Discrete Anomaly Matching
for the Pouliot Type Dualities

TERUHIKO KAWANO

Department of Physics, Kyoto Prefectural University of Medicine, Kyoto 606-0823, Japan

We compute the ’t Hooft anomalies of discrete symmetries in the Pouliot type dual theories and check their anomaly matching conditions. The Pouliot type dual theories we will consider in this paper are two dual pairs; the dual pair of $\mathcal{N} = 1$ supersymmetric theories of a $Spin(7)$ gauge theory with spinors and a $SU(N_f - 4)$ gauge theory with a symmetric tensor, fundamentals and singlets, and the other dual pair of $\mathcal{N} = 1$ supersymmetric theories of a $Spin(10)$ gauge theory with a spinor and vectors and a $SU(N_f - 5)$ gauge theory with a symmetric tensor, fundamentals and singlets. We will show that the both pairs satisfy the discrete anomaly matching conditions.

Contents

1 Introduction 1
2 The dual pair of the $Spin(7)$ theory with spinors 3
3 The dual pair of the $Spin(10)$ with a spinor and vectors 16
4 Discussions 22
A The continuous ’t Hooft anomaly matching of the extended theories of the $Spin(10)$ dual pair 25
1 Introduction

In this paper, we will study the ’t Hooft anomalies of independent discrete symmetries of two dual pairs, an $\mathcal{N} = 1$ supersymmetric $Spin(7)$ gauge theory with $N_f$ spinors and its dual $[11]$, and an $\mathcal{N} = 1$ supersymmetric $Spin(10)$ gauge theory with a spinor and $N_f$ vectors and its dual $[2, 3]$. We will show that the two dual pairs satisfy the ’t Hooft anomaly matching conditions of the independent discrete symmetries.

The concern that quantum gravitational effects might break global symmetries and the facts that discrete symmetries play important roles in phenomenological models have led to the idea of discrete gauge symmetries $[5]$, and further to the anomaly cancellation conditions of them $[6, 7, 8]$. They have been used to derive the anomaly matching conditions of discrete symmetries $[10, 11]$ by extending the discussions given by ’t Hooft $[12]$.

The continuous ’t Hooft anomaly matching conditions are necessary but severe conditions for possible low-energy theories of a strong coupling high-energy theory to satisfy. In particular, they yield very strong evidences for the Seiberg dualities $[13]$. The anomaly matching conditions for discrete symmetries may give more evidences for the conjectures, if a dual pair has independent discrete symmetries. In fact, in the papers $[10, 11]$, for many dual pairs, the discrete anomaly matching have been checked, and some of them do not pass the tests.

Therefore, it is significant to check the discrete anomaly matching for unchecked dual pairs. Among those, we will study the above two pairs, the $Spin(7)$ theory and its dual, and the $Spin(10)$ theory and its dual.

In section 2, we will discuss an independent discrete symmetry of the $Spin(7)$ theory and embed the discrete symmetry into an anomaly free $U(1)$ symmetry by introducing additional fields so as to define the discrete anomalies. We will repeat the same discussions for the dual $SU(N_f - 4)$ theory. In the magnetic theory, the transformation laws of the fields for the discrete symmetry is not uniquely determined. However, we will show that they all saturate the discrete ’t Hooft anomaly matching conditions given by the electric $Spin(7)$ theory. The dual pair of the $Spin(7)$ theory is obtained by the parent dual pair of an $\mathcal{N} = 1$ supersymmetric $Spin(8)$ theory with a spinor and $N_f$ vectors and its dual $[15]$ by higgsing the $Spin(8)$ gauge group by

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1 See $[4]$ for recent discussions.
2 See $[9]$ for recent discussions on discrete (non-)Abelian symmetries.
3 See $[14]$ for a recent excellent review.
the non-zero vacuum expectation value of the spinor. In the Spin(8) theory and the dual theory, there is an anomaly free U(1) global symmetry, and after the higgsing, it is broken into the discrete symmetry of the Spin(7) theory and its dual. The continuous ’t Hooft anomalies which the U(1) symmetry takes part in become the discrete anomalies, and therefore, the fact that the parent dual pair satisfies the continuous ’t Hooft anomaly matching conditions immediately implies that the Spin(7) and its dual satisfy the discrete anomaly matching. However, there is a subtlety upon defining the discrete anomalies from the continuous anomalies because we need to perform a gauge transformation as well to gain the discrete symmetry. We will elaborate on this issue and show that the dual theories surely satisfy the discrete anomaly matching conditions.

In section 3, we will proceed to the dual pair of the Spin(10) theory, and repeat the same procedure as done for the Spin(7) dual pair. We will embed the discrete symmetry into an anomaly free U(1) symmetry by adding fields to the theories and compute the anomalies. Although we can not uniquely determine the transformation laws of the magnetic fields, we will show that they all satisfy the discrete anomaly matching conditions, as for the Spin(7) dual theories. Contrary to the Spin(7) dual theories, there are no known parent dual pair, from which the Spin(10) dual pair can be derived. Therefore, we haven’t found the extended theories with the embedding U(1) symmetry of the dual pair, where all the continuous ’t Hooft anomaly matching conditions including those with the embedding U(1) symmetry are satisfied. However, in Appendix A we will construct extended theories of the Spin(10) dual pair, where the continuous ’t Hooft anomaly matching conditions which the embedding U(1) symmetry take part in are satisfied exactly. It means that the discrete anomaly matching is automatic after the higgsing of the embedding U(1) symmetries on the both sides. The rest of the continuous ’t Hooft anomaly matching are recovered after the decoupling of the additional fields.

Upon computations of anomalies, we will need to use the Dynkin index $T_G(R)$ of a representation $R$ of a group $G$ defined by

$$\text{tr} [T_R^a T_R^b] = T_G(R) \delta^{ab},$$

where $T_R^a (a = 1, \cdots, \dim G)$ are the generators of the group $G$ in the representation $R$. We often omit the subscript $G$ of the Dynkin index, when it is obvious. As for the normalization of $T(R)$, we take $T(\square) = 1$ for the fundamental representation $\square$ of $SU(N)$, and $T(\mathbf{N}) = 2$ for the vector representation of $Spin(N)$. They count the number of zero modes of a single fermion
in the representation of the group, when the one-instanton background is turned on. For the
adjoint representation of $Spin(N)$, we have $T(\text{adj}) = 2(N - 2)$. The properties of the spin
representations frequently depend on the parity of $N$ of $Spin(N)$. For the spin representataion
$2^{n-1}$ of both chiralities of $Spin(2n)$, we have $T(2^{n-1}) = 2^{n-3}$. On the other hand, for the spin
representation $2^n$ of $Spin(2n + 1)$, $T(2^n) = 2^{n-2}$.

2 The dual pair of the $Spin(7)$ theory with spinors

We will consider an $\mathcal{N} = 1$ supersymmetric $Spin(7)$ gauge theory with $N_f$ spinors $Q^i$ ($i = 1, \cdots, N_f$) with no superpotentials [1]. Besides the continuous global symmetries $SU(N_f) \times U(1)_R$, there is an discrete $Z_{2N_f}$ symmetry in the theory. Under the discrete symmetry transformation, the spinors $Q^i$ transform as

$$Q^i \rightarrow \exp\left(\frac{2\pi i}{2N_f}\right)Q^i.$$

The charge assignments of the spinors $Q^i$ are listed in Table 1. Performing the discrete trans-
formation twice, it gives a transformation given by an element of the center $Z_{N_f}$ of the
flavor $SU(N_f)$ symmetry. It implies that the subgroup $Z_{N_f} \subset Z_{2N_f}$ can be identified with the
center of the flavor group $SU(N_f)$. Therefore, we need to take the quotient of $Z_{2N_f}$ by the
subgroup $Z_{N_f}$ to find an independent discrete symmetry.

When $N_f$ is an odd integer, we have $Z_{2N_f} \simeq Z_2 \times Z_{N_f}$, where we may identify the cyclic
group $Z_{N_f}$ with the center of the flavor group $SU(N_f)$. The matter field $Q^i$ transforms into
$-Q^i$ under the remaining subgroup $Z_2$. We may identify it with a gauge transformation. In
fact, we may take the gamma matrices for the gauge group $Spin(7)$,

$$\begin{align*}
\gamma_1 &= \sigma_1 \otimes \sigma_1 \otimes \sigma_1, & \gamma_2 &= \sigma_2 \otimes \sigma_1 \otimes \sigma_1, \\
\gamma_3 &= \sigma_3 \otimes \sigma_1 \otimes \sigma_1, & \gamma_4 &= 1_2 \otimes \sigma_2 \otimes \sigma_1, \\
\gamma_5 &= 1_2 \otimes \sigma_3 \otimes \sigma_1, & \gamma_6 &= 1_2 \otimes 1_2 \otimes \sigma_2, \\
\gamma_7 &= i \gamma_1 \cdots \gamma_6 &= 1_2 \otimes 1_2 \otimes \sigma_3,
\end{align*}$$

Table 1: The charge assignments of the spinors $Q^i$ in the electric $Spin(7)$ theory
so that \( \gamma_{12} = i \sigma_3 \otimes 1_2 \otimes 1_2 \), which is one of the generators of the \( Spin(7) \). Under the gauge transformation generated by \( \gamma_{12} \), the matter fields \( Q^i \) transform as

\[
Q^i \mapsto \exp \left( \frac{1}{2} \phi \gamma_{12} \right) Q^i = \begin{pmatrix} e^{\frac{i}{2} \phi} 1_4 \\ e^{-\frac{i}{2} \phi} 1_4 \end{pmatrix} Q^i.
\]

When \( \phi = 2\pi \), we see that the gauge transformation yields \( Q^i \mapsto -Q^i \). Therefore, we have found that the discrete symmetry \( Z_{2N_f} \) is a subgroup of \( Spin(7) \times SU(N_f) \), when \( N_f \) is an odd integer. It implies that there are no independent discrete symmetries for odd \( N_f \).

We will compute the discrete ’t Hooft anomalies below, whichever the \( Z_{2N_f} \) symmetry is independent or not. As is explained in [10], in order to compute Type II discrete anomalies \( Z_{2N_f}^3, Z_{2N_f} U(1)_{R}, Z_{2N_f}^2 U(1)_{R} \), we need to promote the discrete \( Z_{2N_f} \) symmetry to a anomaly-free continuous \( U(1) \) symmetry. To this end, we will extend the theory by introducing a singlet \( \Phi \) and a spinor \( P \) of the gauge group \( Spin(7) \). Under the promoted \( U(1) \) symmetry transformation, the fields transform as

\[
Q^i \to \exp (i\omega) Q^i, \quad \Phi \to \exp (i2N_f \omega) \Phi, \quad P \to \exp (-iN_f \omega) P,
\]

and \( \Phi \) carries the \( U(1)_R \) charge 0, and \( P \) the \( U(1)_R \) charge one. Then, the \( U(1)_R \) charge assignment for the spinors \( Q^i \) may be kept intact and the \( U(1)_R \) symmetry are still anomaly free. By inspection, we can verify that the promoted \( U(1) \) symmetry is also anomaly free. We will turn on the superpotential \( \Phi PP \). When we will promote the \( U(1) \) symmetry to a gauge symmetry by introducing a \( U(1) \) gauge superfield and turn on the Fayet-Iliopoulos term in the \( D \)-term potential of the gauged \( U(1) \) symmetry, we will find the vacuum \( \langle \Phi \rangle \neq 0 \), where the \( U(1) \) symmetry is broken into the discrete \( Z_{2N_f} \) symmetry. The spinor \( P \) will gain a mass through the superpotential \( \langle \Phi \rangle PP \) and decouple from the rest of the theory in the infrared.

