A note on analytic continuation of characteristic functions

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Abstract We derive necessary and sufficient conditions for a continuous bounded function $f : \mathbb{R} \to \mathbb{C}$ to be a characteristic function of a probability measure. The Cauchy transform $K_f$ of $f$ is used as analytic continuation of $f$ to the upper and lower half-planes in $\mathbb{C}$. The conditions depend on the behavior of $K_f(z)$ and its derivatives on the imaginary axis in $\mathbb{C}$. The main results are given in terms of completely monotonic and absolutely monotonic functions.

Keywords: Characteristic function; complex-valued harmonic function; Cauchy transform; analytic function; completely monotonic function; absolutely monotonic function.

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1 Introduction

Suppose that $\sigma$ is a probability measure on $\mathbb{R}$. Let

$$\hat{\sigma}(x) = \int_{-\infty}^{\infty} e^{-ixt} d\sigma(t)$$

be the Fourier transform of $\sigma$. In the language of probability theory, $f(x) := \hat{\sigma}(-x)$ is called the characteristic function of $\sigma$ (see [4, p. 10]). Any characteristic function $f$ is continuous on $\mathbb{R}$ and satisfies $f(-x) = \overline{f(x)}$ for all $x \in \mathbb{R}$. In particular, such an $f$ is real-valued if and only if it is the Fourier transform of a symmetric distribution $\sigma$ [4, p. 30], i.e. if $\sigma$ satisfies $\sigma(-A) = \sigma(A)$ for any measurable $A \subset \mathbb{R}$.

A function $u = u_1 + iu_2$ in a domain $D \subset \mathbb{C}$ is called complex-valued harmonic if both $u_1$ and $u_2$ are real harmonic functions in $D$. This means that $u_1$ and $u_2$ are twice continuously differentiable on $D$ and satisfy there $\Delta u_1 = \Delta u_2 = 0$, where $\Delta$ is the Laplace operator

$$\Delta = \frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2}.$$

For a function $\varphi : \mathbb{R} \to \mathbb{C}$, the Dirichlet problem on the complex upper half-plane $\mathbb{C}^+ = \{z = x + iy \in \mathbb{C} : y > 0\}$ is to extend $\varphi$ on $\mathbb{C}^+$ to a complex-valued harmonic function $u_\varphi$, so that $u_\varphi(z)$ tends to $\varphi(x_0)$ as $z \in \mathbb{C}^+$ tends to $x_0$ for each $x_0 \in \mathbb{R}$. Let us remember one of the most important case of this problem. If $\varphi$ is a bounded continuous function on $\mathbb{R}$, then

$$u_\varphi(x,y) = \frac{1}{\pi} \int_{-\infty}^{\infty} \frac{y}{(x-t)^2 + y^2} \varphi(t) dt = \left( P_y \ast \varphi \right)(x),$$

is the unique solution to the Dirichlet problem on $\mathbb{C}^+$ that is bounded in $\mathbb{C}^+$. Here

$$P_y(x) = \frac{1}{\pi \frac{x^2}{y^2} + 1}, \quad y > 0,$$
$x \in \mathbb{R}$, is called the Poisson kernel for $\mathbb{C}^+$.

In this paper, we will consider the problem of characterizing of conditions for $\varphi$ to be a characteristic function in terms of its analytic continuation in $\mathbb{C} \setminus \mathbb{R}$. If such a $\varphi$ corresponds to a symmetric distribution on $\mathbb{R}$, then a similar problem for the harmonic continuation (1.2) was solved in [5]. Here and for later use we need the notion of completely monotonic function (see [7, p.p. 144-145]). A function $\omega : (a, b) \to \mathbb{R}$, $-\infty < a < b < \infty$, is said to be completely monotonic if it is infinitely differentiable and

$$(-1)^n \omega^{(n)}(x) \geq 0$$

(1.4)

for each $x \in (a, b)$ and all $n = 0, 1, 2, \ldots$. A function $\omega : [a, b] \to \mathbb{R}$ is called completely monotonic on $[a, b]$ if it is there continuous and completely monotonic on $(a, b)$.

