Optimal Nash Equilibria for Bandwidth Allocation

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Abstract

In bandwidth allocation, competing agents wish to transmit data along paths of links in a network, and each agent’s utility is equal to the minimum bandwidth she receives among all links in her desired path. Recent market mechanisms for this problem have either focused on only Nash welfare \[8\], or ignored strategic behavior \[21\]. We propose a nonlinear variant of the classic trading post mechanism, and show that for almost the entire family of CES welfare functions (which includes maxmin welfare, Nash welfare, and utilitarian welfare), every Nash equilibrium of our mechanism is optimal. Next, we prove that fully strategyproof mechanisms for this problem are impossible in general, with the exception of maxmin welfare. Finally, we show how some of our results can be directly imported to the setting of public decision-making via a reduction due to \[19\].

1 Introduction

Bandwidth allocation is a classic resource allocation problem where competing agents wish to transmit data across paths in a network. Each link has a fixed capacity, and each agent’s utility is equal to the minimum bandwidth she receives among all links in her desired path, i.e., the total amount of flow she routes. The social planner’s goal is to design a mechanism that leads to a “desirable” outcome, for some definition of “desirable”. One of the foundational works in this area is Kelly et al. \[24\], whose pricing scheme results in the allocation maximizing Nash welfare (the product of utilities). We use the standard model proposed by \[24\], where each agent transmits data along a single path that is fixed in advance\(^1\), and there are no monetary payments (i.e., no “real money”).

We study this through the lens of implementation theory. A mechanism is said to Nash-implement a social choice rule \(\Psi\) (for example, \(\Psi\) could denote Nash welfare maximization) if every problem instance has least one Nash equilibrium, and every Nash equilibrium outcome is optimal with respect to \(\Psi\). This is similar to saying that the price of anarchy – the ratio of the optimum and the “worst” Nash equilibrium – of the mechanism is 1.\(^2\) In this paper, we focus on pure Nash equilibria, i.e., we do not consider randomized strategies.

1.1 Trading post

Our main tool will be an augmented version of the trading post mechanism. In the standard trading post mechanism, each agent \(i\) submits a bid \(b_{ij}\) on each good \(j\), with the constraint that \(\sum_j b_{ij} \leq 1\) for each agent \(i\). Let \(x_{ij}\) be the fraction of good \(j\) that agent \(i\) receives; then trading post’s allocation rule is \(x_{ij} = \frac{b_{ij}}{\sum_i b_{ij}}\). In words, each agent receives a share of the good proportional to her share of the aggregate bid on that good. The bids consist of “fake money”: agents have no value for leftover money.

This vanilla version of trading post has limitations. First, when there is a good with supply much larger than other goods\(^3\), trading post may not even have Nash equilibria. A partial solution to this was proposed

\(^1\)This is different from routing games, where each agent strategically chooses which path to use. See Section 1.3 for additional discussion.

\(^2\)The price of anarchy \[25\] concept applies only when \(\Psi\) can be written as the maximization of some cardinal function. This is true when \(\Psi\) denotes Nash welfare maximization, but is not true in general.

\(^3\)Specifically, this occurs when a good has price zero. Having a much larger supply than other goods is sufficient but not necessary for this.
by [8]. For every \( \varepsilon > 0 \), they give a modified version of trading post (parameterized by \( \varepsilon \)) that always has a Nash equilibrium, and where every Nash equilibrium attains at least \( 1 - \varepsilon \) of the maximum possible Nash welfare.\(^4\) Thus their mechanism Nash-implements a \( 1 - \varepsilon \) approximation of Nash welfare; equivalently, the price of anarchy is at most \( \frac{1}{1-\varepsilon} \).

We augment the standard trading post mechanism in two ways. First, we add a special bid of \( \beta \) to handle goods with supply much larger than other goods. This modification allows us to perfectly Nash-implement Nash welfare, i.e., obtain a price of anarchy of 1. Second, we allow for nonlinear bid constraints: instead of \( \sum_j b_{ij} \leq 1 \), we require \( \sum_j f_j(b_{ij}) \leq 1 \), where each \( f_j \) is a nondecreasing function chosen by us. Importantly, all agents are still subject to the same bid constraint. This second modification allows us to Nash-implement many more welfare functions: specifically, almost the entire set of CES welfare functions.

### 1.2 CES welfare functions

For any constant \( \rho \in (-\infty, 0) \cup (0, 1] \), the constant elasticity of substitution (CES) welfare function is defined by

\[
\left( \sum_{i=1}^{n} \frac{u_i^\rho}{\rho} \right)^{1/\rho}
\]

where \( u_i \) is agent \( i \)'s utility, and \( \rho \in \mathbb{R} \) is the elasticity parameter. When \( \rho = 1 \), this is the utilitarian welfare, i.e., sum of utilities. Taking limits as \( \rho \) goes to \( -\infty \) and 0 yields maxmin welfare (the minimum utility) and Nash welfare (the product of utilities), respectively. This class of welfare functions was first discussed by Atkinson [3], although he did not call it by the same name. The closer \( \rho \) gets to \( -\infty \), the more the social planner cares about individual equality (maxmin welfare being the extreme case of this), and the closer \( \rho \) gets to 1, the more the social planner cares about overall societal good (utilitarian welfare being the extreme case of this). The CES welfare function (as opposed to the CES agent utility function) has received almost no attention in the computational economics community, despite being well-studied in the traditional economics literature [3, 5].

These welfare functions also admit an axiomatic characterization:

1. Monotonicity: if one agent’s utility increases while all others are unchanged, the welfare function should prefer the new allocation.
2. Symmetry: the welfare function should treat all agents the same.
3. Continuity: the welfare function should be continuous.
4. Independence of common scale: scaling all agent utilities by the same factor should not affect which allocations have better welfare than others.
5. Independence of unconcerned agents: when comparing the welfare of two allocations, the comparison should not depend on agents who have the same utility in both allocations.
6. The Pigou-Dalton principle: all things being equal, the welfare function should prefer more equitable allocations [12, 34].

Ignoring monotonic transformations of the welfare function (which of course do not affect which allocations have better welfare than others), the set of welfare functions satisfying these axioms is exactly the set of CES welfare functions with \( \rho \in (-\infty, 0) \cup (0, 1] \), including Nash welfare [29].\(^6\) This axiomatic characterization shows that we are not just focusing on an arbitrary class of welfare functions: CES welfare functions are arguably the only reasonable welfare functions.

Recently, [21] showed that for any CES welfare function, nonlinear pricing can be used to obtain market equilibria with optimal CES welfare. However, their equilibrium notion – price curve equilibrium – assumes that agents are not strategic.

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\(^4\)They study Leontief utilities, which is a generalization of bandwidth allocation to the setting where agents may desire goods in different proportions.

\(^5\)Without the Pigou-Dalton principle, \( \rho > 1 \) is also allowed. This can result in unnatural cases where it is optimal to give one agent everything and the rest none, even when this does not maximize the sum of utilities.

\(^6\)This actually does not include maxmin welfare, which obeys weak monotonicity but not strict monotonicity.
1.3 Related work

Trading post and market games. The trading post mechanism – first proposed by Shapley and Shubik [40], and sometimes called the “Shapley-Shubik game”7 – is an example of a strategic market game (for an overview of strategic market games, see [20]). The study of markets has a long history in the economics literature [43, 41, 2, 6]8, but most of this work assumes that agents are price-taking, meaning that they treat the market prices as fixed, and do not behave strategically to affect these prices.9 A market game, however, treats the agents as strategic players who wish to selfishly maximize their own utility. Trading post does not have explicit prices set by a centralized authority: instead, prices arise implicitly from agents’ strategic behavior. In particular, \( \sum_k b_k j \) – the aggregate bid on good \( j \) – functions as the implicit price of good \( j \). Although the trading post mechanism is well-defined for any utility functions, the Nash equilibria are not guaranteed to have many nice properties in general, except in the limit as the number of agents goes to infinity [14] (in this case, the trading post Nash equilibria converge to the price-taking market equilibria).

The paper most relevant to ours is [8], which analyzes the performance of trading post with respect to Nash welfare. They show that for Leontief utilities (which generalize bandwidth allocation), a modified trading post mechanism10 approximates the Nash welfare arbitrarily well. Specifically, for any \( \varepsilon > 0 \), they give a mechanism (parameterized by \( \varepsilon \)) which achieves a \( 1 - \varepsilon \) Nash welfare approximation: there is at least one Nash equilibrium, and every Nash equilibrium has Nash welfare at least \( 1 - \varepsilon \) times the optimal Nash welfare. In the language of implementation theory, this mechanism Nash-implements a \( 1 - \varepsilon \) approximation of Nash welfare. It is worth noting that the authors also consider a broader class of valuations than Leontief, but for this broader class, only a \( 1/2 \) approximation is achieved. Another recent paper gave a strategyproof mechanism achieving a \( 1/e \approx .368 \) approximation of the optimal Nash welfare [11]. Their \( 1/e \) approximation guarantee is weaker than the \( 1/2 \) guarantee of [8] (and their \( 1 - \varepsilon \) guarantee for Leontief), but strategyproofness is sometimes more desirable than Nash implementation. Unfortunately, strategyproofness in the bandwidth allocation setting is generally impossible (Theorem 5.2).

Price-taking markets. The simplest mathematical model of a price-taking market is a Fisher market, due to Irving Fisher [6]. In a Fisher market, there is a set of goods for sale, and each buyer enters the market with a budget she wishes to spend. Each good has a price, and each buyer purchases her favorite bundle among those that are affordable under her budget constraint. Prices are linear, meaning that the cost of a good is proportional to the quantity purchased, and buyers are assumed to have no value for leftover money, so they will always exhaust their entire budgets. A market equilibrium assigns a price to each good so that the demand exactly equals the supply. For a wide class of agent utilities, including bandwidth allocation utilities, an equilibrium is guaranteed to exist [2].11 The seminal work of Eisenberg and Gale showed that for linear prices and a large class of agent utilities (including bandwidth allocation), the market equilibria correspond exactly to the allocations maximizing Nash welfare [15, 16].12 Furthermore, the prices are equal to the optimal Lagrange multipliers in the convex program for maximizing Nash welfare (the Eisenberg-Gale convex program).

Recently, [21] extended this model to allow nonlinear prices, where the cost of a good may be any nondecreasing function of the quantity purchased. These functions are called price curves. They showed that for bandwidth allocation, for any \( \rho \in (-\infty, 1) \), there exist price curves that make every maximum CES welfare allocation a market equilibrium. Furthermore, these prices take a natural form: the cost of purchasing \( x \in \mathbb{R}_{\geq 0} \) of good \( j \) is \( g_j(x) = q_j x^{1-\rho} \), for some nonnegative constants \( q_1 \ldots q_m \). Interestingly, for

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7A plethora of other names have been applied to this mechanism as well, including the proportional share mechanism [17], the Chinese auction [28], and the Tullock contest in rent seeking [9].
8Recently, this topic has garnered significant attention in the computer science community as well (see [42] for an algorithmic exposition).
9There is some work treating price-taking market models as strategic games; see e.g., [1, 7, 8].
10Specifically, they require a minimum allowable bid. This is to handle the issue with vanilla trading post where Nash equilibria may not exist, as described in Section 1.1.
11Specifically, an equilibrium is guaranteed to exist as long agent utilities are continuous, quasi-concave, and non-satiated. The full Arrow-Debreu model also allows for agents to enter to market with goods themselves and not only money; the necessary conditions on utilities are slightly more complex in that setting.
12The conditions for the correspondence between Fisher market equilibria and Nash welfare are slightly stricter than those for market equilibrium existence, but are still quite general. Sufficient criteria were given in [15] and generalized slightly by [22].
\( \rho = 0 \) — which denotes Nash welfare — this function form reduces to a linear price \( q_j \), and we know that linear pricing maximizes Nash welfare. Furthermore, \( q_1 \ldots q_m \) are the optimal Lagrange multipliers in the convex program for maximizing CES welfare.

Trading post with linear bid constraints (\( \sum_j b_{ij} \leq 1 \)) can be thought of as a market game equivalent of the Fisher market model: it implements Nash welfare ([8] proved a \( 1 - \epsilon \) approximation, but we will strengthen this to exact implementation), and the implicit trading post prices (the aggregate bids) are equal to the Fisher market equilibrium prices. Our augmented trading post, with bid constraint \( \sum_j f_j(b_{ij}) \leq 1 \), can be thought of as a market game equivalent of the price curves model. The augmented trading post mechanism we use to implement CES welfare will use \( f_j(b) = b^{1-\rho} \) for each good \( j \), further strengthening this analogy.

Welfare functions. A cardinal welfare function is a function which encapsulates a societal value system by assigning a real number to each possible outcome. This concept was first proposed in 1938 [4], and further developed by [37]. For brevity, these are often referred to as just “welfare functions”. Various welfare functions have been proposed, the most well-studied being utilitarian welfare (the sum of utilities), Nash welfare (the product of utilities) [30, 23], and maxmin welfare (the minimum agent utility) [35, 38, 39]. The class of CES welfare functions was first proposed by [3] and further discussed by [5], although under a different name. See [29] for a modern introduction to welfare functions.

Bandwidth allocation. Bandwidth allocation has been studied both with and without monetary payments; we focus on the later setting, following the model of Kelly et al. [24]. In the bandwidth allocation setting, our augmented trading post mechanism can be thought of as a nonlinear signaling mechanism that provides congestion signals (for example, a packet mark or drop) and a protocol such as TCP [10] can thought of as agent responses. Although it has been known that different marking schemes (such as RED and CHOKe [18, 33]) and versions of TCP lead to different objective functions (eg. [32]), a market-based understanding was developed only for Nash Welfare, starting with the pioneering work of Kelly et al. [24]. Furthermore, the market scheme of Kelly et al. is in the price-taking setting; the only strategic market analysis of bandwidth allocation that we are aware of is the \( 1 - \epsilon \) approximation of Nash welfare due to [8].

Routing games. A related topic is that of routing games. In a routing game, each agent has a fixed source and destination in the network, but chooses which path she uses to get there. Each agent incurs a cost for each link she travels over, and the cost each agent pays is typically nondecreasing function of the total traffic over that link. Each agent wishes to minimize the total cost she incurs by strategically choosing which path to follow. In the standard bandwidth allocation model, each agent has a fixed path, and her goal is to maximize the total amount of flow she is able to send from her source to her destination (which is equal to the minimum bandwidth she receives among links in her path). Instead of choosing which path to follow, each agent’s strategy is how she bids (or more generally, how she interacts with the allocation mechanism). For an overview of routing games, see [36].

Implementation theory. Implementation theory is the study of designing mechanisms whose outcomes coincide with some desirable social choice rule. A social choice rule could be the maximization of a cardinal function, such as a CES welfare function, or something else, such as the set of Pareto optimal allocations. A full survey is outside the scope of this paper; we direct the interested reader to [27].

The “outcome” of a mechanism is not really well-defined; we need to specify a solution concept. The solution concept that we focus on for most of this paper is Nash equilibrium. Possibly the most crucial result regarding implementation in Nash equilibrium (Nash implementation, for short) is due to Maskin [26], who identified a necessary condition for Nash implementation, and a partial converse. He showed that in a very general environment (much broader than bandwidth allocation), any Nash-implementable social choice rule must satisfy what he calls monotonicity. Monotonicity, in combination with a property called no veto power, is sufficient for Nash implementation. In Section 4.3.1, we show that CES welfare functions do not satisfy no veto power, and so cannot be Nash-implemented by Maskin’s approach.
\[ \rho = -\infty \quad \rho \in (-\infty,1) \quad \rho = 1 \]

| Nash-implementable? | \( \rho = -\infty \) | (Thm. 4.1) | \( \rho = 1 \) |
|---------------------|----------------------|-----------|------------------|
| DSE-implementable?   | ✓ (Thm. 5.1)         | ✓         | ✓ (Thm. 5.2)    |

Table 1: A summary of our main implementation results. Here \( \rho = -\infty \) denotes maxmin welfare, \( \rho \in (-\infty,1) \) includes Nash welfare as \( \rho = 0 \), and \( \rho = 1 \) denotes utilitarian welfare. DSE stands for “dominant strategy equilibrium”. “✓” indicates that the type of implementation specified by the row is possible for the social choice rule specified by the column, while “✗” indicates that we give a counterexample, and “?” indicates an open question.

### 1.4 Our results

Our results fall into three categories. Table 1 summarizes the first two categories of results.

**Nash-implementing CES welfare functions.** We view the Nash implementation of CES welfare functions by trading post as our main result (Theorem 4.1). Specifically, for any \( \rho \in (-\infty,1) \), we define an augmented trading post mechanism parametrized by \( \rho \). We denote this mechanism by \( \mathcal{ATP}(\rho) \). We show that \( \mathcal{ATP}(\rho) \) has at least one Nash equilibrium, and that all of its Nash equilibria maximize CES welfare.

Our result improves that of [21] by strengthening their price curve equilibrium (which assumes agents are not strategic) to a strategic equilibrium, and improves that of [8] by generalizing from just Nash welfare to all CES welfare functions (except \( \rho = 1 \)) and strengthening their \( 1 - \varepsilon \) approximation to exact implementation.\(^{13}\) Furthermore, because the price curve equilibria can be computed in polynomial time [21], our Nash equilibria can also be computed in polynomial time.

Our proof makes use of the following results (stated informally):

1. **Theorem 3.1:** Any Nash equilibrium of \( \mathcal{ATP} \) can be converted into an “equivalent” price curve equilibrium.

2. **Theorem 3.2:** Any price curve equilibrium can be converted into an “equivalent” Nash equilibrium of \( \mathcal{ATP} \).

