I describe how integrable quantum field theories in 2 spacetime dimensions are characterized by infinite dimensional quantum group symmetries, namely the q-deformations of affine Lie algebras, and their Yangian limit. These symmetries can provide a new non-perturbative formulation of the theories.

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1. Introduction

In this talk I will review work done in a series of papers with D. Bernard, G. Felder, and F. Smirnov \[1\] \[2\] \[3\]. The subject concerns new symmetry structures of integrable quantum field theories in 2 space-time dimensions.

A class of quantum field theories of special interest are those which possess asymptotic massive particle states. The quantum mechanical scattering of these states is described by the so-called S-matrix. The importance of symmetry in quantum field theory has long been recognized. In 4 space-time dimensions, the possible symmetries of the S-matrix are severely limited by the Coleman-Mandula theorem, which states that such symmetries are necessarily the tensor product of Poincaré symmetry with some internal symmetry. Though an important result, this theorem was rather disappointing, since these allowed symmetries are normally not large enough to provide a non-perturbative solution. One of the hidden assumptions of the Coleman-Mandula theorem is that the symmetry acts on multiparticle states as if they were tensor products of 1-particle states. More precisely if $Q$ is a conserved charge, then it was assumed that its representation on a multiparticle state, which is here denoted as $\Delta(Q)$, is given by

$$\Delta(Q) = Q \otimes 1 \otimes \cdots \otimes 1 + 1 \otimes Q \otimes 1 \otimes \cdots \otimes 1 + \cdots + 1 \otimes \cdots \otimes Q.$$ \hspace{1cm} (1.1)

In modern mathematical terminology, it was assumed that the comultiplication was trivial.

In 4 dimensions the only known exception to the Coleman-Mandula theorem is supersymmetry, and it is due to the possibility of fermionic currents in 4d. However in lower dimensions, the possibility of more exotic statistics of fields implies there can be more non-trivial exceptions to the Coleman-Mandula theorem. The symmetries I will discuss in the following are precisely of this nature. Let me summarize some of the main features. For simplicity of the presentation I will limit myself to the sine-Gordon (SG) and its Gross-Neveu Limit. One can construct explicitly some non-local conserved currents in the SG theory that generate a q-deformation of $sl(2)$ affine Lie algebra $(\hat{sl}_q(2))$ \[4\]. The currents which generate this symmetry have non-trivial, though abelian, braiding relations. At a special value of the SG coupling, where $q = -i$, the $\hat{sl}_q(2)$ symmetry corresponds to a topological extension of $N = 2$ supersymmetry; away from this point the conserved currents have fractional spin, and the $N = 2$ symmetry algebra is deformed into the $\hat{sl}_q(2)$ algebra. It is well known that the SG theory possesses $sl(2)$ symmetry at a special value of the coupling constant. At this value of the coupling the $\hat{sl}_q(2)$ symmetry becomes the
Yangian symmetry. The Yangian is also a deformation of affine Lie algebra, however the deformation preserves the finite Lie subalgebra of the affine Lie algebra. The conserved currents generating the Yangian were studied by Bernard in [4], based on some earlier work of Lüscher[5].

What is remarkable about these symmetries is that they are large enough to provide a non-perturbative solution to the S-matrix of the theory. Requiring that the conserved charges commute with the S-matrix leads to a set of algebraic equations which characterize the S-matrix up to an overall scalar factor. This scalar can be fixed by imposing crossing symmetry, unitarity, and the minimality criterion. In the simple cases we are considering the S-matrices are known[6][7]. These S-matrices were originally found by imposing the Yang-Baxter equation, whereas in our approach the solutions to the symmetry equations automatically satisfy the Yang-Baxter equation. This parallels what was accomplished by Drinfel’d and Jimbo in their work on quantum groups [8][9]. These authors understood that what underlies certain solutions of the Yang-Baxter equation is a Hopf algebra. Indeed the algebraic symmetry equations we obtain are well-known equations in the theory of quantum groups.

The framework based on non-local charges is applicable to many more models than the ones considered in detail here. The analysis of the SG theory extends readily to the affine Toda theories and their Yangian limit[1]. More complicated examples include the many integrable perturbations of rational conformal field theories. For example a restriction of the quantum affine symmetry of the SG theory can be used to derive the S-matrices for the ‘Φ1,3’ perturbations of the c < 1 minimal conformal models; this was developed in detail in [3]. Quantum affine symmetry in perturbed minimal models was also discussed by Mathur[10]. The primary distinction between the latter work and the results presented here is that we deal only with genuine symmetries of a theory, which means symmetries that follow from conserved currents whose charges commute with the S-matrix. The quantum group structure in [10] is not of this type, but rather is more in the spirit of the quantum group symmetry of conformal field theory, as presented in [11].

