STRICT LOG-CONCAVITY OF THE KIRCHHOFF POLYNOMIAL
AND ITS APPLICATIONS TO THE STRONG LEFSCHETZ
PROPERTY

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Abstract. Anari, Gharan, and Vinzant proved (completely) log-concavity of the basis generating functions for all matroids. From the viewpoint of combinatorial Hodge theory, it is natural to ask whether these functions are “strict” log-concave for simple matroids. In this paper, we will show the strictness in the case of simple graphic matroids, that is, Kirchhoff polynomials of simple graphs are strict log-concave. The key observation is that the Kirchhoff polynomials of complete graphs can be seen as the (irreducible) relative invariant of a certain prehomogeneous vector space, which may be independent interesting fact. In application, we will prove that for any \( a_i \in \mathbb{R}_{>0} \), \( a_1 x_1 + \cdots + a_n x_n \in R_M^1 \) satisfies the strong Lefschetz property (moreover, Hodge–Riemann bilinear relation) at degree one of the Artinian Gorenstein algebra \( R_M^1 \) associated to a graphic matroid \( M \), which is defined by Maeno and Numata for all matroids.


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1. Introduction

The Kirchhoff polynomial \( F_\Gamma \) of a graph \( \Gamma = (V, E) \) is a multi-affine homogeneous polynomial of degree \( r \) in \( n \) variables, where \( n = |E| \) and \( r = |V| - 1 \). This is important in several areas like network theory, physics (where it is related to Feymann diagrams) and so on. Also, the Kirchhoff polynomial can be seen as a special case of the basis generating function \( F_M \) for a graphic matroid \( M \). Properties of the basis generating function are extensively studied, for example half-plane property, by many authors ([4]). Recently, in [2], Anari, Gharan, and Vinzant show that \( F_M \) satisfies log-concavity (more precisely, completely log-concavity) on \( \mathbb{R}^n_{\geq 0} \). In other words, they show that \( \log F_M \) is concave on \( \mathbb{R}^n_{\geq 0} \), that is the Hessian matrix \( H_{F_M} \) and the gradient vector \( \nabla F_M \) of \( F_M \) satisfy

\[
(-F_M H_{F_M} + \nabla F_M (\nabla F_M)^T)|_{x=a} \text{ is positive semi-definite}
\]

for any \( a \in \mathbb{R}^n_{\geq 0} \). Their proof is based on the combinatorial Hodge theory developed in [1] and [8]. As we will explain later in this introduction, if (*) is “positive

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definite”, then a certain Hodge–Riemann bilinear form is non-degenerate. Thus, from the view point of the combinatorial Hodge theory, it is important to know whether the basis generating function is strict log-concave on \((\mathbb{R}_{>0})^n\) or not. Our main theorem claims that in the simple graph case, this is true as the following.

**Theorem 1.1** (cf. Theorem 4.12). For any simple graph \(\Gamma\) with \(r+1\) vertices and \(n\) edges, the Kirchhoff polynomial \(F_\Gamma\) is strict log-concave on \((\mathbb{R}_{>0})^n\). In other words, for any \(a \in (\mathbb{R}_{>0})^n\), \(\log F_\Gamma\) is strict concave at \(a\), that is,

\[
(-F_\Gamma H_{\Gamma, i} + \nabla F_\Gamma(\nabla F_\Gamma)^T)|_{x=a} \text{ is positive definite.}
\]

In particular, \(H_{\Gamma, i}|_{x=a}\) is non-degenerate, more precisely, it has \(n-1\) negative eigenvalues and one positive eigenvalue. Thus, \((-1)^{n-1}(\det H_{\Gamma, i})|_{x=a} > 0\).

The proof of our main theorem consists of two steps. First, we reduce our claim to the following determinantal identity of the Hessian of the Kirchhoff polynomial \(F_{K_{r+1}}\) of complete graphs \(K_{r+1}\) (cf. Theorem 4.4).

\[
\det H_{F_{K_{r+1}}} = (-1)^{N-1}c_r(F_{K_{r+1}})^{N-r-1},
\]

where \(c_r > 0\) is a constant, and \(N := (r+1)/2\). Second, we will show the above equality by not computing directly but identifying \(F_{K_{r+1}}\) with the unique irreducible polynomial ("the" relative invariant) associated to a special \(GL_r(\mathbb{C})\) representation, so-called a prehomogeneous vector space. Then by the general theory of a (irreducible) prehomogeneous vector space (13), the Hessian determinant \(\det H_F\) of the relative invariant \(F\) is also a relative invariant of the same representation, so we have

\[
\exists c \in \mathbb{C} \text{ such that } \det H_F = cF^m
\]

by the uniqueness of the relative invariant. We think this method may be useful to prove some (conjectural) Hessian determinantal identity in general.

Here we mention closely related recent work studied by Brändén and Huh. They study that the Hessian of a nonzero Lorentzian polynomial has exactly one positive eigenvalue at any point on the positive orthant (see 3, Section 5 and 7).

In section 5.2 we give some applications of the main theorem to the strong Lefschetz property of the Artinian Gorenstein graded algebra \(R^*_M = \oplus_{r=0}^s R^*_r = \mathbb{R}[x_1, \ldots, x_n]/\text{Ann}(F_\Gamma)\) associated to any simple graph \(\Gamma\) (see Definition 5.1 for the definition). This algebra is defined for any matroid \(M\) by Maeno and Numata (12), and they proved that this algebra has the strong Lefschetz property at all degrees when \(M\) is the projective space \(M(q,n)\) over a finite field (they denote \(R^*_M\) by \(A_M\)). They also conjectured \(R^*_M\) has the strong Lefschetz property for any matroid \(M\) in an extended abstract 11 of the paper 12. As an application of our main theorem, we prove this conjecture at degree one when \(M\) is a graphic matroid as the following.

**Theorem 1.2** (cf. Theorem 5.11). For any simple graph \(\Gamma\) with \(r+1\) vertices and \(n\) edges, and any \(a = (a_1, \ldots, a_n) \in (\mathbb{R}_{>0})^n\), \(L_a := a_1x_1 + \cdots + a_nx_n \in R^*_1\) satisfies the strong Lefschetz property at degree one, that is, the multiplication map

\[
\times L_a^{r-2} : R^*_1 \rightarrow R^*_1^{r-1}
\]

is an isomorphism.

Since the Hodge–Riemann bilinear form (see Definition 5.7 for the definition) of \(R^*_1\) is given by the Hessian \(H_{F_{\Gamma, i}}\), we have the following stronger application.

**Theorem 1.3** (cf. Theorem 5.12). In the above setting, for any \(a \in (\mathbb{R}_{>0})^n\), the Hodge–Riemann bilinear form

\[
\text{HR}^*_a(\Gamma) : R^*_1 \times R^*_1 \rightarrow \mathbb{R}, \quad (\xi_1, \xi_2) \rightarrow \deg(\xi_1 L_a^{r-2}\xi_2)
\]
is non-degenerate, where \( \deg : R_1^r \xrightarrow{\sim} \mathbb{R} \) is a certain isomorphism. Moreover, \( HR^n_0(F_\Gamma) \) has \( n-1 \) negative eigenvalues and one positive eigenvalue.

As we will note at Remark 5.13, our \( HR^n_0(\Gamma) \) is the same as the Hodge–Riemann bilinear form on the degree one part of another algebra \( B^*(M) \) when \( M = \Gamma \), which is studied in [8]. In [8, Remark 15], Huh and Wang consider the Hodge–Riemann bilinear form on \( B^1(M) \) for general simple matroid. Then, our corollary implies the same conclusion for \( B^*(M) \) at degree one as the above theorem (in general, there exists a natural surjection \( B^*(M) \to R^*_M \)).

This paper is organized as follows. In Section 2, we will study the properties of homogeneous polynomials in terms of their Hessian and their log-concavity. In particular, we will collect some propositions on prehomogeneous vector spaces in Subsection 2.2. In Section 3, we will see several definitions and propositions for matroids. In Section 4, we will define the Kirchhoff polynomials of simple graphs, then we prove our main result. In the last half of this section, we will see that the connection between the Kirchhoff polynomials of the complete graphs and certain prehomogeneous vector spaces. In Section 5, we will see that our main result gives applications to the algebra associated to a graphic matroid.

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2. Homogeneous polynomials

Let us consider a homogeneous polynomial \( F \) of degree \( r \) in \( n \) variables with real coefficients, where \( r \geq 2 \). For \( F \), we define the Hessian matrix \( H_F \) and the gradient vector \( \nabla F \) by

\[
H_F = \left( \frac{\partial^2 F}{\partial x_i \partial x_j} \right)_{1 \leq i,j \leq n}, \quad \nabla F = \left( \frac{\partial F}{\partial x_1}, \ldots, \frac{\partial F}{\partial x_n} \right).
\]

We call \( H_F \) the Hessian of \( F \).

In the first half of this section, we will consider the Hessian of \( F \). At first, we will see the identity

\[
\det \left( -FH_F + s(\nabla F)^T \cdot \nabla F \right) = (-1)^{n-1} \frac{r}{r-1} \left( s - \frac{r-1}{r} \right) F^n \det H_F. \tag{1}
\]

Next, for some special polynomial \( F \), we will show the following identity

\[
\det H_F = c' F^{\frac{n(r-1)}{r}}, \tag{2}
\]

where \( c' \) is non-zero.

In the last half of this section, we will consider the property of \( F \) called strict log concavity.

2.1. The Hessians. Here we prove the identity (1). The set of all \( m \times n \) matrices is denoted by \( M_{m \times n} \). \( I_n \) represents the \( n \times n \) identity matrix.

To prove (1), we prepare two lemmas.

