Characterizing total positivity: Single vector tests via linear complementarity, sign non-reversal and variation diminution

Projesh Nath Choudhury

Department of Mathematics, Indian Institute of Science, Bangalore, India

Correspondence
Projesh Nath Choudhury, Department of Mathematics, Indian Institute of science, Bangalore 560012, India.
Email: projeshc@iisc.ac.in, projeshnc@alumni.iitm.ac.in

Funding information
SERB, Grant/Award Number: PDF/2019/000275

Abstract
A matrix $A$ is called totally positive (or totally non-negative) of order $k$, denoted by $TP_k$ (or $TN_k$), if all minors of size at most $k$ are positive (or non-negative). These matrices have featured in diverse areas in mathematics, including algebra, analysis, combinatorics, differential equations and probability theory. The goal of this article is to provide a novel connection between total positivity and optimization/game theory. Specifically, we draw a relationship between totally positive matrices and the linear complementarity problem (LCP), which generalizes and unifies linear and quadratic programming problems and bimatrix games — this connection is unexplored, to the best of our knowledge. We show that $A$ is $TP_k$ if and only if for every submatrix $A_r$ of $A$ formed from $r$ consecutive rows and $r$ consecutive columns (with $r \leq k$), LCP$(A_r, q)$ has a unique solution for each vector $q < 0$. In fact this can be strengthened to check the solution set of the LCP at a single vector for each such square submatrix. These novel characterizations are in the spirit of classical results characterizing $TP$ matrices by Gantmacher–Krein [Compos. Math. 1937] and $P$-matrices by Ingleton [Proc. London Math. Soc. 1966].

Our work contains two other contributions, both of which characterize total positivity using single test...
vectors whose coordinates have alternating signs — that is, lie in a certain open bi-orthant. First, we improve on one of the main results in recent joint work [Bull. London Math. Soc., 2021], which provided a novel characterization of $TP_k$ matrices using sign non-reversal phenomena. We further improve on a classical characterization of total positivity by Brown–Johnstone–MacGibbon [J. Amer. Statist. Assoc. 1981] (following Gantmacher–Krein, 1950) involving the variation diminishing property. Finally, we use a Pólya frequency function of Karlin [Trans. Amer. Math. Soc. 1964] to show that our aforementioned characterizations of total positivity, involving (single) test-vectors drawn from the ‘alternating’ bi-orthant, do not work if these vectors are drawn from any other open orthant.

MSC (2020)
15B48, 90C33 (primary), 15A24 (secondary)

Contents

1. INTRODUCTION AND MAIN RESULTS .................................................. 792
2. THEOREMS A AND B: TOTAL POSITIVITY AND THE LINEAR
   COMPLEMENTARITY PROBLEM ......................................................... 796
3. THEOREM C: SIGN NON-REVERSAL PROPERTY FOR TOTALLY POSITIVE
   MATRICES .............................................................................. 802
4. THEOREM D: VARIATION DIMINUTION AND TOTAL POSITIVITY .......... 803
5. TEST VECTORS FROM ANY OTHER ORTHANT DO NOT WORK ........... 806
ACKNOWLEDGEMENTS................................................................. 810
REFERENCES .............................................................................. 810

1 | INTRODUCTION AND MAIN RESULTS

Given an integer $k \geq 1$, we say that a matrix is \textit{totally positive of order $k$} ($TP_k$) if all its minors of order at most $k$ are positive. A matrix $A$ is \textit{totally positive} ($TP$) if $A$ is $TP_k$ for all $k \geq 1$ that is, all minors of $A$ are positive. Similarly, one defines \textit{totally non-negative} ($TN$) and $TN_k$ matrices for $k \geq 1$. These classes of matrices have important applications in various theoretical and applied branches in mathematics. We mention a few of these topics and some of the experts who worked on them: analysis (Fekete and Pólya [16], Schoenberg [39, 41], Whitney [43]), representation theory (Lusztig [34], Rietsch [37]), cluster algebras (Berenstein, Fomin and Zelevinsky [3, 18]), combinatorics (Brenti [5]), matrix theory (Fallat and Johnson [15], Garloff [22], Pinkus [36]),
differential equations (Karlin [27], Loewner [33]), Gabor analysis (Gröchenig, Romero and Stöckler [23]), integrable systems (Kodama and Williams [30]), probability and statistics (Karlin [27]), interacting particle systems (Gantmacher and Krein [20, 21]) and interpolation theory and splines (de Boor [4], Karlin and Ziegler [28], and Schoenberg with collaborators [13, 40, 42]). We also mention the preprints [1, 2] for preserver problems involving totally positive matrices and Pólya frequency functions/sequences. Given these numerous strong connections to many subfields of the broader mathematical sciences, it is perhaps surprising that a characterization of total positivity in terms of optimization/game theory or an application of total positivity in optimization/game theory remain unexplored, to the best of our knowledge. The main objective of this article is to draw a connection between total positivity and the linear complementarity problem (LCP) which generalizes and unifies linear and quadratic programming problems and bimatrix games. We believe that this connection is novel.

Given a matrix $A \in \mathbb{R}^{n \times n}$ and a vector $q \in \mathbb{R}^n$, the linear complementarity problem (LCP) asks to find, if possible, a vector $x \in \mathbb{R}^n$ such that

$$x \geq 0, \quad y = Ax + q \geq 0, \quad \text{and} \quad x^T y = 0,$$

where $x = (x_1, ..., x_n)^T \geq 0$ means that all $x_i \geq 0$. Any $x$ that satisfies the first two conditions is called a feasible vector, and any feasible vector that satisfies the third condition is called a (complementarity) solution of LCP($A, q$). The set of all solutions of LCP($A, q$) is denoted by $\text{SOL}(A, q)$.

The linear complementarity problem (LCP) has important applications in many different areas, including bimatrix games, convex quadratic programming, fluid mechanics, solution of systems of piecewise linear equations, and variational inequality problems [10–12, 14, 32]. The study of LCPs has resulted in progress along several fronts. For example, the complementary pivot algorithm, which was first developed for solving LCPs, has been generalized in a straightforward way to obtain efficient algorithms for computing Brouwer and Kakutani fixed points, for solving systems of non-linear equations and non-linear programming problems, and for computing economic equilibria. Also, iterative methods developed for solving LCPs are very useful for tackling very large scale linear programs, which cannot be handled with the simplex method because of their large size and numerical difficulties. As far as the bimatrix game is concerned, the LCP formulation was instrumental in the discovery of an efficient constructive method for the computation of a Nash equilibrium point. For more details about LCPs and their applications, we refer to [9, 11, 25].

In this section, we state our first two main results, which provide characterizations of total positivity in terms of the LCP, and thereby connect these two well-studied areas. To state these results, we begin with some preliminary definitions, which we use in this paper without further reference.

**Definition 1.1.** Let $n \geq 1$ be an integer, $A \in \mathbb{R}^{n \times n}$ be a matrix, and $S \subset \mathbb{R}^n$ be a subset.

(i) Define the set $[n] := \{1, 2, ..., n\}$.

(ii) Given $i, j \in [n]$, let $A_{i\setminus j}$ denote the determinant of the $(n-1) \times (n-1)$ submatrix of $A$ obtained by deleting the $i$th row and $j$th column of $A$. If $n = 1$, then define $A_{1\setminus 1} := 1$ to be the determinant of the empty matrix.

(iii) Let $\text{adj}(A)$ denote the adjugate matrix of $A$ and $\{e_i\}$ denote the standard orthonormal basis of $\mathbb{R}^n$. 

(iv) Let $\mathbb{R}^n_{\text{alt}} \subset \mathbb{R}^n$ denote the set of real vectors with all non-zero coordinates and alternating signs.

(v) A submatrix $B$ of $A$ is called a contiguoussubmatrix, if the rows and columns of $B$ are indexed by sets of consecutive integers.

(vi) The matrix $A$ has the sign non-reversal property with respect to $S$, if for all vectors $0 \neq x \in S$, there exists a coordinate $i \in [n]$ such that $x_i(Ax)_i > 0$.

(vii) We also require a non-strict version. The matrix $A$ has the non-strict sign non-reversal property with respect to $S$ if for all vectors $0 \neq x \in S$, there exists a coordinate $i \in [n]$ such that $x_i \neq 0$ and $x_i(Ax)_i \geq 0$.

(viii) We say that a vector $x \in \mathbb{R}^n$ is $\geq 0$ (respectively, $x > 0$, $x \leq 0$, $x < 0$) if every coordinate of $x$ is $\geq 0$ (respectively, $> 0$, $\leq 0$, $< 0$).

We can now state our first main result.

**Theorem A.** Let $m, n \geq k \geq 1$ be integers. Given $A \in \mathbb{R}^{m \times n}$, the following statements are equivalent.

(i) The matrix $A$ is totally positive of order $k$.

(ii) For every square submatrix $A_r$ of $A$ of size $r \in [k]$, LCP($A_r, q$) has a unique solution for all $q \in \mathbb{R}^r$.

(iii) For every contiguoussquare submatrix $A_r$ of $A$ of size $r \in [k]$, LCP($A_r, q$) has a unique solution for all $q \in \mathbb{R}^r$ with $q < 0$.

