BILINEAR FORMS IN WEYL SUMS FOR MODULAR SQUARE ROOTS AND APPLICATIONS

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Dedicated to Bruce Berndt and his penchant for Gauss sums, on the occasion of his 80th birthday.

Abstract. Let \( q \equiv 3 \pmod{4} \) be a prime, \( P \geq 1 \) and let \( N_q(P) \) denote the number of rational primes \( p \leq P \) that split in the imaginary quadratic field \( \mathbb{Q}(\sqrt{-q}) \). The first part of this paper establishes various unconditional and conditional (under existence of a Siegel zero) lower bounds for \( N_q(P) \) in the range \( q^{1/4+\varepsilon} \leq P \leq q \), for any fixed \( \varepsilon > 0 \). This improves upon what is implied by work of Pollack and Benli–Pollack.

The second part of this paper is dedicated to proving an estimate for a bilinear form involving Weyl sums for modular square roots (equivalently Salié sums). Our estimate has a power saving in the so-called Pólya–Vinogradov range, and our methods involve studying an additive energy coming from quadratic residues in \( \mathbb{F}_q \).

This bilinear form is inspired by the recent automorphic motivation: the second moment for twisted \( L \)-functions attached to Kohnen newforms has recently been computed by the first and third authors. So the third part of this paper outlines the arithmetic applications of this bilinear form. These include the equidistribution of quadratic roots of split primes, products of primes, and relaxations of a conjecture of Erdős–Odlyzko–Sárközy.

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1. Introduction

1.1. Motivation and description of our results. Our motivation begins in the early 20th century. This is when I. M. Vinogradov initiated the study of the distribution of both quadratic residues and non-residues modulo a prime $q$. This remains a central theme in classical analytic number theory.
Several fundamental conjectures are still unresolved. Vinogradov’s least quadratic non-residue conjecture asserts that
\[ n_q \leq q^{\alpha(1)}, \tag{1.1} \]
where \( n_q \) denotes the least non-quadratic residue modulo \( q \) (\( n_q \) is necessarily prime). The best unconditional bounds known are
\[ n_q \leq q^{1/4 + o(1)}, \]
largely due to Burgess’ bounds for character sums [Bur57] and ideas of Vinogradov.

It is also natural to consider the least prime quadratic residue \( r_q \). The best unconditional result is due to Linnik and Vinogradov [VL66], where they showed
\[ r_q \leq q^{1/4 + o(1)}. \tag{1.2} \]

Much of the above discussion can be generalised to prime residues and non-residues of an arbitrary Dirichlet character \( \chi \) of order \( k \). One can see the work of Norton [Nor98] for the analogue of (1.1) and of Elliot [Ell71] for the analogue of (1.2).

Given the existence of a small prime quadratic residue and non-residue, the next natural question to ask is how many such quadratic residues and non-residues exist in a given interval \([2,P]\). Making use of reciprocity relations and the sieve, Benli and Pollack [BP19] have results in this direction for quadratic, cubic and biquadratic residues. For general Dirichlet characters one can see work of Pollack [Pol17] and also a more recent work of Benli [Ben19]. The impact and links of results of this type stretch far beyond analytic number theory. For example, Bourgain and Lindenstrauss [BL03, Theorem 5.1], motivated by links with the Arithmetic Quantum Unique Ergodicity Conjecture, have shown that for any \( \varepsilon > 0 \) there exist some \( \delta > 0 \) such that for any sufficiently large \( D \) for the set
\[ \mathcal{R} := \left\{ p \text{ prime} : D^\delta \leq p \leq D^{1/4 + \varepsilon}, \left( \frac{D}{p} \right) = -1 \right\}, \]
we have
\[ \sum_{p \in \mathcal{R}} \frac{1}{p} \geq \frac{1}{2} - \varepsilon. \]
When one relaxes the condition that a non-residue be prime, one does much better, as in the case of Banks, Garaev, Heath-Brown and Shparlinski [BGHBS08] show that for each \( \varepsilon > 0 \) and \( N \geq q^{1/4 + \varepsilon} \), the proportion of quadratic non-residues modulo \( q \) in the interval \([1,N]\) is bounded away from 0 when \( q \) is large enough.
Furthermore, if \( q \equiv 3 \pmod{4} \) be a large prime, then quadratic reciprocity tells us that the following two Legendre symbols are equal:
\[
\left( \frac{-q}{p} \right) = \left( \frac{p}{q} \right).
\]
Thus counting quadratic residues modulo \( q \) is equivalent to counting count small rational primes \( p \leq P \) that split in the imaginary quadratic field \( F := \mathbb{Q}(\sqrt{-q}) \). In particular, throughout this paper,
\[
\chi(\cdot) := \left( \frac{-q}{\cdot} \right)
\]
defined via the Jacobi symbol, always denotes the character attached to \( F \), and \( L(s, \chi) \) denotes the Dirichlet \( L \)-function attached to \( \chi \). Let \( N_q(P) \) denote the number of rational primes \( p \leq P \) that split in \( F \).

A good reference point for the strength of our results is what the Generalised Riemann Hypothesis (GRH) implies. It is well known that under the GRH for \( L(s, \chi) \) that we have \( N_q(P) \geq c_1 P/\log P \) for \( P \geq c_2(\log q)^2 \) for some absolute constants \( c_1, c_2 > 0 \), see, for example, [MV07, Section 13.1, Exercise 5(a)].

Furthermore, a result of Heath-Brown [HB95, Theorem 1] immediately implies that for any fixed \( \varepsilon > 0 \) all but \( o(Q/\log Q) \) primes \( q \leq [Q, 2Q] \), for \( P \geq q^\varepsilon \) we have \( N_q(P) = (1/2 + o(1))P/\log P \) as \( Q \to \infty \). The second theme which we develop in this paper concerns bounds of certain bilinear sums closely related to correlations between values of Salie sums

\[
S(m, n; q) = \sum_{x \in \mathbb{F}_q} \left( \frac{x}{q} \right) e_q(mx + nx)
\]
see [Sal32]. We emphasise that this is closely related to recent work of the first and third authors [DZ19], who have computed a second moment for \( L \)-functions attached to a half-integral weight Kohnen newform, averaged over all primitive characters modulo a prime. Power savings in the error term for such a moment come in part from savings in the bound on correlations between Salie sums (1.3). Our argument gives a direct improvement of [DZ19, Theorem 1.2], see Appendix C. It remains to investigate whether this improvement propagates into a quantitative improvements in the error term of [DZ19, Theorem 1.1] for the second moment of the above \( L \)-functions.

More precisely, given two real numbers \( M, N \) and complex weights \( \alpha = (\alpha_m)_{m \sim M} \) and \( \beta = (\beta_n)_{n \sim N} \), supported on dyadic intervals \( m \sim M \) and \( n \sim N \), where \( a \sim A \) indicates \( a \in [A, 2A] \). Let \( K : \mathbb{F}_q \to \mathbb{C} \) be some function, usually called a kernel. A bilinear form involving \( K \) is
a sum of the shape
\[ \sum_{m \sim M} \sum_{n \sim N} \alpha_m \alpha_n K(mn). \]

Bounds of such sums also have key automorphic and arithmetic applications.

In a series of two recent breakthrough papers using deep algebro-geometric techniques, Kowalski, Michel and Sawin [KMS17, KMS19] have established non-trivial estimates for bilinear forms with Kloosterman sum in [KMS17] and generalised Kloosterman sum in [KMS19]. In particular, their estimates apply below the Pólya–Vinogradov range, that is, when the ranges of summation \( M, N \sim q^{1/2} \), this is where completion and Fourier theoretic methods breakdown. Such bounds are a crucial ingredient to the evaluation of asymptotic moments of \( GL(2) \) \( L \)-functions over primitive Dirichlet characters, with power saving error (as well as many more automorphic applications), see [BFK+17, KMS17, KMS19, Zac17] and references therein.

It is important to note that the bilinear sums we study here do not fall under the umbrella of the results of [KMS19]. The initial approach of [KMS19] (Vinogradov \( ab \) shifting and the Riemann Hypothesis for algebraic curves over finite field) can be used. However, our approach leads to a much stronger result, for comparison see Appendix B.

The above two themes come together in the third direction which we pursue here. Namely we study the distribution of square roots modulo \( q \) of primes \( p \leq P \). In the asymptotic formula we obtain, the main term is controlled by the counting function of primes quadratic residues, while the error term is given by the discrepancy, and depends on the quality of our bounds of certain bilinear sums we estimate in this paper.

We are now ready to state some of the results in this paper. A high level sketch of the ideas and methodology in the proofs is deferred to Section 2.

1.2. Counting split primes in imaginary quadratic extensions. Our first result is an unconditional lower bound for \( N_q(P) \), but with ineffective constant.

**Theorem 1.1.** For any fixed \( \varepsilon > 0 \), any sufficiently large prime \( q \), and any \( P \) with \( q \geq P \geq q^{1/4+\varepsilon} \), we have
\[ N_q(P) \geq c(\varepsilon) \min \left\{ P^{1/2} q^{-\varepsilon/2}, P q^{-1/4-2\varepsilon/3} \right\}, \]
where \( c(\varepsilon) > 0 \) depends only on \( \varepsilon \).
Remark 1.2. As with the results on quadratic residues of Benli and Pollack [Ben19, BP19, Pol17], Siegel’s theorem is used in the proof of Theorem 1.1, and so the constant $c(\varepsilon)$ is ineffective.

Observe that for $\varepsilon > 0$ small and $A > 0$ fixed [Pol17, Theorem 1.3] guarantees that

$$N_q(q^{1/4+\varepsilon}) \geq c(\varepsilon, A)(\log q)^A,$$

for some constant $c(\varepsilon, A) > 0$ that depends only on $\varepsilon$ and $A$. Theorem 1.1 above improves this to a small power of $q$, that is,

$$N_q(q^{1/4+\varepsilon}) \geq c(\varepsilon)q^{\varepsilon/3}.$$

This was also proved independently by Benli [Ben19] (taken with $k = 2$) very recently. Theorem 1.1 also substantially improves the lower bound

$$N_q(P) \geq P^{1/25}q^{-1/50}, \quad q^{1/2+\varepsilon} \leq P \leq q,$$

established by Benli and Pollack [BP19, Theorem 3] and in particular, implies

$$N(q^{1/2+\varepsilon}) \geq q^{1/4}.$$

for any $\varepsilon > 0$, provided that $q$ is large enough.

Our next result is an unconditional bound for $N_q := N_q(q)$, with effective constant.

**Theorem 1.3.** Suppose $q \geq 67$ is prime with $q \equiv 3 \pmod{16}$. Then

$$N_q > \left(\frac{2 - \log(3\sqrt{2})}{2}\right)\frac{\lfloor\sqrt{3q/4}\rfloor}{\log q}.$$

Since

$$\frac{(2 - \log(3\sqrt{2}))\sqrt{3}}{4} = 0.2402\ldots,$$

we see from Theorem 1.3 that there is an effectively computable absolute constant $c_0$ such that for $q \geq c_0$ we have

$$N_q > \frac{0.24\sqrt{q}}{\log q}.$$

In fact one can get a better constant by estimating certain quantities more carefully.

**Remark 1.4.** Unfortunately the argument of the proof of Theorem 1.3 does not scale to estimate $N_q(P)$ with $P < q$. However it actually increases its strength for $P > q$, which is still a meaningful range in the problem of estimating $N_q(P)$. We do not consider this case as small values of $P$ are of our principal interest.
Many famous unsolved conjectures are known to hold under the assumption of a Siegel zero [FI09]. This is because one can sometimes break the parity problem of the sieve with this hypothesis. A notable example is Heath-Brown’s proof [HB83] of the twin prime conjecture assuming Siegel zeros. Continuing this tradition, we prove an essentially sharp lower bound for $N_q(P)$ under the assumption that $L(s, \chi)$ has a mild Siegel zero.

**Theorem 1.5.** Suppose $q \equiv 3 \pmod{4}$ is a large prime and

$$L(1, \chi) = O\left(\frac{1}{(\log q)^{10}}\right).$$

Then for any fixed $\varepsilon > 0$ and $P$ with $q^{1/2+\varepsilon} \leq P \leq q$, we have

$$N_q(P) \geq c(\varepsilon) h(-q) \frac{P}{\sqrt{q(\log q)^2}},$$

where $h(-q)$ is the class number of $F = \mathbb{Q}(\sqrt{-q})$ and $c(\varepsilon) > 0$ depends only on $\varepsilon$.

**Remark 1.6.** The constant $c(\varepsilon)$ in Theorem 1.5 is ineffective.

Counting small split primes in general number fields is a notoriously difficult and fundamental problem. Ellenberg and Venkatesh [EV07] have established a direct connection between counts for small split primes and bounds for $\ell$-torsion in general class groups.

### 1.3. Bilinear forms and equidistribution.

We recall that $a \sim A$ means $a \in [A, 2A)$. Given two real numbers $M, N$ and complex weights

$$\alpha = (\alpha_m)_{m \sim M} \quad \text{and} \quad \beta = (\beta_n)_{n \sim N},$$

we denote

$$\|\alpha\|_\infty := \max_{m \sim M} |\alpha_m| \quad \text{and} \quad \|\alpha\|_\sigma := \left(\sum_{m \sim M} |\alpha_m|^\sigma\right)^{\frac{1}{\sigma}},$$

and similarly for $\beta$.

