Two-Step Estimation of a Strategic Network Formation Model with Clustering

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Abstract

This paper explores strategic network formation under incomplete information using data from a single large network. We allow the utility function to be nonseparable in an individual’s link choices to capture the spillover effects from friends in common. In a network with \( n \) individuals, the nonseparable utility drives an individual to choose between \( 2^{n-1} \) overlapping portfolios of links. We develop a novel approach that applies the Legendre transform to the utility function so that the optimal decision of an individual can be represented as a sequence of correlated binary choices. The link dependence that results from the preference for friends in common is captured by an auxiliary variable introduced by the Legendre transform. We propose a two-step estimator that is consistent and asymptotically normal. We also derive a limiting approximation of the game as \( n \) grows large that can help simplify the computation in very large networks. We apply these methods to favor exchange networks in rural India and find that the direction of support from a mutual link matters in facilitating favor provision.

JEL Codes: C31, C57, D85

Keywords: network formation, strategic interactions, clustering, two-step estimation, limiting game, favor exchange

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1 Introduction

Network formation has attracted considerable interest from economists due to its broad applicability in economics for theorizing phenomena such as job referrals (Beaman and Magruder, 2012), favor exchange (Jackson et al., 2012), interbank lending (Elliott et al., 2014), and production networks (Acemoglu and Azar, 2020). One challenge in characterizing the formation of social and economic networks is that the formation of a link may be influenced by the presence of other links (Jackson, 2008; Jackson et al., 2017). For example, individuals get to know each other through friends of friends (Jackson and Rogers, 2007). Having a friend in common can help support the incentives for establishing a relationship (Jackson et al., 2012). These externalities from indirect connections can often lead to strategic interactions between links that complicate the empirical analysis of network formation.

To account for this link interdependence, empirical models of strategic network formation typically exploit a game-theoretic framework in which the latent utility from forming a link depends on the other links in a network, and the network that individuals form is an equilibrium outcome (see Graham (2020) and de Paula (2020) for surveys). Depending on how we specify the information individuals possess and the strategies they take, significant challenges can arise in the identification, estimation, and computation of model parameters. Miyauchi (2016), de Paula et al. (2018), and Sheng (2020) assumed that individuals form links simultaneously under complete information. Because of the prevalence of multiple equilibria, the parameters in general are partially identified.\footnote{These papers considered undirected networks and used pairwise stability (Jackson and Wolinsky, 1996) as the equilibrium solution.} Mele (2017) and Christakis et al. (2020) circumvented multiplicity by assuming that links in a network are formed in a random sequence. This evolutionary process of network formation provides a particular equilibrium selection mechanism that yields either a unique network or a unique stationary distribution over networks (Jackson and Watts, 2002). Strategic interactions under complete information also generate a complex dependence structure, which makes it difficult to establish asymptotic theorems if one only observes a single large network. de Paula et al. (2018) assumed type-specific unobservables and bounded degrees to achieve asymptotic validity. Leung (2019) and Menzel (2017) proved a LLN and Leung and Moon (2021) proved a CLT with further restrictions on sparsity and preferences.

In this paper, we develop a model of strategic network formation under incomplete information. We assume that individuals know the unobserved (by the researcher) utility shocks for their own potential links, but not the unobserved utility shocks for the potential links of the other individuals. Individuals simultaneously choose the links they wish to form, and the
directed network they form is a Bayesian Nash equilibrium. The Bayesian Nash equilibrium has been widely used in other network-related models. For network formation, we provide evolutionary results resembling those under complete information (Jackson and Watts, 2002; Mele, 2017) that a Bayesian Nash equilibrium can be regarded as a long-term equilibrium in a dynamic process of network formation (Myatt and Wallace, 2003; Myatt and Wallace, 2004; Jackson and Yariv, 2007). Incomplete information offers several advantages over complete information in the econometric analysis. It permits point identification in the presence of multiple equilibria. More importantly, the microfounded assumption of independent private information yields conditional independence between links formed by different individuals that can simplify the asymptotic analysis in a single large network. Leung (2015) pioneered the study of strategic network formation under incomplete information, who assumed that the utility function is additively separable in one’s own links. We extend his work to a more general utility function that is nonseparable in one’s own links.

Our extension to Leung (2015) is motivated by the empirical evidence that social and economic networks typically present high clustering (Jackson, 2008; Jackson et al., 2017; Graham, 2016). This phenomenon occurs in part because two individuals who are connected through a mutual friend may have an increased chance of knowing one another or experience stronger incentives for building a cooperative relationship like risk sharing or favor exchange (Jackson et al., 2012). To capture the preference for friends in common, we allow the utility function to depend on the interaction between an individual’s two link choices. In a network with \( n \) individuals, the nonseparable utility drives an individual to choose between \( 2^n - 1 \) overlapping portfolios of links – a seemingly intractable discrete choice problem, as we assume that \( n \) grows large. We propose a novel approach that applies the Legendre transform (Rockafellar, 1970) to the utility function so that the intractable discrete choice problem is transformed equivalently into a tractable sequence of correlated binary choice problems. The dependence between an individual’s link choices that results from the preference for friends in common is captured by an auxiliary variable introduced by the Legendre transform. From the transformation we can derive the optimal link choices of an individual explicitly.

We propose a two-step estimation procedure where we estimate the link choice probabilities in the first step and estimate the model parameters in the second step. Two-step estimation has been widely used in dynamic discrete choices and games of incomplete infor-

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2Examples include Blume et al. (2015), who developed an incomplete information game of social interactions where individuals do not observe the utility shocks of other individuals in a network, and Galeotti et al. (2010) and Jackson and Yariv (2007), who explored more general games played on a network where individuals do not observe the private costs or degrees of other individuals.

3The clustering considered here is different from that in the statistical literature on community detection, which typically assumes that there is a latent community structure in data (Abbe, 2018; Mele, 2022).

4We are grateful to Terence Tao for suggesting this approach.
We extend this approach to network formation using data from a single large network. The asymptotic analysis is complicated by the fact that the preference for friends in common leads to dependence between an individual’s link choices. The auxiliary variable in the Legendre transform provides a useful tool for investigating how the link dependence affects the asymptotic properties of the estimator. We show that the two-step estimator is consistent and asymptotically normal. The link dependence does not affect the rate of convergence, but does increase the asymptotic variance of the estimator.

While the two-step estimation facilitates computation, accounting for link dependence when computing a link choice probability can be computationally costly in a very large network. To further improve practicality, we show that a link choice probability in a finite-$n$ network converges to a limiting link probability as $n$ approaches infinity. The limiting link probability has an analytical form that is simple to compute. We provide simulation evidence that using the limiting approximation in the second step yields estimates that are similar to those from the finite-$n$ model, provided that the networks are sufficiently large. In addition, we also explore a scenario in which the underlying network is directed, but because of data collection and reporting we only observe undirected relationships.

We apply our methods to favor exchange networks in rural India. We extend Jackson et al. (2012) by investigating the directed relationships in favor exchange – that is, who offers a favor to whom. We find that indirect links have significant effects on favor provision; ignoring these spillover effects will overestimate the homophily effects. More strikingly, we find that the effect of support from a mutual link depends critically on the direction of the support from the provider’s perspective. Individual $i$ is more likely to offer a favor to individual $j$ if $i$ offers a favor to $j$’s favor-exchange companion $k$ instead of being offered a favor by $k$. These directional results complement the findings in Jackson et al. (2012) and shed further light on policy interventions.

The remainder of the paper is organized as follows. Section 2 introduces the model, including the utility, information, and equilibrium. Section 3 derives an explicit expression for the optimal link choices of an individual. Section 4 presents the two-step estimator and its asymptotic properties. Section 5 explores extensions to our approach, including undirected networks and limiting approximation. Section 6 conducts the empirical application. Section 7 concludes the paper. Additional results are presented in the Online Appendix.

Seminal papers on two-step estimation include Hotz and Miller (1993), Bajari et al. (2007), Aguirregabiria and Mira (2007), and Bajari et al. (2010).
2 Model

Consider a set of \( n \) individuals who choose to form a network. Each individual \( i \) is endowed with a vector of observed characteristics \( X_i \) with support \( \mathcal{X} \), and a vector of unobserved link-specific utility shocks \( \epsilon_i = (\epsilon_{i1}, \ldots, \epsilon_{i,i-1}, \epsilon_{i,i+1}, \ldots, \epsilon_{in})' \in \mathbb{R}^{n-1} \), where \( \epsilon_{ij} \) is the utility shock for link \( ij \). Let \( X = (X'_1, \ldots, X'_n)' \in \mathcal{X}^n \) denote the characteristic profile and \( \epsilon = (\epsilon'_1, \ldots, \epsilon'_n)' \in \mathbb{R}^{n(n-1)} \) denote the utility shock profile.

The network formed is denoted by an \( n \times n \) binary matrix \( G \in \mathcal{G} \), where the \( ij \)th entry \( G_{ij} = 1 \) if individual \( i \) forms a link to individual \( j \) and \( G_{ij} = 0 \) otherwise. The diagonal elements \( G_{ii} \) are set to 0 for all \( i \), so there are no self-links. In this paper, we focus on directed networks – that is, \( G_{ij} \) and \( G_{ji} \) can be different. While relationships such as friendships and collaborations are typically undirected, many economic networks are in fact formed as a result of directed individual decisions. Examples include a village resident lending money to another resident, a buyer purchasing a product from a seller, and an employee referring a candidate for a job. Following Bala and Goyal (2000), Mele (2017), and Leung (2015), we consider a noncooperative framework where individual \( i \) can unilaterally decide to form the link \( ij \).\(^6\) In Section 5.1, we extend our analysis to the scenario in which the underlying network is directed, but we only observe undirected relationships in data.

Utility. For a given characteristic profile \( X \) and utility shock vector \( \epsilon_i \), individual \( i \)'s utility in a network \( G \) is given by

\[
U_i(G, X, \epsilon_i; \theta_u) = \frac{1}{n-1} \sum_{j \neq i} G_{ij} \left( u_{ij}(G_{-i}, X; \beta) + \frac{1}{2(n-2)} \sum_{k \neq i,j} G_{ik} v_{ij,k}(G_{-i}, X; \gamma) - \epsilon_{ij} \right),
\]

(2.1)

where \( G_{-i} \) is the submatrix of \( G \) with the \( i \)th row deleted, that is, the links formed by individuals other than \( i \). We assume that the utility function is known up to the parameter \( \theta_u = (\beta', \gamma')' \) in a compact set \( \Theta_u \subset \mathbb{R}^{d_u} \).

In this specification, the term \( u_{ij}(G_{-i}, X; \beta) \) represents individual \( i \)'s incremental utility from linking to individual \( j \) that does not depend on the other links that \( i \) forms. A typical specification of \( u_{ij}(G_{-i}, X; \beta) \) is

\[
u_{ij}(G_{-i}, X; \beta) = \beta_1 + X'_i \beta_2 + d(X_i, X_j)' \beta_3 + G_{ji} \beta_4
\]

\(^6\)Equivalently, we can characterize the formation of a link as a bilateral decision between the provider and the recipient in which the recipient always prefers the link. For example, a village resident always likes to receive a favor, a seller always wants to sell a product (given the price), and a job candidate always wishes to get a job referral.
where \( d(X_i, X_j) \) represents a vector of known functions of \( X_i \) and \( X_j \) that measure the social proximity between \( i \) and \( j \); for example, whether they have the same gender, age, education, and caste. This term captures the homophily effect (Jackson, 2008; Jackson et al., 2017).

The last three terms in equation (2.2) capture the spillover effects from other links that involve \( j \), including the reciprocation effect of link \( ji \) \((\beta_1) \) and the effects of \( j \)'s in-degree \((\beta_5) \) and out-degree \((\beta_6) \). Note that we normalize \( j \)'s in-degree and out-degree by \( n \rightarrow \infty \) so that they do not dominate the other effects asymptotically. The specification in equation (2.2) is similar to that in Leung (2015).

In addition to the utility separable in individual \( i \)'s links, we also allow for the utility to be nonseparable in individual \( i \)'s links. The term \( v_{i,jk}(G_{-i}, X; \gamma) \) represents the incremental utility that \( i \) derives from linking to both individual \( j \) and individual \( k \), if \( j \) and \( k \) are linked to each other. An important example is

\[
v_{i,jk}(G_{-i}, X; \gamma) = (G_{jk} + G_{kj})\gamma_1(X_j, X_k) + \frac{1}{n-3} \sum_{i\neq i,j,k} (G_{jl}G_{lk} + G_{kl}G_{lj})\gamma_2(X_j, X_k). \tag{2.3}
\]

The two terms are motivated by the prevalence of triadic closure \((\gamma_1 > 0) \) and cyclic closure \((\gamma_2 > 0) \), which mean that individual \( i \) is more likely to link to individual \( j \) if \( i \) is linked to a third individual \( k \) who is connected to \( j \) directly or indirectly (Kossinets and Watts, 2006; Jackson, 2008; Jackson et al., 2017).\footnote{The terms \( G_{jk} + G_{kj} \) and \( G_{jl}G_{lk} + G_{kl}G_{lj} \) in equation (2.3) can be replaced by other functions that are symmetric in \( j \) and \( k \). For example, we can replace \( G_{jk} + G_{kj} \) by \( G_{jk}G_{kj} \) (i.e., both \( j \) is linked to \( k \) and \( k \) is linked to \( j \)) or \( 1\{G_{jk} + G_{kj} \geq 1\} \) (i.e., either \( j \) is linked to \( k \) or \( k \) is linked to \( j \)).}

One possible reason for triadic/cyclic closure is that via a mutual friend \( k \), individual \( i \) may have an increased chance to know \( j \). Another reason – one relevant particularly in favor exchange and risk sharing networks – is that being linked to a third individual \( k \) whom \( i \) trusts and who trusts \( j \) may give \( i \) the basis to trust \( j \) (Easley and Kleinberg, 2010; Karlan et al., 2009; Jackson et al., 2012). These mutual-friend effects can also depend on the social proximity between \( j \) and \( k \), as captured by \( \gamma_1(X_j, X_k) \) and \( \gamma_2(X_j, X_k) \), which consist of known nonnegative functions of \( X_j \) and \( X_k \), such as whether \( j \) and \( k \) share certain characteristics and a vector of parameters.\footnote{One example is \( \gamma_1(X_j, X_k) = d_1(X_j, X_k)\gamma_1 \) and \( \gamma_2(X_j, X_k) = d_2(X_j, X_k)^{\gamma_2}, \) where \( d_1(X_j, X_k) \) and \( d_2(X_j, X_k) \) are vectors that measure the social distance between \( j \) and \( k \).}

Note that \( v_{i,jk}(G_{-i}, X; \gamma) \) is symmetric in \( j \) and \( k \) so that the utility function does not depend on how we label the individuals.\footnote{This requires that \( \gamma_1(X_j, X_k) \) and \( \gamma_2(X_j, X_k) \) are symmetric in \( j \) and \( k \).} The second term in equation (2.3) is also normalized to guarantee
its boundedness for large $n$.

**Information.** Most literature on network formation games assumes that individuals have complete information about the game (Jackson and Wolinsky, 1996; Bala and Goyal, 2000; Christakis et al., 2020; Mele, 2017; Miyauchi, 2016; de Paula et al., 2018; Sheng, 2020; Menzel, 2017). While this is appropriate in small networks, in a large network an individual may not observe every aspect of the other individuals. In this paper, we follow Leung (2015) and assume that each individual only has partial information about the other individuals. In particular, we assume that the characteristic profile $X$ is observed by all the individuals, but the utility shock vector $\epsilon_i$ is observed by individual $i$ only.\footnote{This is a standard setup for games of incomplete information (e.g., Bajari et al., 2010).} We also assume that the utility shocks are i.i.d. and are independent of the characteristics. Formally,\footnote{Incomplete information games typically assume independent private information. If we relax Assumption 1(i) and allow $\epsilon_i$ and $\epsilon_j$ to be dependent, linking decisions $G_i$ and $G_j$ become dependent, but their dependence (conditional on $X$) can be fully captured by the dependence between $\epsilon_i$ and $\epsilon_j$. For typical networks where the identity of an individual does not carry any significance, $\epsilon_{ij}$ forms a jointly exchangeable array (Kallenberg, 2005). Asymptotic theorems for such links can be established following the theory of exchangeable arrays (e.g., Menzel, 2021).}

**Assumption 1.** (i) $\epsilon_{ij}$ is i.i.d. with cdf $F_\epsilon(\cdot; \theta_\epsilon)$ known up to the parameter $\theta_\epsilon \in \Theta_\epsilon \subset \mathbb{R}^{d_\epsilon}$. (ii) The distribution of $\epsilon_{ij}$ has a density function $f_\epsilon(\cdot; \theta_\epsilon)$ with respect to Lebesgue measure that is continuously differentiable in $\theta_\epsilon$ and strictly positive on $\mathbb{R}$. (iii) $\epsilon$ and $X$ are independent.

The independence of $\epsilon_i$ across $i$ is a crucial assumption. It enables us to break the link dependence across individuals and reduce the complexity of the model.\footnote{We can relax this assumption by adding an individual-invariant effect, as in Graham (2017).} The independence of $\epsilon_{ij}$ and $\epsilon_{ik}$ is imposed for simplicity.\footnote{The exogeneity of $X$ can be relaxed if the conditional distribution of $\epsilon$ given $X$ is specified.} Assumptions 1(ii)–(iii) are standard regularity assumptions.\footnote{The exogeneity of $X$ can be relaxed if the conditional distribution of $\epsilon$ given $X$ is specified.}

**Equilibrium.** We assume that individuals form links simultaneously. Let $G_i$ be the $i$th row of network $G$, that is, the links formed by individual $i$, and $G_i = \{0, 1\}^{n-1}$ the set of all possible $G_i$. A strategy of individual $i$ is a function $G_i(X, \epsilon_i) : \mathcal{X}^n \times \mathbb{R}^{n-1} \rightarrow G_i$ that maps $i$’s information $(X, \epsilon_i)$ to a row vector of links $G_i$. Denote the strategy profile of all individuals by $G(X, \epsilon) = (G_1(X, \epsilon_1)', \ldots, G_n(X, \epsilon_n)')'$. A Bayesian Nash equilibrium (or an equilibrium for short) of the game is a strategy profile $G(X, \epsilon)$ such that each $G_i(X, \epsilon_i)$ maximizes the expected utility $E[U_i(G_i, G_{-i}, X, \epsilon_i)|X, \epsilon_i]$, where the expectation is taken with respect to the strategies of individuals other than $i$, $G_{-i}$.
For the utility function in (2.1), the expected utility of individual \( i \) is

\[
\mathbb{E}[U_i(G_i, G_{-i}, X, \epsilon_i)|X, \epsilon_i] = \frac{1}{n-1} \sum_{j \neq i} G_{ij} \left( \mathbb{E}[u_{ij}(G_{-i}, X)|X] + \frac{1}{2(n-2)} \sum_{k \neq i,j} G_{ik} \mathbb{E}[v_{i,j,k}(G_{-i}, X)|X] - \epsilon_{ij} \right)
\]

(2.4)

Under the specifications in (2.2)–(2.3), we have

\[
\mathbb{E}[u_{ij}(G_{-i}, X)|X] = \beta_1 + X_i' \beta_2 + d(X_i, X_j)' \beta_3 + \sigma_{ij}(X) \beta_4 + \frac{1}{n-2} \sum_{k \neq i,j} \sigma_{kj}(X) \beta_5 + \frac{1}{n-2} \sum_{k \neq i,j} \sigma_{jk}(X) \beta_6,
\]

(2.5)

and

\[
\mathbb{E}[v_{i,j,k}(G_{-i}, X)|X] = (\sigma_{jk}(X) + \sigma_{kj}(X)) \gamma_1(X_j, X_k) + \frac{1}{n-3} \sum_{l \neq i,j,k} (\sigma_{jl}(X) \sigma_{lk}(X) + \sigma_{kl}(X) \sigma_{lj}(X)) \gamma_2(X_j, X_k),
\]

(2.6)

where \( \sigma_{ij}(X) = \mathbb{E}[G_{ij}|X] \). The expressions for \( \mathbb{E}[u_{ij}(G_{-i}, X)|X] \) and \( \mathbb{E}[v_{i,j,k}(G_{-i}, X)|X] \) follow because the independence of \( \epsilon_i \) across \( i \) implies that \( \epsilon_i \) is independent of the strategies of others \( G_{-i} \) conditional on \( X \) and hence \( \mathbb{E}[u_{ij}(G_{-i}, X)|X] \) and \( \mathbb{E}[v_{i,j,k}(G_{-i}, X)|X] \) only depend on the public information \( X \). Equation (2.6) holds also because, conditional on \( X \), the strategies \( G_j \) (or \( G_k \)) and \( G_i \) are independent.

Following the literature on incomplete information games (Bajari et al., 2010), we can represent an equilibrium in the space of conditional choice probabilities. Given \( X \), let \( \sigma_i(g_i|X) \) denote the conditional probability that individual \( i \) chooses link vector \( g_i \)

\[
\sigma_i(g_i|X) = \mathbb{P}(G_i = g_i|X) = \mathbb{P} \left( \mathbb{E}[U_i(g_i, G_{-i}, X, \epsilon_i)|X, \epsilon_i] \geq \max_{\tilde{g}_i \in \tilde{G}_i} \mathbb{E}[U_i(\tilde{g}_i, G_{-i}, X, \epsilon_i)|X, \epsilon_i] \right)
\]

(2.7)

and \( \sigma(X) = \{\sigma_i(g_i|X), g_i \in \tilde{G}_i, i = 1, \ldots, n\} \) the conditional choice probability (CCP) profile. The right-hand side of equation (2.7) defines a mapping from \( \sigma_{-i}(X) = \{\sigma_j(g_j|X), g_j \in \tilde{G}_j, j \neq i\} \) to \( \sigma_i(g_i|X) \) that we denote by \( \mathcal{P}_i(g_i|X, \sigma_{-i}(X)) \). An equilibrium CCP profile \( \sigma^*(X) \) is a fixed point of the equation

\[
\sigma^*_i(g_i|X) = \mathcal{P}_i(g_i|X, \sigma^*_{-i}(X))
\]

(2.8)

for all \( g_i \in \tilde{G}_i \) and all \( i = 1, \ldots, n \). From an equilibrium CCP profile, we can derive the
equilibrium strategy profile from the optimal decision of each individual $i$ for a given $X$ and $\epsilon_i$. Therefore, we can represent an equilibrium equivalently by the CCP profile.

The Bayesian Nash equilibrium, though less common in network formation, has been widely used in other network-related models. For example, Blume et al. (2015) developed an incomplete information game of social interactions where individuals do not observe the private utility shocks of other individuals in a network. They show that Bayesian Nash equilibria of the game provide a microfoundation that can nest the standard social interaction models such as Manski (1993). Galeotti et al. (2010) and Jackson and Yariv (2007) considered more general games played on a network, where an individual’s payoff depends on the actions taken by neighbors. In such a game, individuals do not observe the private costs or degrees of other individuals. They form beliefs about the degrees of their neighbors based on the degree distribution in the network. Both Galeotti et al. (2010) and Jackson and Yariv (2007) investigated Bayesian Nash equilibria of the game.

One concern regarding the application of the Bayesian Nash equilibrium to network formation is that links may not be formed simultaneously. In network formation games with complete information, equilibrium solutions that assume simultaneous move (e.g., the Nash equilibrium for directed networks and pairwise stability for undirected networks) are usually justified by an evolutionary process that converges to equilibria in the static game. For example, Jackson and Watts (2002) developed a dynamic process of network formation that converges to pairwise stable networks if cycles are ruled out. Mele (2017) considered a similar dynamic process for directed networks that converges to Nash equilibria in the static game. In Online Appendix O.A, we show that similar evolutionary results can be established for the Bayesian Nash equilibrium. Specifically, we construct two dynamic processes of network formation where links are formed over time. The first process assumes that one link is updated in each period as in Myatt and Wallace (2004), and the second process assumes that all links are updated in each period as in Myatt and Wallace (2003). Unlike the dynamic processes in Jackson and Watts (2002) and Mele (2017), where an active individual observes all the links formed in previous periods, we assume that an active individual only observes the distribution of the links formed previously. Following Myatt and Wallace (2004) and Myatt and Wallace (2003), we show that for a sufficiently large network, the first process generates a Markov chain of networks that has a unique limiting distribution with local modes coinciding with stable Bayesian Nash equilibria, and the second process converges to a Bayesian Nash equilibrium in probability. The result in our second process is in line with Jackson and Yariv (2007), who also showed that Bayesian Nash equilibria in a static game are equivalent to steady states of a dynamic process. These evolutionary results suggest that a Bayesian Nash equilibrium can be regarded as a long-term equilibrium in a dynamic
process of network formation.

**Symmetric Equilibrium.** In this paper we focus on symmetric equilibria where observationally identical individuals choose the same strategy. In a symmetric equilibrium, the CCP profile \( \sigma(X) \) satisfies that for any individuals \( i \) and \( j \) with \( X_i = X_j \), we have \( \sigma_i(g_i|X) = \sigma_j(g_j|X) \) for all \( g_i \in G_i \) and \( g_j \in G_j \), with \( g_j \) obtained from \( g_i \) by swapping its \( i \)th and \( j \)th components \( g_{ii} \) and \( g_{ij} \). Simply put, individuals of the same observed characteristics choose a linking decision with the same probability.\(^{14}\) This restriction is motivated by the observation that the utility function is the same for all individuals, so if they hold symmetric beliefs about the decisions of others (as specified in a symmetric CCP profile), then individuals of any given \( X_i \) and \( \epsilon_i \) face the same decision problem. The optimal decision in it is unique with probability one, leading to symmetry in the behaviors. The symmetry of an equilibrium guarantees that the conditional choice probabilities of an individual do not depend on how we label the individuals, a desirable feature in most networks where the identities of individuals do not play any role and individuals are labeled arbitrarily.

In Proposition 2.1, we establish the existence of a symmetric equilibrium. Our proof is similar to that in Leung (2015, Theorem 1). We assume that in observed data, individuals coordinate on a symmetric equilibrium.\(^{15}\) There may be multiple symmetric equilibria that satisfy condition (2.8).

**Proposition 2.1.** Suppose that Assumption 1 is satisfied. For any \( X \), there exists a symmetric equilibrium CCP profile \( \sigma(X) \).

*Proof.* See Appendix A.1. \[ \square \]

**Assumption 2.** The equilibrium \( \sigma \) observed in data is symmetric.

The main challenge in analyzing the model involves characterizing the optimal decision of an individual. Because the expected utility depends on the interaction \( G_{ij}G_{ik} \), an individual no longer chooses between separable links as in Leung (2015), but between portfolios of links. This is a multinomial discrete choice problem with \( 2^{n-1} \) overlapping alternatives. Note that links \( G_{ij} \) and \( G_{ik} \) are strategic complements (substitutes) if \( \gamma_1, \gamma_2 > 0 \) \((< 0)\). The nonseparable decision over links naturally leads to interdependence between the links. In the subsequent section, we develop a novel method to derive the optimal decision of an individual and characterize the link dependence.

\(^{14}\)This restriction does not rule out the possibility that two observationally equivalent individuals form different links in an observed network because they can have different unobserved utility shocks.\(^{15}\)The symmetry implies that the expected utility terms in (2.5) and (2.6) depend on \( i, j \) and \( k \) only through \( X_i, X_j, \) and \( X_k \).
In addition, this challenge can arise in other applications where individuals choose a portfolio of binary actions due to nonseparable utility such as complementarity (Berry et al., 2014). For example, in trade, a firm’s payoff from exporting to one country may depend on whether it exports to another country. In IO, a firm’s payoff from entering a market may depend on whether it enters another market (Zheng, 2016). Because of the spillover effects, the optimal decision of a firm consists of a portfolio of countries to export to or a portfolio of markets to enter—a challenging problem when the number of countries or markets grows large. Our method can be used to analyze the optimal decisions in these problems as well.

3 Optimal Link Choices

In this section, we develop an approach that yields an explicit expression for the optimal link choices of an individual. The idea is to find an auxiliary variable that captures the strategic interactions between an individual’s link choices, so that after the inclusion of this auxiliary variable the link choices become correlated binary choices, with the correlation captured by the auxiliary variable.

Recall that the incremental utility \( v_{i,jk}(G_{i-}, X) \) is symmetric in \( j \) and \( k \). Moreover, in a symmetric equilibrium \( \sigma \) the expected incremental utility \( \mathbb{E}[v_{i,jk}(G_{i-}, X)|X, \sigma] \) depends on \( j \) and \( k \) only through the values of \( X_j \) and \( X_k \). These symmetry properties imply that \( \mathbb{E}[v_{i,jk}(G_{i-}, X)|X, \sigma] \) is a symmetric function of \( X_j \) and \( X_k \).

To facilitate the exposition, we focus on the case where \( X_i \) is discrete. Assume that \( X_i \) takes a finite number of values, which we refer to as the individuals’ types.

**Assumption 3.** \( X_i \) takes \( T < \infty \) distinct values \( x_1, \ldots, x_T \).

Under Assumption 3, we can represent the expected utility in (2.4) in matrix form. For \( 1 \leq s, t \leq T \), let \( V_{i, st}(X, \sigma) \) denote the value of \( \mathbb{E}[v_{i,jk}(G_{i-}, X)|X, \sigma] \) if individuals \( j \) and \( k \) are of types \( x_s \) and \( x_t \) respectively; that is, \( V_{i, st}(X, \sigma) = \mathbb{E}[v_{i,jk}(G_{i-}, X)|X_j = x_s, X_k = x_t, X, \sigma] \). Arrange the \( T^2 \) type-specific expected incremental utilities \( V_{i, st}(X, \sigma) \) in a \( T \times T \) matrix \( V_i(X, \sigma) = (V_{i, st}(X, \sigma)) \in \mathbb{R}^{T \times T} \). Because \( V_{i, st}(X, \sigma) \) is symmetric in \( s \) and \( t \), \( V_i(X, \sigma) \) is a symmetric matrix. Using the matrix notation, we can represent the expected utility in (2.4) as

\[
\mathbb{E}[U_i(G_i, G_{i-}, X, \epsilon_i)|X, \epsilon_i, \sigma] = \frac{1}{n - 1} \sum_{j \neq i} G_{ij}(U_{ij}(X, \sigma) - \epsilon_{ij})
\]

\(^{16}\) The inclusion of \( \sigma \) in the notation indicates that the expectation is taken according to \( \sigma \).

\(^{17}\) We can potentially relax this assumption and allow for continuous \( X \), but this setup complicates the derivation of the optimal link choices because it requires replacing the matrix notation with linear operators. For simplicity, we focus on discrete \( X \) in the paper and leave continuous \( X \) to future research.
\[ + \frac{1}{2(n-1)(n-2)} \sum_{j \neq i} \sum_{k \neq i} G_{ij} G_{ik} Z_j' V_i(X, \sigma) Z_k, \quad (3.1) \]

where \( Z_j = (\{X_j = x_1\}, \ldots, \{X_j = x_T\})' \) is a \( T \times 1 \) vector of binary variables that indicates the type of individual \( j \), and 
\[
U_{ij}(X, \sigma) = \mathbb{E}[u_{ij}(G_{-i}, X)|X, \sigma] - \frac{1}{2(n-2)} Z_j' V_i(X, \sigma) Z_j.
\]
The term \( Z_j' V_i(X, \sigma) Z_k \) represents the expected incremental utility that individual \( i \) receives from linking to both \( j \) and \( k \).

To derive the optimal decision of individual \( i \), we "linearize" the quadratic term in (3.1) using the Legendre transform (Rockafellar, 1970). Consider a \( T \times T \) symmetric matrix \( V \in \mathbb{R}^{T \times T} \) and a \( T \times 1 \) vector \( y \in \mathbb{R}^T \). If \( V \) is positive semi-definite, \( \frac{1}{2} y' V y \) is a convex function of \( y \) and hence has the Legendre transform
\[
\frac{1}{2} y' V y = \max_{\omega \in \mathbb{R}^T} \{ y' V \omega - \frac{1}{2} \omega' V \omega \}, \quad (3.2)
\]
where \( \omega \in \mathbb{R}^T \) is an auxiliary variable. Under the assumption that the matrix \( V_i(X, \sigma) \) is positive semi-definite, by choosing \( y = \frac{1}{n-1} \sum_{j \neq i} G_{ij} Z_j \) and \( V = V_i(X, \sigma) \), we can replace the quadratic term in (3.1) by the maximization on the right-hand side of equation (3.2). This maximization has an objective function that is linear in \( y \) and thus linear in the link choices \( G_{ij} \). The transformation of the expected utility is presented in Lemma 3.1.

**Assumption 4.** Given \( X \), the smallest eigenvalue of the matrix \( V_i(X, \sigma) \) is nonnegative for all \( i \), symmetric equilibria \( \sigma \) and \( \theta_u \in \Theta_u \).

Assumption 4 is satisfied if link preferences present a large degree of homophily and nonnegative clustering. See Example 3.1 for an illustration. In Remark 3.2 we discuss how to relax it.

**Example 3.1.** Consider the expected utility in (2.6) with constant \( \gamma_1 \) and \( \gamma_2 = 0 \). The matrix \( V_i(X, \sigma) \) is of the form
\[
V_i(X, \sigma) = \gamma_1 \begin{bmatrix}
2p_{11}(X) & \cdots & p_{1T}(X) + p_{T1}(X) \\
\vdots & \ddots & \vdots \\
p_{1T}(X) + p_{T1}(X) & \cdots & 2p_{TT}(X)
\end{bmatrix}, \quad (3.3)
\]
where \( p_{st}(X) = \Pr(G_{ij} = 1|X_i = x_s, X_j = x_t, X) \) denotes the type-specific link probability, \( 1 \leq s, t \leq T \). A sufficient condition for \( V_i(X, \sigma) \) to be positive semi-definite is that \( \gamma_1 \geq 0 \) and the matrix (3.3) is diagonally dominant (Horn and Johnson, 1985); that is, \( 2p_{tt}(X) \geq \sum_{s=1, s \neq t}^T (p_{st}(X) + p_{ts}(X)) \) for \( 1 \leq t \leq T \). This condition means that the likelihood that an
individual connects to someone of her type is greater than the overall likelihood that she connects to someone of any other type.