We can utilize the \( U(1) \) gauge superfield to compute the Type II discrete anomalies \( Z_{2N_f}^3, Z_{2N_f} U(1)_{R}, Z_{2N_f}^2 U(1)_{R} \) by calculating the ’t Hooft anomalies \( U(1)^3, U(1)U(1)^2_R, U(1)^2 U(1)_{R} \). For the computations, we will also multiply the \( U(1)_R \) charges by \( N_f \) to make them integers.

Before proceeding to the computations, we notice that the field with the discrete charge \( q \) in the representation \( R \) of the flavor \( SU(N_f) \) group and in the representation \( R_g \) of the gauge \( Spin(7) \) group contributes to the discrete anomalies except for the \( Z_{2N_f} SU(N_f)^2 \) anomaly by a mutiple of \( q \dim R \dim R_g \). We see from Table [2] that the combination \( q \dim R \dim R_g \) is a multiple of \( 2N_f \). Since we count the discrete anomalies modulo \( 2N_f \), all the discrete anomalies...
Table 2: The charge assignments of the fields in the extended $\text{Spin}(7)$ theory

|                | $\text{Spin}(7)$ | $SU(N_f)$ | $U(1)_R$ | $\mathbb{Z}_{2N_f}$ |
|----------------|-------------------|-----------|----------|---------------------|
| $Q^i$          | 8                 |           | $1 - 5/N_f$ | 1                   |
| $\Phi$         | 1                 | 1         | 0        | $2N_f$              |
| $P$            | 8                 | 1         | 1        | $-N_f$              |

except for the $\mathbb{Z}_{2N_f}SU(N_f)^2$ should be zero modulo $2N_f$. In fact, our computations give the discrete 't Hooft anomalies,

- $\mathbb{Z}_{2N_f}SU(N_f)^2$: 8,
- $\mathbb{Z}_{2N_f}(\text{gravity})^2$: $1 \times 8N_f + 2N_f \times 1 + (N_f) \times 8 = 2N_f$
- $\mathbb{Z}^3_{2N_f}$: $(1)^3 \times 8N_f + (2N_f)^3 \times 1 + (N_f)^3 \times 8 = 4 \times 2N_f$
- $\mathbb{Z}_{2N_f}U(1)_R^2$: $1 \times (-5)^2 \times 8N_f + 2N_f \times (-N_f)^2 = (N_f^2 + 100) \times 2N_f$
- $\mathbb{Z}^2_{2N_f}U(1)_R$: $(1)^2 \times (N_f^2)(8N_f^2) \times (-5) = (100 + 2N_f^2) \times 2N_f$.

Note that we can introduce a vector $P'$ of the gauge group $\text{Spin}(7)$ instead of the spinor $P$ to cancel the $U(1)$ gauge anomaly and add the term $\Phi P'P'$ to the superpotential, which becomes the Majorana mass term after the $U(1)$ symmetry breaking $\langle \Phi \rangle \neq 0$. Since the vector $P'$ carries the $U(1)_R$ charge 1, it contributes only to the discrete 't Hooft anomalies $\mathbb{Z}_{2N_f}(\text{gravity})^2$ and $\mathbb{Z}^3_{2N_f}$. Its contributions increase the anomalies $\mathbb{Z}_{2N_f}(\text{gravity})^2$ and $\mathbb{Z}^3_{2N_f}$ computed for the spinor $P$ by $N_f$ and $N^3_f$, respectively.

When $N_f = 6$, the $\text{Spin}(7)$ theory is in the confining phase without chiral symmetry breaking $\mathbb{1}$, and the low-energy physics is described by the mesons $M^{ij} \sim Q^i Q^j$ and the baryons $B \sim Q^4$ with the superpotential

$$\det M - M^{ik}M^{jl}B_{ij}B_{kl} - \text{Pf}B.$$  \(3\)

In order to embed the discrete symmetry into an anomaly free $U(1)$ symmetry, we will introduce a singlet $X$, as listed in Table 3 and replace the superpotential (3) by

$$X \left( \det M - M^{ik}M^{jl}B_{ij}B_{kl} - \text{Pf}B \right).$$

Upon the $U(1)$ symmetry breaking to the discrete symmetry by $\langle X \rangle \neq 0$, rescaling the mesons $M^{ij}$ and the baryons $B_{ij}$, it is reduced into the original low energy theory of $M^{ij}$ and $B_{ij}$. Using
Table 3: The low-energy theory extended for $N_f = 6$

|      | $SU(N_f = 6)$ | $U(1)_R$ | $Z_{2N_f}$ |
|------|---------------|-----------|-------------|
| $M^{ij}$ | □            | 2 − $5/3$ | 2           |
| $B$    | □            | 4 − $10/3$ | 4           |
| $X$    | 1            | 0         | $-12$       |

The fields $M^{ij}$, $B_{ij}$ and $X$, we find that the discrete ’t Hooft anomalies are given by

- $Z_{2N_f}SU(N_f)^2$: $2 \times 8 + 4 \times 4 = 8 + 2 \times 12$,
- $Z_{2N_f}(\text{gravity})^2$: $2 \times \frac{6 \times 7}{2} + 4 \times \frac{6 \times 5}{2} - 12 = 6 + 7 \times 12$,
- $Z_{2N_f}^3$: $2^3 \times \frac{6 \times 7}{2} + 4^3 \times \frac{6 \times 5}{2} + (-12)^3 = -50 \times 12$, \quad ($6^3 = 18 \times 12$),
- $Z_{2N_f}U(1)_R^2$: $2 \times (-4)^2 \times \frac{6 \times 7}{2} + 4 \times (-2)^2 \times \frac{6 \times 5}{2} - 12 \times (-1)^2 = 75 \times 12$,
- $Z_{2N_f}^2U(1)_R$: $(2)^2 \times (-4) \times \frac{6 \times 7}{2} + (4)^2 \times (-2) \times \frac{6 \times 5}{2} + (-12)^2 \times (-1) = -80 \times 12$.

They saturate the discrete ’t Hooft anomalies of the $Spin(7)$ theory extended by the vector $P'$ instead of the spinor $P$. Thus, the low-energy theory of the meson $M^{ij}$ and the baryons $B$ passes the discrete ’t Hooft anomaly matching tests.

The dual of the $Spin(7)$ theory for $7 \leq N_f \leq 14$ is an $\mathcal{N} = 1$ supersymmetric $SU(N_f - 4)$ gauge theory with a $s$ in □, $N_f \tilde{q}_i$ in □, and singlets $M^{ij}$ with the superpotential

$$M^{ij}\tilde{q}_i \cdot s \cdot \tilde{q}_j + \det s.$$  

The charge assignments of the fields are listed in Table 4

|      | $SU(N_f - 4)$ | $SU(N_f)$ | $U(1)_R$ |
|------|---------------|-----------|-----------|
| $s$  | □            | 1         | $2/(N_f - 4)$ |
| $\tilde{q}_i$ | □         | □         | $5/N_f - 1/(N_f - 4)$ |
| $M^{ij}$ | 1          | □         | $2 - 10/N_f$ |

Table 4: The field content of the dual of the $Spin(7)$ theory

The correspondence of the gauge invariant operators $M^{ij} \sim Q^i \cdot Q^j$, $Q^4 \sim \tilde{q}^{N_f - 4}$ and the
invariance of the superpotential determine the transformation laws of the discrete symmetry

\[ M^{ij} \rightarrow \exp \left( 2\pi i \frac{1}{N_f} \right) M^{ij}, \quad \tilde{q}_i \rightarrow \exp \left( 2\pi i \frac{2 + Nfp}{N_f(N_f - 4)} \right) \tilde{q}_i, \quad s \rightarrow \exp \left( -2\pi i \frac{1 + 2p}{N_f - 4} \right) s, \]

up to an integer \( p \in \mathbb{Z} \). The anomaly free condition of the discrete symmetry by the gauge interaction is

\[ \frac{2 + Nfp}{N_f(N_f - 4)} \times N_f - \frac{1 + 2p}{N_f - 4} \times (N_f - 2) = -(p + 1) \in \mathbb{Z}. \]

The anomaly free condition for the discrete symmetry is obviously satisfied for any integer \( p \), and it is sufficient for computations of Type I discrete ’t Hooft anomalies, but, in order to compute Type II discrete anomalies, we need to embed the discrete group into an anomaly free \( U(1) \) group, as is done in the electric theory. To this end, we set \( p = -1 \) so that the gauge anomaly for the discrete symmetry is strictly vanishing\(^4\). We thus find the transformation laws of the discrete symmetry

\[ \tilde{q}_i \rightarrow \exp \left( -2\pi i \frac{N_f - 2}{N_f(N_f - 4)} \right) \tilde{q}_i, \quad s \rightarrow \exp \left( \frac{2\pi i}{N_f - 4} \right) s, \quad M^{ij} \rightarrow \exp \left( \frac{2\pi i}{N_f} \right) M^{ij}. \]

We see that the discrete symmetry group is a subgroup of the cyclic group \( \mathbb{Z}_{N_f(N_f - 4)} \).

Looking at the exponent of the transformed \( \tilde{q}_i \),

\[ - \frac{N_f - 2}{N_f(N_f - 4)} = - \frac{1}{2N_f} - \frac{1}{2(N_f - 4)}, \]

when we perform the same transformation twice, we see that the resulting transformation can be given by an element of the center of the flavor group \( SU(N_f) \) and an element of the center of the gauge group \( SU(N_f - 4) \). Therefore, it is not an independent discrete symmetry. We have seen that this is also the case for the discrete \( \mathbb{Z}_{2N_f} \) symmetry in the electric theory. In this sense, the independent discrete symmetries on the both side of the duality are given by the quotient group \( \mathbb{Z}_2 \). When \( N_f \) is an odd integer, we have seen that the quotient group \( \mathbb{Z}_2 \) is given by the center of the gauge group \( Spin(7) \) in the electric theory, and therefore, no independent discrete symmetry in the electric theory are found. In the magnetic theory, when \( N_f \) is an odd integer, \( i.e., N_f = 2k + 1 \ (k \in \mathbb{Z}) \), the exponent of the transformed \( \tilde{q}_i \) may be rewritten into

\[ - \frac{N_f - 2}{N_f(N_f - 4)} = \frac{k}{N_f} - \frac{k - 1}{N_f - 4}, \]

\(^4\)This is the simplest solution for embedding of the discrete symmetry into a \( U(1) \) symmetry. In order to cancel the gauge anomaly, we may introduce an appropriate set of massive fields in non-trivial representations of the gauge group, as we will see below.
and also we may rewrite the exponents of the other transformed fields into

\[
\exp \left( 2\pi i \frac{1}{N_f - 4} \right) s = \exp \left( 2\pi i \frac{N_f - 3}{N_f - 4} \right) s = \exp \left( 2\pi i \frac{2(k - 1)}{N_f - 4} \right) s,
\]

\[
\exp \left( 2\pi i \frac{1}{N_f} \right) M^{ij} = \exp \left( 2\pi i \frac{-N_f + 1}{N_f} \right) M^{ij} = \exp \left( 2\pi i \frac{-2k}{N_f} \right) M^{ij}.
\]

We find that the transformation laws of the discrete symmetry can be given by an element of the center of the flavor group \(SU(N_f)\) and an element of the center of the gauge group \(SU(N_f - 4)\). Therefore, there is no independent discrete symmetry in the magnetic theory as well for odd \(N_f\).

