The method of upper and lower solutions to impulsive differential equations with integral boundary conditions

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Abstract
This paper considers a second-order impulsive differential equation with integral boundary conditions. Some sufficient conditions for the existence of solutions are proposed by using the method of upper and lower solutions and Leray-Schauder degree theory.

Keywords: impulsive differential equation; lower (upper) solution; integral boundary condition; Leray-Schauder degree

1 Introduction
The theory of impulsive differential equations describes processes which experience a sudden change of their state at certain moments. Processes with such a character arise naturally and often, especially in phenomena studied in physics, chemical technology, population dynamics, biotechnology, and economics. For an introduction of the basic theory of impulsive differential equations in $R^n$, see Lakshmikantham et al. [1], Bainov and Simeonov [2], Samoilenko and Perestyuk [3], and the references therein. The theory of impulsive differential equations has become an important area of investigation in recent years, and it is much richer than the corresponding theory of differential equations (see, for instance, [4–6] and the references therein).

On the other hand, the theory of boundary value problems with integral boundary conditions for ordinary differential equations arises in different areas of applied mathematics and physics. For example, heat conduction, chemical engineering, underground water flow, thermo-elasticity, and plasma physics can be reduced to nonlocal problems with integral boundary conditions. For boundary value problems with integral boundary conditions and comments on their importance, we refer the reader to the papers in Gallardo [7–9] and the references therein. For more information about the general theory of integral equations and their relation with boundary value problems, we refer to the book of Corduneanu [10] and Agarwal and O'Regan [11].

Recently, some well-known works, such as Hao et al. [12], Zhang et al. [13] and Ding and Wang [14], deal with impulsive differential equations with integral boundary conditions. However, most of these results are obtained by using the fixed point theorem in cones. It is well known that the method of upper and lower solutions is a powerful tool for proving the existence results for a large class of boundary value problems, see [15–17].
In [17], Shen and Wang applied the method of upper and lower solutions to solve impulsive differential equations with nonlinear boundary conditions as follows:

\[
\begin{cases}
    x''(t) = f(t, x(t), x'(t)), & t \in J, t \neq t_k, \\
    \Delta x(t_k) = I_k(x(t_k)), & k = 1, 2, \ldots, p, \\
    \Delta x'(t_k) = J_k(x(t_k), x'(t_k)), & k = 1, 2, \ldots, p, \\
    g(x(0), x'(0)) = 0, & h(x(1), x'(1)) = 0.
\end{cases}
\]

Motivated by the works mentioned above, in this paper, we shall employ the method of upper and lower solutions together with Leray-Schauder degree theory to study the existence of a solution of the impulsive BVP

\[
\begin{cases}
    x''(t) + f(t, x(t), x'(t)) = 0, & t \in J^*, \\
    \Delta x(t_k) = I_k(x(t_k)), & k = 1, 2, \ldots, p, \\
    \Delta x'(t_k) = J_k(x(t_k), x'(t_k)), & k = 1, 2, \ldots, p, \\
    x(0) = x(1) = \int_0^1 g(s)x(s) ds,
\end{cases}
\tag{1.1}
\]

where \( J = [0, 1], J^* = \bigcap \{ [t_i, t_{i+1}), t_i < t_2 < \cdots < t_p < t_{p+1} = 1 \}, \Delta x(t_k) = x(t_k^+) - x(t_k^-) \) denotes the jump of \( x(t) \) at \( t = t_k \), and \( x(t_k^+) \) and \( x(t_k^-) \) represent the right and left limits of \( x(t) \) at \( t = t_k \), respectively. \( \Delta x'(t_k) \) has a similar meaning for \( x'(t) \).

### 2 Preliminaries

Define \( PC(I) = \{ x : J \rightarrow R \mid x \in C(J^*), x(t_i^+) \text{ and } x(t_i^-) \text{ exist, and } x(t_i^-) = x(t_i), i = 1, 2, \ldots, p \} \). \( PC^1(I) = \{ x \in PC(I) : x|_{[t_i, t_{i+1})} \in C^1(t_i, t_{i+1}), x'(t_i^+) \text{ and } x'(t_i^-) \text{ exist, and } x'(t_i^-) = x'(t_i), i = 1, 2, \ldots, p \} \). Note that \( PC(I) \) and \( PC^1(I) \) are Banach spaces with the respective norms

\[
\| x \|_\infty = \sup \{ |x(t)| : t \in J \}, \quad \| x \| = \max \{ \| x \|_\infty, \| x' \|_\infty \}.
\]

A function \( x \in E = PC^1(I) \cap C^2(J^*) \) is called a solution of (1.1) if it satisfies the differential equation

\[
x''(t) + f(t, x(t), x'(t)) = 0, \quad t \in J^*,
\]

the impulsive conditions \( \Delta x(t_k) = I_k(x(t_k)), \Delta x'(t_k) = J_k(x(t_k), x'(t_k)) \) and the boundary conditions \( x(0) = x(1) = \int_0^1 g(s)x(s) ds \).

