Field-Dependent BRST-antiBRST Lagrangian Transformations

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Abstract

We continue our study of finite BRST-antiBRST transformations for general gauge theories in Lagrangian formalism initiated in [arXiv:1405.0790[hep-th] and arXi v:1406.0179[hep-th]], with a doublet \( \lambda_a, a = 1, 2 \), of anticommuting Grassmann parameters and prove the correctness of the explicit Jacobian in the partition function, announced in [arXiv:1406.0179[hep-th]], which corresponds to a change of variables with functionally-dependent parameters \( \lambda_a = U_a \Lambda \), induced by a finite Bosonic functional \( \Lambda(\phi, \pi, \lambda) \) and by the anticommuting generators \( U_a \) of BRST-antiBRST transformations in the space of fields \( \phi \) and auxiliary variables \( \pi, \lambda \). On this basis, we derive a Ward identity depending on the field-dependent parameters \( \lambda_a \) and study the problem of gauge dependence, including the case of Yang–Mills theories. We consider a formulation with BRST-antiBRST symmetry breaking terms, additively introduced to the quantum action constructed by the Sp(2)-covariant Lagrangian quantization, obtain the Ward identity and investigate the gauge-independence of the corresponding generating functional of Green’s functions. A formulation with BRST symmetry breaking terms is developed as well. It is argued that the gauge independence of the above generating functionals is fulfilled in the BRST and BRST-antiBRST settings. We apply these concepts to average effective action in Yang–Mills theories within functional renormalization group approach.

Keywords: general gauge theory, BRST-antiBRST Lagrangian quantization, field-dependent BRST-antiBRST transformations, Yang–Mills theory, average effective action, BRST and BRST-antiBRST symmetry breaking

1 Introduction

In our recent works [1, 2, 3], we have proposed an extension of BRST-antiBRST transformations to the case of finite (both global and field-dependent) parameters for Yang–Mills and general gauge theories within the BRST-antiBRST Lagrangian [4, 5, 6] and generalized Hamiltonian [7, 8] quantization methods; see also [9]. The notion of “finiteness” employs the inclusion into finite transformations of a new term, being quadratic in the parameters \( \mu_a \). First of all, this makes it possible to realize the complete BRST-antiBRST invariance of the integrand in the vacuum functional. Second, the functionally-dependent parameters \( \lambda_a = s_a \Lambda \), induced by a Bosonic functional \( \Lambda \), provide an explicit correspondence (due to the compensation equation for the corresponding Jacobian) between a choice of \( \Lambda \) and a transition from the partition function of a theory in a certain gauge, determined by a gauge Boson functional \( F_0 \),
to the same theory in a different gauge, given by another gauge Boson $F$. This becomes a key instrument of a BRST-antiBRST approach that allows one to determine the Gribov horizon functional [10] – which is initially given by the Landau gauge in the Gribov–Zwanziger theory [11] – by using any other gauge, including the $R_\xi$-gauges, eliminating residual gauge invariance in the deep IR region. In this connection, it should be noted that we do not consider here, and have not considered earlier in [1, 2, 3], the case of BRST [12, 13] and antiBRST transformations with one and the same anticommuting Grassmann parameter $\mu = \delta \Lambda$, as suggested in the first paper [14] devoted to finite BRST transformations (see Eqs. (2.3a), (2.3b) therein) in the infinitesimal and finite forms for the global and field-dependent cases in Yang–Mills theories. In fact, such considerations contradict to the ghost number distribution used in [1, 4, 5] for field variables and also contradict to the standard definition of BRST-antiBRST symmetry transformations [15, 16, 17], which implies the presence in BRST transformed fields of a Grassmann-odd parameter and an independent Grassmann-odd parameter for antiBRST transformed fields, $\bar{\mu}$, with the ghost number opposite to that of $\mu$. Second, our consideration is based on special global two-parametric supersymmetries, realized on equal footing in [4, 5, 6, 7, 8]. At the same time, finite field-dependent BRST and antiBRST transformations in Yang–Mills theories and reducible gauge theories with Abelian gauge groups have been recently considered in [18, 19] using a different manner at different stages of quantization, whereas the so-called finite “mixed BRST-antiBRST transformations” – by the terminology of [1, 2, 3, 4, 5, 6, 7, 8] for BRST-antiBRST transformations – do not contain the polynomial term $\Theta_1 \Theta_2 \neq 0$, thus affecting the “finiteness” of such finite BRST-antiBRST transformations. Namely, this eliminates the capability of such finite BRST-antiBRST transformations to be symmetry transformations relating the partition function of a gauge theory in one gauge to the same theory in any other gauge within perturbation theory. Instead, by making a change of variables related to such field-dependent transformations in the corresponding vacuum functional (even in Abelian gauge theories) it is impossible to preserve the quantum action for the same theory and to obtain this theory in another gauge, thus making impossible the gauge independence of the vacuum functional and that of the physical $S$-matrix for a finite change of the gauge condition.

For the sake of completeness, note that finite field-dependent BRST transformations for general gauge theories in the BV quantization scheme [28] have been considered in [27], and earlier in [26]. A construction of finite field-dependent BRST-antiBRST transformations in the Sp(2)-covariant generalized Hamiltonian formalism [7, 8] has been recently developed [2] for arbitrary dynamical systems subject to first-class constraints, along with an explicit construction of the parameters $\lambda_a$ generating a change of the gauge for Yang–Mills theories in the class of $R_\xi$-like gauges. In the case of BRST–BFV symmetry [22], a study of finite field-dependent BRST–BFV transformations in the generalized Hamiltonian formalism [23, 24] has been presented in [25]. In all of these articles, the crucial point has been the so-called compensation equation, first suggested for finite BRST transformations in Yang-Mills theory [29], within the Faddeev–Popov quantization rules [21], which establishes a one-to-one correspondence of field-dependent parameter(s) of BRST(-antiBRST) transformations with a finite change of the gauge condition.

In the Discussion of [3], namely, see (6.2), (6.3) therein, we have announced an explicit Jacobian in the partition function $Z_F$ which corresponds to a change of variables with field-dependent (and functionally-dependent) parameters $\lambda_a$ of finite BRST-antiBRST transformations, on the basis of which we solve the compensation equation in order to find $\lambda_a = \lambda_a(\Delta F)$, thus providing gauge independence, $Z_F = Z_{F+\Delta F}$, derivation of the Ward identities and investigate the problem of gauge dependence; see (6.5)–(6.10) in [3]. This concept was used in [3] to relate on a base of field-dependent BRST-antiBRST transformations quantum BRST-antiBRST invariant actions for the Freedman–Townsend model (of antisymmetric non-Abelian tensor field of not Yang–Mills type with reducible gauge symmetry) in two different gauges determined by a quadratic (in fields) gauge Boson. On the other hand, some problems examined for Yang–Mills theories

\[1\] Calculation of Jacobians corresponding to BRST-antiBRST transformations being linear in finite field-dependent parameters for Yang–Mills and more general gauge theories with an open gauge algebra, as well as transformations with polynomial, however, not being functionally-dependent parameters $\lambda_a$, makes an essential feature of our future research [20].
in [1] have remained unsolved. In addition, the topical problems of BRST-antiBRST symmetry breaking in the Sp(2)-covariant Lagrangian quantization on the basis of finite field-dependent BRST-antiBRST transformations, as well as the study of BRST symmetry breaking in the BV quantization, initiated in [30,31,26] on the basis of finite BRST–BV transformations [27], have not been considered.

Based on these reasons, we intend to address the following problems related to gauge theories in Lagrangian formalism:

1. calculation of the Jacobian for a change of variables in the partition function related to finite field-dependent BRST-antiBRST transformations being polynomial in powers of an Sp (2)-doublet of Grassmann-odd (and functionally-dependent) parameters \( \lambda_a = s_a \Lambda \), generated by a finite Grassmann-even functional \( \Lambda(\phi, \pi, \lambda) \) and by the Grassmann-odd generators \( s_a \) of BRST-antiBRST transformations;

2. derivation of the Ward identities and consideration of gauge dependence on the base of the compensation equation for an unknown functional \( \Lambda(\phi, \pi, \lambda) \) generating the Sp (2)-doublet \( \lambda_a \), in order to establish a relation of the partition function \( Z_F \) (having the quantum action \( S_F \) in a certain gauge determined by a gauge Boson \( F \)) with another partition function \( Z_{F+\Delta F} \) (having the quantum action \( S_{F+\Delta F} \) in a different gauge \( F + \Delta F \));

3. application of the above considerations to obtain a new form of the Ward identities and investigation of gauge dependence in gauge theories with a closed algebra of rank 1, including Yang–Mills theories;

4. introduction of the concept of BRST-antiBRST symmetry breaking in the Sp(2)-covariant Lagrangian quantization, derivation of the Ward identities and the study of gauge dependence on the basis of finite field-dependent BRST-antiBRST transformations;

5. consideration of a new form (as compared to [30,31]) of BRST symmetry breaking in the BV quantization, derivation of the Ward identities and the study of gauge dependence on the basis of finite field-dependent BRST-BV transformations;

6. application of BRST-antiBRST symmetry breaking concept to average effective action in Yang–Mills theories within functional renormalization group approach.

The work is organized as follows. In Section 2 we bring to mind the general setup of finite BRST-antiBRST Lagrangian transformations and prove our conjecture [3] as to the Jacobian, which corresponds to such a change of variables with functionally-dependent parameters, \( \lambda_a = s_a \Lambda \). In Section 3 we examine a compensation equation and derive a new form of the Ward identities, depending on the functionals \( \lambda_a \), investigate the problem of gauge dependence, including the case of Yang–Mills theories. In Section 4 we also obtain the Ward identities using the field-dependent BRST-antiBRST transformations and study the gauge dependence of the generating functionals of Green’s functions for a general gauge theory in Section 3.1 and for Yang–Mills theories in Section 3.2. In Section 4 we introduce the notion of BRST-antiBRST symmetry breaking in the Sp(2)-covariant Lagrangian quantization, derive the Ward identities and study gauge dependence on the basis of finite field-dependent BRST-antiBRST transformations. In Appendix A we reconsider the notion of BRST symmetry breaking within the BV quantization scheme, first developed in [30,31]. We use the notation of our previous works [1,3]. The average effective action concept in the way compatible with gauge independence for the conventional S-matrix for Yang–Mills theory in different gauges is considered in Appendix B. Unless otherwise specified by an arrow, derivatives with respect to the fields are taken from the right, and those with respect to the corresponding antifields are taken from the left. The raising and lowering of Sp (2) indices, \( s^a = \varepsilon^{ab} s_b \), \( s_a = \varepsilon_{ab} s^b \), is carried out with the help of a constant antisymmetric metric tensor \( \varepsilon^{ab} \), \( \varepsilon^{ac} \varepsilon_{cb} = \delta_b^a \), subject to the normalization condition \( \varepsilon^{12} = 1 \).
2 Finite Field-Dependent BRST-antiBRST Transformation and its Jacobian

In the Discussion of our recent work [1], namely, in (6.4), (6.5), we have announced the form of finite BRST-antiBRST transformations $\Delta \Gamma^p$ for a general gauge theory in Lagrangian formalism and subsequently proved [3] for constant finite anticommuting parameters $\lambda_a$ that it actually leaves the integrand $I_1^{(F)}$ in the partition function $Z_F = \int d\Gamma I_1^{(F)}$ invariant to all orders in powers of $\lambda_a$. Namely, the finite BRST-antiBRST transformations

$$\Delta \lambda \Gamma^p = \Gamma^p \left( \frac{\delta}{\delta \phi^A} \pi^A \lambda_2 + \frac{1}{4} \frac{\delta}{\delta \phi^A} \right) \longrightarrow I_1^{(F)} + \Delta \Gamma^p = I_1^{(F)},$$

where

$$s^a = \frac{\delta}{\delta \phi^A} \pi^A \lambda_2 + \frac{1}{4} \frac{\delta}{\delta \phi^A} S, - \frac{1}{2} \frac{\delta}{\delta \phi^A} \lambda^A, \quad s^2 = s^a s_a, \quad (2.1)$$

are realized on the coordinates $\Gamma^p = (\phi^A, \phi^*_A, \bar{\phi}_A, \pi^A, \lambda^A)$ of the space of fields $\phi^A$, antifields $(\phi^*_A, \bar{\phi}_A)$ and auxiliary variables $(\pi^A, \lambda^A)$ used in the Sp(2)-covariant Lagrangian quantization [3] [4], with the following distribution of Grassmann parity and ghost number:

$$\epsilon (\Gamma^p) = (\epsilon_A, \epsilon_A + 1, \epsilon_A, \epsilon_A + 1, \epsilon_A), \quad \text{gh} (\Gamma^p) = (\text{gh}(\phi^A), (-1)^a - \text{gh}(\phi^A), -\text{gh}(\phi^A), (-1)^{a+1} + \text{gh}(\phi^A), \text{gh}(\phi^A)). \quad (2.2)$$

In terms of the components, the transformations (2.1) are given by $(S_A \equiv \delta S/\delta \phi^A)$

$$\Delta \lambda \phi^A = \pi^A \lambda_2 + \frac{1}{2} \lambda^A \lambda_2, \quad \Delta \lambda \phi^*_A = \epsilon^{ab} \phi^*_a \phi^*_b + \frac{1}{2} S_A \lambda_2, \quad \Delta \lambda \bar{\phi}_A = -\epsilon^{ab} \lambda_A \lambda_b, \quad \Delta \lambda \lambda^A = 0, \quad (2.3)$$

The operators $s^a$ in (2.1) are the generators of (infinitesimal $\lambda_2 \equiv \mu_a$) BRST-antiBRST transformations [4] for the integrand, $I_1^{(F)} + \delta \Gamma^p = I_1^{(F)}$, with the restriction $\delta \Gamma^p = \mu_a = \delta (\phi^A, \phi^*_A, \phi^*_A, \pi^A, \lambda^A) = \left( \pi^A, \pi^*_A, - \epsilon^{ab} \pi^*_A \lambda^A, \lambda^A \right)$, which may be regarded as integrability conditions for the validity of $I_1^{(F)} + \Delta \Gamma^p = I_1^{(F)}$ to all orders in the parameters $\lambda_a$, with the restriction $\delta \Gamma^p = \mu_a = \delta (\phi^A, \phi^*_A, \phi^*_A, \pi^A, \lambda^A) = \left( \pi^A, \pi^*_A, - \epsilon^{ab} \pi^*_A \lambda^A, \lambda^A \right)$, being anticommuting and nilpotent:

$$\delta \Gamma^p = \mu_a = \left( \pi^A, \pi^*_A, - \epsilon^{ab} \pi^*_A \lambda^A, \lambda^A \right) \quad (2.4)$$