Since the non-local conserved charges virtually characterize the S-matrix completely, it is natural to suppose that they will provide non-perturbative off-shell information as well. Fields are completely characterized by their form factors, i.e. matrix elements on the space of states [12]. For the usual internal symmetries based on finite dimensional Lie algebras, the implication of the symmetry is that fields are classified according to finite dimensional representations of the Lie algebra. In [2], this notion was generalized to
theories with an arbitrary Hopf algebra symmetry, by introducing an adjoint action of the Hopf algebra on the space of fields. Similar algebraic constructions were made in a lattice context in [13]. For the case of infinite Hopf algebra symmetry that we are interested in, the fields were shown to comprise infinite dimensional Verma-module representations.

I will begin by reviewing some general features of non-local charges in 2d quantum field theory[4][1]. In particular it will be shown how the comultiplication for the charges arises from the braiding of the currents with other fields. This connection was also observed by Gomez and Sierra in their study of finite dimensional quantum group symmetry of conformal field theory[11]. I will then construct explicitly the $\hat{sl}_q(2)$ currents of the SG theory and determine their algebra[1][3]. The construction of [2] will then be presented.

2. General Aspects of Non-Local Charges

Consider some conserved currents $\partial_\mu J^a_\mu = 0$, defining some conserved charges $Q^a = \frac{1}{2\pi i} \int dx J^a_0(x, t)$, and let us suppose the following braiding relations with a set of fields $\Phi^i(x)$:

$$J^a_\mu(x, t) \Phi^k(y, t) = \sum_{b, l} R^{ak}_{bl} \Phi^l(y, t) J^b_\mu(x, t) \quad ; \quad \text{for } x < y. \quad (2.1)$$

The action of the charges $Q^a$ on the fields, which we will denote as $\text{ad}_{Q^a}$, can be defined as follows

$$\text{ad}_{Q^a} \left( \Phi^k(y) \right) = \frac{1}{2\pi i} \int_{\gamma(y)} dz e^{\nu\mu} J^a_\mu(z) \Phi^k(y), \quad (2.2)$$

where the contour $\gamma(y)$ begins and ends at $-\infty$ surrounding the point $y$. Using the braiding relations (2.1) one derives

$$\text{ad}_{Q^a} \left( \Phi^k(y) \right) = Q^a \Phi^k(y) - R^{ak}_{bl} \Phi^l(y) Q^b. \quad (2.3)$$

If $R^{ab}_{dc}$ is the braiding matrix of the currents with themselves as in (2.1), then the integrated version of (2.3) is

$$\text{ad}_{Q^a} \left( Q^b \right) = Q^a Q^b - R^{ab}_{dc} Q^c Q^d. \quad (2.4)$$

Under special conditions the contour on the right hand side of (2.2) can be closed to yield a new operator. In the case of $\text{ad}_{Q^a} \left( Q^b \right)$ closure of the contour yields an algebra; we will encounter such a situation below.

The adjoint action of $Q^a$ on a product of two fields at different spacial locations is again defined as in (2.2), where now the contour surrounds the locations of both fields.
Using the braiding relations to pass the current through the first field before acting on the second, one finds that this action has a non-trivial comultiplication
\[
\Delta (Q^a) = Q^a \otimes 1 + \Theta^a_b \otimes Q^b, \tag{2.5}
\]
where \(\Theta^a_b\) is the braiding operator which acts on the vector space spanned by the fields \(\Phi^i\), i.e. \(\Theta^a_b\) has the matrix elements \(\Theta^a_b = R^a_b\).

Let us apply the above ideas to the non-local charges that arise in a general perturbation of a conformal field theory (CFT). Consider a CFT perturbed by a relevant operator with zero Lorentz spin. The perturbing field can be represented by \(\Phi_{\text{pert}}(z, \bar{z}) = \phi_{\text{pert}}(z)\phi_{\text{pert}}(\bar{z})\). The Euclidean action is taken to be
\[
S = S_{\text{CFT}} + \frac{\lambda}{2\pi} \int d^2z \, \Phi_{\text{pert}}(z, \bar{z}) \tag{2.6}
\]
where \(\lambda\) is a generally dimensionful parameter that measures the strength of the perturbation away from the conformal limit. Here \(z\) and \(\bar{z}\) denote Euclidean coordinates. Chiral fields \(F(z, \bar{z}), \bar{F}(z, \bar{z})\) satisfy \(\partial_{\bar{z}} F(z, \bar{z}) = \partial_z \bar{F}(z, \bar{z}) = 0\) in the conformal limit. Equations of motion for the perturbed chiral fields which are local with respect to the perturbing field can be deduced to first order in perturbation theory using Zamolodchikov’s approach
\cite{14}:
\[
\partial_{\bar{z}} F(z, \bar{z}) = \lambda \oint_{\bar{z}} \frac{dw}{2\pi i} \, \Phi_{\text{pert}}(w, \bar{z}) F(z) \tag{2.7}
\]
\[
\partial_z \bar{F}(z, \bar{z}) = \lambda \oint_{\bar{z}} \frac{dw}{2\pi i} \, \Phi_{\text{pert}}(z, w) \bar{F}(z).
\]
Equations of motion to first order can be exact to all orders in perturbation theory, which can be seen from scaling arguments.

