A GEOMETRIC CONSTRUCTION OF INTEGRABLE
HAMILTONIAN HIERARCHIES ASSOCIATED WITH THE
CLASSICAL AFFINE $\mathcal{W}$-ALGEBRAS

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Abstract. A class of classical affine $\mathcal{W}$-algebras are shown to be isomorphic
as differential algebras to the coordinate rings of double coset spaces of certain
pronilpotent proalgebraic groups. As an application, integrable Hamiltonian
hierarchies associated with them are constructed geometrically, generalizing the
result of Feigin-Frenkel and Enriquez-Frenkel for the principal cases.

1. Introduction

Since the discovery of the Drinfel’d-Sokolov hierarchy [14], a number of integrable
systems have been constructed in the same spirit cf. [4, 6, 18, 21, 22]. Those
integrable systems are not of finite dimension, which are formulated by Poisson
algebras, but of infinite dimension, reflecting classical field theory as their origin.

In [2], Barakat, De Sole and Kac used Poisson vertex algebras, which play a role
of Poisson algebras for integrable systems as above, and introduced the notion of
integrable Hamiltonian hierarchies as a framework of integrability for Poisson vertex
algebras. Since then De Sole, Kac, and Valeri e.g. [9, 10, 11, 12, 13] have studied
systematically integrable Hamiltonian hierarchies associated with Poisson vertex
algebras, called the classical (affine) $\mathcal{W}$-algebras, which are obtained as classical
limit of vertex algebras, called the (affine) $\mathcal{W}$-algebras [7, 20], see also [8, 30].

The $\mathcal{W}$-algebras are parametrized by a triple $(\mathfrak{g}, f, k)$ consisting of a finite di-
mensional simple Lie algebra $\mathfrak{g}$ over $\mathbb{C}$, a nonzero nilpotent element $f \in \mathfrak{g}$, and a
complex number $k \in \mathbb{C}$, called the level and are denoted by $\mathcal{W}^k(\mathfrak{g}, f)$. The classical
$\mathcal{W}$-algebras are also parametrized by the same data and thus we denote them also
by $\mathcal{W}^k(\mathfrak{g}, f)$. De Sole, Kac and Valeri recovered the earlier results on the integrable
Hamiltonian hierarchies mentioned above in their study and obtained a culminating
result in [13] on the existence of integrable Hamiltonian hierarchies associated with
$\mathcal{W}^k(\mathfrak{g}, f)$, which states: the classical affine $\mathcal{W}$-algebra $\mathcal{W}^k(\mathfrak{g}, f)$ admits an integrable
Hamiltonian hierarchy for any finite dimensional simple Lie algebra $\mathfrak{g}$ of classical
type, nonzero nilpotent element $f$, and $k \neq 0$.

Besides the algebraic theory of integrable Hamiltonian hierarchies mentioned
above, there is a result of Feigin-Frenkel [19, 20] and Enriquez-Frenkel [10] which
constructs the Drinfel’d-Sokolov hierarchies by using a geometric realization of the
classical affine $\mathcal{W}$-algebra $\mathcal{W}^1(\mathfrak{g}, f_{\text{prin}})$, where $f_{\text{prin}}$ denotes the principal nilpotent
element in $\mathfrak{g}$. More precisely, they proved that $\mathcal{W}^1(\mathfrak{g}, f_{\text{prin}})$ are isomorphic as differ-
etial algebras to the coordinate rings of certain double coset spaces of pronilpotent
proalgebraic groups and that the Drinfel’d-Sokolov hierarchies are induced from
natural group actions on these spaces.

The aim of this paper is to generalize the those results to the classical $\mathcal{W}$-algebras
$\mathcal{W}^k(\mathfrak{g}, f)$ when $f$ satisfies certain properties (see the condition (F) in the below)
and the level $k \in \mathbb{C}$ is generic. We note that the choice of $f$ is a special case of the
so-called Type I in [4, 6, 18]. The integrable Hamiltonian hierarchies obtained in
the paper for the case $k = 1$ coincide with special cases of those considered in [3] by construction, and are special cases of the result [9].

Let $\mathfrak{g}$ be a finite dimensional Lie algebra over $\mathbb{C}$, and $f \in \mathfrak{g}$ be a non-zero nilpotent element. We fix an $\mathfrak{sl}_2$-triple $\{e, h = 2x, f\}$ containing $f$. We denote by $\Gamma : \mathfrak{g} = \oplus_j g_j$ the $\mathfrak{sl}_2$-grading and by $\mathfrak{g} = h \oplus (\oplus_{\alpha \in \Delta} \mathfrak{g}_\alpha)$ a root space decomposition of $\mathfrak{g}$ which is homogeneous with respect to $\Gamma$. Then the root system $\Delta$ admits an induced grading $\Delta = \bigcup_{j \geq 0} \Delta_j$. Let $\Pi$ denote the subset of $\Delta_{\geq 0} = \bigcup_{j > 0} \Delta_j$ consisting of the elements indecomposable in $\Delta_{\geq 0}$. Let $V^k(\mathfrak{g}_0)$ denote the universal affine Poisson vertex algebra associated with the reductive Lie algebra $\mathfrak{g}_0$ at level $k$ and $F(\mathfrak{g}_{1/2})$ the $\beta\gamma$-system Poisson vertex algebra associated with the symplectic vector space $\mathfrak{g}_{1/2}$. Then the $W$-algebra $W^k(\mathfrak{g}, f)$ is defined as a 0-th cohomology of the BRST complex $C^k(\mathfrak{g}, f)$, see Section 3.1 for details.

For generic level $k$, the $W$-algebras are realized as the joint kernel of certain screening operators [25]. Our first result (Theorem 3.6) is a Poisson vertex algebra analogue of this result. Namely, for generic $k \in \mathbb{C}$, we realize the classical affine $W$-algebra $W^k(\mathfrak{g}, f)$ as the Poisson vertex subalgebra of $V^k(\mathfrak{g}_0) \otimes F(\mathfrak{g}_{1/2})$ invariant under the derivations $Q^W_\alpha : (\alpha \in \Pi)$:

\[(1.1) \quad W^k(\mathfrak{g}, f) \cong \cap_{\alpha \in \Pi} \ker \left( Q^W_\alpha : V^k(\mathfrak{g}_0) \otimes F(\mathfrak{g}_{1/2}) \to V^k(\mathfrak{g}_0) \otimes F(\mathfrak{g}_{1/2}) \right).\]

See [3.19] and [3.17] for the definition of $Q^W_\alpha$.

We now suppose that the pair $(\mathfrak{g}, f)$ satisfies the following condition (F):

(F1) The grading $\Gamma$ is a $\mathbb{Z}$-grading.

(F2) There exists an element $y \in \mathfrak{g}_d$ such that $s = f + yt^{-1} \in \mathfrak{g}[t^\pm 1]$ is semisimple.

(F3) The Lie subalgebra $\ker(ad_s) \subset \mathfrak{g}[t^\pm 1]$ is abelian and $\text{Im}(ad_s) \cap \mathfrak{g}[t^\pm 1] \ni 0 = \mathfrak{g}_0$.

Here $\mathfrak{g}[t^\pm 1] = \mathfrak{g} \otimes \mathbb{C}[t, t^{-1}]$ and $\mathfrak{g}[t^{\pm 1}] = \oplus_{j \in \mathbb{Z}} \mathfrak{g}[t^{\pm 1}]_j$ is the $\mathbb{Z}$-grading given by $\deg(Xt^n) = j + (d + 1)n$, $(X \in \mathfrak{g}_j)$. The nilpotent elements $s$ satisfying (F1)-(F2) are called Type I in the literature e.g. [18, 4, 6]. We consider the completion $L\mathfrak{g} = \mathfrak{g} \otimes \mathbb{C}((t))$. It has subalgebras $L\mathfrak{g}_\pm$, which is the completion of $\mathfrak{g}[t^{\pm 1}]_{\geq 0}$, and $L\mathfrak{g}_\pm = \mathfrak{g}[t^{\pm 1}]_{\geq 0}$. We have the corresponding closed subgroups $LG_+, LG_-$ of the loop group $LG$ of $\mathfrak{g}$.

Let $\mathfrak{a}$ denote the completion of $\ker(ad_s)$ in $L\mathfrak{g}$, and $\mathfrak{a}_\pm = \mathfrak{a} \cap L\mathfrak{g}_\pm$. Let $A$, (resp. $A_\pm$) be the closed subgroup of $LG$ corresponding to $\mathfrak{a}$, (resp. $\mathfrak{a}_\pm$).

The right $A$-action $LG_- \times LG_+ \times A \to LG_- \times LG_+ \times A$, induces a Lie algebra homomorphism $\mathfrak{a} \to \text{Der}(\mathbb{C}[LG_- \times LG_+/A_+])$, and so that $\mathfrak{a} \to \text{Der}(\mathbb{C}[LG_+/A_+])$ since $LG_+/A_+ \subset LG_- \times LG_+ \times A_+$ is an open subset. We denote this action by $a \to a^R$. In particular, we obtain a differential algebra $(\mathbb{C}[LG_+/A_+], s^R)$. On the other hand, the natural left $LG_+$-action $LG_+ \times LG_+ \times A_+ \to LG_+ \times A_+$ induces a Lie algebra homomorphism $Lg_+ \to \text{Der}(\mathbb{C}[LG_+/A_+])$, which we denote by $X \to X^L$.

Our second result (Theorem 4.3) is the following geometric interpretation of (1.11) under the condition (F): there exists an isomorphism of differential algebras

\[\Psi_k : V^k(\mathfrak{g}_0) \to \mathbb{C}[LG_+/A_+],\]

such that the derivation $Q^W_\alpha$ is identified with $X^L_\alpha$ for some element $X_\alpha \in \mathfrak{g}_+$, ($\alpha \in \Pi$). Since such root vectors $X_\alpha$ generate a Lie algebra $\mathfrak{g}_+ = \oplus_{\alpha > 0} \mathfrak{g}_\alpha$, it implies that $W^k(\mathfrak{g}, f)$ is isomorphic to $\mathbb{C}[G_+ \times LG_+/A_+]$ as a differential algebra (Corollary 4.4). Here $G_+$ is the closed subgroup of $LG_+$ corresponding to $\mathfrak{g}_+$.

The action $\mathfrak{a} \to \text{Der}(\mathbb{C}[LG_+/A_+])$ preserves the subalgebra $\mathbb{C}[G_+ \times LG_+/A_+]$. Thus we obtain a space of mutually commutative derivations of $W^k(\mathfrak{g}, f)$ as the image of $\mathfrak{a}$. We denote this space by $\mathcal{H}^k(\mathfrak{g}, f)$. Our third result (Theorem 5.3) states that $\mathcal{H}^k(\mathfrak{g}, f)$ is an integrable Hamiltonian hierarchy associated with $W^k(\mathfrak{g}, f)$. 
The double coset spaces $G_+ \backslash LG_+ / A_+$ considered here are embedded into the abelianized Grassmanians $G[t^{-1}] \backslash G((t)) / A_+$ as a Zariski open subset. The abelianized Grassmanians have been used to construct Drinfeld-Sokolov hierarchies geometrically [3]. As pointed in loc.cit., such a construction implies a strong compatibility of integrable Hamiltonian systems associated with classical affine $W$-algebras and Hitchin systems. The author hopes to investigate the relationship between classical affine $W$-algebras and Hitchin systems in future works.

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2. Poisson vertex superalgebras

2.1. Poisson vertex superalgebra. We recall here some basics about Poisson vertex superalgebras and their relation to the theory of integrable systems, following [2, 30]. We remove the prefix “super” whenever we consider the non-super cases.

A differential $\mathbb{C}$-superalgebra is a pair $(V, \partial)$ consisting of a supercommutative $\mathbb{C}$-superalgebra $V$ and an even derivation $\partial$ on it. We denote by $\bar{a}$ the parity of For $a \in V$ and by $\text{Der}(V)$ the set of super derivations of $V$. A differential $\mathbb{C}$-superalgebra of the form

$$V = \mathbb{C}[u_i^{(n)} | i \in I_0, n \geq 0] \otimes \bigotimes_{i \in I_1} \mathbb{C}u_i^{(n)}$$

as $\mathbb{C}$-superalgebras for some index set $I = I_0 \sqcup I_1$ where $u_i^{(n)} = \partial^n u_i$, is called the superalgebra of differential polynomials in the variables $u_i$, $(i \in I)$.

A Poisson vertex superalgebra is a triple $(V, \partial, \{ -\lambda - \})$ consisting of a differential $\mathbb{C}$-superalgebra $(V, \partial)$ and an even $\mathbb{C}$-bilinear map

$$\{ -\lambda - \} : V \times V \rightarrow V[\lambda], \quad (f, g) \mapsto \{ f_{\lambda} g \} = \sum_{n \geq 0} \lambda^n \frac{1}{n!} f_{(n)} g,$$

called the $\lambda$-bracket, satisfying

$$\{ \partial f_{\lambda} g \} = -\lambda \{ f_{\lambda} g \}, \quad \{ f_{\lambda} \partial g \} = (\lambda + \partial) \{ f_{\lambda} g \},$$

$$\{ g a f \} = -(-1)^{\bar{f} \bar{g}} \{ f_{\partial - \lambda} g \},$$

$$\{ f_{\lambda} \{ g_{\mu} h \} \} - (-1)^{\bar{f} \bar{g}} \{ g_{\mu} \{ f_{\lambda} h \} \} = \{ \{ f_{\lambda} g \}_{\lambda + \mu} h \},$$

$$\{ f_{\lambda} g h \} = \{ f_{\lambda} g \} h + (-1)^{\bar{g}} \{ f_{\lambda} h \} g,$$

$$\{ f g_{\lambda} h \} = (-1)^{\bar{g} \bar{h}} \{ f_{\lambda + \partial} h \} + g + (-1)^{\bar{f} \bar{g} + \bar{h}} \{ g_{\lambda + \partial} h \} \rightarrow f,$$

for $f, g, h \in V$. Here we denote

$$\{ f_{\lambda} g \}_- = \sum_{1}^{1} \frac{1}{n!} f_{(n)} g \lambda^n, \quad \{ f_{\lambda} g \} = \sum_{n \geq 0} \lambda^n \frac{1}{n!} f_{(n)} g.$$

**Remark 2.1.** Given an algebra of differential polynomials $V$ in the variables $u_i$, $(i \in I)$, a linear map $F : \text{span}_\mathbb{C} \{ u_i \}_{i \in I} \rightarrow V[\lambda]$ uniquely extends to $F : V \rightarrow V[\lambda]$ by (2.3), (2.7).

