Search for gauge symmetry generators of singular Lagrangian theory.

A.A. Deriglazov*

Dept. de Matematica, ICE, Universidade Federal de Juiz de Fora, MG, Brazil.

Abstract

We propose a procedure which allows one to construct local symmetry generators of general quadratic Lagrangian theory. Manifest recurrence relations for generators in terms of so-called structure matrices of the Dirac formalism are obtained. The procedure fulfilled in terms of initial variables of the theory, and do not implies either separation of constraints on first and second class subsets or any other choice of basis for constraints.

1 Introduction

Relativistic theories are usually formulated in manifestly covariant form, i.e. in the form with linearly realized Lorentz group. It is achieved by using of some auxiliary variables, which implies appearance of local (gauge) symmetries in the corresponding Lagrangian action. Investigation of the symmetries is essential part of analysis of both classical and quantum versions of a theory. Starting from pioneer works on canonical quantization of singular theories [1-3], one of the most intriguing problems is search for constructive procedure which allows one to find the local symmetries from known Lagrangian or Hamiltonian formulation [4-10]. For a theory with first class constraints only, such a kind procedure has been proposed in [4, 5]. Symmetry structure of a general singular theory has been described in recent works [6-8]. In particular, it was shown how one can find gauge symmetries of Hamiltonian action for quadratic theory [7], as well as for general singular theory [8].

*alexei@ice.ufjf.br On leave of absence from Dept. Math. Phys., Tomsk Polytechnical University, Tomsk, Russia.
In the present work we propose an alternative procedure to construct Lagrangian local symmetries for the case of general quadratic theory. Our method is based on analysis of Noether identities in the Hamiltonian form, the latter has been obtained in our works [9, 10]. Let us enumerate some characteristic properties of the procedure presented below.

1) The procedure do not implies separation of Hamiltonian constraints on first and second class subsets, which is may be the most surprising result of the work.

2) The procedure do not requires choice of some special basis for constraints.

3) All the analysis is fulfilled in terms of initial variables of a theory.

Rather schematically, final result of our work can be described as follows. Let \( H = H_0 + \Phi_\alpha v^\alpha \) be Hamiltonian of a theory, where \( v^\alpha \) denotes Lagrangian multipliers to all primary constraints \( \Phi_\alpha \). Let \( \Phi^{(s-1)}_\mu \) be constraints of \( (s-1) \)-stage of the Dirac procedure. On the next stage one studies the equations \( \{ \Phi^{(s-1)}_\mu, H \} = 0 \) for revealing of \( s \)-stage constraints. \( s \)-stage Dirac functions \( \{ \Phi_{(s-1)}^\mu, H \} \) can be rewritten in the form

\[
\{ \Phi_{(s-1)}^\mu, H \} = A^\mu_{(s-1)} \left( \begin{array}{c} \pi^{(s)}_i(v) \\ \Phi^{(s)}_\zeta (\pi^{(s-1)}_1, \ldots, \pi^{(2)}) \\ C_\alpha (\Phi^{(s-1)}_1, \ldots, \Phi^{(2)}) + D_\alpha (\pi^{(s-1)}_1, \ldots, \pi^{(2)}) \end{array} \right)
\]  

(1)

where \( \pi^{(s)}_i(v) = 0 \) represent equations for determining of the Lagrangian multipliers of these stage, \( \Phi^{(s)}_\zeta \) are \( s \)-stage constraints, and \( A, B, C, D \) will be called \( s \)-stage structure matrices. It may happens that some components of the column \( \{ \} \) do not represent independent restrictions on the variables \( (q, p, v) \) of the theory. The number \([a]\) of these components will be called defect of \( s \)-stage system. Then \([a]\) independent local symmetries of the Lagrangian action can be constructed

\[
\delta q^A = \sum_{\mu=0}^{s-2} \epsilon^\mu a R^\mu_a A(q, \dot{q}),
\]

(2)

where generators \( R \) are specified in terms of the structure matrices in an algebraic way. Recurrence relations for obtaining of the generators are presented below (see Eqs. (95)-(97) for \( s \)-stage symmetries and Eqs. (101)-(103) for lower-stage symmetries).
Thus, knowledge of a structure of $s$-stage Dirac functions (1) is equivalent to knowledge of $s$-stage local symmetries (2). Total number of independent symmetries, which can be find by using of our procedure, coincides with the number of Lagrangian multipliers remaining undetermined in the Dirac procedure.

2 Setting up

We consider Lagrangian theory with action being $(A = 1, 2, \cdots [A])$

$$S = \int d\tau L(q^A, \dot{q}^A),$$

(3)

It is supposed that the theory is singular

$$\text{rank} - \frac{\partial^2 L}{\partial \dot{q}^A \partial \dot{q}^B} = [i] < [A].$$

(4)

According to Dirac [1], Hamiltonian formulation of the theory is obtained as follow. First stage of Hamiltonization procedure is to define equations for the momenta $p_A$: $p_A = \frac{\partial L}{\partial \dot{q}^A}$. Being considered as algebraic equations for determining of velocities $\dot{q}^A$, $[i]$ equations can be resolved for $\dot{q}^i$ and then substituted into the remaining ones. By construction, the resulting equations do not depend on $\dot{q}^A$ and are called primary constraints $\Phi_\alpha(q, p)$ of the Hamiltonian formulation. The equations $p_A = \frac{\partial L}{\partial \dot{q}^A}$ are then equivalent to the following system

$$\dot{q}^i = v^i(q^A, p^i, \dot{q}^\alpha),$$

(5)

$$\Phi_\alpha \equiv p_\alpha - f_\alpha(q^A, p_j) = 0.$$  

(6)

Then one introduces an extended phase space with the coordinates $(q^A, p_A, v_\alpha)$. By definition, Hamiltonian formulation of the theory (3) is the following system of equations on this space

$$\dot{q}^A = \{q^A, H\}, \ \dot{p}_A = \{p_A, H\}, \ \Phi_\alpha(q^A, p_B) = 0,$$

(7)

where $\{ , \}$ is the Poisson bracket, and it was denoted

$$H(q^A, p_A, v^\alpha) = H_0(q^A, p_j) + v^\alpha \Phi_\alpha(q^A, p_B),$$

(8)
\[ H_0 = \left. \left( p_i \dot{q}^i - L + \dot{q}^i \frac{\partial L}{\partial \dot{q}^i} \right) \right|_{\dot{q}^i \rightarrow v^i(q^A, p_j, \dot{q}^\alpha)}. \]  

The variables \( v^\alpha \) are called Lagrangian multipliers to the primary constraints. It is known [11] that the formulations (3) and (7) are equivalent.

Second stage of the Dirac procedure consist in analysis of second stage equations \( \{ \Phi_\alpha, H \} = 0 \), the latter are algebraic consequences of the system (7). Some of second-stage equations can be used for determining of a subgroup of Lagrangian multipliers in an algebraic way. Among the remaining equations one takes functionally independent subsystem, the latter represent secondary Dirac constraints \( \Phi_\alpha^{(2)}(q^A, p_j) = 0 \). They imply third-stage equations, an so on. We suppose that the theory has constraints up to at most N stage: \( \{ \Phi_\alpha, \Phi_\alpha^{(2)}, \ldots, \Phi_\alpha^{(N)} \} \).

3 Sufficient conditions for existence of local symmetry of Lagrangian action

Let us consider infinitesimal transformation

\[ q^A \rightarrow q'^A = q^A + \delta q^A, \quad \delta q^A = \sum_{p=0}^{[p]} \frac{\partial^p}{\partial \tau^p} R^{(p)}(q, \dot{q}, \ddot{q}, \ldots) \],

where parameter \( \epsilon(\tau) \) is arbitrary function of time \( \tau \), and it was denoted \( \frac{\partial^p}{\partial \tau^p} R \). The transformation is local (or gauge) symmetry of an action \( S \), if it leaves \( S \) invariant up to surface term \( \delta L = \frac{d}{d\tau} \omega \),

\[ \delta L = \frac{d}{d\tau} \omega, \]

with some functions \( \omega(q, \epsilon) \). Operators \( \sum_{p=0}^{[p]} R^{(p)}(q, \dot{q}, \ddot{q}, \ldots) \) will be called generators of the local symmetry. Local symmetry implies appearance of identities among equations of motion of the theory. For a theory without higher derivatives and generators of the form\(^2\)

\(^1\)Transformations which involve variation of the evolution parameter: \( \delta \tau, \delta q^A \) are included into the scheme. Actually, with any such transformation it is associated unambiguously transformation of the form \( \dot{q} \): \( \delta \tau = 0, \delta q^A = -q^A \delta \tau + \dot{\delta q}^A \). If \( \delta q \) is a symmetry of the action, the same will be true for \( \delta q \).

\(^2\)Analysis of this work shows that gauge generators can be find in this form.
$R^{(p)A}(q, \dot{q})$, the identities were analyzed in some details in our recent work [10]. Hamiltonian form of the identities has been obtained, then necessary and sufficient conditions for existence of local symmetry of the Lagrangian action were formulated on this ground. Namely, the Hamiltonian identities can be considered as a system of partial differential equations for the Hamiltonian counterparts of the functions $R^{(p)A}(q, \dot{q})$. In the present work we propose pure algebraic procedure to solve these equations. So, let us present the relevant result of the work [10] in a form convenient for subsequent analysis.

For given integer number $s$, let us construct generating functions $T^{(p)}$, $p = 2, 3, \ldots, s$ according to the recurrence relations ($T^{(1)} = 0$)

$$T^{(p)} = Q^{(p)\alpha}(\Phi, H) + \{H, T^{(p-1)}\},$$

where the coefficients $Q^{(p)\alpha}(q^A, p_j, v^\alpha)$ are some functions. Then one can prove [10] the following

**Statement 1.** Let the coefficients $Q^{(p)\alpha}, p = 2, 3, \ldots, s$ have been chosen in such a way that the following equations:

$$\frac{\partial}{\partial v^\alpha} T^{(p)} = 0, \quad p = 2, 3 \ldots, s - 1,$$

$$T^{(s)} = 0,$$

hold. Using these $Q$, let us construct the Hamiltonian functions $R^{(p)A}(q^A, p_j, v^\alpha), p = 0, 1, 2, \ldots, s - 2$

$$R^{(p)\alpha} = Q^{(s-p)\alpha}, \quad R^{(p)i} = \{q^i, \Phi\} R^{(p)\alpha} - \{q^i, T^{(s-1-p)}\},$$

and then the Lagrangian functions

$$R^{(p)A}(q, \dot{q}) \equiv R^{(p)A}(q^A, p_j, v^\alpha) \bigg|_{p_j \rightarrow \frac{\partial L}{\partial \dot{q}^A}} \bigg|_{v^A \rightarrow \dot{q}^A}.$$

Then the transformation

$$\delta q^A = \sum_{p=0}^{s-2} (p) R^{(p)A}(q, \dot{q}),$$

3Equations [13]. [14] has been obtained in [10] starting from hypothesis that the action is invariant, and by making substitution $v^i(q^A, p_j, v^\alpha)$ into the first order identities, i.e. as necessary conditions for existence of gauge symmetry. As it is explained in [12], this substitution is change of variables on configuration-velocity space, which implies that [12], [14] are sufficient conditions also.
is local symmetry of the Lagrangian action.

Search for the symmetry (17) (the latter involves derivatives of the parameter \( \epsilon \) up to order \( s-2 \)) is directly related with \( s \)-stage of the Dirac procedure, see below. On this reason the symmetry (17) will be called \( s \)-stage symmetry. Then the set \( T^{(2)}, T^{(3)}, \ldots, T^{(s)} \) can be called \( s \)-stage generating functions. Some relevant comments are in order.

1) From Eq.(15) it follows that only \( R^{(p)\alpha} \)-block of Hamiltonian generators is essential quantity. On this reason, only this block will be discussed below.

2) Hamiltonian functions (15) can be used also to construct a local symmetry of the Hamiltonian action. Expressions for the corresponding transformations \( \delta q^A, \delta p^A, \delta v^\alpha \) can be find in [10].