Let us now suppose that there are currents conserved to first order in perturbation theory:
\[
\partial_{\bar{z}} J^a(z, \bar{z}) = \partial_z H^a(z, \bar{z}), \quad \partial_z \bar{J}^a(z, \bar{z}) = \partial_{\bar{z}} \bar{H}^a(z, \bar{z}). \tag{2.8}
\]
We assume that in the conformal theory these currents are chiral fields; i.e. when \(\lambda = 0\) they satisfy \(\partial_{\bar{z}} J^a = \partial_z \bar{J}^a = 0\). The condition for the currents to be conserved to first order in perturbation theory is then a condition on the residue of the operator product expansion (OPE) between them and the perturbing field. Namely, the conservation laws \eqref{2.8} hold if the residues of these OPE’s are total derivatives:
\[
\text{Res}_{z=w} \left( \phi_{\text{pert}}(w) \, J^a(z) \right) = \partial_z h^a(z) \tag{2.9a}
\]
\[
\text{Res}_{\bar{z}=\bar{w}} \left( \phi_{\text{pert}}(\bar{w}) \, \bar{J}^a(z) \right) = \partial_{\bar{z}} \bar{h}^a(\bar{z}). \tag{2.9b}
\]
The conditions (2.9) follow from Zamolodchikov’s equation of motion (2.7). In (2.8) the fields \( H^a \) are then
\[
H^a(z, \bar{z}) = \lambda h^a(z) \phi_{\text{pert}}(\bar{z}),
\] (2.10)
and similarly for \( \overline{H}^a(z, \bar{z}) \). From the conserved currents (2.8) we define the conserved charges,
\[
Q^a = \frac{1}{2\pi i} \left( \int dzJ^a + \int d\bar{z}H^a \right), \quad \overline{Q}^a = \frac{1}{2\pi i} \left( \int d\bar{z}\overline{J}^a + \int dz\overline{H}^a \right). \tag{2.11}
\]

Since the currents \( J^a \) and \( \overline{J}^a \) can be non-local, we allow for non-trivial braiding between them:
\[
J^a(x, t) \overline{J}^b(y, t) = R^{ab}_{bb} \overline{J}^b(y, t) J^b(x, t); \quad x < y. \tag{2.12}
\]
There is no contradiction in having the above braiding relations defined for all \( x, y \); we will give explicit examples in the sequel. To find the commutation relations between the charges \( Q^a \) and \( \overline{Q}^a \) associated to these currents, we apply the general framework explained above. Using (2.3) it is easy to compute \( \text{ad}_{Q^a} \left( \overline{Q}^a \right) \) to lowest non-trivial order in perturbation theory. The result is:
\[
Q^a \overline{Q}^a - R^{ab}_{bb} \overline{Q}^b Q^b = \frac{\lambda}{2\pi i} \int dz \partial_z + d\bar{z} \partial_{\bar{z}} \left[ h^a(z) h^a(\bar{z}) \right]. \tag{2.13}
\]
The right hand side of the above equation is a topological charge, and may be understood as a generalization of the topological extensions of supersymmetry in two-dimensions [13].

### 3. The \( \widehat{sl}(2) \) Currents in the Sine-Gordon Theory

I now review how the \( \mathcal{U}_q \left( \widehat{sl}(2) \right) \) loop algebra symmetry is realized in the sine-Gordon theory [1]. The SG theory may be treated as a massive perturbation of the \( c = 1 \) conformal field theory corresponding to a single real scalar field:
\[
S = \frac{1}{4\pi} \int d^2z \partial z \Phi \partial z \Phi + \frac{\lambda}{\pi} \int d^2z : \cos (\hat{\beta} \Phi) :. \tag{3.1}
\]
The theory has a well-known conserved topological current
\[
J^{\text{top}}(x) = \hat{\beta} \frac{1}{2\pi} \epsilon_{\mu
u} \partial \nu \Phi(x)
\]
which defines the topological charge
\[ T = \frac{\hat{\beta}}{2\pi} \int_{-\infty}^{\infty} dx \partial_x \Phi(x). \] (3.2)

With the above \( \hat{\beta} \)-dependent normalization of the topological charge, the soliton states are known to have \( T = \pm 1 \) \[11\] \[12\].