This can be understood in a different way. For odd \(N_f\), \(N_f\) and \(N_f - 4\) are relatively prime to each other\(^5\). In the same way, \(N_f\) and \(N_f - 2\) are also relatively prime to each other, and shifting \(N_f\) by \(-2\), we see that \(N_f - 2\) and \(N_f - 4\) are relatively prime to each other as well. Therefore, since \(N_f - 2\) is prime to \(N_f(N_f - 4)\), the discrete symmetry is a cyclic group \(\mathbb{Z}_{N_f(N_f - 4)}\). Furthermore, as \(N_f\) and \(N_f - 4\) are relatively prime to each other, the group \(\mathbb{Z}_{N_f(N_f - 4)}\) is isomorphic to \(\mathbb{Z}_{N_f} \times \mathbb{Z}_{N_f - 4}\), the product group of the center \(\mathbb{Z}_{N_f}\) of the flavor group \(SU(N_f)\) and the center \(\mathbb{Z}_{N_f - 4}\) of the gauge group \(SU(N_f - 4)\).

Although there are no independent discrete symmetries for odd \(N_f\), we will treat the cases for both odd and even \(N_f\) on the same footing. We will now promote the discrete \(\mathbb{Z}_{N_f(N_f - 4)}\) symmetry to an anomaly free \(U(1)\) symmetry by introducing a singlet chiral superfield \(X\) and replacing the term \(\det s\) in the superpotential by \(X \det s\). We suppose that the singlet \(X\) carries no \(U(1)_R\) charge to leave the \(U(1)_R\) charge of the symmetric tensor \(s\) unchanged.

The anomaly free condition of the \(U(1)\) symmetry by the \(SU(N_f - 4)\) gauge interaction and the invariance of the superpotential suggest that the fields under the \(U(1)\) transformation should transform as

\[
M^{ij} \rightarrow e^{(N_f - 4)i\omega} M^{ij}, \quad \tilde{q}_i \rightarrow e^{-(N_f - 2)i\omega} \tilde{q}_i, \quad s \rightarrow e^{N_f i\omega} s, \quad X \rightarrow e^{-N_f(N_f - 4)i\omega} X,
\]

with a transformation parameter \(\omega\). Note that the \(U(1)\) transformation is distinct from the \(U(1)\) transformation in the electric theory, as we can verify from the gauge invariant operators

\(^5\)When \(N_f\) is odd, it is obvious that \(N_f\) is prime to \(2^m\), for a positive integer \(m\). It means that there are two integers \(p, q\) satisfying \(N_f p + 2^m q = 1\). Rewriting it into \(N_f(p + q) + (N_f - 2^m)(-q) = 1\), we see that \(N_f\) and \(N_f - 2^m\) are relatively prime to each other.
$M^{ij} \sim Q^i Q^j$, $Q^4 \sim \tilde{q}^{N_f-4}$. However, the reason that we embed the discrete symmetry into an anomaly free $U(1)$ symmetry is just to define Type II discrete 't Hooft anomalies, but not to find the dual theory of the extended electric theory with the extra $U(1)$ symmetry.

Let us introduce a $U(1)$ gauge superfield to promote the $U(1)$ symmetry to a gauge symmetry and introduce the Fayet-Iliopoulos term in the $D$-term potential of the $U(1)$ gauge symmetry so that there exists a vacuum $\langle X \rangle \neq 0$, where the $U(1)$ symmetry is broken into the original discrete $Z_{N_f(N_f-4)}$ symmetry. As was done for the electric theory, we can make use of the background $U(1)$ gauge field to compute Type II discrete anomalies $Z_{2N_f}^3$, $Z_{2N_f} U(1)_R^2$, $Z_{2N_f}^2 U(1)_R$ by computing the 't Hooft anomalies $U(1)_3$, $U(1)U(1)_R^2$, $U(1)_2 U(1)_R$. We will multiply the $U(1)_R$ charges by $N_f$. However, the $U(1)_R$ charges are not all integers even after multiplying them by $N_f$, contrary to the electric theory.

|        | $SU(N_f-4)$ | $SU(N_f)$ | $U(1)_R$ | $Z_{N_f(N_f-4)}$ |
|--------|------------|-----------|----------|------------------|
| $s$    |            | 1         | 2/(N_f-4)| $N_f$            |
| $\tilde{q}_i$ |          |           | 5/N_f - 1/(N_f-4) | $-(N_f-2)$ |
| $M^{ij}$ | 1         |           | 2-10/N_f | $N_f-4$          |
| $X$    | 1         | 1         | 0        | $-N_f(N_f-4)$    |

Table 5: The field content in the extended magnetic theory

We compute the 't Hooft anomalies for the magnetic theory,

- $Z_{2N_f} SU(N_f)^2 : -(N_f-2) \times (N_f-4) + (N_f-4) \times (N_f+2) = 4(N_f-4)$,
- $Z_{2N_f}(\text{gravity})^2 : N_f \times \frac{(N_f-4)(N_f-3)}{2} + (-(N_f-2)) \times N_f \times (N_f-4) + (N_f-4) \times \frac{N_f(N_f+1)}{2} - N_f(N_f-4) = 0$,
- $Z_{2N_f}^3 : N_f^3 \times \frac{(N_f-4)(N_f-3)}{2} + (-(N_f-2))^3 \times N_f \times (N_f-4) + (N_f-4)^3 \times \frac{N_f(N_f+1)}{2} + (-(N_f(N_f-4))^3 = -(N_f^2-1)(N_f-4)^2 \times N_f(N_f-4)$,
\[ \bullet \mathbb{Z}_{2N_f} U(1)_R^2: \ N_f \times \frac{(N_f - 4)(N_f - 3)}{2} \times \left( -N_f + \frac{2N_f}{N_f - 4} \right)^2 \\
\quad + \left( -(N_f - 2) \right) \times N_f \times (N_f - 4) \times \left( -N_f + \frac{2N_f}{N_f - 4} \right)^2 \\
\quad + (N_f - 4) \times \frac{N_f(N_f + 1)}{2} \times (N_f - 10)^2 - N_f(N_f - 4) \times (-N_f)^2 \\
= -4 \left( N_f^2 - 25 \right) \times N_f(N_f - 4), \]

\[ \bullet \mathbb{Z}^2_{2N_f} U(1)_R: \ N_f^2 \times \frac{(N_f - 4)(N_f - 3)}{2} \times \left( -N_f + \frac{2N_f}{N_f - 4} \right) \\
\quad + \left( -(N_f - 2) \right)^2 \times N_f \times (N_f - 4) \times \left( -N_f + \frac{N_f}{N_f - 4} \right) \\
\quad + (N_f - 4)^2 \times \frac{N_f(N_f + 1)}{2} \times (N_f - 10) + (N_f(N_f - 4))^2 \times (-N_f) \\
= -2 \left( N_f^2 + 5 \right) (N_f - 4) \times N_f(N_f - 4). \]

In order to examine whether the dual theories satisfy the discrete ’t Hooft anomaly matching conditions, we will embed both of the discrete \( \mathbb{Z}_{2N_f} \) group in the electric theory and the discrete \( \mathbb{Z}_{N_f(N_f - 4)} \) group into a \( \mathbb{Z}_{2N_f(N_f - 4)} \) group by multiplying the discrete charges in the electric theory by \( N_f - 4 \) and those in the magnetic theory by 2, respectively. Then, we find that the \( \mathbb{Z}_{2N_f(N_f - 4)} SU(N_f)^2 \) anomaly in the electric theory exactly matches the one in the magnetic theory. For the other discrete anomalies, we see that the magnetic theory saturates the discrete ’t Hooft anomaly matching conditions from the electric theory, modulo \( 2N_f(N_f - 4) \).

|                | \( SU(N_f - 4) \) | \( SU(N_f) \) | \( U(1)_R \) | \( \mathbb{Z}_{2N_f(N_f - 4)} \) |
|----------------|------------------|--------------|--------------|------------------|
| \( s \)       | [ ]              | 1            | 2/(N_f - 4)  | -2(2p + 1)N_f   |
| \( \tilde{q}_i \) | [ ]             | [ ]          | 5/N_f - 1/(N_f - 4) | 2(N_f p + 2)   |
| \( M^{ij} \)  | 1                | [ ]          | 2 - 10/N_f   | 2(N_f - 4)      |
| \( X \)       | 1                | 1            | 0            | 2(2p + 1)N_f(N_f - 4) |
| \( \tilde{X} \) | 1                | 1            | 0            | -2(p + 1)N_f(N_f - 4)   |
| \( F \)       | [ ]              | 1            | 1 + 1/(N_f - 4) | (p + 1)N_f(N_f - 4) - (1 + 2p)N_f |
| \( \tilde{F} \) | [ ]             | 1            | 1 - 1/(N_f - 4) | (p + 1)N_f(N_f - 4) + (1 + 2p)N_f |

Table 6: The field content in the extended magnetic theory for \( p \neq -1 \)
We have seen the ambiguity of the magnetic discrete symmetry by an integer $p$, and we have taken $p = -1$ as the simplest choice for an anomaly free $U(1)$ symmetry. We will here consider the case $p \neq -1$ and add additional matter fields for an anomaly free $U(1)$ symmetry. We read a $U(1)$ symmetry transformation for $p \neq -1$,

\begin{align*}
M^{ij} &\to e^{i(N_f - 4)\omega} M^{ij}, \\
\tilde{q}_i &\to e^{i(2 + N_f p)\omega} \tilde{q}_i, \\
s &\to e^{-i(2p + 1)N_f \omega} s, \\
X &\to e^{i(2p + 1)N_f (N_f - 1)\omega} X,
\end{align*}

with a transformation parameter $\omega$. Since the $U(1)$ symmetry is anomalous, we will add additional fields $\tilde{X}, F$ and $\tilde{F}$, listed in Table 6, to make it anomaly free. We will add the term $\tilde{X}F\tilde{F}$ to the superpotential so as to make them massive after the $U(1)$ symmetry breaking. Then, the discrete symmetry group is a subgroup of a cyclic group $\mathbb{Z}_{2N_f(N_f - 4)}$, due to the presence of the extra fundamental pair $F, \tilde{F}$. The discrete charges of the magnetic fields are listed in Table 6.

