CMB constraints on spacetime noncommutativity in inflationary scenario

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We investigate the primordial power spectrum of the density perturbations based on the assumption that spacetime is noncommutative in the early stage of inflation. Due to the spacetime noncommutativity, the primordial power spectrum can lose rotational invariance. Using the k-inflation model and slow-roll approximation, we show that the deviation from rotational invariance of the primordial power spectrum depends on the size of noncommutative length scale $L_s$, but not on sound speed. We constrain the contributions from the spacetime noncommutativity to the CMB fluctuations using five-year WMAP CMB maps. We find that the upper bound for $L_s$ depends on the product of sound speed and slow-roll parameter. Estimating this product using cosmological parameters from the five-year WMAP results, the upper bound for $L_s$ is estimated to be less than $10^{-27}$ cm at 99.7% confidence level.

I. INTRODUCTION

The inflationary cosmology \cite{1} is the scenario of the very early universe. It provides a successful mechanism for generating nearly scale invariant primordial density perturbations, that give rise to galaxy formation and temperature anisotropies in the CMB which are in agreement with observation \cite{2}. If the period of inflation is sufficiently longer than that required for solving the horizon and flatness problems, such that the wavelengths of perturbations which are observed today emerged from the Planck regime in the early stages of inflation, the physics on trans-Planckian scales should leave an imprint on the primordial density perturbations \cite{2}.

Near the Planck scale, the properties of spacetime are expected to be modified due to the quantum nature of gravity \cite{4}. It has been shown that a consequence of string theory which is a promising candidate of quantum gravity, is that the spacetime is noncommutative \cite{5}.

For the $\mu$th and $\nu$th position, \beg \begin{align*} [x^\mu, x^\nu] = i \Theta^{\mu\nu}(x), \end{align*} \end{aligned} \end{equation}

where $\Theta^{\mu\nu}$ is an antisymmetric tensor.

The influences of spacetime noncommutativity on the feature of power spectrum of primordial fluctuations have been studied by many authors \cite{3, 6}. If $\Theta^{\mu\nu} = 0$ but $\Theta^\mu = 0$ \cite{7, 8}, the primordial power spectrum can become direction-dependent, and consequently the statistics of CMB fluctuations becomes anisotropic. We are interested in this spacetime noncommutativity induced statistical anisotropy.

Usually, the statistics of the CMB temperature fluctuations is supposed to be isotropic. However, recently there are many attempts to check whether the statistics of the CMB fluctuations is perfectly isotropic by searching for the statistical anisotropy contributions in the CMB sky maps \cite{10, 11, 12}. In the case where the statistics of the CMB fluctuations is anisotropic, the angular power spectrum does not contain all the information about the statistical properties of the CMB fluctuations even when the Gaussianity of the CMB fluctuations is assumed. Some of the estimators for quantifying the statistical anisotropy contributions in the CMB fluctuations have been proposed in \cite{13, 14, 15}. According to \cite{10, 11}, the statistics of the observed CMB fluctuations does not deviate from isotropy significantly.

In this work, we constrain the contributions from spacetime noncommutativity to CMB temperature fluctuations using five-year WMAP CMB maps.

II. THE CONTRIBUTIONS FROM SPACETIME NONCOMMUTATIVITY

In this section, we investigate the contributions from spacetime noncommutativity to the primordial power spectrum in the k-inflation model \cite{16}.

We start with the general action of the inflaton of the form

\begin{equation} S = \frac{1}{2} \int d^4x \sqrt{-g} [R + 2P(X, \phi)] , \end{equation}

where $\phi$ is the inflaton field and $X = -(1/2)\partial_\mu \phi \partial^\mu \phi$. Here, we have set the reduced Planck mass $(8\pi G)^{-1/2}$ = 1. To study the evolution of density perturbations during inflation, one expands the action \cite{2} around the homogeneous and isotropic background. In our consideration, we use the ADM metric formalism in which the line element is given by \cite{10, 11, 12}.

\begin{equation} ds^2 = -N^2d^2t + h_{ij}(dx^i + N_i^0 dt)(dx^j + N^0_j dt) . \end{equation}

