Principal Lyapunov Exponents and Principal Floquet Spaces of Positive Random Dynamical Systems. II. Finite-dimensional Systems

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Abstract

This is the second part in a series of papers concerned with principal Lyapunov exponents and principal Floquet subspaces of positive random dynamical systems in ordered Banach spaces. The current part focuses on applications of general theory, developed in the authors’ paper \textit{Principal Lyapunov exponents and principal Floquet spaces of positive random dynamical systems. I. General theory}, Trans. Amer. Math. Soc., in press, to positive random dynamical systems on finite-dimensional ordered Banach spaces. It is shown under some quite general assumptions that measurable linear skew-product semidynamical systems generated by measurable families of positive matrices and by strongly cooperative or type-$K$ strongly monotone systems of linear ordinary differential equations admit measurable families of generalized principal Floquet subspaces, generalized principal Lyapunov exponents, and generalized exponential separations.

Keywords: Random dynamical system, skew-product linear semidynamical system, principal Lyapunov exponent, principal Floquet subspace, entire positive orbit, random Leslie matrix model, cooperative system of ordinary differential equations, type-$K$ monotone system of ordinary differential equations

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1. Introduction

This is the second part of a series of several papers. The series is devoted to the study of principal Lyapunov exponents and principal Floquet subspaces of positive random dynamical systems in ordered Banach spaces.

Lyapunov exponents play an important role in the study of asymptotic dynamics of linear and nonlinear random evolution systems. Oseledets obtained in [19] important results on Lyapunov exponents and measurable invariant families of subspaces for finite-dimensional dynamical systems, which are called now the Oseledets multiplicative ergodic theorem. Since then a huge amount...
of research has been carried out toward alternative proofs of the Oseledets multiplicative ergodic theorem (see [2], [9], [10], [15], [18], [20], [21] and the references contained therein) and extensions of the Oseledets multiplicative theorem for finite dimensional systems to certain infinite dimensional ones (see [2], [9], [10], [13], [15], [18], [20], [21], [22], [24], and references therein).

The largest finite Lyapunov exponents (or top Lyapunov exponents) and the associated invariant subspaces of both deterministic and random dynamical systems play special roles in the applications to nonlinear systems. Classically, the top finite Lyapunov exponent of a positive deterministic or random dynamical system in an ordered Banach space is called the principal Lyapunov exponent if the associated invariant family of subspaces corresponding to it consists of one-dimensional subspaces spanned by a positive vector (in such case, invariant subspaces are called the principal Floquet subspaces). For more on those subjects see [17].

In the first part of the series, [17], we introduced the notions of generalized principal Floquet subspaces, generalized principal Lyapunov exponents, and generalized exponential separations, which extend the corresponding classical notions. The classical theory of principal Lyapunov exponents, principal Floquet subspaces, and exponential separations for strongly positive and compact deterministic systems is extended to quite general positive random dynamical systems in ordered Banach spaces.

In the present, second part of the series, we consider applications of the general theory developed in [17] to positive random dynamical systems arising from a variety of random mappings and ordinary differential equations. To be more specific, let \((\Omega, \mathcal{F}, \mathbb{P}), \theta_t\) be an ergodic metric dynamical system. We investigate positive random matrix models of the form \(\left((U_\omega(n))_{\omega \in \Omega, n \in \mathbb{Z}^+}, (\theta_n)_{n \in \mathbb{Z}}\right)\) (including random Leslie matrix models) (see Section 3), where

\[
U_\omega(1)u = \begin{pmatrix}
    s_{11}(\omega) & s_{12}(\omega) & \cdots & s_{1N}(\omega) \\
    s_{21}(\omega) & s_{22}(\omega) & \cdots & s_{2N}(\omega) \\
    \vdots & \vdots & \ddots & \vdots \\
    s_{N1}(\omega) & s_{N2}(\omega) & \cdots & s_{NN}(\omega)
\end{pmatrix}u, \quad u \in \mathbb{R}^N,
\]

where \(s_{ij}(\omega) \geq 0\) for \(i, j = 1, 2, \ldots, N\) and \(\omega \in \Omega\); random cooperative systems of ordinary differential equations of the form (see Subsection 4.1)

\[
\dot{u}(t) = A(\theta_t \omega)u(t), \quad \omega \in \Omega, \ t \in \mathbb{R}, \ u \in \mathbb{R}^N,
\]

where

\[
A(\omega) = \begin{pmatrix}
    a_{11}(\omega) & a_{12}(\omega) & \cdots & a_{1N}(\omega) \\
    a_{21}(\omega) & a_{22}(\omega) & \cdots & a_{2N}(\omega) \\
    \vdots & \vdots & \ddots & \vdots \\
    a_{N1}(\omega) & a_{N2}(\omega) & \cdots & a_{NN}(\omega)
\end{pmatrix},
\]

and \(a_{ij}(\omega) \geq 0\) for \(i \neq j, i, j = 1, 2, \ldots, N\) and \(\omega \in \Omega\); and random type-\(K\) monotone systems of ordinary differential equations (see Subsection 4.2)

\[
\dot{u}(t) = B(\theta_t \omega)u(t), \quad \omega \in \Omega, \ t \in \mathbb{R}, \ u \in \mathbb{R}^N,
\]
where for each $\omega \in \Omega$,

$$B(\omega) = \begin{pmatrix}
    b_{11}(\omega) & b_{12}(\omega) & \cdots & b_{1N}(\omega) \\
    b_{21}(\omega) & b_{22}(\omega) & \cdots & b_{2N}(\omega) \\
    \vdots & \vdots & \ddots & \vdots \\
    b_{N1}(\omega) & b_{N2}(\omega) & \cdots & b_{NN}(\omega)
\end{pmatrix},$$

and there are $1 \leq k, l \leq N$ such that $k + l = N$, $b_{ij}(\omega) \geq 0$ for $i \neq j$ and $i, j \in \{1, 2, \ldots, k\}$ or $i, j \in \{k + 1, k + 2, \ldots, k + l\}$, and $b_{ij}(\omega) \leq 0$ for $i \in \{1, \ldots, k\}$ and $j \in \{k + 1, \ldots, k + l\}$ or $i \in \{k + 1, \ldots, k + l\}$ and $j \in \{1, \ldots, k\}$.

We remark that, biologically, (1.1) describes discrete-time age-structured population models, (1.2) models a system of $N$ species which is cooperative, and (1.3) models a community of $N$ species which can be divided into two subcommunities, one subcommunity consisting of $k$ species and the other consisting of $l$ species, such that the interactions between every pair of species in either subcommunity are cooperative, whereas the interactions between the species belonging to different subcommunities are competitive. The study of (1.1), (1.2), and (1.3) will provide some basic tool for the study of random discrete-time age-structured nonlinear population models and random cooperative or type-$K$ monotone systems of nonlinear ordinary differential equations. The reader is referred to [4], [6], [7], [8], [11], [14], [25], [26], [29], and references therein for the study of discrete-time age-structured population models and time periodic cooperative and type-$K$ monotone systems of nonlinear ordinary differential equations.

Under quite general conditions, (1.1), (1.2), generate measurable linear skew-product semidynamical systems on $\Omega \times \mathbb{R}^N$, preserving the natural ordering on $\mathbb{R}^N$ (i.e., the order generated by the cone $(\mathbb{R}^N)^+ = \{ (u = (u_1, \ldots, u_N)^\top : u_i \geq 0, i = 1, \ldots, N \}$), and (1.3) generates a measurable linear skew-product semidynamical system on $\Omega \times \mathbb{R}^N$, preserving the type-$K$ ordering on $\mathbb{R}^N$ generated by $(\mathbb{R}^k)^+ \times (\mathbb{R}^l)^-$ (obtained by $(\mathbb{R}^k)^+ \times (\mathbb{R}^l)^-$). Observe that by the following variable change:

$$u_i \mapsto u_i \text{ for } i = 1, \ldots, k \text{ and } u_i \mapsto -u_i \text{ for } i = k + 1, \ldots, k + l(= N),$$

the random type-$K$ monotone system (1.3) becomes a random cooperative system of form (1.2) (see Subsection 4.2 for detail). We will therefore focus on the study of (1.1) and (1.2). Applying the general theory developed in Part I (17), we obtain the following results.

1. Under some general positivity assumptions, (1.1), (1.2), have nontrivial entire positive orbits (see Theorems 3.1(1), 4.1(1) for detail);

2. Assume the general positivity and some focusing property. (1.1), (1.2) have measurable invariant families of one-dimensional subspaces $\{E_1(\omega)\}$ spanned by positive vectors (generalized principal Floquet subspaces) and whose associated Lyapunov exponent is the top Lyapunov exponent of the system (generalized principal Lyapunov exponent) (see Theorems 3.1(2), 4.1(2) for detail);

3. Assume the general positivity and some strong focusing property. (1.1), (1.2) have also measurable invariant families of one-codimensional subspaces which are exponentially separated from the generalized principal Floquet subspaces (see Theorems 3.1(3), 4.1(3) for detail);
(4) Assume the general positivity with some strong positivity in one direction and some strong focusing property. Then the generalized principal Lyapunov exponent for (1.1), (1.2) is finite, so those equations admit principal Floquet subspaces, principal Lyapunov exponents, and exponential separation in the classical sense (see Theorems 3.4, 4.14 and for detail).

The results (1)–(3) are new. The result (4) recovers many existing results on principal Floquet subspaces and principal Lyapunov exponents for (1.1), (1.2) known in the literature (see [2], [3], [16], etc.).

We remark that the generalized principal Lyapunov exponents in (2) may be $-\infty$. In such a case, when generalized exponential separation holds, the (nontrivial) invariant measurable decomposition associated with the generalized exponential separation is essentially finer than the (trivial) decomposition in the Oseledets multiplicative ergodic theorem (see Subsection 4.3).

The results obtained in this paper have important biological implications. For example, the result (3) implies that the population densities of all species in a random cooperative system of $N$ species increase or decrease at the same rate which equals the generalized principal eigenvalue of the system along the same direction which is the direction of the generalized principal Floquet subspace (see Remarks 3.3 and 4.1).

The rest of this paper is organized as follows. First, for the reader’s convenience, in Section 2 we put the notions, assumptions, definitions, and main results of Part I ([17]) in the context of finite-dimensional systems. We then consider random systems arising from random families of matrices and cooperative and type-K monotone systems of ordinary differential equations in Sections 3 and 4, respectively.

2. General Theory

2.1. Notions, Assumptions, and Definitions

In this subsection, we introduce the notions, assumptions, and definitions introduced in Part I. The reader is referred to Part I ([17]) for detail.

First, we introduce some notions.

For a metric space $Y$, $\mathcal{B}(Y)$ stands for the $\sigma$-algebra of all Borel subsets of $Y$. A pair $(\Omega, \mathcal{F})$ is called a measurable space if $\Omega$ is a set and $\mathcal{F}$ is a $\sigma$-algebra of its subsets.

By a probability space we understand a triple $(\Omega, \mathcal{F}, P)$, where $(\Omega, \mathcal{F})$ is a measurable space and $P$ is a measure defined for all $F \in \mathcal{F}$, with $P(\Omega) = 1$ we call $(\Omega, \mathcal{F}, \mu)$ a probability space.

All Banach spaces considered in the paper are real. $X$ will stand for a finite-dimensional Banach space $X$, with norm $\| \cdot \|$.

By $X^*$ we will denote the dual of $X$, and by $\langle \cdot, \cdot \rangle$ we denote the standard duality pairing (that is, for $u \in X$ and $u^* \in X^*$ the symbol $\langle u, u^* \rangle$ denotes the value of the bounded linear functional $u^*$ at $u$). Without further mention, we understand that the norm in $X^*$ is given by $\|u^*\| = \sup\{ |\langle u, u^* \rangle| : \|u\| \leq 1 \}$.

$\mathbb{T}$ stands for either $\mathbb{Z}$ or $\mathbb{R}$.

For a metric dynamical system $((\Omega, \mathcal{F}, P), (\theta_t)_{t \in \mathbb{T}})$, $\Omega' \subset \Omega$ is invariant if $\theta_t(\Omega') = \Omega'$ for all $t \in \mathbb{T}$. $((\Omega, \mathcal{F}, P), (\theta_t)_{t \in \mathbb{T}})$ is said to be ergodic if for any invariant $F \in \mathcal{F}$, either $P(F) = 1$ or $P(F) = 0$.

From now on, $((\Omega, \mathcal{F}, P), (\theta_t)_{t \in \mathbb{T}})$ (we may simply write it as $(\theta_t)_{t \in \mathbb{T}}$) denotes an ergodic metric dynamical system.
For $\mathbb{T} = \mathbb{R}$ we write $\mathbb{T}^+ \text{ for } [0, \infty)$. For $\mathbb{T} = \mathbb{Z}$ we write $\mathbb{T}^+ \text{ for } \{0, 1, 2, 3, \ldots\}$. By a measurable linear skew-product semidynamical system $\Phi = (U_\omega(t))_{\omega \in \Omega, t \in \mathbb{T}^+}$ we understand a $(\mathcal{B}(\mathbb{T}^+) \otimes \mathcal{F} \otimes \mathcal{B}(X), \mathcal{B}(X))$-measurable mapping

$$[\mathbb{T}^+ \times \Omega \times X \ni (t, \omega, u) \mapsto U_\omega(t)u \in X]$$

satisfying the following:

- $U_\omega(0) = \text{Id}_X \quad \forall \omega \in \Omega,$
- $U_{\theta \omega}(t) \circ U_\omega(s) = U_\omega(t + s) \quad \forall \omega \in \Omega, \quad t, s \in \mathbb{T}^+$;

- for each $\omega \in \Omega$ and $t \in \mathbb{T}^+$, $[X \ni u \mapsto U_\omega(t)u \in X] \in \mathcal{L}(X)$.