**Lemma 3.1.** Suppose that Assumptions 1–4 are satisfied. The expected utility in (3.1) satisfies

\[ E[U_i(G_i, G_{-i}, X, \epsilon_i)|X, \epsilon_i, \sigma] = \frac{1}{n-1} \sum_{j \neq i} G_{ij}(U_{ij}(X, \sigma) - \epsilon_{ij}) + \frac{n-1}{n-2} \max_{\omega \in \mathbb{R}^T} \left\{ \left( \frac{1}{n-1} \sum_{j \neq i} G_{ij}Z_j^T \right) V_i(X, \sigma) \omega - \frac{1}{2} \omega' V_i(X, \sigma) \omega \right\}. \] (3.4)

The optimal decision of individual \( i \) is a link vector \( G_i \in G_i \) that maximizes her expected utility. By Lemma 3.1, the expected utility can be expressed as the optimal value from a maximization over the auxiliary variable \( \omega \in \mathbb{R}^T \). By interchanging the maximization over \( G_i \) and the maximization over \( \omega \), we can solve for the optimal \( G_i \) first from a simple maximization whose objective function is linear in \( G_i \). This optimal \( G_i \) is a function of \( \omega \). By solving for the optimal \( \omega \) next and evaluating the optimal \( G_i \) at the optimal \( \omega \), we can derive the optimal decision that maximizes the expected utility. The results on optimal link choices are established in Theorem 3.1.

**Theorem 3.1.** Suppose that Assumptions 1–4 are satisfied. For each \( i \), the optimal decision \( G_i(\epsilon_i, X, \sigma) \in G_i \) is given by

\[ G_{ij}(\epsilon_i, X, \sigma) = 1 \left\{ U_{ij}(X, \sigma) + \frac{n-1}{n} Z_j^T V_i(X, \sigma) \omega_i(\epsilon_i, X, \sigma) \geq \epsilon_{ij} \right\}, \quad \forall j \neq i. \] (3.5)

The \( T \times 1 \) vector \( \omega_i(\epsilon_i, X, \sigma) \in \mathbb{R}^T \) in (3.5) is an optimal solution to the maximization problem

\[ \max_{\omega \in \mathbb{R}^T} \frac{1}{n-1} \sum_{j \neq i} \left[ U_{ij}(X, \sigma) + \frac{n-1}{n} Z_j^T V_i(X, \sigma) \omega - \epsilon_{ij} \right]_+ - \frac{n-1}{2(n-2)} \omega' V_i(X, \sigma) \omega, \] (3.6)

where \( [\cdot]_+ = \max\{\cdot, 0\} \). Moreover, both \( G_i(\epsilon_i, X, \sigma) \) and \( V_i(X, \sigma) \omega_i(\epsilon_i, X, \sigma) \) are unique almost surely.\(^{18}\)

**Proof.** See Appendix A.2. \( \square \)

\(^{18}\)The optimal \( \omega_i(\epsilon_i, X, \sigma) \) is indeterminate if the matrix \( V_i(X, \sigma) \) is singular. This indeterminacy, however, is irrelevant for the optimal decision \( G_i \) because it is \( V_i(X, \sigma) \omega_i(\epsilon_i, X, \sigma) \), which is unique almost surely, that enters (3.5). For this reason we work on \( V_i(X, \sigma) \omega_i(\epsilon_i, X, \sigma) \) instead of \( \omega_i(\epsilon_i, X, \sigma) \) in the asymptotic analysis.
To gain some intuition about the auxiliary variable $\omega_i(\epsilon_i, X, \sigma)$ in (3.5), from the first-order condition of problem (3.6) (Lemma A.1) we derive

$$\frac{n - 1}{n - 2} Z_j' V_i(X, \sigma) \omega_i(\epsilon_i, X, \sigma) = \frac{1}{n - 2} \sum_{k \neq i} G_{ik}(\epsilon_i, X, \sigma) Z_j' V_i(X, \sigma) Z_k$$

almost surely. The left-hand side of (3.7) is the component added to the latent utility in (3.5). The right-hand side of (3.7) interprets this component as the expected incremental utility from friends in common. We interpret it as such because if individual $i$ contemplates a link to individual $j$, she anticipates that her friend $k$ can potentially become a mutual friend with $j$. If individual $j$ is of type $x_s$ and individual $i$’s friend $k$ is of type $x_t$, then $i$’s expected utility from this potential friend in common is $V_{i, st}(X, \sigma)$. Taking the average over all friends of individual $i$, we obtain the expected incremental utility from friends in common if individual $i$ is linked to individual $j$. By adding this component to the latent utility, we internalize the strategic interactions between the link choices due to the preference for friends in common, so that the optimal decision breaks down into a collection of binary choices.

The auxiliary variable $\omega_i(\epsilon_i, X, \sigma)$ provides an explicit expression for the dependence of the links formed by individual $i$. Note that $\omega_i(\epsilon_i, X, \sigma)$ is a function of $\epsilon_i$ because it is an optimal solution to problem (3.6) whose objective function depends on $\epsilon_i$. The randomness in $\omega_i(\epsilon_i, X, \sigma)$ leads to dependence between the link choices. From (3.5) we can see that two link choices $G_{ij}$ and $G_{ik}$ are dependent either through the presence of $\omega_i(\epsilon_i, X, \sigma)$ in both links, or through the dependence between the utility shock $\epsilon_{ij}$ and $\omega_i(\epsilon_i, X, \sigma)$ in $G_{ik}$ or symmetrically between the utility shock $\epsilon_{ik}$ and $\omega_i(\epsilon_i, X, \sigma)$ in $G_{ij}$. This explicit characterization of the link dependence is useful for studying the asymptotic properties of an estimator.

Remark 3.1. In the special case where there is no effect from friends in common ($\gamma_1, \gamma_2 = 0$), we have $V_i(X, \sigma) = 0$ and the utility function is separable in one’s own links. In this case, the optimal decision in (3.5) reduces to $G_{ij} = 1\{E[u_{ij}(G_{-i}, X)|X, \sigma] \geq \epsilon_{ij}\}$ for $j \neq i$. Because there are no strategic interactions between the link choices of an individual, each link choice is a separate binary choice (Leung, 2015).

Remark 3.2. Assumption 4 is not crucial to our approach, but simplifies the analysis. In Online Appendix O.B, we explore a general setting that allows the eigenvalues of $V_i(X, \sigma)$ to be negative. In this setting, the optimal decision of an individual takes a similar form, but the auxiliary variable is solved from a maximin problem (Proposition O.B.1).
4 Estimation

We now turn our attention to estimating the parameter $\theta$. We propose a two-step estimation procedure, where we estimate the conditional link probabilities in the first step and estimate the parameter $\theta$ in the second step (Leung, 2015). The asymptotic analysis of the estimator is complicated by the fact that link choices of an individual are correlated due to the preference for friends in common. We exploit the representation of optimal link choices in Theorem 3.1 to investigate the link dependence and derive the asymptotic properties of the estimator.

We start with the data generating process. We consider the scenario in which a single large network is observed. In the asymptotic analysis, we assume that the number of individuals in the network $n$ goes to infinity. Because the network depends on $n$, we denote it by $G_n = (G_{n,ij})$ hereafter. We assume that links in a network are generated as follows. First, we draw a vector of characteristics $X = (X'_1, \ldots X'_n)'$ from a joint discrete distribution, where $X_i$ represents the observed characteristics of individual $i$. Because $X$ is ancillary, we treat it as deterministic. Note that $X_i$ can be dependent across $i$. Next, we draw an $(n-1) \times 1$ vector of unobserved preferences $\epsilon_i \in \mathbb{R}^{n-1}$ for each $i$, independently across $i$. After that, each individual chooses to form links, and an equilibrium (a fixed point of (2.8)) emerges. There can be multiple equilibria, and nature selects one equilibrium $\sigma_n$ among the equilibria. The network $G_n$ observed in the data is obtained from the optimal links chosen under $\sigma_n$.

First step. The optimal link choices in (3.5) depend on the equilibrium $\sigma_n$ only through the conditional probabilities of forming each link, denoted by $p_{n,ij} = \mathbb{E}[G_{n,ij}|X]$, $1 \leq i \neq j \leq n$. Moreover, the symmetry of the equilibrium (Assumption 2) implies that each $p_{n,ij}$ depends on $i$ and $j$ only through their types $X_i$ and $X_j$. Under Assumption 3, it is thus sufficient to consider the type-specific conditional link probabilities $p_{n,(st)} = \mathbb{E}[G_{n,ij}|X_i = x_s, X_j = x_t, X]$ for $1 \leq s, t \leq T$. Denote $p_n = (p_{n,(st)}, 1 \leq s, t \leq T)'$. This is the parameter we need to estimate in the first step.

Specifically, for each $1 \leq s, t \leq T$, we estimate $p_{n,(st)}$ by the relative frequency of forming a link among the pairs of individuals that are of types $x_s$ and $x_t$

$$\hat{p}_{n,(st)} = \frac{\sum_i \sum_{j \neq i} G_{n,ij} 1\{X_i = x_s, X_j = x_t\}}{\sum_i \sum_{j \neq i} 1\{X_i = x_s, X_j = x_t\}}. \quad (4.1)$$

Let $\hat{p}_n = (\hat{p}_{n,(st)}, 1 \leq s, t \leq T)'$ denote the first-step estimator.

Second step. To estimate $\theta_0$, we define $P_{n,ij}(\theta, p) = \mathbb{E}[G_{n,ij}(\epsilon_i, X, \theta, p)|X]$ as the model-predicted probability that $i$ forms a link to $j$, where $G_{n,ij}(\epsilon_i, X, \theta, p)$ represents the optimal
link choice in (3.5) given $\theta$ and $p$. The equilibrium condition in (2.8) yields a set of conditional moment restrictions

$$E[G_{n,ij} - P_{n,ij}(\theta_0, p_n)|X] = 0, \quad (4.2)$$

where each value of $(X_i, X_j)$ gives one moment restriction. Based on (4.2), we can construct a GMM estimator for $\theta_0$. Let $q_{n,ij} = q_n(X_i, X_j)$ denote a $d_\theta \times 1$ vector of instruments that can depend on $X$ as well as $\theta_0$ and $p_n$, and $\hat{q}_{n,ij}$ denote an estimator of $q_{n,ij}$. Define

$$\hat{m}_n(\theta, \hat{p}_n) = \frac{1}{n(n-1)} \sum_i \sum_{j \neq i} \hat{q}_{n,ij}(G_{n,ij} - P_{n,ij}(\theta, \hat{p}_n)) \quad (4.3)$$

to be the sample moment, where $\hat{p}_n$ is the first-step estimator. The minimizer of the function $\hat{m}_n(\theta, \hat{p}_n)'\hat{m}_n(\theta, \hat{p}_n)$ gives a GMM estimator $\hat{\theta}_n$. Suppose that the estimator $\hat{\theta}_n$ satisfies $\hat{m}_n(\hat{\theta}_n, \hat{p}_n) = o_p(n^{-1})$.

Remark 4.1 (The advantage of single-market asymptotics). The model in general has multiple equilibria. Nevertheless, because we estimate $p_n$ using links from a single network, which are generated from the same equilibrium (the one observed in data), we do not need to impose restrictions on the equilibrium selection mechanism. This stands in contrast to the multiple-market asymptotics, where in order to achieve identification, we typically need to specify an equilibrium selection mechanism (if using nested fixed point algorithm (Rust, 1987)) or assume that the same equilibrium is selected in all the markets (if using two-step estimation (Bajari et al., 2007)).

Asymptotic analysis. We now investigate the asymptotic properties of the estimator $\hat{\theta}_n$. Given $X$, define the population counterpart of $\hat{m}_n(\theta, p)$ by

$$m_n(\theta, p) = \frac{1}{n(n-1)} \sum_i \sum_{j \neq i} q_{n,ij}(E[G_{n,ij}|X] - P_{n,ij}(\theta, p)). \quad (4.4)$$

Equation (4.2) implies that $m_n(\theta_0, p_n) = 0$. Because $p_n$ is uniquely determined in the first step, we assume that $m_n(\theta, p_n) = 0$ has a unique solution at $\theta_0$. Stacking the moments in the first and second steps then uniquely identifies $\theta_0$ and $p_n$. With abuse of notation, we

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19 We formulate the instrument in a way so that the weighting matrix is absorbed into the instrument. See Newey and McFadden (1994, Section 5.4) for justification of this general formulation.

20 The population moment $m_n(\theta, p)$ has the subscript $n$ because the probability of forming a link depends on the network size.

21 The first step can be characterized by the moment restrictions $E[G_{n,ij} - P_{n,ij}|X] = 0$; or equivalently, $E[G_{n,ij} - P_{n,ij,x} | X_s = x_s, X_t = x_t, X] = 0$ for all $1 \leq s, t \leq T$, where $p_n$ is a unique solution.

22 The identification can hold even in the presence of multiple equilibria, because any equilibrium other than $p_n$ will yield link choice probabilities that do not fit the observed link probabilities in data and thus will not satisfy the moment restrictions in the first step.
write \((\theta, p)\) for \((\theta', p')\).

With the addition of Assumption 5, we show that \((\hat{\theta}_n, \hat{p}_n)\) is consistent for \((\theta_0, p_n)\).

**Assumption 5.** (i) The parameter \(\theta\) lies in a compact set \(\Theta \subseteq \mathbb{R}^{d_\theta}\). (ii) For \(n\) sufficiently large, \(m_n(\theta, p_n) \neq 0\) for all \(\theta \neq \theta_0\). (iii) The instrument \(q_{n,ij}\) and its estimator \(\hat{q}_{n,ij}\) satisfy \(\max_{1 \leq i,j \leq n} \|q_{n,ij} - q_{n,ij}\| = o_p(1)\) and \(\max_{1 \leq i,j \leq n} \|q_{n,ij}\| \leq C_q < \infty\). (iv) \(\lim \inf_{n \to \infty} \frac{1}{n(n-1)} \sum_i \sum_{j \neq i} 1\{X_i = x_s, X_j = x_t\} > 0\) for all \(1 \leq s, t \leq T\).

Assumption 5(i) is a standard regularity condition. Assumption 5(ii) is the identification condition previously discussed. This assumption requires that \(\mathbb{E}[G_{n,ij} - P_{n,ij}(\theta, p_n)|X] = 0\) has a unique solution at \(\theta_0\). Equivalently, we can consider the model-predicted type-specific link choice probabilities \(P_{n,(st)}(\theta, p) = \mathbb{E}[G_{n,ij}(\epsilon_i, X, \theta, p)|X_i = x_s, X_j = x_t, X]\), \(1 \leq s, t \leq T\). The assumption requires that for any \(\theta \neq \theta_0\), there exist \(1 \leq s, t \leq T\) such that \(P_{n,(st)}(\theta, p_n) \neq P_{n,(st)}(\theta_0, p_n)\). When there is no effect from friends in common \((\gamma_1, \gamma_2 = 0)\), this assumption reduces to a standard rank condition that the regressors in (2.5) evaluated at \(p_n\) are linearly independent (Leung, 2015). Assumption 5(iii) is a standard assumption that the instrument is bounded and its estimator is consistent, both uniformly over \(i\) and \(j\). Assumption 5(iv) imposes a mild restriction on \(X\). It requires that the fractions of pairs of each type remain positive as \(n \to \infty\), so that the numbers of pairs of each type grow without bounds, and we can identify and estimate each \(p_{n,(st)}\). The assumption is satisfied if \(X_i\) is i.i.d. or has limited dependence across \(i\).

**Theorem 4.1 (Consistency).** Suppose that Assumptions 1–5 are satisfied. Given \(X\), we have \(\hat{\theta}_n - \theta_0 = o_p(1)\) and \(\hat{p}_n - p_n = o_p(1)\).

**Proof.** See Appendix A.3. \(\square\)

The consistency is established as a result of the fact that given \(X\) links formed by different individuals are independent, although links formed by the same individual are correlated. The conditional independence allows us to establish a uniform LLN for the stacked sample moment, which together with the identification condition yields consistency.

Analyzing the asymptotic distribution of \(\hat{\theta}_n\) is more complicated because links formed by an individual are correlated. Theorem 3.1 shows that link choices \(G_{n,ij}\) and \(G_{n,ik}\) are correlated because they both depend on the auxiliary variable \(\omega_{ni}(\epsilon_i)\), which is a maximizer of the function

\[
\Pi_{ni}(\omega, \epsilon_i) = \frac{1}{n - 1} \sum_{j \neq i} \left[ U_{n,ij} + \frac{n - 1}{n - 2} Z_j' V_{ni'} \omega - \varepsilon_{ij} \right] - \frac{n - 1}{2(n - 2)} \omega' V_{ni'} \omega. \tag{4.5}
\]
In the expression, we add subscript \( n \) to \( \omega_{ni}, \Pi_{ni}, U_{n,ij} \) and \( V_{ni} \) to indicate their dependence on \( n \), and all of the terms are evaluated at \((\theta_0, p_n)\), abbreviated for simplicity. To investigate how the link dependence will affect the asymptotic distribution of \( \hat{\theta}_n \), we represent \( \omega_{ni}(\epsilon_i) \) in an asymptotically linear form. Specifically, let \( \Pi_{ni}(\omega) \) denote the population counterpart of \( \Pi_{ni}(\omega, \epsilon_i) \) given \( X \)

\[
\Pi_{ni}^*(\omega) = \frac{1}{n-1} \sum_{j \neq i} \mathbb{E} \left[ U_{n,ij} + \frac{n-1}{n-2} Z_j V_{ni} \omega - \epsilon_{ij} \right] + X = \frac{n-1}{2(n-2)} \omega' V_{ni} \omega, \quad (4.6)
\]

and \( \omega_{ni}^* \) denote a maximizer of \( \Pi_{ni}(\omega) \). Under the regularity conditions in Assumption O.E.1, we show in Lemma O.E.5 that \( \omega_{ni}(\epsilon_i) \) has an asymptotically linear representation

\[
V_{ni}(\omega_{ni}(\epsilon_i) - \omega_{ni}^*) = \frac{1}{n-1} \sum_{j \neq i} \phi^{\omega}_{n,ij}(\omega_{ni}^*, \epsilon_{ij}) + o_p(n^{-1/2}), \quad (4.7)
\]

where \( \phi^{\omega}_{n,ij}(\omega_{ni}^*, \epsilon_{ij}) \) is an influence function defined in the lemma. Observe that \( \omega_{ni}^* \) is deterministic, so link choices evaluated at \( \omega_{ni}^* \) are independent. The representation indicates that the link dependence due to \( \omega_{ni}(\epsilon_i) \) vanishes at the rate of \( n^{-1/2} \), which is crucial in determining the asymptotic distribution of \( \hat{\theta}_n \).

With the addition of Assumptions 6 and O.E.1, we show that \( \hat{\theta}_n \) is asymptotically normal.

**Assumption 6.** (i) For \( n \) sufficiently large, \( P_{n,ij}(\theta, p) \) is continuously differentiable with respect to \( \theta \) and \( p \) in a neighborhood of \((\theta_0, p_n)\), \( 1 \leq i \neq j \leq n \). (ii) For \( n \) sufficiently large, the \( d_\theta \times d_\theta \) matrix \( J_n = \frac{1}{n(n-1)} \sum \sum_{j \neq i} q_{n,ij} \nabla_{\theta} P_{n,ij}(\theta_0, p_n) \) is nonsingular.

Assumption 6(i) imposes a smoothness restriction on \( P_{n,ij}(\theta, p) \). We show in Lemma O.E.2 that \( P_{n,ij}(\theta, p) \) is continuous in \( \theta \) and \( p \), by exploiting the fact that there is a one-to-one mapping between the optimal decisions of an individual and a partition of the \( \epsilon_i \) space \( \mathbb{R}^{n-1} \) (Online Appendix O.C), where the boundary of each set in the partition is continuous in \( \theta \) and \( p \). The proof suggests that \( P_{n,ij}(\theta, p) \) can have kinks if the binding inequalities that define the partition vary with \( \theta \) and \( p \). This assumption requires that there is a neighborhood of \((\theta_0, p_n)\) that has no kinks. In fact, we show in Proposition 5.1 that \( P_{n,ij}(\theta, p) \) converges to a limit as \( n \to \infty \), which is continuously differentiable in \( \theta \) and \( p \). Therefore, the assumption is less of a concern for larger \( n \). Assumption 6(ii) is a standard regularity condition for \( \hat{\theta}_n \) to have a well-behaved asymptotic distribution. It also ensures that \((\theta_0, p_n)\) is locally identified in a small neighborhood of \((\theta_0, p_n)\). Assumption O.E.1 imposes additional regularity conditions on the auxiliary variable \( \omega \) so that we can derive the asymptotically linear representation in (4.7) as well as other needed asymptotic properties of \( \omega_{ni}(\epsilon_i) \).
**Theorem 4.2 (Asymptotic Distribution).** Suppose that Assumptions 1–6 and O.E.1 are satisfied. Given $X$, we have $\sqrt{n(n-1)}\Sigma_n^{-1/2}J_n(\hat{\theta}_n - \theta_0) \xrightarrow{d} N(0, I_d)$, where $I_d$ is the $d \times d$ identity matrix, $\Sigma_n = \frac{1}{n(n-1)} \sum_i \sum_{j \neq i} E[\phi_{n,ij}^\theta \phi_{n,ij}^{\theta'} | X]$, and $\phi_{n,ij}^\theta$ is a $d \times 1$ vector defined by (A.8) in the proof.

**Proof.** See Appendix A.3. \hfill \Box

We derive the asymptotic distribution by decomposing the sample moment into two leading terms, which correspond to the two components in the influence function $\phi_{n,ij}^\theta$. The first component captures the sampling variation in link choices that does not take into account the link dependence due to $\omega_{ni}(\epsilon_i)$. The second component captures the contribution of the link dependence to the asymptotic distribution. Note that $\hat{\theta}_n$ converges to $\theta_0$ at the rate of $n$ – the square root of the sample size. The link dependence vanishes sufficiently fast so that it does not slow down the rate at which $\hat{\theta}_n$ converges, but does increase its asymptotic variance.

The asymptotic variance of $\hat{\theta}_n$ can be calculated as $\frac{1}{n(n-1)} \sum_i \sum_{j \neq i} J_n^{-1} \nabla_\theta \psi_{n,ij}(\theta, \hat{\theta}_n) \psi_{n,ij}(\theta, \hat{\theta}_n)$. We can estimate the asymptotic variance consistently by a plug-in estimator, where we replace $\theta_0$ and $p_n$ by their estimators $\hat{\theta}_n$ and $\hat{p}_n$. 

**Remark 4.2.** The link choice probability $P_{n,ij}(\theta, p)$ is an $n-1$ dimensional integral that has no closed form and must be computed by simulation. Specifically, we draw $\epsilon_i$ independently $R$ times, and for each simulated $\epsilon_{i,r}, r = 1, \ldots, R$, we compute $\omega_{ni}(\epsilon_{i,r})$ and $G_{n,ij}(\epsilon_{i,r}, \theta, \hat{p}_n)$ in (3.5). The sample average of the $R$ simulated $G_{n,ij}(\epsilon_{i,r}, \theta, \hat{p}_n)$ gives a simulated link choice probability. The simulation does not affect the consistency and asymptotic normality of the estimator, but increases the asymptotic variance by $1 + R^{-1}$ fold (Pakes and Pollard, 1989).

**Instrument.** In practice, we need to choose an instrument. We suggest using the instrument derived from quasi-maximum likelihood estimation (QMLE).\(^{23}\) Let $L_n(\theta, \hat{p}_n)$ denote the log of the quasi-likelihood function evaluated at the first-step estimator $\hat{p}_n$.\(^{24}\)

$$L_n(\theta, \hat{p}_n) = \sum_i \sum_{j \neq i} G_{n,ij} \ln P_{n,ij}(\theta, p) + (1 - G_{n,ij}) \ln(1 - P_{n,ij}(\theta, \hat{p}_n)). \quad (4.8)$$

Taking the derivative with respect to $\theta$, we obtain the quasi-likelihood equation

$$\frac{1}{n(n-1)} \sum_i \sum_{j \neq i} \frac{\nabla_\theta P_{n,ij}(\theta, \hat{p}_n)}{P_{n,ij}(\theta, \hat{p}_n)(1 - P_{n,ij}(\theta, \hat{p}_n))}(G_{n,ij} - P_{n,ij}(\theta, \hat{p}_n)) = 0.$$

\(^{23}\)This is also the optimal instrument given the conditional moment restrictions in (4.2) (Chamberlain, 1987).

\(^{24}\)The quasi-likelihood function does not take into account the joint distribution of link choices $G_{n,ij}$ and $G_{n,ik}$, which can be informative about $\theta$. 

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Comparing this equation with the sample moment in (4.3) suggests the instrument

\[ \hat{q}_{n,ij}(\theta) = \frac{\nabla_{\theta} P_{n,ij}(\theta, \hat{p}_n)}{P_{n,ij}(\theta, \hat{p}_n)(1 - P_{n,ij}(\theta, \hat{p}_n))}, \]

(4.9)

Note that the instrument depends on \( \theta \). We can either construct a preliminary estimator of \( \theta \) using an initial instrument\(^{25}\) or use continuous updating, as in Hansen et al. (1996).\(^{26}\)

5 Extensions

5.1 Undirected Networks

In some applications, the networks that a researcher observes are undirected. In this section, we explore a scenario in which the underlying network that individuals form is in fact directed, but because of data collection and reporting we only observe undirected connections. For example, we observe that there is a favor exchange between two village residents, but we may not know who offers the favor to whom. Because an individual’s utility depends on directed links, individuals choose their optimal directed links in the manner described in Section 2, and the same equilibrium condition applies. What differs from our previous analysis is that we do not observe the directed links, but rather the undirected links induced by them. This situation gives rise to a missing data problem, which poses a challenge for the identification and estimation of the parameters.\(^{27}\)

Let \( G_{n,ij}^u \) denote an undirected link between individuals \( i \) and \( j \). We assume that \( G_{n,ij}^u = r(G_{n,ij}, G_{n,ji}) \), where \( r \) is a known function that is symmetric in its first and second arguments. The function \( r \) represents the mechanism of data collection and reporting that generates undirected links. For example, if we observe an undirected link between \( i \) and \( j \) when both \( i \) links to \( j \) and \( j \) links to \( i \), then \( r \) takes the form \( r(G_{n,ij}, G_{n,ji}) = G_{n,ij}G_{n,ji} \).

\(^{25}\)For example, we can use powers and interactions of \( X_i \) and \( X_j \) to construct an initial instrument.

\(^{26}\)Using instrument (4.9) with continuous mapping is equivalent to QMLE based on (4.8). However, GMM provides more flexibility in choosing the instrument, especially when the link choice probability \( P_{n,ij}(\theta, p) \) is not fully differentiable in \( \theta \) and \( p \). See Section 5.2 for more details.

\(^{27}\)To clarify, we do not consider the scenario in which the underlying network that individuals form is undirected, such as friendship and coauthorship networks (Jackson, 2008). In this case, individuals propose the links they wish to form as in a link announcement game (Myerson, 1991), and a link is formed if both propose to form it. Jackson and Wolinsky (1996) proposed pairwise stability as the equilibrium solution to this network formation game under complete information, which is preferable to Nash equilibrium for accounting for the coordination between a pair of individuals. By the same logic, under incomplete information individuals could have incentives to coordinate on their beliefs about the link choices of others, thereby rendering Bayesian Nash equilibrium an inappropriate solution. We will need an alternative equilibrium solution that can account for the coordination on equilibrium beliefs to characterize the linking behaviors properly.
If the undirected link is observed when either $i$ links to $j$ or $j$ links to $i$, then $r$ can be written as $r(G_{n,ij}, G_{n,ji}) = G_{n,ij} + G_{n,ji} - G_{n,ij}G_{n,ji}$.\footnote{If we ignore the direction of a directed link and allow for the strength of an undirected link as measured by the number of directed links, then $r(G_{n,ij}, G_{n,ji}) = G_{n,ij} + G_{n,ji}$.} Under Assumption 1, these forms of $r$ satisfy $E[r(G_{n,ij}, G_{n,ji})|X] = r(E[G_{n,ij}|X], E[G_{n,ji}|X])$. Hence, we obtain a set of conditional moment restrictions

$$E[G_{n,ij}^u|X] = r(P_{n,ij}(\theta_0, p_n), P_{n,ji}(\theta_0, p_n)), \quad (5.1)$$

where $P_{n,ij}(\theta, p) = E[G_{n,ij}(\epsilon_i, X, \theta, p)|X]$ is the model-predicted probability that $i$ forms a directed link to $j$. Because the directed links are not observed, we treat $p_n = (p_{n,(st)}, 1 \leq s, t \leq T)'$ as additional $T^2$ parameters to be estimated in addition to $\theta_0$, yielding $d_\theta + T^2$ unknowns.

To investigate the identification of $(\theta_0, p_n)$, we denote $p_{n,ij} = \sum_{s=1}^T \sum_{t=1}^T p_{n,(st)}1\{X_i = x_s, X_j = x_t\}$. Mirroring the first step in Section 4, we can derive an additional set of conditional moment restrictions

$$E[G_{n,ij}^u|X] = r(p_{n,ij}, p_{n,ji}), \quad (5.2)$$

Observe that the equations in (5.1) and (5.2) are symmetric in $i$ and $j$ and depend on $i$ and $j$ only through their types $X_i$ and $X_j$. Stacking the equations in (5.1) and (5.2) yields a system of $T(T + 1)$ equations in $d_\theta + T^2$ unknowns.\footnote{For instance, if $T = 2$ we have 6 equations in $4 + d_\theta$ unknowns. For $T = 3$ we have 12 equations in $9 + d_\theta$ unknowns.} The order condition for identification is therefore satisfied, provided that the number of types exceeds the dimension of $\theta$ ($T \geq d_\theta$).

To further derive a sufficient condition for identification, let $\pi_{n,st} = E[G_{n,ij}^u|X_i = x_s, X_j = x_t, X]$ denote the conditional probability that an individual $i$ of type $x_s$ and an individual $j$ of type $x_t$ form an undirected link, and $\pi_n = (\pi_{n,st}, 1 \leq s \leq t \leq T)'$ denote the $T(T + 1)/2 \times 1$ vector of all type-specific conditional probabilities of forming an undirected link. Similarly, we can collect the right-hand sides of (5.1) and (5.2) into $T(T + 1)/2 \times 1$ vectors $r(P_n(\theta, p)) = (r(P_n,(st)(\theta, p), P_n,(ts)(\theta, p)), 1 \leq s \leq t \leq T)'$ and $r(p) = (r(p,(st), p,(ts)), 1 \leq s \leq t \leq T)'$. Rewrite (5.1) and (5.2) in terms of the type-specific moment restrictions

$$\pi_n = r(P_n(\theta_0, p_n))$$

$$\pi_n = r(p_n). \quad (5.3)$$

The identification of $(\theta_0, p_n)$ requires that system (5.3) has a unique solution at $(\theta_0, p_n)$. In Lemma 5.1, we provide an equivalent condition for the identification of $(\theta_0, p_n)$. For any $p \neq \tilde{p}$, we say that $p$ and $\tilde{p}$ are observationally equivalent if $r(p) = r(\tilde{p})$; that is, they yield
the same conditional probabilities over undirected links.

**Lemma 5.1.** The parameter $(\theta_0, p_n)$ is identified from system (5.3) if and only if (i) for any $\theta \neq \theta_0$, $r(P_n(\theta, p_n)) \neq r(P_n(\theta_0, p_n))$, and (ii) for any $p \neq p_n$, if $r(p) = r(p_n)$, then $r(p) \neq r(P_n(\theta, p))$ for all $\theta \in \Theta$.

**Proof.** See Appendix A.4.

Condition (i) indicates that given $p_n$, the conditional choice probabilities of observing an undirected link under $\theta \neq \theta_0$ must be different from those under $\theta_0$. This condition is the undirected counterpart of the identification condition for directed networks. Condition (ii) is an additional restriction that we need to achieve identification, because directed links are not observed and the second equation in (5.3) cannot uniquely pin down $p_n$. This condition requires that if $p \neq p_n$ are observationally equivalent, $p$ cannot be an equilibrium under any $\theta \in \Theta$. In other words, if $p \neq p_n$ is an equilibrium under some $\theta \in \Theta$, $p$ and $p_n$ cannot be observationally equivalent. In particular, if under $\theta_0$ the model predicts multiple equilibria and $p \neq p_n$ is another equilibrium, the two equilibria $p$ and $p_n$ cannot be observationally equivalent. In fact, because the model only predicts a finite number of equilibria under each $\theta$, and the function $r$ puts a particular restriction for two distinct $p$ and $p'$ to be observationally equivalent, it is unlikely for two equilibria to be observationally equivalent.

Once identification is achieved, we can estimate $\theta_0$ and $p_n$ jointly from (5.1) and (5.2) by one-step GMM. The asymptotic properties of the estimator can be analyzed similarly as in directed networks and are omitted.

### 5.2 Limiting Game

How would the network formation game evolve when the number of individuals $n$ grows large? We demonstrate in this section that under certain conditions, a link choice probability in the finite-$n$ game in fact converges to a certain limit as $n$ approaches infinity. In contrast to its finite-$n$ counterpart, the limiting link probability is continuously differentiable in the parameters and can be calculated analytically. It can provide a useful approximation to facilitate the estimation and computation of the parameters.

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30 Such a $p$ must exist because the second equation in (5.3) is rank deficient.

31 For any $p$ that lies in a neighborhood of $p_n$, condition (ii) is satisfied if the matrix $(\nabla_p r'(P_n(\theta_0, p_n)), \nabla_p r'(p_n))'$ has full column rank. Because $\partial r(p_{(s \bar{t})}, p_{(\bar{s} t)})/\partial p_{(s \bar{t})} = 0$ for all $s, \bar{t} \neq s, t$ and $r(p_{(s t)}, p_{(s t)})$ uniquely determines $p_{(s t)}$ for all $t$, a sufficient condition is that for each $s < t$, the two columns $(\partial r'(P_n(\theta_0, p_n))/\partial p_{(s \bar{t})}, \partial r'(p_n)/\partial p_{(s \bar{t})})'$ and $(\partial r'(P_n(\theta_0, p_n))/\partial p_{(s \bar{t})}, \partial r'(p_n)/\partial p_{(s \bar{t})})'$ are linearly independent.
Given a characteristic profile \( X \) and equilibrium \( p = (p_{st}), 1 \leq s, t \leq T \)', recall that the probability that individual \( i \) forms a link to individual \( j \) is given by

\[
P_{n,ij}(X, p) = \Pr \left( U_{n,ij}(X, p) + \frac{n-1}{n} Z_j^i V_{ni}(X, p) \omega_{ni}(\epsilon_i, X, p) \geq \epsilon_{ij} \bigg| X \right), \quad (5.4)
\]

where the auxiliary variable \( \omega_{ni}(\epsilon_i, X, p) \) maximizes the objective function \( \Pi_{ni}(\omega, \epsilon_i, X, p) \) in problem (3.6). Let \( n \to \infty \). Suppose that the expected utility terms \( U_{n,ij}(X, p) \) and \( V_{ni}(X, p) \) converge to some limits, denoted by \( U^*(X_i, X_j, p) \) and \( V^*(X_i, p) \). Moreover, let \( \omega^*(X_i, p) \in \mathbb{R}^T \) denote an optimal solution to the maximization problem

\[
\max_{\omega \in \Omega} \mathbb{E}[U^*(X_i, X_j, p) + Z_j^i V^*(X_i, p) \omega^*(X_i, p) - \epsilon_{ij}]_+ \bigg| X_i \bigg] - \frac{1}{2} \omega'V^*(X_i, p)\omega, \quad (5.5)
\]

Let \( \Pi^*(\omega, X_i, p) \) denote the objective function in (5.5), where we treat \( X_i \) as fixed and take expectation with respect to \( X_j \) and \( \epsilon_{ij} \). Assuming that \( X_i \) is i.i.d., we can regard problem (5.5) as the limiting counterpart of problem (3.6) and show that the finite-\( n \) optimizer \( \omega_{ni}(\epsilon_i, X, p) \) converges to the limiting optimizer \( \omega^*(X_i, p) \) as a result. From these results, we can derive that the finite-\( n \) link probability \( P_{n,ij}(X, p) \) converges to a limit defined by

\[
P^*(X_i, X_j, p) = \Pr (U^*(X_i, X_j, p) + Z_j^i V^*(X_i, p) \omega^*(X_i, p) \geq \epsilon_{ij} \bigg| X_i, X_j \bigg). \quad (5.6)
\]

We refer to \( P^*(X_i, X_j, p) \) as the limiting link probability.

