CHAOTICITY AND REGULAR ACTION OF
DIFFEOMORPHISMS GROUP OF \( \mathbb{K}^n \)

YAHYA N’DAO AND ADlene Ayadi

Abstract. In this paper, we introduce the notion of regular action of any subgroup \( G \) of \( Diff^r(\mathbb{K}^n) \), \( r \geq 1 \) on \( \mathbb{K}^n \), \( (\mathbb{K} = \mathbb{R} \) or \( \mathbb{C} \) (i.e. the closure of every orbit of \( G \) in some open set is a topological sub-manifold of \( \mathbb{K} \) over \( \mathbb{R} \)). We prove that if \( G \) is a Lie group with a regular action then its action cannot be chaotic. Moreover, we prove that the action of any abelian lie group is regular and it cannot be chaotic.

1. Introduction

Denote by \( Diff^r(\mathbb{K}^n) \), \( r \geq 1 \), \( (\mathbb{K} = \mathbb{R} \) or \( \mathbb{C} \) the group of all \( C^r \)-diffeomorphisms of \( \mathbb{K}^n \). Let \( G \) be a subgroup of \( Diff^r(\mathbb{K}^n) \), \( r \geq 1 \). There is a natural action \( G \times \mathbb{K}^n \rightarrow \mathbb{K}^n \). \( (f, x) \mapsto f(x) \). For a point \( x \in \mathbb{K}^n \), denote by \( G(x) = \{ f(x), f \in G \} \subset \mathbb{K}^n \) the orbit of \( G \) through \( x \). A subset \( E \subset \mathbb{K}^n \) is called \( G \)-invariant if \( f(E) \subset E \) for any \( f \in G \); that is \( E \) is a union of orbits. Denote by \( \overline{E} \) (resp. \( \text{int} E \)) the closure (resp. interior) of \( E \). A topological space \( X \) is called a topological manifold with dimension \( r \geq 0 \) if every point has a neighborhood homeomorphic to \( \mathbb{R}^r \). This means that the image of any topological manifold by a homeomorphism is a topological manifold with the same dimension. An orbit \( \gamma \) is called regular with order \( \text{ord}(\gamma) = m \) if for every \( y \in \gamma \) there exists an open set \( O \) containing \( y \) such that \( \gamma \cap O \) is a topological sub-manifold of \( \mathbb{K}^n \) with dimension \( m \) over \( \mathbb{K} \). In particular, \( \gamma \) is locally dense in \( \mathbb{K}^n \) if and only if \( m = 2n \), and it is discrete if and only if \( m = 0 \). Notice that, the closure of a regular orbit is not necessary a manifold (see example 8.4). We say that the action of \( G \) is regular on \( \mathbb{K}^n \) if every orbit of \( G \) is regular. The action of \( G \) is called chaotic if \( G \) has a dense orbit and the union of periodic orbits (orbit finite) is dense in \( \mathbb{K}^n \) (cf. [11], [12], [15]). The action of \( G \) is called pseudo-chaotic if \( G \) has a dense orbit and the union of regular orbits with order 0 is dense in \( \mathbb{K}^n \). See that every chaotic action is pseudo-action. Here, the question to investigate is the following:

A regular action of any subgroup of \( Diff(\mathbb{K}^n) \) can be chaotic?

The notion of regular orbit is a generalization of non exceptional orbit defined for the action of any group of diffeomorphisms on \( \mathbb{K}^n \). A nonempty subset \( E \subset \mathbb{K}^n \) is a minimal set if for every \( y \in E \) the orbit of \( y \) is dense in \( E \). An orbit with its closure

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is a Cantor set is called an exceptional orbit. Their dynamics were recently initiated for some classes in different point of view, (see for instance, [3],[4],[5],[6],[7],[9]).

In [14], A.C. Naolekar and P. Sankaran construct chaotic actions of certain finitely generated abelian groups on even-dimensional spheres, and of finite index subgroups of $SL(n,\mathbb{Z})$ on tori. They also study chaotic group actions via compactly supported homeomorphisms on open manifolds.