3. **Lemma 4.3** [21]: If \( x \) is a maximum CES welfare allocation, then there exist price curves \( g \) of the form \( g_j(x) = q_j x^{1-\rho} \) such that \( (x,g) \) is a price curve equilibrium.

4. **Lemma 4.4:** If \( (x,g) \) is a price curve equilibrium and each \( g_j \) has the form \( g_j(x) = q_j x^{1-\rho} \), then \( x \) is a maximum CES welfare allocation.

Lemmas 4.3 and 4.4 together imply that \( x \) is a maximum CES welfare allocation if and only if it is a price curve equilibrium with respect to some price curves \( g \) of the form \( g_j(x) = q_j x^{1-\rho} \) (where \( q_1 \ldots q_m \) are nonnegative constants). Theorems 3.1 and 3.2 allow us to convert between price curve equilibria and Nash equilibria of \( \mathcal{ATP} \), and thus enable us to apply Lemmas 4.3 and 4.4 to the Nash equilibria of \( \mathcal{ATP}(\rho) \). Specifically, Theorem 3.1 in combination with Lemma 4.4 will show that any Nash equilibrium of \( \mathcal{ATP}(\rho) \) maximizes CES welfare, and Theorem 3.2 in combination with Lemma 4.3 will show that \( \mathcal{ATP}(\rho) \) has at least one Nash equilibrium.

Section 3 is devoted to proving our reduction between price curve equilibrium and Nash equilibria of trading post: Theorems 3.1 and 3.2. This reduction is the main tool we use to Nash-implement CES welfare maximization. Section 4 then uses this reduction, in combination with Lemmas 4.3 and 4.4, to prove our main theorem: Theorem 4.1.

Our trading post approach breaks down for \( \rho = 1 \), but our results also do not rule out the possibility of Nash implementation for \( \rho = 1 \). We leave this as an open question.

\(^{13}\)It is worth noting that the result of [8] holds for Leontief utilities, a generalization of bandwidth allocation utilities.
Results for dominant strategy implementation. A natural question is whether these results can be improved from Nash implementation to implementation in dominant strategy equilibrium (DSE). Section 5 shows that the answer is mostly no: for any \( \rho \in (-\infty, 1] \), there is no mechanism which DSE-implements CES welfare maximization (Theorem 5.2). We do this by showing that there is no strategyproof mechanism for this problem: the revelation principle tells us that DSE-implementability implies strategyproofness, so impossibility of strategyproofness implies impossibility of DSE implementation.

On the positive side, we show that maxmin welfare (\( \rho = -\infty \)) can in fact be DSE-implemented by a simple revelation mechanism (Theorem 5.1). This is actually stronger than strategyproofness: strategyproofness requires truth-telling to be a DSE, but does not rule out the possibility of additional dominant strategy equilibria that are not optimal. In contrast, DSE implementation requires every DSE to be optimal.

Although every DSE is also a Nash equilibrium, DSE-implementability does not imply Nash-implementability [13]. A DSE implementation requires every DSE to be optimal, but there could be Nash equilibria (which are not dominant strategy equilibria) that are not optimal. This means that Theorem 5.1 does not imply Nash-implementability of maxmin welfare. In fact, our revelation mechanism which DSE-implements maxmin welfare is not a Nash implementation: there exist Nash equilibria which are not optimal (see Section 5.1.1 for an example). Thus our results do not resolve whether maxmin welfare is Nash-implementable: we leave this as another open question.

Results for public decision-making. In private goods resource allocation (which includes bandwidth allocation), each agent’s utility depends only on the set of resources she receives. In public decision-making, the group makes a single decision that affects everyone. Recently, [19] showed that under some assumptions (See Section 6), any public decision-making instance can be reduced to an “equivalent” private goods instance. In Section 6, we use this result to lift some of our (private goods) bandwidth allocation results to the public decision-making setting. Specifically we obtain the following two results for public decision-making: 1) a trading post mechanism that Nash-implements CES welfare for any \( \rho \in (-\infty, 1) \), and 2) a strategyproof mechanism that computes maxmin welfare.\(^{14}\) These results hold when agents have the public decision-making equivalent of bandwidth allocation utilities.

The rest of the paper is structured as follows. Section 2 formally defines the models of bandwidth allocation, price curves, trading post, and implementation theory. Section 3 presents our reduction between price curves and our augmented trading post mechanism. In Section 4, we use this reduction to Nash-implement CES welfare maximization for \( \rho \in (-\infty, 1) \). Section 5 handles DSE-implementation, and Section 6 presents our results for public decision-making.

2 Model

Let \( N = \{1, 2, \ldots, n\} \) be a set of agents, and let \( M = \{1, 2, \ldots, m\} \) be a set of divisible goods, each representing a link in a network. Throughout the paper, we use \( i \) and \( k \) to refer to agents and \( j \) and \( \ell \) to refer to goods. Let \( s_j \) denote the available supply of good \( s_j \). The social planner needs to determine an allocation \( \mathbf{x} \in \mathbb{R}^{n \times m}_{\geq 0} \), where \( x_{ij} \in \mathbb{R}^m_{\geq 0} \) is the bundle of agent \( i \), and \( x_{ij} \in [0, s_j] \) is the quantity of good \( j \) allocated to agent \( i \). An allocation cannot allocate more than the available supply: \( \mathbf{x} \) is a valid allocation if and only if \( \sum_{i} x_{ij} \leq s_j \) for all \( j \).

Agent \( i \)'s utility for a bundle \( x_i \) is denoted by \( u_i(x_i) \in \mathbb{R}_{\geq 0} \). We assume agents have bandwidth allocation utilities, which take the form:

\[
u_i(x_i) = \min_{j \in R_{i}} x_{ij}
\]

where \( R_{i} \) is the set of links that agent \( i \) requires. We assume that \( R_{i} \neq \emptyset \) for all \( i \), i.e., each agent desires at least one good. It will sometimes be useful to define weights \( w_{ij} \) where \( w_{ij} = 1 \) if \( j \in R_{i} \), and 0 otherwise.

Just as agents have utilities over the bundles they receive, we can imagine a social planner who wishes to design a mechanism to maximize some societal welfare function \( \Phi(\mathbf{x}) \). One can think of \( \Phi \) as the social planner’s utility function, which takes as input the agent utilities, instead of a bundle of goods.

\(^{14}\)The second result is weaker than the corresponding private goods analog, since strategyproofness is weaker than full DSE implementation. We are also unable to lift our negative DSE implementation result. These two issues are discussed at the end of Section 6.
The most well-studied welfare functions are the **maxmin welfare** \( \Phi(x) = \min_{i \in N} u_i(x_i) \), the **Nash welfare** \( \Phi(x) = \left( \prod_{i \in N} u_i(x_i) \right)^{1/n} \), and the **utilitarian welfare** \( \Phi(x) = \sum_{i \in N} u_i(x_i) \). These three welfare functions can be generalized by a CES welfare function:

\[
\Phi_\rho(x) = \left( \sum_{i \in N} u_i(x_i)^\rho \right)^{1/\rho}
\]

where \( \rho \) is a constant in \((-\infty, 0) \cup (0, 1]\). The limits as \( \rho \to -\infty \) and \( \rho \to 0 \) yield maxmin welfare and Nash welfare, respectively. Throughout the paper, we will use \( \rho = -\infty \) and \( \rho = 0 \) to denote maxmin welfare and Nash welfare (e.g., “This theorem holds for \( \rho \in (\infty, 1) \)” would include Nash welfare but not maxmin welfare).

For \( \rho \neq 1 \), this function is strictly concave in \( u_1(x_1) \ldots u_n(x_n) \), so every optimal allocation \( x \) has the same utility vector.\(^{15}\)

### 2.1 Price curves

Price curves, introduced by [21], generalize the well-studied Fisher market model. In a Fisher market [6], each good is available for sale and each agent enters the market with a fixed budget she wishes to spend. Each good \( j \) has a price \( p_j \), and the cost of purchasing \( x \in \mathbb{R}_{\geq 0} \) of good \( j \) is \( p_j \cdot x \). Price curves allow the cost of a good to be any nondecreasing function of the quantity purchased. When each price curve \( g_j \) is defined by \( g_j(x) = p_j \cdot x \), this reduces to the Fisher market model. We do not consider strategic behavior in the price curves model; instead, we use this model as a tool for analyzing the Nash equilibria of our augmented trading post.

Like the Fisher market model, the price curves model assumes that agents have no value for leftover money; this will imply that each agent always spends her entire budget. Throughout the paper, we will assume that all agents have the same budget, and normalize all budgets to 1 without loss of generality. In this paper, we will assume that price curves are either strictly increasing, or identically zero (denoted \( g_j(0) = 0 \))\(^{16}\). We also assume that price curves are normalized (\( g_j(0) = 0 \)) and continuous.

Informally, a **price curve equilibrium** assigns a price curve \( g_j : \mathbb{R}_{\geq 0} \to \mathbb{R}_{\geq 0} \) to each good \( j \) so that the agents’ demand equals supply. Formally, for price curves \( g = (g_1 \ldots g_m) \), the cost of a bundle \( x_i \) is

\[
C_g(x_i) = \sum_{j \in M} g_j(x_{ij})
\]

and the **demand set** \( D_i(g) \) is the set of agent \( i \)’s favorite affordable bundles:

\[
D_i(g) = \arg \max_{x_i \in \mathbb{R}_{\geq 0}^n} \ u_i(x_i) \ 	ext{s.t.} \ C_g(x_i) \leq 1
\]

If \( g_j \) is strictly increasing for all \( j \in M \), an agent with bandwidth allocation utility will only purchase goods in her set \( R_i \), and will purchase the exact same quantity of each. When \( g_j \equiv 0 \), any agent can add more of that good at no additional cost: this cannot improve her utility, but it also cannot hurt it. This complication with zero-price goods is discussed more in Section 2.2.1.

A price curve equilibrium (PCE) \((x, g)\) is an allocation \( x \) and price curves \( g \) such that

1. Each agent receives a bundle in her demand set: \( x_i \in D_i(g) \).
2. The market clears: for all \( j \in M \), \( \sum_{i \in N} x_{ij} \leq s_j \), and \( \sum_{i \in N} x_{ij} = s_j \) whenever \( g_j \neq 0 \).\(^{17}\)

The second condition states that the demand never exceeds the supply, and that any good whose supply is not completely exhausted must have a price of zero. This implies that no agent has utility for the leftover goods; otherwise she would simply buy more at no additional cost.

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\(^{15}\)There could be multiple optimal allocations, however. For example, consider one agent who desires two goods with supply \( s_1 \) and \( s_2 > s_1 \). The agent’s optimal utility will be \( s_1 \), but we can either allocate the rest of the second good anyway, or leave some unallocated; the utility is unaffected.

\(^{16}\)This assumption is not made in [21], but is helpful for our purposes.

\(^{17}\)The definition given in [21] omits this condition, because that paper is also interested in equilibria where the supply is not exhausted. Such equilibria cannot be optimal for our purposes, so we disallow them in our definition.
2.2 The trading post mechanism

In the standard trading post mechanism, each agent $i$ places a bid $b_i \in \mathbb{R}_{\geq 0}^m$, where $b_{ij} \in \mathbb{R}_{\geq 0}$ is the amount of bids on good $j$. Each agent $i$ must obey the constraint $\sum_{j \in M} b_{ij} \leq 1$. We use $b \in \mathbb{R}_{\geq 0}^{m \times n}$ to represent the matrix of all bids.

Each agent receives a fraction of the good in proportion to the fraction of the total bid on that good. Formally,

$$x_{ij} = \frac{b_{ij}}{\sum_{k \in N} b_{kj}} \cdot s_j$$

As in the Fisher market model, we assume that agents have no value for leftover money. The aggregate bid on good $j$ is $\sum_{k \in N} b_{ij}$, and can be thought of as the “price” of good $j$: in fact, this analogy will be crucial in our proofs.

We augment the standard trading post mechanism in two ways. The first is necessary in order to ensure the existence of equilibrium when goods have price zero, and the second is to extend this mechanism to implement CES welfare functions beyond Nash welfare.

2.2.1 Handling goods with price zero

In Fisher markets, it is possible for some goods to have price zero. This occurs when that good is not the “rate-limiting factor”, i.e., there is enough of that good for everyone and the supply constraint is not tight. This is a problem for standard trading post: in order to receive any amount of good $j$, agent $i$ must bid $b_{ij} > 0$. But if the supply constraint is not tight in the Fisher market setting, there will be at least one agent receiving more of the good than they need. Such an agent will decrease their bid so that they are only receiving what they need. However, this process will continue infinitely, with agents repeatedly decreasing their bids on this good, but never reaching bid 0.

To handle this, we present the following modified allocation rule. We allow an additional special bid of $\beta$ so that $b_{ij} \in \mathbb{R}_{\geq 0} \cup \{\beta\}$. Conceptually, a bid of 0 indicates that the agent actually does not want the good; bidding $\beta$ indicates that the agent desires the good, but is hoping to get it for free, so to speak. We treat $\beta$ as zero in arithmetic, for example, in the constraint $\sum_{j \in M} b_{ij} \leq 1$. Similarly, we interpret $b_{ij} > 0$ to mean $b_{ij} \notin \{0, \beta\}$.

Our modified allocation rule follows this series of steps:

1. If at least one agent bids a positive (i.e., neither 0 nor $\beta$) amount on good $j$, we follow the standard trading post rule: $x_{ij} = \frac{b_{ij}}{\sum_{k \in N} b_{kj}} \cdot s_j$.

2. However, if all agents bid 0 either or $\beta$ on good $j$, then we allow each agent to have as much good $j$ as they want. Specifically, for any agent $i$ with $b_{ij} = \beta$, let $\ell_i$ be an arbitrary good with $b_{i\ell_i} > 0$. Then we allocate $x_{ij} = x_{i\ell_i}$. For completeness, if there is no good $\ell$ with $b_{i\ell} > 0$ (although this will never happen at equilibrium), we set $x_{ij} = 0$. For agents $i$ bidding 0 on good $j$, we set $x_{ij} = 0$.

3. After following the above steps, for any good $\ell$ where $\sum_{i \in N} x_{i\ell} > s_{\ell}$ (violating the supply constraint), for all $i \in N$ bidding $\beta$ on good $\ell$, we set $x_{ij} = 0$ for all $j \in M$ as a penalty. In words, if so many agents try to get good $j$ for free that the supply constraint is violated, they are all penalized by receiving nothing. Not to worry: this will never happen at equilibrium.

This modification will allow us to simulate a good having price zero.

It is important that we allow separate bids of 0 and $\beta$. Consider a good $j$ where $b_{kj} \in \{0, \beta\}$ for all $k \in N$. Suppose some agent $i$ does not need good $j$, and bidding $\beta$ would cause the supply constraint to be violated and the Step 3 penalty to be invoked. Such an agent can bid 0 on good $j$, which allows her to still spend no money on this good, without the possibility of invoking the Step 3 penalty.

2.2.2 Allowing nonlinear constraints

It will turn out that trading post with the standard constraint of $\sum_{j \in M} b_{ij} \leq 1$ implements Nash welfare. To implement other CES welfare functions, let $f = (f_1 \ldots f_m)$ be nondecreasing functions from $\mathbb{R}_{\geq 0}$ to $\mathbb{R}_{\geq 0}$. 

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We call $f$ the \textit{constraint curves}. Like price curves, we assume that each $f_j$ is continuous and normalized. Unlike price curves, we require each $f_j$ to be strictly increasing: $f_j \equiv 0$ is not allowed. Throughout the paper, we will use $f, f'$ to denote constraint curves and $g, g'$ to denote price curves.

The mechanism $\text{ATP}(f)$ is the same as normal trading post, except each agent’s constraint is now

$$\sum_{j \in M} f_j(b_{ij}) \leq 1$$

We can define $C(f(b_i))$ like we defined $C_g(x_i)$ for price curves $g$ and a bundle $x_i$. Specifically, $C(f(b_i)) = \sum_{j \in M} f_j(b_{ij})$. Thus each agent’s constraint is $C_g(x_i) \leq 1$ in the price curves model, and is $C_f(b_i) \leq 1$ in the trading post model.

The most natural case will be when $f_1 \ldots f_m$ are all the same function. In particular, let $\text{ATP}(\rho)$ be the mechanism where $f(j) = b^{1-\rho}$ for all $j \in M$. In general, we will use $\text{ATP}(f, b)$ to denote the allocation $x$ produced by the mechanism $\text{ATP}(f)$ when agents bid $b$.

### 2.3 Implementation theory

This section covers only the basic concepts of implementation theory; we direct the reader to [27] for a broad overview of this area.

A \textit{social choice rule} $\Psi$ takes as input a utility profile $u = u_1 \ldots u_n$ and returns a set of “optimal” outcomes. In our case, $\Psi$ will represent maximizing a CES welfare function. Define $\Psi(\rho(u))$ by

$$\Psi(\rho(u)) = \arg\max_{x \in \mathbb{R}^n_0} \left( \sum_{i \in N} u_i(x_i)^\rho \right)^{1/\rho}$$

In general, a social choice rule need not express the maximization of any cardinal function.

