Abstract. N=2 heterotic strings may provide a window into the physics of M-theory radically different than that found via the other supersymmetric string theories. In addition to their supersymmetric structure, these strings carry a four-dimensional self-dual structure, and appear to be completely integrable systems with a stringy density of states. These lectures give an overview of N=2 heterotic strings, as well as a brief discussion of possible applications of both ordinary and heterotic N=2 strings to D-branes and matrix theory.

1. Introduction

A few years ago, if asked to describe string theory, the average practitioner would have classified its different manifestations according to their various worldsheet gauge principles. On the 1+1 dimensional worldsheet, there can be \((p,q)\) supersymmetries that square to translations along the (left,right)-handed light cone; one says that the worldsheet has \((p,q)\) gauged supersymmetry. The bosonic string has no supersymmetry; \(p = q = 0\). The supersymmetric string theories have, say, \(q = 1\). Thus type IIA/B string theory has \((1,1)\) supersymmetry. The type I/IA strings are the orbifold of these by worldsheet parity, and the heterotic strings are in the class \((0,1)\). Remarkably, we now understand that all the supersymmetric string theories – type IIA/B, type I, and heterotic – appear to describe asymptotic expansions of a single nonperturbative master theory: M-theory. This theory has many miraculous duality properties that are now only beginning to be unravelled; other lecturers at this school will review the current state of affairs. In these lectures, I will give an overview of a relatively unexplored corner of string theory, namely the N=2 strings \([1, 2, 3]\) (more specifically strings with \((2,2)\) or \((2,1)\) gauged worldsheet supersymmetry).
The driving force behind the recent unification has been the recognition of the fundamental importance of the spacetime (as opposed to worldsheet) supersymmetry algebra. The small (BPS) representations of the supersymmetry algebra form a quasi-topological sector of the theory. By tracking this BPS spectrum across moduli space, one can deduce the interconnections of the various string limits. The issue we wish to address is the role of the heterotic (2,1) string, which is also a stringy realization of spacetime supersymmetry and therefore ought to play a role somehow. As we will see [4, 5], many of the basic objects of M-theory are realized in the (2,1) string. Self-duality and integrability are further features of the (2,1) string, arising from the chiral sector with N=2 worldsheet supersymmetry.

We will see that the chiral critical dimensions of the (2,1) string are $d = 4$ (2 space, 2 time) for the N=2 sector, and $d = 12$ (10 space, 2 time) for the N=1 sector. To see the relation to M-theory, consider standard heterotic target space geometry. Here this is a 2+2 dimensional base manifold for the dimensions common to both chiralities, with the additional left-movers parametrizing an eight-dimensional torus of stringy dimensions fibered over it (see figure 1).

![Figure 1. (2,1) heterotic geometry.](image)

The geometrical fields are the graviton $h$, antisymmetric tensor field $b$, gravitino $\chi$, and the gauge connection $\varepsilon$ on the fiber. These fields are further restricted by the extra constraints of N=2 local supersymmetry on the worldsheet, giving rise to prepotentials

\begin{align*}
h_{\mu\nu} + b_{\mu\nu} &\rightarrow I_\lambda^\mu \partial_\lambda a_\nu \\
\varepsilon_\mu^a &\rightarrow I_\mu^a \partial_\lambda \varphi^a \\
\chi_\mu^a &\rightarrow I_\mu^a \partial_\lambda \psi^a.
\end{align*}

(1)
Here $I^a_\mu$ is an almost complex structure on the base space, which is required by the local N=2 worldsheet supersymmetry. Under linearized gauge transformations, these restricted fields transform as

$$
\delta(h + b) = \partial \xi \quad \to \quad \delta a \sim \xi \\
\delta \varepsilon = \partial \Lambda \quad \to \quad \delta \varphi \sim \Lambda \\
\delta \chi = \partial \eta \quad \to \quad \delta \psi \sim \eta.
$$

In other words, the remnant fields are Nambu-Goldstone fields of spontaneously broken symmetries (spacetime antisymmetric tensor field gauge transformations, translations, supersymmetries) on a D-brane. More precisely, one has in a complex coordinate basis $\varepsilon = i(\bar{\partial} - \partial)\varphi + (\partial + \bar{\partial})\theta = \partial(\theta + i\varphi) + \bar{\partial}(\theta - i\varphi)$, where the (real) gauge symmetry is $\delta \theta = \Lambda$. However, one can go to a holomorphic basis by complexifying the gauge group $G$; then $\varphi$ is a coordinate on $G_C/G$ whose dynamics is determined by holomorphic gauge symmetry much as in the 2d WZW model [6, 7, 8]. This virtually guarantees us a connection to brane physics, since brane dynamics is almost by definition given by the nonlinear Lagrangian of the spacetime symmetries broken by the brane. A picture of this aspect of (2,1) string theory is shown in figure 2. The (2,1) string worldsheet maps into the worldvolume of a D-brane, which is itself embedded in spacetime. The quanta of this brane are the (2,1) strings themselves. Since the transverse fluctuations of the brane can be traced to those of the fiber connection $\varepsilon \sim \partial \varphi$, while the longitudinal directions are those of the base space, we see that spacetime itself is the total space of the heterotic geometry of figure 1.

![Figure 2. Chain of brane embeddings implied by (2,1) string states.](image)

The dimension of the brane is determined by the number of independent momentum components in string vertex operators $O_\alpha e^{ik\cdot x}$. When the (2,1) string target space is $\mathbb{R}^{2,2} \times T^8$, this kinematics is 1+1 or 2+1 dimensional;
when the spatial dimensions are further compactified, the kinematics is more or less 9+1 dimensional – the target is a kind of ninebrane.

As mentioned above, an additional geometric structure is self-duality. The almost complex structure $I^A_\mu$ is one of a triplet of such structures preserved by the target space geometry (these almost complex structures are not integrable, since $(\partial - \bar{\partial})I = db = \text{torsion}$). Thus one can bring to bear the machinery of twistor theory to characterize the classical solution space. One of the characteristic properties of self-dual gravity is the symmetry group of area preserving diffeomorphisms $SDiff^2_2 \sim SU(\infty)$, suggesting a connection to matrix theory [9].

There are indeed a number of intriguing analogies among the matrix model of M-theory, (2,1) strings, and other matrix models:

1. The (BFSS) matrix model of M-theory [9] realizes the membrane as a collective phenomenon of D0-brane supergravitons, i.e. as a state in the large N collective field theory. In a sense, the BFSS matrix model gives a map from a single noncommutative torus into ‘spacetime’. In the graviton limit, the matrices approximately commute; in the membrane limit, the commutators are large. Similarly, the (2,1) string describes a map from a single brane into spacetime.

2. Other examples where large N collective field theory generates string theory as an asymptotic expansion around a particular master field include:
   (a) The 1+1d noncritical string based on the original matrix model of [10].
   (b) 2d Yang-Mills [11].
   (c) 2+1d SU(k) Chern-Simons theory as $k,N \to \infty$ [12].
   (d) 2+2d self-dual gravity [13],[1]. This connection will be outlined in section 2 below.

3. (2,1) string perturbation theory is an expansion around the infinite tension limit; $T_{\text{brane}} \sim g_s^{-2}$. In other words, it is an expansion in low energy relative to the brane tension. This is similar to 1+1d noncritical string theory, where the effective expansion parameter is the energy of excitations relative to some scale (in that case the cosmological constant $\mu$ of Liouville theory, or equivalently the fermi energy of the matrix partons).

Thus one sees common themes cropping up in diverse settings.

The focus of these lectures will be the structure of N=2 strings, and their possible relation to matrix dynamics. We begin in section 2 with the somewhat simpler (2,2) string, in order to introduce the novel features of local N=2 worldsheet supersymmetry and its associated self-duality structure. Included are an overview of the relation between the self-dual gravity
of the (2,2) string and $SDiff_2$ dynamics; as well as an aside on the D-brane spectrum of (2,2) strings, which leads to yet another interesting connection to matrix theory. Section 3 introduces (2,1) strings – their worldsheet gauge algebra, spectrum, and simple compactifications. The target space effective action for (2,1) strings is derived in section 4; parallels with matrix theory are explored in section 5.

