Quantizations of $D = 3$ Lorentz symmetry

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Abstract

Using the isomorphism $\mathfrak{so}(3; \mathbb{C}) \simeq \mathfrak{sl}(2; \mathbb{C})$ we develop a new simple algebraic technique for complete classification of quantum deformations (the classical $r$-matrices) for real forms $\mathfrak{so}(3)$ and $\mathfrak{so}(2,1)$ of the complex Lie algebra $\mathfrak{so}(3; \mathbb{C})$ in terms of real forms of $\mathfrak{sl}(2; \mathbb{C})$: $\mathfrak{su}(2)$, $\mathfrak{su}(1,1)$ and $\mathfrak{sl}(2; \mathbb{R})$. We prove that the $D = 3$ Lorentz symmetry $\mathfrak{so}(2,1) \simeq \mathfrak{su}(1,1) \simeq \mathfrak{sl}(2; \mathbb{R})$ has three different Hopf-algebraic quantum deformations which are expressed in the simplest way by two standard $\mathfrak{su}(1,1)$ and $\mathfrak{sl}(2; \mathbb{R})$ $q$-analogs and by simple Jordanian $\mathfrak{sl}(2; \mathbb{R})$ twist deformations. These quantizations are presented in terms of the quantum Cartan-Weyl generators for the quantized algebras $\mathfrak{su}(1,1)$ and $\mathfrak{sl}(2; \mathbb{R})$ as well as in terms of quantum Cartesian generators for the quantized algebra $\mathfrak{so}(2,1)$. Finally, some applications of the deformed $D = 3$ Lorentz symmetry are mentioned.

1 Introduction

The search for quantum gravity is linked with studies of noncommutative space-times and quantum deformations of space-time symmetries. The considerations of simple dynamical models in quantized gravitational background (see e.g. [1, 2]) indicate that the presence of quantum gravity effects generates noncommutativity of space-time coordinates, and as well the Lie-algebraic space-time symmetries (e.g. Lorentz, Poincaré) are modified into quantum symmetries, described by noncommutative Hopf algebras, named after Drinfeld quantum deformations or quantum group [3]. We recall that in relativistic theories the basic role is played by Lorentz symmetries and Lorentz algebra, i.e. all aspects of their quantum deformations should be studied in very detailed and careful way.

For classifications, constructions and applications of quantum Hopf deformations of an universal enveloping algebra $U(g)$ of a Lie algebra $g$, Lie bialgebras $(g, \delta)$ play an essential role (see e.g. [3, 4] and [5, 6]). Here the cobracket $\delta$ is a linear skew-symmetric map $g \rightarrow g \wedge g$ with
the relations consisted with the Lie bracket in $\mathfrak{g}$:

$$\delta([x, y]) = [x \otimes 1 + 1 \otimes x, \delta(y)] - [y \otimes 1 + 1 \otimes y, \delta(x)],$$

$$\delta \otimes \text{id}) \delta(x) + \text{cycle} = 0$$

for any $x, y \in \mathfrak{g}$. The first relation in (1.1) is a condition of the 1-cocycle and the second one is the co-Jacobi identity (see [3, 6]). The Lie bialgebra $(\mathfrak{g}, \delta)$ is a correct infinitesimalisation of the quantum Hopf deformation of $U(\mathfrak{g})$ and the operation $\delta$ is a an infinitesimal part of difference between a coproduct $\Delta$ and an opposite coproduct $\tilde{\Delta}$ in the Hopf algebra, $\delta(x) = h^{-1}(\Delta - \tilde{\Delta})$ mod $h$ where $h$ is a deformation parameter. Any two Lie bialgebras $(\mathfrak{g}, \delta)$ and $(\mathfrak{g}, \delta')$ are isomorphic (equivalent) if they are connected by a $\mathfrak{g}$-automorphism $\varphi$ satisfying the condition

$$\delta(x) = (\varphi \otimes \varphi)\delta'(\varphi^{-1}(x))$$

for any $x \in \mathfrak{g}$. Of special our interest here are the quasitriangular Lie bialgebras $(\mathfrak{g}, \delta_{(r)}) := (\mathfrak{g}, \delta, r)$, where the cobracket $\delta_{(r)}$ is given by the classical $r$-matrix $r \in \mathfrak{g} \wedge \mathfrak{g}$ as follows:

$$\delta_{(r)}(x) = [x \otimes 1 + 1 \otimes x, r].$$

It is easy to see from (1.2) and (1.3) that two quasitriangular Lie bialgebras $(\mathfrak{g}, \delta_{(r)})$ and $(\mathfrak{g}, \delta_{(r')})$ are isomorphic iff the classical $r$-matrices $r$ and $r'$ are isomorphic, i.e. $(\varphi \otimes \varphi)r' = r$. Therefore for a classification of all nonequivalent quasitriangular Lie bialgebras $(\mathfrak{g}, \delta_{(r)})$ of the given Lie algebra $\mathfrak{g}$ we need find all nonequivalent (nonisomorphic) classical $r$-matrices. Because nonequivalent quasitriangular Lie bialgebras uniquely determine non-equivalent quasitriangular quantum deformations (Hopf algebras) of $U(\mathfrak{g})$ (see [3, 4]) therefore the classification of all nonequivalent quasitriangular Hopf algebras is reduced to the classification of the nonequivalent classical $r$-matrices.

In this paper we investigate the quantum deformations of $D = 3$ Lorentz symmetry. Firstly, following [7], we obtain the complete classifications of the nonequivalent (nonisomorphic) classical $r$-matrices for complex Lie algebra $\mathfrak{sl}(2; \mathbb{C})$ and its real forms $\mathfrak{su}(2)$, $\mathfrak{su}(1, 1)$ and $\mathfrak{sl}(2; \mathbb{R})$ with the help of explicite formulas for the automorphisms of these Lie algebras in terms of the Cartan-Weyl bases. In the case of $\mathfrak{sl}(2; \mathbb{C})$ there are two nonequivlalent classical $r$-matrices - standard and Jordanian ones. For $\mathfrak{su}(2)$ algebra there is only the standard nonequivalent $r$-matrix. These results are well known. For the $\mathfrak{su}(1, 1)$ case we obtained three nonequivalent $r$-matrices - standard, quasi-standard and quasi-Jordanian ones. In the case of $\mathfrak{sl}(2; \mathbb{R})$ we find also three nonequivalent $r$-matrices - standard, quasi-standard and Jordanian ones. Then using isomorphisms $\mathfrak{o}(2, 1) \simeq \mathfrak{su}(1, 1) \simeq \mathfrak{sl}(2; \mathbb{R})$ we express these $r$-matrices in terms of the Cartan basis of the $D = 3$ Lorentz algebra $\mathfrak{o}(2, 1)$ and we see that two systems with three $r$-matrices for $\mathfrak{su}(1, 1)$ and $\mathfrak{sl}(2; \mathbb{R})$ algebras coincides. Thus we obtain that the isomorphic Lie algebras $\mathfrak{su}(1, 1)$ and $\mathfrak{sl}(2; \mathbb{R})$ have the isomorphic systems of their quasitriangle Lie bialgebras. In the case of $\mathfrak{o}(2, 1)$ we obtain that the $D = 3$ Lorentz algebra has two standard $q$-deformations and one Jordanian. These Hopf deformations are presented in explicite form in terms of the quantum Cartan-Weyl generators for the quantized universal enveloping algebras of $\mathfrak{su}(1, 1)$ and $\mathfrak{sl}(2; \mathbb{R})$ and also in the terms of the quantum Cartan generators.

It should be noted that the full list of the nonequivalent classical $r$-matrices for $\mathfrak{sl}(2; \mathbb{R})$ and $\mathfrak{o}(2, 1)$ Lie algebras has been obtained early by different methods [8, 9] (see also [10, 11, 12]), however the complete list of the nonequivalent Hopf quantisations for these Lie algebras has not been presented in the literature. Furthermore, there was put forward an incorrect hypothesis.
that the isomorphic Lie algebra $\mathfrak{su}(1, 1)$ and $\mathfrak{sl}(2; \mathbb{R})$ do not have any isomorphic quasitriangular Lie bialgebras (see [13]).

The isomorphic Lie algebras $\mathfrak{o}(2, 1), \mathfrak{sl}(2; \mathbb{R}), \mathfrak{su}(1, 1)$ and their quantum deformations play very important role in physics as well as in mathematical considerations, so the structure of these deformations should be understood with full clarity. The $\mathfrak{o}(2, 1)$ Lie algebra has been used as $D = 1$ conformal algebra describing basic symmetries in conformal classical and quantum mechanics [14]; in such a case $\mathfrak{o}(2, 1)$ algebra is realized as a nonlinear realization on the one-dimensional time axis [15, 16] and can be extended to $\mathfrak{osp}(1|2)$ describing $D = 1 N = 2$ supersymmetric conformal algebra [17]. In field-theoretic framework the $\mathfrak{o}(2, 1)$ Lie algebra describes Lorentz symmetries of three-dimensional relativistic systems with planar $d = 2$ space sector, which are often discussed as simplified version of the four-dimensional relativistic case. Due to the isomorphism $\mathfrak{o}(2, 2) \simeq \mathfrak{o}(2, 1) \oplus \mathfrak{o}(2, 1)$ our results can be also applied to the description of $D = 3$ AdS symmetries [18]. We recall that $\mathfrak{o}(2, 2)$ symmetry has been employed in Chern-Simons formulation of $D = 3$ gravity [19, 20, 21], with Lorentzian signature and non-vanishing negative cosmological constant. Subsequently, the quantum deformations of $D = 3$ Chern-Simons theory have been used for the description of $D = 3$ quantum gravity as deformed $D = 3$ topological QFT [22, 23]. Three-dimensional deformed space-time geometry is also a basis of historical Ponzano-Regge formulation of $D = 3$ quantum gravity [24], which was further developed into spin foam [25] and causal triangulation [26] approaches.

