ON THE CONNECTION FORMULAS OF THE THIRD PAINLEVÉ TRANSCENDENT

R. Wong
Liu Bie Ju Centre for Mathematical Sciences
City University of Hong Kong, Tat Chee Avenue
Kowloon, Hong Kong, China

H. Y. Zhang
Department of Mathematics
City University of Hong Kong, Tat Chee Avenue
Kowloon, Hong Kong, China

Dedicated to Professor Ta-Tsien Li on the occasion of his 70th birthday

Abstract. We consider the connection problem for the sine-Gordon PIII equation

\[ u_{xx} + \frac{1}{x} u_x + \sin u = 0, \]

which is the most commonly studied case among all general third Painlevé transcendents. The connection formulas are derived by the method of “uniform asymptotics” proposed by Bassom, Clarkson, Law and McLeod (Arch. Rat. Mech. Anal., 1998).

1. Introduction. In this paper we consider the nonlinear differential equation

\[ u_{xx} + \frac{1}{x} u_x + \sin u = 0, \]  

(1.1)

which is known as the sine-Gordon PIII equation. It is deduced from the third Painlevé equation (PIII)

\[ w'' = \left(\frac{w'}{w}\right)^2 - \frac{w'}{x} + \frac{1}{x} (\alpha w^2 + \beta) + \gamma w^3 + \frac{\delta}{w} \]  

(1.2)

via the transformation

\[ w(x) = \exp \left(\frac{iu(x)}{2}\right), \]  

(1.3)

when \( \alpha = \beta = 0 \) and \( \gamma = -\delta = -\frac{1}{4} \). In view of its simplicity and its similarity to the sine-Gordon equation, (1.1) is the most commonly studied equation among all general PIII equations.

It has been known for some time that for any given real numbers \( r \) and \( s \), there exists a solution of equation (1.1) which satisfies

\[ u(x) = r \ln x + s + O(x^2) \text{ as } x \to 0; \]

(1.4)

2000 Mathematics Subject Classification. Primary: 33E17, 34M55; Secondary: 35Q53.

Key words and phrases. Connection formulas, uniform asymptotics, the third Painlevé transcendent (PIII), sine-Gordon PIII equation, parabolic cylinder functions, Bessel functions.
It is also known that with the given asymptotic behavior at the origin, the solution of equation (1.1) behaves like

$$u(x) = 2\pi k + \alpha x^{-1/2} \cos \left( x - \frac{\alpha^2}{16} \ln x + \beta \right) + o(x^{-1/2}) \text{ as } x \to \infty,$$  

(1.5)

where $\alpha$, $\beta$ are real constants and $k$ is an integer; see [4, p.470]. Furthermore, the asymptotic formula (1.5) can be differentiated with respect to $x$.

The relationship between the parameters $\alpha$, $\beta$ in (1.5) and the parameters $r$, $s$ in (1.4) is provided by the connection formulas

$$\alpha^2 = -\frac{16}{\pi} \ln \left( \frac{\text{Re} \, A}{\pi} \right),$$  

(1.6)

$$\beta = -\frac{\alpha^2}{8} \ln 2 - \arg \Gamma \left( -\frac{i \alpha^2}{16} \right) - \arg p + \frac{3\pi}{4} + 2n\pi,$$  

(1.7)

where

$$A = 2^{3ir/2} e^{is/2} \Gamma^{-2} \left( \frac{1}{2} + \frac{ir}{4} \right), \quad B = 2^{-3ir/2} e^{-is/2} \Gamma^{-2} \left( \frac{1}{2} - \frac{ir}{4} \right),$$  

$$p = \frac{ Ae^{-\pi r/4} - Be^{\pi r/4} }{A + B},$$  

(1.8)

and $n$ is an integer. Moreover, the integer $k$ in (1.5) is given by

$$k = \text{integer part of } \left\{ \frac{1}{2\pi} \left[ \pi - s + 3r \ln 2 - 4 \arg \Gamma \left( \frac{1}{2} + \frac{ir}{4} \right) \right] \right\}.$$  

(1.9)

The above connection formulas can be established by using the method of isomonodromic deformation; see [4, p.446-448]. For convenience, we recall some of the relevant facts from the isomonodromy formalism for the sine-Gordon PIII reduction. The Lax pair of the sine-Gordon PIII reduction is the system of linear ordinary differential equations

$$\frac{\partial \Psi}{\partial x} = \left\{ -\frac{iu_x}{2} \sigma_1 - \frac{ix\lambda}{8} \sigma_3 \right\} \Psi,$$  

(1.10)

$$\frac{\partial \Psi}{\partial \lambda} = \left\{ -\frac{i x^2}{16} \sigma_3 - \frac{1}{\lambda} \frac{ixu_x}{4} \sigma_1 + \frac{i \cos u}{\lambda^2} \sigma_3 - \frac{i \sin u}{\lambda^2} \sigma_2 \right\} \Psi,$$  

(1.11)

where $\sigma_1$, $\sigma_2$ and $\sigma_3$ are the Pauli matrices

$$\sigma_1 = \left( \begin{array}{cc} 0 & 1 \\ 1 & 0 \end{array} \right), \quad \sigma_2 = \left( \begin{array}{cc} 0 & -i \\ i & 0 \end{array} \right), \quad \sigma_3 = \left( \begin{array}{cc} 1 & 0 \\ 0 & -1 \end{array} \right),$$  

(1.12)

and $u$, $u_x$ are complex parameters. It is easily verified that the Lax pair satisfies the compatibility condition $\frac{\partial^2 \Psi}{\partial x \partial \lambda} = \frac{\partial^2 \Psi}{\partial \lambda \partial x}$ if and only if $u$ is a solution of the sine-Gordon PIII equation. In the neighborhood of infinity and of the origin, we define, respectively, the canonical sectors

$$\Omega^{(\infty)} = \{ \lambda : -\pi < \arg \lambda < \pi, \ |\lambda| > r_1 \}$$  

(1.13)

and

$$\Omega^{(0)} = \{ \lambda : -\pi < \arg \lambda < \pi, \ |\lambda| < r_2 \},$$  

(1.14)

where $r_1$ and $r_2$ are positive constants with $r_1 < r_2$, and we assume that $x \in \mathbb{R}$. Since the Lax pair (1.10) – (1.11) has two irregular singular points of the first rank, one at $\lambda = 0$ and the other at $\lambda = \infty$, we may suppose that the system (1.10) – (1.11)
The matrix $Q(\Omega)$ solutions of the Lax pair (1.18) done in \[ only one irregular singular point. If we only calculate the Stokes multipliers as was the Lax pair for the sine-Gordon PIII equation is two, while the PII equation has differences in the details. For instance, the number of irregular singular points in the method of “uniform asymptotics” proposed in \[ make rigorous, a comment made in \[ which is equivalent to saying that the problem is isomonodromic in $Q(\Omega) = 0$; i.e., $Q(x)$ is a constant matrix, which is equivalent to saying that the problem is isomonodromic in $x$.

The connection problem (1.6) – (1.9) has previously been solved by computing, for both large and small values of $x$, the asymptotic behaviors of the fundamental solutions as $\lambda \to \infty$ and also as $\lambda \to 0$ (in the canonical sectors $\Omega(\infty)$ and $\Omega(0)$, respectively). In an overlapping region, one then uses the WKB asymptotics to do the matching; see [4, pp.465-468]. This procedure is complicated, and is difficult to make rigorous, a comment made in [2, p.245].

In this paper we shall solve the connection problem for equation (1.1) by using the method of “uniform asymptotics” proposed in [2] and applied to the second Painlevé (PII) equation. Although the basic idea is taken from [2], there are some differences in the details. For instance, the number of irregular singular points in the Lax pair for the sine-Gordon PIII equation is two, while the PII equation has only one irregular singular point. If we only calculate the Stokes multipliers as was done in [2], we will miss some of the important information for the monodromy data we need. This inspired us to consider the connection matrix $Q(x)$ which, as we shall see, involves two parameters, instead of the Stokes multiplier which involves only one parameter. The difficulty in extending the techniques for PII to other transcendents is also acknowledged by the authors of [2, p.244, lines 21-22].

The material in this paper is arranged as follows: In Section 2, we express the solutions of the Lax pair (1.10) – (1.11) in terms of the parabolic cylinder functions $D_\nu(z)$ and $D_{-\nu-1}(iz)$. Based on the asymptotic behavior of these latter functions, we work out in Section 3 the asymptotic formulas of the fundamental solutions $\Psi(\infty)(x, \lambda)$ and $\Psi(0)(x, \lambda)$ as $x \to \infty$, which hold uniformly with respect to $\lambda$. Once this is done, there is no further rigorous analysis required. All we need to do is to use the known asymptotic formula of $u(x)$ given in (1.5) to obtain the monodromy data; this is also done in Section 3. The last section is devoted to deriving corresponding results as $x \to 0$. Since the procedure of getting asymptotic formulas for the fundamental solutions and the monodromy data as $x \to 0$, as given
in [4, pp.462-463], is very simple and rigorous, we will just follow that method and correct several mistakes (typographic errors) in the calculations presented there.

2. Uniform asymptotics as \( x \to \infty \). In order to explain our method clearer, we use the same notation as given in [4, p.452], and write equation (1.17) as

\[
\begin{pmatrix}
\psi_{11}^{(\infty)} & \psi_{12}^{(\infty)} \\
\psi_{21}^{(\infty)} & \psi_{22}^{(\infty)}
\end{pmatrix} =
\begin{pmatrix}
\psi_{11}^{(0)} & \psi_{12}^{(0)} \\
\psi_{21}^{(0)} & \psi_{22}^{(0)}
\end{pmatrix}
\begin{pmatrix}
q_{11} & q_{12} \\
q_{21} & q_{22}
\end{pmatrix},
\]

(2.1)

It is shown in [4, pp.450-452] that the connection matrix \( Q \) can be expressed as

\[
Q = \begin{pmatrix}
q_{11} & q_{12} \\
q_{21} & q_{22}
\end{pmatrix}
= (1 + pq)^{-\frac{1}{2}} \begin{pmatrix} 1 & p \\ -q & 1 \end{pmatrix},
\]

(2.2)

where \( p \) and \( q \) are constants.

