GEOMETRIC QUANTIZATION OF THE MODULI SPACE OF THE SELF-DUALITY EQUATIONS ON A RIEMANN SURFACE

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Abstract. The self-duality equations on a Riemann surface arise as dimensional reduction of self-dual Yang-Mills equations. Hitchin had showed that the moduli space \( \mathcal{M} \) of solutions of the self-duality equations on a compact Riemann surface of genus \( g > 1 \) has a hyperKähler structure. In particular \( \mathcal{M} \) is a symplectic manifold. In this paper we elaborate on one of the symplectic structures, the details of which is missing in Hitchin’s paper. Next we apply Quillen’s determinant line bundle construction to show that \( \mathcal{M} \) admits a prequantum line bundle. The Quillen curvature is shown to be proportional to the symplectic form mentioned above. We do it in two ways, one of them is a bit unnatural (published in R.O.M.P.) and a second way which is more natural.

Keywords: Geometric quantization, Quillen determinant line bundle, moment map.

1. Introduction

Geometric prequantization is a construction of a Hilbert space, namely the square integrable sections of a prequantum line bundle on a symplectic manifold \( (\mathcal{M}, \Omega) \) and a correspondence between functions on \( \mathcal{M} \) (the classical observables are functions on the phase space \( \mathcal{M} \)) and operators on the Hilbert space such that the Poisson bracket of two functions corresponds to the commutator of the operators. The latter is ensured by the fact that the curvature of the prequantum line bundle is precisely the symplectic form \( \Omega \) \([19]\). Let \( f \in C^\infty(\mathcal{M}) \). Let \( X_f \) be the vector field defined by \( \Omega(X_f, \cdot) = -df \). Let \( \theta \) be the symplectic potential corresponding to \( \Omega \). Then we can define the operator corresponding to the function \( f \) to be \( \hat{f} = -ih[X_f - \frac{i}{2} \theta(X_f)] + f \). Then if \( f_1, f_2 \in C^\infty(\mathcal{M}) \) and \( f_3 = \{f_1, f_2\} \), Poisson bracket of the two, then \([\hat{f}_1, \hat{f}_2] = -ih\hat{f}_3\).

A relevant example would be geometric quantization of the moduli space of flat connections. The moduli space of flat connections of a principal \( G \)-bundle on a Riemann surface has been quantized by Witten by a construction of the determinant line bundle of the Cauchy-Riemann operator, namely, \( \mathcal{L} = \wedge^{\text{top}}(\text{Ker} \bar{\partial}_A)^* \otimes \wedge^{\text{top}}(\text{Coker} \bar{\partial}_A) \), \([17]\). It carries the Quillen metric such that the canonical unitary connection has a curvature form which coincides with the natural Kähler form on the moduli space of flat connections on vector bundles over the Riemann surface of a given rank \([14]\).

Inspired by \([2]\), applying Quillen’s determinant line bundle construction we construct prequantum line bundles on the moduli space of solutions to the vortex equations \([7]\) and the moduli space the self-duality equations over a Riemann surface.
which is a hyperKähler manifold [10]. In this paper we quantize one of the symplectic forms in two ways. In [6] we show the quantization of the full hyperKähler structure.

The self-duality equations on a Riemann surface arise from dimensional reduction of self-dual Yang-Mills equations from 4 to 2 dimensions [10]. They have been studied extensively in [10]. They are as follows. Let $M$ be a compact Riemann surface of genus $g > 1$ and let $P$ be a principal unitary $U(n)$-bundle over $M$. Let $A$ be a unitary connection on $P$, i.e. $A = A^{(1,0)} + A^{(0,1)}$ such that $A^{(1,0)} = -A^{(0,1)*}$, where $*$ denotes conjugate transpose [8], [13]. Let $\Phi$ be a complex Higgs field, $\Phi \in \mathcal{H} = \Omega^{1,0}(M; \text{ad}P \otimes \mathbb{C})$. The pair $(A, \Phi)$ will be said to satisfy the self-duality equations if

\begin{align*}
(1) & \quad F = -[\Phi, \Phi^*], \\
(2) & \quad d''A\Phi = 0.
\end{align*}

Here $\Phi^* = \phi^*d\bar{z}$ where $\phi^*$ is taking conjugate transpose of the matrix of $\phi$. Let the solution space to (1) $-$ (2) be denoted by $S$. There is a gauge group acting on the space of $(A, \Phi)$ which leave the equations invariant. If $g$ is an $U(n)$ gauge transformation then $(A_1, \Phi_1)$ and $(A_2, \Phi_2)$ are gauge equivalent if $dA_2g = gdA_1$ and $\Phi_2g = g\Phi_1$ [10], page 69. Taking the quotient by the gauge group of the solution space to (1) and (2) gives the moduli space of solutions to these equations and is denoted by $\mathcal{M}$. Hitchin shows that there is a natural metric on the moduli space $\mathcal{M}$ and further proves that the metric is hyperKähler [10].

In the next section, we will elaborate on the natural metric and one of the symplectic forms, explicit mention of which is missing in [10]. In the section after that we will construct a determinant line bundle on the moduli space $\mathcal{M}$ such that its Quillen curvature is precisely this symplectic form. This will put us in the context of geometric quantization.

We will do this in two different ways $-$ the first one using $\bar{\partial} + A_0^{(1,0)} + \Phi^{(1,0)}$ which gauge transforms like $\bar{g}(\bar{\partial} + A_0^{(1,0)} + \Phi^{(1,0)})\bar{g}^{-1}$. This is a bit unnatural since the action is with $\bar{g}$. (This approach was published in R.O.M.P.)