**Definition 2.1** The function \( \alpha \in E \) is said to be a lower solution for boundary value problem (1.1) if

\[
\alpha''(t) + f(t, \alpha(t), \alpha'(t)) \geq 0, \quad t \in J^*,
\]

\[
\Delta \alpha(t_k) = I_k(\alpha(t_k)), \quad \Delta \alpha'(t_k) = J_k(\alpha(t_k), \alpha'(t_k)), \quad k = 1, 2, \ldots, p,
\]

\[
\alpha(0) \leq \int_0^1 g(s)\alpha(s) ds, \quad \alpha(1) \leq \int_0^1 g(s)\alpha(s) ds.
\]
The function $\beta \in E$ is said to be an upper solution for boundary value problem (1.1) if

\[
\beta''(t) + f(t, \beta(t), \beta'(t)) \leq 0, \quad t \in J^*,
\]

\[
\Delta \beta(t_k) = I_k(\beta(t_k)), \quad \Delta \beta'(t_k) = J_k(\beta(t_k), \beta'(t_k)), \quad k = 1, 2, \ldots, p,
\]

\[
\beta(0) \geq \int_0^1 g(s)\beta(s) \, ds, \quad \beta(1) \geq \int_0^1 g(s)\beta(s) \, ds.
\]

For $\alpha, \beta \in E$, we write $\alpha \leq \beta$ if $\alpha(t) \leq \beta(t)$ for all $t \in J$.

**Definition 2.2** Let $\alpha, \beta \in E$ be such that $\alpha \leq \beta$ on $J$. We say that $f$ satisfies the Nagumo condition relative to $\alpha, \beta$ if for

\[
\omega = \max_{1 \leq k \leq p} \left\{ \frac{|\alpha(t_{k+1}) - \beta(t'_k)|}{t_{k+1} - t_k}, \frac{|\beta(t_{k+1}) - \alpha(t'_k)|}{t_{k+1} - t_k} \right\},
\]

there exists a constant $D$ such that

\[
D > \max \{\omega, \|\alpha'\|_\infty, \|\beta'\|_\infty\},
\]

a continuous function $\Psi : [0, \infty) \to (0, \infty)$, and constants $A \geq 0$, $B \geq 0$ such that

\[
|f(t, x, y)| \leq A|y|\Psi(|y|) + B, \quad t \in J, \alpha \leq x \leq \beta, y \in R
\]

and

\[
\int_{\omega}^D \frac{1}{\Psi(s)} \, ds > A \left[ \max_{t \in J} \beta(t) - \min_{t \in J} \alpha(t) \right] + B \max_{s \geq \omega} \Psi^{-1}(s).
\]

In addition, we assume that the following conditions hold:

(H1) $f$ satisfies the Nagumo condition relative to $\alpha, \beta$;

(H2) $I_k, J_k \in C(R, R$. $J_k(y, z)$ is nondecreasing in $z \in [-D, D]$ for all $1 \leq k \leq p$;

(H3) $0 < \int_0^1 g(s) \, ds < 1$.

We consider the modified problem

\[
x''(t) + F(t, x(t), \frac{d}{dt}m_{\alpha, \beta}(t, x)) = 0, \quad t \in J^*,
\]

\[
\Delta x(t_k) = I_k(m_{\alpha, \beta}(x(t_k))), \quad k = 1, 2, \ldots, p,
\]

\[
\Delta x'(t_k) = J_k(m_{\alpha, \beta}(x(t_k), n(t_k))), \quad k = 1, 2, \ldots, p,
\]

\[
x(0) = x(1) = \int_0^1 g(s)x(s) \, ds
\]

where

\[
F(t, x, y) = f(t, m_{\alpha, \beta}(t, x), n(t, y)) + \frac{m_{\alpha, \beta}(t, x) - x}{1 + (x - m_{\alpha, \beta}(t, x))^2}
\]
and

\[ m_{\alpha,\beta}(t,x) = \max\{\alpha(t), \min\{x, \beta(t)\}\}, \]

\[ n(t,x) = \max\{-D, \min\{x,D\}\}. \]

One can find the next result, with its proof, in [18].

