Contrasting Classical and Quantum Vacuum States in Non-Inertial Frames

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Abstract

Classical electron theory with classical electromagnetic zero-point radiation (stochastic electrodynamics) is the classical theory which most closely approximates quantum electrodynamics. Indeed, in inertial frames, there is a general connection between classical field theories with classical zero-point radiation and quantum field theories. However, this connection does not extend to noninertial frames where the time parameter is not a geodesic coordinate. Quantum field theory applies the canonical quantization procedure (depending on the local time coordinate) to a mirror-walled box, and, in general, each non-inertial coordinate frame has its own vacuum state. In particular, there is a distinction between the "Minkowski vacuum" for a box at rest in an inertial frame and a "Rindler vacuum" for an accelerating box which has fixed spatial coordinates in an (accelerating) Rindler frame. In complete contrast, the spectrum of random classical zero-point radiation is based upon symmetry principles of relativistic spacetime; in empty space, the correlation functions depend upon only the geodesic separations (and their coordinate derivatives) between the spacetime points. The behavior of classical zero-point radiation in a noninertial frame is found by tensor transformations and still depends only upon the geodesic separations, now expressed in the non-inertial coordinates. It makes no difference whether a box of classical zero-point radiation is gradually or suddenly set into uniform acceleration; the radiation in the interior retains the same correlation function except for small end-point (Casimir) corrections. Thus in classical theory where zero-point radiation is defined in terms of geodesic separations, there is nothing physically comparable to the quantum distinction between the Minkowski and Rindler vacuum states. It is also noted that relativistic classical systems with internal potential energy must be spatially extended and can not be point systems. Based upon the classical analysis, it is suggested that the claimed heating effects of acceleration through the vacuum may not exist in nature.
I. INTRODUCTION

Classical electron theory with classical electromagnetic zero-point radiation (stochastic electrodynamics) is the classical theory which comes closest to quantum electrodynamics. However, there seems to be little interest in the physical interpretations provided by this classical theory. This lack of interest in the related classical theory holds even when quantum theory ventures into untested areas involving noninertial coordinate frames such as appear in connection with black holes and acceleration through the vacuum. In this article, we illustrate the contrasting classical and quantum interpretations surrounding vacuum behavior in an inertial and in a noninertial (Rindler) frame. Although the ideas are believed to have much wider implications, the illustrations here focus on a massless relativistic scalar field in two spacetime dimensions in flat spacetime.

There is a general connection between the classical and quantum field theories in an inertial frame. However, this connection does not extend to noninertial frames where the time parameter is not a geodesic coordinate. Irrespective of the spacetime metric, quantum field theory regards one box as good as another when applying the canonical quantization procedure to a mirror-walled box. In general, each non-inertial coordinate frame has its own vacuum state. In particular, there is a distinction between the "Minkowski vacuum" for a box at rest in an inertial frame and a "Rindler vacuum" for an accelerating box which has fixed spatial coordinates in an (accelerating) Rindler frame. It has been claimed that the radiation in a box in the Minkowski vacuum which is very gradually speeded up to become a box in uniform acceleration, will end up in the Rindler vacuum state; on the other hand, if the box in the Minkowski vacuum is suddenly accelerated, then the box will contain Rindler quanta. This quantum situation is completely different from that found in classical physics. In the first place, the spectrum of random classical zero-point radiation is based upon symmetry principles of relativistic spacetime; the spectrum is such as to give correlation functions which depend only upon the geodesic separations (and their coordinate derivatives) between the spacetime points. In an inertial frame, the zero-point radiation spectrum is Lorentz invariant, scale invariant, and conformal invariant. The behavior of zero-point radiation in a noninertial frame is found by tensor transformations to the non-inertial coordinates. In particular, we can calculate the spectrum of classical zero-point radiation in an accelerating box, and we find that, except for small endpoint (Casimir)
effects, the spectrum and correlation functions are the same as observed by a Rindler observer accelerating through zero-point radiation. It makes no difference whether or not the box of classical zero-point radiation is gradually or suddenly set into uniform acceleration; the radiation in the interior retains the same zero-point spectrum. In classical theory where zero-point radiation is defined in terms of geodesic separations, there is nothing physically comparable to the quantum distinction between the Minkowski and Rindler vacuum states.

The work presented here involves only the free scalar field in a box with Dirichlet boundary conditions in one spatial dimension. Also, we will be interested only in the large-box approximation and will not treat the Casimir effects associated with a the discrete normal mode structure of the box. We start out in an inertial frame. We review the determination of the classical zero-point spectrum in the box and also the canonical quantization procedure for the corresponding quantum scalar field in the same box. Then we turn to the situation of thermal equilibrium in the box and note the contrasting classical and quantum points of view for thermal radiation. All of this work confirms the general connection between classical and quantum free fields in an inertial frame in two spacetime dimensions. This connection was treated earlier in four spacetime dimensions for electromagnetic fields and for scalar fields. Next we turn to the situation for a coordinate frame undergoing uniform proper acceleration through Minkowski spacetime (a Rindler frame). Quantum field theory introduces a canonical quantization in a box at rest in a Rindler frame which parallels that in an inertial frame, without making any adjustment because of the nongeodesic time coordinate involved in the quantization. In complete contrast, classical theory takes the correlation function for zero-point radiation as dependent only upon the geodesic separations of the field points, with tensor coordinate transformations between various coordinate frames. In the limit of a large Rindler-frame box, the classical radiation inside the box is shown to agree exactly with the empty-space zero-point radiation of an inertial frame. However, in the limit of a large Rindler-frame box, the quantum vacuum remains distinct from the quantum empty-space inertial vacuum. It is also emphasized that relativistic classical systems with internal potential energy must be spatially extended and can not be point systems. In contrast, systems used within quantum theory are often described as small (point) systems. Based upon the classical analysis, it is suggested that the claimed "heating effects of acceleration through the vacuum" may not exist in nature.
II. THE VACUUM STATE IN AN INERTIAL FRAME

A. Scalar Field in Two Spacetime Dimensions

We will consider a relativistic massless scalar field $\phi$ which is a function of $(ct, x)$ in an inertial frame with spacetime metric $ds^2 = g_{\mu \nu} dx^\mu dx^\nu$, where the indices $\mu$ and $\nu$ run over 0 and 1, $x^0 = ct$, $x^1 = x$, and

$$ds^2 = c^2 dt^2 - dx^2.$$  \hspace{1cm} (1)

The behavior of the field $\phi$ follows from the Lagrangian density $\mathcal{L} = (1/8\pi) \frac{1}{c^2} \left( \frac{\partial \phi}{\partial t} \right)^2 - \left( \frac{\partial \phi}{\partial x} \right)^2$. \hspace{1cm} (2)

The wave equation $\partial_\mu [\partial \mathcal{L}/\partial(\partial_\mu \phi)] = 0$ for the field is

$$\frac{1}{c^2} \left( \frac{\partial^2 \phi}{\partial t^2} \right) - \left( \frac{\partial^2 \phi}{\partial x^2} \right) = 0.$$  \hspace{1cm} (3)

The associated stress-energy-momentum tensor density $T^{\mu \nu} = [\partial \mathcal{L}/\partial(\partial_{\mu} \phi)] \partial^\nu \phi - g^{\mu \nu} \mathcal{L}$ gives the energy density $u$ as

$$u = T^{00} = T^{11} = \frac{1}{8\pi} \left[ \frac{1}{c^2} \left( \frac{\partial \phi}{\partial t} \right)^2 + \left( \frac{\partial \phi}{\partial x} \right)^2 \right],$$  \hspace{1cm} (4)

and the momentum density as

$$T^{01} = T^{10} = -\frac{1}{4\pi c} \frac{\partial \phi}{\partial t} \frac{\partial \phi}{\partial x}.$$  \hspace{1cm} (5)

The energy $U$ in the field in a one-dimensional box extending from $x = a$ to $x = b$ is

$$U = \int_a^b dx \frac{1}{8\pi} \left[ \frac{1}{c^2} \left( \frac{\partial \phi}{\partial t} \right)^2 + \left( \frac{\partial \phi}{\partial x} \right)^2 \right].$$  \hspace{1cm} (6)

B. Radiation Spectrum in a Box

Both classical and quantum field theories start with the normal mode structure of the radiation field in a box. We consider standing wave solutions which vanish at the walls $x = a$ and $x = b$ of the box (Dirichlet boundary conditions) so that a normalized normal mode can be written as

$$\phi_n(ct, x) = f_n \left( \frac{2}{b-a} \right)^{1/2} \sin \left[ \frac{n\pi}{b-a}(x-a) \right] \cos \left[ \frac{n\pi}{b-a}ct - \theta_n \right].$$  \hspace{1cm} (7)
where $f_n$ is the amplitude of the normal mode. The radiation field in the box can be written as a sum over all the normal modes

$$
\phi(ct, x) = \sum_{n=1}^{\infty} f_n \left( \frac{2}{b - a} \right)^{1/2} \sin \left[ \frac{n\pi}{b - a} (x - a) \right] \cos \left[ \frac{n\pi}{b - a} ct - \theta_n \right],
$$

(8)

where $\theta_n$ is an appropriate phase. From Eq. (4) we find that each mode $\phi_n(ct, x)$ has the time-average spatial energy density

$$
u_n(x) = \left\langle \frac{1}{8\pi} \left[ \frac{1}{c^2} \left( \frac{\partial \phi_n}{\partial t} \right)^2 + \left( \frac{\partial \phi_n}{\partial x} \right)^2 \right] \right\rangle_{\text{time}}
$$

$$= \frac{1}{8\pi} \left( \frac{n\pi}{b - a} \right)^2 f_n^2 \left[ \frac{2}{b - a} \right] \sin^2 \left[ \frac{n\pi}{b - a} (x - a) \right] \cos^2 \left[ \frac{n\pi}{b - a} ct - \theta_n \right] \left\langle \cos^2 \left[ \frac{n\pi}{b - a} ct - \theta_n \right] \right\rangle_{\text{time}}
$$

$$+ \cos^2 \left[ \frac{n\pi}{b - a} (x - a) \right] \sin^2 \left[ \frac{n\pi}{b - a} ct - \theta_n \right] \left\langle \sin^2 \left[ \frac{n\pi}{b - a} ct - \theta_n \right] \right\rangle_{\text{time}}
$$

$$= \frac{1}{8\pi} \left( \frac{n\pi}{b - a} \right)^2 f_n^2 \frac{b - a}{b - a},
$$

(9)

which is uniform in space. The total mode energy $U_n$ found by integrating over the length of the box is given by

$$U_n = \frac{1}{8\pi} \left( \frac{n\pi}{b - a} \right)^2 f_n^2.
$$

(10)

where the wave amplitude $f_n$ must be determined by some additional physical considerations.

