ON THE KPZ EQUATION WITH FRACTIONAL DIFFUSION: GLOBAL REGULARITY AND EXISTENCE RESULTS

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Abstract. In this work we analyze the existence of solutions to the fractional quasilinear problem,

\[
\begin{cases}
    u_t + (-\Delta)^s u &= |\nabla u|^{\alpha} + f & \text{in } \Omega_T \equiv \Omega \times (0, T), \\
    u(x, t) &= 0 & \text{in } (\mathbb{R}^N \setminus \Omega) \times [0, T), \\
    u(x, 0) &= u_0(x) & \text{in } \Omega,
\end{cases}
\]

where $\Omega$ is a $C^{1,1}$ bounded domain in $\mathbb{R}^N$, $N > 2s$ and $\frac{1}{2} < s < 1$. We will assume that $f$ and $u_0$ are non-negative functions satisfying some additional hypotheses that will be specified later on.

Assuming certain regularity on $f$, we will prove the existence of a solution to problem (P) for values $\alpha < \frac{N}{1+s}$, as well as the non-existence of such a solution when $\alpha > \frac{N}{1-s}$. This behavior clearly exhibits a deep difference with the local case.

To Ireneo, our teacher, colleague and friend, in memoriam.

Contents

1. Introduction 1
2. Preliminaries and functional setting 5
3. Gradient regularity of the solutions to the linear problem with $\frac{1}{2} < s < 1$. 7
4. Non existence result. 35
5. The existence results. 38
5.1. Existence result for $L^1$ data and $\alpha < \frac{N + 2s}{N + 1}$. 38
5.2. Existence results for $\frac{N + 2s}{N + 1} \leq \alpha$. Proofs of Theorem 1.3 and Theorem 1.4 40
6. Comparison principle and a partial uniqueness result for a problem with a drift term. Applications to the quasi-linear problem (1.1) 48
6.1. Applications 52
6.2. Some remarks on asymptotic behavior 56
References 57

1. Introduction

In the paper [35], Kardar, Parisi and Zhang describe the following model for the growth of surfaces

\[ u_t - \Delta u = c \sqrt{1 + |\nabla u|^2} + f, \]

where $f$ represents in general a stochastic process. After a Taylor expansion for small size of the gradient, they consider instead the so-called KPZ equation,

\[ u_t - \Delta u = c|\nabla u|^2 + g. \]

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During the final stages of this work, Ireneo Peral sadly passed away after a very short period of illness. He worked very hard and made fundamental contributions to this project, but the responsibility for the correctness of the presentation rests solely and exclusively with the other three authors. The paper is dedicated to him as a tribute for his guidance and teachings.
It is important to say that, from a physical point of view, the KPZ equation is a relevant case of study because, among other things, it defines a new universality class for a lot of models in Statistical Mechanics (see for instance [11] and [23]). In that sense, the behavior of the so called Hopf-Cole class of solutions has been deeply researched in the seminal paper by M. Hairer [31].

We will restrict ourselves to the deterministic setting, that is, when the source term is a function with a suitable summability. In the local critical case \( \alpha = 2 \) with \( s = 1 \) there is a large literature of results. For instance, in the paper [3] (see also the references therein) a classification of the solutions was found showing in particular an extreme case of non-uniqueness.

The relevant facts in the KPZ model are that the growth is driven in the direction of the gradient of the interface while the diffusion comes from the classical Laplacian (remember that behind all this one finds always a Brownian motion).

There is another question to take into account, which is that the diffusion to consider may change according with the medium. For instance in the paper by Barenblatt, Bertsch, Chertock, and Prostokishin, [12], the growth in a porous medium was considered. This model has been studied for instance in the papers, [6] and [7]. The diffusion for a power law in the gradient (the \( p \)-Laplacian) has been also studied, see for instance [4], [17] and the references therein.

The main goal of this work is to study a non local version of the Kardar-Parisi-Zhang equation. More precisely, the idea is to consider diffusion driven by the fractional Laplacian (so that behind it what one finds is a Lévy process). More specifically, we deal with the problem

\[
\begin{align*}
 u_t + (-\Delta)^s u &= |\nabla u|^\alpha + f & \text{in } \Omega_T \equiv \Omega \times (0, T), \\
 u(x, t) &= 0 & \text{in } (\mathbb{R}^N \setminus \Omega) \times [0, T), \\
 u(x, 0) &= u_0(x) & \text{in } \Omega,
\end{align*}
\]

where \( \Omega \) is a \( C^{1,1} \) bounded domain in \( \mathbb{R}^N \), \( N > 2s \) and \( \frac{1}{2} < s < 1 \). We suppose that \( f \) and \( u_0 \) are non-negative functions satisfying some hypotheses that we will precise later.

By \( (-\Delta)^s \) we mean the fractional Laplacian operator of order \( 2s \) given as the multiplier of the Fourier transform with symbol \( |\xi|^{2s} \). That is, for every \( u \in \mathcal{S}(\mathbb{R}^N) \), the Schwartz class,

\[
(-\Delta)^s u = \mathcal{F}^{-1}(|\xi|^{2s} \mathcal{F}(u)), \quad \xi \in \mathbb{R}^N, \ s \in (0, 1),
\]

where \( \mathcal{F} \) denotes Fourier transform and \( \mathcal{F}^{-1} \) its inverse.

As was indicated by M. Riesz in his foundational paper [49], since the formal homogeneous kernel corresponding to the multiplier \( |\xi|^{2s} \) is a constant multiple of \( |x|^{-N-2s} \), therefore not in \( L^{1}_{\text{loc}} \), the definition cannot be a convolution but rather a principal value given by the following expression

\[
(-\Delta)^s u(x) := a_{N,s} \text{ P.V. } \int_{\mathbb{R}^N} \frac{u(x) - u(y)}{|x-y|^{N+2s}} dy, \quad s \in (0, 1),
\]

where

\[
a_{N,s} := 2^{2s-1} \frac{\Gamma\left(\frac{N+2s}{2}\right)}{\Gamma(-s)} \pi^{-\frac{N}{2}}
\]

is the normalization constant related to the definition through the Fourier transform. This formula is obtained by analytic continuation of the Riesz potentials in the complex plane. See the details for instance in [38] and [46]. The hypothesis \( s > \frac{1}{2} \) is a natural assumption to allow the presence of a power of the gradient as a nonlinear perturbation. The stationary problem associated to problem (1.1) has recently been studied in [22] and [5].

The interest of the fractional Laplacian is motivated, in addition to the mathematical relevance, by the fact that it has recently been used in a number of equations modeling concrete phenomena. Among others, we mention crystal dislocation [25], [26], mathematical finances [10] and quantum mechanics, see [39].
For the nonlocal case $s \in (\frac{1}{2}, 1)$ and for regular data, the authors in [34, 51] proved the existence of a regular solution using semi-group theory and probabilistic tools. More precisely, the authors in [34] treat the case $f = 0$ and $\Omega = \mathbb{R}^N$, under regular assumption on $u_0$, in order to get global estimates. In their approach, the fact that $\Omega = \mathbb{R}^N$ turns out to be a fundamental key, which is lost in the case of a bounded domain. As we will see later, the work on bounded domains, $\Omega$, generates a loss of regularity near the boundary and, as a consequence, non existence results holds for large values of $q$.

The main goal of this paper is to consider a general class of data. It is important to remark that monotony arguments have serious limitations in order to pass to the limit in the approximating problems. To overcome these difficulties we will follow the arguments used for the elliptic case in [5]. In particular, we will use apriori estimates and the Schauder fixed point theorem, which in the stationary case are inspired by results in [43] and [45] for local operators.

We briefly sketch now the main results in the paper. First, via a fixed point argument we obtain the following results for general data.

**Theorem 1.1.** Suppose in the problem (1.1) that $\alpha < \frac{N + 2s}{N + 1}$, then there exists $T := T(\Omega, s) > 0$ such that for all $(f, u_0) \in L^m(\Omega_T) \times L^1(\Omega)$ with $1 \leq m < \frac{1}{s}$, problem (1.1) has a solution $u \in L^q(0, T; W^{1,q}_s(\Omega))$ for all $q < \frac{N + 2s}{N + 1}$ and $T_k(u) \in L^2(0, T; H^1_0(\Omega))$ for all $k > 0$, where $T_k(\sigma)$ is given by

$$T_k(\sigma) = \begin{cases} \sigma, & \text{if } |\sigma| \leq k; \\ k \frac{\sigma}{|\sigma|}, & \text{if } |\sigma| > k; \end{cases}$$

Notice that, even for $f \in L^m(\Omega_T)$ with $1 < m < \frac{1}{s}$, the existence result holds with the same assumption on $\alpha$ as in $L^1$ data. This assumption does not appear in the local case $s = 1$ where the relation $\alpha \leftrightarrow m$ is strictly increasing. In the non local case, this limitation is due to the fact that the global regularity for the gradient term imposes many restrictions on the parameters $s, m, \alpha$ and makes a fundamental difference between the local and the nonlocal case.

In the case of $L^1$-data the above existence result is optimal, in the sense that if $\alpha > \frac{N + 2s}{N + 1}$, then we can find $f \in L^1(\Omega_T)$ or $u_0 \in L^1(\Omega)$ such that problem (1.1) has no solution in the space $L^\infty(0, T; W^{1,\alpha}_s(\Omega))$.

For large value of $\alpha$ a serious limitation appears as a consequence of the lack of regularity for the linear problem near the boundary. This loss of regularity allows us to get the following non existence result which makes more significant the difference between the local and the nonlocal case. However, this is coherent with the local case; indeed, one sees in the threshold that $\frac{1}{1-s} \to \infty$ as $s \to 1$.

**Theorem 1.2.** Suppose that $\alpha > \frac{1}{1-s}$, and let $(f, u_0) \in L^\infty(\Omega_T) \times L^\infty(\Omega)$ be nonnegative functions with $(f, u_0) \neq (0,0)$. Then, the problem

$$\begin{cases} u_t + (-\Delta)^s u &= |\nabla u|^q + f \quad \text{in } \Omega_T \equiv \Omega \times (0, T), \\
 u(x, t) &= 0 \quad \text{in } (\mathbb{R}^N \setminus \Omega) \times (0, T), \\
u(x, 0) &= u_0(x) \quad \text{in } \Omega, \end{cases}$$

has no solution $u$ such that $u \in L^\infty(0, T; W^{1,\alpha}_s(\Omega))$.

In the same direction we prove a general non existence result of weak solutions in a suitable weighted Sobolev space for a range of the parameter $\alpha$. This gives a fundamental difference related to the local case, $s = 1$, where existence of a solution is proved for all $\alpha > 1$ under suitable regularity assumptions on the data, see for example [14].

For the existence result, we will distinguish two types of problems according to the integrability of the gradient term. In the first case we look for the global integrability of the gradient term in the whole domain $\Omega_T$. 

Theorem 1.3. Assume that \( \frac{2s-1}{1-s} > \frac{(N+2s)^2}{N+1} \) and that \( \frac{N+2s}{N+1} \leq \alpha < \frac{2s-1}{(1-s)(N+2s)} \).

Suppose that \( u_0 = 0, f \in L^m(\Omega_T) \) with \( m \geq \frac{1}{s} \) satisfies one of the following conditions:

(I) either \( \frac{N+2s}{2s-1} \leq m \),

(II) or \( \frac{N+2s}{2s-1} \leq \frac{1}{\alpha} \left( \frac{2s}{s(2s-1)} - \frac{1}{s-\alpha} \right) \),

then there exists \( T := T(\Omega,s) > 0 \) such that problem (1.1) has a solution \( u \in L^\alpha(0,T;W_0^{1,\alpha}(\Omega)) \) and moreover \( u \in L^\gamma(0,T;W_0^{1,\gamma}(\Omega)) \) for all \( \gamma < \frac{1}{1-s} \) if (I) holds and \( u \in L^\gamma(0,T;W_0^{1,\gamma}(\Omega)) \) for all \( \gamma \leq \frac{1}{(N+2s)(m(1-s) + 1) - m(2s-1)} \) if (II) holds.

Notice that the hypothesis on \( s \) means that \( s \) must be close to 1 and \( \alpha \ll \frac{s}{1-s} \). The above conditions are needed to ensure the global integrability of the gradient term in the whole \( \Omega_T \).

For the complete range of the parameter \( \alpha \), that is \( \frac{N+2s}{N+1} \leq \alpha \ll \frac{s}{1-s} \) and without any restriction in the order of the operator \( s > 1 \), to have a weak solution to the problem requires a natural weight that in fact is a power of the distance to the boundary of \( \Omega \). For simplicity, throughout this paper we denote \( \delta(x) := \text{dist}(x,\partial\Omega) \), with \( x \in \Omega \), such a distance. Hence, the existence of a distributional solution in this case will be obtained in a weighted Sobolev space. More precisely we have the following result.

Theorem 1.4. For every \( s \in \left( \frac{1}{2}, 1 \right) \), assume that \( \frac{N+2s}{N+1} \leq \alpha \ll \frac{s}{1-s} \). Let \( f \) be a nonnegative function such that \( f \in L^m(\Omega) \) with \( m > \max \left\{ \frac{N+2s}{s(2s-1)} - \frac{N+2s}{s-\alpha(1-s)} \right\} \), then there exists \( T := T(\Omega,s) > 0 \) such that problem (1.1) has a distributional solution \( u \in L^\alpha(0,T;W_0^{1,\alpha}(\Omega)) \) and \( L^1(0,T;W_0^{1,1}(\Omega)) \). Moreover \( u\delta^{1-s} \in L^\alpha(0,T;W_0^{1,\alpha}(\Omega)) \).

In the common values of \( \alpha \) in Theorem 1.3 we require \( s \) to be close to 1. Also, the integrability of the datum, \( m \), is bigger than in Theorem 1.4, where there is no restriction in \( s \). The optimality of the results remain open.

If the source term \( f \) is null, then we can prove the existence of a solution using a suitable change of function and Theorem 1.3. More precisely we have

Theorem 1.5. In the problem (1.1), let us consider \( f = 0 \). Assume that \( \frac{2s-1}{1-s} > \frac{(N+2s)^2}{N} \) and that \( \frac{N+2s}{N+1} \leq \alpha \ll \frac{2s}{(1-s)(N+2s) + 1} \). Let \( u_0 \) be a nonnegative measurable function such that \( u_0 \in L^\sigma(\Omega) \) with \( \sigma > \frac{(2s-\alpha) - \alpha(1-s)(N+2s)}{(2s-\alpha)(\alpha-1)N} \), then there exists \( T := T(\Omega,s) > 0 \) such that the problem

\[
\begin{align*}
\begin{cases}
u_t + (-\Delta)^s u & = |\nabla u|^\sigma & \text{in } \Omega_T, \\
u(x,t) & = 0 & \text{in } (\mathbb{R}^N \setminus \Omega) \times [0,T), \\
u(x,0) & = u_0(x) & \text{in } \Omega,
\end{cases}
\end{align*}
\]

has a solution \( u \in L^\gamma(0,T;W_0^{1,\gamma}(\Omega)) \), for all \( \gamma < \frac{\sigma(N+2s)}{(1-s)\sigma(N+2s) + N+\sigma} \).

A similar result with weights could be obtained by application of Theorem 1.4. This problem will be studied in a forthcoming paper, in which we will also look for the asymptotic behavior of the solutions with respect to \( t \).
As a direct application of the arguments developed, we will treat the problem with drift, that is, the nonlinearity on the gradient is substituted by a term of the form \( B(x,t) \cdot \nabla u \). The existence of a solution is known in the literature under regularity conditions on the data \( f \) and \( u_0 \).

Here we will prove the existence of a solution under natural condition on the field \( B \) and general data \((f, u_0) \in L^1(\Omega_T) \times L^1(\Omega)\). Using a suitable Harnack inequality we are able to prove a comparison principle and then the uniqueness of the solution follows for the linear problem with drift. This will be the key in order to show the uniqueness of a good solution to problem (1.1).

**Remark 1.6.** Notice that the above existence results also hold true without any additional assumption on \( T \) if we alternatively assume that the corresponding norm of the data are small. This can be seen if we substitute the data \( f, u_0 \) with \( \lambda f, \lambda u_0 \) with \( |\lambda| \) small. However in the case of Theorem 1.1, using the uniqueness result proved in Theorem 6.6, we are able to show the existence of a minimal solution (and in some cases a unique solution) without any rescription on the norm of the data. See Theorem 6.7 below.

The paper is organized as follows. In Section 2 we collect some tools that will be used systematically in the paper. We begin by specifying the sense in which the solutions are understood, and we state some tools as the Kato’s inequality and the gradient regularity for an associated elliptic problem.

In order to prove gradient regularity for the solution based on the representation formula, we need to show gradient regularity for the fractional heat kernel. This is done in Section 3 where we also consider the general linear fractional heat equation with \( L^1 \) data. In this case we are able to show the uniqueness of the solution and to prove the strong convergence of the solution of the approximating problems in a suitable Sobolev space without using the Landes regularizing approximation. General regularity results of the gradient and the Hardy-Sobolev term \( \frac{u}{\delta^s} \) are obtained, in particular, close to \( \partial \Omega \times (0, T) \). This will be useful in order to complete the regularity schema of this class of equations, as is done in [5] for the elliptic equation.

As a consequence of the results in Section 3, Section 4 is devoted to prove the surprising nonexistence result. We consider the Problem (1.1) in Section 5. We begin with the case of \( L^1 \) data and for all \( \alpha < \alpha_0 = \frac{N + 2s}{N + 1} \) and prove the existence of a weak solution with suitable regularity. We also prove that the condition on \( \alpha \) is optimal in the sense that for \( \alpha > \alpha_0 \), then there exists \( f \in L^1(\Omega_T) \) such that problem (1.1) has no solution.

Problem (1.1) with general \( \alpha < \frac{s}{1 - s} \) is treated in Subsection 5.2. Under suitable hypotheses on \( f \), we are able to show the existence of a weak solution. We also treat the case where \( f \equiv 0 \) and \( u_0 \geq 0 \). Following closely the ideas as in the case \( f \equiv 0 \), we prove the existence of a solution that lives in a suitable Sobolev space.

In the last section we consider the linear problem with drift. The existence of a weak solution is proved for all data in \( L^1 \). According to additional hypotheses on the drift term, we are able to show the uniqueness of the weak solution. As a consequence we prove a general comparison principle, using a suitable singular Gronwall-Bellman inequality, that allows us to show the existence of a minimal solution to problem (1.1) under suitable hypothesis on \( \alpha \).

We thank A. Younes for pointing us some misprints in an earlier version of this work.

2. Preliminaries and functional setting

Let us begin by some results from fractional Sobolev spaces that will be used in this paper. We refer to [24] for more details and proofs.

Assume that \( s \in (0, 1) \) and \( p > 1 \). For a measurable \( \Omega \subset \mathbb{R}^N \), the fractional Sobolev Space \( W^{s,p}(\Omega) \) is defined by

\[
W^{s,p}(\Omega) \equiv \left\{ \phi \in L^p(\Omega) : \int_\Omega \int_\Omega |\phi(x) - \phi(y)|^p d\nu < +\infty \right\},
\]

where, for simplicity, we set

\[
d\nu = \frac{dx \, dy}{|x - y|^{N + ps}}.
\]
Notice that $W^{s,p}(\Omega)$ is a Banach Space endowed with the norm
\[
||\phi||_{W^{s,p}(\Omega)} = \left( \int_{\Omega} |\phi(x)|^p \, dx \right)^{\frac{1}{p}} + \left( \int_{\Omega} \int_{\Omega} |\phi(x) - \phi(y)|^p \, d\nu \right)^{\frac{1}{p}}.
\]
The space $W^{s,p}_0(\Omega)$ is defined as the completion of $C_0^\infty(\Omega)$ with respect to the previous norm.

If $\Omega$ is a bounded regular domain, we can endow $W^{s,p}_0(\Omega)$ with the equivalent norm
\[
||\phi||_{W^{s,p}_0(\Omega)} = \left( \int_{\Omega} \int_{\Omega} |\phi(x) - \phi(y)|^p \, d\nu \right)^{\frac{1}{p}}.
\]

The next Sobolev inequality is proved in [8], see also [24] and [47] for an elementary proof.

**Theorem 2.1. (Fractional Sobolev inequality)** Assume that $0 < s < 1, p > 1$ satisfy $ps < N$. There exists a positive constant $S \equiv S(N, s, p)$ such that for all $v \in C_0^\infty(\mathbb{R}^N)$,
\[
\int_{\mathbb{R}^N} |v(x) - v(y)|^p \, dx dy \geq S \left( \int_{\mathbb{R}^N} |v(x)|^{p_*} \, dx \right)^{\frac{1}{p_*}},
\]
where
\[
p_* = \frac{pN}{N - ps}.
\]

We will denote by $H^s(\mathbb{R}^N)$ the Hilbert space $W^{s,2}(\mathbb{R}^N)$. If $u \in H^s(\mathbb{R}^N)$, we define
\[
(-\Delta)^s u(x) = P.V. \int_{\mathbb{R}^N} \frac{u(x) - u(y)}{|x - y|^{N + 2s}} \, dy.
\]

For $w, v \in H^s(\mathbb{R}^N)$, we have
\[
\langle (-\Delta)^s w, v \rangle = \frac{1}{2} \int_{\mathbb{R}^N} \frac{(w(x) - w(y))(v(x) - v(y)) \, dx dy}{|x - y|^{N + 2s}}.
\]

If $H^s_0(\Omega)$ is the closure of $C_0^\infty(\Omega)$ with respect to the norm $H^s(\mathbb{R}^N)$ and if $w, v \in H^s_0(\Omega)$, then
\[
\langle (-\Delta)^s w, v \rangle = \frac{1}{2} \int_{D_\Omega} \frac{(w(x) - w(y))(v(x) - v(y)) \, dx dy}{|x - y|^{N + 2s}},
\]
where $D_\Omega = (\mathbb{R}^N \times \mathbb{R}^N) \setminus (\mathcal{C} \times \mathcal{C})$.

Since we are considering parabolic problems, we need to define the corresponding parabolic spaces. For $q \geq 1$, the space $L^q(0, T; W^{s,p}_0(\Omega))$ is defined as the set of functions $\phi$ such that $\phi \in L^q(\Omega_T)$ with $||\phi||_{L^q(0, T; W^{s,p}_0(\Omega))} < \infty$ where
\[
||\phi||_{L^q(0, T; W^{s,p}_0(\Omega))} = \left( \int_0^T \int_{D_\Omega} |\phi(x, t) - \phi(y, t)|^q \, d\nu dt \right)^{\frac{1}{q}}.
\]

It is clear that $L^q(0, T; W^{s,p}_0(\Omega))$ is a Banach Space.

Consider now the linear problem
\[
\begin{cases}
u_t + (-\Delta)^s u = f & \text{in } \Omega_T = \Omega \times (0, T), \\
u = 0 & \text{in } (\mathbb{R}^N \setminus \Omega) \times (0, T), \\
u(x, 0) = u_0(x) & \text{in } \Omega,
\end{cases}
\]
where $\Omega \subset \mathbb{R}^N$ is a bounded regular domain. If the data $(f, u_0) \in L^2(\Omega_T) \times L^2(\Omega)$, then we can deal with energy solution. More precisely we have the next definition.

**Definition 2.2.** Assume $(f, u_0) \in L^2(\Omega_T) \times L^2(\Omega)$, then we say that $u$ is an energy solution to problem (2.2) if $u \in L^2(0, T; H^s_0(\Omega)) \cap C([0, T], L^2(\Omega))$, $u_t \in L^2(0, T; H^{-s}(\Omega))$, and for all $v \in L^2(0, T; H^s_0(\Omega))$ we have
\[
\int_0^T \langle u_t, v \rangle dt + \frac{1}{2} \int_0^T \int_{\Omega_T} (u(x, t) - u(y, t))(v(x, t) - v(y, t)) \, d\nu \, dt \\
= \int_{\Omega_T} f v \, dx \, dt
\]
and $u(x, \cdot) \to u_0$ strongly in $L^2(\Omega)$, as $t \to 0$. 
Theorem 2.4. Suppose that \( u, \phi \in C^\infty(\Omega \times (0,T)) \cap C^{\alpha,\beta}(\Omega \times (0,T)) \), \( \phi = 0 \) in \( (\mathbb{R}^N \setminus \Omega) \times (0,T) \), \( \phi(x,T) = 0 \) in \( \Omega \). From [41], we know that if \( \phi \in \mathcal{T} \), then \( \phi \in C^\infty(\Omega \times (0,T)) \) and \( \phi \in \mathcal{T} \) satisfies the equation in a pointwise sense.

We are now able to state the meaning of weak solution.

**Definition 2.3.** Assume that \((f, u_0) \in L^1(\Omega_T) \times L^1(\Omega)\). We say that \( u \in C([0,T]; L^1(\Omega)) \) is a weak solution to problem (2.2) if for all \( \phi \in \mathcal{T} \) we have

\[
\int_{\Omega_T} u(-\phi_t + (-\Delta)^s \phi) \, dx \, dt = \int_{\Omega_T} f \phi \, dx \, dt + \int_{\Omega} u_0(x) \phi(x,0) \, dx. \tag{2.3}
\]

The next existence result is proved in [41], (see also [2] and [15] for some different approaches.)

**Theorem 2.4.** Suppose that \((f, u_0) \in L^1(\Omega_T) \times L^1(\Omega)\), then problem (2.2) has a unique weak solution \( u \) such that \( u \in C([0,T]; L^1(\Omega)) \cap L^m(\Omega_T) \) for all \( m \in [1, \frac{N+2s}{N-2s}) \), \( \|(-\Delta)^{\frac{s}{2}} u\| \in L^r(\Omega_T) \) for all \( r \in [1, \frac{N+2s}{N-2s}) \) and \( T_k(u) \in L^2(0,T; H_0^s(\Omega)) \) for all \( k > 0 \) where \( T_k(\sigma) = \max\{k, \min\{k, \sigma\} \} \). Moreover \( u \in L^q(0,T, W_{loc}^{s,q}(\Omega)) \) for all \( 1 \le q < \frac{N+2s}{N-2s} \). In addition we have

\[
\|u\|_{C([0,T];L^1(\Omega))} + \|u\|_{L^m(\Omega_T)} + \|(-\Delta)^{\frac{s}{2}} u\|_{L^r(\Omega_T)} + \|u\|_{L^q(0,T, W_{loc}^{s,q}(\Omega))} \\
\le C(\Omega_T) \left( \|f\|_{L^1(\Omega_T)} + \|u_0\|_{L^1(\Omega)} \right). \tag{2.4}
\]

**Remark 2.5.** The regularity condition obtained in Theorem 2.4 is optimal in the sense that if \( m \ge \frac{N+2s}{N} \), then we can find \( f \in L^1(\Omega_T) \) and \( u_0 \in L^1(\Omega) \) such that \( u^m \notin L^1(\Omega_T) \). This fact will be used in Theorem 1.1 in order to show the optimality of the condition imposed on \( \alpha \).

In the case of having data in \( L^1_{loc}(\Omega) \), the natural concept is the usual distributional solution, that is, given by the following definition.

**Definition 2.6.** Assume that \((f, u_0) \in L^1_{loc}(\Omega_T) \times L^1_{loc}(\Omega)\). We say that \( u \in C^\infty(\Omega_T) \cap C([0,T], L^1_{loc}(\Omega)) \) is a distributional solution to Problem (2.2) if for all \( \varphi \in C^\infty(\Omega_T) \), for all \( \eta \in C^\infty_0(\Omega) \), we have

\[
\int_{\Omega_T} u(-\varphi_t + (-\Delta)^s \varphi) \, dx \, dt = \int_{\Omega_T} f(x,t)\varphi(x,t) \, dx \, dt,
\]

and

\[
\int_{\Omega} u(x,t) \eta(x) \, dx \to \int_{\Omega} u_0(x) \eta(x) \, dx \text{ as } t \to 0.
\]

Finally, the next Kato type inequality will be useful in order to show a priori estimates and the positivity of the solution. The proof follows exactly as in the elliptic case proved in [21]. (See also [46]).

**Theorem 2.7.** Let \( \phi \in C^2(\mathbb{R}) \) be a convex function. Assume \( u \in L^2(0,T; H_0^s(\Omega)) \cap C([0,T], L^2(\Omega)) \). Define \( v = \phi(u) \) and suppose that \( |v_t + (-\Delta)^s v| \in L^1(\Omega_T) \), then

\[
v_t + (-\Delta)^s v \le \phi'(u)(u_t + (-\Delta)^s u). \tag{2.5}
\]

3. Gradient Regularity of the solutions to the linear problem with \( \frac{1}{2} < s < 1 \).

In this section the principal goal is to study the gradient regularity of the solution to the linear problem

\[
\begin{align*}
u_t + (-\Delta)^s u &= f & \text{in } \Omega_T = \Omega \times (0,T), \\
u &= 0 & \text{in } (\mathbb{R}^N \setminus \Omega) \times (0,T), \\
u(x,0) &= u_0(x) & \text{in } \Omega,
\end{align*}
\tag{3.1}
\]
Lemma 3.1. Assume that $s \in (\frac{1}{2}, 1)$, then for all $x, y \in \Omega$ and for all $0 < t < T$,
\[
P_\Omega(x, y, t) \lesssim \left(1 + \frac{\delta^s(x)}{\sqrt{t}}\right) \times \left(1 + \frac{\delta^s(y)}{\sqrt{t}}\right) \times \left(t^\frac{s}{2s - 1} \wedge \frac{t}{|x - y|^{N + 2s}}\right)
\] (3.2)
and
\[
|\nabla_x P_\Omega(x, y, t)| \leq C \left(\frac{1}{(\delta^s(x) \wedge t^{\frac{s}{2s - 1}})}\right) P_\Omega(x, y, t).
\] (3.3)

Setting
\[
\int_0^\infty P_\Omega(x, y, t) \, dt = G_s(x, y)
\] (3.4)
where $G_s(x, y)$ is the Green function of the fractional Laplacian operator with Dirichlet condition, that is,
\[
G_s(x, y) \simeq \frac{1}{|x - y|^{N - 2s}} \left(\frac{\delta^s(x)}{|x - y|^s} \wedge 1\right) \left(\frac{\delta^s(y)}{|x - y|^s} \wedge 1\right).
\] (3.5)

Remark 3.2. Notice that
\[
P_\Omega(x, y, t) \lesssim t^{-\frac{N}{2s}} \wedge \frac{t}{|x - y|^{N + 2s}},
\]
therefore
\[
P_\Omega(x, y, t) \lesssim \frac{2t}{t^{-\frac{N}{2s}} + |x - y|^{N + 2s}} \leq C(s, N) \frac{t}{(t^{\frac{s}{2s - 1}} + |x - y|^{N + 2s})}.
\] (3.6)

Hereafter we will call
\[
H(x, t) := \frac{t}{(t^{\frac{s}{2s - 1}} + |x|^{N + 2s})}.
\] (3.7)

We begin by proving the following basic result.

