A Quantum Analogue of the Boson-Fermion Correspondence

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Abstract

We review the classical boson-fermion correspondence in the context of the $\hat{sl}(2)$ current algebra at level 2. This particular algebra is ideal to exhibit this correspondence because it can be realized either in terms of three real bosonic fields or in terms of one real and one complex fermionic fields. We also derive a fermionic realization of the quantum current algebra $U_q(\hat{sl}(2))$ at level 2 and by comparing this realization with the existing bosonic one we extend the classical correspondence to the quantum case.

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1 Introduction

It is now well established that the quantum affine algebras play a key role in the description of the dynamics of two-dimensional quantum integrable models such as the XXZ Heisenberg spin chain \([1]\), the sine-Gordon and the Thirring models \([2]\).

For practical computational purposes however, these infinite-dimensional symmetries are hardly directly useful. The reason is that the relevant representations are also infinite-dimensional and their intertwiners or vertex operators, which are the main objects that explicitly appear in the computation of physical quantities like correlation functions, form factors and S matrix elements, have nontrivial fusion rules (operator product expansions.) This makes their normal ordering highly complicated and very difficult to use for the computation of the above physical quantities. There are no simple analogues here of the Wick theorem which allows to perform the normal ordering of operators constructed in terms of free fields. However, one can still hope to apply the Wick theorem to vertex operators if one succeeds to realize these intertwiners in terms of free fields. This has been the main motivation for obtaining the free field realizations of all the infinite-dimensional symmetry algebras that appear in conformal field theories and more recently, for the extension of these realizations to quantum affine algebras. (See Ref. \([3]\) for a general review, more references will be given in the subsequent sections of this paper.)

In contrast to the situation with the classical affine algebras, so far, all the existing free field realizations of quantum affine algebras have only been given in terms of bosonic fields. There is a single exception to this statement: a fermionic construction of these algebras at level zero has been derived in Ref. \([4]\), it does not seem possible however to generalize this construction to higher levels. Since quantum affine algebras with positive levels are these that play a major role in physical models through their highest weight representations, it is therefore of interest to investigate fermionic realizations of these algebras so as to provide alternative models to the existing bosonic ones.

In this paper, we construct a fermionic realization for \(U_q(\widehat{sl}(2))\) at the particular level 2. The main feature of this algebra is that its fermionic realization requires one real and one complex fermionic fields. Since we know already the bosonization of this algebra, we
may therefore derive the equivalence between the two and obtain a quantum analogue of the important boson-fermion correspondence that plays a major role in conformal field theories \[3, 4\].

This paper is organized as follows. In section 2, we first review the classical \( \widehat{sl}(2) \) algebra at arbitrary level in order to establish our notations. Then, we describe both bosonic and fermionic realizations of this algebra at the particular level 2 and discuss their equivalence, i.e., the classical boson-fermion correspondence. We give in section 3 the quantum analogues of the results of section 2. We first define the \( U_q(\widehat{sl}(2)) \) quantum algebra at arbitrary level and describe one of its bosonizations at level 2. Then, we present the main new results of this paper, that is, the fermionic realization of \( U_q(\widehat{sl}(2)) \) at level 2 and its equivalence to the bosonic one, i.e., the quantum analogue of the boson-fermion correspondence. Conclusions are found in section 4.

2 Classical boson-fermion correspondence

2.1 The \( \widehat{sl}(2) \) current algebra

The \( \widehat{sl}(2) \) affine algebra is generated by the Chevalley basis elements \( \{ e_i, f_i, h_i, d; i = 0, 1 \} \) subjected to

\[
egin{align*}
[h_i, h_j] & = 0, \\
[h_i, e_j] & = a_{ij}e_j, \\
[h_i, f_j] & = -a_{ij}f_j, \\
[e_i, f_j] & = \delta_{ij}h_i, \\
[d, h_i] & = 0, \\
[d, e_i] & = \delta_{i0}e_i, \\
[d, f_i] & = -\delta_{i0}f_i, \\
[k, x] & = 0, \quad \forall x \in \widehat{sl}(2), \quad i, j = 0, 1; \tag{2.1}
\end{align*}
\]

where,

\[
(a_{ij}) = \begin{pmatrix} 2 & -2 \\ -2 & 2 \end{pmatrix} \tag{2.2}
\]
is the $\hat{sl}(2)$ generalized Cartan matrix and the element $k = h_1 + h_0$ referred to as the level. The Serre relations should be added to the defining relations (2.1). As is well known, the above algebra is isomorphic to the one generated by the elements \( \{ E_\pm^n, H_n, k, d; n \in \mathbb{Z} \} \) which obey:

\[
\begin{align*}
[H_n, H_m] &= nk\delta_{n+m,0}, \\
[H_n, E_\pm^m] &= \pm\sqrt{2}E_\pm^{n+m}, \\
[E_\pm^n, E^-_m] &= \sqrt{2}H_{n+m} + nk\delta_{n+m,0}, \\
[d, E_\pm^n] &= nE_\pm^n, \\
[d, H_n] &= nH_n, \\
[k, x] &= 0, \quad \forall x \in \hat{sl}(2); \quad n, m \in \mathbb{Z}.
\end{align*}
\]

(2.3)

The isomorphism is realized through the identifications

\[
\begin{align*}
h_1 &\equiv \sqrt{2}H_0, \quad h_0 \equiv k - \sqrt{2}H_0, \\
e_1 &\equiv E_0^+, \quad e_0 \equiv E_1^-, \\
f_1 &\equiv E_0^-, \quad f_0 \equiv E_1^+.
\end{align*}
\]

(2.4)

Note that the elements $k$ and $d$ are not redefined in this second form of $\hat{sl}(2)$. There is yet a third form of this algebra which is isomorphic to the previous two and which we refer to as the $\hat{sl}(2)$ current algebra. It is most useful for the purpose of discussing free field realizations as will be clear throughout the remainder of this paper. The structure relations are given as operator product expansions (OPE’s) among the currents (i.e., formal generating functions in a complex variable $z$) $E^\pm(z) = \sum_{n \in \mathbb{Z}} E^\pm_n z^{-n-1}$ and $H(z) = \sum_{n \in \mathbb{Z}} H_n z^{-n-1}$. They read

\[
\begin{align*}
H(z).H(w) &\sim \frac{k}{(z-w)^2}, \quad |z| > |w|, \\
H(z).E^\pm(w) &\sim \pm\frac{\sqrt{2}E^\pm(w)}{z-w}, \quad |z| > |w|, \\
E^+(z).E^-(w) &\sim \frac{\sqrt{2}H(w)}{z-w} + \frac{k}{(z-w)^2}, \quad |z| > |w|, \\
E^-(z).E^+(w) &\sim \frac{\sqrt{2}H(w)}{z-w} + \frac{k}{(z-w)^2}, \quad |z| > |w|.
\end{align*}
\]

(2.5 - 2.8)

Here the symbol $\sim$ means an equality up to regular terms as $z$ approaches $w$. The relations involving both elements $k$ and $d$ are the same as in (2.3).
2.2 Bosonic realization of $\widehat{sl}(2)$ at level 2

