FIELDS OF DEFINITION FOR ADMISSIBLE GROUPS

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Abstract. A finite group $G$ is admissible over a field $M$ if there is a division algebra whose center is $M$ with a maximal subfield $G$-Galois over $M$. We consider nine possible notions of being admissible over $M$ with respect to a subfield $K$ of $M$, where the division algebra, the maximal subfield or the Galois group are asserted to be defined over $K$. We completely determine the logical implications between all variants.

1. Introduction

A group $G$ is admissible over a field $M$ if there is a $G$-crossed product $M$-division algebra, namely a division algebra $D$ whose center is $M$ with a maximal subfield $L$ which is Galois over $M$ with Galois group $G$.

Given a subfield $K$ of $M$ and a group $G$ which is admissible over $M$, one may ask how well can the admissibility be realized over $K$. For example, $G$ can be already $K$-admissible, with a $G$-crossed product over $K$ which remains a division algebra after scalar extension to $M$. Failing this strong assumption, it is still possible that $G$ is both $K$ and $M$-admissible; that the $G$-crossed product $D$ is defined over $K$ (namely, $D = D_0 \otimes_K M$ for a suitable division algebra over $K$); that $L$ is defined and Galois over $K$ (namely $L = L_0 \otimes_K M$ where $L_0/K$ is $G$-Galois); or that $L$ is merely defined over $K$.

This paper studies nine variations of $M$-admissibility of a group $G$, with respect to a fixed subfield $K$ of $M$. We provide a complete diagram of implications between those conditions (see Section 2). Furthermore, we provide counterexamples to every implication which is not proved before with $G$ being a $p$-group and $M$ a number field (see Section 5).

It turns out that for $G$ cyclic and $M$ a number field, eight of the nine conditions are satisfied (see Section 3). We shall also consider tame admissibility which is the type of admissibility that is best understood (see e.g. [5]) and show that these eight variants coincide with respect to tame admissibility.

The difference between tame and wild admissibility is an essential ingredient in the construction of counterexamples in Section 5.

2. Conditions on the field of definition

2.1. The nine variations. Let $K$ be a field and $G$ a finite group. We shall say that a field $L$ is $M$-adequate if it is a maximal subfield in some division algebra whose center is $K$. We shall say that $L$ is a $G$-extension of $K$ if $L/K$ is a Galois extension with Galois group Gal$(L/K) \cong G$.

Let $M/K$ a finite extension. One way to study the condition

(1) $G$ is $M$-admissible
is by refining it to require that the crossed-product division algebra or its maximal subfield are defined over $K$ (we say that a field or an algebra over $M$ is defined over $K$ if it is obtained by scalar extension from $K$ to $M$).

Condition (1) requires the existence of an $M$-adequate $G$-extension $L/M$. Three ways in which this field can be related to $K$ provide the following variants:

(2) there is an $M$-adequate $G$-extension $L/M$ for which $L$ is Galois over $K$;

(3) there is an $M$-adequate $G$-extension $L/M$ for which $L$ is defined over $K$;

(4) there is an $M$-adequate $G$-extension $L/M$ so that $L = L_0 \otimes M$ and $\text{Gal}(L_0/K) \cong G$.

For the algebra $D$ to be defined over $K$, we may require that:

(5) there is a $K$-division algebra $D_0$ and a $G$-extension $L/M$ for which $L$ is a maximal subfield of $D_0 \otimes M$;

(6) there is a $K$-division algebra $D_0$ and a maximal subfield $L_0$ which is a $G$-extension of $K$ so that $L_0 \cap M = K$ and $L = L_0M$ is a maximal subfield of the division algebra $D = D_0 \otimes M$.

If $L = L_0 \otimes_K M$, the interaction between $L_0$ and $L$ may involve the division algebras:

(7) there is a $K$-adequate $G$-extension $L_0/K$ for which $L_0M$ is an $M$-adequate $G$-extension;

(8) there is a $K$-adequate $G$-extension $L_0/K$ for which $L_0M$ is an $M$-adequate $G$-extension.

And finally we have the double condition

(9) $G$ is both $K$-admissible and $M$-admissible.

