski’s sense (cf. Definition 4.4 of [Her]). Other notions relevant to finiteness were studied, for example, in [Cruz]. The term **truly finite** was suggested by K. Kunen in a private communication with E. Wajch.

Let us establish three distinct notions of countability.

**Definition 2.2.** A set $X$ is called:

(i) **countable** or **T-countable** (truly countable) if $X$ is equipollent with a subset of $\omega$;
(ii) **D-countable** if every D-infinite subset of \(X\) is equipollent with \(X\);

(iii) **W-countable** if every well-orderable subset of \(X\) is D-countable.

To each notion of countability \(Q\) corresponds a notion of uncountability.

**Definition 2.3.** We say that a set is **\(Q\)-uncountable** if it is not \(Q\)-countable where \(Q\) stands for T, D or W. Sets that are T-uncountable are called **uncountable**.

Let us denote by \(\text{CC}(D\text{-fin})\) the following statement: every non-void countable collection of pairwise disjoint non-void D-finite sets has a choice function. As usual, \(\text{CC}(\text{fin})\) is the statement: every non-void countable collection of pairwise disjoint non-void finite sets has a choice function.

**Proposition 2.4.** The following conditions are equivalent:

(i) \(\text{CC}(D\text{-fin})\);

(ii) every D-countable set is countable.

**Proof.** Let \(X\) be a set. Assume that \(X\) is D-countable. If \(X\) is D-infinite, then \(X\) is countable (cf. [W], p. 48). Assume that \(X\) D-finite. Then if (i) holds, it follows from E13 of Section 4.1 of [Her] that the set \(X\) is finite, so countable. Hence (i) implies (ii). Now, assume that (ii) holds and that \(X\) is D-finite. Then \(X\) is D-countable, so countable. This implies that \(X\) is equipollent with a finite subset of \(\omega\) and, in consequence, \(X\) is finite. By E13 of Section 4.1 of [Her], (ii) implies (i). \(\square\)

**Fact 2.5.** For every D-finite set \(X\), the following conditions are equivalent:

(i) \(X\) is finite;

(ii) \(X \cup \omega\) is D-countable.

**Corollary 2.6.** If \(X\) is an infinite D-finite set, then the set \(X \cup \omega\) is D-uncountable.

**Fact 2.7.** A set \(X\) is countable if and only if \(X \cup \omega\) is D-countable.

**Corollary 2.8.** In every model \(M\) for ZF such that there is in \(M\) an infinite D-finite subset of \(\mathbb{R}\), the collection of all D-countable subsets of \(\mathbb{R}\) is not a bornology.
Fact 2.9. In every set $X$, the following collections are bornologies:

(i) the collection $FB(X)$ of all finite subsets of $X$;
(ii) the collection of all $D$-finite subsets of $X$;
(iii) the collection of all countable subsets of $X$;
(iv) the collection of all $W$-countable subsets of $X$.

Several remarks on $D$-countability can be found in [W].

3 Second-countable bornological biuniverses

One may deduce wrongly from Theorem 5.5 of [Hu] that every base of a second-countable boundedness $B$ certainly contains a countable base for $B$. However, we are going to prove that Theorem 5.5 of [Hu] is relatively independent of ZF. To do this, let us consider the following bornologies in $\mathbb{R}$:

$UB(\mathbb{R}) = \{ A \subseteq \mathbb{R} : \exists r \in \mathbb{R} A \subseteq (-\infty; r) \}$

$LB(\mathbb{R}) = \{ A \subseteq \mathbb{R} : \exists r \in \mathbb{R} A \subseteq (r; +\infty) \}$

Of course, $UB(\mathbb{R})$ and $LB(\mathbb{R})$ are second-countable.

Proposition 3.1. Equivalent are:

(i) $CC(\mathbb{R})$;

(ii) for every unbounded to the right subset $D$ of $\mathbb{R}$, the collection $A(D) = \{ (-\infty; d) : d \in D \}$ contains a countable base for $UB(\mathbb{R})$;

(iii) for every unbounded to the left subset $D$ of $\mathbb{R}$, the collection $A(D) = \{ (d; +\infty) : d \in D \}$ contains a countable base for $LB(\mathbb{R})$;

Proof. First, assume that $CC(\mathbb{R})$ holds and that $D$ is an unbounded to the right subset of $\mathbb{R}$. It follows from Theorem 3.8 of [Her] that there exists an unbounded sequence $(d_n)_{n \in \omega}$ of elements of $D \cap [0; +\infty)$. Then $\{ (-\infty; d_n) : n \in \omega \}$ is a countable base for $UB(\mathbb{R})$.

Now, suppose that $CC(\mathbb{R})$ does not hold. By Theorem 3.8 of [Her], there exists an unbounded subset $B$ of $\mathbb{R}$ which does not contain any unbounded sequence. Then the set $D = B \cup \{ -x : x \in B \}$ does not contain any
unbounded sequence. The collection \( \mathcal{A}(D) \) is a base for \( \text{UB}(\mathbb{R}) \) such that \( \mathcal{A}(D) \) does not contain any countable base for \( \text{UB}(\mathbb{R}) \). Hence (i) implies (ii).

To show that (ii) and (iii) are equivalent, it suffices to make a suitable use of the mapping \( f : \mathbb{R} \to \mathbb{R} \) defined by \( f(x) = -x \) for \( x \in \mathbb{R} \).

**Corollary 3.2.** Let \( M \) be any model for \( \text{ZF} \) such that \( \text{CC}(\mathbb{R}) \) fails in \( M \). Then the bornology \( \text{UB}(\mathbb{R}) \) has a base which does not contain any countable base for \( \text{UB}(\mathbb{R}) \). In consequence, Theorem 5.5 of [Hu] is false in \( M \).

**Proposition 3.3.** \( \text{(ZF + CC)} \) If a boundedness \( \mathcal{B} \) in \( X \) has a countable base, then every base for \( \mathcal{B} \) contains a countable base for \( \mathcal{B} \).

**Proof.** Let \( \mathcal{A} \) be a base for \( \mathcal{B} \). Consider an arbitrary countable base \( \mathcal{C} \) for \( \mathcal{B} \). Then \( \mathcal{C} \neq \emptyset \). For \( C \in \mathcal{C} \), let \( \mathcal{A}(C) = \{ A \in \mathcal{A} : C \subseteq A \} \). Since \( \mathcal{A} \) is a base for \( \mathcal{B} \), we have \( \mathcal{A}(C) \neq \emptyset \) whenever \( C \in \mathcal{C} \). Using \( \text{CC} \), we deduce that there exists \( x \in \prod_{C \in \mathcal{C}} \mathcal{A}(C) \). Then \( A_0 = \{ x(C) : C \in \mathcal{C} \} \subseteq \mathcal{A} \) and \( A_0 \) is a countable base for \( \mathcal{B} \).

We can get the following correct modification in \( \text{ZF} \) of Theorem 5.5 of [Hu]:

**Proposition 3.4.** Let \( \mathcal{C} \) be a countable base for a boundedness \( \mathcal{B} \) in \( X \) such that \( \mathcal{B} \) does not have a maximal set with respect to inclusion. Then there exists a strictly increasing sequence \( \{ A_n : n \in \omega \} \) of members of \( \mathcal{C} \) such that the collection \( \{ A_n : n \in \omega \} \) is a base for \( \mathcal{B} \).

**Proof.** It follows from the countability of \( \mathcal{C} \) that we can write \( \mathcal{C} = \{ C_n : n \in \omega \} \). Let \( A_0 = C_0 \). Since \( \mathcal{B} \) does not contain maximal bounded sets, there exists \( B \in \mathcal{B} \) such that \( B \) is not a subset of \( A_0 \cup C_1 \) and there exists \( C \in \mathcal{C} \) such that \( A_0 \cup B \subset C \). This proves that there exists \( C \in \mathcal{C} \) such that \( A_0 \cup C_1 \neq C \) and \( A_0 \cup C_1 \subseteq C \). Let \( n_1 = \min \{ n \in \omega : A_0 \cup C_1 \subset C_n \} \) and \( A_1 = C_{n_1} \). Of course, we use the symbol \( \subset \) for strict inclusion. Suppose that, for \( m \in \omega \setminus \{ 0 \} \), we have already defined the set \( A_m \in \mathcal{C} \). In much the same way as above, we take \( n_{m+1} = \min \{ n \in \omega : A_m \cup C_{n+1} \subset C_n \} \) and \( A_{m+1} = C_{n_{m+1}} \). The sequence \( \{ A_n \} \) has the required properties.

Although Theorem 5.5 of [Hu] is unprovable in \( \text{ZF} \), the following proposition about bornological biuniverses clearly shows that Theorem 5.6 of [Hu] holds true in \( \text{ZF} \); however, in the proof of Theorem 5.6 in [Hu], an illegal in \( \text{ZF} \) countable choice was involved. Therefore, we offer its more careful proof in \( \text{ZF} \).
Proposition 3.5. Let us suppose that $(X, \tau_1, \tau_2)$ is a bitopological space, while $\mathcal{B}$ is a second-countable $(\tau_1, \tau_2)$-proper boundedness in $X$ such that $\mathcal{B}$ does not have maximal sets with respect to inclusion. Then there exists a strictly increasing sequence $(A_n)$ of $\tau_1$-open sets such that $A = \{A_n : n \in \omega\}$ is a base for $\mathcal{B}$ such that $\text{cl}_{\tau_2} A_n \subset A_{n+1}$ for each $n \in \omega$.

Proof. Take, by Proposition 3.4, a strictly increasing countable base $C = \{C_n : n \in \omega\}$ for $\mathcal{B}$. Let $A_0 = \text{int}_{\tau_1} C_0$. Suppose that, for $m \in \omega$, we have already defined a $\tau_1$-open set $A_m \in \mathcal{B}$. We use similar arguments to the ones given in the proof to Proposition 3.4 with the exception that, since $\mathcal{B}$ is $(\tau_1, \tau_2)$-proper, we may define $n_{m+1} = \min\{n \in \omega : \text{cl}_{\tau_2}(A_m \cup C_{n+1}) \subset \text{int}_{\tau_1} C_{n+1}\}$ and $A_{m+1} = \text{int}_{\tau_1} C_{n_{m+1}}$. \hfill $\square$

4 Quasi-metrization theorems for bornological biuniverses

If $\tau$ is a topology on $X$ and if $A \subseteq X$, we denote $\tau |_A = \{A \cap V : V \in \tau\}$. For the real line $\mathbb{R}$, the topology $u = \{\emptyset, \mathbb{R}\} \cup \{(-\infty; a) : a \in \mathbb{R}\}$ is called the upper topology on $\mathbb{R}$, while the topology $l = \{\emptyset, \mathbb{R}\} \cup \{(a; +\infty) : a \in \mathbb{R}\}$ is called the lower topology on $\mathbb{R}$ (cf. [FL], [Sal]). If $A \subseteq \mathbb{R}$, then we use $(A, u, l)$ as an abbreviation of $(A, u |_A, l |_A)$ where $u = u |_A$ and $l = l |_A$.

Definition 4.1. Suppose that $(X, \tau_1^X, \tau_2^X)$ and $(Y, \tau_1^Y, \tau_2^Y)$ are bitopological spaces. A mapping $f : X \to Y$ is called bicontinuous with respect to $(\tau_1^X, \tau_2^X, \tau_1^Y, \tau_2^Y)$ (in abbreviation: bicontinuous) if

$$\{f^{-1}(V) : V \in \tau_i^X\} \subseteq \tau_i^Y$$

for each $i \in \{1, 2\}$.

A crucial role in the study of bornologies is played by a concept of a characteristic function of a bornology which is also called a forcing function (cf. [Hu], [Be]). We need to extend this concept to bornological biuniverses.

Definition 4.2. Let $(X, \tau_1, \tau_2)$ be a bitopological space. Then a $(\tau_1, \tau_2)$-characteristic function for a bornology $\mathcal{B}$ in $X$, is a bicontinuous function $f : (X, \tau_1, \tau_2) \to ([0; +\infty), u, l)$ such that

$$\mathcal{B} = \{A \subseteq X : \sup\{f(x) : x \in A\} < +\infty\}.$$
Fact 4.3 (cf. 4.1 of [Kel]). Let $d$ be a quasi-metric on $X$ and let $x_0 \in X$. Define $f(x) = d(x_0, x)$ for $x \in X$. Then the function $f : (X, \tau(d), \tau(d^{-1})) \rightarrow ([0; +\infty], u, l)$ is bicontinuous.

Definition 4.4. We say that a (quasi)-metric $d$ induces a bornological biuniverse $((X, \tau_1, \tau_2), B)$ if $\tau_1 = \tau(d)$, $\tau_2 = \tau(d^{-1})$ and $B = B_d(X)$.

Proposition 4.5. Suppose that a bornological biuniverse $((X, \tau_1, \tau_2), B)$ is (quasi)-metrizable. Then there exists a $(\tau_1, \tau_2)$-characteristic function for the bornology $B$.

Proof. Let us consider an arbitrary point $x_0 \in X$ and any (quasi)-metric $d$ such that $d$ induces the bornological biuniverse $((X, \tau_1, \tau_2), B)$. Then, by Fact 4.3, a $(\tau_1, \tau_2)$-characteristic function for $B$ is the function $f : X \rightarrow \mathbb{R}$ where $f(x) = d(x_0, x)$ for $x \in X$. □

Proposition 4.6. Suppose that a bornological biuniverse $((X, \tau_1, \tau_2), B)$ is such that $B$ has a $(\tau_1, \tau_2)$-characteristic function. Then $B$ is both second-countable and $(\tau_1, \tau_2)$-proper.

Proof. Let $f$ be any $(\tau_1, \tau_2)$-characteristic function for $B$. For $n \in \omega$, let $A_n = f^{-1}((-\infty, n])$. Then the collection $\{A_n : n \in \omega\}$ is a countable base for $B$ such that $\text{cl}_{\tau_2} A_n \subseteq \text{int}_{\tau_1} A_{n+1}$. □

Theorem 4.7. Let us suppose that $(X, \tau_1, \tau_2)$ is a (quasi)-metrizable bitopological space and that $B$ is a bornology in $X$. Then the following conditions are all equivalent:

(i) the bornological biuniverse $((X, \tau_1, \tau_2), B)$ is (quasi)-metrizable;

(ii) there exists a $(\tau_1, \tau_2)$-characteristic function for $B$;

(iii) the bornology $B$ is $(\tau_1, \tau_2)$-proper and it has a countable base.