We now briefly review the bosonization of the $\widehat{sl}(2)$ current algebra given in (2.5)-(2.8). This bosonization is called the Wakimoto realization and is valid for a generic level $k$. Here, we shall only present the case $k = 2$ which is in fact very similar to the general one. It requires three free real bosonic fields $\phi^1(z)$, $\phi^2(z)$ and $\phi^3(z)$ with OPE’s

$$\phi^j(z)\phi^j(w) = -(1)^{j-1}\delta^j \ln|z - w| + :\phi^j(z)\phi^j(w):, \quad |z| > |w|. \quad (2.9)$$

The mode expansions of these fields are given by

$$\phi^j(z) = \phi^j - i\phi^j_0 \ln z + i \sum_{n \neq 0} \frac{\phi^j_n}{n} z^{-n}, \quad j = 1, 2, 3. \quad (2.10)$$

The bosonic modes $\{\phi^j_n, \phi^j; n \in \mathbb{Z}; j = 1, 2, 3\}$ generate a Heisenberg algebra with non-vanishing commutators given by

$$[\phi^j_n, \phi^\ell_m] = (1)^{j-1}n\delta^j \delta_{n+m,0},$$
$$[\phi^j, \phi^\ell_0] = (1)^{j-1}i\delta^j \delta_{n,0}, \quad \ell = 1, 2, 3. \quad (2.11)$$

In (2.9), the symbol :: denotes the bosonic normal ordering which is defined by

$$:\phi^1_n\phi^1_m : = \phi^1_m\phi^1_n, \quad n > 0,$$
$$:\phi^1_n\phi^1_m : = \phi^1_m\phi^1_n, \quad n < 0,$$
$$:\phi^1_0\phi^1 : = \phi^1\phi^1_0,$$
$$:\phi^1\phi^1_0 : = \phi^1\phi^1_0. \quad (2.12)$$

This means that the creation modes $\{\phi^j_n; n < 0\}$ and the shift mode $\phi^j_0$ are always moved to the left of the annihilation modes $\{\phi^j_n; n > 0\}$ and of the momentum mode $\phi^j_0$ in products of the above bosonic fields. With these definitions it can easily be checked that the ‘Heisenberg mode algebra’ (2.11) is equivalent to the ‘Heisenberg current algebra’ (2.9). As usual in the classical case (i.e. $q = 1$), a field or a product of fields defined at the same point $z$ is understood to be normal ordered and the symbol :: is omitted. The Wakimoto bosonization reads

$$H(z) = i\sqrt{2}\partial\phi^1(z),$$
$$E^\pm(z) = \left(\pm i\partial\phi^2(z) + i\sqrt{2}\partial\phi^3(z)\right)\exp\{\pm i(\phi^1(z) + \phi^2(z))\}. \quad (2.13)$$

2.3 Fermionic realization of $\hat{sl}(2)$ at level 2

Unlike the Wakimoto bosonization, the fermionic realization of the $\hat{sl}(2)$ current algebra (2.3)-(2.8) exists only for the particular cases of $k = 0, 1, 2$. Let us review it when $k = 2$.

It requires one real and one complex fermionic fields, which we denote respectively by $\psi(z)$ and $\chi(z)$. These fields satisfy the following OPE's:

\[
\begin{align*}
\psi(z)\psi(w) &= \frac{1}{z-w} + :\psi(z)\psi(w):, \quad |z| > |w|, \\
\chi(z)\chi^\dagger(w) &= \frac{1}{z-w} + :\chi(z)\chi^\dagger(w):, \quad |z| > |w|, \\
\chi^\dagger(z)\chi(w) &= \frac{1}{z-w} + :\chi^\dagger(z)\chi(w):, \quad |z| > |w|, \\
\chi(z)\chi(w) &= :\chi(z)\chi(w):, \\
\chi^\dagger(z)\chi^\dagger(w) &= :\chi^\dagger(z)\chi^\dagger(w):,
\end{align*}
\]

where $\chi^\dagger(z)$ stands for the Hermitian conjugate of $\chi(z)$. The mode expansions of these fields are given by

\[
\begin{align*}
\psi(z) &= \sum_{r \in \mathbb{Z}+1/2} \psi_r z^{-r-1/2}, \\
\chi(z) &= \sum_{r \in \mathbb{Z}+1/2} \chi_r z^{-r-1/2}, \\
\chi^\dagger(z) &= \sum_{r \in \mathbb{Z}+1/2} \chi^\dagger_r z^{-r-1/2}.
\end{align*}
\]  

(2.15)

The fermionic modes $\{\psi_r, \chi_r, \chi^\dagger_r; r \in \mathbb{Z}+1/2\}$ satisfy the following anticommutation relations:

\[
\begin{align*}
\{\psi_r, \psi_s\} &= \delta_{r+s,0}, \\
\{\chi_r, \chi^\dagger_s\} &= \delta_{r+s,0}, \\
\{\chi_r, \chi_s\} &= 0, \\
\{\chi^\dagger_r, \chi^\dagger_s\} &= 0, \quad r, s \in \mathbb{Z}+1/2;
\end{align*}
\]

(2.16)

with $\psi_r$ commuting with $\{\chi_r, \chi^\dagger_r\}$. Note that the above ‘Clifford mode algebra’ (2.16) is equivalent to the ‘Clifford current algebra’ (2.14) if the fermionic normal ordering introduced in (2.14) is defined as in the bosonic case, that is, in products of the fermionic fields, the creation modes $\{\psi_r, \chi_r, \chi^\dagger_r; r \leq -1/2\}$ are always moved to the left of the annihilation modes $\{\psi_r, \chi_r, \chi^\dagger_r; r \geq 1/2\}$. Here also we adopt the standard convention that a field or a product of fields defined at the same point $z$ is understood to be normal ordered, in which case the symbol :: will be omitted. It can easily be checked that with this normal ordering the following relation is satisfied:

\[
: \psi(z)\psi(z) := 0.
\]

(2.17)
With the above definitions and conventions, the fermionic realization of the $\hat{sl}(2)$ current algebra at level 2 (2.5)-(2.8) is given by

$$
H(z) = \sqrt{2} \chi(z) \chi^\dagger(z),
$$
$$
E^+(z) = \sqrt{2} \psi(z) \chi(z),
$$
$$
E^-(z) = \sqrt{2} \psi(z) \chi^\dagger(z).
$$

Let us show for example how (2.7) is simply recovered from these identifications:

$$
E^+(z).E^-(w) = 2 \psi(z) \chi(z).\psi(w) \chi^\dagger(w) \\
= 2 \left( \frac{1}{z-w} : \psi(z) \psi(w) : \right) \left( \frac{1}{z-w} + : \chi(z) \chi^\dagger(w) : \right) \\
= \frac{2}{(z-w)^2} + \frac{2 \chi(w) \chi^\dagger(w)}{z-w} + \text{regular as } z \to w \\
= \frac{2}{(z-w)^2} + \sqrt{2} H(w) \frac{z-w}{z-w} + \text{regular as } z \to w.
$$

Note that relation (2.17) has been used in this derivation.