Write \( \Psi = (\psi_1, \psi_2)^T \). From (1.11), we have

\[
\frac{d \psi_1}{d \lambda} = \left( -\frac{ix^2}{16} + \frac{i \cos u}{\lambda^2} \right) \psi_1 + \left( -\frac{ixu}{4\lambda} - \frac{\sin u}{\lambda^2} \right) \psi_2
\]

(2.3)

and

\[
\frac{d \psi_2}{d \lambda} = \left( -\frac{ixu}{4\lambda} + \frac{\sin u}{\lambda^2} \right) \psi_1 + \left( \frac{ix^2}{16} - \frac{i \cos u}{\lambda^2} \right) \psi_2.
\]

(2.4)

In terms of the new variables

\[
\xi = \frac{x}{4} \text{ and } \eta = \frac{x\lambda}{4},
\]

(2.5)

equations (2.3) and (2.4) become

\[
\frac{d \psi_1}{d \eta} = \xi \left[ \left( -i + \frac{i \cos u}{\eta^2} \right) \psi_1 + \left( -\frac{ixu}{\eta} - \frac{\sin u}{\eta^2} \right) \psi_2 \right]
\]

(2.6)

and

\[
\frac{d \psi_2}{d \eta} = \xi \left[ \left( -\frac{ixu}{\eta} + \frac{\sin u}{\eta^2} \right) \psi_1 + \left( i - \frac{i \cos u}{\eta^2} \right) \psi_2 \right].
\]

(2.7)

Differentiating (2.7) with respect to \( \eta \) and eliminating \( \psi_1 \), we get

\[
\frac{d^2 \psi_2}{d \eta^2} = \left( \frac{ixu}{\eta^2} - \frac{2 \sin u}{\eta^3} \right) \left( -\frac{ixu}{\eta} + \frac{\sin u}{\eta^2} \right)^{-1} \frac{d \psi_2}{d \eta} + \left\{ \xi^2 \left( -1 - \frac{1}{\eta^4} + \frac{u^2}{\eta^2} + \frac{2 \cos u}{\eta^2} \right) - \xi \left( \frac{ixu}{\eta^2} - \frac{2 \sin u}{\eta^3} \right) \right\} \psi_2.
\]

(2.8)

To remove the term \( d \psi_2/d \eta \) in (2.8), we set

\[
\phi = \left( -\frac{ixu}{\eta} + \frac{\sin u}{\eta^2} \right)^{-1/2} \psi_2.
\]

(2.9)
and obtain
\[
\frac{d^2 \phi}{d\eta^2} = \xi^2 \left\{ -1 - \frac{1}{\eta^2} \frac{u^2}{\eta^2} + \frac{2 \cos u}{\eta^2} + \frac{2i \cos u}{\xi \eta^3} \right. \\
\left. - \frac{1}{\xi} \left( \frac{iu_x}{\eta^2} \frac{2 \sin u}{\eta^3} \right) \left( \frac{1}{\eta^2} - i \frac{\cos u}{\eta^2} \right) \left( \frac{iu_x}{\eta^2} + \frac{\sin u}{\eta^2} \right)^{-1} \right. \\
\left. + \frac{3}{4\xi^2} \left( \frac{iu_x}{\eta^2} \frac{2 \sin u}{\eta^3} \right)^2 \left( \frac{iu_x}{\eta^2} + \frac{\sin u}{\eta^2} \right)^{-2} \right. \\
\left. - \frac{1}{2\xi^2} \left( \frac{-2iu_x}{\eta^3} + \frac{6 \sin u}{\eta^4} \right) \left( \frac{iu_x}{\eta^2} + \frac{\sin u}{\eta^2} \right)^{-1} \right\} \phi. \tag{2.10}
\]

We rewrite (2.10) as
\[
\frac{d^2 \phi}{d\eta^2} = -\xi^2 \left\{ \left( \frac{1}{\eta^2} - 1 \right)^2 - \frac{F_1(\eta, \xi)}{\xi} - \frac{F_2(\eta, \xi)}{\xi^2} \right\} \phi, \tag{2.11}
\]
where
\[
F_1(\eta, \xi) = -\frac{\xi u^2}{\eta^2} + \frac{2 \xi (\cos u - 1)}{\eta^2} + \frac{2i}{\eta} - i \left( \frac{1}{\eta^2} - \frac{\cos u}{\eta^2} \right) \left( \frac{\sin u}{iu_x} \right)^{-1}, \tag{2.12}
\]
and
\[
F_2(\eta, \xi) = \frac{3}{4} \left( \frac{iu_x}{\eta^2} - \frac{2 \sin u}{\eta^3} \right)^2 \left( \frac{iu_x}{\eta^2} + \frac{\sin u}{\eta^2} \right)^{-2} \\
- \frac{1}{2} \left( \frac{-2iu_x}{\eta^3} + \frac{6 \sin u}{\eta^4} \right) \left( \frac{iu_x}{\eta^2} + \frac{\sin u}{\eta^2} \right)^{-1}. \tag{2.13}
\]

For large values of \( \xi \) (or, equivalently, \( x \)), an approximate equation to that of (2.11) is
\[
\frac{d^2 \phi}{d\eta^2} = -\xi^2 \left( \frac{1}{\eta^2} - 1 \right)^2 \phi. \tag{2.14}
\]
Thus, by the WKB approximation [7, p.203, eq.(5.03)], two linearly independent asymptotic solutions of (2.11) are given by
\[
\left( \frac{1}{\eta^2} - 1 \right)^{-1/2} \exp \left\{ \pm i\xi \int_0^\eta (\sigma^{-2} - 1) d\sigma \right\} = \left( \frac{1}{\eta^2} - 1 \right)^{-1/2} \exp \left\{ \pm i\xi \left( \frac{1}{\eta} - \eta \right) \right\}, \tag{2.15}
\]
from which it follows that the anti-Stokes lines of the solution \( \phi \) to (2.11) are the positive and the negative real lines in the \( \eta \)–plane. Note that the negative line is the cut which we have taken in the definitions of the canonical sectors \( \Omega^{(\infty)} \) and \( \Omega^{(0)} \) in (1.13) and (1.14), respectively. (Recall: \( \eta = \lambda x/4 \) and \( x > 0 \).)

Analyzing (2.11) further, one finds that there are two coalescing turning points near \( \eta = 1 \), which is one of the zeros of the dominant term in (2.11), i.e., the coefficient function of \( \phi \) in (2.14). The other two coalescing turning points are close to \( \eta = -1 \), which is the other zero of the dominant term. For convenience, let us put
\[
F(\eta, \xi) = \left( \frac{1}{\eta^2} - 1 \right)^2 - \frac{F_1(\eta, \xi)}{\xi} - \frac{F_2(\eta, \xi)}{\xi^2}, \tag{2.16}
\]
and denote by \( \eta_1 \) and \( \eta_2 \) the two turning points of \( F \) near \( \eta = 1 \), i.e., \( F(\eta_1, \xi) = F(\eta_2, \xi) = 0 \). Substituting \( \eta_1 \) and \( \eta_2 \) into (2.16) gives

\[
\left( \frac{1}{\eta_j} + 1 \right)^2 \left( \frac{1}{\eta_j} - 1 \right)^2 = \frac{F_1(\eta_j, \xi)}{\xi} + \frac{F_2(\eta_j, \xi)}{\xi^2}, \quad j = 1, 2. \tag{2.17}
\]

Let \( u_k := u - 2k\pi \). By the asymptotic formula in (1.5), we have \( \sin u \sim u_k \) and \( \cos u = 1 - 2\sin^2(u/2) \sim 1 - \frac{1}{2}u_k^2 \). It can be shown as in [5, pp.41-43] that when \( x \to \infty \),

\[
u_k = -\alpha x^{-1/2} \sin \left\{ x - \frac{\alpha^2}{16} \ln x + \beta \right\} + o(x^{-1/2}). \tag{2.18}
\]

Coupling (2.18) and the approximation for \( \cos u \), we obtain when \( \eta \) is bounded away from 0,

\[
F_1(\eta, \xi) = \left\{ -\frac{\alpha^2}{4\eta^2} + \frac{2i}{\eta} - i \left( 1 - \frac{1}{\eta^2} \right) \left( \eta - \sin \frac{u}{iu_x} \right)^{-1} \right\} + \frac{1}{\eta^2} o(1) \tag{2.19}
\]

and

\[
F_2(\eta, \xi) = O(1) \tag{2.20}
\]

as \( \xi \to \infty \). Hence, when \( \eta_1 \) and \( \eta_2 \) approach 1, we have \( \eta_j^{-1} + 1 \to 2 \) (\( j = 1, 2 \)) and

\[
F_1(1, \xi) \sim -\frac{\alpha^2}{4} + 2i, \tag{2.21}
\]

From this, it follows that the two turning points have the asymptotic formulas

\[
\eta_j^{-1} \sim 1 \pm \left( -\frac{\alpha^2}{16} + \frac{i}{2} \right)^{1/2} \xi^{-1/2}, \quad j = 1, 2. \tag{2.22}
\]

Since we are only concerned with the two turning points near \( \eta = 1 \), the similarity of the approximate equation (2.14) and Weber’s equation

\[
\frac{d^2y}{dz^2} = \left[ \frac{z^2}{4} - \left( \nu + \frac{1}{2} \right) \right] y \tag{2.23}
\]

prompts us to compare solutions of equation (2.11) with the parabolic cylinder functions \( D_\nu(z) \) and \( D_{-\nu-1}(iz) \), which are two linearly independent solutions of (2.23); see [1, p.686]. Set \( z = e^{\pi i/4} \sqrt{2\xi} \). Then Weber’s equation becomes

\[
\frac{d^2y}{dz^2} = -\xi^2 (\zeta^2 - \theta^2)y, \tag{2.24}
\]

where

\[
\theta^2 = \frac{2\nu + 1}{i\xi}. \tag{2.25}
\]