Let $V$ be a superalgebra of differential polynomials in the variables $u_i$, $(i \in I)$. We denote by

$$\frac{\partial}{\partial u_i^{(n)}}, \frac{\partial R}{\partial R u_i^{(n)}} \in \text{Der}(V)$$

denote the derivation with respect to $u_i^{(n)}$ from the left and the right respectively, (which coincide if $V$ is non-super). Then the $\lambda$-bracket on $V$ is determined by the values $\{ u_i \lambda u_j \}$ in the following sense.
Theorem 2.2 \((2 \text{[30]})\). Let \((V, \partial)\) be a superalgebra of differential polynomials in the variables \(u_i, \ (i \in I)\), and \(H_{ij}(\lambda), \ (i, j \in I)\), an element of \(V[\lambda]\) with the same parity as \(u_i u_j\). Then there is a unique Poisson vertex superalgebra structure on \(V\) satisfying \(\{u_i u_j\} = H_{ij}(\lambda)\) if and only if the \(\lambda\)-bracket satisfies \((2.11)\) and \((2.10)\) for \(u_i, \ (i \in I)\). Moreover, the \(\lambda\)-bracket is given by
\[
\{f, g\} = \sum_{i, j \in I, m, n \geq 0} (-1)^{i \bar{j} + i j} \frac{\partial g}{\partial_R u_j^{(n)}}^m (\lambda + \partial)^n H_{ji}(\lambda + \partial)^m \frac{\partial f}{\partial u_i^{(m)}}, \quad f, g \in V.
\]

Example 2.3 (Universal affine Poisson vertex algebra). Given a finite dimensional Lie algebra \(L\) over \(\mathbb{C}\) and a nondegenerate symmetric invariant bilinear form \(\kappa\) on \(L\), let \(V^\kappa(L)\) denote the algebra of differential polynomials in the variables given by a basis of \(L\). Then a \(\lambda\)-bracket \(\{-\lambda-\} : L \times L \to V^\kappa(L)[\lambda]\) given by \(\{u, v\} = [u, v] + \kappa(u, v)\lambda, \ (u, v \in L)\), defines a Poisson vertex algebra structure on \(V^\kappa(L)\). This is called the universal \(\text{affine Poisson vertex algebra}\) associated with \(L\) at level \(\kappa\).

Given a Poisson vertex superalgebra \(V\), a Poisson \(\text{vertex module}\) over \(V\) (cf. \([1]\)) is a vector superspace \(M\) which is a \(V\)-module as a supercommutative algebra \(V\) and endowed with an even \(\mathbb{C}\)-bilinear map \(\{-\lambda-\} : V \times M \to M[\lambda]\) satisfying
\[
(2.8) \quad \{\partial f, m\} = -\lambda \{f, m\},
\]
\[
(2.9) \quad \{f, \{g, m\}\} = (-1)^{|g|} \{g, \{f, m\}\} = \{\{f, g\}, \lambda + \partial m\},
\]
\[
(2.10) \quad \{f, g \cdot m\} = \{f, g\} \cdot m + (-1)^{|g|} g \cdot \{f, m\},
\]
\[
(2.11) \quad \{f \cdot g, m\} = (-1)^{|\lambda|} \{f, g \cdot m\} + (-1)^{|g|} g \{f, \lambda + \partial m\} + f \{g, \lambda + \partial m\}
\]
for \(f, g \in V, \ m \in M\). Here the right \(V\)-action \(M \times V \to M\) is defined by the action of \(V\) as a supercommutative algebra. In this case, we define the \(\lambda\)-bracket
\[
\{-\lambda\} : M \times V \to M[\lambda], \quad (m, f) \mapsto \{m, f\} = (-1)^{|m|} \frac{\partial f}{\partial \lambda}, \ \{f - \partial, \lambda\} m
\]
for \(m \in M\), the linear map \(\{m, -\} : V \to M[\lambda]\) is called an \(\text{intertwining operator}\) and \(m\) is called the Hamiltonian.

For a Poisson vertex superalgebra \(V\), the vector superspace \(\text{Lie}(V) = V/\partial V\) is called the \(\text{space of local functionals}\). We denote by
\[
\tilde{f} : V \to \text{Lie}(V), \quad f \mapsto \tilde{f}
\]
the canonical projection.

Proposition 2.4 \((2 \text{[30])}\).

(1) The bilinear map
\[
\text{Lie}(V) \times \text{Lie}(V) \to \text{Lie}(V), \quad (\tilde{f}, \tilde{g}) \mapsto \tilde{f} \{\tilde{g}\}_{\lambda=0}
\]
is well-defined and defines a Lie superalgebra structure. Moreover, if \(V\) is even and an algebra of differential polynomials in the variables \(\{u_i\}_{i \in I}\), then
\[
[\tilde{f}, \tilde{g}] = \sum_{i, j \in I} \int \frac{\delta g}{\delta_R u_j^{(n)}} u_i \delta f_{u_i^{(n)}} + \delta f_{u_i^{(n)}} \frac{\delta g}{\delta_R u_j^{(n)}},
\]
where \(\frac{\delta f}{\delta_R u_i^{(n)}} = \sum_{n \geq 0} (-\partial)^n \frac{\partial f}{\partial u_i^{(n)}}\) and \(\frac{\delta g}{\delta_R u_i^{(n)}} = \sum_{n \geq 0} (-\partial)^n \frac{\partial g}{\partial u_i^{(n)}}\) denote the left and right variational derivative of \(f\) with respect to \(u_i\).

(2) The Lie superalgebra \(\text{Lie}(V)\) acts on \(V\) by
\[
\eta : \text{Lie}(V) \to \text{Der}(V), \quad \tilde{f} \mapsto \{f, -\}_{\lambda=0}.
\]

We use the following lemma in the below.
Lemma 2.5 (cf. [2 Proposition 1.33]). Let $L$ be a finite dimensional Lie algebra over $\mathbb{C}$ equipped with a nondegenerate symmetric invariant bilinear form $\kappa$. For the universal affine Poisson vertex algebra $V^\kappa(L)$, the kernel of $\eta : \text{Lie}(V^\kappa(L)) \to \text{Der}(V^\kappa(L))$ is
\[
\text{Ker}(\eta) = \text{span}_\mathbb{C} \{ \int f | f \in Z(L) \},
\]
where $Z(L)$ is the center of $L$.

Proof. Let $\{u_i\}_{i \in I}$ be a basis of $L$. Suppose $\int F \in \text{Ker}(\eta) \subset \text{Lie}(V^\kappa(L))$. By Theorem 2.2
\[
\eta(\int F) = \sum_{i,j,n} \partial^n \left( ([u_i, u_j] + \kappa(u_i, u_j)\partial) \frac{\delta F}{\delta u_i} \right) \frac{\partial}{\partial u_j}.
\]
It follows $M_j = \sum_i ([u_i, u_j] + \kappa(u_i, u_j)\partial) \frac{\delta F}{\delta u_i} = 0$ for $j \in I$. Define a degree on $V^\kappa(L)$ by $\text{deg}(u_i^{(n)}) = n$ and $\text{deg}(AB) = \text{deg}(A) + \text{deg}(B)$. Then $\delta / \delta u$ preserves the degree.

Let $G_i$ be the top degree component of $\delta F / \delta u$. Then the top component $M_j^{top} = \sum_i \kappa(u_i, u_j)\partial G_i$. Since $\kappa$ is nondegenerate, we obtain $\partial G_i = 0$, which implies $G_i \in \mathbb{C}$. Since $V^\kappa(L)$ is an algebra of differential polynomials, we conclude $F \in \mathbb{C} \oplus L$. (See the proof of [2 Proposition 1.5].) Set $F = a + \sum b_i u_i$, $(a, b_i \in \mathbb{C})$. Then $M_j = \sum_i b_i u_i$, $u_j = 0$, $(j \in I)$, which implies $\sum_i b_i u_i \in Z(L)$.

Finally, given a Poisson vertex superalgebra $V$, an element $\int f \in \text{Lie}(V)$ is called integrable if there exists an infinite dimensional abelian Lie subsuperalgebra $H$ of $\text{Lie}(V)$ which contains $\int f$. In this case, $H$ is called an integrable Hamiltonian hierarchy associated with $V$.

2.2. Differential graded Poisson vertex superalgebra. A differential graded Poisson vertex superalgebra (d.g. Poisson vertex superalgebra) is a pair $(V, d)$ consisting of a Poisson vertex superalgebra $V = \oplus_{n \in \mathbb{Z}} V_n$ and a linear map $d : V \to V$, called the differential, satisfying

- $V$ is a $\mathbb{Z}$-graded Poisson vertex superalgebra, i.e., $V = \oplus_{n \in \mathbb{Z}} V_n$ as a vector superspace satisfying
  \[
  V_n \cdot V_m \subset V_{n+m}, \quad \{ V_{n\lambda} V_m \} \subset V_{n+m}[\lambda],
  \]
- the linear map $d$ is of homogeneous parity and satisfies $d^2 = 0$,
\[
d : V_n \rightarrow V_{n+1},
\]
\[
d(ab) = d(a) \cdot b + (-1)^{\tilde{a} \tilde{b}} a \cdot d(b), \quad d\{a_{\lambda} b\} = \{d(a)_{\lambda} b\} + (-1)^{\tilde{a}} \{a_{\lambda} d(b)\}.
\]

The cohomology $H^*(V; d) = \oplus_{n \in \mathbb{Z}} H^n(V; d)$ inherits a $\mathbb{Z}$-graded Poisson vertex algebra structure. Moreover, $H^0(V; d)$ is a Poisson vertex subsuperalgebra and $H^n(V; d)$, $(n \in \mathbb{Z})$, is a Poisson vertex module over $H^0(V; d)$. In the sequel, we also use the notion of a differential graded vertex superalgebra. The definition is similar and therefore we omit the details.

2.3. Classical limit. Let $V$ be a vertex superalgebra over a polynomial ring $\mathbb{C}[\epsilon]$. Suppose that $V$ is free as a $\mathbb{C}[\epsilon]$-module and the $\lambda$-bracket satisfies
\[
[V_{\lambda} V] \subset \epsilon V.
\]

Define the vector superspace $V^{cl} = V / \epsilon V$ and let $V \rightarrow V^{cl}$, $(f \mapsto \bar{f})$ denote the canonical projection. Then
\[
V^{cl} \times V^{cl} \rightarrow V^{cl}, \quad (\bar{f}, \bar{g}) \mapsto \bar{f}(-1)\bar{g}
\]
is well-defined and defines an associative supercommutative algebra structure on $V^{cl}$. Since the translation operator $\partial$ of $V$ preserves $\epsilon V$, it induces a linear map
\[
\partial : V^{cl} \rightarrow V^{cl}, \quad \partial \bar{f} \mapsto \bar{\partial(\bar{f})},
\]
which is a derivation of \( V^{cl} \). Since the \( \varepsilon V \) is an ideal of \( V \) by (2.12), the \( \lambda \)-bracket of \( V \) induces a bilinear map 

\[
\{-\lambda-\} : V^{cl} \times V^{cl} \rightarrow V^{cl}[\lambda], \quad (fg) \mapsto [\lambda f].
\]

The triple \((V^{cl}, \partial, \{-\lambda-\})\) defines a Poisson vertex superalgebra, called the classical limit of \( V \) (cf. [11, 23]).

3. Screening operators for classical affine \( \mathcal{W} \)-algebras

In this section, we describe the classical affine \( \mathcal{W} \)-algebras by using screening operators. They will be obtained as a classical limit of the screening operators for the affine \( \mathcal{W} \)-algebras obtained in [25]. We will use the same notation for Poisson vertex algebras as vertex algebras since there will be no confusion.