### 2.4 Classical boson-fermion correspondence

In this section, the bosonic and the fermionic realizations are seen to be equivalent through identifications among the fermionic fields and certain vertex operators constructed in terms of the bosonic free fields. This is known as the boson-fermion correspondence. We prove to start that the complex fermionic field $\chi(z)$, its conjugate $\chi^\dagger(z)$ and their quadratic normal ordered products can be bosonized as follows:

$$
\chi(z) \equiv e^{i \phi^1(z)},
$$
$$
\chi^\dagger(z) \equiv e^{-i \phi^1(z)},
$$
$$
: \chi(z) \chi^\dagger(w) : \equiv \frac{1}{z-w} \left( : e^{i \phi^1(z)-i \phi^1(w)} : -1 \right),
$$
$$
: \chi^\dagger(z) \chi(w) : \equiv \frac{1}{z-w} \left( : e^{-i \phi^1(z)+i \phi^1(w)} : -1 \right),
$$
$$
: \chi(z) \chi(w) : \equiv (z-w) : e^{i \phi^1(z)+i \phi^1(w)} :,
$$
$$
: \chi^\dagger(z) \chi^\dagger(w) : \equiv (z-w) : e^{-i \phi^1(z)-i \phi^1(w)} :.
$$

Indeed, using the relation (which describes the normal ordering of pure exponential vertex operators):

$$
e^{ia \phi^1(z)} . e^{ib \phi^1(w)} = (z-w)^{ab} : e^{ia \phi^1(z)+ib \phi^1(w)} :, \quad a, b \in R,
$$
one can easily show that the current algebra generated by the vertex operators \( e^{i\phi^1(z)} \) and \( e^{-i\phi^1(z)} \) is the same as the one generated by the fermionic fields \( \chi(z) \) and \( \chi^\dagger(z) \) (2.14). More explicitly, we have

\[
\begin{align*}
e^{i\phi^1(z)}e^{-i\phi^1(w)} &= \frac{1}{z-w} + \frac{1}{z-w} \left( :e^{i\phi^1(z)-i\phi^1(w)} : -1 \right), \\
e^{-i\phi^1(z)}e^{i\phi^1(w)} &= \frac{1}{z-w} + \frac{1}{z-w} \left( :e^{-i\phi^1(z)+i\phi^1(w)} : -1 \right), \\
e^{i\phi^1(z)}e^{i\phi^1(w)} &= (z-w) : e^{i\phi^1(z)+i\phi^1(w)} :, \\
e^{-i\phi^1(z)}e^{-i\phi^1(w)} &= (z-w) : e^{-i\phi^1(z)-i\phi^1(w)} :.
\end{align*}
\] (2.22)

Note that if we expand both sides of the relation (2.20) around \( z = w \) we get at the zeroth order the relation

\[
: \chi(w)\chi^\dagger(w) := i\partial\phi^1(w).
\] (2.23)

Re-written in terms of modes, this relation leads to

\[
\begin{align*}
\phi^1_n &= \sum_{r \in \mathbb{Z}+1/2} : \chi_r\chi^\dagger_{n-r} : \\
&= \sum_{r \leq -1/2} \chi_r\chi^\dagger_{n-r} - \sum_{r \geq 1/2} \chi^\dagger_r\chi_r, \quad n \in \mathbb{Z}.
\end{align*}
\] (2.24)

Therefore a Heisenberg mode algebra generated by \( \phi^1_n \) can always be realized in terms of a Clifford algebra generated by \( \chi_r \) and \( \chi^\dagger_r \). This explains the correspondence between a free real bosonic field and a free complex fermionic field.

Let us make here the following remark: since \( \chi^\dagger(z) \) is the Hermitian conjugate of \( \chi(z) \) and because of the above identifications, we expect the vertex operator \( e^{-i\phi^1(z)} \) to be also the Hermitian conjugate of \( e^{i\phi^1(z)} \). To check this, recall the general definition of the adjoint operator of a vertex operator. Let

\[
U_\alpha(z) = :e^{i\alpha\phi^1(z)} : = e^{i\alpha\phi(z)}e^{-\alpha\sum_{n<0} \frac{\phi^1}{} z^{-n}} e^{-\alpha\sum_{n>0} \frac{\phi^1}{} z^{-n}} e^{\alpha\phi_0 \ln z}
\] (2.25)

be a vertex operator with a ‘charge’ \( \alpha \) (in our case \( \alpha = 1 \).) Due to nontrivial scaling dimension (conformal dimension) of \( U_\alpha(z) \), its Hermitian conjugate is defined by \[6, 7\]

\[
[U_\alpha(z)]^\dagger = U_{-\alpha}(\frac{1}{z^*})\frac{1}{z^*\alpha}.
\] (2.26)

In analogy with classical functions, we might think of \( U_{-\alpha}(z) \) as the Hermitian conjugate of \( U_\alpha(z) \) provided that \( \phi^1(z) \) is real, that is,

\[
[\phi^1(z)]^\dagger = \phi^1(\frac{1}{z^*}),
\] (2.27)
since the scaling dimension of $\phi_1^1(z)$ is equal to zero. The latter relation reads in terms of modes as

$$
\begin{align*}
\phi_1^{1\dagger} &= \phi_1^1, \\
\phi_0^{1\dagger} &= \phi_0^1, \\
\phi_n^{1\dagger} &= \phi_{-n}^1.
\end{align*}
$$

(2.28)

With these definitions of the adjoint modes $\{\phi_1^{1\dagger}, \phi_n^{1\dagger}; n \in \mathbb{Z}\}$, we can easily verify that $U_{-\alpha}(z)$ is the Hermitian conjugate of $U_\alpha(z)$, i.e., that the relation (2.26) is satisfied. Indeed,

$$
[U_\alpha(z)]^\dagger = [e^{i\alpha \phi_1^1} e^{-\alpha \sum_{n<0} \frac{\phi_1^n}{n} z^{-n}} e^{-\alpha \sum_{n>0} \frac{\phi_1^n}{n} z^{-n}} e^{\alpha \phi_0^1 \ln z}]^\dagger
$$

$$
= e^{\alpha \phi_0^1 \ln z^*} e^{-i\alpha \phi_1^1} e^{-\alpha \sum_{n>0} \frac{\phi_1^n}{n} z^{-n}} e^{-\alpha \sum_{n<0} \frac{\phi_1^n}{n} z^{-n}}
$$

$$
= z^{-\alpha^2} e^{-i\alpha \phi_1^1} e^{\alpha \sum_{n<0} \frac{\phi_1^n}{n} z^{-n}} e^{\alpha \sum_{n>0} \frac{\phi_1^n}{n} z^{-n}} e^{-\alpha \phi_0^1 \ln z^{-1}}
$$

$$
= U_{-\alpha}(\frac{1}{z}) U_{-\alpha}(z).
$$

(2.29)

For consistency, the adjoint operation as defined in (2.28) must be an antiautomorphism of the Heisenberg algebra (2.11); it is readily checked that this is the case.

