Abstract. We study the Chow motive (with rational coefficients) of a hypersurface $X$ in the projective space by using the variety $F(X)$ of $l$-dimensional planes contained in $X$. If the degree of $X$ is sufficiently small, we show that the primitive part of the motive of $X$ is the tensor product of a direct summand in the motive of a suitable complete intersection in $F(X)$ and the $l$-th twist $\mathbb{Q}(-l)$ of the Lefschetz motive.

Introduction

Let $X$ be a smooth hypersurface of degree $d$ in the projective space $\mathbb{P}^n_k$ over a field $k$. In this paper we study the Chow motive (with rational coefficients) of $X$ provided that $d$ is sufficiently small.

Roitman has shown that the Chow group of zero-dimensional cycles of degree 0 is a torsion group if $d \leq n$ [4]. For higher dimensional cycles it is known [1, Theorem 4.6] that the Chow groups satisfy

$$\text{CH}_{l'}(X) \otimes \mathbb{Q} = \text{CH}_{l'}(\mathbb{P}^n) \otimes \mathbb{Q} = \mathbb{Q}$$

for $0 \leq l' \leq l - 1$ if $n \geq (\frac{d+1}{l+1})$ and $d \geq 3$. The identity also holds if $X$ is covered by $l$-dimensional planes [3, Theorem 9.28], or more generally if $X$ is a hyperplane section of a hypersurface $Y$ which is covered by $l$-dimensional planes [3].

Results on triviality of Chow groups give rise to a decomposition of the motive $h(X)$ associated with $X$. In our case, we get

$$h(X) \cong M_D \otimes \mathbb{Q}(-l) \oplus \bigoplus_{i=0}^{l-1} \mathbb{Q}(-i),$$

where $\mathbb{Q}(-1)$ is the Lefschetz motive and $M_D$ is a direct summand of the motive $h(D)$ of some variety $D$. Our purpose is to describe $M_D$.

In order to state the theorem we need the following notation. An $l$-dimensional plane $E$ in $\mathbb{P}^n$ is called an osculating plane if the intersection $E \cap X$ is an $(l-1)$-dimensional plane or if $E$ is contained in $X$. We say that $X$ has sufficiently many osculating planes if there exists an osculating plane through every closed point of $X$ (the planes may be defined over a field extension of $k$).
Theorem (see Theorem 2.5). Let \( n, d, l \) be numbers such that a general hypersurface of degree \( d \) in the projective space \( \mathbb{P}^n \) has sufficiently many osculating \( l \)-dimensional planes. Let \( X \subset \mathbb{P}^n \) be a smooth hypersurface of degree \( d \) such that the Fano variety \( F(X) \) of \( l \)-dimensional planes contained in \( X \) is smooth and has the expected dimension. Furthermore, let \( HF(X) \subset F(X) \) be a smooth complete intersection of hyperplanes (in the Plücker embedding) with \( \dim HF(X) = n - 2l - 1 \). Then there is an isomorphism in the category of Chow motives with rational coefficients:

\[
h(X) \cong M_{HF(X)} \otimes \mathbb{Q}(-l) \oplus \bigoplus_{i=0}^{n-1} \mathbb{Q}(-i),
\]

where \( M_{HF(X)} \) is a direct summand in the motive of \( HF(X) \).

The conditions on \( n, d, l \) hold if \( n \geq \frac{(1+d-1)+l-1}{l} \). For a finite field \( k \) there may be no smooth complete intersection \( HF(X) \) of hyperplanes over \( k \). In this case the variety \( HF(X) \) exists over a suitable finite field extension of \( k \).

Let us sketch the idea of the proof. We consider the family of planes

\[
\Xi = \{(x, E) \in X \times HF(X) \mid x \in E\} \subset X \times HF(X)
\]

over \( HF(X) \). The cycle \( \Xi \) defines a correspondence \( \phi_1: HF(X) \otimes \mathbb{Q}(-l) \to X \), resp. \( \phi_2: X \to HF(X) \otimes \mathbb{Q}(-l) \). The composite \( \phi_1 \circ \phi_2 \) is the cycle

\[
Z_X = \{(x, y) \in X \times X \mid x, y \in E, E \in HF(X)\}
\]

in \( CH^{n-1}(X \times X) \). The most important step is to show that

\[
(0.0.1) \quad Z_X = \tau^*(a) + m \cdot \Delta_X,
\]

for some \( a \in CH^{n-1}(\mathbb{P}^n \times \mathbb{P}^n) \), some nonzero integer \( m \), the inclusion \( \tau: X^2 \hookrightarrow (\mathbb{P}^n)^2 \) and the diagonal \( \Delta_X \). In order to prove (0.0.1) we introduce the doubled incidence variety of degree \( d \) hypersurfaces together with two points:

\[
\Sigma = \{(x, y, Y) \in (\mathbb{P}^n)^2 \times \mathbb{P}(\text{Sym}^d(k^{n+1})) \mid x, y \in Y\}.
\]