In mathematics and mathematical physics the importance of $\mathfrak{o}(2, 1)$ and its deformations follows also from the unique role of the $\mathfrak{o}(2, 1)$ algebra as the lowest-dimensional rank one noncompact simple Lie algebra, endowed only with unitary infinite-dimensional representations. One can point out that the programm of deformations of infinite-dimensional modules of quantum-deformed $U(\mathfrak{su}(1, 1))$ algebra has been initiated already more than twenty years ago (see e.g. [27]). The $(2 + 1)$-dimensional models are also important in the theory of classical and quantum integrable systems [28, 29] with their symmetries described by Poisson-Lie groups in classical case and after quantization by quantum groups. In particular recently, using sigma model formulation of (super)string actions (see e.g. [30]), there were introduced the integrable deformations of string target (super)spaces obtained by Yang-Baxter deformations [31–34] of the principal as well as coset sigma models with symmetries, which may contain $AdS_2 \simeq \mathfrak{o}(2; 1)$ and $AdS_3 \simeq \mathfrak{o}(2, 2)$ factors [35–37].

The plan of our paper is the following. In Sect. 2 we consider the complex Lie algebra $\mathfrak{o}(3; \mathbb{C})$ and its all real forms: $\mathfrak{o}(3) \simeq \mathfrak{su}(2)$ and $\mathfrak{o}(2, 1) \simeq \mathfrak{su}(1, 1) \simeq \mathfrak{sl}(2; \mathbb{R})$. In Sect. 3 we classify all classical $r$-matrices for these real forms and in Sect. 4 we provide the explicite isomorphisms between the real $\mathfrak{su}(1, 1), \mathfrak{sl}(2; \mathbb{R})$ and $\mathfrak{o}(2, 1)$ bialgebras. In Sect. 5 all three Hopf-algebraic quantizations (explicite quantum deformations) of the real $D = 3$ Lorentz symmetry are presented in detail: quantized bases, coproducts and universal $R$-matrices are given. In Sect. 6 we present short summary and outlook.

2 Complex $D = 3$ Euclidean Lie algebra $\mathfrak{o}(3; \mathbb{C})$ and its real forms

We first remind different most popular bases of the complex $D = 3$ Euclidean Lie algebra $\mathfrak{o}(3; \mathbb{C})$: metric, Cartesian and Cartan-Weyl bases (see [7]).

The metric basis contains in its commutation relations an explicite metric, namely, the complex $D = 3$ Euclidean Lie algebra $\mathfrak{o}(3; \mathbb{C})$ is generated by three Euclidean basis elements
$L_{ij} = -L_{ji} \in \mathfrak{o}(3; \mathbb{C})$ ($i, j = 1, 2, 3$) satisfying the relations

$$[L_{ij}, L_{kl}] = g_{jk} L_{il} - g_{jl} L_{ik} + g_{il} L_{jk} - g_{ik} L_{jl}, \quad (2.1)$$

where $g_{ij}$ is the Euclidean metric: $g_{ij} = \text{diag}(1, 1, 1)$. The Euclidean algebra $\mathfrak{o}(3; \mathbb{C})$, as a linear space, is a linear envelope of the basis $\{L_{ij}\}$ over $\mathbb{C}$.

The Cartesian (or physical) basis of $\mathfrak{o}(3; \mathbb{C})$ is related with the generators $L_{ij}$ as follows

$$I_i := -\frac{1}{2} \varepsilon_{ijk} L_{jk} \quad (i, j, k = 1, 2, 3). \quad (2.2)$$

From (2.1) and (2.2) we get

$$[I_i, I_j] = \varepsilon_{ijk} I_k. \quad (2.3)$$

If we consider a Lie algebra over $\mathbb{R}$ with the commutation relations (2.3) then we get the compact real form $\mathfrak{o}(3) := \mathfrak{o}(3; \mathbb{R})$ with the anti-Hermitian basis

$$I_i^* = -I_i \quad (i = 1, 2, 3) \quad \text{for } \mathfrak{o}(3). \quad (2.4)$$

The real form $\mathfrak{o}(2, 1)$ is given by the formulas:

$$I_i^\dagger = (-1)^{i-1} I_i \quad (i = 1, 2, 3) \quad \text{for } \mathfrak{o}(2, 1). \quad (2.5)$$

For the description of quantum deformations and in particular for the classification of classical $r$-matrices of the complex Euclidean algebra $\mathfrak{o}(3; \mathbb{C})$ and its real forms $\mathfrak{o}(3)$ and $\mathfrak{o}(2, 1)$ it is convenient to use the Cartan–Weyl (CW) basis of the isomorphic complex Lie algebra $\mathfrak{sl}(2; \mathbb{C})$ and its real forms $\mathfrak{su}(2)$, $\mathfrak{sl}(1, 1)$ and $\mathfrak{sl}(2, \mathbb{R})$. In the case of $\mathfrak{o}(3)$ the $\mathfrak{su}(2)$ Cartan–Weyl basis can be chosen as follows

$$H := i I_3, \quad E_\pm := i I_1 \mp I_2,$$

$$[H, E_\pm] = \pm E_\pm, \quad [E_+, E_-] = 2H, \quad \text{for } \mathfrak{su}(1, 1), \quad (2.6)$$

where the conjugation ($^*$) is the same as in (2.4)

For the real form $\mathfrak{o}(2, 1)$ we will use two CW bases of $\mathfrak{sl}(2; \mathbb{C})$ real forms: $\mathfrak{sl}(1, 1)$ and $\mathfrak{sl}(2, \mathbb{R})$. Such bases are given by

$$H := i I_2, \quad E_\pm := i I_1 \pm I_3, \quad \text{for } \mathfrak{su}(1, 1), \quad (2.7)$$

$$H' := i I_3, \quad E'_\pm := i I_1 \mp I_2, \quad \text{for } \mathfrak{sl}(2, \mathbb{R}). \quad (2.8)$$

Both bases $\{E_\pm, H\}$ and $\{E'_\pm, H'\}$ have the same commutation relations but they have different reality properties, namely

$$H^\dagger = H, \quad E_\pm^\dagger = -E_\mp \quad \text{for } \mathfrak{su}(1, 1), \quad (2.9)$$

$$H'^\dagger = -H', \quad E'_\pm^\dagger = -E'_\mp \quad \text{for } \mathfrak{sl}(2, \mathbb{R}). \quad (2.10)$$

\footnote{The basis elements $E_\pm, H$ over $\mathbb{C}$ with the defining relations in the second line of (2.6) generates the complex Lie algebra $\mathfrak{sl}(2; \mathbb{C})$. The relations in the first line of (2.6) reproduce the isomorphism between $\mathfrak{o}(3; \mathbb{C})$ and $\mathfrak{sl}(2; \mathbb{C})$.}
where the conjugation \((\dagger)\) is the same as in (2.5)\(^2\). The relations between the \(\mathfrak{su}(1,1)\) and \(\mathfrak{su}(2,\mathbb{R})\) bases look as follows

\[
H = -\frac{i}{2}(E'_+ - E'_-),
\]

\[
E_{\pm} = \mp i H' + \frac{1}{2}(E'_+ + E'_-). \tag{2.11}
\]

3 Classical \(r\)-matrices of \(\mathfrak{sl}(2;\mathbb{C})\) and its real forms: \(\mathfrak{su}(2), \mathfrak{su}(1,1)\) and \(\mathfrak{sl}(2;\mathbb{R})\)

By definition any classical \(r\)-matrix of arbitrary complex or real Lie algebra \(\mathfrak{g}, r \in \mathfrak{g} \wedge \mathfrak{g}\), satisfy the classical Yang-Baxter equation (CYBE):

\[
[[r, r]] = \Omega. \tag{3.1}
\]

Here \([[\cdot, \cdot]]\) is the Schouten bracket which for any monomial skew-symmetric two-tensors \(r_1 = x \wedge y\) and \(r_2 = u \wedge v\) \((x, y, u, v \in \mathfrak{g})\) is given by\(^3\)

\[
[[x \wedge y, u \wedge v]] := \ x \wedge ([y, u] \wedge v + u \wedge [y, v])
\]
\[
- y \wedge ([x, u] \wedge v + u \wedge [x, v])
\]
\[
= [[u \wedge v, x \wedge y]] \tag{3.2}
\]

and \(\Omega\) is the \(\mathfrak{g}\)-invariant element which in the case of \(\mathfrak{g} := \mathfrak{sl}(2;\mathbb{C})\) looks as follows:

\[
\Omega = \gamma \Omega(\mathfrak{sl}(2;\mathbb{C})) = \gamma (4E_- \wedge H \wedge E_+) \tag{3.3}
\]

where \(\gamma \in \mathbb{C}\), and \(E_- , H\) is the CW basis of \(\mathfrak{sl}(2;\mathbb{C})\) with the defining relations on the second line of (2.6).

Firstly we show that \textit{any two-tensor of} \(\mathfrak{sl}(2;\mathbb{C}) \wedge \mathfrak{sl}(2;\mathbb{C})\) \textit{is a classical} \(\mathfrak{sl}(2;\mathbb{C})\) \(r\)-\textit{matrix}. Indeed, let

\[
r := \beta_+ r_+ + \beta_0 r_0 + \beta_- r_- \quad (\beta_+, \beta_0, \beta_- \in \mathbb{C}) \tag{3.4}
\]

be an arbitrary element of \(\mathfrak{sl}(2;\mathbb{C}) \wedge \mathfrak{sl}(2;\mathbb{C})\), where

\[
r_+ := E_+ \wedge H, \quad r_0 := E_+ \wedge E_-, \quad r_- := H \wedge E_- \tag{3.5}
\]

are the basis elements of \(\mathfrak{sl}(2;\mathbb{C}) \wedge \mathfrak{sl}(2;\mathbb{C})\). Because all terms (3.5) are classical \(r\)-matrices, moreover \([[r_\pm, r_\pm]] = 0\), as well as the Schouten brackets of the elements \(r_\pm\) with \(r_0\) are also equal to zero, \([[r_\pm, r_0]] = 0\), and we have

\[
[[r, r]] = 2\beta_+ \beta_- [[r_+, r_-]] + \beta_0^2 [[r_0, r_0]]
\]
\[
= (\beta_0^2 + \beta_+ \beta_-)(4E_- \wedge H \wedge E_+) \equiv \gamma \Omega. \tag{3.6}
\]

\(^2\)It should be noted that in the case of \(\mathfrak{su}(1,1)\) the Cartan generator \(H\) is compact while for the case \(\mathfrak{su}(2,\mathbb{R})\) the generator \(H'\) is noncompact.