3.1. Affine \( \mathcal{W} \)-algebras. Let \( g \) be a finite dimensional simple Lie algebra over \( \mathbb{C} \) with the normalized symmetric invariant bilinear form \( \kappa = \langle \cdot, \cdot \rangle \). Let \( f \in g \) be a nonzero nilpotent element, fix an \( sl_2 \)-triple \( \{e, h, f\} \) containing \( f \) and denote by \( \Gamma : g = \oplus_{d \in \mathbb{Z}} g_d \) the \( \mathbb{Z} \)-grading given by \( \text{ad}_x \), with \( d \) the largest number such that \( g_d \neq 0 \). We fix a triangular decomposition \( g = n_+ \oplus h \oplus n_\cdot \) so that \( x \in h \), \( g_{>0} = \oplus_{j>0} g_j \subset n_+ \), and \( g_{<0} = \oplus_{j<0} g_j \subset n_\cdot \). Let \( g = h \oplus \oplus_{\alpha \in \Delta} g_{\alpha} \) be a root space decomposition, \( \Delta = \{ \alpha \in \Delta | g_{\alpha} \subseteq g_j \} \), and \( \Delta_{<0} = \cup_{j>0} \Delta_{j} \). Fix a nonzero root vector \( e_{\alpha} \) in \( g_{\alpha} \) and a basis \( e_i \), \( i \in I \), of \( h \). Then \( e_{\alpha} \), \( \alpha \in \Delta \), form a basis of \( g \). We denote by \( \{e_{\alpha}\}_{\alpha \in \Delta} \) its dual basis of \( g \) with respect to \( \kappa \). Let \( \lambda_{\alpha, \beta} \) denote the structure constants of \( g \), i.e., \( [e_{\alpha}, e_{\beta}] = \sum_{\gamma \in \Delta \cap \Delta} \lambda_{\alpha, \beta} e_{\gamma} \).

Let \( V^k(g) \) be the universal affine vertex algebra of \( g \) at level \( k \), generated by the even elements \( e_{\alpha} \), \( \alpha \in \Delta \cup \Delta_{<0} \), with \( \lambda \)-bracket \( [e_{\alpha} e_{\beta}] = [e_{\alpha}, e_{\beta}] + k(e_{\alpha} e_{\beta}) \lambda \). Let \( F^k(g_{\geq 0}) \) be the charged free fermion vertex superalgebra associated with the symplectic odd vector superspace \( g_{>0} \oplus g_{\geq 0} \), generated by the odd elements \( \varphi_{\alpha} \), \( \varphi_{\alpha}^\alpha \), \( \alpha \in \Delta_{>0} \), with \( \lambda \)-bracket \( [\varphi_{\alpha} \lambda, \varphi_{\beta}^\beta] = \delta_{\alpha, \beta}, [\varphi_{\alpha} \lambda \varphi_{\beta}^\beta] = [\varphi_{\alpha}^\alpha \varphi_{\beta}^\beta] = 0 \). Let \( F(g_{1/2}) \) be the \( \beta \)-system vertex algebra associated with the symplectic vector space \( g_{1/2} \), generated by \( \Phi_{\alpha} \), \( \alpha \in \Delta_{1/2} \), with \( \lambda \)-bracket \( [\Phi_{\alpha} \lambda \Phi_{\beta}] = \chi([e_{\alpha}, \beta]) \), where \( \chi(-) = (f \mapsto f) \).

The affine \( \mathcal{W} \)-algebra \( W^k(g, f) \) associated with the triple \( (g, f, k) \), \( k \in \mathbb{C} \), is the vertex algebra defined as the 0-th cohomology of the differential graded vertex algebra

\[
C^k(g, f) = V^k(g) \otimes F^k(g_{\geq 0}) \otimes F(g_{1/2}),
\]

with differential

\[
d(0) = \sum_{\alpha \in \Delta_{>0}} ((e_{\alpha} + \Phi_{\alpha} + \chi(e_{\alpha})) \varphi_{\alpha} - \frac{1}{2} \sum_{\alpha, \beta, \gamma \in \Delta_{>0}} \lambda_{\alpha, \beta} \gamma \varphi_{\alpha}^\gamma \varphi_{\beta}^\beta \Big|_{\lambda=0},
\]

called the BRST complex. The grading \( C^k(g, f) = \oplus_{n \in \mathbb{Z}} C^k_n(g, f) \) is given by \( \text{gr}(e_{\alpha}) = \text{gr}(\Phi_{\beta}) = 0 \), \( \text{gr}(\varphi_{\alpha}) = -\text{gr}(\varphi_{\alpha}) = 1 \) with \( \text{gr}(AB) = \text{gr}(A) + \text{gr}(B) \) and \( \text{gr}(\partial A) = \text{gr}(A) \). Then we have \( d(0) : C^k_n(g, f) \rightarrow C^k_{n+1}(g, f) \). We have \( H^n(C^k(g, f)) = 0 \) for \( n \neq 0 \) (see [27, 28]).

3.2. Classical limit. By [27, 28], we have vertex subsuperalgebras \( C^k_-(g, f) \), which gives a decomposition of a complex \( C^k(g, f) = C^k_+(g, f) \otimes C^k_-(g, f) \) and satisfies \( H^n(C^k_+(g, f)) \cong \delta_{n,0} \mathcal{C} \). Thus \( H^0(C^k_-(g, f)) \cong W^k(g, f) \). As a vertex superalgebra, \( C^k_-(g, f) \) is generated by

\[
J^u = u + \sum_{\beta, \gamma \in \Delta_{>0}} c^u_{\beta, \gamma} \varphi_{\beta} \varphi_{\gamma}, \quad (u \in g_{\leq 0}),
\]

\[
\Phi_{\alpha}, (\alpha \in \Delta_{1/2}), \varphi^\alpha, (\alpha \in \Delta_{>0}).
\]

Following [25], we introduce the classical affine \( \mathcal{W} \)-algebra as the cohomology of the differential graded Poisson vertex algebra in the classical limit of \( C^k_+(g, f) \).
Suppose \( k + h^\vee \neq 0 \). Set \( \epsilon = \frac{k'}{k + h^\vee} \), \( (k' \in \mathbb{C}\setminus\{0\}) \), \( \tilde{J}^u = \epsilon J^u \), \( (u \in \mathfrak{g}_{\leq 0}) \), and \( \Phi_\alpha = \epsilon \Phi_\alpha \), \( (\alpha \in \Delta_{1/2}) \). Then we have

\[
(3.13) \quad [\tilde{J}_\lambda^\beta, \tilde{J}_\gamma^\alpha] = \epsilon (\tilde{J}^{[u,v]} + (k'(u|v) + o(\epsilon))\lambda), \quad [\varphi_\lambda^\alpha \tilde{J}_\gamma^\alpha] = \epsilon \sum_{\beta \in \Delta_{> 0}} c_{\alpha, \beta}^\alpha \varphi_\beta.
\]

(3.14) \quad \[\Phi_\alpha \lambda \tilde{J}_\beta^\alpha = \epsilon \chi([(\epsilon_\alpha, \epsilon_\beta]), \quad [\tilde{J}_\lambda^\alpha \Phi_\alpha] = [\varphi_\alpha^\alpha \varphi_\beta] = [\varphi_\alpha^\alpha \Phi_\beta] = 0, \]

(cf. \[25\]). Viewing \( \epsilon \) as an indeterminate in \( (3.13), (3.14) \), we obtain a vertex superalgebra \( \tilde{C}_k^k \) over the polynomial ring \( \mathbb{C}[\epsilon] \). By Section 3.2, we obtain a Poisson vertex superalgebra \( \tilde{C}_k^k / \epsilon \tilde{C}_k^k \), which we denote by \( C_k^{cl} = C_k^{cl}(g, f) \). We have an isomorphism

\[
C_k^{cl} \cong V^k(\mathfrak{g}_{\leq 0}) \otimes F(\mathfrak{g}_{\geq 0}) \otimes \text{Sym}(\mathbb{C}[\partial] \mathfrak{g}_{> 0})
\]

of Poisson vertex superalgebra where \( V^k(\mathfrak{g}_{< 0}) \) is the universal affine Poisson vertex algebra generated by \( \mathfrak{g}_{< 0} \) with \( \lambda \)-bracket \( \{ u, v \} = [u, v] + k'(u|v)\lambda \), \( F(\mathfrak{g}_{1/2}) \) the \( \beta \gamma \)-system Poisson vertex algebra generated by \( \Phi_\alpha \), \( (\alpha \in \Delta_{1/2}) \), with \( \lambda \)-bracket \( \{ \Phi_\alpha \lambda \Phi_\beta \} = (f[(\epsilon_\alpha, \epsilon_\beta)]) \), and \( \text{Sym}(\mathbb{C}[\partial] \mathfrak{g}_{> 0}) \) the differential \( \mathbb{C} \)-superalgebra generated by odd elements \( \varphi_\alpha \), \( (\alpha \in \Delta_{> 0}) \), which satisfy

\[
\{ \varphi_\alpha^\alpha u \} = \sum_{\beta \in \Delta_{> 0}} c_{\alpha, \beta}^\alpha \varphi_\beta, \quad \{ \varphi_\lambda^\alpha \varphi_\beta \} = \{ \varphi_\lambda^\alpha \Phi_\beta \} = 0.
\]

Decompose the differential \( d(0) \) as \( d(0) = d_{st}(0) + d_{ne}(0) + d_{\lambda}(0) \) where

\[
d_{st}(0) = \sum_{\alpha \in \Delta_{> 0}} e_\alpha \varphi_\alpha - \frac{1}{2} \sum_{\alpha, \beta \in \Delta_{> 0}, \gamma \in \Delta_{> 0}} c_{\alpha, \beta}^\gamma \varphi_\alpha \varphi_\beta, \quad d_{ne}(0) = \sum_{\alpha \in \Delta_{1/2}} \Phi_\alpha \varphi_\alpha, \quad d_{\lambda}(0) = \sum_{\alpha \in \Delta_{> 0}} \chi(\epsilon_\alpha) \varphi_\alpha.
\]

Then we have

\[
[d_{st} \lambda \tilde{J}_\lambda^\alpha] = - \sum_{\alpha \in \Delta_{1/2}, \beta \in \Delta_{> 0}} c_{\alpha, \beta}^\alpha \tilde{J}_\beta^\alpha + \sum_{\beta \in \Delta_{> 0}} (k'(u|v)(\partial + \lambda) + o(\epsilon)) \varphi_\beta,
\]

\[
[d_{ne} \lambda \tilde{J}_\lambda^\alpha] = \sum_{\alpha \in \Delta_{1/2}, \beta \in \Delta_{> 0}} c_{\beta, \gamma}^\alpha \tilde{J}_\alpha^\beta \varphi_\gamma, \quad [d_{ne} \lambda \Phi_\alpha] = \sum_{\beta \in \Delta_{1/2}} \chi(\epsilon_\beta, \epsilon_\alpha) \varphi_\beta,
\]

\[
[d_{st} \lambda \varphi_\alpha] = - \frac{1}{2} \sum_{\beta, \gamma \in \Delta_{> 0}} c_{\beta, \gamma}^\alpha \varphi_\beta \varphi_\gamma, \quad d_{ne} \lambda \Phi_\alpha = \sum_{\beta \in \Delta_{1/2}} \chi(\epsilon_\beta, \epsilon_\alpha) \varphi_\beta.
\]

The differential \( d(0) = \{ d_{st} + d_{ne} + d_{\lambda} \} \), \( \lambda = 0 \) is given by

\[
\{ d_{st} \lambda \}, \{ d_{ne} \lambda \}, \{ d_{\lambda} \lambda \} : C_k^{cl}(g, f) \to C_k^{cl}(g, f)[\lambda],
\]

which satisfy

\[
\{ d_{st} \lambda u \} = - \sum_{\alpha \in \Delta_{1/2}, \beta \in \Delta_{> 0}} c_{\alpha, \beta}^\alpha e_\alpha \varphi_\beta + \sum_{\beta \in \Delta_{> 0}} k(u|\epsilon_\beta)(\partial + \lambda) \varphi_\beta,
\]

\[
\{ d_{st} \lambda \epsilon_\alpha \} = 0, \quad \{ d_{st} \lambda \varphi_\alpha \} = - \frac{1}{2} \sum_{\beta, \gamma \in \Delta_{> 0}} c_{\beta, \gamma}^\alpha \varphi_\beta \varphi_\gamma,
\]

\[
\{ d_{ne} \lambda \epsilon_\alpha \} = \sum_{\alpha \in \Delta_{1/2}} (f[(\epsilon_\beta, \epsilon_\alpha)]) \varphi_\beta, \quad \{ d_{ne} \lambda \varphi_\alpha \} = 0,
\]

and

\[
\{ d_{\lambda} \lambda u \} = \sum_{\beta \in \Delta_{> 0}} (f[u|\epsilon_\beta]) \varphi_\beta, \quad \{ d_{\lambda} \lambda \epsilon_\alpha \} = 0.
\]
and are extended to $C_k^1$ by \[23], \[2.6] (see Remark \[24]). The 0-th cohomology of $C_k^1(g, f)$, which we denote by $\mathcal{W}^k(g, f) = H^0(C_k^1(g, f))$, is a Poisson vertex algebra called the classical affine $W$-algebra associated with $(g, f, k)$ (\[25\]). Note that $H^n(C_k^1(g, f)) = 0$ holds for all $n \neq 0$.

3.3. Screening operators. Introduce another grading $\text{wt}$ on $C_k^1$ by

$$\text{wt}(u) = -2j, (u \in g), \quad \text{wt}(\Phi_\alpha) = 0, (\alpha \in \Delta_{1/2}), \quad \text{wt}(\varphi^\alpha) = 2j (\alpha \in \Delta),$$

and a decreasing filtration $\{F_p C_k^1\}_{p \geq 0}$ on $C_k^1$ by

$$F_p C_k^1 = \text{span}\{ A \in C_k^1 | \text{wt}(A) \geq p \},$$

This filtration is exhaustive, separated, and compatible with the grading of $C_k^1$ as a complex. The associated spectral sequence $\{E_r, d_r\}_{r \geq 0}$ has the differentials

$$d_0 = d_{st(0)}, \quad d_1 = d_{ne(0)}, \quad d_2 = d_{\chi(0)}, \quad d_r = 0, (r \geq 3),$$

and thus converges at $r = 3$. We will describe $\mathcal{W}^k(g, f) = H^0(C_k^1(g, f))$ by using it. Since the calculation is straightforward, we omit the details. (The analogous argument for vertex algebras can be found in \[25\].)