This make it possible to present the generating functional $Z_F(J)$ of Green’s functions in [3] [4], depending on external sources $J_A$, with $\epsilon (J_A) = \epsilon_A$, gh$(J_A) = -\text{gh}(\phi^A)$, as the functional integral

$$Z_F(J) = \int \text{d} \Gamma \exp \left\{ \left( i / \hbar \right) \left[ S_F (\Gamma) + J_A \phi^A \right] \right\}, \quad S_F = S - (1/2) F \left( U^2 \right), \quad U^2 = U_a \lambda^A \lambda^A, \quad \lambda^A \lambda^A = 0, \quad F = F (\phi). \quad (2.6)$$

Here, $\hbar$ is the Planck constant; the configuration space $\phi^A$, containing the classical fields $A^i$ and the Sp(2)-symmetric ghost-antighost and Nakanishi–Lautrup fields, depends on the irreducible [4] or reducible [5] character of a given
gauge theory, whereas the auxiliary fields \((\pi^A, \lambda^A)\) are required in order to introduce the gauge by using a gauge-fixing Bosonic functional \(F(\phi)\) with a vanishing ghost number. In its turn, the Bosonic functional \(S(\phi, \phi^*, \phi)\) with a vanishing ghost number is a solution of the generating equations
\[
\frac{1}{2} (S, S)^a + V^a S = \frac{i}{\hbar} \Delta^a S \Leftrightarrow \left( \Delta^a + \frac{i}{\hbar} V^a \right) \exp \left( \frac{i}{\hbar} S \right) = 0 , \quad (2.7)
\]
with a boundary condition for vanishing antifields \(\phi^{*A}, \phi_A\) given by the classical action \(S_0(A)\). In \(2.7\), the extended antibracket \((G, H)^a\) for arbitrary functionals \(G, H\) and the operators \(\Delta^a, V^a\) are given by
\[
(G, H)^a = G \left( \frac{\delta}{\delta \phi^A}, \frac{\delta}{\delta \phi_A^*} \right) - \frac{\delta}{\delta \phi_A^*} \frac{\delta}{\delta \phi^A} (H) \right) , \quad \Delta^a = (-1)^{\varepsilon_A} \frac{\delta}{\delta \phi^A} \frac{\delta}{\delta \phi_A^*} , \quad V^a = \varepsilon^{ab} \phi^A \frac{\delta}{\delta \phi_A} . \quad (2.8)
\]
The invariance \(3\) of the integrand \(I_4^{(F)} = d\Gamma \exp \left( (i/\hbar) S_F (\Gamma) \right) \) in \(2.6\) with vanishing sources \(J_A = 0\) under the global finite BRST-antiBRST transformations \(2.1\) can be established by using the generating equations \(2.7\), while taking into account the nilpotency \(\widetilde{U}^a \widetilde{U}^b \widetilde{U}^c = 0\) of the operators \(\widetilde{U}^a\) in \(2.6\) and using the explicit change \(3\)
\[
\Delta_\lambda G = G \left( \frac{s^a \lambda_a}{\hbar} + \frac{1}{4} \lambda^2 \right) . \quad (2.9)
\]
under the transformations \(2.1\) of an arbitrary functional \(G(\Gamma)\) expandable as a power series in \(\Gamma^p\),
\[
G (\Gamma + \Delta_\lambda \Gamma) = G (\Gamma) + G_p (\Gamma) \Delta_\lambda \Gamma^p + (1/2) G_p \Delta_\lambda \Gamma \Delta_\lambda \Gamma^p = G (\Gamma) + \Delta_\lambda G (\Gamma) \, , \quad \text{with} \quad G_p \equiv G \frac{\delta}{\delta \Gamma^p} , \quad (2.10)
\]
with allowance for the explicit form \(3\) of the Jacobian \(\text{Sdet} \left\| (\Gamma^p + \Delta_\lambda \Gamma^p)^{\frac{\delta}{\delta \Gamma^p}} \right\| \equiv \exp \left( \mathfrak{3} \right) \)
\[
\exp (\mathfrak{3}) = \exp \left[ - (\Delta^a S) \lambda_a - \frac{1}{4} (\Delta^a S) \frac{s^a}{s^2} \lambda^2 \right] \quad (2.11)
\]
corresponding to the transformation of the integration measure \(d\Gamma\) with respect to the change of variables \(\Gamma \rightarrow \tilde{\Gamma} = \Gamma + \Delta_\lambda \Gamma\), namely,
\[
d\tilde{\Gamma} = \text{Sdet} \left\| (\Gamma^p + \Delta_\lambda \Gamma^p)^{\frac{\delta}{\delta \Gamma^p}} \right\| = d\Gamma \exp \left( \text{Str ln} (\mathbb{I} + M) \right) \equiv d\Gamma \exp (\mathfrak{3}) \, ,
\]
where \(\mathfrak{3} = \text{Str ln} (\mathbb{I} + M) = - \sum_{n=1}^{\infty} \frac{(-1)^n}{n} \text{Str} M^n \). \quad (2.12)

In \(3\), we have announced that the Jacobian \(\exp (\mathfrak{3})\) and the related integration measure \(d\tilde{\Gamma}\) corresponding to a finite change of variables \(\Delta \Gamma^p\) with the choice of field-dependent parameters \(\lambda_a = s_a \Lambda\) for \(\Lambda = \Lambda (\phi, \pi, \lambda)\), inspired by the infinitesimal field-dependent BRST-antiBRST transformations of \(3\), should take the form
\[
\exp (\mathfrak{3}) = \exp \left[ - (\Delta^a S) \lambda_a - \frac{1}{4} (\Delta^a S) \frac{s^a}{s^2} \lambda^2 \right] \exp \left( \text{ln} (1 + f)^{-2} \right) , \quad \text{with} \quad f = - \frac{1}{2} \Lambda \frac{s^2}{s^2} , \quad (2.13)
\]
\[
d\tilde{\Gamma} = d\Gamma \exp \left( \frac{i}{\hbar} (-ih\mathfrak{3}) \right) = d\Gamma \exp \left\{ \frac{i}{\hbar} \left[ -ih (\Delta^a S) \lambda_a + \frac{ih}{4} (\Delta^a S) \frac{s^a}{s^2} \lambda^2 + \frac{i}{2} (1 - \frac{1}{2} \Lambda \frac{s^2}{s^2}) \right] \right\} , \quad (2.14)
\]
where \(\Lambda (\phi, \pi, \lambda)\) is a certain Bosonic potential with a vanishing ghost number. Therefore, in order to prove the above statement \(2.1, 2.1\), let us examine the general case of a finite BRST-antiBRST transformation \(2.1\) parameterized by \(\lambda_a (\Gamma)\) and consider the even matrix \(M\) in \(2.12\) with the elements \(M^p q, \varepsilon (M^p q) = \varepsilon_p + \varepsilon_q\),
\[
M^p q = \Delta \Gamma^p \frac{\delta}{\delta \Gamma^q} = U^p q + V^p q + W^p q \, , \quad \text{with} \quad V^p q = (V_1^p q + (V_2^p q) \, , \quad (2.15)
\]
for \(U^p q = X^{pa} \lambda_a q \), \(V_1^p q = \lambda_a X^{pa} q (-1)^{\varepsilon + 1} \), \(V_2^p q = \lambda_a Y^p q \lambda^a q (-1)^{\varepsilon + 1} \), \(W^p q = - \frac{1}{2} \lambda^2 Y^p q \). \quad (2.16)
taking account of the notation \( \delta \)

\[
X^{pa} \equiv \Gamma^{\lambda}_{\dot{\lambda}} \text{ and } Y^p \equiv (1/2) X^{qa}_{\dot{q}} X^{qb}_{\dot{b}} \varepsilon_{ba} = - (1/2) \Gamma^{\lambda}_{\dot{\lambda}} = -1/2. 
\]  

(2.17)

First of all, let us establish a useful relation between the matrices \( V_1 \) and \( W \) in (2.15). To do so, we use the generating equations (2.7) and represent the condition of invariance of the integrand \( I^F_1 \) in (2.6) under the BRST-antiBRST transformations \( \delta \Gamma^p = \Gamma^{\lambda}_{\dot{\lambda}} \mu_a = X^{pa} \mu_a \) given by (2.4) in the form

\[
\mathcal{S}_{F,p} X^{pa} = i h X^{pa}_p, \text{ where } X^{pa}_p = -\Delta^a S. 
\]

(2.18)

Let us now write identically:

\[
\text{Str} (V_1) + \text{Str} (W) - \frac{1}{2} \text{Str} (V_1^2) = \left[ (V_1)^p_p + W^p_p - \frac{1}{2} (V_1)^p_q (V_1)_q^p \right] (-1)^{\varepsilon_p} 
\]

\[
= X^{pa}_p \lambda_a - \frac{1}{2} (-1)^{\varepsilon_p} \left( Y^{p}_p - \frac{1}{2} X^{pa}_p X^{qb}_{\dot{b}} \varepsilon_{ba} \right) \lambda^2. 
\]

Considering

\[
Y^{p}_p - \frac{1}{2} X^{pa}_q X^{qb}_{\dot{b}} \varepsilon_{ba} = \frac{1}{2} \varepsilon_{ba} \left( X^{pa}_q X^{qb}_{\dot{b}} (-1)^{\varepsilon_p} + X^{pa}_q X^{qb}_{\dot{b}} \right) - \frac{1}{2} \varepsilon_{ba} X^{pa}_q X^{qb}_{\dot{b}} 
\]

\[
= \frac{1}{2} \varepsilon_{ba} \left( X^{pa}_q X^{qb}_{\dot{b}} (-1)^{\varepsilon_p} + X^{pa}_q X^{qb}_{\dot{b}} - X^{pa}_q X^{qb}_{\dot{b}} \right) \frac{1}{2} \varepsilon_{ba} X^{pa}_q X^{qb}_{\dot{b}} (-1)^{\varepsilon_p}, 
\]

we arrive at

\[
\text{Str} (V_1) + \text{Str} (W) - \frac{1}{2} \text{Str} (V_1^2) = X^{pa}_p \lambda_a + \frac{1}{4} \varepsilon_{ab} X^{pa}_{\dot{a}q} X^{qb} \lambda^2, 
\]

(2.19)

where (2.18) implies \( (G^{\lambda}_{\dot{\lambda}} = s^a G) \)

\[
X^{pa} = -\Delta^a S, \quad X^{pa}_{\dot{q}} X^{qb}_{\dot{b}} = - (\Delta^a S)_{\dot{p}} X^{qb}_{\dot{b}} = - s^b (\Delta^a S), \text{ with } G_{\dot{p}} X^{pa} = G_{\dot{p}} (s^a \Gamma^p) = s^a G. 
\]

(2.20)

Therefore,

\[
\text{Str} (V_1) + \text{Str} (W) - \frac{1}{2} \text{Str} (V_1^2) = - (\Delta^a S) \lambda_a - \frac{1}{4} (s_a \Delta^a S) \lambda^2. 
\]

(2.21)

Taking account of the relation between the matrices \( V_1 \) and \( W \) in (2.15), established for arbitrary \( \lambda_a (\Gamma) \), we now proceed to the case of field-dependent parameters \( \lambda_a = \Lambda^{\lambda}_{\dot{\lambda}} a \) in (2.4), determined by a Bosonic potential \( \Lambda (\phi, \pi, \lambda) \), which implies \( \lambda_a = \Lambda \Lambda^{\lambda}_{\dot{\lambda}} a \), in view of (2.3). To do so, using the property \( \text{Str} (AB) = \text{Str} (BA) \) of arbitrary even matrices \( A, B \) and the fact that the occurrence of \( W \sim \lambda^2 \) in \( \text{Str} (M^n) \) more than once yields zero, \( \lambda^4 = 0 \), we have

\[
\text{Str} (M^n) = \text{Str} (U + V + W)^n = \sum_{k=0}^{n} C^n_k \text{Str} \left[ (U + V)^{n-k} W^k \right], \quad C^n_k = \frac{n!}{k! (n-k)!}. 
\]

(2.22)

Furthermore,

\[
\text{Str} (U + V + W)^n = \text{Str} (U + V)^n + n \text{Str} \left[ (U + V)^{n-1} W \right] = \text{Str} (U + V)^n + n \text{Str} (U^{n-1} W), 
\]

(2.23)

since any occurrence of \( W \sim \lambda^2 \) and \( V \sim \lambda_a \) simultaneously entering \( \text{Str} (M^n) \) yields zero, owing to \( \lambda_a \lambda^2 = 0 \), as a consequence of which \( W \) can only be coupled with \( U^{n-1} \). Having established (2.23), let us examine \( \text{Str} (U^{n-1} W) \), namely,

\[
\text{Str} (U^{n-1} W) = \begin{cases} 
\text{Str} (W), & n = 1, \\
0, & n > 1.
\end{cases} 
\]