To avoid using lower indices, we usually write $\Phi = (U_\omega(t), \theta_t)$.

When $\mathbb{T}^+ = [0, \infty)$ we call a measurable linear skew-product semidynamical system a (measurable linear skew-product) semiflow. To emphasize the situation when $\mathbb{T}^+ = \{0, 1, 2, \ldots\}$, we speak of (measurable linear skew-product) discrete-time semidynamical system.

Let $\Phi = ((U_\omega(t))_{\omega \in \Omega, t \in \mathbb{T}^+}, (\theta_t)_{t \in \mathbb{T}^+})$ be a measurable linear skew-product semidynamical system on $X$ covering $(\theta_t)_{t \in \mathbb{T}}$. For $\omega \in \Omega$, $t \in \mathbb{T}^+$ and $u^* \in X^*$ we define $U^*_\omega(t)u^*$ by

$$(u, U^*_\omega(t)u^*) = \langle u, U_\omega(t)u^* \rangle \quad \text{for each } u \in X$$

(2.3)

(in other words, $U^*_\omega(t)$ is the mapping dual to $U_{\theta_t \omega}(t)$). The mapping

$$[\mathbb{T}^+ \times \Omega \times X^* \ni (t, \omega, u^*) \mapsto U^*_\omega(t)u^* \in X^*]$$

is $(\mathcal{B}(\mathbb{T}^+) \otimes \mathcal{F} \otimes \mathcal{B}(X^*), \mathcal{B}(X^*))$-measurable. We call the measurable linear skew-product semidynamical system $\Phi^* = (U^*_\omega(t), \theta_{-t})$ on $X^*$ covering $\theta_{-t}$ the dual of $\Phi$.

By a cone in $X$ we understand a closed convex set $X^+$ such that

(C1) $\alpha \geq 0$ and $u \in X^+$ imply $\alpha u \in X^+$, and

(C2) $X^+ \cap (-X^+) = \{0\}$.

A pair $(X, X^+)$, where $X^+$ is a cone in $X$, is referred to as an ordered Banach space.

If $(X, X^+)$ is an ordered Banach space, for $u, v \in X$ we write $u \leq v$ if $v - u \in X^+$, and $u < v$ if $u \leq v$ and $u \neq v$. The symbols $\geq$ and $>$ are used in an analogous way.

Let $X^+$ be a cone in $X$. $X^+$ is called total if $\text{cl}(X^+ - X^+) = X$, reproducing if $X^+ - X^+ = X$, and solid if the interior $X^{++}$ of $X^+$ is nonempty. $X^+$ is called normal if there exists $C > 0$ such that for any $u, v \in X$ satisfying $0 \leq u \leq v$ there holds $\|u\| \leq C\|v\|$. An ordered Banach space $(X, X^+)$ is called strongly ordered if $X^+$ is solid.

The following lemma is well known.

**Lemma 2.1.** Let $X$ be a cone in a finite-dimensional Banach space $X$ with norm $\|\cdot\|$.

1. $X^+$ is normal.
2. $X^+$ is solid iff $X^+$ is reproducing iff $X^+$ is total.

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(3) If $X^+$ is reproducing, then there exists $K \geq 1$ with the property that for each $u \in X$ there are $u^+, u^- \in X^+$ such that $u = u^+ - u^-$, $\|u^+\| \leq K\|u\|$, $\|u^-\| \leq K\|u\|$.

Observe that Lemma 2.1(3) holds for general Banach spaces (see \cite[Theorem 2.2]{1}).

Let $X^+$ be a solid cone in a finite-dimensional Banach space $X$. By Lemma 2.1(1), $X^+$ is normal. By appropriately renorming $X$ we can assume that the norm $\|\cdot\|$ has the property that for any $u, v \in X$, $0 \leq u \leq v$ implies $\|u\| \leq \|v\|$ (see \cite[V.3.1, p. 216]{23}). Such a norm is called monotonic. From now on, when speaking of a strongly ordered Banach space we assume that the norm on $X$ is monotonic.

For an ordered Banach space $(X, X^+)$ denote by $(X^*)^+$ the set of all $u^* \in X^*$ such that $\langle u, u^* \rangle \geq 0$ for all $u \in X^+$. If $X^+$ is solid then $(X^*)^+$ is a solid cone in $X^*$.

Assume that $(X, X^+)$ is an ordered Banach space. We say that $u, v \in X^+ \setminus \{0\}$ are comparable, written $u \sim v$, if there are positive numbers $\underline{u}, \underline{v}$ such that $\underline{u} v \leq u \leq \underline{v} v$. The $\sim$ relation is clearly an equivalence relation. For a nonzero $u \in X^+$ we understand by the component of $u$, denoted by $C_u$, the equivalence class of $u$, $C_u = \{ v \in X^+ \setminus \{0\} : v \sim u \}$.

Example 2.1. Let

$$X = \{ u = (u_1, \ldots, u_N)^\top : u_i \in \mathbb{R} \text{ for all } 1 \leq i \leq N \}.$$ 

By the standard cone in $X$ we understand

$$X^+ = \{ u = (u_1, \ldots, u_N)^\top \in X : u_i \geq 0 \text{ for all } 1 \leq i \leq N \}.$$ 

$X^+$ is a solid cone, with interior

$$X^{++} = \{ u = (u_1, \ldots, u_N)^\top \in X : u_i > 0 \text{ for all } 1 \leq i \leq N \}.$$ 

Furthermore, $(X, X^+)$ is a lattice: any two $u, v \in X$ have a least upper bound $u \vee v$, $(u \vee v)_i = \max\{u_i, v_i\}$, $1 \leq i \leq N$, and a greatest lower bound $u \wedge v$, $(u \wedge v)_i = \min\{u_i, v_i\}$, $1 \leq i \leq N$.

In the notation of Lemma 2.1(3) we specify $u^+$ to be equal to $u \wedge 0$, and $u^-$ to be equal to $(-u) \vee 0$.

All $p$-norms, $1 \leq p \leq \infty$, on $X$ are monotonic. Indeed, if $1 \leq p < \infty$ then for any $0 \leq u \leq v$ one has

$$\|u\|_p^p = \sum_{i=1}^N |u_i|^p \leq \sum_{i=1}^N |v_i|^p = \|v\|_p^p.$$ 

Moreover, for those norms the constant $K$ in Lemma 2.1(3) can be taken to be $1$:

$$\|u^+\|_p^p = \sum_{i=1}^N |\max\{u_i, 0\}|^p = \sum_{i \in \{1, \ldots, N\} \setminus \{u_i > 0\}} |u_i|^p \leq \sum_{i=1}^N |u_i|^p = \|u\|_p^p,$$

and similarly for $u^-$. The case $p = \infty$ is considered in an analogous way.

When we identify the dual space $X^*$ with $\mathbb{R}^N$ and the duality pairing $\langle u, u^* \rangle$ with $u^\top u^*$, the cone $(X^*)^+$ is given by the same formula as $X^+$. 


Now we introduce our assumptions.

**(B0)** (Ordered finite-dimensional space) $X^+$ is a reproducing cone in a finite-dimensional Banach space $X$ (this is equivalent to saying that $(X, X^+)$ is a strongly ordered finite-dimensional Banach space), with dim $X \geq 2$.

(Compare (A0)(i)–(iii) in [17]). We remark that (A0)(i) and (ii) in [17] are automatically satisfied for a cone $X^+$ in finite-dimensional Banach space. (A0)(iii) in [17] assumes that $(X, X^+)$ is a Banach lattice, which implies that $X^+$ is reproducing or equivalently solid and hence (B0) is weaker than (A0)(iii). In general, the main results in [17] still hold if the assumption that $(X, X^+)$ is a Banach lattice is replaced by the assumption that $X^+$ is reproducing (see Remarks 2.2 and 2.3).

**(B1)** (Integrability/injectivity) $\Phi = ((U_\omega(t))_{\omega \in \Omega, t \in T^+}, (\theta_t)_{t \in T})$ is a measurable linear skew-product semidynamical system on a finite-dimensional Banach space $X$ covering an ergodic metric dynamical system $(\theta_t)_{t \in T}$ on $(\Omega, \mathcal{F}, \mathbb{P})$, with the complete measure $\mathbb{P}$ in the case of $T = \mathbb{R}$, satisfying the following:

(i) **(Integrability)**

- In the discrete-time case: The function 
  \[ \Omega \ni \omega \mapsto \ln^+ \|U_\omega(1)\| \in [0, \infty) \] 
  \[ \in L_1((\Omega, \mathcal{F}, \mathbb{P})). \]

- In the continuous-time case: The functions 
  \[ \Omega \ni \omega \mapsto \sup_{0 \leq s \leq 1} \ln^+ \|U_\omega(s)\| \in [0, \infty) \] 
  \[ \in L_1((\Omega, \mathcal{F}, \mathbb{P})). \]

and 
  \[ \Omega \ni \omega \mapsto \sup_{0 \leq s \leq 1} \ln^+ \|U_{\theta_\omega}(1 - s)\| \in [0, \infty) \] 
  \[ \in L_1((\Omega, \mathcal{F}, \mathbb{P})). \]

(ii) **(Injectivity)** For each $\omega \in \Omega$ the linear operator $U_\omega(1)$ is injective.

(Compare (A1)(i)–(iii) in [17]. Note that $U_\omega(1)$ is automatically completely continuous in the finite-dimensional case).

**(B2)** (Positivity) $(X, X^+)$ is an ordered finite-dimensional Banach space and $\Phi = ((U_\omega(t))_{\omega \in \Omega, t \in T^+}, (\theta_t)_{t \in T})$ is a measurable linear skew-product semidynamical system on $X$ covering an ergodic metric dynamical system $(\theta_t)_{t \in T}$ on $(\Omega, \mathcal{F}, \mathbb{P})$, satisfying the following:

\[ U_\omega(t)u_1 \leq U_\omega(t)u_2 \]

for any $\omega \in \Omega$, $t \in T^+$ and $u_1, u_2 \in X$ with $u_1 \leq u_2$.

(Compare (A2) in [17]).

It follows immediately that if (B2) is satisfied then there holds

\[ U_\omega^*(t)u_1^* \leq U_\omega^*(t)u_2^* \]

for any $\omega \in \Omega$, $t \in T^+$ and $u_1^*, u_2^* \in X^*$ with $u_1^* \leq u_2^*$. 

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(B3) (Focusing) (B2) is satisfied and there are $e \in X^+$ with $\|e\| = 1$ and an $(\mathcal{F}, \mathcal{B}(\mathbb{R}))$-measurable function $\kappa : \Omega \to [1, \infty)$ with $\ln^+ \ln \kappa \in L_1((\Omega, \mathcal{F}, \mathbb{P}))$ such that for any $\omega \in \Omega$ and any nonzero $u \in X^+$ there is $\beta(\omega, u) > 0$ with the property that

$$\beta(\omega, u)e \leq U_\omega(1)u \leq \kappa(\omega)\beta(\omega, u)e.$$ 

(Compare (A3) in [17]).

(B3)* (Focusing) (B2) is satisfied and there are $e^* \in (X^+)^*$ with $\|e^*\| = 1$ and an $(\mathcal{F}, \mathcal{B}(\mathbb{R}))$-measurable function $\kappa^* : \Omega \to [1, \infty)$ with $\ln^+ \ln \kappa^* \in L_1((\Omega, \mathcal{F}, \mathbb{P}))$ such that for any $\omega \in \Omega$ and any nonzero $u^* \in (X^+)^*$ there is $\beta^*(\omega, u^*) > 0$ with the property that

$$\beta^*(\omega, u^*)e^* \leq U^*_\omega(1)u^* \leq \kappa^*(\omega)\beta^*(\omega, u^*)e^*.$$ 

(Compare (A4) in [17]).

(B4) (Strong focusing) (B3), (B3)* are satisfied and $\ln \kappa \in L_1((\Omega, \mathcal{F}, \mathbb{P}))$, $\ln \kappa^* \in L_1((\Omega, \mathcal{F}, \mathbb{P}))$, and $(e, e^*) > 0$.

(Compare (A5) in [17]).

(B5) (Strong positivity in one direction) There are $\overline{e} \in X^+$ with $\|\overline{e}\| = 1$ and an $(\mathcal{F}, \mathcal{B}(\mathbb{R}))$-measurable function $\nu : \Omega \to (0, \infty)$, with $\ln^+ \nu \in L_1((\Omega, \mathcal{F}, \mathbb{P}))$, such that

$$U_\omega(1)\overline{e} \geq \nu(\omega)\overline{e} \quad \forall \omega \in \Omega.$$ 

(Compare (A5)* in [17]).

(B5)* (Strong positivity in one direction) There are $\overline{e}^* \in (X^+)^*$ with $\|\overline{e}^*\| = 1$ and an $(\mathcal{F}, \mathcal{B}(\mathbb{R}))$-measurable function $\nu^* : \Omega \to (0, \infty)$, with $\ln^+ \nu^* \in L_1((\Omega, \mathcal{F}, \mathbb{P}))$, such that

$$U^*_\omega(1)\overline{e}^* \geq \nu^*(\omega)\overline{e}^* \quad \forall \omega \in \Omega.$$ 

(Compare (A5)* in [17]).

**Remark 2.1.** We can replace time 1 with some nonzero $T$ belonging to $T^+$ in (B1), (B3), (B4), and (B3)*.

We now state the definitions introduced in Part I. Throughout the rest of this subsection, we assume $(X, X^+)$ is a finite-dimensional ordered Banach space, and (B2).

**Definition 2.1** (Entire orbit). For $\omega \in \Omega$, by an entire orbit of $U_\omega$ we understand a mapping $v_\omega : T \to X$ such that $v_\omega(s + t) = U_{\theta_{\omega}(t)}v_\omega(s)$ for any $s \in T$ and $t \in T^+$. The function constantly equal to zero is referred to as the trivial entire orbit.

