Abstract:

We discuss the physics of the $3 + 1$ dimensional $\lambda \Phi^4$ quantum field theory in terms of the statistical mechanics of a gas of particles ('atoms') that interact via a $-1/r^3$-plus-hard-core potential. The hard-core potential, $\delta^{(3)}(r)$, arises from the bare vertex diagram, while the attractive, long-range $-1/r^3$ potential is due to exchange of a particle pair via the $t, u$-channel “fish” diagram. (Higher-order diagrams preserve this form of the interparticle potential.) For sufficiently small atom mass, the lowest-energy state is not the ‘empty’ state with no atoms, but a state with a non-zero density of spontaneously created atoms, Bose-condensed in the zero-momentum mode. This corresponds to the spontaneous-symmetry-breaking phase transition, and the ‘phonon’ excitations of the Bose condensate correspond to Higgs particles. The important point is that the phase transition happens while the atom’s physical mass $m$ is still positive: it does not wait until $m^2$ passes through zero and becomes negative, contrary to the assumption of a second-order transition, on which renormalization-group-improved perturbation theory is based.
1. Outline

In this paper we consider the physics of \((\lambda \Phi^4)_{3+1}\) theory from a “particle” viewpoint – in the same sense that quantum electrodynamics (QED) can be viewed, not as an abstract field theory, but as a theory of charged particles interacting via a \(1/r\) potential. Our aim is to gain some really ‘earthy’ intuition about the physics. We begin by outlining the whole story, the component parts of which will be discussed in the remainder of the paper.

It is well known that the non-relativistic (NR) limit of \(\lambda \Phi^4\) theory corresponds to spinless particles that interact via a \(\delta(3)(r)\) potential \[1\]. Our key observation is that a qualitatively different potential arises from the exchange of a particle-antiparticle pair via the “fish” diagram. For massless exchanged particles this gives rise to an attractive, long-range, \(-1/r^3\) potential. (Including the mass yields an additional factor of \(2mrK_1(2mr)\) which supplies an \(\exp(-2mr)\) factor at large \(r\).) Thus, the net interparticle potential has a “\(-1/r^3\)-plus-hard-core” form. The hard core, a regularized form of the \(\delta(3)(r)\) term, has large positive potential for \(r < r_0\), where \(1/r_0\) acts as an ultraviolet cutoff.

We therefore consider the statistical mechanics of a dilute gas of \(N\) such bosons in a box of volume \(V\). Without the \(-1/r^3\) term this would be the classic hard-sphere Bose-gas problem \[2, 3, 4\]. There is interesting physics – Bose-Einstein condensation, corresponding to SSB from a symmetric phase with ‘atoms’ to a broken phase where ‘phonons’ are the excitations. The spectrum of the phonons differs from that of the atoms. With NR kinematics, where the atom spectrum is \(k^2/2m\), the phonons have a Bogoliubov spectrum \((k/2m)\sqrt{k^2 + 16\pi na}\), where \(a\) is the atoms’ scattering length and \(n\) is their number density, \(N/V\). Modifying the analysis to allow for relativistic kinematics yields a phonon spectrum that, for \(m\) small, is of normal relativistic form \(E(k) = \sqrt{k^2 + 8\pi na}\). The parameter

\[ M(n) = \sqrt{8\pi na} \]  

is then the phonon mass, and we can identify the ‘phonons’ as Higgs particles. The non-trivial change of spectrum survives even in the limit \(a \to 0\), provided that \(na\) remains finite \[5\]. Although this requires \(n \to \infty\), the gas is still \textit{dilute} because \(na^3\) vanishes; i.e., the spacing of the atoms is vastly greater than their scattering length. In such a limit the approximations \(na^3 \ll 1\) and \(ka \ll 1\) required by the analysis become exact. One then has a “trivial”-but-not-entirely trivial theory \[5\]: the phonons are exactly non-interacting, but are non-trivial coherent states of indefinitely many atoms, and their spectrum is manifestly not the original atom spectrum.

All of this assumes, however, that the system is constrained to have \(N\) particles (with
the interesting physics requiring $n$ large $\sim 1/a$). Relativistically, though, the ‘atoms’ can mutually annihilate, so the lowest-energy state would just be an empty box.

However, inclusion of the $-1/r^3$ long-range attraction changes the situation. It provides a potential-energy gain that could offset the rest-mass cost associated with $N$ particles and the energy cost of their mutual hard-core repulsion. In that case, an empty box could become unstable to spontaneous particle creation, leading to a new vacuum state with a non-zero density of ‘atoms’. We find that, indeed, this phenomenon does occur once the atom’s mass $m$ is sufficiently small.

The attractive potential gives a negative energy contribution $\frac{4}{3}N^2\bar{u}$, where $\bar{u}$ is the average potential energy of a pair. For a $-1/r^3$ potential $\bar{u}$ is logarithmically divergent, since one basically finds $\bar{u} \sim \int d^3r (-1/r^3)/V$. Obviously, the atomic mass $m$ would cut off the integral at $1/(2m)$, but if we make $m$ smaller and smaller then eventually a more important consideration is that the long-distance interaction between two particles will be “screened” by intervening particles. The effective $r_{\text{max}}$ will then involve the density $n = N/V$. Specifically, we argue that the two virtual particles being exchanged have to propagate through the background density of atoms and multiple collisions effectively convert their propagators from ‘atom’ to ‘phonon’ propagators. Thus, $r_{\text{max}}$ is not $1/(2m)$ but $1/(2M(n))$, once we are in a regime where $M(n)$ is much greater than $m$.

The energy of the gas at zero temperature (when almost all particles have zero 3-momentum) then consists of three terms (i) rest masses, (ii) repulsion energy (obtained in the hard-sphere Bose-gas analysis [2, 3, 4]), and (iii) energy gain from attraction:

$$E = Nm + c_1 \frac{Nna}{m} + \frac{4}{3}N^2\bar{u},$$  \hspace{1cm} (1.2)

where $c_1$ is a numerical coefficient and $\bar{u}$ is $1/V$ times the strength of the $-1/r^3$ potential (proportional to $-\lambda^2/m^2$), times $\ln(r_{\text{max}}^2/r_0^2) \sim -\ln(nar_0^2)$. Dividing by the volume, this gives an energy density for a state of density $n$ of the form

$$\mathcal{E} \equiv E/V = nm + c_1 \frac{n^2a}{m} + c_2 \frac{\lambda^2}{m^2}n^2\ln(nar_0^2),$$  \hspace{1cm} (1.3)

where $c_2$ is another numerical coefficient. The absence of $n^3$ or higher terms reflects the assumption that the gas is dilute ($na^3 \ll 1$), rendering 3-body interactions negligible. Indeed, the last term, resulting from pairwise attractive interactions, would have been a simple $n^2$ term (absorbable into a re-definition of $a$) if the long-range potential had fallen off faster than $-1/r^3$. As it is, the incipient divergence results in the $\ln n$ factor.

Upon minimizing $\mathcal{E}$ with respect to $n$, we see that, once $m$ is small enough, a state with non-zero density, $n_v$, will be energetically favoured over the ‘empty’ vacuum, $n = 0$. The vi-
tal point is that the phase transition is first-order; it happens while \( m^2 \) is still positive, and involves a discontinuous change in the density from 0 to \( n_v \). The usual ‘renormalization-group-improved perturbation theory’ picture assumes a second-order phase transition occurring only when \( m^2 \) (the renormalized mass-squared of the symmetric vacuum) reaches zero; i.e., when the \( n = 0 \) symmetric vacuum has tachyonic excitations, rendering it locally unstable.

The above description has a direct translation into quantum-field-theory (QFT) language; the density \( n \) corresponds to \( \frac{1}{2} m \phi^2 \), where \( \phi \) is the expectation value of the field, and the scattering length \( a \) is related to the coupling strength by \( a = \lambda/(8\pi m) \). (Hence, \( M^2(n) = 8\pi a n \) translates to \( \frac{1}{2} \lambda \phi^2 \), as expected.) The energy density as a function of density translates directly into the effective potential of QFT. It consists of a \( \frac{1}{2} m^2 \phi^2 \) term, a \( \lambda \phi^4 \) term, and a \( \lambda^2 \phi^4 \ln\left(\frac{1}{2} \lambda \phi^2 / \Lambda^2\right) \) term, where the ultraviolet cutoff \( \Lambda \) is proportional to \( 1/r_0 \).

The effective potential thus has the same form as the famous “one-loop” result \([6]\). Conventional wisdom holds that in \( \lambda \phi^4 \) theory the one-loop result is not reliable — and, indeed, from a loop-expansion perspective it cannot be justified. Nevertheless, as we have argued previously \([5],[7]—[11]\), this form is actually effectively exact. The particle-gas picture clarifies the reasons for this exactness and provides new physical insight.

In order for the Higgs mass \( M_h \equiv M(n = n_v) \) to be finite we need \( \lambda \) to tend to zero as \( 1/\ln(\Lambda/M_h) \) \([3],[6]—[11]\). For the phase transition to occur, the mass term \( nm = \frac{1}{2} m^2 \phi^2 \) must not dominate the other terms; this requires \( m^2 < \lambda M_h^2 \); i.e., \( m \) is infinitesimally small in the physical units defined by \( M_h \). It follows that \( a \) will tend to zero, giving “triviality.” The new ground state is still dilute (\( n_v a \) is finite, but \( a \to 0 \), so \( n_v a^3 \to 0 \)), and so the approximations involved all become exact. Moreover, since \( \lambda \) is of order \( 1/\ln \Lambda \), higher-order diagrams cannot materially alter the “\(-1/r^3\)-plus-hard-core” form of the interparticle potential (see Sect. 9). To incorporate all-orders effects we simply let ‘\( \lambda \)’ be the effective strength of the short-range repulsion (such that \( a = \lambda/(8\pi m) \) is the actual, not the Born-approximation, scattering length).

The “particle-gas” approach gives a clue to what goes wrong with the ‘renormalization-group-improved perturbation theory’ approach. That approach assumes that loop diagrams basically renormalize the strength of the \( \delta^{(3)}(r) \) vertex, turning the bare coupling into a running coupling. However, that is only part of the story; the \( t, u \)-channel “fish” diagram produces qualitatively different physics; long-range attraction rather than short-range repulsion. Higher orders then serve to renormalize both those interactions. It is as
though there were two operators in the game, not one.

2. Preliminaries

We now turn to a detailed exposition of the picture just outlined. We shall consider the single-component $\lambda \Phi^4$ theory, in which there is only a discrete symmetry $\Phi \rightarrow -\Phi$, and the particle is its own antiparticle. (Generalization to the $O(N)$ case should be straightforward.) The Hamiltonian is:

$$H = \int d^3x \left[ \frac{1}{2} (\dot{\Phi}^2 + (\nabla \Phi)^2 + m^2 \Phi^2) + \frac{\lambda}{4!} \Phi^4 \right],$$

(2.1)

where $\ldots$ indicates normal ordering so that the parameter $m$ is the physical mass of the ‘atoms.’ We use the term ‘atoms’ as shorthand for “particles of the symmetric phase” (i.e., elementary excitations above the $\langle \Phi \rangle = 0$ vacuum).

The aim of this paper is to gain physical insight into results previously obtained using field-theory formalism. What we are after is not a better calculation, but some visceral physical intuition. For calculational purposes the “field language” is preferable, but for intuitive insight the “particle language” is uniquely valuable. We basically seek the “translation” between these two different languages. The “particle language” is obviously natural for describing the NR limit of the theory, and in this sense we are following the lead of Bég and Furlong, Huang, and Jackiw [1]. However, we wish to use the particle language without necessarily invoking the NR limit. As always, this is somewhat problematic, since “relativistic quantum mechanics” is not a completely well-defined theoretical framework. We shall not worry overmuch about numerical coefficients, since it is the general form of the result that is the key point.

To illustrate the above remarks, we briefly mention a precedent in QED. Consider the electron-nucleus bound-state problem: the NR limit is simple in “particle” language — one just needs to solve the Schrödinger equation with a $-1/r$ potential. To incorporate relativistic kinematics and spin effects one can replace the Schrödinger equation with the Dirac equation for a $-1/r$ potential. However, that does not account for all relativistic effects; it omits effects due to virtual particle-antiparticle pairs, which give rise to the Lamb shift. However, even those effects can be patched into the particle language; one first calculates how virtual pairs modify the $-1/r$ potential at short distances, and then considers the additional potential term as a perturbation. Such a particle-language treatment, originally due to Bethe [12], captures the essential physics, though it is hard to make it into a fully satisfactory calculation (see [13], Sect. 8.7). While a field-language approach is
preferable for achieving a systematic calculation, the particle-language approach remains precious for the insight it provides.

The plan of the paper is as follows: Sect. 3 discusses the interparticle potential; Sect. 4 reviews the hard-sphere Bose gas and its “relativization”; and Sect. 5 considers the effect of including a $-1/r^3$ potential. These ingredients are combined to obtain the energy-density expression in Sect. 6. This is not meant to be a proper calculation; rather, it is a collage, cut-and-pasted from well-established sources, intended to reveal the essential physics. We shamelessly ignore complications that only affect numerical factors, such as identical-particle factors of 2.