For $a, h \in \mathbb{F}_q^\times$ we consider bilinear forms in Weyl sums for square roots

$$W_{a,q}(\alpha, \beta; h, M, N) = \sum_{m \sim M} \sum_{n \sim N} \alpha_m \beta_n \sum_{x \in \mathbb{F}_q^{\times}} e_q(hx).$$

(1.5)

We notice that this is equivalent to studying bilinear forms with Salié sums, given by (1.3), thanks to the evaluation in [Sal32], see also [IK04,
Lemma 12.4] and [Sar90, Lemma 4.4]:
\[
\frac{1}{\sqrt{q}} S(m, n; q) = \frac{1}{\sqrt{q}} S(1, mn; q) = \varepsilon_q \left( \frac{n}{q} \right) \sum_{x \in \mathbb{F}_q} e_q(2x) \tag{1.6}
\]
where \( \varepsilon_q = 1 \) if \( q \equiv 1 \pmod{4} \) and \( \varepsilon_q = i \) if \( q \equiv 3 \pmod{4} \), (note that if \( (mn/q) = -1 \), the Salié sums vanishes). The tuple of characters \((1, (\cdot / q))\) is Kummer induced (cf. [KMS19, Section 2]), and so the results of [KMS19] do not apply to (1.5).

Our goal is to improve the trivial bound
\[
W_{a,q}(\alpha, \beta; h, M, N) = O \left( \|\alpha\|_1 \|\beta\|_1 \right),
\]
in the Pólya–Vinogradov range. In arithmetic applications the case when weights satisfy
\[
\|\alpha\|_\infty, \|\beta\|_\infty = q^{o(1)} \tag{1.7}
\]
is most important. However, our bounds are valid for more general weights.

Making use of weighted additive energies, the methods of Weyl for quadratic exponential sums and ideas from [CCG+14], we prove the following result.

**Theorem 1.7.** For any positive integers \( M, N \leq q \) and any weights \( \alpha \) and \( \beta \) as in (1.4), we have
\[
|W_{a,q}(\alpha, \beta; h, M, N)| \leq \|\alpha\|_2 \|\beta\|_1^{3/4} \|\beta\|_\infty^{1/4} q^{1/8+o(1)} \left( M^{1/2} q^{-1/8} + M^{5/16} \right) \left( N^{1/4} q^{-1/8} + N^{1/16} \right).
\]

When the weights \( \alpha \) and \( \beta \) satisfy (1.7), we can re-write the bound of Theorem 1.7 in the following simplified form
\[
|W_{a,q}(\alpha, \beta; h, M, N)| \leq (MN)^{13/16} q^{1/8+o(1)} \left( M^{3/16} q^{-1/8} + 1 \right) \left( N^{3/16} q^{-1/8} + 1 \right). \tag{1.8}
\]

Power savings in the error term of an asymptotic formula for a second moment of certain \( L \)-functions [DZ19] comes from savings in the bound on \( W_{a,q}(\alpha, \beta; h, M, N) \) in the Pólya–Vinogradov range, as well as other ranges in which spectral techniques are used. Additional ideas are also needed in [DZ19] to make the moment unconditional, because \( \alpha_m \) and \( \beta_n \) are coefficients of a fixed normalised Kohnen newform. It is not known yet that they satisfy (1.7) (in this context, the condition (1.7) is the Ramanujan–Petersson conjecture, also equivalent to the Lindelöf hypothesis for the twisted \( L \)-function attached to the Shimura lift). The best known bound is \( \alpha_m = O \left( m^{1/6+\varepsilon} \right) \) due to Conrey and Iwaniec [CI00].
We now outline the arithmetic applications of Theorem 1.7.

We recall that the discrepancy $D(N)$ of a sequence in $\xi_1, \ldots, \xi_N \in [0, 1)$ is defined as

$$D_N = \sup_{0 \leq \gamma \leq 1} |\#\{1 \leq n \leq N : \xi_n \in [0, \gamma)\} - \gamma N|.$$  \hfill (1.9)

For positive integers $P$ and $Q$ we denote by $\Delta_q(P, R)$ the discrepancy of the sequence (multiset) of points

$$\{x/q : x^2 \equiv pr \pmod{q} \text{ for some primes } p \leq P, \ r \leq R\}.$$

Combining the bound (1.8) with the classical Erdős–Turán inequality, see Lemma 3.1 below, we derive an equidistribution of the modular square roots of products of two primes.

**Corollary 1.8.** For $1 \leq P, R \leq q$ we have

$$\Delta_q(P, R) \leq (PR)^{13/16} q^{1/8+o(1)} \left(P^{3/16} q^{-1/8} + 1\right) \left(R^{3/16} q^{-1/8} + 1\right).$$

Our next application is to a relaxed version of the still open problem of Erdős, Odlyzko and Sárközy [EOS87] on the representation of all reduced classes modulo an integer $m$ as the products $pr$ of two primes $p, r \leq m$. This question has turned out to be too hard even for the Generalised Riemann Hypothesis, thus various relaxations have been considered, see [Shp19] for a short overview of currently available results in this direction.

Here we obtain the following variant about products of two small primes and a small square. To simplify the result we assume that $p, r \leq q^{2/3}$. In this case one easily derives similarly to Corollary 1.8 the following result.

**Corollary 1.9.** Let $P, R \leq q^{2/3}$ be real numbers such that the interval $[2, P]$ contains $P^{1+o(1)}$ of both prime quadratic residues and non-residues. If $q \geq S \geq 1$ is a real number such that

$$(PR)^{3/16} S \geq q^{9/8+\varepsilon},$$

then any reduced residue class modulo $q$ can be represented as $prs^2$ for two primes $p \leq P, \ r \leq R$ with some real $P, R \leq q^{2/3}$ and a positive integer $s \leq S$.

In particular, if $P = R = S$ then the result of Corollary 1.9 is nontrivial if $P \geq q^{9/11+\varepsilon}$, while for $P = R = q$ we need $S \geq q^{3/4+\varepsilon}$.

1.4. Distribution of roots of primes. For a positive integer $P$ we denote by $\Gamma_q(P)$ the discrepancy of the sequence (multiset) of points

$$\{x/q : x^2 \equiv p \pmod{q} \text{ for some prime } p \leq P\}.$$
Theorem 1.7 in combination with the classical Erdős–Turán inequality (see Lemma 3.1) and the Heath-Brown identity [HB82] yield the following result on the equidistribution of square roots of split primes.

**Theorem 1.10.** We have

\[
\Gamma_q(P) \leq P^{13/16} q^{1/8 + o(1)}.
\]

If the interval \([2, P]\) contains \(P^{1+o(1)}\) prime quadratic residues, which is certainly expected and is known under the GRH, see Sections 1.1 and 1.2, then Theorem 1.10 is nontrivial for \(P \geq q^{2/3 + \varepsilon}\) with some fixed \(\varepsilon > 0\), while for \(P = q\) we get \(\Gamma_q(q) \leq q^{15/16 + o(1)}\).

We also point out that the equidistribution here is the opposite situation considered in [DFI95, DFI12, Hom08, LM15, Tót00], where the modulus is prime and varies.

2. **High level sketch of the methods**

Here we outline the main ideas and methods behind the proofs in this paper, without paying too much attention to technical detail. Let \(q \geq P \geq q^{1/4 + \varepsilon}\). The starting point in Theorem 1.1 is a certain linear combination of logarithms that biases split primes. In particular, consider

\[
Q(P) := \sum_{n \sim P} r(n) \log n \in \mathbb{R},
\]

where \(r(n) = R_{-q}(n)/2\), and \(R_{-q}(n)\) is the number of representations of \(n\) by a complete set of inequivalent positive definite quadratic forms of discriminant \(-q\).

The behaviour of the character \(\chi(n) := (-q/n)\) governs \(r(n)\), the mean value of \(r(n)\) and hence \(Q(P)\). Observe that by [IK04, Equation (22.22)], for \(q \geq 5\) we have

\[
R_{-q}(n) = 2r(n) = 2 \prod_{p \mid |n} \left(1 + \chi(p) + \cdots + \chi^f(p)\right), \quad \gcd(n, q) = 1.
\]

(2.1)

Classical work of Linnik and Vinogradov [VL66], and subsequent refinements by Pollack [Pol17] give a mean value asymptotic of the shape

\[
\sum_{n \leq x} r(n) \sim L(1, \chi)x, \quad x \geq q^{1/4 + \varepsilon} \quad \text{and} \quad x \to \infty.
\]

(2.2)

Thus

\[
Q(P) \sim \frac{1}{2} L(1, \chi) P \log P \quad \text{as} \quad P \to \infty.
\]

(2.3)

In reality, (2.2) comes equipped with a power saving error term \(O(x^{1-\eta})\) for some \(\eta > 0\) depending only on \(\varepsilon\), but our arguments require the main
term to dominate. Thus an application of Siegel’s theorem renders (2.2) and (2.3) ineffective, and so Theorem 1.1 has an ineffective constant. It is worth pointing out that the condition $x \geq q^{1/4+\varepsilon}$ in (2.2) comes directly from Burgess bounds for character sums.

From (2.1), it is clear that $r$ is a multiplicative function that spikes powers of large split primes. Thus we would expect a large asymptotic contribution to $Q(P)$ from integers divisible by powers of large split primes.

To quantify this, one can write

$$Q(P) = \sum_p c_p(P) \log p, \quad c_p(P) \in \mathbb{Z}_{\geq 0}. \quad (2.4)$$

For a parameter $2 \leq y \leq P$, one can further consider separately the sum in (2.4) over split primes $p \leq y$, inert primes $2 \leq p \leq P$ and over split primes $y < p \leq P$. The rest of the proof chooses the largest possible $y$ such that the sum over split primes $y < p \leq P$ contributes at least $\varepsilon L(1, \chi)P \log P$ to (2.3). From there it is routine to show that this implies a lower bound of $yq^{o(1)}$ split primes $y < p \leq P$.

The coefficients $c_p(P)$ are approximately mean values of the $r(b)$ for $b \leq P/p$. Thus to bring (2.3) into play, we need $Pq^{-1/4-o(1)} \geq y$, and so we see that Burgess bounds directly impact our method. On the other hand, to control the asymptotic contribution of small split primes, we need $y \leq P^{1/2-o(1)}$. See Section 4.2 for the full proof.

Theorem 1.3 is an effective lower bound for the number of split primes $P < q$. We consider the principal form

$$Q(U, V) = U^2 + \frac{q+1}{4} V^2.$$ 

Our strategy here is to consider the product

$$R_q := \prod_{1 \leq n \leq t} Q(n, 1), \quad t = \left\lceil \frac{\sqrt{3q}}{4} \right\rceil.$$ 

Each prime $p \mid R_q$ is split and satisfies $2 \leq p \leq q$. We obtain a lower bound on $\omega(R_q)$ by estimating $\text{ord}_p(R_q)$. We use a combination of Hensel lifting and Stirling’s formula to do this. This can be found in Section 4.3.

We now shift our attention to Theorem 1.5. Now $q \geq P \geq q^{1/2+\varepsilon}$ and we want to count split primes in $[2, P]$. A natural starting point would be to search for primes represented by one the $h(-q)$ classes of binary quadratic forms. Suppose

$$F(U, V) = AU^2 + BUV + CV^2.$$
is a such a reduced form, that is, 
\[ \gcd(A, B, C) = 1 \quad \text{and} \quad |B| \leq A \leq C, \]
and \( B^2 - 4AC = -q \) and define 
\[ \pi_F(x) := \{ p \leq x : p = f(u, v) \text{ for some } (u, v) \in \mathbb{Z}^2 \}. \]

The Chebatorev Density Theorem [Tsc26], see also [LO77], implies that primes are asymptotically equidistributed amongst form classes 
\[ \pi_F(x) \sim \frac{\delta_F x}{h(-q) \log x} \quad \text{as} \quad x \to \infty, \quad (2.5) \]
where \( \delta_F = 1/2 \) or 1 depending on whether \( F(U, V) \) and \( F(U, -V) \) are \( SL_2(\mathbb{Z}) \)-equivalent in the class group or not. The asymptotic (2.5) in [LO77] requires \( x \) exponentially larger than \( q \), which certainly is insufficient for the range we are interested. If one assumes the GRH for Hecke \( L \)-functions, then the same arguments of Lagarias and Odlyzko extend (2.5) to the range \( x \geq q^{1+\varepsilon} \), referred to as the \textit{GRH range}. This narrowly misses the range we are interested in.

However, the GRH range does not take into account the size of the coefficients \( F \), and one might expect primes to appear with relative frequency when \( x \geq C^{1+\varepsilon} \), referred to as the \textit{optimal range} (cf. [Zam18]). Zaman [Zam18, Theorem 1.1] has established upper bounds for \( \pi_F(x) \) in the optimal range under the GRH for \( L(s, \chi) \), but we are far away from any type of unconditional lower bound. One can see a recent impressive work of Thorner and Zaman [TZ19] that in particular shows the the least prime represented by \( F \) is \( O(q^{700}) \) (improving work of [Fog62, Wei83, KM02]).

Duke’s equidistribution theorem on Heegner points on the modular surface [Duk88] gives a positive proportion of reduced forms whose coefficients satisfy \( \max\{|A|, |B|, |C|\} = O(\sqrt{q}) \) (note that Duke’s theorem contains an ineffective constant, which is one source of ineffectivity of Theorem 1.5). The proof of Theorem 1.5 proceeds initially in the spirit above: the \( \beta \)-sieve, as in [FI10, Theorem 11.13], is applied to each form. For some absolute constant \( c > 0 \), each form produces at least \( cP/\sqrt{q} \log^2 q \) integers with a bounded number of prime factors (one can also see [Zam18, Theorem 1.5], however we sieve in only one variable for simplicity).

One runs immediately runs into the parity problem of the sieve, and this is where the Siegel zero comes in. It is necessary for us to collect each sieved set produced from each form into a single set, \( \mathcal{F}_{-q} \), of size 
\[ \#\mathcal{F}_{-q} \geq c h(-q) \frac{P}{\sqrt{q} \log^2 q} \]
for some absolute constant $c > 0$. Now $\mathcal{F}_{-q}$ is large enough for us to use the Siegel zero. For a split prime of medium size, it remains sieve out integers of the form $n = pm$ from $\mathcal{F}_{-q}$, where $R_{-q}(m) \geq 1$. Observe that (2.2) tells us that there are fewer than normal such $n$, approximately $O(L(1, \chi)P/p)$. A similar argument using (2.2) tells us there are fewer than normal split primes $p \leq P^{1/2}$, roughly $O(L(1, \chi)^{1/3}P^{1/2})$ many. This turns out to be enough to break parity, and the full proof of Theorem 1.5 can be found Section 5.