In the next section we will do it using $\bar{\partial} + A_0^{(0,1)} + \Phi^{(0,1)}$ which gauge transforms like $g(\bar{\partial} + A_0^{(0,1)} + \Phi^{(0,1)})g^{-1}$ which is more natural.

In the end we discuss the holomorphicity of the prequantum line bundle w.r.t. the first complex structure.

In the paper where we construct prequantum line bundles for the full hyperKähler structure of the moduli space [6] we mention the second approach to quantizing the first symplectic form.

Papers which may be of interest in this context are [3], [4], [11], [16] [18]. These papers use algebraic geometry and algebraic topology and may provide alternative methods to quantizing the hyperKähler system. Our method, in contrast, is very elementary and we explicitly construct the prequantum line bundles. The only machinery we use is Quillen’s construction of the determinant line bundle. [14].

Note: After writing the paper the author found Kapustin and Witten’s paper [12] where they have applied Beilinson and Drinfeld’s quantization of the Hitchin system to study the geometric Langlands programme.
2. SYMPLECTIC STRUCTURE OF THE MODULI SPACE

This section is an elaboration of what is implicit in [10]. Following the ideas in [10] we give a proof that \( \Omega \) below is a symplectic form on \( \mathcal{M} \).

Let the configuration space be defined as \( \mathcal{C} = \{ (A, \Phi) | A \in \mathcal{A}, \Phi \in \mathcal{H} \} \) where \( \mathcal{A} \) is the space of unitary connections on \( P \) and \( \mathcal{H} = \Omega^{(1,0)}(M, \text{ad}P \otimes \mathbb{C}) \) is the space of Higgs field. Unitary connections satisfy \( A = A^{(1,0)} + A^{(0,1)} \) where \( A^{(1,0)} = -A^{(0,1)} \), where * is conjugate transpose.

Let us define a metric on the complex configuration space

\[
g((\alpha, \gamma^{(1,0)}), (\beta, \delta^{(1,0)})) = -\text{Tr} \int_M (\alpha \wedge \ast_1 \beta) - 2\text{ImTr} \int_M (\gamma^{(1,0)} \wedge \ast_2 \delta^{(1,0)tr})
\]

where \( \alpha, \beta \in T_A \mathcal{A} = \{ \alpha \in \Omega^1(M, \text{ad}P) | \alpha^{(1,0)*} = -\alpha^{(0,1)} \} \) and \( \gamma^{(1,0)}, \delta^{(1,0)} \in T_\Phi \mathcal{H} = \Omega^{1,0}(M; \text{ad}P \otimes \mathbb{C}) \).

Here the superscript \( \text{tr} \) stands for transpose in the Lie algebra of \( U(n) \), \( \ast_1 \) denotes the Hodge star taking \( dx \) forms to \( dy \) forms and \( dy \) forms to \( -dx \) forms (i.e., \( \ast_1(\eta dz) = -i\eta dz \) and \( \ast_1(\eta d\bar{z}) = i\eta d\bar{z} \) and \( \ast_2 \) denotes the operation (another Hodge star), such that \( \ast_2(\eta dz) = \bar{\eta}dz \) and \( \ast_2(\eta d\bar{z}) = -\bar{\eta}d\bar{z} \).

We check that this coincides with the metric on the moduli space \( \mathcal{M} \) given by [10], page 79 and page 88. Hitchin identifies \( A \) with its \( A^{(0,1)} \) part. (Since it is a unitary connection, \( A^{(1,0)} \) is determined by \( A^{(0,1)} \) by the formula \( A^{(1,0)} = -A^{(0,1)*} \). This is different from our point of view where in the connection part we keep both \( A^{(1,0)} \) and \( A^{(0,1)} \) even though they are related. On \( T_{(\mathcal{A}, \Phi)} \mathcal{C} = T_A \mathcal{A} \times T_\Phi \mathcal{H} \) which is \( \Omega^{(0,1)}(M, \text{ad}P \otimes \mathbb{C}) \times \Omega^{(1,0)}(M, \text{ad}P \otimes \mathbb{C}) \) for him, Hitchin defines a metric \( g_1 \) such that \( g_1((\alpha^{(0,1)}, \gamma^{(1,0)}), (\alpha^{(0,1)}, \gamma^{(1,0)})) = 2i\text{Tr} \int_M (\alpha^{(0,1)*} \wedge \alpha^{(0,1)}) + 2i\text{Tr} \int_M (\gamma^{(1,0)} \wedge \gamma^{(1,0)*}) \ast \) denotes conjugate transpose as usual. Let \( \gamma^{(1,0)} = cdz, \) where \( c \) is a matrix.

On \( T_{(\mathcal{A}, \Phi)} \mathcal{C} \), our metric

\[
g((\alpha, \gamma^{(1,0)}), (\alpha, \gamma^{(1,0)}))
\]

\[
= -\text{Tr} \int_M (\alpha \wedge \ast_1 \alpha) - 2\text{ImTr} \int_M (\gamma^{(1,0)} \wedge \ast_2 \gamma^{(1,0)tr})
\]

\[
= -\text{Tr} \int_M (\alpha^{(1,0)} + \alpha^{(0,1)}) \wedge (-i\alpha^{(1,0)} + i\alpha^{(0,1)})
\]

\[
-2\text{ImTr} \int_M (c dz \wedge c^* \bar{d}z)
\]

\[
= 2i\text{Tr} \int_M (\alpha^{(0,1)} \wedge \alpha^{(1,0)}) - 2Im \int_M (-2i)\text{Tr}(c c^*)dx \wedge dy
\]

