We establish and analyze a new relationship between the matrix functions describing spin fields of a spin $s$, where $2s \in \mathbb{Z}^+$, and $\mathbb{C}P^{2s}$ two-dimensional Euclidean sigma models. The spin matrices are constructed from the rank-1 Hermitian projectors of the sigma models or from the anti-Hermitian immersion functions of their soliton surfaces in the $\mathfrak{su}(2s + 1)$ algebra. We provide a geometric interpretation of this construction. For the spin fields which can be represented as linear combinations of the generalized Pauli matrices, we find the dynamics equation satisfied by their coefficients. This equation is identical to the stationary equation of a two-dimensional Heisenberg model. We show that the same holds for matrices congruent to the generalized Pauli matrices through any coordinate-independent unitary linear transformation. These properties allow for new interpretations of the spins as compositions of more elementary objects. They also open up the possibility of future applications of the sigma models to the situations which depend on spin behaviour, including spintronics, spin glasses and quantum computing.
vectors always yield spins when combined in the above way. Possible applications of these results, which include quantum computing, are mentioned in section 5.

2. Basics of the \( \mathbb{C}P^{2s} \) sigma models

The main feature of nonlinear sigma models in field theory is that the transformed field admits a very simple effective Lagrangian density, defined in \( \mathbb{C} \), which assumes values in some manifold [11]

\[
\mathcal{L} = \partial_{\mu} \phi^2 \partial^{\mu} \phi,
\]

with appropriate algebraic constraints on the field \( \phi \). This way, the complexity of their dynamics is contained in the geometry of the target space. Such an approach has found many applications [2–9]. Even for very simple \( \mathbb{C}P^{N-1} \) models, where the target is a single complex \( N \)-dimensional sphere, their properties are highly nontrivial [11–18]. The models described by the Lagrangian (1) are the starting point of the present study.

As a rule, the domain is parametrized in terms of the complex variables \( \xi = x + iy \in \mathbb{C} \), while the target manifold variables are either vectors \( z \) of a complex unit sphere, \( z' \cdot z = 1 \), embedded in \( \mathbb{C}^N \), or the Grassmannian homogeneous variables \( f \) for which \( z = f/(f^\dagger \cdot f)^{1/2} \) (the dagger superscript denotes the Hermitian conjugate). Another convenient choice for the variables may be projectors \( P \in GL^2(\mathbb{C}) \) mapping on the directions of \( z \) (and \( f \)), namely

\[
P = z \otimes z^\dagger = f \otimes f^\dagger /
\]

\[
f^\dagger \cdot f,
\]

where \( \otimes \) is the tensor product. This description in terms of projectors proves to be simpler and more useful than that in terms of the variables \( f \) or \( z \). The action corresponding to the Lagrangian density (1) integrated over the complex plane \( \mathbb{C} \) becomes (with a constant factor for convenience)

\[
\mathcal{A} = \frac{1}{2} \int_{\mathbb{C}} d\xi d\bar{\xi} \text{tr}(\partial P \cdot \bar{\partial} P),
\]

under the idempotency condition

\[
P^2 - P = 0, \quad P^\dagger = P, \quad \text{tr} P = 1,
\]

where \( \partial \) and \( \bar{\partial} \) are the derivatives with respect to \( \xi \) and \( \bar{\xi} \) respectively. The Euler–Lagrange (E–L) equations are simply [11]

\[
[P, \bar{\partial} P] = 0,
\]

where the square bracket denotes the commutator. The solutions of equation (5), satisfying the condition (4), may be obtained through a recurrence procedure. In [12, 13], it was shown that all solutions corresponding to the finite action (3), expressed in terms of the homogeneous variables \( f_0 \), result from the successive application of a raising operator

\[
f_{k+1} = \mathcal{P}^+ f_k = \left( I_{2s+1} - \frac{f_k \otimes f_k^\dagger}{f_k^\dagger \cdot f_k} \right) \cdot \partial f_k, \quad k = 0, \ldots, 2s,
\]

to some holomorphic solution \( f_0 \), where \( I_N \) is the \( N \times N \) unit matrix. The last nontrivial vector \( f_{2s+1} \) is the antiholomorphic solution \( f_{2s} \) (the action of \( \mathcal{P}^+ \) on an antiholomorphic vector obviously yields a zero vector). Similarly all solutions can be obtained from an antiholomorphic solution \( f_{2s} \), by acting on it with an analogous lowering operator \( \mathcal{P}^- \).

These raising and lowering operators have their counterparts for projectors [14, 16], namely

\[
P_{k+1} = \Pi^+ (P_k) = t_{k+1} \bar{\partial} P_k \cdot P_k \cdot \bar{\partial} P_k, \quad P_{k-1} = \Pi^- (P_k) = t_k \partial P_k \cdot P_k \cdot \partial P_k,
\]

\[
k = 0, \ldots, 2s,
\]

with the real scalars \( t_k \) equal to

\[
t_j(\xi, \bar{\xi}) = \left[ \text{tr}(\partial P_j \cdot P_j \cdot \partial P_j) \right]^{-1} = \left[ \text{tr}(\bar{\partial} P_{j-1} \cdot P_{j-1} \cdot \bar{\partial} P_{j-1}) \right]^{-1}
\]

for \( j = 1, \ldots, 2s \) and \( t_0 = t_{2s+1} = 0 \).

