A singularity-free WEC-respecting time machine

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Abstract
A time machine (TM) is constructed whose creating in contrast to all TMs known so far requires neither singularities, nor violation of the weak energy condition (WEC). The spacetime exterior to the TM closely resembles the Friedmann universe.

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1 Introduction
This paper concerns an aspect of the long-standing question: How to create a time machine (or why is it impossible)? We begin with the following

Definition. Let $N$ be an inextendible acausal spacetime. We call $L_N \subset N$ a time machine (created in the universe $M$) if

1. $L_N$ comprises the causality violating set $V$:

   $L_N \supset V \equiv \{ P | P \in N, J^+(P) \cap J^-(P) \neq P \}$

2. $N \setminus J^+(L_N)$ is isometric to $M \setminus J^+(L_M)$, where $M$ is some inextendible causal spacetime and $L_M \subset M$ is compact.

It is meant here that depending on whether we decide to make a time machine or not the world and our laboratory must be described by $N$ and $L_N$, or by $M$ and $L_M$, respectively. We require the compactness of $L_M$ to differentiate TMs, which supposedly can be built by some advanced civilization, and causality violations of a cosmological nature such as the Gödel universe or the Gott “time machine” [1].

A few important facts are known about time machines. Among them:

1. Time machines with compact $V$ (compactly generated TMs, CTMs) evolving from a noncompact partial Cauchy surface must violate WEC [2],

2. Creation of a CTM leads to singularity formation unless some energy condition (slightly stronger than WEC) is violated [1] [3].

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1 Moreover, it is not clear whether singularities can be avoided at all, even at the sacrifice of WEC. Absence of singularities has not been proven, as far as I know, for any of TMs considered so far.
Though strictly speaking these facts do not prove that CTMs are impossible altogether, they, at least, can be used as starting point for the search for a mechanism protecting causality from CTMs [2].

CTMs, however, only constitute a specific class of TMs. Noncompactly generated TMs (NTMs) seem to be every bit as interesting as CTMs. Sometimes (see e.g. [3]) they are barred from consideration on the basis that some unpredictable information could enter noncompact V from a singularity or from infinity. This is so indeed, but compactness does not eliminate this trouble (in fact, compactness does not even eliminate singularities as a possible source of unpredictable information, as in the Misner space). The creation of a time machine is inherently connected with a loss of predictability (cf. [4]). One inevitably risks meeting something unexpected (i.e. not fixed by the data on the initial surface) as soon as one intersects a Cauchy horizon (by the very definition of the horizon). So, CTMs and NTMs are not too different in this regard.

It therefore seems important to find out whether there are any similar obstacles to creating NTMs. The example of the Deutsch-Politzer TM [5] showed that item 1 is not true in the case of NTMs. This TM, however, possesses singularities (and though mild, they are of such a nature that one cannot “smooth” them out [6]). Thus the following questions remained unanswered:

1. Do time machines without singularities exist?
2. Are the weak energy condition and the absence of singularities mutually exclusive for (noncompactly generated) time machines?

Our aim in this paper is to give the answers to both these questions (positive to the first and negative to the second). We make no attempt to discuss possible consequences of these answers. In particular, being interested only in the very existence of the desired TM we consider the fact that it may be created (see the Definition) in the spacetime resembling the Friedmann universe, only as a pleasant surprise.

2 Construction of the time machine

To construct a singularity-free TM it would be natural to start from the Deutsch-Politzer TM and to look for an appropriate conformal transformation of its metric which would move the dangerous points to infinity. However, it is not easy in the four-dimensional case to find a transformation (if it exists) yielding both WEC fulfilment and b- (or BA-) completeness. So, we shall use somewhat different means [7]. First, by a conformal transformation we make a part of the two-dimensional Deutsch-Politzer spacetime locally complete (the spacetime $Q_N$ below), then compose $L_N$ obeying WEC from $Q_N$ and some $S$ (chosen so that it does not spoil the completeness), and finally embed the resulting TM in an appropriate $N$.