Proof. Let us consider any quasi-metric $\sigma$ on $X$ such that $\tau_1 = \tau(\sigma)$ and $\tau_2 = \tau(\sigma^{-1})$. Put $d(x, y) = \min\{\sigma(x, y), 1\}$ for $x, y \in X$. It is easy to observe that if $X \in B$, then all conditions (i) – (iii) are fulfilled. Assume that $X \notin B$. It follows from Proposition 4.5 that (i) implies (ii). Assume (ii) and suppose that $f$ is a $(\tau_1, \tau_2)$-characteristic function for $B$. For $x, y \in X$, let $\rho(x, y) = d(x, y) + \max\{f(y) - f(x), 0\}$. Then the quasi-metric $\rho$ induces the bornological biuniverse $((X, \tau_1, \tau_2), B)$. In the case when $\sigma$ is a metric,
we can put \( \rho(x, y) = d(x, y) + |f(y) - f(x)| \) to obtain a metric that induces \(((X, \tau_1, \tau_2), \mathcal{B})\). Hence (ii) implies (i).

Now, assume that (iii) holds. Since \( X \notin \mathcal{B} \), it follows from Proposition 3.5 that there exists a base \( \{A_n : n \in \omega\} \) for \( \mathcal{B} \) such that \( \text{cl}_{\tau_2}A_n \) is a proper subset of \( \text{int}_{\tau_1}A_{n+1} \) for each \( n \in \omega \). We may assume that \( A_0 = \emptyset \). For \( n \in \omega \setminus \{0\} \) and \( x \in X \), let \( f_n(x) = d(\text{cl}_{\tau_2}A_n, x) \) and \( g_n(x) = d(x, X \setminus \text{int}_{\tau_1}A_{n+1}) \). Then \( f_n : (X, \tau_1, \tau_2) \to ([0; +\infty), u, l) \) and \( g_n : (X, \tau_1, \tau_2) \to ([0; +\infty), l, u) \) are bicontinuous. For each \( x \in X \), we have \( f_n(x) + g_n(x) \neq 0 \), so we can put \( h_n(x) = \frac{f_n(x)}{f_n(x) + g_n(x)} \). Moreover, we define \( h_0(x) = 1 \) for each \( x \in X \). It is easy to check that the function \( h_n : (X, \tau_1, \tau_2) \to ([0; 1], u, l) \) is bicontinuous for each \( n \in \omega \) (cf. the proof to Corollary 2.2.16 in [Sal]). Let \( \psi(x) = h_n(x) + n \) when \( x \in \text{int}_{\tau_1}A_{n+1} \setminus \text{int}_{\tau_1}A_n \). We are going to prove that the function \( \psi \) is bicontinuous with respect to \((\tau_1, \tau_2, u, l)\).

Let \( x \in \text{int}_{\tau_1}A_{n+1} \setminus \text{int}_{\tau_1}A_n \) and \( y \in \text{int}_{\tau_1}A_{m+1} \setminus \text{int}_{\tau_1}A_m \). Consider any real numbers \( r, s \) such that \( r < \psi(x) < s \). We assume that \( n \neq 0 \). There exists \( U_s \in \tau_1 \) such that \( x \in U_s \subseteq \text{int}_{\tau_1}A_{n+1} \) and if \( y \in U_s \), then \( h_n(y) + n < s \). There exists \( V_r \in \tau_2 \) such that \( x \in V_r \subseteq X \setminus \text{cl}_{\tau_2}A_{n-1} \) and if \( y \in V_r \), then \( h_n(y) + n > r \). Of course, if \( m = n \), then \( \psi(y) < s \) when \( y \in U_s \), while \( \psi(y) > r \) when \( y \in V_r \). Let us assume that \( m \neq n \). Suppose that \( y \in U_s \). Then \( m < n \), so \( \psi(y) \leq 1 + m \leq n \leq \psi(x) < s \).

Suppose that \( y \in V_r \). If \( m > n \), we have \( \psi(y) \geq m \geq 1 + n \geq \psi(x) > r \). Let \( m < n \). Since \( y \notin \text{int}_{\tau_1}A_{m-1} \), we have \( m + 1 \geq n \). As \( m + 1 \leq n \), we have \( m + 1 = n \). If \( x \notin \text{cl}_{\tau_2}A_n \) we could take \( V_r^* = V_r \cap (X \setminus \text{cl}_{\tau_2}A_n) \in \tau_2 \) and observe that if \( y \in V_r^* \), then \( m \geq n \) and \( \psi(y) > r \). Let us consider the case when \( m < n \) and \( x \in \text{cl}_{\tau_2}A_n \). Then \( \psi(x) = n \). We take a positive real number \( \epsilon \) such that \( n - \epsilon > r \). Since \( h_{n-1}(x) = 1 \), there exists \( W_\epsilon \in \tau_2 \) such that \( x \in W_\epsilon \) and \( h_{n-1}(t) > 1 - \epsilon \) for each \( t \in W_\epsilon \). If \( y \in W_\epsilon \cap V_r \) and \( m + 1 = n \), then \( \psi(y) = h_{n-1}(y) + n - 1 > 1 - \epsilon + n - 1 > r \). The case when \( n = 0 \) is also obvious now. This completes the proof that \( \psi \) is bicontinuous with respect to \((\tau_1, \tau_2, u, l)\). It is easy to check that \( \mathcal{B} = \{A \subseteq X : \sup \psi(A) < +\infty\} \), so \( \psi \) is a \((\tau_1, \tau_2)\)-characteristic function for \( \mathcal{B} \). Hence (ii) follows from (iii). To complete the proof, it suffices to apply Proposition 4.6.

\[ \square \]

Corollary 4.8. Theorem 1.10 is true.

Corollary 4.9. The assumption of ZFC can be weakened to ZF in Theorem 1.9.

Corollary 4.10. Let us suppose that \((X, \tau)\) is a topological space and \( \mathcal{B} \) is
a bornology in $X$. Then the bornological biuniverse $((X, \tau, \tau), \mathcal{B})$ is quasi-metrizable if and only if the bornological universe $((X, \tau), \mathcal{B})$ is metrizable.

**Proof.** It suffices to prove that if there exists a quasi-metric which induces $((X, \tau, \tau), \mathcal{B})$, then $((X, \tau), \mathcal{B})$ is metrizable. Let $d$ be a quasi-metric in $X$ such that $\tau = \tau(d) = \tau(d^{-1})$ and $\mathcal{B} = \mathcal{B}_d(X)$. Define $\rho = \max\{d, d^{-1}\}$. Then $\rho$ is a metric in $X$ such that $\tau(\rho) = \tau$. Moreover, by Theorem 4.7, the bornology $\mathcal{B}$ is second-countable and $\tau$-proper; hence, the bornological universe $((X, \tau), \mathcal{B})$ is metrizable by Theorem 4.7. \hfill $\square$

**Example 4.11.** Let $d$ be the quasi-metric from Example 1.6. Then $\tau(d) = \tau(d^{-1}) = \mathcal{P}(\omega)$. Moreover, $\mathcal{B}_d(\omega) = \mathcal{P}(\omega)$ and $\mathcal{B}_{d^{-1}}(\omega) = \mathcal{FB}(\omega)$. The metric $\rho = \max\{d, d^{-1}\}$ does not induce $((\omega, \mathcal{P}(\omega), \mathcal{P}(\omega)), \mathcal{B}_d(\omega))$; however, $\rho$ induces $((\omega, \mathcal{P}(\omega), \mathcal{P}(\omega)), \mathcal{FB}(\omega))$.

**Example 4.12.** Let $\tau_{S,r}$ be the right half-open interval topology in $\mathbb{R}$ and let $\tau_{S,l}$ be the left half-open interval topology in $\mathbb{R}$. Then $((\mathbb{R}, \tau_{S,r})$ is the Sorgenfrey line.

(i) The bornological biuniverse $((\mathbb{R}, \tau_{S,r}, \tau_{S,l}), \mathcal{UB}(\mathbb{R}))$ is not metrizable but it is quasi-metrizable by the following quasi-metric $\rho_S$:

$$
\rho_S(x, y) = \begin{cases} 
  y - x, & x \leq y \\
  1, & x > y.
\end{cases}
$$

Let us notice that $\mathcal{B}_{\rho_S^{-1}}(\mathbb{R}) = \mathcal{LB}(\mathbb{R})$ and the quasi-metric $\rho_S$ does not induce the bornological biuniverse $((\mathbb{R}, \tau_{S,r}, \tau_{S,l}), \mathcal{LB}(\mathbb{R}))$. However, the bornological biuniverse $((\mathbb{R}, \tau_{S,r}, \tau_{S,l}), \mathcal{LB}(\mathbb{R}))$ is induced by the quasi-metric $\rho_L$ defined as follows:

$$
\rho_L(x, y) = \begin{cases} 
  \min\{y - x, 1\}, & x \leq y \\
  1 + x - y, & x > y.
\end{cases}
$$

(ii) The non-metrizable bornological biuniverse $((\mathbb{R}, \tau_{S,r}, \tau_{S,l}), \mathcal{P}(\mathbb{R}))$ is quasi-metrizable by the quasi-metric $\rho_{S,1}$ defined as follows:

$$
\rho_{S,1}(x, y) = \begin{cases} 
  \min\{1, y - x\}, & x \leq y \\
  1, & x > y.
\end{cases}
$$
Example 4.13. We consider the following hedgehog-like scheme. Let $(X, d)$ be a quasi-metric space such that $X$ has at least two distinct points. Let $S$ be a non-empty set. We fix $x_0 \in X$ and put $Y_s = (X \setminus \{x_0\}) \times \{s\}$ for $s \in S$. Let us fix $p \notin \bigcup_{s \in S} Y_s$ and put $Y = \{p\} \cup \bigcup_{s \in S} Y_s$. Let $x, y \in X \setminus \{x_0\}$ and $s, s' \in S$. We define $\rho(p, p) = 0$, $\rho((x, s), p) = d(x, x_0)$, $\rho(p, (x, s)) = d(x_0, x)$ and $\rho((x, s), (y, s)) = d(x, y)$. If $s \neq s'$, we put $\rho((x, s), (y, s')) = d(x, x_0) + d(x_0, y)$. Let us consider the collection $\mathcal{B}$ of all sets $A \subseteq Y$ such that there are finite $S(A) \subseteq S$ such that $A \subseteq \{p\} \cup \bigcup_{s \in S(A)} Y_s$. Then $\mathcal{B}$ is a bornology in $Y$. If $S$ is countable, then $\mathcal{B}$ is second-countable. If $S$ is infinite and, simultaneously, $x_0$ is an accumulation point of $(X, \tau(d))$, then the bornology $\mathcal{B}$ is not $(\tau(\rho), \tau(\rho^{-1}))$-proper. Let us denote the bornological biuniverse $((Y, \tau(\rho), \tau(\rho^{-1})), \mathcal{B})$ by $J(X, d, x_0, S)$ and let $Y(X, d, x_0, S) = (Y, \tau(\rho))$. We can apply $J(X, d, x_0, S)$ as follows.

(i) If $X = [0; 1]$ and $d(x, y) = |x - y|$ for $x, y \in X$, then the bornological universe $J(X, d, 0, \omega)$ is not quasi-metrizable although its bornology is second-countable. In this case, $Y(X, d, x_0, \omega)$ is the hedgehog space of spininess $\omega$ (cf. 4.1.5 of [En]), so we can call $((Y, \tau(\rho)), \mathcal{B})$ the bornological hedgehog space of spininess $\omega$.

(ii) If $\rho_S$ is the quasi-metric defined in Example 4.12 (i), then the bornological biuniverse $J(\mathbb{R}, \rho_S, 0, \omega)$ is not quasi-metrizable but its bornology has a countable base.

(iii) Let $C$ be the unit circle in $\mathbb{R}^2$. We fix $x_0 \in C$ and we consider the Euclidean metric $d_e$ in $C$. The bornological biuniverse $J(C, d_e, x_0, \omega)$ is not quasi-metrizable although its bornology has a countable base. We can call $J(C, d_e, x_0, \omega)$ the bornological metric wedge sum of circles. In this case, the topological space $Y(C, d_e, x_0, \omega)$ is not compact.

(iv) It is worthwhile to compare $J(C, d_e, x_0, \omega)$ with the bornological Hawaiian earring $(H, \mathcal{B}_H)$ where $H = \bigcup_{n \in \omega \setminus \{0\}} H_n$ is considered with its natural topology inherited from $\mathbb{R}^2$ and, for each $n \in \omega \setminus \{0\}$, the set $H_n$ is the circle with centre $(\frac{1}{n}, 0)$ and radius $\frac{1}{n}$, while $\mathcal{B}_H$ is the collection of all sets $A \subseteq H$ such that there exist sets $n(A) \in \omega$ such that $A \subseteq \bigcup_{n \in n(A) \setminus \{0\}} H_n$. Then $H$ is compact and the bornology $\mathcal{B}_H$ has a countable base. Since there does not exist $A \in \mathcal{B}_H$ such that
(0, 0) ∈ int_{d^*}A, it follows from Theorem 4.7 that the bornological universe \((H, \mathcal{B}_H)\) is not quasi-metrizable.

In view of the examples above, when \(d\) is a quasi-metric in \(X\) and \(\mathcal{B}\) is a bornology in \(X\) but \(d\) does not induce the bornological biuniverse \((X, \mathcal{B}) = ((X, \tau(d)), \tau(d^{-1})), \mathcal{B})\), it might be interesting to find, in terms of \(d\), necessary and sufficient conditions for \((X, \mathcal{B})\) to be quasi-metrizable. To do this, we need the following concept:

**Definition 4.14.** Let \(d\) be a quasi-pseudometric in a set \(X\) and let \(\delta \in (0; +\infty)\). For a set \(A \subseteq X\), the \(\delta\)-neighbourhood of \(A\) with respect to \(d\) is the set \([A]_d^\delta = \bigcup_{a \in A} B_d(a, \delta)\).