We provide a diagrammatic description of each condition, for easy reference. Inclusion is denoted by a vertical line, and diagonal lines show the extension of scalars from $K$ to $M$. A vertical line is decorated by $G$ if the field extension is $G$-Galois. Note that in some cases (4), (6) and (7) the fact that the extension $L_0/K$ is Galois implies the same condition on the extension $L/M$.

\[
\begin{array}{cccc}
D & D & D & D \\
\mid & \mid & \mid & \mid \\
L & L & L & L \\
\mid & \mid & \mid & \mid \\
M & M & M & M \\
\mid & \mid & \mid & \mid \\
K & K & K & K \\
\end{array}
\]
We shall say that a triple \((K, M, G)\) satisfies Condition (m) if there are \(L_0, L, D_0\) and \(D\) as required in this condition. In such case we shall also say \((L_0, L, D_0, D)\) realizes Condition (m), omitting \(L_0\) or \(D_0\) if they are not needed.

**Remark 2.1.** Let \(M/K\) be a finite extension of fields and \(G\) a finite group. One might also consider the condition

\[
\text{(10) there is a } G\text{-crossed product } K\text{-division algebra } D_0, \text{ for which } D = D_0 \otimes M \text{ is also a } G\text{-crossed product division algebra.}
\]

Here there is no explicit assumption that the maximal subfields be related; in the spirit of previous diagrams, this condition is described by

However, (10) is equivalent to Condition (6). Indeed, suppose that \((L_0, L, D_0, D)\) realizes (10). Then \(D\) is of index \(|G|\) and \(D\) is also split by \(L' = ML_0\). Therefore \([L':M] = |G|\), \(L_0 \cap M = K\) and hence we can take \(L'\) to be the required maximal \(G\)-subfield of \(D\). Thus, \((L_0, L', D_0, D)\) realizes (6). The converse implication is obvious, taking \(L = L_0 \otimes_K M \subset D_0 \otimes_K M = D\).

**2.2. The logical implications.** The following theorem describes the relation between the nine variants:
Theorem 2.2. Let $M/K$ be a finite extension of fields and $G$ a finite group. Then the implications in Diagram (2.1) hold, but no others.

In Section 5 we give counterexamples for the false implications, with $G$ being a $p$-group and $K$ a number field in each case. Let us go over the implications in diagram (2.1).

Proof of positive part of Theorem 2.2. Fix $K$, $M$ and $G$. Clearly if $(L_0, L, D_0, D)$ realizes Condition (6), $(L_0, L, D_0, D)$ also realizes (7) and $(L, D_0, D)$ realizes (5), so that $(6) \Rightarrow (5), (7)$.

If $(L_0, L, D_0, D)$ realizes (7) then $L_0/K$ is a $G$-extension and hence $L = L_0M/M$ is also a $G$-extension (since $L_0 \cap M = K$). Thus, $(L_0, L, D_0, D)$ realizes (8). It is clear that $L_0$ is a field of definition of $L$ (and $\text{Gal}(L_0/K) = G$) and hence $(L_0, L, D)$ realizes (3). As $L_0$ is a $K$-adequate $G$-extension and $L$ is an $M$-adequate $G$-extension, $(L_0, L, D_0, D)$ realizes (9). Therefore $(7) \Rightarrow (1), (9), (8)$.

If $(L_0, L, D)$ realizes Condition (1) then $\text{Gal}(L_0/K) = G$, $\text{Gal}(L/M) = G$ (since $L_0 \cap M = K$) and hence $(L_0, L, D_0, D)$ realizes Condition (3). If $(L_0, L, D_0, D)$ realizes condition (3), clearly $L_0$ is a field of definition of $L$ and hence $(L_0, L, D)$ realizes Condition (3).

Clearly when $(K, M, G)$ satisfies either of the conditions (2), (3), (5), (9), $G$ is $M$-admissible and hence (2), (3), (5), (9) $\Rightarrow (1)$.

3. Cyclic groups over number fields

For a prime $v$ of a number field $K$, we denote by $K_v$ the completion of $K$ with respect to $v$. If $L/K$ is a finite Galois extension, $L_v$ denotes the completion of $L$ with respect to some prime divisor of $v$ in $L$.

The basic criterion for admissibility over number fields is due to Schacher:

Theorem 3.1 ([10]). Let $K$ be a number field and $G$ a finite group. Then $G$ is $K$-admissible if and only if there exists a Galois $G$-extension $L/K$ such that for every rational prime $p$ dividing $|G|$, there is a pair of primes $v_1, v_2$ of $K$ such that each of $\text{Gal}(L_{v_1}/K_{v_1})$ contains a $p$-Sylow subgroup of $G$.

