Tidal effects around higher-dimensional black holes

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In four-dimensional spacetime, moons around black holes generate low-amplitude tides, and the energy extracted from the hole’s rotation is always smaller than the gravitational radiation lost to infinity. Thus, moons orbiting a black hole inspiral and eventually merge. However, it has been conjectured that in higher-dimensional spacetimes orbiting bodies generate much stronger tides, which backreact by tidally accelerating the body 

\[ \omega < m \Omega_H \]

Outside the innermost stable circular orbit \( \Omega_H \) is met, it is possible to imagine the existence of floating orbits around BHs. Generically, orbiting bodies around BHs spiral 

\[ \text{inwards} \]

as a consequence of gravitational-wave emission. When the condition for superradiance is met, it is possible to imagine the existence of floating orbits, i.e., orbits in which the energy radiated to infinity by the body is entirely compensated by the energy extracted from the hole. Within general relativity in four dimensions, tidal effects are in general completely washed-out by gravitational-wave emission and orbiting bodies always spiral inwards \( \text{inwards} \). However, when coupling to scalar fields is allowed, an induced dipole moment produces a tidal acceleration (or polarization acceleration \( \text{acceleration} \)) which might be orders of magnitude stronger than tidal quadrupolar effects. Furthermore, in theories where massive scalar fields are present, the coupling of the scalar field to matter can produce resonances in the scalar energy flux, which can lead to floating orbits outside the innermost stable circular orbit \( \text{orbit} \).

It was recently argued via a tidal analysis framework that higher-dimensional BHs in general relativity should be prone to strong tidal effects \( \text{effects} \). One of the consequences of those studies was that orbiting bodies around higher-dimensional rotating BHs always spiral \n
\[ \text{outwards} \]

if the tidal acceleration (or, equivalently, the superradiance condition) is met. In this paper, we use a fully relativistic analysis, albeit in the test-particle limit, to prove this behavior. For simplicity, we consider the coupling of massless scalar fields to matter around a rotating BH in higher-dimensional spacetimes. We show that, for spacetime dimensions \( D > 5 \), tidal effects are so strong, that the energy extracted from the BH is greater than the energy radiated to infinity. Higher-dimensional spacetimes are of interest in a number of theories and scenarios \( \text{scenarios} \).
Throughout the paper we use as discussed in Section V. We conclude in Section VI.

In Section IV we solve the wave equation, then compare our analytical results with those obtained for the energy fluxes. In Section V we solve the wave equation in terms of BH perturbations sourced by a test-particle in circular orbit around a spinning BH. We derive the Teukolsky equations and the expressions for the energy fluxes. In Section VI we solve the wave equation analytically in the low-frequency regime. We then compare our analytical results with those obtained by a direct numerical integration of the wave equation, as discussed in Section VII. We conclude in Section VIII.

Throughout the paper we use $G = c = 1$ units, except in Sec. II where, for clarity, we show $G$ and $c$ explicitly.

II. TIDES FOR CHARGED INTERACTIONS IN $4 + n$ DIMENSIONS

The flux emitted by a particle orbiting a spinning BH can be estimated at newtonian level in terms of BH tidal acceleration and by applying the membrane paradigm. In this section, we generalize the computation sketched in Ref. to higher dimensions and to scalar fields.

Let us consider the interaction of a particle with scalar charge $q_p$ and gravitational mass $m_p$ orbiting a neutral central object of mass $M$ and radius $R$. If the object has a dielectric constant $\varepsilon = \varepsilon_0$, the particle external field induces a polarization surface charge density on the central object and a dipole moment which are given, respectively, by

$$\sigma_{\text{pol}} = (3 + n) \varepsilon_0 \beta E_0 \cos \theta,$$

$$p = \Omega_{(n+3)} \varepsilon_0 \beta R^{3+n} E_0,$$

where

$$E_0 = \frac{q_p}{\Omega_{(n+3)} \varepsilon_0 r_0^{3+n}},$$

and $r_0$ is the orbital distance, $\Omega_{n+3}$ is the solid angle of the $(n + 3)$-sphere, $\beta$ is some constant that depends on the relative dielectric constant of the object and $\theta$ is the polar angle with respect to single axis of rotation of the central object.

Assuming circular orbits, the tangential force on the charge $q_p$ due to the induced electric field is given by

$$F_\theta = \frac{q_p p}{\Omega_{(n+3)} \varepsilon_0 r_0^{3+n} \sin \theta}.$$

Without dissipation, the dipole moment would be aligned with the particle’s position vector. Here, we consider that dissipation introduces a small time lag $\tau$, such that the dipole moment leads the particle’s position vector by a constant angle $\phi$ given by (see II, III for details)

$$\phi = (\Omega - \Omega_H) \tau,$$

where $\Omega_H$ and $\Omega$ are the rotational angular velocity and the orbital angular velocity, respectively. At first order in $\phi$, the tangential component of the force reads

$$F_\theta \sim \frac{q_p p}{\Omega_{(n+3)} \varepsilon_0 r_0^{3+n} (\Omega - \Omega_H) \tau}.$$

This exerts a torque $r_0 F_\theta$ and the change in orbital energy over one orbit reads

$$\dot{E}_{\text{orbital}} = \frac{1}{2\pi} \int_0^{2\pi} r_0 F_\theta d\theta = \Omega r_0 F_\theta = \frac{\beta^2 q_p^2 R^{3+n} \Omega (\Omega - \Omega_H) \tau}{\Omega_{(n+3)} \varepsilon_0 r_0^{3+n}}.$$

where, in the last step, we used Eq. (2).

Remarkably, the equation above qualitatively describes the energy flux across the horizon of a rotating BH if one identifies $\Omega_H$ with the angular velocity of the BH and the lag $\tau$ with the light-crossing time, $\tau \sim R/c$, where $R^{1+n} = G_D M/[(n+1)c^2]$, $G_D$ is the D-dimensional gravitational constant and $M$ is the BH mass. Accordingly, a particle orbiting a rotating BH in $4 + n$ dimensions dissipates energy at the event horizon at a rate of roughly

$$\dot{E}_H = \frac{\beta^2 q_p^2 G_D}{\Omega_{(n+3)} \varepsilon_0 c^{n+3} (n+1) \Omega (\Omega - \Omega_H)}.$$

On the other hand, charged accelerating particles radiate to infinity according to Larmor’s formula, which in $4 + n$ dimensions reads (this will be also derived below, see Eq. (22))

$$\dot{E}_\infty = \frac{\gamma^2}{c^{3+n}} \Omega^{4+n} r_0^2,$$

where $\gamma$ is some coupling constant.