(2.24)
Indeed, due to the contraction property $U^2 = f \cdot U \implies U^l = f^{l-1} \cdot U$, where $f$ is a Bosonic parameter (for details, see (2.36) below), we have

$$\text{Str} \left( U^{n-1}W \right) = f^{n-2}\text{Str} \left( UW \right) \quad , \quad n > 1 , \quad \text{(2.26)}$$

$$\text{Str} \left( UW \right) = \text{Str} \left( WU \right) = (WU)^{p}(-1)^{\varepsilon_p} = W_{q}^{p}U_{p}^{q}(-1)^{\varepsilon_p} = -\frac{1}{2}\lambda^{2} \left( V_{q}^{a}X_{q}^{a} \right) \lambda_{a,p}(-1)^{\varepsilon_p} = 0 , \quad \text{(2.27)}$$

since, taking account of the restricted dependence of $\lambda_{a} \left( \phi , \pi , \lambda \right)$ on $\Gamma^{p}$, the nilpotency of the operators $\vec{\lambda}^{a}$, being the restriction of $\vec{X}^{a}$ to the subspace $\left( \phi^{A} , \pi^{A} , \lambda^{A} \right)$, and using the definitions (2.17), we have

$$\left( V_{q}^{a}X_{q}^{a} \right) \lambda_{a,p}(-1)^{\varepsilon_p} = -\frac{1}{2} \left( \Gamma^{p}\lambda_{s}^{x} \lambda_{a}^{p} \right) \lambda_{a,p}(-1)^{\varepsilon_p} = -\frac{1}{2} \left( \Gamma^{p}\lambda_{s}^{x} \lambda_{a}^{p} \right) \lambda_{a,p}(-1)^{\varepsilon_p} = 0 ,$$

which implies

$$\text{Str} \left( M^n \right) = \text{Str} \left( U + V \right)^n + n \text{Str} \left( U^{n-1}W \right) = \left\{ \begin{array}{ll}
\text{Str} \left( U + V \right) + \text{Str} \left( W \right) , & \quad n = 1 , \\
\text{Str} \left( U + V \right)^n , & \quad n > 1 ,
\end{array} \right. \quad \text{(2.28)}$$

so that $W$ drops out of $\text{Str} \left( M^n \right)$, $n > 1$, and enters the Jacobian only as $\text{Str} \left( W \right)$. Next, as we examine the contribution $\text{Str} \left( U + V \right)^n$ in (2.28), we notice that an occurrence of $V \sim \lambda_a$ more then twice yields zero, $\lambda_a \lambda_0 \lambda_c \equiv 0$. Direct calculations for $n = 2, 3$ lead to

$$\text{Str} \left( U + V \right)^n = \sum_{k=0}^{n} C_{n}^{k}\text{Str} \left( U^{n-k}V^{k} \right) = \text{Str} \left( U^n + nU^{n-1}V + C_{n}^{2}P^{n-2}V^2 \right) . \quad \text{(2.29)}$$

Using considerations identical with those presented in Appendix B.2 of [H], we can prove by induction, taking account of the fact $V \sim \lambda_a$ and the contraction property $U^2 = f \cdot U$, established in (2.36), that for any $n \geq 4$ we have

$$\text{Str} \left( U + V \right)^n = \text{Str} \left( U^n + nU^{n-1}V + nU^{n-2}V^2 + K_nU^{n-3}UVU \right) , \quad \text{(2.30)}$$

where the coefficients $K_n$ are given by

$$K_n = C_{n}^{2} - n , \quad C_{n}^{2} = n \left( n - 1 \right) / 2 \implies K_n = n \left( n - 3 \right) / 2 , \quad \text{(2.31)}$$

which implies

$$\frac{C_{n}^{2}}{n} - \frac{K_n}{n} = 1 , \quad \frac{C_{n}^{2}}{n} - \frac{K_{n+1}}{n+1} = \frac{1}{2} . \quad \text{(2.32)}$$

According to the previous considerations,

$$\text{Str} \left( M^n \right) = \sum_{k=0}^{1} C_{n}^{k}\text{Str} \left( U^{n-k}V^{k} \right) + D_n , \quad n \geq 1 , \quad \text{(2.33)}$$

for $D_n = \left\{ \begin{array}{ll}
\text{Str} \left( W \right) , & \quad n = 1 , \\
C_{n}^{3}\text{Str} \left( U^{n-2}V^2 \right) , & \quad n = 2 , 3 , \\
\left( C_{n}^{2} - K_n \right) \text{Str} \left( U^{n-2}V^2 \right) + K_n\text{Str} \left( U^{n-3}UVU \right) , & \quad n > 3 ,
\end{array} \right. \quad \text{(2.34)}$

we have

$$\text{Str} \left( M^n \right) = \left\{ \begin{array}{ll}
\text{Str} \left( U \right) + \text{Str} \left( V \right) + \text{Str} \left( W \right) , & \quad n = 1 , \\
\text{Str} \left( U^n \right) + C_{n}^{1}\text{Str} \left( U^{n-1}V \right) + C_{n}^{2}\text{Str} \left( U^{n-2}V^2 \right) , & \quad n = 2 , 3 , \\
\text{Str} \left( U^n \right) + C_{n}^{1}\text{Str} \left( U^{n-1}V \right) + \left( C_{n}^{2} - K_n \right) \text{Str} \left( U^{n-2}V^2 \right) + K_n\text{Str} \left( U^{n-3}UVU \right) , & \quad n > 3 .
\end{array} \right. \quad \text{(2.35)}$$
First of all, the calculation of the Jacobian is based on the previously established relation (2.22) between the matrices $V_1$, $W$, and therefore we will take account of the related combination

$$\text{Str} (V_1) + \text{Str} (W) - \frac{1}{2} \text{Str} (V_1^2) .$$

Besides, recalling that $\lambda_a = \Lambda \overrightarrow{U}_a$, we can deduce the additional property

$$U^2 = f \cdot U , \quad f = -\frac{1}{2} \text{Str} (U) , \tag{2.36}$$

where the quantity $f$ is given by

$$\lambda_{b,p} X^{pa} = \lambda_b \overrightarrow{U}^a = \delta_b^a f \implies f = \frac{1}{2} \lambda_a \overrightarrow{U}^a = -\frac{1}{2} \Lambda \overrightarrow{U}^2 . \tag{2.37}$$

Indeed,

$$(U^2)^p_q = (U)^p_q (U)^r_s \lambda_{b,q} = f \cdot \delta_b^a X^{pa} \lambda_{b,q} = f \cdot (U)^p_q ,$$

$$\lambda_{a,q} X^{qb} = \lambda_a \overrightarrow{U}^b = \Lambda \overrightarrow{U} \overrightarrow{U}^b = \delta_a^b f , \quad f = -F_{aA} \lambda^A - (1/2) \varepsilon_{ab} \pi Aa F_{ab} \pi Bb ,$$

$$f = \frac{1}{2} \lambda_{a,p} X^{pa} = -\frac{1}{2} U^2_p (-1)^{\varepsilon_p} = -\frac{1}{2} \text{Str} (U) . \tag{2.38}$$

As a consequence,

$$\text{Str} (UV) = \text{Str} (VU) = (1 + f) \text{Str} (V_2) , \tag{2.39}$$

$$\text{Str} (UV^2) = \text{Str} (VUV) = (1 + f) \text{Str} [(V_1 + V_2) V_2] , \tag{2.40}$$

$$\text{Str} (UVUV) = \text{Str} (VUUV) = (1 + f)^2 \text{Str} (V_2^2) . \tag{2.41}$$

Indeed, due to the relations (2.24), (2.27) and their consequences ($G \overrightarrow{U}^a \equiv U^a G$)

$$U^a U^b \varepsilon_{ab} = -\varepsilon_{ab} Y^2 , \quad U^a \lambda^b = -\varepsilon_{ab} f , \tag{2.42}$$

with account taken of the notation

$$\Gamma^a_E = (\phi^A, \pi^A_a, \pi^A) , \tag{2.43}$$

we have

$$\text{Str} VU = \text{Str} (U) (1 + f), \quad \text{Str} V^2 U = \text{Str} (V_2) ,$$

$$\text{Str} VUV = \text{Str} (VUV) = (1 + f)^2 \text{Str} (V_2^2) .$$

which proves (2.39). In a similar way, using (2.42), (2.43) and the property of nilpotency $\lambda_a \lambda_b \lambda_c = 0$, it is straightforward to verify the remaining properties (2.40), (2.41) of the matrices $U$, $V$. As a consequence of (2.36) and (2.39)–(2.41), we have

$$\text{Str} (U^n) = f^{n-1} \text{Str} (U) = -2f^n , \quad n \geq 1 ,$$

$$\text{Str} (U^{n-1} V) = \begin{cases} \text{Str} (V) = \text{Str} (V_1) + \text{Str} (V_2) , & n = 1 , \\
\text{Str} (V) = \text{Str} (V_1) + \text{Str} (V_2) , & n > 1 , \end{cases}$$

$$\text{Str} (U^{n-2} V^2) = \begin{cases} \text{Str} (V) = \text{Str} (V_1) + 2 \text{Str} (V_2) + \text{Str} (V_2^2) , & n = 2 , \\
\text{Str} (V) = \text{Str} (V_1) + \text{Str} (V_2) , & n > 2 , \end{cases}$$

$$\text{Str} (U^{n-3} VUV) = \text{Str} (UVUV) = (1 + f)^2 \text{Str} (V_2^2) , \quad n > 3 .$$
We further notice that \( \text{Str} (V_1 V_2) \neq 0 \). Indeed, due to the definitions (2.17) and the nilpotency property \( U^a U^b U^c = 0 \), we have

\[
(V_1 V_2)_p^p (-1)^{r_p} = \lambda_a X_{ap}^a Y_q^p (\lambda^2)_p = \frac{1}{2} \lambda_a \left( X_{ap}^a X_{qb}^b \right) X_{\varepsilon db} X_{db} (\lambda^2)_p \\
= \frac{1}{2} \lambda_a \left[ \left( X_{ap}^a X_{qb}^b \right)_r - \left( X_{ap}^a X_{qb}^b \right)_r (1)^{r_\varepsilon (\varepsilon + 1)} \right] X_{\varepsilon db} X_{\varepsilon db} (\lambda^2)_p \\
= \frac{1}{2} \lambda_a \left[ (U^a U^b U^c)_p - X_{ap}^a X_{qb}^b X_{\varepsilon db} (1)^{r_\varepsilon (\varepsilon + 1)} \right] X_{\varepsilon db} (\lambda^2)_p \\
= -\frac{1}{2} \varepsilon_{ba} X_{ap}^a X_{qb}^b X_{\varepsilon db} X_{db} \lambda_a (\lambda^2)_p .
\]  
(2.46)

Besides,

\[
\text{Str} (V_2^2) = \text{Str}^2 (V_2) \neq 0 .
\]  
(2.47)

Indeed,

\[
(V_2)_p^p (-1)^{r_p} = \lambda_a Y_p^a \lambda_p ,
\]  
(2.48)

\[
(Q_2)_p^p (Q_2)_p^p (-1)^{r_p} = \left( \lambda_a Y_p^a \lambda_p \right) \left( \lambda_a Y^a \lambda_q \right) .
\]  
(2.49)

Therefore, \( \mathbb{S} \) in the expression (2.22) for the Jacobian \( \exp (\mathbb{S}) \) has the general structure

\[
\mathbb{S} = A (f, V_1, W) + B (f | V_2) + C (f | V_1 V_2) ,
\]  
(2.50)

for \( B (f | V_2) = b_1 (f) \text{ Str} (V_2) + b_2 (f) \text{ Str} (V_2^2) = [b_1 (f) + b_2 (f) \text{ Str} (V_2)] \text{ Str} (V_2) \), and \( C (f | V_1 V_2) = c (f) \text{ Str} (V_1 V_2) \).

Let us examine \( A (f, V_1, W) \). Namely, in view of (2.22) and \( \text{Str} (U^n) = -2^n \), we have

\[
A (f, V_1, W) = \text{Str} (V_1) + \text{Str} (W) - \frac{1}{2} \text{Str} (V_1^2) + 2 \sum_{n=1}^{\infty} \frac{(-1)^n}{n} f^n \\
= - (\Delta^a S) \lambda_a - \frac{1}{4} (\Delta^a S) \frac{\varepsilon a}{\varepsilon a} \lambda^2 - 2 \ln (1 + f) .
\]  
(2.51)

Let us examine the explicit structure of the series related to \( b_1 (f) \): the quantity \( \text{Str} (V_2) \) derives from \( \text{Str} (U^{n-1} V) \) for \( n \geq 1 \) in (2.44), and is coupled with the combinatorial coefficient \( C^1_n \). The part of \( \mathbb{S} \) containing \( \text{Str} (V_2) \) is given by

\[
b_1 (f) \text{ Str} (V_2) = C^1_n \text{ Str} (V_2) - \sum_{n=2}^{\infty} \frac{(-1)^n}{n} C^1_n f^{n-2} (1 + f) \text{ Str} (V_2) ,
\]  
(2.52)

whence

\[
b_1 (f) = 1 - (1 + f) \sum_{n=0}^{\infty} (-1)^n f^n .
\]  
(2.53)

Let us examine the explicit structure of the series related to \( b_2 (f) \): the quantity \( \text{Str}^2 (V_2) \) derives from \( \text{Str} (U^{n-2} V^2) \) for \( n \geq 2 \) in (2.44), coupled with the combinatorial coefficients \( C^2_n \) for \( n = 2, 3 \) and \( (C^2_n - K_n) \) for \( n > 3 \), and also derives from \( \text{Str} (U^{n-3} V U V) \) for \( n > 3 \) in (2.44), coupled with the combinatorial coefficients \( K_n \). The part of \( \mathbb{S} \) containing \( \text{Str}^2 (V_2) \) reads

\[
b_2 (f) \text{ Str}^2 (V_2) = \frac{(-1)^2}{2} C^2_n \text{ Str}^2 (V_2) - \frac{(-1)^3}{3} C^2 (1 + f) \text{ Str}^2 (V_2) \\
- \sum_{n=4}^{\infty} \frac{(-1)^n}{n} (C^2_n - K_n) f^{n-3} (1 + f) \text{ Str}^2 (V_2) - \sum_{n=4}^{\infty} \frac{(-1)^n}{n} K_n f^{n-4} (1 + f)^2 \text{ Str}^2 (V_2) ,
\]  
(2.54)
whence
\[ b_2 (f) = - \frac{1}{2} + (1 + f) - \sum_{n=4}^{\infty} \frac{(-1)^n}{n} \left[ \left( C_n^2 - K_n \right) f^{n-3} (1 + f) + K_n f^{n-4} (1 + f)^2 \right]. \]  
(2.55)

Let us examine the explicit structure of the series related to \( c (f) \): the quantity \( \text{Str} (V_1 V_2) \) derives from \( \text{Str} (U^{-2} V^2) \) for \( n \geq 2 \) in (2.45), and is coupled with the combinatorial coefficients \( C_n^2 \), for \( n = 2, 3 \), and \( C_n^2 - K_n \), for \( n > 3 \). The part of \( \Im \) containing \( \text{Str} (V_1 V_2) \) is given by
\[ c (f) \text{Str} (V_1 V_2) = - \frac{(-1)^2}{2} C_2^2 \text{Str} (2V_1 V_2) = \frac{(-1)^3}{3} C_3^2 (1 + f) \text{Str} (V_1 V_2) \]
\[ - \sum_{n=4}^{\infty} \frac{(-1)^n}{n} \left( C_n^2 - K_n \right) f^{n-3} (1 + f) \text{Str} (V_1 V_2), \]  
whence
\[ c (f) = -1 + (1 + f) - \sum_{n=4}^{\infty} \frac{(-1)^n}{n} \left( C_n^2 - K_n \right) f^{n-3} (1 + f) = f - (1 + f) \sum_{n=4}^{\infty} \frac{(-1)^n}{n} \left( \frac{C_n^2}{n} - \frac{K_n}{n} \right) f^{n-3}. \]  
(2.57)

By virtue of (2.32), one can show [1] that the series \( b_1 (f), b_2 (f), c (f) \), given by (2.53), (2.55), (2.56), vanish identically:
\[ b_1 (f) = b_2 (f) = c (f) \equiv 0. \]  
(2.58)

From the vanishing of all the coefficients \( b_1 (f), b_2 (f), c (f) \), we conclude that
\[ B (f|V_2) = b_1 (f) \text{Str} (V_2) + b_2 (f) \text{Str} (V_2^2) \equiv 0 \quad \text{and} \quad C (f|V_1 V_2) = c (f) \text{Str} (V_1 V_2) \equiv 0, \]  
(2.59)

and therefore the Jacobian \( \exp (\Im) \) is finally given by
\[ \Im = A (f, V_1, W) + B (f|V_2) + C (f|V_1 V_2) = A (f, V_1, W) = - (\Delta^a S) \lambda_a - \frac{1}{4} (\Delta^a S) \frac{\tau_3}{8} a \lambda^2 - 2 \ln (1 + f), \]
for \( f = - (1/2) \Lambda \dot{U}^2 = - (1/2) \Lambda \dot{\pi}^2, \quad \Lambda = \Lambda (\phi, \pi, \lambda). \)

which is identical with (2.13) and therefore proves (2.14).