Define the quasi-chiral components \( \varphi, \varphi \) of \( \Phi \) as
\[
\varphi(x, t) = \frac{1}{2} \left( \Phi(x, t) + \int_{-\infty}^{x} dy \partial_t \Phi(y, t) \right),
\]
\[
\varphi(x, t) = \frac{1}{2} \left( \Phi(x, t) - \int_{-\infty}^{x} dy \partial_t \Phi(y, t) \right),
\] (3.3)
such that \( \Phi = \varphi + \varphi \). When \( \lambda = 0 \), \( \varphi = \varphi(z) \), and \( <\varphi(z)\varphi(w)> = -\log(z - w) \). Similarly for \( \varphi \). We will make use of the following braiding relations:
\[
\exp(i\varphi(x, t)) \exp(ib\varphi(y, t)) = e^{\pm i\pi ab} \exp(i\varphi(x, t)) \exp(ib\varphi(y, t)), \quad \text{for } x > y
\]
\[
\exp(i\varphi(x, t)) \exp(ib\varphi(y, t)) = e^{\pm i\pi ab} \exp(ib\varphi(x, t)) \exp(i\varphi(y, t)), \quad \text{for } x < y
\] (3.4)
\[
\exp(i\varphi(x, t)) \exp(ib\varphi(y, t)) = e^{i\pi ab} \exp(ib\varphi(y, t)) \exp(i\varphi(x, t)), \quad \forall x, y.
\]

The topological charge of these fields follows from the relation
\[
[T, \exp(i\varphi + ib\varphi)] = \hat{\beta}(a - b) \exp(i\varphi + ib\varphi). \] (3.5)

The conformal dimensions \((h, \bar{h})\) of the field \( \exp(i\varphi + ib\varphi) \) are \((a^2/2, b^2/2)\), and its Lorentz spin (eigenvalue under Lorentz boosts) is \( h - \bar{h} \).

Using the general results of the last section, it can be shown that the model \((3.1)\) has the following non-local quantum conserved currents: \( \partial \varphi J_\pm = \partial_z H_\pm; \partial \varphi \bar{J}_\pm = \partial_{\varphi \bar{H}}_\pm \), where
\[
J_\pm(x, t) = \exp \left( \pm \frac{2i}{\beta} \varphi(x, t) \right), \quad \bar{J}_\pm(x, t) = \exp \left( \mp \frac{2i}{\beta} \varphi(x, t) \right)
\] (3.6a)
\[
H_\pm(x, t) = \lambda \frac{\hat{\beta}^2}{\beta^2 - 2} \exp \left[ \pm i \left( \frac{2}{\beta} - \hat{\beta} \right) \varphi(x, t) + i\hat{\beta} \varphi(x, t) \right]
\] (3.6b)
\[
\bar{H}_\pm(x, t) = \lambda \frac{\hat{\beta}^2}{\beta^2 - 2} \exp \left[ \mp i \left( \frac{2}{\beta} - \hat{\beta} \right) \varphi(x, t) \pm i\hat{\beta} \varphi(x, t) \right].
\] (3.6c)
The coefficient $2/\hat{\beta}$ in the exponent for the components $J_\pm, \mathcal{J}_\pm$ of the currents follows from the residue condition (2.9).

These currents define the conserved charges $Q_\pm, \overline{Q}_\pm$. The conserved charges have non-trivial Lorentz spin:

$$\frac{1}{\gamma} \equiv \text{spin}(Q_\pm) = -\text{spin}(\overline{Q}_\pm) = \frac{2}{\beta^2} - 1. \quad (3.7)$$

Consequently the non-local conserved currents have non-trivial braiding relations among themselves and with other fields. In particular the braiding relations (3.4) imply:

$$J_\pm(x, t) \mathcal{J}_\mp(y, t) = q^{-2} \mathcal{J}_\mp(y, t) J_\pm(x, t) \quad ; \forall \ x, y$$

$$J_\pm(x, t) \mathcal{J}_\pm(y, t) = q^2 \mathcal{J}_\pm(y, t) J_\pm(x, t) \quad ; \forall \ x, y \quad (3.8)$$

where

$$q = \exp(-2\pi i/\beta^2) = -\exp(-i\pi/\gamma). \quad (3.9)$$

Using the above braiding relations and the equation (2.13) one finds that the conserved charges satisfy the relations

$$Q_\pm \overline{Q}_\pm - q^2 \overline{Q}_\pm Q_\pm = 0 \quad (3.10)$$

$$Q_\pm \overline{Q}_\mp - q^{-2} \overline{Q}_\mp Q_\pm = \frac{\lambda}{2\pi i} \gamma^2 \int_t dx \partial_x \left[ \exp \left( \pm i \left( \frac{2}{\beta^2} - \frac{1}{\beta} \right) \Phi(x, t) \right) \right].$$