2. (2,2) Strings

The gauged N=2 supersymmetry on the string worldsheet consists of currents $T$ (the stress tensor), $G^\pm$ (the two supercurrents), and $J$ (a $U(1)_R$-symmetry current). Their algebra is schematically

\[
\begin{align*}
[T,G^\pm] &\sim \frac{3}{2}G^\pm \\
[T,J] &\sim J \\
[G^+,G^-] &\sim T + J \\
[J,G^\pm] &\sim \pm G^\pm.
\end{align*}
\]

In the conformal gauge, the contributions to the Virasoro central charge from the Faddeev-Popov ghosts is $-26 + 2 \cdot 11 - 2 = -6$, so that the critical dimension is $d_{\text{crit}} = 2/3 c_{\text{matter}} = 4$. A free field representation of the above currents is

\[
\begin{align*}
T &= -\frac{1}{2} \partial X \cdot \partial X - \psi \partial \psi \\
J &= I_{\mu\nu} \psi^\mu \psi^\nu, \quad \mu = 0, ..., 3 \\
G^\pm &= (\eta_{\mu\nu} \pm I_{\mu\nu}) \psi^\mu \partial X^\nu = G_{N=1} \pm G_2.
\end{align*}
\]

Here, $I_{\mu\nu}$ is a self-dual tensor which acts as a complex structure. Preserving this structure under analytic continuation requires the signature to be Euclidean (4+0) or ultrahyperbolic (2+2). Even though the indefinite signature case has more than one negative metric string coordinate, there are no negative metric states. Each gauge invariance removes two fields’ worth of degrees of freedom; in fact, the two bosonic ($T,J$) and two fermionic ($G^\pm$) gauge symmetries kill all the oscillator modes of the string.

One of the powerful consequences of this fact is the triviality of the string S-matrix. On the one hand, there can be no Regge behavior in scattering amplitudes, since the sequence of poles in the S-matrix are associated with physical oscillator excitations of the string, of which there are none. On the other hand, the covariant formalism generates a Koba-Nielsen (integral) representation for the S-matrix amplitude, which exhibits such poles. The tension between these two properties is resolved by the vanishing of any amplitude beyond the three-point function (the three-point function is
protected because it does not involve Koba-Nielsen integrals). The single nontrivial S-matrix element is
\[ \langle V(1)V(2)V(3) \rangle = (k_1 \cdot I \cdot k_2)_{\ell}(k_1 \cdot I \cdot k_2)_{r} . \]
Since the only center of mass of the string can be excited, it is possible to find a representative of the string vertex operator which is a simple exponential
\[ V = \bar{\Sigma}gh \Sigma gh e^{ik \cdot x} \]
apart from ghost (measure) factors \( \Sigma gh \) (see Appendix A for a brief summary). A BRST-equivalent representative (or ‘picture’) of the vertex represents it as an integral over (2,2) superspace,
\[ V = \int d^2\theta d^2\bar{\theta} e^{ik \cdot X} \]
i.e. from the N=1 point of view the single physical state is a graviton whose only physical polarization is a fluctuation in the Kahler potential –
\[ g_{ij} = \partial_i \partial_j K \] in complex coordinates.
Another important feature of the (2,2) string is the absence of target space supersymmetry. Consider a Wilson line of the U(1) R-current, \( \exp[i\alpha \int z J] \). Parallel transport of a complex fermion \( \psi \) around the point \( z \) picks up a phase \( \exp[i\alpha] \); however \( J \) is gauged, and so the phase cannot have physical significance. In particular, there is no physical distinction between periodic (NS) and antiperiodic (R) boundary conditions, hence no target space fermions and no target space supersymmetry.

The effective action which generates the S-matrix (5) is
\[ S_{\text{eff}} = \frac{1}{g_{\text{str}}^2} \int d^4x \left[ \frac{1}{2} \partial K \cdot \bar{\partial} K + \frac{1}{2} K \partial \bar{\partial} K \wedge \partial \bar{\partial} K \right] , \]
from which one obtains the equation of motion
\[ I \wedge \partial \bar{\partial} K + \partial \bar{\partial} K \wedge \partial \bar{\partial} K = 0 , \]
known as the Plebanski equation. This is the equation that governs the dynamics of self-dual gravity. It is a straightforward exercise in 2+2 kinematics to show that the quartic S-matrix computed from the action (8) vanishes [1].

An alternate route to the Plebanski equation proceeds from the generalized beta-function of the background sigma model for string propagation.

\[ ^1 \text{Similar to the 1+1d noncritical string, where the ‘bulk’ S-matrix is trivial, the effective action is cubic (see below).} \]
In (2,2) supersymmetry, there are two kinds of scalar superfields: chiral and twisted chiral [14]. The simplest situation has the sigma model background described entirely in terms of chiral superfields; then the target space holonomy lies in U(d). The sigma model beta function equations are the conditions for SU(d) holonomy

\[ R_{\mu\nu} \Gamma = 2 \nabla_\mu \nabla_\nu \Phi \]  

\[ (\Gamma^\mu_{\nu\lambda} = \{^\mu_{\nu\lambda} \} - \frac{1}{2} H^\mu_{\nu\lambda} \text{ is the connection with torsion } H) \]. These equations may be integrated in complex coordinates to yield

\[ \log \det[g_{ij}] = 2\Phi + f(x) + \bar{f}(\bar{x}) \]  

A further analysis of the conditions for (2,2) supersymmetry in the sigma model coupled to worldsheet gravity [15] shows that the dilaton \( \Phi \) is locally the real part of a holomorphic function; thus locally one can choose coordinates such that the dynamical equation is

\[ \det[g_{ij}] = 1 \]  

Some solutions of these equations have been described in [15]. The single physical degree of freedom of the N=2 string is the center of mass mode \( K(x, \bar{x}) \) describing fluctuations of the Kähler potential \( g_{ij} = I_{ij} + \partial_i \partial_j K \), where \( I \) is the background Kähler form. Expanding (12) in this way yields the Plebanski equation (9). Note that although \( K \) contains the field-theoretic degrees of freedom of the string, there may be additional moduli in the global modes of the metric, antisymmetric tensor, and dilaton – for instance the action (8) on K3 depends implicitly on the full 80 moduli of the conformal field theory, as well as the string coupling \( \kappa = e^{-2\Phi} \). These are analogues of the special states of 2d noncritical string theory [16] that exist only for discrete momenta.

Backgrounds involving one chiral and one twisted chiral superfield must have two commuting complex structures [14] and are thus essentially trivial [17] – the quasi-Kähler potential \( \tilde{K} \) describing the background geometry satisfies a free field equation. When spacetime has a translational Killing vector field with compact orbits, these backgrounds are related to the above self-dual geometry by T-duality [14, 15]. Thus one might call the theory described by (8) the N=2B theory, and the theory described by the free quasi-Kähler potential \( \tilde{K} \) the N=2A string.

There are also interesting solutions which fall outside the class described by constant dilaton, for example the NS instanton [18],[15]

\[ e^{2\Phi} = e^{2\Phi_0} + \frac{n\alpha'}{r^2} \]  

\[ H_{\mu\nu\lambda} = -\epsilon_{\mu\nu\lambda}^\rho \nabla_\rho \Phi \]  

\[ G_{\mu\nu} = e^{2\Phi} \delta_{\mu\nu} \]  

\[ (13) \]
which is the counterpart of the NS five-brane in the usual 10d theory (an instanton is the magnetic dual to a string in 4d). In particular, this object is not that same as the D-instanton encountered below.

The Ricci-flat four dimensional geometry required by the N=2 string admits a hyperKahler structure, and the metric is self-dual. It is easy to see that the N=2 local supersymmetry generates N=4 global supersymmetry in the critical dimension $d_{\text{crit}} = 4$. The canonical normalization of the U(1) R-current

$$J(z)J(w) \sim \frac{1}{(z-w)^2}$$

(14)

implies that the exponential fields $J^\pm(z) = \exp[\pm i \sqrt{2} \int^z J]$, together with $J(z)$, form an SU(2) current algebra; furthermore, $J^\pm G^\pm \sim (z-w)^{-1} \tilde{G}^\pm$ are two additional supersymmetry currents. These enlarge the N=2 currents (3) to the N=4 current multiplet. Since any solution to the background field equations of motion preserves an SU(2) symmetry in the tangent space (in Euclidean signature; SL(2,R) in ultrahyperbolic signature), the target space is hyperKahler. There is a two-sphere’s worth of choices of which U(1) inside this global SU(2) is the gauged local R-symmetry. Parametrizing this choice is the twistor variable $\zeta \in \mathbb{CP}^1 = SU(2)/U(1)$ of self-dual geometry. For more on the relation of twistor geometry to the (2,2) string, see for instance [1].

