Reentrance during nonequilibrium relaxation

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Abstract. – We consider nonequilibrium critical dynamics of the two-dimensional Ising model for which the initial state is prepared by switching on random fields with zero mean and variance $H$. In the initial state there is no magnetic order but the clusters of parallel spins have a percolation transition for small enough $H$. Using heat-bath dynamics we measure the relaxation of the magnetization which shows a reentrance in time. Due to cluster dissolution in the early time regime there is a decrease of the magnetization, followed by an increase due to nonequilibrium domain growth which itself turns to a decrease due to equilibrium relaxation. The power law decay of the nonequilibrium autocorrelation function is not influenced by the percolation in the initial state.

Introduction. – In nonequilibrium critical dynamics [1–4] the system under consideration is prepared in some initial state from which it is quenched to the critical temperature, $T_c$, and then let to evolve in time according to the given dynamical rules. Generally one is interested in the relaxation of the magnetization, $m(t) = \langle \sigma(t) \rangle$, and in the behavior of the autocorrelation function, $G(t,s) = \langle \sigma(t)\sigma(s) \rangle$. Here $\sigma(t)$ denotes the operator of the order parameter and $s$ and $t$ are the waiting time and the observation time, respectively.

In most of the studied cases the initial state is of two kinds: it is either the (completely) ordered one, or the (completely) disordered one. Starting from the ordered state with the initial temperature $T_i = 0$, the critical relaxation process involves only equilibrium critical exponents. For example the decay of the magnetization is asymptotically given by

$$m(t) \simeq t^{-x/z}, \quad T_i = 0,$$

where $x = \beta/\nu$ is the anomalous dimension of the magnetization and $z$ is the dynamical exponent. The decay of the autocorrelation function follows the rule:

$$G(t,s) \sim (t-s)^{-2x/z} g(t/s), \quad T_i = 0,$$

where the scaling function $g(y) \sim y^{x/z}$ for $y \gg 1$ and tends to a constant otherwise. This means that $G(t,s)$ decreases as $t^{-x/z}$ for $s \ll t$, similar to the magnetization in Eq. (1), whereas for $t/s = O(1)$ the decay is the same as for the equilibrium autocorrelation function.

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If on the contrary the initial state is the completely disordered one with \( T_i \to \infty \), new type of nonequilibrium exponents appear in the relaxation process. Starting with a small initial value, \( m_i \), the magnetization behaves as \[ m(t) \sim m_i t^\theta, \quad t < t_i, \quad T_i = \infty. \] (3)

Here \( \theta \) is the initial slip exponent, which is zero in mean-field theory but generally positive in realistic systems. Consequently in the initial period, \( t < t_i \), the order in the system is increasing due to critical fluctuations. The limiting time-scale \( t_i \) can be obtained from the condition that for \( t > t_i \) the relaxation goes over to the decay of Eq. (1), leading to the estimate:

\[ t_i \sim m_i^{-z/z_i}, \quad x_i = \theta z + x. \] (4)

Here \( x_i \) is called the anomalous dimension of the initial magnetization. Note that \( t_i \) tends to infinity as \( m_i \) goes to zero. Another new feature of nonequilibrium dynamics starting from a disordered initial state is given by the fact that the autocorrelation function, which is measured in the state with \( m_i = 0 \), is non-stationary: \( G(t, s) \) depends both on \( t \) and \( s \) and not only on the time difference \( t - s \). In the limit \( t \gg s \) the decay is given by \[ G(t, s) \sim t^{-\lambda/z}, \quad t \gg s, \quad T = \infty, \] (5)

and the autocorrelation exponent \( \lambda \) satisfies the scaling relation [7] \( \lambda = d - \theta z = d - x_i + x \).

At this point we can note a formal analogy between nonequilibrium critical dynamics and static critical behavior in semi-infinite systems [8–11]. In the former problem translational invariance is broken in time due to the quench, whereas in the latter phenomenon translational invariance is broken in space in the direction perpendicular to the surface of the system. The initial state in dynamics with \( T_i = 0 \) (\( T_i = \infty \)) corresponds to a semi-infinite system with fixed (open) boundary conditions. In this respect the exponent \( x_i \) is analogous to the surface magnetization exponent \( x_1 \) at an ordinary surface transition.