C. Canonical Quantization of the Quantum Scalar Field

Classical and quantum theories take different points of view regarding the vacuum radiation field. Quantum field theory follows the canonical quantization procedure which rewrites the cosine time dependence in terms of complex exponentials (the positive and negative frequency aspects) and introduces annihilation and creation operators $\pi_n, \pi_n^+$ for each normal mode $n$ so that the field becomes an operator field $\overline{\phi}(ct, x)$ with a vacuum energy

$$U_n = (1/2)\hbar \omega = (1/2)\hbar c n \pi / (b - a)
$$

(11)

per normal mode. Thus from Eqs. (8), (10), and (11) the quantum field is

$$\overline{\phi}(ct, x) = \sum_{n=1}^{\infty} \left( \frac{8\pi \hbar c (b - a)}{n\pi} \right)^{1/2} \left( \frac{2}{b - a} \right)^{1/2} \sin \left[ \frac{n\pi}{b - a} (x - a) \right] \times \frac{1}{2} \left\{ \pi_n \exp \left[ i \frac{n\pi}{b - a} ct \right] + \pi_n^+ \exp \left[ -i \frac{n\pi}{b - a} ct \right] \right\}
$$

(12)
Here the operator $\overline{a}_n$ annihilates the vacuum, $\overline{a}_n|0> = 0$, and the operator commutation relations are $[\overline{a}_n, a_m] = [\overline{a}_n, a_m^+] = 0$, $[\overline{a}_n, a_m^+] = 1$.

In the quantum vacuum state $|0>$ in the inertial frame, the two-point vacuum expectation value which is symmetrized in operator order is easily calculated and takes the form

$$<0|\frac{1}{2}\{\overline{\phi}(ct, x)\overline{\phi}(ct', x') + \overline{\phi}(ct', x')\overline{\phi}(ct, x)}|0> = \sum_{n=1}^{\infty} \frac{4\hbar c}{n} \sin \left[ \frac{n\pi}{b-a} (x-a) \right] \sin \left[ \frac{n\pi}{b-a} (x'-a) \right] \cos \left[ \frac{n\pi}{b-a} c(t-t') \right]$$

(13)

**D. Zero-Point Radiation for the Classical Scalar Field**

The vacuum state for the classical scalar field involves random classical zero-point radiation which is featureless, so that its correlation functions depend only on the geodesic separations (and coordinate derivatives) between the field points. In an inertial frame, the zero-point radiation is Lorentz invariant, scale invariant, and indeed conformal invariant.

Random classical radiation can be written in the form given by Eq. (8) with the phases $\theta_n$ randomly distributed in the interval $[0, 2\pi)$ and independently distributed for each $n$. In an inertial frame, the invariance properties of the spectrum can be shown to lead to a spectral form corresponding to an energy per normal mode which is a multiple of the frequency with an undetermined multiplicative constant, $U_n = const \times \omega_n$. In order to give a close connection between the classical and quantum theories, we choose the energy per normal mode to agree with that used in the quantum theory as given in Eq. (11). In order to make the classical and quantum field expressions look as similar as possible, we rewrite Eq. (8) in the form parallel to Eq. (12), (note the change from $8\pi$ over to $4\pi$),

$$\phi_0(ct, x) = \sum_{n=1}^{\infty} \left( 4\pi \hbar c \frac{(b-a)}{n\pi} \right)^{1/2} \left( \frac{2}{b-a} \right)^{1/2} \sin \left[ \frac{n\pi}{b-a} (x-a) \right] \cos \left[ \frac{n\pi}{b-a} ct - \theta_n \right]$$

$$= \sum_{n=1}^{\infty} \left( 4\pi \hbar c \frac{(b-a)}{n\pi} \right)^{1/2} \left( \frac{2}{b-a} \right)^{1/2} \sin \left[ \frac{n\pi}{b-a} (x-a) \right]$$

$$\times \frac{1}{2} \left\{ e^{-i\theta_n} \exp \left[ i \frac{n\pi}{b-a} ct \right] + e^{i\theta_n} \exp \left[ -i \frac{n\pi}{b-a} ct \right] \right\}$$

(14)

It is convenient to characterize random classical radiation by the two-point correlation function $\langle \phi(ct, x)\phi(ct', x') \rangle$ obtained by averaging over the random phases as $\langle \cos \theta_n \sin \theta_{n'} \rangle = 0$, $\langle \cos \theta_n \cos \theta_{n'} \rangle = \langle \sin \theta_n \sin \theta_{n'} \rangle = (1/2)\delta_{n,n'}$, or as $\langle \exp[\theta_n] \exp[\theta_{n'}] \rangle = \langle \exp[\theta_n] \exp[\theta_{n'}] \rangle = \langle \exp[i\theta_n + i\theta_{n'}] \rangle = \langle \exp[i(\theta_n + \theta_{n'})] \rangle = \langle \exp[i\theta_n] \exp[i\theta_{n'}] \rangle$.
\langle \exp[-\theta_n] \exp[-\theta_{n'}] \rangle = 0, \quad \langle \exp[\theta_n] \exp[-\theta_{n'}] \rangle = \delta_{n,n'}

From these relations, we can easily show, for example, that \( \langle \cos(A + \theta_n) \cos(B + \theta_{n'}) \rangle = \cos(A - B)(1/2)\delta_{nn'} \). The two-point correlation function for a general distribution of random classical scalar waves is found by averaging over the random phases \( \theta_n \)

\[
\langle \phi_{0_{\text{box}}} (ct, x) \phi_{0_{\text{box}}} (ct', x') \rangle = \langle \phi_{0_{\text{mirror}}} (ct, x) \phi_{0_{\text{mirror}}} (ct', x') \rangle = 4\hbar c \int_{k=0}^{k=\infty} \frac{dk}{k} \sin[k(x - a)] \sin[k(x' - a)] \cos[kc(t - t')] .
\]

This integral is convergent. It can be rewritten as a sum of terms of the form \( \int (dk/k) \cos(ka) \) and evaluated as an indefinite integral. Thus we find

\[
\langle \phi_{0_{\text{mirror}}} (ct, x) \phi_{0_{\text{mirror}}} (ct', x') \rangle = -\hbar c \ln \left| \frac{[(x - x') - c(t - t')][(x - x') - c(t - t')] - [x + x' - 2a - c(t - t')] [(x + x' - 2a) - c(t - t')]}{(x + x' - 2a)^2 - c^2(t - t')^2} \right|.
\]

The correlation function for empty space can be found by moving the mirror at the left-hand edge \( x = a \) of the box out to spatial infinity, \( a \to -\infty \). However, this procedure introduces a divergence going as \( \hbar c \ln |(2a)^2| \). One way to eliminate this divergence is to
take the spatial derivatives of the correlation function. Indeed, we can go back to the integral of Eq. (16) and use the identity \(2 \sin A \sin B = \cos(A - B) - \cos(A + B)\) to rewrite the correlation function as

\[
\langle \phi_0\text{mirror}(ct, x)\phi_0\text{mirror}(ct', x') \rangle = 2\hbar c \int_{k=0}^{k=\infty} \frac{dk}{k} \cos[k(x - x')] \cos[kc(t - t')] \\
-2\hbar c \int_{k=0}^{k=\infty} \frac{dk}{k} \cos[k(x + x' - 2a)] \cos[kc(t - t')] \quad (18)
\]

Both integrals in Eq. (18) are divergent at \(k \to 0\). In the limit \(a \to -\infty\), corresponding to moving the left-hand reflecting mirror at \(x = a\) out to spatial minus infinity, we can drop the second line in Eq. (19) as a very rapidly oscillating cosine function. Thus for a box extending infinitely far in both directions, we find the free-space correlation function

\[
\langle \phi_0(ct, x)\phi_0(ct', x') \rangle = 2\hbar c \int_{k=0}^{k=\infty} \frac{dk}{k} \cos[k(x - x')] \cos[kc(t - t')] \\
= \hbar c \int_{k=0}^{k=\infty} \frac{dk}{k} \cos[k{(x - x')} + c(t - t')] \\
+ \hbar c \int_{k=0}^{k=\infty} \frac{dk}{k} \cos[k{(x - x')} - c(t - t')] \\
= \hbar c \int_{k=-\infty}^{k=\infty} \frac{dk}{|k|} \cos[k(x - x')] - |k|c(t - t')] \quad (19)
\]

where in the second line and third lines we have used the identity \(2 \cos A \cos B = \cos(A + B) + \cos(A - B)\) and in the last line have incorporated both the sum and difference cosine terms by extending the integral over negative values of \(k\).