Tidal acceleration occurs when the orbit of the particle is pushed outwards due to energy dissipation in the central object. This is only possible if two conditions are satisfied: (i) $\Omega < \Omega_H$, so that $\dot{E}_H < 0$ and the energy flows out of the BH; and (ii) $|\dot{E}_H| > \dot{E}_\infty$, i.e. the rate at which energy is dissipated to infinity must be smaller than the rate at which energy is extracted from the BH.
From Eqs. (5) and (6), and using $\Omega \sim r_0^{-3(n+1)/2}$ [17], we find

$$\frac{\left| \mathcal{E}_H \right|}{\mathcal{E}_\infty} = \frac{\beta q^2 G_D}{\Omega^{(n+3)\epsilon_0/2} c^{(n+3)(2-n)/(n+1)^{n+1}} (n+1)^{(n+3)/n}} \times$$

$$= \frac{M^{(n+3)/n}}{\Omega^{(n+3)/(n+1)n+1)} (n+1)^{(n+3)/n}} \bigg( \Omega_H - \Omega \bigg) \sim \frac{v}{c} \frac{(n-1)(n+3)}{(n+1)^{n+1}} . \quad (10)$$

where we have assumed $\Omega_H \gg \Omega$ and we have defined the orbital velocity

$$v = [M(n+1)]^{-\frac{1}{n+3}} \Omega^{\frac{1}{n+1}} . \quad (11)$$

At large distance, $v \sim r_0 \Omega$ and, when $n = 0$, we recover the standard definition, $v = (M\Omega)^{1/3}$. Surprisingly, for $n > 1$ ($D > 3$) tidal acceleration dominates at large distances. This simple argument suggests that test-particles orbiting rotating BHs in dimensions greater than five would generically extract energy from the BH horizon at a larger rate than the energy emitted in gravitational waves to infinity. As a consequence, the orbital separation will increase in time, i.e. the system will "outspiral". In the next sections we shall prove this is indeed the case, by computing the linear response of a higher-dimensional spinning BH to a test-particle in circular orbit.

### III. Scalar Perturbations of Singly-Spinning Myers-Perry Black Holes

#### A. The Background Metric

In four dimensions, there is only one possible angular momentum parameter for an axisymmetric spacetime and rotating BH solutions are uniquely described by the Kerr family. In higher dimensions there are several choices of rotation axis, which correspond to a multitude of angular momentum parameters [18]. Here we shall focus on the simplest case, where there is only a single axis of rotation. In the following we shall adopt the notation used in Refs. [18, 21], to which we refer for details.

The metric of a $4+n$ dimensional Kerr-Myers-Perry BH with only one nonzero angular momentum parameter is given in Boyer-Lindquist coordinates by [18]

$$ds^2 = -\Delta - a^2 \sin^2 \vartheta dt^2 - \frac{2a(r^2 + a^2 - \Delta) \sin^2 \vartheta}{\Sigma} dt d\phi + \frac{(r^2 + a^2)^2 - \Delta a^2 \sin^2 \vartheta}{\Sigma} \sin^2 \vartheta d\varphi^2 + \Sigma dr^2$$

$$+ \Sigma d\vartheta^2 + r^2 \cos^2 \vartheta d\Omega_n^2 , \quad (12)$$

where

$$\Sigma = r^2 + a^2 \cos^2 \vartheta , \quad \Delta = r^2 + a^2 - 2Mr^{1-n} , \quad (13)$$

and $d\Omega_n^2$ denotes the standard line element of the unit $n$-sphere. This metric describes a rotating BH in asymptotically flat, vacuum spacetime, whose physical mass $\mathcal{M}$ and angular momentum $\mathcal{J}$ (transverse to the $r\phi$ plane) respectively read

$$\mathcal{M} = \frac{(n+2)A_{n+2}}{8\pi} M , \quad \mathcal{J} = \frac{2}{n+2} Ma , \quad (14)$$

where $A_{n+2} = 2\pi^{n+3/2}/\Gamma((n+3)/2)$.

The event horizon is located at $r = r_H$, defined as the largest real root of $\Delta$. In four dimensions, an event horizon exists only for $a \leq M$. In five dimensions, an event horizon exists only for $a \leq \sqrt{2}M$, and the BH area shrinks to zero in the extremal limit $a \rightarrow \sqrt{2}M$. On the other hand, when $D > 5$, there is no upper bound on the BH spin and a horizon exists for any $a$.

#### B. Setup

We consider a small object orbiting a spinning BH and a massless scalar field coupled to matter. At first order in perturbation theory, the scalar field equation in the background [12] reads

$$\Box \varphi = \frac{1}{\sqrt{-g}} \frac{\partial}{\partial x^\mu} \left( \sqrt{-g} g^{\mu \nu} \frac{\partial}{\partial x^\nu} \varphi \right) = \alpha T , \quad (15)$$

where $\alpha$ is some coupling constant. For simplicity we focus on source terms of the form

$$T = \int \frac{dr}{\sqrt{-g}} m_p \delta^{(4+n)}(x - X(\tau)) , \quad (16)$$

which corresponds to the trace of the stress-energy tensor of a point particle with mass $m_p$. We also restrict to equatorial circular orbits ($\dot{\vartheta} = 0$, $\dot{\varphi} = \pi/2$), which is an unrealistic approximation in higher dimensions: generic circular orbits are unstable, with an instability timescale of order of the orbital period [17]. Nevertheless, our purpose here is to show that tidal effects can dominate, it is not clear what the overall combined effect of tidal acceleration and circular geodesic motion instability is. Extending the present analysis to generic orbits and relaxing the test-particle approximation are interesting future developments.