To formally establish the convergence result, we impose the following assumptions.

**Assumption 7.** (i) The auxiliary variable \( \omega \) lies in a compact set \( \Omega \subseteq \mathbb{R}^T \). (ii) For any \( X_i \) and \( p \), any \( \omega^*(X_i, p) \) that maximizes \( \Pi^*(\omega, X_i, p) \) yields a unique \( V^*(X_i, p) \omega^*(X_i, p) \). (iii) \( X_i \) is i.i.d. across \( i \). (iv) There exist \( U^*(X_i, X_j, p) \in \mathbb{R} \) and \( T \times T \) matrix \( V^*(X_i, p) \in \mathbb{R}^{T \times T} \) such that for any \( X_i \) and \( p \), \( \max_{j \not= i} |U_{n,ij}(X, p) - U^*(X_i, X_j, p)| = o_p(1) \) and \( ||V_{ni}(X, p) - V^*(X_i, p)|| = o_p(1) \).

Assumption 7(i) is a standard regularity condition. Because \( \frac{\partial}{\partial c} \mathbb{E}[c - \epsilon]_+ = \frac{\partial}{\partial c} f_\epsilon(c - \epsilon) f_\epsilon(c) dc = F_\epsilon(c) \), problem (5.5) has the first-order condition \( V^*(X_i, p) \mathbb{E}[Z_j F_\epsilon(U^*(X_i, X_j, p) + Z_j^i V^*(X_i, p) \omega) \big| X_i] = V^*(X_i, p) \omega \). Any optimal solution to the first-order condition must be bounded. Therefore, without loss of generality we assume that \( \omega \) lies in a compact set \( \Omega \subseteq \mathbb{R}^T \).\(^{32}\) Assumption 7(ii) is an identification condition.\(^{33}\) It is to ensure that \( \omega_{ni}(\epsilon_i, X, p) \)

\(^{32}\)This assumption is also imposed in Assumption O.E.1(i) to derive the asymptotic properties of \( \omega_{ni}(\epsilon_i) \) under fixed \( X \).

\(^{33}\)This assumption resembles Assumption O.E.1(ii) for fixed \( X \). See the discussion there for justification of the assumption.
Our analysis so far has avoided an assumption on how \( X_i \) are generated. In Assumption 7(iii), we assume that \( X_i \) is i.i.d. so that the finite-\( n \) objective function \( \Pi_{ni}(\omega, \epsilon_i, X, p) \) converges to the limiting objective function \( \Pi^*(\omega, X_i, p) \), which is also necessary for the convergence of \( \omega_{ni}(\epsilon_i, X, p) \). Assumption 7(iv) assumes the convergence of the expected utilities. In Example 5.1, we verify that for the expected utility specified in (2.5)--(2.6), Assumption 7(iv) is satisfied under Assumption 7(iii).

**Example 5.1.** Consider the expected utility in (2.5)--(2.6). Fix \( X_i \) and \( X_j \). Let \( p = (p(x_s, x_t), 1 \leq s, t \leq T)' \) be an equilibrium. Define

\[
U^*(X_i, X_j, p) = \beta_1 + X'_i\beta_2 + d(X_i, X_j)'\beta_3 + p(X_j, X_i)\beta_4
+ \mathbb{E}[p(X_k, X_j)|X_j]\beta_5 + \mathbb{E}[p(X_j, X_k)|X_j]\beta_6,
\]

where the expectations in (5.7) are taken with respect to \( X_k \). Further, define the \( T \times T \) matrix \( V^*(p) = (V^*_st(p)) \), where the \( st \)th entry is given by

\[
V^*_st(p) = (p(x_s, x_t) + p(x_t, x_s))\gamma_2(x_s, x_t)
+ \mathbb{E}[p(x_s, X_i)p(X_t, x_t) + p(x_t, X_i)p(X_i, x_s)]\gamma_2(x_s, x_t).
\]

The expectation in (5.8) is taken with respect to \( X_i \). We show in Lemma A.3 that \( U_{n,ij}(X, p) \) and \( V_{ni}(X, p) \) converge to \( U^*(X_i, X_j, p) \) and \( V^*(p) \) respectively under Assumption 7(iii).

Intuitively, because the spillover effects in (2.5)--(2.6) take the form of a sample average, they converge by the law of large numbers.

Proposition 5.1 shows that the finite-\( n \) link probabilities converge in probability to the limiting link probabilities as \( n \to \infty \).

**Proposition 5.1.** Under Assumptions 1-4 and 7, for any \( X_i, X_j \) and \( p \), we have \( P_{n,ij}(X, p) \to P^*(X_i, X_j, p) = o_p(1) \).

**Proof.** See Appendix A.4. \( \square \)

**Link choices in the limiting game.** The limiting link probability \( P^*(X_i, X_j, p) \) in (5.6) represents the probability of forming a link when there are infinitely many players in the game, which we refer to as the limiting game. The formula suggests that an individual \( i \)

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34 We require \( V^*(X_i, p)\omega^*(X_i, p) \) instead of \( \omega^*(X_i, p) \) to be unique because \( V^*(X_i, p) \) may be singular, but it is \( V^*(X_i, p)\omega^*(X_i, p) \) that affects the link choices.

35 It is possible to relax Assumption 7(iii) to allow for non-i.i.d. \( X_i \), but for expositional simplicity we do not pursue it here.
forms a link to $j$ in the limiting game following the binary choice problem

$$G_{ij} = 1\{U^*(X_i, X_j, p) + Z_j'V^*(X_i, p)\omega^*(X_i, p) \geq \varepsilon_{ij}\}, \forall j \neq i. \quad (5.9)$$

In this binary choice, the latent utility from forming the link depends on the equilibrium $p$, indicating that the strategic interactions among link choices do not vanish in the limit. The spillover effects in the finite-$n$ game that take the form of a sample average (equations (2.5)–(2.6)) are now replaced by the population means (equations (5.7)–(5.8)). The presence of the limiting auxiliary variable $\omega^*(X_i, p)$ also indicates that even in the limit, the strategic interactions due to the preference for friends in common remain. By including $\omega^*(X_i, p)$ in the latent utility, we are able to internalize the limiting spillovers caused by this preference and represent the optimal decision of an individual as an infinite collection of myopic binary choices. Because $\omega^*(X_i, p)$ does not depend on $\epsilon_i$, given $X_i$ individual $i$’s link choices in the limiting game are independent.

Large market approximation has been widely used in the literature as a simplification for finite-$n$ markets which may otherwise be hard to analyze (mostly because of complicated equilibria). For example, Menzel (2015) discovered the limiting approximation of a one-to-one matching model under non-transferable utility. Azevedo and Leshno (2016) characterized an equilibrium in a many-to-one matching model under non-transferable utility using cutoffs and established the convergence of the equilibrium cutoffs as the market size grows large. We extend this research to network formation and develop the limiting approximation of a network formation game that presents a preference for friends in common.

**Advantage of the limiting approximation.** The two-step estimator proposed in Section 4 requires an instrument in the second stage. We suggested using the instrument derived from quasi-maximum likelihood (equation (4.9)); however, this instrument involves the derivative of a link choice probability. Because the limiting auxiliary variable $\omega^*(X_i, p)$ does not depend on $\epsilon_i$, the limiting link probability $P^*(X_i, X_j, p)$ is continuously differentiable in the parameters. Therefore, we can use the derivative of a limiting link probability to construct the instrument, addressing the concern that the finite-$n$ link probabilities may have kinks. Given that the finite-$n$ and limiting link probabilities are asymptotically close (Proposition 5.1), the instrument based on the limiting link probabilities should achieve similar asymptotic efficiency compared to that which is based on the finite-$n$ ones.\(^{36}\)

We can further simplify the moment condition by replacing the finite-$n$ link probabilities in the moment function with the limiting ones. This approximation leads to a computational

\(^{36}\)Because we only approximate the instrument, the consistency of the estimator is not affected.
improvement because the limiting link probabilities can be computed without simulation. The approximated moment function yields a misspecified model, although the misspecification vanishes asymptotically. To analyze the asymptotic properties of the estimator, we must investigate to what extent the limiting link probabilities evaluated at a finite-\(n\) equilibrium differ from that equilibrium. In the presence of multiple equilibria, additional assumptions that guarantee the sequence of equilibrium selection mechanisms to converge would be needed to achieve the consistency of the estimator.\(^{37}\)

**Simulation evidence.** Given the scope of this paper, we do not investigate the theoretical properties of the limiting approximation. However, we provide some simulation evidence on its performance. In Online Appendix O.D, we evaluate our approach in a simulation study, where we use the limiting link probabilities to approximate the instrument and/or the moment function. The estimates that use the limiting link probabilities for the instrument (Table O.D.1 Case (ii)) are similar to those that use the finite-\(n\) ones (Table O.D.1 Case (i)), although they are biased and have larger root MSEs in small networks (\(n \leq 25\)). The estimates that use the limiting link probabilities for both the moment function and instrument (Table O.D.1 Case (iii)) are the most biased and have the largest root MSEs in small networks, but once the networks become moderately large (\(n \geq 100\)), they perform similarly to the other estimates – they are unbiased with comparable root MSEs. These results suggest that the limiting link probabilities provide a useful approximation provided that the networks are sufficiently large.

6 Empirical Application

**Data and setup.** We apply our approach to investigate favor exchange networks in rural India. The data were collected from 75 rural villages in southern India as part of a study of a microfinance program (see Jackson et al. (2012) and Banerjee et al. (2013) for detailed descriptions of the data). The respondents in the survey were asked whether they provided monetary, in-kind (kerorice), advisory, or medical help to – or received such favors from – other individuals surveyed in the same village. Because providing and receiving a favor represent different decisions, we keep the directed relationships and construct a directed network of favor exchange in each village.

In particular, we say that individual \(i\) lends money or kerorice to individual \(j\) if either \(i\) reports lending money or kerorice to \(j\) or \(j\) reports borrowing money or kerorice from \(i\). Similarly, we say that individual \(i\) gives advice or medical help to individual \(j\) if either \(i\)

\(^{37}\)This situation is related to the convergence of equilibria explored in Menzel (2016).
reports providing such help to \( j \) or \( j \) reports receiving such help from \( i \). We say that individual \( i \) does a favor to individual \( j \) if \( i \) lends money or kerorice or gives advice or medical help to \( j \), and this relationship defines a directed link from \( i \) to \( j \) in the favor exchange network.

Our empirical study is motivated by Jackson et al. (2012), who found that the provision of a favor is supported by mutual relationships with other individuals. We aim to provide further evidence on self-support within a network using the structural model of network formation that we consider in the paper. Due to the intrinsic nature of a favor, we assume that whoever receives a favor accepts it, so that the presence of a favor is the unilateral decision of the provider. The finding in Jackson et al. (2012) suggests that the marginal utility of individual \( i \) from providing a favor to individual \( j \) may depend on the support from the mutual relationships that both \( i \) and \( j \) have with another individual \( k \). Note that the support measure in Jackson et al. (2012) is defined for undirected networks. We extend it to directed networks by considering links \( G_{jk} \) and \( G_{kj} \) separately. We also distinguish between inward support and outward support. Inward support includes the mutual relationships in which individual \( i \) receives a favor from another individual \( k \), while outward support includes the mutual relationships in which individual \( i \) provides a favor to \( k \). In a directed network of favor exchange, inward support and outward support may play different roles in facilitating the provision of a favor.

Specifically, we consider the utility function in (2.1), where the separable utility \( u_{ij} \) includes the inward support \( \frac{1}{n-2} \sum_{k \neq i,j} G_{ki}(G_{jk} + G_{kj}) \) and the nonseparable utility includes the outward support \( \frac{1}{n-2} \sum_{k \neq i,j} G_{ik}(G_{jk} + G_{kj}) \), with \( v_{i,jk} = (G_{jk} + G_{kj})^{\gamma_1} \). These measures are a linear and directed version of the support measure in Jackson et al. (2012). Following Leung (2015), we also include in \( u_{ij} \) other spillover effects such as reciprocation \( (G_{ji}) \), recipient’s in-degree \( \frac{1}{n-2} \sum_{k \neq i,j} G_{kj} \) and recipient’s out-degree \( \frac{1}{n-2} \sum_{k \neq i,j} G_{jk} \). While the spillover effects in \( u_{ij} \) can be estimated by Leung (2015), in order to estimate the effect of outward support, we need our approach to deal with the nonseparability of links.

In addition to the spillover effects, we include in \( u_{ij} \) the homophily effects, measured by the indicators for whether a pair of individuals has the same gender, age, education, and caste. We also allow \( u_{ij} \) to depend on the characteristics of the provider, including gender, age, education, and caste. We discretize age into three categories (under 29, 30–49 and over 50) and education into two categories (below and above the median\(^{39}\)). The castes are classified into three categories: scheduled (including scheduled castes and scheduled tribes),

\(^{38}\)These directed relationships are constructed using the variables Borrow-money, Lend-money, Borrow-kerorice, Lend-kerorice, Advice-come, Advice-go and Medical-help in the data. See Jackson et al. (2012) for detailed descriptions of these variables.

\(^{39}\)The median number of years of schooling in the sample is 5.
other backward class (OBC), and general. The discretization and categorization give a type space of 36 types ($T = 36$). The unobservable $\epsilon_{ij}$ is assumed to follow a logistic distribution.

**Estimation and inference.** We apply the two-step procedure to estimate the utility parameters. In the first step, we estimate the probability that individual $i$ does a favor to individual $j$ given the characteristics of $i$ and $j$. This link choice probability is village-specific because different villages can have different equilibria. In the data, pairs of certain types may not exist in a village, so instead of a village-by-village frequency estimator (as considered in Section 4), we use a smoothed nonparametric estimator. Given that the links are binary, we use a logit series estimator. Specifically, we pool the links across villages into one estimation sample and run a logit regression of favor provision on the characteristics of the provider and recipient – including their gender, age, education, and caste – and all the interaction terms up to order 2. To account for the village-specific heterogeneity, we also control for village fixed effects. The predicted link choice probabilities for each type in each village give the first-step estimates. In addition, to verify Assumption 4, we use the estimates to calculate the matrix in (3.3) without the parameter $\gamma_1$. For our specification, this matrix is positive semi-definite in all the 75 villages. Therefore, Assumption 4 is satisfied provided that $\gamma_1$ is nonnegative.

In the second step, we estimate the utility parameters in $u_{ij}$ and $v_{i,jk}$ by GMM. We use the moment in (4.3), with the instrument given by (4.9).\(^40\) The finite-$n$ link choice probability $P_{n,ij}$ in (4.3) must be calculated by simulation. To ease the computational burden, we approximate it using a version of the limiting approximation developed in Section 5.2, where we keep all the terms as in the finite-$n$ link choice probability, except that the auxiliary variable $\omega_{ni}(\epsilon_i)$ that maximizes (4.5) is replaced by its population counterpart $\omega_{ni}^*$, which maximizes (4.6). The population $\omega_{ni}^*$ does not involve unobservables and can be calculated without simulation.\(^41\) Like in the first step, we pool the links across villages and add village fixed effects to allow for village-specific heterogeneity in the constant.

In practice, to get an educated guess for the initial values of the parameters, we first estimate the parameters approximately by logit, where we use the right-hand side of (3.7) – calculated based on the observed links – to approximate the auxiliary term on the left-hand side.\(^42\) Given that the logit approximation runs quite fast, it is also useful to researchers

\(^40\)In practice, we implement GMM by weighted nonlinear least squares (NLS), where we use the optimal weight $1/(P_{n,ij}(1 - P_{n,ij}))$ for link $G_{ij}$. The first-order condition of the weighted NLS coincide with that of GMM, so the estimates should be equivalent. An advantage of weighted NLS is that we can use the built-in command in MATLAB *nlmefit* to calculate the estimates.

\(^41\)On an 8-core CPU a single evaluation of the approximated link probabilities for all the 1296 pair types in all the 75 villages takes two seconds.

\(^42\)The estimates from the logit approximation are actually similar to those from GMM. The results are
who want to experiment with specifications.\footnote{For our specifications with outward support – the specification that is most costly in computation – on an 8-core CPU, the logit approximation takes less than two minutes, while the GMM estimation, implemented by weighted NLS, takes about three hours.}

The standard errors of the estimators are calculated using the asymptotic variance in Theorem 4.2 with some modifications. First, the use of the population $\omega_{ni}^*$ implies that the links are conditional independent. This property simplifies the asymptotic distribution in Theorem 4.2, and only the first term in the influence function $\phi_{n,ij}^\theta$ remains. Moreover, because the first step is estimated by logit series, the contribution of the first step to the asymptotic variance is adjusted accordingly. In addition, because we pool the links across villages into one estimation sample, the sample size is given by the number of links in all the villages.\footnote{Omitting the links between individuals with missing covariates, we get a sample of 4,179,414 directed links.} The sum terms in Theorem 4.2 are also modified to include the summation over villages.

\textbf{Results.} Table 6.1 presents the second-step GMM estimates and their standard errors. For comparison, we consider four specifications of the spillover effects. Column 1 assumes no spillover effects. Column 2 allows for four separable spillover effects, including reciprocation, recipient’s in-degree and out-degree, and inward support.\footnote{To estimate the effect of inward support, we need a first-step estimate for $\mathbb{E}[G_{ki},G_{kj} | X]$, that is, the conditional probability that individual $k$ forms a link with both $i$ and $j$. Under the limiting approximation, the two links are conditional independent. Therefore, we estimate $\mathbb{E}[G_{ki},G_{kj} | X]$ approximately by the products of the estimated $\mathbb{E}[G_{ki} | X]$ and $\mathbb{E}[G_{kj} | X]$.} Column 3 allows for outward support only. Column 4 considers all five spillover effects. In all the four specifications, we control for the homophily measures, the provider’s characteristics, and the village fixed effects. We find that individuals with the same gender, age, education, and caste are more likely to exchange favors, with caste and gender similarity having the greatest impacts. Moreover, individuals are more likely to provide a favor if they are male, older, better educated, and in a higher caste. These findings are consistent with the evidence documented in the literature (Jackson et al., 2012). We also observe that the homophily effects tend to be smaller in the specifications with spillovers (Column 1 vs Columns 2-4). This finding suggests that we may over-estimate the homophily effects if we ignore the spillover effects and estimate a dyadic model.

Besides the covariate effects, Table 6.1 also provides evidence on the spillover effects. In Columns 2 and 4, we see a positive reciprocation effect: that is, an individual is more willing to do a favor to someone who also does her a favor. Moreover, the effects of recipient’s in-degree and out-degree are both negative, indicating that an individual having more re-
Table 6.1: Two-Step GMM Estimation of Favor Provision

|                                | (1)     | (2)     | (3)     | (4)     |
|--------------------------------|---------|---------|---------|---------|
| Homophily effects              |         |         |         |         |
| Same gender                    | 0.988   | 0.867   | 0.875   | 0.816   |
|                                | (0.008) | (0.013) | (0.007) | (0.020) |
| Same age                       | 0.231   | 0.212   | 0.218   | 0.212   |
|                                | (0.008) | (0.008) | (0.007) | (0.009) |
| Same education                 | 0.169   | 0.143   | 0.154   | 0.147   |
|                                | (0.008) | (0.007) | (0.007) | (0.007) |
| Same caste                     | 1.632   | 1.551   | 1.389   | 1.436   |
|                                | (0.010) | (0.017) | (0.009) | (0.065) |
| Provider’s characteristics     |         |         |         |         |
| Female                         | -0.321  | -0.334  | -0.217  | -0.253  |
|                                | (0.008) | (0.009) | (0.005) | (0.012) |
| Age 30–49                      | 0.261   | 0.258   | 0.199   | 0.176   |
|                                | (0.011) | (0.011) | (0.008) | (0.029) |
| Age 50+                        | 0.399   | 0.396   | 0.307   | 0.274   |
|                                | (0.013) | (0.014) | (0.009) | (0.021) |
| Education > Median             | 0.092   | 0.093   | 0.067   | 0.062   |
|                                | (0.009) | (0.009) | (0.006) | (0.012) |
| Scheduled                      | -0.098  | -0.108  | -0.034  | -0.028  |
|                                | (0.015) | (0.017) | (0.010) | (0.075) |
| OBC                            | -0.428  | -0.411  | -0.317  | -0.355  |
|                                | (0.015) | (0.017) | (0.011) | (0.122) |
| Spillover effects              |         |         |         |         |
| Reciprocation                  | 6.454   |         | 4.612   |         |
|                                | (0.485) |         | (1.148) |         |
| Recipient’s in-degree          | -19.961 | -20.872 |         |         |
|                                | (2.277) | (7.265) |         |         |
| Recipient’s out-degree         | -13.093 | -14.981 |         |         |
|                                | (1.715) | (7.894) |         |         |
| Inward support                 | -85.505 | -170.449|         |         |
|                                | (13.059)| (69.328)|         |         |
| Outward support                |         |         | 155.161 | 158.974 |
|                                |         |         | (1.992) | (3.550) |
| Village Fixed Effects          | Yes     | Yes     | Yes     | Yes     |
| Observations                   | 4,179,414 | 4,179,414 | 4,179,414 | 4,179,414 |

Note: Standard errors are in parentheses. The dependent variable is an indicator for whether resident $i$ provides a favor to resident $j$. The moment function and instrument in the second step are constructed using the limiting link probabilities.
tionships with others – by either helping or being helped by others – is less likely to receive a favor. A better-connected individual has more resources to rely on, so she is in less need of help from any particular friend.

The most striking result in Table 6.1 is that the effects of inward support and outward support are both significant but in opposite directions: inward support has a negative effect, while outward support has a positive effect (Columns 2-4). The estimates are similar no matter whether we control for either of the supports or both. Although having mutual relationships with a third individual indeed matters in the provision of a favor (as discovered in Jackson et al. (2012)), the direction of the mutual relationships from the provider’s perspective is crucial. If there is another individual k who exchanges favors with individuals i and j, i is less likely to do a favor to j if i receives a favor from k. On the contrary, i is more likely to do a favor to j if i also does a favor to k. Outward support, rather than inward support, better explains an individual’s incentive to provide a favor. The two measures provide contrasting policy implications for the facilitation of favor exchange networks.

Predicting support distribution. In addition to the estimation, we investigate the prediction performance of the model in terms of the support measure under different values of the spillover effects. To this end, we take the estimates in Column 4 of Table 6.1, and simulate eight directed networks in each village, where we set the effects of inward support, outward support, and/or other spillovers (reciprocation, recipient’s in-degree and out-degree) to zero. Then for each predicted network G, we calculate a linear version of the support measure as defined in Jackson et al. (2012)

$$\text{Supp}(G) = \frac{\sum_i \sum_{j \neq i} G_{ij} \left( \frac{1}{n-2} \sum_{k \neq i,j} (G_{ik} \lor G_{ki}) \land (G_{jk} \lor G_{kj}) \right)}{\sum_i \sum_{j \neq i} G_{ij}},$$

where $x \lor y = \max\{x, y\}$ and $x \land y = \min\{x, y\}$. For each link $G_{ij}$ in network G, the link-wise support is given by the fraction of individuals that have mutual relationships with individuals i and j, where the supporting relationships can be in any directions. The average over the link-wise supports for all the links in network G gives the support measure of the network $\text{Supp}(G)$.

In Figure 6.1, we plot the distributions of the support measure in the observed and predicted networks across the 75 villages. The top panel includes the predictions from the four specifications of inward and outward supports without the other spillovers, and the bottom panel includes those from the four specifications with the other spillovers. We can see

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46 We fix the first-step estimates when predicting a network under alternative parameter values. The predicted links do not reflect the potential change in the equilibrium.
that the predictions from the specifications without outward support in general understate the support distribution in the data. In particular, the inclusion of inward support does not improve the performance. The specification that best matches the support distribution in the data is the one with reciprocation, recipient’s degrees, and outward support. These findings further underscore the distinction between inward support and outward support and highlight the importance of outward support in matching the support distribution in the data.

In addition, compared to the specifications with reciprocation and recipient’s degrees, the specifications without these effects generate support distributions with heavy upper tails that do not appear in the data (top panel). It is important to include these spillovers to better match the support distribution in the data.
7 Conclusion

In this paper, we develop an econometric methodology for strategic network formation under incomplete information using data from a single large network. The utility function can be nonseparable in an individual’s link choices because of the spillover effects from friends in common. We develop a novel approach that applies the Legendre transform to the utility function so that the optimal decision of an individual can be represented equivalently as a sequence of correlated binary choices. We propose a two-step estimation procedure, where we estimate the link choice probabilities in the first step and estimate the model parameters in the second step. We show that the two-step estimator is consistent and asymptotically normal. The link dependence due to the preference for friends in common does not affect the rate of convergence, but does increase the asymptotic variance of the estimator. We also explore a scenario of undirected networks and derive a limiting approximation of the game that can help simplify the computation in very large networks.

There are a few more extensions of our approach that might be of interest. We may relax the i.i.d. assumption on the utility shocks by adding an individual-invariant heterogeneity, as in Graham (2017). Both the individual heterogeneity and the strategic interactions considered in this paper can generate link interdependence. It would be valuable to investigate the extent to which each of them accounts for the link interdependence in network data. A recent strand of literature explores social interactions in endogenous networks where the endogeneity of a network is characterized through a network formation model (Goldsmith-Pinkham and Imbens, 2013; Hsieh and Lee, 2016; Johnsson and Moon, 2021; Auerbach, 2022). These studies typically model network formation by a dyadic regression or a sequential process. Our paper provides an alternative model of network formation that is simple to analyze and allows for strategic interactions.47

A Appendix

Notation We use $\| \cdot \|$ to denote the Euclidean norm. For an $n \times 1$ vector $x \in \mathbb{R}^n$ and an $n \times n$ matrix $A \in \mathbb{R}^{n^2}$, we have $\|x\| = (\sum_{i=1}^{n} x_i^2)^{1/2}$ and $\|A\| = (\text{tr}(AA'))^{1/2} = (\sum_{i=1}^{n} \sum_{j=1}^{n} a_{ij}^2)^{1/2}$.

47Other related studies along this line include Badev (2021), who developed a joint model of network formation and individual outcomes, and Battaglini et al. (2021), who developed a model of network formation to recover unobserved social networks using only observable outcomes.
A.1 Proofs in Section 2

Proof of Proposition 2.1. We follow the proof in Leung (2015, Theorem 1). Organize the conditional choice probabilities in an \( n \times 2^{n-1} \) matrix \( \sigma(X) \). The \( i \)’s conditional choice probabilities \( \sigma_i(X) = \{\sigma_i(g_i|X), g_i \in \mathcal{G}_i\} \). The elements in the row sum to 1. Denote the set of all such matrices by \( \Sigma(X) \). With row \( i \) of \( \sigma(X) \) we associate \( X_i \). Let \( \Sigma^*(X) \subseteq \Sigma(X) \) denote the subset of matrices of conditional choice probabilities such that if \( X_i = X_j \) then \( \sigma_i(X) = \sigma_j(X) \), that is, \( \sigma_i(g_i|X) = \sigma_i(g_j|X) \) for \( g_i \in \mathcal{G}_i \) and \( g_j \in \mathcal{G}_j \) where \( g_j \) is obtained from \( g_i \) by swapping the \( i \)th and \( j \)th components of \( g_i \). If we organize the conditional choice probabilities in (2.7) in an \( n \times 2^{n-1} \) matrix \( P(X, \sigma) \), it maps the matrix \( \sigma \) to a matrix of conditional choice probabilities in \( \Sigma(X) \). An equilibrium is a fixed point of this mapping. Because we focus on the symmetric equilibria in \( \Sigma^*(X) \), we must show that \( P(X, \sigma) \) is a continuous mapping from \( \Sigma^*(X) \) to \( \Sigma^*(X) \) and that the set \( \Sigma^*(X) \) is convex and compact.

First, the mapping \( P(X, \sigma) \) maps \( \Sigma^*(X) \) to itself. Let \( \sigma(X) \in \Sigma^*(X) \). If \( X_i = X_j \), then individuals \( i \) and \( j \) have the same expected incremental utilities in (2.5) and (2.6). Because \( \epsilon_i \) and \( \epsilon_j \) follow the same distribution, rows \( i \) and \( j \) of \( P(X, \sigma(X)) \) are identical, so indeed \( P(X, \sigma(X)) \in \Sigma^*(X) \). Second, a convex combination of matrices \( \sigma(X), \tilde{\sigma}(X) \in \Sigma^*(X) \) is a matrix with rows that sum to 1 and that rows \( i \) and \( j \) are identical if \( X_i = X_j \). The convex combination is therefore in \( \Sigma^*(X) \). Third, \( \Sigma^*(X) \) is bounded. It is also closed. Let \( \{\sigma^k(X), k = 1, 2, \ldots\} \) be a sequence in \( \Sigma^*(X) \) that converges to a limit. Then for all \( k \) the rows of \( \sigma^k(X) \) sum to 1 and rows \( i \) and \( j \) are identical if \( X_i = X_j \). So the limit has the same properties and is therefore in \( \Sigma^*(X) \). Finally, the mapping \( P(X, \sigma) \) is continuous on \( \Sigma^*(X) \), which is proved in Lemma O.E.2. We conclude that by Brouwer’s fixed point theorem, \( P(X, \sigma) \) has a fixed point in \( \Sigma^*(X) \).

A.2 Proofs in Section 3

Proof of Theorem 3.1. By Lemma 3.1, for any \( G_i \in \mathcal{G}_i \) we can write the expected utility as

\[
\mathbb{E}[U_i(G_i, G_{-i}, X, \epsilon_i)|X, \epsilon_i, \sigma] = \frac{1}{n-1} \sum_{j \neq i} G_{ij}(U_{ij}(X, \sigma) - \epsilon_{ij}) + \frac{n - 1}{n - 2} \max_{\omega} \left\{ \left( \frac{1}{n - 1} \sum_{j \neq i} G_{ij} Z'_j \right) V_i(X, \sigma)\omega - \frac{1}{2} \omega' V_i(X, \sigma)\omega \right\} \\
= \max_{\omega} \frac{1}{n - 1} \sum_{j \neq i} G_{ij} \left( U_{ij}(X, \sigma) + \frac{n - 1}{n - 2} Z'_j V_i(X, \sigma)\omega - \epsilon_{ij} \right) - \frac{n - 1}{2(n - 2)} \omega' V_i(X, \sigma)\omega.
\]  

(A.1)
Let $\tilde{\Pi}_i(G_i, \omega, \epsilon_i, X, \sigma)$ denote the objective function of the last maximization problem in (A.1). We can derive the maximum expected utility from

$$\max G_i \mathbb{E}[U_i(G_i, G_{-i}, X, \epsilon_i)|X, \epsilon_i, \sigma] = \max G_i \max \tilde{\Pi}_i(G_i, \omega, \epsilon_i, X, \sigma)$$

$$= \max \tilde{\Pi}_i(G_i, \omega, \epsilon_i, X, \sigma)$$

$$= \max \Pi_i(\omega, \epsilon_i, X, \sigma), \quad (A.2)$$

where $\Pi_i(\omega, \epsilon_i, X, \sigma)$ denotes the objective function in problem (3.6). The second equality above follows because $\max_{\omega} \tilde{\Pi}_i(G_i, \omega, \cdot) \leq \max_{\omega} \max G_i \tilde{\Pi}_i(G_i, \omega, \cdot)$ for all $G_i$, so we have $\max_{G_i} \max_{\omega} \tilde{\Pi}_i(G_i, \omega, \cdot) \leq \max_{\omega} \max G_i \tilde{\Pi}_i(G_i, \omega, \cdot)$, and similarly for the other direction. The last equality follows because for any $\omega$, we have $\max_{G_i} \tilde{\Pi}_i(G_i, \omega, \cdot) = \Pi_i(\omega, \cdot)$. The equivalence result in (A.2) shows that the maximum expected utility can be derived by solving the last maximization problem in (A.2), or equivalently by solving problem (3.6). By the definitions of $\omega_i(\epsilon_i, X, \sigma)$ and $G_i(\epsilon_i, X, \sigma)$, we further derive

$$\max \omega \Pi_i(\omega, \epsilon_i, X, \sigma) = \tilde{\Pi}_i(G_i(\epsilon_i, X, \sigma), \omega_i(\epsilon_i, X, \sigma), \epsilon_i, X, \sigma)$$

$$\leq \max \omega \Pi_i(G_i(\epsilon_i, X, \sigma), \omega, \epsilon_i, X, \sigma)$$

$$= \mathbb{E}[G_i(\epsilon_i, X, \sigma), G_{-i}, X, \epsilon_i)|X, \epsilon_i, \sigma], \quad (A.3)$$

where the last equality comes from (A.1). Combining (A.2) and (A.3), we see that the inequality in (A.3) becomes an equality. Therefore, $G_i(\epsilon_i, X, \sigma)$ is an optimal solution.

