Coassociative 4-folds with Conical Singularities

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1 Introduction

This paper is dedicated to the study of deformations of coassociative 4-folds in a $G_2$ manifold which have conical singularities. Understanding the deformations of such singular coassociative 4-folds should be a useful step towards attempting to prove a 7-dimensional analogue of the SYZ conjecture. The research detailed here is motivated by the work on the deformation theory of special Lagrangian $m$-folds with conical singularities by Joyce in the series of papers [6], [7], [8], [9] and [10], and the work of the author in [15] on deformations of asymptotically conical coassociative 4-folds.

We begin, in Section 2, by discussing the notions of $G_2$ structures, $G_2$ manifolds and coassociative 4-folds. In Section 3 we introduce a distinguished class of singular manifolds known as CS manifolds. CS manifolds have conical singularities and their nonsingular part is a noncompact Riemannian manifold. We also define what we mean by CS coassociative 4-folds.

In order that we may employ various analytic techniques in the course of our study, we choose to use weighted Banach spaces of forms on the nonsingular part of a CS manifold. These spaces are described in §4. We then focus, in Section 5, on a particular linear, elliptic, first-order differential operator acting between weighted Banach spaces in the case of a 4-dimensional CS manifold. The Fredholm and index theory of this operator is discussed using the theory developed in [14].

In Section 6 we stratify the types of deformations allowed into three problems, each with an associated nonlinear first-order differential operator whose kernel gives a local description of the moduli space. The main result for each problem, given in §7, states that the moduli space is locally homeomorphic to the kernel of a smooth map between smooth manifolds. In each case, the map in question can be considered as a projection from the infinitesimal deforma-
tion space onto the obstruction space. Thus, when there are no obstructions the moduli space is a smooth manifold. Furthermore, using the material in § helps to provide a lower bound on the expected dimension of the moduli space.

The last section shows that, in weakening the condition on the $G_2$ structure of the ambient 7-manifold, there is a generic smoothness result for the moduli spaces of deformations corresponding to our second and third problems.

Notes

(a) Manifolds are taken to be nonsingular and submanifolds to be embedded, for convenience, unless stated otherwise.

(b) We use the convention that the natural numbers $N = \{0, 1, 2, \ldots\}$.

2 Coassociative 4-folds

The key to defining coassociative 4-folds lies with the introduction of a distinguished 3-form on $\mathbb{R}^7$.

Definition 2.1 Let $(x_1, \ldots, x_7)$ be coordinates on $\mathbb{R}^7$ and write $dx_{ij\ldots k}$ for the form $dx_i \wedge dx_j \wedge \ldots \wedge dx_k$. Define a 3-form $\varphi_0$ by:

$$
\varphi_0 = dx_{123} + dx_{145} + dx_{167} + dx_{246} - dx_{257} - dx_{347} - dx_{356}.
$$

(1)

The 4-form $\star \varphi_0$, where $\varphi_0$ and $\star \varphi_0$ are related by the Hodge star, is given by:

$$
\star \varphi_0 = dx_{4567} + dx_{2367} + dx_{2345} + dx_{1357} - dx_{1346} - dx_{1256} - dx_{1247}.
$$

(2)

Our choice of expression (1) for $\varphi_0$ follows that of [5, Chapter 10]. This form is sometimes known as the $G_2$ 3-form because the Lie group $G_2$ is the subgroup of $GL(7, \mathbb{R})$ preserving $\varphi_0$.

Definition 2.2 A 4-dimensional submanifold $N$ of $\mathbb{R}^7$ is coassociative if and only if $\varphi_0|_N \equiv 0$ and $\star \varphi_0|_N > 0$.

This definition is not standard but is equivalent to the usual definition in the language of calibrated geometry by [3, Proposition IV.4.5 & Theorem IV.4.6].

Remark The condition $\varphi_0|_N \equiv 0$ forces $\star \varphi_0$ to be a nonvanishing 4-form on $N$. Thus, the positivity of $\star \varphi_0|_N$ is equivalent to a choice of orientation on $N$.

So that we may describe coassociative submanifolds of more general 7-manifolds, we make two definitions following [2, p. 7] and [5, p. 243].
**Definition 2.3** Let $M$ be an oriented 7-manifold. For each $x \in M$ there exists an orientation preserving isomorphism $\iota_x : T_x M \rightarrow \mathbb{R}^7$. Since $\dim G_2 = 14$, $\dim \text{GL}_+(T_x M) = 49$ and $\dim \Lambda^3 T_x^* M = 35$, the $\text{GL}_+(T_x M)$ orbit of $\iota_x^*(\phi_0)$ in $\Lambda^3 T_x^* M$, denoted $\Lambda^3_x T_x^* M$, is open. A 3-form $\phi$ on $M$ is definite, or positive, if $\phi|_{T_x M} \in \Lambda^3_x T_x^* M$ for all $x \in M$. Denote the bundle of definite 3-forms $\Lambda^3 T^* M$. It is a bundle with fibre $\text{GL}_+(7, \mathbb{R})/G_2$ which is not a vector subbundle of $\Lambda^3 T^* M$.

Essentially, a definite 3-form is identified with the $G_2$ 3-form on $\mathbb{R}^7$ at each point in $M$. Therefore, to each definite 3-form $\phi$ we can uniquely associate a 4-form $*\phi$ and a metric $g$ on $M$ such that the triple $(\phi, *\phi, g)$ corresponds to $(\phi_0, *\phi_0, g_0)$ at each point. This leads us to our next definition.

**Definition 2.4** Let $M$ be an oriented 7-manifold, let $\phi$ be a definite 3-form on $M$ and let $g$ be the metric associated to $\phi$. We call $(\phi, g)$ a $G_2$ structure on $M$. If $\phi$ is closed (or coclosed) then $(\phi, g)$ is a closed (or coclosed) $G_2$ structure. A closed and coclosed $G_2$ structure is called torsion-free.

Our choice of notation here agrees with [2].

**Remark** There is a 1-1 correspondence between pairs $(\phi, g)$ and principal $G_2$ subbundles of the frame bundle.

Our definition of torsion-free $G_2$ structure is not standard, but agrees with other definitions by the following result [19, Lemma 11.5].

**Proposition 2.5** Let $(\phi, g)$ be a $G_2$ structure and let $\nabla$ be the Levi–Civita connection of $g$. The following are equivalent:

\[
d\phi = d^* \phi = 0; \quad \nabla \phi = 0; \quad \text{and} \quad \text{Hol}(g) \subseteq G_2 \text{ with } \phi \text{ as the associated 3-form.}
\]

**Definition 2.6** Let $M$ be an oriented 7-manifold endowed with a $G_2$ structure $(\phi, g)$, denoted $(M, \phi, g)$. We say that $(M, \phi, g)$ is a $\phi$-closed, or $\phi$-coclosed, 7-manifold if $(\phi, g)$ is a closed, respectively coclosed, $G_2$ structure. If $(\phi, g)$ is torsion-free, we call $(M, \phi, g)$ a $G_2$ manifold.

We are now able to complete our definitions.

**Definition 2.7** A 4-dimensional submanifold $N$ of $(M, \phi, g)$ is coassociative if and only if $\phi|_N \equiv 0$ and $*\phi|_N > 0$.

We end this section with a result, which follows from [16, Proposition 4.2], that is invaluable in describing the deformation theory of coassociative 4-folds.
Proposition 2.8 Let $N$ be a coassociative 4-fold in $(M, \varphi, g)$. There is an isomorphism between the normal bundle $\nu(N)$ of $N$ in $M$ and $\Lambda^2 T^* N$ given by $v \mapsto (v \cdot \varphi)|_{TN}$.

3 Conical singularities

3.1 CS manifolds

Definition 3.1 Let $M$ be a connected Hausdorff topological space and let $z_1, \ldots, z_s \in M$. Suppose that $\hat{M} = M \setminus \{z_1, \ldots, z_s\}$ has the structure of a (nonsingular) $n$-dimensional Riemannian manifold, with Riemannian metric $g$, compatible with its topology. Then $M$ is a manifold with conical singularities (at $z_1, \ldots, z_s$ with rate $\lambda$) if there exist constants $\epsilon > 0$ and $\lambda > 1$, a compact $(n-1)$-dimensional Riemannian manifold $(\Sigma_i, h_i)$, an open set $U_i \ni z_i$ in $M$ with $U_i \cap U_j = \emptyset$ for $j \neq i$ and a diffeomorphism $\Psi_i : (0, \epsilon) \times \Sigma_i \to U_i \setminus \{z_i\} \subseteq \hat{M}$, for $i = 1, \ldots, s$, such that

$$|\nabla^j_i (\Psi^*_i (g) - g_i) = O(r_i^{\lambda-1-j})$$

for $j \in \mathbb{N}$ as $r_i \to 0$, (3)

where $r_i$ is the coordinate on $(0, \infty)$ on the cone $C_i = (0, \infty) \times \Sigma_i$, $g_i = dr_i^2 + r_i^2 h_i$ is the conical metric on $C_i$, $\nabla_i$ is the Levi-Civita connection derived from $g_i$ and $|.|$ is calculated using $g_i$. We call $C_i$ the cone at the singularity $z_i$ and let the ends $\hat{M}_\infty$ of $\hat{M}$ be the disjoint union

$$\hat{M}_\infty = \bigsqcup_{i=1}^s U_i \setminus \{z_i\}.$$

We say that $M$ is CS or a CS manifold (with rate $\lambda$) if it is a manifold with conical singularities which have rate $\lambda$ and it is compact as a topological space. In these circumstances it may be written as the disjoint union

$$M = K \sqcup \bigsqcup_{i=1}^s U_i,$$

where $K$ is compact as it is closed in $M$.

The condition $\lambda > 1$ guarantees that the metric on $\hat{M}$ genuinely converges to the conical metric on $C_i$, as is evident from (3). Since $M$ is supposed to be Hausdorff, the set $U_i \setminus \{z_i\}$ is open in $\hat{M}$ for all $i$. Moreover, the condition that the $U_i$ are disjoint may be easily satisfied since, if $i \neq j$, $z_i$ and $z_j$ may be separated by two disjoint open sets and, by hypothesis, there are only a finite number of singularities.
Remark If \( M \) is a CS manifold, \( \tilde{M} \) is a \textit{noncompact} manifold.

\textbf{Definition 3.2} Let \( M \) be a CS manifold. Using the notation of Definition 3.1, a \textit{radius function} on \( \tilde{M} \) is a smooth function \( \rho : \tilde{M} \to (0, 1] \), bounded below by a positive constant on \( \tilde{M} \setminus \tilde{M}_\infty \), such that there exist positive constants \( c_1 < 1 \) and \( c_2 > 1 \) with
\[
c_1 r_i < \Psi_i^* (\rho) < c_2 r_i
\]
on \((0, \varepsilon) \times \Sigma_i \) for \( i = 1, \ldots, s \).

If \( M \) is CS we may construct a radius function on \( \tilde{M} \) as follows. Let \( \rho(x) = 1 \) for all \( x \in \tilde{M} \setminus \tilde{M}_\infty \). Define \( \rho_i : \Psi_i((0, \varepsilon/2) \times \Sigma_i) \to (0, 1) \) to be equal to \( r_i/\varepsilon \) for \( i = 1, \ldots, s \) and then define \( \rho \) by interpolating smoothly between its definition on \( \tilde{M} \setminus \tilde{M}_\infty \) and \( \rho_i \) on each of the disjoint sets \( \Psi_i((\varepsilon/2, \varepsilon) \times \Sigma_i) \).

\textbf{3.2 CS coassociative 4-folds}

Let \( B(0; \eta) \) denote the open ball about 0 in \( \mathbb{R}^7 \) with radius \( \eta > 0 \), i.e. \( B(0; \eta) = \{ v \in \mathbb{R}^7 : |v| < \eta \} \). We define a preferred choice of local coordinates on a \( G_2 \) manifold near a finite set of points.

\textbf{Definition 3.3} Let \( (M, \varphi, g) \) be a \( G_2 \) manifold as in Definition 3.1 and let \( z_1, \ldots, z_s \) be points in \( M \). There exist a constant \( \eta > 0 \), an open set \( V_i \ni z_i \) in \( M \) with \( V_i \cap V_j = \emptyset \) for \( j \neq i \) and a diffeomorphism \( \chi_i : B(0; \eta) \subseteq \mathbb{R}^7 \to V_i \) with \( \chi_i(0) = z_i \), for \( i = 1, \ldots, s \), such that \( \zeta_i = d\chi_i|_0 : \mathbb{R}^7 \to T_z M \) is an isomorphism identifying the standard \( G_2 \) structure \((\varphi_0, g_0)\) on \( \mathbb{R}^7 \) with the pair \((\varphi|_{T_{z_i}M}, g|_{T_{z_i}M})\). We call the set \( \{ \chi_i : B(0; \eta) \to V_i : i = 1, \ldots, s \} \) a \( G_2 \) coordinate system near \( z_1, \ldots, z_s \).

We say that two \( G_2 \) coordinate systems near \( z_1, \ldots, z_s \), with maps \( \chi_i \) and \( \tilde{\chi}_i \) for \( i = 1, \ldots, s \) respectively, are \textit{equivalent} if \( d\tilde{\chi}_i|_0 = d\chi_i|_0 = \zeta_i \) for all \( i \).

The definition above is an analogue of the local coordinate system for almost Calabi–Yau manifolds used by Joyce [6, Definition 3.6]. Although the family of \( G_2 \) coordinate systems near \( z_1, \ldots, z_s \) is clearly infinite-dimensional, there are only finitely many equivalence classes, given by the number of possible sets \( \{ \zeta_1, \ldots, \zeta_s \} \). Moreover, the family of choices for each \( \zeta_i \) is isomorphic to \( G_2 \).

\textbf{Note} Definition 3.3 does not require the \( G_2 \) structure \((\varphi, g)\) to be \textit{torsion-free}.

\textbf{Definition 3.4} Let \( (M, \varphi, g) \) be a \( G_2 \) manifold, let \( N \subseteq M \) be compact and connected and let \( z_1, \ldots, z_s \in N \). We say that \( N \) is a 4-fold in \( M \) with \textit{conical singularities} at \( z_1, \ldots, z_s \) with rate \( \lambda \), denoted a \textit{CS 4-fold}, if \( \tilde{N} = N \setminus \{ z_1, \ldots, z_s \} \)
is a (nonsingular) 4-dimensional submanifold of $M$ and there exist constants $0 < \epsilon < \eta$ and $\lambda > 1$, a compact 3-dimensional Riemannian submanifold $(\Sigma_i, h_i)$ of $S^6 \subseteq \mathbb{R}^7$, where $h_i$ is the restriction of the round metric on $S^6$ to $\Sigma_i$, an open set $U_i \ni z_i$ in $N$ with $U_i \subseteq V_i$ and a smooth map $\Phi_i : (0, \epsilon) \times \Sigma_i \to B(0; \eta) \subseteq \mathbb{R}^7$, for $i = 1, \ldots, s$, such that $\Psi_i = \chi_i \circ \Phi_i : (0, \epsilon) \times \Sigma_i \to U_i \setminus \{z_i\}$ is a diffeomorphism and $\Phi_i$ satisfies
\begin{equation}
|\nabla^j_i(\Phi_i(r_i, \sigma_i) - \nu_i(r_i, \sigma_i))| = O(r_i^{\lambda - j}) \quad \text{for } j \in \mathbb{N} \text{ as } r_i \to 0,
\end{equation}
where $\nu_i(r_i, \sigma_i) = r_i \sigma_i \in B(0; \eta)$, $\nabla_i$ is the Levi–Civita connection of the cone metric $g_i = dr_i^2 + r_i^2 h_i$ on $C_i = (0, \infty) \times \Sigma_i$ coupled with partial differentiation on $\mathbb{R}^7$, $|.|$ is calculated with respect to $g_i$ and $\{\chi_i : B(0; \eta) \to V_i : i = 1, \ldots, s\}$ is a $G_2$ coordinate system near $z_1, \ldots, z_s$.

We call $C_i$ the cone at the singularity $z_i$ and $\Sigma_i$ the link of the cone $C_i$. We may write $N$ as the disjoint union

$$N = K \sqcup \bigcup_{i=1}^s U_i,$$

where $K$ is compact.

If $\tilde{N}$ is coassociative in $M$, we say that $N$ is a CS coassociative 4-fold.

Suppose $N$ is a CS 4-fold at $z_1, \ldots, z_s$ with rate $\lambda$ in $(M, \varphi, g)$ and use the notation of Definition 3.4. The induced metric on $\tilde{N}$, $g|_{\tilde{N}}$, makes $\tilde{N}$ into a Riemannian manifold. Moreover, it is clear from 4 that the maps $\Psi_i$ satisfy 3 in Definition 3.1 with the same constant $\lambda$. Thus, $N$ may be considered as a CS manifold with rate $\lambda$.

It is important to note that, if $\lambda \in (1, 2)$, Definition 3.4 is independent of the choice of $G_2$ coordinate system near the singularities, up to equivalence. Suppose we have two equivalent coordinate systems defined using maps $\chi_i$ and $\tilde{\chi}_i$. These maps must agree up to second order since the zero and first order behaviour of each is prescribed, as stated in Definition 3.3. Therefore, the transformed maps $\tilde{\Phi}_i$ corresponding to $\tilde{\chi}_i$ such that $\tilde{\Psi}_i = \tilde{\chi}_i \circ \tilde{\Phi}_i = \chi_i \circ \Phi_i = \Psi_i$ are defined by:

$$\tilde{\Phi}_i = (\tilde{\chi}_i^{-1} \circ \chi_i) \circ \Phi_i.$$

Hence
\begin{equation}
|\nabla^j_i(\tilde{\Phi}_i(r_i, \sigma_i) - \Phi_i(r_i, \sigma_i))| = O(r_i^{-j}) \quad \text{for } j \in \mathbb{N} \text{ as } r_i \to 0,
\end{equation}
where $\nabla_i$ and $|.|$ are calculated as in Definition 3.4. Thus, in order that the terms generated by the transformation of the $G_2$ coordinate system neither dominate nor be of equal magnitude to the $O(r_i^{\lambda-j})$ terms given in 4, we need $\lambda < 2$. 

6
We now make a definition which also depends only on equivalence classes of $G_2$ coordinate systems near the singularities.

**Definition 3.5** Let $N$ be a CS 4-fold at $z_1, \ldots, z_s$ in a $G_2$ manifold $(M, \varphi, g)$. Use the notation of Definitions 3.3 and 3.4. For $i = 1, \ldots, s$ define a cone $\hat{C}_i$ in $T_{z_i}M$ by $\hat{C}_i = (\zeta_i \circ \iota_i)(C_i)$. We call $\hat{C}_i$ the tangent cone at $z_i$.

One can show that $\hat{C}_i$ is a tangent cone to $N$ at $z_i$ in the sense of geometric measure theory (see, for example, [4, p. 233]). We also have a straightforward result relating to the tangent cones at singular points of CS coassociative 4-folds.

**Proposition 3.6** Let $N$ be a CS coassociative 4-fold at $z_1, \ldots, z_s$ in a $G_2$ manifold $(M, \varphi, g)$. The tangent cones at $z_1, \ldots, z_s$ are coassociative.

**Proof**: Use the notation of Definitions 3.3 and 3.4.

It is enough to show that $\iota_i(C_i)$ is coassociative in $\mathbb{R}^7$ for all $i$, since $\zeta_i : \mathbb{R}^7 \to T_{z_i}M$ is an isomorphism identifying $(\varphi_0, g_0)$ with $(\varphi|_{T_{z_i}M}, g|_{T_{z_i}M})$. This is equivalent to the condition $\iota_i^*(\varphi_0) \equiv 0$ for $i = 1, \ldots, s$.

