A hypothesis concerning “quantal” Hilbert space criterion of chaos in nonlinear dynamical systems

Krzysztof Kowalski

Department of Theoretical Physics, University of Łódź, ul. Pomorska 149/153, 90-236 Łódź, Poland

Based on the Hilbert space approach to the theory of nonlinear dynamical systems developed by the author a hypothesis is formulated concerning the “quantal” criterion for classical ordinary differential systems to exhibit chaotic behaviour.

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1. Introduction

As so many natural laws and models of natural phenomena are described by nonlinear dynamical systems \( \frac{dx}{dt} = F(x) \), it is no exaggeration to say that they abound in modern science and technology. Physical and numerical experiments show that deterministic chaos in such systems is ubiquitous. On the other hand, the recent observations concerning possibility of controlling chaos [1] indicate its new promising applications in engineering and medical sciences [2]. No wonder that the interest in the theory of chaotic dynamical systems is steadily increasing.

The object of the present paper is to formulate a hypothesis concerning “quantal” Hilbert space criterion of chaos in nonlinear systems \( \frac{dx}{dt} = F(x) \), where \( F \) is analytic in \( x \), with the use of the formalism developed by the author [3-9], relying on reduction of nonlinear dynamical systems to the linear, abstract, Schrödinger-like equation in Hilbert space. Namely, it is suggested that the chaotic behaviour of the nonlinear dynamical systems can be related to the growth of the “quantal” entropy which can be naturally introduced within the Hilbert space approach resembling quantum mechanics.

2. Hilbert space approach

In this section we briefly outline the Hilbert space description of nonlinear ordinary differential systems [3]. Consider the analytic system (complex or real)

\[
\frac{dz}{dt} = F(z), \quad z(0) = z_0, \tag{1}
\]

where \( F: \mathbb{C}^k \to \mathbb{C}^k \) is analytic in \( z \).

The vectors \( |z, t\rangle \) defined by

\[
|z, t\rangle = \exp\left[\frac{i}{2}(|z|^2 - |z_0|^2)\right]|z\rangle, \tag{2}
\]

where \( |z\rangle \) is a normalized coherent state (see appendix) and \( z \) fulfils (1), satisfy the following linear, Schrödinger-like equation in Hilbert space:

\[
\frac{d}{dt}|z, t\rangle = M|z, t\rangle, \quad |z, 0\rangle = |z_0\rangle, \tag{3}
\]

where \( M \) is the boson operator such that

\[
M = a^\dagger \cdot F(a). \tag{4}
\]

In view of (2) we arrive at the following eigenvalue equation:

\[
a|z_0, t\rangle = z(z_0, t)|z_0, t\rangle \tag{5}
\]
relating the solution \( z(z_0, t) \) of (1) and the solution \(|z_0, t\rangle\) of (3). Thus, it turns out that the nonlinear dynamical system (1) can be cast into the linear, abstract, Schrödinger-like equation (3). We note that the algorithm can be immediately extended to the case of nonautonomous systems such that the corresponding vector field is analytic in \( z \)-variables. We also remark that the eigenvalue equation (5) suggests the “quantization scheme” of the form \( z \rightarrow a \), where \( z \) fulfils (1), within introduced formalism resembling quantum mechanics.

Observe that (3) and (5) correspond to the “Schrödinger picture” within the actual “quantal” treatment. We end this brief account of the Hilbert space approach with discussion of the “Heisenberg picture”. The “Heisenberg equations of motion” obeyed by the time-dependent Bose annihilation operators are of the form

\[
\frac{da}{dt} = [a, M], \quad a(0) = a,
\]

where \( M \) is the “Hamiltonian” given by (4). Since the “Hamiltonian” in “Schrödinger picture” \( M \) is independent of time, therefore it coincides with the “Hamiltonian” in “Heisenberg picture”.

The formal solution of (6) can be written as

\[
a(t) = V(t)^{-1}aV(t),
\]

where \( V(t) = e^{tM} \) is the evolution operator. It is clear that the following relation holds true:

\[
|z_0, t\rangle = V(t)|z_0\rangle,
\]

where \(|z_0, t\rangle\) is the solution of (3).

Now eqs. (5), (8) and (7) taken together yield

\[
a(t)|z_0\rangle = z(z_0, t)|z_0\rangle.
\]

Hence we find that the solutions of the systems (1) coincide with the expectation values (covariant symbols) of the time-dependent Bose annihilation operators, i.e.

\[
z(z_0, t) = \langle z_0|a(t)|z_0\rangle.
\]

It should be noted that whenever the “quantization scheme” \( z \rightarrow a \) is assumed, where \( z \) fulfils (1), then (10) forms the “Ehrenfest’s theorem” within the actual “quantal” approach.