For prograde orbits around a singly-spinning Myers-Perry BH [12] the energy, angular momentum and frequency of the point particle with mass $m_p$ orbiting at $r = r_0$ read [17]

$$E_p = \frac{\mathcal{E}}{m_p} = \frac{a \sqrt{(n+1)M + r_0^{2n} - 2Mr_0^{1-n}}}{r_0^{2n+1} \sqrt{2a \sqrt{(n+1)M + r_0^{2n} - (n+3)M} r_0^{1-n}}} . \quad (17)$$

$$L_p = \frac{\mathcal{J}}{m_p} = \frac{\sqrt{(n+1)M \left(r_0^{2n} - 2a \sqrt{\frac{M}{n+1} + r_0^{2n} - (n+3)M} r_0^{1-n}\right)}}{r_0^{(n+1)(n+1) - 1} \sqrt{2a \sqrt{(n+1)M + r_0^{2n} - (n+3)M} r_0^{1-n}}} , \quad (18)$$
where $\Omega_p = \frac{\sqrt{(n+1)M}}{a\sqrt{(n+1)M + r_0^a}}$. \hfill (19)

The only nonvanishing components of the (4+n)-velocity $U^\nu$ of the particle on a timelike geodesic are given by

$$m_p \Delta_{r=r_0} U^t = \left( \frac{r_0^2}{r_0^2 + a^2} + \frac{2Ma^2}{r_0^{a+1}} \right) E_p - \frac{2MaE_p}{r_0^{a+1}}$$ \hfill (20)$$m_p \Delta_{r=r_0} U^\varphi = \frac{2MaE_p}{r_0^{a+1}} + \left( 1 - \frac{2M}{r_0^{a+1}} \right) L_p \hfill (21)

C. The wave equation

Because of the coupling to matter, the orbiting object emits scalar radiation which is governed by Eq. (15). To separate Eq. (15), we consider the ansatz

$$\varphi(t, r, \vartheta, \phi) = \sum_{l,m,j} \int d\omega e^{im\varphi - i\omega t} R(r) S_{lmj}(\vartheta) Y_j, \hfill (22)$$

where $Y_j$ are hyperspherical harmonics \cite{20, 22} on the n-sphere with eigenvalues given by $-j(n - j)$ and $j$ being a non-negative integer. The radial and angular equations read

$$r^n \frac{d}{dr} \left( r^n \Delta \frac{dR}{dr} \right) + \left\{ \frac{\left[ \omega(r^2 + a^2) - ma \right]^2}{\Delta} - \frac{j(j + n - 1)a^2}{r^2} - \lambda \right\} R = T_{lmj}, \hfill (23)$$

and

$$\frac{1}{\sin \vartheta \cos^n \vartheta} \frac{d}{d\vartheta} \left( \sin \vartheta \cos^n \vartheta \frac{dS_{lmj}}{d\vartheta} \right) + \left[ \omega^2 a^2 \cos^2 \vartheta - \frac{m^2}{\sin^2 \vartheta} - \frac{j(j + n - 1)}{\cos^2 \vartheta} + A_{lmj} \right] S_{lmj} = 0, \hfill (24)$$

where $\lambda = A_{lmj} - 2m\omega a + \omega^2 a^2$ and

$$T_{lmj} = \frac{m_p \alpha}{U_{\varphi \varphi}} S_{lmj}(\pi/2) Y_j^*(\pi/2, \pi/2, \ldots) \times \delta(r - r_0) \delta(m\Omega_p - \omega), \hfill (25)$$

which has been derived from the stress-energy tensor of the point particle. Defining a new radial function $X_{lmj}(r)$

$$X_{lmj} = r^{-n/2}(r^2 + a^2)^{1/2} R, \hfill (26)$$

we get the non-homogeneous equation for the scalar field

$$\left[ \frac{d^2}{dr_*^2} + V \right] X_{lmj}(r^*) = \frac{\Delta}{(r^2 + a^2)^{3/2}} r^{n/2} T_{lmj}, \hfill (27)$$

where $dr/dr_* = \Delta/(r^2 + a^2)$ defines the standard tortoise coordinates and the effective potential $V$ reads

$$V = \omega^2 + \frac{3r^2 \Delta^2}{(r^2 + a^2)^4} - \frac{\Delta \left[ 3r^2 + a^2 - 2Mr^{1-n}(2 - n) \right]}{(r^2 + a^2)^3} + \frac{1}{(r^2 + a^2)^2} \left\{ a^2 m^2 - 4M + \frac{am\omega}{n-1} - \Delta \left( \omega^2 a^2 + A_{lmj} \right) + \Delta \left[ \frac{n(2 - n)\Delta}{4r^2} - n + \frac{2n(1 - n)M}{2r^{n+1}} \right] - \frac{j(j + n - 1)a^2}{r^2} \right\}. \hfill (28)$$

In the low frequency limit the angular equation (24) can be solved exactly. At first order in $a\omega$, the eigenvalues can be computed analytically \cite{22}

$$A_{lmj} = (2k + j + |m|)(2k + j + |m| + n + 1) + O(a\omega). \hfill (29)$$

By setting $2k = l - (j + |m|)$, the eigenvalues above take the form $A_{lmj} = l(l + n + 1)$ and $l$ is such that $l \geq (j + |m|)$, which generalizes the four-dimensional case. An important difference from the four-dimensional case is that regularity of the angular eigenfunctions requires $k$ to be a non-negative integer, i.e. for given $j$ and $m$ only specific values of $l$ are admissible. In fact, it is convenient to label the eigenfunctions and the eigenvalues with the “quantum numbers” ($k, j, m$) rather than with ($l, j, m$) as in the four dimensional case. The (non-normalized) 0th-order eigenfunctions are given in terms of hypergeometric functions \cite{20, 22}

$$S_{kjm} \propto \sin(\vartheta)^{|m|}x^j F\left[-k, k + j + |m| + \frac{n + 1}{2}, \frac{n + 1}{2}, x^2 \right], \hfill (30)$$

where $x = \cos(\vartheta)$. We adopt the following normalization condition

$$\int_0^{\pi/2} d\vartheta \sin \vartheta \cos^n \vartheta S_{kjm} S_{kjm}^* = 1, \hfill (31)$$

where the integration domain has been chosen in order to have a nonvanishing measure also in the case of odd dimensions. Note that this normalization differs from that adopted in Ref. \cite{22}.