**Lemma 2.1** For each \( x \in E \), the following two properties hold:

(i) \( \frac{d}{dt} m_{\alpha,\beta}(t,x) \) exists for a.e. \( t \in J^* \);

(ii) if \( x, x_m \in E \) and \( x_m \rightarrow x \) in \( E \), then

\[ \frac{d}{dt} m_{\alpha,\beta}(t,x(t)) \rightarrow \frac{d}{dt} m_{\alpha,\beta}(t,x(t)) \] for a.e. \( t \in J^* \).

**Lemma 2.2** For any \( v(t) \in C(J) \), the following boundary value problem

\[
\begin{cases}
  x''(t) + v(t) = 0, & t \in J^*, \\
  \Delta x(t_k) = I_k(x(t_k)), & k = 1, 2, \ldots, p, \\
  \Delta x'(t_k) = J_k(x(t_k),x'(t_k)), & k = 1, 2, \ldots, p, \\
  x(0) = x(1) = \int_0^1 g(s)x(s) \, ds
\end{cases}
\]

has a unique solution as follows:

\[
x(t) = \int_0^1 H(t,s)v(s) \, ds + \sum_{0<s<t} \left[ I_k(x(t_k)) + (t - t_k)J_k(x(t_k),x'(t_k)) \right] \\
- t \sum_{k=1}^p \left[ I_k(x(t_k)) + (1 - t_k)J_k(x(t_k),x'(t_k)) \right] \\
+ \frac{1}{1-\mu} \int_0^1 \sum_{0<s<t} \left[ I_k(x(t_k)) + (s - t_k)J_k(x(t_k),x'(t_k)) \right] ds \\
- \frac{\nu}{1-\mu} \sum_{k=1}^p \left[ I_k(x(t_k)) + (1 - t_k)J_k(x(t_k),x'(t_k)) \right],
\]

where

\[ H(t,s) = G(t,s) + \frac{1}{1-\mu} \int_0^1 G(s,\tau)g(\tau) \, d\tau, \]

\[ G(t,s) = \begin{cases} 
  t(1-s), & 0 \leq t \leq s \leq 1, \\
  s(1-t), & 0 \leq s \leq t \leq 1,
\end{cases} \]

and

\[ \mu = \int_0^1 g(s) \, ds, \quad \nu = \int_0^1 sg(s) \, ds. \]

**Lemma 2.3** If \( x \) is a solution of BVP (2.1), \( \alpha(t) \) and \( \beta(t) \) are lower and upper solutions of (1.1), respectively, \( \alpha \leq \beta \), and

\[ I_k(\alpha(t_k)) \leq I_k(x) \leq I_k(\beta(t_k)), \quad k = 1, \ldots, p, \]

(2.3)
for
\[ \alpha(t_k) \leq x \leq \beta(t_k), \]
then
\[ \alpha(t) \leq x(t) \leq \beta(t), \quad t \in [0, 1]. \]

**Proof** Denote \( y(t) = x(t) - \beta(t) \), we will only see that \( x(t) \leq \beta(t) \) for every \( t \in J \). An analogous reasoning shows that \( x(t) \geq \alpha(t) \) for all \( t \in J \). Otherwise, if \( x(t) \leq \beta(t), \ t \in [0, 1] \), does not hold, then \( \sup_{0 \leq t \leq 1} (x(t) - \beta(t)) > 0 \), there are three cases.

Case 1. Suppose that \( \max_{0 \leq t \leq 1} y(t) = \sup_{0 \leq t \leq 1} (x(t) - \beta(t)) = y(0) \), or \( \max_{0 \leq t \leq 1} y(t) = y(1) \), we only see that \( \max_{0 \leq t \leq 1} y(t) = y(0) \). Easily, it holds that \( y(0) > 0 \). From Definition 2.1 and (H3), we have
\[
\begin{align*}
y(0) &= x(0) - \beta(0) \leq \int_0^1 g(s)(x(s) - \beta(s)) \, ds \\
&\leq \max_{s \in [0, 1]} (x(s) - \beta(s)) \int_0^1 g(s) \, ds \\
&\leq \max_{s \in [0, 1]} (x(s) - \beta(s)) = \max_{s \in [0, 1]} y(s),
\end{align*}
\]
which is a contradiction.