Note that $\varphi|_{\hat{N}} \equiv 0$ implies that, for all $i$, $\varphi|_{U_i \setminus \{z_i\}} \equiv 0$. Hence, $\Psi_i^*(\varphi) = \Phi_i^*(\chi_i^*(\varphi))$ vanishes on $C_i$ for all $i$. Using (4),

$$|\Phi_i^*(\chi_i^*(\varphi)) - \iota_i^*(\chi_i^*(\varphi))| = O(r_i^{\lambda-1}) \quad \text{as } r_i \to 0$$

for all $i$. Moreover,

$$|\iota_i^*(\chi_i^*(\varphi)) - \iota_i^*(\varphi_0)| = O(r_i) \quad \text{as } r_i \to 0$$

since

$$\chi_i^*(\varphi) = \varphi_0 + O(r_i) \quad \text{and} \quad |\nabla \iota_i| = O(1) \quad \text{as } r_i \to 0.$$ 

Therefore, because $\lambda > 1$,

$$|\iota_i^*(\varphi_0)| \to 0 \quad \text{as } r_i \to 0$$

for all $i$. As $T_{r_i, \sigma_i} \iota_i(C_i) = T_{\sigma_i} \iota_i(C_i)$ for all $(r_i, \sigma_i) \in C_i$, $|\iota_i^*(\varphi_0)|$ is independent of $r_i$ and thus vanishes for all $i$ as required. □

### 4 Weighted Banach spaces

For this section let $M$ be an $n$-dimensional CS manifold and let $\hat{M}$ be its nonsingular part as in Definition 3.1. We define *weighted* Banach spaces of forms as in [1] §1, as well as the usual ‘unweighted’ spaces.
**Definition 4.1** Let \( p \geq 1 \) and let \( k, m \in \mathbb{N} \) with \( m \leq n \). The Sobolev space \( L^p_k(\Lambda^m T^* \hat{M}) \) is the set of \( m \)-forms \( \xi \) on \( \hat{M} \) which are \( k \) times weakly differentiable and such that the norm

\[
\| \xi \|_{L^p_k} = \left( \sum_{j=0}^{k} \int_{\hat{M}} |\nabla^j \xi|^p dV_g \right)^{\frac{1}{p}}
\]  

(5)

is finite. The normed vector space \( L^p_k(\Lambda^m T^* \hat{M}) \) is a Banach space for all \( p \geq 1 \) and \( L^2_k(\Lambda^m T^* \hat{M}) \) is a Hilbert space.

We introduce the space of \( m \)-forms \( L^p_k, \text{loc}(\Lambda^m T^* \hat{M}) = \{ \xi : f \xi \in L^p_k(\Lambda^m T^* \hat{M}) \text{ for all } f \in C_\infty^{\text{cs}}(\hat{M}) \} \)

where \( C_\infty^{\text{cs}}(\hat{M}) \) is the space of smooth functions on \( \hat{M} \) with compact support.

Let \( \mu \in \mathbb{R} \) and let \( \rho \) be a radius function on \( \hat{M} \). The weighted Sobolev space \( L^p_k, \mu(\Lambda^m T^* \hat{M}) \) of \( m \)-forms \( \xi \) on \( \hat{M} \) is the subspace of \( L^p_k, \text{loc}(\Lambda^m T^* \hat{M}) \) such that the norm

\[
\| \xi \|_{L^p_k, \mu} = \left( \sum_{j=0}^{k} \int_{\hat{M}} |\rho^j \nabla^j \xi|^p dV_g \right)^{\frac{1}{p}}
\]  

(6)

is finite. Then \( L^p_k, \mu(\Lambda^m T^* \hat{M}) \) is a Banach space and \( L^2_k, \mu(\Lambda^m T^* \hat{M}) \) is a Hilbert space.

We may note here, trivially, that \( L^0_0(\Lambda^m T^* \hat{M}) \) is equal to the standard \( L^p \)-space of \( m \)-forms on \( \hat{M} \). Further, by comparing equations (5) and (6) for the respective norms, \( L^p(\Lambda^m T^* \hat{M}) = L^p_0, -\frac{n}{p}(\Lambda^m T^* \hat{M}) \). In particular,

\[
L^2(\Lambda^m T^* \hat{M}) = L^2_0, -\frac{n}{2}(\Lambda^m T^* \hat{M}).
\]  

(7)

For the following two definitions we take \( C^k_{\text{loc}}(\Lambda^m T^* \hat{M}) \) to be the vector space of \( k \) times continuously differentiable \( m \)-forms.

**Definition 4.2** Let \( \rho \) be a radius function on \( \hat{M} \), let \( \mu \in \mathbb{R} \) and let \( k, m \in \mathbb{N} \) with \( m \leq n \). The weighted \( C^k \)-space \( C^k_{\mu}(\Lambda^m T^* \hat{M}) \) of \( m \)-forms \( \xi \) on \( \hat{M} \) is the subspace of \( C^k_{\text{loc}}(\Lambda^m T^* \hat{M}) \) such that the norm

\[
\| \xi \|_{C^k_{\mu}} = \sum_{j=0}^{k} \sup_{\hat{M}} |\rho^j \nabla^j \xi|
\]

is finite. We also define

\[
C^\infty_{\mu}(\Lambda^m T^* \hat{M}) = \bigcap_{k \geq 0} C^k_{\mu}(\Lambda^m T^* \hat{M}).
\]
Then $C^k_\mu(\Lambda^m T^* \hat{M})$ is a Banach space but in general $C^\infty_\mu(\Lambda^m T^* \hat{M})$ is not.

In the next definition we refer to the usual normed vector space $C^k(\Lambda^m T^* \hat{M})$ of $k$ times continuously differentiable $m$-forms such that the following norm is finite:

$$\|\xi\|_{C^k} = \sum_{j=0}^k \sup_{M} |\nabla^j \xi|.$$  

**Definition 4.3** Let $d(x, y)$ be the geodesic distance between points $x, y \in \hat{M}$ and let $\rho$ be a radius function on $\hat{M}$. Let $a \in (0, 1)$ and let $k, m \in \mathbb{N}$ with $m \leq n$. Let

$$H = \{(x, y) \in \hat{M} \times \hat{M} : x \neq y, c_1 \rho(x) \leq \rho(y) \leq c_2 \rho(x) \text{ and there exists a geodesic in } \hat{M} \text{ of length } d(x, y) \text{ from } x \text{ to } y\},$$

where $0 < c_1 < 1 < c_2$ are constant. A section $s$ of a vector bundle $V$ on $\hat{M}$, endowed with a connection, is Hölder continuous (with exponent $a$) if

$$[s]^a = \sup_{(x, y) \in H} \frac{|s(x) - s(y)|_V}{d(x, y)^a} < \infty.$$  

We understand the quantity $|s(x) - s(y)|_V$ as follows. Given $(x, y) \in H$, there exists a geodesic $\gamma$ of length $d(x, y)$ connecting $x$ and $y$. Parallel translation along $\gamma$ using the connection on $V$ identifies the fibres over $x$ and $y$ and the metrics on them. Thus, with this identification, $|s(x) - s(y)|_V$ is well-defined.

The Hölder space $C^{k, a}(\Lambda^m T^* \hat{M})$ is the set of $\xi \in C^k(\Lambda^m T^* \hat{M})$ such that $\nabla^k \xi$ is Hölder continuous (with exponent $a$) and the norm

$$\|\xi\|_{C^{k, a}} = \|\xi\|_{C^k} + [\nabla^k \xi]^a$$

is finite. The normed vector space $C^{k, a}(\Lambda^m T^* \hat{M})$ is a Banach space.

We also introduce the notation

$$C^{k, a}_{\text{loc}}(\Lambda^m T^* \hat{M}) = \{\xi \in C^{k, a}_{\text{loc}}(\Lambda^m T^* \hat{M}) : f \xi \in C^{k, a}_{\mu}(\Lambda^m T^* \hat{M}) \text{ for all } f \in C^\infty_{\text{cs}}(\hat{M})\}.$$  

Let $\mu \in \mathbb{R}$. The weighted Hölder space $C^{k, a}_{\mu}(\Lambda^m T^* \hat{M})$ of $m$-forms $\xi$ on $\hat{M}$ is the subspace of $C^{k, a}_{\text{loc}}(\Lambda^m T^* \hat{M})$ such that the norm

$$\|\xi\|_{C^{k, a}_{\mu}} = \|\xi\|_{C^k_{\mu}} + [\xi]^k_{\mu}$$

is finite, where

$$[\xi]^k_{\mu} = [\rho^{k+a-\mu} \nabla^k \xi]^a.$$
Then $C^{k, a}_\mu(\Lambda^m T^* \hat{M})$ is a Banach space. It is clear that we have an embedding $C^{k, a}_\mu(\Lambda^m T^* \hat{M}) \hookrightarrow C^l_\mu(\Lambda^m T^* \hat{M})$ whenever $l \leq k$.

We shall need the analogue of the Sobolev Embedding Theorem for weighted spaces, which is adapted from [14, Lemma 7.2] and [1, Theorem 1.2].

**Theorem 4.4 (Weighted Sobolev Embedding Theorem)** Let $p, q \geq 1$, $\alpha \in (0, 1)$, $\mu, \nu \in \mathbb{R}$ and $k, l, m \in \mathbb{N}$ with $m \leq n$.

(a) If $k \geq l$, $k - \frac{n}{p} \geq l - \frac{n}{q}$ and either

(i) $p \leq q$ and $\mu \geq \nu$ or

(ii) $p > q$ and $\mu > \nu$,

there is a continuous embedding $L^p_{k, \mu}(\Lambda^m T^* \hat{M}) \hookrightarrow L^q_{l, \nu}(\Lambda^m T^* \hat{M})$.

(b) If $k - \frac{n}{p} \geq l + \alpha$, there is a continuous embedding $L^p_{k, \mu}(\Lambda^m T^* \hat{M}) \hookrightarrow C^{l + \alpha, a}_\mu(\Lambda^m T^* \hat{M})$.

We shall also require an Implicit Function Theorem for Banach spaces, which follows immediately from [12, Chapter 6, Theorem 2.1].

**Theorem 4.5 (Implicit Function Theorem)** Let $X$ and $Y$ be Banach spaces and let $W \subseteq X$ be an open neighbourhood of 0. Let $G : W \rightarrow Y$ be a $C^k$ map ($k \geq 1$) such that $G(0) = 0$. Suppose further that $dG|_0 : X \rightarrow Y$ is surjective with kernel $K$ such that $X = K \oplus A$ for some closed subspace $A$ of $X$. There exist open sets $V \subseteq K$ and $V' \subseteq A$, both containing 0, with $V \times V' \subseteq W$, and a unique $C^k$ map $V : V' \rightarrow V'$ such that

$$\text{Ker } G \cap (V \times V') = \{(x, V(x)) : x \in V\}$$

in $X = K \oplus A$.

5 The operator $d + d^*$

In this section we let $M$ be a 4-dimensional CS manifold and let $\hat{M}$ be as in Definition 3.1. An essential part of our study is the use of the Fredholm and index theory for the elliptic operator $d + d^*$ acting from $\Lambda^2_+ T^* \hat{M} \oplus \Lambda^4 T^* \hat{M}$ to $\Lambda^3 T^* \hat{M}$. We therefore consider

$$d + d^* : L^p_{k+1, \mu}(\Lambda^2_+ T^* \hat{M} \oplus \Lambda^4 T^* \hat{M}) \rightarrow L^p_{k, \mu-1}(\Lambda^3 T^* \hat{M}),$$

where $p \geq 2$, $k \in \mathbb{N}$ and $\mu \in \mathbb{R}$. 

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5.1 Fredholm theory

Our first result follows from [14, Theorem 1.1 & Theorem 6.1].

**Proposition 5.1** There exists a countable discrete set $D \subseteq \mathbb{R}$ such that \((8)\) is Fredholm if and only if $\mu \notin D$.

Moreover, we can give an explicit description of $D$ by a similar argument to [15, p. 13-14], which is for asymptotically conical (AC) manifolds, as follows.

Recall the notation of Definition 3.1. Transform the metric on $\hat{M}$ to a conformally equivalent metric which is asymptotically cylindrical on the ends $\hat{M}_\infty$ of $\hat{M}$; that is, if $(t_i, \sigma_i)$ are coordinates on $(0, \infty) \times \Sigma_i$, the metric is asymptotic to $dt_i^2 + h_i$. With respect to this new metric, $d + d^*$ corresponds to

$$(d + d^*)_\infty = e^{mt}(d + e^{-2t}d^*)e^{-mt}$$

acting on $m$-forms on $\hat{M}$.

Let

$$\Sigma = \bigcup_{i=1}^s \Sigma_i.$$ 

If $\pi : (0, \infty) \times \Sigma \to \Sigma$ is the natural projection map, the action of $(d + d^*)_\infty$ on $\pi^*(\Lambda^2 T^*\Sigma) \oplus \pi^*(\Lambda^{\text{odd}} T^*\Sigma)$ is:

$$(d + d^*)_\infty = \begin{pmatrix} d + d^* & \frac{\partial}{\partial t} + 3 - m \\ -\frac{\partial}{\partial t} + m & -(d + d^*) \end{pmatrix}$$

(9)

where $m$ denotes the operator which multiplies $m$-forms by a factor $m$. However, we wish only to consider elements of $\Lambda^1 T^*\Sigma \oplus \Lambda^2 T^*\Sigma$ which correspond to self-dual 2-forms on $\hat{M}$, so we define $V_\Sigma \subseteq \Lambda^2 T^*\hat{M} \oplus \Lambda^4 T^*\hat{M}$ by

$$V_\Sigma = \bigcup_{i=1}^s \{(\alpha, *\alpha + \beta) : \alpha \in \Lambda^2 T^*\Sigma_i, \beta \in \Lambda^3 T^*\Sigma_i\}.$$

Then $\pi^*(V_\Sigma)$ corresponds to $\Lambda^2 T^*\hat{M} \oplus \Lambda^4 T^*\hat{M}$.

For $w \in \mathbb{C}$ define a map $(d + d^*)_\infty(w)$ by:

$$(d + d^*)_\infty(w) = \begin{pmatrix} d + d^* & -w + 3 - m \\ w - m & -(d + d^*) \end{pmatrix}$$

(10)

acting on $V_\Sigma \otimes \mathbb{C}$. Notice that we have formally substituted $w$ for $-\frac{\partial}{\partial t}$ in (9).

Let

$$W_\Sigma = \bigcup_{i=1}^s \{(*\alpha + \beta, \alpha) : \alpha \in \Lambda^2 T^*\Sigma_i, \beta \in \Lambda^3 T^*\Sigma_i\} \subseteq \Lambda^{\text{odd}} T^*\Sigma \oplus \Lambda^2 T^*\Sigma.$$
Define $C \subseteq \mathbb{C}$ as the set of $w$ for which the map 
\[
(d + d^*)_{\infty}(w) : L^p_{k+1}(V \otimes \mathbb{C}) \rightarrow L^p_{k}(W \otimes \mathbb{C})
\]
is not an isomorphism. By the proof of [14, Theorem 1.1], $\mathcal{D} = \{\text{Re } w : w \in C\}$. By [17, Lemma 6.1.13], the corresponding sets $\mathcal{C}((\Delta^m))$, where $\Delta^m$ is the Laplacian on $m$-forms, are all real for an asymptotically conical manifold. Since the same will be true for the CS case, we deduce that $C \subseteq \mathbb{R}$. Hence $C = \mathcal{D}$.

The symbol, hence the index $\text{ind}_w$, of $(d + d^*)_{\infty}(w)$ is independent of $w$.

Furthermore, $(d + d^*)_{\infty}(w)$ is an isomorphism for generic values of $w$ since $\mathcal{D}$ is countable and discrete. Therefore $\text{ind}_w = 0$ for all $w \in C$; that is,
\[
\dim \text{Ker}(d + d^*)_{\infty}(w) = \dim \text{Coker}(d + d^*)_{\infty}(w),
\]
so that (10) is not an isomorphism precisely when it is not injective.

The condition $(d + d^*)_{\infty}(w) = 0$, using (10), corresponds to the existence of $\alpha \in C^\infty(L^2 T^* \Sigma_i)$ and $\beta \in C^\infty(L^3 T^* \Sigma_i)$, for some $i$, satisfying
\[
d\alpha = w\beta \quad \text{and} \quad d\ast \alpha + d^\ast \beta = (w - 2)\alpha.
\]

Notes

(a) The equations above imply that
\[
\text{dd}^* \beta = \Delta \beta = w(w - 2)\beta.
\]

Since eigenvalues of the Laplacian on $\Sigma_i$ must necessarily be positive, $\beta = 0$ if $w \in (0, 2)$.

(b) If $w = 0$ and we take $\alpha = 0$, (11) forces $\beta$ to be coclosed. As there are nontrivial coclosed 3-forms on $\Sigma_i$, $(d + d^*)_{\infty}(0)$ is not injective, so $0 \in \mathcal{D}$.

(c) Suppose that $w = 2$ lies in $\mathcal{D}$. Then (11) gives $[\beta] = 0$ in $H^3_{\text{DR}}(\Sigma_i)$. We know that $\beta$ is harmonic so, by Hodge theory, $\beta = 0$. Therefore $2 \in \mathcal{D}$ if and only if there exists a nonzero closed and coclosed 2-form on $\Sigma_i$ for some $i$.

We state a proposition which follows from the work above.

Proposition 5.2 Let $M$ be a 4-dimensional CS manifold. Use the notation of Definition 3.1. For $i = 1, \ldots, s$ let $D(\mu, i) = \{(\alpha, \beta) \in C^\infty(L^2 T^* \Sigma_i \oplus L^3 T^* \Sigma_i) : d\alpha = \mu \beta, \ d\ast \alpha + d^\ast \beta = (\mu - 2)\alpha\}$. The set $\mathcal{D}$ of real numbers $\mu$ such that $\mathcal{S}$ is not Fredholm is given by:
\[
\mathcal{D} = \bigcup_{i=1}^{s}\{\mu \in \mathbb{R} : D(\mu, i) \neq 0\}.
\]
Remark A perhaps more illuminating way to characterise $D(\mu, i)$ is by:

$$(\alpha, \beta) \in D(\mu, i) \iff \xi = (r^{\mu-2}\alpha + r^{\mu-1}dr \wedge \ast \alpha, r^{\mu-3}dr \wedge \beta)$$

is an $O(r^\mu)$ solution of $(d + d^*)\xi = 0$ on $C_i$,

using the notation of Definition 3.1.

Lockhart and McOwen [14, §10] study the Laplacian on $m$-forms on a manifold with a conical singularity. From this work, which can easily be extended to manifolds with more than one singularity, we can make an important observation about the set $\mathcal{D}$.

Proposition 5.3 In the notation of Proposition 5.2, $\mathcal{D} \cap (-2, -1] = \emptyset$.

Proof: Let

$$\Delta^m : L^p_{k+1, \mu}(\Lambda^m T^* \hat{M}) \to L^p_{k-1, \mu-2}(\Lambda^m T^* \hat{M})$$

be the Laplacian on $m$-forms and denote the set of $\mu$ such that it is not Fredholm by $\mathcal{D}(\Delta^m)$. Since $\mu > -2$ and $p \geq 2$ we see that $L^p_{k+1, \mu} \hookrightarrow L^2_{0, -2} = L^2$ by Theorem 4.4 and (7).

We then apply [14, Theorem 10.2] for the Laplacian on 2-forms and 4-forms on a 4-dimensional CS manifold to see that

$$\mathcal{D}(\Delta^2) \cap (-2, -1] = \mathcal{D}(\Delta^4) \cap (-2, -1] = \emptyset.$$ 

Note that our rate $\mu$ is related to the weighting factor in [14, §10], which we may denote as $\nu$, by $\mu = -\nu - 2$. As it is clear that $\mathcal{D} \subseteq (\mathcal{D}(\Delta^2) \cup \mathcal{D}(\Delta^4))$, the result follows. □

5.2 Index theory

We begin with some definitions following [14].

Definition 5.4 Use the notation of §5.1. Let $\mu \in \mathcal{D}$. Define $d(\mu)$ to be the dimension of the vector space of solutions of $(d + d^*)\infty\xi = 0$ of the form

$$\xi(t, \sigma) = e^{-\mu t}p(t, \sigma)$$

where $p(t, \sigma)$ is a polynomial in $t \in (0, \infty)$ with coefficients in $C^\infty(\Sigma \otimes \mathbb{C})$.

The next result is immediate from [14, Theorem 1.2].
Theorem 5.5 Let $\lambda, \lambda' \not\in D$ with $\lambda' \leq \lambda$, where $D$ is given in Proposition 5.2. For any $\mu \not\in D$ let $\text{ind}_\mu(d + d^*)$ denote the Fredholm index of $[\mathfrak{u}]$. Then

$$\text{ind}_{\lambda'}(d + d^*) - \text{ind}_{\lambda}(d + d^*) = \sum_{\mu \in D \cap (\lambda', \lambda)} d(\mu).$$

We make a key observation, which shall be used on a number of occasions in later sections.

