POLYNOMIAL KNOTS

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Abstract. A polynomial knot is a smooth embedding $\kappa : \mathbb{R} \to \mathbb{R}^n$ whose components are polynomials. The case $n = 3$ is of particular interest. It is both an object of real algebraic geometry as well as being an open ended topological knot. This paper contains basic results for these knots as well as many examples.

1. Introduction

Definition 1. A polynomial knot is a smooth embedding $\kappa : \mathbb{R} \to \mathbb{R}^n$ whose components are polynomials.

A polynomial knot thus has the properties that $\kappa'(t) \neq 0$ for all $t$, and $\kappa(s) \neq \kappa(t)$ for all $s \neq t$. Polynomial knots were first investigated by Shastri [33] in connection with a conjecture of Abhyankar. He found a simple representation of the trefoil (overhand) knot:

\begin{align*}
  x(t) &= t^3 - 3t \\
  y(t) &= t^4 - 4t^2 \\
  z(t) &= t^5 - 10t.
\end{align*}

This elegant example has remained at the base of the subject. He also found equations for the figure-eight knot. We give this knot and many other examples of equations for topological knots in §5.1. It is not easy to find these equations.

A topological knot is usually understood to be an embedded circle in the three sphere. Polynomial knots are not of this type, but rather a “long” or “open ended” knot. The one point compactification of a polynomial knot is a knot in the usual sense. If the compactification is obtained by using stereographic projection, then easy arguments show that the compactified polynomial knot is tame, and also smoothly embedded except possibly at the pole of the sphere where it has a nicely behaved algebraic singular point. We also show that a family of polynomial knots is transverse to a suitably large sphere about the origin. This material is in §2.

Section 3 is concerned with approximation theorems. We first show that if $\alpha$ is an embedding of a compact interval in $\mathbb{R}^n$ and $\beta$ is another
map close to both the locus and derivative of $\alpha$, then $\beta$ is an embedding isotopic to $\alpha$. A corollary is that a smooth embedding $\alpha$ on a compact interval can be approximated by a polynomial $\beta$ which has the same “knot type”, and that the same is true if $n = 2$ and the map $\alpha$ has transverse intersections. This result is obtained using the Weierstrass approximation theorem and integration.

Shastri proved that for every topological knot there is a polynomial knot equivalent to it. The approximation results above fill in some details of his proof. These methods also show that a topological knot can be approximated by a finite Fourier series as well as a rational function. A constructive proof of Shastri’s result has been provided by Wright [40].

Akbulut and King [1] prove a related result, namely if $K \subset S^3 \subset \mathbb{R}^4$ is a compact smooth knot then there is a real algebraic variety $Z \subset \mathbb{R}^4$ such that $Z \cap S^3$ is knot-equivalent to $K$. Also Mond and van Straten [26] show that any knot type can be realized as the real Milnor fiber of an appropriate space curve singularity. Note that Shastri’s result guarantees a polynomial parameterization for any knot type.

In addition to their usual equivalence as topological knots, polynomial knots have two stronger types of equivalence coming from the algebraic structure given by the coefficients of their defining polynomials. These coefficients form an open subset of a high-dimension Euclidean space. We call two polynomial knots path equivalent if they are in the same connected component of of this subset. We show that two path-equivalent polynomial knots are topologically equivalent. This argument is not trivial since it is conceivable that an area of knottedness might slide off to infinity as the knot moves in a family.

Left-right equivalence is defined by applying orientation-preserving linear automorphisms to the domain $\mathbb{R}^1$ and range $\mathbb{R}^n$. A knot obtained by applying such transformations is easily shown to be both topologically and path equivalent to the original knot. More generally, one can use polynomial automorphisms. These results are in §4.

In §5.2 we discuss how the equations defining a polynomial knot can be reduced to ones of simpler form. In fact, a polynomial knot can be reduced, using the standard operations of linear algebra (adding one row to another and so forth), to one whose coefficient matrix is in row echelon form. A stronger operation is available, that of adding polynomial multiples of one row to another. This may reduce the degree of the knot. (The degree of a polynomial knot is defined as the maximum degree of its component polynomials.) Note that the first operation is available in projective space, whereas the second is not.
These results are used to find simple equations for a polynomial knot representing a given topological knot.

In §5 we give many examples, most of which are due to students with whom we have worked under the auspices of the National Science Foundation Research Experiences for Undergraduates (REU) program. We also show that the degree of a polynomial representative of a topological knot is bounded below by expressions in the crossing number, bridge number, and superbridge number. The first two of these were found by Lee Rudolph, and the third by our students. These restrictions allows us to list and analyze the topological types represented by polynomial knots degree by degree. In this task the first result mentioned is the most useful.

In this section we also discuss curvature and minimal lexicographical orderings, and we compare the results for polynomial knots with those for polygonal knots.

In §6 we discuss the topological structure of spaces of polynomial knots. These results, mostly due to Vassiliev [39], are augmented by results due to our REU students. Here again we compare with the situation for polynomial knots.

Section 7 contains remarks on the situation over the complex numbers. We discuss the question of whether an embedding \( \mathbb{C} \to \mathbb{C}^n \) is rectifiable (left equivalent to a linear embedding). We also mention a topic from algebraic geometry, the standard correspondence of maps \( \mathbb{C}^1 \to \mathbb{C}^n \) with maps \( \mathbb{C}[x_1, x_2, \ldots, x_n] \to \mathbb{C}[t] \), as well as the correspondence of maps which are embeddings to surjective ring maps.

Many other special types of knots have been studied: rational knots (whose components are rational functions), trigonometric knots (whose components are finite Fourier series), holonomic knots (knots of the form \((f, f', f'')\) for some function \(f\)), polygonal or stick knots (whose components are piecewise-linear embeddings \( \mathbb{R} \to \mathbb{R}^3 \)), knots on a cubic lattice, thick knots (made of rope of a certain radius), and so forth. Polynomial knots are an addition to these boutique knots; they are attractive because of their connections with algebraic geometry. Knots whose components are rational functions will be the subject of a later paper.

Many of our results are stated for embeddings \( \mathbb{R} \to \mathbb{R}^n \) where \( n \geq 2 \), which we will also loosely refer to as polynomial knots. There is no extra cost in doing this, and some of the results are interesting in dimensions other than three.

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2. Basic results

2.1. Topological knots. We first review some basic concepts of knot theory. A topological knot is a continuous embedding $\alpha$ of the circle $S^1$ in the $n$-sphere $S^n$, where $n \geq 2$. (Usually one takes $n = 3$, but it will be useful to consider the other cases as well.) Let $S^1$ and $S^n$ have orientations coming from the inclusion $\mathbb{R}^k \subset S^k$, where $\mathbb{R}^k$ has its standard orientation.

Topological knots $\alpha_0$ and $\alpha_1$ are by definition topologically equivalent (“have the same knot type”) if there are orientation-preserving homeomorphisms $f$ of $S^1$ and $g$ of $S^n$ such that

$$\alpha_0 = g \circ \alpha_1 \circ f.$$  

Topological equivalence thus preserves the orientations on both the domain and the range.

Using stereographic projection we may identify the one point compactification of real $n$-space with the $n$-sphere. This map takes the point of compactification to the north pole of the $n$-sphere. A polynomial embedding $\kappa : \mathbb{R} \to \mathbb{R}^n$ extends to a unique map $\overline{\kappa} : S^1 \to S^n$ taking poles to poles. The map $\overline{\kappa}$ is a continuous embedding, but not necessarily smooth because there is an algebraic singularity at the north pole which may or may not be $C^1$. The details may be found in §2.2. Even if it has a singularity, a polynomial knot is a knot in the classical sense:

**Proposition 1.** The completed polynomial knot $\overline{\kappa} : S^1 \to S^n$ is tame.

Recall that a topological knot is tame if it is topologically equivalent to a polygonal knot. Tame knots in dimension three are the subject of classical knot theory; in dimensions greater than three all tame knots are trivial. The above proposition follows from the next two lemmas.

**Lemma 1.** The knot $\overline{\kappa}$ is piecewise differentiable.

By “piecewise differentiable” we mean that the domain can be divided into a finite number of segments such that the map is differentiable on the interior of each segment and has a limiting tangent line at each end. This follows from the fact that the curve has an isolated algebraic singularity in a neighborhood of the point at infinity.
Lemma 2. A piecewise differentiable knot is tame.