**Theorem 1** [5, Theorem 2]. Suppose that $\varphi : \mathbb{R} \to \mathbb{R}$ is a bounded continuous even function and $\varphi(0) = 1$. Then $\varphi$ is a characteristic function if and only if the function $y \to u\varphi(0, y)$ is completely monotonic on $[0, \infty)$.

Other form of this theorem (in other terms) has been shown by Egorov [1]. Moreover, [1] deals with absolutely integrable and infinitely differentiable $\varphi$ satisfying more other conditions.

If $\varphi$ is a complex-valued function, then it is easy to see that Theorem 1 fails in general. Indeed, suppose that

$$\varphi(x) = (1 - \alpha)e^{-ix} + \alpha e^{ix},$$

(1.5)

where $\alpha \in \mathbb{R}$. Since

$$\hat{P}_y(\xi) = e^{-y|\xi|}, \; \xi \in \mathbb{R},$$

for all $y > 0$, we obtain by straightforward calculation in (1.2) that $u\varphi(0, y) = e^{-y}$. Therefore, $u\varphi(0, y)$ is completely monotonic on $[0, \infty)$ for all $\alpha \in \mathbb{R}$. On the other hand, by the Bochner theorem for characteristic function, (1.5) is characteristic if and only if $\alpha \in [0, 1]$.

Here we will extend Theorem 1 for complex-valued characteristic functions. To this end, let us introduce some notions and basic facts.

If $u$ is a complex-valued harmonic function in a domain $D \subset \mathbb{C}$, then another complex-valued harmonic function $v$ in $D$ is harmonic conjugate of $u$ if $u + iv$ is analytic in $D$. Recall that if $D \neq \emptyset$, then the harmonic conjugate $v$ of $u$ is unique, up to adding a constant.

Let $f : \mathbb{R} \to \mathbb{C}$ be a continuous bounded function. The integral

$$v_f(x, y) = \frac{1}{\pi} \int_{-\infty}^{\infty} \frac{x - t}{(x - t)^2 + y^2} f(t) dt$$

can be chosen as the harmonic conjugate of $u_f$ in $\mathbb{C}^+$ (see, for example, [3, p. 108]). In that case $u_f + iv_f$ coincides with the usual Cauchy transform (the Cauchy integral) of $f$

$$k_f(z) = \frac{i}{\pi} \int_{-\infty}^{\infty} \frac{f(t)}{z - t} dt.$$ 

(1.6)

Here the integral is absolutely convergent as long as

$$\int_{-\infty}^{\infty} \frac{|f(t)|}{1 + |t|} dt < \infty.$$ 

(1.7)
In general, if \( f \in L^p(\mathbb{R}), 1 \leq p < \infty \), but not in the case of an arbitrary \( f \in L^\infty(\mathbb{R}) \).

In general, if \( f \in L^p(\mathbb{R}), 1 \leq p \leq \infty \), then we shall use the following harmonic conjugate of (1.2)

\[
V_f(x,y) = \frac{1}{\pi} \int_{-\infty}^{\infty} \left( \frac{x-t}{(x-t)^2+y^2} + \frac{t}{t^2+1} \right) f(t) \, dt
\]

(see [3, p.p. 108-109]). Denote

\[
K_f(z) = \frac{i}{\pi} \int_{-\infty}^{\infty} \left( \frac{1}{z-t} + \frac{t}{t^2+1} \right) f(t) \, dt,
\]

(1.8)

\( z \in \mathbb{C} \setminus \mathbb{R} \). We call \( K_f \) the modified Cauchy transform of \( f \). Both integrals in (1.6) and in (1.8) define analytic functions in \( \mathbb{C} \setminus \mathbb{R} = \mathbb{C}^+ \cup \mathbb{C}^- \) (e.g., see [6, p.p. 144-145]), where \( \mathbb{C}^- \) denotes the open lower half-plane in \( \mathbb{C} \). Let us write \( k_f^{(+)}(z) = k_f(z) \) for \( z \in \mathbb{C}^+ \), and \( K_f(z) = K_f(z) \) for \( z \in \mathbb{C}^- \).