Let $C$ be a solution concept (e.g., Nash equilibrium), $H$ be a mechanism (sometimes called a “game form”), and $H(u)$ be the induced game for utility profile $u$.\footnote{In general, the difference between a game and a mechanism is that the game definition includes the agent utilities, whereas a mechanism does not.} Let $C(H(u))$ be the set of strategy profiles\footnote{A strategy profile is a list of strategies $S_1 \ldots S_n$, where $S_i$ is the strategy played by agent $i$. For trading post, a strategy is $b_i \in \mathbb{R}^m_0$ and a strategy profile is $b \in \mathbb{R}^m_0^{\times n}$.} satisfying $C$ for that game. For example, if $C$ denotes Nash equilibrium, then $C(H(u))$ would be the set of Nash equilibria of the game $H(u)$. To distinguish between equilibrium strategies (e.g., what agents bid) and equilibrium outcomes (e.g., the resulting allocation), we use $C_X(H(u))$ to denote the set of outcomes resulting from strategy profiles satisfying $C$.

\begin{definition}
A mechanism $H$ implements a social choice rule $\Psi$ if for any utility profile $u$,

$$\emptyset \neq C_X(H(u)) \subseteq \Psi(u)$$

Using the running example of Nash equilibrium, $H$ Nash-implements $\Psi$ if for any utility profile $u$, there is at least one Nash equilibrium, and every Nash equilibrium of $H(u)$ results in an outcome that is optimal under $\Psi$. We denote the set of Nash equilibria of $H(u)$ by $NE(H(u))$, and the set of outcomes resulting from some Nash equilibrium by $NE_X(H(u))$. When only a single utility profile $u$ is under consideration, we will frequently leave $\Psi$ implicit and write $NE(H)$.

It is worth noting that some of the literature refers to Definition 2.1 as \emph{weak implementation}, where full \emph{implementation} requires that $C_X(H(u)) = \Psi(u)$, i.e., every outcome that is optimal under $\Psi$ should be a Nash equilibrium outcome of $H(u)$. We feel that this distinction is not important in our case, since the utility vector in $\Psi(\rho(u))$ is unique (with the exception of $\rho = 1$, which we do not Nash implement anyway): thus allocations $x \in \Psi(\rho(u))$ differ only in what they do with leftover supply, i.e., supply that will not affect anyone’s utility. If one truly cared about this distinction, our augmented trading post mechanism could be further augmented by allowing each agent another special bid that indicated how much of the leftover supply they wanted. Since these special bids would not affect the utilities, the Nash equilibrium utilities would not be affected, and there would be a combination of leftover supply bids that achieves any maximum CES welfare allocation.\footnote{We would also need to include another penalty step if the leftover supply bids lead to a supply constraint being violated.}

We remind the reader of the following standard definitions:
1. Nash equilibrium: a strategy profile where no agent can strictly improve her utility by unilaterally changing her strategy. We consider only pure Nash equilibria, i.e., we do not allow randomized strategies.

2. Dominant strategy: a strategy that is optimal regardless of what other agents do.

3. Dominant strategy equilibrium (DSE): a strategy profile where each agent plays a dominant strategy.

4. Strategyproofness: A revelation mechanism (i.e., a mechanism that asks each agent to report her utility function) is strategyproof if telling the truth is a dominant strategy for every agent.

DSE-implementability implies strategyproofness via the revelation principle\textsuperscript{21}, but it is not generally true that any strategyproof social choice rule is DSE-implementable. Strategyproofness ensures that truth-telling is a dominant strategy equilibrium, but there could also be bad equilibria that are not consistent with \( \Psi \).

By definition, every DSE is also a Nash equilibrium. However, it is not generally true that DSE-implementability implies Nash-implantability [13]. DSE-implementability requires that every DSE of the mechanism be optimal under \( \Psi \), but the mechanism might have additional Nash equilibria (that are not dominant strategy equilibria) that are not consistent with \( \Psi \). We will need to take both this and the previous paragraph into account when studying DSE implementation.

We now move on to our results, beginning with our reduction between price curves and \( \mathcal{ATP} \). This reduction will be the main tool we use to show that \( \mathcal{ATP} \) Nash-implement CES welfare maximization.

### 3 Reduction between price curves and augmented trading post

In this section, we show that any equilibrium of our augmented trading post mechanism can be transformed into a price curve equilibrium, and vice versa. Section 4 will use this result (along with the existence of price curve equilibria maximizing CES welfare, due to [21]) to prove that the \( \mathcal{ATP}(\rho) \) mechanism Nash-implements CES welfare maximization.

#### 3.1 Intuition behind the reduction

First, notice that augmented trading post and price curves have similar-looking constraints: \( \sum_{j \in M} f_j(b_{ij}) \leq 1 \) and \( \sum_{j \in M} g_j(x_{ij}) \leq 1 \). If \( f = g \), these constraints become identical, so \( b_i \) is a feasible bid if and only if \( x_i \) is a feasible purchase subject to price curves \( g \). Suppose that \( (x, g) \) is a price curve equilibrium. For now, assume each \( g_j \) is strictly increasing (the formal proof will also handle the possibility of \( g_j \equiv 0 \)). Let \( x' \) be the outcome of \( \mathcal{ATP}(f) \) when agents bid \( b \) (i.e., \( x' = \mathcal{ATP}(f, b) \)), and suppose that \( b_{ij} = x_{ij} \) for all \( i, j \):

\[
x'_{ij} = \frac{b_{ij}}{\sum_{k \in N} b_{kj}} s_{j} = \frac{x_{ij}}{\sum_{k \in N} x_{kj}} x_{ij}
\]

where the last equality uses the fact that \( \sum_{k \in N} x_{ij} = s_{j} \) when \( (x, g) \) is a PCE and \( g_j \not\equiv 0 \).

Thus the allocation resulting from \( \mathcal{ATP}(f) \) under bids \( b \) is in fact \( x \). Furthermore, since \( (x, g) \) is a price curve equilibrium, each agent exhausts her price curve constraint: \( C_g(x) = 1 \). Since \( f = g = x \), this implies that \( C_f(b_i) = 1 \) for all \( i \in N \). Furthermore, in any price curve equilibrium with all nonzero prices, each agent should be spending exclusively on goods in her set \( R_i \), and purchasing them in equal amounts. Thus in the trading post outcome \( x' \), each agent \( i \) also also spending exclusively on \( j \in R_i \) and acquiring them in equal amounts.

We claim that \( b \) is a Nash equilibrium of \( \mathcal{ATP}(f) \). Suppose the opposite: then there must exist an agent \( i \) and an alternate bid \( b'_i \) such that bidding \( b'_i \) instead of \( b_i \) increases her utility. Thus under \( b'_i \), she receives strictly more of all goods in \( R_i \). But this means that she must be bidding strictly more on each of these goods, which would violate her bid constraint, since \( C_f(b_i) = 1 \) is already tight. Therefore \( b \) must be a Nash equilibrium of \( \mathcal{ATP}(f) \).

The above is an informal proof of one direction of the reduction: transforming price curve equilibria into trading post equilibria. Similarly, if we are given a Nash equilibrium \( b \) of \( \mathcal{ATP}(f) \), we can let \( g = f \)

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\textsuperscript{21}See Chapter 9 of [31] for an introduction to the revelation principle.
(actually, $g$ will be a scaled version of $f$) and $x = \mathcal{ATP}(f, b)$, and use the same intuition to show that $(x, g)$ is a price curve equilibrium.

There are several additional complications. The largest of these is dealing with goods that have price zero in $g$; indeed, this is the issue that prevents vanilla trading post from implementing Nash welfare maximization [8]. Another difficulty is that in trading post, what you bid depends on others’ bids (whereas for price curves, it only depends on $g$). However, due to the nature of bandwidth allocation utilities, agents will always purchase in proportion to their weights $w_{ij}$, and the outcomes at equilibrium will correspond. We will end up with the following two theorems:

**Theorem 3.1.** Let $f$ be constraint curves where each $f_j$ is homogenous of degree $\alpha_j$ for some $\alpha_j > 0$. For bids $b \in NE(\mathcal{ATP}(f))$, define nonnegative constants $a_1 \ldots a_m$ by $a_j = (\sum_{k \in N} b_{kj}/s_j)^{\alpha_j}$. Define price curves $g$ by

$$g_j(x) = \begin{cases} 0 & \text{if } b_{ij} \in \{0, \beta\} \forall i \in N \\ a_j f_j(x) & \text{otherwise} \end{cases}$$

Let $x = \mathcal{ATP}(f, b)$. Then $(x, g)$ is a price curve equilibrium.

**Theorem 3.2.** Let $h$ be any constraint curve. Let $(x, g)$ be a price curve equilibrium, and define $f$ and $b$ by

$$f_j(b) = \begin{cases} h(b) & \text{if } g_j \equiv 0 \\ g_j(b) & \text{otherwise} \end{cases} \quad b_{ij} = \begin{cases} \beta & \text{if } g_j \equiv 0 \text{ and } j \in R_i \\ 0 & \text{if } g_j \equiv 0 \text{ and } j \not\in R_i \\ x_{ij} & \text{otherwise} \end{cases}$$

Then $b$ is a Nash equilibrium of $\mathcal{ATP}(f)$.

Section 3.2 presents some useful necessary and sufficient conditions for price curve equilibrium and trading post Nash equilibrium. Section 3.3 shows that any trading post Nash equilibrium can be transformed into a price curve equilibrium (Theorem 3.1), and Section 3.4 shows that any price curve equilibrium can be transformed into a trading post Nash equilibrium (Theorem 3.2).

### 3.2 Equilibrium conditions for price curves and trading post

Recall that $w_{ij} = 1$ if $j \in R_i$, and 0 otherwise. The following lemma for trading post states a useful necessary and sufficient condition for Nash equilibria of $\mathcal{ATP}(f)$.

**Lemma 3.1.** Let $x = \mathcal{ATP}(f, b)$. Then $b \in NE(\mathcal{ATP}(f))$ if and only if all of the following hold:

1. For all $i \in N$, $x_{ij} = w_{ij} u_i(x_i)$ for all $j \in M$ where there exists $k \in N$ with $b_{kj} > 0$.
2. For all $i \in N$, $C_T(b_i) = 1$.

**Proof.** ($\Rightarrow$) Assume that the two conditions of the lemma are true. First, we claim that $b_{ij} \in \{0, \beta\}$ for all $j \not\in R_i$: agent $i$ only spends money on goods in $R_i$. This is because $w_{ij} u_i(x_i) = 0$ for $j \not\in R_i$, but $b_{ij} > 0$ ensures that $x_{ij} > 0$, so $x_{ij} = w_{ij} u_i(x_i)$ would be impossible. Therefore $C_T(b_i) = \sum_{j \in M} f_j(b_{ij}) = \sum_{j \in R_i} f_j(b_{ij}) = 1$.

Now suppose that $b$ is not a Nash equilibrium: then there exists an agent $i$ and bid $b_i'$ such that $u_i(x_i') > u_i(x_i)$, where $x'$ is the resulting allocation when agent $i$ bids $b_i'$ and every agent $k \neq i$ still bids $b_k$. Condition 1 implies that $x_{ij} = u_i(x_i)$ for all $j \in R_i$ with $b_{ij} > 0$ (since $w_{ij} = 1$ for $j \in R_i$). Since $b_{ij} > 0$ only when $j \in R_i$, we have $x_{ij} = u_i(x_i')$ whenever $b_{ij} > 0$. Thus $u_i(x_i') > x_{ij}$ whenever $b_{ij} > 0$. Since $x_{ij}' > u_i(x_i')$ for all $j \in R_i$, we have $x_{ij}' > x_{ij}$ when $b_{ij} > 0$.

We next claim that $b_{ij}' > b_{ij}$ whenever $b_{ij} > 0$. If there exists $k \neq i$ with $b_{kj} > 0$, then $b_{ij}' > b_{ij}$ is necessary to ensure that $x_{ij}' > x_{ij}$. The only other possibility is that $b_{ij} > 0$, but $b_{ij}' = \beta$, and $b_{kj} \in \{0, \beta\}$ for all $k \neq i$. But in this case, following Step 1 of $\mathcal{ATP}$’s allocation rule, $x_{ij} = s_j$. Then $u_i(x_i) = s_j$. This is the highest utility agent $i$ could ever have, since $u_i(x_i) \leq s_j$ for all $j \in R_i$. This contradicts $u_i(x_i') > u_i(x_i)$. We conclude that $b_{ij}' > b_{ij}$ whenever $b_{ij} > 0$. 

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Therefore, since each \( f_j \) is strictly increasing,

\[
\sum_{j \in M} f_j(b_{ij}) = \sum_{j: b_{ij} > 0} f_j(b_{ij}) < \sum_{j: b_{ij} > 0} f_j(b'_{ij}) \leq \sum_{j \in M} f_j(b'_{ij})
\]

Since \( C_{T}(b_i) = \sum_{j \in M} f_j(b_{ij}) = 1 \) by assumption, we have \( \sum_{j \in M} f_j(b'_{ij}) > 1 \). This means that \( b'_i \) violates the bid constraint, and so is not a valid bid. Therefore \( b \) is a Nash equilibrium.

(\( \Rightarrow \)) Suppose that \( b \) is a Nash equilibrium of \( \mathcal{ATP}(f) \). If \( C_{T}(b_i) > 1 \), \( b_i \) violates the supply constraint, so \( b \) cannot be a Nash equilibrium. If \( C_{T}(b_i) < 1 \), agent \( i \) can improve her utility by bidding slightly more on every good (and thus receiving slightly more of every good). Thus \( C_{T}(b_i) = 1 \) must hold.

Suppose \( x_{i\ell} \neq w_{i\ell} u_i(x_i) \) for some \( \ell \in M \) where there exists \( k \in N \) with \( b_{k\ell} > 0 \). By definition of \( u_i \), \( u_i(x_i)w_{i\ell} > x_{i\ell} \) is impossible, so we must have \( x_{i\ell} > w_{i\ell} u_i(x_i) \). Consider a new bid \( b'_i \) where \( b'_{ij} = b_{ij} \) for all \( j \neq \ell \), \( b_{i\ell}' \) is such that \( x_{i\ell} = w_{i\ell} u_i(x_i) \) (where \( x' \) is the resulting allocation when \( i \) bids \( b'_i \) and each \( k \neq i \) bids \( b_k \)). Thus \( b_{i\ell}' < b_{i\ell} \).

By definition of \( u_i \), we have \( u_i(x'_i) = u_i(x_i) \), but \( C_{T}(b'_i) < C_{T}(b_i) = 1 \), since \( f_j(b'_i) \leq f_j(b_{ij}) \) for all \( j \in M \), and \( f_{i\ell}(b'_i) < f_{i\ell}(b_{i\ell}) \). Thus there must exist a bundle \( b''_i \) with \( b''_{ij} > b'_{ij} \) for all \( j, \) but \( C_{T}(b''_i) \leq 1 \), i.e., \( b''_i \) obeys the bid constraint. Furthermore, let \( x'' \) be the resulting allocation when \( i \) bids \( b''_i \) and each \( k \neq i \) bids \( b_k \); then \( x''_{ij} > x_{ij} \) for all \( j \in M \). Therefore \( u_i(x''_i) > u_i(x'_i) = u_i(x_i) \). Thus this means \( b \) cannot be a Nash equilibrium, which is a contradiction.

Next, we give an analogous lemma for price curve equilibrium. Note that the last condition in Lemma 3.2 is simply one of the conditions in the definition of PCE.

**Lemma 3.2.** An allocation \( x \) and price curves \( g \) are a PCE if and only if all of the following hold:

1. For all \( i \in N \), \( x_{ij} = w_{ij} u_i(x_i) \) whenever \( g_j \neq 0 \).
2. For all \( i \in N \), \( C_{g}(x_i) = 1 \).
3. For all \( j \in M \), \( \sum_{i \in N} x_{ij} \leq s_j \), and \( \sum_{i \in N} x_{ij} = s_j \) whenever \( g_j \neq 0 \).

**Proof.** The third condition is simply one of the two conditions in the definition of PCE. The other requirement for PCE is that \( x_i \in D_i(g) \) for all \( i \in N \), so it suffices to show that \( x_i \in D_i(g) \) if and only if \( C_{g}(x_i) = 1 \) and \( x_{ij} = w_{ij} u_i(x_i) \) whenever \( g_j \neq 0 \).

(\( \Rightarrow \)) Suppose that \( C_{g}(x_i) = 1 \), and \( x_{ij} = w_{ij} u_i(x_i) \) whenever \( g_j \neq 0 \). We first claim that agent \( i \) only spends money on goods in \( R_i \). This is because \( u_i(x_i) = 0 \) for \( j \notin R_i \) (because \( w_{ij} = 0 \) for \( j \notin R_i \)), and spending money implies that \( g_j \neq 0 \) and \( x_{ij} > 0 \), which makes \( x_{ij} = w_{ij} u_i(x_i) \) impossible. Thus \( C_{g}(x_i) = \sum_{j \in R_i} g_j(x_{ij}) = \sum_{j \in R_i: g_j \neq 0} g_j(x_{ij}) \).

Now suppose for sake of contradiction that there exists another bundle \( x'_i \) that is also affordable, and \( u_i(x'_i) > u_i(x_i) \). For all \( j \in R_i \) with \( g_j \neq 0 \), we have \( x_{ij} = w_{ij} u_i(x_i) = u_i(x_i) \) (because \( w_{ij} = 1 \) for \( j \in R_i \), so \( u_i(x'_i) > x_{ij} \) for \( j \in R_i \), \( g_j \neq 0 \). Therefore

\[
C_{g}(x_i) = \sum_{j \in R_i: g_j \neq 0} g_j(x_{ij}) < \sum_{j \in R_i: g_j \neq 0} g_j(x'_{ij}) \leq \sum_{j \in M} g_j(x'_{ij}) = C_{g}(x'_i)
\]

Since, \( C_{g}(x_i) = 1 \), we have \( C_{g}(x'_i) > 1 \). But this implies that \( x'_i \) is not affordable, which is a contradiction. Therefore \( x_i \in D_i(g) \).