The projection \( p: \Sigma \to (\mathbb{P}^n)^2 \) is a projective bundle over \( (\mathbb{P}^n)^2 - \Delta_{\mathbb{P}^n} \) and \( \Delta_{\mathbb{P}^n} \), so that \( CH^{n-1}(\Sigma) \) can be calculated by using the projective bundle formula and the localization sequence. The cycle \( p^{-1}(\Delta_{\mathbb{P}^n}) \in CH^{n-1}(\Sigma) \) maps to the diagonal \( \Delta_X \) by the pullback map \( \gamma^* \) of the inclusion \( j: X \times X \to \Sigma \). One defines a cycle \( Z \in CH^{n-1}(\Sigma) \) with \( \gamma^*(Z) = Z_X \), and by comparing \( Z \) with \( p^{-1}(\Delta_{\mathbb{P}^n}) \) we obtain (0.0.1) after applying \( \gamma^* \).

1. **Cycles in the doubled incidence variety**

1.1. Let \( k \) be a field and let \( \mathbb{P}(\text{Sym}^d(k^{n+1})) \) denote the hypersurfaces of degree \( d \) in the projective space \( \mathbb{P}^n \). We denote by \( \Sigma \) the doubled incidence variety

\[
\Sigma = \{(x, y, X) \in (\mathbb{P}^n)^2 \times \mathbb{P}(\text{Sym}^d(k^{n+1})) \mid x, y \in X\}.
\]

Letting \( p: \Sigma \to (\mathbb{P}^n)^2 \) be the projection, we define \( \Sigma_1 := p^{-1}(\Delta_{\mathbb{P}^n}) \) and \( \Sigma_0 := \Sigma - \Sigma_1 \). In the diagram

\[
\begin{array}{ccc}
\Sigma_0 & \xrightarrow{p} & \Sigma \\
\downarrow & & \downarrow \\
(\mathbb{P}^n)^2 - \Delta_{\mathbb{P}^n} & \xleftarrow{p} & (\mathbb{P}^n)^2 - \Delta_{\mathbb{P}^n}
\end{array}
\]
the varieties \( \Sigma_0 \), resp. \( \Sigma_1 \), are projective bundles with fiber dimension \( N - 2 \), resp. \( N - 1 \), where \( N = \dim \mathbb{P}(\text{Sym}^d(k^{n+1})) \). It is easy to see that \( \text{Sing}(\Sigma) = \{(x, x, X) \mid x \in \text{Sing}(X)\} \). The singular locus is a projective bundle over \( \Delta_p \) with fiber dimension \( N - 1 - n \).

1.2. There is an exact sequence

\[
(1.2.1) \quad \text{CH}^0(\Sigma_1) \to \text{CH}^{n-1}(\Sigma) \to \text{CH}^{n-1}(\Sigma_0) \to 0
\]

from the localization sequence of Chow groups (\( \text{CH}^i \) denotes the group of \( i \) codimensional cycles modulo rational equivalence). Moreover, there is a natural splitting defined as follows.

Let \( \phi : \Sigma \to \mathbb{P}(\text{Sym}^d(k^{n+1})) \) be the projection and set \( c = \phi^*(c_1(\mathcal{O}(1))) \). For the other projection \( p : \Sigma \to (\mathbb{P}^n)^2 \) we may define \( p^* \) to be the composite \( \epsilon \circ \text{pr}\), where \( \epsilon : \Sigma \to (\mathbb{P}^n)^2 \times \mathbb{P}(\text{Sym}^d(k^{n+1})) \) is the regular embedding and \( \text{pr} \) is the projection to \( (\mathbb{P}^n)^2 \). From the projective bundle formula we see that

\[
\bigoplus_{i=0}^{n-1} (n-1-i)^* \text{CH}^i((\mathbb{P}^n)^2) \subset \text{CH}^{n-1}(\Sigma)
\]

splits the sequence (1.2.1), so that every class \( Z \) in \( \text{CH}^{n-1}(\Sigma) \) can be written as

\[
Z = \sum_{i=0}^{n-1} (n-1-i)^* \cdot p^*(a_i) + m \cdot [\Sigma_1]
\]

with \( a_i \in \text{CH}^i((\mathbb{P}^n)^2) \) and \( m \) is an integer.