In [16] P.W.Michor and C.Vizman proved that some groups of diffeomorphisms of a manifold $M$ act $n$-transitively for each finite $n$ (i.e. for any two ordered sets of $n$ different points $(x_1,\ldots,x_n)$ and $(y_1,\ldots,y_n)$ in $M$ there is a smooth diffeomorphism $f$ in the group such that $f(x_i) = y_i$ for each $i$).

In [13], A.Ayadi, H.Marzougui and Y.N’dao studied the dynamic of the affine diffeomorphisms groups, whose generalize the structure’s theorem given in [1] for abelian linear group.

For a subset $E \subset \mathbb{K}^n$, denote by $\text{vect}(E)$ the vector subspace of $\mathbb{K}^n$ generated by all elements of $E$.

Our principal results can be stated as follows:

**Theorem 1.1.** Let $G$ be a lie subgroup of $\text{Diff}^r(\mathbb{K}^n)$, $r \geq 1$. If the natural action of $G$ on $\mathbb{K}^n$ is regular then it can not be pseudo-chaotic. In particular, it can not be chaotic.

For the abelian group there are the important results for the case $\text{Fix}(G) \neq \emptyset$.

**Theorem 1.2.** Let $G$ be an abelian lie subgroup of $\text{Diff}^r(\mathbb{K}^n)$, $r \geq 1$. Then the natural action of $G$ on $\mathbb{K}^n$ is regular.

**Corollary 1.3.** Let $G$ be an abelian lie subgroup of $\text{Diff}^r(\mathbb{K}^n)$, $r \geq 1$. Then the natural action of $G$ on $\mathbb{K}^n$ can not be chaotic.

As a directly consequence of Theorem 1.1, we prove the regularity action of any abelian linear group on $\mathbb{K}^n$.

**Corollary 1.4.** Let $L$ be an abelian subgroup of $\text{GL}(n,\mathbb{K})$. Then the natural action of $L$ on $\mathbb{K}^n$ is regular.

This paper is organized as follows: In Section 2, we proof the Theorem 1.1. In Section 3, we introduce algebra results for abelian lie group. The section 4, is devoted to study the regular action of abelian lie group, moreover, we prove the Theorem 1.2 and Corollaries 1.3 and 1.4.
2. Proof of Theorem [1.1]

2.1. Exponential map. In this section, we illustrate the theory developed of the group $Diff(\mathbb{R}^n)$ of diffeomorphisms of $\mathbb{R}^n$. For simplicity, throughout this section we only consider the case of $\mathbb{K} = \mathbb{R}$; however, all results also hold for complexes case. The group $Diff(\mathbb{R}^n)$ is not a Lie group (it is infinite-dimensional), but in many way it is similar to Lie groups. For example, it easy to define what a smooth map from some Lie group $G$ to $Diff(\mathbb{R}^n)$ is: it is the same as an action of $G$ on $\mathbb{R}^n$ by diffeomorphisms. Ignoring the technical problem with infinite-dimensionality for now, let us try to see what is the natural analog of the Lie algebra $\mathfrak{g}$ for the group $G$. It should be the tangent space at the identity; thus, its elements are derivatives of one-parameter families of diffeomorphisms.

Let $\varphi^t : G \to G$ be one-parameter family of diffeomorphisms. Then, for every point $m \in G$, $\varphi^t(m)$ is a curve in $G$ and thus $\frac{d}{dt} \varphi^t(m)_{/t=0} = \xi(m) \in T_mG$ is a tangent vector to $G$ at $m$. In other words, $\frac{d}{dt} \varphi^t$ is a vector field on $G$.

The exponential map $exp : g \to G$ is defined by $exp(x) = \gamma_x(1)$ where $\gamma_x(t)$ is the one-parameter subgroup with tangent vector at 1 equal to $x$.