Let us now describe the bosonization of the real fermion $\psi(z)$. This case is more subtle than the previous one because if we compare both the bosonic and fermionic realizations of $\widetilde{sl}(2)$ as given respectively by (2.13) and (2.18) and if we take into account the bosonization of the complex fermions $\chi(z)$ and $\chi^{\dagger}(z)$ (2.20), it is then clear that there are two different ways of bosonizing the real fermionic field $\psi(z)$. They are given by

$$
\psi(z) \equiv \psi^\pm(z) = \left( \pm \frac{i}{\sqrt{2}} \partial \phi^2(z) + i \partial \phi^3(z) \right) e^{\pm i\phi^2(z)},
$$

(2.30)

where we have introduced the notation $\psi^\pm(z)$ in order to distinguish these two possible bosonizations of $\psi(z)$ from each other. This peculiar feature—the existence of two possible bosonizations of the same real fermion $\psi(z)$—deserves some explanation. Given the form of the vertex operators arising in the bosonizations of $\psi^\pm(z)$ and the fact that the OPE’s of the fields $\phi^2(z)$ and $\phi^3(z)$ involve a ‘Lorentzian metric,’ that is,

$$
\begin{align*}
\phi^2(z) &. \phi^2(w) = \ln(z-w) + :\phi^2(z)\phi^2(w):, \\
\phi^3(z) &. \phi^3(w) = -\ln(z-w) + :\phi^3(z)\phi^3(w):,
\end{align*}
$$

(2.31)

we need to derive a bosonic normal ordering between two generic vertex operators of the form $V(z,a,a') = ia.\partial \phi(z) \exp ia'.\phi(z)$ and $V(z,b,b') = ib.\partial \phi(z) \exp ib'.\phi(z)$. Here $\phi(z) \equiv$
The symbols \(a, a', b\) and \(b'\) stand for two-dimensional vectors with real number entries and their scalar product is defined through the metric \(\text{diag}(-;+).\) This normal ordering is given by

\[
V(z, a, a').V(w, b, b') = :\left\{ \frac{a.b-(a.a')(a'.b)}{(z-w)^2} + i\frac{(a.b')b\partial\phi(w)-(a.a')b\partial\phi(z)}{z-w} \right\} + (i a\partial\phi(z))(i b\partial\phi(w))(z-w)^{a'.b'} e^{i a'.\phi(z)+ib'.\phi(w)} :, \tag{2.32}
\]

\(\psi^+(z)\) and \(\psi^-(z)\) can be respectively identified with \(V(z, a, a')\) and \(V(z, b, b')\) if we set

\[
\begin{align*}
a &= (-\frac{1}{\sqrt{2}}; 1), \\
a' &= (-1; 0), \\
b &= (\frac{1}{\sqrt{2}}; 1), \\
b' &= (1; 0). \\
\end{align*} \tag{2.33}
\]

Using these identifications and the vertex operator normal ordering (2.32) we get the following OPE's:

\[
\begin{align*}
\psi^+(z)\psi^-(w) &\equiv V(z, a, a').V(w, b, b') \\
&= \frac{1}{z-w} \text{ regular as } z \to w, \\
\psi^-(z)\psi^+(w) &\equiv V(z, b, b').V(w, a, a') \\
&= \frac{1}{z-w} \text{ regular as } z \to w, \\
\psi^+(z)\psi^+(w) &\equiv V(z, a, a').V(w, a, a') \\
&= \frac{I^+(w)}{z-w} \text{ regular as } z \to w, \\
\psi^-(z)\psi^-(w) &\equiv V(z, b, b').V(w, b, b') \\
&= \frac{I^-(w)}{z-w} \text{ regular as } z \to w,
\end{align*} \tag{2.34}
\]

where the fields \(I^\pm(w)\) are given by

\[
\begin{align*}
I^+(w) &= :\left( (ia\partial\phi(w))^2 + \frac{i}{\sqrt{2}} a\partial^2\phi(w) \right) e^{2ia'.\phi(w)} :, \\
I^-(w) &= :\left( (ib\partial\phi(w))^2 + \frac{i}{\sqrt{2}} b\partial^2\phi(w) \right) e^{2ib'.\phi(w)} :. \tag{2.35}
\end{align*}
\]

Therefore, the real fermionic field \(\psi(z)\) can be bosonized in two different ways if the following conditions are satisfied:

\[
I^\pm(w) \equiv 1. \tag{2.36}
\]

This is indeed the case and is a general feature of the bosonizations of conformal field theories with background charges [8]. The background charge can here be easily read off
from the bosonization of the energy-momentum tensor \( T(z) \), which is in turn obtained from
the bosonization of the currents \( H(z) \) and \( E^\pm(z) \) through the Sugawara construction. It
reads
\[
T(z) = -\frac{1}{2}(\partial \phi^1(z))^2 - \frac{1}{2}(\partial \phi(z))^2 + i\alpha_0.\partial^2 \phi(z),
\]
with the two dimensional background charge \( \alpha_0 \) given by
\[
\alpha_0 = (0; -\frac{1}{2\sqrt{2}}).
\]
In fact, with this background charge, we can already check that the fields \( I^\pm(w) \) have the
same conformal dimension as the identity field 1. Let us recall for this purpose that the con-
formal dimension \( \Delta \) of a vertex field of the form \( ix.\partial^n\phi(w) \exp iy.\phi(w) \) or \( (ix.\partial\phi(w))^n \exp iy.\phi(w) \),
with \( x \) and \( y \) arbitrary two-dimensional vectors, is given by
\[
\Delta = n + \frac{y^2}{2} - \alpha_0.y.
\]
Applying this formula to \( I^+(w) \) and \( I^-(w) \) we find that both have zero conformal dimension
just as the identity field.

To sum up this analysis of the classical case: the \( \widehat{sl}(2) \) current algebra at level 2 admits
both bosonic and fermionic realizations, which are described respectively by the relations
\( (2.13) \) and \( (2.18) \); the equivalence between the two, the so-called boson-fermion correspon-
dence, is given by the relations \( (2.20) \) and \( (2.30) \).

### 3 Quantum boson-fermion correspondence

#### 3.1 The \( U_q(\widehat{sl}(2)) \) quantum current algebra

The \( U_q(\widehat{sl}(2)) \) quantum algebra can be presented as follows in terms of the Chevalley basis
elements \( \{e_i, f_i, t_i^{\pm 1}, q^{\pm d}; i = 0, 1\} \) [9, 10, 11]. The defining relations are taken to be
\[ t_i t_j = t_j t_i, \]
\[ t_i e_j t_i^{-1} = q^{a_{ij}} e_j, \]
\[ t_i f_j t_i^{-1} = q^{-a_{ij}} f_j, \]
\[ [e_i, f_j] = \delta_{ij} \frac{t_i - t_i^{-1}}{q^{\delta_{ij} - 1}}, \]
\[ q^d e_i q^{-d} = q^{\delta_{ij}} e_i, \]
\[ q^d f_i q^{-d} = q^{-\delta_{ij}} f_i, \]
\[ q^d t_i q^{-d} = t_i, \]

with \((a_{ij})\) the \(\hat{sl}(2)\) generalized Cartan matrix, and supplemented with quantum analogues of the Serre relations. This algebra is an associative Hopf algebra with comultiplication

\[
\begin{align*}
\Delta(e_i) &= e_i \otimes 1 + t_i \otimes e_i, \\
\Delta(f_i) &= f_i \otimes t_i^{-1} + 1 \otimes f_i, \\
\Delta(t_i^{\pm 1}) &= t_i^{\pm 1} \otimes t_i^{\mp 1}, \\
\Delta(q^{\pm d}) &= q^{\mp d} \otimes q^{\pm d},
\end{align*}
\]  

and antipode

\[
\begin{align*}
S(e_i) &= -t_i^{-1} e_i, \\
S(f_i) &= -f_i t_i, \\
S(t_i^{\pm 1}) &= t_i^{\mp 1}, \\
S(q^{\pm d}) &= q^{\mp d}.
\end{align*}
\]