We use this criterion in the construction of counterexamples in Section 5 and to prove the following proposition:

Proposition 3.2. Let $G$ be a cyclic group. Then Conditions (1) and (3)–(9) are satisfied for any extension of number fields $M/K$. 
Definition 4.1. Following this observation one defines: $\zeta \in L$ and so $L$ prime to $t$ tame subextension of $L/K$ group $\text{Br}(L/K)$. Let $L_0/K$ be the maximal subfield, and supported by $\{v_1, v_2\}$. As $v_i$ split completely in $M$ we have $L = L_0 M$ satisfies $\text{Gal}(L_{v_i}/M_{v_i}) = G$ for $i = 1, 2$ and hence $\text{Gal}(L/M) = G$. Finally $D = D_0 \otimes M$ is a division algebra by the choice of the $v_i$. Thus $L$ is $M$-adequate and $(K, M, G)$ satisfies $(6)$. 

Example 3.3. If $M/K$ is not normal, $(6)$ does not necessarily hold for a cyclic group $G$. Let $n \geq 2$ and $M/K$ be an extension of degree $n$ whose Galois closure $M'$ has Galois group $\text{Gal}(M'/K) = S_n$. Then any field $L \supseteq M$, which is Galois over $K$, must contain $M'$ and hence there is no (adequate) $C_n$-extension $L/M$ for which $L/K$ is Galois. In particular, $(6) \neq (2)$.

Remark 3.4. If $F_1$ and $F_2$ are field extensions of $F$ such that $L = F_1 \otimes_F F_2$ is a field, and $F_1/F$ and $L/F_1$ are Galois, then $L$ is Galois over $F$.

Remark 3.5. This shows that if $M/K$ is Galois then $(6) \Rightarrow (2)$. In particular $(2)$ holds for $G$ cyclic.

We mention in this context the ‘linear disjointness’ (LD) of number fields, as defined and established in [3 Prop. 2.7]: for every finite extension $M/K$ in characteristic 0, any central simple algebra over $K$ contains a maximal separable subfield $P$ that is linearly disjoint from $M$ over $K$. This notion can be bypassed by appealing to the Chebotarev density, as above.

4. TAME ADMISSIBILITY

The conditions of Section 2 can also be considered with respect to tame $K$-admissibility. Let us recall the definition of tame admissibility.

For an extension of fields $L/K$, $\text{Br}(L/K)$ denotes the kernel of the restriction map $\text{res} : \text{Br}(K) \to \text{Br}(L)$. Let $\text{Br}(L/K)_{\text{tr}}$ be the subgroup of the relative Brauer group $\text{Br}(L/K)$ that consists of the Brauer classes which are split by the maximal tame subextension of $L_v/K_v$, for every prime $v$ of $L$.

Over a number field $K$, the exponent of a division algebra is equal to its index, and so $L$ is $K$-adequate if and only if there is an element of order $[L:K]$ in $\text{Br}(L/K)$ ([10 Proposition 2.1]). Following this observation one defines:

Definition 4.1. Let $K$ be a number field. We say that a finite extension $L$ of $K$ is tamely $K$-adequate if there is an element of order $[L:K]$ in $\text{Br}(L/K)_M$.

Likewise, a finite group $G$ is tamely $K$-admissible if there is a tamely $K$-adequate $G$-extension $L/K$.

4.1. Liedahl’s condition. Let $\mu_n$ denote the set of $n$-th roots of unity in $\mathbb{C}$. For $t$ prime to $n$, let $\sigma_{t,n}$ be the automorphism of $\mathbb{Q}(\mu_n)/\mathbb{Q}$ defined by $\sigma_{t,n}(\zeta) = \zeta^t$ for $\zeta \in \mu_n$.
Definition 4.2. We say that a metacyclic $p$-group $G$ satisfies Liedahl’s condition (first defined in [3]) with respect to $K$, if it has a presentation
\begin{equation}
G = \langle x, y \mid x^m = y^t, \ y^n = 1, \ x^{-1}yx = y^t \rangle
\end{equation}
such that $\sigma_{t,n}$ fixes $K \cap \mathbb{Q}(\mu_n)$.

It follows from [3] (see also [4, Corollary 2.1.7]) that tamely $K$-admissible groups $G$ have metacyclic $p$-Sylow subgroups that satisfy Liedahl’s condition for every prime divisor $p$ of $|G|$. There are no known counterexamples to the opposite implication. In fact if a metacyclic $p$-group satisfies Liedahl’s condition over $K$ then it is realizable over infinitely many completion of $K$ (see [3]).

Remark 4.3. Note that if a metacyclic $p$-group $G$ satisfies Liedahl’s condition over $M$, then it satisfies the condition over every subfield $K$.