A comparison of the two equations (2.11) and (2.24) suggests that we may assume the solutions of (2.11) are of the form

\[
\phi_\nu(\eta) = \rho(\eta)D_\nu \left( e^{\pi i/4} \sqrt{2\xi} \zeta(\eta) \right) := \rho(\eta)G_\nu(\zeta(\eta)), \tag{2.26}
\]

where \( \rho(\eta) \) and \( \zeta(\eta) \) are functions to be determined and \( G_\nu(\zeta) \) satisfies equation (2.24). To determine \( \rho(\eta) \) and \( \zeta(\eta) \), we substitute (2.26) into (2.11) to obtain

\[
\rho'' G_\nu + 2\rho' \zeta \frac{dG_\nu}{d\zeta} + \rho \left( \zeta' \right)^2 \frac{d^2G_\nu}{d\zeta^2} + \zeta'' \frac{dG_\nu}{d\zeta} = -\xi^2 F(\eta, \xi) \rho G_\nu. \tag{2.27}
\]

Since the coefficient of \( dG_\nu/d\zeta \) is supposedly equal to zero, (2.27) gives

\[
2\rho' \zeta' + \rho \zeta'' = 0, \tag{2.28}
\]
from which it follows that
\[ \rho = (\zeta')^{-1/2}, \]  
where we have taken the constant of integration to be 0 since its exact value is inessential. With the terms involving \( dG_{\nu}/d\zeta \) removed, and since \( G_{\nu} \) satisfies (2.24), we have from (2.27)
\[ (\zeta^2 - \theta^2)\zeta^2 = F(\eta, \xi) + \frac{\rho''}{\xi^2 \rho}. \]  
Furthermore, since the function \( F(\eta, \xi) \) already contains a term of order \( O(\xi^{-2}) \), the last term on the right-hand side of (2.30) can be absorbed into \( F(\eta, \xi) \), so that (2.30) simplifies to
\[ (\zeta^2 - \theta^2)\zeta^2 = F(\eta, \xi). \]  
In order for the mapping from \( \zeta \to \eta \) to be one-to-one, neither \( d\zeta/d\eta \) nor \( d\eta/d\zeta \) can vanish. This suggests that we should require the zeros of \( \zeta^2 - \theta^2 \), i.e., \( \zeta = \pm \theta \), to correspond to the zeros of \( F(\eta, \xi) \) near 1. Thus, we define the variable \( \theta \) by the formula:
\[ \int_{-\theta}^{\theta} (\zeta^2 - \theta^2)^{1/2} d\zeta = \int_{\eta_1}^{\eta_2} F^{1/2}(\eta, \xi) d\eta, \]  
where \( \eta_1 \) and \( \eta_2 \) are the two zeros of \( F(\eta, \xi) \) near 1, whose asymptotic behavior is given in (2.22). The cut for the integrand on the left-hand side is the line segment joining \( -\theta \) to \( \theta \). Taking the path of integration along the lower edge of the cut, we obtain
\[ \int_{-\theta}^{\theta} \sqrt{(\zeta - \theta)(\zeta + \theta)} d\zeta = e^{-i\pi/2} \int_{-\theta}^{\theta} \sqrt{(\theta - \zeta)(\zeta + \theta)} d\zeta. \]  
Hence
\[ \int_{-\theta}^{\theta} (\zeta^2 - \theta^2)^{1/2} d\zeta = -\frac{1}{2} \pi i \theta^2. \]  
By (2.16), the right-hand side of (2.32) can be written as
\[ \int_{\eta_1}^{\eta_2} F^{1/2}(\eta, \xi) d\eta = \int_{\eta_1}^{\eta_2} \left[ \left( \frac{1}{\eta^2} - 1 \right)^2 - \frac{F_1(\eta, \xi)}{\xi} - \frac{F_2(\eta, \xi)}{\xi^2} \right]^{1/2} d\eta. \]  
Since both upper and lower limits of integration are close to 1, we can use the approximations of \( F_1(\eta, \xi) \) and \( F_2(\eta, \xi) \) given in (2.19) and (2.20), respectively. Let \( \eta = 1 + t\xi^{-1/2} \), and then
\[ \int_{\eta_1}^{\eta_2} \left[ \left( \frac{1}{\eta^2} - 1 \right)^2 - \frac{F_1(\eta, \xi)}{\xi} - \frac{F_2(\eta, \xi)}{\xi^2} \right]^{1/2} d\eta \]
\[ = \int_{-k}^{k} \left[ \frac{4}{(t\xi^{-1/2})^2} - \frac{F_1(1 + t\xi^{-1/2}, \xi)}{\xi} + o(\xi^{-1}) \right]^{1/2} \xi^{-1/2} dt \]
\[ = \frac{2}{\xi} \int_{-k}^{k} \left[ (-t)^2 - \frac{F_1(1, \xi)}{4} \right]^{1/2} dt + o(\xi^{-1}), \]  
where \( k \sim (-\alpha^2 + \frac{i}{2})^{1/2} \); see (2.22). In view of (2.21), the last integral is equal to
\[ \int_{-k}^{k} ((-t)^2 - k^2)^{1/2} dt + o(1) = i \int_{-k}^{k} \sqrt{(k-t)(k+t)} dt + o(1) \]
\[ = \frac{1}{2} \pi ik^2 + o(1), \]  
\[ \]
where, as in (2.34), the cut for the above integrand is the line segment joining \( k \) and \( -k \), and the path of integration is along the lower edge of the cut. A combination of (2.32) and (2.34)-(2.37) gives

\[
\theta^2 \sim -\frac{2}{\xi} \left( -\frac{\alpha^2}{16} + \frac{i}{2} \right).
\]  

(2.38)

Coupling (2.25) and (2.38) determines the approximate value

\[
\nu \sim \frac{\alpha^2}{16} 
\]  

(2.39)

for the order of the parabolic cylinder function \( D_{\nu} \left( e^{\pi i/4} \sqrt{2\xi} \zeta(\eta) \right) \) in (2.26).

With \( \theta \) so chosen, we can define \( \zeta(\eta) \) by

\[
\int_{\theta}^{\xi} (\tau^2 - \theta^2)^{1/2} d\tau = \int_{\eta_2}^{\eta} F^{1/2}(\sigma, \xi) d\sigma.
\]  

(2.40)

For large values of \( \xi \) (or, equivalently, \( x \)), two linearly independent asymptotic solutions of equation (2.11), that are uniform with respect to \( \eta \) (or, equivalently, \( \lambda \)), are

\[
(\zeta')^{-1/2} D_{\nu} \left( e^{\pi i/4} \sqrt{2\xi} \zeta(\eta) \right) \text{ and } (\zeta')^{-1/2} D_{-\nu-1} \left( e^{3\pi i/4} \sqrt{2\xi} \zeta(\eta) \right).
\]  

(2.41)

By virtue of (2.31), they can also be written as

\[
\phi_{\nu} = \left( \frac{\zeta^2 - \theta^2}{F(\eta, \xi)} \right)^{1/4} D_{\nu} \left( e^{\pi i/4} \sqrt{2\xi} \zeta(\eta) \right),
\]  

(2.42)

and

\[
\tilde{\phi}_{-\nu-1} = \left( \frac{\zeta^2 - \theta^2}{F(\eta, \xi)} \right)^{1/4} D_{-\nu-1} \left( e^{3\pi i/4} \sqrt{2\xi} \zeta(\eta) \right).
\]  

(2.43)

3. Monodromy data as \( x \to +\infty \). To work out the asymptotic behavior of the two fundamental solutions to equations (1.10) − (1.11) that satisfy the boundary conditions (1.15) and (1.16), we must first find the behavior of \( \zeta(\eta) \) as \( \eta \to \infty \) and \( \eta \to 0 \).

Returning to (2.40), we have by using integration by parts and trigonometric substitution

\[
\int_{\theta}^{\xi} (\tau^2 - \theta^2)^{1/2} d\tau = \frac{1}{2} \left\{ \zeta (\zeta^2 - \theta^2)^{1/2} - \theta^2 \log (\zeta + (\zeta^2 - \theta^2)^{1/2}) + \theta^2 \log \theta \right\};
\]  

(3.1)

cf. [1, p.13, eq.(3.3.41)]. As \( |\zeta| \to \infty \), we obtain

\[
\int_{\theta}^{\xi} (\tau^2 - \theta^2)^{1/2} d\tau \sim \frac{1}{2} \left( \zeta^2 - \frac{1}{2} \theta^2 - \theta^2 \log 2\zeta + \theta^2 \log \theta \right),
\]  

(3.2)

when the cut for the integrand is again the line segment joining \(-\theta\) to \( \theta \).

Now we calculate the integral on the right-hand side of (2.40), and concentrate on \( |\eta| \to \infty \) first. Put \( \eta^* = (1 - \xi^{-1/2+\varepsilon})^{-1} \), where \( \varepsilon \) is a small positive number. The integral can be written as

\[
I = \int_{\eta_2}^{\eta} F^{1/2}(\sigma, \xi) d\sigma = \left( \int_{\eta_2}^{\eta^*} + \int_{\eta^*}^{\eta} \right) F^{1/2}(\sigma, \xi) d\sigma := I_1 + I_2.
\]  

(3.3)
By (2.16),
\[
I_1 = \int_{\eta_2}^{\eta^*} \left[ \left( \frac{1}{\sigma} + 1 \right)^2 \left( \frac{1}{\sigma} - 1 \right)^2 - \frac{F_1(\sigma, \xi)}{\xi} - \frac{F_2(\sigma, \xi)}{\xi^2} \right]^{1/2} \, d\sigma.
\]  (3.4)
Since the integration path stays away from the origin, we can use the approximations of \(F_1(\eta, \xi)\) and \(F_2(\eta, \xi)\) in (2.19) and (2.20), respectively. Let \(\sigma = 1 + t\xi^{-1/2}\). Then \(I_1\) becomes
\[
I_1 = \int_{k}^{l} \left[ 4 \left( -t\xi^{-1/2} \right)^2 - \frac{F_1(1 + t\xi^{-1/2}, \xi)}{\xi} + o(\xi^{-1}) \right]^{1/2} \xi^{-1/2} \, dt,
\]  (3.5)
where
\[
k = (\eta_2 - 1)\xi^{1/2} \sim \left( -\frac{\alpha^2}{16} + \frac{i}{2} \right) \xi^{1/2} \text{ and } l = (\eta^* - 1)\xi^{1/2} \sim \xi^*.
\]  (3.6)
Furthermore, by the same reasoning, the term \(F_1(1 + t\xi^{-1/2}, \xi)\) in (3.5) can be replaced by \(F_1(1, \xi)\), and we have
\[
I_1 = \frac{2}{\xi} \int_{k}^{l} \left( (t)^2 - k^2 \right)^{1/2} \, dt + o(\xi^{-1}),
\]  (3.7)
where as before the cut for the integrand is the line segment joining \(-k\) to \(k\). Since \(\sqrt{-l + k} = e^{\pi/2} \sqrt{t - k}\) and \(\sqrt{-t - k} = e^{\pi/2} \sqrt{t + k}\), it follows that
\[
I_1 = \frac{2}{\xi} e^{\pi i} \int_{k}^{l} \sqrt{t^2 - k^2} \, dt + o(\xi^{-1})
\]
\[
= -\frac{1}{\xi} \left\{ l(l^2 - k^2)^{1/2} - k^2 \log(l + (l^2 - k^2)^{1/2}) + k^2 \log k \right\} + o(\xi^{-1})
\]  (3.8)
\[
= -\frac{1}{\xi} \left\{ \xi^{2*} - \frac{1}{2} k^2 - k^2 \log(2\xi^*) + k^2 \log k \right\} + o(\xi^{-1});
\]
cf. (3.1) and (3.8).