The heart of Theorem 1.7 relies on a non-trivial estimate for a certain weighted additive energy, which can be found in Section 6.1. For a complex weight $\beta$ as in (1.4) and $j \in \mathbb{F}_q^\times$, we study

$$E_{q,j}(\beta) := \sum_{(u,v,x,y) \in \mathbb{F}_q^4, u+y=x+v} \beta_{ju^2} \beta_{jv^2} \beta_{jx^2} \beta_{jy^2},$$

where $ju^2, jv^2, jx^2, jy^2$ are all computed modulo $q$ and take the value of the reduced residue between 1 and $q$. We omit the subscript $j$ when $j = 1$. Quantities of this type are well known in additive combinatorics under the name of additive energy. Algebraic manipulations reduce such a problem to counting the number $\mathbb{F}_q$-rational points to curve of the form

$$y^2 = f(x), \quad f \in \mathbb{F}_q[x] \text{ with } \deg f = 2,$$

all lying in a small box. Since the degree of $f$ is small, this situation is well handled by methods of Weyl for quadratic exponential sums, and worked out in [CCG+14, Theorem 5]. One should also note a similar principles are implicit in the recent work of Dunn and Zaharescu [DZ19], where instead of studying energies, the distribution of $\alpha n^2$ is used.

It is interesting to note that in the Type I setting, that is when for the weight $\beta$ we have $\beta_n = 1$ identically, Theorem 1.7 (in the Pólya–Vinogradov range) surpasses the bound implied by the standard argument using Vinogradov’s shifting by $ab$-trick and Weil’s Riemann hypothesis for curves over a finite field. The details of this are worked out in Appendix B, and are of independent interest.

3. Preliminaries

3.1. Notation. Throughout the paper, the notation $U = O(V)$, $U \ll V$ and $V \gg U$ are equivalent to $|U| \leq cV$ for some positive constant $c$, which throughout the paper may depend on small real positive parameter $\varepsilon$. 
For any quantity $V > 1$ we write $U = V^{o(1)}$ (as $V \to \infty$) to indicate a function of $V$ which satisfies $|U| \leq V^\varepsilon$ for any $\varepsilon > 0$, provided $V$ is large enough.

For a real $A > 0$ we write $a \sim A$ to indicate that $a$ is in the dyadic interval $A \leq a < 2A$.

For $\xi \in \mathbb{R}$, and $m \in \mathbb{N}$ we denote
\[ e(\xi) = \exp(2\pi i \xi) \quad \text{and} \quad e_m(\xi) = \exp(2\pi i \xi/m). \]

We also use $(k/q)$ to denote the Legendre symbol of $k$ modulo $q$, a prime.

We always use the letter $p$, with or without subscript, to denote a prime number.

We use $\mathbb{F}_q$ to denote the finite field of $q$ elements, which we assume to be represented by the set $\{0, \ldots, q-1\}$ for $q$ prime.

As usual, for an integer $a$ with $\gcd(a, q) = 1$ we define $\overline{a}$ by the conditions
\[ a\overline{a} \equiv 1 \pmod{q} \quad \text{and} \quad \overline{a} \in \{1, \ldots, q-1\}. \]

We also use $1_S$ to denote the characteristic function of a set $S$.

3.2. Exponential sums and discrepancy. We start with recalling the classical Erdős–Turán inequality (see, for instance, [KN74, Theorem 2.5]).

**Lemma 3.1.** Let $x_n, n \in \mathbb{N}$, be a sequence in $[0, 1)$. Then for any $H \in \mathbb{N}$, the discrepancy $D_N$ given by (1.9), we have
\[ D_N \leq 3 \left( \frac{N}{H+1} + \frac{1}{H} \left| \sum_{n=1}^{N} e(h \xi_n) \right| \right). \]

It is now useful to recall the definition of the Gauss sum for odd prime modulus
\[ G_q(a, b, q) := \sum_{x \in \mathbb{F}_q} e_q(ax^2 + bx), \quad (a, b) \in \mathbb{F}_q^\times \times \mathbb{F}_q. \quad (3.1) \]

The standard evaluation in [BEW98, Section 1.5] or [Dav00, Section 2] or [IK04, Theorem 3.3] leads to the formula
\[ G_q(a, b) = e_q\left( -\frac{4ab^2}{q} \right) \varepsilon_q \sqrt{q} \left( \frac{a}{q} \right). \quad (3.2) \]

where
\[ \varepsilon_q = \begin{cases} 1 & \text{if } q \equiv 1 \pmod{4}, \\ i & \text{if } q \equiv -1 \pmod{4}. \end{cases} \]
We also need the following bound for exponential sums over square roots.

**Lemma 3.2.** For any integer $W \leq q$ we have

$$\sum_{w=1}^{W} \sum_{x \in \mathbb{F}_q, x^2 = w} e_q(ax) \ll q^{1/2+o(1)}.$$  

**Proof.** Completing the exponential sum as in [IK04, Section 12.2] gives that

$$\sum_{w=1}^{W} \sum_{x \in \mathbb{F}_q, x^2 = aw} e_q(hx) \ll \log q \max_{0 \leq t \leq q-1} |G_q(t)\alpha|.$$  \hspace{1cm} (3.3)

The value $t = 0$ does not contribute anything and for any $t \in \mathbb{F}_q^\times$ use (3.2).  

3.3. **Siegel’s theorem.** We recall that the celebrated result of Siegel gives the following lower bound on $L(1, \chi)$, see [Dav00, Chapter 21].

**Lemma 3.3.** For any $\delta > 0$, there is a constant $C(\delta) > 0$ such that

$$L(1, \chi) \geq C(\delta) q^{-\delta}.$$ 

Any ineffectiveness in our results come only from Lemma 3.3 since the constant $C(\delta)$ is not ineffective.

4. **Counting small split primes unconditionally: Proofs of Theorems 1.1 and 1.3**

4.1. **Preparations.** Here we recall an asymptotic formula due to Pollack [Pol17, Proposition 3.1] that builds on some work of Linnik and Vinogradov [VL66, Theorem 2]. It is used extensively in the proofs of Theorems 1.1 and 1.5. Recalling that $\chi(\cdot) := (-q/\cdot)$, let

$$r(n) := \sum_{d|n} \chi(d).$$

**Lemma 4.1.** For each $\varepsilon > 0$, there is a constant $\eta > 0$ for which the following holds: if $x \geq q^{1/4+\varepsilon}$, then the sum

$$\sum_{n \leq x} r(n) = L(1, \chi)x + O(x^{1-\eta}).$$

where the implied constant depends only on $\varepsilon$.  

Thus by Lemma 4.1 and [IK04, Equations (22.14) and (22.22)] we have
\[
\sum_{n \leq x, \gcd(n, q) = 1} R_{-q}(n) = 2L(1, \chi)x + O(x^{1-\eta}),
\]
for \( x \geq q^{1/4+\varepsilon} \) and \( q \geq 5 \),
where the implied constant depends only on \( \varepsilon \).

### 4.2. Ineffective lower bounds: Proof of Theorem 1.1.

#### 4.2.1. Preliminaries for the proof.

We recall that \( n \sim P \) means \( n \in [P/2, P] \cap \mathbb{N} \) and consider the sum
\[
Q(P) := \sum_{n \sim P} r(n) \log n. \tag{4.2}
\]

Observe that Lemma 4.1 immediately implies that
\[
Q(P) = \frac{1}{2} L(1, \chi)P \log P + O(P^{1-\eta}).
\]
for some \( \eta > 0 \) depending only on \( \varepsilon > 0 \). By Siegel’s theorem, see Lemma 3.3, in a similar way as before, we see the main term in (4.3) dominates. Thus
\[
Q(P) = \left( \frac{1}{2} + o(1) \right) L(1, \chi)P \log P. \tag{4.3}
\]

We now observe that each summand is positive. Factoring each \( n \sim P \) and collecting together contributions from each prime \( p \leq P \) we write (4.2) as
\[
Q(P) = \sum_{2 \leq p \leq P} c_p(P) \log p,
\]
with some integer coefficients \( c_p(P) \geq 0 \).

We fix some parameter \( y \geq 1 \) to be chosen later. In order to use Lemma 4.1 in the coming arguments, we need
\[
P/y \geq q^{1/4+2\varepsilon/3}. \tag{4.4}
\]

We write \( Q(P) \) as
\[
Q(P) = Q_1(P, y) + Q_2(P, y) + Q_3(P, y),
\]
where
\[
Q_1(P, y) := \sum_{\substack{p \leq y \\\chi(p) = 1}} c_p(P) \log p, \quad Q_2(P, y) := \sum_{\substack{2 \leq p \leq P \\\chi(p) = -1}} c_p(P) \log p,
\]
and
\[
Q_3(P, y) := \sum_{\substack{2 \leq p \leq P \\\chi(p) = 1}} c_p(P) \log p.
\]
and

$$Q_3(P, y) := \sum_{y < p \leq P \atop \chi(p) = 1} c_p(P) \log p.$$  

Below we obtain upper bounds on the sums $Q_1(P, y)$ and $Q_2(P, y)$ which together with (4.3) implies a lower bound on $Q_3(P, y)$. Thus estimating $c_p(P)$ we obtain a lower bound on the support of $Q_3(P, y)$ which is formed by split primes between $y$ and $P$ and which is exactly the quantity we want to estimate.

4.2.2. Estimate for $Q_1(P, y)$. Let $\text{ord}_p n$ denote the $p$-adic order of $n$. Hence each $n \in \mathbb{N}$ can be uniquely written as

$$n = bp^\ell \quad \text{and} \quad \ell = \text{ord}_p n. \quad (4.5)$$

Then (2.1) implies

$$r(n) = r(p^\ell b) = (\ell + 1)r(b) \quad \text{if} \quad \chi(p) = 1. \quad (4.6)$$

When $\ell = 1$ for a split prime, the $\ell + 1 = 2$ on the right side of (4.6) heavily influences the choice of $y$ (see also (4.10) below) that we eventually make.

For a given prime $p \leq y$ with $\chi(p) = 1$, we compute $c_p(P)$. Clearly for each $n \sim P$ written in the form (4.5), $b$ does not contribute to $c_p(P)$ and this using (4.6) we see that the total contribution is

$$\ell r(p^\ell b) = \ell(\ell + 1)r(b).$$

Hence, if $\chi(p) = 1$ then we have

$$c_p(P) = \sum_{\ell=1}^{\infty} \ell(\ell + 1) \sum_{b \sim P/p^\ell \atop \gcd(b, p) = 1} r(b).$$

At this point we drop the conditions

$$\chi(p) = 1 \quad \text{and} \quad \gcd(b, p) = 1,$$

which is possible by the nonnegativity of $r(b)$. We do this partially for typographical simplicity, but more importantly because we reuse the same argument to estimate $Q_2(P, y)$ below.

Hence we write

$$Q_1(P) \leq \sum_{p \leq y} \log p \sum_{\ell=1}^{\infty} \ell(\ell + 1) \sum_{b \sim P/p^\ell} r(b)$$

$$= \sum_{\ell=1}^{\infty} \ell(\ell + 1) \sum_{p \leq y} \log p \sum_{b \sim P/p^\ell} r(b). \quad (4.7)$$
We now estimate the right side of (4.7). First consider the $\ell = 1$ term in (4.7). We derive

$$2 \sum_{p \leq y} \log p \sum_{b \sim P/p} r(b) \leq 2 \sum_{p \leq y} \log p \sum_{b \sim P/p} r(b)$$

$$= 2 \left( \frac{1}{2} L(1, \chi) P + O \left( P^{1-\eta y^\eta} \right) \right) \sum_{p \leq y} \frac{\log p}{p}.$$  \hspace{1cm} (4.8)

for some $\eta > 0$ depending only on $\varepsilon$, where the last equality follows from Lemma 4.1 (which applies with $2\varepsilon/3$ instead of $\varepsilon$, since $y$ satisfies (4.4)). Recalling the Mertens formula, see [IK04, Equation (2.14)], we obtain

$$\sum_{p \leq y} \frac{\log p}{p} = (1 + o(1)) \log y.$$  \hspace{1cm} (4.9)

We now impose a second constraint

$$y \leq P^{1/2-\varepsilon/2}.$$  \hspace{1cm} (4.10)

Note that by (4.10) we have $P^{1-\eta y^\eta} \leq P^{1-(1+\varepsilon)\eta/2}$. Thus we have

$$P^{1-\eta y^\eta} \log P \ll P^{1-\eta/2}.$$  

Thus (4.8) and (4.9) tell us that

$$2 \sum_{p \leq y} \log p \sum_{b \sim P/p} r(b)$$

$$\leq \left( \frac{1 - \varepsilon}{2} + o(1) \right) L(1, \chi) P \log P + O \left( P^{1-\eta/2} \right).$$  \hspace{1cm} (4.11)

Now consider the summands in (4.7) corresponding to $\ell \geq 2$.

First we switch the order of summations between $p$ and $\ell$ again and move the summation over $p$ outside. We the consider contributions from the terms with

$$P/p^\ell \geq q^{1/4+\varepsilon/3} \quad \text{and} \quad P/p^\ell < q^{1/4+\varepsilon/3}$$

separately.