**Lemma 2.1.** For an \( n \times n \) matrix \( N \) of rank one, we have

\[
\det(I_n - sN) = 1 - s(\text{tr} N).
\]

**Lemma 2.2** (Euler’s identity). For a homogeneous polynomial \( F \) of degree \( r \) in \( n \) variables, where \( r \geq 2 \), we have

\[
r(r-1)F = x^T H_F x,
\]
$$\begin{align*}
(r - 1)(\nabla F)^T &= H_F x,
\end{align*}$$

where $x = (x_1, \ldots, x_n)^T$.

Lemma 2.1 is straightforward. The proof of Lemma 2.2 is in [2 Corollary 4.3].

**Proposition 2.3.** For a homogeneous polynomial $F$ of degree $r$ in $n$ variables, where $r \geq 2$, we have

$$\det (-FH_F + s(\nabla F)^T \cdot \nabla F) = (-1)^{n-1} \frac{r}{r-1} \left( s - \frac{r-1}{r} \right) F^n \det H_F.$$ 

**Proof.** Let $A$ be a regular $n \times n$ matrix and $v$ a column vector of size $n$ such that $v^T Av \neq 0$. We set

$$N = \frac{1}{v^T Av} (Av)(Av)^T A^{-1}.$$ 

In this case, we have rank $N = 1$ and tr $N = 1$. By Lemma 2.1

$$\det \left( I_n - \frac{1}{v^T Av} (Av)(Av)^T A^{-1} s \right) = 1 - s.$$ 

Multiplying $\det A$ from the right,

$$\det \left( A - \frac{1}{v^T Av} (Av)(Av)^T s \right) = (1 - s) \det A.$$ 

Multiplying the left hand side by $(-v^T Av)^n$,

$$(*) \quad \det (-v^T Av) A + (Av)(Av)^T s = (1 - s)(-v^T Av)^n \det A = (-1)^{n-1} (s - 1)(-v^T Av)^n \det A.$$ 

By Lemma 2.2 we have the following identity

$$-FH_F + s(\nabla F)^T \cdot \nabla F = \frac{1}{r(r-1)} \left\{ - (x^T H_F x) H_F + \frac{sr}{r-1} (H_F x)(H_F x)^T \right\}.$$ 

Thus, applying $(*)$ as $A = H_F$ and $v = x$, we obtain the desired equation. $\square$

By Proposition 2.3 we obtain the identity (1).

**2.2. Prehomogeneous vector spaces.** Here we prove the identity (2) for the relative invariant of an irreducible prehomogeneous vector space (Corollary 2.16). To prove it, we introduce the notion of prehomogeneous vector space developed by Kimura and Sato [15] and many authors. To be self-contained, we collect some useful propositions in [15] and give their proofs. Basically, we will follow [15]. We also use notations in [9].

**Definition 2.4** (Prehomogeneous vector space cf. [15] Definition 1 in Section 2 & p.36]). Let $(G, \rho, V)$ be a triplet of a connected linear algebraic group $G$, a finite dimensional vector space $V$, and a rational representation $\rho$ of $G$ on $V$, all defined over $\mathbb{C}$. We call $(G, \rho, V)$ a prehomogeneous vector space if there exists a proper algebraic $G$-invariant subset $S \subset V$ such that $V \setminus S$ is a single $G$-orbit. Then, we say that $S$ is the singular set of $(G, \rho, V)$. We say that $(G, \rho, V)$ is irreducible when $\rho$ is an irreducible representation.

**Definition 2.5** (Relative invariants cf. [15] Definition 2 in Section 4]). Let $(G, \rho, V)$ be a prehomogeneous vector space. A not identically zero rational function $F \in \mathbb{C}(V)$ is called a relative invariant (with respect to $\chi$) of $(G, \rho, V)$ if there exists a rational character $\chi \in \text{Hom}(G, \mathbb{C}^*)$ which satisfies the following:

$$F(\rho(g)x) = \chi(g) F(x) \quad (g \in G, x \in V).$$

In this case, we write $F \leftrightarrow \chi$. 

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Note that a relative invariant is a rational function on $V$, not necessarily a polynomial on $V$. We define a subgroup $X_1(G)$ of $\text{Hom}(G, \mathbb{C}^*)$ by

$$X_1(G) := \{ \chi \in \text{Hom}(G, \mathbb{C}^*) \mid \exists F \in \mathbb{C}(V) \text{ such that } F \leftrightarrow \chi \}.$$  

**Remark 2.6.** For any $\chi \in X_1(G)$, if $\rho(g_1) = \rho(g_2)$, then $\chi(g_1) = \chi(g_2)$. In particular, we can think as $X_1(G) \subset \text{Hom}(\rho(G), \mathbb{C}^*)$ by the natural inclusion $\text{Hom}(\rho(G), \mathbb{C}^*) \hookrightarrow \text{Hom}(G, \mathbb{C}^*)$ induced from $G \twoheadrightarrow \rho(G)$.

**Proposition 2.7 ([L5] Proposition 3 in Section 2).** Let $(G, \rho, V)$ be a prehomogeneous vector space. Then, any $G$-invariant rational function $F \in \mathbb{C}(V)^G$ is constant.

**Proof.** By definition, there exists a proper algebraic subset $S \subset V$ whose complement $V \setminus S$ is a single open dense $G$-orbit. Then, by assumption, $F$ is a constant function on some open dense subset of $V$. This implies that $F$ is constant. □

**Proposition 2.8 ([L5] Proposition 3 in Section 4).** Let $(G, \rho, V)$ be a prehomogeneous vector space. A relative invariant $F$ is uniquely determined up to a constant multiple by its corresponding character. In other words, if $F_1 \leftrightarrow \chi$ and $F_2 \leftrightarrow \chi$, then $F_1 = cF_2$ for some $c \in \mathbb{C}^*$. In particular, any relative invariant is a homogeneous rational function.

**Proof.** If $F_1 \leftrightarrow \chi$ and $F_2 \leftrightarrow \chi$ for some $\chi$, then clearly, $F_2/F_1$ is a $G$-invariant rational function. Thus, by Proposition 2.7, it is a constant. Let $F$ be a relative invariant corresponding to $\chi$. Then, for each $t \in \mathbb{C}^*$, we have clearly $F_1(x) := F(tx) \leftrightarrow \chi$. Thus, there exists a constant $c_t \in \mathbb{C}^*$ such that $F(tx) = c_t \cdot F(x)$ in $\mathbb{C}(V)$. This implies that $F$ is homogeneous. □

As we state the following, the Hessian determinant of any relative invariant is also a relative invariant.

**Lemma 2.9.** Let $(G, \rho, V)$ be a prehomogeneous vector space. If $F$ is a relative invariant corresponding to some character $\chi$, then $\det H_F$ is a relative invariant corresponding to the character $\chi^N \cdot (\det)^{-2}$, where $N = \dim V$ and $\det : G \to \mathbb{C}^* : g \mapsto \det(\rho(g))$.

**Proof.** (cf. [L5] Proof of Proposition 8 in Section 4) By choosing a basis of $V$, we may assume that $V = \mathbb{C}^N$ and $G \subseteq GL_N(\mathbb{C})$. For $g = (g_{k\ell}) \in G$, we have

$$\frac{\partial^2}{\partial x_i \partial x_j} (F(gx)) = \frac{\partial F}{\partial x_i} \sum_{k=1}^N g_{k\ell} \frac{\partial F}{\partial x_k} (gx) \cdot \frac{\partial \left( \sum_{\ell=1}^N g_{k\ell} x_\ell \right)}{\partial x_j}$$

$$= \sum_{k=1}^N g_{kj} \frac{\partial F}{\partial x_i} (gx)$$

$$= \sum_{k, \ell} g_{\ell, i} \cdot \frac{\partial^2 F}{\partial x_\ell \partial x_k} (gx) \cdot g_{kj}.$$  

Then, as a matrix, we have

$$\left( \frac{\partial^2}{\partial x_i \partial x_j} (F(gx)) \right)_{i,j} = g^T \left( \frac{\partial^2 F}{\partial x_k \partial x_\ell} (gx) \right)_{k, \ell} g.$$  

Since $F(gx) = \chi(g) F(x)$, the Hessian matrix $H_F(gx)$ is

$$H_F(gx) := \left( \frac{\partial^2 F}{\partial x_i \partial x_j} (gx) \right)_{i,j} = \chi(g) \cdot (g^T)^{-1} \left( \frac{\partial^2 F}{\partial x_i \partial x_j} (x) \right) g^{-1}.$$  


Then we have \( \det H_F(gx) = (\chi(g))^N \cdot (\det g)^{-2} \det H_F(x) \). This means \( \det H_F(x) \) is a relative invariant corresponding to the character \( \chi^N \cdot (\det)^{-2} \).

Below, let \( \langle \chi_1, \ldots, \chi_\ell \rangle \) be the abelian group generated by characters \( \chi_1, \ldots, \chi_\ell \). We say that \( \chi_1, \ldots, \chi_\ell \) are multiplicative independent if \( \langle \chi_1, \ldots, \chi_\ell \rangle \) is a free abelian group of rank \( \ell \).

**Lemma 2.10** (cf. [15] Lemma 4 in Section 4]). Let \((G, \rho, V)\) be a triplet and \(F_1, \ldots, F_\ell\) be relative invariants corresponding to some characters \(\chi_1, \ldots, \chi_\ell \in\text{Hom}(G, \mathbb{C}^*)\) respectively. If \(\chi_1, \ldots, \chi_\ell\) are multiplicatively independent, then \(F_1, \ldots, F_\ell\) is algebraically independent over \(\mathbb{C}\).