The introduction of a $U(1)$ gauge superfield to promote the $U(1)$ symmetry to a $U(1)$ gauge group is the same as for $p = -1$, and we will find the vacuum where $\langle X \rangle \neq 0, \langle \tilde{X} \rangle \neq 0$, breaking the $U(1)$ symmetry to the discrete $\mathbb{Z}_{2N_f(N_f - 4)}$ symmetry. Multiplying the $U(1)_R$ charges by $N_f$, we compute the discrete 't Hooft anomalies. We take the $U(1)_R$ charges of $F$ and $\tilde{F}$ to be the values in Table 6 in order for the $\mathbb{Z}_{2N_f(N_f - 4)} U(1)_R$ anomaly to be a multiple of $2N_f(N_f - 4)$, therefore, saturating the matching condition of the anomaly in the electric theory, modulo $2N_f(N_f - 4)$, as we will see soon.

To compare these anomalies with those in the electric theory, we will embed the electric $\mathbb{Z}_{2N_f}$ group into the $\mathbb{Z}_{2N_f(N_f - 4)}$ group by multiplying the discrete charges by $N_f - 4$, and we find that all the discrete 't Hooft anomalies of both electric and magnetic theories match modulo $2N_f(N_f - 4)$.

In the previous sections, we have promoted the discrete symmetries into anomaly free $U(1)$ symmetries by extending both of the dual theories to larger theories. However, those extended theories are not dual to each other. But, as we will see below, there is an ideal dual pair of extended theories, an $\mathcal{N} = 1$ supersymmetric $Spin(8)$ gauge theory with a spinor $S$ and $N_f$ vectors $Q^i (i = 1, \cdots, N_f)$ with no superpotential, and its dual theory. See Table 7 for the field content of the $Spin(8)$ theory.
Table 7: The $\text{Spin}(8)$ theory with a spinor and $N_f$ vectors

|       | $\text{Spin}(8)$ | $\text{SU}(N_f)$ | $U(1)$ | $U(1)_R$ |
|-------|------------------|------------------|--------|----------|
| $S$   | $8_s$            | 1                | $-N_f$ | 0        |
| $Q^i$ | $8_v$            | $\square$       | 1      | $1 - 5/N_f$ |

Although there are no independent discrete symmetries on the both sides of the duality, they have an anomaly free $U(1)$ symmetry. After higgsing the $\text{Spin}(8)$ to $\text{Spin}(7)$ by $\langle S \rangle \neq 0$, $N_f$ vectors $Q^i$ of $\text{Spin}(8)$ become $N_f$ spinors of $\text{Spin}(7)$. The dual of the $\text{Spin}(8)$ theory is reduced to the dual of the $\text{Spin}(7)$ theory. Then, the non-zero vacuum expectation value $\langle S \rangle \neq 0$ also breaks the anomaly free $U(1)$ symmetry to the discrete symmetry of the $\text{Spin}(7)$ theory. Before the higgsing of the $\text{Spin}(8)$ gauge group, all the continuous ’t Hooft anomaly matching conditions are satisfied by the $\text{Spin}(8)$ theory and its dual theory. Therefore, all the discrete ’t Hooft anomaly matching conditions should be satisfied by the $\text{Spin}(7)$ theory and its dual $\text{SU}(N_f - 4)$ theory. However, it is somewhat less trivial to confirm this, as we will see below.

The magnetic theory of the $\text{Spin}(8)$ theory is an $\mathcal{N} = 1$ supersymmetry $\text{SU}(N_f - 4)$ theory with a symmetric tensor $s$, $N_f$ antifundamentals $\tilde{q}_i$ ($i = 1, \cdots, N_f$), and singlets $M^{ij}$, $X$ with the superpotential

$$M^{ij} \tilde{q}_i \cdot s \cdot \tilde{q}_j + X \det s.$$  

The singlets $M^{ij}$, $X$ correspond to the gauge invariant operators of the electric theory, $M^{ij} \sim Q^i \cdot Q^j$, $X \sim S \cdot S$. There is also another gauge invariant operator $B \sim S^2 Q^4 \sim \tilde{q}^{N_f - 4}$. See Table 8 for the field content of the magnetic theory.

The magnetic theory reminds us of the extended dual theory of the $\text{Spin}(7)$ theory with $p = -1$. In order to go down to the dual theory of the $\text{Spin}(7)$ theory, we take the vacuum with $\langle X \rangle \neq 0$, which breaks the $U(1)$ symmetry to the discrete symmetry, under which the fields transform as

$$s \rightarrow \exp \left( \frac{2\pi i}{N_f - 4} \right) s, \quad \tilde{q}_i \rightarrow \exp \left[ -2\pi i \left( \frac{1}{2N_f} + \frac{1}{2(N_f - 4)} \right) \right] \tilde{q}_i,$$

$$M^{ij} \rightarrow \exp \left( \frac{2\pi i}{N_f} \right) M^{ij},$$

(4)

same $U(1)$ symmetry, the continuous ’t Hooft anomaly matching conditions are not satisfied.
|       | \(SU(N_f - 4)\) | \(SU(N_f)\) | \(U(1)\) | \(U(1)_R\) |
|-------|----------------|------------|---------|-------------|
| \(s\) | \[\square\]    | 1          | \(2N_f/(N_f - 4)\) | \(2/(N_f - 4)\) |
| \(\tilde{q}_i\) | \[\square\] | \[\square\] | \(-1 - N_f/(N_f - 4)\) | \(5/N_f - 1/(N_f - 4)\) |
| \(M^{ij}\) | 1          | \[\square\] | 2       | \(2 - 10/N_f\) |
| \(X\) | 1          | 1          | \(-2N_f\) | 0           |

Table 8: The field content of the dual of the \(Spin(8)\) theory

which is identical to the discrete symmetry for \(p = -1\). Rescaling \(s\), \(\tilde{q}_i\) and \(M^{ij}\), the superpotential is reduced into \(M^{ij}\tilde{q}_i \cdot s \cdot \tilde{q}_j + \det s\). Thus, we obtain the dual of the \(Spin(7)\) theory.

On the other hand, in the electric \(Spin(8)\) theory, the vacuum \(\langle X \rangle \neq 0\) corresponds to the vacuum \(\langle S \rangle \neq 0\) via the correspondence \(X \sim S^2\). Then, on the vacuum \(\langle S \rangle \neq 0\), the gauge group \(Spin(8)\) is broken to \(Spin(7)\). Since \(X\) and \(S\) transform under the global \(U(1)\) symmetry transformation as

\[
X \rightarrow \exp(-4\pi iN_f \omega)X, \quad S \rightarrow \exp(-2\pi iN_f \omega)S,
\]

with a transformation parameter \(\omega\), the discrete symmetry \([\square]\) corresponds to

\[
S \rightarrow \exp(-\pi i)S, \quad Q^i \rightarrow \exp\left(\frac{2\pi i}{2N_f}\right)Q^i. \tag{5}
\]

In order to leave the vacuum expectation value \(\langle S \rangle \neq 0\) invariant, we need to perform the gauge transformation

\[
S \rightarrow -S, \quad Q^i \rightarrow Q^i, \tag{6}
\]

at the same time, which is in a \(\mathbb{Z}_2\) subgroup of the center of the \(Spin(8)\) group. Let us take a closer look at the gauge transformation \([\square]\). To this end, we take the gamma matrices for the gauge group \(Spin(8)\)

\[
\Gamma_m = \gamma_m \otimes \sigma_1, \quad (m = 1, \cdots, 7), \quad \Gamma_8 = \mathbf{1}_8 \otimes \sigma_2
\]

and the chirality matrix \(\Gamma_9 = \Gamma_1 \Gamma_2 \cdots \Gamma_8 = \mathbf{1}_8 \otimes \sigma_3\), where \(\gamma_m (m = 1, \cdots, 7)\) are the gamma matrices of the \(Spin(7)\) group in \([\square]\). Let us take the spinor \(S\) to be of positive chirality.
Γ_9S = S. Then, the gauge transformation generated by Γ_{12} transforms the spinor S and the
vectors Q^i into

\[ S \rightarrow \begin{pmatrix} e^{\frac{i}{2}\phi} 1_4 \\ e^{-\frac{i}{2}\phi} 1_4 \end{pmatrix} S, \quad Q^i \rightarrow \begin{pmatrix} \cos \phi & -\sin \phi \\ \sin \phi & \cos \phi \end{pmatrix} \begin{pmatrix} 1_6 \end{pmatrix}, \tag{7} \]

and the Spin(8) gaugino λ into

\[ \lambda \rightarrow \begin{pmatrix} \cos \phi & -\sin \phi \\ \sin \phi & \cos \phi \end{pmatrix} \lambda \begin{pmatrix} \cos \phi & \sin \phi \\ -\sin \phi & \cos \phi \end{pmatrix} \begin{pmatrix} 1_6 \end{pmatrix}. \]

We may define the gauge transformation (6) as the limit φ → 2π, and then it is convenient
to decompose the representations of the Spin(8) group under Spin(2) × Spin(6), where the
Spin(2) group is generated by the Γ_{12}. The spinor S is decomposed into \( 4_{1/2} \oplus 4_{-1/2} \), and the
vectors Q^i into \( 1_1 \oplus 1_{-1} \oplus 6_0 \). The gaugino is decomposed into \( 1_0 \oplus 6_1 \oplus 6_{-1} \oplus 15_0 \). We may
regard the charges of the Spin(2) as additional discrete charges to accompany with the discrete
transformation (5) in order to leave \( \langle S \rangle \neq 0 \) invariant, a more refined definition of (6). The
discrete transformation (5) is anomaly free,

\[ -\frac{1}{2} \times T(8_s) + \frac{1}{2N_f} \times N_f \times T(8_v) = 0. \]

Although the naive definition (6) of the accompanying gauge transformation may be anomalous,
we can verify that the refined one keeps it anomaly free. For example, the spinor S contributes
to the gauge anomaly as

\[ \frac{1}{2} \times \left[ T_{Spin(6)}(4) + \left( \frac{1}{2} \right)^2 \right] - \frac{1}{2} \times \left[ T_{Spin(6)}(\bar{4}) + \left( -\frac{1}{2} \right)^2 \right] = 0, \]

with the Dynkin indices \( T_{Spin(6)}(4) \) and \( T_{Spin(6)}(\bar{4}) \) of the representations 4 and \( \bar{4} \), respectively,
of the Spin(6) group, where we have used the fact that the generators of the Spin(6) are
traceless. In the same way, we can show that the contributions from the vectors Q^i and the
gaugino λ to the gauge anomaly also vanish. The global symmetries \( SU(N_f) \times U(1)_R \times U(1) \) of
the Spin(8) theory are anomaly free, and the generators of the SU(N_f) group are traceless. The
Spin(2) charges are ‘traceless’ in each of the irreducible representations of the Spin(8) group.
They imply that the contributions from the Spin(2) charges, i.e., the accompanying gauge
transformation, to the discrete 't Hooft anomalies are canceled to give nothing. For example, the $\mathbb{Z}_{2N_f}^3$ anomaly is computed by $(U(1) + \text{Spin}(2))^3$, where $U(1)$ denotes the global $U(1)$ charges, which are the same as the discrete charges appearing in the discrete transformation \(5\) and $\text{Spin}(2)$ means the $\text{Spin}(2)$ charges of the accompanying gauge transformation. Then, $U(1)\text{Spin}(2)^2$, $U(1)^2\text{Spin}(2)$, and $\text{Spin}(2)^3$ are zero, and it becomes $U(1)^3$. Thus, we may ignore the $\text{Spin}(2)$ charges to compute the discrete anomalies, and therefore, the discrete 't Hooft anomalies of the $\text{Spin}(7)$ theory should be given by the continuous 't Hooft anomalies of the $\text{Spin}(8)$ theory. For the magnetic theories, there is no such subtleties to compute the 't Hooft anomalies, and the discrete 't Hooft anomalies of the dual of the $\text{Spin}(7)$ theory are also given by the continuous 't Hooft anomalies of the dual of the $\text{Spin}(8)$ theory. Since the continuous 't Hooft anomaly matching conditions are satisfied \([15]\) by the $\text{Spin}(8)$ theory and the dual $SU(N_f - 4)$ theory, the discrete 't Hooft anomaly matching conditions are obviously satisfied by the $\text{Spin}(7)$ theory and its dual theory. In fact, we can verify that the discrete 't Hooft matching conditions are satisfied by the direct computations