\[
= -2i\text{Tr} \int_M (\alpha^{(0,1)} \wedge \alpha^{(0,1)*}) + 4 \int_M \text{Re}(\text{Tr}(c c^*))dx \wedge dy
\]

\[
= 2i\text{Tr} \int_M (\alpha^{(0,1)*} \wedge \alpha^{(0,1)}) + 2i \int_M (-2i)\text{Tr}(c c^*) dx \wedge dy
\]

\[
= 2i\text{Tr} \int_M (\alpha^{(0,1)*} \wedge \alpha^{(0,1)}) + 2i \int_M \text{Tr}(c c^*) dz \wedge d\bar{z}
\]

\[
= 2i\text{Tr} \int_M (\alpha^{(0,1)*} \wedge \alpha^{(0,1)}) + 2i \int_M \gamma^{(1,0)} \wedge \gamma^{(1,0)*}
\]
where we have used the fact that $\alpha^{(1,0)} = -\alpha^{(0,1)}$ and that $\text{Tr}(cc^*)$ is real. Thus we get the same metric as Hitchin does.

The symmetry as $\alpha$ is interchanged with $\beta$ and $\gamma$ is interchanged with $\delta$ is as follows. In the first term $\alpha \wedge s_1 \beta = (\alpha^{(1,0)} + \alpha^{(0,1)}) \wedge (s_1 \beta^{(1,0)} + s_1 \beta^{(0,1)}) = i(\alpha^{(1,0)} \wedge \beta^{(0,1)}) - i(\alpha^{(0,1)} \wedge \beta^{(1,0)})$. It is easy to check that $\beta \wedge s_1 \alpha$ is also exactly the same. In the second term, first note that $\text{Re} \text{Tr}(AB^*) = \text{Re} \text{Tr}(BA^*)$ for matrices $A,B$. But we have matrix valued one forms. Thus if $\gamma^{(1,0)} = \text{Ad}_z, \delta^{(1,0)} = \text{Bdz}$, $A,B$ are matrices, then $*_2 \delta^{(1,0)} tr = B^* d\bar{z}$. Then the second term in the metric is the integral of $-2i\text{Im} \text{Tr}(\text{Ad}_z B^* d\bar{z}) = -2i\text{Im} \text{Tr}(AB^* dz \wedge d\bar{z}) = -2i\text{Im} \text{Tr}(AB^*)(-2i) dx \wedge dy = -2 \text{Re} \text{Tr}(AB^*)(-2) dx \wedge dy$ Interchanging $\gamma$ with $\delta$ amounts to interchanging $A$ and $B$ which does not change the term.

There is an almost complex structure on $\mathcal{C}$, namely, $\mathcal{I} = \begin{bmatrix} s_1 & 0 \\ 0 & i \end{bmatrix}$. Thus $\mathcal{I}(\beta, \delta^{(0,1)}) = (s_1 \beta, i\delta^{(0,1)})$.

(If one identifies $T_A \mathcal{A}$ with $\Omega^{(0,1)}(M, \text{ad} P \otimes \mathbb{C})$ as Hitchin does, then $\mathcal{I} = \begin{bmatrix} i & 0 \\ 0 & i \end{bmatrix}$ i.e., $\mathcal{I}(\beta^{(0,1)}, \delta^{(0,1)}) = (i\beta^{(0,1)}, i\delta^{(0,1)})$. This is the viewpoint we will take in a paper sequel to this one. The metric and the symplectic forms will remain the same.)

We can define a symplectic form

$$\Omega((\alpha, \gamma^{(1,0)}), (\beta, \delta^{(1,0)})) = g((\alpha, \gamma^{(1,0)}), \mathcal{I}(\beta, \delta^{(1,0)}))$$

Then

$$\Omega((\alpha, \gamma^{(1,0)}), (\beta, \delta^{(1,0)})) = \text{Tr} \int_M (\alpha \wedge \beta) - 2i\text{Im} \text{Tr} \int_M (\gamma^{(1,0)} \wedge *_2 (i\delta^{(1,0)\text{tr}})) = \text{Tr} \int_M (\alpha \wedge \beta) - 2i\text{Im} \text{Tr} \int_M ((-i)\gamma^{(1,0)} \wedge *_2 (\delta^{(1,0)\text{tr}})) = \text{Tr} \int_M (\alpha \wedge \beta) + 2\text{Re} \text{Tr} \int_M (\gamma^{(1,0)} \wedge \delta^{(1,0)*}) = \text{Tr} \int_M (\alpha \wedge \beta) - \text{Tr} \int_M (\gamma \wedge \delta)$$

This is because $*_2 i^* = -1$, $*_2 (i\delta^{(1,0)\text{tr}}) = -i*_2 (\delta^{(1,0)\text{tr}})$, $\text{Re}(-iz) = \text{Im} z$ and finally $*_2 (\delta^{(1,0)\text{tr}}) = \delta^{(1,0)*}$. Note that the first term of this symplectic form appears also in [11], page 587.

We will show by a moment map construction that this form descends to a symplectic form on the moduli space $\mathcal{M}$.