**Definition 2.2** (Entire positive orbit). An entire orbit $v_\omega$ of $U_\omega$ is called an entire positive orbit if $v_\omega(t) \in X^+$ for all $t \in T$. An entire positive orbit is nontrivial if $v_\omega(t) \in X^+ \setminus \{0\}$ for all $t \in T$. 

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Entire (positive) orbits of $\Phi^+$ are defined in a similar way.

A family $\{E(\omega)\}_{\omega \in \Omega_0}$ of $l$-dimensional vector subspaces of $X$ is **measurable** if there are $(\mathcal{F}, \mathcal{B}(X))$-measurable functions $v_1, \ldots, v_l: \Omega_0 \to X$, $\mathbb{P}(\Omega_0) = 1$, such that $(v_1(\omega), \ldots, v_l(\omega))$ forms a basis of $E(\omega)$ for each $\omega \in \Omega_0$.

Let $\{E(\omega)\}_{\omega \in \Omega_0}$ be a family of $l$-dimensional vector subspaces of $X$, and let $\{F(\omega)\}_{\omega \in \Omega_0}$ be a family of $l$-codimensional vector subspaces of $X$, such that $E(\omega) \oplus F(\omega) = X$ for all $\omega \in \Omega_0$, $\mathbb{P}(\Omega_0) = 1$. We define the family of projections associated with the decomposition $E(\omega) \oplus F(\omega) = X$ as $\{P(\omega)\}_{\omega \in \Omega_0}$, where $P(\omega)$ is the linear projection of $X$ onto $F(\omega)$ along $E(\omega)$, for each $\omega \in \Omega_0$.

The following remark is in order regarding measurability of decomposition. In Part I ([17]) of the series, as well as in, for example, [13], when a decomposition $E(\omega) \oplus F(\omega) = X$ of a (perhaps infinite-dimensional) Banach space $X$ is considered, where $E(\omega)$ has finite dimension $l$, the assumption is that the family $\{E(\omega)\}_{\omega \in \Omega_0}$ is measurable and that the family of projections $\{P(\omega)\}_{\omega \in \Omega_0}$ is **strongly measurable**: for each $u \in X$ the mapping $[\Omega_0 \ni \omega \mapsto P(\omega)u \in X]$ is $(\mathcal{F}, \mathcal{B}(X))$-measurable. But in the finite-dimensional case this is equivalent to saying that both families $\{E(\omega)\}_{\omega \in \Omega_0}$ and $\{F(\omega)\}_{\omega \in \Omega_0}$ are measurable. We call such a decomposition a **measurable decomposition**.

We say that the decomposition $E(\omega) \oplus F(\omega) = X$ is **invariant** if $\Omega_0$ is invariant, $U_\omega(t)E(\omega) = E(\theta_t\omega)$ and $U_\omega(t)F(\omega) = F(\theta_t\omega)$, for each $t \in \mathbb{T}^+$. A (strongly measurable) family of projections associated with the invariant measurable decomposition $E(\omega) \oplus F(\omega) = X$ is referred to as **tempered** if

$$
\lim_{t \to \pm \infty} \frac{\ln \|P(\theta_t\omega)\|}{t} = 0 \quad \text{\(\mathbb{P}\)-a.s. on } \Omega_0.
$$

**Definition 2.3** (Generalized principal Floquet subspaces and principal Lyapunov exponent). A family of one-dimensional subspaces $\{\tilde{E}(\omega)\}_{\omega \in \tilde{\Omega}}$ of $X$ is called a family of generalized principal Floquet subspaces of $\Phi = (U_\omega(t), \theta_t)$ if $\tilde{\Omega} \subset \Omega$ is invariant, $\mathbb{P}(\tilde{\Omega}) = 1$, and

(i) $\tilde{E}(\omega) = \text{span } \{w(\omega)\}$ with $w: \tilde{\Omega} \to X^+ \setminus \{0\}$ being $(\mathcal{F}, \mathcal{B}(X))$-measurable,

(ii) $U_\omega(t)\tilde{E}(\omega) = \tilde{E}(\theta_t\omega)$, for any $\omega \in \tilde{\Omega}$ and any $t \in \mathbb{T}^+$,

(iii) there is $\lambda \in [-\infty, \infty)$ such that

$$
\lambda = \lim_{t \to \pm \infty} \frac{1}{t} \ln \|U_\omega(t)w(\omega)\| \quad \forall \omega \in \tilde{\Omega},
$$

and

(iv) $$
\limsup_{t \to \pm \infty} \frac{1}{t} \ln \|U_\omega(t)u\| \leq \lambda \quad \forall \omega \in \tilde{\Omega} \text{ and } \forall u \in X \setminus \{0\}.
$$

$\lambda$ is called the **generalized principal Lyapunov exponent** of $\Phi$ associated to the generalized principal Floquet subspaces $\{E(\omega)\}_{\omega \in \tilde{\Omega}}$. 

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Observe that if \( \{ \tilde{E}(\omega) \}_{\omega \in \tilde{\Omega}} \) is a family of generalized principal Floquet subspaces of \( (U_\omega(t), \theta_t) \), then for any \( \omega \in \tilde{\Omega} \) the function \( v_\omega : T \to X^+ \setminus \{0\} \),

\[
v_\omega(t) = \begin{cases} 
U_\omega(t)w(\omega), & t \in T, \ t \geq 0 \\
\frac{\|w(\omega)\|}{\|U_{\theta_t \omega}(-t)w(\theta_t \omega)\|}w(\theta_t \omega), & t \in T, \ t < 0
\end{cases}
\] (2.4)

is a nontrivial entire positive orbit of \( U_\omega \).

**Definition 2.4** (Generalized exponential separation). \( \Phi = (U_\omega(t), \theta_t) \) is said to admit a generalized exponential separation if there are a family of generalized principal Floquet subspaces \( \{ \tilde{E}(\omega) \}_{\omega \in \tilde{\Omega}} \) and a measurable family of one-codimensional subspaces \( \{ \tilde{F}(\omega) \}_{\omega \in \tilde{\Omega}} \) of \( X \) satisfying the following

(i) \( \tilde{F}(\omega) \cap X^+ = \{0\} \) for any \( \omega \in \tilde{\Omega} \),

(ii) \( X = \tilde{E}(\omega) \oplus \tilde{F}(\omega) \) for any \( \omega \in \tilde{\Omega} \), where the decomposition is invariant, and the family of projections associated with this decomposition is tempered,

(iii) there exists \( \tilde{\sigma} \in (0, \infty) \) such that

\[
\lim_{t \to \infty} \frac{1}{t} \ln \frac{\|U_\omega(t)\|}{\|U_\omega(t)w(\omega)\|} = -\tilde{\sigma} \quad \forall \omega \in \tilde{\Omega}.
\]

We say that \( \{ \tilde{E}(\cdot), \tilde{F}(\cdot), \tilde{\sigma} \} \) generates a generalized exponential separation.

We end this subsection with the following theorem, which follows from the Oseledets-type theorems proved in [21].

**Theorem 2.1.** Let \( X \) be a finite-dimensional Banach space, and let \( \Phi = (U_\omega(t), \theta_t) \) be a measurable linear skew-product semidynamical system satisfying (B1)(i). Then there exists an invariant \( \Omega_0 \subset \Omega \), \( \mathbb{P}(\Omega_0) = 1 \), and \( \lambda_1 \in [-\infty, \infty) \) with the property that

\[
\lim_{t \to \infty} \frac{\ln \|U_\omega(t)\|}{t} = \lambda_1 \quad \forall \omega \in \Omega_0.
\] (2.5)

Moreover, either

(i) \( \lim_{t \to \infty} \frac{\ln \|U_\omega(t)u\|}{t} = \lambda_1 \quad \forall \omega \in \Omega_0 \) and \( \forall u \in X \setminus \{0\} \),

or

(ii) there exist \( \hat{\lambda}_2 < \lambda_1 \), and a measurable family of linear subspaces \( \{ \hat{F}_1(\omega) \}_{\omega \in \Omega_0} \) of \( X \) such that

- \( U_\omega(t)\hat{F}_1(\omega) \subset \hat{F}_1(\theta_t \omega) \), for all \( \omega \in \Omega_0 \) and all \( t \in \mathbb{T}^+ \),
By the assumptions, the mappings \( U \) satisfy:

\[
\lim_{t \to \infty} \frac{\ln \| U_\omega(t)u \|}{t} = \lambda_1 \quad \forall \omega \in \Omega_0 \text{ and } \forall u \in X \setminus \bar{F}_1(\omega),
\]

\[
\lim_{t \to \infty} \frac{\ln \| U_\omega(t)|_{\bar{F}_1(\omega)} \|}{t} = \lambda_2 \quad \forall \omega \in \Omega_0.
\]

Moreover, if (B1)(ii) holds, then there is a measurable family of linear subspaces \( \{ E_1(\omega) \}_{\omega \in \Omega_0} \) of \( X \) such that:

- \( U_\omega(t)E_1(\omega) = E_1(\theta_t \omega) \), for all \( \omega \in \Omega_0 \) and all \( t \in \mathbb{T}^+ \),
- \( X = E_1(\omega) \oplus \bar{F}_1(\omega) \) for any \( \omega \in \Omega_0 \), where the family of projections associated with this invariant decomposition is tempered,

\[
\lim_{t \to \infty} \frac{\ln \| U_\omega(t)u \|}{t} = \lambda_1 \quad \forall \omega \in \Omega_0 \text{ and } \forall u \in E_1(\omega) \setminus \{0\}.
\]

### 2.2. General Theorems

In this subsection, we state some general theorems, most of which are established in Part I ([17]). Throughout this subsection, we assume that \( (X, X^+) \) is an ordered finite-dimensional Banach space. The first theorem concerns the existence of entire positive solutions, which partially follows from [17, Theorem 3.5].

**Theorem 2.2** (Entire positive orbits). Assume (B1)(i) and (B2).

(i) Let \( U_\omega(t)(X^+ \setminus \{0\}) \subset X^+ \setminus \{0\} \) for all \( \omega \in \Omega \) and all \( t \in \mathbb{T}^+ \). Then for each \( \omega \in \Omega \) there exists an entire positive orbit \( v_\omega: \mathbb{T} \to X^+ \setminus \{0\} \) of \( U_\omega \).

(ii) Assume moreover that \( X^+ \) is reproducing. Let (B1)(ii) be satisfied. If (ii) in Theorem [2.1] holds then for each \( \omega \in \Omega_0 \) an entire positive orbit can be chosen so that \( v_\omega(t) \in X^+ \cap E_1(\theta_t \omega) \) for all \( t \in \mathbb{T} \).

**Proof.** (i) Denote \( S_1(X^+) := \{ u \in X^+ : \| u \| = 1 \} \).

Fix \( \omega \in \Omega \). For \((m,n) \in \mathbb{Z}^2 \) such that \( 0 \leq n \leq m \) define the mapping \( \mathcal{U}(m,n): S_1(X^+) \to S_1(X^+) \) by the formula

\[
\mathcal{U}(m,n)(u) := \frac{U_{\theta^{-n}\omega}(m-n)u}{\| U_{\theta^{-n}\omega}(m-n)u \|}.
\]

By the assumptions, the mappings \( \mathcal{U}(m,n) \) are well defined and continuous. It follows from (2.4) that \( \mathcal{U}(m_2,n) = \mathcal{U}(m_1,n) \circ \mathcal{U}(m_2,m_1) \), consequently \( \text{Im}(\mathcal{U}(m_2,n)) \subset \text{Im}(\mathcal{U}(m_1,n)) \), for any \( 0 \leq n \leq m_2 \leq m_1 \).

For \( n = 0, 1, 2, \ldots \) let

\[
G_n := \bigcap_{m=n}^{\infty} \text{Im}(\mathcal{U}(m,n)).
\]

\( G_n \), as the intersection of a nonincreasing family of compact nonempty sets, is compact and nonempty, too.
Notice that \( u \in S_1(X^+) \) belongs to \( G_n \) if and only if there is a sequence \( (u^{(m)})_{m=n}^{\infty} \) with \( u^{(n)} = u \) such that \( u^{(m)} \in S_1(X^+) \) and \( u^{(m)} = U(m + 1; m)u^{(m+1)} \) for each \( m = n, n + 1, n + 2, \ldots \).

It suffices now to pick one \( u \in G_0 \) and \( (u^{(m)})_{m=0}^{\infty} \subset S_1(X^+) \) with \( u^{(0)} = u \) and \( u^{(m)} = U(m + 1; m)u^{(m+1)} \) for each \( m = 0, 1, 2, \ldots \). Then put

\[
v_\omega(n) := \begin{cases} U_\omega(n)u^{(0)} & \text{for } n = 0, 1, 2, \ldots \\ \|U_{\theta_{-1}\omega}(1)u^{(1)}\|\|U_{\theta_{-2}\omega}(1)u^{(2)}\|\ldots\|U_{\theta_{n}\omega}(1)u^{(-n)}\| & \text{for } n = -1, -2, \ldots \end{cases}
\]

This concludes the proof of (i) in the discrete-time case. In the continuous-time case one puts

\[
v_\omega(t) := U_{\theta_{[t]}}(t - [t])v_\omega([t]) \text{ for any } \omega \in \Omega \text{ and any } t \in \mathbb{R} \setminus \mathbb{Z}.
\]

(ii) follows from [17, Theorem 3.6].

The next theorem shows the existence of generalized Floquet subspaces and principal Lyapunov exponent, which partially follows from [17, Theorem 3.5].