\(^3\)For general polynomial (a sum of monomials) two-tensors \(r_1\) and \(r_2\) one can use the bilinearity of the Schouten bracket.
Thus an arbitrary element (3.4) is a classical $r$-matrix, and if its coefficients $\beta_\pm, \beta_0$ satisfy the condition $\gamma := \beta_0^2 + \beta_+ \beta_- = 0$ then it satisfies the homogeneous CYBE, if $\gamma := \beta_0^2 + \beta_+ \beta_- \neq 0$ it satisfies the non-homogeneous CYBE.

We shall call the parameter $\gamma = \beta_0^2 + \beta_+ \beta_-$ in (3.6) the $\gamma$-characteristic of the classical $r$-matrix (3.4). It is evident that the $\gamma$-characteristic of the classical $r$-matrix $r$ is invariant under the $\mathfrak{sl}(2; \mathbb{C})$-automorphisms, i.e. any two $r$-matrices $r$ and $r'$, which are connected by a $\mathfrak{sl}(2; \mathbb{C})$-automorphism, have the same $\gamma$-characteristic, $\gamma = \gamma'$. We can show also that any two $\mathfrak{sl}(2; \mathbb{C})$ $r$-matrices $r$ and $r'$ with the same $\gamma$-characteristic can be connected by a $\mathfrak{sl}(2; \mathbb{C})$-automorphism.

There are two types of explicite $\mathfrak{sl}(2; \mathbb{C})$-automorphisms which were presented in [7]. First type connecting the classical $r$-matrices with zero $\gamma$-characteristic is given by the formulas (see (3.15) in [7])\(^4\):

$$
\varphi_0(E_+)=\chi(\tilde{\beta}_+E_+−2\tilde{\beta}_0H+\tilde{\beta}_-E_-),
\varphi_0(E_-)=\chi^{-1}(\tilde{\beta}_-E_+−2\kappa\tilde{\beta}_0H+\tilde{\beta}_+E_-),
\varphi_0(H)=\tilde{\beta}_0E_++(\kappa\tilde{\beta}_++\tilde{\beta}_-)H+\kappa\tilde{\beta}_0E_-, \tag{3.7}
$$

where $\chi$ is a non-zero rescaling parameter (including $\chi = 1$), $\kappa$ takes two values $+1$ or $−1$, and the parameters $\tilde{\beta}_i (i = +, 0, −)$ satisfy the conditions:

$$
\tilde{\beta}_0^2 + \tilde{\beta}_+ \tilde{\beta}_- = 0, \quad \kappa \tilde{\beta}_+ - \tilde{\beta}_- = 1. \tag{3.8}
$$

Let us consider two general $r$-matrices with zero $\gamma$-characteristics:

$$
r := \beta_+ E_+ \wedge H + \beta_0 E_+ \wedge E_- + \beta_- H \wedge E_-, \quad r' := \beta'_+ E_+ \wedge H + \beta'_0 E_+ \wedge E_- + \beta'_- H \wedge E_-,
\tag{3.9}
$$

where $\beta_0^2 + \beta_+ \beta_- = 0$ and $\beta'_0^2 + \beta'_+ \beta'_- = 0$. Moreover, we suppose that the parameters $\beta_\pm$ and $\beta'_\pm$ satisfy the additional relations:

$$
\kappa \beta_+ - \beta_- = \chi \beta'_+ - \chi^{-1} \kappa \beta'_- \neq 0, \tag{3.10}
$$

where the parameters $\kappa$ and $\chi$ are the same as in (3.7). One can check that the following formula is valid:

$$
\begin{align*}
\beta_+ E_+ \wedge H &+ \beta_0 E_+ \wedge E_- + \beta_- H \wedge E_-
= \beta'_+ \varphi_0(E_+) \wedge \varphi_0(H) + \beta'_0 \varphi_0(E_+) \wedge \varphi_0(E_-) \\
&+ \beta'_- \varphi_0(H) \wedge \varphi_0(E_-), \tag{3.11}
\end{align*}
$$

where $\varphi_0$ is the $\mathfrak{sl}(2; \mathbb{C})$-automorphism (3.7) with the following parameters:

$$
\begin{align*}
\tilde{\beta}_0 &= \frac{\beta_0(\chi \beta'_+ + \chi^{-1} \kappa \beta'_-)}{(\kappa \beta_+ - \beta_-)(\chi \beta'_+ - \chi^{-1} \kappa \beta'_-)} - \beta'_0(\kappa \beta_+ + \beta_-), \\
\tilde{\beta}_+ &= \frac{\kappa(\kappa \beta_+ + \beta_-)(\chi \beta'_+ + \chi^{-1} \kappa \beta'_-)}{2(\kappa \beta_+ - \beta_-)(\chi \beta'_+ - \chi^{-1} \kappa \beta'_-)} + \frac{\kappa}{2}, \\
\tilde{\beta}_- &= \frac{(\kappa \beta_+ + \beta_-)(\chi \beta'_+ + \chi^{-1} \kappa \beta'_-) + 4 \beta_0 \beta'_0}{2(\kappa \beta_+ - \beta_-)(\chi \beta'_+ - \chi^{-1} \kappa \beta'_-)} - \frac{1}{2}.
\tag{3.12}
\end{align*}
$$

\(^4\)The formulas (3.7) are obtained from (3.15) in [7] by the substitution: $\beta_0/(k\beta_+ - \beta_-) = -2\tilde{\beta}_0, \beta_\pm/(k\beta_+ - \beta_-) = \tilde{\beta}_\pm.$
It is easy to check that as expected the formulas (3.12) satisfy the conditions (3.8).

Let us assume in (3.9), (3.11) and (3.12) that the parameters $\beta'_0$ and $\beta'_-$ are equal to zero. Then the general classical $r$-matrix $r$ in (3.9), satisfying the homogeneous CYBE, is reduced to usual Jordanian form by the automorphism (3.7) with the parameters:

$$
\tilde{\beta}_0 = \frac{\beta_0}{\kappa \beta_+ - \beta_-}, \quad \tilde{\beta}_\pm = \frac{\beta_\pm}{\kappa \beta_+ - \beta_-}.
$$

(3.13)

Second type of $\mathfrak{sl}(2; \mathbb{C})$-automorphism connecting the classical $r$-matrices with non-zero $\gamma$-characteristic is given as follows$^5$

$$
\varphi_1(E_+) = \frac{\chi}{2}((\tilde{\beta}_0 + 1)E_+ + 2\tilde{\beta}_-H - \frac{\tilde{\beta}_0^2}{\tilde{\beta}_0 + 1}E_-),
$$

(3.14)

$$
\varphi_1(E_-) = \frac{\chi^{-1}}{2}\left(\frac{-\tilde{\beta}_0^2}{\tilde{\beta}_0 + 1}E_+ + 2\tilde{\beta}_+H + (\tilde{\beta}_0 + 1)E_-\right),
$$

$$
\varphi_1(H) = \frac{1}{2}(-\tilde{\beta}_+E_+ + 2\tilde{\beta}_0H - \tilde{\beta}_-E_-),
$$

where $\chi$ is a non-zero rescaling parameter, and $\tilde{\beta}_0^2 + \tilde{\beta}_+\tilde{\beta}_- = 1$.

Let us consider two general $r$-matrices with non-zero $\gamma$-characteristics:

$$
r := \beta_+E_+ \wedge H + \beta_0E_+ \wedge E_- + \beta_-H \wedge E_-,
$$

(3.15)

$$
r' := \beta'_+E_+ \wedge H + \beta'_0E_+ \wedge E_- + \beta'_-H \wedge E_-,
$$

where the parameters $\beta_\pm, \beta_0$ and $\beta'_\pm, \beta'_0$ can be equal to zero provided that $\gamma = \beta_0^2 + \beta_+\beta_- = \gamma' = (\beta'_0)^2 + \beta'_+\beta'_- \neq 0$, i.e. both $r$-matrices $r$ and $r'$ have the same non-zero $\gamma$-characteristic $\gamma = \gamma' \neq 0$. One can check the following relation:

$$
\beta_+E_+ \wedge H + \beta_0E_+ \wedge E_- + \beta_-H \wedge E_- = \beta'_+\varphi_1(E_+) \wedge \varphi_1(H) + \beta'_0\varphi_1(E_+) \wedge \varphi_0(E_-)
$$

(3.16)

$$
+ \beta'_-\varphi_1(H) \wedge \varphi_1(E_-),
$$

where $\varphi_1$ is the $\mathfrak{sl}(2; \mathbb{C})$-automorphism (3.14) with the parameters:

$$
\tilde{\beta}_0 = \frac{(\beta_0 + \beta'_0)^2 - (\beta_+ - \chi\beta'_+)(\beta_- - \chi^{-1}\beta'_-)}{(\beta_0 + \beta'_0)^2 + (\beta_+ - \chi\beta'_+)(\beta_- - \chi^{-1}\beta'_-)},
$$

(3.17)

$$
\tilde{\beta}_\pm = \frac{2(\beta_0 + \beta'_0)(\beta_\pm - \chi^{\pm 1}\beta'_\pm)}{(\beta_0 + \beta'_0)^2 + (\beta_+ - \chi\beta'_+)(\beta_- - \chi^{-1}\beta'_-)}.
$$

It is easy to check that the formulas (3.17) satisfy the condition $\tilde{\beta}_0^2 + \tilde{\beta}_+\tilde{\beta}_- = 1$.

If we assume in (3.15)–(3.17) that the parameters $\beta'_\pm$ are equal to zero then the general classical $r$-matrix $r$ in (3.15), satisfying the non-homogeneous CYBE, is reduced to the usual standard form by the automorphism (3.14) with the following parameters:

$$
\tilde{\beta}_0 = \frac{\beta_0}{\beta'_0}, \quad \tilde{\beta}_\pm = \frac{\beta_\pm}{\beta'_0}.
$$

(3.18)

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$^5$The formulas (3.8) are obtained from (3.14) in [7] by the substitution: $\beta_0 = 2\tilde{\beta}_0, \beta_\pm = -\tilde{\beta}_\pm, D = 4$. 