This self-duality structure leads to an infinite number of conservation laws, and it is likely that this integrability underlies the triviality of the S-matrix seen above. Another way to exhibit these conservation laws reveals a connection to the area-preserving diffeomorphism group $SDiff_2$ [13]. Consider the zero-curvature condition in two dimensions

$$[\partial_t + \lambda A_t, \partial_x + \lambda A_x] = 0,$$

(15)

where $\lambda$ is an arbitrary (spectral) parameter, and $A_\alpha (\alpha = 1, 2)$ is an $SDiff_2$-valued connection. One can represent $SDiff_2$ as the algebra of canonical transformations of a one-dimensional phase space parametrized by coordinates $(p, q)$. Then (15) at order $\lambda$ implies

$$A_t(x, t; p, q) = - (\partial_t \partial_p \Omega) \partial_q + (\partial_t \partial_q \Omega) \partial_p$$
$$A_x(x, t; p, q) = - (\partial_x \partial_p \Omega) \partial_q + (\partial_x \partial_q \Omega) \partial_p ;$$

(16)

the $o(\lambda^2)$ equations then read (in an appropriate choice of coordinates)

$$\left(\partial_t \partial_p \Omega \right) \left(\partial_x \partial_q \Omega \right) - \left(\partial_t \partial_q \Omega \right) \left(\partial_x \partial_p \Omega \right) = 1,$$

(17)

i.e. the Plebanski equation. The symmetry group of (15) is the loop group of $SDiff_2$; the self-dual gravity equations are essentially the same as the
two-dimensional chiral model equations for the group $SDiff_2$, where the four-dimensional spacetime is parametrized by $(x,t,p,q)$.

2.1. OPEN STRINGS, D-BRANES, AND MATRIX THEORY

The connection of the target space dynamics of the $(2,2)$ string to $SDiff_2$ leads to an intriguing toy version of the circle of ideas underlying matrix theory. $N=2$ strings have very little dynamics; the $N=(2,2)$ string has no physical transverse excitations. The center-of-mass mode of the closed string describes fluctuations of the Kähler potential of the self-dual metric; the corresponding mode of the open string describes fluctuations of a self-dual Yang-Mills field. The open string S-matrix is again essentially trivial, vanishing beyond the three-point function. In such a simplified dynamical setting, one might expect to be able to make exact statements about $N=2$ string D-brane dynamics as well, perhaps even nonperturbatively. The D-branes should strongly affect the strong-coupling behavior of the theory, since they become light in this regime.

In a sense, in $N=2$ strings self-duality plays a role similar to the BPS condition. There is no BPS structure per se, due to the absence of spacetime supersymmetry; the almost topological nature of the dynamics makes up for this, however. This is because spacetime supersymmetry of the usual superstring is intimately related to the spectral flow that analytically relates the NS and R sectors in worldsheet $N=2$ supersymmetry [19]. In the $N=2$ string, this spectral flow is gauged, so in a sense one has only the ‘BPS’ sector\(^2\). The spectral flow also generates the global $SU(2)$ symmetry whose preservation is tantamount to the hyperKähler condition on the target space, and hence self-duality.

The fascinating proposal of matrix theory [9] casts the full light-cone gauge dynamics of M-theory in terms of the quantum mechanics of the BPS sector of zero-branes. Only the unexcited open strings stretching between the zero-branes are kept, yet one recaptures a remarkable amount of the complete theory in this way. One might hope that $N=2$ string D-branes serve as a toy model of this dynamics, since $N=2$ local supersymmetry permits only such states in the physical state space of open strings – all open string oscillations are unphysical.

Appendix B is devoted to a cursory explanation of the open string sector of $(2,2)$ strings. The upshot is that there are D-branes, just as in the $(1,1)$ and $(0,0)$ strings. The latter lead to conventional (super)Yang-Mills dynamics on the brane, dimensionally reduced from 10 or 26 dimensions. Thus it is not surprising that, since the $(2,2)$ open string sector describes

\(^2\)More precisely, the sort of BPS states that preserve the self-dual supercharge, i.e. $Q_\alpha$ and not $\dot{Q}_\alpha$.
four dimensional self-dual gauge theory, the D-branes are governed by the
dimensional reduction of the self-duality equation:

| worldvolume dimension | integrable system | equation |
|-----------------------|------------------|----------|
| 4                     | SDYM             | $F + \tilde{F} = 0$ |
| 3                     | Bogomolny        | $F \_{ij} + \epsilon_ {ijk} D_k \Phi = 0$ |
| 2                     | Hitchin          | $F + \left[ X, \bar{X} \right] = 0$ |
| 1                     | Nahm             | $D_t X^i + \epsilon_ {ijk} \left[ X^j, X^k \right] = 0$ |
| 0                     | ADHM             | $\left[ X^\mu, X^\nu \right] + \epsilon_ {\mu\nu\lambda\sigma} \left[ X^\lambda, X^\sigma \right] = 0$ |

We have denoted the branes by their total worldvolume dimensions rather
than the space dimension, since that notation is a bit ambiguous in signature $2+2$. Curiously, for worldvolume dimension equal to two (D-strings),
the equation is very similar to (15). In fact one can approximate matrix
theory rather closely, in the following sense. Consider the limit of $N$ co-
incident D-strings, so that the $[X, \bar{X}]$ term in the Hitchin equation drops
out. then the equation governing the D-string is a zero-curvature condi-
tion for its two-dimensional gauge field. In the limit $N \to \infty$, one has the
equations of motion of the two dimensional chiral model with gauge group
$SU(\infty) \sim SDiff_2$; this can be shown to be equivalent to the self-duality
equations (c.f. the last of refs. [13]). Thus one has a situation where one
can recover the low-energy theory corresponding to the $N=2$ string from the
large-N limit of one of its D-branes. The principal difference is the lack of a
connection to the infinite momentum frame, and the corresponding appear-
ance of an extra dimension at strong coupling. Also, the transverse space to
the D-branes does not appear to be related to the extra dimensions arising
from the matrix phase space; one starts from a two-dimensional object in
a four-dimensional space whose transverse position is nondynamical, and
grows a different two dimensions by taking the large N limit. It may be that
the appropriate starting point is a six-dimensional theory [20]. Perhaps the
$(0+2)$-brane is close to an analogue of the D-particle in the construction of
[9] since it spans both time directions of $2+2$ spacetime. It would indeed
be intriguing if one could realize self-dual gravity in terms of D-brane ‘con-
stituents’; perhaps this would provide an interpretation of the ‘entropy’ of
the nuts and bolts of compactified self-dual solutions [21].

Independent of matrix theoretic ideas, the D-branes are the light objects
of the theory at strong $N=2$ string coupling; they, together with the NS
instanton (13), should dominate the nonperturbative behavior of self-dual
gravity.
The N=2 string D-brane system is also a convenient laboratory for the exploration of the Nahm transform; T-duality relates the different branes upon, for instance, compactification on $T^4$ (mirror symmetry plays an analogous role on K3). Another intriguing point is that the moduli space of $k$ instantons in SU($N$) gauge theory has been proposed as the bosonic part of the configuration space of matrix noncritical strings [22]. N=2 strings generate exactly this moduli space as the space of physical deformations, with the D-instantons regulating the singularity at zero instanton scale size; and manifestly incorporate T-duality. Thus there might be a close relationship to the dynamics of this so-far mysterious 6d noncritical string theory to quantum mechanics on the moduli space of N=2 strings.

3. (2,1) Strings

Heterotic (2,1) strings [2] combine the self-dual, integrable structure of (2,2) strings with the spacetime supersymmetry present in strings with N=1 gauged worldsheet supersymmetry. A free field representation for the N=1 currents is

\begin{align}
T &= -\frac{1}{2} \partial_x \cdot \partial x - \psi \cdot \partial \psi - \frac{1}{2} \partial y \cdot \partial y - \lambda \cdot \partial \lambda \\
G &= \psi \cdot \partial x + \lambda \cdot \partial y.
\end{align}

(18)

Here $y^a, a = 1, \ldots, 8$ is a chiral boson and $\lambda^a$ a Majorana-Weyl fermion.\footnote{Our conventions are as follows: the N=1 chiral sector will be the left-moving oscillations; the N=2 chiral sector will be right-movers. The indices are $\mu = 0, 1, 2, 3$ for $x^\mu \in \mathbb{R}^{2,2}$; $a = 1, \ldots, 8$ for $y^a \in T^8$; often we will use a combined index $M = (\mu, a) = 0, \ldots, 11$ when dealing with purely left-moving quantities.}

Using these currents as gauge constraints in conformal gauge is not sufficient to remove both of the two timelike oscillator mode towers of $x$ and $\psi$; one needs another bosonic and fermionic gauge current. The simplest choice [2] is a U(1) supercurrent

\begin{align}
J &= v \cdot \partial x + v_{\text{int}} \cdot \partial y \\
\Psi &= v \cdot \psi + v_{\text{int}} \cdot \lambda.
\end{align}

(19)

In order to gauge this current, it must be anomaly free, which forces $v = (v, v_{\text{int}})$ to be a null vector: $v^2 + v_{\text{int}}^2 = 0$. The ghosts for $J, \Psi$ have $c = -3$; therefore, they increase the critical dimension by two, so $d_{\text{crit}} = 12 = 10 + 2$.