**Theorem 2.3** (Generalized principal Floquet subspace and Lyapunov exponent). Assume (B1)(i), (B2) and (B3). Then there exist an invariant set \( \tilde{\Omega}_1 \subset \Omega \), \( \mathbb{P}(\tilde{\Omega}_1) = 1 \), and an \( (\mathcal{G}, \mathcal{B}(X)) \)-measurable function \( w: \tilde{\Omega}_1 \to X, w(\omega) \in C_\infty \) and \( \|w(\omega)\| = 1 \) for all \( \omega \in \tilde{\Omega}_1 \), having the following properties:

1. \( w(\theta_t \omega) = \frac{U_\omega(t)w(\omega)}{\|U_\omega(t)w(\omega)\|} \) for any \( \omega \in \tilde{\Omega}_1 \) and \( t \in \mathbb{T}^+ \).

2. Let for some \( \omega \in \tilde{\Omega}_1 \) a function \( v_\omega: \mathbb{T} \to X^+ \setminus \{0\} \) be an entire orbit of \( U_\omega \). Then \( v_\omega(t) = \|v_\omega(0)\|w_\omega(t) \) for all \( t \in \mathbb{T} \), where

\[
w_\omega(t) := \begin{cases} (U_{\theta_t \omega}(-t)|\tilde{E}_1(\theta_t \omega))^{-1}w(\omega) & \text{for } t \in \mathbb{T}, \ t < 0 \\ U_\omega(t)w(\omega) & \text{for } t \in \mathbb{T}^+,
\end{cases}
\]

with \( \tilde{E}_1(\omega) = \text{span}\{w(\omega)\} \).

3. There exists \( \tilde{\lambda}_1 \in [-\infty, \infty) \) such that

\[
\tilde{\lambda}_1 = \lim_{t \to \infty} \frac{1}{t} \ln \rho_t(\omega) = \int_{\Omega} \ln \rho_1(\omega) \, d\mathbb{P}
\]

for each \( \omega \in \tilde{\Omega}_1 \), where

\[
\rho_t(\omega) := \begin{cases} \|U_\omega(t)w(\omega)\| & \text{for } t \geq 0, \\ 1/\|U_{\theta_t \omega}(-t)w(\theta_t \omega)\| & \text{for } t < 0.
\end{cases}
\]

4. Assume, moreover, (B0). Then \( \tilde{\lambda}_1 = \lambda_1 \), where \( \lambda_1 \) is as in Theorem 2.1. In particular, for any \( u \in X \setminus \{0\} \),

\[
\limsup_{t \to \infty} \frac{1}{t} \ln \|U_\omega(t)u\| \leq \tilde{\lambda}_1,
\]

and then \( \{\tilde{E}_1(\omega)\}_{\omega \in \tilde{\Omega}_1} \) is a family of generalized Floquet subspaces.
Proof. Parts (1), (2) and (3) just correspond to parts (1), (2) and (3) of \cite[Theorem 3.6]{17}.

(4) By \cite[Proposition 5.5(2)]{17}, there exists $\sigma_1 > 0$ such that
\[
\limsup_{t \to \infty} \frac{1}{t} \ln \left\| U_\omega(t)u \right\| \leq -\sigma_1
\]
for any nonzero $u \in X^+$, $\mathbb{P}$-a.e. on $\Omega$, which, combined with (3), gives that
\[
\limsup_{t \to \infty} \frac{1}{t} \ln \left\| U_\omega(t)u \right\| \leq \lambda_1
\]
for any nonzero $u \in X^+$. Since $X = X^+ - X^+$, the above inequality is satisfied for any nonzero $u \in X$. It suffices now to apply Theorem 2.2. \hfill \Box

Remark 2.2. Theorem 2.3(4) is apparently a stronger version of \cite[Theorem 3.6(5)]{17}: In \cite, Theorem 3.6(5)\] $X, X^+$ is assumed to be a Banach lattice, which implies that $X^+$ is reproducing. We point out that \cite[Theorem 3.6(5)]{17} in fact also holds if the assumption that $(X, X^+)$ is a Banach lattice is replaced by the assumption that $X^+$ is reproducing.

If (B1)(i), (B2) and (B3)* are satisfied, the counterpart of Theorem 2.3 for the dual $\Phi^*$ states, among others, the existence of an invariant $\tilde{\Omega}_1 \subset \tilde{\Omega}$, $\mathbb{P}(\tilde{\Omega}_1) = 1$, and an $(\tilde{\mathcal{S}}, \mathcal{B}(\Phi^*))$-measurable function $w^* : \tilde{\Omega}_1 \to X^*$, $w^*(\omega) \in C_\omega$, and $\|w^*(\omega)\| = 1$ for all $\omega \in \tilde{\Omega}_1$, satisfying $w^*(\theta_{-t}\omega) = U_{\omega}^*(t)w^*(\omega)/\|U_{\omega}^*(t)w^*(\omega)\|$ for all $\omega \in \tilde{\Omega}_1$ and all $t \in \mathbb{T}^+$. For $\omega \in \tilde{\Omega}_1$, define $\tilde{F}_1(\omega) := \{ u \in X : \langle u, w^*(\omega) \rangle = 0 \}$. Then $\{ \tilde{F}_1(\omega) \}_{\omega \in \tilde{\Omega}_1}$ is a family of one-codimensional subspaces of $X$, such that $U_{\omega}(t)\tilde{F}_1(\omega) \subset \tilde{F}_1(\theta_{-t}\omega)$ for any $\omega \in \tilde{\Omega}_1$ and any $t \in \mathbb{T}^+$.

The last theorem is about the existence of generalized exponential separation, which partially follows from \cite[Theorem 3.8]{17}.

Theorem 2.4 (Generalized exponential separation). Assume (B0), (B1)(i), (B2), and (B4). Then there is an invariant set $\tilde{\Omega}_0$, $\mathbb{P}(\tilde{\Omega}_0) = 1$, having the following properties.

1. The family $\{ \tilde{P}(\omega) \}_{\omega \in \tilde{\Omega}_0}$ of projections associated with the measurable invariant decomposition $\tilde{E}_1(\omega) \oplus \tilde{F}_1(\omega) = X$ is tempered.

2. $\tilde{F}_1(\omega) \cap X^+ = \{0\}$ for any $\omega \in \tilde{\Omega}_0$.

3. For any $\omega \in \tilde{\Omega}_0$ and any $u \in X \setminus \tilde{F}_1(\omega)$ (in particular, for any nonzero $u \in X^+$) there holds
\[
\lim_{t \to \infty} \frac{1}{t} \ln \| U_\omega(t) \| = \lim_{t \to \infty} \frac{1}{t} \ln \| U_\omega(t)u \| = \lambda_1.
\]

4. There exist $\tilde{\sigma} \in (0, \infty]$ and $\tilde{\lambda}_2 \in [-\infty, \infty)$, $\tilde{\lambda}_2 = \tilde{\lambda}_1 - \tilde{\sigma}$, such that
\[
\lim_{t \to \infty} \frac{1}{t} \ln \frac{\| U_\omega(t)\tilde{P}(\omega) \|}{\| U_\omega(t)w(\omega) \|} = -\tilde{\sigma}
\]
and 

\[ \lim_{t \to \infty} \frac{1}{t} \ln \| U_\omega(t) \| \| F_1(\omega) \| = \tilde{\lambda}_2 \]

for each \( \omega \in \Omega_0 \). Hence \( \Phi \) admits a generalized exponential separation.

(5) Assume that (ii) in Theorem 2.1 holds. Then \( \tilde{\lambda}_2 = \lambda_2 < \tilde{\lambda}_1 \) and \( \tilde{F}_1(\omega) = F_1(\omega) \) for \( \mathbb{P} \)-a.e. \( \omega \in \Omega_0 \). If, moreover, (B1)(ii) holds then \( E_1(\omega) = \tilde{E}_1(\omega) \) for \( \mathbb{P} \)-a.e. \( \omega \in \Omega_0 \).

(6) If (B5) or (B5)* holds, then \( \lambda_1 > -\infty \).

Proof. The proofs of parts (1) through (4) and of (6) go along the lines of proofs of the corresponding parts of [17, Theorem 3.8], the only difference being that in the present paper we do not assume that \((X, X^+)\) is a Banach lattice. However, by Lemma 2.13, there exists \( K > 0 \) such that any \( u \in X \) with \( \| u \| = 1 \) can be written as \( u^+ - u^- \), with \( u^+, u^- \in X^+, \| u^+ \| \leq K, \| u^- \| \leq K \) (cf. the proof of [17, Proposition 5.11], where this property is needed).

Regarding the first part of (5), observe that nowhere in the proof in [17] of the fact that \( \hat{F}_1(\omega) = \tilde{F}_1(\omega) \) \( \mathbb{P} \)-a.e. the injectivity ((B1)(ii)) is needed. \( \square \)

Remark 2.3. Theorem 2.1 is apparently a stronger version of [17, Theorem 3.8]: It is assumed in [17, Theorem 3.8] that \((X, X^+)\) is a Banach lattice, which implies that \( X^+ \) is reproducing. We point out that [17, Theorem 3.8] also holds if the assumption that \((X, X^+)\) is a Banach lattice is replaced by the assumption that \( X^+ \) is reproducing.

3. Positive Matrix Random Dynamical Systems

In the present section we consider applications of the general results stated in Section 2 to discrete-time finite-dimensional random dynamical systems arising from positive random matrices.

Throughout this section \( X \) stands for the \( N \)-dimensional real vector space of column vectors, with \( N \geq 2 \).

We assume that the dual space \( X^* \) consists of column vectors, too, and that the duality pairing between \( X \) and \( X^* \) is given by the standard inner product (denoted also by the symbol \( \langle \cdot, \cdot \rangle \)). Consequently, \( \langle u, u^* \rangle = u^T u \), for any \( u \in X \), \( u^* \in X^* \), where \( T \) denotes the matrix transpose.

\( X^+ \) is the standard nonnegative cone, \( X^+ = \{ u = (u_1, \ldots, u_N)^T : u_i \geq 0 \text{ for all } 1 \leq i \leq N \} \), and \( X^{++} \) is its interior, \( X^{++} = \{ u = (u_1, \ldots, u_N)^T : u_i > 0 \text{ for all } 1 \leq i \leq N \} \).

Denote by \( (e_1, \ldots, e_N) \) the standard basis of \( X \).

We will identify a discrete-time measurable linear skew-product semidynamical system \( \Phi = ((U_\omega(n))_{\omega \in \Omega}, (\theta_n)_{n \in \mathbb{Z}^+}) \) on \( X \) covering a metric dynamical system \((\theta_n)_{n \in \mathbb{Z}}\) with a \((\mathbb{P}, \mathcal{B}(\mathbb{R}^N \times N))\)-measurable family \( (S(\omega))_{\omega \in \Omega} \) of \( N \times N \) matrices

\[
S(\omega) = \begin{pmatrix}
  s_{11}(\omega) & s_{12}(\omega) & \cdots & s_{1N}(\omega) \\
s_{21}(\omega) & s_{22}(\omega) & \cdots & s_{2N}(\omega) \\
  \vdots & \vdots & \ddots & \vdots \\
s_{N1}(\omega) & s_{N2}(\omega) & \cdots & s_{NN}(\omega)
\end{pmatrix},
\]

(3.1)

that is, \( U_\omega(1) = S(\omega) \). Also, we will write \( \theta \) instead of \( \theta_1 \).
We will use the notation

\[ S^{(n)}(\omega) := S(\theta^{n-1}\omega)S(\theta^{n-2}\omega)\cdots S(\omega), \quad \omega \in \Omega, \quad n \geq 1 \]

(commonly \( S^{(0)}(\omega) \) equals the identity matrix). Eq. (2.2) takes now the form

\[ S^{(m+n)}(\omega) = S^{(m)}(\theta^m\omega)S^{(n)}(\omega), \quad \omega \in \Omega, \quad m, n \in \mathbb{Z}^+. \]

(3.2)

For the dual system \((S^*(\omega))_{\omega \in \Omega}\) it follows from (2.3) that \( S^*(\omega) = (S(\theta^{-1}\omega))^\top \).

Let

\[ m_{c,i}(\omega) := \min_{1 \leq j \leq N} s_{ji}(\omega), \quad M_{c,i}(\omega) = \max_{1 \leq j \leq N} s_{ji}(\omega), \]

and

\[ m_{r,i}(\omega) := \min_{1 \leq j \leq N} s_{ij}(\omega), \quad M_{r,i}(\omega) := \max_{1 \leq j \leq N} s_{ij}(\omega) \]

for \( \omega \in \Omega \) and \( i = 1, 2, \ldots, N \). Further, let

\[ m_r(\omega) := \min_{1 \leq i \leq N} \sum_{j=1}^N s_{ij}(\omega), \quad m_c(\omega) := \min_{1 \leq i \leq N} \sum_{j=1}^N s_{ij}(\omega). \]

Finally, let

\[ m(\omega) := \min_{1 \leq i, j \leq N} s_{ij}(\omega), \quad M(\omega) := \max_{1 \leq i, j \leq N} s_{ij}(\omega). \]

We state the following standing assumptions.

(D1) (Positivity, injectivity and integrability) \((\theta^n)_{n \in \mathbb{Z}}\) is an ergodic metric dynamical system on \((\Omega, \mathcal{F}, \mathbb{P})\), and \(S = (s_{ij})_{i,j=1}^N : \Omega \to \mathbb{R}^{N \times N}\) is an \((\mathcal{F}, \mathfrak{B}(\mathbb{R}^{N \times N}))\)-measurable matrix function such that

(i) \( s_{ij}(\omega) \geq 0 \) for all \( \omega \in \Omega \) and \( i, j = 1, 2, \ldots, N \).

(ii) For each \( \omega \in \Omega \), \( S(\omega) \) is a non-singular matrix.

(iii) \((i)\) holds and \( \ln^+ M \in L_1((\Omega, \mathcal{F}, \mathbb{P})) \).