In the ADM formulation $h_{ij}$ and $\phi$ are the dynamical variables, while $N$ and $N_i^0$ are Lagrange multipliers. To compute the perturbed action in the slow-roll approximation, it is convenient to use the uniform curvature gauge in which \cite{15} $\delta \phi \equiv \varphi(t, \mathbf{x})$ and $h_{ij} = a^2 \delta_{ij}$, where $a$ is the cosmic scale factor, $\delta \phi(t, \mathbf{x}) = \phi(t, \mathbf{x}) - \phi_0(t)$ is the perturbation in the inflaton field and the subscript 0 represents the background value.
In order to find the second order action for \( \varphi \), one first computes the constraint equations for \( N \) and \( N^i \) from the action (2) and solve these equations for \( N \) and \( N^i \) to first order of \( \varphi \) (3). Substituting the results back in the action, expanding the action to the second order of perturbation and keeping the lowest order of slow-roll parameter, the second order action for perturbation is given by

\[
\delta S^{(2)} = \int d^4x \frac{a^3}{2} \left( P_{,X_0}X_0 (\phi_0 \varphi)^2 - P_{,X_0} \partial_\mu \varphi \partial^\mu \varphi \right) \tag{4}
\]

\[
= \int d^4x \frac{a^3}{2} \rho_{,X_0} \left( \varphi^2 - c^2_0 (\partial \varphi)^2 \right), \tag{5}
\]

where \( c^2_0 = P_X / \rho_X \) is the sound speed, the subscript \( X \) denotes a derivative with respect to \( X \), \( \rho \) is the energy density of the inflaton and \( \rho_X = P_X + 2P_{XX} \). In the above calculation, we neglect the terms in the perturbed action that are multiplied by the metric perturbation because these terms are subleading in slow-roll parameter (4, 17).

We now study how the spacetime noncommutativity influences the action for the field perturbations. In order to take the effect of spacetime noncommutativity into account, we replace the ordinary products in the action with the star products. In curved spacetime, the star product can be expanded as (8)

\[
f \star g = \sum_{k=1}^{\infty} \frac{1}{k!} \left( \frac{i}{2} \right)^k \Theta^{\mu_1 \nu_1} \cdots \Theta^{\mu_k \nu_k} (D_{\mu_1} \cdots D_{\mu_k} f) (D_{\nu_1} \cdots D_{\nu_k} g), \tag{6}
\]

where \( D_{\mu} \) is the covariant derivative. In our consideration, we suppose that the nonzero components of \( \Theta^{\mu \nu} \) are \( \Theta^{12} = -\Theta^{21} = L_s^2 / a^2 \), where \( L_s \) is the noncommutative length scale (3). Since the terms in the second order action that are multiplied by metric perturbation are subleading in slow-roll parameter, the spacetime noncommutativity effect can be incorporated by replacing the products between fields in the Lagrangian by the star products, and expanding the Lagrangian to the second order in the field perturbation. In the calculation of star product, we also ignore the metric perturbation because it gives rise to the terms that are subleading in slow-roll parameter. Since it is possible to study the effect of spacetime noncommutativity perturbatively (3), we do the calculation up to the lowest non trivial order in \( \Theta^{\mu \nu} \). Up to the second order in \( \Theta^{\mu \nu} \), the star products of the multiple functions can be written as (9)

\[
f_1 \star \cdots \star f_n = \left( 1 + \frac{i}{2} \Theta^{\mu \nu} \sum_{a<b} D_{\mu}^a D_{\nu}^b - \frac{1}{8} \Theta^{\mu \nu} \Theta^{\rho \sigma} \sum_{a,b,c,d} C_{\mu}^{a} C_{\nu}^{b} C_{\rho}^{c} C_{\sigma}^{d} \right) f_1 \cdots f_n, \tag{7}
\]

where \( a, b, c, d \) run over \( 1, \ldots, n \).