1.3. Let \( \mathcal{H} \) be a smooth, connected projective \( k \)-scheme and \( \Xi \subset \mathcal{H} \times \mathbb{P}^n \) be a family of \( \kappa \)-dimensional subschemes of \( \mathbb{P}^n \), flat over \( \mathcal{H} \). We assume that \( \kappa \geq 1 \) and that the sheaf \( \text{pr}_{1*}(\mathcal{O}_\Xi \otimes \text{pr}_2^*\mathcal{O}_{\mathbb{P}^n}(d)) \) on \( \mathcal{H} \) is locally free. Furthermore, we assume that the map

\[
(1.3.1) \quad \text{Sym}^d((k^{n+1})^\vee) \otimes \mathcal{O}_{\mathcal{H}} = \text{pr}_{1*}\text{pr}_2^*\mathcal{O}_{\mathbb{P}^n}(d) \to \text{pr}_{1*}(\mathcal{O}_\Xi \otimes \text{pr}_2^*\mathcal{O}_{\mathbb{P}^n}(d)),
\]

induced by \( \mathcal{O}_{\mathcal{H} \times \mathbb{P}^n} \to \mathcal{O}_\Xi \) and \( H^0(\mathbb{P}^n, \mathcal{O}_{\mathbb{P}^n}(d)) = \text{Sym}^d((k^{n+1})^\vee) \), is surjective.

We denote by \( E \) the kernel of (1.3.1). It is convenient to write \( QE \) for the sheaf \( \text{pr}_{1*}(\mathcal{O}_\Xi \otimes \text{pr}_2^*\mathcal{O}_{\mathbb{P}^n}(d)) \). In the following commutative diagram we fix the notation for the various maps:

\[
\begin{array}{cccccc}
\Xi \times \mathcal{H} & \Xi \times \mathcal{H} & \mathcal{H}(\mathcal{E}^\vee) & f_E & \Sigma & \mathbb{P}(\mathbb{P}^n)^2 \\
\downarrow f_{\mathcal{H}(\mathcal{E}^\vee)} & \downarrow f_{\mathcal{H}(\mathcal{E}^\vee)} & \downarrow f_{\mathcal{H}(\mathcal{E}^\vee)} & \downarrow f_{\mathcal{H}(\mathcal{E}^\vee)} & \downarrow f_{\mathcal{H}(\mathcal{E}^\vee)} \\
\mathcal{H}(\mathcal{E}^\vee) & \mathcal{H}(\mathcal{E}^\vee) & \mathcal{H}(\mathcal{E}^\vee) & \downarrow \psi & \mathbb{P}(\text{Sym}^d(k^{n+1})) \\
\downarrow \psi & \downarrow \psi & \downarrow \psi & \downarrow \psi & \downarrow \psi \\
\mathcal{H} & \mathcal{H} & \mathcal{H} & \mathcal{H} & \mathcal{H} \\
\end{array}
\]

Here \( \psi : \mathcal{H}(\mathcal{E}^\vee) \to \mathbb{P}(\text{Sym}^d(k^{n+1})) \) is the composite

\[
\mathcal{H}(\mathcal{E}^\vee) \xrightarrow{\psi} \mathcal{H} \times \mathbb{P}(\text{Sym}^d((k^{n+1})^\vee)) \xrightarrow{\text{pr}_2} \mathbb{P}(\text{Sym}^d((k^{n+1})^\vee)),
\]
and \( \psi \) is induced by the surjective map \( \text{Sym}^d((k^{n+1})^\vee) \otimes \mathcal{O}_\mathcal{H} \to E^\vee \). By definition, \( \mathbb{P}_\mathcal{H}(E^\vee) \) is the incidence variety \( \{(L, X) \in \mathcal{H} \times \mathbb{P}(\text{Sym}^d((k^{n+1})^\vee)) \mid L \subset X \} \). The map \( f_\Sigma \) is defined as follows. By using \( \Xi \subset \mathcal{H} \times \mathbb{P}^n \) we get

\[
\Xi \times \mathcal{H} \Xi \times \mathcal{H} \mathbb{P}_\mathcal{H}(E^\vee) \longrightarrow (\mathcal{H} \times \mathbb{P}^n) \times_\mathcal{H} (\mathcal{H} \times \mathbb{P}^n) \times_\mathcal{H} \mathbb{P}_\mathcal{H}(E^\vee)
\]

and the image of \( \tilde{f}_\Sigma \) is the incidence variety \( \{(x, y, X) \mid \exists L \in \mathcal{H} x, y \in L \subset X \} \). In particular, \( \tilde{f}_\Sigma \) factors through \( \Sigma \), and we denote this map by \( f_\Sigma \).

Defining

\[
Z := f_\Sigma \circ f_{\mathbb{P}_\mathcal{H}(E^\vee)} \circ f_\mathcal{H} : \text{CH}^{n-\epsilon}(\mathcal{H}) \to \text{CH}^*(\Sigma),
\]

we will be mainly interested in cycles \( Z(a) \in \text{CH}^{n-1}(\Sigma) \) and their pullback to \( X \times X \subset \Sigma \) for a hypersurface \( X \).