2.1 Curved Deutsch-Politzer time machine $Q_N$

Let $Q_f$ be a square $q_m$

$$q_m \equiv \{\chi, \tau \mid m > |\chi|, |\tau|\}$$ (1)
endowed with the following metric:

\[ ds^2 = f^{-2}(\tau, \chi)(-d\tau^2 + d\chi^2) \]  

(2)

Here \( f \) is a smooth bounded function defined on \( \mathbb{R}^2 \supset q_m \) such that

\[ f(\tau, \chi) = 0 \iff \{ \tau = \pm h, \chi = \pm h \} \]  

(3)

with \( 0 < h < m/2 \). The four points \( f^{-1}(0) \) bound two segments

\[ l_{\pm} \equiv \{ \chi, \tau \mid \tau = \pm h, |\chi| < h \} \]

and we require that

\[ f(\tau, \chi)|_{U^+} = f(\tau - 2h, \chi), \]  

(4)

where \( U^+ \) is some neighborhood of \( l^+ \).

Now (as is done with the Minkowski plane in the case of the “usual” Deutsch-Politzer spacetime) remove the points \( f^{-1}(0) \) from \( q_m \), make cuts along \( l^+ \) and \( l^- \), and glue the upper bank of each cut with the lower bank of the other (see Fig. 1). The resulting spacetime \( Q_N \) (“curved Deutsch-Politzer TM”) is not, of course, diffeomorphic to \( Q_f \). Nevertheless, for simplicity of notation we shall continue to use the “old coordinates” \( \tau, \chi \) for its points.

### 2.2 The time machine \( L_N \)

Let \( S \) be a two-sphere with the standard metric:

\[ ds^2 = R^2(d\theta^2 + \sin^2 \theta d\varphi^2) \]  

(5)

where \( R \) is a constant satisfying (to bring about WEC fulfillment, see below)

\[ R^{-2} \geq \max_{\tilde{q}_m}(f(\chi, \chi) - f(\tau, \tau) + f_{\tau}^2 - f_{\chi}^2) \]  

(6)

Then

\[ L_N \equiv Q_N \times S \]  

(7)

is just the desired time machine.
2.3 Exemplary spacetimes $N, M$

To find an appropriate $N$ require in addition to (3, 4, 6)

$$f(P) = \text{const} \equiv f_0 \quad \text{when } P \notin q_m$$

and denote by $\tilde{Q}$ the spacetime obtained by replacing $q_m \rightarrow \mathbb{R}^2$ in the definition of $Q_N$. It follows from what is proven in the next section that the spacetime $\tilde{Q} \times S$ is inextendible and could thus be taken as $N$ (with, for example, $L_M \equiv Q_{f_0} \times S$). We would like, however, to construct another, more “realistic” $N$.

Consider a manifold $\mathbb{R}^1 \times S^3$ with the metric

$$ds^2 = a^2[-d\tau^2 + d\chi^2 + \rho^2(d\theta^2 + \sin^2 \theta d\varphi^2)]$$

(9)

Here $\tau$ is a coordinate on $\mathbb{R}^1$ and $\theta, \varphi, \chi (-\pi/2 \leq \chi \leq \pi/2)$ are polar coordinates on $S^3$. Impose the following conditions on $a, \rho$ (it suffices to choose $m < \pi/2$, $f_0 < R^{-1} \cos m$ for their feasibility):

on $q_m$ \quad $a = 1/f$, \quad $\rho = fR$ \quad (10a)

exterior to $q_m$ \quad $a = \hat{a}(\tau)$, \quad $\rho = \hat{\rho}(\chi)$, \quad (10b)

where $\hat{\rho}, \hat{a}$ are convex positive functions and for some $n \in (m, \pi/2)$ holds $\hat{\rho}|_{|\chi|>n} = \cos \chi$.

It is easy to see that the region $|\tau|, |\chi| < m$ of this manifold is $Q_f \times S$. So

![Figure 2: “Almost Friedmann” time machine. Shaded regions are parts of the Friedmann universe. The thick horizontal lines depict cosmological singularities.](image)

we can repeat the manipulations with cuts and obtain a TM with the metric (10a) on $N \setminus L_N$ (see Fig. 2), that is the TM is created in a spacetime with the metric of the Friedmann universe outside some spherical layer and some time interval.