Case 2. Suppose that there exist \( k \in \{0, 1, \ldots, p\} \) and \( \tau \in (t_k, t_k + 1) \) such that
\[
\sup_{t \in (t_k, t_k+1]} \ y(t) = y(\tau) = x(\tau) - \beta(\tau) > 0.
\]

Then \( y'(\tau) = 0 \) and \( y''(\tau) \leq 0 \). On the other hand,
\[
y'(\tau) \geq -F(\tau, x(\tau), \frac{d}{dt} m_{a,\beta}(\tau, x)) + f(\tau, \beta(\tau), \beta'(\tau))
\]
\[
= -f(\tau, m(\tau, x(\tau)), \frac{d}{dt} m_{a,\beta}(\tau, x)) - \frac{m_{a,\beta}(\tau, x(\tau)) - x(\tau)}{1 + (x(\tau) - m_{a,\beta}(\tau, x(\tau)))^2} + f(\tau, \beta(\tau), \beta'(\tau))
\]
\[
= -f(\tau, \beta(\tau), \beta'(\tau)) - \frac{m_{a,\beta}(\tau, x(\tau)) - x(\tau)}{1 + (x(\tau) - m_{a,\beta}(\tau, x(\tau)))^2} + f(\tau, \beta(\tau), \beta'(\tau))
\]
\[
= \frac{y(\tau)}{1 + y(\tau)^2} > 0,
\]
which is a contradiction. Hence the function \( y \) cannot have any positive maximum interior to the interval \((t_k, t_k + 1)\) for \( k = 1, 2, \ldots, p \).

Case 3. According to Case 2, if \( \sup_{0 \leq t \leq 1} y(t) > 0 \), then \( \sup_{0 \leq t \leq 1} y(t) = y(t^*_1) \), or \( \sup_{0 \leq t \leq 1} y(t) = y(t^*_1) = y(t_k) \), we only prove that \( \sup_{0 \leq t \leq 1} y(t) = y(t^*_k), \ k = 1, 2, \ldots, p \). Suppose that \( \sup_{0 \leq t \leq 1} y(t) = y(t^*_1) \), easily, \( y'(t_1) = y'(t^*_1) \geq 0 \).
From (2.1) and (2.3), we have
\[ x(t_k^+) - x(t_k) = x(t_k^+) - x(t_k^-) = I_1(m_{a,\beta}(t_k, x(t_k))) \]
\[ = I_1(\beta(t_k)) - \beta(t_k^-) = \beta(t_k^+) - \beta(t_k), \]
\[ x'(t_k^+) - x'(t_k^-) = I_1(m_{a,\beta}(t_k, x(t_k)), n(t_k, x'(t_k))) \]
\[ \geq I_1(\beta(t_k), \beta'(t_k)) \geq \beta'(t_k) - \beta(t_k). \]

Hence
\[ y(t_k^-) = x(t_k^+) - \beta(t_k^+) = x(t_k) - \beta(t_k) > 0, \]
\[ y'(t_k^-) = x'(t_k^+) - \beta'(t_k^+) \geq x'(t_k) - \beta'(t_k) \geq 0. \]

Suppose that \( y'(t_k^-) = 0 \) and \( y \) is nonincreasing on some interval \( (t_k, t_{k+1} + \gamma) \subset (t_k, t_{k+1}) \), where \( \gamma > 0 \) is sufficiently small such that \( y(t_k) > 0 \) on \( t \in (t_k, t_{k+1} + \gamma) \). For \( t \in (t_k, t_{k+1} + \gamma) \),
\[ x''(t) - \beta''(t) \geq -f\left(t, \beta(t), \frac{d}{dt} m_{a,\beta}(t, x)\right) - \frac{m_{a,\beta}(t, x) - x}{1 + (x - m_{a,\beta}(t, x))^2} + f\left(t, \beta(t), \beta'(t)\right) \]
\[ = \frac{m_{a,\beta}(t, x) - x}{1 + (x - m_{a,\beta}(t, x))^2} > 0, \]
which contradicts the assumption of monotonicity of \( y \). Thus, we obtain
\[ 0 < y(t_k^-) < y(t_k) = y(t_k), \]
\[ y'(t_k^-) = y'(t_k) \geq 0. \]