We note that at $\vartheta = \pi/2$ only hyperspherical harmonics with $j = 0$ are non-vanishing. Thus, in order to calculate the fluxes on circular orbits, one only needs to consider terms with $j = 0$. In this case, the hyperspherical harmonics $Y_0$ are constant.

D. Green function approach and energy fluxes

To solve the wave equation, let us choose two independent solutions $X_{kjm}$ and $X_{kjm}^\infty$ of the homogeneous equation which satisfy the following boundary conditions

$$X_{kjm}^\infty \sim e^{ik\omega r_*}, \hfill (32)$$

$$X_{kjm}^{r_H} \sim A_{out} e^{-ik\omega r_*} + A_{in} e^{ik\omega r_*}, \hfill (33)$$

$$X_{kjm}^\infty \sim B_{out} e^{ik\omega r_*} + B_{in} e^{-ik\omega r_*}, \hfill (34)$$

$$X_{kjm}^{r_H} \sim e^{-ik\omega r_*}, \hfill (35)$$

where $r_H$ is the horizon radius.
Here \( k_H = \omega - m\Omega_H, k_\infty = \omega \) and \( \Omega_H = -\lim_{r\to r_H} g_{t\phi}/g_{\phi\phi} = a/(r_H^2 + a^2) \) is the angular velocity at the horizon of locally nonrotating observers. The Wronskian of the two linearly independent solutions reads
\[
W = X_{k_j m}^{r_H} \frac{dX_{k_j m}^{r_H}}{dr} - X_{k_j m}^{\infty} \frac{dX_{k_j m}^{\infty}}{dr} = 2ik_\infty A_{m} ,
\]
and it is constant by virtue of the field equations. Finally, Eq. \( 27 \) can be solved in terms of the Green function \( 23 \)
\[
X_{k_j m}(r) = \frac{X_{k_j m}^{\infty}}{W} \int_{-\infty}^{r} T_{k_j m}(r') \frac{\Delta r'^{n/2}}{(r'^2 + a^2)^{3/2}} X_{k_j m}^{r_H} dr' + \frac{X_{k_j m}^{r_H}}{W} \int_{r}^{\infty} T_{k_j m}(r') \frac{\Delta r'^{n/2}}{(r'^2 + a^2)^{3/2}} X_{k_j m}^{\infty} dr' .
\]
For very large values of \( r \) the above equation has the following asymptotic form
\[
X_{k_j m}(r \to \infty) = \frac{e^{ik_\infty r}}{2ik_\infty A_{m}} \int_{-\infty}^{\infty} T_{k_j m}(r') X_{k_j m}^{r_H}(r') \frac{\Delta r'^{n/2}}{(r'^2 + a^2)^{3/2}} dr' = Z_{k_j m}^{\infty} \delta(\omega - m\Omega_p) e^{ik_\infty r} ,
\]
where, using Eq. \( 23 \),
\[
Z_{k_j m}^{\infty} = -\alpha X_{k_j m}^{r_H}(r_0) S_{k_j m}^{\infty}(\pi/2) Y_{\theta}^{\pi/2, \pi/2, \ldots} \left( \frac{m}{\sqrt{r_0^2 + a^2 r_0^{n/2}}} \right) m_p .
\]
Likewise, at the horizon we get
\[
X_{k_j m}(r_* \to -\infty) = Z_{k_j m}^{r_H}(\omega - m\Omega_p) e^{-ik_\infty r_*} ,
\]
where, \( Z_{k_j m}^{r_H} = -\alpha X_{k_j m}^{\infty}(r_0) S_{k_j m}(\pi/2) Y_{\theta}^{\pi/2, \pi/2, \ldots} \left( \frac{m}{\sqrt{r_0^2 + a^2 r_0^{n/2}}} \right) m_p \).

The scalar energy flux at the horizon and at infinity are defined as
\[
\dot{E}_{H,\infty} = \lim_{r \to r_H, \infty} \int d\omega d\phi \prod_{i=1}^{n} d\theta_i \sqrt{-g} \hat{G}_{i} ,
\]
where the stress tensor reads \( T_{\mu\nu} = (\nabla_\mu \varphi \nabla_\nu \varphi^* - \frac{1}{2} g_{\mu\nu} \nabla_\alpha \varphi \nabla^\alpha \varphi^* ) \).
\[
\text{Finally, using Eqs. } 23 \text{ and } 35, \text{ we get}
\]
\[
\dot{E}_{H,\infty} = \sum_{k_j m} m\Omega_p k_{H,\infty} |Z_{k_j m}^{r_H, \infty}|^2 .
\]
The equation above shows that, if the superradiant condition \( k_H < 0 \) (\( \omega < m\Omega_H \)) is met, the energy flux at the horizon can be negative, \( \dot{E}_H < 0 \), i.e. energy can be extracted from a spinning BH \( 3, 21 \). In four dimensions, \( |\dot{E}_H| \ll \dot{E}_\infty \) and the superradiance extraction is generically negligible. As we show in the next section, in higher dimensions the opposite is true, \( |\dot{E}_H| \gg \dot{E}_\infty \) and superradiance dominates over gravitational-wave emission.

### IV. ANALYTICAL SOLUTION AT LOW FREQUENCIES

The scalar flux can be evaluated analytically in the low-frequency regime (see e.g. \( 25, 26 \)). Let us first focus on the solution \( X_{k_j m}^{r_H} \), which is regular at the horizon.