A function \( \omega(x) \) is said to be absolutely monotonic on \((a, b)\) if and only if \( \omega(-x) \) is completely monotonic on \((-b, -a)\) (see [7, p.p. 144-145]). It is obvious that such a function \( \omega(x) \) can be characterized by the inequalities

\[
\omega^{(n)}(x) \geq 0,
\]

(1.9)

\( n = 0, 1, 2, \ldots \).

The main results are the following theorems:

**Theorem 2.** Suppose that \( f : \mathbb{R} \to \mathbb{C} \) is a bounded continuous function and \( f(0) = 1 \). Then \( f \) is a characteristic function if and only if there is a constant \( a_f \in \mathbb{R} \) such that:

(i) \( a_f + K_f^{(+)}(iy) \) is completely monotonic for \( y \in (0, \infty) \), and

(ii) \( -\left( a_f + K_f^{(-)}(iy) \right) \) is absolutely monotonic for \( y \in (-\infty, 0) \).

**Theorem 3.** Suppose that \( f \) is as in Theorem 2 and satisfies (1.7). Then \( f \) is a characteristic function if and only if:

(i) \( k_f^{(+)}(iy) \) is completely monotonic for \( y \in (0, \infty) \), and

(ii) \( -k_f^{(-)}(iy) \) is absolutely monotonic for \( y \in (-\infty, 0) \).

## 2 Proofs

A function \( \varphi \) on \( \mathbb{R} \) is said to be positive definite if

\[
\sum_{i,j=1}^{n} \varphi(x_i - x_j)c_i c_j \geq 0
\]

for every choice of \( x_1, \ldots, x_n \in \mathbb{R} \), for every choice of \( c_1, \ldots, c_n \in \mathbb{C} \), and all \( n \in \mathbb{N} \). By the Bochner theorem (e.g., see [2, Theorem 33.3]), we have that a continuous function \( \varphi \) is positive definite if and only if there exists a non-negative finite measure \( \mu \) on \( \mathbb{R} \) such that \( \varphi(x) = \hat{\mu}(-x) \).

We will need to use later the following lemma:

**Lemma 1** [2, Theorem 33.10]. Let \( \varphi \) be a continuous positive definite function on \( \mathbb{R} \) such that \( \varphi \in L^1(\mathbb{R}) \). Then \( \hat{\varphi} \) is nonnegative, \( \hat{\varphi} \) is in \( L^1(\mathbb{R}) \), and \( \hat{\varphi}(x) = \varphi(x) \) for all \( x \in \mathbb{R} \).

Here

\[
\varphi(t) = \frac{1}{2\pi} \int_{-\infty}^{\infty} e^{it\xi} \varphi(\xi) \, d\xi
\]

is the inverse Fourier transform of \( \varphi \). We first prove our Theorem 3.
Proof of Theorem 3. Suppose that \( f \) is a characteristic function. Let \( z = x + iy \in \mathbb{C} \) with \( y \neq 0 \). According to (1.7), the integral in (1.6) converges absolutely. Moreover, it is easily checked [6, p.p. 144-145] that

\[
\frac{d^n}{dy^n} k_f(iy) = \frac{i}{\pi} \int_{-\infty}^{\infty} \frac{\partial^n}{\partial y^n} \left( \frac{1}{iy - t} \right) f(t) dt
\]

(2.1)

for all \( m = 0, 1, 2, \ldots \).

Let now \( z \in \mathbb{C}^+ \). Then by direct calculation we obtain that

\[
\int_0^\infty x^n e^{-xt} e^{-iyt} dx = (-1)^n \frac{\partial^n}{\partial y^n} \left( \frac{i}{iy - t} \right).
\]

This, together with Bochner’s theorem shows that for any \( y > 0 \) the right side of the previous equality is a continuous positive definite as a function of \( t \in \mathbb{R} \). Set

\[
\varphi_n(t) = (-1)^n \frac{\partial^n}{\partial y^n} \left( \frac{i}{iy - t} \right) f(t).
\]

Under the condition (1.7), we have that \( \varphi_n \) satisfies the hypotheses of Lemma 1. Hence

\[
(-1)^n \int_0^\infty \frac{\partial^n}{\partial y^n} \left( \frac{i}{iy - t} \right) f(t) dt = \int_0^\infty \varphi_n(t) dt = \hat{\varphi}_n(0) \geq 0
\]

for all \( m = 0, 1, 2, \ldots \). Now, by (2.1), we get that the function \( y \rightarrow k_f^{(+)}(iy) \) satisfies (1.4), i.e., it is completely monotonic on \((0, \infty)\).