There is a second equivalent definition of \(U_q(\hat{sl}(2))\), which was first given by Drinfeld [12]. It is generated by the elements \(\{E^\pm_n, H_m, q^{\pm \sqrt{2} H_0}, q^{\pm d}, \gamma^{\pm 1/2}; n \in \mathbb{Z}, m \in \mathbb{Z} \neq 0\}\) that obey

\[
\begin{align*}
[H_n, H_m] &= [2n]_2 \frac{\gamma^{nk} - \gamma^{-nk}}{q^{-q^{1/2}} - 1} \delta_{n+m,0}, \quad n \neq 0, \\
[q^{\pm \sqrt{2} H_0}, H_m] &= 0, \\
[H_n, E^\pm_m] &= \pm \sqrt{2} \frac{\gamma^{kn} [2m]_2}{2n} E^\pm_{n+m}, \quad n \neq 0, \\
q^{\sqrt{2} H_0} E^\pm_n q^{-\sqrt{2} H_0} &= q^{\pm 2} E^\pm_n, \\
[E^+_n, E^-_m] &= \gamma^{k(n-m)/2} \Phi_{n+m} - \gamma^{-k(n-m)/2} \Phi_{n+m}, \\
q^d E^\pm_n q^{-d} &= q^{n} E^\pm_n, \\
q^d H_n q^{-d} &= q^{n} H_n, \\
[\gamma^{\pm 1/2}, x] &= 0, \quad \forall x \in U_q(\hat{sl}(2)).
\end{align*}
\]
We are using the notation \([n] \equiv (q^n - q^{-n})/(q - q^{-1})\), \(k\) is again called the level, and, \(\Psi_n\) and \(\Phi_n\) are given by the mode expansions of the fields \(\Psi(z)\) and \(\Phi(z)\), which are themselves defined by

\[
\Psi(z) = \sum_{n \geq 0} \Psi_n z^{-n} = q^{\sqrt{2}H_0} \exp\{\sqrt{2}(q - q^{-1}) \sum_{n > 0} H_n z^{-n}\},
\]

\[
\Phi(z) = \sum_{n \leq 0} \Phi_n z^{-n} = q^{-\sqrt{2}H_0} \exp\{-\sqrt{2}(q - q^{-1}) \sum_{n < 0} H_n z^{-n}\}.
\] (3.44)

The isomorphism between these two presentations of \(U_q(sl(2))\) is provided by the following identifications:

\[
t_0 \equiv \gamma^k q^{-\sqrt{2}H_0},
\]

\[
t_1 \equiv q^{\sqrt{2}H_0},
\]

\[
e_0 \equiv E_1 q^{-\sqrt{2}H_0},
\]

\[
e_1 \equiv E_0^+, \quad (3.45)
\]

\[
f_0 \equiv q^{\sqrt{2}H_0} E_{-1}^+,
\]

\[
f_1 \equiv E_{-1}^-.
\]

Using these assignments, the comultiplication (3.41) can be rewritten as follows in terms of the Drinfeld generators:

\[
\Delta(E_n^+) = E_n^+ \otimes \gamma^{kn} + \gamma^{2kn} q^{\sqrt{2}H_0} \otimes E_n^+ + \sum_{i=0}^{n-1} \gamma^{k(n+3i)/2} \Psi_{n-i} \otimes \gamma^{k(n-i)} E_i^+ \mod N_- \otimes N_+^2,
\]

\[
\Delta(E_{-n}^+) = E_{-n}^+ \otimes \gamma^{-kn} + q^{-\sqrt{2}H_0} \otimes E_{-n}^+ + \sum_{i=0}^{n-1} \gamma^{k(n-i)} \Phi_{n+i} \otimes \gamma^{k(i-m)} E_{-i}^+ \mod N_- \otimes N_+^2,
\]

\[
\Delta(E_n^-) = E_n^- \otimes \gamma^{-2kn} q^{-\sqrt{2}H_0} + \gamma^{-kn} \otimes E_n^- + \sum_{i=0}^{n-1} \gamma^{-k(n-i)} E_i^- \otimes \gamma^{k(n+i)} \Phi_{-n-i} \mod N_- \otimes N_+^2,
\]

\[
\Delta(E_{-m}^-) = \gamma^{km} \otimes E_m^- + E_m^- \otimes q^{\sqrt{2}H_0} + \sum_{i=1}^{m-1} \gamma^{k(m-1)} E_m^- \otimes \gamma^{-k(m-i)/2} \psi_{m-i} \mod N_- \otimes N_+,
\]

\[
\Delta(H_m) = H_m \otimes \gamma^{km/2} + \gamma^{3km/2} \otimes H_m \mod N_- \otimes N_+,
\]

\[
\Delta(H_{-m}) = H_{-m} \otimes \gamma^{-3km/2} + \gamma^{-km/2} \otimes H_{-m} \mod N_- \otimes N_+,
\]

\[
\Delta(q^{\pm\sqrt{2}H_0}) = q^{\pm\sqrt{2}H_0} \otimes q^{\pm\sqrt{2}H_0},
\]

\[
\Delta(\gamma^{\pm\frac{1}{2}}) = \gamma^{\pm\frac{1}{2}} \otimes \gamma^{\pm\frac{1}{2}},
\]

\[
\Delta(q^{\pm d}) = q^{\pm d} \otimes q^{\pm d},
\] (3.46)

where \(m > 0\), \(n \geq 0\), and \(N_+ \) and \(N_+^2\) are left \(Q(q)[\gamma^\pm, \Psi_m, \Phi_{-m}; m, n \in \mathbb{Z}_{\geq 0}]\)-modules generated by \(\{E_m^+; m \in \mathbb{Z}\}\) and \(\{E_m^- E_n^+; m, n \in \mathbb{Z}\}\) respectively [13, 14]. The second presentation of \(U_q(sl(2))\) is the quantum analogue of the one given in (2.3) of the \(\hat{sl}(2)\) classical algebra.
There is yet a third description of $U_q(\widehat{sl}(2))$, which we shall call the quantum current algebra and which is the quantum analogue of the $\widehat{sl}(2)$ current algebra defined in (2.3)-(2.8). This third form will again be most convenient when dealing with free field realizations. It is defined through the following OPE’s:

$$\Psi(z).\Phi(w) = \frac{(z-wq^{2k})(z-wq^{-2-k})}{(z-wq^{2-k})(z-wq^{-2+k})} \Phi(w).\Psi(z).$$

$$\Psi(z).E^\pm(w) = q^{\pm 2} \frac{z-wq^{\pm(2k/2)}}{z-wq^{\pm(2-2k/2)}} E^\pm(w).\Psi(z).$$