(iv) For every contiguoussquare submatrix $A_r$ of $A$ of size $2 \leq r \leq k$ and for all $q \in \mathbb{R}^r$ with $q < 0$, SOL($A_r, q$) does not simultaneously contain two vectors with sign pattern

$$
\begin{pmatrix}
+ \\
0 \\
+ \\
0 \\
\vdots
\end{pmatrix}
\quad \text{and} \quad
\begin{pmatrix}
0 \\
+ \\
0 \\
+ \\
\vdots
\end{pmatrix}.
$$

Moreover for $r = 1$ and all $i \in [m]$, $j \in [n]$, LCP($a_{ij_{1 \times 1}}, q$) has a solution for some scalar $q < 0$.

In fact, we improve this result by characterizing total positivity in terms of the number of solutions of the LCP at a single vector, for each contiguoussubmatrix.

**Theorem B.** Let $m, n \geq k \geq 1$ be integers. Given $A \in \mathbb{R}^{m \times n}$, the following statements are equivalent.

(i) The matrix $A$ is totally positive of order $k$.

(ii) For every $r \in [k]$ and contiguous $r \times r$ submatrix $A_r$ of $A$, define the vectors

$$
x^{A_r} := (A_r^{11}, 0, A_r^{13}, 0, ...)^T, \quad q^{A_r} := -A_r x^{A_r}. \quad (1.2)
$$

Then $x^{A_r}$ is the only solution of LCP($A_r, q^{A_r}$).

Here, the determinant of the empty matrix is defined to be one.
An immediate application of Theorem A is a novel characterization of Pólya frequency sequences of order \( k \) via linear complementarity. These are real sequences \( (c_n)_{n \in \mathbb{Z}} \) such that for all integers

\[
1 \leq r \leq k, \quad m_1 < \cdots < m_r, \quad n_1 < \cdots < n_r,
\]

the determinant \( \det(c_{m_i-n_j})_{i,j=1}^r \geq 0 \). If all such determinants are positive, we say that the sequence is a \( TP_k \) Pólya frequency sequence.

**Corollary 1.2.** Let \( k \geq 1 \) be an integer. A real sequence \( (c_n)_{n \in \mathbb{Z}} \) is a \( TP_k \) Pólya frequency sequence, if and only if for all integers \( r \in [k] \) and \( l \in \mathbb{Z} \), and all vectors \( q \in \mathbb{R}^r \) with \( q < 0 \), there exists a unique \( x \in [0,\infty)^r \) such that

\[
y_i := \sum_{j=1}^r c_{l+i-j} x_j + q_i \geq 0, \quad x_i y_i = 0, \quad \forall i \in [r]. \tag{1.3}
\]

This can be shown by applying Theorem A to the square submatrices

\[
\begin{pmatrix}
  c_l & c_{l-1} & \cdots & c_{l-r+1} \\
  c_{l+1} & c_l & \cdots & c_{l-r+2} \\
  \vdots & \vdots & \ddots & \vdots \\
  c_{l+r-1} & c_{l+r-2} & \cdots & c_l
\end{pmatrix}, \quad l \in \mathbb{Z}.
\]

We now explain the organization of the paper, including two additional results below. In the next section we prove Theorems A and B. In addition to classical results by Fekete, Ingleton, Samelson–Thrall–Wesler, and Schoenberg, the key new ingredient is a novel characterization of total positivity (in fact of \( TP_k \)), which uses the sign non-reversal property and which was not known until our recent joint work [8].

In Section 3, we return to our recent joint work [8], in which we showed that \( TN_k \) matrices are characterized by the sign non-reversal property at a single vector (for each square submatrix). A similar result for \( TP_k \) remained elusive, though we had shown a characterization involving the sign non-reversal property at a single vector, but with an (uncountable) additional set of constraints. Our next main result, Theorem C, addresses this gap and provides truly a single test vector for each square submatrix of a \( TP_k \) matrix without additional conditions.

In Section 4 we return to an even earlier, fundamental result of Gantmacher and Krein in 1950 [21] (see also its stronger version in [7]). This is a well-known characterization of total positivity, in terms of the variation diminishing property on the test set of all real vectors, which has numerous applications (see [27, 35]). Our final main result, Theorem D, improves on this by reducing the test set to a single vector for each square submatrix.

In the final section, we take a second look at our results in the previous two Sections 3 and 4. We showed in these two sections (and previous work) that the variation diminishing property and the sign non-reversal property, each at a single test vector (for each square submatrix of \( A \)) suffices to prove the total positivity of the matrix \( A \). The proofs reveal that these test vectors necessarily have coordinates with alternating signs. We now show that such ‘single test vectors’ must have alternating-signed coordinates. Namely, any \( TP_{n-1} \) or \( TN_{n-1} \) matrix in \( \mathbb{R}^{n \times n} \), even one with a negative determinant, satisfies the variation diminishing property and the sign non-reversal property on every vector in every other (open) orthant in \( \mathbb{R}^n \). We also provide a similar observation about the LCP.
We conclude this section with some general remarks. In 1937, Gantmacher–Krein \[20\] gave a fundamental characterization of totally positive matrices of order \(k\) by the positivity of the spectra of all submatrices of size at most \(k\). There is a well-known article \[17\] by Fomin–Zelevinsky about tests for totally positive matrices; there have been numerous subsequent papers along this theme, for example, \[6\]. The present paper may be regarded as being similar in spirit.

2 \ THEOREMS A AND B: TOTAL POSITIVITY AND THE LINEAR COMPLEMENTARITY PROBLEM

In this section we prove Theorems A and B. To proceed, we require two preliminary results. The first result establishes a connection between the LCP and matrices with positive principal minors (these are known as \(P\)-matrices):

**Theorem 2.1** (Ingleton \[24\], Samelson–Thrall–Wesler \[38\]). A matrix \(A \in \mathbb{R}^{n \times n}\) has all principal minors positive if and only if LCP\((A, q)\) has a unique solution for all \(q \in \mathbb{R}^n\).

The proof of Theorem A also uses the following result, proved in recent joint work, which characterizes total positivity in terms of the sign non-reversal phenomenon.

**Theorem 2.2** \[8\]. Let \(m, n \geq k \geq 1\) be integers. Given \(A \in \mathbb{R}^{m \times n}\), the following statements are equivalent.

(i) The matrix \(A\) is totally positive of order \(k\).
(ii) Every square submatrix of \(A\) of size \(r \in [k]\) has the sign non-reversal property with respect to \(\mathbb{R}^r\).
(iii) Every contiguous square submatrix of \(A\) of size \(r \in [k]\) has the sign non-reversal property with respect to \(\mathbb{R}^r_{\text{alt}}\).

We will also revisit and strengthen this result in Theorem C below.

**Proof of Theorem A.** That (i) \(\Rightarrow\) (ii) follows from Theorem 2.1, while (ii) \(\Rightarrow\) (iii) \(\Rightarrow\) (iv) is immediate. To show (iv) \(\Rightarrow\) (i), we first claim that all entries of \(A\) are positive. Indeed, if \(a_{ij} \leq 0\) for some \(i \in [m], j \in [n]\), then LCP\((a_{ij})_{1 \times 1}, q)\) does not have a solution for any scalar \(q < 0\).

Next, we show that all the minors of \(A\) of size \(2 \leq r \leq k\) are positive. By Theorem 2.2, it suffices to show that every contiguous square submatrix of \(A\) of size \(r \in [k] \setminus \{1\}\) has the sign non-reversal property with respect to \(\mathbb{R}^r_{\text{alt}}\). Let \(r \in [k] \setminus \{1\}\) and suppose for contradiction that \(A_r\) is an \(r \times r\) contiguous submatrix of \(A\) such that \(A_r\) does not satisfy the sign non-reversal property with respect to \(\mathbb{R}^r_{\text{alt}}\). Then there exists \(x \in \mathbb{R}^r_{\text{alt}}\) such that \(x_i(A_r)x_i \leq 0\) for all \(i \in [r]\). Let \(x^+ := \frac{1}{2}(|x| + x)\) and \(x^- := \frac{1}{2}(|x| - x)\), where we define \(|(x_1, \ldots, x_n)^T| := (|x_1|, \ldots, |x_n|)^T\). Then \(x^+\) has sign pattern

\[
\begin{pmatrix}
+ & 0 \\
+ & 0 \\
0 & + \\
\vdots & \vdots
\end{pmatrix}
\]

and \(x^-\) has sign pattern

\[
\begin{pmatrix}
0 & + \\
0 & + \\
\vdots & \vdots
\end{pmatrix}
\]
Let \( v = A_r x \) and set \( v^\pm := \frac{1}{2}(|v| \pm v) \). Note that \( x = x^+ - x^- \) and \( A_r x = v^+ - v^- \). Define

\[
q := v^+ - A_r x^+ = v^- - A_r x^-.
\]

Since \( x^+, x^- \geq 0 \) and \( A_r > 0 \), we have \( q < 0 \). Also, \( (x^+)^T v^+ = 0 \) and \( (x^-)^T v^- = 0 \), since \( x_i v_i \leq 0 \) for all \( i \in [r] \). Thus \( \text{LCP}(A_r, q) \) has solutions having sign patterns

\[
\begin{bmatrix}
+ \\
0 \\
+ \\
0 \\
\vdots \\
+
\end{bmatrix}
\quad \text{and} \quad
\begin{bmatrix}
0 \\
+ \\
0 \\
+ \\
\vdots
\end{bmatrix}.
\]

This yields the desired contradiction. □

To prove Theorem B, we require the well-known 1912 result of Fekete for \( TP \) matrices, which was extended in 1955 by Schoenberg to \( TP_k \) matrices.