Let us notice that if \(\emptyset \neq A \subseteq X\), then \([A]^\delta_d = \{x \in X : d(A, x) < \delta\}\).

**Theorem 4.15.** For every bornological biuniverse \(((X, \tau_1, \tau_2), \mathcal{B})\), the following conditions are equivalent:

(i) \(((X, \tau_1, \tau_2), \mathcal{B})\) is (quasi)-metrizable;

(ii) there exists a (quasi)-metric \(d\) in \(X\) such that \(\tau_1 = \tau(d)\), \(\tau_2 = \tau(d^{-1})\) and \(\mathcal{B}\) has a base \(\{B_n : n \in \omega\}\) with the following property:

\[\forall n \in \omega \exists \delta \in (0; +\infty) [B_n]^\delta_d \subseteq B_{n+1}\]

**Proof.** Let \(((X, \tau_1, \tau_2), \mathcal{B})\) be a bornological biuniverse. Suppose that (i) holds and that \(d\) is a (quasi)-metric in \(X\) such that \(d\) induces \(((X, \tau_1, \tau_2), \mathcal{B})\). We consider an arbitrary \(x_0 \in X\) and, for \(n \in \omega\), we define \(B_n = B_d(x_0, n+1)\). Since \([B_n]^\uparrow_d \subseteq B_{n+1}\), we infer that (ii) follows from (i).

Assume that (ii) is satisfied. Let \(C \subseteq X\) and \(D \subseteq X\) be such that, for some \(\delta \in (0; +\infty)\), the inclusion \([C]^\delta_d \subseteq D\) holds. Let \(x \in \text{cl}_{\tau_2} C\). There exists \(y \in C \cap B_{d^{-1}}(x, \delta)\). Then \(d(y, x) < \delta\), so \(x \in [C]^\delta_d\). Therefore \(\text{cl}_{\tau_2} C \subseteq [C]^\delta_d\). Of course, since \([C]^\delta_d \subseteq D\), we have \([C]^\delta_d \subseteq \text{int}_{\tau_1} D\). In consequence, \(\text{cl}_{\tau_2} C \subseteq \text{int}_{\tau_1} D\). Now, we deduce from Theorem 4.7 that (ii) implies (i).

**Corollary 4.16.** For every bornological universe \(((X, \tau), \mathcal{B})\), the following conditions are equivalent:

(i) the universe \(((X, \tau), \mathcal{B})\) is (quasi)-metrizable;
(ii) there exists a (quasi)-metric $d$ in $X$ such that $\tau = \tau(d)$ and, simultaneously, $\mathcal{B}$ has a base $\{B_n : n \in \omega\}$ with the following property:

$$\forall n \in \omega \exists \delta \in (0; +\infty) [B_n]_d^\delta \subseteq B_{n+1};$$

The following example shows that the sets $B_n$ can be $d$-unbounded in Theorem 4.15 and Corollary 4.16.

**Example 4.17.** For the bornological biuniverse $((\mathbb{R}, \tau_{S,r}, \tau_{S,l}), \mathcal{L}B(\mathbb{R}))$ and for the quasi-metric $\rho_S$ from Example 4.12, the sets $B_n = [-n; +\infty)$ with $n \in \omega$ satisfy condition (ii) of Theorem 4.15 if we put $d = \rho_S$ and $\delta = 1$. However, the sets $B_n$ are all $\rho_S$-unbounded.

We offer a number of other relevant examples in Section 10.

## 5 The kernel of a boundedness

If $X$ is a topological space and $\mathcal{B}$ is a boundedness in $X$, a notion of a kernel of the universe $(X, \mathcal{B})$ was introduced in Definition 6.3 in [Hu]. We adapt this notion to our needs.

**Definition 5.1.** Let $\tau$ be a topology in a set $X$. If $\mathcal{B}$ is a boundedness in $X$, then the $\tau$-kernel of $\mathcal{B}$ is the set

$$\Lambda_{\tau}(\mathcal{B}) = \bigcup \{\text{int}_{\tau} A : A \in \mathcal{B}\}.$$  

**Definition 5.2.** Let $(X, \tau_1, \tau_2)$ be a bitopological space. Suppose that $\mathcal{B}$ is a boundedness in $X$ and put $\Lambda = \Lambda_{\tau_1}(\mathcal{B})$. Let $\mathcal{B}_\Lambda = \{A \cap \Lambda : A \in \mathcal{B}\}$. Then the ordered pair $((\Lambda, \tau_1 |_{\Lambda}, \tau_2 |_{\Lambda}), (\mathcal{B}_\Lambda))$ will be called the bornological biuniverse induced by $\mathcal{B}$.

**Fact 5.3.** Suppose that $(X, \tau_1, \tau_2)$ is a bitopological space. If $\mathcal{B}$ is a $(\tau_1, \tau_2)$-proper boundedness in $X$ and if $\Lambda = \Lambda_{\tau_1}(\mathcal{B})$, then the bornology $\mathcal{B}_\Lambda$ in $\Lambda$ is $(\tau_1 |_{\Lambda}, \tau_2 |_{\Lambda})$-proper.

For a topological space $X = (X, \tau)$ and a boundedness $\mathcal{B}$ in $X$, when $\Lambda = \Lambda_{\tau}(\mathcal{B})$, Theorem 13.5 of [Hu] concerns the problem of the metrizability of the bornological universe $(\Lambda, \mathcal{B}_\Lambda)$ under the assumption that $\Lambda$ is a separable metrizable subspace of $X$. However, the proof to Theorem 13.5 in [Hu] is not in ZF. We give a generalization to bornological universes of Theorem 13.5 of [Hu] and show its proof in ZF. We also show that the assumption of separability is needless in Theorem 13.5 of [Hu].
Theorem 5.4. Assume that \((X, \tau_1, \tau_2)\) is a bitopological space and that \(B\) is a second-countable \((\tau_1, \tau_2)\)-proper boundedness in \(X\). Let \(\Lambda\) be the \(\tau_1\)-kernel of \(B\) and suppose that the bitopological space \((\Lambda, \tau_1 \mid_\Lambda, \tau_2 \mid_\Lambda)\) is quasi-metrizable. Then there exists a quasi-metric \(\rho\) on \(\Lambda\) such that the following conditions are satisfied:

(i) \(\tau_1 \mid_\Lambda = \tau(\rho)\) and \(\tau_2 \mid_\Lambda = \tau(\rho^{-1})\);

(ii) \(B\) is the collection of all \(\rho\)-bounded subsets of \(\Lambda\);

(iii) for each pair of points \(x_0 \in \Lambda, x_* \in X \setminus \Lambda\) and for each positive real number \(b\), there exists \(G \in \tau_2\) such that \(x_* \in G\) and \(\rho(x_0, x) > b\) whenever \(x \in G \cap \Lambda\).

Proof. Since \(B\) is \((\tau_1, \tau_2)\)-proper, we have \(B = B_\Lambda\). In view of Fact 5.3, \(B\) is \((\tau_1 \mid_\Lambda, \tau_2 \mid_\Lambda)\)-proper. In the light of Theorem 4.7, there exists in \(ZF\) a quasi-metric \(\rho\) in \(\Lambda\) such that both conditions (i) and (ii) are satisfied. Let \(x_0 \in \Lambda\) and \(x_* \in X \setminus \Lambda\). Consider an arbitrary positive real number \(b\). Put \(B = \{x \in \Lambda : \rho(x_0, x) \leq b\}\). Of course, \(B \in B\). Since \(B\) is \((\tau_1, \tau_2)\)-proper, there exists \(U \in \tau_1\) such that \(B \subseteq U\). Using the assumption that \(B\) is \((\tau_1, \tau_2)\)-proper once again, we deduce that \(\text{cl}_{\tau_2} U \subseteq \Lambda\). Let \(G = X \setminus \text{cl}_{\tau_2} U\). Then \(G \in \tau_2\), \(G \cap \Lambda \subseteq \Lambda \setminus B\) and \(x_* \in G\). It is evident that if \(x \in G \cap \Lambda\), then \(\rho(x_0, x) > b\). \(\square\)

Now, we can immediately deduce in \(ZF\) the following improvement of Theorem 13.5 of \(\[\text{Hu}\]\):

Corollary 5.5. If \(B\) is a second-countable proper boundedness in a topological space \(X\) such that the set \(\Lambda = \bigcup B\) is a metrizable subspace of \(X\), then there exists a metric \(\rho\) on \(\Lambda\) such that the following conditions are satisfied:

(i) the topology of \(\Lambda\) as a subspace of \(X\) is induced by \(\rho\);

(ii) \(B = \{A \subseteq \Lambda : \text{diam}_\rho(A) < +\infty\}\);

(iii) for each pair of points \(x_0 \in \Lambda, x_* \in X \setminus \Lambda\) and for each positive real number \(b\), there exists an open set \(G\) in \(X\) such that \(x_* \in G\) and \(\rho(x_0, x) > b\) whenever \(x \in G \cap \Lambda\).
6 Uniformly quasi-metrizable bornologies

This section has been inspired by the necessary and sufficient conditions for a bornology to be uniformly metrizable given in [GM]. We adapt the conditions of Theorem 2.4 of [GM] to bornologies in quasi-metric spaces.

For \( x, y \in \mathbb{R} \), let us put

\[
\rho_u(x,y) = \max\{y-x,0\}, \quad \rho_l(x,y) = \max\{x-y,0\}.
\]

Then \( \rho_u, \rho_l \) are quasi-pseudometrics in \( \mathbb{R} \) such that \( \rho_u^{-1} = \rho_l \); moreover, \( \tau(\rho_u) \) is the upper topology \( u \) in \( \mathbb{R} \), while \( \tau(\rho_l) \) is the lower topology \( l \) in \( \mathbb{R} \).

**Definition 6.1.** Let \( d_X, d_Y \) be quasi-pseudometrics in sets \( X \) and \( Y \), respectively. We say that a mapping \( f : X \to Y \) is \((d_X, d_Y)\)-uniformly continuous if the following condition is satisfied:

\[
\forall \varepsilon \in (0;+\infty) \exists \delta \in (0;+\infty) \forall x_1, x_2 \in X \left[ d_X(x_1, x_2) < \delta \Rightarrow d_Y(f(x_1), f(x_2)) < \varepsilon \right].
\]

**Definition 6.2.** Quasi-pseudometrics \( d_0, d_1 \) in a set \( X \) are called uniformly equivalent if the following condition holds:

\[
\forall \varepsilon \in (0;+\infty) \exists \delta_0, \delta_1 \in (0;+\infty) \forall x, y \in X \forall i \in \{0, 1\} \left[ d_i(x, y) < \delta_i \Rightarrow d_{1-i}(x, y) < \varepsilon \right].
\]

**Definition 6.3.** Suppose that \((X, d)\) is a quasi-metric space and that \( B \) is a bornology in \( X \). We say that \( B \) is uniformly quasi-metrizable with respect to \( d \) if there exists a quasi-metric \( \rho \) in \( X \) such that \( d \) and \( \rho \) are uniformly equivalent, while \( B \) is the collection of all \( \rho \)-bounded sets.

**Definition 6.4.** We say that quasi-metrics \( d, \rho \) in \( X \) are uniformly locally identical if they are uniformly equivalent and there exists \( \delta \in (0;+\infty) \) such that, for all \( x, y \in X \), we have \( \rho(x, y) = d(x, y) \) whenever \( d(x, y) < \delta \) (cf. [WJ] and Remark 2.5 of [GM]).

**Theorem 6.5.** Suppose that \((X, d)\) is a quasi-metric space and that \( B \) is a bornology in \( X \). Then the following conditions are all equivalent:

(i) \( B \) is uniformly quasi-metrizable with respect to \( d \);

(ii) \( B \) has a base \( \{B_n : n \in \omega\} \) such that, for some \( \delta \in (0;+\infty) \) and for each \( n \in \omega \), the inclusion \( B_n^{\delta} \subseteq B_{n+1} \) holds;
(iii) there exists a quasi-metric \( \rho \) in \( X \) such that \( d, \rho \) are uniformly locally identical and \( \mathcal{B} \) is the collection of all \( \rho \)-bounded sets.

(iv) there exists a \((d, \rho_u)\)-uniformly continuous \((\tau(d), \tau(d^{-1}))\)-characteristic function for \( \mathcal{B} \).

**Proof.** Assume (i) and suppose that \( \rho \) is a uniformly equivalent with \( d \) quasi-metric in \( X \) such that \( \mathcal{B} \) is the collection of all \( \rho \)-bounded sets. Let \( x_0 \in X \) and, for \( n \in \omega \), let \( B_n = B_\rho(x_0, n+1) \). We choose \( \delta \in (0; +\infty) \) such that \( \rho(x, y) < \frac{1}{2} \) whenever \( d(x, y) < \delta \). Then \( [B_n]_d^\rho \subseteq [B_n]^\rho \subseteq B_{n+1} \) for each \( n \in \omega \), so (ii) follows from (i).