The path of the second integral in (3.3) also stays away from the origin. Hence, the approximations of \(F_1(\eta, \xi)\) and \(F_2(\eta, \xi)\) in (2.19) and (2.20), respectively, can again be used, and we have
\[
I_2 = \int_{\eta^*}^{\eta} \left( \frac{1}{\sigma^2} - 1 \right) \left[ 1 - \frac{F_1(\sigma, \xi)}{\xi(\sigma^{-2} - 1)} + o(\xi^{-1}) \right]^{1/2} \, d\sigma.
\]  (3.9)
By the binomial expansion,
\[
I_2 = \int_{\eta^*}^{\eta} \left( \frac{1}{\sigma^2} - 1 \right) \, d\sigma - \int_{\eta^*}^{\eta} \left[ \frac{F_1(\sigma, \xi)}{2\xi(\sigma^{-2} - 1)} + o(\xi^{-1}) \right] \, d\sigma := I_{2,1} + I_{2,2}.
\]  (3.10)
The first integral in (3.10) can be evaluated as follows:
\[
I_{2,1} = \int_{\eta^*}^{1} \left( \frac{1}{\sigma^2} - 1 \right) \, d\sigma - \int_{1}^{\eta} \left( \frac{1}{\sigma^2} - 1 \right) \, d\sigma
\]
\[
= \xi^{-1+2*} + \frac{1}{(-\eta)} - \eta + 2 + o(\xi^{-1}).
\]  (3.11)
By using the approximation in (2.19), we obtain for the second integral in (3.10)
\[
I_{2,2} = -\frac{1}{2\xi} \int_{\eta^*}^{\eta} \left[ \frac{\alpha^2/8 - i}{\sigma - 1} - \frac{\alpha^2/8 + i}{\sigma + 1} + i \left( \sigma - \frac{\sin u}{iu^*} \right)^{-1} \right] \, d\sigma + o(\xi^{-1}),
\]  (3.12)
from which it follows that

\[ I_{2,2} = \frac{1}{2\xi} \left\{ \left( -\frac{\alpha^2}{8} + i \right) \log \left( \frac{\eta - 1}{\xi^{1/2}} \right) + \left( \frac{\alpha^2}{8} + i \right) \log \left( \frac{\eta + 1}{2} \right) \\
- i \log \left[ \left( \eta - \frac{\sin u}{iu_x} \right) \left( 1 - \frac{\sin u}{iu_x} \right)^{-1} \right] \right\} + o(\xi^{-1}). \]  

(3.13)

Since

\[ \int_{\theta}^{\zeta} (r^2 - \theta^2)^{1/2} d\tau = I_1 + I_{2,1} + I_{2,2} \]  

(3.14)

by (2.40), a combination of (3.2), (3.8), (3.11) and (3.13) gives

\[
\zeta^2 + \frac{2}{\xi} \left( -\frac{\alpha^2}{16} + \frac{i}{2} \right) \log \zeta \\
= -2\eta + \frac{2}{(-\eta)} + 4 + \frac{1}{\xi} \left\{ \left( -\frac{\alpha^2}{8} + i \right) \log (\eta - 1) + \left( \frac{\alpha^2}{8} + i \right) \log \left( \frac{\eta + 1}{2} \right) \\
- i \log \left[ \left( \eta - \frac{\sin u}{iu_x} \right) \left( 1 - \frac{\sin u}{iu_x} \right)^{-1} \right] + \left( -\frac{\alpha^2}{16} + \frac{i}{2} \right) \log 2 - \left( \frac{\alpha^2}{16} - \frac{i}{2} \right) \pi i \right\} + o(\xi^{-1}).
\]

(3.15)

When \(|\eta| \to \infty\), (3.15) simplifies to

\[
\zeta^2 + \frac{2}{\xi} \left( -\frac{\alpha^2}{16} + \frac{i}{2} \right) \log \zeta \\
\sim -2\eta + 4 + \frac{1}{\xi} \left[ i \log \eta - \left( \frac{3\alpha^2}{16} + \frac{i}{2} \right) \log 2 + i \log \left( 1 - \frac{\sin u}{iu_x} \right) - \frac{\alpha^2}{16} \pi i - \frac{\pi}{2} \right].
\]

(3.16)

Next, we consider the case when \(|\eta| \to 0\). Let \(\eta_* = 1 - \zeta^{-1/2+\varepsilon}\), where \(\varepsilon\) is a small positive constant, and split the integral \(I\) in (3.3) into three parts

\[ I = \left( \int_{\eta_2}^{\eta_1} + \int_{\eta_1}^{\eta_*} + \int_{\eta_*}^{\eta} \right) F^{1/2}(\sigma, \xi) d\sigma := I_0 + I_1 + I_2. \]  

(3.17)

On account of (2.36) – (2.37),

\[ I_0 = \int_{\eta_2}^{\eta_1} F^{1/2}(\sigma, \xi) d\sigma = -\frac{1}{\xi} k^2 \pi i + o(\xi^{-1}). \]  

(3.18)

To calculate the integral

\[ I_1 = \int_{\eta_1}^{\eta_*} \left[ \left( \frac{1}{\sigma + 1} \right)^2 - \left( \frac{1}{\sigma - 1} \right)^2 \right] \left( \frac{F_1(\sigma, \xi)}{\xi} - \frac{F_2(\sigma, \xi)}{\xi^2} \right)^{1/2} d\sigma, \]  

(3.19)
we again let $\sigma = 1 + t\xi^{-1/2}$; cf. (3.4). As in (3.5) – (3.8), we obtain

$$I_1 = \int_{-k}^{-l} \left[ 4 (-t \xi^{-1/2})^2 - \frac{F_1(1, \xi)}{\xi} + o(\xi^{-1}) \right]^{1/2} \xi^{-1/2} dt$$

$$= \frac{2}{\xi} \int_{-k}^{-l} (-t^2 - k^2)^{1/2} dt + o(\xi^{-1})$$

$$= -\frac{2}{\xi} \int_k^{\infty} (r^2 - k^2)^{1/2} dr + o(\xi^{-1})$$

$$= -\frac{1}{\xi} \left\{ l(l^2 - k^2)^{1/2} - k^2 \log(l + (l^2 - k^2)^{1/2}) + k^2 \log k \right\} + o(\xi^{-1})$$

$$= -\frac{1}{\xi} \left\{ \xi^{2\epsilon} - \frac{1}{2} k^2 - k^2 \log(2\xi^\epsilon) + k^2 \log k \right\} + o(\xi^{-1}), \quad (3.20)$$

where $k = (\eta_2 - 1)\xi^{1/2} \sim (-\frac{\alpha^2}{16} + \frac{\xi}{2})^{1/2}$ (or $k = -(\eta_1 - 1)\xi^{1/2}$) and $l \sim \xi^\epsilon$; see (3.8).

For the third integral $I_2$ in (3.17), we can not use the approximations in (2.19), as was done in (3.4) and (3.19). However, according to (2.12) and (2.13), we have $F_1(\eta, \xi) \sim O(\eta^{-2})$ and $F_2(\eta, \xi) \sim O(\eta^{-2})$ as $|\eta| \to 0$. Thus, we obtain from (2.16)

$$I_2 = \int_{\eta,}^{\eta,} \left( \frac{1}{\sigma^2} - 1 \right) \left[ 1 - \frac{F_1(\sigma, \xi)}{\xi} \left( \frac{1}{\sigma^2} - 1 \right)^{-2} + O \left( \frac{\sigma^2}{\xi^2} \right) \right]^{1/2} d\sigma$$

$$= \int_{\eta,}^{\eta,} \left( \frac{1}{\sigma^2} - 1 \right) \left[ 1 - \frac{F_1(\sigma, \xi)}{2\xi} \left( \frac{1}{\sigma^2} - 1 \right)^{-2} + O \left( \frac{\sigma^2}{\xi^2} \right) \right] d\sigma$$

$$= \int_{\eta,}^{\eta,} \left( \frac{1}{\sigma^2} - 1 \right) d\sigma - \int_{\eta,}^{\eta,} \left[ \frac{F_1(\sigma, \xi)}{2\xi} \left( \frac{1}{\sigma^2} - 1 \right)^{-1} + O(\xi^{-2}) \right] d\sigma$$

$$:= I_{2,1} + I_{2,2}. \quad (3.21)$$

The first integral on the right can be evaluated as in (3.11), and we have

$$I_{2,1} = \int_{\eta,}^{1} \left( \frac{1}{\sigma^2} - 1 \right) d\sigma + \int_{1}^{\eta,} \left( \frac{1}{\sigma^2} - 1 \right) d\sigma$$

$$= \xi^{-1+2\epsilon} - \eta + \frac{1}{(-\eta)} + 2 + o(\xi^{-1}). \quad (3.22)$$