(\( \Leftarrow \)) Suppose \( x_i \in D_i(g) \). If \( C_{g}(x_i) > 1 \), \( x_i \) is not affordable, which is impossible. If \( C_{g}(x_i) < 1 \), agent \( i \) can improve her utility by purchasing slightly more of every good. Thus \( \sum_{j \in M} g_j(x_{ij}) = 1 \) must hold.

Suppose \( x_{i\ell} \neq w_{i\ell} u_i(x_i) \) for some \( \ell \in M \) where \( g_j \neq 0 \). By definition, \( u_i(x_i)w_{i\ell} > x_{i\ell} \) is impossible, so we must have \( x_{i\ell} > w_{i\ell} u_i(x_i) \). Consider a bundle \( x'_i \) where \( x'_{ij} = x_{ij} \) for all \( j \neq \ell \), but \( x'_{i\ell} = w_{i\ell} u_i(x_i) \). Then \( u_i(x'_i) = u_i(x_i) \). Furthermore, \( g_j(x'_i) < g_j(x_i) \), so \( C_{g}(x'_i) < C_{g}(x_i) \leq 1 \). Consider another bundle \( x''_i \) where \( x''_{ij} > x'_{ij} \) for all \( j \in M \), but \( C_{g}(x''_i) \leq 1 \): this is always possible because each \( g_j \) is continuous, and \( C_{g}(x''_i) < 1 \). Then \( x''_i \) is affordable, but \( u_i(x''_i) > u_i(x'_i) = u_i(x_i) \). This contradicts \( x_i \in D_i(g) \).

We are now ready to move on to the reduction itself.
3.3 Transforming trading post equilibria into price curve equilibria

This direction of the reduction will require an additional mild condition, involving the following definition.

**Definition 3.1.** We say that a function \( f : \mathbb{R}_{\geq 0} \to \mathbb{R}_{\geq 0} \) is homogenous of degree \( \alpha > 0 \) if for any \( b, c \in \mathbb{R}_{\geq 0} \), \( f(c \cdot b) = c^\alpha f(b) \).

Our main result of this section is the following theorem:

**Theorem 3.1.** Let \( f \) be constraint curves where each \( f_j \) is homogenous of degree \( \alpha_j > 0 \). For bids \( b \in NE(\mathcal{ATP}(f)) \), define nonnegative constants \( a_1 \ldots a_m \) by \( a_j = (\sum_{k \in N} b_{kj}/s_j)^{\alpha_j} \). Define price curves \( g \) by

\[
g_j(x) = \begin{cases} 0 & \text{if } b_{ij} \in \{0, \beta\} \forall i \in N \\ a_jf_j(x) & \text{otherwise} \end{cases}
\]

Let \( x = \mathcal{ATP}(f, b) \). Then \((x, g)\) is a price curve equilibrium.

Before proving Theorem 3.1, we prove several helpful lemmas (Lemmas 3.3–3.5). Throughout Lemmas 3.3–3.5, we assume \( x, g, \) and \( a_1 \ldots a_m \) are defined as in Theorem 3.1. We also assume that \( b \in NE(\mathcal{ATP}(f)) \). Let \( x' \) be the intermediate allocation after Step 2 of \( \mathcal{ATP} \)'s allocation rule.

Our first lemma simply states that all agents end up with positive utility.

**Lemma 3.3.** For all \( i \in N \), \( u_i(x_i) > 0 \).

**Proof.** It is always possible for each agent to bid a nonzero amount on each good and obtain nonzero utility. Thus any Nash equilibrium must give each agent nonzero utility. \( \square \)

The following lemma states that the intermediate allocation after Step 2 is in fact the final allocation.

**Lemma 3.4.** We have \( x = x' \).

**Proof.** We need to show that Step 3 of \( \mathcal{ATP} \)'s allocation rule is not invoked. Suppose it were invoked: then there is an agent \( i \) who ends up with \( x_{ij} = 0 \) for all \( j \), and thus \( u_i(x_i) = 0 \). But this contradicts Lemma 3.3. We conclude that \( x = x' \). \( \square \)

Lemma 3.5 states that under these constraint curves and bids, the bid constraint is equivalent to the price curves constraint.

**Lemma 3.5.** For all \( i \in N \), \( C_g(x_i) = C_f(b_i) \).

**Proof.** By the allocation rule of \( \mathcal{ATP} \), for all \( j \in M \) where there exists \( k \in N \) with \( b_{kj} > 0 \), for all \( i \in N \), we have

\[
x'_{ij} = \frac{b_{ij}}{\sum_{k \in N} b_{kj}} s_j
\]

Lemma 3.4 implies that \( x = x' \). Also, since \( a_j = (\sum_{k \in N} b_{kj}/s_j)^{\alpha_j} \), we have \( s_j/\sum_{k \in N} b_{kj} = a_j^{-1/\alpha_j} \), so

\[
x_{ij} = \frac{b_{ij}}{\sum_{k \in N} b_{kj}} s_j = b_{ij} a_j^{-1/\alpha_j}
\]

whenever there exists \( k \in N \) with \( b_{kj} > 0 \). By the definition of \( g \), \( g_j \neq 0 \) if and only if there exists \( k \in N \) with \( b_{kj} > 0 \) (since constraint curves are assumed to be strictly increasing). Therefore

\[
C_g(x_i) = \sum_{j \in M} g_j(x_{ij})
= \sum_{j: g_j \equiv 0} g_j(x_{ij}) + \sum_{j: g_j \neq 0} g_j(x_{ij})
= \sum_{j: g_j \neq 0} a_j f_j(x_{ij})
\]

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and conclude that \( b \) also gives us \( \sum_{j : g_j \neq 0} f_j(b_{ij}) \) as required. Summing this across agents gives us

\[
\sum_{j : g_j \neq 0} f_j(b_{ij}) = \sum_{j \in M} f_j(b_{ij}) = C_f(b_i)
\]

as required.

We are now ready to prove the main result of this section.

**Theorem 3.1.** Let \( f \) be constraint curves where each \( f_j \) is homogenous of degree \( \alpha_j \) for some \( \alpha_j > 0 \). For bids \( b \in NE(\text{ATP}(f)) \), define nonnegative constants \( a_1 \ldots a_m \) by \( a_j = (\sum_{k \in N} b_{kj}/s_j)^{\alpha_j} \). Define price curves \( g \) by

\[
g_j(x) = \begin{cases} 
  0 & \text{if } g_j \equiv 0 \ \forall i \in N \\
  a_j f_j(x) & \text{otherwise}
\end{cases}
\]

Let \( x = \text{ATP}(f, b) \). Then \((x, g)\) is a price curve equilibrium.

**Proof.** Since \( b \in NE(\text{ATP}(f)) \), we have \( C_f(b_j) = 1 \) for all \( i \in N \) by Lemma 3.1. This implies \( C_g(x_i) = 1 \) by Lemma 3.1 also gives us \( x_{ij} = w_{ij} u_i(x_i) \) whenever there exists \( k \in N \) with \( b_{kj} > 0 \). As before, \( g_j \neq 0 \) if and only if there exists \( k \in N \) with \( b_{kj} > 0 \). Therefore \( x_{ij} = w_{ij} u_i(x_i) \) whenever \( g_j \neq 0 \).

Thus in order to apply Lemma 3.2, we just need to show that \( \sum_{i \in N} x_{ij} \leq s_j \) for all \( j \in M \), and that \( \sum_{i \in N} x_{ij} = s_j \) whenever \( g_j \neq 0 \). Since \( x \) is a valid allocation, we immediately have \( \sum_{i \in N} x_{ij} \leq s_j \) for all \( j \in M \). Consider an arbitrary good \( j \) with \( g_j \neq 0 \): then by the definition of \( g_j \), there exists \( k \in N \) with \( b_{kj} > 0 \). Thus good \( j \) is allocated according to Step 1 of \( \text{ATP} \)'s allocation rule, and we get

\[
x'_{ij} = \frac{b_{ij}}{\sum_{k \in N} b_{kj}} \cdot s_j
\]

Summing this across agents gives us

\[
\sum_{i \in N} x'_{ij} = \sum_{i \in N} \frac{b_{ij}}{\sum_{k \in N} b_{kj}} \cdot s_j = s_j
\]

Thus by Lemma 3.4, \( \sum_{i \in N} x_{ij} = s_j \), as required. Therefore we can apply Lemma 3.2 and conclude that \((x, g)\) is a PCE.

**3.4 Transforming price curve equilibria into trading post equilibria**

Our main result of this section is the following theorem:

**Theorem 3.2.** Let \( h \) be any constraint curve. Let \((x, g)\) be a price curve equilibrium, and define \( f \) and \( b \) by

\[
f_j(b) = \begin{cases} 
  h(b) & \text{if } g_j \equiv 0 \\
  g_j(b) & \text{otherwise}
\end{cases}
\]

\[
b_{ij} = \begin{cases} 
  \beta & \text{if } g_j \equiv 0 \text{ and } j \in R_i \\
  0 & \text{if } g_j \equiv 0 \text{ and } j \not\in R_i \\
  x_{ij} & \text{otherwise}
\end{cases}
\]

Then \( b \) is a Nash equilibrium of \( \text{ATP}(f) \).
The proof of this theorem is slightly more involved than the proof of Theorem 3.1, but the intuition is the same. As before, we prove this theorem via a series of lemmas (Lemmas 3.6–3.11). Let $x' = ATP(f, b)$ be the final allocation resulting from bids $b$, and let $x''$ be the allocation resulting from bids $b$ after Step 2 of $ATP$’s allocation rule. We use these definitions and assume that $(x, g)$ is a PCE for the remainder of Section 3.4.

As in the other direction of the reduction, our first lemma states that all agents end up with positive utility.

**Lemma 3.6.** For all $i \in N$, $u_i(x_i) > 0$.

**Proof.** Regardless of the price curves, it is always possible for each agent to buy a nonzero amount of each good and obtain nonzero utility. Since $x_i \in D_i(g)$, $x_i$ must give agent $i$ nonzero utility.

We next claim that for all goods with nonzero price, the intermediate allocation after Step 2 of $ATP(f)$ is equal to $x$, the allocation from the price curve equilibrium.

**Lemma 3.7.** For all $i \in N$, $x''_{ij} = x_{ij}$ whenever $g_j \neq 0$.

**Proof.** When $g_j \neq 0$, $b_{ij} = x_{ij}$. Since each good is required by at least one agent, and $u_i(x_i) > 0$ for all $i$ by Lemma 3.6, there exists $k \in N$ where $x_{kj} > 0$. Therefore $b_{kj} > 0$, so we follow Step 1 of $ATP$’s allocation rule. Thus for all $i \in N$,

$$x''_{ij} = \frac{b_{ij}}{\sum_{k \in N} b_{kj}} s_j = \frac{x_{ij}}{\sum_{k \in N} x_{kj}} s_j$$

Since $(x, g)$ is a PCE, Lemma 3.2 gives us $\sum_{k \in N} x_{kj} = s_j$ whenever $g_j \neq 0$. Therefore

$$x''_{ij} = \frac{x_{ij}}{\sum_{k \in N} x_{kj}} s_j = x_{ij}$$

as required.

The next lemma states that for all goods where some agent is bidding a positive amount, every agent’s bundle in $x''$ matches up exactly with her weights and her utility for $x_i$.

**Lemma 3.8.** For all $j \in M$ where there exists $k \in N$ with $b_{kj} > 0$, we have $x''_{ij} = w_{ij} u_i(x_i)$.

**Proof.** By the definition of $b$, if $b_{kj} > 0$ for some $k \in N$, then $g_j \neq 0$. Since $(x, g)$ is a price curve equilibrium, we then have $x_{ij} = w_{ij} u_i(x_i)$ by Lemma 3.2. Lemma 3.7 gives us $x''_{ij} = x_{ij}$, so $x''_{ij} = w_{ij} u_i(x_i)$.

Next, we show that each agent’s utility for her bundle after Step 2 is equal to her utility for $x_i$.

**Lemma 3.9.** For all $i \in N$, $u_i(x''_i) = u_i(x_i)$.

**Proof.** It suffices to show that for all $j \in R_i$, $x''_{ij} = u_i(x_i)$.

Case 1: $j \in R_i$ and $g_j \neq 0$. Lemma 3.7 implies that $x''_{ij} = x_{ij}$ in this case. Since $(x, g)$ is a price curve equilibrium, Lemma 3.2 implies that $x_{ij} = w_{ij} u_i(x_i)$. Since $w_{ij} = 1$ for $j \in R_i$, $x''_{ij} = u_i(x_i)$, as required.

Case 2: $j \in R_i$ and $g_j = 0$. If $g_j = 0$, the definition of $b$ implies that all agents bid either $\beta$ or 0 on $j$. Thus will be following Step 2 of $ATP$’s allocation rule. By definition of $b$, $b_{ij} = \beta$ in this case. Following Step 2 of the $ATP$ allocation rule, let $t_j$ be a good with $b_{jt_j} > 0$: then $x''_{ij} = x''_{jt_j}$. Since $b_{jt_j} > 0$ by assumption, we have $x''_{jt_j} = w_{jt_j} u_i(x_i)$ by Lemma 3.8. Furthermore, $b_{jt_j} > 0$ implies $x''_{jt_j} > 0$, so $w_{jt_j} u_i(x_i) > 0$. Thus we must have $w_{jt_j} = 1$, which implies $x''_{ij} = x''_{jt_j} = u_i(x_i)$.

Therefore $x''_{ij} = u_i(x_i)$ for all $j \in R_i$, so $u_i(x''_i) = u_i(x_i)$, as required.

The next lemma states that the intermediate allocation after Step 2 is equal to the final allocation produced by $ATP(f, b)$.

**Lemma 3.10.** We have $x' = x''$.
Proof. We need to show that the penalty in Step 3 is not invoked. Suppose it is invoked: then there is a good \( j \) allocated by Step 2 where \( \sum_{i \in N} x_{ij}^\prime > s_j \). For each \( i \in N \) bidding \( \beta \) on good \( j \), define \( \ell_i \) as usual: then \( x_{ij}^\prime = x_{i\ell_i}^\prime \). Since \( b_{i\ell_i} > 0 \), Lemma 3.8 implies that \( x_{i\ell_i}^\prime = w_{i\ell_i} u_i(x_i) \) whenever \( b_{ij} = \beta \).

\[
 s_j < \sum_{i \in N} x_{ij}^\prime = \sum_{i : b_{ij} = \beta} x_{i\ell_i}^\prime = \sum_{i : b_{ij} = \beta} w_{i\ell_i} u_i(x_i)
\]

By definition of \( b \), we must have \( g_j \equiv 0 \): that is the only situation where agents bid \( \beta \). Furthermore, \( b_{ij} = \beta \) if and only if \( j \in R_i \). Also using \( w_{i\ell_i} \leq 1 \) (in reality, \( w_{i\ell_i} = 1 \) exactly, but we only need the inequality), gives us

\[
 s_j < \sum_{i : j \in R_i} w_{i\ell_i} u_i(x_i) = \sum_{i : j \in R_i} u_i(x_i)
\]

Using \( w_{ij} = 1 \) if and only if \( j \in R_i \) then gives us

\[
 s_j < \sum_{i : j \in R_i} u_i(x_i) = \sum_{i : j \in R_i} w_{ij} u_i(x_i) = \sum_{i \in N} w_{ij} u_i(x_i)
\]

By definition of \( u_i \), \( x_{ij} \geq w_{ij} u_i(x_i) \) for all \( j \in M \). Therefore

\[
 \sum_{i \in N} x_{ij} \geq \sum_{i \in N} w_{ij} u_i(x_i) > s_j
\]

But this implies that \( x \) is not a valid allocation, which is a contradiction. We conclude that Step 3 is not invoked, and thus \( u_i(x'_i) = u_i(x''_i) \), which is equal to \( u_i(x_i) \) by Lemma 3.9.

Next, we show that the price curves constraint and bid constraint coincide.

Lemma 3.11. For all \( i \in N \), \( C_F(b_i) = C_g(x_i) \).

Proof. By the definition of \( b \), \( b_{ij} \in \{ 0, \beta \} \) when \( g_j \equiv 0 \). Therefore:

\[
 C_F(b_i) = \sum_{j \in M} f_j(b_{ij})
 = \sum_{j : g_j \equiv 0} f_j(b_{ij}) + \sum_{j : g_j \equiv 0} f_j(b_{ij})
 = \sum_{j : g_j \equiv 0} f_j(b_{ij})
 = \sum_{j : g_j \equiv 0} g_j(x_{ij})
 = \sum_{j \in M} g_j(x_{ij})
 = C_g(x_i)
\]

We are now ready to prove the main result of this section.

Theorem 3.2. Let \( h \) be any constraint curve. Let \( (x, g) \) be a price curve equilibrium, and define \( f \) and \( b \) by

\[
 f_j(b) = \begin{cases} h(b) & \text{if } g_j \equiv 0 \\ g_j(b) & \text{otherwise} \end{cases} \quad b_{ij} = \begin{cases} \beta & \text{if } g_j \equiv 0 \text{ and } j \in R_i \\ 0 & \text{if } g_j \equiv 0 \text{ and } j \not\in R_i \\ x_{ij} & \text{otherwise} \end{cases}
\]

Then \( b \) is a Nash equilibrium of \( ATP(f) \).
Proof. Suppose \((x, g)\) is a price curve equilibrium. By Lemma 3.2, we have \(C_S(x_i) = 1\) for all \(i \in N\). Thus Lemma 3.11 implies that \(C_T(b_i) = 1\) as well, which satisfies condition 2 of Lemma 3.1.