3) According to the statement, symmetries of different stages \( s \) can be looked for separately. To find 2-stage symmetries \( \delta a_2 q^A = \epsilon a_2 R^{(0)A}_{a_2}, \) one look for solutions \( Q^{(2)\alpha}_{a_2} \) of the equation

\[
T^{(2)} = Q^{(2)\alpha}_{a_2} \{ \Phi_{\alpha}, H \} = 0.
\]

Note that it implies analysis of second-stage Dirac functions \( \{ \Phi_{\alpha}, H \} \). 3-stage symmetries \( \delta a_3 q^A = \epsilon a_3 R^{(0)A}_{a_3} + \dot{\epsilon} a_3 R^{(1)A}_{a_3} \) are constructed from solutions \( Q^{(2)\alpha}_{a_3}, Q^{(3)\alpha}_{a_3} \) of the equations

\[
\frac{\partial}{\partial v^\beta} T^{(2)} = \frac{\partial}{\partial v^\beta} (Q^{(2)\alpha}_{a_2} \{ \Phi_{\alpha}, H \}) = 0, \\
T^{(3)} \equiv Q^{(3)\alpha}_{a_3} \{ \Phi_{\alpha}, H \} + \{ Q^{(2)\alpha}_{a_2} \Phi_{\alpha}, H \}, H \} = 0,
\]

and so on. In a theory with at most \( N \)-stage Dirac constraints presented, the procedure stops for \( s = N + 1 \), see Sect. 9 below.

4) Generating equations (13), (14), are related with \( s \)-stage of the Dirac procedure in the following sense. In Sect. 6 we demonstrate that the coefficients \( Q^{(p)}, p = 2, 3, \ldots, s \) can be chosen in such a way that each generating function \( T^{(p)} \) is linear combination of constraints \( \Phi_{\alpha k} \) of the stages \( k = 2, 3, \ldots, p \). In particular, \( T^{(s-1)} = \sum_{p=2}^{s-1} \epsilon^p \Phi_{\alpha p}, \) then \( T^{(s)} \sim \{ T^{(s-1)}, H \} \) in Eq.(14) involves the Dirac functions up to \( s \)-stage: \( \{ \Phi_{\alpha p}, H \}, p = 2, 3, \ldots, s-1 \). So, search for \( s \)-stage symmetries implies analysis of \( s \)-stage of the Dirac procedure.

5) Since the generating equations (13), (14) do not involve the momenta \( p^A \), one can search for solutions in the form \( Q^{(p)\alpha}(q^A, p_j, v^\alpha) \).

\[\text{Coefficients } Q \text{ of different stages are independent.}\]
As a result, Hamiltonian generators do not depend on $p_\alpha$. In this case, passage to the Lagrangian first order formulation is change of variables [12]: $(q^A, p_i, v^\alpha) \leftrightarrow (q^A, v^i, v^\alpha)$. This change has been performed in Eq. (16).

6) Analysis of the Hamiltonian equations (13), (14) turns out to be a more simple task as compared to the corresponding Lagrangian (or first order) version. Besides the fact that one is able to use well developed Dirac method, crucial simplification is linearity on $v^\alpha$ of all the Hamiltonian quantities appeared in the analysis. On this reason, search for solutions of the generating equations can be reduced to solving of some system of linear inhomogeneous equations, see below. Lagrangian version of our procedure will imply looking for solutions of some non linear algebraic equations on each step of the procedure.

4 Analysis of second-stage Dirac functions

As it was discussed in the previous section, expression for generating function $T^{(k)}$ involve the Dirac functions $\{\Phi_{\alpha p}, H\}$, $p = 1, 2, \ldots, k - 1$. One needs to know detailed structure of them to solve the generating equations. Let us point that this part of analysis is, in fact, part of the Dirac procedure for revealing of higher-stage constraints. The only difference is that in the Dirac procedure one studies the equations $\{\Phi_{\alpha p}, H\} = 0$, where constraints and equations for Lagrangian multipliers of previous stages can be used. Since our generating equations must be satisfied by $Q$ for any $q, p, v$, one needs now to study the Dirac functions outside of extremal surface. Below we suppose that matrices $\{\Phi_{\alpha p}, \Phi_{\alpha q}\}$ have constant rank in vicinity of phase space point under consideration. In particular, it is true for quadratic theory. In the next section we formulate an induction procedure to represent $p$-stage Dirac functions in normal form convenient for subsequent analysis, see Eq. (15) below. On each stage, it will be necessary to divide some groups of functions on subgroups. Here we present detailed analysis of second stage, with the aim to clarify notations which will be necessary to work out $p$-stage Dirac functions and the corresponding generating equations.

With a group of quantities appeared on first stage of the Dirac procedure we assign number of the stage, the latter replace corresponding index (the number will be called index of the group below).
Then the primary constraints are $\Phi_\alpha \equiv \Phi_1$, and the Lagrangian multipliers are denoted as $v^\alpha \equiv v^1$. Number of functions in a group is denoted as $[1] \equiv [\alpha]$. For the second stage Dirac functions one writes

$$\{\Phi_\alpha, \Phi_\beta v^\beta + H_0\} \rightarrow \{\Phi_1, \Phi_1' v^1' + H_0\} = \{\Phi_1, \Phi_1' v^1' + \{\Phi_1, H_0\} \equiv \triangle_{(2)11'} v^1' + H_{(2)1}. \quad (20)$$

So, repeated up and down number of stage imply summation over the corresponding indices. With quantities first appeared on second stage has been assigned number of the stage: $\triangle_{(2)}$, $H_{(2)}$ (where confusion is not possible, it can be omitted).

Let us describe a procedure to represent second-stage Dirac functions (20) in the normal form. Suppose that $rank \triangle_{(2)11'} = [2]$, then one finds $[2] = [1] - [\overline{2}]$ independent null-vectors $\tilde{K}_{(2)\overline{2}}$ of the matrix $\triangle_{(2)}$ with components $K_{(2)\overline{2}}$. Let $K_{(2)\overline{1}}$ be any completion of the set $\tilde{K}_{(2)\overline{2}}$ up to a basis of $[1]$-dimensional space. By construction, the matrix

$$K_{(2)\overline{1}} = \begin{pmatrix} K_{(2)\overline{2}} \\ K_{(2)\overline{2}}' \end{pmatrix}, \quad (21)$$

is invertible, with the inverse matrix being $\tilde{K}_{(2)1} \hat{K}_{(2)\overline{1}} = \delta_{1}'. \quad (22)$

Since $K_{(2)\overline{1}} \triangle_{(2)11'} = 0$, the conversion matrix can be used to separate the Dirac functions on $v$-dependent and $v$-independent parts: $\{\Phi_1, H\} = \tilde{K} K \{\Phi_1, H\}$, with the result being

$$\{\Phi_1, H\} = \tilde{K}_{(2)1} \hat{1} \begin{pmatrix} \pi_{\overline{1}'}(v^1) \\ \Phi_2(q^1, p_2) \end{pmatrix}, \quad (23)$$

$$\pi_{\overline{1}'}(v^1) \equiv X_{(2)\overline{1}} v^1 + Y_{(2)\overline{1}},$$

$$\Phi_2 \equiv K_{(2)\overline{1}} H_{(2)1}. \quad (24)$$

Here it was denoted

$$X_{(2)\overline{1}} = K_{(2)\overline{1}}' \triangle_{(2)11'}, \quad Y_{(2)\overline{1}} = K_{(2)\overline{1}} H_{(2)1}. \quad (24)$$
Let us analyze the functions $\pi_2(v^1)$. By construction, the matrix $X$ has maximum rank equal $[\underline{2}]$. Without loss of generality, we suppose that from the beginning $v^1$ has been chosen such that the rank columns appear on the left: $X_{(2)21} = (X_{(2)\underline{2}2}, X_{(2)\underline{2}\underline{2}})$. So, the Lagrangian multipliers are divided on two groups $v^1 = (v^2, v^\underline{2})$, one writes

$$\pi_2 = X_{(2)\underline{2}2}v^\underline{2} + X_{(2)\underline{2}\underline{2}}v^2 + Y_{(2)\underline{2}}. \quad (25)$$

Then $v^2$ can be identically rewritten in terms of $v^\underline{2}, \pi_\underline{2}$ as follows:

$$v^\underline{2} \equiv \tilde{X}_{(2)}\underline{2}\underline{2} \pi_\underline{2}(v^1) + \Lambda_{(2)}\underline{2}v^2 + W_{(2)}\underline{2}, \quad (26)$$

where

$$\Lambda_{(2)}\underline{2} = -\tilde{X}_{(2)}\underline{2}\underline{2} X_{(2)\underline{2}\underline{2}}, \quad W_{(2)}\underline{2} = -\tilde{X}_{(2)}\underline{2}\underline{2} Y_{(2)\underline{2}}. \quad (27)$$

We stress that Eq.(26) is an identity. It will be necessary to analyze third-stage Dirac functions below.

Let us analyze the functions $\Phi_{\underline{2}}$ in Eq.(23). By construction, they depend on the phase space variables $z_1 \equiv (q_i, p_j)$. According to Dirac, functionally independent functions among $\Phi_{\underline{2}}$ are called secondary constraints, and the equations $\Phi_{\underline{2}} = 0$ can be used to express a part $\bar{z}_{\underline{2}}$ of the phase space variables $z_1 = (\bar{z}_2, z_2)$ in terms of $z_2$. Let us suppose

$$\text{rank} \frac{\partial \Phi_{\underline{2}}}{\partial z_1}_{\Phi_{\underline{2}}} = \text{rank} \frac{\partial \Phi_{\underline{2}}}{\partial \bar{z}_{\underline{2}}}_{\Phi_{\underline{2}}} = [\bar{z}_{\underline{2}}]. \quad (28)$$

We demonstrate that the functions $\Phi_{\underline{2}}$ can be identically rewritten in the form

$$\Phi_{\underline{2}}(z_1) = U_{(2)\underline{2}}\bar{z}_{\underline{2}} \Phi_{\underline{2}}(z_1), \quad [\bar{2}] = [\underline{2}] - [2], \quad \det U \neq 0, \quad (29)$$

where index $\bar{2}$ is divided on two groups $\bar{2} = (2, \tilde{2})$, and $\Phi_2$ are functionally independent.

---

5On this stage one has $[\underline{2}] = [2]$, but it will not be true for higher stages. On this reason we adopt different notations for these groups.
Actually, under the conditions (28) there is exist the representation [11] (see also Appendix A)

\[ \Phi_2 = \Lambda(\tilde{2}) \times (2) \Phi_2(\tilde{z}_1), \quad \text{rank} \Lambda = [2], \quad \text{rank} \frac{\partial \Phi_2}{\partial \tilde{z}_2} \bigg|_{\Phi_2} = [\tilde{z}_2]. \tag{30} \]

Using invertible numerical matrix \( Q \), lines of the matrix \( \Lambda \) can be rearranged

\[ \Phi_2 = Q(\tilde{2}) \times (\tilde{2}) \begin{pmatrix} \Lambda'(2) \times (2) \\ \Lambda''(2) \times (2) \end{pmatrix} \Phi_2, \tag{31} \]

in such a way that \( \det \Lambda' \neq 0 \). Then one writes identically

\[ \Phi_2 = Q \begin{pmatrix} \Lambda' & 0 \\ \Lambda'' & 1 \end{pmatrix} \begin{pmatrix} \Phi_2 \\ 0 \end{pmatrix} = U(2) \begin{pmatrix} \Phi_2 \\ 0 \end{pmatrix}, \tag{32} \]

where \( U(2) \) the invertible matrix.

Substitution of this result into Eq.(23) gives the normal form of second-stage Dirac functions

\[ \{ \Phi_1, H \} = A(2)_{1} \hat{1}(q^A, p_j) \begin{pmatrix} \pi_\Sigma(v^1) \\ \Phi_2(q^A, p_j) \end{pmatrix} \tag{33} \]

where \( A \) is invertible matrix

\[ A(2)_{1} \hat{1} = \hat{K}(2) \begin{pmatrix} 1(\Sigma) \times (\Sigma) \\ 0 \\ 0 \end{pmatrix} U(\tilde{2}) \tilde{2} \tag{34} \]

and functions \( \pi_\Sigma(v^1) \) are given by Eq.(25). In the process, the Lagrangian multipliers \( v^1 \) have been divided on subgroups \((v^\Sigma, v^\Sigma)\), where \( v^\Sigma \) can be identically rewritten in terms of \( v^2, \pi^2 \) according to Eq.(23). The functions \( \Phi_\alpha = p_\alpha - f_\alpha(q^A, p_j), \Phi_2(q^A, p_j) \) are functionally independent, \( \Phi_2 \) represent all secondary constraints of the theory. By construction, \( \pi_\Sigma(v^1) = 0 \) turn out to be equations for determining of the Lagrangian multipliers \( v^\Sigma \).