Now, let us suppose that (ii) holds. We may assume that \( \delta \in (0; +\infty) \) and that \( \{B_n : n \in \omega \} \) is a base for \( \mathcal{B} \) such that \( B_0 = \emptyset, B_1 \neq \emptyset \) and \([B_n]_d^\rho \subseteq B_{n+1} \) for each \( n \in \omega \). We shall mimic the proof to Proposition 2.2 in [GM] and change parts of it to show that (iii) follows from (ii). We define \( \phi_0(x) = 1 \) for each \( x \in X \). If \( n \in \omega \setminus \{0\} \), we define \( \phi_n(x) = \min\{1, \frac{1}{n}d(B_n, x)\} \) for each \( x \in X \). It is easy to check that the function \( \phi_n \) is \((d, \rho_u)\)-uniformly continuous; moreover, \( \phi_n(B_n) \subseteq \{0\} \) and \( \phi_n(X \setminus B_{n+1}) \subseteq \{1\} \). Let us consider the function \( \chi : X \to [0; +\infty) \) defined by

\[
\chi(x) = n - 2 + \phi_{n-1}(x)
\]

for each \( x \in B_n \setminus B_{n-1} \) and for each \( n \in \omega \setminus \{0\} \). To prove that \( \chi \) is \((d, \rho_u)\)-uniformly continuous, let us consider an arbitrary pair \( x, y \) of points of \( X \) such that \( d(x, y) < \delta \). Let \( n \in \omega \) be the unique natural number such that \( x \in B_n \setminus B_{n-1} \). If \( z \in X \setminus B_{n+1} \), then \( d(x, z) \geq \delta \). This implies that \( y \in B_{n+1} \). Let \( m \in \omega \setminus \{0\} \) be the unique natural number such that \( y \in B_m \setminus B_{m-1} \). Then \( m \leq n + 1 \). We have \( \chi(y) - \chi(x) = m - n + \phi_{m-1}(y) - \phi_{n-1}(x) \). If \( m = n + 1 \), then \( \chi(y) - \chi(x) = 1 + \phi_n(y) - \phi_{n-1}(x) = \phi_n(y) - \phi_n(x) + \phi_{n-1}(y) - \phi_{n-1}(x) \leq \frac{2}{\delta}d(x, y) \). If \( m = n \), then \( \chi(y) - \chi(x) = \phi_{n-1}(y) - \phi_{n-1}(x) \leq \frac{1}{\delta}d(x, y) \). Suppose that \( m < n \). Then \( m - n + 1 \leq 0, x \notin B_m, y \in B_{n-1} \) and \( \chi(y) - \chi(x) = m - n + 1 + \phi_{m-1}(y) - \phi_{m-1}(x) + \phi_{n-1}(y) - \phi_{n-1}(x) \leq \frac{2}{\delta}d(x, y) \). In consequence, \( \chi \) is \((d, \rho_u)\)-uniformly continuous. Therefore, \( \chi : (X, \tau(d), \tau(d^{-1})) \to (\mathbb{R}, u, l) \) is bicontinuous. Since, for \( A \subseteq X \), we have that \( A \in \mathcal{B} \) if and only if \( \chi \) is bounded on \( A \), the function \( \chi \) is a \((\tau(d), \tau(d^{-1}))\)-characteristic function for \( \mathcal{B} \). In much the same way as in Remark 2.5 of [GM], we can define \( \rho(x, y) = \max\{\min\{d(x, y), 1\}, \frac{1}{2} \max\{\chi(y) - \chi(x), 0\}\} \) to get a quasi-metric \( \rho \) uniformly locally identical with \( d \) such that \( \mathcal{B} \) is the collection of all \( \rho \)-bounded sets. Thus (ii) implies (iii).
Let us assume that \((\text{iii})\) holds. We take a quasi-metric \(\rho\) in \(X\) such that \(d\) and \(\rho\) are uniformly locally identical and \(\mathcal{B} = \mathcal{B}_\rho(X)\). We fix \(x_0 \in X\) and define \(f(x) = \rho(x_0, x)\) for \(x \in X\) to get a \((\tau(\rho), \tau(\rho^{-1}))\)-characteristic function \(f\) for \(\mathcal{B}\) such that \(f\) is \((d, \rho_u)\)-uniformly continuous. Hence \((\text{iii})\) implies \((\text{iv})\).

Finally, we suppose that \((\text{iv})\) holds and we consider an arbitrary function \(g\) such that \(g\) is a \((\tau(d), \tau(d^{-1}))\)-characteristic function for \(\mathcal{B}\) and \(g\) is \((d, \rho_u)\)-uniformly continuous. For \(x, y \in X\), we can define \(d_g(x, y) = \min\{d(x, y), 1\} + \max\{g(y) - g(x), 0\}\) to see that \((\text{iv})\) implies \((\text{i})\).

\[\square\]

**Corollary 6.6.** *Theorem 2.4 and Remark 2.5 of [GM] hold true in ZF.*

One can use Example 10.16 (i)-(iii) given at the end of Section 10 to see that, for a quasi-metric \(d\) in \(X\) and a bornology \(\mathcal{B}\) in \(X\), it may happen that the bornological universe \((X, \tau(d)), \mathcal{B})\) is quasi-metrizable or even metrizable, while \(\mathcal{B}\) is not uniformly quasi-metrizable with respect to \(d\).

### 7 Applications to independence from ZF

Mimicking [GM], let us consider the following bornologies in a metric space \((X, d)\): the bornology \(\mathbf{FB}(X)\) of all finite subsets of \(X\), the bornology \(\mathbf{CB}_d(X)\) generated by the compact subsets of \((X, d)\), the bornology \(\mathbf{TB}_d(X)\) of all totally bounded subspaces of \((X, d)\), as well as the bornology \(\mathbf{BB}_d(X)\) of all Bourbaki-bounded sets. Several theorems about equivalents of the uniform metrizability of the bornologies \(\mathbf{FB}(X), \mathbf{CB}_d(X), \mathbf{TB}_d(X)\) and \(\mathbf{BB}_d(X)\) in ZFC were proved in [GM]. We are going to show that some of the above-mentioned theorems of [GM] are independent of ZF, while other theorems of [GM] can be proved in ZF. Clearly, we have already shown in the previous section that both Proposition 2.2 and Theorem 2.4 of [GM] hold true in ZF.

The following theorem will be helpful:

**Theorem 7.1.** *Equivalent are:

\[(i) \text{ CC(fin)};\]

\[(ii) \text{ for every discrete space } X, \text{ the bornological universe } (X, \mathbf{FB}(X)) \text{ is metrizable (in the sense of Hu)} \text{ if and only if } X \text{ is countable.}\]*
Proof. Assume (i) and let $X$ be any discrete space such that the bornological universe $(X, FB(X))$ is metrizable. It follows from Theorem 4.7 that $FB(X)$ has a countable base. If $A$ is a countable base for $FB(X)$, then $X = \bigcup A$, so, by Proposition 3.5 of [Her], $X$ is countable if $CC(\text{fin})$ holds. It is obvious that if $X$ is a countable discrete space, then the bornological universe $(X, FB(X))$ is metrizable in $ZF$ by Theorem 4.7.

Now, assume that $CC(\text{fin})$ fails. Then, in view of Proposition 3.5 of [Her], there exists a sequence $(A_n)_{n \in \omega}$ of pairwise disjoint non-void finite sets such that the set $Z = \bigcup_{n \in \omega} A_n$ is uncountable. Let us equip $Z$ with its discrete topology. Then the collection $\{\bigcup_{m \in n} A_m : n \in \omega\}$ is a countable base for $FB(Z)$. Then, by Theorem 4.7, the bornological universe $(Z, FB(Z))$ is metrizable. This contradicts (ii).

For a set $X$ and a cardinal number $\kappa$, let us use the notation $[X]^{\leq \kappa}$ for the collection of all subsets $A$ of $X$ such that $A$ is of cardinality at most $\kappa$ and the notation $[X]<\kappa$ for the collection of all subsets of $X$ that are of cardinality $<\kappa$. (cf. Definition I.13.19 of [Ku2]). Then $[X]^{\leq \omega} = FB(X)$, while $[X]<\omega$ is the bornology of all at most countable subsets of $X$.

The proof to the following interesting theorem is somewhat more complicated than to Theorem 7.1.

**Theorem 7.2.** Equivalent are:

(i) for every sequence $(X_n)_{n \in \omega}$ of non-void at most countable sets $X_n$, the product $\prod_{n \in \omega} X_n$ is non-void;

(ii) for every discrete space $X$, the bornological universe $(X, [X]^{\leq \omega})$ is metrizable if and only if $X$ is countable.

Proof. Assume (i). Let $X$ be a discrete space such that the bornological universe $(X, [X]^{\leq \omega})$ is metrizable. Then, by Theorem 4.7, there exists a countable base $B = \{X_n : n \in \omega\}$ for the bornology $[X]^{\leq \omega}$. Suppose that $X$ is uncountable. We may assume that $X_n \subseteq X_{n+1}$ and that $X_n \neq X_{n+1}$ for each $n \in \omega$. By (i), there exists $x \in \prod_{n \in \omega} (X_{n+1} \setminus X_n)$. Then, for such an $x$, if $A = \{x(n) : n \in \omega\}$, then $A \in [X]^{\leq \omega}$, while there does not exist $n \in \omega$ such that $A \subseteq X_n$. This is impossible because $B$ is a base for $[X]^{\leq \omega}$. Therefore, (i) implies (ii).

Now, let us suppose that (i) is false. Consider any sequence $(X_n)_{n \in \omega}$ of non-empty countable sets such that $\prod_{n \in \omega} X_n = \emptyset$. For each $n \in \omega$, the set $Y_n = \prod_{i \in n+1} X_i$ is countable and non-empty. In much the same way as in

\[20\]
the proof to Theorem 2.12 of [Her], we can show that there does not exist an infinite set \( M \subseteq \omega \) such that \( \prod_{n \in M} Y_n \neq \emptyset \). Let \( Y = \bigcup_{n \in \omega} Y_n \) and let \( f : \omega \to Y \) be an injection. Then the set \( M_f = \{ n \in \omega : f(\omega) \cap Y_n \neq \emptyset \} \) is finite. This proves that \( Y \) is uncountable and if \( A_n = \bigcup_{m \in n+1} Y_m \) for \( n \in \omega \), then the collection \( A = \{ A_n : n \in \omega \} \) is a countable base for \( [Y]^{\leq \omega} \). If we equip \( Y \) with its discrete topology, we will obtain that (ii) is false. Hence (ii) implies (i).

\[ \square \]

Remark 7.3. It is unknown to us whether there is a model for \( \text{ZF} \) in which \( \text{CUT} \) fails (cf. [Her]), while condition (i) of Theorem 7.2 is satisfied.

Remark 7.4. It is evident that conditions (1) and (2) of Theorem 2.6 of [GM] are equivalent in \( \text{ZF} \). In view of our Theorem 6.5 and the proof of (3) \( \Rightarrow \) (1) of Theorem 2.6 given in [GM], we have that (3) \( \Rightarrow \) (1) of Theorem 2.6 of [GM] holds true in \( \text{ZF} \). Since \( \text{CC(fin)} \) is relatively independent of \( \text{ZF} \), it follows from our Theorem 7.1 that Theorem 2.6 of [GM] is relatively independent of \( \text{ZF} \). If \( M \) is a model for \( \text{ZF} + \neg \text{CC(fin)} \), then Theorems 7.1 and 6.5 show that there exists in \( M \) an uncountable metric space \( (X,d) \) such that \( \text{FB}(X) \) is uniformly metrizable with respect to \( d \), so Theorem 2.6 of [GM] fails in \( M \). Now, we can deduce from Proposition 3.5 of [Her] that Theorem 2.6 of [GM] is equivalent with \( \text{CC(fin)} \).

Remark 7.5. Let us notice that both (1) \( \Leftrightarrow \) (2) and (3) \( \Rightarrow \) (1) of Theorem 3.1 of [GM] hold true in \( \text{ZF} \). Unfortunately, Theorem 3.1 of [GM] is relatively independent of \( \text{ZF} \). Namely, in much the same way as in Remark 7.4, we can show that in every model \( M \) for \( \text{ZF} + \neg \text{CC(fin)} \), there exists an uncountable set \( X \) such that, for the discrete metric \( d \) in \( X \), the bornology \( \text{CB}_d(X) \) is uniformly metrizable with respect to \( d \), while \( (X,d) \) is not Lindelöf but it is obviously uniformly locally compact.

Remark 7.6. As Gutierres showed in [Gut], while working with completions of metric spaces, one must be more careful in \( \text{ZF} \) than in \( \text{ZF} + \text{CC} \). Let us observe that if \( M \) is a model for \( \text{ZF} \) such that there is in \( M \) an uncountable set \( X \) such that \( \text{FB}(X) \) is uniformly metrizable with respect to the discrete metric \( d \) in \( X \), then \( \text{TB}_d(X) = \text{FB}(X) = \text{BB}_d(X) \) is uniformly metrizable, while \( (X,d) \) is neither Lindelöf nor Bourbaki-separable. Therefore, Theorems 4.2 and 5.8 of [GM] fail in \( M \). In the light of our Theorem 7.1 and of the fact that \( \text{CC(fin)} \) is relatively independent of \( \text{ZF} \), Theorems 4.2 and 5.8 of [GM] are relatively independent of \( \text{ZF} \).
Since many articles about bornologies have been published so far, it may take a lot of time to investigate which of the theorems in the articles can fail in some models for ZF. There are theorems about connections between bornologies and realcompactifications that have already appeared in print (cf. [Vr2]) and they seem to be unprovable in ZF. In view of Theorem 10.12 of [PW], perhaps, some of them can be proved in ZF + UFT where UFT stands for the Ultrafilter Theorem (cf. [Her]). Let us leave it as an open problem which of the theorems about bornologies that have been proved by other authors in ZFC may fail in models for ZF and which of them can be proved under weaker assumptions than ZFC. We have given only a partial solution to this problem.

8 Compact bornologies in quasi-metric spaces

In the light of Remark 7.5, Theorem 3.1 of [GM] may fail in a model for ZF. We are going to prove in ZF its modified version for compact bornologies in quasi-metric spaces.

For a topological space \((X, \tau)\), let \(\mathbb{CB}_\tau(X)\) be the bornology in \(X\) generated by the collection of all compact subsets of \((X, \tau)\). If it is useful, we shall use the notation \(\mathbb{CB}(X, \tau)\) for \(\mathbb{CB}_\tau(X)\).

**Definition 8.1.** Let \(d\) be a quasi-metric in \(X\).

(i) We denote by \(\mathbb{CB}_d(X)\) the bornology \(\mathbb{CB}_{\tau(d)}(X)\).

(ii) We say that \(X\) is **uniformly locally compact with respect to** \(d\) if there exists \(\delta \in (0; +\infty)\) such that \(B_d(x, \delta) \in \mathbb{CB}_d(X)\) for each \(x \in X\).

The following example shows that, contrary to compact bornologies in metric spaces, it may happen that, for a quasi-metric \(d\) in \(X\), there is a set \(A \in \mathbb{CB}_d(X)\) such that \(\text{cl}_{\tau(d)}A \notin \mathbb{CB}_d(X)\).