The following is shown in [3, Theorem 30] for $G$ a $p$-group, and in [4, Theorem 2.3.1] for $G$ solvable.

Theorem 4.4. Let $K$ be a number field and $G$ a solvable group whose Sylow subgroups satisfy Liedahl’s condition. Then $G$ is tamely $K$-admissible.

Remark 4.5. In fact the proof of [4, Theorem 2.3.1] shows that there is a $G$-extension $L_0/K$ and $D_0 \in \Br(L_0/K)_G$ such that $D := D_0 \otimes_K L$ remains a division algebra.

In particular $L := L_0 \otimes_K M$ is an $M$-adequate field which is a $G$-extension of $M$. Thus, not only $G$ is $M$-admissible but there is also a $G$-crossed product division algebra $D$ and a maximal subfield $L$ so that both are defined compatibly over $\mathbb{Q}$.

As a corollary one has (see [4]):

Corollary 4.6. Let $K$ be a number field. Let $G$ be a solvable group such that the rational prime divisors of $|G|$ do not decompose (i.e. have a unique prime divisor) in $K$. Then $G$ is $K$-admissible if and only if its Sylow subgroups are metacyclic and satisfy Liedahl’s condition.

4.2. Fields of definition for tame admissibility. The conditions of Section 2 can also be considered with respect to tame $K$-admissibility. Let $G$ be a solvable group and $K, M$ number fields. By Proposition 4.3 if $G$ is tamely $M$-admissible then there is a tamely $K$-adequate $G$-extension $L_0/K$ for which $L = L_0M$ is $M$-adequate (and hence tamely $M$-adequate). For $m = 1, \ldots, 6$ let $(m^*)$ denote the condition $(m)$, where every adequate extension is assumed to be tamely adequate, and an admissible group is assumed tamely admissible. More precisely for $m = 5, 6$ we consider

(5) there is a $K$-division algebra $D_0$ and a $G$-extension $L/M$ for which $[D] = [D_0 \otimes M] \in \Br(L/M)_G$ and $L$ is a maximal subfield of $D$,

and

(6) there is a $K$-division algebra $D_0$ and a maximal subfield $L_0$ which is a $G$-extension of $K$ so that $L_0 \cap M = K$, $D_0 \in \Br(L_0/K)_G$, and $L = L_0M$ is a maximal subfield of $D = D_0 \otimes M$ (and hence $[D] \in \Br(L/M)_G$).

Corollary 4.7. Let $G$ be a solvable group and $M/K$ a finite extension of number fields. Then the conditions $(1^*)$ and $(3^*)-(6^*)$ are all equivalent.
Proof. With the added conditions the implications given in (2.1) clearly continue to hold. But by Remark 4.3 the implication (1) \( \Rightarrow (3) \) also holds.

\[ \square \]

5. Examples

In this section we give counterexamples for all the implications not claimed in Theorem 2.2. In all the examples, the group \( G \) is a \( p \)-group. This shows that Diagram 2.1 describes all the correct implications even for \( p \)-groups.

Let us first show that none of the conditions (2), (5) or (9) imply any other condition except (1). For this, by the implication Diagram 2.1, it is sufficient to notice that (2) \( \Rightarrow (3) \). It suffices to notice that (2) \( \Rightarrow (3) \). We will show that (3) \( \Rightarrow (5) \) by demonstrating that (7) \( \Rightarrow (5) \). In fact an example for (7) \( \Rightarrow (5) \) will show that no other condition, except (4), implies Condition (5). To complete the proof we should also prove (2) \( \Rightarrow (3) \), (8) \( \Rightarrow (4) \), (1) \( \Rightarrow (9) \) and (1) \( \Rightarrow (8) \).

Remark 5.1. Note that (6) \( \Rightarrow (2) \) follows from Remark 3.3.

Example 5.2 ((2) \( \Rightarrow (3) \), (5) \( \Rightarrow (3) \), (9) \( \Rightarrow (3) \)). Let \( p \equiv 1 \pmod{4} \), \( G = (\mathbb{Z}/p\mathbb{Z})^3 \) and \( K = \mathbb{Q}(i, \sqrt{p}) \). Note that \( p \) splits in \( K \). Denote the prime divisors of \( p \) in \( K \) by \( v_1, v_2 \).

Let \( K_{v_i}(p)_{ab} \) be the maximal abelian pro-\( p \) extension of \( K_{v_i} \). By local class field theory the Galois group \( \text{Gal}(K_{v_i}(p)_{ab}/K_{v_i}) \) is isomorphic to the pro-\( p \) completion of the group \( K_{v_i}^* \) which is \( \mathbb{Z}_p^n \) where \( n = [K_{v_i} : \mathbb{Q}_p] + 1 = 3 \) (see [12], Chapter 14, Section 6).