\[ \bullet \ Z_{2N_f}^{SU(N_f)^2} = U(1)SU(N_f)^2 \]

the electric theory: \( \frac{1}{2N_f} \times 8 = \frac{4}{N_f} \),

the magnetic theory: \(-\left( \frac{1}{2N_f} + \frac{1}{2(N_f - 4)} \right) \times (N_f - 4) + \frac{1}{N_f} \times (N_f + 2) = \frac{4}{N_f} \),

\[ \bullet \ Z_{2N_f}^{(\text{gravity})^2} = U(1)(\text{gravity})^2 \]

the electric theory: \(-\frac{1}{2} \times 8 + \frac{1}{2N_f} \times N_f \times 8 = 0, \)

the magnetic theory: \(\frac{1}{N_f - 4} \times \frac{(N_f - 4)(N_f - 3)}{2} - \left( \frac{1}{2N_f} + \frac{1}{2(N_f - 4)} \right) \times (N_f - 4) \times N_f + \frac{1}{N_f} \times \frac{N_f(N_f + 1)}{2} - 1 = 0, \)

\[ \bullet \ Z_{2N_f}^3 = U(1)^3 \]

the electric theory: \(\left( -\frac{1}{2} \right)^3 \times 8 + \left( \frac{1}{2N_f} \right)^3 \times N_f \times 8 = \frac{1}{N_f^2} - 1, \)

the magnetic theory: \(\left( \frac{1}{N_f - 4} \right)^3 \times \frac{(N_f - 4)(N_f - 3)}{2} - \left( \frac{1}{2N_f} + \frac{1}{2(N_f - 4)} \right)^3 \times (N_f - 4) \times N_f + \left( \frac{1}{N_f} \right)^3 \times \frac{N_f(N_f + 1)}{2} - 1 = \frac{1}{N_f^2} - 1. \)
\[ Z_{2N_f} U(1)_R^2 = U(1)^2 U(1)_R \]

the electric theory: \(- \frac{1}{2} \times (-1)^2 \times 8 + \frac{1}{2N_f} \times \left( -\frac{5}{N_f} \right)^2 \times N_f \times 8 = \frac{100}{N_f^2} - 4, \]

the magnetic theory: \[ \frac{1}{N_f - 4} \times \left( -1 + \frac{2}{N_f - 4} \right)^2 \times \left( \frac{N_f - 4}{2} \right)^2 \times \left( N_f - 4 \right) \times N_f \]
\[ - \left( -\frac{1}{2N_f} - \frac{1}{2(N_f - 4)} \right)^2 \times \left( -1 + \frac{5}{N_f} - \frac{1}{N_f - 4} \right)^2 \times \left( N_f - 4 \right) \times N_f \]
\[ + \frac{1}{N_f} \times \left( 1 - \frac{10}{N_f} \right)^2 \times \frac{N_f(N_f + 1)}{2} - 1 \times (-1)^2 = \frac{100}{N_f^2} - 4, \]

\[ Z_{2N_f}^2 U(1)_R = U(1)^2 U(1)_R \]

the electric theory: \[ \left( -\frac{1}{2} \right)^2 \times (-1) \times 8 + \left( \frac{1}{2N_f} \right)^2 \times \left( -\frac{5}{N_f} \right) \times N_f \times 8 = -\frac{10}{N_f^2} - 2, \]

the magnetic theory: \[ \left( \frac{1}{N_f - 4} \right)^2 \times \left( -1 + \frac{2}{N_f - 4} \right)^2 \times \left( \frac{N_f - 4}{2} \right)^2 \times \left( N_f - 4 \right) \times N_f \]
\[ + \left( -\frac{1}{2N_f} - \frac{1}{2(N_f - 4)} \right)^2 \times \left( -1 + \frac{5}{N_f} - \frac{1}{N_f - 4} \right)^2 \times \left( N_f - 4 \right) \times N_f \]
\[ + \left( \frac{1}{N_f} \right)^2 \times \left( 1 - \frac{10}{N_f} \right) \times \frac{N_f(N_f + 1)}{2} + (-1)^2 \times (-1) = -\frac{10}{N_f^2} - 2. \]

Under the gauge transformation (7), the gauge invariant operator \( S^2Q^4 \) is left invariant by definition. Therefore, under the discrete transformation (5), it transforms as
\[ S^2Q^4 \rightarrow \exp \left[ 2\pi i \left( \frac{4}{2N_f} - 1 \right) \right] S^2Q^4. \]

The contribution from the spinor \( S \) to the discrete charge of \( S^2Q^4 \) accounts for the discrepancy of the \( U(1) \) symmetries between the \( Spin(7) \) theory and its dual, upon the promotion of the discrete symmetries into anomaly free \( U(1) \) symmetries, when choosing \( p = -1 \), as we have seen in the previous sections.

In summary, we have seen that the discrete ’t Hooft anomaly matching between the \( Spin(7) \) theory and its dual can beautifully be shown by using the parent dual pair, the \( Spin(8) \) theory and its dual.

### 3 The dual pair of the \( Spin(10) \) with a spinor and vectors

We will consider an \( \mathcal{N} = 1 \) supersymmetric \( Spin(10) \) gauge theory with a single spinor \( S \) and \( N_f \) vectors \( Q^i \) \( (i = 1, \cdots, N_f) \) with no superpotentials and its dual theory[2, 3]. See Table
for the charge assignments of the matter fields in the electric $Spin(10)$ theory. There is an independent $\mathbb{Z}_{2N_f}$ symmetry in the theory. The discrete $\mathbb{Z}_{2N_f}$ symmetry transformation act on the vectors $Q^i$ and the spinor $S$ as

$$Q^i \rightarrow \exp\left(\frac{2\pi i}{2N_f}\right)Q^i, \quad S \rightarrow S,$$

which is anomaly free discrete symmetry. We will introduce a singlet $\Phi$ and a vector $P$ with the superpotential $\Phi PP$ in order to promote the discrete symmetry $\mathbb{Z}_{2N_f}$ to an anomaly free $U(1)$ symmetry for computations of Type II discrete ’t Hooft anomalies so that the fields $\Phi$, $P$ cancel the gauge anomaly of the promoted $U(1)$ symmetry by the $Spin(10)$ gauge interactions.

|       | $Spin(10)$ | $SU(N_f)$ | $U(1)$ | $U(1)_R$ | $\mathbb{Z}_{2N_f}$ |
|-------|------------|-----------|--------|---------|---------------------|
| $S$   | 16         | 1         | $-N_f$ | 1       | 0                   |
| $Q^i$ | 10         |           | 2      | $1 - 8/N_f$ | 1                   |
| $\Phi$| 1          | 1         | 0      | 0       | $2N_f$              |
| $P$   | 10         |           | 0      | 1       | $-N_f$              |

Table 9: The electric $Spin(10)$ theory with a spinor and $N_f$ vectors

In order to make the $U(1)_R$ charges integers, we will multiply them by $N_f$. Since the spinor $S$ does not give any contributions to the discrete anomalies, we may divide the $U(1)$ charges by two. Each of the fields gives a multiple of $q \dim R \dim R_g$ as its contribution to all the discrete anomalies except for $\mathbb{Z}_{2N_f}$, $SU(N_f)^2$, where $q$ is the discrete charge of the field, $\dim R$ is the dimension of its representation of the flavor symmetry group $SU(N_f)$ and $\dim R_g$ is the dimension of its representation of the gauge group $Spin(10)$. As we can see from Table 9, the quantity $q \dim R \dim R_g$ for each of all the fields is always a multiple of $2N_f$, and therefore, all the discrete anomalies except for $\mathbb{Z}_{2N_f}SU(N_f)^2$ are zero modulo $2N_f$. In fact, we compute the discrete ’t Hooft anomalies:

- $\mathbb{Z}_{2N_f}SU(N_f)^2 : 10,$
- $\mathbb{Z}_{2N_f}(\text{gravity})^2 : 1 \times 10N_f + 2N_f \times 1 + (-N_f) \times 10 = 2N_f,$
- $\mathbb{Z}_{2N_f}^2 : 1^3 \times 10N_f + (2N_f)^3 \times 1 + (-N_f)^3 \times 10 = (-2N_f^2 + 5) \times 2N_f,$
- $\mathbb{Z}_{2N_f}U(1)_R^2 : 1 \times (-8)^2 \times 10N_f + 2N_f \times (-N_f)^2 = (N_f^2 + 320) \times 2N_f,$
• \( \mathbb{Z}_{2N_f}U(1)^2 : 1 \times 1^2 \times 10N_f = 5 \times 2N_f, \)
• \( \mathbb{Z}_{2N_f}U(1)_RU(1) : 1 \times (-8) \times 1 \times 10N_f = -40 \times 2N_f, \)
• \( \mathbb{Z}_{2N_f}^2U(1)_R : 1^2 \times (-8) \times 10N_f + (2N_f)^2 \times (-N_f) = -(40 + 2N_f^2) \times 2N_f. \)
• \( \mathbb{Z}_{2N_f}^2U(1) : 1^2 \times 1 \times 10N_f = 5 \times 2N_f. \)

The magnetic theory exists for \( 7 \leq N_f \leq 21, \) and it is an \( \mathcal{N} = 1 \) supersymmetric \( SU(N_f-5) \) gauge theory with \( N_f \) antifundamentals \( \tilde{q}_i, \) a single fundamental \( q, \) a symmetric tensor \( s \) and singlets \( M^{ij}, Y^i, \) with the superpotential [2, 3]

\[
M^{ij} \tilde{q}_i \cdot s \cdot \tilde{q}_j + Y^i \tilde{q}_i \cdot q + \det s.
\]

The charge assignments for the magnetic fields are listed in Table 10. The gauge invariant

| \( SU(N_f - 5) \) | \( SU(N_f) \) | \( U(1) \) | \( U(1)_R \) | \( \mathbb{Z}_{2N_f(N_f-5)} \) |
|---|---|---|---|---|
| \( \tilde{q}_i \) | \( \square \) | \( \square \) | -2 | 8/N_f - 1/(N_f - 5) |
| \( q \) | \( \square \) | 1 | 2N_f | -1 + 1/(N_f - 5) |
| \( s \) | \( \square \) | 1 | 0 | 2/(N_f - 5) |
| \( M^{ij} \) | 1 | \( \square \) | 4 | 2 - 16/N_f |
| \( Y^i \) | 1 | \( \square \) | -2(N_f - 1) | 3 - 8/N_f |
| \( X \) | 1 | 1 | 0 | 0 |