The cycle \( \Xi \in \text{CH}^{n-\epsilon}(\mathcal{H} \times \mathbb{P}^n) \) has a unique representation,

\[
[\Xi] = \sum_{i=0}^{n-\kappa} \xi_{n-\kappa-i} \otimes H^i,
\]

with \( \xi_j \in \text{CH}^j(\mathcal{H}) \) and \( H = c_1(\mathcal{O}_{\mathbb{P}^n}(1)) \).

**Lemma 1.4.** For \( a \in \text{CH}^{n-1-\epsilon}(\mathcal{H}) \), let \( Z(a)|_{\Sigma_0} = \sum_{i=0}^{n-1} c^{n-1-i} \cdot p^*(a_i) \) be the pullback of \( Z(a) \) to \( \Sigma_0 \). The classes \( a_i \) can be computed as follows:

\[
\sum_i a_i = \sum_{0 \leq s, t \leq n-\kappa} b_{s,t}H^s \otimes H^t \]

in \( \text{CH}^*(\mathbb{P}^n)^2 - \Delta_{\mathbb{P}^n} \) and where

\[
b_{s,t} = \int_H \xi_{n-\kappa-s} \cdot (1 + d \otimes H)(1 + H \otimes d).
\]

**Proof:** On \( U := (\mathbb{P}^n)^2 - \Delta_{\mathbb{P}^n} \) the evaluation morphism

\[
\text{Sym}^d((k^{n+1})^\vee) \otimes \mathcal{O}_U \to \text{pr}_1^* \mathcal{O}_{\mathbb{P}^n}(d) \oplus \text{pr}_2^* \mathcal{O}_{\mathbb{P}^n}(d)
\]

is surjective; let \( G \) be the kernel. We have \( \Sigma_0 = \mathbb{P}_U(G^\vee) \) and \( c|_{\Sigma_0} = c_1(\mathcal{O}_{\mathbb{P}_U(G^\vee)}(1)) \), so that

\[
p_a(\frac{1}{1 - c} Z(a)|_{\Sigma_0}) = \sum_{j \geq 0} s_j(G) \cdot \sum_{i=0}^{n-1} a_i,
\]

where \( s_j(G) \) are the Segre classes. Since \( G \) is the kernel of \ref{iso}, we see that

\[
(1 + d \cdot \text{pr}_1^* c_1(\mathcal{O}_{\mathbb{P}^n}(1))) (1 + d \cdot \text{pr}_2^* c_1(\mathcal{O}_{\mathbb{P}^n}(1))) \sum_{j \geq 0} s_j(G) = 1.
\]

Define

\[
T_i(a) = p_a(c^{n-1+i}Z(a))
\]
and let \( j : U \subset (\mathbb{P}^n)^2 \) be the open immersion. It follows from \((1.4.2)\) and \((1.4.3)\) that
\[
\sum_{i=0}^{n-1} a_i = \frac{\sum_{i\geq 0} f^* T_i(a)}{(1 + d \otimes \mathbb{H})(1 + \mathbb{H} \otimes d)}.
\]

Let us now compute the \( T_i \):
\[
p_* (c^{N-n-1+i} Z(a)) = p_* f_{\Sigma *} ((\phi \circ f_{\Sigma})^* c_1 (\mathcal{O}(1))^N \cdot f_{\mathbb{H}} \circ f_{\mathbb{H}} (E^*)^*(a))
\]
\[
= p_* f_{\Sigma *} f_{\mathbb{H}} (E^*) \psi^* c_1 (\mathcal{O}(1))^N \cdot f_{\mathbb{H}} (a).
\]
The map \( T = p_* f_{\Sigma *} f_{\mathbb{H}} (E^*) \) is given by the correspondence \( \mathbb{P}(E^') \times \mathbb{H} \Sigma \times \mathbb{H} \Sigma \) in \( \mathbb{P}(E^') \times (\mathbb{P}^n)^2 \), and using \((1.3.3)\), we see that
\[
[\Sigma \times \mathbb{H} \Sigma] = \sum_{0 \leq s, t \leq n - \kappa} (\xi_{n-\kappa-s} \cdot \xi_{n-\kappa-t}) \otimes H^s \otimes H^t
\]
in \( \text{CH}^*(\mathbb{H} \times (\mathbb{P}^n)^2) \), so that the coefficient of \( H^s \otimes H^t \) in \( T_i(a) \) is
\[
\int_{\mathbb{P}(E^')} f_{\mathbb{H}}^* (\xi_{n-\kappa-s} \cdot f_{\mathbb{H}}^* \xi_{n-\kappa-t}) \cdot \psi^* c_1 (\mathcal{O}(1))^N \cdot f_{\mathbb{H}}^* (a)
\]
\[
= \int_{\mathbb{H}} \xi_{n-\kappa-s} \cdot \xi_{n-\kappa-t} \cdot s_{N-n-\rk(E)+i}(E) \cdot a.
\]
If \( Z(a) \in \text{CH}^{n-1}(\Sigma) \), then \( T_i(a) \in \text{CH}^1((\mathbb{P}^n)^2) \) by definition so that the coefficient of \( H^s \otimes H^t \) in \( T_i(a) \) vanishes if \( s + t \neq i \). The identity \( s_{N-n-\rk(E)+i}(E) = c_{\rk(QE)-n-\rk(E)+i}(E) \) completes the proof. \( \square \)