Lemma 3.8 implies that \(x''_{ij} = w_{ij}u_i(x_{ij})\) whenever there exists \(k \in N\) with \(b_{kj} > 0\). Combining this with Lemmas 3.9 and 3.10 gives us \(x'_{ij} = w_{ij}u_i(x'_{ij})\) whenever there exists \(k \in N\) with \(b_{kj} > 0\). This satisfies condition 1 of Lemma 3.1. Therefore by Lemma 3.1, \(b\) is a Nash equilibrium of \(ATP(f)\).

\[\square\]

## 4 Nash-implementing CES welfare functions with trading post

In this section, we use the reduction between price curves and augmented trading post to show that for any \(\rho \in (-\infty, 1)\), \(ATP(\rho)\) Nash-implements CES welfare maximization. Recall that \(ATP(\rho)\) is the augmented trading post mechanism where \(f_j(b) = b^{1-\rho}\) for all \(j \in M\). Our key tools will be the reduction from Section 3, and a result regarding price curve equilibria from [21] which we describe in Section 4.2. The final result is Theorem 4.1:

**Theorem 4.1.** For any \(\rho \in (-\infty, 1)\), the mechanism \(ATP(\rho)\) Nash-implements the maximum CES welfare social choice rule.

Before we can prove Theorem 4.1, we need a few additional properties. Section 4.1 shows that scaling the constraint curves does not affect the set of Nash equilibrium outcomes. In Section 4.2, we discuss necessary and sufficient conditions for CES optimality based on price curves. Finally, we prove the main theorem in Section 4.3.

### 4.1 Nash equilibria of trading post are invariant to scaling of constraint curves

In order to use the reduction from Section 3, we would like to set \(f_j(b) = g_j(b) = q_jb^{1-\rho}\). However, this would not be a valid mechanism: \(q_1 \ldots q_m\) depend on the utility profile \(u\), and the mechanism cannot depend on \(u\). In this section, we show that scaling by \(q_1 \ldots q_m\) does not affect the Nash equilibrium outcomes of \(ATP\). This will allow us to use the mechanism \(ATP(\rho)\) instead, which does not depend on \(u\).

Recall that for the mechanism \(ATP(f)\), \(NE(ATP(f))\) is the set of Nash equilibrium bids \(b\), and \(NE_X(ATP(f))\) is set of allocations \(x\) resulting from some \(b \in NE(ATP(f))\).

**Lemma 4.1.** Let \(a_1 \ldots a_m\) be positive constants and let \(f\) be constraint curves where each \(f_j\) is homogenous of degree \(\alpha_j > 0\). Define \(f'\) by \(f'_j(b) = a_jf_j(b)\). Then \(NE_X(ATP(f)) \subseteq NE_X(ATP(f'))\).

**Proof.** Let \(x\) be an arbitrary allocation in \(NE_X(ATP(f))\); we will show that \(x \in NE_X(ATP(f'))\). By definition, there exist bids \(b \in NE(ATP(f))\) such that \(x = ATP(f, b)\). Define \(b'\) by \(b'_{ij} = a_j^{-1/\alpha_j}b_{ij}\) when \(b_{ij} > 0\) and \(b'_{ij} = b_{ij}\) otherwise. We first show that \(C_T(b') = C_T(b)\):

\[
\sum_{j \in M} f'_j(b'_{ij}) = \sum_{j \in M} a_jf_j(a_j^{-1/\alpha_j}b_{ij}) = \sum_{j \in M} a_j(a_j^{-1/\alpha_j})^{\alpha_j}f_j(b_{ij}) = \sum_{j \in M} f_j(b_{ij})
\]

Let \(x' = ATP(f', b')\). For any good \(j\) where \(b'_{kj} > 0\) for some \(k \in N\) (and thus also \(b_{kj} > 0\)),

\[
x'_{ij} = \frac{b'_{ij}}{\sum_{k \in N} b'_{kj}} = \frac{a_j^{-1/\alpha_j}b_{ij}}{\sum_{k \in N} a_j^{-1/\alpha_j}b_{kj}} = \frac{b_{ij}}{\sum_{k \in N} b_{kj}} = x_{ij}
\]

Thus for any good \(j\) where \(b'_{ij} > 0\) for some \(k \in N\), we have \(x'_{ij} = x_{ij}\). For any good \(j\) where \(b'_{ij} \in \{0, \beta\}\) for all \(k\), we also have \(b_{kj} \in \{0, \beta\}\) for all \(k\). Thus in both cases we follow Step 2 of \(ATP\)'s allocation rule. Since
\(x_i' = x_{ij}\) for the good \(j\) where \(b'_{kj} > 0\) for some \(k\), Step 2 results in \(x_{ij}' = x_{ij}\) for goods where \(b'_{kj} \in \{0, \beta\}\) for all \(k\). Therefore \(x' = x\).

This implies that \(u_i(x_i) = u_i(x_i')\) for all \(i \in N\). Therefore \(x_{ij} = w_{ij}u_i(x_i)\) if and only if \(x_{ij}' = w_{ij}u_i(x_i')\). Thus the conditions of Lemma 3.1 hold for \(b, f\) if and only if they hold for \(b', f'\). Therefore since \(b \in NE(ATP(f))\), we have \(b' \in NE(ATP(f'))\), and thus \(x = x' \in NE_X(ATP(f')).\) We conclude that \(NE_X(ATP(f)) \subseteq NE_X(ATP(f'))\).

\[\square\]

**Lemma 4.2.** Let \(a_1 \ldots a_m\) be positive scalars and let \(f\) be constraint curves where each \(f_j\) is homogenous of degree \(\alpha_j\). Define \(f'\) by \(f'_j(b) = a_j f_j(b)\). Then \(NE_X(ATP(f')) = NE_X(ATP(f))\).

**Proof.** Lemma 4.1 gives us \(NE_X(ATP(f)) \subseteq NE_X(ATP(f'))\), so it remains only to show that \(NE_X(ATP(f')) \subseteq NE_X(ATP(f))\). We can actually do this by symmetry. Define \(a'_1 \ldots a'_m\) by \(a'_j = 1/a_j\). Then \(a'_1 \ldots a'_m\) are positive scalars such that \(f_j(b) = a'_j f'_j(b)\). Each \(f'_j\) is also homogenous of degree \(\alpha_j\):

\[f'_j(c \cdot b) = a_j f_j(c \cdot b) = c^{\alpha_j} a_j f_j(b) = c^{\alpha_j} f'_j(b)\]

Then we can apply Lemma 4.1 with the roles of \(f'\) and \(f\) swapped to give us \(NE_X(ATP(f')) \subseteq NE_X(ATP(f))\), which completes the proof.

\[\square\]

### 4.2 Price curves and CES optimality

The following lemma was proven by [21], who introduced the notion of price curves.

**Lemma 4.3 ([21]).** For utility profile \(u, \rho \in (-\infty, 1), \text{ and } x \in \Psi_\rho(u),\) there exist price curves \(g\) such that \((x, g)\) is a price curve equilibrium. Furthermore, for each \(j \in M, g_j\) takes the form \(g_j(x) = q_j x^{1-\rho}\) for some nonnegative constants \(q_1 \ldots q_m\).

Lemma 4.3 states that for any maximum CES welfare allocation \(x\), there exist price curves of the form \(g_j(x) = q_j x^{1-\rho}\) where \((x, g)\) is a PCE. Lemma 4.4 states the converse: if \(g\) takes the form \(g_j(x) = q_j x^{1-\rho}\) for nonnegative constants \(q_1 \ldots q_m,\) and \((x, g)\) is a PCE, then \(x\) is a maximum CES welfare allocation. Furthermore, \(q_1 \ldots q_m\) are the Lagrange multipliers of the convex program for maximizing CES welfare, so \(q_1 \ldots q_m\) can be computed in polynomial time.

**Lemma 4.4.** Suppose \(\rho \in (-\infty, 1)\) and that price curves \(g\) take the form \(g_j(x) = q_j x^{1-\rho}\) for each \(j \in M,\) for some nonnegative constants \(q_1 \ldots q_m\). Then if \((x, g)\) is a PCE, \(x \in \Psi_\rho(u)\).

Lemma 4.4 and its proof do not appear in the version of [21] currently on arXiv, and that paper is not yet published. For this reason, we include the proof in this paper (see Appendix A), but this proof will appear in the final version of [21], not in this paper.

### 4.3 Main theorem

We are now finally ready to prove that \(ATP(\rho)\) Nash-implements \(\Psi_\rho\). We will make of the follow results from previous sections:

1. **Theorem 3.1:** Any Nash equilibrium of \(ATP\) can be converted into an “equivalent” price curve equilibrium.

2. **Theorem 3.2:** Any price curve equilibrium can be converted into an “equivalent” Nash equilibrium of \(ATP\).

3. **Lemma 4.2:** The set of Nash equilibrium outcomes of \(ATP\) is invariant to constant scaling of the constraint curves.

4. **Lemma 4.3 [21]:** For any maximum CES welfare allocation \(x\), there exist price curves \(g\) of the form \(g_j(x) = q_j x^{1-\rho}\) such that \((x, g)\) is a PCE.

5. **Lemma 4.4:** If there exist price curves \(g\) of the form \(g_j(x) = q_j x^{1-\rho}\) such that \((x, g)\) is a PCE, then \(x\) is a maximum CES welfare allocation.
Recall that for a utility profile $\mathbf{u}$, the induced game of mechanism $\text{ATP}(\rho)$ is denoted by $\text{ATP}(\rho)(\mathbf{u})$. We left $\mathbf{u}$ implicit when dealing with Nash equilibria in previous sections, but we make it explicit here.

**Theorem 4.1.** For any $\rho \in (-\infty, 1)$, the mechanism $\text{ATP}(\rho)$ Nash-implements the maximum CES welfare social choice rule.

**Proof.** We need to show that for any utility profile $\mathbf{u}$, $\emptyset \neq \text{NE}_X(\text{ATP}(\rho)(\mathbf{u})) \subseteq \Psi_\rho(\mathbf{u})$: in words, for any $\mathbf{u}$, there is at least one Nash equilibrium, and every Nash equilibrium allocation of $\text{ATP}(\rho)(\mathbf{u})$ is a maximum CES welfare allocation with respect to $\rho$ and $\mathbf{u}$.

Pick any $x^* \in \Psi_\rho(\mathbf{u})$, and define $q_1 \ldots q_m$ and $\mathbf{g}$ as in Lemma 4.3. Define $\mathbf{f}$ by $f_j(b) = q_j b^{1-\rho}$ when $q_j \neq 0$, and $f_j(b) = b^{1-\rho}$ when $q_j = 0$. By Lemma 4.2, we have $\text{NE}_X(\text{ATP}(\rho)(\mathbf{u})) = \text{NE}_X(\text{ATP}(\mathbf{f})(\mathbf{u}))$. Thus it suffices to show that $\emptyset \neq \text{NE}_X(\text{ATP}(\mathbf{f})(\mathbf{u})) \subseteq \Psi_\rho(\mathbf{u})$.

We first show that $\text{NE}_X(\text{ATP}(\mathbf{f})(\mathbf{u})) \neq \emptyset$, i.e., $\text{ATP}(\mathbf{f})(\mathbf{u})$ has at least one Nash equilibrium. By Lemma 4.3, $(x^*, \mathbf{g})$ is a PCE. Since $g_j(x) = q_j x^{1-\rho}$ by Lemma 4.3, we have $f_j(b) = g_j(b)$ whenever $q_j \neq 0$ (which is equivalent to $g_j \equiv 0$). When $g_j \equiv 0$, $f_j(b) = b^{1-\rho}$, which is strictly increasing. Thus $\mathbf{f}$ satisfies the requirements of Theorem 3.2. If we define $\mathbf{b}$ as a function of $x^*$ as in Theorem 3.2, then by Theorem 3.2, $\mathbf{b} \in \text{NE}(\text{ATP}(\mathbf{f})(\mathbf{u}))$. Therefore $\text{ATP}(\mathbf{f})(\mathbf{u})$ has at least one Nash equilibrium.

It remains to show that $\text{NE}_X(\text{ATP}(\mathbf{f})(\mathbf{u})) \subseteq \Psi_\rho(\mathbf{u})$, i.e., every Nash equilibrium outcome of $\text{ATP}(\mathbf{f})(\mathbf{u})$ is a maximum CES welfare allocation. Consider an arbitrary $\mathbf{x} \in \text{NE}_X(\text{ATP}(\mathbf{f})(\mathbf{u}))$. Then there exists $\mathbf{b} \in \text{NE}(\text{ATP}(\mathbf{f})(\mathbf{u}))$ such that $\mathbf{x} = \text{ATP}(\mathbf{f}, \mathbf{b})$. Noting that each $f_j$ is homogenous of degree $1 - \rho$, define $\mathbf{g}'$ as a function of $\mathbf{f}$ and $a_1 \ldots a_m$ as a function of $\mathbf{b}$ as in Theorem 3.1:

$$a_j = \left( \frac{\sum_{k \in N} b_{kj}}{s_j} \right)^{1-\rho} \quad \text{and} \quad g'_j(x) = \begin{cases} 0 & \text{if } b_{ij} \in \{0, \beta\} \forall i \in N \\ a_j f_j(x) & \text{otherwise} \end{cases}$$

By Theorem 3.1, $(\mathbf{x}, \mathbf{g}')$ is a PCE. Furthermore, we can write each $g'_j(x) = q'_j x^{1-\rho}$ for nonnegative constants $q'_1 \ldots q'_m$. Therefore by Lemma 4.4, $\mathbf{x} \in \Psi_\rho(\mathbf{u})$.

Thus we have shown that $\mathbf{x} \in \Psi_\rho(\mathbf{u})$ for all $\mathbf{x} \in \text{NE}_X(\text{ATP}(\mathbf{f})(\mathbf{u}))$, so $\text{NE}_X(\text{ATP}(\rho)(\mathbf{u})) \subseteq \Psi_\rho(\mathbf{u})$. Since $\text{NE}_X(\text{ATP}(\rho)(\mathbf{u})) = \text{NE}_X(\text{ATP}(\mathbf{f})(\mathbf{u}))$, we conclude that $\emptyset \neq \text{NE}_X(\text{ATP}(\rho)(\mathbf{u})) \subseteq \Psi_\rho(\mathbf{u})$. \qed

Finally, we note that a Nash equilibrium $\mathbf{b} \in \text{NE}(\text{ATP}(\rho)(\mathbf{u}))$ can be computed in polynomial time. Since $q_1 \ldots q_m$ are the Lagrange multipliers of the convex program for maximizing CES welfare, they can be computed in polynomial time. Then Theorem 3.2 can be applied to obtain $\mathbf{b}' \in \text{NE}(\text{ATP}(\mathbf{f})(\mathbf{u}))$, and finally Lemma 4.2 yields an equivalent $\mathbf{b} \in \text{NE}(\text{ATP}(\rho)(\mathbf{u}))$.

### 4.3.1 Maskin’s approach and no veto power

As discussed in Section 1.3, Maskin proved that in a very general environment, any social choice rule satisfying monotonicity and no veto power is Nash-implementable [26]. We briefly show that bandwidth allocation does not satisfy no veto power for any $\rho \in (-\infty, 1)$, and thus is not conducive to Maskin’s approach.

**Definition 4.1.** A social choice rule $\Psi$ satisfies no veto power if whenever there exists an allocation $\mathbf{x}$ where for all $i \in N$ except at most 1, $u_i(x_i) \geq u_i(y_i)$ for all allocations $\mathbf{y}$, we have $\mathbf{x} \in \Psi(\mathbf{u})$.

In words, if there is a single allocation that everyone (except at most one agent) agrees is their favorite, then that allocation should be optimal under $\Psi$ (the last agent should not be able to “veto” this allocation).

In general, agents will not agree on a favorite allocation: each agent would like to receive all of the resources herself. However, when agents’ $R_i$ sets are pairwise disjoint, it is possible for all agents to agree on a favorite allocation.

Consider an instance with $n$ agents and $n$ goods, each with supply 1. For all $i \in N$, let $R_i = \{i\}$: each agent just desires a single good. Consider the allocation $\mathbf{x}$ where for all $i \in \{1 \ldots n-1\}$, $x_{ii} = 1$, but $x_{nn} = 0$ (and $x_{ij} = 0$ otherwise). For agents $1 \ldots n-1$, this is the most utility they can possibly get, so this satisfies the preconditions of Definition 4.1. However, for any $\rho \in (-\infty, 1)$, $\mathbf{x} \not\in \Psi_\rho(\mathbf{u})$, because the CES welfare can be improved by increasing $x_{nn}$. Specifically, for every $\rho$, the unique optimal CES allocation has $x_{ii} = 1$ for all $i \in \{1 \ldots n\}$. 

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5 Dominant strategy implementation and strategyproofness

In Section 4, we showed that for every \( \rho \in (-\infty, 1) \), CES welfare maximization is Nash-implementable. A natural question to ask is whether this result can be improved to dominant strategy equilibrium implementation (DSE implementation, for short). In this section, we show that \( \rho = -\infty \) (maxmin welfare) is DSE-implementable (Theorem 5.1), but for every \( \rho \in (-\infty, 1] \), \( \Psi_\rho \) is not DSE-implementable (Theorem 5.2).

A related question is whether these welfare functions can be computed in a strategyproof manner. Recall that a mechanism is strategyproof when honestly reporting one’s preferences is always a dominant strategy. As discussed in Section 2.3, DSE-implementability implies strategyproofness by the revelation principle, but the converse is not necessarily true: strategyproofness ensures that truth-telling is a dominant strategy equilibrium, but there could also be bad dominant strategy equilibria. For our positive result (Theorem 5.1), we will show that our mechanism is strategyproof, and also that there are no bad dominant strategy equilibria. For our negative result (Theorem 5.2), we show that the social choice rule in question is not strategyproof, which implies that it is not DSE-implementable.