3. The criterion of chaos

We now formulate the hypothesis concerning the criterion for the nonlinear dynamical systems (1) to show chaotic behaviour. Consider the system (1). We introduce the operator \( \rho \) of the form (see (A.2)):

\[
\rho = \int_{\mathbb{R}^{2k}} d\mu(w) e^{-|w-z_0|^2} |w\rangle\langle w|.
\]
Evidently, the operator $\rho$ is Hermitian and nonnegative, i.e.

$$\rho^\dagger = \rho,$$  \hspace{1cm} (12a)

$$\rho \geq 0.$$  \hspace{1cm} (12b)

Furthermore, using (A.6) we immediately obtain

$$\text{Tr} \rho = 1.$$  \hspace{1cm} (12c)

Finally, taking into account (11), (A.2), (A.5) and (10) we get

$$\langle a(t) \rangle = \text{Tr}(\rho a(t)) = z(t),$$  \hspace{1cm} (12d)

where $a(t)$ are the time-dependent Bose annihilation operators and $z(t)$ fulfills (1).

We have thus shown that if the “quantization scheme” $z \rightarrow a$, where $z$ satisfies (1) is assumed, then $\rho$ plays the role of the density matrix within the “quantal” Hilbert space approach. Prompted by this analogy, we define the “entropy” as

$$S = -\text{Tr}(\rho \ln \rho).$$  \hspace{1cm} (13)

On using the relation

$$\text{Tr} \rho^n = \frac{1}{(2^n - 1)^k}, \quad n \geq 1,$$  \hspace{1cm} (14)

following directly from (11), (A.2) and (A.4), we find

$$S = -\sum_{i=0}^{\infty} \binom{i + k - 1}{i} \frac{1}{2^{i+k}} \ln \frac{1}{2^{i+k}} = 2k \ln 2.$$  \hspace{1cm} (15)

Since an arbitrary vector in a Fock space can be specified by an infinite sequence of $n$-vectors of 1’s and 0’s and the number of different $n$-vectors is $2^n$, therefore the obtained value of the “entropy” can be regarded as an averaged amount of the information necessary to fix the coherent state $|z_0\rangle$ in a Hilbert space of states.

Now in analogy to quantum mechanics, we introduce the time-dependent “density matrix” such that

$$\rho(t) = \int_{\mathbb{R}^{2k}} d\mu(w) e^{-|w-z_0|^2} e^{tM} |w\rangle\langle w| e^{tM^\dagger},$$  \hspace{1cm} (16)

where $M$ is the “Hamiltonian” related to the system (1).

We note that $\rho(t)$ is Hermitian and nonnegative at any time. The time-dependent “entropy” corresponding to (16) is given by

$$S(t) = -\text{Tr}[\rho(t) \ln \rho(t)].$$  \hspace{1cm} (17)

We are now in a position to propound our hypothesis. We assert that if the following conditions hold:

$$\exists t_0, t > t_0 \quad |z(z_0, t)|^2 < C,$$  \hspace{1cm} (18a)
where $z(z_0, t)$ is the solution to (1); $C > 0$ is a constant,

$$\forall t > t_*, \frac{dS(t)}{dt} > 0,$$

then the system (1) is chaotic one.

4. Discussion

We have formulated in this work a hypothesis concerning a new, “quantal” Hilbert space criterion of chaos in nonlinear dynamical systems. Note that the condition (18a) ensures that the “density matrix” $\rho(t)$ is of trace class for all $t > t_*$. Indeed, from (16) and the relation

$$\langle z_0, t|z_0, t\rangle = \exp(|z(z_0, t)|^2 - |z_0|^2),$$

which is an immediate consequence of (2), it follows that

$$\text{Tr}\rho(t) = \int_{\mathbb{R}^{2k}} d\mu(w) \exp(-|w - z_0|^2) \exp(|z(w, t)|^2 - |w|^2),$$

where $z(z_0, t)$ is the solution to (1).

On the other hand, the condition (18a) is consistent with the exceptional role of dissipativity of systems showing chaotic behaviour. We remark that the nonunitary evolution implied by the Schrödinger-like equation (3) is crucial for the actual treatment. In fact, as in quantum mechanics, the unitary evolution implies $S(t) = S = \text{const}$. We note at the same time that the evolution operator $V(t) = e^{tM}$ defined by (8) is unitary only in the case of the linear system (1) with a skew-Hermitian fundamental matrix. Finally, we point out that the “quantal” Hilbert space approach recognizes the divergence of orbits in the phase space of the system (1). Namely, we have the formula

$$\frac{|\langle z_0, t|z_0 + \delta z_0, t\rangle|^2}{\langle z_0, t|z_0 + \delta z_0, t\rangle \langle z_0 + \delta z_0, t|z_0 + \delta z_0, t\rangle} = \exp(-|z(z_0 + \delta z_0, t) - z(z_0, t)|^2)$$

which can be easily obtained from (2) and (A.1). Therefore, the growth of the distance between the two trajectories of the system (1) starting from $z_0$ and $z_0 + \delta z_0$ implies the decay of the corresponding transition probability from the left-hand side of (21).