To calculate $E_1 = H^1(C_k^1; d_0)$, notice that $d_0$ acts by 0 on $V^k(g_0) \otimes F(g_{1/2})$ and that $(\wedge(\oplus_{\alpha \in \Delta_{>0}} \mathbb{C}^{\varphi^\alpha}), d_0)$ is a subcomplex isomorphic to the Chevalley-Eilenberg complex of the Lie algebra $g_{>0}$ with coefficients in the trivial representation $\mathbb{C}$. Thus $(V^k(g_0) \otimes F(g_{1/2}) \otimes (\wedge(\oplus_{\alpha \in \Delta_{>0}} \mathbb{C}^{\varphi^\alpha}), d_0)$ is a subcomplex, whose cohomology is $V^k(g_0) \otimes F(g_{1/2}) \otimes H^*(g_{>0}; \mathbb{C})$.

Lemma 3.1.

1. The natural map $V^k(g_0) \otimes F(g_{1/2}) \otimes H^*(g_{>0}; \mathbb{C}) \rightarrow E_1$ is an isomorphism of graded vector spaces for generic $k \in \mathbb{C}$.
2. The isomorphism $E_1^{(0)} \cong V^k(g_0) \otimes F(g_{1/2})$ is an isomorphism of Poisson vertex algebras.
3. Each cohomology $E_1^{(n)}$ is a Poisson vertex module over $E_1^{(0)}$. Moreover, $E_1^{(n)}$ is isomorphic to $V^k(g_0) \otimes F(g_{1/2}) \otimes H^*(g_{>0}; \mathbb{C})$ as vector spaces.

Let us describe the Poisson vertex modules $E_1^{(n)}$ more explicitly. Recall that the coadjoint representation on $g_{>0}^*$ of $g_0$ induces a representation of $g_0$ on $H^*(g_{>0}, \mathbb{C})$ as described as follows. (cf. \[29\] Chapter 3]) Let $W$ denote the Weyl group of $g$ and set $J_0 = \{ i \in I | \alpha_i \in \Delta \}$, $W_0^* = \{ w \in W | w\Delta_0^+ \subset \Delta^+ \}$. Let $*: W \times h^* \rightarrow h^*$ denote the shifted action of $W$ and $l: W \rightarrow \mathbb{Z}_{\geq 0}$ the length function. Then there is an isomorphism of $g_0$-modules

$$H^*(g_{>0}; \mathbb{C}) \cong \bigoplus_{w \in W_0^*, l(w) = n} L_0(w^{-1} * 0),$$

where $L_0(w^{-1} * 0)$ is the integrable highest weight $g_0$-module with highest weight $w^{-1} * 0$.

For a $g_0$-module $M$, set $M^k = V^k(g_0) \otimes_{\mathbb{C}} M$. The space $M^k$ has a unique Poisson vertex module over $M^k$ such that $V^k(g_0)$ acts as a commutative algebra by multiplication on the first component and the $\lambda$-bracket $\{-,-\}: V^k(g_0) \otimes M^k \rightarrow M^k[\lambda]$ satisfies $\{a_\lambda b \otimes m\} = \{a \lambda b\} \otimes m + b \otimes a \cdot m$ for $a \in g_0$, $b \in V^k(g_0)$, and $m \in M$. We denote by $L_0^k(w^{-1} * 0)$ the Poisson vertex module obtained from $L_0(w^{-1} * 0)$.

Lemma 3.2. There is an isomorphism

$$E_1^{(n)} \cong \bigoplus_{w \in W_0^*, l(w) = n} L_0^k(w^{-1} * 0) \otimes F(g_{1/2}),$$
as Poisson vertex modules over $V^k(g_0) \otimes F(g_{1/2})$.

In particular, we have $E_1^{(1)} = \bigoplus_{i \in I \setminus I_0} L_0^k(-\alpha_i) \otimes F(g_{1/2})$ and the subspace $L_0(-\alpha_i)$ is identified as

$$L_0(-\alpha_i) = \bigoplus_{\beta \in [\alpha_i]} \mathbb{C} \varphi^\beta \subset E_1^{(1)},$$

where $[\alpha_i] = \Delta_{>0} \cap (\alpha_i + Q_0)$ and $Q_0$ denotes the root lattice of $\Delta_0$. For $\alpha \in [\alpha_i]$ with $i \in I \setminus I_0$, we have

$$\partial \varphi^\alpha = \frac{1}{k} \sum_{\beta \in [\alpha_i], \gamma \in I \setminus \Delta_0} c^\alpha_{\beta, \gamma} e^\gamma \varphi^\beta.$$

Let us describe the differentials on $E_1$ induced from $d_{\text{ne}(0)}$ and $d_{\chi}(0)$. Consider the intertwining operators $Q^W_i : V^k(g_0) \otimes F(g_{1/2}) \to L_0^k(-\alpha_i) \otimes F(g_{1/2})$ given by

$$Q^W_i = \begin{cases} \sum_{\beta \in [\alpha_i]} \{ \Phi_{\beta} \varphi^\beta \} |_{\lambda=0}, & (i \in I_{1/2}), \\ \sum_{\beta \in [\alpha_i]} \{ (f_{i\beta}) \varphi^\beta \} |_{\lambda=0}, & (i \in I_1). \end{cases}$$

Then we have:

**Lemma 3.3.** The differentials on $E_1$ induced by $d_{\text{ne}(0)}$ and $d_{\chi}(0)$ are given by

$$d_{\text{ne}(0)} = \sum_{i \in I_{1/2}} Q^W_i$$ and $$d_{\chi}(0) = \sum_{i \in I_1} Q^W_i.$$

Recall that the complex $C_k^cl = C_k^cl(g, f)$ is $\mathbb{Z}_{\geq 0}$-graded and that $H^\alpha(C_k^cl(g, f)) \cong \delta_{\alpha,0} W_k^\alpha(g, f)$. Then we see that $W_k^\alpha(g, f)$ is a subalgebra of the 0-th degree $C_{k,0}^cl = V^k(g_{\leq 0}) \otimes F(g_{1/2})$. Since $\mathcal{I} = V^k(g_{< 0}) \otimes F(g_{1/2})$ is a Poisson vertex ideal of $C_k^cl$, we obtain a homomorphism of Poisson vertex algebras

$$W_k^\alpha(g, f) \to C_{k,0}^cl/\mathcal{I} \cong V^k(g_0) \otimes F(g_{1/2}).$$

It is injective and, by using the differentials $d_{\text{ne}(0)}$ and $d_{\chi}(0)$, the image is described as in the following theorem.

**Theorem 3.4.** For generic $k \in \mathbb{C}$, there is an isomorphism

$$j : W_k^\alpha(g, f) \cong \bigcap_{i \in I_{1/2} \cup I_1} \text{Ker} \left( Q^W_i : V^k(g_0) \otimes F(g_{1/2}) \to L_0^k(-\alpha_i) \otimes F(g_{1/2}) \right),$$

of Poisson vertex algebras.

The operators $\{ Q^W_i \}_{i \in I_{1/2} \cup I_1}$ are called the screening operators for $W_k^\alpha(g, f)$. We note that the level $k = 1$ is generic [24]. The inclusion $j : W_k^\alpha(g, f) \to V^k(g_0) \otimes F(g_{1/2})$ in Theorem 3.4 induces a Lie algebra homomorphism between their spaces of local functionals

$$j_* : \text{Lie}(W_k^\alpha(g, f)) \to \text{Lie}(V^k(g_0) \otimes F(g_{1/2})).$$

(See Proposition 2.4)

**Lemma 3.5.** The Lie algebra homomorphism $j_*$ is injective.

**Proof.** It is easy to see

$$\text{Ker} \left( \partial : L_0^k(-\alpha_i) \otimes F(g_{1/2}) \to L_0^k(-\alpha_i) \otimes F(g_{1/2}) \right) = 0, \quad (i \in I_{1/2} \cup I_1).$$

Take an element $f \in W_k^\alpha(g, f)$ such that $\int f \in \text{Ker} j_*$. Then there exists an element $G \in V^k(g_0)$ such that $j(f) = \partial g$. Let $H^W_{i\lambda}$ denote the Hamiltonian of $Q^W_i$. Then we have

$$0 = Q^W_i(j(f)) = \{ H^W_{i\lambda} j(f) \} |_{\lambda=0} = \{ H^W_{i\lambda} \partial g \} |_{\lambda=0} = \partial \{ H^W_{i\lambda} g \} |_{\lambda=0},$$

as required.
and so that \( \{H_{g,\beta}^W\}_{\lambda=0} \in \text{Ker}(\partial : L_0^k(\alpha_i) \otimes F(g_{1/2})) \to L_0^k(\alpha_i) \otimes F(g_{1/2})) = 0. \) Therefore, we obtain \( g \in j(W^k(g, f)) \) and so that \( j f = 0. \)

Let \( Q^W_\alpha : V^k(g_0) \otimes F(g_{1/2}) \to V^k(g_0) \otimes F(g_{1/2}) \), \( (\alpha \in \Pi) \), be the derivation determined by

\[
\begin{align}
Q^W_\alpha e_\beta &= \sum_{\gamma \in [\alpha]} c_{\alpha,\beta}^\gamma \Phi_\gamma, \\
Q^W_\alpha \Phi_\beta &= (f|e_\alpha, e_\beta|), \quad (\alpha \in \Pi_1), \\
Q^W_\alpha e_\beta &= (f|e_\beta, e_\alpha|), \\
Q^W_\alpha \Phi_\beta &= 0, \quad (\alpha \in \Pi_1),
\end{align}
\]

Therefore, \( 0 = (f|e_\alpha, e_\beta|) \).

\[
\begin{align}
[Q^W_\alpha, \partial] = \frac{1}{k} \sum_{\beta \in I \cup \Delta_0} c_{\alpha,\beta}^\gamma e_\beta Q^W_\gamma.
\end{align}
\]

**Theorem 3.6.** For generic \( k \in \mathbb{C} \), the classical affine \( W \)-algebra \( W^k(g, f) \) is isomorphic to the Poisson vertex subalgebra

\[
W^k(g, f) \cong \bigcap_{\alpha \in \Pi} \text{Ker}(Q^W_\alpha : V^k(g_0) \otimes F(g_{1/2}) \to V^k(g_0) \otimes F(g_{1/2})),
\]

of \( V^k(g_0) \otimes F(g_{1/2}) \) invariant under the derivations \( Q^W_\alpha \), \( (\alpha \in \Pi) \).

We call the level \( k \in \mathbb{C} \) generic when (3.18) holds.

**Proof.** Since \( Q^W_\alpha : V^k(g_0) \otimes F(g_{1/2}) \to L_0^k(-\alpha_i) \otimes F(g_{1/2}) \) acts by derivation, it decomposes as \( Q^W_\alpha = \sum_{\alpha \in [\alpha]} \phi^\alpha Q^W_\alpha \), where \( Q^W_\alpha \in \text{Der}(V^k(g_0) \otimes F(g_{1/2})) \) and \( \phi^\alpha \) is the multiplication by \( \phi^\alpha \). We check that \( Q^W_\alpha \) satisfies (3.17) and (3.10).

By direct calculation, (3.10) follows from the definition of \( Q^W_\alpha \). To show (3.17), recall that \( V^k(g_0) \) has a Virasoro element \( L = \frac{1}{2k} \sum_{\alpha \in I \cup \Delta_0} e_\alpha e_\alpha \), i.e., it satisfies \( \{L_\lambda L\} = (\lambda + \partial) L + c_k L \) for some \( c_k \in \mathbb{C} \) and \( \partial = L(0) \). Then we have \( [Q^W_\alpha, \partial] = \{(Q^W_\alpha L)\}_{\lambda=0}, \{L_{\mu}^\alpha\}_{\mu=0} = \{Q^W_\alpha, \partial)\}_{\lambda=0} = 0 \). Here, we have used (2.25) in the second, (2.3) in the third, and (2.23) in the last equality. Therefore, \( 0 = \sum_{\alpha \in [\alpha]} \phi^\alpha(Q^W_\alpha, \partial) = (\partial \phi^\alpha) Q^W_\alpha \).

Now (3.17) follows from this by (3.15).\( \square \)

4. GEOMETRIC REALIZATION OF \( W^k(g, f) \)

4.1. Double coset space. Consider the Lie algebra \( g[t^{\pm 1}] = g \otimes \mathbb{C}[t^{\pm 1}] \). By abuse of notation, we denote by \( \kappa = (\cdot | -) \) the invariant bilinear form on \( g[t^{\pm 1}] \) given by \( (at^n)[bt^m] = (a\bar{b})\delta_{n+m,0} \), which extends the one on \( g \). We extend the grading \( \Gamma \) on \( g \) to \( g[t^{\pm 1}] \) by setting \( \text{deg}(at^n) = \text{deg}(a) + (d + 1)n \) (see Section 3.1) and fix a homogeneous basis \( e_\alpha \), \( (\alpha \in I \cup \Delta) \), extending the basis \( e_\alpha \), \( (\alpha \in I \cup \Delta) \), of \( g \). We denote \( |\alpha| = \text{deg}(e_\alpha) \) for simplicity and \( c_{\alpha,\beta}^\gamma \) the structure constants, i.e., \( [e_\alpha, e_\beta] = \sum_{\gamma} c_{\alpha,\beta}^\gamma e_\gamma \). Let \( e_\alpha \), \( (\alpha \in I \cup \Delta) \), denote the dual basis of \( e_\alpha \), \( (\alpha \in I \cup \Delta) \), with respect to \( \kappa \).