The integrals in Eqs. (18) and (19) are divergent as \(k \to 0\). This divergence can be removed by considering the coordinate derivatives of the correlation functions. Thus in free space, we consider \(\langle \phi_0(ct, x)\partial_{ct'}\phi_0(cf', x') \rangle\) and \(\langle \phi_0(ct, x)\partial_{x'}\phi_0(cf', x') \rangle\). The resulting expressions are convergent as \(k \to 0\) but now divergent as \(k \to \infty\). However, the divergence at large values of \(k\) involves oscillating sine functions. Thus we may introduce a convergence factor such as \(\exp[-\Lambda k]\) into the integrand, carry out the integrals in terms of exponentials, and then take the no-cutoff limit \(\Lambda \to 0\) to obtain the singular Fourier sine transforms of
the form
\[ \int_0^\infty dk \, k^{2m} \sin(bk) = \frac{(-1)^{2m}(2m)!}{b^{2m+1}} \]  \hspace{-2cm} (20)

In this fashion we obtain the closed-form expression

\[ \frac{\partial}{\partial ct} \langle \phi_0(ct, x) \phi_0(ct', x') \rangle = \frac{\partial}{\partial ct'} \left\{ -\frac{\ln|c^2(t-t')^2 - (x-x')^2|}{c^2(t-t')^2 - (x-x')^2} \right\} = 2\hbar c \frac{c(t-t')}{c^2(t-t')^2 - (x-x')^2} \]  \hspace{-2cm} (21)

and similarly obtain

\[ \frac{\partial}{\partial x'} \langle \phi_0(ct, x) \phi_0(ct', x') \rangle = \frac{\partial}{\partial x'} \left\{ -\frac{\ln|c^2(t-t')^2 - (x-x')^2|}{c^2(t-t')^2 - (x-x')^2} \right\} = 2\hbar c \frac{-(x-x')}{c^2(t-t')^2 - (x-x')^2} \]  \hspace{-2cm} (22)

both of which agree with the limit \( a \to -\infty \) in Eq. (17). We note that in empty space there is no length or time parameter which is singled out by the zero-point radiation in an inertial frame. The zero-point correlation functions depend upon the geodesic separation \( c^2(t-t')^2 - (x-x')^2 \) between the field points \((ct, x)\) and \((ct', x')\).

For later comparisons, it is useful to have the closed-form expressions for the zero-point correlation functions in empty space as a function of time at a single spatial coordinate \( x = x' \) and as a function of space at a single time \( t = t' \). Thus we have for the non-vanishing correlations from Eqs. (21) and (22)

\[ \langle \phi_0(ct, x) \partial_{ct'} \phi_0(ct', x') \rangle_{x' = x} = 2\hbar c \frac{1}{c(t-t')} \]  \hspace{-2cm} (23)

and

\[ \langle \phi_0(ct, x) \partial_{x'} \phi_0(ct', x') \rangle_{t = t'} = 2\hbar c \frac{1}{(x-x')} \]  \hspace{-2cm} (24)

The spatial derivatives of the correlation function for a mirror at \( x = a \) at the left-hand end
of the spatial region can be written explicitly as

\[
\left\langle \phi_0^{\text{mirror}}(ct, x) \partial_{ct'} \phi_0^{\text{mirror}}(ct', x') \right\rangle = 2hc \left\{ \frac{c(t-t')}{c^2(t-t')^2 - (x-x')^2} - \frac{c(t-t')}{c^2(t-t')^2 - (x+x'-2a)^2} \right\}
\]

\[
= \partial_{ct'} \left\{ \frac{-h}{c^2(t-t')^2 - (x-x')^2} \right\} - \partial_{ct'} \left\{ \frac{-h}{c^2(t-t')^2 - (x+x'-2a)^2} \right\}
\]

\[
\langle \phi_0^{\text{mirror}}(ct, x) \partial_{x'} \phi_0^{\text{mirror}}(ct', x') \rangle = 2hc \left\{ \frac{-(x-x')}{c^2(t-t')^2 - (x-x')^2} + \frac{-(x+x'-2a)}{c^2(t-t')^2 - (x+x'-2a)^2} \right\}
\]

\[
= \partial_{x'} \left\{ \frac{-h}{c^2(t-t')^2 - (x-x')^2} \right\} - \partial_{x'} \left\{ \frac{-h}{c^2(t-t')^2 - (x+x'-2a)^2} \right\}
\]

(25)

(26)

E. Thermal Scalar Radiation

Within classical theory with classical zero-point radiation, zero-point radiation represents real radiation which is always present, and thermal radiation is additional random radiation above the zero-point value. Thus if \( U(\omega, T) \) is the energy per normal mode at frequency \( \omega \) and temperature \( T \), the thermal energy contribution \( U_T(\omega, T) \) is found by subtracting off the zero-point energy, \( U_T(\omega, T) = U(\omega, T) - U(\omega, 0) \). The additional thermal energy is distributed across the low-frequency modes of the radiation field. The (finite) total thermal energy \( U_T(T) \) in a box is found by summing the thermal energy per normal mode \( U_T(\omega, T) \) over all the normal modes at temperature \( T \) in a box of finite size. The spatial density of thermal energy is given by \( u(T) = U_T(T)/(b-a) = a_{ss} T^2 \) where \( a_{ss} \) is the constant for one-spatial-dimension scalar radiation corresponding to Stefan's constant for electromagnetic radiation.[9] Classical thermal radiation is described in exactly the same random-phase fashion as the zero-point radiation except that the spectrum is changed. The thermal radiation spectrum for massless scalar radiation can be derived from classical theory involving zero-point radiation and the structure of spacetime.[2][10][11] One finds for the energy per normal mode at frequency \( \omega \) and temperature \( T \)

\[
U(\omega, T) = (1/2)\hbar \omega \coth[\hbar \omega/(2k_B T)]
\]

(27)

The calculation for the classical two-point field correlation function at finite temperature accordingly takes exactly the same form as given above in Eqs. [15], except that the
spectrum is changed so that now

\[
\langle \phi_T(\text{box})(ct,x)\phi_T(\text{box})(ct',x') \rangle = \sum_{n=1}^{\infty} \frac{2\hbar c}{n} \coth \left[ \frac{\hbar m \pi}{2(b-a)} \right] \sin \left[ \frac{n\pi}{b-a} (x - a) \right] \sin \left[ \frac{n\pi}{b-a} (x' - a) \right] \cos \left[ \frac{n\pi}{b-a} c(t - t') \right]
\]

The quantum point of view regarding thermal radiation is strikingly different from the classical viewpoint. The vacuum of the quantum scalar field is said to involve fluctuations but no quanta, no elementary excitations, no scalar photons, whereas the thermal radiation field involves a distinct pattern of scalar photons. If the index \( m \) is used to label the normal modes in a one-dimensional box, the quantum expectation values correspond to an incoherent sum over the expectation values for the fields for all numbers \( n_m \) of photons of frequency \( \omega_m = m\pi c/(b - a) \) with a weighting given by the Boltzmann factor \( \exp[-n_m \hbar \omega_m/(k_B T)] \).

Thus the quantum two-point field correlation function for our example involving a box in one spatial dimension is given by [2]

\[
\langle\langle 1/2 \{ \overline{\phi}(ct,x)\overline{\phi}(ct',x') + \phi(ct',x')\overline{\phi}(ct,x) \}\rangle \rangle_T
= \sum_{m=1}^{\infty} \sum_{n_m=0}^{\infty} 1 \cdot \frac{Z\{\hbar c \pi/[2(b-a)k_B T]\}}{Z(\hbar c \pi/[2(b-a)k_B T])} \exp\left[-n \frac{\hbar c m \pi}{(b-a)k_B T} \right]
\times \langle n_m \{1/2 \{ \overline{\phi}(ct,r)\overline{\phi}(ct',r') + \phi(ct',r')\overline{\phi}(ct,r) \}\} \rangle_{n_m}
= \sum_{m=1}^{\infty} 2\hbar c \coth \left[ \frac{\hbar m \pi}{2(b-a)} \right] \sin \left[ \frac{m\pi}{b-a} (x - a) \right] \sin \left[ \frac{m\pi}{b-a} (x' - a) \right] \cos \left[ \frac{m\pi}{b-a} c(t - t') \right] \]

(29)

where we have noted that

\[
\frac{1}{2} \coth \frac{x}{2} = \frac{\sum_{n=0}^{\infty} (n + 1/2) \exp[-nx]}{\sum_{n=0}^{\infty} \exp[-nx]}
\]

(30)

and have defined

\[
Z(x) = \sum_{n=0}^{\infty} \exp[-nx]
\]

(31)

Thus for symmetrized products of quantum fields, the quantum expectation value in Eq. (29) is in exact agreement with the corresponding classical average value found in Eq. (28). Again the agreement holds for higher order correlation functions provided the quantum operator order is completely symmetrized [2].
The agreement between the classical and quantum correlation functions remains in the limits of a large box \( b \to \infty \) analogous to the transition from Eq. (15) over to Eq. (16) and in the removal of the left-hand mirror to negative spatial infinity as in the transition from Eq. (16) over to Eq. (19).

It should be emphasized that although there is complete agreement between the correlation functions arising in classical theory and the symmetrized expectation values in quantum theory, the interpretations in terms of fluctuations arising from classical wave interference or in terms of fluctuations arising from the presence of photons are completely different between the theories.\[12\] The contrast in interpretations and indeed in predictions becomes even more striking when an accelerating coordinate frame is involved.