We first make the following change of variable
\[
h = \frac{\Delta}{r^2 + a^2} \Rightarrow \frac{dh}{dr} = (1 - h) r \frac{A(r)}{\sqrt{r^2 + a^2}} ,
\]
where \( A(r) = (n + 1) + (n - 1)a^2/r^2 \). Then, near the horizon \( r \sim r_H \), the radial equation \( 23 \) can be written as
\[
h(1 - h)\frac{dR}{dh} + (1 - D_s h)\frac{dR}{dh} + \left[ \frac{P^2}{A(r_H)^2 h(1 - h)} - \frac{\Lambda}{r_H^2 A(r_H)^2 (1 - h)} \right] R = 0 ,
\]
where
\[
P = \omega(r_H + a^2/r_H) - ma/r_H ,
\]
\[
\Lambda = |l(l + n + 1) + j(j + n - 1)a^2/r_H^2|(r_H^2 + a^2) ,
\]
\[
D_s = 1 - \frac{4a^2 r_H^2}{[(n + 1)r_H^2 + (n - 1)a^2]^2} .
\]
Using the redefinition \( R(h) = h^\alpha (1 - h)^\beta F(h) \) the above equation takes the form
\[
h(1 - h)\frac{d^2 F}{dh^2} + [c - (a + b + 1)h]\frac{dF}{dh} - (ab)F = 0 ,
\]
with
\[
a = \alpha + \beta + D_s - 1 , \quad b = \alpha + \beta , \quad c = 1 + 2\alpha ,
\]
and \( \alpha \) and \( \beta \) must satisfy the following algebraic equations
\[
\alpha^2 + \frac{P^2}{A(r_H)^2} = 0 ,
\]
\[
\beta^2 + \beta(D_s - 2) + \frac{P^2}{A(r_H)^2} - \frac{\Lambda}{r_H^2 A(r_H)^2} = 0 ,
\]
whose solutions read
\[
\alpha_\pm = \pm i \frac{P}{A(r_H)} ,
\]
\[
\beta_\pm = \frac{1}{2} \left[ (2 - D_s) \pm \sqrt{(D_s - 2)^2 - 4\frac{P^2}{A(r_H)^2} + 4\frac{\Lambda}{r_H^2 A(r_H)^2}} \right] .
\]
The two linearly independent solutions of Eq. \( 35 \) are \( F(a, b; c; h) \) and \( h^{1-\alpha}F(a + 1 - c, b + 1 - c; 2 - c; h) \), where \( F \) is the hypergeometric function. Convergence requires

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*Note: The text is a transcription of the content from the image, ensuring clear and natural readability.*
Re\([c - a - b] > 0\), which can be only obtained if the minus
sign is chosen in the solutions above. In the following,
we shall identify \(\beta \equiv \beta_\perp \) and \(\alpha \equiv \alpha_\perp\). The general
solution of Eq. (51) is then
\[
R(h) = A_1 h^\alpha (1 - h)^{\beta} F(a, b, c; h) + B_1 h^{-\alpha} (1 - h)^{\beta} F(a + 1 + c, b + 1 - c; 2 - c; h) .
\]
(51)

Expanding the above result near the horizon, we get
\[
R(h) = A_1 h^{-\alpha} (1 - h)^{\beta} \frac{\Gamma[1 + 2\alpha] \Gamma[2 - D_\perp - 2\beta]}{\Gamma[2 - D_\perp + \alpha - \beta] \Gamma[1 + \alpha - \beta]} \times F(a, b, a + b - 1; 1 - h) + B_1 h^\alpha (1 - h)^{2 - 2D_\perp - \beta} \times \frac{\Gamma[1 + 2\alpha] \Gamma[2\beta + D_\perp - 2\beta]}{\Gamma[\alpha + \beta + D_\perp - 1] \Gamma[\alpha + \beta]} \times F(c - a, c - b, c - a - b + 1; 1 - h) .
\]
(53)

We can now expand this result in the low-frequency
regime and for small values of \(a/r_{H}\), in the region where
\(1 - h \ll 1\) and \(r \gg r_{H}\),
\[
R \sim \frac{X_{kjm}^{rH}}{r^{1 + n/2}} \sim \frac{r^j}{(2M)^{2(n+1)+2} r_H^{2(D + \alpha - \beta) - 1 + n}} \times \frac{\Gamma[1 + 2\alpha] \Gamma[2 - D_\perp - 2\beta]}{\Gamma[2 - D_\perp + \alpha - \beta] \Gamma[1 + \alpha - \beta]} .
\]
(54)

We will follow Poisson to solve the wave equation
at large distances. It is useful to rewrite the radial
equation (27) in terms of the dimensionless variable \(z = \omega r\).
At large distances Eq. (27) reads
\[
\left[ f \frac{d^2}{dz^2} + \frac{(n + 1) \epsilon}{2z^{n+1}} \frac{d}{dz} + 1 - \frac{l(l + n + 1) + \frac{n}{2}(1 + \frac{n}{2})}{z^2} - \epsilon \left[ 1 + \frac{n}{2}(n + 1) \right] \right] X_{kjm}(z) = 0 ,
\]
(55)
where \(f = 1 - \epsilon/z^{n+1}\), \(\epsilon = 2M\omega^{n+1}\) is a dimensionless
parameter, and we used the fact that at large distances the
eigenvalues take the form \(A_{kjm} = l(l + n + 1)\). We can rewrite it in a simpler form if we define the quantum
number \(J(J + 1) = l(l + n + 1) + \frac{n}{2}(1 + \frac{n}{2})\). Solving for \(J\), and assuming \(J\) is a non-negative number, we get
\[
J = l + \frac{n}{2} .
\]
(56)

In the limit \(\epsilon \ll 1\), Eq. (55) reads
\[
\left[ \frac{d^2}{dz^2} + 1 - \frac{J(J + 1)}{z^2} \right] X_{kjm}(z) = 0 .
\]
(57)

The solution can be written in terms of a linear combina-
tion of Riccati-Bessel functions, \(\sqrt{z}J_{J+1/2}(z)\) and
\(\sqrt{z}N_{J+1/2}(z)\). The requirement that \(X_{kjm}^{rH}\) be regular
at the horizon demands
\[
X_{kjm}^{rH}(z) = B \sqrt{z}J_{J+1/2}(z) ,
\]
(58)
where \(B\) is a constant. The asymptotic expansions for
the Bessel functions are well known and read
\[
X_{kjm}^{rH}(z \ll 1) \sim \frac{B_{J+1/2}^{J+1}}{2J+1/2\Gamma[J + 3/2]} \left[ 1 + O(z^2) \right] .
\]
(59)