**Case A:** $P/p^\ell \geq q^{1/4+\varepsilon/3}$.

Reducing $\eta$ if necessary (to correspond to $\varepsilon/3$ rather than to $2\varepsilon/3$ as
in our original choice), we see that Lemma 4.1 implies that
\[
\sum_{p \leq y} \log p \sum_{\ell \geq 2 \atop P/p^\ell \geq q^{1/4+\varepsilon/3}} \ell (\ell + 1) \sum_{b \sim P/p^\ell} r(b) = \sum_{p \leq y} \log p \sum_{\ell \geq 2 \atop P/p^\ell \geq q^{1/4+\varepsilon/3}} \ell (\ell + 1)
\]
\[
= \sum_{p \leq y} \log p \sum_{\ell \geq 2 \atop P/p^\ell \geq q^{1/4+\varepsilon/3}} \ell (\ell + 1) \left( \frac{1}{2} L(1, \chi) P/p^\ell + O \left( (P/p^\ell)^{1-\eta} \right) \right). \tag{4.12}
\]
Furthermore
\[
\sum_{\ell \geq 2 \atop P/p^\ell \geq q^{1/4+\varepsilon/3}} \ell (\ell + 1) \left( \frac{1}{2} L(1, \chi) P/p^\ell + O \left( (P/p^\ell)^{1-\eta} \right) \right) \ll L(1, \chi) P \sum_{\ell \geq 2} \frac{\ell (\ell + 1)}{P^\ell} + P^{1-\eta} \sum_{\ell \geq 2} \frac{\ell (\ell + 1)}{P^{\ell(1-\eta)}}
\]
\[
\leq L(1, \chi) P \sum_{\ell \geq 2} k^{-2+o(1)} + P^{1-\eta} \sum_{k \leq y} k^{-2(1-\eta)+o(1)} \ll L(1, \chi) P + P^{1-\eta}. \tag{4.13}
\]

Now, dropping the primality condition (and assuming without loss of generality that \(\eta < 1/2\)), we derive from (4.12) that
\[
\sum_{p \leq y} \log p \sum_{\ell \geq 2 \atop P/p^\ell \geq q^{1/4+\varepsilon/3}} \ell (\ell + 1) \sum_{b \sim P/p^\ell} r(b) \leq L(1, \chi) P \sum_{k \leq y} k^{-2+o(1)} + P^{1-\eta} \sum_{k \leq y} k^{-2(1-\eta)+o(1)} \ll L(1, \chi) P + P^{1-\eta}. \tag{4.13}
\]

**Case B:** \(P/p^\ell < q^{1/4+\varepsilon/3}\).
For these terms, we first observe that using from (2.1) and the well-known bound on the divisor function \(\tau(b)\), see [IK04, Equation (1.81)], we have
\[
0 \leq r(b) \leq \tau(b) = b^{o(1)}. \tag{4.14}
\]
We now drop the condition \(\gcd(b, p) = 1\) and derive
\[
\sum_{p \leq y} \log p \sum_{\ell \geq 2 \atop P/p^\ell \geq q^{1/4+\varepsilon/3}} \ell (\ell + 1) \sum_{b \sim P/p^\ell} r(b) \leq \sum_{p \leq y} \log p \sum_{\ell \geq 2 \atop P/p^\ell \geq q^{1/4+\varepsilon/3}} \ell (\ell + 1) \sum_{b \sim P/p^\ell} \tau(b).
\]
We see from (4.14) that inner sum over $b$ is bounded by
\[(P/p^\ell)^{1+o(1)} \leq q^{1/4+\varepsilon/3+o(1)}.
\]
We also observe that it vanishes unless $p^\ell \leq P$. Hence
\[
\sum_{p \leq y} \log p \sum_{\ell \geq 2 \atop p^\ell < q^{1/4+\varepsilon/3}} \ell (\ell + 1) \sum_{b \sim P/p^\ell} r(b) \leq q^{1/4+\varepsilon/3+o(1)} \sum_{p \leq y} \log p \sum_{\ell \geq 2 \atop p^\ell < P} \ell^2 \leq yq^{1/4+\varepsilon/3+o(1)}.
\]
Using (4.4) we see that $yq^{1/4+\varepsilon/3+o(1)} \leq Pq^{-\varepsilon/3+o(1)} \ll P^{1-\varepsilon/4}$ and we arrive to the estimate
\[
\sum_{p \leq y} \log p \sum_{\ell \geq 2 \atop p^\ell < q^{1/4+\varepsilon/3}} \ell (\ell + 1) \sum_{b \sim P/p^\ell} r(b) \ll P^{1-\varepsilon/4}.
\] (4.15)

Combining (4.7) and (4.11), (4.13) and (4.15) we have
\[
Q_1(P, y) \leq \left(\frac{1-\varepsilon}{2} + o(1)\right) L(1, \chi) P \log P + O(P^{1-\eta/2} + P^{1-\varepsilon/4}).
\] (4.16)
We recall that Siegel’s theorem given in Lemma 3.3. Thus the main term eventually dominates the error term on the left side of (4.16). Thus,
\[
Q_1(P, y) \leq \frac{1}{2} (1 - \varepsilon + o(1)) L(1, \chi) P \log P.
\] (4.17)

4.2.3. Estimate for $Q_2(P, y)$. Suppose $p \mid n$ and $\chi(p) = -1$. Then $\text{ord}_p n$ must be even, otherwise $r(n) = 0$ by (2.1). Applying (2.1), for even $\ell$ we have
\[
r(p^\ell b) = r(b).
\]
Observe that
\[
Q_2(P, y) = \sum_{p \leq \sqrt{P}} \log p \sum_{\ell \geq 2 \atop \chi(p) = -1} \ell \sum_{b \sim P/p^\ell \atop \gcd(b, p) = 1} r(b).
\]
At this point we drop the conditions
\[
\chi(p) = -1 \quad \text{and} \quad \gcd(b, p) = 1.
\]
Hence, we write
\[
Q_2(P, y) \leq \sum_{p \leq \sqrt{P}} \log p \sum_{\ell \geq 2 \atop b \sim P/p^\ell} \ell \sum_{b \sim P/p^\ell} r(b).
\] (4.18)
For terms with $P/p^\ell \geq q^{1/4+\varepsilon/3}$ replace the condition $2 \mid \ell$ with $\ell \geq 2$ and arrive at the same sum as in (4.13), except with $y$ replaced by $\sqrt{P}$ which does not change the bound (which in fact does not depend on $y$). Hence we now obtain

\begin{equation}
\sum_{p \leq \sqrt{P}} \log p \sum_{\ell \geq 2 \text{ even}} \sum_{b \sim \sqrt{P}} r(b) \ll L(1, \chi)P + P^{1-\eta}. \quad (4.19)
\end{equation}

For terms with $P/p^\ell < q^{1/4+\varepsilon/3}$, we use (4.14) and derive

\begin{equation}
\sum_{p \leq \sqrt{P}} \log p \sum_{\ell \geq 2 \text{ even}} \sum_{b \sim P/p^\ell} \sigma(b) \ll \sum_{p \leq \sqrt{P}} \log p \sum_{\ell \geq 2 \text{ even}} \sum_{b \sim P/p^\ell} \sigma(b).
\end{equation}

We see from (4.14) that inner sum over $b$ is bounded by

\begin{equation}
(P/p^\ell)^{1+o(1)} \leq q^{1/4+\varepsilon/3+o(1)}.
\end{equation}

Hence we obtain

\begin{equation}
\sum_{p \leq \sqrt{P}} \log p \sum_{\ell \geq 2 \text{ even}} \sum_{b \sim P/p^\ell} \sigma(b) \ll P^{1+o(1)} \sum_{p \leq \sqrt{P}} \log p \sum_{\ell \geq 2 \text{ even}} \frac{\ell}{p^\ell}
\end{equation}

\begin{equation}
= P^{1+o(1)} \sum_{p \leq \sqrt{P}} \sum_{\ell \geq 2 \text{ even}} \frac{\log(p^\ell)}{p^\ell}
\end{equation}

\begin{equation}
\leq P^{1+o(1)} \sum_{k > P^{1/2}q^{-1/4-\varepsilon/3}} \frac{\log(k^2)}{k^2}
\end{equation}

\begin{equation}
\leq P^{1/2}q^{1/4+\varepsilon/3+o(1)}.
\end{equation}

We see that the condition $P \geq q^{1/4+\varepsilon}$ implies

\begin{equation}
P^{1/2}q^{1/4+\varepsilon/3+o(1)} \leq P^{1/2}q^{1/4+\varepsilon/3+o(1)} \ll P^{1-\varepsilon/4}
\end{equation}

and we arrive to the estimate

\begin{equation}
\sum_{p \leq \sqrt{P}} \log p \sum_{\ell \geq 2 \text{ even}} \sum_{b \sim P/p^\ell} \sigma(b) \ll P^{1-\varepsilon/4}. \quad (4.20)
\end{equation}

Hence, combining (4.19) and (4.20) and Siegel’s theorem, given in Lemma 3.3, we derive from (4.18) that

\begin{equation}
Q_2(P, y) \ll L(1, \chi)P. \quad (4.21)
\end{equation}
4.2.4. **Concluding the proof.** Comparing (4.3), (4.17) and (4.21), we conclude that

\[ Q_3(P, y) \geq \frac{1}{2} (\varepsilon + o(1)) L(1, \chi) P \log P. \]  

\[ (4.22) \]

Observe that

\[ Q_3(P, y) = \sum_{y < p \leq P} c_p(P) \log p, \]  

\[ (4.23) \]

where

\[ c_p(P) = \sum_{\ell=1}^{\infty} \ell(\ell + 1) \sum_{b \sim P/p^\ell, (b, p) = 1} r(b). \]

Estimating \( r(n) \) via the divisor function as in (4.14), we infer that for \( y < p \leq P \),

\[ c_p(P) \log p \leq P y^{-1} q^{o(1)}, \]  

\[ (4.24) \]

Comparing (4.22), (4.23) and (4.24), we see that the sum \( Q_3(P, y) \) is supported on at least

\[ Q_3(P, y) P^{-1} y q^{o(1)} \geq \frac{1}{2} (\varepsilon + o(1)) L(1, \chi) y q^{o(1)} \]

split primes between \( y \) and \( P \). Choosing the largest \( y \) that satisfies both (4.4) and (4.10), that is

\[ y = \min \left\{ P q^{-1/4-2\varepsilon/3}, P^{1/2-\varepsilon/2} \right\} \]

as well as applying Siegel’s theorem given in Lemma 3.3, concludes the proof.

4.3. **Effective lower bounds: Proof of Theorem 1.3.** Let

\[ Q(X, Y) = X^2 + \frac{q+1}{4} Y^2 \]

be the principal form of discriminant \(-q\). We define

\[ P(X) := Q(X, 1), \]

and consider the product

\[ R_q := \prod_{1 \leq n \leq t} P(n), \]

where

\[ t = \left\lfloor \sqrt{3q/2} \right\rfloor. \]

Observe that that for \( 1 \leq n \leq t \) we have

\[ q/4 < P(n) \leq q, \]
and
\[(q/4)^t < \sqrt{R_q} < q^t. \tag{4.25}\]

Recall that an integer \(k\) is represented by some quadratic form of discriminant \(-q\) if and only if there is a solution \(b\) to \(b^2 \equiv -q \pmod{4k}\), see [IK04, Equation (22.21)]. Thus if \(p \mid k\), then \(b^2 \equiv -q \pmod{p}\) has a solution and so \(p\) must be split. Thus the prime factorisation of \(R_q\) contains only split primes. Hence our strategy is to obtain a lower bound for
\[\omega(R_q) \leq N_q, \tag{4.26}\]
the number of distinct prime factors of \(R_q\).

Clearly, for each prime \(p \geq 3\), the congruence 
\[P(x) \equiv 0 \pmod{p} \tag{4.27}\]
has two distinct non-zero solutions in \(\mathbb{F}_p\) which are not roots of the derivative \(P'(X) = 2X\). Hence, applying Hensel’s lifting these roots can be uniquely lifted to \(p\)-adic solutions \(\alpha_p, \beta_p \in \mathbb{Z}_p\). Let \(\kappa_p \in \mathbb{N}\) be the least positive integer such that \(p^{\kappa_p+1} > t\) and let \(1 \leq a_p < b_p < p^{\kappa_p}\) be the unique integers such that
\[\ord_p (\alpha_p - a_p), \ord_p (\beta_p - b_p) \geq \kappa_p. \tag{4.28}\]
where, as before, \(\ord_p n\) denotes the \(p\)-adic order of \(n \in \mathbb{Z}_p\). Since for \(1 \leq n \leq t\) and \(n \neq a_p, b_p\) we trivially have
\[\ord_p(n - a_p), \ord_p(n - b_p) < \kappa_p. \tag{4.29}\]
Using (4.28) and (4.29), we derive
\[\ord_p P(n) = \ord_p(n - \alpha_p) + \ord_p(n - \beta_p) = \ord_p(n - a_p - \alpha_p - a_p) + \ord_p(n - b_p - \beta_p + b_p) \tag{4.30}\]
\[= \ord_p(n - a_p) + \ord_p(n - b_p).\]

We observe that \(\ord_p(a_p - b_p) = 0\) as the underlying to the congruence (4.27) are distinct modulo \(p\). This ensures that for any given \(n \in \mathbb{N}\), both terms on the right side of (4.30) cannot be simultaneously non-zero. Thus
\[\ord_p R_q = \ord_p P(a_p) + \sum_{1 \leq n \leq t, n \neq a_p} \ord_p(n - a_p)\]
\[+ \ord_p P(b_p) + \sum_{1 \leq n \leq t, n \neq b_p} \ord_p(n - b_p).\]
We now write
\[
\sum_{1 \leq n \leq t} \operatorname{ord}_p (n - a_p) = \sum_{1 \leq n < a_p} \operatorname{ord}_p (n - a_p) + \sum_{a_p + 1 \leq n \leq t} \operatorname{ord}_p (n - a_p)
\]
\[
= \sum_{1 \leq n < a_p} \operatorname{ord}_p (n) + \sum_{1 \leq n \leq t - a_p} \operatorname{ord}_p (n)
\]
\[
\leq \operatorname{ord}_p (a_p - 1)! + \operatorname{ord}_p (t - a_p)! \leq \operatorname{ord}_p (t - 1)!
\]
and similarly for the other sum. Hence, using the trivial bound
\[
p^{\operatorname{ord}_p (a_p)} \leq \mathcal{P} (a_p) \leq \mathcal{P} (p^{\omega_R}) \leq \mathcal{P} (t) = t^2 + \frac{q + 1}{4} \leq q
\]
we obtain
\[
\operatorname{ord}_p R_q \leq \operatorname{ord}_p \left( \mathcal{P} (a_p) \mathcal{P} (b_p) ((t - 1)!)^2 \right)
\]
\[
\leq 2 \frac{\log q}{\log p} + 2 \operatorname{ord}_p (t - 1)!. \tag{4.31}
\]
Since \( q \equiv 3 \pmod{16} \), we have
\[
\operatorname{ord}_2 \mathcal{P} (n) = \begin{cases} 
0 & \text{if } n \text{ is even}, \\
1 & \text{if } n \text{ is odd}, 
\end{cases}
\]
thus
\[
\operatorname{ord}_2 R_q \leq t/2. \tag{4.32}
\]
Combining (4.31) and (4.32) to bases \( p \) and 2 respectively, and then taking the product over all primes, we derive
\[
R_q < 2^{t/2} q^{2 \omega_R ((t - 1)!)^2}. \tag{4.33}
\]
Recalling the lower bound from (4.25), then (4.33) implies that
\[
(q/4)^t < 2^{t/2} q^{2 \omega_R ((t - 1)!)^2}. \tag{4.34}
\]
By the Stirling formula, we have
\[
(t - 1)! \leq (t - 1)^{t-1/2} e^{-t+2} \leq (t/e)^t
\]
provided \( t > 7 \) which holds for \( q \geq 67 \).