3 Ward Identities and Gauge Dependence Problem

In this section, we touch upon the consequences (Subsection 3.1) implied by a solution of the compensation equation [3] for an unknown functional \( \Lambda (\phi, \pi, \lambda) \) which determines a field-dependent BRST-antiBRST transformation that amounts to a precise change of the gauge-fixing functional for an arbitrary gauge theory. The modified Ward identities and gauge dependence for Yang–Mills theories and, more generally, for gauge theories with a closed algebra of rank 1 are examined in Subsection 3.2.

3.1 General Gauge Theory

Let us bring to mind the results of [3], based on the representation (2.14) for the Jacobian established in Section 2 for the change of variables (2.1), given by a field-dependent BRST-antiBRST transformation, \( \Gamma \rightarrow \Gamma = \Gamma + \Delta \chi \Gamma \). First of all, making in \( Z_F \) a change of variables \( \Gamma \rightarrow \Gamma, \) with \( \lambda_a = s_a \Lambda \) and \( \Lambda = \Lambda (\phi, \pi, \lambda), \) while taking account of the Jacobian (2.14), the change of the action \( S_F \) according to (2.14), and also using the invariance condition (2.18),
related to the generating equations \((2.27)\), and its differential consequence resulting from applying the operator \(\tilde{s}_a\), with allowance made for the BRST-antiBRST invariance of the term \(F\), and therefore the corresponding field-dependent parameters have the form
\[
H_\phi(\Lambda, \pi, \lambda) = \int d\Gamma \exp \left\{ \frac{i}{\hbar} \left[ S_F + (S_F \tilde{s}_a + i\hbar \Delta^a S) \lambda_a + \frac{1}{4} (S_F \tilde{s}_a \tilde{s}_a + i\hbar \Delta^a S \tilde{s}_a) \lambda^2 + i\hbar \ln \left( 1 - \frac{1}{2} \Lambda \tilde{s}_a \right) \right] \right\} .
\]

The solution of this equation for an unknown functional \(\Lambda = \Lambda(\phi, \pi, \lambda)\):
\[
i\hbar \ln \left( 1 - \frac{1}{2} \Lambda \tilde{U}^2 \right) = -\frac{1}{2} \Delta F \tilde{U}^2 ,
\]
where allowance has been made for the equality \(\tilde{s}_a = \tilde{U}^a\), which takes place for the operators \(\tilde{s}_a\) restricted to the variables \((\phi, \pi_a, \lambda)\), or, equivalently,
\[
\frac{1}{2} \Lambda \tilde{U}^2 = 1 - \exp \left( \frac{i}{\hbar} \Delta F \tilde{U}^2 \right) .
\]
The solution of this equation for an unknown functional \(\Lambda(\phi, \pi, \lambda)\), which determines \(\lambda_a = \Lambda \tilde{U}_a\), with accuracy up to BRST-antiBRST exact terms \((s^a\) being restricted to \(\phi, \pi_a, \lambda)\), is given by
\[
\Lambda(\Gamma|\Delta F) = \frac{i}{2\hbar} g(y) \Delta F , \quad \text{for } g(y) = \left[ 1 - \exp(y) \right] / y \quad \text{and} \quad y \equiv \frac{i}{4\hbar} \Delta F \tilde{U}^2 ,
\]
and therefore the corresponding field-dependent parameters have the form
\[
\lambda_a(\Gamma|\Delta F) = \frac{i}{2\hbar} g(y) \left( \Delta F \tilde{U}_a \right) ,
\]
whose approximation linear in \(\Delta F\) is given by
\[
\lambda_a(\Gamma|\Delta F) = \frac{i}{2\hbar} \left( \Delta F \tilde{U}_a \right) + o(\Delta F) .
\]
Consequently, for any change \(\Delta F\) of the gauge condition \(F \rightarrow F + \Delta F\), we can construct a unique field-dependent BRST-antiBRST transformation with functionally-dependent parameters \((3.5)\) that allows one to preserve the form of the partition function \((3.1)\) for the same gauge theory. On the other hand, if we consider the compensation equation \((3.2)\) for an unknown gauge variation \(\Delta F\) with a given \(\Lambda(\phi, \pi, \lambda)\), we can present it in the form
\[
4i\hbar \ln \left( 1 - \frac{1}{2} \Lambda \tilde{U}^2 \right) = -\Delta F \tilde{U}^2 \iff 4i\hbar \left[ \sum_{n=1}^\infty \frac{(-1)^n}{2^n n} \left( \Lambda \tilde{U}^2 \right)^{n-1} \Lambda \right] \tilde{U}^2 = \Delta F \tilde{U}^2 ,
\]
whose solution, with accuracy up to \(\tilde{U}^a\)-exact terms, is given by
\[
\Delta F(\Gamma|\Lambda) = 4i\hbar \left[ \sum_{n=1}^\infty \frac{(-1)^n}{2^n n} \left( \Lambda \tilde{U}^2 \right)^{n-1} \Lambda \right] = 4i\hbar \left[ \sum_{n=1}^\infty \frac{(-1)^n}{2^n n} \left( \lambda^a \tilde{U}_a \right)^{n-1} \Lambda \right] .
\]
Consequently, for any change of variables in the partition function \(Z_F\) given by finite field-dependent BRST-antiBRST transformations with the parameters \(\lambda^a = \Lambda \tilde{U}_a\), we obtain the same partition function \(Z_{F+\Delta F}\), however, evaluated in a gauge determined by the Bosonic functional \(F + \Delta F\), in accordance with \((3.8)\).\footnote{In [3], we make a transformation parameterized by \(\Lambda\) to perform a transition from \(Z_{F+\Delta F}\) to \(Z_F\), which accounts for the opposite sign at \(\Delta F\) in \((3.2)\) and related formulas.}
Making in (2.7) a field-dependent BRST-antiBRST transformation (2.1) and using the relations (2.11) and (3.4), we obtain a modified Ward (Slavnov–Taylor) identity:

\[
\left\{1 + \frac{i}{\hbar} J_A \phi^A \left[ \overline{U^a_2} \lambda_a (\Lambda) + \frac{1}{2} (\overline{U^2} \lambda^2 (\Lambda)) \right] - \frac{1}{4} \left( \frac{i}{\hbar} \right)^2 J_A \phi^A \left[ \overline{U_{AB}} \lambda_{AB} \lambda^2 (\Lambda) \right] \right\}_{F,J} = 1 . \quad (3.9)
\]

Here, the symbol \( (O)_{F,J} \) for a quantity \( O = O(\Gamma) \) stands for the source-dependent average expectation value corresponding to a gauge-fixing \( F(\phi) \)

\[
\langle O \rangle_{F,J} = Z_F^{-1}(J) \int d\Gamma \ O(\Gamma) \exp \left\{ \frac{i}{\hbar} \left[ S_F(\Gamma) + J_A \phi^A \right] \right\} , \quad \text{with} \quad (1)_{F,J} = 1 . \quad (3.10)
\]

Due to the presence of \( \Lambda(\Gamma) \), which implies \( \lambda_a(\Lambda) \), the modified Ward identity depends on a choice of the gauge Boson \( F(\phi) \) for non-vanishing \( J_A \), according to (3.3), (3.5), and therefore the corresponding Ward identities for Green’s functions, obtained by differentiating (3.9) with respect to the sources, contain the functionals \( \lambda_a(\Lambda) \) and their derivatives as weight functionals. By virtue of (3.9) for constant \( \lambda_a \), the usual Ward identities follow from the first order in \( \lambda_a \) and a new Ward identity [3] from the second order in \( \lambda_a \):

\[
J_A \left( \phi^A \overline{U^a_2} \right)_{F,J} = 0 , \quad J_A \left( \phi^A \overline{U^2} - \overline{U^a_2} (i/\hbar) J_B(\phi^B \overline{U^a_2}) \right)_{F,J} = 0 . \quad (3.11)
\]

In conclusion, taking account of (3.5), we find that (3.9) implies a relation which describes the gauge dependence of \( Z_F(J) \) for a finite change of the gauge, \( F \to F + \Delta F \):

\[
Z_{F+\Delta F}(J) = Z_F(J) \left\{ 1 + \left( \frac{i}{\hbar} \right)^2 J_A \phi^A \left[ \overline{U^a_2} \lambda_a (\Gamma) + \frac{1}{4} (\overline{U^2} \lambda^2 (\Gamma)) \right] \right\}.
\]

\[
S_{F}(\phi) = S_0(A) - (1/2) F^{F_{AB}} = S_0(A) + F_{AB} X^A \varepsilon_{ab} X^B , \quad (3.14)
\]

### 3.2 Yang–Mills Theory

Let us apply the modified Ward identities (3.9) and the representation (3.12) for gauge dependence to the generating functional \( Z_F(J) \) of irreducible gauge theories of rank 1 with a closed gauge algebra of the generators of gauge transformations, \( R_a(A) \), \( \varepsilon(R_a) = \varepsilon_i + \varepsilon_a \), including the case of Yang–Mills theories. The corresponding configuration space \( \phi^A = (A^i, B^a, C^{\alpha a}) \) contains the classical fields, the Nakanishi–Lautrup fields, and the ghost-antighost fields [4].

First of all, in accordance with [1], the generating functional of Green’s functions \( Z_F(J) \) and the corresponding quantum action \( S_F(\phi) \) are readily obtained from (2.6), with \( S \) being a solution of (2.7), by integrating over \( (\phi^a, \bar{\phi}, \pi^a, \lambda) \),

\[
Z_F(J) = \int d\phi \ \exp \left\{ \frac{i}{\hbar} \left[ S_F(\phi) + J_A \phi^A \right] \right\} , \quad (3.13)
\]

\[
S_F(\phi) = S_0(A) - (1/2) F^{F_{AB}} = S_0(A) + F_{AB} X^A \varepsilon_{ab} X^B , \quad (3.14)
\]

where

\[
X^A = \left( X_1^A, X_2^A, X_3^A \right) , \quad Y^A = \left( Y_1^A, Y_2^A, \varepsilon_{ab} X^A \right) , \quad (3.15)
\]

\[
X_1^A = R_a^i C_{i a} , \quad X_2^a = -\frac{1}{2} F_{\beta}^\alpha C_{\beta}^\gamma \varepsilon_{ab} \left( 2F_{\gamma j}^\beta R_j^\rho + F_{\gamma a}^\rho \right) C_{\rho b} C_{i a} \varepsilon_{ab} , \quad X_3^{\alpha a} = -\varepsilon_{ab} B^a - \frac{1}{2} (-1)^{\varepsilon_b} F_{\beta}^\alpha C_{\beta}^a C_{i b} \varepsilon_{ab} , \quad (3.16)
\]

\[
Y_1^A = R_a^i B^a + \frac{1}{2} (-1)^{\varepsilon_a} R_{\alpha j}^i R_{\beta j}^a C_{\beta}^a \varepsilon_{ab} , \quad Y_2^a = 0 , \quad Y_3^{\alpha a} = -2X_3^{\alpha a} ,
\]
and there hold the subsidiary conditions \( X^A_a = 0 \). The nilpotent, \( \tilde{s}^a \), \( \tilde{s}^b \), \( \tilde{s}^c \) determine the finite BRST-antiBRST transformations:

\[
\Delta \lambda \phi^A = X^A_a \lambda_a - \frac{1}{2} Y^A \lambda^2 = \phi^A \left( \frac{1}{6} \lambda^2 \tilde{s}^a \lambda_a + \frac{1}{4} \tilde{s}^a \lambda^2 \right),
\]

so that the corresponding Jacobian Sdet \( \left| (\phi^A + \Delta \lambda \phi^A) \frac{\delta}{\delta \phi^A} \right| \) of a change of variables \( \phi \to \phi (1 + \tilde{s}^a \lambda_a + \frac{1}{2} \tilde{s}^2 \lambda^2) \) takes the form \( (X^A_a = 0) \)

\[
\text{Sdet} \left| (\phi^A + \Delta \lambda \phi^A) \frac{\delta}{\delta \phi^A} \right| = \exp \left[ X^A_a \lambda_a + \frac{1}{4} \varepsilon_{ab} X^A_{AB} X^{Bb} \lambda^2 + \ln \left( 1 - \frac{1}{2} \tilde{s}^2 \lambda^2 \right)^{-2} \right], \quad \text{for } \lambda_a = \Lambda \tilde{s}^a,
\]

\[
\text{Sdet} \left| (\phi^A + \Delta \lambda \phi^A) \frac{\delta}{\delta \phi^A} \right| = \exp \left[ X^A_a \lambda_a + \frac{1}{4} \varepsilon_{ab} X^A_{AB} X^{Bb} \lambda^2 \right] = 1 , \quad \text{for } \lambda_a = \text{const} .
\]

The conditions \( X^A_a = 0 \) imply the relations \( R^i_{a_1} (A) = F^a_{\alpha \beta} (A) = 0 \) for the gauge generators and structure functions in \( R^i_{a_1 j} (A) R^j_{a_2} (A) = (\varepsilon_{a_1} \varepsilon_{a_2}) R^i_{a_2} (A) R^j (A) = -R^i (A) F^a_{\alpha \beta} (A) \). The first Jacobian \( [3.18] \), corresponding to field-dependent (and functionally-dependent) parameters \( \lambda_a (\phi) = (\Lambda \tilde{s}^a) (\phi) \), is identical with that of \([1]\).