Proposition 3.3. Assume that $s \in (\frac{1}{2}, 1)$, then for all $q < \frac{N + 2s}{N + 1}$, we have
\[
\int_0^T \int_\Omega |\nabla_x P_\Omega|^q \, dx \, dy \, dt \leq C(\Omega) T^{\frac{N + 2s - (N + s)}{2s - 1}} + T^{\frac{N + 2s - (N + 1)}{2s - 1}}.
\]

Proof. From (3.2) and (3.3) we obtain that
\[
|\nabla_x P_\Omega(x, y, t)| \leq C \left(\frac{1}{(\delta^s(x) \wedge t^{\frac{s}{2s - 1}})}\right) \left(1 + \frac{\delta^s(y)}{\sqrt{t}}\right) \times \left(1 + \frac{\delta^s(y)}{\sqrt{t}}\right) \times \frac{t}{(t^{\frac{s}{2s - 1}} + |x - y|^{N + 2s})}.
\]

Hence, according with (3.6),
\[
|\nabla_x P_\Omega(x, y, t)| \leq \begin{cases} 
\left(1 + \frac{\delta^s(y)}{\sqrt{t}}\right) \frac{\sqrt{t}}{(\delta(x))^{1-s}} \frac{C}{(t^{\frac{s}{2s - 1}} + |x - y|^{N + 2s})}, & \text{if } \delta(x) < t^{\frac{s}{2s - 1}}, \\
C \left(1 + \frac{\delta^s(y)}{\sqrt{t}}\right) \frac{t^{\frac{s}{2s - 1}}}{(t^{\frac{s}{2s - 1}} + |x - y|^{N + 2s})}, & \text{if } \delta(x) \geq t^{\frac{s}{2s - 1}}.
\end{cases}
\] (3.8)
Thus
\[
\iint_{\Omega} |\nabla_z P_{\Omega}|^q \, dx \, dy \, dt \leq \iint_{(0,T) \times \Omega \cap \{\delta (x) < 1\}} |\nabla_z P_{\Omega}|^q \, dx \, dy \, dt \quad \text{and} \quad \iint_{(0,T) \times \Omega \cap \{\delta (x) \geq 1\}} |\nabla_z P_{\Omega}|^q \, dx \, dy \, dt := I_1 + I_2.
\]

Using (3.8), it holds that
\[
I_1 \leq \int_{\Omega} \frac{dx}{(\delta (x))^{(1-s)q}} \left( \int_{\Omega} \frac{dy}{(t^{\frac{N}{q}} + |x-y|)^q (N+2s)} \right) \leq \int_{\Omega} \frac{dx}{(\delta (x))^{(1-s)q}} \left( \int_{\mathbb{R}^N} \frac{t^{\frac{N}{q}} dy}{(1 + |x-y|)^q (N+2s)} \right)
\]
\[
= C \int_{\Omega} \frac{dx}{(\delta (x))^{(1-s)q}} \int_{0}^{T} \frac{t^{\frac{N}{q}} \cdot \frac{q^s}{(1 + |x-y|)^q (2(N+2s))}}{dt} \, dt \int_{0}^{\infty} \frac{\theta^{N-1}}{(1 + \theta)^{q(N+2s)}} \, d\theta,
\]
with \( \theta = \frac{|x-y|}{t^{\frac{N}{q}}} \). Since \( q < \frac{N+2s}{N+s} < \frac{1}{1-s} \), then \((1-s)q < 1\) and \( \frac{N}{2s} + \frac{q(N + 2s)}{2s} > 1 \). Hence
\[
I_1 \leq C(\Omega)T^{\frac{N+2s-q(N+s)}{2s}}.
\]

Respect to \( I_2 \), we have
\[
I_2 \leq C \int_{0}^{T} \int_{\Omega} \int_{\mathbb{R}^N} \frac{t^{\frac{N}{q}} dy}{(t^{\frac{N}{q}} + |x-y|)^{(N+2s)q}} \, dx \, dy \, dt
\]
\[
= C \left( \int_{0}^{T} \frac{t^{\frac{N}{q}} \cdot j((2s-1) + \frac{N-(N+2s)}{2s})}{dt} \, dt \right) \int_{0}^{\infty} \frac{\theta^{N-1}}{(1 + \theta)^{q(N+2s)}} \, d\theta.
\]
Since \( q < \frac{N+2s}{N+1} \), then \( \frac{2(s-1) + N-(N+2s)}{2s} > -1 \). Hence
\[
I_2 \leq C(\Omega)T^{\frac{N+2s-q(N+1)}{2s}}.
\]

Combining the above estimates on \( I_1 \) and \( I_2 \), the result follows. \( \square \)

We start with the following elementary result about the linear problem (3.1) without source term.

**Proposition 3.4.** Suppose that \( f \equiv 0 \) and \( u_0 \in L^p(\Omega) \). If \( u \) is the unique weak solution to problem (3.1), then for all \( r \geq \rho \) and for all \( t > 0 \), we have
\[
||u(\cdot, t)||_{L^r(\Omega)} \leq C(\Omega) T^{\frac{N}{2}(\frac{1}{2} - \frac{1}{r})} ||u_0||_{L^\rho(\Omega)}.
\]  

**Proof.** We have the representation formula for the solution,
\[
u(x, t) = \int_{\Omega} u_0(y) P_{\Omega}(x, y, t) \, dy.
\]

From Theorem 2.4 we obtain that \( u \in L^m(\Omega_T) \) for all \( m < \frac{N+2s}{1} \). To prove the estimate (3.9) we take advantage of the linearity of the problem and use a duality argument. Let \( \phi \in C_0^\infty(\Omega) \), then
\[
||u(\cdot, t)||_{L^r(\Omega)} = \sup_{||\phi||_{L^{r'}(\Omega)} = 1} \int_{\Omega} \phi(x) u(x, t) \, dx = \sup_{||\phi||_{L^{r'}(\Omega)} = 1} \int_{\Omega} \phi(x) \int_{\Omega} u_0(y) P_{\Omega}(x, y, t) \, dy \, dx.
\]

Using estimate (3.2), it holds that
\[
||u(\cdot, t)||_{L^r(\Omega)} \leq \sup_{||\phi||_{L^{r'}(\Omega)} \leq 1} \int_{\Omega \times \Omega} |u_0(y)||\phi(x)| \frac{t}{(t^{\frac{N}{q}} + |x-y|)(N+2s)} \, dy \, dx
\]
\[
= \sup_{||\phi||_{L^{r'}(\Omega)} \leq 1} \int_{\Omega \times \Omega} |\phi(x)||u_0(y)| H(x-y, t) \, dy \, dx
\]

where
\[
H(x-y, t) = \frac{1}{(t^{\frac{N}{q}} + |x-y|)(N+2s)}.
\]
with \( H(x, \sigma) \) defined in (3.7). Using Young inequality, we get
\[
||u(., t)||_{L^r(\Omega)} \leq C \sup_{\phi \in (L^r(\Omega))'} ||\phi||_{L^r(\Omega)} ||u_0||_{L^r(\Omega)} ||H(., t)||_{L^s(\Omega)}
\]
with \( \frac{1}{r} + \frac{1}{p} + \frac{1}{s} = 2 \). By a direct computations we reach that
\[
||H(., t)||_{L^s(\Omega)}^2 \leq ||H(., t)||_{L^s(\Omega)}^2 \leq t^{a-\frac{s(N+2)}{2N}} \int_{\mathbb{R}^N} \left( \left( \frac{|x|}{t} + \frac{1}{r} \right)^{s(N+2)} \right) \ dx.
\]
Setting \( z = \frac{x}{t^a} \), then \( ||H(., t)||_{L^s(\Omega)}^q \leq Ct^{a+\frac{s(N+2)}{2N}} \). Thus
\[
||H(., t)||_{L^s(\Omega)} \leq C(\Omega) t^{\frac{a}{2N}(\frac{1}{r}-\frac{1}{s})} = Ct^{\frac{a}{2N}(\frac{1}{r}-\frac{1}{s})}.
\]
Hence
\[
||u(., t)||_{L^r(\Omega)} \leq C(\Omega) t^{\frac{a}{2N}(\frac{1}{r}-\frac{1}{s})} ||u_0||_{L^r(\Omega)}.
\]

Next we state the following compactness result that could be seen as the parabolic extension of the result in [22] and which have an interest in itself.

**Theorem 3.5.** Assume that \((f, u_0) \in L^1(\Omega_T) \times L^1(\Omega)\). Let \( u \) be the unique solution to problem (3.1), then for all \( q \leq \frac{N+2}{N+1} \),
\[
||u||_{C([0, T]; L^q(\Omega_T))} + ||\nabla u||_{L^q(\Omega_T)} \leq C(q, \Omega_T) \left( ||f||_{L^1(\Omega_T)} + ||u_0||_{L^1(\Omega)} \right).
\]
Moreover, for \( q \leq \frac{N+2}{N+1} \) fixed, setting \( \tilde{K} : L^1(\Omega_T) \times L^1(\Omega) \to L^q(0, T; W_0^{1,q}(\Omega)) \), \( \tilde{K}(f, u_0) = u \), the unique solution to problem (3.1), then \( \tilde{K} \) is a compact operator.

**Proof.** Without loss of generality we can assume that the data \( u_0, f \) are nonnegative, since thanks to the linearity of the operator the general case can be obtained by decomposing the datum into its positive and negative parts and then dealing with two data separately.

Since \((u_0, f) \in L^1(\Omega) \times L^1(\Omega_T)\), then \( u \in L^m(\Omega_T) \) for all \( m \leq \frac{N+2}{N+1} \) (see [41]). From the representation formula, setting \( \Omega_t = \Omega \times (0, t) \) with \( t < T \), we get
\[
u(x, t) = \int_{\Omega} u_0(y)P_{\Omega}(x, y, t) \ dy + \int_{\Omega} f(y, \sigma)P_{\Omega}(x, y, t - \sigma) \ dy \ d\sigma.
\]
Hence
\[
|\nabla u(x, t)|^q \leq C(\Omega_T) \left( \int_{\Omega} u_0(y)|\nabla_x P_{\Omega}(x, y, t)| \ dy + \int_{\Omega} f(y, \sigma)|\nabla_x P_{\Omega}(x, y, t - \sigma)| \ dy \ d\sigma \right)^q
\]
\[
\leq C(\Omega_T) \left( \int_{\Omega} u_0(y) \left( \frac{\nabla_x P_{\Omega}(x, y, t)}{P_{\Omega}(x, y, t)} \right) \ P_{\Omega}(x, y, t) \ dy \right)
\]
\[+ \int_{\Omega} f(y, \sigma) \left( \frac{\nabla_x P_{\Omega}(x, y, t - \sigma)}{P_{\Omega}(x, y, t - \sigma)} \right) \ P_{\Omega}(x, y, t - \sigma) \ dy \ d\sigma \right)^q
\]
\[
\leq C(\Omega_T) \left( \int_{\Omega} u_0(y) h(x, y, t) P_{\Omega}(x, y, t) \ dy \right)^q
\]
\[+ C(\Omega_T) \left( \int_{\Omega} f(y, \sigma) h(x, y, t - \sigma) P_{\Omega}(x, y, t - \sigma) \ dy \ d\sigma \right)^q
\]
\[= J_1(x, t) + J_2(x, t)
\]
with \( h(x, y, t) = \frac{\nabla_x P_{\Omega}(x, y, t)}{P_{\Omega}(x, y, t)} \leq C \left( \frac{1}{\delta(x) \land t^{\frac{N}{2}}}. \right) \).
To estimate $J_1$, we decompose the integral as follows

$$J_1(x, t) = C \left( \int_{\Omega} u_0(y) h(x, y, t) P_\Omega(x, y, t) \, dy \right)^q$$

$$\leq C \left( \int_{\{x \in \Omega | (x,y) > \frac{t}{2} \}} u_0(y) h(x, y, t) P_\Omega(x, y, t) \, dy \right)^q$$

$$+ \left( \int_{\{x \in \Omega | (x,y) \leq \frac{t}{2} \}} u_0(y) h(x, y, t) P_\Omega(x, y, t) \, dy \right)^q$$

$$\leq \frac{C}{t^\frac{q}{2}} \left( \int_{\{x \in \Omega | (x,y) > \frac{t}{2} \}} u_0(y) P_\Omega(x, y, t) \, dy \right)^q + \frac{C}{\delta^q(x)} \left( \int_{\{x \in \Omega | (x,y) \leq \frac{t}{2} \}} u_0(y) P_\Omega(x, y, t) \, dy \right)^q$$

$$= J_{11}(x, t) + J_{12}(x, t).$$

By estimate (3.2) and Hölder inequality, we obtain that

$$J_{11}(x, t) = \frac{C}{t^\frac{q}{2}} \left( \int_{\{x \in \Omega | (x,y) > \frac{t}{2} \}} u_0(y) P_\Omega(x, y, t) \, dy \right)^q$$

$$\leq C \left( \int_{\{x \in \Omega | (x,y) > \frac{t}{2} \}} u_0(y) \frac{t_1^\frac{1}{2}}{(t^\frac{1}{2} + |x-y|)^{N+2s}} \, dy \right)^q$$

$$\leq C ||u_0||_{L^\frac{q}{2}(\Omega)} \int_{\Omega} u_0(y) \frac{t_1^\frac{1}{2} \gamma y}{(t^\frac{1}{2} + |x-y|)^{q(N+2s)}} \, dy.$$
Thus
\[
\iint_{\Omega_T} J_{12}(x,t) dx dt \leq C \|u_0\|^q_{L^1(\Omega)} \int_{\Omega} u_0(y) \left( \int_{\Omega} \frac{1}{(\delta(x))^{q(\ell + 1) - N}} \left( \int_0^T P_{\Omega}(x,y,t) dt \right) dx \right) dy.
\]
Recall that, from (3.4),
\[
\int_0^\infty P_{\Omega}(x,y,t) dt = G_s(x,y),
\]
the Green function of the fractional Laplacian in \(\Omega\). Then
\[
\iint_{\Omega_T} J_{12}(x,t) dx dt \leq C \|u_0\|^q_{L^1(\Omega)} \int_{\Omega} u_0(y) \left( \int_{\Omega} \frac{G_s(x,y)}{(\delta(x))^{q(\ell + 1) - N}} dx \right) dy.
\]
Let \(\varphi(x) := \int \frac{G_s(x,y)}{\delta(q(\ell + 1) - N)} dx\), then \(\varphi\) is the unique solution to the problem
\[
\begin{aligned}
(\Delta)^{\frac{q}{2}} \varphi &= \frac{1}{(\delta(x))^{q(\ell + 1) - N}} \quad \text{in } \Omega, \\
\varphi &= 0 \quad \text{in } (\mathbb{R}^N \setminus \Omega).
\end{aligned}
\]
(3.13)

Since \(q(\ell + 1) - N < 2s\), from [5], see also [9], it follows that \(\varphi \in L^\infty(\Omega)\). Hence we conclude that
\[
\iint_{\Omega_T} J_{12}(x,t) dx dt \leq C \|u_0\|^q_{L^1(\Omega)}.
\]
Combing the above estimate we deduce that
\[
\iint_{\Omega_T} J_1(x,t) dx dt \leq C \|u_0\|^q_{L^1(\Omega)}.
\]
We treat now \(J_2\). As in the previous estimates, we have
\[
J_2(x,t) = \left( \iint_{\Omega \times (0,t]} f(y,\sigma) h(x,y,t-\sigma) P_{\Omega}(x,y,t-\sigma) dy d\sigma \right)^q
\]
\[
= \left( \iint_{\Omega \times (0,t] \cap \{\delta(x) > (t-\sigma)^{\frac{1}{2s}}\}} f(y,\sigma) h(x,y,t-\sigma) P_{\Omega}(x,y,t-\sigma) dy d\sigma \right)^q
\]
\[
+ \left( \iint_{\Omega \times (0,t] \cap \{\delta(x) \leq (t-\sigma)^{\frac{1}{2s}}\}} f(y,\sigma) h(x,y,t-\sigma) P_{\Omega}(x,y,t-\sigma) dy d\sigma \right)^q
\]
\[
\leq C \left( \left( \iint_{\Omega \times (0,t] \cap \{\delta(x) > (t-\sigma)^{\frac{1}{2s}}\}} f(y,\sigma) \frac{P_{\Omega}(x,y,t-\sigma)}{(t-\sigma)^{\frac{1}{2s}}} dy d\sigma \right)^q
\]
\[
+ \frac{C}{\delta(\sigma)} \left( \int_{\Omega \times (0,t] \cap \{\delta(x) \leq (t-\sigma)^{\frac{1}{2s}}\}} f(y,\sigma) P_{\Omega}(x,y,t-\sigma) dy d\sigma \right)^q
\]
\[
= J_{21}(x,t) + J_{22}(x,t).
\]
Using estimate (3.2) and by Hölder inequality, we get
\[
J_{21}(x,t) = \left( \iint_{\Omega \times (0,t] \cap \{\delta(x) > (t-\sigma)^{\frac{1}{2s}}\}} f(y,\sigma) \frac{(t-\sigma)^{1 - \frac{1}{2s}}}{\left((t-\sigma)^{\frac{1}{2s}} + |x-y|\right)^N + 2s} dy d\sigma \right)^q
\]
\[
\leq C \|f\|^q_{L^1(\Omega_T)} \int_{\Omega \times (0,t]} f(y,\sigma) \frac{(t-\sigma)^{q(1 - \frac{1}{2s})}}{((t-\sigma)^{\frac{1}{2s}} + |x-y|)^{q(N + 2s)}} dy d\sigma.
\]
Integrating in \(\Omega_T\), following closely the computations used in the estimate of the term \(J_{11}\), we deduce that
\[
\iint_{\Omega_T} J_{21}(x,t) dx dt \leq CT^{\gamma_1 + 2} \|f\|^q_{L^1(\Omega_T)}
\]
with \(\gamma_1 = \frac{N}{2s} - q \frac{N + 1}{2s} > -1\).
Now respect to $J_{22}$, using estimate (3.11) and by Hölder inequality, it follows that

$$
J_{22}(x, t) \leq \frac{C ||f||_{L^1(T)}^{q-1}}{\delta^q(x)} \int_{\Omega \times (0,t)} f(y, \sigma) P_1(x, y, t - \sigma) \, dy \, d\sigma
$$

$$
\leq C ||f||_{L^1(T)}^{q-1} \int_{\Omega \times (0,t)} \frac{f(y, \sigma)}{(\delta(x))^{q(N-1)}} P_1(x, y, t - \sigma) \, dy \, d\sigma
$$

(3.14)

Integrating in $\Omega_T$ as above, there results that

$$
\int_{\Omega_T} J_{22}(x, t) \, dx \, dt \leq C ||f||_{L^1(T)}^{q-1} \int_{\Omega_T} f(y, \sigma) \left( \int_{\Omega} \frac{1}{(\delta(x))^{q(N-1)}} \int_{\sigma}^{T} P_1(x, y, t - \sigma) \, dt \right) \, dy \, d\sigma.
$$

It is clear that, from (3.4),

$$
\int_{\sigma}^{T} P_1(x, y, t - \sigma) \, dt = \int_{0}^{T - \sigma} P_1(x, y, \eta) \, d\eta \leq G_s(x, y).
$$

Thus

$$
\int_{\Omega_T} J_{22}(x, t) \, dx \, dt \leq C ||f||_{L^1(T)}^{q-1} \int_{\Omega_T} f(y, \sigma) \, \varphi(y) \, dy,
$$

where $\varphi$ is the unique solution to problem (3.13). Since $q(N + 1) - N < 2s$, we have that $\varphi \in L^\infty(\Omega)$ and then

$$
\int_{\Omega_T} J_{22}(x, t) \, dx \, dt \leq C ||f||_{L^1(T)}^{q-1}.
$$

Hence

$$
\int_{\Omega_T} J_{2}(x, t) \, dx \, dt \leq C ||f||_{L^1(T)}^{q}.
$$

Therefore we conclude that

$$
\left( \int_{\Omega_T} |\nabla u(x, t)|^q \, dx \, dt \right)^{\frac{1}{q}} \leq \left( \int_{\Omega_T} \left( J_1(x, t) + J_2(x, t) \right) \, dx \, dt \right)^{\frac{1}{q}}
$$

(3.15)

$$
\leq C(\Omega, T) \left( \|f\|_{L^1(\Omega_T)} + \|u_0\|_{L^1(\Omega)} \right).
$$

Fixed $q_0 < \frac{N + 2s}{N - 1}$, we define

$$
\hat{K} : L^1(\Omega_T) \times L^1(\Omega) \rightarrow L^0(0, T; W^{1, \phi}_0(\Omega)),
$$

by $\hat{K}(f, u_0) = u$, where $u$ is the unique solution to (3.1). From (3.15) we deduce that $\hat{K}$ is well defined and continuous. Let show that $\hat{K}$ is a compact operator.

Denote $w$ and $\tilde{w}$ the unique solutions to the problems

$$
\begin{cases}
  w_t + (-\Delta)^s w = 0 & \text{in } \Omega_T, \\
  w = 0 & \text{in } (\mathbb{R}^N \setminus \Omega) \times (0, T), \\
  w(x, 0) = u_0 & \text{in } \Omega,
\end{cases}
$$

(3.16)

and

$$
\begin{cases}
  \tilde{w}_t + (-\Delta)^s \tilde{w} = f & \text{in } \Omega_T, \\
  \tilde{w} = 0 & \text{in } (\mathbb{R}^N \setminus \Omega) \times (0, T), \\
  \tilde{w}(x, 0) = 0 & \text{in } \Omega,
\end{cases}
$$

(3.17)

respectively. It is clear that $w + \tilde{w} = u$.

From Proposition 3.4, we know that $w \in L^m(\Omega_T)$ for all $m < \frac{N + 2s}{N}$ and for all $\theta > 1$,

$$
t^{\frac{\theta}{2}(1 - \frac{1}{\theta})} \|w(\cdot, t)\|_{L^\theta} \leq C \|u_0\|_{L^1(\Omega)}.
$$

Then

$$
\sup_{t \in [0, T]} t^{\frac{\theta}{2}(\theta - 1)} \int_\Omega w^\theta(x, t) \, dx \leq C(\Omega_T) \|u_0\|^\theta_{L^1(\Omega)}.
$$

(3.18)
Now, going back to the definition of $J_{11}$, we get

$$ J_{11}(x,t) = \frac{C}{t^\frac{q}{2s}} \left( \int_{\{x\geq t^{\frac{1}{2s}}\}} u_0(y)P_\Omega(x,y,t) \, dy \right)^q \leq \frac{C}{t^\frac{q}{2s}} w^q(x,t). $$

Choosing $\theta = q$ in (3.18),

$$ \int_{\Omega_T} J_{11}(x,t) \, dx \, dt \leq C \int_0^T \int_{\Omega_T} \left( \frac{t^{-\frac{N(q-1)}{2s}}}{\int \frac{\int_{\Omega_T} w^q(x,t) t^{-\frac{N(q-1)}{2s}} \, dx \, dt}{t^\frac{q}{2s}} \right) \, dt. $$

By hypothesis $q < \frac{N+2s}{N+1}$, that is, $\frac{q}{2s} + \frac{N(q-1)}{2s} < 1$, then since

$$ \left( \frac{\int_{\Omega_T} w^q(x,t) t^{-\frac{N(q-1)}{2s}} \, dx \, dt}{t^\frac{q}{2s}} \right) \in L^\infty(0,T), $$

there exists $a > 1$ such that $a\left(\frac{q}{2s} + \frac{N(q-1)}{2s}\right) < 1$ and

$$ \int_{\Omega_T} J_{11}(x,t) \, dx \, dt \leq C \left( \int_0^T \left( \int_{\Omega_T} w^q(x,t) t^{-\frac{N(q-1)}{2s}} \, dx \, dt \right) \right) \leq C \int_0^T \left( \int_{\Omega_T} w^q(x,t) t^{-\frac{N(q-1)}{2s}} \, dx \, dt \right) \, dt. $$

Respect to the term $J_{21}$, we have

$$ J_{21}(x,t) = C \left( \int \int_{\Omega_T \cap \{x>(t-\sigma)^{\frac{1}{2s}}\}} f(y,\sigma) \frac{P_\Omega(x,y,t-\sigma)}{(t-\sigma)^{\frac{N}{2s}}} \, dy \, d\sigma \right)^q \leq C \left( \int \int_{\Omega_T \cap \{x>(t-\sigma)^{\frac{1}{2s}}\}} \frac{f(y,\sigma) P_\Omega(x,y,t-\sigma)}{(t-\sigma)^{\frac{N}{2s}}} \, dy \, d\sigma \right) \leq C a^{q-1}(x,t) \int_{\Omega_T \cap \{x>(t-\sigma)^{\frac{1}{2s}}\}} f(y,\sigma) \frac{P_\Omega(x,y,t-\sigma)}{(t-\sigma)^{\frac{N}{2s}}} \, dy \, d\sigma. $$

Thus integrating in $\Omega_T$ and by estimate (3.2), we obtain that

$$ \int_{\Omega_T} J_{21}(x,t) \, dx \, dt \leq C \int_{\Omega_T} f(y,\sigma) \left( \int_{\Omega_T \cap \{x>(t-\sigma)^{\frac{1}{2s}}\}} \tilde{w}(x,t)^{q-1} \frac{(t-\sigma)^{\frac{N}{2s}}}{(t-\sigma)^{\frac{N}{2s}} + |x-y|^{N+2s}} \, dy \, d\sigma \right) \, dy \, d\sigma. $$

Since $q < \frac{N+2s}{N+1}$ and using the fact that $s > \frac{1}{2}$, we get the existence of $\frac{2s(q-1)}{N+1} < r < \frac{N+2s}{N}$ such that $q < \frac{N+2s+2r}{N+2s+r}$. Hence $r > q-1$ and

$$ \frac{r}{r-(q-1)} \left( \frac{2s-q}{2s} - \frac{N+2s}{2s} \right) + \frac{N}{2s} > -1. $$

Therefore, using Hölder inequality

$$ \int_{\Omega_T} J_{21}(x,t) \, dx \, dt \leq C \int_{\Omega_T} f(y,\sigma) \int_{\Omega_T} \frac{\tilde{w}(x,t)^{q-1}}{(t-\sigma)^{\frac{N}{2s}} + |x-y|^{N+2s}} \, dy \, d\sigma. $$

Setting $z = \frac{x-y}{(t-\sigma)^{\frac{1}{2s}}}$, it holds that

$$ \int_{\Omega_T} J_{21}(x,t) \, dx \, dt \leq C ||\tilde{w}||_{L^r(\Omega_T)}^{q-1} \int_{\Omega_T} f(y,\sigma) \left( \int_{\sigma}^T (t-\sigma)^{\gamma_2} \int_{R^N} \frac{1}{(1+|z|)^{(N+2s)\frac{r}{r-(q-1)}}} \, dz \, dt \right)^{\frac{r(q-1)}{2}} \, dy \, d\sigma, $$
where $\gamma_2 = \frac{r}{r-(q-1)} \left( \frac{2s-q}{2s} - \frac{N+2s}{2s} \right) + \frac{N}{2s}$. Since $\gamma_2 > -1$, then

$$\int_{\Omega_T} J_{21}(x,t) dx dt \leq CT^{\frac{N-1}{N+1}} ||f||_{L^1(\Omega_T)} ||\tilde{u}||_{L^q(\Omega_T)}^{q-1}.$$  \hspace{1cm} (3.20)

Respect to the terms $J_{12}$ and $J_{22}$ we have that

$$J_{12}(x,t) = \frac{C}{\delta^3(x)} \left( \int_{\{x: \beta(x) < \epsilon(x) \}} u_0(y) P_{\Omega}(x,y,t) dy \right)^q = C \frac{u^q(x,t)}{\delta^q(x)},$$  \hspace{1cm} (3.21)

and

$$J_{22}(x,t) = \frac{C}{\delta^3(x)} \left( \int_{\{(x,t) \cap \{ \beta(x) < \epsilon(x) \} \}} f(y,\sigma) P_{\Omega}(x,y,t-\sigma) dy d\sigma \right)^q = C \frac{\tilde{u}^q(x,t)}{\delta^q(x)}.$$  \hspace{1cm} (3.22)

Let $\{f_n,u_{n0}\}$ be a bounded sequence in $L^1(\Omega_T) \times L^1(\Omega)$ and define $u_n = \tilde{K}(f_n,u_{n0})$. Using the previous estimates, it follows that, for all $q < \frac{N+2s}{N+1}$,

$$||\nabla u_n||_{L^q(\Omega_T)} \leq C(q,\Omega)(||f_n||_{L^1(\Omega_T)} + ||u_{n0}||_{L^1(\Omega)} \leq C.$$ 

Hence, there exists $u \in L^q(0,T;W_0^{1,q}(\Omega))$ for all $q < \frac{N+2s}{N+1}$ such that, up to a subsequence, $u_n \rightharpoonup u$ weakly in $L^q(0,T;W_0^{1,q}(\Omega))$, $u_n \rightarrow u$ strongly in $L^q(\Omega_T)$ for all $\sigma < \frac{N+2s}{N}$ and $u_n \rightarrow u$ a.e in $\Omega_T$. Fixing the above subsequence, we define $(w_n,\tilde{w}_n)$ be the solutions to problems (3.16) and (3.17) with $u_{n0}$ and $f_n$ respectively. Then the sequences $\{w_n^g(x,t)\}_{n}, \{w_n\}_{n}$ and $\{\tilde{w}_n\}_{n}$ are bounded in $L^\infty((0,T);L^q(\Omega))$, $L^q(\Omega_T)$ and in $L^q(\Omega_T)$ for all $\sigma < \frac{N+2s}{N+1}$ and $r < \frac{N+2s}{N}$ respectively. Hence using Vitali’s lemma we deduce that the sequences $\{w_n^g(x,t)\}_{n}, \{w_n\}_{n}$ and $\{\tilde{w}_n\}_{n}$ are strongly converging in $L^q((0,T);L^q(\Omega))$, in $L^q(\Omega_T)$ and in $L^q(\Omega_T)$ for all $\sigma < \frac{N+2s}{N+1}$, for all $r < \frac{N+2s}{N}$ and for all $a > 1$.

By the linearity of the operator, it follows that $(u_i - u_j)$ solves

$$\begin{cases}
(u_i - u_j)_t + (-\Delta)^a (u_i - u_j) = f_i - f_j & \text{in } \Omega_T, \\
u_i - u_j = 0 & \text{in } (\mathbb{R}^N \setminus \Omega) \times (0,T), \\
(u_i - u_j)(x,0) = u_{i0} - u_{j0} & \text{in } \Omega.
\end{cases}$$  \hspace{1cm} (3.23)

Going back to the first formula in (3.15) and by estimates (3.19), (3.20), (3.21), (3.22), we get

$$\left( \int_{\Omega_T} |\nabla(u_i - u_j)(x,t)|^q dx dt \right)^{\frac{1}{q}} \leq C(\Omega,T) \left( \int_0^T \left( \int_{\Omega} |w_i(x,t) - w_j(x,t)|^q \frac{N+2s-Nq}{2s} dx \right)^{\frac{(q-1)}{q}} dt \right)^{\frac{1}{q}} + ||f_i - f_j||_{L^1(\Omega_T)} ||\tilde{w}_i - \tilde{w}_j||_{L^{q-1}(\Omega_T)}$$

$$+ ||\frac{w_i - w_j}{\delta}||_{L^q(\Omega_T)} + ||\frac{\tilde{w}_i - \tilde{w}_j}{\delta}||_{L^q(\Omega_T)}.$$

Letting $i,j \rightarrow \infty$, it holds that

$$\left( \int_{\Omega_T} |\nabla(u_i - u_j)(x,t)|^q dx dt \right)^{\frac{1}{q}} \rightarrow 0.$$

Then the operator $\tilde{K}$ is compact. \hspace{1cm} \Box

**Remark 3.6.**

(1) Thanks to the above computations, we can prove that the constant $C(T,\Omega)$, that appears in estimate (3.15) satisfies $C(T,\Omega) \rightarrow 0$ as $T \rightarrow 0$. This fact will be used below in order to show existence result for problem (1.1) using a Fixed Point Theorem.
(2) Using an approximation argument and by the linearity of the operator, we can prove that the result of Theorem 3.5 holds if \( f \) is a bounded Radon measure.

(3) It is worthy to point out that the same arguments give the elliptic case by a slightly different method as in [22].

Following the same representation argument as above we get the next technical regularity result.

**Proposition 3.7.** Assume that \( (f, u_0) \in L^1(\Omega_T) \times L^1(\Omega) \) and let \( u \) be the unique solution to the problem (3.1), then \( T_k(u) \in L^2(0, T; W_{0, \sigma}^1(\Omega)) \) for all \( 1 \leq \sigma < 2s \). Moreover we have

\[
\|T_k(u)\|_{L^2(0, T; W_{0, \sigma}^1(\Omega))} \leq C(\Omega, T)k^{\sigma^{-1}}(\|u_0\|_{L^1(\Omega)} + \|f\|_{L^1(\Omega_T)}).
\]

In addition, if \( u_n = \bar{K}(f_n, u_{n0}) \), then, up to a subsequence, it follows that \( T_k(u_n) \to T_k(u) \) strongly in \( L^2(0, T; W_{0, \sigma}^1(\Omega)) \), for all \( 1 \leq \sigma < 2s \).