Sect. 7 discusses the ‘translations’ relating the scattering length $a$ and density $n$ to the QFT quantities $\lambda$ and $\phi^2$. The translated results then yield the QFT effective potential. Sect. 8 examines the phase transition, and the continuum limit (how $m$ and $\lambda$ must scale in the cutoff $\rightarrow \infty$ limit, and what this means for $n$ and $a$). Sect. 9 shows that higher-order diagrams do not materially alter the interparticle potential. Our conclusions and outlook are summarized in Sect. 10.

Appendices A and B review quantum-mechanical (QM) scattering from a short-range potential, and from a $-1/r^3$-plus-hard-core potential, respectively. Appendix C discusses field re-scaling in particle-gas language. Appendix D offers an alternative version of the calculation which, if less physically insightful, is tidier and has the numerical factors properly in place.

## 3. The equivalent interparticle potential

The equivalence between photon exchange and the $1/r$ Coulomb potential (and between pion exchange and the Yukawa potential $e^{-mr/r}$) is well known. Quite generally, for a given QFT matrix element $\mathcal{M}$ for $2 \rightarrow 2$ scattering, we can ask “What interparticle potential $V(r)$, when used in QM scattering theory, would yield the same scattering amplitude?” A sophisticated discussion of this question is given in the review of Feinberg et al [14]. For our purposes a rather simple-minded version will suffice. Consider the elastic scattering of equal mass particles, $\mathbf{p}_1, \mathbf{p}_2 \rightarrow \mathbf{p}_1', \mathbf{p}_2'$, in the centre-of-mass frame ($\mathbf{p}_1 = -\mathbf{p}_2 \equiv \mathbf{p}$). Define $E = \sqrt{p^2 + m^2}$ and let $\mathbf{q}$ be the momentum transfer $\mathbf{p}_1' - \mathbf{p}_1$, and the scattering angle be $\theta$. Let us compare the QFT formula for the differential cross section [13]:

$$\frac{d\sigma}{d\Omega} = \frac{1}{256\pi^2} \frac{1}{E^2} |\mathcal{M}|^2,$$

(3.1)
with the corresponding formula in QM:
\[ \frac{d\sigma}{d\Omega} = |f(\theta)|^2, \]  
(3.2)
where \( f(\theta) \), the scattering amplitude, is given, in Born approximation, by
\[ f(\theta) = -\frac{E/2}{2\pi} \int d^3r \ e^{-i\mathbf{q} \cdot \mathbf{r}} V(\mathbf{r}). \]  
(3.3)
Note that we have “relativized” the familiar NR QM result by replacing the reduced mass \( \mu \equiv m/2 \) by \( E/2 \). This corresponds to replacing the Schrödinger equation with the “relativized” version \( \hat{E}_1 + \hat{E}_2 + V(\mathbf{r}) \), with \( \hat{E}_i \equiv \sqrt{\hat{p}_i^2 + m^2} \), where \( \hat{p}_i \) is particle \( i \)'s momentum operator [14, 16].

Comparing the two formulas (and fixing the phase factor to get a positive result in (3.6) below) gives
\[ \int d^3r \ e^{-i\mathbf{q} \cdot \mathbf{r}} V(\mathbf{r}) = \frac{1}{4E^2} \mathcal{M}. \]  
(3.4)
Taking the 3-dimensional Fourier transform of each side gives
\[ V(\mathbf{r}) = \frac{1}{4E^2} \int \frac{d^3q}{(2\pi)^3} e^{i\mathbf{q} \cdot \mathbf{r}} \mathcal{M}. \]  
(3.5)
Note that this “equivalent interparticle potential” is energy dependent, so it is an “effective” concept only. However, this caveat becomes unimportant in the NR regime, \( E \sim m \). Following Feinberg et al [14] we regard the \( 1/(4E^2) \) factor as independent of the momentum transfer \( \mathbf{q} \). Hence, \( V(\mathbf{r}) \) is just a function of the relative position \( \mathbf{r} \) (conjugate to \( \mathbf{q} \)) that also depends, parametrically, on \( E \).

For \( \lambda \Phi^4 \) theory the bare vertex diagram gives \( \mathcal{M} = \lambda \) and the resulting interparticle potential is
\[ V(\mathbf{r}) = \frac{1}{4E^2} \lambda \delta^{(3)}(\mathbf{r}). \]  
(3.6)
We now ask what interparticle potential is produced by the \( t \)-channel “fish” diagram. First we note that if \( \mathcal{M} \) depends only on \( q \equiv |\mathbf{q}| \) then the angular integrations in (3.5) yield
\[ V(\mathbf{r}) = \frac{1}{8\pi^2E^2} \frac{1}{r^3} \int_0^{\infty} dq \frac{2 \sin qr}{qr} \mathcal{M}(q) \]  
(3.7)
\[ = \frac{1}{8\pi^2E^2} \frac{1}{r^3} \int_0^{\infty} dy y \sin y \mathcal{M}(q = y/r). \]  
(3.8)
Note that \( V(\mathbf{r}) \) is spherically symmetric and naturally has a \( 1/r^3 \) factor.

The matrix element for the \( t \)-channel “fish” diagram, neglecting the masses of the exchanged particles for now, is just
\[ \mathcal{M}_{t-exch}(q) = \frac{\lambda^2}{16\pi^2} \ln(q/\Lambda), \]  
(3.9)
where $\Lambda$ is an ultraviolet cutoff. Substituting in Eq. (3.8) yields an integral that is not properly convergent, but which can be made well defined by including a convergence factor $e^{-\epsilon y}$ and then taking the limit $\epsilon \to 0$. In this sense we have

$$\int_0^\infty dy y \sin y = 0,$$

(3.10)

and

$$\int_0^\infty dy y \ln y \sin y = -\frac{\pi}{2}.$$

(3.11)

(We made use of formulas 3.944.5 and 4.441.1 from Gradshteyn and Rhyzik (GR) [17]. Note also GR 3.761.2, 3.761.4, and 4.422.)

Eq. (3.10) implies that any constant term in the matrix element will give no contribution to the potential at $r \neq 0$. Such constant terms do, however, lead to a $\delta^{(3)}(r)$ contribution, as one sees by returning to the form (3.3). These delta-function contributions can be viewed as renormalizing the bare coupling strength. (The $s$-channel diagram also contributes solely to the coupling-constant renormalization.)

Another consequence of (3.10) is that no dependence on $\Lambda$ survives in the $-1/r^3$ potential. The result of substituting (3.9) into (3.8) is thus:

$$V_{t-exch}(r) = -\frac{\lambda^2}{256\pi^3 E^2} \frac{1}{r^3}.$$

(3.12)

Allowing for the mass of the exchanged particles leads to

$$\mathcal{M}_{t-exch}(q) = \frac{\lambda^2}{32\pi^2} \left[ f(q) \ln \left( \frac{f(q) + 1}{f(q) - 1} \right) + \text{const.} \right],$$

(3.13)

where $f(q) \equiv \sqrt{1 + 4m^2/q^2}$ and the constant term contains a $\ln \Lambda^2$ divergence, as in (3.9). Substituting in (3.8) we obtain the result in (3.12) multiplied by

$$I \equiv -\frac{1}{\pi} \int_0^\infty dy \sin y \sqrt{y^2 + \sigma^2} \ln \left( \frac{\sqrt{y^2 + \sigma^2} + y}{\sqrt{y^2 + \sigma^2} - y} \right),$$

(3.14)

where $\sigma = 2mr$. Thanks to (3.10) this integral satisfies

$$\frac{\partial I}{\partial \sigma} = -\frac{\sigma}{\pi} \int_0^\infty dy \frac{\sin y}{\sqrt{y^2 + \sigma^2}} \ln \left( \frac{\sqrt{y^2 + \sigma^2} + y}{\sqrt{y^2 + \sigma^2} - y} \right),$$

(3.15)

which can be evaluated by using GR 3.775.1 and noting that $X^{\nu} - Y^{\nu} \to \nu \ln(X/Y)$ as $\nu \to 0$. The result is $-\sigma K_0(\sigma)$, where $K_0$ is the modified Bessel function of order zero. Integrating with respect to $\sigma$ gives

$$I = \sigma K_1(\sigma).$$

(3.16)
We used formula 9.6.26 of Ref. [19]. The constant of integration must be zero to agree with the massless case ($\sigma = 0$) evaluated earlier.) Therefore the massive case yields Eq. (3.12) times a factor $2mrK_1(2mr)$. This factor becomes unity when $mr \to 0$, while for large $mr$ it tends to $\sqrt{\pi mr} e^{-2mr}$. The exponential factor makes good sense; it is just like the Yukawa factor except that, as there are two particles exchanged, we get $e^{-2mr}$ rather than $e^{-mr}$.

One can also evaluate $V_{t-exch}$ from Feinberg et al’s formulas: Their spectral function $\rho_2$ is given as $-I(t)\phi(s,t)/(4\pi)$ (see their Eqs. (2.52)–(2.54)), where the $\phi$ factor here reduces to just $\lambda^2$, and $I(t)$ for equal masses becomes $\sqrt{t-4m^2}/(8\sqrt{t})$. Inserting this into their Eq. (2.28b) (times $m^2/E^2$ to convert their $U(r)$ to $V(r)$) gives

$$V_{t-exch} = \frac{1}{16\pi^2 E^2} \frac{1}{r} \int_{4mr}^{\infty} dt \left( \frac{\lambda^2}{32\pi} \right) \frac{\sqrt{t-4m^2}}{\sqrt{t}} e^{-\sqrt{tr}}. \quad (3.17)$$

Changing the integration variable to $z = r\sqrt{t}$ leads to

$$V_{t-exch} = -\frac{\lambda^2}{256\pi^3 E^2} \frac{1}{r^3} \int_{2mr}^{\infty} dz \sqrt{z^2-4m^2r^2} e^{-z}. \quad (3.18)$$

The integral trivially gives unity for $m = 0$ and is easily evaluated in general using GR 3.387.6 to give $2mrK_1(2mr)$, confirming our previous result.

In summary; for $m$ small the potential consists of a $\delta^{(3)}(r)$ ‘hard core’ plus an attractive $-1/r^3$ tail out to distances of order $1/(2m)$, beyond which the potential is exponentially suppressed.

4. The hard-sphere Bose gas

In this section we temporarily ignore the $-1/r^3$ part of the potential and consider a gas of ‘atoms’ interacting via a short-range, repulsive potential. We follow the hard-sphere-Bose-gas analysis given in the textbook of K. Huang [3]. (See also [4] and the original references [2].) This analysis describes Bose-Einstein condensation in a dilute atomic gas; a phenomenon only recently observed in a tour de force of experimental technique [20].

Consider a box of volume $V$ containing $N$ identical bosons (‘atoms’). A convenient variable is the number density, $n \equiv N/V$. [Huang uses the volume per particle, $v = V/N$, which is just $1/n$.] The atoms interact via an interatomic potential, assumed to be short range, but which may have any shape. The point is that, in quantum mechanics, low-energy scattering is insensitive to the shape of the potential and can be characterized by a single parameter, the scattering length $a$ — which is just (minus) the low-energy limit of the scattering amplitude [21].
The analysis assumes small $a$ in two senses:

\begin{align}
\text{diluteness: } \quad na^3 &\ll 1, \\
\text{‘low energy’: } \quad ka &\ll 1,
\end{align}

(4.1)

(4.2)

where $k$ is the wavenumber (or momentum, since we set $\hbar$ to unity). The ‘low energy’ assumption means that the scattering is essentially pure $s$-wave.

An important technical simplification, explained in detail by Huang, is that, in this approximation, one may replace the short-range, repulsive potential by a $\delta$-function “pseudopotential” [3]:

\[ V_{\text{pseud}}(r) = \frac{4\pi a}{m} \delta^{(3)}(r) \left( \frac{\partial}{\partial r} \right) r, \]

(4.3)

where the differential operator $\left( \frac{\partial}{\partial r} \right) r$ acts to “weed out” any $1/r$ singularity in the wavefunction to which it is applied. The Hamiltonian of the system [see Huang (10.124) and (13.86)] is then:

\[ H = \frac{-1}{2m} \sum_{j=1}^{N} \nabla_j^2 + \frac{4\pi a}{m} \sum_{i<j} \delta^{(3)}(r_i - r_j) \frac{\partial}{\partial r_{ij}} r_{ij}. \]

(4.4)

This pseudopotential Hamiltonian is intended to be used in Born approximation only. This is important because, if treated as a real potential, a repulsive $\delta$-function, of whatever strength, produces zero scattering in 3+1 dimensions. (See Appendix A). The normalization of the $\delta$-function term is such that, in Born approximation, it gives a scattering amplitude $-a$, and thus mimics the full effect of the true potential.