3 Proofs

3.1 Weak energy condition

The metric of the time machine $L_N$ due to (10a) is given by (2, 5) and the condition (3) guarantees that the weak (and even the dominant) energy conditions hold there (see Fig. 2 for details).
In the outer space $N \setminus L_N$ the metric is given by (9). Introducing the quantities
\[ \Phi \equiv \Lambda \equiv \ln a, \quad r \equiv a \rho \]
we bring it to the form (14.49) of ref. [8]. The fact that by (10b) $a$ and $\rho$ are positive and convex gives us:
\[ \ddot{\Phi} \leq 0, \quad \rho''/\rho \leq 0, \quad 1 - \rho'^2 \geq 0, \] (11)
in the last inequality we have also used that $\rho'(\pm \pi/2) = \mp 1$.) Hence (see [8] for notation)
\[ E = \bar{E} = a^{-2} \ddot{\Phi} \leq 0, \quad H = 0, \]
\[ F = r^{-2}(1 - \rho'^2) + a^{-2} \ddot{\Phi}^2 \geq 0, \]
\[ \bar{F} = a^{-2}(\ddot{\Phi}^2 - \rho''/\rho) \geq 0 \]
So (see (14.52) of ref. [8]), WEC holds in this region too.

3.2 Completeness

The results of [7] prove that there are no “BA-singularities” in $L_N$, that is any timelike inextendible (in $N$) curve $\gamma \subset L_N$ with bounded acceleration has infinite proper length. There is a popular idea, however, that only $b$-complete regions may be accepted as singularity-free. So, the remainder of the article is devoted to the proof of the fact that $L_N$ (not the whole $N$, where cosmological singularities $\dot{a} = 0$ present) has no “$b$-singularities.” We shall use the following new notation:
\[ x^1 \equiv \tau, \quad x^2 \equiv \chi, \quad x^3 \equiv \theta, \quad x^4 \equiv \varphi, \]
\[ \alpha \equiv \chi + \tau, \quad \beta \equiv \chi - \tau, \quad \dot{} \equiv d/ds. \]

Let $\gamma(s) = x^i(s), \, i = 1, \ldots, 4$ be a $C^1$ curve in $L_N$. It defines two other curves (its projections onto $Q_N$ and $S$):
\[ Q_N \ni \gamma_Q(s) \equiv x^k(s), \quad k = 1, 2 \] (12)
\[ S \ni \gamma_S(s) \equiv x^j(s), \quad j = 3, 4 \] (13)

Lying in $L_N$ and $Q_N$ the curves $\gamma$ and $\gamma_Q$ can be considered at the same time as lying in $N$ and $Q$, respectively. We shall call such curves inextendible if they are inextendible in those “larger” spacetimes. Let $\{e_{(i)}(s)\}$ be an orthonormal basis in the point $\gamma(s)$, obtained from $\{e_{(i)}(0)\}$ by parallel propagating along $\gamma$ and $e_{(\alpha)} \equiv e_{(1)} + e_{(2)}, \quad e_{(\beta)} \equiv e_{(1)} - e_{(2)}$. Choosing $e_{(i)}(0) \sim \partial_s$ and solving the equations $\nabla_s e_{(i)} = 0$ one immediately finds
\[ \nabla_{\gamma_Q} e_{(k)} = 0, \quad \nabla_{\gamma_S} e_{(j)} = 0 \] (14)
and
\[ e_{(i)}^i(s) = e_{(i)}(0) \exp \left\{ 2 \int_0^s \dot{\phi}_{(\mu)} \mu ds' \right\} \quad (\mu \equiv \alpha, \beta) \] (15)
where $\phi \equiv \ln f$. 

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The “affine length” $\mu_N$ of $\gamma$ is by definition
\begin{equation}
\mu_N[\gamma] \equiv \int_0^1 \left( \sum_i \langle \dot{\gamma}, e_i(\gamma) \rangle^2 \right)^{1/2} ds
\end{equation}

We define affine lengths $\mu_Q[\gamma_Q]$ and $\mu_S[\gamma_S]$ by changing $i$ in (16) to $k$ and $j$, respectively. Due to (14) these definitions are consistent. Obviously,
\begin{equation}
\mu_N[\gamma] \geq 1/2 \left( \mu_Q[\gamma_Q] + \mu_S[\gamma_S] \right) \tag{17}
\end{equation}