Note that ”evolution” of the index 1 of previous stage during the second stage can be resumed as follow: it can either be divided

\[ \text{Let us point that only on this place of the Dirac procedure some nonlinear algebraic equations appear. As it will be shown below, analysis of the generating equations implies solution of linear systems only. In this sense, our procedure can be applied to any theory where the Dirac procedure works.} \]
on two subgroups: \(1 = (\bar{2}, 2)\), or can be converted into \(\hat{1}\) and then divided on three subgroups: \(1 \rightarrow \hat{1} = (\bar{2}, 2, \bar{2})\). Dimensions of the indices are related with rank properties of second stage Dirac system as follow:

\[ \bar{2} \] is number of Lagrangian multipliers which can be determined on the second stage;

\[ 2 \] is number of multipliers which remains undetermined after the second stage;

\[ 2 \] is number of secondary constraints;

\[ \bar{2} = [1] - \bar{2} - 2 \] is called defect of the system.

5 Notations

Discussion of previous section on second stage of the Dirac procedure excuses the following notations.

a) Notations for phase space variables \((p_\alpha, \ldots)\) are not included:

\((q_A, p_j) \equiv z_1\). On second stage of the Dirac procedure the group can be divided on two subgroups \(z_1 = (\bar{z}_2, z_2)\), where \(\bar{z}_2\) are variables which can be presented through \(z_2\) using the secondary constraints. On third stage one has \(z_2 = (\bar{z}_3, z_3)\), and so on. In the end of Dirac procedure one obtains the following division: \(z_1 = (\bar{z}_2, \bar{z}_3, \ldots, \bar{z}_N, z_N)\).

b) For the Lagrangian multipliers to the primary constraints we assign ”covariant” index \(v^\alpha \equiv v^1\). On second stage of the Dirac procedure the group can be divided on two subgroups \(v^1 = (v^\bar{2}, v^{\bar{2}})\), where \(v^\bar{2}\) represents subgroup which can be presented through \(v^{\bar{2}}\) on this stage. On the next stages one has \(v^{\bar{2}} = (v^\bar{3}, v^{\bar{3}}), \ldots, v^{\bar{N}} = (v^{\bar{N}+1}, v^{N+1})\). Thus the symbols \(v^1, v^{\bar{2}}, 1 < k < p - 1\) appeared on the stage \(p\) mean:

\[ v^k = (v^{k+1}, v^{k+2}, \ldots, v^{p-1}, v^{p-1}), \quad (35) \]

where \(v^{p-1}\) can be further divided during this stage: \(v^{p-1} = (v^{\bar{2}}, v^\bar{2})\). In the end of the Dirac procedure one obtains the following division: \(v^1 = (v^{\bar{2}}, v^{\bar{3}}, \ldots, v^{N+1}, v^{N+1})\), where \(v^{N+1}\) are Lagrangian multipliers remaining undetermined in the process, see next section.

c) With group of \(p\)-stage Dirac constraints (see next section) we assign ”contravariant” index \(\Phi^\alpha_p \equiv \Phi_p\). Then complete set of functionally independent constraints of the theory is \(\Phi_1, \Phi_2, \ldots, \Phi_N\).
d) With a group of functions \( \{ \psi \} \), first appeared on the stage \( p \), we assign the symbol \( (p) \): \( \psi_{(p)} \) (when confusion is not possible, it can be omitted).

e) According to these notations, \( p \)-stage Dirac functions are
\[
\begin{align*}
\{ \Phi_{p-1}, \Phi_1 v^1 + H_0 \} & \longrightarrow \{ \Phi_{p-1}, \Phi_1 v^1 + H_0 \} = \\
\{ \Phi_{p-1}, \Phi_1 \} v^1 + \{ \Phi_{p-1}, H_0 \} & \equiv \triangle_{p} v^1 + H_{p-1} \text{,} \\
\end{align*}
\]
where (see next section) \( [p] \) is number of Lagrangian multipliers determined on the stage \( p \), and \( \bar{[p]} \) is number of multipliers which remains undetermined after the stage \( p \).

g) On each stage \( p \) it is appear invertible matrix with natural block structure:
\[
K_{(p)p-1} = 
\begin{pmatrix}
K_{(p)p-1} & K_{(p)p-1} \\
K_{(p)p-1} & K_{(p)p-1}
\end{pmatrix}
\]
It can be used to convert any quantity \( \psi_{p-1} \) into \( \hat{\psi}_{p-1} \) as follow:
\[
\psi_{p-1} \rightarrow \hat{\psi}_{p-1} = K_{(p)p-1}^{-1} \psi_{p-1} = (\hat{\psi}_{p}, \hat{\psi}_{p})
\]
Where confusion is not possible, we write the quantity without hat: \( \hat{\psi}_{p-1} \equiv \psi_{p-1} \). Index \( \bar{p} \) can be further divided
\[
\hat{\psi}_{p} = (\hat{\psi}_{p}, \hat{\psi}_{p}), \quad \Rightarrow \quad \hat{\psi}_{p-1} = (\hat{\psi}_{p}, \hat{\psi}_{p}) \]
where \( [p] = [\Phi_p] \), and \( \bar{[p]} = [p-1] - [p] - [\bar{p}] \).

h) Repeated up and down number of stage imply summation over the corresponding indices. In contrast, summation over stages always indicated explicitly, for example
\[
\sum_{n=2}^{p} Q^{(p-n)\alpha_n} \Phi_{\alpha_n} = \sum_{n=2}^{p} \left( \sum_{\alpha_n=1}^{[\alpha_n]} Q^{(p-n)\alpha_n} \Phi_{\alpha_n} \right) \text{.}
\]
In resume, evolution of indices during the stage $p$ can be described as follow:
For $k < p - 1$, the notations $\psi_{(p)1}, \psi_{(p)k}$ are explained in f).
Index $p - 1$ of previous stage can be divided on two subgroups

$$ p - 1 = (\overline{p}, p). \quad (43) $$

Index $p - 1$ of previous stage can be converted into $\widehat{p - 1}$ and then divided on three subgroups

$$ p - 1 \rightarrow \widehat{p - 1} = (\overline{p}, \overline{p}, \overline{\overline{p}}). \quad (44) $$

As it will be shown in the next section, dimensions of the indices are related with rank properties of $p$-stage Dirac system as follows:
$\overline{p}$ is number of Lagrangian multipliers which are determined on the stage $p$;
$[p]$ is number of multipliers which remains undetermined after the stage $p$;
$[p]$ is number of $p$-stage Dirac constraints;
$[\overline{p}] = [p - 1] - \overline{p} - [p]$ is called defect of $p$-stage Dirac system (36).
Number of independent (but possibly reducible) $p$-stage symmetries, which can be find by our procedure, coincides with the defect $[\overline{p}]$, see below.

6 Normal form of $p$-stage Dirac functions

Primary Dirac constraints have been specified in section 2. Aim of this section is to give formal definition for $p$-stage Dirac constrains by induction. The definition is based on possibility to rewrite $p$-stage Dirac functions $\{\Phi_{p-1}, H\}$ in special form (45) which will be called normal form. The normal form will be our basic expression for analysis of the generating equations below.

**Definition.** 1) First-stage constraints are (functionally independent) primary constraints: $\Phi_1 \equiv \Phi_\alpha$.
2) Let $\Phi_1, \Phi_2, \ldots, \Phi_{p-1}$ is set of constraints of previous stages. Suppose that $p$-stage Dirac functions have been identically rewritten in the form

$$ \{\Phi_{p-1}, H\} = $$
\begin{equation}
A_{(p)p-1}^{p-1} \left( \begin{array}{c}
\Phi_p(q^A, p_j) + B_{(p)p}(\pi_{p-1}^{p-1}, \pi_T) \\
C_{(p)p}(\Phi_{p-1}, \Phi_2) + D_{(p)p}(\pi_{p-1}^{p-1}, \pi_T)
\end{array} \right)
\end{equation}

where

\text{a) Functions } \pi_k^{(v_k-1)}, k = 2, 3, \ldots, p \text{ have the structure}
\begin{align*}
\pi_2 &= X(2)\bar{v}_1^1 + Y(2), \\
\pi_k &= X(k) v_k^{k-1} + Y(k), \\
\text{rank} \left| \frac{\partial \pi_k}{\partial v_k} \right| &= [\pi_k] = [v_k],
\end{align*}

\text{with some coefficients } X(z_1), Y(z_1).

\text{b) } B_{(p)p}(\pi_{p-1}^{p-1}, \pi_T), \ C_{(p)p}(\Phi_{p-1}, \Phi_2), \ D_{(p)p}(\pi_{p-1}^{p-1}, \pi_T) \text{ are linear homogeneous functions of indicated variables, with coefficients dependent on } z_1 \equiv (q^A, p_j) \text{ only.}

\text{c) The matrix } A_{(p)p-1}^{p-1}(q^A, p_j) \text{ is invertible.}

\text{d) The group } z_1 \text{ is divided on } z_1 = (\bar{z}_2, \bar{z}_3, \ldots, \bar{z}_p, z_p), \text{ such that}

\begin{align*}
\text{rank} \left| \frac{\partial \Phi_k}{\partial z_1} \right| &= \text{rank} \left| \frac{\partial \Phi_k}{\partial \bar{z}_k} \right| = [\bar{z}_k] = [\Phi_k], \ k = 2, 3, \ldots, p, \\
\text{rank} \left| \frac{\partial (\Phi_2, \ldots, \Phi_p)}{\partial z_1} \right| &= \text{rank} \left| \frac{\partial (\bar{z}_2, \ldots, \bar{z}_p)}{\partial \bar{z}_2} \right| = [\bar{z}_2] + \ldots + [\bar{z}_p] = [\Phi],
\end{align*}

where } \Phi = (\Phi_2, \ldots, \Phi_p). \text{ It means that } \Phi_p(z_1) \text{ are functionally independent, and } \Phi_k, k = (1,2,\ldots,p) \text{ are functionally independent functions also (note that the primary constraints are included).}

Then the functions } \Phi_p(q^A, p_j) \text{ are called } p\text{-stage constraints.}

The matrix } A \text{ and matrices which form } B, C, D \text{ will be called } p\text{-stage structure matrices.}

To confirm the definition, we use induction over number of stage } p \text{ to prove that the Dirac functions can be actually rewritten in the normal form (45).

1) It was demonstrated in section 3 that second-stage Dirac functions can be presented in the form (45), see Eqs. (33), (23). On the case, one has } B = C = D = 0.

2) Suppose the Dirac functions of stages } k = 2, 3, \ldots, p-1 \text{ have been presented in the normal form, and thus the constraints } \Phi_1, \Phi_2,
... $\Phi_{p-1}$ are specified according to the definition. Let us consider $p$-stage Dirac functions

$$\{\Phi_{p-1}, \Phi_1 v^1 + H_0\} = \{\Phi_{p-1}, \Phi_1\} v^1 + \{\Phi_{p-1}, H_0\} = \nabla_{(p)p-1} v^1 + H_{(p)p-1}. \quad (48)$$

According to the induction hypothesis a), one has the division $v^1 = (v^\mathcal{F}, v^\mathcal{A}), v^2 = (v^\mathcal{F}, v^\mathcal{A}), \ldots, v^{p-2} = (v^{p-1}, v^{p-1})$, where, using Eq. (46), each $v^k$ can be identically written in the form (see also Lemma 1 in Appendix A)

$$v^k \equiv X(k) \pi_p^{-1} + \Lambda(k) \pi_p^{-1} + W(k)\pi_p^{-1}. \quad (49)$$

Using this representation in Eq. (48) one obtains

$$\{\Phi_{p-1}, H\} = \Delta_{(p)p-1} v^{p-1} + L_{(p)p-1} + M_{(p)p-1}(\pi_{p-1}, \ldots, \pi_{p-1}), \quad (50)$$

where $\Delta, L, M$ can be find in terms of $\nabla_{(p)}, H_{(p)}$ with help of recurrence relations, see Lemma 2 in Appendix A. Further, the conversion matrix $K_{(p)}$, constructed starting from $\Delta_{(p)}$, can be used to separate $v$-dependent and $v$-independent functions among (50). According to Lemma 1 of Appendix A one obtains

$$\{\Phi_{p-1}, H\} = K_{(p)p-1}^{p-1} \left( \Phi_p(q^A, p_j) + N_p(\pi_{p-1}, \ldots, \pi_{p-1}) \right), \quad (51)$$

where

$$\pi_p = X_{(p)p-1} v^{p-1} + Y_{(p)p-1}, \quad \Phi_p \equiv K_{(p)p-1}^{p-1} L_{(p)p-1}, \quad N_p = K_{(p)p-1}^{p-1} M_{(p)p-1}. \quad (52)$$

Here it was denoted

$$X_{(p)p-1} = K_{(p)p-1}^{p-1} \Delta_{(p)p-1}, \quad rank X_{(p)} = [\pi_p], \quad Y_{(p)p-1} = K_{(p)p-1}^{p-1} L_{(p)p-1}. \quad (53)$$

Let us analyze the functions $\Phi_p(z_1)$ in Eq. (51). Some of them may be functionally independent on the constraints of previous stages as well as functionally independent among themselves. Induction hypothesis d) allows one to write the representation (see Lemma 3 in Appendix A)

$$\Phi_p(z_1) = U_{(p)p}^{p-1} \left( C_p(\Phi_{p-1}, \ldots, \Phi_2) \right),$$
\[
\text{rank } \frac{\partial \Phi_p}{\partial \bar{z}_{p-1}} \bigg|_{\Phi} = [\Phi_p], \quad \det U \neq 0, \quad (54)
\]

where index \( \tilde{p} \) was divided on two groups \( \tilde{p} = (p, \tilde{p}) \), and the rank condition implies that \( z_{p-1} \) can be divided: \( z_{p-1} = (\bar{z}_p, z_p) \), \( [\bar{z}_p] = [\Phi_p] \), \( \bar{z}_p \) can be find through \( z_p \) from the equations \( \Phi_p = 0 \).