At large distance, the second term within the square
brackets is subdominant and we shall ignore it. Matching
(59) to (54) we get
\[
B = \frac{2^{J+1/2} \Gamma[J + 3/2] \Gamma[1 + 2\alpha] \Gamma[2 - D_\perp - 2\beta]}{\epsilon^{1/(n+1)} \Gamma[1 + \alpha - \beta] \Gamma[2 - D_\perp + \alpha - \beta] \Gamma[2 - D_\perp - 1]} \times \left( \frac{(2M)^{1/(2n+2)}}{r_H^{1/2}} \right)^{(J+1)} \Gamma[1 + O(\epsilon)] .
\]
(60)

The parameter \(A_{in}\) can be extracted from the behavior of
the function near \(z = \infty\). Recalling the large-argument of the
Bessel functions, \(X_{kjm}^{rH}(z \to \infty) \sim B/\sqrt{2/\pi \sin(z - J\pi/2)}\) and, using Eq. (53), it follows that
\[
A_{in} = \frac{2^J \Gamma[J + 3/2] \Gamma[1 + 2\alpha] \Gamma[2 - D_\perp - 2\beta]}{\sqrt{\pi} \Gamma[2 - D_\perp + \alpha - \beta] \Gamma[1 + \alpha - \beta] \Gamma[2 - D_\perp - 1]} \times \left( \frac{i}{\epsilon^{1/(n+1)}} \right)^{(J+1)} \left( \frac{(2M)^{1/(2n+2)}}{r_H^{1/2}} \right)^{(J+1)} \Gamma[1 + O(\epsilon)] .
\]
(61)

With all of this at hand, we can now compute the flux
at infinity in the low-frequency regime. From Eqs. (54) and (60) we get
\[
\hat{E}_\infty = m^2 \omega^2 |Z_{kjm}^{\infty}|^2
\]
\[
= m^{2 + 2l + n} \left[ \frac{\epsilon \omega^{n+1}}{2l + n + 1} \right] \Gamma(l + n + 1 + \frac{n}{2}) \times \left( \frac{\epsilon \omega^{n+1}}{2l + n + 1 + \frac{n}{2}} \right) |S_{kjm}(\pi/2)|^2 |Y_{j}(\pi/2, \pi/2, 2, \ldots)|^2
\]
\[
\times \frac{1}{r_0^{2/(n+1)+2/3}} \times \frac{(n + 1)! M^{l+n/2+1} |M|^{l+n/2+1}}{2^{l+n+1} \pi^{l+n/2+1}} \times \frac{1}{r_0^{2/(n+1)+2/3}}
\]
(62)

where we used the fact that for small frequencies (large
distances) \(r^2 + a^2 \sim r^2\), \(U^t \sim 1\) and \(\omega = m\Omega_\perp \sim m\sqrt{(n + 1)M/r_0^{(3+n)/2}}\).

Let us now perform the same calculation for the solution
\(X_{kjm}^{H}\), which satisfies outgoing-wave boundary
conditions at infinity. The method is analogous to that al-
ready described above. However, since for this case the
boundary condition is imposed at infinity we do not re-
quire regularity at the horizon. In the limit \(\epsilon \ll 1\), \(X_{kjm}^{\infty}\)
can be identified, up to a normalization constant, with
\[
X_{kjm}^{\infty}(z) = C \sqrt{z} H_{J+1/2}^{(1)}(z) ,
\]
(63)

where \(H_{J+1/2}^{(1)}(z) = J_{J+1/2}(z) + i N_{J+1/2}(z)\) is the Hankel
function. To determine the constant \(C\) we match this
solution in the limit \( z \to \infty \) to the required boundary condition

\[
X_{kj,m}^\infty (z \to \infty) \sim e^{i z}.
\]  

(64)

Recalling the asymptotic behavior of the Hankel functions, we have

\[
C\sqrt{2} H_{J+1/2}(z) \to C \sqrt{\frac{2}{\pi}} e^{i \frac{\pi}{4} z} \left[ (-i)^{J+1} + O(1/z) \right],
\]

(65)

from which we get \( C = i^{J+1} \sqrt{\pi/2} \).

The small-argument behavior of the Bessel functions reads (when \( J > 0 \))

\[
J_{J+1/2}(z \ll 1) \sim \frac{z^{J+1/2}}{2^{J+1/2} \Gamma[J + 3/2]} \left[ 1 + O(z^2) \right],
\]

\[
N_{J+1/2}(z \ll 1) \sim \frac{z^{J+1/2}}{\pi} \left[ 1 + O(z^2) \right].
\]

(66)

At leading order and near \( z = 0 \), the function \( N_{J+1/2} \) dominates over \( J_{J+1/2} \). Hence, we get

\[
X_{kj,m}^\infty (z \ll 1) \sim i \frac{2^J}{\sqrt{\pi}} \Gamma[J + 1/2] z^{-J}.
\]

(67)

We can now compute the flux across the horizon. Using Eqs. (66) and (69) we get

\[
\dot{E}_H = m \Omega_p k_H \left| Z_{kj,m}^r \right|^2
\]

\[
= mk_H (\alpha m_p)^2 \Gamma_1 \left( \frac{n + 1}{2} \right)^{1/2} r_H (2M) \frac{2^{l+2} + 2}{n+1} \times
\]

\[
|S_{kj,m}(\pi/2)|^2 |Y_j(\pi/2, \pi/2, \ldots)|^2 \times \frac{r_0^{-(l+5)n+7}}{2},
\]

(68)

where \( \Gamma_1 = \Gamma[2(l+n/2+1)/2 - D_2 - \alpha - \beta] \Gamma[l+1-\alpha-\beta] \Gamma[l+1-\alpha-\beta] \).