For illustration, we suppose that \( P(X, \phi) \) can be written as \( P(X, \phi) = F(\phi)G(X) = \phi^m X^n \), and use eq. (17) to compute the star product between field variables. We first consider the case where the derivatives in eq. (9) act on the terms in \( G(X) \) only. In this case, the contribution from spacetime noncommutativity starts to appear at the second order of \( \Theta^{\mu \nu} \), i.e.,

\[
\delta_{\phi} G = -\frac{1}{8} \Theta^{\mu \nu} \Theta^{\rho \sigma} \sum_{a,b,c,d} D_{\mu}^a D_{\nu}^b D_{\rho}^c D_{\sigma}^d \frac{1}{2} \partial_\gamma \partial^\gamma \phi \partial^\rho \phi \partial^\sigma \phi \partial^\sigma \phi, \tag{9}
\]

where \( \delta_{\phi} G = \int d^4x \delta^{(2)}(X) \) is the first order perturbation in \( X \). This equation, we omit the terms that are proportional to \( D_{\mu} D_{\nu} \partial_\gamma \partial^\gamma \phi \partial^\rho \phi \partial^\sigma \phi \) because these terms vanish after integration by parts. Using similar consideration, one can show that if the derivatives in eq. (7) act on the terms in \( F(\phi) \) only, the noncommutative contribution will be proportional to \( P_{\phi_0 \phi_0} \), and if the derivatives act on the terms in both \( F(\phi) \) and \( G(X) \), the noncommutative contribution will be proportional to \( P_{\phi_0 \phi_0} \).

In our case, both contributions can be neglected because the terms that are proportional to \( P_{\phi_0 \phi_0} \) are subleading in slow-roll parameter. Hence, the spacetime noncommutativity modifies the action in leading order of slow-roll parameter as

\[
\delta_{\phi} S^{(2)} = \delta S^{(2)} + \delta_{\phi} S_X^{(2)}, \tag{9}
\]

where \( \delta_{\phi} S^{(2)} \) is the action for the perturbed field in eq. (8) and \( \delta_{\phi} S_X^{(2)} = \int d^4x \delta^{(2)}(X) \). The action \( \delta_{\phi} S_X^{(2)} \) can be split into three parts as \( \delta_{\phi} S_X^{(2)} = \delta_{\phi} S_{X_1} + \delta_{\phi} S_{X_2} + \delta_{\phi} S_{X_3} \), where the subscripts 1, 2, 3 denote the contribution from the first, second and third terms on the RHS of eq. (8). The expression for \( \delta_{\phi} G \) in eq. (9) is valid for a generic \( P(X, \phi) \), because one always do Taylor’s expansion of \( X, \phi \) around the background when applying the star product in the action for perturbation. Moreover, the action (9) can be obtained by just replacing products with star products in the action (1).