Finally for $\mathfrak{sl}(2,C)$ we obtain the well-known result:

For the complex Lie algebra $\mathfrak{sl}(2,C)$ there exists up to $\mathfrak{sl}(2,C)$ automorphisms two solutions of CYBE, namely Jordanian $r_J$ and standard $r_{st}$:

\[
\begin{align*}
r_J &= \beta E_+ \wedge H, \quad [[r_J, r_J]] = 0, \\
r_{st} &= \beta' E_+ \wedge E_-, \quad [[r_{st}, r_{st}]] = \beta'^2 \Omega,
\end{align*}
\]

where the complex parameter $\beta$ in (3.19) can be removed by the rescaling automorphism: $\varphi(E_+) = \beta^{-1} E_+$, $\varphi(E_-) = \beta E_-, \varphi(H) = H$; in (3.20) the parameter $\beta' = e^{i\phi} |\beta'|$ for $|\phi| \leq \frac{\pi}{2}$ is effective.

The general non-reduced expression (3.4) is convenient for the application of reality conditions:

\[
r^* := \beta^*_+ E^*_+ \wedge H^* + \beta^*_0 E^*_+ \wedge E^*_+ + \beta^*_+ H^* \wedge E^*_+ = -r,
\]

where $\ast$ is the conjugation associated with corresponding real form ($\ast = \ast, \dagger$), and $\beta^*_i$ ($i = +, 0, -)$ means the complex conjugation of the number $\beta_i$. It should be noted that for any classical $r$-matrix $r$, $r^\ast$ is again a classical $r$-matrix. Moreover, if $r$-matrix is anti-real (anti-Hermitian)\(^6\), i.e. it satisfies the condition (3.21), then its $\gamma$-characteristic is real. Indeed, applying the conjugation $\ast$ to the relation (3.6) we have for the left-side: $[[r, r]]^\ast = -[[r^\ast, r^\ast]] = -[[r, r]]$ and for the right-side: $(\gamma \Omega)^\ast = -\gamma^\ast \Omega$ for all real forms $\mathfrak{su}(2)$, $\mathfrak{su}(1,1)$, $\mathfrak{su}(2; \mathbb{R})$. It follows that the parameter $\gamma$ is real, $\gamma^\ast = \gamma$.

I. The compact real form $\mathfrak{su}(2)$ ($H^\ast = H, E^\ast_\pm = E_\pm$).

In this case it follows from (3.21) that

\[
\beta_0^* = \beta_0, \quad \beta^*_\pm = \beta_\pm.
\]

If in (3.4) $\gamma = \beta^2_0 + \beta_+ \beta_-$ then $\beta_0 \beta^*_0 + \beta_+ \beta^*_+ = 0$ and it follows that $\beta_0 = \beta_+ = \beta_-$, i.e. any classical $r$-matrix which satisfies the homogeneous CYBE and the $\mathfrak{su}(2)$ reality condition, is equal zero.

If in (3.4) $\gamma = \beta^2_0 + \beta_+ \beta_- \neq 0$ we have three possible $\mathfrak{su}(2)$ real classical $r$-matrices:

\[
\begin{align*}
r_1 &:= \beta_0 E_+ \wedge E_-, \\
r_2 &:= \beta^*_+ E_+ \wedge H + \beta^*_+ H \wedge E_-, \\
r_3 &:= \beta^*_+ E_+ \wedge H + \beta_0 E_+ \wedge E_- + \beta^*_+ H \wedge E_-,
\end{align*}
\]

where $\beta_0$ and $\beta'_0$ are real numbers and we use the conditions (3.22). The $r$-matrices $r_i$ ($i = 1, 2, 3$) satisfy the non-homogeneous CYBE

\[
[[r_i, r_i]] = \gamma_i \Omega,
\]

where all $\gamma_i$ ($i = 1, 2, 3$) are positive: $\gamma_1 = \beta^2_0 > 0$, $\gamma_2 = \beta_+ \beta^*_+ > 0$, $\gamma_3 = \beta^2_0 + \beta'_+ \beta^*_+ > 0$.

Let the classical $r$-matrices (3.15) be $\mathfrak{su}(2)$-antireal, i.e. their parameters satisfy the reality conditions (3.22). It follows that the functions (3.17) for $\chi = e^{i\phi}$ have the same conjugation properties, i.e. $\beta^*_0 = \beta_0$, $\beta^*_\pm = \beta_\pm$, and we obtain that the automorphism (3.14) with such parameters is $\mathfrak{su}(2)$-real, i.e.:

\[
\begin{align*}
\varphi_1(E^\pm_+) & = \varphi_1(E^*_\pm) = \varphi_1(E_\mp), \\
\varphi_1(H^\pm) & = \varphi_1(H^*) = \varphi_1(H).
\end{align*}
\]

---

\(^6\)The anti-real property $r^* = -r$ is a direct consequence of the reality condition for the co-bracket $\delta(x) := [x \otimes 1 + 1 \otimes x, r]$, namely $\delta(x)^* = \delta(x^*)$ for $\forall x \in \mathfrak{g}^\ast(= \{\mathfrak{su}(2), \mathfrak{su}(1,1), \mathfrak{sl}(2,\mathbb{R})\})$. 

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We see that the \( r \)-matrices \( r_2 \) and \( r_3 \) in (3.23) can be reduced to the standard \( r \)-matrix \( r_{st} := r_1 \) using the formula (3.16).

It is easy to see that the standard \( r \)-matrix \( r_{st} = r_1 \) in (3.23) effectively depends only on positive values of the parameter \( \alpha := \beta_0 \). Indeed, we see that

\[
\alpha \varphi(E_+) \wedge \varphi(E_-) = -\alpha E_+ \wedge E_-,
\]

where \( \varphi \) is the simple \( \mathfrak{su}(2) \) automorphism: \( \varphi(E_{\pm}) = E_{\mp}, \varphi(H) = -H \), i.e. any negative value of parameter \( \alpha \) in \( r_{st} \) can be replaced by the positive one.

We obtain the following result:

For the compact real form \( \mathfrak{su}(2) \) there exists up to the \( \mathfrak{su}(2) \) automorphisms only one solution of CYBE and this solution is the usual standard classical \( r \)-matrix \( r_{st} \):

\[
r_{st} := \alpha E_+ \wedge E_-, \quad [r_{st}, r_{st}] = \gamma \Omega,
\]

where the effective parameter \( \alpha \) is a positive number, and \( \gamma = \alpha^2 \).

II. The non-compact real form \( \mathfrak{su}(1, 1) \) \( (H^1 = H, E_{\pm}^1 = -E_{\mp}) \).

In this case it follows from (3.21) that

\[
\beta_0^* = \beta_0, \quad \beta_+^* = -\beta_-, \quad \beta_-^* = -\beta_+, \quad (3.28)
\]

If \( \beta_0^2 + \beta_+ \beta_- = 0 \) in (3.4) then \( \beta_0 \beta_0^* - \beta_+^* \beta_-^* = 0 \), i.e. \( \beta_{\pm} = \pm e^{\pm i \phi} |\beta_0| \), and we have the following \( \phi \)-family of \( \mathfrak{su}(1, 1) \) homogeneous CYBE solutions:

\[
r_\phi := \beta_0 \left( e^{i \phi} \frac{|\beta_0|}{\beta_0} E_+ \wedge H + E_+ \wedge E_- - e^{-i \phi} \frac{|\beta_0|}{\beta_0} H \wedge E_- \right), \quad (3.29)
\]

where \( \beta_0 \) is real. By using the \( \mathfrak{su}(1, 1) \)-real rescaling automorphism \( \varphi(E_{\pm}) = (-i e^{i \phi} |\beta_0|) \frac{1}{\beta_0} E_{\pm}, \varphi(H) = H \) we can reduce the \( \phi \)-family (3.29) to \( r_{qJ} := \beta_0(\imath E_+ \wedge H + E_+ \wedge E_- + \imath H \wedge E_-) \):

\[
r_\phi = \beta_0 \left( e^{i \phi} \frac{|\beta_0|}{\beta_0} E_+ \wedge H + E_+ \wedge E_- - e^{-i \phi} \frac{|\beta_0|}{\beta_0} H \wedge E_- \right) = \beta_0 \left( \imath (\varphi(E_+ - \varphi(E_-)) \wedge \varphi(H) + \varphi(E_+) \wedge \varphi(E_-) \right), \quad (3.30)
\]

We shall call a \( \mathfrak{su}(1, 1) \)-real \( r \)-matrix “quasi-Jordanian” if it can not be reduced to Jordanian form by a \( \mathfrak{su}(1, 1) \)-real automorphism, but after complexification of \( \mathfrak{su}(1, 1) \) it can be reduced to Jordanian form by an appropriate complex \( \mathfrak{sl}(2, \mathbb{C}) \)-automorphism. Thus all \( r \)-matrices in the \( \phi \)-family (3.29) are quasi-Jordanian and they are connected with each other by the \( \mathfrak{su}(1, 1) \)-real rescaling automorphism. We take \( r_{qJ} \) as an representative of the \( \phi \)-family. It is easy to see that the quasi-Jordanian \( r \)-matrix \( r_{qJ} \) effectively depends only on positive values of the parameter \( \beta_0 \), indeed,

\[
r_{qJ} = \beta_0(E_+ \wedge H + E_+ \wedge E_- + H \wedge E_-) = -\beta_0((\varphi(E_+) \wedge \varphi(H) + \varphi(E_+) \wedge \varphi(E_-) + \varphi(H) \wedge \varphi(E_-)), \quad (3.31)
\]

where \( \varphi \) is the simple \( \mathfrak{su}(1, 1) \) automorphism \( \varphi(E_{\pm}) = E_{\mp}, \varphi(H) = -H \), i.e. any negative value of parameter \( \beta_0 \) in \( r_{qJ} \) can be changed into a positive one.
In the case $\beta_0^2 + \beta_+ \beta_- \neq 0$ in (3.4) we have four versions of $\mathfrak{su}(1,1)$-real classical $r$-matrices. Two of them are characterized by positive value of $\gamma_i$, $(i = 1, 2)$:

$$r_1 := \beta_0 E_+ \wedge E_-, \qquad r_2 := \beta_+ E_+ \wedge H + \beta_0^* E_+ \wedge E_- - \beta^*_+ H \wedge E_-,$$

$$\{[r_i, r_i]\} := \gamma_i \Omega \quad (i = 1, 2),$$

where $\beta_0$ and $\beta_0^*$ are real (see (3.28)), and $\gamma_1 = \beta_0^2 > 0$, $\gamma_2 = \beta_0^2 \beta_0^* - \beta^*_+ \beta^*_+ > 0$. The remaining two are with negative values of $\gamma_i$, $(i = 3, 4)$:

$$r_3 := \beta^*_+ E_+ \wedge H - \beta^*_+ H \wedge E_-, \qquad r_4 := \beta^*_+ E_+ \wedge H + \beta_0^* E_+ \wedge E_- - \beta^*_+ H \wedge E_-,$$

$$\{[r_i, r_i]\} := \gamma_i \Omega \quad (i = 3, 4),$$

where $\beta_0^*$ is real (see (3.28)), and $\gamma_3 = -\beta_0^* \beta^*_+ < 0$, $\gamma_4 = \beta^*_+ \beta_0^* - \beta^*_+ \beta^*_+ < 0$.