(D2) (Focusing)

(i) \( s_{ij}(\omega) > 0 \) for all \( \omega \in \Omega \) and \( i, j = 1, 2, \ldots, N \) and \( \ln^+ (\ln M_{c,i} - \ln m_{c,i}) \in L_1((\Omega, \mathcal{F}, \mathbb{P})) \) for \( i = 1, 2, \ldots, N \).

(ii) \( s_{ij}(\omega) > 0 \) for all \( \omega \in \Omega \) and \( i, j = 1, 2, \ldots, N \) and \( \ln^+ (\ln M_{r,i} - \ln m_{r,i}) \in L_1((\Omega, \mathcal{F}, \mathbb{P})) \) for \( i = 1, 2, \ldots, N \).

(iii) \( s_{ij}(\omega) > 0 \) for all \( \omega \in \Omega \) and \( i, j = 1, 2, \ldots, N \) and \( \ln M_{c,i} - \ln m_{c,i} \in L_1((\Omega, \mathcal{F}, \mathbb{P})) \), \( \ln M_{r,i} - \ln m_{r,i} \in L_1((\Omega, \mathcal{F}, \mathbb{P})) \) for \( i = 1, 2, \ldots, N \).

(D3) (Strong positivity in one direction)

(i) \( m_r(\omega) > 0 \) for each \( \omega \in \Omega \) and \( \ln^+ m_r \in L_1((\Omega, \mathcal{F}, \mathbb{P})) \)

(ii) \( m_c(\omega) > 0 \) for each \( \omega \in \Omega \) and \( \ln^+ m_c \in L_1((\Omega, \mathcal{F}, \mathbb{P})) \).
Proposition 3.1 (Positivity, injectivity and integrability). (1) (B2) holds if and only if (D1)(i) holds.

(2) (B1)(ii) holds if and only if (D1)(ii) holds.

(3) Assume (D1)(i). Then (B1)(i) holds if and only if (D1)(iii) holds.

Proof. (1) and (2) are obvious.

(3) Since $M(\omega) \leq \|S(\omega)\| \leq \sqrt{N}M(\omega)$, the satisfaction of (B1)(i) is equivalent to $\ln^+ M \in L_1((\Omega, \mathcal{F}, \mathbb{P}))$. □

Proposition 3.2 (Focusing). Assume that (D1)(i) holds.

(1) (B3) holds with $e \in X^+$ if and only if (D2)(i) holds.

(2) (B3)* holds with $e^* \in X^+$ if and only if (D2)(ii) holds.

(3) (B4) holds with $e, e^* \in X^+$ if and only if (D2)(iii) holds.

Proof. (1) Suppose that (B3) is satisfied with some $e \in X^+$. In particular, there holds

$$\beta(\omega, e_i) e \leq S(\omega)e_i \leq \kappa(\omega)\beta(\omega, e_i) e.$$ 

Note that there are positive reals $\delta \leq \overline{\delta}$ such that

$$\delta(1, 1, \ldots, 1)^T \leq e \leq \overline{\delta}(1, 1, \ldots, 1)^T,$$ 

consequently

$$\delta\beta(\omega, e_i)(1, 1, \ldots, 1)^T \leq S(\omega)e_i \leq \overline{\delta}\kappa(\omega)\beta(\omega, e_i)(1, 1, \ldots, 1)^T.$$ 

Since

$$S(\omega)e_i = (s_{1i}(\omega), s_{2i}(\omega), \ldots, s_{Ni}(\omega))^T,$$ 

one has

$$\beta(\omega, e_i) \leq \frac{m_{e,i}(\omega)}{\delta}, \quad \kappa(\omega)\beta(\omega, e_i) \geq \frac{M_{e,i}(\omega)}{\overline{\delta}},$$ 

hence

$$\frac{M_{e,i}(\omega)}{m_{e,i}(\omega)} \leq \frac{\overline{\delta}}{\delta}\kappa(\omega) \quad \forall i = 1, 2, \ldots, N.$$ 

It then follows that $\ln^+(\ln M_{c,i} - \ln m_{c,i}) \in L_1((\Omega, \mathcal{F}, \mathbb{P}))$ for $i = 1, 2, \ldots, N$.

Conversely, suppose that $s_{ij}(\omega) > 0$ for $i, j = 1, \ldots, N$ and $\ln^+(\ln M_{c,i} - \ln m_{c,i}) \in L_1((\Omega, \mathcal{F}, \mathbb{P}))$ for $i = 1, \ldots, N$. Note that for any $u = (u_1, \ldots, u_n)^T \in X^+ \setminus \{0\}$, $u = u_1e_1 + \cdots + u_ne_N$. Hence

$$u_1m_{c,1}(\omega) + \cdots + uNm_{c,N}(\omega) \leq (S(\omega)u)_i \leq u_1M_{c,1}(\omega) + \cdots + u_NM_{c,N}(\omega), \quad 1 \leq i \leq N.$$ 

Put $e = (1, \ldots, 1)^T/\sqrt{N}$. Let

$$\beta(\omega, u) = \sqrt{N} (u_1m_{c,1}(\omega) + \cdots + u_Nm_{c,N}(\omega)),$$ 

and

$$\kappa(\omega) = N \max_{1 \leq i \leq N} \frac{M_{c,i}(\omega)}{m_{c,i}(\omega)}.$$
Then \( \varkappa \) is measurable and

\[
\beta(\omega, u)e \leq S(\omega)u \leq \varkappa(\omega)\beta(\omega, u)e
\]

and

\[
\ln^+ \ln \varkappa(\omega) \leq \ln N + \max_{1 \leq i \leq N} \ln^+(\ln M_{c,i}(\omega) - \ln m_{c,i}(\omega)).
\]

Therefore \( \ln^+ \ln \varkappa \in L_1((\Omega, \mathcal{F}, \mathbb{P})) \).

(2) It can be proved by arguments similar to those in the proof of (1).

(3) First, assume that (B4) holds.

Copying the proof of (1) we obtain that \( s_{ij}(\omega) > 0 \) for all \( \omega \in \Omega, i, j = 1, 2, \ldots, N \), and

\[
\frac{M_{c,i}(\omega)}{m_{c,i}(\omega)} \leq \frac{\delta}{\delta} \varkappa(\omega) \quad \forall \omega \in \Omega, \ i = 1, 2, \ldots, N.
\]

Hence

\[
\ln M_{c,i}(\omega) - \ln m_{c,i}(\omega) \leq \ln \delta - \ln \delta + \ln \varkappa(\omega) \quad \forall \omega \in \Omega, \ i = 1, 2, \ldots, N.
\]

Therefore, \( \ln M_{c,i}(\omega) - \ln m_{c,i}(\omega) \in L_1((\Omega, \mathcal{F}, \mathbb{P})) \).

Similarly, it can be proved that \( \ln M_{r,i}(\omega) - \ln m_{r,i}(\omega) \in L_1((\Omega, \mathcal{F}, \mathbb{P})) \). Therefore (D2)(iii) holds.

Next, we assume that (D2)(iii) holds. Copying the proof of (1) we see that (B3) holds with

\[
e = (1, \ldots, 1)^{\top}/\sqrt{N}, \quad \varkappa(\omega) = N \max_{1 \leq i \leq N} \frac{M_{c,i}(\omega)}{m_{c,i}(\omega)},
\]

Therefore,

\[
\ln \varkappa(\omega) \leq \ln N + \max_{1 \leq i \leq N} (\ln M_{c,i}(\omega) - \ln m_{c,i}(\omega)).
\]

It then follows that \( \ln \varkappa \in L_1((\Omega, \mathcal{F}, \mathbb{P})) \).

Similarly, it can be proved that \( e^* = e \) and \( \ln \varkappa \in L_1((\Omega, \mathcal{F}, \mathbb{P})) \).

By \( e^* = e, \langle e, e^* \rangle > 0 \). Therefore (B4) holds.

\[\]
consequently
\[ \nu(\omega) \leq \frac{\delta}{\delta} m_r(\omega). \]

This implies that \( m_r(\omega) > 0 \) for each \( \omega \in \Omega \) and \( \ln^{-} m_r \in L_1((\Omega, F, P)) \). Hence (D3)(i) holds.

Conversely, assume that (D3)(i) holds. Put \( \bar{\sigma} = (1, \ldots, 1)^{T} / \sqrt{N} \). Let
\[ \nu(\omega) = m_r(\omega). \]

By arguments as above,
\[ S(\omega) \bar{\sigma} \geq \nu(\omega) \bar{\sigma} \quad \forall \omega \in \Omega. \]

This implies that (B5) holds with \( \bar{\sigma} \in X^{++} \).

(2) It can be proved by similar arguments. \( \square \)

**Remark 3.1.** Assume \( s_{ij}(\omega) > 0 \) for all \( i, j = 1, 2, \ldots, N \).

1. If \( \ln^{+} (\ln M - \ln m) \in L_1((\Omega, F, P)) \), then \( \ln^{+} (\ln M_{c,i} - \ln m_{c,i}) \in L_1((\Omega, F, P)) \), \( \ln^{+} (\ln M_{r,i} - \ln m_{r,i}) \in L_1((\Omega, F, P)) \), and hence (D2)(i)–(ii) holds.

2. If \( \ln M - \ln m \in L_1((\Omega, F, P)) \), then \( \ln M_{c,i} - \ln m_{c,i} \in L_1((\Omega, F, P)) \), \( \ln M_{r,i} - \ln m_{r,i} \in L_1((\Omega, F, P)) \), and hence (D2)(iii) holds.

3. If \( \ln^{+} m \in L_1((\Omega, F, P)) \), then (D3)(i)–(ii) holds.

**Theorem 3.1.** (1) (Entire positive orbits) Assume (D1)(iii) and that \( S(\omega)(X^{+} \setminus \{0\}) \subset X^{+} \setminus \{0\} \) for all \( \omega \in \Omega \). Then, for any \( \omega \in \tilde{\Omega} \), there is an entire positive orbit \( v_{\omega}: Z \to X^{+} \setminus \{0\} \) of \( U_{\omega} \).

2. (Generalized principal Floquet subspaces and Lyapunov exponents) Assume (D1)(iii) and (D2)(ii)–(iii). Then \( \Phi \) and \( \Phi^{*} \) admit families of generalized principal Floquet subspaces \( \hat{E}_{1}(\omega)_{\omega \in \tilde{\Omega}_{1}} \) \( = \{ \text{span} \{ w(\omega) \}_{\omega \in \tilde{\Omega}_{1}} \} \) and \( \hat{E}^{*}_{1}(\omega)_{\omega \in \tilde{\Omega}_{1}} = \{ \text{span} \{ w^{*}(\omega) \}_{\omega \in \tilde{\Omega}_{1}} \} \), with \( w(\omega), w^{*}(\omega) \in X^{++} \) for all \( \omega \in \tilde{\Omega}_{1} \).

3. (Generalized exponential separation) Assume (D1)(iii) and (D2)(i)–(iii). Then there is \( \tilde{\sigma} \in (0, \infty) \) such that the triple \( \{ \hat{E}_{1}(\cdot), \hat{F}_{1}(\cdot), \tilde{\sigma} \} \) generates a generalized exponential separation, where \( \hat{F}_{1}(\omega) := \{ u \in X : \langle u, w^{*}(\omega) \rangle = 0 \} \). If we assume moreover (D1)(ii), then \( S(\omega) \hat{F}_{1}(\omega) = \hat{F}_{1}(\theta \omega) \) for any \( \omega \in \tilde{\Omega}_{1} \).

4. (Finiteness of principal Lyapunov exponent) Assume (D1)(iii), (D2)(i)–(iii), and (D3)(i) or (ii). Then \( \lambda_{1} > -\infty \), where \( \lambda_{1} \) is the generalized principal Lyapunov exponent associated to \( \{ \hat{E}_{1}(\omega) \} \).

**Proof of Theorem 3.1.** (1) It follows from Theorem 2.2(1).

(2) Parts of it follow from Propositions 3.1(1), (3), 3.2(1), (2), Theorem 2.3, and its analog for the dual system.

(3) It follows from Propositions 3.1(1), (3), 3.2(1)–(3) and Theorem 2.4. The last sentence follows from (D1)(ii).

(4) It follows from Proposition 3.3 and Theorem 2.4. \( \square \)
Remark 3.2. In [3] the authors investigated the principal Lyapunov exponent and principal Floquet subspaces for the random dynamical system generated by $S(\omega)$ satisfying
\[
\ln^- m \in L_1((\Omega, \mathcal{F}, \mathbb{P})), \quad \ln^+ M \in L_1((\Omega, \mathcal{F}, \mathbb{P})).
\] (3.3)

Theorem 3.1 extends the results in [3] in the following aspects.

1. The result in Theorem 3.1(1) is new.

2. The conditions in Theorem 3.1(2) and (3) are weaker than the conditions posed in [3] and the results in Theorem 3.1(2) and (3) are not covered in [3]. Observe that $\lambda_1$ in Theorem 3.1(2) and (3) may be $-\infty$.

3. Theorem 3.1(4) recovers the results in [3].

Remark 3.3. Theorem 3.1(3) implies that for any $u_0 \in X^+ \setminus \{0\}$ and $\omega \in \tilde{\Omega}_1$,
\[
\lim_{n \to \infty} \frac{1}{n} \ln \|S^n(\omega)u_0\| = \hat{\lambda}_1
\]
and
\[
\lim_{n \to \infty} \frac{1}{n} \ln \left\| \frac{S^n(\omega)u_0}{\|S^n(\omega)u_0\|} - w(\theta^n\omega) \right\| \leq -\tilde{\sigma}.
\]
Hence $S^n(\omega)u_0$ decreases or increases exponentially at the rate $\hat{\lambda}_1$, and its direction converges exponentially at the rate at least $\tilde{\sigma}$ toward the direction of $w(\theta^n\omega)$.