**Proof.** Without loss of generality we can assume that \( f \geq 0 \) in \( \Omega_T \) and \( u_0 \geq 0 \) in \( \Omega \). Fix \( 1 < \rho < 2s \), as in the proof of Theorem 3.5, we have

\[
|\nabla u(x, t)|^\rho \leq C \left( \int_\Omega u_0(y) P_\Omega(x, y, t) \, dy \right) \rho \left( \int_\Omega h^\rho(x, y, t)|u_0(y)| P_\Omega(x, y, t) \, dy \right)^{\frac{\rho}{\rho - 1}}
\]

\[
+ C \left( \int_\Omega \int_{\Omega_T} f(y, \sigma) P_\Omega(x, y, t - \sigma) \, dy \, d\sigma \right)^{\frac{\rho}{\rho - 1}} \left( \int_\Omega \int_{\Omega_T} h^\rho(x, y, t - \sigma) f(y, \sigma) P_\Omega(x, y, t - \sigma) \, dy \, d\sigma \right)^{\frac{\rho - 1}{\rho}}
\]

\[
\leq C u^{\frac{\rho}{\rho - 1}}(x, t) \left( \int_\Omega h^\rho(x, y, t)|u_0(y)| P_\Omega(x, y, t) \, dy \right) + \left( \int_\Omega \int_{\Omega_T} h^\rho(x, y, t - \sigma) f(y, \sigma) P_\Omega(x, y, t - \sigma) \, dy \, d\sigma \right) \chi_{(u < k)}.
\]

Hence

\[
\int_{\Omega_T} |\nabla T_k(u(x, t))|^\rho \, dx \, dt 
\]

\[
\leq C(\Omega, T)k^{\rho^{-1}} \left( \int_\Omega u_0(y) \left( \int_{\Omega_T} h^\rho(x, y, t) P_\Omega(x, y, t) \, dx \, dt \right) \, dy \right) \rho
\]

\[
+ \int_{\Omega_T} f(y, \sigma) \left( \int_\sigma^T \int_\Omega h^\rho(x, y, t - \sigma) P_\Omega(x, y, t - \sigma) \, dx \, dt \right) \, dy \, d\sigma
\]

\[
\leq Ck^{\rho^{-1}}(J_1 + J_2).
\]

Recall that \( h(x, y, t) \leq C \left( \frac{1}{\delta(x) \wedge t^{\frac{\rho}{\rho - 1}}} \right) \), hence using the fact that \( \rho < 2s \), we reach that

\[
J_1 = \int_\Omega u_0(y) \left( \int_{\{\Omega_T \cap \{\delta(x) \leq t^{\frac{\rho}{\rho - 1}}\}} h^\rho P_\Omega(x, y, t) \, dx \, dt \right) \, dy
\]

\[
+ \int_\Omega u_0(y) \left( \int_{\{\Omega_T \cap \{\delta(x) < t^{\frac{\rho}{\rho - 1}}\}} h^\rho(x, y, t) P_\Omega(x, y, t) \, dx \, dt \right) \, dy
\]

\[
= I_1 + I_2.
\]

Now, by Lemma 3.1, it holds that

\[
I_1 \leq C \int_\Omega u_0(y) \left( \int_{\Omega_T} \frac{1}{t^{\frac{\rho}{\rho - 1}}} \frac{t}{(t^{\frac{\rho}{\rho - 1}} + |x - y|)^{N + 2s}} \, dx \, dy \right)
\]

Setting, \( z = \frac{x - y}{t^{\frac{\rho}{\rho - 1}}} \), it follows that

\[
I_1 \leq C\|u_0\|_{L^1(\Omega)} \int_0^T t^{1 - \frac{\rho}{\rho - 1} - \frac{2s}{N + 2s}} dt \int_{\mathbb{R}^N} \frac{dz}{(1 + |z|)^{(N + 2s)}}
\]

\[
\leq C\|u_0\|_{L^1(\Omega)} \int_0^T t^{\frac{\rho}{\rho - 1}} dt.
\]
Since $\frac{m}{N} < 1$, then

$$I_1 \leq CT^{1-\frac{m}{N}}||u_0||_{L^1(\Omega)}.$$ 

We deal now with $I_2$. We have

$$I_2 \leq C \int_\Omega u_0(y) \int_{\Omega_T} \frac{1}{\theta^\rho(x)} P_\Omega(x, y, t) \, dx \, dt \, dy$$

$$\leq C \int_\Omega u_0(y) \int_{\Omega_T} \frac{1}{\theta^\rho(x)} \left( \int_0^\infty P_\Omega(x, y, t) \, dt \right) \, dx \, dy.$$ 

Recall that

$$\int_0^\infty P_\Omega(x, y, t) \, dt = G_\sigma(x, y),$$

then

$$I_2 \leq C \int_\Omega u_0(y) \int_{\Omega_T} \frac{1}{\theta^\rho(x)} G_\sigma(x, y) \, dx \, dy = C \int_\Omega u_0(y) \varphi(y) \, dy,$$

where $\varphi$ is the solution to problem (3.13). Since $\rho < 2s$, then $\varphi \in L^\infty(\Omega)$, hence

$$I_2 \leq C||u_0||_{L^1(\Omega)}.$$ 

Hence we conclude that $J_1 \leq C(\Omega, T)||u_0||_{L^1(\Omega)}$.

In the same way we obtain that

$$J_2 \leq C(\Omega, T)||f||_{L^1(\Omega_T)}.$$ 

Therefore

$$\int_{\Omega_T} |\nabla T_k(u(x, t))|^q \, dx \, dt \leq C(\Omega, T) k^{\sigma-1} (||u_0||_{L^1(\Omega)} + ||f||_{L^1(\Omega_T)}).$$ 

Define $u_n = \hat{K}(f_n, u_{n0})$, then according with the proof of Theorem 3.5, we know that, up to a subsequence, $u_n \rightharpoonup u$ strongly in $L^q(0, T; W^{1,q}_0(\Omega))$ for all $q < \frac{N + 2s}{N + 1}$. Now, using the fact that the sequence $\{T_k(u_n)\}_n$ is bounded in $L^\sigma(0, T; W^{1,\sigma}_0(\Omega))$ for all $1 \leq \sigma < 2s$ and using Vitali’s Lemma, it follows that $T_k(u_n) \rightarrow T_k(u)$ strongly in $L^\sigma(0, T; W^{1,\sigma}_0(\Omega))$. \hfill \Box

**Remark 3.8.** If $u_0 = 0$ and $f \in L^m(\Omega_T)$ with $m > 1$, we can improve the regularity results obtained previously. Notice that, related to the fractional Laplacian, a kind of regularity holds in the local gradient where we are close to the boundary. This can be seen in the explicit example $w(x) = (1 - |x|^2)^{\gamma}$, which solves

$$(-\Delta)^{\gamma} w = 1 \text{ in } B_1(0) \text{ and } w = 0 \text{ in } \mathbb{R}^N \setminus B_1(0).$$

However, as it was observed in [5], we can show a complete regularity schema using in a suitable weighted Sobolev space. The main tool will be a universal control of the term $\frac{u}{\delta^\rho}$ that holds for any $s \in (0, 1)$ and without using the classical Hardy-Sobolev inequality.

Before considering the case $m > 1$, we enunciate the next regularity result that will be used through the paper and that clarifies the regularity of the solution in the space for fixed time.

**Proposition 3.9.** Suppose that $f \in L^1(\Omega_T)$ and $u_0 = 0$. Let $u$ be the unique weak solution to problem (3.1), then for all $1 < q < \frac{N + 2s}{N + 1}$ and for all $\eta > 0$,

$$\int_\Omega |\nabla u(x, t)|^{q} \, dx \leq C(\Omega_T)||f||_{L^1(\Omega_T)} \int_0^t \int_\Omega |f(y, \sigma)|(t - \sigma)^{\frac{\gamma}{q} - \eta} \, dy \, d\sigma,$$

where $\hat{\gamma} := \frac{N}{2s} - q \frac{N + 1}{2s} \in (-1, 0)$. In particular, we obtain that for all $\eta > 0$,

$$\left( \int_\Omega |\nabla u(x, t)|^{q} \, dx \right)^{\frac{1}{q}} \leq C(\Omega_T) \left( \int_0^t \int_\Omega |f(y, \sigma)|(t - \sigma)^{\frac{\gamma}{q} - \eta} \, dy \, d\sigma \right)^{\frac{1}{\hat{\gamma}}}.$$
Proof. From Theorem 3.5, we know that \( u \in L^q((0, T), W_0^{1,q}(\Omega)) \) for all \( q < \frac{N+2s}{N+1} \). Then we have

\[
u(x, t) = \int_0^t \int_{\Omega} f(y, \sigma) P_\Omega(x, y, t - \sigma) \, dy \, d\sigma.
\]

Hence

\[
|\nabla u(x, t)| \leq \int_0^t \int_{\Omega} |f(y, \sigma)| \frac{|\nabla_x P_\Omega(x, y, t - \sigma)|}{P_\Omega(x, y, t - \sigma)} P_\Omega(x, y, t - \sigma) \, dy \, d\sigma
\]

\[
\leq C \left( \int_{\Omega \times (0, t) \cap \{\delta(x) > (t - \sigma)^{\frac{1}{N}}\}} |f(y, \sigma)| P_\Omega(x, y, t - \sigma) \, (t - \sigma)^{1 - \frac{1}{N}} dy \, d\sigma \right)^q + \left( \int_{\Omega \times (0, t) \cap \{\delta(x) \leq (t - \sigma)^{\frac{1}{N}}\}} |f(y, \sigma)| P_\Omega(x, y, t - \sigma) \, (t - \sigma)^{q(1 - \frac{1}{N})} dy \, d\sigma \right)^q.
\]

Therefore, fixing \( 1 < q < \frac{N+2s}{N+1} \), we reach that

\[
|\nabla u(x, t)|^q \leq C \left( \int_{\Omega \times (0, t) \cap \{\delta(x) > (t - \sigma)^{\frac{1}{N}}\}} |f(y, \sigma)| P_\Omega(x, y, t - \sigma) \, (t - \sigma)^{1 - \frac{1}{N}} dy \, d\sigma \right)^q + \left( \int_{\Omega \times (0, t) \cap \{\delta(x) \leq (t - \sigma)^{\frac{1}{N}}\}} |f(y, \sigma)| P_\Omega(x, y, t - \sigma) \, (t - \sigma)^{q(1 - \frac{1}{N})} dy \, d\sigma \right)^q
\]

(3.27)

Using estimate (3.6) and by Hölder inequality, we get

\[
I_1(x, t) = \left( \int_{\Omega \times (0, t) \cap \{\delta(x) > (t - \sigma)^{\frac{1}{N}}\}} |f(y, \sigma)| \frac{(t - \sigma)^{1 - \frac{1}{N}}}{(t - \sigma)^{\frac{1}{N}} + |x - y|^N} dy \, d\sigma \right)^q
\]

\[
\leq C ||f||_{L^1(\Omega)}^{q-1} \int_{\Omega \times (0, t)} |f(y, \sigma)| \frac{(t - \sigma)^{q(1 - \frac{1}{N})}}{(t - \sigma)^{\frac{1}{N}} + |x - y|^{q(N+2s)}} dy \, d\sigma.
\]

Integrating in \( \Omega \), following the same change of variable as in the proof of Theorem 3.5,

\[
\int_\Omega I_1(x, t) \, dx \leq C ||f||_{L^1(\Omega)}^{q-1} \int_0^t \int_\Omega |f(y, \sigma)| \frac{(t - \sigma)^{q(1 - \frac{1}{N})}}{(t - \sigma)^{\frac{1}{N}} + |x - y|^{q(N+2s)}} dx \, dy \, d\sigma
\]

\[
\leq C ||f||_{L^1(\Omega)}^{q-1} \int_0^t \int_\Omega |f(y, \sigma)| \, (t - \sigma)^{1 - \frac{1}{N}} \, dy \, d\sigma.
\]

Setting \( \gamma := \frac{N}{2s} - q \frac{N+1}{2s} \in (-1, 0) \), then

\[
\int_\Omega I_1(x, t) \, dx \leq C ||f||_{L^1(\Omega)}^{q-1} \int_0^t \int_\Omega |f(y, \sigma)| \, (t - \sigma)^\gamma \, dy \, d\sigma.
\]

(3.28)

We treat now \( I_2 \). As above, we get

\[
I_2(x, t) \leq \frac{C ||f||_{L^1(\Omega)}^{q-1}}{\delta^q(x)} \int_{\Omega \cap \{|x - y| < \frac{1}{2}\delta(x)\}} |f(y, \sigma)| P_\Omega(x, y, t - \sigma) \, dy \, d\sigma
\]

\[
\leq \frac{C ||f||_{L^1(\Omega)}^{q-1}}{\delta^q(x)} \int_{\Omega \cap \{|x - y| < \frac{1}{4}\delta(x)\}} |f(y, \sigma)| P_\Omega(x, y, t - \sigma) \, dy \, d\sigma
\]

\[
+ \frac{C ||f||_{L^1(\Omega)}^{q-1}}{\delta^q(x)} \int_{\Omega \cap \{|x - y| > \frac{1}{4}\delta(x)\}} |f(y, \sigma)| P_\Omega(x, y, t - \sigma) \, dy \, d\sigma = I_{21}(x, t) + I_{22}(x, t).
\]
We begin by estimating $I_{21}$. Notice that $|\delta(y) - \delta(x)| \leq |x - y|$, then $\delta(y) \leq \frac{3}{2} \delta(x)$. Thus $\frac{\delta(y)}{(t - \sigma)^{q}} \leq C(\Omega_{T})$. Hence using (3.6),

$$P_{\Omega}^{q}(x, y, t - \sigma) \leq C(\Omega_{T}) \frac{(t - \sigma)^{q}}{\left(\frac{1}{q} \frac{1}{q} + |x - y|\right)^{q}(N + 2s)}.$$  \hspace{1cm} (3.28)

Since $|x - y| \leq \frac{1}{2} \delta(x)$, then

$$P_{\Omega}^{q}(x, y, t - \sigma) \leq C(\Omega_{T}) \frac{(t - \sigma)^{q}}{|x - y|^{q}(t - \sigma)^{q} + |x - y|^{q}(N + 2s)}.$$  \hspace{1cm} (3.29)

Going back to the definition of $I_{21}$, setting $z = \frac{x - y}{(t - \sigma)^{\frac{1}{q}}}$ and integrating in $\Omega$, there results that

$$\int_{\Omega} I_{21}(x, t)dx \leq C(\Omega_{T}) \left(\int_{\Omega} \left(\frac{1}{\frac{1}{q} + |x - y|}\right)^{\frac{t}{q}} dx \right) dy d\sigma. \hspace{1cm} (3.29')$$

Respect to $I_{22}$, since $\delta(x) \leq 2|x - y|$, we get $\delta(y) \leq C|x - y|$, then, in this case

$$P_{\Omega}^{q}(x, y, t - \sigma) \leq C(\Omega_{T}) \left(1 \leq \frac{\delta(x)}{(t - \sigma)} \right)^{q} \left(1 \leq \frac{\delta(y)}{(t - \sigma)} \right)^{q} \frac{(t - \sigma)^{q}}{(t - \sigma)^{q} + |x - y|}.$$  \hspace{1cm} (3.30)

Since, for all $\theta \in (0, 1)$,

$$\left(1 \leq \frac{\delta(x)}{(t - \sigma)} \right)^{q} \leq \left(1 \leq \frac{\delta(x)}{(t - \sigma)} \right)^{q} \quad \text{and} \quad \left(1 \leq \frac{\delta(y)}{(t - \sigma)} \right)^{q} \leq \left(1 \leq \frac{\delta(y)}{(t - \sigma)} \right)^{q}.$$

Choosing $\theta = \frac{1}{q}$, we deduce that

$$P_{\Omega}^{q}(x, y, t - \sigma) \leq C(\Omega_{T}) \frac{\delta^{q}(x)}{(t - \sigma)^{q}} \leq \frac{\delta^{q}(y)}{(t - \sigma)^{q}} \frac{(t - \sigma)^{q}}{(t - \sigma)^{q} + |x - y|}.$$  \hspace{1cm} (3.31)

Thus, for $\eta \in (0, 1)$ small enough,

$$\frac{P_{\Omega}^{q}(x, y, t - \sigma)}{\delta^{q}(x)} \leq C(\Omega_{T}) \frac{\delta^{q}(x) \delta^{q}(y)}{|x - y|^{N}} \frac{1}{(t - \sigma)^{q}} \frac{(t - \sigma)^{q}}{(t - \sigma)^{q} + |x - y|^{q(1 - \eta)}}.$$  \hspace{1cm} (3.32)

From (3.5), we deduce that, in this case,

$$\mathcal{G}_{n}(x, y) \simeq \frac{\delta^{q}(x) \delta^{q}(y)}{|x - y|^{N}}.$$
Hence
\[ \frac{P_{\Omega}^2(x,y,t-\sigma)}{\delta^q(x)} \leq C(\Omega_T) \frac{G_s(x,y)}{(\delta(x))^{2s(1-\eta)}}(t-\sigma)^{\gamma-\eta}. \]

Therefore
\[
\int_{\Omega} I_{22}(x,t)dx \leq C(\Omega_T) \|f\|^q_{L^q_t(\Omega_t)} \int_0^t \int_{\Omega} |f(y,\sigma)|(t-\sigma)^{\gamma-\eta} \left( \int_{\Omega} \frac{G_s(x,y)}{(\delta(x))^{2s(1-\eta)}}dx \right) dyd\sigma \\
\leq C(\Omega_T) \|f\|^q_{L^q_t(\Omega_t)} \int_0^t \int_{\Omega} |f(y,\sigma)|(t-\sigma)^{\gamma-\eta}\varphi(y)dyd\sigma,
\]

where \( \varphi(y) := \int_{\Omega} \frac{G_s(x,y)}{(\delta(x))^{2s(1-\eta)}}dx \). It is clear that \( \varphi \) solves the problem
\[
\begin{cases}
(-\Delta)^s \varphi = \frac{1}{(\delta(x))^{2s(1-\eta)}} & \text{in } \Omega, \\
\varphi = 0 & \text{in } (\mathbb{R}^N \setminus \Omega).
\end{cases}
\]

Since \( 2s(1-\eta) < 2s \), then \( \varphi \in L^2(\Omega) \). Thus, for all \( \eta > 0 \),
\[
\int_{\Omega} I_{22}(x,t)dx \leq C(\Omega_T) \|f\|^q_{L^q_t(\Omega_t)} \int_0^t \int_{\Omega} |f(y,\sigma)|(t-\sigma)^{\gamma-\eta}dyd\sigma. \tag{3.30}
\]

Combining estimates (3.28),(3.29),(3.30), going back to (3.56) and integrating in \( \Omega \), we conclude that for all \( \eta > 0 \),
\[
\int_{\Omega} |\nabla u(x,t)|^q dx \leq C(\Omega_T) \|f\|^q_{L^q_t(\Omega_t)} \int_0^t \int_{\Omega} |f(y,\sigma)|((t-\sigma)^{\gamma-\eta} + (t-\sigma)^{\gamma})dyd\sigma.
\]

Hence
\[
\int_{\Omega} |\nabla u(x,t)|^q dx \leq C(\Omega_T) \|f\|^q_{L^q_t(\Omega_t)} \int_0^t \int_{\Omega} |f(y,\sigma)|(t-\sigma)^{\gamma-\eta}dyd\sigma.
\]

Now using the fact that
\[
\|f\|_{L^q_t(\Omega_t)} \leq C(\Omega_T) \int_0^t \int_{\Omega} |f(y,\sigma)|(t-\sigma)^{\gamma-\eta}dyd\sigma,
\]

we conclude that
\[
\left( \int_{\Omega} |\nabla u(x,t)|^q dx \right)^\frac{1}{q} \leq C(\Omega_T) \int_0^t \int_{\Omega} |f(y,\sigma)|(t-\sigma)^{\gamma-\eta}dyd\sigma.
\]

\[ \square \]

If \( m > 1 \), we can prove a regularity result in a suitable weighted Sobolev space whose weight is a power of the distance to the boundary.

This result will be a consequence of the next two Theorems.

**Theorem 3.10.** Assume that \( u_0 \equiv 0 \), \( f \in L^m(\Omega_T) \) for some \( m > 1 \) and let \( u \) be the unique weak solution to problem (3.1), then \( \frac{u}{\delta^s} \in L^\theta(\Omega_T) \) for all \( \theta > 1 \) such that \( \frac{1}{\theta} > \frac{1}{m} - \frac{s}{N+2s} \). Moreover,
\[
\left\| \frac{u}{\delta^s} \right\|_{L^\theta(\Omega_T)} \leq C \|f\|_{L^m(\Omega_T)}, \tag{3.31}
\]

with
\[
\begin{cases}
\theta < \infty & \text{if } m \geq \frac{N+2s}{s}, \\
\theta < \frac{m(N+2s)}{N+2s-ms} & \text{if } m < \frac{N+2s}{s}.
\end{cases}
\]
Proof. Without loss of generality, we can assume that \( f \geq 0 \), hence \( u \geq 0 \) in \( \mathbb{R}^N \times (0, T) \). By the representation formula we have that

\[
u(x, t) = \int_0^1 \int_\Omega f(y, \sigma) P_\sigma(x, y, t - \sigma) \, dy \, d\sigma,
\]

then using the properties of \( P_\sigma \), and (3.6) it holds that

\[
\frac{\nu(x, t)}{\delta^s(x)} \leq C \int_0^T \int_\Omega f(y, \sigma) \frac{(t - \sigma)^{\frac{s}{2}}}{((t - \sigma)^{\frac{2}{2}} + |x - y|)^{N+2s}} \, dy \, d\sigma.
\]

As above, consider \( \phi \in \mathcal{C}_0^\infty(\Omega_T) \), then

\[
\left\| \frac{\nu}{\delta^s} \right\|_{L^p(\Omega_T)} = \sup_{\left\| \phi \right\|_{L^p(\Omega_T)} \leq 1} \left\| \int_\Omega \phi(x, t) \frac{\nu(x, t)}{\delta^s(x)} \, dx \right\| \leq C \sup_{\left\| \phi \right\|_{L^p(\Omega_T)} \leq 1} \int_0^T \int_\Omega |\phi(x, t)| \int_0^t \int_\Omega f(y, \sigma) \frac{(t - \sigma)^{\frac{s}{2}}}{((t - \sigma)^{\frac{2}{2}} + |x - y|)^{N+2s}} \, dy \, d\sigma \, dx \, dt,
\]

where

\[
H(|x - y|, \sigma) = \frac{(t - \sigma)^{\frac{s}{2}}}{((t - \sigma)^{\frac{2}{2}} + |x - y|)^{N+2s}}.
\]

Using Young inequality, it holds that

\[
\left\| \frac{\nu}{\delta^s} \right\|_{L^p(\Omega_T)} \leq C \sup_{\left\| \phi \right\|_{L^p(\Omega_T)} \leq 1} \int_0^T \int_\Omega |\phi(x, t)| \left\| f(\cdot, \sigma) \right\|_{L^m(\Omega)} \left\| H(\cdot, t - \sigma) \right\|_{L^\gamma(\Omega)} \, dx \, dt,
\]

with \( \frac{1}{\theta} + \frac{1}{m} + \frac{1}{a} = 2 \).

As in the computation of the term defined in (3.6) and by (3.10), we get

\[
\left\| H(\cdot, t - \sigma) \right\|_{L^\gamma(\Omega)} \leq C(t - \sigma)^{-\frac{1}{2} + \frac{N}{2\gamma} - \frac{N}{2s}}.
\]

Substituting in (3.32),

\[
\left\| \frac{\nu}{\delta^s} \right\|_{L^p(\Omega_T)} \leq C \sup_{\left\| \phi \right\|_{L^p(\Omega_T)} \leq 1} \int_0^T \int_\Omega |\phi(x, t)| \left\| f(\cdot, \sigma) \right\|_{L^m(\Omega)} \left( \int_0^T \hat{H}^\gamma(t) \, dt \right)^{\frac{1}{\gamma}} \, dx \, dt,
\]

where \( \gamma \geq 1 \) and \( \frac{1}{\theta} + \frac{1}{m} + \frac{1}{\gamma} = 2 \). It is clear that \( a = \gamma \).

Hence

\[
\int_0^T \hat{H}^\gamma(t) \, dt = \int_0^T t^{(-\frac{1}{2} + \frac{N}{2\gamma} - \frac{N}{2s})} \, dt.
\]

The previous integral is finite if and only if

\[
\gamma(-\frac{1}{2} + \frac{N}{2\gamma} - \frac{N}{2s}) > -1.
\]
Hence $\gamma < \frac{N+2s}{m+2s}$. Using the fact that $\frac{1}{\theta} + \frac{1}{m} + \frac{1}{\gamma} = 2$, it holds that $\frac{1}{\theta} > \frac{1}{m} - \frac{s}{N+2s}$ and hence $\theta < \frac{m(N+2s)}{(N+2s-2m+1)}$. Then we conclude.

**Theorem 3.11.** Assume that the conditions of Theorem 3.10 hold. Let $u$ be the unique weak solution to problem (3.1), then $|\nabla u|^{1-s} \in L^p(\Omega_T)$ for all $p \geq 1$ such that $\frac{1}{p} > \frac{1}{m} - \frac{2s-1}{N+2s}$. Moreover

$$\|\nabla u|^{1-s}\|_{L^p(\Omega_T)} \leq C\|f\|_{L^m(\Omega_T)},$$

(3.33)

with

$$p < \infty \quad \text{if } m \geq \frac{N+2s}{2s-1},$$

$$p < \frac{m(N+2s)}{N+2s-2m+1} \quad \text{if } m < \frac{N+2s}{2s-1}.$$  

**Proof.** We follow the same technique as in the proof of Theorem 3.10. By the representation formula and by using (3.3) and setting $\Omega_t = \Omega \times (0,t)$, we have

$$|\nabla u(x,t)| \leq C \int_{\Omega_t} f(y,\sigma)|\nabla_x P_{\Omega}(x,y,t-\sigma)| \, dy \, d\sigma$$

$$\leq C \int_{\Omega_t} f(y,\sigma) \frac{\nabla_x P_{\Omega}(x,y,t-\sigma)}{P_{\Omega}(x,y,t-\sigma)} P_{\Omega}(x,y,t-\sigma) \, dy \, d\sigma$$

$$\leq \frac{C}{\delta(x)} \int_{\{\delta(x) \leq (t-\sigma)^{\frac{1}{m-1}}\}} f(y,\sigma) \, dy \, d\sigma$$

$$+ C \int_{\{\delta(x) > (t-\sigma)^{\frac{1}{m-1}}\}} f(y,\sigma) \frac{(t-\sigma)^{\frac{2s-1}{m-1}}}{((t-\sigma)^{\frac{2s-1}{m-1}} + |x-y|)^{N+2s}} \, dy \, d\sigma.$$  

(3.34)

Hence

$$|\nabla u(x,t)|^{1-s}(x) \leq \frac{C u(x,t)}{\delta(x)} + C \int_{\Omega_t} f(y,\sigma) \frac{(t-\sigma)^{\frac{2s-1}{m-1}}}{((t-\sigma)^{\frac{2s-1}{m-1}} + |x-y|)^{N+2s}} \, dy \, d\sigma$$

$$:= J_1(x,t) + J_2(x,t).$$  

(3.35)

From Theorem 3.10, it holds that $J_1 \in L^\theta(\Omega_T)$ for all $\theta < \frac{m(N+2s)}{(N+2s-2m+1)}$ and

$$\|J_1\|_{L^\theta(\Omega_T)} \leq C\|f\|_{L^m(\Omega_T)}.$$  

(3.36)

We deal with $J_2$. As above, we will use a duality argument. Let $\phi \in C_0^\infty(\Omega_T)$, and define

$$\tilde{H}(x,t) := \frac{(t^{\frac{2s-1}{m-1}})}{(t^{\frac{2s-1}{m-1}} + |x|)^{N+2s}},$$

then

$$\|J_2\|_{L^p(\Omega_T)} = \sup_{\|\phi\|_{L^{p'}(\Omega_T)} \leq 1} \int_{\Omega_T} \phi(x,t) J_2(x,t) \, dx \, dt$$

$$\leq \sup_{\|\phi\|_{L^{p'}(\Omega_T)} \leq 1} \int_{\Omega_T} |\phi(x,t)| \int_0^T \int_\Omega f(y,\sigma) \frac{(t-\sigma)^{\frac{2s-1}{m-1}}}{((t-\sigma)^{\frac{2s-1}{m-1}} + |x-y|)^{N+2s}} \, dy \, d\sigma \, dx \, dt$$

$$\leq \sup_{\|\phi\|_{L^{p'}(\Omega_T)} \leq 1} \int_0^T \int_\Omega \int_\Omega |\phi(x,t)| \tilde{H}(x-y,t-\sigma) f(y,\sigma) \, dy \, dx \, dt.$$
Hence using Young inequality, we obtain that
\begin{equation}
\|J_2\|_{L^p(\Omega_T)} = \sup_{\|\phi\|_{L^p'(\Omega_T)} \leq 1} \int_0^T \|\phi(.,t)\|_{L^p'(\Omega)} \int_0^t \|\overline{H}(.,t - \sigma)\|_{L^p(\Omega)} \|f(.,\sigma)\|_{L^m(\Omega)} d\sigma dt,
\end{equation}
with \(\frac{1}{p'} + \frac{1}{m} + \frac{1}{\gamma} = 2\). By direct computations, we have
\[\|\overline{H}(.,t - \sigma)\|_{L^p(\Omega)} \leq C(t - \sigma)^\frac{2m+1}{2p} \|\phi(.,t)\|_{L^p'(\Omega)} \leq C(t - \sigma)^\frac{2m+1}{2p} \frac{N+2s}{2},\]
Going back to (3.37), we conclude that
\[\|J_2\|_{L^p(\Omega_T)} \leq C \sup_{\|\phi\|_{L^p'(\Omega_T)} \leq 1} \int_0^T \|\phi(.,t)\|_{L^p'(\Omega)} \int_0^t \|f(.,\sigma)\|_{L^m(\Omega)} (t - \sigma)^\frac{2m+1}{2p} \frac{N+2s}{2} d\sigma dt\]
Thus, using again Young inequality, we get
\[\|J_2\|_{L^p(\Omega_T)} \leq C\|f\|_{L^m(\Omega_T)} \sup_{\|\phi\|_{L^p'(\Omega_T)} \leq 1} \|\phi\|_{L^p'(\Omega_T)} \left(\int_0^T |t|^\frac{\gamma}{\gamma} \frac{2m+1}{2p} \frac{N+2s}{2} dt\right)^\frac{1}{\gamma},\]
where \(\frac{1}{p'} + \frac{1}{m} + \frac{1}{\gamma} = 2\). Hence \(\gamma = a\). It is clear that the last integral is finite if and only if \(\gamma < \frac{N+2s}{N+a}\).
Since \(\frac{1}{p'} + \frac{1}{m} + \frac{1}{\gamma} = 2\), then \(\frac{1}{p} > \frac{1}{m} = \frac{2s-1}{N+2s} + \frac{2s-1}{N+2s}\). Hence
\[\|J_2\|_{L^p(\Omega_T)} \leq C\|f\|_{L^m(\Omega_T)}.
\]
If \(m \geq \frac{N+2s}{2s-1}\), then the above condition holds for all \(p > 1\). Since \(s \in (\frac{1}{2}, 1)\), \(\frac{N+2s}{2s-1} > \frac{N+2s}{s}\),
and then combining (3.36) and (3.38), we conclude that, for all \(p < \infty\),
\[\|\nabla \phi\|_{L^p(\Omega_T)} \leq C\|f\|_{L^m(\Omega_T)}.
\]
If \(m < \frac{N+2s}{2s-1}\), then \(p < \frac{m(N+2s)}{N+2s - m(2s-1)}\). Since \(\frac{m(N+2s)}{N+2s - m(2s-1)} < \frac{m(N+2s)}{(N+2s - ms)}\),
using (3.36) and (3.38), we obtain that
\[\|\nabla \phi\|_{L^p(\Omega_T)} \leq C\|f\|_{L^m(\Omega_T)}
\]
for all \(p\) which satisfies \(\frac{1}{p} > \frac{1}{m} = \frac{2s-1}{N+2s}\). \(\square\)

**Remark 3.12.** Let \(u(x,t) = tw\) where \(w(x) = (1 - |x|^2)^s\), solves the problem
\[(-\Delta)^s w = 1 \text{ in } B_1(0) \text{ and } w = 0 \text{ in } \mathbb{R}^N \setminus B_1(0).\]
Then
\[u_t + (-\Delta)^s u = w + t := f(x,t) \text{ in } B_1(0) \times (0,T).\]
Notice that \(f \in C(\overline{\Omega_T})\), however \(|\nabla u(x)| = 2st|x|(1 - |x|^2)^{s-1}\) in \(B_1(0) \times (0,T)\), then \(|\nabla u|^{\alpha} \in L^\infty(B_1(0) \times (0,T)) \text{ if and only if } \alpha \geq 1 - s\) which show in some way the optimality of the regularity result obtained in Theorem 3.11.

**Corollary 3.13.**

1. By the result of Theorems 3.10 and 3.11 and since \(|\nabla \delta| = 1 \text{ a.e. in } \Omega\), it holds that if \(u\) is the unique weak solution to problem (3.1), then \((u\delta^{1-s}) \in L^p(0,T; W_0^{1,p}(\Omega))\) for all \(p\) such that \(\frac{1}{p} > \frac{1}{m} = \frac{2s-1}{N+2s}\) and
\[\|u\delta^{1-s}\|_{L^p(0,T; W_0^{1,p}(\Omega))} \leq C\|f\|_{L^m(\Omega)}.
\]
Moreover, if \(m < \frac{N+2s}{2s-1}\), then the above estimate holds for all \(p < \frac{m(N+2s)}{N+2s - m(2s-1)}\).
(2) Assume that \( f \in L^m(\Omega_T) \) for some \( m > 1 \) and let \( u \) be the unique weak solution to problem (3.1), then

(a) If \( m \geq \frac{N + 2s}{2s - 1} \), then \( \int_{\Omega_T} |\nabla u|^a dx < \infty \) for all \( a < \frac{1}{1 - s} \).