As for the uniqueness, $G_i(\epsilon_i, X, \sigma)$ is unique almost surely because $\epsilon_i$ has a continuous distribution by Assumption 1(i), so two linking decisions achieve the same expected utility with probability zero. To show the uniqueness of $V_i(X, \sigma)\omega_i(\epsilon_i, X, \sigma)$, by Lemma A.1, $\omega_i(\epsilon_i, X, \sigma)$ satisfies the first-order condition $V_i(X, \sigma)\omega_i(\epsilon_i, X, \sigma) = \frac{1}{n-1} \sum_{j \neq i} G_{ij}(\epsilon_i, X, \sigma)V_i(X, \sigma)Z_j$ almost surely. Since $G_i(\epsilon_i, X, \sigma)$ is unique almost surely, so is $V_i(X, \sigma)\omega_i(\epsilon_i, X, \sigma)$.

**Lemma A.1.** Suppose that Assumptions 1–4 are satisfied. An optimal $\omega_i(\epsilon_i, X, \sigma)$ that solves problem (3.6) satisfies the first-order condition

$$V_i(X, \sigma)\omega = \frac{1}{n-1} \sum_{j \neq i} 1 \left\{ U_{ij}(X, \sigma) + \frac{n-1}{n-2} Z_j V_i(X, \sigma) \geq \epsilon_{ij} \right\} V_i(X, \sigma)Z_j, \quad a.s.$$  

**Proof.** Observe that the objective function $\Pi_i(\omega, \epsilon_i, X, \sigma)$ in problem (3.6) is sub-differentiable in $\omega$.

By the optimality of $\omega_i(\epsilon_i, X, \sigma)$, $\Pi_i(\omega, \epsilon_i, X, \sigma)$ has subgradient 0 at $\omega_i(\epsilon_i, X, \sigma)$, that

\[ \text{Note that the function max}\{x, 0\} \text{ is differentiable at } x \neq 0 \text{ and sub-differentiable at } x = 0 \text{ with sub-derivatives in } [0, 1]. \]
is, \( \omega_i(\epsilon_i, X, \sigma) \) satisfies the first-order condition

\[
\frac{1}{n-1} \sum_{j \neq i} 1 \left\{ U_{ij}(X, \sigma) + \frac{n-1}{n-2} Z_j^t V_i(X, \sigma) \omega > \epsilon_{ij} \right\} V_i(X, \sigma) Z_j - V_i(X, \sigma) \omega \\
= - \frac{1}{n-1} \sum_{j \neq i} 1 \left\{ U_{ij}(X, \sigma) + \frac{n-1}{n-2} Z_j^t V_i(X, \sigma) \omega = \epsilon_{ij} \right\} \text{diag}(\tau) V_i(X, \sigma) Z_j,
\]

(A.4)

for some \( \tau = (\tau_1, \ldots, \tau_T) \in [0, 1]^T \). Define the right-hand side of (A.4) as \( \Delta_n(\omega, \epsilon_i, X, \sigma) \). For any \( \omega \),

\[
\Pr(\|\Delta_n(\omega, \epsilon_i, X, \sigma)\| > 0 | X) \\
\leq \Pr \left( \exists j \neq i, U_{ij}(X, \sigma) + \frac{n-1}{n-2} Z_j^t V_i(X, \sigma) \omega = \epsilon_{ij} \left| X \right. \right) \\
\leq \sum_{j \neq i} \Pr \left( U_{ij}(X, \sigma) + \frac{n-1}{n-2} Z_j^t V_i(X, \sigma) \omega = \epsilon_{ij} \bigg| X \right) = 0,
\]

(A.5)

where the last equality follows because \( \epsilon_{ij} \) has a continuous distribution. Hence the first-order condition (A.4) holds with \( \Delta_n(\omega, \epsilon_i, X, \sigma) \) replaced by 0 with probability one. By (A.5) again, we can replace \( > \) in the indicator on the left-hand side of (A.4) by \( \geq \) with probability one and the lemma is proved. \( \square \)

### A.3 Proofs in Section 4

**Proof of Theorem 4.1.** For \( p = (p_{st}), 1 \leq s, t \leq T \) \( \in \mathcal{P} = [0, 1]^T \), define the \( T^2 \times 1 \) vector function \( \hat{h}_n(p) = (\hat{h}_{n,st}(p), 1 \leq s, t \leq T)' \), where

\[
\hat{h}_{n,st}(p) = \frac{1}{n(n-1)} \sum_i \sum_{j \neq i} 1 \{ X_i = x_s, X_j = x_t \} (G_{n,ij} - p_{st}).
\]

Provided that \( \frac{1}{n(n-1)} \sum_i \sum_{j \neq i} 1 \{ X_i = x_s, X_j = x_t \} > 0 \), which holds for \( n \) sufficiently large by Assumption 5(iv), the first-step estimator \( \hat{p}_n \) satisfies that \( \hat{h}_n(\hat{p}_n) = 0 \). Define the population counterpart of \( \hat{h}_n(p) \) by \( h_n(p) = (h_{n,st}(p), 1 \leq s, t \leq T)' \), where

\[
h_{n,st}(p) = \frac{1}{n(n-1)} \sum_i \sum_{j \neq i} 1 \{ X_i = x_s, X_j = x_t \} (\mathbb{E}[G_{n,ij}|X] - p_{st}).
\]

We can write the moment restrictions in the first step as \( h_n(p_n) = 0 \). Provided that \( \frac{1}{n(n-1)} \sum_i \sum_{j \neq i} 1 \{ X_i = x_s, X_j = x_t \} > 0 \), \( p_n \) is a unique solution. Stack the moments in the first and second steps and define \( \hat{m}_n(\theta, p) = [m_n(\theta, p)', h_n(p)']' \) and \( \hat{m}_n(\theta, p) = [\hat{m}_n(\theta, p)', \hat{h}_n(p)']' \). The identification condition in Assumption 5(ii) implies that for \( n \) suffi-
ciently large, \( \tilde{m}_n(\theta, p) \) has a unique solution at \((\theta_0, p_0)\). Moreover, the optimality of \( \hat{\theta}_n \) and \( \hat{h}_n(\hat{\theta}_n) = 0 \) imply that \( \tilde{m}_n(\hat{\theta}_n, \hat{p}_n) = o_p(1) \).

We prove consistency following Newey and McFadden (1994). Fix \( \delta > 0 \). Let \( B_0(\delta) = \{ (\theta, p) \in \Theta \times \mathcal{P} : \| (\theta, p) - (\theta_0, p_0) \| < \delta \} \) be an open \( \delta \)-ball centered at \((\theta_0, p_0)\). We have

\[
\Pr(\| (\hat{\theta}_n, \hat{p}_n) - (\theta_0, p_0) \| < \delta | X) \geq \Pr \left( \| \tilde{m}_n(\hat{\theta}_n, \hat{p}_n) \| < \inf_{(\theta, p) \in (\Theta \times \mathcal{P}) \setminus B_0(\delta)} \| \tilde{m}_n(\theta, p) \| \right) \). \tag{A.6}
\]

By the triangle inequality and \( \tilde{m}_n(\hat{\theta}_n, \hat{p}_n) = o_p(1) \), we obtain

\[
\| \tilde{m}_n(\hat{\theta}_n, \hat{p}_n) \| \leq \| \tilde{m}_n(\theta, p) - m_n(\theta, p) \| + \| m_n(\theta, p) \|
\]

\[
\leq \sup_{(\theta, p) \in \Theta \times \mathcal{P}} \| \tilde{m}_n(\theta, p) - m_n(\theta, p) \| + o_p(1).
\]

The uniform LLN in Lemma O.E.1 shows that \( \sup_{(\theta, p) \in \Theta \times \mathcal{P}} \| \tilde{m}_n(\theta, p) - m_n(\theta, p) \| = o_p(1) \). Moreover, observe that for each \( 1 \leq s, t \leq T \)

\[
\hat{h}_{n,st}(p) - h_{n,st}(p) = \frac{1}{n(n-1)} \sum_i \sum_{j \neq i} 1\{ X_i = x_s, X_j = x_t \} (G_{n,ij} - E[G_{n,ij}|X]),
\]

which does not depend on \( p \). Because given \( X, G_{n,i} = (G_{n,ij}, j \neq i) \) are independent across \( i \), \( E[(\hat{h}_{n,st}(p) - h_{n,st}(p))^2 | X] \) is given by \( (n(n-1))^{-2} \) times

\[
\sum_i \sum_{j \neq i} 1\{ X_i = x_s, X_j = x_t \} E[(G_{n,ij} - E[G_{n,ij}|X])^2 | X]
\]

\[
+ \sum_i \sum_{j \neq i} \sum_{k \neq i,j} 1\{ X_i = x_s, X_j = x_t, X_k = x_t \} E[(G_{n,ij} - E[G_{n,ij}|X])(G_{n,ik} - E[G_{n,ik}|X]) | X].
\]

Because both \( E[(G_{n,ij} - E[G_{n,ij}|X])^2 | X] \) and \( E[(G_{n,ij} - E[G_{n,ij}|X])(G_{n,ik} - E[G_{n,ik}|X]) | X] \) are bounded by 1, we can bound \( E[(\hat{h}_{n,st}(p) - h_{n,st}(p))^2 | X] \) by \( (n(n-1))^{-2} (n(n-1) + n(n-1)(n-2)) = o(1) \). Therefore, \( E[\| \hat{h}_n(p) - h_n(p) \|^2 | X] = \sum_{s=1}^T \sum_{t=1}^T E[(\hat{h}_{n,st}(p) - h_{n,st}(p))^2 | X] = o(1) \).

By Markov’s inequality, we obtain \( \sup_{p \in \mathcal{P}} \| \hat{m}_n(p) - m_n(p) \| = o_p(1) \). Combining the results yields \( \sup_{p \in \mathcal{P}} \| \hat{m}_n(p) - m_n(p) \| = o_p(1) \) and thus \( \tilde{m}_n(\hat{\theta}_n, \hat{p}_n) = o_p(1) \).

For \( n \) sufficiently large, by the uniqueness of \((\theta_0, p_0)\) we have \( \| \tilde{m}_n(\theta, p) \| > 0 \) for all \( (\theta, p) \neq (\theta_0, p_0) \). The compactness of \((\Theta \times \mathcal{P}) \setminus B_0(\delta) \) (Assumption 5(i)) and the continuity of \( \tilde{m}_n(\theta, p) \) in \( \theta \) and \( p \) (Lemma O.E.2) imply that \( \inf_{(\theta, p) \in (\Theta \times \mathcal{P}) \setminus B_0(\delta)} \| \tilde{m}_n(\theta, p) \| > 0 \) for \( n \) sufficiently large. Combining the results we can see that the right-hand side of equation (A.6) goes to 1 and the consistency is proved.

\[\square\]

Proof of Theorem 4.2. Because \( \max_{1 \leq i,j \leq n} \| q_{n,ij} - q_{n,ij} \| = o_p(1) \) (Assumption 5(iii)), the
sampling variation in $\hat{q}_{n,ij}$ has no effect on the asymptotic distribution of $\hat{\theta}_n$. Hence, $\hat{\theta}_n$ satisfies the moment condition with $\hat{q}_{n,ij}$ replaced by $q_{n,ij}$, that is, $\frac{1}{n(n-1)} \sum_i \sum_{j \neq i} q_{n,ij}(G_{n,ij} - P_{n,ij}(\hat{\theta}_n, \hat{p}_n)) = o_p(n^{-1})$.

By the consistency of $\hat{\theta}_n$ and $\hat{p}_n$, for $n$ sufficiently large ($\hat{\theta}_n, \hat{p}_n$) lies in a neighborhood of $(\theta_0, p_0)$ where $P_{n,ij}(\theta, p)$ is continuously differentiable (Assumption 6(i)). From the Taylor expansion $P_{n,ij}(\hat{\theta}_n, \hat{p}_n) = P_{n,ij}(\theta_0, p_0) + \nabla \theta^{\top} P_{n,ij}(\theta_0, p_0)(\hat{\theta}_n - \theta_0) + \nabla p_i^{\top} P_{n,ij}(\theta_0, p_0)(\hat{p}_n - p_0) + o_p(||(\hat{\theta}_n, \hat{p}_n) - (\theta_0, p_0)||)$, and upon rearranging the terms we derive

$$\frac{1}{n(n-1)} \sum_i \sum_{j \neq i} q_{n,ij} \nabla \theta^{\top} P_{n,ij}(\theta_0, p_0)(\hat{\theta}_n - \theta_0)$$

$$= \frac{1}{n(n-1)} \sum_i \sum_{j \neq i} q_{n,ij} (G_{n,ij} - P_{n,ij}(\theta_0, p_0) - \nabla \theta^{\top} P_{n,ij}(\theta_0, p_0)(\hat{\theta}_n - \theta_0))$$

$$- \frac{1}{n(n-1)} \sum_i \sum_{j \neq i} q_{n,ij} o_p(||(\hat{\theta}_n, \hat{p}_n) - (\theta_0, p_0)||) + o_p(n^{-1}). \quad (A.7)$$

The first-step estimator $\hat{p}_n = (\hat{p}_{n,(st)}, \hat{p}_{n,(tt)})$ satisfies

$$\hat{p}_n - p_n = \frac{1}{n(n-1)} \sum_i \sum_{j \neq i} w_{n,ij}(G_{n,ij} - P_{n,ij}(\theta_0, p_0)),$$

where $w_{n,ij} = (w_{n,ij,(11)}, \ldots, w_{n,ij,(tt)})' \in \mathbb{R}^{T^2}$ is a $T^2 \times 1$ vector and $w_{n,ij,(st)} = 1\{X_i = x_s, X_j = x_t\}/\frac{1}{n(n-1)} \sum_i \sum_{j \neq i} 1\{X_i = x_s, X_j = x_t\}$ for $1 \leq s, t \leq T$. Hence, we can write the first term on the right-hand side of (A.7) in a sample average form

$$\frac{1}{n(n-1)} \sum_i \sum_{j \neq i} \tilde{q}_{n,ij} (G_{n,ij} - P_{n,ij}(\theta_0, p_0)),$$

where $\tilde{q}_{n,ij} = q_{n,ij} - (\frac{1}{n(n-1)} \sum_i \sum_{j \neq k} q_{n,kl} \nabla p_i P_{n,kl}(\theta_0, p_0))w_{n,ij}$ is the augmented instrument that incorporates the weight $w_{n,ij}$ in the first step. Because $\max_{1 \leq i, j \leq n} ||q_{n,ij}|| < \infty$ (Assumption 5(iii)) and for $n$ sufficiently large, $\max_{1 \leq i, j \leq n} ||w_{n,ij}|| < \infty$ (Assumption 5(iv)) and $\max_{1 \leq i, j \leq n} ||\nabla p_i P_{n,ij}(\theta_0, p_0)|| < \infty$ (Assumption 6(i)), we have $\max_{1 \leq i, j \leq n} ||\tilde{q}_{n,ij}|| < \infty$.

Applying Lemma O.E.8 for $\tilde{q}_{n,ij}$, we obtain

$$\frac{1}{n(n-1)} \sum_i \sum_{j \neq i} \tilde{q}_{n,ij} (G_{n,ij} - P_{n,ij}(\theta_0, p_0)) = O_p(n^{-1}).$$

Observe that $\hat{p}_n - p_n$ has a similar form with $w_{n,ij}$ in place of $\tilde{q}_{n,ij}$. Similarly as in Lemma O.E.8, we can prove that $\hat{p}_n - p_n = O_p(n^{-1})$. Because $||\hat{p}_n - p_n|| \leq ||\hat{\theta}_n - \theta_0|| + ||\hat{p}_n - p_n||$ and $\max_{1 \leq i, j \leq n} ||q_{n,ij}|| < \infty$, we can bound the second term on the right-hand side of (A.7) by $o_p(||\hat{\theta}_n - \theta_0|| + ||\hat{p}_n - p_n||) = o_p(||\hat{\theta}_n - \theta_0||) + o_p(||\hat{p}_n - p_n||) = o_p(||\hat{\theta}_n - \theta_0||) + o_p(n^{-1})$. Moreover, the left-hand side of (A.7) is given by $J_n(\hat{\theta}_n - \theta_0)$. By Assumption 6(ii), we have the bound $||J_n(\theta - \theta_0)||^2 \geq c^2||\theta - \theta_0||^2$, where $c^2 = \lambda_{\min}(J_n^\top J_n) > 0$. (A.7)
for \(n\) sufficiently large. Combining the results we derive from equation (A.7) that \(\|\hat{\theta}_n - \theta_0\| (c + o_p(1)) \leq O_p(n^{-1}) + o_p(n^{-1})\). This implies that \(\hat{\theta}_n - \theta_0 = O_p(n^{-1})\), that is, \(\hat{\theta}_n\) is \(n\)-consistent for \(\theta_0\).

To derive the asymptotic distribution of \(\hat{\theta}_n\), write equation (A.7) as

\[
\sqrt{n(n-1)} J_n (\hat{\theta}_n - \theta_0) = \frac{1}{\sqrt{n(n-1)}} \sum_i \sum_{j \neq i} \tilde{q}_{n,ij} (G_{n,ij} - P_{n,ij}(\theta_0,p_n)) + o_p(1),
\]

and apply Lemma O.E.8 to the leading term on the right-hand side. Define the \(d_\theta \times 1\) vector

\[
\phi^\theta_{n,ij} = \tilde{q}_{n,ij} (g_{n,ij}(\omega_{ni}^\ast, \epsilon_{ij}) - P_{n,ij}^\ast(\omega_{ni}^\ast)) + J_{n}^\omega(\omega_{ni}^\ast, \tilde{q}_{n,ij}) \phi_{n,ij}^\ast(\omega_{ni}^\ast, \epsilon_{ij}), \quad (A.8)
\]

where \(g_{n,ij}(\omega_{ni}^\ast, \epsilon_{ij}) = 1\{U_{n,ij} + \frac{n-1}{n-2} Z_j' V_{ni} \omega_{ni}^\ast \geq \epsilon_{ij}\}\), \(P_{n,ij}^\ast(\omega_{ni}^\ast) = F_c(U_{n,ij} + \frac{n-1}{n-2} Z_j' V_{ni} \omega_{ni}^\ast)\), \(J_{n}^\omega(\omega_{ni}^\ast, \tilde{q}_{n,ij}) = \frac{1}{n-1} \sum_{j \neq i} \tilde{q}_{n,ij} \nabla_{\omega^\ast} P_{n,ij}^\ast(\omega_{ni}^\ast)\), and \(\phi_{n,ij}^\ast(\omega_{ni}^\ast, \epsilon_{ij}) \in \mathbb{R}^T\) is the influence function defined in Lemma O.E.5. Define the \(d_\theta \times d_\theta\) variance matrix \(\Sigma_n = \frac{1}{n(n-1)} \sum_i \sum_{j \neq i} \mathbb{E}[\phi^\theta_{n,ij} \phi^\theta_{n,ij}] \). By Lemma O.E.8, we obtain \(\sqrt{n(n-1)} \Sigma_n^{-1/2} J_n (\hat{\theta}_n - \theta_0) \xrightarrow{d} N(0, I_{d_\theta})\). \(\square\)

### A.4 Proofs in Section 5

**Proof of Lemma 5.1.** Suppose that \((\theta_0, p_n)\) is identified. For any \(\theta \neq \theta_0\), because \((\theta, p_n)\) satisfies the second equation in (5.3), it should not satisfy the first equation in (5.3), that is, \(r(P_n(\theta, p_n)) = r(P_n(\theta_0, p_n))\), and thus condition (i) is satisfied. For any \(p \neq p_n\) such that \(r(p) = r(p_n)\), the second equation in (5.3) is satisfied. Therefore, the first equation in (5.3) should not be satisfied for any \((\theta, p)\) with such \(p\). This implies condition (ii).

To prove the reverse, suppose that condition (i) or (ii) does not hold. If there is \(\theta \neq \theta_0\) such that \(r(P_n(\theta, p_n)) = r(P_n(\theta_0, p_n))\), then \((\theta, p_n)\) satisfies system (5.3) and \((\theta_0, p_n)\) is not identified. If there is \(p \neq p_n\) such that \(r(p) = r(p_n)\) and \(r(p) = r(P_n(\theta, p))\) for some \(\theta\), then \((\theta, p)\) satisfies system (5.3) and \((\theta_0, p_n)\) is not identified. \(\square\)

**Proof of Proposition 5.1.** Denote \(U_{ij}^\ast(p) = U^\ast(X_i, X_j, p), V_i^\ast(p) = V^\ast(X_i, p)\) and \(\omega_i^\ast(p) = \omega^\ast(X_i, p)\). Suppress the argument \((X, p)\) or \(p\) whenever possible. By definition

\[
P_{n,ij}(X, p) - P^\ast(X_i, X_j, p) = \mathbb{E}[1\{U_{n,ij} + \frac{n-1}{n-2} Z_j' V_{ni} \omega_{ni} \geq \epsilon_{ij}\} - 1\{U_{ij}^\ast + Z_j' V_i^\ast \omega_i \geq \epsilon_{ij}\}] \quad (A.9)
\]

The right-hand side of (A.9) is bounded by the probability that \(\epsilon_{ij}\) lies between \(U_{n,ij} + \frac{n-1}{n-2} Z_j' V_{ni} \omega_{ni} \) and \(U_{ij}^\ast + Z_j' V_i^\ast \omega_i\). Define \(\Delta_{n,ij}(\epsilon_{ij}) = U_{n,ij} + \frac{n-1}{n-2} Z_j' V_{ni} \omega_{ni} - (U_{ij}^\ast + Z_j' V_i^\ast \omega_i)\). If \(J_n\) is nonsingular, then \(J_n' J_n\) is positive definite and \(\lambda_{\min}(J_n' J_n) > 0\).
\[ Z'_i V'_i \omega^*_i = (U_{n,ij} - U'_{ij}) + Z'_j (V_{ni} - V'_i) \omega_{ni}(\epsilon_i) + Z'_j V'_i (\omega_{ni}(\epsilon_i) - \omega^*_i) + \frac{1}{n-2} Z'_j V_{ni} \omega_{ni}(\epsilon_i). \]

For any \( \delta_n > 0 \), if \( \epsilon_{ij} \) lies between \( U_{n,ij} + \frac{n-1}{n-2} Z'_j V_{ni} \omega_{ni}(\epsilon_i) \) and \( U'_{ij} + Z'_j V'_i \omega^*_i \), and if their difference \( |\Delta_{n,ij}(\epsilon_i)| \) is at most \( \delta_n \), then \( \epsilon_{ij} \) must lie between \( U'_{ij} + Z'_j V'_i \omega^*_i - \delta_n \) and \( U'_{ij} + Z'_j V'_i \omega^*_i + \delta_n \). Therefore, we can further bound the right-hand side of (A.9) by

\[
\Pr(|\Delta_{n,ij}(\epsilon_i)| > \delta_n |X) + \Pr(U'_{ij} + Z'_j V'_i \omega^*_i - \delta_n \leq \epsilon_{ij} \leq U'_{ij} + Z'_j V'_i \omega^*_i + \delta_n |X). \tag{A.10}
\]

Consider the first term in (A.10). By Lemma A.2, \( V'_i (\omega_{ni}(\epsilon_i) - \omega^*_i) = o_p(1). \)

Moreover, \( U_{n,ij} - U'_{ij} = o_p(1) \) and \( \|V_{ni} - V'_i\| = o_p(1) \) by Assumption 7(iv), and \( V_{ni} \) and \( \omega_{ni}(\epsilon_i) \) are bounded. Hence, for any \( \delta_n > 0 \), we have \( \Pr(|\Delta_{n,ij}(\epsilon_i)| > \delta_n |X,i, X_j) \to 0 \) as \( n \to \infty \). By the law of iterated expectations \( \Pr(|\Delta_{n,ij}(\epsilon_i)| > \delta_n |X) = \mathbb{E}[\Pr(|\Delta_{n,ij}(\epsilon_i)| > \delta_n |X)|X_i, X_j] \) and Markov’s inequality, given \( X_i \) and \( X_j \) we must have \( \Pr(|\Delta_{n,ij}(\epsilon_i)| > \delta_n |X) = o_p(1) \).

For the second term in (A.10), by the mean-value theorem we derive \( \Pr(U'_{ij} + Z'_j V'_i \omega^*_i - \delta_n \leq \epsilon_{ij} \leq U'_{ij} + Z'_j V'_i \omega^*_i + \delta_n |X) = f(\epsilon(U'_{ij} + Z'_j V'_i \omega^*_i + t_{n,ij} \delta_n)) \delta_n, \) for some \( -1 \leq t_{n,ij} \leq 1 \). Given \( X_i \) and \( X_j \), because \( f(U'_{ij} + Z'_j V'_i \omega^*_i + t_{n,ij} \delta_n) \) is bounded, by choosing \( \delta_n > 0 \) with \( \delta_n \downarrow 0 \) as \( n \to \infty \), we derive that the second term in (A.10) is \( o(1) \). Combining the results proves the proposition. \( \square \)

**Lemma A.2** (Consistency of \( \omega_{ni}(\epsilon_i) \) for \( \omega^*_i \)). Suppose that Assumptions 1–4 and 7 are satisfied. Given \( X_i \), we have \( V'_i (\omega_{ni}(\epsilon_i) - \omega^*_i) = o_p(1) \).

**Proof.** Recall that \( \omega^*_i \) solves problem (5.5). Denote the objective function in (5.5) by \( \Pi^*_i(\omega) \). Following the proof of Lemma O.E.3 with \( \Pi^*_n(\omega) \) replaced by \( \Pi^*_i(\omega) \), which is continuous in \( \omega \), it suffices to show that \( \sup_{\omega \in \Omega} |\Pi_{ni}(\omega, \epsilon_i) - \Pi^*_i(\omega)| = o_p(1) \). By the triangle inequality

\[
\sup_{\omega \in \Omega} |\Pi_{ni}(\omega, \epsilon_i) - \Pi^*_i(\omega)| \\
\leq \sup_{\omega \in \Omega} \left| \frac{1}{n-1} \sum_{j \neq i} (U_{n,ij} + \frac{n-1}{n-2} Z'_j V_{ni} \omega - \epsilon_{ij} - U'_{ij} - Z'_j V'_i \omega^*_i + \epsilon_{ij}) \right| \\
+ \sup_{\omega \in \Omega} \left| \frac{1}{n-1} \sum_{j \neq i} [U'_{ij} + Z'_j V'_i \omega - \epsilon_{ij} + \mathbb{E}[[U'_{ij} + Z'_j V'_i \omega - \epsilon_{ij}] |X_i]] \right| \\
+ \sup_{\omega \in \Omega} \frac{1}{2} \left| \frac{n-1}{n-2} \omega' V_{ni} \omega - \omega' V'_i \omega^*_i \right|. \tag{A.11}
\]

Because \( |x| - |y| \leq |x - y| \), the first term on the right-hand side can be bounded by

\[
\sup_{\omega \in \Omega} \frac{1}{n-1} \sum_{j \neq i} |U_{n,ij} - U'_{ij} + Z'_j (V_{ni} - V'_i) \omega + \frac{1}{n-2} Z'_j V_{ni} \omega| \leq \max_{j \neq i} |U_{n,ij} - U'_{ij}| + \|V_{ni} -
\]

\[50\]In Lemma A.2, we fix \( X_i \) and treat \((X_k, k \neq i)\) as random. The result also holds if we fix \( X_j \) in addition to \( X_i \).
\( V_i^* \| \sup_{\omega \in \Omega} \| \omega \| + \frac{1}{n-2} \| V_{ni} \| \sup_{\omega \in \Omega} \| \omega \| = o_p(1) \), where the last equality follows by Assumption 7(iv) and the boundedness of \( \| V_{ni} \| \). Similarly, we can bound the last term on the right-hand side of (A.11) by \( \sup_{\omega \in \Omega} \frac{1}{2} | \omega'(V_{ni} - V_i^*)\omega| + \sup_{\omega \in \Omega} \frac{1}{2(n-2)} | \omega'V_{ni}\omega| \leq \frac{1}{2} \| V_{ni} - V_i^* \| \sup_{\omega \in \Omega} \| \omega \|^2 + \frac{1}{2(n-2)} \| V_{ni} \| \sup_{\omega \in \Omega} \| \omega \|^2 = o_p(1) \).

As for the second term on the right-hand side of (A.11), observe that given \( X_i \), the function \([U_{ij}^* + Z_j'V_i^*\omega - \epsilon_{ij}]_+\) is i.i.d. across \( j \) (Assumption 7(iii)). This function is continuous in \( \omega \) on a compact set \( \Omega \) (Assumption 7(i)). Moreover, for all \( \omega \in \Omega \), \([U_{ij}^* + Z_j'V_i^*\omega - \epsilon_{ij}]_+ \leq [U_{ij}^* + \| V_i^* \| \sup_{\omega \in \Omega} \| \omega \| - \epsilon_{ij}]_+\) with \( \mathbb{E}[[U_{ij}^* + \| V_i^* \| \sup_{\omega \in \Omega} \| \omega \| - \epsilon_{ij}]_+ | X_i] < \infty \). Therefore, the conditions of the uniform LLN are satisfied (Jennrich, 1969) and \( \sup_{\omega \in \Omega} \frac{1}{n-1} \sum_{j \neq i} [U_{ij}^* + Z_j'V_i^*\omega - \epsilon_{ij}]_+ - \mathbb{E}[[U_{ij}^* + Z_j'V_i^*\omega - \epsilon_{ij}]_+ | X_i] = o_p(1) \). The lemma is proved.

**Lemma A.3 (Example 5.1).** Under Assumption 7(iii), given \( X_i \) and \( p \), Assumption 7(iv) is satisfied for the specification in (2.5)-(2.6) and \( U^*(X_i, X_j, p) \) and \( V^*(X_i, p) \) defined in Example 5.1.

**Proof.** By definition

\[
U_{n,ij}(X, p) - U^*(X_i, X_j, p) = \frac{1}{n-2} \sum_{k \neq i,j} (p(X_k, X_j) - \mathbb{E}[p(X_k, X_j) | X_j]) \beta_5 \\
+ \frac{1}{n-2} \sum_{k \neq i,j} (p(X_j, X_k) - \mathbb{E}[p(X_j, X_k) | X_j]) \beta_6 \\
- \frac{1}{n-2} Z_j'V_{ni}(X, p) Z_j.
\]

Given \( X_j \), by Assumption 7(iii) both \( p(X_k, X_j) \) and \( p(X_j, X_k) \) are i.i.d. across \( k \). Hence, by the LLN the first two terms on the right-hand side are \( o_p(1) \). The last term on the right-hand side is \( o_p(1) \) by the boundedness of \( V_{ni} \). Because \( X_j \) takes \( T \) values, we obtain \( \max_{j \neq i} |U_{n,ij}(X, p) - U^*(X_i, X_j, p)| = o_p(1) \).

For \( 1 \leq s, t \leq T \),

\[
V_{ni,st}(X, p) - V_{st}^*(p) = \frac{1}{n-3} \sum_{l \neq i,j,k} (p(x_s, X_l)p(X_t, x_t) + p(x_t, X_l)p(X_t, x_s) \\
- \mathbb{E}[p(x_s, X_l)p(X_t, x_t) + p(x_t, X_l)p(X_t, x_s)] ) \gamma_2(x_s, x_t)
\]

By Assumption 7(iii) \( p(x_s, X_l)p(X_t, x_t) + p(x_t, X_l)p(X_t, x_s) \) is i.i.d. across \( l \). By the LLN, we obtain \( |V_{ni,st}(X, p) - V_{st}(p)| = o_p(1) \) and hence \( \| V_{ni}(X, p) - V^*(X_i, p) \| = o_p(1) \).

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O.A Dynamic Processes of Network Formation

O.A.1 Sequential Updating

In this section, we follow Myatt and Wallace (2004) and construct a dynamic process of network formation, where links in a network are formed and updated in a random sequence. Assume that the network is directed. We consider a simplified version of the utility in (2.1)–(2.3), where there exists no $X$, nor the effect of friends in common ($v_{ijk}(G_{i-}) = 0$). Hence, individual $i$’s utility is given by $U_i(G, \epsilon_i) = 1/n - 1 \sum_{j \neq i} G_{ij} (\beta_1 + G_{ji} \beta_4 + 1/n \sum_{k \neq i,j} G_{kj} \beta_5 + 1/n \sum_{k \neq i,j} G_{jk} \beta_6)$.

We construct the dynamic process as follows. In an initial period, individuals simultaneously form links, based on their arbitrary beliefs about the links formed by other individuals $G_{i-}$. After the initial period, individuals update links sequentially. Specifically, in each period, a link $G_{ij}$ is selected at random; individual $i$, who determines the link, receives a new draw of $\epsilon_{ij}$. The active individual observes $\epsilon_{ij}$ as well as the distribution of the links formed in the past, and updates the link accordingly.

Let $Z \in \mathbb{Z} = \{0, 1, \ldots, n(n-1)\}$ denote the total number of links formed in the network. Because individuals are of the same type (no $X$), $Z$ captures the state of play. Given state $z$, an active individual $i$ forms a belief about link $G_{jk}$ by the fraction of links that are formed in the network $\mathbb{E}[G_{jk}|Z = z] = \frac{z}{n(n-1)}$ for all $j \neq i$ and $k \neq j$. The optimal choice for link $G_{ij}$ is $G_{ij} = 1\{\beta_1 + \frac{z}{n(n-1)}(\beta_4 + \beta_5 + \beta_6) \geq \epsilon_{ij}\}$. The probability that individual $i$ forms link $G_{ij}$ is $F_\epsilon(\beta_1 + \frac{z}{n(n-1)}(\beta_4 + \beta_5 + \beta_6)) = \Psi(\frac{z}{n(n-1)})$, where we define $\Psi(p) = F_\epsilon(\beta_1 + p(\beta_4 + \beta_5 + \beta_6))$.

To analyze this dynamic process, we calculate the transition probabilities between states. Let $p_{z,z'} = \Pr(Z_{t+1} = z'|Z_t = z)$ denote the probability of a transition from state $z$ to state $z'$. Given state $z$, the state in the next period can be $z+1$, $z$, or $z-1$. The state moves from $z$ to $z+1$ if an unformed link $G_{ij} = 0$ is selected and individual $i$ decides to form the link. The probability from $z$ to $z+1$ is $p_{z,z+1} = 1 - \frac{z}{n(n-1)} \Psi(\frac{z}{n(n-1)})$. Similarly, the state
moves from $z$ to $z-1$ if a formed link $G_{ij} = 1$ is selected and individual $i$ decides to sever the link. The probability from $z$ to $z-1$ is $p_{z,z-1} = \frac{z}{n(n-1)} (1 - \Psi\left(\frac{z}{n(n-1)}\right))$. The state remains at $z$ if either a formed link is selected and remains formed or an unformed link is selected and remains unformed. The probability of remaining at $z$ is $p_{z,z} = \frac{z}{n(n-1)} \Psi\left(\frac{z}{n(n-1)}\right) + \left(1 - \frac{z}{n(n-1)}\right) (1 - \Psi\left(\frac{z}{n(n-1)}\right))$.

