Universal time-fluctuations in near-critical out-of-equilibrium quantum dynamics

Lorenzo Campos Venuti and Paolo Zanardi

Department of Physics and Astronomy and Center for Quantum Information Science & Technology,
University of Southern California, Los Angeles, CA 90089-0484, USA

Out of equilibrium quantum systems, on top of quantum fluctuations, display complex temporal patterns. Such time fluctuations are generically exponentially small in the system volume and can be therefore safely ignored in most of the cases. However, if one consider small quench experiments, time fluctuations can be greatly enhanced. We show that time fluctuations may become stronger than other forms of equilibrium quantum fluctuations if the quench is performed close to a critical point. For sufficiently relevant operators the full distribution function of dynamically evolving observable expectation values, becomes a universal function uniquely characterized by the critical exponents and the boundary conditions. At regular points of the phase diagram and for non sufficiently relevant operators the distribution becomes Gaussian. Our predictions are confirmed by an explicit calculation on the quantum Ising model.

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Introduction Low temperature quantum matter at equilibrium organizes itself in different phases separated by critical regions featuring enhanced quantum fluctuations [1, 2]. This is due to the existence of competing interaction terms in the system’s Hamiltonian each of which strives to order the system according different symmetry patterns. When the system is taken out-of-equilibrium e.g., by a sudden change of the Hamiltonian parameters, on top of these quantum fluctuations, temporal fluctuations are present as well. In a series of papers [3–6] we have shown that the full time-statistics of a dynamically evolving expectation value $A(t)$ of a quantum observable $A$ provides a wealth of information about the equilibration properties of the system, finite-size precursors of quantum criticality [3] as well as a tool to single out quantum integrability in an operational fashion [6].

Let us start by setting the general stage of our investigations. Consider a quantum system driven out of equilibrium by an Hamiltonian $H$ and evolving unitarily according to the Schrödinger equation $\rho(t) = e^{-itH}\rho_0 e^{itH}$. For definiteness we will focus on a many-body quantum system defined on a $d$-dimensional lattice of volume $V$, and $A$ being a local extensive observable. A first natural question is now: what is the typical size of temporal fluctuations $\mathcal{A}(t)$ is now: what is the typical size of temporal fluctuations $\mathcal{A}(t) = \text{tr}[A_p(t)]$? Reimann in [7, 8] has proven, assuming the non-resonant condition for the energy gaps, the following bound on the temporal variance of $A$ [27]:

$$\Delta A^2 := \langle (A(t) - \bar{A})^2 \rangle \leq \text{diam}(A)^2 \text{tr}(\rho^2),$$

where $\text{diam}(A)$ is the diameter of the spectrum of $A$, a measures of the strength of $A$. As shown in [6] (see also [9, 10]), for clustering initial states $\mathcal{A}$, \text{tr}(\rho^2) \lesssim e^{-\alpha V}$ so that the normalized fluctuations $\Delta A/\mathcal{A}$ are bounded by $\Delta A/\mathcal{A} \lesssim e^{-\alpha V}$. This shows that in this general situation time fluctuations are practically absent [29] and one can safely replace dynamically evolving quantities $A(t)$ with their averages, i.e., $\mathcal{A}(t) \approx \mathcal{A}$ i.e., equilibration is achieved.

However, there are at least two situations in which time fluctuations are greatly enhanced and in some cases may become even stronger than equilibrium ones. One possibility is to consider systems of non-interacting particles. The bound [1] does not apply in this case and rather a “Gaussian equilibration” scenario sets in whereby time fluctuations are seen to scale as $\Delta A/\mathcal{A} \sim V^{-1/2}$ [6] (see also [10–13]). This seems to be a precise prediction of the folklore according to which free systems show poor equilibration.

Another possibility is offered by a small quench experiment. With this we mean to tune the initial state to be the ground state of a given Hamiltonian and then perform a small sudden change in the Hamiltonian parameters. Intuitively, if the quench is sufficiently small only relatively few quasi-particles get excited and contribute to the equilibration process. This in turn results in poor equilibration property, i.e. large time fluctuations. Roughly speaking this is a region of parameters for which $\alpha V \lesssim 1$ and so the bound [1] becomes ineffective. As shown in [4] this situation can be used to locate precursor of criticality on small systems by looking at dynamically evolving quantities.

In this Letter we will analyze this infinitesimal quench scenario in detail and show that the full time-probability distributions of (properly rescaled) expectation values of observables feature a novel type of universality in the infinite-volume limit. For sufficiently relevant operators the full distribution function become a universal function uniquely characterized by the critical exponents and the boundary conditions. Whereas, for non sufficiently relevant operators or at regular point of the phase diagram the distribution becomes Gaussian.