(b) If \( \frac{1}{s} \leq m < \frac{N + 2s}{2s - 1} \), then \( \int_0^T \int_{\Omega} |\nabla u|^a dx < \infty \), for all \( a < \frac{m(N + 2s)}{m(N + 2s) - m(2s - 1)} \).

(c) If \( 1 < m < \frac{1}{s} \), then \( \int_{\Omega_T} |\nabla u|^a dx < \infty \) for all \( a < \frac{N + 2s}{N + 1} \).

**Proof.** Let us begin with the first statement. Define \( v(x, t) = u(x, t)\delta^{1-s}(x) \), then

\[
\nabla v(x, t) = \delta^{1-s}(x)\nabla u + (1 - s)\frac{u(x, t)}{\delta^s(x)}\nabla \delta(x).
\]

Using the fact that \( |\nabla \delta(x)| = 1 \) a.e. in \( \Omega \), it holds that

\[
|\nabla v(x, t)| \leq \delta^{1-s}(x)|\nabla u(x, t)| + (1 - s)|\frac{u(x, t)}{\delta^s(x)}|.
\]

Hence the desired estimate follows combining the two estimates obtained in Theorems 3.10 and 3.11.

We prove the second point that provides a global regularity for the gradient term without using any weight. It is clear that \( u \in L^p(0, T; W^{1, a}_0(\Omega)) \) for all \( a < \frac{N + 2s}{N + 1} \). Now, using Theorem 3.11, we reach that \( |\nabla u|\delta^{1-s} \in L^p(\Omega_T) \) with \( p \geq 1 \) which satisfies \( \frac{1}{p} > \frac{1}{m} - \frac{2s - 1}{N + 2s} \). Hence using Hölder inequality, we get

\[
\int_{\Omega_T} |\nabla u|^a dx dt = \int_{\Omega_T} \left(||\nabla ||^{\alpha(a - 1)}\right)^{a}(\delta^{1-s}) dx dt \\
\leq \left(\int_{\Omega_T} (|\nabla u|^{\alpha - 1}) dx dt\right)^{\frac{a}{\alpha}} \left(\int_{\Omega_T} \delta^{\alpha(1-s)} dx dt\right)^{\frac{\alpha - a}{\alpha}},
\]

where \( p > a \) to be chosen later. The last integral is finite if and only if \( \frac{ap}{p - a} (1 - s) < 1 \), that is, if \( a < \frac{m}{(1 - s)p + 1} \). Notice that in particular, \( a < \frac{1}{1 - s} \).

If \( m \geq \frac{N + 2s}{2s - 1} \), then by Theorem 3.11, we know that \( |\nabla u|\delta^{1-s} \in L^p(\Omega_T) \) for all \( p < \infty \). Hence the condition \( \frac{ap}{p - a} (1 - s) < 1 \) holds if \( a(1 - s) < 1 \) and then we conclude.

Assume that \( m < \frac{N + 2s}{2s - 1} \), since \( p < \frac{m(N + 2s)}{N + 2s - ms} \), we get

\[
a < \frac{m(N + 2s)}{m(N + 2s) - m(2s - 1)}.
\]

It is clear that \( \frac{m(N + 2s)}{m(N + 2s) - m(2s - 1)} \geq 1 \) if \( m \geq \frac{1}{s} \). Thus we conclude. \( \square \)

In the next result, under suitable hypotheses on \( s, m \) and \( p \), we improve the integrability of the gradient of the solution without the degenerate weight or with a less degenerate weight at the boundary.

**Theorem 3.14.** Assume that the conditions of Theorem 3.10 hold. Let \( u \) be the unique weak solution to problem (3.1), then:

(1) Let \( m_1 = \min\{\frac{s}{1 - s}, m\} \). Then \( u \in L^p(0, T; W^{1, p}_0(\Omega)) \) for \( p < \frac{m_1(N + 2s)}{N + s + (1 - s)m_1} \), and

\[
||\nabla u||_{L^p(\Omega_T)} \leq C(\Omega_T, s, N, p)||f||_{L^m(\Omega_T)}
\]
Precisely, it holds that $\hat{p} > m$ where we have used the fact that
\[
\alpha > \frac{1}{1-s} \left( \left( \frac{1}{m} - \frac{1}{p} \right) (N + 2s) + (1 - s) - \frac{s}{m} \right),
\]
we have $|\nabla u|^\alpha \in L^p(\Omega_T)$ and
\[
|||\nabla u|^\alpha|||_{L^p(\Omega_T)} \leq C(\Omega_T, s, N, p)||f||_{L^m(\Omega_T)}.
\]

Proof. As in the proof of Theorem 3.11, from estimate (3.34), we have
\[
|\nabla u(x, t)| \leq \frac{C}{\delta(x)} \int_{\{\Omega_t \cap \{\delta(x) \leq (t-\sigma) \frac{1}{T}\}} f(y, \sigma) P_{\Omega}(x, y, t-\sigma) dy d\sigma + C \int_{\Omega_t} f(y, \sigma) \frac{(t-\sigma)^{2\alpha - 1}}{((t-\sigma)^{\frac{1}{2\alpha}} + |x-y|)^{N+2s}} dy d\sigma.
\]
(3.41)

The estimate of the term $J_2(x, t)$ is similar to the estimate of the term $J_2$ in the proof of Theorem 3.11. Precisely, it holds that $J_2 \in L^p(\Omega_T)$ for $p \geq 1$ satisfying $\frac{1}{p} > \frac{1}{m} - \frac{2s - 1}{N + 2s}$ and
\[
|||J_2|||_{L^p(\Omega_T)} \leq C||f||_{L^m(\Omega_T)}.
\]
(3.42)

We deal with $J_1$, in this case we follow the argument as in the proof of Theorem 3.5. Since $f \in L^m(\Omega_T)$ with $m > 1$, then using H"older inequality we deduce that
\[
J_1(x, t) \leq \frac{C}{\delta(x)} \left( \int_{\{\Omega_t \cap \{\delta(x) \leq (t-\sigma) \frac{1}{T}\}} f^m(y, \sigma) P_{\Omega}(x, y, t-\sigma) dy d\sigma \right)^{\frac{1}{m}}
\]
\[
\times \left( \int_{\{\Omega_t \cap \{\delta(x) \leq (t-\sigma) \frac{1}{T}\}} P_{\Omega}(x, y, t-\sigma) dy d\sigma \right)^{\frac{1}{m}}
\]
\[
\leq \frac{C}{\delta(x)^{\frac{1}{m}} \alpha} \left( \int_{\{\Omega_t \cap \{\delta(x) \leq (t-\sigma) \frac{1}{T}\}} f^m(y, \sigma) P_{\Omega}(x, y, t-\sigma) dy d\sigma \right)^{\frac{1}{m}}.
\]
(3.43)

where we have used the fact that
\[
\int_{\{\Omega_t \cap \{\delta(x) \leq (t-\sigma) \frac{1}{T}\}} P_{\Omega}(x, y, t-\sigma) dy d\sigma \leq C(\Omega, T) \delta^s(x) \text{ for } (x, t) \in \Omega_T.
\]

Fix $p > m$ to be chosen later, then
\[
J_1^p(x, t) \leq \frac{C}{\delta(x)^{p(1 - \frac{1}{m})}} \left( \int_{\{\Omega_t \cap \{\delta(x) \leq (t-\sigma) \frac{1}{T}\}} f^m(y, \sigma) P_{\Omega}(x, y, t-\sigma) dy d\sigma \right)^{\frac{1}{m}}
\]
\[
\leq \frac{C}{\delta(x)^{p(1 - \frac{1}{m})}} \left( \int_{\{\Omega_t \cap \{\delta(x) \leq (t-\sigma) \frac{1}{T}\}} f^m(y, \sigma) P_{\Omega}^{\frac{m}{p}}(x, y, t-\sigma) dy d\sigma \right) \left( \int_{\Omega_t} f^m(y, \sigma) dy d\sigma \right)^{\frac{p(m)}{m}}
\]
\[
\leq \frac{C||f||_{L^m(\Omega_T)}^{p-m}}{\delta(x)^{p(1 - \frac{1}{m})}} \int_{\{\Omega_t \cap \{\delta(x) \leq (t-\sigma) \frac{1}{T}\}} f^m(y, \sigma) P_{\Omega}^{\frac{m}{p}}(x, y, t-\sigma) dy d\sigma.
\]
(3.44)
Recall that \( t^{\frac{s}{m}} P_\Omega(x,y,t) \leq C := C(\Omega, s, N, T) \), thus \( \left( t^{\frac{s}{m}} P_\Omega(x,y,t) \right)^{\frac{1}{s}} \leq C \). Hence, if \( \delta(x) \leq (t - \sigma)^{\frac{s}{m}} \), it follows that

\[
P_\Omega^{\frac{1}{s}}(x, y, t - \sigma) \leq C(t - \sigma)^{-(\frac{s}{m} - 1)} P_\Omega(x, y, t - \sigma) \leq \frac{C}{(\delta(x))^{(\frac{s}{m} - 1)}} P_\Omega(x, y, t - \sigma).
\]

Therefore we obtain that

\[
J_1^p(x, t) \leq \frac{C||f||_{L^p(\Omega_T)}^{p-m}}{(\delta(x))^{p(1 - \frac{1}{m}) + N(\frac{1}{m} - 1)}} \iint_{\Omega_T} f^m(y, \sigma) P_\Omega(x, y, t - \sigma) dy d\sigma.
\] (3.45)

Let \( \beta = p(1 - \frac{s}{m'}) + N(\frac{p}{m} - 1) \), then according to the value of \( \beta \), we will consider two cases:

**The first case** \( \beta < 2s \): Notice that in this case if \( m_1 < \frac{s}{1 - s} \) then \( m_1 < p < \frac{m_1(N + 2s)}{N + s + (1 - s)m_1} \).

Moreover, \( \frac{m_1(N + 2s)}{N + s + (1 - s)m_1} < \frac{m_1(N + 2s)}{N + 2s - m_1(2s - 1)} \) defined in Theorem 3.11.

Integrating in \( \Omega_T \), there results that

\[
\iint_{\Omega_T} J_1^p(x, t) dx dt \leq C||f||_{L^p(\Omega_T)}^{p-m_1} \iint_{\Omega_T} f^{m_1}(y, \sigma) \left( \int_0^{\delta(x)} \int_\sigma^T P_\Omega(x, y, t - \sigma) dt dx \right) dy d\sigma.
\]

Recall that

\[
\int_\sigma^T P_\Omega(x, y, t - \sigma) dt \leq \int_0^{T-\sigma} P_\Omega(x, y, \eta) d\eta \leq G_s(x, y).
\]

Hence

\[
\iint_{\Omega_T} J_1^p(x, t) dx dt \leq C||f||_{L^m(\Omega_T)}^{p-m_1} \iint_{\Omega_T} f^{m_1}(y, \sigma) \varphi(y) dy d\sigma.
\]

where \( \varphi(y) = \int_{\Omega_T} \frac{G_s(x, y)}{(\delta(x))^{n_0}} dx \). Since \( \beta < 2s \), then \( \varphi \in L^\infty(\Omega) \) and then

\[
\iint_{\Omega_T} J_1^p(x, t) dx dt \leq C||f||_{L^m(\Omega_T)}^p.
\]

If \( m > \frac{s}{1 - s} \) the final estimate follows by the Hölder inequality. Notice that in this case, \( p < \frac{s}{1 - s} \).

**The second case** \( \beta \geq 2s \): Consider \( \frac{m(N + 2s)}{N + s + (1 - s)m} \leq p < \frac{m(N + 2s)}{N + s - m(N + 2s - 1)} \).

Since \( \beta \geq 2s \),

\[
(\frac{1}{m} - \frac{1}{p})(N + 2s) < \frac{s}{m}.
\]

Let

\[
\Upsilon := (\frac{1}{m} - \frac{1}{p})(N + 2s) + (1 - s) - \frac{s}{m},
\]

then \( 0 < \Upsilon < 1 - s \). Fix \( 0 < \alpha < 1 \) such that \( \frac{\Upsilon}{1 - \alpha} < \alpha < 1 \), thus we reach that \( \beta - \alpha_0(1 - s) < 2s \).

Going back to (3.45), it holds that

\[
J_1^p(x, t)(\delta(x))^{\alpha_0(1-s)} \leq \frac{C||f||_{L^m(\Omega_T)}^{p-m}}{(\delta(x))^{p(1 - \frac{1}{m}) + N(\frac{1}{m} - 1) - \alpha_0(1-s)}} \iint_{\Omega_T} f^m(y, \sigma) P_\Omega(x, y, t - \sigma) dy d\sigma.
\] (3.46)

Setting \( \hat{\beta} = p\left(1 - \frac{s}{m'}\right) + N(\frac{p}{m} - 1) - \alpha_0(1-s) \), then \( \hat{\beta} = \frac{p}{m}(s + (1 - s)m + N - \alpha m(1 - s)) - N \).

By the above condition on \( p \) and \( \alpha \), we deduce that \( \hat{\beta} < 2s \). Repeating the argument used in the first case, it holds that

\[
\iint_{\Omega_T} J_1^p(x, t)(\delta(x))^{\alpha_0(1-s)} dx dt \leq C||f||_{L^m(\Omega_T)}^p,
\]

and then we conclude. 
\( \square \)
Remark 3.15. Notice that \( u \in L^{2s}(0, T; W^{1,2s}_0(\Omega)) \) if \( m > \frac{2s(N+2s)}{N + 2s^2} \).

Now, if \( f \in L^1(\Omega_T) \cap L^m(K \times (0, T)) \), where \( m > 1 \) and \( K \subset \subset \Omega \) is any compact set of \( \Omega \), then the regularity result of Theorem 3.11 holds locally in \( \Omega \times (0, T) \). More precisely we have

Proposition 3.16. Suppose that \( m > 1 \) and assume that \( f \in L^1(\Omega_T) \cap L^m(K \times (0, T)) \) for every compact set \( K \subset \subset \Omega \). Let \( u \) be the unique weak solution to problem (3.1) and consider \( \Omega_1 \subset \subset \Omega \) with \( \text{dist}(\Omega_1, \partial \Omega) > 0 \). Let \( K_1 \subset \subset \Omega_1 \) be a compact set of \( \Omega \) such that \( \Omega_1 \subset \subset K_1 \subset \subset \Omega \). Then \( u \in L^p(\Omega_1 \times (0, T)) \) for all \( p < \frac{m(N+2s)}{N + 2s - m(2s-1)} \). Moreover,

\[
\|u\|_{L^p(\Omega_1 \times (0,T))} \leq C(\|f\|_{L^m(K_1 \times (0,T))} + \|f\|_{L^1(\Omega_T)}),
\]

and

\[
\|\nabla u\|_{L^p(\Omega_1 \times (0,T))} \leq C(\|f\|_{L^m(K_1 \times (0,T))} + \|f\|_{L^1(\Omega_T)}),
\]

where \( C := C(K_1, \Omega_1, \Omega, T, N, m) \).

Proof. Since \( f \in L^1(\Omega_T) \), then \( \|\nabla u\| \in L^q(\Omega_T) \) for all \( q < \frac{N+2s}{N+1} \). Now we closely follow the proofs of Theorem 3.10 and Theorem 3.11. We have

\[
\frac{u(x,t)}{\delta^s(x)} \leq C \int_{0}^{t} \int_{\Omega} f(y,\sigma) \frac{(t-\sigma)}{((t-\sigma)^{\frac{N}{2s}} + |x-y|)^{N+2s}} \, dy \, d\sigma.
\]

Fix \( \Omega_1 \subset \subset \Omega \) with \( \text{dist}(\Omega_1, \partial \Omega) = c_0 > 0 \) and let \( K_1 \) be a compact set of \( \Omega \) such that \( \Omega_1 \subset \subset K_1 \subset \subset \Omega \). Let \( x \in \Omega_1 \), then

\[
u(x,t) \leq C(\Omega_1, c_0, C) \int_{0}^{t} \int_{\Omega_1} f(y,\sigma) \frac{(t-\sigma)}{((t-\sigma)^{\frac{N}{2s}} + |x-y|)^{N+2s}} \, dy \, d\sigma
\]

\[
\leq C(\Omega_1, c_0, C) \left\{ \int_{0}^{t} \int_{K_1} f(y,\sigma) \frac{(t-\sigma)}{((t-\sigma)^{\frac{N}{2s}} + |x-y|)^{N+2s}} \, dy \, d\sigma + \int_{0}^{t} \int_{(\Omega \setminus K_1)} f(y,\sigma) \frac{(t-\sigma)}{((t-\sigma)^{\frac{N}{2s}} + |x-y|)^{N+2s}} \, dy \, d\sigma \right\}
\]

Since \( x \in \Omega_1 \subset \subset K_1 \), then for all \( y \in \Omega \setminus K_1, |y-x| > \hat{c} > 0 \). Thus

\[
\int_{0}^{t} \int_{(\Omega \setminus K_1)} f(y,\sigma) \frac{(t-\sigma)}{((t-\sigma)^{\frac{N}{2s}} + |x-y|)^{N+2s}} \, dy \, d\sigma
\]

\[
\leq C(\|f\|_{L^1(\Omega_T)}).
\]

Therefore we conclude that

\[
u(x,t) \leq C(\Omega_1, c_0, C) \left( \int_{0}^{t} \int_{K_1} f(y,\sigma) \frac{(t-\sigma)}{((t-\sigma)^{\frac{N}{2s}} + |x-y|)^{N+2s}} \, dy \, d\sigma + \|f\|_{L^1(\Omega_T)} \right).
\]

Since \( f \in L^m(K_1 \times (0, T)) \), then the rest of the proof follows exactly from the same duality argument as in the proof of Theorem 3.10. Hence, estimate (3.47) holds. In a similar way, we prove estimate (3.48).

\[\Box\]

Remarks 3.17.

(1) For \( a \geq 1 \), we define the space \( L^{m}_{\text{loc}}(\Omega_T) \) as the set of measurable functions \( u \) such that \( u \in L^{m}(\Omega_T) \), for any \( \eta \in C^{\infty}_{0}(\Omega) \). Then the result of Proposition 3.16 affirms that if \( u \) is the unique solution to problem (3.1), then \( u \in L^p_{\text{loc}}(\Omega_T) \) for all \( p < \frac{m(N+2s)}{N + 2s - m(2s-1)} \) and \( |\nabla u| \in L^{p}_{\text{loc}}(\Omega_T) \) for all \( p < \frac{m(N+2s)}{(N + 2s - m(2s-1))} \).

(2) The result of Proposition 3.16 will be useful in order to get \( C^1 \) regularity using a bootstrap argument if, in addition, we have global bounds in \( L^1 \) and a local family of bounds in a suitable \( L^{m}_{\text{loc}} \) space.
We deal now with the case $f \equiv 0$ and $u_0 \in L^\theta(\Omega)$ with $\theta \geq 1$. Following the same computations as above, we get the next results.

**Theorem 3.18.** Suppose that $f \equiv 0$ and $u_0 \in L^\rho(\Omega)$ with $\rho \geq 1$. If $u$ is the unique weak solution to problem (3.1), then $\frac{u}{\delta^s} \in L^\theta(\Omega_T)$ for all $\theta < \frac{\rho(N + 2s)}{N + \rho}$ and $|\nabla \delta^{-s}| \in L^p(\Omega)$ for all $p < \frac{\rho(N + 2s)}{N + \rho}$.

Moreover,

$$
\left\| \frac{u}{\delta^s} \right\|_{L^\theta(\Omega_T)} + \left\| |\nabla \delta^{-s}| \right\|_{L^p(\Omega_T)} \leq C(\Omega_T, p, \theta) \left\| u_0 \right\|_{L^\rho(\Omega)}.
$$

**Proof.** For the reader convenience, we include here some details for the estimate of the term $\frac{u}{\delta^s}$.

$$
\left\| \frac{u}{\delta^s} \right\|_{L^\theta(\Omega_T)} = \sup_{\|\phi\|_{L^\theta(\Omega_T)} \leq 1} \int_{\Omega_T} \phi(x, t) \frac{u(x, t)}{\delta^s(x)} \, dx dt
\leq \sup_{\|\phi\|_{L^\theta(\Omega_T)} \leq 1} \int_{\Omega_T} |\phi(x, t)| \int_\Omega u_0(y) \frac{t^\frac{1}{2}}{(t^\frac{1}{2} + |x - y|)^{N+2s}} \, dy \, dx dt
\leq \sup_{\|\phi\|_{L^\theta(\Omega_T)} \leq 1} \int_0^T \int_\Omega |\phi(x, t)| H(x - y, t) u_0(y) dy dx dt
$$

with $\theta' = \frac{\theta}{\theta - 1}$. Similarly to the proofs above, by Young’s inequality, we have that

$$
\left\| \frac{u}{\delta^s} \right\|_{L^\theta(\Omega_T)} \leq C \sup_{\|\phi\|_{L^\theta(\Omega_T)} \leq 1} \int_0^T \|\phi(., .)\|_{L^\theta(\Omega)} \left\| u_0 \right\|_{L^\rho(\Omega)} \|H(., .)\|_{L^\gamma(\Omega)} \, d\tau dt,
$$

where

$$
H(|x - y|, t) = \frac{t^\frac{1}{2}}{(t^\frac{1}{2} + |x - y|)^{N+2s}}
$$

and $\frac{1}{\theta'} + \frac{1}{\rho} + \frac{1}{\alpha} = 2$. Notice that

$$
\|H(., .)\|_{L^\gamma(\Omega)} \leq C t^{-\frac{1}{\theta'} + \frac{2s}{\theta} - \frac{\theta}{N}}.
$$

Therefore,

$$
\left\| \frac{u}{\delta^s} \right\|_{L^\theta(\Omega_T)} \leq C \|u_0\|_{L^\rho(\Omega)} \sup_{\|\phi\|_{L^\theta(\Omega_T)} \leq 1} \int_0^T \|\phi(., .)\|_{L^\theta(\Omega)} t^{-\frac{1}{\theta'} + \frac{2s}{\theta} - \frac{\theta}{N}} \, dt
$$

and by using the Hölder inequality, we get

$$
\left\| \frac{u}{\delta^s} \right\|_{L^\theta(\Omega_T)} \leq C \|u_0\|_{L^\rho(\Omega)} \sup_{\|\phi\|_{L^\theta(\Omega_T)} \leq 1} \|\phi\|_{L^\theta(\Omega)} \left( \int_0^T \theta^\theta (t^{-\frac{1}{\theta'} + \frac{2s}{\theta} - \frac{\theta}{N}}) \, dt \right)^\frac{1}{\theta}.
$$

The last integral is finite if and only if $\theta (-\frac{1}{2} + \frac{N}{2s} - \frac{N}{2s}) > -1$, thus

$$
\frac{N}{2s} (1 - \frac{1}{\alpha}) < \frac{1}{\theta} - \frac{1}{2}.
$$

Since $\frac{1}{\rho} + \frac{1}{\alpha} = 1 + \frac{1}{\theta}$ then $1 - \frac{1}{\alpha} = \frac{1}{\rho} - \frac{1}{\theta}$. Substituting in the previous inequality, we conclude that

$$
\theta < \frac{\rho(N + 2s)}{N + \rho s}.
$$

To estimate the gradient term we consider that, by the representation formula, we have

$$
|\nabla u(x, t)| \leq C(\Omega_T) \int_{\Omega} u_0(y) |\nabla_x P_\Omega(x, y, t)| \, dy \leq C(\Omega_T) \int_{\Omega} u_0(y) \left| \frac{\nabla_x P_\Omega(x, y, t)}{P_\Omega(x, y, t)} \right| P_\Omega(x, y, t) dy
\leq C(\Omega_T) \int_{\Omega} u_0(y) h(x, y, t) P_\Omega(x, y, t) dy
$$

with $h(x, y, t) = \frac{|\nabla_x P_\Omega(x, y, t)|}{P_\Omega(x, y, t)} \leq C \left( \frac{1}{\delta(x) \wedge t^{\frac{1}{2}}} \right)$.
Hence
\[
|\nabla u(x,t)| \leq \frac{C}{|\delta(x)|^{1+s}} \int_{\Omega} u_0(y) P_\Omega(x,y,t) \, dy + \frac{C}{\delta(x)^s} \int_{\Omega} u_0(y) P_\Omega(x,y,t) \, dy
\]
\[
= J_{11}(x,t) + J_{12}(x,t).
\]
We again use the duality argument, starting by the estimation of the term \(J_{11}\). By estimate (3.2) and Hölder inequality, we obtain that
\[
\|J_{11}\|_{L^p(\Omega_T)} = \sup_{\{\|\phi\|_{L^{p'}}(\Omega_T) \leq 1\}} \int_{\Omega_T} \phi(x,t) J_{11}(x,t) \, dx \, dt
\]
\[
\leq \sup_{\{\|\phi\|_{L^{p'}}(\Omega_T) \leq 1\}} \int_{\Omega_T} |\phi(x,t)| \int_{\Omega} u_0(y) \frac{t^{1-\frac{1}{p}}}{(t^{\frac{1}{p}} + |x-y|)^{N+2s}} \, dy \, dx \, dt
\]
\[
\leq \sup_{\{\|\phi\|_{L^{p'}}(\Omega_T) \leq 1\}} \int_{0}^{T} \int_{\Omega} |\phi(x,t)| H(x,y) u_0(y) \, dy \, dx \, dt
\]
with \(H(x,t) = \frac{t^{1-\frac{1}{p}}}{(t^{\frac{1}{p}} + |x|)^{N+2s}}\). Using Young inequality, it holds that
\[
\|J_{11}\|_{L^p(\Omega_T)} \leq C \|u_0\|_{L^{p}(\Omega)} \sup_{\{\|\phi\|_{L^{p'}}(\Omega_T) \leq 1\}} \int_{0}^{T} \|\phi(.,t)\|_{L^{p'}(\Omega)} \|H(.,t)\|_{L^1(\Omega)} d\sigma dt,
\]
with \(\frac{1}{p} + \frac{1}{\rho} + \frac{1}{a} = 2\). Notice that by a direct calculation
\[
\|H(.,t)\|_{L^1(\Omega)} \leq C t^{(1-\frac{1}{p}) + \frac{N}{(N+2s)}}. \]
Therefore we reach that
\[
\|J_{11}\|_{L^p(\Omega_T)} \leq C \|u_0\|_{L^{p}(\Omega)} \sup_{\{\|\phi\|_{L^{p'}}(\Omega_T) \leq 1\}} \int_{0}^{T} \|\phi(.,t)\|_{L^{p'}(\Omega)} t^{(1-\frac{1}{p}) + \frac{N}{(N+2s)}} \, dt.
\]
Using Hölder inequality, we get
\[
\|J_{11}\|_{L^p(\Omega_T)} \leq C \|u_0\|_{L^{p}(\Omega)} \sup_{\{\|\phi\|_{L^{p'}}(\Omega_T) \leq 1\}} \|\phi\|_{L^{p'}(\Omega_T)} \left( \int_{0}^{T} t^{p((1-\frac{1}{p}) + \frac{N}{(N+2s)})} \, dt \right)^{\frac{1}{p}}.
\]
The last integral is finite if and only if \(p((1-\frac{1}{2s}) + \frac{N}{2a} - \frac{N + 2s}{2s}) > -1\). Thus
\[
\frac{1}{2s} + \frac{N}{2a} - \frac{1}{1 - \frac{1}{p}} < 0.
\]
Since \(\frac{1}{p} + \frac{1}{\rho} + \frac{1}{a} = 2\) we have \(\frac{1}{p} + \frac{1}{a} = 1 + \frac{1}{p}\). Then \(1 - \frac{1}{a} = \frac{1}{p} - \frac{1}{p}\). Going back to the previous inequality, we conclude that
\[
\frac{1}{2s} + \frac{N}{2a} - \frac{1}{1 - \frac{1}{p}} \leq \frac{1}{p} (1 + \frac{N}{2s}).
\]
Hence \(J_{11} \in L^p(\Omega_T)\) for all \(p < \frac{\rho(N + 2s)}{N + p}\).

We deal now with \(J_{12}\). We have
\[
J_{12}(x,t) = \frac{C}{\delta(x)^s} \int_{\{x \in \Omega, \, |x-y| \leq t^{\frac{1}{s}}\}} u_0(y) P_\Omega(x,y,t) \, dy
\]
\[
\leq C \frac{C}{\delta(x)^s} \int_{\Omega} u_0(y) \frac{t^{1-\frac{1}{s}}}{(t^{\frac{1}{s}} + |x-y|)^{N+2s}} \, dy.
\]
Hence
\[
J_{12}(x,t) \delta^{1-s}(x) \leq C \int_{\Omega} u_0(y) \frac{t^{\frac{1}{s}}}{(t^{\frac{1}{s}} + |x-y|)^{N+2s}} \, dy,
\]
that is the same term estimated above. Hence we conclude that $J_{12} \delta^{1-s} \in L^p(\Omega_T)$ for all $p < \frac{\rho(N + 2s)}{N + s \rho}$.

Since $\frac{\rho(N + 2s)}{N + s \rho} \geq \frac{\rho(N + 2s)}{N + \rho}$, then using the fact that $\Omega_T$ is bounded, it follows that $|\nabla u|\delta^{1-s}(x) \in L^p(\Omega_T)$ for all $p < \frac{\rho(N + 2s)}{N + \rho}$. \hfill \Box

As in Corollary 3.13, we have the next regularity for the gradient.

**Corollary 3.19.** Assume that $f \equiv 0$ and $u_0 \in L^\rho(\Omega)$ with $\rho \geq 1$. If $u$ is the unique weak solution to problem (3.1), then $|\nabla u| \in L^a(\Omega)$ for all $a < \bar{U} := \frac{\rho(N + 2s)}{(1-s)(N + 2s) + N + \rho}$. Moreover $\bar{U} > \frac{N + 2s}{N + 1}$ if $\frac{N + 2s}{N} < \frac{1}{1-s} \text{ and } \rho > \frac{N}{sN - 2s(1-s)}$.

As a conclusion, from Proposition 3.4 and using the same approach as in the proof of Theorem 3.18, keeping the power in the time variable, we get the next results.

**Proposition 3.20.** Suppose that $f \equiv 0$ and $u_0 \in L^\rho(\Omega)$. If $u$ is the unique weak solution to problem (3.1), then for all $r > 1$ and for all $t > 0$, we have

$$||u(\cdot, t)||_{L^r(\Omega)} \leq C t^{-\frac{N}{\rho} + \frac{1}{r} - \frac{1}{s}} ||u_0||_{L^\rho(\Omega)},$$

(3.49)

and

$$||u(\cdot, t)||_{L^r(\Omega)} \leq C t^{-\frac{N}{\rho} + \frac{1}{r} - \frac{1}{s}} ||u_0||_{L^\rho(\Omega)},$$

(3.50)

Moreover, $u \in L^\sigma(\Omega_T)$ for all $\sigma < \frac{\rho(N + 2s)}{N} \text{ and } \frac{u}{\delta^\sigma} ||\nabla u||_{L^\gamma(\Omega_T)}$ for all $\gamma < \frac{\rho(N + 2s)}{N + \rho}$.

Within the same framework of the Theorem 3.14, in order to get regularity for the gradient term without any degenerate weight we have the next result.

**Theorem 3.21.** Suppose that $f \equiv 0$ and $u_0 \in L^\rho(\Omega)$ and let $\rho_1$ be defined by $\rho_1 = \min\{\rho, 2s\}$. If $u$ is the unique weak solution to problem (3.1), then $|\nabla u| \in L^\theta(\Omega)$ for all $q < \frac{\rho_1(N + 2s)}{N + \rho_1}$. Moreover,

$$|||\nabla u|||_{L^\theta(\Omega_T)} \leq C(\Omega_T, p, \theta)||u_0||_{L^\rho(\Omega)}.$$

**Proof.** To estimate the gradient term we consider that, by the representation formula, we have

$$\frac{|\nabla u(x, t)|}{t^\tau} \leq C(\Omega) \int_\Omega u_0(y)|\nabla_x P_\Omega(x, y, t)| dy \leq C(\Omega_T) \int_\Omega u_0(y) \frac{|\nabla_x P_\Omega(x, y, t)|}{P_\Omega(x, y, t)} P_\Omega(x, y, t) dy$$

$$\leq C \gamma_{\delta(x) > t^{1+\tau}} \int_\Omega u_0(y) P_\Omega(x, y, t) dy + \frac{C}{\delta(x)} \gamma_{\delta(x) \leq t^{1+\tau}} \int_\Omega u_0(y) P_\Omega(x, y, t) dy$$

$$= \tilde{J}_1(x, t) + \tilde{J}_2(x, t).$$

As in the proof of Theorem 3.18, using the duality argument we obtain that, for all $q < \frac{\rho(N + 2s)}{N + \rho}$,

$$||\tilde{J}_1||_{L^\theta(\Omega_T)} \leq C ||u_0||_{L^\rho(\Omega)}.$$

We treat now $\tilde{J}_2$. We have

$$\tilde{J}_2(x, t) = \frac{C}{\delta(x)} \gamma_{\delta(x) \leq t^{1+\tau}} \int_\Omega u_0(y) P_\Omega(x, y, t) dy$$

$$\leq \frac{C}{\delta(x)} \gamma_{\delta(x) \leq t^{1+\tau}} \left( \int_\Omega u_0(y) P_\Omega(x, y, t) dy \right)^{\frac{1}{p'}} \left( \int_\Omega P_\Omega(x, y, t) dy \right)^{\frac{1}{p'}}$$

$$\leq \frac{C}{\delta(x)} \gamma_{\delta(x) \leq t^{1+\tau}} \left( \int_\Omega u_0(y) P_\Omega(x, y, t) dy \right)^{\frac{1}{p'}}.$$
where we have used the fact that

$$\int_\Omega P_{\Omega}(x,y,t) \, dy \leq C \frac{\delta^\rho(x)}{\sqrt{t}} \leq C$$

in the set \( \{ \delta(x) \leq \frac{1}{\sqrt{t}} \} \).