In fact, Appendix I of [9] shows that that a differentiable knot is tame, and the proof easily can be modified: If $p$ is a cusp point, then pull apart the branches of the cusp so that they are no longer tangent to each other, then make each cusp point point the vertex of a cone. We can also compactify $\mathbb{R}^n$ by projective $n$-space $\mathbb{P}^n$. The space $\mathbb{P}^n - \mathbb{R}^n$ is $\mathbb{H}^n$, the hyperplane at infinity. (Note that $H^1_\infty$ is a point.) A polynomial map $\kappa : \mathbb{R} \to \mathbb{R}^n$ extends to a unique map $\bar{\kappa} : \mathbb{P}^1 \to \mathbb{P}^n$ taking $H^1_\infty$ to a point on $H^\infty_n$. The projection $\mathbb{P}^n \to S^n$ collapsing the hyperplane $H^\infty_n$ to the north pole of $S^n$ takes $\bar{\kappa}$ to $\bar{\kappa}$. Projective compactifications will be discussed more fully in the forthcoming paper [10].

2.2. Stereographic projection and the singularity at infinity. In this section we use stereographic projection to give an analytic chart for $\mathbb{R}^n$ at infinity, then use this to describe the singularity of a polynomial knot at this point.

Now let $S^n \subset \mathbb{R}^{n+1}$ be the unit sphere centered at the origin, $n \geq 1$. The map

$$\sigma : \mathbb{R}^n \times \{0\} \to S^n$$

with formula

$$(x, 0) \mapsto \left( \frac{2x}{|x|^2 + 1}, \frac{|x|^2 - 1}{|x|^2 + 1} \right),$$

yields the stereographic projection map

$$\sigma : \mathbb{R}^n \cup \{\infty\} \to S^n,$$

taking the point of compactification $\infty$ to the north pole of $S^n$. We use this map to provide an analytic chart for $\mathbb{R}^n$ in a neighborhood of $\infty$.

Let $\kappa$ be a polynomial map of degree $d > 0$. The compactification $\bar{\kappa}$ of $\kappa$ under stereographic projection is the closure of the composition $\sigma \circ \kappa$. As $t$ tends to $\pm \infty$, the composition $\sigma \circ \kappa(t)$ tends to the north pole. Suppose that

$$\kappa(t) = a_d t^d + a_{d-1} t^{d-1} + \ldots + a_1 t + a_0$$

where $a_d, a_{d-1}, \ldots, a_0 \in \mathbb{R}^n$ and $a_d \neq 0$. The standard coordinates on $\mathbb{R}^n$ near the origin give local coordinates on $S^n$ near the north pole. In terms of these, the map $\bar{\kappa}$ is given by

$$t \mapsto \frac{2\kappa(t)}{\left|\kappa(t)\right|^2 + 1}.$$
Setting $s = 1/t$, the above expression becomes
\[
\frac{2(a_d s^d + a_{d-1} s^{d+1} + \ldots + a_0 s^{2d})}{a_d \cdot a_d + 2a_d \cdot a_{d-1} s + \ldots + (a_0 \cdot a_0 + 1)s^{2d}}
\]
which simplifies to
\[
\frac{2a_d}{|a_d|^2} s^d + [h].
\]
(Here, and below, $[h]$ denotes higher order terms, and $[l]$ lower order terms.) The computations above establish the following result.

**Proposition 2.** Let $n \geq 1$, let $\kappa : \mathbb{R} \to \mathbb{R}^n$ be a polynomial map of degree $d$ and let $\sigma : \mathbb{R}^n \cup \{\infty\} \to S^n$ be stereographic projection. The compactification $\overline{\kappa} = \sigma \circ \kappa$ is a curve in $S^n$ with an analytic singularity of order $d$ in local coordinates at the north pole.

For example, the Shastri trefoil $\kappa(t) = (t^3 - 3t, t^4 - 4t^2, t^5 - 10t)$ has singularity at the pole with expansion $2(s^7, s^6, s^5) + [h]$. More generally, a polynomial knot of the form $\kappa(t) = (t^p, t^q, t^r) + [l]$ with $p \leq q \leq r$ has local analytic expansion $2(s^{2r-p}, s^{2r-q}, s^r) + [h]$ at the north pole. In particular a polynomial knot with $d > 1$ will always have an algebraic singularity at infinity. It may however be $C^1$ there.

2.3. **Transversality.** In this section we show that polynomial maps are well-behaved at infinity, in fact transverse to all sufficiently large spheres about the origin.

**Proposition 3.** (a). If $n \geq 1$ and $\alpha : \mathbb{R} \to \mathbb{R}^n$ is a polynomial map, then as $t \to +\infty$ the angle between the vectors $\alpha(t)$ and $\alpha'(t)$ approaches 0, and as $t \to -\infty$ this angle approaches $\pi$. In particular, there exists $r_0 \gg 0$ such that the $(n-1)$-sphere of radius $|\alpha(t)|$ about the origin is transverse to $\alpha'(t)$ for all $|t| \geq r_0$.

(b). Suppose that $\{\alpha_u\}$ is a family of maps depending continuously on a parameter $0 \leq u \leq 1$. If the degree of $\alpha_u$ is constant (independent of $u$), then there is an $r_0$ such that Part (a) is true for all $\alpha_u$.

**Proof.** (a). Suppose that $\alpha(t)$ is as in Equation [4]. The cosine of the angle between the vectors $\alpha(t)$ and $\alpha'(t)$ is
\[
\frac{\alpha(t)}{|\alpha(t)|} \cdot \frac{\alpha'(t)}{|\alpha'(t)|}
\]
which becomes
\[
\pm \frac{a_d + \frac{1}{t} a_{d-1} + \ldots + \frac{1}{t^{d-1}} a_0}{|a_d + \frac{1}{t} a_{d-1} + \ldots + \frac{1}{t^{d-1}} a_0|} \cdot \frac{d a_d + \frac{d-1}{t} a_{d-1} + \ldots + \frac{1}{t^{d-1}} a_0}{|a_d + \frac{d-1}{t} a_{d-1} + \ldots + \frac{1}{t^{d-1}} a_0|}.
\]
As $t \to \pm \infty$ this expression approaches
\[ \pm \frac{a_d \cdot da_d}{|a_d \cdot da_d|} = \pm 1. \]

(b). The coefficients of $\alpha_u(t)$ are functions of $u$. Since $a_d \neq 0$ the denominators can be bounded below by a nonzero constant. The above argument then continues to hold. \qed

3. POLYNOMIAL APPROXIMATIONS OF SMOOTH KNOTS

3.1. Approximation. Let $M \subset \mathbb{R}$ be a compact connected interval.

In this section we show that given a smooth embedding $\alpha : M \to \mathbb{R}^n$, and another map $\beta : M \to \mathbb{R}^n$ close to it (both the function and its derivative), then $\beta$ is an embedding which is isotopic to $\alpha$.

Lemma 3. Suppose that $\gamma : M \to \mathbb{R}^n$ is $C^2$, where $n \geq 1$.
(a). If $\gamma'(t) \neq 0$ for all $t \in M$, then there are positive numbers $r$ and $\delta$ such that $|s - t| \leq r$ implies that $|\gamma(s) - \gamma(t)| \geq \delta |s - t|$.
(b). If $\gamma(s) \neq \gamma(t)$ for all $s \neq t \in M$, then for each $r > 0$ there is a $\delta$ such that $|s - t| \geq r$ implies that $|\gamma(s) - \gamma(t)| \geq \delta$.

Proof. (a). Since $M$ is compact and $\gamma'(t) \neq 0$ for all $t \in M$, there is a $\delta > 0$ such that $|\gamma'(t)| \geq 2\delta$ for all $t \in M$. Let $m$ be the maximum value of $(1/2)|\gamma''(t)|$ for $t \in M$. If $m = 0$ then $\gamma$ is linear and the result is clear. Otherwise let $r = \delta/m$. By Taylor’s theorem, for each $s, t \in M$ there exists $\tau$ between $s$ and $t$ such that
\[ |\gamma(s) - \gamma(t)| = |\gamma'(t)(s - t) + (1/2)\gamma''(\tau)(s - t)^2| \]
\[ \geq |s - t||\gamma'(t)| - |(1/2)\gamma''(\tau)(s - t)| \]
\[ \geq \delta |s - t|. \]

The last line follows since $|s - t| \leq r$ implies that $(1/2)|\gamma''(\tau)||s - t| \leq mr = \delta$.
(b). This follows since the set $K = \{(s, t) \in M \times M : |s - t| \geq r\}$ is compact so the function $|\gamma(s) - \gamma(t)|$ on $K$ is bounded below. \qed

Definition 2. Let $M \subset \mathbb{R}$ and let $\alpha, \beta : M \to \mathbb{R}^n$ be $C^1$ maps. The maps $\alpha$ and $\beta$ are $\epsilon$-close if $|\alpha(t) - \beta(t)| < \epsilon$ and $|\alpha'(t) - \beta'(t)| < \epsilon$, for all $t \in M$.