III. RADIATION IN A RINDLER FRAME

A. Rindler Coordinate Frame

Although there is close agreement between classical and quantum field theories in an inertial frame, the two theories part company in noninertial frames. The noninertial frame which we will consider in this article is a Rindler coordinate frame accelerating through Minkowski spacetime in two spacetime dimensions.\[13\][14] If the coordinates of a spacetime point in an inertial frame are given by \((ct, x)\), then the coordinates \((\eta, \xi)\) of the spacetime point in the Rindler frame which is at rest with respect to the inertial frame at time \(t = 0 = \eta\) are given by

\[
ct = \xi \sinh \eta \\
x = \xi \cosh \eta
\]

(32)

(33)

with \(-\infty < \eta < \infty\), and \(0 < \xi\). Using the relation \(\cosh^2 \eta - \sinh^2 \eta = 1\), it follows that a point with fixed spatial coordinate \(\xi\) in the Rindler frame has coordinates \(x_\xi(t)\) in the inertial frame given by

\[
x_\xi(t) = \xi \cosh \eta = (\xi^2 + \xi^2 \sinh \eta)^{1/2} = (\xi^2 + c^2 t^2)^{1/2}
\]

(34)

and so moves with acceleration \(a_\xi = d^2x/dt^2 = c^2/\xi\) at time \(t = 0\), and indeed in the Rindler frame has constant proper acceleration

\[
a_\xi = c^2/\xi
\]

(35)
at all times. Thus for large coordinates $\xi$, the acceleration $a_\xi$ becomes small whereas for small $\xi$, the proper acceleration diverges. The point $\xi = 0$ is termed the "event horizon" for the Rindler coordinate frame.

The metric in the Rindler frame can be obtained from Eqs. (32) and (33) as

$$ds^2 = dt^2 - dx^2 = \xi^2 d\eta^2 - d\xi^2$$

(36)

It is clear from this expression that the time coordinate $\eta$ in the Rindler frame is not a geodesic coordinate. Indeed, the geodesic separation between two spacetime points which takes the form $c^2(t - t')^2 - (x - x')^2$ in the geodesic coordinates of the inertial frame becomes in Rindler coordinates

$$c^2(t - t')^2 - (x - x')^2 = (\xi \sinh \eta - \xi' \sinh \eta')^2 - (\xi \cosh \eta - \xi' \cosh \eta')^2$$

$$= 2\xi\xi' \cosh(\eta - \eta') - \xi^2 - \xi'^2$$

(37)

B. Normal Modes in a Box in a Rindler Frame

We now consider the spectrum of random radiation as observed in the Rindler frame. First we obtain the radiation normal modes. The wave equation (3) in an inertial frame can be transformed to the wave equation in the Rindler frame by using the transformations (32) and (33) together with the scalar behavior of the field $\phi$ under a coordinate transformation. The scalar field takes the same value in any coordinate frame. Thus the field $\phi(\eta, \xi)$ in the Rindler frame is equal to the field $\phi(ct, x)$ in the inertial frame at the same spacetime point,

$$\phi(\eta, \xi) = \phi(ct, x) = \phi(\xi \sinh \eta, \xi \cosh \eta).$$

(38)

If we use the usual rules for partial derivatives, we find that Eq. (3) becomes in the Rindler frame

$$\left(\frac{\partial^2 \phi}{\partial \xi^2}\right) + \frac{1}{\xi} \left(\frac{\partial \phi}{\partial \xi}\right) - \frac{1}{\xi^2} \left(\frac{\partial^2 \phi}{\partial \eta^2}\right) = 0.$$  

(39)

The solutions of Eq. (39) take the form $H(\ln \xi \pm \eta)$ where $H$ is an arbitrary function. Thus, whereas the general solution of the scalar wave equation (3) in an inertial frame is $\phi(ct, x) = h_+(x - ct) + h_-(x + ct)$ where $h_+$ and $h_-$ are arbitrary functions, the general solution in a Rindler frame is $\phi(\eta, \xi) = H_+(\ln \xi - \eta) + H_-(\ln \xi + \eta)$ where $H_+$ and $H_-$ are arbitrary functions. The normal mode solutions of the wave equation in the Rindler frame
for a box extending from $0 < \xi = a$ to $\xi = b$ with Dirichlet boundary conditions can be obtained by separation of variables and expressed as a time-Fourier series

$$\varphi_n(\eta, \xi) = \mathcal{F}_n \left( \frac{2}{\ln(b/a)} \right)^{1/2} \sin \left[ \frac{n \pi}{\ln(b/a)} \ln \left( \frac{\xi}{a} \right) \right] \cos \left[ \frac{n \pi}{\ln(b/a)} \eta + \theta_n \right], \quad (n = 1, 2, 3 \ldots),$$

(40)

where $\mathcal{F}_n$ is the amplitude of the normal mode and the spatial functions

$$\psi_n(\xi) = \left( \frac{2}{\ln(b/a)} \right)^{1/2} \sin \left[ \frac{n \pi}{\ln(b/a)} \ln \left( \frac{\xi}{a} \right) \right],$$

(41)

arise from a Sturm-Liouville system[15] and form a complete orthonormal set with weight $1/\xi$ on the interval $a < \xi < b$. Thus we find

$$\int_a^b d\xi \frac{d\xi}{\xi} \psi_n(\xi) \psi_m(\xi) = \int_a^b d\xi \frac{2}{\xi \ln(b/a)} \sin \left[ \frac{n \pi}{\ln(b/a)} \ln \left( \frac{\xi}{a} \right) \right] \sin \left[ \frac{m \pi}{\ln(b/a)} \ln \left( \frac{\xi}{a} \right) \right]$$

$$= \int_{v=0}^{v=\pi} \frac{d\xi}{\pi} \frac{2}{\ln(b/a)} \sin(nv) \sin(mv) = \delta_{nm},$$

(42)

where we have used the substitution $v = [\pi \ln(\xi/a)]/\ln(b/a)$ in evaluating the integral. For a radiation normal mode, the Rindler time parameter $\eta$ agrees with all local clocks when adjusted by $\xi$, and thus the time $\tau = \xi \eta$ gives the proper time of a clock located at fixed Rindler spatial coordinate $\xi$.

For time-stationary random radiation in the Rindler frame with an unknown time-spectral amplitude $\mathcal{F}_n$, the field $\varphi(\eta, \xi)$ can be written as a sum over the normal modes $\varphi_n(\eta, \xi)$ in Eq. (40) with random phases $\theta_n$ distributed randomly over the interval $[0, 2\pi)$ and distributed independently for each value of $n$.

$$\varphi_{\text{box}}(\eta, \xi) = \sum_{n=1}^{\infty} \mathcal{F}_n \left( \frac{2}{\ln(b/a)} \right)^{1/2} \sin \left[ \frac{n \pi}{\ln(b/a)} \ln \left( \frac{\xi}{a} \right) \right] \cos \left[ \frac{n \pi}{\ln(b/a)} \eta + \theta_n \right]$$

(43)

Then the two-field correlation function is obtained in analogy with Eqs. (14)–(15)

$$\langle \varphi_{\text{box}}(\eta, \xi) \varphi_{\text{box}}(\eta', \xi') \rangle = \int_{\eta=0}^{\eta=\infty} d\kappa \mathcal{F}^2(\kappa) \sin \left[ \frac{n \pi}{\ln(b/a)} \ln \left( \frac{\xi}{a} \right) \right] \sin \left[ \frac{n \pi}{\ln(b/a)} \ln \left( \frac{\xi'}{a} \right) \right] \cos \left[ \frac{n \pi}{\ln(b/a)} \eta + \theta_n \right].$$

(44)

For a large box $b \to \infty$, The normal mode frequencies $\kappa_n = n \pi / \ln(b/a)$ become continuous and the sum in Eq. (44) becomes the integral for the correlation function for a mirror at the left-hand edge $\xi = a$ of the box

$$\langle \varphi_{\text{mirror}}(\eta, \xi) \varphi_{\text{mirror}}(\eta', \xi') \rangle = \frac{1}{\pi} \int_{\kappa=0}^{\infty} d\kappa \mathcal{F}^2(\kappa) \sin \left[ \kappa \ln \left( \frac{\xi}{a} \right) \right] \sin \left[ \kappa \ln \left( \frac{\xi'}{a} \right) \right] \cos \left[ \kappa (\eta - \eta') \right]$$

(45)
The expression (45) can be rewritten in the form

\[
\langle \varphi_{\text{mirror}}(\eta, \xi) \varphi_{\text{mirror}}(\eta', \xi') \rangle = \frac{1}{2\pi} \int_{-\infty}^{\infty} d\kappa F^2(\kappa) \cos [\kappa(\ln \xi - \ln \xi')] \cos [\kappa(\eta - \eta')]
- \frac{1}{2\pi} \int_{\kappa=0}^{\infty} d\kappa F^2(\kappa) \cos [\kappa(\ln \xi + \ln \xi' - 2 \ln a)] \cos [\kappa(\eta - \eta')]
\] (46)

In the limit \(a \to 0\) in which the mirror at \(\xi = a\) is moved to the event horizon, the last integral in Eq. (46) involves a rapidly oscillating cosine function; it can be taken to vanish when considering the time derivative at \(\xi = \xi'\). Thus we find the free-space expression

\[
\langle \varphi(\eta, \xi) \partial_{\eta'} \varphi(\eta', \xi') \rangle_{\xi = \xi'} = \frac{1}{4\pi} \int_{-\infty}^{\infty} d\kappa F^2(\kappa) \kappa \sin [\kappa(\eta - \eta')]
\] (47)

where the spectral amplitude \(F(\kappa)\) of the random radiation is still unspecified.