**Proof.** Assume \(F_1, \ldots, F_\ell\) are algebraically dependent. By definition, there exist monomials \(\Phi_k(F_1, \ldots, F_\ell) = a_k \ell_1^{d_1} \cdots \ell_\ell^{d_\ell} \) \((1 \leq k \leq s)\) of \(F_1, \ldots, F_\ell\) such that they are linearly dependent over \(\mathbb{C}\) and (we can assume) any \(s - 1\) of them are linearly independent over \(\mathbb{C}\) \((s \geq 2)\). Then, \(\Phi_k(F_1, \ldots, F_\ell)\) is a relative invariant corresponding to the character \(\mu_k := \chi_1^{d_1} \cdots \chi_\ell^{d_\ell}\). This implies that if \((c_1, \ldots, c_s) \in W := \{ (c_1, \ldots, c_s) \in \mathbb{C}^s | \sum_{k=1}^s c_k \Phi_k(F_1, \ldots, F_\ell) = 0 \}\), then \((c_1 \mu_1(g), \ldots, c_s \mu_s(g)) \in W\) \((g \in G)\). Since \(\dim W = 1\), we have \(\mu_1 = \cdots = \mu_s\). On the other hand, any \(s - 1\) of \(\Phi_k(F_1, \ldots, F_\ell)\) \((1 \leq k \leq s)\) are linearly independent, in particular, for any \(1 \leq p \neq q \leq s\), we have \((d_1, \ldots, d_p) \neq (d_1, \ldots, d_q)\). Then, by assumption, \(\chi_1, \ldots, \chi_\ell\) are multiplicatively independent, in particular, if \(1 \leq p \neq q \leq s\), then \(\mu_p \neq \mu_q\). This is a contradiction. \(\square\)

**Proposition 2.11** (cf. [15] Proposition 5 in Section 4)). Let \((G, \rho, V)\) be a pre-homogeneous vector space and \(S\) be its singular set. Let \(S_1, \ldots, S_\ell\) be all codimension 1 irreducible components of \(S\) and \(F_i\) be the defining irreducible polynomial of each \(S_i\). Then, \(F_1, \ldots, F_\ell\) are relative invariants corresponding to some multiplicatively independent characters \(\chi_1, \ldots, \chi_\ell\), in particular, \(F_1, \ldots, F_\ell\) are algebraically independent over \(\mathbb{C}\). Moreover, any relative invariant \(F\) can be expressed as \(F = cF_1^{m_1} \cdots F_\ell^{m_\ell} \) \((c \in \mathbb{C}, m_i \in \mathbb{Z})\). In particular, \(X_1(G) = \langle \chi_1, \ldots, \chi_\ell \rangle\) is a free abelian group of rank \(\ell\).

**Proof.** First, we will prove that each \(F_i\) is a relative invariant. Since \(G\) is connected (i.e., irreducible) and \(S_i\) is irreducible, the Zariski closure \(\overline{\rho(G)} \cdot S_i\) of the image of the multiplication morphism \(G \times S_i \to S\) is also irreducible. Since \((S_i \subseteq \rho(G)) \cdot S_i\) is irreducible, we have \(\rho(G) \cdot S_i = S_i\), in particular, \(\rho(G) \cdot S_i = S_i\). This implies that for each \(g \in G\), the vanishing loci of two irreducible polynomials \(F_i(gx)\) and \(F_j(g)\) \((x)\) are same. For each \(g \in G\), there exists \(\chi_i(g) \in \mathbb{C}^*\) such that \(F_i(g) = \chi_i(g) F_i(x)\). Then, \(\chi_i\) is a character, and \(F_i\) is a relative invariant corresponding to \(\chi_i\). Next, we will show that \(\chi_1, \ldots, \chi_\ell\) are multiplicatively independent. If not so, there exists a \(\langle d_1, \ldots, d_\ell \rangle \in \mathbb{Z}^\ell \setminus \{0\}\) such that \(\chi_1^{d_1} \cdots \chi_\ell^{d_\ell} = 1\). We may assume \(d_i \neq 0\). Then, \(F_1^{-d_1} \cdots F_\ell^{-d_\ell}\) are relative invariants corresponding to the same character \(\chi_1^{-d_1} \cdots \chi_\ell^{-d_\ell}\). By Proposition 2.8, \(F_i^{-d_i} \cdots F_\ell^{-d_\ell}\) are same up to constant multiple, however this contradicts to the irreducibility of \(F_i\) and \(F_i \neq F_j\) \((i \neq j)\). The algebraically independence of \(F_1, \ldots, F_\ell\) is followed by the above lemma. Since the vanishing locus of any relative invariant \(F\) is \(G\)-invariant proper subset of \(V\), this is a subset of \(S\). This implies that \(F\) is some products of \(F_1, \ldots, F_\ell\). \(\square\)

To consider when \((G, \rho, V)\) is an irreducible representation later, we note the following fundamental theorem by Cartan on irreducible representations.

**Theorem 2.12** (Cartan cf. [15] Theorem 1 in Section 1)). Let \((G, \rho, V)\) be a triplet. Assume that \(dp : g \to gl(V)\) is an irreducible representation. Then, its image \(dp(g)\) is reductive, and \(dp(g)\) is isomorphic to one of the following:
Definition 2.17. \((\text{strict) log-concavity})\) we define strict log-concavity. That

Corollary 2.13. In the above setting, \(\rho(G)\) is reductive and it is isomorphic to one of the following:

1. \(\text{GL}_1(\mathbb{C}) \times G_1 \times \cdots \times G_s\), where \(G_i\) is an algebraic group whose Lie algebra is simple (in particular, its center \(Z(G_i)\) is finite).
2. \(G_1 \times \cdots \times G_s\), where \(G_i\) is an algebraic group whose Lie algebra is simple (in particular, its center \(Z(G_i)\) is finite).

As noted at Remark 2.6 (2), we can think \(X_1(G)\) is a subgroup of \(\text{Hom}(\rho(G), \mathbb{C}^*)\). Since \(\rho(G)\) is reductive, the quotient \(\rho(G)/Z(\rho(G))\) by the center \(Z(\rho(G))\) is semi-simple algebraic group. Then, the following is exact:

\[
0 \rightarrow \text{Hom}(\rho(G)/Z(\rho(G)), \mathbb{C}^*) \rightarrow \text{Hom}(\rho(G), \mathbb{C}^*) \rightarrow \text{Hom}(Z(\rho(G)), \mathbb{C}^*)
\]

As the character group of semi-simple group is trivial, the natural linear map \(\text{Hom}(\rho(G), \mathbb{C}^*) \rightarrow \text{Hom}(Z(\rho(G)), \mathbb{C}^*)\) is injective. Thus, we can think as \(X_1(G) \subset \text{Hom}(Z(\rho(G)), \mathbb{C}^*)\). Now, since \(\rho\) is irreducible, by Corollary 2.13 we have

\[
\text{Hom}(\rho(G), \mathbb{C}^*) \cong \mathbb{Z} \times G_{\text{finite}} \text{ or } G_{\text{finite}},
\]

where \(G_{\text{finite}}\) is a finite abelian group. As \(X_1(G)\) is a free abelian group of rank \(\ell\), where \(\ell\) is the number of codimension one irreducible components of the singular set \(S\). In particular, we have the following.

Proposition 2.14 (cf. [15] Proposition 12 in Section 4]). Let \((G, \rho, V)\) be an irreducible prehomogeneous vector space. Then there is at most one irreducible relative invariant \(F\) up to constant multiple. In particular, any relative invariant is in the form of \(cF^m\) for \(c \in \mathbb{C}\) and \(m \in \mathbb{Z}\).

Definition 2.15 (cf. [15] Definition 13 in Section 4]). Let \((G, \rho, V)\) be an irreducible prehomogeneous vector space. We call \(F\) (appeared in Proposition 2.14) the relative invariant of \((G, \rho, V)\), which is defined up to constant multiple.

We say a prehomogeneous vector space \((G, \rho, V)\) is regular when there exists a relative invariant \(F \in \mathbb{C}(V)\) such that its Hessian determinant \(\det H_F\) is not identically zero on \(V\) ([15] Definition 7 in Section 4]). Then by Lemma 2.9 we have the following key determinantal identity of the Hessian of the relative invariant when \((G, \rho, V)\) is regular. We learn this corollary from [5] Remark 3.5.

Corollary 2.16. Let \((G, \rho, V)\) be a regular irreducible prehomogeneous vector space of dimension \(n\). Assume that the degree of the relative invariant \(F\) is \(r\). Then, the Hessian determinant of \(F\) is in the form of

\[
\det H_F = cF^{rac{\alpha(r+2)}{r}},
\]

where \(c \in \mathbb{C}^*\) is a constant.

2.3. Strict log-concavity of homogeneous polynomials. Let \(F\) be a homogeneous polynomial of degree \(r\) in \(n\) variables with real coefficients, where \(r \geq 3\). Here we consider log-concavity of \(F\). For a symmetric matrix \(A\), \(A \succeq 0\) denotes that \(A\) is positive semi-definite, and \(A > 0\) denotes that \(A\) is positive definite. Now we define strict log-concavity.

Definition 2.17 ((strict) log-concavity). We say that \(F\) is log-concave (resp. strict log-concave) at \(a \in \mathbb{R}^n\) if

\[
(-FH_F + (\nabla F)^T(\nabla F))|_{x=a} \succeq 0 \text{ (resp. } > 0).\]
By a technical reason, we also introduce strict “homogeneous” log concavity which is stronger than strict log-concavity. We will not, however, use this notion essentially until the final section, so it is no problem to replace (strict) homogeneous log-concavity by (strict) log-concavity until then.

**Definition 2.18** (strict) homogeneous log-concavity. We say that $F$ is *homogeneous log-concave* (resp. *strict homogeneous log-concave*) at $\mathbf{a} \in \mathbb{R}^n$ if for any $s \geq \frac{r-1}{r}$ (resp. $s > \frac{r-1}{r}$),

$$(-FH_F + s(\nabla F)^T(\nabla F))|_{x=a} \geq 0 \ (\text{resp. }> 0).$$

As remarked in [8] Example 1.11.2, $F$ is (strict) homogeneous log-concave at $\mathbf{a} \in \mathbb{R}^n$ if and only if $F^k$ is log-concave at $\mathbf{a}$ for any $k > r$.