Furthermore, as also discussed in Section 2.3, DSE-implementability does not imply Nash-implementability: DSE-implementability requires every DSE to be consistent with \( \Psi \), but the mechanism might have additional (non-DSE) Nash equilibria that are not consistent with \( \Psi \). Thus Theorem 5.1 actually does not imply that maxmin welfare is Nash-implementable. Consequently, our results do not resolve whether maxmin welfare is Nash-implementable; we leave this an open question. We do show, however, that our DSE implementation of maxmin welfare is not a Nash implementation (see Section 5.1.1).

We briefly discuss a subtlety relating to uniqueness (and lack thereof). In a sense, all strategyproof mechanisms that implement a social choice rule \( \Psi \) are the same: they all ask agents to report their utility functions \( u_1 \ldots u_n \), then compute an outcome \( x \in \Psi(u) \). However, if \( \Psi(u) \) contains multiple elements (i.e., there are multiple optimal allocations), it matters which is chosen. If leftover supply is allocated arbitrarily, it can be hard to reason about the optimal allocation under different utility profiles. Furthermore, not even the optimal vector of agent utilities is unique for maxmin welfare and utilitarian welfare (although it is for \( \rho \in (-\infty, 1) \)).

Consequently, for our positive result (Theorem 5.1), we will specify our mechanism such that it selects a unique allocation for each utility profile \( u \) (even when they are multiple optimal allocations). For our negative result (Theorem 5.2), we will give an instance where an agent lying makes her utility in every new optimal allocation strictly larger than her utility in every optimal allocation under a truthful report.

5.1 Maxmin welfare is DSE-implementable

**Theorem 5.1.** Maxmin welfare is DSE-implementable.

**Proof.** We begin by writing the following convex program for maxmin welfare:

\[
\begin{aligned}
\max \quad & \gamma \\
\text{s.t.} \quad & u_i = \gamma \quad & \forall i \in N \\
& u_i w_{ij} = x_{ij} \quad & \forall i \in N, j \in M \\
& \sum_{i \in N} x_{ij} \leq s_j \quad & \forall j \in M \\
& x_{ij} \geq 0 \quad & \forall i, j \in M
\end{aligned}
\]

We are using \( u_i \) as a variable in the convex program, but we will reserve \( u_i(x_i) \) for denoting agent \( i \)'s utility for a bundle \( x_i \).

We could have used \( u_i \geq \gamma \) and \( u_i w_{ij} \leq x_{ij} \) for our first two constraints, but requiring \( u_i = \gamma \) and \( u_i w_{ij} = x_{ij} \) ensures a unique solution (and does not affect the optimal value). To see that this program correctly computes an optimal maxmin allocation, let \( x \) be a solution and suppose there exists \( y \) such that \( \min_{k \in N} u_k(y_k) > \min_{k \in N} u_k(x_k) \). Consider the allocation \( x' \) where \( x'_{ij} = w_{ij} \min_{k \in N} u_k(y_k) \) for all \( i, j \).

---

\[22\]This is in the setting where no payments are involved, like this paper. If payments are allowed, these mechanisms can of course differ in what agents are asked to pay.
Then \( u_i(x_i') = \min_{k \in N} u_k(y_k) \) for all \( i \in N \). Furthermore, \( y_{ij} \geq u_{ij} u_i(y_i) \) by definition of \( u_i(y_i) \), and \( u_i(y_i) \geq u_i(x_i') \), so
\[
y_{ij} \geq u_{ij} u_i(y_i) \geq u_{ij} u_i(x_i') = x_{ij}'
\]
Thus since \( y \) is a valid allocation, so is \( x' \). Therefore \( x' \) is feasible for our convex program, and \( \min_{i \in N} u_i(x_i') = \min_{i \in N} u_i(y_i) > \min_{i \in N} u_i(x_i) \), so \( x \) could not have been an optimal solution to our convex program.

Now consider the mechanism where we ask each agent \( i \) to honestly report her set \( R_i \) (which specifies her weights \( w_{i1} \ldots w_{im} \)), and compute an allocation by the convex program above. First, we will show that this mechanism is strategyproof. Second, we will show that there are no “bad” dominant strategy equilibria, i.e., every DSE results in an optimal maxmin allocation.

**Strategyproofness.** Suppose for sake of contradiction that there exists an instance where an agent \( i \) can increase her utility by reporting some \( R_i' \neq R_i \). Let \( x_i \) and \( x_i' \) be agent \( i \)’s bundles when she reports \( R_i \) and \( R_i' \), respectively (assuming other agents make the same reports in both cases). Due to the constraint \( u_i w_{ij} = x_{ij} \), our mechanism will set \( x_{ij} = 0 \) for all \( j \notin R_i \). If \( R_i' \subseteq R_i \), then there exists a \( j \in R_i \) where \( j \notin R_i' \). Thus \( x_{ij}' = 0 \), which implies that \( u_i(x_i') = 0 \), since \( j \in R_i \).

Suppose \( R_i \supsetneq R_i' \), and let \( w_{i1}' \ldots w_{im}' \) be the weights associated with \( R_i' \). In this case, there exists a \( j \in R_i \setminus R_i' \), so \( w_{ij}' = 1 \) and \( w_{ij} = 0 \). We claim that any utility vector \( u_1 \ldots u_n \) that is feasible in the original program (when agent \( i \) reports \( R_i \)) is also feasible in the new program (when agent \( i \) reports \( R_i' \)).

Let \( x_{kj} = u_k w_{kj} \) and \( x_{kj}' = u_k w_{kj}' \). Since \( w_{kj}' \geq w_{kj} \) for all \( k,j \), we have \( x_{kj}' \geq x_{kj} \) for all \( j \in M \), so \( \sum_{k \in N} x_{kj} \leq x_{kj}' \leq s_j \). Thus if \( x' \) and \( u_1 \ldots u_n \) are feasible together, so are \( x \) and \( u_1 \ldots u_n \). This means that the optimal value of the new program is at most the optimal value of the original program: the objective functions are the same, and the feasible set for the new program is a subset of that of the original program. Since each agent’s utility is equal to the objective value of the convex program, this means that agent \( i \)’s utility when she reports \( R_i' \) cannot improve.

Thus we have shown that reporting \( R_i' \neq R_i \) cannot improve agent \( i \)’s utility. We conclude that this mechanism is strategyproof.

**No “bad” dominant strategy equilibria.** We now show that any DSE results in an optimal maxmin allocation. We claim that in any DSE, the vector of utilities is the same as in the truthful DSE, which we know has optimal maxmin welfare. Since this is a revelation mechanism, each agent just reports a utility function \( u_i \). Let \( u = u_1 \ldots u_n \) be the true utility profile, and let \( u' = u_1' \ldots u_n' \) be an arbitrary DSE. Since the mechanism is strategyproof, we know that \( u \) is also a DSE. Thus for every agent \( i \), \( u_i \) and \( u_i' \) are both dominant strategies (it is possible that \( u_i = u_i' \)).

Define another utility profile \( u' \) where each agent \( i \in \{1 \ldots r-1 \} \) reports \( u_i \), and each agent \( i \in \{r \ldots n \} \) reports \( u_i' \). Suppose every agent \( i \neq r \) reports \( u_i' \): then if agent \( r \) truthfully reports \( u_r \), the resulting utility profile is \( u' + 1 \), and if agent \( r \) reports \( u_r' \), the resulting profile is \( u' \). Let \( x' + 1 \) be the resulting allocation in the former case, and \( x' \) be the resulting allocation in the latter case. Since reporting \( u_r \) and reporting \( u_r' \) are both dominant strategies for agent \( r \), she must be indifferent between \( x' + 1 \) and \( x' \) (according to her true utility function, \( u_r \)). Formally, \( u_r(x_i') = u_r(x_i'^{+1}) \).

Next, by the definition of the mechanism, each agent has the same utility for her resulting bundle (according to the utility function she reports). Let \( \gamma' \) be every agent’s utility for the allocation \( x' \), according to her reported utility function \( u_i' : u_i'(x_i'^{+1}) = \gamma' \). Our next claim is that \( u_i(x_i) = \gamma' \), i.e., each agent’s true utility for \( x' \) is \( \gamma' \). As before, we know that \( R_i \subseteq R_i' \) for each agent \( i \): reporting \( R_i' \subseteq R_i \) always results in getting zero utility. This means that \( R_i \subseteq R_i' \). Furthermore, the convex program ensures that each agent \( i \) receives the same amount of every good in her reported set \( R_i' \). Thus we have
\[
u_i(x_i') = \min_{j \in R_i'} x_{ij}' = \min_{j \in R_i} x_{ij} = u_i(x_i')
\]
Therefore \( u_i(x_i') = \gamma' \), i.e., each agent’s true utility for \( x' \) is \( \gamma' \). In particular, \( u_r(x_i') = \gamma' \) and \( u_r(x_i'^{+1}) = \gamma'^{+1} \).

We showed above that \( u_r(x_i') = u_r(x_i'^{+1}) \), so we now have \( \gamma'^{+1} = \gamma' \) for all \( r \). This implies \( \gamma^1 = \gamma^{n+1} \). Thus each agent’s true utility for \( x^1 \) (which is \( \gamma^1 \)) is the same as each agent’s true utility for \( x^{n+1} \) (which is \( \gamma^{n+1} \)). Recall that \( x^{n+1} \) is the resulting allocation when each agent truthfully reports \( u_i \), and that \( x^1 \) is the resulting allocation when each agent reports \( u_i' \). Thus we have shown that each agent’s utility is the same in these two allocations.
Since \( \mathbf{u}' \) was an arbitrary DSE, we have shown that in any DSE, every agent’s utility is the same as in the truthful outcome. Therefore the outcome of any DSE is an optimal maxmin allocation.

### 5.1.1 Our revelation mechanism does not Nash-implement maxmin welfare

In this section, we show that our DSE implementation of maxmin welfare is not a Nash implementation, i.e., there may be Nash equilibria that are not optimal. Consider an instance with \( n \) agents and \( n \) goods, where each agent \( i \)'s true set of desired goods is \( R_i = \{i\} \). Assume each good has supply 1. The allocation with optimal maxmin welfare has \( x_{ii} = 1 \) for all \( i \in N \) and \( x_{ij} = 0 \) for \( j \neq i \), i.e., it gives the entirety of each good to the unique agent who desires it. This results in each agent having utility 1, and thus maxmin welfare of 1.

Now consider the strategy profile where each agent \( i \) reports that she desires every good, i.e., reports \( M \). The resulting allocation will give each agent exactly \( 1/n \) of each good, resulting in each agent’s utility (according to her true utility function) being 1/n. We claim that this is a Nash equilibrium. Suppose an agent \( i \) reports \( R'_i \) instead of \( M \). If \( R'_i = \emptyset \), agent \( i \) receives nothing, so that cannot increase her utility. Thus let \( j \) be any good in \( R'_i \). Since the other \( n-1 \) agents are also reporting that they desire good \( j \), our mechanism would divide \( j \) evenly across the all the agents, resulting in agent \( i \) receiving \( x_{ij} = 1/n \). Since our mechanism gives each agent equal the same quantity of each good in their reported set, agent \( i \) does not receive more than \( 1/n \) of any good, so her utility is at most 1/n. Thus agent \( i \) cannot improve her utility by bidding some \( R'_i \neq M \), so the strategy profile where each agent reports \( M \) is a Nash equilibrium. Furthermore, the maxmin welfare is \( 1/n \), which is actually a factor of \( n \) worse than the optimal maxmin welfare of 1.

### 5.2 For all \( \rho \in (-\infty, 1] \), CES welfare maximization is not DSE-implementable

To show impossibility of DSE implementation, it is sufficient to show impossibility of strategyproofness. Our counterexample will be the following instance with 5 agents and 7 goods, where each row is an agent, each column is a good, and the cell in the \( i \)th row and \( j \)th column gives \( w_{ij} \):

|        | \( g_1 \) | \( g_2 \) | \( g_3 \) | \( g_4 \) | \( g_5 \) | \( g_6 \) | \( g_7 \) |
|--------|----------|----------|----------|----------|----------|----------|----------|
| agent 1| 1        | 1        | 0        | 0        | 0        | 0        | 1        |
| agent 2| 0        | 0        | 1        | 1        | 0        | 0        | 1        |
| agent 3| 0        | 0        | 0        | 0        | 1        | 1        | 1        |
| agent 4| 1        | 0        | 1        | 0        | 1        | 0        | 0        |
| agent 5| 0        | 1        | 0        | 1        | 0        | 1        | 0        |

Let the supply of good 7 be 2, and let all other goods have supply 1. Notice that agents 1, 2, and 3 all conflict on good 7, but otherwise are not in competition. Agents 4 and 5 are not in competition with each other, but each conflicts with each of agents 1, 2, and 3. Let \( \mathbf{u} \) denote this utility profile, and \( \mathbf{u}' \) denote the utility profile where \( R'_4 = \{1, 3, 5, 7\} \) instead of \( R_4 = \{1, 3, 5\} \), and all other utilities are unchanged. We will claim that under utility profile \( \mathbf{u} \), agent 4 can increase her utility by misreporting \( R'_4 \) instead of \( R_4 \).

We will prove this using two main lemmas. Lemma 5.1 states that when agent 4 truthfully reports \( R_4 \), her utility is strictly less than \( 1/2 \). Lemma 5.3 states that when agent 4 lies and reports \( R'_4 \) instead, her utility is at least \( 1/2 \). (Lemma 5.2 is a tool used in the proof of Lemma 5.3). Note that each lemma is referring to agent 4’s true utility function \( u_4 \).

**Lemma 5.1.** For every \( \rho \in (-\infty, 1] \), every \( \mathbf{x} \in \Psi_\rho(\mathbf{u}) \) has \( u_4(x_4) < 1/2 \).

**Proof.** For \( \rho = 0 \), an optimal Nash welfare allocation can be computed explicitly, and any such allocation \( \mathbf{x} \) will have \( u_4(x_4) < 1/2 \). Recall that although the optimal allocation may not be unique, the optimal utility vector is, since Nash welfare is strictly concave.

Let \( A = \{1, 2, 3\} \) and \( B = \{4, 5\} \). For \( \rho \in (-\infty, 0) \cup (0, 1] \), we write the following convex program for maximizing CES welfare:

\[
\max_{u_1, u_2, u_3, u_4, u_5 \in \mathbb{R}_{\geq 0}} (u_4^\rho + u_2^\rho + u_3^\rho + u_4^\rho + u_5^\rho)^{1/\rho}
\]
have evaluated at \( u \). Furthermore, an identical argument applies to the agents in feasible, so we must have \( u \) to the above convex program. Thus this program correctly maximizes CES welfare. This implies that for every \( u \in \Psi_\rho(u) \), there exists an optimal solution to the above convex program \( u^*_1 \ldots u^*_5 \) such that \( u_i(x_i) = u^*_i \) for all \( i \in N \). We proceed by case analysis.

Case 1: \( u^*_1 + u^*_2 + u^*_3 = 2 \). In this case, one of those three agents must have utility at least 2/3. Since agent 4 is in competition with each of those agents for a good with supply 1, this implies that \( u^*_4 = 1/3 < 1/2 \).

Case 2: \( u^*_1 + u^*_2 + u^*_3 \neq 2 \). Because of the convex program’s second constraint, \( u^*_1 + u^*_2 + u^*_3 > 2 \) is not feasible, so we must have \( u^*_1 + u^*_2 + u^*_3 < 2 \). In this case, the convex program above reduces to:

\[
\max_{u_1, u_2, u_3, u_4, u_5 \in \mathbb{R}_{\geq 0}} (u_0^0 + u_3^0 + u_0^1 + u_0^2)^{1/\rho}
\]

s.t. \( u_i + u_k \leq 1 \) \( \forall i \in A, k \in B \)

We next claim that there exists \( u_A^* \) and \( u_B^* \) such that \( u_i^* = u_A^* \) for all \( i \in A \), and \( u_i^* = u_B^* \) for all \( i \in B \). Suppose there exists \( i, i' \in A \) such that \( u_i^* < u_{i'}^* \). Then we could increase \( u_i^* \) while still obeying the constraints of this program, and increase \( \frac{1}{\rho}(u_i^0 + u_{i'}^0 + u_3^0 + u_4^0 + u_5^0) \). That would imply that \( u^*_1 \ldots u^*_5 \) could not be optimal. Furthermore, an identical argument applies to the agents in \( B \).

Therefore there exists \( u_A^* \) and \( u_B^* \) such that \( u_i^* = u_A^* \) for all \( i \in A \), and \( u_i^* = u_B^* \) for all \( i \in B \), so we can further rewrite the convex program as

\[
\max_{u_A, u_B \in \mathbb{R}_{\geq 0}} (3u_A^0 + 2u_B^0)^{1/\rho}
\]

s.t. \( u_A + u_B \leq 1 \)

where \( (u_A^*, u_B^*) \) is the optimal solution of this program. Clearly we must have \( u_B^* = 1 - u_A^* \). Furthermore, we claim that we can change the objective function from \((3u_A^0 + 2u_B^0)^{1/\rho}\) to \( \frac{1}{\rho}(3u_A^0 + 2u_B^0) \), and that this changes the optimal value of the program, but does not change the optimal solution, i.e., the argmax. This because when \( \rho > 0 \), maximizing \((3u_A^0 + 2u_B^0)^{1/\rho}\) is equivalent to maximizing \(3u_A^0 + 2u_B^0\), and when \( \rho < 0 \), maximizing \((3u_A^0 + 2u_B^0)^{1/\rho}\) is equivalent to minimizing \(3u_A^0 + 2u_B^0\), which is equivalent to maximizing \( \frac{1}{\rho}(3u_A^0 + 2u_B^0) \).