Consider the completion \( L_g = \lim_{\lambda \to 0} (g[t^{\pm 1}]/g[t^{\pm 1}]_{>-\lambda}) \), which we call the loop algebra of \( g \). The grading on \( g[t^{\pm 1}] \) induces a decomposition \( L_g = L_{g,+} \oplus L_{g,-} \) where \( L_{g,+} = \lim_{\lambda \to 0} (g[t^{\pm 1}]_{>-\lambda}/g[t^{\pm 1}]_{>-\lambda}) \) and \( L_{g,-} = g[t^{\pm 1}]_{\leq 0} \). We decompose \( X \in L_g \) as \( X = X_+ + X_- \in L_{g,+} \oplus L_{g,-} \). The subspace \( L_{g,+} \) is an affine scheme of infinite type. By setting \( z_0 \) the coordinates of the basis \( e_\alpha \), \( (\alpha \in \Delta_{>0} \subset \Delta) \), of \( L_{g,+} \), we have \( \mathbb{C}[L_{g,+}] = \mathbb{C}[z_0 \mid \alpha \in \Delta_{>0}] \).

Let \( G \) be the connected simply-connected algebraic group whose Lie algebra is \( g \) and \( LG \) the loop group of \( G \), whose Lie algebra is \( L_g \). We have closed subgroups \( LG_\pm \) of \( LG \) whose Lie algebra is \( L_{g,\pm} \).

Let us consider the quotient space \( LG_- / LG \), which is an ind-scheme. It has an open subscheme \( LG_+ \). The \( \mathbb{C} \)-points of \( LG_+ \) is identified with the \( LG_+ \)-orbits of the image of the identity in \( LG_- / LG \). Since \( LG_+ \) is a pronipotent proalgebraic
group, the exponential map \( \exp : \mathfrak{L}G_+ \to LG_+ \) is an isomorphism of schemes. Thus the coordinate ring \( \mathbb{C}[LG_+] \) is identified with \( \mathbb{C}[\mathfrak{L}G_+] = \mathbb{C}[z_\alpha | \alpha \in \Delta_{>0}] \).

The left multiplication \( LG_+ \times LG_+ \to LG_+, ((g_1, g_2) \mapsto g_1^{-1}g_2) \) induces a \( \mathfrak{L}g_+ \)-action on \( \mathbb{C}[LG_+] \) as derivations

\[
(4.19) \quad \xi^L : \mathfrak{L}g_+ \to \text{Der}(\mathbb{C}[LG_+]), \quad \phi \mapsto \phi^L.
\]

Since \( LG_+ \) is embedded into \( LG_+ \) as an open subscheme, the right multiplication \( LG_- \setminus LG \times LG_- \to LG_- \setminus LG, ((g_1, g_2) \mapsto [g_1g_2]), \) induces a \( \mathfrak{L}g_- \)-action on \( \mathbb{C}[LG_+] \) as derivations

\[
(4.20) \quad \xi^R : \mathfrak{L}g \to \text{Der}(\mathbb{C}[LG_+]), \quad \phi \mapsto \phi^R.
\]

The commutator of these actions \( \xi^L, \xi^R \) can be expressed by using the distinguished element \( K = \exp(\sum_{\alpha \in \Delta_{>0}} z_\alpha e_\alpha) \in LG_+(\mathbb{C}[LG_+]). \) Note that this element is coordinate independent since it is identified with the identity morphism under the correspondence

\[
LG_+(\mathbb{C}[LG_+]) = \text{Hom}(\mathbb{C}[LG_+], \mathbb{C}[LG_+]).
\]

**Lemma 4.1.** For \( u \in \mathfrak{L}g_+ \) and \( v \in \mathfrak{L}g_+, [u^L, v^R] = [u, (KvK^{-1})_-]_L^L \) holds.

**Proof.** We use the notation \( e^X = \exp(X) \) for brevity. Let \( \epsilon_i \) be dual numbers, i.e., satisfies \( \epsilon^2_i = 0. \) For \( F \in \mathbb{C}[LG_+] \) and \( g \in LG_+ \), we have

\[
(v^R F)(g) = \text{the } \epsilon_2\text{-linear term of } F(g e^{\epsilon_2 v})
\]

and

\[
(u^L v^R F)(g) = \text{the } \epsilon_1\text{-linear term of } (v^R F)(e^{-\epsilon_1 u} g)
\]

\[
= \text{the } \epsilon_1 \epsilon_2\text{-linear term of } F((e^{(\epsilon_1 u + g v g^{-1})} e^{-\epsilon_1 u} g)
\]

\[
= \text{the } \epsilon_1 \epsilon_2\text{-linear term of } F(g_1(\epsilon_1, \epsilon_2)),
\]

where

\[
g_1(\epsilon_1, \epsilon_2) = e^{-u\epsilon_1 + (gv g^{-1})_+ \epsilon_2 - ([u, g v g^{-1}]+(gv g^{-1})_+ u)\epsilon_1 \epsilon_2} g.
\]

Similarly, we obtain

\[
(u^R v^L F)(g) = \text{the } \epsilon_1 \epsilon_2\text{-linear term of } F(g_2(\epsilon_1, \epsilon_2))
\]

where

\[
g_2(\epsilon_1, \epsilon_2) = e^{-u\epsilon_1 + (gv g^{-1})_+ \epsilon_2 - u(g v g^{-1})_+ \epsilon_1 \epsilon_2} g.
\]

Since

\[
g_1(\epsilon_1, \epsilon_2) = e^{-[u, (gv g^{-1})_-]_L^L \epsilon_1 \epsilon_2} g_2(\epsilon_1, \epsilon_2),
\]

we obtain \([u^L, v^R] F(g) = [u, (gv g^{-1})_-]_L^L F(g). \)

In the sequel, we assume that \((\mathfrak{g}, f)\) satisfies the condition \((F)\) introduced in Section 1:

(F1) The grading \( \Gamma \) is a \( \mathbb{Z}\)-grading.

(F2) There exists an element \( y \in \mathfrak{g}_s \) such that \( s = f + yt^{-1} \in \mathfrak{g}[t^\pm 1] \) is semisimple.

(F3) The Lie subalgebra \( \text{Ker}(\text{ad}_s) \subset \mathfrak{g}[t^\pm 1] \) is abelian and \( \text{Im}(\text{ad}_s) \cap \mathfrak{g}[t^\pm 1]_0 = \mathfrak{g}_0. \)

**Example 4.2.** The following pairs \((\mathfrak{g}, f)\) satisfy \((F)\).

1. \((\mathfrak{g}, f)\) with \( f \) principal nilpotent element.
2. \((\mathfrak{g}(C_n), f_{2^m})\) and \((\mathfrak{g}(C_{2n}), f_{(2^n)})\), \((n \geq 2)\).
3. \((\mathfrak{g}(X_n), f)\) with \( X = E, F, G \) listed in Table...
Here we have used the classification of nilpotent elements of \( g(C_n) \) by symplectic partitions and that of \( g(X_n) \), \( (X = E, F, G) \), by the weighted Dynkin diagram, (see [3].)

Let \( \mathfrak{a} \) denote the completion of \( \text{Ker}(\text{ad}_a) \) in \( Lg \). It is abelian by (F2). Let \( A \subset LG \), \( (\text{resp. } A_\pm \subset LG) \) be the closed subgroup of \( LG \) whose Lie algebra is \( \mathfrak{a} \), \( (\text{resp. } \mathfrak{a}_\pm = \mathfrak{a} \cap Lg_\pm) \). We consider the quotient space \( LG_+/A_+ \). It admits a left \( LG_+ \)-action

\[
LG_+ \times LG_+/A_+ \rightarrow LG_+/A_+, \quad (g, hA_+) \mapsto g^{-1}hA_+,
\]

and a right \( A_- \)-action

\[
LG_+/A_+ \times A_- \rightarrow LG_- \backslash LG_+/A_+, \quad (hA_+, g) \mapsto hgA_+.
\]

The right \( A_- \)-action is well-defined since \( A \) is abelian. These actions induce infinitesimal actions \( \xi^L : Lg_+ \rightarrow \text{Der} \mathbb{C}[LG_+/A_+] \), \( (\phi \mapsto \phi^L) \) and \( \xi^R : \mathfrak{a}_- \rightarrow \mathbb{C}[LG_+/A_+] \), \( (\phi \mapsto \phi^R) \) respectively. In particular, \( \mathbb{C}[LG_+/A_+] \) becomes a differential algebra by letting \( s^R \) be the differential.

Set

\[
E_\alpha = k(e_\alpha[KsK^{-1}]) \in \mathbb{C}[LG_+], \quad (\alpha \in I \cup \Delta_0).
\]

We have \( E_\alpha \in \mathbb{C}[LG_+/A_+] \) since

\[
a^R(e_\alpha[KsK^{-1}]) = \text{the } \epsilon \text{-linear term of } (e_\alpha[K^\epsilon s(K^\epsilon)^{-1}]) = (e_\alpha[K[a, s]K^{-1}]) = 0,
\]

for \( a \in \mathfrak{a}_+ \). Since the Poisson vertex algebra \( V^k(g_0) \) is an algebra of differential polynomials, there exists a unique homomorphism of differential algebras

\[
\Psi_k : V^k(g_0) \rightarrow \mathbb{C}[LG_+/A_+], \quad e_\alpha \mapsto E_\alpha.
\]

**Theorem 4.3.** Suppose that \((g, f)\) satisfies (F) and \( k \in \mathbb{C} \) is generic. Then \( \Psi_k \) is an isomorphism of differential algebras and satisfies

\[
(4.22) \quad \Psi_k^W e_\alpha \Psi_k^{-1} = -\frac{1}{k} e_\alpha^L, \quad \alpha \in \Pi.
\]

**Proof.** Since \( V^k(g_0) \) and \( \mathbb{C}[LG_+/A_+] \) are polynomial rings, it suffices to show that the linear map \( d_0 \Psi_k : T^*_0 \text{Spec} V^k(g_0) \rightarrow T^*_{[c]} LG_+/A_+ \) between the cotangent spaces is an isomorphism. By the identification \( Lg_+^* \cong Lg_{<0} \) and \( \mathfrak{a}^*_+ \cong \mathfrak{a}_- \) induced by \( \kappa \), we obtain \( T^*_{[c]} LG_+/A_+ \cong (Lg_+/\mathfrak{a}_+)^* \cong Lg_{<0}/\mathfrak{a}_- \). This isomorphism is given by

| Dynkin diagram | weights of vertices | Dynkin diagram | weights of vertices |
|---------------|-------------------|---------------|-------------------|
| \( G_2 \)     | 2 0               | \( E_6 \) (continued) | 2 2 2 2 2 2        |
|               | 2 2               | \( E_7 \)      | 2 0 2 2 2 2        |
| \( F_4 \)     | 0 2 0 0           | \( E_8 \)      | 0 0 2 0 0 2 0      |
|               | 2 2 2 2           | \( \Psi \)      | 2 0 2 0 2 0 2      |
| \( E_6 \)     | 2 0 2 2 2         |               | 2 2 2 2 2 2        |
|               | 2 2 2 2           |               |                   |
Under this isomorphism, we have $d_0\Psi_k(\partial^ne_\alpha) = k\text{ad}_a^{n+1}e_\alpha$, as proved by inductive use of the $n = 0$ case:

$$d_0\Psi_k(e_\alpha) = kd(e_\alpha | KsK^{-1}) = k\sum_{|\beta|=1} (s, e_\alpha)[e_\beta,s]$$

$$= \sum_{|\beta|=1} k((s, e_\alpha)e_\beta) = k\sum_{|\beta|=1} ([s, e_\alpha][e_\beta])e_\beta$$

$$= k[s, e_\alpha].$$

It follows from (F2) and (F3) that $\Psi_k$ is an isomorphism.

To show (4.22), it suffices to show

$$u_\alpha^L E_\beta = (f[[e_\beta, e_\alpha]], [u_\alpha^L, s]^R) = \frac{1}{k}\sum_{\rho \in I \cup \Delta_0} c_{\alpha,\rho}^e E_\rho u_\gamma^R,$$

where $u_\alpha = -\frac{1}{k}e_\alpha$ by (3.17), (3.18). For the first one, we have

$$u_\alpha^L E_\beta = k(e_\beta[[\frac{1}{k}e_\alpha, KsK^{-1}]] = ([e_\beta, e_\alpha][KsK^{-1}])$$

$$= ([e_\beta, e_\alpha][s]) = ([e_\beta, e_\alpha][f]),$$

The second one follows from Lemma 4.1 since

$$[u_\alpha^L, s]^R = [u_\alpha, (KsK^{-1})_0]^R = \frac{1}{k^2} \sum_{\rho \in I \cup \Delta_0} c_{\alpha,\rho}^e E_\rho e_\gamma = \frac{1}{k} \sum_{\rho \in I \cup \Delta_0} c_{\alpha,\rho}^e E_\rho u_\gamma^R.$$

Note that the Lie subalgebra $\mathfrak{g}_+ = \oplus_{j>0} \mathfrak{g}_j$ is generated by the subspace $\oplus_{\alpha \in \Pi} \mathfrak{g}_\alpha$.

Let $G_r \subset LG_+$ denote the closed subgroup whose Lie algebra is $\mathfrak{g}_+$.

**Corollary 4.4.** The isomorphism $\Psi_k$ restricts to an isomorphism of differential algebras $W^k(\mathfrak{g}, f) \cong \mathbb{C}[G_+ \setminus LG_+/ A_+]$.