Observable dynamics for small quench Consider then the following small quench scenario. The system is prepared in the ground state of the Hamiltonian $H_0$ for $t < 0$. At time $t = 0$ one suddenly switches on a small perturbation $B$ such that the evolution Hamiltonian becomes $H = H_0 + \delta \lambda B$, with $\delta \lambda$ a small parameter [30]. Expanding $\mathcal{A}(t)$ up to first order in $\delta \lambda$ using Dyson expansion and the spectral resolution
where the first, time-independent term is the average of $A(t)$ and with $Z_n := A_{n,B_{n,0}} / (E_n - E_0)$. The leading contribution to the temporal variance is therefore at second order and assuming that the gaps $E_n - E_0$ are non-degenerate one obtains

$$\Delta A_B^2 = 2\delta \lambda^2 \sum_{n > 0} |Z_n|^2 + O(\delta \lambda^3). \quad (3)$$

We added a subscript $B$ to recall that the variance is computed with perturbation $B$. Eq. (2) shows a striking similarity to the zero temperature equilibrium isothermal susceptibility defined by $\langle \psi(\delta \lambda)|A|\psi(\delta \lambda)\rangle = \langle \psi(0)|A|\psi(0)\rangle + \delta \lambda \chi_{AB} + O(\delta \lambda^2)$ ($\langle \psi(\delta \lambda)\rangle$ being the ground state of $H$). Indeed we can write $\chi_{AB} = 2 \sum_{n > 0} \Re e Z_n$. Inasmuch a super-extensive scaling of the susceptibility can be used to detect criticality the same can be said for the time fluctuations.

Using Eq. (2) we can actually obtain the full probability distribution of the variable $A$. Assuming rational independence (RI) of the gaps $E_n - E_0$ and using the theorem of averages we can express the time average as a phase space average over a large dimensional torus. We then obtain, for the characteristic function of $A$,

$$\langle e^{i s(A - \bar{A})/\delta \lambda} \rangle = \prod_{n > 0} J_0(2s |Z_n|) := J^A(s), \quad (4)$$

where $J_0$ is the Bessel function of the first kind. So the probability distribution of $A$ is completely encoded in the function $J^A(s)$. Let us also define the Wick rotated function $I^A(s) = J^A(is)$ which has the advantage of being positive for real $s$. We can then define the coefficients $a_n$ by the series $\ln \langle J_0(s) \rangle = \sum_{n=1}^{\infty} a_n s^n/n!$ which converges absolutely in a neighborhood of the origin. Note that $a_{2p} = 0$ for $p$ odd. The cumulant of the variable $(A - \bar{A})/\delta \lambda$ are given by $\kappa_{2p} = a_{2p} 2^{2p} Q_{2p}$ with $Q_{2p} := \sum_{n > 0} |Z_n|^{2p}$ (odd cumulants are zero). Under the assumption of convergence the probability distribution of $A$ is uniquely characterized by the coefficients $Q_{2p}$. Conversely the probability distribution uniquely defines the coefficients $Q_{2p}$, which are generalizations of the variance Eq. (3). Intuitively, at critical points the cumulants $\kappa_{2p}$ (through the coefficients $Q_{2p}$) tend to diverge with the system size, higher cumulant being more divergent.

Let us analyze the behavior of $Q_{2p}$ close to quantum criticality. In this case $\lambda \Delta \lambda = |\lambda - \lambda_c|$ measures the distance from the critical point $\lambda_c$. Using standard scaling arguments one can show that $Q_{2p} \propto \xi^{2p+\alpha}$ with $\alpha = 2d + \delta - \Delta_B - \Delta_A$ (see Supplemental Material). Here $\Delta_A/\delta \lambda$ are the scaling dimensions of the observables $A/B$ that we assumed extensive and $\xi$ is the dynamical critical exponent. Instead, away from criticality the expectation is $Q_{2p} \propto L^\Delta$. Requiring that, at finite size, $Q_{2p}$ is analytic in the system parameters and matches the above scaling, one can predict the behavior of $Q_{2p}$ close to the critical point both in the critical region $\xi \gg L$ and in the off-critical one $\xi \ll L$.

$$\kappa_{2p} \sim Q_{2p} \sim \left\{ \begin{array}{ll} L^{2p+\alpha} & \xi \gg L \\ \delta \lambda^{d-2+\alpha} L^d & \xi \ll L. \end{array} \right. \quad (5)$$

Let us compare the strength of the temporal fluctuations encoded in Eq. (5) with other familiar forms of quantum fluctuations close to criticality. Equilibrium quantum fluctuations of an observable $A$ in a state $|\psi\rangle$ are encoded in the cumulants $\langle A^n\rangle$, where the subscript $n$ denotes connected averages with respect to $|\psi\rangle$. In the critical region the singular part of these cumulants scales as $L^{n(d-\Delta_A)}$ where $\Delta_A$ is the scaling dimension of $A$ [31]. When one is interested in the response to a perturbation $A$ encoded in a Hamiltonian $H = H_0 + \lambda A$, other generalized cumulants are given by the higher order susceptibilities $\partial^n E/\partial \lambda^n$ ($E$ is the ground state of $H$, or the free energy at positive temperature) [31]. At criticality such generalized susceptibilities grow as $L^{-\alpha n/\nu}$, in particular one as $L^{2d+\delta - 2\Delta_A}$ for $n = 2$. Comparing with Eq. (6) we see that temporal fluctuations—which are basically absent for general quenches—become the strongest fluctuations for small quenches close to criticality. Indeed, looking at the scaling of the cumulants (and setting $A = B$ for simplicity) the exponents for the temporal variance, susceptibilities and quantum variance satisfy $2(2d + \delta - 2\Delta_A) > 2d + \delta - 2\Delta_A > 2d - 2\Delta_A$. As noted in [44] this opens up the possibility of observing dynamical manifestations of criticality on small systems.