Proposition 4. (a). Suppose that $n \geq 1$, and that $M$ is a compact connected interval. Let $\alpha : M \to \mathbb{R}^n$ be a $C^2$ embedding and let $\beta : M \to \mathbb{R}^n$ be a $C^2$ map $\epsilon$-close to $\alpha$, where $\epsilon > 0$ is a suitably small number. Then $\beta$ is an embedding, and $\beta$ is isotopic to $\alpha$. 
(b). If $n = 2$, and the word “embedding” is replaced by “has transversal self-intersections”, then the same proposition holds.

The number $\epsilon$ will be determined in the course of the proof.

Proof. Let

$$\gamma_u(t) = u\beta(t) + (1 - u)\alpha(t)$$

where $0 \leq u \leq 1$. Note that $\gamma_0 = \alpha$ and $\gamma_1 = \beta$. Fix $u$. We will show that $\gamma_u$ is an embedding. The family $\gamma_u$ thus provides an isotopy of $\alpha$ to $\beta$.

First we show that $\gamma_u'(t) \neq 0$, for all $t \in M$:

$$|\gamma_u'(t)| = |u\beta'(t) + (1 - u)\alpha'(t)|$$

$$\geq |\alpha'(t)| - u|\beta'(t) - \alpha'(t)|$$

Since $M$ is compact and $\alpha' \neq 0$ for $t \in M$, there is a $\delta_1 > 0$ such that $|\alpha'(t)| \geq \delta_1$. If $|\beta'(t) - \alpha'(t)| \leq \delta_1/2$ then the above expression is at least $\delta_1/2$ and hence nonzero.

Next we show that $s \neq t$ implies that $\gamma_u(s) \neq \gamma_u(t)$. Since $\gamma_u'(t) \neq 0$, Lemma 3(a) implies that there is an $r > 0$ such that such that this is true if $|s - t| \leq r$. Now suppose that $|s - t| \geq r$. Then

$$|\gamma_u(s) - \gamma_u(t)| = |u\beta(s) + (1 - u)\alpha(s)| - |u\beta(t) + (1 - u)\alpha(t)|$$

$$\geq |\alpha(s) - \alpha(t)| + u\left|\left(\beta(s) - \alpha(s)\right) + \left(\alpha(t) - \beta(t)\right)\right|$$

By Lemma 3(b), there is a $\delta_2 > 0$ such that $|s - t| \geq r$ implies $|\alpha(s) - \alpha(t)| \geq \delta_2$. If $|\beta(t) - \alpha(t)| \leq \delta_2/4$, then the above expression is nonzero. If $0 < \epsilon \leq \min\{\delta_1/2, \delta_2/4\}$ the then statement (a) follows. The proof of statement (b) is essentially the same.

3.2. Polynomial knots. We now apply the above results to fill in some details in Shastri’s result that for every topological knot there is a polynomial knot of the same knot type. The polynomial knot constructed here will not be $\epsilon$-close to the original knot; in fact the original is a compact subset of $\mathbb{R}^n$ whereas the polynomial knot will pass through the point at infinity.

An alternate proof of this approximation theorem is due to Wright [40]. The method here is first to replace the $z$-coordinate of the knot
by an interpolating polynomial which goes under and over at the same times. The result is then turned on its side so that the $x$-coordinate is up, and this is also approximated by an interpolating polynomial. This is repeated with the $y$-coordinate. The final result is a polynomial knot of the same knot type as the original. This method is algorithmic, and can be turned into a computer program.

In addition, Mui [27, 28] describes a method to find height functions of low degree for a given knot projection.

**Proposition 5.** (a). If $n \geq 1$ and $\alpha : [a, b] \to \mathbb{R}^n$ is a $C^2$ embedding, then there is a polynomial embedding $\beta : [a, b] \to \mathbb{R}^n$ isotopic to $\alpha$. Furthermore the maps $\alpha$ and $\beta$ can be taken $\epsilon$-close for any small $\epsilon > 0$.

(b). If $n = 2$, and the word “embedding” is replaced by “has transversal self-intersections”, then the same proposition holds.

**Proof.** Let $\xi$ be a small positive number. By the Weierstrass approximation theorem there is a polynomial $\gamma$ such that $|\gamma(t) - \alpha'(t)| < \xi$ for all $t \in [a, b]$. If

$$\beta(t) = \int_a^t \gamma(s)ds + \alpha(a)$$

then $\beta'(t) = \gamma(t)$, and

$$|\beta(t) - \alpha(t)| = \left| \left[ \int_a^t \gamma(s)ds + \alpha(a) \right] - \left[ \int_a^t \alpha'(s)ds + \alpha(a) \right] \right|$$

$$= \left| \int_a^t \gamma(s)ds - \int_a^t \alpha'(s)ds \right|$$

$$\leq (b - a) \sup_{s \in [a, b]} |\gamma(s) - \alpha'(s)|$$

$$\leq (b - a)\xi.$$  

If $\xi$ is small enough so that $\epsilon = \max\{\xi, (b-a)\xi\}$ satisfies the hypotheses of Proposition 4, then part (a) follows. The proof of part (b) is similar. □

The above proposition fills in a detail in the following result due to Shastri:

**Proposition 6.** For every $C^2$ knot there is a polynomial knot of the same knot type.

In fact, let

$$\alpha(t) = (\alpha_1(t), \alpha_2(t), \alpha_3(t)) : \mathbb{RP}^1 \to \mathbb{R}^3$$

be a $C^2$ embedding with the required knot type. (We use the real projective line $\mathbb{RP}^1$ rather than the circle $S^1$ in order to have the inclusion
Without loss of generality we may assume that the map

\[ t \to (\alpha_1(t), \alpha_2(t)) : \mathbb{RP}^1 \to \mathbb{R}^2 \]

has only double points. Choose an interval \([a, b] \subset \mathbb{R}^1 \subset \mathbb{RP}^1\) containing these double points. Find a polynomial function \(\beta_3 : \mathbb{R}^1 \to \mathbb{R}^3\) which represents the height function \(\alpha_3\). There is \(c < a < b < d\) with \(c, d \in \mathbb{R}\) such that \(\beta_3\) is monotonic on \((-\infty, c)\) and \((d, +\infty)\). By Proposition \(\text{5(b)}\) there is a polynomial map \((\beta_1, \beta_2) : [c, d] \to \mathbb{R}^2\) which has transverse intersections and which is isotopic to the map \((\alpha_1, \alpha_2)\). The spherical completion of

\[ \beta = (\beta_1, \beta_2, \beta_3) : \mathbb{R}^1 \to \mathbb{R}^3 \]

has the same knot type as \(\alpha\).

If the polynomial approximation \((\beta_1, \beta_2)\) to the plane projection had been chosen before the height function \(\beta_3\), then the result may not have the correct knot type since \((\beta_1, \beta_2)\) may have additional intersections outside of the interval of approximation.

### 3.3. Trigonometric knots.

Suppose that \(n \geq 1\). A finite Fourier series \(\beta : [0, 2\pi] \to \mathbb{R}^n\) is a map of the form

\[ \beta(t) = \sum_{k=0}^{m} a_k \sin kt + b_k \cos kt \]

where \(a_k\) and \(b_k\) are real vectors of dimension \(n\).

**Proposition 7.** Let \(\epsilon > 0\). If \(\alpha : \mathbb{R} \to \mathbb{R}^n\) is a \(C^2\) embedding, and if \(\alpha\) is periodic with period \(2\pi\), then there is a finite Fourier series which is an \(\epsilon\)-close embedding isotopic to \(\alpha\).

The proof is similar to the corresponding proof for polynomials, using the theorem that \(\alpha\) has a Fourier series whose partial sums converge uniformly to \(\alpha\).

### 3.4. Rational knots.

**Proposition 8.** Let \(\epsilon > 0\). If \(\alpha : \mathbb{RP}^1 \to \mathbb{R}^n\) is a \(C^2\) embedding, then there is a rational function which is an \(\epsilon\)-close embedding isotopic to \(\alpha\).

By Section \(\text{3.2}\) there is a polynomial knot of the same knot type as \(\alpha\), but this approximation may not be \(\epsilon\)-close.