Hence, as in the estimate of the term \( \tilde{J}_1 \) in Theorem 3.14, we deduce that

$$J_2^\rho(x,t) \leq \frac{C}{\delta^\rho(x)} \chi_{\{ \delta(x) \leq \frac{1}{\sqrt{t}} \}} \left( \int_\Omega u_0^\rho(y) P_{\Omega}(x,y,t) \, dy \right)^{\frac{\rho}{\rho-1}}$$

$$\leq \frac{C}{\delta^\rho(x)} \chi_{\{ \delta(x) \leq \frac{1}{\sqrt{t}} \}} \left( \int_\Omega u_0^\rho(y) P_{\Omega}^\rho(x,y,t) \, dy \right) \left( \int_\Omega \rho_0^\rho(y) \rho_0^\rho(y) \, dy \right)^{\frac{\rho}{\rho-1}}$$

$$\leq \frac{C \| u_0 \|_{L^\rho(\Omega)}^{\frac{\rho}{\rho-1}}}{(\delta^\rho(x))^\frac{\rho}{\rho-1}} \chi_{\{ \delta(x) \leq \frac{1}{\sqrt{t}} \}} \int_\Omega u_0^\rho(y) P_{\Omega}(x,y,t) \, dy.$$

Setting \( \beta_0 = q + (\frac{\rho}{2} - 1)N \), then \( \beta < 2s \) if and only if \( q < \frac{\beta(N+2s)}{N+\rho} \). It is clear that \( \frac{\beta(N+2s)}{N+\rho} > \rho \) if and only if \( \rho < 2s \). Thus assuming the above assumptions and integrating in \( \Omega_T \), we obtain that

$$\int_{\Omega_T} J_2^\rho(x,t) \, dx \, dt \leq C \| u_0 \|_{L^\rho(\Omega)}^{\frac{\rho}{\rho-1}} \int_\Omega u_0^\rho(y) \phi(y) \, dy,$$

where, as above, \( \phi(y) = \frac{Q_0(x,y)}{(\delta(x))^\infty} \, dy \leq C \). Hence

$$\int_{\Omega_T} J_2^\rho(x,t) \, dx \, dt \leq C \| u_0 \|_{L^\rho(\Omega)}^{\frac{\rho}{\rho-1}}.$$

Thus \( u \in L^q((0,T);W_0^{1,q}(\Omega)) \) and

$$\| \nabla u \|_{L^q(\Omega_T)} \leq C(\Omega_T, N, s, p) \| u_0 \|_{L^\rho(\Omega)}.$$

Notice that, from [16], see also [34], working in the whole space \( \mathbb{R}^N \), the above regularity result on the gradient holds globally. However, when working in a bounded domain, the term \( \delta^{1-s} \) appears in a natural way describing the gradient’s behavior.

Under the local summability condition \( f \delta^\beta \in L^1(\Omega_T) \) for some \( \beta < 2s - 1 \), it is possible to show the existence of weak solution with the same range of regularity. This will be the key in order to analyze problem (1.1) for large value of \( \alpha \).

More precisely, we have the following existence result.

**Theorem 3.22.** Assume that \( f, u_0 \) are measurable functions such that \( f \delta^\beta \in L^1(\Omega_T) \) for some \( \beta < (2s - 1) \) and \( u_0 \in L^1(\Omega) \).

Then problem (3.1) has a unique weak solution \( u \) such that for all \( q < \frac{N + 2s}{N + \beta + 1} \),

$$\| u \|_{C([0,T],L^1(\Omega,\delta^\beta \rho \, dx))} + \| \nabla u \|_{L^q(\Omega_T)} \leq C(q, \beta, \Omega_T) \left( \| f \delta^\beta \|_{L^1(\Omega_T)} + \| u_0 \|_{L^1(\Omega)} \right).$$

Moreover, for \( q < \frac{N + 2s}{N + \beta + 1} \) fixed, setting \( \hat{K} : L^q(\Omega_T, \delta^\beta(x) \, dx) \times L^1(\Omega) \rightarrow L^q(0,T; W_0^{1,q}(\Omega)) \), \( \hat{K}(f,u_0) = u \), the unique solution to problem (3.1), then \( \hat{K} \) is a compact operator.
Proof. Without loss of generality, we can assume that $u_0 = 0$ and that $f \geq 0$ in $\Omega_T$. We begin by proving the existence part. Let $f_n = T_n(f)$ and define $u_n$ to be the unique energy solution to the approximating problem

$$
\begin{cases}
  u_{nt} + (-\Delta)^s u_n &= f_n \quad \text{in } \Omega_T, \\
  u_n &= 0 \quad \text{in } (\mathbb{R}^N \setminus \Omega) \times (0, T), \\
  u(x, 0) &= 0 \quad \text{in } \Omega.
\end{cases}
$$

(3.52)

It is clear that $\{u_n\}_n$ is an increasing sequence in $n$. Let $\psi$ be the solution to the problem

$$
\begin{cases}
  (-\Delta)^s \psi &= \frac{1}{\delta^{2s-\beta}} \quad \text{in } \Omega, \\
  \psi &= 0 \quad \text{in } \mathbb{R}^N \setminus \Omega,
\end{cases}
$$

(3.53)

whose existence is a consequence of [22]. By the results in [1] and [9] extending the results in [50], we find that $\psi \approx C\delta^\beta$. Hence, by an approximation argument, we can use $\psi$ as a test function in (3.52) to obtain that

$$
\sup_{t \in [0, T]} \int_{\Omega} u_n(x, t) \delta^\beta(x) dx + \int_0^T \int_{\Omega} \frac{u_n(x, t)}{\delta^{2s-\beta}(x)} dx dt \leq C(\Omega, \beta, s) \int_0^T \int_{\Omega} f \delta^\beta(x) dx dt.
$$

Hence there exists a measurable function $u \in L^\infty(0, T; L^1(\Omega, \delta^\beta dx)) \cap L^1(\Omega_T)$, such that $\frac{u_n}{\delta^{2s-\beta}} \to \frac{u}{\delta^{2s-\beta}}$ strongly in $L^1(\Omega_T)$.

We claim that the sequence $\{u_n\}_n$ is bounded in $L^\theta(\Omega_T)$, for all $\theta < \frac{N+2s}{N+1}$.

To prove the claim consider the representation formula,

$$
u_n(x, t) = \int_0^t \int_\Omega f_n(y, \sigma) P_{\Omega}(x, y, t - \sigma) dy d\sigma,
$$

then using the properties of $P_{\Omega}$, it holds that

$$
u_n(x, t) \leq C \int_0^t \int_\Omega \frac{\delta^\beta(y)}{\sqrt{t - \sigma}} f(y, \sigma) \frac{(t - \sigma)}{((t - \sigma) \frac{2s-\beta}{2s} + |x - y|)^{N+2s}} dy d\sigma
\leq C \int_0^t \int_\Omega \frac{\delta^\beta(y)}{\sqrt{t - \sigma}} \frac{1}{\delta^\beta(y)} f(y, \sigma) \frac{(t - \sigma)}{((t - \sigma) \frac{2s-\beta}{2s} + |x - y|)^{N+2s}} dy d\sigma.
$$

Since

$$
\left(1 + \frac{\delta^\beta(y)}{\sqrt{t - \sigma}}\right) \frac{1}{\delta^\beta(y)} \leq C(\Omega_T, s, \beta) (t - \sigma)^{-\frac{2s-\beta}{2s}} \text{ in } \Omega_T,
$$

we conclude that

$$
u_n(x, t) \leq C \int_0^t \int_\Omega f(y, \sigma) \frac{(t - \sigma)^{\frac{2s-\beta}{2s}}}{((t - \sigma) \frac{2s-\beta}{2s} + |x - y|)^{N+2s}} dy d\sigma.
$$

Setting $g(y, \sigma) = f(y, \sigma) \delta^\beta(y)$, then $g \in L^1(\Omega_T)$. We use the duality argument as in the proof of Theorem 3.10. For the reader convenience we include here some details.

Let $\phi \in C_0^\infty(\Omega_T)$, then

$$
\|u_n\|_{L^\infty(\Omega_T)} = \sup_{\|\phi\|_{L^\infty(\Omega_T)} \leq 1} \int_{\Omega_T} \phi(x, t) u_n(x, t) dx dt
\leq \sup_{\|\phi\|_{L^\infty(\Omega_T)} \leq 1} \int_{\Omega_T} |\phi(x, t)| \int_0^t \int_\Omega g(y, \sigma) \delta^\beta(y) \frac{(t - \sigma)^{\frac{2s-\beta}{2s}}}{((t - \sigma) \frac{2s-\beta}{2s} + |x - y|)^{N+2s}} dy d\sigma
\leq \sup_{\|\phi\|_{L^\infty(\Omega_T)} \leq 1} \int_0^T \int_\Omega \int_0^t |\phi(x, t)| H(x - y, t - \sigma) g(y, \sigma) dy dx d\sigma dt,
$$

where $H(x, \sigma) = \frac{\sigma^{\frac{2s-\beta}{2s}}}{(\sigma \frac{2s-\beta}{2s} + |x|)^{N+2s}}$. 

Let us begin by estimating

\[ I = \int_0^T \|\phi(., t)\|_{L^p(\Omega)} \int_0^t \|g(., \sigma)\|_{L^1(\Omega)} \|H(., t - \sigma)\|_{L^q(\Omega)} d\sigma dt, \]

with \( \frac{1}{q'} + \frac{1}{a} = 1 \) and then \( a = \theta. \)

By a direct computation we deduce that \( \|H(., t - \sigma)\|_{L^q(\Omega)} \leq C(t - \sigma)^{\frac{2s - \beta}{2s} + \frac{N}{2a} - \frac{(N + 2s)}{2s}}. \) Thus

\[ \|u_n\|_{L^p(\Omega_T)} \leq C \sup_{\{||\phi||_{L^p(\Omega_T)} \leq 1\}} \|\phi(., t)\|_{L^p(\Omega)} \int_0^t \|g(., \sigma)\|_{L^1(\Omega)} \|H(., t - \sigma)\|_{L^q(\Omega)} d\sigma dt, \]

Using Hölder inequality, we get

\[ \|u_n\|_{L^p(\Omega_T)} \leq C \sup_{\{||\phi||_{L^p(\Omega_T)} \leq 1\}} \|\phi(., t)\|_{L^p(\Omega)} \left( \int_0^t (t - \sigma)^{\theta\left(\frac{2s - \beta}{2s} + \frac{N}{2a} - \frac{(N + 2s)}{2s}\right)} d\sigma \right)^{\frac{1}{\theta}} dt. \]

The last integral is finite if and only if \( \theta \frac{2s - \beta}{2s} + \frac{N}{2a} - \frac{(N + 2s)}{2s} > -1. \) Since \( \frac{1}{q'} + \frac{1}{a} = 1, \) then the above condition holds if \( \theta < \frac{N + 2s}{N + \beta} \) and then the claim follows. Thus

\[ \|u_n\|_{L^p(\Omega_T)} \leq C(\Omega_T, s, \beta) \|g\|_{L^1(\Omega_T)} = C\|f\|_{L^p(\Omega_T)}. \]

In the same way we can prove that the sequence \( \{u_n\}_n \) is bounded in \( L^p(\Omega_T) \) for all \( \theta < \frac{N + 2s}{N + \beta + 1}. \)

To show that the sequence \( \{\|\nabla u_n\|\}_n \) is bounded in \( L^p(\Omega_T) \) for all \( 1 \leq p < \frac{N + 2s}{N + \beta + 1}, \) we use the same arguments as in the proof of Theorem 3.11. We have

\[ \|\nabla u_n(x, t)\| \leq C \int_0^t \int_{\Omega} f_n(y, \sigma) \|\nabla_x P_\sigma(x, y, t - \sigma)\| dy d\sigma \]

\[ \leq C \left( \int_{\Omega \times (0,t)} \int_{\Omega \times (0,t)} f_n(y, \sigma) \|\nabla_x P_\sigma(x, y, t - \sigma)\| P_\sigma(x, y, t - \sigma) \|\nabla_x P_\sigma(x, y, t - \sigma)\| dy d\sigma \right)^{\frac{1}{2}} \left( \int_{\Omega \times (0,t)} \int_{\Omega \times (0,t)} f_n(y, \sigma) \|\nabla_x P_\sigma(x, y, t - \sigma)\| \|\nabla_x P_\sigma(x, y, t - \sigma)\| dy d\sigma \right)^{\frac{1}{2}} \]

\[ = I_{1n}(x, t) + I_{2n}(x, t). \]

Let us begin by estimating \( I_{1n}. \) Using estimate (3.2) and by Hölder inequality, we get

\[ I_{1n}^q(x, t) = \left( \int_{\Omega \times (0,t)} \int_{\Omega \times (0,t)} f_n(y, \sigma) \delta(\sigma) \left( (t - \sigma)^{\frac{1}{2} - \frac{\beta}{2s}} + |x - y| \right)^{N + \beta + 1} dy d\sigma \right)^q \]

\[ \leq C \|f_n\|_{L^{q-1}(\Omega_T)} \left( \int_{\Omega \times (0,t)} \int_{\Omega \times (0,t)} f_n(y, \sigma) \delta(\sigma) \left( (t - \sigma)^{\frac{1}{2} - \frac{\beta}{2s}} + |x - y| \right)^{N + \beta + 1} dy d\sigma \right)^{\frac{1}{q}} \]

Integrating in \( \Omega_T, \) as in the proof of Theorem 3.11, we deduce that

\[ \int_{\Omega_T} I_{1n}^q(x, t) dt \leq CT^{\gamma_1 + 1} \|f_n\|_{L^\infty(\Omega_T)}^{\frac{q}{q - 1}} \]

with \( \gamma_1 = \frac{N}{2s} - q \frac{N + \beta + 1}{2s} > -1. \)
Now respect to $I_{2n}$, using estimate (3.11) and by H"older inequality, it follows that
\[
I_{2n}^q(x,t) = \frac{C}{\delta^q(x)} \left( \int_{\{x \in (0,t) \cap \delta(x) \leq t + \frac{1}{4}\}} f_n(y,\sigma) P_\Omega(x,y,t-\sigma) \, dy \, d\sigma \right)^q
\]
\[
\leq \frac{C||f_n\delta^\beta||_{L^1(\Omega_T)}^q}{\delta^q(x)} \int_{\{x \in (0,t) \cap \delta(x) \leq t + \frac{1}{4}\}} f_n(y,\sigma) P_\Omega(x,y,t-\sigma) \, dy \, d\sigma
\]
\[
\leq C||f_n\delta^\beta||_{L^1(\Omega_T)} \int_{\{x \in (0,t) \cap \delta(x) \leq t + \frac{1}{4}\}} f_n(y,\sigma) \frac{P_\Omega(x,y,t-\sigma)}{\delta^q(y)} \, dy \, d\sigma.
\]
Integrating in $\Omega_T$,
\[
\int_{\Omega_T} I_{2n}^q(x,t) \, dx \, dt \leq C||f_n\delta^\beta||_{L^1(\Omega_T)} \int_{\Omega_T} f_n(y,\sigma) \frac{\varphi(y)}{\delta^q(y)} \, dy \, d\sigma.
\]
Recalling that
\[
\int_0^T P_\Omega(x,y,t-\sigma) \, dt \leq \int_0^{T-\sigma} P_\Omega(x,y,\eta) \, d\eta \leq G_\varsigma(x,y),
\]
we find that
\[
\int_{\Omega_T} I_{2n}^q(x,t) \, dx \, dt \leq C||f_n\delta^\beta||_{L^1(\Omega_T)} \int_{\Omega_T} f_n(y,\sigma) \varphi(y) \, dy,
\]
where $\varphi$ is the unique solution to problem (3.13). Since $s < q(N+1) - N < 2s$ then from [1] and [9], it follows that $\varphi \in L^\infty(\Omega)$ and $\varphi(y) \simeq (\delta(y))^{2s-(q(N+1)-N)}$. Thus
\[
\varphi(y) \simeq (\delta(y))^{2s-(q(N+1)-N)}\delta^\beta(y).
\]
It is clear that $2s - (q(N+1) - N) - q\beta > 0$ if and only if $q < \frac{N+2s}{N+1+\beta}$, which is the hypothesis. Thus
\[
\int_{\Omega_T} I_{2n}^q(x,t) \, dx \, dt \leq C||f_n\delta^\beta||_{L^1(\Omega_T)}^q.
\]
As a conclusion, we have proved that for all $q < \frac{N+2s}{N+1+\beta}$,
\[
||\nabla u_n||_{L^q(\Omega_T)} \leq C(\Omega_T)||f_n\delta^\beta||_{L^1(\Omega_T)}.
\]
Hence there exists a solution $u$ to problem (3.1) in the sense of distributions such that
\[
u \in L^q(0,T; W^{1,q}_0(\Omega)) \cap C([0,T], L^1(\Omega, \delta^\beta \, dx)), \text{ for all } q < \frac{N+2s}{N+\beta+1}.
\]
It is clear that if $u_1, u_2 \in L^q(0,T; W^{1,q}_0(\Omega)) \cap C([0,T], L^1(\Omega, \delta^\beta \, dx))$ are solutions to (3.1), then the difference $v = u_1 - u_2$, satisfies $v \in L^q(0,T; W^{1,q}_0(\Omega)) \cap C([0,T], L^1(\Omega, \delta^\beta \, dx))$ and
\[
\begin{cases}
v_t + (-\Delta)^s v = 0 & \text{in } \Omega_T, \\
v = 0 & \text{in } (\mathbb{R}^N \setminus \Omega) \times (0,T), \\
v(x,0) = 0 & \text{in } \Omega.
\end{cases}
\]
Using Kato inequality as in Theorem 2.7, we reach that $v_+ \in L^q(0,T; W^{1,q}_0(\Omega)) \cap C([0,T], L^1(\Omega, \delta^\beta \, dx))$ satisfies
\[
(v_+)_t + (-\Delta)^s v_+ \leq 0 \text{ in } \Omega_T.
\]
Hence using $\phi_1$, the positive first eigenfunction of the fractional Laplacian in $\Omega$, as a test function in (3.57), we reach that $v_+ = 0$. In the same way and since $-v$ is also a solution to (3.56), we conclude that $v_- = 0$. Thus $v = 0$.

Now setting
\[
\hat{K}: L^1(\Omega_T, \delta^\beta \, dx) \times L^1(\Omega) \to L^q(0,T; W^{1,q}_0(\Omega)), q < \frac{N+2s}{N+\beta+1},
\]
where $\hat{K}(f, u_0) = u$ is the unique solution to problem (3.1), then as in the proof of Theorem 3.5, taking advantage of the linearity of the operator, we conclude that $\hat{K}$ is a compact operator. □
As in proposition 3.16, if \( f \in L^1(\Omega_T, \delta^\beta(x)dxdt) \cap L^m(K \times (0, T)) \), with \( m > 1 \) and \( K \subset \subset \Omega \) is any compact set of \( \Omega \), then we have the next general regularity result.

**Proposition 3.23.** Let \( m > 1 \). Assume that \( f \in L^1(\Omega_T, \delta^\beta(x)dxdt) \cap L^m(K \times (0, T)) \) for any compact set \( K \) of \( \Omega \). Define \( u \) to be the unique weak solution to problem (3.1) and let \( \Omega_1 \subset \subset \Omega \) with \( \text{dist}(\Omega_1, \partial \Omega) > 0 \). Consider \( K_1 \subset \subset \Omega \), a compact set of \( \Omega \) such that \( \Omega_1 \subset \subset K_1 \).

Then \( u \in L^p(\Omega_1 \times (0, T)) \) for all \( \theta < \frac{m(N + 2s)}{(N + 2s - m(2s - \beta))_+} \) and \( |\nabla u| \in L^p(\Omega_1 \times (0, T)) \) for all \( p < \frac{m(N + 2s)}{(N + 2s - m(2s - 1 - \beta))_+} \). Moreover

\[
|u||_{L^p(\Omega_1 \times (0, T))} + \|\nabla u||_{L^p(\Omega_1 \times (0, T))} \leq C(||f||_{L^m(K_1 \times (0, T))} + ||f\delta^\beta||_{L^1(\Omega_T)}),
\]

(3.58)

where \( C := C(K_1, \Omega_1, \Omega, T, N, m) \).

**Proof.** Since \( f\delta^\beta \in L^1(\Omega_T) \), then \( |\nabla u| \in L^p(\Omega_T) \) for all \( q < \frac{N + 2s}{N + 2s + 1} \). As in the proof of Proposition 3.16, we have

\[
u_u(x, t) \leq C \int_0^t \int_\Omega f(y, \sigma)\delta^\beta(y) \frac{(t - \sigma)^{\frac{2s-\beta}{2s}}}{((t - \sigma)^{\frac{2s}{2s}} + |x - y|)^{N+2s}} dy d\sigma.\]

Fix \( \Omega_1 \subset \subset \Omega \) with \( \text{dist}(\Omega_1, \partial \Omega) = c_0 > 0 \) and let \( K_1 \) be a compact set of \( \Omega \) such that \( \Omega_1 \subset \subset K_1 \subset \subset \Omega \). Let \( x \in \Omega_1 \), then

\[
u(x) \leq C(\Omega_1, c_0, C) \left\{ \int_0^t \int_\Omega \theta f(y, \sigma)\delta^\beta(y) \frac{(t - \sigma)^{\frac{2s-\beta}{2s}}}{((t - \sigma)^{\frac{2s}{2s}} + |x - y|)^{N+2s}} dy d\sigma + \int_\Omega \int_\Omega f(y, \sigma)\delta^\beta(y) \frac{(t - \sigma)^{\frac{2s-\beta}{2s}}}{((t - \sigma)^{\frac{2s}{2s}} + |x - y|)^{N+2s}} dy d\sigma \right\}.
\]

If \((x, y) \in (\Omega_1 \subset \subset K_1) \times (\Omega \setminus K_1)\), then \(|x - y| > \delta > 0\). Hence

\[
\int_0^t \int_{\Omega \setminus K_1} f(y, \sigma)\delta^\beta(y) \frac{(t - \sigma)^{\frac{2s-\beta}{2s}}}{((t - \sigma)^{\frac{2s}{2s}} + |x - y|)^{N+2s}} dy d\sigma \leq C||f\delta^\beta||_{L^1(\Omega_T)}.
\]

Thus

\[
u(x) \leq C(\Omega_1, c_0, C) \left( \int_0^t \int_{K_1} f(y, \sigma)\delta^\beta(y) \frac{(t - \sigma)^{\frac{2s-\beta}{2s}}}{((t - \sigma)^{\frac{2s}{2s}} + |x - y|)^{N+2s}} dy d\sigma + ||f\delta^\beta||_{L^1(\Omega_T)} \right).
\]

Since \( f\delta^\beta \in L^m(K_1 \times (0, T)) \), then we exactly follow the same duality argument as in the proof of Theorem 3.10. Hence estimate (3.58) follows. In a similar way we prove estimate in the gradient. □

**Remarks 3.24.** To obtain the above regularity result we have used the fact that if

\[
g(x, t) := \int_0^t \int_\Omega f(y, \sigma) \frac{(t - \sigma)^{\gamma}}{((t - \sigma)^{\frac{2s}{2s}} + |x - y|)^{N+2s}} dy d\sigma
\]

with \( f \in L^m(\Omega_T) \), \( m \geq 1 \) and \( a > 0 \), then \( g \in L^\gamma(\Omega_T) \) where \( \gamma \) satisfies

\[
\frac{1}{\gamma} \geq \frac{1}{m} - \frac{2sa}{N + 2s}.
\]

This result will be used systematically in what follows.

4. Non existence result.

In the local case, \( s = 1 \), existence of solution holds for all \( \alpha > 0 \) at least for \( f \in L^\infty(\Omega_T) \) and \( u_0 \in L^\infty(\Omega) \), see for instance [14], [29] and [48]. However in the nonlocal case, \( s < 1 \), and by the lack of regularity near of the boundary \( \partial \Omega \), a threshold on \( \alpha \) appears for the existence. This behavior represents a deep difference with the local case, though it is stable when \( s \to 1 \). Let us recall and prove the non existence result stated in the introduction.
Theorem 4.1. Assume that \( \alpha \geq \frac{1}{1-s} \), then for all nonnegative data \((f, u_0) \in L^\infty(\Omega_T) \times L^\infty(\Omega)\) with \((f, u_0) \neq (0, 0)\), the problem

\[
\begin{align*}
  u_t + (-\Delta)^s u &= |\nabla u|^\alpha + f & \text{in } \Omega_T, \\
  u(x, t) &= 0 & \text{in } (\mathbb{R}^N \setminus \Omega) \times [0, T), \\
  u(x, 0) &= u_0(x) & \text{in } \Omega,
\end{align*}
\]  

has no weak solution \( u \) in the sense of Definition 2.3 with \( u \in L^\alpha(0, T; W^{1,\alpha}_0(\Omega)) \).

Before starting with the proof of Theorem 4.1, we need the following result that extends to the fractional framework the one proved in [42] for the heat equation. The result is proved in [13] using apriori estimates. We give here a different proof using the properties of the Dirichlet heat kernel.

Proposition 4.2. Assume that the condition on \((f, u_0)\) of the above Theorem holds. Let \( w \) be the unique solution to the problem

\[
\begin{align*}
  w_t + (-\Delta)^s w &= f & \text{in } \Omega_T, \\
  w(x, t) &= 0 & \text{in } (\mathbb{R}^N \setminus \Omega) \times [0, T), \\
  w(x, 0) &= u_0(x) & \text{in } \Omega,
\end{align*}
\]  

with either \( u_0 \geq 0 \) or \( f \geq 0 \) in \( \Omega \times (0, t_0) \), \( t_0 < T \) being fixed. Then there exists \( C := C(t_0, \Omega, u_0, f) > 0 \) such that

\[
w(x, t_0) \geq C \delta^s(x) \text{ in } \Omega.
\]  

Proof. Notice that

\[
w(x, t) = \int_\Omega u_0(y) P_\Omega(x, y, t) \, dy + \int_0^t \int_\Omega f(y, \sigma) P_\Omega(x, y, t - \sigma) \, dy \, d\sigma
\]

\[
\geq \int_\Omega u_0(y) P_\Omega(x, y, t) \, dy.
\]

Suppose in the first case that \( u_0 \geq 0 \), then

\[
w(x, t) \geq \int_\Omega u_0(y) P_\Omega(x, y, t) \, dy.
\]

Now fix \( t_0 \in (0, T) \), using the estimate on \( P_\Omega \) given in (3.2), we deduce that

\[
w(x, t_0) \geq C \left( 1 \land \frac{\delta^s(x)}{t_0} \right) \int_\Omega \left( 1 \land \frac{\delta^s(y)}{t_0} \right) \times \left( t_0 \land \frac{t_0}{|x - y|^{N+2s}} \right) u_0(y) \, dy.
\]

Notice that for all \( x, y \in \Omega \), we have

\[
\left( t_0 \land \frac{t_0}{|x - y|^{N+2s}} \right) \geq C \frac{t_0}{(t_0^2 + |x - y|^{N+2s})} \geq C \frac{t_0}{(t_0^2 + \text{diam}(\Omega))^{N+2s}},
\]

and

\[
\left( 1 \land \frac{\delta^s(x)}{t_0} \right) \geq \delta^s(x) \min\left\{ \frac{1}{\sqrt{t_0}}, \frac{1}{\text{diam}^s(\Omega)} \right\} = \delta^s(x)C(t_0, \Omega).
\]

Hence

\[
w(x, t_0) \geq R(t_0) \delta^s(x) \int_\Omega u_0(y) \delta^s(y) \, dy,
\]

with

\[
R(t) = \left( \frac{t_0}{(t_0^2 + \text{diam}(\Omega))^{N+2s}} \left( \min\left\{ \frac{1}{\sqrt{t_0}}, \frac{1}{\text{diam}^s(\Omega)} \right\} \right)^2 \right)^{1/2}.
\]

Notice that

\[
R(t) \geq \begin{cases} 
  C(\Omega) & \text{if } t \geq (\text{diam}(\Omega))^{2s} \\
  C(\Omega)t & \text{if } t \leq (\text{diam}(\Omega))^{2s}.
\end{cases}
\]

Thus \( w(x, t_0) \geq C(t_0, u_0, \Omega)\delta^s(x) \) which is the desire estimate.
Consider now the case where \( f \geq 0 \) in \( \Omega \times (0, t_0) \). As above we deduce that
\[
 w(x, t) \geq \int_0^t \int_{\Omega} f(y, \sigma) P_\Omega(x, y, t - \sigma) \, dy \, d\sigma.
\]
For \( t_0 > 0 \) fixed, using the properties of the kernel \( P_\Omega \), given in (3.2), it holds that
\[
 w(x, t_0) \geq C \int_0^{t_0} \int_{\Omega} \left( 1 \wedge \frac{\delta^s(x)}{\sqrt{t_0 - \sigma}} \right) \left( 1 \wedge \frac{\delta^s(y)}{\sqrt{t_0 - \sigma}} \right) \times \left( (t_0 - \sigma)^{-\frac{\alpha}{p}} \wedge \frac{t_0 - \sigma}{|x - y|^N} \right) f(y, \sigma) \, dy \, d\sigma.
\]
As in the first case, it follows that
\[
 w(x, t_0) \geq C \delta^s(x) \int_0^{t_0} \int_{\Omega} R(t_0 - \sigma) f(y, \sigma) \, dy \, d\sigma.
\]
Since \( R_0(t_0 - \sigma) \geq C(T, \Omega)(t_0 - \sigma) \), then using the fact that \( f \geq 0 \) in \( \Omega \times (0, t_0) \), it follows that
\[
 \int_0^{t_0} \int_{\Omega} R(t_0 - \sigma) f(y, \sigma) \, dy \, d\sigma = C(t_0, f, \Omega) > 0.
\]
Thus we conclude. \( \square \)

**Remark 4.3.** It is clear that if \( t \in (t_1, t_2) \subset (0, T) \), by Proposition 4.2 we have that, for all \( (x, t) \in \Omega \times (t_1, t_2) \),
\[
 w(x, t) \geq C(t_0, t_1, \Omega) \delta^s(x) \int_\Omega u_0(y) \delta^s(y) dy.
\]

We are now in position to prove the non existence result in Theorem 4.1.

**Proof of Theorem 4.1.** We argue by contradiction. Assume that \( \alpha \geq \frac{1}{1 - s} \) and suppose that problem (4.1) has a solution \( u \in L^\alpha(0, T; W_0^{1, \alpha}(\Omega)) \). Fix \( (t_1, t_2) \subset (0, T) \), then by Proposition 4.2, we reach that
\[
 u(x, t) \geq C(t_0, t_1 \Omega) \delta^s(x) \int_\Omega u_0(y) \delta^s(y) dy \quad \text{for all } (x, t) \in \Omega \times (t_1, t_2).
\]
Since \( u \in L^\alpha(0, T; W_0^{1, \alpha}(\Omega)) \), using Hardy inequality it holds that \( \frac{u}{\delta} \in L^\alpha(\Omega \times (t_1, t_2)) \). Thus
\[
 C(t_0, t_1 \Omega) \int_{t_1}^{t_2} \int_\Omega \frac{\delta^{\alpha s}(x)}{\delta^s(x)} dx \leq C \int_{t_1}^{t_2} \int_\Omega \frac{u^\alpha(x, t)}{\delta^s(x)} dx < \infty.
\]
Since \( \alpha \geq \frac{1}{1 - s} \), then \( \alpha(1 - s) \geq 1 \) and then we reach a contradiction. Hence we conclude. \( \square \)

If we are dealing with weak solution in the sense of Definition 2.3, we can prove a non existence result for a suitable range of \( \alpha \). Before starting with the non existence result, we recall the weighted Hardy inequality proved in [44] (see also [40]).