**Example 8.2.** Let us consider the set \(X = X_1 \cup X_2\) where \(X_1 = \{\frac{1}{2^n} : n \in \omega\}\) and \(X_2 = \{\frac{1}{2^{n+1}} : n \in \omega\}\). Let \(x, y \in X\). If \(x = y\), we put \(d(x, y) = 0\). When \(x \neq y\), we put \(d(x, y) = 1\) if either \(x, y \in X_1\) or \(x, y \in X_2\), or \(x \in X_1, y \in X_2\); moreover, we put \(d(x, y) = y\) if \(x \in X_2, y \in X_1\). In this way, we have defined a quasi-metric on \(X\) such that, for each \(y \in X_2\), the set \(A_y = \{y\} \cup X_1\) is compact in \((X, \tau(d))\), while \(\text{cl}_{\tau(d)}A_y = X \notin \mathbb{CB}_d(X)\).
**Definition 8.3.** We say that a topological space \((X, \tau)\) is \(\sigma\)-CB if there exists a countable collection \(\mathcal{A} \subseteq \mathbf{CB}_\tau(X)\) such that \(X = \bigcup \mathcal{A}\).

**Remark 8.4.** Clearly, it holds true in ZF that every \(\sigma\)-compact space is \(\sigma\)-CB and every \(\sigma\)-CB Hausdorff space is \(\sigma\)-compact. In every model for ZF + CC, a topological space is \(\sigma\)-compact if and only if it is \(\sigma\)-CB. We do not know whether there is a model for ZF + \(\neg\)CC in which a topological space can be simultaneously \(\sigma\)-CB and not \(\sigma\)-compact.

**Theorem 8.5.** Let \(d\) be a (quasi)-metric in \(X\). Then the following conditions are equivalent:

1. \(\mathbf{CB}_d(X)\) is uniformly (quasi)-metrizable with respect to \(d\);
2. \(X\) is uniformly locally compact with respect to \(d\) and \((X, \tau(d))\) is \(\sigma\)-CB.

**Proof.** Assume (i). Let \(\rho\) be a uniformly equivalent with \(d\) quasi-metric in \(X\) such that \(\mathbf{CB}_d(X)\) is the collection of all \(\rho\)-bounded sets. There exists \(\delta \in (0; +\infty)\) such that \(\rho(x, y) < 1\) whenever \(d(x, y) < \delta\). Then, for each \(x \in X\), we have \(B_d(x, \delta) \subseteq B_\rho(x, 1) \in \mathbf{CB}_d(X)\), so \(X\) is uniformly locally compact with respect to \(d\). Moreover, since, by Theorem 4.7, \(\mathbf{CB}_d(X)\) has a countable base, we deduce that \((X, \tau(d))\) is \(\sigma\)-CB.

Now, assume (ii). Let \(\delta \in (0; +\infty)\) be such that, for each \(x \in X\), we have \(B_d(x, \delta) \in \mathbf{CB}_d(X)\). Let \(C\) be compact in \((X, \tau(d))\). It follows from the compactness of \(C\) that there exists a finite set \(K \subseteq C\) such that \(C \subseteq \bigcup_{x \in K} B_d(x, \frac{\delta}{2})\). Then \([C]_d^\frac{\delta}{2} \subseteq \bigcup_{x \in K} [B_d(x, \frac{\delta}{2})]_d^\frac{\delta}{2} \subseteq \bigcup_{x \in K} B_d(x, \delta) \in \mathbf{CB}_d(X)\). Therefore, \([C]_d^\frac{\delta}{2} \in \mathbf{CB}_d(X)\) (cf. proof to 3.1 in [GM]). This implies that \([C]_d^\frac{\delta}{2} \in \mathbf{CB}_d(X)\) whenever \(C \in \mathbf{CB}_d(X)\).

Let \(\mathcal{A} = \{A_n : n \in \omega\} \subseteq \mathbf{CB}_d(X)\) be such that \(X = \bigcup \mathcal{A}\). We may assume that \(A_n \subseteq A_{n+1}\) for each \(n \in \omega\). If \(C\) is compact in \((X, \tau(d))\) then, since \(C \subseteq \bigcup_{n \in \omega} [A_n]_d^\frac{\delta}{2}\), there exists \(m \in \omega\) such that \(C \subseteq \bigcup_{n \in m+1} [A_n]_d^\frac{\delta}{2} = [A_m]_d^\frac{\delta}{2} \in \mathbf{CB}_d(X)\). Therefore, the collection \(\{[A_n]_d^\frac{\delta}{2} : n \in \omega\}\), is a countable base for \(\mathbf{CB}_d(X)\). This, together with the fact that \([C]_d^\frac{\delta}{2} \in \mathbf{CB}_d(X)\) whenever \(C \in \mathbf{CB}_d(X)\), implies that there exists a subsequence \((B_n)_{n \in \omega}\) of the sequence \(([A_n]_d^\frac{\delta}{2})_{n \in \omega}\) such that \([B_n]_d^\frac{\delta}{2} \subseteq B_{n+1}\) for each \(n \in \omega\). Then \(\{B_n : n \in \omega\}\) is a base for \(\mathbf{CB}_d(X)\). This, together with Theorem 6.5, implies that (i) follows from (ii). \(\square\)
Example 8.6. Let \((X, d)\) be the quasi-metric space from Example 8.2. Then condition (ii) of Theorem 8.5 is satisfied; hence, the bornology \(\mathbf{CB}_d(X)\) is uniformly quasi-metrizable with respect to \(d\). We can also define a uniformly locally identical with \(d\) quasi-metric \(\rho\) in \(X\) such that \(\mathbf{CB}_d(X) = \mathcal{B}_\rho(X)\). To do this, let us consider \(x, y \in X\). We put \(\rho(x, y) = 0\) if \(x = y\). Now, suppose that \(x \neq y\). Then \(\rho(x, y) = 1\) if \(x, y \in X_1\). For \(x \in X_2\) and \(y \in X_1\), we define \(\rho(x, y) = y\). Finally, for \(x \in X\) and \(y \in X_2\), we put \(\rho(x, y) = \frac{1}{y}\).

Using similar arguments to the ones of the proof to Theorem 8.5, we deduce the following corollary:

Corollary 8.7 (cf. [WJ], Theorem 3.1 of [GM] and Corollary 3.3 of [GM]). For every metric space \((X, d)\), it holds true in \(ZF\) that \(\mathbf{CB}_d(X)\) is uniformly metrizable with respect to \(d\) if and only if \((X, d)\) is both \(\sigma\)-compact and uniformly locally compact.

9 Fundamental bornologies in gtses

A new problem is to find an appropriate definition of (quasi)-metrizability for a generalized topological space (in abbreviation: a gts) in the sense of Delfs and Knebusch. Since the notion of a gts in this sense is rather complicated (cf. [DK], [P1]) and it seems that it is still not commonly known to the mathematical community, let us recall it to make our paper more legible.

Definition 9.1 (cf. Definition 2.2.2 in [P1]). A generalized topological space in the sense of Delfs and Knebusch (abbreviated to gts) is a triple \((X, \mathcal{O}_X, \mathcal{C}_X)\) where \(X\) is a set for which \(\mathcal{O}_X \subseteq \mathcal{P}(X)\), while \(\mathcal{C}_X \subseteq \mathcal{P}(\mathcal{O}_X)\) and the following conditions are satisfied:

(i) if \(U \subseteq \mathcal{O}_X\) and \(U\) is finite, then \(\bigcup U \in \mathcal{O}_X, \bigcap U \in \mathcal{O}_X\) and \(U \in \mathcal{C}_X\);

(ii) if \(U \in \mathcal{C}_X, V \in \mathcal{O}_X\) and \(V \subseteq \bigcup U\), then \(\{U \cap V : U \in U\} \in \mathcal{C}_X\);

(iii) if \(U \in \mathcal{C}_X\) and, for each \(U \in U\), we have \(\mathcal{V}(U) \in \mathcal{C}_X\) such that \(\bigcup \mathcal{V}(U) = U\), then \(\bigcup_{U \in U} \mathcal{V}(U) \in \mathcal{C}_X\);

(iv) if \(U \subseteq \mathcal{O}_X\) and \(V \in \mathcal{C}_X\) are such that \(\bigcup V = \bigcup U\) and, for each \(V \in V\) there exists \(U \in U\) such that \(V \subseteq U\), then \(U \in \mathcal{C}_X\);
(v) If \( U \in \text{Cov}_X \), \( V \subseteq \bigcup_{U \in \mathcal{U}} U \) and, for each \( U \in \mathcal{U} \), we have \( V \cap U \in \text{Op}_X \), then \( V \in \text{Op}_X \).

Remark 9.2. If \((X, \text{Op}_X, \text{Cov}_X)\) is a gts, then \( \text{Op}_X = \bigcup \text{Cov}_X \) and, therefore, we can identify the gts with the ordered pair \((X, \text{Cov}_X)\) (cf. [P1], [PW]). If this is not misleading, we shall denote a gts \((X, \text{Cov}_X)\) by \( X \).

As far as gtses are concerned, we shall use the terminology of [DK], [P1]-[P2] and [PW].

**Definition 9.3** (cf. [P1]). If \( X = (X, \text{Cov}_X) \) and \( Y = (Y, \text{Cov}_Y) \) are gtses, then:

(i) A set \( U \subseteq X \) is called **open** in the gts \( X \) if \( U \in \text{Op}_X \);

(ii) The collection \( \text{Cov}_X \) is the **generalized topology** in \( X \);

(iii) An **admissible open family** in the gts \( X \) is a member of \( \text{Cov}_X \);

(iv) A mapping \( f : Y \to X \) is \((\text{Cov}_Y, \text{Cov}_X)\)-**strictly continuous** (in abbreviation: strictly continuous) if, for each \( U \in \text{Cov}_X \), we have \( \{f^{-1}(U) : U \in \mathcal{U}\} \in \text{Cov}_Y \).

In this section, let us have a brief look at very natural bornologies in generalized topological spaces. In the next section, we apply the bornologies in gtses to our concepts of (quasi)-metrizability in the category \( \text{GTS} \) of generalized topological spaces and strictly continuous mappings.

**Definition 9.4** (cf. Definitions 2.2.13 and 2.2.25 of [P1]). If \( K \) is a subset of a set \( X \), then we say that a family \( \mathcal{U} \subseteq \mathcal{P}(X) \) is **essentially finite on** \( K \) if there exists a finite \( V \subseteq \mathcal{U} \) such that \( K \cap \bigcup \mathcal{U} \subseteq \bigcup V \).

**Definition 9.5** (cf. Definition 2.2.25 of [P1]). If \( X = (X, \text{Cov}_X) \) is a gts, then a set \( K \subseteq X \) is called **small in the gts** \( X \) if each family \( \mathcal{U} \in \text{Cov}_X \) is essentially finite on \( K \).

The collection of all small sets of a gts \( X \) is a bornology in \( X \) (cf. Fact 2.2.30 of [P1]).

**Definition 9.6.** For a gts \( X \), the **small bornology** of \( X \) is the collection \( \text{Sm}(X) \) of all small sets in \( X \).
**Sm**(*X*) was denoted by **Sm**<sub>X</sub> in [P1] but, since we are inspired by [GM] and we use the notation of [GM], we have replaced **Sm**<sub>X</sub> by **Sm**(*X*) partly for elegance, partly for convenience.

**Definition 9.7** (cf. Definition 3.2 of [PW]). If **X** is a gts, we call a set **A** ⊆ **X** admissibly compact in **X** if, for each \(U \in \text{Cov}_X\) such that **A** ⊆ \(\bigcup U\), there exists a finite \(V \subseteq U\) such that **A** ⊆ \(\bigcup V\).

**Definition 9.8.** For a gts **X**, the admissibly compact bornology of **X** is the collection \(ACB(X)\) of all subsets of admissibly compact sets of the gts **X**.

For a collection \(\mathcal{A}\) of subsets of a set **X**, we denote by \(\tau(\mathcal{A})\) the weakest among all topologies in **X** that contain \(\mathcal{A}\). For a gts \((\text{X}, \text{Op}_X, \text{Cov}_X)\), we call the topological space \(X_{top} = (X, \tau(\text{Op}_X))\) the **topologization of the gts** **X** (cf. [PW]).

**Definition 9.9.** Let **X** be a gts. We say that a set **A** is topologically compact in **X** if **A** is compact in \(X_{top}\) (cf. Definition 3.2 of [PW]). The compact bornology \(CB(X_{top})\) will be called the **compact bornology of the gts** **X** and it will be denoted by \(CB(X)\).

**Fact 9.10.** For every gts **X**, the inclusion \((\text{Sm}(X) \cup CB(X)) \subseteq ACB(X)\) holds.

In general, the collections \(\text{Sm}(X) \cup CB(X)\) and \(ACB(X)\) can be distinct and neither \(\text{Sm}(X) \subseteq CB(X)\) nor \(CB(X) \subseteq \text{Sm}(X)\).

**Example 9.11.** For \(X = \mathbb{R} \times \{0, 1\}\), let \(\text{Op}_X\) be the natural topology in **X** inherited from the usual topology of \(\mathbb{R}\) and let \(\text{Cov}_X\) be the collection of all families \(\mathcal{U} \subseteq \text{Op}_X\) such that \(\mathcal{U}\) is essentially finite on \(\mathbb{R} \times \{0\}\). Then, for \(A = [0; 1] \times \{1\}\) and \(B = \mathbb{R} \times \{0\}\), we have \(A \in \text{CB}(X)\) \(\backslash\) \(\text{Sm}(X)\) and \(B \in \text{Sm}(X)\) \(\backslash\) \(\text{CB}(X)\), while \(A \cup B \in ACB(X) \backslash (\text{CB}(X) \cup \text{Sm}(X))\).

For a set **X** and a collection \(\Psi \subseteq \mathcal{P}^2(X)\), we denote by \((\Psi)_X\) the smallest among generalized topologies in **X** that contain \(\Psi\). If \(\mathcal{A} \subseteq \mathcal{P}(X)\), let \(\text{EssCount}(\mathcal{A})\) be the collection of all essentially countable subfamilies of \(\mathcal{A}\). We recall that \(\text{EssFin}(\mathcal{A})\) is the collection of all essentially finite subfamilies of \(\mathcal{A}\) (cf. [P1]-[P2] and [PW]).
Fact 9.12 (cf. Examples 2.2.35 and 2.2.14(8) of [P1]). Let \((X, \tau)\) be a topological space. That \(\text{EssFin}(\tau)\) is a generalized topology in \(X\) is true in \(ZF\). That \(\text{EssCount}(\tau)\) is a generalized topology in \(X\) is true in \(ZF + CC\).