**Proof.** By Proposition \(\text{7}\) there is a finite Fourier series \(\beta(t)\) which is an embedding isotopic to \(\alpha\). The standard substitution

\[ t = \frac{1}{2} \arctan x \]
changes this into a rational function. (For details see Clark [7].)

The rational functions produced by this method appear to have needlessly high degree, so it would be nice to have a direct proof of this proposition.

4. Equivalence of polynomial knots

The polynomial knots in this section are of the form \( \kappa : \mathbb{R} \to \mathbb{R}^n \) for \( n \geq 2 \). We give three different definitions of equivalence between two polynomial knots \( \kappa_0 \) and \( \kappa_1 \) and describe connections among them.

The three types of equivalence are: (1) have the same topological knot type, (2) are in the same connected component of the coefficient space, and (3) if transformations of the range and domain take one to the other. If they are equivalent in either the second or third sense, then they are equivalent in the first sense. Whether the converse is true is not as clear, as is the relation between the last two.

Definition 3. Two polynomial knots \( \kappa_0 \) and \( \kappa_1 \) are topologically equivalent if there are orientation-preserving homeomorphisms \( f \) of \( S^1 \) and \( g \) of \( S^n \) such that

\[
\bar{\kappa}_0 = g \circ \bar{\kappa}_1 \circ f.
\]

They are thus topologically equivalent if their spherical completions are equivalent as topological knots. Recall that polynomial knots are tame, so the above is the usual knot equivalence. Topologically equivalent polynomial knots may have different degrees; for example one can replace the parameter \( t \) by \( t^k + t \) where \( k \) is an odd integer.

The next definition of equivalence makes use of the parameter space of these knots. Let \( \mathcal{M}_d^n \) be the space of all polynomial maps \( \mathbb{R} \to \mathbb{R}^n \) of degree \( d \). This space is isomorphic to a proper subset of \( (\mathbb{R}^{d+1})^n \) and inherits its topology. (The inclusion is proper since \( a_d \neq 0 \).) Let \( \Sigma_d^n \subset \mathcal{M}_d^n \) be the space of maps \( \alpha \) with either double points \( (r \neq s \text{ such that } \alpha(r) = \alpha(s)) \) or critical points \( (t \text{ such that } \alpha'(t) = 0) \). This is a closed subset of \( \mathcal{M}_d^n \). Let \( \mathcal{K}_d^n = \mathcal{M}_d^n - \Sigma_d^n \); this is the parameter space of polynomial knots of degree \( d \).

Definition 4. Two polynomial knots \( \kappa_0 \) and \( \kappa_1 \) of degree \( d \) are path equivalent if they are in the same connected component of \( \mathcal{K}_d^n \).

Proposition 9. If two polynomial knots are path equivalent, then they are topologically equivalent.

Proof. Let \( \kappa_0 \) and \( \kappa_1 \) be knots in the same path component of \( \mathcal{K}_d^n \), and let \( \{ \kappa_u, 0 \leq u \leq 1 \} \) be a family of knots connecting them. Since \( \mathcal{K}_d^n \) is open we may assume without loss of generality that this family is
smooth in $u$. Let $\Theta : \mathbb{R} \times I \to \mathbb{R}^n \times I$ be defined by $\Theta(t, u) = (\kappa_u(t), u)$. (Here $I$ is the unit interval.) Let $K$ be the image of $\Theta$. This is a smooth submanifold of $\mathbb{R}^n \times I$, since each $\kappa_u$ has no singularities. We identify the one point compactification $\mathbb{R}^n \cup \{\infty\}$ with $S^n$. Let $N$ be the north pole of $S^n$. Let $\bar{K} \subset S^n \times I$ be the closure of $K$. By Proposition 3 the stratification $S^3 \times I - \bar{K} \supset K \supset \{(N) \times I}$ satisfies the Whitney conditions. Furthermore the map $\pi : S^3 \times I \to I$ is proper, and for each stratum $S$ the map $\pi|S : S \to I$ is a submersion. By Thom’s first isotopy lemma ([35], [20]), this bundle is locally trivial ($C^0$, not necessarily $C^\infty$). Thus the topological knots $\bar{\kappa}_0$ and $\bar{\kappa}_1$ are equivalent (preserving orientations).

□

**Question.** Is the converse true? In other words, can two topologically-equivalent knots be connected by a path of polynomial knots? This is an attractive question whose answer is probably negative, though no examples are known. For example, in dimensions greater than three all topological knots are unknotted, but there may be more that one component of unknotted polynomial knots.

**Remark.** Suppose that $\gamma_1 : \mathbb{R} \to \mathbb{R}^3$ is a open-ended topological trefoil with the property that $\gamma(t) = (t, 0, 0)$ for sufficiently large $|t|$. Let $\gamma_u = \gamma_1 + (1/u - 1, 0, 0)$ for $0 < u \leq 1$ be this knot translated to the right, and let $\gamma_0(t) = (t, 0, 0)$. In this family the knot $\gamma_1$ is translated to infinity and becomes unknotted. Here the stratification above does not satisfy the Whitney conditions, so the proposition does not hold.

The next type of equivalence combines left equivalence (algebraic transformations of the range) and right equivalence (algebraic transformations of the domain). The first moves the knot around, and the second reparameterizes it.

The simplest type of transformation that can be used is an affine automorphism, a map of the form $T(x) + c$ for $x \in \mathbb{R}^n$, where $T : \mathbb{R}^n \to \mathbb{R}^n$ is a bijective linear transformation and $c$ a constant. More generally, $T$ can be a map of the form $T(x) = (T_1(x), T_2(x), \ldots T_n(x))$ where each $T_i$ is a polynomial in $x$; this map is a *polynomial automorphism* if it if has an inverse of this form. For example, the map $(x, y) \mapsto (x, y + x^2)$ of $\mathbb{R}^2$ is a polynomial automorphism. If $T_l$ and $T_r$ are polynomial automorphisms and $\kappa$ is a polynomial knot, then so is $T_l \circ \kappa \circ T_r$. Note that a bijective polynomial map with non-zero differential at each point of $\mathbb{R}^n$ need not be a polynomial automorphism, for example the map $t \mapsto t^k + t$ of the real numbers with $k$ odd. Also note that a polynomial automorphism may not extend to projective space; this makes the theory of affine knots different from the theory of projective knots.
Definition 5. Two polynomial knots $\kappa_0, \kappa_1 : \mathbb{R} \to \mathbb{R}^n$ are left-right (LR) equivalent if there are orientation-preserving polynomial automorphisms $T_r$ of $\mathbb{R}^1$ and $T_l$ of $\mathbb{R}^n$ such that

$$\kappa_0 = T_l \circ \kappa_1 \circ T_r.$$ 

Note also that an affine automorphism of $\mathbb{R}^n$ preserves the degree of a polynomial knot, whereas a polynomial automorphism may not (for example the transformation $(x, y) \mapsto (x, y + x^2)$ applied to the trivial knot $t \mapsto (t, 0)$).

Of course there can be many different variations on this type of equivalence: equivalence using linear transformations, equivalence using transformations which are not necessarily orientable, left and right equivalence separately, and so forth. We specify these as needed.

Proposition 10. If two polynomial knots are LR-equivalent, then they are topologically equivalent.

Proof. The map $\bar{T}_l : S^3 \to S^3$, where the bar denotes completion, takes $\bar{\kappa}_0$ to $\bar{\kappa}_1 \circ (\bar{T}_r)^{-1}$. The latter is a reparameterization of $\bar{\kappa}_1$ and hence of the same knot type. \qed

Proposition 11. If two polynomial knots are LR-equivalent by (orientation-preserving) affine transformations, then they are path equivalent. This follows since the group of such transformations is connected.

Question. In general, the relation between path equivalence and LR-equivalence is not clear. If two polynomial knots of the same degree are LR-equivalent by polynomial transformations, are they path equivalent? This is true if the transformations of the domain and range are tame. (Tame transformations are by definition a composition of maps which add a polynomial multiple of one row to another row. Not all polynomial automorphisms are tame; see [32].) Conversely, if two polynomial knots are path equivalent, are they LR-equivalent? This is probably not true, though no examples are known. For the complex case of these two questions see [7].

5. Examples and restrictions

In this section we examine the following questions:

1. Given a topological knot, find a polynomial knot such that its spherical compactification has the same knot type.
2. Given a topological knot, find a lower bound for the degree of a polynomial representation of this knot.
3. Find a polynomial representative of this degree.
We also examine the above questions using the lexicographic ordering on the degrees of \((x(t), y(t), z(t))\). We compare all these results with the corresponding situation for stick knots. Finally, we relate the total curvature of the knot with its degree.