Example 3.1. (Random Leslie matrices). Assume that, for each $\omega \in \Omega$, $S(\omega)$ is a random Leslie matrix, i.e.,
\[
S(\omega) = \begin{pmatrix}
m_1(\omega) & m_2(\omega) & m_3(\omega) & \cdots & m_{N-1}(\omega) & m_N(\omega) \\
b_1(\omega) & 0 & 0 & \cdots & 0 & 0 \\
0 & b_2(\omega) & 0 & \cdots & 0 & 0 \\
0 & 0 & b_3(\omega) & \cdots & 0 & 0 \\
\vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\
0 & 0 & 0 & \cdots & b_{N-1}(\omega) & 0
\end{pmatrix},
\]
where $m_j$ $(>0)$ $(j = 1, 2, \ldots, N)$ represent the number of offspring an individual in age group $j$ in the current time contributes to the first age group at next time, $b_j$ $(>0)$ $(j = 1, 2, \ldots, N-1)$ denote the proportion of individuals of age $j$ in the current time surviving to age $j+1$ at next time.

We have

Theorem 3.2. Assume that $\ln^- m_0$, $\ln^+ M_0 \in L_1((\Omega, \mathcal{F}, \mathbb{P}))$, where
\[
m_0(\omega) = \min\{m_1(\omega), m_2(\omega), \ldots, m_N(\omega), b_1(\omega), \ldots, b_{N-1}(\omega)\}
\]
and
\[
M_0(\omega) = \max\{m_1(\omega), m_2(\omega), \ldots, m_N(\omega), b_1(\omega), \ldots, b_{N-1}(\omega)\}.
\]
Then (D1)(i)–(iii), (D2)(i)–(iii) and (D3)(i)–(ii) are satisfied with $S(\omega)$ replaced by $S^{(N)}(\omega)$. Consequently, there are
\[ \tilde{\Omega}_1 \subset \Omega \text{ with } P(\tilde{\Omega}_1) = 1 \text{ and } \theta(\tilde{\Omega}_1) = \tilde{\Omega}_1, \]
\[ \{ \tilde{E}_1(\omega) \}_{\omega \in \tilde{\Omega}_1} = \{ \text{span} \{ w(\omega) \} \}_{\omega \in \tilde{\Omega}_1} \text{ with } w: \tilde{\Omega}_1 \to X^{++} \text{ being } (\mathfrak{g}, \mathcal{B}(X))-\text{measurable}, \]
\[ \{ \tilde{F}_1(\omega) \}_{\omega \in \tilde{\Omega}_1} \text{ with } \tilde{F}_1(\omega) = \{ u \in X : \langle u, w^*(\omega) \rangle = 0 \} \text{ for each } \omega \in \tilde{\Omega}_1, \]
\[ w^*: \tilde{\Omega}_1 \to X^{++} \text{ being } (\mathfrak{g}, \mathcal{B}(X))-\text{measurable}, \]
\[ \tilde{\lambda}_1 \in (-\infty, \infty), \text{ and } \tilde{\sigma} \in (0, \infty], \]
such that

- \{ \tilde{E}_1(\omega) \}_{\omega \in \tilde{\Omega}_1} \text{ is a family of generalized principal Floquet subspaces of } S(\cdot) \text{ and } \tilde{\lambda}_1 \text{ is the generalized principal Lyapunov exponent associated to } \{ \tilde{E}_1(\omega) \}_{\omega \in \tilde{\Omega}_1};
- \{ \tilde{E}_1(\omega), \tilde{F}_1(\omega), \tilde{\sigma} \} \text{ generates a generalized exponential separation of } S(\cdot);
- \{ S(\omega)\tilde{E}_1(\omega) = \tilde{E}_1(\theta \omega) \} \text{ and } S(\omega)\tilde{F}_1(\omega) = \tilde{F}_1(\theta \omega) \text{ for each } \omega \in \tilde{\Omega}_1.

\text{Indication of proof.} \text{ The theorem follows from Theorem } 3.4.1 \text{ and Remark 3.6.1.} \qed

4. Random Cooperative and Type-K Monotone Systems of Ordinary Differential Equations

In this section we consider applications of the general results stated in Section 2 to random dynamical systems generated by linear random cooperative systems and linear random type-K monotone systems.

It is a standing assumption in Subsections 4.1 and 4.2 that \((\theta_t)_{t \in \mathbb{R}}\) is an ergodic metric dynamical system on \((\Omega, \mathfrak{g}, P)\), where the probability measure \(P\) is complete.

Further, in Subsections 4.1 and 4.2 we make the following standing assumption.

\textbf{(OP0) (Measurability)} \(C: \Omega \to \mathbb{R}^{N \times N}\), where \(C\) stands for \(A\) in Subsection 4.1 and for \(B\) in Subsection 4.2 is \((\mathfrak{g}, \mathcal{B}(\mathbb{R}^{N \times N}))-\text{measurable}, \text{ and } [\mathbb{R} \ni t \mapsto \|C(\theta \omega)\| \in \mathbb{R}] \text{ belongs to } L_{1,\text{loc}}(\mathbb{R}) \) for all \(\omega \in \Omega\).

We remark here that (OP0) is satisfied if \(C \in L_1((\Omega, \mathfrak{g}, P), \mathbb{R}^{N \times N})\) (see [2, Example 2.2.8]).

Under the assumption (OP0), for any \(\omega \in \Omega\) and \(u_0 \in X\) there exists a unique solution \([\mathbb{R} \ni t \mapsto u(t; \omega, u_0) \in X]\) of the linear system
\[ u' = C(\theta \omega)u \] (4.1)
satisfying the initial condition
\[ u(0; \omega, u_0) = u_0. \] (4.2)

The solution is understood in the Carathéodory sense: The function \([t \mapsto u(t; \omega, u_0)]\) is absolutely continuous, \(u(0; \omega, u_0) = u_0\), and for Lebesgue-a.e. \(t \in \mathbb{R}\) there holds
\[ \frac{du(t; \omega, u_0)}{dt} = C(\theta t \omega)u(t; \omega, u_0). \]

Define
\[ U_\omega(t)u_0 := u(t; \omega, u_0), \quad (t \in \mathbb{R}, \ \omega \in \Omega, \ u_0 \in X). \]
\[ ((U_\omega(t))_{\omega \in \Omega, t \in [0, \infty)}, (\theta_t)_{t \in \mathbb{R}}) \] is a measurable linear skew-product semiflow on \( X \) covering \( \theta \) (for proofs see, e.g., [2, Section 2.2]). Indeed, all the relations in the definition are satisfied for \( t \in \mathbb{R} \), and we can legitimately call \( \Phi = ((U_\omega(t))_{\omega \in \Omega, t \in \mathbb{R}}, (\theta_t)_{t \in \mathbb{R}}) \) a measurable linear skew-product flow on \( X \) generated by \( (B1) \). In particular, \( (B1)(ii) \) holds.

4.1. Random Cooperative Systems of Ordinary Differential Equations

In this subsection, we consider applications of the general results stated in Section 2 to the following cooperative system of ordinary differential equations

\[ \dot{u}(t) = A(\theta_t \omega)u(t), \quad \omega \in \Omega, \ t \in \mathbb{R}, \ u \in \mathbb{R}^N, \] (4.3)

where

\[ A(\omega) = \begin{pmatrix} a_{11}(\omega) & a_{12}(\omega) & \cdots & a_{1N}(\omega) \\ a_{21}(\omega) & a_{22}(\omega) & \cdots & a_{2N}(\omega) \\ \vdots & \vdots & \ddots & \vdots \\ a_{N1}(\omega) & a_{N2}(\omega) & \cdots & a_{NN}(\omega) \end{pmatrix}. \]

All the notations of \( X, X^+, \ldots \), etc., are as in Section 3. Recall that \( \|\cdot\|_1 \) is the \( \ell_1 \)-norm in \( X \), \( \|u\|_1 = |u_1| + \cdots + |u_N| \) for \( u = (u_1, \ldots, u_N)^T \).

In the rest of this subsection we assume that \( (OP0) \) holds. We state the standing assumptions on \( A(\omega) \).

Let

\[ \tilde{a}_{ii}(\omega) := \min_{0 \leq t \leq 1} \int_0^t a_{ii}(r, \omega) \, dr, \quad \tilde{a}_{ij}(\omega) := \min_{0 \leq s \leq 1} \int_s^1 a_{ij}(r, \omega) \, dr. \] (4.4)

(O1) Cooperativity \( a_{ij}(\omega) \geq 0 \) for all \( i \neq j, i, j = 1, 2, \ldots, N \) and \( \omega \in \Omega \).

(O2) Integrability The function \([\Omega \ni \omega \mapsto \max_{1 \leq i, j \leq N} a_{ij}(\omega)]\) is in \( L_1((\Omega, \mathcal{F}, \mathbb{P})) \).

(O3) Irreducibility There is an \( (\mathfrak{F}, \mathcal{B}(\mathbb{R})) \)-measurable function \( \delta: \Omega \to (0, \infty) \) such that for each \( \omega \in \Omega \) and \( i \in \{1, 2, \ldots, N\} \) there are \( j_1 = i, j_2, j_3, \ldots, j_N \in \{1, 2, \ldots, N\} \) satisfying

(i) \( \{j_1, j_2, \ldots, j_N\} = \{1, 2, \ldots, N\} \) and \( a_{j_i j_{i+1}}(\theta_t \omega) \geq \delta(\omega) \) for Lebesgue-a.e. \( 0 \leq t \leq 1 \) and \( l = 1, 2, \ldots, N - 1 \).

(ii) \( \ln^+ \ln(\mathfrak{F}/\beta) \in L_1((\Omega, \mathcal{F}, \mathbb{P})) \), where

\[ \mathfrak{F}(\omega) = \exp \left( \int_0^1 \left( \sum_{l=1}^N \max_{1 \leq l, j \leq N} a_{lj}(\theta_r \omega) \right) \, dr \right), \quad \beta(\omega) = \min_{1 \leq i \leq N} \beta_i(\omega), \]

and

\[ \beta_i(\omega) = \min \left\{ \exp(\tilde{a}_{jj_1}(\omega)), \exp(\tilde{a}_{j_1 j_2}(\omega) + \tilde{a}_{j_2 j_3}(\omega))\delta(\omega), \right. \]

\[ \exp(\tilde{a}_{j_1 j_2}(\omega) + \tilde{a}_{j_2 j_3}(\omega) + \tilde{a}_{j_3 j_4}(\omega))\frac{\delta^2(\omega)}{2!}, \quad \ldots, \]

\[ \left. \exp(\tilde{a}_{j_1 j_2}(\omega) + \tilde{a}_{j_2 j_3}(\omega) + \tilde{a}_{j_3 j_4}(\omega) + \cdots + \tilde{a}_{j_N j_1}(\omega))\frac{\delta^N(\omega)}{N!} \right\} \]

with \( j_1 = i \).
(iii) \( \ln (\beta/\bar{\beta}) \in L_1((\Omega, \mathcal{F}, P)) \), where \( \beta \) and \( \bar{\beta} \) are as in (ii).
(iv) \( \ln \frac{-\beta}{\bar{\beta}} \in L_1((\Omega, \mathcal{F}, P)) \), where \( \beta \) is as in (ii).

(O3)' (Off-diagonal positivity)

(i) There is an \((\mathfrak{F}, \mathcal{B}(\mathbb{R}))\)-measurable function \( \hat{\delta} : \Omega \to (0, \infty) \) such that for any \( \omega \in \Omega \) and any \( i \neq j \) there holds \( a_{ij}(\theta t \omega) \geq \hat{\delta}(\omega) \) for Lebesgue-a.e. \( t \in [0, 1] \).

(ii) \( \ln^+ \ln (\beta/\bar{\beta}) \in L_1((\Omega, \mathcal{F}, P)) \), where

\[
\bar{\beta}(\omega) = \exp \left( \int_0^1 \left( \sum_{l=1}^N \max_{1 \leq j \leq N} a_{lj}(\theta \tau \omega) \right) d\tau \right), \quad \hat{\beta}(\omega) = \min_{1 \leq i \leq N} \hat{\beta}_i(\omega),
\]

and

\[
\hat{\beta}_i(\omega) = \min \left\{ \exp(\hat{\delta}_{ii}(\omega)), \left( \min_{1 \leq j \neq N} \exp(\hat{\delta}_{ij}(\omega) + \bar{\delta}_{ij}(\omega)) \right) \right\}.
\]

(iii) \( \ln (\beta/\bar{\beta}) \in L_1((\Omega, \mathcal{F}, P)) \), where \( \beta \) and \( \bar{\beta} \) are as in (ii).

(iv) \( \ln^{-} \bar{\beta} \in L_1((\Omega, \mathcal{F}, P)) \), where \( \bar{\beta} \) is as in (ii).

Proposition 4.1 (Positivity). Assume (O1). Then \( \Phi \) satisfies (B2).

Proof. See [27, Thm. 1].

Proposition 4.2 (Integrability). Assume (O1) and (O2). Then (B1)(i) is satisfied.