Proposition 5.6 Let $\lambda, \lambda' \in \mathbb{R}$ such that $\lambda' \leq \lambda$ and $[\lambda', \lambda] \cap D = \emptyset$. The kernels, and cokernels, of $[\mathfrak{u}]$ when $\mu = \lambda$ and $\mu = \lambda'$ are equal.

Proof: Denote the dimensions of the kernel and cokernel of $[\mathfrak{u}]$, for $\mu \not\in D$, by $k(\mu)$ and $c(\mu)$ respectively. Since $[\lambda', \lambda] \cap D = \emptyset$, $k(\lambda) - c(\lambda) = k(\lambda') - c(\lambda')$ and hence

$$k(\lambda) - k(\lambda') = c(\lambda) - c(\lambda').$$

We know that $k(\lambda) \leq k(\lambda')$ because $L^p_{k+1, \lambda} \hookrightarrow L^p_{k+1, \lambda'}$ by Theorem 4.4 as $\lambda \geq \lambda'$. Similarly, since $c(\mu)$ is equal to the dimension of the kernel of the formal adjoint operator acting on a Sobolev space with weight $-3 - \mu$, $c(\lambda) \geq c(\lambda')$. Noting that the right-hand side of (12) is non-negative and the left-hand side is less than or equal to zero, we conclude that both must be zero. The result follows from the fact that the kernel of $d + d^*$ in $L^p_{k+1, \lambda}$ is contained in the kernel of $d + d^*$ in $L^p_{k+1, \lambda'}$, and vice versa for the cokernels. □

We can now go further and give a more explicit description of the quantity $d(\mu)$ in Definition 5.4.

Proposition 5.7 Using the notation of Proposition 5.2 and Definition 5.4,

$$d(\mu) = \sum_{i=1}^s \dim D(\mu, i) \text{ for } \mu \in D.$$

This is an analogue of [15, Proposition 5.4], which is for the AC scenario, and can be proved in exactly the same manner.

6 The deformation problems

We have a common notation for the next three sections. Let $N$ be a CS coassociative 4-fold at $z_1, \ldots, z_s$ with rate $\lambda$ in a $G_2$ manifold $(M, \varphi, g)$. Suppose $\lambda \in (1, 2) \setminus D$, where $D$ is defined in Proposition 5.2 and the cone at $z_i$ is $C_i$ with link $\Sigma_i$. We shall then use the notation of Definitions 3.4 and 3.5. In particular, we let $\{\chi_i : B(0; \eta) \to V_i : i = 1, \ldots, s\}$, with $d\chi_i|_0 = \zeta_i$ for all $i$. 

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be the $G_2$ coordinate system near $z_1, \ldots, z_s$ used to define $N$ and let $\mathcal{C}_i$ be the tangent cone at $z_i$. Recalling that $N$ is a CS manifold, in the sense of Definition 3.1, we have a radius function $\rho$ on $\hat{N}$ as in Definition 3.2.

We consider deformations of $N$ which are CS coassociative 4-folds at $s$ points with rate $\lambda$ in $(M, \varphi, g)$ with the same cones at the singularities as $N$, but the singularities need not be at the same points, nor have identical tangent cone. We also, eventually, consider deforming the $G_2$ structure on the ambient 7-manifold $M$.

6.1 Problem 1: fixed singularities and $G_2$ structure

The first deformation problem we consider is where the deformations of $N$ have identical singular points to $N$ with the same rate, cones and tangent cones, and the $G_2$ structure of $M$ is fixed.

**Definition 6.1** The moduli space of deformations $\mathcal{M}_1(N, \lambda)$ for Problem 1 is the set of $N'$ in $(M, \varphi, g)$ which are CS coassociative 4-folds at $z_1, \ldots, z_s$ with rate $\lambda$, having cone $C_i$ and tangent cone $\hat{C}_i$ at $z_i$ for all $i$, such that there exists a homeomorphism $h : N \to N'$, isotopic to the identity, with $h(z_i) = z_i$ for $i = 1, \ldots, s$ and such that $h|_{\hat{N}} : \hat{N} \to N' \setminus \{z_1, \ldots, z_s\}$ is a diffeomorphism.

We begin our formulation of a local description of $\mathcal{M}_1(N, \lambda)$ with a result which is immediate from the proof of [11, Chapter IV, Theorem 9] since $M$ is a Riemannian manifold.

**Theorem 6.2** Let $P$ be a closed embedded submanifold of $M$. There exist an open subset $V$ of the normal bundle $\nu(P)$ of $P$ in $M$, containing the zero section, and an open set $S$ in $M$ containing $P$, such that the exponential map $\exp|_V : V \to S$ is a diffeomorphism.

**Note** The proof of this result relies entirely on the observation that $\exp|_{\nu(P)}$ is a local isomorphism upon the zero section.

This information provides us with a useful corollary.

**Corollary 6.3** For $i = 1, \ldots, s$ choose $\Phi_i : (0, \varepsilon) \times \Sigma_i \to B(0; \eta) \subseteq \mathbb{R}^7$ uniquely by imposing the condition that

$$\Phi_i(r_i, \sigma_i) - \iota_i(r_i, \sigma_i) \in (T_{r_i, \sigma_i} \iota_i(C_i))^{\perp}$$

for all $(r_i, \sigma_i) \in (0, \varepsilon) \times \Sigma_i$, which can be achieved by making $\varepsilon$ smaller and $K$ larger if necessary. Let $P_i = \iota_i((0, \varepsilon) \times \Sigma_i)$, $Q_i = \Phi_i((0, \varepsilon) \times \Sigma_i)$ and define
Moreover, to the metric $\chi$ of $\Phi_i$ along the cone $C_i$ such that $n_i|_{\hat{V}_i} : \hat{V}_i \to \hat{S}_i$ is a diffeomorphism. We can ensure that $\hat{S}_i$ lies in $B(0; \eta)$ by making $\hat{V}_i$ smaller if necessary.

Furthermore, since $\Phi_i - \iota_i$ is orthogonal to $(0, \epsilon) \times \Sigma_i$, it can be identified with a small section of the normal bundle and hence $P_i$ lies in $\hat{S}_i$ as long as $\hat{S}_i$ grows at $O(r_i)$ as $r_i \to 0$. As we can form $\hat{S}_i$ and $\hat{V}_i$ in a translation equivariant way because we are working on a portion of the cone $C_i$, we can construct our sets with this decay rate as $r_i \to 0$ and such that they do not collapse as $r_i \to \epsilon$.

\[ \square \]

Corollary 6.3 helps us in establishing the next proposition.

**Proposition 6.4** There exist an open set $\hat{U} \subseteq \Lambda^2_+ T^* \hat{N}$ containing the zero section, an open set $\hat{T} \subseteq M$ containing $\hat{N}$ and a diffeomorphism $\delta : \hat{U} \to \hat{T}$ which takes the zero section to $\hat{N}$. Moreover, $\hat{U}$ and $\hat{T}$ can be chosen to grow with order $O(\rho)$ as $\rho \to 0$ and $\delta$ is compatible with the identifications $U_i \{ z_i \} \cong (0, \epsilon) \times \Sigma_i$ for all $i$ and the isomorphism $\nu : \nu(\hat{N}) \to \Lambda^2_+ T^* \hat{N}$ given in Proposition 2.

\[ \square \]

**Proof:** Use the notation of Corollary 6.3 and define $\hat{T}_i = \chi_i(\hat{S}_i)$. Then $\hat{T}_i$ is an open set in $M$ such that $U_i \{ z_i \} \subseteq \hat{T}_i \subseteq \hat{V}_i$, since $\chi_i(Q_i) = U_i \{ z_i \}$, and which grows with order $O(\rho)$ as $\rho \to 0$.

Consider the bundle $(\Lambda^2_+)^{(0, \epsilon) \times \Sigma_i}_h T^*((0, \epsilon) \times \Sigma_i)$, where the notation $\Lambda^2_+ h$ indicates that the Hodge star is calculated using the metric $h$ and we consider $(0, \epsilon) \times \Sigma_i \cong P_i \subseteq \mathbb{R}^7$. Then

\[ j_i : \nu(P_i) \longrightarrow (\Lambda^2_+)^{(0, \epsilon) \times \Sigma_i}_h T^* P_i \]

\[ v|_{r_i, \sigma_i} \longmapsto (v|_{r_i, \sigma_i} \cdot \chi^*_i(\varphi)|_{\Phi_i(r_i, \sigma_i)}) |_{T_{r_i, \sigma_i} P_i} \]

is an isomorphism because $U_i \{ z_i \}$ is coassociative and thus $P_i$ is, with respect to the metric $\chi^*_i(g)$ and 3-form $\chi^*_i(\varphi)$, and hence we may apply Proposition 2.
Note also that

$$\Psi^\ast_i : (\Lambda^2_+ g T^* (U_i \setminus \{z_i\})) \longrightarrow (\Lambda^2_+ \chi^\ast_i g T^* ((0, \epsilon) \times \Sigma_i))$$

is clearly a diffeomorphism. Therefore, let \( \hat{U}_i \subseteq (\Lambda^2_+ g T^* (U_i \setminus \{z_i\})) \) be such that \( \Psi^\ast_i (\hat{U}_i) = \gamma_i (\hat{V}_i) \). Note, by construction, that \( \hat{U}_i \) grows with order \( O(\rho) \) as \( \rho \to 0 \).

Define a diffeomorphism \( \delta_i : \hat{U}_i \to \hat{T}_i \) such that the following diagram commutes:

$$\begin{array}{ccc}
\hat{U}_i & \xrightarrow{\Psi^\ast_i} & j_i (\hat{V}_i) \\
\downarrow{\delta_i} & & \downarrow{j_i^{-1}} \\
\hat{T}_i & \xleftarrow{\chi_i} & S_i.
\end{array} \quad (13)
$$

Interpolating smoothly over \( K \), we extend \( \bigcup_{i=1}^n \hat{U}_i \) and \( \bigcup_{i=1}^n \hat{T}_i \) to \( \hat{U} \) and \( \hat{T} \) as required and extend the diffeomorphisms \( \delta_i \) smoothly to a diffeomorphism \( \delta : \hat{U} \to \hat{T} \) such that \( \delta \) acts as the identity on \( \hat{N} \), which is identified with the zero section in \( \Lambda^2_+ T^* \hat{N} \).

Note that we have a splitting \( T\hat{U}|_{(x,0)} = T_x \hat{N} \oplus \Lambda^2_+ T^*_x \hat{N} \) for all \( x \in \hat{N} \). Thus we can consider \( d\delta \) at \( \hat{N} \) as a map from \( T\hat{N} \oplus \Lambda^2_+ T^* \hat{N} \) to \( T\hat{N} \oplus \nu(\hat{N}) \cong TM|_{\hat{N}} \).

Hence, we require in our extension of \( \delta_i \) to \( \delta \) to ensure that, in matrix notation,

$$d\delta|_{\hat{N}} = \begin{pmatrix} I & A \\ 0 & j^{-1} \end{pmatrix}, \quad (14)$$

where \( I \) is the identity and \( A \) is arbitrary. This can be achieved because of the definition of \( \delta_i \).

The compatibility of \( \delta \) with \( j \) and \( \Psi_i \) for all \( i \), mentioned in the statement of the proposition, is given by \( [13] \) and the behaviour of \( d\delta|_{\hat{N}} \) stipulated in \( [14] \).

We now define our deformation map for Problem 1. Let \( C^k_{\text{loc}}(\hat{U}) = \{ \alpha \in C^k_{\text{loc}}(\Lambda^2_+ T^* \hat{N}) : \alpha \in \hat{U} \} \), where \( \hat{U} \) is given in Proposition 6.3 and adopt similar notation to define subsets of the spaces of forms described in \( \S \).

**Definition 6.5** Use the notation of Proposition 6.3. Let \( \Gamma_\alpha \) be the graph of \( \alpha \in C^1_{\text{loc}}(\hat{U}) \) and let \( \pi_\alpha : \hat{N} \to \Gamma_\alpha \) be given by \( \pi_\alpha(x) = (x, \alpha(x)) \). Let \( f_\alpha = \delta \circ \pi_\alpha \) and let \( \hat{N}_\alpha = f_\alpha(\hat{N}) \subseteq \hat{T} \). Define a map \( F_1 \) from \( C^1_{\text{loc}}(\hat{U}) \) to \( C^0_{\text{loc}}(\Lambda^3 T^* \hat{N}) \) by:

$$F_1(\alpha) = f_\alpha^\ast \left( \varphi|_{\hat{N}_\alpha} \right).$$

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By \cite[p. 731]{16}, which we are allowed to use by our choice of $\delta$, the linearisation of $F_1$ at 0 is
\[ dF_1|_0(\alpha) = L_1(\alpha) = d\alpha \]
for all $\alpha \in C^1_{\text{loc}}(\Lambda^2_+ T^* \hat{N})$.

**Remark** The operator $L_1$ is not elliptic.

By Proposition\cite[24]{24} $F_1$ is the set of $\alpha \in C^1_{\text{loc}}(\hat{U})$ such that $\hat{N}_\alpha$ is coassociative.

However, we want CS coassociative deformations with singularities at the same points with the same tangent cones. Suppose $\alpha \in C^1_{\text{loc}}(\hat{U})$ and $N_\alpha = \hat{N}_\alpha \cup \{z_1, \ldots, z_s\}$ is such a deformation. Then there exist smooth maps $(\Phi_\alpha)_i : (0, \epsilon) \times \Sigma_i \to B(0; \eta)$ satisfying \cite{24} such that $(\Psi_\alpha)_i = \chi_i \circ (\Phi_\alpha)_i$ is a diffeomorphism onto an open subset of $\hat{N}_\alpha$ for all $i$ as in Definition\cite[3.4]{3.4} Note that we are free to use $\chi_i$ because the tangent cones at the singularities of $N_\alpha$ must be the same as for $N$, so any $G_2$ coordinate system near the singularities used to define $N_\alpha$ must be equivalent to the one given by $\chi_i$ for $i = 1, \ldots, s$. Choose $(\Phi_\alpha)_i$ uniquely such that
\[ (\Phi_\alpha)_i(r_i, \sigma_i) - \iota_i(r_i, \sigma_i) \in (T_{r_i, \sigma_i} \iota_i(C_i)) \]
for all $(r_i, \sigma_i) \in (0, \epsilon) \times \Sigma_i$.

Use the notation of Corollary\cite[6.3]{6.3} and the proof of Proposition\cite[6.4]{6.4} Since
\[ \Phi_i(r_i, \sigma_i) - \iota_i(r_i, \sigma_i) \in (T_{r_i, \sigma_i} P_i) \simeq \nu_{r_i, \sigma_i}(P_i), \]
$\Phi_i - \iota_i$ can be identified using $j_i$ with the graph of $\beta_i \in (\Lambda^2_+ \chi^i_\ast(g) T^*((0, \epsilon) \times \Sigma_i))$. Thus,
\[ |\nabla^j_i \beta_i| = O(r_i^{\lambda_j - i}) \quad \text{for } j \in \mathbb{N} \text{ as } r_i \to 0 \quad (15) \]
by \cite{24} and therefore $\beta_i \in C^\infty((\Lambda^2_+ \chi^i_\ast(g) T^*((0, \epsilon) \times \Sigma_i))$.

We may similarly deduce, by the definition of $\delta$, $\Phi_i$ and $(\Phi_\alpha)_i$, that $(\Phi_\alpha)_i - \iota_i = ((\Phi_\alpha)_i - \Phi_i) + (\Phi_i - \iota_i)$ corresponds to the graph of $\Psi_i^\ast(\alpha) + \beta_i$ on $(0, \epsilon) \times \Sigma_i$, recalling that
\[ \Psi_i^\ast : \Lambda^2_+ T^*(U_i \setminus \{z_i\}) \to (\Lambda^2_+ \chi^i_\ast(g) T^*((0, \epsilon) \times \Sigma_i)) \]
is a diffeomorphism for all $i$. Since $N_\alpha$ has the same types of singularities as $N$, both $\beta_i$ and $\Psi_i^\ast(\alpha) + \beta_i$ lie in $C^\infty((\Lambda^2_+ \chi^i_\ast(g) T^*((0, \epsilon) \times \Sigma_i))$ for each $i$. Thus $\alpha$ must lie in $C^\infty((\Lambda^2_+ T^* \hat{N})$.

We conclude that $\hat{N}_\alpha$ is a sufficiently nearby deformation of $\hat{N}$ with the same conical singularities if and only if $\alpha \in C^\infty(\hat{U}) \subseteq C^\infty((\Lambda^2_+ T^* \hat{N})$. We state this as a proposition.
Proposition 6.6 The moduli space of deformations for Problem 1 is locally homeomorphic to $\text{Ker} F_1 \equiv \{ \alpha \in C^∞_Λ(\hat{U}) : F_1(\alpha) = 0 \}$.

We define an associated map $G_1$ to $F_1$.

Definition 6.7 Define $G_1 : C^1_{\text{loc}}(\hat{U}) \times C^1_{\text{loc}}(Λ^4T^*\hat{N}) \to C^0_{\text{loc}}(Λ^3T^*\hat{N})$ by:

$$G_1(\alpha, \beta) = F_1(\alpha) + d^*\beta.$$  

Then $G_1$ is a first order elliptic operator at $(0, 0)$ since

$$dG_1|_{(0,0)} = d + d^* : C^1_{\text{loc}}(Λ^2_N^*\hat{N} \oplus Λ^4T^*\hat{N}) \to C^0_{\text{loc}}(Λ^3T^*\hat{N}).$$

Note If $G_1(\alpha, \beta) = 0$ and $\beta \in C^∞_Λ(Λ^4T^*\hat{N})$, $*\beta$ is a harmonic function which decays with order $O(\rho^λ)$ as $ρ \to 0$. Since $\lambda > 1$, $*\beta \to 0$ as $ρ \to 0$ and hence, by the Maximum Principle for harmonic functions, it must be 0.

We therefore deduce the following.

Proposition 6.8 Ker $F_1 \equiv \{ (\alpha, \beta) \in C^∞_Λ(\hat{U}) \times C^∞(Λ^4T^*\hat{N}) : G_1(\alpha, \beta) = 0 \}$.

We conclude this subsection by stating and proving two results on regularity which are analogous to [15] Proposition 4.3 and the argument in [15] p. 22-24 respectively.

Proposition 6.9 The map $F_1$ given in Definition 6.6 can be written as

$$F_1(\alpha)(x) = da(x) + P_{F_1}(x, \alpha(x), \nabla \alpha(x)) \quad (16)$$

for $x \in \hat{N}$, where $P_{F_1} : \{(x, y, z) : (x, y) \in \hat{U}, z \in T^*_x\hat{N} \oplus Λ^2_T^*\hat{N}\} \to Λ^P^*\hat{N}$ is a smooth map such that $P_{F_1}(x, y, z) \in Λ^P^*\hat{N}$. For $\alpha \in C^∞_Λ(\hat{U})$ with $\|\alpha\|_{C^1_1}$ sufficiently small, denoting $P_{F_1}(x, \alpha(x), \nabla \alpha(x)) = P_{F_1}(\alpha)(x)$, $P_{F_1}(\alpha) \in C^∞_Σ(Λ^3T^*\hat{N}) \subseteq C^∞_∀(Λ^3T^*\hat{N})$, as $\lambda > 1$. Moreover, for each $k \in \mathbb{N}$, if $\alpha \in C^k_Λ(\hat{U})$ and $\|\alpha\|_{C^1_1}$ is sufficiently small, $P_{F_1}(\alpha) \in C^{k}_Σ(Λ^3T^*\hat{N})$ and there exists a constant $c_k > 0$ such that

$$\|P_{F_1}(\alpha)\|_{C^{k-2}_Σ} \leq c_k \|\alpha\|^2_{C^{k+1}_Σ}.$$  

Proof: Firstly, by the definition of $F_1$, $F_1(\alpha)(x)$ relates to the tangent space to $Γ_α$ at $π_α(x)$. Note that $T_{π_α(x)}Γ_α$ depends on both $α(x)$ and $\nabla α(x)$ and hence so must $F_1(\alpha)(x)$. We may then define $P_{F_1}$ by [16] such that it is a smooth function of its arguments as claimed.
We argued above that we may identify \( \Phi_i - \iota_i \) on \((0, \epsilon) \times \Sigma_i \) with
\[
\beta_i \in C^\infty((\Lambda^2_\epsilon) \chi^*_i g) T^*((0, \epsilon) \times \Sigma_i))
\]
for \( i = 1, \ldots, s \). Recall that
\[
\Psi^*_i : \Lambda^2_\epsilon T^*(U_i \setminus \{z_i\}) \to (\Lambda^2_\epsilon) \chi^*_i g) T^*((0, \epsilon) \times \Sigma_i)
\]
is a diffeomorphism. Let \( k \in \mathbb{N}, \alpha \in C^{k+1}(\hat{U}), \alpha_i = \alpha|_{U_i \setminus \{z_i\}} \) and \( \gamma_i = \Psi^*_i(\alpha_i) \).