One appears to have a ‘superstring in d=12’, but not really; the additional gauge constraints project momenta and polarizations to be orthogonal to $v$. If $x \in \mathbb{R}^{2,2}$, then the $(y, \lambda)$ system must form a $c = 12$ holomorphic superconformal field theory (just as the internal sector of the (1,0) string in $\mathbb{R}^{9,1}$ must form a $c = 16$ conformal field theory). The unique choice

$\lambda^0, a = 1, \ldots, 8$ is a chiral boson and $\lambda^a$ a Majorana-Weyl fermion.\footnote{Our conventions are as follows: the N=1 chiral sector will be the left-moving oscillations; the N=2 chiral sector will be right-movers. The indices are $\mu = 0, 1, 2, 3$ for $x^\mu \in \mathbb{R}^{2,2}$; $a = 1, \ldots, 8$ for $y^a \in T^8$; often we will use a combined index $M = (\mu, a) = 0, \ldots, 11$ when dealing with purely left-moving quantities.}
preserving spacetime supersymmetry, modular invariance, etc., is for \( y \) to live on the Cartan torus of \( E_8 \).

Massless vertex operators are BRST equivalent to

\[
\begin{align*}
V_{NS}^{\text{grav}} &= (\sum_{gh}^{(N=2)})(\sum_{NS}^{(N=1)}) \xi_{\mu}(k) \psi^{\mu} e^{i k \cdot x} \\
V_{NS}^{\text{gauge}} &= (\sum_{gh}^{(N=2)})(\sum_{NS}^{(N=1)}) \xi_{a}(k) \lambda^{a} e^{ik \cdot x} \\
V_{R} &= (\sum_{gh}^{(N=2)})(\sum_{R}^{(N=1)}) u^{a}(k) S_{a} e^{ik \cdot x}.
\end{align*}
\] (20)

As in the (2,2) case, an equivalent representative of the graviton is

\[
V = \int d\theta d\bar{\theta} e^{ik \cdot X} = \int d\theta d\bar{\theta} |_{N=1} I_{\mu}^{\lambda} k_{\lambda} \xi_{\mu} (\bar{D}_{N=1} X^{\mu} D X^{\nu}) \exp[i k_{\mu} X^{\mu} + i k_{a} Y^{a}],
\] (21)

so that the graviton/antisymmetric tensor has a special polarization \( h_{\mu\nu} + b_{\mu\nu} \sim I_{\mu}^{\lambda} \partial_{\lambda} \xi_{\nu} \), verifying the claim (1) of the introduction. Similarly, the gauge field \( \varepsilon^{a}_{\mu} \sim I_{a}^{\lambda} \partial_{\lambda} \varphi^{a} \) and gravitino \( \chi^{a}_{\mu} \sim I_{\mu}^{\lambda} \partial_{\lambda} \psi^{a} \).

If the target space is \( \mathbb{R}^{2,2} \times T^{8}_{\text{int}} \), only the states (20) satisfy level matching

\[
\begin{align*}
\bar{L}_{0} &= k_{\mu} k^{\mu} = 0 \\
L_{0} &= k_{M} k^{M} + N_{\ell} = 0
\end{align*}
\] (22)

In this case the only physical states are at the massless level, \( N_{\ell} = 0 \), much like the (2,2) string; with no momentum in the internal \( T^{8} \) directions, \( k_{a} = 0 \).

The Virasoro/null current constraints impose restrictions on the polarizations and momenta:

\[
\begin{align*}
\text{Virasoro} & \quad k_{M} \xi^{M} = 0 \quad \xi \sim \xi + \alpha k \quad \text{NS} \\
& \quad k u = 0 \quad \text{R} \\
\text{Null current} & \quad v_{M} \xi^{M} = 0 \quad \xi \sim \xi + \alpha v \quad \text{NS} \\
& \quad v u = 0 \quad \text{R} .
\end{align*}
\]

These constraints reduce the 12 polarizations of the NS vector to 8 transverse, and the 32 components of the Majorana-Weyl R sector spinor to 8 physical states, as expected. The spacetime interpretation of these physical states depends on the specific orientation of the null vector \( v \), see figure 3.
Figure 3. Choices of null constraint on the left-movers.

In figure 3a, the null vector is oriented entirely within the $\mathbb{R}^{2,2}$ base space. The kinematics consists of 1+1 dimensional momenta $k$ (recall that the level matching constraint eliminates any momentum components in the $E_8$ directions), with the physical polarizations consisting of a 1+1 gauge field $a_\mu$, 8 scalars $\phi^a$, and 8 fermions $\psi^\alpha$. The spectrum is that of the type IIB D-string! On the other hand, if the null vector has its time component in $\mathbb{R}^{2,2}$ and its space component in $T^8$ (see figure 3b), then the kinematics is 2+1 dimensional, and the physical polarizations are the 2+1 gauge potential $a_\mu$, 7 scalars $\phi^a$, and 8 fermions $\psi^\alpha$. The spectrum in this case is that of the type IIA D2-brane.

One always has sixteen supersymmetries

$$Q_\alpha = \int \Sigma^{(N=1)}_{gh,R} S_{\alpha} \quad , \quad \gamma Q = 0 \quad , \quad (23)$$

just as a type II D-brane breaking half the supersymmetries. The algebra of these supercharges is (on-shell, since all considerations are modulo BRST equivalence of the (2,1) string)

$$\{Q_\alpha, Q_\beta\} = (\gamma^{MN})_{\alpha \beta} P_M v_N \quad . \quad (24)$$

If our interpretation is correct, one would expect there to be an additional sixteen supersymmetries which are spontaneously broken by the brane. These supersymmetries would be nonlinearly realized in the worldvolume theory, and hence not visible in the single-string Hilbert space. One might see them in a careful study of the vertex algebra. In any event, the (2,1) string would appear to describe D1- or D2-branes in static gauge, stretched
across the noncompact spatial directions of $\mathbb{R}^{2,2}$. The transverse directions appear to be compactified on a torus, probably the $E_8$ Cartan torus.

3.1. TOROIDAL COMPACTIFICATION OF (2,1) STRINGS

Qualitatively new features arise upon further compactification of the (2,1) string, since the level-matching constraints (22) prove to be much less restrictive. In fact, the physical spectrum consists of a stringy tower of ‘Dabholkar-Harvey’ states – ground states on the right-moving $N=2$ side, oscillator excitation at arbitrary level on the left-moving $N=1$ side, with momentum and winding to compensate. Note that since it is the worldsheet chirality that carries spacetime supersymmetry that must be excited, states with $N_{\ell} \neq 0$ break target space supersymmetry; i.e. these states are not BPS-saturated.

Consider then the further compactification of the spatial $x^2, x^3$ coordinates of $\mathbb{R}^{2,2}$, so that the target is $\mathbb{R}^2(\text{time}) \times T^2 \times T^8_{\text{int}}$. More generally, one could consider an arbitrary spatial torus corresponding to a point in the Narain moduli space $\mathcal{N}^{10,2}$, but a product torus will suffice for illustrative purposes. The (2,1) string will in general have both momentum and winding

$$p_{\ell,r}^i = \frac{n^i}{R_i} \pm \frac{m^i R_i}{2}, \quad i = 2, 3.$$  \hspace{1cm} (25)

Reexamining the level-matching constraints

$$0 = -p_0^2 - p_1^2 + p_2^2 + p_3^2, $$

$$0 = -p_0^2 - p_1^2 + p_2^2 + p_3^2 + p_T^2 + 2N_{\ell}, \hspace{1cm} (26)$$

one sees that many more states are now available – one need only satisfy the mass shell condition, the level-matching constraint, and the null constraint (as well as the highest weight condition under the current algebra):

$$0 = -p_0^2 - p_1^2 + \left(\frac{n^i}{R_i}\right)^2 + \left(\frac{1}{4} m^i R_i\right)^2 + \frac{1}{2} p_T^2 + N_{\ell} $$

$$0 = 2m^i n^i + p_T^2 + 2N_{\ell} $$

$$0 = \mathbf{v} \cdot p_{\ell}. $$  \hspace{1cm} (27)

Note that the constraints now allow the internal momenta $p_T$ to be arbitrarily excited, so that the kinematics is more or less ten dimensional. These states are similar to the perturbative BPS states of the type II superstring; for instance, the level density grows exponentially. Even with this change of circumstances, one can check that the four-point function continues to
vanish, indicating that the triviality of the S-matrix continues to hold. T-duality of the (2,1) string seems to imply that there is a minimum ‘size’ to the target circle(s) over which the target D-brane is stretched: $R_{\text{min}} \sim R_{\text{str}}^{(2,1)}$.