Clearly, (strict) homogeneous log-concavity implies (homogeneous) log-concavity respectively.

From here, we assume that $F$ is a multi-affine polynomial with positive coefficients, where a *multi-affine polynomial* is a sum of square-free monomials. One of the important properties of a strict log-concave multi-affine polynomial $F$ with positive coefficients is its Hessian $H_F$ is non-degenerate, moreover it has only one positive eigenvalue. To prove this, we note the Cauchy’s interlacing theorem.

**Theorem 2.19** (Cauchy’s interlacing Theorem [7] Corollary 4.3.9]). For a real symmetric $n \times n$ matrix $A$ with eigenvalues $\alpha_1 \geq \cdots \geq \alpha_n$ and a vector $\mathbf{v} \in \mathbb{R}^n$, the eigenvalues $\alpha_1 \geq \cdots \geq \alpha_n$ interlace the eigenvalues $\beta_1 \geq \cdots \geq \beta_n$ of $B := A + \mathbf{v}\mathbf{v}^T$. That is,

$$\beta_1 \geq \alpha_1 \geq \beta_2 \geq \cdots \geq \alpha_{n-1} \geq \beta_n \geq \alpha_n.$$

**Corollary 2.20.** If $F$ is strict log-concave at $\mathbf{a} \in (\mathbb{R}_{>0})^n$, then $H_F|_{x=a}$ has exactly $n-1$ negative eigenvalues and exactly one positive eigenvalue. In particular, $(-1)^{n-1}(\det H_F)|_{x=a} > 0$.

**Proof.** We set $A = (-FH_F)|_{x=a}$ and $B = (-FH_F + (\nabla F)^T(\nabla F))|_{x=a}$, and we denote their eigenvalues as $\alpha_1 \geq \cdots \geq \alpha_n$ and $\beta_1 \geq \cdots \geq \beta_n$ respectively. Since $F$ is strict log-concave at $\mathbf{a} \in (\mathbb{R}_{>0})^n$, we have $\beta_n > 0$. Hence it follows from Cauchy’s interlacing theorem that eigenvalues $\alpha_1, \ldots, \alpha_{n-1}$ are positive. On the other hand, since $F$ is a multi-affine polynomial, all diagonal components of its Hessian are zero. In particular, we have $0 = \text{tr} A = \sum_{i=1}^n \alpha_i$. Thus, $\alpha_n$ should be negative. Hence $(-FH_F)|_{x=a}$ has exactly $n-1$ positive eigenvalues and exactly one negative eigenvalue. Since $F$ is a polynomial with positive coefficients, we have $F(\mathbf{a}) > 0$ for any point $\mathbf{a} \in (\mathbb{R}_{>0})^n$. Therefore $H_F|_{x=a}$ has exactly $n-1$ negative eigenvalues and exactly one positive eigenvalue. \hfill \Box

For $F$, we define

$$F_0 = F|_{x_k = 0} \in \mathbb{R}[x_1, \ldots, x_k, \ldots, x_N],$$

$$F_k = \frac{\partial F}{\partial x_k} \in \mathbb{R}[x_1, \ldots, x_k, \ldots, x_N].$$

Note that $F = F_0 + x_kF_k$ in this case.

The following lemma looks rather technical, however one can see that this gives a relationship between (strict) homogeneous log-concavity of $F$ and (strict) homogeneous log-concavity of $F_0$ and $F_k$.

**Lemma 2.21.** If $F_0(a_1, \ldots, a_k, \ldots, a_N) \neq 0$ and $F_k(a_1, \ldots, a_k, \ldots, a_N) \neq 0$ for $\mathbf{a} \in \mathbb{R}^N_{>0}$, then the following are equivalent for any $s \geq \frac{r-1}{r}$ (resp. $s > \frac{r-1}{r}$).

(i) $(-FH_F + s(\nabla F)^T(\nabla F))|_{x=a} \geq 0$ (resp. $> 0$).
Then, for any \(\tilde{\alpha}\gamma\) may assume positive definiteness (the argument is similar for positive semi-definiteness). We may assume \(k = 1\). Since we have \(F = F_0 + x_1F_1\), we can compute \(-FH_F + s(\nabla F)^T(\nabla F)\) as the following. Here, note that \((1,1)\)-component of the Hessian matrix of \(F\) is 0 since \(F\) is multi-affine.

\[
- FH_F + s(\nabla F)^T(\nabla F) = -F \begin{pmatrix} 0 & \nabla F_1 \\ (\nabla F_1)^T & H_{F_0} + x_1H_{F_1} \end{pmatrix}
+ s \begin{pmatrix} F_1^2 & F_1(\nabla F_0 + x_1\nabla F_1)^T \\ (\nabla F_0 + x_1\nabla F_1)^T & (\nabla F_0 + x_1\nabla F_1)^T(\nabla F_0 + x_1\nabla F_1) \end{pmatrix}
\]

Then, for any \(\tilde{y} = (y_1 | \ y) \in \mathbb{R}^N \setminus \{0\}\), we have

\[
\tilde{y}^T(-FH_F + s(\nabla F)^T(\nabla F))\tilde{y}
= -F\{2y_1(\nabla F_1 y) + y^T H_{F_0} y + x_1(y^T H_{F_1} y)\}
+ s\{F_1^2 y_1^2 + 2y_1 F_1(\nabla F_0 y + x_1(\nabla F_1 y)) + (\nabla F_0 y + x_1(\nabla F_1 y))^2\}
= (sF_1^2)\delta^2 + 2\{-F(\nabla F_1 y) + sF_1(\nabla F_0 y + x_1(\nabla F_1 y))\}y_1
+ y^T(-F(H_{F_0} + x_1H_{F_1}) + s(\nabla F_0 + x_1\nabla F_1)^T(\nabla F_0 + x_1\nabla F_1)) y.
\]

If \(y = 0\), then \(y_1 \neq 0\), so \(\tilde{y}^T(-FH_F + s(\nabla F)^T(\nabla F))\tilde{y} = (sF_1^2)\delta^2 > 0\). Thus, \((-FH_F + s(\nabla F)^T(\nabla F))\) is positive definite if and only if for any \(y_1 \in \mathbb{R}\) and \(y \neq 0\), \(\alpha\gamma + 2\beta|y_1| + \gamma > 0\), where

\[
\alpha = sF_1^2 > 0,
\beta = -F(\nabla F_1 y) + sF_1(\nabla F_0 y + x_1(\nabla F_1 y)),
\gamma = y^T(-F(H_{F_0} + x_1H_{F_1}) + s(\nabla F_0 + x_1\nabla F_1)^T(\nabla F_0 + x_1\nabla F_1)) y.
\]

Since this is equivalent to \(\alpha\gamma - \beta^2 > 0\) for any \(y \in \mathbb{R}^{n-1} \setminus 0\), then we have

\[
0 < (sF_1^2)\{-F(\nabla F_0 y + x_1\nabla F_1 y) + s(\nabla F_0 y + x_1\nabla F_1 y)^2\}
- \{-F(\nabla F_0 y + x_1(\nabla F_1 y))\}^2
= -(sF_1^2)F(\nabla F_0 y + x_1\nabla F_1 y) - F^2(\nabla F_1 y)^2
+ 2sF_1 F(\nabla F_1 y)(\nabla F_0 y + x_1(\nabla F_1 y)).
\]
Dividing both sides by $F$, we have
\[
0 < -(sF_1^2)(y^T H F_2 y + x_1 y^T H F_1 y) - (F_0 + x_1 F_1)(\nabla F_1 y)^2 \\
+ 2sF_1(\nabla F_1 y)(\nabla F_0 y + x_1(\nabla F_1 y))
\]
\[
= s x_1 F_1 \left\{- F_1(y^T H F_1 y) + \frac{2s - 1}{s}(\nabla F_1 y)^2 \right\} \\
+ \left\{- sF_1^2(y^T H F_0 y) - F_0(\nabla F_1 y)^2 + 2sF_1(\nabla F_1 y)(\nabla F_0 y) \right\}
\]
\[
= s x_1 F_1 \left\{- F_1(y^T H F_1 y) + \frac{2s - 1}{s}(\nabla F_1 y)^2 \right\} \\
+ \frac{sF_1^2}{F_0} \left\{- F_0(y^T H F_0 y) + s(\nabla F_0 y)(\nabla F_0 y) \right\} \\
- \frac{s^2F_1^2}{F_0}(\nabla F_0 y)^2 - F_0(\nabla F_1 y)^2 + 2sF_1(\nabla F_1 y)(\nabla F_0 y)
\]
\[
= s x_1 F_1 \left\{- F_1(y^T H F_1 y) + \frac{2s - 1}{s}(\nabla F_1 y)^2 \right\} \\
+ \frac{sF_1^2}{F_0} \left\{- F_0(y^T H F_0 y) + s(\nabla F_0 y)(\nabla F_0 y) \right\} \\
- \frac{1}{F_0} \left\{sF_1(\nabla F_0 y) - F_0(\nabla F_1 y)^2 \right\}
\]
\[
= \frac{1}{F_0} y^T \left( \frac{s x_1 F_0 F_1}{F_0} \left( - F_1 H F_1 + \frac{2s - 1}{s} (\nabla F_1)^2 \right) + sF_1^2 \left( - F_0 H F_0 + s(\nabla F_0)^2 \nabla F_0 \right) \\
- (sF_1) \nabla F_0 + (F_0 - F_0 \nabla F_1) \right) y.
\]

Multiplying both sides by $F_0$, we complete the proof of the equivalence of (i) and (ii). \qed

By Lemma 2.21, we can prove the following which is important in the proof of our main theorem (Theorem 3.11).