To evaluate the second integral on the right-hand side of (3.21), we use (2.19) and proceed as in (3.13). The result is

$$I_{2,2} = \frac{-1}{2\xi} \int_{\eta,}^{\eta,} \left[ \frac{\alpha^2/8 - i}{\sigma - 1} - \frac{\alpha^2/8 + i}{\sigma + 1} + i \left( \sigma - \frac{\sin u}{iu_{\sigma x}} \right)^{-1} + O(\xi^{-2}) \right] d\sigma$$

$$= \frac{1}{2\xi} \left\{ \left( \frac{\alpha^2}{8} + i \right) \log \left( \frac{1 - \eta}{\xi^{1/2}} \right) + \left( \frac{\alpha^2}{8} + i \right) \log \left( \frac{\eta + 1}{2} \right) \right. \right.$$

$$- i \log \left( \eta - \frac{\sin u}{iu_{\sigma x}} \right) \left( 1 - \frac{\sin u}{iu_{\sigma x}} \right)^{-1} \right\} + O(\xi^{-2}). \quad (3.23)$$
On account of (3.2) and (3.20)-(3.23), it follows from (2.40) that

\[
\begin{align*}
\zeta^2 + 2 \left( -\frac{\alpha^2}{16} + \frac{i}{2} \right) \log \zeta \\
= -2\eta + \frac{2}{(\eta)} + 4 + \frac{1}{\xi} \left\{ \left( -\frac{\alpha^2}{8} + i \right) \log (1 - \eta) + \left( \frac{\alpha^2}{8} + i \right) \log \left( \frac{\eta + 1}{2} \right) \right\} \\
- i \log \left[ \left( \eta - \sin u \right) \left( 1 - \sin u \right)^{-1} \right] + \left( \frac{\alpha^2}{16} + \frac{i}{2} \right) \log 2 + \left( \frac{\alpha^2}{16} - \frac{i}{2} \right) \pi i \right\} \\
+ o(\xi^{-1}).
\end{align*}
\]

(3.24)

where use has also been made of (2.38). As \(|\eta| \to 0\), (3.24) gives

\[
\begin{align*}
\zeta^2 + 2 \left( -\frac{\alpha^2}{16} + \frac{i}{2} \right) \log \zeta \\
\sim -2\eta + 4 + \frac{1}{\xi} \left\{ \left( -\frac{3\alpha^2}{16} + \frac{i}{2} \right) \log 2 + i \log \left[ \left( \eta - \sin u \right) \left( 1 - \sin u \right)^{-1} \right] + \left( \frac{\alpha^2}{16} + \frac{i}{2} \right) \log 2 + \left( \frac{\alpha^2}{16} - \frac{i}{2} \right) \pi i \right\}.
\end{align*}
\]

(3.25)

We now recall the asymptotic behavior of \(D_\nu(z)\). From [3, p.132], we have:

\[
\begin{align*}
D_\nu(z) &\sim \begin{cases} 
\alpha \nu e^{-i\xi^2/4}, & \text{arg } z \in (-\frac{3\pi}{4}, \frac{3\pi}{4}), \\
\nu e^{-i\xi^2/4} - \frac{\sqrt{2\pi}}{\Gamma(-\nu)} e^{i\pi \nu} z^{-\nu-1} e^{i\xi^2/4}, & \text{arg } z \in (\frac{\pi}{4}, \frac{3\pi}{4}), \\
e^{-2i\pi \nu} \nu e^{-i\xi^2/4} - \frac{\sqrt{2\pi}}{\Gamma(-\nu)} e^{i\pi \nu} z^{-\nu-1} e^{i\xi^2/4}, & \text{arg } z \in (\frac{3\pi}{4}, \frac{5\pi}{4})
\end{cases}
\end{align*}
\]

(3.26)

as \(|z| \to \infty\). For arg \(\lambda \in (-\pi, \pi)\), we have arg \(\eta \in (-\pi, \pi)\). Since \(\zeta^2 \sim -2\eta\) as \(|\eta| \to \infty\), if we take \(-1 = e^{\pi i}\), it follows that arg \(\zeta \in (0, \pi)\). Therefore, arg \((e^{\pi i/4} \sqrt{2\xi\zeta}) \in (\frac{\pi}{2}, \frac{3\pi}{2})\) and arg \((e^{3\pi i/4} \sqrt{2\xi\zeta}) \in (\frac{3\pi}{2}, \frac{5\pi}{2})\). Again, taking \(-1 = e^{\pi i}\), from (2.16) we have \(F_{\nu} \sim e^{\pi i/2}\) as \(|\eta| \to \infty\). Since \((\zeta^2 - \theta^2)^{1/4} \sim 1/2\zeta^1/2\) as \(|\eta| \to \infty\), by using the appropriate asymptotic formulas of \(D_\nu(z)\) in (3.26), we obtain from (2.42)

\[
\begin{align*}
\phi_\nu(\eta, \xi) &\sim \frac{\alpha \nu \eta^{\frac{1}{2}+\nu} e^{-i\xi^2/2}}{\Gamma(-\nu)} \left( \frac{\pi i}{\sqrt{2\xi}} \right)^\nu e^{-\pi i/2} \\
&- \frac{\sqrt{2\pi}}{\Gamma(-\nu)} e^{i\pi \nu} \xi^{-\nu-1} e^{i\xi^2/2} \left( \frac{\pi i}{\sqrt{2\xi}} \right)^{-\nu-1}
\end{align*}
\]

(3.27)

as \(|\eta| \to \infty\). To express \(e^{\pm i\xi^2/2}\) in terms of \(\eta\), we multiply both sides of (3.16) by \(\pm i\xi/2\) and then take exponentials. Since \(\frac{\alpha \nu i}{\theta} \sim \nu\) by (2.39), this yields

\[
\begin{align*}
\phi_\nu(\eta, \xi) &\sim e^{i\xi \eta^{\frac{1}{2}}} e^{\nu \pi i} A_0 + \frac{\sqrt{2\pi}}{\Gamma(-\nu)} e^{\frac{i\pi i/4 + \frac{3\pi}{2}}{2}} e^{-i\xi \eta^{\frac{1}{2}}} B_0,
\end{align*}
\]

(3.28)

as \(|\eta| \to \infty\), where

\[
\begin{align*}
A_0 &= e^{-2i\xi^2} \frac{\alpha \nu i}{\theta} e^{-\pi i/2} \left( \frac{\pi i}{\sqrt{2\xi}} \right)^\nu e^{i\pi \nu} \left( 1 - \frac{\sin u}{iu_x} \right)^\frac{1}{2}
\end{align*}
\]

(3.29)

and

\[
\begin{align*}
B_0 &= e^{-2i\xi^2} \frac{\alpha \nu i}{\theta} e^{\frac{3\pi i}{2}} e^{-\pi i/2} \left( \frac{\pi i}{\sqrt{2\xi}} \right)^{-\nu-1} e^{i\pi \nu} \left( 1 - \frac{\sin u}{iu_x} \right)^{-\frac{1}{2}}.
\end{align*}
\]

(3.30)
Similarly, we have from (2.43)
\[ \tilde{\phi}_{-\nu-1}(\eta, \xi) \sim e^{-i\zeta \eta} \eta^\frac{1}{2} e^{\nu \pi i + \pi i} B_0 + \frac{\sqrt{2\pi}}{\Gamma(1+\nu)} e^{\nu \pi i} \eta^\frac{1}{2} A_0. \] (3.31)

By (2.9), the entries \( \psi^{(\infty)}_{21} \) and \( \psi^{(\infty)}_{22} \) in the fundamental solution \( \Psi^{(\infty)}(x, \lambda) \) given in (1.17) are linear combinations of the solutions
\[ \left( -\frac{iu_x}{\eta} + \frac{\sin u}{\eta^2} \right)^{1/2} \phi_\nu(\eta, \xi) \] and
\[ \left( -\frac{iu_x}{\eta} + \frac{\sin u}{\eta^2} \right)^{1/2} \tilde{\phi}_{-\nu-1}(\eta, \xi), \]
to equation (2.11); see also (2.1). Moreover, they have the asymptotic behavior prescribed in (1.15) when \( |\eta| \to \infty \). Thus, we have
\[ \psi^{(\infty)}_{21} = \left( -\frac{iu_x}{\eta} + \frac{\sin u}{\eta^2} \right)^{1/2} \left[ c_1 \phi_\nu(\eta, \xi) + c_2 \tilde{\phi}_{-\nu-1}(\eta, \xi) \right] \sim \frac{u_x}{2\eta} e^{-i\zeta \eta}, \] (3.32)
\[ \psi^{(\infty)}_{22} = \left( -\frac{iu_x}{\eta} + \frac{\sin u}{\eta^2} \right)^{1/2} \left[ c_3 \phi_\nu(\eta, \xi) + c_4 \tilde{\phi}_{-\nu-1}(\eta, \xi) \right] \sim e^{i\xi \eta}. \] (3.33)

Comparing the coefficients of \( e^{-i\zeta \eta} \) and \( e^{i\xi \eta} \) on both sides of the above two (asymptotic) equations, we obtain
\[ c_1 = \frac{i}{2} \frac{\sqrt{2\pi}}{\Gamma(1+\nu)} e^{\nu \pi i} B_0^{-1}(-iu_x)^{1/2}, \quad c_2 = -\frac{i}{2} e^{\nu \pi i} B_0^{-1}(-iu_x)^{1/2}; \]
\[ c_3 = e^{\nu \pi i} A_0^{-1}(-iu_x)^{-1/2}, \quad c_4 = \frac{i}{2} \frac{\sqrt{2\pi}}{\Gamma(1+\nu)} e^{\nu \pi i} A_0^{-1}(-iu_x)^{-1/2}. \] (3.34)

When \( |\eta| \to 0, \zeta^2 \sim \frac{2}{\eta} \). Again take \( -1 = e^{\pi i} \); then \( \arg \zeta \in (-\pi, 0) \), from which it follows that \( \arg (e^{\pi i/4} \sqrt{2\pi} \xi) \in (-\frac{3\pi}{4}, \frac{\pi}{4}) \) and \( \arg (e^{3\pi i/4} \sqrt{2\pi} \xi) \in (-\frac{\pi}{4}, \frac{3\pi}{4}) \). By (2.16), \( F^{-\frac{1}{4}} \sim \eta \) as \( |\eta| \to 0 \). By using the asymptotic formulas of \( D_\nu(z) \) in (3.26), and the relationship between \( \zeta \) and \( \eta \) given in (3.25), we obtain as in (3.28) and (3.31) the following asymptotic behavior of \( \phi_\nu(\eta, \xi) \) and \( \tilde{\phi}_{-\nu-1}(\eta, \xi) \) as \( |\eta| \to 0 \):
\[ \phi_\nu(\eta, \xi) \sim e^{i\xi \eta} \eta A_0 \left( -\frac{iu_x}{\sin u} \right)^{\frac{1}{2}}, \] (3.35)

and
\[ \tilde{\phi}_{-\nu-1}(\eta, \xi) \sim e^{-i\xi \eta} \eta B_0 \left( -\frac{iu_x}{\sin u} \right)^{-\frac{1}{2}}. \] (3.36)