Let the classical $r$-matrices (3.15) be $\mathfrak{su}(1,1)$-antireal, i.e. their parameters satisfy the reality conditions (3.28). In such a case the functions (3.17) for $\chi = e^{i \phi}$ have the same conjugation properties, i.e. $\beta_0^* = \overline{\beta}_0$, $\beta_+^* = -\beta_-^*$, and we obtain that the automorphism (3.14) with these parameters is $\mathfrak{su}(1,1)$-real, i.e.:

$$\varphi_1(E_+^\dagger) = \varphi_1(E_+^\dagger) = -\varphi_1(E_+), \quad \varphi_1(H^\dagger) = \varphi_1(H),$$

(3.34)

It allows to reduce the $r$-matrix $r_2$ to the standard $r$-matrix $r_{st} := r_1$ for $\gamma_1 = \gamma_2 > 0$ and the $r$-matrix $r_3$ to the $r$-matrix $r_4$ for $\gamma_3 = \gamma_4 < 0$ by use of the formula (3.16). By analogy to the notation of quasi-Jordanian $r$-matrices we shall call the $r$-matrices $r_3$ and $r_4$ as quasi-standard ones and take $r_{qst} := \alpha(E_+ + E_-) \wedge H$ as their representative.

Finally for $\mathfrak{su}(1,1)$ we obtain:

For the non-compact real form $\mathfrak{su}(1,1)$ there exists up to $\mathfrak{su}(1,1)$ automorphisms three solutions of CYBE, namely quasi-Jordanian $r_{qJ}$, standard $r_{st}$ and quasi-standard $r_{qst}$:

$$r_{qJ} = \frac{\alpha}{2}(i(E_+ - E_-) \wedge H + E_+ \wedge E_-), \quad \{[r_{qJ}, r_{qJ}]\} = 0,$$  

$$r_{st} = \alpha E_+ \wedge E_- \quad \{[r_{st}, r_{st}]\} = \alpha^2 \Omega,$$  

$$r_{qst} = \alpha(E_+ + E_-) \wedge H \quad \{[r_{qst}, r_{qst}]\} = -\alpha^2 \Omega,$$

(3.35), (3.36), (3.37)

where $\alpha$ effectively is a positive number.

III. The non-compact real form $\mathfrak{sl}(2; \mathbb{R})$ ($H^\dagger = -H'$, $E_+^\dagger = -E_-'$). In this case from (3.21) we obtain

$$\beta_0^\ast = -\beta_0, \quad \beta_+^\ast = -\beta_-,$$

(3.38)

i.e. all parameters $\beta_i$ $(i = +, 0, -)$ are purely imaginary.

Consider the case $\beta_0^2 + \beta_+ \beta_- = 0$ in (3.4). We have three $\mathfrak{su}(2; \mathbb{R})$ solutions of the homogeneous CYBE:

$$r_1' = \beta_+ E_+ \wedge H', \quad r_2' = \beta_- H' \wedge E'_-, \quad r_3' = \beta'_+ E_+ \wedge H' + \beta_0 E'_+ \wedge E'_- + \beta'_- H' \wedge E'_-,$$

(3.39)

The $r$-matrix $r_{qst}$ is connected with $r_3$ (3.33) by the following way. Substituting $\beta_+ = |\beta_+| e^{i \phi}$ in $r_3$ (3.33) and using the $\mathfrak{su}(1,1)$-real rescaling automorphism $\varphi(E_\pm) = e^{\pm i \phi} E_\pm$, $\varphi(H) = H$ we obtain $r_{qst}$ with $\alpha = |\beta_+|$.\]
where all parameters $\beta_i$ ($i = +, -, \beta'_i$ ($i = +, 0, -$) are purely imaginary, and $\beta_0^2 + \beta'_+ \beta'_- = 0$.

If the classical $r$-matrices (3.9), where all generators $H, E_\pm$ are replaced by $H', E', \text{are sl}(2; \mathbb{R})$-antireal, i.e. their parameters satisfy the reality conditions (3.38), then for the real parameter $\chi$ all functions (3.12) are real, i.e. $\beta_0^* = \tilde{\beta}_0, \beta_\pm^* = \tilde{\beta}_\pm$. We obtain that the automorphism of the type (3.7) with such parameters is $\text{sl}(2; \mathbb{R})$-real, i.e.:

$$
\varphi_0(E'\pm) = -\varphi_0(E'\mp),
$$

$$
\varphi_0(H') = -\varphi_0(H'),
$$

(3.40)

It allows to reduce the $r$-matrices $r'_{2}$ and $r'_{3}$ in (3.39) to the Jordanian $r$-matrix $r'_f := r'_1$ by using the formula (3.11).

In the case $\beta_0^2 + \beta_\pm \neq 0$ in (3.4) we have seven versions of $\text{sl}(2; \mathbb{R})$-real classical $r$-matrices. Five of them are with negative values of $\gamma_i$, ($i = 1, 2, \ldots, 5$):

$$
r'_1 := \beta_0 E'_+ \wedge E'_-,
$$

$$
r'_2 := \beta_+ E'_+ \wedge H' + \beta_0 E'_+ \wedge E'_-,
$$

$$
r'_3 := \beta_0 E'_+ \wedge E'_+ + \beta_- H' \wedge E'_-, 
$$

$$
r'_4 := \beta'_+ E'_+ \wedge H' + \beta'_- H' \wedge E'_-,
$$

$$
r'_5 := \beta_{+}'' E'_+ \wedge H' + \beta_{-}'' E'_+ \wedge E'_- + \beta_{+}'' H' \wedge E'_-, 
$$

$$
[[r'_i, r'_j]] := \gamma_i \Omega' \quad (i = 1, 2, \ldots, 5),
$$

(3.41)

where all parameters $\beta$ are purely imaginary, and $\gamma_1 = \gamma_2 = \gamma_3 = \beta_0^2 < 0, \gamma_4 = \beta'_+ \beta'_- < 0, \gamma_5 = \beta_0^2 + \beta'_+ \beta'_- < 0; \Omega'$ is the $\text{sl}(2; \mathbb{R})$-invariant element$^8$. $\Omega' = \gamma (4E'_- \wedge H' \wedge E'_+ )$. The remaining two $r$-matrices $r'_i$ ($i = 6, 7$) have positive values of $\gamma_i$:

$$
r'_6 := \beta_{+}'' E'_+ \wedge H' + \beta_{-}'' H' \wedge E'_- ,
$$

$$
r'_7 := \beta_{+}'' E'_+ \wedge H' + \beta_{-}'' E'_+ \wedge E'_- + \beta_{+}'' H' \wedge E'_-, 
$$

$$
[[r'_i, r'_j]] := \gamma_i \Omega' \quad (i = 6, 7),
$$

(3.42)

where $\gamma_6 = \beta''_+ \beta''_- > 0$ and $\gamma_7 = \beta_{0}'' + \beta''_+ \beta''_-$.

Let the classical $r$-matrices (3.15) be $\text{sl}(2; \mathbb{R})$-antireal, i.e. with their parameters satisfying the reality conditions (3.38). In such way the functions (3.17) for real $\chi$ are real, i.e. $\beta_0^* = \tilde{\beta}_0, \beta_\pm^* = \tilde{\beta}_\pm$, and we obtain that the automorphism (3.14) with such parameters is $\text{sl}(2; \mathbb{R})$-real. We can conclude that for the case of the negative $\gamma$-characteristics $\gamma_i < 0$ ($i = 1, \ldots, 5$) all $r$-matrices $r_i$ ($i = 2, \ldots, 5$) in (3.41) are reduced to the standard formula $r'_{st} := r'_1$ and in the case of the positive $\gamma$-characteristics $\gamma_i > 0$ ($i = 6, 7$) the classical $r$-matrix $r'_7$ in (3.42) is reduced to the quasi-standard $r$-matrix $r'_{qst} := r'_6$.

Let us show that the $r$-matrix $r'_{qst}$ effectively depend only on one positive parameter. Indeed, it is easy to see that

$$
r'_{qst} = \sqrt{\beta_+ \beta_-} \left( \frac{\beta_+}{\sqrt{\beta_+ \beta_-}} E'_+ \wedge H' + \frac{\beta_-}{\sqrt{\beta_+ \beta_-}} H' \wedge E'_- \right),
$$

(3.43)

$^8$Using (2.11) it is easy to check that $\Omega' = \Omega$ (see the formula (3.3)).
where $\varphi$ is the $\mathfrak{sl}(2, \mathbb{R})$-real automorphism: $\varphi(E'_\pm) = \frac{\mp \beta_\pm}{\sqrt{\beta_+ \beta_-}} E'_\pm$, $\varphi(H') = H'$, and $\alpha = \sqrt{\beta_+ \beta_-}$ is positive.