We suppose that \( D_{\mu} \Theta^{\mu \nu} = 0 \), and do integration by parts for the action \( \delta_{\phi} S_X^{(2)} \) in the similar way as (3). After doing integration by parts and evaluating the covariant
derivatives, each part of this action becomes
\[ \delta \Theta S_X = \int d^4x \frac{aH^2L_s^4}{8} P_{XX} X \partial_\mu \partial_\nu \partial_\rho \partial_\sigma \phi, \]
(10)
\[ \delta \Theta S_X = \int d^4x \frac{aH^2L_s^4}{16} P_X \left( \partial_\mu \partial_\nu \partial_\sigma \partial_\rho \phi \right) \]
\[ + 2H \partial_\mu \partial_\nu \partial_\sigma \partial_\rho \phi - H^2 \partial_\mu \partial_\nu \partial_\sigma \partial_\rho \phi, \]
(11)
\[ \delta \Theta S_X = - \int d^4x \frac{aH^2L_s^4}{8} P_{XX} 2X \partial_\mu \partial_\nu \left( \partial_\rho \partial_\sigma \phi + H \partial_\rho \phi \right), \]
where \( p = 1, 2 \) and \( H = \dot{a}/a \) is the Hubble parameter. In the above three equations and following calculation, we omit the subscript 0 for \( P, \rho \) and \( X \) because from now on we will treat them as the background quantities. We note that in the calculation of \( \delta \Theta S_X \) we suppose that \( \dot{X} < X \) due to the slow-roll approximation. From these results, we get
\[ \delta \Theta S_X^{(2)} = \frac{1}{16} \int d^4x aH^2L_s^4 P_X \left[ \beta \partial_\mu \partial_\nu \partial_\sigma \partial_\rho \phi \right] \]
\[ + 2\epsilon^2 \partial_\mu \partial_\nu \partial_\sigma \partial_\rho \phi + 2\beta \partial_\mu \partial_\nu \partial_\sigma \partial_\rho \phi - c^2 H^2 \partial_\mu \partial_\nu \partial_\sigma \partial_\rho \phi, \]
(12)
where \( \beta = 2c^2 - 1 \). In order to find the evolution equation and the corresponding solution from action (12), it is convenient to write the action in the terms of the variable \( v = z\zeta \), where \( z = a \sqrt{2X \rho_X / H} \). The variable \( \zeta \) is the curvature perturbation in comoving gauge which has a direct connection with the generation of large scale structure and CMB fluctuations. This quantity is related to \( \phi \) through \( \zeta = H \phi / \dot{\phi}_0 \). Expressing the action (12) in terms of \( v \) and expanding \( v \) in Fourier space as \( v(t, \mathbf{x}) = \int (d^3k/(2\pi)^{3/2}) (a_k \phi(t)e^{ik \cdot x} + h.c.) \), where \( a_k, \phi \) satisfy the canonical commutation relations, the action (12), evaluated in the vacuum after normal ordering, becomes
\[ S = \int d\eta d^3k \left[ \frac{|v'_k|^2}{2} + \left( \frac{a''}{a} - c^2 k^2 \right) |v_k|^2 \right. \]
\[ - \frac{L_s^4 H^2 k^2}{8a^2} \left( \beta |v'_k|^2 - c^2 k^2 |v_k|^2 \right) \]
\[ + \left( \beta \left( 6 \frac{a''}{a} - 10 (H a)^2 \right) - \frac{1}{2} (H a)^2 \right) |v'_k|^2 \right], \]
(13)
where the prime denotes derivative with respect to the conformal time \( \eta = \int da/a \) and \( k^2 = k_x^2 + k_y^2 \). In the above action, we have neglected the terms \( c^2 \) and \( \dot{\rho}_X \) because these terms are subleading in slow-roll parameter. Up to the lowest order of slow-roll parameter, we can write \( a'' = 1/(H \eta)^2 + \mathcal{O}(\epsilon) \approx 1/(H \eta)^2 \) and \( a''/a \approx 2/\eta \), where \( \epsilon = -H/2 \) is the slow-roll parameter. Using these approximations and writing \( v_k \) in terms of the new variable \( y_k(\eta) = (1-\beta c^2 k^2)^{1/2} v_k(\eta) \), where \( k^2 = H^4 L_s^4/8 \), the action (13) up to first order in \( k^2 \) takes the simple form
\[ S = \int d\eta d^3k \left[ y'_k^2 - \left( c^2 k^2 - \frac{2}{\eta^2} \right) y_k^2 \right. \]
\[ + \beta k^2 k^2 - \frac{1}{2} \right] \left| y_k^2 \right| \right]. \]
(14)
In the derivation of this action, we suppose that \( k\eta^2 \eta^2 < 1 \) and \( |c^2| \) do not differ much from unity, so that one can expand \( 1/(1-\beta k^2 k^2 \eta^2) \approx 1 + \beta k^2 k^2 \eta^2 + \ldots \) and neglect the terms that are proportional to \( \beta k^2 k^2 \eta^2 \). This implies that this action is valid only if \( \eta^2 < \eta^2 \sim 8/(H^4 L_s^4 k^2) \). For a suitable choice of initial conditions, we have \( -\infty < \eta < 0 \) during inflation, so that the perturbation mode \( k \) exits the horizon and sound horizon at \( \eta_c = 1/k \) and \( \eta_s = 1/(kc_s) \) respectively. Hence, we will be able to follow the evolution of the perturbation mode \( k \) well before its horizon (sound horizon) exit, i.e., \( \eta < \eta_c (\eta < \eta_s) \), if the noncommutative length scale is required to be smaller than the Hubble radius, \( L_s < H^{-1} \).