**Proposition 4.4.** Assume that \( \alpha > 1 \) and \( 0 < \sigma < \alpha - 1 \), then for all \( \phi \in W_0^{1, \alpha}(\Omega) \), we have
\[
 \int_\Omega \frac{\phi(x)}{\delta^{\alpha - \sigma}(x)} dx \leq C \int_\Omega |\nabla \phi(x)|^\alpha \delta^\sigma(x) dx,
\]
where \( C := C(\Omega, p, N) \).

Define the space
\[
 \tilde{W}_{\alpha, \sigma}(\Omega) := \{ \varphi \in W_0^{1,1}(\Omega) \text{ with } \int_\Omega |\nabla \varphi|^\alpha \delta^\sigma dx < \infty \}.
\]
If \( \sigma + 1 < \alpha \), then using Hölder inequality, the space \( \tilde{W}_{\alpha, \sigma}(\Omega) \) can be endowed with the norm
\[
 ||\varphi||_{\tilde{W}_{\alpha, \sigma}(\Omega)} = \left( \int_\Omega |\nabla \varphi|^\alpha \delta^\sigma dx \right)^{\frac{1}{\alpha}}.
\]
Let \( \tilde{H}_{\alpha, \sigma}(\Omega) \) be the completion of \( C_0^\infty(\Omega) \) with respect to the norm of \( \tilde{W}_{\alpha, \sigma}(\Omega) \).
Notice that $\delta^\sigma$ belongs to the Muckenhoupt class $A_2$ if $0 < (\sigma + 1) < \alpha$, see for instance Theorem 3.1 in [28]. Hence by the results of [52], we reach that $\tilde{H}_{\alpha,\sigma}(\Omega) = \tilde{W}_{\alpha,\sigma}(\Omega)$. Thus, for all $\varphi \in \tilde{W}_{\alpha,\sigma}(\Omega)$ and by Proposition 4.4, for all $\varphi \in \tilde{W}_{\alpha,\sigma}(\Omega)$, we have
\begin{equation}
C(\Omega, N) \int_{\Omega} \frac{|\varphi(x)|^\alpha}{\delta^{\alpha-\beta}(x)} dx \leq \int_{\Omega} |\nabla \varphi|^\alpha \delta^\sigma dx. \tag{4.5}
\end{equation}

As a consequence, we have the next general non existence result.

**Theorem 4.5.** Assume that $\alpha \geq \frac{1 + \beta}{1 - s}$ for some $\beta > 0$. Then for all nonnegative data $(f, u_0) \in L^\infty(\Omega_T) \times L^\infty(\Omega)$ with $(f, u_0) \neq (0, 0)$, the problem (4.1) has no weak solution $u$ in the sense of Definition 2.3 such that $u \in L^1(0, T; W^{1,1}_{\omega,\sigma}(\Omega))$ with $|\nabla u|^\alpha \delta^\sigma \in L^1(\Omega_T)$.

**Proof.** We argue by contradiction. Assume that the above conditions hold and that problem (1.1) has a weak solution $u$ with $u \in L^1(0, T; W^{1,1}_{\omega,\sigma}(\Omega))$ and $|\nabla u|^\alpha \delta^\sigma \in L^1(\Omega_T)$. Since $\beta + 1 < \alpha$, then $u \in \tilde{W}_{\alpha,\sigma}(\Omega)$. Fix $(t_1, t_2) \subset (0, T)$, then by Proposition 4.2, we reach that
\begin{equation}
u(x, t) \geq C(t_0, t_1, t_2) \delta^{\alpha}(x) \text{ for all } (x, t) \in (0, T). \tag{4.6}
\end{equation}

Since $u \in L^\alpha(0, T; \tilde{W}_{\alpha,\sigma}(\Omega))$, using the weighted Hardy inequality (4.5), it holds that
\begin{equation}
C(\Omega, N) \int_{\Omega} \frac{u^\alpha(x, t)}{\delta^{\alpha-\beta}(x)} dx \leq \int_{\Omega} |\nabla u(x, t)|^\alpha \delta^\sigma dx \text{ a.e. in } (0, T).
\end{equation}

Integrating in the time,
\begin{equation}
C(\Omega, N) \int_{t_0}^{t_1} \int_{\Omega} \frac{u^\alpha(x, t)}{\delta^{\alpha-\beta}(x)} dx \leq \int_{t_0}^{t_1} \int_{\Omega} |\nabla u(x, t)|^\alpha \delta^\sigma dx < \infty.
\end{equation}

Hence by estimate (4.6) we reach that
\begin{equation}
C(t_0, t_1, \Omega) \int_{\Omega} \frac{1}{\delta^{\alpha(1-s)-\beta}(x)} dx \leq \int_{t_0}^{t_1} \int_{\Omega} |\nabla u(x, t)|^\alpha \delta^{1-s} dx < \infty.
\end{equation}

Using the fact that $\alpha(1-s) - \beta \geq 1$, we reach a contradiction. \hfill \square

**Corollary 4.6.**

1. Assume that $\alpha \geq \frac{1 + s}{1 - s}$, then problem (4.1) has no weak solution $u$ in the sense of Definition 2.3 with $u \in L^1(0, T; W^{1,1}_{0,\omega}(\Omega))$ and $|\nabla u|^\alpha \delta^\sigma \in L^1(\Omega_T)$.

2. Assume that $\alpha \geq \frac{2s}{1 - s}$, then problem (4.1) has no weak solution $u$ in the sense of Definition 2.3 with $u \in L^1(0, T; W^{1,1}_{0,\omega}(\Omega))$ and $|\nabla u|^\alpha \delta^\sigma \in L^1(\Omega_T)$ for some $\beta < 2s - 1$.

**Remarks 4.7.** The above non existence results make a significative difference with respect to the local case $s = 1$, where an existence result holds for all $\alpha > 1$ under suitable condition of the data. See for example [14] and the reference therein.

5. The existence results.

The goal of this section is to prove an existence result for problem (1.1) under suitable condition on $\alpha$ and the data $f$ as was established in the Introduction.

5.1. **Existence result for $L^1$ data and $\alpha < \frac{N + 2s}{N + 1}$.** In this subsection we will prove Theorem 1.1. We assume that $(f, u_0) \in L^m(\Omega_T) \times L^1(\Omega)$ where $1 \leq m < \frac{1}{s}$. By the regularity result in Theorem 3.5 and the second point in Corollary 3.13, it follows that the condition $\alpha < \frac{N + 2s}{N + 1}$ is natural in order to get the existence of a solution to problem (1.1).
Proof of Theorem 1.1. Let $T < T'$ to be chosen later and define $E_q(\Omega_T) = L^q(0, T; W_0^{1,q}(\Omega))$ for $q \geq 1$. Assume that $1 \leq m < \frac{1}{s}$ and fix $\alpha < \frac{N + 2s}{N + 1}$. Let $l > 0$ to be chosen later. Define the set

$$E(\Omega_T) = \{ v \in E_1(\Omega_T) \text{ such that } v \in E_r(\Omega_T) \text{ with } \alpha < r < \frac{N + 2s}{N + 1} \text{ and } \| v \|_{E_r(\Omega_T)} \leq l^{\frac{\alpha}{s}} \}.$$ 

It is easy to check that $E(\Omega_T)$ is a closed convex set of $E_1(\Omega_T)$. For $(f, u_0) \in L^m(\Omega_T) \times L^1(\Omega)$ fixed, we consider the operator

$$K : E(\Omega_T) \to E_1(\Omega_T)$$

$$v \to T(v) = u$$

where $u$ is the unique solution to problem

$$\begin{aligned}
\{ 
 u_t + (-\Delta)^s u &= |\nabla v|^\alpha + f &\text{in } \Omega_T \equiv \Omega \times (0, T), \\
 u(x, t) &= 0 &\text{in } (\mathbb{R}^N \setminus \Omega) \times [0, T), \\
 u(x, 0) &= u_0(x) &\text{in } \Omega.
\end{aligned}$$

(5.1)

Since $\alpha < r$, then $|\nabla v|^\alpha + f \in L^1(\Omega_T)$, thus the existence of $u$ is a consequence of [41] and moreover $u \in L^q(0, T; W_0^{1,q}(\Omega)) \cap C([0, T]; L^1(\Omega))$ for all $q < \frac{N + 2s}{N + 1}$. Hence $K$ is well defined.

We claim that:

1. There exists $l > 0$ such that $K(E(\Omega_T)) \subset E(\Omega_T)$.
2. $K$ is a continuous and compact operator on $E(\Omega_T)$.

Proof of the claim. First, we prove that we can find $l > 0$ such that $K(E(\Omega_T)) \subset E(\Omega_T)$. In fact, thanks to Theorem 3.5, we have that $u \in E_q(\Omega_T)$ for all $q < \frac{N + 2s}{N + 1}$. In particular, $u \in E_r(\Omega_T)$.

Now, fixed $q < \frac{N + 2s}{N + 1}$, it follows that

$$\| u \|_{E_q(\Omega_T)} \leq C(T, \Omega) \left( \| f + |\nabla v|^\alpha \|_{L^1(\Omega_T)} + \| u_0 \|_{L^1(\Omega)} \right).$$

Recall that, by Remark 3.6, we know that $C(\Omega, T) \to 0$ as $T \to 0$, then choosing $T$ small, there exists $l > 0$ such that $C(T, \Omega) \left( \| f \|_{L^1(\Omega_T)} + \| u_0 \|_{L^1(\Omega)} + l \right) \leq l^{\frac{\alpha}{s}}$. Hence

$$\| u \|_{E_r(\Omega_T)} \leq l^{\frac{\alpha}{s}},$$

and then $u \in E(\Omega_T)$. Thus $K(E(\Omega_T)) \subset E(\Omega_T)$. Hence, from now, we fix $T < T'$ such that the above conclusions hold.

To prove the continuity of $K$ with respect to the topology of $E_1(\Omega_T)$, we consider $\{ v_n \} \subset E(\Omega_T)$ such that $v_n \to v$ strongly in $E_1(\Omega_T)$. Define $u_n = K(v_n)$, $u = K(v)$ and $w_n = u_n - u$. We have to show that $w_n \to u$ strongly in $E_1(\Omega_T)$. Since

$$w_{nt} + (-\Delta)^s w_n = |\nabla v_n|^\alpha - |\nabla v|^\alpha,$$

to show that $w_n \to 0$ strongly in $E_1(\Omega_T)$, we have to prove that $\| \nabla v_n - \nabla v \|_{L^\infty(\Omega_T)} \to 0$ as $n \to \infty$.

Recall that $\{ v_n \} \subset E(\Omega_T)$ and $\| v_n - v \|_{E_1(\Omega_T)} \to 0$ as $n \to \infty$, then $\nabla v_n \to \nabla v$ strongly in $(L^1(\Omega_T))^N$ and $\| \nabla v_n \|_{L^\infty(\Omega_T)} \leq C$. 

Since $1 < \alpha < r$, then using Hölder inequality, we reach that
\[
\|\nabla v_n - \nabla v\|_{L^s(\Omega_T)} \leq \|\nabla v_n - \nabla v\|_{L^1(\Omega_T)} \|\nabla v\|_{L^r(\Omega_T)} \leq C\|\nabla v\|_{L^1(\Omega_T)} \rightarrow 0 \text{ as } n \rightarrow \infty.
\]
Now, by using the definition of $u_n$ and $u$, there results that $u_n \to u$ strongly in $E_1(\Omega_T)$. Thus $K$ is continuous.

To finish we have just to show that $K$ is a compact operator with respect to the topology of $E_1(\Omega_T)$. Let $\{v_n\}_n \subset E(\Omega_T)$ be such that $\|v_n\|_{E_1(\Omega_T)} \leq C$. Since $\{v_n\}_n \subset E_r(\Omega_T)$, then $\|v_n\|_{E_r} \leq C$ and therefore up to a subsequence, $v_n \rightharpoonup v$ in $E_r(\Omega_T)$.

Since $\alpha < r$, then there exists $\delta > 0$ such that the sequence $\{\|v_n\|_\alpha\}_n$ is bounded in $L^{1+\delta}(\Omega_T)$. Thus, up to a subsequence,
\[
\|\nabla v_n\|_\alpha \to g \text{ weakly in } L^{1+\delta}(\Omega_T).
\]
Let $u$ to be the unique solution to the problem
\[
\begin{aligned}
&u_t + (-\Delta)^su = g + f \quad \text{in } \Omega_T, \\
u(x,0) = 0 \quad \text{in } (\mathbb{R}^N \setminus \Omega) \times [0,T), \\
u(x,0) = u_0(x) \quad \text{in } \Omega,
\end{aligned}
\]
using the compactness result of Theorem 3.5, we conclude that, up to a subsequence, $u_n \to u$ strongly in $E_0(\Omega_T)$ for all $q < \frac{N + 2s}{N + 1}$. In particular $u_n \to u$ strongly in $E_r(\Omega_T)$, hence the claim follows.

As a conclusion and using the Schauder Fixed Point Theorem, there exists $u \in E(\Omega_T)$ such that $K(u) = u$, then $u \in L^q(0,T;W^{1,q}_0(\Omega))$ and $u$ solves (1.1). It is not difficult to show that $T_k(u) \in L^2(0,T;H^s_0(\Omega))$ for all $k > 0$. \hfill \Box

**Remark 5.1.**

1. Using a suitable approximation argument we can prove that the existence result of Theorem 1.1 holds if $f$ is a bounded Radon measure.

2. The existence result of Theorem 1.1 is optimal in the sense that if $\alpha > \frac{N + 2s}{N + 1}$, then we can find $f \in L^1(\Omega_T)$ or $u_0 \in L^1(\Omega)$ such that problem (1.1) has no solution in the space $L^{\sigma}(0,T;W^{1,\sigma}_0(\Omega))$. To see the optimality condition, we will use Remark 2.5.

Fix $0 \leq f \in L^1(\Omega_T)$ or $0 \leq u_0 \in L^1(\Omega)$ such that problem (3.1) has a solution $v$ with $v_m \notin L^1(\Omega_T)$, being $m = \frac{N + 2s}{N}$. Suppose now that problem (1.1) has a solution $u$ with $\alpha \geq \frac{N + 2s}{N + 1}$. By the comparison principle we easily reach that $u \geq v$. Since $u \in L^{\alpha}(0,T;W^{1,\alpha}_0(\Omega)) \cap C([0,T];L^1(\Omega))$, then we obtain that $u \in L^\sigma(\Omega_T)$ where $\sigma = \alpha\frac{N + 1}{N}$. Since $v \leq u$, then $v \in L^\sigma(\Omega_T)$. It is clear that $\sigma > \frac{N + 2s}{N}$, hence we reach a contradiction with the hypotheses on $v$.

3. For the uniqueness and the global existence in the time, we refer to Theorem 6.7.

5.2. Existence results for $\frac{N + 2s}{N + 1} < \alpha$. Proofs of Theorem 1.3 and Theorem 1.4. In this subsection we assume that $\frac{N + 2s}{N + 1} < \alpha$. According to the regularity of $f$ and $u_0$, we are able to show the existence of a solution that is in a suitable Sobolev space, under suitable condition on $m$ and $s$. For simplicity of presentation we will consider two cases:

- $f \neq 0$, $u_0 = 0$ and
- $f = 0$, $u_0 \neq 0$.

**Proof of Theorem 1.3.**

Recall that $\frac{2s - 1}{1 - s} > \frac{(N + 2s)^2}{N + 1}$ and $\frac{N + 2s}{N + 1} < \alpha < \frac{2s - 1}{(1 - s)(N + 2s)}$.

Assume that $f \in L^m(\Omega_T)$ with $m \geq \frac{1}{s}$. We present a proof in the case $m < \frac{N + 2s}{2s - 1}$. The other case follows in a similar way.
Since \( \frac{N+2s}{2s-1} \frac{1}{(1-s)(N+2s)} < m \), then \( \alpha < \frac{(N + 2s)}{(N + 2s)(m(1 - s) + 1) - m(2s - 1)} \). It is clear that

\[
m > \frac{2s - 1}{2s - 1 - (1 - s)(N + 2s)}.
\]

Hence using the fact that

\[
\frac{N + 2s}{N + 1} \leq \alpha < \frac{(N + 2s)}{(N + 2s)(m(1 - s) + 1) - m(2s - 1)},
\]

we get the existence of \( r \) such that \( ma < r < \frac{m(N + 2s)}{(N + 2s)(m(1 - s) + 1) - m(2s - 1)} = \hat{P} \) defined in Corollary 3.13.

Recall that \( E_{\sigma} (\Omega_T) \equiv L^s (0, T; W^{1,\sigma}_0 (\Omega)) \). Define the set

\[
E (\Omega_T) = \{ v \in E_1 (\Omega_T) \text{ such that } v \in E_r (\Omega_T) \text{ with } \| v \|_{E_{\sigma} (\Omega_T)} \leq \frac{1}{\hat{P}} \}.
\]

(5.3)

Then \( E (\Omega_T) \) is a closed convex set of \( E_1 (\Omega_T) \). Setting

\[
K : E (\Omega_T) \rightarrow E_1 (\Omega_T)
\]

\[
v \mapsto K (v) = u
\]

where \( u \) is the unique solution to problem

\[
\left\{ \begin{array}{ll}
\frac{\partial u}{\partial t} + (-\Delta)^{\alpha} u &= |\nabla u|^s + f & \text{in } \Omega_T, \\
u(x,t) &= 0 & \text{in } (\mathbb{R}^N \setminus \Omega) \times [0, T), \\
u(x,0) &= 0 & \text{in } \Omega.
\end{array} \right.
\]

(5.4)

Since \( ma < r < \frac{m(N + 2s)}{(N + 2s)(m(1 - s) + 1) - m(2s - 1)} \), using the regularity result in Corollary 3.13 and as in the proof of Theorem 1.1, there exists \( T > 0 \) such that \( K (E (\Omega_T)) \subset E (\Omega_T) \) and that \( K \) is a continuous, compact operator on \( E (\Omega_T) \). Using the Schauder Fixed Point Theorem, we get the existence of \( u \in E (\Omega_T) \) such that \( K (u) = u \), then \( u \in L^1 (0, T; W^{1,\sigma}_0 (\Omega)) \) and \( u \) solves (1.1).

\( \square \)

Remarks 5.2. Notice that

1. If \( m > \frac{N + 2s}{2s - 1} \), then the critical growth range \( \alpha = 2s \) is covered by the existence result of Theorem 1.3 if \( \frac{2s - 1}{1 - s} > 2s(N + 2s) \), which holds at least for \( s \) close to 1.

2. If \( \frac{1}{s} < m \leq \frac{N + 2s}{2s - 1} \), then the critical growth range \( \alpha = 2s \) is covered by the existence result of Theorem 1.3 if

\[
m > \frac{N + 2s}{2s - 1} \frac{2s - 1}{(2s - 1) - (1 - s)(N + 2s)}.
\]

(5.5)

Since in this case \( m \leq \frac{N + 2s}{2s - 1} \), then we deduce that from (5.5), we have \( \frac{2s - 1}{1 - s} > 2s(N + 2s) \).

We deal now with the complete range of the parameter \( \alpha \), \( \frac{N + 2s}{N + 1} \leq \alpha < \frac{s}{1 - s} \). In this case, under suitable hypothesis on \( m \), we will show the existence of a distributional solution that is in a suitable weighted Sobolev space. This result shows the influence of the singularity of the kernel on the boundary, which is a relevant difference with the heat equation.

Proof of Theorem 1.4. Without loss of generality we can assume that \( N \geq 2 \).

Fix \( \alpha \) such that \( \frac{N + 2s}{N + 1} \leq \alpha < \frac{s}{1 - s} \) and suppose that \( f \in L^m (\Omega_T) \) with

\[
m > \max \left\{ \frac{N + 2s}{s(2s - 1)}, s - \alpha(1 - s) \right\}.
\]

Recall that if \( v \) solves the problem

\[
\left\{ \begin{array}{ll}
v_t + (-\Delta)^{\alpha} v &= f & \text{in } \Omega_T, \\
v(x,t) &= 0 & \text{in } (\mathbb{R}^N \setminus \Omega) \times [0, T), \\
v(x,0) &= 0 & \text{in } \Omega,
\end{array} \right.
\]

such that \( (\Omega, K) \) is the unique solution to problem (5.1).
then by Theorem 3.11, for all $p < \infty$, we have

$$\|\|\nabla v|^{\delta - s}\|_{L^p(\Omega_T)} \leq C_0\|f\|_{L^m(\Omega_T)},$$

and then

$$\|\nabla (v^{\delta - s})\|_{L^p(\Omega_T)} \leq \hat{C}_0\|f\|_{L^m(\Omega_T)}.$$ (5.7)

Since $s > \frac{1}{2}$, we can chose $T > 0$ such that for some universal constant $\hat{C}$ depending only on $\Omega_T$, $N$, $s$, there exists $l > 0$ such that

$$\hat{C}(l + \|f\|_{L^m(\Omega_T)}) = l^\frac{1}{2}.$$ (5.8)

Fix $T, l > 0$ as above and define the set

$$E = \{ v \in E_1(\Omega) : v^{\delta - s} \in L^{m_0}(0, T; W^{1,m_0}_0(\Omega)) \text{ and } \left( \int\int_{\Omega_T} |\nabla (v^{\delta - s})|^{m_0} dx dt \right)^\frac{1}{m_0} \leq l^\frac{1}{2} \}.$$ (5.9)

It is clear that $E$ is a closed convex set of $E_1(\Omega_T)$. Using Hardy inequality we reach that if $v \in E$, then

$$\left( \int\int_{\Omega_T} |\nabla v|^{m_0} \delta^{m_0(1-s)} dx \right)^\frac{1}{m_0} \leq \hat{C}_0 l^\frac{1}{2}.$$ (5.10)

Consider the operator

$$\mathcal{T} : E \rightarrow E_1(\Omega_T),
\quad v \mapsto \mathcal{T}(v) = u$$

where $u$ is the unique solution to problem

$$\begin{cases}
  u_t + (-\Delta) u = |\nabla v|^\alpha f & \text{in } \Omega_T, \\
  u(x, t) = 0 & \text{in } (\mathbb{R}^N \setminus \Omega) \times [0, T), \\
  u(x, 0) = u_0(x) & \text{in } \Omega.
\end{cases}$$ (5.10)

**First step:** We prove that $\mathcal{T}$ is well defined. By Theorem 3.22, to get the desired result, we just have to show the existence of $\beta < 2s - 1$ such that $|\nabla v|^\alpha \delta^\beta \in L^1(\Omega_T)$. Since $v \in E$, then $|\nabla v|^\alpha \in L^1_{loc}(\Omega)$. We have

$$\int\int_{\Omega_T} |\nabla v|^\alpha \delta^\beta dx dt = \int\int_{\Omega_T} |\nabla v|^\alpha \delta^{\alpha(1-s)} \delta^{\beta - \alpha(1-s)} dx dt \leq \left( \int\int_{\Omega_T} |\nabla v|^{m_0} \delta^{m_0(1-s)} dx dt \right)^\frac{1}{m_0} \left( \int\int_{\Omega_T} \delta^{(\beta - \alpha(1-s)) m'} dx dt \right)^\frac{1}{m'}.$$ (5.11)

If $\alpha(1-s) < 2s - 1$, we can chose $\beta < 2s - 1$ such that $\alpha(1-s) < \beta$. Hence $\int (\delta(x))^{(\beta - \alpha(1-s)) m'} dx < \infty$.

Assume that $\alpha(1-s) \geq 2s - 1$, then $s \in \left( \frac{1}{2}, \frac{1}{2} \right]$. Since $m > \frac{N + 2s}{s - \alpha(1-s)}$ and $\alpha < \frac{s}{1-s}$, then $(\alpha(1-s) - (2s-1)) m' < 1$. Hence we get easily the existence of $\beta < 2s - 1$ such that $(\alpha(1-s) - \beta) m' < 1$. Thus we conclude.

Therefore, since $|\nabla v|^\alpha \delta^\beta + f \in L^1(\Omega_T)$, using Theorem 3.22, there exists $u$ that solves problem (5.10) with $u \in E_0(\Omega_T)$ for all $a < \frac{N + 2s}{N + \beta + 1}$. Hence $\mathcal{T}$ is well defined.

**Second step:**
We claim that:

1. For $l > 0$ as above, $\mathcal{T}(E) \subset E$,
2. $\mathcal{T}$ is a continuous and compact operator on $E$.

**Proof of (1).**
We have

$$u(x, t) = \int_0^t \int_{\Omega} |\nabla v(y, \sigma)|^{\alpha} P_\Omega(x, y, t - \sigma) dy d\sigma + \int_0^t \int_{\Omega} f(y, \sigma) P_\Omega(x, y, t - \sigma) dy d\sigma$$

$$= J_1(x, t) + J_2(x, t).$$

Let us begin by estimating $J_1$. Notice that, for all $\theta \in [0, 1]$,

$$\left( 1 - \frac{\delta^\beta(y)}{\sqrt{t-\sigma}} \right) \leq \delta^{\theta}(y)(t - \sigma)^{-\frac{1}{2}} \text{ in } \Omega_T,$$ (5.11)
and
\[
(1 + \frac{\delta^s(x)}{\sqrt{t - \sigma}}) \leq \delta^{\alpha(1 - \theta)}(x)(t - \sigma)^{-\frac{(1-s)}{2}} \text{ in } \Omega_T. \tag{5.12}
\]
Choosing \( \theta = \frac{\alpha(1-s)}{s} < 1 \) and using the properties of \( P_{\Omega} \), it holds that
\[
J_1(x, t) \leq C(\delta(x))^{s(\alpha+1) - \alpha} \int_0^t \int_{\Omega} |\nabla v(y, \sigma)|^\alpha \delta^{\alpha(1-s)}(y) \frac{(t - \sigma)^{\frac{1}{2}}}{((t - \sigma)^{\frac{1}{2}} + |x - y|)^{N + 2s}} dy d\sigma.
\]
Since \( |\nabla v|^{\alpha}\delta^{\alpha(1-s)} \in L^m(\Omega_T) \), then by Remark 3.24 and since \( m \geq \frac{N + 2s}{s(2s - 1)} \), then \( \frac{J_1}{\delta s - \alpha(1-s)} \in L^\gamma(\Omega_T) \) for all \( \gamma < \infty \) and
\[
\left\| \frac{J_1}{\delta s - \alpha(1-s)} \right\|_{L^\gamma(\Omega_T)} \leq C\left\| |\nabla v|^{\alpha}\delta^{\alpha(1-s)} \right\|_{L^m(\Omega_T)}.
\]
As in the proof of Theorem 3.10 and since \( f \in L^m(\Omega_T) \), we obtain that
\[
\left\| \frac{J_2}{\delta s - \alpha(1-s)} \right\|_{L^\gamma(\Omega_T)} \leq C\left\| \|f\|_{L^m(\Omega_T)} \right\|.
\]
Hence we conclude that
\[
\left\| \frac{u}{\delta s - \alpha(1-s)} \right\|_{L^\gamma(\Omega_T)} \leq C\left\| |\nabla v|^{\alpha}\delta^{\alpha(1-s)} \right\|_{L^m(\Omega_T)} + \|f\|_{L^m(\Omega_T)}.
\]
We deal now with the gradient term.
We have that
\[
|\nabla u(x, t)| \leq C \left\{ \int_0^t \int_{\Omega} |\nabla v(y, \sigma)|^\alpha |\nabla P_{\Omega}(x, y, t - \sigma)| dy d\sigma + \int_0^t \int_{\Omega} f(y, \sigma) |\nabla P_{\Omega}(x, y, t - \sigma)| dy d\sigma \right\} = I_1(x, t) + I_2(x, t).
\]
Notice that following the same computation as in the proof of Theorem 3.11 and using the fact that \( m \geq \frac{N + 2s}{s(2s - 1)} \), we find that
\[
\left\| I_2 \delta^{1-s} \right\|_{L^\gamma(\Omega_T)} \leq C\left\| \|f\|_{L^m(\Omega_T)} \right\|,
\]
for all \( \gamma < \infty \). Hence we have just to estimate \( I_1 \). Notice that
\[
I_1(x, t) \leq \int_0^t \int_{\Omega} |\nabla v(y, \sigma)|^\alpha \frac{1}{\delta(x)} \left( \frac{1}{|\nabla P_{\Omega}(x, y, t - \sigma)|} \right) dy d\sigma.
\]
Thus
\[
I_1(x, t) \delta^{1-s}(x) \leq \delta^{1-s}(x) \int_0^t \int_{\Omega} |\nabla v(y, \sigma)|^\alpha \left( \frac{1}{\delta(x)} \right) \frac{1}{|\nabla P_{\Omega}(x, y, t - \sigma)|} dy d\sigma
\]
\[
\leq C \left\{ \delta^{s(\alpha-1)} \int_{\Omega_T \cap \{\delta(x) < t - \sigma\}} |\nabla v(y, \sigma)|^\alpha P_{\Omega}(x, y, t - \sigma) dy d\sigma
\]
\[
+ \delta^{1-s}(x) \int_{\Omega_T \cap \{\delta(x) \geq t - \sigma\}} |\nabla v(y, \sigma)|^\alpha \frac{P_{\Omega}(x, y, t - \sigma)}{(t - \sigma)^{\frac{1}{2}}} dy d\sigma
\]
\[
= L_1(x, t) + L_2(x, t).
\]
By the estimates (5.11) and (5.12), we deduce that
\[
L_1(x, t) \leq \frac{1}{\delta^{s(1-\theta)}(x)} \int_{\Omega_T \cap \{\delta(x) < t - \sigma\}} |\nabla v(y, \sigma)|^\alpha \delta^{\alpha(1-s)}(y) \frac{(t - \sigma)^{\frac{2s - \alpha(1-s)}{2} - \frac{s}{2}}}{((t - \sigma)^{\frac{1}{2}} + |x - y|)^{N + 2s}} dy d\sigma.
\]
Setting
\[
g(x, t) := \int_{\Omega_T \cap \{\delta(x) < t - \sigma\}} |\nabla v(y, \sigma)|^\alpha \delta^{\alpha(1-s)}(y) \frac{(t - \sigma)^{\frac{2s - \alpha(1-s)}{2} - \frac{s}{2}}}{((t - \sigma)^{\frac{1}{2}} + |x - y|)^{N + 2s}} dy d\sigma,
\]
then by Remark 3.24 and since $|\nabla v(y,\sigma)|^{\alpha} \delta^{\alpha(1-s)}(y) \in L^m(\Omega_T)$, it follows that $g \in L^{m}(\Omega_T)$ with $\gamma_\theta$ that satisfies

$$\frac{1}{\gamma_\theta} > \frac{1}{m} - \frac{2s(2s - \alpha(1 - s) \theta)}{N + 2s} = \frac{1}{m} - \frac{2s - \alpha(1 - s) - s\theta}{N + 2s}.$$  

(5.15)

It is clear that for all $\varepsilon > 0$, we can chose $\theta$ closed to 1 such that $L_1(x, t) \in L^{\gamma_{\varepsilon}}(\Omega_T)$. Hence to show that $L_1(x, t) \in L^{m}(\Omega_T)$, we have just to show that $\gamma_\theta > m\alpha$ for $\theta$ close to 1. Since $m > \frac{N + 2s}{s - \alpha(1 - s)}$, then $\frac{1}{m} > \frac{1}{m} - \frac{2s - \alpha(1 - s) - s}{N + 2s}$. Hence, for any $\theta < 1$, we have $\frac{1}{m} > \frac{1}{m} - \frac{2s - \alpha(1 - s) - s}{N + 2s}$ and then we conclude. As a consequence we deduce that $L_1 \in L^{m \gamma_{\alpha}}(\Omega_T)$ for some $\rho > 0$ and

$$||L_1||_{L^{m \gamma_{\alpha}}(\Omega_T)} \leq C(\Omega_T)|||\nabla v|^{\alpha} \delta^{\alpha(1-s)}||_{L^m(\Omega_T)}.$$  

(5.16)

We deal now with $L_2(x, t)$.

$$L_2(x, t) = \delta^{1-s}(x) \int_{\Omega_T \cap \delta(x) \geq (t-\sigma)^{\frac{1}{s}}} |\nabla v(y, \sigma)|^{\alpha} \frac{P_\sigma(x, y, t - \sigma)}{(t - \sigma)^{\frac{1}{s}} \delta} dyd\sigma$$

$$= \delta^{1-s}(x) \int_{\Omega_T \cap \delta(x) \geq (t-\sigma)^{\frac{1}{s}}} \frac{|\nabla v(y, \sigma)|^{\alpha} \delta^{\alpha(1-s)}(y)}{(t - \sigma)^{1 - \frac{1}{s} - \theta}} dyd\sigma + \delta^{1-s}(x) \int_{\Omega_T \cap \delta(x) \geq (t-\sigma)^{\frac{1}{s}}} \frac{|\nabla v(y, \sigma)|^{\alpha} \delta^{\alpha(1-s)}(y)}{(t - \sigma)^{1 - \frac{1}{s} - \theta} + (t - x)(t - y)^{N+2s}} dyd\sigma.$$

$$= L_{21}(x, t) + L_{22}(x, t) + L_{23}(x, t).$$

Let us begin by estimating $L_{21}$. Since $\delta$ is a Lipschitz function, it holds that, if $(x, y) \in \{|x-y| < \frac{1}{2}\delta(x)\}$, then $|\delta(y) - \delta(x)| \leq |x - y| \leq \frac{1}{2}\delta(x)$. Thus $\frac{1}{2}\delta(x) \leq \delta(y) \leq \frac{3}{2}\delta(x)$. Hence we obtain that

$$L_{21}(x, t) \leq C \int_{\Omega_T \cap \delta(x) \geq (t-\sigma)^{\frac{1}{s}}} |\nabla v(y, \sigma)|^{\alpha} \delta^{\alpha(1-s)}(y) \frac{(t - \sigma)^{1 - \frac{1}{s} - \frac{\alpha}{s}}}{(t - \sigma)^{1 - \frac{1}{s} - \theta} + (t - x)(t - y)^{N+2s}} dyd\sigma.$$

Choosing $\theta = \frac{(\alpha-1)(1-s)}{s} < 1$, we get

$$L_{21}(x, t) \leq C \int_{\Omega_T \cap \delta(x) \geq (t-\sigma)^{\frac{1}{s}}} |\nabla v(y, \sigma)|^{\alpha} \delta^{\alpha(1-s)}(y) \frac{(t - \sigma)^{1 - \frac{1}{s} - \frac{\alpha}{s}}}{(t - \sigma)^{1 - \frac{1}{s} - \theta} + (t - x)(t - y)^{N+2s}} dyd\sigma.$$

Using Remark 3.24, we conclude that $L_{21} \in L^{\gamma}(\Omega)$ for all $\gamma$ such that $\frac{1}{\gamma} > \frac{1}{m} - \frac{s - \alpha(1 - s)}{N + 2s}$ and

$$||L_{21}||_{L^{\gamma}(\Omega_T)} \leq C(\Omega_T)|||\nabla v|^{\alpha} \delta^{\alpha(1-s)}||_{L^m(\Omega_T)}.$$  

(5.17)

Since $m > \frac{N + 2s}{s - \alpha(1 - s)}$, then the above estimate holds for all $\gamma$ and then we conclude.