In nonequilibrium critical dynamics there are a few examples in which the initial state contains nontrivial power-law correlations. Here we mention relaxation in the 2d XY-model [12, 13] when the initial state is prepared at a temperature \( T_1 > 0 \) and the quench is made to a temperature \( T_2 > T_1 \) with both temperatures in the critical phase, i.e. \( T_2 < T_{KT} \), where \( T_{KT} \) is the Kosterlitz-Thouless temperature. According to spin-wave theory for \( T_2 \ll T_{KT} \) the autocorrelation function in Eq. (2) is modified by replacing \( x \) by \( x(T_2) - x(T_1) \), where \( x(T) \) is the temperature dependent anomalous dimension. This theoretical prediction has been checked numerically [14]. In another work the nonequilibrium dynamics of the d-dimensional spherical model is studied starting from an initial state which has long-range correlations [15]. Depending on \( d \) and the correlation exponent \( \alpha \), different types of critical ageing behavior are found. In both examples described above there is quasi-long-range order in the initial state which is characterized by a fractal dimension \( d_f^i \) larger than the fractal dimension \( d_f = d - x \) of the critical state.

In the present paper we consider a type of initial states in which the disordering effect is not due to temperature, but is due to switching on random fields of zero mean and variance \( H \). In the limiting case of strong fields, \( H \gg J \), \( J \) being the coupling of the system, we recover the completely disordered initial state with \( T_i = \infty \). In the other limiting case, \( H \to 0 \), the initial state is completely ordered as for \( T_i = 0 \). We are interested in the behavior of nonequilibrium dynamics for intermediate values of \( \Delta = H/J = O(1) \). To be concrete we consider the two-dimensional Ising model for which random fields of any strength destroy the magnetic long-range order [16]. However the structure of the initial state on the level of clusters of
parallel spins shows interesting features [17]. For strong enough fields, \( \Delta > \Delta_{\text{perc}} \), these clusters have a finite extent, whereas for weaker fields, below \( \Delta_{\text{perc}} \), there is a percolation phenomenon: there are giant clusters for both spin directions which percolate the sample. According to numerical results [17, 18] these clusters are isomorphic with those of short range percolation having the same fractal dimension [19] \( d_{\text{perc}} = \frac{91}{48} \).

In this paper we want to clarify the effect of the change in the initial state topology on the properties of the nonequilibrium dynamics. In particular we want to see how the percolation of parallel spins is manifested in the magnetization relaxation.

**Numerical results.** – The initial state used in our nonequilibrium relaxation measurements is generated by switching on random fields, so that the Hamiltonian of the system is defined by:

\[
H = -J \sum_{\langle ij \rangle} \sigma_i \sigma_j - \sum_i h_i \sigma_i ,
\]

where \( \sigma_i = \pm 1 \) is an Ising spin located at site \( i \) of a square lattice. The nearest-neighbor coupling, \( J > 0 \), is ferromagnetic whereas the \( h_i \) random fields are taken from a symmetric Gaussian distribution:

\[
P(h_i) = \frac{1}{\sqrt{2\pi H^2}} \exp \left[ -\frac{h_i^2}{2H^2} \right] ,
\]

which has a variance \( H \). The ground state of the system for a given realization of the random fields is exactly calculated by a combinatorial optimization method [20] which works in strongly polynomial time. With help of this effective algorithm systems as large as \( L = 500 \) are treated numerically and averaging is performed over at least 80000 realizations.

As we have already mentioned in the introduction the magnetic correlation function in the initial state, \( \langle \sigma_i \sigma_j \rangle \), is short-ranged, so that the magnetic correlation length \( \xi_m \) is finite. However, considering the topology of clusters of parallel spins other length-scales can be defined, which can diverge by varying the parameter \( \Delta = H/J \). One of these lengths is the so-called breaking-up length, \( l_b \), which measures the typical fluctuations of the interface separating the different spin orientations. In other words in a finite system with \( L < l_b \) the ground state is homogeneous, but for \( L > l_b \) both spin orientations are present in the ground state and one can use a coarse-grained description with effective cells of size \( \sim l_b \). For weak random fields the breaking-up length asymptotically behaves as [21]

\[
l_b \sim \exp(A/\Delta^2) ,
\]

thus it is divergent as \( \Delta \to 0 \). The other length appearing in the geometry of the initial state is the percolation correlation length, \( \xi_{\text{perc}} \), which is the measure of the typical size of the largest clusters in the system. \( \xi_{\text{perc}} \) is divergent below the percolation transition point, which is estimated as [17] \( \Delta_{\text{perc}} \approx 1.65 \).