The \( \mathbb{C}P^{2s} \) sigma models with finite action are completely integrable [13]. Furthermore, the E-L equations can be written in the form of a conservation law

\[
\partial \left[ \bar{\partial} P \cdot P \right] + \bar{\partial} \left[ \partial P \cdot P \right] = 0,
\]

which shows that a total differential can be constructed out of the commutators appearing in (9). The integral of the total differential over any contour \( \gamma \) (with the constant of integration chosen so as to ensure tracelessness) is an anti–Hermitian immersion function \( X_t(\xi, \bar{\xi}) \) of a two-dimensional surface in a \( \mathfrak{su}(N) \) Lie algebra [17].
Moreover, these immersion functions can be explicitly expressed as linear combinations of the projectors $P_k$, $k = 0, \ldots, 2s$, [17]. The definition and the explicit form of the immersion functions in terms of the projectors $P_k$ are given by

$$X_k = i \int_\gamma (-[\partial P, P] d\xi + [\dot{\partial} P, P] d\dot{\xi}) = -i \left( P_k + 2 \sum_{j=0}^{k-1} P_j \right) + i c_k \bar{n}_{2k+1},$$

(10)

Here $c_k = (2k + 1)/(2s + 1)$ is the constant ensuring the tracelessness. These functions describe two-dimensional soliton surfaces whose conditions of immersion are the E-L equations (5). Their dynamics is governed by an action integral analogous to (3) [15]

$$A_X = \frac{1}{2} \int d\xi d\dot{\xi} \, \text{tr}(\partial X_k \cdot \partial X_k), \quad k = 0, \ldots, N - 1,$$

(11)

but the condition is defined by a different minimal polynomial, namely

$$(X_k - ic_k \bar{n})(X_k - i(c_k - 1)\bar{n})(X_k - i(c_k - 2)\bar{n}) = 0, \quad c_k = \frac{1 + 2k}{N}.$$  

(12)

Nevertheless $X_k$ satisfy the same E-L equations (5) as the original projectors $P_k$

$$[\partial \bar{n} X_k, X_k] = 0.$$  

(13)

Since the Killing form defining the metric in the $su(N)$ Lie algebra

$$\langle X, Y \rangle = -\frac{1}{2} \text{tr}(XY), \quad X, Y \in suN$$

(14)

is identical to the Lagrangian density of (11), the $X_k$ define the conditional minimal two-dimensional surfaces in that algebra (the condition is (12)). These surfaces have no common points, except for the case of $CP^1$, having only two surfaces $X_0$ and $X_1$ which coincide [19]. The immersion function matrices $X_k$ span a Cartan subalgebra of the $su(N)$ algebra. Moreover, regarding the Killing form as the scalar product, we find that the angles between the immersion functions $X_k \in su(N)$ are independent of the coordinates $(\xi, \bar{\xi})$ and the initial $P_0$, being functions of $N = 2s + 1$ and $k$ only, namely, for $k < m$ [19]

$$\cos \phi_{km} = \frac{c_k(2 - c_m)}{\left( c_k(2 - c_m) - \frac{1}{N}(c_m(2 - c_m) - \frac{1}{N}) \right)^{1/2}} \in (0, 1),$$

(15)

This fact is relevant for the construction of spins, which will be discussed in the next section.

Since the $P_k$ are mutually orthogonal projectors of rank 1, it follows that each of them has only one nonzero eigenvalue equal to 1 and together they constitute a partition of unity. Hence an appropriate linear combination of these projectors may have any required set of eigenvalues. A matrix corresponding to a component of a spin vector, say $S^z$, has eigenvalues $-s, -s + 1, \ldots, s$ where $s$ is a positive integer or half-integer ($2s \in \mathbb{Z}^+$), with the exception of the trivial case $s = 0$. A matrix having the same eigenvalues as a spin operator may be constructed from the $CP^1$ projectors as

$$S^z = \sum_{k=0}^{2s} (k - s) P_k.$$  

(16)

In the next section we will show that this combination, regarded as a function of $(\xi, \bar{\xi}) \in \mathbb{C}$, has many more properties of a spin field than just the proper set of eigenvalues. Therefore, we will refer to the matrices (16) as spin matrices.

3. Spin matrices

3.1. Geometric picture

Up to a constant factor, the combination (16) is equal to the sum of the immersion functions of the disjoint soliton surfaces $X_k$ [15]

$$S^z = (-i/2) \sum_{k=0}^{2s} X_k,$$

(17)

which suggests an interpretation of the spin as a composite phenomenon. Specifically, the field of the spin matrices is a vector function $\mathbb{C} \to su(2s + 1)$, multiplied by $(-i/2).$ The constancy of the angles between the
$X_k$ makes the spin matrix a vector sum of side edges of a pyramid (directed from the apex to the base vertices). The dimensionality of the pyramid is $N - 1 = 2s$ rather than $N$, due to the linear dependence of the surfaces $X_k$, which follows from their construction of the projectors $(10)$, namely

$$\sum_{k=0}^{N-1} (-1)^k X_k = 0. \tag{18}$$

The resultant spin vector $(17)$ would vary with $(\zeta, \xi)$, preserving the rigid structure of the pyramid (see figure 1).