We can now obtain an expression for the ratio of the fluxes on the horizon and at infinity for general \( l, m \) and \( n \). Using the expressions for \( \dot{E}_H \) and \( \dot{E}_\infty \) calculated above, we find

\[
\frac{\dot{E}_H}{\dot{E}_\infty} = \frac{k_H r_H}{2} \frac{2^{l+n/2+1} \Gamma[l+n/2+3/2]^2}{\pi m^{2+n+1}} \Gamma_1^2
\]

\[
\times \left( \frac{2}{n+1} \right)^{\frac{2^{l+n} + 1}{2}} 2^{(n-1)(2(l+n)+1)} \frac{r_0^{-(n-1)(n+1+2)}}{2},
\]

(69)

This can be written as a function of the orbital velocity,

\[
\frac{\dot{E}_H}{\dot{E}_\infty} = \frac{k_H r_H}{2} \frac{2^{l+n+2+1} \Gamma[l+n/2+3/2]^2}{\pi m^{2+n+1}} \Gamma_1^2
\]

\[
\times \left( \frac{2}{n+1} \right)^{\frac{2^{l+n} + 1}{2}} 2^{(n-1)(2(l+n)+1)} \frac{r_0^{-(n-1)(n+1+2)}}{2},
\]

(70)

where we have used Eq. (11) at large distance. For sufficiently small orbital frequencies, such that the superradiance condition is met, the flux at the horizon is negative, we then find that the ratio between the fluxes grows in magnitude with \( r_0 \) and the particle is tidally accelerationed outwards. For the dipolar mode, \( l = 1 \), this expression is in complete agreement with the expected behavior derived from a newtonian tidal analysis, cf. Eq. (10).

Note that these results were derived under the assumption of slow rotation, \( a \ll r_H \). This approximation is particularly severe in the near-extremal, five-dimensional case, where \( r_H \to 0 \). Nevertheless, as we discuss in the next section, our method captures the correct scaling of the energy fluxes for any spin and it even gives overall coefficients which are in very good agreement with the numerical ones in the slowly-rotating case. This is shown in Fig. 1 where we compare the analytical results of this section with the numerical fluxes computed in Sec. V.

![FIG. 1. Comparison between the flux ratio \( \rho = \dot{E}_H / \dot{E}_\infty \) (in absolute value) calculated analytically and numerically, as a function of the particle velocity \( v \) for \( n = 0, 1, 2, 3, 4 \) \((D = 4, 5, 6, 7, 8)\) and \( a = 0.2M^{1/(l+n)} \). The straight curves correspond to the analytical formula with \( l = 1 \) and the dots are the numerical results discussed in Sec. V. In the slowly-rotating regime, numerical results are in very good agreement with the analytical formula.](image)

V. NUMERICAL RESULTS

The Green function approach described above can be implemented numerically using standard methods [10, 12, 23]. For given values of \( r_0, a \) and \( n \), we can compute the fluxes by truncating the sum in Eq. (69) to some \( k_{\text{max}}, m_{\text{max}} \) and \( j_{\text{max}} \). As discussed before, for circular orbits only \( j = 0 \) terms give a nonvanishing contribution.

For small and moderately large orbital velocities, the sum converges rapidly even for small truncation orders and we typically set \( k_{\text{max}} = 3 \) and \( m_{\text{max}} = 6 \). However, the convergence is very poor when the orbital velocity approaches the speed of light, i.e. when the orbit is close to the prograde null circular geodesic. Recall that circular orbits around Myers-Perry BHs in higher dimensions are unstable [13] and, in particular there is no innermost stable circular orbit for \( D > 4 \). Thus, for our purposes we
could in principle consider particles in circular orbit up to the light-ring, which exists for any dimension [17]. As the particle approaches the light-ring, the flux is dominated by increasingly higher multipoles, thus affecting the convergence properties of the series (39). For this reason, the plots presented below are extended up to some value of the velocity that guarantees good convergence.

Furthermore, for large orbital velocity and highly spinning BHs, the zeroth order angular eigenfunctions and the corresponding eigenfrequencies (28) might be not accurate. Therefore, when \( \omega \gg 1 \), we have used exact numerical values of \( A_{kmj} \) obtained by solving Eq. (24) with the continued fraction method [22]. We note however that Eq. (28) reproduces the exact results surprisingly well, even when \( \omega \sim 1 \).

We checked our method by reproducing the results of Ref. [12] for the massless case in four dimensions. In addition, we can compute the energy flux in any number of dimensions. The fluxes \( \dot{E}_H \) and \( \dot{E}_\infty \) for \( D = 5 \) and \( D = 6 \) are shown in Tables I and II for \( r_0 = 10r_H \). We show the total flux as well as the first multipolar contributions.

**TABLE I.** Fluxes across the horizon and to infinity for \( n = 1 \) (\( D = 5 \)), \( a = M^{1/(1+n)} \) and \( r_0 = 10r_H \). In the last row we show the total flux obtained summing up to \( k_{\text{max}} = 3 \) and \( m_{\text{max}} = 6 \).

| \( k \) | \( m \) | \( j \) | \( r_0/r_H \) | \( \dot{E}_H (\alpha m_p)^{-2} \) | \( \dot{E}_\infty (\alpha m_p)^{-2} \) | \( |\dot{E}_H|/|\dot{E}_\infty| \) |
|------|------|------|----------------|----------------|----------------|----------------|
| 0    | 0    | 10   | \(-1.4759 \times 10^{-9}\) | \(1.5790 \times 10^{-9}\) | 0.9347         |
| 0    | 2    | 10   | \(-8.3175 \times 10^{-11}\) | \(1.6288 \times 10^{-10}\) | 0.5107         |
| 0    | 3    | 10   | \(-3.1691 \times 10^{-12}\) | \(1.0080 \times 10^{-11}\) | 0.3144         |
| 1    | 1    | 10   | \(-1.2718 \times 10^{-14}\) | \(5.1192 \times 10^{-16}\) | 24.844         |
| \(\sum_{kmj}\) | 10   | \(-3.1248 \times 10^{-9}\) | \(3.5052 \times 10^{-9}\) | 0.8915         |