In order to study the influence of spacetime noncommutativity on the behavior of density perturbation, we compute the power spectrum of the curvature perturbation \( \zeta \) from action (14). From this action, one can show that the evolution equation for \( y_k \) is
\[ y''_k + \left( c^2 k^2 \gamma^2 - \frac{2}{\eta^2} \right) y_k = 0, \]
(15)
where \( \gamma^2 = 1 + \kappa^2 \sin^2(\theta) \) and \( \theta = \sin^{-1}(k/k_s) \) denotes the angle between the vectors \( k \) and \( k_s \). Following standard procedure, the solution of eq. (15) is given by
\[ y_k = \frac{e^{-ikc_s \gamma}}{\sqrt{2kc_s \gamma}} \left( 1 + \frac{i}{kc_s \gamma} \right), \]
(16)
so that the primordial power spectrum is given by
\[ P_k = \frac{k^3 |v_k|^2}{2\pi^2} = \frac{k^3 |v_k|^2}{2\pi^2} \approx \frac{H^2}{\pi m_p^2 c_s \epsilon} \left( 1 - \frac{3}{2} \kappa^2 \sin^2(\theta) \right), \]
(17)
where \( m_p = G^{-1/2} \) is the Planck mass. It can be seen that the obtained power spectrum is direction-dependent due to the spacetime noncommutativity. The magnitude of the noncommutative contribution depends on the angle between the vectors \( \mathbf{k} \) and \( \mathbf{k}_s \), but not on the sound speed. From the expression for \( \kappa \), it can be seen that the deviation from rotational invariance of the power spectrum also depends on the noncommutative length scale and Hubble parameter during inflation. The amplitude of this power spectrum can be defined as \( A_s = H^2/(\pi m_p^2 c_s \epsilon) \). This quantity as well as \( \kappa \) are evaluated when the perturbation mode \( k \) exits the horizon. If we neglect the contributions from spacetime noncommutativity, the above power spectrum gives rise to the usual power spectrum as in (18).
III. THE CMB CONSTRAINTS

In this section, we will constrain the contributions from spacetime noncommutativity to CMB anisotropies using the CMB data. It is well known that if the primordial power spectrum is direction-dependent, the statistics of CMB will become anisotropic, i.e., the two-point function of the temperature fluctuations is no longer rotationally invariant. Here, we consider only the two-point function because we assume the non-Gaussianity of the CMB fluctuations to be negligible. In addition to spacetime noncommutativity, the direction-dependent primordial power spectrum can also be a consequence of many phenomena in the early universe, for example see [19].

To compare theoretical prediction with the observation, it is convenient to expand the temperature anisotropy into spherical harmonics

$$\Delta T(\hat{n}) = T(\hat{n}) - T_0 = \sum_{l,m} a_{lm} Y_{lm}(\hat{n}),$$  

(18)

where $T_0$ is the mean temperature of the CMB. If we write the direction-dependent primordial power spectrum as [15]

$$P(k) = A(k) \left[ 1 + \sum_{LM} P_{LM} Y_{LM}(k) \right],$$  

(19)

the covariance matrix of $a_{lm}$ will be

$$\langle a_{l_1m_1} a_{l_2m_2}^\ast \rangle = \delta_{l_12} \delta_{m_1m_2} C_{l_1} + \sum_{LM} \Xi^{l_1m_1}_{l_2m_2} D_{l_1l_2}^{LM}. \tag{20}$$

Here, $C_{l_1}$ is the CMB angular power spectrum, given by

$$C_{l_1} = (4\pi)^2 \int_0^\infty dk k^2 A(k) |T_{l_1}(k)|^2, \tag{21}$$

where $T_{l_1}$ is the CMB transfer function which is taken to be rotationally invariant. It is known that if the two-point function of the temperature anisotropy is not rotationally invariant, the covariance matrix of $a_{lm}$ will not be diagonal. The off diagonal elements of the covariance matrix appear in the second term of eq. (20). This term is given by

$$D_{l_1l_2}^{LM} = (4\pi)^2 (-i)^{l_1-l_2} \int_0^\infty dk k^2 A(k) P_{LM} T_{l_1}(k) T_{l_2}(k),$$  