Before proceeding to our main results, we summarize a few simple properties of the spin matrices associated with the $\mathbb{C}P^2$ models expressed in terms of rank-1 Hermitian projectors $P_k$. All the discussed properties of the spin matrices $S^z$ will follow from the defining relation $(16)$ and from the fact that the Hermitian matrices $P_k$ map onto one-dimensional subspaces of $\mathbb{C}^N$.

### 3.2. Properties of the spin matrices

**Property 1.** If a Hermitian rank-1 projector $P_k$ maps onto a one-dimensional subspace of $\mathbb{C}^N$, $N = 2s + 1$, then the trace and the rank of the spin matrix $S^z$ $(16)$ are

$$\text{tr } S^z = 0, \quad \text{rank } S^z = \begin{cases} N & \text{for } N = 2n, \\ N - 1 & \text{for } N = 2n + 1, \end{cases} \quad n \in \mathbb{Z}^+. \tag{19}$$

The fact that, for even $N$, the ranks of the matrices $S^z$ are equal to $N$, while for odd $N$, the ranks of the matrices $S^z$ are equal to $N - 1$, is due to the presence of a zero eigenvalue corresponding to the eigenvector $P_k$, where $k = s = \frac{s}{2}(N - 1)$.

**Property 2.** The quadratic form corresponding to the Killing form in $\mathfrak{su}(N)$ for the matrix $S^z$ is constant and may be defined as

$$\langle S^z, S^z \rangle = \frac{1}{N} \text{tr}(S^z \cdot S^z) = \frac{N^2 - 1}{12} = \frac{s(s + 1)}{3}, \tag{20}$$

which corresponds to the expected value for the squared length of one component of a quantum-mechanical spin vector (the multiplier $1/N$ instead of the usual $1/2$ is a natural consequence of representing the $\mathfrak{su}(2)$ spin matrix by an $\mathfrak{su}(N)$ matrix $S^z$). This correct value of the length is another spin property of our matrices.

**Property 3.** At each point $(\zeta, \xi)$, the spin matrix $S^z$ satisfies an algebraic condition determined by the characteristic polynomial corresponding to the eigenvalue problem.
where $I_N$ is the $N \times N$ identity matrix. As all eigenvalues are different, (22) is the minimal polynomial.

**Property 4.** According to our earlier result (equation (47) of [16]), if the rank-1 projectors $P_k$ satisfy the E-L equations (3), then any linear combination of these projectors is also a solution of (5) (this fact is nontrivial because the E-L equations are nonlinear). Hence the field of the spin matrices $S^2(\zeta, \bar{\zeta})$, obtained from (16), also satisfies the same E-L equations (5), except that the constraint is (22) (instead of $P^2 = P$).

**Property 5.** It follows from property 4 that the spin matrix fields $S^2$ are conditional stationary points of the same action integral (3) as the projectors and the condition is (22) (rather than $P^2 = P$). The immersion functions of the soliton surfaces $X_k$ have the same Lagrangian density of (3), so they represent the conditional minimal surfaces in $su(2s + 1)$, under the condition (12). The spin matrix is a sum (17) at each point $(\zeta, \bar{\zeta})$. Thus $iS^2(\zeta, \bar{\zeta})$ represents a surface, which is minimal under the condition (22).

### 3.3. $S^2$ matrices as spins

A 3-dimensional basis for spin matrices describing a system of spin $s = (N − 1)/2$ particles consists of three $N \times N$ generalized Hermitian Pauli matrices whose elements read [20]

\[
\begin{align*}
(\sigma^x)_{mn} &= (\delta_{m,n+1} + \delta_{m+1,n}) \sqrt{s(m+n+1) - m n} \\
(\sigma^y)_{mn} &= i(\delta_{m,n+1} - \delta_{m+1,n}) \sqrt{s(m+n+1) - m n} \\
(\sigma^z)_{mn} &= 2(s-m)\delta_{mn}
\end{align*}
\]

where $0 \leq m, n \leq N − 1$. These matrices generate an irreducible representation of $su(2)$ in $\mathbb{C}^N$.

We know from the proof given in [10] that a special role is played by the solutions of the $\mathbb{C}P^{2i}$ sigma models which stem from the Veronese sequence of holomorphic functions

\[
f_k(\zeta, \bar{\zeta}) = \sum_{j=0}^{2s} \binom{2s}{j} \zeta^j \bar{\zeta}^{2s-j}.
\]

The functions $f_0$ (24) and $f_k$, $k = 1, \ldots, 2s$, obtained from $f_0$ by the recurrence formulae (6), define the corresponding rank-1 projectors by means of (2). The spin matrix $S^2$ obtained from these projectors (16) is tridiagonal and can be uniquely decomposed into a linear combination of matrices $\sigma^x$, $\sigma^y$, $\sigma^z$ (23). These solutions will be discussed in detail in the next section. Unfortunately, not all spin matrices (16) have this property.

**Example.** A simple counterexample may be constructed from the recurrence formulae applied e.g. to the following holomorphic vector of a $\mathbb{C}P^{2i}$ model

\[
f_0(\zeta) = (1, \zeta, \zeta^2, \ldots, \zeta^{2s}).
\]

The resulting spin matrix $S^2$ is obtained from the projectors $P_0$ (2) as their linear combination (16) where the projectors follow from the recurrence formulae (7) applied to $P_0 = f_0 \otimes f_0 / (f_0^* \cdot f_0)$. This matrix does not have to be tridiagonal, whereas any combination of the diagonal matrix $\sigma^2$ and tridiagonal matrices $\sigma^x, \sigma^y$ has to be tridiagonal. In the simplest case of the $\mathbb{C}P^2$ model, the $3 \times 3$ matrix has a nonzero element

\[
(S^2)_{33} = -3\xi^3 + \xi^2 + 1 \xi + \bar{\xi} (\xi^2 + 4\xi \bar{\xi} + 1)^{-1}.
\]

Obviously, $(S^2)_{11}$ is also nonzero as it is the complex conjugate of (26). Hence the matrix is not tridiagonal.