The topological charges on the right hand side of (3.10) can be expressed in terms of the usual topological charge $T$ in (3.2). A soliton configuration can be taken to satisfy $\Phi(x = \infty) = 0$; the classical soliton solutions do in fact satisfy this. Integrating (3.10) we obtain the algebra to lowest non-trivial order in perturbation theory:

$$Q_\pm \overline{Q}_\pm - q^2 \overline{Q}_\pm Q_\pm = 0 \quad (3.11a)$$

$$Q_\pm \overline{Q}_\mp - q^{-2} \overline{Q}_\mp Q_\pm = a \left( 1 - q^{\pm 2T} \right) \quad (3.11b)$$

$$[T, Q_\pm] = \pm 2 \ Q_\pm, \quad [T, \overline{Q}_\pm] = \pm 2 \overline{Q}_\pm \quad (3.11c)$$

where $a \equiv \lambda \gamma^2 / 2\pi i$. In deriving the last equations we have used (3.5). It can be shown using scaling arguments that this algebra is exact to all orders. Note that when $q = -i$, ($\hat{\beta} = 2/\sqrt{3}$), the algebra (3.11) is a topological extension of the $N = 2$ supersymmetry algebra.
The above algebra (3.11) is isomorphic to the $\hat{sl}_q(2)$ loop algebra. Let $E_i$, $F_i$, $H_i$, $i = 0, 1$, denote the Chevaley basis for the $\hat{sl}_q(2)$ algebra. They satisfy the following defining relations:

\[
\begin{align*}
[H_i, E_j] &= a_{ij} E_j \\
[H_i, F_j] &= -a_{ij} F_j \\
[E_i, F_j] &= \delta_{ij} \frac{q^{H_i} - q^{-H_i}}{q - q^{-1}}
\end{align*}
\tag{3.12}
\]

with $a_{ij}$ the Cartan matrix of the affine Kac-Moody algebra $\hat{sl}_q(2)$. The relations between the non-local charges $Q_\pm$ and $\overline{Q}_\pm$ and these generators are:

\[
\begin{align*}
Q_+ &= c E_1 q^{H_1/2} \\
Q_- &= c E_0 q^{H_0/2} \\
\overline{Q}_- &= c F_1 q^{H_1/2} \\
\overline{Q}_+ &= c F_0 q^{H_0/2} \\
T &= H_1 = -H_0
\end{align*}
\tag{3.13}
\]

where $c$ is a constant ($c^2 = \frac{1}{2\pi^2} \gamma^2 (q^{-2} - 1)$). The last equation in (3.13) reflects the fact that the center of $sl_q(2)$ is zero, and we are actually dealing with a deformation of the loop algebra.

The complete set of relations for the $\hat{sl}_q(2)$ algebra include the additional deformed Serre relations. They are in this case

\[
\text{ad}_{Q_\pm}^3 (Q_\pm) = \text{ad}_{\overline{Q}_\pm}^3 (\overline{Q}_\pm) = 0.
\tag{3.14}
\]

These relations were proven in [3] by studying the conditions under which the contours in the expression $\text{ad}_{Q_\pm}^n (J^-_\mu (y, t))$ could be closed. Using the braiding relations of the currents one finds they can be closed precisely when $n = 3$. Using scaling arguments and the short distance properties of the ultraviolet CFT, one then deduces that the operator obtained upon closing the contours is necessarily zero.

I will now describe how one can use the above $\hat{sl}_q(2)$ symmetry to derive the S-matrix for the scattering of solitons. To do this one needs to know how the symmetry algebra is represented on the space of asymptotic states. The asymptotic soliton states of topological charge $\pm 1$ are denoted $| \pm 1/2, \theta \rangle$, where $\theta$ is the rapidity:

\[
p_0(\theta) = m \cosh(\theta) \quad p_1(\theta) = m \sinh(\theta),
\tag{3.15}
\]

\[1 \quad a_{11} = a_{00} = -a_{10} = -a_{01} = 2\]
and \(| \pm 1/2, \theta \rangle = | \pm 1/2, \theta \rangle\). A set of chiral fields of topological charge \(\pm 1\) with non-vanishing matrix elements between the states and the vacuum can be taken to be
\[
\Psi_{\pm}(x,t) = \exp\left(\pm i \frac{\varphi(x,t)}{\beta}\right), \quad \overline{\Psi}_{\pm}(x,t) = \exp\left(\mp i \frac{\varphi(x,t)}{\beta}\right).
\]

The fields \(\Psi_{\pm}\) and \(\overline{\Psi}_{\pm}\) do not create independent particle states, for the usual reasons. The representation of \(U_q\left(\hat{sl}(2)\right)\) on the space of one-particle states can be shown to be
\[
Q_{\pm} = c e^{\theta/\gamma} \sigma_\pm q^{\pm \sigma_3/2}, \quad \overline{Q}_{\pm} = c e^{-\theta/\gamma} \sigma_\pm q^{\mp \sigma_3/2}, \quad T = \sigma_3,
\]
where \(\sigma_\pm, \sigma_3\) are the Pauli spin matrices. The on-shell operators \(\exp(\theta/\gamma)\) are a consequence of the Lorentz spin of the conserved charges, since the Lorentz boost generator is represented as \(-\partial_\theta\) on-shell. The above representation of the \(U_q\left(\hat{sl}(2)\right)\) loop algebra is in the so-called principal gradation.