**Corollary 2.22.** Let $F \in \mathbb{R}[x_1, \ldots, x_N]$ be a multi-affine homogeneous polynomial of deg $F = r \geq 3$ with positive coefficients. For a subset $I$ of $[N]$ and $0 \leq k \leq N$, we define

\[
C_{N-k}^I = \{ (z_{k+1}, \ldots, z_N) \in \mathbb{R}_{>0}^{N-k} \mid z_j \geq 0 (j \notin I), z_i > 0 (i \in I) \}.
\]

We assume that $F$ is strict homogeneous log-concave on $C_{N-k}^I$. If

\[
(3) \quad \frac{\partial F}{\partial x_i} \neq 0, \quad \frac{\partial F}{\partial x_2} \neq 0, \ldots, \quad \frac{\partial F}{\partial x_k} \neq 0
\]

hold as polynomials for some $0 \leq k \leq N - r$, then $F|_{x_1 = \ldots = x_k = 0} \in \mathbb{R}[x_{k+1}, \ldots, x_N]$ is strict homogeneous log-concave on $C_{N-k}^I$.

**Proof.** We will show this by induction on $k$. In the case where $k = 0$, the claim is obvious by the assumption. For $1 \leq k \leq N - r$, by the induction hypothesis, $F|_{x_1 = \ldots = x_{k-1} = 0}$ is strict homogeneous log-concave on $C_{N-k}^{I+1}$. Let

\[
f = F|_{x_1 = \ldots = x_{k-1} = 0} \in \mathbb{R}[x_k, \ldots, x_N].
\]

Applying Lemma 2.21 to $f$ and $a = (0 \mid \mathfrak{z})^T \in C_{N-k}^{I+1}$ for any $\mathfrak{z} \in C_{N-k}^I$, we have

\[
\left( sF_1^2( - f_0 H f_0 + s(\nabla f_0)^T (\nabla f_0) ) \\
- (sF_1) \nabla f_0 + (f_0 - f_0 \nabla f_k) \right) \mid_{(x_{k+1}, \ldots, x_n) = \mathfrak{z}} > 0,
\]
where \( f_0 := F|_{x_1=\cdots=x_k=0}, f_k := \frac{\partial F|_{x_1=\cdots=x_k=0}}{\partial x_k} \) (they are not identically zero as polynomials by assumption). In particular, for any \( \mathbf{z} \in \mathbb{C}^{N-k} \), we have

\[
\left(-f_0 H_f + s(\nabla f_0^T (\nabla f_0))\right)_{(x_k+\ldots,x_N)=\mathbf{z}} \geq 0.
\]

This completes the proof. \( \square \)

3. **Matroids**

In this section, we will give basic terms of a matroid. We will use [14] for terms of a matroid.

**Definition 3.1** (Matroid). A matroid \( M \) is an ordered pair \( (E, B) \) consisting of a finite set \( E \) and a collection \( B \) of subsets of \( E \) satisfying the following properties:

- \( B \neq \emptyset \).
- If \( B_1 \) and \( B_2 \) are in \( B \) and \( x \in B_1 \setminus B_2 \), then there is an element \( y \in B_2 \setminus B_1 \) such that \( \{y\} \cup (B_1 \setminus \{x\}) \in B \).

In this case, we call each \( B \in B \) a basis of \( M \).

**Example 3.2** (Graphic matroid). For any finite graph \( \Gamma = (V, E) \) with the vertex set \( V \) and the edge set \( E \), we call a subgraph \( T \subseteq \Gamma \) a spanning tree in \( \Gamma \) if \( T \) does not contain any cycles and \( T \) passes through all vertices of \( \Gamma \). Let \( B_\Gamma \) be the set of all spanning trees in \( \Gamma \). Then \( M(\Gamma) = (E, B_\Gamma) \) is a matroid. We call such matroids graphic matroids.

**Remark 3.3.** If \( M \) is a graphic matroid, then there exists a connected graph \( \Gamma \) such that \( M(\Gamma) \) is isomorphic to \( M \).

**Example 3.4** (Submatroid). Let \( M = (E, B) \) be a matroid. For \( E' \subseteq E \), we define \( B' \) by \( B' = \{ B \in B \mid B \subseteq E' \} \). Then \( M' = (E', B') \) is a matroid. We call \( M' \) a submatroid of \( M \).

Let \( M = (E, B) \) be a matroid. We call each subset of a basis of \( M \) a independent set of \( M \) and call each subset of \( E \) not contained in any basis a dependent set of \( M \). A minimal dependent set of \( M \) is called a circuit of \( M \). We say that \( C \) is a \( n \)-circuit if \( C \) is a circuit and \( C \) has \( n \) elements. In particular we call each 1-circuit a loop. We call an element \( e \) coloop of \( M \) if \( \{e\} \) is contained in each basis of \( M \). We say that a matroid \( M \) is simple if there is neither 1-circuit nor 2-circuit.

We can directly prove the following from the definition of the basis.

**Proposition 3.5.** Let \( M \) be a matroid with the basis set \( B \). If \( B \) and \( B' \) are basis of \( M \), then the number of elements of them are the same. In other words, if \( B, B' \in B \), then \( |B| = |B'| \).

We say that a matroid \( M \) has rank \( r \) if the number of element of a basis of \( M \) is \( r \). The rank of \( M \) is denoted by rank \( M \).

**Definition 3.6** (Basis generating function). For any matroid \( M = (E, B) \), we define the basis generating function \( F_M(x) \) of \( M \) by

\[
F_M(x) = \sum_{B \in B} \prod_{i \in B} x_i.
\]

By definition and Proposition 3.5 for a matroid \( M = (E, B) \) of rank \( r \), its basis generating function \( F_M(x) \) is a multi-affine homogeneous polynomial of degree \( r \) in \( |E| \) variables with positive coefficients. Moreover, for any \( e \in E \) which is not a loop and a coloop, we have

\[
F_M(x) = F_{M\setminus e}(x) + x_e F_{M/e}(x),
\]
where $M \setminus e$ (resp. $M/e$) is the deletion (resp. contraction) of $M$ with respect to $e$ (see [14] for the definitions of them). In particular, if a matroid $M_0$ is obtained by deleting some elements $e_1, \ldots, e_k \in E$ from $M$, then we have

$$F_{M_0} = F_M|x_{e_1}=\ldots=x_{e_k}=0.$$  

We note that for any matroid $M$ on $[n] = \{1, 2, \ldots, n\}$, (not necessarily strict) homogeneous log-concavity of $F_M(x)$ on $\mathbb{R}^n_{\geq 0}$ is already known in [2, Theorem 4.2] as below. Precisely speaking, they show log-concavity in their paper, however by reading their proof carefully, one can easily show homogeneous log-concavity of $F_M(x)$ on $\mathbb{R}^n_{\geq 0}$.

**Theorem 3.7 ([2, Theorem 4.2]).** For any matroid $M$, $F_M(x)$ is homogeneous log-concave on $\mathbb{R}^n_{\geq 0}$. In other words,

$$-F_MH_{F_M} + s(\nabla F_M)^T(\nabla F_M) \succeq 0$$

for any $a \in \mathbb{R}^n_{\geq 0}$ and $s \geq \frac{r-1}{r}$.

**Remark 3.8.** In [2, Theorem 4.2], they also show $F_M(x)$ satisfies completely log-concavity, i.e., for any $v_1, \ldots, v_k \in \mathbb{R}^n_{\geq 0}$ $(0 \leq k \leq r - 2)$, $\partial_{v_1} \cdots \partial_{v_k} F_M(x)$ is log-concave on $\mathbb{R}^n_{\geq 0}$.

**Remark 3.9.** If $M$ is not simple, then $\det(-F_MH_{F_M} + s(\nabla F_M)^T(\nabla F_M))$ is identically zero, in particular, it cannot be positive definite at any point in $\mathbb{R}^n$. In fact, we assume $M$ has a loop $e$ or parallel elements $\{e_1, e_2\}$. In the former case, by definition, $\frac{\partial}{\partial x_e} F_M = 0$, in particular, $\det H_{F_M} = 0$. In the latter case, we can express $F_M$ like

$$F_M = F_M|_{x_e=x_{e_2}=0} + (x_{e_1} + x_{e_2})G,$$

where

$$G = G(x_1, \ldots, x_{e_1}, \ldots, x_{e_2}, \ldots, x_n) = \frac{\partial F_M}{\partial x_{e_1}} - \frac{\partial F_M}{\partial x_{e_2}}.$$

Thus we have $\det H_{F_M} = 0$. In both cases, we have $\det H_{F_M} = 0$. As we have seen at Proposition 2.3, this implies that $\det(-F_MH_{F_M} + s(\nabla F_M)^T(\nabla F_M)) = 0$.

In the rest of this section, we prepare some lemmas for our main theorem.