The dynamic process defines a Markov chain over the state space $\mathbb{Z}$. It is irreducible (because there is a positive probability of moving between any two states in a finite number of steps) and aperiodic (because there is a positive probability of remaining in a state). Therefore, the Markov chain is ergodic, and hence by the Ergodic Theorem it has a unique stationary distribution. Let $\pi = (\pi_z, z \in \mathbb{Z})$ denote the stationary distribution. Following Myatt and Wallace (2004), we can show that for any state $0 \leq z < n(n-1)$,

$$
\frac{\pi_z}{\pi_{z+1}} = \frac{p_{z+1,z}}{p_{z,z+1}} = \frac{\frac{z+1}{n(n-1)} \left(1 - \Psi\left(\frac{z+1}{n(n-1)}\right)\right)}{\left(1 - \frac{z}{n(n-1)}\right) \Psi\left(\frac{z}{n(n-1)}\right)}.
$$

Consider Bayesian Nash equilibria (BNE) in the static model. Let $p = \Pr(G_{ij} = 1)$ denote the probability of forming a link. Because links are independent (because of the absence of $X$ and separability of utility), a BNE can be represented by the link choice probability $p$. In particular, an equilibrium $p$ solves the fixed point equation $p = \Psi(p).$\textsuperscript{51} We show that the BNE in the static model can be related to the stationary distribution in the dynamic model. In particular, specific, we follow Myatt and Wallace (2004, Proposition 1) and show that the local maxima in the stationary distribution coincide with the stable Bayesian Nash equilibria in the static model.

**Proposition O.A.1.** The local maxima (modes) of the stationary distribution $\pi$ coincide with the stable Bayesian Nash equilibria of the static model. The local minima of the stationary distribution $\pi$ coincide with the unstable Bayesian Nash equilibria of the static model. Formally, let $\lfloor x \rfloor$ denote the largest integer below $x$, and $\lceil x \rceil$ denote the smallest integer above $x$. For sufficiently large $n$, we have $\pi_{\lfloor pn(n-1) \rfloor} < \pi_{\lfloor pn(n-1) \rfloor}$ for $p < \Psi(p)$ and $\pi_{\lceil pn(n-1) \rceil} > \pi_{\lfloor pn(n-1) \rfloor}$ for $p > \Psi(p)$.

**Proof.** For any $p$,

$$
\frac{\pi_{\lfloor pn(n-1) \rfloor}}{\pi_{\lfloor pn(n-1) \rfloor}} = \frac{p_{\lfloor pn(n-1) \rfloor, \lfloor pn(n-1) \rfloor}}{p_{\lfloor pn(n-1) \rfloor, \lfloor pn(n-1) \rfloor}} = \frac{\lfloor pn(n-1) \rfloor}{n(n-1)} \left(1 - \Psi\left(\frac{\lfloor pn(n-1) \rfloor}{n(n-1)}\right)\right) \left(1 - \frac{\lfloor pn(n-1) \rfloor}{n(n-1)}\right) \Psi\left(\frac{\lceil pn(n-1) \rceil}{n(n-1)}\right).
$$

(O.A.1)

Note that $\frac{\lfloor pn(n-1) \rfloor}{n(n-1)}$ and $\frac{\lfloor pn(n-1) \rfloor}{n(n-1)} \to p$ as $n \to \infty$. Because $\Psi(p)$ is continuous in $p$, for

\textsuperscript{51}This equation does not depend on $n$, nor does an equilibrium $p$.

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sufficiently large \( n \), the ratio in (O.A.1) can be made arbitrarily close to \( \frac{p(1-\Psi(p))}{(1-p)\Psi(p)} \). Since \( \frac{p(1-\Psi(p))}{(1-p)\Psi(p)} < 1 \) if and only if \( p < \Psi(p) \), we derive that for sufficiently large \( n \), \( \frac{\pi_{pn(n-1)}}{\pi_{pn(n-1)}} < 1 \) for \( p < \Psi(p) \) and \( \frac{\pi_{pn(n-1)}}{\pi_{pn(n-1)}} > 1 \) for \( p > \Psi(p) \).

\[ \square \]

O.A.2 Simultaneous Updating

We follow Myatt and Wallace (2003) to construct the second dynamic process of network formation, where all the links in a network are updated in each period. We follow the setting in the first process and assume that there is no \( X \) nor the effect from friends in common.

The initial period is the same as that in the first process: individuals simultaneously form links based on their arbitrary beliefs. After the initial period, individuals simultaneously update their links in each period. Specifically, in period \( t \), each individual \( i \) receives a new draw of \( \epsilon_{i,t} = (\epsilon_{ij,t}, j \neq i) \) and observes the distribution of links formed in period \( t-1 \). Based on the information, each \( i \) updates links \( G_{ij,t} = (G_{ij,t}, j \neq i) \) accordingly.

Let \( Z_{t-1} \in \mathbb{Z} = \{0, 1, \ldots, n(n-1)\} \) denote the total number of links formed in period \( t-1 \). In period \( t \), individual \( i \) forms a belief about link \( G_{jk,t} \) by the fraction of links formed in the previous period \( \mathbb{E}[G_{jk,t}|Z_{t-1}] = \frac{Z_{t-1}}{n(n-1)} \) for all \( j \neq i \) and \( k \neq j \). The optimal choice for link \( G_{ij,t} \) is \( G_{ij,t} = \{\beta_1 + \frac{Z_{t-1}}{n(n-1)}(\beta_4 + \beta_5 + \beta_6) - \epsilon_{ij,t} \geq 0\} \). The probability that individual \( i \) forms link \( G_{ij,t} \) is \( \Pr(G_{ij,t} = 1|Z_{t-1}) = F_\epsilon(\beta_1 + \frac{Z_{t-1}}{n(n-1)}(\beta_4 + \beta_5 + \beta_6)) = \Psi(\frac{Z_{t-1}}{n(n-1)}) \), where we define \( \Psi(p) = F_\epsilon(\beta_1 + p(\beta_4 + \beta_5 + \beta_6)) \).

Let \( p_t = \frac{Z_{t-1}}{n(n-1)} = \frac{1}{n(n-1)} \sum_i \sum_{j \neq i} G_{ij,t} \) denote the fraction of links formed in period \( t \). Proposition O.A.2 shows that the link fraction \( p_t \) converges to a Bayesian Nash equilibrium (BNE) in the static model. Recall that a BNE \( p^* \) solves the equation \( p^* = \Psi(p^*) \).

**Proposition O.A.2.** Let \( \tilde{p}_t, t \geq 0, \) be an iterative process defined by \( \tilde{p}_t = \Psi(\tilde{p}_{t-1}) \) with \( \tilde{p}_0 = p_0 \). If there exists an equilibrium \( p^* \) such that \( p^* = \Psi(p^*) \) and \( \tilde{p}_t \to p^* \) as \( t \to \infty \), then \( p_t - p^* = o_p(1) \) as \( n, t \to \infty \).

**Proof.** Conditional on \( p_{t-1} \), the link fraction \( p_t \) has expectation \( \mathbb{E}[p_t|p_{t-1}] = \Psi(p_{t-1}) \). By the law of large numbers \( p_t = \Psi(p_{t-1}) + o_p(1) \) as \( n \to \infty \). Because \( \Psi(p) \) is continuous in \( p \), applying the continuous mapping theorem iteratively we derive \( p_t = \tilde{p}_t + o_p(1), t = 1, 2, \ldots \). That is, each link fraction \( p_t \) is asymptotically close to its deterministic counterpart \( \tilde{p}_t \) as \( n \to \infty \). By the triangle inequality \( |p_t - p^*| \leq |p_t - \tilde{p}_t| + |\tilde{p}_t - p^*| \), where \( |p_t - \tilde{p}_t| \) for all \( t \) as \( n \to \infty \), and \( |\tilde{p}_t| \) as \( t \to \infty \). Therefore, as \( n, t \to \infty \), we have \( p_t - p^* = o_p(1) \). \( \square \)
O.B General $V_i$

In this section, we generalize Theorem 3.1 by allowing for a general matrix $V_i(X, \sigma)$. Consider the expected utility in (3.1). For a general $V_i(X, \sigma)$, we cannot directly apply the Legendre transform in (3.2). Instead, we first exploit a spectral decomposition of $V_i(X, \sigma)$. Observe that $V_i(X, \sigma)$ is real and symmetric and thus has a real spectral decomposition

$$V_i(X, \sigma) = \Phi_i(X, \sigma) \Lambda_i(X, \sigma) \Phi_i'(X, \sigma),$$

where $\Lambda_i(X, \sigma) = \text{diag}(\lambda_{i1}(X, \sigma), \ldots, \lambda_{iT}(X, \sigma))$ denotes the $T \times T$ diagonal matrix of eigenvalues in $\mathbb{R}$ and $\Phi_i(X, \sigma) = (\phi_{i1}(X, \sigma), \ldots, \phi_{iT}(X, \sigma))$ denotes the $T \times T$ orthogonal matrix of eigenvectors in $\mathbb{R}^T$. Using the spectral decomposition, we can express the quadratic term in (3.1) as a function of the squares of $\frac{1}{n-1} \sum_{j \neq i} G_{ij} Z_j' \phi_{it}(X, \sigma)$, $t = 1, \ldots, T$, which are linear in link choices $G_{ij}$.

Next, we “linearize” these squares of linear functions using a special case of the Legendre transform. In particular, for any scalar $y \in \mathbb{R}$, we have

$$\frac{1}{2} y^2 = \max_{\omega \in \mathbb{R}} \left\{ y \omega - \frac{1}{2} \omega^2 \right\},$$

where $\omega \in \mathbb{R}$ is a scalar auxiliary variable. By choosing $y = \frac{1}{n-1} \sum_{j \neq i} G_{ij} Z_j' \phi_{it}(X, \sigma)$, we can replace its square by the maximization on the right-hand-side of (O.B.1). The transformation of the expected utility is presented in Lemma O.B.1. Based on the transformed expected utility, we derive optimal link choices in Proposition O.B.1.

**Lemma O.B.1.** Suppose that Assumptions 1–3 are satisfied. The expected utility in (3.1) satisfies

$$\mathbb{E}[U_i(G_i, G_{-i}, X, \epsilon_i)|X, \epsilon_i, \sigma]$$

$$= \frac{1}{n-1} \sum_{j \neq i} G_{ij} (U_{ij}(X, \sigma) - \epsilon_{ij})$$

$$+ \frac{n-1}{2(n-2)} \sum_{t=1}^T \lambda_{it}(X, \sigma) \left( \frac{1}{n-1} \sum_{j \neq i} G_{ij} Z_j' \phi_{it}(X, \sigma) \right)^2$$

$$= \frac{1}{n-1} \sum_{j \neq i} G_{ij} (U_{ij}(X, \sigma) - \epsilon_{ij})$$

$$+ \frac{n-1}{n-2} \sum_{t=1}^T \lambda_{it}(X, \sigma) \max_{\omega_t \in \mathbb{R}} \left\{ \left( \frac{1}{n-1} \sum_{j \neq i} G_{ij} Z_j' \phi_{it}(X, \sigma) \omega_t - \frac{1}{2} \omega_t^2 \right) \right\}. \quad (\text{O.B.2})$$

Online Appendix 4
Proof. The first equality follows because by the spectral decomposition we have

\[
\sum_{j \neq i} \sum_{k \neq i} G_{ij} G_{ik} Z_j' V_i(X, \sigma) Z_k = \left( \sum_{j \neq i} G_{ij} Z_j' \Phi_i(X, \sigma) \right) \Lambda_i(X, \sigma) \left( \sum_{k \neq i} G_{ik} \Phi_i(X, \sigma) Z_k \right)
\]

\[
= (n-1)^2 \sum_{t=1}^{T} \lambda_t(X, \sigma) \left( \frac{1}{n-1} \sum_{j \neq i} G_{ij} Z_j' \phi_t(X, \sigma) \right)^2.
\]

The second equality follows from (O.B.1).

\[\square\]

Proposition O.B.1. Suppose that Assumptions 1–3 are satisfied. For each \(i\), the optimal decision \(G_i(\epsilon_i, X, \sigma) \in \mathcal{G}_i\) is given by

\[G_{ij}(\epsilon_i, X, \sigma) = 1 \left\{ U_{ij}(X, \sigma) + \frac{n-1}{n-2} Z_j' \Phi_i(X, \sigma) \Lambda_i(X, \sigma) \omega_i(\epsilon_i, X, \sigma) \geq \epsilon_{ij} \right\}, \forall j \neq i, \quad \text{(O.B.3)}\]

almost surely. The \(T \times 1\) vector \(\omega_i(\epsilon_i, X, \sigma) \in \mathbb{R}^T\) in (O.B.3) is an optimal solution to the maximin problem

\[
\max_{\omega_t, t \in T_i^+} \min_{\omega_t, t \in T_i^-} \frac{1}{n-1} \sum_{j \neq i} \left[ U_{ij}(X, \sigma) + \frac{n-1}{n-2} Z_j' \Phi_i(X, \sigma) \Lambda_i(X, \sigma) \omega_t - \epsilon_{ij} \right] + \frac{n-1}{2(n-2)} \omega' \Lambda_i(X, \sigma) \omega,
\]

\[\text{(O.B.4)}\]

where \(T_i^+ = \{1 \leq t \leq T : \lambda_t(X, \sigma) > 0\}\) and \(T_i^- = \{1 \leq t \leq T : \lambda_t(X, \sigma) < 0\}\). We set \(\omega_t(\epsilon_i, X, \sigma) = 0\) if \(\lambda_t(\epsilon_i, X, \sigma) = 0\). Moreover, both \(G_i(\epsilon_i, X, \sigma)\) and \(\omega(\epsilon_i, X, \sigma)\) are unique almost surely.

Proof. We prove the proposition in the case where both \(T_i^+\) and \(T_i^-\) are nonempty. The proof holds for special cases where \(T_i^+\) is empty (negative semi-definite \(V_i(X, \sigma)\)) or \(T_i^-\) is empty (positive semi-definite \(V_i(X, \sigma)\)) without modification. Note that the latter has been proved in Theorem 3.1.

By Lemma O.B.1, the expected utility satisfies

\[
\mathbb{E}[U_i(G_{i, G_{-i}}, X, \epsilon_i)|X, \epsilon_i, \sigma] = \max_{\omega_t, t \in T_i^+} \min_{\omega_t, t \in T_i^-} \frac{1}{n-1} \sum_{j \neq i} G_{ij} \left( U_{ij}(X, \sigma) + \frac{n-1}{n-2} Z_j' \sum_{t=1}^{T} \phi_t(X, \sigma) \lambda_t(X, \sigma) \omega_t - \epsilon_{ij} \right)
\]

\[
- \frac{n-1}{2(n-2)} \sum_{t=1}^{T} \lambda_t(X, \sigma) \omega_t^2
\]

\[
= \max_{\omega_t, t \in T_i^+} \min_{\omega_t, t \in T_i^-} \frac{1}{n-1} \sum_{j \neq i} G_{ij} \left( U_{ij}(X, \sigma) + \frac{n-1}{n-2} Z_j' \Phi_i(X, \sigma) \Lambda_i(X, \sigma) \omega - \epsilon_{ij} \right)
\]

\[
- \frac{n-1}{2(n-2)} \omega' \Lambda_i(X, \sigma) \omega.
\]

\[\text{(O.B.5)}\]

Online Appendix 5
The first equality follows because if we move an eigenvalue $\lambda_{it}$ inside a maximization, it remains a maximization if $\lambda_{it} \geq 0$ and switches to a minimization if $\lambda_{it} < 0$.

Let $\tilde{\Pi}(G_i, \omega, \epsilon_i, X, \sigma)$ denote the objective function of the last maximin problem in (O.B.5). We have

$$\max_{G_i} \mathbb{E}[U_i(G_i, G_{-i}, X, \epsilon_i)|X, \epsilon_i, \sigma] = \max_{G_i} \max_{\omega_t, t \in T_i^+} \min_{\omega_t, t \in T_i^-} \tilde{\Pi}_i(G_i, \omega, \epsilon_i, X, \sigma)$$

$$\leq \max_{\omega_t, t \in T_i^+} \min_{\omega_t, t \in T_i^-} \max_{G_i} \tilde{\Pi}_i(G_i, \omega, \epsilon_i, X, \sigma)$$

$$= \max_{\omega_t, t \in T_i^+} \min_{\omega_t, t \in T_i^-} \Pi_i(\omega, \epsilon_i, X, \sigma),$$

where $\Pi_i(\omega, \epsilon_i, X, \sigma)$ denotes the objective function in (O.B.4). The inequality follows because $\max_{\omega_t, t \in T_i^+} \min_{\omega_t, t \in T_i^-} \tilde{\Pi}_i(G_i, \omega, \cdot) \leq \max_{\omega_t, t \in T_i^+} \min_{\omega_t, t \in T_i^-} \max_{G_i} \tilde{\Pi}_i(G_i, \omega, \cdot)$ for all $G_i$ and hence the maximum of the left-hand side over $G_i$ is bounded by the right-hand side. The last equality in (O.B.6) holds because for any $\omega$, $\tilde{\Pi}_i(G_i, \omega, \cdot)$ is separable in each $G_{ij}$ so the optimal $G_{ij}$ is given by (O.B.3) with $\omega_i(\epsilon_i, X, \sigma)$ replaced by $\omega$ and $\max_{G_i} \tilde{\Pi}_i(G_i, \omega, \cdot) = \Pi_i(\omega, \cdot)$.

Next we show that the inequality in (O.B.6) is an equality. Since $\omega_i(X, \epsilon_i, \sigma)$ is an optimal solution to problem (O.B.4), similarly as in Lemma A.1 it satisfies the first-order condition

$$\Lambda_i(X, \sigma)\omega_i(\epsilon_i, X, \sigma) = \frac{1}{n-1} \sum_{j \neq i} \left\{ U_{ij}(X, \sigma) + \frac{n-1}{n-2} Z_j' \Phi_i(X, \sigma) \Lambda_i(X, \sigma) \omega_i(\epsilon_i, X, \sigma) \geq \epsilon_{ij} \right\} \cdot \Lambda_i(X, \sigma) \Phi_i'(X, \sigma) Z_j, \ a.s.. \quad (O.B.7)$$

Multiplying both sides by $\Phi_i(X, \sigma)$ gives

$$\Phi_i(X, \sigma)\Lambda_i(X, \sigma)\omega_i(\epsilon_i, X, \sigma) = \frac{1}{n-1} V_i(X, \sigma) \sum_{j \neq i} G_{ij}(\epsilon_i, X, \sigma) Z_j, \ a.s., \quad (O.B.8)$$

where $G_{ij}(\epsilon_i, X, \sigma)$ is given in (O.B.3). By the definition of $G_i(\epsilon_i, X, \sigma)$ and $\omega_i(\epsilon_i, X, \sigma)$, the maximin value of $\Pi(\omega, \epsilon_i, X, \sigma)$ is given by

$$\max_{\omega_t, t \in T_i^+} \min_{\omega_t, t \in T_i^-} \Pi_i(\omega, \epsilon_i, X, \sigma)$$

$$= \frac{1}{n-1} \sum_{j \neq i} G_{ij}(\epsilon_i, X, \sigma)(U_{ij}(X, \sigma) - \epsilon_{ij})$$

$$+ \frac{1}{n-2} \sum_{j \neq i} G_{ij}(\epsilon_i, X, \sigma) Z_j' \Phi_i(X, \sigma) \Lambda_i(X, \sigma) \omega_i(\epsilon_i, X, \sigma)$$

$$- \frac{n-1}{2(n-2)} \omega_i(\epsilon_i, X, \sigma)' \Lambda_i(X, \sigma) \omega_i(\epsilon_i, X, \sigma)$$

$$= \frac{1}{n-1} \sum_{j \neq i} G_{ij}(\epsilon_i, X, \sigma)(U_{ij}(X, \sigma) - \epsilon_{ij})$$

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where the second equality follows from the first-order condition (O.B.7) and the third equality follows from equation (O.B.8).

Combining (O.B.6) and (O.B.9) yields

$$
\max_{\mathcal{G}_i} \mathbb{E}[U_i(G_i, G_{-i}, X, \epsilon_i)|X, \epsilon_i, \sigma] \leq \max_{\omega \in \mathcal{T}_+} \min_{\epsilon \in \mathcal{E}} \Pi_i(\omega, \epsilon, X, \sigma)
$$

$$
= \mathbb{E}[U_i(G_i(\epsilon_i, X, \sigma), G_{-i}, X, \epsilon_i)|X, \epsilon_i, \sigma], \text{ a.s.}
$$

Because \( \max_{G_i} \mathbb{E}[U_i(G_i, G_{-i}, X, \epsilon_i)|X, \epsilon_i, \sigma] \geq \mathbb{E}[U_i(G_i(\epsilon_i, X, \sigma), G_{-i}, X, \epsilon_i)|X, \epsilon_i, \sigma] \), the inequality becomes an equality, and all the terms are equal almost surely. Hence, \( G_i(\epsilon_i, X, \sigma) \) is an optimal solution almost surely.

As for the uniqueness, \( G_i(\epsilon_i, X, \sigma) \) is unique almost surely because \( \epsilon_i \) has a continuous distribution, so two decisions achieve the same utility with probability zero. The uniqueness of \( \Lambda_i(X, \sigma)\omega_i(\epsilon_i, X, \sigma) \) follows from the uniqueness of \( G_i(\epsilon_i, X, \sigma) \), equation (O.B.8) and the invertibility of \( \Phi_i(X, \sigma) \).

Proposition O.B.1 is established as a result of the fact that the transformed expected utility in (O.B.2) is separable in each maximization; hence a maximization over \( \omega_t \) remains unchanged if \( \lambda_{it}(X, \sigma) > 0 \) and switches to a minimization if \( \lambda_{it}(X, \sigma) < 0 \). If \( \lambda_{it}(X, \sigma) = 0 \), the function \( \Pi_i(\omega, \epsilon, X, \sigma) \) does not depend on \( \omega_i \), so we set \( \omega_{it}(\epsilon_i, X, \sigma) = 0 \). The separability also implies that the order of the maximizations and minimizations does not matter.

To gain some intuition regarding the role of the eigenvalues of \( V_i(X, \sigma) \), we consider an example with two types (\( T = 2 \)).

**Example O.B.1.** Suppose that the utility function takes the form \((2.1)-(2.3)\), with a constant \( \gamma_1 > 0 \) and \( \gamma_2 = 0 \). Suppress the argument \((X, \sigma)\) for simplicity. The matrix \( V_i \) is given by \((3.3)\). Suppose that \( V_{i,12} > 0 \). Let \( \lambda_{i1} \geq \lambda_{i2} \) denote the eigenvalues of \( V_i \). We can show that \( \lambda_{i1} > 0 \).\(^{52}\) Assume that \( \lambda_{i2} \neq 0 \). Let \( \phi_{i1} = (\phi_{i1,1}, \phi_{i1,2})' \) and \( \phi_{i2} = (\phi_{i2,1}, \phi_{i2,2})' \) denote

\(^{52}\)The eigenvalues are given by \( \lambda_{i1}, \lambda_{i2} = \frac{1}{2}(V_{i,11} + V_{i,22} \pm \sqrt{V_{i,11}^2 + V_{i,22}^2 + 4V_{i,12}^2 - 2V_{i,11}V_{i,22}}) \). Since \( V_{i,12} > 0 \), they satisfy \( \lambda_{i1} > \max\{V_{i,11}, V_{i,22}\} \geq 0 \) and \( \lambda_{i2} < \min\{V_{i,11}, V_{i,22}\} \).
the corresponding eigenvectors. It can be shown that the elements of \( \phi_{11} \) have the same sign, and the elements of \( \phi_{12} \) have opposite signs.\(^{53}\)

From the first-order condition (O.B.7), the optimal \( \omega_i \) satisfies
\[
\omega_{it} = \phi_{it}' \frac{1}{n-1} \sum_{j \neq i} G_{ij} Z_j, \quad t = 1, 2,
\]
almost surely. Note that \( \sum_{j \neq i} G_{ij} Z_j \) is a \( 2 \times 1 \) vector of the numbers of friends that individual \( i \) has of each type. Therefore, \( \omega_{it} \) is a weighted sum of the numbers of friends of each type, with weights given by the components of \( \phi_{it} \). A larger \( |\omega_{11}| \) is associated with more friends of each type,\(^{54}\) and a larger \( |\omega_{12}| \) is associated with a circle of friends that is of one type.

Following the proof of Proposition O.B.1, we can express the expected utility in (3.1) as
\[
E[U_i|X, \epsilon_i, \sigma] = \frac{1}{n-1} \sum_{j \neq i} G_{ij} (U_{ij} - \epsilon_{ij}) + \frac{n-1}{2(n-2)} \omega_i' \Lambda_i \omega_i
\]
almost surely. Because \( \lambda_{11} > 0 \), individual \( i \) prefers a larger \( |\omega_{11}| \), implying that she prefers to have more friends of any type, with the type that corresponds to the larger value of \( |\phi_{i,11}| \) and \( |\phi_{i,12}| \) being most preferred. If \( \lambda_{12} > 0 \), individual \( i \) prefers a larger \( |\omega_{12}| \), implying that she prefers her circle of friends to be of one type. If \( \lambda_{12} < 0 \), individual \( i \) prefers an \( \omega_{12} \) closer to 0, implying that she prefers an integrated circle of friends.

In a special case where only agents of the opposite type link (\( V_{i,11} = V_{i,22} = 0 \)), we have \( \lambda_{11} = V_{i,12}, \lambda_{12} = -V_{i,12}, \phi_{i1} = (\frac{1}{\sqrt{2}}, \frac{1}{\sqrt{2}})' \) and \( \phi_{i2} = (\frac{1}{\sqrt{2}}, -\frac{1}{\sqrt{2}})' \). Therefore, \( \omega_{11} = \frac{1}{\sqrt{2(n-1)}} \sum_{j \neq i} G_{ij} (Z_{j1} + Z_{j2}) \) and \( \omega_{12} = \frac{1}{\sqrt{2(n-1)}} \sum_{j \neq i} G_{ij} (Z_{j1} - Z_{j2}) \). Intuitively, if a network only allows for cross-type links, an individual would prefer to make as many friends as she could and to choose an equal number of friends of each type.

### O.C Partition Representation of Optimal Decisions

In this section, we establish a one-to-one mapping between the optimal decisions of an individual and a partition of the \( \epsilon_i \) space \( \mathbb{R}^{n-1} \). The partition representation of optimal decisions is useful for analyzing the properties of link choice probabilities.

Recall that for the expected utility in (2.4), the expected marginal utility of individual \( i \) from forming a link to individual \( j \) is given by
\[
\frac{1}{n-1} (E[u_{ij}|X] + \frac{1}{n-2} \sum_{k \neq i,j} G_{ik} E[v_{ijk}|X] - \epsilon_{ij}),
\]
\(^{53}\)By definition \( V_i \phi_{i1} = \lambda_{i1} \phi_{i1} \), so \( (\lambda_{i1} - V_{i,11}) \phi_{i1} = V_{i,12} \phi_{i,12} \) and \( V_{i,12} \phi_{i,11} = (\lambda_{i1} - V_{i,22}) \phi_{i,12} \). Since \( \lambda_{i1} > \max\{V_{i,11}, V_{i,22}\} \) and \( V_{i,12} > 0 \), these equations imply that \( \phi_{i,11} \) and \( \phi_{i,12} \) must have the same sign, that is, \( \phi_{i,11} \phi_{i,12} > 0 \). Similarly, we can show \( \phi_{i,21} \phi_{i,22} < 0 \).

\(^{54}\)The type that corresponds to the larger value of \( |\phi_{i1}| \) and \( |\phi_{i2}| \) makes a larger contribution.
where we have used the symmetry of \( v_{i,jk} \) in \( j \) and \( k \) \((v_{i,jk} = v_{i,kj})\). If \( G_i \in G_i \) is an optimal decision, it must satisfy that for each \( j \neq i \), \( G_{ij} = 1 \) if and only if the expected marginal utility from the link is nonnegative. This yields the system of equations

\[
G_{ij} = 1 \left\{ E[u_{ij}|X] + \frac{1}{n-2} \sum_{k \neq i,j} G_{ik}E[v_{i,jk}|X] \geq \epsilon_{ij} \right\}, \forall j \neq i. \quad \text{(O.C.1)}
\]

There may be multiple solutions to (O.C.1). For example, assume that \( E[v_{i,jk}|X] > 0 \), so link choices are strategic complements. For \( \epsilon_i \in \mathbb{R}^{n-1} \) such that \( E[u_{ij}|X] < \epsilon_{ij} \leq E[u_{ij}|X] + \frac{1}{n-2} \sum_{k \neq i,j} E[v_{i,jk}|X] \), \( \forall j \neq i \), we find that both \( G_i^1 = (G_{ij} = 1, \forall j \neq i) \) and \( G_i^0 = (G_{ij} = 0, \forall j \neq i) \) are solutions to (O.C.1). We refer to (O.C.1) as the local optimal condition.

Among the local solutions to (O.C.1), an optimal decision \( G_i \) achieves the highest expected utility, that is,

\[
\frac{1}{n-1} \sum_{j \neq i} G_{ij} \left( E[u_{ij}|X] + \frac{1}{2(n-2)} \sum_{k \neq i,j} G_{ik}E[v_{i,jk}|X] - \epsilon_{ij} \right) \geq \max\limits_{\tilde{g}_i \in G_i} \frac{1}{n-1} \sum_{j \neq i} \tilde{g}_{ij} \left( E[u_{ij}|X] + \frac{1}{2(n-2)} \sum_{k \neq i,j} \tilde{g}_{jk}E[v_{i,jk}|X] - \epsilon_{ij} \right). \quad \text{(O.C.2)}
\]

Because \( \epsilon_i \) follows a continuous distribution (Assumption 1(ii)), two decisions achieve the same expected utility with probability zero. Therefore, there is a unique solution to (O.C.2) with probability one. We refer to (O.C.2) as the global optimal condition.

For each \( g_i \in G_i \), define the set

\[
E_i(g_i, X) = \{ \epsilon_i \in \mathbb{R}^{n-1} : g_i \text{ satisfies conditions (O.C.1) and (O.C.2)} \}. \quad \text{(O.C.3)}
\]

It represents the collection of \( \epsilon_i \in \mathbb{R}^{n-1} \) such that \( g_i \) is the optimal solution. Note that since \( \epsilon_i \) has an unbounded support \( \mathbb{R}^{n-1} \), the set \( E_i(g_i, X) \) is nonempty for all \( g_i \in G_i \). Because there is a unique optimal decision for almost all \( \epsilon_i \in \mathbb{R}^{n-1} \), the sets \( \{E_i(g_i, X), g_i \in G_i\} \) form a partition of \( \mathbb{R}^{n-1} \) with probability one. The results are summarized in Proposition O.C.1.

**Proposition O.C.1.** Suppose that Assumption 1 is satisfied. For each \( i \), an optimal decision \( G_i \in G_i \) satisfies conditions (O.C.1) and (O.C.2). Moreover, the sets \( \{E_i(g_i, X), g_i \in G_i\} \) form a partition of \( \mathbb{R}^{n-1} \) with probability one.

The local optimal condition (O.C.1) resembles pure-strategy Nash equilibria in entry games with complete information (Ciliberto and Tamer, 2009). The strategic interactions between link choices arise from the effect of friends in common \((E[v_{i,jk}|X] \neq 0)\). The multiplic-
ity of the local solutions to (O.C.1) resembles the multiplicity of equilibria, and the global optimal condition (O.C.2) resembles an equilibrium selection mechanism (Ciliberto and Tamer, 2009).

**O.D Monte Carlo Simulations**

In this section, we evaluate our approach in a simulation study. We consider the utility specification

\[
U_i(G, X, \epsilon_i) = \frac{1}{n-1} \sum_{j \neq i} G_{ij} \left( \beta_1 + X_i \beta_2 + |X_i - X_j| \beta_3 + \frac{1}{n-2} \sum_{k \neq i, j} G_{jk} \beta_4 \right) + \frac{1}{2(n-2)} \sum_{k \neq i, j} G_{ik}(G_{jk} + G_{kj}) \gamma - \epsilon_{ij},
\]

where \(X_i\) is an i.i.d. binary variable with equal probability of being 0 or 1, and \(\epsilon_{ij}\) is i.i.d. following \(N(0,1)\). The true parameter values are given by \((\beta_1, \beta_2, \beta_3, \beta_4, \gamma) = (-1,1,-2,1,1)\), where \(\beta_3\) represents the homophily effect, \(\beta_4\) represents the effect of friends of friends, and \(\gamma\) represents the effect of friends in common. We consider a variety of network sizes \(n = 10, 25, 50, 100, 250, 500\).

For each \(n\), we generate a single directed network as follows. First, we generate a characteristic profile \(X\) and compute a Bayesian Nash equilibrium \(\sigma\).\(^{55}\) Second, we use the equilibrium \(\sigma\) to compute \(U_{n,ij}(X,\sigma)\) and \(V_{ni}(X,\sigma)\). Third, we compute \(\omega_{ni}(\epsilon_i)\) for a simulated \(\epsilon_i\) and obtain the optimal link choices from equation (3.5).\(^{56}\) Each experiment is repeated 100 times.

We estimate the parameters by two-step GMM, where we first estimate the link choice probabilities by the frequency estimator and then estimate the parameters by GMM. We use the moment function in (4.3) and the instrument in (4.9) and consider three varieties of the second step. (i) Both the moment function and instrument are constructed using the finite-\(n\) link probabilities. (ii) The moment function is constructed using the finite-\(n\) link probabilities, but the instrument is constructed using the limiting link probabilities. (iii) Both the

\(^{55}\)We compute an equilibrium by iterating equation (2.8) from an initial value. The initial value we use is an equilibrium in the limiting game, which is computed by solving for the limiting version of equation (2.8), where we replace the finite-\(n\) choice probability on the right-hand side of (2.8) by its limiting counterpart.

\(^{56}\)For small networks, it could be computationally competitive to maximize expected utility (2.4) directly by quadratic integer programming (QIP). In our simulation study, we compute the optimal link choices using QIP for \(n \leq 100\) and equation (3.5) for \(n > 100\). We solve QIP using the solver cplexmiqp provided in CPLEX. QIP is also a check on whether the link choices in (3.5) maximize the expected utility. We find that the simulated link choice probabilities based on QIP and those based on (3.5) are indeed the same.
moment function and instrument are constructed using the limiting link probabilities. Table O.D.1 reports the average biases and root MSEs of the estimates in the three cases.