Remark 9.13. It is unprovable in \(ZF\) that, for every topological space \((X, \tau)\), the collection \(\text{EssCount}(\tau)\) is a generalized topology in \(X\). Namely, let \(M\) be a model for \(ZF + \neg CC(\text{fin})\). In view of the proof to Theorem 7.1, there exists in \(M\) an uncountable set \(X\) such that \(X\) is a countable union of finite sets. Let \(\tau = P(X)\). If \(\text{EssCount}(\tau)\) were a generalized topology in \(X\), the family of all singletons of \(X\) would belong to \(\text{EssCount}(\tau)\) which is impossible since \(X\) is uncountable.

Let us observe that, for the gts \(X\) from Example 9.11, the admissibly compact bornology of \(X\) is generated by \(\text{CB}(X) \cup \text{Sm}(X)\). That not every gts may share this property is shown by the following example:

Example 9.14. (ZF+CC) For \(X = \omega_1\), let \(\text{Op}_X\) be the topology induced by the usual linear order in \(\omega_1\) and let \(\text{Cov}_X = \text{EssCount}(\text{Op}_X)\). Then \(\text{Sm}(X) = \text{FB}(X) \neq \text{CB}(X) \neq \text{ACB}(X) = P(X)\).

In what follows, for sets \(X, Y\) with \(Y \subseteq X\) and for \(\Psi \subseteq \mathcal{P}^2(X)\), we use the notation \(\Psi \cap_2 Y\) from [P1] for the collection of all families \(U \cap_1 Y = \{U \cap Y : U \in \mathcal{U}\}\) where \(\mathcal{U} \in \Psi\). We want to describe \(\langle \Psi \cap_2 Y\rangle_Y\) more precisely in the case when \(\Psi \cap_2 Y \subseteq \text{EssFin}(\mathcal{P}(Y))\). To do this, we need the concept of a complete ring of sets in \(Y\) that was of frequent use in [PW]. Namely, a complete ring in \(Y\) is a collection \(C \subseteq \mathcal{P}(Y)\) such that \(\emptyset, Y \in C\), while \(C\) is closed under finite unions and under finite intersections. For \(A \subseteq \mathcal{P}(Y)\), let \(L_Y[A]\) be the intersection of all complete rings in \(Y\) that contain \(A\).

Proposition 9.15. For a set \(X\), let \(\Psi \subseteq \mathcal{P}^2(X)\). Suppose that \(Y \subseteq X\) and that each family from \(\Psi\) is essentially finite on \(Y\). Then the following conditions are satisfied:

\((i)\) \(\langle \Psi \cap_2 Y\rangle_Y = \text{EssFin}(L_Y[\bigcup(\Psi \cap_2 Y)]) = \text{EssFin}(\bigcup(\langle \Psi \cap_2 Y\rangle_Y) = \text{EssFin}(\bigcup(\langle \Psi \rangle_X) \cap_2 Y;\)

\((ii)\) each family from \(\langle \Psi \rangle_X\) is essentially finite on \(Y\).

Proof. By applying Proposition 2.2.37 of [P1] to the mapping id\(_{Y}\) : \(Y \to X\), we obtain the inclusion \(\langle \Psi \rangle_X \cap_2 Y \subseteq \langle \Psi \cap_2 Y\rangle_Y\) which, together with \((i)\), implies \((ii)\). To prove \((i)\), let us put \(\mathcal{G}_0 = \langle \Psi \cap_2 Y\rangle_Y, \mathcal{G}_1 = \text{EssFin}(L_Y[\bigcup(\Psi \cap_2 Y)\bigcup(\langle \Psi \rangle_X) \cap_2 Y;\)

}
\( \mathcal{G}_2 = \text{EssFin}(\bigcup \mathcal{G}_0) \) and \( \mathcal{G}_3 = \text{EssFin}(\bigcup (\Psi)_X) \cap_2 Y \). Obviously, \( \mathcal{G}_0, \mathcal{G}_1 \) and \( \mathcal{G}_2 \) are generalized topologies in \( Y \). By Proposition 2.2.53 of \([P1]\), the collection \( \mathcal{G}_3 \) is also a generalized topology in \( Y \). Since \( \Psi \cap_2 Y \subseteq \mathcal{G}_1 \) and \( L_Y[\bigcup (\Psi \cap_2 Y)] \subseteq \bigcup \mathcal{G}_0 \), we have \( \mathcal{G}_0 \subseteq \mathcal{G}_1 \subseteq \mathcal{G}_2 \subseteq \mathcal{G}_0 \). It follows from the inclusion \( \langle \Psi \rangle_X \cap_2 Y \subseteq \mathcal{G}_0 \) that \( \mathcal{G}_3 \subseteq \mathcal{G}_0 \). Since \( \bigcup (\langle \Psi \rangle_X \cap_2 Y) \) is a complete ring of subsets of \( Y \), we get \( \mathcal{G}_0 \subseteq \mathcal{G}_1 \subseteq \mathcal{G}_2 \subseteq \mathcal{G}_0 \). This completes our proof to (i).

**Definition 9.16.** If \( X = (X, \text{Op}_X, \text{Cov}_X) \) is a gts, then:

(i) the partial topologization of \( X \) is the gts \( X_{pt} = (X, (\text{Op})_{pt}, (\text{Cov})_{pt}) \) where \( (\text{Op})_{pt} = \tau(\text{Op}) \) and \( (\text{Cov})_{pt} = \langle \text{Cov} \cup \text{EssFin}(\tau(\text{Op})) \rangle_X \) (cf. Definition 4.1 of \([PW]\));

(ii) the gts \( X \) is called partially topological if \( X = X_{pt} \) (cf. Definition 2.2.4 of \([P1]\));

(iii) \( \text{GTS}_{pt} \) is the category of all partially topological spaces and strictly continuous mappings, while the mapping \( pt : \text{GTS} \rightarrow \text{GTS}_{pt} \) is the functor of partial topologization defined by: \( pt(X) = X_{pt} \) for every gts \( X \) and \( pt(f) = f \) for every morphism in \( \text{GTS} \) (cf. \([AHS]\), \([P1]\) and Definition 4.2 of \([PW]\)).

**Proposition 9.17.** Let \( X \) be a gts. Then \( \text{Sm}(X) = \text{Sm}(X_{pt}), \text{CB}(X) = \text{CB}(X_{pt}) \) and \( \text{ACB}(X_{pt}) \subseteq \text{ACB}(X) \).

**Proof.** The equality \( \text{CB}(X) = \text{CB}(X_{pt}) \) and both the inclusions \( \text{Sm}(X_{pt}) \subseteq \text{Sm}(X) \) and \( \text{ACB}(X_{pt}) \subseteq \text{ACB}(X) \) are trivial. Let \( X = (X, \text{Op}_X, \text{Cov}_X) \) and let \( \Psi = \text{Cov}_X \cup \text{EssFin}(\tau(\text{Op}_X)) \). Suppose that \( Y \in \text{Sm}(X) \). Since each family from \( \Psi \) is essentially finite on \( Y \), we infer from Proposition 9.15 that \( Y \in \text{Sm}(X_{pt}) \).

**Definition 9.18.** A generalized bornological universe is an ordered pair \( ((X, \text{Op}_X, \text{Cov}_X), \mathcal{B}) \) such that \( (X, \text{Op}_X, \text{Cov}_X) \) is a gts, while \( \mathcal{B} \) is a bornology in \( X \).

**Definition 9.19** (cf. Proposition 2.2.71 of \([P1]\)). Let \( \text{Op}_X \) be a complete ring of subsets of a set \( X \). Then:

(i) for a collection \( \mathcal{B} \subseteq \mathcal{P}(X) \), we define \( \text{EF}(\text{Op}_X, \mathcal{B}) = \{ U \subseteq \text{Op}_X : \forall A \in \mathcal{B} \{ A \cap U : U \in U \} \in \text{EssFin}(\mathcal{P}(A)) \} \);
(ii) for a topology \( \tau \) in \( X \) and for a bornology \( \mathcal{B} \) in \( X \), the gts induced by the bornological universe \( ((X, \tau), \mathcal{B}) \) is the triple \( \text{gts}((X, \tau), \mathcal{B}) = (X, \tau, \text{EF}(\tau, \mathcal{B})) \).

In the light of the proof to Proposition 2.1.31 in [P2], we have the following fact:

**Fact 9.20.** Suppose that \( ((X, \tau), \mathcal{B}) \) is a bornological universe such that \( \tau \cap \mathcal{B} \) is a base for \( \mathcal{B} \). Then \( \text{Sm}((X, \tau, \text{EF}(\tau, \mathcal{B}))) = \mathcal{B} \).

**Definition 9.21** (cf. Example 2.1.12 of [P2]). For a (quasi)-metric \( d \) on a set \( X \), the triple \( (X, \tau(d), \text{EF}(\tau(d), \mathcal{B}_d(X))) \) will be called the gts induced by the (quasi)-metric \( d \).

**Fact 9.22** (cf. Example 2.1.12 of [P2]). If \( d \) is a quasi-metric on a set \( X \), then \( \text{EF}(\tau(d), \mathcal{B}_d(X)) \) is a generalized topology in \( X \) and

\[ \text{Sm}((X, \text{EF}(\tau(d), \mathcal{B}_d(X)))) = \mathcal{B}_d(X). \]

10 \( \mathcal{B} \)-(quasi)-metrization of gtses

**Definition 10.1.** Suppose that \( (X, \mathcal{B}) \) is a generalized bornological universe. Then we say that the gts \( X \) is \( \mathcal{B} \)-(quasi)-metrizable or (quasi)-metrizable with respect to \( \mathcal{B} \) if the bornological universe \( (X_{\text{top}}, \mathcal{B}) \) is (quasi)-metrizable.

**Definition 10.2.** Let \( X \) be a gts and let \( S \) be either \( \text{CB} \) or \( \text{ACB} \), or \( \text{Sm} \). Then we say that \( X \) is \( S \)-(quasi)-metrizable if \( X \) is (quasi)-metrizable with respect to \( S(X) \).

With Proposition 9.17 in hand, we can immediately deduce that the following proposition holds:

**Proposition 10.3.** Let \( X \) be a gts and let \( S \) be either \( \text{CB} \) or \( \text{Sm} \). Then \( X \) is \( S \)-(quasi)-metrizable if and only if \( X_{\text{pt}} \) is \( S \)-(quasi)-metrizable.

**Remark 10.4.** If \( X \) is a gts, then the \( \text{ACB} \)-(quasi)-metrizability of \( X_{\text{pt}} \) is the (quasi)-metrizability of \( X_{\text{pt}} \) with respect to \( \text{ACB}(X_{\text{pt}}) \), while the \( \text{ACB} \)-quasi-metrizability of \( X \) is equivalent to the (quasi)-metrizability of \( X_{\text{pt}} \) with respect to \( \text{ACB}(X) \). We do not know whether the \( \text{ACB} \)-(quasi)-metrizability of \( X \) is equivalent to the \( \text{ACB} \)-(quasi)-metrizability of \( X_{\text{pt}} \).
Definition 10.5. A gts $X = (X, \text{Op}_X, \text{Cov}_X)$ is called:

(i) **locally small** if there exists $U \in \text{Cov}_X$ such that $U \subseteq \text{Sm}(X)$ and $X = \bigcup U$ (cf. Definition 2.1.1 of [P2]);

(ii) **weakly locally small** if there exists a collection $U \subseteq \text{Op}_X \cap \text{Sm}(X)$ such that $X = \bigcup U$.

Our next theorem says about the form of the partial topologization of an $\text{Sm}$-(quasi)-metrizable gts $X$ when $X_{pt}$ is locally small.

Theorem 10.6. Suppose that $X = (X, \text{Op}, \text{Cov})$ is a gts such that its partial topologization $X_{pt} = (X, \text{Op}_{pt}, \text{Cov}_{pt})$ is locally small. Then the following conditions are equivalent:

(i) $X$ is $\text{Sm}$-(quasi)-metrizable;

(ii) $X_{pt}$ is induced by some (quasi)-metric $d$.

Proof. In view of Proposition 9.17, we have $\text{Sm}(X) = \text{Sm}(X_{pt})$. In consequence, it is obvious that if $X_{pt}$ is induced by a (quasi)-metric $d$, then $X$ is $\text{Sm}$-(quasi)-metrizable. Assume that $X$ is $\text{Sm}$-(quasi)-metrizable and that $d$ is a (quasi)-metric on $X$ such that $\tau(\text{Op}) = \tau(d)$ and $\text{Sm}(X_{pt})$ is the collection of all $d$-bounded sets. Since $X_{pt}$ is locally small, it follows from Proposition 2.1.18 of [P2] that $X_{pt}$ is induced by $d$. 

Fact 10.7. If a gts $X$ is induced by a (quasi)-metric, then $X$ is locally small and partially topological.

Fact 10.8. (i) If $X$ is a locally small gts, then $X_{pt}$ is locally small.

(ii) If a gts $X$ is such that $X_{pt}$ is locally small, then $X$ is weakly locally small.

(iii) A gts $X$ is weakly locally small if and only if $X_{pt}$ is weakly locally small.