In the case where \( z \in \mathbb{C}^- \), we have

\[
\int_{-\infty}^0 |x|^n e^{-xt} e^{-iyt} dx = -\frac{\partial^n}{\partial y^n} \left( \frac{i}{iy - t} \right)
\]

for \( n = 0, 1, 2, \ldots \). Again, we see that the right side of this equality is continuous and positive definite as a function of \( t \in \mathbb{R} \). Applying now Lemma 1 to

\[
\varphi_n(t) = -\frac{\partial^n}{\partial y^n} \left( \frac{i}{iy - t} \right) f(t),
\]

we get as in the previous case that the function \( y \rightarrow -k_f^{(-)}(iy) \) satisfies (1.9). Thus, it is absolutely monotonic on \((-\infty, 0)\).

Suppose that \( k_f^{(+)}(iy) \) is completely monotonic for \( y \in (0, \infty) \). By the Bernstein-Widder theorem (see [7, p. 116]), there exists a nonnegative (not necessarily finite) measure \( \mu \) supported on \([0, \infty)\) such that

\[
k_f^{(+)}(iy) = \int_0^\infty e^{-yt} d\mu(t),
\]

(2.2)

where the integral converges for all \( y \in (0, \infty) \). This means that for any given \( \tau > 0 \), the function

\[
u_{\tau}(iy) := k_f^{(+)}(iy + \tau)
\]

(2.3)

is completely monotonic for \( y \in [0, \infty) \). Now, there is a finite nonnegative measure \( \mu_{\tau} \) on \([0, \infty)\) such that

\[
u_{\tau}(iy) = \int_0^\infty e^{-yt} d\mu_{\tau}(t),
\]

\( y \in [0, \infty) \). Any such \( \nu_{\tau} \) can be continued analytically to \( \mathbb{C}^+ \) as the Laplace transform of finite \( \mu_{\tau} \). According to (2.3), we have that the Laplace transform of \( \mu \)

\[
L_{\mu}(z) = \int_0^\infty e^{zt} d\mu(t)
\]

(2.4)
is well-defined and is also analytic in $\mathbb{C}^+$. With equation (2.2) in mind, the applications of the uniqueness theorem for analytic functions (1.6) and (2.4) in $\mathbb{C}^+$ yields
\[ \frac{i}{\pi} \int_{-\infty}^{\infty} \frac{f(t)}{z-t} \, dt = k_f(z) = k_f^{(-)}(z) = \int_0^\infty e^{\text{ist}} \, d\mu(t), \tag{2.5} \]
for $z \in \mathbb{C}^+$.

Let $y \rightarrow -k_f^{(-)}(iy)$ be absolutely monotonic on $(-\infty,0)$. By definition, the function $-k_f^{(-)}(-iy)$ is completely monotonic for $y \in (0,\infty)$. Using the same argument as before, we have that there exists a nonnegative measure $\eta$ on $(-\infty,0]$ such that
\[ k_f(z) = k_f^{(-)}(z) = -\int_{-\infty}^0 e^{\text{ist}} \, d\eta(t), \tag{2.6} \]
where the integral is absolutely convergent for each $z \in \mathbb{C}^-$.

Fix $y > 0$. Using (2.5) and (2.6), we get
\[ \left( P_y * f \right)(x) = \frac{1}{2} \left[ k_f(x+iy) - k_f(x-iy) \right] = \int_{-\infty}^{\infty} e^{\text{ist}} \, d\vartheta_y(t), \tag{2.7} \]
where
\[ \vartheta_y(t) = \frac{1}{2} e^{-|y|} \left[ \mu(t) + \eta(t) \right]. \]
Since the integrals (2.4) and (2.6) are absolutely convergent, it follows that $\vartheta_y$ is a finite measure on $\mathbb{R}$. Now, applying the Bochner theorem to (2.7), we see that $(P_y * f)(x)$ is continuous positive definite function on $\mathbb{R}$. Recall that the family of Poisson kernels $(P_y)_{y>0}$ form an approximate unit in $L^1(\mathbb{R})$ (e.g., see [3, p. 111]). Hence
\[ \lim_{y \rightarrow 0} \left( P_y * f \right)(x_0) = f(x_0) \]
for any point $x = x_0$ of continuity of $f$. According to the fact that the pointwise limit of positive definite functions also is a positive definite, we have that $f$ is positive definite on $\mathbb{R}$. This proves Theorem 3.