By the same reasoning as before, the entries \( \psi^{(0)}_{21} \) and \( \psi^{(0)}_{22} \) in the fundamental solution \( \Psi^{(0)}(x, \lambda) \) given in (1.17) are linear combinations of the solutions
\[ \left( -\frac{iu_x}{\eta} + \frac{\sin u}{\eta^2} \right)^{1/2} \phi_\nu(\eta, \xi) \] and
\[ \left( -\frac{iu_x}{\eta} + \frac{\sin u}{\eta^2} \right)^{1/2} \tilde{\phi}_{-\nu-1}(\eta, \xi), \]
to equation (2.11); see also (2.1). Furthermore, they have the asymptotic prescribed by (1.16). Thus, we obtain
\[ \psi^{(0)}_{21} = \left( -\frac{iu_x}{\eta} + \frac{\sin u}{\eta^2} \right)^{1/2} \left[ c_5 \phi_\nu(\eta, \xi) + c_6 \tilde{\phi}_{-\nu-1}(\eta, \xi) \right] \sim -i \frac{u}{2} e^{-i\xi \eta}, \] (3.37)
\[ \psi^{(0)}_{22} = \left( -\frac{iu_x}{\eta} + \frac{\sin u}{\eta^2} \right)^{1/2} \left[ c_7 \phi_\nu(\eta, \xi) + c_8 \tilde{\phi}_{-\nu-1}(\eta, \xi) \right] \sim \cos \frac{u}{2} e^{i\xi \eta}. \] (3.38)
Comparing the coefficients of \(e^{-i\xi}\eta\) and \(e^{i\xi}\eta\) on both sides of the above two asymptotic equations, we have
\[
c_5 = 0, \quad c_6 = -\frac{i}{2} B_0^{-1}(-iu_x)^{1/2}; \\
c_7 = \left(\cos \frac{u}{2}\right) A_0^{-1}(-iu_x)^{-1/2}, \quad c_8 = 0.
\] (3.39)

According to (2.1), \(\psi^{(\infty)}_{21} = q_{11}\psi_{21}^{(0)} + q_{21}\psi_{22}^{(0)}\). Hence, from (3.32), (3.37) and (3.38), it follows that
\[
\frac{i}{2} \sqrt{\frac{2\pi}{\Gamma(1 + \nu)}} e^{\nu i/2} B_0^{-1}(-iu_x)^{1/2} \phi_\nu - \frac{i}{2} e^{\nu i} B_0^{-1}(-iu_x)^{1/2} \bar{\phi}_{-\nu - 1}
\]
\[
= -q_{11} \left(\frac{i}{2}\right) B_0^{-1}(-iu_x)^{1/2} \bar{\phi}_{-\nu - 1} + q_{21} \left(\cos \frac{u}{2}\right) A_0^{-1}(-iu_x)^{-1/2} \phi_\nu,
\] (3.40)
which in turn gives
\[
q_{11} = e^{\nu i} \quad \text{and} \quad q_{21} = \frac{u_x}{2} \sqrt{\frac{2\pi}{\Gamma(\nu + 1)}} e^{\nu i/2} \left(\cos \frac{u}{2}\right)^{-1} A_0 \frac{B_0}{B_0}.
\] (3.41)

By the definition of \(q\) in (2.2), we have
\[
q = -q_{21}/q_{11}
\]
\[
\frac{u_x}{2} \sqrt{\frac{2\pi}{\Gamma(\nu + 1)}} e^{-\nu i/2} \left(\cos \frac{u}{2}\right)^{-1} A_0 \frac{B_0}{B_0}.
\] (3.42)

Inserting (3.29) and (3.30) into (3.42), we obtain from (1.5) and (2.18)
\[
q \sim \frac{4}{\alpha} \sqrt{\frac{2\pi}{\Gamma(-\alpha^2 i/16)}} e^{\alpha^2 \pi/32 - \alpha^2 i/8} e^{3\pi i/4} e^{-i\beta},
\] (3.43)
where we have also made use of (3.39). In the same manner, since \(\psi^{(\infty)}_{22} = q_{12}\psi_{21}^{(0)} + q_{22}\psi_{22}^{(0)}\), by (2.1) we obtain from (3.33), (3.37) and (3.38),
\[
e^{\nu i} A_0^{-1}(-iu_x)^{-1/2} \phi_\nu + i \sqrt{\frac{2\pi}{\Gamma(-\nu)}} e^{\nu i/2} A_0^{-1}(-iu_x)^{-1/2} \bar{\phi}_{-\nu - 1}
\]
\[
= -q_{12} \left(\frac{i}{2}\right) B_0^{-1}(-iu_x)^{1/2} \bar{\phi}_{-\nu - 1} + q_{22} \left(\cos \frac{u}{2}\right) A_0^{-1}(-iu_x)^{-1/2} \phi_\nu,
\] (3.44)
from which it follows that
\[
q_{12} = -\frac{2i}{u_x} \sqrt{\frac{2\pi}{\Gamma(-\nu)}} e^{\nu i/2} B_0 A_0 \quad \text{and} \quad q_{22} = e^{\nu i} \left(\cos \frac{u}{2}\right)^{-1}.
\] (3.45)

By the definition of \(p\) in (2.2), we have
\[
p = q_{12}/q_{22}
\]
\[
\sim \frac{4}{\alpha} \sqrt{\frac{2\pi}{\Gamma(-\alpha^2 i/16)}} e^{\alpha^2 \pi/32 - \alpha^2 i/8} e^{3\pi i/4} e^{-i\beta},
\] (3.46)
on account of (3.29), (3.30), (1.5) and (2.18). From (3.43) and (3.46), it is evident that \(p\) and \(q\) are indeed independent of \(x\), and that they are complex conjugates of each other. Furthermore, (3.46) gives
\[
|p| = \frac{4}{\alpha} \frac{\sqrt{2\pi e^{\alpha^2 \pi/32}}}{\Gamma(-\alpha^2 i/16)}
\] (3.47)
and
\[ \arg p = - \arg \Gamma(-\frac{i\alpha^2}{16}) - \frac{\alpha^2}{8} \ln 2 + \frac{3}{4} \pi - \beta + 2n\pi, \] (3.48)
where \( n \) is an integer. Since \( |\Gamma(iy)|^2 = \pi/(y \sinh \pi y) \), we conclude
\[ \alpha^2 = \frac{8}{\pi} \ln(1 + |p|^2), \] (3.49)
and
\[ \beta = -\frac{\alpha^2}{8} \ln 2 - \arg \Gamma(-\frac{i\alpha^2}{16}) - \arg p + \frac{3}{4} \pi + 2n\pi. \] (3.50)

4. Monodromy data as \( x \to 0 \). As explained in Section 1, we shall calculate the monodromy data as \( x \to 0 \) by following the method given in [4] and [6]. First, we take \( 1 \ll r_1 < r_2 \ll x^{-2} \) in (1.13) and (1.14) so that the two sectors \( \Omega^{(\infty)} \) and \( \Omega^{(0)} \) are overlapping. Again, we prescribe the asymptotic behavior of the two fundamental solutions \( \Psi^{(\infty)} \) and \( \Psi^{(0)} \) in these sectors by (1.15) and (1.16), respectively. Since \( u(x) \to \infty \) as \( x \to 0^+ \) by (1.4), we can choose a sequence of values of \( u \) satisfying \( \cos u = 1 \) and \( \sin u = 0 \). Furthermore, since the connection matrix \( Q \) is independent of \( \lambda \), we may without loss of generality restrict \( \lambda \) to be real and positive (i.e., \( \arg \lambda = 0 \)).

Recall the Bessel equation
\[ w'' + \frac{1}{z} w' + \left( 1 - \frac{\nu^2}{z^2} \right) w = 0, \] (4.1)
which has the two linearly independent solutions \( H^{(1)}_\nu(z) \) and \( H^{(2)}_\nu(z) \). With the change of variable \( y = \sqrt{z} w \), equation (4.1) becomes
\[ y'' = \left( -1 + \frac{\nu^2 - 1}{z^2} \right) y, \] (4.2)
two linearly independent solutions of which are \( \sqrt{z} H^{(1)}_\nu(z) \) and \( \sqrt{z} H^{(2)}_\nu(z) \).