For each \( i \), define a function \( F_{C_i}(\gamma_i + \beta_i) \) on \((0, \epsilon) \times \Sigma_i \) by
\[
F_{C_i}(\gamma_i + \beta_i)(r_i, \sigma_i) = F_i(\alpha_i)(\Psi_i(r_i, \sigma_i)).
\]
(17)

Define a smooth function \( P_{C_i} \) by an equation analogous to (16):
\[
F_{C_i}(\gamma_i + \beta_i)(r_i, \sigma_i) = d(\gamma_i + \beta_i)(r_i, \sigma_i)
+ P_{C_i}(r_i, \sigma_i, (\gamma_i + \beta_i)(r_i, \sigma_i), \nabla(\gamma_i + \beta_i)(r_i, \sigma_i)).
\]
(18)

We notice that \( F_{C_i} \) and \( P_{C_i} \) are only dependent on the cone \( C_i \) and, rather trivially, on \( \epsilon \). Therefore, because of this fact and our choice of \( \delta \) in Proposition 5.3, these functions have scale equivariance properties. We may therefore derive equations and inequalities on \( \{\epsilon\} \times \Sigma_i \) and deduce the result on all of \((0, \epsilon) \times \Sigma_i \) by introducing an appropriate scaling factor of \( r \).

Now, since \( \alpha = 0 \) corresponds to our coassociative 4-fold \( \hat{N} \), \( F_{I}(0) = 0 \). So, by (17),
\[
F_{C_i}(\beta_i) = d\beta_i + P_{C_i}(\beta_i) = 0,
\]
(19)

adopting similar notation for \( P_{C_i}(\beta_i) \) as for \( F_{I}(\alpha_i) \). Using (16)-(19), we deduce that
\[
P_{F_i}(\alpha_i) = d\beta_i + P_{C_i}(\gamma_i + \beta_i) = d\beta_i + P_{C_i}(\gamma_i + \beta_i) - (d\beta_i + P_{C_i}(\beta_i))
= P_{C_i}(\gamma_i + \beta_i) - P_{C_i}(\beta_i).
\]
(20)

We then calculate
\[
P_{C_i}(\gamma_i + \beta_i) - P_{C_i}(\beta_i) = \int_0^1 \frac{d}{dt} P_{C_i}(t\gamma_i + \beta_i) \, dt
= \int_0^1 \gamma_i \cdot \frac{\partial P_{C_i}}{\partial y}(t\gamma_i + \beta_i) + \nabla\gamma_i \cdot \frac{\partial P_{C_i}}{\partial z}(t\gamma_i + \beta_i) \, dt,
\]
(21)

recalling that \( P_{C_i} \) is a function of three variables \( x, y \) and \( z \). Using Taylor’s Theorem,
\[
P_{C_i}(\gamma_i + \beta_i) = P_{C_i}(\beta_i) + \gamma_i \frac{\partial P_{C_i}}{\partial y}(\beta_i) + \nabla\gamma_i \cdot \frac{\partial P_{C_i}}{\partial z}(\beta_i) + O(r^{-2}||\gamma_i||^2 + ||\nabla\gamma_i||^2)
\]
(22)
when $|\gamma_i|$ and $|\nabla \gamma_i|$ are small. Since $dF_1|_0(\alpha_i) = d\alpha_i$, $dF_C,|_{\beta_i}(\gamma_i + \beta_i) = d\gamma_i$ and hence $dP_{C,}|_t \beta_i = 0$. Thus, the first derivatives of $P_{C,}$ with respect to $y$ and $z$ must vanish at $\beta_i$ by (22). Therefore, given small $\nu > 0$ there exists a constant $A_0 > 0$ such that

$$\left| \frac{\partial P_{C,}(t\gamma_i + \beta_i)}{\partial y} \right| \leq A_0(r^{-2}|\gamma_i| + r^{-1}|\nabla \gamma_i|); \quad \text{and}$$

$$\left| \frac{\partial P_{C,}(t\gamma_i + \beta_i)}{\partial z} \right| \leq A_0(r^{-1}|\gamma_i| + |\nabla \gamma_i|)$$

for $t \in [0, 1]$ whenever

$$r^{-1}|\gamma_i|, r^{-1}|\beta_i|, |\nabla \gamma_i| \text{ and } |\nabla \beta_i| \leq \nu. \quad (24)$$

By (16), $r^{-1}|\beta_i|$ and $|\nabla \beta_i|$ tend to zero as $r \to 0$. We can thus ensure that (24) is satisfied by the $\beta_i$ components by making $\epsilon$ smaller. Hence, (24) holds if $||\gamma_i||_{C^1_i} \leq \nu$. Therefore, putting estimates (23) in (21) and using (20),

$$|P_{F_1}(\alpha_i)| = |P_{C,}(\gamma_i + \beta_i) - P_{C,}(\beta_i)| \leq A_0(r^{-1}|\gamma_i| + |\nabla \gamma_i|)^2 \quad (25)$$

whenever $||\gamma_i||_{C^1_i} \leq \nu$. As $r \to 0$ the terms in the bracket on the right-hand side of (25) are of order $O(r^{\lambda-1})$ by (15). Thus, $|P_{F_1}(\alpha_i)|$ is of order $O(r^{2\lambda-2})$, hence $O(r^{\lambda-1})$ since $\lambda > 1$, as $r \to 0$ for $i = 1, \ldots, s$. We deduce that $|P_{F_1}(\alpha)|$ is of order $O(\rho^{2\lambda-2})$ as $\rho \to 0$ for all $\alpha \in C^1_\lambda(\bar{U})$ with $||\alpha||_{C^1_i}$ sufficiently small.

Similar calculations give analogous results to (24) for derivatives of $P_{F_1}$, but we shall explain the method by considering the first derivative. From (21) we calculate

$$\nabla(P_{C,}(\gamma_i + \beta_i) - P_{C,}(\beta_i))$$

$$= \int_0^1 \nabla \left( \gamma_i \frac{\partial P_{C,}}{\partial y}(t\gamma_i + \beta_i) + \nabla \gamma_i \cdot \frac{\partial P_{C,}}{\partial z}(t\gamma_i + \beta_i) \right) dt$$

$$= \int_0^1 \nabla \gamma_i \cdot \frac{\partial P_{C,}}{\partial y} + \gamma_i \cdot \left( \nabla(t\gamma_i + \beta_i) \cdot \frac{\partial^2 P_{C,}}{\partial y^2} + \nabla^2(t\gamma_i + \beta_i) \cdot \frac{\partial^2 P_{C,}}{\partial y^2} \right)$$

$$+ \nabla^2 \gamma_i \cdot \frac{\partial P_{C,}}{\partial z} + \gamma_i \cdot \left( \nabla(t\gamma_i + \beta_i) \cdot \frac{\partial^2 P_{C,}}{\partial z^2} + \nabla^2(t\gamma_i + \beta_i) \cdot \frac{\partial^2 P_{C,}}{\partial z^2} \right) dt.$$

Whenever $||\gamma_i||_{C^1_i} \leq \nu$ there exists a constant $A_1 > 0$ such that (23) holds with $A_0$ replaced by $A_1$ and, for $t \in [0, 1],$

$$\left| \frac{\partial^2 P_{C,}}{\partial y^2}(t\gamma_i + \beta_i) \right|, \left| \frac{\partial^2 P_{C,}}{\partial y^2}(t\gamma_i + \beta_i) \right| \text{ and } \left| \frac{\partial^2 P_{C,}}{\partial z^2}(t\gamma_i + \beta_i) \right| \leq A_1.$$
since the second derivatives of $P_{C_i}$ are continuous functions defined on the closed bounded set given by $\|\gamma_i\|_{C^1} \leq \nu$. We deduce that

$$|\nabla (P_{F_i}(\alpha_i))| = |\nabla (P_{C_i}(\gamma_i + \beta_i) - P_{C_i}(\beta_i))| \leq A_i \left( \sum_{j=0}^{2} r^{j-2} |\nabla^j \gamma_i| \right)$$

whenever $\|\gamma_i\|_{C^1} \leq \nu$. Therefore $|\nabla (P_{F_i}(\alpha_i))|$ is of order $O(r^{2\lambda-3})$, hence $O(r^{\lambda-2})$, as $r \to 0$.

In general we have the estimate

$$|\nabla^l (P_{F_i}(\alpha_i))| \leq A_i \left( \sum_{j=0}^{l+1} r^{-j(l+1)} |\nabla^j \gamma_i| \right)$$

for some $A_i > 0$ whenever $\|\gamma_i\|_{C^1} \leq \nu$. The result follows. \[\square\]

We now consider the regularity of solutions to the nonlinear elliptic equation $G_1(\alpha, \beta) = 0$ near $(0, 0)$.

**Proposition 6.10** Let $(\alpha, \beta) \in L^p_{k+1, \lambda}(\hat{U}) \times L^p_{k+1, \lambda}(\Lambda^4 T^* \hat{N})$ for some $p > 4$ and $k \geq 2$. If $G_1(\alpha, \beta) = 0$ and $\|\alpha\|_{C^1}$ is sufficiently small, $(\alpha, \beta) \in C^\infty_\Lambda(\hat{U}) \times C^\infty_\Lambda(\Lambda^4 T^* \hat{N})$.

**Proof:** Suppose that $(\alpha, \beta) \in L^p_{k+1, \lambda}(\hat{U}) \times L^p_{k+1, \lambda}(\Lambda^4 T^* \hat{N})$ for some $p > 4$ and $k \geq 2$. Then $\alpha$ and $\beta$ lie in $C^1_{\text{loc}}$ by Theorem 1.3 since $\frac{k}{p} > \frac{1}{2}$.

Suppose further that $G_1(\alpha, \beta) = 0$ and that $\|\alpha\|_{C^1}$ is sufficiently small. Since $F_1$ smoothly depends on $\alpha$ and $\nabla \alpha$, $G_1$ is a smooth function of $\alpha$, $\beta$, $\nabla \alpha$ and $\nabla \beta$. We apply [18, Theorem 6.8.1], which is a general regularity result for nonlinear elliptic equations, to conclude that $\alpha$ and $\beta$ are smooth. However, we want more than this: the derivatives of $\alpha$ and $\beta$ must decay at the required rates.

Recall the note after Definition 6.7 that $G_1(\alpha, \beta) = 0$ implies that $\beta = 0$. Thus $\beta \in C^\infty_\Lambda(\Lambda^4 T^* \hat{N})$ trivially.

For the following argument we find it useful to work with weighted Hölder spaces. By Theorem 1.3, $\alpha \in C^{1, a}_{\Lambda}(\hat{U})$ with $a = 1 - 4/p \in (0, 1)$ since $p > 4$. We also know that $d^*(G_1(\alpha, \beta)) = d^*(F_1(\alpha)) = 0$, which is a nonlinear elliptic equation on $\alpha$. Using the notation and results of Proposition 6.9, $d^* \alpha + d^*(P_{F_1}(\alpha)) = 0$ and $d^*(P_{F_1}(\alpha)) \in C^{k-2, a}_{2\lambda-3}(\Lambda^2 T^* \hat{N})$. We see that

$$d^*(F_1(\alpha))(x) = R(x, \alpha(x), \nabla \alpha(x))\nabla^2 \alpha(x) + E(x, \alpha(x), \nabla \alpha(x)),$$

where $R(x, \alpha(x), \nabla \alpha(x))$ and $E(x, \alpha(x), \nabla \alpha(x))$ are smooth functions of their arguments. Define

$$S_{\alpha}(\gamma)(x) = R(x, \alpha(x), \nabla \alpha(x))\nabla^2 \gamma(x)$$
for $\gamma \in C^2_{\text{loc}}(\Lambda_2^2 T^* N)$. Then $S_\alpha$ is a smooth, linear, elliptic, second-order operator, if $\|\alpha\|_{C^1}$ is sufficiently small, whose coefficients depend on $x$, $\alpha(x)$ and $\nabla \alpha(x)$. These coefficients therefore lie in $C^{k-1,a}_{\text{loc}}$. We also notice that

$$S_\alpha(\alpha)(x) = -E(x, \alpha(x), \nabla \alpha(x)) \in C^{k-2,a}_{2\lambda-3}(\Lambda^2 T^* N) \subseteq C^{k-2,a}_{\lambda-2}(\Lambda^2 T^* N),$$

since $\lambda > 1$. However, $E(x, \alpha(x), \nabla \alpha(x))$ only depends on $\alpha$ and $\nabla \alpha$, and is at worst quadratic in these quantities by Proposition 6.9, so it must in fact lie in $C^{k-1,a}_{\lambda-2}(\Lambda^2 T^* N)$ since we are given control on the decay of the first $k$ derivatives of $\alpha$ as $\rho \to 0$.

The work in [17, §6.1.1] on asymptotically conical manifolds gives regularity results for smooth linear elliptic operators acting between weighted Hölder spaces. These results can easily be adapted to the CS scenario. In particular, if $\gamma \in C^2_{\text{loc}}(\Lambda_2^2 T^* N)$ and $S_\alpha(\gamma) \in C^{k-1,a}_{\lambda-2}(\Lambda^2 T^* N)$, we have that $\gamma \in C^{k+1,a}_{\lambda}(\Lambda_2^2 T^* N)$. Since $k \geq 2$, $\alpha$ and $S_\alpha(\alpha)$ satisfy these conditions by the discussion above. We deduce that $\alpha \in C^{k+1,a}_{\lambda}(\Lambda_2^2 T^* N)$ only knowing a priori that $\alpha \in C^{k,a}_{\lambda}(\Lambda_2^2 T^* N)$. We proceed by induction to show that $\alpha \in C^{k,a}_{\lambda}(\Lambda_2^2 T^* N)$ for all $l \geq 2$.

6.2 Problem 2: moving singularities and fixed $G_2$ structure

For this problem we again consider deformations of $N$ in $(M, \phi, g)$ which are CS coassociative 4-folds at $s$ points with the same rate and cones at the singularities, but now we allow the singular points and tangent cones at those points to differ from those of $N$. However, we still assume that the $G_2$ structure on $M$ is fixed.

**Definition 6.11** The **moduli space of deformations** $\mathcal{M}_2(N, \lambda)$ for Problem 2 is the set of $N'$ in $(M, \phi, g)$ which are CS coassociative 4-folds at $z'_1, \ldots, z'_s$ with rate $\lambda$, having cone $C_i$ and tangent cone $\hat{C}_i$ at $z'_i$ for all $i$, such that there exists a homeomorphism $h : N \to N'$, isotopic to the identity, with $h(z_i) = z'_i$ for $i = 1, \ldots, s$ and such that $h|_{S} : \hat{N} \to N' \setminus \{z'_1, \ldots, z'_s\}$ is a diffeomorphism.

Here it is more difficult to create a local description of the moduli space which is compatible with the analytic framework in which our study is made. What one would consider more ‘intuitive’ approaches do not, as far as the author is aware, bear fruit. We therefore follow what is, at first sight, a slightly indirect route.

For each $i = 1, \ldots, s$ let $B_i$ be an open set in $M$ containing $z_i$ such that $B_i \cap B_j = \emptyset$ for $i \neq j$. Let $B = \bigcap_{i=1}^s B_i$. For each $z' = (z'_1, \ldots, z'_s) \in B,$
we have a family \(I(z')\) of choices of \(s\)-tuples \(\zeta' = (\zeta'_1, \ldots, \zeta'_s)\) of isomorphisms \(\zeta': \mathbb{R}^7 \to T_{z'}M\) identifying \((\varphi_0, g_0)\) with \((\varphi|_{T_zM}, g|_{T_zM})\). Clearly, for each \(z' \in B\), \(I(z') \cong G_2\). We thus make the following definition.

**Definition 6.12** The translation space is

\[
\mathcal{T} = \{(z', \zeta') : z' \in B, \zeta' \in I(z')\}.
\]

It is a principal \(G_2\) bundle over \(B\) and hence is a smooth manifold.

Let \(H_i\) denote the Lie subgroup of \(G_2\) preserving \(t_i(C_i)\) in \(\mathbb{R}^7\) for \(i = 1, \ldots, s\) and let \(H = \prod_{i=1}^s H_i \subseteq G_2\). Then \(H\) acts freely on \(\mathcal{T}\) by

\[
(z', \zeta') \mapsto (z', (\zeta'_1 \circ A_1^{-1}, \ldots, \zeta'_s \circ A_s^{-1})),
\]

where \((A_1, \ldots, A_s) \in H\). Thus there exists an \(H\)-orbit through \((z, \zeta)\) in \(\mathcal{T}\), where \(z = (z_1, \ldots, z_s)\) and \(\zeta = (\zeta_1, \ldots, \zeta_s)\).

Define \(\mathcal{T}'\) to be a small open ball in \(\mathbb{R}^n\) containing 0, where \(n = \text{dim } \mathcal{T} - \text{dim } H\), and let \(h_{\mathcal{T}'} : \mathcal{T}' \to \mathcal{T}\) be an embedding with \(h_{\mathcal{T}'}(0) = (z, \zeta)\) such that \(h_{\mathcal{T}'}(\mathcal{T}')\) is transverse to the \(H\)-orbit through \((z, \zeta)\). Write \(h_{\mathcal{T}'}(t) = (z(t), \zeta(t))\) for \(t \in \mathcal{T}'\), with \(z(0) = z\) and \(\zeta(0) = \zeta\).

**Notes**

(a) If \(t, t' \in \mathcal{T}'\), with \(t \neq t'\), are such that \(z(t) = z(t')\), the \(s\)-tuples of tangent cones, \(\{C_1(t), \ldots, C_s(t)\}\) and \(\{\hat{C}_1(t'), \ldots, \hat{C}_s(t')\}\), are distinct.

(b) \(\mathcal{T}'\) is an open ball in \(\mathbb{R}^n \cong T_0 \mathcal{T}\) and hence can be considered as an open subset of \(T_0 \mathcal{T}\).

We use \(\mathcal{T}'\) to extend \(N\) to a family of nearby CS 4-folds and provide an analogue to Proposition [6.3](#) for Problem 2. In defining \(N\) we chose a \(G_2\) coordinate system \(\{\chi_i : B(0; \eta) \to V_i : i = 1, \ldots, s\}\) with \(d\chi_i|_0 = \zeta_i\) for \(i = 1, \ldots, s\). Extend this to a smooth family of \(G_2\) coordinate systems

\[
\left\{\{\chi_i(t) : B(0; \eta) \to V_i(t) : i = 1, \ldots, s\} : t \in \mathcal{T}'\right\},
\]

where \(V_i(t)\) is an open set in \(M\) containing \(z_i(t)\), \(\chi_i(t)(0) = z_i(t)\), \(d\chi_i(t)|_0 = \zeta_i(t)\), \(\chi_i(0) = \chi_i\) and \(V_i(0) = V_i\) for \(i = 1, \ldots, s\).

**Proposition 6.13** Use the notation of Proposition [6.3](#) and Definition [6.12](#)
(a) There exists a family $N = \{ N(t) : t \in \hat{T} \}$ of CS 4-folds in $M$, with $N(0) = N$, such that $N(t)$ has singularities at $z_1(t), \ldots, z_s(t)$ with rate $\lambda$, cones $C_1, \ldots, C_s$ and tangent cones $\hat{C}_1(t), \ldots, \hat{C}_s(t)$ defined by $\hat{C}_i(t) = (\zeta_i(t) \circ \iota_i)(C_i)$.