C. Classical Zero-Point Radiation in the Rindler-Frame Box

It was noted earlier that the spectrum of classical zero-point radiation follows from the assumed symmetry properties of the vacuum. Thus the spectrum of random classical radiation in empty space is assumed to be featureless; the two-point correlation function can depend upon only the geodesic separation (and its coordinate derivatives) between the spacetime points. This dependence upon the geodesic separation has been exhibited in earlier articles for the relativistic scalar and electromagnetic fields in four spacetime dimensions.\[5\] [6] For the example of two spacetime dimensions used in the present article, the derivative correlation functions (21) and (22) involve the partial derivatives of the logarithm of the spacetime separation \(|c^2(t - t')^2 - (x - x')^2|\) between the spacetime points \((ct, x)\) and \((ct', x')\).

In classical theory, the zero-point radiation is physically present. There is no notion of "virtual" photons which may come into and then out of existence. Thus in empty space, the spectrum of radiation which is found in the Rindler frame follows directly by tensor transformation from the radiation found in the inertial frame. We find for a scalar field
that the correlation function is the same in the inertial frame and the Rindler frame for the
same spacetime points

\[ \langle \varphi(\eta, \xi) \varphi(\eta', \xi') \rangle = \langle \phi(ct, x) \phi(ct, x) \rangle = \langle \phi(\xi \sinh \eta, \xi \cosh \eta) \phi(\xi' \sinh \eta', \xi' \cosh \eta') \rangle \] (48)

However, it is clear from this equation (48) that the functional dependence of the correla-
tion function upon \( \xi, \xi', \eta, \eta' \) will in general be quite different from the dependence upon
\( x, x', t, t' \) since from Eq. (37), the geodesic separation takes the form 
\[ c^2(t - t')^2 - (x - x')^2 = 2\xi\xi' \cosh(\eta - \eta') - \xi^2 - \xi'^2, \]
and the Rindler frame time parameter \( \eta \) is not a geodesic co-
ordinate. In empty space, the closed form expressions for the spatial derivatives of the
correlation function in the Rindler frame follow from Eqs. (21), (22), (37) and (48) as

\[ \langle \varphi_0(\eta, \xi) \partial_{\eta'} \varphi_0(\eta', \xi') \rangle = \partial_{\eta'} \{-\hbar c \ln |2\xi\xi' \cosh(\eta - \eta') - \xi^2 - \xi'^2| \} \] (49)

\[ \langle \varphi_0(\eta, \xi) \partial_{\xi'} \varphi_0(\eta', \xi') \rangle = \partial_{\xi'} \{-\hbar c \ln |2\xi\xi' \cosh(\eta - \eta') - \xi^2 - \xi'^2| \} \] (50)

The time-spectrum found in the Rindler frame may be obtained by taking the singular
Fourier sine transform of the time correlation at a single spatial coordinate \( \xi = \xi' \). Thus
from Eq. (47) and (49), we find for the spectral function corresponding to classical zero-point
radiation [16]

\[ \mathcal{F}_\omega^2(\kappa) = \frac{4}{\kappa} \int_0^\infty d(\eta - \eta') \sin[\kappa(\eta - \eta')] \langle \varphi_0(\eta, \xi) \partial_{\eta'} \varphi_0(\eta', \xi') \rangle_{\xi=\xi'} \]
\[ = \frac{4}{\kappa} \int_0^\infty d(\eta - \eta') \sin[\kappa(\eta - \eta')] \frac{\hbar c \sinh(\eta - \eta')}{\cosh(\eta - \eta') - 1} \]
\[ = \frac{4\hbar c}{\kappa} \int_0^\infty du \sin(\kappa u) \coth(\frac{u}{2}) = \frac{4\hbar c}{\kappa} \pi \coth[\kappa\pi] \] (51)

In a Rindler frame box of finite length, this spectral function (51) is restricted to the allowed
normal modes \( \kappa_n = n\pi / \ln(b/a) \), so that

\[ \varphi_{\text{box}}(\eta, \xi) = \sum_{n=1}^{n=\infty} \left( 4\pi \frac{\hbar c \ln(b/a)}{n\pi} \coth \left[ \frac{n\pi^2}{\ln(b/a)} \right] \right)^{1/2} \left( \frac{2}{\ln(b/a)} \right)^{1/2} \sin \left[ \frac{n\pi}{\ln(b/a)} \ln \left( \frac{\xi}{a} \right) \right] \]
\[ \times \cos \left[ \frac{n\pi}{\ln(b/a)} \eta + \theta_n \right] \] (52)
and the two-point correlation function in the box is given by

\[
\langle \varphi_{0\text{box}}(\eta, \xi) \varphi_{0\text{box}}(\eta', \xi') \rangle \\
= \sum_{n=1}^{\infty} 4\pi \frac{hc \ln(b/a)}{n\pi} \coth \left[ \frac{n\pi^2}{\ln(b/a)} \right] \left( \frac{1}{\ln(b/a)} \right) \sin \left[ \frac{n\pi}{\ln(b/a)} \ln \left( \frac{\xi}{a} \right) \right] \sin \left[ \frac{n\pi}{\ln(b/a)} \ln \left( \frac{\xi'}{a} \right) \right] \\
\times \cos \left[ \frac{n\pi(\eta - \eta')}{\ln(b/a)} \right]
\] (53)

In the limit as \( b \to \infty \), corresponding to the right-hand edge of the box going to positive spatial infinity, the normal mode frequencies \( \kappa_n = n\pi / \ln(b/a) \) become continuous, and the correlation function (53) becomes that for a mirror at the left-hand edge \( \xi = a \) of the box,

\[
\langle \varphi_{0\text{mirror}}(\eta, \xi) \varphi_{0\text{mirror}}(\eta', \xi') \rangle \\
= 4hc \int_{\kappa=0}^{\infty} \frac{d\kappa}{\kappa} \coth [\kappa \pi] \sin \left[ \kappa \ln \left( \frac{\xi}{a} \right) \right] \sin \left[ \kappa \ln \left( \frac{\xi'}{a} \right) \right] \cos [\kappa(\eta - \eta')] \\
\] (54)

This is a convergent integral which can be evaluated as [16]

\[
\langle \varphi_{0\text{mirror}}(\eta, \xi) \varphi_{0\text{mirror}}(\eta', \xi') \rangle \\
= -hc \ln \frac{\sinh\left[\left\{\ln(\xi/a) - \ln(\xi'/a) + (\eta - \eta')\right\}/2\right]\sinh\left[\left\{\ln(\xi/a) - \ln(\xi'/a) - (\eta - \eta')\right\}/2\right]}{\sinh\left[\left\{\ln(\xi/a) + \ln(\xi'/a) + (\eta - \eta')\right\}/2\right]\sinh\left[\left\{\ln(\xi/a) + \ln(\xi'/a) - (\eta - \eta')\right\}/2\right]} \\
= -hc \ln \frac{\sinh\left[\left\{\ln(\xi'/a^2) + (\eta - \eta')\right\}/2\right]\sinh\left[\left\{\ln(\xi'/a^2) - (\eta - \eta')\right\}/2\right]}{\sinh\left[\left\{\ln(\xi/a^2) + (\eta - \eta')\right\}/2\right]\sinh\left[\left\{\ln(\xi/a^2) - (\eta - \eta')\right\}/2\right]} \\
\] (55)

In the limit where the mirror at \( \xi = a \) is moved to the event horizon, \( a \to 0 \), the correlation function in Eq. (55) diverges as \( hc \ln |\xi'/a|^2| = hc \ln |\xi' - hc \ln (a)|^2| \), which appears similar to the divergence in Eq. (49), except that in previous case \( a \to -\infty \) whereas here \( a \to 0 \). Just as was done earlier, the divergence can be eliminated by taking coordinate derivatives. In this limit, the correlation function (55) should correspond to that for empty space since as the mirror goes to the event horizon of the Rindler frame, the phases of waves change very rapidly with distance, and we expect that the phases of the incident and reflected waves should become uncoupled. In the limit \( a \to 0 \), the correlation function for the mirror (55) becomes (with divergence-eliminating coordinate derivatives)

\[
\partial_{\nu'} \langle \varphi_0(\eta, \xi) \varphi_0(\eta', \xi') \rangle \\
= \partial_{\nu'} \left\{ -hc \ln \left| 4\xi' \sinh\left[\left\{\ln(\xi'/\xi) + (\eta - \eta')\right\}/2\right]\sinh\left[\left\{\ln(\xi'/\xi) - (\eta - \eta')\right\}/2\right]\right\} \\
\] (56)

This expression indeed agrees with the correlation functions for empty space given in (49)
and \((50)\) since
\[
-4\xi \xi' \sinh\left\{\ln\left(\frac{\xi}{\xi'} + (\eta - \eta')/2\right)\sinh\left\{\ln\left(\frac{\xi}{\xi'} - (\eta - \eta')/2\right)\right\}\right.
\]
\[
= 2\xi \xi' \cosh(\eta - \eta') - \xi^2 - \xi'^2
\]

(57)

Thus a box with classical zero-point radiation takes on the empty-space zero-point correlation function when the box is expanded to cover the entire Rindler spacetime region (the Rindler wedge). The presence of any reflecting walls on the Rindler box becomes ever less important as the walls recede to the limits of the Rindler region.