In the (NR) quantized-field representation the Hamiltonian is

\[ H = \frac{-1}{2m} \int d^3r \psi^\dagger(r) \nabla^2 \psi(r) + \frac{2\pi a}{m} \int d^3r_1 \int d^3r_2 \psi^\dagger(r_1) \psi^\dagger(r_2) \delta^{(3)}(r_1 - r_2) \frac{\partial}{\partial r_{12}} [r_{12} \psi(r_1) \psi(r_2)]. \]

(4.5)

Substituting

\[ \psi(r) = \sum_k a_k \frac{e^{ikr}}{\sqrt{V}}, \quad \left[ a_k, a^\dagger_{k'} \right] = \delta_{kk'}, \]

(4.6)

where the operators, $a^\dagger_k, a_k$, create and annihilate atoms in the free-particle state of wavenumber $k$, this becomes

\[ H = \frac{1}{2m} \sum_k k^2 a_k^\dagger a_k + \frac{2\pi a}{mV} \sum_{p,q} a_{p}^\dagger a_{q}^\dagger \frac{\partial}{\partial r} \left[r \sum_k e^{ikr} a_{p+kq-k} \right]_{r=0}. \]

(4.7)

For the ground state of the system (and for any low-lying excited states) one expects almost all the atoms to be in the $k = 0$ state. Hence, the occupation numbers should satisfy
\[ n_{k=0} \sim N \] and \[ \sum_{k \neq 0} n_k/N \ll 1. \] Thus, one can effectively make the replacement \( a_0 = a_0^\dagger = \sqrt{N}. \) Dropping terms relatively suppressed by \( 1/N \) leads to an effective Hamiltonian \( \mathcal{H}_\text{eff} \):

\[
\mathcal{H}_\text{eff} = N \frac{2\pi na}{m} + \frac{1}{2m} \sum_{k \neq 0} \left( (k^2 + 8\pi na) a_k^\dagger a_k + 4\pi na (a_k^\dagger a_{-k}^\dagger + a_k a_{-k}) \right). \tag{4.8}
\]

[The prime on the summation has to do with the \((\partial/\partial r)r\) subtlety; see below.]

The next step is to diagonalize \( \mathcal{H}_\text{eff} \) by a Bogoliubov transformation. That is, one introduces new operators \( b_k^\dagger, b_k \) that are suitable linear combinations of \( a_k^\dagger, a_k \):

\[
a_k = \frac{1}{\sqrt{1 - \alpha_k^2}} (b_k - \alpha_k b_{-k}^\dagger). \tag{4.9}
\]
The \( b_k b_{-k} \) and \( b_k^\dagger b_{-k}^\dagger \) terms in \( \mathcal{H}_\text{eff} \) are eliminated by choosing

\[
\alpha_k = 1 + x^2 - x \sqrt{x^2 + 2}, \quad x^2 = \frac{2m\omega_0(k)}{8\pi na}, \tag{4.10}
\]
where \( \omega_0(k) \) is the original atom spectrum, \( k^2/(2m) \). This yields

\[
\mathcal{H}_\text{eff} = E_0 + \sum_{k \neq 0} \omega(k) b_k^\dagger b_k, \tag{4.11}
\]
where

\[
\omega(k) = \omega_0(k) \left( \frac{1 + \alpha_k}{1 - \alpha_k} \right) = \omega_0(k) \sqrt{1 + 2/x^2}, \tag{4.12}
\]
which gives

\[
\omega(k) = \omega_0(k) \sqrt{1 + (16\pi na)/(2m\omega_0(k))}. \tag{4.13}
\]
The new operators, \( b_k^\dagger, b_k \), create and destroy the elementary excitations of the Bose-Einstein condensate — phonons — that are not single-atom excitations, but coherent states involving an indefinite number of atoms. The spectrum of these excitations (their kinetic energy as a function of their momentum) is given by \( \omega(k) \).

The ground-state energy is given by

\[
E_0 = N \frac{2\pi na}{m} - \frac{2\pi na}{m} \sum_{k \neq 0} \omega_k = N \frac{2\pi na}{m} \left( 1 + \mathcal{O}(\sqrt{na^2}) \right). \tag{4.14}
\]
The first term arises simply from the \( p = q = k = 0 \) term in the summation in \((4.7)\). The second term arises in the diagonalization step when one makes the substitution \( b_k b_k^\dagger = b_k^\dagger b_k + 1 \). The \( k \)-summation can be converted to an integral by \( \sum_k \rightarrow V \int d^3k/(2\pi)^3 \). The integral would be linearly divergent but for the prime on the summation, whose effect is to
subtract the divergent term \[23\]. The result is that this second term is suppressed relative to the first by a factor of \(\sqrt{na^3}\).

An important observation \[3\] is that in the limit \(a \to 0\), with \(n \to \infty\) such that \(na\) remains finite, the results remain non-trivial while the approximations (4.1, 4.2) become exact: The gas is then infinitely dilute, since \(na^3 = (na)a^2 \to 0\), and the assumption \(k \ll 1/a\) becomes no restriction. In this limit the atoms’ interaction vanishes, but non-trivial effects survive because there is an infinite density of them. One effect is the non-zero ground-state energy term in (4.11). The other is that the system’s excitations are phonons, non-trivial coherent states of indefinitely many atoms, whose spectrum (4.13) is not the trivial \(k^2/2m\) atom spectrum. The phonons themselves are exactly non-interacting, so the resulting theory is “trivial” in the technical sense. The relevance of this “trivial”-but-not-entirely-trivial limit will become clear later.

Now let us consider the effect of allowing for relativistic kinematics. For actual atoms (Hydrogen, Lithium, etc.) this would be pointless, because their mass \(m\) is always much greater than \(1/a\); consequently, since the analysis assumes \(k \ll 1/a\), one necessarily has \(k\) much, much less than \(m\). However, we shall want to consider “atoms” whose mass is small compared to \(1/a\). Thus, even when the energy is ‘low’ in the sense of \(k \ll 1/a\) it need not be small with respect to \(m\). Thus, for us, relativistic kinematics will be relevant. (We shall speak of ‘low energies’ if \(k \ll 1/a\) and ‘NR energies’ if \(k \ll m\).)

A quick-and-dirty argument for how to “relativize” Huang’s results is the following. (In Appendix D we shall describe a proper relativistic version of the calculation.) The original spectrum \(\omega_0(k)\), instead of being the NR kinetic energy \(k^2/2m\) should be the relativistic kinetic energy \(E_k - m\), where \(E_k \equiv \sqrt{k^2 + m^2}\). (Recall that the relativistic and NR conventions for the zero of kinetic energy differ by \(m\).) As seen in the previous section, the formula for the scattering amplitude in terms of \(V(r)\), (3.3), has a factor of \(E\), while \(V(r)\) in (3.3) has a \(1/(4E^2)\) factor. Therefore, with respect to the NR case, we must scale \(a\) by a factor of \((E/m)(m/E)^2 = m/E\). Hence, \(x^2\) in Eq. (4.10) becomes \(2E_k(E_k - m)/(8\pi na)\), and Eq. (4.13) becomes

\[
\tilde{E}_k = (E_k - m)\sqrt{1 + \frac{8\pi na}{E_k(E_k - m)}}.
\] (4.15)

In the limit \(m \to 0\), where the atom spectrum is \(E_k = k\), the phonon spectrum \(\tilde{E}_k\) tends to \(\sqrt{k^2 + 8\pi na}\) \[24\], which has the usual form for a relativistic particle. We may then identify

\[
M^2(n) \equiv 8\pi na
\] (4.16)
with the mass squared of the ‘phonon’ excitation (which is the “Higgs particle” in QFT language).

5. Effect of the $-1/r^3$ potential

We now turn our attention to the effect of an attractive $-1/r^3$ potential between the atoms. The special property of a $-1/r^3$ potential is actually evident at the classical level and it is instructive to look at this first. Conveniently, the text by Reif [25] discusses the dilute classical gas using an illustrative interatomic potential of the form:

$$ V(r) = \begin{cases} \infty, & r < r_0 \\ -\mathcal{A}/r^s, & r > r_0 \end{cases} \quad (5.1) $$

with an arbitrary power $s$. ($\mathcal{A}$ is a constant, which Reif writes as $u_0 r_0^s$.) The internal energy of the gas receives a contribution from pairwise interactions that is the number of pairs times the average energy of a pair: $\frac{4}{3}N(N-1)\bar{u} \sim \frac{4}{3}N^2\bar{u}$. Because the gas is dilute, 3-body interactions, etc., are negligible, and also the motion of any pair should not be appreciably correlated with the motion of the other atoms, which simply provide a heat bath of temperature $T$. Then, the probability of a given separation $r$ is proportional to the Boltzmann factor, yielding [25]

$$ \bar{u} = \frac{\int d^3r e^{-\beta V(r)} V(r)}{\int d^3r e^{-\beta V(r)}}, \quad (5.2) $$

where $\beta = 1/(k_B T)$, where $k_B$ is Boltzmann’s constant.

In the high-temperature (small $\beta$) limit one might naively say that the exponentials go to unity, giving

$$ \bar{u} = \frac{1}{V} \int d^3r V(r) = \frac{-4\pi\mathcal{A}}{V} \int_{r_0}^{\infty} \frac{dr}{r^{s-2}}, \quad (5.3) $$

which produces a logarithmic divergence in our case, $s = 3$. A more satisfactory derivation of this result [25] is to re-express (5.2) as

$$ \bar{u} = -\frac{\partial}{\partial \beta} \ln \left( \int d^3r e^{-\beta V(r)} \right). \quad (5.4) $$

Adding and subtracting unity from the integrand, one has

$$ \int d^3r e^{-\beta V(r)} = V + I(\beta), \quad (5.5) $$

with

$$ I(\beta) \equiv \int d^3r \left( e^{-\beta V(r)} - 1 \right), \quad (5.6) $$

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so that, up to $O(1/V^2)$ terms,

$$\bar{u} = \frac{1}{V} \frac{\partial I}{\partial \beta}. \quad (5.7)$$

The integral $I(\beta)$ is related directly, by $B_2 = -\frac{1}{4}I(\beta)$, to the second virial coefficient, which represents the first deviation from the ideal-gas law in an expansion of the pressure in powers of the density:

$$\frac{p}{k_BT} = n + B_2n^2 + O(n^3). \quad (5.8)$$

At high temperatures $I(\beta)$ can be evaluated by expanding the exponential, yielding

$$I(\beta) = -\frac{4}{3}r_0^3 + 4\pi A\beta \int_{r_0}^{\infty} \frac{dr}{r^{s-2}}. \quad (5.9)$$

The first term is the excluded volume due to the hard core. Substituting in (5.7) reproduces the naive result (5.3), and we see again the divergence as $s \to 3$.

Now, it might seem odd to focus on the high-temperature limit, when we are really interested in $T = 0$. However, the divergence at $s = 3$ will be present at any temperature; it is just easiest to see it in the high-temperature limit. [For any $\beta$ the exponent $\beta V(r)$ becomes small as $r \to \infty$, allowing us to expand the exponential to obtain the $1/r^{s-2}$ behaviour at large $r$.] Also, the high-temperature limit in the classical analysis has something in common with the $T = 0$ quantum case. If we consider a potential $V(r)$ quantum mechanically as a perturbation we obtain a contribution to the energy:

$$\frac{1}{2} \int d^3r_1 \int d^3r_2 \psi^\dagger(r_1)\psi^\dagger(r_2)V(|r_1 - r_2|)\psi(r_1)\psi(r_2). \quad (5.10)$$

This is in the NR quantized field representation [Cf. Eq. (4.5), but with a potential $V(r)$ in place of the $\delta$-function pseudopotential]. For the zero-temperature ground state of the unperturbed gas we have simply $\psi(r) = \text{const.} = \sqrt{N/V}$ (normalized to $\int d^3r \psi^\dagger \psi = N$). Thus, the above equation gives

$$\frac{1}{2} \left(\frac{N}{V}\right)^2 V \int d^3r V(r), \quad (5.11)$$

which is of the form $\frac{1}{4}N^2\bar{u}$ with

$$\bar{u} = \frac{1}{V} \int d^3r V(r), \quad (5.12)$$

just as in (5.3). The point is that in the unperturbed quantum ground state the particles are evenly distributed over the box, just as they are in the classical, high-temperature case. So in both cases we get an unweighted integral over $V(r)$. This produces the logarithmic divergence when $V(r) \propto -1/r^3$. 

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The crucial question now is: “What cuts off this divergence?” Clearly, the atom mass \( m \) can act as a cutoff, since the actual potential only behaves as \(-1/r^3\) out to distances of order \( 1/(2m) \), due to the \( 2mrK_1(2mr) \) factor. However, when \( m \) becomes very small, and this cutoff distance becomes enormous, another effect comes into play; namely, the possibility of other atoms “getting in the way” of the interaction and tending to “screen” it. The crucial point is that this provides a cutoff that involves the background density \( n \).

A first, over-simplistic approach, which nonetheless brings out the basic point that \( r_{\text{max}} \) will depend on \( n \), is to suppose that the interaction becomes “screened” as soon as it is more likely than not that a third particle intervenes between the two particles that are trying to interact. That would give an \( r_{\text{max}} \) of order the average atom spacing, which is \( n^{-1/3} \).

A better argument is to say that in the “fish” diagram the virtual particles being exchanged have to propagate through the background medium. In so doing they will experience multiple collisions with the \( k = 0 \) background atoms. This produces a propagator \( G' \) that is a sum of all possible numbers of “mass insertions,” where each insertion \(-i\Delta^2\) corresponds to a collision with a background atom:

\[
G' = G + G(-i\Delta^2)G + G(-i\Delta^2)G(-i\Delta^2)G + \ldots ,
\]

where \( G = \frac{i}{p^2-m^2} \), giving \( G' = \frac{i}{p^2-(m^2+\Delta^2)} \). The new propagator must be the phonon propagator, since these are the elementary excitations in the presence of the background density, \( n \), and are the only things that propagate over macroscopic distances. Thus, we can identify \( m^2 + \Delta^2 \) as the phonon mass-squared \( M^2(n) \). This implies a cutoff distance \( r_{\text{max}} \sim 1/(2M(n)) \), rather than \( 1/(2m) \). This assumes that \( M(n) \) is much greater than \( m \); otherwise the atom-mass cutoff \( 1/(2m) \) would take precedence. Recalling that \( M^2(n) \sim 8\pi na \) for \( M(n) \gg m \), we see that \( r_{\text{max}} \propto n^{-1/2}a^{-1/2} \), rather than the \( n^{-1/3} \) of the previous paragraph.