We analyze now the term $L_{22}$. We set

$$A := \Omega_T \cap \{\delta(x) \geq (t-\sigma)^{\frac{1}{s}}\} \cap \left\{\frac{1}{2}\delta(x) < |x - y| \leq \delta(x)\right\},$$

then as above choosing $\theta = \frac{\alpha(1-s)}{s} < 1$, we obtain that

$$L_{22}(x, t) \leq \delta^{1-s}(x) \int_{A} |\nabla v(y, \sigma)|^{\alpha} \delta^{\alpha(1-s)}(y) \frac{(t - \sigma)^{1 - \frac{1}{s} - \frac{\alpha}{s}}}{(t - \sigma)^{1 - \frac{1}{s} - \theta} + (t - x)(t - y)^{N+2s}} dyd\sigma.$$
It is clear that $2s - 1 - \alpha(1 - s) > 0$ if and only if $s > \frac{\alpha + 1}{\alpha + 2}$. Hence, in any case, using Hölder inequality, we get

$$L_{22}(x, t) \leq \delta^{1-s}(x) \left( \iint_A |\nabla v(y, \sigma)|^{m\alpha(\delta(y))} \, d\sigma \right)^{\frac{1}{m}} \left( \iint_A \frac{(t - \sigma)^{m'((2s-1)-\alpha(1-s))}}{((t - \sigma)^{\frac{2}{m'}} + |x - y|)^{m'(N+2s)}} \, d\sigma \right)^{\frac{1}{m'}}.$$

By the definition of the set $A$, we obtain

$$\begin{align*}
\iint_A & \frac{(t - \sigma)^{m'((2s-1)-\alpha(1-s))}}{((t - \sigma)^{\frac{2}{m'}} + |x - y|)^{m'(N+2s)}} \, d\sigma \\
&= \iint_{\{(t - \sigma) \leq \delta((2s-1)-\alpha(1-s))\}} \frac{(t - \sigma)^{m'((2s-1)-\alpha(1-s))}}{((t - \sigma)^{\frac{2}{m'}} + |x - y|)^{m'(N+2s)}} \, d\sigma \\
&= C(N) \iint_{\{(t - \sigma) \leq \delta((2s-1)-\alpha(1-s))\}} \frac{(t - \sigma)^{m'((2s-1)-\alpha(1-s))}}{((t - \sigma)^{\frac{2}{m'}} + |x - y|)^{m'(N+2s)}} \, d\sigma \\
&\leq \frac{C(N, s)}{(\delta(x))^{m'(N+2s)-N}} \int_{\{(t - \sigma) \leq \delta((2s-1)-\alpha(1-s))\}} (t - \sigma)^{m'((2s-1)-\alpha(1-s))} \, d\sigma \\
&\leq \frac{C(N, s)}{(\delta(x))^{m'(N+2s)-N}} \int_0^{\delta((2s-1)-\alpha(1-s))} \rho^{m'((2s-1)-\alpha(1-s))} \, d\rho.
\end{align*}$$

If $s \geq \frac{\alpha + 1}{\alpha + 2}$ then $\frac{m'((2s-1)-\alpha(1-s))}{2s} \geq 0$. If $\frac{1}{2} < s < \frac{\alpha + 1}{\alpha + 2}$, since $m' < \frac{N + 2s}{N + s + \alpha(1-s)}$, then $\frac{m'((2s-1)-\alpha(1-s))}{2s} > -1$. Thus

$$\begin{align*}
\frac{C(N, s)}{(\delta(x))^{m'(N+2s)-N}} \int_0^{\delta((2s-1)-\alpha(1-s))} \rho^{m'((2s-1)-\alpha(1-s))} \, d\rho &\leq (\delta(x))^{N+2s-m'(N+1+\alpha(1-s))}.
\end{align*}$$

Going back to the estimate of $L_{22}$ it holds that

$$L_{22}(x, t) \leq C(\delta(x))^{s-\alpha(1-s) - \frac{N+2s}{m}} \left( \iint_A |\nabla v(y, \sigma)|^{m\alpha(\delta(y))} \, d\sigma \right)^{\frac{1}{m}}.$$

Since $m > \frac{N + 2s}{s - \alpha(1-s)}$, we reach that $s - \alpha(1-s) - \frac{N + 2s}{m} > 0$. Thus $L_{22} \in L^\infty(\Omega_T)$ and for all $\gamma > 1$,

$$||L_{22}||_{L^\gamma(\Omega_T)} \leq C(\Omega_T) \left( \iint_{\Omega_T} |\nabla v(y, \sigma)|^{2sm\delta^{2s\gamma(1-s)}(y)} \, d\sigma \right)^{\frac{1}{m'}}. \quad (5.18)$$

Respect to $L_{23}$, setting

$$A := \Omega_T \cap \{(t - \sigma)^{\frac{1}{m'}} \leq |x - y| \leq \delta(x)\}.$$
Thus, as above, we conclude that
\[ L_{23}(x, t) \leq \delta^{1-s}(x) \left( \int_{A} |\nabla v(y, \sigma)|^{m_{\alpha}} \delta^{m_{\alpha}(1-s)}(y) dyd\sigma \right)^{\frac{1}{m_{\alpha}}} \]
\[ \leq \delta^{1-s}(x) \left( \int_{A} |\nabla v(y, \sigma)|^{m_{\alpha}} \delta^{m_{\alpha}(1-s)}(y) dyd\sigma \right)^{\frac{1}{m_{\alpha}}} \]
\[ \times \left( \int_{(t-\sigma)\leq \delta^{2s}(x)} (t-\sigma)^{2}(2s+1-\alpha(1-s)) + \frac{2}{m_{\alpha}(N+2s)} \int_{(t-\sigma)\leq \delta^{2s}(x)} r^{N-1} + \frac{2}{m_{\alpha}(N+2s)} dr d\sigma \right)^{\frac{1}{m_{\alpha}}} . \]

Thus, as above, we conclude that
\[ L_{23}(x, t) \leq C(\delta(x))^{1-\alpha(1-s)} \left( \int_{A} |\nabla v(y, \sigma)|^{m_{\alpha}} \delta^{m_{\alpha}(1-s)}(y) dyd\sigma \right)^{\frac{1}{m_{\alpha}}} . \]

Then as in the estimate of \( L_{22} \), we reach that \( L_{23} \in L^{\infty}(\Omega_{T}) \) and for all \( \gamma > 1 \),
\[ \|L_{23}\|_{L^{\gamma}(\Omega_{T})} \leq C \left( \left( \int_{\Omega_{T}} |\nabla v(y, \sigma)|^{2m_{\alpha}} \delta^{2m_{\alpha}(1-s)}(y) dyd\sigma \right)^{\frac{1}{2m_{\alpha}}} + \|f\|_{L^{m}(\Omega_{T})} \right) . \]

As a conclusion and combining estimates (5.17), (5.18) and (5.19) we reach that if \( u \) solves (5.10) with \( v \in E \), then
\[ \left( \int_{\Omega_{T}} |\nabla u|^{\gamma(1-s)} dxdt \right)^{\frac{1}{\gamma}} \leq C(\Omega_{T}, N, s) \left( \left( \int_{\Omega_{T}} |\nabla v(y, \sigma)|^{m_{\alpha}} \delta^{m_{\alpha}(1-s)}(y) dyd\sigma \right)^{\frac{1}{m_{\alpha}}} + \|f\|_{L^{m}(\Omega_{T})} \right) , \]
and
\[ \left\| \frac{u}{\delta^{a-\alpha(1-s)}} \right\|_{L^{\gamma}(\Omega_{T})} \leq C \left( \left\| |\nabla v|^{a} \delta^{a(1-s)} \right\|_{L^{\gamma}(\Omega_{T})} + \|f\|_{L^{m}(\Omega_{T})} \right) , \]
for some \( \gamma > m_{\alpha} \). Recall that \( T \) is fixed such that (5.8) holds. Thus \( u \in E \) and then \( T(E) \subset E \).

Now we proof (2). Let begin by proving the continuity of \( T \) respect to the topology of \( L^{1}(0, T; W^{1,1}_{0}(\Omega)) \). Consider \( \{v_{n}\}_{n} \subset E \) such that \( v_{n} \to v \) strongly in \( L^{1}(0, T; W^{1,1}_{0}(\Omega)) \). Define \( u_{n} = T(v_{n}) \), \( u = T(v) \).

We have to show that \( u_{n} \to u \) strongly in \( L^{1}(0, T; W^{1,1}_{0}(\Omega)) \). Using Theorem 3.22, to show the desired result, we have just to prove that
\[ \|\nabla v_{n} - \nabla v\|_{L^{\infty}(d^{\beta}dx, \Omega_{T})} \to 0 \text{ as } n \to \infty, \]
for some \( \beta < 2s - 1 \). As in the proof of the fist step, we get the existence of \( \beta < 2s - 1 \) such that
\[ \int_{\Omega_{T}} |\nabla v_{n}|^{\alpha} \delta^{\beta} dx \leq C \text{ for all } n. \]

Since \( m_{\alpha} > 1 \), then setting \( a = \frac{\alpha(m-1)}{m_{\alpha} - 1} < 1 \), it follows that \( \frac{\alpha - a}{1 - a} = m_{\alpha} \). Hence by Hölder inequality, we conclude that
\[ \|\nabla v_{n} - \nabla v\|_{L^{\infty}(d^{\beta}dx, \Omega_{T})} \leq C\|\nabla v_{n} - \nabla v\|_{L^{2}(d^{\beta}dx, \Omega_{T})} \|\nabla v_{n} - \nabla v\|_{L^{\frac{2m_{\alpha}}{a}}(d^{\beta}dx, \Omega_{T})} \to 0 \text{ as } n \to \infty. \]
Now, using the definition of \( u_{n} \) and \( u \), there results that \( u_{n} \to u \) strongly in \( L^{\gamma}(0, T; W^{1,1}_{0}(\Omega)) \) for some \( \sigma > 1 \) and then we conclude.

We prove now that \( T \) is compact. Let \( \{v_{n}\}_{n} \subset E \) be such that \( \|v_{n}\|_{L^{1}(0, T; W^{1,1}_{0}(\Omega))} \leq C \).
Since \( \{v_n\}_n \subset E \), then \( \|\nabla(v_n \delta^{1-s})\|_{L^m(\Omega_T)} \leq C \). Therefore, up to a subsequence, 
\[ v_{n_k} \delta^{1-s} \rightharpoonup v \delta^{1-s} \text{ weakly in } L^{m\alpha}(0,T;W^{1,m\alpha}_{loc}(\Omega)). \]

It is clear that \( v_{n_k} \rightharpoonup v \) weakly in \( L^{m\alpha}(0,T;W^{1,m\alpha}_{loc}(\Omega)) \).

Let 
\[ F_n = |\nabla v_n|^\alpha + f, F = |\nabla v|^\alpha + f, \]
then, as in the first step, we can prove that \( F_n \delta^{\beta} \) is bounded in \( L^{1+\alpha}(\Omega) \) for some \( \alpha > 0 \) and then \( F_n \delta^{\beta} \rightharpoonup F \delta^{\beta} \) weakly in \( L^{1+\alpha}(\Omega) \). Using the compactness result in Theorem 3.22, we conclude that, up to a subsequence, \( u_{n_k} \rightharpoonup u \) strongly in \( L^q(0,T;W^{1,1}_{loc}(\Omega)) \) for all \( q < \frac{N}{N+2} + 1 \) and then the result follows.

Hence we are in position to use the Schauder Fixed Point Theorem and then there exists \( u \in E \) such that \( T(u) = u \). Thus \( u \delta^{1-s} \in L^{m\alpha}(0,T;W^{1,m\alpha}_{loc}(\Omega)) \) and \( u \) solves (1.1) in the sense of distribution.

\[ \square \]

**Remarks 5.3.** The above arguments can be used to treat the problem 
\[
\begin{align*}
\begin{array}{ll}
\{ & u_t + (-\Delta)^s u + |\nabla u|^\alpha = f & \text{in } \Omega_T, \\
u(x,t) = 0 & \text{in } \mathbb{R}^N \setminus \Omega \times (0,T), \\
u(x,0) = 0 & \text{in } \Omega,
\end{array}
\end{align*}
\]  
(5.20)

where \( \alpha < \frac{s}{1-s} \) and \( f \in L^m(\Omega_T) \) with \( m > \max\left\{ \frac{N+2s}{\alpha(2s-1)} \frac{N+2s}{\alpha(1-s)-s} \right\} \). Then for \( T \) small, problem (5.20) has a distributional solution \( u \in L^\alpha(0,T;W^{1,\alpha}_{loc}(\Omega)) \cap L^1(0,T;W^{1,1}_{loc}(\Omega)) \) with \( u \delta^{1-s} \in L^{m\alpha}(0,T;W^{1,m\alpha}_{loc}(\Omega)) \).

In the case where \( \frac{s}{1-s} \leq \alpha \), then we consider the modified problem 
\[
\begin{align*}
\begin{array}{ll}
\{ & u_t + (-\Delta)^s u = \delta^{\alpha(1-s)-s}(x)|\nabla u|^\alpha + f & \text{in } \Omega_T, \\
u(x,t) = 0 & \text{in } \mathbb{R}^N \setminus \Omega \times (0,T), \\
u(x,0) = 0 & \text{in } \Omega,
\end{array}
\end{align*}
\]  
(5.21)

where \( m = \max\left\{ \frac{N+2s}{\alpha(2s-1)} \frac{N+2s}{\alpha(1-s)-s} \right\} \), then as in the proof of the previous Theorem, existence of solution holds using the Schauder fixed point Theorem in the set 
\[ \hat{E}(\Omega_T) = \{ v \in E_1(\Omega_T) : v \delta^{1-s} \in E_m(\Omega_T) \text{ with } \|v \delta^{1-s}\|_{E_m(\Omega_T)} \leq \frac{1}{2} \}. \]

If \( f = 0 \) and \( u_0 \neq 0 \), we have the result in Theorem 1.5 whose proof follows.

**Proof of Theorem 1.5.**

Recall that \( \frac{2s-1}{1-s} > \frac{(N+2s)^2}{N} \) and that \( \frac{N+2s}{N+\tau} \leq \alpha < \frac{2s}{(1-s)(N+2s)+1} \). Define \( \psi \) to be the unique solution to problem 
\[
\begin{align*}
\begin{array}{ll}
\{ & \psi_t + (-\Delta)^s \psi = 0 & \text{in } \Omega_T, \\
\psi(x,t) = 0 & \text{in } (\mathbb{R}^N \setminus \Omega) \times [0,T), \\
\psi(x,0) = u_0(x) & \text{in } \Omega.
\end{array}
\end{align*}
\]  
(5.21)

Recall that \( u_0 \in L^\tau(\Omega) \) where \( \sigma > \frac{(\alpha-1)N}{(2s-\alpha) - \alpha(1-s)(N+2s)} \), then by the regularity results in Proposition 3.20 and Corollary 3.19, we obtain that \( |\nabla \psi| \in L^\theta(\Omega_T) \) for all \( \theta < m_1 = \frac{\sigma(N+2s)}{(1-s)\sigma(N+2s) + N + \sigma} \).

Notice that if \( v \) solves the problem 
\[
\begin{align*}
\begin{array}{ll}
\{ & \nu_t + (-\Delta)^s \nu = |\nabla(\nu + \psi)|^\alpha & \text{in } \Omega_T, \\
\nu(x,t) = 0 & \text{in } (\mathbb{R}^N \setminus \Omega) \times [0,T), \\
\nu(x,0) = 0 & \text{in } \Omega,
\end{array}
\end{align*}
\]  
(5.22)

then \( u \equiv \nu + \psi \) is a solution to (1.3). Hence we have just to prove the existence of \( \nu \). Notice that 
\[ |\nabla(v + \psi)|^\alpha \leq C_1|\nabla v|^\alpha + C_2|\nabla \psi|^\alpha. \]
Define $f \equiv C_2 |\nabla \psi|^\alpha$, then $f \in L^m(\Omega_T)$ for any $m < \frac{m_1}{\alpha} = \frac{\sigma(N + 2s)}{(1 - s)\sigma(N + 2s) + N + \sigma}$. It is clear that $m < \frac{\sigma(N + 2s)}{2s(1 - s)\sigma(N + 2s) + N + \sigma} < \frac{N + 2s}{2s - 1}$.

Hence, using the fact that $\frac{N + 2s}{2s - 1} \leq \alpha < \frac{2s}{(1 - s)(N + 2s) + 1}$ and $\sigma > \frac{(2s - \alpha - (1 - \alpha)(N + 2s))}{(N + 2s)(N + 2s) + 1}$, then we get the existence of $\frac{1}{s} \leq m < \frac{N + 2s}{2s - 1}$ such that $m\alpha < \frac{m(N + 2s)}{(N + 2s)(m(1 - s) + 1) - m(2s - 1)} = \hat{P}$ defined in Corollary 3.13.

Hence we can fix $r > 1$ such that $m\alpha < r < \frac{m(N + 2s)}{(N + 2s)(m(1 - s) + 1) - m(2s - 1)}$. Now following the same argument as in the proof of Theorem 1.3 we get the desired existence result. \hfill \Box

Remarks 5.4.

(1) Let us consider now the case where $f \geq 0$ and $u_0 \geq 0$ simultaneously. As in the proof of Theorem 1.5, define $\psi$ to be the unique solution to problem (5.21) and let $\vartheta$ the solution to the problem

$$
\begin{align*}
\vartheta_t + (-\Delta)^s \vartheta & = |\nabla (\vartheta + \psi)|^\alpha + f \quad \text{in } \Omega_T, \\
\vartheta(x, t) & = 0 \quad \text{in } \mathbb{R}^N \setminus \Omega \times [0, T), \\
\vartheta(x, 0) & = 0 \quad \text{in } \Omega,
\end{align*}
$$

(5.23)

then $u = \vartheta + \psi$ solves the problem

$$
\begin{align*}
u_t + (-\Delta)^s \nu & = |\nabla \psi|^\alpha + f \quad \text{in } \Omega_T, \\
u(x, t) & = 0 \quad \text{in } \mathbb{R}^N \setminus \Omega \times [0, T), \\
u(x, 0) & = 0 \quad \text{in } \Omega,
\end{align*}
$$

Taking into consideration that

$$
|\nabla (\vartheta + \psi)|^\alpha \leq C_1 |\nabla \vartheta|^\alpha + C_2 |\nabla \psi|^\alpha,
$$

we can reproduce the same approach as in the proof of Theorem 1.5, with $\hat{f} = C_2 |\nabla \psi|^\alpha + f$, to get the existence of a solution $\vartheta$ to problem 5.23 combining the regularity of $f$ and $u_0$. Hence we conclude.

(2) In a forthcoming paper we will treat the problem

$$
\begin{align*}
u_t + (-\Delta)^s \nu & = |\nabla \psi|^\alpha \quad \text{in } \Omega_T, \\
u(x, t) & = 0 \quad \text{in } \mathbb{R}^N \setminus \Omega \times [0, T), \\
u(x, 0) & = u_0(x) \quad \text{in } \Omega,
\end{align*}
$$

(5.24)

under the general condition $\alpha < \frac{s}{1 - s}$. Existence of solutions will proved in a suitable parabolic weighted Sobolev space. Global existence in time or blow-up in finite time will be also analyzed.

6. Comparison principle and a partial uniqueness result for a problem with a drift term. Applications to the quasi-linear problem (1.1)

We will study in this Section an equation with a drift, that is, we substitute the nonlinear term in the gradient by a term of the form $B(x, t) \cdot \nabla u$. Then the existence of a solution is obtained under natural conditions of the field $B$ and the data $f, u_0$. Using the previous arguments, we can prove the existence for the largest class of the data $f, u_0$, under the natural condition on $B$.

Let us begin by considering the next problem

$$
\begin{align*}
u_t + (-\Delta)^s \nu & = B(x, t) \cdot \nabla u + f \quad \text{in } \Omega_T, \\
u(x, t) & = 0 \quad \text{in } \mathbb{R}^N \setminus \Omega \times [0, T), \\
u(x, 0) & = u_0(x) \quad \text{in } \Omega,
\end{align*}
$$

(6.1)

where $B \in (L^m(\Omega_T))^N$ with $m > \frac{N + 2s}{2s - 1}$. We are able to prove the next existence result that extends the one obtained in [51].
Theorem 6.1. Assume that $\Omega \subset \mathbb{R}^N$ is a bounded regular domain and $s \in (\frac{1}{2},1)$. Suppose that $B \in (L^m(\Omega_T))^N$ with $m > \frac{N + 2s}{2s - 1}$, then for all $(f,u_0) \in L^1(\Omega_T) \times L^1(\Omega)$, the problem (6.1) has a weak solution $u \in L^q(0,T;W_0^1,q(\Omega))$, $q < \frac{N + 2s}{N + 1}$ and $T_k(u) \in L^2(0,T;H_0^s(\Omega))$ for all $k > 0$.

Furthermore, If $u_0 \geq 0$ in $\Omega$ and $f \geq 0$ in $\Omega_T$, then $u \geq 0$ in $\Omega_T$.

Moreover, if $B$ satisfies one of the following assumption:

(1) $B$ does not depend on $t$,

(2) $B \in (L^m(\Omega_T))^N$ with $m > \frac{N + 2s}{N + 1}$ for some $1 < q < \frac{N + 2s}{2s - 1}$, then the solution obtained above is unique.

Proof. As in the proof of Theorem 2.4, we define the operator

$$K : \tilde{E}(\Omega_T) \to \tilde{E}_1(\Omega_T),$$

$$v \to T(v) = u,$$

where

$$\tilde{E}(\Omega_T) = \{v \in E_1(\Omega_T) \mid \sigma' < r < \frac{N + 2s}{N + 1} \text{ and } \|v\|_{E_r(\Omega_T)} \leq \frac{1}{r^\frac{s}{r}}\}.$$

and $u$ is the unique solution to problem

$$\begin{cases}
  u_t + (-\Delta)^s u &= B(x,t) \cdot \nabla v + f \text{ in } \Omega_T \equiv \Omega \times (0,T), \\
  u(x,t) &= 0 \text{ in } (\mathbb{R}^N \setminus \Omega) \times [0,T), \\
  u(x,0) &= u_0(x) \text{ in } \Omega.
\end{cases}$$

(6.2)

Let us begin by proving that $K$ is well defined.

Since $B \in (L^m(\Omega_T))^N$ with $m > \frac{N + 2s}{2s - 1}$, then $m' < \frac{N + 2s}{N + 1}$. Fix $r$ such that $m' < r < \frac{N + 2s}{N + 1}$, then for $v \in \tilde{E}(\Omega_T)$, using Hölder inequality, we conclude that

$$\int_0^T \int_{\Omega} |B(x,t) \cdot \nabla v| dxdt \leq C(T,\Omega) \left( \int_0^T \int_{\Omega} |B(x,t)|^m dxdt \right)^{\frac{1}{m}} \left( \int_0^T \int_{\Omega} |\nabla v|^r dxdt \right)^{\frac{1}{r}}.$$

Hence $|B(x,t) \cdot \nabla v| \in L^1(\Omega_T)$ and then $K$ is well defined. Now, the existence result follows using the same compactness arguments as in the proof of Theorem 2.4.

Assume now that $f \geq 0$. To get the existence of a nonnegative solution we consider the next variation of the operator $K$. Namely we define $u$ to be the unique solution to the problem

$$\begin{cases}
  u_t + (-\Delta)^s u &= B(x,t) \cdot \nabla v_+ + f \text{ in } \Omega_T, \\
  u(x,t) &= 0 \text{ in } (\mathbb{R}^N \setminus \Omega) \times [0,T), \\
  u(x,0) &= u_0(x) \text{ in } \Omega.
\end{cases}$$

(6.3)

It is clear that, with the new definition, if $u$ is a fixed point of $K$, then $u$ solves

$$\begin{cases}
  u_t + (-\Delta)^s u &= B(x,t) \cdot \nabla u_+ + f \text{ in } \Omega_T, \\
  u(x,t) &= 0 \text{ in } (\mathbb{R}^N \setminus \Omega) \times [0,T), \\
  u(x,0) &= u_0(x) \text{ in } \Omega.
\end{cases}$$

Using $T_k(u_-)$ as a test function in the previous equation we reach that $T_k(u_-) = 0$ for all $k$. Hence $u_- = 0$ and then $u \geq 0$ in $\Omega_T$. It is clear that in a symmetric way, if $f \leq 0$ in $\Omega_T$ and $u_0 \leq 0$ in $\Omega$, then we get the existence of a $u \leq 0$ in $\Omega_T$.

We prove now that the solution is unique. Assume that $u_1, u_2$ are solutions to problem (6.1), setting $v = u_1 - u_2$, then $v$ solves

$$\begin{cases}
  v_t + (-\Delta)^s v &= B(x,t) \cdot \nabla v \text{ in } \Omega_T, \\
  v(x,t) &= 0 \text{ in } (\mathbb{R}^N \setminus \Omega) \times (0,T), \\
  v(x,0) &= 0 \text{ in } \Omega.
\end{cases}$$

(6.4)

Notice that $v \in L^q(0,T;W_0^1,q(\Omega))$ for all $a < \frac{N + 2s}{N + 1}$ and $T_k(v) \in L^2(0,T;H_0^s(\Omega))$ for all $k > 0$. 

Define $g(x, t) = B(x, t) \cdot \nabla v(x, t)$, since $B \in (L^m(\Omega_T))^N$ with $m > \frac{N + 2s}{2s - 1}$, then we can fix $1 < q < \frac{N + 2s}{N + 1}$ such that $q' < m$. Notice that

$$v(x, t) = \int_0^t \int_\Omega g(y, \sigma) P_\Omega(x, y, t - \sigma) \, dy \, d\sigma,$$

and

$$|\nabla v(x, t)| \leq \int_0^t \int_\Omega |g(y, \sigma)||\nabla_x P_\Omega(x, y, t - \sigma)| \, dy \, d\sigma.$$

From Proposition 3.9 we deduce that for all $\eta > 0$,

$$\left( \int_\Omega |\nabla v(x, t)|^q \, dx \right)^{\frac{1}{q}} \leq C(\Omega_T) \left( \int_0^t \int_\Omega \left|g(y, \sigma)\right|(t - \sigma)^{-\eta} \, dy \, d\sigma \right)^{\frac{1}{\eta}}.$$

Recall that $|g(x, t)| \leq |B(x, t)||\nabla v(x, t)|$, thus

$$\int_\Omega |g(y, \sigma)| \, dy \leq \left( \int_\Omega |\nabla v(x, t)|^q \, dx \right)^{\frac{1}{q}} \left( \int_\Omega |B(y, \sigma)|^q \, dy \right)^{\frac{1}{q}}.$$

Let $\alpha = 1 + \gamma - \eta \in (0, 1)$, then

$$\left( \int_\Omega |\nabla v(x, t)|^q \, dx \right)^{\frac{1}{q}} \leq C(\Omega_T) \int_0^t (t - \sigma)^{\alpha - 1} \left( \int_\Omega |\nabla v(x, t)|^q \, dx \right)^{\frac{1}{q}} \left( \int_\Omega |B(y, \sigma)|^q \, dy \right)^{\frac{1}{q}} \, d\sigma.$$

Setting $Y(t) = \left( \int_\Omega |\nabla v(x, t)|^q \, dx \right)^{\frac{1}{q}}$, then $Y \in L^q(0, T)$ and

$$Y(t) \leq C(\Omega_T) \int_0^t (t - \sigma)^{\alpha - 1} K(\sigma) Y(\sigma) \, d\sigma,$$

with $K(\sigma) = \left( \int_\Omega |B(y, \sigma)|^q \, dy \right)^{\frac{1}{q}}$.

Assume that $B$ depends only on $x$, then $K \in L^\infty(0, T)$. From the singular Bellman-Gronwall inequality proved in [32], Lemma 7.7.1, we deduce that $Y = 0$ in $L^1(0, T)$. Thus

$$\int_0^T \left( \int_\Omega |\nabla v(x, t)|^q \, dx \right)^{\frac{1}{q}} \, dt = 0.$$

Since $v \in L^q(0, T); W^{1,q}_0(\Omega)$, we obtain that $v = 0$ and the result follows.

Now, in second hypothesis, if $B \in (L^m(\Omega_T))^N$ with $m > \max\{q', \frac{2s}{N + 2s - q(N + 1)}\} > \frac{N + 2s}{2s - 1}$, then for $\eta$ small enough, $m\alpha > 1$. We set $\bar{K}(t, \sigma) = (t - \sigma)^{\alpha - 1} K(\sigma)$, then there exists $\theta \in \left(1, \frac{1}{1 - \alpha}\right)$ such that

$$\int_0^T \left( \int_0^t (\bar{K}(t, \sigma))^\theta \, d\sigma \right)^{\frac{1}{\theta}} \, dt < \infty.$$

Since

$$Y(t) \leq C(\Omega_T) \int_0^t \bar{K}(\tau, \sigma) Y(\sigma) \, d\sigma,$$

then from [37], we obtain that $Y = 0$ in $L^1(0, T)$. Hence as above we conclude that $v = 0$. \hfill \Box

**Remarks 6.2.** Under the additional hypothesis on $B$ that ensures the uniqueness of the solution to problem (6.1), we can also see $u$ as mild solution to problem (6.1) in the sense that, if we denote by $P_\Omega$, the Dirichlet heat kernel associated to the operator

$$L(v) := \partial_t v + (-\Delta)^s v - B(x, t) \cdot \nabla v,$$
then
\[ u(x, t) = \int_\Omega u_0(y) \tilde{P}_\Omega(x, y, t) \, dy + \int_0^t \int_\Omega f(y, \sigma) \tilde{P}_\Omega(x, y, t - \sigma) \, dy \, d\sigma. \]

Notice that from [33] (see Example 3, page 335), we get that $B \in K(\eta, Q)$, defined in [33]. Then $\tilde{P}_\Omega \preceq P_\Omega$.

**Remarks 6.3.** Related to the linear problem (6.4), we can show that $v \in C^{1,2\tau}_{t,x}(\Omega \times (0, T))$ for some $\tau \in (0, 1)$. Effectively, let us begin by proving that $g \in L^m_{\text{loc}}(\Omega_T)$ (where $L^m_{\text{loc}}(\Omega_T)$ is defined in Remark 3.17). Recall that $m > \frac{N + 2s}{2s - 1}$, then, using Hölder inequality it holds that $g \in L^1(\Omega_T)$ for $1 < l_1 < \frac{\hat{q}\sigma}{\sigma + \hat{q}}$ where $\hat{q} = \frac{N + 2s}{N + 1}$. Fix $l_1$ as above, by Proposition 3.16, we obtain that $|\nabla v| \in L^{l_1}_{\text{loc}}(\Omega_T)$ with $r_1 = \frac{l_1(N + 2s)}{N + 2s - l_1(2s - 1)}$. Using again Hölder inequality, we reach that $g \in L^{l_2}_{\text{loc}}(\Omega_T)$ with $l_2 = \frac{r_1\sigma}{\sigma + r_1}$. Using again Proposition 3.16, it follows that $|\nabla w| \in L^{l_2}_{\text{loc}}(\Omega_T)$ with $r_2 = \frac{l_2(N + 2s)}{N + 2s - l_2(2s - 1)}$.