As we apply the procedure used to obtain \([3.9]\) to gauge theories with a closed algebra of rank 1, we arrive at a modified Ward identity:

\[
\left\langle 1 + \frac{i}{\hbar} J_A \left[ X^A a \lambda_a (\Lambda) - \frac{1}{2} Y^A \lambda^2 (\Lambda) \right] - \frac{1}{4} \left( \frac{i}{\hbar} \right)^2 \varepsilon_{ab} J_A X^A a J_B X^B b \lambda^2 (\Lambda) \right\rangle_{F,J} = 1 , \quad \text{where the symbol “(\mathcal{O})_{F,J}” for a quantity } \mathcal{O} = \mathcal{O} (\phi) \text{ is determined as in } [3.10], \text{ however, for } \mathcal{Z}_F (J) \text{ in } [3.13]. \text{ The identity } [3.20] \text{ has the same interpretation as } [3.9]: \text{ for constant } \lambda_a \text{ we obtain from } [3.20] \text{ an Sp}(2)\text{-doublet of the usual Ward identities at the first order in } \lambda_a, \text{ and a derivative identity at the second order in } \lambda_a:
\]

\[
J_A \langle X^A_a \rangle_{F,J} = 0, \quad \langle J_A \left[ 2 Y^A + (i/\hbar) \varepsilon_{ab} X^A a J_B X^B b \right] \rangle_{F,J} = 0 .
\]

By virtue of the representation \([3.5]\) for \( \lambda_a (\phi) \Delta F \), applied to gauge theories in question, the Ward identity \([3.20]\) implies a relation which describes the gauge dependence of \( \mathcal{Z}_F (J) \) for a finite change of the gauge \( F \to F + \Delta F \):

\[
\mathcal{Z}_{F + \Delta F} (J) - \mathcal{Z}_F (J) = \mathcal{Z}_F (J) \left\langle \frac{i}{\hbar} J_A \left[ X^A a \lambda_a (\phi) - \Delta F \right] - \frac{1}{2} Y^A \lambda^2 (\phi) - \Delta F \right\rangle_{F,J} - (-1)^{\varepsilon_{ab}} \frac{1}{2 \hbar} J_B J_A \left( X^A a X^B b \right) \varepsilon_{ab} \lambda^2 (\phi) - \Delta F \right\rangle_{F,J} .
\]

For Yang–Mills theories, in which \( X^A_a \equiv 0 \), we obtain a new representation for the modified Ward identity \([3.20]\), with the following identification of \( X^A a \) and \( Y^A \) in \([3.16]\), according to \([1]\) \([4]\):

\[
X_1^{\mu a} = D^{\mu mn} C^{na} , \quad Y_1^{\mu} = D^{\mu mn} B^n + \frac{1}{2} f^{mn} C^{lb} D^{\mu nk} C^{cb} \varepsilon_{ba} ,
\]

\[
X_2^{\mu a} = -\frac{1}{2} f^{mn} B^l C^{ma} - \frac{1}{12} f^{mn} f^{lr} C^{lb} C^{ra} C^{nc} \varepsilon_{cb} , \quad Y_2^{\mu} = 0 ,
\]

\[
X_3^{\mu ab} = -\varepsilon^{ab} B^m - \frac{1}{2} f^{mn} C^{lb} C^{ma} , \quad Y_3^{\mu a} = f^{mn} B^l C^{ma} + \frac{1}{6} f^{mn} f^{lr} C^{lb} C^{ra} C^{nc} \varepsilon_{cb} ,
\]

where \( \phi^A = (A^{\mu a}, B^\mu, C^{ma}) \); the generators of gauge transformations \( R^{mn}_{\mu}(x; y) \), the covariant derivatives \( D_{\mu}^{mn} \), and the structure functions \( F^\gamma_{\alpha \beta} \) are given by \([1]\) \([4]\):

\[
R^{mn}_{\mu}(x; y) = D^{mn}_{\mu}(x) \delta(x - y) , \quad D_{\mu}^{mn} = \partial_{\mu}^{mn} \partial_\mu + f^{mnl} A^l_{\mu} , \quad F^\gamma_{\alpha \beta} = f^{lmn} \delta(x - z) \delta(y - z) .
\]
4 BRST-antiBRST Symmetry Breaking in Gauge Theories

In this section, we introduce the concept of BRST-antiBRST symmetry breaking in gauge theories, inspired by our research [30] within the BV quantization scheme and partially repeating the study of [26]. To this end, let us consider a Bosonic functional \( M = M(\phi, \phi^*, \bar{\phi}) \) having a vanishing ghost number and not necessarily being BRST-antiBRST invariant. This functional may be added to the action \( S_F \) in (3.11), or introduced in a multiplicative way into the partition function, \( m = \exp M \), in order to improve the properties of the path integral as has been done within the functional renormalization group approach \([26, 32, 33, 34, 35]\), by means of the average affective action for the purpose of extraction of residual Gribov copies \([10, 11]\); see also \([37, 38, 39]\). Let us impose the condition that the corresponding functional integral be well-defined and determine the generating functional of Green's functions \( Z_{M,F}(J) \) with broken BRST-antiBRST symmetry as follows:

\[
Z_{M,F}(J) = \int d\Gamma \exp \left\{ (i/\hbar) \left[ S_F(\Gamma) + M(\phi, \phi^*, \bar{\phi}) + J_A \phi^A \right] \right\}, \quad \text{with} \quad Z_{0,F}(J) \equiv Z_F(J), \tag{4.1}
\]

where we have not imposed on \( M \) any equation \( \frac{1}{2}(M, M)^a - V^a M = 0 \) or \( \frac{1}{2}(M, M)^a - V^a M = -i\hbar \Delta^a M \) analogous to the so-called soft BRST symmetry equation \([30, 31]\) for gauge theories in the BV formalism. Another requirement for \( M \) is the following inequality, in terms of the operators \( \bar{s}^a \) in (2.1), with account taken of the representation (2.9) for a finite BRST-antiBRST variation of an arbitrary functional:

\[
M \bar{s}^a \neq 0 \quad \Rightarrow \quad \Delta \lambda M = M \left( \bar{s}^a \lambda_a + \frac{1}{4} \bar{s}^a \lambda_a^2 \right) \neq 0. \tag{4.2}
\]

We shall refer to a gauge theory as having a broken BRST-antiBRST symmetry in the Sp(2)-covariant Lagrangian quantization \([4, 5]\) if the total action of the theory, \( S_{\text{tot}} = S_F + M \), determines the partition function \( Z_{M,F}(0) \) in (4.1) and the broken BRST-antiBRST symmetry condition (4.2) for \( M \) is fulfilled.

It is well known that the introduction of a BRST-antiBRST non-invariant term to the quantum action may lead to the appearance of two fundamental problems for physical quantities (such as the S-matrix) in a theory with broken BRST-antiBRST symmetry, namely, gauge dependence and unitarity failure. For this reason, let us study the suggested construction of a gauge theory with broken BRST-antiBRST symmetry as concerns the dependence of the functional \( Z_{M,F}(J) \) on a choice of the gauge condition.

Let us first obtain the Ward identities for a gauge theory with broken BRST-antiBRST symmetry. To this end, we make in (4.1) a field-dependent BRST-antiBRST transformation (2.1) and use the relations (3.3) and (2.14) to obtain a modified Ward (Slavnov–Taylor for \( Z_{M,F}(J) \)) identity:

\[
\left\langle 1 + \frac{i}{\hbar} [J_A \phi^A + M_F] \left[ \overline{U}^a \lambda_a(\Lambda) + \frac{1}{4} \overline{U}^2 \lambda^2(\Lambda) \right] - \frac{1}{4} \left( \frac{i}{\hbar} \right)^2 [J_A \phi^A + M_F] \overline{U}^a \left[ J_B \phi^B + M_F \right] \overline{U}_a \lambda^2(\Lambda) \right\rangle_{M,F,J} = 1, \tag{4.3}
\]

where account has been taken of the fact that if in some reference frame with a gauge Boson \( F(\phi) \) the broken BRST-antiBRST symmetry term has the form \( M_F = M \), then in a different reference frame determined by a gauge Boson \( (F + \Delta F)(\phi) \) it should be calculated as

\[
M_{F+\Delta F} = M_F + \Delta \lambda M F. \tag{4.4}
\]

\[3\]On the basis of this equation, one can develop a concept of soft BRST-antiBRST symmetry breaking in gauge theories, which may lead to additional Ward identities in terms of the functional \( M \); however, we leave this problem outside of the scope of the present work.
Here, the symbol \(\langle \mathcal{O} \rangle_{M,F,J} \) for a quantity \(\mathcal{O} = \mathcal{O}(\Gamma)\) stands for a source-dependent average expectation value corresponding to a gauge-fixing \(F(\phi)\) with respect to \(Z_{M,F}(J)\):

\[
\langle \mathcal{O} \rangle_{M,F,J} = Z_{M,F}(J)^{-1} \int d\Gamma \mathcal{O}(\Gamma) \exp i \left\{ \frac{i}{\hbar} \left[ S_F(\Gamma) + M_F + J_A \phi^A \right] \right\}, \quad \text{with} \quad \langle 1 \rangle_{M,F,J} = 1.
\]

Due to the presence of \(\lambda_a(\Lambda)\), and therefore \(\Lambda(\Gamma)\), the modified Ward identity depends on a choice of the gauge Boson \(F(\phi)\) for non-vanishing \(J_A\). For constant \(\lambda_a\), the identity (4.3) decomposes in powers of \(\lambda^2\) identities:

\[
\left< \left[ J_A \phi^A + M_F \right] \bar{U}^a \right>_{M,F,J} = 0, \quad \text{(4.6)}
\]

\[
\left< \left[ J_A \phi^A + M_F \right] \bar{U}^2 - \left( \frac{i}{\hbar} \right) \left[ J_A \phi^A + M_F \right] \bar{U}^a \left[ J_B \phi^B + M_F \right] \bar{U}_a \right>_{M,F,J} = 0. \quad \text{(4.7)}
\]

The identities (4.3), (4.6), (4.7) differ from the Ward identities for a gauge theory without BRST-antiBRST symmetry breaking terms.

With account taken of (3.5) and (4.3), the equality (4.3) implies a relation which describes the gauge dependence of \(Z_{M,F}(J)\) for a finite change of the gauge \(F \to F + \Delta F\):

\[
Z_{M,F,\Delta F,F+\Delta F}(J) - Z_{M,F}(J) = Z_{M,F}(J) \left< \left( \frac{i}{\hbar} J_A \phi^A \left[ \bar{U}^a \lambda_a (\Gamma) - \Delta F \right] + \frac{1}{4} \bar{U}^2 \lambda^2 (\Gamma) - \Delta F \right) \right>_{M,F,J} - (-1)^F \left( \frac{i}{2\hbar} \right)^2 J_B J_A \left( \phi^A \bar{U}^a \right) \left( \phi^B \bar{U}_a \right) \lambda^2 (\Gamma) - \Delta F\right>_{M,F,J}. \quad \text{(4.8)}
\]

From (3.12) it follows that upon the shell determined by \(J_A = 0\) a finite change of the generating functional of Green’s functions with a term of broken BRST-antiBRST symmetry does not depend on the choice of the gauge condition with respect to a finite change of the gauge, \(F \to F + \Delta F\). The same statement can be made for the physical \(S\)-matrix, due to the equivalence theorem [40].

5 Conclusion

In the present work, we have proved the correctness of the form of the Jacobian (2.14) of a change of variables in the partition function related to finite field-dependent BRST-antiBRST transformations \(2.1\) announced in [3] being polynomial in powers of an \(Sp(2)\)-doublet of Grassmann-odd with functionally-dependent parameters \(\lambda_a = \Lambda U_a\), generated by a finite even-valued functional \(\Lambda(\phi, \pi^a, \lambda)\) and nilpotent Grassmann-odd operators \(\bar{U}_a\) (2.5) as the restriction of generators \(s_a\) of BRST-antiBRST transformations to the subspace \((\phi^A, \pi^{AB}, \lambda^A)\). Based on representation (2.14) we justify derivation of the compensation equation for parameters \(\lambda_a\) and find its solution (3.3), (5.5) which means that for any finite change of gauge condition \(F \to F + \Delta F\) there exists unique \(Sp(2)\)-duplet \(\lambda_a(\Delta F)\) that gauge independence of vacuum functionals, \(Z_F = Z_{F+\Delta F}\) holds. Conversely, any change of variables in the vacuum functional \(Z_F\) generated by finite field-dependent BRST-antiBRST transformation with functionally dependent parameters \(\lambda_a = \Lambda U_a\) corresponds to finite change of the gauge condition, \(F \to F + \Delta F\), with \(\Delta F(\Gamma|\Lambda)\) determined by solution (3.8) for inverse problem of compensation equation (3.7) so that vacuum functional \(Z_F\) is gauge-independent. That property follows from the fact that both the Jacobian of the such change of variables and the changing of the gauge appear by BRST-antiBRST exact quantities. In case of functionally independent odd parameters \(\lambda_a\) finite BRST-antiBRST

4\text{as it follows from (5.3), (5.5) the finite BRST-antiBRST transformations with constant parameters form a two-parametric Lie supergroup}
transformations generate from corresponding jacobian BRST-antiBRST-not exact terms which may change not BRST-antiBRST part [not corresponding to gauge-fixed action!] of a quantum action of a gauge theory under consideration. We study this problem in our forthcoming paper [20]. Note, that for finite field-dependent BRST transformations there is not such possibility. On a base of proved representation for Jacobian (2.14) we justify derivation of modified Ward identity depending on functionals $\lambda (\Delta F)$ (3.3) and consider a gauge dependence problem for generating functional of Green’s functions $Z_F(J)$ (3.12). As usual, but for finite change of the gauge Boson $\Delta F(\Gamma|\Lambda)$ the functional $Z_F(J)$ does not depend on a choice of the gauge on its extremales determined by $J_A = 0$ that is a basis to study this problem for the effective action.