Table 10: The magnetic theory of the \( Spin(10) \) with a spinor and vectors

operators \( M^{ij} \sim Q^i Q^j, Y^i \sim S^2 Q^i, B \sim S^2 Q^5 \sim \tilde{q}^{N_f-5} \) are transformed under the discrete symmetry transformation of the electric theory as

\[
M^{ij} \rightarrow \exp \left( \frac{2\pi i}{N_f} \right) M^{ij}, \quad Y^i \rightarrow \exp \left( \frac{2\pi i}{2N_f} \right) Y^i, \quad B \rightarrow \exp \left( \frac{2\pi i}{2N_f} \right) B,
\]

which, together with the invariance of the superpotential, determines the discrete symmetry transformation of the other magnetic fields

\[
\tilde{q}_i \rightarrow \exp \left[ 2\pi i \left( \frac{1 + 2p}{2(N_f - 5)} - \frac{1}{2N_f} \right) \right] \tilde{q}_i,
\]

\[
s \rightarrow \exp \left( -2\pi i \frac{1 + 2p}{N_f - 5} \right) s, \quad q \rightarrow \exp \left( -2\pi i \frac{1 + 2p}{2(N_f - 5)} \right) q,
\]

18
up to an integer $p$. The discrete symmetry transformations with different values of $p$ are related to one another by the gauge transformations given by elements of the center of the gauge group $SU(N_f - 5)$, and therefore, there exists only one independent discrete symmetry out of them.

Since our intention for the promotion of the discrete symmetry to an anomaly free $U(1)$ symmetry is just to define Type II discrete ’t Hooft anomalies in the magnetic theory, we do not have to take the $U(1)$ symmetry to be the promoted $U(1)$ symmetry of the electric theory through the correspondence of the gauge invariant operators. When we choose $p = -1$ and promote it to the $U(1)$ symmetry transformation of the magnetic theory

$$M^{ij} \rightarrow \exp \left( \frac{2\pi i}{N_f} \omega \right) M^{ij}, \quad Y^i \rightarrow \exp \left( \frac{2\pi i}{2N_f} \omega \right) Y^i, \quad \tilde{q}_i \rightarrow \exp \left( -2\pi i \frac{2N_f - 5}{2N_f(N_f - 5)} \omega \right) \tilde{q}_i,$$

$$s \rightarrow \exp \left( \frac{2\pi i}{N_f - 5} \omega \right) s, \quad q \rightarrow \exp \left( \frac{2\pi i}{2(N_f - 5)} \omega \right) q,$$

with a transformation parameter $\omega$, we find that the promoted $U(1)$ symmetry is anomaly free, and we need to introduce no more fields to cancel the gauge anomaly. However, the gauge invariant operator $B_{mag} \sim \tilde{q}^{N_f - 5}$ transforms under the magnetic $U(1)$ symmetry into the transform of $B_{ele} \sim S^2 Q^5$ under the electric $U(1)$ symmetry multiplied by $e^{-2\pi i\omega}$.

For the promotion to the anomaly free $U(1)$ symmetry, we will also replace the term $\det s$ in the superpotential by $X \det s$ with a singlet $X$, transforming under the promoted magnetic $U(1)$ symmetry as

$$X \rightarrow \exp (-2\pi i\omega) X.$$

In order to compute the Type II discrete ’t Hooft anomalies, we will introduce a $U(1)$ gauge superfield for the promoted magnetic $U(1)$ symmetry and will turn on the Fayet-Iliopoulos term in the $D$-term potential of the $U(1)$ gauge symmetry. Then, we will find a vacuum with $\langle X \rangle \neq 0$, which breaks the $U(1)$ symmetry down back to the original discrete symmetry in the magnetic theory. In the infrared, the theory is reduced into the original magnetic theory.

The discrete symmetry group is a subgroup of a cyclic group $\mathbb{Z}_{2N_f(N_f - 5)}$. However, when we perform the above discrete symmetry transformation twice, the resulting transformation can be given by an element of the center of the flavor group and an element of the center of the gauge group, and therefore, it is not an independent discrete symmetry, anymore.

As is done for the electric theory, we will multiply the $U(1)_R$ charges by $N_f$ and divide the $U(1)$ charges by two. Let $\Psi$ be one of the magnetic fields with the discrete charge $q$ in
the representation $R$ of the flavor symmetry group $SU(N_f)$ and in the representation $R_g$ of
the gauge symmetry group $SU(N_f - 5)$. The field $\Psi$ gives its contributions to all the discrete
't Hooft anomaly except for $\mathbb{Z}_{2N_f(N_f - 5)}SU(N_f)^2$ by a multiple of $q \dim R \dim R_g$. As we can
see from Table [10] the combination $q \dim R \dim R_g$ is a multiple of $N_f(N_f - 5)$, and therefore,
checking the discrete anomaly matching except for $\mathbb{Z}_{2N_f(N_f - 5)}SU(N_f)^2$ is to examine whether
it is an even or odd multiple of $N_f(N_f - 5)$ for the anomalies including no $U(1)_R$. Although the
$U(1)_R$ charges of the magnetic fields are not all integers, even after multiplying them by $N_f$, we
find that the anomalies including the $U(1)_R$ symmetry are also integers by the computations,

- $\mathbb{Z}_{2N_f(N_f - 5)}SU(N_f)^2$: 
  
  $$(-2N_f + 5) \times (N_f - 5) + 2(N_f - 5) \times (N_f + 2) + (N_f - 5)$$
  
  $$= 10(N_f - 5),$$

- $\mathbb{Z}_{2N_f(N_f - 5)}(\text{gravity})^2$: 
  
  $$(-2N_f + 5) \times N_f(N_f - 5) + N_f \times (N_f - 5)$$
  
  $$+ (2N_f) \times \frac{(N_f - 5)(N_f - 4)}{2} + 2(N_f - 5) \times \frac{N_f(N_f + 1)}{2}$$
  
  $$+ (N_f - 5) \times N_f + (-2N_f(N_f - 5)) = 2N_f(N_f - 5),$$

- $\mathbb{Z}_{2N_f(N_f - 5)}^3$: 
  
  $$(-2N_f + 5)^3 \times N_f(N_f - 5) + N_f^3 \times (N_f - 5) + (2N_f)^3 \times \frac{(N_f - 5)(N_f - 4)}{2}$$
  
  $$+ (2(N_f - 5))^3 \times \frac{N_f(N_f + 1)}{2} + (N_f - 5)^3 \times N_f + (-2N_f(N_f - 5))^3$$
  
  $$= -(N_f - 5)^2(4N_f^2 - 5) \times 2N_f(N_f - 5),$$

- $\mathbb{Z}_{2N_f(N_f - 5)}U(1)^2_R$: 
  
  $$(-2N_f + 5) \times \left(-N_f + 8 - \frac{N_f}{N_f - 5}\right)^2 \times N_f(N_f - 5)$$
  
  $$+ N_f \times \left(-2N_f + \frac{N_f}{N_f - 5}\right)^2 \times (N_f - 5)$$
  
  $$+ (2N_f) \times \left(-N_f + \frac{2N_f}{N_f - 5}\right)^2 \times \frac{(N_f - 5)(N_f - 4)}{2}$$
  
  $$+ 2(N_f - 5) \times (N_f - 16)^2 \times \frac{N_f(N_f + 1)}{2}$$
  
  $$+ (N_f - 5) \times (2N_f - 8)^2 \times N_f + (-2N_f(N_f - 5)) \times (-1)^2$$
  
  $$= 320 \times 2N_f(N_f - 5),$$

- $\mathbb{Z}_{2N_f(N_f - 5)}U(1)^2$: 
  
  $$(-2N_f + 5) \times (-1)^2 \times N_f(N_f - 5) + N_f \times (N_f)^2 \times (N_f - 5)$$
  
  $$+ 2(N_f - 5) \times 2^2 \times \frac{N_f(N_f + 1)}{2} + (N_f - 5) \times (-N_f - 1)^2 \times N_f$$
  
  $$= (N_f^2 + 5) \times 2N_f(N_f - 5),$$
\[ \begin{align*}
\mathbf{Z}_{2N_f(N_f-5)}U(1)_R U(1) : & \quad (-2N_f + 5) \times \left( -N_f + 8 - \frac{N_f}{N_f - 5} \right) \times (-1) \times N_f(N_f - 5) \\
& \quad + N_f \times \left( -2N_f + \frac{N_f}{N_f - 5} \right) \times N_f \times (N_f - 5) \\
& \quad + 2(N_f - 5) \times (N_f - 16) \times 2 \times \frac{N_f(N_f + 1)}{2} \\
& \quad + (N_f - 5) \times (2N_f - 8)^2 \times (-1) \times N_f \\
& = -2(N_f^2 + 20) \times 2N_f(N_f - 5),
\end{align*} \]

\[ \begin{align*}
\mathbf{Z}_{2N_f(N_f-5)}^2 U(1)_R : & \quad (-2N_f + 5)^2 \times \left( -N_f + 8 - \frac{N_f}{N_f - 5} \right) \times N_f(N_f - 5) \\
& \quad + N_f^2 \times \left( -2N_f + \frac{N_f}{N_f - 5} \right) \times (N_f - 5) \\
& \quad + (2N_f)^2 \times \left( -N_f + \frac{2N_f}{N_f - 5} \right) \times \frac{(N_f - 5)(N_f - 4)}{2} \\
& \quad + (2(N_f - 5))^2 \times (N_f - 16) \times \frac{N_f(N_f + 1)}{2} \\
& \quad + (N_f - 5)^2 \times (2N_f - 8) \times N_f + (-2N_f(N_f - 5))^2 \times (-1) \\
& = -4(N_f - 5)(N_f^2 + 10) \times 2N_f(N_f - 5),
\end{align*} \]

\[ \begin{align*}
\mathbf{Z}_{2N_f(N_f-5)}^2 U(1) : & \quad (-2N_f + 5)^2 \times (-1) \times N_f(N_f - 5) + N_f^2 \times (N_f) \times (N_f - 5) \\
& \quad + (2(N_f - 5))^2 \times 2 \times \frac{N_f(N_f + 1)}{2} + (N_f - 5)^2 \times (-1) \times N_f \\
& = 5(N_f - 5) \times 2N_f(N_f - 5).
\end{align*} \]

In order to check the discrete 't Hooft matching conditions between the dual theories, we will embed the \( \mathbf{Z}_{2N_f} \) symmetry group of the electric theory into the \( \mathbf{Z}_{2N_f(N_f-5)} \) group by multiplying the discrete charges of the electric fields by \( N_f - 5 \). Then, we can see that the \( \mathbf{Z}_{2N_f(N_f-5)}SU(N_f)^2 \) anomaly in the electric theory is in agreement with the one in the magnetic theory. Recalling that the other discrete anomalies in the electric theory are zero modulo \( 2N_f(N_f - 5) \), we find that the 't Hooft anomaly matching conditions of them are also satisfied by the magnetic theory.