Columns 1 and 2 present the results in case (i). We compute the finite-\( n \) link choice probabilities by simulation. Specifically, we draw a random sample of \( \epsilon_i \); for each simulated \( \epsilon_i \), we compute \( \omega_{ni}(\epsilon_i) \) and derive the optimal link choices from equation (3.5). The fraction of draws that result in a link gives a simulated link choice probability. We calculate the GMM estimator by continuous updating, where the instrument is computed by simulation\(^{57}\) and the derivative in the numerator in (4.9) is approximated by a numerical derivative.\(^{58}\) The results show that the two-step GMM based on the finite-\( n \) link probabilities performs well. The estimates are close to the true values even for network sizes as small as \( n = 25 \). The root MSEs also decrease as the network size increases, as expected.

Columns 3 and 4 present the results in case (ii). We simulate the finite-\( n \) link probabilities in the moment function as described in case (i), but the limiting link probabilities and their derivatives in the instrument are computed analytically. We find that for small networks (e.g., \( n = 10 \)), the estimates are biased and the root MSEs are larger than those in case (i). But for larger networks, the biases and root MSEs become close to those in case (i). These results suggest that in large networks, the computationally convenient limiting link probabilities can be used to approximate the instrument without sacrificing the estimation precision.

Columns 5 and 6 present the results in case (iii). Because both the moment function and instrument are constructed using the limiting link probabilities, which can be computed without simulation, this case is the most computationally convenient among the three.\(^{59}\) The results show that the estimates present large biases in small networks, but these biases vanish as the network size grows, suggesting that the estimator based on the limiting link probabilities is consistent. The root MSEs are generally larger than those in cases (i) and (ii), but become similar once the networks are sufficiently large (e.g., \( n = 500 \)).

In sum, the two-step estimation procedure based on the finite-\( n \) link probabilities performs well even in relatively small networks. In sufficiently large networks, the estimates based on the limiting link probabilities can perform as well as those based on the finite-\( n \) ones, regardless of whether we use the limiting link probabilities for the instrument or the moment function. This parity suggests that the limiting link probabilities can provide a

\(^{57}\) We simulate the instrument using \( \epsilon_i \) that are drawn independently of those drawn to simulate the link choice probabilities in the moment function.

\(^{58}\) Because the sample moment is not everywhere differentiable, we use the derivative-free optimization solver \texttt{fminsearch} provided in MATLAB when searching for the estimate of \( \theta \).

\(^{59}\) The moment condition in this case is equal to the first-order condition from QMLE based on the limiting link probabilities, so we estimate \( \theta \) equivalently by QMLE based on the limiting link probabilities.
| Para. | Case (i) | Case (ii) | Case (iii) |
|-------|----------|-----------|------------|
| $n=10$ |          |           |            |
| $\beta_1$ | 0.005 | 0.206 | -0.008 | 0.438 | -0.152 | 2.284 |
| $\beta_2$ | -0.042 | 0.191 | 0.273 | 1.010 | 1.806 | 3.004 |
| $\beta_3$ | 0.063 | 0.402 | -0.940 | 3.338 | -4.469 | 3.649 |
| $\beta_4$ | -0.019 | 0.185 | -0.166 | 1.900 | -3.626 | 8.890 |
| $\gamma$ | -0.021 | 0.182 | -0.057 | 1.004 | -1.194 | 6.438 |
| $n=25$ |          |           |            |
| $\beta_1$ | -0.010 | 0.066 | -0.017 | 0.097 | 0.281 | 0.447 |
| $\beta_2$ | 0.006 | 0.109 | 0.012 | 0.185 | 1.639 | 2.029 |
| $\beta_3$ | -0.038 | 0.194 | -0.065 | 0.268 | -1.899 | 2.152 |
| $\beta_4$ | 0.003 | 0.092 | 0.016 | 0.232 | -2.835 | 3.710 |
| $\gamma$ | -0.004 | 0.098 | -0.014 | 0.146 | -1.887 | 3.948 |
| $n=50$ |          |           |            |
| $\beta_1$ | -0.003 | 0.042 | -0.010 | 0.052 | 0.014 | 0.126 |
| $\beta_2$ | -0.001 | 0.065 | -0.005 | 0.070 | 0.058 | 0.499 |
| $\beta_3$ | -0.001 | 0.097 | 0.005 | 0.101 | -0.064 | 0.499 |
| $\beta_4$ | 0.020 | 0.083 | 0.050 | 0.110 | -0.142 | 0.921 |
| $\gamma$ | -0.012 | 0.072 | -0.016 | 0.094 | -0.091 | 0.551 |
| $n=100$ |          |           |            |
| $\beta_1$ | 0.004 | 0.023 | 0.005 | 0.023 | 0.005 | 0.034 |
| $\beta_2$ | -0.007 | 0.036 | -0.009 | 0.040 | 0.008 | 0.084 |
| $\beta_3$ | -0.010 | 0.052 | -0.010 | 0.050 | -0.007 | 0.084 |
| $\beta_4$ | 0.031 | 0.064 | 0.034 | 0.073 | -0.015 | 0.165 |
| $\gamma$ | -0.019 | 0.055 | -0.021 | 0.062 | -0.041 | 0.208 |
| $n=250$ |          |           |            |
| $\beta_1$ | 0.002 | 0.008 | 0.002 | 0.008 | -0.001 | 0.014 |
| $\beta_2$ | -0.001 | 0.017 | 0.000 | 0.018 | 0.004 | 0.039 |
| $\beta_3$ | 0.000 | 0.020 | -0.001 | 0.021 | -0.003 | 0.037 |
| $\beta_4$ | 0.027 | 0.035 | 0.031 | 0.038 | 0.009 | 0.075 |
| $\gamma$ | -0.013 | 0.033 | -0.017 | 0.036 | -0.031 | 0.173 |
| $n=500$ |          |           |            |
| $\beta_1$ | -0.001 | 0.006 | -0.001 | 0.005 | -0.001 | 0.010 |
| $\beta_2$ | 0.007 | 0.011 | 0.010 | 0.013 | 0.001 | 0.022 |
| $\beta_3$ | 0.003 | 0.011 | 0.000 | 0.012 | 0.000 | 0.022 |
| $\beta_4$ | -0.002 | 0.028 | -0.001 | 0.034 | 0.006 | 0.047 |
| $\gamma$ | -0.005 | 0.019 | -0.011 | 0.025 | -0.014 | 0.103 |

Note: Average biases and root MSEs from 100 repeated samples. Case (i) uses the moment function and instrument based on the finite-$n$ link probabilities. Case (ii) uses the moment function based on the finite-$n$ link probabilities and the instrument based on the limiting link probabilities. Case (iii) uses the moment function and instrument based on the limiting link probabilities. The finite-$n$ link probabilities are computed from simulations by either solving quadratic integer programming (for $n \leq 100$) or applying equation (3.5) (for $n > 100$).
useful approximation for reducing the computational burden in large networks.

**O.E Additional Lemmas for Section 4**

**Notation** For any random variable $Z \in \mathbb{R}^n$, $\|Z\|_{\psi|X}$ denotes the conditional Orlicz norm of $Z$ given $X$ for a non-decreasing, convex function $\psi$ with $\psi(0) = 0$; that is, $\|Z\|_{\psi|X} = \inf\{C > 0 : \mathbb{E}[\psi(|Z|/C)|X] \leq 1\}$. Conditional Orlicz norms satisfy the triangle inequality. For conditional Orlicz norms $\|\cdot\|_{\psi|X}$ with the functions $\psi_p(z) = e^{z^p} - 1$ for $p \geq 1$, the bound $z^p \leq \psi_p(z)$ for all $z \geq 0$ implies that $(\mathbb{E}[|Z|^p|X])^{1/p} \leq \|Z\|_{\psi_p|X}$ for all $p \geq 1$.

Moreover, $(\mathbb{E}[|Z|^p|X])^{1/p} \leq p!\|Z\|_{\psi_1|X}$ for all $p \geq 1$.

**O.E.1 Lemmas for the Consistency of $\hat{\theta}_n$ and $\hat{p}_n$**

**Lemma O.E.1** (Uniform LLN of sample moments). Suppose that Assumptions 1–4 and 5(iii) are satisfied. Given $X$, $\sup_{(\theta, p) \in \Theta \times \mathcal{P}} \| \hat{m}_n(\theta, p) - m_n(\theta, p) \| = o_p(1)$.

**Proof.** By the definitions of $\hat{m}_n$ and $m_n$,

$$
\hat{m}_n(\theta, p) - m_n(\theta, p) \\
= \frac{1}{n(n-1)} \sum_i \sum_{j \neq i} (\hat{q}_{n,ij}(G_{n,ij} - P_{n,ij}(\theta, p)) - q_{n,ij}(\mathbb{E}[G_{n,ij}|X] - P_{n,ij}(\theta, p))) \\
= \frac{1}{n(n-1)} \sum_i \sum_{j \neq i} (\hat{q}_{n,ij} - q_{n,ij})(G_{n,ij} - P_{n,ij}(\theta, p)) \\
+ \frac{1}{n(n-1)} \sum_i \sum_{j \neq i} q_{n,ij}(G_{n,ij} - \mathbb{E}[G_{n,ij}|X]). \quad (O.E.1)
$$

The second to last term in (O.E.1) is $o_p(1)$ uniformly in $(\theta, p) \in \Theta \times \mathcal{P}$, because it is uniformly bounded by $(n(n-1))^{-1} \sum_i \sum_{j \neq i} \sup_{(\theta, p) \in \Theta \times \mathcal{P}} \| (\hat{q}_{n,ij} - q_{n,ij})(G_{n,ij} - P_{n,ij}(\theta, p)) \| \leq \max_{1 \leq i, j \leq n} \| \hat{q}_{n,ij} - q_{n,ij} \| = o_p(1)$ (Assumption 5(iii)). Further, define

$$
Y_{ni} = \frac{1}{n-1} \sum_{j \neq i} q_{n,ij}(G_{n,ij} - \mathbb{E}[G_{n,ij}|X]),
$$

so the last term in (O.E.1) is given by $n^{-1} \sum_i Y_{ni}$. This term does not depend on $\theta$ or $p$. We show that it is $o_p(1)$ following a pointwise LLN. Given $X$, $Y_{ni}$, $i = 1, \ldots, n$, are independent.\(^{60}\)

\(^{60}\)This is true because $\mathbb{E}[\psi_p(|Z|^p/\|Z\|_{\psi_p|X})|X] \leq 1 \leq \mathbb{E}[\psi_p(|Z|^p/(\mathbb{E}[|Z|^p|X])^{1/p})|X]$, where the second inequality follows from $z^p \leq \psi_p(z)$.
with mean 0, so $\mathbb{E}[\|n^{-1} \sum_i Y_{ni}\|^2|X] = n^{-2} \sum_i \mathbb{E}[\|Y_{ni}\|^2|X]$. For each $i$, 

$$
\mathbb{E}[\|Y_{ni}\|^2|X] = \frac{1}{(n-1)^2} \sum_{j \neq i} q'_{n,i,j} \mathbb{E}[(G_{n,i,j} - \mathbb{E}[G_{n,i,j}|X])^2|X]q_{n,i,j}
+ \frac{1}{(n-1)^2} \sum_{j \neq i} \sum_{k \neq i,j} q'_{n,i,j} \mathbb{E}[(G_{n,i,j} - \mathbb{E}[G_{n,i,j}|X])(G_{n,ik} - \mathbb{E}[G_{n,ik}|X])|X]q_{n,ik}.
$$

Because both $\mathbb{E}[(G_{n,i,j} - \mathbb{E}[G_{n,i,j}|X])^2|X]$ and $\mathbb{E}[(G_{n,i,j} - \mathbb{E}[G_{n,i,j}|X])(G_{n,ik} - \mathbb{E}[G_{n,ik}|X])|X]$ are bounded by 1, we can bound $\mathbb{E}[\|n^{-1} \sum_i Y_{ni}\|^2|X]$ by $\frac{1}{n^2(n-1)^2} (n(n-1) \max_{1 \leq i,j \leq n} \|q_{n,ij}\|^2 + n(n-1)(n-2) \max_{1 \leq i,j,k \leq n} \|q_{n,ij}\| \|q_{n,ik}\|) = o(1)$, where the last equality holds by Assumption 5(iii). By Markov’s inequality, we conclude that $n^{-1} \sum_i Y_{ni} = o_p(1)$ and hence $\sup_{(\theta,p) \in \Theta \times \mathcal{P}} \|\bar{m}_n(\theta,p) - m_n(\theta,p)\| = o_p(1)$. 

**Lemma O.E.2** (Continuity of CCP). Suppose that Assumptions 1-4 are satisfied. Given $X$, the conditional choice probability $P_{n,ij}(\theta,p)$ is continuous in $\theta$ and $p$.

**Proof.** Theorem 3.1 shows that 

$$
P_{n,ij}(\theta,p) = \int 1 \left\{ U_{n,ij}(\theta,p) + \frac{n-1}{n-2} Z_j V_{ni}(\theta,p) \omega_{ni}(\epsilon_i,\theta,p) \geq \epsilon_{ij} \right\} f_{\epsilon_i}(\epsilon_i;\theta) d\epsilon_i,
$$

where $f_{\epsilon_i}$ represents the density of $\epsilon_i$. By equations (2.4)-(2.6) and (3.1) and Assumption 1(i), $U_{n,ij}(\theta,p)$, $V_{ni}(\theta,p)$, and $f_{\epsilon_i}(\epsilon_i;\theta)$ are continuous in $\theta$ and $p$. The challenge is that $\omega_{ni}(\epsilon_i,\theta,p)$ is a function of $\epsilon_i$ and it depends on $\theta$ and $p$. To establish the continuity of $P_{n,ij}(\theta,p)$, we need to investigate how $\omega_{ni}(\epsilon_i,\theta,p)$ varies with $\theta$ and $p$.

By Lemma A.1, $\omega_{ni}(\epsilon_i,\theta,p)$ satisfies the first-order condition 

$$
V_{ni}(\theta,p)\omega_{ni}(\epsilon_i,\theta,p) = \frac{1}{n-1} \sum_{k \neq i} G_{n,ik}(\epsilon_i,\theta,p)V_{ni}(\theta,p)Z_k, \ a.s.,
$$

where $G_{ni}(\epsilon_i,\theta,p) = (G_{n,ik}(\epsilon_i,\theta,p), k \neq i) \in \mathcal{G}_i$ is the optimal decision in Theorem 3.1. Therefore, we can express $P_{n,ij}(\theta,p)$ equivalently as

$$
P_{n,ij}(\theta,p) = \int 1 \left\{ U_{n,ij}(\theta,p) + \frac{1}{n-2} \sum_{k \neq i} G_{n,ik}(\epsilon_i,\theta,p)Z_j V_{ni}(\theta,p)Z_k \geq \epsilon_{ij} \right\} f_{\epsilon_i}(\epsilon_i;\theta) d\epsilon_i.
$$

By Proposition O.C.1, the optimal decision $G_{ni}(\epsilon_i,\theta,p)$ takes the value $g_i \in \mathcal{G}_i$ if and only if $\epsilon_i \in \mathcal{E}_i(g_i;\theta,p)$, where the set $\mathcal{E}_i(g_i;\theta,p)$ is defined in (O.C.3) as

$$
\mathcal{E}_i(g_i;\theta,p) = \{ \epsilon_i \in \mathbb{R}^{n-1} : g_i \text{ satisfies conditions (O.C.1) and (O.C.2)} \}.
$$
For any \( g_i \in \mathcal{G}_i \), the equations in (O.C.1) define an orthant in \( \mathbb{R}^{n-1} \)

\[
\epsilon_{ij} \begin{cases} < U_{n,ij}^0(\theta, p) + \frac{1}{n-2} \sum_{k \neq i,j} g_{ik} Z'_j V_{ni}(\theta, p) Z_k, & \text{if } g_{ij} = 1, \\ \geq U_{n,ij}^0(\theta, p) + \frac{1}{n-2} \sum_{k \neq i,j} g_{ik} Z'_j V_{ni}(\theta, p) Z_k, & \text{if } g_{ij} = 0, \end{cases} \quad \text{(O.E.2)}
\]

for all \( j \neq i \), where \( U_{n,ij}^0(\theta, p) = \mathbb{E}[u_{ij}(\theta, p)|X] \). Both \( U_{n,ij}^0(\theta, p) \) and \( V_{ni}(\theta, p) \) are continuous in \( \theta \) and \( p \), so the boundary of the orthant is continuous in \( \theta \) and \( p \). Moreover, the inequality in (O.C.2) defines a half-space in \( \mathbb{R}^{n-1} \) given by

\[
\frac{1}{n-1} \sum_{j \neq i} (g_{ij} - \tilde{g}_{ij}) \epsilon_{ij} \leq \max_{\tilde{g}_i \in \tilde{\mathcal{G}}_i} \frac{1}{n-1} \sum_{j \neq i} \left( (g_{ij} - \tilde{g}_{ij}) U_{n,ij}^0(\theta, p) + \frac{1}{2(n-2)} \sum_{k \neq i,j} (g_{ij} g_{ik} - \tilde{g}_{ij} \tilde{g}_{ik}) Z'_j V_{ni}(\theta, p) Z_k \right) \quad \text{(O.E.3)}
\]

Each function inside the maximization on the right-hand side of (O.E.3) is continuous in \( \theta \) and \( p \). While the set of solutions to (O.E.2) for a given \( \epsilon_i \) can be discontinuous in \( \theta \) and \( p \) (e.g., some link choices in an optimal \( g_i \) may switch from 0 to 1 or the opposite as \( \theta \) or \( p \) changes), this event occurs with probability zero because \( \epsilon_i \) follows a continuous distribution under Assumption 1(i). Because max is a continuous operation, with probability one the right-hand side of (O.E.3) is continuous in \( \theta \) and \( p \) and hence the half-space defined by (O.E.3) has a boundary that is continuous in \( \theta \) and \( p \).

For any \( g_i \in \mathcal{G}_i \), the set \( \mathcal{E}_i(g_i, \theta, p) \) is the intersection of the orthant in (O.E.2) and the half-space defined by (O.E.3). Because continuity is preserved under max and min operations, if two sets have boundaries that are continuous in \( \theta \) and \( p \), their intersection must also have a boundary that is continuous in \( \theta \) and \( p \). Therefore, the set \( \mathcal{E}_i(g_i, \theta, p) \) has a boundary that is continuous in \( \theta \) and \( p \) with probability one.

Partitioning the space of \( \epsilon_i \) into a collection of the sets \( \mathcal{E}_i(g_i, \theta, p) \) for all \( g_i \in \mathcal{G}_i \), we can write \( P_{n,ij}(\theta, p) \) as

\[
P_{n,ij}(\theta, p) = \sum_{g_i \in \mathcal{G}_i} \int_{\mathcal{E}_i(g_i, \theta, p)} \left \{ U_{n,ij}(\theta, p) + \frac{1}{n-2} \sum_{k \neq i} g_{ik} Z'_j V_{ni}(\theta, p) Z_k \geq \epsilon_{ij} \right \} f_{\epsilon_i}(\epsilon_i; \theta) d\epsilon_i.
\]

Because each set \( \mathcal{E}_i(g_i, \theta, p) \) has a boundary that is continuous in \( \theta \) and \( p \) with probability one, each integral in the summation is continuous in \( \theta \) and \( p \). Therefore, \( P_{n,ij}(\theta, p) \) is continuous in \( \theta \) and \( p \). \( \square \)
O.E.2 Lemmas on the Asymptotic Properties of $\omega_{ni}(\epsilon_i)$

In this section, we establish in a few lemmas the asymptotic properties of $\omega_{ni}(\epsilon_i)$ that are needed to prove Theorem 4.2. We show that $V_{ni}\omega_{ni}(\epsilon_i)$ is consistent for $V_{ni}\omega_{ni}^*$ (Lemma O.E.3). Moreover, $V_{ni}\omega_{ni}(\epsilon_i)$ has an asymptotically linear representation (Lemma O.E.5) and satisfies certain uniformity properties (Lemma O.E.6). Additional results that are needed to prove these lemmas are given in Lemmas O.E.4 and O.E.7.

We make the following assumptions on the auxiliary variable $\omega$. All the random quantities are evaluated at $(\theta_0,p_n)$, which is suppressed for simplicity.

**Assumption O.E.1.** (i) The auxiliary variable $\omega$ is in a compact set $\Omega \subseteq \mathbb{R}^T$, which contains a compact neighborhood of 0. (ii) For $n$ sufficiently large, there is a $\omega_{ni}^* \in \Omega$ such that $\Pi_{ni}(\omega_{ni}^*) > \Pi_{ni}(\omega)$ for all $\omega \in \Omega$ with $V_{ni}(\omega - \omega_{ni}^*) \neq 0$, $1 \leq i \leq n$. (iii) For all $n$ and $1 \leq i \leq n$, the matrix $H_{ni}^\omega(\omega) = \frac{1}{n-2} \sum_{j \neq i} f_e(U_{n,ij} + \frac{n-1}{n-2}Z_jV_{ni}\omega)V_{ni}Z_jZ_j' - IT$ is nonsingular at $\omega_{ni}^*$, where $IT$ is the $T \times T$ identity matrix.

Assumptions O.E.1(i) and (iii) are standard regularity conditions. Note that by the first-order condition in Lemma A.1, we obtain that $\|V_{ni}\omega_{ni}(\epsilon_i)\| \leq \|V_{ni}\| < \infty$, so $V_{ni}\omega_{ni}(\epsilon_i)$ is bounded almost surely. Without loss of generality, we can assume that $\omega$ lies in a compact set $\Omega \subseteq \mathbb{R}^T$ as in Assumption O.E.1(i). To derive a sufficient condition for Assumption O.E.1(iii), define the matrix $D_{ni}(\omega) = \frac{1}{n-2} \sum_{j \neq i} f_e(U_{n,ij} + \frac{n-1}{n-2}Z_jV_{ni}\omega)V_{ni}Z_jZ_j'$, which is a $T \times T$ diagonal matrix, with the $t$th diagonal element given by $\frac{1}{n-2} \sum_{j \neq i} 1\{X_j = x_t\} f_{e,t}(\omega) > 0$, where $f_{e,t}(\omega) = f_e(U_{n,ij} + \frac{n-1}{n-2}Z_jV_{ni}\omega)$ for $X_j = x_t$. We can write $H_{ni}^\omega(\omega) = V_{ni}D_{ni}(\omega) - IT$. A sufficient condition for $H_{ni}^\omega(\omega_{ni}^*)$ being nonsingular is that $\|V_{ni}\|D_{ni}(\omega_{ni}^*) < 1$.

Assumption O.E.1(ii) is an identification condition. It requires that there is a unique value of $V_{ni}\omega$ such that $\Pi_{ni}^*(\omega)$ achieves its maximum. Note that $\Pi_{ni}^*(\omega)$ depends on $\omega$ only through the value of $V_{ni}\omega$: if $\omega$ and $\tilde{\omega}$ satisfy $V_{ni}(\omega - \tilde{\omega}) = 0$, then $\omega'V_{ni}\omega = \tilde{\omega}'V_{ni}\tilde{\omega}$ and thus $\Pi_{ni}^*(\omega) = \Pi_{ni}^*(\tilde{\omega})$. We assume that $\Pi_{ni}^*(\omega)$ achieves its maximum at a unique value of $V_{ni}\omega$ instead of a unique $\omega$ to account for the fact that $V_{ni}$ may be singular. The assumption becomes trivial if $V_{ni} = 0$. For $V_{ni} \neq 0$, we can show that a related condition $\|V_{ni}\|D_{ni}(\omega) < 1$ for all $\omega \in \Omega$ is sufficient for Assumption O.E.1(ii) to hold.\(^{61}\)

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\(^{61}\) The Taylor expansion of $\Pi_{ni}^*(\omega)$ around $\omega_{ni}^*$ contains a quadratic term given by $(\omega - \omega_{ni}^*)'(V_{ni}D_{ni}(\omega)V_{ni} - V_{ni}(\omega - \omega_{ni}^*))$, where $\tilde{\omega} \in \Omega$ lies between $\omega$ and $\omega_{ni}^*$. Assumption O.E.1(ii) is satisfied if this quadratic term is negative for all $\omega \in \Omega$ with $V_{ni}(\omega - \omega_{ni}^*) \neq 0$. Because $V_{ni}$ is positive semi-definite by Assumption 4, it has a square root $V_{ni}^{1/2}$. We can write $V_{ni}D_{ni}(\omega)V_{ni} - V_{ni} = V_{ni}^{1/2}(V_{ni}^{1/2}D_{ni}(\omega)V_{ni}^{1/2} - IT)V_{ni}^{1/2}$. For any vector $x \in \mathbb{R}^T$, by Cauchy-Schwarz inequality we have $x'V_{ni}^{1/2}D_{ni}(\omega)V_{ni}^{1/2}x \leq \|D_{ni}(\omega)\|\|V_{ni}^{1/2}x\|^2 \leq \|D_{ni}(\omega)\|\|V_{ni}\|\|x\|^2$. Therefore, the quadratic term is less than $\|D_{ni}(\omega)\|\|V_{ni}\| - 1\|V_{ni}^{1/2}(\omega - \omega_{ni}^*)\|^2 < 0$ for $\omega \in \Omega$ with $V_{ni}(\omega - \omega_{ni}^*) \neq 0$ (which implies $V_{ni}^{1/2}(\omega - \omega_{ni}^*) \neq 0$), and thus Assumption O.E.1(ii) is satisfied.

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Lemma O.E.3 (Consistency of $\omega_n(\epsilon_i)$ for $\omega_n^*$). Suppose that Assumptions 1–4 and O.E.1(i)–(ii) are satisfied. Given $X$, we have $V_n(\omega_n(\epsilon_i) - \omega_n^*) = o_p(1)$, $1 \leq i \leq n$.

Proof. If $V_n = 0$, then $V_n(\omega_n(\epsilon_i) - \omega_n^*) = 0$ and the consistency holds trivially. Suppose $V_n \neq 0$. Fix $\delta > 0$. Let $\mathcal{B}_n(\omega_n^*, \delta) = \{\omega \in \Omega : \|V_n(\omega - \omega_n^*)\| < \delta\}$ be a subset of $\Omega$ containing $\omega_n^*$. We have

$$\Pr(\|V_n(\omega_n(\epsilon_i) - \omega_n^*)\| < \delta | X) \geq \Pr\left(\Pi_n^*(\omega_n(\epsilon_i)) > \sup_{\omega \in \Omega \setminus \mathcal{B}_n(\omega_n^*, \delta)} \Pi_n^*(\omega) \bigg| X\right). \quad \text{(O.E.4)}$$

By the optimality of $\omega_n(\epsilon_i)$, we have

$$\Pi_n^*(\omega_n(\epsilon_i)) - \Pi_n^*(\omega_n^*) = \Pi_n^*(\omega_n(\epsilon_i)) - \Pi_n^*(\omega_n^*) + \Pi_n^*(\omega_n^*) - \Pi_n^*(\omega_n^*) \geq \Pi_n^*(\omega_n(\epsilon_i)) - \Pi_n^*(\omega_n^*) + \Pi_n^*(\omega_n^*) - \Pi_n^*(\omega_n^*) \geq -2 \sup_{\omega \in \Omega} |\Pi_n(\omega, \epsilon_i) - \Pi_n^*(\omega)|.$$

The uniform LLN in Lemma O.E.4 implies that $\sup_{\omega \in \Omega} |\Pi_n(\omega, \epsilon_i) - \Pi_n(\omega)| = o_p(1)$. From the compactness of $\Omega \setminus \mathcal{B}_n(\omega_n^*, \delta)$, the continuity of $\Pi_n^*(\omega)$, and the identification condition in Assumption O.E.1(ii), we derive that for $n$ sufficiently large, $\sup_{\omega \in \Omega \setminus \mathcal{B}_n(\omega_n^*, \delta)} \Pi_n^*(\omega) = \Pi_n^*(\omega_\ast) < \Pi_n^*(\omega_n^*)$ for some $\omega_\ast \in \Omega \setminus \mathcal{B}_n(\omega_n^*, \delta)$. Combining the results, we can see that the right-hand size of (O.E.4) goes to 1.

Lemma O.E.4 (Uniform LLN for $\Pi_n$). Suppose that Assumptions 1–4 and O.E.1(i) are satisfied. Given $X$, we have $\sup_{\omega \in \Omega} |\Pi_n(\omega, \epsilon_i) - \Pi_n^*(\omega)| = o_p(1)$, $1 \leq i \leq n$.

Proof. Defining $\pi_{n,ij}(\omega, \epsilon_i) = [U_{n,ij} + \frac{n-1}{2} Z_j' V_n \omega - \epsilon_{ij}]_+$, we can write

$$\Pi_n(\omega, \epsilon_i) - \Pi_n^*(\omega) = \frac{1}{n-1} \sum_{j \neq i} (\pi_{n,ij}(\omega, \epsilon_i) - \mathbb{E}[\pi_{n,ij}(\omega, \epsilon_i) | X]).$$

By Assumption O.E.1(i), we have $|Z_j' V_n \omega| \leq \sup_{\omega \in \Omega} \|V_n \omega\| \leq M < \infty$. Hence, for all $\omega \in \Omega$, $\pi_{n,ij}(\omega, \epsilon_i)^2 \leq (U_{n,ij} + \frac{n-1}{2} M - \epsilon_{ij})^2$, with $\mathbb{E}[(U_{n,ij} + \frac{n-1}{2} M - \epsilon_{ij})^2 | X] < \infty$. Also $\pi_{n,ij}(\omega, \epsilon_i)$ is continuous in $\omega$ on a compact set $\Omega$. Therefore the conditions of the uniform LLN are satisfied (Jennrich, 1969) and the lemma is proved.

Lemma O.E.5 (Asymptotically linear representation of $\omega_n(\epsilon_i)$). Suppose that Assumptions 1–4 and O.E.1 are satisfied. Given $X$, $V_n(\omega_n(\epsilon_i))$ has an asymptotically linear representation

$$V_n(\omega_n(\epsilon_i) - \omega_n^*) = \frac{1}{n-1} \sum_{j \neq i} \phi_{n,ij}^\omega(\omega_n^*, \epsilon_{ij}) + r_{n,ij}^\omega(\epsilon_i). \quad \text{(O.E.5)}$$

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with \( r_n^\omega(\epsilon_i) = o_p(n^{-1/2}) \). In the expression, \( \phi_{n,ij}^\omega(\omega_{ni}^*, \epsilon_{ij}) = -H_{ni}^\omega(\omega_{ni}^*)^{-1}\phi_{n,ij}(\omega_{ni}^*, \epsilon_{ij}) \), where \( \phi_{n,ij}(\omega, \epsilon_{ij}) = 1\{U_{n,ij} + \frac{n-1}{n-2}Z_j'V_n\omega \geq \epsilon_{ij}\}V_{ni}Z_j - V_{ni}\omega \) and \( H_{ni}^\omega(\omega_{ni}^*) = \frac{1}{n-2}\sum_{j\neq i} f_e(U_{n,ij} + \frac{n-1}{n-2}Z_j'V_n\omega_{ni})V_{ni}Z_jZ_j' - I_T \).

**Proof.** By Lemma A.1, \( \omega_{ni}(\epsilon_i) \) satisfies the first-order condition

\[
\frac{1}{n-1}\sum_{j\neq i} \phi_{n,ij}^\omega(\omega, \epsilon_{ij}) = 0, \text{ a.s.} \tag{O.E.6}
\]

Under Assumption O.E.1(ii), \( \Pi_{ni}^*(\omega) \) is maximized at \( \omega_{ni}^* \), so \( \omega_{ni}^* \) satisfies the population counterpart of the first-order condition

\[
\frac{1}{n-1}\sum_{j\neq i} E[\phi_{n,ij}^\omega(\omega, \epsilon_{ij})|X] = 0, \tag{O.E.7}
\]

where \( E[\phi_{n,ij}^\omega(\omega, \epsilon_{ij})|X] = F_e(U_{n,ij} + \frac{n-1}{n-2}Z_j'V_n\omega)V_{ni}Z_j - V_{ni}\omega \). Let \( \Gamma_{ni}(\omega, \epsilon_i) \) denote the left-hand side of equation (O.E.6) and \( \Gamma_{ni}^*(\omega) \) denote the left-hand side of equation (O.E.7).

View \( \Gamma_{ni}^*(\omega) \) as a function of \( V_{ni}\omega \). Expanding it with respect to \( V_{ni}\omega \) at \( V_{ni}\omega_{ni}^* \) yields

\[
\Gamma_{ni}^*(\omega_{ni}(\epsilon_i)) = H_{ni}^\omega(\omega_{ni}^*)V_{ni}(\omega_{ni}(\epsilon_i) - \omega_{ni}^*) + O(||V_{ni}(\omega_{ni}(\epsilon_i) - \omega_{ni}^*)||^2). \tag{O.E.8}
\]

For any \( \omega \in \Omega \), we have

\[
||H_{ni}^\omega(\omega_{ni}^*)V_{ni}(\omega - \omega_{ni}^*)||^2 = (V_{ni}(\omega - \omega_{ni}^*))'H_{ni}^\omega(\omega_{ni}^*)'H_{ni}^\omega(\omega_{ni}^*)V_{ni}(\omega - \omega_{ni}^*) \geq \epsilon_n^2||V_{ni}(\omega - \omega_{ni}^*)||^2, \]

where \( \epsilon_n^2 = \lambda_{min}(H_{ni}^\omega(\omega_{ni}^*)'H_{ni}^\omega(\omega_{ni}^*)) > 0 \) is the smallest eigenvalue of the matrix \( H_{ni}^\omega(\omega_{ni}^*)'H_{ni}^\omega(\omega_{ni}^*) \). Combining this with equation (O.E.8) and the consistency of \( V_{ni}\omega_{ni}(\epsilon_i) \) we obtain

\[
||\Gamma_{ni}^*(\omega_{ni}(\epsilon_i))|| \geq ||V_{ni}(\omega_{ni}(\epsilon_i) - \omega_{ni}^*)||(\epsilon_n + o_p(1)). \tag{O.E.9}
\]

To derive the convergence rate of \( V_{ni}(\omega_{ni}(\epsilon_i) - \omega_{ni}^*) \), it is sufficient to derive the rate of \( \Gamma_{ni}^*(\omega_{ni}(\epsilon_i)) \).