We have refined the properties of gauge theories with a closed algebra of rank 1, including the Yang–Mills type of theories, on the basis of the above results for a general gauge theory and obtained a modified Ward identity depending on the functional $\lambda_a (\Delta F)$ restricted to the field variables $(A^i, B^a, C^{\alpha a})$ and $(A^{\mu m}, B^m, C^{m a})$, respectively, in (3.20). In addition, we have described (3.22) gauge dependence for this kind of theories.

We have introduced the notion of BRST-antiBRST symmetry breaking in the of Sp(2)-covariant Lagrangian quantization, obtained the modified Ward identity (4.3) for the generating functional $Z_{M,F}(J)$ of Green’s functions with broken BRST-antiBRST symmetry and investigated gauge dependence on the basis of finite field-dependent BRST-antiBRST transformations. We have demonstrated that a non-renormalized functional $Z_{M,F}(J)$ does not depend on a choice of the gauge on the extremales given by $J_A = 0$ in (4.8), thus providing a consistent (leaving aside the unitarity problem) manner of introducing into a gauge theory of terms with broken BRST-antiBRST symmetry, which has been applied in [1] to determine in a consistent perturbative manner the Gribov–Zwanziger horizon functional (obtained non-perturbatively in [11]) by using any other gauge beyond the Landau gauge and may be compared with other non-perturbative approaches [38, 39] to the Gribov–Zwanziger horizon functional in various gauges.

We have revised (in comparison with the study of [26, 30, 31]) the problem of BRST symmetry breaking in the BV quantization scheme, obtained the modified Ward identity (A.12) for the generating functional of Green’s functions with broken BRST symmetry $Z_{M,\Lambda}(J)$, without introducing a soft BRST symmetry condition for the BRST non-invariant term $M$, and investigated gauge dependence by using the recently proposed finite field-dependent BRST–BV transformations [27]. We have established that the non-renormalized functional $Z_{M,\Psi}(J)$ does not depend on a choice of the gauge determined by a Fermionic functional $\psi(\phi)$ on the extremales given by $J_A = 0$ in (A.14), thus justifying once again, in addition to [26], the consistency of introduction into a gauge theory of terms with broken BRST symmetry in the framework of the BV quantization.

We have introduced also the concept of effective average action within Functional Renormalization Group approach in Yang–Mills theories as an application of general construction of BRST-antiBRST symmetry breaking in the of Sp(2)-covariant Lagrangian quantization by means . To this end, we introduced regulator action (B.7), determined generating functionals of Green’s functions, including effective average action, derived for them modified (depending on field-dependent odd parameters $\lambda_a$) and usual Ward identities with BRST-antiBRST symmetry breaking terms. We established that the non-renormalized functionals $Z_{F;k}(J)$, $Z_{F;k}(J, \phi_a^*, \tilde{\phi})$ do not depend on a choice of the gauge determined by the functional $F(\phi)$ on the extremales given by $J_A = 0$ and hypersurface $\phi_a^* = 0$ for latter in (B.29), thus providing gauge independent conventional S-matrix for any value of external momentum-shell parameter, $k$. We suggested the form of the regulator action in any gauge (B.33)–(B.37) within $F_\xi$-family (B.31) of gauges corresponding to the standard $R_\xi$-gauges.

Concluding, note that among the interesting problems left out of the scope of the present work, beside the evaluation of Jacobians for arbitrary finite field-dependent BRST-antiBRST transformations, one may turn to the study of the group properties of finite field-dependent BRST-antiBRST transformations.
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Appendix

A On BRST Symmetry Breaking in Gauge Theories

In this Appendix, we touch upon the study of BRST symmetry breaking in gauge theories, which is inspired by our works \[26, 30\] within the BV quantization scheme \[28\], and is based on the recently proposed BRST–BV field-dependent transformations \[27\]. To this end, let us remind that the generating functional of Green’s functions \(Z_\psi(J)\), implying the partition function \(Z = Z_\psi(0)\) is given by \[28\]

\[
Z_\psi(J) = \int d\phi d\phi^* d\lambda \exp \left\{ \frac{i}{\hbar} \left[ S(\phi, \phi^*) + \left( \phi^*_A - \psi(\phi) \frac{\delta}{\delta \phi^A} \right) \lambda^A + J_A \phi^A \right] \right\},
\]
which, in terms of a nilpotent operator \(\tilde{U}\) introduced in \[42\], can be presented in the form

\[
Z_\psi(J) = \int d\phi d\phi^* d\lambda \exp \left\{ \left( i/\hbar \right) \left[ S(\phi, \phi^*) + \left( \phi^*_A \phi^A - \psi(\phi) \right) \tilde{U} + J_A \phi^A \right] \right\} \quad \text{for} \quad \tilde{U} = \frac{\delta}{\delta \phi^A} \lambda^A,
\]
where \(\phi^A, \phi^*_A, \lambda^A, \psi(\phi)\) are the respective fields, antifields, Lagrangian multipliers, and the gauge fermion, which introduces the gauge into the path integral. The quantum action \(S\) is subject to the master equation in terms of the antibracket \((\bullet, \bullet) \equiv (\bullet, \bullet)^1\) and the nilpotent Laplacian \(\Delta \equiv \Delta^1\), given by \[2.8\] with \(a = 1\):

\[
\frac{1}{2} (S, S) = i\hbar \Delta S \iff \Delta \exp \left( \frac{i}{\hbar} S \right) = 0 \quad \text{for} \quad S|_{\phi^*=-\hbar=0} = S_0(A)
\]
For vanishing external sources \(J_A\), the integrand in \[A.1\] or \[A.2\] is invariant under the finite BRST–BV transformations with a constant Fermionic parameter \(\mu (\mu^2 = 0)\)

\[
\Delta_\mu (\phi^A, \phi^*_A, \lambda^A) = \left( \phi^A \tilde{U}, - (\phi^*_A, S), 0 \right) \mu \equiv (\phi^A, \phi^*_A, \lambda^A) \tilde{S} \mu,
\]
whereas the transformation of the vacuum functional \(Z_\psi\) with respect to \((\phi^A, \phi^*_A, \lambda^A) \rightarrow (\phi^A, \phi^*_A, \lambda^A) + \Delta_\mu (\phi^A, \phi^*_A, \lambda^A)\), related to finite field-dependent BRST-BV transformations \[27\] with an arbitrary functional \(\mu = \mu(\phi, \lambda)\) is in one-to-one correspondence with a finite change of the gauge fermion \(\psi \rightarrow \psi + \psi'\),

\[
\psi' = \frac{\hbar}{i} \left[ \sum_{n=1}^{n-1} \frac{(-1)^{n-1}}{n} \left( \mu \tilde{U} \right)^{n-1} \right] \mu,
\]
which justifies the gauge-independence of the vacuum functional, \(Z_\psi = Z_{\psi+\psi'}\). The relation \[A.5\] is given by a solution of the so-called compensation equation (implied by the condition \(Z_\psi = Z_{\psi+\psi'}\) for an unknown functional \(\psi'\),

\[5\]In this Appendix, we use the standard notation \[28\] for Lagrangian multipliers, which differs from the auxiliary fields \(\lambda^A\) of the opposite Grassmann parity in the basic part of the work.
which can be easily obtained from the Jacobian $J = \text{Sdet} \left| z^{p}(1 + \frac{\delta}{\delta z}) \right|$, for $z^{p} = (\phi^{A}, \phi^{*A}, \lambda^{A})$, corresponding to the change of variables \[27\], namely,

$$\ln \left(1 + \mu \hat{U} \right) = \frac{i}{\hbar} \psi \hat{U} \quad \text{with} \quad J = (1 + \mu \hat{U})^{-1}. \quad (A.6)$$

Considering $\mu(\phi, \lambda)$ as an unknown parameter of the field-dependent BRST–BV transformation \[A.4\], which realizes $Z_{\psi} = Z_{\psi + \psi'}$ for a given finite change of the gauge $\psi'$, a solution of \[A.6\] with accuracy up to $\hat{U}$-exact terms reads

$$\mu(\phi, \lambda|\psi') = -\frac{i}{\hbar} \bar{\psi}(y)\psi' , \quad \text{for} \quad y = \frac{i}{\hbar} \psi' \hat{U} , \quad (A.7)$$

with a function $g(y)$ given by \[34\].

Let us now consider a BRST non-invariant Bosonic functional $M_{\psi} = M_{\psi}(\phi, \phi^{*})$, $M_{\psi} = 0$, with $gh(M_{\psi}) = 0$, which may, or may not, be subject to the so-called condition of soft BRST symmetry breaking \[30, 31\]

$$(M_{\psi}, M_{\psi}) = 0, \quad [(M_{\psi}, M_{\psi}) = -2i\hbar \Delta M_{\psi}], \quad (A.8)$$

and is assumed to be such that the path integrals $Z_{M_{\psi}, \psi}(J)$ and $Z_{M_{\psi}, \psi} = Z_{M_{\psi}, \psi}(J)|_{J=0}$

$$Z_{M_{\psi}, \psi}(J) = \int d\phi d\phi^{*} d\lambda \exp \left\{ \left(\frac{i}{\hbar} \left[ S_{M}(\phi, \phi^{*}) + (\phi^{*A}Z\phi^{A} - \psi(\phi)) \right] \hat{U} + J\phi^{A} \right) \right\} , \quad \text{for} \quad Z_{0, \psi}(J) = Z_{\psi}(J), \quad (A.9)$$

with the action

$$S_{M} = S + M_{\phi} , \quad \text{so that} \quad S_{M=0} = S, \quad (A.10)$$

be well-determined in perturbation theory. In this case, we will speak of a gauge theory with BRST symmetry breaking and call $Z_{M_{\psi}, \psi}(J)$ in \[A.9\] the generating functional of Green’s functions with BRST symmetry breaking. It is obvious that the integrand in \[A.9\] for $J = 0$ is not BRST invariant. However, the gauge independence of the vacuum functional $Z_{M_{\psi}, \psi}(0)$ can be restored if we suppose that within the reference frame given by the gauge fermion $\psi + \psi'$ the BRST symmetry breaking term should have a BRST transformed representation, with $\mu(\psi') = \mu(\phi, \lambda|\psi')$ being a solution \[A.7\] of the compensation equation \[A.6\], namely,

$$M_{\psi + \psi'} = M_{\psi} + \Delta_{\mu(\psi')}M_{\psi}. \quad (A.11)$$

Indeed, a modified Ward identity for $Z_{M_{\psi}, \psi}(J)$ is easily obtained by making in \[A.9\] a field-dependent BRST transformation \[A.4\] and using the relations \[A.7\] and the expression \[A.6\] for the Jacobian:

$$\left\langle \left[ 1 + \frac{i}{\hbar} \left[ J\phi^{A} + M_{\psi} \right] \hat{U} \mu(\psi') \right] \right\rangle \left( 1 + \mu(\psi') \hat{U} \right)^{-1} \bigg|_{M, \psi, J} = 1 , \quad (A.12)$$

where the symbol “$\langle \mathcal{O} \rangle_{M, \psi, J}$” for a quantity $\mathcal{O}$ stands for a source-dependent average expectation value with respect to $Z_{M, \psi}(J)$, corresponding to the gauge-fixing $\psi$. Note that \[A.12\] differs from the Ward identity for a gauge theory without BRST symmetry breaking and takes the form with a constant $\mu$

$$\left\langle \left[ J\phi^{A} + M_{\psi} \right] \hat{U} \right\rangle \bigg|_{M, \psi, J} = 0 , \quad (A.13)$$

which is identical, after introducing external antifields $\phi^{*A}$, with the Ward identity for $Z_{M_{\psi}, \psi}(J, \phi^{*})$ in \[26, 30\].

The Ward identity \[A.12\], with allowance made for \[A.7\] and \[A.11\], implies an equation which describes the gauge dependence of $Z_{M, \psi}(J)$ for a finite change of the gauge $\psi \rightarrow \psi + \psi'$, namely,

$$Z_{M_{\psi + \psi'}, \psi + \psi'}(J) - Z_{M, \psi}(J) = Z_{M, \psi}(J) \left\langle \frac{i}{\hbar} J\phi^{A} \hat{U} \mu(\phi, \lambda|\psi') \right\rangle \bigg|_{M, \psi, J} \quad (A.14)$$
As in the case of BRST-antiBRST symmetry \([A.12]\), the relation \([A.14]\) allows one to state, due to the standard arguments of the equivalence theorem \([30]\), that upon the shell determined by \(J \Lambda = 0\) a finite change of the generating functional of Green’s functions with a broken BRST symmetry term does not depend on a choice of the gauge condition with respect to a finite change of the gauge \((\psi \to \psi + \psi')\). The same statement can be made for the physical \(S\)-matrix. We hope that in the case of a renormalized theory the above property is preserved by a renormalized functional \(Z_{M,\psi;R}(J)\) and plan to study this problem in a separate paper.

**B  Effective Average Action in Yang–Mills theories**

Here we apply BRST-antiBRST symmetry breaking concept to effective average action in Yang–Mills theories within functional renormalization group (for review and references see \([36]\)) approach to the BRST-antiBRST Lagrangian quantization of Yang–Mills theories.