If $\xi \in g$ is a vector field, then $exp(t\xi)$ should be one-parameter family of diffeomorphisms whose derivative is vector field $\xi$. So this is the solution of differential equation

$$\frac{d}{dt} \varphi^t(m)_{/t=0} = \xi(m).$$

In other words, $\varphi^t$ is the time $t$ flow of the vector field. Thus, it is natural to define the Lie algebra of $G$ to be the space $g$ of all smooth vector $\xi$ fields on $\mathbb{R}^n$ such that $exp(t\xi) \in G$ for every $t \in \mathbb{R}$.

2.2. Proof of Theorem [1.1]. In this section suppose that the action of the lie group $G$ is regular. Denote by:
- $U_r = \{x \in \mathbb{K}^n : ord(G(x)) \geq r\}$.
- $g$ be the lie algebra associated to $G$.
- The exponential map $exp : g \to G$ the exponential map defined above.

**Lemma 2.1.** Let $x \in \mathbb{K}^n$. Then $G(x)$ is regular with order $r \geq 0$ if and only if there exist an open set $O_x$ containing $G(x)$ such that $G(x) \cap O_x$ is a manifold with dimension $r \geq 0$.

**Proof.** $\implies$ is obvious by definition.

$\Leftarrow$ Let $x \in G(u)$ and $O$ be an open neighborhood of $x$ such that $\overline{G(u)} \cap O_x$ is a manifold with dimension $r \geq 0$ over $\mathbb{K}$. Let $y \in G(u)$, then $y = f(x)$ for some $f \in G$. So $O'_x = f(O_x)$ is an neighborhood of $y$ and satisfying $\overline{G(u)} \cap O'_x = f(\overline{G(u)} \cap O)$ is a manifold with dimension $r \geq 0$ over $\mathbb{K}$. It follows that $G(u)$ is regular with order $r$. 

Denote by:
- $M(x) = \overline{G(x)} \cap O_x$, is a manifold with dimension $ord(G(x))$ and $O_x$ is an open subset of $\mathbb{K}^n$ containing $G(x)$ given by Lemma 2.1.
Lemma 2.2. Let $x \in \mathbb{K}^n$. Then $G(x)$ is regular with order $p$ if and only if there exist $f_1, \ldots, f_p \in g$ such that $\frac{\partial}{\partial t}(\exp(t \ast f_1)(x))/t=0, \ldots, \frac{\partial}{\partial t}(\exp(t \ast f_p)(x))/t=0$ generate $T_x(M(x))$.

Proof. By definition of the tangent space on $M(x)$ we have

$$T_xM(x) = \text{vect} \left\{ \frac{\partial}{\partial t}(\exp(t \ast f)(x))/t=0, \ f \in g \right\}.$$ 

The proof follows from the fact that $\dim(M(x)) = \dim(T_xM(x))$. □

Lemma 2.3. if $U_r \neq \emptyset$ then it is an open subset of $\mathbb{K}^n$.

Proof. Let $x \in U_r$ and $p = \dim(M(x)) \geq r$. By Lemma 2.2 there exist $f_1, \ldots, f_p \in g$ such that $\frac{\partial}{\partial t}(\exp(t \ast f_1)(x))/t=0, \ldots, \frac{\partial}{\partial t}(\exp(t \ast f_p)(x))/t=0$ generate $T_xM(x)$. Denote by $v_k(x) = \frac{\partial}{\partial t}(\exp(t \ast f_k)(x))/t=0$ for every $k = 1, \ldots, p$. Consider, for all $z \in \mathbb{K}^n$, the Gram’s determinant

$$\Delta(z) = \det((v_i(z) \mid v_j(z)))_{1 \leq i,j \leq p}$$

of the vectors $v_1(z), \ldots, v_p(z)$ where $\langle \cdot | \cdot \rangle$ denotes the scalar product in $\mathbb{K}^n$. It is well known that these vectors are independent if and only if $\Delta(z) \neq 0$, in particular $\Delta(x) \neq 0$. Let

$$V_x = \{z \in \mathbb{K}^n, \ \Delta(z) \neq 0\}$$

Since the map $z \mapsto \Delta(z)$ is continuous, $V_x$ is an open set of $\mathbb{K}^n$. Now $\Delta(x) \neq 0$, and so $x \in V_x \subset U_r$. We conclude that $U_r$ is an open set. □