6. The ground-state energy density

We are now in a position to make an estimate of the ground-state energy density of the gas as a function of the atom density \( n \). From the last section, the contribution from the \(-A/r^3 \) interaction is \( \frac{1}{2}N^2\bar{u} \), with

\[
\bar{u} = \frac{1}{V} \int_{r_0}^{r_{\text{max}}} \left(-\frac{A}{r^3}\right) 4\pi r^2 dr = -\frac{4\pi A}{V} \ln\left(r_{\text{max}}/r_0\right),
\]

for the ground-state energy density.
The coefficient of the \(-1/r^3\) interaction, \(\mathcal{A}\), can be inferred from Eq. (3.12). Including a factor of 2 to allow for both \(t\) and \(u\) amplitudes, we have \(\mathcal{A} = \lambda^2/(128\pi^3E^2)\). (For the ground state, with almost all of the particles in the \(k = 0\) mode, we may set \(E = m\).) Setting \(r_{\text{max}}\) equal to \(1/(2M(n)) = 1/(2\sqrt{\pi na})\) we have an energy contribution

\[
\frac{1}{2} N^2 \left( -\frac{4\pi}{V} \frac{\lambda^2}{128\pi^3 m^2} \right) \frac{1}{2} \ln \left( \frac{1}{32\pi nar_0^2} \right). \tag{6.2}
\]

In addition, there is the ground-state energy found in the hard-sphere Bose-gas analysis, namely \(N^2 \frac{2\pi na}{m}\). (See Eq. (4.14).) This represents the energy cost of the hard-sphere repulsions in the quantum ground state. Finally, there is the rest-mass cost of having \(N\) particles (which, of course, is ignored in non-relativistic calculations). In total we have

\[
E_{\text{g.s.}} = Nm + N^2 \frac{2\pi na}{m} + Nn \frac{\lambda^2}{128\pi^2 m^2} \ln(32\pi nar_0^2). \tag{6.3}
\]

Dividing by the volume yields the energy density:

\[
\mathcal{E}_{\text{g.s.}} = nm + \frac{2\pi n^2 a}{m} + n^2 \frac{\lambda^2}{128\pi^2 m^2} \ln(32\pi nar_0^2). \tag{6.4}
\]

We see that this is a sum of \(n\), \(n^2\) and \(n^2 \ln n\) terms; these represent 1-body, 2-body, and “2-body-plus-medium” effects, respectively. Possible 3-body (\(n^3\)) or higher terms are negligible because of the diluteness assumption.

Note that if the attractive potential had fallen off faster than \(1/r^3\) it would have given a simple \(n^2\) term — which could then have been combined with the other \(n^2\) term. That would correspond to replacing \(a\), the scattering length due to the hard-core alone, by the scattering length produced by the whole potential. This observation suggests an alternative way to obtain \(\mathcal{E}_{\text{g.s.}}\): simply replace the “\(a\)” in the hard-sphere term \(2\pi n^2 a/m\) by the “effective scattering length” \(a_{\text{eff}}(r_{\text{max}})\) of the whole potential (hard core plus \(-1/r^3\)), cutting off the infrared divergence at \(r_{\text{max}} = 1/2M(n)\). In Appendix B we show that \(a_{\text{eff}}\) can be expressed as \(a - m\mathcal{A}\ln(r_{\text{max}}/r_0)\), with the parameter \(a\) now representing the scattering length due to the core and the “short-range part” of the \(-1/r^3\) potential. (Thus, \(a\) can naturally be much greater than \(r_0\).) The \(-m\mathcal{A}\ln(r_{\text{max}})\) term in \(a_{\text{eff}}\) agrees precisely with the Born-approximation result — not surprisingly, since it arises from the “long-range tail” of the \(-1/r^3\) potential, where the potential is very weak. Thus, our treatment of the \(-\mathcal{A}/r^3\) interaction using lowest-order perturbation theory in Sect. 5 is effectively exact, provided that we re-interpret \(a\).

Because of the \(n^2 \ln n\) term in \(\mathcal{E}_{\text{g.s.}}\), the system has a first-order phase transition. We shall analyze this in Sect. 8, but first we re-cast the result in QFT language.
7. Translation to QFT language

What do the scattering length \( a \) and the density \( n \) correspond to in QFT terms? Clearly, the scattering length and the coupling \( \lambda \) are related, since both reflect the strength of the interaction. To leading order, we may directly compare the QFT expression for the cross section, (3.1), with the QM result \( \frac{d\sigma}{d\Omega} = a^2 \). Including a factor of 2 for identical-particle reasons \[26\], we have

\[
\frac{(2\lambda)^2}{256\pi^2 E^2} = a^2, \tag{7.1}
\]

or

\[
a = \frac{\lambda}{8\pi E}. \tag{7.2}
\]

In the ground state, where almost all atoms have \( k = 0 \), we may set \( E = m \). The above result also follows by identifying the \( \delta^{(3)}(r) \) potential obtained from the bare vertex diagram, Eq. (3.6), with the \( \delta^{(3)}(r) \) pseudopotential of the hard-sphere analysis, (4.3). The same \( a \leftrightarrow \lambda \) connection (in the context of an O(2) theory, where the numerical factor is different) has been noted in Ref. \[27\].

The other relation we need is between the background density \( n \) and the strength of the background field \( \phi = \langle \Phi \rangle \). In a finite-volume box, where the modes are discrete, one may expand the field \( \Phi(x, t) \) as:

\[
\Phi(x, t) = \sum_k \frac{1}{\sqrt{2V}} \left[ a_k e^{ik \cdot x} + a_k^\dagger e^{-ik \cdot x} \right] \tag{7.3}
\]

where \( a_k \) carries a time-dependent factor \( e^{-iE_k t} \), with \( E_k = \sqrt{k^2 + m^2} \) and

\[
[a_k, a_k^\dagger] = \delta_{k,k'}. \tag{7.4}
\]

The number operator is

\[
\hat{N} = \sum_k a_k^\dagger a_k. \tag{7.5}
\]

Following Ref. \[1\], one argues that in the condensate state almost all the atoms are in the \( k = 0 \) mode, so \( N = a_0^\dagger a_0 \), and one may effectively regard \( a_0 \) (or rather \( a_0^{\text{NR}} \equiv a_0 e^{imt} \)) as the c-number \( \sqrt{N} \). Then from (7.3) one obtains the expectation value

\[
\phi = \langle \Phi \rangle = \frac{1}{\sqrt{2V m}} a_0^{\text{NR}} = \sqrt{\frac{2N}{V m}}, \tag{7.6}
\]

and hence \( n \equiv N/V \) is given by

\[
n = \frac{1}{2} m \phi^2. \tag{7.7}\]
With this identification, the trick of setting \( a_0 = a_0^\dagger = \sqrt{N} \) is equivalent to shifting the field by a constant \( \phi \).

We may now consider both \( M^2 \) and \( \mathcal{E} \) to be functions of \( \phi^2 \), rather than \( n \). Using the translations (7.2) and (7.7) gives

\[
M^2 = 8\pi na = 8\pi \left(4m\phi^2\right) \left(\frac{\lambda}{8\pi m}\right) = 4\lambda \phi^2.
\]  

(7.8)

Similarly, the energy density (6.4) becomes gives

\[
\mathcal{E}_{\text{g.s.}} = \frac{1}{2} m^2 \phi^2 + \frac{1}{16} \lambda \phi^4 + \frac{1}{2} \frac{\lambda^2}{256\pi^2} \phi^4 \ln \left(\frac{2\lambda \phi^2}{r_0^2}\right).
\]  

(7.9)

The energy density, expressed as a function of the field’s expectation value, is of course the effective potential, \( V_{\text{eff}}(\phi) \), in QFT language. We obviously recognise the first term as the usual mass term. Up to numerical factors [28], the second term is the \((\lambda/4!)\phi^4\) term of the classical potential and the last term is the zero-point energy of free-field fluctuations about the background field \( \phi \) (the “Trace-Log” term in the functional evaluation of the effective potential \( \text{à la} \) Jackiw [29]). Recall that \( r_0 \), our short-distance cutoff, is proportional to \( 1/\Lambda \), where \( \Lambda \) is a momentum-space ultraviolet cutoff.

8. The phase transition region

We now study the conditions needed for the system to be at or near the phase transition. (Away from this region the physics would be totally dull.) We prefer to use the QFT variables and then translate back to \( a \) and \( n \). Because we have been somewhat cavalier about numerical coefficients let us start from:

\[
M^2(\phi_B) = c_0 \frac{1}{2} \lambda \phi_B^2,
\]  

(8.1)

and

\[
V_{\text{eff}} = \frac{1}{2} m^2 \phi_B^2 + c_1 \frac{\lambda}{4!} \phi_B^4 + c_2 \frac{\lambda^2}{256\pi^2} \phi_B^4 \ln\left(c_0 \frac{1}{2} \lambda \phi_B^2/\Lambda^2\right),
\]  

(8.2)

where \( c_0, c_1, c_2 \) are numerical coefficients of order 1, whose specific values won’t change the general picture. (We have added a ‘\( B \)’ subscript to \( \phi \) to emphasize that it is the (constant part of the) bare field. Field re-scaling is discussed in Appendix C.)

If \( V_{\text{eff}} \) has non-trivial minima at \( \phi_B = \pm v_B \), then \( v_B \) must be a solution of \( dV_{\text{eff}}/d\phi_B = 0 \), which gives the equation:

\[
m^2 + c_1 \frac{\lambda}{6} v_B^2 + c_2 \frac{\lambda^2}{64\pi^2} v_B^2 \left(\ln\left(c_0 \frac{1}{2} \lambda v_B^2/\Lambda^2\right) + \frac{1}{2}\right) = 0.
\]  

(8.3)
Let \( v_0 \) denote the solution in the special case \( m^2 = 0 \); it is given by

\[
v_0^2 = \frac{2 \Lambda^2}{c_0 \lambda} e^{-1/2} \exp \left( -\frac{32 \pi^2 c_1}{3 \lambda c_2} \right),
\]

(8.4)

The original equation (8.3) can then be re-written as:

\[
f(v_B^2) \equiv -c_2 \frac{\lambda^2}{64 \pi^2} v_B^2 \ln(v_B^2/v_0^2) = m^2.
\]

(8.5)

A graph of \( f(v_B^2) \) climbs steeply from zero, has a maximum, and thereafter decreases, becoming negative when \( v_B^2 > v_0^2 \). Equating this to \( m^2 \) we see that: (i) If \( m^2 \) is positive and larger than the maximum value of \( f \), there will be no solution. This corresponds to \( V_{\text{eff}} \) having a single minimum at \( \phi = 0 \). (ii) If \( m^2 \) is positive, but not too large, there will be two solutions for \( v_B^2 \). This corresponds to \( V_{\text{eff}} \) having a minimum at \( \phi = 0 \), then a maximum (at the smaller \( v_B \) root) and then a minimum (at the larger \( v_B \) root). (iii) If \( m^2 \) is negative there is a unique solution, with \( v_B \) greater than \( v_0 \). Here the origin is a maximum of \( V_{\text{eff}} \) and \( v_B \) is a minimum.

Case (i) is of little interest, having no SSB. Case (iii) is interesting and perfectly acceptable in QFT terms, but with \( m^2 \) negative (tachyonic ‘atoms’) our particle-gas description is inappropriate (or, at least, loses its intuitive appeal). We therefore focus on case (ii), where \( m^2 \) is positive but smaller than the maximum value of \( f \). Since \( f \) is maximum at \( v_B^2 = e^{-1} v_0^2 \), we need

\[
m^2 < c_2 \frac{\lambda^2}{64 \pi^2} v_B^2.
\]

(8.6)

for a non-trivial minimum of \( V_{\text{eff}} \) to exist. A stronger condition is needed for this minimum to be lower than the minimum at the origin; i.e., for \( V_{\text{eff}}(v_B) < 0 \):

\[
m^2 < c_2 \frac{\lambda^2}{128 \pi^2} v_B^2.
\]

(8.7)

Recalling Eq. (8.1) and that \( M_h \equiv M(\phi_B = v_B) \), we see that \( m^2 \) must be of order \( \lambda M_h^2 \) or less.