We need to find out the vector field generated by the action of the gauge group on $\mathcal{C}$ and hence on $\mathcal{S}$ the solution space to (1) - (2). Let $g = e^{\epsilon \zeta}$, $\zeta \in u(n)$, the Lie algebra of the gauge group. i.e. $\zeta^* = -\zeta$. Under the action of the gauge group connection $A \rightarrow A_g = gAg^{-1} + gdg^{-1}$ and $\Phi \rightarrow \Phi_g = g\Phi g^{-1}$. (The action on $A$ can be derived as follows: $(d + A_g) = g(d + A)g^{-1}$, by [10]. Thus, $(d + A_g)s = gd(g^{-1}s) + gAg^{-1}s = ds + gd(g^{-1}s) + gAg^{-1}s$. Thus $A_g = gAg^{-1} + gdg^{-1}$.) Taking $\epsilon$ to be very small, we write $g = 1 + \epsilon \zeta$ and $g^{-1} = 1 - \epsilon \zeta$ up to first order in $\epsilon$. Thus $A \rightarrow A - \epsilon(d\zeta - [\zeta, A])$ Similarly, $\Phi \rightarrow \Phi + \epsilon[\zeta, \Phi]$. Thus on $\mathcal{C}$ there is a vector field generated by the gauge action given by $X_\zeta = (X_1, X_2) = (d\zeta - [\zeta, A]), [\zeta, \Phi])$.

Define a moment map $\mu : \mathcal{C} \rightarrow u(n)^*$ as follows:

$$\mu(A, \Phi) = (F(A) + [\Phi, \Phi^*]).$$
Given $\zeta$ as before, define the Hamiltonian to be

$$H_\zeta = \text{Tr} \int_M (F(A) + [\Phi, \Phi^*])\zeta.$$ 

Now $F(A) = dA + A \wedge A$. Thus $F' = \lim_{t \to 0} \frac{F(A+\beta t) - F(A)}{t} = d\beta + [\beta, A]$ where $\beta \in T_A A$. (Note that for adP valued $p, q$ forms $[\omega^p, \omega^q] = (-1)^{pq+1}[\omega^q, \omega^p]$, [1], page 546.) Let

$$h_\zeta = \text{Tr} \int_M F(A)\zeta = \text{Tr} \int_M \zeta F(A)$$

(since $\text{Tr}(AB) = \text{Tr}(BA)$). Now for $u, v, w \in \Omega^* (M, \text{ad} P)$, $[u, v] \wedge w = u \wedge [v, w]$, [1], page 546. Thus $\text{Tr}(\zeta \wedge [\beta, A]) = \text{Tr}([\zeta, \beta] \wedge A) = \text{Tr}(-A \wedge [\zeta, \beta]) = \text{Tr}(-[A, \zeta] \wedge \beta)$. Thus if $\beta \in T_A A = \Omega^1(M, \text{ad} P)$,

$$dh_\zeta(\beta) = \text{Tr} \int_M \zeta (d\beta + [\beta, A])$$

$$= \text{Tr} \int_M (-d\zeta \wedge \beta + [\beta, A])$$

$$= \text{Tr} \int_M (- (d\zeta - [\zeta, A]) \wedge \beta)$$

$$= \text{Tr} \int_M (X_1 \wedge \beta). \quad (4)$$

Let

$$f_\zeta(\Phi) = \text{Tr} \int_M ([\Phi, \Phi^*])\zeta,$$

and $\delta^{(1,0)} \in \Omega^{(1,0)} (M, \text{ad} P \otimes \mathbb{C})$, the tangent space to $\mathcal{H}$. Let $\delta^{(1,0)} = edz, \Phi = \phi dz, \Phi^* = \phi^* d\bar{z}$, $\delta^{(1,0)*} = e^* d\bar{z}$, where $e$ and $\phi$ are matrices.

$$df_\zeta(\delta^{(1,0)}) = \text{Tr} \int_M ([\delta^{(1,0)}, \Phi^*])\zeta + [\Phi, \delta^{(1,0)*}]\zeta$$

$$= 2\text{Re} \text{Tr} \int_M ([\zeta, \Phi] \wedge \delta^{(1,0)*})$$

$$= 2\text{Re} \text{Tr} \int_M (X_2 \wedge \delta^{(1,0)*}). \quad (5)$$

This follows from the fact that

$$2\text{Re}(\text{Tr}([\zeta, \Phi] \wedge \delta^{(1,0)*})) = 2\text{Re}(\text{Tr}([\zeta, \phi] e^*) dz \wedge d\bar{z})$$

$$= 2\text{Re}(\text{Tr}[\zeta, \phi] e^*(-2idx \wedge dy))$$

$$= 4\text{Im}(\text{Tr}([\zeta, \phi] e^*)) dx \wedge dy$$

Now,

$$\text{Im} \text{Tr}([\zeta, \phi] e^*) = \text{Tr}([\zeta, \phi] e^* - e[\zeta, \phi^*]) / 2i$$

$$= \text{Tr}([\zeta, \phi] e^* + e[\phi^*, \zeta]) / 2i$$

$$= \text{Tr}([\phi, e^*] \zeta + [e, \phi^*] \zeta) / 2i.$$ 

Here we have used the fact that $\zeta^* = -\zeta$ since $\zeta \in \mathfrak{u}(n)$ and $\text{Tr}([A, B]C) = \text{Tr}([B, C]A)$. 


Thus,\[4\text{Im} \text{Tr}([\zeta, \phi] e^*) dx \wedge dy = \text{Tr}([\phi, e^*] \zeta + [e, \phi^*] \zeta) dz \wedge d\bar{\zeta} = \text{Tr}([\Phi, \delta^{(1,0)*}] \zeta + [\delta^{(1,0)}, \Phi^*] \zeta)\]

Thus (5) follows.

From (4) and (5) it follows that \(dH_\zeta((\beta, \delta^{(1,0)})) = dh_\zeta(\beta) + df_\zeta(\delta^{(1,0)})\)
\[
= \text{Tr} \int_M (X_1 \wedge \beta) + 2\text{Re} \text{Tr} \int_M (X_2 \wedge \delta^{(1,0)*})
= \Omega(X_\zeta, (\beta, \delta^{(1,0)}))
\]

Therefore,
\[dH_\zeta(Y) = \Omega(X_\zeta, Y)\]

Thus the gauge group action on \(\mathcal{C}\) is Hamiltonian and arises from a moment map.