Taking the logarithm of both sides of (4.34) yields
\[
\log q - \log 4 < \frac{1}{2} \log 2 + \frac{2 \omega(R_q) \log q}{t} + 2 \log t - 2.
\]
Since
\[
\log q - 2 \log t = \log q - \log t^2 \geq \log q - \log (3q/4) = - \log (3/4)
\]
and
\[
2 - \log 4 - \frac{1}{2} \log 2 - \log (3/4) = 2 - \log (3\sqrt{2})
\]
we see that
\[ \omega(R_q) \geq \frac{(2 - \log(3\sqrt{2}))}{2} \frac{t}{\log q}. \]

Recalling (4.26) we conclude the proof.

5. Counting split primes conditionally: Proof of Theorem 1.5

5.1. Values of quadratic forms free of small prime divisors.
Suppose
\[ f(U, V) = AU^2 + BUV + CV^2 \in \mathbb{Z}[U, V], \]
is a binary quadratic form with discriminant \( B^2 - 4AC = -q \), we refer to [IK04, Section 22.1] for a general background. Suppose the form is reduced, that is,
\[ \gcd(A, B, C) = 1 \quad \text{and} \quad |B| \leq A \leq C. \] (5.1)
This implies that \( 3AC \leq 4AC - B^2 = q \). Hence
\[ A \leq \sqrt{\frac{q}{3}} \quad \text{and} \quad AC \leq \frac{q}{3}. \] (5.2)

Let \( P \geq 1 \) be any number such that \( P \geq Cq^2 \). For each
\[ 1 \leq v \leq \sqrt{P/C} \quad \text{with} \quad \gcd(v, A) = 1, \]
consider
\[ F_v(U) := f(U, v) = AU^2 + BUV + Cv^2 \in \mathbb{Z}[U]. \]

We still have
\[ \gcd(A, Bv, Cv^2) = 1. \] (5.3)

Thus the reduction of \( F_v \in \mathbb{Z}[U] \) modulo each odd prime \( p \) is non-zero. Only \( p = 2 \) could be a common factor.

Now, suppose 4 divides the following 3 consecutive values of \( F_v \)
\[ 4 \mid F_v(0), \quad 4 \mid F_v(1), \quad 4 \mid F_v(2) \] (5.4)
or equivalently
\[ 4 \mid Cv^2, \quad 4 \mid A + Bv + Cv^2, \quad 4 \mid 4A + 2Bv + Cv^2. \]

From the first and the third divisibility we concluded that \( 2 \mid Bv. \) Since we have (5.3), we see that \( A \) must be odd and so \( F_v(1) = A + Bv + Cv^2 \) is odd. Thus the divisibilities (5.4) are not possible simultaneously. Hence one can always choose \( n_0 \in \{0, 1, 2\} \) such that \( 4 \nmid F_v(n_0) \). Let \( e_v \in \{0, 1\} \) be such that \( 2^{e_v} \mid F_v(n_0) \) but \( 2^{e_v+1} \nmid F_v(n_0) \). Then the polynomial
\[ G_v(x) = \frac{1}{2^{e_v}} F_v(4x + n_0) \]
produces only odd values that are also produced by the original polynomial $F_v$.

Define the sequence
\[ A_v := \{ G_v(n) \}_{1 \leq n \leq \sqrt{P/A}}. \]
All the elements $A_v$ are less than an absolute constant times $P$. Observe that
\[ G_v(x) \equiv 0 \pmod{p} \]
has no solutions when $p = 2$ by construction and at most two solutions when $p$ is odd by Lagrange’s theorem. Thus we have
\[ \#A_v(p) = g_v(p)\#A_v + r_v(p) \]
where
\[ 0 \leq g_v(p) \leq 2/p \quad \text{and} \quad |r_v(p)| \leq 2. \tag{5.5} \]
We can extend $g_v(p)$ to a multiplicative function $g_v(d)$ supported on squarefree $d$ and using the Chinese Remainder Theorem, we obtain
\[ \#A_v(d) = g_v(d)\#A_v + r_v(d) \quad \text{and} \quad |r_v(p)| \leq 2^{\omega(d)} \tag{5.6} \]
with
\[ g_v(d) := \prod_{p \mid d \text{ prime}} g_v(p) \quad \text{and} \quad \omega(d) := \sum_{p \mid d \text{ prime}} 1. \]

We now need a lower bound on the number of elements in the sequence $A_v$ which are free of small prime divisors. To do this we appeal to [FI10, Theorem 11.13] and thus below we try to match the notation from [FI10]. Namely, we are interested in a good lower bound on cardinality of the set
\[ S(A_v, z) := \{ n \in [1, \sqrt{P/A}] : p \mid G_v(n) \implies p \geq z \}. \]
From (5.5) and the Mertens formula, see [FI10, Proposition 2.2], we see that for any $2 \leq w < z$ we have
\[ \prod_{w \leq p < z} (1 - g_v(p))^{-1} < \left( \frac{\log z}{\log w} \right)^2 \left( 1 + O\left( (\log w)^{-1} \right) \right). \]
Hence [FI10, Equation 11.129] is satisfied with $\kappa = 2$ and thus by the table in [FI10, Section 11.19] we can take $\beta = 4.833986 \ldots$ in [FI10, Theorem 11.13] and thus we conclude that for any fixed $s \geq \beta$ there is a constant $c(s) > 0$ such that and $D = z^s$ we derive
\[ \#S(A_v, z) \geq (c(s) + o(1)) V_v(z) X + R(D, z), \]
where
\[ X := \sqrt{P/A}, \quad V_v(z) := \prod_{p < z} (1 - g_v(p)), \quad R(D, z) := \sum_{d < D \ p|d \Rightarrow p < z} |r_v(d)|. \]

Using the trivial bound \( 2^\omega(d) \leq \tau(d) \) we see that from (5.6) and the bound on the divisor function [IK04, Equation (1.81)] that \( |r_v(d)| = d^{\alpha(1)} \), Hence we have a trivial bound
\[ R(D, z) \leq D^{1+o(1)} = z^{s+o(1)}. \]

Also as in above, by the Mertens formula, see [FI10, Proposition 2.2],
\[ V_v(z) \gg \frac{1}{(\log z)^2}. \]

Taking \( s = 4.85 > \beta \), and \( z = X^{10/49} \) we conclude that
\[ \#S(A_v, X^{10/49}) \gg \frac{X}{(\log X)^2}. \quad (5.7) \]

Hence \( A_v \) contains at least \( c_0 \sqrt{P/A}/(\log q)^2 \) elements with at most 5 prime factors, for some absolute constant \( c_0 > 0 \). Note that the choice \( P \) ensures that \( X \geq \sqrt{P/A} \geq \sqrt{P/C} = q^{\epsilon/2} \).

Thus the right side of (5.7) is meaningful.

5.2. Construction of the set \( F_{-q} \). We start with emphasising that the implied constant in (5.7) is absolute.

Consider a complete set of \( h(-q) \) inequivalent forms for the class group of discriminant \(-q\). Denote them
\[ f_t(U, V) = A_t U^2 + B_t U V + C_t V^2 \in \mathbb{Z}[U, V], \quad t = 1, \ldots, h(-q), \quad (5.8) \]

where \( h(-q) \) is the class number, see [IK04, Section 22.2].

The primitive binary quadratic forms in (5.8) are in bijection with the set of Heegner points
\[ \Lambda_{-q} := \left\{ zQ_t := \frac{-B_t + \sqrt{-q}}{2A_t} : B_t^2 - 4A_tC_t = -q, \ zQ_t \in \mathcal{D} \right\}, \]

where \( \mathcal{D} \) is the standard fundamental domain for the modular group \( \Gamma \setminus \mathbb{H} \). Consider
\[ \Omega := \{ \tau = x + iy : -1/2 \leq x \leq 1/2, \quad 1 \leq y \leq 10 \} \subseteq \mathcal{D}. \]
The equidistribution theorem of Duke [Duk88, Theorem 1] yields
\[ \frac{\#(\Lambda_{-q} \cap \Omega)}{\#\Lambda_{-q}} = \frac{27}{10\pi} + O(q^{-\delta}), \quad \text{as} \quad q \to \infty, \quad (5.9) \]
where \( \delta > 0 \) depends only on \( \Omega \), and the implied constant depends only on \( \Omega \) and \( \delta > 0 \), but is ineffective. Note that (5.9) is taken with respect to the normalised hyperbolic area measure

\[
d\mu(\tau) = \frac{3}{\pi} dxdy/y^2,
\]
where as in the above \( \tau = x + iy \).

Suppose the forms in (5.8) corresponding to Heegner points in \( \Omega \) are indexed by \( t \in S_{-q} \). Thus (5.9) guarantees

\[
\#S_{-q} \geq \left( \frac{27}{10\pi} + o(1) \right) h(-q). \tag{5.10}
\]

The binary quadratic forms with \( t \in S_{-q} \) satisfy (5.1), (5.2) as well as

\[
1 \leq \text{Im} z_{Q_t} = \frac{\sqrt{q}}{2A_t} \leq 10.
\]

Thus

\[
\max(|A_t|, |B_t|, |C_t|) \leq \frac{20}{3}\sqrt{q}, \quad t \in S_{-q}. \tag{5.11}
\]

Now we restrict our attention to \( \{f_t\}_{t \in S_{-q}} \) in (5.8). For each such \( f_t \), one can form the polynomials \( G_u(t) \) and sequences \( A_u(t) \) as above. Thus by (5.7) we have

\[
\#S^{\sharp}(A_u(t), X^{10/49}) \gg \frac{X_t}{(\log X_t)^2}, \quad t \in S_{-q}. \tag{5.12}
\]

with implied constant independent of \( t \) and \( u \), where \( X_t := \sqrt{P/A_t} \) and \( S^{\sharp} \) indicates that only square-free elements of a set \( S \subseteq \mathbb{Z} \) are included.

Let

\[
F_{-q} := \bigcup_{t \in S_{-q}} \bigcup_{u=1}^P S^{\sharp}(A_u(t), X^{10/49}_t).
\]

We now need a lower bound for \( \#F_{-q} \). Recall that by our construction, all elements of \( F_{-q} \) are odd. Since each \( n \in F_{-q} \) has at most 5 prime factors, we see that (2.1) implies

\[
0 < R_{-q}(n) \leq 2^6.
\]

Thus, recalling (5.10), (5.11) and (5.12), we derive

\[
\#F_{-q} \gg \sum_{t \in S_{-q}} \frac{P}{\sqrt{A_tC_t\log^2 q}} \gg h(-q) \frac{P}{\sqrt{q\log^2 q}}. \tag{5.13}
\]
5.3. Descent to split primes. We recall an integer \( n \) is represented by some quadratic form of discriminant \(-q\) if and only if there is a solution \( b \) to \( b^2 \equiv -q \pmod{4n} \), see [IK04, Equation (22.21)]. Thus if \( p \mid n \), then \( b^2 \equiv -q \pmod{p} \) has a solution and so \( p \) must be split. Each \( n \in \mathcal{F}_q \) is squarefree and odd, so if \( p \mid n \), then \( n = pm \) where \( p \) is split and \( m \) is represented by a quadratic form of discriminant \(-q\).

Now we come to the heart of the argument, where we sift \( \mathcal{F}_q \) down to just primes using the Siegel zero. Cover the interval \([P^{10/19}, P^{1/2}]\) into \( O(\log q) \) dyadic intervals \([A, 2A]\) and let \( \nu \in \{2, 3\} \) be the least integer such that \( A\nu > P^{1/2} \).

Let \( Q_{-q}(A) \) denote the set of primes \( p \in [A, 2A] \) that are split. Our goal is to show that the number of elements \( n \in \mathcal{F}_q \) that are divisible by some \( p \in Q_{-q}(A) \) is significantly less than the lower bound for \#\( \mathcal{F}_q \) established in (5.13). The remaining elements of \( \mathcal{F}_q \) are primes.