Proof of Theorem 1.1. Suppose that the action of the lie group $G$ is chaotic, then $G$ has a dense orbit denoted by $G(x), x \in \mathbb{K}^n$. So $G(x)$ is regular with order $n$. By Lemma 2.3 $U_n$ is a non empty open subset of $\mathbb{K}^n$. By construction, for every $y \in U_n$, $G(y)$ is regular with order $n$, so it is locally dense in $\mathbb{K}^n$. This means that if $\mathcal{P}$ is the union of all pseudo-periodic orbits then $U_r \cap \mathcal{P} = \emptyset$, so $\mathcal{P}$ can not be dense in $\mathbb{K}^n$. The proof is completed. □

3. Algebra results for abelian lie group

For a subset $E \subset \mathbb{K}^n$, denote by $\text{vect}(E)$ the vector subspace of $\mathbb{K}^n$ generated by all elements of $E$. Set $\mathcal{A}(G)$ be the algebra generated by $G$. In general, $\mathcal{A}(G)$ is not commutative. For a fixed vector $x \in \mathbb{K}^n \backslash \{0\}$, denote by:
- $\Phi_x : \mathcal{A}(G) \rightarrow \Phi_x(\mathcal{A}(G)) \subset \mathbb{K}^n$ the linear map given by $\Phi_x(f) = f(x)$.
- $E(x) = \Phi_x(\mathcal{A}(G))$.

Lemma 3.1. Let $G$ be an abelian subgroup of $\text{Diff}^r(\mathbb{K}^n)$, $r \geq 1$ and $x \in \mathbb{K}^n$. Then $E(x)$ is $G$-invariant.
It is clear that $E(x)$ is a vector space since $\Phi_x$ is a linear map. Suppose that $E(x)$ is generated by $f_1(x), \ldots, f_p(x)$, with $f_k \in G$, $k = 1, \ldots, p$. Let $y = \sum_{k=1}^{p} \alpha_k f_k(x) \in E(x)$ and $f \in G$, then $y = g(x)$, with $g = \sum_{k=1}^{p} \alpha_k f_k \in \mathcal{A}(G)$. Therefore $f(y) = f \circ g(x) = \Phi_x(f \circ g) \in E(x)$, since $f \circ g \in \mathcal{A}(G)$. 

4. Regularity action of abelian lie group

4.1. Whitney Topology on $C^0(\mathbb{K}^n, \mathbb{K}^n)$. For each open subset $U \subset \mathbb{K}^n \times \mathbb{K}^n$ let $\overline{U} \subset C^0(\mathbb{K}^n, \mathbb{K}^n)$ be the set of continuous functions $g$, whose graphs $\{(x, g(x)) \in \mathbb{K}^n \times \mathbb{K}^n, x \in \mathbb{K}^n\}$ is contained in $U$. It is not difficult to verify the family $U_j$ given in the following, defines a topology in $C^0(\mathbb{K}^n, \mathbb{K}^n)$: $U_j$ is the compact family of the balls $U_{q, r} = \{x \in \mathbb{K}^n, \|x - q\| < \frac{1}{j}\}$, (resp. $U_{p, q, r} = \{x \in \mathbb{K}^n, \|x - p - iq\| < \frac{1}{j}\}$), $p, q \in \mathbb{Q}^n$, $r \in \mathbb{N}^*$. 

We want to construct a neighborhood basis of each function $f \in C^0(\mathbb{K}^n, \mathbb{K}^n)$. Let $K_j = \{x \in \mathbb{K}^n, \|x\| \leq j\}$ be a countable family of compact sets (closed balls with center 0) covering $\mathbb{K}^n$ such that $K_j$ is contained in the interior of $K_{j+1}$. Consider then the compact subsets $L_j = K_j \setminus \text{Int}(K_{j-1})$, which are compact sets, too. Let $\varepsilon = (\varepsilon_j)_j$ be a sequence of positive numbers and then define 

$$V_{(f, \varepsilon)} = \{f \in C^0(\mathbb{K}^n, \mathbb{K}^n) : \|f(x) - g(x)\| < \varepsilon_j, \text{ for any } x \in L_j, \forall j\}.$$ 