We use the preceding procedure and deduce by induction that
\[ y(t_k) > 0, \quad y'(t_k) \geq 0, \quad k = 1, 2, \ldots, p + 1, \]
which contradicts that \( y(1) \) cannot be the maximum point. By the same analysis, we can get that \( y(t_k) > 0, k = 2, 3, \ldots, p, \) cannot hold. \qed

**Lemma 2.4** \(-D \leq x'(t) \leq D \) on \( J \), where \( x(t) \) is the solution of (2.1).

**Proof** Here we only show \( x'(t) \leq D \). Suppose that there exists \( \rho \in (t_k, t_{k+1}) \) with \( x'(\rho) = \frac{x(t_{k+1}) - x(t_k)}{t_{k+1} - t_k} \), \( k \in \{1, 2, \ldots, p\} \), and as a result,
\[ -D < -\omega \leq \frac{\alpha(t_{k+1}) - \beta(t_k)}{t_{k+1} - t_k} \leq x'(\rho) \leq \frac{\beta(t_{k+1}) - \alpha(t_k)}{t_{k+1} - t_k} \leq \omega < D. \]

Therefore, there exist \( r_1, r_2 \in (t_k, t_{k+1}) \) such that \( x'(r_1) = \omega, x'(r_2) = D \) and either
\[ \omega \leq x'(t) \leq D, \quad t \in (r_1, r_2), \]
or
\[ \omega \leq x'(t) \leq D, \quad t \in (r_2, r_1). \]
We only consider the first case, since the other case can be handled similarly. It follows from the assumption that
\[ x''(t) \leq Ax'(t)\Psi(x'(t)) + B \quad \text{for} \ t \in (r_1, r_2). \]

This implies that
\[
\int_{r_1}^{r_2} \frac{x''(t)}{\Psi(x'(t))} \, dt \leq A \int_{r_1}^{r_2} x'(t) \, dt + B \int_{r_1}^{r_2} \Psi^{-1}(x'(t)) \, dt
\]
\[
= A(x(r_2) - x(r_1)) + B \int_{r_1}^{r_2} \Psi^{-1}(x'(t)) \, dt,
\]
which yields
\[
\int_{0}^{D} \frac{ds}{\Psi(s)} = \int_{x(r_1)}^{x(r_2)} \frac{ds}{\Psi(s)} \leq A(x(r_2) - x(r_1)) + B(r_2 - r_1) \max_{s \geq 0} \Psi^{-1}(s)
\]
\[
\leq A\left(\max_{t \in I} \beta(t) - \min_{t \in I} \alpha(t)\right) + B \max_{s \geq 0} \Psi^{-1}(s).
\]
This, obviously, contradicts the choice of \( D \). The proof is complete. \( \square \)

**Lemma 2.5** (Schauder’s fixed point theorem) *Let \( K \) be a convex subset of a normed linear space \( E \). Each continuous, compact map \( L : K \to K \) has a fixed point.*

### 3 Existence results

**Theorem 3.1** *Suppose that conditions \((H_1)-(H_3)\) hold. Then BVP (1.1) has at least one solution \( x \in E \cap C^2(J^*) \) such that*

\[ \alpha(t) \leq x(t) \leq \beta(t), \quad -D \leq x'(t) \leq D, \quad t \in [0,1]. \]

**Proof** Solving (2.1) is equivalent to finding \( x \in E \) which satisfies

\[
x(t) = \int_{0}^{1} H(t,s)F\left(s, x(s), \frac{d}{ds}m_{s,\beta}(s, x)\right) \, ds + \sum_{0 \leq t_k \leq t} \left[ I_k^x(x(t_k)) + (t - t_k)I_k^\gamma(x(t_k), x'(t_k))\right]
\]

\[
- t \sum_{k=1}^{p} \left[ I_k^x(x(t_k)) + (1 - t_k)I_k^\gamma(x(t_k), x'(t_k))\right] + \frac{1}{1 - \mu} \int_{0}^{1} \sum_{0 \leq t_k \leq s} \left[ I_k^x(x(t_k))\right] \, ds - \frac{\nu}{1 - \mu} \sum_{k=1}^{p} \left[ I_k^x(x(t_k)) + (1 - t_k)I_k^\gamma(x(t_k), x'(t_k))\right],
\]

where \( I_k^x(x(t_k)) = I_k(m_{s,\beta}(t_k, x(t_k))), I_k^\gamma(x(t_k), x'(t_k)) = J_k(m_{s,\beta}(t_k, x(t_k)), n(t_k, x'(t_k))). \)