5.1. **Examples.** Let us start with the first question. Although every topological knot has a polynomial representation \((\S 3)\), less is known about constructing polynomial representatives in low degree. In this section we give many examples which have low degree equations. (See \([36]\) for a selected list.) Some have elegant equations, others messy. The latter arise by constantly adjusting coefficients until the required knot type is obtained.

5.1.1. *The trefoil knot.* Shastri’s equations have degree \((3,4,5)\). These are the simplest set of equations for a nontrivial knot.

\[
\begin{align*}
x(t) &= t^3 - 3t \\
y(t) &= t^4 - 4t^2 \\
z(t) &= t^5 - 10t.
\end{align*}
\]

5.1.2. *The figure-eight knot.* Shastri also found the following equations of degree \((3,5,7)\):

\[
\begin{align*}
x(t) &= t^3 - 3t \\
y(t) &= t(t^2 - 1)(t^2 - 4) \\
z(t) &= t^7 - 42t.
\end{align*}
\]

Equations of the same degree but with just two terms were found by Brown \([4]\):

\[
\begin{align*}
x(t) &= t^3 - 5t \\
y(t) &= t^5 - 28t \\
z(t) &= t^7 - 32t^3.
\end{align*}
\]

The following equations of degree \((4,5,6)\) were found by McFeron \([21]\):

\[
\begin{align*}
x(t) &= t^4 - 7t \\
y(t) &= t^5 - 43t \\
z(t) &= t^6 - 125t.
\end{align*}
\]
\[ x(t) = -t^4 + 2.279283653t^3 + 5t^2 - 8.63068748t + 0.35140383 \\
y(t) = t^5 - 5t^3 + 4t \\
z(t) = (t + 2.06) \cdot (t + 1.916737670) \cdot (t + 0.2122155248) \cdot (t - 1.379221313) \cdot (t - 2.05) \cdot (t + 10). \]

5.1.3. Knots with five crossings. The 5\textsubscript{1} knot (the torus knot of type (2,5)):

- Auerbach \[2\] (degree (4,5,7)):
  \[
x(t) = (t^2 + 5t + 4)(t^2 - 7t + 10) \\
y(t) = (t^2 - 5.2t)(t^2 - 9)(t + 4.7) \\
z(t) = t^7 - 3.90763t^6 - 25.8835t^5 + 83.4739t^4 + 176.691t^3 \\
  - 364.064t^2 - 331.888t + 321.285. \]

- REU 1998 (degree (5,6,7)):
  \[
x = 1000t^5 - 541000t^3 + 44100000t \\
y = 100t^6 - 80400t^4 + 17126400t^2 - 792985600 \\
z = t^7 - 13433 \frac{16}{16} t^6 + \frac{783769}{4} t^5 - 9363600 t. \]

- Brown \[4\] (degree (4,5,7)):
  \[
x(t) = t^4 - 24t^2 \\
y(t) = t^5 - 36t^3 + 260t \\
z(t) = t^7 - 31t^5 + 168t^3 + 560t. \]

The 5\textsubscript{2} knot (REU 2006), equations of degree (4, 5, 9):

\[
x(t) = t^4 - 12t^2 \\
y(t) = t(t^2 - 4)(t^2 - 11) \\
z(t) = (t - 0.5)(t - 10)(t - 12.5)(t - 20) \cdot \\
  (t - 28)(t - 29)(t - 40)(t - 50.2)(t - 50.6). \]

5.1.4. Knots with six crossings. The 6\textsubscript{1} knot (REU 2006), equations of degree (4, 7, 11):

\[
x(t) = (t^2 - 0.5)(t^2 - 15.3) \\
y(t) = t(t^2 - 16)(t^2 - 7)(t^2 - 4.5) \\
z(t) = t(t^2 - 15.2)(t^2 - 9)(t^2 - 6.25)(t^2 - 1)(t^2 - 0.25). \]
The $6_2$ knot \cite{1}, equations of degree $(4, 5, 11)$:

\[ x(t) = t^4 - 12t^2 \]
\[ y(t) = t(t^2 - 4)(t^2 - 11) \]
\[ z(t) = t(t^2 - 1)(t^2 - 9)(t^2 - \frac{49}{16})(t^2 - \frac{169}{16})(t^2 - \frac{100}{9}). \]

The $6_3$ knot (Curry, REU 1998), equations of degree $(3, 8, 10)$:

\[ x(t) = t^3 - 100t \]
\[ y(t) = t \cdot (t + 4) \cdot (t - 6) \cdot (t - 8) \cdot (t - 9) \cdot (t + 9) \cdot (t - 11) \cdot (t + 11) \]
\[ z(t) = (t + 1) \cdot (t^2 - 4) \cdot (t - 6) \cdot (t + 7) \cdot (t + 9.2) \cdot (t - 9.5) \]
\[ \cdot (t + 10) \cdot (t - 10.5) \cdot (t + 11.5). \]

5.1.5. Other knots. Many other examples of polynomial knots are known: Auerbach \cite{2} found equations of degree $(3,10,11)$ for the $(2,7)$ torus knot, and of degree $(3,16,17)$ for the $(2,9)$ torus knot. Curry (REU 1998) found the following equations of degree $(7, 6, 7)$ for the $(4,3)$ torus knot (the knot $8_{19}$):

\[ x(t) = t(t^2 - 12^2)(t^2 - 28^2)(t^2 - 30^2) \]
\[ y(t) = (t^2 - 6^2)(t^2 - 23^2)(t^2 - 29^2) \]
\[ z(t) = -t(t^2 - 8^2)(t^2 - 12.2^2)(t^2 - 29^2). \]

Equations for the other knots with the same projection as $8_{19}$ can be found in \cite{17}. Mishra and others in a series of papers \cite{30, 24, 25, 18, 19} find polynomial representations of torus and two-bridge knots. Wright \cite{40} gives an algorithmic method for finding a polynomial representation of any knot (see \S 3.2).

5.2. Reduced forms. We now show that polynomial knots can be reduced to ones of simpler form. This procedure has been used extensively when finding equations for a specific knot (\S 5) and in fact can be used to reduce these equations even more. It has also been used in \cite{25}. Our reductions will use polynomial automorphisms of the range; we refer to this as “left equivalence”. Also recall that path equivalence was defined in \S 4. Let

\[ \kappa(t) = a_d t^d + a_{d-1} t^{d-1} + \ldots + a_1 t + a_0 \]

be a polynomial knot of degree $d$, where $a_d, a_{d-1}, \ldots, a_0 \in \mathbb{R}^n$ for $n \geq 1$ are vectors, and $a_d \neq 0$. The next result follows by a translation of the range.
Reduction 1. The knot $\kappa$ is both left and path equivalent to one with $a_0 = 0$.

In what follows we assume that $a_0 = 0$. We define the coefficient matrix of $\kappa$ to be

$$A = [a_d \ a_{d-1} \cdots a_1]$$

where the $a_i$ are regarded as column vectors. The next two reductions are based on the simple observation that

$$L \circ \kappa(t) = LA \ t,$$

where $t$ is the column vector $(t^d, t^{d-1}, \ldots, t)$ and $L$ is a linear transformation of $\mathbb{R}^n$. In other words, linear transformations of the range are the same as left multiplication of the coefficient matrix.

Reduction 2. The knot $\kappa$ is both left and path equivalent to one with $a_d = (1, 1, \ldots, 1)$. (Thus this knot is in $\mathcal{V}_d^0$; see [10].)

This follows if $L$ is an orientation-preserving linear transformation with the property that $L(a_d) = (1, 1, \ldots, 1)$. Let $(d_1, d_2, \ldots, d_n)$ be the vector degree of $\kappa$, where $d_i$ is the degree of the $i$-th component of $\kappa$.

Reduction 3a. The knot $\kappa$ is left equivalent to a knot of vector degree $(d'_1, d'_2, \ldots, d'_n)$ with $d'_1 < d'_2 < \cdots < d'_n = d$ and with leading coefficients 1.

In fact, let $L$ be the linear transformation which reduces the coefficient matrix to reduced row echelon form. This reduced form is unique. Note, however, that the transformation $L$ may not necessarily preserve orientation. Using an orientation-preserving transformation gives the following result:

Reduction 3b. The knot $\kappa$ is both left and path equivalent to a knot of vector degree $(d'_1, d'_2, \ldots, d'_n)$ with $d'_1 < d'_2 < \cdots < d'_n = d$ and with leading coefficients 1, except for the last row, where the leading coefficient may be $\pm 1$.