To show this, we denote \( Q_{-q}(A) = \#Q_{-q}(A) \).

Consider all square-free products of \( \nu \) such primes. There are
\[
\binom{Q_{-q}(A)}{\nu} \gg Q_{-q}(A)^\nu
\]
such products. If \( n \) is a product of split primes then \( x^2 \equiv -q \pmod{n} \) is solvable, and since \( q \equiv 3 \pmod{4} \), we can lift the congruence to \( x^2 \equiv -q \pmod{4n} \), so \( R_{-q}(n) \geq 1 \).

In what follows all implied constants may depend on \( \varepsilon > 0 \) (but are ineffective).

Since \( A^\nu > P^{1/2} > q^{1/4+\varepsilon/2} \), the asymptotic formula (4.1) and the inequality (5.14) imply that
\[
Q_{-q}(A) \ll L(1, \chi)^{1/\nu} A + O(A^{1-\eta}).
\]

We now recall Siegel’s theorem, see Lemma 3.3. Since \( A^\nu \geq q^{1/4} \), taking \( \delta = \eta/8 \) in Lemma 3.3 we obtain
\[
L(1, \chi) \geq C(\eta/8)q^{-\eta/8} \geq C(\eta/8)\, A^{-\nu\eta/2}.
\]
Thus, if \( q \) is large enough the first term on the right hand side of (5.15) dominates and we derive
\[
Q_{-q}(A) \ll L(1, \chi)^{1/\nu} A.
\]
Thus by the hypothesis on the size of \( L(1, \chi) \) we have
\[
\frac{Q_{-q}(A)}{A} \ll \frac{1}{(\log q)^{10/3}}.
\]
By the discussion at the start of Section 5.3, if \( p \in \mathcal{Q}_q(A) \) divides \( n \in \mathcal{F}_q \), then \( n = pm \) where \( R_q(m) \geq 1 \). Thus the number of elements of \( \mathcal{F}_q \) that are divisible by a \( p \in \mathcal{Q}_q(A) \) is bounded from above by

\[
\sum_{m \leq P/p} R_q(m).
\]

Since \( P/p \geq q^{1/4 + \varepsilon/2} \), the asymptotic formula (4.1) implies that

\[
\sum_{m \leq P/p} R_q(m) = 2L(1, \chi) P + O \left( (Pp^{-1})^{1-\eta} \right) \ll L(1, \chi) \frac{P}{p},
\]

where the last inequality follows from Lemma 3.3 (we recall that the implied constants may depend on \( \varepsilon \)). Summing this contribution over all \( p \in \mathcal{Q}_q(A) \), using (5.16) and Dirichlet’s class number formula yields

\[
\# \{ n \in \mathcal{F}_q : \exists p \in \mathcal{Q}_q(A) \text{ such that } p \mid n \} \ll L(1, \chi) \frac{P}{p} \frac{Q_q(A)}{A} \ll \frac{h(-q)P}{\sqrt{q} \left( \log q \right)^{10/3}}.
\]

Summing the last display over each of the \( O(\log q) \) dyadic intervals \([A, 2A]\), we see that the number of composite integers in \( \mathcal{F}_q \) is

\[
O \left( h(-q)P/\sqrt{q} \left( \log q \right)^{7/3} \right).
\]

Comparing this with (5.13) we conclude the proof.

6. **Bounds of bilinear Weyl sum with square roots: Proof of Theorem 1.7**

6.1. **Additive energy of modular square roots.** Our argument is based on a weighted additive energy for modular squares, which can be of independent interest.

For a complex weight \( \beta \) as in (1.4) and \( j \in \mathbb{F}_q^x \) we define the **weighted additive energy**

\[
E_{q,j}(\beta) := \sum_{(u,v,x,y) \in \mathbb{F}_q^4 \atop u+v=x+y} \beta_{ju^2} \overline{\beta_{jv^2}} \beta_{jx^2} \overline{\beta_{jy^2}}.
\]

Recall that \( ju^2, jv^2, jx^2, jy^2 \) are all computed modulo \( q \) and take the value of the reduced residue between 1 and \( q \). We omit the subscript \( j \) when \( j = 1 \). Quantities of this type are well known in additive combinatorics under the name of **additive energy**.
Lemma 6.1. For a weight $\beta$ as in (1.4) supported on $[N, 2N]$ with $2N \leq q$, we have

$$E_{q,j}(\beta) \ll \|\beta\|_\infty^2 \|\beta\|^2_2 (N^2/q + N^{1/2}q^{o(1)}).$$

Proof. The contribution from solutions with $u + y = x + v = 0$ is $O(\|\beta\|^4_2)$. For each $\lambda \in \mathbb{F}_q^\times$ we define

$$Q_{\lambda,j}(\beta) := \sum_{(u,v) \in \mathbb{F}_q^2 \atop u - v = \lambda} \beta_{ju^2} \overline{\beta}_{jv^2}.$$  (6.1)

Thus

$$E_{q,j}(\beta) = \sum_{\lambda \in \mathbb{F}_q^\times} |Q_{\lambda,j}(\beta)|^2 + O(\|\beta\|^4_2).$$

from which we derive

$$|E_{q,j}(\beta)| \leq \max_{\lambda \in \mathbb{F}_q^\times} |Q_{\lambda,j}(\beta)| \cdot \sum_{\lambda \in \mathbb{F}_q^\times} |Q_{\lambda,j}(\beta)| + O(\|\beta\|^4_2).$$  (6.2)

The triangle inequality gives

$$\sum_{\lambda \in \mathbb{F}_q^\times} |Q_{\lambda,j}(\beta)| = O(\|\beta\|^2_2).$$  (6.3)

Now,

$$\max_{\lambda \in \mathbb{F}_q^\times} |Q_{\lambda,j}(\beta)| \leq \|\beta\|_\infty^2 \max_{\lambda \in \mathbb{F}_q^\times} Q_{\lambda,j}(1_{[N,2N]}).$$  (6.4)

Next, we show that

$$Q_{\lambda,j}(1_{[N,2N]}) \leq 4 \cdot \#\{(Z,V) \in [-2N, 2N] \times [N, 2N] : (Z - j\lambda^2)^2 = 4\lambda^2 jV\}.$$  (6.5)

Indeed, recall that

$$Q_{\lambda,j}(1_{[N,2N]}) = \sum_{(u,v) \in \mathbb{F}_q^2 \atop u - v = \lambda \atop ju^2, jv^2 \in [N,2N]} 1.$$  

Making a change of variables

$$U = ju^2 \quad \text{and} \quad V = jv^2,$$

and using the linear equation $u - v = \lambda$, we see that

$$U - V = j\lambda(2v + \lambda).$$

Rearranging and squaring, we obtain

$$(U - V - j\lambda^2)^2 = 4\lambda^2 jV.$$
Making the linear change in variables

\[ Z := U - V, \]

we obtain

\[ (Z - j\lambda^2)^2 = 4j\lambda^2V. \quad (6.6) \]

Given any solution \((Z, V)\) to (6.6), this corresponds to at most 4 pairs \((u, v) \in \mathbb{F}_q^2\), and so this establishes (6.5). We apply [CCG+14, Theorem 5], which in the special case of quadratic polynomials implies

\[ Q_{\lambda,j}(1_{[N,2N]}) \ll N^2/q + N^{1/2}q^{o(1)} \]

uniformly with respect to \(j, \lambda \in \mathbb{F}_q^\times\). Combining this bound with (6.4) gives

\[ \max_{\lambda \in \mathbb{F}_q^\times} |Q_{\lambda,j}(\beta)| \ll \|\beta\|_2^2 \left( N^2/q + N^{1/2}q^{o(1)} \right) + \|\beta\|_4^4. \quad (6.7) \]

Using (6.3) and (6.7) in (6.2) we obtain

\[ |E_{q,j}(\beta)| \ll \|\beta\|_2^2 \||\beta\|_\infty^2 \left( N^2/q + N^{1/2}q^{o(1)} \right) + \|\beta\|_2^4. \]

Since trivially \(\|\beta\|_2 \leq \|\beta\|_1^{1/2} \|\beta\|_\infty^{1/2}\) we now derive the desired result. ■

We further remark that for small \(N\) yet another bound on additive energy of modular square roots is possible, improving that of Lemma 6.1. This bound does not improve our main results but since the question is of independent interest we present it in Appendix A.

### 6.2. Bounding bilinear sums via additive energy

Applying the Cauchy inequality and interchanging summation after expanding the second square gives

\[ |W_{a,q}(\alpha, \beta; h, M, N)|^2 \leq \|\alpha\|_2^2 \sum_{n_1, n_2 \sim N} \beta_{n_1} \overline{\beta}_{n_2} \sum_{m \sim M} \sum_{u, v \in \mathbb{F}_q} e_q(h(u - v)). \]

We now write

\[ |W_{a,q}(\alpha, \beta; h, M, N)|^2 \leq \|\alpha\|_2^2 (R_1 + R_{-1}) \quad (6.8) \]

where

\[ R_j := \sum_{n_1, n_2 \sim N} \beta_{n_1} \overline{\beta}_{n_2} \sum_{m \sim M} \sum_{u, v \in \mathbb{F}_q} e_q(h(u - v)). \]

\[ \text{with} \quad \left( \frac{n_1}{q} \right) = \left( \frac{n_2}{q} \right) = j \]

\[ \text{and} \quad \left( \frac{am}{q} \right) = j \] for \( \left( \frac{u^2}{q} \right) = \left( \frac{v^2}{q} \right) = j \).
Both sums can be estimated analogously, so we only concentrate on $R_1$. Simplifying $R_1$, we obtain

$$R_1 = \sum_{n_1, n_2} \beta_{n_1} \beta_{n_2} \sum_{m \sim M} \sum_{t \in \mathbb{F}_q} e_q(ht(u - v))$$

Collecting the terms with the same value of $\lambda = u - v$ we obtain

$$R_1 = \sum_{\lambda \in \mathbb{F}_q} A_{h,\lambda,a} Q_{\lambda,1}(\beta),$$

where

$$A_{h,\lambda,a} = \sum_{m \sim M} \sum_{t \in \mathbb{F}_q} e_q(ht\lambda)$$

and $Q_{\lambda,1}(\beta)$ is defined in (6.1).

We also notice that we have

$$\sum_{\lambda \in \mathbb{F}_q} |A_{h,\lambda,a}|^4 \ll qE_{q,b}(1_{[M,2M]}),$$

(6.9)

where $b$ is defined be $ab \equiv 1 \pmod q$, $1 \leq b < q$.

Thus, writing $|Q_{\lambda,1}(\beta)| = (|Q_{\lambda,1}(\beta)|^2)^{1/4} |Q_{\lambda,1}(\beta)|^{1/2}$ by the Hölder inequality, using (6.3) (the bound (6.3) also holds when sum is over all $\lambda \in \mathbb{F}_q$) and (6.9) we derive

$$|R_1|^4 \ll q E_{q,b}(1_{[M,2M]}) \sum_{\lambda \in \mathbb{F}_q} |Q_{\lambda,1}(\beta)|^2 \left( \sum_{\lambda \in \mathbb{F}_q} |Q_{\lambda,1}(\beta)| \right)^2$$

(6.10)

$$\ll q \|\beta\|_1^4 E_{q,b}(1_{[M,2M]}) E_{q,1}(\beta).$$

as well as a full analogue of (6.10) for $R_{-1}$.

Now using Lemma 6.1, we obtain

$$|R_1|^4 \ll q^{1+o(1)} \|\beta\|_1^2 \|\beta\|_1^6$$

$$\left( M^4 q^{-1} + M^5/2 \right) \left( N^2 q^{-1} + N^1/2 \right),$$

(6.11)

which now together with (6.8) implies the desired bound.

**Remark 6.2.** By orthogonality we have,

$$\sum_{\lambda \in \mathbb{F}_q} |A_{h,\lambda,a}|^2 \ll qM.$$  

Hence, instead of (6.10) we have

$$|R_1|^2 \ll qM \sum_{\lambda \in \mathbb{F}_q} |Q_{\lambda,1}(\beta)|^2.$$
From Lemma 6.1 we now derive

\[ |W_{a,q}(\alpha, \beta; h, M, N)| \leq M^{1/4} \|\alpha\|_2 \|\beta\|_1^{1/2} \|\beta\|_{\infty}^{1/2} q^{o(1)} (N^{1/2} + N^{1/8} q^{1/4 + o(1)}). \]

We also discuss some alternative approaches to bound bilinear sums in Appendix B.

7. Equidistribution of roots of primes: Proof of Theorem 1.10

7.1. Preliminary transformations. For \( P \leq q \), let \( \mathcal{P}_q(P) \) be the set of primes \( p \leq P \), \( p \neq q \), such that \( p \) is a quadratic residue modulo \( q \). To study the discrepancy of the roots of these quadratic congruences, we introduce the exponential sum

\[ S_q(h, P) := \sum_{p \in \mathcal{P}_q(P)} \sum_{x \in \mathbb{F}_q} e_q(hx) = \sum_{x^2 = p} \sum_{x \in \mathbb{F}_q} e_q(hx) + O(1) \]

(where the term \( O(1) \) accounts for \( q = p \) which has possible been added to the sum). We see that Lemma 3.1 reduces the discrepancy question to estimating the sums \( S_q(h, P) \).

In fact, following the standard principle, we also introduce the sums

\[ \tilde{S}_q(h, P) = \sum_{k=1}^{\log P/\log H} \Lambda(k) \sum_{x^2 = k} e_q(hx) \]

(7.1)

where, as usual we use \( \Lambda \) to denote the von Mangoldt function. Clearly, one can bound the sums \( S_q(h, P) \) via the sums \( \tilde{S}_q(h, t) \), \( t \leq P \), using partial summation.