**TABLE II.** Same as in Table I but for \( n = 2 \) (\( D = 6 \)).

| \( k \) | \( m \) | \( j \) | \( r_0/r_H \) | \( \dot{E}_H (\alpha m_p)^{-2} \) | \( \dot{E}_\infty (\alpha m_p)^{-2} \) | \( |\dot{E}_H|/|\dot{E}_\infty| \) |
|------|------|------|----------------|----------------|----------------|----------------|
| 0    | 0    | 10   | \(-4.1768 \times 10^{-12}\) | \(1.3915 \times 10^{-12}\) | 300.155        |
| 0    | 2    | 10   | \(-3.0159 \times 10^{-13}\) | \(3.8070 \times 10^{-14}\) | 792.180        |
| 0    | 3    | 10   | \(-1.3677 \times 10^{-14}\) | \(4.8651 \times 10^{-15}\) | 2811.35        |
| 1    | 1    | 10   | \(-2.8126 \times 10^{-16}\) | \(1.0497 \times 10^{-22}\) | 2.6796 \times 10^6 |
| \(\sum_{kmj}\) | 10   | \(-8.9858 \times 10^{-12}\) | \(2.8601 \times 10^{-12}\) | 314.176 |

Tables I and II confirm our analytical expectations that the behavior for \( n > 1 \) (\( D > 5 \)) is qualitatively different: the energy flux across the horizon is larger (in modulus) than the flux at infinity. This is shown in Fig. 2, where we compare the flux ratio \( \rho = \dot{E}_H/\dot{E}_\infty \) as a function of the orbital velocity \( v \) for \( a = 0.99 M^{1/(1+n)} \) in various dimensions. This figure is analogous to Fig. 1 but for \( a = 0.99 M^{1/(1+n)} \), i.e. a regime that is not well described by the analytical formula [40]. For \( D = 4 \), we find the usual behavior, i.e. the flux at the horizon is usually negligible with respect to that at infinity and the ratio decreases rapidly at large distance. The case \( D = 5 \) marks a transition, because \( \rho \) is constant at large distance. This is better shown in the left panel of Fig. 3.

On the other hand, for any \( D > 5 \) the flux across the horizon generically dominates over the flux at infinity.

![FIG. 2. The flux ratio \( \rho = \dot{E}_H/\dot{E}_\infty \) (in absolute value) as a function of the particle velocity \( v \) defined in Eq. (11) for \( n = 0, 1, 2, 3, 4 \) (\( D = 4, 5, 6, 7, 8 \)) and \( a = 0.99 M^{1/(1+n)} \).](attachment:image.png)

In Fig. 3, we show the flux ratio \( \rho \) for some selected value of the spin parameter \( a \) in five dimensions (left panel) and in six dimensions (right panel). When \( D = 5 \), the ratio is constant in the small \( v \) region and it approaches unity in the extremal limit, \( a \to \sqrt{2}M \). As shown in the right panel of Fig. 3, when \( D = 6 \) there exist some orbital velocity for which \( -\dot{\rho} = 1 \), corresponding to a total vanishing flux, \( \dot{E}_H + \dot{E}_\infty = 0 \). These orbital frequencies correspond to “floating” orbits [21, 10]. Although in the right panel of Fig. 3 this is shown only for \( a/\sqrt{M^{1/3}} = 0.1, 0.2, 0.3 \), we expect this to be a generic feature also for larger values of the spin. The poor convergence properties of the series (39) prevent us to extend the curves to larger values of \( v \), where floating orbits for \( a > 0.3 M^{1/3} \) are expected to occur. At smaller velocity, the energy flux contribution dominates and the motion of the test-particle is generically dominated by tidal acceleration. Similar results can be obtained for any \( D \geq 6 \).

**VI. CONCLUSIONS**

We computed the rate at which the energy is extracted from a singly-spinning, higher-dimensional BH when a massless scalar field is coupled to a test-particle in circular orbit. We showed that, for dimensions greater than five and small orbital velocities, the energy flux radiated to infinity becomes negligible compared to the energy extracted from the BH via superradiance.

Although we considered scalar-wave emission, we expect our results to be generic in higher dimensions. In particular, superradiance should be a dominant effect also for gravitational radiation. At leading order, the ratio \( |\dot{E}_H|/|\dot{E}_\infty| \) for gravitational radiation should scale
FIG. 3. The ratio $\rho = \dot{E}_H^\text{Tot} / \dot{E}_\infty^\text{Tot}$ as a function of the orbital velocity defined in Eq. (11) for several values of $a$. Left panel: when $D = 5$, the ratio is constant in the small $v$ region and it approaches unity in the extremal limit, $a \to \sqrt{2M}$. Right panel: when $D = 6$, the flux at the horizon can exceed the flux at infinity. For each curve, the intersection with the horizontal line corresponds to a floating orbit, $-\rho = 1$. Note that, at large orbital velocity, the superradiant condition is not met and $\dot{E}_H > 0$.

with the velocity as described by Eq. (70). The dominant quadrupole term ($l = 2$) reads [6]

$$|\dot{E}_H| \sim \frac{v^{-(n-1)(n+5)}}{n+1}.$$  

By comparing the formula above to Eq. (70) with $l = 1$, we note that dipolar effects are dominant over their quadrupolar counterpart. Nevertheless, even in the purely gravitational case, tidal acceleration and floating orbits around spinning BHs are generic and distinctive effects of higher dimensions.

In principle, gravitational waveforms would carry a clear signature of floating orbits [10, 12]. Does floating or these strong tidal effects have any significance in higher-dimensional BH physics? We should start by stressing that circular geodesics in higher dimensions are unstable, on a timescale comparable to the one discussed here [17]; however, our analysis suggests that, while more pronounced for circular orbits, tidal acceleration is generic and in no way dependent on the stability of the orbit under consideration. We are thus led to conjecture that tidal effects are crucial to determine binary evolution in higher dimensions. It is possible that tidal effects already play a role in the numerical simulations of the kind recently reported in Refs. [13, 51, 92], but further study is necessary. One of the consequences of our results for those type of simulations is, for instance, that in higher dimensional BH collisions the amount of gravitational radiation accretion might play an important role. It would certainly be an interesting topic for further study to understand tidal effects for generic orbits, and to include finite-size effects in the calculations.

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