**Proposition.** If $\gamma \subset L_N$ is inextendible then $\mu_N[\gamma] = \infty$.
If $\gamma$ is inextendible, than either $\gamma_S$ or $\gamma_Q$ (or both) are inextendible, too. But $S$ is obviously $b$-complete. So the Proposition follows from (17) coupled with the following

**Lemma.** If $\gamma_Q$ is inextendible, then $\mu_Q[\gamma_Q] = \infty$.
Let us introduce a function $\Phi$ on $\gamma_Q$: $\Phi \equiv \int_0^s (\phi, \dot{\alpha} - \phi, \dot{\beta}) ds'$ and let us split $\gamma_Q$ on segments $\gamma_n = \gamma[s_n, s_{n+1}]$ so that the sign of $\Phi$ does not change on $\gamma_n$: $\gamma_Q = \bigcup_n \gamma_n$, $\gamma_n : \Phi(s_n) = 0$, $\Phi(s_n < s < s_{n+1}) \leq 0$ (or $\geq 0$). \tag{18}

Denote the contribution of a segment $\gamma_n$ in $\mu_Q[\gamma_Q]$ by $\mu_n$. Since
\begin{equation}
(\langle \dot{\gamma}, e_{(1)} \rangle^2 + \langle \dot{\gamma}, e_{(2)} \rangle^2)^{1/2} \geq 1/2 \left( |\langle \dot{\gamma}, e_{(\alpha)} \rangle| + |\langle \dot{\gamma}, e_{(\beta)} \rangle| \right)
\end{equation}
we can write for $\mu_n$ (cf. 13):
\begin{equation}
\mu_n \geq C_1 \int_{s_n}^{s_{n+1}} f^{-2} \left[ |\dot{\alpha}| \exp \left( 2 \int_{s_n}^s \phi_{\beta} \dot{\beta} ds' \right) + |\dot{\beta}| \exp \left( 2 \int_{s_n}^s \phi_{\alpha} \dot{\alpha} ds' \right) \right] ds \tag{19}
\end{equation}

Here and subsequently we denote by $C_p$, $p = 1, \ldots$ some irrelevant positive constants factored out from the integrand. Using
\begin{equation}
[f(s)]^{-2} = [f(0)]^{-2} \exp \left( -2 \int_0^s (\phi_{\alpha} \dot{\alpha} + \phi_{\beta} \dot{\beta}) ds' \right) \tag{20}
\end{equation}
we can rewrite (13) as
\begin{equation}
\mu_n \geq C_2 \int_{s_n}^{s_{n+1}} \left( |\dot{\alpha}| \exp \left( -2 \int_{s_n}^s \phi_{\alpha} \dot{\alpha} ds' \right) + |\dot{\beta}| \exp \left( -2 \int_{s_n}^s \phi_{\beta} \dot{\beta} ds' \right) \right) ds \tag{21}
\end{equation}

For definiteness let $\Phi \leq 0$ on $\gamma_n$. Then the first exponent in (21) is greater than the second and we can replace it by their geometric mean, that is [see (20)] by
\[ f(0)/f(s) \text{. So,} \]
\[
\mu_n \geq C_3 \int_{s_n}^{s_{n+1}} |\dot{\alpha}/f| \, ds \geq C_4 \int_{s_n}^{s_{n+1}} |\phi,\alpha| \, ds \\
\geq -C_5 \int_{s_n}^{s_{n+1}} (\phi,\alpha + \phi,\beta) \, ds \geq C_5 \left( \phi(s_n) - \phi(s_{n+1}) \right) \quad (22) \]

(The third inequality follows again from \( \Phi \leq 0 \).) Clearly, (22) also holds for those \( \gamma_n \) where \( \Phi \geq 0 \) and hence summing over \( n \) gives:
\[
\mu_Q[\gamma_Q] \geq C_5 \left( \phi(0) - \phi(s) \right), \quad \forall s \in [0, 1) \quad (23) \]

There are no closed null geodesics in \( \tilde{Q} \). So, it is “locally complete” [10]. That is any inextendible \( \gamma_Q \) either has \( \mu_Q[\gamma_Q] = \infty \), or leaves any compact subset of \( \tilde{Q} \). But in the latter case (recall that \( \gamma_Q \in Q_N \) \( \phi(s) \) is unbounded below, which due to (23) again gives
\[ \mu_Q[\gamma_Q] = \infty. \]

\[ \square \]

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