(b) Let $\hat{N}(t) = N(t) \setminus \{ z_1(t), \ldots, z_s(t) \}$ and write

$$N(t) = K(t) \cup \bigcup_{i=1}^s U_i(t)$$

where $K(t)$ is compact and $U_i(t) \setminus \{ z_i(t) \} \cong (0, \epsilon) \times \Sigma_i$ for all $i$, in the obvious way, ensuring that $K(0) = K$ and $U_i(0) = U_i$. For $t \in \hat{T}$, there exist open sets $\hat{T}(t) \subseteq M$ containing $\hat{N}(t)$ and diffeomorphisms $\delta(t) : \hat{U} \to \hat{T}(t)$ taking the zero section to $\hat{N}(t)$, varying smoothly in $t$, with $\hat{T}(0) = \hat{T}$ and $\delta(0) = \delta$. Moreover, $\hat{T}(t)$ can be chosen to grow with order $O(\rho)$ as $\rho \to 0$ and $\delta(t)$ is compatible with the identifications $U_i(t) \setminus \{ z_i(t) \} \cong (0, \epsilon) \times \Sigma_i$ for all $i$.

**Remark** The family $N$ does not necessarily consist of CS coassociative 4-folds and $\delta(t)$ is not required to be compatible with the isomorphism $\nu(\hat{N}) \cong \Lambda^2_+ T^* \hat{N}$ for $t \neq 0$.

**Proof:** Use the notation from the proof of Proposition 6. For $t \in \hat{T}$, define $\hat{T}_i(t) = \chi_i(t)(\hat{S}_i)$ and

$$U_i(t) = \left( \chi_i(t) \circ \Phi_i((0, \epsilon) \times \Sigma_i) \right) \cup \{ z_i(t) \}$$

for $i = 1, \ldots, s$. Then $\hat{T}_i(t)$ contains $U_i(t) \setminus \{ z_i(t) \}$. Define a diffeomorphism $\delta_i(t)$ such that the following diagram commutes:

$$
\begin{array}{ccc}
\hat{U}_i & \xrightarrow{\Phi_i^*} & j_i(\hat{V}_i) \\
\downarrow \delta_i(t) & & \downarrow j_i^{-1} \\
\hat{T}_i(t) & \xleftarrow{\chi_i(t)} & \hat{S}_i.
\end{array}
$$

We then interpolate smoothly over $K$ to extend $\bigcup_{i=1}^s \hat{T}_i(t)$ to $\hat{T}(t)$ and $\delta_i(t)$ to $\delta(t)$ as required. Note by construction that $\hat{T}(t)$ grows with order $O(\rho)$ as $\rho \to 0$. 

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Let \( e(t) = \delta(t)|_{\hat{N}} \) and define \( \hat{N}(t) = e(t)(\hat{N}) \). Then \( e(t) : \hat{N} \to \hat{N}(t) \) is a diffeomorphism for all \( t \in \bar{T} \) and \( e(0) \) is the identity. Let \( N(t) = \hat{N}(t) \cup \{z_1(t), \ldots, z_s(t)\} \). We then have a family \( \mathcal{N} = \{N(t) : t \in \bar{T}\} \) as claimed. Note that \( K(t) = e(t)(K) \).

By the construction of \( \delta(t) \) and the family \( \mathcal{N} \), it is clear that the proposition is proved, where the compatibility conditions on \( \delta(t) \) are given by (20). \( \square \)

The next definition is analogous to Definition 6.5.

Definition 6.14 Use the notation of Proposition 6.13. Let \( \Gamma_\alpha \) be the graph of \( \alpha \in C^1_{\text{loc}}(\hat{U}) \) and let \( \pi_\alpha : \hat{N} \to \Gamma_\alpha \) be given by \( \pi_\alpha(x) = (x, \alpha(x)) \). For \( t \in \bar{T} \), let \( f_\alpha(t) = \delta(t) \circ \pi_\alpha \) and let \( \hat{N}_\alpha(t) = f_\alpha(t)(\hat{N}) \). Define \( F_2 \) from \( C^1_{\text{loc}}(\hat{U}) \times \bar{T} \) to \( C^0_{\text{loc}}(\Lambda^3 T^* \hat{N}) \) by:

\[
F_2(\alpha, t) = f_\alpha(t)^* \left( \varphi|_{\hat{N}_\alpha(t)} \right).
\]

The linearisation of \( F_2 \) at \((0, 0)\) acts as

\[
dF_2|_{(0, 0)} : (\alpha, t) \mapsto d\alpha + L_2(t),
\]

where \( \alpha \in C^1_{\text{loc}}(\Lambda^2_+ T^* \hat{N}) \), \( t \in T_0 \bar{T} \) and \( L_2 \) is a linear map into the space of smooth exact 3-forms on \( \hat{N} \) since \( \varphi \) is exact near \( \hat{N} \).

Remark By construction \( F_2(\alpha, 0) = F_1(\alpha) \) as given in Definition 6.5.

Clearly, \( \ker F_2 \) is the set of \( \alpha \in C^1_{\text{loc}}(\hat{U}) \) and \( t \in \bar{T} \) such that \( \hat{N}_\alpha(t) \) is coassociative. However, we have not yet encoded the information that \( N_\alpha(t) \) is CS with rate \( \lambda \). This is the subject of the next proposition.

Proposition 6.15 The moduli space of deformations for Problem 2 is locally homeomorphic to \( \ker F_2 = \{ (\alpha, t) \in C^\infty_{\text{loc}}(\hat{U}) \times \bar{T} : F_2(\alpha, t) = 0 \} \).

Proof: For each \( t \in \bar{T} \), we are in the situation of Problem 1 in the sense that we want coassociative deformations \( \hat{N}_\alpha(t) \) of \( \hat{N}(t) \), defined by a self-dual 2-form \( \alpha \), which have the same singular points, cones and tangent cones as \( \hat{N}(t) \). It is thus clear that \( \alpha \in C^\infty_{\text{loc}}(\hat{U}) \) by Proposition 6.6. \( \square \)

We now introduce an associated map \( G_2 \) to \( F_2 \).

Definition 6.16 Define \( G_2 : C^1_{\text{loc}}(\hat{U}) \times C^1_{\text{loc}}(\Lambda^4 T^* \hat{N}) \times \bar{T} \to C^0_{\text{loc}}(\Lambda^3 T^* \hat{N}) \) by:

\[
G_2(\alpha, \beta, t) = F_2(\alpha, t) + d^* \beta.
\]

Then \( dG_2|_{(0, 0, 0)} : (\alpha, \beta, t) \mapsto d\alpha + d^* \beta + L_2(t) \), in the notation of Definition 6.14.

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We then have an analogous result to Proposition 6.8, which follows in exactly the same fashion because $F_2(\alpha, t)$ is exact.

**Proposition 6.17**

$$\text{Ker } F_2 \cong \{ (\alpha, \beta, t) \in C^\infty(\hat{U}) \times C^\infty(\Lambda^4 T^* \hat{N}) \times \hat{T} : G_2(\alpha, \beta, t) = 0 \}.$$

The next result studies the regularity of the kernel of $G_2$ near $(0, 0, 0)$ and is the analogue of Proposition 6.10.

**Proposition 6.18** Let $(\alpha, \beta, t) \in L^p_{k+1, \lambda}(\hat{U}) \times L^p_{k+1, \lambda}(\Lambda^4 T^* \hat{N}) \times \hat{T}$, where $p > 4$ and $k \geq 2$. If $G_2(\alpha, \beta, t) = 0$ and $\|\alpha\|_{C_1^2}$ and $t$ are sufficiently small, $(\alpha, \beta) \in C^\infty(\hat{U}) \times C^\infty(\Lambda^4 T^* \hat{N})$.

**Proof:** Note that $dG_2(\alpha, \beta, t) = \Delta \beta = 0$ implies that $\beta = 0$ by the Maximum Principle for harmonic functions and $d^* G_2(\alpha, \beta, t) = d^* F_2(\alpha, t) = 0$ is an elliptic equation at 0 on $\alpha$. Using similar notation to the proof of Proposition 6.10,

$$d^* F_2(\alpha, t)(x) = R_t(x, \alpha(x), \nabla \alpha(x)) \nabla^2 \alpha(x) + E_t(x, \alpha(x), \nabla \alpha(x)),$$

where $R_t$ and $E_t$ are smooth functions of their arguments. If we define

$$S_{(\alpha, t)}(\gamma)(x) = R_t(x, \alpha(x), \nabla \alpha(x)) \nabla^2 \gamma(x),$$

then $S_{(\alpha, t)}$ is a smooth linear differential operator on $\gamma \in C^2_{\text{loc}}(\Lambda^4 T^* \hat{N})$. The ellipticity of $S_{\alpha} = S_{(\alpha, 0)}$ results from the coassociativity of $\hat{N}$. Ellipticity is an open condition so, although $\hat{N}(t)$ is not necessarily coassociative, the fact that it is ‘close’ to being coassociative means that $S_{(\alpha, t)}$ is elliptic, as long as we shrink $\hat{T}$ as necessary to make $t$ sufficiently small.

The regularity results for $S_{(\alpha, t)}$ follow in the same way as in the proof of Proposition 6.10 since $F_2(\alpha, t)$ depends smoothly on $t$ and $\hat{N}(t)$ is asymptotically coassociative near the singular points, which validates the use of the theory from [17] §6.1.1. Recall that $L^p_{k+1, \lambda} \hookrightarrow C^{k, a}_{\lambda}$ where $a = 1 - 4/p$. Thus, if $S_{(\alpha, t)}(\gamma) \in C^{k-1,a}_{\lambda-2}$ and $\gamma \in C^4_{\lambda}(\hat{U})$, then $\gamma \in C^{k+1,a}_{\lambda}(\hat{U})$.

Since $E_0 = E$ maps into $C^{k-1,a}_{\lambda-2}$, as argued in the proof of Proposition 6.10 and $F_2$ depends smoothly on $t$, $E_t$ maps into $C^{k-1,a}_{\lambda-2}$ for $t$ sufficiently small. Hence,

$$S_{(\alpha, t)}(\alpha)(x) = -E_t(x, \alpha(x), \nabla \alpha(x)) \in C^{k-1,a}_{\lambda-2}.$$

We deduce that $\alpha \in C^{k+1,a}_{\lambda}$, given only that $\alpha \in C^k_{\lambda}$. Induction gives the result.

$\square$
6.3 Problem 3: moving singularities and varying $G_2$ structure

For our final problem we consider CS deformations $N'$ of $N$ with the same rate and cones at $s$ singularities, but with possibly different singular points and tangent cones there, such that $N'$ is coassociative under a deformation of the $G_2$ structure on $M$.

We begin with the following.

**Proposition 6.19** Use the notation of Proposition 6.4. Let

$$T = \hat{T} \cup \bigcup_{i=1}^{s} V_i \supseteq N.$$

By making $\hat{T}$ and $V_i$, for $i = 1, \ldots, s$, smaller if necessary, $T$ retracts onto $N$.

There exists an isomorphism $\Xi : H^3_{\text{dR}}(T) \to H^3_{\text{cs}}(\hat{N})$.

**Proof:** Let $[\xi] \in H^3_{\text{dR}}(T)$. Since the sets $V_i$ retract onto $\{z_i\}$ for $i = 1, \ldots, s$, $\xi$ can be chosen such that $\xi|_{V_i} = 0$. Therefore, $\xi|_{U \setminus \{z_i\}} = 0$ which implies that the support of $\xi|_{\hat{N}}$ is contained in $K$, which is compact. Hence $[\xi|_{\hat{N}}]$ is a well-defined element of $H^3_{\text{cs}}(\hat{N})$. Define $\Xi$ by $[\xi] \mapsto [\xi|_{\hat{N}}]$. We show that $\Xi$ is well-defined. Suppose that $\xi' = \xi + dv$, for $v \in C^\infty(\Lambda^2 T^* T)$, such that $\xi'|_{V_i} = 0$ for all $i$. Then $dv|_{V_i} = 0$ for all $i$. Since $V_i$ retracts onto $\{z_i\}$ we can choose $v$ such that $v|_{V_i} = 0$ without affecting $dv$ by smoothly interpolating over $\hat{T}$. Thus $v|_{\hat{N}}$ is compactly supported on $\hat{N}$ and $\xi|_{\hat{N}} + d(v|_{\hat{N}}) = \xi'|_{\hat{N}}$. Hence $\Xi$ is well-defined and injective.

Any closed form on $\hat{N}$ with support in $K$ can be extended smoothly to a closed form on $T$ which vanishes on $V_i$ for all $i$. Thus, any cohomology class in $H^3_{\text{cs}}(\hat{N})$ has a representative $\gamma$ that can be lifted to a form $\xi$ on $T$ such that $\Xi([\xi]) = [\gamma]$, which implies that $\Xi$ is surjective. \(\square\)

**Notes** The reason for this result is two-fold.

(a) The condition that $\Xi([\varphi|_{T}]) = 0$ in $H^3_{\text{dR}}(\hat{N})$ is implied by the coassociativity of $\hat{N}$ and it forces $[\varphi|_{\hat{N}}] = 0$ in $H^3_{\text{cs}}(\hat{N})$. This is stronger than the seemingly more natural condition of $[\varphi|_{\hat{N}}] = 0$ in $H^3_{\text{dR}}(\hat{N})$, which would be the correct requirement if $\hat{N}$ were compact by the work of McLean \[16\].

(b) If a $G_2$ structure $(\varphi', g')$ on $M$ is such that $\Xi([\varphi'|_{T}]) \neq 0$ then $\varphi'|_{\hat{N}'} \neq 0$ for any nearby deformation $\hat{N}'$ of $\hat{N}$, so there are no coassociative deformations.
Proposition 6.19 allows us to define a distinguished family of ‘nearby’ $G_2$ structures to $(\varphi, g)$.

**Definition 6.20** Let $\hat{F}$ be a small open ball about 0 in $\mathbb{R}^m$ for some $m$. Let

$$\mathcal{F} = \{(\varphi^f, g^f) : f \in \hat{F}\}$$

be a family of torsion-free $G_2$ structures, with $(\varphi^0, g^0) = (\varphi, g)$, such that $\Xi((\varphi^f|_T)) = 0$ in $H^3_{cs}(\hat{N})$ and the map $h_{\hat{F}} : \hat{F} \to \mathcal{F}$ given by $h_{\hat{F}}(f) = (\varphi^f, g^f)$ is an embedding.

**Note** $\hat{F}$ can be considered as an open subset of $T_0 \hat{F}$.

We now describe the moduli space for Problem 3.

**Definition 6.21** The moduli space of deformations $M_3(N, \lambda)$ for Problem 3 is the set of pairs $(N', f)$ of $f \in \hat{F}$ and $N'$ in $(M, \varphi^f, g^f)$ which are CS coassociative 4-folds at $z'_1, \ldots, z'_s$ with rate $\lambda$, having cone $C_i$ and tangent cone $\hat{C}'_i$ at $z'_i$ for all $i$, such that there exists a homeomorphism $h : N \to N'$, isotopic to the identity, with $h(z_i) = z'_i$ for $i = 1, \ldots, s$ and such that $h|_{\hat{N}} : \hat{N} \to N' \setminus \{z'_1, \ldots, z'_s\}$ is a diffeomorphism.

We have a projection map $\pi_{\hat{F}} : M_3(N, \lambda) \to \hat{F}$, with $\pi_{\hat{F}}(N', f) = f$, whose fibres $\pi_{\hat{F}}^{-1}(f)$ are equal to the moduli space for Problem 2 defined using the $G_2$ structure $(\varphi^f, g^f)$.

We must adapt our translation space from Problem 2 to incorporate the varying $G_2$ structure.

**Definition 6.22** Use the notation of Definitions 6.12 and 6.20. For $f \in \hat{F}$ and $z' \in B$ let $I^f(z')$ denote the family of choices of $s$-tuples $\zeta' = (\zeta'_1, \ldots, \zeta'_s)$ of isomorphisms $\zeta'_i : \mathbb{R}^7 \to T_{z'_i}M$ identifying $(\varphi_0, g_0)$ with $(\varphi^f|_{T_{z'_i}M}, g^f|_{T_{z'_i}M})$.

The translation space corresponding to $\hat{F}$ is

$$\mathcal{T}_{\hat{F}} = \{(z', \zeta', f) : z' \in B, f \in \hat{F}, \zeta' \in I^f(z')\}.$$  

It is a principal $G_2^s$ bundle over $B \times \hat{F}$.

There is a natural free action of $H$ on $\mathcal{T}_{\hat{F}}$ and hence an $H$-orbit through $(z, \zeta, 0)$. Therefore, we may embed $\hat{T} \times \hat{F}$ into $\mathcal{T}_{\hat{F}}$ by $h_{\hat{T} \times \hat{F}} : (t, f) \mapsto (z(t, f), \zeta(t, f), f)$ such that $h_{\hat{T} \times \hat{F}}(\hat{T} \times \hat{F})$ is transverse to this $H$-orbit, $h_{\hat{T} \times \hat{F}}(t, 0) = h_{\hat{F}}(t)$ for all $t$ and $z(0, f) = z$ for all $f$.

Use the notation introduced before Proposition 6.13. Extend the $G_2$ coordinate system near $z_1, \ldots, z_s$ used to define $N$ to a smooth family of $G_2$ coordinate
systems

\[ \{ \chi_i(t,f) : B(0; \eta) \to V_i(t,f) : i = 1, \ldots, s \} : (t,f) \in \hat{T} \times \hat{F} \]

such that \( V_i(t,f) \) is an open set in \( M \) containing \( z_i(t,f) \), \( \chi_i(t,f)(0) = z_i(t,f) \), \( d\chi_i(t,f)|_0 = \zeta_i(t,f) \), \( \chi_i(t,0) = \chi_i(t) \), \( V_i(0,f) = V_i \) and \( V_i(t,0) = V_i(t) \) for \( i = 1, \ldots, s \). We state the analogue of Proposition 6.13.

**Proposition 6.23** Use the notation of Propositions 6.4 and 6.13 and Definition 6.22.

(a) There exists a family \( \hat{N} = \{ N(t,f) : (t,f) \in \hat{T} \times \hat{F} \} \) of CS 4-folds in \( M \), with \( N(0,0) = N \) and \( N(t,0) = N(t) \), such that \( N(t,f) \) has singularities at \( z_1(t,f), \ldots, z_s(t,f) \) with rate \( \lambda_i \), cones \( C_1, \ldots, C_s \) and tangent cones \( \hat{C}_1(t,f), \ldots, \hat{C}_s(t,f) \) defined by \( \hat{C}_i(t,f) = (\zeta_i(t,f) \circ i_\lambda)(C_i) \).

(b) Let \( \hat{N}(t,f) = N(t,f) \setminus \{ z_1(t,f), \ldots, z_s(t,f) \} \) and write

\[ N(t,f) = K(t,f) \sqcup \bigsqcup_{i=1}^s U_i(t,f) \]

where \( K(t,f) \) is compact and \( U_i(t,f) \setminus \{ z_i(t,f) \} \cong (0,\epsilon) \times \Sigma_i \) for all \( i \), in the obvious way, ensuring that \( K(0,f) = K \), \( K(t,0) = K(t) \), \( U_i(0,f) = U_i \) and \( U_i(t,0) = U_i(t) \). For \( (t,f) \in \hat{T} \times \hat{F} \), there exist open sets \( \hat{T}(t,f) \subseteq M \) containing \( \hat{N}(t,f) \) and diffeomorphisms \( \delta(t,f) : \hat{U} \to \hat{T}(t,f) \) taking the zero section to \( \hat{N}(t,f) \), varying smoothly in \( t \) and \( f \), with \( \hat{T}(0,f) = \hat{T} \), \( \hat{T}(t,0) = \hat{T}(t) \) and \( \delta(t,0) = \delta(t) \). Moreover, \( \hat{T}(t,f) \) can be chosen to grow with order \( O(\rho) \) as \( \rho \to 0 \) and \( \delta(t,f) \) is compatible with the identifications \( U_i(t,f) \setminus \{ z_i(t,f) \} \cong (0,\epsilon) \times \Sigma_i \) for \( i = 1, \ldots, s \).

The proof is almost identical to that of Proposition 6.13 and so we omit it. The compatibility conditions on \( \delta(t,f) \) are given by similar commutative diagrams to 6.13.