Appendix. Coherent states
We append the basic facts about coherent states. Recall first that the Bose creation ($a^\dagger$) and annihilation ($a$) operators, where $a^\dagger = (a_1^\dagger, \ldots, a_k^\dagger)$, $a = (a_1, \ldots, a_k)$, obey the Heisenberg algebra

\[
[a_i, a_j^\dagger] = \delta_{ij} I, \quad [a_i, a_j] = [a_i^\dagger, a_j^\dagger] = 0, \quad i, j = 1, \ldots, k.
\]

The coherent states $|z\rangle$, where $z \in \mathbb{C}^k$, are usually defined as eigenvectors of the annihilation operators, that is

\[
a |z\rangle = z |z\rangle.
\]

The normalized coherent states can be defined as

\[
|z\rangle = \exp\left( -\frac{1}{2}|z|^2 \right) \exp(z \cdot a^\dagger) |0\rangle.
\]

where $|z|^2 = \sum_{i=1}^k |z_i|^2$, $u \cdot v = \sum_{i=1}^k u_i v_i$ and $|0\rangle$ is the vacuum vector satisfying

\[
a |0\rangle = 0.
\]

The coherent states are not orthogonal, namely

\[
\langle z | w \rangle = \exp\left( -\frac{1}{2}(|z|^2 + |w|^2 - 2z^* \cdot w) \right),
\]

where the asterisk designates the complex conjugation and $z^* = (z_1^*, \ldots, z_k^*)$.

Hence we find

\[
|\langle z | w \rangle|^2 = \exp(-|z - w|^2). \quad (A.1)
\]

These states form the complete (overcomplete) set. The resolution of the identity for the coherent states is given by

\[
\int_{\mathbb{R}^{2k}} d\mu(z) |z\rangle\langle z| = I, \quad (A.2)
\]

where

\[
d\mu(z) = \prod_{i=1}^k \frac{1}{\pi} d(\text{Re} z_i) d(\text{Im} z_i).
\]

Now let $|\phi\rangle$ be an arbitrary state. It can be easily shown that the function (symbol of the vector) $\phi(z^*) = \langle z | \phi \rangle$ is of the form

\[
\phi(z^*) = \tilde{\phi}(z^*) \exp\left( -\frac{1}{2}|z|^2 \right), \quad (A.3)
\]

where $\tilde{\phi}(z^*)$ is an analytic (entire) function.

Taking into account (A.2) and (A.3) we get

\[
\langle \phi | \psi \rangle = \int_{\mathbb{R}^{2k}} d\mu(z) \exp(-|z|^2) (\tilde{\phi}(z^*))^* \bar{\psi}(z^*).
\]
Thus the abstract vectors can be represented by analytic functions. This representation is usually known as the Bargmann representation. The action of the Bose operators in the Bargmann representation has the following form:

$$a\tilde{\phi}(z^*) = \frac{\partial}{\partial z^*}\tilde{\phi}(z^*), \quad a^\dagger\tilde{\phi}(z^*) = z^*\tilde{\phi}(z^*).$$

On using (A.2) and (A.3) we arrive at the following reproducing property of coherent states:

$$\tilde{\phi}(w^*) = \int_{\mathbb{R}^{2k}} d\mu(z) \exp(-|z|^2)K(w^*, z)\tilde{\phi}(z^*), \quad (A.4)$$

where the reproducing kernel (Bergman reproducing kernel) is

$$K(w^*, z) = \exp(w^* \cdot z).$$

Taking the Hermitian conjugate of (A.4) we obtain the following form of the reproducing property:

$$\tilde{\psi}(w) = \int_{\mathbb{R}^{2k}} d\mu(z) \exp(-|z|^2) \exp(w \cdot z^*)\tilde{\psi}(z). \quad (A.5)$$

We finally remark that the coherent states are the convenient tool for the study of operators. For example, the trace of a linear operator $L$ can be expressed by

$$\text{Tr}L = \int_{\mathbb{R}^{2k}} d\mu(z) L(z^*, z), \quad (A.6)$$

where

$$L(z^*, z) = \langle z|L|z \rangle$$

is called the covariant symbol of the operator $L$ [10].
References

[1] E. Ott, C. Grebogi and J.A. Yorke, Phys. Rev. Lett. 64 (1990) 1196.
[2] A. Garfinkel, M.L. Spano, W.L. Ditto and J.N. Weiss, Science 257 (1992) 1230.
[3] K. Kowalski, Physica A 145 (1987) 408.
[4] K. Kowalski, Physica A 152 (1988) 98.
[5] K. Kowalski and W.-H. Steeb, Progr. Theor. Phys. 85 (1991) 713.
[6] K. Kowalski and W.-H. Steeb, Progr. Theor. Phys. 85 (1991) 975.
[7] K. Kowalski and W.-H. Steeb, Nonlinear Dynamical Systems and Carleman Linearization (World Scientific, Singapore, 1991).
[8] K. Kowalski, Physica A 195 (1993) 137.
[9] K. Kowalski, Physica A 198 (1993) 493.
[10] F.A. Berezin, Mat. Sb. 86 (1971) 578 (Russian).