We claim this is a neighborhood system of the function $f$ in $C^0(\mathbb{K}^n, \mathbb{K}^n)$. Since $L_i$ is compact, the set $U = \{(x, y) \in \mathbb{K}^n \times \mathbb{K}^n : \|f(x) - g(x)\| < \varepsilon_j, \text{ if } x \in L_j\}$ is open. Thus, $V_{(f, \varepsilon)} = \overline{U}$ is an open neighborhood of $f$. On the other hand, if $O$ is an open subset of $\mathbb{K}^n \times \mathbb{K}^n$ which contains the graph of $f$, then since $L_j$ is compact, it follows that there exists $\varepsilon_j > 0$ such that if $x \in L_j$ and $\|y - f(x)\| < \varepsilon_j$, then $(x, y) \in O$. Thus, taking $\bar{\varepsilon} = (\varepsilon_j)_j$ we have $V_{(f, \varepsilon)} \subset \overline{V}$, so we have obtained the family $V_{(f, \varepsilon)}$ is a neighborhood system of $f$. Moreover, for each given $\varepsilon = (\varepsilon_j)_j$, we can find a $C^\infty$-function $\varepsilon : \mathbb{K}^n \rightarrow \mathbb{R}_+$, such that $\varepsilon(x) < \varepsilon_j$ for any $x \in L_j$. It follows that the family $V_{(f, \varepsilon)} = \{(x, y) \in \mathbb{K}^n \times \mathbb{K}^n : \|f(x) - g(x)\| < \varepsilon(x)\}$ is also a neighborhood system.

**Lemma 4.1.** The linear map $\Phi_x : \mathcal{A}(G) \rightarrow E(x)$ is continuous.

**Proof.** Firstly, we take the restriction of the Whitney topology to $\mathcal{A}(G)$. Secondly, let $f \in \mathcal{A}(G)$ and $\varepsilon > 0$. Then for $\varepsilon = (\varepsilon_i)_i$ with $\varepsilon_i = \varepsilon$ and for $V_{(f, \varepsilon)}$ be a neighborhood system of $f$, we obtain: for every $g \in V_{(f, \varepsilon)} \cap \mathcal{A}(G)$ for every $y \in L_i$, $\|f(y) - g(y)\| < \varepsilon$, $\forall i$. In particular for $i = i_0$ in which $x \in L_{i_0}$, we have $\|f(x) - g(x)\| < \varepsilon$, so $\|\Phi_x(f) - \Phi_x(g)\| < \varepsilon$. It follows that $\Phi_x$ is continuous.

Denote by $g$ the lie algebra of $G$ and $p = \text{dim}(g)$. Since $G$ is abelian so is $g$. Set $f_1, \ldots, f_p \in g$ be the generators of $g$. We let:

- $\exp : g \rightarrow G$ the lie exponential map associated to $G$.
- $G_0$ be the connected component of $G$ containing the identity map $\text{id}$. So $G_0$ is generated by $\exp(g)$ and it is an abelian lie subgroup of $G$. Since $g$ is abelian, $G_0 = \exp(g)$.

For a fixed point $x \in \mathbb{K}^n$, denote by:
- $\mathcal{G}_x = \{ f \in \mathcal{G}_0, \ f(x) = x \}$ the stabilizer of $\mathcal{G}_0$ on the point $x$. It is a lie subgroup of $\mathcal{G}_0$.

**Proposition 4.2.** (\cite{1}, Theorem 3.29) Let $\widetilde{\mathcal{G}}$ be a Lie group acting on $\mathbb{K}^n$ with lie algebra $\mathfrak{g}$ and let $u \in \mathbb{K}^n$.

(i) The stabilizer $\mathcal{G}_u = \{ B \in \mathcal{G} : \ Bu = u \}$ is a closed Lie subgroup in $\mathcal{G}$, with Lie algebra $\mathfrak{h}_u = \{ B \in \mathfrak{g} : \ Bu = 0 \}$.