**Remark** \( \delta(t,f) \) is not required to be compatible with the isomorphism \( \nu(\hat{N}) \cong \Lambda^*_T T^* \hat{N} \) for \( (t,f) \neq (0,0) \).

We proceed by defining our final deformation map.

**Definition 6.24** Use the notation of Proposition 6.23. Let \( \Gamma_\alpha \) be the graph of \( \alpha \in C^1_{\text{loc}}(\hat{U}) \) and let \( \pi_\alpha : \hat{N} \to \Gamma_\alpha \) be given by \( \pi_\alpha(x) = (x,\alpha(x)) \). For
Proposition 6.28

Let \( p > 4 \) where \( G \) is generalised to the map \( F \) and let \( \tilde{N}_\alpha(t,f) = f_\alpha(t,f)(\hat{N}) \). Define \( F_3 \) from \( C^1_{\text{loc}}(\bar{U}) \times \hat{T} \times \hat{F} \) to \( C^0_{\text{loc}}(\Lambda^3 T^* \hat{N}) \) by:

\[
F_3(\alpha, t, f) = f_\alpha(t,f)^* \left( \varphi^f |_{\tilde{N}_\alpha(t,f)} \right),
\]

The linearisation of \( F_3 \) at \((0,0,0)\) acts as

\[
dF_3|_{(0,0,0)} : (\alpha, t, f) \mapsto d\alpha + L_2(t) + L_3(f),
\]

where \( \alpha \in C^1_{\text{loc}}(\Lambda^2_+ T^* \hat{N}) \), \((t,f) \in T_0 \hat{T} \oplus T_0 \hat{F} \), \( L_2 \) is given in Definition 6.24 and \( L_3 \) is a linear map into the space of smooth exact 3-forms on \( \hat{N} \) by the condition imposed on \( \varphi^f \) in Definition 6.20.

Note: \( F_3(\alpha, t, 0) = F_2(\alpha, t) \) as given in Definition 6.24.

Now, \( \text{Ker} \ F_3 \) corresponds to choices of \( \tilde{N}_\alpha(t,f) \) which are coassociative with respect to \( (\varphi^f, g^f) \). The next result is then clear from considering the proof of Proposition 6.15.

Proposition 6.25 The moduli space of deformations for Problem 3 is locally homeomorphic to \( \text{Ker} \ F_3 = \{ (\alpha, t, f) \in C^\infty(\bar{U}) \times \hat{T} \times \hat{F} : F_3(\alpha, t, f) = 0 \} \).

We again have an associated map to our deformation map.

Definition 6.26 Define \( G_3 : C^1_{\text{loc}}(\bar{U}) \times C^1_{\text{loc}}(\Lambda^4 T^* \hat{N}) \times \hat{T} \times \hat{F} \rightarrow C^0_{\text{loc}}(\Lambda^3 T^* \hat{N}) \) by:

\[
G_3(\alpha, \beta, t, f) = F_3(\alpha, t, f) + d^* \beta.
\]

Then \( dG_3|_{(0,0,0,0)} : (\alpha, \beta, t, f) \mapsto d\alpha + d^* \beta + L_2(t) + L_3(f), \) in the notation of Definition 6.24.

The next result is analogous to Propositions 6.18 and 6.17 and may be immediately deduced from the exactness of \( F_3(\alpha, t, f) \), which follows from the condition imposed on \( \varphi^f \) in Definition 6.20.

Proposition 6.27

\[
\text{Ker} \ F_3 \cong \{ (\alpha, \beta, t, f) \in C^\infty(\bar{U}) \times C^\infty(\Lambda^4 T^* \hat{N}) \times \hat{T} \times \hat{F} : G_3(\alpha, \beta, t, f) = 0 \}.
\]

The argument used to prove the regularity result Proposition 6.18 is easily generalised to the map \( G_3 \), so we end the section with the following.

Proposition 6.28 Let \( (\alpha, \beta, t, f) \in L^p_{k+1, \lambda}(\bar{U}) \times L^p_{k+1, \lambda}(\Lambda^4 T^* \hat{N}) \times \hat{T} \times \hat{F} \), where \( p > 4 \) and \( k \geq 2 \). If \( G_3(\alpha, \beta, t, f) = 0 \) and \( \| \alpha \|_{C^1_\hat{U}} \), \( t \) and \( f \) are sufficiently small, \( (\alpha, \beta) \in C^\infty(\bar{U}) \times C^\infty(\Lambda^4 T^* \hat{N}) \).
7 The deformation and obstruction spaces

In this section we describe the infinitesimal deformation and obstruction spaces for each of our problems and show in each scenario that, if the obstruction space is zero, we get a smooth moduli space of deformations. We recollect the common notation introduced at the start of §6. In addition, fix some $p > 4$ and integer $k \geq 2$.

7.1 Problem 1

Recall the maps $F_1$ and $G_1$ given in Definitions 6.5 and 6.7 respectively. Their kernels give a local description for the moduli space $M_1(N, \lambda)$ by Propositions 6.6 and 6.8. Therefore the kernels of $dF_1|_0$ and $dG_1|_{(0,0)}$ describe the infinitesimal deformations.

Definition 7.1 The infinitesimal deformation space for Problem 1 is

$$\mathcal{I}_1(N, \lambda) = \{ \alpha \in C_\infty^k(\Lambda^2 T^* \hat{N}) : d\alpha = 0 \}$$

$$\cong \{ (\alpha, \beta) \in C_\infty^k(\Lambda^2_+ T^* \hat{N} \oplus \Lambda^4 T^* \hat{N}) : d\alpha + d^* \beta = 0 \}.$$

The equivalence of the spaces follows by Proposition 6.8 or, more simply, by the Maximum Principle for harmonic functions.

Using Proposition 6.10

$$\mathcal{I}_1(N, \lambda) \cong \{ (\alpha, \beta) \in L^p_{k+1, \lambda}(\Lambda^2 T^* \hat{N} \oplus \Lambda^4 T^* \hat{N}) : d\alpha + d^* \beta = 0 \}.$$

Therefore, $\mathcal{I}_1(N, \lambda)$ is finite-dimensional.

We turn to possible obstructions to the deformation theory and start with the following.

Proposition 7.2 The map $F_1$ takes $L^p_{k+1, \lambda}(\hat{U})$ into $d(L^p_{k+1, \lambda}(\Lambda^2 T^* \hat{N}))$.

Proof: Let $\alpha \in L^p_{k+1, \lambda}(\hat{U})$ and let $T$ be as in Proposition 6.19. As noted after that proposition, $[\varphi|_T] = 0$ in $H^3_{dR}(T)$ and hence $\varphi|_T$ is exact. Thus, $\varphi|_T = d\psi$ for some $\psi \in C_\infty(\Lambda^2 T^* T)$. However, we want to select $\psi$ in a particular way near the singularities. On $B(0; \eta) \subseteq \mathbb{R}^7$, for each $i = 1, \ldots, s$,

$$\chi_i^\ast(\varphi) = \varphi_0 + O(r_i).$$

If $v$ is the dilation vector field on $\mathbb{R}^7$, given in coordinates $(x_1, \ldots, x_7)$ by

$$v = x_1 \frac{\partial}{\partial x_1} + \ldots + x_7 \frac{\partial}{\partial x_7},$$

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we can choose \( \psi \) to satisfy
\[
\chi_i^*(\psi) = \frac{1}{3}(v \cdot \varphi_0) + O(r_i^2)
\]
on \( V_i \), since \( d(v \cdot \varphi_0) = 3\varphi_0 \), then extend \( \psi \) smoothly to a form on \( T \) such that \( d\psi = \varphi|_T \). Note that
\[
(v \cdot \varphi_0)|_{\iota_i(C_i)} = v \cdot (\varphi_0)|_{\iota_i(C_i)} = 0
\]as \( v \in T(\iota_i(C_i)) \). Hence \( \chi_i^*(\psi) = O(r_i^2) \) on \( \iota_i(C_i) \), for all \( i \), and similar results hold for the derivatives of \( \psi \). Define
\[
H_1(\alpha) = f_\alpha^*(\psi|_{\mathcal{N}_\alpha})
\]
so that \( F_1(\alpha) = d(H_1(\alpha)) \). Note that \( \chi_i^*(\psi)|_{\iota_i(C_i)} = O(r_i^2) \) is dominated by \( O(r_i^2) \) terms as \( r_i \to 0 \) since \( \lambda < 2 \). Further, \( f_\alpha^*(\psi|_{\mathcal{N}_\alpha}) \) has the same growth as \( \chi_i^*(\psi)|_{(\Phi_\alpha)_i,((0,\varepsilon)\times \Sigma_i)} \) as \( r_i \to 0 \), using the notation preceding Proposition 6.6.

However,
\[
\chi_i^*(\psi)|_{(\Phi_\alpha)_i,((0,\varepsilon)\times \Sigma_i)} = \chi_i^*(\psi)|_{(\Phi_\alpha)_i,((0,\varepsilon)\times \Sigma_i)} + \chi_i^*(\psi)|_{\iota_i((0,\varepsilon)\times \Sigma_i)}.
\]
The first term on the right-hand side depends on \( ||(\Phi_\alpha)_i - \iota_i|| \) and hence is \( O(r_i^2) \) as \( r_i \to 0 \). This dominates the second term by our observation above. Hence, \( H_1(\alpha) \in L_{k,\lambda}^p \) because \( H_1 \) depends on \( \alpha \) and \( \nabla \alpha \). Note that \( H_1(\alpha) \) has one degree of differentiability less than expected.

Recalling that \( \lambda \notin \mathcal{D} \), we deduce that \( F_1(\alpha) \) lies in \( d(L_{k,\lambda}^p(\Lambda^2T^*\tilde{N})) \) and hence is \( L^2 \)-orthogonal to elements of the kernel of
\[
d + d^* : L_{l+1,-3-\lambda}(\Lambda^3T^*\tilde{N}) \to L_{l,-4-\lambda}(\Lambda^2T^*\tilde{N} + \Lambda^4T^*\tilde{N}),
\]
where \( q > 1 \) such that \( 1/p + 1/q = 1 \). We show that
\[
d(L_{k,\lambda}^p(\Lambda^2T^*\tilde{N})) \oplus d^*(L_{k,\lambda}^p(\Lambda^4T^*\tilde{N})) \subseteq L_{k-1,\lambda-1}^p(\Lambda^3T^*\tilde{N})
\]
is characterised as the subspace which is \( L^2 \)-orthogonal to this kernel.

Consider
\[
d + d^* : L_{k,\lambda}^p(\Lambda^{even}T^*\tilde{N}) \to L_{k-1,\lambda-1}^p(\Lambda^{odd}T^*\tilde{N}).
\]
This elliptic map has image which comprises precisely of those elements of \( L_{k-1,\lambda-1}^p(\Lambda^{odd}T^*\tilde{N}) \) which are \( L^2 \)-orthogonal to the kernel \( \mathcal{K} \) of
\[
d + d^* : L_{l+1,-3-\lambda}(\Lambda^{odd}T^*\tilde{N}) \to L_{l,-4-\lambda}(\Lambda^{even}T^*\tilde{N}).
\]
The space $\mathcal{K}$ can be written as the direct sum $\mathcal{K} = \mathcal{K}^{1} \oplus \mathcal{K}^{3} \oplus \mathcal{K}^{m}$, where

$$\mathcal{K}^{j} = \mathcal{K} \cap L_{k+1}^{p}, -3-\lambda(A^{j}T^{*}\hat{N})$$

for $j = 1$ and $3$ and $\mathcal{K}^{m}$ is some transverse subspace. Then

$$d(L_{k, \lambda}^{p}(A^{2}T^{*}\hat{N})) \oplus d'(L_{k, \lambda}^{p}(A^{4}T^{*}\hat{N})) = \{\alpha_{3} : \exists \alpha_{1} \text{ such that } (\alpha_{1}, \alpha_{3}) \in \mathcal{K}^{1}\},$$

where we take the orthogonal complement in $L_{k, \lambda-1}^{p}$. Note that the projection $\pi_{1}(\mathcal{K}^{m})$ of $\mathcal{K}^{m}$ onto the space of $1$-forms must meet $\mathcal{K}^{1}$ in the zero form since, if $(\alpha_{1}, \alpha_{3}) \in \mathcal{K}^{m}$ and $\alpha_{1} \in \mathcal{K}^{1}$ then $\alpha_{3} \in \mathcal{K}^{3}$, which contradicts the direct sum decomposition of $\mathcal{K}$. Therefore, $\pi_{1}(\mathcal{K}^{m})$ and $\mathcal{K}^{1}$ are transverse finite-dimensional subspaces of $L_{k+1, -3-\lambda}^{p}(A^{3}T^{*}\hat{N})$. Hence, there exists a space $\mathcal{A}$ of smooth compactly supported $1$-forms on $\hat{N}$ which is $L^{2}$-orthogonal to $\mathcal{K}^{1}$ and such that $\mathcal{A} \times \mathcal{K}^{m} \to \mathbb{R}$ given by $(\gamma, \xi) \to (\gamma, 0) \cdot \xi$ is a dual pairing. If $\alpha_{3} \in L_{k, \lambda-1}^{p}(A^{3}T^{*}\hat{N})$ such that $\alpha_{3} \in (\mathcal{K}^{3})^\perp$, there exists a unique $\alpha_{1} \in \mathcal{A}$ such that $(\alpha_{1}, 0) \cdot \xi = (0, \alpha_{3}) \cdot \xi$ for all $\xi \in \mathcal{K}^{m}$, which implies that $(\alpha_{1}, \alpha_{3}) \in (\mathcal{K}^{m})^\perp$.

We conclude that

$$(\mathcal{K}^{3})^\perp = \{\alpha_{3} \in (\mathcal{K}^{3})^\perp : \exists \alpha_{1} \in \mathcal{K}^{1} \text{ such that } (\alpha_{1}, \alpha_{3}) \in \mathcal{K}^{m}\} = \{\alpha_{3} : \exists \alpha_{1} \text{ such that } (\alpha_{1}, \alpha_{3}) \in \mathcal{K}^{1}\} = d(L_{k, \lambda}^{p}(A^{2}T^{*}\hat{N})) \oplus d'(L_{k, \lambda}^{p}(A^{4}T^{*}\hat{N})) \subseteq L_{k-1, \lambda-1}^{p}(A^{3}T^{*}\hat{N}).$$

However, $\mathcal{K}^{3}$ is independent of $k$, and hence $F_{1}(\alpha)$ must lie in the image of $d + d'$ from $L_{k+1, \lambda}^{p}$, since $F_{1}(\alpha)$ lies in $L_{k, \lambda-1}^{p}$. We may thus write $F_{1}(\alpha) = d\gamma + d^{*}\beta$ for some $\gamma \in L_{k+1, \lambda}^{p}(A^{2}T^{*}\hat{N})$ and $\beta \in L_{k+1, \lambda}^{p}(A^{4}T^{*}\hat{N})$. Moreover, $d\gamma \beta = 0$ and so $\beta$ is harmonic and $O(\rho^{\lambda})$ as $\rho \to 0$. By the Maximum Principle, (noting that $\ast\beta$ is a harmonic function on $\hat{N}$), $\beta = 0$. The proposition is thus proved.

We deduce from Propositions 6.1, 6.5, 6.10, and 7.2 that $\mathcal{M}_{1}(N, \lambda)$ is locally homeomorphic to the kernel of

$$G_{1} : L_{k+1, \lambda}^{p}(\hat{U}) \times L_{k+1, \lambda}^{p}(A^{4}T^{*}\hat{N}) \to d(L_{k+1, \lambda}^{p}(A^{2}T^{*}\hat{N})) \oplus d'(L_{k+1, \lambda}^{p}(A^{4}T^{*}\hat{N})).$$

Therefore, our deformation theory will be obstructed if and only if the map

$$d : L_{k+1, \lambda}^{p}(A^{2}T^{*}\hat{N}) \to d(L_{k+1, \lambda}^{p}(A^{2}T^{*}\hat{N}))$$

is not surjective. This leads us to the next result and definition.

**Proposition 7.3** There exists a finite-dimensional subspace $\mathcal{O}_{1}(N, \lambda)$ of $L_{k, \lambda-1}^{p}(A^{3}T^{*}\hat{N})$ such that

$$d(L_{k+1, \lambda}^{p}(A^{2}T^{*}\hat{N})) = d(L_{k+1, \lambda}^{p}(A^{2}T^{*}\hat{N})) \oplus \mathcal{O}_{1}(N, \lambda).$$

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Proof: The Fredholmness of \(d + d^*\) implies that the images of \(L_{k+1,\lambda}^p(\Lambda^2 T^*\hat{N}) \oplus L_{k+1,\lambda}^p(\Lambda^4 T^*\hat{N})\) and \(L_{k+1,\lambda}^p(\Lambda^4 T^*\hat{N}) \oplus L_{k+1,\lambda}^p(\Lambda^4 T^*\hat{N})\) under \(d + d^*\) are both closed and have finite codimension in \(L_{k,\lambda-1}^p(\Lambda^3 T^*\hat{N})\). Since
\[
\{0\} = d(L_{k+1,\lambda}^p(\Lambda^2 T^*\hat{N})) \cap d^*(L_{k+1,\lambda}^p(\Lambda^4 T^*\hat{N})) = d(L_{k+1,\lambda}^p(\Lambda^2 T^*\hat{N})) \cap d^*(L_{k+1,\lambda}^p(\Lambda^4 T^*\hat{N}))
\]
by the Maximum Principle, we deduce that \(d(L_{k+1,\lambda}^p(\Lambda^2 T^*\hat{N}))\) and \(d(L_{k+1,\lambda}^p(\Lambda^2 T^*\hat{N}))\) are both closed and that the former has finite codimension in the latter. Thus, \(O_1(N,\lambda)\) can be chosen as stated.

**Definition 7.4** The obstruction space for Problem 1 is
\[
O_1(N,\lambda) \equiv \frac{d(L_{k+1,\lambda}^p(\Lambda^2 T^*\hat{N}))}{d(L_{k+1,\lambda}^p(\Lambda^2 T^*\hat{N}))}.
\]

We proceed as follows. Define
\[
\begin{align*}
U_1 &= L_{k+1,\lambda}^p(\hat{U}) \times L_{k+1,\lambda}^p(\Lambda^2 T^*\hat{N}), \\
X_1 &= L_{k+1,\lambda}^p(\Lambda^2 T^*\hat{N}) \oplus \Lambda^4 T^*\hat{N}, \\
Y_1 &= O_1(N,\lambda) \subseteq L_{k,\lambda-1}^p(\Lambda^3 T^*\hat{N}) \text{ and} \\
Z_1 &= d(L_{k+1,\lambda}^p(\Lambda^2 T^*\hat{N})) \oplus d^*(L_{k+1,\lambda}^p(\Lambda^4 T^*\hat{N})).
\end{align*}
\]

Then \(X_1, Y_1\) and \(Z_1\) are Banach spaces and \(U_1\) is an open neighbourhood of \((0,0)\) in \(X_1\) because \(L_{k+1,\lambda}^p \hookrightarrow C_\lambda^0\) by Theorem 6.4 and \(\hat{U}\) grows with order \(O(\rho)\) as \(\rho \to 0\) by Proposition 6.4. Thus, \(W_1 = U_1 \times Y_1\) is an open neighbourhood of \((0,0,0)\) in \(X_1 \times Y_1\). Define \(G_1 : W_1 \to Z_1\) by:
\[
G_1(\alpha,\beta,\gamma) = G_1(\alpha,\beta) + \gamma.
\]

Then \(G_1\) is well-defined by Propositions 7.2 and 7.3 and its derivative at \((0,0,0)\) acts from \(X_1 \times Y_1\) to \(Z_1\) as
\[
dG_1|_{(0,0,0)} : (\alpha,\beta,\gamma) \mapsto d\alpha + d^*\beta + \gamma.
\]

Clearly, \(dG_1|_{(0,0,0)}\) is surjective by construction and its kernel, using the fact that \((d + d^*)(X_1) \cap Y_1 = \{0\}\), is given by:
\[
\Ker dG_1|_{(0,0,0)} = \{(\alpha,\beta,\gamma) \in X_1 \times Y_1 : d\alpha + d^*\beta + \gamma = 0\}
\]
\[
\cong \{(\alpha,\beta) \in X_1 : d\alpha + d^*\beta = 0\} \cong \mathcal{I}_1(N,\lambda).
\]
The conclusion, by implementing the Implicit Function Theorem for Banach spaces (Theorem 7.5), is that $\ker G_1$ is a smooth manifold near zero which may be identified with an open neighbourhood $\hat{\mathcal{M}}_1(N, \lambda)$ of 0 in $\mathcal{I}_1(N, \lambda)$. Formally, if we write $X_1 = \mathcal{I}_1(N, \lambda) \oplus A$ for some closed subspace $A$ of $X_1$, there exist open sets $\hat{\mathcal{M}}_1(N, \lambda) \subseteq \mathcal{I}_1(N, \lambda)$, $V_A \subseteq A$, $V_Y \subseteq Y_1$, all containing 0, with $\hat{\mathcal{M}}_1(N, \lambda) \times V_A \subseteq U_1$, and smooth maps $V_A : \hat{\mathcal{M}}_1(N, \lambda) \to V_A$ and $V_Y : \hat{\mathcal{M}}_1(N, \lambda) \to V_Y$ such that

$$\ker G_1 \cap (\hat{\mathcal{M}}_1(N, \lambda) \times V_A \times V_Y) = \{ (x, V_A(x), V_Y(x)) : x \in \hat{\mathcal{M}}_1(N, \lambda) \}.$$ 

If we define a smooth map $\pi_1 : \hat{\mathcal{M}}_1(N, \lambda) \to O_1(N, \lambda)$ by $\pi_1(x) = V_Y(x)$, the moduli space $\mathcal{M}_1(N, \lambda)$ near $N$ is locally homeomorphic to the kernel of $\pi_1$ near 0. We can think of $\pi_1$ as a map on an open neighbourhood of $(0, 0, 0)$ in $\ker G_1$ which projects onto the obstruction space. We write these results as a theorem.