**Lemma 2.1.** \(\Omega\) is a symplectic form on \(\mathcal{M}\).

**Proof.** We saw that equation (1) is a moment map \(\mu = 0\).

About equation (2) it is easy to check that \(\mathcal{I}X\) satisfies the linearization of equation (2) iff \(X\) satisfies it. (This basically is as follows. Eq(2) is \((\bar{\delta} + A^{(0,1)})\Phi = 0\).

Its linearization is \(\partial \delta^{(1,0)} + \alpha^{(0,1)} \wedge \Phi + A^{(0,1)} \wedge \gamma^{(1,0)} = 0\), with \(X = (\alpha, \gamma^{(1,0)}) \in T_{(A, \Phi)}\mathcal{C}\) as before. \(\mathcal{I}X = (-i\alpha^{(1,0)}, i\alpha^{(0,1)}, i\gamma^{(1,0)})\). Since in \(\mathcal{I}X\) both \(\alpha^{(0,1)}\) and \(\gamma^{(1,0)}\) just get multiplied by \(i\), linearization of eq(2) is satisfied by \(\mathcal{I}X\) iff it is satisfied by \(X\).

If \(S\) be the solution space to equation (1) and (2) and \(X \in T_p S\) then \(\mathcal{I}X \in T_p S\) iff \(X \in T_p S\) is orthogonal to the gauge orbit \(O_p = G \cdot p\). The reason is as follows. We let \(X_\zeta \in T_p O_p\), \(g(X_\zeta, X) = \Omega(X_\zeta, \mathcal{I}X) = \text{Tr} \int_M \zeta \cdot d\mu(\mathcal{I}X)\), and therefore \(\mathcal{I}X\) satisfies the linearization of equation (1) iff \(d\mu(\mathcal{I}X) = 0\) iff \(g(X_\zeta, X) = 0\) for all \(\zeta\).

Similarly, \(g(X_\zeta, \mathcal{I}X) = -\Omega(X_\zeta, X) = -\text{Tr} \int_M \zeta d\mu(X)\). Thus \(X \in T_p S\) implies \(\mathcal{I}X\) is orthogonal to \(X_\zeta\) for all \(\zeta\). Thus \(X \in T_p S\) and \(X\) orthogonal to \(X_\zeta\) implies that \(\mathcal{I}X \in T_p S\) and \(\mathcal{I}X\) is orthogonal to \(X_\zeta\).

Now we are ready to show that \(\mathcal{M}\) has a natural symplectic structure and an almost complex structure compatible with the symplectic form \(\Omega\) and the metric \(g\).

First we show that the almost complex structure descends to \(\mathcal{M}\). Then using this and the symplectic quotient construction we will show that \(\Omega\) gives a symplectic structure on \(\mathcal{M}\). To show that \(\mathcal{I}\) descends as an almost complex structure we let \(\text{pr} : S \to S/G = \mathcal{M}\) be the projection map and set \([p] = \text{pr}(p)\). Then we can naturally identify \(T_{[p]} \mathcal{M}\) with the quotient space \(T_p S/T_p O_p\), where \(O_p = G \cdot p\) is the gauge orbit. Using the metric \(g\) on \(S\) we can realize \(T_{[p]} \mathcal{M}\) as a subspace in \(T_p S\) orthogonal to \(T_p O_p\). Then by what is said before, this subspace is invariant under \(\mathcal{I}\). Thus \(T_{[p]} \mathcal{M}\) descends to \(\mathcal{M}\) and \(\mathcal{I}\) is \(G\)-invariant.

The symplectic structure \(\Omega\) descends to \(\mu^{-1}(0)/G\), (by what we said before and by the Marsden-Wiener symplectic quotient construction [9], [10]) since the leaves of the characteristic foliation are the gauge orbits. Now, as a 2-form \(\Omega\) descends to \(\mathcal{M}\) and so does the metric \(g\). We check that equation (2) does not give rise to new degeneracy of \(\Omega\) (i.e. the only degeneracy of \(\Omega\) is due to (1) but along gauge orbits). Thus \(\Omega\) is symplectic on \(\mathcal{M}\). \(\square\)
3. Prequantum line bundles

A very clear description of the determinant line bundle can be found in [3] and [14]. Here we mention the formula for the Quillen curvature of the determinant line bundle $\Lambda^\top(Ker\partial_A)^* \otimes \Lambda^\top(Coker\partial_A) = \det(\partial_A)$, where $\partial_A = \partial + A^{(0,1)}$, given the canonical unitary connection $\nabla_Q$, induced by the Quillen metric [14]. Namely, recall that the affine space $A$ (notation as in [14]) is an infinite-dimensional Kähler manifold. Here each connection is identified with its $(0,1)$ part. Since the total connection is unitary (i.e. of the form $A = A^{(1,0)} + A^{(0,1)}$, where $A^{(1,0)} = -A^{(0,1)*}$) this identification is easy. In fact, for every $A \in A$, $T_A(A) = \Omega^{0,1}(M, adP)$ and the corresponding Kähler form is given by

$$F(\alpha, \beta) = \text{Re} \text{Tr} \int_M (\alpha^{(0,1)} \wedge \beta^{(0,1)*}) = -\text{Re} \text{Tr} \int_M (\alpha^{(0,1)} \wedge \beta^{(1,0)})$$