7.2. The Heath-Brown identity. To estimate the sum (7.1) we apply the Heath-Brown identity in the form given by [FKM14, Lemma 4.1] (see also [IK04, Proposition 13.3]) as well as a smooth partition of unity from [Fou85, Lemme 2] (or [FKM14, Lemma 4.3]).

We also fix a parameter \( H \geq 1 \), to be optimised later and define

\[ J = \lceil \log P/\log H \rceil. \]

(7.2)

We always assume that \( H \) exceed some fixed small power of \( q \) so we always have \( J \ll 1 \).
Now, as in [FKM14, Lemma 4.3], we decompose \( \tilde{S}_q(h, P) \) into a linear combination of \( O(\log^2 q) \) sums with coefficients bounded by \( O(\log q) \),

\[
\Sigma(\mathbf{V}) := \sum_{m_1, \ldots, m_J = 1}^{\infty} \gamma_1(m_1) \ldots \gamma_J(m_J) \sum_{n_1, \ldots, n_J = 1}^{\infty} \sum_{x \in \mathbb{F}_q} e_q(hx),
\]

where

\[
\mathbf{V} := (M_1, \ldots, M_J, N_1, \ldots, N_J) \in [1/2, 2P]^{2J}
\]

is a \( 2J \)-tuple of parameters satisfying

\[
N_1 \geq \ldots \geq N_J, \quad M_1, \ldots, M_J \leq P^{1/J}, \quad P \ll Q \ll P,
\]

(7.3)

(7.4)

(7.5)

and

- the arithmetic functions \( m_i \mapsto \gamma_i(m_i) \) are bounded and supported in \([M_i/2, 2M_i]\);
- the smooth functions \( x_i \mapsto V_i(x) \) have support in \([1/2, 2]\) and satisfy

\[
V^{(j)}(x) \ll q^{j\varepsilon}
\]

for all integers \( j \geq 0 \), where the implied constant may depend on \( j \) and \( \varepsilon \).

In particular, it is convenient to introduce the notation \( a \sim A \) as an equivalent of \( a \in [A/2, 2A] \). Hence we can rewrite the sums \( \Sigma(\mathbf{V}) \) in the following form

\[
\Sigma(\mathbf{V}) = \sum_{m_i \sim M_i, n_i \sim N_i}^{m_i, \ldots, m_J, n_i, \ldots, n_J} \gamma_1(m_1) \ldots \gamma_J(m_J) V_1 \left( \frac{n_1}{N_1} \right) \ldots V_J \left( \frac{n_J}{N_J} \right) \sum_{x \in \mathbb{F}_q} e_q(hx).
\]

In particular, we see that the sums \( \Sigma(\mathbf{V}) \) are supported on a finite set. We now collect various bound on the sums \( \Sigma(\mathbf{V}) \) which we derive in various ranges of parameters \( M_1, \ldots, M_J, N_1, \ldots, N_J \) until we cover the whole range in (7.3).
7.3. **Bounds of multilinear sums.** To estimate the multilinear sums \( \Sigma(V) \), we put \( N_1 \) in ranges which we call “small”, “moderate”, “large” and “huge”. We the further split the “moderate” range in further subranges depending on “small” and “large” values of \( N_2 \).

Our main tool is the bound (1.8) where it is convenient to observe that \( M^{3/16} q^{-1/8} + 1 \ll 1 \) for \( M \ll q^{2/3} \) and similarly for the other term involving \( N \). It is also convenient to assume that

\[
P \geq q^{2/3},
\]

as otherwise the result is trivial.

**Case I: Small \( N_1 \).**

First we consider the case when \( N_1 \leq H \).

From the definition of \( J \) in (7.2) and the condition (7.4) we see that

\[
M_1, \ldots, M_J \leq H.
\]

We see that if (7.7) holds then we can choose two arbitrary sets \( \mathcal{I}, \mathcal{J} \subseteq \{1, \ldots, J\} \) such for

\[
M = \prod_{i \in \mathcal{I}} M_i \prod_{j \in \mathcal{J}} N_j \quad \text{and} \quad N = Q/M,
\]

where \( Q \) is given by (7.5) we have

\[
P^{1/2} \ll N \ll H^{1/2} P^{1/2}.
\]

Indeed, we simply start multiplying consecutive elements of the sequence \( M_1, \ldots, M_J, N_1, \ldots, N_J \) until the product \( Q_+ \) exceeds \( P^{1/2} \) while the previous product \( Q_- \) is at most \( H \), we have \( Q_+ < HQ_- \). Hence

- either we have \( P^{1/2} \leq Q_+ \leq P^{1/2} H^{1/2} \) and then we set \( M = Q/Q_+ \) and \( N = Q_+ \);
- or we have \( P^{1/2} > Q_- > H^{-1/2} P^{1/2} \) and then we set \( M = Q_- \) and \( N = Q/Q_- \).

Hence in either case the corresponding \( N \) satisfies the upper bound in (7.9).

In this case, since for \( N \gg P^{1/2} \) we have \( M \ll P/N \ll P^{1/2} \ll q^{2/3} \), the bound (1.8), implies

\[
|\Sigma(V)| \ll P^{13/16} q^{1/8+o(1)} \left( H^{3/32} P^{3/32} q^{-1/8} + 1 \right) = H^{3/32} P^{29/32} q^{o(1)} + P^{13/16} q^{1/8+o(1)}.
\]

(7.10)
Case II: Moderate $N_1$.
We now consider the case
\[ H < N_1 \leq q^{1/3}. \]  
(7.11)
We further split it into two subcases, depending on the size of $N_2$.

- **Subcase II.1: Moderate $N_1$ and small $N_2$.**
  If we have
  \[ q^{1/3} \geq N_1 \geq H > N_2 \]
then we again start multiplying $N_1$ by other elements from the sequence $M_1, \ldots, M_J, N_2, \ldots, N_J$ and prepare $M$ and $N$ with (7.9), so again we have the bound (7.10).

- **Subcase II.2: Moderate $N_1$, and large $N_2$.**
  It remains to consider the case when
  \[ q^{1/3} \geq N_1 \geq N_2 \geq H. \]
In this case, we define $M$ and $N$ as
\[
M = \prod_{i=1}^{J} M_i \prod_{j=3}^{J} N_j \quad \text{and} \quad N = N_1 N_2.
\]
thus we have
\[ q^{2/3} \geq N \geq H^2. \]
We also note that
\[
M^{3/16} q^{-1/8} + 1 \ll (P/N)^{3/16} q^{-1/8} + 1 \leq (P/H^2)^{3/16} q^{-1/8} + 1 = H^{-3/8} P^{3/16} q^{-1/8} + 1
\]
and since $N \leq N_1^2 \leq q^{2/3}$ we also have
\[ N^{3/16} q^{-1/8} + 1 \ll 1. \]
Hence, the bound (1.8), implies
\[
|\Sigma(V)| \leq H^{-3/8} P q^{o(1)} + P^{13/16} q^{1/8 + o(1)}. \quad (7.12)
\]

Case III: Large $N_1$.
In the case when
\[ q^{2/3} \geq N_1 \geq q^{1/3} \]  
(7.13)
we set
\[
M = \prod_{i=1}^{J} M_i \prod_{j=2}^{J} N_j \quad \text{and} \quad N = N_1,
\]
With the above choice, under the condition (7.13) we have the bounds
\[
(M^{3/16}q^{-1/8} + 1) \left( N^{3/16}q^{-1/8} + 1 \right) \ll M^{3/16}q^{-1/8} + 1
\ll \left( P/q^{1/3} \right)^{3/16}q^{-1/8} + 1 = P^{3/16}q^{-3/16} + 1.
\]
Therefore, we see that the bound (1.8) implies
\[
|\Sigma(V)| \leq Pq^{-1/16+o(1)} + P^{13/16}q^{1/8+o(1)} \leq P^{13/16}q^{1/8+o(1)}.
\]
\[\text{(7.14)}\]

**Case IV: Huge }N_1\text{.**

We now consider the case when
\[
q^{2/3} < N_1 \leq P.
\]
\[\text{(7.15)}\]

In this case we write
\[
\Sigma(V) = \sum_{m_i \sim M_i}^{m_1 \sim M_1} \ldots \gamma_J(m_J) \sum_{n_i \sim N_i}^{n_1 \sim N_1} \ldots \sum_{i=2,\ldots,J} V_2 \left( \frac{n_2}{N_2} \right) \ldots V_J \left( \frac{n_J}{N_J} \right)
\]
\[
\sum_{n_1 \sim N_1} V_1 \left( \frac{n_1}{N_1} \right) \sum_{x \in F_q} \sum_{x^2 = a m_1 \ldots m_J n_1 \ldots n_J} e_q(hx).
\]

For each fixed choice of }m_1, \ldots, m_J\text{ and }n_2, \ldots, n_J\text{ we set
\[
a = m_1 \ldots, m_J n_2, \ldots, n_J,
\]
and we complete the innermost summation over }n_1\text{ using the standard completion technique, see [IK04, Section 12.2]. More precisely, partial summation gives
\[
\sum_{n_1 \sim N_1} V_1 \left( \frac{n_1}{N_1} \right) \sum_{x \in F_q} e_q(hx)
\]
\[
= -\frac{1}{N_1} \int_{N_1/2}^{2N_1} \frac{V'(u)}{N_1} \sum_{x^2 = aw, 2 \leq w < u} e_q(hx) du.
\]
Recalling Lemma 3.2, we conclude that
\[
\sum_{n_1 \sim N_1} V_1 \left( \frac{n_1}{N_1} \right) \sum_{x \in F_q} e_q(hx) \leq q^{1/2+o(1)}.
\]

Therefore,
\[
|\Sigma(V)| \leq M_1 \ldots M_J N_2 \ldots N_J q^{1/2+o(1)}
\]
\[\leq P N_1^{-1} q^{1/2+o(1)} \leq P q^{-1/6+o(1)}.
\]
\[\text{(7.16)}\]
7.4. Optimisation and concluding the proof. Observe that the bounds (7.10), (7.12), (7.14) and (7.16) cover all four possibilities (7.7), (7.11), (7.13) and (7.15).

We now choose $H$ to balance its contribution to the bounds (7.10) and (7.12). This leads us to the equation

$$H^{3/32} P^{29/32} = H^{-3/8} P$$

Thus, we choose

$$H = P^{1/5}. \tag{7.17}$$

Recalling (7.6), we have $H \geq q^{2/15}$ and hence $J \ll 1$ as required.

With the choice of $H$ as in (7.17), the above bounds can be combined as

$$|\Sigma(V)| \leq P^{37/40} q^{o(1)} + P^{13/16} q^{1/8+o(1)} + P^{-1/6+o(1)} \tag{7.18}$$

Clearly

$$P^{13/16} q^{1/8+o(1)} \geq P^{37/40} \geq P^{-1/6}$$

for $P \leq q$. Hence the bound (7.18) can be simplified as

$$|\Sigma(V)| \leq P^{13/16} q^{1/8+o(1)}. \tag{7.19}$$

Together with Lemma 3.1 we now derive from (7.19) that

$$\Gamma_q(P) \leq P^{13/16} q^{1/8+o(1)},$$

which concludes the proof of Theorem 1.10.

**Appendix A. Additive energy bounds of modular square roots**

For small $N$ we have an improvement of Lemma 6.1. To emphasise the ideas we consider the following special case of the quantity $E_{q,j}(\beta)$ from Section 6.1. For an integer $N$ we define

$$E_q(N) := \# \{(u, v, x, y) \in \mathbb{F}_q : u^2, v^2, x^2, y^2 \sim N \text{ and } u + v = x + y\}$$

(recall that $u^2, v^2, x^2, y^2$ are all computed modulo $q$).

**Proposition A.1.** For any positive integer $N \leq q^{1/2}$, we have

$$E_q(N) \leq N^6 q^{-1+o(1)} + N^2 q^{o(1)}.$$

**Proof.** Squaring both sides of the congruence $u + v \equiv x + y \pmod{q}$, we obtain

$$u^2 + 2uv + v^2 \equiv x^2 + 2xy + y^2 \pmod{q}. \tag{A.1}$$

We denote

$$z = x^2 + y^2 - u^2 - v^2,$$
write the congruence (A.1) as
\[ 2(uv - xy) \equiv z \pmod{q} \]
with \( z \in [-6N, 6N] \). We square it again and arrive to
\[ 8uvxy \equiv w \pmod{q} \]
with
\[ w = 4u^2v^2 + 4x^2y^2 - z^2 \in [-8N^2, 8N^2]. \]
Since \( u^2, v^2, x^2, y^2 \sim N \) and \( z \in [-6N, 6N] \), we see that
\[ w \in [-4N^2, 32N^2]. \]
Thus the product \( uvxy \) falls in \( O(N^2) \) arithmetic progressions modulo \( q \) and thus so does the product \( UVXY \leq 256N^4 \).

Note that for \( N \leq q^{1/4+o(1)} \), Proposition A.1 implies an essentially optimal bound \( E_q(N) \leq N^{2+o(1)} \).

Appendix B. Some related sums

B.1. Type-I and Type-II sums. The sums \( V_{a,q}(\alpha, \beta; h, M, N) \) with two nontrivial weights are usually called Type-II sums.

However for some applications sums with only one nontrivial weight, such as
\[ V_{a,q}(\alpha, \beta; h, M, N) = \sum_{m \sim M} \sum_{n \sim N} \alpha_m \sum_{x \in \mathbb{F}_q} e_q(hx). \]
are also important and are usually called Type-I sums.

Typically Type-I sums admit easier treatment with stronger bounds. For example, the sums \( V_{a,q}(\alpha, \beta; h, M, N) \) one can apply some ideas of Blomer, Fouvry, Kowalski, Michel, and Milićević [BFK+17] with a follow up application of the Bombieri bound [Bom66] for exponential sums along a curve. Unfortunately the resulting bound
\[ |V_{a,q}(\alpha; h, M, N)| \leq \sqrt{||\alpha||_1 \|\alpha\|_2} M^{1/12} N^{7/12} q^{1/4+o(1)}, \] (B.1)
which also requires the additional conditions
\[ MN \leq q^{3/2} \quad \text{and} \quad M \leq N^2, \] (B.2)
do not improve a combination of Theorem 1.7 and the bound
\[ |V_{a,q}(\alpha, \beta; h, M, N)| \leq Mq^{1/2+o(1)} \]
which can be obtained via the completion method exactly as (7.16).