Hence, we define the two sequences $\{l_n\}_n$ and $\{r_n\}_n$ by

\[
\begin{align*}
1 < l_1 < \hat{q} &= \frac{N + 2s}{N + 1}, \\
\frac{r_i}{\sigma + r_i}, i \geq 1, \\
l_{i+1} &= \frac{r_i\sigma}{\sigma + r_i}, i \geq 1.
\end{align*}
\]

Thus
\[ r_{i+1} = \frac{m(N + 2s)r_i}{m(N + 2s) - r_i((2s - 1)m - (N + 2s))}. \]

It is clear that $r_{i+1} > r_i$. Let show that there exists $i_0$ such that $r_{i_0} \geq \frac{\sigma(N + 2s)}{(2s - 1)m - (N + 2s)}$. We argue by contradiction. Assume that $r_i < \frac{m(N + 2s)}{m(N + 2s) - r_i((2s - 1)m - (N + 2s))}$ for all $i$. Since $\{r_i\}_i$ is an increasing sequence, there exists a $\tilde{r}$ such that $r_i \uparrow \tilde{r} \leq \frac{m(N + 2s)}{m(N + 2s) - (2s - 1)m - (N + 2s)}$.

Thus $\tilde{r} = \frac{m(N + 2s)}{m(N + 2s) - \tilde{r}((2s - 1)m - (N + 2s))}$, hence $\tilde{r} = 0$, a contradiction with the fact that $\{r_i\}_i$ is an increasing sequence.

Therefore, there exists $i_0 \in N$ such that $r_{i_0} \geq \frac{m(N + 2s)}{m(N + 2s) - (2s - 1)m - (N + 2s)}$. Thus, using Hölder inequality, we conclude that $g \in L^{\frac{N + 2s}{2s - 1}}_{\text{loc}}(\Omega_T) \cap L^1(\Omega_T)$. Hence by the result of Proposition 3.16, we obtain that $|\nabla v| \in L^{\frac{m}{2s - 1}}_{\text{loc}}(\Omega_T)$ for all $a > 1$. Thus $g \in L^m_{\text{loc}}(\Omega_T)$ and the claim follows.

Therefore, by the regularity results in [19], [30] and [33], we obtain that then $v \in C^{1,2\tau}_{t,x}(\Omega \times (0, T))$ for some $\tau \in (0, 1)$.

Under the hypotheses
\[ f \delta^3 \in L^1(\Omega), B\delta^3 \in (L^\sigma(\Omega_T))^N \quad \text{with } \sigma > \frac{N + 2s}{2s - \beta - 1} \quad \text{for some } 0 < \beta < 2s - 1, \]
then as in Theorem 6.1, we have the next existence result.

**Theorem 6.4.** Assume that the hypotheses (6.5) on $f$ and $B$ hold and $u_0 \in L^1(\Omega)$. Then the problem (6.1) has a distributional solution $u \in L^a(0, T; W^{1,a}_0(\Omega))$, $a < \frac{N + 2s}{N + \beta + 1}$. Moreover, if in addition $B$ satisfies one of the following conditions:

1. $B$ does not depend on $t$,
(2) \( B \in (L^m(\Omega_T))^N \) with \( m > \max\{q', \frac{2s}{(N + 2s) - q(N + \beta + 1)} \} > \frac{N + 2s}{2s - \beta - 1} \) for some \( 1 < q < \frac{N + 2s}{N + \beta + 1} \),

then the solution is unique.

**Proof.** As in the proof of Theorem 6.1, consider the set

\[
E_\beta(\Omega_T) = \{ v \in E_1(\Omega_T) \text{ such that } v \in E_r(\Omega_T) \text{ with } \sigma' < r < \frac{N + 2s}{N + \beta + 1} \text{ and } \|v\|_{E_r(\Omega_T)} \leq l^{\frac{1}{2}} \}.
\]

We define the operator

\[
K : E_\beta(\Omega_T) \rightarrow E_1(\Omega_T)
\]

\[
v \rightarrow T(v) = u
\]

where \( u \) is the unique solution to problem

\[
\begin{align*}
    u_t + (-\Delta)u &= B(x,t) \cdot \nabla v + f \text{ in } \Omega_T, \\
    u(x,t) &= 0 \text{ in } (\mathbb{R}^N \setminus \Omega) \times [0,T), \\
    u(x,0) &= u_0(x) \text{ in } \Omega.
\end{align*}
\]

Since, for \( v \in E_\beta(\Omega_T), \)

\[
\int_{\Omega_T} |B(x,t)||\nabla v|^p dx dt \leq C(T, \Omega) \left( \int_0^T \int_\Omega |B(x,t)|^p dx dt \right)^{\frac{1}{p}} \left( \int_0^T \int_\Omega |\nabla v|^p dx dt \right)^{\frac{1}{p}},
\]

we obtain that \( |(B(x,t), \nabla v) + f|^{\beta} \in L^1(\Omega_T). \) Thus using Theorem 3.22, we get the existence of a unique \( u \in E_a(\Omega_T) \) for all \( a < \frac{N + 2s}{N + \beta + 1} \) and

\[
\|u\|_{E_a(\Omega_T)} \leq C(\Omega_T) \left( \|f\|_{L^1(\Omega_T)} + \left( \int_{\Omega_T} |B(x,t)|^q \delta dx dt \right)^{\frac{1}{q}} \left( \int_{\Omega_T} |\nabla v|^r dx dt \right)^{\frac{1}{r}} \right).
\]

Hence we conclude that \( K \) is well defined and that \( K(E_\beta(\Omega_T)) \subset E_\beta(\Omega_T). \) Now the rest of the proof follows exactly as the proof of Theorem 6.1.

To prove the uniqueness part under the additional hypotheses on \( B, \) we follow exactly the same arguments as in the proof of uniqueness part in Theorem 6.1. \( \square \)

### 6.1. Applications

In this subsection we will obtain some applications of Theorem 6.1 in order to prove a comparison principle and, as a consequence, a uniqueness result for some particular cases of the quasi-linear problem. We begin by showing the next comparison principle.

**Theorem 6.5. (Comparison Principle)** Let \( w_1, w_2 \in L^q(0,T; W_0^{1,q}(\Omega)) \) for all \( q < \frac{N + 2s}{N + \beta + 1}, \) be such that

\[
\begin{align*}
    (w_1)_t + (-\Delta)^s w_1 &= H_1(x,t, w_1, \nabla w_1) \text{ in } \Omega_T, \\
    w_1 &= 0 \text{ in } (\mathbb{R}^N \setminus \Omega) \times (0,T), \\
    w(x,0) &= u_0(x) \text{ in } \Omega,
\end{align*}
\]

and

\[
\begin{align*}
    (w_2)_t + (-\Delta)^s w_2 &= H_2(x,t, w_2, \nabla w_2) \text{ in } \Omega_T, \\
    w_2 &= 0 \text{ in } (\mathbb{R}^N \setminus \Omega) \times (0,T), \\
    w(x,0) &= \tilde{u}_0(x) \text{ in } \Omega,
\end{align*}
\]

where \( H_1(x,t, w_1, \nabla w_1), H_2(x,t, w_2, \nabla w_2) \in L^1(\Omega_T) \) and \( u_0, \tilde{u}_0 \in L^1(\Omega). \)

(1) \( H_1(x,t, w_1, \nabla w_1), H_2(x,t, w_2, \nabla w_2) \in L^1(\Omega_T) \) and \( u_0, \tilde{u}_0 \in L^1(\Omega). \)

(2) \( H_1(x,t, w_1, \nabla w_1) - H_1(x,t, w_2, \nabla w_2) = \langle B(x,t, w_1, w_2), \nabla (w_1 - w_2) \rangle + f(x,t, w_1, w_2) \in \Omega_T \)

where \( B \in (L^m(\Omega_T))^N \) with \( m > \max\{q', \frac{2s}{(N + 2s) - q(N + 1)} \} > \frac{N + 2s}{2s - \beta - 1} \) for some \( 1 < q < \frac{N + 2s}{N + \beta + 1} \) and \( f \leq 0 \) in \( \Omega_T. \)

(3) \( u_0 \leq \tilde{u}_0 \in L^1(\Omega). \)

Then \( w_1 \leq w_2 \) in \( \Omega_T. \)
Proof. Let \( v = (u_1 - u_2) \), then \( v \) solves
\[
\begin{aligned}
&v_t + (-\Delta)^s v = B(x,t,w_1,w_2) \cdot \nabla v + f(x,t,w_1,w_2) \quad \text{in } \Omega_T, \\
v(x,t) = 0 \quad \text{in } (\mathbb{R}^N \setminus \Omega) \times (0,T), \\
v(x,0) = v_0 - v_0 \quad \text{in } \Omega.
\end{aligned}
\] (6.7)
Since \( B \) satisfies the hypotheses of Theorem 6.1, it follows that problem (6.7) has a unique solution. Now, using the fact that \( f \leq 0 \) in \( \Omega_T \) and \( v_0 \leq 0 \), then by Theorem 6.1, we reach that \( v \leq 0 \) in \( \Omega_T \) and then we conclude. \( \square \)

As a consequence, we get the next comparison result for approximated problems.

**Theorem 6.6.** Assume that \( a > 0 \) and \( \alpha > 1 \). Then for all \( (f,u_0) \in L^1(\Omega_T) \times L^1(\Omega) \), the problem
\[
\begin{aligned}
u_t + (-\Delta)^s v &= \frac{|\nabla u_1|^\alpha}{a_1 + |\nabla u_1|^\alpha} + \frac{|\nabla u_2|^\alpha}{a_2 + |\nabla u_2|^\alpha} + f \quad \text{in } \Omega_T, \\
u(x,t) &= 0 \quad \text{in } (\mathbb{R}^N \setminus \Omega) \times [0,T), \\
u(x,0) &= u_0(x) \quad \text{in } \Omega.
\end{aligned}
\] (6.8)
has a unique solution \( u_a \) such that \( u_a \in L^2(0,T;W^{1,q}_0(\Omega)) \) for all \( q < \frac{N + 2s}{N + 1} \) and \( T_k(u_a) \in L^2(0,T;H^1_0(\Omega)) \), for all \( k > 0 \). Moreover, if \( 0 < a_1 < a_2 \), then \( u_{a_1} \geq u_{a_2} \).

**Proof.** Since the dependance on the gradient term is bounded, then the existence follows using the arguments as in the proof of Theorem (3.5). Let show the comparison result which gives apriori the uniqueness part. Let \( 0 < a_1 \leq a_2 \) and consider \( u_{a_1}, u_{a_2} \) the solutions to (6.8) with \( a = a_1, a_2 \) respectively. Setting \( v = (u_{a_1} - u_{a_2}) \), then \( v \in L^r((0,T),W^{1,\tau}_0(\Omega)) \), for all \( \tau < \frac{N + 2s}{N - 1} \), and \( v \) solves
\[
v_t + (-\Delta)^s v = \frac{|\nabla u_1|^\alpha}{a_1 + |\nabla u_1|^\alpha} - \frac{|\nabla u_2|^\alpha}{a_2 + |\nabla u_2|^\alpha}.
\]
Let \( H(\rho) = \frac{\rho^\alpha}{a_1 + \rho^\alpha} \), \( \rho \geq 0 \), then
\[
v_t + (-\Delta)^s v = H(|\nabla u_1|) - H(|\nabla u_2|) - \frac{|\nabla u_2|^\alpha}{a_1 + |\nabla u_1|^\alpha} - \frac{|\nabla u_2|^\alpha}{a_2 + |\nabla u_2|^\alpha}.
\]
It is clear that \( h(x,t) := \left( \frac{|\nabla u_2|^\alpha}{a_1 + |\nabla u_1|^\alpha} - \frac{|\nabla u_2|^\alpha}{a_2 + |\nabla u_2|^\alpha} \right) \geq 0 \) a.e. in \( \Omega_T \). Thus \( v \) satisfies
\[
\begin{aligned}
v_t + (-\Delta)^s v &= B(x,t) \cdot \nabla v + h(x,t) \quad \text{in } \Omega_T, \\
v(x,t) &= 0 \quad \text{in } (\mathbb{R}^N \setminus \Omega) \times [0,T), \\
v(x,0) &= u_0(x) \quad \text{in } \Omega,
\end{aligned}
\]
where
\[
B(x,t) = \begin{cases}
0 & \text{if } |\nabla u_{a_1} - \nabla u_{a_2}| = 0, \\
\frac{\nabla u_{a_1} - \nabla u_{a_2}}{|\nabla u_{a_1} - \nabla u_{a_2}|^2} & \text{if } |\nabla u_{a_1} - \nabla u_{a_2}| \neq 0.
\end{cases}
\]
One can easily see that \( |B(x,t)| \leq C \). Therefore by Theorem 6.1 we obtain that \( v \leq 0 \) and then we conclude. \( \square \)

**Theorem 6.7.** Consider the problem
\[
\begin{aligned}
u_t + (-\Delta)^s u &= |\nabla u|^\alpha + f \quad \text{in } \Omega_T, \\
u(x,t) &= 0 \quad \text{in } (\mathbb{R}^N \setminus \Omega) \times [0,T), \\
u(x,0) &= u_0(x) \quad \text{in } \Omega,
\end{aligned}
\] (6.9)
where \( \Omega \subset \mathbb{R}^N \) is a bounded regular domain with \( N > 2s \) and \( \frac{1}{2} < s < 1 \). Assume that \( \alpha < \frac{N + 2s}{N + 1} \) and \( (f,u_0) \in L^1(\Omega_T) \times L^1(\Omega) \) are non negative functions. Then problem (6.9) has a minimal solution \( u \in L^q(0,T;W^{1,q}_0(\Omega)) \) for all \( q < \frac{N + 2s}{N + 1} \). In addition, if \( \alpha < \frac{(N + 2s)^2 + 2s(N + 1)}{(N + 1)(N + 4s)} < \frac{N + 2s}{N + 1} \), then the solution is unique.
Proof. Using Theorem 2.1 we get the existence of \( T_0 \leq T \) such that problem (6.9) has a solution 
\( u \in L^q(0, T_0; W_0^{1,q}) \) for all \( q < \frac{N + 2s}{N + 1} \).

To show the existence of a minimal solution we consider \( f_n = T_n(f) \) and \( u_{0n} = T_n(u_0) \). Define \( u_n \) to be the unique solution to problem

\[
\begin{align*}
& u_{nt} + (-\Delta)^s u_n = \frac{|\nabla u_n|^\alpha}{1 + \frac{1}{n}|\nabla u_n|^\alpha} + f_n & \text{in } \Omega T_0 \equiv \Omega \times (0, T_0), \\
& u_n(x, t) = 0 & \text{in } (\mathbb{R}^N \setminus \Omega) \times [0, T_0), \\
& u_n(x, 0) = u_{0n}(x) & \text{in } \Omega.
\end{align*}
\]  

(6.10)

It is clear that \( \{u_n\}_n \) is an increasing sequence of \( n \). If \( \hat{u} \) is a nonnegative solution to problem (6.9) with \( \hat{u} \in L^q(0, T_0; W_0^{1,q}) \) for all \( q < \frac{N + 2s}{N + 1} \), then by the comparison principle in Theorem 6.6 we deduce that \( u_n \leq \hat{u} \) for all \( n \). Hence we get the existence of \( u = \lim_{n \to \infty} u_n \) such that \( u \leq \hat{u} \). Thus \( u \in L^r(\Omega T_0) \) for all \( r < \frac{N + 2s}{N} \). To finish we have just to show that \( u \) is a solution to problem (6.9).

We claim that the sequence \( \{\frac{|\nabla u_n|^\alpha}{1 + \frac{1}{n}|\nabla u_n|^\alpha}\}_n \) is bounded in \( L^1(\Omega T_0) \). For simplicity of tipping we set \( g_n(x, t) = \frac{|\nabla u_n|^\alpha}{1 + \frac{1}{n}|\nabla u_n|^\alpha} \) and we define \( w_n \) to be the unique solution to the problem

\[
\begin{align*}
& w_{nt} + (-\Delta)^s w_n = f_n & \text{in } \Omega T_0 \equiv \Omega \times (0, T_0), \\
& w_n(x, t) = 0 & \text{in } (\mathbb{R}^N \setminus \Omega) \times [0, T_0), \\
& w_n(x, 0) = u_{0n}(x) & \text{in } \Omega.
\end{align*}
\]

By Theorem 2.4, we obtain that the sequence \( \{w_n\} \) is bounded in \( L^q(0, T_0; W_0^{1,q}) \) for all \( q < \frac{N + 2s}{N + 1} \) and that \( w_n \uparrow w \), the unique weak solution to the problem

\[
\begin{align*}
& w_t + (-\Delta)^s w = f & \text{in } \Omega T_0, \\
& w(x, t) = 0 & \text{in } (\mathbb{R}^N \setminus \Omega) \times [0, T_0), \\
& w(x, 0) = u_0(x) & \text{in } \Omega.
\end{align*}
\]

Thus

\[ u_n(x, t) = \int_0^t \int_{\Omega} g_n(y, \sigma) P_{\Omega}(x, y, t - \sigma) \, dy \, d\sigma + w_n(x, t) \]

and

\[ |\nabla u_n(x, t)| \leq \int_0^t \int_{\Omega} g_n(y, \sigma)|\nabla_x P_{\Omega}(x, y, t - \sigma)| \, dy \, d\sigma + |\nabla w_n(x, t)|. \]

Fixed \( q \in \left(\alpha, \frac{N + 2s}{N + 1}\right) \), then

\[ |\nabla u_n(x, t)|^q \leq C \left( \int_0^t \int_{\Omega} g_n(y, \sigma)|\nabla_x P_{\Omega}(x, y, t - \sigma)| \, dy \, d\sigma \right)^q + C|\nabla w_n(x, t)|^q. \]  

(6.11)

Setting

\[ D_n(x, t) = \left( \int_0^t \int_{\Omega} g_n(y, \sigma)|\nabla_x P_{\Omega}(x, y, t - \sigma)| \, dy \, d\sigma \right)^q, \]

and taking into consideration that \( \{|\nabla w_n|^\alpha\}_n \) is bounded in \( L^1(\Omega) \), to prove the claim we have just to show that \( \{D_n\} \) is bounded in \( L^1(\Omega_T) \).

As in the proof of the compactness part in Theorem 2.4, we have

\[
D_n(x, t) = \left( \int_0^t \int_{\Omega} g_n(y, \sigma)|\nabla_x P_{\Omega}(x, y, t - \sigma)| \, dy \, d\sigma \right)^q \leq C \left( \int_{\Omega \times (0, t) \cap \{t(x) > (t - \sigma)^{\frac{1}{q'}}\}} g_n(y, \sigma) \frac{P_{\Omega}(x, y, t - \sigma)}{(t - \sigma)^{\frac{1}{q'}}} \, dy \, d\sigma \right)^q + \frac{C}{\delta^q} \left( \int_{\Omega \times (0, t) \cap \{t(x) \leq (t - \sigma)^{\frac{1}{q'}}\}} g_n(y, \sigma) P_{\Omega}(x, y, t - \sigma) \, dy \, d\sigma \right)^q = D_{n1}(x, t) + D_{n2}(x, t).
\]
Similar to estimating the terms $J_{21}$ and $J_{22}$ in (3.20), (3.20) respectively, we have that
\[
\int_{\Omega_T} D_n(x,t) dx dt \leq C(\Omega_T) T^{r + \frac{q - 1}{2}} T^\frac{q - 1}{q} \|u_n\|_{L^q(\Omega_T)} (6.12)
\]
and
\[
D_n(x,t) = \frac{C}{\delta(x)} \left( \int_{\{0 < t \leq (\delta(x) \leq \frac{1}{T})\}} g_n(y,\sigma) P_{\Omega}(x,y,t-\sigma) dy d\sigma \right)^q \leq C \frac{\hat{u}^q(x,t)}{\delta(x)}, (6.13)
\]
where $r < \frac{N+2s}{N+1}$. Thus
\[
\int_{\Omega_T} D_n(x,t) dx dt \leq C(\Omega_T) \int_{\Omega_T} \frac{\hat{u}^q(x,t)}{\delta(x)} dx dt + T^{r + \frac{q - 1}{2}} T^\frac{q - 1}{q} \|u_n\|_{L^q(\Omega_T)} (6.13)
\]
Using the fact that $\hat{u}^q(x,t) \in L^1(\Omega_T)$ for all $q < \frac{N+2s}{N+1}$, $\hat{u} \in L^r(\Omega_T)$ for all $r < \frac{N+2s}{N+1}$, and going back to (6.11), it holds that
\[
\int_{\Omega_T} |\nabla u_n(x,t)|^q dx dt \leq C(\Omega_T) \|g_n\|_{L^1(\Omega)} + C(\Omega_T) \int_{\Omega_T} |\nabla u_n(x,t)|^q dx dt + C(\Omega_T).
\]
Using the fact that $\alpha < q$ and by Young inequality we reach that
\[
\int_{\Omega_T} |\nabla u_n(x,t)|^q dx dt \leq C(\Omega_T) \|g_n\|_{L^1(\Omega)} + C(\Omega_T) \|g_n\|_{L^q(\Omega_T)} (6.13)
\]
Using the fact that $u_{n+1}, u_{n+2} \in \Omega_T$ and the claim follows. Therefore, using Theorem 2.4 we conclude that, up to a subsequence, $u_n \rightarrow u$ strongly in $L^q(0,T;W^{1,q}_0(\Omega))$ for all $q < \frac{N+2s}{N+1}$. Thus $u$ is the minimal solution to problem (6.9) in $\Omega_T$.

We prove now that the minimal solution $u$ can be defined in the set $\Omega_T$. According to Theorem 1.1, the existence result holds for $L^1$ data in the set $\Omega_0 \times (t_1,t_2)$ if $t_2 - t_1 < \bar{C} := C(\Omega,s,N)$. Let $u$ the minimal solution obtained above in the set $\Omega_0 \times (t_0,t_0) and suppose that $T_0 < T$. Consider $T_1 = T_0 - \varepsilon$ with $\varepsilon > 0$ is chosen such that $0 < \varepsilon < \bar{C}$. Then $u_1, T_1 \in L^1(\Omega)$ and then the problem
\[
\begin{cases}
     v_t + (-\Delta)^s v &= |\nabla v|^q + f &\text{in } \Omega \times (T_1, T_1 + \bar{C}), \\
     v(x,t) &= 0 &\text{in } (\mathbb{R}^N \setminus \Omega) \times (T_1, T_1 + \bar{C}), \\
     v(x,T_1) &= u(x,T_1) &\text{in } \Omega,
\end{cases}
(6.14)
\]
has a minimal solution $v$. It is clear that $u$ is a solution of the same problem as $v$ in the set $\Omega \times [T_1, T_0]$. Hence $u = u$ in the set $\Omega \times [T_1, T_0]$. Setting
\[
\overline{v}(x,t) = \begin{cases} 
    u(x,t) &\text{if } (x,t) \in \Omega \times [0,T_1], \\
    v(x,t) &\text{if } (x,t) \in \Omega \times [T_1, T_1 + \bar{C}],
\end{cases}
\]
then $\overline{v}$ is the minimal solution to the problem
\[
\begin{cases}
     \overline{v}_t + (-\Delta)^s \overline{v} &= |\nabla \overline{v}|^q + f &\text{in } \Omega \times (0,T_1 + \bar{C}), \\
     \overline{v}(x,t) &= 0 &\text{in } (\mathbb{R}^N \setminus \Omega) \times (0,T_1 + \bar{C}), \\
    \overline{v}(x,0) &= u(x) &\text{in } \Omega.
\end{cases}
(6.15)
\]
Repeating the above argument in a finite time of steps we get the existence of a minimal solution $u$ to the problem (6.9) defined in the set $\Omega_T$ with $u \in L^q(0,T;W^{1,q}_0(\Omega))$ for all $q < \frac{N+2s}{N+1}$.

Finally to show the uniqueness under the condition $\alpha < \frac{(N+2s)^2 + 2s(N+1)}{(N+1)(N+4s)}$. Notice that
\[
\frac{(N+2s)^2 + 2s(N+1)}{(N+1)(N+4s)} < \frac{N+2s}{N+1} \text{ if and only if } 2s > 1 \text{ which is our main hypothesis.}
\]
We will use the comparison principle in Theorem 6.5. If $u_1, u_2$ are two solutions to problem (6.9) with $u_1, u_2 \in L^q(0,T;W^{1,q}_0(\Omega))$ for all $q < \frac{N+2s}{N+1}$. Then $v = u_1 - u_2$ solves
\[
\begin{cases}
     v_t + (-\Delta)^s v &= \langle B(x,t), \nabla v \rangle &\text{in } \Omega_T, \\
     v(x,t) &= 0 &\text{in } (\mathbb{R}^N \setminus \Omega) \times [0,T), \\
    v(x,0) &= 0 &\text{in } \Omega,
\end{cases}
(6.16)
\]
where $v \in L^q(0,T;W_0^{1,q}(\Omega))$, $q < \frac{N + 2s}{N + 1}$ and $|B(x,t)| \leq C(|\nabla u_1|^{\alpha-1} + |\nabla u_2|^{\alpha-1})$. It is clear that $B \in L^q(\Omega_T)$ for all $m < \frac{N + 2s}{(N + 1)(\alpha - 1)}$. Recall that $\alpha < \frac{N + 2s}{N + 1}$, then $\alpha' < \frac{N + 2s}{(N + 1)(\alpha - 1)}$. Since $\alpha < \frac{(N + 2s)^2 + 2s(N + 1)}{(N + 1)(N + 4s)}$, then

$$\frac{N + 2s}{(N + 1)(\alpha - 1)} > \max\{\alpha', \frac{2s}{(N + 2s) - \alpha(N + 1)}\} > \frac{N + 2s}{2s - 1}.$$ 

Hence we can choose $m < \frac{N + 2s}{(N + 1)(\alpha - 1)}$ such that $m > \max\{\alpha', \frac{2s}{(N + 2s) - \alpha(N + 1)}\} > \frac{N + 2s}{2s - 1}$. Hence by the comparison principle in Theorem 6.5 we deduce that $v = 0$ and then we conclude.

Under additional regularity hypothesis on the solution, we can prove the next uniqueness result.

**Theorem 6.8.** Assume that $\alpha > 1$, then the problem (6.9) has at most one solution $u$ such that $u \in L^q(0,T;W_0^{1,q}(\Omega))$ with $\frac{q}{\alpha - 1} > \max\{\beta', \frac{2s}{(N + 2s) - \beta(N + 1)}\}$ for some $1 < \beta < \frac{N + 2s}{N + 1}$. In particular, problem (6.9) has at most one solution $u \in C^1(\Omega_T)$.

**Proof.** If $u_1, u_2$ are two solution with the above regularity, then wetting $v = u_1 - u_2$, it holds that $v$ solves the problem

$$\begin{cases}
  v_t + (-\Delta)^{\alpha}v = B(x,t) \cdot \nabla v & \text{in } \Omega_T, \\
  v(x,t) = 0 & \text{in } (\mathbb{R}^N \setminus \Omega) \times [0,T), \\
  v(x,0) = 0 & \text{in } \Omega,
\end{cases}
$$

where $v \in L^q(0,T;W_0^{1,q}(\Omega))$ and $|B(x,t)| \leq C(|\nabla u_1|^{\alpha-1} + |\nabla u_2|^{\alpha-1})$. According to the regularity hypothesis on $u_1, u_2$, we obtain that $B \in L^q(\Omega_T)$ with $m = \frac{q}{\alpha - 1} > \max\{\beta', \frac{2s}{(N + 2s) - \beta(N + 1)}\}$ for some $1 < \beta < \frac{N + 2s}{N + 1}$.

Thus using the comparison principle in Theorem 6.5 we obtain that $u_1 = u_2$ and then we conclude. 

6.2. Some remarks on asymptotic behavior. We now deal with asymptotic behavior of the solutions. Let us begin by the next global existence result for the Cauchy problem given in [27].

**Theorem 6.9.** Assume that $\alpha < \frac{N + 2s}{N + 1}$ and $u_0 \in W^{1,\infty}(\mathbb{R}^N) \cap L^1(\mathbb{R}^N)$, then the problem

$$\begin{cases}
  u_t + (-\Delta)^{\alpha}u = |\nabla u|^\alpha & \text{in } \mathbb{R}^N \times (0,T), \\
  u(x,0) = u_0(x) & \text{in } \mathbb{R}^N,
\end{cases}
$$

has a unique global solution $u$ such that $u \in C([0,T],W^{1,\infty}(\mathbb{R}^N))$ for all $T > 0$. Moreover if $\alpha > \frac{N + 2s}{N + 1}$ and either $\|u_0\|_{L^1(\mathbb{R}^N)}$ or $\|\nabla u_0\|_{L^1(\Omega)}$ is small, then $\|u(.,t)\|_{L^1(\Omega)} \leq C$ for all $t$.

It is clear that if $u$ is a solution to problem (6.18) with $u_0$ satisfying the conditions of Theorem 6.9, then $u$ is globally defined in $t$. We refer to [27] and [34] for the proof.

In our case and according to the value of $\alpha$, we can prove the next partial blow up result.

**Theorem 6.10.** Assume that $s \in \left(\frac{\sqrt{5} - 1}{2}, 1\right]$ and suppose that $1 + s < \alpha < \frac{s}{1 - s}$. Then for all data $f \in L^\infty(\Omega \times (0,\infty))$, $f \geq 0$, the solution $u$ to problem (1.1) obtained in Theorem 1.4 blows-up in a finite time in the sense that

$$\int_\Omega u(x,t)\delta^*(x)dx \rightarrow \infty \text{ for } t \rightarrow T^*.$$ 

**Proof.** Since $s > \frac{\sqrt{5} - 1}{2}$ the interval of $\alpha$ is non empty. We will use a convexity argument. Let $\phi_1$ be the first positive bounded eigenfunction of the fractional Laplacian, then $\phi_1$ satisfies

$$\begin{cases}
  (-\Delta)^{\alpha}\phi_1 = \lambda_1\phi_1 & \text{in } \Omega, \\
  \phi_1 > 0 & \text{in } \Omega, \\
  \phi_1 = 0 & \text{in } \mathbb{R}^N \setminus \Omega,
\end{cases}$$

where $\phi_1 \neq 0$. Since $\lambda_1 > 0$, we have that $\phi_1$ is a positive function. Therefore the solution $u$ to problem (1.1) is a super-solution and hence $u(x,t) \rightarrow \infty$ as $t \rightarrow T^*$. \qed
and $\phi_1(x) \preceq \delta^s(x)$. Using $\phi_1$ as a test function in the problem of $u$ and integrating in $x$, we reach that
\[
\frac{d}{dt} \int_{\Omega} u(x,t)\phi_1(x)dx + \lambda_1 \int_{\Omega} u(x,t)\phi_1(x)dx = \int_{\Omega} |\nabla u|^\alpha \phi_1(x)dx \geq C \int_{\Omega} |\nabla u|^\alpha \delta^s(x)dx.
\]
Since $s < \alpha - 1$, then using the weighted Hardy inequality (4.4), we conclude that
\[
\frac{d}{dt} \int_{\Omega} u(x,t)\phi_1(x)dx + \lambda_1 \int_{\Omega} u(x,t)\phi_1(x)dx \geq C(\Omega, s) \int_{\Omega} u^\alpha(x,t)\phi_1(x)dx.
\]
On the other hand $\alpha > s$, hence $\int_{\Omega} u^\alpha(x,t)\phi_1(x)dx \geq C(\Omega, s) \int_{\Omega} u^\alpha(x,t)\phi_1(x)dx$. Therefore using Jensen’s inequality it follows that
\[
\frac{d}{dt} \int_{\Omega} u(x,t)\phi_1(x)dx + \lambda_1 \int_{\Omega} u(x,t)\phi_1(x)dx \geq C(\Omega, s) \int_{\Omega} u^\alpha(x,t)\phi_1(x)dx
\]
\[
\geq C(\Omega, s, \alpha) \left( \int_{\Omega} u(x,t)\phi_1(x)dx \right)^\alpha.
\]
Define $Y(t) = \int_{\Omega} u(x,t)\phi_1(x)dx$, then
\[
Y'(t) + \lambda_1 Y(t) \geq C(\Omega, s, \alpha) Y^\alpha(t).
\]
A simple convex argument allows us to get the existence of $A$ such that if $Y(0) > A$, then $Y(t) \to \infty$ if $t \to T^*$ depending only on the data. Hence we conclude.

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