**Theorem 7.5** Use the notation of Definitions 6.1, 7.1 and 7.4. There exists a smooth manifold $\hat{\mathcal{M}}_1(N, \lambda)$, which is an open neighbourhood of 0 in $\mathcal{I}_1(N, \lambda)$, and a smooth map $\pi_1 : \hat{\mathcal{M}}_1(N, \lambda) \to O_1(N, \lambda)$, with $\pi_1(0) = 0$, such that an open neighbourhood of 0 in $\ker \pi_1$ is homeomorphic to an open neighbourhood of $N$ in $\mathcal{M}_1(N, \lambda)$.

We deduce from this theorem that, if the obstruction space is zero, the moduli space is a smooth manifold near $N$ of dimension equal to that of the infinitesimal deformation space. We expect the obstruction space to be zero for generic choices of $N$ and the $G_2$ structure on $M$.

### 7.2 Problem 2

Recall the notation introduced in Definitions 6.12, 6.14 and 6.16. We begin by defining the infinitesimal deformation space for this problem.

**Definition 7.6** The infinitesimal deformation space for Problem 2 is

$$\mathcal{I}_2(N, \lambda) = \{ (\alpha, t) \in C_\Lambda^\infty(\Lambda_+^2 T^* \hat{N} + T_0 \hat{T}) : d\alpha + L_2(t) = 0 \} \cong \{ (\alpha, \beta, t) \in C_\Lambda^\infty(\Lambda_+^2 T^* \hat{N} \oplus \Lambda^4 T^* \tilde{N}) \oplus T_0 \hat{T} : d\alpha + d^* \beta + L_2(t) = 0 \}.$$ 

The equivalence in the definition follows from Proposition 6.17 or from the observation that $d\alpha + L_2(t)$ is exact and so $\beta = 0$ by the Maximum Principle.

By Proposition 6.18,

$$\mathcal{I}_2(N, \lambda) \cong \{ (\alpha, \beta, t) \in \Lambda^p_{k+1, \lambda}(\Lambda_+^2 T^* \hat{N} \oplus \Lambda^4 T^* \tilde{N}) \oplus T_0 \hat{T} : d\alpha + d^* \beta + L_2(t) = 0 \}.$$ 

Therefore, $\mathcal{I}_2(N, \lambda)$ is finite-dimensional.
Note There is a subspace of $\mathcal{I}_2(N,\lambda)$ which is isomorphic to $\mathcal{I}_1(N,\lambda)$.

To start our consideration of obstructions, we have the generalisation of Proposition 7.2.

**Proposition 7.7** The map $F_2$ takes $L^p_{k+1,\lambda}(\hat{U}) \times \hat{T}$ into $d(L^p_{k+1,\lambda}(\Lambda^2T^*\hat{N}))$.

**Proof:** Use the notation from Proposition 6.13 and its proof and from the proof of Proposition 7.2. Recall that we have an open set $T \supseteq \hat{T}$ in $M$ containing $N$, which retracts onto $N$, and $\psi \in C^\infty(\Lambda^2T^*T)$ such that $d\psi = \varphi|_T$. We may similarly construct open sets $T(t) \supseteq \hat{T}(t)$ in $M$, with $T(0) = T$, which contain $N(t)$ and retract onto it, varying smoothly with $t$. We also have $\psi(t) \in C^\infty(\Lambda^2T^*T(t))$, with $\psi(0) = \psi$, such that $d\psi(t) = \varphi|_{T(t)}$, using the fact that $\varphi$ is exact on $N(t)$. Again, the $\psi(t)$ vary smoothly with $t$. Formally, let $T(t) = \hat{T}(t) \cup \bigcup_{i=1}^s V_i(t)$.

By making $\hat{T}(t)$ and $V_i(t)$ smaller if necessary, $T(t)$ will be an open set as stated. We may choose $\psi(t)$ such that $\chi_i(t)^*(\psi(t)) = \frac{1}{3}(v \cdot \varphi_0) + O(r_i^2)$ on $V_i(t)$ and then extend smoothly to a form $\psi(t)$ on $T(t)$ as required. Define $H_2(\alpha, t) = f_\alpha(t)^* \left( \psi(t)|_{\hat{N}_\alpha(t)} \right)$.

Then $d(H_2(\alpha, t)) = F_2(\alpha, t)$. Moreover, by the same reasoning that $H_1(\alpha) \in L^p_{k,\lambda}$ in the proof of Proposition 7.2, $H_2(\alpha, t)$ lies in $L^p_{k,\lambda}$. Therefore, $F_2(\alpha, t)$ lies in $d(L^p_{k+1,\lambda}(\Lambda^2T^*\hat{N}))$. However, because $F_2(\alpha, t) \in L^p_{k,\lambda-1}(\Lambda^3T^*\hat{N})$, the argument at the end of the proof of Proposition 7.2 implies that $F_2(\alpha, t) \in d(L^p_{k+1,\lambda}(\Lambda^2T^*\hat{N}))$ as required. □

We now define the obstruction space.

**Definition 7.8** From Propositions 5.8 and 5.10, since $L_2$ is a linear map on a finite-dimensional vector space, there exists a finite-dimensional subspace $O_2(N,\lambda)$ of $L^p_{k+1,\lambda-1}(\Lambda^2T^*\hat{N})$ such that

$$d(L^p_{k+1,\lambda}(\Lambda^2T^*\hat{N})) = (d(L^p_{k+1,\lambda}(\Lambda^2T^*\hat{N})) + L_2(T(t))) \oplus O_2(N,\lambda).$$

We define $O_2(N,\lambda)$ to be the *obstruction space* for Problem 2.
Note $\mathcal{O}_2(N, \lambda)$ may be chosen to be contained in $\mathcal{O}_1(N, \lambda)$.

Following the scheme for Problem 1, we let

$$U_2 = L^p_{k+1, \lambda}(\check{U}) \times L^p_{k+1, \lambda}(\Lambda^2 T^* \check{N}) \times \check{T},$$

$$X_2 = L^p_{k+1, \lambda}(\Lambda^2 T^* \check{N} \oplus \Lambda^4 T^* \check{N}) \oplus T_0 \check{T},$$

$$Y_2 = \mathcal{O}_2(N, \lambda) \subseteq L^p_{k, \lambda-1}(\Lambda^3 T^* \check{N})$$

and

$$Z_2 = d(L^p_{k+1, \lambda}(\Lambda^2 T^* \check{N})) \oplus d^*(L^p_{k+1, \lambda}(\Lambda^4 T^* \check{N})).$$

Recall that $\check{T} \subseteq \mathbb{R}^n \cong T_0 \check{T}$ is open. Then $X_2$, $Y_2$ and $Z_2$ are Banach spaces, $U_2$ is an open neighbourhood of $(0, 0, 0)$ in $X_2$ and hence $W_2 = U_2 \times Y_2$ is an open neighbourhood of $(0, 0, 0, 0)$ in $X_2 \times Y_2$. Define $\mathcal{G}_2 : W_2 \rightarrow Z_2$ by:

$$\mathcal{G}_2(\alpha, \beta, t, \gamma) = G_2(\alpha, \beta, t) + \gamma.$$

Then $d\mathcal{G}_2|_{(0, 0, 0, 0)} : X_2 \times Y_2 \rightarrow Z_2$ acts as

$$(\alpha, \beta, t, \gamma) \mapsto d\alpha + d^* \beta + L_2(t) + \gamma.$$

By construction, $d\mathcal{G}_2|_{(0, 0, 0, 0)}$ is surjective and, using the fact that the image of $d\mathcal{G}_2|_{(0, 0, 0, 0)}$ meets $Y_2$ at 0 only,

$$\text{Ker } d\mathcal{G}_2|_{(0, 0, 0, 0)} = \{ (\alpha, \beta, t, \gamma) \in X_2 \times Y_2 : d\alpha + d^* \beta + L_2(t) + \gamma = 0 \}
\cong \{ (\alpha, \beta, t) \in X_2 : d\alpha + d^* \beta + L_2(t) = 0 \} \cong \mathcal{I}_2(N, \lambda).$$

As for Problem 1, Theorem 7.9 gives us that Ker $\mathcal{G}_2$ is a smooth manifold near zero which may be identified with an open neighbourhood $\mathcal{M}_2(N, \lambda)$ of $(0, 0)$ in $\mathcal{I}_2(N, \lambda)$. We can again define a smooth map $\pi_2 : \mathcal{M}_2(N, \lambda) \rightarrow \mathcal{O}_2(N, \lambda)$ such that Ker $\pi_2$ is locally homeomorphic near $(0, 0)$ to an open neighbourhood of $N$ in $\mathcal{M}_2(N, \lambda)$. We thus have the following theorem.

**Theorem 7.9** Use the notation of Definitions 6.11 and 7.8. There exists a smooth manifold $\mathcal{M}_2(N, \lambda)$, which is an open neighbourhood of $(0, 0)$ in $\mathcal{I}_2(N, \lambda)$, and a smooth map $\pi_2 : \mathcal{M}_2(N, \lambda) \rightarrow \mathcal{O}_2(N, \lambda)$, with $\pi_2(0, 0) = 0$, such that an open neighbourhood of zero in Ker $\pi_2$ is homeomorphic to an open neighbourhood of $N$ in $\mathcal{M}_2(N, \lambda)$.

We deduce that, if $\mathcal{O}_2(N, \lambda) = \{0\}$, the moduli space for Problem 2 is a smooth manifold near $N$ of dimension $\dim \mathcal{I}_2(N, \lambda) = \dim \mathcal{I}_1(N, \lambda) + \dim \check{T}$, which we expect to occur for generic choices of $N$ and the torsion-free $G_2$ structure on $M$. We shall see, in §8 that if we choose a suitable generic closed $G_2$ structure on $M$ we may drop the assumption that $N$ is generic and still obtain a smooth moduli space.
7.3 Problem 3

We presume in this subsection that the reader is sufficiently familiar with the schemata we have used in the previous two subsections to be able to generalise them to Problem 3. This allows us to present a tidier treatment of the problem.

Recall the notation of Definitions 6.20, 6.24 and 6.26.

Definition 7.10 The infinitesimal deformation space \( \mathcal{I}_3(N, \lambda) \) for Problem 3 is
\[
\mathcal{I}_3(N, \lambda) = \left\{ (\alpha, t, f) \in C_\infty^\infty(\Lambda_2^2 T^* \hat{N}) \oplus T_0 \hat{T} \oplus T_0 \hat{F} : d\alpha + L_2(t) + L_3(f) = 0 \right\}
\]
\[
\cong \left\{ (\alpha, \beta, t, f) \in C_\infty^\infty(\Lambda_2^2 T^* \hat{N} \oplus \Lambda^4 T^* \hat{N}) \oplus T_0 \hat{T} \oplus T_0 \hat{F} : d\alpha + d^* \beta + L_2(t) + L_3(f) = 0 \right\}.
\]

By Proposition 6.28,
\[
\mathcal{I}_3(N, \lambda) \cong \left\{ (\alpha, \beta, t, f) \in L_{k+1, \lambda}^p(\Lambda_2^2 T^* \hat{N}) \oplus \Lambda^4 T^* \hat{N}) \oplus T_0 \hat{T} \oplus T_0 \hat{F} : d\alpha + d^* \beta + L_2(t) + L_3(f) = 0 \right\}.
\]

In considering obstructions, we first have the generalisation of Propositions 7.2 and 7.7.

Proposition 7.11 \( F_3(L_{k+1, \lambda}^p(\hat{U}) \times \hat{T} \times \hat{F}) \subseteq d(L_{k+1, \lambda}^p(\Lambda_2^2 T^* \hat{N})). \)

The proposition is proved in a similar way to Proposition 7.7 and so we omit the details. The result leads us to define our final obstruction space.

Definition 7.12 From Propositions 7.8 and 7.11 since \( L_2 \) and \( L_3 \) are linear maps on finite-dimensional vector spaces, there exists a finite-dimensional subspace \( \mathcal{O}_3(N, \lambda) \) of \( L_{k+1, \lambda}^p(\Lambda_2^2 T^* \hat{N}) \) such that
\[
d(L_{k+1, \lambda}^p(\Lambda_2^2 T^* \hat{N})) = (d(L_{k+1, \lambda}^p(\Lambda_2^2 T^* \hat{N}))) + L_2(T_0 \hat{T}) + L_3(T_0 \hat{F}) \oplus \mathcal{O}_3(N, \lambda).
\]

We define \( \mathcal{O}_3(N, \lambda) \) to be the obstruction space for Problem 3.

Note We may choose our obstruction spaces such that \( \mathcal{O}_3(N, \lambda) \subseteq \mathcal{O}_2(N, \lambda) \subseteq \mathcal{O}_1(N, \lambda) \).

The use of the Implicit Function Theorem (Theorem 4.5) in the derivation of Theorems 7.5 and 7.9 can be easily generalised to give the following.

Theorem 7.13 Use the notation of Definitions 6.21, 7.10 and 7.12. There exists a smooth manifold \( \mathcal{M}_3(N, \lambda) \), which is an open neighbourhood of \( (0, 0, 0) \) in \( \mathcal{I}_3(N, \lambda) \), and a smooth map \( \pi_3 : \mathcal{M}_3(N, \lambda) \to \mathcal{O}_3(N, \lambda) \), with \( \pi_3(0, 0, 0) = 0 \), such that an open neighbourhood of zero in \( \text{Ker} \pi_3 \) is homeomorphic to an open neighbourhood of \( (N, 0) \) in \( \mathcal{M}_3(N, \lambda) \).
We deduce that, if \( O_3(N, \lambda) = \{0\} \), \( M_3(N, \lambda) \) is a smooth manifold near \((N, 0)\) of dimension \( \dim I_3(N, \lambda) = \dim I_2(N, \lambda) + \dim \hat{F} \). Moreover, the projection map \( \pi_\hat{F} : M_3(N, \lambda) \rightarrow \hat{F} \) is smooth near \((N, 0)\). We expect this to occur for generic choices of \( N \) and the torsion-free \( G_2 \) structure on \( M \). If we allow ourselves to work with closed \( G_2 \) structures on \( M \), we shall show in \( \S 9 \) that we may drop our genericity assumptions for \( N \) and \((\varphi, g)\) and still get a smooth moduli space.

### 8 Dimension calculations

We shall relate the expected dimension of the moduli space for Problem 1 to the index of \( d + d^* \) as discussed in \( \S 5.2 \). Recall that \( p > 4 \), \( k \geq 2 \) and \( \lambda \in (1, 2) \setminus \mathcal{D} \).

**Definition 8.1** Define

\[
H^m = \{ \xi \in L^2(\Lambda^m T^* \hat{N}) : d\xi = d^*\xi = 0 \}.
\]

The Hodge star maps \( H^2 \) into itself, so there is a splitting \( H^2 = H^2_+ \oplus H^2_- \) where

\[
H^2_+ = H^2 \cap C^\infty(\Lambda^2_+ T^* \hat{N})
\]

Let \( \mathcal{J} = j \left( H^2_{cs}(\hat{N}) \right) \), where \( H^m_{cs}(\hat{N}) \) is the \( m \)th compactly supported cohomology group on \( \hat{N} \) and \( j : H^2_{cs}(\hat{N}) \rightarrow H^2_{dR}(\hat{N}) \) is the inclusion map. If \( \alpha, \beta \in \mathcal{J} \), there exist compactly supported closed 2-forms \( \xi \) and \( \eta \) such that \( \alpha = [\xi] \) and \( \beta = [\eta] \). We define a product on \( \mathcal{J} \times \mathcal{J} \) by

\[
\alpha \cup \beta = \int_{\hat{N}} \xi \wedge \eta.
\]

Suppose that \( \xi' \) and \( \eta' \) are also compactly supported with \( \alpha = [\xi'] \) and \( \beta = [\eta'] \). Then there exist 1-forms \( \chi \) and \( \zeta \) such that \( \xi - \xi' = d\chi \) and \( \eta - \eta' = d\zeta \). Therefore,

\[
\int_S \xi' \wedge \eta' = \int_S (\xi - d\chi) \wedge (\eta - d\zeta) = \int_S \xi \wedge \eta - d\chi \wedge \eta - \xi' \wedge d\zeta
\]

\[
= \int_S \xi \wedge \eta - d(\chi \wedge \eta) - d(\xi' \wedge \zeta) = \int_S \xi \wedge \eta,
\]

as both \( \chi \wedge \eta \) and \( \xi' \wedge \zeta \) have compact support. The product \( \cup \) on \( \mathcal{J} \times \mathcal{J} \) is thus well-defined and is a symmetric topological product with a signature \((a, b)\).

By \[13\] Example (0.16), \( H^2 \cong \mathcal{J} \) and the isomorphism is given by \( \xi \mapsto [\xi] \). Thus, \( \dim H^2_+ = a \) and hence is a topological number.
\begin{definition}
\[(d_+ + d^*)_\lambda = d + d^* : L^p_{k+1, \lambda}(\Lambda^2 T^* \hat{N} \oplus \Lambda^4 T^* \hat{N}) \to L^p_{k, \lambda-1}(\Lambda^2 T^* \hat{N}). \]
\end{definition}

By Definition \ref{thm} \(L_1(N, \lambda)\) is isomorphic to the kernel of this map. Define the adjoint map by
\[(d_+^* + d^*)_{-3-\lambda} = d_+^* + d : L^q_{l+1, -3-\lambda}(\Lambda^2 T^* \hat{N}) \to L^q_{l, -4-\lambda}(\Lambda^2 T^* \hat{N} \oplus \Lambda^4 T^* \hat{N}),\]
where \(q > 1\) such that \(1/p + 1/q = 1\) and \(l \geq 4\). The cokernel of \((d_+ + d^*)_\lambda\) is then isomorphic to the kernel of \((d_+^* + d^*)_{-3-\lambda}\).

\begin{note}
The choice of \(l \geq 4\) in Definition \ref{thm} ensures that \(L^q_{l+1, -3-\lambda} \hookrightarrow C^{1,a}_{-3-\lambda}\) for \(0 < a \leq 4 - \frac{1}{q} = \frac{4}{p} < 1\) by Theorem \ref{thm}.
\end{note}

We now study the dimension of the kernel and cokernel of \((d_+ + d^*)_\mu\).

\begin{proposition}
The kernel of \((d_+ + d^*)_2\) is isomorphic to \(\mathcal{H}_+^2\). Furthermore, if \(\mu > -2\) is such that \((-2, \mu] \cap D = \emptyset\), \(\dim \ker (d_+ + d^*)_\mu = \dim \mathcal{H}_+^2\).
\end{proposition}

\begin{proof}
Using \ref{thm} and the Maximum Principle,
\[
\mathcal{H}_+^2 = \{ \alpha \in L^2(\Lambda^2 T^* \hat{N}) \cap C^\infty(\Lambda^2 T^* \hat{N}) : d\alpha = d^*\alpha = 0 \}
= \{ \alpha \in L^0_{0, -2}(\Lambda^2 T^* \hat{N}) \cap C^\infty(\Lambda^2 T^* \hat{N}) : d\alpha = 0 \}
\cong \{ (\alpha, \beta) \in L^2_{0, -2}(\Lambda^2 T^* \hat{N} \oplus \Lambda^4 T^* \hat{N}) : \alpha \in C^\infty(\Lambda^2 T^* \hat{N}), \ d\alpha + d^*\beta = 0 \}.
\]
This gives the first part of the proposition.