3.2. HETEROTIC/TYPE I CONSTRUCTION

It is now well-understood that the heterotic and type IA string theories are particular asymptotic limits in the moduli space of $\mathbb{Z}_2$ orientifolds of M-theory, see figure 4.

\[\text{E8 x E8 heterotic} \quad 11\text{d sugra on } \mathbb{R}/\mathbb{Z}_2\]

\[\text{SO(32) heterotic/type I} \quad \text{type IA}\]

**Figure 4.** Moduli space of $S^1 \times S^1/\mathbb{Z}_2$ compactifications of M-theory.

The simplest situation has 16 ninebranes at each of the two orientifold planes to cancel anomalies; the low-energy dynamics consists of 11d supergravity in the bulk, and 10d SYM on these walls. Since the (2,1) string seems only to describe a brane in spacetime, to realize these vacua as a (2,1) string background one might look for a wrapped membrane/string that sees this structure, see figure 5.

\[\text{x}_3=0 \quad \text{x}_3=\pi R_3\]

**Figure 5.** The heterotic membrane geometry generated by the (2,1) string.
In other words, one looks to describe a cylindrical, open membrane stretched between the orientifold planes \( x_3 = 0, \pi R_3 \). Its boundaries should have 1+1 dimensional fields describing rank eight current algebra, such that one obtains a heterotic string when \( R_2 \gg R_3 \); on the other hand, when \( R_2 \ll R_3 \), these fields will be frozen, leaving the finite number of Chan-Paton labels of type IA theory. The orientifold symmetry acts as \( x_3 \rightarrow -x_3 \), \( A_{MNP} \rightarrow -A_{MNP} \). The sign flip of the three-form gauge field indicates orientation reversal of the M-theory membrane worldvolume. Since one appears to be in static gauge, the orbifold \( x_3 \rightarrow -x_3 \) of the (2,1) string will accomplish both the orbifold of spacetime and the orientation reversal of the brane worldvolume. One simply wants to find a consistent supersymmetric orbifold of the (2,1) string with this operation as part of the orbifold group. A straightforward chain of logic leads to a unique answer satisfying the orbifold level-matching constraints, preserving half the sixteen space-time supersymmetries, and treating all the internal coordinates \( y \) on the same footing. Right-moving worldsheet supersymmetry is preserved only if, in addition to \( x_3 \rightarrow -x_3 \), one simultaneously flips \( \bar{\psi}_3 \rightarrow -\bar{\psi}_3 \). Now consider the right-moving U(1) R-current, e.g. \( \bar{J} = \bar{\psi}^0 \psi^3 + \bar{\psi}^1 \psi^2 \); there are two choices for the \( \mathbb{Z}_2 \) to have a well-defined action on the gauge algebra: (1) \( \bar{\psi}_0 \rightarrow -\bar{\psi}_0 \), which one can show does not lead to a spacetime supersymmetric solution; and (2) \( \bar{\psi}_1 \rightarrow -\bar{\psi}_1 \), which gives the twisted N=2 algebra \[ (23) \]
\[ \bar{J} \rightarrow -\bar{J} \quad , \quad \bar{G}^+ \leftrightarrow \bar{G}^- . \] Proceeding along these lines, one finds a unique \( \mathbb{Z}_2 \) satisfying the above requirements:

\[
\begin{align*}
(x_1, x_3; y_1, \ldots, y_8) & \rightarrow -(x_1, x_3; y_1, \ldots, y_8) \\
(\psi_1, \psi_3; \lambda_1, \ldots, \lambda_8) & \rightarrow -(\psi_1, \psi_3; \lambda_1, \ldots, \lambda_8) \\
(\bar{\psi}_1, \bar{\psi}_3) & \rightarrow - (\bar{\psi}_1, \bar{\psi}_3) ,
\end{align*}
\]
with all other coordinates invariant.

Vertex operators describe physical fluctuations of the target brane, and have the form \( \mathcal{O} \cos[k_3 x^3] \) if the polarization operator \( \mathcal{O} \) is \( \mathbb{Z}_2 \) even, and \( \mathcal{O} \sin[k_3 x^3] \) if \( \mathcal{O} \) is \( \mathbb{Z}_2 \) odd. At the massless level, one finds Neumann boundary conditions \( \cos[k_3 x^3] \) for polarizations along \((0, 2; 4, \ldots, 11)\), and Dirichlet boundary conditions \( \sin[k_3 x^3] \) for polarizations along \((1, 3)\). Thus, in space there are nine Neumann and one Dirichlet boundary condition; in time, one Dirichlet and one Neumann. Since we wish to avoid thinking about what a Dirichlet boundary condition means in physical time, we place the time component of the vector \( \mathbf{v} \) (defining the null projection \( J \)) in the \( x_1 \) direction; thus its space component must also be one of the \( \mathbb{Z}_2 \) twisted directions \((2, 4, \ldots, 11)\). Choosing the latter in one of the internal directions, one finds now an open membrane stretched between eight-branes.
in a type IIA description; that is, when the target \((x_2, x_3)\) torus orbifold is a large cylinder, in the low-energy limit one has a 2+1 target space gauge field coupled to 7 scalars with Neumann boundary conditions. This is precisely the bulk dynamics of an open membrane stretched between D8-branes. One can lift to this effective theory to eleven dimensions by dualizing the vector to another scalar (which one easily checks also has Neumann boundary conditions) to find an open membrane stretched between orientifold nine-branes.

The boundary dynamics comes from the twisted sectors of the (2,1) string orbifold, as these states are pinned to the fixed points \(x_3 = 0, \pi R_3\) which are the orientifold planes. Since the orbifold is an asymmetric one, the number of states at each of the two fixed points \(x_3 = 0, \pi R_3\) is the square root of the number of fixed points of the internal \(T^8/Z_2\) orbifold, \(\sqrt{2^8} = 16\). Consistency of the operator algebra shows that these states appear in the Ramond sector, and an analysis of the BRST constraints shows that they are chiral in the target space. Thus there are sixteen massless fermion fields living on each boundary of the open target membrane, which describe a rank eight current algebra, and are inert under spacetime supersymmetry. Half the supersymmetry charges \((23)\) are broken by the orbifold. Note that this is exactly the spectrum found on the D8-brane boundaries of the matrix theory description of the heterotic string, a fact that will be important for us below.

There is a very similar orbifold [5], also breaking half the supersymmetry, obtained if one relaxes the condition that all internal coordinates are treated identically. Twisting half rather than all the \(y^a, \lambda^a\)

\[
(x_1, x_3; y_1, \ldots, y_4) \rightarrow -(x_1, x_3; y_1, \ldots, y_4) \\
(\psi_1, \psi_3; \lambda_1, \ldots, \lambda_4) \rightarrow -(\psi_1, \psi_3; \lambda_1, \ldots, \lambda_4) \\
(\tilde{\psi}_1, \tilde{\psi}_3) \rightarrow -(\tilde{\psi}_1, \tilde{\psi}_3),
\]

is also a consistent asymmetric orbifold. The twisted sector ground states transform as a hypermultiplet under the eight remaining supersymmetries.

Finally, it should be noted that we have only described the low-energy spectrum; there is again a stringy tower of left-moving states just as in the untwisted, toroidally compactified (2,1) string.

### 4. S-matrix, effective action and geometry of (2,1) strings

The tree-level, three-point S-matrix factorizes between left- and right-movers, hence we can immediately deduce the answer by combining (2,2) string result (5) with that of the superstring; for three massless metric perturbations, the result is

\[
\langle V_\xi(1) V_\xi(2) V_\xi(3) \rangle = (k_1 \cdot I \cdot k_3) \cdot (\xi_1 \cdot \xi_2 \cdot k_1 \cdot \xi_3) + \text{cyclic},
\]

(29)
leading to the cubic effective Lagrangian

$$L^{(3)}_{\text{eff}} = \frac{1}{2} F_{\mu\rho} F_{\nu\lambda} F_{\sigma\mu} - \frac{1}{8} (F_{\alpha\beta} F_{\beta\alpha}) (F_{\mu\nu} F_{\nu\mu})$$

(30)

with $F_{\mu\nu} = \partial_{(\mu} F_{\nu)}$. Curiously, this is a term in the expansion of the Born-Infeld Lagrangian around the background $F_{\mu\nu} = I_{\mu\nu}$. One can incorporate the effects of the internal scalars $\varphi$ simply by letting the polarization indices in (30) run over 10d (or 12d, since we are imposing the null constraints by hand), while keeping the kinematics 1+1 or 2+1 dimensional. Regarding the fermions, their S-matrix is

$$\langle V_{\psi}^{(1)} V_{\xi}^{(2)} V_{\psi}^{(3)} + V_{\psi}^{(1)} V_{\varphi}^{(2)} V_{\psi}^{(3)} \rangle = (k_1 \cdot I \cdot k_3) [\bar{u}_1 (\xi^\mu \Gamma^\mu + \zeta^a \Gamma^a) u_3] \ell,$$