We first investigate the asymptotic behavior of the solution \( \Psi^{(\infty)} \). Let \( \eta_1 = x^2 \lambda/16 \), and consider the system:
\[ \frac{\partial \Phi}{\partial \eta_1} = \left\{ -i\sigma_3 - \frac{i\lambda}{4\eta_1} \sigma_1 \right\} \Phi, \] (4.3)
which is an approximate equation of (1.11). Write \( \Phi = (\phi_1, \phi_2)^T \), and define \( \phi_+ = \phi_1 + \phi_2, \phi_- = \phi_1 - \phi_2 \). It is readily verifiable that
\[ \frac{\partial \phi_+}{\partial \eta_1} = -i\phi_- - \frac{i\lambda}{4\eta_1} \phi_+ \] (4.4)
and
\[ \frac{\partial \phi_-}{\partial \eta_1} = -i\phi_+ + \frac{i\lambda}{4\eta_1} \phi_- \] (4.5)
Differentiating each of the above two equations one more time with respect to \( \eta_1 \) gives
\[ \frac{\partial^2 \phi_+}{\partial \eta_1^2} = \left( -1 + \frac{4i\lambda x u_x - (x u_x)^2}{16\eta_1^2} \right) \phi_+ \] (4.6)
and
\[ \frac{\partial^2 \phi_-}{\partial \eta_1^2} = \left( -1 + \frac{-4i\lambda x u_x - (x u_x)^2}{16\eta_1^2} \right) \phi_- \] (4.7)
If we let
\[ \nu = \frac{ixu_x}{4} + \frac{1}{2} \quad \text{and} \quad \mu = \nu - 1 = \frac{ixu_x}{4} - \frac{1}{2}, \]
then we find that both \( \phi_+ \) and \( \phi_- \) satisfy a Bessel equation (4.2) and we may take
\[ \phi_+ = \sqrt{\eta_1} H^{(2)}_{\nu}(\eta_1) \quad \text{or} \quad \sqrt{\eta_1} H^{(1)}_{\nu}(\eta_1), \]
\[ \phi_- = \sqrt{\eta_1} H^{(2)}_{\mu}(\eta_1) \quad \text{or} \quad \sqrt{\eta_1} H^{(1)}_{\mu}(\eta_1). \]

Let \( \Phi^{(1)}, \Phi^{(2)} \) be two linearly independent solutions of equation (4.3), and put
\[ \Lambda = (\Phi^{(1)}, \Phi^{(2)}). \] Then \( \Lambda \) is a \( 2 \times 2 \) matrix, and is of the form
\[ \Lambda = \begin{pmatrix} \delta_1 \sqrt{\eta_1} H^{(2)}_{\nu}(\eta_1) + \tilde{\delta}_1 \sqrt{\eta_1} H^{(2)}_{\mu}(\eta_1) & \delta_2 \sqrt{\eta_1} H^{(1)}_{\nu}(\eta_1) - \tilde{\delta}_2 \sqrt{\eta_1} H^{(1)}_{\mu}(\eta_1) \\ \delta_1 \sqrt{\eta_1} H^{(2)}_{\nu}(\eta_1) - \tilde{\delta}_1 \sqrt{\eta_1} H^{(2)}_{\mu}(\eta_1) & \delta_2 \sqrt{\eta_1} H^{(1)}_{\nu}(\eta_1) + \tilde{\delta}_2 \sqrt{\eta_1} H^{(1)}_{\mu}(\eta_1) \end{pmatrix}, \]
where \( \delta_1, \tilde{\delta}_1, \delta_2 \) and \( \tilde{\delta}_2 \) are constants to be determined. Furthermore, we may always write
\[ \Psi^{(\infty)} = \Lambda(1 + \varepsilon_1(x, \lambda)). \]
Substituting (4.12) into (1.11), and using equation (4.3) and the method of successive approximations (see [7, pp.193-196]), it can be shown that
\[ \varepsilon_1(x, \lambda) = O(\lambda^{-1}) \ll 1 \text{ in } \Omega^{(\infty)}. \]
Thus, we obtain
\[ \Psi^{(\infty)} \sim \Lambda. \]

To determine the constants \( \delta_1, \tilde{\delta}_1, \delta_2 \) and \( \tilde{\delta}_2 \) in (4.11), we recall the well-known asymptotic formulas for large values of \( z \)
\[ H^{(1)}_{\nu}(z) \sim \left( \frac{2}{\pi z} \right)^{1/2} e^{i(z - \frac{\pi}{4} - \frac{\nu}{2} \pi)} (-\pi < \text{ph } z < 2\pi), \]
\[ H^{(2)}_{\nu}(z) \sim \left( \frac{2}{\pi z} \right)^{1/2} e^{-i(z - \frac{\pi}{4} + \frac{\nu}{2} \pi)} (-2\pi < \text{ph } z < \pi); \]
cf. [7, p.238]. Inserting (4.15) - (4.16) into (4.11), we get from (4.14)
\[ \psi^{(\infty)}_{11} \sim \delta_1 \sqrt{\eta_1} H^{(2)}_{\nu}(\eta_1) + \tilde{\delta}_1 \sqrt{\eta_1} H^{(2)}_{\mu-1}(\eta_1) \]
\[ \sim \left( \frac{2}{\pi} \right)^{1/2} e^{-i\eta_1} \left( \delta_1 e^{\frac{i}{8} \pi i - xu_x \pi/8} + \tilde{\delta}_1 e^{-xu_x \pi/8} \right), \]
and
\[ \psi^{(\infty)}_{22} \sim \delta_2 \sqrt{\eta_1} H^{(1)}_{\nu}(\eta_1) + \tilde{\delta}_2 \sqrt{\eta_1} H^{(1)}_{\mu-1}(\eta_1) \]
\[ \sim \left( \frac{2}{\pi} \right)^{1/2} e^{i\eta_1} \left( \delta_2 e^{-\frac{i}{8} \pi i + xu_x \pi/8} + \tilde{\delta}_2 e^{xu_x \pi/8} \right). \]
Comparing (4.17) and (4.18) with the asymptotic behavior of \( \Psi^{(\infty)} \) prescribed in (1.15), i.e.,
\[ \psi^{(\infty)}_{11} \sim e^{-ix^2 \lambda/16} = e^{-i\eta_1}, \quad \psi^{(\infty)}_{22} \sim e^{ix^2 \lambda/16} = e^{i\eta_1} \]
as \( |\lambda| \to \infty \), we obtain
\[ \delta_1 = \frac{1}{2} \left( \frac{\pi}{2} \right)^{1/2} e^{-\frac{i}{8} \pi i + xu_x \pi/8}, \quad \tilde{\delta}_1 = \frac{1}{2} \left( \frac{\pi}{2} \right)^{1/2} e^{xu_x \pi/8}, \]
\[ \delta_2 = \frac{1}{2} \left( \frac{\pi}{2} \right)^{1/2} e^{\frac{i}{8} \pi i - xu_x \pi/8}, \quad \tilde{\delta}_2 = \frac{1}{2} \left( \frac{\pi}{2} \right)^{1/2} e^{-xu_x \pi/8}. \]
By the reflection formula
\[ H^{(1)}_\nu(\eta_1) = e^{-\nu\pi i}H^{(1)}_{-\nu}(\eta_1), \]
we conclude
\[ \Psi^{(\infty)} \sim \frac{1}{2} \left( \frac{\pi}{2} \right)^{1/2} e^{xu_\pi i / \sqrt{\eta_1}} \times \begin{pmatrix} -iH^{(2)}_\nu(\eta_1) + H^{(2)}_{\nu-1}(\eta_1) & H^{(1)}_\nu(\eta_1) - iH^{(1)}_{1-\nu}(\eta_1) \\ -iH^{(2)}_\nu(\eta_1) - H^{(2)}_{\nu-1}(\eta_1) & H^{(1)}_\nu(\eta_1) + iH^{(1)}_{1-\nu}(\eta_1) \end{pmatrix}. \]

This matrix is given in [4, p.462] and [6, p.99], except for several typographical errors.

Next, we investigate the asymptotic behavior of the solution \( \Psi^{(0)} \). Let \( \eta_2 = \frac{1}{\lambda} \), and consider the system:
\[ \frac{\partial \Phi}{\partial \eta_2} = \begin{pmatrix} -i\sigma_3 + \frac{iux}{4\eta_2}\sigma_1 \end{pmatrix} \Phi, \]
which is again the approximate equation of (2.2). Write \( \Phi = (\phi_1, \phi_2)^T \), and put \( \phi_+ = \phi_1 + \phi_2 \) and \( \phi_- = \phi_1 - \phi_2 \). One can easily verify that
\[ \frac{\partial \phi_+}{\partial \eta_2} = -i\phi_- + \frac{iux}{4\eta_2} \phi_+ \]
and
\[ \frac{\partial \phi_-}{\partial \eta_2} = -i\phi_+ - \frac{iux}{4\eta_2} \phi_- \]

Differentiating each of the above two equations one more time with respect to \( \eta_2 \), we obtain
\[ \frac{\partial^2 \phi_+}{\partial \eta_2^2} = \left( -1 + \frac{-4iux - (ux)^2}{16\eta_2^2} \right) \phi_+ \]
and
\[ \frac{\partial^2 \phi_-}{\partial \eta_2^2} = \left( -1 + \frac{4iux - (ux)^2}{16\eta_2^2} \right) \phi_- \]
cf. (4.2). With \( \nu \) and \( \mu \) defined as in (4.8), it is evident that we can again take
\[ \phi_+ = \sqrt{\eta_2}H^{(2)}_\nu(\eta_2) \text{ or } \sqrt{\eta_2}H^{(1)}_\mu(\eta_2), \]
\[ \phi_- = \sqrt{\eta_2}H^{(2)}_{\nu-1}(\eta_2) \text{ or } \sqrt{\eta_2}H^{(1)}_{\mu-1}(\eta_2). \]

Let \( \Phi^{(1)} \), \( \Phi^{(2)} \) be two linearly independent solutions of equation (4.23), and put \( \Lambda = (\Phi^{(1)}, \Phi^{(2)}) \). Then \( \Lambda \) is a 2 \times 2 matrix, and as in (4.11) may be expressed as
\[ \Lambda = \begin{pmatrix} \delta_1\sqrt{\eta_2}H^{(2)}_\nu(\eta_2) + \delta_1\sqrt{\eta_2}H^{(2)}_\mu(\eta_2) & \delta_2\sqrt{\eta_2}H^{(1)}_\nu(\eta_2) + \delta_2\sqrt{\eta_2}H^{(1)}_\mu(\eta_2) \\ -\delta_1\sqrt{\eta_2}H^{(2)}_{\nu-1}(\eta_2) + \delta_1\sqrt{\eta_2}H^{(2)}_{\mu-1}(\eta_2) & -\delta_2\sqrt{\eta_2}H^{(1)}_{\nu-1}(\eta_2) + \delta_2\sqrt{\eta_2}H^{(1)}_{\mu-1}(\eta_2) \end{pmatrix}, \]
where \( \delta_1, \delta_1, \delta_2 \) and \( \delta_2 \) are constants to be determined. Write
\[ \Psi^{(0)} = \Lambda(1 + \varepsilon_2(x, \lambda)), \]
and substitute it into (1.11). By using the method of successive approximations, we get as in (4.13) and (4.14)
\[ \varepsilon_2(x, \lambda) = O(x^2 \lambda) \ll 1 \text{ in } \Omega^{(0)}, \]
and

\[ \Psi^{(0)} \sim \tilde{\Lambda}. \]  \hspace{1cm} (4.33)