**Lemma 3.10.** Let $M$ be a matroid on $[N]$ of rank $M = r \geq 2$ with no loops (we don’t assume $M$ is simple). We consider its basis generating function $F_M(x)$. For any basis $B = \{i_{N-r+1}, \ldots, i_N\} \in \mathcal{B}$ of $M$ and its complement $\{j_1, \ldots, j_{N-r}\}$, $F_M(x)$ satisfies the following (1 \leq k \leq N-r).

$$\frac{\partial F_M|_{x_{i_{j_1}}=\ldots=x_{i_{j_{k-1}}}=0}}{\partial x_{i_k}} \neq 0. \quad (4)$$

**Proof.** By the definition of $F_M$, we only have to show that for each $k$, there exists a basis $B_0 \in \mathcal{B}$ such that $B_0 \cap \{i_1, \ldots, i_{k-1}\} = \emptyset$ and $i_k \in B_0$. Below, we will show that there exists some $\ell$ such that we can take $\{i_k\} \cup \{j_{N-r+1}, \ldots, j_\ell, \ldots, j_N\}$ as $B_0$. In fact, by [13, Corollary 1.2.6] there is a unique circuit $C(i_k, B)$ which is contained in $B \cup \{i_k\}$ (so-called the fundamental circuit). Since by definition and assumption, $i_k \in C(i_k, B)$, $i_k$ is not a loop, and $C(i_k, B)$ contains some $j_\ell$. Then by [13, Exercise 1.2.5], $B_0 := \{i_k\} \cup \{j_{N-r+1}, \ldots, j_\ell, \ldots, j_N\}$ is a basis. \hfill $\Box$

Since the basis generating function $F_M(x)$ of any simple matroid $M$ satisfies the condition (4) by Corollary 2.22 and Lemma 3.10 we have the following.
Theorem 3.11. Let \( M \) be a simple matroid on \([N]\) of rank \(M = r \geq 3\). For any basis \(B\), we assume that \( F_M \) is strict homogeneous log-concave on \( C_{B>0}^{N} (\subseteq (\mathbb{R}_{>0})^{N}) \). Then for any submatroid \( M_0 := M \setminus \{j_1, \ldots, j_k\}\) of rank \(r\), \( F_{M_0} \) is strict homogeneous log-concave on \( C_{B_0>0}^{N-k} (\subseteq (\mathbb{R}_{>0})^{N-k}) \) for any basis \( B_0 \) of \( M_0 \).

Proof. Let \( B_0 := \{i_N, \ldots, i_N\} \) be a basis of \( M_0 \) (and \( M \)). Since \( F_M \) satisfies the condition (*) for \( x_{j_1}, \ldots, x_{j_k} \) by Lemma 3.10, the polynomial \( F_{M_0} = F_M |_{x_{j_1}=\cdots=x_{j_k}=0} \) is strict homogeneous log-concave on \( C_{B_0>0}^{N-k} (\subseteq (\mathbb{R}_{>0})^{N-k}) \).

4. Main result

In this section, we show our main result. Our main result is that the Kirchhoff polynomial of each simple graph is strict log-concave on \( \mathbb{R}^{n}_{>0} \) (Theorem 4.2).

First we define the Kirchhoff polynomial of a graph.

Definition 4.1 (Kirchhoff polynomial). For a connected graph \( \Gamma = (V,E) \) with \(|E| = n\), we define the Kirchhoff polynomial of \( \Gamma \) by

\[
F_\Gamma(x_1, \ldots, x_n) = \sum_{T \in \mathcal{B}_\Gamma} \prod_{i \in T} x_i,
\]

where \( \mathcal{B}_\Gamma \) is the set of spanning trees in \( \Gamma \).

One can find that the Kirchhoff polynomial can be seen as the special case of the basis generating function of a matroid by Example 4.2.

Theorem 4.2 (Main result). For any simple graph \( \Gamma = (V,E) \) with \(|V| = r+1 \geq 3 \) and \(|E| = n \geq 3\), the Kirchhoff polynomial \( F_\Gamma(\mathbf{x}) \) is strict homogeneous log-concave on \( (\mathbb{R}_{>0})^{n} \). In other words,

\[
(-F_\Gamma \cdot H_{F_\Gamma} + s(\nabla F_\Gamma)^{\top} \nabla F_\Gamma)|_{\mathbf{x}=\mathbf{a}} > 0
\]

for any \( \mathbf{a} \in (\mathbb{R}_{>0})^{n} \) and \( s > \frac{1}{r-1} \). In particular, \( H_{F_\Gamma}|_{\mathbf{x}=\mathbf{a}} \) is non-degenerate, more precisely, it has exactly \( n-1 \) negative eigenvalues and exactly one positive eigenvalue. Thus, \((-1)^{n-1}(\det H_{F_\Gamma})|_{\mathbf{x}=\mathbf{a}} > 0\). Moreover, for each spanning tree \( T \) in \( \Gamma \), \( F_T \) is strict homogeneous log-concave on \( C_{T>0}^{n} \), where

\[
C_{T>0}^{n} = \{ \mathbf{a} \in \mathbb{R}^{n}_{>0} \mid z_i > 0 \ (i \in T), \ z_j \geq 0 \ (j \notin T) \} \ (\supseteq (\mathbb{R}_{>0})^{n}).
\]

Now we prove our main theorem: The Kirchhoff polynomial can be seen as the special case of the basis generating function of a matroid. Hence we have log-concavity of the Kirchhoff polynomial by Theorem 3.7. So we only have to check the strictness. Since we have Proposition 2.1, the Kirchhoff polynomial is strictly log-concave if and only if its Hessian does not vanish. Every simple graph is obtained from the complete graph with same number vertices by cutting edges. In other words, every simple graphic matroid is a submatroid of the graphic matroid of the complete graph. We can find easily the following corollary by Theorem 3.11.

Corollary 4.3. Let \( \Gamma = (V,E) \) be a simple graph with \(|V| = r + 1 \geq 3 \) and \(|E| = n \geq 3\). For each spanning tree \( T \) in \( \Gamma \), we assume that \( F_T \) is strict homogeneous log-concave on \( C_{T>0}^{n} \). Then for any connected subgraph \( \Gamma' = (V',E') \) with \(|V'| = r + 1 \) and \(|E'| = n - k\), \( F_{\Gamma'} \) is strict homogeneous log-concave on \( C_{T'>0}^{n-k} \ (\supseteq (\mathbb{R}_{>0})^{n-k}) \) for any basis \( T' \) in \( \Gamma' \).

Since we have Corollary 4.3 we will only have to show the Hessian does not vanish in the case of the complete graph. As stated in Section 2, for the relative invariant of an irreducible prehomogeneous vector space, its Hessian is in the form of \( C_{T_m}^{n} \). We can show that the Kirchhoff polynomial of the complete graph can be realized as the relative invariant. Then we have the following.
Theorem 4.4. Let $N = \binom{r+1}{2}$. We have
\[ \det H_{F_{K_{r+1}}} = (-1)^{N-1} c_r (F_{K_{r+1}})^{N-r-1}, \]
where $c_r = 2^{N-r}(r-1)$.

Theorem 4.4 implies that for any spanning tree $T$, the Kirchhoff polynomial is
strictly log-concave on $C_T > 0$. Hence we obtain our main result from Corollary 4.3.

In the rest of this section, we study more precisely the Kirchhoff polynomials
and we give a proof of Theorem 4.4.

In general, if a connected graph $\Gamma$ has $r+1$ vertices, then a spanning tree in $\Gamma$ has $r$ edges. Hence the Kirchhoff polynomial of $\Gamma$ with $r+1$ vertices is
a homogeneous polynomial of degree $r$. Moreover the Kirchhoff polynomial is a
multi-affine polynomial that each coefficient is one.

Example 4.5. Consider two graphs as follows.

![Figure 1. $K_4$](image1)

![Figure 2. $K_4 \setminus \{2,3\}$](image2)

The number of spanning trees in $K_4$ and $K_4 \setminus \{2,3\}$ are sixteen and eight, respectively. Then the Kirchhoff polynomial of $K_4$ is as follows:

$$F_{K_4}(x) = x_{12}x_{13}x_{14} + x_{12}x_{14}x_{23} + x_{13}x_{14}x_{23} + x_{12}x_{13}x_{24}$$

$$+ x_{13}x_{14}x_{24} + x_{12}x_{23}x_{24} + x_{13}x_{23}x_{24} + x_{14}x_{23}x_{24}$$

$$+ x_{12}x_{13}x_{34} + x_{12}x_{14}x_{34} + x_{12}x_{23}x_{34} + x_{13}x_{23}x_{34}$$

$$+ x_{14}x_{23}x_{34} + x_{12}x_{24}x_{34} + x_{13}x_{24}x_{34} + x_{14}x_{24}x_{34}.$$

And the Kirchhoff polynomial of $K_4 \setminus \{2,3\}$ is as follows:

$$F_{K_4 \setminus \{2,3\}}(x) = x_{12}x_{13}x_{14} + x_{12}x_{13}x_{24} + x_{13}x_{14}x_{24} + x_{12}x_{13}x_{34}$$

$$+ x_{12}x_{14}x_{34} + x_{12}x_{24}x_{34} + x_{13}x_{24}x_{34} + x_{14}x_{24}x_{34}.$$

In Example 4.5, we can see that the Kirchhoff polynomial of $K_4 \setminus \{2,3\}$ is equal
to the Kirchhoff polynomial of $K_4$ substituting zero to the variable $x_{23}$. In general,
every Kirchhoff polynomial is obtained from the Kirchhoff polynomial of the
complete graph with same number vertices by substituting zero to some variables.

Next we will see that the Kirchhoff polynomial is realized as the determinant of
some matrix. This is called the Matrix-tree theorem. Let $(E_r)_{ij}$ be a $r \times r$ matrix
whose the $(i,j)$-component is one and the others are zero. If the size of the matrix
is obvious from the context, then we will drop the size as $E_{ij} = (E_r)_{ij}$. For a
matrix $X$, $X^{(ij)}$ denotes the submatrix of $X$ obtained from removing the $i$th row
and the $j$th column.

Definition 4.6 (Laplacian). For a graph $\Gamma = (V,E)$ with $|V| = r$, we associate a
variable $x_e$ to each edge $e \in E$, and we define the Laplacian $L_\Gamma$ of $\Gamma$ indexed by
vertices by

$$L_\Gamma = \sum_{e=(i,j) \in E} x_e (E_{ii} - E_{ij} - E_{ji} + E_{jj}).$$

The following theorem is well-known fact. For example, see [14] Theorem VI.29
for the detail.
Theorem 4.7 (The Matrix-Tree Theorem). For a graph \( \Gamma \), its Kirchhoff polynomial \( F_\Gamma \) is equal to any cofactor of its Laplacian \( L_\Gamma \). In other words, for a graph \( \Gamma = (V, E) \) with \( |V| = r \),

\[
F_\Gamma = (-1)^{i+j} \det(L_\Gamma^{(ij)})
\]

for any \( 1 \leq i, j \leq r \).