where $\alpha^{(0,1)}, \beta^{(0,1)} \in \Omega^{0,1}(M, adP)$, and $\beta^{(1,0)} = -\beta^{(0,1)*}$. It is skew symmetric if you interchange $\alpha^{(0,1)} = Ad\bar{z}$ and $\beta^{(0,1)} = Bd\bar{z}$ (follows from the fact that $\text{Im}(\text{Tr}(AB^*)) = -\text{Im}(\text{Tr}(BA^*))$ for matrices $A$ and $B$, using once again $d\bar{z} \wedge dz$ is imaginary). Let $\alpha = \alpha^{(0,1)} + \alpha^{(1,0)}$, $\beta = \beta^{(0,1)} + \beta^{(1,0)}$. It is clear from the fact that $\alpha^{(1,0)} = -\alpha^{(0,1)*}$ and $\beta^{(1,0)} = -\beta^{(0,1)*}$ that

$$F(\alpha, \beta) = -\frac{1}{2} \text{Tr} \int_M \alpha \wedge \beta.$$

Here we have used the fact that

$$2\text{Re} \text{Tr} \int_M \alpha^{(0,1)} \wedge \beta^{(1,0)}$$

$$= \text{Tr} \int_M \alpha^{(0,1)} \wedge \beta^{(1,0)} + \text{Tr} \int_M \overline{\alpha^{(0,1)}} \wedge \overline{\beta^{(1,0)}}$$

$$= \text{Tr} \int_M \alpha^{(0,1)} \wedge \beta^{(1,0)} + \text{Tr} \int_M (-\overline{\alpha^{(0,1)}}) \wedge (-\overline{\beta^{(1,0)}})$$

$$= \text{Tr} \int_M \alpha^{(0,1)} \wedge \beta^{(1,0)} + \text{Tr} \int_M \alpha^{(1,0)} \wedge \beta^{(0,1)}$$

$$= \text{Tr} \int_M \alpha \wedge \beta.$$

Then one has $F(\nabla_Q) = \frac{1}{2} F$.

3.1. Prequantization of the moduli space $\mathcal{M}$. In this section we show that $\mathcal{M}$ admits a prequantum line bundle, i.e. a line bundle whose curvature is the symplectic form $\Omega$.

**Theorem 3.1.** The moduli space $\mathcal{M}$ of solutions to (1) and (2) admits a prequantum line bundle $P$ whose Quillen curvature $F = \frac{i}{2} \Omega$ where $\Omega$ is the natural symplectic form on $\mathcal{M}$ as in (3).

First we note that to the connection $A$ we can add any one form and still obtain a derivative operator. To a connection $A_0$ whose gauge equivalence class is fixed, we will add $\Phi$ to obtain new connections which will appear in the Cauchy-Riemann derivative operators.

**Definitions:** Let us denote by $L = \det(\partial + A^{(0,1)})$ a determinant bundle on $\mathcal{A}$.

Let $R = \det(\partial + A_0^{(1,0)} + \Phi^{(1,0)})$ where $A_0$ is a connection whose gauge equivalence class is fixed, i.e. $A_0$ is allowed to change only in the gauge direction. The $R$ is a line...
bundle on \( C \) defined such that the fiber on \((A, \Phi)\) is that of \( \text{det}(\overline{\partial} + A_0^{(1,0)} + \Phi^{(1,0)})\), but the fiber over gauge equivalent \((A_g, \Phi_g)\) is that of \( \text{det}(\overline{\partial} + A_0^{(1,0)} + \Phi^{(1,0)})\overline{g}^{-1}\). This is because the gauge group \( G \) acts on all of \( A \) simultaneously, so that when \( A \to A_g, A_0 \to A_{0g} \).

(In the next section we have a better approach where we donot have to deal with \( \overline{g} \)).

Let \( P = \mathcal{L}^{-2} \oplus \mathbb{R}^2 \) denote a line bundle over \( C \). We will show that this line bundle is well defined on \( M \) and has Quillen curvature a constant multiple of the symplectic form \( \Omega \).

**Lemma 3.2.** \( P \) is a well-defined line bundle over \( M \subset C/G \), where \( G \) is the gauge group.

**Proof.** First consider the Cauchy-Riemann operator \( D = \overline{\partial} + A^{(0,1)} \). Under gauge transformation \( D = \overline{\partial} + A^{(0,1)} \to D_g = g(\overline{\partial} + A^{(0,1)})g^{-1} \). We can show that the operators \( D \) and \( D_g \) have isomorphic kernel and cokernel and their corresponding Laplacians have the same spectrum and the eigenspaces are of the same dimension. Let \( \Delta \) denote the Laplacian corresponding to \( D \), and \( \Delta_g \) that corresponding to \( D_g \). Then \( \Delta_g = g\Delta g^{-1} \). Thus the isomorphism of eigenspaces is \( s \to gs \). Thus when one identifies \( \wedge^\text{top}(KerD)^* \otimes \wedge^\text{top}(CokerD) \) with \( \wedge^\text{top}(K^\alpha(\Delta_{g}))^* \otimes \wedge^\text{top}(D(K^\alpha(\Delta_g))) \) where \( K^\alpha(\Delta) \) is the direct sum of eigenspaces of the operator \( \Delta \) of eigenvalue \( s < a \), over the open subset \( U^a = \{A|a \notin \text{Spec} \Delta \} \) of the affine space \( A \) (see [5], [14] for more details), there is an isomorphism of the fibers as \( D \to D_g \). Thus one can identify

\[
\wedge^\text{top}(K^\alpha(\Delta))^* \otimes \wedge^\text{top}(D(K^\alpha(\Delta))) \equiv \wedge^\text{top}(K^\alpha(\Delta_g))^* \otimes \wedge^\text{top}(D(K^\alpha(\Delta_g))).
\]

By extending this definition from \( U^a \) to \( V^a = \{(A, \Phi)|a \notin \text{Spec} \Delta \}, \) an open subset of \( C \), we can define the fiber over the quotient space \( C/G \) to be the equivalence class of this fiber.