Finally for $\mathfrak{sl}(2, \mathbb{R})$ we obtain the following result:

For the non-compact real form $\mathfrak{sl}(2, \mathbb{R})$ there exists up to $\mathfrak{sl}(2, \mathbb{R})$ automorphisms three solutions of CYBE, namely Jordanian $r'_J$, standard $r'_st$ and quasi-standard $r'_qst$:

\[
r'_J = i \alpha E'_+ \wedge H', \quad [[r'_J, r'_J]] = 0, \tag{3.44}
\]

\[
r'_st = i \alpha E'_+ \wedge E'_-, \quad [[r'_st, r'_st]] = -\alpha^2 \Omega', \tag{3.45}
\]

\[
r'_qst = i \alpha (E'_+ + E'_-) \wedge H', \quad [[r'_qst, r'_qst]] = \alpha^2 \Omega', \tag{3.46}
\]

where the parameter $\alpha$ is a positive number.

4 Explicit isomorphism between $\mathfrak{su}(1, 1)$ and $\mathfrak{sl}(2; \mathbb{R})$ bialgebras and its application to $\mathfrak{o}(2, 1)$ quantizitions

Using the formulas (2.7) and (2.8) we express the triplets of the classical $\mathfrak{su}(1, 1)$ and $\mathfrak{sl}(2; \mathbb{R})$ $r$-matrices in terms of the $\mathfrak{o}(2, 1)$ basis (2.3), (2.5). We get the following results.

(i) The $\mathfrak{su}(1, 1)$ case:

\[
r_{qJ} = \frac{\alpha}{2} (i(E_+ - E_-) \wedge H + E_+ \wedge E_-) = -\alpha (i I_1 - I_2) \wedge I_3, \quad [[r_{qJ}, r_{qJ}]] = 0, \tag{4.1}
\]

\[
r_{st} = \alpha E_+ \wedge E_- = -2i \alpha I_1 \wedge I_3, \quad [[r_{st}, r_{st}]] = \alpha^2 \Omega, \tag{4.2}
\]

\[
r_{qst} = \alpha (E_+ + E_-) \wedge H = -2 \alpha I_1 \wedge I_2, \quad [[r_{qst}, r_{qst}]] = -\alpha^2 \Omega, \tag{4.3}
\]

where the $\mathfrak{o}(2, 1)$-invariant element $\Omega$ expressed in terms of the Cartesian basis (2.3) satisfying the reality condition (2.5) looks as follows

\[
\Omega = -8 I_1 \wedge I_2 \wedge I_3. \tag{4.4}
\]

(ii) The $\mathfrak{su}(2; \mathbb{R})$ case:

\[
r'_J = i \alpha E'_+ \wedge H' = -\alpha (i I_1 - I_2) \wedge I_3), \quad [[r'_J, r'_J]] = 0, \tag{4.5}
\]

\[
r'_st = i \alpha E'_+ \wedge E'_- = -2 \alpha I_1 \wedge I_2, \quad [[r'_st, r'_st]] = -\alpha^2 \Omega'. \tag{4.6}
\]

\[
r'_qst = i \alpha (E'_+ + E'_-) \wedge H' = -2 \alpha I_1 \wedge I_3, \quad [[r'_qst, r'_qst]] = \alpha^2 \Omega', \tag{4.7}
\]

where $\Omega' = \Omega$. 

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Comparing the $r$-matrix expressions (4.1)–(4.3) with (4.5)–(4.7) we obtain that
\begin{align*}
    r_{qJ} &= r'_{J} = -\alpha(I_1 - I_2) \land I_3, \\
    r_{st} &= r'_{qst} = -2\alpha I_1 \land I_3, \\
    r_{qst} &= r'_{st} = -2\alpha I_1 \land I_2.
\end{align*}

We see that the quasi-Jordanian $r$-matrix $r_{qJ}$ in the $\mathfrak{su}(1, 1)$ basis is the same as the Jordanian $r$-matrix $r'_{J}$ in the $\mathfrak{sl}(2; \mathbb{R})$ basis, and the standard $r$-matrix $r_{st}$ in the $\mathfrak{su}(1, 1)$ basis becomes the quasi-standard $r$-matrix $r'_{qst}$ in the $\mathfrak{sl}(2; \mathbb{R})$ basis. Conversely, the quasi-standard $r$-matrix $r_{qst}$ in the $\mathfrak{su}(1, 1)$ basis is the same as the standard $r$-matrix $r'_{st}$ in the $\mathfrak{sl}(2; \mathbb{R})$ basis.

The relations (4.8)–(4.10) show that the $\mathfrak{su}(1, 1)$ and $\mathfrak{sl}(2; \mathbb{R})$ bialgebras are isomorphic. This result finally resolves the doubts about isomorphism of these two bialgebras (for example, see [13]).

Using the isomorphisms of the $\mathfrak{su}(1, 1)$ and $\mathfrak{sl}(2; \mathbb{R})$ bialgebras we take as basic $r$-matrices for the $D = 3$ Lorentz algebra $\mathfrak{o}(2, 1)$ the following ones:
\begin{align*}
    r_{st} &= -2\alpha I_1 \land I_3 = \alpha E_+ \land E_-, \\
    r'_{st} &= -2\alpha I_1 \land I_2 = \alpha E'_+ \land E'_-, \\
    r'_{J} &= -\alpha(I_1 - I_2) \land I_3 = \alpha E'_+ \land H'.
\end{align*}

The first two $r$-matrices $r_{st}$ and $r'_{st}$ with the effective positive parameter $\alpha$ correspond to the $q$-analogs of $\mathfrak{su}(1, 1)$ and $\mathfrak{sl}(2; \mathbb{R})$ real algebras, the third $r$-matrix $r'_{J}$ presents the Jordanian twist deformation of $\mathfrak{sl}(2; \mathbb{R})$. In the next section we shall show how to quantize the $r$-matrices (4.11)–(4.13) in an explicite form.

5 Quantizations of the $D = 3$ Lorentz symmetry

The $q$-analogs of the universal enveloping algebras $U(\mathfrak{g})$ for the real Lie algebras $\mathfrak{g} = \mathfrak{su}(1, 1)$, $\mathfrak{sl}(2; \mathbb{R})$ were already considered (see e.g. [6, 27, 38]) and they are given as follows. The quantum deformation ($q$-analogue) of $U(\mathfrak{g})$ is an unital associative algebra $U_q(\mathfrak{g})$ with generators $X_\pm$, $q^{\pm X_0}$ and the defining relations:
\begin{align*}
    q^{X_0-q^{-X_0}} = q^{-X_0}q^{X_0} = 1, \\
    q^{X_0}X_\pm = q^{X_0}X_\pm q^{X_0}, \\
    [X_+, X_-] = \frac{q^{2X_0} - q^{-2X_0}}{q - q^{-1}},
\end{align*}

with the reality conditions:
\begin{align*}
    (i) \quad X_\pm^\dagger = -X_\mp, \quad (q^{X_0})^\dagger = q^{X_0}, \quad q := e^\alpha \quad \text{for } U_q(\mathfrak{su}(1, 1)), \\
    (ii) \quad X_\pm^\dagger = -X_\mp, \quad (q^{X_0})^\dagger = q^{-X_0}, \quad q := e^{-\alpha} \quad \text{for } U_q(\mathfrak{sl}(2; \mathbb{R})),
\end{align*}

where $\alpha$ is real in accordance with (4.11) and (4.12).

A Hopf structure on $U_q(\mathfrak{g})$ ($\mathfrak{g} = \mathfrak{su}(1, 1)$, $\mathfrak{sl}(2; \mathbb{R})$) is defined with help of three additional operations: coproduct (comultiplication) $\Delta_q$, antipode $S_q$ and counit $\epsilon_q$:
\begin{align*}
    \Delta_q(q^{\pm X_0}) &= q^{\pm X_0} \otimes q^{\pm X_0}, \\
    \Delta_q(X_\pm) &= X_\pm \otimes q^{X_0} + q^{-X_0} \otimes X_\pm, \\
    S_q(q^{\pm X_0}) &= q^{\mp X_0}, \quad S_q(X_\pm) = -q^{\mp X_\pm}, \\
    \epsilon_q(q^{\pm X_0}) &= 1, \quad \epsilon_q(X_\pm) = 0.
\end{align*}
with the reality conditions$^9$:

\[
\Delta_q^\dagger(X) = \Delta_q(X^\dagger), \quad S_q^\dagger(X) = S_q^{-1}(X^\dagger), \quad \epsilon_q^\dagger(X) = \epsilon_q(X^\dagger) \tag{5.4}
\]

for any $X \in U_q(\mathfrak{g})$. The quantum algebra $U_q(\mathfrak{g})$ is endowed also with the opposite Hopf structure: opposite coproduct $\tilde{\Delta}_q^{10}$, corresponding antipode $\tilde{S}_q^\dagger$ and counit $\tilde{\epsilon}_q^\dagger$.

An invertible element $R_q := R_q(\mathfrak{g})$ which satisfies the relations:

\[
R_q \Delta_q(X) = \tilde{\Delta}_q(X) R_q, \quad \forall X \in U_q(\mathfrak{g}),
\]

\[
(\Delta_q \otimes \text{id}) R_q = R_q^{13} R_q^{23}, \quad (\text{id} \otimes \Delta_q) R_q = R_q^{12} R_q^{13}
\]  

as well as, due to (5.5), the quantum Yang-Baxter equation (QYBE)

\[
R_q^{12} R_q^{13} R_q^{23} = R_q^{23} R_q^{13} R_q^{12} \tag{5.6}
\]

is called the universal R-matrix. Let $U_q(\mathfrak{b}_+)$ and $U_q(\mathfrak{b}_-)$ be quantum Borel subalgebras of $U_q(\mathfrak{g})$, generated by $X_+, q^\pm X_0$ and $X_-, q^\pm X_0$ respectively. We denote by $T_q(\mathfrak{b}_+ \otimes \mathfrak{b}_-)$ the Taylor extension of $U_q(\mathfrak{b}_+) \otimes U_q(\mathfrak{b}_-)^{11}$. One can show (see [39, 40]) that there exists unique solution of equations (5.5) in the space $T_q(\mathfrak{b}_+ \otimes \mathfrak{b}_-)$ and such solution has the following form

\[
R_q(\mathfrak{g}) := R_q^\tau = \exp_{q^2} \left( (q - q^{-1}) X_+ q^{-X_0} \otimes q^{X_0} X_- \right) q^{2X_0 \otimes X_0}
\]

\[
= q^{2X_0 \otimes X_0} \exp_{q^2} \left( (q - q^{-1}) X_+ q^{X_0} \otimes q^{-X_0} X_- \right),
\]  

where $q = e^\alpha$ for $U_q(\mathfrak{su}(1, 1))$ and $q = e^{i\alpha}$ for $U_q(\mathfrak{sl}(2, \mathbb{R}))$. Here we use the standard definition of the $q$-exponential:

\[
\exp_q(x) := \sum_{n \geq 0} \frac{x^n}{(n)_q!}, \quad (n)_q := \frac{1 - q^n}{1 - q},
\]

\[
(n)_q! := (1)_q (2)_q \cdots (n)_q.
\]

Analogously, there exists unique solution of equations (5.5) in the space $T_q(\mathfrak{b}_- \otimes \mathfrak{b}_+) = \tau \circ T_q(\mathfrak{b}_+ \otimes \mathfrak{b}_-)$ and such solution is given by the formula

\[
R_q(\mathfrak{g}) := R_q^\tau = \exp_{q^2} \left( (q^{-1} - q) X_- q^{-X_0} \otimes q^{X_0} X_+ \right) q^{-2X_0 \otimes X_0}
\]

\[
= q^{-2X_0 \otimes X_0} \exp_{q^2} \left( (q^{-1} - q) X_- q^{X_0} \otimes q^{-X_0} X_+ \right),
\]  

where $q$ satisfies the conditions (5.2).