Now, define the following operator \( T : E \to E \) by

\[
(Tx)(t) = \int_{0}^{1} H(t,s)F\left(s, x(s), \frac{d}{ds}m_{s,\beta}(s, x)\right) \, ds
\]

\[
+ \sum_{0 \leq t_k \leq t} \left[ I_k^x(x(t_k)) + (t - t_k)I_k^\gamma(x(t_k), x'(t_k))\right]
\]
which is a solution of BVP (2.1). And by Lemma 2.2 and Lemma 2.3, we know that \( T \) is a solution of BVP (1.1). The proof is complete.

Suppose that conditions (H1)-(H3) hold. Assume that there exist two lower solutions \( \alpha_1 \) and \( \alpha_2 \) and two upper solutions \( \beta_1 \) and \( \beta_2 \) for problem (1.1), satisfying the following:

(i) \( \alpha_1 \leq \alpha_2 \leq \beta_2 \);
(ii) \( \alpha_1 \leq \beta_1 \leq \beta_2 \);
(iii) \( \alpha_2 \not\leq \beta_1 \), which means that there exists \( t \in [0,1] \) such that \( \alpha_2 > \beta_1 \);
(iv) if \( x \) is a solution of (1.1) with \( x \geq \alpha_2 \), then \( x > \alpha_2 \) on (0,1);
(v) if \( x \) is a solution of (1.1) with \( x \leq \beta_1 \), then \( x < \beta_1 \) on (0,1).

If \( f \) satisfies the Nagumo condition with respect to \( \alpha_1 \), \( \alpha_2 \), then problem (1.1) has at least three solutions \( x_1 \), \( x_2 \) and \( x_3 \) satisfying

\[
\alpha_1 \leq x_1 \leq \alpha_2, \quad \alpha_2 \leq x_2 \leq \beta_2, \quad x_3 \not\leq \beta_1, \quad x_3 \not\geq \alpha_2.
\]

**Proof** We consider the following modified problem:

\[
x''(t) + F\left(t, x(t), \frac{d}{dt} m_{\alpha_1, \beta_2}(t, x)\right) = 0, \quad t \in J^*,
\]

\[
\Delta x(t_k) = \bar{I}_k(x(t_k)), \quad k = 1, 2, \ldots, p,
\]

\[
\Delta x'(t_k) = \bar{J}_k(x(t_k), x'(t_k)), \quad k = 1, 2, \ldots, p,
\]

\[
x(0) = x(1) = \int_0^1 g(s) x(s) \, ds,
\]

where \( \bar{I}_k(x(t_k)) = I_k(m_{\alpha_1, \beta_2}(t_k, x(t_k))), \bar{J}_k(x(t_k), x'(t_k)) = I_k(m_{\alpha_1, \beta_2}(t_k, x(t_k)), n(t_k, x'(t_k))). \)

Now define the following operator \( \overline{T} : E \to E \) by

\[
(\overline{T} x)(t) = \int_0^1 H(t, s) F\left(s, x(s), \frac{d}{ds} m_{\alpha_1, \beta_2}(s, x)\right) \, ds + \sum_{0 < t_k < t} \left[ \bar{I}_k(x(t_k)) + (t - t_k) \bar{J}_k(x(t_k), x'(t_k)) \right] \]

\[
- t \sum_{k=1}^p \left[ I_k(x(t_k)) + (1 - t_k) \bar{I}_k(x(t_k), x'(t_k)) \right] + \frac{1}{1 - \mu} \int_0^1 \sum_{0 < t_l < t} \left[ \bar{I}_k(x(t_k)) \right] \]

\[
+ (s - t_k) \bar{J}_k(x(t_k), x'(t_k)) \right] \, ds - \frac{\nu}{1 - \mu} \sum_{k=1}^p \left[ I_k(x(t_k)) + (1 - t_k) \bar{I}_k(x(t_k), x'(t_k)) \right].
\]
Since the functions \( m_{\alpha_1, \beta_2} \) and \( n \) are continuous and bounded, we obtain that there exists \( M_F > 0 \) such that
\[
|F(t, x(t), x'(t))| \leq M_F.
\]
Let
\[
M = \frac{2M_F}{1 - \mu} + \frac{3 + \nu - 2\mu}{1 - \mu} \sum_{k=1}^{\infty} \left[ \max_{x \in [-d, d]} |l_k(x)| + \max_{x, y \in [-d, d]} |I_k(x, y)| \right],
\]
where \( d = 1 + \max_{x \in \mathbb{R}} \{ \|\alpha_1\|_\infty, \|\beta_2\|_\infty \} \) and
\[
\Omega = \left\{ x \in E, \|x\|_\infty \leq M \right\}.
\]
It is standard that \( \mathcal{T} : E \rightarrow E \) is completely continuous. It is immediate from the argument above that \( \mathcal{T}(\Omega) \subset \Omega \).