Substitution of Eq. (54) into Eq. (51) gives the normal form (45) of second-stage Dirac functions, with the quantities \( A(p), B(p), D(p) \) being

\[
A(p)p_{p-1} = \tilde{K}(p) \begin{pmatrix} 1_{(p)\times(p)} & 0 \\ 0 & U(p)p_{p} \end{pmatrix}, \quad (55)
\]

\[
B(p)p = (\tilde{U}(p)N)_p, \quad D(p)\tilde{p} = (\tilde{U}(p)N)\tilde{p}. \quad (56)
\]

It finishes the proof. We have described procedure to represent \( p \)-stage Dirac functions in the normal form (45).

It is easy to see that our definition of \( p \)-stage constraints is equivalent to the standard one. Actually, according to Dirac, to reveal \( p \)-stage constraints one studies the equations \( \{ \Phi_{p-1}, H \} = 0 \) on the surface of previously determined constraints and Lagrangian multipliers. But, according to our proof, on this surface the equations are equivalent to the system \( \pi_\bar{p}(v) = 0, \quad \Phi_p(q^A, p_j) = 0 \), where the first equation determines some of Lagrangian multipliers while the second one represents our \( p \)-stage constraints. Note that division on subgroups has been made in accordance with rank properties of \( p \)-stage Dirac system (45), which determines dimensions of subgroups as they were described in the end of section 5.

7 Normal form of second and third stage generating functions

In the next section we develop procedure to rewrite the set of \( s \)-stage generating functions (12) in the normal form (12), i.e. as combination of constraints. Before doing this, it is instructive to see how the procedure works for lower stages.

Let us consider second-stage generating function

\[
T^{(2)} = Q^{(2)}1 \{\Phi_1, H\}, \quad (57)
\]
where the coefficients \(Q^{(2)1}(q^A, p_j)\) are arbitrary functions. Using Eq. (33) one obtains

\[
T^{(2)} = \hat{Q}^{(2)\overline{1}} \pi_{\overline{2}} + \hat{Q}^{(2)2} \Phi_2 + \hat{Q}^{(2)\overline{2}} 0_2, \tag{58}
\]

where \(\hat{Q}\) represent blocks of converted \(Q\):

\[
Q^{(2)1} A^{(2)1} \hat{1} = \hat{Q}^{(2)\overline{1}} = (\hat{Q}^{(2)\overline{1}}_1, \hat{Q}^{(2)2}, \hat{Q}^{(2)\overline{2}}). \tag{59}
\]

Let us take \(\hat{Q}^{(2)\overline{1}} = 0\) and denote \(\hat{Q}^{(2)2} = -Q^{(2)2}\). It leads to the desired result

\[
T^{(2)} = -Q^{(2)2} \Phi_2. \tag{60}
\]

It was achieved by the following choice:

\[
Q^{(2)1} = (0, -Q^{(2)2}, \hat{Q}^{(2)\overline{2}} \overline{A}^{(2)1}), \tag{61}
\]

where \(Q^{(2)2}, \hat{Q}^{(2)\overline{2}}\) remains arbitrary functions.

Taking further \(Q^{(2)2} = 0\), one obtains solution (61) of second-stage generating equations (18). According to Statement 1, one writes out immediately \([2]\) independent local symmetries of the Lagrangian action, see Sect. 9. Number of second-stage symmetries coincides with the defect of second-stage Dirac system (33). Note also that the symmetries are specified, in fact, by second-stage structure matrix \(A_{(2)}\).

Let us consider set of third-stage generating functions

\[
T^{(2)} = Q^{(2)1} \{\Phi_1, H\}, \tag{62}
\]

\[
T^{(3)} = Q^{(3)1} \{\Phi_1, H\} + \{H, T^{(2)}\}, \tag{63}
\]

where the coefficients \(Q^{(2)1}(q^A, p_j, v^\alpha)\), \(Q^{(3)1}(q^A, p_j, v^\alpha)\) are some functions. As in the previous case, making the choice (61), one writes \(T^{(2)}\) in the normal form (61). Using this expression for \(T^{(2)}\) as well as Eq. (45), one obtains the following expression for \(T^{(3)}\)

\[
T^{(3)} = Q^{(3)1} \{\Phi_1, H\} - \{H, Q^{(2)2} \Phi_2\} =
\]

\[
\hat{Q}^{(3)\overline{1}} \left( \begin{array}{c}
\pi_{\overline{2}} \\
\Phi_2 \\
0_2
\end{array} \right) - \{H, Q^{(2)2} \} \Phi_2 + \hat{Q}^{(2)\overline{2}} \left( \begin{array}{c}
\pi_{\overline{2}} \\
\Phi_3 + B^{(3)\overline{3}}(\pi_{\overline{2}}) \\
C^{(3)\overline{3}}(\Phi_2) + D^{(3)\overline{3}}(\pi_{\overline{2}})
\end{array} \right) \tag{64}
\]

\[17\]
where
\[ Q^{(2)2} A_{(3)2} \hat{=} \hat{Q}^{(2)2} = (\hat{Q}^{(2)3}, \hat{Q}^{(2)3}, \hat{Q}^{(2)3}), \quad (65) \]
\[ Q^{(3)1} A_{(2)1} \hat{=} \hat{Q}^{(3)1} = (\hat{Q}^{(3)2}, \hat{Q}^{(3)2}, \hat{Q}^{(3)2}). \]
Collecting similar terms in Eq. (64) one has
\[ T^{(3)} = (\hat{Q}^{(3)2} + \hat{Q}^{(2)3} B_{(3)3} \bar{\Pi} + \hat{Q}^{(2)3} D_{(3)3} \bar{\Pi}) \pi_{\bar{\Pi}} + \hat{Q}^{(2)3} \pi_{\bar{\Pi}} + \left( \hat{Q}^{(3)2} - \{H, Q^{(2)2}\} + \hat{Q}^{(2)3} C_{(3)3} \right) \bar{\Phi}_2 + \hat{Q}^{(2)3} \bar{\Phi}_3 + \hat{Q}^{(3)2} 0_2. \]
Then the following choice
\[ \hat{Q}^{(2)2} = 0, \quad \hat{Q}^{(3)1} = -\hat{Q}^{(2)3} B_{(3)3} \bar{\Pi} - \hat{Q}^{(2)3} D_{(3)3} \bar{\Pi}, \quad (67) \]
\[ \hat{Q}^{(2)3} = -\hat{Q}^{(2)3}, \quad \hat{Q}^{(3)2} = -\hat{Q}^{(3)2} + \{H, \hat{Q}^{(2)3} A_{(3)3} \} - \hat{Q}^{(2)3} C_{(3)3}, \]
with arbitrary functions \( Q^{(2)3}, Q^{(3)2} \), gives \( T^{(3)} \) in the normal form
\[ T^{(3)} = -\hat{Q}^{(3)2} \Phi_2 - \hat{Q}^{(2)3} \Phi_3. \]
Thus the normal form \((60), (68)\) for the third-stage generating functions is supplied by special choice of \( Q^{(2)1}, Q^{(3)1} \). The coefficients have been divided on the following groups:
\[ Q^{(2)1} = \left( 0_{\bar{\Pi}}, 0_{\bar{\Pi}}, \hat{Q}^{(2)3}, \hat{Q}^{(2)3}, \hat{A}_{(3)2}^{(2)3}, \hat{A}_{(2)1}^{(2)3} \right), \]
\[ Q^{(3)1} = \left( \hat{Q}^{(3)2}, \hat{Q}^{(3)2}, \hat{Q}^{(3)2}, \hat{Q}^{(3)2}, \hat{A}_{(2)1}^{(3)2}, \hat{A}_{(2)1}^{(2)1} \right). \]
To describe structure of the groups, it will be convenient to use the following triangle table
\[ Q^{(2)1} \sim 0_{\bar{\Pi}}, 0_{\bar{\Pi}}, Q^{(2)3}, \hat{Q}^{(2)3}, \hat{Q}^{(2)3}, \hat{Q}^{(2)3}, \hat{Q}^{(2)3}, \hat{Q}^{(2)3}, \hat{Q}^{(3)2}, \hat{Q}^{(3)2}, \hat{Q}^{(3)2}, \hat{Q}^{(3)2}, \hat{Q}^{(3)2}, \hat{Q}^{(3)2}, \hat{Q}^{(3)2}, \hat{Q}^{(3)2}. \]
Writting out such a kind tables below, we omit the conversion matrices and write arbitrary coefficients \( Q^k \) instead of \( \hat{Q}^k \) in central column of the table. Then the central column and columns on r.h.s. of it represent coefficients which remains arbitrary. Detailed expression for division of \( Q^{(p)1} \) on subgroups of \( s \)-stage can be find in Appendix B, see Eq. (135).
Taking \( Q^{(2)3} = 0, Q^{(3)2} = 0 \), one obtains solution \( (70), (67) \) of third-stage generating equations \( (19) \). According to Statement
1, it implies \( \exists \) independent third-stage local symmetries of the Lagrangian action, see Sect. 9. Number of the symmetries coincides with the defect of third-stage Dirac system \( \{15\} \). The symmetries are specified, in fact, by the structure matrices \( A(2), A(3), B(3), C(3), D(3) \).

8 Normal form of \( s \)-stage generating functions

We demonstrate here that the set of \( s \)-stage generating functions can be written in the normal form \( \{72\} \).

**Statement 3.** Consider the Lagrangian theory with the Hamiltonian \( H \), and with constraints at most \( N \) stage appeared in the Hamiltonian formulation: \( \Phi_1, \Phi_2, \ldots, \Phi_N \). For some fixed integer number \( s \), \( 2 \leq s \leq N + 1 \), let us construct the generating functions according to recurrence relations \( (T(1) = 0)\)

\[
T(p) = Q^{(p)} \{ \Phi_1, H \} + \{ H, T^{(p-1)} \}, \quad p = 2, 3, \ldots, s. \quad (71)
\]

Then the coefficients \( Q^{(p)}(q^A, p, v^a) \) can be chosen in such a way, that all \( T(p) \) turn out to be linear combinations of the constraints

\[
T(p) = - \sum_{n=2}^{p} Q^{(p+2-n)n} \Phi_n, \quad p = 2, 3, \ldots, s. \quad (72)
\]

Choice of \( Q^{(p)} \), which supplies the normal form can be described as follows:

a) For any \( k = 2, 3, \ldots, s, \quad n = 1, 2, \ldots, (s + 1 - k) \), \( Q^{(k)n} \) is divided on three subgroups with help of the structure matrix of \( n + 1 \) stage \( A_{(n+1)} \)

\[
Q^{(k)n} A_{(n+1)} \equiv \hat{Q}^{(k)n} = (\hat{Q}^{(k)n+1}, \hat{Q}^{(k)n+1}, \hat{Q}^{(k)n+1}), \quad (73)
\]

where for any \( p = 2, 3, \ldots, s, \quad n = 2, 3, \ldots, p \) one has

\[
\hat{Q}^{(p+2-n)n} = -Q^{(p+2-n)n} + \{ H, \hat{Q}^{(p+1-n)n} A_{(n+1)n} \} - \sum_{m=n}^{p-1} \hat{Q}^{(p+1-m)n+1} C_{(m+1)m+1}^{n}, \quad \Rightarrow \hat{Q}^{(2)p} = -Q^{(2)p}, \quad (74)
\]

\[
\hat{Q}^{(p+2-n)p} = - \sum_{m=n}^{p-1} (\hat{Q}^{(p+1-m)n+1} B_{(m+1)m+1}^{n+1} + \ldots
\]

19
b) The coefficients \( \hat{Q}(n) \), \( k = 2, 3, \ldots, s \), \( n = 1, 2, \ldots, (s + 1 - k) \) remain arbitrary.
c) The coefficients \( Q^{(s+2-n)n}, n = 2, 3, \ldots, s \) remain arbitrary.