If we consider the zero-point correlation function in free space as a function of space for a single time \(\eta = \eta'\) or as a function of time for a single coordinate \(\xi = \xi'\) in the Rindler frame, then we find the non-vanishing two-point correlations in free space from Eqs. (49) and (50),
\[
\langle \varphi_0(\eta, \xi) \partial_\eta' \varphi_0(\eta', \xi') \rangle_{\xi = \xi'} = 2\hbar \frac{2 \sinh[(\eta - \eta')/2] \cosh[(\eta - \eta')/2]}{4 \sinh^2[(\eta - \eta')/2]}
\]
\[
\quad = \hbar c \coth\left(\frac{\eta - \eta'}{2}\right)
\]

(58)

and
\[
\langle \varphi_0(\eta, \xi) \partial_\xi' \varphi_0(\eta', \xi') \rangle_{\eta = \eta'} = \frac{2\hbar c}{\xi - \xi'}
\]

(59)

\[
\langle \varphi_0(\eta, \xi) \partial_\xi' \varphi_0(\eta', \xi') \rangle_{\xi = \xi'} = -\frac{\hbar c}{\xi}
\]

(60)

D. Canonical Quantization in a Rindler-Frame Box

Quantum theory regards canonical quantization as a fundamental procedure which can be followed in any box, no matter whether the box is at rest in an inertial frame or is at rest in a noninertial coordinate frame. Thus for a box in a Rindler frame, the quantum field can be expressed in a form parallel to Eq. (12) as
\[
\Psi_{\text{box}}(\eta, \xi) = \sum_{n=1}^{n=\infty} \left(8\pi \hbar c \frac{\ln(b/a)}{n\pi}\right)^{1/2} \left(2 \frac{2}{\ln(b/a)}\right)^{1/2} \sin \left[\frac{n\pi}{\ln(b/a)} \ln\left(\frac{\xi}{a}\right)\right]
\]
\[
\times \frac{1}{2} \left\{ \bar{b}_n \exp \left[ i \frac{n\pi}{\ln(b/a)} \eta \right] + \bar{b}^+_n \exp \left[ -i \frac{n\pi}{\ln(b/a)} \eta \right]\right\}
\]

(61)

where \(\bar{b}_n\) and \(\bar{b}^+_n\) are the annihilation and creation operators for particles in the Rindler-frame box. Notice that the amplitude appearing in the sum is the same factor involving
the square root of $8\pi\hbar c$ times the wave number, just as in Eq. 12 in the inertial frame in empty space. In contrast, the classical theory involves the amplitude factor $F_0(\kappa)$ given in Eq. 51 in order to compensate for the fact that the time coordinate for the normal modes is not a geodesic coordinate. In quantum theory, there is a Rindler-frame vacuum state $|0_R>$ which is annihilated by the Rindler operator $\overline{b}_n$. The two-point Rindler-vacuum expectation value for the symmetrized product of the field operators gives the result parallel to Eq. 13 as

$$<0_R|\frac{1}{2}\{\overline{\varphi}_{\text{box}}(\eta, \xi)\overline{\varphi}_{\text{box}}(\eta', \xi') + \overline{\varphi}_{\text{box}}(\eta', \xi')\overline{\varphi}_{\text{box}}(\eta, \xi)}|0_R>$$

$$= \sum_{n=1}^{\infty} \frac{4\hbar c}{n} \sin \left[ \frac{n\pi}{\ln(b/a)} \ln \left( \frac{\xi}{a} \right) \right] \sin \left[ \frac{n\pi}{\ln(b/a)} \ln \left( \frac{\xi'}{a} \right) \right] \cos \left[ \frac{n\pi(\eta - \eta')}{\ln(b/a)} \right]$$

(62)

In the limit as $b \to \infty$, this expression becomes the Rindler-vacuum expectation value for the situation of continuous normal mode frequencies $\kappa_n = n\pi/\ln(b/a)$ and a mirror at $\xi = a$, analogous to Eqs. 16, and 17,

$$<0_R|\frac{1}{2}\{\overline{\varphi}_{\text{mirror}}(\eta, \xi)\overline{\varphi}_{\text{mirror}}(\eta', \xi') + \overline{\varphi}_{\text{mirror}}(\eta', \xi')\overline{\varphi}_{\text{mirror}}(\eta, \xi)}|0_R>$$

$$= 2\hbar c \int_0^\infty \frac{d\kappa}{\kappa} \sin \left[ \kappa \ln \left( \frac{\xi}{a} \right) \right] \sin \left[ \kappa \ln \left( \frac{\xi'}{a} \right) \right] \cos[\kappa(\eta - \eta')]$$

$$= -\hbar c \ln \left| \frac{\{\ln(\xi/a) - \ln(\xi'/a)\} - c(t - t')}{\{\ln(\xi/a) + \ln(\xi'/a)\} - c(t - t')} \right|$$

$$= -\hbar c \ln \left| \frac{\{\ln(\xi/a)\}^2 - c^2(t - t')^2}{\{\ln(\xi/a) + \ln(\xi'/a)\}^2 - c^2(t - t')^2} \right|$$

(63)

In the limit $a \to 0$ that the mirror is moved to the event horizon, the expectation value for the quantum fields in the Rindler vacuum becomes divergent as $2\hbar c \ln[2\ln(a)]$. Again the divergence can be eliminated by taking coordinate derivatives

$$\partial_\mu <0_R|\frac{1}{2}\{\varphi(\eta, \xi)\varphi(\eta', \xi') + \varphi(\eta', \xi')\varphi(\eta, \xi)}|0_R>$$

$$= \partial_\mu \left\{ -\hbar c \ln \left| \{\ln(\xi/\xi')\}^2 - c^2(t - t')^2 \right| \right\}$$

(64)

Thus we obtain

$$<0_R|\frac{1}{2}\{\varphi(\eta, \xi)\partial_\eta \varphi(\eta', \xi') + \partial_\eta \varphi(\eta', \xi')\varphi(\eta, \xi)}|0_R>$$

$$= 2\hbar c \frac{(\eta - \eta')}{(\eta - \eta')^2 - (\ln \xi - \ln \xi')^2}$$

(65)
If we consider the spatial dependence at a single time and the time dependence at a single
spatial point, we find for the symmetrized expectation value for the Rindler vacuum that
the non-vanishing values from Eqs. (65) and (66) are
\[
< 0_R \frac{1}{2} \{ \tilde{\varphi}(\eta, \xi) \partial_{\xi'} \tilde{\varphi}(\eta', \xi') + \partial_{\xi'} \tilde{\varphi}(\eta', \xi') \tilde{\varphi}(\eta, \xi) \} | 0_R > = 2 \hbar c \frac{(\ln \xi - \ln \xi')}{(\eta - \eta')^2 - (\ln \xi - \ln \xi')^2}
\]
(66)

and
\[
< 0_R \frac{1}{2} \{ \tilde{\varphi}(\eta, \xi) \partial_{\eta'} \tilde{\varphi}(\eta', \xi') + \partial_{\eta'} \tilde{\varphi}(\eta', \xi') \tilde{\varphi}(\eta, \xi) \} | 0_R >_{\xi=\xi'} = 2 \hbar c \frac{1}{(\eta - \eta')}
\]
(67)

and
\[
< 0_R \frac{1}{2} \{ \tilde{\varphi}(\eta, \xi) \partial_{\eta'} \tilde{\varphi}(\eta', \xi') + \partial_{\eta'} \tilde{\varphi}(\eta', \xi') \tilde{\varphi}(\eta, \xi) \} | 0_R >_{\eta=\eta'} = 2 \hbar c \frac{1}{(\ln \xi - \ln \xi')}
\]
(68)

The Rindler vacuum expectation value in (67) with its dependence upon the inverse time
separation is analogous to the free-space inertial frame vacuum expectation value (23) in
an inertial frame. However, the Rindler vacuum expectation value (68) with its logarithmic
dependence on \( \xi \) and \( \xi' \) has no analogue in an inertial frame. The "Rindler vacuum" is
different from the "Minkowski vacuum" under canonical quantization.

E. Contrasting Classical-Quantum Viewpoints in a Rindler Frame

Although the classical zero-point correlation functions and the quantum symmetrized
vacuum expectation values agree in inertial frames, they are no longer in agreement in non-
inertial frames. The vacuum states arise from very different concepts in the classical and the
quantum theories. The essential feature of classical zero-point radiation is that the spectrum
of random radiation is featureless. Therefore in empty space, classical zero-point radiation
depends only upon the geodesic separation of the field points. The spectrum obtained from
the continuous frequencies of empty space is then restricted to the allowed normal mode
frequencies in a box of finite size. In the limit where the sides of the box are moved to the
limits of the spacetime, the spectrum in the box becomes that of empty space. Thus a box
with walls at rest in an inertial frame and a box at rest with respect to the coordinates of a
Rindler frame have very different normal modes, and the spectral amplitudes are readjusted
to reflect the change from a geodesic to non-geodesic time coordinate. In terms of a geodesic
time coordinate such as appears in an inertial frame, the spectrum of zero-point radiation is given by \( f_0^2(k) = 4\pi\hbar c/|k| \) where the constant is chosen to give an energy \((1/2)\hbar c|k|\) per normal mode. In terms of the non-geodesic time coordinate \( \eta \) appearing in a Rindler frame, the spectrum of zero-point radiation is given by \( F_0^2(\kappa) = (4\pi\hbar c/\kappa) \coth(\pi\kappa) \). If the walls of the box are moved to the limits of the Rindler wedge, the random radiation in the Rindler space is exactly that of the inertial space. The classical vacuum is unique.