(31)

where $\Gamma^\mu$ are twelve dimensional gamma matrices. One obtains the effective Lagrangian

$$L^{(3)}_{\text{eff}} = (\bar{\psi} \Gamma^\mu \partial_\nu \psi) I^{\nu\lambda} F_{\lambda\mu} + (\bar{\psi} \Gamma^a \partial_\nu \psi) I^{\nu\lambda} \partial_\lambda \varphi^a.$$ 

(32)

These are the only nonzero S-matrices\(^4\) in $\mathbb{R}^{2,2} \times T^8$, but this does not mean that we have found the full effective action. It happens that the iteration of these cubic field-theoretic vertices generates a four-point S-matrix, such that kinematics in $\mathbb{R}^{2,2}$ allows the pole terms to cancel among $s$, $t$, and $u$ channels, leaving an explicit four-point contact term. For the full S-matrix to vanish, one must add a cancelling quartic term in the effective action. Continued iteration yields terms to all orders in small fluctuations. This situation contrasts with the (2,2) string, where the symmetry is powerful enough to cause the potential four-point contact term to vanish, and allows the effective action to cubic order to in fact be the exact theory.

The simplest way to extract the full answer is to realize that the iteration of the cubic terms will generate contact terms of the form $k^n (\delta \phi)^n$ at $n^{\text{th}}$ order in fluctuations, while fluctuations in the geometrical fields $h, b, \varepsilon$ are all of the form $\partial (\delta \phi)$ due to the (2,1) supersymmetry constraints. Hence the effective action found from the beta function will come entirely from one loop (higher loops will come with extra factors of $\alpha' k^2$, and hence more powers of momenta than small fluctuations). The beta function equations (10) can be integrated once to give

$$\Gamma^\mu_{\nu\rho} F^\nu_{\nu\rho} = 0$$

(33)

with the effective action

$$S^{\text{grav}}_{\text{eff}} = \frac{1}{g_{\text{str}}^2} \int d^4x \det \frac{1}{2} [\eta_{ij} + F_{ij}] ,$$

(34)

\(^4\)Hence the analogy with the 1+1d noncritical string made in the introduction, wherein the ‘bulk’ S-matrix is also trivial.
at least for the gravity sector. The gauge fields may be added via an analysis of sigma model anomalies \[24,8\]; the result is

\[ S_{\text{eff}} = \frac{1}{g_{\text{str}}^2} \int d^4 x \det \frac{1}{2} \left[ \eta_{ij} + F_{ij} + \alpha' \partial_i \varphi^a \partial_j \varphi^a \right]. \] (35)

It has been checked \[24\] that this action reproduces all cubic and induced quartic vertices involving only bosons, and that the S-matrix vanishes at quartic order. The action is remarkably similar to the static gauge effective action for D-branes. There are two significant differences, however: (1) For target space \( \mathbb{R}^{2,2} \times T^8 \), it is exact to all orders in \( \alpha' = \ell_{\text{str}}^2 \), presumably related to the fact that there are no oscillator excitations of the underlying (2,1) string; and (2) the dynamics is integrable. The vanishing of the S-matrix in the fully compactified case indicates that this integrability continues to hold even when the kinematics is essentially ten-dimensional. It would be very interesting to work out the generalization of (35) in this case. It would also be interesting to explore the possibility of nontrivial solutions along the lines of (13).

The terms in the effective action involving fermions should in principle be determined by supersymmetry. However, it is difficult to compute quantities involving nontrivial Ramond vertex backgrounds (there is no sigma-model approach). A better understanding of the spacetime supersymmetry current algebra and its anomaly structure would be helpful, since it was essentially this structure which enabled us to determine the dependence of the effective action on the scalars \( \varphi^a \). As mentioned in the introduction, the physical fields on the target brane are all Nambu-Goldstone modes of the spacetime symmetries spontaneously broken by the brane, and therefore the effective action is expected to be largely determined by the various broken and unbroken symmetries. Because the bosonic structure is so similar to the Dirac-Born-Infeld/Nambu-Goto action, it is natural to guess that the full answer including fermions has a structure similar to that of Dp-branes in static gauge

\[ S^{(p)} = \int d^{p+1} \sigma \det \frac{1}{2} \left[ \eta_{\mu \nu} + F_{\mu \nu} + \partial_\mu \phi^a \partial_\nu \phi^a \right. \]

\[ -2 \bar{\psi} \left( \Gamma_\mu + \Gamma_\alpha \partial_\mu \phi^a \right) \partial_\nu \phi + \left( \bar{\psi} \Gamma^a \partial_\mu \phi \right) \left( \bar{\psi} \Gamma^a \partial_\nu \phi \right) \]. \] (36)

The sorts of terms appearing in the expansion of this action are compatible with with the cubic S-matrix (32), but the full structure is far from understood. This fermionic completion of of the action (35) represents a rather nontrivial coupling of SDYM to self-dual gravity with torsion, with an additional fermionic symmetry whose geometry ought to be quite intriguing.
5. Connections between (2,1) strings and matrix theory

In this final section of these lectures, I would like to present some evidence for the idea that (2,1) strings are closely related to the formulation of M-theory on $T^9$ as a matrix model. A striking feature of supergravity U-duality is the appearance of exceptional groups in less than six noncompact spacetime dimensions. Naive extrapolation leads to a duality group $E_{9(9)}(\mathbb{Z})$ in two dimensions [25]. However, the solutions to low-energy supergravity which exhibit the various BPS charges permuted under duality, become more and more singular the lower the dimension. For instance, in 2+1 dimensions, $n$ fundamental strings wound around a given cycle of the internal $T^7$ has a dilaton background of the form [26]

$$e^{-\Phi} = e^{-\Phi_0} - \sum_{i=1}^{n} 8G_N \ell_{st}^{-2} \ln |\vec{r} - \vec{r}_i|;$$

one can introduce an arbitrary number of such sources, but the dynamics becomes strongly coupled near the core of each one. Presumably the correct solution receives modifications due to the nonperturbative states that become light there (c.f. [27]). This is supported by the observation that, in the IIB theory, one can wrap a D7-brane around the $T^7$; these are in the same U-duality multiplet as the fundamental string, yet they have a rather different classical solution – the core is nonsingular (from the viewpoint of F-theory), but one cannot introduce more than 24 sources. In 1+1 dimensions, the situation is even worse [28]; any source generates a response in the dilaton of the form

$$e^{-\Phi(x^+)} = e^{-\Phi_0} - \int \int_{-\infty}^{x^+} T_{++} ;$$

positive stress-energy of the source forces a singularity in the dilaton at finite $x^+$, so again information of a more nonperturbative nature is needed. Finally, if one considers the velocity-dependent forces between these objects in low dimensions, eventually they become confining even between the BPS states. For instance, in matrix theory the velocity dependent force between two matrix partons on $\mathbb{R}^{10-d,1} \times T^d$ is $v^4/r^{7-d}$, becoming confining below 3+1 dimensions.

It is not clear what to make of these observations. Two possibilities are

- Supergravity makes sense as a low-energy theory, but that the branes that have been our guide to defining matrix theory do not make sense
– or at least cannot be present in the arbitrary numbers needed to define a large-N continuum limit.\footnote{Also the branes have a strong back-reaction on spacetime, perhaps precluding the existence of the null Killing vector needed to define the light cone gauge.}

– Matrix theory exists in all dimensions; however, the confining potential between matrix partons means that there will be no moduli space for the gauge dynamics defining matrix theory in low dimensions, therefore no low-energy approximation that one could call spacetime, therefore no supergravity.

The connection between (2,1) strings and brane dynamics cries out for some connection to M-theory. Could (2,1) strings describe the ‘low-dimensional phase’ of matrix M-theory conjectured above? There are a number of reasons to think so:

1. At least eight coordinates are compact on the scale $\ell_{\text{str}}^{(2,1)}$.
2. Massless physical states of the fully compactified (2,1) string have interactions similar to what one would expect from SYM with gauge group $SDiff_2$.\footnote{In matrix theory on $T^6$, the theory cannot quite be SYM even at low energies, due to anomalies.}
3. The $\mathbb{Z}_2$ orbifold that gives the heterotic string acts as expected in matrix theory [29].
4. In the fully compactified (2,1) string, the decompactification limits are analogous to the limits which yield matrix IIA/B strings.