In the numerical calculations we considered the disorder in such a range for which the coarse-grained description is applicable, i.e. the breaking-up length satisfies the relation: \( \xi, L \gg l_b(\Delta) \). In this way we made the calculations for \( 1.25 \leq \Delta \leq 4 \). For example, for \( \Delta = 1.4 \), deeply in the percolation regime, the breaking-up length is \( l_b \approx 25 \) [17].

Having prepared the system in the initial state with a small magnetization \( m_i \), we quench it to the critical point and study the nonequilibrium relaxation process, thereby using the standard heat-bath algorithm. As shown in Figure 4 the magnetization displays an intriguing reentrance in time for not too large values of \( \Delta \): immediately after the quench the magnetization first decreases (regime 1), reaches a minimum at time \( t_{\text{min}} \) (the value of which depends on \( \Delta \) and \( m_i \)), increases in the regime 2 with \( t_{\text{min}} < t < t_{\text{max}} \), before decreasing again in the
large time limit with $t > t_{\text{max}}$ (regime 3). This non-monotoneous behaviour of the magnetization is understood by looking at the snapshots of Figure 2. At $t = 0$ the system is formed by homogeneous cells wherein the spins have the same orientation. The typical linear size of these cells is $l_b$. Immediately after the quench these compact cells are dissolved which leads to a decrease of the magnetization (see the snapshot for $t = 5$). As this process is exactly the same as when starting from a totally ordered initial state (corresponding to one large, the system filling, homogeneous cell) the decay of the magnetization follows the power law $t_1$. This dissolution stops when the total magnetic moment of the cells is of the order $O(1)$, i.e. when the magnetization has been reduced by a factor $l_b^d$ where $d = 2$ is the dimensionality of the system. This yields the prediction

$$t_{\text{min}}^{-x/z} \sim t_b^d$$

and therefore

$$\ln t_{\text{min}} \sim \ln t_b \sim \frac{1}{\Delta^2}.$$  

(10)

After the cells have been dissolved, the usual nonequilibrium domain growth sets in yielding an increasing magnetization in the regime 2. This increase of order (see the snapshot in Figure 2 at $t = 150$) is due to the critical fluctuations and is therefore similar to that observed when starting from a completely disordered initial state with a small magnetization. We therefore expect that the temporal evolution of the magnetization in the regime 2 is also governed by the initial slip exponent $\theta$, see Eq. (3), and that the domain growth continues up to a time $t_{\text{max}}$ with $\ln t_{\text{max}} \sim \ln t_{\text{min}} + \ln t_i$ where $t_i$ is given in Eq. (4). For longer times (regime 3) the relaxation again goes over to the well-known decay $t_1$.

We have studied this intriguing behaviour of the magnetization in a systematic way by varying the variance of the random fields over a large range. Some of our main findings are summarized in Figure 3 where we present results obtained for the initial magnetization $m_i = 0.0390625$. We checked that our conclusions do not depend on the chosen value of $m_i$ by simulating systems with other values of the initial magnetization.
Fig. 2 – Snapshots of the spin configuration of a system with $128 \times 128$ spins and $m_i = 0.0390625$ at three different times: ground state at $t=0$ with $\Delta = 1.7$, close to the minimum at $t = 5$, and in the increasing part at $t = 150$.

Figure 3 shows the temporal evolution of the magnetization at short times for various values of the variance $\Delta$. Starting with the smallest value one observes that by increasing $\Delta$ the minimum gets shallower and its position is shifted to smaller $t$. At a threshold value $\Delta_c \approx 2.0$ (the exact value of $\Delta_c$ depends on the value of $m_i$) the minimum disappears and the magnetization increases monotonically at early times. As mentioned before, the existence of a minimum follows from the competition of two processes, cluster dissolution and domain growth. When $\Delta$ increases, the breaking-up length (and therefore also the effective length of the cells) decreases, making the cluster dissolution less and less effective, up to the point where domain growth is the only relevant mechanism, yielding an increasing magnetization immediately after the quench.