In complete contrast, the vacuum of quantum field theory arises from a prescriptive process which takes no account of the spacetime metric. In any box, the amplitude for the normal modes is fixed, and annihilation and creation operators are introduced for the positive and negative time aspects. Thus a box with walls at rest in an inertial frame and a box at rest with respect to the coordinates of a Rindler frame have very different normal modes but the same spectral amplitude, and accordingly have very different vacuum states. If the walls of the Rindler box are moved out to the limits of the Rindler spacetime wedge, the quantum fluctuations associated with the Rindler vacuum state remain quite different from the quantum fluctuations associated with the inertial vacuum state. The "Rindler vacuum" is different from the "Minkowski vacuum" even for a large box. There is a non-uniqueness for the quantum vacuum in non-inertial frames.

Of course, one can apply tensor transformations to the vacuum expectation values of the symmetrized quantum operators which were found in an inertial frame. Since the symmetrized quantum expectation values agree exactly with the corresponding classical correlation functions in an inertial frame, we obtain exactly the same expressions as found for the classical correlation functions in the Rindler frame. The spatial dependence on the geodesic coordinate \( \xi \) found in Eq. (59) for the correlation function at a single time \( \eta = \eta' \) agrees exactly with that found in the corresponding expression (24) in an inertial frame (for \( x = \xi, x' = \xi' \)), as we indeed expect since a fixed time \( \eta = \eta' \) corresponds to a single time \( t = t' \) in the momentarily comoving inertial reference frame, and all inertial frames have the same correlation functions for zero-point radiation. The absence of any spatial correlation length in Eq. (59) corresponds to zero-temperature \( T = 0 \). However, the time dependence in Eq. (58) for the correlation function at a single spatial coordinate \( \xi = \xi' \) is quite different from the time dependence (23) found in an inertial frame. Indeed, The appearance of the hyperbolic cotangent function for the time-Fourier spectrum in Eq. (51) has led some physicists to speak of the "thermal effects
of acceleration through the vacuum" [17][18][19][20][21] with temperature $T = \hbar a/(2\pi c k_B)$. After all, the hyperbolic cotangent function appeared in Eq. (27) for the spectrum of thermal radiation in an inertial frame. Thus the spectra in the Rindler frame can be used to suggest either finite temperature $T = \hbar a/(2\pi c k_B)$ or zero-temperature $T = 0$ depending upon one’s point of view. This ambiguity arises precisely because the Rindler frame is not an inertial frame and the Rindler time parameter $\eta$ is not a geodesic coordinate. Indeed one may inquire as to just what spectrum corresponds to thermal radiation in a noninertial frame. Within classical physics, this question has been discussed in connection with time-dilating conformal transformations which allow us to derive the Planck spectrum from the structure of relativistic spacetime.[9][10][11]

Despite the classical-quantum agreement of the tensor-transformed inertial expectation values, the quantum viewpoint is more complicated since quantum theory introduces a new vacuum state associated with canonical quantization in the Rindler frame. Canonical quantization within a box in a Rindler frame leads to field fluctuations which are quite different from those found from quantization in an inertial frame. Thus the time dependence of the symmetrized Rindler vacuum expectation value at a single spatial coordinate in (67) (with its inverse time dependence) is indeed analogous to the inverse time dependence found in (23) for the inertial frame. However, the logarithmic spatial dependence of the Rindler vacuum expectation value at a single time in (68) is quite different from that given in (24) for the inertial frame. Thus the quantum vacuum in a Rindler frame has quite different properties from the quantum vacuum in an inertial frame. Indeed, over 30 years ago, Fulling called attention to this "Nonuniqueness of Canonical Quantization in Riemannian Space-Time."[17]

And what is the physical meaning of the "Rindler vacuum state" which is different from the familiar "Minkowski vacuum state"? According to some quantum field theorists, the vacuum is established by the walls of the box which confine the radiation. If the walls of the box are established at temperature $T = 0$ in the inertial frame vacuum and then the box has its acceleration slowly increased to the final acceleration, its interior will be in the Rindler vacuum. On the other hand, if the box at temperature $T = 0$ in the inertial frame is suddenly accelerated, the box will contain Rindler excitations corresponding to the Fulling-Davies-Unruh temperature $T = \hbar a/(2\pi c k_B)$ as measured in the Rindler frame where the Rindler vacuum is the lowest energy state.[3]
The classical theory with zero-point radiation lends no support to this quantum interpretation. The classical vacuum state involving classical zero-point radiation is unique; its description between any two coordinate frames is found by tensor transformations. In particular, classical physics has nothing like the scenario described above for a box of zero-point radiation which is moved from an inertial to a Rindler frame. According to classical theory, (except for small Casimir effects) it matters not how the box of (featureless) classical zero-point radiation is moved from the inertial frame into the accelerating Rindler frame; the box of radiation will always correspond to zero-point radiation as described by tensor transformation from an inertial frame. This statement seems to come as a surprise to many physicists who are misled by their experience with spectra involving finite total energy. The invariant result for a box of zero-point radiation follows from the very special character of the zero-point spectrum which has no structure other than that which is given to it by the coordinates associated with the metric of the spacetime.

In an inertial frame in empty space, the zero-point radiation spectrum is Lorentz invariant and scale invariant; it depends only upon the separation (and coordinate derivatives) between the two spacetime points measured along a geodesic between the points. Perhaps the reader can obtain a better sense of the special character of zero-point radiation from the following considerations. We saw in Eqs. (21) and (22) that the spectrum of random classical zero-point radiation for the scalar field in an inertial frame depends upon the logarithm of the invariant separation between the two spacetime points. Since we are dealing with a scalar field, the correlation function takes the same value in the Rindler frame. If we transform the Minkowski coordinates to Rindler coordinates, as given in Eqs. (49) and (50), we find that the correlation function is time stationary; it depends upon only the time difference \((\eta - \eta')\) and not on any initial time. There is no spectrum of finite energy density which has such behavior; time-translation invariance both in all inertial frames and in all Rindler frames is a property unique to the zero-point radiation spectrum.

The solutions for the wave equations (3) and (39) are unique for boundary conditions which specify both the function and its first time derivative at a single time coordinate. We can imagine a box of zero-point radiation which is at rest in an inertial frame and then is suddenly accelerated so as to remain at the fixed coordinates of a Rindler frame. If we have a box at rest with respect to the coordinates of a Rindler frame, it will be instantaneously
at rest with respect to some inertial frame. Within the classical theory, the zero-point radiation within the box differs from the zero-point radiation in the inertial frame by simply the fact that the box modes are restricted to the normal modes of the box rather than being the continuous modes of empty space. As was proved in our analysis above, the zero-point radiation in a Rindler box whose walls are moved to the limits of the Rindler wedge is in complete agreement with the radiation in the empty-space Rindler frame and the radiation in the empty-space inertial frame. Thus the only difference between the radiation inside the box and the radiation of the empty-space inertial frame outside the box are the Casimir aspects associated with the discreteness of the normal mode spectrum of a finite box. For a large box, the zero-point radiation can be accelerated without changing its spectrum.

In work published earlier,[9][10] it has been pointed out that the Planck spectrum for classical thermal radiation arises naturally by considering the time-dilation symmetry of thermal radiation in a Rindler frame. Thus in an inertial frame, a time-dilating conformal transformation carries thermal radiation at temperature $T$ into thermal radiation at temperature $\sigma T$ where $\sigma$ is a positive real number. Under such a transformation, zero-point radiation in an inertial frame remains zero-point radiation. However, in a Rindler frame, a time-dilating conformal transformation carries zero-point radiation into thermal radiation at a non-zero-temperature.[11] The perspective from classical physics suggests that the canonical quantization procedure in a non-inertial frame may be predicting results which have no realization in nature.

**F. Detectors Accelerating through Classical Zero-Point Radiation**

Although during the 1970’s there were discussions as to whether or not acceleration through the quantum Minkowski vacuum turned virtual photons into real photons, today quantum field theory claims merely that ”detectors” accelerating through the quantum vacuum behave as through they were in a thermal bath.[21] Indeed one quantum theorist has asserted that on acceleration through the vacuum, ”Steaks will cook, eggs will fry.”[22] Of course, there is no experimental basis for such an assertion. And our suggestion is that such an assertion may be wrong.

Quantum theorist often speak of using a very small system which would not be affected by gravity in order to examine the thermal bath behavior of mechanical systems.[7] Indeed,
within classical physics, there are calculations for point harmonic oscillators and point magnetic dipole rotators accelerated through classical zero-point radiation; these systems indeed take on values for the average energy as though they were located in an inertial frame in a thermal bath with temperature \( T = \frac{\hbar a}{2\pi c k_B} \). Point systems respond simply to the time correlation function and so do not sample anything regarding spatial extent. Indeed, by using time-dilating conformal transformations it can be shown that if we consider only the correlations in time at a fixed spatial coordinate without measuring anything involving spatial extent, then we can not separate out the effects of acceleration from those of non-zero temperature.

However, are point mechanical systems reliable indicators of thermal behavior? We suggest that point systems with internal structure are not relativistic systems and can not be expected to illustrate accurately the ideas of a relativistic field theory. Point systems do not exist as relativistic systems except for point masses. Point systems (such as a harmonic oscillator of vanishing spatial extent) which contain potential energy have no mechanism to show the dependence of the supporting force on the internal potential energy of the system when the system is located in a gravitational field or in an accelerating coordinate frame. This situation is in complete contrast with electromagnetic systems of charged particles; such systems must have finite spatial extent and will be affected by gravity. When the mechanical system contains electromagnetic energy, then the mechanism for the connecting the supporting force to the system potential energy in a gravitational field (or in an accelerating coordinate frame) involves the droop of the electromagnetic field lines. However, a mechanical system with electromagnetic potential energy, such as a classical hydrogen atom, must have finite spatial extent, and therefore responds to both the temporal and spatial correlation aspects of the fluctuating field.