From (8.1) at \( \phi_B = v_B \), we see that \( M_h^2 \) is of order \( \lambda v_B^2 \), and \( v_B^2 \) (for the root corresponding to a minimum of \( V_{\text{eff}} \)) lies between \( e^{-1} v_0^2 \) and \( v_0^2 \). Thus, from (8.4) we find that

\[
M_h^2 = \mathcal{O} \left( \Lambda^2 \exp \left( -\frac{32 \pi^2 c_1}{3 \lambda c_2} \right) \right).
\]

(8.8)

We want to take the cutoff \( \Lambda \) to infinity, but we want to keep \( M_h^2 \) finite, so that the phonons (Higgs bosons) correspond to finite-mass particles. The only way to achieve
these requirements is to have $\lambda$ tend to zero as $\Lambda \to \infty$. In fact, just by re-arranging (8.8) we see that $\lambda$ needs to behave as:

$$\frac{\lambda}{16\pi^2} \sim \frac{2}{3} \frac{c_1}{c_2} \frac{1}{\ln(\Lambda^2/M_h^2)}.$$  

Thus, $\lambda$ tends to zero as $1/\ln \Lambda$. The result of the previous paragraph that $m^2 \sim \lambda M_h^2$ means that $m^2$ is much, much smaller than $M_h^2$. Since we require $M_h^2$ to be finite, which defines our physical mass scale, we need $m^2$ to go to zero as $1/\ln \Lambda$. We also see that $v_B^2$ must go to infinity, like $\ln \Lambda$ so that $\lambda v_B^2$, and hence $M_h^2$, is finite.

This behaviour, namely

$$\lambda = \mathcal{O}(1/\ln \Lambda), \quad m^2 = \mathcal{O}(1/\ln \Lambda), \quad v_B^2 = \mathcal{O}(\ln \Lambda),$$  

(8.10)

(for $\Lambda$ in units of $M_h$) is needed if the physics is to remain interesting as $\Lambda \to \infty$. If $\lambda$ were larger, then $M_h$ would go to infinity (in particular, $M_h = \mathcal{O}(\Lambda)$ for any finite $\lambda$). If $m^2$ were larger, then the $\frac{4}{3} m^2 \phi^2$ term would completely dominate $V_{\text{eff}}$, and we would simply have a theory of non-interacting atoms. The interesting region is where one is sufficiently close to the phase transition, where the symmetric vacuum and the $v_B$ vacuum are in close competition. The unusual feature is that the elementary excitations of these vacua, atoms and phonons, respectively, have vastly different masses; to keep the phonon mass finite, we must let the atom mass be vanishingly small.

Using the translations $a = \lambda/(8\pi m)$ and $n = \frac{4}{3} m \phi_B^2$, we see that

$$a = \mathcal{O}(1/\sqrt{\ln \Lambda}), \quad n = \mathcal{O}(\sqrt{\ln \Lambda}),$$  

(8.11)

where $n_v$ is the atom density in the SSB vacuum $\phi_B = v_B$. Thus, the scattering length $a$ tends to zero (though it is much greater than the core size $r_0 \sim 1/\Lambda$.) This result reflects the “triviality” property of the $\lambda \Phi^4$ theory [30, 31]. However, $n_v a$ remains finite, because of course $M_h^2$ was required to be finite. Thus, the situation mentioned earlier [3] is automatically realized; the ‘diluteness’ and ‘low energy’ approximations (4.1, 4.2) become exact ($n_v a^3 \to 0$, and $ka \to 0$ for any finite momentum $k$). The phonons are exactly non-interacting in this limit, so the theory is “trivial” in the technical sense. However, the phonons are non-trivial coherent states of the atoms, so some interesting physics is still present.

Moreover, this interesting physics can show up ‘experimentally.’ The energy density difference between the SSB vacuum and the $\phi = 0$ vacuum is finite in the physical units defined by $M_h$. Therefore, if we heat the walls of our box to some critical temperature, finite in units of $M_h$, we can expect to see a phase transition. In the high-temperature phase there is no Bose condensate and the excitations are the massless atoms.

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9. All-orders considerations

We began by considering the interparticle potential produced by the simplest diagrams: The tree-level vertex gave a \( \delta^{(3)}(r) \) hard-core interaction, while the 1-loop “fish” diagram gave an additional \(-1/r^3\) interaction (times \(2mnrK_1(2nr)\) if the mass is retained). What about higher-order diagrams? What contributions to the interparticle potential do they give? The answer is that in the region of interest, namely for a \( \lambda \) that is \( O(1/\ln(\Lambda/M_h)) \), they basically give only contributions of the same type, namely \( \delta^{(3)}(r) \) and \(-1/r^3\). Thus, while higher orders can change the coefficients of these terms, they cannot modify the basic form of the interparticle potential.

To see this, consider a general \( n \)-loop 4-point diagram, neglecting the mass \( m \). In general this will be a sum of leading, sub-leading, \ldots logarithms of \( \Lambda/q \), where \( q \) is the momentum transfer:

\[
\mathcal{M}_n = \lambda^{n+1} \left[ A_n(\ln \Lambda/q)^n + B_n(\ln \Lambda/q)^{n-1} + \ldots \right],
\]

where \( A_n, B_n, \ldots \) are some numerical coefficients. It will be convenient here to use units with \( M_h = 1 \), so that \( \ln \Lambda/M_h \) may be abbreviated as \( \ln \Lambda \), etc. Substituting \( \ln \Lambda/q = \ln \Lambda - \ln q \), and recalling that we are interested in the case where \( \lambda \) is proportional to \( 1/\ln \Lambda \), we see that \( \mathcal{M} \) consists of (i) \( q \)-independent terms with one or more \( 1/\ln \Lambda \) factors; (ii) \( \ln q \) terms with two or more \( 1/\ln \Lambda \) factors; (iii) \( \ln^2 q \) terms with three or more \( 1/\ln \Lambda \) factors; (iv) etc. These must be inserted in the 3-dimensional Fourier-transform integral, \((3.5)\) or \((3.8)\), to find their contributions to the interparticle potential, \( V(r) \). We consider each in turn. (i) The \( q \)-independent terms clearly produce \( \delta^{(3)}(r) \) contributions. (ii) The \( \ln q \) terms will produce \(-1/r^3\) contributions, exactly as in Sect. 3. (iii) The \( \ln^2 q \) terms, when we set \( q = y/r \), involve \( \ln^2 y - 2 \ln y \ln r + \ln^2 r \); when inserted in the \( y \) integration \((3.8)\) these give, respectively, a \( 1/r^3 \) term, a \((\ln r)/r^3\) term, and zero (for \( r \neq 0 \)). Thus, the only new contribution to \( V(r) \) is a \((\ln r)/r^3\) term, which however is suppressed by one more power of \( 1/\ln \Lambda \) than the \(-1/r^3\) terms. (iv) Similarly, the \( \ln^p q \) terms in \( \mathcal{M}_n \) can generate \((\ln^{p-1} r)/r^3\) terms in \( V(r) \), but these are suppressed by \( p - 1 \) powers of \( 1/\ln \Lambda \), relative to the \(-1/r^3\) terms.

Thus, in aggregate, there is no change in the \( -1/r^3 + \delta^{(3)}(r) \) form of the potential, except at very short distances, \( r \sim 1/\Lambda \) (or extremely large distances, \( r \sim \Lambda/M_h^2 \)). The modification at very small distances is not unexpected; it will smooth the join between the hard-core and the \(-1/r^3\) pieces of \( V(r) \). (Any modification at extremely large distances, \( r \sim \Lambda/M_h^2 \), would be physically irrelevant since the interaction is effectively cut off at
distances of order $r_{\text{max}} \sim 1/(2M_{h})$.

To summarize: higher orders give contributions comparable in size to the lowest-order terms, but, up to negligible corrections, those contributions maintain the basic 
$-1/r^3 + \delta^{(3)}(r)$ form of the interparticle potential. Our previous analysis remains valid provided that we understand $\lambda$ as an effective coupling strength incorporating these all-order effects. We need not worry about how to actually calculate this $\lambda$ in terms of the original bare coupling(s) \[32\]. This is exactly like the NR Bose gas, where the short-range potential, whatever its shape, can be parametrized by a single parameter $a$, the actual scattering length, and one does not concern oneself with calculating $a$ from the original potential. Note, however, that we should not require the effective $\lambda$ to be a finite parameter; it and $a$ have to be infinitesimally small.

Finally, we note that higher orders should not upset the relation between the coefficient $\mathcal{A}$ of the long-range $-\mathcal{A}/r^3$ potential and $\lambda$. The long-range potential arises physically from two short-range repulsive interactions linked by quasi-free propagation of two virtual particles. If $\lambda$ represents the actual strength of the short-range interaction, then $\mathcal{A}$ is proportional to its square.

10. Conclusions and outlook

Since our opening section provides an outline of the whole picture, we give only the briefest of summaries here: The attractive $-1/r^3$ interparticle potential leads to an $n^2 \ln n$ term in the energy density, and hence to a $\phi^4 \ln \phi^2$ term in the effective potential, and hence to a first-order phase transition, occurring before $m^2$ reaches zero.

The form of effective potential obtained here agrees with our previous work \[\text{[7, 5]}\] and is, of course, just what one would obtain in the one-loop approximation \[\text{[6]}\]. However, our point is that this form is effectively exact (in the relevant case $\lambda = \mathcal{O}(1/\ln \Lambda)$, as $\Lambda \to \infty$). This is not because higher-loop contributions are negligible: in fact, all terms in the loop expansion then have the same size. However, as explained in the previous section, higher-loop contributions, with $\lambda = \mathcal{O}(1/\ln \Lambda)$, give contributions of the same form as at one loop.

Just as in the NR Bose-gas analysis, the result hinges, not on any perturbative assumption, but on the diluteness ($na^3 \ll 1$) and ‘low energy’ ($ka \ll 1$) approximations \[\text{[3, 4]}\], both of which become exact in the continuum limit. It might appear that our treatment of the $-1/r^3$ interaction’s effect in Sect. 5 relied on perturbation theory: However, as explained in Appendix B, the Born approximation is effectively exact because the crucial
ln(r_{max}) behaviour arises from the tail of the potential, where it is very weak. An alternative calculation that is directly a relativistic version of Ref. 3's Bose-gas analysis is presented in Appendix D.

The origin of the crucial $\phi^4 \ln \phi$ term can be viewed in at least two ways. In this paper it was obtained as the effect of a $-1/r^3$ interaction — a relativistic-quantum modification of the original interparticle potential. In the usual QFT calculation, however, it arises from the zero-point energy of the fluctuation field $\Phi(x) - \phi$. These two descriptions, in “particle” and “field” languages, respectively, are just two complementary ways of looking at the same physics. There is a parallel with the Lamb shift in QED which can be viewed as the effect on the energy levels of a relativistic-quantum modification of the electron-nucleus potential [12]. However, it can also be viewed as arising from the interaction of the electron with the zero-point fluctuations of the electromagnetic field [13].

Both in our previous QFT calculation and here we find that the theory is “trivial,” in agreement with a large body of evidence [30, 31]. That is, we find that scattering amplitudes (for ‘atoms’ or ‘phonons’) vanish as the cutoff goes to infinity. However, because the physics of the vacuum involves infinitely many particles interacting infinitesimally weakly, there is a finite effect on $V_{\text{eff}}$. In QFT terms this effect is just the zero-point energy of the free fluctuation field $\Phi(x) - \phi$. It makes perfect sense that the effective potential of a “trivial” theory should be just the classical potential plus the zero-point energy of the “trivial” fluctuations. This predicted form of the effective potential has been confirmed to high precision by recent lattice data [34, 35, 36].

As so often, the difficulty is not in understanding new ideas, but in freeing ourselves from the tyranny of old ones. The conventional view (see, eg., [1, 37, 38]) has it that higher-loop effects change the $\phi^4 \ln \phi$ term in the effective potential to something like $\phi^4/|\ln \phi|$. There is then no SSB until $m^2$ goes negative, and the transition is second order. This so-called “RG improvement” arises from a re-summation of the geometric series of “leading logs” to obtain a “renormalized coupling constant” that runs according to $\lambda_R(\phi) = \lambda_R(\mu)/[1 - b\lambda_R(\mu)\ln(\phi/\mu)]$, where $\mu$ is some finite renormalization scale and $b = 3/(16\pi^2)$. Absorbing the 1 into the scale of the logarithm gives $\lambda_R(\phi) = 1/(b\ln(\tilde{\mu}/\phi))$, where $\tilde{\mu}$ is some finite mass scale. The effective potential in this approach is then basically $\lambda_R(\phi)\phi^4$. However, the Landau-pole problem, due to the absence of asymptotic freedom in perturbation theory, renders this approach unavoidably inconsistent, as we discuss in detail in Ref. 9. Moreover, precise lattice data for the effective potential cannot be fitted by any version of the “RG-improved” formula [34, 35, 36].
In the particle-gas picture, if one attempted to follow the “RG-improvement” program, one would apply leading-log re-summation to the 4-point function, thereby obtaining a scattering matrix element $M(q)$ with a Landau pole. Its 3-dimensional Fourier transform would then not exist, and one could not obtain any sensible result for the interparticle potential, $V(r)$.

The particle-gas language provides a clue to what is missing in current applications of the RG method to the $\lambda\Phi^4$ case. Those approaches assume that the main effect of higher-order radiative corrections is to make the coupling constant “run”; i.e., the form of the action remains basically the same at all scales, just with a $\lambda$ coupling that evolves. However, while some higher-order effects do indeed renormalize the strength of the bare interaction, there are other contributions — starting with the $t,u$-channel “fish” diagram — that produce qualitatively different physics, namely long-range attraction. This effect will appear in different guises in different formalisms, but its inclusion is crucial.