Since \( K_{v_1} = K_{v_2} = \mathbb{Q}_p(\sqrt{p}) \) this shows \( G \) is realizable over \( K_{v_1}, K_{v_2} \). By the Grunwald-Wang Theorem there a \( (\mathbb{Z}/p^2\mathbb{Z})^3 \)-extension \( \hat{M}/K \) such that \( M_{v_i} \) is the maximal abelian extension of exponent \( p^2 \) of \( K_{v_i} \), namely the unique \( (\mathbb{Z}/p^2\mathbb{Z})^3 \)-extension of \( K_{v_i} \). Let \( M = \hat{M}^G \), so that \( \text{Gal}(M/K) \approx G \).

Since \( \hat{M}/M \) and \( M/K \) both have full local degrees at \( v_1, v_2 \), both are adequate \( G \)-extensions. Note that \( \hat{M} \) is also Galois over \( K \). By choosing \( L = M \), we deduce that \( (K, M, G) \) satisfies conditions (2) and (9). To show that \( (K, M, G) \) satisfies (3) it suffices to notice that \( v_1, v_2 \) have unique prime divisors \( w_1, w_2 \) in \( M \). Every division algebra \( D \) whose invariants are supported in \( \{w_1, w_2\} \) is \( K \)-uniformly distributed and hence \( D \in \text{Im}(\text{res}^M_K) \). Take \( D \) with

\[ \text{inv}_{w_1}(D) = \frac{1}{p^3}, \quad \text{inv}_{w_2}(D) = -\frac{1}{p^3} \]

and \( \text{inv}_w(D) = 0 \) for any other prime \( w \) of \( M \). We then have \( D \in \text{Im}(\text{res}^M_K) \), \( D \) is a \( G \)-crossed product division algebra and hence \( (K, M, G) \) satisfies (3).

Let us show (3) is not satisfied. Suppose on the contrary that there is a triple \( (L_0, L, D) \) realizing (3). By Remark 4.4 \( L/K \) is Galois and

\[ \text{Gal}(L/K) \cong \text{Gal}(L/L_0) \times \text{Gal}(L/M) \cong G \ltimes \phi G' \]

via some homomorphism \( \phi : G \rightarrow \text{Aut}(G) = \text{GL}_3(\mathbb{F}_p) \). As \( G \) is a \( p \)-group, \( \phi \) is a homomorphism into some \( p \)-Sylow subgroup \( P \) of \( \text{GL}_3(\mathbb{F}_p) \). These are all conjugate, so we can choose a basis \( \{v_1, v_2, v_3\} \) of \( \mathbb{F}_p^3 \) for which \( P \) is the Heisenberg group (in...
other words the unipotent radical of the standard Borel subgroup), generated by the transformations:
\[ \phi_x(a, b, c) = (a + b, b, c), \quad \phi_y(a, b, c) = (a, b + c, c), \quad \phi_z(a, b, c) = (a + c, b, c) \]
which correspond to the matrices
\[ x = \begin{pmatrix} 1 & 1 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}, \quad y = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 1 \\ 0 & 0 & 1 \end{pmatrix}, \quad u = \begin{pmatrix} 1 & 0 & 1 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}. \]
Note that \( P \) has the presentation
\[ P = \langle x, y, u \mid x^p = y^p = u^p = [x, u] = [y, u] = 1, [y, x] = u \rangle. \]

Every subgroup of the form \( \mathbb{F}_p^2 \rtimes G \) is a maximal subgroup of \( G \rtimes G \) and thus the Frattini subgroup \( \Phi \) of \( G \) is contained in \( 1 \rtimes G \). Now the subgroup \( H = \langle v_1, v_2 \rangle \leq G \) is invariant under the action of \( P \) and hence under the action of \( G \) via \( \phi \). So, \( G \rtimes_\phi H \leq G \rtimes_\phi G \) is a maximal subgroup and \( \Phi \leq 1 \rtimes H \). This shows that \( \dim_{\mathbb{F}_p} G/\Phi \geq 4 \) and thus \( G \rtimes_\phi G \) is not generated by less than 4 elements. Therefore \( G \rtimes G \) is not realizable over \( \mathbb{Q}_p(\sqrt[p]{p}) \).