Before carrying out a proof, let us confirm that the recurrence relations \( (74), (75) \) actually determines the coefficients. According to the statement, \( Q^{(p)1} \) is converted into \( \hat{Q}^{(p)1} \), and then is divided on subgroups \( \hat{Q}^{(p)2}, \hat{Q}^{(p)3} \). Then \( Q^{(p)2} \) is picked out from \( \hat{Q}^{(p)2} \) according to Eq. (73). The coefficient \( Q^{(p)2} \) will be further converted and divided, creating \( Q^{(p)3} \), and so on. Resulting structure of the coefficients \( Q^{(p)1} \) can be described by the following table (first line represents \( Q^{(2)1} \), second line represents \( Q^{(3)1} \), and so on, up to \( Q^{(s)1} \)):

\[
\begin{array}{cccccccccccc}
0^2 & 0^3 & 0^3 & \cdots & 0^3 & Q^{(2)s} & \hat{Q}^{(2)s} & \cdots & \hat{Q}^{(2)3} & \hat{Q}^{(2)2} \\
Q^{(3)2} & Q^{(3)3} & \cdots & Q^{(3)s-1} & Q^{(3)s-1} & \cdots & \hat{Q}^{(3)s-1} & \cdots & \hat{Q}^{(3)s-1} & \cdots \\
\cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots \\
\hat{Q}^{(p)2} & \cdots & \hat{Q}^{(p)s+2-p} & Q^{(p)s+2-p} & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots \\
\cdots & \cdots & \hat{Q}^{(s-1)2} & Q^{(s-1)3} & \hat{Q}^{(s-1)3} & \cdots & \hat{Q}^{(s-1)3} & \cdots & \hat{Q}^{(s-1)3} & \cdots \\
\end{array}
\]

Note that any group \( Q^{(p)n} \) with \( n \neq s+2-p \) is presented on the table by the interval of \( p \)-line between \( \hat{Q}^{(p)n+1} \) and \( \hat{Q}^{(p)n+1} \). Manifest form for division of \( Q^{(p)1} \) on subgroups of \( s \)-stage can be find in Appendix B, see Eq. (135).

To analyse the expressions \( (74), (75) \) it is convenient to fix number of line: \( p + 2 - n = k \), then they can be written in the form

\[
\hat{Q}^{(k)n} = -Q^{(k)n} + \{H, \hat{Q}^{(k-1)n} \hat{A}_{(n+1)n} \} - \sum_{m=2}^{k-1} \hat{Q}^{(m)k+n-m} C_{(k+n-m)k+n-m}, \quad (77)
\]

\[
\hat{Q}^{(k)p} = -\sum_{m=2}^{k-1} \left( \hat{Q}^{(m)k+n-m} B_{(k+n-m)k+n-m} + \hat{Q}^{(m)k+n-m} D_{(k+n-m)k+n-m} \right), \quad (78)
\]
where \( k = 2, 3, \ldots, s, \) \( n = 2, 3, \ldots, (s + 2 - k) \). From this it follows that any group \( \hat{Q}^{(k)n} \) of the line \( k \) of the triangle is expressed through some groups placed in previous lines on r.h.s. of \( \hat{Q}^{(k)n} \). Any group \( \hat{Q}^{(k)n} \) is presented through the interval \([ \hat{Q}^{(k)n} + 1, \hat{Q}^{(k)n} + 1 \]) of the line \( k \) as well as through some groups of previous lines placed on r.h.s. of \( n \) column. After all, all the coefficients are expressed through \( \hat{Q}^{(k)n} \), \( Q^{(n)s+2-n} \equiv Q^{(n)s+2-n} \), which remains arbitrary functions (the latters are placed in the central column and on r.h.s. of it in the triangle). Note that all arbitrary functions \( Q^{(s+2-n)n} \) appear in the expression for higher generating function \( T^{(s)} \).

**Proof.** The statement 3 will be demonstrated by induction over \( s, 2 \leq s \leq N + 1 \). It was shown in section 7 (see Eqs. (60), (61)) that the statement is true for \( s = 2 \). Supposing that the statement is true for \( s = p - 1 \), let us consider generating functions of the stage \( s = p \): \( T^{(2)}, T^{(3)}, \ldots, T^{(p-1)}, T^{(p)} \). According to induction hypothesis, \( T^{(2)}, T^{(3)}, \ldots, T^{(p-1)} \) can be written in the normal form

\[
T^{(k)} = - \sum_{n=2}^{k} Q^{(k+2-n)n} \Phi_n, \quad k = 2, 3, \ldots, p - 1.
\]

where the coefficients \( Q^{(k)} \), \( k = 2, 3, \ldots, p \) have the structure

\[
\begin{array}{ccccccc}
0^{\hat{Q}} & 0^{\hat{Q}} & \ldots & 0^{\hat{Q}} & 0^{\hat{Q}} & \ldots & 0^{\hat{Q}} \\
\hat{Q}^{(3)\hat{Q}} & \hat{Q}^{(3)\hat{Q}} & \ldots & \hat{Q}^{(3)\hat{Q}} & \hat{Q}^{(3)\hat{Q}} & \ldots & \hat{Q}^{(3)\hat{Q}} \\
\ldots & \ldots & \ldots & \ldots & \ldots & \ldots & \ldots \\
\hat{Q}^{(p-1)\hat{Q}} & \hat{Q}^{(p-1)\hat{Q}} & \ldots & \hat{Q}^{(p-1)\hat{Q}} & \hat{Q}^{(p-1)\hat{Q}} & \ldots & \hat{Q}^{(p-1)\hat{Q}} \\
Q^{(p)\hat{Q}} & Q^{(p)\hat{Q}} & \ldots & Q^{(p)\hat{Q}} & Q^{(p)\hat{Q}} & \ldots & Q^{(p)\hat{Q}}
\end{array}
\]

with arbitrary functions placed in the central column and on r.h.s. of it. Let us consider the remaining generating function

\[
T^{(p)} = Q^{(p)\hat{Q}} \{ \Phi_1, H \} - \{ H, \sum_{n=2}^{p-1} Q^{(p+1-n)n} \Phi_n \},
\]

Note that only \( Q \) of the central column are presented in this expression. Being arbitrary functions, these \( Q \) can be used to write the expression in the normal form, as it is demonstrated in Lemma 4 of Appendix B. According to the Lemma, each coefficient of the central column in Eq. (80) is divided on three subgroups (73). From comparison of Eqs. (71), (72) with corresponding equations (126), (127)
of Lemma 4 one obtains rules to transform quantities of Lemma 4 into our case

\[ Q^n \rightarrow Q^{(p+1-n)n}, \quad \hat{Q}^n \rightarrow \hat{Q}^{(p+1-n)n}, \quad Q^n \rightarrow Q^{(p+2-n)n}. \] (82)

It implies that \( \hat{Q}^{(p+2-n)n} \), \( \hat{Q}^{(p+2-n)n} \) are determined by Eqs. (74), (75). Note that division (73) has been made in accordance with rank properties of \( p \)-stage Dirac system (45), the latter determines dimensions of subgroups (see description of the dimensions after Eq. (44)). ♠

For completeness, let us present normal form of lower-stage generating functions.

**Normal form of second-stage generating function.** For \( s = 2 \) one obtains immediately (see also section 7)

\[ T^{(2)} = \hat{Q}^{(2)} \{ \Phi_1, H \} = -\hat{Q}^{(2)} \Phi_2, \] (83)

where

\[ \hat{Q}^{(2)} = \begin{pmatrix} 0 & -\hat{Q}^{(2)} & \hat{Q}^{(2)} \end{pmatrix} \tilde{A}^{(2)}_{1}, \] (84)

with arbitrary functions \( \hat{Q}^{(2)}(q^A, p_j), \hat{Q}^{(2)}(q^A, p_j) \).

**Normal form of third-stage generating functions.**

\[ T^{(2)} = \hat{Q}^{(2)} \{ \Phi_1, H \} = -\hat{Q}^{(2)} \Phi_2, \]
\[ T^{(3)} = \hat{Q}^{(3)} \{ \Phi_1, H \} + \{ H, T^{(2)} \} = \hat{Q}^{(3)} \{ \Phi_1, H \} - \{ H, \hat{Q}^{(2)} \Phi_2 \} = -\hat{Q}^{(3)} \Phi_2 - \hat{Q}^{(2)} \Phi_3. \] (85)

Division of the initial coefficients can be described by the triangle

\[ \begin{array}{cccc}
Q^{(2)} & \sim & 0^2 & 0^3 & \hat{Q}^{(2)} & \hat{Q}^{(2)} & \hat{Q}^{(2)} \\
Q^{(3)} & \sim & \hat{Q}^{(3)} & \hat{Q}^{(3)} & \hat{Q}^{(3)} & \hat{Q}^{(3)} & \hat{Q}^{(3)}
\end{array} \] (86)

where the central column and columns on r.h.s. of it represent functions which remain arbitrary. Manifest form of the coefficients which provides the normal form (85) is as follows:

\[ \begin{align*}
Q^{(2)} &= \begin{pmatrix} 0^2, \hat{Q}^{(2)} & \hat{Q}^{(2)} \end{pmatrix} \tilde{A}^{(2)}_{1}, \\
Q^{(3)} &= \begin{pmatrix} \hat{Q}^{(3)} & \hat{Q}^{(3)} & \hat{Q}^{(3)} \end{pmatrix} \tilde{A}^{(2)}_{1}, \\
\hat{Q}^{(2)} &= -\hat{Q}^{(2)}.
\end{align*} \] (87)
\[ \hat{Q}^{(3)\overrightarrow{2}} = Q^{(2)3} B_{(3)3} \overrightarrow{3} - \hat{Q}^{(2)3} D_{(3)3} \overrightarrow{3}, \]
\[ \hat{Q}^{(3)2} = -Q^{(3)2} + \{H, (0^\overrightarrow{3}, -Q^{(2)3}, \hat{Q}^{(2)3}) \} \hat{A}_{(3)2} \overrightarrow{2} - \hat{Q}^{(2)3} C_{(3)3} \overrightarrow{2}. \quad (88) \]

Here all \( Q \) on r.h.s. of Eq. (88) are arbitrary functions.

The coefficients appeared in \( T^{(3)} \) remains arbitrary functions, while \( Q^{(2)2} \) in \( T^{(2)} \) is
\[ Q^{(2)2} = (0^\overrightarrow{3}, -Q^{(2)3}, \hat{Q}^{(2)3}) \hat{A}_{(3)2} \overrightarrow{2}. \quad (89) \]

**Normal form of 4-stage generating functions.**

\[
T^{(2)} = Q^{(2)1} \{\Phi_1, H\} = -Q^{(2)2} \Phi_2, \quad T^{(3)} = Q^{(3)1} \{\Phi_1, H\} + \{H, T^{(2)}\} = Q^{(3)2} \{\Phi_1, H\} - \{H, Q^{(2)2} \Phi_2\} = -Q^{(3)2} \Phi_2 - Q^{(2)3} \Phi_3, \quad (90) \\
T^{(4)} = Q^{(4)1} \{\Phi_1, H\} + \{H, T^{(3)}\} = Q^{(4)2} \{\Phi_1, H\} - \{H, Q^{(3)2} \Phi_2 + Q^{(2)3} \Phi_3\} = -Q^{(4)2} \Phi_2 - Q^{(3)3} \Phi_3 - Q^{(2)4} \Phi_4. \\
\]