**Proof.** The claim follows from Theorem 4.3 since

$$W^k(\mathfrak{g}, f) \cong \bigcap_{\alpha \in \Pi_1} \text{Ker}(Q_\alpha^W : V^k(\mathfrak{g}_0) \to V^k(\mathfrak{g}_0)),$$

by Theorem 3.6 and (F1). \qed

### 4.2. Mutually commutative derivations on $W^k(\mathfrak{g}, f)$

By (F3), we have a Lie algebra homomorphism $\xi^R : a_- \to \text{Der}(\mathbb{C}[LG_+/A_+]), (a \mapsto a^R)$.

**Proposition 4.5.** The action $\xi^R$ preserves $\mathbb{C}[G_+ \setminus LG_+/ A_+] \subset \mathbb{C}[LG_+/A_+]$.

**Proof.** By Lemma 4.1 we have, for $e_\alpha \in \mathfrak{g}$, and $a \in a_-,

$$[e_\alpha^L, a^R] = [e_\alpha, (KaK^{-1})_0]^L = [e_\alpha, (KaK^{-1})_0]^L = \sum_{\beta \in I \cup \Delta_0} F_\beta[e_\alpha, e_\beta]^L,$$

for some polynomials $F_\beta \in \mathbb{C}[LG_+/A_+]$. Hence, for $G \in \mathbb{C}[G_+ \setminus LG_+/ A_+]$, we have

$$e_\alpha^L(a^R G) = [e_\alpha^L, a^R]G = \sum_{\beta \in I \cup \Delta_0} F_\beta[e_\alpha, e_\beta]^L G = 0.$$

Since $a_-$ is abelian, its image $\xi^R(a_-) \subset \text{Der}(\mathbb{C}[LG_+/A_+])$ is also abelian. By Theorem 4.3 $\xi^R(a_-)$ is identified with a set of commutative derivations of $W^k(\mathfrak{g}, f)$, which we denote by $\mathcal{H}^k(\mathfrak{g}, f)$. In the next section, we will prove that $\mathcal{H}^k(\mathfrak{g}, f)$ is an integrable Hamiltonian hierarchy associated with the classical affine $W$-algebra $W^k(\mathfrak{g}, f)$. 


5. $H^k(g,f)$ as an integrable Hamiltonian hierarchy

In this section, we always assume that the condition (F) holds and $k \in \mathbb{C}$ is generic, and identify $V^k(g_0)$ with $\mathbb{C}[L G_+/A_+]$ by $\Psi_k$, (Theorem 4.3).

5.1. Construction of Hamiltonians. Let $\Omega_{dR}(LG_+) = \mathbb{C}[L G_+] \otimes \wedge LG^*$ denote the algebraic de Rham complex of $LG_+$. Here $LG^*$ is the vector space dual to the space $LG_+$ of the right invariant vector fields $L G^\nu_+$ on $LG_+$. We take a basis $\varphi^\alpha$, $(\alpha \in \Delta > 0)$, of $LG^*$ so that $\varphi^\alpha(e) = \delta_{\alpha, \beta}$, $(\alpha, \beta \in \Delta > 0)$, holds. Note that the complex $\Omega_{dR}(LG_+)$ coincides with the Chevalley-Eilenberg complex of the Lie algebra $LG$ with coefficients in $(\mathbb{C}[L G_+], \xi^{L})$. Similarly, let $\Omega_{dR}(A_+) = \mathbb{C}[A_+] \otimes \wedge a^*_+$ denote the algebraic de Rham complex of $A_+$. Then the inclusion $A_+ \hookrightarrow LG_+$ induces the projection $\Omega_{dR}(LG_+) \rightarrow \Omega_{dR}(A_+)$. It restricts to $a^*_R$-invariant subcomplexes:

\begin{equation}
\pi : \mathbb{C}[L G_+/A_+] \otimes \wedge LG^* \rightarrow \mathbb{C} \otimes \wedge a^*_+.
\end{equation}

As a $LG_+$-module, $\mathbb{C}[L G_+/A_+]$ is isomorphic to the $LG_+$-module $\text{Coind}_{a_+}LG \mathbb{C}$ coinduced from the trivial $a_+$-module $\mathbb{C}$. Then $\pi$ induces an isomorphism

\[ H^*(LG_+; \mathbb{C}[L G_+/A_+]) \cong H^*(LG_+; \text{Coind}_{a_+}LG \mathbb{C}) \cong H^*(a_+; \mathbb{C}) \]

by Shapiro's lemma, (cf. [24]). The right hand side is isomorphic to $\wedge a^*_+$ since $a_+$ is abelian. The action $\xi^R$ of $a_-$ induces $a_-$-actions on the complexes $\Omega_{dR}(LG_+)$, $\Omega_{dR}(A_+)$, which commute with $\pi$. Since $a$ is abelian, $a_-$ acts on $\Omega_{dR}(A_+)$ trivially. Thus $a_-$ acts on the cohomology $H^*(LG_+; \mathbb{C}[L G_+/A_+])$ trivially. In particular, $s$ acts on $H^*(LG_+; \mathbb{C}[L G_+/A_+])$ trivially. By abuse of notation, we write the above $a_-$-actions by $\xi^R$.

Consider the double complex $\mathcal{C}$

\[ \mathcal{C} \xrightarrow{\iota} \Omega_{dR}(LG_+) \rightarrow \Omega_{dR}(LG_+) \rightarrow \mathcal{C}. \]

Here $\iota$ is the unit morphism and $\epsilon$ the counit morphism. It has the following shape:

\[
\begin{array}{cccc}
\mathbb{C} & \rightarrow & \Omega_{dR}(LG_+) & \rightarrow & \Omega_{dR}(LG_+) & \rightarrow & \mathbb{C} \\
\epsilon & & & & & & \\
0_{(-1,1)} & \rightarrow & \mathbb{C}[L G_+/A_+](0,1) & \rightarrow & \mathbb{C}[L G_+/A_+] \otimes LG^*_0(1,1) & \rightarrow & \cdots \\
\partial = s^R & & & & & & \\
0_{(-1,0)} & \rightarrow & \mathbb{C}[L G_+/A_+](0,0) & \rightarrow & \mathbb{C}[L G_+/A_+] \otimes LG^*_0(1,0) & \rightarrow & \cdots \\
\iota & & & & & & \\
\mathbb{C} & \rightarrow & \mathbb{C} & \rightarrow & \mathbb{C} & \rightarrow & \mathbb{C} \\
\end{array}
\]

where the subscript $(i, j)$ denotes the bidegree of $\mathcal{C}$. Then the calculation of the cohomology $H^*(\mathcal{C})$ via spectral sequences gives the isomorphisms

\begin{equation}
\Delta^* \cong H^1(\mathcal{C}) \cong \text{Ker}(d : \mathbb{C}[L G_+/A_+] / \mathbb{C} \oplus \text{Im}(s^R) \rightarrow \mathbb{C}[L G_+/A_+] \otimes LG^*_0 / \text{Im}(s^R)).
\end{equation}

By Theorem 4.3 we have an isomorphism $\mathbb{C}[L G_+/A_+] / \mathbb{C} \oplus \text{Im}(s^R) \cong \text{Lie}(V^k(g_0)) / \mathbb{C}$ as vector spaces. Since $V^k(g_0)$ is an algebra of differential polynomials, $f : \mathbb{C} \rightarrow \text{Lie}(V^k(g_0))$ is injective. Thus we may lift an element $\int f \in \text{Lie}(V^k(g_0)) / \mathbb{C}$ to $\int \tilde{f} \in \text{Lie}(V^k(g_0)) / \mathbb{C}$ which is without the constant term $\tilde{f}(0) = 0$. Identifying $a_-$ with $\Delta^*$ by $\kappa$, we obtain an isomorphism

\[ a_- \cong \Delta^* \cong \left\{ \int f \in \text{Lie}(V^k(g_0)) | f(0) = 0, d \int f = 0 \right\}, \quad a \mapsto \int H(a). \]
The statement of the following proposition makes sense by Lemma 3.5.

**Proposition 5.1.** The image \( \int H(a_-) \) lies in \( \text{Lie}(W^k(\mathfrak{g}, f)) \).

**Proof.** We may assume that the basis \( \{e_\alpha\}_{\alpha \in \Delta > 0} \) of \( L\mathfrak{g}_+ \) respects the decomposition \( L\mathfrak{g}_+ = \text{Im}(ad_\alpha) \oplus \text{Ker}(ad_\lambda) \). For \( \alpha \in a_- \), the element \( \int H(a) \in \text{Lie}(V^k(\mathfrak{g}_0)) \) satisfies, by construction, \( dH(a) = s^R A \) for some element \( A = \sum_{\alpha \in \Delta > 0} F_\alpha \otimes \varphi^\alpha \in \mathbb{C}[L\mathfrak{g}_+/A_+] \otimes L\mathfrak{g}_+^* \). Then,

\[
dH(a) = s^R A = s^R \sum_{\alpha \in \Delta > 0} F_\alpha \otimes \varphi^\alpha
\]

by Theorem 4.3.

It follows that

On the other hand, by the definition of a unique Poisson vertex superalgebra structure, which satisfies \( dH = \sum_{\alpha \in \Delta > 0} dH_\alpha \) and \( dH_\alpha \) satisfies, by construction, \( dH_\alpha = \sum_{\beta \in \Pi_1} F_\beta \otimes \varphi^\beta \), for some element \( \tilde{F}_\alpha \in \mathbb{C}[L\mathfrak{g}_+/A_+] \). In the last equality, we have used Lemma 5.2 below. By the identification \( \mathbb{C}[L\mathfrak{g}_+/A_+] \cong V^k(\mathfrak{g}_0) \) and \( \mathbb{C}[L\mathfrak{g}_+/A_+]|_{\varphi^\alpha} \cong L^k_\mathfrak{g}_0(-\lambda), (\alpha \in \Pi_1) \), we obtain

\[
\sum_{\alpha \in \Pi_1} \left( \partial F_\alpha + \sum_{\beta \in \Pi_1, \gamma \in I \cup \Delta_0} F_\beta \delta_{\alpha, \gamma} E_\gamma \right) \otimes \varphi^\alpha = \partial \left( \sum_{\alpha \in \Pi_1} F_\alpha \otimes \varphi^\alpha \right) = 0 \in \bigoplus_{\alpha \in \Pi_1} \text{Lie}(L^k_{\mathfrak{g}_0}(-\lambda)).
\]

On the other hand, by the definition of \( d \), we have \( dH(a) = \sum_{\alpha \in \Delta > 0} e^L_\alpha H(a) \otimes \varphi^\alpha \).

It follows that \( e^L_\alpha \int H(a) = 0 \) for all \( \alpha \in \Pi_1 \), which implies \( \int H(a) \in \text{Lie}(W^k(\mathfrak{g}, f)) \) by Theorem 4.3.

**Lemma 5.2.** The following formula holds

\[
(5.25) \quad s^R \varphi^\alpha = \frac{1}{k} \sum_{\beta \in \Delta > 0, \gamma \in I \cup \Delta_0} E_\gamma e^\alpha_{\beta, \gamma} \varphi^\beta + \sum_{\beta \in \Delta > 0} e^\alpha_{\beta, s} \varphi^\beta, \quad \alpha \in \Delta > 0.
\]

**Proof.** By Lemma 4.1, we have

\[
(5.26) \quad [s^R, e^L_\alpha] = -[e_\alpha, (Ksk^{-1})_{-}]^L_{+} = \frac{1}{k} \sum_{\beta \in I \cup \Delta_0} E_\gamma e_\alpha e^-_{\beta} - [e_\alpha, s]^L_{+}.
\]

Then, from the definition of \( L\mathfrak{g}_+^* \), we have

\[
(s^R \varphi^\alpha)(e_\beta) = -\varphi^\alpha(\partial e_\beta) = \frac{1}{k} \sum_{\gamma \in I \cup \Delta_0} E_\gamma c^\alpha_{\beta, \gamma} + c^\alpha_{\beta, s}.
\]

The claim (5.25) follows from it.

\[
\Box
\]

5.2. **Poisson vertex superalgebra structure on** \( \mathbb{C}[L\mathfrak{g}_+/A_+] \otimes \bigwedge L\mathfrak{g}_+^* \). We extend the Poisson vertex algebra structure on \( \mathbb{C}[L\mathfrak{g}_+/A_+] \) given by Theorem 4.3 to the whole differential superalgebra \( \mathbb{C}[L\mathfrak{g}_+/A_+] \otimes \bigwedge L\mathfrak{g}_+^*, s^R \).