$$\Phi(z).E^\pm(w) = q^{\pm 2} \frac{z-wq^{\pm(2-k/2)}}{z-wq^{\pm(2+k/2)}} E^\pm(w).\Phi(z).$$

$$E^+(z).E^-(w) \sim \frac{1}{w(q-q^{-1})} \left\{ \frac{\Psi(wq^{k/2})}{z-wq^k} - \frac{\Phi(wq^{-k/2})}{z-wq^{-k}} \right\}, \quad |z| > |wq^{\pm k}|, \quad (3.50)$$

$$E^-(z).E^+(w) \sim \frac{1}{w(q-q^{-1})} \left\{ \frac{\Psi(wq^{-k/2})}{z-wq^{-k}} - \frac{\Phi(wq^{k/2})}{z-wq^k} \right\}, \quad |z| > |wq^{\pm k}|, \quad (3.51)$$

$$E^\pm(z).E^\pm(w) = \frac{(z-wq^{\pm 2} - w)}{z-wq^{\pm 2}} E^\pm(w).E^\pm(z). \quad (3.52)$$

Here, the quantum currents $E^\pm(z)$ are the following generating functions of the Drinfeld generators:

$$E^\pm(z) = \sum_{n \in \mathbb{Z}} E^\pm_n z^{-n-1}, \quad (3.53)$$

and $\Psi(z)$ and $\Phi(z)$ are given by (3.44). The combination $(\Psi(z) - \Phi(z))/\sqrt{2z(q-q^{-1})}$ is the quantum analogue of the classical current $H(z)$. Note that we have used the same symbol, $E^\pm(z)$, to denote both the classical and quantum currents but it should be clear from the context which ones are being considered. In order to find a free field realization of the $U_q(\widehat{sl}(2))$ currents $E^\pm(z)$, $\Psi(z)$ and $\Phi(z)$ one therefore needs to resolve the relations (3.47)-(3.52) in terms of free fields, either bosonic or fermionic.

### 3.2 Bosonic realization of $U_q(\widehat{sl}(2))$ at level 2

The bosonization of $U_q(\widehat{sl}(2))$ has recently attracted a lot of interest. First, Frenkel and Jing derived a bosonization of this algebra at level $k = 1$ in terms of a single deformed free bosonic field [15]. Then, many authors generalized this bosonization to a generic level $k$ in terms of three deformed free bosonic fields. These results are referred to as q-deformations.
of the Wakimoto bosonization (see [10] for a complete list of references.) Finally, the q-

deformation of the Wakimoto bosonization of $U_q(\hat{sl}(n))$ has been constructed in Ref. [17],

and realizations of the $su(n)$ finite Lie algebra have been independently achieved in both

Refs. [18] and [19]. Let us now briefly review the bosonization introduced in [20] (see also

[16]) since it is the direct quantum analogue of the classical bosonization given in (2.13).

For the case that we are interested in, i.e., $k = 2$, great simplifications occur although most

features of the general case (such as the need for three deformed bosonic fields) are retained.

This bosonization is given by [20]:

$$
\Psi(z) = \exp(i\varphi_1^{1,+}(z) - i\varphi_1^{1,-}(z)) = q^{2\varphi_1^0} \exp(2(q - q^{-1}) \sum_{n>0} \varphi_1^{1,-} z^{-n}),
$$

$$
\Phi(z) = \exp(i\varphi_1^{1,+}(z) - i\varphi_1^{1,-}(z)) = q^{-2\varphi_1^0} \exp(-2(q - q^{-1}) \sum_{n<0} \varphi_1^{1,-} z^{-n}),
$$

$$
E^\pm(z) = \frac{\exp(\pm i\varphi_1^{1,\pm}(z))}{z^{(q-q^{-1})}} (X^\pm(z) - Y^\pm(z)),
$$

where

$$
X^\pm(z) = \exp\{\pm i\varphi_2^2(z) + \frac{i}{\sqrt{2}}(\varphi_3^2(zq^2) - \varphi_3^2(z))\},
$$

$$
Y^\pm(z) = \exp\{\pm i\varphi_2^2(zq^{-1}) + \frac{i}{\sqrt{2}}(\varphi_3^2(zq^{-2}) - \varphi_3^2(z))\}.
$$

The three deformed free bosonic fields $\varphi_1^{1,\pm}(z)$, $\varphi_2^2(z)$ and $\varphi_3^2(z)$ are defined by

$$
\varphi_1^{1,\pm}(z) = \varphi_1 - i\varphi_0^1 \ln z + 2i \sum_{n\neq 0} \frac{q^{\pm|n|}}{|2n|} \varphi_1^{1,-} z^{-n},
$$

$$
\varphi_j^2(z) = \varphi_j - i\varphi_0^2 \ln z + i \sum_{n\neq 0} \frac{z^{-n}}{n} \varphi_n^j, \quad j = 2, 3.
$$

Their modes $\{\varphi_j^i, \varphi_n^j, \quad j = 1, 2, 3\}$ generate three deformed Heisenberg algebras with the

following commutation relations:

$$
[\varphi_n^j, \varphi_m^\ell] = (-1)^{j-1} n I_j(n) \delta^\ell_j \delta_{n+m,0},
$$

$$
[\varphi_j^i, \varphi_0^\ell] = (-1)^{j-1} i \delta^\ell_j \delta_{j,\ell} \quad j, \ell = 1, 2, 3,
$$

where

$$
I_1(n) = \frac{|2n|^2}{4n^2},
$$

$$
I_2(n) = \frac{1}{2} \frac{|4n|}{|2n|},
$$

$$
I_3(n) = 1.
$$

The fields $\varphi^i(z)$ and their modes are normalized so that in the limit $q \to 1$ they tend
to the classical fields $\phi^i(z)$ and their modes introduced in (2.10) and (2.11). It can also
be checked that in this same limit, the bosonization of the quantum currents $E^\pm(z)$ and 
$(\Psi(z) - \Phi(z))/(\sqrt{2}z(q - q^{-1}))$ reduces to that of the classical currents $E^\pm(z)$ and $H(z)$ given in (2.13).