(22)

and

$$\Xi^{l_1m_1}_{l_2m_2} = \frac{\sqrt{(2l_1+1)(2l_2+1)/(4\pi(2l_3+1))} c^{l_1l_2}_{l_3,m_1,m_2}}{c^{l_1l_2}_{l_3,m_1,m_2}},$$

where $c^{l_1l_2}_{l_3,m_1,m_2}$ are the Clebsch-Gordan coefficients. Comparing the power spectrum in eq. (17) with the one in eq. (21), we find that the non zero components of $P_{LM}$ are

$$P_{00} = -\frac{4\sqrt{\frac{3}{5}}}{3} \kappa^2 \quad \text{and} \quad P_{20} = \frac{4}{3} \left(\frac{\sqrt{\frac{3}{5}}}{2}\right)^2 \kappa^2, \tag{23}$$

We see that the contributions from spacetime noncommutativity also influence the diagonal elements of the covariance matrix, i.e. they modify the amplitude of $C_{l_1}$. Substituting $P_{LM}$ from eq. (23) into eq. (20), we obtain the covariance matrix

$$\langle a_{l_1m_1} a_{l_2m_2}^\ast \rangle \simeq C_{l_1} \delta_{m_1m_2} \left\{ \delta_{l_1l_2} \left[ 1 + \frac{3\kappa^2}{2} \left( \frac{(l_1+1)^2 - m_1^2}{4l_1^2 - 1} + \frac{l_2^2 - m_2^2}{4l_2^2 - 1} - 1 \right) \right] - \left( \delta_{l_1-2l_2} \times \right. \right.$$  

$$\left. 3\kappa^2 C_{l_1l_2} \frac{2(2l_1 - 3)(2l_1 + 1)}{4l_1^2 - 1} \right\}, \tag{24}$$

where $C_{l_1l_2} = (4\pi)^2 \int_0^\infty dk k^2 A(k) T_{l_1}(k) T_{l_2}(k)/C_{l_1}$. We next constrain the contributions from spacetime noncommutativity in this covariance matrix using the five-year WMAP foreground-reduced maps [20]. Since the spacetime noncommutativity also influences the diagonal elements of the covariance matrix and its contribution depends on $H^2$, we write the noncommutative contribution in terms of the amplitude of the primordial power spectrum $A_s$ as

$$\frac{3}{2} \kappa^2 = A_s^2 \Sigma, \tag{25}$$

where $\Sigma = 3\pi^2 m^2 c^2 L_s^4 / 16$. We adopt the procedures in [11] to compute the posterior probability of the parameters $\Sigma$ and $A_s$, given the observed temperature anisotropies $a$,

$$P(\sigma|a) \propto L(a|\sigma) P(\sigma), \tag{26}$$

where $\sigma = \{ \Sigma, A_s \}$ is the set of parameters, $L(a|\sigma)$ is the likelihood and $P(\sigma)$ is the prior. Since the galactic contamination cannot be completely removed from some regions of the sky, one does not have the full-sky CMB maps with well-defined error properties. To reduce the galactic contamination, one masks the contaminated regions as $c_i = M_i \Delta T_i$, where $i$ is the pixel index, $c_i$ is the masked CMB map, $\Delta T_i$ is the full-sky map and $M_i$ is a mask which is zero at the contaminated points and is one elsewhere. This relation can be written in harmonic space as

$$c_{lm} = M_{lm} b_{lm'}, \tag{27}$$

where the matrix $M_{lm,l'm'}$ is given by $M_{lm,l'm'} = \sum_{LM} M_{LM} \Xi_{lm,l'm'}^{LM}$. Here, $c_{lm}$, $M_{lm}$ and $b_{lm}$ are the spherical harmonic coefficients of $c_i$, $M_i$ and $\Delta T_i$ respectively. Moreover, due to the instrument noise, the finite beam resolution and the discreteness of the temperature maps, the contributions to the unmasked CMB map come from the sum of the instrument noise with the convolution between the signal of the CMB anisotropies and the window function, such that

$$\Delta T = W a + N, \tag{28}$$

where $W$ is the window function and $N$ is the instrument noise. We suppose that the contributions from the
beam and the pixel asymmetries are negligible. Using eqs. (27) and (28), the covariance matrix of the masked temperature multipoles can be written as

$$C_{lm,l'm'} = \sum_{l_1,m_1,l_2,m_2} M_{lm,l_1m_1} \left[ W_{l_1}(a_{l_1m_1}a^*_{l_2m_2})W_{l_2} + N_{l_1m_1,l_2m_2} \right] \delta{A}_{l,m,l',m'},$$  