On the other hand both \( L/M \) and \( M/K \) have full rank at \( w_i \) and \( v_i \) and hence \( \text{Gal}(L_{wi}/K_{vi}) = G \rtimes G \) which is a contradiction as \( G \rtimes G \) is not realizable over \( K_{vi} \). Thus, \( (K, M, G) \) does not satisfy Condition (4).

**Example 5.3 ((2) \not\Rightarrow (3), (3) \not\Rightarrow (5), (5) \not\Rightarrow (1))**. Let \( p \equiv 1 \pmod{4} \), \( K = \mathbb{Q}(i) \) and \( v_1, v_2 \) the two prime divisors of \( p \) in \( K \). Let \( G = \mathbb{F}_p^2 \) and \( P = \mathbb{F}_p \langle \mathbb{Z}/p\mathbb{Z} \rangle \) so that \( P = G \rtimes \langle x \rangle \) where \( x^p = 1 \).

The maximal \( p \)-extension \( \mathbb{Q}_p(p) \) has Galois group \( G_{\mathbb{Q}_p(p)} := \text{Gal}(\mathbb{Q}_p(p)/\mathbb{Q}_p) \) which is a free pro-\( p \)-group on two generators. As \( P \) is generated by two elements it is realizable over \( \mathbb{Q}_p \). Since \( P \) is a wreath product of abelian groups it has a generic extension over \( K \) and hence by (9) there is a \( P \)-extension \( L/K \) for which \( \text{Gal}(L_{wi}/K_{vi}) = P \) for \( i = 1, 2 \). Let us choose \( M = L^2 \) the \( G \)-fixed subfield of \( L \).

Then clearly \( L/M \) is an \( M \)-adequate extension which is defined over \( K \) since
\[ \text{Gal}(L/K) \cong \text{Gal}(M/K) \rtimes \text{Gal}(L/M). \]
The subfield \( L_0 = L^{(x)} \) is \( K \)-adequate since \( [(L_0)_{v_i}:K_{v_i}] = p^i \) for \( i = 1, 2 \) and hence \( (K, M, G) \) satisfies Condition (8).
(We write \( (L_0)_{v_i} \) even though \( L_0/K \) is not Galois, since \( v_i \) has a unique prime divisor in \( L_0 \) for \( i = 1, 2 \).)

Now since \( G \) is an abelian group of rank \( p > 2 \), \( G \) is not realizable over \( K_{v_1}, K_{v_2} \cong \mathbb{Q}_p \) and hence not \( K \)-admissible. It follows that \( (K, M, G) \) does not satisfy Condition (9). In order for \( (K, M, G) \) to satisfy Condition (4) there should be a \( G \)-extension \( L_0/K \) for which \( L_0M \) is \( M \)-adequate. In particular, \( \text{Gal}((L_0M)_{v_1}/M_{v_1}) \cong G \) and hence \( \text{Gal}((L_0)_{v_1}/K_{v_1}) \cong G \) which contradicts the fact that \( G \) is not realizable over \( K_{v_1} \cong \mathbb{Q}_p \). Thus \( (K, M, G) \) does not satisfy Condition (4) either.

By Remark 3.3, as \( M/K \) is Galois, (8) \(\Rightarrow\) (3) \(\Rightarrow\) (2) and hence \( (K, M, G) \) also satisfies (2).

**Example 5.4 ((1) \not\Rightarrow (3), (4) \not\Rightarrow (8))**. Let \( p \equiv 1 \pmod{4} \) and \( v \) be its unique prime divisor in \( K = \mathbb{Q}(\sqrt[p]{p}) \). Let \( M = \mathbb{Q}(\sqrt[p]{p}, i) \) and \( G = (\mathbb{Z}/p\mathbb{Z})^2 \).
By the Grunwald-Wang Theorem, there is a Galois $G$-extension $L_0/K$ for which $\text{Gal}(L_0)_v/K_v = G$. Thus $L = L_0M$ is a Galois $G$-extension of $M$ such that $\text{Gal}(L_v/M_v) = G$ for each of the two prime divisors $v_1, v_2$ of $v$ in $M$. It follows that $L$ is $M$-adequate and $(K, M, G)$ satisfies Condition (4). But as $p$ has a unique prime divisor in $K$ and $G$ is not metacyclic, $G$ is not $K$-admissible and hence $(K, M, G)$ does not satisfy Condition (9).