However since the argument may have other applications we sketch it here with a brief outline of main steps and then give an short outline of further modifications which can achieved within this approach and to which the method of proof of Theorem 1.7 does not apply. We also believe that Proposition B.1 below is of independent interest and may have further applications.

B.2. Preliminary transformations. Let $K : \mathbb{F}_q \rightarrow \mathbb{C}$ is an arbitrary function with $|K(x)| \ll 1$, which is usually called the kernel. Consider the sum

$$
S = \sum_{m \sim M} \alpha_m \sum_{n \sim N} K(mn).
$$

Choose real parameters $A, B \geq 1$ such that $AB \leq N$, and $2AM < q$. (B.3)
as in [BFK+17, Equation (5.8)] we have

$$
ABS \ll \sqrt{||\alpha||_1 ||\alpha||_2} (AN)^{3/4} q^{o(1)}
$$

(B.4)

where $\eta_b$ are complex numbers satisfying $|\eta_b| \leq 1$, $b \sim B$. Expanding the fourth power in (B.4), the innermost sum in the second factor becomes

$$
\sum_{r \in \mathbb{F}_q} \sum_{1 \leq s \leq 2AM} \left| \sum_{B < b \leq 2B} \eta(b) K(s(r + b)) \right|^4 = \sum_{b \in B} \eta(b) \Sigma(K, b),
$$

(B.5)

where $B$ denotes the set of quadruples $b = (b_1, b_2, b_3, b_4)$ of integers satisfying $B < b_j \leq 2B$, $j = 1, 2, 3, 4$, the coefficients $\eta(b)$ satisfy $|\eta(b)| \leq 1$ for all $b \in B$, and

$$
\Sigma(K, b) = \sum_{r \in \mathbb{F}_q} \sum_{1 \leq s \leq 2AM} K(s(r + b_1)) K(s(r + b_2))
$$

$$
\times K(s(r + b_3)) K(s(r + b_4)).
$$

(B.6)

Let $B^\Delta$ be the subset of $b$ admitting a subset of two entries matching the entries of the complement (for instance, such as $b_1 = b_2$ and $b_3 = b_4$ or $b_1 = b_3$ and $b_2 = b_4$). For such tuples $b$ we use the trivial bound

$$
\sum_{b \in B^\Delta} |\Sigma(K, b)| \ll AB^2 M q.
$$

(B.7)
For \( b \not\in B^\Delta \), we complete the sum over \( s \) in (B.6) using additive characters, see [IK04, Section 12.2] and derive, similarly to (3.3),
\[
\Sigma(K, b) \ll \log q \max_{t \in \mathbb{F}_q} |\Sigma(K, b, t)|,
\]
where
\[
\Sigma(K, b, t) := \sum_{r,s \in \mathbb{F}_q} e_q(st) K(s(r+b_1)) K(s(r+b_2))
\times K(s(r+b_3)) K(s(r+b_4)).
\]

**B.3. Reduction to exponential sums along a curve.** Now consider the kernel of our interest:
\[
K(x) := \sum_{u \in \mathbb{F}_q \atop u^2 = ax} e_q(hu).
\]

**Proposition B.1.** For all \( t \in \mathbb{F}_q \) and all \( b \not\in B^\Delta \) above we have
\[
\Sigma(K, b, t) \ll q.
\]

**Proof.** Since \( b \not\in B^\Delta \) there is a least one value among \( b_1, b_2, b_3, b_4 \) which is not repeated among other values. Without loss of generality we can assume that
\[
b_1 \neq b_2, b_3, b_4.
\]

Substituting (B.10) into (B.9) we obtain
\[
\Sigma(K, b, t) = \sum_{r \in \mathbb{F}_q} \sum_{s \in \mathbb{F}_q} e_q(u_1 + v_1 - u_2 - v_2 + st),
\]
where \( \mathcal{Z}_{b,r,s} \) is the set of solutions \((u_1, v_1, u_2, v_2) \in \mathbb{F}_q^4\) to
\[
\begin{align*}
  u_1^2 &= s(r + b_1), \quad v_1^2 = s(r + b_2),
  u_2^2 &= s(r + b_3), \quad v_2^2 = s(r + b_4).
\end{align*}
\]

When \( r = -b_1 \) and \( s \in \mathbb{F}_q \), then \( \#\mathcal{W}_{b,r,s} = O(1) \), and so the contribution to (B.12) is \( O(q) \).

So now we can assume that \( r \neq -b_1 \). We see that if \( u_1 = 0 \), then \( s = 0 \) and so also \( v_1 = u_2 = v_2 = 0 \). Eliminating \( s \) from (B.13), and writing
\[
(u_1, v_1, u_2, v_2) = (w, wx, wy, wz),
\]
we see that (B.12) becomes
\[
\Sigma(K, b, t) = \Sigma^*(K, b, t) + O(q),
\]
where
\[
\Sigma^*(K, b, t) := \sum_{r \in \mathbb{F}_q} \sum_{s \in \mathbb{F}_q} e_q(u_1 + v_1 - u_2 - v_2 + st).
\]
where
\[ \Sigma^*(K, b, t) = \sum_{(r,x,y,z) \in V_b^\ast} \sum_{w \in \mathbb{F}_q} e_q \left( w(1 + x - y - z) + (r + b_1)tw^2 \right), \]
and sums is taken over the set \( V_b^\ast \) of solutions \((r, x, y, z) \in \mathbb{F}_q^4\) with \( r \neq -b_1 \)
\[ (r + b_1)x^2 = (r + b_2), \quad (r + b_1)y^2 = (r + b_3), \quad (r + b_1)z^2 = (r + b_4). \]
We now change the variable \( r \to u - b_1, \ u \in \mathbb{F}_q^\times \), and write
\[ \Sigma^*(K, b, t) = \sum_{(u,x,y,z) \in U_b^\ast} \sum_{w \in \mathbb{F}_q} e_q \left( w(1 + x - y - z) + tuw^2 \right), \quad (B.15) \]
where \( U_b^\ast \) of solutions \((u, x, y, z) \in \mathbb{F}_q^4\) with \( u \neq 0 \)
\[ x^2 = 1 + c_1u, \quad y^2 = 1 + c_2u, \quad z^2 = 1 + c_3u \]
where
\[ c_i = b_{i+1} - b_1, \quad i = 1, 2, 3. \]
Observe that by (B.11) we have
\[ c_i \neq 0, \quad i = 1, 2, 3. \]
The case \( t \in \mathbb{F}_q^\times \). Recalling the definition of the Gauss sums (3.1) we write
\[ \Sigma^*(K, b, t) = \sum_{(u,x,y,z) \in U_b^\ast} G_q(tu, 1 + x - y - z). \quad (B.17) \]
Evaluating the Gauss sums as in (3.2) we see that (B.17) becomes
\[ \Sigma^*(K, b, t) = \varepsilon_q \sqrt{q} \sum_{(u,x,y,z) \in U_b^\ast} \left( \frac{tu}{q} \right) e_q \left( -4tu(1 + x - y - z)^2 \right), \]
where we can now extend \( U_b^\ast \) to the set \( U_b \) which also allows the value \( u = 0 \) (this value is eliminated automatically by the pole in the \( 4tu \) term).
Recalling the definition of the Legendre symbol, we represent mixed sum \( \Sigma^*(K, b, t) \) with multiplicative and additive characters as a linear combination of two pure exponential sums with rational functions
\[ \Sigma^*(K, b, t) := 2\Sigma_2(K, b, t) - \Sigma_1(K, b, t) \quad (B.18) \]
where
\[
\begin{align*}
\Sigma_1(K, b, t) &:= \varepsilon_q \sqrt{q} \sum_{(u, x, y, z) \in U_b} e_q \left( -\frac{4tu}{q}(1 + x - y - z)^2 \right), \\
\Sigma_2(K, b, t) &:= \varepsilon_q \sqrt{q} \sum_{(w, x, y, z) \in W_{b, t}} e_q \left( -\frac{w^2}{q}(1 + x - y - z)^2 \right),
\end{align*}
\]
where the set $U_b$ is the same as $U^*_b$ in which also allows the value $u = 0$ (this value is eliminated automatically by the pole in the $\frac{4tu}{q}$ term), and $W_{b, t}$ is the set of solutions $(w, x, y, z) \in \mathbb{F}_q^4$ to
\[
x^2 = 1 + c_1 t w^2, \quad y^2 = 1 + c_2 t w^2, \quad z^2 = 1 + c_3 t w^2.
\]

A simple argument using the Weil bound on character sums (see, for example, [IK04, Theorem 11.23]) shows that each of the varieties $W_{b, t}$ and $U_b$ has $A(c_1, c_2, c_3)q + O(q^{1/2})$ rational points over $\mathbb{F}_q$ where $A(c_1, c_2, c_3) = 1$ if all $c_i$ distinct, 2 if exactly two of the $c_i$ are equal and 4 if all the $c_i$ are equal. Thus the variety is of dimension 1 by the Lang–Weil theorem (that is, an algebraic curve over $\mathbb{F}_q$). Elementary but somewhat tedious calculations show that the function $\frac{4tu}{q}(1 + x - y - z)^2$ is not constant on $U_b$ and the function $\frac{w^2}{q}(1 + x - y - z)^2$ is not constant on $W_{b, t}$.

Therefore the bound of Bombieri [Bom66, Theorem 6]) applies to both sums and yields
\[
\Sigma_{1,2}(K, b, t) \ll q.
\]

Now using this bound in (B.18) and recalling (B.14), we see that
\[
\Sigma(K, b, t) \ll q, \quad t \in \mathbb{F}_q^\times. \quad (B.19)
\]

The case $t = 0$. In this case, we see from (B.15) that
\[
\Sigma^*(K, b, 0) = \sum_{(u, x, y, z) \in U^*_b} \sum_{w \in \mathbb{F}_q} e_q (w(1 + x - y - z)) = qT,
\]
where $T$ is the number of solutions $(u, x, y, z) \in \mathbb{F}_q^4$ to (B.16) with $1 + x = y + z$. Direct elimination of variables shows that $T = O(1)$ and we obtain
\[
\Sigma(K, b, 0) \ll q. \quad (B.20)
\]

Combining (B.19) and (B.20) we derive the desired result. 

B.4. **Concluding the argument.** Substituting the bound of Proposition B.1 in (B.8) we obtain
\[
\Sigma(K, b) \ll q \log q,
\]
with the contribution from $b \in B^\Delta$ to in (B.5), bounded as (B.7), we see that (B.4) becomes

$$V_{a,q}(\alpha; h, M, N) \leq (AB)^{-1} \sqrt{\parallel\alpha\parallel_1 \parallel\alpha\parallel_2} (AN)^{3/4} (AB^2 Mq + B^4 q)^{1/4} q^{o(1)}.$$  

We now choose

$$A = \frac{1}{2} M^{-1/3} N^{2/3} \quad \text{and} \quad B = (MN)^{1/3}$$

to balance the above estimate, and note that the conditions (B.2) imply $A, B \geq 1$ as well as (B.3), which implies (B.1).

**B.5. Further possibilities.** One of the obvious ways to try to improve the bound (B.1) is to use higher powers as in [KMS19]. However studying exponential sums over more general higher dimensional varieties can be quite challenging.

This however leads to some further possibilities where the above method can be more competitive is an extension of the summation over $m$ in the sum $V_{a,q}(\alpha, \beta; h, M, N)$ from a dyadic interval $m \sim M$ to an arbitrary set $m \in M$ with $M \subseteq \mathbb{F}_q$. More precisely, the method of [KMS19] rests on bounds for the second moment of the quantity

$$\nu(r, s) = \sum_{a \sim A, \, m \sim M, \, n \in [0, 6N]} |\alpha_m|.$$  

It has been shown in [BS19] that one can obtain good bounds on this quantity even if $m$ runs through an arbitrary set $M \subseteq \mathbb{F}_q$.

**Appendix C. Correlation between Salié sums**

The identity (1.6) links Salié sums to sums over modular square roots and plays an important role in the proof of [DZ19, Theorem 1.2]. Using our argument, we now are able to obtain the following improvement of the bound of [DZ19, Theorem 1.2] on sums of Salié sums (1.3).

**Proposition C.1.** For any positive integers $M, N \leq q$ and any integer $a$ with $\gcd(a, q) = 1$, we have

$$\sum_{n_1, n_2 \sim N} \left| \sum_{m \sim M} S(m, an_1; q) S(m, an_2; q) \right| \leq q^{5/4 + o(1)} \left( Mq^{-1/4} + M^{5/8} \right) \left( N^2 q^{-1/4} + N^{13/8} \right).$$
Proof. From [DZ19, Equations (8.1) and (8.2)] we infer that
\[
\sum_{n_1,n_2 \sim N} S(m, an_1; q)S(m, an_2; q) \leq q(R_1 + R_{-1}),
\]
where \( R_j, j = \pm 1, \) are as in (6.8). Using the bound (6.11) we derive the desired result. \( \square \)

For example, for \( M, N \leq q^{2/3} \), the bound of Proposition C.1 simplifies as
\[
\sum_{n_1,n_2 \sim N} S(m, an_1; q)S(m, an_2; q) \leq MN^2 q \left( \frac{q^2}{M^3 N^3} \right)^{1/8}
\]
which improves the trivial bound \( MN^2 q \) whenever \( MN \geq q^{2/3 + \varepsilon} \) for any fixed \( \varepsilon > 0 \).

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