Thus,
\[
\deg(I - \mathcal{T}, \Omega, 0) = 1.
\]
Let
\[
\Omega_{\alpha_2} = \left\{ x \in \Omega : x > \alpha_2 \text{ on } (0, 1) \right\}, \quad \Omega_{\beta_1} = \left\{ x \in \Omega : x < \beta_1 \text{ on } (0, 1) \right\}.
\]
Since \( \alpha_2 \leq \beta_1, \alpha_2 > -\overline{M}, \beta_1 < \overline{M} \) (i.e., choose \( \overline{M} \) such that \( \|\alpha_2\|_\infty, \|\beta_1\|_\infty < M \)). It follows that \( \Omega_{\beta_1} \neq \emptyset \neq \Omega_{\alpha_2} \), \( \Omega_{\beta_1} \cap \Omega_{\alpha_2} = \emptyset \), and \( \Omega \setminus (\Omega_{\beta_1} \cup \Omega_{\alpha_2}) \neq \emptyset \). By assumptions (iv) and (v), there are no solutions in \( \partial \Omega_{\beta_1} \cup \partial \Omega_{\alpha_2} \). Thus,
\[
\deg(I - \mathcal{T}, \Omega, 0) = \deg(I - \mathcal{T}, \Omega \setminus (\Omega_{\beta_1} \cup \Omega_{\alpha_2}), 0) + \deg(I - \mathcal{T}, \Omega_{\beta_1}, 0) + \deg(I - \mathcal{T}, \Omega_{\alpha_2}, 0).
\]
We show that \( \deg(I - \mathcal{T}, \Omega_{\beta_1}, 0) = \deg(I - \mathcal{T}, \Omega_{\alpha_2}, 0) = 1 \), then
\[
\deg(I - \mathcal{T}, \Omega \setminus (\Omega_{\beta_1} \cup \Omega_{\alpha_2}), 0) = -1,
\]
and there are solutions in \( \Omega_{\beta_1} \cup \Omega_{\alpha_2}, \) \( \Omega_{\beta_1}, \) \( \Omega_{\alpha_2}, \) as required.

We now show \( \deg(I - \mathcal{T}, \Omega_{\alpha_2}, 0) = 1 \). The proof that \( \deg(I - \mathcal{T}, \Omega_{\beta_1}, 0) = 1 \) is similar and hence omitted. We define \( I - \mathcal{W} \), the extension to \( \overline{\Omega} \) of the restriction of \( I - \mathcal{T} \) to \( \overline{\Omega}_{\alpha_2} \) as follows.

Let
\[
w(t, x, y) = f\left(t, m_{\alpha_2, \beta_2}(t, x), n(t, y)\right) + \frac{m_{\alpha_2, \beta_2}(t, x) - x}{1 + (x - m_{\alpha_2, \beta_2}(t, x))^2}.
\]
Thus, \( w \) is a continuous function on \( [0, 1] \times \mathbb{R}^2 \) and satisfies
\[
|w(t, x, y)| \leq A|y|\Psi(|y|) + B \quad \text{for } |y| < D,
\]
\[
|w(t, x, y)| \leq M_1 \quad \text{for } (t, x, y) \in [0, 1] \times \mathbb{R}^2,
\]
for some constants \( M_1 \). Moreover, we may choose \( M_1 \) so that \( \|\alpha_2\|_\infty, \|\beta_1\|_\infty < M_1 \).
Consider the following problem:

\[
x''(t) + w(t, x(t), \frac{d}{dt} m_{u_2, \beta_2}(t, x)) = 0, \quad t \in J^*,
\]

\[
\Delta x(t_k) = \bar{I}_k(x(t_k)), \quad k = 1, 2, \ldots, p,
\]

\[
\Delta x'(t_k) = \tilde{I}_k(x(t_k), x'(t_k)), \quad k = 1, 2, \ldots, p,
\]