**Theorem 2.3** (Fekete [16], Schoenberg [41]). Let \( m, n \geq k \geq 1 \) be integers. Then \( A \in \mathbb{R}^{m \times n} \) is \( TP_k \) if and only if every contiguous square submatrix of \( A \) of size \( r \in [k] \) has positive determinant.

**Proof of Theorem B.** That \( (i) \Rightarrow (ii) \) follows from Theorem A. To show \( (ii) \Rightarrow (i) \), by the Fekete–Schoenberg Theorem 2.3, it suffices to show all contiguous minors of \( A \) of size \( r \in [k] \) are positive. This is shown by induction on \( r \). Let \( r = 1 \) and let \( A_1 = (a_{ij}) \) for some \( i \in [m], \ j \in [n] \). Then \( x^{A_1} = (1) \) and \( q^{A_1} = -(a_{ij}) \). If \( a_{ij} = 0 \), then \( \text{LCP}(A_1, q^{A_1}) \) has infinitely many solutions, a contradiction. If \( a_{ij} < 0 \), then \( \text{LCP}(A_1, q^{A_1}) \) has two solutions, again a contradiction. Thus all \( 1 \times 1 \) minors of \( A \) are positive. Let \( r \in [k] \setminus \{1\} \) and suppose that all contiguous minors of \( A \) of size at most \( (r-1) \) are positive. Let \( A_r \) be an \( r \times r \) contiguous submatrix of \( A \), and define

\[
x^{A_r} := (A_{11}^r, 0, A_{13}^r, 0, \ldots)^T, \quad z^{A_r} := (0, A_{12}^r, 0, A_{14}^r, \ldots)^T, \quad q^{A_r} := -A_r x^{A_r}.
\]

By the Fekete–Schoenberg Theorem 2.3, all proper minors of \( A_r \) are positive and so \( x^{A_r}, z^{A_r} \geq 0 \). Thus \( x^{A_r} \) is a solution of \( \text{LCP}(A_r, q^{A_r}) \).

Next, observe that, if \( A_r \) is singular, then

\[
A_r \begin{pmatrix}
A_{11}^r \\
0 \\
A_{13}^r \\
0 \\
\vdots
\end{pmatrix} = A_r \begin{pmatrix}
0 \\
A_{12}^r \\
0 \\
A_{14}^r \\
\vdots
\end{pmatrix},
\]

so \( z^{A_r} \) is another solution of \( \text{LCP}(A_r, q^{A_r}) \), a contradiction. Thus \( A_r \) is invertible. We now claim that \( \det A_r > 0 \). Indeed, suppose \( \det A_r < 0 \). Then

\[
y = A_r z^{A_r} + q^{A_r} = -(\det A_r)e_1 \geq 0 \quad \text{and} \quad y^T z^{A_r} = 0.
\]
Thus, $z^{A_r}$ is again another solution of $LCP(A_r, q^{A_r})$, a contradiction by (ii). Hence $\det A_r > 0$ and the proof is complete. □

**Remark 2.4.** In Theorem B, instead of the vector $x^{A_r} = (A^1_{r1}, 0, A^1_{r3}, 0, \ldots)^T$, we can take $x^{A_r i} := (A^1_{r1}, 0, A^1_{r3}, 0, \ldots)^T$ for odd $i \in [r]$, or a positive linear combination of some of these vectors. The proof is similar to that of Theorem B, where we define $q^{A_r}$ similarly as in (1.2).

## 2.1  Totally non-negative matrices and the LCP

We now turn our attention to identifying $TN_k$ matrices via the LCP. For a totally non-negative matrix $A \in \mathbb{R}^{n \times n}$, $LCP(A, q)$ need not have a solution for some $q \in \mathbb{R}^n$. For instance, $A = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}$ is a totally non-negative matrix, but $LCP(A, q)$ has no solution for $q = \begin{pmatrix} 0 \\ -1 \end{pmatrix}$.

For an $n \times n$ real matrix $A$, let $Q_A$ denote the set of all $q \in \mathbb{R}^n$ for which $SOL(A, q) \neq \emptyset$. If $A \in \mathbb{R}^{n \times n}$ is a matrix with non-negative entries, then $Q_A = \mathbb{R}^n$ if and only if all the diagonal entries of $A$ are positive [11, Chapter 3.8]. Our next result gives a sufficient condition for total non-negativity via the LCP. To proceed further, we need a basic result characterizing $TN_k$, which was surprisingly discovered only recently.

**Theorem 2.5** [8]. Let $m, n \geq k \geq 1$ be integers. Given $A \in \mathbb{R}^{m \times n}$, the following statements are equivalent.

(i) The matrix $A$ is totally non-negative of order $k$.

(ii) Every square submatrix of $A$ of size $r \in [k]$ has the non-strict sign non-reversal property with respect to $\mathbb{R}^r$.

(iii) Every square submatrix of $A$ of size $r \in [k]$ has the non-strict sign non-reversal property with respect to $\mathbb{R}_r$.

We now show how to apply the LCP to deduce total non-negativity.

**Proposition 2.6.** Let $m, n \geq k \geq 1$ be integers and let $A \in \mathbb{R}^{m \times n}$. Then $A$ is totally non-negative of order $k$ if the following two conditions hold, for $1 \leq r \leq k$.

(i) ($2 \leq r \leq k$): For every $r \times r$ submatrix $A_r$ of $A$ and for all $q \in Q_{A_r}$ with $q \leq 0$, if $z^1 = (x_1, 0, x_3, ...)^T$ and $z^2 = (0, x_2, 0, x_4, ...)^T$ (where all $x_i > 0$) are two solutions of $LCP(A_r, q)$, then $A_r z^1 = A_r z^2$.

(ii) ($r = 1$): For every $1 \times 1$ submatrix $A_1$ of $A$ and for all scalars $q \in Q_{A_1}$, if $z^1$ and $z^2$ are two solutions of $LCP(A_1, q)$, then $A_1 z^1 = A_1 z^2$.

**Proof.** Firstly we show that all $1 \times 1$ minors of $A$ are non-negative. Let $A_1 = (a_{ij})$ for some $i \in [m], j \in [n]$. If $a_{ij} = 0$, then we are done. If $a_{ij} < 0$, then $LCP(A_1, q)$ has two solutions $z^1 = (1)$ and $z^2 = (0)$, where $q = -(a_{ij})$, but $A_1 z^1 \neq A_1 z^2$, a contradiction. Thus the matrix $A$ has non-negative entries.

Next we claim that the determinant of every square submatrix $A_r$ of $A$ of size $r \in [k] \setminus \{1\}$ is non-negative. Fix $r \in [k] \setminus \{1\}$ and let $A_r$ be an $r \times r$ submatrix of $A$. By Theorem 2.5, it is sufficient
to show that $A_r$ has the non-strict sign non-reversal property with respect to $\mathbb{R}^r_{alt}$. Suppose that $A_r$ does not satisfy this property. Then there exists $x \in \mathbb{R}^r_{alt}$ such that $x_i(A_r x)_i < 0$ for all $i \in [r]$. Defining $x^+, x^-, v^+, v^-$ and $q$ as in the proof of Theorem A, we conclude that $x^+$ and $x^-$ are two solutions of $LCP(A_r, q)$. Also, $x^+, x^-$ have sign patterns

$$
\begin{pmatrix} + \\ 0 \\
+ \\
0 \\
\vdots 
\end{pmatrix} \quad \text{and} \quad \begin{pmatrix} 0 \\ + \\
0 \\
+ \\
\vdots 
\end{pmatrix},
$$

respectively (or vice versa), so $q \leq 0$ (since $A \geq 0$). Since $Ax \in \mathbb{R}^r_{alt}$, by (2.1), we have

$$A_r x^+ + q = v^+ \neq v^- = A_r x^- + q.$$

Thus $A_r x^+ \neq A_r x^-$, a contradiction. Hence $A$ is $TN_k$. □

The converse of the above result need not be true. We illustrate this with an example.

**Example 2.7.** Let

$$A = \begin{pmatrix} 2 & 1 & 1 \\
2 & 1 & 1 \\
1 & 1 & 1 
\end{pmatrix} \quad \text{and} \quad q = \begin{pmatrix} -3 \\
-3 \\
-2 
\end{pmatrix}.$$

Then $A$ is a totally non-negative matrix and

$$z^1 = \begin{pmatrix} 1 \\
0 \\
1 
\end{pmatrix} \quad \text{and} \quad z^2 = \begin{pmatrix} 0 \\
3 \\
0 
\end{pmatrix}$$

are two solutions of $LCP(A, q)$, but $A z^1 \neq A z^2$.

The next result improves on the previous one, by giving a sufficient condition for total non-negativity using the LCP at a single vector $q$ (for each square submatrix of $A$).

**Proposition 2.8.** Let $m, n \geq k \geq 1$ be integers and $A \in \mathbb{R}^{m \times n}$. Then $A$ is $TN_k$ if for every square submatrix $A_r$ of $A$ of size $r \in [k]$, if $z^1$ and $z^2$ are two solutions of $LCP(A_r, q^{A_r})$, then $A_r z^1 = A_r z^2$, where $q^{A_r}$ is defined as in (1.2).

**Proof.** We show that $\det A_r \geq 0$ for all $r \times r$ submatrices $A_r$ of $A$, by induction on $r \in [k]$. The base case $r = 1$ can be proved similarly to Proposition 2.6.