The action of the conserved charges on the multiparticle states is provided by the comultiplication which follows from (2.15) and the braiding of the non-local currents with the soliton fields. From the braiding relations
\[
J_{\pm}(x,t) \overline{\Psi}(y,t) = q^{\pm T} \overline{\Psi}(y,t) J_{\pm}(x,t) ; \quad \forall x, y
\]
\[
\overline{J}_{\pm}(x,t) \Psi(y,t) = q^{T} \Psi(y,t) \overline{J}_{\pm}(x,t) ; \quad \forall x, y ,
\]
one thereby deduces that the comultiplication is:
\[
\Delta (Q_{\pm}) = Q_{\pm} \otimes 1 + q^{\pm T} \otimes Q_{\pm}
\]
\[
\Delta (\overline{Q}_{\pm}) = \overline{Q}_{\pm} \otimes 1 + q^{T} \otimes \overline{Q}_{\pm}
\]
\[
\Delta (T) = T \otimes 1 + T \otimes 1.
\]
and is equivalent to the usual one. The two-particle to two-particle S-matrix \(\hat{S}\) is an operator from \(V_1 \otimes V_2\) to \(V_2 \otimes V_1\), where \(V_i\) are the vector spaces spanned by \(| \pm 1/2, \theta_i \rangle\). The \(U_q\left(\hat{sl}(2)\right)\) symmetry of the S-matrix is the condition
\[
\hat{S}_{12}(\theta_1 - \theta_2; q) \Delta_{12}(a) = \Delta_{21}(a) \hat{S}_{12}(\theta_1 - \theta_2; q) \quad a \in U_q\left(\hat{sl}(2)\right).
\]
These equations are familiar in the theory of quantum groups, and show that the S-matrix is just the universal \(\mathcal{R}\)-matrix specialized to the rapidity-dependent representations of \(\hat{sl}_q(2)\). The minimal solution to these symmetry equations is the conjectured S-matrix of SG solitons[6].
4. General Hopf Algebra Symmetry of Fields

We have seen how the infinite dimensional quantum groups are represented on the space of asymptotic states though finite dimensional rapidity-dependent representations. In this section I will address how the symmetry is realized on the space of fields of the theory. We already described how to act on fields with conserved charges in (2.2). In order to determine how the fields obtained via this adjoint action transform under the symmetry, we first express this adjoint action in a more suitable way\[2\]. Let $A$ be a Hopf algebra equipped with comultiplication $\Delta: A \to A \otimes A$, counit $\epsilon: A \to \mathbb{C}$, and antipode $s: A \to A$, with the following properties:

\begin{align}
\Delta(ab) &= \Delta(a)\Delta(b) \quad (4.1a) \\
(\Delta \otimes id) \Delta(a) &= (id \otimes \Delta) \Delta(a) \quad (4.1b) \\
(\epsilon \otimes id) \Delta(a) &= (id \otimes \epsilon) \Delta(a) = a \quad (4.1c) \\
m(s \otimes id) \Delta(a) &= m(id \otimes s) \Delta(a) = \epsilon(a) \quad (4.1d)
\end{align}

for $a, b \in A$ and $m$ the multiplication map: $m(a \otimes b) = ab$. Eq. (4.1a) implies $\Delta$ is a homomorphism of $A$ to $A \otimes A$, (4.1b) is the coassociativity, and (4.1c, d) are the defining properties of the counit and antipode. Let us chose a basis $\{e_a\}$ in $A$, and define the multiplication and comultiplication in terms of structure constants $m_{ab}^c$ and $\mu_{c}^{ab}$:

\begin{align}
e_a e_b &= m_{ab}^c e_c \quad (4.2a) \\
\Delta(e_a) &= \mu_{a}^{bc} e_b \otimes e_c \quad (4.2b)
\end{align}

For our purposes we will also need the concept of the quantum double of $A$, which is a construction due originally to Drinfel’d [18] [19]. Let $A^*$ denote a Hopf algebra dual to $A$ in the quantum double sense, with basis $\{e^a\}$. The elements of the dual basis are defined to satisfy the following relations:

\begin{align}
e^a e^b &= \mu_{c}^{ab} e^c \quad (4.3a) \\
\Delta(e^a) &= m_{bc}^a e^c \otimes e^b \quad (4.3b)
\end{align}