Reduction 4. The knot $\kappa$ is left equivalent to a knot of vector degree $(d'_1, d'_2, \ldots, d'_n)$ with $d'_1 < d'_2 < \ldots d'_n \leq d$, where $d'_i$ is not in the semigroup generated by nonnegative integral combinations of $d'_1, d'_2, \ldots, d'_{i-1}$ for $2 \leq i \leq n$.

The transformation used here is a nonlinear shear. This reduction is a more refined version of the one above, and it may reduce the degree of $\kappa$; such reductions are not possible for knots in projective space.
Proof. For simplicity we prove this in the case \( n = 3 \); the general case is similar. Suppose that \( \kappa \) has vector degree \((p, q, r)\). We may assume that \( \kappa \) is in the form of the previous reduction. If \( q = mp \) for a positive integer \( m \), let \( A : \mathbb{R}^3 \mapsto \mathbb{R}^3 \) be the orientation preserving polynomial map

\[
P(x, y, z) = (x, y - x^m, z).
\]

Note that \( P \) has a polynomial inverse \( Q(x', y', z') = (x', y' + x'^m, z') \) and that the leading terms in \( P \circ \kappa \) have vector degree \((p, q', d)\) with \( q' \) strictly less than \( q \). Similarly, if \( d = mp + nq \) with \( m, n \) nonnegative integers, not both zero, consider the polynomial map \( P : \mathbb{R}^3 \mapsto \mathbb{R}^3 \) defined by

\[
P(x, y, z) = (x, y, z - x^m y^n).
\]

Again, \( P \) is an automorphism. The leading terms in \( P \circ \kappa \) now have degree \( p, q, r \) with \( r \) strictly less than \( d \). \(\square\)

These transformations are tame (see the end of §4). If only an even number of row switches are involved then these knots are path equivalent. (See the remarks at the end of §4.)

5.3. Finding lower bounds on the degree. Next we assemble some tools to examine the second question of finding lower bounds for degree of a polynomial representation of a given topological knot. In general these lower bounds are rather weak and more work needs to be done here.

5.3.1. The crossing number. Recall that the crossing number of a topological knot \( K \) (thought of as a subset of \( \mathbb{R}^3 \)) is the least number of crossings in any planar projection of any \( K' \) with the same knot type as \( K \). The “crossing number” of a polynomial knot refers to the crossing number of its one-point compactification; the same applies to “bridge number” and so forth.

The following two results are due to Lee Rudolph (Mount Holyoke seminar lecture, 1995 (see [2]).

**Lemma 4.** Let \( \alpha(t) = (x(t), y(t)) \) be a parameterized curve in the real plane, where \( x(t) \) and \( y(t) \) are polynomials in \( t \) of degree \( d_x \) and \( d_x \) respectively. Assume that \( \alpha'(t) \neq 0 \) for all \( t \), and that the self-intersections of this curve are double points (transverse intersections). Then this curve has at most \((1/2)(d_x - 1)(d_y - 1)\) double points.

**Proof.** This curve has a double point at \((x_0, y_0)\) if and only if there are \( r \neq s \) so that \( x(r) = x(s) = x_0 \) and \( y(r) = y(s) = y_0 \). Such \( r \) and \( s \) are
solutions to the equations
\[ \frac{x(r) - x(s)}{r - s} = 0 \]
\[ \frac{y(r) - y(s)}{r - s} = 0. \]

The first equation has degree \( d_x - 1 \), and the second \( d_y - 1 \), so by Bezout’s theorem these equations have at most \((d_x - 1)(d_y - 1)\) intersections in the real plane. Thus there are at most \((d_x - 1)(d_y - 1)\) ordered pairs \((r, s)\) giving crossings. Since a crossing is specified by an unordered pair, there are at most \((1/2)(d_x - 1)(d_y - 1)\) crossings. □

**Proposition 12.** If \( \kappa(t) \) is a polynomial knot of degree \( d \) and crossing number \( c \), then
\[ c \leq (1/2)(d - 2)(d - 3). \]

**Proof.** Without loss of generality we may assume that \( z(t) \) has degree \( d \). By Reductions 3 and 4 there is a polynomial knot \( \kappa_1(t) = (x_1(t), y_1(t), z(t)) \) with the same knot type as \( \kappa \) and \( x_1(t) \) of degree at most \( d - 1 \) and \( y_1(t) \) of degree at most \( d - 2 \). The result then follows from the above lemma. □

The knot projection in the complex projective plane has exactly \((1/2)(d - 1)(d - 2)\) intersections; we return to this topic in \[10\].

5.3.2. The bridge number. Recall that the bridge number of a topological knot \( K \) can be defined as the minimum number of local maxima of \( K' \) in the direction \( v \), over all \( K' \) with the same knot type as \( K \), and all directions \( v \).

Schubert [31, Satz 10] shows that a torus knot of type \((p, q)\) with \( p < q \) has bridge number \( p \), and also (Satz 7) that the bridge number of a connected sum of knots is the sum less one of their bridge numbers.

The following proposition is also due to Lee Rudolph (see [2]). Unfortunately it is rather weak since many knots have bridge number two.

**Proposition 13.** If \( \kappa \) is a polynomial knot of degree \( d \) and bridge number \( b \), then
\[ b \leq (1/2)(d - 1). \]

**Proof.** We find an upper bound for the bridge number taking the \( z \) axis as the direction. Let \( e \) be the degree of \( z(t) \). By Reduction 3 above we may assume that \( e \leq d - 2 \). Without loss of generality we may assume that the coefficient of the highest term in \( z(t) \) is positive. If \( e \) is odd then the polynomial \( z(t) \) has at most \((1/2)(e - 1)\) local maxima, and
if \( e \) is even it has at most \((1/2)e - 1\). Hence the spherical completion \( \bar{\kappa} \) of \( \kappa \) has at most \((1/2)(e + 1) \leq (1/2)(d - 1)\) local maxima in the \( z \) direction (since it has a maximum at infinity). □

5.3.3. The superbridge number. Kuiper \cite{16} defines the superbridge number of a topological knot \( K \) as the minimum, over all \( K' \) of the same knot type as \( K \), of the maximum number of local maxima of \( K' \) in the direction \( v \), for all directions \( v \). He then shows that a torus knot of type \( (p, q) \) has superbridge number \( q \) if \( p < q < 2p \), and has superbridge number \( 2p \) if \( 2p < q \).

**Proposition 14.** (REU 1998) If \( \kappa \) is a polynomial knot of degree \( d \) and superbridge number \( s \), then

\[
s \leq (1/2)(d + 1).
\]

**Proof.** We find an upper bound for \( s \) using the knot \( \bar{\kappa} \). For any rotation of \( \bar{\kappa} \) the degree of the \( z \) coordinate is at most \( d \). An argument as above proves the proposition. □

5.4. Applications. Now let examine Questions 2 and 3, proceeding degree by degree. Let \( \kappa \) be a polynomial knot. (The unknot can be represented in all degrees by \( \kappa(t) = (t^d, t, t) \).

- \( \text{deg}(\kappa) \leq 4 \): The crossing number (and bridge number) of \( \kappa \) is at most one, so \( \kappa \) is the unknot. This also follows since the space of such polynomial knots is path connected (§6).
- \( \text{deg}(\kappa) \leq 5 \): The crossing number of \( \kappa \) is at most three, hence \( \kappa \) is either the trefoil or the unknot. Shastri’s equations for the trefoil are of degree five. Equations for the mirror image of the trefoil are obtained by changing the sign of one equation, or switching two of them.
- \( \text{deg}(\kappa) \leq 6 \): The crossing number of \( \kappa \) is at most 10. McFerons’ equations above for the figure-eight knot have degree six. It seems unlikely that knots of five crossings or more can be represented by equations of degree six, though further methods are needed here.
- \( \text{deg}(\kappa) = 7 \): By Proposition 14 this is the minimum degree for equations of the (2,5)-torus knot, since it has superbridge number four. Seven is also the minimum degree for equations of the torus knot of type \( (3,4) \), since it has bridge number 3. Equations for these knots are given above.
5.5. **Further methods.** Let $\kappa(t) = (x(t), y(t), z(t))$ be a polynomial knot, $u(t)$ a polynomial and $\epsilon > 0$ a suitably small number. The REU 2006 group showed that $\tilde{\kappa}(t) = (x(t), y(t), z(t) + \epsilon u(t))$ is a polynomial knot with the same topological knot type as $\kappa(t)$. Starting with $\kappa$ it is then possible to do the above and then apply Reduction 4 to obtain equations of lower degree with the same knot type as $\kappa$.