The next lemma computes the pullback of \( Z(a) \) to \( X \times X \) for a smooth hypersurface \( X \). We write \( i_{\Sigma}^* \) for the inclusion \( X \times X \to \Sigma \) and \( i_{(\mathbb{P}^n)^2}^* \) for the inclusion \( X \times X \to (\mathbb{P}^n)^2 \). Note that both inclusions are locally complete intersections; thus the pullback is well-defined. Indeed, since \( X \) is smooth the image \( i_{\Sigma}^*(X \times X) \) is contained in the open smooth part of \( \Sigma \) (see \((1.1)\)).

**Lemma 1.5.** For \( Z(a) \in \text{CH}^{n-1}(\Sigma) \) and \( Z(a)|_{\Sigma_n} = \sum_{i=0}^{n-1} c^{n-1-i} \cdot p^*(a_i) \), we have
\[
i_{\Sigma}^* Z(a) = i_{(\mathbb{P}^n)^2}^*(a_{n-1}) - m \cdot \Delta_X,
\]
where
\[
m = d \cdot \sum_{j=\kappa-1}^{n-1} (-d)^j \int_{\mathbb{H}} \xi_{n-\kappa} \cdot \xi_{j-\kappa+1} \cdot c_{\rk(QE)-2-j}(QE) \cdot a.
\]

**Proof.** We know that
\[
Z(a) = \sum_{i=0}^{n-1} c^{n-1-i} \cdot p^*(a_i) - m \cdot [\Sigma_1]
\]
for some \( m \). The line bundle \( i_{\Sigma}^* \phi^* \mathcal{O}(1) \) is trivial and \( i_{\Sigma}^*[\Sigma_1] = \Delta_X \); therefore
\[
i_{\Sigma}^* Z(a) = i_{(\mathbb{P}^n)^2}^*(a_{n-1}) - m \cdot \Delta_X.
\]

We claim that
\[
i_{(\mathbb{P}^n)^2}^* \cdot i_{\Sigma}^* \beta = p_* (c^{N} \cdot \beta)
\]
for every class \( \beta \in \text{CH}^*(\Sigma) \). This follows from the following fact. If \( g : D \subset Y \) is a Cartier divisor on \( Y \) and \( L \) is the associated line bundle, then \( g_* g^*(\beta) = c_1(L) \cdot \beta \).
2.1. In the following we work with the Grassmannian of 

2.2. We will be interested in cycles 

By applying \( i_{(p^n)} \) to (2.2.2) and using \( i_{(p^n)}^* i_{(p^n)_2}^* a_{n-1} = d^2 (H \otimes H) \cdot a_{n-1} \) and \( i_{(p^n)}^* \Delta X = d \cdot \sum_{i \geq 0} H^{i+1} \otimes H^{n-i} \), we find that \( m = d \cdot \gamma \), as claimed. 

2. Motives of hypersurfaces and their Fano varieties

2.1. In the following we work with the Grassmannian of \( \kappa \)-planes \( \mathcal{H} = \text{Gr}_\kappa \) in projective space where \( \Xi \subseteq \text{Gr}_\kappa \times \mathbb{P}^n \) is the universal family. We denote by \( V \), resp. \( QV \), the tautological bundle \( V \subseteq O_{\text{Gr}^1} \), resp. the quotient \( O_{\text{Gr}^1}/V \). The family \( \Xi \) is the projective bundle \( \Xi = P_{\text{Gr}_\kappa}(V') \), and it is easy to see that

\[
[\Xi] = \sum_{i=0}^{n-k} c_{n-k-i}(QV) \otimes H^i
\]

in \( \text{CH}^{n-k}(\text{Gr} \times \mathbb{P}^n) \). Furthermore we have that \( QE = pr_{1*}(O_{\Xi} \otimes pr_2^* O_{\mathbb{P}^n} (d)) = \text{Sym}^d (V') \).