(29)

where $N_{l_1m_1,l_2m_2}$ is the pixel noise covariance matrix, given by $N_{l_1m_1,l_2m_2} = \Delta a \sum_{LM} \xi_{LM}^2 N_{LM}$. Here, $\Delta a$ is the area of each pixel in the temperature map, and $N_{LM}$ is defined as $N_{LM} = \sum_{i} \Delta \sigma_i (\sigma^2_i/n^2_{i,obs}) Y_{LM}(\hat{r}_i)$, where $\sigma_i$ is the rms noise of a single observation and $n^2_{i,obs}$ is the number of observations of pixel $i$.

In order to compute the likelihood, the inversion of the covariance matrix is required. Since the inversion of the large matrix is time consuming, we avoid to inverse the large covariance matrix by writing the likelihood function in terms of the reduced bipolar coefficients instead of $c_{lm}$. The reduced bipolar coefficients are defined as

$$A_{LM} = \sum_{l_1,l_2,m_2} (-1)^{m_2} d_{l_1m_1} d^*_{l_2m_2} \Sigma_{l_1m_1,l_2m_2} A^M, \quad (30)$$

where $d_{lm}$ are the harmonic coefficients of the temperature anisotropies. For the full-sky and noiseless case, the mean of the reduced bipolar coefficients for the noncommutative $k$-inflation can be computed using eq. (21), and the non zero components are

$$\langle A_{00} \rangle = \sum_l (-1)^l C_l \sqrt{2l+1} \left( 1 - \frac{2}{3} A^2 \right), \quad (31)$$

$$\langle A_{20} \rangle = -2A^2 \sum_l (-1)^l C_l \sqrt{\frac{l(l+1)(2l+1)}{45(2l-1)(2l+3)}} \times \quad (32)$$

$$\langle A_{LM} \rangle = \sum_{l_1,l_2,m_2} (-1)^{m_2} C_{l_1m_1,l_2m_2} \Sigma_{l_1m_1,l_2m_2} A^M, \quad (33)$$

It can be seen from the above equation that due to the effect of the mask, $\langle A_{00} \rangle$ and $\langle A_{20} \rangle$ will not be the only non zero components of $A_{LM}$.

We define $\delta A_{LM} = A_{LM} - \langle A_{LM} \rangle$, and write the covariance matrix for $\delta A_{LM}$ as

$$C_{LM,L'M'} = \sum_{l_1,m_1,l_2,m_2} \Sigma_{l_1m_1,l_2m_2} \Sigma_{l_1m_1,l_2m_2} A^M \times$$

$$\left( C_{l_1m_1,l_3m_3} C_{l_2m_2,l_4m_4} + C_{l_1m_1,l_4m_4} C_{l_2m_2,l_3m_3} \right), \quad (34)$$

where $C_{LM,L'M'} = \langle \delta A_{LM} \delta A^*_{L'M'} \rangle$. Therefore, the posterior probability function takes the form

$$P(\sigma | A_{LM}) \propto \exp \left( -\frac{1}{2} Z^2 \right) \det \frac{1}{2C} P(A_s), \quad (35)$$