Let us also show that $(K, M, G)$ does not satisfy Condition (8). Assume, on the contrary, that $(L_0, L, D_0, D)$ realizes (8). Then, as $L_0$ is $K$-adequate there are two primes $w_1, w_2$ of $K$ for which $[(L_0)_{w_i}:K_{w_i}] = |G|$. Without loss of generality we assume $w_1 \neq v$ (otherwise take $w_2$). Then \( \text{Gal}(L_{w_1}/M_{w_1}) \cong G \) since \( (L_0)_{w_1} \cap M_{w_1} = K_{w_1} \). This is a contradiction since tamely ramified extensions (such as $L_{w_1}/M_{w_1}$) have metacyclic Galois groups. Thus $(K, M, G)$ does not satisfy Condition (8).

Remark 5.5. Let us also show that $(K, M, G)$ does not satisfy (5) (so that this example will also show that (4) $\not\Rightarrow$ (5)). Assume on the contrary that there is a $(L, D_0, D)$ that realizes (5). Since $D$ contains $L$ as a maximal subfield, $\text{Gal}(L_v/M_v) = G$ and $\text{inv}_v(D) = \frac{m_u}{p^x}$ where $(m_i, p) = 1$, for $i = 1, 2$. Note that $G$ is realizable over $M_v$ only for divisors $v$ of $p$, so that $\text{inv}_u(D) = \frac{m_u}{p^x}$ for suitable $m_v \in \mathbb{Z}$ for any $u \neq v_1, v_2$. Now, since $D$ is in the image of the restriction, we have $m_1 = m_2$. The sum of $M$-invariants of $D$ is an integer and hence $p \mid m_1 + m_2 = 2m_1$ which contradicts $(m_i, p) = 1$.

Example 5.6 ((5) $\not\Rightarrow$ (3)). Let $p$ be any odd prime, and $q$ a prime $\equiv 1 \pmod{p}$. Let $K = \mathbb{Q}(\sqrt{p})$, so that $q$ splits (completely) in $K$. Let $v$ be the prime divisor of $p$ in $K$ and $w$ a prime divisor of $q$ in $K$. Let $M$ be a $\mathbb{Z}/p\mathbb{Z}$-extension of $K$ in which $v$ splits completely and $w$ is inert. Let $G = (\mathbb{Z}/p\mathbb{Z})^3$.

Consider the $K$-division algebra $D_0$ whose invariants are:

$$\text{inv}_v(D_0) = \frac{1}{p^2}, \quad \text{inv}_w(D_0) = -\frac{1}{p^3}$$

and $\text{inv}_u(D_0) = 0$ for any other prime $u$ of $K$. Now $D = D_0 \otimes_K M$ has $M$-invariants $\text{inv}_u(D) = \frac{1}{p^3}$ for the prime divisors $v_1, v_2, \ldots, v_p$ of $v$ in $M$, $\text{inv}_w(D) = -\frac{1}{p^3}$ for the prime divisor $w'$ of $w$ and $\text{inv}_u(D) = 0$ for any other prime $u$ of $M$. Note that $G$ is realizable over $M_v \cong K_v$ and since $q \equiv 1 \pmod{p}$, $(\mathbb{Z}/p\mathbb{Z})^3$ is realizable over $M_{w'}$. By the Grunwald-Wang Theorem, there is a Galois $G$-extension $L/M$ for which:

$$\text{Gal}(L_v/M_v) = G \text{ for } i = 1, \ldots, p,$$ 

and $\text{Gal}(L_{w'}/M_{w'}) = (\mathbb{Z}/p\mathbb{Z})^2$.

Thus $L$ is a maximal subfield of $D$ and $(K, M, G)$ satisfies condition (5). Since $p$ has a unique prime divisor in $K$ and $G$ is not metacyclic we deduce $G$ is not $K$-admissible and hence $(K, M, G)$ does not satisfy Condition (9).

Example 5.7 ((2) $\not\Rightarrow$ (3), (7) $\not\Rightarrow$ (5)). Let $p \geq 13$ be a prime such that $p \equiv 1 \pmod{4}$. Let $K = \mathbb{Q}(\mu_p)$ and $M = \mathbb{Q}(\mu_{p^2}) = \mathbb{Q}(i, \mu_{p^2})$. Let $G$ be the following metacyclic group of order $p^3$:

$$G = \left\langle x, y \mid x^p = y^{p^2} = 1, x^{-1}yx = y^{p+1} \right\rangle.$$ 

Note that $p$ splits in $\mathbb{Q}(i)$ and has exactly two prime divisors $v_1, v_2$ in $M$. Let $u$ be the unique prime divisor of $p$ in $K$. 