**Proposition 5.3.** The differential superalgebra \( \mathbb{C}[L\mathfrak{g}_+/A_+] \otimes \bigwedge L\mathfrak{g}_+^*, s^R \) admits a unique Poisson vertex superalgebra structure, which satisfies

1. \( \{u, v \} = [u, v] + k(u/v)\lambda \), \( (u, v \in \mathfrak{g}_0) \),
2. \( \{ \varphi^\alpha, \varphi^\beta \} = 0 \), \( (\varphi^\alpha, \varphi^\beta \in L\mathfrak{g}_+^* \cup a^*_+, u \in \mathfrak{g}_0) \),
3. \( \{ u, \varphi^\alpha \} = -\sum c^\alpha_{u, \beta} \varphi^\beta \), \( (\varphi^\alpha \in L\mathfrak{g}_+^* \cup a^*_+, u \in \mathfrak{g}_0) \).
It satisfies
\[ u_{\lambda} \varphi^\alpha = - \sum c_{\alpha,\beta}^\alpha \varphi^\beta, \quad (v^\alpha \in Lg^*, u \in g_0). \]

**Proof.** It follows from Lemma 5.2 that \( C[LG_+/A_+] \otimes \wedge Lg_+^* \) is an algebra of differential polynomials in the variables given by the union of bases of \( g_0 \) and \( Lg_1 \cup a_+^* \). By Theorem 2.2, it suffices to show (2.4), (2.5) for these variables in order to prove (1) by the inductive use of the \( n \)-th differential, \( \partial \). By (2.4), they reduce to the special case
\[ (5.28) \quad \{ \epsilon_{\alpha \lambda} \{ \epsilon_{\beta \mu} \varphi^\gamma \} \} = \{ \epsilon_{\alpha \lambda} \epsilon_{\beta \mu} \varphi^\gamma \} = \{ \epsilon_{\alpha \lambda} \epsilon_{\beta \mu} \varphi^\gamma \}. \]

Since it coincides with the Jacobi identity \( (\text{ad}^*(\epsilon_{\alpha})) \text{ad}^*(\epsilon_{\beta}) \text{ad}^*(\epsilon_{\alpha})) \varphi^\gamma = \text{ad}^*(\epsilon_{\alpha}) (\epsilon_{\beta}) (\epsilon_{\alpha}) \varphi^\gamma \) of the coadjoint \( g_0^* \)-action on \( Lg_1^* \), (5.28) holds.

Next, we show (5.27). Since \( \text{ad}_s \) is an isomorphism on \( \text{Im}(\text{ad}_s) \subset Lg \), we have its inverse \( \text{ad}_s^{-1} \) on \( \text{Im}(\text{ad}_s) \) and on its dual space \( \text{Im}(\text{ad}_s)^* \). Then we have a decomposition \( Lg_{>0} = \text{ad}_s^{-1} \text{Im}(\text{ad}_s)_{>0} \oplus (Lg_1 \cup a_+) \). We show (5.27) for \( \varphi^\alpha \in \text{Im}(\text{ad}_s) \) by induction on degree. From Lemma 5.2, we have
\[ \text{ad}_s^{-1} \varphi^\alpha = \partial \varphi^\alpha - \frac{1}{k} \sum_{\alpha, \beta} c_{\alpha, \beta}^\alpha \varphi^\beta, \quad \varphi^\alpha \in \text{Im}(\text{ad}_s)_{>0}. \]

Here the sum with respect to Greek letters (resp. Roman letters) is taken over \( I \sqcup \Delta_0 \) (resp. \( \bar{\Delta}_{>0} \)). We use the same rule below. Then, for \( \epsilon_{\alpha} \in g_0 \) and \( \varphi^\alpha \in \text{Im}(\text{ad}_s)_{>0} \), we have
\[ \{ \epsilon_{\alpha \lambda} (\text{ad}_s^{-1})^n \varphi^\beta \} = - \sum_{|b|=|a|} c_{\alpha, b}^\alpha (\text{ad}_s^{-1})^n \varphi^b, \]
which is proved by the inductive use of the \( n = 1 \) case:
\[ \{ \epsilon_{\alpha \lambda} \text{ad}_s^{-1} \varphi^\alpha \} = \{ \epsilon_{\alpha \lambda} \partial \varphi^\alpha \} - \frac{1}{k} \sum_{\beta, b} c_{\alpha, b}^\alpha \{ \epsilon_{\alpha \lambda} \epsilon_{\beta} \varphi^b \} = \]
\[ = (\partial + \lambda) \{ \epsilon_{\alpha \lambda} \partial \varphi^\alpha \} - \frac{1}{k} \sum_{\beta, b} c_{\alpha, b}^\alpha \{ \epsilon_{\alpha \lambda} \epsilon_{\beta} \varphi^b \} = \]
\[ = -(\partial + \lambda) \sum_{b} c_{\alpha, b}^\alpha \varphi^b - \frac{1}{k} \sum_{\alpha, b} c_{\alpha, b}^\alpha \left( (\epsilon_{\alpha} + k(\epsilon_{\alpha} | \epsilon_{\beta})) \varphi^b - c_{\alpha, b}^\alpha \epsilon_{\beta} \varphi^b \right) = \]
\[ = -\lambda \left( \sum_{b} c_{\alpha, b}^\alpha \varphi^b + \sum_{\beta, b} c_{\alpha, b}^\alpha (\epsilon_{\alpha} | \epsilon_{\beta}) \varphi^b \right) - \left( \sum_{b} c_{\alpha, b}^\alpha \partial \varphi^b + \frac{1}{k} \sum_{\beta, b} c_{\alpha, b}^\alpha (\epsilon_{\alpha} | \epsilon_{\beta}) \varphi^b - \right. \sum_{c} c_{\alpha, c}^\alpha \epsilon_{\beta} \varphi^c \right) = \]
\[ = - \sum_{b} c_{\alpha, b}^\alpha \text{ad}_s^{-1} \varphi^b. \]

In the last equality, we used \((\epsilon_{\alpha} | \epsilon_{\beta}) = \delta_{\alpha, \beta}\) for the \( \lambda \)-linear term and use Lemma 5.2 \( c_{\alpha, b}^\alpha = \epsilon_{\beta}^b \), and the Jacobi identity of the Lie bracket for the constant term. This implies (5.27) for \( \varphi^\alpha \in \text{Im}(\text{ad}_s)^* \). Then (5.27) follows from this and (3). \( \square \)

The de Rham differential \( d = \sum_{\alpha \in \Delta_0} \epsilon_{\alpha}^* \otimes \varphi^\alpha \) on \( C[LG_+/A_+] \) can be described in terms of the Poisson vertex superalgebra structure in Proposition 5.3.

**Lemma 5.4.** The differential \( d \) is determined uniquely by
(i) \( d \epsilon_{\beta} = -k \sum_{\alpha \in \Delta_0} (\epsilon_{\beta}, s | \epsilon_{\alpha}) \otimes \varphi^\alpha \), \( (\beta \in I \sqcup \Delta_0) \),
(ii) \( [d, \partial] = 0 \).
Proof. (1) follows immediately from the formula \(e^L_\beta E_\alpha = -k([e_\alpha, s]\ e_\beta)\) (see also ([4.21]). (2) follows from ([5.20] and Lemma 5.2). \(\square\)

Let \(s^* \in Lg^*_+\) denote the element corresponding to \(s \in Lg^*_+\) by the identification \(\kappa = (-|-): Lg^*_\pm \cong Lg^*_\pm\) and \(\bar{s} \in Lg^*_+\) the element corresponding to \(s^*\).

**Proposition 5.5.** Under the isomorphism \(\Psi_k: V^k(g_0) \cong \mathbb{C}[LG_+/A_+]\),

\[
d = -k\{s^*_\lambda\}\}_{\lambda=0},
\]

holds on \(\mathbb{C}[LG_+/A_+]\).

Proof. It suffices to show that \(-k\{s^*_\lambda\}\}_{\lambda=0}\) satisfies (i), (ii) in Lemma 5.4. (i) follows from the definition of \(s^*\) and Proposition 5.3 (3). (ii) follows from ([2.3]). \(\square\)

The following property of the derivation \(\eta_a = \eta(\int H(a)) \in \text{Der}(\mathbb{C}[LG_+/A_+])\), \((a \in a_-)\), will be used.

**Lemma 5.6.** For \(a \in a_-\), there exist some polynomials \(F_\alpha(a) \in \mathbb{C}[LG_+/A_+]\), \((a \in I \sqcup \Delta_0)\), which satisfies

\[
[u^L, \eta_a] = \sum_{\alpha \in I \sqcup \Delta_0} F_\alpha(a)[u, e_\alpha]^L, \quad u \in Lg_1.
\]

Proof. By construction of \(\int H(a)\), we have \(d \int H(a) = \int dH(a) = 0 \in \text{Lie}(\mathbb{C}[LG_+/A_+] \otimes \bigwedge Lg^*_+\) and thus \(dH(a)|_{\lambda=0} = 0\). By ([2.6]), we have

\[
\{dH(a)|_{\lambda=0}\} = \sum_{\alpha \in \Delta_0} e^L_\alpha \otimes \varphi^\alpha, \eta_a
\]

It follows from them that

\[
[e^L_\alpha, \eta_a] = \sum_{\beta \in \Delta_0} (e_\alpha | \eta_a(\varphi^\beta))e^L_\beta,
\]

where \((-|-): Lg_+ \times Lg^*_+ \to \mathbb{C}\) denotes the canonical pairing. By Proposition 5.3, we have

\[
\eta_a(\varphi^\beta) = [H(a)|_{\lambda=0}e^\beta] = \sum_{\gamma \in I \sqcup \Delta_0} (e_\gamma | \varphi^\beta) e^\gamma_{\lambda=0} = \sum_{\gamma \in I \sqcup \Delta_0} \sum_{\rho \in \Delta_0} c^\beta_{\gamma, \rho} \frac{\delta H(a)}{\delta e_\gamma}.
\]

Hence, for \(a \in Lg_1\) we obtain

\[
[e^L_\alpha, \eta_a] = \sum_{\beta \in \Delta_0} (e_\alpha | \eta_a(\varphi^\beta))e^L_\beta = \sum_{\beta \in \Delta_0} \sum_{\gamma \in I \sqcup \Delta_0} \sum_{\rho \in \Delta_0} c^\beta_{\gamma, \rho} \frac{\delta H(a)}{\delta e_\gamma} (e_\alpha | \varphi^\rho)e^L_\beta
\]

This completes the proof. \(\square\)

### 5.3. Integrable Hamiltonian hierarchy

Let \(\text{Vect}(LG_+)\) denote the Lie algebra of the algebraic vector fields on \(LG_+\). Recall the infinitesimal \(Lg_+\)-actions ([1.10], [4.21]).

**Lemma 5.7.** We have \(\text{Vect}(LG_+)Lg^L_+ = Lg^R_+\) and \(\text{Vect}(LG_+)Lg^R_+ = Lg^L_+\).
Proof. Although this is well-known, we give a proof for the completeness of the paper. The pairing
\[ \mathcal{U}(Lg^+) \times \mathbb{C}[LG_+] \to \mathbb{C}, \quad (X_1X_2 \cdots X_n, F) \mapsto (X_1^L X_2^L \cdots X_n^L F)(e), \]
defined by is \( Lg^+ \)-invariant and nondegenerate. Here \( \mathcal{U}(Lg^+) \) denotes the universal enveloping algebra of \( Lg^+ \). Hence we have \( \mathbb{C}[LG_+]^{Lg^+} \cong \mathcal{U}(Lg^+)/Lg^+ \mathcal{U}(Lg^+) \cong \mathbb{C}. \) Since \( Lg^+ \) commutes with \( Lg^+ \) and \( \text{Vect}(LG^+) \cong \mathbb{C}[LG_+] \otimes \mathbb{C} Lg^+ \), we obtain
\[ \text{Vect}(LG^+)^{Lg^+} \cong (\mathbb{C}[LG_+] \otimes \mathbb{C} Lg^+) Lg^+ \cong \mathbb{C}[LG^+] Lg^+ \otimes \mathbb{C} Lg^+ \cong Lg^+ R. \]
The latter claim is proved similarly. \qed

We say that an element \( X \in \text{Vect}(LG^+) \) satisfies \( (P) \) if
\[ [u^L, X] \in \sum_{\alpha \in L \cup \Delta_0} \mathbb{C}[LG_+] [u, e_\alpha]^L, \quad u \in Lg_1. \]

Let \( \mathcal{L} \) denote the set of elements in \( \text{Vect}(LG^+) \) satisfying \( (P) \). It is straightforward to show that \( \mathcal{L} \subset \text{Vect}(LG^+) \) form a Lie subalgebra. The Witt algebra \( \text{Witt} = \mathbb{C}(t) \frac{\partial}{\partial t} \) acts on \( Lg \) by derivations with respect to \( t \). Since the subalgebra \( \text{Witt}_- = \mathbb{C}[t^{-1}] \frac{\partial}{\partial t} \) preserves \( Lg_0 \), it acts on \( B \backslash G(t) \) infinitesimally and therefore on the open dense subset \( LG^+ \). The induced \( \text{Witt}_- \)-action on \( \mathbb{C}[LG^+] \) is given by
\[ L_n \exp \left( \sum_{\alpha \in \Delta_0} z_\alpha e_\alpha \right) = \epsilon_\text{linear term of} \exp \left( \sum_{\alpha \in \Delta_0} z_\alpha e_\alpha \right) e^{t L_n}, \]
where \( L_n = -t^{n+1} \frac{\partial}{\partial t} \in \text{Witt}_- \).

**Lemma 5.8.** We have \( \mathcal{L} = \text{Witt}_- \rtimes Lg^R. \)

**Proof.** By Lemma 4.1 we have \( Lg^R \subset \mathcal{L} \). In particular, this implies that \( \mathcal{L} \) is a \( Lg \)-module. The inclusion \( \text{Witt}_- \subset \mathcal{L} \) is shown in the same way as in the proof of Lemma 4.1. It is obvious that these two actions give an action of their semidirect product \( \text{Witt}_- \rtimes Lg^R \subset \mathcal{L} \).