Division of the initial coefficients can be described by the triangle
\[
\begin{align*}
Q^{(2)1} & \sim 0^\overrightarrow{2} & 0^\overrightarrow{3} & Q^{(2)4} & \hat{Q}^{(2)4} & \hat{Q}^{(2)3} & \hat{Q}^{(2)2} \\
Q^{(3)1} & \sim \hat{Q}^{(3)\overrightarrow{2}} & \hat{Q}^{(3)\overrightarrow{3}} & Q^{(3)3} & \hat{Q}^{(3)3} & \hat{Q}^{(3)2} & \hat{Q}^{(3)1} \\
Q^{(4)1} & \sim \hat{Q}^{(4)\overrightarrow{2}} & \hat{Q}^{(4)\overrightarrow{3}} & Q^{(4)2} & \hat{Q}^{(4)2} & \hat{Q}^{(4)1} & \hat{Q}^{(4)1} \\
\end{align*}
\]
where the central column and columns on r.h.s. of it are arbitrary functions. Manifest form of the coefficients which provides the normal form (90) is as follows:

\[
Q^{(2)1} = (0^\overrightarrow{3}, (0^\overrightarrow{3}, \hat{Q}^{(2)4}, \hat{Q}^{(2)4}) \hat{A}_{(4)3} \overrightarrow{3}, \hat{Q}^{(2)3}) \hat{A}_{(3)2} \overrightarrow{2}, \hat{Q}^{(2)2} \hat{A}_{(2)1} \overrightarrow{1}, \\
Q^{(3)1} = (\hat{Q}^{(3)\overrightarrow{2}}, (\hat{Q}^{(3)\overrightarrow{3}}, \hat{Q}^{(3)\overrightarrow{3}}, \hat{Q}^{(3)3}) \hat{A}_{(3)2} \overrightarrow{2}, \hat{Q}^{(3)2}) \hat{A}_{(2)1} \overrightarrow{1}, \\
Q^{(4)1} = (\hat{Q}^{(4)\overrightarrow{2}}, \hat{Q}^{(4)\overrightarrow{3}} \hat{Q}^{(4)2}) \hat{A}_{(2)1} \overrightarrow{1}. \quad (91) \\
\]

\[
\begin{align*}
\hat{Q}^{(2)4} & = -Q^{(2)4}, \\
\hat{Q}^{(3)\overrightarrow{2}} & = (0^\overrightarrow{3}, -Q^{(2)4}, \hat{Q}^{(2)4}) \hat{A}_{(4)3} \overrightarrow{3}, B_{(3)3} \overrightarrow{3} - \hat{Q}^{(2)3} D_{(3)3} \overrightarrow{3}, \\
\hat{Q}^{(3)\overrightarrow{3}} & = Q^{(2)4} B_{(4)4} \overrightarrow{3} - \hat{Q}^{(2)4} D_{(4)4} \overrightarrow{3}, \\
\hat{Q}^{(3)3} & = -Q^{(3)3} + \{H, (0^\overrightarrow{3}, -Q^{(2)4}, \hat{Q}^{(2)4}) \} \hat{A}_{(4)3} \overrightarrow{3} - \hat{Q}^{(2)4} C_{(3)3} \overrightarrow{3}, \\
\hat{Q}^{(4)\overrightarrow{2}} & = Q^{(2)4} B_{(4)4} \overrightarrow{3} - \hat{Q}^{(2)4} D_{(4)4} \overrightarrow{3} - (\hat{Q}^{(3)3} + \\
\{H, (0^\overrightarrow{3}, -Q^{(2)4}, \hat{Q}^{(2)4}) \} \hat{A}_{(4)3} \overrightarrow{3} - \hat{Q}^{(2)4} C_{(3)3} \overrightarrow{3} - \hat{Q}^{(2)4} D_{(3)3} \overrightarrow{3}, \\
\hat{Q}^{(4)\overrightarrow{3}} & = -Q^{(3)3} + \{H, (0^\overrightarrow{3}, -Q^{(2)4}, \hat{Q}^{(2)4}) \} \hat{A}_{(4)3} \overrightarrow{3} - \hat{Q}^{(2)4} C_{(4)4} \overrightarrow{3}, \\
\hat{Q}^{(3)\overrightarrow{3}} & = (H, (0^\overrightarrow{3}, -Q^{(2)4}, \hat{Q}^{(2)4}) \} \hat{A}_{(4)3} \overrightarrow{3} - \hat{Q}^{(2)4} C_{(4)4} \overrightarrow{3}. \quad (93) \\
\end{align*}
\]
Here all $Q$ on r.h.s. of Eq. (93) are arbitrary functions.

The coefficients $Q$ in $T^{(4)}$ remain arbitrary functions. The line between $0^\sigma$ and $\tilde{Q}^{(2)3}$ in $Q^{(2)1}$ represents the coefficient $Q^{(2)2}$ which appears in $T^{(2)}$, similarly can be find all the coefficients appeared in $T^{(3)}$. One obtains their manifest form as follows:

\begin{align*}
Q^{(2)2} &= \left(0^\sigma, 0^\tau, -Q^{(2)4}, Q^{(2)4}\right) \tilde{\mathcal{A}}^{(4)3}_3, \\
Q^{(2)3} &= \left(0^\sigma, -Q^{(2)4}, Q^{(2)4}\right) \tilde{\mathcal{A}}^{(4)3}_3, \\
Q^{(3)2} &= \left(Q^{(2)4}B^{(4)4}_4 - \tilde{Q}^{(2)4}D^{(4)4}_4, -Q^{(3)3} + \right. \\
&\left. \{H, \left(0^\sigma, -Q^{(2)4}, Q^{(2)4}\right) \tilde{\mathcal{A}}^{(4)3}_3\} - \tilde{Q}^{(2)4}C^{(4)4}_4, \tilde{Q}^{(3)3}\right) \tilde{\mathcal{A}}^{(3)2}_3. \\
\end{align*}

(94)

9 Gauge symmetries of quadratic theory

Suppose that in the Hamiltonian formulation of our theory there are appear constraints up to at most $N$-stage. According to the Statement 1, symmetries of different stages are looked for separately. Generators of $s$-stage local symmetries (15)-(17) can be constructed starting from any solution of generating equations (13)-(14). Using normal form (72)-(78) of generating functions, one concludes that Eq. (14) is satisfied by taking $Q^{(p), s+2-p} = 0$, $p = 2, 3, \ldots, s$, i.e. all the coefficients in the central column of the triangle (76) must be zeros. Eq. (13) states that generating functions $T^{(p)}$ with $p = 2, 3, \ldots, s - 1$ do not depend on the Lagrangian multipliers. Dependence on $v^1$ can appear only due to second term in Eq. (74) (or, equivalently, in Eq. (77)). Thus one needs to kill this term, which can be easily achieved in a theory with all the structure matrices $\tilde{\mathcal{A}}$ (see Eq. (15)) being numerical matrices. It happens, in particular, in any quadratic theory (then all the structure matrices $A, B, C, D$ in Eq. (15) turn out to be numerical matrices). We analyse this case in the present section. For the case, it is consistent to look for solutions with $Q = const$, then the second term in Eq. (74) disappears, and the generating equations (13)-(14) are trivially satisfied.

Thus for any quadratic Lagrangian theory it is sufficient to take elements of central column of the triangle (76) be zeros, and elements on r.h.s. of it be arbitrary numbers, to obtain some local symmetry of the Lagrangian action (16), (17). In the case, the generators turn out to be numerical matrices.

Let us discuss some particular set of generators constructed as
follows. On the stage $s$ of the Dirac procedure, one takes $[\tilde{s}]$ sets of $\hat{Q}^{(2)}\tilde{s}$, namely $\hat{Q}^{(2)}_{\tilde{s}} = \delta_{\tilde{s}}\tilde{s}$, where $[\tilde{s}]$ is defect of the system (45).

Remaining arbitrary coefficients on r.h.s. of the triangle (76) are taken vanishing. Then these $[\tilde{s}]$ solutions $Q^{(p)}\tilde{s}, p = 2, 3, \ldots, s$ of generating equations have the form (135) of Appendix C, where one needs to substitute

$$Q^{(p)}\tilde{s} + 2 - p = 0, \quad \hat{Q}^{(p)}\tilde{s} = \delta_{\tilde{s}}\tilde{s}, \quad p \neq 2, \quad k \neq s,$$

while others coefficients can be find from Eqs. (77), (78), the latter acquire the form

$$\hat{Q}^{(2)}\tilde{s}^{n} = -Q^{(2)}\tilde{s}^{n}, \quad \hat{Q}^{(k)}\tilde{s}^{n} = -Q^{(k)}\tilde{s}^{n} - \delta_{s}k^{n-2}C(s)s^{n},$$

$$\delta_{s}k^{n-2}(\hat{Q}(s)\tilde{s}^{n} + \sum_{m=2}^{k-1}Q^{(m)}k+n-m B(k+n-m)k+n-m + \sum_{m=2}^{k-1}C(s)s^{m+n-m}B(k+n-m)k+n-m).$$

It gives a set of $s$-stage local symmetries (17), number of them coincides with defect $[\tilde{s}]$ of $s$-stage Dirac system (45)

$$\delta_{s}q^{A} = \sum^{s-2}_{p=0} (p)\hat{s} R^{(p)}_{\tilde{s}}, \quad R^{(p)}_{\tilde{s}} = Q^{(s-p)}\tilde{s}.$$  \hspace{1cm} (97)

One notes that

$$\delta_{s}q^{1} = (s-2)\hat{s}Q^{(2)}\tilde{s} + \ldots + (s-2)\hat{s}\tilde{s} + \ldots, \quad \tilde{s}E(s)\tilde{s}^{1} = A(s)\tilde{s}^{s-2}A(s-1)\tilde{s}^{s-2} \ldots A(2)\tilde{s}^{1},$$

where by construction

$$\text{rank} \tilde{E}(s)\tilde{s}^{1} = [\tilde{s}] = \text{max}.$$  \hspace{1cm} (98)

It implies that the symmetries obtained are independent.

Let us construct these symmetries for $s = 2, 3, \ldots, N + 1$. The procedure stops on the stage $s = N + 1$, since the structure matrices $A, B, C, D$ are not defined for $N + 2$. Then total number of the symmetries which can be constructed by using of our procedure is

$$\sum_{s=2}^{N+1} [\tilde{s}] = \sum_{s=2}^{N+1} [s - 1] - \sum_{s=2}^{N+1} [s] - \sum_{s=2}^{N} [s] =$$

25
\[
\sum_{s=1}^{N} [s] - \sum_{s=2}^{N+1} [s] = [1] - \sum_{s=2}^{N+1} [s] = \left[ q^{N+1} \right], \tag{100}
\]
i.e. coincides with the number of Lagrangian multipliers remaining undetermined in the Dirac procedure. All the symmetries obtained are independent in the sense that matrix constructed from the blocks \( R_s^{(s-2)} \), \( s = 2, 3, \ldots, N + 1 \) has maximum rank by construction.

Using Eqs. (95), (96), it is not difficult to write manifest form of lower-stage symmetries, namely

**Second-stage symmetries**

\[
\delta_2 q^1 = \epsilon^2 \bar{A}_{(2)} \bar{q} \tag{101}
\]

**Third-stage symmetries**

\[
\delta_3 q^1 = -\epsilon^3 \left( D_{(3)} \bar{\bar{q}} \bar{A}_{(2)} + C_{(3)} \bar{A}_{(2)}^2 \right) + \epsilon^3 \bar{A}_{(3)} \bar{A}_{(2)} \bar{q} \tag{102}
\]

**4-stage symmetries**

\[
\delta_4 q^1 = -\epsilon^4 \left( \left( D_{(4)} \bar{q} - C_{(4)} \bar{A}_{(3)} \bar{A}_{(2)} \right) \bar{A}_{(2)} + C_{(4)} \bar{A}_{(2)}^2 \right) + \epsilon^4 \left( \bar{A}_{(4)} \bar{A}_{(3)} \bar{A}_{(2)} \right) + \epsilon^4 \left( \bar{A}_{(4)} \bar{A}_{(3)} \bar{A}_{(2)} \right) \bar{q} \tag{103}
\]

Thus knowledge of structure matrices \( A, B, C, D \) of the Dirac procedure determines independent local symmetries of the Lagrangian action. Number of the symmetries coincides with number of Lagrangian multipliers remaining arbitrary in the end of Dirac procedure. Surprising conclusion following from the present analysis is that search for gauge symmetries in quadratic theory do not requires separation of the Dirac constraints on first and second class subsets.

10 **Acknowledgments**

Author would like to thank the Brazilian foundations CNPq and FAPEMIG for financial support.

11 **Appendix A**

We present here three Lemmas which are used in section 5 to rewrite \( p \)-stage Dirac functions in the normal form (45).
Suppose $\triangle_{p-1, p-1}(z_1)$ be a matrix with $\text{rank}\triangle = [\overline{p}], [p - 1] - [\overline{p}] = [\overline{p}]$. Let $K_p^{p-1}$ represents complete set of independent null-vectors of $\triangle$ and $\mathcal{K}_p^{p-1}$ be any completion of the set $\mathcal{K}_p^{p-1}$ up to base of $[p - 1]$-dimensional space. By construction, the matrix

$$K_{p-1}^{p-1} = \begin{pmatrix} K_p^{p-1} \\ K_p^{p-1} \end{pmatrix},$$

(104)

is invertible, with the inverse matrix being $\mathcal{K}_p^{p-1}K_p^{p-1} = \delta_{p-1}^{p-1}$.