We are going to present a pair of weakly locally small but not locally small gtses. For $\Psi \subseteq \mathcal{P}^2(X)$, we put $\Psi_0 = \Psi$ and, for $n \in \omega$, assuming that the collection $\Psi_n \subseteq \mathcal{P}^2(X)$ has been defined, we put $\Psi_{n+1} = (\Psi_n)^+$ where $^+$ is the operator described in the proof of Proposition 2.2.37 in [P1]. Then $\langle \Psi \rangle_X = \bigcup_{n \in \omega} \Psi_n$. The symbols $\bigcup_1, \cap_1, \bigcup_2, \cap_2$ have the same meaning as in [P1].
Example 10.9. [ZF + CC]. Suppose that $Y$ is an uncountable set. For
$n \in \omega$, we put $Y_n = Y \times \{n\}$. Let $X = \bigcup_{n \in \omega} Y_n$, $O_p X = \text{FB}(X) \cup \{X\}$
and $\text{Cov}_X = \text{EF}(\text{Op}_X, \{Y_n : n \in \omega\})$. The gts $X = (X, \text{Op}_X, \text{Cov}_X)$ is
weakly locally small and not small. If $X$ were locally small, then $Y_0$ would
be a subset of a small open set (Fact 2.1.21 in $\text{P2}$), so $Y_0$ would be finite.
Hence, $X$ is not locally small. We have $\{Y_n : n \in \omega\} \in \text{EF}(\text{Op}_X, \{Y_n : n \in \omega\})$ and all the sets $Y_n$ are small and open in $(X, \text{EF}(\text{Op}_X, \{Y_n : n \in \omega\}))$, so the gts $(X, \text{EF}(\text{Op}_X, \{Y_n : n \in \omega\}))$ is locally small. We put
$\Psi = \text{Cov}_X \cup \text{EssFin}(\text{Op}_X)$. Then $\text{pt}(\text{Cov}_X) = \langle \Psi \rangle_X$ is the generalized
topology of $X_{pt}$. By Proposition 9.15, $\langle \Psi \rangle_X \subseteq \text{EF}(\text{Op}_X, \{Y_n : n \in \omega\})$.
Surprisingly, if CC holds, then $X_{pt}$ is not locally small and, in consequence,
$\langle \Psi \rangle_X \not\subseteq \text{EF}(\text{Op}_X, \{Y_n : n \in \omega\})$. To prove this, let us assume $\text{ZF + CC}$.
It is easy to observe the following facts:

Fact 1. $X \notin [X]^{\leq \omega} \cup_1 \text{Sm}(X)$.

Fact 2. Each $\Psi_n$ is closed with respect to restriction: $\Psi_n \cap_2 A \subseteq \Psi_n$ for
$A \subseteq X$.

For $\mathcal{W} \subseteq \mathcal{P}(X)$, let us consider the following property:

$\text{P}(\mathcal{W})$: $\mathcal{W}$ has an uncountable member and $\mathcal{W} \subseteq [X]^{\leq \omega} \cup_1 \text{Sm}(X)$.

For $n \in \omega$, let $T(n)$ be the statement:

$T(n)$: if $\mathcal{W} \in \Psi_n$ has $\text{P}(\mathcal{W})$, then $\mathcal{W}$ is essentially finite on $X \setminus A$ for
some countable $A \subseteq X$.

We are going to prove by induction that the following fact holds:

Fact 3. $T(n)$ is true for each $n \in \omega$.

Proof. Let $\mathcal{W} \in \Psi_0$ have property $\text{P}(\mathcal{W})$. Then, by Fact 1, $X \notin \mathcal{W}$. Thus
$\mathcal{W} \in \text{EssFin}(\text{Op}_X)$. Hence $T(0)$ holds. Suppose that $T(n)$ is true. The
finiteness, stability, and regularity induction steps from the proof of Propo-
sition 2.2.37 in $\text{P1}$ are obvious.

Transitivity step. Let $\mathcal{W} \in \Psi_{n+1}$ have property $\text{P}(\mathcal{W})$. Suppose that
$\mathcal{U} \in \Psi_n$ and $\{\mathcal{V}(U) : U \in \mathcal{U}\} \subseteq \Psi_n$ are such that $\mathcal{W} = \bigcup_{U \in \mathcal{U}} \mathcal{V}(U)$ and,
for each $U \in \mathcal{U}$, we have $U = \bigcup \mathcal{V}(U)$. Consider any $U \in \mathcal{U}$. If every
member of $\mathcal{V}(U)$ is countable, then $U \in [X]^{\leq \omega}$ because CC holds and $\mathcal{V}(U)$
is essentially countable. Suppose $\mathcal{V}(U)$ has an uncountable member. Since
$\mathcal{V}(U)$ has property $\text{P}(\mathcal{V}(U))$, it follows from the inductive assumption that
there is a countable set $A(U) \subseteq X$ such that $\mathcal{V}(U)$ is essentially finite on
$X \setminus A(U)$. Then $U \in [X]^{\leq \omega} \cup_1 \text{Sm}(X)$ and $U$ is uncountable. The above
implies that $\mathcal{U}$ has property $\text{P}(\mathcal{U})$. By the assumption, there is a countable
$A \subseteq X$ such that $\mathcal{U}$ is essentially finite on $X \setminus A$. Let $\mathcal{U}^* \subseteq \mathcal{U}$ be a finite family
such that \( \bigcup \mathcal{U} \setminus A = \bigcup \mathcal{U} \setminus A \). For each \( U \in \mathcal{U}^* \), the set \( U \) is countable or \( \mathcal{V}(U) \) is essentially finite on \( U \setminus A(U) \). This implies that there is a countable \( A(\mathcal{W}) \) such that \( \mathcal{W} \) is essentially finite on \( X \setminus A(\mathcal{W}) \).

**Saturation step.** Suppose that there exists \( V \in \Psi_n \) such that \( \bigcup \mathcal{V} = \bigcup \mathcal{W} \) and, for each \( V \in \mathcal{V} \), there is \( W(V) \in \mathcal{W} \) such that \( V \subseteq W(V) \). Since \( \mathcal{W} \subseteq [X]^{\leq \omega} \cup_1 \text{Sm}(X) \), we have \( \mathcal{V} \subseteq [X]^{\leq \omega} \cup_1 \text{Sm}(X) \). Since \( \mathcal{W} \) has an uncountable member and \( \mathcal{V} \) is essentially countable, also \( \mathcal{V} \) has an uncountable member and has property \( \mathbf{P}(\mathcal{V}) \). By the inductive assumption, there exists a countable \( A(\mathcal{V}) \) such that \( \mathcal{V} \) is essentially finite on \( X \setminus A(\mathcal{V}) \). Then \( \mathcal{W} \) is essentially finite on \( X \setminus A(\mathcal{V}) \), too.

Suppose that \( X_{pt} \) is locally small. There exists \( W \in pt(\text{Cov}_X) \) such that \( \mathcal{W} \subseteq \text{Sm}(X) \) and \( X = \bigcup \mathcal{W} \). Since \( X \) is uncountable and \( \mathcal{W} \) is essentially countable, at least one member of \( \mathcal{W} \) is uncountable, so \( \mathbf{P}(\mathcal{W}) \) holds true. By Fact 3, there exists a countable \( A(\mathcal{W}) \) such that \( \mathcal{W} \) is essentially finite on \( X \setminus A(\mathcal{W}) \). Then \( X \setminus A(\mathcal{W}) \in \text{Sm}(X) \). This is impossible by Fact 1.

The example above is not a solution to the following open problem:

**Problem 10.10.** Is it true in \( \text{ZF} \) that if the partial topologization of a gts \( X \) is locally small, then so is \( X \)?

**Proposition 10.11.** Suppose that \( X = (X, \text{Op}_X, \text{Cov}_X) \) is a gts and \( \mathcal{B} \) is a bornology in \( X \). Then the following conditions are equivalent:

\[
(i) \text{ the gts } X \text{ is (quasi)-metrizable with respect to } \mathcal{B}; \\
(ii) \text{ the gts } (X, \text{EF}(\tau(\text{Op}_X), \mathcal{B})) \text{ is Sm-(quasi)-metrizable and } \tau(\text{Op}_X) \cap \mathcal{B} \text{ is a base for } \mathcal{B}. 
\]

**Proof.** Assume that \( (i) \) holds. Then, by Theorem 4.7, the collection \( \tau(\text{Op}_X) \cap \mathcal{B} \) is a base for \( \mathcal{B} \). It follows from Fact 9.20 that \( \mathcal{B} = \text{Sm}((X, \text{EF}(\tau(\text{Op}_X), \mathcal{B}))) \). In consequence, \( (i) \) implies \( (ii) \). On the other hand, we can use Fact 9.20 with both Definitions 9.19 and 10.1 to infer that \( (i) \) follows from \( (ii) \). \qed

**Definition 10.12.** Suppose that \( (X, \mathcal{B}) \) is a generalized bornological universe where \( X = (X, \text{Op}_X, \text{Cov}_X) \). Let us say that \( X \) is **strongly \( \mathcal{B} \)-(quasi)-metrizable** if there exists a (quasi)-metric \( d \) on \( X \) such that \( \mathcal{B} \) is the collection of all \( d \)-bounded sets and \( \text{Op}_X = L_X[\{B_d(x, r) : x \in X \land r \in (0; +\infty)]}. \)
Definition 10.13. A (quasi)-metric gts is an ordered pair \((X, d)\) where \(X = (X, O_pX, CovX)\) is a gts and \(d\) is a (quasi)-metric in \(X\) such that \(\tau(d) = \tau(O_pX)\).

Definition 10.14. Suppose that \((X, d)\) is a (quasi)-metric gts and that \(\mathcal{B}\) is a bornology in \(X\). We say that \((X, d)\) is uniformly \(\mathcal{B}\)-(quasi)-metrizable or uniformly (quasi)-metrizable with respect to \(\mathcal{B}\) if the bornology \(\mathcal{B}\) is uniformly (quasi)-metrizable with respect to \(d\).

Remark 10.15. For a bornology \(\mathcal{B}\) in a gts \(X\), one can find results in the previous sections that deliver necessary and sufficient conditions for \(X\) to be (quasi)-metrizable with respect to \(\mathcal{B}\) (see Theorems 4.7 and 4.15, as well as Corollaries 4.10 and 4.16) and for a metric gts \((X, d)\) to be uniformly (quasi)-metrizable with respect to \(\mathcal{B}\) (see Theorems 6.5 and 8.5).

Let us use the real lines described in Definition 1.2 of \([PW]\) as our illuminating examples for the notions of (uniform) \(\mathcal{B}\)-(quasi)-metrizability in the category GTS.

Example 10.16. Let \(\tau_{nat}\) be the natural topology in \(\mathbb{R}\). For \(x, y \in \mathbb{R}\), we put \(d_n(x, y) = |x - y|, d_{n,1}(x, y) = \min\{d_n(x, y), 1\}\) and

\[
d_n^+(x, y) = d_n(\Phi(x), \Phi(y)) \text{ where } \Phi(x) = \begin{cases} e^x, & x < 0, \\ 1 + x, & x \geq 0. \end{cases}
\]

Moreover, we define \(d_{n,1}^+(x, y) = \min\{d_n^+(x, y), 1\}\). Let us observe that the metrics \(d_n\) and \(d_n^+\) are equivalent but not uniformly equivalent.

(i) We have \(\mathcal{B}_{d_n}(\mathbb{R}) = \mathbb{C}B_{\tau_{nat}}(\mathbb{R})\) and \(\mathcal{B}_{d_n^+}(\mathbb{R}) = \mathbb{U}B(\mathbb{R})\). Let us observe that, for a fixed \(\delta \in (0; +\infty)\), there exists \(n(\delta) \in \omega\) such that if \(C_m = [-m; m]\) for \(m \in \omega\) with \(m > n(\delta)\), then \((-\infty; m) \subseteq [C_m]_{d_n^+}\). This, together with Theorem 6.5, implies that \(\mathcal{B}_{d_n}(\mathbb{R})\) is not uniformly quasi-metrizable with respect to \(d_n^+\).

(ii) For the usual topological real line \(\mathbb{R}_{ut}\) (cf. Definition 1.2(i) of \([PW]\)), we have \(\mathbf{FB} = \mathbf{S}m \subseteq \mathbf{C}B = \mathbf{A}CB\) and \(\text{int}_{\tau_{nat}}A = \emptyset\) for each \(A \in \mathbf{S}m(\mathbb{R}_{ut})\), so the gts \(\mathbb{R}_{ut}\) is not \(\mathbf{S}m\)-quasi-metrizable and it is \(\mathbf{A}CB\)-metrizable by \(d_n\). The metric gtses \((\mathbb{R}_{ut}, d_n)\) and \((\mathbb{R}_{ut}, d_{n,1})\) are \(\mathbf{A}CB\)-uniformly metrizable. It follows from (i) that the metric gtses \((\mathbb{R}_{ut}, d_n^+)\) is \((\mathbb{R}_{ut}, d_{n,1}^+)\) are not uniformly \(\mathbf{A}CB\)-quasi-metrizable.
(iii) For the real lines $\mathbb{R}_{lst}$ and $\mathbb{R}_{lom}$ (cf. Definition 1.2(iv)-(v) of [PW]), we have $pt(\mathbb{R}_{lom}) = \mathbb{R}_{lst}$ and $Sm = CB = ACB = B_{d_n}(\mathbb{R})$. The metric gtses $(\mathbb{R}_{lst}, d_n)$ and $(\mathbb{R}_{lom}, d_n)$ are both uniformly $Sm$-metrizable; however, none of the metric gtses $(\mathbb{R}_{lom}, d_n^+)$ and $(\mathbb{R}_{lst}, d_n^+)$ is uniformly $Sm$-metrizable (see (i)).

(iv) For the real lines $\mathbb{R}_{l+om}$ and $\mathbb{R}_{l+st}$ (cf. Definition 1.2(vii)-(viii) of [PW]), we have $pt(\mathbb{R}_{l+om}) = \mathbb{R}_{l+st}$ and $CB = CB_{\tau_{nat}}(\mathbb{R}) \subset Sm = ACB = B_{d_n}(\mathbb{R})$. Now, it is obvious that both the metric gtses $(\mathbb{R}_{l+om}, d_n^+)$ and $(\mathbb{R}_{l+st}, d_n^+)$ are uniformly $ACB$-metrizable by the metric $d_n^+$. The gtses $\mathbb{R}_{l+om}$ and $\mathbb{R}_{l+st}$ are $Sm$-metrizable. The metric gtses $(\mathbb{R}_{l+om}, d_n)$ and $(\mathbb{R}_{l+st}, d_n)$ are uniformly $Sm$-metrizable and uniformly $ACB$-metrizable by $d_u(x, y) = d_{n,1}(x, y) + |\max(y, 0) - \max(x, 0)|$.