Incidentally, let us consider the discrete 't Hooft anomalies for the other cases with \( p \neq -1 \). When \( n \equiv p+1 \neq 0 \), the invariance of the term \( X \) 's in the superpotential requires the discrete charge of \( X \) to be \( -2N_f(N_f - 5)(1 - 2n) \), as in Table \[11\]. Then, the promoted \( U(1) \) symmetry is anomalous by the \( SU(N_f - 5) \) gauge interactions, and therefore, we need to introduce more matter fields listed in Table \[11\] to cancel the gauge anomaly in order to promote the discrete \( \mathbf{Z}_{2N_f(N_f-5)} \) symmetry to an anomaly \( U(1) \) symmetry. We will also add the term \( \tilde{X}F\tilde{F} \) to the
superpotential so that the extra matter fields $\tilde{X}$, $F$, $\tilde{F}$ decouple in the vacuum $\langle \tilde{X} \rangle \neq 0$ at the low energies. The discrete charges of $F$ and $\tilde{F}$ are determined by cancellation of the gauge anomaly of the promoted $U(1)$ symmetry and by the requirement that performing the discrete symmetry transformation twice gives the gauge transformation of the element of the center of the gauge group $SU(N_f - 5)$. We have chosen the $U(1)_R$ charges of $F$ and $\tilde{F}$ so as to saturate the ’t Hooft anomaly matching condition for $Z_{2N_f(N_f - 5)}U(1)_R^2$.

|          | $SU(N_f - 5)$ | $SU(N_f)$ | $U(1)$ | $U(1)_R$ | $Z_{2N_f(N_f - 5)}$ |
|----------|---------------|------------|--------|----------|-------------------|
| $\tilde{X}$ | 1             | 1          | 0      | 0        | $-2N_f(N_f - 5)n$ |
| $F$      | $\Box$        | 1          | 0      | $1 + 1/(N_f - 5)$ | $nN_f(N_f - 5) + N_f$ |
| $\tilde{F}$ | $\Box$    | 1          | 0      | $1 - 1/(N_f - 5)$ | $nN_f(N_f - 5) - N_f$ |
| $X$      | 1             | 1          | 0      | 0        | $-2N_f(N_f - 5)(1 - 2n)$ |

Table 11: The discrete charge of $X$ and the additional fields $\tilde{X}$, $F$, $\tilde{F}$ for $p \neq -1$ in the magnetic dual of the $Spin(10)$ theory, where $n = p + 1$.

The remaining procedure we have to carry out for the computation of the discrete anomalies is almost the same as what was done for $p = -1$, except that we take the vacuum such that $\langle X \rangle \neq 0$, $\langle \tilde{X} \rangle \neq 0$.

The $Z_{2N_f(N_f - 5)}SU(N_f)^2$ anomaly for $p \neq -1$ is computed to give

- $Z_{2N_f(N_f - 5)}SU(N_f)^2 : 10(N_f - 5) + 2N_f(N_f - 5)n$,

which is equal modulo $2N_f(N_f - 5)$ to the one for the case $p = -1$ ($n = 0$). For the rest of the discrete anomalies, multiplying the $U(1)_R$ charges by $N_f$ and dividing the $U(1)$ charges by two, we find that they are all zero modulo $2N_f(N_f - 5)$, satisfying all the ’t Hooft anomaly matching conditions.

4 Discussions

We have studied the discrete anomaly matching of the two dual pairs. One of them is the $Spin(7)$ gauge theory with spinors and the $SU(N_f - 4)$ gauge theory with a symmetric tensor, fundamentals and singlets [11]. The other is the $Spin(10)$ gauge theory with a spinor and vectors.
and the $SU(N_f - 5)$ gauge theory with a symmetric tensor, fundamentals and singlets \cite{2,3}. We have shown that both of the dual pairs satisfy the discrete anomaly matching conditions.

For the dual pair of the $Spin(7)$ theory, we have done this in two ways. In one way, we have embedded the discrete symmetries into an anomaly free $U(1)$ symmetries by additional fields, which decouple after the $U(1)$ symmetry breaking into the discrete symmetries, on the both sides of the duality. The extended theories are not dual to each other, and we have to compute the continuous 't Hooft anomalies in order to ensure the anomaly matching conditions. In the other way, we take another dual pair \cite{15} of the $Spin(8)$ gauge theory with a spinor and vectors and the $SU(N_f - 4)$ gauge theory with a symmetric tensor, anti-fundamentals and singlets, which is reduced to the $Spin(7)$ dual pair by higgsing the $Spin(8)$ gauge group to $Spin(7)$. The $Spin(8)$ dual pair has an anomaly free $U(1)$ symmetry, which is broken to the discrete symmetries of the $Spin(7)$ dual pair upon the higgsing of $Spin(8)$. Since the continuous 't Hooft anomaly matching is satisfied by the $Spin(8)$ dual pair and since the discrete anomalies of the $Spin(7)$ dual pair are given by the continuous anomalies of the $Spin(8)$ dual pair, we have seen that the discrete anomaly matching conditions are obviously satisfied by the $Spin(7)$ dual pair, modulo subtleties with the gauge transformation involved in the actual computations.

For the $Spin(10)$ dual pair, we have no known parent dual pair and we have computed the discrete anomalies by embedding the discrete symmetries on the both sides of the duality into anomaly free $U(1)$ symmetries. In Appendix A, we have constructed the extended theories with the embedding $U(1)$ symmetries, which satisfy the continuous 't Hooft anomaly matching conditions for the anomalies with the $U(1)$ symmetries. Although the $U(1)$ symmetries of the both sides of the duality are different from each other, the continuous anomalies with the $U(1)$ symmetries become the discrete anomalies after the higgsing of the $U(1)$ symmetries, and we easily see that the discrete anomaly matching is achieved.

Another dual pair of an $\mathcal{N} = 1$ supersymmetric $G_2$ gauge theory with fundamentals in the representation 7 and an $\mathcal{N} = 1$ supersymmetric $SU(N_f - 3)$ gauge theory with a symmetric tensor, fundamentals and singlets is reduced from the $Spin(7)$ dual pair \cite{11} by higgsing the $Spin(7)$ gauge group. Therefore, it should be straightforward to check the discrete anomaly matching in a similar way to what we have done for the $Spin(8)$ dual pair. This is also the case for other Pouliot type dualities \cite{17} reduced from the $Spin(10)$ dual pair.

\footnote{See also \cite{10}, for the confining phases of the $G_2$ gauge theory.}
There are no known dual pair, from which the \( Spin(10) \) dual pair is derived. Although we wish that the extended theories in Appendix A would give an insight into the discovery of such a parent dual pair, we guess that the gauge group of the electric \( Spin(10) \) theory should be larger than the \( Spin(10) \) group \[8\], so that the higgsing in the electric theory should correspond to the decoupling of massive states in the magnetic theory.

Finally, we may extend the studies in this paper to a dual pair of an \( \mathcal{N} = 1 \) supersymmetric \( Spin(10) \) gauge theory with more than one spinor and vectors, and the dual theory [19]. However, we will leave this subject for future investigation.

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\[8\] See [18] for the confining phases of \( \mathcal{N} = 1 \) supersymmetric \( Spin(11) \) and \( Spin(12) \) gauge theories with a spinor and vectors.
Appendix

A The continuous ’t Hooft anomaly matching of the extended theories of the $\text{Spin}(10)$ dual pair

Contrary to the $\text{Spin}(7)$ dual pair, there is no known parent dual pair, which is reduced into the $\text{Spin}(10)$ theory and the dual $SU(N_f - 5)$ theory by higgsing or decoupling of massive states. Therefore, there are no reasons to expect that the extended theories of the $\text{Spin}(10)$ theory and its dual satisfy all the continuous ’t Hooft anomaly matching conditions. The extended theories have the $U(1)$ symmetries, into which the discrete symmetries are embedded. We will denote the $U(1)$ symmetries as $U(1)_X$, although the $U(1)_X$ symmetry of the electric theory is not the same as the one of the magnetic theory, for $p \neq 0$.

For $p = -1$, we will compute the continuous ’t Hooft anomalies of the both extended theories, and will find that there are discrepancies in the anomalies with the $U(1)_X$ charges between the electric theory and the magnetic theory. However, we will show that we can eliminate the discrepancies by adding a set of fields to the electric theory, as shown in Table 12 and to the magnetic theory as listed in Table 13. After higgsing the $U(1)_X$ symmetry down to the discrete symmetry, the additional fields gain masses and decouple in the infrared. Then, the anomalies with the $U(1)_X$ charges become the discrete ’t Hooft anomalies of the original dual theories, and we can ensure that they satisfy the discrete ’t Hooft anomaly matching conditions.

The anomalies without the $U(1)_X$ charges are the same as those of the original dual theories, except for the $U(1)_R^3$ and $U(1)_R(\text{gravity})^2$ anomalies. Therefore, they satisfy the ’t Hooft anomaly matching conditions. For the $U(1)_R^3$ and $U(1)_R(\text{gravity})^2$ anomalies, there are additional contributions from the Higgs fields $\Phi$ in the electric theory and $X$ in the magnetic theory, respectively. However, their contributions are the same, and the $U(1)_R^3$ and $U(1)_R(\text{gravity})^2$ anomalies also satisfy the anomaly matching conditions between the electric theory and the magnetic theory. The remaining anomalies are those which the $U(1)_X$ charges take part in, and our results of the computations of them are the following:
\section*{Equations}

\begin{itemize}
\item $U(1)_X SU(N_f)^2$:
\begin{align*}
\text{the electric side: } \frac{5}{N_f}, \\
\text{the magnetic side: } & \left( -\frac{1}{2N_f} - \frac{1}{2(N_f - 5)} \right) (N_f - 5) + \frac{2}{2N_f} (N_f + 2) + \frac{1}{2N_f} = \frac{5}{N_f}.
\end{align*}

\item $U(1)_X (\text{gravity})^2$:
\begin{align*}
\text{the electric side: } & \frac{1}{2N_f} \times 10N_f + 1 - \frac{1}{2} \times 10 = 1, \\
\text{the magnetic side: } & \left( -\frac{1}{2N_f} - \frac{1}{2(N_f - 5)} \right) N_f(N_f - 5) + \left( \frac{1}{2(N_f - 5)} \right) (N_f - 5) \\
& + \frac{2}{2(N_f - 5)} \frac{(N_f - 5)(N_f - 4)}{2} + \frac{2}{2N_f} \frac{N_f(N_f + 1)}{2} + \frac{1}{2N_f} N_f - 1 \\
& = 1.
\end{align*}

\item $U(1)_X^3$:
\begin{align*}
\text{the electric side: } & \left( \frac{1}{2N_f} \right)^3 \times 10N_f + 1 + \left( -\frac{1}{2} \right)^3 \times 10 = \frac{5}{4N_f^2} - \frac{1}{4}, \\
\text{the magnetic side: } & \left( -\frac{1}{2N_f} - \frac{1}{2(N_f - 5)} \right)^3 N_f(N_f - 5) + \left( \frac{1}{2(N_f - 5)} \right)^3 (N_f - 5) \\
& + \frac{2}{2(N_f - 5)} \frac{(N_f - 5)(N_f - 4)}{2} + \left( \frac{2}{2N_f} \right)^3 \frac{N_f(N_f + 1)}{2} \\
& + \left( \frac{1}{2N_f} \right)^3 N_f - 1 = \frac{5}{4N_f^2} - 1.
\end{align*}