Proof. We claim that for any \( \omega \in \Omega \) and any \( u_0 \in X^+ \) there holds

\[
\|U_\omega(\cdot)u_0\|_1 = \|u(\cdot; \omega, u_0)\|_1 \leq \exp \left( \int_0^1 \left( \sum_{i=1}^N \max_{1 \leq j \leq N} a_{ij}(\theta \tau \omega) \right) d\tau \right) \|u_0\|_1 \quad (4.5)
\]

for all \( t \geq 0 \). Indeed, fix \( \omega \in \Omega \) and \( u_0 \in X^+ \) with \( \|u_0\|_1 = 1 \), and denote \( u(\cdot) = (u_1(\cdot), \ldots, u_N(\cdot)) := u(\cdot; \omega, u_0) \). For each \( 1 \leq i \leq N \) we estimate

\[
\frac{du_i(t)}{dt} = \sum_{j=1}^N a_{ij}(\theta t \omega) u_j(t) \leq \max_{1 \leq j \leq N} a_{ij}(\theta t \omega) \cdot \sum_{k=1}^N u_k(t),
\]

consequently, in view of Proposition 4.1

\[
\frac{d}{dt} \|u(t)\|_1 \leq \sum_{i=1}^N \max_{1 \leq j \leq N} a_{ij}(\theta t \omega) \cdot \|u(t)\|_1,
\]

for Lebesgue-a.e. \( t \in \mathbb{R} \). The estimate (4.5) follows by comparison theorems for Carathéodory solutions, see, e.g., [12, Thm. 1.10.1].

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Since, by remarks in Example 2, any $u_0 \in X$ can be written as $u_0^+ - u_0^-$ with $\|u_0^+\|_1 \leq \|u_0\|_1$, $\|u_0^-\|_1 \leq \|u_0\|_1$, and, moreover, $\|u_0\|_1 = \|u_0^+\|_1 + \|u_0^-\|_1$, we have that

$$\|U_\omega(t)u_0\|_1 \leq \|U_\omega(t)u_0^+\|_1 + \|U_\omega(t)u_0^-\|_1 \leq \exp \left( \int_0^t \left( \sum_{i=1}^N \max_{1 \leq j \leq N} a_{ij}(\theta_r) \right) d\tau \right) \|u_0\|_1 \quad (4.6)$$

for all $\omega \in \Omega$, $u_0 \in X$ and $t \geq 0$. As the integrand in the rightmost term in (4.6) is nonnegative, we infer that

$$\ln^+ \|U_\omega(t)\|_1 \leq \int_0^t \left( \sum_{i=1}^N \max_{1 \leq j \leq N} a_{ij}(\theta_r) \right) d\tau$$

for all $\omega \in \Omega$ and $t \geq 0$. Since the norms $\|\cdot\|$ and $\|\cdot\|_1$ are equivalent, the assertion follows. \(\square\)

Proposition 4.3 (Focusing). (1) Assume (O1) and (O3)\((i)-(ii)\), or (O1) and (O3)\'(i)-(ii). Then (B3), (B3)* hold.

(2) Assume (O1) and (O3)\((i),(iii)\), or (O1) and (O3)\'(i), (iii). Then (B4) holds.

Proof. Observe that, since in (B3) or (B4) the suitable properties are to hold for $U_\omega(1)$ and $U_\omega^*(1)$ only, we can (and do) apply the results in Section 3.

First, assume (O1) and (O3)\((i)-(ii)\). Fix $\omega \in \Omega$ and $i \in \{1, \ldots, N\}$. By (O3)(i), there are $j_1, j_2, \ldots, j_N$ with $j_1 = i$ such that $\{j_1, j_2, \ldots, j_N\} = \{1, 2, \ldots, N\}$ and

$$a_{j_k, j_{k+1}}(\theta_\tau) \geq \delta(\omega) \quad \text{for} \quad t \in [0, 1]. \quad (4.7)$$

Note that

$$\frac{du_i(t; \omega, e_i)}{dt} \geq a_{ii}(\theta_t) u_i(t; \omega, e_i)$$

for Lebesgue-a.e. $t \in \mathbb{R}$ and $u_i(0; \omega, e_i) = 1$, so, by comparison results for Carathéodory solutions ([12, Thm. 1.10.1]),

$$u_i(t; \omega, e_i) \geq \exp \left( \int_0^t a_{ii}(\theta_r) d\tau \right) \quad (4.8)$$

for $t > 0$.

Similarly,

$$\frac{du_{ji}(t; \omega, e_i)}{dt} \geq a_{ji, j_l}(\theta_t) u_{ji}(t; \omega, e_i) + a_{ji-1, j_l}(\theta_t) u_{ji-1}(t; \omega, e_i), \quad l = 2, 3, \ldots, N,$$

for Lebesgue-a.e. $t \in \mathbb{R}$ and $u_{ji}(0; \omega, e_i) = 0$, so, by comparison results for Carathéodory solutions,

$$u_{ji}(t; \omega, e_i) \geq \int_0^t \exp \left( \int_s^t a_{ji, j_l}(\theta_r) d\tau \right) a_{ji-1, j_l}(\theta_s) u_{ji-1}(s; \omega, e_i) ds \quad (4.9)$$

for $t > 0$ and $l = 2, 3, \ldots, N$.  

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By (4.8) and (4.9), there holds

\[
\begin{align*}
    u_{ii}(\omega) & \geq \exp(\tilde{a}_{i,j_i}(\omega)) \\
    u_{jj}(\omega) & \geq \exp(\tilde{a}_{j,j_i}(\omega) + \bar{a}_{j,j}(\omega))d(\omega) \\
    u_{jj}(\omega) & \geq \exp(\tilde{a}_{j,j_i}(\omega) + \bar{a}_{j,j}(\omega) + \bar{a}_{j,j}(\omega))d^2(\omega) \\
    \vdots & \\
    u_{jj}(\omega) & \geq \exp(\tilde{a}_{j,j_i}(\omega) + \bar{a}_{j,j}(\omega) + \bar{a}_{j,j}(\omega) + \cdots + \bar{a}_{j,j}(\omega))d^N(\omega) \frac{N!}{N!}
\end{align*}
\]

(4.10)

where \(\tilde{a}_{j,j}(\omega)\) and \(\bar{a}_{j,j}(\omega)\) are as in (4.4), and \(u_{kl}(\omega) = u_k(1; \omega, e_l)\) for \(k, l \in \{1, \ldots, N\}\).

Further we have

\[
u_{ji}(\omega) \leq \|u(1; \omega, e_i)\|_1 \leq \exp \left( \int_0^1 \left( \sum_{i=1}^N \max_{1 \leq k \leq N} a_{ik}(\theta, \omega) \right) d\tau \right) \text{ for } 1 \leq j \leq N \text{ (by (4.4)).}
\]

We have proved that

\[
\beta(\omega) \leq u_{ij}(\omega) \leq \beta(\omega)
\]

(4.11)

for all \(\omega \in \Omega\) and \(i, j \in \{1, 2, \ldots, N\}\), where \(\beta(\omega)\) and \(\beta(\omega)\) are as in (O3). Consequently, (D2)(i)–(ii) hold with

\[
\beta(\omega) \leq \max_{i,j} \beta(\omega) \leq \beta(\omega) \leq \max_{i,j} \beta(\omega) \leq \beta(\omega)
\]

for all \(\omega \in \Omega, i = 1, 2, \ldots, N\), which gives with the help of Proposition 3.21–(2) that (B3) and (B3)* hold.

Next, assume (O1) and (O3)'(i)–(ii). Fix \(\omega \in \Omega\) and \(i \in \{1, \ldots, N\}\). As above, we estimate

\[
u_i(t; \omega, e_i) \geq \exp \left( \int_0^t a_{ii}(\theta, \omega) d\tau \right), \quad t \geq 0.
\]

(4.12)

For \(j \in \{1, \ldots, N\}, j \neq i\), we have

\[
\frac{du_{ij}(t; \omega, e_i)}{dt} \geq a_{jj}(\theta, \omega) u_j(t; \omega, e_i) + a_{ji}(\theta, \omega) u_i(t; \omega, e_i)
\]

for Lebesgue-a.e. \(t \in \mathbb{R}\) and \(u_j(0; \omega, e_i) = 0\), which gives

\[
u_j(t; \omega, e_i) \geq \int_0^t \exp \left( \int_s^t a_{jj}(\theta, \omega) d\tau \right) a_{ji}(\theta, \omega) u_i(s; \omega, e_i) ds
\]

(4.13)

for \(t \geq 0\). By (4.12) and (4.13), there holds

\[
u_{ii}(\omega) \geq \exp(\tilde{a}_{i,i}(\omega)) \text{ and } u_{ji}(\omega) \geq \exp(\tilde{a}_{i,i}(\omega) + \bar{a}_{i,j}(\omega))d(\omega), j \neq i,
\]

(4.14)

from which it follows that

\[
\tilde{\beta}(\omega) \leq u_{ij}(\omega) \leq \beta(\omega)
\]

(4.15)
for all \( \omega \in \Omega \) and \( i, j \in \{1, 2, \ldots, N\} \), where \( \tilde{\beta}(\omega) \) and \( \beta(\omega) \) are as in \((O3)'\). The rest goes along the lines of the proof in the above case.

This completes the proof of (1). The proof of (2) goes in a similar way. \( \square \)

**Proposition 4.4** (Strong positivity in one direction). Assume \((O1)\) and \((O3)(i), (iv)\), or \((O1)\) and \((O3)(i), (iv)\). Then \((B5)\) and \((B5)^*\) hold.

**Proof.** As in the proof of Proposition 4.3 we use \((4.11)\) (or \((4.15)\)) to show that \((D3)(i)-(ii)\) hold with \( m_r(\omega) \geq N\tilde{\beta}(\omega) \) and \( m_r(\omega) \geq N\beta(\omega) \) (or with \( m_r(\omega) \geq \beta(\omega) \) and \( m_r(\omega) \geq \beta(\omega) \)), and apply Proposition 3.3. \( \square \)

For \( \omega \in \Omega \), by an **entire positive solution** of \((4.3)\), we understand an entire positive orbit of \( U_\omega \), that is, a function \( v_\omega : \mathbb{R} \to X^+ \) such that

\[
U_\omega(s)(v_\omega(t)) = v_\omega(t + s) \quad \text{for all} \quad t \in \mathbb{R}, \ s \in [0, \infty).
\]

An entire positive solution \( v_\omega \) is **nontrivial** if \( v_\omega(t) \in X^+ \setminus \{0\} \) for each \( t \in \mathbb{R} \).

**Theorem 4.1.** (1) (Entire positive solution) Assume \((O1)\). For any \( \omega \in \Omega \) there exists a nontrivial entire positive solution of \((4.3)\).

(2) (Generalized principal Floquet subspaces and Lyapunov exponent) Let \((O1)\) and \((O2)\) be satisfied. Moreover, assume \((B3)\) and \((B3)^*\) (for instance, assume \((O3)(i)-(ii)\) or \((O3)'(i)-(ii)\)). Then \( \Phi \) and \( \Phi^* \) admit families of generalized principal Floquet subspaces \( \{ \tilde{E}_1(\omega) \}_{\omega \in \Omega} = \{ \text{span} \{ \tilde{w}(\omega) \} \}_{\omega \in \Omega} \) and \( \{ \tilde{E}_1^*(\omega) \}_{\omega \in \Omega} = \{ \text{span} \{ \tilde{w}^*(\omega) \} \}_{\omega \in \Omega} \). Moreover,

\[
\tilde{\lambda}_1 = \int_{\Omega} \kappa(\omega) \, d\tilde{\nu}(\omega), \tag{4.16}
\]

where \( \kappa(\omega) = \langle A(\omega)w(\omega), w(\omega) \rangle \), and \( \tilde{\lambda}_1 \) is the generalized principal Lyapunov exponent associated to \( \{ \tilde{E}_1(\omega) \}_{\omega \in \Omega} \).

(3) (Generalized exponential separation) Let \((O1)\) and \((O2)\) be satisfied. Moreover, assume \((B4)\) (for instance, assume \((O3)(i)\) and \((iii)\), or \((O3)'(i)\) and \((iii)\)). Then there is \( \tilde{\sigma} \in (0, \infty] \) such that the triple \( \{ \tilde{E}_1(\omega), \tilde{F}_1(\omega), \tilde{\sigma} \} \) generates a generalized exponential separation of \( S(\omega) \), where \( \tilde{F}_1(\omega) := \{ u \in X : \langle u, \tilde{w}^*(\omega) \rangle = 0 \} \). Moreover, \( U_\omega(t)\tilde{F}_1(\omega) = \tilde{F}_1(\theta_1\omega) \) for any \( \omega \in \Omega_1 \).

(4) (Finiteness of principal Lyapunov exponent) Let \((O1)\) and \((O2)\) be satisfied. Moreover, assume \((B5)\) or \((B5)^*\) (for instance, assume \((O3)(i)\), (iii) and (iv), or \((O3)'(i)\), (iii) and (iv)). Then \( \tilde{\lambda}_1 > -\infty \).