Let $r \in [k] \setminus \{1\}$ and suppose that all the minors of $A$ of size at most $(r - 1)$ are non-negative. Let $A_r$ be a square submatrix of $A$ of size $r$. If $\det A_r = 0$, then we are done. If $\det A_r < 0$, repeating
the proof of Theorem B, once again we have

\[
x^A_r = \begin{pmatrix} A_{11}^r \\ 0 \\ A_{13}^r \\ 0 \\ \vdots \\ 0 \\ A_{14}^r \\ \vdots \end{pmatrix} \quad \text{and} \quad z^A_r = \begin{pmatrix} 0 \\ A_{12}^r \\ 0 \\ A_{14}^r \\ \vdots \\ 0 \\ A_{14}^r \\ \vdots \end{pmatrix}
\]

are two distinct solutions of LCP\((A_r, q^A_r)\), but \(Ax^A_r \neq Az^A_r\). Thus \(\det A_r > 0\) and hence \(A\) is \(TN_k\). □

**Remark 2.9.** The converse of the preceding proposition need not be true. For instance,

\[
A = \begin{pmatrix} 0 & 0 & 0 \\ 2 & 2 & 1 \\ 1 & 1 & 1 \end{pmatrix}
\]

is a totally non-negative matrix and

\[
q^A = -\begin{pmatrix} 0 \\ 2 \\ 1 \end{pmatrix}
\]

Then

\[
z^1 = \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix} \quad \text{and} \quad z^2 = \begin{pmatrix} 2 \\ 0 \\ 0 \end{pmatrix}
\]

are two solutions of LCP\((A, q^A)\), but \(Az^1 \neq Az^2\).

In the last part of this section, we discuss the solution set of LCP\((A, q)\), with \(A \in \mathbb{R}^{n \times n}\) being a totally non-negative matrix. Firstly we recall a 1968 result of Karlin for non-singular totally non-negative matrices.

**Lemma 2.10** (Karlin, [27]). Let \(A \in \mathbb{R}^{n \times n}\) be a non-singular totally non-negative matrix. Then all the principal minors of \(A\) are positive.

Thus, if \(A\) is a non-singular totally non-negative matrix, by Theorem 2.1, LCP\((A, q)\) has a unique solution for all \(q \in \mathbb{R}^n\).

Given a matrix \(A \in \mathbb{R}^{n \times n}\), a solution \(x\) of LCP\((A, q)\) is called a non-degenerate solution if \(x_i \neq (Ax + q)_i\) for all \(i \in [n]\). Note that a solution \(x\) of the LCP\((A, q)\) is non-degenerate if and only if the support of \(x\) and the support of \(y := Ax + q\) are complementary index sets in \([n]\). A vector \(q \in \mathbb{R}^n\) is called non-degenerate with respect to \(A\) if all the solutions of LCP\((A, q)\) are non-degenerate. With this information in hand, we now present a partial converse of Proposition 2.6.
Lemma 2.11. Let $A \in \mathbb{R}^{n \times n}$ be a totally non-negative matrix and let $q \in \mathbb{R}^n$ be a non-degenerate vector with respect to $A$. If $z^1 = (x_1, 0, x_3, \ldots)^T$ and $z^2 = (0, x_2, 0, x_4, \ldots)^T$ are two solutions of LCP($A, q$) with all $x_i > 0$, then $A_z z^1 = A_z z^2$.

Proof. Suppose that $A \in \mathbb{R}^{n \times n}$ is a totally non-negative matrix and $q$ is a non-degenerate vector with respect to $A$. Let $z^1 = (x_1, 0, x_3, \ldots)^T$ and $z^2 = (0, x_2, 0, x_4, \ldots)^T$ be two solutions of LCP($A, q$) such that all $x_i > 0$ and $Az^1 \neq Az^2$. Let $y^1 = Az^1 + q$ for $i = 1, 2$ and let $z = z^1 - z^2$ and $y = y^1 - y^2$. Then $y = Az$ and $y, z \in \mathbb{R}^n_{\text{alt}}$, since $q$ is non-degenerate. Since $z^1$ and $z^2$ are solutions of LCP($A, q$), $\text{sgn}(y_j) = -\text{sgn}(z_j)$ for all $j \in [n]$. Thus $z_i (Az)_i < 0$ for all $i \in [n]$, a contradiction by Theorem 2.5. Hence $Az^1 = Az^2$. \qed

In the next result we put certain conditions on the matrix $A$ instead of the vector $q$, and present another partial converse of Proposition 2.6.

Theorem 2.12. Let $A \in \mathbb{R}^{n \times n}$ be a totally non-negative matrix such that whenever a set of columns of $A$ forms a basis of its column space, the corresponding principal submatrix is invertible. For all $q \in Q_A$, if $z^1$ and $z^2$ are two solutions of LCP($A, q$), then $Az^1 = Az^2$.

Proof. We prove this by contradiction. Let $q \in Q_A$ and let $z^1$ and $z^2$ be two distinct solutions of LCP($A, q$) such that $Az^1 \neq Az^2$. Let $z = z^1 - z^2$ and $y = y^1 - y^2$, where $y^1 := Az^1 + q$ and $y^2 := Az^2 + q$. By the definition of the LCP, $z^*_i y^*_i = 0$ and $z^*_i, y^*_i \geq 0$ for $k = 1, 2$ and all $i$, so

$$z_i y_i = (z^1_i - z^2_i)(y^1_i - y^2_i) = -z^1_i y^2_i - z^2_i y^1_i \leq 0 \text{ for } i \in [n].$$

Also, $y \neq 0$, since $Az^1 \neq Az^2$. Let $J_z \subseteq [n]$ denote the support of $z$. Then

$$y_i \leq 0 \text{ (} > 0 \text{) whenever } z_i > 0 \text{ (} < 0 \text{) for all } i \in J_z. \quad (2.2)$$

We first claim that there exists $x \in \mathbb{R}^n$ with support $J_x \subseteq J_z$ such that $\text{sgn}(z_i) = \text{sgn}(x_i)$ for $i \in J_x$, $Ax = y$, and the columns of $A$ corresponding to $J_x$ are linearly independent. If the columns of $A$ corresponding to $J_z$ are linearly independent, then we are done. Otherwise there exists a non-zero vector $v \in \mathbb{R}^n$ such that $Av = 0$ and $v_j = 0$ for $j \in [n] \setminus J_z$. Define

$$\alpha := \min \left\{ \frac{|z_i|}{|v_i|} : i \in J_z \text{ and } v_i \neq 0 \right\}, \quad x^1 := z - \text{sgn}(z_{i_0}) \text{sgn}(v_{i_0}) \alpha v, \quad (2.3)$$

where $i_0 \in J_z$ is the index where the minimum $\alpha$ is attained. Then $Ax^1 = y$ and $\text{sgn}(z_i) = \text{sgn}(x^1_i)$ for $i \in J_{x^1}$. Since $x^1_{i_0} = \text{sgn}(z_{i_0}) \frac{|z_{i_0}| - \text{sgn}(v_{i_0}) \alpha v_{i_0}|}{|v_{i_0}|}$, for at least one component $i \in J_z, x^1_i = 0$. Now, if the columns of $A$ corresponding to $J_{x^1}$ are linearly independent, then we are done. Otherwise, we apply the same technique to the new vector and keep repeating. Thus there exists $x \in \mathbb{R}^n$ with support $J_x \subseteq J_z$ such that

$$\text{sgn}(z_i) = \text{sgn}(x_i) \text{ for } i \in J_x, \quad Ax = y, \quad (2.4)$$
and the columns of $A$ corresponding to $J_x$ are linearly independent. Since $A$ is totally non-negative, by the hypothesis and Lemma 2.10, all the principal minors of $A_{J_x}$ are positive, where $A_{J_x}$ denotes the principal submatrix of $A$ whose rows and columns are indexed by $J_x$. By (2.2) and (2.4), $A_{J_x}$ reverses the sign of $x_{J_x}$, a contradiction by [19, Theorem 2]. Thus $Az^1 = Az^2$. □

The next remark, which is a standalone observation that may be of independent interest, suggests steps to solve the problem $LCP(A, q)$ with $A$ being a TP/TN matrix.

Remark 2.13. Let $A \in \mathbb{R}^{n \times n}$ be a totally positive/non-negative matrix and let $q \in \mathbb{R}^n$. If a coordinate of $q$ is non-negative, say $q_1 \geq 0$, then we obtain the submatrix $B$ from $A$ by deleting the corresponding row and column of $A$, and similarly we obtain a vector $q^2$ from $q$. Next we try to solve the new linear complementarity problem $LCP(B, q^2)$. If $x^2$ is a solution of the new problem $LCP(B, q^2)$, then $x := (0, (x^2)^T)^T$ is a solution of the original problem $LCP(A, q)$.

3 | THEOREM C: SIGN NON-REVERSAL PROPERTY FOR TOTALLY POSITIVE MATRICES

In recent joint work [8], Theorem 2.5 above had a fourth part in terms of a single vector, for characterizing $TN_k$ matrices. Similarly, we gave a new test for total positivity using the sign non-reversal property at a single vector, but under certain additional conditions.

**Theorem 3.1** [8]. Let $m, n \geq k \geq 1$ be integers. Given $A \in \mathbb{R}^{m \times n}$, the following statements are equivalent.