To determine the constants \( \delta_1, \delta_1, \delta_2 \) and \( \delta_2 \), we insert (4.15) and (4.16) into (4.30), and use the prescribed behavior of \( \Psi^{(0)} \) given in (1.16). The result is

\[
\Psi^{(0)} \sim \frac{1}{2} \left( \frac{\pi}{2} \right)^{1/2} e^{iux/\pi} \sqrt{\eta_2} \times \left( 
\begin{array}{cc}
-ie^{i\varphi} H^{(2)}_\nu(\eta_2) + e^{-i\varphi} H^{(2)}_{\nu-1}(\eta_2) & -e^{i\varphi} H^{(1)}_{\nu}(\eta_2) + ie^{-i\varphi} H^{(1)}_{1-\nu}(\eta_2) \\
-ie^{i\varphi} H^{(2)}_\nu(\eta_2) + e^{-i\varphi} H^{(2)}_{\nu-1}(\eta_2) & e^{i\varphi} H^{(1)}_{\nu}(\eta_2) + ie^{-i\varphi} H^{(1)}_{1-\nu}(\eta_2)
\end{array}
\right).
\]  \hspace{1cm} (4.34)

Returning to (2.1), we have

\[
\psi^{(\infty)}_{21} = q_{11} \psi^{(0)}_{21} + q_{21} \psi^{(0)}_{22},
\]

\[
\psi^{(\infty)}_{22} = q_{12} \psi^{(0)}_{21} + q_{22} \psi^{(0)}_{22}.
\]  \hspace{1cm} (4.35)

Now consider these equations for \( \lambda \) in the overlapping region

\[ \Omega := \{ \lambda : 1 \ll r_1 < |\lambda| < r_2 \ll x^{-2}, -\pi < \arg \lambda < \pi \}; \]

see the beginning of Section 4. In this region, both \( \eta_1 \) and \( \eta_2 \) tend to zero. Inserting (4.20) into (4.11), and using (4.14), we obtain from (4.34)

\[
\sqrt{\eta_1} \left[ -iH^{(2)}_\nu(\eta_1) - H^{(2)}_{\nu-1}(\eta_1) \right] = \sqrt{\eta_2} \left[ q_{11} \left( ie^{i/2\varphi} H^{(2)}_\nu(\eta_2) + e^{-i/2\varphi} H^{(2)}_{\nu-1}(\eta_2) \right) + q_{21} \left( e^{i/2\varphi} H^{(2)}_{\nu}(\eta_2) + ie^{-i/2\varphi} H^{(2)}_{1-\nu}(\eta_2) \right) \right].
\]  \hspace{1cm} (4.37)

By (1.4), \( xux = r + O(x^2) \) as \( x \to 0 \). Thus, asymptotically, we can replace \( xux \) by \( r \) in (4.8), (4.22) and (4.34). When \( |\eta_i| \to 0 \), we have

\[ H^{(1)}_\nu(\eta_i) \sim -H^{(2)}_\nu(\eta_i) \sim -\left( \frac{1}{\pi} \right) \Gamma(\nu) \left( \frac{\eta_i}{2} \right)^{-\nu}, \text{ Re } \nu > 0, \]  \hspace{1cm} (4.38)

for \( i = 1, 2 \); see [1, p.360]. If \( \nu < 0 \), corresponding results can be given by using the reflection formulas

\[ H^{(1)}_\nu(\eta_i) = e^{-\nu \varpi} H^{(1)}_{1-\nu}(\eta_i), \quad H^{(2)}_\nu(\eta_i) = e^{\nu \varpi} H^{(2)}_{1-\nu}(\eta_i); \]  \hspace{1cm} (4.39)

see [7, p.239]. Applying (4.38) and (4.39), one can readily show that the left-hand side (LHS) of (4.36) is equal to

\[
-\frac{1}{\pi} \Gamma(\nu) \lambda^{-\varphi} x^{-\nu} 2^{\varphi+\frac{1}{2}} + ie^{-\nu \varphi} \frac{1}{\pi} \Gamma(1-\nu) \lambda^{\varphi} x^{-\nu} 2^{-\varphi+\frac{1}{2}}.
\]  \hspace{1cm} (4.40)

Here we have also made use of the facts that \( \nu \sim (\frac{1}{2} + \frac{i}{2} \varphi) \) and \( \eta_1 = x^2 \lambda/16 \); see (4.8) and (4.3), respectively. Similarly, it can be verified that the right-hand side (RHS) of (4.36) is equal to

\[
q_{11} \left( ie^{\varphi} x^{-\frac{1}{2}} \frac{1}{\pi} \Gamma(\nu) \lambda^{\varphi} 2^{\varphi+\frac{1}{2}} - ie^{-\nu \varphi} x^{-\nu} e^{-\varphi} \frac{1}{\pi} \Gamma(1-\nu) \lambda^{-\varphi} 2^{-\varphi+\frac{1}{2}} \right)
\]

\[
- q_{21} \left( ie^{\varphi} x^{-\frac{1}{2}} \frac{1}{\pi} \Gamma(\nu) \lambda^{\varphi} 2^{\varphi+\frac{1}{2}} + ie^{-\nu \varphi} x^{-\nu} e^{-\varphi} \frac{1}{\pi} \Gamma(1-\nu) \lambda^{-\varphi} 2^{-\varphi+\frac{1}{2}} \right).
\]  \hspace{1cm} (4.41)
Equating the coefficients of $x^{\nu+\frac{1}{2}}\lambda^\frac{1}{4}$ and $x^{-\nu-\frac{1}{2}}\lambda^{-\frac{1}{4}}$ in (4.40) and (4.41), we obtain respectively
\[ e^{-\frac{r\pi}{4}}\Gamma(1-\nu)2^{-\frac{\nu}{4}} = q_{111}e^{\frac{r\pi}{4}}\Gamma(\nu)2^{\nu} - q_{211}e^{-\frac{r\pi}{4}}\Gamma(\nu)2^{\nu} \] (4.42)
and
\[ \Gamma(\nu)2^{\nu} = q_{111}e^{-\frac{r\pi}{4}}e^{-\frac{r\pi}{4}}\Gamma(1-\nu)2^{-\frac{\nu}{4}} + q_{211}e^{-\frac{r\pi}{4}}\Gamma(1-\nu)2^{-\frac{\nu}{4}} \] (4.43)
Since $\nu \sim \frac{1}{2} + \frac{1}{4}$, equations (4.42) – (4.43) can be expressed in terms of the constants $A$ and $B$ in (1.8) and we have
\[ B = q_{111}\Gamma\left(\frac{1}{2} + \frac{ir}{4}\right)\Gamma\left(\frac{1}{2} - \frac{ir}{4}\right)e^{\frac{r\pi}{4}} - q_{211}\Gamma\left(\frac{1}{2} + \frac{ir}{4}\right)\Gamma\left(\frac{1}{2} - \frac{ir}{4}\right) \] (4.44)
and
\[ A = q_{111}\Gamma\left(\frac{1}{2} + \frac{ir}{4}\right)\Gamma\left(\frac{1}{2} - \frac{ir}{4}\right)e^{-\frac{r\pi}{4}} + q_{211}\Gamma\left(\frac{1}{2} + \frac{ir}{4}\right)\Gamma\left(\frac{1}{2} - \frac{ir}{4}\right). \] (4.45)
Addition of (4.44) and (4.45) gives
\[ A + B = q_{111}\left(e^{\frac{r\pi}{4}} + e^{-\frac{r\pi}{4}}\right) \frac{\pi}{\cosh(r\pi/4)} = 2\pi q_{111}. \] (4.46)
From (4.44) and (4.45), we also have
\[ Ae^{\frac{r\pi}{4}} - Be^{-\frac{r\pi}{4}} = q_{211}\left(e^{\frac{r\pi}{4}} + e^{-\frac{r\pi}{4}}\right) \frac{\pi}{\cosh(r\pi/4)} = 2\pi q_{211}. \] (4.47)
Thus,
\[ q_{111} = \frac{A + B}{2\pi}, \quad q_{211} = \frac{1}{2\pi} \left(Ae^{\frac{r\pi}{4}} - Be^{-\frac{r\pi}{4}}\right). \] (4.48)
Recall the statement following (3.46) that $p$ and $q$ are complex conjugates of each other. Furthermore, from (1.8) it is evident that $A$ and $B$ are also complex conjugates of each other. Hence, it follows from (2.2) that
\[ q_{12} = \frac{1}{2\pi} \left(Ae^{-\frac{r\pi}{4}} - Be^{\frac{r\pi}{4}}\right), \quad q_{22} = \frac{A + B}{2\pi}. \] (4.49)
Also from (2.2), we have
\[ p = \frac{Ae^{-\frac{r\pi}{4}} - Be^{\frac{r\pi}{4}}}{A + B}, \quad q = \frac{-Ae^{\frac{r\pi}{4}} + Be^{-\frac{r\pi}{4}}}{A + B}. \] (4.50)
and
\[ \frac{1}{\sqrt{1 + |p|^2}} = \frac{1}{\sqrt{1 + pq}} = \frac{A + B}{2\pi} = \frac{\text{Re } A}{\pi}. \] (4.51)
Together with (3.49) and (3.50), we have established (1.6), (1.7) and (1.8). Formula (1.9) is relatively easier to verify, and a proof can be found in [4, pp.473-474].

REFERENCES
[1] M. Abramowitz and I. A. Stegun, “Handbook of Mathematical Functions with Formulas, Graphs, and Mathematical Tables,” Dover Publications, 1965.
[2] A. P. Bassom, P. A. Clarkson, C. K. Law and J. B. McLeod, Application of uniform asymptotics to the second Painlevé transcendent, Arch. Rational Mech. Anal., 143 (1998), 241–271.
[3] C. M. Bender and S. A. Orszag, “Advanced Mathematical Methods for Scientists and Engineers,” McGraw-Hill Book Co., New York, 1978.
[4] A. S. Fokas, A. R. Its, A. A. Kapaev and V. Y. Novokshenov, “Painlevé Transcendents: The Riemann Hilbert Approach,” American Mathematical Society, Providence, RI, 2006.
Received November 2007; revised February 2008.

E-mail address: racwong@cityu.edu.hk
E-mail address: h.y.zhang@student.cityu.edu.hk