In this case, the generating functional \([2.6]\), with \(S\) being a solution of \([2.7]\), by integrating over \((\phi^*_{\alpha}, \phi, \pi^a, \lambda)\), is reduced to \(Z_F(J)\) \([3.13]\) with BRST-antiBRST invariant quantum action \(S_F(\phi)\) \([3.14]\) under finite BRST-antiBRST transformations, \(\phi^A \to A + \Delta \phi^A\). \([3.17]\). We consider extended by external antifields \(\delta^*_\alpha, \delta\) the generating functional of Green’s functions \(Z_F(J, \phi^*_{\alpha}, \bar{\phi})\) then being written without “tilde” as \(Z_F(J, \phi^*_{\alpha}, \bar{\phi}) = Z_F(J, \phi^*_{\alpha}, \bar{\phi})\),

\[
Z_F(J, \phi^*_{\alpha}, \bar{\phi}) = \int d\phi \exp \left\{ \frac{i}{\hbar} \left[ S_F(\phi, \phi^*_{\alpha}, \bar{\phi}) + J_A \phi^A \right] \right\}, \quad \text{with} \quad S_F = S_F + \phi^*_{\alpha} a^a \phi^A - \frac{1}{2} \bar{\phi} a^a \phi^A, \quad (B.1)
\]

for \(Z_F(J, 0, 0) = Z_F(J)\). The generating functional of vertex Green’s functions (effective action) \(\Gamma_F(\phi, \phi^*_{\alpha}, \bar{\phi})\) is determined in BRST-antiBRST quantization \([4, 5]\) as the Legendre transformation of \(\hbar \ln Z_F(J, \phi^*_{\alpha}, \bar{\phi})\) with respect to sources \(J_A\),

\[
\Gamma_F(\phi, \phi^*_{\alpha}, \bar{\phi}) = \left\{ \frac{\hbar}{\iota} \ln Z_F(J, \phi^*_{\alpha}, \bar{\phi}) - J_A \phi^A, \quad \phi^A = \frac{\hbar \delta}{\iota \delta J_A} \ln Z_F(J, \phi^*_{\alpha}, \bar{\phi}), \quad (B.2)\right.
\]

with \(J_A\) expressed in terms of average fields \(\phi^A\) from \(J = -(\delta \Gamma_F)/(\delta \phi)\). Note, that nonrenormalizable \(\Gamma_F(\phi, \phi^*_{\alpha}, \bar{\phi})\), first, satisfies to usual Ward identities in terms of the extended antibracket,

\[
\frac{1}{2} \left( \Gamma_F(\phi, \phi^*_{\alpha}, \bar{\phi}), \Gamma_F(\phi, \phi^*_{\alpha}, \bar{\phi}) \right) = V^a \Gamma_F(\phi, \phi^*_{\alpha}, \bar{\phi}) = 0, \quad (B.3)
\]

being rewritten by means of \([B.2]\) from the Ward identities for \(Z_F(J, \phi^*_{\alpha}, \bar{\phi})\): \((J_A \delta / \delta \phi^A - V^a) Z_F(J, \phi^*_{\alpha}, \bar{\phi}) = 0\), second, for vanishing \(\phi^*_{\alpha}, \bar{\phi}\) we obtain usual effective action for Yang–Mills theory in BRST-antiBRST formalism, \(\Gamma_F(\phi, 0, 0) = \Gamma_F(\phi)\). The actions \(\Gamma_F(\phi, \phi^*_{\alpha}, \bar{\phi}), \Gamma_F(\phi)\) are invariant respectively under finite BRST-antiBRST transformations for average fields,

\[
\Delta \lambda(\phi^A, \phi^*_{\alpha}, \bar{\phi}) = \left( \phi^A \left[ \exp \left( \frac{\pi^a \lambda_a}{\hbar} \right) - 1 \right], 0, \left( \hbar e^{a} \bar{\phi} A \right) \right) \quad \text{for} \quad \left( \frac{\phi - \phi^*}{\phi^*} \right) = \left( \phi^A \right) \quad (B.4)
\]

and \(\phi^A \to A + \Delta \phi^A|_{\phi^*_{\alpha}, \phi^A} = 0 \iff \phi^A \to \phi^A \phi^A \phi^A = \phi^A \exp \left( \frac{\pi^a \lambda_a}{\hbar} \right) \right|_{\phi^A} \phi^A \phi^A = \phi^A \phi^A\). \(\phi^A\) and \(\phi^A\) are inverse respectively under finite BRST-antiBRST transformations for average fields,

\[
\phi^A \phi^A \phi^A = \frac{1}{2} \delta \left( \frac{\phi - \phi^*}{\phi^*} \right) \quad \text{for} \quad J = 0 \quad \text{is invariant with account for the identity:} \quad s^a s^b = (1/2) \epsilon^{a b} s^2 \quad \text{for} \quad a, b = 1, 2.
\]

Now, we extend the construction of FRG earlier applied within Lagrangian BRST quantization only to the case of BRST-antiBRST quantization scheme. Doing so, we introduce, instead of \(\Gamma_F\), the so-called effective average action \(\Gamma_{F,k}\) with an external momentum-shell parameter, \(k\) smoothly related to \(\Gamma_F\)

\[
\lim_{k \to 0} \Gamma_{F,k}(\phi, \phi^*_{\alpha}, \bar{\phi}) = \Gamma_F(\phi, \phi^*_{\alpha}, \bar{\phi}), \quad (B.6)
\]
in such a way that the action $S_F(\phi, \phi^*_a, \bar{\phi})$ (for vanishing antifields the action $S_F(\phi)$) for Yang–Mills theories should be extended by means of BRST-antiBRST symmetry breaking terms, $M_F(\phi)$, having the form of the regulator action $S_k = S_{F,k}$, quadratic in fields $A^i = A^i(x)$ and $Sp(2)$-duplet of ghost-antighost fields $C^{\alpha a} = C^{\alpha a}(x)$ for $M(\phi) = S_k(\phi)$,

$$S_{F,k}(\phi) = \frac{1}{2} A^i A^j (R_{F,k})_{ij} + \frac{1}{2} \varepsilon_{ab} (R_{F,k,gh})_{a\beta} \bar{C}^{\beta b} C^{\alpha a}(-1) \varepsilon^a = \int d^Dx \left\{ \frac{1}{2} A^i A^j (R_{F,k})^{mn} (x) A^m (x) + \frac{1}{2} \varepsilon_{ab} C^{\alpha a} (x)(R_{F,k,gh})^{mn} (x) C^{\beta b} (x) \right\}, \tag{B.7}$$

with Lorentz indices $\mu, \nu = 0, 1, ..., D - 1$ of Minkowski space $R^{1,D-1}$ with metric $\eta_{\mu\nu} = \text{diag}(-, +, ..., +)$, and indices $m, n, l = 1, ..., N^2 - 1$ of Lie algebra $su(N)$ (see as well $\text{(3.23)}$, $\text{(3.24)}$ in Section $3.2$). In $\text{(B.7)}$, we have specified the condensed notations, so the regulator quantities $(R_{F,k}, (R_{F,k,gh})$ determined in the reference frame with gauge Boson $F$ have no dependence on the fields, obey the property $(R_{F,k})_{ij} = (-1)^{\varepsilon^i \varepsilon_j} (R_{F,k})_{ji}$, $(R_{F,k,gh})_{a\beta} = (-1)^{\varepsilon_a \varepsilon_\beta} (R_{F,k,gh})_{b\alpha}$ and vanish as the parameter $k$ tends to zero.

By definition, the regulator action $S_{F,k}$ is not BRST-antiBRST invariant,

$$S_{F,k} \mathcal{S}^a = \int d^Dx \left\{ A^{mn} (R_{F,k})_{\mu \nu} D^{\mu \nu} C^{\alpha a} + \varepsilon_{bc} (R_{F,k,gh})_{a\beta} C^{\beta b} \right\} - \frac{1}{2} \int f_{mn} C^{\alpha a} \left( \varepsilon^a B^m - \frac{1}{2} f^{mno} C^{\alpha a} \right) \int d^Dx \left\{ -2 \varepsilon_{ab} C^{\alpha a} (R_{F,k,gh})^{mn} f_{nl} \left( B^o C^{db} + \frac{1}{2} f^{ors} C^{oc} C^{rb} \right) \right\}, \tag{B.8}$$

where to derive $\text{(B.8)}$, $\text{(B.9)}$ we have used easily checked Leibnitz-like properties of the generators of BRST-antiBRST transformations, $s^a$ and $s^2$, acting on the product of any functionals $A, B$ with definite Grassmann parities,

$$(AB) \mathcal{S}^a = (A \mathcal{S}^a) B (-1)^{\varepsilon_B} A (B \mathcal{S}^a), \quad (AB) \mathcal{S}^2 = (A \mathcal{S}^2) B - 2 (A \mathcal{S}^2) (B \mathcal{S}^2) (-1)^{\varepsilon_B} A (B \mathcal{S}^2), \tag{B.9}$$

We determine an extended $Z_{F,k}(J, \phi^*_a, \bar{\phi})$ and usual $Z_{F,k}(J)$ generating functionals of Green’s functions coinciding respectively for $k \to 0$ with $Z_F(J, \phi^*_a, \bar{\phi})$ and $Z_F(J)$ $\text{(B.1)}$ according to $\text{(4.1)}$,

$$Z_{F,k}(J, \phi^*_a, \bar{\phi}) = \int d\phi \exp \left\{ \frac{i}{\hbar} \left[ S_F(\phi, \phi^*_a, \bar{\phi}) + S_{F,k} + J A \phi^A \right] \right\}, \tag{B.12}$$

Before taking the limit $k \to 0$, the integrand in $\text{(B.12)}$ for $J = 0$ is not BRST-antiBRST invariant due to $\text{(B.8)}$, $\text{(B.10)}$ whereas in the limit $k \to 0$ the functionals $Z_{F,k}$, take correct value, coinciding with the usual generating functionals $Z_F$. The average effective action $\Gamma_{F,k} = \Gamma_{F,k}(\phi, \phi^*_a, \bar{\phi})$ (the generating functional of vertex functions in the presence of regulators) is introduced according to the rule described by Eq. $\text{(4.2)}$, namely,

$$\Gamma_{F,k}(\phi, \phi^*_a, \bar{\phi}) = \frac{\hbar}{i} \ln Z_{F,k}(J, \phi^*_a, \bar{\phi}) - J A \phi^A, \quad \phi^A = \frac{\hbar}{i \delta A} \ln Z_{F,k}(J, \phi^*_a, \bar{\phi}), \tag{B.13}$$

with the obvious consequences, $J = - (\delta \Gamma_{F,k})/(\delta \phi)$, for the Legendre transformation $\text{(B.13)}$.

Note, first of all, that the average effective action, satisfies the functional integro-differential equation

$$\exp \left\{ \frac{i}{\hbar} \Gamma_{F,k}(\phi, \phi^*_a, \bar{\phi}) \right\} = \int d\varphi \exp \left\{ \frac{i}{\hbar} \left[ S_F(\phi + \hbar \varphi, \phi^*_a, \bar{\phi}) + S_{F,k}(\phi + \hbar \varphi) - \frac{\delta \Gamma_{F,k}(\phi, \phi^*_a, \bar{\phi})}{\delta \phi} \hbar \varphi \right] \right\}, \tag{B.14}$$
determining the loop expansion $\Gamma_{F;k} = \sum_{n \geq 0} \hbar^n \Gamma_{nF;k}$. Thus, the tree-level (zero-loop) and one-loop approximations of Eq. (14) correspond to

\[ \Gamma_{0F;k}(\phi, \phi_a^*, \bar{\phi}) = S_{F0}(\phi, \phi_a^*, \bar{\phi}) + S_{0F;k}(\phi), \]

\[ \Gamma_{1F;k}(\phi, \phi_a^*, \bar{\phi}) = S_{F1}(\phi, \phi_a^*, \bar{\phi}) + S_{1F;k}(\phi) - \frac{1}{2} \ln \text{Sdet} \left[ (S_{F0} + S_{0F;k})_{AB} \right] \]

For vanishing $\phi_a^*, \bar{\phi}$ from Eqs. (14)–(16) it follows an equation and zero-loop and one-loop approximations being analogous for an usual average effective action $\Gamma_{F;k}(\phi) = \Gamma_{F;k}(\phi, \phi_a^*, \bar{\phi})$ in BRST-antiBRST quantization.

Second, as to the regulator functions, we suppose that they model the non-perturbative contributions to the self-energy part of the Feynman diagrams, so that the dependence on the parameter $k$ enables one to extract some additional information about the scale dependence of the theory beyond the loop expansion [35]. Third, the modified Ward (Slavnov–Taylor) identities for the functionals $Z_{F;k}(J)$ and $\Gamma_{F;k}(\phi)$ are easily obtained from the general result (13) and take the form for $Z_{F;k}(J)$ identity:

\[ \langle \mathcal{O} \rangle_{F;k,J} = Z_{F;k}^{-1}(J) \int d\phi \mathcal{O}(\phi) \exp \left\{ \frac{i}{\hbar} \left[ S_{F}(\phi) + S_{F;k} + J_A \phi^A \right] \right\}, \quad \text{with} \quad (1)_{F;k,J} = 1. \]

In Eq. (17) account has been taken of the fact that if in some reference frame with a gauge Boson $F(\phi)$ the regulator action has the form $S_{F;k}$, then in a different reference frame determined by a gauge Boson $(F + DF)(\phi)$ it should be calculated in accordance with (14) and (18)–(20) as

\[ S_{F + DF;k} = (1 + \Delta \lambda(DF)) S_{F;k} = S_{F;k} \exp \left( \frac{\lambda^a}{\Lambda} \lambda_a(DF) \right). \]

For constant $\lambda_a$, the identity (17) decomposes in powers of $\lambda_a$ and assumes the form of two independent and one dependent (at $\Lambda^2$) identities identical to (10), (17) where the substitution $(M_F, \bar{U}^a, \langle \cdots \rangle_{M,F,J}) \rightarrow (S_{F;k}, \bar{S}^a, \langle \cdots \rangle_{F;k,J})$ should be done.