Consider now the rescaled random variable $R(t) = (A(t) - \bar{A})/\Delta A$ whose cumulants are given by $\kappa_{2n}^R = \kappa_{2n}^A/\kappa_{2n}^A$ for $n \geq 1$ whereas odd cumulants are zero. The probability distribution of $R$ is uniquely determined by the ratios $R_{2p} = Q_{2p}/(Q_2)^p$. From Eq. (5) we see that in the quasi-critical regime, these ratios are scale independent and define some presumably universal constants. Let us now find these constants. With the help of density of states $\rho(E) = \text{tr} (\delta (H - E))$ we can write $Q_p = \int \rho(E) \rho(E) \, dE$. Since $\rho(E) \, dE$ is scale invariant, from $Q_p \propto E^p$ we derive $Q_p(E) \propto E^{p(\Delta+/\nu)}$. In order to proceed further we must assume the form of the low energy dispersion. The simplest possibility is a rotationally invariant spectrum at small momentum $E \approx C|k|^\zeta = C(\sum_j k_j^2)^{\zeta/2}$ where $k$ is a quasi-momentum vector. In one dimension this is essentially the only possibility but for $d > 1$ one can also have anisotropic transitions where the form of the dispersion depends on the direction. Using the isotropic assumption we obtain $Q_p \propto C' \sum_k |k|^p \propto \xi^{p\zeta}$. In doing so we have essentially restricted the sum over $n$ to the one-particle contribution. This is expected to be the leading contribution whereas higher particle sectors contribute at most to the extensive, regular term. At this point, a part from an irrelevant constant $C'$, the behavior of the cumulants is uniquely specified by the critical exponent $\alpha$ and the boundary conditions that specify $k$. More precisely the probability distribution of the rescaled variable $R(t)$ is a universal function which depends only
on $\alpha$ and the boundary conditions. Let us be more explicit. Assume for concreteness that the lattice is a hyper-cube of size $L$ and the boundary conditions (BC) are such that moments are quantized according to $k = (2\pi/L)(n + b)$ with $n_i = 1, \ldots, L$. The BC on the direction $i$ are fixed by $b_i \in [0, 1/2]$ which interpolate between periodic (PBC, $b_i = 0$) and anti-periodic (ABC, $b_i = 1/2$) BC. Define now the generalized $d$-dimensional Hurwitz-Epstein $\zeta$-function as $\zeta_b (\alpha) = \sum_{n=1}^{\infty} \cdot \sum_{n_j=1}^{\infty} |n + b|^{-\alpha}$. The cumulants of $\mathcal{R}$ depend on the universal ratios $R_{2p} = Q_{2p}/(Q_2)^p$ which are given by (letting $L$ go to infinity) $R_{2p} = \zeta_b (2p\alpha)/\zeta_b (2\alpha)^p$. For example, in $d = 1$ and for PBC one has $R_{2p} = \zeta (2p\alpha)/\zeta (2\alpha)^p$ where $\zeta (\alpha)$ is the familiar Riemann zeta function. Clearly, anisotropic energy dispersion can also be treated introducing even more general zeta functions with different exponents in different directions. So far the exponent $\alpha$ has been quite arbitrary. Indeed $\alpha$ can even become negative if $A$ and $B$ are not sufficiently relevant. For example $\alpha = -\zeta$ if both $A$ and $B$ are marginal operators. Apparently the moments of $\mathcal{R}$ become non-normalizable in this case. The correct procedure is to keep the sum over $k$ finite, compute the leading finite size correction, calculate the ratios $R_{2p}$ and then take the infinite volume limit. The result is

$$
\lim_{L \to \infty} R_{2p} = \begin{cases} 
\delta_{p,1} \
\zeta_b (2p\alpha)/\zeta_b (2\alpha)^p \quad 2\alpha \leq d \\
2\alpha > d.
\end{cases}
$$

(6)

For $2\alpha \leq d$ the characteristic function of $\mathcal{R}(t)$ becomes $e^{-x^2/2}$ in the thermodynamic limit and so $\mathcal{R}$ tends in distribution to Gaussian. Clearly the Gaussian behavior observed here for not sufficiently relevant operators, is also to be expected at regular points of the phase diagram. A discussion of the regular points as well as a comparison of the dynamical central limit type theorem here discussed and the one for quantum fluctuations at equilibrium can be found in the Supplemental Material.