Mui [28] finds bounds in the cases where the polynomials are sparse or have few monomials.

5.6. **Lexicographic ordering.** One could also ask Questions 2 and 3 for the vector degree of $(x(t), y(t), z(t))$ (cf. Reduction 4 of the previous section). Mishra [25] considers lexicographical (dictionary) order on the degree vector of a polynomial knot, and finds the minimal vector degree of torus knots and two-bridge knots (see the references in 5.1.5).

Note that a polynomial representation of minimal lexicographical order may not be of minimal degree. For example, Shastri’s representation of the figure-eight knot, with vector degree $(3, 5, 7)$, has lower lexicographical order than McFeron’s representation, which has vector degree $(4, 5, 6)$, but the latter has lower degree.

5.7. **Curvature.** Let $k$ be the total absolute curvature of a polynomial knot of degree $d$. Brutt [5] shows that $k \leq \pi(d - 1)$; she uses Milnor’s result [23, Thm 3.1] that the average number of local extrema of a curve in $\mathbb{R}^n$ is $(1/\pi)\text{vol}(S^{n-1})$.

5.8. **Polygonal knots.** It is interesting to compare the above results for polynomial knots with the corresponding results for polygonal (stick) knots. (A *polygonal knot* is an embedded polygon in $\mathbb{R}^3$; this is the classical type of knot.) Let $e(K)$ be the number of edges in a stick knot $K$. For the following results, see for instance [6, 13, 22, 29]; the methods of proof are in general quite different from those used for polynomial knots. For the structure of the space of polygonal knots, see the next section.

- $e(K) \leq 5$: $K$ is the unknot.
- $e(K) = 6$: $K$ is the trefoil or the unknot.
- $e(K) = 7$: $K$ is the figure-eight or one of the above.
- $e(K) = 8$: $K$ is either a prime knot of six or fewer crossings, a square or granny knot, the knot $8_{19}$ (the $(3,4)$-torus knot) or the knot $8_{20}$.
- All knots with crossing number seven can be constructed with nine edges.
6. Spaces of knots

6.1. Polynomial knots. In this section we describe various results on the topology of the space of polynomial knots. Recall that $\mathcal{K}_d^n$ is the space of polynomial knots $\mathbb{R}^1 \to \mathbb{R}^n$ of the form

$$\begin{align*}
x_1(t) &= a_1^d t^d + a_1^{d-1} t^{d-1} + \ldots + a_1^1 t + a_1^0 \\
x_2(t) &= a_2^d t^d + a_2^{d-1} t^{d-1} + \ldots + a_2^1 t + a_2^0 \\
& \vdots \\
x_n(t) &= a_n^d t^d + a_n^{d-1} t^{d-1} + \ldots + a_n^1 t + a_n^0.
\end{align*}$$

with $a_k^d \neq 0$ for some $k$. Let $\mathcal{V}_d^n \subset \mathcal{K}_d^n$ denote the subspace of knots of the form

$$\begin{align*}
x_1(t) &= t^d + a_1^{d-1} t^{d-1} + \ldots + a_1^1 t \\
x_2(t) &= t^d + a_2^{d-1} t^{d-1} + \ldots + a_2^1 t \\
& \vdots \\
x_n(t) &= t^d + a_n^{d-1} t^{d-1} + \ldots + a_n^1 t
\end{align*}$$

with $d \geq 1$. Both these spaces have a finite number of path components since they are semi-algebraic sets (see for instance [3, Theorem 2.2.1]). Vassiliev [39], by analyzing the discriminant set (the set of singular knots), proves the following results:

- For $n \geq 2$ the space $\mathcal{V}_3^n$ is contractible (see also [11]).
- For $n \geq 2$ the space $\mathcal{V}_4^n$ is homology equivalent to $S^{n-2}$. They are homotopy equivalent if $n \geq 4$.
- For even $d$ there is a product decomposition $\mathcal{V}_d^n = X \times S^1$ for some space $X$.
- For $n \geq 5$ and all $d$ the space $\mathcal{V}_d^n$ is simply-connected.

If $d \leq 4$ these results show that $K \in \mathcal{V}_d^n$ is unknotted. (For a simpler proof see [5.4].)

Let $\mathcal{V}_{d_1,d_2,\ldots,d_n}^n$ denote the space of maps $\mathbb{R}^1 \to \mathbb{R}^n$ of the form

$$\begin{align*}
x_1(t) &= t^{d_1} + a_1^{d_1-1} t^{d_1-1} + \ldots + a_1^1 t \\
x_2(t) &= t^{d_2} + a_2^{d_2-1} t^{d_2-1} + \ldots + a_2^1 t \\
& \vdots \\
x_n(t) &= t^{d_n} + a_n^{d_n-1} t^{d_n-1} + \ldots + a_n^1 t.
\end{align*}$$

Kim, Stemkoski and Yuen (REU 2000) show that this space has a conical structure:
Proposition 15. Each polynomial knot $\kappa$ in $V_{d_1,d_2,...,d_n}$ lies on a curve of polynomial knots which are left-right equivalent to $\kappa$. These curves have endpoint the map $t \mapsto (t^{d_1}, t^{d_2}, \ldots, t^{d_n})$.

Proof. For each $u \geq 0$, let $\kappa_u$ be given by
\[
\begin{align*}
x_1(t) &= t^{d_1} + a_{d_1-1}^{1} ut^{d_1-1} + \cdots + a_{1}^{1} u^{d_1-1}t \\
x_2(t) &= t^{d_2} + a_{d_2-1}^{2} ut^{d_2-1} + \cdots + a_{1}^{2} u^{d_2-1}t \\
&\vdots \\
x_n(t) &= t^{d_n} + a_{d_n-1}^{n} ut^{d_n-1} + \cdots + a_{1}^{n} u^{d_n-1}t.
\end{align*}
\]

For $u > 0$ we have
\[
\kappa_u(t) = (u^{d_1} x_1(t/u), u^{d_2} x_2(t/u), \ldots, u^{d_n} x_n(t/u)).
\]
Thus $\kappa_1 = \kappa$ and $\kappa_u$ is left-right equivalent to $\kappa_1$ for $0 < u \leq 1$. For $u = 0$ we have
\[
\kappa_0(t) = (t^{d_1}, t^{d_2}, \ldots, t^{d_n}).
\]

Note that this proof also shows that every knot in $V_{d_1,d_2,...,d_n}$ lies in the deformation space of the map $t \mapsto (t^{d_1}, t^{d_2}, \ldots, t^{d_n})$.

6.2. Polynomial knots of degree five. Kim, Stemkoski and Yuen \[14\] analyze the structure of the space $K^3_5$ of polynomial knots of degree five in dimension three. In this section we sketch some of their results.

We call a polynomial knot of degree five an “S-trefoil” if it is topologically equivalent (in the sense of §4) to the Shastri trefoil of the introduction. (In fact the Shastri trefoil is right-handed.) A polynomial knot topologically equivalent to the mirror image of the Shastri trefoil will be called an “\(\bar{S}\)-trefoil”.

Let $\mathcal{V}_{3,4,5}$ be polynomial knots of the form
\[
\begin{align*}
x(t) &= t^3 + a_2 t^2 + a_1 t \\
y(t) &= t^4 + b_3 t^3 + b_2 t^2 + b_1 t \\
z(t) &= t^5 + c_4 t^4 + c_3 t^3 + c_2 t^2 + c_1 t
\end{align*}
\]
and let
\[
\mathcal{V}^s_{3,4,5} \subset \mathcal{V}_{3,4,5}
\]
be the knots with $a_2 = b_3 = c_3 = c_4 = 0$.

Lemma 5. The region of knots in $\mathcal{V}^s_{3,4,5}$ whose projection is a trefoil is nonempty and contractible.

This lemma is proved by direct computation. The next lemma follows by an analysis of the knots with these projections.
Lemma 6. The region of $S$-trefoils in $\mathcal{V}_{3,4,5}^s$ is nonempty and contractible. There are no $\bar{S}$-trefoils in $\mathcal{V}_{3,4,5}^s$.