On the other hand, it is evident that any diagonalizable $N \times N$ matrix having the proper eigenvalues is congruent to $\sigma^2/2$ (and also to $\sigma^x/2$ or $\sigma^y/2$, by other congruency transformations). The spin-like linear combination of projectors (16) is Hermitian, so it is diagonalizable by a unitary matrix. Let $U$ be the unitary diagonalizing matrix for $S^2$ (16), where both $S^2$ and $U$ are functions of $(\zeta, \bar{\zeta})$. Then

\[
\begin{align*}
(\sigma^x) &= 2U^{-1} \cdot S^2 \cdot U, \\
(\sigma^y) &= \frac{1}{2}U \cdot \sigma^2 \cdot U^{-1}.
\end{align*}
\]

If $U$ acts on $\sigma^x$ and $\sigma^y$ in the same way, we obtain three matrices which constitute a basis for another irreducible representation of $su(2)$ in $\mathbb{C}^N$ (it is straightforward to show that they span a Lie subalgebra of $su(N)$).
In the context of (27b), the whole dynamics of the spin matrix lies in the unitary transformation \( U \). On the other hand, in some situations, we can analyze the dynamics of spin matrices without referring to the transformation (27).

Let us start with the matrices \( S^2 \) which are combinations of the generalized Pauli matrices (23)

\[
S^2(\xi, \tilde{\xi}) = \alpha^x(\xi, \tilde{\xi}) \sigma^x + \alpha^y(\xi, \tilde{\xi}) \sigma^y + \alpha^z(\xi, \tilde{\xi}) \sigma^z, \quad (\alpha^x)^2 + (\alpha^y)^2 + (\alpha^z)^2 = 1/4.
\]

The coefficients in (28) are the coordinates of the spin vector in the basis (23). For such matrices \( S^2 \) we have

**Proposition 1.** Let \( \alpha \) be a vector whose components are the coefficients \( \alpha^x, \alpha^y, \alpha^z \) of equation (28). Then \( \alpha \) satisfies the equation

\[
\alpha \times \partial \partial \alpha = 0,
\]

where \( \times \) denotes the usual vector product in \( \mathbb{C}^3 \). The vector \( \alpha \) is subject to the normalization condition

\[
4 \alpha \cdot \alpha - 1 = 0.
\]

Equation (29) represents the E-L equations (5) expressed in terms of the vector \( \alpha \).

**Proof.** According to Property 4, the spin matrices \( S^2 \) satisfy the E-L equations (5). Substituting (28) into those equations, we obtain the coordinates of (29) in the basis (23).

Equation (29) together with the constraint (30) describe stationary states of the two-dimensional Heisenberg model (see appendix).

It is worthwhile to note, that a correspondence between hyperbolic sigma models and the Heisenberg model was described in [21, 22] as early as 2001. However that relation is different from ours and it is limited to a two-dimensional target space. To our best knowledge, these are the only articles linking these models.

According to Property 4, the E-L equations for the \( S^2 \) matrices follow from the same action integral (3) as the equations for the projectors (5) determining the conditional stationary point of the action integral (3) under the condition (22). Similarly, the spin-dynamic equations (29) can be derived as conditional stationary points of the action integral over the complex plane

\[
\mathcal{A}_\alpha = \int_\mathbb{C} d\xi \cdot d\tilde{\xi} [\partial \alpha \cdot \partial \alpha - \mu(\xi, \tilde{\xi})(4 \alpha \cdot \alpha - 1)].
\]

The Lagrange multiplier \( \mu \) in the action integral is introduced in order to comply with the constraint (30). Under the variation of the action (31), we have the following:

**Proposition 2.** The spin dynamic equations (29) are defined by the stationary points of the action integral (31).

**Proof.** The proof is straightforward if we take the Fréchet derivative of (31) with respect to \( \alpha(\xi, \tilde{\xi}) \). This yields

\[
\partial \partial \alpha - 4 \mu \alpha = 0.
\]

Performing the vector multiplication by \( \alpha \) on both sides of (32), we obtain (29).

**Corollary.** It is evident from the above proof that equation (29) is merely the necessary condition for \( \alpha \) to follow spin dynamics. It has to be supplemented by the normalization condition (30). For the action (31), we can get the complete E-L equations by calculating the scalar product of (32) with \( \alpha \), which allows us to calculate the multiplier \( \mu \) explicitly from the \( \alpha \)-normalization condition (30), thus getting \( \mu = \alpha \cdot \partial \partial \alpha \). Substituting this value into (32), we obtain the equation free from extra conditions, namely

\[
(\mathbb{I}_3 - 4 \alpha \otimes \alpha) \cdot \partial \partial \alpha = 0,
\]

where \( \mathbb{I}_3 \) is the three-dimensional identity tensor.