**Proof.** (1) It follows from Theorem 2.2(1).

(2) Parts of it follow from Proposition 4.3(1), Theorem 2.3, and its counterpart for the dual system. By Theorem 2.3,

\[
\tilde{\lambda}_1 = \lim_{t \to \infty} \frac{\ln \|U_\omega(t)w(\theta_1\omega)\|}{t}
\]
for $\mathbb{P}$-a.e. $\omega \in \Omega$. Differentiating formally we obtain
\[
\frac{d}{dt} \ln \|U_\omega(t)w(\theta_t\omega)\| = \frac{1}{2} \frac{d}{dt} \ln \langle U_\omega(t)w(\theta_t\omega), U_\omega(t)w(\theta_t\omega) \rangle
\]
\[
= \frac{\langle A(\theta_s\omega)(U_\omega(s))w(\theta_s\omega), U_\omega(s)w(\theta_s\omega) \rangle}{\|U_\omega(s)w(\theta_s\omega)\|^2}
\]
\[
= \langle A(\theta_s\omega)w(\theta_s\omega), w(\theta_s\omega) \rangle = \kappa(\theta_s\omega).
\]

It follows from (OP0) that for each $\omega \in \Omega$ the function $R \ni t \mapsto A(\theta_t\omega)(U_\omega(t))w(\theta_t\omega) \in \mathbb{R}^N$ belongs to $L^1_{1,loc}(\mathbb{R}, \mathbb{R}^N)$. Consequently we have
\[
\ln \|U_\omega(t)w(\theta_t\omega)\| = \int_0^t \kappa(\theta_s\omega) \, ds \quad \forall t \geq 0.
\]

For $\omega \in \Omega$ let $w(\omega) = (w_1(\omega), \ldots, w_N(\omega))^\top$. We estimate
\[
\langle A(\omega)w(\omega), w(\omega) \rangle = \sum_{i=1}^N \left( \sum_{j=1}^N a_{ij}(\omega)w_j(\omega) \right) w_i(\omega)
\]
\[
\leq \left( \max_{1 \leq i \leq N} \sum_{j=1}^N a_{ij}(\omega)w_j(\omega) \right) \|w(\omega)\|_1 \leq \sqrt{N} \max_{1 \leq i \leq N} \sum_{j=1}^N a_{ij}(\omega)w_j(\omega)
\]
\[
\leq \sqrt{N} \max_{1 \leq i \leq N} \left( \max_{1 \leq j \leq N} a_{ij}(\omega) \|w(\omega)\|_1 \right) \leq N \sum_{i=1}^N \max_{1 \leq j \leq N} a_{ij}(\omega).
\]

Hence, by (O1) and (O2), $\kappa^+ \in L_1((\Omega, \mathcal{F}, \mathbb{P}))$, which allows us to apply the Birkhoff Ergodic Theorem to get (4.16).

(3) It follows from Proposition 4.3(2) and Theorem 2.4. The last sentence follows from [17, Theorem 3.8(5)].

(4) It follows from Proposition 4.4 and Theorem 2.4.

\[\square\]

Remark 4.1. Theorem 4.1(3) implies that for any $u_0 \in X^+ \setminus \{0\}$ and $\omega \in \tilde{\Omega}_1$,
\[
\lim_{t \to \infty} \frac{1}{t} \ln \|U_\omega(t)u_0\| = \hat{\lambda}_1
\]
and
\[
\limsup_{t \to \infty} \frac{1}{t} \ln \left\| \frac{U_\omega(t)u_0}{\|U_\omega(t)u_0\|} - w(\theta_t\omega) \right\| \leq -\hat{\sigma}.
\]

Hence $U_\omega(t)u_0$ decreases or increases exponentially at the rate $\hat{\lambda}_1$, and its direction converges exponentially at the rate at least $\hat{\sigma}$ toward the direction of $w(\theta_t\omega)$. 

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4.2. Type-K Monotone Systems of Ordinary Differential Equations

In this subsection, we consider applications of the general results stated in Section 2 to the following type-K monotone systems of ordinary differential equations

\[ \dot{u}(t) = B(\theta t)u(t), \quad \omega \in \Omega, \ t \in \mathbb{R}, \ u \in \mathbb{R}^N, \]  

where

\[
B(\omega) = \begin{pmatrix}
  b_{11}(\omega) & b_{12}(\omega) & \cdots & b_{1N}(\omega) \\
  b_{21}(\omega) & b_{22}(\omega) & \cdots & b_{2N}(\omega) \\
  \vdots & \vdots & \ddots & \vdots \\
  b_{N1}(\omega) & b_{N2}(\omega) & \cdots & b_{NN}(\omega)
\end{pmatrix}
\]

satisfies the following assumptions.

(P1) (Type-K monotonicity) There are \(1 \leq k, l \leq N\) such that \(k + l = N\), \(b_{ij}(\omega) \geq 0\) for \(i \neq j\) and \(i, j \in \{1, 2, \ldots, k\}\) or \(i, j \in \{k + 1, k + 2, \ldots, k + l\}\), and \(b_{ij}(\omega) \leq 0\) for \(i \in \{1, 2, \ldots, k\}\) and \(j \in \{k + 1, k + 2, \ldots, k + l\}\) or \(i \in \{k + 1, k + 2, \ldots, k + l\}\) and \(j \in \{1, 2, \ldots, k\}\).

(P2) (Integrability) The function \([\Omega \ni \omega \mapsto \max_{1 \leq i, j \leq N} |b_{ij}(\omega)|]\) is in \(L_1((\Omega, \mathcal{F}, \mathbb{P}))\).

To simplify the notation, we write \(K\) for \(\{1, \ldots, k\}\) and \(L\) for \(\{k + 1, \ldots, n\}\).

Make the following change of variables, \(u \mapsto v\), where

\[
v_i = \begin{cases}
  u_i & \text{if } i \in K \\
  -u_i & \text{if } i \in L.
\end{cases}
\]

Then (4.17) becomes

\[ \dot{v}(t) = A(\theta t)v, \]  

where \(A(\theta t) = (a_{ij}(\theta t))_{i,j=1}^n\) with

\[
a_{ij}(\omega) = \begin{cases}
  b_{ij}(\omega) & \text{if } i, j \in K \text{ or } i, j \in L \\
  -b_{ij}(\omega) & \text{if } i \in K \text{ and } j \in L \text{ or } i \in L \text{ and } j \in K.
\end{cases}
\]

By (P1)–(P2), \(A(\omega)\) satisfies (O1)–(O2). Let

\[ \tilde{X}^+ := \{ u = (u_1, \ldots, u_N)^\top : u_i \geq 0 \text{ for } i \in K \text{ and } u_i \leq 0 \text{ for } i \in L \}. \]

Then \(\tilde{X}^+\) is a solid cone in \(X\), \((X, \tilde{X}^+)\) satisfies (B0), and \(\Phi\) satisfies (B2) with respect to the order induced by \(\tilde{X}^+\). We say an entire solution \([\mathbb{R} \ni t \mapsto v(t) \in X]\) of (4.17) is positive if \(v(t) \in \tilde{X}^+\) for any \(t \in \mathbb{R}\).

In the rest of this section, we assume (P1)–(P2) and that \(A(\omega)\) is as in (4.18). The order in \(X\) is referred to the order generated by \(\tilde{X}^+\). By Theorem 4.1 and the relation between (4.17) and (4.18), we have

**Theorem 4.2.** (1) (Entire positive solution) For any \(\omega \in \Omega\) there exists a nontrivial entire positive solution of (4.17).
(2) (Generalized principal Floquet subspaces and Lyapunov exponent) Assume (B3) and (B3)* (for instance, assume that $A(\omega)$ satisfies (O3) (i)–(ii) or (O3)' (i)–(ii)). Then $\Phi$ and $\Phi^*$ admit families of generalized principal Floquet subspaces \( \{ E_1(\omega) \}_{\omega \in \Omega} = \{ \text{span} \{ w(\omega) \} \}_{\omega \in \Omega} \) and \( \{ \tilde{E}_1(\omega) \}_{\omega \in \tilde{\Omega}_1} = \{ \text{span} \{ w^*(\omega) \} \}_{\omega \in \tilde{\Omega}_1} \). Moreover,
\[
\tilde{\lambda}_1 = \int_{\tilde{\Omega}} \kappa(\omega) \, d\tilde{\mathbb{P}}(\omega), \tag{4.19}
\]
where $\kappa(\omega) = (B(\omega)w(\omega), w(\omega))$, and $\tilde{\lambda}_1$ is the generalized principal Lyapunov exponent associated to $\{ \tilde{E}_1(\omega) \}_{\omega \in \tilde{\Omega}_1}$.

(3) (Generalized exponential separation) Assume (B4) (for instance, assume $A(\omega)$ satisfies (O3) (i) and (iii), or (O3)' (i) and (iii)). Then there is $\tilde{\sigma} \in (0, \infty]$ such that the triple \( \{ \tilde{E}_1(\omega), \tilde{F}_1(\omega), \tilde{\sigma} \} \) generates a generalized exponential separation of $S(\omega)$, where $\tilde{F}_1(\omega) := \{ u \in X : (u, w^*(\omega)) = 0 \}$. Moreover, $U_\omega(t)\tilde{F}_1(\omega) = \tilde{F}_1(\theta_t\omega)$ for any $\omega \in \Omega_1$.

(4) (Finiteness of principal Lyapunov exponent) Assume (B5) or (B5)* (for instance, assume that $A(\omega)$ satisfies (O3) (i), (iii) and (iv), or (O3)' (i), (iii) and (iv)). Then $\tilde{\lambda}_1 > -\infty$.

4.3. An Example

In the present subsection we give an example showing that the invariant decomposition provided by a generalized exponential separation may be finer than the Oseledets decomposition given in Theorem 2.1.

We start with the two-dimensional torus, written as $[0, 1] \times [0, 1]$. Choose an irrational number $\rho \in (0, 1)$. Define
\[
\theta_t(\omega_1, \omega_2) := (\omega_1 + t, \omega_2 + \rho t), \quad t \in \mathbb{R}, \quad (\omega_1, \omega_2) \in [0, 1] \times [0, 1],
\]
where addition is understood modulo 1.

Let $\Omega$ be the set of those $\omega \in [0, 1] \times [0, 1]$ for which $\theta_t\omega \neq (1, 1)$ for any $t \in \mathbb{R}$. $\mathfrak{F}$ equals the family of Lebesgue-measurable subsets of $\Omega$, and $\mathbb{P}$ is the normalized Lebesgue measure on $\Omega$. $((\Omega, \mathfrak{F}), (\theta_t))_{t \in \mathbb{R}}$ is an ergodic metric flow, with complete $\mathbb{P}$.

Define an ($\mathfrak{F}, \mathcal{B}(\mathbb{R})$)-measurable function $a: \Omega \to \mathbb{R}$,
\[
a(\omega) = a(\omega_1, \omega_2) := -\frac{1}{(\omega_1 + \omega_2)^2}.
\]

For $\omega \in \Omega$ put
\[
A(\omega) = \begin{pmatrix} a(\omega_1, \omega_2) & 1 \\ 1 & a(\omega) \end{pmatrix}.
\]

The first part of (OP0) is satisfied. Further, for each $\omega \in \Omega$ the discontinuity points of the function $[\mathbb{R} \ni t \mapsto A(\theta_t\omega) \in \mathbb{R}^{2 \times 2}]$ are precisely those $t \in \mathbb{R}$ at which either $(\theta_t\omega)_1 = 1$ or $(\theta_t\omega)_2 = 1$ (but not both, thanks to our choice of $\Omega$). At any of such points the function is left- or right-continuous, with finite limits. Therefore its is locally bounded, hence locally integrable, and the second part of (OP0) is satisfied, too.

(O1) and (O2) are obvious. It follows from Propositions 4.1 and 4.2 that (B1)(i) and (B2) hold.
Observe that for each $\omega \in \Omega$ one has

$$U_\omega(t) = \exp\left( \int_0^t a(\theta_\tau \omega) \, d\tau \right) e^{tB}, \quad t \in \mathbb{R},$$

(4.20)

where $B = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$, consequently

$$U_\omega(t) = \exp\left( \int_0^t a(\theta_\tau \omega) \, d\tau \right) \begin{pmatrix} \cosh t & \sinh t \\ \sinh t & \cosh t \end{pmatrix}, \quad t \in \mathbb{R}.$$

It is straightforward that

$$(\sinh 1)e \leq e^B e_i \leq (\cosh 1)e, \quad i = 1, 2,$$

which gives that for each $\omega \in \Omega$ and each $u \in X^+ \setminus \{0\}$, $u^* \in (X^*)^+ \setminus \{0\}$, there are $\beta(\omega, u) > 0$, $\beta^*(\omega, u^*) > 0$ such that

$$\beta(\omega, u)e \leq U_\omega(t)u \leq (\coth 1)\beta(\omega, u)e,$$
$$\beta^*(\omega, u^*)e^* \leq U^*_{\omega}(t)u^* \leq (\coth 1)\beta^*(\omega, u^*)e^*.$$

Consequently, (B4) holds with $\kappa$ and $\kappa^*$ constantly equal to $\coth 1$.

Due to Eq. (4.20), for each $\omega \in \Omega$ the subspace $\tilde{E}_1(\omega)$, provided by Theorem 4.1(3), equals the invariant subspace of $B$ corresponding to the principal eigenvalue of $B$, that is, $\tilde{E}_1(\omega) = \text{span}\{(1, 1)^T\}$, whereas $\tilde{F}_1(\omega)$ equals the complementary invariant subspace of $B$, that is, $\tilde{F}_1(\omega) = \text{span}\{(1, -1)^T\}$. In particular, $w(\omega) = (1, 1)^T / \sqrt{2}$ for any $\omega \in \Omega$.

We apply (4.16) to calculate $\tilde{\lambda}_1$. There holds $\kappa(\omega) = 1 + a(\omega)$ for each $\omega \in \Omega$. We estimate

$$\hat{\lambda}_1 = \int_\Omega (1 + a(\omega)) \, dP(\omega) \leq 1 - \int_0^1 \int_0^{1-\omega_1} \int_0^{d\omega_2} \frac{d\omega_2}{(\omega_1 + \omega_2)^2} = -\infty.$$

Obviously, $\hat{\lambda}_2 = \hat{\lambda}_1 = -\infty$. To calculate $\hat{\sigma}$, observe that, by (4.20),

$$\hat{\sigma} = -\lim_{t \to \infty} \frac{1}{t} \ln \frac{\|U_\omega(t)|_{\tilde{F}_1(\omega)}\|}{\|U_\omega(t)w(\omega)\|} = -\lim_{t \to \infty} \frac{1}{t} \ln \frac{\|e^{tB}|_{\text{span}\{(1, 1)^T\}}\|}{\|e^{tB}|_{\text{span}\{(1, -1)^T\}}\|} = 2 \quad \forall \omega \in \Omega.$$

Notice that Case (i) in Theorem 2.1 is satisfied and no invariant families $\{E_1(\omega)\}$, $\{\tilde{F}_1(\omega)\}$, can be defined in terms of exponential rates of convergence.

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