Remark 4.8 will be important key.

Remark 4.8. For the complete graph \( K_{r+1} \) with \( r+1 \) vertices, we denote \( x_{ij} = x_e \) for each edge \( e = \{i, j\} \). In particular, \( x_{ij} = x_{ji} \). Then the entries in Laplacian \( L_{K_{r+1}} = (\ell_{ij})_{1 \leq i, j \leq r+1} \) is

\[
\ell_{ij} = \begin{cases}
\left( \sum_{k=1}^{r+1} x_{ki} \right) - x_{ii} & \text{(if } i = j\text{)}, \\
-x_{ij} & \text{(otherwise)}.
\end{cases}
\]

Let \( \text{Sym}(r, \mathbb{C}) \) be the set of symmetric matrices of size \( r \) with complex entries. Note that \( L_{K_{r+1}}^{(11)} \) is a symmetric matrix and \( \{x_{ij}\}_{1 \leq i < j \leq r+1} \) gives a coordinate of \( \text{Sym}(r, \mathbb{C}) \). Hence we have

\[
\{ L_{K_{r+1}}^{(11)} \mid x_{ij} \in \mathbb{C} \} = \text{Sym}(r, \mathbb{C}).
\]

Therefore the Kirchhoff polynomial \( F_{K_{r+1}} \) can be regarded as a function from \( \text{Sym}(r, \mathbb{C}) \) to \( \mathbb{C} \). In other words, we can regard the Kirchhoff polynomial as the following function:

\[
F_\Gamma = \det : \text{Sym}(r, \mathbb{C}) \to \mathbb{C}.
\]

In [15], the classification of irreducible prehomogeneous vector spaces has already done. Here, we focus on the following prehomogeneous vector space whose the relative invariant is given by the Kirchhoff polynomial of complete graphs. See [15, Proposition 3 in Section 5] or [15, Section 7, I-(2)] for the detail of Proposition 4.9.

Proposition 4.9. Let \( \rho \) be the representation of \( GL_r(\mathbb{C}) \) on \( \text{Sym}(r, \mathbb{C}) \) such that

\[
\rho(P)X = PXP^T \quad (P \in GL_r(\mathbb{C})).
\]

Then \( (GL_r(\mathbb{C}), \rho, \text{Sym}(r, \mathbb{C})) \) is a regular irreducible prehomogeneous vector space. Moreover, the relative invariant is given by \( \det : \text{Sym}(r, \mathbb{C}) \to \mathbb{C} \).

As remarked at Remark 4.8, the Kirchhoff polynomial \( F_{K_{r+1}}(x) \) of the complete graph \( K_{r+1} \) is the relative invariant of the prehomogeneous vector space in Proposition 4.9.

On the other hand, it is known that the evaluation of \( (\det H_{F_{K_{r+1}}})_{x=(1,1,\ldots,1)} \) by the second author [18]. Note that we used Cayley’s theorem \( F_{K_{r+1}}(1,1,\ldots,1) = (r+1)^{r-1} \) at the second equality in Proposition 4.10 (see [13] Theorem VI.30 for the detail of Cayley’s theorem).

Proposition 4.10 (Yazawa [18 Theorem 3.3]). For the complete graph \( K_{r+1}, \)

\[
(\det H_{F_{K_{r+1}}})_{x=(1,1,\ldots,1)} = (-1)^{N-1}2^N(r-r^r+N(r-4))(r-2)
= (-1)^{N-1}2^N(r-1)(F_{K_{r+1}}(1,1,\ldots,1))^{N-r-1},
\]

where \( N = \binom{r+1}{2} \).

By Corollary 2.16 and Propositions 4.9, 4.10, we have Theorem 4.4.
5. Applications

In this section, we define an graded Artinian Gorenstein algebra $R$ associated to a graph $\Gamma$ (more generally, to a matroid), which is introduced by Maeno and Numata in [12]. Then by using strict (homogeneous) log-concavity of $F_{\Gamma}$ at any $a \in (\mathbb{R}_{>0})^n$, we will prove that $L_a := a_1x_1 + \cdots + a_n x_n \in R_1^{\Gamma}$, satisfies the strong Lefschetz property at $R_1^{\Gamma}$. We will also mention the relation between our result and known results by Huh and Wang in [8] (Remark 5.13). In the rest of this paper, let $k$ be a field of characteristic zero.

5.1. Artinian Gorenstein algebras. First, we define an Artinian Gorenstein algebra associated to each homogeneous polynomial.

Definition 5.1. Let $F$ be a homogenenous polynomial of $F \in k[x_1, \ldots, x_n]$. We define an ideal $\text{Ann}(F)$ and a quotient algebra $R^*_F$ by

$$\text{Ann}(F) = \left\{ P \in k[x_1, \ldots, x_n] \mid P \left( \frac{\partial}{\partial x_1}, \ldots, \frac{\partial}{\partial x_n} \right) F = 0 \right\},$$

$$R^*_F = k[x_1, \ldots, x_n]/\text{Ann}(F).$$

Definition 5.2 (Poincaré duality algebra cf. [13, Definition 2.1]). A finite-dimensional graded $k$-algebra $R^* = \bigoplus_{\ell=0}^{r} R^\ell$ is called the Poincaré duality algebra if $\dim_k R^r = 1$ and the bilinear pairing induced by the multiplication $R^\ell \times R^{r-\ell} \to R^r$ is non-degenerate for $\ell = 0, \ldots, \lfloor \frac{r}{2} \rfloor$.

These rings $R^*_F$ can represent all (standard) graded Artinian Gorenstein algebras as the following.

Theorem 5.3 (cf. [13, Proposition 2.1, Theorem 2.1 and Remark 2.3]). Let $I$ be an homogenenous ideal of $k[x_1, \ldots, x_n]$ and $R^* := k[x_1, \ldots, x_n]/I$ the quotient algebra. Then the following are equivalent:

- The $k$-algebra $R^*$ is an Artinian Gorenstein algebra.
- There exists a homogeneous polynomial $F \in k[x_1, \ldots, x_n]$ such that $I = \text{Ann}(F)$.
- $R^*$ is a (Artinian) Poincaré duality algebra.

We recall the notion of the strong Lefschetz property and the Hodge–Riemann bilinear form.

Definition 5.4 (The strong Lefschetz property). Let $R^* = \bigoplus_{\ell=0}^{r} R^\ell, R^r \neq 0$, be a graded Artinian algebra. We say that $L \in R^1$ satisfies the strong Lefschetz property at degree $\ell$ (or $R^\ell$) if the multiplication map $\times L^{r-2\ell} : R^{\ell} \to R^{r-\ell}$ is bijective.

Remark 5.5. Our definition of the strong Lefschetz property is the strong Lefschetz property in the narrow sense in [13, Definition 2.1].

We will use the following criterion which is the special case of the general criterion in [13, Theorem 3.1] and [17, Theorem 4].

Theorem 5.6 (The Hessian criterion of the strong Lefschetz property cf. [13, Theorem 3.1], [17, Theorem 4]). Assume that $x_1, \ldots, x_n \in R_1^{\Gamma}$ is a basis. An element $L_a := a_1x_1 + \cdots + a_n x_n \in R_1^{\Gamma}$ satisfies the strong Lefschetz property at degree one if and only if $F(a_1, \ldots, a_n) \neq 0$ and

$$\det H_{F|x=a} \neq 0.$$
where $H_F := \left( \frac{\partial^2 F}{\partial x_i \partial x_j} \right)_{1 \leq i, j \leq n}$ is the Hessian matrix of $F$.

**Definition 5.7** (Hodge–Riemann relation). Let $R^* := \oplus_{\ell=0}^r R^\ell$, $R^\ell \neq 0$, be a graded Artinian Gorenstein algebra. We say that $L \subseteq R^I$ satisfies the Hodge–Riemann relation at degree $\ell$ (or $R_L$) if for the map $Q^\ell_L : R^\ell \times R^\ell \to k$, $Q^\ell_L(x, y) = (-1)^\ell \deg(xL - 2\ell y)$, the map $Q^\ell_L$ is positive definite on $\text{Ker}(L^{r+1-2\ell})$, where $\deg : R_F \twoheadrightarrow k$ is a fixed isomorphism.

**Remark 5.8.** For a Poincaré duality algebra, the Hodge–Riemann bilinear form at degree one is non-degenerate if and only if $L_a$ satisfies the strong Lefschetz property at degree one.

**Remark 5.9.** Assume that $x_1, \ldots, x_n$ forms a basis of $R^1_F$. Then $HR^1_a(F)$ is given by the Hessian matrix $H_F|_{x=a}$ with respect to $x_1, \ldots, x_n \in R^1_F$. In fact, by definition, we have

$$HR^1_a(x_i, x_j) = \text{deg}(x_i L^2 a x_j)$$

$$= \left( a_1 \frac{\partial}{\partial x_1} + \cdots + a_n \frac{\partial}{\partial x_n} \right)^{r-2} \left( \frac{\partial^2}{\partial x_i \partial x_j} \right) F$$

$$= \left. \frac{\partial^2 F}{\partial x_i \partial x_j} \right|_{x=a}.$$

(For the detail, see the proof of [13, Theorem 3.1].)

**Remark 5.10.** In general, $x_1, \ldots, x_n$ is not necessarily linearly independent in $R^1_F$. For example, $F := x_1 x_2 + x_1 x_3 + 4 x_1 x_4 + x_2 x_3 + x_2 x_4 + x_3 x_4$ satisfies $-\frac{\partial F}{\partial x_1} + 2 \frac{\partial F}{\partial x_2} + 2 \frac{\partial F}{\partial x_3} - \frac{\partial F}{\partial x_4} = 0$. In this case, $H_F$ is identically zero.