This result can be generalized to the matrices congruent to \( \sigma_x, \sigma_y, \sigma_z \) by a coordinate-independent unitary transformation.

**Proposition 3.** Let

\[
S^2 = \alpha \cdot \mathbf{s} := \sum_{k \in \{x,y,z\}} \alpha^k s^k,
\]

where \( \alpha^k \in \mathbb{R} \), the Euclidean norm is \( |\alpha| = 1/2 \) and \( s^k \) are congruent to \( \sigma^k \) by a constant unitary matrix \( U \):

\[
s^k = U \cdot \sigma^k \cdot U^{-1}, \quad k \in \{x, y, z\}.
\]

Then if the commutator \( [S^2, \partial \partial S^2] \) vanishes, the vector \( \alpha \) satisfies equation (29).
A simple proof follows from the fact that all commutators \([s^k, s^m]\), \(k, m \in \{x, y, z\}\) are congruent to \([\sigma^k, \sigma^m]\) through the same transformation matrix \(U\).

**Remark 1.** Proposition 3 is trivial for \(s = \frac{1}{2}\) (i.e. \(N = 2\)) due to the isomorphism of \(SU(2)\) with \(SO(3)\), which makes the transformation a rotation of the vector \(\alpha\) by a constant angle. It is nontrivial for higher spins \((N > 2)\) as the set of constant \(U\) transformations is much richer.

**Remark 2.** In the general case, proposition 3 is not true if the transformation matrix depends on the coordinates. This is apparent because in all of the proven cases, the \(S^+\) matrix depends on \(\xi, \bar{\xi}\) through the coefficients \(\alpha^+, \alpha^-, \alpha^z\) which are two algebraically independent functions only (note the normalization condition (30)). In general, the system (5) can have more degrees of freedom.

### 4. Spins from the Veronese vectors

The E-L equations (5) expressed in terms of the homogeneous variables \(f_k\), \(k = 0, \ldots, 2s\) have the form

\[
\begin{aligned}
\left( \frac{2s+1}{2s} \right) f_k - \frac{f_k \otimes f_k}{f_k \cdot f_k} \cdot \left[ \partial \eta f_k - \frac{1}{f_k \cdot f_k}((f_k \cdot \partial f_k) \partial f_k + (f_k \cdot \partial f_k) \partial f_k) \right] = 0.
\end{aligned}
\]  

(36)

One of the most useful holomorphic solutions (i.e. for \(k = 0\)) is given by

\[
\begin{aligned}
f_0 = \left( 1; \left(\frac{2s}{r} \right)^{1/2} \xi, \ldots, \left(\frac{2s}{r} \right)^{1/2} \xi^i, \ldots, \xi^{2s} \right) \in \mathbb{C}^{2s+1}\}\{\emptyset\}.
\end{aligned}
\]  

(37)

Starting from this solution, a sequence of solutions for \(k = 1, \ldots, 2s\) may be obtained through the recurrence relations (6). The sequence \(\{f_0, f_1, \ldots, f_k\}\) is called the Veronese sequence [23].

In this section we consider the \(\mathbb{C}P^{2s}\) models in which the vectors \(f_0, f_1, \ldots, f_{N-1}\), make a Veronese sequence. All these vectors \(f_0, \ldots, f_{N-1}\), \(N = 2s + 1\), may be expressed in terms of the Krawchouk orthogonal polynomials [10].

\[
\begin{aligned}
(f_k)_j = \frac{(2s)!}{(2s - k)!} \left( \frac{-\bar{\xi}}{1 + \xi} \right)^k \left( \frac{2s}{r} \right)^{1/2} \xi^k K_j(k; p; 2s), \quad 0 \leq k, j \leq 2s,
\end{aligned}
\]  

(38)

where \(p\) is the stereographic projection variable

\[
\begin{aligned}
0 < p = \frac{\xi \bar{\xi}}{1 + \xi \bar{\xi}} < 1.
\end{aligned}
\]  

(39)

Here \((f_k)_j\) is the \(j\)th component of the vector \(f_k \in \mathbb{C}^{2s+1}\}\{\emptyset\}\) and \(K_j(k; p, 2s)\) are the Krawtchouk polynomials for which we use the convention that for \(k = 0\)

\[
\begin{aligned}
K_0(0; p, 2s) = 1.
\end{aligned}
\]  

(40)

The Krawtchouk polynomials can be expressed in terms of the hypergeometric functions [24]

\[
\begin{aligned}
K_j(k; p, 2s) = 2F_1(-j, -k; -2s; 1/p), \quad 0 \leq k \leq 2s.
\end{aligned}
\]  

(41)

The element in the \(i\)th row and \(j\)th column of the rank-1 Hermitian projector \(P_k\) as given by (2) has the form [10]

\[
\begin{aligned}
(P_k)_{ij} = \frac{(2s)!}{(2s - k)!} \left( \frac{(\xi \bar{\xi})^j}{(1 + \xi \bar{\xi})^{2s}} \xi^i \right) \sqrt{\frac{(2s)!}{(2s - j)!}} K_i(k; k),
\end{aligned}
\]  

(42)

where we have omitted the dependence of \(K_i\) on \(p\) and \(2s\).