Consider again the vertex operators of the fully compactified (2,1) string

$$V_{NS} = (\text{ghosts})(\xi_M(p_\ell)\psi^M) \exp[ip_\ell \cdot x_\ell + ip_r \cdot x_r] \equiv A_M(x_\ell, x_r)$$

$$V_R = (\text{ghosts})(u_A(p_\ell)S^A) \exp[ip_\ell \cdot x_\ell + ip_r \cdot x_r] \equiv \lambda_A(x_\ell, x_r) \cdot (39)$$

The three-point function of, for instance, the bosons (29) generates the cubic coupling

$$\mathcal{L}^{(3)} = g_{\text{str}} \partial^M \partial^N \{A_M, A_N\} ,$$

where $\{F,G\} = (\partial F/\partial x_\ell^\mu)I^\mu(\partial G/\partial x_r^{\nu})$ is a Poisson bracket induced by the complex structure of the right-moving N=2 supersymmetry. This cubic vertex has the same structure as Yang-Mills with a gauge group of symplectic diffeomorphisms. In this interpretation, we wish to regard $x_\ell$ as transverse ‘spacetime’ coordinates, and $x_r$ as phase space coordinates on the Lie algebra of $SDiff_2 \text{ a la matrix theory.}$ In fact, the left- and right-moving degrees of freedom are not independent; they are coupled through the level-matching constraints (26). Moreover, the lattice of momenta $\Gamma^{10,2}$ is generically not factorized as $\Gamma^1_{\ell} \times \Gamma^2_r$. Nevertheless, there may be a sense in which a structure of $SDiff_2$ gauge theory is present off-shell [30]. It is
possible that this gluing of gauge and spacetime fluctuations is an artifact of the expansion about a particular classical solution, much as a monopole ties rotations in space and isospin.

To see the action of $SDiff_2$, consider the right-moving chiral part of the vertex operators, taking sets which are mutually local in their OPE’s; this means

$$p_i, \quad i = 1, 2, 3 : \quad p_i^2 = \sum_{i=1}^{3} p_i = 0 . \quad (41)$$

The generic construction of such sets is as follows: Take two orthogonal null vectors $n_{(1)}, n_{(2)}$ (i.e. $n_{(1)}^2 = n_{(2)}^2 = n_{(1)} \cdot n_{(2)} = 0$, and $n_{(1)} \neq \alpha n_{(2)}$). This is possible in signature 2+2. The $n_{(a)}$ span a self-dual null plane $N$. Now focus on momenta $p = an_{(1)} + bn_{(2)}$ which are arbitrary linear combinations of the two basis vectors. The chiral vertex operators $J_p = \oint dz \int d^2\bar{\theta} \exp[ip \cdot x_r]$ satisfy the algebra

$$[J_p, J_q] = p \cdot I \cdot q J_{p+q} \quad (42)$$

modulo BRST equivalence. Momenta $p \notin N$ do not have mutually local OPE’s, and hence no well-defined algebra structure. These on-shell $p_r \in N$ are completely determined by their spatial components; the null condition forces the length of the spacelike part to equal that of the timelike part, so that the triple $(p, q, -p - q)$ form congruent triangles in the temporal $x_0-x_1$ and spatial $x_2-x_3$ planes (an overall relative orientation of these two triangles is determined by the choice of $N$). The upshot is that the algebra (42) is an algebra of two-dimensional and not four-dimensional symplectic diffeomorphisms, i.e. $SDiff_2 \sim SU(\infty)$.

Thus the structure of the fully compactified (2,1) string strongly resembles that of matrix theory on $T^9$, although there are profound differences due to the constraints relating left- and right-moving coordinates ($x_t, x_r$). Further support for this idea is provided by the fact that the decompactification limit of the (2,1) string is described by a D-brane style Lagrangian for either strings or membranes (depending on the orientation of the null vector $v$); in fact it is precisely the D-string action if the null vector $v \in \mathbb{R}^{2,2}$. D-string dynamics is precisely what one finds in matrix theory if one shrinks a compactified circle to a radius $R_i \ll \ell_{pl}$; then the dual circle on which the SYM dynamics takes place lives on a circle of radius $\Sigma \sim \ell_{pl}^3/R_i R$, so that the IR dynamics becomes 1+1d SYM in the limit [22].

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7This choice is not unique; in fact the set of self-dual null planes is parametrized by a coordinate $\zeta \in \mathbb{R}P^2$ which is the twistor parameter.

8The left-right constraints are most powerful here, since they restrict all momenta to $p_\ell = p_r$. It would seem then that this limit describes only a single D-string and not $N \to \infty$ of them. It may be that this freezing of $U(N)$ down to $U(1)$ is an artifact of the string perturbation expansion, which takes place about a particular classical solution; perhaps this symmetry is broken in the particular vacuum seen by the (2,1) string.
The heterotic/type I construction in section 3.2 follows precisely the pattern one would expect of matrix theory on $T^9$. In matrix theory, the heterotic/type I theory on $T^d$ arises from a SYM orientifold on $\tilde{T}^d \times \tilde{S}^1$; the orientifold $\mathbb{Z}_2$ acts simultaneously on the torus and acts as the involution that sends $SU(N) \to SO(N)$. One must also add by hand 32 fermionic matter multiplets in order to cancel gauge anomalies; these represent the couplings of the zero-brane/supergravitons to the nine-branes of the background spacetime. All of this structure has a direct parallel in the (2,1) string. The orbifold group of section 3.2 acts on the left-moving ‘spacetime’ coordinates $x_\ell$ by reflecting nine spatial coordinates (of which one is removed by null projection) while leaving one untouched – in other words, as though the parameter space were $\tilde{T}^8/\mathbb{Z}_2 \times \tilde{S}^1$. Simultaneously, it acts to flip the right-moving $U(1)$ R-current $J \to -J$. Thus the $SDiff_2$ algebra (42) will undergo a $\mathbb{Z}_2$ quotient which one can perhaps think of as $SO(\infty)$. Finally, one finds the requisite 32 fermionic states in the twisted sector (and here they are not simply put in by hand). Decompactification of a circle again resembles the construction of the matrix heterotic string [29], although again the left-right gluing hides the gauge group structure in this limit.

The alternate orbifold presented at the end of section 3.2 seems to be a $\tilde{T}^4/\mathbb{Z}_2 \times \tilde{T}^5$ orbifold, but rather than sixteen hypermultiplets appearing in the twisted sector (as expected in matrix theory on this space), we find only one.

At this point several remarks are in order. First, the (2,1) string appears to lack the parameters needed to explore the full 128-parameter moduli space of matrix theory on $T^9$. Eight circles are fixed to the scale $\ell_{\text{str}}^{(2,1)}$, so it is not clear exactly how to connect the construction to the rest of M-theory. This aim might be furthered by understanding how to construct U-duality multiplets in the context of (2,1) string theory. The perturbative string states will be the momentum states of the SYM theory under the present interpretation (which are longitudinally wrapped membranes in matrix theory). The duality group is probably $E_{9(9)}(\mathbb{Z})$ [25] and does not act entirely within the perturbative theory. In fact, one expects [32] states whose masses scale as arbitrarily high powers of $1/g_{\text{YM}}^2$. Since $g_{\text{YM}} \propto g_{\text{str}}$, we are not going to find these easily.

Appendices

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9This corrects an erroneous interpretation in [31], where an attempt was made to connect the duality group to the symmetries of the internal $E_8$ torus.
A. Technology of local N=2 worldsheet supersymmetry

The ghost operators in superstring vertices provide the appropriate superconformally covariant geometrical measure for their integration over the worldsheet (for details, see [33, 34]). The superghosts have a number of different ‘pictures’, or BRST representatives, because the splitting of the supersymmetry ghosts into creation and annihilation operators is ambiguous. For the N=1 superconformal algebra, there is a single pair of such ghosts β, γ; for N=2 one has two – β±, γ± for the two supersymmetries G±. The ghost current may be bosonized as
\[
\partial \phi = \beta_\pm \gamma_\pm.
\]
The exponentials \( \exp[\alpha \phi_\pm] \) interpolate between the various vacua; \( \phi_\pm \) are free fields.