For the average effective action $\Gamma_{F;k}(\phi)$ the modified Ward identity (17) takes the form,

\[ \langle [1 + \frac{i}{\hbar} \left[ -\frac{\delta \Gamma_{F;k}}{\delta \phi^A} \phi^A + S_{F;k} \right] \bar{S}^a \lambda_a(\lambda) + \frac{1}{4} \bar{S}^2 \lambda^2(\lambda) \rangle - \frac{1}{4} \left( \frac{i}{\hbar} \right)^2 \left[ -\frac{\delta \Gamma_{F;k}}{\delta \phi^A} \phi^A + S_{F;k} \right] \bar{S}^a \lambda^a \lambda_a(\lambda) \langle (1 - \frac{1}{2} \lambda^2(\lambda))^{-2} \rangle_{F;k,\phi} = 1, \]

where the operators $\bar{S}^a$, $\bar{S}^2$ do not act on $(\delta \Gamma_{F;k})/\delta \phi$ and $\langle \cdots \rangle_{F;k,\phi}$ denotes an average expectation value corresponding to a gauge-fixing $\phi(\phi)$ with respect to $\Gamma_{F;k}(\phi)$.

Repeating the arguments of the Section 4 and with use of the results for Yang–Mills theory for gauge dependence problem (3.22) in Section 3.2 we obtain a relation which describes the gauge dependence of $Z_{F;k}(J)$ for a finite change of the gauge $F \rightarrow F + DF$ with account taken of (3.19) and (3.20) specified below (3.22) for Yang–Mills theory [1], the equality (17) implies a relation which describes the gauge dependence of $Z_{F;k}(J)$ for a finite change of the gauge $F \rightarrow F + DF$:

\[ Z_{F + DF;k}(J) - Z_{F;k}(J) = Z_{F;k}(J) \left[ \left( \frac{i}{\hbar} J_A \phi^A \right) \left[ \bar{S}^a \lambda_a (\phi) - \Delta F \right] + \frac{1}{4} \bar{S}^2 \lambda^2 (\phi) \right] - (-1)^F \left( \frac{i}{2\hbar} \right)^2 J_B J_A \left( \phi^A \phi^{\alpha(a)} \right) \lambda^2 (\phi - DF) \right]_{F;k,J}, \]

(21)
where

\[\lambda_a (\phi | - \Delta F) = - \frac{1}{2i \hbar} [\Delta F, S_{\alpha}] \sum_{n=0}^{\infty} \frac{(-1)^n}{n!} \left( \frac{1}{4\hbar} \Delta F S_{\alpha}^2 \right)^n.\] 

(B.22)

From (B.21) it follows that upon the shell determined by \(J_A = 0\) a finite change of the generating functional of Green’s functions with a BRST-antiBRST not-invariant regulator action \(S_k\) does not depend on the choice of the gauge condition with respect to a finite change of the gauge, \(F \to F + \Delta F\) for any value of an external momentum-shell parameter, \(k\).

For extended generating functional \(Z_{F;k}(J, \phi_a^*, \phi)\) making finite (group) BRST-antiBRST transformations \(\phi^A \to \tilde{\phi}^A = \phi^A \exp \left( \mp S_{\alpha} \lambda_a \right)\) in the sector of only the fields \(\phi^A\) with an arbitrary functional \(\lambda_a(\phi) = \Lambda(\phi) S_{\alpha}\) modified Ward identity, take the form,

\[
\begin{align*}
\left\{ \left[ J_A + \delta S_{F;k} A \right] \frac{\delta}{\delta \phi^A_{\alpha}} \lambda_a (A) - \frac{1}{2} \frac{\delta}{\delta \phi^A_{\alpha}} \tilde{\lambda}^2 (A) \right\} - V^a \hat{\lambda}_a (A) - \frac{\varepsilon_{ab}}{4} \left\{ \left[ J_A + \delta S_{F;k} A \right] \frac{\delta}{\delta \phi^B_{\beta}} V^b \right\} \tilde{\lambda}^2 (A) \right)
\times \left[ J_B + \delta S_{F;k} B \right] \frac{\delta}{\delta \phi^B_{\beta} \phi^A_{\alpha}} (-1)^{\varepsilon_{ab}+1} + \hat{\lambda}_a (A) \right) - 2 Z_{F;k,J} = 0,
\end{align*}
\]

where the notations for \(\tilde{S}_{F;k}, \lambda_a, \tilde{\lambda}_a, \tilde{\lambda}_B, A\) are determined by the rules,

\[
\hat{S}_{F;k,A} = \frac{\delta S_{F;k} (\phi)}{\delta \phi^A_{\alpha}} \bigg|_{\phi \to \frac{\phi}{\varepsilon^{\frac{1}{2}}}}; \quad \hat{\lambda}_a = \lambda_a (\phi) \bigg|_{\phi \to \frac{\phi}{\varepsilon^{\frac{1}{2}}}}; \quad \hat{\lambda}_{B,A} = \frac{\delta^2 \Lambda (\phi)}{\delta \phi^B_{\beta} \delta \phi^A_{\alpha}} \bigg|_{\phi \to \frac{\phi}{\varepsilon^{\frac{1}{2}}}}.
\]

(B.24)

Again for constant \(\lambda_a\), the identity \((B.23)\) decomposes in powers of \(\lambda_a\) and assumes the form of only two independent (at \(\lambda_a\)) Ward identities

\[
\tilde{q}^a Z_{F;k,J} = 0 \quad \text{for} \quad \tilde{q}^a = \left[ J_A + \delta S_{F;k} A \right] \frac{\delta}{\delta \phi^A_{\alpha}} - V^a,
\]

with nilpotent \(\tilde{q}^a: \tilde{q}^a \tilde{q}^b + \tilde{q}^b \tilde{q}^a = 0\), whereas at \(\lambda^2\) an expected Ward identity is identically vanish due to identity

\[
\frac{\varepsilon_{ab}}{4} \tilde{q}^a \tilde{q}^b = - \frac{1}{2} \left[ J_A + \delta S_{F;k} A \right] \frac{\delta}{\delta \phi^A_{\alpha}}.
\]

(B.26)

In turn, for the extended average effective action \(\Gamma_{F;k}(J, \phi_a^*, \phi)\) the modified Ward identity depending on odd functionally dependent functionals \(\lambda_a = \Lambda S_{\alpha}\) may be derived from (B.23) in a way analogical to trick in (B.26) but for \(\Gamma_{F;k}(J, \phi_a^*, \phi)\) is easily obtained in the form

\[
\frac{1}{2} \left( \Gamma_{F;k}(\phi, \phi_a^*, \phi), \Gamma_{F;k}(\phi, \phi_a^*, \phi) \right) + V^a \Gamma_{F;k}(\phi, \phi_a^*, \phi) - S_{F;k,A}(\phi) \frac{\delta}{\delta \phi^A_{\alpha}} \Gamma_{F;k}(\phi, \phi_a^*, \phi) = 0,
\]

(B.27)

which in limit \(k \to 0\) coincide with (B.23) for usual effective action \(\Gamma_{F}(\phi, \phi_a^*, \phi)\). Here the quantities, \(\tilde{S}_{F;k,A}(\phi)\) for operatorial fields \(\tilde{\phi}^A\) are determined by the rule,

\[
S_{F;k,A}(\phi) = \frac{\delta S_{F;k} (\phi)}{\delta \phi^A} \bigg|_{\phi \to \tilde{\phi}} \quad \text{with} \quad \tilde{\phi}^A = \phi^A + i \hbar (\Gamma_{F;k}^{-1})_{AB} \frac{\delta}{\delta \phi^B}, \quad \text{for} \quad (\Gamma_{F;k}^{-1})_{AC} (\Gamma_{F;k})_{CB} = \delta^B_{A},
\]

(B.28)

and \((\Gamma_{F;k}^{-1})_{AB} = \delta_{A}^{\delta_{B}^{\delta}} \delta_{\delta_{A}^{\delta} \delta_{B}^{\delta}} \Gamma_{F;k}\).

Finally, a relation which describes the gauge dependence of the extended functional \(Z_{F;k}(J, \phi_a^*, \phi)\) for a finite change of the gauge \(F \to F + \Delta F\) \((B.22)\) follows from the equality \((B.23)\) with account of \((B.24)\): 

\[
Z_{F+\Delta F;k}(J, \phi_a^*, \phi) - Z_{F;k}(J \phi_a^*, \phi) = \left\{ J_A \left[ \frac{\delta}{\delta \phi^A_{\alpha}} \hat{\lambda}_a (-\Delta F) - \frac{1}{2} \frac{\delta}{\delta \phi^A_{\alpha}} \hat{\lambda}^2 (-\Delta F) \right] - V^a \hat{\lambda}_a (-\Delta F) \right)
\]

\[
- \frac{\varepsilon_{ab}}{4} (-1)^{\varepsilon_{ab}} J_B J_A \left[ \frac{\delta}{\delta \phi^B_{\beta}} - V^b \right] \left[ \frac{\delta}{\delta \phi^A_{\alpha}} - V^a \right] \Lambda^2 (-\Delta F) \right\} Z_{F;k}(J \phi_a^*, \phi).
\]

(B.29)
From (B.29) it follows that upon the shell determined by $J_A = 0$ on the hypersurface $\phi^{\star A}_{\lambda A} = 0$ a finite change of the extended generating functional with a BRST-antiBRST not-invariant regulator action $S_k$ does not depend on the choice of the gauge condition with respect to a finite change of the gauge, $F \to F + \Delta F$ for any value of an external momentum-shell parameter, $k$.

The consistency of the FRG method, based on the introduction of Eq. (B.8), means that the values of the effective average actions $\Gamma_{F:k}$, $\Gamma_{F+\Delta F:k}$ calculated for two different gauges determined by $\chi^a$ and $\chi^a + \Delta \chi^a$ (see [14] for details) corresponding, to the gauge functionals $F$ and $F + \Delta F$ should coincide on the mass-shell for vanishing antifields $\phi^{\star A}_{\lambda A}$ for any value of the parameter $k$ (i.e., along the FRG trajectory, but not only in its boundary points). For completeness, the form of FRG flow equation for $\Gamma_{F:k}$ that describe the FRG trajectory (coinciding by the form with one in [34]) reads, with account for the notation, $\partial_t = k \frac{d}{dx}$:

$$\partial_t \Gamma_{F:k} = \partial_t S_{F:k} - \frac{\hbar}{i} \left\{ \frac{1}{2} \partial_t (R_{F:k})^{\mu \nu}_{\mu \nu} \left( \Gamma_{F:k}^{\nu -1} \right)^{(\mu \nu)} + \partial_t (R_{F:k,gh})^{\mu \nu}_{\mu \nu} \left( \Gamma_{F:k}^{\nu -1} \right)^{(\mu \nu)} \right\}$$ (B.30)

and has the same form for $\phi^{\star A}_{\lambda A}$ and $\phi_A$-independent part of $\Gamma_{F:k}$ due to parametric dependence on $\phi^{\star A}_{\lambda A}$, $\tilde{\phi}_A$ for all the terms in (B.30).

Let us consider concrete choice of the initial gauge Bosonic functional $F = F_\xi$ from the $F_\xi$-family of the gauge-fixing functionals $F_\xi = F_\xi (A, C)$ corresponding to an $R_\xi$-like gauge introduced within BRST-antiBRST Lagrangian quantization in [1]

$$F_\xi = \frac{1}{2} \int d^D x \left( -A^m_{\mu} A^{m \mu} + \frac{\xi}{2} \varepsilon_{ab} C^{ma} C^{mb} \right),$$ (B.31)

$$F_0 = -\frac{1}{2} \int d^D x A^m_{\mu} A^{m \mu} \quad \text{and} \quad F_1 = \frac{1}{2} \int d^D x \left( -A^m_{\mu} A^{m \mu} + \frac{1}{2} \varepsilon_{ab} C^{ma} C^{mb} \right),$$ (B.32)

where the gauge-fixing functional $F_{(0)} (A)$ induces the contribution $S_{F_{(0)}:k}$ to the quantum action that arises in the case of the Landau gauge $\chi (A) = \partial^\mu A^m_{\mu} = 0$ whereas the functional $F_1 (A, C)$ corresponds to the Feynman (covariant) gauge $\chi (A, B) = \partial^\mu A^m_{\mu} + (1/2) B^m = 0$. The parameters $\lambda_a (\phi \Delta F_\xi) = s_a \Lambda$ of a finite field-dependent BRST-antiBRST transformation that connects an $R_\xi$ gauge with an $R_{\xi+\Delta \xi}$ gauge, according to (B.22) have the form,

$$\lambda_a (\phi \Delta F_\xi) = \frac{\Delta \xi}{4 \hbar} \varepsilon_{ab} \int d^D x B^m C^{mb} \sum_{n=0}^{\infty} \frac{1}{(n+1)!} \left[ \frac{\Delta \xi}{4 \hbar} \int d^D y \left( B^a B^n - \frac{1}{24} f^{uwt} f^{irs} C^{ac} C^{rp} C^{wd} C^{ua} \varepsilon_{cd} \varepsilon_{pq} \right) \right]^n$$ (B.33)

where we have taken account of the expression for $\Delta F = \Delta F_\xi$

$$\Delta F_\xi = F_{\xi + \Delta \xi} - F_\xi = \frac{\Delta \xi}{4} \varepsilon_{ab} \int d^D x C^{ma} C^{mb}.$$ (B.34)

and for $\Delta F_\xi^{\xi a}$, $\Delta F_\xi^{\xi 2}$ respectively

$$\Delta F_\xi^{\xi a} = \frac{\Delta \xi}{2} \varepsilon_{bc} \int d^D x \varepsilon^{ca} B^m C^{mb},$$ (B.35)

$$\Delta F_\xi^{\xi 2} = 2 \int d^D x \left\{ \left( \partial^\mu A^m_{\mu} \right) + \frac{\xi}{2} D^m \right\} B^m + \frac{1}{2} \left( \partial^\mu C^{ma} \right) D^{mn} C^{mb} \varepsilon_{ab} - \frac{\xi}{48} f^{mnt} f^{irs} C^{ma} C^{rc} C^{mb} C^{md} \varepsilon_{cd} \varepsilon_{pq} \right\}.$$ (B.36)

Now, according to the representation (B.19) for the regulator action $S_{F_{\xi:k}}$ in $R_{\xi+\Delta \xi}$-gauge one may find its form in a way respecting gauge independence problem described by (B.21), (B.29) if $S_{F_{\xi+\Delta F_{\xi}:k}}$ is explicitly calculated by the rule

$$S_{F_{\xi+\Delta F_{\xi}:k}} = S_{F_{\xi:k}} \exp \left( \frac{\xi}{2} \lambda_a (\Delta F_\xi) \right),$$ (B.37)

with account of the formulae (B.8)–(B.10).
References

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