\[
x(0) = x(1) = \int_0^1 g(s)x(s) \, ds,
\]

where \( \bar{I}_k(x(t_k)) = I_k(m_{u_2, \beta_2}(t_k, x(t_k))), \tilde{I}_k(x(t_k), x'(t_k)) = I_k(m_{u_2, \beta_2}(t_k, x(t_k)), n(t_k, x'(t_k))). \) Now define the following operator:

\[
(Wx)(t) = \int_0^1 H(t, s)w(s, x(s), \frac{d}{ds} m_{u_2, \beta_2}(s, x)) \, ds \\
+ \sum_{0< t_k < t} \left[ \bar{I}_k(x(t_k)) + (t - t_k)\tilde{I}_k(x(t_k), x'(t_k)) \right] \\
- \sum_{k=1}^p \left[ \bar{I}_k(x(t_k)) + (1 - t_k)\tilde{I}_k(x(t_k), x'(t_k)) \right] \\
+ \frac{1}{1 - \mu} \int_0^1 \sum_{0< t_k < t} \left[ \bar{I}_k(x(t_k)) + (s - t_k)\tilde{I}_k(x(t_k), x'(t_k)) \right] \, ds \\
- \frac{v}{1 - \mu} \sum_{k=1}^p \left[ \bar{I}_k(x(t_k)) + (1 - t_k)\tilde{I}_k(x(t_k), x'(t_k)) \right].
\]

Again, it is easy to check that \( x \) is a solution of (3.3) if \( x \in \Omega_{\alpha_2} \) and \( Wx = x \) (note that \( W : E \rightarrow E \) is compact). Thus, \( \deg(I - T, \Omega_{\alpha_2}, 0) = 0 \). Moreover, it is easy to see that \( W(\Omega) \subseteq \Omega \). By assumptions (iv) and (v), there are no solutions in \( \partial \Omega_{\alpha_2} \cup \partial \Omega_{u_2} \). So,

\[
\deg(I - T, \Omega_{\alpha_2}, 0) = \deg(I - W, \Omega \setminus \overline{\Omega}_{\alpha_2}, 0) + \deg(I - W, \Omega_{\alpha_2}, 0) = \deg(I - W, \Omega, 0) = 1.
\]

Thus there are three solutions, as required. The proof is complete. \( \square \)

**Theorem 3.3** Suppose that conditions (H1)-(H3) hold. Assume that there exist two lower solutions \( \alpha_1 \) and \( \alpha_2 \) and two upper solutions \( \beta_1 \) and \( \beta_2 \) for problem (1.1), satisfying

(i) \( \alpha_1 \leq \alpha_2 \leq \beta_2 \);

(ii) \( \alpha_1 \leq \beta_1 \leq \beta_2 \);

(iii) \( \alpha_2 \not< \beta_1 \);

(iv) there exists \( 0 < \bar{\epsilon} < \min_{\epsilon \in [0, 1]} [\alpha_2(t) - \alpha_1(t), \beta_2(t) - \beta_1(t)] \) such that all \( \epsilon \in (0, \bar{\epsilon}] \), the functions \( \alpha_2 - \epsilon \) and \( \beta_1 + \epsilon \) are, respectively, lower and upper solutions of (1.1);

(v) \( \alpha_2 - \epsilon \not< \beta_1 + \bar{\epsilon} \).

If \( f \) satisfies the Nagumo condition with respect to \( \alpha_1, \beta_2 \), then problem (1.1) has at least three solutions \( x_1, x_2, \) and \( x_3 \) satisfying

\[
\alpha_1 \leq x_1 \leq \beta_1, \quad \alpha_2 \leq x_2 \leq \beta_2, \quad x_3 \not< \beta_1, \quad x_3 \not> \alpha_2.
\]
Proof In the proof of Theorem 3.3, define
\[
\Omega_{\alpha_2} = \{ u \in \Omega : u > \alpha_2 - \bar{\epsilon} \text{ on } (0, 1) \},
\]
\[
\Omega_{\beta_1} = \{ u \in \Omega : u < \beta_1 + \bar{\epsilon} \text{ on } (0, 1) \},
\]
where $\Omega$ is defined in Theorem 3.2.

Competing interests
The authors declare that they have no competing interests.

Authors' contributions
The authors declare that the study was realized in collaboration with the same responsibility. All authors read and approved the final manuscript.

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