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Appendix A: QM scattering from a short-range potential

We briefly review quantum-mechanical scattering theory for a short-range repulsive potential, which is important background to our whole discussion. The scattering of two equal-mass particles is equivalent to the scattering of a particle of reduced mass $\mu = \frac{m}{2}$ from a fixed potential [38]. We consider a potential

\[
V(r) = \begin{cases} 
V_0 & r < r_0, \\
0 & r > r_0,
\end{cases}
\]  

(A.1)

with a high ‘aspect ratio’ $2\mu V_0 r_0^2 > 1$ [40]. Ultimately, we shall be interested in the limit $r_0 \to 0$ where the interaction becomes pointlike. In this limit the potential becomes

\[
V(r) \to W\delta^{(3)}(r), \quad W \equiv \frac{4}{3}\pi r_0^3 V_0.
\]  

(A.2)

For ‘low energies’ $k \ll 1/r_0$, the scattering is entirely $s$-wave. The cross section is then given by a single term of the partial-wave expansion:

\[
\frac{d\sigma}{d\Omega} = |f(\theta)|^2 = \frac{1}{k^2} \left| e^{i\delta_0} \sin \delta_0 \right|^2,
\]  

(A.3)

where $\delta_0$ is the $s$-wave phase shift, defined by the asymptotic behaviour of the radial wavefunction:

\[
\chi(r) \equiv r\psi(r) \propto \sin(kr + \delta_0) \quad \text{as } r \to \infty.
\]  

(A.4)

A low-energy expansion of $k \cot \delta_0$ is the famous “effective-range formula:”

\[
k \cot \delta_0 = -\frac{1}{a} + \frac{1}{2} r_{\text{eff}} k^2 + O(k^4),
\]  

(A.5)

where $a$ is the scattering length and $r_{\text{eff}}$ is the effective range. We may neglect the $r_{\text{eff}}$ term at ‘low energies’ $k \ll 1/a$ [40]. At these low energies $\delta_0$ is small and negative, $\delta_0 = -ka$, giving a scattering amplitude $f(\theta) = -a$.

For the specific potential (A.1) the Schrödinger equation for energy $E = k^2/(2\mu)$ yields

\[
\chi(r) = \begin{cases} 
A \sinh \alpha r + B \cosh \alpha r & r < r_0, \\
C \sin kr + D \cos kr & r > r_0,
\end{cases}
\]  

(A.6)

where $A, B, C, D$ are constants and

\[
\alpha = \alpha(k) \equiv (2\mu(V_0 - E))^{1/2}.
\]  

(A.7)

The condition that $\psi(r)$ is regular at $r = 0$ requires $\chi(r = 0) = 0$, and hence fixes $B = 0$. Matching conditions at $r = r_0$ give

\[
A \sinh \alpha r_0 = C \sin kr_0 + D \cos kr_0
\]  

(A.8)
\[ \alpha A \cosh \alpha r_0 = k \left( C \cos kr_0 - D \sin kr_0 \right). \] (A.9)

From the ratio of these two equations we obtain
\[ \frac{D}{C} = \left[ \frac{k \cos kr_0 \sinh \alpha r_0 - \alpha \sin kr_0 \cosh \alpha r_0}{k \sin kr_0 \sinh \alpha r_0 + \alpha \cos kr_0 \cosh \alpha r_0} \right]. \] (A.10)

Comparing (A.4) and (A.6) we see that \( \tan \delta_0 = \frac{D}{C} \). Hence, from (A.5) we have
\[ a = \lim_{k \to 0} \left( -\frac{1}{k \cot \delta_0} \right) = \lim_{k \to 0} \left( -\frac{D}{kC} \right). \] (A.11)

Hence, we obtain
\[ a = r_0 \left( 1 - \frac{\tanh \alpha r_0}{\alpha r_0} \right), \] (A.12)

with \( \alpha \equiv \alpha(0) = (2\mu V_0)^{1/2} \). Since the function \( 1 - \tanh x/x \) lies between 0 and 1, we see that \( a \leq r_0 \). No matter how large we make \( V_0 \), the strength of the potential, the scattering length can never exceed \( r_0 \). Thus, one has “triviality” — zero scattering amplitude for all finite energies — in the limit \( r_0 \to 0 \).

In particular, in the \( \delta \)-function limit, \( V(r) \to W\delta^{(3)}(r) \) one has \( \alpha r_0 \sim \sqrt{W/r_0} \to \infty \) and so \( a = r_0 \to 0 \). The fact that a \( \delta \)-function potential produces zero scattering in 3 (or more) dimensions reflects the “triviality” of (NR) \( \lambda \Phi^4 \) theory \[1\].

In Born approximation, however, one would get
\[ a = \frac{\mu}{2\pi} \int d^3r e^{-ikr} V(r) = \frac{\mu}{2\pi} \frac{4}{3} \pi r_0^3 V_0 = \frac{\mu}{2\pi} W \] (A.13)

which is finite if \( W \) is finite. One may re-write the Born result as
\[ a = \frac{1}{3}(\alpha r_0)^2 r_0 \] (A.14)

and recognize that perturbation theory corresponds to an expansion of the result (A.12) for small \( \alpha r_0 \). However, a \( \delta \)-function potential actually entails a infinitely large \( \alpha r_0 \), as noted above.

**Appendix B: QM scattering from a \(-1/r^3\) potential**

In this appendix we treat quantum-mechanical scattering from a \(-1/r^3\)-plus-hard-core potential. We shall denote the scattering length for such a potential by \( a_{\text{eff}}(r_{\text{max}}) \), since it is necessary to cut off the potential at large distances to avoid an infrared divergence:
\[ V(r) = \begin{cases} V_0 & r < r_0, \\ -A/r^3 & r_0 < r < r_{\text{max}}, \\ 0 & r > r_{\text{max}}, \end{cases} \] (B.1)
In the region $r_0 < r < r_{\text{max}}$ the Schrödinger equation for zero energy ($\chi(r) = r\psi(r)$):

$$\frac{d^2\chi}{dr^2} + \frac{2\mu A}{r^3} \chi = 0 \quad (\text{B.2})$$

can be solved exactly. By defining

$$z \equiv 2 \left( \frac{2\mu A}{r} \right)^{1/2} \quad (\text{B.3})$$

and

$$\chi(r) = w(z)/z, \quad (\text{B.4})$$

the equation is transformed into

$$z^2 \frac{d^2 w}{dz^2} + z \frac{dw}{dz} + (z^2 - 1)w = 0, \quad (\text{B.5})$$

whose solutions are the Bessel functions $J_1(z)$ and $Y_1(z)$ \[19\].

The zero-energy solution for $\chi(r)$ in the three regions is then

$$\chi(r) = \begin{cases} 
A \sinh \alpha r & r < r_0, \\
(FJ_1(z) + GY_1(z))/z & r_0 < r < r_{\text{max}}, \\
\tilde{C}r + D & r > r_{\text{max}},
\end{cases} \quad (\text{B.6})$$

where $A, F, G, \tilde{C}, D$ are constants and $\alpha \equiv \sqrt{2\mu V_0}$. The solution at large $r$ is to be regarded as the $k \to 0$ limit of $C \sin kr + D \cos kr$, and so $\tilde{C}$ is identified with $kC$. Recall from the previous appendix that the scattering length is the $k \to 0$ limit of $-D/(kC)$, which is $-D/\tilde{C}$. This slight cheat allows to obtain the scattering length without needing to solve the Schrödinger equation for a general energy.

Matching the solutions at $r_{\text{max}}$ gives

$$\tilde{C}r_{\text{max}} + D = \frac{1}{z_m} (FJ_1(z_m) + GY_1(z_m)) \quad (\text{B.7})$$

$$\tilde{C} = \frac{1}{2r_{\text{max}}} (FJ_2(z_m) + GY_2(z_m)), \quad (\text{B.8})$$

where $z_m$ is $z$ at $r = r_{\text{max}}$. From the ratio of these equations we obtain $a_{\text{eff}} = -D/\tilde{C}$ as

$$a_{\text{eff}} = r_{\text{max}} \left[ 1 - \frac{2}{z_m} \frac{(FJ_1(z_m) + GY_1(z_m))}{FJ_2(z_m) + GY_2(z_m)} \right]. \quad (\text{B.9})$$

For large $r_{\text{max}} \gg 2\mu A$ we may expand the Bessel functions for $z$ small \[19\]:

$$J_1(z) = \frac{1}{2} z + O(z^2), \quad Y_1(z) = \frac{1}{\pi} \left( -\frac{2}{z} + z[\ln(z/2) + \gamma - \frac{1}{2}] + O(z^3) \right)$$

$$J_2(z) = O(z^2), \quad Y_2(z) = \frac{1}{\pi} \left( -\frac{4}{z^2} - 1 + O(z^2 \ln z) \right). \quad (\text{B.10})$$
A cancellation of leading terms leaves
\[ a_{\text{eff}} = r_{\text{max}}^{2} \frac{z_m^2}{4} \left( 2 \ln \left( \frac{z_m}{2} \right) + 2 \gamma + \pi \frac{F}{G} \right). \] (B.11)

The ratio \( F/G \) can be found by a similar matching at \( r = r_0 \):
\[ \frac{F}{G} = \frac{z_0 \tanh \alpha r_0 Y_2(z_0) - 2 \alpha r_0 Y_1(z_0)}{2 \alpha r_0 J_1(z_0) - z_0 \tanh \alpha r_0 J_2(z_0)}. \] (B.12)

For small \( r_0 \) (\( r_0 \ll 2 \mu A \)) we may expand the Bessel functions for large \( z \) to obtain
\[ \frac{F}{G} = \frac{\kappa + \tan(z_0 - \frac{3 \pi}{4})}{\kappa \tan(z_0 - \frac{3 \pi}{4}) - 1}, \] (B.13)
where \( \kappa \equiv z_0 \tanh \alpha r_0 / (2 \alpha r_0) \). Note that we have not assumed that \( \alpha r_0 \) is small. Because of the \( \tan(z_0 - \frac{3 \pi}{4}) \) factors, with \( z_0 \) large, the result is sensitive to \( z_0 \) modulo \( 2 \pi \), and so, from the definition of \( z \), depends on the precise ratio of \( 2 \mu A \) to \( r_0 \). Thus, the ratio \( F/G \) can take any value, and depends sensitively on the precise details of the hard-core potential and how it joins on to the \( -A/r^3 \) potential.

Substituting for \( z_m \) we may write the result (B.11) as
\[ a_{\text{eff}} = a - 2 \mu A \ln(r_{\text{max}}/r_0), \] (B.14)
where \( a \equiv 2 \mu A [\ln(2 \mu A/r_0) + 2 \gamma + \pi F/G] \). This can be compared with the result in the Born approximation which is
\[ a_{\text{eff}} = \frac{\mu}{2\pi} \int d^3 r V(r) = a_{\text{Born}} - 2 \mu A \ln(r_{\text{max}}/r_0), \] (B.15)
where \( a_{\text{Born}} \equiv \frac{2}{3} \mu V_0 r_0^3 \) (see (A.13)). Thus, we may effectively use the Born-approximation result provided we replace \( a_{\text{Born}} \) by the free parameter \( a \). We may think of \( a \) as the actual scattering length due to the core. (More precisely, it is due to the core and the “un-Born part” of the \( -1/r^3 \) potential [11].) Just as in the pseudopotential approach of [3], all the details of the short-range potential can be parametrized by the single parameter \( a \).

The fact that the Born approximation yields the correct \( \ln r_{\text{max}} \) term is natural, since this term arises from the large-\( r \) tail where the potential is weak.

In the \( \lambda \Phi^4 \) context the \( -A/r^3 \) potential arises from two hard-core interactions connected by particle-pair exchange. It follows that \( A \) will be proportional to \( a^2 \). In other words, if the short-range interaction is parametrized by a coupling \( \lambda \equiv 8 \pi m a \), there will be a \( -1/r^3 \) interaction of order \( \lambda^2 \). We can correctly treat the \( -1/r^3 \) interaction in Born approximation even though we cannot rely on Born approximation to relate \( \lambda \) to the original bare \( \lambda \).
Appendix C: Field re-scaling in ‘particle-gas’ language

In the phase-transition region, our effective potential (8.2), written in terms of $\phi_B$ is an extremely flat function, since the $\phi_B = 0$ vacuum and the $\phi_B = \pm \nu_B$ vacua are infinitely far apart ($\nu_B^2 = O(\ln \Lambda)$), but differ in energy density only by a finite amount. It is therefore natural to want to re-define the scale of the horizontal axis; i.e., to define a “renormalized” or “re-scaled” field $\phi_R$.

To do this we introduced the following procedure [7, 5, 10]: First we decompose the full field $\Phi_B(x)$ into its zero- and finite-momentum pieces:

$$\Phi_B(x) = \phi_B + h(x),$$

where $\int d^4x\, h(x) = 0$. Then we re-scale the zero-momentum part of the field:

$$\phi_B^2 = Z\phi\phi_R^2.$$  \hspace{1cm} (C.2)

The finite-momentum modes, however, are not re-scaled. The point is that for a “trivial” theory, as here, scattering theory and the Lehmann spectral decomposition require the wavefunction renormalization constant $Z_h$ (in $h(x) = Z_h h_R(x)$) to be unity. However, there is no constraint on $Z_\phi$, since there is no scattering theory for a zero-momentum mode — the incident particles never reach each other. For the purpose of rendering $V_{\text{eff}}$ a finite function of $\phi_R$, any $Z_\phi$ of order $\ln \Lambda$ would do: What determines the absolute normalization is the requirement that

$$\left.\frac{d^2 V_{\text{eff}}}{d\phi_R^2}\right|_{\phi_R=\nu_R} = M_h^2.$$  \hspace{1cm} (C.3)

This is a standard renormalization condition, but the context here is rather unconventional. In this Appendix we clarify its meaning using the ‘particle-gas’ language.