\item $U(1)_X U(1)_R^2$:
\begin{align*}
\text{the electric side: } & \frac{1}{2N_f} \times \left( -\frac{8}{N_f} \right)^2 \times 10N_f + 1 \times (-1)^2 = 320 \frac{1}{N_f^2} + 1, \\
\text{the magnetic side: } & \left( -\frac{1}{2N_f} - \frac{1}{2(N_f - 5)} \right) \left( -1 + \frac{8}{N_f} - \frac{1}{N_f - 5} \right)^2 N_f(N_f - 5) \\
& + \left( \frac{1}{2(N_f - 5)} \right) \left( -2 + \frac{1}{N_f - 5} \right)^2 (N_f - 5) \\
& + \left( \frac{2}{2(N_f - 5)} \right) \left( -1 + \frac{2}{N_f - 5} \right)^2 \frac{(N_f - 5)(N_f - 4)}{2} \\
& + \frac{2}{2N_f} \left( 1 - \frac{16}{N_f} \right)^2 \frac{N_f(N_f + 1)}{2} + \frac{1}{2N_f} \left( 2 - \frac{8}{N_f} \right)^2 N_f - 1 \times (-1)^2 \\
& = 320 \frac{1}{N_f^2}.
\end{align*}
\end{itemize}
\( U(1)_X U(1)^2 \):

the electric side: \( \frac{1}{2N_f} \times 2^2 \times 10N_f = 20 \),

the magnetic side:
\[
\begin{align*}
&\left( -\frac{1}{2N_f} - \frac{1}{2(N_f - 5)} \right) (-2)^2 N_f(N_f - 5) \\
&+ \left( \frac{1}{2(N_f - 5)} \right) (2N_f)^2 (N_f - 5) + \frac{2}{2N_f} \times 4^2 \times \frac{N_f(N_f + 1)}{2} \\
&+ \frac{1}{2N_f} \left( -2(N_f - 1) \right)^2 N_f = 20 + 4N_f^2.
\end{align*}
\]

\( U(1)_X U(1)_R U(1) \):

the electric side: \( \frac{1}{2N_f} \times 2 \times \left( -\frac{8}{N_f} \right) \times 10N_f = -\frac{80}{N_f} \),

the magnetic side:
\[
\begin{align*}
&\left( -\frac{1}{2N_f} - \frac{1}{2(N_f - 5)} \right) (-2) \left( -1 + \frac{8}{N_f} - \frac{1}{N_f - 5} \right) N_f(N_f - 5) \\
&+ \left( \frac{1}{2(N_f - 5)} \right) (2N_f) \left( -2 + \frac{1}{N_f - 5} \right) (N_f - 5) \\
&+ \frac{2}{2N_f} 4 \left( 1 - \frac{16}{N_f} \right) \frac{N_f(N_f + 1)}{2} + \frac{1}{2N_f} (-2(N_f - 1)) \left( 2 - \frac{8}{N_f} \right) N_f \\
&= -\frac{80}{N_f} - 4N_f.
\end{align*}
\]

\( U(1)^2 X U(1)_R \):

the electric side: \( \left( \frac{1}{2N_f} \right)^2 \times \left( -\frac{8}{N_f} \right) \times 10N_f = -\frac{20}{N_f^2} \),

the magnetic side:
\[
\begin{align*}
&\left( -\frac{1}{2N_f} - \frac{1}{2(N_f - 5)} \right)^2 \left( -1 + \frac{8}{N_f} - \frac{1}{N_f - 5} \right) N_f(N_f - 5) \\
&+ \left( \frac{1}{2(N_f - 5)} \right)^2 \left( -2 + \frac{1}{N_f - 5} \right) (N_f - 5) \\
&+ \left( \frac{2}{2(N_f - 5)} \right)^2 \left( -1 + \frac{2}{N_f - 5} \right) \frac{(N_f - 5)(N_f - 4)}{2} \\
&+ \left( \frac{2}{2N_f} \right)^2 \left( 1 - \frac{16}{N_f} \right) \frac{N_f(N_f + 1)}{2} + \left( \frac{1}{2N_f} \right)^2 \left( 2 - \frac{8}{N_f} \right) N_f \\
&= -\frac{20}{N_f^2} - 1.
\end{align*}
\]
• $U(1)_X^2 U(1)$:

the electric side: \( \left( \frac{1}{2N_f} \right)^2 \times 2 \times 10N_f + 1^2 \times (-1) = \frac{5}{N_f} - 1, \)

the magnetic side: \( \left( -\frac{1}{2N_f} - \frac{1}{2(N_f-5)} \right)^2 (-2) N_f(N_f-5) \)
\[+ \left( \frac{1}{2(N_f-5)} \right)^2 (2N_f)(N_f-5) + \left( \frac{2}{2N_f} \right)^2 \frac{4N_f(N_f+1)}{2} \]
\[+ \left( \frac{1}{2N_f} \right)^2 \left( -2(N_f-1) \right) N_f + (-1)^2 \times (-1) = \frac{5}{N_f} - 1. \]

| $Spin(10)$ | $SU(N_f)$ | $U(1)$ | $U(1)_R$ | $U(1)_X$ |
|-----------|-----------|--------|----------|----------|
| $S$       | 16        | $-N_f$ | 1        | 0        |
| $Q^i$     | 10        | $\Box$ | 2        | $1 - 8/N_f$ | $1/(2N_f)$ |
| $\Phi$    | 1         | 1      | 0        | 0        | $-1/2$ |
| $P$       | 10        | 1      | 0        | 1        | -1/2   |
| $\bar{\Phi}$ | 1       | 1      | 0        | 0        | -2     |
| $\Psi$    | 1         | 1      | 0        | 1        | $1/2$  |
| $\bar{\Psi}$ | 1       | 1      | 0        | 1        | $1/2$  |
| $\Xi$     | 1         | 1      | 0        | 0        | -2     |
| $G$       | 1         | $-N_f$ | 2        | 2        |
| $H$       | 1         | $N_f$  | 0        | 0        |
| $\tilde{\Xi}$ | 1       | 1      | 0        | 0        | -2     |
| $\tilde{G}$ | 1        | 1      | $N_f$   | 0        | 2      |
| $\tilde{H}$ | 1        | 1      | $-N_f$  | 2        | 0      |

Table 12: The original fields and the set of the fields added to the extended $Spin(10)$ theory to fill the gaps in the continuous ’t Hooft anomalies

We see that there are discrepancies in the anomalies except for $U(1)_X SU(N_f)$, $U(1)_X (gravity)^2$ and $U(1)_X^2 U(1)$, between both the sides. We will add the singlet fields $\bar{\Phi}$, $\Psi$ and $\bar{\Psi}$, listed in Table 12 and the term $\bar{\Phi}\Psi\bar{\Psi}$ to the superpotential in the electric theory. The fields contribute
to the ’t Hooft anomalies by

\[
\begin{align*}
\bullet & \ U(1)_X SU(N_f)^2 : 0, \\
\bullet & \ U(1)_X U(1)_R^2 : 1, \\
\bullet & \ U(1)_X U(1)_R : 0, \\
\bullet & \ U(1)_X^3 : 0, \\
\bullet & \ U(1)_X^2 U(1)_R : 0, \\
\bullet & \ U(1)_X^3 U(1)_R : 0, \\
\end{align*}
\]

which fill the gap in the anomalies \( U(1)_X^3, U(1)_X U(1)_R^2, U(1)_X^3 U(1)_R \).

There still remain discrepancies in the anomalies \( U(1)_X U(1)_R^2 \) and \( U(1)_X U(1)_R U(1) \). To fill the gap, we introduce two sets of fields \((\Xi, G, H), (\tilde{\Xi}, \tilde{G}, \tilde{H})\), and add the terms \( \Xi G H + \tilde{\Xi} \tilde{G} \tilde{H} \) to the superpotential in the electric theory. Their contributions to the ’t Hooft anomalies are the following:

\[
\begin{align*}
\bullet & \ U(1)_X SU(N_f)^2 : 0, \\
\bullet & \ U(1)_X : 0, \\
\bullet & \ U(1)_X^3 : 0, \\
\bullet & \ U(1)_X U(1)_R^2 : 0, \\
\bullet & \ U(1)_X U(1)_R : 0, \\
\bullet & \ U(1)_X^2 U(1)_R : 0, \\
\end{align*}
\]

and we find that they fill the gap in the anomalies \( U(1)_X U(1)_R^2, U(1)_X U(1)_R U(1) \). However, unfortunately, they make a gap again in the anomaly \( U(1)_X^3 U(1)_R \), and all the other anomalies satisfy the matching conditions.

In order to fill the gap in the anomaly \( U(1)_X^3 U(1)_R \), we will add three fields \( \tilde{X}', F', \tilde{F}' \), listed in Table 13 and add the term \( \tilde{X}' F' \tilde{F}' \) to the superpotential in the magnetic extended theory. They contribute to the anomaly \( U(1)_X^3 U(1)_R \) by \(-8\), and finally, all the anomalies with the \( U(1)_X \) charges satisfy the continuous ’t Hooft anomaly matching conditions.

However, the additional fields to the both sides cause the discrepancies in the anomalies \( U(1)_R^3 \) and \( U(1)_R gravity^2 \). Upon the higgsing of the \( U(1)_X \) symmetries by the vacuum expectation values \( \langle \Phi \rangle, \langle \tilde{\Phi} \rangle, \langle \Xi \rangle, \langle \Xi \rangle \) on the electric side, \( \langle X \rangle, \langle X' \rangle \) on the magnetic side, those extended theories are reduced into the original dual pair in the infrared, after the decoupling of the massive fields, and we find that the continuous ’t Hooft anomalies without the \( U(1)_X \) charges match with each other. Since the continuous ’t Hooft anomalies with the \( U(1)_X \) charges yields the discrete ’t Hooft anomalies of the original dual theories in the infrared, it is obvious from the above computations that they satisfy the discrete ’t Hooft anomaly matching conditions.
SU(N_f - 5)  SU(N_f)  U(1)  U(1)_R  U(1)_X
\hline
q_i  0  0  -2  \frac{8}{N_f} - 1/(N_f - 5)  -1/(2N_f) - 1/(2(N_f - 5))
q  0  1  2N_f  -1 + 1/(N_f - 5)  1/(2(N_f - 5))
s  0  1  0  2/(N_f - 5)  1/(N_f - 5)
M^{ij}  1  0  4  2 - 16/N_f  1/N_f
Y^i  1  0  -2(N_f - 1)  3 - 8/N_f  1/(2N_f)
X  1  1  0  0  0
\hline
X'  1  1  0  0  0
F'  1  1  0  0  2
F'  1  1  0  2  0
\hline
Table 13: The original magnetic fields and the set of the fields added to the extended $SU(N_f - 5)$ theory to fill the gaps in the continuous ’t Hooft anomalies

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