(ii) The map $\mathcal{G} \times \mathcal{G}_u \to \mathbb{K}^n$ given by $B, \mathcal{G}_u \mapsto Bu$ is an immersion. Thus, the orbit $\mathcal{G}(u)$ is an immersed submanifold in $\mathbb{K}^n$. In particular $\dim(\mathcal{G}(u)) = \dim(\mathfrak{g}) - \dim(\mathfrak{h}_u)$.

Denote by:

- $H$ be the algebra associated to $\mathcal{G}_x$ and $F_x$ is the supplement of $H_x$ in $\mathfrak{g}$ (i.e. $F_x \oplus H_x = \mathfrak{g}$). By Proposition 4.2 we have $H_x = \{ f \in \mathfrak{g}, \ f(x) = 0 \}$ and $G_0 = \exp(F_x) \circ \exp(H_x)$.

In particular $G_0(x) = \Phi_x(\exp(F_x))$. The restriction $\Phi_x(\exp(F_x) \cap V) \to \Phi_x(\exp(F_x) \cap V) \subset G_0(x)$ of $\Phi_x$ is an immersed submanifold in $\mathbb{K}^n$.

**Proposition 4.3.** Let $G$ be an abelian subgroup of $\text{Diff}^r(\mathbb{K}^n)$, $r \geq 1$ such that $0 \in \text{Fix}(G)$ and $x \in \mathbb{K}^n$. Then:

(i) $G_0(x)$ is the connected component of $G(x)$ containing $x$.

(ii) The restriction $\Phi_x^{(1)} : \exp(F_x) \cap V \to \Phi_x(\exp(F_x) \cap V) \subset G_0(x)$ of $\Phi_x$ to $\exp(F_x) \cap V$ is an homogeneous group.

**Proof.** (i) By Lemma 4.1 the map $\Phi_x : A(G) \to E(x) \subset \mathbb{K}^n$ is a continuous surjective linear map. The proof follows then from the fact that $G_0(x) = \Phi_x(G_0)$.

(ii) By Lemma 4.1 the map $\Phi_x^{(1)}$ is continuous, surjective.

It is injective: Indeed, if $f, g \in \exp(F_x) \cap V$ such that $\Phi_x^{(1)}(f) = \Phi_x^{(1)}(g)$, then $f(x) = g(x)$, so $g^{-1} \circ f(x) = x$. Hence $g^{-1} \circ f \in G_x \cap \exp(F_x) = \{ id \}$ (Lemma,...).

It follows that $f = g$.

$(\Phi_x^{(1)})^{-1} : \Phi_x(\exp(F_x) \cap V) \to \exp(F_x) \cap V$ is continuous; indeed, let $y = \exp(f)(x) \in \Phi_x(\exp(F_x) \cap V), f \in F_x$ and $(y_m)$ be a sequence in $\Phi_x(\exp(F_x) \cap V)$ tending to $y$. Suppose that $f_1, \ldots, f_q$ generate $F_x$, for some $q \leq p$. Set $y_m = \exp(t_1 f_1 + \cdots + t_q f_q)(x)$ and $y = \exp(t_1 f_1 + \cdots + t_q f_q)(x)$, with $|t_k| < 1$ and $|t_k, m| < 1$. We can assume (leaving to take a subsequence) that $\lim_{m \to +\infty} t_k, m = s_k, $ with $|s_k| \leq 1$ for every $k = 1, \ldots, q$. Write $y = \exp(s_1 f_1 + \cdots + s_q f_q)$ and $y_m = \exp(t_1 f_1 + \cdots + t_q f_q)$. By continuity of the exponential map we have $(g_m)_m$ tends to $g$ when $m \to +\infty$. By continuity of $\Phi_x$ we obtain $y_m = \Phi_x(y_m)$ tends to $y = \Phi_x(g)$, so $s_k = t_k$ for every $k = 1, \ldots, p$. As $g = (\Phi_x^{(1)})^{-1}(y)$ and $y_m = (\Phi_x^{(1)})^{-1}(y_m)$, it follows that $(\Phi_x^{(1)})^{-1}(y_m)$ tends to $(\Phi_x^{(1)})^{-1}(y)$. This completes the proof.