As formal Taylor series the solutions (5.7) and (5.9) are independent and they are related by the relation

\[
R_q^\tau = \tau \circ R_q^{-1}. \tag{5.10}
\]

It should be noted also that

\[
(R_q^\tau)^{-1} = R_q^\tau, \quad (R_q^{-1})^{-1} = R_q^{-1}. \tag{5.11}
\]

---

$^9 \Delta_q^\dagger(X) := (\Delta_q(X))^{1 \otimes 1}.$

$^{10}$The opposite (transformed) coproduct $\tilde{\Delta}_q(\cdot)$ is a coproduct with permuted components, i.e. $\tilde{\Delta}_q(\cdot) = \tau \circ \Delta_q(\cdot)$ where $\tau$ is the flip operator: $\tau \circ \sum X_{(1)} \otimes X_{(2)} = \sum X_{(2)} \otimes X_{(1)}$.

$^{11}$The opposite (transformed) coproduct $\tilde{\Delta}_q(\cdot)$ is a coproduct with permuted components, i.e. $\tilde{\Delta}_q(\cdot) = \tau \circ \Delta_q(\cdot)$ where $\tau$ is the flip operator: $\tau \circ \sum X_{(1)} \otimes X_{(2)} = \sum X_{(2)} \otimes X_{(1)}$.
From the explicite forms (5.7) and (5.9) we see that
\[
(R_q)\dagger = \tau \circ R_q = (R_q)^{-1}, \quad (R_q^\tau)\dagger = \tau \circ R_q = (R_q^\tau)^{-1} \quad \text{for} \quad U_q(\mathfrak{su}(1,1)),
\]
\[
(R_q^\tau)\dagger = (R_q)\dagger, \quad (R_q^\tau)\dagger = (R_q^\tau)^{-1} \quad \text{for} \quad U_q(\mathfrak{sl}(2;\mathbb{R})),
\]
i.e. in the case \(U_q(\mathfrak{sl}(2;\mathbb{R}))\) both \(R\)-matrices \(R_q^\tau\), \(R_q^\tau\) are unitary and in the case \(U_q(\mathfrak{su}(1,1))\) they can be called "flip-Hermitian" or "\(r\)-Hermitian".

In the limit \(\alpha \to 0 \ (q \to 1)\) we obtain for the \(R\)-matrix (5.5)
\[
R_q(\mathfrak{g}) = 1 + r_{BD} + O(\alpha^2).
\]
(5.13)

Here \(r_{BD}\) is the classical Belavin-Drinfeld \(r\)-matrix:
\[
r_{BD} = 2\beta(X_+ \otimes X_- + X_0 \otimes X_0),
\]
(5.14)

where \(\beta = \alpha, \ X_\pm = E_\pm, \ X_0 = H\) for the case \(\mathfrak{g} = \mathfrak{su}(1,1)\), and \(\beta = \alpha, \ X_\pm = E'_\pm, \ X_0 = H'\) for the case \(\mathfrak{g} = \mathfrak{sl}(2;\mathbb{R})\). The \(r\)-matrix \(r_{BD}\) is not skew-symmetric and it satisfies the standard CYBE
\[
[r_{BD}^{12}, r_{BD}^{13} + r_{BD}^{23}] + [r_{BD}^{13}, r_{BD}^{23}] = 0
\]
(5.15)

which is obtained from QYBE (5.6) in the limit (5.13). The standard \(r\)-matrix (4.11) or (4.12) is the skew-symmetric part of \(r_{BD}\), namely
\[
r_{BD} = \frac{1}{2} \check{r}_{st} + \frac{1}{2} \check{C}_2
\]
(5.16)

where \(\check{r}_{st} = r_{BD}^{12} - r_{BD}^{23}\) is the standard \(r\)-matrix (4.11) or (4.12) and \(\check{C}_2 = 2\beta C_2 = r_{BD}^{12} + r_{BD}^{21}\)
where \(C_2\) is the split Casimir element of \(\mathfrak{su}(1,1)\) or \(\mathfrak{sl}(2;\mathbb{R})\).

We can introduce the quantum Cartesian generators by the formulas: \(X_\pm = iJ_1 \pm J_3, \ q^{\pm iJ_2}\)).12 In terms of these generators the quantum algebra \(U_q(\mathfrak{su}(1,1))\), which will be denoted by \(U_{(r_{st})}(\mathfrak{o}(2,1))\), can be reformulated as follows. The quantum deformation of \(U(\mathfrak{o}(2,1))\), corresponding to the classical \(r\)-matrix (4.11), is an unital associative algebra \(U_{(r_{st})}(\mathfrak{o}(2,1))\) with the generators \(\{J_1, J_3, q^{\pm iJ_2}\}\) and the defining relations \((k = 1, 3)\):
\[
q^{iJ_2}q^{-iJ_2} = q^{-iJ_2}q^{iJ_2} = 1, \quad [J_1, J_3] = \frac{i(q^{2iJ_2} - q^{-2iJ_2})}{2(q - q^{-1})},
\]
\[
q^{\pm iJ_2}J_k = \frac{1}{2}(q + q^{-1})J_k q^{\pm iJ_2} \pm \frac{i}{2}(q - q^{-1})\varepsilon_{k2i}J_1 q^{\pm iJ_2}
\]
(5.17)

with the reality condition \(J_1^\dagger = J_1, \ J_3^\dagger = J_3, \ (q^{\pm iJ_2})^\dagger = q^{\mp iJ_2}, \ q^* = q (q := e^\alpha, \ \alpha \in \mathbb{R})\). These relations are the \(q\)-analog of the relations (2.3) with the reality condition (2.5). The Hopf algebra structure on \(U_{(r_{st})}(\mathfrak{o}(2,1))\) is given as follows \((k = 1, 3)\):
\[
\Delta_q(J_k) = J_k \otimes q^{iJ_2} + q^{-iJ_2} \otimes J_k,
\]
\[
\Delta_q(q^{\pm iJ_2}) = q^{\pm iJ_2} \otimes q^{\pm iJ_2}, \quad S_q(q^{\pm iJ_2}) = q^{\mp iJ_2},
\]
(5.18)
\[
S_q(J_k) = -\frac{1}{2}(q + q^{-1})J_k + \frac{i}{2}(q - q^{-1})\varepsilon_{k2i}J_1, \quad \epsilon_q(q^{\pm iJ_2}) = 1, \quad \epsilon_q(J_k) = 0,
\]

---

12The generators \(J_i = (-1)^{i-1}J_i^\dagger \ (i = 1, 2, 3)\) are \(q\)-analog of the Cartesian basis (2.3), (2.5) \((\lim_{q \to 1} J_i \to I_i)\).
Substituting in the formulas (5.7) and (5.9) the expressions \(X_\pm = iJ_1 \pm J_2\), \(q^{\pm X_0} = q^{\pm iJ_3}\), we obtain the universal \(R\)-matrix in terms of the quantum Cartesian generators \(J_i\) \((i = 1, 2, 3)\) with the defining relations (5.17).

We can also introduce another quantum Cartesian generators by the formulas: \(X_\pm = iJ_1^\prime \mp J_2^\prime\), \(q^{\pm X_0} = q^{\pm iJ_3^\prime}\). In terms of these generators the quantum algebra \(U_q(\mathfrak{sl}(2, \mathbb{R}))\), which will be denoted by \(U'(\alpha, 1)\), can be reformulated as follows. The quantum deformation of \(U(\alpha, 1)\), corresponding to the classical \(r\)-matrix (4.12), is a unital associative algebra \(U'(\alpha, 1)\) with the generators \(\{J_1^\prime, J_2^\prime, q^{\pm iJ_3^\prime}\}\) and the defining relations \((k = 1, 2)\):

\[
q^{iJ_1^\prime}q^{-iJ_3^\prime} = q^{-iJ_3^\prime}q^{iJ_1^\prime} = 1, \quad [J_1^\prime, J_2^\prime] = -\frac{i(q^{2iJ_3^\prime} - q^{-2iJ_3^\prime})}{2(q - q^{-1})}, \tag{5.19}
\]

with the reality conditions \(J_1^\dagger = J_1^\prime, J_2^\dagger = -J_2^\prime, (q^{J_3^\prime})^\dagger = q^{J_3^\prime}, q^* = q^{-1}\) \((q := e^{i\alpha}, \alpha \in \mathbb{R})\). The Hopf structure on \(U'(\alpha, 1)\) is provided by the formulae \((k = 1, 2)\):

\[
\begin{align*}
\Delta_q(J_k^\prime) &= J_k^\dagger \otimes q^{iJ_3^\prime} + q^{-iJ_3^\prime} \otimes J_k^\prime, \\
\Delta_q(q^{iJ_3^\prime}) &= q^{iJ_3^\prime} \otimes q^{iJ_3^\prime}, \\
S_q(J_k^\prime) &= -\frac{1}{2}(q + q^{-1})J_k^\prime + \frac{i}{2}(q - q^{-1})\epsilon_{k3l}J_l^\prime, \\
\epsilon_q(q^{iJ_3^\prime}) &= 1, \quad \epsilon_q(J_k^\prime) = 0.
\end{align*}
\]

Substituting in the formulas (5.7) and (5.9) the expressions \(X_\pm = iJ_1^\prime \pm J_2^\prime\), \(q^{X_0} = q^{iJ_3^\prime}\) we obtain the universal \(R\)-matrix in terms of the physical generators \(J_i^\prime\) \((i = 1, 2, 3)\) with the defining relations (5.19).