For the Veronese sequence solutions of the \(\mathbb{C}P^{2s}\) models, the analytic recurrence relations can be replaced by simpler algebraic ones. It is convenient to use the combinations of the \(x\) and \(y\) components of the spin matrices \(S^x = S^x \pm i S^y\) and \(\sigma^x = \sigma^x \pm i \sigma^\pm\), rather than the components themselves. The spin matrix \(S^z\) may be simply represented by a combination of the diagonal matrix \(\sigma^z\) and tridiagonal matrices \(\sigma^+, \sigma^-, \sigma^z\), namely [10]

\[
\begin{aligned}
S^z = \frac{1}{2(1 + \xi \bar{\xi})}[(\xi \bar{\xi}) - 1] \sigma^z - \xi \sigma^- - \bar{\xi} \sigma^+.
\end{aligned}
\]  

(43)

The \(su(2)\) commutation relations

\[
\begin{aligned}
[S^x, S^z] = \pm S^z, \quad [S^+, S^-] = 2S^z
\end{aligned}
\]  

(44)

(identical to the relations satisfied by the respective combinations \(\sigma^\pm\)), suggest the following forms of the components \(S^x\) and \(S^z\)
\[
S^+ = \frac{1}{2(1 + \xi^2)}(2 \xi \sigma^z + \xi^2 \sigma^x - \sigma^y), \\
S^- = (S^+)^* = \frac{1}{2(1 + \xi^2)}(2 \xi \sigma^z - \sigma^x + \xi^2 \sigma^y)
\]

(our matrices \(\sigma^z\) and \(\sigma^\pm\) are twice as large as those of [10] in order to match their notion as the generalized Pauli matrices (23)).

It is easy to check that the components of the spin \(S^z, S^\pm\) do indeed satisfy the commutation relations (44). Moreover [10], \(S^\pm\) play the role of the creation and annihilation operators for \(f_k\), namely

\[
S^+ f_k = -(1 + \xi) f_{k+1}, \\
S^- f_k = k(k - 1 - 2s) f_{k-1},
\]

where, by convention, \(f_{s+1} = f_{2s+1} = 0\) (see [10] for the proof).

Similarly, we get the algebraic recurrence relations for the projectors, namely

\[
P_{k+1} = \frac{S^+ P_k S^-}{\text{tr}(S^+ P_k S^-)}, \quad \text{for } k = 0, \ldots, 2s - 1 \\
P_{k-1} = \frac{S^- P_k S^+}{\text{tr}(S^- P_k S^+)} \quad \text{for } k = 1, \ldots, 2s.
\]

Consequently, the algebraic recurrence relations for the immersion functions \(X_k\) satisfy the algebraic conditions

\[
X_{k+1} = X_k - i \left( \frac{S^+ P_k S^-}{\text{tr}(S^+ P_k S^-)} + P_k - \frac{2}{2s + 1} \lfloor 2s \rfloor \right), \\
X_{k-1} = X_k + i \left( \frac{S^- P_k S^+}{\text{tr}(S^- P_k S^+)} + P_k - \frac{2}{2s + 1} \lfloor 2s \rfloor \right).
\]

The algebraic recurrence relations allow us to recursively construct the Veronese sequence of solutions \(f_k\) (or rank-1 projectors \(P_k\)) from the holomorphic solution \(f_0\) (or \(P_0\)) in a simpler way than the analytic relations (6).

The spin matrices corresponding to the Veronese vectors look particularly simple if we express them in terms of the spherical coordinates of the vector \(\alpha\) from (28). Let

\[
\alpha^x = \frac{1}{2} \cos \theta, \quad \alpha^y = \frac{1}{2} \sin \theta \cos \varphi, \quad \alpha^z = \frac{1}{2} \sin \theta \sin \varphi,
\]

where \(0 \leq \theta \leq \pi, \ 0 \leq \varphi < 2\pi\). Then a straightforward calculation leads to

\[
\theta = 2 \arctan|\xi|, \quad \varphi = -\arg \xi \pmod{2\pi}.
\]

Thus the angle between the vector \(\alpha\) and the \(z\)-direction depends on the modulus \(|\xi|\) only, while a change in the phase of \(\xi\) yields an identical rotation of the spin vector about the \(z\) axis.

An interesting property of the Veronese vector solution is that the Lagrangian density is constant when transformed from \(C\) onto the Riemann sphere \(S^2\) by a stereographic transformation. Specifically, if \(u, v, w\) are the Cartesian coordinates of points on the projection sphere, then the transformation reads

\[
\xi = \frac{u + iv}{1 - w}, \quad \text{or} \quad u = \frac{\xi + \bar{\xi}}{\xi \bar{\xi} + 1}, \quad v = -i \frac{\xi - \bar{\xi}}{\xi \bar{\xi} + 1}, \quad w = \frac{\xi \bar{\xi} - 1}{\xi \bar{\xi} + 1}.
\]

Then the Lagrangian density on the sphere is equal to

\[
\mathcal{L}_S = \frac{1}{2}(s + 2sk - k^2),
\]

which yields the action through simple multiplication by \(4\pi\): \(A = 2\pi(s + 2sk - k^2)\). It increases with \(k\), attaining its maximum at \(k = s\) for integer spins or \(k = s \pm \frac{1}{2}\) for half-integer spins. For larger \(k\), it decreases symmetrically, down to \(2\pi s\), i.e. the same value as for \(k = 0\).