3.3 Fermionic realization of $U_q(sl(2))$ at level 2

In this section, we construct the quantum currents $E^\pm(z)$ and $(\Psi(zq) - \Phi(zq^{-1}))/z(q - q^{-1})$ in terms of one deformed free real fermionic field and one deformed ‘interactive’ complex fermionic field so that the quantum current algebra (3.47)-(3.52) is satisfied. Here again we shall use the same notation as in the classical case and denote the deformed real fermionic field by $\psi(z)$, and the deformed complex fermionic field and its Hermitian conjugate by $\chi(z)$ and $\chi^\dagger(z)$ respectively. Although we have originally used the quantum analogue of the boson-fermion correspondence to derive the fermion realization of $U_q(sl(2))$, we will now in analogy with the classical case, follow the opposite path by first stating our results and justifying them afterwards in the next section where we elaborate on this quantum boson-fermion correspondence. Our fermionic realization will be given by

$$
E^+(z) = \sqrt{2}|\psi(z)\chi(z),
$$
$$
E^-(z) = \sqrt{2}|\psi(z)\chi^\dagger(z),
$$

where $\psi(z)$, $\chi(z)$ and $\chi^\dagger(z)$ satisfy the following quantum analogue of the Clifford current algebra:

$$
\psi(z)\psi(w) = \frac{z - w}{(z - wq^2)(z - wq^{-2})} + :\psi(z)\psi(w):, \ |z| > |wq^{\pm 2}|, \quad (3.60)
$$
$$
\chi(z)\chi^\dagger(w) = \frac{I(w)}{z - w} + :\chi(z)\chi^\dagger(w):, \ |z| > |w|, \quad (3.61)
$$
$$
\chi^\dagger(z)\chi(w) = \frac{I(w)}{z - w} + :\chi^\dagger(z)\chi(w):, \ |z| > |w|, \quad (3.62)
$$
$$
\chi(z)\chi(w) = -q^2\frac{(z - wq^{-2})}{z - wq^2}\chi(w)\chi(z), \quad (3.63)
$$
$$
\chi^\dagger(z)\chi^\dagger(w) = -q^{-2}\frac{(z - wq^2)}{z - wq^{-2}}\chi^\dagger(w)\chi^\dagger(z), \quad (3.64)
$$
$$
I(z)\chi(w) = q^2\frac{(z - wq^{-2})}{z - wq^2}\chi(w)I(z), \quad (3.65)
$$
\[ I(z) \chi^\dagger(w) = q^{-2}(z - wq^2) \frac{z - wq^2 - z}{z - wq^2} \chi^\dagger(w) I(z), \]  

(3.66)

and where ‘the quantum analogue of the identity operator \( I(z) \)’ is defined by

\[
I(z) \equiv \Psi(zq) - zq(q - q^{-1}) : \chi(zq^2) \chi^\dagger(z) : \\
\equiv \Phi(zq^{-1}) + zq^{-1}(q - q^{-1}) : \chi(zq^{-2}) \chi^\dagger(z) : .
\]  

(3.67)

The mode expansions of these fermionic fields and \( I(z) \) are given by

\[
\psi(z) = \sum_{r \in \mathbb{Z} + 1/2} \psi_r z^{r-1/2}, \\
\chi(z) = \sum_{r \in \mathbb{Z} + 1/2} \chi_r z^{r-1/2}, \\
\chi^\dagger(z) = \sum_{r \in \mathbb{Z} + 1/2} \chi_r^\dagger z^{r-1/2}, \\
I(z) = \sum_{n \in \mathbb{Z}} I_n z^{-n}.
\]  

(3.68)

The fermionic normal ordering is again the same as in the classical case. Consequently, one can easily show that the relation (2.17) is also valid in the quantum case, i.e.,

\[ : \psi(z) \psi(z) : = 0. \]  

(3.69)

Using this fermionic realization we can show for instance that relation (3.63) of the \( U_q(\widehat{\mathfrak{sl}}(2)) \) current algebra is indeed satisfied (this example is the quantum analogue of the one given in (2.19)):

\[
E^+(z).E^-(w) = [2] \psi(z)\chi(z).\psi(w)\chi^\dagger(w) \\
= [2] \left( \frac{z}{z-wq^2(z-wq^{-2})} + : \psi(z)\psi(w) : \left( \frac{I(w)}{z-w} + : \chi(z)\chi^\dagger(w) : \right) \right) \\
= [2] \left( \frac{I(w)}{z-wq^2(z-wq^{-2})} \right) + \text{regular} \\
= \frac{[2]}{w(q^2-q^{-2})} \left( \frac{I(w)+wq(q-q^{-1})\chi(wq^2)\chi^\dagger(w)}{z-wq^2} - \frac{I(w)-wq^{-1}(q-q^{-1})\chi(wq^{-2})\chi^\dagger(w)}{z-wq^{-2}} \right) + \text{regular} \\
= \frac{1}{w(q-q^{-1})} \left( \frac{\Psi(wq)}{z-wq^2} - \Phi(wq^{-1}) \right) + \text{regular}.
\]  

(3.70)

The relations (3.67) and (3.69) have been used to carry the intermediate steps. The above Clifford quantum current algebra (3.60)-(3.66) amounts to the following in terms of modes:

\[
\{ \psi_r, \psi_s \} = \frac{q^{2r+q-2r}}{[2]} \delta_{r+s,0}, \\
\{ \chi_r, \chi_s^\dagger \} = I_{r+s}, \\
\chi_{r+1}\chi_s + q^2 \chi_s\chi_{r+1} = \chi_{s+1}\chi_r + q^2 \chi_r\chi_{s+1}, \\
\chi_{r+1}\chi_s^\dagger + q^{-2} \chi_s^\dagger\chi_{r+1} = \chi_{s+1}\chi_r^\dagger + q^{-2} \chi_r^\dagger\chi_{s+1}, \\
I_{n+1}\chi_r - q^2 \chi_r I_{n+1} = -\chi_{r+1}I_n + q^2 I_{n}\chi_{r+1}, \\
I_{n+1}\chi_r - q^{-2} \chi_r^\dagger I_{n+1} = -\chi_{r+1}^\dagger I_n + q^{-2} I_{n}\chi_{r+1}, \quad n \in \mathbb{Z}; \quad r, s \in \mathbb{Z} + 1/2.
\]  

(3.71)
It is clear from these relations, that the real quantum fermion field is a free field just as its classical analogue, since its modes can be redefined so as to satisfy the classical Clifford mode algebra (2.16). However, since the modes of the quantum complex fields $\chi(z)$ and $\chi^\dagger(z)$ cannot be redefined so as to satisfy the classical Clifford mode algebra (2.16), we may therefore consider $\chi(z)$ and $\chi^\dagger(z)$ as ‘interactive’ fields. This is a major difference from the classical case. Note that the real fermionic field was first defined in Ref. [21] for the purpose of obtaining a mixed bosonic-fermionic realization of $U_q(so(2n + 1))$ at level 1.

3.4 Quantum boson-fermion correspondence

We will elaborate in this section on the equivalence between the bosonic and fermionic realizations of $U_q(sl(2))$ at level 2. Contrary to the classical case it is now the bosonization of the complex fermionic field $\chi(z)$ that is more subtle. Let us therefore discuss first the bosonization of the real fermionic field $\psi(z)$. As in the classical case, it is clear that the real fermionic field $\psi(z)$ can be bosonized in two different but equivalent forms, which we denote by $\psi^\pm(z)$:

$$\psi(z) \equiv \psi^\pm(z) = \frac{1}{z\sqrt{[2](q - q^{-1})}}(X^\pm(z) - Y^\pm(z)), \quad (3.72)$$

where $X^\pm(z)$ and $Y^\pm(z)$ are defined by (3.55). Using the basic OPE’s

$$\varphi^2(z)\varphi^2(w) = \frac{1}{2}\ln(z - wq^2)(z - wq^{-2}) + :\varphi^2(z)\varphi^2(w):,$$

$$\varphi^3(z)\varphi^3(w) = - \ln(z - w) + :\varphi^3(z)\varphi^3(w):, \quad (3.73)$$