**Proof of Theorem 2.** Fix $z \in \mathbb{C} \setminus \mathbb{R}$. Since the modified Cauchy kernel
\[ \gamma(z,t) = \frac{i}{\pi} \left( \frac{1}{z-t} + \frac{t}{t^2+1} \right) \]
is an integrable function, it follows that its Fourier transform is well-defined and is also continuous function. Let $\xi_A$ denote the indicator function of a measurable subset $A \subset \mathbb{R}$. If $z = iy$, $y \neq 0$, then by direct calculation we obtain that
\[ \widehat{\gamma}(iy;-x) = \int_{-\infty}^{\infty} \gamma(iy;t) e^{\text{ist}} \, dt = \begin{cases} 2 \xi_{[0,\infty)}(x) e^{-|x|} - \text{sign}(x) e^{-|x|}, & \text{if } x \neq 0, \\ 1, & \text{if } x = 0, \end{cases} \tag{2.8} \]
for $y > 0$, and
\[ \widehat{\gamma}(iy;-x) = \begin{cases} -2 \xi_{(-\infty,0]}(x) e^{-|x|} - \text{sign}(x) e^{-|x|}, & \text{if } x \neq 0, \\ -1, & \text{if } x = 0, \end{cases} \tag{2.9} \]
for $y < 0$.

To prove the necessity implication, suppose that $f$ is a characteristic function. Then
\[ f(t) = \int_{-\infty}^{\infty} e^{\text{ist}} \, d\sigma(x), \tag{2.10} \]
where $\sigma$ is a probability measure on $\mathbb{R}$. Since the function (2.8) is positive for small $y$ and is negative for large $y$, we have that $\gamma(iy;t)$ is not necessary positive definite as a function of $x \in \mathbb{R}$ in general. Therefore, we cannot apply Lemma 1, as in the proof of Theorem 3. But, on the other hand, both (2.8) and (2.9) are integrable functions for $x \in \mathbb{R}$. 

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Let $y > 0$. Substituting (2.10) into (1.8), applying (2.8) and using Fubini’s theorem, we get

$$K_f(iy) = \int_{-\infty}^{\infty} \gamma(iy; t) f(t) dt = \int_{-\infty}^{\infty} \hat{\gamma}(iy; -x) d\sigma(x) = \int_0^{\infty} e^{-\xi x} d\sigma_1(x) - a_f,$$

where $\sigma_1$ denotes the nonnegative finite measure $2\sigma \cdot \zeta_{[0,\infty)} - \sigma \{0\}$, while

$$a_f = \int_{-\infty}^{\infty} \text{sign}(x) e^{-|x|} d\sigma(x). \tag{2.11}$$

Here $\sigma \{0\}$ is the measure $\sigma$ of the one-point set $\{0\}$. So by the Bernstein-Widder theorem [7, p. 116], the function $K_f(iy) + a_f$ is completely monotonic for $y > 0$.

If $y < 0$, then combining (1.8), (2.9), and (2.10) we obtain

$$-K_f(iy) = \int_{-\infty}^{0} e^{-\xi x} d\sigma_2(x) + a_f$$

with $\sigma_2 = 2\sigma \cdot \zeta_{(-\infty,0]} - \sigma \{0\}$ and $a_f$ defined in (2.11). Finally, it is easy to verify by straightforward calculation of derivatives that the function

$$-(a_f + K_f^{(-)}(iy)) = \int_{-\infty}^{0} e^{-\xi x} d\sigma_2(x)$$

satisfies (1.9) for $y \in (-\infty,0)$.

The sufficiency can be proved in a manner similar to the proof of the sufficiency of Theorem 3.

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