Thus our new convex program is

\[
\max_{u_A \in [0, 1]} \frac{1}{\rho}(3u_A^0 + 2(1 - u_A)^\rho)
\]

This is a program we can analyze. For \( \rho = 1 \), the objective function becomes \( u_A + 2 \), so we immediately have \( u_A^* = 1 \) and thus \( u_B^* = 0 \). For \( \rho < 1 \), we take the derivative with respect to \( u_A \) should be 0 when evaluated at \( u_A^* \):

\[
3u_A^\rho - 2(1 - u_A^*)^{\rho-1} = 0
\]

\[
3u_A^\rho = 2(1 - u_A^*)^{\rho-1}
\]

\[
3 - \frac{2}{\rho}u_A^* = 2 - \frac{2}{\rho}u_A^*
\]

\[
(3 - \frac{2}{\rho} + 2\frac{1}{\rho})u_A^* = 2 - \frac{2}{\rho}
\]

\[
u_A^* = \frac{2}{3 - \frac{2}{\rho} + 2\frac{1}{\rho}}
\]

\[
u_A^* = \frac{1}{(3/2)\rho-1} + 1
\]
Since $\rho < 1$, $\rho - 1$ is negative, so $\frac{1}{\rho - 1}$ is negative. Then since $3/2 > 1$, $(3/2)^{1/\rho} < 1$. Altogether, this implies that
\[ u_1^* = u_2^* = u_3^* = u_4^* > 1/2 \quad \text{and} \quad u_4^* = u_5^* = u_6^* < 1/2 \]
as required.

The following lemma is a standard property of strictly concave and differentiable functions: it essentially states that any such function is bounded above by any tangent line. This lemma is sometimes called the “Rooftop Theorem”. To avoid confusion with $u'$ and $x'$, we use $Dh$ to denote the derivative of $h$, instead of $h'$ (this is Euler’s notation for derivatives).

**Lemma 5.2.** Let $h : \mathbb{R} \to \mathbb{R}$ be strictly concave and differentiable. Then for all $a, b \in \mathbb{R}$ where $a \neq b$, $h(a) < h(b) + (Dh(b))(a - b)$.

**Lemma 5.3.** For any $\rho \in (-\infty, 1]$, for any $x' \in \Psi_\rho(u')$, $u_4(x'_4) \geq 1/2$.

**Proof.** It suffices to show that $u'_4(x'_4) \geq 1/2$: since $R_4 \subset R'_4$, we have
\[ u_4(x'_4) = \min_{j \in R_4} x'_{4j} = \min_{j \in R'_4} x'_{4j} = u'_4(x'_4) \]

For $\rho = 1$, the set of optimal allocations can be computed explicitly to see that for all $x' \in \Psi_{\rho=1}(u')$, $u'_i(x'_i) = 1/2$ for all $i \in N$.

We now use this to show that for any $\rho \in (-\infty, 1)$, for any $x' \in \Psi_\rho(u')$, $u'_i(x'_i) = 1/2$ for all $i \in N$. Intuitively, the larger $\rho$ is, the more we care about efficiency and the less we care about fairness. But if the most efficient solution (i.e., the optimal allocation for $\rho = 1$) coincides with the most fair solution (i.e., having all utilities equal), then no matter how much we care about efficiency vs fairness, we should get the same outcome.

Let $x^*$ be any allocation in $\Psi_{\rho=1}(u')$: then $u'_i(x^*_i) = 1/2$ for all $i \in N$. Fix a $\rho \in (-\infty, 1)$ and let $h(a) = \frac{1}{\rho} a^\rho$ if $\rho \neq 0$, and $h(a) = \log(a)$ if $\rho = 1$. In both of these $h$ is strictly concave and differentiable. Consider any allocation $y$ where for some $i \in N$, $u_i(y_i) \neq 1/2$. For brevity, let $u_i^* = u'_i(x^*_i)$ and $u_i^y = u'_i(y_i)$.

For all such $i$, Lemma 5.2 implies that $h(u_i^y) < h(u_i^*) + (Dh(u_i^*)) (u_i^y - u_i^*)$ for $u_i^y \neq u_i^*$. When $u_i^y = 1/2 = u_i^*$, we have $h(u_i^y) = h(u_i^*) + (Dh(u_i^*)) (u_i^y - u_i^*) = 0$. Thus for $\rho \neq 0$, we have
\[
\frac{1}{\rho} \sum_{i \in N} u_i^y \rho \leq \sum_{i \in N} h(u_i^y) = \frac{1}{\rho} \left( \frac{1}{\rho} \sum_{i \in N} (u_i^*)^\rho + (Dh(1/2)) (u_i^y - u_i^*) \right) = \frac{1}{\rho} \sum_{i \in N} (u_i^*)^\rho + (Dh(1/2)) \left( \sum_{i \in N} u_i^y - \sum_{i \in N} u_i^* \right)
\]
Since $x^* \in \Psi_{\rho=1}(u')$, $\sum_{i \in N} u_i^* \geq \sum_{i \in N} u_i^y$. Therefore $(Dh(1/2)) \left( \sum_{i \in N} u_i^y - \sum_{i \in N} u_i^* \right) \leq 0$, so
\[
\frac{1}{\rho} \sum_{i \in N} u_i^y \rho \leq \frac{1}{\rho} \sum_{i \in N} (u_i^*)^\rho
\]
As before, this implies that $(\sum_{i \in N} u_i^y)^{1/\rho} < (\sum_{i \in N} u_i^*)^{1/\rho}$. The analysis for $\rho = 0$ (i.e., Nash welfare) is the same, except we end up with $\sum_{i \in N} \log(u_i^y) < \sum_{i \in N} \log(u_i^*)$ instead, which implies $\prod_{i \in N} u_i^y < \prod_{i \in N} u_i^*$.

Thus for any allocation $y$ where there exists $i \in N$ with $u'_i(y_i) \neq 1/2$, the CES welfare of $x^*$ is better than the CES welfare of $y$. This implies that for any $\rho \in (-\infty, 1]$, any $x' \in \Psi_\rho(u')$ must have $u'_i(x'_i) = 1/2$ for all $i \in N$. \qed
Theorem 5.2. For all $\rho \in (-\infty, 1)$, $\Psi_{\rho}$ is not DSE-implementable.

Proof. Similar to the proof of Theorem 5.1, it suffices to show that $\Psi_{\rho}$ cannot be computed in a strategyproof mechanism. Suppose there were a strategyproof mechanism $H$: then for utility profile $u$, $H$ must return an allocation $x \in \Psi_{\rho}(u)$, and for utility profile $u'$, $H$ must return an allocation $x' \in \Psi_{\rho}(u')$. By Lemmas 5.1 and 5.3, we have $u(x_4) < 1/2$ and $u(x'_4) \geq 1/2$. If agent 4 reports $R'_4 = \{1, 3, 5, 7\}$ instead of $R_4 = \{1, 3, 5\}$, she alters the utility profile from $u$ to $u'$, which resulting in her receiving a bundle with higher utility. Therefore $H$ is not strategyproof.

6 Implications for public decision-making

Recently, [19] investigated connections between private goods and public decision-making. For the allocation of private goods, which includes bandwidth allocation, each agent’s utility depends only on the bundle of resources that she receives. In contrast, in public decision-making, the group chooses a single outcome that affects everyone.

It was shown in [19] that it is possible to reduce any public decision-making instance into a private goods instance by a technique they call pairwise issue expansion. Specifically, for any public decision-making instance, they construct a private goods instance such that the private goods market equilibria in the constructed instance correspond to market equilibria in the original public decision-making instance. The authors focus on the case of linear prices and Nash welfare, but they also prove that for any welfare function $\Phi$, the $\Phi$-maximizing outcomes of the private goods instance and public decision-making instance correspond.

This reduction allows results for private goods instances to be directly imported to the public decision-making setting. For example, if one has an algorithm for computing a $\Phi$-maximizing outcome in private goods instances, this reduction yields an algorithm for computing a $\Phi$-maximizing outcome in public-decisions instances. In this section, we combine their reduction with our results to obtain Nash implementation and strategyproofness results for public decision-making. Specifically, we Nash-implement $\Psi_{\rho}$ for $\rho \in (-\infty, 1)$ (Theorem 6.1), and we give a strategyproof mechanism for maxmin welfare (Theorem 6.2).

6.1 The public decision-making problem

In the public decision-making problem, there is a set of agents $N = \{1 \ldots n\}$ and a set of issues $M = \{1 \ldots m\}$, where each agent $i$ has a preferred alternative $a_{ij}$ for each issue $j$. When issues are binary, as assumed by [19], $a_{ij} \in \{0, 1\}$. Although agents’ alternatives as assumed to be in $\{0, 1\}$, they allow randomized outcomes, so any point in $[0, 1]$ is a valid decision for each issue. An outcome $z \in [0, 1]^m$ selects a point $z^j \in [0, 1]$ for each issue $j$. Let $x_{ij} = 1 - |a_{ij} - z^j|$. Since $|a_{ij} - z^j|$ is the distance between $i$’s preferred alternative and the decision made, $x_{ij}$ is maximized when $a_{ij} = z^j$: $x_{ij}$ represents the extent to which agent $i$ is satisfied on issue $j$. We call $x_i = (x_{i1}, \ldots, x_{im}) \in [0, 1]^m$ agent $i$’s public bundle.

Each agent may also value different combinations of issues differently, denoted by a utility function $u_i(x_i) \in \mathbb{R}_{\geq 0}$. Although [19] handles a wide range of utility functions, we are focused on bandwidth allocation utilities, which can be defined in the same way:

$$u_i(x_i) = \min_{j \in R_i} x_{ij}$$

where $R_i$ is the set of issues that agent $i$ cares about. In the case of public issues, this utility function no longer admits the interpretation of routing flow in a network, but is still well-defined mathematically.

6.1.1 The reduction

The reduction given by [19] takes any public decision-making instance and transforms it into an “equivalent” private goods instance. If the public decision-making instance has $n$ agents and $m$ goods, the private goods instance has $n$ agents and $O(mn^2)$ goods. If the utility functions in the public decision-making instance are in class $H$, then the utility functions in the private goods instance are nested $H$-Leontief. The definition of nested $H$-Leontief is not important for our purposes; what matters for us is that nested bandwidth-allocation-Leontief utilities are also bandwidth allocation utilities.
We will use $T$ to denote the reduction transformation. Specifically, for utility profile $u = u_1 \ldots u_n$ over public issues $M = \{1 \ldots m\}$, let $T(M) = \{1 \ldots O(mn^2)\}$ be a set of private goods, and let $T(u) = T(u_1) \ldots T(u_n)$ denote the corresponding utility profile over $T(M)$. If $u$ are bandwidth allocation utilities, then $T(u)$ are bandwidth allocation utilities as well.

The reduction transformation also gives a way to convert between public decision-making outcomes and private goods outcomes: for a public decision-making outcome $z \in [0, 1]^m$, let $T(z)$ denote the corresponding private goods allocation, and for a private goods allocation $x \in \mathbb{R}_{\geq 0}^{m \times n}$, let $T^{-1}(x)$ denote the corresponding public decision-making outcome.

How the reduction works is not important for us; what is important is the following two lemmas, proven by [19] (with notation adapted to our context). The first states that the reduction preserves each agent’s individual utility, and the second states that the reduction preserves welfare maximization.

**Lemma 6.1** ([19]). For any private goods allocation $x$, $T(u_i)(x_i) = u_i(T^{-1}(x))$.

**Lemma 6.2** ([19]). Let $x$ be a private goods allocation and $\Phi$ be any welfare function. Then $T^{-1}(x)$ is a $\Phi$-maximizing outcome with respect to $u$ if and only if $x$ is a $\Phi$-maximizing allocation with respect to $T(u)$.

### 6.2 Our results for public decision-making

Combining Lemmas 6.1 and 6.2 with our results allows us to prove the following two results for public decision-making with bandwidth allocation utilities.

**Theorem 6.1.** Assume agents have bandwidth allocation utilities. Then for any $\rho \in (-\infty, 1)$, $\Psi_\rho$ is Nash-implementable for public decision-making.

**Proof.** Consider the following mechanism $H(\rho)$ for public decision-making. Each agent $i$ submits a bid $b_{ij}$ for each good $j \in T(M)$. The bids must obey the constraint that $C_f(b_i) \leq 1$, where $f_j(b) = b_j^{1-\rho}$ for all $j \in T(M)$. The mechanism then computes $x = AT(p, b)$, and returns $T^{-1}(x)$ as the public decision-making outcome. We claim that this mechanism Nash-implements $\Psi_\rho$ for public decision-making.

We need to show that for any public decision-making utility profile $u$, $0 \neq NE_X(H(\rho)(u)) \subseteq \Psi_\rho(u)$. We first claim that $b \in NE(H(\rho)(u))$ if and only if $b \in NE(AT(p)(T(u)))$.

($\Rightarrow$) Suppose $b \in NE(H(\rho)(u))$ and suppose for sake of contradiction that there is agent $i$ and bid $b_i'$ where agent $i$ could improve her final utility (in the public decisions instance) by bidding $b_i'$ instead. Let $x = AT(p, b)$ be the allocation resulting from $AT(p)$ when agents bid $b$, and let $x'$ be the allocation resulting from $AT(p)$ when agent $i$ bids $b_i'$ and all other agents bid $b$. Then the resulting public decision is $T^{-1}(x')$. Since we assumed that agent $i$ is improving her final utility, we have $u_i(T^{-1}(x')) > u_i(T^{-1}(x))$. Then Lemma 6.1 implies that $T(u_i)(x'_i) > T(u_i)(x_i)$. But this means that in $AT(p)$, agent $i$ could improve her private goods utility by bidding $b_i'$ instead, which contradicts that $b \in NE(AT(p)(T(u)))$. Therefore $b \in NE(H(\rho)(u))$.

($\Leftarrow$) Suppose that $b \in NE(H(\rho)(u))$ and suppose that there is agent $i$ and bid $b_i'$ where agent $i$ could improve her private goods utility in $AT(p)$ by bidding $b_i'$ instead. Define $x$ and $x'$ as above. Then by assumption, $T(u_i)(x'_i) > T(u_i)(x_i)$. Lemma 6.1 then implies that $u_i(T^{-1}(x')) > u_i(T^{-1}(x))$. But this means that in $H(\rho)$, agent $i$ could improve her final utility (in the public decisions instance) by bidding $b_i'$ instead, which contradicts that $b \in NE(H(\rho)(u))$. Therefore $b \in NE(AT(p)(u))$.

Thus we have shown that $b \in NE(H(\rho)(u))$ if and only if $b \in NE(AT(p)(T(u)))$. Theorem 4.1 implies that $NE(AT(p)(T(u)))$ is nonempty, so $NE(H(\rho)(u))$ is nonempty as well. It remain to show that $NE_X(H(\rho)(u)) \subseteq \Psi_\rho(u)$. Consider an arbitrary $z \in NE_X(H(\rho)(u))$: then there exists $b \in NE(H(\rho)(u))$ such that $z = H(\rho, b)$. Thus $b \in NE(AT(p)(T(u)))$ as well. Let $x = AT(p, b)$: then by Theorem 4.1, $x \in \Psi_\rho(T(u))$. Furthermore, by the definition of $H(\rho)$, $z = T^{-1}(x)$. Thus by Lemma 6.2, $z \in \Psi_\rho(u)$. Therefore $NE_X(H(\rho)(u)) \subseteq \Psi_\rho(u)$, which completes the proof. 

**Theorem 6.2.** Assume agents have bandwidth allocation utilities. Then for public decision-making, maxmin welfare can be computed by a strategyproof mechanism.

**Proof.** The general logic here is the same as for Theorem 6.1. Let $H$ be the mechanism for private goods maxmin welfare from Theorem 5.1. Define a new mechanism $T^{-1}(H)$ as follows. First, each agent reports
her utility function $u_i$. Next, we convert $u = u_1 \ldots u_n$ to $T(u)$ and use $H$ to compute a private goods allocation $x$ for utility profile $T(u)$. The mechanism $T^{-1}(H)$ then returns $T^{-1}(x)$. We claim that $T^{-1}(H)$ is strategyproof.

Suppose the opposite: then there is an instance where some for some agent $i$, she can improve her utility by reporting $R'_i$ instead of $R_i$. Let $x'$ denote the private goods allocation by $T^{-1}(H)$ when agent $i$ reports $R'_i$ instead. Then we have assumed that $u_i(T^{-1}(x')) > u_i(T^{-1}(x))$. Thus Lemma 6.1 implies that $T(u_i(x'_i)) > T(u_i(x_i))$. This means that in the private goods instance, agent $i$ can improve her utility lying about her preferences (she can report whatever set $T(R'_i)$ in the private goods instance corresponds to $R'_i$ in the public goods instance). This contradicts the strategyproofness of the mechanism from Theorem 5.1. Therefore $T^{-1}(H)$ is strategyproof.