Lemma 7. There is a deformation retraction
\[ \mathcal{V}_{3,4,5}^s \leftarrow \mathcal{V}_{3,4,5}. \]
In fact, this deformation retraction is constructed using left-right equivalences. Thus the knots in $\mathcal{V}_{3,4,5}$ are topologically equivalent to those in $\mathcal{V}_{3,4,5}^s$.

Corollary 1. The region of $S$-trefoils in $\mathcal{V}_{3,4,5}$ is nonempty and contractible. There are no $\bar{S}$-trefoils in $\mathcal{V}_{3,4,5}$.

Let $\tilde{\mathcal{V}}_{d_1,d_2,5}$ be the image of $\mathcal{V}_{d_1,d_2,5}$ under the map $x \rightarrow -x$. Since this map takes a trefoil to its mirror image, the following is also true:

Corollary 2. The region of $\bar{S}$-trefoils in $\tilde{\mathcal{V}}_{3,4,5}$ is nonempty and contractible. There are no $S$-trefoils in $\tilde{\mathcal{V}}_{3,4,5}$.

A principal consequence of the above results is the following proposition and its corollary:

Proposition 16. The region of $S$-trefoils in $\mathcal{K}_5^3$ is nonempty and connected.

Corollary 3. The region of $S$-trefoils in $\mathcal{K}_5^3$ is nonempty and connected.

The following argument proves the above proposition: Let $\kappa \in \mathcal{K}_5^3$, where $\mathcal{K}_5^S = \{ \kappa \in \mathcal{K}_5^3 : \kappa \text{ is an } S\text{-trefoil} \}$. By Reduction 3b (Section 5.2), $\kappa$ is both left and path equivalent to a knot $\tilde{\kappa}$ in $\mathcal{V}_{d_1,d_2,5}$ or $\tilde{\mathcal{V}}_{d_1,d_2,5}$, where $d_1 < d_2 < 5$. Since left-equivalence (and path-equivalence) implies topological equivalence (§4), the path of knots from $\kappa$ to $\tilde{\kappa}$, including the endpoints, lies in $\mathcal{K}_5^S$. In fact $\tilde{\kappa}$ is in $\mathcal{V}_{3,4,5}^s$ or $\tilde{\mathcal{V}}_{3,4,5}$, since otherwise it would be unknotted (§5.4). Since $\tilde{\kappa}$ is an $S$-trefoil, Corollary 1 implies that it must be in $\mathcal{V}_{3,4,5}^s$. If $\kappa' \in \mathcal{K}_5$ is another $S$-trefoil, by the above argument it is also connected by a path of knots in $\mathcal{K}_5^S$ to an $S$-trefoil $\tilde{\kappa}' \in \mathcal{V}_{3,4,5}^s$. Since the region $\mathcal{V}_{3,4,5}^s$ is connected (Corollary 1), the proposition follows. The corollary is a simple consequence.

6.3. Polygonal knots. It is interesting to compare these results with similar ones for the space $\mathcal{P}_k$ of oriented polygonal knots in $\mathbb{R}^3$ with $k$ edges.

- $\mathcal{P}_3$, $\mathcal{P}_4$, and $\mathcal{P}_5$ are connected. (This result is attributed to Kuiper.)
- $\mathcal{P}_6$ has 5 path components: one of unknots, and two each of right-handed and left-handed trefoils [6].
• $P_7$ has 5 path components: unknots, right-handed trefoils, left-handed trefoils, and two of figure-eight knots [6].

• $P_8$ has at least 20 path components [6].

7. Complex polynomial knots

Let $\alpha : \mathbb{C}^1 \to \mathbb{C}^n$ be a polynomial map; in particular $\alpha$ can be the complexification of a real polynomial map. (Maps of this type in the projective case have been studied by Viro [37].) We say that $\alpha$ is an embedding if it is an injection and $\alpha'(t) \neq 0$ for all $t \in \mathbb{C}$; in case we call $\alpha$ a complex polynomial knot. Note that the set of polynomial knots of degree $d$ such that its complexification is an embedding is dense in $K_n^d$. For $n \geq 3$ complex polynomial knots are topologically unknotted. Furthermore the parameter space of such knots of degree $d$ is connected since the subset of maps with singularities has codimension two.

The question of left-right equivalence is complicated. A polynomial embedding $\alpha : \mathbb{C} \to \mathbb{C}^n$ is rectifiable if there is a polynomial automorphism $P$ of $\mathbb{C}^n$ such that $P \circ \alpha = \iota$, where $\iota(t) = (t, 0, \cdots, 0)$. In other words, it is rectifiable if it is left equivalent to a linear map. A basic question is whether every polynomial embedding is rectifiable. This is true for $n = 2$ by the theorem of Abhyankar-Moh, a surprising result since topological knotting is possible in this dimension. It is also true for $n \geq 4$ [8, 12]. For $n = 3$ this is apparently unknown. Shastri’s work initiated in this question; he conjectured that the trefoil knot of the introduction is not rectifiable. For this circle of ideas see for example [38].

Next we describe some connections with commutative algebra and the standard correspondences of algebraic geometry. If $\alpha = (\alpha_1, \ldots, \alpha_n) : \mathbb{C} \to \mathbb{C}^n$ is a polynomial map, let $\hat{\alpha} : \mathbb{C}[x_1, \ldots, x_n] \to \mathbb{C}[t]$ be defined by $\hat{\alpha}(x_i) = \alpha_i(t)$. Conversely, if $\beta : \mathbb{C}[x_1, \ldots, x_n] \to \mathbb{C}[t]$ is a map of rings, then $\hat{\beta} = (\hat{\beta}_1, \ldots, \hat{\beta}_n)$ is defined by $\hat{\beta}_i(t) = \beta(x_i)$. This provides a 1-1 correspondence between polynomial maps $\mathbb{C} \to \mathbb{C}^n$ and maps of rings $\mathbb{C}[x_1, x_2, \ldots, x_n] \to \mathbb{C}[t]$.

**Proposition 17.** The map $\alpha$ is an embedding if and only if the map $\hat{\alpha}$ is surjective.

This proposition makes Gröbner basis methods useful for determining whether a complex map is an embedding. (See also [34].)

**Proof.** Let $V \subset \mathbb{C}^n$ denote the image of $\alpha$, and let $I \subset \mathbb{C}[x_1, \ldots x_n]$ be its ideal. Let $\alpha^* : \mathbb{C}[x_1, \ldots, x_n]/I \to \mathbb{C}[t]$ be the map dual to $\alpha$. Proposition A.2.12 (p. 417) of [15] asserts that $\alpha^*$ is surjective exactly when $V \subset \mathbb{C}^n$ is a closed subvariety and $\alpha : \mathbb{C} \to V$ is an isomorphism.
Since the map $\hat{\alpha}$ factors through $\alpha^*$, the former is surjective exactly when the latter is. Also $\alpha$ is an isomorphism exactly when it is bijective and its derivative $D_t\alpha$ is nonsingular for all $t \in \mathbb{C}$. Finally, $V$ is always a closed subvariety since $\alpha$ is a polynomial map. \hfill $\square$

Here is a simpler proof of the forwards implication: Let $\alpha$ be as above, and $t \in \mathbb{C}$. Since $\hat{\alpha}$ is surjective, there is a polynomial $p \in \mathbb{C}[X_1, \ldots, X_n]$ such that $p(\alpha(t)) = t$. Taking derivative with respect to $t$ gives $(Dp) \circ \alpha' = 1$. Thus $\alpha'(t) \neq 0$ for all $t$. Next we show that $\alpha$ is injective. Let $s, t \in \mathbb{C}$ with the property that $\alpha(s) = \alpha(t)$. Applying $p$ to both sides gives $s = t$.

**Example.** (1) The Shastri trefoil extended to $\mathbb{C}$ is an embedding since $yz - x^3 - 5xy + 2z - 7x$ maps to $t$. Similarly for the complexified figure-eight knot the polynomial $x^2z - xy^2 - 7x^2y - 23x^3 - 3z + 22y + 71x$ maps to $t$. (See [33].)

(2). The cusp $\alpha(t) = (t^2, t^3)$ is not an embedding; the map is defined by $x \mapsto t^2$ and $y \mapsto t^3$ is not surjective. The same is true of the double point $\alpha(t) = (t^2, t^3 - 1)$. Also the proposition is not true over $\mathbb{R}$; for example $\alpha(t) = t^3 + t$ is an embedding but $\hat{\alpha}$ is not surjective. (Jason Starr.)

Papers by Mount Holyoke REU students cited in the references below can be found at www.mtholyoke.edu/acad/math/reu.

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