1. The matrix $A$ is totally positive of order $k$.
2. For every $r \in [k]$ and contiguous $r \times r$ submatrix $A_r$ of $A$, define the vectors
   \[
   d^{[r]} := (1, -1, 1, \ldots, (-1)^{r-1})^T, \quad z^{A_r} := (\det A_r) \text{adj}(A_r)d^{[r]}.
   \]

   Now: (i) $A_rx \neq 0$ for all $x \in \mathbb{R}^r_{\text{alt}}$; and (ii) $A_r$ has the non-strict sign non-reversal property with respect to $z^{A_r}$.

In the next result, we drop the condition (i) that is, $A_rx \neq 0$ for all $x \in \mathbb{R}^r_{\text{alt}}$, and identify a new test vector for the sign non-reversal property which is simpler than (3.1). In particular, we are able to characterize total positive matrices using the sign non-reversal property truly at a single vector.

**Theorem C.** Let $m, n \geq k \geq 1$ be integers. Given $A \in \mathbb{R}^{m \times n}$, the following statements are equivalent.

1. The matrix $A$ is totally positive of order $k$.
2. For every $r \in [n]$ and contiguous $r \times r$ submatrix $A_r$ of $A$, define the vector
   \[
   x^{A_r} := (A_{11}^r, -A_{12}^r, \ldots, (-1)^{r-1}A_{rr}^r)^T.
   \]

   Then $A_r$ has the sign non-reversal property with respect to $x^{A_r}$.

**Proof.** That (i) $\Rightarrow$ (ii) is immediate from Theorem 2.2 (iii).
We will now prove \((ii) \Rightarrow (i)\) using induction on the size of the contiguous minors of \(A\) (by the Fekete–Schoenberg Theorem 2.3). The base case \(r = 1\) directly follows from the hypothesis in \((ii)\). For the induction step, assume that all contiguous minors of \(A\) of size at most \(r - 1\) (where \(2 \leq r \leq k\)) are positive. Let \(A_r\) be an \(r \times r\) contiguous submatrix of \(A\). By the induction hypothesis, all the proper contiguous minors of \(A_r\) are positive. Thus all the proper minors of \(A_r\) are positive by the Fekete–Schoenberg Theorem 2.3. Define the vector \(x^{A_r}\) as in (3.2). Then \(x^{A_r} \in \mathbb{R}_{alt}^r\), and
\[
A_r x^{A_r} = (\det A_r)e_1.
\]
Now we claim that \(\det A_r > 0\). By hypothesis, there exists \(i \in [r]\) such that
\[
0 < x^{A_r}_i (A_r x^{A_r})_i = (\det A_r) x^{A_r}_i e_1^i.
\]
Thus \(i = 1\) and \(\det A_r > 0\).

\[\square\]

Remark 3.2. Note that the vector \(x^{A_r}\) as defined in (3.2) is the first column of \(\text{adj}(A_r)\). Instead of \(\text{adj}(A_r)e_1\) one can take \(x^{A_r} = \text{adj}(A_r)e^j\) for any \(j \in [r]\), or even \(x^{A_r} := \text{adj}(A_r)\alpha\), where \(\alpha = (\alpha_1, -\alpha_2, \alpha_3, \ldots, (-1)^{r-1}\alpha_r)^T\) is an arbitrary non-zero vector in the orthant where all \(\alpha_i \geq 0\) (or \(\leq 0\)). Then Theorem C still holds, with a similar proof.

Remark 3.3. Recently in [8], we discussed a new characterization of totally non-negative matrices in terms of the non-strict sign non-reversal property. We proved that \(A \in \mathbb{R}^{m \times n}\) is totally non-negative of order \(k\) if and only if every submatrix \(A_r\) of \(A\) of size \(r \in [k]\) has the non-strict sign reversal property with respect to a single vector of the form \(z^{A_r} = (\det A_r)\text{adj}(A_r)d^{[r]}\) as in (3.1). It is easy to verify that the result is still true if we take any non-negative integer power of \(\det A_r\). More generally, the result holds if we take \(z^{A_r} := \text{adj}(A_r)\alpha\) for arbitrary fixed \(\alpha \in \mathbb{R}_{alt}^r\).

4 | THEOREM D: VARIATION DIMINUTION AND TOTAL POSITIVITY

A very important and widely used characterization of totally positive and totally non-negative matrices is in terms of their variation diminishing property. The term ‘variation diminishing’ was coined by Pólya in correspondence with Fekete in 1912 [16] to prove the following result (stated by Laguerre [31]) using Pólya frequency sequences and their variation diminishing property: given a polynomial \(f(x)\) and an integer \(s \geq 0\), the number \(\text{var}(e^{sx} f(x))\) of variations in the Maclaurin coefficients of \(e^{sx} f(x)\) is non-increasing in \(s\), hence is bounded above by \(\text{var}(f) < \infty\). The variation diminishing property of totally non-negative matrices was first studied by Schoenberg [39] in 1930. In 1950 [21], Gantmacher–Krein made fundamental contributions relating total positivity and variation diminution. To proceed, we need some notation.

Definition 4.1. Given a vector \(x \in \mathbb{R}^n\), let \(S^-(x)\) denote the number of changes in sign after deleting all zero entries in \(x\). Next, the zero entries of \(x\) are arbitrarily assigned a value of \(\pm 1\), and we denote by \(S^+(x)\) the maximum possible number of sign changes in the resulting sequence. For \(0 \in \mathbb{R}^n\), we set \(S^+(0) := n\) and \(S^-(0) := 0\).
The following result of Brown–Johnstone–MacGibbon [7] (see also Gantmacher–Krein [21, Chapter V]) gives a characterization of totally positive matrices in terms of the variation diminishing property (cited from Pinkus’s book).

**Theorem 4.2** [36, Theorem 3.3]. Given a real $m \times n$ matrix $A$, the following statements are equivalent.

(i) $A$ is totally positive.

(ii) For all $0 \neq x \in \mathbb{R}^n$, $S^+(Ax) \leq S^-(x)$. If moreover equality occurs and $Ax \neq 0$, the first (last) component of $Ax$ (if zero, the unique sign required to determine $S^+(Ax)$) has the same sign as the first (last) non-zero component of $x$.

We next recall the analogous characterization for totally non-negative matrices using the variation diminishing property.

**Theorem 4.3** [36, Theorem 3.4]. Given a real $m \times n$ matrix $A$, the following statements are equivalent.

(i) $A$ is totally non-negative.

(ii) For all $x \in \mathbb{R}^n$, $S^-(Ax) \leq S^-(x)$. If moreover equality occurs and $Ax \neq 0$, the first (last) non-zero component of $Ax$ has the same sign as the first (last) non-zero component of $x$.

Observe that the set of test vectors in the second statements of Theorems 4.2 and 4.3 is uncountable. It is natural to ask if this can be reduced to a finite set of test vectors? Our next result provides a positive answer — in fact, a single vector for each submatrix.

**Theorem D.** Given a real $m \times n$ matrix $A$, the following statements are equivalent.

(i) $A$ is totally positive.

(ii) For all $0 \neq x \in \mathbb{R}^n$, $S^+(Ax) \leq S^-(x)$. If moreover equality occurs and $Ax \neq 0$, the first (last) component of $Ax$ (if zero, then the unique sign required to determine $S^+(Ax)$) has the same sign as the first (last) non-zero component of $x$.

(iii) For every $r \in [\min\{m, n\}]$ and contiguous $r \times r$ submatrix $A_r$ of $A$, define the vector

$$x^{A_r} := (A_r^{11}, -A_r^{12}, \ldots, (-1)^{r-1} A_r^{rr})^T.$$  \hspace{1cm} (4.1)

Then $S^+(A_r x^{A_r}) \leq S^-(x^{A_r})$. If equality holds here, then the first (last) component of $A_r x^{A_r}$ (if zero, the unique sign required to determine $S^+(A_r x^{A_r})$) has the same sign as the first (last) non-zero component of $x^{A_r}$.

**Proof.** That $(i) \Rightarrow (ii)$ was Theorem 4.2. We first show $(ii) \Rightarrow (iii)$. Let $k = \min\{m, n\}$, fix $r \in [k]$ and let $A_r$ be a contiguous submatrix of $A$, say $A_r = A_{IJ}$ for contiguous sets of indices $I \subseteq [m]$ and $J \subseteq [n]$ with $|I| = |J| = r$. Define $x^{A_r}$ as in (4.1); note that this is non-zero. We extend $x^{A_r}$ to $x \in \mathbb{R}^n$ by embedding in positions $J$ and padding by zeroes elsewhere. Then

$$S^-(x) = S^-(x^{A_r}) \text{ and } S^+(Ax) \geq S^+(A_r x^{A_r}).$$ \hspace{1cm} (4.2)

Thus $S^+(A_r x^{A_r}) \leq S^-(x^{A_r})$, since $S^+(Ax) \leq S^-(x)$.


Suppose that $S^+(A_r x^{A_r}) = S^-(x^{A_r})$. Then $S^+(Ax) = S^-(x)$. Without loss of generality, assume that the first and last non-zero entries of $x^{A_r}$ are in positions $s, t \in [r]$, respectively. Enumerate the indices in $i \in [r]$ by $i_1 < i_2 < \cdots < i_r$. By the hypothesis, it follows that all coordinates of $Ax$ in positions $1, 2, \ldots, i_s$ (respectively, $i_t, \ldots, m$) have the same sign and this sign agrees with that of $x^{A_r}$ (respectively, $x^{A_r}$). This concludes $(ii) \Rightarrow (iii)$.