where $Z^2 = \sum_{LM,L'M'} \delta A_{LM} \left( C^{-1} \right)_{LM,L'M'} \delta A^*_{L'M'}$. Here, we have used a flat prior on $\Sigma$ and a Gaussian prior on $A_s$. For the Gaussian prior, the mean and variance of $A_s$ are taken from the five-year WMAP results [21]. Since the contributions from spacetime noncommutativity mainly appear in the $\langle A_{00} \rangle$ and $\langle A_{20} \rangle$, we restrict the multipole index $L$ of $A_{LM}$ to be less than 4. We note that the parallel computing of covariance matrices $C_{l_1m_1,l_2m_2}$ and $C_{LM,L'M'}$ can be easily implemented in the Healpix package [22]. The data that are used to computed $\delta A_{LM}$ are the five-year WMAP foreground-reduced V2 and W1 differential assembly temperature maps [20]. These maps are masked using the band-limited masks in [11]. According to [11], we limit the multipole index of $c_{lm}$ and $M_{LM}$ to be $l \leq 62$ and $L \leq 92$ respectively. The covariance matrix $C$ can be computed from the covariance matrix in eq. [24]. Since the spacetime noncommutativity does not affect the cosmic evolution after the inflationary epoch, we can use CMBEASY [23] to compute the CMB transfer function and $C_l$ by supposing that the cosmic evolution after inflation obeys the $\Lambda$CDM model whose parameters are taken from the best fit value of the five-year WMAP results [21]. However, recall that the amplitude of the primordial power spectrum is treated as a free parameter. We compute the posterior probability function for $\sigma$ and marginalize it over $A_s$ to obtain the marginalized posterior probability function for $\Sigma$. The marginalized posterior probability functions for $\Sigma$ obtained from V2 and W1 maps have a peak at negative $\Sigma$, which can occur if $c^2_s$ is negative. However, we cannot use the above analysis to constrain parameters $L_s$, $c^2_s$ and $\epsilon$ simultaneously due to degeneracy among these parameters. Since we are interested in the constraint on $L_s$, we suppose that values of $c^2_s$ and $\epsilon$ are known. For canonical scalar field, $\Sigma > 0$ because $c^2_s = 1$. Hence, we restrict ourselves to the case where $\Sigma > 0$, i.e. $c^2_s > 0$. In this case, the confidence intervals for $\Sigma$ can be obtained from the areas under the curves of The marginalized posterior probability functions in figure 1.

In order to estimate the upper bound for $L_s$ from the upper bound for $\Sigma$ in table 1, we further assume that the values of the product $\epsilon c_s$ are known precisely, so that the upper bound for $L_s$ can be written as $L_s < 1.4 \times 10^4 [c^2_s \epsilon c_s]^{-1/2} \text{GeV}^{-1} \sim 1.4 \times 10^{-28} [c^2_s \epsilon c_s]^{-1/2} \text{cm}$ at 99.7% confidence level. As shown in [16], the ratio of the tensor to scalar perturbations amplitudes $r$ depends on $\epsilon c_s$. We roughly estimate the values of $\epsilon c_s$ using the values of $r$ from the five-year WMAP results, and obtain $L_s < 10^{-27} \text{cm}$ at 99.7% confidence level.
TABLE I: The upper bound for the parameter $\Sigma \times 10^{-18}$ from W1 and V2 maps.

|       | 68%CL | 95%CL | 99.9%CL |
|-------|-------|-------|---------|
| W1    | 0.024 | 0.048 | 0.070   |
| V2    | 0.029 | 0.055 | 0.078   |

FIG. 1: The marginalized posterior probability functions for $\Sigma$. The solid line represents the probability function from W1 map, while the long dashed line represents the probability function from V2 map. Here, $\Sigma$ is restricted to be positive.

IV. CONCLUSIONS

In this work, we study the effect of spacetime noncommutativity on the rotational invariance of the primordial power spectrum, and constrain the contributions from this effect using CMB data. In the slow-roll approximation, the deviation from rotational invariance of the primordial power spectrum due to the spacetime noncommutativity effect depends on the factor $3H^4L_s^4/16$, but not on sound speed. Here, $H$ is evaluated at the time when perturbation mode $k$ crosses the horizon during inflation.

Since the primordial power spectrum is direction-dependent, the covariance matrix for the harmonic coefficients of the CMB temperature anisotropies has off diagonal elements. As is well known, this implies that statistics of the CMB anisotropies become anisotropic. The spacetime noncommutativity also contributes to the diagonal elements of the covariant matrix suggesting that the noncommutative contribution also modifies the amplitude of the CMB angular power spectrum.

Both contributions from spacetime noncommutativity are simultaneously constrained using five-year WMAP foreground-reduced V2 and W1 maps. The upper bound for the quantity $L_s\{\epsilon_{sa}\}^{1/2}$ is approximately $1.4 \times 10^{-27}$ cm at 99.7% confidence level. Estimating the values of $\epsilon_{sa}$ using the tensor to scalar perturbations amplitudes from the five-year WMAP results, we obtain $L_s < 10^{-27}$ cm at 99.7% confidence level.

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