First, consider a slight perturbation of the symmetric vacuum state (“empty box”). We add a very small density $n$ of atoms, each with zero 3-momentum. The energy density is now:

$$\mathcal{E}(n) = 0 + nm + O(n^2 \ln n),$$  \hspace{1cm} (C.4)

where the first term is the energy of the unperturbed vacuum state (zero); the second term is the rest-mass cost of introducing $N$ particles, divided by the volume; and the third term is negligible if we consider a sufficiently small density $n$. Thus, we obviously have the relation

$$\left.\frac{\partial \mathcal{E}}{\partial n}\right|_{n=0} = m.$$  \hspace{1cm} (C.5)
The equivalent in field language follows from the translation

\[ n = \frac{1}{2} m \phi_B^2 \]  

and is that

\[ \mathcal{E}(\phi_B) \equiv V_{\text{eff}}(\phi_B) = 0 + \frac{1}{2} m^2 \phi_B^2 + \mathcal{O}(\phi_B^4 \ln \phi_B^2), \]  

so that

\[ \left. \frac{\partial^2 V_{\text{eff}}}{\partial \phi_B^2} \right|_{\phi_B=0} = m^2. \]  

Now, let us consider a slight perturbation of the SSB vacuum (the box filled with a spontaneously-generated condensate). Before we perturb it, this state has a density \( n_v \) of atoms, where \( n_v \) is a (local) minimum of \( \mathcal{E}(n) \). From (C.6) we have the translation \( n_v = \frac{1}{2} m v_B^2 \). This vacuum state, though complicated in terms of atoms, is simple in terms of phonons: By definition it is just the state with no phonons. We now perturb it by adding a small density \( n' \) of phonons, each with negligible kinetic energy. The energy density of the perturbed state is then

\[ \mathcal{E}(n') = \mathcal{E}(n' = 0) + n' M_h + \ldots, \]  

where the first term is the energy density of the unperturbed state; the second term is the rest-mass cost of the added phonons; and any other terms from phonon interactions are negligible if \( n' \) is small enough. Thus, paralleling (C.5) we have

\[ \left. \frac{\partial \mathcal{E}}{\partial n'} \right|_{n'=0} = M_h. \]  

It is now natural to define a “phonon field” whose constant part, \( f \), is related to the phonon density \( n' \) by

\[ n' \equiv \frac{1}{2} M_h f^2, \]  

in analogy to (C.6). The “renormalized field” \( \phi_R \) is simply this \( f \) plus a constant. A constant must be added if we want to have \( \phi_R \) proportional to \( \phi_B \). Since, by definition, \( f = 0 \) when \( \phi_B = v_B \), we need

\[ \phi_R \equiv f + v_R \]  

with

\[ \frac{v_R}{v_B} = \frac{\phi_R}{\phi_B} \equiv \frac{1}{Z_{\phi}^{1/2}}. \]  

Now we can eliminate \( f \) in favour of \( \phi_R \) and re-write (C.11) as

\[ n' = \frac{1}{2} M_h (\phi_R - v_R)^2. \]
Hence, (C.9) can be re-written in field language as
\[ \mathcal{E}(\phi_R) \equiv V_{\text{eff}}(\phi_R) = V_{\text{eff}}(\phi_R = v_R) + \frac{1}{2} M_h^2 (\phi_R - v_R)^2 + \ldots \]  \hspace{1cm} (C.15)

The crucial condition, Eq. (C.3), follows directly from this. It is simply the field-language equivalent of “phonon-language” equation, (C.10), and just says that the phonon mass is, self-consistently, \( M_h \). It is also, of course, the SSB-vacuum counterpart to Eq. (C.8) for the symmetric vacuum.

The moral of this story is that the constant field \( \phi_R - v_R \) is related to phonon density \( n' \) in the same fashion that \( \phi_B \) is related to the atom density \( n \). Note that there is a duality under

\[
\begin{align*}
\text{atoms} & \leftrightarrow \text{phonons}, \quad \phi_B \leftrightarrow \phi_R - v_R, \quad n \leftrightarrow n', \quad Z_\phi \leftrightarrow Z_\phi^{-1}.
\end{align*}
\]  \hspace{1cm} (C.16)

Physically, this means that, we may use either ‘atom’ or ‘phonon’ degrees of freedom to describe the theory. Small excitations about the ‘phonon vacuum’ are easily described in terms of phonons, but are complicated in terms of atoms — and vice versa. Moreover, just as we can describe the ‘phonon vacuum’ as a complicated coherent state of atoms, we may also view the ‘atom vacuum’ (empty box) as a complicated coherent state of phonons.

One can obtain various formulas for \( Z_\phi \) from the above considerations. There is no way to express \( Z_\phi \) just in terms of phonon properties of the phonon vacuum, nor in terms of atom properties of the atom vacuum. \( Z_\phi \) is about the connection between these two states, and requires some knowledge of the atom properties of the phonon vacuum, or, dually, the phonon properties of the atom vacuum. For instance, in field language, one may write

\[
Z_{\phi}^{-1} = \frac{1}{M_h^2} \left. \frac{\partial^2 V_{\text{eff}}}{\partial \phi_B^2} \right|_{\phi_B = v_B}.
\]  \hspace{1cm} (C.17)

Translated to particle language this gives

\[
Z_{\phi}^{-1} = \frac{2 m_n v}{M_h^2} \left. \frac{\partial^2 \mathcal{E}}{\partial n^2} \right|_{n = n_v},
\]  \hspace{1cm} (C.18)

involving how the energy density varies around the phonon vacuum when we vary the atom density \( n \). (There is also a ‘dual’ version of both these formulas.) Note that the first derivative \( \frac{\partial \mathcal{E}}{\partial n} \bigg|_{n = n_v} \), vanishes, since \( \mathcal{E}(n) \) has a minimum at \( n = n_v \). However, in terms of phonons, when we vary \( n' \) about \( n' = 0 \) (which corresponds to the state \( n = n_v \)), we get Eq. (C.10). It follows that we must have \( \frac{\partial \mathcal{E}}{\partial n'} \bigg|_{n = n_v} = 0 \); i.e., around the phonon vacuum,
a small variation in atom density causes no first-order change in phonon density (nor in
the energy density). One can express $Z_\phi$ in terms of the second derivative as:

$$Z_\phi^{-1} = \frac{2mn_v}{M_h} \frac{\partial^2 n'}{\partial n^2}\bigg|_{n=n_v}. \quad (C.19)$$

(Again, there is a ‘dual’ formula).

$Z_\phi$ can be written in yet more ways. Substituting $(\phi_R - v_R)^2 = (\phi_B - v_B)^2/Z_\phi$ into
(C.14), and then using $\phi_B = \sqrt{2n/m}$ from (C.6) yields

$$Z_\phi = \frac{M_h (\sqrt{n} - \sqrt{n_v})^2}{m n'/n|_{n=0}}. \quad (C.20)$$

Evaluated at $n = n_v$ this gives $0/0$, but evaluated at $n = 0$ it gives

$$Z_\phi = \frac{M_h n_v}{m n'|_{n=0}}, \quad (C.21)$$

which is a ‘self-dual’ formula involving the ratio (density of atoms in the phonon vac-

uum)/(density of phonons in the atom vacuum). The correctness of the formula be-
comes obvious if we use the $n, \phi_B$ and $n', \phi_R$ translations, since it then reduces to just

$$Z_\phi = v_B^2/v_R^2.$$

**Appendix D: A tidier calculation**

Finally, to bring together all the pieces of the picture, we describe a relativistic version of
Huang’s hard-sphere Bose-gas analysis. We start with the Hamiltonian of the relativistic
field theory, Eq. (2.1). It is convenient to subtract a term $\mu \hat{N}$, where $\hat{N}$ is the number
operator $\sum a_k^\dagger a_k$. In the end we shall set $\mu = 0$. However, when comparing with the NR
analysis we must set $\mu = m$ to convert to the NR convention that particle rest masses do
not count in the kinetic energy — so that in the total energy of the system, $H$, we must
subtract $m$ for each particle in the system; i.e., $H_{\text{NR}} = H - m\hat{N}$.

The calculation then proceeds as in Sect. 4. Instead of Eq. (4.6) for the NR field
we have Eq. (7.3) for the relativistic field [14]. Making this substitution and writing
$a_0 \sim a_0^\dagger \sim \sqrt{N}$ (which is equivalent to shifting the field by a constant, $\phi$) leads to

$$H - \mu \hat{N} = V \left( (m - \mu)n + \frac{\lambda n^2}{6m^2} \right) + \sum_{k \neq 0} \left[ a_k^\dagger a_k \left( E_k - \mu + \frac{\xi}{2E_k} \right) + \frac{\xi}{4E_k} \left( a_k a_{-k} + a_k^\dagger a_{-k}^\dagger \right) \right] \quad (D.1)$$

with

$$\xi = \frac{\lambda n}{m} = \frac{\phi^2}{\lambda} \quad (D.2)$$
Some comments on this are in order: Our $H$ was normal ordered (with respect to mass $m$), so the $\lambda$-independent terms are rather obvious. The other terms come from $\int d^3x : \Phi^4 :$. Just as in the NR calculation, one needs to keep terms with at least two $a_0$ or $a_0^\dagger$ factors; other terms will be relatively suppressed by $1/\sqrt{N}$ factors. That is (in the words of Ref. [4]) we need only take into account interaction of particles in the condensate with each other and interaction of “excited” particles with particles in the condensate, neglecting interaction of “excited” particles with each other. Finally, we observe that in the NR case, substituting $\mu = m$, and $E_k \sim m$, and $E_k - m \approx k^2/(2m)$ this equation reproduces Eq. (1.8) (except for a different ground-state-energy term [13]) if we identify $\xi$ with $8\pi na$. This implies the identification $\lambda = 8\pi ma$, in agreement with (7.2) above.

We now proceed to introduce $b_k, b_k^\dagger$, just as in Eq. (4.9). Requiring the $b_k b_{-k}$ terms to cancel gives

$$\alpha_k = 1 + x^2 - x\sqrt{x^2 + 2} = \frac{2}{\xi} E_k (E_k - \mu), \quad (D.3)$$

which has the same form as (4.10), except for the modified definition of $x^2$ (which has the form anticipated in the final paragraph of Sect.4). We are then left (after a little algebra) with

$$H_{\text{eff}} - \mu \hat{N} = \text{const} + \frac{\xi}{4} \sum_{k \neq 0} \left[ b_k^\dagger b_k \frac{1}{E_k \alpha_k} - b_k b_{k}^\dagger \frac{\alpha_k}{E_k} \right], \quad (D.4)$$

and hence

$$H_{\text{eff}} - \mu \hat{N} = E_0 + \sum_{k \neq 0} \tilde{E}(k) b_k^\dagger b_k, \quad (D.5)$$

which is the analog of (4.11). The new spectrum $\tilde{E}_k$ is given by

$$\tilde{E}_k = (E_k - \mu) \sqrt{1 + \frac{\xi}{E_k (E_k - \mu)}} \quad (D.6)$$

For $\mu = m$ this gives the right NR limit. With $\mu = 0$ one gets

$$\tilde{E}(k) = \sqrt{E_k^2 + \xi} = \sqrt{k^2 + m^2 + \xi} \quad (D.7)$$

which has the expected relativistic form. We may then identify the ‘phonon’ (‘Higgs’) mass squared as

$$M^2(\phi) = m^2 + \xi = m^2 + \frac{1}{2} \lambda \phi^2. \quad (D.8)$$

This is the same as before, except that we have been more careful and retained the $m^2$ term. However, this makes no real difference, since in the interesting region $m^2$ is infinitesimal in comparison with $\xi = \frac{1}{2} \lambda \phi^2$. 32
The $E_0$ term in (D.5) is

$$E_0 = V \left[ (m - \mu)n + \frac{\lambda n^2}{6m^2} \right] - \frac{\xi}{4} \sum_{k \neq 0} \frac{\alpha_k}{E_k},$$

(D.9)

where the term in square brackets (for $\mu = 0$) is just the classical potential, as one sees by substituting $n = \frac{1}{4}m\phi^2$. The last term, which arises in the step from (D.4) to (D.5), is the direct analog of the second term in (4.14) in the NR calculation. The NR version has $m$ rather than $E_k$ in the denominator, and its $k$ summation would be linearly divergent, but for the $(\partial/\partial r)$ pseudopotential subtlety. The relativistic analog, because of the extra $1/E_k$ factor is only logarithmically divergent. It represents a quantum contribution to the ground-state energy — and it is not hard to recognize it as the effect we have discussed, arising physically from the quantum-induced $-1/r^3$ interaction between the atoms.