**Loschmidt echo** Let us now extend the formalism by considering a particular, non-extensive observable $A = |\psi_0\rangle\langle \psi_0|$. In this case $A(t)$ becomes the so-called the Loschmidt echo (LE) or survival probability given by $\mathcal{L}(t) = |\langle \psi_0 | e^{-itH} | \psi_0 \rangle|^2$. The Loschmidt echo is essentially the Fourier transform of the work distribution function and it is currently at the center of much theoretical work. We will show that it is possible to obtain its full time statistics exactly for a general initial state. Using the spectral resolution of $H$ the LE can be written as $\mathcal{L}(t) = |G(t)|^2$ with $G(t) = \sum_{n} p_n e^{-itE_n} := X(t) + iY(t)$ where $p_n = |\langle n | \psi_0 \rangle|^2$ and $X(t), Y(t)$ are its real and imaginary part. Let us start by noticing that

$$
\text{Prob} \left( \mathcal{L} < r \right) = \int_{x^2+y^2<r} P_{X,Y}(x,y) \, dx \, dy
$$

(7)

where $P_{X,Y}(x,y)$ is the joint probability density of $X$ and $Y$. i.e. $P_{X,Y}(x,y) = \delta(X-x) \delta(Y-y)$. The related, joint characteristic function $\chi(\xi, \eta) = e^{i\xi X + i\eta Y}$ can again be computed as a phase-space average assuming rational independence of the energies $E_n$. Expressing Eq. (7) in terms of $\chi(\xi, \eta)$, integrating over $x$ and $y$ and differentiating with respect to $\tau$ we obtain the following expression for the probability density of the Loschmidt echo (more details in the Supplemental Material)

$$
P_{\mathcal{L}}(x) = \int_{0}^{\infty} K(x,\rho) J^\mathcal{L}(\rho) \, d\rho
$$

(8)

$$
K(x,\rho) = \frac{J_1 (\sqrt{x}\rho)}{2\sqrt{x}} + \frac{\rho}{4} [J_0 (\sqrt{x}\rho) - J_2 (\sqrt{x}\rho)]
$$

(9)

and $J^\mathcal{L}(\rho) := \prod_{n} J_0 (p_n \rho)$. The probability distribution of the LE is completely encoded in the function $J^\mathcal{L}$. Proceeding as previously we realize that $J^\mathcal{L}(\rho) = \exp \left[ \sum_{n} a_n \text{tr} \left( \overline{\rho}^n \right) (-i\rho)^n / n! \right]$ where $\overline{\rho}$ is the infinite time average of $\rho(t)$. Again, under assumption of convergence, the probability distribution of $\mathcal{L}$ is uniquely specified by the numbers $\text{tr} \left( \overline{\rho}^n \right)$ and vice-versa. Since $\overline{\rho}$ is a positive operator this uniquely specifies the spectrum of $\overline{\rho}$. Note that

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![Figure 1: Upper panel: Universal probability distribution for the Loschmidt echo in one dimension for different $\nu$ and PBC moments i.e. $k = 2\pi n/L$. Solid curve $\nu = 1$ ($\alpha = 2$) dashed $\nu = 4/3$ ($\alpha = 3/2$). The curves are obtained integrating numerically Eq. (9). Lower panel: Critical probability distribution for the quantity $\langle |M(t)| - M_0 \rangle/L^d h$ close to the Ising critical point $h = 1$. The inset shows the characteristic function $J(s) = \prod_{n=0}^{\infty} J_0 (s/(2n + 1))$. The dashed lines are obtained using quasimoments of the form $k = 2\pi n/L$. The histogram is computed performing a numerical experiment on the Ising model on a chain of $L = 1006$ sites with periodic boundary conditions. The quench parameters are $h_1 = 1, h_2 = 1.0003$ and $\gamma_1 = \gamma_2 = 1$. The statistics is obtained sampling 600,000 random times uniformly sampled in $[0, T]$ with $T = 600,000$. The distribution is unchanged using a different $\gamma_1 = \gamma_2 = 0$ as implied by universality.
the average LE is given by the first term in the expansion of $J^2(\rho)$ and is $\mathcal{L} = \text{tr} (\hat{p}^2)$. Let us first investigate the case of small quench close to a critical point. It has been shown already that, as a function of energy the weights $p(E_n) := p_n$ behave as $p(E) \sim \delta \lambda^2 E^{-2/(\zeta \nu)}$ in the quasi-critical regime $4 \leq \nu \leq 5$. Using again the isotropic assumption $E \sim ||k||^2$ we find $p_k \sim \delta \lambda^2 ||k||^{-2/\nu}$. Let us now study the rescaled LE $\mathcal{L}(t)/\mathcal{L}$. Its distribution is determined by $J^L(\rho/\sqrt{\mathcal{L}}) = \exp \left[ \sum_{n=1}^\infty a_{2n} T_{2n} (\rho^2)^n / (2n!) \right]$ with $T_{2n} := \text{tr} (\hat{p}^{2n}) / \text{tr} (\hat{p}^2)^n$. Once again, at criticality the probability distribution is uniquely determined by the scaling exponent $\nu$ and the quantization of the quasi-momenta $k$. Moreover, taking the thermodynamic limit ($L \to \infty$) in the quasi-critical region, the rescaled cumulants $T_{2n}$ become exactly $T_{2n} = R_{2n}$ of Eq. (6) with $\alpha = 2/\nu$. In particular we see that for $4 \nu \leq d \lim_{L \to \infty} J^L(\rho/\sqrt{\mathcal{L}}) = \exp [-\rho^2 / 4]$. Using this expression and Eq. (8) one gets the probability distribution of the rescaled LE: $\lim_{L \to \infty} P_{\mathcal{L}/\mathcal{L}} (x) = \theta (x) e^{-x}$ i.e. a Poissonian distribution. This in turns implies, for the un-rescaled variable, $P_{\mathcal{L}} (x) \sim \theta (x) e^{-x/\mathcal{L}}/\mathcal{L}$ a result which has been observed in $[3]$ (see also the recent $[13]$). Clearly the Gaussian behavior of $J^L$ predicted for not sufficiently relevant operators $\nu \leq d/4$ is expected to be seen also in the off-critical region. In fact, the same considerations regarding the $R$ variable apply in this case. A small quench in the vicinity but not at the critical point has the effect of opening up a mass gap. This in turn cures the infra-red divergences (UV divergences are cured by the lattice) and one obtains that $\lim_{L \to \infty} T_{2n} = \delta_{n,1}$. One also expects that this behavior extends a fortiori for more general quenches. Indeed we conjecture that $\lim_{L \to \infty} T_{2n} = \delta_{n,1}$ for any sufficiently clustering initial state $|\psi_0\rangle$. A plot of the universal, critical distribution of the LE is shown in Fig. 4