Similarly one can deal with the other case of \( \text{det}(\overline{\partial} + A_0^{(1,0)} + \Phi^{(1,0)}) \), because under gauge transformation, \( \overline{\partial} + A_0^{(1,0)} + \Phi^{(1,0)} \to \overline{g}(\overline{\partial} + A_0^{(1,0)} + \Phi^{(1,0)})\overline{g}^{-1} \). Let \( ([A], [\Phi]) \in C/G \), where \( [A], [\Phi] \) are gauge equivalence classes of \( A, \Phi \), respectively. Then associated to the equivalence class \( ([A], [\Phi]) \) in the base space, there is an equivalence class of fibers coming from the identifications of \( \text{det}(\overline{\partial} + A_0^{(1,0)} + \Phi^{(1,0)}) \) with \( \text{det}(\overline{g}(\overline{\partial} + A_0^{(1,0)} + \Phi^{(1,0)})\overline{g}^{-1}) \) as mentioned in the previous case. Note that in this case, the equivalence class of the fiber above \( ([A], [\Phi]) \) is changing as \( [\Phi] \) is changing. It is unaffected by change in \( [A] \), since \( [A_0] \) is not changing.

This way one can prove that \( P \) is well defined on \( C/G \). Then we restrict it to \( M \subset C/G \).

**Curvature and symplectic form:**

Recall \( \alpha \in \Omega^2(M, \text{ad} P) \) has the decomposition \( \alpha = \alpha^{(1,0)} + \alpha^{(0,1)} \), where \( \alpha^{(1,0)} = -\alpha^{(0,1)} \). Similar decomposition holds for \( \beta, \gamma, \delta \in \Omega^1(M, \text{ad} P) \).

Let \( p = (A, \Phi) \in S \) where \( S \) is the space of solutions to Hitchin equations (1) and (2). Let \( X, Y \in T_pM \). We write \( X = (\alpha, \gamma) \) and \( Y = (\beta, \delta) \), where \( \alpha^{(0,1)}, \beta^{(0,1)} \in T_A(A^{(0,1)}) = \Omega^{(0,1)}(M, \text{ad} P \otimes \mathbb{C}) \) and \( \gamma^{(1,0)}, \delta^{(1,0)} \in T_\Phi \mathcal{H} = \Omega^{(1,0)}(M, \text{ad} P \otimes \mathbb{C}) \).

Since \( T_pM \) can be identified with a subspace in \( T_pS \) orthogonal to \( T_pO_p \) (the tangent space to the gauge orbit) then \( X, Y \) can be said to satisfy (a) \( X, Y \in T_pS \)
i.e. they satisfy linearization of (1) and (2) and (b) $X, Y$ are orthogonal to $T_pO_p$, the tangent space to the gauge orbit.

Let $F_{\mathcal{L}^2}, F_{\mathcal{R}^2}$ denote the Quillen curvatures of the determinant line bundles $\mathcal{L}^2, \mathcal{R}^2$, respectively. Then,

\[
F_{\mathcal{L}^2}((\alpha, \gamma^{(1,0)}), (\beta, \delta^{(1,0)})) = -2F_{\mathcal{L}}((\alpha, \gamma^{(1,0)}), (\beta, \delta^{(1,0)}))
\]

\[
= -2\frac{i}{\pi} \text{Re} \text{Tr} \int_M (\alpha^{(0,1)} \wedge \beta^{(0,1)*})
\]

\[
= \frac{i}{\pi} \text{Tr} \int_M \alpha \wedge \beta
\]

\[
F_{\mathcal{R}^2}((\alpha, \gamma^{(1,0)}), (\beta, \delta^{(1,0)})) = 2F_{\mathcal{R}}((\alpha, \gamma^{(1,0)}), (\beta, \delta^{(1,0)}))
\]

\[
= 2\frac{i}{\pi} \text{Re} \text{Tr} \int_M (\gamma^{(1,0)} \wedge \delta^{(1,0)*})
\]

\[
= \frac{2i}{\pi} \text{Re} \text{Tr} \int_M \gamma^{(1,0)} \wedge \delta^{(1,0)*}
\]

\[
= -\frac{i}{\pi} \text{Tr} \int_M \gamma \wedge \delta
\]

Note: $\overline{\gamma^{(1,0)}}$ and $\overline{\delta^{(1,0)*}}$ contributes because of the term $\overline{\Phi^{(1,0)}}$ in $\mathcal{R}$. $\alpha, \beta$ donot contribute to this curvature because in the definition of $\mathcal{R}$ the gauge equivalence class of $A_0$ is fixed.

**Lemma 3.3.** The Quillen curvature of $\mathcal{P} = \mathcal{L}^2 \otimes \mathcal{R}^2$ on $\mathcal{M}$ is $\frac{i}{\pi} \Omega$.

**Proof.** It is easy to check that $F_{\mathcal{L}^2} + F_{\mathcal{R}^2} = \frac{2i}{\pi} \Omega$. □

These lemmas prove the theorem (3.1).

4. A MORE NATURAL APPROACH FOR THE QUANTIZATION

We define $\Phi^{(0,1)} = -\Phi^{(1,0)*}$. We call the old $\Phi$ in the Hitchin equations by $\Phi^{(1,0)}$.