Rewriting the previous equation (for $\mu = 0$), the energy density of the ground state as a function of $n$, or equivalently, $\phi$, is:

$$V_{\text{eff}} = E_0/V = \frac{1}{2} m^2 \phi^2 + \frac{\lambda}{4!} \phi^4 - \frac{1}{8} \lambda \phi^2 \int \frac{d^3k}{(2\pi)^3} \frac{\alpha_k}{E_k}.$$  

(D.10)

Substituting for $\alpha_k$ from (D.3), and noting that $x\sqrt{x^2 + 2} = 4E_k \tilde{E}_k/(\lambda \phi^2)$ one has

$$V_{\text{eff}} = \frac{1}{2} m^2 \phi^2 + \frac{\lambda}{4!} \phi^4 + I_1(M) - I_1(m) - \frac{1}{4} \lambda \phi^2 I_0(m),$$

(D.11)

where

$$I_1(M) \equiv \int \frac{d^3k}{(2\pi)^3} \frac{1}{2} \tilde{E}_k, \quad I_1(m) \equiv \int \frac{d^3k}{(2\pi)^3} \frac{1}{2} E_k, \quad I_0(m) \equiv \int \frac{d^3k}{(2\pi)^3} \frac{1}{2 E_k}.$$  

(D.12)

In field language $I_1(M)$ is just the zero-point energy contribution from a free field with a mass $M(\phi)$, while the last two terms in (D.11) are just the subtractions implied by normal ordering. These terms remove the quartic and quadratic divergences in $I_1(M)$, leaving a logarithmic divergence. Explicitly, in the $m \to 0$ limit the result is

$$I_1(M) - I_1(m) - \frac{1}{4} \lambda \phi^2 I_0(m) \to \frac{\lambda^2}{256\pi^2} \phi^4 \left[ \ln\left(\frac{\lambda \phi^2}{\Lambda^2}\right) - \frac{3}{2} \right].$$

(D.13)

Thus, the result for $V_{\text{eff}}$ is our old friend the “one-loop” result.

The crucial point is that this form of $V_{\text{eff}}$ is effectively exact. Just as in the NR calculation the validity of the result requires only the ‘diluteness’ ($n a^3 \to 0$) and ‘low-energy’ ($k a \to 0$) approximations. The only subtlety is that the scattering length $a$ should be the actual scattering length. The calculation of the actual $a$ in terms of the original potential may be very hard, but is of no importance. In fact the actual shape of the
original potential is irrelevant, (provided only that it is short-range and repulsive); its effect is entirely parametrized by the scattering length it gives rise to. Similarly, here the ‘\(\lambda\)’ has to be understood as some effective coupling strength, incorporating the effects of the original QFT interaction (which could include \(\Phi^6, \Phi^8, \ldots\), terms) to all orders. In the QFT case, if a continuum limit is to exist (i.e., if finite physics is to exist in the limit \(\Lambda \to \infty\)), then \(a\) and \(\lambda\) must be infinitesimally small, which corresponds to “triviality” of the theory. However, there is still non-trivial physics for the ground state, due to an infinite number of infinitesimally weakly interacting particles.
References

[1] M. A. B. Bég and R. C. Furlong, Phys. Rev. D 31, 1370 (1985); K. Huang, Intl. J. Mod. Phys. A 4, 1037 (1989); R. Jackiw, in Bég Memorial Volume, eds. A. Ali and P. Hoodbhoy (World Scientific, Singapore, 1991).

[2] N. N. Bogoliubov, J. Phys. USSR 11 (1947) 23; T. D. Lee, K. Huang, and C. N. Yang, ibid 106 (1957) 1135; K. Huang, C. N. Yang, and J. M. Luttinger, Phys. Rev., 105 (1957) 776; K. A. Brueckner and K. Sawada, ibid 106 (1957) 1117.

[3] K. Huang, Statistical Mechanics, 2nd Edition (John Wiley, New York, 1987). The relevant sections are, principally, 10.5, 12.3-12.5, 13.8, A.2, A.3.

[4] A. A. Abrikosov, L. P. Gorkov, and I. E. Dzyaloshinski, Methods of Quantum Field Theory in Statistical Physics (edited and translated by R. A. Silverman) (Dover Publications, New York, 1975). The most relevant sections are 4 and 25. Note especially footnote 24 in Sect. 4.

[5] M. Consoli and P. M. Stevenson, Z. Phys. C. 63 (1994) 427.

[6] S. Coleman and E. Weinberg, Phys. Rev. D 7 (1973) 1888. This classic paper computed the 1-loop effective potential for \( \lambda \Phi^4 \) and other theories. It also pointed out quite correctly that for \( \lambda \Phi^4 \) theory the 1-loop result cannot be justified on the basis of a well-convergent loop expansion. (In other theories the loop expansion is better behaved.) The problem is that at the SSB minimum the 1-loop ‘correction’ becomes as large as the classical term, and one can infer that higher-loop ‘corrections’ will be equally large. The solution advocated was to use a “RG-improvement” procedure – a leading-log re-summation – which converts the result to a \( \phi^4/|\ln \phi| \) form. This was a very reasonable conclusion in 1973, but we now know that \( \lambda \Phi^4 \) has a “triviality” property, quite unlike other QFT’s. Our point [5]-[10] is that, for nonperturbative reasons independent of the loop-expansion mindset, the “one-loop” form is effectively exact in \( \lambda \Phi^4 \) theory.

[7] M. Consoli and P. M. Stevenson, hep-ph/9303256; hep-ph/9407334.

[8] U. Ritschel, Phys. Lett. B 318 (1993) 617.

[9] M. Consoli and P. M. Stevenson, Mod. Phys. Lett. A 11 (1996) 2511.

[10] M. Consoli and P. M. Stevenson, Phys. Lett. B 391 (1997) 144.
Our work had its origins in studies of the Gaussian effective potential. Some key papers of this ‘pre-history’ include Ref. [24] and P. M. Stevenson and R. Tarrach, Phys. Lett. B 176 (1986) 436; V. Branchina, P. Castorina, M. Consoli, and D. Zappalá, Phys. Rev. D 42 (1990) 3587; R. Ibañez-Meier and P. M. Stevenson, Phys. Lett. B 297 (1992) 144; V. Branchina, M. Consoli and N. M. Stivala, Z. Phys. C 57 (1993) 251.

H. A. Bethe, Phys. Rev. 72 (1947) 339.

J. D. Bjorken and S. D. Drell, Relativistic Quantum Mechanics (McGraw-Hill, New York, 1965).

G. Feinberg, J. Sucher, and C. K. Au, Phys. Rep. 180 (1989) 85.

Note that we are ignoring identical-particle complications here.

One may also understand the $E/m$ factor as follows: In obtaining the Born-approximation cross section via Fermi’s Golden Rule, the only changes are to the flux factor and to the density of states. The flux factor, appearing in the denominator, is proportional to velocity, which is now $p/E$ instead of $p/m$. In the free-particle density of states the counting of states in an interval $dp$ goes as usual, but when relating $dp$ to $dE$ we must use $E^2 = p^2 + m^2$ rather than the NR form $E - m = p^2 / (2m)$, giving $pdp = E dE$ rather than $pdp = mdE$. We therefore obtain two factors $E/m$ in the cross section, and hence one factor of $E/m$ in the scattering amplitude.

I. S. Gradshteyn and I. M. Ryzhik, Table of Integrals, Series, and Products (Corrected and Enlarged Edition) (Academic Press, San Diego, 1980).

See, eg., P. Ramond, Field Theory: a Modern Primer (Addison-Wesley, 1981), Eq. (4.4.16).

M. Abramowitz and I. A. Stegun, Handbook of Mathematical Functions (Dover Publications, New York, 1965).

M. H. Anderson et al, Science 269 (1995) 198; C. C. Bradley et al, Phys. Rev. Lett. 75 (1995) 1687; K. B. Davis et al, ibid (1995)3969.

In this sense, any short-range potential with scattering length $a$ is equivalent to a “hard-sphere” potential $V(r) = \infty$ for $r < a$ and $V(r) = 0$ for $r > a$. This corresponds to viewing the atoms as hard spheres with radius $a/2$, the closest approach distance being twice the radius. However, in general $a$ depends on the strength of the
potential as well as its spatial extent. It will be important later not to confuse $a$ with the hard-core radius, $r_0$, of our $-1/r^3$-plus-hard-core potential.

[22] The equation quoted is (13.94) of Huang [3]. Beware some misprints in the preceding equation, (13.93).

[23] The primed summation is defined by $\sum' f_k = \lim_{r \to 0} \frac{\partial}{\partial r} (r \sum f_k e^{ikr})$. For any $f_k$ that leads to a convergent integral, the prime makes no difference. However, for $f_k = 1/k^2$, after performing the angular integration, and changing the variable to $u = kr$, one gets $\frac{\partial}{\partial r}$ of an $r$-independent quantity.

[24] M. Consoli and A. Ciancitto, Nucl. Phys. B 254 (1985) 653.

[25] F. Reif, Fundamentals of Statistical and Thermal Physics (McGraw-Hill, New York, 1965), Sect. 10.3, 10.4.

[26] Identical-particle factor-of-2 issues are a headache that we shall not attempt to resolve fully. However, we argue that $|M|^2$ in (3.1) should become $|M(\theta) + M(\pi - \theta)|^2 = (2\lambda)^2$. The corresponding modification of the QM formula (3.2) seems to be already allowed for in Huang’s $a$; that is, $|f(\theta) + f(\pi - \theta)|^2 \equiv a^2$.

[27] K. M. Benson, J. Bernstein, and S. Dodelson, Phys. Rev. D 44 (1991) 2480; J. Bernstein and S. Dodelson, Phys. Rev. Lett. 66 (1991) 683.

[28] The $2/3$ factor missing in the $\lambda \phi^4$ term is probably attributable to the $s$-channel (annihilation/re-creation) interaction, which is not considered in the NR analysis. In our case, where the atom is its own antiparticle, the $s$-channel would contribute to the scattering length $a$ but presumably not to the hard-sphere repulsion energy, since it allows the atoms to overlap. The unwanted factor $1/2$ in the last term is presumably due to our inadequate treatment of identical-particle complications. See Appendix D for a calculation where the numerical factors are under control.

[29] R. Jackiw, Phys. Rev. D 9 (1974) 1686.

[30] J. Fröhlich, Nucl. Phys. B 200(FS4) (1982) 281; M. Aizenman, Phys. Rev. Lett. 47 (1981) 1; A. Sokal, Ann. Inst. H. Poincaré, 37 (1982) 317; R. Fernández, J. Fröhlich, and A. D. Sokal, Random Walks, Critical Phenomena, and Triviality in Quantum Field Theory (Springer-Verlag, Berlin, 1992).
[31] K. G. Wilson and J. Kogut, Phys. Rep. C12 (1974) 75; G. A. Baker and J. M. Kincaid, Phys. Rev. Lett. 42 (1979) 1431; B. Freedman, P. Smolensky, and D. Weingarten, Phys. Lett. B 113 (1982) 481; D. J. E. Callaway and R. Petronzio, Nucl. Phys. B 240 (1984) 577; I. A. Fox and I. G. Halliday, Phys. Lett. B 159 (1985) 148; C. B. Lang, Nucl. Phys. B 265 (1986) 630.

[32] We could allow the original Lagrangian to include $\Phi^6$, $\Phi^8$, ..., terms, which would also modify the effective $\lambda$ but would not change the physics.

[33] T. A. Weldon, Phys. Rev. 74 (1948) 1157; See Ref. [13] pp. 58-60.

[34] A. Agodi, G. Andronico, and M. Consoli, Z. Phys. C 66 (1995) 439.

[35] A. Agodi, G. Andronico, P. Cea, M. Consoli, L. Cosmai, R. Fiore, and P. M. Stevenson, Mod. Phys. Lett. A 12 (1997) 1011.

[36] A. Agodi, G. Andronico, P. Cea, M. Consoli, and L. Cosmai, contribution to the Lattice ’97 conference, Edinburgh, July 1997.

[37] M. E. Peskin and D. V. Schroeder, An Introduction to Quantum Field Theory (Addison-Wesley, 1995).

[38] C. Itzykson and J. M. Drouffe, Statistical field theory, (Cambridge University Press, 1989); Sect. 5.4.2.

[39] We shall again ignore indistinguishable-particle complications.

[40] The condition $2\mu V_0 r_0^2 > 1$ means that for ‘low energies’ $k \ll 1/r_0$ the particle’s kinetic energy cannot exceed the barrier height $V_0$. This precludes obtaining an effective range $r_{\text{eff}} \gg r_0$.

[41] Note that it is perfectly possible to have $a \gg r_0$. In fact, one may check that the behaviour envisaged in Sect. 8, Eqs. [8.10], [8.11], in which $a \sim \mathcal{O}(1/\sqrt{\ln r_0})$, is consistent with the generic case that $F/G$ is an order-1 number.

[42] The connection between the relativistic and NR fields is discussed by T. S. Evans, Proceedings of the 4th Workshop on Thermal Field Theories and their Applications (World Scientific, 1996) [hep-ph/9510298]. (His discussion is for a complex $\Phi$ field, but is easily adapted to the real-$\Phi$ case.) If $\Phi(x)$ is the relativistic field and $\Pi(x)$ is its canonical conjugate, then the NR field is $\psi = \sqrt{\omega^2/2} \Phi + i / \sqrt{2\omega} \Pi$, where $\omega = \omega(\nabla)$.
is the differential operator $\sqrt{\nabla^2 + m^2}$. In general, the transformation is a non-local one, but in the NR limit where $\omega \to m$ it becomes approximately local. To allow for the NR convention for the zero of energy being different by $m$ one would also have to include an $e^{imt}$ factor.

[43] This appears to bear out our comments on Eq. (7.9) in footnote [28] concerning the role of the $s$-channel interaction.