**Corollary 4.4.** (Under notations of Proposition 4.3) The set $B(x) = \Phi_x(\exp(F_x) \cap V)$ is a topological submanifold of $\mathbb{K}^n$ containing $x$. Moreover, there exists an open subset $W$ of $\mathbb{K}^n$ such that $W \cap G(x) = B(x)$. 
Proof. By construction $\exp(F_x)$ is a lie subgroup of $G_0$, so it is a topological manifold. \textcolor{red}{By Proposition 4.3, $B(x)$ is homeomorphic to $\exp(F_x) \cap V$ which is an open subset of $\exp(F_x)$. Then $B(x)$ is a topological manifold with dimension equal to $\dim(\exp(F_x))$.} On the other hand, by (i), $G_0(x) = \Phi_x(\exp(F_x))$ is a connected component of $G(x)$ containing $x$, then there exists an open subset $O$ of $\mathbb{K}^n$ such that $O \cap G(x) = G_0(x)$. Since $\dim(B(x)) = \dim(\exp(F_x)) = \dim(F_x)$. By Proposition 4.2, $G_0(x)$ is an immersed submanifold of $\mathbb{K}^n$ with dimension $\dim(F_x) = \dim(g) - \dim(H_x)$ because $g$ is also the lie algebra of $G_0$. Therefore $\dim(B(x)) = \dim(G_0(x))$, so $B(x)$ is an open subset of $G_0(x)$. Then there exists an open subset $W$ of $\mathbb{K}^n$ containing $x$ and contained in $O$ such that $G_0(x) \cap W = B(x)$. It follows that $W \cap G(x) = G_0(x) \cap W = B(x)$. The proof is completed.

\textbf{Lemma 4.5.} For every neighborhood $W$ of a point $x \in \mathbb{K}^n$, we have $G(x) \cap W = G(x) \cap W \cap W$.

\textbf{Proof.} It is clear that $G(x) \cap W \cap W \subset G(x) \cap W$. Now, let $y \in G(x) \cap W$ then there exists a sequence $(y_m)_m$ in $G(x)$ tending to $y$. So $y_m \in W$ from some row $m_0$. Thus $y \in G(x) \cap W \cap W$. \hfill $\square$

\textbf{Proof of Theorem 4.3} \textbf{Let $G$ be an abelian subgroup of $Diff^r(\mathbb{K}^n)$, $r \geq 1$ with $Fix(G) \neq \emptyset$. We can assume that $0 \in Fix(G)$, otherwise, we replace $G$ by $T_{-a} \circ G \circ T_a$, for some vector $a \in Fix(G)$. By Corollary 4.4, there exists an open subset $W$ of $\mathbb{K}^n$ such that $W \cap G(x) = B(x)$ is a submanifold of $\mathbb{K}^n$. So $B(x)$ is locally closed, we can assume that $B(x) \cap W = B(x)$. Therefore, by Lemma 4.3 we have $G(x) \cap W = G(x) \cap W \cap W$, so $B(x) \subset G(x) \cap W = G(x) \cap W \cap W = B(x)$ Hence $G(x) \cap W = B(x)$ is a topological manifold. By Lemma 2.1 it follows that $G(x)$ is regular. We conclude that the action of $G$ is regular. \hfill $\square$

\textbf{Proof of Corollary 4.3} The proof results directly from Theorem 4.1 and Theorem 4.2 \hfill $\square$

\textbf{Proof of Corollary 4.4} The proof results directly from Theorem 4.2 \hfill $\square$

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Yahya N’dao, University of Moncton, of mathematics and statistics, Canada  
E-mail address: yahiandao@yahoo.fr

Adlene Ayadi, University of Gafsa, Faculty of sciences, Department of Mathematics, Gafsa, Tunisia.  
E-mail address: adlenesoo@yahoo.com