To show its equality, consider the quotient \( Lg \)-module \( M := \mathcal{L}/\text{Witt}_- \rtimes Lg^R \). Let us show that \( M \) belongs to the category \( \mathcal{O} \) of the affine Lie algebra \( \tilde{g} = g[t^\pm 1] \oplus \mathbb{C} \oplus L_0 \) at level 0. Let \( \tilde{g} = \tilde{n}_+ \oplus \tilde{h} \oplus \tilde{n}_- \) be the standard triangular decomposition of \( \tilde{g} \). Indeed, the action of the Cartan subalgebra \( \tilde{h} = h \oplus \mathbb{C} \oplus L_0 \) on \( \text{Vect}(LG^+) \) is diagonalizable and, moreover, preserves \( \mathcal{L} \) and \( \text{Witt}_- \rtimes Lg^R \). Hence it is diagonalizable on \( M = \mathcal{L}/\text{Witt}_- \rtimes Lg^R \). Let us show that the dimension \( \dim \mathcal{U}(\tilde{n}_+ w) \) is finite for any \( w \in M \). Let \( \tilde{w} \in \mathcal{L} \) denote a lift of \( w \). The decomposition \( \tilde{n}_+ = g^+_0 \times \tilde{n}_>0 \) where \( g^0_0 = \bigoplus_{\alpha \in \Delta_0} g_{0\alpha} \) and \( \tilde{n}_>0 = \tilde{n}_+ \cap L_0 \) induces a decomposition of the universal enveloping algebra \( \mathcal{U}(\tilde{n}_+) \cong \mathcal{U}(g^0_0) \otimes \mathcal{U}(\tilde{n}_>0) \) as vector spaces. By definition of \( \mathcal{L} \), \( \tilde{w} \) satisfies, for any \( u \in Lg_1 \),
\[ [u^L, w] = \sum_{\alpha \in L \cup \Delta_0} F_{\alpha}(w)[e_\alpha, u]^L \]
for some element \( F_{\alpha}(w) \in \mathbb{C}[LG_+] \).

Since \( \tilde{n}_>0 \) and \( \tilde{n}_>0^R \) commute, for any \( Z \in \mathcal{U}(\tilde{n}_>0) \),
\[ [u^L, Z^R(w)] = Z^R([u^L, w]) = \sum_{\alpha \in L \cup \Delta_0} Z^R(F_{\alpha}(w))[e_\alpha, u]^L. \]

If \( Z^R(F_{\alpha}(w)) = 0 \) for all \( \alpha \in L \cup \Delta_0 \) and \( u \in Lg_1 \), then by Lemma 5.7 we have \( [Z^R, w] \in \tilde{n}_>0^R \), which is zero in \( M \). Considering the \( \tilde{h} \)-weights of \( Z^R(F_{\alpha}(w)) \), we conclude that \( \dim \mathcal{U}(\tilde{n}_>0)^R(F_{\alpha}(w)) < \infty \) for each \( \alpha \) and hence \( \dim \mathcal{U}(\tilde{n}_>0)^R w < \infty \). Thus it remains to show \( \dim \mathcal{U}(g^0_0)w < \infty \). Take \( u \in g^0_0 \) and \( v \in g^0_+ \), we have
Therefore, by the action of $\text{Witt}_- \bowtie Lg^R + Cw$ by a trivial 1-dimensional $\bar{\mathfrak{g}}$-module Witt$- \bowtie Lg^R$ for $N$ large enough, it follows that $[u^L, [v^R, [v^R, \ldots, [v^R, w] \ldots]] = 0$ for $n$ sufficiently large and hence $[v^L, [v^R, \ldots, [v^R, w] \ldots]] \in \mathfrak{n}_{>0}$, which is zero in $M$. Therefore, we obtain $\dim \mathfrak{h}(\mathfrak{g}^R)w < \infty$. Thus we conclude that $M$ belongs to the category $O$.

Suppose $M \neq 0$. Then it contains a nonzero highest weight vector $w$. Let $\mathfrak{n} \in \mathcal{L}$ denote a lift of $w$. Then Witt$_- : Lg^R + Cw$ is an extension of $\mathfrak{n}$-module Witt$_- : Lg^R$. Since the action $Lg^R$ on $C[LG_+]$ is faithful, $Lg^R \cong Lg$ as $\mathfrak{n}$-modules. On the other hand, the cohomology $H^n(\mathfrak{n}; \text{Witt}_- \bowtie Lg^R)$ is described as

$$H^n(\mathfrak{n}; \text{Witt}_- \bowtie Lg) \cong H^{n-1}(\mathfrak{n}; \mathbb{C}) \otimes \text{Witt}_{> 0},$$

where Witt$_{>0} = \mathbb{C}[[t]]/t^2$. This isomorphism is $\mathfrak{h}$-equivariant. In particular, all the degrees of the homogeneous elements with respect to the action of $x+(d+1)L_0$ in the first cohomology $H^1(\mathfrak{n}; \text{Witt}_- \bowtie Lg) \cong \text{Witt}_{>0}$ are positive. (See Section 1 for the definition of $x$ and $d$.) On the other hand, the cocycle corresponding to $Cw$ is negative due to the definition of $\mathcal{L}$, which is a contradiction. Therefore, we conclude $M = 0$. This completes the proof.

**Theorem 5.9.** Suppose that $(\mathfrak{g}, f)$ satisfies (F) and $k \in \mathbb{C}$ is generic. Then, $\mathcal{H}^k(\mathfrak{g}, f)$ is an integrable Hamiltonian hierarchy associated with $\mathcal{W}^k(\mathfrak{g}, f)$.

**Proof.** By abuse of notation, let $\eta_{\alpha} \in \text{Der}(V^k(\mathfrak{g}))$ also denote the corresponding element in $\text{Vect}(LG_+/A_+)$ by $\Psi_k$ in Theorem 4.3. Since $LG_+ \cong (LG_+/A_+) \times A_+$ as affine schemes in an $A_+^{R,+}$-equivariant way, the elements of $\text{Vect}(LG_+/A_+)$ naturally lift to elements of $\text{Vect}(LG_+)$ which commute with $a_+^R$. Again, let $\eta_{\alpha}$ denote the lift of $\eta_{\alpha}$ in $\text{Vect}(LG_+)$. Then by Lemma 4.8, the lift $\eta_{\alpha}$ lies in $\mathcal{L}$ and, moreover,

$$[u^L, \eta_{\alpha}] = \sum_{\alpha \in \Delta_0} F_{\alpha}(\eta_{\alpha})[u, e_{\alpha}]^L \text{ with } F_{\alpha}(\eta_{\alpha}) \in \mathbb{C}[LG_+/A_+] \text{ for } u \in Lg_1.$$

Since $\mathfrak{n}_1 \in \mathcal{L}$, we obtain $[\bar{s}^R, \eta_{\alpha}] \in \text{Vect}(LG_+)^{Lg^R}_+ = Lg^R_+$. The degree of $[\bar{s}^R, \eta_{\alpha}]$ is positive and the degrees of the elements in $Lg^R_+$ are positive with respect to the action of $x+(d+1)L_0$. This implies $[\bar{s}^R, \eta_{\alpha}] = 0$, i.e., $\eta_{\alpha} \in \mathcal{L}^R$. On the other hand, we have $\mathcal{L}^R = \text{Witt}^R_- \bowtie (Lg^R)^R$ since $[\bar{s}^R, \text{Witt}^-] \subset [a, \text{Witt}^-]^R \subset a^R$ and $[\bar{s}^R, Lg^R] \cap a^R = 0$. Since $\bar{s} \in Lg$ is not an element in $\mathfrak{g}$, we have $\text{Witt}^R_- = 0$. From the definition of $\bar{s}$ (and $s$), we obtain $Lg^R = a^R$. Thus $\mathcal{L}^R = a^R$ holds. Therefore, $\eta_{\alpha}$ lies in $a^R$, which means $\int H(a_-) \subset a^R$. Considering the degree given by the action of $x+(d+1)L_0$, we see $\int H(a_-) \subset a^R$. By Lemma 4.8, it is easy to
show that the map $\int$ on $H(a_-)$ is injective. Since the map $a \mapsto \int H(a)$ preserves the degree with respect to the action of $x + (d+1)L_0$, we obtain $\int H(a_-) = a^R$. □

References

[1] T. Arakawa, Introduction to W-algebras and their representation theory, Perspectives in Lie theory, Springer INdAM Ser., 19, 179–250, Springer, Cham, 2017.

[2] A. Barakat and A. De Sole and V. Kac, Poisson vertex algebras in the theory of Hamiltonian equations, Jpn. J. Math., 4, 2009, (2), 141–252.

[3] D. Ben-Zvi, E. Frenkel, Spectral curves, opers and integrable systems, Publ. Math. Inst. Hautes Études Sci., 94, 2001, 87–159.

[4] N. Burroughs and M. de Groot and T. Hollowood and J. Miramontes, Generalized Drinfel'd-Sokolov hierarchies. II. The Hamiltonian structures, Comm. Math. Phys., 153, 1993, (1), 187–215.

[5] D. Collingwood and W. McGovern, Nilpotent orbits in semisimple Lie algebras, Van Nostrand Reinhold Mathematics Series, Van Nostrand Reinhold Co., New York, 1993, xiv+186.

[6] M. de Groot and T. Hollowood and J. Miramontes, Generalized Drinfel’d-Sokolov hierarchies, Comm. Math. Phys., 145, 1992, (1), 57–84.

[7] A. De Sole and V. Kac, Finite vs affine W-algebras, Jpn. J. Math., 1, 2006, (1), 137–261.

[8] A. De Sole and V. Kac, The variational Poisson cohomology, Jpn. J. Math., 8, 2013, (1), 1–145.

[9] A. De Sole and V. Kac and D. Valeri, Classical W-algebras and generalized Drinfel’d-Sokolov bi-Hamiltonian systems within the theory of Poisson vertex algebras, Comm. Math. Phys., 323, 2013, (2), 663–3616.

[10] A. De Sole and V. Kac and D. Valeri, Adler-Gelfand-Dickey approach to classical W-algebras within the theory of Poisson vertex algebras, Int. Math. Res. Not. IMRN, 21, 2015, 11186–11235.

[11] A. De Sole and V. Kac and D. Valeri, Structure of classical (finite and affine) W-algebras, J. Eur. Math. Soc. (JEMS), 18, 2016, (9), 1873–1908.

[12] A. De Sole and V. Kac and D. Valeri, Classical affine W-algebras for gl_N and associated integrable Hamiltonian hierarchies, Comm. Math. Phys., 348, 2016, (1), 265–319.

[13] A. De Sole and V. Kac and D. Valeri, Classical affine W-algebras and the associated integrable Hamiltonian hierarchies for classical Lie algebras, Comm. Math. Phys., 360, 2018, (3), 851–918.

[14] V. Drinfel’d and V. Sokolov, Lie algebras and equations of Korteweg-de Vries type, Current problems in mathematics, Vol. 24, Itogi Nauki i Tekhniki, 1984, 81–180.

[15] A. Elashvili and V. Kac, Classification of good gradings of simple Lie algebras, Lie groups and invariant theory, Amer. Math. Soc. Transl. Ser. 2, 2005, 213, 85–104.

[16] B. Enriquez and E. Frenkel, Equivalence of two approaches to integrable hierarchies of KdV type, Comm. Math., Phys., 185, 1997, (1), 211–230.

[17] L. Fehér and J. Harland and I. Marshall, Generalized Drinfel’d-Sokolov reductions and KdV type hierarchy, Comm. Math. Phys., 154, 1993, (1), 181–214.

[18] B. Feigin and E. Frenkel, Kac-Moody groups and integrability of soliton equations, Invent. Math., 120, 1995, (2), 379–408.

[19] B. Feigin and E. Frenkel, Integrals of motion and quantum groups, Integrable systems and quantum and classical (finite and affine) W-algebras, VIII J. A. Swieca Summer School on Particles and Fields (Rio de Janeiro, 1995), Lecture Notes in Math., 1620, 1996, 349–418.

[20] C. Fernández-Pousa and M. Gallas and J. Miramontes and J. Sánchez, Integrable systems and W-algebras, VIII J. A. Swieca Summer School on Particles and Fields (Rio de Janeiro, 1995), 1996, 475–479.

[21] C. Fernández-Pousa and M. Gallas and V. Manuel and J. Miramontes and J. Sánchez, W-algebras from soliton equations and Heisenberg subalgebras, Ann. Physics, 243, 1995, (2), 372–419.

[22] E. Frenkel and D. Ben-Zvi, Vertex algebras and algebraic curves, Mathematical Surveys and Monographs, 88, American Mathematical Society, Providence, RI, 2001, xii+348.

[23] D. Fuks, Cohomology of infinite-dimensional Lie algebras, Contemporary Soviet Mathematics, 1986, xii+339.

[24] N. Genra, Screening operators for W-algebras, el. Math. New. Ser. (publishes online)), 25, 2017, (3), 2157–2202.

[25] V. Kac and M. Wakimoto, Quantum reduction for affine superalgebras, Comm. Math. Phys., 211, 2000, (2), 307–342.

[26] V. Kac and M. Wakimoto, Quantum reduction and representation theory of superconformal algebras, Adv. Math., 185, 2004, (2), 400–458.
[28] V. Kac and M. Wakimoto, Corrigendum to: “Quantum reduction and representation theory of superconformal algebras” [Adv. Math. 185 (2004), no. 2, 400–458; MR2060475], Adv. Math., 193, 2005, (2), 453–455.

[29] S. Kumar, Kac-Moody groups, their flag varieties and representation theory, Progress in Mathematics, 204, Birkhäuser Boston, Inc., Boston, MA, 2002, xvi+606.

[30] U. Suh, Classical Affine W-Superalgebras via Generalized Drinfel’d–Sokolov Reductions and Related Integrable Systems, Comm. Math. Phys., 358, 2018, (1), 199–236.

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