Computing critical distribution functions is a very hard task at equilibrium as one needs the exact analytic form of the characteristic function. For this reasons results are essentially limited to models that can be mapped to a system of non-interacting particles such as the 2D classical Ising model and its 1D quantum counterpart (see e.g. $[16]$). The situation for the temporal fluctuation is analogous and in order to exemplify the formalism we consider now the 1D quantum XY model with PBC. We perform a small quench in the transverse field close to the Ising critical point at $h \approx 1$. The expectation value of the total magnetization takes the form $M(t) = \sum_j (\sigma_j^z (t)) = \mathcal{M} + \sum_k \sin (\delta \theta_k) \sin (\delta \theta_k) \cos (t \theta_k)$. The function $W_k := \sin (\delta \theta_k^2) \sin (\delta \theta_k) \sin (\delta \theta_k)$ plays the role of $Z_k$ in Eq. (5). The characteristic function can be computed assuming independent rational dependence of the one particle energies and one obtains

$$e^{i M (t) / \Delta M} = \exp \sum_k \ln \left[ J_0 (\lambda W_k / \Delta M) \right]. \quad (10)$$

The scaling dimensions are in this case $d = \zeta = \Delta_A = \Delta_B = 1$ implying that $\sum_{n>0} |Z_n|^2 \sim L^2$. This can be in fact proven analytically as, for small quench, $W_k$ is singular at $k = \pi$, where it behaves as $W_k \sim 1 / |\gamma (k - \pi)|$. Expanding the argument of the exponential in series one realizes that only the divergent part of $W_k$ is needed when computing the limit $L \to \infty$. Given that the fact, that for large $L$, $\Delta M^2 = 2^{1-1} (L/2\pi)^2 \sum_{n>0} (n+1)^{-2} / \sum_{n>0} (n+1)^{-2}$, we then obtain $\lim_{L \to \infty} \sum_k \ln \left[ J_0 (\lambda W_k / \Delta M) \right] = \sum_{n>0} \ln J_0 (\lambda \alpha_n)$, where $\alpha_n := \sqrt{\sin (2\pi / n) / (n+1)^2} [32]$. In figure 1 we plot the exact, critical, probability distribution of the transverse magnetization obtained Fourier transforming numerically Eq. (10). We also show very good agreement with a numerical, small quench experiment performed on a XY chain. The critical distribution is observed as long as $\xi \approx \delta h^{-1} \gg L$ and $L \gtrsim 20$. For shorter sizes, $M$ is a sum of few random variables and can be well approximated by retaining the two dominant variables $[4]$. In the off-critical region $\xi \ll L$ one obtains a Gaussian distribution $[6]$.

Applications We would like now to point out the possible use of the time-fluctuations formalism to distinguish critical or gapped regions in non-homogeneous systems. A concrete realization of these systems is offered by optical lattices of cold atoms in harmonic traps. Traditionally $[17-20]$ Mott-insulating regions are distinguished from superfluid ones by looking either at the quantum fluctuations of the particle densities $\langle n_i^2 \rangle - \langle n_i \rangle^2$ or at the local compressibility (or suitably averaged version thereof) $\partial^2 (n_i) / \partial \mu_i$, i.e. a susceptibility. Small (resp. large) fluctuations correspond to Mott-insulating, “gapped” (resp. superfluid, “critical”) regions. Indeed, as expected by the larger scaling and confirmed in $[18]$ the local compressibility is so far the best indicator of insulating/superfluid region. The study of temporal fluctuations for small quenches in trapped cold atom systems, may offer an experimentally accessible yet powerful way to investigate such non-homogeneous systems. The feasibility of such an approach is currently under investigation $[21]$.