To show $(iii) \Rightarrow (i)$, by Theorem 2.3, it suffices to show that the determinants of all $r \times r$ contiguous submatrices are positive, for $1 \leq r \leq \min\{m, n\}$. We prove this by induction on $r$. The case $r = 1$ is immediate from $(iii)$. For the induction step, suppose that all contiguous minors of $A$ of size at most $(r-1)$ are positive and $A_r$ is an $r \times r$ contiguous submatrix of $A$. By the Fekete–Schoenberg Theorem 2.3, $A_r$ is $TP_{r-1}$. Define the vector $x^{A_r}$ as in (4.1). Then $x^{A_r} \in \mathbb{R}^r_{alt}$ and $S^-(x^{A_r}) = r - 1$.

We first show that $A_r$ is invertible. Indeed suppose that $A_r$ is singular. Then $A_r x^{A_r} = 0$ and $r = S^+(A_r x^{A_r}) > S^-(x^{A_r})$, a contradiction. Thus $A_r$ is invertible.

Next we show that $det A_r > 0$. Since $A_r x^{A_r} = (det A_r)e^1$, we have

$$r - 1 = S^+(A_r x^{A_r}) = S^-(x^{A_r}).$$

Thus by $(iii)$, the first component of $A_r x^{A_r}$ has the same sign as the first non-zero component of $x^{A_r}$. Hence $det A_r > 0$ and the induction step is complete. \(\square\)

**Remark 4.4.** Remark 3.2 applies verbatim to Theorem D.

We conclude this section with a similar improvement to the classical characterization of TN matrices via variation diminution.

**Theorem 4.5.** Given a real $m \times n$ matrix $A$, the following statements are equivalent.

(i) $A$ is totally non-negative.

(ii) For all $x \in \mathbb{R}^n$, $S^-(Ax) \leq S^-(x)$. If moreover equality occurs and $Ax \neq 0$, then the first (last) non-zero component of $Ax$ has the same sign as the first (last) non-zero component of $x$.

(iii) For every square submatrix $A_r$ of $A$ of size $r \in [\min\{m, n\}]$ define the vector

$$y^{A_r} := \text{adj}(A_r) \alpha, \text{ for arbitrary fixed } \alpha \in \mathbb{R}^r_{alt}.$$ (4.3)

Then $S^-(A_r y^{A_r}) \leq S^-(y^{A_r})$. If equality holds here and $A_r y^{A_r} \neq 0$, the first (last) non-zero component of $A_r y^{A_r}$ has the same sign as the first (last) non-zero component of $y^{A_r}$.

**Proof.** That $(i) \Rightarrow (ii)$ was Theorem 4.3. To show $(ii) \Rightarrow (iii)$, repeat the proof of Theorem D, but working with arbitrary $r \times r$ submatrices $A_r$ of $A$, where $r \in [\min\{m, n\}]$ and the vector $y^{A_r}$ from (4.3) is used in place of $x^{A_r}$.

Next we show $(iii) \Rightarrow (i)$. Let $k = \min\{m, n\}$ and $r \in [k]$. We show that $det A_r \geq 0$ for all square submatrices $A_r$ of $A$ of size $r$. We prove this by induction on $r$, with the base case $r = 1$ immediate. Now suppose that all minors of $A$ of size at most $(r-1)$ are non-negative and $A_r$ is an $r \times r$ submatrix of $A$. If $det A_r = 0$, then we are done. Let $det A_r = 0$, $\alpha = (\alpha_1, \alpha_2, \ldots, \alpha_r)^T \in \mathbb{R}^r_{alt}$ and define $y^{A_r}$ as in (4.3). Then no row of $\text{adj}(A_r)$ is zero and $y^{A_r} \in \mathbb{R}^r_{alt}$. Since $A_r y^{A_r} = (det A_r) \alpha$, we have

$$S^-(A_r y^{A_r}) = S^-(y^{A_r}) = r - 1.$$
Also, the first entry of $A_r y^{A_r}$ is $\alpha_1 \det A_r$ and the first entry of $y^{A_r}$ is
\[
\sum_{j=1}^{r} (-1)^{j-1}\alpha_j A_r^{j1},
\]
where the summation is positive (or negative) if and only if $\alpha_1$ is positive (or negative) by the induction hypothesis. Thus $\det A_r > 0$ by assumption.

5 | TEST VECTORS FROM ANY OTHER ORTHANT DO NOT WORK

In the previous two sections, we have seen that $TP_k$ matrices are characterized by their contiguous square submatrices $A_r \ (r \in [k])$ satisfying either the variation diminishing property, or the sign non-reversal property, on the entire open bi-orthant $\mathbb{R}_{alt}^r$ for each $r \in [k]$ — or on single test vectors which turn out to lie in this bi-orthant. We conclude by explaining the sense in which these results are ‘best possible’. Informally, we claim that if $x \in \mathbb{R}^r$ lies in any other open orthant (that is, all $x_j \neq 0$ and there are two successive $x_j$ of the same sign), then every $TP_{r-1}$ matrix $A_r$ of size $r$ satisfies the variation diminishing property and the sign non-reversal property with respect to $x$. (In particular, since there exist $TP_{r-1}$ matrices $A_r$ that are not $TP$, the aforementioned characterizations cannot hold with test vectors in any open bi-orthant other than in $\mathbb{R}_{alt}^r$.)

**Theorem 5.1.** Suppose that $x \in \mathbb{R}^r$ has non-zero coordinates, and at least two successive coordinates have common sign. Let $A_r \in \mathbb{R}^{r \times r}$ be $TP_{r-1}$. Then:

1. $A_r$ satisfies the variation diminishing property with respect to $x$. In other words, $S^+(A_r x) \leq S^-(x)$. If moreover equality holds, then the first (last) component of $Ax$ (if zero, the unique sign required to determine $S^+(Ax)$) has the same sign as the first (last) component of $x$.

2. $A_r$ satisfies the sign non-reversal property with respect to $x$. In other words, there exists a coordinate $i \in [r]$ such that $x_i (A_r x)_i > 0$.

It remains to observe that there do exist matrices which are $TP_{n-1}$ but not $TP_n$, for each $n$. This follows from the analysis of a Pólya frequency function studied by Karlin [26] in 1964; this analysis was only recently carried out by Khare [29]. See Example 5.2 below, for details.

**Proof of Theorem 5.1.** Let $x \in \mathbb{R}^r$ such that all the coordinates of $x$ are non-zero, and at least two successive coordinates have common sign. Decompose $x$ into contiguous coordinates of like signs:

\[
(x_{1}, \ldots, x_{s_1}), \ (x_{s_1+1}, \ldots, x_{s_2}), \ \ldots (x_{s_{k+1}}, \ldots, x_{r}),
\]

with all coordinates in the $i$th component having the same sign, which we choose to be $(-1)^{i-1}$ without loss of generality. Set $s_0 = 0$ and $s_{k+1} = r$ and observe that $k \leq r - 2$. Let $a^1, \ldots, a^r \in \mathbb{R}^r$ denote the columns of $A_r$, and define

\[
b^i := \sum_{j=s_{i-1}+1}^{s_i} |x_j| a^j, \quad \text{for } i \in [k+1].
\]
We claim that the matrix $B := [b^1, ..., b^{k+1}] \in \mathbb{R}^{r \times (k+1)}$ is totally positive. Indeed, since all $x_j$ are non-zero, and all proper minors of $A_r$ are positive, given an integer $p \in [k + 1]$ and $p$-element subsets $I \subset [r], J \subseteq [k + 1]$ using standard properties of determinants, we have

$$
\det B_{I \times J} = \sum_{l_1 = s_{j_1} - 1 + 1}^{s_{j_1}} \prod_{l_p = s_{j_p} - 1 + 1}^{|x_{l_1}| \cdots |x_{l_p}|} \det A_{r \times L} > 0,
$$

where $L = \{l_1, ..., l_p\}$, and $B_{I \times J}$ denotes the submatrix of $B$ whose rows and columns are indexed by $I, J$, respectively. Thus $B$ is $TP$; and we also define $y := Ax = Bd^{[k+1]}$, where $d^{[k+1]} := (1, -1, 1, \ldots, (-1)^{k})^T \in \mathbb{R}^{k+1}$.

With this analysis in hand, we can now prove the theorem.

(1) Note that $S^-(x) = k$. If $S^+(Ax) > S^-(x)$, then there exist indices $i_1 < i_2 < \cdots < i_{k+2} \in [r]$ and a sign $\epsilon = \pm 1$ such that $(-1)^{t-1}\epsilon y_{i_t} \geq 0$ for $t \in [k + 2]$. Moreover at least two of the $y_{i_t}$ are non-zero, since $B$ is $TP$. Define the $(k+2) \times (k+2)$ matrix

$$
M := [y_I | B_{I \times [k+1]}], \quad \text{where } I = \{i_1, ..., i_{k+2}\}.
$$

Then $\det M = 0$, since the first column of $M$ is an alternating sum of the rest. Expanding along the first column,

$$
0 = \sum_{l=1}^{k+2} (-1)^{l-1} y_{i_l} \det B_I \setminus [i_l] \times [k+1],
$$

a contradiction, since all terms $(-1)^{l-1}y_{i_l}$ have the same sign, at least two $y_{i_l}$ are non-zero, and all minors of $B$ are positive. Thus $S^+(Ax) \leq S^-(x)$.