By equations (O.E.6) and (O.E.7), we can write

\[
\Gamma_{ni}^*(\omega_{ni}(\epsilon_i)) = -\Gamma_{ni}(\omega_{ni}^*, \epsilon_i) - (\Gamma_{ni}(\omega_{ni}(\epsilon_i), \epsilon_i) - \Gamma_{ni}^*(\omega_{ni}(\epsilon_i))) - (\Gamma_{ni}(\omega_{ni}^*, \epsilon_i) - \Gamma_{ni}^*(\omega_{ni}(\epsilon_i))), \text{ a.s.} \tag{O.E.10}
\]

Note that \( \Gamma_{ni}(\omega, \epsilon_i) - \Gamma_{ni}^*(\omega) = \frac{1}{n-1}\sum_{j\neq i}(\phi_{n,ij}^\gamma(\omega, \epsilon_{ij}) - E[\phi_{n,ij}^\gamma(\omega, \epsilon_{ij})|X]), \) where \( \phi_{n,ij}^\gamma(\omega, \epsilon_{ij}) = 1\{U_{n,ij} + \frac{n-1}{n-2}Z_j'V_n\omega \geq \epsilon_{ij}\}V_{ni}Z_j \). Define the empirical process

\[
\mathbb{G}_n \phi_{n,ij}^\gamma(\omega, \epsilon_{ij}) = \frac{1}{\sqrt{n-1}}\sum_{j\neq i}(\phi_{n,ij}^\gamma(\omega, \epsilon_{ij}) - E[\phi_{n,ij}^\gamma(\omega, \epsilon_{ij})|X]), \omega \in \Omega. \tag{O.E.11}
\]
We can represent the last term in (O.E.10) as $(n-1)^{-1/2}(G_n, \phi_{ni}^*(\omega_{ni}(\epsilon_i), \epsilon_i) - G_n, \phi_{ni}^*(\omega_{ni}^*, \epsilon_i))$. By Lemma O.E.7(i), this term satisfies

$$\Gamma_{ni}(\omega_{ni}(\epsilon_i), \epsilon_i) - \Gamma_{ni}^*(\omega_{ni}(\epsilon_i)) - (\Gamma_{ni}(\omega_{ni}^*, \epsilon_i) - \Gamma_{ni}^*(\omega_{ni}^*)) = o_p(n^{-1/2}).$$  
(O.E.12)

and is thus negligible.

The first term on the right-hand side of (O.E.10) $\Gamma_{ni}(\omega_{ni}^*, \epsilon_i)$ is the leading term. To derive its rate of convergence, define the random variable $Y_{n,ij} \in \mathbb{R}^T$

$$Y_{n,ij} = \frac{1}{\sqrt{n-1}}(\phi_{n,ij}^*(\omega_{ni}^*, \epsilon_{ij}) - \mathbb{E}[\phi_{n,ij}^*(\omega_{ni}^*, \epsilon_{ij})|X]).$$

Then $\Gamma_{ni}(\omega_{ni}^*, \epsilon_i) = \frac{1}{\sqrt{n-1}} \sum_{j \neq i}^n Y_{n,ij}$ Note that \{Y_{n,ij}, j \neq i\} is a triangular array. We apply the Lindeberg-Feller CLT to show that $\Gamma_{ni}(\omega_{ni}^*, \epsilon_i) = O_p(n^{-1/2})$. By the Cramer-Wold device it suffices to show that $a' \sum_{j \neq i} Y_{n,ij}$ satisfies the Lindeberg condition for any $T \times 1$ vector of constants $a \in \mathbb{R}^T$. The Lindeberg condition is that for any $\xi > 0$

$$\lim_{n \to \infty} \sum_{j \neq i} \mathbb{E} \left[ \frac{(a'Y_{n,ij})^2}{a'S_{n,i}^2} \mathbf{1}\{a'Y_{n,ij} \geq \xi \sqrt{a'S_{n,i}^2} \} \mid X \right] = 0,$$

(O.E.13)

where

$$S_{n,i}^2 = \sum_{j \neq i}^n \mathbb{E}[Y_{n,ij}Y_{n,ij}'|X]$$

$$= \frac{1}{n-1} \sum_{j \neq i} F_\epsilon \left( U_{n,ij} + \frac{n-1}{n-2}Z_jV_{ni}\omega_{ni}^* \right) \left( 1 - F_\epsilon \left( U_{n,ij} + \frac{n-1}{n-2}Z_j'V_{ni}\omega_{ni}^* \right) \right) V_{ni}Z_jZ_j'V_{ni}.$$ 

Observe that the sum in (O.E.13) is bounded by $\mathbb{E}[(a'S_{n,i}^2)^{-1} \sum_{j \neq i} (a'Y_{n,ij})^2 \mathbf{1}\{a'Y_{n,ij} \geq \xi \sqrt{a'S_{n,i}^2} \} \mid X]$. The random variable $(a'S_{n,i}^2)^{-1} \sum_{j \neq i} (a'Y_{n,ij})^2$ has a finite expectation and is therefore $O_p(1)$. Hence, if

$$\frac{\max_{j \neq i} |a'Y_{n,ij}|}{\sqrt{a'S_{n,i}^2}} = o_p(1),$$  
(O.E.14)

then $(a'S_{n,i}^2)^{-1} \sum_{j \neq i} (a'Y_{n,ij})^2 \mathbf{1}\{a'Y_{n,ij} \geq \xi \sqrt{a'S_{n,i}^2} \} = O_p(1) = o_p(1)$. This random variable is bounded by $(a'S_{n,i}^2)^{-1} \sum_{j \neq i} (a'Y_{n,ij})^2$ which has a finite expectation. We conclude that by dominated convergence the Lindeberg condition is satisfied if (O.E.14) holds.

By Markov’s inequality, equation (O.E.14) holds if $\mathbb{E}[\max_{j \neq i} (a'Y_{n,ij})^2 | X] = o(1)$. By the maximal inequality in van der Vaart and Wellner (1996, Lemma 2.2.2), we have the bound

$$\mathbb{E}[\max_{j \neq i} (a'Y_{n,ij})^2 | X] \leq \| \max_{j \neq i} (a'Y_{n,ij})^2 \|_{\psi_1} \leq K \ln(n+1) \max_{j \neq i} \| (a'Y_{n,ij})^2 \|_{\psi_1} | X,$$

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$K < \infty$ is a constant. Note that the random variable $a'Y_{n,ij}$ is bounded by $\|a\|\|Y_{n,ij}\| \leq 2\|a\|\|V_{ni}\|/\sqrt{n-1} \leq M/\sqrt{n-1} < \infty$, so by Hoeffding’s inequality for bounded random variables (Boucheron et al., 2013, Theorem 2.8) $\Pr((a'Y_{n,ij})^2 \geq t|X) = \Pr(a'Y_{n,ij} \geq \sqrt{t}|X) + \Pr(-a'Y_{n,ij} \geq \sqrt{t}|X) \leq 2\exp(-\frac{n}{2M^2})$, and hence by van der Vaart and Wellner (1996, Lemma 2.2.1) we can bound $\|(a'Y_{n,ij})^2\|_{\psi_1|X} \leq 6M^2/(n-1)$. Combining these results yields $\mathbb{E}[\max_{j \neq i}(a'Y_{n,ij})^2|X] \leq 6M^2K\ln(n+1)/(n-1) = o(1)$, so the Lindeberg condition holds. We conclude that $\sum_{j \neq i}Y_{n,ij} = O_p(1)$ and thus $\Gamma_n(\omega^*_{ni}, \epsilon_i) = O_p(n^{-1/2})$.

Combining equations (O.E.9), (O.E.10) and (O.E.12), we obtain $\|V_{ni}(\omega_{ni}(\epsilon_i) - \omega^*_{ni})\|(c_n + o_p(1)) \leq O_p(n^{-1/2})$. This implies that

$$V_{ni}(\omega_{ni}(\epsilon_i) - \omega^*_{ni}) = O_p(n^{-1/2}),$$

that is, $V_{ni}\omega_{ni}(\epsilon_i)$ converges to $V_{ni}\omega^*_{ni}$ at the rate of $n^{-1/2}$. Combining equations (O.E.8), (O.E.10) and (O.E.12) we derive that $H_{ni}^{\omega}(\omega^*_{ni})V_{ni}(\omega_{ni}(\epsilon_i) - \omega^*_{ni}) = -\Gamma_n(\omega^*_{ni}, \epsilon_i) + o_p(n^{-1/2})$.

By Assumption O.E.1(iii), the matrix $H_{ni}^{\omega}(\omega^*_{ni})$ is invertible. Multiplying both sides by the inverse $H_{ni}^{\omega}(\omega^*_{ni})^{-1}$, we obtain (O.E.5).

**Lemma O.E.6 (Uniform Properties of $\omega_{ni}(\epsilon_i)$).** Suppose that Assumptions 1–4 and O.E.1 are satisfied. Given $X$, we have (i) $\max_{1 \leq i \leq n} \|V_{ni}(\omega_{ni}(\epsilon_i) - \omega^*_{ni})\|^2_{\psi_1|X} = o(n^{-1/2})$, and (ii) the remainder term $r_{ni}^{\omega}(\epsilon_i)$ in equation (O.E.5) satisfies $\max_{1 \leq i \leq n} \|r_{ni}^{\omega}(\epsilon_i)\|_{\psi_1|X} = o(n^{-1/2})$.

**Proof.** Part (i): By the maximal inequality in van der Vaart and Wellner (1996, Lemma 2.2.2) we can bound $\max_i \|V_{ni}(\omega_{ni}(\epsilon_i) - \omega^*_{ni})\|^2_{\psi_1|X} \leq K\ln(n+1) \max_i \|V_{ni}(\omega_{ni}(\epsilon_i) - \omega^*_{ni})\|^2_{\psi_1|X}$, where $K < \infty$ is a constant. For any random variable $Z \in \mathbb{R}$ and constant $C > 0$, we have $\mathbb{E}[\psi_1(Z^2/C^2)|X] = \mathbb{E}[\psi_2(|Z|/C)|X]$, where $\psi_1(z) = e^z - 1$ and $\psi_2(z) = e^{z^2} - 1$. This implies that $\|Z^2\|_{\psi_1|X} = \|Z\|_{\psi_2|X}$ and hence $\|V_{ni}(\omega_{ni}(\epsilon_i) - \omega^*_{ni})\|^2_{\psi_1|X} = \|V_{ni}(\omega_{ni}(\epsilon_i) - \omega^*_{ni})\|^2_{\psi_2|X}$. Combining these results, we can bound $\max_i \|V_{ni}(\omega_{ni}(\epsilon_i) - \omega^*_{ni})\|^2_{\psi_1|X}$ by $K\ln(n+1) \max_i \|V_{ni}(\omega_{ni}(\epsilon_i) - \omega^*_{ni})\|^2_{\psi_2|X}$.

To further bound $\|V_{ni}(\omega_{ni}(\epsilon_i) - \omega^*_{ni})\|_{\psi_2|X}$ uniformly in $i$, by comparing equations (O.E.8) and (O.E.10), we derive the remainder $r_{ni}^{\omega}(\epsilon_i)$ in equation (O.E.5) as

$$r_{ni}^{\omega}(\epsilon_i) = -H_{ni}^{\omega}(\omega^*_{ni})^{-1}(O(\|V_{ni}(\omega_{ni}(\epsilon_i) - \omega^*_{ni})\|^2) + \frac{1}{\sqrt{n-1}}(\mathcal{G}_n\phi_{ni}^{\gamma}(\omega_{ni}(\epsilon_i), \epsilon_i) - \mathcal{G}_n\phi_{ni}^{\gamma}(\omega^*_{ni}, \epsilon_i))).$$

(O.E.15)

Substituting $r_{ni}^{\omega}(\epsilon_i)$ in (O.E.5) with (O.E.15), we obtain

$$\|V_{ni}(\omega_{ni}(\epsilon_i) - \omega^*_{ni})(1 + o_p(1))\|_{\psi_2|X} \leq \left| \frac{1}{n-1} \sum_{j \neq i} \phi_{n,ij}^{\omega}(\omega^*_{ni}, \epsilon_{ij}) \right|_{\psi_2|X}$$

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where we have used the triangle inequality for the Orlicz norm and the boundedness of $H_{n}^{i}(\omega^{*}_{ni})^{-1}$. Note that $\|V_{ni}(\omega_{ni}(\epsilon_{i}) - \omega^{*}_{ni}(1 + o_{p}(1))\|_{\psi_{2}|X} = \|V_{ni}(\omega_{ni}(\epsilon_{i}) - \omega^{*}_{ni})\|_{\psi_{2}|X}(1 + o(1))$.\(^{62}\)

Consider the first term on the right-hand side of (O.E.16). Recall that $\phi_{ni,j,t}^{\omega}(\omega^{*}_{ni}, \epsilon_{ij}) \in \mathbb{R}^{T}$ is the influence function in (O.E.5). For $1 \leq t \leq T$, let $\phi_{ni,j,t}^{\omega}(\omega^{*}_{ni}, \epsilon_{ij})$ denote the $t$th component of $\phi_{ni,j}(\omega^{*}_{ni}, \epsilon_{ij})$. Write $\phi_{ni,j,t}^{\omega}(\omega^{*}_{ni}, \epsilon_{ij})$ and $\phi_{ni,j,t}^{\omega}(\omega^{*}_{ni}, \epsilon_{ij})$ as $\phi_{ni,j}^{\omega}$ and $\phi_{ni,j,t}^{\omega}$. Note that

$$
\left|\frac{1}{n-1} \sum_{j \neq i} \phi_{ni,j}^{\omega}(\omega^{*}_{ni}, \epsilon_{ij})\right| = \left(\sum_{t} \left(\frac{1}{n-1} \sum_{j \neq i} \phi_{ni,j,t}^{\omega}(\omega^{*}_{ni}, \epsilon_{ij})\right)^{2}\right)^{1/2} \leq \sum_{t} \left|\frac{1}{n-1} \sum_{j \neq i} \phi_{ni,j,t}^{\omega}(\omega^{*}_{ni}, \epsilon_{ij})\right|,
$$

for any $\kappa > 0$, we have

$$
\text{Pr}\left(\left|\frac{1}{n-1} \sum_{j \neq i} \phi_{ni,j}^{\omega}(\omega^{*}_{ni}, \epsilon_{ij})\right| > \kappa \right) \leq \sum_{t} \text{Pr}\left(\left|\frac{1}{n-1} \sum_{j \neq i} \phi_{ni,j,t}^{\omega}(\omega^{*}_{ni}, \epsilon_{ij})\right| > \kappa \right) \leq \sum_{t} \text{Pr}\left(\left|\frac{1}{n-1} \sum_{j \neq i} \phi_{ni,j,t}^{\omega}(\omega^{*}_{ni}, \epsilon_{ij})\right| > \kappa \right) \leq 2T\exp\left(-\frac{(n-1)^{2}}{2M^{2}T^{2}}\right).
$$

Therefore, by van der Vaart and Wellner (1996, Lemma 2.2.1), we get the bound

$$
\frac{1}{n-1} \sum_{j \neq i} \phi_{ni,j}^{\omega}(\omega^{*}_{ni}, \epsilon_{ij}) \leq \sqrt{2(2T + 1)TM/(n-1)},
$$

which is constant across $i$.

By (O.E.19) in Lemma O.E.7, $\|G_{n}\phi_{ni}^{\gamma}(\omega_{ni}(\epsilon_{i}), \epsilon_{i}) - G_{n}\phi_{ni}^{\gamma}(\omega^{*}_{ni}, \epsilon_{i})\|_{\psi_{1}|X} = o(1)$. Following the proof for (O.E.19) and applying Theorems 2.14.5 and 2.14.1 in van der Vaart and Wellner (1996) for $p = 2$, we can derive similarly $\|G_{n}\phi_{ni}^{\gamma}(\omega_{ni}(\epsilon_{i}), \epsilon_{i}) - G_{n}\phi_{ni}^{\gamma}(\omega^{*}_{ni}, \epsilon_{i})\|_{\psi_{1}|X} = o(1)$, so the second term on the right-hand side of equation (O.E.16) is $o(1)/\sqrt{n - 1}$. Combining these results yields $\max_{i} \|V_{ni}(\omega_{ni}(\epsilon_{i}) - \omega^{*}_{ni})\|_{\psi_{1}|X} \leq K\ln(n + 1)((\sqrt{2(2T + 1)TM} + o(1))/\sqrt{n - 1})^{2} = o(n^{1/2})$. Part (i) is proved.

Part (ii): By (O.E.19) in Lemma O.E.7, $\max_{i} \|G_{n}\phi_{ni}^{\gamma}(\omega_{ni}(\epsilon_{i}), \epsilon_{i}) - G_{n}\phi_{ni}^{\gamma}(\omega^{*}_{ni}, \epsilon_{i})\|_{\psi_{1}|X} = o(1)$. Combining this with part (i) and using $\max_{i} H_{n}^{i}(\omega^{*}_{ni})^{-1} \leq M < \infty$, we derive that

$$
\sum_{i} \max_{i} \|r_{ni}^{\omega}(\epsilon_{i})\|_{\psi_{1}|X} \leq \max_{i} \|H_{n}^{i}(\omega^{*}_{ni})^{-1}\| \|O(\sum_{i} \|V_{ni}(\omega_{ni}(\epsilon_{i}) - \omega^{*}_{ni})\|_{\psi_{1}|X})\|

+ (n - 1)^{-1/2} \max_{i} \|G_{n}\phi_{ni}^{\gamma}(\omega_{ni}(\epsilon_{i}), \epsilon_{i}) - G_{n}\phi_{ni}^{\gamma}(\omega^{*}_{ni}, \epsilon_{i})\|_{\psi_{1}|X} = o(1).
$$

\hspace{1cm} \Box

Lemma O.E.7 (Stochastic Equicontinuity). Suppose that Assumptions 1–4 and O.E.1 are satisfied. Given $X$, $G_{n}\phi_{ni}^{\gamma}(\omega, \epsilon_{i})$ defined in (O.E.11) satisfies that (i) if $V_{ni}(\omega_{ni}(\epsilon_{i}) - \omega^{*}_{ni}) = \mathcal{O}(Z)$ and conditional Orlicz norm $\||Z|\|_{\psi|X}$, we have $\|Z_{\omega_{ni}}(1)\|_{\psi|X} = o(\|Z\|_{\psi|X})$. This is because for any sequence $\delta_{n} \downarrow 0$, if there were $M < \infty$ such that $\|Z\|_{\psi|X} \leq M(\|Z_{\omega_{ni}}(1)\|_{\psi|X} - \delta_{n})$ for $n$ sufficiently large, then since $\|Z_{\omega_{ni}}(1)\|_{\psi|X} = o(1)$, we have for sufficiently large $n$, $1 \leq E[\psi((Z_{\omega_{ni}}(1))/\|Z_{\omega_{ni}}(1)\|_{\psi|X} - \delta_{n})]|X] \leq E\[\psi(M|Z_{\omega_{ni}}(1)\|\|Z\|_{\psi|X})|X]\rightarrow 0$ by dominated convergence, a contradiction. Therefore, $\|Z_{\omega_{ni}}(1)\|_{\psi|X} = o(1)$ and hence $\|Z_{\omega_{ni}}(1)\|_{\psi|X} = o(\|Z\|_{\psi|X})$. Online Appendix 21
because by Assumption (\(\eta\)) norm of an envelope of
where
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where
suffices to show
Proof. Part (i): By the consistency of \(V_{ni}(\omega_{ni}(\epsilon_i) - \omega_{ni}^*)\), we can define \(h_{ni} = r_{ni}V_{ni}(\omega_{ni}(\epsilon_i) - \omega_{ni}^*)\) for some \(r_{ni} \to \infty\) such that \(h_{ni} \in \Omega\) if \(n\) is sufficiently large,\(^{63}\) because by Assumption O.E.1(i) \(\Omega\) contains a compact neighborhood of 0.

Recall that \(\phi_{ni,j}(\omega, \epsilon_{ij}) = 1\{U_{ni,ij} + \frac{n-1}{n-2}Z_j^i V_{ni}\omega \geq \epsilon_{ij}\}\) View it as a function of \(\omega^v = V_{ni}\omega \in \Omega\), and define \(\phi_{ni,j}(\omega^v, \epsilon_{ij}) = 1\{U_{ni,ij} + \frac{n-1}{n-2}Z_j^i \omega^v \geq \epsilon_{ij}\}\) \(V_{ni}\omega\). We can write \(G_n(\phi_{ni,j}(\omega^v, \epsilon_{ij}) = 1\{U_{ni,ij} + \frac{n-1}{n-2}Z_j^i \omega^v \geq \epsilon_{ij}\}\) \(V_{ni}\omega_j\). By Markov’s inequality and the change of variable, it suffices to show \(\mathbb{E}[\sup_{\omega^v, h \in \Omega} \|G_n(\phi_{ni,j}(\omega^v, r_{ni}^{-1}h, \epsilon_{ij}) - \phi_{ni,j}(\omega^v, \epsilon_{ij})\|X\} = o(1)\). For simplicity, we write \(G_n(\phi_{ni,j}(\omega^v + r_{ni}^{-1}h, \epsilon_{ij}) - \phi_{ni,j}(\omega^v, \epsilon_{ij})\) as \(\gamma_n(\omega^v, r_{ni}^{-1}h)\).

To show this, we need to prove that the empirical process \(\gamma_n(\omega^v, r_{ni}^{-1}h)\), indexed by \(\omega^v, h \in \Omega\), is stochastically equicontinuous. This is a triangular array with function \(\phi_{ni,j}\) that varies across \(j\), so most of the ready-to-use results for stochastic equicontinuity (Andrews, 1994) are not applicable. Instead, we apply the maximal inequalities in van der Vaart and Wellner (1996) to directly prove the stochastic equicontinuity.

For any \(\omega^v, \tilde{\omega}^v \in \Omega\), we can bound the function \(\phi_{ni,j}(\omega^v, \epsilon_{ij}) - \phi_{ni,j}(\tilde{\omega}^v, \epsilon_{ij})\) by \(\|\phi_{ni,j}(\omega^v, \epsilon_{ij}) - \phi_{ni,j}(\tilde{\omega}^v, \epsilon_{ij})\| \leq \|V_{ni}\||\{U_{ni,ij} + \frac{n-1}{n-2}Z_j^i \omega^v \geq \epsilon_{ij}\} - 1\{U_{ni,ij} + \frac{n-1}{n-2}Z_j^i \tilde{\omega}^v \geq \epsilon_{ij}\}| \leq \eta_{ni,j}(\omega^v, \tilde{\omega}^v, \epsilon_{ij})\), where \(\eta_{ni,j}(\omega^v, \tilde{\omega}^v, \epsilon_{ij}) = \|V_{ni}\|\) if \(\epsilon_{ij}\) lies between \(U_{ni,ij} + \frac{n-1}{n-2}Z_j^i \omega^v\) and \(U_{ni,ij} + \frac{n-1}{n-2}Z_j^i \tilde{\omega}^v\), and 0 otherwise. By Theorem 2.14.1 in van der Vaart and Wellner (1996) with \(p = 1\), we derive the maximal inequality

\[
\mathbb{E} \left[ \sup_{\omega^v, h \in \Omega} \|\gamma_n(\omega^v, r_{ni}^{-1}h)\| \right] \leq K \mathbb{E} \left[ J(1, F_{ni}^\epsilon) \sup_{\omega^v, h \in \Omega} \|\eta_{ni}(\omega^v + r_{ni}^{-1}h, \omega^v, \epsilon_{ij})\| n \right] \{X\},
\]

(O.E.17)

where \(K > 0\) is a constant, \(\eta_{ni}(\omega^v, r_{ni}^{-1}h, \omega^v, \epsilon_{ij})\) is the empirical \(L_2\) norm of \(\eta_{ni}\), and \(J(1, F_{ni}^\epsilon)\) is the uniform entropy integral of the set of arrays \(F_{ni}^\epsilon = \{(\phi_{ni,j}(\omega^v + r_{ni}^{-1}h, \epsilon_{ij}) - \phi_{ni,j}(\omega^v, \epsilon_{ij}), \tilde{\omega}^v, \epsilon_{ij}) : \omega^v, h \in \Omega\}\).\(^{64}\) That is,

\[
J(1, F_{ni}^\epsilon) = \int_0^1 \sup_{\alpha \in \mathbb{R}^{n-1}} \sqrt{\ln D(\alpha \cap \eta_{ni}(\omega^v) n \cap \alpha \cap F_{ni}(\epsilon), \|[\alpha \cap \eta_{ni}(\omega^v) n \cap \alpha \cap F_{ni}^\epsilon, 1\| n) d\xi,
\]

(O.E.18)

where \(\eta_{ni}(\epsilon) = \sup_{\omega^v, h \in \Omega} \eta_{ni}(\omega^v + r_{ni}^{-1}h, \omega^v, \epsilon_{ij})\) is an \((n-1) \times 1\) vector of envelope functions

\(^{63}\)This implies that \(r_{ni}\) diverges more slowly than \(V_{ni}(\omega_{ni}(\epsilon_i) - \omega_{ni}^*)\) converges to zero.

\(^{64}\)From the proof of Theorem 2.14.1 in van der Vaart and Wellner (1996) it follows that the empirical \(L_2\) norm of an envelope of \(F_{ni}(\epsilon)\) can be replaced by the supremum of the empirical \(L_2\) norm of the \(n-1\) bounding functions in \(\eta_{ni}(\omega^v + r_{ni}^{-1}h, \omega^v, \epsilon_{ij})\). Also the theorem holds for a triangular array with independent but non-identically distributed observations.
of $\mathcal{F}_{ni}(\epsilon_i)$, $\alpha$ is an $(n - 1) \times 1$ vector of nonnegative constants, $\alpha \odot \tilde{\eta}_{ni}(\epsilon_i)$ is the Hadamard product of $\alpha$ and $\tilde{\eta}_{ni}(\epsilon_i)$, $\alpha \odot \mathcal{F}_{ni}(\epsilon_i)$ is the set of Hadamard products of $\alpha$ and the functions in $\mathcal{F}_{ni}(\epsilon_i)$, $\|\|_n$ is the empirical $L_2$ norm defined earlier, and $D(\xi\|\alpha \odot \tilde{\eta}_{ni}(\epsilon_i)\|_n, \alpha \odot \mathcal{F}_{ni}(\epsilon_i), \|\cdot\|_n)$ is the packing number, that is, the maximum number of points in the set $\alpha \odot \mathcal{F}_{ni}(\epsilon_i)$ that are separated by the distance $\xi\|\alpha \odot \tilde{\eta}_{ni}(\epsilon_i)\|_n$ for the norm $\|\cdot\|_n$. The sup inside the integral is taken over all $(n - 1) \times 1$ vectors $\alpha$ of nonnegative constants.

To show that the uniform entropy integral $J(1, \mathcal{F}_{ni}(\epsilon_i))$ is finite, consider the function $g_{n,ij}^v(\omega^v, \epsilon_{ij}) = 1\{U_{n,ij} + \frac{n-1}{n} Z_j \omega^v \geq \epsilon_{ij}\}$. It is an indicator with the argument being a linear function of $\omega^v$. We can show that the set $\{g_{n,ij}^v(\omega^v, \epsilon_{ij}), j \neq i : \omega^v \in \Omega\}$ has a pseudo-dimension of at most $T$, so it is Euclidean (Pollard, 1990, Corollary 4.10). Note that $\phi_{n,ij}^v(\omega^v, \epsilon_{ij}) = g_{n,ij}^v(\omega^v, \epsilon_{ij}) V_{ni} Z_j$, and $V_{ni} Z_j$ is a $T \times 1$ vector that does not depend on $\omega^v$. From the stability results in Pollard (1990, Section 5), each component of the doubly indexed process $\{\phi_{n,ij}^v(\omega^v + r_{ni}^{-1} h, \epsilon_i) - \phi_{n,ij}^v(\omega^v, \epsilon_i), j \neq i : \omega^v, h \in \Omega\}$ is Euclidean. Therefore, the set $\mathcal{F}_{ni}(\epsilon_i)$ has a finite uniform entropy integral, that is, $J(1, \mathcal{F}_{ni}(\epsilon_i)) \leq \bar{J}$ uniformly in $\epsilon_i$ and $n$ for some $\bar{J} < \infty$.

Next, we bound the empirical $L_2$ norm term in (O.E.17). By Cauchy-Schwarz inequality inequality

$$\mathbb{E}[\sup_{\omega^v, h \in \Omega} \|\hat{\eta}_{ni}(\omega^v + r_{ni}^{-1} h, \omega^v, \epsilon_i)\|_n | X] \leq \left( \mathbb{E}[\sup_{\omega^v, h \in \Omega} \frac{1}{n-1} \sum_{j \neq i} \|\hat{\eta}_{n,ij}(\omega^v + r_{ni}^{-1} h, \omega^v, \epsilon_{ij})\|_n^2 | X] \right)^{1/2}.$$

Consider the empirical process $G_n \hat{\eta}_{n,ij}^2(\omega^v + r_{ni}^{-1} h, \omega^v, \epsilon_i) = \frac{1}{\sqrt{n-1}} \sum_{j \neq i} \{\hat{\eta}_{n,ij}(\omega^v + r_{ni}^{-1} h, \omega^v, \epsilon_{ij}) - \mathbb{E}[\hat{\eta}_{n,ij}^2(\omega^v + r_{ni}^{-1} h, \omega^v, \epsilon_{ij}) | X]\}$, indexed by $\omega^v, h \in \Omega$. Note that each $\hat{\eta}_{n,ij}^2$ is bounded by $\|V_{ni}\|^2 \leq \bar{\eta}^2 < \infty$. Similarly to equation (O.E.17), we apply Theorem 2.14.1 in van der Vaart and Wellner (1996) with $p = 1$ to get the upper bound

$$\mathbb{E} \left[ \sup_{\omega^v, h \in \Omega} \|G_n \hat{\eta}_{n,ij}^2(\omega^v + r_{ni}^{-1} h, \omega^v, \epsilon_i)\|_n | X \right] \leq K^v \mathbb{E}[J(1, \mathcal{F}_{ni}(\epsilon_i))] \|\bar{\eta}\|_n | X],$$

where $K^v < \infty$ is a constant, $\|\bar{\eta}\|_n = \left( \frac{1}{n-1} \sum_{j \neq i} \|\tilde{\eta}_i^4\|^{1/2} = \bar{\eta}^2$, and $J(1, \mathcal{F}_{ni}(\epsilon_i))$ is the uniform entropy integral of the set of arrays $\mathcal{F}_{ni}(\epsilon_i) = \{(\eta_{n,ij}(\omega^v + r_{ni}^{-1} h, \omega^v, \epsilon_{ij}), j \neq i) : \omega^v, h \in \Omega\}$. Similarly to the argument for the set $\mathcal{F}_{ni}(\epsilon_i)$, we can show that the set $\mathcal{F}_{ni}(\epsilon_i)$ has a finite

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65To see this, by the definition of pseudo-dimension, it suffices to show that for each index set $I = \{j_1, \ldots, j_{T+1}\} \in \{1, \ldots, n\}\{i\}$ and each point $c \in \mathbb{R}^{T+1}$, there is a subset $J \subseteq I$ such that no $\omega^v \in \Omega$ can satisfy the inequalities $g_{n,ij}^v(\omega^v, \epsilon_{ij}) > c_j$ for $j \in J$ and $c_j < c_j$ for $j \in I \setminus J$. If $c$ has a component $c_j$ that lies outside of $(0, 1)$, we can choose $J$ such that $j \in J$ if $c_j \geq 1$ and $j \in I \setminus J$ if $c_j \leq 0$ so no $\omega^v$ can satisfy the inequalities above. It thus suffices to consider $c$ with all the components in $(0, 1)$ and for such $c$ the inequalities reduce to $U_{n,ij} + \frac{n-1}{n} Z_j \omega^v \geq \epsilon_{ij}$ for $j \in J$ and $< 0$ for $j \in I \setminus J$. Since $Z_j \in \mathbb{R}^T$ for all $j$, there exists a non-zero vector $\tau = (\tau_1, \ldots, \tau_{T+1}) \in \mathbb{R}^{T+1}$ such that $\sum_{t=1}^{T+1} \tau_t Z_j \omega^v = 0$, so $\sum_{t=1}^{T+1} \tau_t \frac{n-1}{n} Z_j \omega^v = 0$ for all $\omega \in \Omega$. We may assume that $\tau_t > 0$ for at least one $t$. If $\sum_{t=1}^{T+1} \tau_t U_{n,ij} - \epsilon_{n,ij} \geq 0$, it is impossible to find a $\omega^v \in \Omega$ satisfying these inequalities for the choice $J = \{j \in I : \tau_t \leq 0\}$, because this would lead to the contradiction $\sum_{t=1}^{T+1} \tau_t U_{n,ij} - \epsilon_{n,ij} = \sum_{t=1}^{T+1} \tau_t U_{n,ij} - \epsilon_{n,ij} + \sum_{t=1}^{T+1} \tau_t \frac{n-1}{n} Z_j \omega^v = \sum_{t=1}^{T+1} \tau_t U_{n,ij} + \frac{n-1}{n} Z_j \omega^v - \epsilon_{n,ij} < 0$. If $\sum_{t=1}^{T+1} \tau_t U_{n,ij} - \epsilon_{n,ij} < 0$, we could choose $J = \{j \in I : \tau_t \geq 0\}$ to reach a similar contradiction.
uniform entropy integral $J(1, F_{n_i}^\eta(\epsilon_i)) \leq \tilde{J}^n < \infty$. From these results we obtain

$$
\mathbb{E} \left[ \sup_{\omega^v, h \in \Omega} \frac{1}{n - 1} \sum_{j \neq i} \eta_{n,ij}^2(\omega^v + r_{n_i}^{-1}h, \omega^v, \epsilon_{ij}) \right] X
\leq \frac{1}{\sqrt{n - 1}} \mathbb{E} \left[ \sup_{\omega^v, h \in \Omega} |G_n \eta_{n_i}^2(\omega^v + r_{n_i}^{-1}h, \omega^v, \epsilon_i)| \right] X \leq \frac{K^n \tilde{J}^n \eta^2}{\sqrt{n - 1}}.
$$

Moreover, by the mean-value theorem, for any $\omega^v, h \in \Omega$ and any $j \neq i$, we have

$$
\mathbb{E} \left[ \eta_{n,ij}^2(\omega^v + r_{n_i}^{-1}h, \omega^v, \epsilon_{ij}) \right] X
= \left| F_\epsilon(U_{n,ij} + \frac{n - 1}{n - 2} Z_j(\omega^v + r_{n_i}^{-1}h)) - F_\epsilon(U_{n,ij}) \right| \|V_n Z_j\|^2
= \frac{r_{n_i}^{-1} f_\epsilon(U_{n,ij} + \frac{n - 1}{n - 2} Z_j(\omega^v + t_{n,ij} r_{n_i}^{-1}h))} \|V_n Z_j\|^2,
$$

for some $t_{n,ij} \in [0, 1]$. By the boundedness of $f_\epsilon$ under Assumption 1(ii) and $\sup_{h \in \Omega} \|h\| < \infty$, there is a $M < \infty$ such that $\mathbb{E} \left[ \eta_{n,ij}^2(\omega^v + r_{n_i}^{-1}h, \omega^v, \epsilon_{ij}) \right] X \leq r_{n_i}^{-1} M$ for all $\omega^v, h \in \Omega$ and all $j$. From these results we derive $\mathbb{E} \left[ \sup_{\omega^v, h \in \Omega} \left\| G_{n_i}^\gamma(\omega^v + r_{n_i}^{-1}h) \right\| X \right] \leq (n - 1)^{-1/2} K^n \tilde{J}^n \eta^2 + r_{n_i}^{-1} M$.