The quantization of \(U(\mathfrak{sl}(2, \mathbb{R}))\) corresponding to the classical Jordanian \(r\)-matrix (4.13) is well known for a long time [41, 42, 43] and it is defined by the twist \(F\) (see [42]):

\[
F = \exp(H' \otimes \sigma), \quad \sigma = \ln(1 + i\alpha E_+'). \tag{5.21}
\]

The two-tensor \(F\) satisfies the 2-cocycle condition

\[
F^{12}(\Delta \otimes \text{id})(F) = F^{23}(\text{id} \otimes \Delta)(F), \tag{5.22}
\]

and the "unital" normalization

\[
(\varepsilon \otimes \text{id})(F) = (\text{id} \otimes \varepsilon)(F) = 1. \tag{5.23}
\]

It is evident that the twist (5.21) is unitary

\[
F^* = F^{-1}. \tag{5.24}
\]

The twisting element \(F\) defines a deformation of the universal enveloping algebra \(U(\mathfrak{sl}(2; \mathbb{R}))\) considered as a Hopf algebra. The new deformed coproduct and antipode are given as follows

\[
\Delta^{(F)}(X) = F\Delta(X)F^{-1} , \quad S^{(F)}(X) = uS(X)u^{-1} \tag{5.25}
\]

The generators \(J_i^\prime\) \((i = 1, 2, 3)\) are also the \(q\)-analog of the Cartesian basis given by (2.3), (2.5) \((\lim_{q \to 1} J_i^\prime \to I_i)\).
for any $X \in U(\mathfrak{sl}(2; \mathbb{R}))$, where $\Delta(X)$ and $S(X)$ are the coproduct and the antipode before twisting: $\Delta(X) = X \otimes 1 + 1 \otimes X$, $S(X) = -X$; and

$$u = m(id \otimes S)(F) = \exp(-i\alpha H'E_+).$$  \hfill (5.26)

It is easy to see that we get the $*$-Hopf algebra, i.e.

$$(\Delta^{(F)}(X))^* = \Delta^{(F)}(X^*), \quad (S^{(F)}(X))^* = S^{(F)}(X^*)$$  \hfill (5.27)

for any $X \in U(\mathfrak{sl}(2; \mathbb{R}))$. One can calculate the following formulae for the deformed coproducts $\Delta^{(F)}$ (see [42]):

$$\Delta^{(F)}(H') = H' \otimes e^{-\sigma} + 1 \otimes H',$$

$$\Delta^{(F)}(E_+') = E_+'^* \otimes e^\sigma + 1 \otimes E_+',$$

$$\Delta^{(F)}(E_-') = E_-'^* \otimes e^{-\sigma} + 1 \otimes E_-' + 2i\alpha H' \otimes H'e^{-\sigma} + \alpha^2 H'(H' - 1) \otimes E_+'^* e^{-2\sigma}.$$  \hfill (5.28)

Using (5.25) and (5.26) one gets the formulae for the deformed antipode $S^{(F)}$:

$$S^{(F)}(H') = -H' e^{-\sigma}, \quad S^{(F)}(E_+') = -E_+'^* e^{-\sigma},$$

$$S^{(F)}(E_-') = -E_-'^* e^\sigma + 2i\alpha H'^2 e^\sigma - \alpha^2 H'(H' - 1)E_+'^* e^\sigma.$$  \hfill (5.29)

It is easy to see the universal $R$-matrix $R^{(F)}$ for this twisted deformation looks as follows

$$R^{(F)} = \bar{F} F^{-1}, \quad (R^{(F)})^* = (R^{(F)})^{-1}.$$  \hfill (5.30)

In the limit $\alpha \to 0$ we obtain for the $R$-matrix (5.23)

$$R^{(F)} = 1 + r_j + O(\alpha^2),$$  \hfill (5.31)

where $r_j$ is the classical Jordanian $r$-matrix (4.13). Using the relations (2.6) we can express all the formulas (5.28)–(5.30) in terms of the Cartesian basis (2.3) and (2.5)).

We add that the Jordanian deformation has been described as well in a deformed $\mathfrak{sl}(2; \mathbb{R})$ algebra basis [44, 45].

### 6 Short Summary and Outlook

By using the three-fold isomorphism of classical Lie algebras $\mathfrak{so}(2, 1) \simeq \mathfrak{sl}(2; \mathbb{R}) \simeq \mathfrak{su}(1, 1)$ one can express the infinitesimal versions of the $D = 3$ Lorentz quantum deformations in terms of classical $\mathfrak{so}(2, 1)$, $\mathfrak{sl}(2; \mathbb{R})$ and $\mathfrak{su}(1, 1)$ $r$-matrices. The first aim of our paper was to derive $\mathfrak{so}(2, 1)$, $\mathfrak{su}(1, 1)$ and $\mathfrak{sl}(2; \mathbb{R})$ bialgebras using representation-independent purely algebraic method (see Sect. 3) and further to provide the explicite maps which relate them (see Sect. 4). We start in Sect. 3 with the derivation of known pair of inequivalent complex $\mathfrak{so}(3; \mathbb{C}) \simeq \mathfrak{sl}(2; \mathbb{C})$ $r$-matrices - the Jordanian (nonstandard) one and the Drinfeld-Jimbo (standard) $r$-matrix. Passing from $\mathfrak{sl}(2; \mathbb{C})$ to $\mathfrak{sl}(2; \mathbb{R})$ we obtain three independent $\mathfrak{sl}(2; \mathbb{R})$ $r$-matrices. First two of them are the real forms of two basic complex $\mathfrak{sl}(2; \mathbb{C})$ $r$-matrices, the third $\mathfrak{sl}(2; \mathbb{R})$ $r$-matrix, which we called quasi-standard (see 3.46), is the sum of two skew-symmetric 2-tensors. We do not know however how to obtain directly the universal $R$-matrix from the third $r$-matrix. We show that
there is however a way out (see Sect. 4): the quasi-standard \( r \)-matrix (3.46) (see also (3.7)) can be transformed by the map (2.11) into the standard \( r \)-matrix in \( \mathfrak{su}(1, 1) \) basis, with known universal \( R \)-matrix (see e.g. [6]). In such a way we can derive the effective quantization of all three \( D = 3 \) Lorentz \( r \)-matrices, however we recall that for such a purpose it is necessary to use both \( \mathfrak{sl}(2; \mathbb{R}) \) and \( \mathfrak{su}(1, 1) \) bases.

In second part of Introduction we mentioned main applications of \( D = 3 \) Lorentz symmetries and their deformations, but still more important for the description of noncommutative \( D = 3 \) space-time geometry and \( D = 3 \) quantum gravity are the quantum deformations of \( D = 3 \) Poincaré algebra, with noncommutative translations sector. These quantum deformations were classified (see e.g. [46]) in terms of classical \( r \)-matrices, but systematic studies of their Hopf quantizations still should be completed. There were considered also the quantum deformations of \( D = 3 \) de-Sitter (\( dS \)) and anti-de-Sitter (\( AdS \)) space-times, with nonvanishing cosmological constant \( \Lambda \). In \( D = 3 \) \( dS \) case (\( \Lambda > 0 \)) all Hopf-algebraic quantizations are known, because they were studied as the quantum deformations of \( D = 4 \) Lorentz algebra \( \mathfrak{o}(3, 1) \) [47]. In \( D = 3 \) \( AdS \) case (\( \Lambda < 0 \)) with \( \mathfrak{o}(2, 2) \) symmetry some Hopf-algebraic quantum deformations were also given, but recently there was presented complete classification of real \( \mathfrak{o}(2, 2) \) \( r \)-matrices\(^{14}\).

For physical applications it is very important to consider subsequently the quantum space-time deformations for \( D = 4, 5, 6 \). The deformations of physical \( D = 4 \) space-time and \( D = 4 \) Poincaré algebra were extensively studied for more than a quarter of the century [52]–[56], but it should be observed that the complete list of \( D = 4 \) Poincaré \( r \)-matrices (\( D = 4 \) Poincaré bialgebras) is still not complete\(^{15}\). The next task could be to describe all deformations of \( D = 4 \) space-times with constant curvature and arbitrary signature, which would classify all possible \( D = 4 \) quantum \( dS \) and \( AdS \) algebras as well as the quantum-deformed \( D = 5 \) Euclidean \( \mathfrak{o}(5) \) symmetries. For such a purpose one can look for the extension of algebraic methods used to classify the deformations of \( \mathfrak{o}(4; \mathbb{C}) \) and its real forms (see [7]) to the case of \( \mathfrak{o}(5; \mathbb{C}) \) and the real forms \( \mathfrak{o}(5) \), \( \mathfrak{o}(4, 1) \) and \( \mathfrak{o}(3, 2) \). Finally the systematic study of deformations of \( \mathfrak{o}(6; \mathbb{C}) \) is another important challenge, in particular because the deformations of its real form \( \mathfrak{o}(4, 2) \simeq \mathfrak{su}(2, 2) \) will provide the list of quantum \( D = 4 \) conformal algebras.

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\(^{14}\) See [7] and Addendum (to be published); for earlier efforts to describe \( \mathfrak{o}(2, 2) \) quantum deformations see e.g. [48]. We recall that \( \mathfrak{o}(D, D) \) algebras describe the symmetries of double geometry [49, 50], which were used recently e.g. in the description of self-dual models in so-called metastring theory [51].

\(^{15}\) The classification problems of \( D = 4 \) Poincaré bialgebras described in [56] still remain unsolved.
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