In the NS sector there are two canonical pictures for a given chiral sector of the usual N=1 superstring; the N=2 string doubles this. Consider a superfield, with expansion in components
\[
O = O^{(0)} + \theta_+ O^{(1)} + \theta_- O^{(1)} + \bar{\theta}_+ \bar{O}^{(2)}
\]
under the right-moving N=2 supersymmetry. For example, the exponential \( \exp[i k \cdot X] \) expands as
\[
e^{i k \cdot X} = e^{ik \cdot x} [1 + i \theta_+ k \cdot (1 + I) \cdot \bar{\psi} + i \theta_- k \cdot (1 - I) \cdot \bar{\psi}]
+ \bar{\theta}_+ \bar{\theta}_-(k \cdot I \cdot \bar{\partial} x - (k \cdot \bar{\psi})(k \cdot I \cdot \bar{\psi})].
\]

BRST invariant vertex operators are then
\[
V^{(0)} = e^{-\phi - \bar{\phi}} e^{ik \cdot x}
\]
\[
V^{(1,\pm)} = e^{-\phi_\pm} [k \cdot (1 \mp I) \cdot \bar{\psi}] e^{ik \cdot x}
\]
\[
V^{(2)} = [k \cdot I \cdot \bar{\partial} x - (k \cdot \bar{\psi})(k \cdot I \cdot \bar{\psi})] e^{ik \cdot x}.
\]

Any combination of these with \((\pm)\) ghost charges adding up to \(-2\) (to cancel the background charge on the sphere) will yield the same on-shell S-matrix amplitude (5). The N=1 structure is as usual [33] in terms of the bosonization \( \partial \phi = \beta \gamma \), with additional ghost factors in the Ramond sector from the spin field for the supercurrent \( \Psi \), equation (19). These are again constructed in the standard way: the corresponding spin 1/2 ghosts \( \tilde{\beta}, \tilde{\gamma} \) are bosonized via \( \partial \rho = \tilde{\beta} \tilde{\gamma} \), and the standard picture for the Ramond sector vertices involves a factor \( \Sigma^{(N=1)}_{gh,R} = e^{-\frac{1}{2} \phi + \frac{1}{2} \rho} \). The NS vertex operators in equations (6),(20) have been written in the zero picture for both left- and right-movers.

\(^{10}\)For the twisted N=2 algebra, it is more convenient to bosonize the twist eigenstates \( \beta_+ \gamma_+ \pm \beta_- \gamma_- \) rather than the charge eigenstates.
B. (2,2) open strings and D-branes

The open string sector describes self-dual Yang-Mills, in a formulation due to Yang [1],[35]. The equations of motion are

\[ F^{(2,0)} = F^{(0,2)} = 0 \]
\[ k \wedge F = 0 \]  

(47)

The single physical string mode is again the center-of-mass, which is an adjoint scalar \( \varphi \) related to the gauge field by

\[ A = (De^{-\varphi})e^{\varphi}, \quad \bar{A} = (\bar{D}e^\varphi)e^{-\varphi}, \] 

(48)

with \( D \) a background covariant derivative. This ansatz automatically solves the (2,0) and (0,2) components of (47), and converts the (1,1) part into a wave equation for \( \varphi \). In the modern interpretation of gauge charge, the Chan-Paton indices are carried by branes filling spacetime, whose number may be determined by closed string tadpole cancellation (of course, such restrictions are not active at open string tree level, where we can formally consider any classical Chan-Paton group). A calculation of the tadpole [35] indicates a gauge group \( G=SO(2) \); however, it should be noted that there are many massless tadpoles in N=2 string loop amplitudes, whose interpretation is currently unclear, rendering the determination of \( G \) somewhat ambiguous.

B.1. BOUNDARY CONDITIONS AND D-BRANES

The open string boundary conditions of N=2 superconformal field theory have been investigated by [36]. They fall into two classes:

\[ A - \text{type} : \quad J_\ell = -J_r, \quad G^+_\ell = \pm G^-_r \]
\[ B - \text{type} : \quad J_\ell = +J_r, \quad G^+_\ell = \pm G^-_r \]  

(49)

where \( d \) is the complex dimension of the target space and \( p + 1 \) is the brane dimension\(^{11}\). These two types of boundary condition are related to the two types of N=2 superfield [14]: If a chiral field obeys type B boundary conditions, then its T-dual is a twisted chiral superfield satisfying type A boundary conditions, and vice-versa. Also, for a single free superfield, type B boundary conditions arise when the scalar components are either both Neumann or both Dirichlet, whereas type A has one Dirichlet and one Neumann boundary condition.

\(^{11}\)Unfortunately, this convention is not particularly well-adapted to N=2 strings with signature 2+2; nevertheless we will adhere to it, using the notation \((s+t)\)-brane for a brane with \( s \) space and \( t \) time dimensions when necessary.
The dynamics of type II branes in ten dimensions is, at long wavelengths, the dimensional reduction of the Yang-Mills theory governing 10d open strings, the dynamics of the Dirichlet coordinates being “frozen” on the brane. Correspondingly, one expects the dynamics of N=2 string D-branes to be governed by the dimensional reduction of the self-dual equations (47). For instance, the D-string equations are the Hitchin equations

$$D_A X = D_A \bar{X} = 0 \quad , \quad F_A + [X, \bar{X}] = 0 . \quad (50)$$

Here the dynamical field is a 2d scalar $\varphi$ in the adjoint of $U(N_2)$ determining the reduced gauge field (48). The covariantly constant (and therefore nondynamical) Higgs field $X$ describes the transverse positions of the $N_2^2$ D-strings. When $N_2$ D-strings coincide, these equations become the 2d chiral model equations. Similarly, one can show that the D-particle is described by the Nahm equations

$$DX^i = \epsilon^{ijk}[X^j, X^k] , \quad (51)$$

and the D2-brane by the Bogomolny equations (for recent related work, see [37])

$$F_{ij} = \epsilon_{ijk} D_k X . \quad (52)$$

There are other possible reductions, for instance imposing Dirichlet boundary conditions along a null direction, or wrapping branes around homology cycles of nontrivial self-dual four-manifolds such as K3; we will not discuss them here. Note that the equations (50)-(52) are all in the family of Uhlenbeck-Yau type equations describing BPS configurations of D-branes (c.f. [38]), another indication of the resemblance of the dynamics to the BPS sector of superstrings.

The D-instanton is also of some interest, in that its amplitudes may serve to define an off-shell continuation of N=2 string dynamics, and thus an off-shell quantum theory of self-dual gravity and Yang-Mills. These amplitudes are in some respects similar to those of macroscopic loops in noncritical string theory [39].

One may also consider composite systems of $p$- and $p'$-branes. For example D-instantons in the open string theory are composites of a (0+0)-brane and a (4+0)-brane which regularize an abelian instanton $^{12}$. The 0-4 strings should regulate the moduli space of ADHM instantons (although this needs to be checked).

$^{12}$Assuming the tadpole cancellation giving gauge group SO(2); otherwise – for instance at open string tree level – it may be consistent to consider multiple (4+0)-branes and thus nonabelian (e.g. SU(N_{4+0}) self-dual YM gauge group. Then the D-instantons carry fundamental representation gauge quantum numbers.
B.2. COUPLING D-BRANES TO SELF-DUAL GRAVITY

The SDYM open string dynamics couples to the self-dual gravity of N=2 closed strings, since as usual open string loops factorize on closed string exchange. Moreover, performing a Kähler gauge transformation \( \delta K = \Lambda (X) + \Lambda^* (X^*) \) on the worldsheet action

\[
S = \int_{\Sigma} d^2 z d^2 \theta d^2 \bar{\theta} \ K(X, X^*) + \oint_{\partial \Sigma} ds d^2 \theta \varphi (X, X^*) ,
\]

one sees that \( \Lambda \) can be pushed onto gauge transformations of the U(1) part of the prepotential \( \varphi \) which encodes the self-dual U(N)=SU(N) \times U(1) open string gauge background (48).

Marcus has shown [35] that the leading correction to the closed string equation of motion (12) is

\[
\det [g_{ij}] = 1 - \frac{2 \kappa^2}{g^2} \text{Tr} [F_{ij} F^{ij}] + O(g^4),
\]

where \( g \) and \( \kappa \) are the open and closed string couplings, related by \( \kappa \sim \sqrt{\hbar g^2} \). Using the open string equation of motion \( k \wedge F = 0 \), one may rewrite this in a way that manifests the above gauge symmetry:

\[
\frac{1}{N} \text{Tr} [(k + g N^{R^2} F) \wedge (k + g N^{R^2} F)] = \Omega \wedge \Omega^* \]

where \( \Omega \) is the holomorphic two-form.

It seems, however, that D-branes will not act as linearized sources for static gravitational and/or axion-dilaton fields. A static string solution, as for instance that of the fundamental string [26] is not self-dual (as one may see for instance by the set of supersymmetries it preserves).

The D-branes of ordinary 10d string theory carry charges under the antisymmetric tensor fields of the R-R sector. In the N=2 string these fields are gauge equivalent to NS-NS fields by spectral flow in the N=2 U(1); effectively, there are no R-R gauge fields, and the D-branes serve as sources for the NS sector fields (as one sees by inserting closed string vertex operators on the disk). There appear to be no gauge charges carried by N=2 string D-branes, apart from those of the self-dual gravitational field.

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