It remains to prove the remainder of the assertion (1). We continue to employ the notation in the preceding discussion, now using $k + 1$ in place of $k + 2$. We claim that, if $S^+(Ax) = S^-(x) = k$ with $Ax \neq 0$, and if moreover $(-1)^{t-1} \epsilon y_{i_t} \geq 0$ for $t \in [k + 1]$, then $\epsilon = 1$.

To show this we use the fact that the submatrix $B_{I \times [k+1]}$ is $TP$, where $I = \{i_1, ..., i_{k+1}\}$. Also, $B_{I \times [k+1]}d^{[k+1]} = y_I$, since $Bd^{[k+1]} = Ax$. By Cramer’s rule, the first coordinate of $d^{[k+1]}$ is

$$
1 = \frac{\det[y_I | B_{I \times [k+1]] \setminus [1]}]}{\det B_{I \times [k+1]}},
$$

Multiply both sides by $\epsilon \det B_{I \times [k+1]}$ and expand the numerator along the first column. This yields:

$$
\epsilon \det B_{I \times [k+1]} = \sum_{l=1}^{k+1} (-1)^{l-1} \epsilon y_{i_l} \det B_I \setminus [i_l] \times [k+1] \setminus [1].
$$

Since each summand on the right side is non-negative with at least one positive and $B$ is $TP$, it implies that $\epsilon = 1$.

(2) We prove this by contradiction. Let $x_i y_i \leq 0$ for all $i \in [r]$. Consider the index set

$$
I = \{i_1, ..., i_{k+1}\}, \quad \text{where } i_t \in [s_{t-1} + 1, s_t].
$$
Then the matrix $B_{I^{[k+1]}}$ is totally positive and $B_{I^{[k+1]}}d_{[k+1]} = y_I$. Thus $B_{I^{[k+1]}}$ reverses the signs of $d_{[k+1]}$, a contradiction by Theorem 2.2.

**Example 5.2.** We now explain how to construct a multi-parameter family of real $n \times n$ matrices for every integer $n \geq 3$, each of which is $TP_{n-1}$ but has negative determinant. This construction involves non-integer powers of a certain Pólya frequency function, studied by Karlin in 1964 [26]:

$$\Omega(x) := \begin{cases} xe^{-x}, & \text{if } x > 0; \\ 0, & \text{otherwise}. \end{cases} \quad (5.2)$$

Karlin showed that if $\alpha \in \mathbb{Z}_{\geq 0} \cup [k-2, \infty)$, then $\Omega(x)^\alpha$ is $TN_k$, that is, given real $x_1 < \ldots < x_k$ and $y_1 < \ldots < y_k$, the matrix $B := (\Omega(x_i - y_j)^\alpha)_{i,j=1}^k$ is $TN$.

Recently in [29], Khare showed that if $y_1 < \ldots < y_k < x_1 < \ldots < x_k$, then $(\Omega(x_i - y_j)^\alpha)_{i,j=1}^k$ is $TP_k$ if $\alpha > k-2$ and not $TN_k$ if $\alpha \in (0, k-2) \setminus \mathbb{Z}$. Thus, consider $n \geq 3$ and $\alpha \in (n-3, n-2)$, and choose real scalars $y_1 < \ldots < y_n < x_1 < \ldots < x_n$. Then the matrix $A := (\Omega(x_i - y_j)^\alpha)_{i,j=1}^n$ is $TP_{n-1}$ but not $TN_n$, whence $\det A < 0$.

An analogue of Theorem 5.1 holds for $TN_{r-1}$ matrices.

**Theorem 5.3.** Let $x \in \mathbb{R}^r \setminus \mathbb{R}_{alt}^r$ with all $x_i \neq 0$ and $A_r \in \mathbb{R}^{r \times r}$ be a $TN_{r-1}$ matrix. Then:

1. $A_r$ satisfies the variation diminishing property with respect to $x$. In other words, $S^-(A_rx) \leq S^-(x)$. If moreover equality holds and $Ax \neq 0$, then the first (last) non-zero component of $Ax$ has the same sign as the first (last) component of $x$.
2. $A_r$ satisfies the non-strict sign non-reversal property with respect to $x$. In other words, there exists a coordinate $i \in [r]$ such that $x_i(A_rx)_i \geq 0$.

The proof requires Whitney’s density result for totally positive matrices and a lemma on sign changes of limits of vectors.

**Theorem 5.4 (Whitney, [43]).** Given integers $m, n \geq k \geq 1$, the set of $m \times n$ $TP_k$ matrices is dense in the set of $m \times n$ $TN_k$ matrices.

**Lemma 5.5** [36]. Given $x = (x_1, \ldots, x_n)^T \in \mathbb{R}^n \setminus \{0\}$, define the vector

$$\bar{x} := (x_1, -x_2, x_3, \ldots, (-1)^{n-1}x_n) \in \mathbb{R}^n.$$ 

Then $S^+(x) + S^-(\bar{x}) = n - 1$. Moreover, if $\lim_{p \to \infty} x_p = x$, then

$$\liminf_{p \to \infty} S^-(x_p) \geq S^-(x), \quad \limsup_{p \to \infty} S^+(x_p) \leq S^+(x).$$

We can now prove the above properties of $TN_{r-1}$ matrices.

**Proof of Theorem 5.3.** (1) Let $x \in \mathbb{R}^r \setminus \mathbb{R}_{alt}^r$ with all $x_i \neq 0$. Since $A_r$ is totally non-negative of order $r-1$, by Whitney’s density Theorem 5.4, there exists a sequence $A_r^{(l)}$ of totally positive matrices
of order $r - 1$ such that

$$\lim_{l \to \infty} A_r^{(l)} = A_r.$$ 

Now use Theorem 5.1 and Lemma 5.5 to compute:

$$S^{-}(A_r x) \leq \liminf_{l \to \infty} S^{-}(A_r^{(l)} x) \leq \liminf_{l \to \infty} S^{+}(A_r^{(l)} x) \leq \liminf_{l \to \infty} S^{-}(x) = S^{-}(x).$$ 

Next, if equality occurs and $A_r x \neq 0$, then for all $l$ large enough, we have

$$S^{-}(A_r x) \leq S^{-}(A_r^{(l)} x) \leq S^{+}(A_r^{(l)} x) \leq S^{-}(x),$$ 

by Theorem 5.1 and Lemma 5.5. Thus $S^{-}(A_r^{(l)} x) = S^{+}(A_r^{(l)} x) = S^{-}(x)$ for $l$ sufficiently large. This implies that (for large $l$) the sign changes in $A_r^{(l)} x$ have no dependence on the zero entries. Thus the non-zero sign patterns of $A_r^{(l)} x$ agree with those of $A_r x$. Also, by Theorem 5.1, both $x$ and $A_r^{(l)} x$ admit partitions of the form (5.1) with alternating signs, with precisely $S^{-}(x)$-many sign changes. Hence the same holds for the sign patterns of $A_r x$ and $x$.

(2). By Theorem 5.4, there exists a sequence $A_r^{(l)} \to A_r$ of $TP_{r-1}$ matrices. Now $A_r^{(l)}$ is $TP_{r-1}$, so by Theorem 5.1(2) there exists $i_l \in [r]$ such that $x_{i_l}(A_r^{(l)} x)_{i_l} > 0$. Hence there exists $i_0 \in [r]$ and an increasing subsequence of positive integers $l_p$ such that $i_{l_p} = i_0$ for all $p \geq 1$. Now (2) follows:

$$x_{i_0}(A_r x)_{i_0} = \lim_{p \to \infty} x_{i_{l_p}}(A_r^{(l_p)} x)_{i_{l_p}} \geq 0. \quad \Box$$

By Example 5.2, we have a $TN_{r-1}$ matrix which is not $TN_r$. This gives us the following proposition.

**Proposition 5.6.** Test-vectors from any open orthant apart from the open bi-orthant $\mathbb{R}_{\text{alt}}^r$ cannot be used to characterize total non-negativity via either variation diminution or sign non-reversal.

We conclude with a similar observation about the LCP: Theorem A shows that for $TP_k$ matrices $A$, the solution sets to certain LCPs cannot simultaneously contain two vectors with alternately zero and positive coordinates (and disjoint supports). Our final result shows that if $A \in \mathbb{R}^{r \times r}$ is merely $TP_{r-1}$, the same holds when ‘alternating’ is replaced by ‘not always alternating’. Thus, the ‘alternation’ is also distinguished for the LCP-characterization of total positivity.

**Proposition 5.7.** If $A \in \mathbb{R}^{r \times r}$ with $r \geq 2$ is a $TP_{r-1}$ matrix, then SOL($A_r, q$) with $q < 0$ does not simultaneously contain two vectors which have disjoint supports, and at least one of which has two consecutive positive coordinates.

The proof is analogous to Theorem A using Theorem 5.1 (2). Hence by Example 5.2, totally positive matrices cannot be identified by the solution sets SOL($A, q$) of LCP, which does not simultaneously contain two vectors with disjoint supports, and at least one of which has two consecutive positive coordinates.
ACKNOWLEDGEMENTS
I thank the referee for carefully going through the paper and for their suggestions. I also thank Jürgen Garloff and Apoorva Khare for a detailed reading of an earlier draft and for providing valuable feedback. This work is supported by National Post-Doctoral Fellowship (PDF/2019/000275) from SERB (Govt. of India).