If \(-2 \notin D\), \([-2, \mu] \cap D = \emptyset\) and thus, by Proposition \ref{thm} \(\dim \ker (d_+ + d^*)_\mu = \dim \ker (d_+ + d^*)_2\).

Suppose now that \(-2 \in D\) and that \((\alpha, \beta)\) corresponds to a self-dual 2-form and 4-form on \(\hat{N}\) which are subtracted from the kernel of \((d_+ + d^*)_\nu\) as \(\nu\) crosses \(-2\) from below. By the work in \ref{thm} \S3 & \S4 this occurs if and only if \((\alpha, \beta)\) is asymptotic to an \(O(r^{-2})\) form \(\xi\) on \(C_i\), for some \(i\), satisfying \((d + d^*)\xi = 0\). (The form \(\xi\) is determined by an element of \(D(-2, i)\), using the notation of Proposition \ref{thm}). Therefore, \((\alpha, \beta)\) is of order \(O(\rho^{-2})\) as \(\rho \to 0\) and thus lies in \(L^2\). We deduce that \((\alpha, \beta) \in \ker (d_+ + d^*)_2\), implying that the function \(k(\nu) = \ker (d_+ + d^*)_\nu\) is upper semi-continuous at \(-2\) by Proposition \ref{thm}.

The second part of the proposition is thus proved.
\end{proof}

\begin{proposition}
If \(\mu < -1\) is such that \([\mu, -1] \cap D = \emptyset\), the cokernel of \((d_+ + d^*)_\mu\) is isomorphic to \(H^3_{dR}(\hat{N})\).
\end{proposition}
Proof: By Theorem 5.3 there exists a countable discrete subset $\mathcal{D}'$ of rates $\nu$ such that

$$d + d^* : L^p_{k+1, \nu}(\Lambda^\text{even} T^* \hat{N}) \to L^p_{k, \nu-1}(\Lambda^\text{odd} T^* \hat{N})$$

(28)

is not Fredholm. Clearly, $\mathcal{D}' \supseteq \mathcal{D}$. For $\nu \notin \mathcal{D}'$ with $\nu < -1$, so that $-3 - \nu > \nu - 1$,

$$L^p_{k, \nu-1}(\Lambda^\text{odd} T^* \hat{N}) = (d + d^*)(L^p_{k+1, \nu}(\Lambda^\text{even} T^* \hat{N})) \oplus \mathcal{K},$$

where $\mathcal{K}$ is the kernel of the adjoint map

$$d + d^* : L^q_{l+1, -\nu-3}(\Lambda^\text{odd} T^* \hat{N}) \to L^q_{l, -\nu-4}(\Lambda^\text{even} T^* \hat{N}),$$

for $1/p + 1/q = 1$ and $l \geq 4$, which is graded and closed under the Hodge star.

If $\gamma \in L^p_{k, \nu-1}(\Lambda^3 T^* \hat{N})$ then $(\gamma, \gamma) \in L^p_{k, \nu-1}(\Lambda^\text{odd} T^* \hat{N})$ and hence there exist some $\gamma_m \in L^p_{k+1, \nu}(\Lambda^m T^* \hat{N})$, for $m = 0, 2, 4$, and $\eta \in \mathcal{K}$ such that

$$(\gamma, \gamma) = (d + d^*)(\gamma_0, \gamma_2, \gamma_4) + \eta.$$

By applying the Hodge star,

$$(\gamma, \gamma) = (d + d^*)(\gamma_4, *\gamma_2, *\gamma_0) + *\eta.$$

Adding the above formulae and averaging gives:

$$\gamma = d\left(\frac{\gamma_2 + *\gamma_2}{2}\right) + d^*\left(\frac{\gamma_0 + \gamma_4}{2}\right) + \tilde{\eta},$$

where $\tilde{\eta} \in \mathcal{K} \cap L^p_{k, \nu-1}(\Lambda^3 T^* \hat{N})$. We deduce that

$$L^p_{k, \nu-1}(\Lambda^3 T^* \hat{N}) = \left( (d(L^p_{k+1, \nu}(\Lambda^2 T^* \hat{N})) + d^*(L^p_{k+1, \nu}(\Lambda^3 T^* \hat{N})) \right) \oplus \mathcal{K}^3,$$

where $\mathcal{K}^3 = \mathcal{K} \cap L^p_{k, \nu-1}(\Lambda^3 T^* \hat{N})$. Moreover, for $\nu \notin \mathcal{D}'$, $\nu < -1$,

$$d(L^p_{k+1, \lambda}(\Lambda^2 T^* \hat{N})) = d(L^p_{k+1, \lambda}(\Lambda^3 T^* \hat{N})).$$

We must surely have that the images are equal for $\nu \notin \mathcal{D}, \nu < -1$, as well.

Thus, the cokernel of $(d_+ + d^*)_{\mu}$ is isomorphic to the kernel of

$$(d^* + d)_{-3-\mu} = d^* + d : L^q_{l+1, -3-\mu}(\Lambda^3 T^* \hat{N}) \to L^q_{l, -4-\mu}(\Lambda^2 T^* \hat{N} \oplus \Lambda^3 T^* \hat{N}).$$

(29)

Using (7) as in the proof of Proposition 5.3, the kernel of $(d^* + d)_{-3-(-1)} = (d^* + d)_{-2}$ is isomorphic to $\mathcal{H}^3$. By [13], Example (0.16)], $\mathcal{H}^3 \cong H^1_{\text{dR}}(\hat{N})$ and the isomorphism is given by $\gamma \mapsto [\gamma]$. Since $[\mu, -1] \cap \mathcal{D} = \emptyset$, there are no changes in the cokernel in $[\mu, -1]$ by Proposition 5.6. Moreover, the dimension of the
cokernel is lower semi-continuous in $\mu$ at $-1$; this fact can be demonstrated using similar methods to those employed in the proof of Proposition 8.3. The result follows.

By Proposition 5.3, $(-2, -1] \cap D = \emptyset$. Therefore, for any $\mu \in (-2, -1]$, $\dim \ker (d_+ + d^*)_\mu = \dim \mathcal{H}_+^2$ and $\dim \text{coker} (d_+ + d^*)_\mu = b^1(\hat{N})$, using Propositions 8.3 and 8.4. Knowing the index of $(d_+ + d^*)_\mu$ for $\mu \in (-2, -1]$, we can calculate it for all growth rates using Theorem 5.5.

**Proposition 8.5** Use the notation of Propositions 5.2 and 5.7. If $\lambda \in (1, 2)$, $\lambda \notin D$, the index of $(d_+ + d^*)_\lambda$ is given by:

$$\text{ind} (d_+ + d^*)_\lambda = \dim \mathcal{H}_+^2 - b^1(\hat{N}) - \sum_{\mu \in (-1, \lambda) \cap D} d(\mu).$$

However, the obstruction space $\mathcal{O}_1(\hat{N}, \lambda)$ given in Definition 7.4 is a subspace of the cokernel of $(d_+ + d^*)_\lambda$, so we must relate their dimensions.

**Proposition 8.6** The following inequality holds:

$$\dim \mathcal{O}_1(\hat{N}, \lambda) \leq \dim \text{coker} (d_+ + d^*)_\lambda - b^1(\hat{N}).$$

**Proof:** From the proof of Proposition 7.2, the image of $(d_+ + d^*)_\lambda = d + d^* : L^p_{k+1, \lambda}(\Lambda^2 T^* \hat{N} \oplus \Lambda^4 T^* \hat{N}) \to L^p_{k, \lambda-1}(\Lambda^3 T^* \hat{N})$

is characterised as the subspace of $L^p_{k, \lambda-1}(\Lambda^3 T^* \hat{N})$ which is $L^2$-orthogonal to the kernel $\mathcal{K}$ of $(d^* + d)_{-3-\lambda}$ defined by (29). Furthermore, as noticed in the proof of Proposition 8.3, Image $(d + d^*)_\lambda$ has finite codimension in $L^p_{k, \lambda-1}(\Lambda^3 T^* \hat{N})$. Therefore, we may choose a finite-dimensional space $\mathcal{C}$ of smooth compactly supported 3-forms on $\hat{N}$ such that

$$L^p_{k, \lambda-1}(\Lambda^3 T^* \hat{N}) = \text{Image} (d + d^*)_\lambda \oplus \mathcal{C}$$

and so that the product $\mathcal{C} \times \mathcal{K} \to \mathbb{R}$ given by $(\gamma, \eta) \mapsto \langle \gamma, \eta \rangle_{L^2}$ is nondegenerate.

We may similarly deduce that the image of $(d_+ + d^*)_\lambda$ is the subspace of $L^p_{k, \lambda-1}(\Lambda^3 T^* \hat{N})$ which is $L^2$-orthogonal to the kernel $\mathcal{K}'$ of $(d^*_+ + d)_{-3-\lambda}$. Then $\mathcal{K}' \supseteq \mathcal{K}$ and $\mathcal{K}$ consists of closed and coclosed 3-forms, whereas $\mathcal{K}'$ consists of 3-forms $\eta$ such that $d\eta = d^*_+ \eta = 0$. Hence, we may choose a subspace $\mathcal{K}''$ of $\mathcal{K}'$, transverse to $\mathcal{K}$, comprising 3-forms which are not coclosed and such that $\mathcal{K}' = \mathcal{K} \oplus \mathcal{K}''$. 

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The next stage is to extend $C$ to a space $C' = C \oplus C''$, where $C''$ consists of smooth exact compactly supported 3-forms on $\hat{N}$, such that

\[
L^p_{k, \lambda-1}(\Lambda^3 T^* \hat{N}) = \text{Image} (d_+ + d^*)_\lambda \oplus C'
\]

and such that the product $C'' \times K'' \to \mathbb{R}$ given by $(\gamma, \eta) \mapsto \langle \gamma, \eta \rangle_{L^2}$ is nondegenerate, which is possible as $K''$ comprises forms which are not coclosed. By construction, $C''$ is a valid choice for $O_1(N, \lambda)$ by Proposition 7.3. Therefore,

\[
\dim O_1(N, \lambda) = \dim C' - \dim C = \dim \text{Coker} (d_+ + d^*)_\lambda - \dim K.
\]

If $\gamma$ lies in the kernel of (29) for rate $\mu = -1$ then $\gamma \in K$ for $\lambda \in (1, 2)$ by Theorem 4.4. Thus, the map from $K$ to $H^1_{dR}(\hat{N})$ given by $\gamma \mapsto [\gamma]$ is surjective. This gives the result. \hfill \Box

We may now calculate a lower bound for the expected dimension of $\mathcal{M}_1(N, \lambda)$ using Propositions 8.5 and 8.6.

**Proposition 8.7** Using the notation of Propositions 5.2 and 5.7,

\[
\dim I_1(N, \lambda) - \dim O_1(N, \lambda) \geq \dim \mathcal{H}_+^2 - \sum_{\mu \in (-1, \lambda) \cap \mathcal{D}} d(\mu).
\]

Recalling that the dimension of $\mathcal{T}$ given in Definition 6.12 is $21\mathfrak{s}$, we derive analogous results for our other problems.

**Proposition 8.8** Using the notation of Definitions 6.12 and 6.20 and Propositions 5.2 and 5.7,

\[
\dim I_2(N, \lambda) - \dim O_2(N, \lambda) \geq \dim \mathcal{H}_+^2 + 21\mathfrak{s} - \dim \mathcal{H} - \sum_{\mu \in (-1, \lambda) \cap \mathcal{D}} d(\mu).
\]

and

\[
\dim I_3(N, \lambda) - \dim O_3(N, \lambda) \geq \dim \mathcal{H}_+^2 + 21\mathfrak{s} - \dim \mathcal{H} + \dim \mathfrak{F} - \sum_{\mu \in (-1, \lambda) \cap \mathcal{D}} d(\mu).
\]

We note that Propositions 5.2, 8.4, 8.6 and 8.7 imply the following bound on $\dim O_1(N, \lambda)$.

**Proposition 8.9** In the notation of Propositions 5.2 and 5.7,

\[
\dim O_1(N, \lambda) \leq \sum_{\mu \in (-1, \lambda) \cap \mathcal{D}} d(\mu)
\]

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We also know that, in Problem 2, we remove the obstructions which correspond to translations of the singularities and $G_2$ transformations of the tangent cones. These obstructions occur, respectively, at rates 0 and 1. Hence, $d(0) \geq 7s$, $d(1) \geq 14s - \dim H$ and we have the following stronger bound on the dimension of $\mathcal{O}_2(N, \lambda)$.

**Proposition 8.10** In the notation of Definition 6.12 and Propositions 5.2 and 5.7,

$$
\dim \mathcal{O}_2(N, \lambda) \leq -21s + \dim H + \sum_{\mu \in (-1, \lambda) \cap \mathcal{D}} d(\mu).
$$

### 9 $\varphi$-Closed 7-manifolds

For our deformation problems we have assumed the ambient manifold $(M, \varphi, g)$ is a $G_2$ manifold; that is, $M$ is endowed with a $G_2$ structure such that $d\varphi = d^\ast \varphi = 0$. However, the results of McLean [16] we have used, which are based upon the linearisation of the map we denoted $F_1$ in Definition 6.5, still hold if this condition on $\varphi$ is relaxed to just $d\varphi = 0$. Thus, our deformation theory results hold if $(M, \varphi, g)$ is a $\varphi$-closed 7-manifold in the sense of Definition 2.6.

**Remark** The effect of $d\varphi$ not being closed on $M$ means that coassociative 4-folds in $M$ are no longer necessarily volume minimizing in their homology class. This does not, however, affect our discussion.

The use of $\varphi$-closed 7-manifolds $(M, \varphi, g)$ is that closed $G_2$ structures occur in infinite-dimensional families, since the set of closed definite 3-forms on $M$, in the sense of Definition 2.3, is open. We show that we can choose a family $\mathcal{F}$, in a similar fashion to Definition 6.20 of Problem 3, of closed $G_2$ structures on $M$ such that $\dim \mathcal{F} = \dim \mathcal{O}_1(N, \lambda)$ and, further, such that $\mathcal{O}_3(N, \lambda) = \{0\}$. In other words, we have enough freedom in our choice of $\mathcal{F}$ to ensure that $dF_3|_{(0,0,0)}$, as given in Definition 6.24 maps onto $d(L^P_{k+1, \lambda}(\Lambda^2 T^* N))$. Then $\mathcal{M}_3(N, \lambda)$ is a smooth manifold near $(N, 0)$ by Theorem 7.13 and $\pi_{\mathcal{F}} : \mathcal{M}_3(N, \lambda) \to \tilde{\mathcal{F}}$ is a smooth map near $(N, 0)$.

Sard’s Theorem [11, p. 173] states that, if $f : X \to Y$ is a smooth map between finite-dimensional manifolds, the set of $y \in Y$ with some $x \in f^{-1}(y)$ such that $df|_x : T_x X \to T_y Y$ is not surjective is of measure zero in $Y$. Therefore, $f^{-1}(y)$ is a submanifold of $X$ for almost all $y \in Y$.

By Sard’s Theorem, $\pi^{-1}_\mathcal{F}(f)$ is a smooth manifold near $(N, f)$ for almost all $f \in \tilde{\mathcal{F}}$. As observed in Definition 6.24, $\pi^{-1}_\mathcal{F}(f)$ corresponds to the moduli space of deformations for Problem 2 defined using the $G_2$ structure $(\varphi^f, g^f)$. Thus,
for any given $N$, a generic perturbation of the closed $G_2$ structure within $\mathcal{F}$ ensures that $\mathcal{M}_2(N, \lambda)$ is smooth near $N$.

We thus prove the following, which is similar to the result [7, Theorem 9.1].

**Theorem 9.1** Let $(M, \varphi, g)$ be a $\varphi$-closed 7-manifold in the sense of Definition 2.6 and let $N$ in $(M, \varphi, g)$ be a CS coassociative 4-fold at $z_1, \ldots, z_s$ with rate $\lambda \in (1, 2) \setminus \mathcal{D}$, where $\mathcal{D}$ is defined in Proposition 5.2. Use the notation of Definitions 6.12, 7.4 and Proposition 5.7. Let $m = \dim \mathcal{O}_1(N, \lambda)$ and let $\hat{F}$ be an open ball about 0 in $\mathbb{R}^m$. There exists a smooth family $\mathcal{F} = \{\varphi_f, g_f\} : f \in \hat{F}$ of closed $G_2$ structures on $M$ such that $\mathcal{O}_1(N, \lambda) = \{0\}$. Hence, the moduli space of deformations for Problem 3 is a smooth manifold near $(N, 0)$ of dimension greater than or equal to

$$\dim \mathcal{H}_+^2 + 21s - \dim \mathcal{H} + \dim \mathcal{O}_1(N, \lambda) - \sum_{\mu \in (-1, \lambda) \cap \mathcal{D}} d(\mu).$$

Moreover, for generic $f \in \hat{F}$, the moduli space of deformations in $(M, \varphi_f, g_f)$ for Problem 2 is a smooth manifold near $N$ of dimension greater than or equal to

$$\dim \mathcal{H}_+^2 + 21s - \dim \mathcal{H} - \sum_{\mu \in (-1, \lambda) \cap \mathcal{D}} d(\mu).$$

**Proof:** Use the notation in the proof of Proposition 8.6. Recall that we have a subspace $K''$ of $L^q_{t+1, -3-\lambda}(\Lambda^3 T^* \hat{N})$ consisting of forms $\eta$ such that $d\eta = d^* \eta = 0$ but $d^* \eta \neq 0$. Moreover, $\mathcal{O}_1(N, \lambda)$ can be chosen to be a space of smooth compactly supported exact 3-forms $\gamma$ such that $\langle \gamma, \eta \rangle_{L^2} = 0$ for all $\eta \in K'' \setminus \{0\}$ implies that $\gamma = 0$. Therefore $K'' \cong (\mathcal{O}_1(N, \lambda))^*$ and hence has dimension $m$.

Let $\{\eta_1, \ldots, \eta_m\}$ be a basis for $K''$ and choose a basis $\{dv_1, \ldots, dv_m\}$ for $\mathcal{O}_1(N, \lambda)$, where $v_j$ is a smooth compactly supported 2-form for all $j$, such that $\langle dv_1, \eta_j \rangle_{L^2} = \delta_{ij}$. This is possible because the $L^2$ product on $\mathcal{O}_1(N, \lambda) \times K''$ is nondegenerate. For $f = (f_1, \ldots, f_m) \in \mathbb{R}^m$ define

$$v_f = \sum_{j=1}^m f_j v_j.$$  

Using the notation of Proposition 6.13 define $(\varphi_f, g_f)$, for $f$ in a sufficiently small open ball $\hat{F}$ about 0 in $\mathbb{R}^m$, to be a closed $G_2$ structure on $M$ such that $\Xi(\varphi_f|_{\hat{F}}) = 0$ in $H^3_{cs}(\hat{N})$ and $\varphi_f|_{\hat{N}} = dv_f$. Recall from Definitions 6.24 and 7.12 that we have a linear map $L_3 : T_0 \hat{F} \cong \mathbb{R}^m \to d(L^q_{t+1, -1}(\Lambda^2 T^* \hat{N}))$ arising from $dF_3|_{(0,0,0)}$. By construction, $L_3(f) = dv_f$ for $f \in \mathbb{R}^m$ and hence $L_3$ maps...
onto $O_1(N, \lambda)$. Proposition 7.3 and Definition 7.12 imply that $O_3(N, \lambda) = \{0\}$ as required.

The latter parts of the theorem follow from the discussion preceding it and Proposition 5.3.

\[ \square \]

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