(v) Let us consider the gtses $\mathbb{R}_{om}, \mathbb{R}_{slom}, \mathbb{R}_{rom}$ and $\mathbb{R}_{st}$ (cf. Definition 1.2(ii),(iii), (vi) and (x) of [PW]). We have $pt(\mathbb{R}_{om}) = pt(\mathbb{R}_{slom}) = pt(\mathbb{R}_{rom}) = \mathbb{R}_{st}$ and $CB \subset Sm = ACB = \mathcal{P}(\mathbb{R})$. The real lines $\mathbb{R}_{om}, \mathbb{R}_{slom}, \mathbb{R}_{rom}$ and $\mathbb{R}_{st}$ are $Sm$-metrizable by the metric $d_{n,1}$ and they are $CB$-metrizable by the metric $d_n$.

(vi) The gts $\mathbb{R}_{om}$ (cf. Definition 1.2(ii) of [PW]) is strongly $Sm$-metrizable by $d_{n,1}$.

In connection with strong $Sm$-(quasi)-metrizability, let us pose the following open problem:

**Problem 10.17.** Find useful simultaneously necessary and sufficient conditions for a gts to be strongly $Sm$-(quasi)-metrizable.

It might be helpful to have a look at several simple examples of gtses of type $(X, EF(\tau, \mathcal{B}))$ and compare them with Proposition 10.11.

**Example 10.18. (Gtses from the Sorgenfrey line.)** Let us use the topologies $\tau_{S,r}$ and $\tau_{S,l}$ considered in Example 4.12, as well as the quasi-metrics $\rho_S, \rho_{S,1}$ and $\rho_L$ defined in Example 4.12.

(i) The gts $(\mathbb{R}, EF(\tau_{S,r}, CB_{\tau_{nat}}(\mathbb{R})))$ is $Sm$-quasi-metrizable by the quasi-metric $\rho_0$ defined as follows:

$$
\rho_0(x, y) = \begin{cases} y - x, & x \leq y \\ 1 + x - y, & x > y. \end{cases}
$$
(ii) The gts \((\mathbb{R}, EF(\tau_{S,r}, UB(\mathbb{R})))\) is \(S_{\text{m}}\)-quasi-metrizable by \(\rho_S\), while the gts \((\mathbb{R}, EF(\tau_{S,r}, LB(\mathbb{R})))\) is \(S_{\text{m}}\)-quasi-metrizable by \(\rho_L\).

(iii) The gts \((\mathbb{R}, EF(\tau_{S,r}, \mathcal{P}(\mathbb{R})))\) is \(S_{\text{m}}\)-quasi-metrizable by \(\rho_{S,1}\).

(iv) It follows from Theorem 4.7 that the gtses \((\mathbb{R}, EF(\tau_{S,r}, FB(\mathbb{R})))\) and \((\mathbb{R}, EF(\tau_{nat}, FB(\mathbb{R})))\) are not \(S_{\text{m}}\)-quasi-metrizable because \(\tau_{S,r} \cap FB(\mathbb{R})\) is not a base for \(FB(\mathbb{R})\).

**Example 10.19.** (Quasi-metric gtses from the Sorgenfrey line.) We use the same notation as in Example 10.18.

(i) The quasi-metric gts \(((\mathbb{R}, EF(\tau_{S,r}, CB_{\tau_{nat}}(\mathbb{R}))), \rho_S)\) is uniformly \(S_{\text{m}}\)-quasi-metrizable by \(\rho_0\).

(ii) The quasi-metric gts \(((\mathbb{R}, EF(\tau_{S,r}, UB(\mathbb{R}))), \rho_0)\) is uniformly \(S_{\text{m}}\)-quasi-metrizable by \(\rho_S\), while the quasi-metric gts \(((\mathbb{R}, EF(\tau_{S,r}, LB(\mathbb{R}))), \rho_0)\) is uniformly \(S_{\text{m}}\)-quasi-metrizable by \(\rho_L\).

(iii) The quasi-metric gts \(((\mathbb{R}, EF(\tau_{S,r}, \mathcal{P}(\mathbb{R}))), \rho_0)\) is uniformly \(S_{\text{m}}\)-quasi-metrizable by \(\min\{\rho_0, 1\}\).

**Example 10.20.** Let us put \(J = [0; 1] \times \{0\} \) and \(J_q = \{q\} \times [0; 1]\). For \(S = [0; 1] \cap \mathbb{Q}\), let \(X = J \cup \bigcup_{q \in S} J_q\). We consider the collection \(\mathcal{B}\) of all sets \(A \subseteq X\) that have the property: there exists a finite \(S(A) \subseteq S\) such that \(A \subseteq J \cup \bigcup_{q \in S(A)} J_q\).

(i) Let \(d_e\) be the Euclidean metric in \(X\). Then, for each \(A \in \mathcal{B}\), we have \(\text{int}_{\tau_{d_e}} A = \emptyset\), so, for every topology \(\tau_2\) in \(X\), the bornology \(\mathcal{B}\) is not \((\tau_{d_e}, \tau_2)\)-proper. In consequence, the gts \((X, EF(\tau(d_e), \mathcal{B}))\) is not \(S_{\text{m}}\)-quasi-metrizable.

(ii) We define another metric \(\rho\) in \(X\) as follows. For \(x, y \in [0; 1]\) and \(q, q' \in S\) with \(q \neq q'\), we put \(\rho((x, 0), (y, 0)) = |x - y|, \rho((q, x), (q, y)) = |x - y|\) and \(\rho((q, x), (q', y)) = x + |q - q'| + y\). Then, for each \(q \in S\) and for any \(a, b \in [0; 1]\) with \(a < b\), we have \(\{q\} \times (a; b) = \text{int}_{\tau_{\rho}} [\{q\} \times (a; b)] \in \mathcal{B}\). Since there does not exist \(A \in \mathcal{B}\) such that \(J \subseteq \text{int}_{\tau_{\rho}} A\), we deduce that the gts \((X, EF(\tau(\rho), \mathcal{B}))\) is not \(S_{\text{m}}\)-quasi-metrizable. The space \((X, \tau(\rho))\) can be called the comb with its hand \(J\) and teeth \(J_q\), \(q \in \mathbb{Q}\) (compare with Example IV.4.7 of [Kn]).
Remark 10.21. One can easily reformulate Theorems 4.7 and 4.15 to get simultaneously necessary and sufficient conditions for a bornological bimiverse to be quasi-pseudometrizable. One can also use quasi-pseudometrics instead of quasi-metrics in Theorem 6.5 to obtain conditions equivalent with the uniform quasi-pseudometrizability of a bornology with respect to a given quasi-pseudometric.

Example 10.22. The topological space \((\mathbb{R}, u)\) is not quasi-metrizable (since it is not \(T_1\)) but it is quasi-pseudometrizable by \(\rho_u\) (see Section 6).

(i) The gts \((\mathbb{R}, EF(u, UB(\mathbb{R})))\) is \(\text{Sm}\)-quasi-pseudometrizable by \(\rho_u\).

(ii) For the gts \(\mathcal{R}_{ul} = (\mathbb{R}, EF(u, LB(\mathbb{R})))\) we have \(\text{Sm}(\mathcal{R}_{ul}) = \mathcal{P}(\mathbb{R})\). This is why \(\mathcal{R}_{ul}\) is \(\text{Sm}\)-quasi-pseudometrizable by \(\rho_{u,1} = \min\{1, \rho_u\}\).

(iii) For the gts \(\mathcal{R}_{ub} = (\mathbb{R}, EF(u, UB(\mathbb{R}) \cap LB(\mathbb{R})))\) we have \(\text{Sm}(\mathcal{R}_{ub}) = UB(\mathbb{R})\). This is why \(\mathcal{R}_{ub}\) is \(\text{Sm}\)-quasi-pseudometrizable by \(\rho_u\).

(iv) The gts \(\mathcal{R}_{uf} = (\mathbb{R}, EF(u, FB(\mathbb{R})))\) is not \(\text{LB}(\mathbb{R})\)-quasi-pseudometrizable because \(\text{int}_u A = \emptyset\) for each \(A \in \text{LB}(\mathbb{R})\). Here \(\text{Sm}(\mathcal{R}_{uf})\) is the collection of all sets \(A \in UB(\mathbb{R})\) such that every non-empty subset of \(A\) has a maximal element. Similarly, \(\mathcal{R}_{uf}\) is not \(\text{Sm}\)-quasi-pseudometrizable. Since \(\text{ACB}(\mathcal{R}_{uf}) = \text{CB}(\mathcal{R}_{uf}) = UB(\mathbb{R})\), the gts \(\mathcal{R}_{uf}\) is \(\text{ACB}\)-quasi-pseudometrizable by \(\rho_u\).

11 New topological categories

The table of categories in [AHS], among other categories, says about the category \(\text{Top}\) of topological spaces, the category \(\text{BiTop}\) of bitopological spaces and about the category \(\text{Bor}\) of bornological sets. The categories \(\text{GTS}, \text{GTS}_{pt}, \text{SS}\) of small generalized topological spaces and \(\text{LSS}\) of locally small generalized topological spaces, as well as \(\text{SS}_{pt}\) and \(\text{LSS}_{pt}\), were introduced in [P1] and [P2]. We pointed out in [PW] that, while working with categories and proper classes, a modification of \(\text{ZF}\) is required. We assume a suitably modified version of \(\text{ZF}\) suggested in [PW].

In the light of Proposition 9.17 and Fact 10.8, we can state the following:

Fact 11.1. The functor \(\text{pt}\) of partial topologization preserves smallness and local smallness. More precisely:
(i) pt restricted to SS maps SS onto SS\textsubscript{pt};

(ii) pt restricted to LSS maps LSS onto LSS\textsubscript{pt}.

All the categories Top, BiTop, GTS, GTS\textsubscript{pt}, SS, SS\textsubscript{pt} and Bor are topological constructs (cf. [AHS], [Sal], [P1], [P2], [PW] and [H-N]). Since Top and Bor are topological constructs, it is obvious that the category Ubor of bornological universes (cf. Remark 2.2.70 of [P1]) is a topological construct, too. Let us define several more categories and answer the question whether they are topological constructs.

**Definition 11.2** (cf. 1.2.1 in [H-N]). Let $B_X$ be a boundedness in a set $X$ and let $B_Y$ be a boundedness in a set $Y$. We say that a mapping $f : X \to Y$ is $(B_X, B_Y)$-bounded (in abbreviation: bounded) if, for each $A \in B_X$, we have $f(A) \in B_Y$.

**Definition 11.3.** Suppose that $((X, \tau^X_1, \tau^X_2), B_X)$ and $((Y, \tau^Y_1, \tau^Y_2), B_Y)$ are bornological biuniverses. We say that a mapping $f : X \to Y$ is a bounded bicontinuous mapping from $((X, \tau^X_1, \tau^X_2), B_X)$ to $((Y, \tau^Y_1, \tau^Y_2), B_Y)$ if $f$ is bicontinuous with respect to $(\tau^X_1, \tau^X_2, \tau^Y_1, \tau^Y_2)$ and $f$ is $(B_X, B_Y)$-bounded.

**Definition 11.4.** Suppose that $((X, \text{Cov}_X), B_X)$ and $((Y, \text{Cov}_Y), B_Y)$ are generalized bornological universes. We say that a mapping $f : X \to Y$ is a bounded strictly continuous mapping from $((X, \text{Cov}_X), B_X)$ to $((Y, \text{Cov}_Y), B_Y)$ if $f$ is both $(B_X, B_Y)$-bounded and $(\text{Cov}_X, \text{Cov}_Y)$-strictly continuous.

**Definition 11.5.** A generalized bornological universe $((X, \text{Cov}_X), B)$ is called:

(i) partially topological if the gts $(X, \text{Cov}_X)$ is partially topological;

(ii) small if the gts $(X, \text{Cov}_X)$ is small.

**Definition 11.6.** We define the following categories:

(i) BiUBor where objects are bornological biuniverses and morphisms are bounded bicontinuous mappings;

(ii) GeUBor where objects are generalized bornological universes and morphisms are bounded strictly continuous mappings;
(iii) \( \text{Ge}_{\text{pt}} \text{UBor} \) where objects are partially topological generalized bornological universes and morphisms are bounded strictly continuous mappings;

(iv) \( \text{SmUBor} \) where objects are small generalized bornological universes and morphisms are bounded strictly continuous mappings;

(v) \( \text{Sm}_{\text{pt}} \text{UBor} \) where objects are partially topological small generalized bornological universes and morphisms are bounded strictly continuous mappings.

**Proposition 11.7.** The categories defined in 11.6 are all topological constructs.

**Proof.** To check that, for instance, \( \text{Ge}_{\text{pt}} \text{UBor} \) is a topological construct, we mimic the proof to Theorem 4.4 of [PW]. Namely, let us consider a source \( F = \{ f_i : i \in I \} \) of mappings \( f_i : X \to Y_i \) indexed by a class \( I \) where every \( Y_i \) is a partially topological generalized bornological universe and \( Y_i = (X_i, \text{Cov}_i, B_i) \). Let \( \text{Cov}_X \) be the \( \text{GTS} \)-initial generalized topology for \( F \) in \( X \) (cf. Definition 4.3 of [PW]) and let \( B_X = \bigcap_{i \in I} \{ A \subseteq X : f_i(A) \in B_i \} \).

For \( X = ((X, \text{Cov}_X), B_X) \), let \( X_{\text{pt}} = (\text{pt}((X, \text{Cov}_X)), B_X) \). The canonical morphism \( \text{id} : X_{\text{pt}} \to X \) is such that all mappings \( f_i \circ \text{id} \) are morphisms in \( \text{Ge}_{\text{pt}} \text{UBor} \). For any object \( Z \) of \( \text{Ge}_{\text{pt}} \text{UBor} \) and a mapping \( h : Z \to X_{\text{pt}} \), we can observe that if all \( f_i \circ \text{id} \circ h \) with \( i \in I \) are morphisms, then \( \text{id} \circ h \) is a morphism of \( \text{GTS} \), so \( \text{pt}(h) = h \) is a morphism of \( \text{GTS}_{\text{pt}} \). If all \( f_i \circ \text{id} \circ h \) are bounded, then \( \text{pt}(h) = h \) is bounded, too.

That \( \text{BiUBor}, \text{GeUBor}, \text{SmUBor} \) and \( \text{Sm}_{\text{pt}} \text{UBor} \) are topological can be proved by using more or less similar arguments. \( \square \)

Some other topological constructs relevant to bornologies or to quasi-pseudometrics were considered in [CL] and [Vr1].

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