Let us first show that \((K, M, G)\) does not satisfy Condition (5). As \(M\) does not satisfy Liedahl’s condition, \(G\) is not realizable over \(M_v\) for any \(v \neq v_1, v_2\). Assume on the contrary there is an \(M\)-adequate \(G\)-extension \(L/M\) and an \(M\)-division algebra \(D\) which is defined over \(K\) and has a maximal subfield \(L\). Then necessarily: \(\text{inv}_{v_1}(D) = \text{inv}_{v_2}(D) = \frac{m}{p^r}\) for some \((a, p) = 1\). But as the sum of invariants of \(D\) is 0 and \(G\) is not realizable over any other \(v\) we have \(p | 2a\) or \(p | a\). Contradiction.

To prove that \((K, M, G)\) satisfies Condition (7) we shall need the following lemma:

**Lemma 5.8.** Let \(p \geq 11\) be a prime, \(k = \mathbb{Q}_p(\mu_p)\) and \(G\) the group defined in (5.1). Then, given a \(G\)-extension \(m/k\), there is a \(G\)-extension \(l/k\) for which \(m \cap l = k\).

**Proof.** For any \(G\)-extension \(l/k\) we note that \(\text{Gal}(l \cap m/k)\) is an epimorphic image of \(G\) and as such it is either \(G\) or an abelian group. Thus if \(l\) intersects with \(m\) non-trivially then it also intersects with \(m' = m^{(p^r)}\) (the fixed field of \(y^r\) which also corresponds to the abelianization of \(G\)). We note that \(\text{Gal}(m'/k) = (\mathbb{Z}/p\mathbb{Z}) \times (\mathbb{Z}/p\mathbb{Z})\). The maximal abelian group realizable over \(k\) is of rank \(p - 1\), and since \(\frac{p-1}{2} \geq 4\) there is a \((\mathbb{Z}/p\mathbb{Z})^2\)-extension \(l'/k\) which is disjoint from \(m'\) and for which the epimorphism \(\pi : G_k \to \text{Gal}(l'/k)\) splits through a free pro-\(p\) group of rank \(\frac{p-1}{2}\). Thus \(l'\) is disjoint from \(m'\) and hence to \(m\). Embedding \(l'\) into a \(G\)-extension produces a \(G\)-extension which is disjoint to \(m\). This is possible since the following embedding problem for \(G_k\):

\[
\begin{array}{ccc}
G_k & \longrightarrow & \mathbb{Z}/p\mathbb{Z}^2 \\
\downarrow & & \downarrow \\
\mathbb{F}_p(z^{p-1}) & \longrightarrow & 0
\end{array}
\]

splits through a free pro-\(p\) group of large enough rank and hence has a surjective solution. \(\square\)

Let us prove Condition (7) is satisfied. Let \(\sigma_{p+1} \in \text{Gal}(\mathbb{Q}(\mu_{p^2})/\mathbb{Q})\) be the automorphism that sends \(\sigma_{p+1}(\zeta) = \zeta^{p+1}\) where \(\zeta\) is a primitive root of unity of order \(p^2\). Thus \(\sigma_{p+1}\) fixes \(\mu_p\) and hence \(\sigma_{p+1} \in \text{Gal}(\mathbb{Q}(\mu_{p^2})/K)\). As \(G\) satisfies Liedahl’s condition over \(K\), \(G\) is realizable over infinitely many primes of \(K\) (see the proof of [3 Theorem 29] or [1 Theorem 2.3.1]), so choose one such prime \(w\) which is not a divisor of \(p\). Since \([K_w : \mathbb{Q}_p] = p - 1 \geq 11\), it follows from Lemma 5.8 that \(G\) is also realizable over \(K_w\) and furthermore there is a \(G\)-extension \(L_{0w}/K_w\) for which \(M_u \cap L_{0w} = K_u\).

By Theorems 6.4(b) and 2.5 of [7] (see also [4 Proposition 1.2.13]), there is a \(G\)-extension \(L_0/K\) for which \(\text{Gal}(L_0/K) = G\) and \((L_0)_u = L_{0w}\). Hence \(L_0\) is \(K\)-adequate. Let \(L = L_0M\). As \((M_u \cap L_{0w}) = K_u\) we have \(\text{Gal}(L_v/M_v) = G\) for \(i = 1, 2\). Thus \(L/M\) is an \(M\)-adequate \(G\)-extension and \((K, M, G)\) satisfies Condition (7).

By Remark 3.5, as \(M/K\) is Galois, (7) \(\implies\) (2). Thus, \((K, M, G)\) also satisfies (2). This concludes the proof of Example 5.7.
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