which are easily derived from the quantum Heisenberg algebras (3.57) and a normal ordering similar to the one introduced in the classical case, we find

$$\psi^\pm(z)\psi^\mp(w) = \frac{1}{(z - wq^2)(z - wq^{-2})} + \text{regular as } z \to wq^\pm 2,$$

$$\psi^\pm(z)\psi^\pm(w) = \frac{q}{2z - wq^2}I^\pm_1(w) + \frac{q^{-1}}{2z - wq^{-2}}I^\pm_2(w) + \text{regular as } z \to wq^\pm 2, \quad (3.74)$$

where

$$I^\pm_1(w) = \frac{w^2(q + q^{-1})(q - q^{-1})^2}{w^2(q + q^{-1})(q - q^{-1})^2}(q^{-1} : X^\pm(wq^2)X^\pm(w) :$$

$$-(q + q^{-1}) : Y^\pm(wq^2)X^\pm(w) : + q : Y^\pm(wq^2)Y^\pm(w) :),$$

$$I^\pm_2(w) = \frac{q^2}{w^2(q + q^{-1})(q - q^{-1})^2}(q^{-1} : X^\pm(wq^{-2})X^\pm(w)$$

$$-(q + q^{-1}) : X^\pm(wq^{-2})Y^\pm(w) : + q : Y^\pm(wq^{-2})Y^\pm(w) :). \quad (3.75)$$
It therefore follows from the above relations, that we need to make the identifications:

\[ I_1^\pm (w) \equiv I_2^\pm (w) \equiv 1, \]  

(3.76)

in order to have

\[ \psi^\pm (z) \psi^\pm (w) = \frac{z - w}{(z - w q^2)(z - w q^{-2})} + \text{regular as } z \rightarrow w q^\pm. \]  

(3.77)

The identifications (3.76) can be thought of as the quantum analogues of the conditions (2.36) that were arrived at in the classical situation.

We now turn to the bosonization of the complex fermionic fields \( \chi(z) \) and \( \chi^\dagger(z) \). Looking at the bosonization and the fermionization of \( E^\pm(z) \), we are led to set

\[ \chi(z) \equiv e^{i \varphi^+(z)}, \]

\[ \chi^\dagger(z) \equiv e^{-i \varphi^-(z)}. \]  

(3.78)

Furthermore, by comparing the relations (3.64) and (3.65) respectively with

\[ e^{i \varphi^+(z)} e^{-i \varphi^-(w)} = \frac{z - w}{z - w} + \frac{e^{i \varphi^+(z) - i \varphi^-(w)}}{z - w}, \]  

(3.79)

\[ e^{-i \varphi^-(z)} e^{i \varphi^+(w)} = \frac{z - w}{z - w} + \frac{e^{-i \varphi^-(z) + i \varphi^+(w)}}{z - w}, \]  

(3.80)

we get the identifications

\[ I(w) \equiv e^{i \varphi^+(w) - i \varphi^-(w)} ; \]

(3.81)

\[ : \chi(z) \chi^\dagger(w) : \equiv \frac{e^{i \varphi^+(z) - i \varphi^-(w)}}{z - w} ; \]

(3.82)

\[ : \chi^\dagger(z) \chi(w) : \equiv \frac{e^{-i \varphi^-(z) + i \varphi^+(w)}}{z - w} ; \]

(3.83)

Note that the terms on the right hand side of (3.82) and (3.83) are regular as they should since those on the left hand side are regular by definition. So far, we have described the quantum boson-fermion correspondence for the currents \( E^\pm(z) \). Let us now consider the currents \( \Psi(z) \), \( \Phi(z) \) and \( I(z) \). We have given the bosonization of \( \Psi(z) \) and \( \Phi(z) \) in (3.54).
As for the fermionization, it is only for the combination \((\Psi(zq) - \Phi(zq^{-1}))/z(q - q^{-1})\) that we have been at this point able to derive it—see (3.59). The equivalence of the bosonization and the fermionization of the latter combination is due to the identifications (3.82) and (3.83). As for \(I(z)\), this equivalence is due to (3.81). Because of this and in view of (3.67), we can explicitly construct all the modes \(\{\Psi_0 - \Phi_0, \Psi_{n>0}, \Phi_{n<0}, I_{n\neq 0}; n \in \mathbb{Z} \neq 0\}\) in terms of the complex fermionic ones \(\{\chi_r, \chi_r^\dagger; r \in \mathbb{Z} + 1/2\}\). Explicitly, we have

\[
\begin{align*}
\Psi_n &= q^n(q - q^{-1}) \sum_{r \in \mathbb{Z}+1/2} (q^{2r} + q^{-2r}) : \chi_r \chi_{n-r}^\dagger : , \quad n > 0, \\
\Phi_n &= -q^{-n}(q - q^{-1}) \sum_{r \in \mathbb{Z}+1/2} (q^{2r} + q^{-2r}) : \chi_r \chi_{n-r}^\dagger : , \quad n < 0, \\
\Psi_0 - \Phi_0 &= \Psi_0 - \Psi_0^{-1} = (q - q^{-1}) \sum_{r \in \mathbb{Z}+1/2} (q^{2r} + q^{-2r}) : \chi_r \chi_{n-r}^\dagger : , \quad (3.84) \\
I_n &= q^n \Psi_n - (q - q^{-1}) \sum_{r \in \mathbb{Z}+1/2} q^{-2r} : \chi_r \chi_{n-r}^\dagger : , \quad n > 0, \\
I_n &= q^n \Phi_n + (q - q^{-1}) \sum_{r \in \mathbb{Z}+1/2} q^{2r} : \chi_r \chi_{n-r}^\dagger : , \quad n < 0.
\end{align*}
\]

Note that the first three relations of (3.84) are the quantum analogues of the classical relations (2.24).

4 Conclusions

In this paper, we have used the \(U_q(\hat{sl}(2))\) quantum algebra at level 2 as a framework to discuss the quantum analogue of the boson-fermion correspondence. The main virtue of this algebra for our purposes, is that it allows both boson-real fermion and boson-complex fermion correspondences. We have shown that the real quantum fermion is free like its classical analogue since it can be redefined to satisfy the same anticommutation relations as the classical ones. However, we have stressed that the natural complex quantum fermion which arises in this construction, unlike its classical counterpart, is not free, since its fusion with its conjugate gives rise to the new field \(I(z)\). Therefore, it cannot be redefined so as to satisfy the classical anticommutation relations. In our fermionic realization of \(U_q(\hat{sl}(2))\) at level 2, the modes of the currents \(E^\pm(z)\), \((\Psi(z) - \Phi(z))/z(q - q^{-1})\) and \(I(z)\) are all explicitly constructed in terms of the fermionic operators. Only the zeroth mode of \(I(z)\) could not be realized in terms of the fermionic modes. The reason for this is still unclear and it would be interesting to elucidate this point. Another important issue worth looking at is the realization of the complex interactive fermion field in terms of free fermion fields.
This would considerably simplify the construction of the fermionic Fock spaces and of the fermionic vertex operators which intertwine them. Moreover, this would provide another method for computing important physical quantities like the correlation functions, the form factors and the S matrix elements of the spin-1 XXZ Heisenberg chain and its continuum limit.

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