Indeed, this question of finite spatial extent has direct relevance to the arguments given previously regarding "sudden" versus "adiabatic" acceleration of boxes of radiation. We can imagine a mechanical system located at a fixed position in the interior of a box of radiation which is moved from an inertial frame over to a Rindler frame. Within classical theory, this mechanical system takes on the same value whether alone and accelerated through the zero-point radiation of a Minkowski frame or whether at rest inside a (large) box in an accelerating Rindler frame because the spectrum of classical zero-point radiation is the same inside or outside the (large) box. However, quantum theory might suggest different behavior.
for the mechanical system in these two cases; in the first case the system is responding to the
tensor transformations of the fluctuations of the Minkowski vacuum and in the second case
the system is (presumably) responding to the fluctuations of the Rindler vacuum. Indeed
a point system will simply respond to the local time-fluctuations of the radiation inside the
box. This is not true for a hydrogen atom or any spatially extended relativistic system.
The field lines of a Coulomb potential "droop" in a gravitational field and the extent of the
"droop" is a measure of the strength of the gravitational (or acceleration) field. The final
droop of the field lines of a Coulomb potential in a Rindler frame has nothing to do with the
way in which the potential may have been moved from an inertial to the Rindler frame. When
a point harmonic oscillator is moved up and down in thermal radiation in a gravitational (or
acceleration) field, it can be used to violated fundamental laws of thermodynamics precisely
because it does not readjust to the gravitational field. A hydrogen atom, which is truly a
relativistic system, will readjust to the gravitational field by the droop of the field lines as
it is moved up or down in a Rindler frame. Only relativistic systems should be considered
seriously when dealing with relativistic situations. It seems possible that all the claims that
acceleration through the vacuum provides a thermal bath may be in error.

IV. CLOSING SUMMARY

Although quantum field theory and classical field theory with classical zero-point radia-
tion have related vacuum states in inertial frames, the theories part company in non-inertial
frames. The vacuum correlation functions of the classical theory depend upon geodesic
separations in the spacetime whereas the expectation values of the quantum theory depend
upon a canonical quantization procedure which makes no distinction between geodesic and
non-geodesic coordinates. The classical vacuum is unique. The non-uniqueness of the
quantum vacuum was noted by Fulling over thirty years ago. This contrast invites deeper
exploration. [28]

[1] For a recent brief review, see T. H. Boyer, "Any classical description of nature requires classical
electromagnetic zero-point radiation," Am. J. Phys. 79, 1163-1167 (2011). For an extensive
review of work before 1996, see L. de la Pena and A. M. Cetto, The Quantum Dice: An
Introduction to Stochastic Electrodynamics (Kluwer, Boston 1996). See also, T. H. Boyer, "Random electrodynamics: The theory of classical electrodynamics with classical electromagnetic zero-point radiation," Phys. Rev. 11, 790-808 (1975).

[2] T. H. Boyer, "General connection between random electrodynamics and quantum electrodynamics for free electromagnetic fields and for dipole oscillator systems," Phys. Rev. D 11, 809-830 (1975).

[3] An anonymous referee for Ref. 6 wrote: "It is also important to note that, in quantum field theory, if a box with totally reflecting walls starts off at rest with no acceleration, and with its interior in the vacuum state, then has its acceleration slowly increased to the final acceleration, its interior will be in the 'Rindler vacuum', not the thermal bath. Those boundaries to the box make a huge difference." A different referee wrote: "So suppose that we now take the mirror to be inertial for $t < 0$, and then to accelerate uniformly for $t > 0$. Then surely the mirror again introduces extra correlations and breaks some of the supposed symmetries of the zero-point radiation. If two such mirrors form a box of size $L$, then after a proper time $L/c$ has elapsed along either mirror, one would expect even the radiation in the deep interior of the box to be affected by the mirrors motion."

[4] T. W. Marshall, "Statistical Electrodynamics," Proc. Camb. Phil. Soc. 61, 537-546 (1965). T. H. Boyer, "Derivation of the Blackbody Radiation Spectrum without Quantum Assumptions," Phys. Rev. 182, 1374-11383 (1969).

[5] T. H. Boyer, "Conformal Symmetry of Classical Electromagnetic Zero-Point Radiation," Found. Phys. 19, 349-365 (1989).

[6] T. H. Boyer, "Classical and quantum interpretations regarding thermal behavior in a coordinate frame accelerating through zero-point radiation," arXiv physics 1011.1426.

[7] A typical comment is that of an anonymous referee for Ref. 6, "...(again assuming that the body is small enough so that the gravitational field does not affect it), ...

[8] See, for example, H. Goldstein, "Classical Mechanics 2nd edn," (Addison-Wesley, Reading, MA 1981), pp. 575-578. We are using unrationaled units.

[9] T. H. Boyer, "Classical physics of thermal scalar radiation in two spacetime dimensions," Am. J. Phys. 79, 644-656 (2011).

[10] T. H. Boyer, "Derivation of the Planck spectrum for relativistic classical scalar radiation from thermal equilibrium in an accelerating frame," Phys. Rev. D 81, 105024 (2010).
T. H. Boyer, "The blackbody radiation spectrum follows from zero-point radiation and the structure of relativistic spacetime in classical physics," Found. Phys. 42, 595-614 (2012).

T. H. Boyer, "Classical Statistical Thermodynamics and Electromagnetic Zero-Point Radiation," Phys. Rev. 186, 1304-1318 (1969).

See for example, W. Rindler, *Essential Relativity: Special, General, and Cosmological 2nd ed* (Springer-Verlag, New York 1977), p. 59-51, 156. W. Rindler, "Kruskal space and the uniformly accelerated frame," Am. J. Phys. 34, 1174-1178 (1966).

See, for example, B. F. Schutz, *A First Course in General Relativity* (Cambridge, London 1985), p.150 or J. D. Hamilton, "The uniformly accelerated reference frame," Am. J. Phys. 46, 83-89 (1978) or J. R. Van Meter, S. Carlip, and F. V. Hartemann, "Reflection of plane waves from a uniformly accelerating mirror," Am J. Phys. 69, 783-787 (2001).

See, for example, M. D. Greenberg, *Advanced Engineering Mathematics*, 2nd ed. (Prentice Hall, Upper Saddle River, NJ, 1998), Sec. 17.7, or J. Matthews and R. L. Walker, *Mathematical Methods of Physics*, 2nd ed. (Benjamin/Cummins, Reading, MA, 1970), pp. 264, 338.

I. S. Gradshteyn and I. M. Ryzhik, *Tables of Integrals, Series, and Products* (Academic Press, New York, 1965), p. 494, \( \int_0^\infty dx x^{2m} \sin bx/(e^x - 1) = (-1)^m \partial^{2m}/\partial b^{2m} [(\pi/2) \coth b\pi - (1/2b)] \), \( b > 0 \).

S. A. Fulling, "Nonuniqueness of canonical field quantization in Riemannian space-time," Phys. Rev. D 7, 2850-2862 (1973).

P. C. Davies, "Scalar particle production in Schwarzschild and Rindler metrics," J. Phys. A 8, 609-616 (1975).

W. G. Unruh, "Notes on blackhole evaporation," Phys. Rev. D 14, 870-892 (1976).

P. M. Alsing and P. W. Milonni, "Simplified derivation of the Hawking-Unruh temperature for an accelerated observer in vacuum," Am. J. Phys. 72, 1524-1529 (2004).

See the recent review by L. C. B. Crispino, A. Higuchi, G. E. A. Matsas, "The Unruh effect and its applications," Rev. Mod. Phys. 80, 787-838 (2008).

An anonymous referee for Ref. 6 wrote: "It is important ... to realize that the claim that an accelerated observer see a thermal bath is based on the fact that such an observer, carrying a thermometer (which is insensitive in its operation to the presence of a strong gravitational field) will find it reading a temperature. Steaks will cook, eggs will fry." Later this same referee wrote: "I cannot recommend a paper that denies a quantum field theory direct consequence,
namely, that under uniform acceleration a small thermometer will register a temperature. Accelerated enough steaks in Minkowski vacuum will cook, and eggs will fry.”

[23] T. H. Boyer, "Thermal effects of acceleration through random classical radiation," Phys. Rev. D 21, 2137-2148 (1980). T. H. Boyer, "Thermal effects of acceleration for a classical dipole oscillator in classical electromagnetic zero-point radiation," Phys. Rev. D 29, 1089-1095 (1984).

[24] D. C. Cole, "Properties of a classical charged harmonic oscillator accelerated through classical electromagnetic zero-point radiation," Phys. Rev. D 31, 1972-1981 (1985).

[25] T. H. Boyer, "Thermal effects of acceleration for a classical spinning magnetic dipole in classical electromagnetic zero-point radiation," Phys. Rev. D 30, 1228-1232 (1984).

[26] See ref. 10, p.105024-9.

[27] T. H. Boyer, "Example of mass-energy relation: Classical hydrogen atom accelerated or supported in a gravitational field," Am. J. Phys. 66, 872-876 (1998).

[28] This point of view was not shared by a referee for Ref. 6 who declared, "If there is any disagreement with the standard quantum treatment then this approach is surely wrong.”