JOURNAL INFORMATION
The Bulletin of the London Mathematical Society is wholly owned and managed by the London Mathematical Society, a not-for-profit Charity registered with the UK Charity Commission. All surplus income from its publishing programme is used to support mathematicians and mathematics research in the form of research grants, conference grants, prizes, initiatives for early career researchers and the promotion of mathematics.

REFERENCES
1. A. Belton, D. Guillot, A. Khare, and M. Putinar, *Totally positive kernels, Polya frequency functions, and their transforms*, https://arxiv.org/abs/2006.16213, 2020.
2. A. Belton, D. Guillot, A. Khare, and M. Putinar, *Hirschman-Widder densities*, https://arxiv.org/abs/2101.02129, 2021.
3. A. Berenstein, S. Fomin, and A. Zelevinsky, *Parametrizations of canonical bases and totally positive matrices*, Adv. Math. **122** (1996), 49–149. https://doi.org/10.1006/aima.1996.0057.
4. C. de Boor, *On calculating with B-splines*, J. Approx. Theory **6** (1972), no. 1, 50–62. https://doi.org/10.1016/0021-9045(72)90080-9.
5. F. Brenti, *Combinatorics and total positivity*, J. Combin. Theory Ser. A **71** (1995), no. 2, 175–218. https://doi.org/10.1016/0097-3165(95)90000-4.
6. A. Brosowsky, S. Chepuri, and A. Mason, *Parametrizations of k-non-negative matrices: cluster algebras and k-positivity tests*, J. Combin. Theory Ser. A **174** (2020), art. 105217, 25 pp. https://doi.org/10.1016/j.jcta.2020.105217.
7. L. D. Brown, I. M. Johnstone, and K. B. MacGibbon, *Variation diminishing transformations: a direct approach to total positivity and its statistical applications*, J. Amer. Statist. Assoc. **76** (1981), no. 376, 824–832. https://doi.org/10.1287/moor.12475.
8. R. W. Cottle, *On a problem in linear inequalities*, J. London Math. Soc. **43** (1968), 378–384. https://doi.org/10.1112/jlms/s1-43.1.378.
9. R. W. Cottle, G. B. Dantzig, *Complementary pivot theory of mathematical programming*, Linear Algebra Appl. **1** (1968), no. 1, 103–125. https://doi.org/10.1016/0024-3795(68)90052-9.
10. R. W. Cottle, J-S Pang, and R. E. Stone, *The linear complementarity problem*, Classics in applied mathematics, SIAM, Philadelphia, PA, 2009. https://doi.org/10.1137/1.9780898719000.
11. C. W. Cryer, *The solution of a quadratic programming problem using systematic overrelaxation*, SIAM J. Control 9 (1971), 385–392. https://epubs.siam.org/doi/pdf/10.1137/0309028.
12. H. B. Curry and I. J. Schoenberg, *On Polya frequency functions IV: the fundamental spline functions and their limits*, J. Anal. Math. **17** (1966), 71–107. https://doi.org/10.1007/BF02788655.
13. B. C. Eaves and H. Scarf, *The solution of systems of piecewise linear equations*, Math. Oper. Res. **1** (1976), no. 1, 1–27. https://doi.org/10.1287/moor.1.1.1.
14. S. M. Fallat and C. R. Johnson, *Totally non-negative matrices*, Princeton Series in Applied Mathematics, Princeton University Press, Princeton, 2011. https://press.princeton.edu/books/hardcover/9780691121574/totally-non-negative-matrices.
15. M. Fekete and G. Pólya, *Über ein Problem von Laguerre*, Rend. Circ. Mat. Palermo **34** (1912), 89–120. https://doi.org/10.1007/BF03015009.
16. S. Fomin and A. Zelevinsky, *Total positivity: tests and parametrizations*, Math. Intelligencer **22** (2000), no. 1, 23–33. https://doi.org/10.1007/BF03024444.
18. S. Fomin and A. Zelevinsky, Cluster algebras. I. Foundations, J. Amer. Math. Soc. 15 (2002), no. 2, 497–529. https://doi.org/10.1090/S0894-0347-01-00385-X.

19. D. Gale and H. Nikaido, The Jacobian matrix and global univalence of mappings, Math. Ann. 159 (1965), 81–93. https://doi.org/10.1007/BF01360282.

20. F. R. Gantmacher and M. G. Krein, Sur les matrices complètement nonnégatives et oscillatoires, Compositio Math. 4 (1937), 445–476. http://www.numdam.org/item?id=CM_1937__4__445_0.

21. F. R. Gantmacher and M. G. Krein, Oscillyaciony matricy i yadra i malye kolebaniya mehaničeskikh sistem, 2nd ed., Gosudarstv. Isdat. Tehn.-Teor. Lit., Moscow-Leningrad, 1950.

22. J. Garloff and D. G. Wagner, Hadamard products of stable polynomials are stable, J. Math. Anal. Appl. 202 (1996), no. 3, 797–809. https://doi.org/10.1006/jmaa.1996.0348.

23. K. Gröchenig, J. L. Romero, and J. Stöckler, Sampling theorems for shift-invariant spaces, Gabor frames, and totally positive functions, Invent. Math. 211 (2018), 1119–1148. https://doi.org/10.1007/s00222-017-0760-2.

24. A. W. Ingleton, A problem in linear inequalities, Proc. London Math. Soc. (3) 16 (1966), 519–536. https://doi.org/10.1112/plms/s3-16.1.519.

25. A. W. Ingleton, The linear complementarity problem, J. London Math. Soc. (2) 2 (1970), 330–336. https://doi.org/10.1112/jlms/s2-2.2.330.

26. S. Karlin, Total positivity, absorption probabilities and applications, Trans. Amer. Math. Soc. 111 (1964), 33–107. https://doi.org/10.2307/1993667.

27. S. Karlin, Total positivity. Vol. I, Stanford University Press, Stanford, CA, 1968.

28. S. Karlin and Z. Ziegler, Chebyshevian spline functions, SIAM J. Numer. Anal. 3 (1966), no. 3, 514–543. https://doi.org/10.1137/0703044.

29. A. Khare, Critical exponents for total positivity, individual kernel encoders, and the Jain-Karlin-Schoenberg kernel, Preprint, https://arxiv.org/abs/2008.05121v1, 2020.

30. Y. Kodama and L. Williams, KP solitons and total positivity for the Grassmannian, Invent. Math. 198 (2014), no. 3, 637–699. https://doi.org/10.1007/s00222-014-0506-3.

31. E. Laguerre, Mémoire sur la théorie des équations numériques, J. Math. Pures Appl. 9 (1883), 9–146. http://sites.mathdoc.fr/JMPA/PDF/JMPA_1883_9_A5_0.pdf.

32. C. E. Lemke, Bimatrix equilibrium points and mathematical programming, Manage. Sci. 11 (1965), no. 7, 681–689. https://doi.org/10.1287/mnsc.11.7.681.

33. C. Loewner, On totally positive matrices, Math. Z. 63 (1955), 338–340. https://doi.org/10.1007/BF01187945.

34. G. Lusztig, Total positivity in reductive groups, Lie theory and geometry, vol. 123 of Progress in Mathematics, Birkhäuser, Boston, MA, 1994, pp. 531–568.

35. M. Margaliot and E. D. Sontag, Revisiting totally positive differential systems: a tutorial and new results, Automatica J. IFAC 101 (2019), 1–14. https://doi.org/10.1016/j.automatica.2018.11.016.

36. A. Pinkus, Totally positive matrices, Cambridge Tracts in Mathematics, vol. 181, Cambridge University Press, Cambridge, 2010. https://doi.org/10.1017/CBO9780511691713.

37. K. C. Riemenschneider, On totally positive Toeplitz matrices and quantum cohomology of partial flag varieties, J. Amer. Math. Soc. 16 (2003), no. 2, 363–392. https://doi.org/10.1090/S0894-0347-02-00412-5.

38. H. Samelson, R. M. Thrall, and O. Wesler, A partition theorem for Euclidean n-space, Proc. Amer. Math. Soc. 9 (1958), 805–807. https://doi.org/10.2307/2033091.

39. I. J. Schoenberg, Über variationsvermindernde lineare Transformationen, Math. Z. 32 (1930), 321–328. https://doi.org/10.1007/BF0194637.

40. I. J. Schoenberg, Contributions to the problem of approximation of equidistant data by analytic functions. Part A. On the problem of smoothing or graduation, Quart. Appl. Math. 4 (1946), no. 1, 45–99. A first class of analytic approximation formulae. https://doi.org/10.1090/qam/15914.

41. I. J. Schoenberg, On the zeros of the generating functions of multiply positive sequences and functions, Ann. of Math. (2) 62 (1955), no. 3, 447–471. https://doi.org/10.2307/1970073.

42. I. J. Schoenberg and A. M. Whitney, On Pólya frequency functions. III. The positivity of translation determinants with an application to the interpolation problem by spline curves, Trans. Amer. Math. Soc. 74 (1953), 246–259. https://doi.org/10.1090/S0002-9947-1953-0053981.

43. A. M. Whitney, A reduction theorem for totally positive matrices, J. Anal. Math. 2 (1952), no. 1, 88–92. https://doi.org/10.1007/BF02786969.