Theorem 6.2 is weaker than the private goods analog (Theorem 5.1) in that Theorem 5.1 gives full DSE implementation, whereas Theorem 6.2 only gives strategyproofness. In contrast, Theorem 6.1 gives full Nash implementation. The reason for this is the following. In Theorem 6.1, we were able to prove that the set of Nash equilibrium bids was same for both the public decision-making and private goods mechanisms. For Theorem 6.2, we would like to prove a similar correspondence between dominant strategy equilibria of $H$, the private goods mechanism, and dominant strategy equilibria of $T^{-1}(H)$, the public decision-making mechanism. The issue here is that there are some private goods utility functions $u_i$ that do not correspond to any public decision-making utility function. That is, there may be no public decision-making utility function $u'_i$ where $T(u'_i) = u_i$. This means that the strategy spaces of $H$ and $T^{-1}(H)$ are actually not isomorphic, and proving this correspondence becomes difficult. In contrast, the two strategy spaces in the proof of Theorem 6.1 are the same.

Finally, we note that Theorem 5.2 -- our result that $\Psi_\rho$ is not DSE-implementable for $\rho \in (-\infty, 1]$ -- cannot be directly imported to the public decision-making context. This is because it is not clear whether there exists a public decision-making instance whose private goods analog is equal to the instance we used to prove Theorem 5.2.

7 Conclusion and future work

In this paper, we showed that every CES welfare function except $\rho = 1$ can be Nash-implemented by an augmented trading post mechanism. This strengthened previous results which only handled Nash welfare [8] or assumed agents did not behave strategically [21]. Next, we showed that DSE implementation for this problem is generally impossible, with the exception of maximin welfare, where a simple revelation mechanism does indeed DSE-implement maxmin welfare. Finally, we used a reduction from [19] to lift some of our results to the public decision-making setting.

We were not able to resolve whether utilitarian welfare is Nash-implementable for bandwidth allocation. Our trading post mechanism breaks down in this setting, since $f_j(b) = b^{k_j-1} = 1$ is not a valid constraint curve. Maskin’s monotonicity approach is not viable either, since utilitarian welfare does not satisfy no veto power. We were also not able to resolve whether maximin welfare is Nash-implementable, although we did show that our DSE implementation of maximin welfare is not a Nash implementation. We leave both of these as open questions.

Another interesting direction would be to extend these results to a wider range of utility functions. Our reduction between price curves and trading post means that if price curve equilibria maximizing CES welfare were shown to exist for a wider range of utility functions, it seems likely that our Nash implementation results would carry over as well (depending on the form of the price curves).

It would also be interesting to consider another dimension of strategic behavior by allowing agents to choose which path in the network to use. In this case, we could write each agent’s utility function as $u_i(x_i) = \max_{p \in P_i} \min_{j \in p} x_{ij}$, where $P_i$ is the set of paths from agent $i$’s desired source to desired destination. This is reminiscent of routing games, in that agents are strategically choosing their paths, but still distinct, in that each agent may use the same link in different quantities (i.e., receive different amounts of bandwidth).

More broadly, we feel that trading post is a powerful mechanism that is able to simulate a price-taking market while also handling strategic behavior. We wonder if trading post, or variants thereof, may be useful in designing mechanisms for other resource allocation problems as well.
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References

[1] Bharat Adsul, Ch. Sobhan Babu, Jugal Garg, Ruta Mehta, and Milind Sohoni. Nash equilibria in fisher market. In *Proceedings of the Third International Conference on Algorithmic Game Theory*, SAGT’10, pages 30–41, Berlin, Heidelberg, 2010. Springer-Verlag.

[2] Kenneth J. Arrow and Gerard Debreu. Existence of an equilibrium for a competitive economy. *Econometrica*, 22(3):265–290, 1954.

[3] Anthony B Atkinson. On the measurement of inequality. *Journal of Economic Theory*, 2(3):244 – 263, 1970.

[4] Abram Bergson. A reformulation of certain aspects of welfare economics. *The Quarterly Journal of Economics*, 52(2):310–334, 1938.

[5] Charles Blackorby and David Donaldson. Measures of relative equality and their meaning in terms of social welfare. *Journal of Economic Theory*, 18(1):59 – 80, 1978.

[6] William C. Brainard and Herbert E. Scarf. How to compute equilibrium prices in 1891. *American Journal of Economics and Sociology*, 64(1):57–83, 2005.

[7] Simina Brânzei, Yiling Chen, Xiaotie Deng, Aris Filos-Ratsikas, Soren Kristoffer Stiil Frederiksen, and Jie Zhang. The fisher market game: Equilibrium and welfare. In *Proceedings of the Twenty-Eighth AAAI Conference on Artificial Intelligence*, AAAI’14, pages 587–593. AAAI Press, 2014.

[8] Simina Branzei, Vasilis Gkatzelis, and Ruta Mehta. Nash social welfare approximation for strategic agents. In *Proceedings of the 2017 ACM Conference on Economics and Computation*, EC ’17, pages 611–628, New York, NY, USA, 2017. ACM.

[9] James M Buchanan, Robert D Tollison, and Gordon Tullock. *Toward a theory of the rent-seeking society*. Number 4. Texas A & M Univ Pr, 1980.

[10] V. Cerf and R. Kahn. A protocol for packet network intercommunication. *IEEE Transactions on Communications*, 22(5):637–648, May 1974.

[11] Richard Cole, Vasilis Gkatzelis, and Gagan Goel. Mechanism design for fair division: Allocating divisible items without payments. In *EC 2013*, 2013.

[12] Hugh Dalton. The measurement of the inequality of incomes. *The Economic Journal*, 30(119):348–361, 1920.

[13] Partha Dasgupta, Peter Hammond, and Eric Maskin. The implementation of social choice rules: Some general results on incentive compatibility. *The Review of Economic Studies*, 46(2):185–216, 1979.

[14] Pradeep Dubey and Martin Shubik. A theory of money and financial institutions, 28. the non-cooperative equilibria of a closed trading economy with market supply and bidding strategies. *Journal of Economic Theory*, 17(1):1 – 20, 1978.

[15] E. Eisenberg. Aggregation of Utility Functions. *Management Science*, 7(4):337–350, July 1961.

[16] Edmund Eisenberg and David Gale. Consensus of Subjective Probabilities: The Pari-Mutuel Method. *The Annals of Mathematical Statistics*, 30(1):165–168, March 1959.
[17] M. Feldman, K. Lai, and L. Zhang. The proportional-share allocation market for computational resources. *IEEE Transactions on Parallel and Distributed Systems*, 20(8):1075–1088, Aug 2009.

[18] S. Floyd and V. Jacobson. Random early detection gateways for congestion avoidance. *IEEE/ACM Transactions on Networking*, 1(4):397–413, Aug 1993.

[19] Nikhil Garg, Ashish Goel, and Benjamin Plaut. Markets for public decision-making. In *Proceedings of the 14th Conference on Web and Internet Economics*, WINE ’18, 2018.

[20] Gaël Giraud. Strategic market games: an introduction. *Journal of Mathematical Economics*, 39(5):355 – 375, 2003. Strategic Market Games.

[21] Ashish Goel, Reyna Hulett, and Benjamin Plaut. Markets beyond nash welfare for leontief utilities. *CoRR*, abs/1807.05293, 2018.

[22] Kamal Jain and Vijay V. Vazirani. Eisenberg–Gale markets: Algorithms and game-theoretic properties. *Games and Economic Behavior*, 70(1):84–106, September 2010.

[23] Mamoru Kaneko and Kenjiro Nakamura. The nash social welfare function. *Econometrica*, 47(2):423–35, 1979.

[24] F P Kelly, A K Maulloo, and D K H Tan. Rate control for communication networks: shadow prices, proportional fairness and stability. *Journal of the Operational Research Society*, 49(3):237–252, Mar 1998.

[25] Elias Koutsoupias and Christos Papadimitriou. Worst-case equilibria. In *Proceedings of the 16th Annual Conference on Theoretical Aspects of Computer Science*, STACS’99, pages 404–413, Berlin, Heidelberg, 1999. Springer-Verlag.

[26] Eric Maskin. Nash equilibrium and welfare optimality. *The Review of Economic Studies*, 66(1):23–38, 1999.

[27] Eric Maskin and T. Sjöström. *Implementation Theory*, pages 237–288. North Holland, Amsterdam, 2002.

[28] Alexander Matros. Chinese auctions. In *GAME THEORY AND MANAGEMENT. Collected abstracts of papers presented on the Fifth International Conference Game Theory and Management/Editors Leon A. Petrosyan and Nikolay A. Zenkevich.–SP.: Graduate School of Management SPbU, 2011.–268 p. The collection contains abstracts of papers accepted for the Fifth International*, page 153, 2011.

[29] Hervé Moulin. *Fair Division and Collective Welfare*, chapter 3. MIT Press, January 2003.

[30] John Nash. The Bargaining Problem. *Econometrica*, 18(2):155–162, April 1950.

[31] Noam Nisan, Tim Roughgarden, Éva Tardos, and Vijay V Vazirani, editors. *Algorithmic Game Theory*. Cambridge University Press, 2007.

[32] Jitendra Padhye, Victor Firoiu, Don Towsley, and Jim Kurose. Modeling tcp throughput: A simple model and its empirical validation. *SIGCOMM Comput. Commun. Rev.*, 28(4):303–314, October 1998.

[33] Rong Pan, B. Prabhakar, and K. Psounis. Choke - a stateless active queue management scheme for approximating fair bandwidth allocation. In *Proceedings IEEE INFOCOM 2000. Conference on Computer Communications. Nineteenth Annual Joint Conference of the IEEE Computer and Communications Societies (Cat. No.00CH37064)*, volume 2, pages 942–951 vol.2, 2000.

[34] A.C. Pigou. *Wealth and Welfare*. PCMI collection. Macmillan and Company, limited, 1912.

[35] John Rawls. *A Theory of Justice*. Belknap Press of Harvard University Press, Cambridge, Massachussetts, 1 edition, 1971.

[36] Tim Roughgarden. *Selfish Routing and the Price of Anarchy*. The MIT Press, 2005.
A Proof of Lemma 4.4

In this section, we provide a proof of Lemma 4.4. As noted in Section 4.2, this lemma will appear in the final version of [21], not this paper.

Lemma 4.4. Suppose $\rho \in (-\infty, 1)$ and that price curves $g$ take the form $g_j(x) = q_j x^{1-\rho}$ for each $j \in M$, for some nonnegative constants $q_1 \ldots q_m$. Then if $(x, g)$ is a PCE, $x \in \Psi_\rho(u)$.

Proof. First, for Nash welfare ($\rho = 0$), this is exactly Eisenberg and Gale’s result: the linear-pricing equilibrium allocations are exactly the allocations maximizing Nash welfare [15, 16]. Thus for the rest of this proof, we assume $\rho \neq 0$.

The proof follows a duality argument. We begin by writing the following program to maximize CES welfare:

$$
\max_{x \in \mathbb{R}^n_{\geq 0}, u \in \mathbb{R}^m_{\geq 0}} \left( \sum_{i \in N} u_i^\rho \right)^{1/\rho}
\text{s.t. } x_{ij} \geq w_{ij} u_i \quad \forall i \in N, j \in M \text{ where } w_{ij} \neq 0
\sum_{i \in N} x_{ij} \leq s_j \quad \forall j \in M
$$

The objective $(\sum_{i \in N} u_i^\rho)^{1/\rho}$ is concave for any $\rho \in (-\infty, 0) \cup (0, 1)$, so the resulting program is convex.

Next, we claim that we change the objective function to $\frac{1}{\rho} \sum_{i \in N} u_i^\rho$ without affecting the optimal point, i.e., the arg max (although the optimal value will certainly be affected). When $\rho > 0$, maximizing $(\sum_{i \in N} u_i^\rho)^{1/\rho}$ is equivalent to maximizing $\sum_{i \in N} u_i^\rho$, which is equivalent to maximizing $\frac{1}{\rho} \sum_{i \in N} u_i^\rho$. When $\rho < 0$, maximizing $(\sum_{i \in N} u_i^\rho)^{1/\rho}$ is equivalent to minimizing $\sum_{i \in N} u_i^\rho$, which is equivalent to maximizing $\frac{1}{\rho} \sum_{i \in N} u_i^\rho$. Thus we now focus on the convex program with the same constraints, but with the new objective function $\frac{1}{\rho} \sum_{i \in N} u_i^\rho$.

Next, we write the Lagrangian of this new program. Let $\lambda_{ij}$ be the Lagrange multiplier associated with the constraint $x_{ij} \geq w_{ij} u_i$ and let $q_j$ be the Lagrange multiplier associated with the constraint $\sum_{i \in N} x_{ij} \leq s_j$. We will use $\lambda$ and $q$ to denote the vectors of all such Lagrange multipliers. Then the Lagrangian is given by

$$
L(x, u, \lambda, q) = \frac{1}{\rho} \sum_{i \in N} u_i^\rho - \sum_{i \in N} \sum_{j \in M} \lambda_{ij} (x_{ij} - w_{ij} u_i) - \sum_{j \in M} q_j (\sum_{i \in N} x_{ij} - s_j)
$$

Now suppose $(x^*, g)$ is a PCE, where $g_j(x) = q_j x^{1-\rho}$ for all $j \in M$ for nonnegative constants $q_1^* \ldots q_m^*$. Let $u_i^* = u_i(x_i^*)$ be agent $i$’s utility for $x^*$, and let $\lambda_{ij}^* = q_j^* w_{ij}$. Let $u^* = u_1^* \ldots u_n^*$, let $q^* = q_1^* \ldots q_m^*$, and let $\lambda^*$ represent the vector of all $\lambda_{ij}^*$’s.
The above convex program satisfies strong duality by Slater’s condition. Therefore the Karush-Kuhn-Tucker (KKT) conditions are both necessary and sufficient for optimality. Specifically, if we can show that \((x^*, u^*, \lambda^*, q^*)\) satisfies the KKT conditions, then \((x^*, u^*)\) is optimal for the primal. The KKT conditions are primal feasibility, dual feasibility, complementary slackness, and stationarity. Since \(x^*\) is a valid allocation and \(u_i^* = u_i(x_i^*)\) for all \(i \in N\), primal feasibility of \((x^*, u^*)\) immediately follows. Since \(q_j^* \geq 0\) for all \(j \in M\) by assumption and \(\lambda_{ij}^* \geq 0\) by definition, we have dual feasibility as well.

Complementary slackness requires that for every constraint, either the constraint is tight, or the corresponding dual variable is equal to 0. For the supply constraints, we need to show that for all \(j \in M\), either \(\sum_{j} x_{ij}^* = s_j\), or \(q_j^* = 0\). Since \((x^*, g)\) is a PCE, Lemma 3.2 immediately implies this, since \(q_j^* = 0\) if and only if \(g_j = 0\). For the other constraints, we need to show that for all \(i, j\), either \(\lambda_{ij}^* = 0\) or \(x_{ij}^* = w_{ij}u_i(x_i^*)\). Since \(\lambda_{ij}^* = q_j^*w_{ij}\), we have \(\lambda_{ij}^* = 0\) when \(q_j^* = 0\). When \(q_j^* \neq 0\), we have \(x_{ij}^* = w_{ij}u_i(x_i^*)\) by Lemma 3.2. This satisfies the complementary slackness conditions.

For stationarity, we need to show that the gradient of \(L\) with respect to \(x\) and \(u\) vanishes at \((x^*, u^*, \lambda^*, q^*)\). Specifically, we need to show that \(\frac{\partial L}{\partial u_i}(x^*, u^*, \lambda^*, q^*) = u_i^{\rho-1} - \sum_{j \in M} \lambda_{ij}^* = 0\), and \(\frac{\partial L}{\partial x_{ij}}(x^*, u^*, \lambda^*, q^*) = \frac{\lambda_{ij}^*}{w_{ij}} - q_j^* = 0\). The latter follows immediately from the definition of \(\lambda_{ij}^*\), so it remains to show the former.

Since \((x^*, g)\) is a PCE, Lemma 3.2 implies that for all \(i \in N\),

\[
C_g(x_i^*) = \sum_{j \in M} g_j(x_{ij}^*) = \sum_{j \in M} q_j^*x_{ij}^ {1-\rho} = 1
\]

Furthermore, by Lemma 3.2, we have \(x_{ij}^* = w_{ij}u_i^*\) whenever \(g_j \neq 0\), which in this case is equivalent to \(q_j^* \neq 0\). Therefore

\[
\sum_{j \in M} q_j^*x_{ij}^ {1-\rho} = \sum_{j:q_j^* \neq 0} q_j^*(w_{ij}u_i^*)^{1-\rho} = u_i^{1-\rho} \sum_{j:q_j^* \neq 0} q_j^*w_{ij}^{1-\rho} = u_i^{1-\rho} \sum_{j:q_j^* \neq 0} q_j^*w_{ij}
\]

where the last equality is because \(w_{ij} \in \{0, 1\}\). Therefore we have

\[
u_i^{1-\rho} \sum_{j:q_j^* \neq 0} q_j^*w_{ij} = 1
\]

\[
\sum_{j:q_j^* \neq 0} q_j^*w_{ij} = u_i^{\rho-1}
\]

\[
u_i^{\rho-1} - \sum_{j:q_j^* \neq 0} q_j^*w_{ij} = 0
\]

\[
u_i^{\rho-1} - \sum_{j \in M} q_j^*w_{ij} = 0
\]

\[
u_i^{\rho-1} - \sum_{j \in M} \lambda_{ij}^* = 0
\]

Therefore \(\frac{\partial L}{\partial u_i}(x^*, u^*, \lambda^*, q^*) = u_i^{\rho-1} - \sum_{j \in M} \lambda_{ij}^*\) is indeed 0. Thus the KKT conditions are satisfied. Therefore \((x^*, u^*, \lambda^*, q^*)\) is optimal for \(L\), which implies that \((x^*, u^*)\) is optimal for the primal: in other words, \(x^*\) is a maximum CES welfare allocation. \(\Box\)