Conclusions In this Letter we have shown that the temporal fluctuations of quantum observables for a small Hamiltonian quench near a critical points feature a novel type of universality that mirrors the one of quantum fluctuations at equilibrium. The initial quantum state is chosen to be the ground state of a Hamiltonian $H_0$ which is then slightly perturbed to $H = H_0 + \delta \lambda B$. Given the observable $A$ the temporal probability distribution $P_{\text{dyn}}(a) := \delta (\langle A (t) \rangle - a)$ (overline denotes the infinite time average) becomes Gaussian for regular points of the phase diagram whereas it acquires a universal form at critical points. Assuming hyperscaling, the critical distribution function $P_{\text{dyn}}(a)$ is uniquely characterized by the critical exponents and the boundary conditions it is hence even more universal than the the equilibrium case. Moreover universal dynamical distributions are observed even for less relevant operators. A byproduct of this analysis is that, in the critical regime, temporal fluctuations are stronger than other forms of equilibrium quantum fluctuations. This opens up the possibility of assessing the critical character of non-homogeneous systems by performing quench experiments.

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[27] We use here outline to indicate (infinite) time average which we define as $\bar{T} \equiv \lim_{T \to \infty} F^{T/2} \int_0^T f(t) dt$.

[28] By clustering we mean here the property that, for operators $A, B$ localized around $x_A, x_B$, the correlation $\langle AB \rangle$ decays sufficiently fast (i.e. exponentially) with the distance $\|x_A - x_B\|$.

[29] It is important to notice that these time fluctuations are extremely small compared to the quantum fluctuations which, for clustering state $|\psi\rangle$ are of the order of $\Delta A(|\langle A|\rangle) \sim V^{-1/2}$

[30] The precise definition of “small” is given in the Supplemental Material

[31] The term generalized cumulants can be understood given that these time fluctuations are quantities derivatives of a --generalized-- cumulant generating function, the free energy, i.e. $tr \exp(-\beta H_0 + \lambda A) = \exp(-\beta F) = \exp(-\sum B^2 F/\beta A^2)$.

[32] Note that the equilibrium analogue of the above distribution $P_{eq}(x) = \langle \delta (M-x) \rangle$ instead is Gaussian for large size because the $M$ is not sufficiently relevant ($2(d-\Delta M) = 0$ if $d = 1$) as shown in [25].

Supplemental Material

Small quench regime and universality

The small quench regime can be encoded by the relation $tr (\hat{p}^2) \simeq 1$. If this condition is met the bound [1] becomes ineffective. For small quench $tr (\hat{p}^2)$ can be related to the fidelity between initial and final ground state and its fidelity susceptibility $\chi_F$ [22]. Considering the scaling of the fidelity susceptibility $\chi_F$ in this regime (see [23]) one obtains $\max_{\lambda} \Delta \lambda \{ L^d, L^{2/\nu} \} \ll 1$ or $\Delta \lambda \ll \min \{ L^{-d/2}, L^{-1/2} \}$ where $\Delta \lambda = \lambda_2 - \lambda_1$ is the quench amplitude. As is often the case, the symbol ”$\ll$” indicates a conservative estimates and $\Delta \lambda \sim \min \{ L^{-d/2}, L^{-1/2} \}$ indicates the region where
a crossover takes place. Once the small quench condition is satisfied, universal behavior in the full time statistics of observables expectation values is expected in the quasi-critical region when $\xi^{(2)} \gg L$ where $\xi^{(2)} \simeq |\lambda_2 - \lambda_n|^{-\nu}$ is the correlation length of the evolution Hamiltonian. Roughly speaking the condition to observe universal distribution can be written compactly as $\xi^{(2)} \gg L$ with $j = 1, 2, \ldots$. Moreover $L$ should be large enough such that i) the universality in the function $Q_{2p}(E)$ sets in and ii) the finite-size corrections to the zeta function results are small. Both of these conditions depend on the critical exponent $2\alpha$. For larger values of $2\alpha - d$, a critical, universal distribution can be observed for smaller sizes. In the opposite, off-critical region $\xi^{(2)} \ll L$, temporal distributions are expected to be of Gaussian shape [4,5].

We have verified universality in temporal distributions on the hand of the XY model in transverse field given by the Hamiltonian

$$H = \sum_{j=1}^{L} \left[ \frac{1+\gamma}{2} \sigma_j^x \sigma_{j+1}^x + \frac{1-\gamma}{2} \sigma_j^y \sigma_{j+1}^y + \hbar \sigma_j^z \right],$$  \hspace{1cm} (11)

defined on a chain of $L$ sites with periodic boundary conditions for the spins, i.e. $\sigma_L = \sigma_1$. The system is initialized in the ground state of Hamiltonian (11) with parameters $\hbar = \hbar_1$ and $\gamma = \gamma_1$ which are then suddenly changed to $\hbar_2$ and $\gamma_2$. The transverse magnetization at time $t$ has the form

$$\mathcal{M}(t) = \sum_j \langle \sigma_j^x \rangle(t) = \sum_k \left[ \cos(\vartheta_k^{(2)}) \cos(\delta \vartheta_k) + \sin(\vartheta_k^{(2)}) \sin(\delta \vartheta_k) \cos(t \Lambda_k) \right],$$

where $k$ are ABC momenta for the fermions: $k = 2\pi/L (n + 1/2)$, $n = 0, 1, \ldots, L - 1$ and the Bogoliubov angles are given by $\vartheta_k = \sqrt{\lambda} \sinh(k) / \cosh(k)$ and $\delta \vartheta_k = \vartheta_k^{(2)} - \vartheta_k^{(1)}$.