2.2. We will be interested in cycles \( Z(c_1(V')^s) \in \text{CH}^{n-1}(\Sigma) \), for \( s \geq 0 \) (notation as in (1.3.2)). By counting dimensions we see that

\[
\text{dim } \Xi - \text{rk}(QE) - (n - 1) = \kappa(n - \kappa) - \left( \frac{d + \kappa}{\kappa} \right) + \kappa + 1.
\]

Let us consider the variety

\[
\{(x, E_{\kappa-1}, E_{\kappa}, X) \in \mathbb{P}^n \times \text{Gr}_{\kappa-1} \times \text{Gr}_{\kappa} \times \mathbb{P}(\text{Sym}^d (k^{n+1})) \mid x \in E_{\kappa-1} \subseteq E_{\kappa}, E_{\kappa} \cap X = E_{\kappa-1} \text{ or } E_{\kappa} \subseteq X \}\.
\]

More formally, this variety is defined as follows. On \( \Xi = P_{\text{Gr}_\kappa}(V') \) there is an exact sequence of vector bundles

\[
0 \to V_1' \to V' \otimes O_{\Xi} \to O_{\Xi}(1) \to 0,
\]

and the points of \( P_{\Xi}(V_1) \) are \( \{(x, E_{\kappa-1}, E_{\kappa}) \mid x \in E_{\kappa-1} \subseteq E_{\kappa} \} \).

Since \( O_{P_{\Xi}(V_1)(-1)} \subset V_1' \otimes O_{P_{\Xi}(V_1)} \subset V' \otimes O_{P_{\Xi}(V_1)} \), we can define \( G \) to be the kernel of

\[
\text{Sym}^d (k^{n+1})' \otimes O_{P_{\Xi}(V_1)} \to \text{Sym}^d (V')/O_{P_{\Xi}(V_1)}(-d).
\]

Then \( P_{\Xi}(V_1)(G') \) is the variety (2.2.2).

The following condition will imply that the diagonal \( \Delta_X \), for a hypersurface \( X \), can be written in terms of the pullback of \( Z(c_1(V')^s) \) to \( X \times X \) (i.e. \( m \neq 0 \) in Lemma 1.5).

(B) The following map is surjective:

\[
P_{P_{\Xi}(V_1)}(G') \to \{(x, X) \in \mathbb{P}^n \times \mathbb{P}(\text{Sym}^d (k^{n+1})) \mid x \in X \},
\]

\[
(x, E_{\kappa-1}, E_{\kappa}, X) \mapsto (x, X).
\]

By counting dimensions we see that a necessary condition for (B) is \( s \geq 0 \) (with \( s \) as in (2.2.1)). If \( d = 2 \), then \( s \geq 0 \) is not sufficient; the first example is \( \kappa = 3 \).
and $n = 5$. In fact, (B) is equivalent to $n \geq 2 \cdot \kappa$ if $d = 2$, which can be checked by using the following lemma. However, we don’t know any counterexamples to (2.2.4) for $d > 2$. If $\kappa = 1$, then (2.2.4) is true. This also holds for

$$(n, d, \kappa) = (6, 3, 2), (8, 4, 2), (11, 5, 2), (9, 3, 3).$$

It is known [1 Lemma 1.1 + Lemma 4.2] that (B) is true if $d \geq 3$ and

$$n - \kappa + 1 \geq \left( \frac{\kappa - 1 + d}{\kappa} \right).$$

**Lemma 2.3.** Condition (B) holds if and only if

$$m = d \cdot \sum_{j=\kappa-1}^{n-1} (-d)^j \int_K \xi_{n-\kappa} \cdot \xi_{j-\kappa+1} \cdot c_{\text{rk}(QE) - 2 - j}(QE) \cdot c_1(V^\vee)^s$$

is nonzero.

**Proof.** From the construction of $\mathbb{P}_{\Xi(V_1)}(G^\vee)$ we have the maps

$$\mathbb{P}_{\Xi(V_1)}(G^\vee) \xrightarrow{f} \mathbb{P}_{\Xi}(V_1) \xrightarrow{g} \Xi \xrightarrow{h} \text{Gr}_{\kappa}.$$ We claim that

$$(h \circ g \circ f)_*(c_1(O_\Xi(1))^{n-1} \cdot c_1(O_{\mathbb{P}_{\Xi(V_1)}(G^\vee)}(1))^N) = (-1)^{n-1}d \cdot \sum_{j=\kappa-1}^{n-1} (-d)^j \xi_{n-\kappa} \cdot \xi_{j-\kappa+1} \cdot c_{\text{rk}(QE) - 2 - j}(QE)$$

$$(N = \dim \mathbb{P}(\text{Sym}^d(k^{n+1}))).$$ Indeed, $g_*(f_*(c_1(O_{\mathbb{P}_{\Xi(V_1)}(G^\vee)}(1))^N)) = g_s c_{\text{rk}(QE) - 1}(G) = g_s c_{\text{rk}(QE) - 1}(QE/O_{\mathbb{P}_{\Xi(V_1)}}(-d)) = g_s \sum_i d^i c_{\text{rk}(QE) - i - 1}(QE) \cdot c_1(O_{\mathbb{P}_{\Xi(V_1)}}(1)^i) = \sum_i d^i c_{\text{rk}(QE) - i - 1}(QE) \cdot s_{i-\kappa+1}(V_1^\vee),$

and from (2.2.3) we obtain $s_j(V_1^\vee) = s_j(V^\vee) + c_1(O_\Xi(1)) \cdot s_{j-1}(V^\vee)$ for all $j$. Thus, (2.3.2)

$$(h \circ g \circ f)_*(c_1(O_\Xi(1))^{n-1} \cdot c_1(O_{\mathbb{P}_{\Xi(V_1)}(G^\vee)}(1))^N) = \sum_i d^i c_{\text{rk}(QE) - i - 1}(QE) \cdot (s_{n-\kappa-1}(V)s_{i-\kappa+1}(V^\vee) + s_{n-\kappa}(V)s_{i-\kappa}(V^\vee)).$$