5.2. **The strong Lefschetz property of the Artinian Gorenstein algebras associated to simple graphic matroids.** Here we will consider the Artinian Gorenstein algebra $R_{F_M}$ associated to the basis generating function $F_M$ of a simple matroid $M$, in particular, the Kirchhoff polynomial $F_\Gamma$ of a simple graph $\Gamma$. In these cases, we will often abbreviate $R^*_{F_M}$ to $R^*_M$ and $R^1_F$ to $R^*_F$, and we consider these algebras when $k = \mathbb{R}$.

Maeno and Numata conjectured that for any matroid $M$, the algebra $R^*_M$ has the strong Lefschetz property at all degrees in $[11]$.

By our main result Theorem 5.11, we have the following.

**Theorem 5.11** (The strong Lefschetz property of $R^*_F$ at degree one). For any simple graph $\Gamma = (V, E)$ with $|V| = r + 1 \geq 3$ and $|E| = n \geq 3$, and any $a = (a_1, \ldots, a_n) \in (\mathbb{R}_{>0})^n$, $x_1 \cdots + n x_n \in R^*_F$ satisfies the strong Lefschetz property at degree one.

**Theorem 5.12** (The non-degeneracy of the Hodge–Riemann bilinear form in the graphic case). In the above setting, for any $a \in (\mathbb{R}_{>0})^n$, the Hodge–Riemann bilinear form $HR^1_a(F_\Gamma)$ is non-degenerate. Moreover, $HR^1_a(F_\Gamma)$ has $n - 1$ negative eigenvalues and one positive eigenvalue.

We will explain that our result is not followed from related known results in [8].

**Remark 5.13.** In [12], Maeno and Numata introduce an Artinian (generally non-Gorenstein) graded $k$-algebra $k[x_1, \ldots, x_n]/J_M$ associated to each matroid $M$, where $J_M$ is a certain ideal such that $J_M \subseteq \text{Ann} F_M$. Then, they show that the strong Lefschetz property of $x_1 \cdots + n x_n$ at every degree when $M = M(q, n)$ is the projective space over a finite field $F_q$ in this case, $J_M = \text{Ann} F_M$ (see [12] Example 2.3 & Theorem 4.3 (2))). In [8], Huh and Wang denote this ring by $B^*(M) = \oplus_{\ell=0}^r B^\ell(M)$, and they study this ring associated to each general simple matroid. They show that
this satisfies the “injective” Lefschetz property when \( M \) is a representable matroid \( M \), i.e., for any \( 0 \leq \ell \leq \lfloor \frac{n}{2} \rfloor \), the multiplication map \( \times L^{r-2\ell} \) : \( B^\ell(M) \to B^{r-\ell}(M) \) of the element \( L := x_1 + \cdots + x_n \) is injective, where \( M \) is a representable matroid on \([n]\) of rank \( M = r \). Since we have the natural surjection \( B^*(M) \twoheadrightarrow R_M^* \), we have the following commutative diagram:

\[
\begin{array}{ccc}
B^1(M) & \xrightarrow{\times L^{r-2}} & B^{r-1}(M) \\
\downarrow & & \downarrow \\
R_M^1 & \xrightarrow{\times L^{r-2}} & R_M^{r-1}
\end{array}
\]

This diagram would not imply our Corollary 5.11 that is, \( \times L^{r-2} : R_M^1 \to R_M^{r-1} \) is an isomorphism in the graphic case \( M = M_\Gamma \). On the other hand, by the non-degeneracy of the Hessian of \( F_\Gamma \), in particular, we know that \( \{x_1, \ldots, x_n\} \) is a basis of \( R_M^1 \). Since \( x_1, \ldots, x_n \) is a basis of \( B^1(M) \) by the definition of \( B^*(M) \), this implies that \( B^1(M) = R_M^1 \), and the Hodge–Riemann bilinear form at \( B^1(M) \) and \( R_M^1 \) are the same. Thus Corollary 5.12 implies the Hodge–Riemann bilinear form at \( B^1(M_\Gamma) \) with respect to \( L_\alpha \) is non-degenerate, moreover it has \( n - 1 \) negative eigenvalues and one positive eigenvalue.

5.3. The strong Lefschetz Property of elementary symmetric functions. Here we will show that for any simple graph \( \Gamma \) with \( r \) vertices and \( n \) edges, \( 1 \leq \ell \leq r - 2 \), and \( a \in (\mathbb{R}_{>0})^n \), \( \partial a \ell F_\Gamma \) is strict homogeneous log-concave at \( a \), in particular strict log-concave at \( a \). As an application, we will prove that for elementary symmetric functions \( e_n - \ell(x_1, \ldots, x_n) \) (\( 0 \leq \ell \leq n - 2 \), \( x_1 + \cdots + x_n \)) satisfies the strong Lefschetz property at degree one in \( R_{n-\ell}^* \).

First, we note the following general property of (strict) homogeneous log-concave polynomial. In the proof, we use essentially the assumption of (strict) “homogeneous” log-concavity. Note that (strict) log-concave does not imply Lemma 5.14.

**Lemma 5.14.** Let \( F \in \mathbb{R}[x_1, \ldots, x_n] \) be a homogeneous polynomial of deg \( F = r \geq 3 \). For any \( a \in \mathbb{R}^n \), the following are equivalent:

(i) \( F \) is homogeneous log-concave (resp. strict homogeneous log-concave) at \( a \), i.e., for any \( s \geq \frac{r-1}{r-2} \) (resp. \( s > \frac{r-1}{r-2} \)),

\[
(-FH_F + s\nabla F)^T(\nabla F))|_{x=a} \geq 0 \quad \text{(resp.} \quad > 0)\.
\]

(ii) \( \partial a F := \sum_{i=1}^n a_i \frac{\partial F}{\partial x_i} \) is homogeneous log-concave (resp. strict homogeneous log-concave) at \( a \), i.e., for any \( s' \geq \frac{r-1}{r-2} \) (resp. \( s' > \frac{r-1}{r-2} \))

\[
(-\partial a F)H_{\partial a F} + s'(\nabla \partial a F)^T(\nabla \partial a F))|_{x=a} \geq 0 \quad \text{(resp.} \quad > 0)\.
\]

**Proof.** Assume that \( \partial a F := \sum_{i=1}^n a_i \frac{\partial F}{\partial x_i} \) is strict homogeneous log-concave at \( a \). By Euler’s identity, we have the following identities.

\[
r F(a) = (\partial a F)(a),
\]

\[
(r - 1)\nabla F|_{x=a} = \nabla (\partial a F)|_{x=a},
\]

\[
(r - 2)H_F|_{x=a} = H_{\partial a F}|_{x=a}.
\]

Then, we have

\[
(-FH_F + s\nabla F)^T(\nabla F))|_{x=a} = \frac{1}{r(r-2)} \left\{ -\partial a F)H_{\partial a F} + s \cdot \frac{r(r-2)}{(r-1)^2} (\nabla \partial a F)^T(\nabla \partial a F) \right\}|_{x=a}.
\]

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If we set
\[ s' = s - \frac{r(r - 2)}{(r - 1)^2}, \]
then by some easy computations, we have \( s > \frac{r - 2}{r} \) if and only if \( s' > \frac{r - 2}{r - 1} \). This implies the equivalence of (i) and (ii).

**Corollary 5.15.** Let \( F \in \mathbb{R}[x_1, \ldots, x_n] \) be a homogeneous polynomial of degree \( r \geq 3 \). If \( F \) is strict homogeneous log-concave at \( a \in \mathbb{R}^n \), then for any \( 0 \leq \ell \leq r - 2 \), \( \partial_a^\ell F := (\partial_a^\ell F) \) is also strict homogeneous log-concave at \( a \in \mathbb{R}^n \).

Then, by Corollary 2.20, we have the following corollary.

**Corollary 5.16.** For any simple graph \( \Gamma = (V, E) \) with \( |V| = r + 1 \geq 3 \) and \( |E| = n \geq 3 \), and any \( a \in (\mathbb{R}_{>0})^n \), \( \partial_a^\ell F_{\Gamma} \) is strict homogeneous log-concave at \( a \), where \( F_{\Gamma} \) is the Kirchhoff polynomial of \( \Gamma \). In particular, \((-1)^{n-r-1} \det \partial_a^\ell F_{\Gamma}|x=a > 0\), and \( a_1x_1 + \cdots + a_nx_n \in R_{\ell}^{1} \) satisfies the strong Lefschetz property at degree one.

Let \( e_{n-\ell} = e_{n-\ell}(x_1, \ldots, x_n) \) be the \((n-\ell)\)-th elementary symmetric polynomial in \( n \) variables. Then one can easily show the following identity:
\[ e_{n-\ell}(x_1, \ldots, x_n) = \ell \partial_a^\ell e_n(x_1, \ldots, x_n), \]
where \( a = (1, 1, \ldots, 1)^T \). Since \( e_n(x_1, \ldots, x_n) = x_1 \cdots x_n \) is the Kirchhoff polynomial of a tree with \( n + 1 \) vertices, we have the following by Corollary 5.16.

**Corollary 5.17.** For the elementary symmetric polynomial \( e_{n-\ell} = e_{n-\ell}(x_1, \ldots, x_n) \) \((0 \leq \ell \leq n - 2)\), the element \( x_1 + \cdots + x_n \in R_{n-\ell}^{1} \) satisfies the strong Lefschetz property at degree one.

**Remark 5.18.** In [10, Theorem 4.3], Maeno and Numata showed that for the \( e_k(x_1, \ldots, x_n) \), the element \( x_1 + \cdots + x_n \) satisfies the strong Lefschetz property at all degrees. They used the Hessian criterion (see Theorem 5.0), and they showed the non-degeneracy of the Hessian matrix of \( e_k(x_1, \ldots, x_n) \) at \((x_1, \ldots, x_n) = (1, \ldots, 1)\) by the non-degeneracy of the Poincaré duality of some Gorenstein algebra.

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