Scaling behavior

Let us write $Q_p = Q_p^0 \text{tr} \Omega_p$, where we have defined $\Omega := \Omega^{-1} \text{tr} \Omega^p$. Since $\Omega$ is dimensionless the scaling behavior of $Q_p$ is clearly dictated by the scaling dimension of $Q_1^p$ this latter in turn is just $p$ times the one of $Q_1$. Now $Q_1 \geq |Q_1|$ where $Q_1 := \sum_{n=0}^{L} Z_{n} = \sum_{n=0}^{L} \langle 0\vert A(n) \langle n \vert B(0) \rangle / (E_n - E_0)$, therefore the scaling dimension $\Delta Q_1$ of $Q_1$ is lower bounded by the one of $Q_1$. In formulae $\Delta Q_p = \Delta Q_1^p = p \Delta Q_1 \geq p \Delta Q_1$, Now we observe that $Q_1$ can be written as the time integral of a connected (imaginary time) two point cross-correlation function of the observables $A$ and $B$:

$$\hat{Q}_1 = \sum_{n=0}^{\infty} \int_{0}^{\infty} d\tau e^{-\tau(E_n - E_0)} \langle 0\vert A(n) \langle n \vert B(0) \rangle$$

$$= \sum_{n=0}^{\infty} \int_{0}^{\infty} d\tau \langle 0\vert e^{\tau H} A e^{-\tau H} \langle n \vert B(0) \rangle$$

$$= \int_{0}^{\infty} d\tau \langle A(\tau) B(0) \rangle_c$$

(15)

where $\langle A(\tau) B(0) \rangle_c := \langle 0\vert A(\tau) B(0) \rangle - \langle 0\vert A(0) \langle 0 \vert B(0) \rangle$. Assuming now that these operators are local i.e., $X = \sum_j X_j$, $(X = A, B)$ in the continuous limit one finds $\hat{Q}_1 \rightarrow \int d^4x d^4y \int d\tau G_{A,B}(x,\tau; y, 0)$ where $G_{A,B}(x,\tau; y, 0) := \langle A(x,\tau) B(y, \tau) \rangle_c$. Performing the scaling transformation $x, y, \tau \rightarrow \lambda x, \lambda y, \lambda^\alpha \tau$ and using the definition of scaling dimension of $A$ and $B$ i.e., $G_{A,B} \rightarrow \lambda^{-\Delta A - \Delta B}$. One finds that $\Delta Q_1 = 2d + \zeta - \Delta A - \Delta B$. Finally assuming that $\Delta Q_1 = \Delta Q_1 =: \alpha$ we recover the key scaling relation used in the main text i.e., $\Delta Q_{2p} = 2p\alpha$.

Proof of Eq. (6)

Let us define the truncated version of $\zeta_b(\alpha)$ as

$$\zeta_b(\alpha, L) := \sum_{n=1}^{L} \cdots \sum_{n_{d-1}=1}^{L} \| n + b \|^{-\alpha}.$$  \hspace{1cm} (16)

Note that $\zeta_b(\alpha, L) = \zeta_b(\alpha) - \zeta_b(\alpha, L)$ with $L = (L, L, \ldots, L)$. Now, for $\alpha \neq d$ one has

$$\zeta_b(\alpha, L) = \zeta_b(\alpha) + L^{d-\alpha} \left( \frac{C_b}{d-\alpha} + O(L^{-1}) \right)$$

(17)

where $C_b$ is a constant independent of $L$. For $\alpha = d$ the scaling gets modified to

$$\zeta_b(d, L) = C_b \ln L + O(1).$$

(18)

The finite size ratios are given by $R_{2p} = \zeta_b(2p\alpha, L) / \zeta_b(2\alpha, L)$. Plugging in Eqns. (17) and (18) and taking the limit $L \rightarrow \infty$ one recovers Eq. (6).

Regular points

Let us then analyze the universal cumulant ratios $R_p$ at gapped region of the phase diagram. Using norm inequalities one can only show that $R_{2p} \leq 1$ for all $p$ whereas to prove the central limit theorem (CLT) one would need $\lim_{L \rightarrow \infty} R_{2p} = 0$ for $p \geq 2$. Actually, using Lyapunov condition, it suffices to show that $R_4 \rightarrow 0$. Now at regular point of the phase diagram, the infrared divergence is cured by the gap $E \geq \Delta$. Moreover, quantum lattice model do not have divergence in the UV as they have a natural cutoff. For example, a quasi-relativistic, phenomenological one-particle
dispersion often used to model interacting lattice models is given by $\epsilon_k = \sqrt{\sin(k)^2 + m^2}$. Now, close but not exactly at the critical point, the contribution to $Q_x$ coming from the one-particle excitations is $Q_x \sim \sum_k (\sin(k)^2 + m^2)^{-p_{\alpha}}$ ($\zeta = 1$ in this case). Since $1/\epsilon_k$ is a bounded function of $k$ we conclude that $Q_{2\rho} \propto L$ implying the CLT for the rescaled variable $\mathcal{R}$ as claimed in the main text.