On $\Xi$ the natural morphism $QE \otimes O_\Xi(-d) \to O_\Xi$ is surjective, so that the top Chern class of $QE \otimes O_\Xi(-d)$ vanishes:

$$h_*(c_{\text{rk}(QE)}(QE \otimes O_\Xi(-d)) = h_*(\sum_i (-d)^i c_{\text{rk}(QE) - i}(QE) \cdot c_1(O_\Xi(1)^i)$$

$$= \sum_i (-d)^i c_{\text{rk}(QE) - i}(QE)s_{i-\kappa}(V) = (-1)^{n-1}d \cdot \sum_i d^i c_{\text{rk}(QE) - i - 1}(QE)s_{i+1-\kappa}(V^\vee)$$

is zero and together with (2.3.2) we obtain

$$(h \circ g \circ f)_*(c_1(O_\Xi(1))^{n-1} \cdot c_1(O_{\mathbb{P}_{\Xi(V_1)}(G^\vee)}(1))^N) = d \cdot \sum_i d^i c_{\text{rk}(QE) - i - 2}(QE) \cdot s_{n-\kappa}(V)s_{i+1-\kappa}(V^\vee).$$

Then, $(-1)^j s_j(V^\vee) = s_j(V) = c_j(QV) = \xi_j$ proves the claim.
Let \( \pi \) be the map in condition (B). For a general closed point \((x, X)\) the irreducible components of \( \pi^{-1}(x, X) \) map generically one to one to \( \text{Gr}_\kappa \). The class \( c_1(V^\vee) = c_1(\Lambda^{\kappa+1}V^\vee) \) is the class of an ample line bundle, and \( s \) is the dimension of the generic fiber of \( \pi \) if \( \pi \) is surjective. Thus, \( \pi \) is surjective if and only if 

\[
\int_{\mathbb{P}^{\mathbb{P}(V_1)}(G')} c_1(V^\vee)^* \cdot [\pi^{-1}(x, X)] > 0.
\]

Since \( \mathcal{O}_{\Xi}(1) = \pi^*\mathcal{O}_{\mathbb{P}^n}(1) \) and \( \mathcal{O}_{\mathbb{P}^{\mathbb{P}(V_1)}(G')} \) is a generically finite surjective morphism which resolves the singularities of the pullback and the pushforward commute [2, Theorem 6.2, Remark 6.2.1], because some sufficiently general points \((x_i, X_i)\) for some sufficiently general points \((x_i, X_i)\), and \( c_1(\mathcal{O}_{\Xi}(1))^{n-1} \cdot c_1(\mathcal{O}_{\mathbb{P}^{\mathbb{P}(V_1)}(G')} \mathcal{O}_{\mathbb{P}(\text{Sym}^d(k^{n+1}))}) = \sum_i \pi^*(x_i, X_i). \) Now (2.3.1) implies the lemma.

In the following theorem we work with the category of (pure) Chow motives with rational coefficients (see [2, Chapter 16]), where \( \mathbb{Q}(-1) \) denotes the Lefschetz motive.

**Theorem 2.5.** Let \( n, d, \kappa \) be numbers satisfying (B). Let \( X \subset \mathbb{P}^n \) be a smooth hypersurface of degree \( d \) such that the Fano variety \( F_\kappa(X) \) of \( \kappa \)-dimensional planes contained in \( X \) has the expected dimension (i.e. \( \dim F_\kappa(X) = (\kappa+1)(n-\kappa) - \binom{\kappa+1}{\kappa} \)), and let \( HF_\kappa(X) \subset F_\kappa(X) \) be a complete intersection of hyperplanes (in the Plücker embedding) with \( \dim HF_\kappa(X) = n - 2\kappa - 1 \). Furthermore, let \( \psi : HF_\kappa(X) \rightarrow HF_n(X) \) be a generically finite surjective morphism which resolves the singularities of \( HF_\kappa(X) \). Then there is an isomorphism in the category of Chow motives with rational coefficients

\[
(X, \text{id}_X) \cong (\widetilde{HF_\kappa(X)}, P) \otimes \mathbb{Q}(-\kappa) \oplus \bigoplus_{i=0}^{n-1} \mathbb{Q}(-i)
\]

for a suitable projector \( P \). We give an explicit formula for \( P \) in the proof.