The quantum double is a Hopf algebra structure on the space $A \otimes A^*$. The relations between the elements $e_a$ and the dual elements $e^a$ are defined as follows:

\begin{align}e_a e_b &= \mu_{a}^{kcl} m_{idk}^b s_{i}^l e^d e_c \quad (4.4a) \\
e^b e_a &= \mu_{a}^{lck} m_{kdi}^b s_{i}^l e_c e^d \quad (4.4b)
\end{align}
where $\mu^l_{ck} = \mu^l_i \mu^i_{ck}$; $m^b_{kdm} = m^i_{kd} m^b_{im}$, and $s'(e_a) = s^b_a e_b$ is the skew antipode. The universal $\mathcal{R}$-matrix is an element of $\mathcal{A} \otimes \mathcal{A}^*$ defined by

$$\mathcal{R} = \sum_a e_a \otimes e^a, \quad (4.5)$$

satisfying

$$\mathcal{R} \Delta(e_a) = \Delta'(e_a) \mathcal{R}, \quad (4.6)$$

for all $e_a$, where $\Delta'$ is the permuted comultiplication. $\mathcal{R}$ also satisfies the Yang-Baxter equation.

Let us now define the adjoint action on a field or product of fields as

$$\text{ad}_{e_a}(\Phi(x_1) \cdots \Phi(x_n)) = \mu^{bc}_a e_b \Phi(x_1) \cdots \Phi(x_n) s(e_c). \quad (4.7)$$

This adjoint action generalizes the ordinary commutator in Lie algebra symmetry to an arbitrary Hopf algebra, and has the appropriate properties. In particular,

$$\text{ad}_{e_a} \text{ad}_{e_b} = \text{ad}_{e_a e_b}. \quad (4.8)$$

This latter property implies that fields related through adjoint action form a representation of $\mathcal{A}$. More precisely let $\Phi_\Lambda(x)$ denote the set of fields so obtained, and let $\rho_\Lambda(a)$ denote the representation of $\mathcal{A}$. Then

$$\text{ad}_{e_a}(\Phi_\Lambda(x)) = \rho_\Lambda(e_a)\Phi_\Lambda(x). \quad (4.9)$$

The adjoint action also satisfies the important property

$$\text{ad}_{e_a}(\Phi_{\Lambda_1}(x_1) \Phi_{\Lambda_2}(x_2)) = \mu^{bc}_a \text{ad}_{e_b}(\Phi_{\Lambda_1}(x_1)) \text{ad}_{e_c}(\Phi_{\Lambda_2}(x_2)). \quad (4.10)$$

These properties are all proven using the Hopf algebra properties (4.1).

An important consequence of (4.10) is that the braiding of the multiplets of fields is given by the universal $\mathcal{R}$-matrix:

$$\Phi_{\Lambda_2}(y, t) \Phi_{\Lambda_1}(x, t) = \mathcal{R}_{\rho_{\Lambda_1}, \rho_{\Lambda_2}} \Phi_{\Lambda_1}(x) \Phi_{\Lambda_2}(y) \quad x < y, \quad (4.11)$$

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2 The adjoint action in (4.7) differs slightly from the one used in [2]. The origin of the difference is the same as ‘right’ versus ‘left’ action in Lie group theory, i.e. the adjoint action in [2] satisfied $\text{ad}_{e_a} \text{ad}_{e_b} = \text{ad}_{e_b e_a}$. 

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where $R_{\rho_1,\rho_2}$ is the universal $R$ matrix specialized to the representations $\rho_{\Lambda_1,2}$ of the fields $\Phi_{\Lambda_1,2}$. The above relation is easily proven by applying $ad_{e_a}$ to both sides of (4.11), and using (4.10) to prove that $R$ must satisfy its defining relations (4.6).

It can be shown that the above abstract adjoint action is equivalent to the adjoint action defined via the contours in the quantum field theory (2.2). The proof goes as follows [2]. We first define an adjoint representation $\rho_{adj}$ of $\mathcal{A}, \mathcal{A}^*$:

$$\langle b | \rho_{adj}(e_a) | c \rangle = m_{ac}^b \quad (4.12a)$$

$$\langle c | \rho_{adj}(e^a) | b \rangle = \mu_{cb}^a \quad (4.12b).$$

That the structure constants form a representation of the algebra is a consequence of the associativity and coassociativity of $\mathcal{A}$. We associate the conserved currents $J^\mu_a(x)$ for the charge $e_a$ to the fields in $\Phi_{\rho_{adj}}(x)$. The braiding of these currents with other fields can then be deduced from (4.11). One finds

$$J^\mu_a(y) \Phi_{\Lambda}(x) = \rho_{\Lambda} \left( R^b_a \right) \Phi_{\Lambda}(x) J^\mu_b(y) \quad x < y, \quad (4.13)$$

where

$$R^b_a \equiv \mu_{cb}^a e_c. \quad (4.14)$$

Using these braiding relations in (2.3), one finds that (2.3) is equivalent to (1.7).