Up to a multiplicative constant, the Lagrangian is the norm of the mean curvature \(-\frac{1}{2}\text{tr} \partial X_k \cdot \bar{\partial} X_k\) of its soliton surface \(X_k\). Hence the surfaces have mean curvatures of constant norm.

5. Possible applications and conclusions

The present paper provides a new interpretation of spin operators as structured entities, specifically, for spin \(s\) particles, they are pyramids in \(su(2s + 1)\) algebra (17). The corresponding spin fields preserve angles between
the edges of the pyramids while the point \((\xi, \bar{\xi})\) varies. This imposes a constraint on the possible dependence of the spins on the position. It may be interesting to look for an experiment which would reveal this structure of the spin. We postpone this problem to future research.

The results presented in this paper complement and extend our previous studies [14, 16, 19] on the theory of two-dimensional Euclidean sigma models. The inclusion of the \(su(2)\) spin-\(s\) representations may have a significant impact on many problems with physical applications. Since the equations (29) describe stationary states of the two-dimensional Heisenberg model (see appendix), our result describe ferro-, antiferro- and ferrimagnets of this model in terms of \(\mathbb{C}P^2\) sigma models. This allows for the calculation of properties of these magnetic materials by means of the apparently unconnected methods of particle physics. The complete integrability of the \(\mathbb{C}P^2\) sigma models provides a useful tool for solving problems arising in physics of these magnetic materials, see e.g. [25]. In particular, the Veronese solutions of the \(\mathbb{C}P^2\) sigma models allow us to build the corresponding spin fields explicitly.

A shortcoming of the description in terms of the \(\mathbb{C}P^2\) sigma models is the limited class of spin fields that can be described. The building blocks, which are solutions \(\psi_f\) of these sigma models, are obtained from a holomorphic function by successive application of the creation operators (6). On the other hand, the class of holomorphic functions is sufficiently broad as to allow for many nontrivial spin fields.

An interesting shortcoming of our approach is the limitation of the possible results obtained in this way to static solutions of the Heisenberg model. A problem worth future research is an extension of a \(\mathbb{C}P^2\) model to a one-parameter family of such models which would include the time dependence into the Heisenberg model. It is a promising direction of looking for the mutual relation between these models.

Our description also expresses the spin-\(s\) matrix fields in terms of the \(\mathbb{C}P^2\) projectors, which are orthogonal vectors. In the special case of the Veronese solutions, these vectors turn out to be sequences of Krawtchouk polynomials. Such a representation using the Fourier-Krawtchouk transformation was recently introduced in [26] to achieve quantum information processing in constant time (see also [27]). Moreover, this constant-time signal-evolution analysis works on finite strings of arbitrary length [26, 28]. The transformation represents the transformed function limited to a finite interval in terms of the Krawtchouk polynomials multiplied by the Fourier variable \(\exp\left(-\frac{2\pi}{l}(l-k)\right)\), \(k, l \in \mathbb{Z}\). It has its counterpart in the splitting of the spin component \(S^z\) into the rank-1 projectors proportional to products of two Krawtchouk polynomials with \(\xi\) and \(\bar{\xi}\) in consecutive powers. Our more general scheme encompasses such a possibility. It is very likely that a discretization (sampling), followed by appropriate transformations of this kind, may broaden the scope of efficient quantum computations. Further possible fields of applications of these polynomials include digital image processing [29] as well as medical image reconstruction and recognition [26, 30].

Since the present results provide a novel representation of the spin matrices in terms of \(\mathbb{C}P^2\) projectors or, equivalently, in terms of immersion functions of soliton surfaces, we expect that our formalism can offer a new geometrical interpretation of the spin phenomena in physically relevant theories and applications such as spin generation in spintronics [31] or looking for long-range order in spin-glasses [32]. Furthermore, we hope that special descriptions of \(\mathbb{C}P^2\) projectors, for instance in terms of Krawtchouk polynomials, will facilitate the calculations in the above-mentioned theories. It is a promising direction of possible research. This task will be undertaken in our future work.

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Appendix

Consider the two-dimensional Heisenberg model consisting of spins \(s_{lm}\), located in the positions \((x_k, y_m)\), \(k, m \in \mathbb{Z}\) of a rectangular lattice whose cell size is \(a \times b\), i.e. \(x_k = ka, y_m = mb, a, b \in \mathbb{R}^+\). Its Hamiltonian is given by

\[
H = \sum_{l,m} \left[ J_1 s_{lm} \cdot s_{l+1,m} + J_2 s_{lm} \cdot s_{l,m+1} + J_3 (s_{lm} \cdot s_{l+1,m+1} + s_{l+1,m} \cdot s_{lm+1}) \right],
\]

(A.1)

where \(J_1, J_2 \in \mathbb{R}\), are the coupling constants along the \(x\) and \(y\) directions respectively and \(J_3 \in \mathbb{R}\) is the coupling constant over the cell diagonals; the summation encompasses all nodes of the lattice.

We go to the continuous limit by defining \(s_{lm} = \alpha (x_k, y_m)\), and expanding the Hamiltonian to second order in the lattice constants \(a, b\). The first order terms vanish due to the fact that the spins are perpendicular to their
first derivatives. The continuous Hamiltonian, up to a constant, reads
\[ H = a^2(f_1 - f_2 - 2f_3)\alpha_x^2 + b^2(-h + f_2 - 2f_3)\alpha_y^2 + (j_1 + j_2 + 2j_3)\alpha \cdot (a^2\alpha_{xx} + b^2\alpha_{yy}), \] (A.2)

(where the subscripts \(x, y\) denote differentiation).