**Proof.** From Lemma 1.5 and Lemma 2.3 we obtain

\[
(2.5.1) \quad \Delta_X = -\frac{1}{m} \iota^*_\Sigma Z(c_1(V^\vee)^*) + \frac{1}{m} \iota^*_{\mathbb{P}^n} (a_{n-1}).
\]

In the Cartesian diagram

\[
\begin{array}{ccc}
\Xi \times F_\kappa(X) & \xrightarrow{f_{\times X}} & X \times X \\
\downarrow & & \downarrow \pi_X \\
\Xi \times \text{Gr}_\kappa & \xrightarrow{f_{\times \text{Gr}_\kappa}} & \Sigma
\end{array}
\]

the pullback and the pushforward commute [2, Theorem 6.2, Remark 6.2.1], because \( F_\kappa(X) \) has the expected dimension. Therefore

\[
(2.5.2) \quad \iota^*_\Sigma Z(c_1(V^\vee)^*) = f_{\times X} \ast (c_1(V^\vee)^*) = (f_{\times X} \circ \eta)_\ast ([\Xi \times HF_\kappa(X)] \Xi),
\]

where \( \eta : HF_\kappa(X) \rightarrow F_\kappa(X) \) is a complete intersection of \( s \) hyperplanes (thus the dimension is \( \dim HF_\kappa(X) = n - 2\kappa - 1 \)). It is more convenient to write \( H = HF_\kappa(X) \) and \( \tilde{H} = \tilde{HF}_\kappa(X) \).
There is a cycle \( Y \in \text{CH}_{n-2k-1}(\hat{H}) \otimes \mathbb{Q} \) (i.e. \( Y \) is a rational linear combination of connected components of \( \hat{H} \)) such that \( \psi_*(Y) = [H] \).

Let \( \phi_1 \in \text{Cor}(\hat{H} \otimes \mathbb{Q}(-k), X) \) (resp. \( \phi_2 \in \text{Cor}(X, \hat{H} \otimes \mathbb{Q}(-k)) \)) be the correspondence defined by the cycle \( \hat{H} \times_H \Xi \) in \( \hat{H} \times X \) (resp. \( [\Xi \times_H \hat{H}] \cdot \text{pr}_H^*(Y) \) in \( X \times \hat{H} \), where \( \text{pr}_H : X \times \hat{H} \to \hat{H} \) is the projection).

We consider the commutative diagram

\[
\begin{array}{ccc}
\Xi \times \hat{H} & \xrightarrow{\psi'} & \Xi \\
\downarrow_{\text{id} \times \psi \times \text{id}} & & \downarrow_{f \times X \circ \eta} \\
X \times \hat{H} & \xrightarrow{\text{pr}_H} & X 	imes X
\end{array}
\]

It is easy to see that \( \phi_1 \circ \phi_2 = (f \times X \circ \eta \circ \psi')_*([\Xi \times_H \hat{H}] \cdot \text{pr}_H^*(Y)) \), and \( \psi'_*([\Xi \times_H \hat{H} \times \Xi] \cdot \text{pr}_H^*(Y)) = [\Xi \times_H \hat{H}] \) together with (2.5.2) yields

\[
(2.5.3) \quad \phi_1 \circ \phi_2 = Z(c_1(V^V)^*).
\]

If \( H \) is the class of a hyperplane in \( \mathbb{P}^n \), then we write \( P_i \) for the pullback of \( \frac{1}{n} H^{n-1-i} \otimes H^i \in \text{CH}^{n-1}(\mathbb{P}^n \times \mathbb{P}^n) \otimes \mathbb{Q} \) to \( X \times X \). The correspondences \( P_0, \ldots, P_{n-1} \) are idempotent and orthogonal. We may write \( a_{n-1} = \frac{m}{n} \sum_{i=0}^{n-1} \beta_i H^{n-1-i} \otimes H^i \), and it follows from (2.5.1) and (2.5.3) that

\[
\Delta_X - \sum_i P_i = -\frac{1}{m} \phi_1 \circ \phi_2 + \sum_{i=0}^{n-1} (\beta_i - 1) P_i.
\]

Composition with \( P_i \) shows that \( \frac{1}{m} \phi_1 \circ \phi_2 \circ P_i = (\beta_i - 1) P_i \) and

\[
(2.5.4) \quad \Delta_X - \sum_i P_i = \phi_1 \circ (-\frac{1}{m} \phi_2 + \frac{1}{m} \sum_{i=0}^{n-1} \phi_2 \circ P_i).
\]

Since \( (X, P_i) \cong \mathbb{Q}(-i) \) and \( (X, \Delta_X - \sum_i P_i) \cong (\hat{H}, -\frac{1}{m} \phi_2 \circ (\Delta_X - \sum_i P_i) \circ \phi_1) \) by (2.5.4), this proves the theorem.

\[ \square \]

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