Combining the results we conclude that $\mathbb{E} \left[ \sup_{\omega^v, h \in \Omega} \left\| G_{n_i}^\gamma(\omega^v + r_{n_i}^{-1}h) \right\| X \right]$ is bounded by $\delta K \tilde{J} ((n - 1)^{-1/2} K^n \tilde{J}^n \eta^2 + r_{n_i}^{-1} M)^{1/2} = o(1)$. Part (i) is proved.

Part (ii): Because $V_{n_i}(\omega_{n_i}(\epsilon_i) - \omega_{n_i}^*) = O_p(n^{-1/2})$, we can define $h_{n_i} = n^\kappa V_{n_i}(\omega_{n_i}(\epsilon_i) - \omega_{n_i}^*)$ for $0 < \kappa < 1/2$ such that $h_{n_i} \in \Omega$ if $n$ is sufficiently large. Recall from part (i) that by the change of variable $G_n(\phi_{n_i}^\gamma(\omega_{n_i}(\epsilon_i), \epsilon_i)) - G_n(\phi_{n_i}^\gamma(\omega_{n_i}(\epsilon_i), \epsilon_i)) = G_n(\phi_{n_i}^\gamma(\omega_{n_i}(\epsilon_i), \epsilon_i) - \phi_{n_i}^\gamma(\omega_{n_i}(\epsilon_i), \epsilon_i))$. By Markov's inequality, it suffices to show that $\mathbb{E} \left[ \sup_{\omega^v, h \in \Omega} \left\| G_n(\phi_{n_i}^\gamma(\omega^v + n^{-\kappa}h, \epsilon_i) - \phi_{n_i}^\gamma(\omega^v, \epsilon_i)) \right\| X \right] = o(1)$. For simplicity, we write $G_n(\phi_{n_i}^\gamma(\omega^v + n^{-\kappa}h, \epsilon_i) - \phi_{n_i}^\gamma(\omega^v, \epsilon_i))$ as $G_{n_i}(\omega^v, n^{-\kappa}h)$.

By the maximal inequality in van der Vaart and Wellner (1996, Lemma 2.2.2), we derive the bound $\mathbb{E} \left[ \sup_{\omega^v, h \in \Omega} \left\| G_{n_i}^\gamma(\omega^v, n^{-\kappa}h) \right\| X \right] \leq \max_i \sup_{\omega^v, h \in \Omega} \left\| G_{n_i}^\gamma(\omega^v, n^{-\kappa}h) \right\|_{\psi_1 | X} \leq K \ln(n + 1) \max_i \sup_{\omega^v, h \in \Omega} \left\| G_{n_i}^\gamma(\omega^v, n^{-\kappa}h) \right\|_{\psi_1 | X}$ for constant $K < \infty$. Moreover, by Theorem 2.14.5 (with $p = 1$) and Lemma 2.2.2 in van der Vaart and Wellner (1996), for each $1 \leq i \leq n$ we can bound

$$
\sup_{\omega^v, h \in \Omega} \left\| G_{n_i}^\gamma(\omega^v, n^{-\kappa}h) \right\|_{\psi_1 | X} \leq K_1 \left( \mathbb{E} \left[ \sup_{\omega^v, h \in \Omega} \left\| G_{n_i}^\gamma(\omega^v, n^{-\kappa}h) \right\| X \right] \right).
$$

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and are satisfied. Let \( \text{Lemma O.E.8} \)

\[
K \leq \sup_{\omega^v, h \in \Omega} \| \eta_{n,ij}(\omega^v + n^{-\kappa}h, \omega^v, \epsilon_{ij}) \|_{\psi_1|X}
\]

for constant \( K_1 < \infty \), and \( \eta_{n,ij} \) is defined in part (i). For the first term on the right-hand side, from part (i) we can derive \( \mathbb{E}[\sup_{\omega^v, h \in \Omega} \| G^\gamma_n(\omega^v, n^{-\kappa}h) \| |X] \leq K\bar{J}(n-1)^{-1/2}K^n\bar{n}\bar{n}^2 + n^{-\kappa}M)^{1/2} \). As for the second term on the right-hand side, the definition of \( \eta_{n,ij} \) implies that \( \max_{j \neq i} \sup_{\omega^v, h \in \Omega} \| \eta_{n,ij}(\omega^v + n^{-\kappa}h, \omega^v, \epsilon_{ij}) \|_{\psi_1|X} \leq \| V_{ni} \| \leq \bar{n} < \infty \). Therefore, for all \( 1 \leq i \leq n \) we have the uniform bound

\[
\left\| \sup_{\omega^v, h \in \Omega} \| G^\gamma_n(\omega^v, n^{-\kappa}h) \| \right\|_{\psi_1|X} \leq K_1 \left( K\bar{J} \left[ \frac{K^n\bar{n}\bar{n}^2}{\sqrt{n-1}} + \frac{M}{n^{\kappa}} + \frac{\bar{n}\ln n}{\sqrt{n-1}} \right] \right).
\]  

(O.E.19)

Combining the results, we conclude that \( \mathbb{E}[\max_i \sup_{\omega^v, h \in \Omega} \| G^\gamma_n(\omega^v, n^{-\kappa}h) \| |X] \) is bounded by \( \delta^{-1}K \ln(n+1)K_1(K\bar{J}(n-1)^{-1/2}K^n\bar{n}\bar{n}^2 + n^{-\kappa}M)^{1/2} + \bar{n}(n-1)^{-1/2}\ln n = o(1) \). \( \square \)

**O.E.3 Lemmas for the Asymptotic Distribution of \( \hat{\theta}_n \)**

**Lemma O.E.8 (Asymptotic normality of the sample moments).** Suppose that Assumptions 1–4 and O.E.1 are satisfied. Let

\[
Y_n = \frac{1}{\sqrt{n(n-1)}} \sum_i \sum_{j \neq i} q_{n,ij}(G_{n,ij} - P_{n,ij}(\theta_0, p_n)),
\]

where \( q_{n,ij} \in \mathbb{R}^d \) is a \( d \times 1 \) vector of instruments that is a function of \( X \) and satisfies \( \max_{1 \leq i, j \leq n} \| q_{n,ij} \| \leq \bar{q} < \infty \). Define the \( d \times 1 \) vector

\[
\phi^y_{n,ij} = q_{n,ij}(g_{n,ij}(\omega_{ni}^*, \epsilon_{ij}) - P_{n,ij}(\omega_{ni}^*)) + J^\omega_{ni}(\omega_{ni}^*, q_{ni}) \phi^\omega_{n,ij}(\omega_{ni}^*, \epsilon_{ij}),
\]

where \( g_{n,ij}(\omega, \epsilon_{ij}) = 1\{U_{n,ij} + \frac{n-1}{n-2}Z_jV_{ni}\omega \geq \epsilon_{ij}\} \) is the indicator function defined in equation (3.5), \( P_{n,ij}(\omega_{ni}^*) = F_c(U_{n,ij} + \frac{n-1}{n-2}Z_jV_{ni}\omega_{ni}^*), J^\omega_{ni}(\omega_{ni}^*, q_{ni}) = \frac{1}{n-1} \sum_{j \neq i} q_{n,ij} \nabla_{\omega^c}P_{n,ij}(\omega_{ni}^*) \) is the \( d \times T \) weighted Jacobian matrix, where \( q_{ni} = (q_{n,ij}, \forall j \neq i) \), and \( \phi^\omega_{n,ij}(\omega_{ni}^*, \epsilon_{ij}) \in \mathbb{R}^T \) is the influence function defined in Lemma O.E.5. Define the \( d \times d \) variance matrix \( \Sigma_n = \frac{1}{n(n-1)} \sum_i \sum_{j \neq i} \mathbb{E}[\phi^y_{n,ij}(\phi^y_{n,ij})'|X] \). Given \( X \), we have

\[
\Sigma_n^{-1/2} Y_n \xrightarrow{d} N(0, I_d),
\]

where \( I_d \) is the \( d \times d \) identity matrix.

**Proof.** By Theorem 3.1, an observed link \( G_{n,ij} \) is given by the link indicator \( g_{n,ij}(\omega, \epsilon_{ij}) \)

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evaluated at \( \omega_{ni}(\epsilon_i) \), that is, \( G_{n,ij} = g_{n,ij}(\omega_{ni}(\epsilon_i), \epsilon_{ij}) \), where \( \omega_{ni}(\epsilon_i) \) maximizes the function \( \Pi_{ni}(\omega) \). Moreover, \( P_{n,ij}(\theta_0, p_n) = \mathbb{E}[g_{n,ij}(\omega_{ni}(\epsilon_i), \epsilon_{ij})|X] \). Therefore, \( Y_n \) can be represented as

\[
Y_n = \frac{1}{\sqrt{n(n-1)}} \sum_i \sum_{j \neq i} q_{n,ij} \left( g_{n,ij}(\omega_{ni}(\epsilon_i), \epsilon_{ij}) - \mathbb{E}[g_{n,ij}(\omega_{ni}(\epsilon_i), \epsilon_{ij})|X] \right).
\]

The challenge in deriving the asymptotic distribution of \( Y_n \) lies in the fact that link choices of an individual are correlated through \( \omega_{ni}(\epsilon_i) \). To account for the correlation, we decompose \( Y_n \) into four parts \( Y_n = T_{1n} + T_{2n} + T_{3n} + T_{4n} \), where

\[
T_{1n} = \frac{1}{\sqrt{n(n-1)}} \sum_i \sum_{j \neq i} q_{n,ij} \left( g_{n,ij}(\omega_{ni}^*, \epsilon_{ij}) - P_{n,ij}^*(\omega_{ni}^*) \right)
\]

\[
T_{2n} = \frac{1}{\sqrt{n(n-1)}} \sum_i \sum_{j \neq i} q_{n,ij} \left( g_{n,ij}(\omega_{ni}(\epsilon_i), \epsilon_{ij}) - g_{n,ij}(\omega_{ni}^*, \epsilon_{ij}) - (P_{n,ij}^*(\omega_{ni}(\epsilon_i)) - P_{n,ij}^*(\omega_{ni}^*)) \right)
\]

\[
T_{3n} = \frac{1}{\sqrt{n(n-1)}} \sum_i \sum_{j \neq i} q_{n,ij} \left( P_{n,ij}^*(\omega_{ni}(\epsilon_i)) - \mathbb{E}[P_{n,ij}^*(\omega_{ni}(\epsilon_i))|X] \right)
\]

\[
T_{4n} = \frac{1}{\sqrt{n(n-1)}} \sum_i \sum_{j \neq i} q_{n,ij} \left( \mathbb{E}[P_{n,ij}^*(\omega_{ni}(\epsilon_i))|X] - \mathbb{E}[g_{n,ij}(\omega_{ni}(\epsilon_i), \epsilon_{ij})|X] \right).
\]

The four terms in the decomposition can be interpreted as follows. The first term \( T_{1n} \) is the sample moment if we replace \( \omega_{ni}(\epsilon_i) \) by its limit \( \omega_{ni}^* \). This substitution removes the correlation between the link choices of an individual. The second term \( T_{2n} \) is the difference between the dependent sample moment and the independent one in \( T_{1n} \). The fact that this term is shown to be negligible indicates that the correlation between link choices vanishes as \( n \) grows large. The sampling variation in \( \omega_{ni}(\epsilon_i) \) is captured by the third term \( T_{3n} \) which contributes to the asymptotic variance of the moment function. Finally, the fourth term \( T_{4n} \) satisfies \( T_{4n} = -\mathbb{E}[T_{2n}|X] \) and hence is asymptotically negligible.

Let us now examine the four terms in (O.E.20).

**Step 1:** \( T_{1n} \). The term \( T_{1n} \) is a normalized sum of link indicators that are evaluated at \( \omega_{ni}^* \) rather than \( \omega_{ni}(\epsilon_i) \) and thus are independent. This is a leading term in \( Y_n \) and has an asymptotically normal distribution because the CLT applies. It captures the sampling variation in link choices due to \( \epsilon_{ij} \).

**Step 2:** \( T_{2n} \). We show that \( T_{2n} \) is \( o_p(1) \). For each \( i \), define the empirical process \( \mathbb{G}_n q_{ni} g_{ni}(\omega, \epsilon_i) = \frac{1}{\sqrt{n-1}} \sum_{j \neq i} q_{n,ij} \left( g_{n,ij}(\omega, \epsilon_{ij}) - P_{n,ij}^*(\omega) \right) \), \( \omega \in \Omega \). Then \( T_{2n} \) is a normalized average of these empirical processes for all \( i \),

\[
T_{2n} = \frac{1}{\sqrt{n}} \sum_i \mathbb{G}_n q_{ni} \left( g_{ni}(\omega_{ni}(\epsilon_i), \epsilon_i) - g_{ni}(\omega_{ni}^*, \epsilon_i) \right).
\]
While each empirical process in $T_{2n}$ is $o_p(1)$ by establishing stochastic equicontinuity, we cannot directly invoke a stochastic equicontinuity argument to show that their normalized average $T_{2n}$ is $o_p(1)$. Instead, we use an maximal inequality to derive a uniform bound on the $L_2$ norm of each empirical process.

Note that each $\mathbb{G}_n g_{ni}(\omega_{ni}(\epsilon_i), \epsilon_i) - g_{ni}(\omega_{ni}^*, \epsilon_i))$ only involves $\epsilon_i$, so given $X$ they are independent across $i$. Moreover, by Lemma O.E.5 we have $V_{ni}(\omega_{ni}(\epsilon_i) - \omega_{ni}^*) = O_p(n^{-1/2})$, so if we define $h_{ni} = n^\kappa V_{ni}(\omega_{ni}(\epsilon_i) - \omega_{ni}^*)$ for $0 < \kappa < 1/2$, then $h_{ni} \in \Omega$ if $n$ is sufficiently large, because by Assumption O.E.1(i) $\Omega$ contains a compact neighborhood of 0. Further, view $g_{n,ij}(\omega, \epsilon_{ij})$ as a function of $\omega^v = V_{ni}\omega \in \Omega$, and define $g_{n,ij}(\omega^v, \epsilon_{ij}) = 1\{U_{n,ij} + \frac{n-1}{n-2}Z_j^v \omega^v \geq \epsilon_{ij}\}$. Then $g_{ni}(\omega_{ni}(\epsilon_i), \epsilon_i) - g_{ni}(\omega_{ni}^*, \epsilon_i) = g_{ni}(\omega_{ni}(\epsilon_i), \epsilon_i) - g_{ni}(\omega_{ni}^*, \epsilon_i)$, where $\omega_{ni}(\epsilon_i) = V_{ni}\omega_{ni}(\epsilon_i)$ and $\omega_{ni}^* = V_{ni}\omega_{ni}^*$. Combining these results we obtain the bound

$$
\mathbb{E}[||T_{2n}||^2 | X] \leq n^{-1} \sum_i \mathbb{E} \left[ \sup_{\omega^v, h \in \Omega} \left|\mathbb{G}_n g_{ni}(g_{ni}(\omega_{ni}(\epsilon_i) + n^{-\kappa}h, \epsilon_i) - g_{ni}(\omega_{ni}^*, \epsilon_i))\right| \right]^2 | X | . \quad (O.E.21)
$$

Below we derive a bound on each term in the summation in (O.E.21) that is uniform in $i$. Observe that for any $\omega^v, \tilde{\omega}^v \in \Omega$, the function $q_{n,ij}(g_{n,ij}(\omega^v, \epsilon_{ij}) - g_{n,ij}(\tilde{\omega}^v, \epsilon_{ij}))$ can be bounded by $\|q_{n,ij}(g_{n,ij}(\omega^v, \epsilon_{ij}) - g_{n,ij}(\tilde{\omega}^v, \epsilon_{ij}))\| \leq \|q_{n,ij}||1\{U_{n,ij} + \frac{n-1}{n-2}Z_j^v \tilde{\omega}^v \geq \epsilon_{ij}\} - 1\{U_{n,ij} + \frac{n-1}{n-2}Z_j^v \omega^v \geq \epsilon_{ij}\}| \leq \eta_{ni,j}(\omega^v, \tilde{\omega}^v, \epsilon_{ij})$, where $\eta_{ni,j}(\omega^v, \tilde{\omega}^v, \epsilon_{ij}) = \|q_{n,ij}||$ if $\epsilon_{ij} \in$ between $U_{n,ij} + \frac{n-1}{n-2}Z_j^v \omega^v$ and $U_{n,ij} + \frac{n-1}{n-2}Z_j^v \tilde{\omega}^v$, and 0 otherwise. By Theorem 2.14.1 in van der Vaart and Wellner (1996) with $p = 2$ we derive the maximal inequality for each $i$ (see footnote 64)

$$
\mathbb{E} \left[ \sup_{\omega^v, h \in \Omega} \left|\mathbb{G}_n q_{ni}(g_{ni}(\omega_{ni}(\epsilon_i) + n^{-\kappa}h, \epsilon_i) - g_{ni}(\omega_{ni}^*, \epsilon_i))\right| \right]^2 | X | \leq K \mathbb{E} \left[ J(1, F_{ni}(\epsilon_i)) \right]^2 \sup_{\omega^v, h \in \Omega} \left|\eta_{ni}(\omega_{ni}(\epsilon_i) + n^{-\kappa}h, \omega^v, \epsilon_i)\right| | X | \right] . \quad (O.E.22)
$$

for constant $K < \infty$, where $\|\eta_{ni}\|_n = (\frac{1}{n-1} \sum_{j \neq i} \eta_{ni,j}^2)^{1/2}$ is the empirical $L_2$ norm of $\eta_{ni}$, and $J(1, F_{ni}(\epsilon_i))$ is the uniform entropy integral (defined as in (O.E.18) for the set of arrays $F_{ni}(\epsilon_i) = \{(q_{n,ij}(g_{n,ij}(\omega^v, \epsilon_{ij}) - g_{n,ij}(\omega^v, \epsilon_{ij})), j \neq i) : \omega^v, h \in \Omega\}$. Similarly as in Lemma O.E.7, we can show that $J(1, F_{ni}(\epsilon_i)) \leq \bar{J}$ for some $\bar{J} < \infty$.

We follow the argument in Lemma O.E.7 to further bound the empirical $L_2$ norm term in (O.E.22). Note that $\eta_{ni,j}^2$ is bounded by $\|q_{n,ij}\|^2 \leq \max_{1 \leq i, j \leq n} \|q_{n,ij}\|^2 \leq \bar{q}^2 < \infty$. Similarly to the argument in Lemma O.E.7, we can show that the set of arrays $\{(\eta_{ni,j}^2(\omega^v + n^{-\kappa}h, \omega^v, \epsilon_{ij}), j \neq i) : \omega^v, h \in \Omega\}$ has a finite uniform entropy integral bounded by some $\bar{J}^n < \infty$. By Theorem 2.14.1 in van der Vaart and Wellner (1996) with $p = 1$ we have the bound $\mathbb{E} \left[ \sup_{\omega^v, h \in \Omega} \left|\mathbb{G}_n \eta_{ni}(\omega_{ni}(\epsilon_i) + n^{-\kappa}h, \omega^v, \epsilon_i)\right| | X | \right] \leq K^n \bar{J}^n \bar{q}^2$, with constant $K^n < \infty$. 

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Therefore,

$$
\mathbb{E} \left[ \sup_{\omega^v, h \in \Omega} \frac{1}{n-1} \sum_{j \neq i} \eta_{n,ij}^2 (\omega^v + n^{-\kappa}h, \omega^v, \epsilon_{ij}) \right] \leq \frac{K^n \bar{J}^n \bar{W}^2}{\sqrt{n-1}}.
$$

Moreover, by the mean-value theorem, for any \( \omega^v, h \in \Omega \) and any \( j \neq i \), we have

$$
\mathbb{E}[\eta_{n,ij}^2 (\omega^v + n^{-\kappa}h, \omega^v, \epsilon_{ij})|X] = F_i(U_{n,ij} + \frac{n-1}{n-2} Z_i^j(\omega^v + n^{-\kappa}h)) - F_i(U_{n,ij} + \frac{n-1}{n-2} Z_i^j(\omega^v)) \frac{\langle q_{n,ij} \rangle^2}{2}.
$$

for some \( t_{n,ij} \in [0,1] \). By the boundedness of \( f_i \) under Assumption 1(ii), sup_{\omega, h \in \Omega} \|h\| < \infty, \text{ and } \|q_{n,ij}\|^2 \leq \bar{q}^2 < \infty, \text{ there is a } M < \infty \text{ such that } \mathbb{E}[\eta_{n,ij}^2 (\omega^v + n^{-\kappa}h, \omega^v, \epsilon_{ij})|X] \leq \frac{n^{\kappa}M}{n-1} \text{ for all } \omega^v, h \in \Omega \text{ and all } i, j. \text{ Hence, sup}_{\omega^v, h \in \Omega} \frac{1}{n-1} \sum_{j \neq i} \mathbb{E}[\eta_{n,ij}^2 (\omega^v + n^{-\kappa}h, \omega^v, \epsilon_{ij})|X] \leq \frac{n^{\kappa}M}{n-1}. \text{ From these results we derive } \mathbb{E}[\sup_{\omega^v, h \in \Omega} \frac{1}{n-1} \sum_{j \neq i} \eta_{n,ij}^2 (\omega^v + n^{-\kappa}h, \omega^v, \epsilon_{ij})|X] \leq (n-1)^{-1/2}K^n \bar{J}^n \bar{W}^2 + n^{\kappa}M \text{ for each } i. \text{ Note that the bound is constant across } i. \text{ Combining the results we can bound } \mathbb{E}[\|T_{2n}\|^2|X] \text{ by } K^n \bar{J}^2((n-1)^{-1/2}K^n \bar{J}^n \bar{W}^2 + n^{\kappa}M) = o(1) \text{ and hence } T_{2n} = o_p(1) \text{ by Markov’s inequality.}

\textbf{Step 3: } T_{3n}. \text{ View } P_{n,ij}^*(\omega) \text{ as a function of } V_{ni}\omega. \text{ By Taylor expansion, we have}

$$
P_{n,ij}^*(\omega_{ni}(\epsilon_i)) = P_{n,ij}^*(\omega_{ni}) + \nabla_{\omega^v} P_{n,ij}^*(\omega_{ni}) V_{ni}(\omega_{ni}(\epsilon_i) - \omega_{ni}) + O(\|V_{ni}(\omega_{ni}(\epsilon_i) - \omega_{ni})\|^2),
$$

where \( \nabla_{\omega^v} P_{n,ij}^*(\omega_{ni}) = \frac{n-1}{n-2} f_i(U_{n,ij} + \frac{n-1}{n-2} Z_i^j(\omega_{ni})) Z_j \) is the derivative of \( P_{n,ij}^*(\omega) \) with respect to \( V_{ni}\omega_{ni} \). Lemma O.E.5 shows that \( V_{ni}(\omega_{ni}(\epsilon_i) - \omega_{ni}) \) has an asymptotically linear representation \( V_{ni}(\omega_{ni}(\epsilon_i) - \omega_{ni}) = \frac{1}{n-1} \sum_{j \neq i} \phi_{n,ij}^* (\omega_{ni}, \epsilon_i) + r_{ni}^* (\epsilon_i). \) Let \( \bar{q}_{ni} = \frac{1}{n-1} \sum_{j \neq i} q_{n,ij} \) be the \( d_0 \times 1 \) vector of instruments averaged over \( j \). By the asymptotically linear representation, we can decompose \( T_{3n} \) into three parts

$$
T_{3n} = T_{3n}^l + (r_{1n} - \mathbb{E}[r_{1n}|X]) + (r_{2n} - \mathbb{E}[r_{2n}|X]),
$$

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where

\[
T_{3n}^l = \frac{1}{\sqrt{n(n-1)}} \sum_i \sum_{j \neq i} J_{ni}^\omega(\omega^*_ni, q_{ni}) \phi_{n,ij}^\omega(\omega^*_ni, \epsilon_{ij})
\]

\[
r_{1n} = \sqrt{\frac{n-1}{n}} \sum_i J_{ni}^\omega(\omega^*_ni, q_{ni}) r_{ni}^\omega(\epsilon_i)
\]

\[
r_{2n} = \sqrt{\frac{n-1}{n}} \sum_i \tilde{q}_{ni} O(\|V_{ni}(\omega_{ni}(\epsilon_i) - \omega^*_ni)\|^2).
\]

\(T_{3n}^l\) is a leading term that contributes to the asymptotic distribution of \(Y_n\). It captures the sampling variation due to \(\omega_{ni}(\epsilon_i)\). We will combine it with \(T_1n\) to derive the asymptotic distribution of \(Y_n\). Below we show that the two centered remainders \(r_{1n} - \mathbb{E}[r_{1n}|X]\) and \(r_{2n} - \mathbb{E}[r_{2n}|X]\) are both \(o_p(1)\).

Given \(X\), each \(r_{ni}^\omega(\epsilon_i)\) only depends on \(\epsilon_i\), so they are independent across \(i\) (Assumption 1(i)). Hence,

\[
\mathbb{E}[\|r_{1n} - \mathbb{E}[r_{1n}|X]\|^2|X] = \frac{n-1}{n} \sum_i \mathbb{E}[\|J_{ni}^\omega(\omega^*_ni, q_{ni})(r_{ni}^\omega(\epsilon_i) - \mathbb{E}[r_{ni}^\omega(\epsilon_i)|X])\|^2|X]
\]

\[
\leq (n-1) \max_i \|J_{ni}^\omega(\omega^*_ni, q_{ni})\|^2 \max_i \mathbb{E}[\|r_{ni}^\omega(\epsilon_i) - \mathbb{E}[r_{ni}^\omega(\epsilon_i)|X]\|^2|X].
\]

For any random variable \(Z\), recall that \(\mathbb{E}[|Z| |X| \leq |Z| \psi_1|X|\) and \(\mathbb{E}[Z^2|X] \leq 4|Z|^2 \psi_1|X|\). From \(|\max_i \|r_{ni}^\omega(\epsilon_i)\|\psi_1|X| = o(n^{-1/2})\) (Lemma O.E.6(ii)) we thus derive that \(\mathbb{E}[\max_i \|r_{ni}^\omega(\epsilon_i)\||X] = o(n^{-1/2})\) and \(\mathbb{E}[\max_i \|r_{ni}^\omega(\epsilon_i)\|^2|X] = o(n^{-1})\). Therefore, \(\max_i \mathbb{E}[\|r_{ni}^\omega(\epsilon_i) - \mathbb{E}[r_{ni}^\omega(\epsilon_i)|X]\|^2|X] \leq \max_i \mathbb{E}[\|r_{ni}^\omega(\epsilon_i)\|^2|X] + 3(\mathbb{E}[\max_i \|r_{ni}^\omega(\epsilon_i)\|^2|X])^2 = o(n^{-1})\).

Because \(J_{ni}^\omega(\omega^*_ni, q_{ni})\) is bounded uniformly in \(i\), we obtain \(\mathbb{E}[\|r_{1n} - \mathbb{E}[r_{1n}|X]\|^2|X] = o(1)\) and thus \(r_{1n} - \mathbb{E}[r_{1n}|X] = o_p(1)\) by Markov’s inequality.

Similarly, with \(O(\|V_{ni}(\omega_{ni}(\epsilon_i) - \omega^*_ni)\|^2)\) in place of \(r_{ni}^\omega(\epsilon_i)\) and \(\bar{q}_{ni}\) in place of \(J_{ni}^\omega(\omega^*_ni, q_{ni})\) and by \(\|max_i \|V_{ni}(\omega_{ni}(\epsilon_i) - \omega^*_ni)\|^2\psi_1|X = o(n^{-1/2})\) (Lemma O.E.6(ii)), we can show that \(r_{2n} - \mathbb{E}[r_{2n}|X] = o_p(1)\).

**Step 4: \(T_4n\)**. Recall that \(T_4n = -\mathbb{E}[T_{2n}|X]\). In Step 2 we showed that \(\mathbb{E}[\|T_{2n}\|^2|X] = o(1)\). Because \(\mathbb{E}[\|T_{2n}|X]\) \(\leq \mathbb{E}[\|T_{2n}\|^2|X] \leq (\mathbb{E}[\|T_{2n}\|^2|X])^{1/2}\), we have \(T_{4n} = o(1)\).

Combing the four steps, we derive that \(Y_{ni} = T_{1n} + T_{3n} + T_{4n} + o_p(1) = \sum_i \phi_{ni}^y + o_p(1)\), where for each \(i\),

\[
\phi_{ni}^y = \frac{1}{\sqrt{n(n-1)}} \sum_{j \neq i} \phi_{n,ij}^y.
\]

Given \(X\), \(\phi_{ni}^y\), \(i = 1, \ldots, n\), are independent but not identically distributed. We apply the
Lindeberg-Feller CLT to derive the asymptotic distribution of \( \sum_i \phi_{ni}^y \). Note that \( \mathbb{E}[\phi_{ni}^y | \mathbf{X}] = 0 \) for all \( i \). By the Cramer-Wold device it suffices to show that \( a^t \sum_i \phi_{ni}^y \) satisfies the Lindeberg condition for any \( d_\theta \times 1 \) vector of constants \( a \in \mathbb{R}^{d_\theta} \). The Lindeberg condition is that for any \( \xi > 0 \)

\[
\lim_{n \to \infty} \frac{1}{d^t \Sigma_n a} \sum_i \mathbb{E}[(a^t \phi_{ni}^y)^2 1\{a^t \phi_{ni}^y \geq \xi \sqrt{d^t \Sigma_n a}\} | \mathbf{X}] = 0,
\]

(\text{O.E.23})

where \( \Sigma_n = \sum_i \mathbb{E}[\phi_{ni}^y (\phi_{ni}^y)^t] | \mathbf{X} = \frac{1}{n(n-1)} \sum_i \sum_{j \neq i} \mathbb{E}[\phi_{ni}^y (\phi_{nj}^y)^t] | \mathbf{X} \). Following the argument in the proof of Lemma O.E.5, the Lindeberg condition holds if

\[
\frac{\max_i (a^t \phi_{ni}^y)}{\sqrt{d^t \Sigma_n a}} = o_p(1).
\]

(O.E.24)

By Markov’s inequality, equation (O.E.24) holds if \( \mathbb{E}[\max_i (a^t \phi_{ni}^y)^2 | \mathbf{X}] = o(1) \). By the maximal inequality in van der Vaart and Wellner (1996, Lemma 2.2.2), we have the bound

\[
\mathbb{E}[\max_i (a^t \phi_{ni}^y)^2 | \mathbf{X}] \leq \max_i (a^t \phi_{ni}^y)^2 \mathbb{E}[|\phi_{ni}^y|] \leq K \ln(n + 1) \max_i (a^t \phi_{ni}^y)^2 \mathbb{E}[|\phi_{ni}^y|],
\]

where \( K < \infty \) is a constant. Moreover, note that \( a^t \phi_{ni}^y = (n(n-1))^{-1/2} \sum_{j \neq i} a^t \phi_{nj,i}^y \), and each \( a^t \phi_{nj,i}^y \) is bounded by \( \|a\| (2\|q_{n,i,j}\| + \|J_m(\omega_n, q_n)|\| |\phi_{nj,i}(\omega_n, \epsilon_{ij})|) = M_{n,i,j} \leq M_n < \infty \). By Hoeffding’s inequality for bounded random variables Boucheron et al. (2013, Theorem 2.8), we derive that \( \Pr((a^t \phi_{ni}^y)^2 \geq t | \mathbf{X}) = \Pr(a^t \phi_{ni}^y \geq \sqrt{t}| \mathbf{X}) + \Pr(-a^t \phi_{ni}^y \geq \sqrt{t}| \mathbf{X}) \leq 2 \exp(-\frac{n(n-1)t}{2 \sum_{j \neq i} M_{n,i,j}}) \), and hence by Lemma 2.2.1 in van der Vaart and Wellner (1996) we can bound

\[
\mathbb{E}[\max_i (a^t \phi_{ni}^y)^2 | \mathbf{X}] \leq \frac{6}{n(n-1)} \sum_{j \neq i} M_{n,i,j}^2 \leq \frac{6}{n} M_n^2.
\]

Each \( \mathbb{E}[\phi_{ni}^y (\phi_{nj}^y)^t] | \mathbf{X} \) is positive definite, so \( a^t \Sigma_n a > 0 \). Combining these results we can bound \( \mathbb{E}[\max_i (a^t \phi_{ni}^y)^2 | \mathbf{X}] \) by \( 6KM_n^2 \ln(n + 1)/n = o(1) \), so the Lindeberg condition holds.

By the Lindeberg-Feller CLT \( (a^t \Sigma_n d - 1/2) a^t \sum_i \phi_{ni}^y \overset{d}{\rightarrow} N(0,1) \). Because \( \Sigma_n \) is positive definite, there is a nonsingular symmetric matrix \( \Sigma_n^{1/2} \) such that \( \Sigma_n^{1/2} \Sigma_n^{1/2} = \Sigma_n \). Let \( \tilde{a} = \Sigma_n^{1/2} a \), then \( a^t \sum_i \phi_{ni}^y = a^t \Sigma_n^{-1/2} \sum_i \phi_{ni}^y \) and \( a^t \Sigma_n a = a^t \Sigma_n^{-1/2} \Sigma_n^{1/2} \Sigma_n^{1/2} a = \tilde{a}^t \tilde{a} \). Note that \( \Sigma_n \) is nonsingular, so \( \tilde{a} \) is also an arbitrary vector in \( \mathbb{R}^{d_\theta} \). The previous result implies that \( \tilde{a}^t \Sigma_n^{-1/2} \sum_i \phi_{ni}^y \overset{d}{\rightarrow} N(0, \tilde{a}^t \tilde{a}) \). By the Cramer-Wold device, \( \Sigma_n^{-1/2} \sum_i \phi_{ni}^y d \overset{d}{\rightarrow} N(0, I_{d_\theta}) \), where \( I_{d_\theta} \) is the \( d_\theta \times d_\theta \) identity matrix. Because \( Y_n = \sum_i \phi_{ni}^y + o_p(1) \), we conclude that by Slutsky’s theorem \( Y_n \) has the asymptotic distribution \( \Sigma_n^{-1/2} Y_n \overset{d}{\rightarrow} N(0, I_{d_\theta}) \).