Figure 3(b) gives a closer look at the location of the minimum. Plotting $\ln t_{\text{min}}$ as a function of $1/\Delta^2$ for $1.25 \leq \Delta \leq 1.95$. For variances $\Delta \geq \Delta_c \approx 1.65$ the data follow a straight line with slope 9.55. Deviations appear for $\Delta < \Delta_c$. Error bars are smaller than the size of the symbols. (c) Magnetization vs time in the regime 2 for $\Delta = 1.4, 1.6, 1.8, 2.0, 2.2, 2.6, 3.0$ (full lines from bottom to top) in a log-log plot. The dashed line is obtained when one starts from an infinite temperature initial state. All curves yield the same slope for large $t$. 

Fig. 3 – (a) Magnetization vs time for various values of the variance $\Delta$: 1.4, 1.5, 1.6, 1.7, 1.8, 2.0, 2.2, 2.6, 3.0 (from bottom to top). (b) Logarithm of $t_{\text{min}}$ as function of $1/\Delta^2$ for $1.25 \leq \Delta \leq 1.95$. For variances $\Delta \geq \Delta_c \approx 1.65$ the data follow a straight line with slope 9.55. Deviations appear for $\Delta < \Delta_c$. Error bars are smaller than the size of the symbols. (c) Magnetization vs time in the regime 2 for $\Delta = 1.4, 1.6, 1.8, 2.0, 2.2, 2.6, 3.0$ (full lines from bottom to top) in a log-log plot. The dashed line is obtained when one starts from an infinite temperature initial state. All curves yield the same slope for large $t$. 

Fig. 3 – Snapshots of the spin configuration of a system with $128 \times 128$ spins and $m_i = 0.0390625$ at three different times: ground state at $t=0$ with $\Delta = 1.7$, close to the minimum at $t = 5$, and in the increasing part at $t = 150$. 

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Fig. 4 - Temporal evolution of the autocorrelation function $G(t, s = 0)$ for different values of $\Delta$: 1.4, 1.6, 1.8, 2.0, 4.0 (from top to bottom). The grey line results from an infinite temperature initial state. The inset shows the same quantity for $\Delta = 1.4$ and systems with different linear extend $L$: 128, 200, 500 (from bottom to top).

We end the presentation of our numerical data with a discussion of the autocorrelation function. As already mentioned, see Eq. (5), the autocorrelation follows a simple power law in the long time limit when quenched from infinite temperature to the critical point, with an exponent $\lambda/z$ when $\Delta$ independent, value of $\theta$. We therefore expect to regain for the autocorrelation the same value for $\lambda/z$ for any value of $\Delta$, even so the non-trivial initial state may have a long lasting impact on the spin configurations. This is indeed what we observe, as shown in Figure 4. Plotting $\ln G(t, 0)$ as a function of $\ln t$ yields at late times straight lines with a common slope $\lambda/z \approx 0.73$. Corrections at early times are more important for smaller values of $\lambda$, i.e. for larger effective cell sizes in the initial state, as expected. We finally mention that for small values of $\Delta$ strong finite-size effects appear, see the inset of Figure 4 which forced us to simulate systems containing as many as $500 \times 500$ spins.
Discussion. – We have reported in this work an intriguing reentrance in time in critical relaxation measurements which start from a non-trivial initial state given by a ground state of the RFIM model. Competition between two different mechanisms, the cluster dissolution of the compact cells and the usual domain growth, is responsible for this novel feature in nonequilibrium critical dynamics. Note that by varying the strength of the random field, $\Delta$, and the value of the initial magnetization, $m_i$, one can tune the borders of the different regimes in Figure 1 so that both $t_{\min}$ and $t_i$ can be macroscopic. Interestingly, the percolation point in the initial states, given by a critical variance of the random fields, has an impact on the nonequilibrium dynamics by introducing new finite time and length scales.

Finally, we note that nonequilibrium relaxation based on random fields can be applied for other systems, too. For example for the 3d Ising model the initial state is either ordered (for weak random fields) or disordered (for strong random fields) [22]. Most interesting is, however, the borderline situation when at the critical value of the strength of the random fields the initial state is critical and described by a random fixed point.

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