For the infinite dimensional Hopf algebras we are considering, the multiplets of fields $\Phi_{\Lambda}(x)$ can be seen to comprise infinite dimensional Verma-module representations. This was shown in [2] by studying the form-factors of these fields. Consider the matrix element of a field $\phi(x)$ on the space of asymptotic states (form factor):

$$\langle \alpha_1, \ldots, \alpha_m | \phi(0) \ | \beta_1, \ldots, \beta_1 \rangle, \quad (4.15)$$

where $\alpha_i, \beta_i$ are rapidites. Given the representation of $\mathcal{A}$ on the asymptotic states, it is clear that the form factor of any other field obtained from adjoint action on $\phi$ is explicitly computable from the knowledge of the form factors of $\phi(x)$, since the elements of $\mathcal{A}$ on the left or right of the field in (1.7) give a known transformation on states.

Let me illustrate this for the case of the Yangian symmetry. The SG theory at the coupling $\hat{\beta} = \sqrt{2}$ is equivalent to the chiral sl(2) Gross-Neveu model, and the spectrum consists of a doublet of solitons. At this point the S-matrix is sl(2) invariant, and is a well-known rational solution of the Yang-Baxter equation:

$$S_{12}(\theta) = s_0(\theta) \left( \beta - i \pi P_{12} \right), \quad (4.16)$$
where $P_{12}$ is the permutation operator:

$$P_{12} = \frac{1}{2} \left( \sum_{a=1}^{3} \sigma^a \otimes \sigma^a + 1 \right),$$  \hfill (4.17)

($\sigma^a$ are the Pauli spin matrices), and

$$s_0(\beta) = \frac{\Gamma(1/2 + \beta/2\pi i) \Gamma(-\beta/2\pi i)}{\Gamma(1/2 - \beta/2\pi i) \Gamma(\beta/2\pi i)}.$$  \hfill (4.18)

Let $Q^a_0, a = 1, 2, 3$ denote the global sl(2) generator. The S-matrix is invariant under some additional symmetries generated by $Q^a_1$, which are represented on the asymptotic 1-particle states as $Q^a_1 |\theta > = \theta t^a |\theta >$, where $t^a$ is a 2-dimensional representation of sl(2). These charges satisfy the algebra

$$[Q^a_0, Q^b_0] = f^{abc} Q^c_0, \quad [Q^a_0, Q^b_1] = f^{abc} Q^c_1.$$  \hfill (4.19)

They have the following comultiplication:

$$\Delta(Q^a_0) = Q^a_0 \otimes 1 + 1 \otimes Q^a_0$$ \hfill (4.20a)

$$\Delta(Q^a_1) = Q^a_1 \otimes 1 + 1 \otimes Q^a_1 + \alpha f^{abc} Q^b_0 \otimes Q^c_0,$$ \hfill (4.20b)

where $\alpha = -i\pi$. The extra term in the comultiplication for $Q^a_1$ is required for invariance of the S-matrix.

These are the defining relations of the Yangian. This symmetry may be thought of as arising in the $q \to -1$ limit of the sl$_q$(2) symmetry. The Yangian, as originally defined by Drinfel’d, is a deformation of half of the sl$_q$(2) affine Lie algebra that preserves the finite sl(2) subalgebra. In order to determine the universal $\mathcal{R}$-matrix, it was necessary in [2] to extend the Yangian to its quantum double. It was then shown that the deformation of the remaining half of the affine Lie algebra resides in this quantum double.

By studying the form factors of the adjoint action of $Q^a_1$ on the energy-momentum tensor $T_{\mu\nu}$ one can derive the fundamental identity:

$$[Q^a_1, T_{\mu\nu}(x)] = -\frac{1}{2} \left( \epsilon_{\mu\alpha} \partial_{\alpha} J^a_\nu(x) + \epsilon_{\nu\alpha} \partial_{\alpha} J^a_\mu(x) \right),$$ \hfill (4.21)

where $J^a_\mu(x)$ is the global current for the sl(2) charge $Q^a_0$. Thus we begin to see how non-trivial the Yangian symmetry is: the energy-momentum tensor and global current share the same Yangian multiplet.
5. Conclusions

It is clear from the above discussion that the infinite quantum group symmetry of integrable massive models can provide new non-perturbative results. I believe it is possible to completely define the models and determine their main properties from these symmetries alone. However much remains to be understood toward the completion of this program. In particular the form factors have not been completely understood purely in terms of the symmetry, unlike the S-matrix. Some steps in this direction have been taken in the works\cite{20} \cite{21}, where it was shown how one of the axioms of the form factor bootstrap can be understood as a deformed Knizhnik-Zamolodchikov equation. The issue of how the complete set of fields in a theory is classified according to the symmetry is another important problem that has not been completely solved, and this information is necessary for a formulation of the correlation functions.

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