**Lemma 1** For some set of variables $v^{p-1}$, let us consider linear functions $\triangle_{p-1, p-1}v^{p-1} + L_{p-1}(z_1) + M_{p-1}(z_1, \Pi)$, where $\Pi$ is group of some variables. Then

a) The functions can be identically rewritten in the form

$$\triangle_{p-1, p-1}v^{p-1} + L_{p-1} + M_{p-1} = \mathcal{K}_{p-1}^{p-1}\left( \pi_p (v^{p-1}) + \Phi_p(z_1) + N_p(z_1, \Pi) \right),$$

(105)

$$\pi_p \equiv X_{p-1}^{p-1}v^{p-1} + Y_{p},$$

$$\Phi_p \equiv \mathcal{K}_p^{p-1}L_{p-1},$$

$$N_p \equiv \mathcal{K}_p^{p-1}M_{p-1}.$$  

(106)

Here it was denoted

$$X_{p-1}^{p-1} = K_p^{p-1}\triangle_{p-1, p-1}, \quad \text{rank} X = [\pi_p],$$

$$Y_{p} = K_p^{p-1}L_{p-1}.$$  

(107)

b) $v^{p-1}$ can be divided on groups $v^p, v^\mathcal{P}$ in such a way that there is identity

$$v^{p-1} = \mathcal{X}^{p, p}v^p + \mathcal{Y}_{\mathcal{P}}v^\mathcal{P} + W^p.$$  

(108)

**Proof.** Computing product of $1 = K\mathcal{K}$ with $\triangle v + L + M$ one obtains Eqs. (105)-(107). Let us show that $\text{rank} X = [\pi_p]$. Suppose that $\text{rank} X < [\pi_p]$, then there is null-vector $\xi_p^{p-1}\triangle_{p-1, p-1} = 0$. If $\xi_p^{p-1}K_p^{p-1} = 0$, the vectors $\mathcal{K}_p^p$ are linearly dependent, which is contradiction. If $[\xi_p^{p-1}K_p^{p-1}] \triangle_{p-1, p-1} = 0$, the linear combination $\xi_p^{p-1}\mathcal{K}_p^{p}$ of the vectors $\mathcal{K}_p^p$ is null vector of $\triangle$. It is impossible, since $\mathcal{K}_p^p$ form basis of null-vectors and $\mathcal{K}_p^p$ are linearly independent of them by construction.
Since \( \text{rank}X = [\pi]\) = \( \max \), the first equation in (106) can be resolved in relation of some subgroup \( \pi^p \) of the group \( v^p = (v^p, v^2) \), which implies the identity Eq. (108). ♠

Suppose a group \( v^1 \) is divided on subgroups as follows: \( v^1 = (v^\pi, v^2), v^2 = (v^\pi, v^3), \ldots, v^{p-2} = (v^{p-1}, v^{p-1}), \) with the resulting division being \( v^1 = (v^\pi, v^2, \ldots, v^{p-1}, v^{p-1}) \). Consider the functions

\[
\pi^p(v^{p-1}) = X_{(n)}\pi^{n-1}v^{p-1} + Y_{(n)}\pi,
\]

\[
\text{rank}X = [\pi], \quad n = 2, 3, \ldots, p - 1.
\] (109)

As it was discussed in the proof of the Lemma 1, one writes the identities

\[
v^\pi \equiv \tilde{X}_{(n)}^{\pi\pi'} \pi^\pi (v^{p-1}) + \Lambda_{(n)}^{\pi\pi'} v^n + W_{(n)}^{\pi}. \tag{110}
\]

**Lemma 2** Let \( 3 \leq k \leq p \) be some fixed number and \( \nabla_{p-1}(z_1) \), \( H_{p-1}(z_1) \) are some quantities. Then there is identity

\[
\nabla_{p-1,1}v^1 + H_{p-1} = \\
\Delta_{(k)p-1,k-1}v^{k-1} + L_{(k)p-1} + M_{(k)p-1}(\pi_{k-1}, \ldots, \pi_{\pi}), \tag{111}
\]

where the quantities \( \Delta, \ L, \ M \) can be find from the following recurrence relations (for \( k = 2 \) one takes \( \Delta_{(2)p-1,1} = \nabla_{p-1,1}, L_{(2)p-1} = H_{p-1}, M_{(2)p-1} = 0 \)):

\[
\Delta_{(k)p-1,k-1} = \Delta_{(k-1)p-1,k-1} + \Delta_{(k-1)p-1,k-1}L_{(1)p-1}^{\pi_{k-1}} + W_{(k-1)}^{\pi_{k-1}} \tag{112}
\]

**Proof.** (Induction over \( k \).) For \( k = 3 \) one has

\[
\nabla_{p-1,1}v^{1} + H_{p-1} = \nabla_{p-2}v^{2} + \nabla_{p-1,\pi}^{\pi}(\tilde{X}_{(2)}^{\pi\pi'} \pi_{\pi'} + \Lambda_{(2)}^{\pi\pi'} v^{\pi} + W_{(2)}^{\pi}) \\
H_{p-1} = (\nabla_{p-2}v^{2} + \nabla_{p-1,\pi}^{\pi} \Lambda_{(2)}^{\pi\pi'} v^{\pi}) v^{2} + (H_{p-1} + \nabla_{p-1,\pi}^{\pi} W_{(2)}^{\pi}) \tag{113}
\]

\[
\nabla_{p-1,\pi}^{\pi} \tilde{X}_{(2)}^{\pi\pi'} \pi_{\pi'} \equiv \Delta_{(3)p-1,2}v^{2} + L_{(3)p-1} + M_{(3)p-1}(\pi_{\pi}),
\]

in accordance with (112). Now, let us suppose that the Lemma is true for \( k - 1 \). According to the induction hypothesis, one writes

\[
\nabla_{p-1,1}v^{1} + H_{p-1} = \\
\Delta_{(k-1)p-1,k-2}v^{k-2} + L_{(k-1)p-1} + M_{(k-1)p-1}.
\]
Substitution of $v^{k-2}$ in the form $v^{k-1}, v^{k-1}$, where $v^{k-1}$ is given by Eq. (111) leads, similarly to the previous computation, to Eqs. (111), (112).

Let $\Phi(z_1)$ be system of functionally independent functions, and $z_1$ is divided on $(\bar{z}_2, z_2)$ in such a way that

$$\text{rank} \frac{\partial \Phi}{\partial z_1}|_{\Phi} = \text{rank} \frac{\partial \Phi}{\partial \bar{z}_2}|_{\Phi} = [\Phi] = [\bar{z}_2].$$

(114)

Let $\Phi_p(z_1)$ are some functions and

$$\text{rank} \frac{\partial(\Phi, \Phi_p)}{\partial z_1}|_{\Phi, \Phi_p} = a.$$  

(115)

**Lemma 3.** There is the representation

$$\Phi_p(z_1) = U_p \tilde{\Phi}_p(\Phi, C_p(\Phi)),$$

(116)

where

a) $U(z_1)$ is invertible matrix.

b) $C_p(\Phi)$ are linear homogeneous functions with the coefficients dependent on $z_1$ only.

c) $z_2 = (\bar{z}_3, z_3)$ such that

$$\text{rank} \frac{\partial \Phi_p}{\partial z_1}|_{\Phi_p} = \text{rank} \frac{\partial \Phi_p}{\partial \bar{z}_3}|_{\Phi_p} = [\Phi_p] = [\bar{z}_3] = a - [\Phi].$$

(117)

d) Functions $\Phi$, $\Phi_p$ are functionally independent.

This result means that the system of independent functions $\Phi$, $\Phi_p$ is equivalent to the initial system $\Phi$, $\Phi_p$.

**Proof.** Eqs. (114), (115) imply that the matrix $\frac{\partial(\Phi, \Phi_p)}{\partial z_1}$ has rank minor composed of lines $\Phi$ and some of lines of $\Phi_p$. One uses numeric invertible matrix $Q$ to rearrange the lines

$$\begin{pmatrix}
\Phi \\
\Phi_p
\end{pmatrix} = 
\begin{pmatrix}
1_{[\Phi] \times [\Phi]} & 0 \\
0 & Q_{[\Phi] \times [\Phi]}
\end{pmatrix}
\begin{pmatrix}
\Phi \\
\Phi_p
\end{pmatrix}$$

(118)

in such a way that $(\Phi, \Phi_p')$ are functionally independent

$$\text{rank} \frac{\partial(\Phi, \Phi_p')}{\partial z_1}|_{\Phi, \Phi_p'} = a, \quad \text{rank} \frac{\partial \Phi_p'}{\partial z_1}|_{\Phi, \Phi_p'} = [\Phi_p'] = a - [\Phi].$$

(119)
Under these conditions the right column in Eq. (118) has the representation
\[ \begin{bmatrix} \Phi'(ar{z}_2, z_2) \\ \Phi_p'(z_2) \end{bmatrix} = \begin{bmatrix} E & F \\ G & H \end{bmatrix} \begin{bmatrix} \Phi' \\ \Phi_p' \end{bmatrix}(\bar{z}_2, z_2) \]
(120)
where \((S)\) is the block matrix.

\[ \text{rank} S|_{\Phi, \Phi_p} = a, \quad \text{rank} \frac{\partial \Phi'}{\partial z_1}|_{\Phi'} = \text{rank} \frac{\partial \Phi}{\partial \bar{z}_2}|_{\Phi} = [\Phi'] = [\bar{z}_2], \quad \text{rank} \frac{\partial \Phi_p'}{\partial z_2}|_{\Phi_p'} = [\Phi_p'] = a - [\Phi]. \]  
(121)

According to these conditions, the functions \(\Phi'(ar{z}_2, z_2), \Phi_p'(z_2)\) are functionally independent, and \(z_2\) can be divided on \((\bar{z}_3, z_3)\), \([\bar{z}_3] = [\Phi_p']\), such that \(\bar{z}_2, \bar{z}_3\) can be find from equations \(\Phi'(ar{z}_2, z_2) = 0, \Phi_p'(z_2) = 0\) in terms of \(z_3\).

Since \((\Phi, \Phi_p), (\Phi', \Phi_p')\) and \(\Phi\) are systems of functionally independent functions, one has

\[ \det \begin{bmatrix} E & F \\ G & H \end{bmatrix} \neq 0, \quad \det E \neq 0. \]
(122)

It allows one to write Eq. (120) in the form

\[ \begin{bmatrix} \Phi \\ \Phi_p' \end{bmatrix} = \left( \begin{array}{cc} 1 & 0 \\ \Lambda_1 & \Lambda_2 \\ \Lambda_3 & \Lambda_4 \end{array} \right) \left( \begin{array}{c} E\Phi' + F\Phi_p' \\ \Phi_p' \end{array} \right). \]  
(123)

where \(\Lambda_1 = G\bar{E}, \Lambda_2 = H - G\bar{E}F, \Lambda_3 = M\bar{E}, \Lambda_4 = N - M\bar{E}F\), and \(\Lambda_2\) is invertible. One concludes \(E\Phi' + F\Phi_p' = \Phi\), then Eq. (123) implies

\[ \begin{bmatrix} \Phi_p' \\ \Phi_p'' \end{bmatrix} = \left( \begin{array}{cc} \Lambda_2 & 0 \\ \Lambda_4 & 1 \end{array} \right) \left( \begin{array}{c} \Phi_p' + \bar{\Lambda}_2\Lambda_1\Phi \\ \Lambda_3 - \Lambda_4\bar{\Lambda}_2\Lambda_1 \end{array} \right). \]
(124)

\[ ^7 \text{a) Let } \phi = K\psi \text{ with functionally independent sets } \phi \text{ and } \psi, \text{ and } [\phi] = [\psi]. \text{ Suppose } \det K = 0, \text{ then one uses null-vectors of } K \text{ (similarly to section 4) to write } \phi = Q \begin{bmatrix} \psi' \\ 0 \end{bmatrix}, \text{ det } Q \neq 0, \text{ which is in contradiction with functionally independence of } \phi. \]

\[ ^7 \text{b) From the first line of Eq. (120) one has } \Phi = E\Phi' + F\Phi_p', \text{ then } 0 \neq \det \frac{\partial \Phi}{\partial \bar{z}_2}|_{\Phi, \Phi_p'}, \text{ then } \det E \neq 0. \]
Let us denote

\[ \Phi_p \equiv \Phi'_p + \Lambda_2 \Lambda_1 \Phi, \quad C \equiv \Lambda_3 - \Lambda_4 \Lambda_1. \]  

(125)

Since \( \Phi'_p \equiv \Lambda_2 \Phi_p \) with \( \Lambda_2 \) invertible, the systems are equivalent, and

\[ \text{rank} \left| \frac{\partial \Phi_p}{\partial z_1} \right|_{\Phi_p} = [\Phi_p] = [\Phi'_p] = a - [\Phi]. \]  

By construction, the \((\Phi, \Phi_p)\) are functionally independent. Combining Eqs. (118), (124) one finds the desired result (116), where \( U = Q \begin{pmatrix} \Lambda_2 & 0 \\ \Lambda_4 & 1 \end{pmatrix} \) is invertible.