The equations defining a conditional stationary point of the Hamiltonian read
\[ a^2(f_1 + 2f_3)\alpha_{xx} + b^2(j_1 + 2j_3)\alpha_{yy} - 2\mu \alpha = 0, \] (A.3)

where the Lagrange multiplier \(\mu(x, y)\) corresponds to the constraint \(\alpha^2 = \text{const}\.\) The substitution
\[ \xi = \frac{x}{a^2(2f_1 + 2f_2)} + \frac{iy}{b^2(2f_1 + 2f_3)} \] (A.4)
yields equation (32) and also the stationary two-dimensional Heisenberg equation (29) with constraints (30) or (33).

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References

[1] Gell-Mann M and Lévy M 1969 The axial vector current in beta decay Nuovo Cimento 16 1729–50
[2] Callan G G et al 1969 Structure of phenomenological Lagrangians II Phys Rev B 177 2247–50
[3] Coleman S et al 1969 Structure of phenomenological Lagrangians I Phys Rev B 177 2239–47
[4] Davydov A 1999 Solitons in Molecular Systems (New York: Kluwer)
[5] Landolfi G 2003 On the Canham–Helfrich membrane model J Phys A: Math Gen 36 4699–715
[6] Rajaraman R 2002 CPT^N solitons in quantum Hall systems Eur Phys J B 29 157–62
[7] Seiberg N 2010 Modifying the sum over topological sectors and constraints on supergravity J. High Energ. Phys. 2010 70
[8] Viswanathan K and Parthasarathy R 1995 QCD strings as a constrained Grassmannian sigma model Phys Rev D 51 3830–8
[9] Zhitnitsky A R 1992 Non-perturbative effects in 2d CPT^N model and 4d YM theory A Image N toron approach Nucl Phys B 374 183–222
[10] Crampé N and Grundland A M 2019 CPT^N Sigma Models Described Through Hypergeometric Orthogonal Polynomials Ann Henri Poincaré 20 3365–87
[11] Zakrzewski W J 1989 Low Dimensional Sigma Models (Bristol: Adam Hilger) pp 46–74
[12] Din A M et al 1984 The Riemann–Hilbert problem and finite action CPT^N–1 solutions Nucl Phys B 233 269–99
[13] Din A M and Zakrzewski W J 1980 General classical solutions of the CPT^N–1 model Nucl Phys B 174 397–403
[14] Goldstein P P and Grundland A M 2010 Invariant recurrence relations for CPT^N–1 models J Phys A: Math Theor. 43 265206
[15] Goldstein P P, Grundland A M and Post S 2012 Soliton surfaces associated with CPT^N–1 sigma models: differential and algebraic aspects J Phys A: Math Theor 45 395208
[16] Goldstein P P and Grundland A M 2018 On a stack of surfaces obtained from the CPT^N–1 sigma models J Phys A: Math Theor 51 095201
[17] Grundland A M and Yurdusen I 2008 Surfaces obtained from CPT^N–1 sigma models Int J Mod Phys A 23 5137–57
[18] Zakrzewski W J 2007 Surfaces in R^{(N–1)}; based on harmonic maps S^2 → CPT^N–1 J Math Phys 48 113520
[19] Goldstein P P and Grundland A M 2011 On the surfaces associated with CPT^N–1 models J Phys Conf Ser. 284 012031
[20] Merzbacher E 1998 Quantum Mechanics (New York: Wiley) p 423
[21] Ward R S and Winn A E 1998 Integrable systems admitting topological solitons J Phys A: Math Gen 31 L261–6
[22] Winn A E 2001 A Pivotal model for the (1 + 1)–dimensional Heisenberg and Sigma models J Nord Phys 8 294–9
[23] Bolton J et al 1988 On conformal minimal immersion of S^2 into CPT^N Math Ann 279 599–620
[24] Koornwinder T et al 1982 Krawtchouk polynomials, a unification of two different group theoretic interpretation SIAM J Math Anal 13 1011–1023
[25] Levanuy A P and Garcia N 1992 The two-dimensional Heisenberg ferromagnet with various types of interactions: temperature dependence of magnetic parameters J Phys Condens. Matter 4 10177–94
[26] Stobinska H et al 2019 Quantum interference enables constant–time quantum information processing Sci Adv 5 eaaw9674 (5 pp) (2019)
[27] Blais A, Girvin S M and Oliver W D 2020 Quantum information processing and quantum optics with circuit quantum electrodynamics Nat. Phys. 16 247–56
[28] Atlashiyev N M and Wolf K B 1997 Fractional Fourier–Krawtchouk transform J Opt Soc Am A 14 1467–77
[29] Yap P T et al 2003 Image analysis by Krawtchouk moments Eur Phys J B 29 157–62
[30] Gauthier G et al 2017 Facial expression recognition using Krawtchouk moments and support vector machine classifier Fourth IEEE International Conf. on Image Information Processing (ICIIP) 1–6
[31] Hirohata A et al 2020 Reviews on spintronics: principles and device applications J Magn Magn Mater 509 166711 Ch, 1
[32] Stein D L and Newman C M 2013 Spin Glasses and Complexity Series: Primers in Complex Systems (Princeton, NJ: Princeton University Press)