**CLT: comparison with the equilibrium case**

Let us now compare the origin of the central limit theorem and of universality for temporal fluctuations and equilibrium fluctuations. In the equilibrium framework, outside criticality, an extensive observable can be considered as a sum of weakly dependent random variables. The CLT arises from the linked cluster expansion \[24\] and the fact that, outside criticality the dependent random variables. The CLT arises from the linked cluster expansion \[24\] and the fact that, outside criticality the dependent random variables.

Substituting $r^2 \to r$ and differentiating with respect to $r$ we get a very convenient form of the probability density for the Loschmidt echo, i.e. Eq. \(8\)

\[
P_{\mathcal{E}}(x) = \int_0^\infty K \left(\sqrt{x}, \rho \right) J(\rho) d\rho
\]

\[
K \left(\sqrt{x}, \rho \right) = \frac{J_1 \left(\sqrt{x} \rho \right)}{2 \sqrt{x}} + \frac{\rho}{4} \left[ J_0 \left(\sqrt{x} \rho \right) - J_2 \left(\sqrt{x} \rho \right) \right]
\]

\[
J^\mathcal{E}(\rho) = \prod_n J_0 \left(p_n \rho \right).
\]

**Quasi-free systems**

The formalism developed in the main text does not directly apply to quasi-free systems because the many-body energies are massively rationally dependent. In this case the analysis has been carried out in \[5\]. If the initial state has covariance matrix $R_{x,y} = \langle e^{i\lambda c_x} \rangle$, and the Hamiltonian is $H = \sum_{x,y} \epsilon_x c_x c_y$, the LE can be expressed as $\mathcal{L}(t) = |\det(1 - R + R e^{-i t M})|^2$. Now, if $[M,R] = 0$ as it happens for quenches, both matrices can be diagonalized simultaneously and one gets

\[
\mathcal{L}(t) = \prod_k \left(1 - \alpha_k \sin^2 \left(\frac{t \epsilon_k}{2}\right)\right)
\]

\[
\mathcal{L}(t) = \prod_k \left(1 - \alpha_k \sin^2 \left(\frac{t \epsilon_k}{2}\right)\right)
\]

where $\epsilon_k$ are the one-particle energies, and $\alpha_k = 4r_k(1 - r_k)$ and $r_k$ are the eigenvalues of $R$. In this case it’s easier to get the probability distribution for the logarithmic LE $\mathcal{G} = \ln \mathcal{L}$. Assuming now rational independence for the one-particle energies, we can obtain its characteristic function

\[
e^{i \lambda \mathcal{G}} = \prod_k F_1 \left(\frac{1}{2}, -i \lambda, 1, \alpha_k\right).
\]

Now for any finite quench, all the cumulants of $\mathcal{G}$ are extensive so that the rescaled variable $(\mathcal{G} - \mathcal{G})/\Delta \mathcal{G}$ tends in distribution to a Gaussian in the thermodynamic limit \[5\]. In the critical, small quench scenario, a good approximation to the distribution of $\mathcal{G}$ is obtained retaining few (e.g. 2) lowest weights $\alpha_k$ and it acquires a double peaked form as shown in \[5\].

**Proof of Eq. \(8\)**

Assuming rational independence of the many-body energies $E_n$, the joint characteristic function can be computed via

\[
\chi(\xi, \eta) = e^{i \xi X + i \eta Y} = \prod_n \int \frac{d\varphi_n}{2\pi} e^{i \xi p_n \cos(\varphi_n) + i \eta p_n \sin(\varphi_n)} = \prod_n J_0 \left(p_n \sqrt{\xi^2 + \eta^2}\right).
\]

Let us now compute the cumulative distribution of the LE,

\[
Prob \left(\mathcal{L} < x^2\right) = \int \frac{d\xi}{(2\pi)} \frac{d\eta}{(2\pi)} \int_{x^2 + y^2 < x^2} dxdy e^{-i \xi \eta - i \eta^2} \chi(\xi, \eta)
\]

\[
= \int \frac{d\xi}{(2\pi)} \int_0^\infty \rho d\rho \int_0^{2\pi} d\phi e^{-i \xi \rho \cos \phi - i \eta \rho \sin \phi} \frac{\chi(\xi, \eta)}{(2\pi)^2}
\]

\[
= \int \frac{d\xi}{(2\pi)} \int_0^\infty \rho d\rho J_0 \left(\rho \sqrt{\xi^2 + \eta^2}\right) \frac{\chi(\xi, \eta)}{(2\pi)}
\]