12 Appendix B

We present here basic Lemma which was used in section 8 to rewrite \( s \)-stage generating functions in the normal form (72).

**Lemma 4.** Consider an expression

\[ T^{(p)} = Q^1 \{ \Phi_1, H \} - \{ H, \sum_{n=2}^{p-1} Q^n \Phi_n \}, \]  

(126)

with arbitrary functions \( Q^n(q^1, p_j), \ n = 1, 2, \ldots, p - 1 \). Then \( Q^n \) can be chosen in such a way, that \( T^{(p)} \) will be linear combinations of the constraints

\[ T^{(p)} = - \sum_{n=2}^{p} Q^n \Phi_n, \]  

(127)

where \( Q^n \) are arbitrary functions. Choice of \( Q^n \), which supplies the normal form can be described as follow:

a) For any \( n, \ Q^n \) is divided on three subgroups with help of the structure matrix \( A_{(n+1)} \) (see Eq.(55))

\[ Q^n A_{(n+1)} \overset{\sim}{=}, \hat{\Phi} = (\hat{Q}^{n+1}, \hat{Q}^{n+1}, \hat{Q}^{n+1}) \]  

(128)

where for any \( n = 2, 3, \ldots, p \) one has

\[ \hat{Q}^{n} = - Q^n + \{ H, \hat{Q}^{n+1} A_{(n+1)} \} - \sum_{m=n}^{p-1} \hat{Q}^{m+1} C_{(m+1)m}^{n+1}, \quad \Rightarrow \hat{Q}^{p} = - Q^{p}, \]  

(129)
\[ \hat{Q}^n = - \sum_{m=n}^{p-1} \left( \hat{Q}^{m+1} B_{(m+1)m+1} \pi + \hat{Q}^{m+1} D_{(m+1)m+1} \pi \right) \]  
\[ \hat{Q}^{m+1} D_{(m+1)m+1} \pi \]  
\[ \implies \hat{Q}^p = 0, \]  \hspace{1cm} (130)

and \( B, C, D \) are structure matrix of Eq.(145).

b) The coefficients \( \hat{Q}^n, n = 2, 3, \ldots, p \) remains arbitrary.

**Proof.** Using normal form of the Dirac functions given by Eq.(45), one obtains the following expression for \( T^{(p)} \)

\[ T^{(p)} = \hat{Q}^2 \left( \begin{array}{c} \pi_2 \\ \Phi_2 \\ 0_2 \end{array} \right) - \sum_{n=2}^{p-1} \{H, Q^n\} \Phi_n + \]  \hspace{1cm} (131)

\[ \sum_{n=2}^{p-1} \hat{Q}^n \left( \begin{array}{c} \Phi_{n+1} + B_{(n+1)n+1} (\pi_{n+1}, \ldots, \pi_{n+2}) \\ C_{(n+1)n+1} (\Phi_{n+1}, \ldots, \Phi_2) + D_{(n+1)n+1} (\pi_{n+1}, \ldots, \pi_{n+2}) \end{array} \right) \]

where \( \hat{Q} \) are given by Eq.(128). Collecting similar terms in Eq.(131) and using the identity

\[ \sum_{n=2}^{p-1} \sum_{m=2}^{n} K(n)^{\text{max}} \pi = \sum_{n=2}^{p-1} \left( \sum_{m=n}^{p-1} K(m) \pi \right) \pi, \]  \hspace{1cm} (132)

one obtains

\[ T^{(p)} = \]  \hspace{1cm} (133)

\[ \sum_{n=2}^{p-1} \left( \hat{Q}^n + \sum_{m=n}^{p-1} \left( \hat{Q}^{m+1} B_{(m+1)m+1} \pi + \hat{Q}^{m+1} D_{(m+1)m+1} \pi \right) \right) \pi + \]  \[ \hat{Q}^p \pi + \]  \[ \sum_{n=2}^{p-1} \left( \hat{Q}^n - \{H, Q^n\} + \sum_{m=n}^{p-1} \hat{Q}^{m+1} C_{(m+1)m+1} \pi \right) \Phi_n + \]  \[ \hat{Q}^p \Phi_p + \hat{Q}^2 0_2. \]

Then the choice of \( Q \) written in Eqs.(129), (130) leads to the normal form (127) for \( T^{(p)} \). ♠

Evolution of the functions \( Q \) of Eq.(128) can be described schemat-
ially as follows:

\[
\begin{pmatrix}
Q^{p-1} \\
\vdots \\
Q^2 \\
Q^1
\end{pmatrix}
\rightarrow
\begin{pmatrix}
\hat{Q}^p & \hat{Q}^{p-1} & \hat{Q}^p \\
\hat{Q}^2 & \hat{Q}^{2-1} & \hat{Q}^2 \\
\hat{Q}^1 & \hat{Q}^{1-1} & \hat{Q}^1
\end{pmatrix}
\rightarrow
\begin{pmatrix}
\hat{Q}^p & Q^p & \hat{Q}^{p-1} \\
\hat{Q}^2 & Q^2 & \hat{Q}^{2-1} \\
0 & Q^2 & \hat{Q}^{2-1}
\end{pmatrix}
\tag{134}
\]

Being arranged in this order, the lines correspond to the ones in Eq. (7), see Eq. (82). Functions placed in second and third columns of the last matrix remain arbitrary. According to Eq. (130), any group \(\hat{Q}^n\) of the line \(n\) is presented through functions of previous lines placed in second and third columns. According to Eq. (129), any \(Q^n\) is presented in terms of arbitrary function \(Q^m\) as well as through functions of previous lines.

It is convenient to write manifest form for division of the coefficients \(Q^{(p)1}\) on \(s\)-stage of the Dirac procedure, namely

\[
Q^{(2)1} = \left(0^2, 0^2, \ldots, \left(0^2, \hat{Q}^{(2)s}\hat{Q}^{(2)s}\right)^{s-1}, \hat{Q}^{(2)s-1}\hat{Q}^{(2)s-1}\hat{Q}^{(2)s-1}\hat{Q}^{(2)s-1}\right)^{s-2}
\]

\[
\tilde{A}_{(s-1)s-2}^{\hat{Q}^{(2)s-1}}, \hat{A}_{(s-1)s-2}^{\hat{Q}^{(2)s-1}}\hat{A}_{(s-1)s-2}^{\hat{Q}^{(2)s-1}}\hat{A}_{(s-1)s-2}^{\hat{Q}^{(2)s-1}}\hat{A}_{(s-1)s-2}^{\hat{Q}^{(2)s-1}}\right)^{\hat{A}_{(s-1)s-2}^{\hat{Q}^{(2)s-1}}}
\]

\[
Q^{(3)1} = \left(\hat{Q}^{(3)s}, \ldots, \left(\hat{Q}^{(3)s-2}, \hat{Q}^{(3)s-2}, \hat{Q}^{(3)s-2}, \hat{Q}^{(3)s-2}\right)^{s-2}, \hat{Q}^{(3)s-2}\hat{Q}^{(3)s-2}\hat{Q}^{(3)s-2}\hat{Q}^{(3)s-2}\right)^{\hat{A}_{(s-1)s-2}^{\hat{Q}^{(3)s-2}}}
\]

\[
\tilde{A}_{(s-2)s-3}^{\hat{Q}^{(3)s-2}}, \hat{A}_{(s-2)s-3}^{\hat{Q}^{(3)s-2}}\hat{A}_{(s-2)s-3}^{\hat{Q}^{(3)s-2}}\hat{A}_{(s-2)s-3}^{\hat{Q}^{(3)s-2}}\hat{A}_{(s-2)s-3}^{\hat{Q}^{(3)s-2}}\right)^{\hat{A}_{(s-2)s-3}^{\hat{Q}^{(3)s-2}}}
\]

\[
Q^{(p)1} = \left(\hat{Q}^{(p)s}, \ldots, \left(\hat{Q}^{(p)s+2-p}, \hat{Q}^{(p)s+2-p}, \hat{Q}^{(p)s+2-p}\right)^{s+1-p}, \hat{Q}^{(p)s+1-p}\hat{Q}^{(p)s+1-p}\hat{Q}^{(p)s+1-p}\hat{Q}^{(p)s+1-p}\right)^{\hat{A}_{(s+1-p)s-p}^{\hat{Q}^{(p)s+1-p}}}
\]

\[
\tilde{A}_{(s+1-p)s-p}^{\hat{Q}^{(p)s+1-p}}, \hat{A}_{(s+1-p)s-p}^{\hat{Q}^{(p)s+1-p}}\hat{A}_{(s+1-p)s-p}^{\hat{Q}^{(p)s+1-p}}\hat{A}_{(s+1-p)s-p}^{\hat{Q}^{(p)s+1-p}}\hat{A}_{(s+1-p)s-p}^{\hat{Q}^{(p)s+1-p}}\right)^{\hat{A}_{(s+1-p)s-p}^{\hat{Q}^{(p)s+1-p}}}
\]

\[
Q^{(p)2} = \left(\hat{Q}^{(p)s+2-p}, \hat{Q}^{(p)s+2-p}, \hat{Q}^{(p)s+2-p}\right)^{s+1-p}, \hat{Q}^{(p)s+1-p}\hat{Q}^{(p)s+1-p}\hat{Q}^{(p)s+1-p}\hat{Q}^{(p)s+1-p}\right)^{\hat{A}_{(s+1-p)s-p}^{\hat{Q}^{(p)s+1-p}}}
\]

\[
\tilde{A}_{(s+1-p)s-p}^{\hat{Q}^{(p)s+1-p}}, \hat{A}_{(s+1-p)s-p}^{\hat{Q}^{(p)s+1-p}}\hat{A}_{(s+1-p)s-p}^{\hat{Q}^{(p)s+1-p}}\hat{A}_{(s+1-p)s-p}^{\hat{Q}^{(p)s+1-p}}\hat{A}_{(s+1-p)s-p}^{\hat{Q}^{(p)s+1-p}}\right)^{\hat{A}_{(s+1-p)s-p}^{\hat{Q}^{(p)s+1-p}}}
\]

\]

33
\begin{align}
Q^{(s-1)1} &= \left( \hat{Q}^{(s-1)\overline{3}}, \hat{Q}^{(s-1)\overline{3}} \hat{\bar{Q}}^{(s-1)\overline{3}} \right)^{\hat{\tin\overline{3}}} \hat{\bar{A}}^{\tin\overline{3}1}, \\
Q^{(s)1} &= \left( \hat{Q}^{(s)\overline{3}}, \hat{Q}^{(s)\overline{2}}, \hat{Q}^{(s)\overline{2}} \right)^{\hat{\tin\overline{2}}} \hat{\bar{A}}^{\tin\overline{2}1},
\end{align}

References

[1] P.A.M. Dirac, Can. J. Math. 2 (1950) 129; Lectures on Quantum Mechanics (Yeshiva Univ., New York, 1964).
[2] J.L. Anderson and P.G. Bergmann, Phys. Rev. 83 (1951) 1018.
[3] P.G. Bergmann and I. Goldberg, Phys. Rev. 98 (1955) 531.
[4] M. Henneaux, C. Teitelboim and J. Zanelli, Nucl. Phys. B332 (1990) 169.
[5] G. Barnich, F. Brandt and M. Henneaux, Commun. Math. Phys. 174 (1995) 57; M. Henneaux and C. Teitelboim, Quantization of Gauge Systems (Princeton: Princeton Univ. Press, 1992).
[6] V. A. Borovkov and I. V. Tyutin, Physics of Atomic Nuclei 61 (1998) 1603; Physics of Atomic Nuclei 62 (1999) 1070;
[7] D. M. Gitman and I. V. Tyutin, J.Phys. A 38 (2005) 5581.
[8] D. M. Gitman and I. V. Tyutin, Symmetries and physical functions in general gauge theory, hep-th/0503218.
[9] A. A. Deriglazov, Note on Lagrangian and Hamiltonian symmetries, hep-th/9412244.
[10] A. A. Deriglazov, and K.E. Evdokimov, Int. J. Mod. Phys. A 15 (2000) 4045 hep-th/9912179.
[11] D. M. Gitman and I. V. Tyutin, Quantization of Fields with Constraints (Berlin: Springer-Verlag, 1990).
[12] A. A. Deriglazov, Analysis of constrained theories without use of primary constraints, Phys. Lett. B (1995), in press hep-th/0506187.