Let us denote by $L = \det(\bar{\partial} + A^{(0,1)})$ a determinant bundle on $\mathcal{A}$.

Let $R = \det(\bar{\partial} + A^{(0,1)} + \Phi^{(0,1)})$ where $A_0$ is a connection whose gauge equivalence class is fixed, i.e. $A_0$ is allowed to change only in the gauge direction.

Let $\mathcal{P} = \mathcal{L}^2 \otimes \mathcal{R}^2$ denote a line bundle over $\mathcal{C} = \mathcal{A} \times \mathcal{H}$.

(This combination will give the prequantum line bundle corresponding to $\Omega$).

**Lemma 4.1.** $\mathcal{P}$ is a well-defined line bundle over $\mathcal{M} \subset \mathcal{C}/\mathcal{G}$, where $\mathcal{G}$ is the gauge group.

**Proof.** Let us consider the Cauchy-Riemann operator $D = \bar{\partial} + A^{(0,1)} + \Phi^{(0,1)}$ which appears in $R$. The other case is analogous. Under gauge transformation $D = \bar{\partial} + A_0^{(0,1)} + \Phi^{(0,1)} \rightarrow D_g = g(\bar{\partial} + A_0^{(0,1)} + \Phi^{(0,1)})g^{-1}$ since it is the $(0,1)$ part of the connection operator $d + A_0 + \Phi$ which transforms in the same way. We can show that the operators $D$ and $D_g$ have isomorphic kernel and cokernel and their corresponding Laplacians have the same spectrum and the eigenspaces are of the same dimension. Let $\Delta$ denote the Laplacian corresponding to $D$ and $\Delta_g$ that corresponding to $D_g$.The Laplacian is $\Delta = DD$ where $\bar{D} = \bar{\partial} + A_0^{(1,0)} + \Phi^{(1,0)}$, where recall $A_0^{(1,0)*} = -A_0^{(0,1)}$ and $\Phi^{(1,0)*} = -\Phi^{(0,1)}$. Note that $\bar{D} \rightarrow \bar{D}_g = g\bar{D}g^{-1}$ under gauge transformation since it is the $(1,0)$ part of the connection operator.
\( d + A_0 + \Phi \) which transforms in the same way. Thus \( \Delta_g = g \Delta g^{-1} \). Thus the isomorphism of eigenspaces of \( \Delta \) and \( \Delta_g \) is \( s \rightarrow gs \). Thus when one identifies 
\[
\det D = \wedge^{\text{top}}(\text{Ker} D)^* \otimes \wedge^{\text{top}}(\text{Coker} D) \text{ with } \wedge^{\text{top}}(K^a(\Delta))^* \otimes \wedge^{\text{top}}(D(K^a(\Delta)))
\]
where \( K^a(\Delta) \) is the direct sum of eigenspaces of the operator \( \Delta \) of eigenvalues \( < a \), over the open subset \( U^a = \{(A^{(0,1)}, \Phi^{(0,1)})|a \notin \text{Spec}\Delta\} \) of \( C \) (see [5], [14] for more details), there is an isomorphism of the fibers as \( D \rightarrow D_g \). Thus one can identify
\[
\wedge^{\text{top}}(K^a(\Delta))^* \otimes \wedge^{\text{top}}(D(K^a(\Delta))) \equiv \wedge^{\text{top}}(K^a(\Delta_g))^* \otimes \wedge^{\text{top}}(D(K^a(\Delta_g)))
\]
We can define the fiber over the quotient space \( U^a/\mathcal{G} \) to be the equivalence class of this fiber. Covering \( C \) by open sets of the type \( U^a \) enables us to define it on \( C/\mathcal{G} \).
Then we restrict it to the moduli space \( \mathcal{M} \subset C/\mathcal{G} \).

\( \mathcal{L} \) also descends to the moduli space in the same spirit. \( \square \)

**Curvatures and symplectic forms**

Recall \( \alpha \in \Omega^1(M, \text{ad} P) \) has the decomposition \( \alpha = \alpha^{(1,0)} + \alpha^{(0,1)} \), where \( \alpha^{(1,0)} = -\alpha^{(0,1)*} \). Similar decomposition holds for \( \beta, \gamma, \delta \in \Omega^1(M, \text{ad} P) \).

Let \( p = (A, \Phi) \in S \) where \( S \) is the space of solutions to Hitchin equations (1) and (2). Let \( X, Y \in T_p \mathcal{M} \). We write \( X = (\alpha, \gamma) \) and \( Y = (\beta, \delta) \), where \( \alpha^{(0,1)}, \beta^{(0,1)} \in T_A(\mathcal{A}^{(0,1)}) = \Omega^{(0,1)}(M, \text{ad} P \otimes \mathbb{C}) \) and \( \gamma^{(1,0)}, \delta^{(1,0)} \in T_p \mathcal{H} = \Omega^{(1,0)}(M, \text{ad} P \otimes \mathbb{C}) \). Since \( T_p \mathcal{M} \) can be identified with a subspace in \( T_p S \) orthogonal to \( T_p O_p \) (the tangent space to the gauge orbit) then \( X, Y \) can be said to satisfy (a) \( X, Y \in T_p S \) i.e. they satisfy linearization of (1) and (2) and (b) \( X, Y \) are orthogonal to \( T_p O_p \), the tangent space to the gauge orbit.

Let \( \mathcal{F}_{\mathcal{L}^{-2}}, \mathcal{F}_{\mathcal{R}^2} \), denote the Quillen curvatures of the determinant line bundles \( \mathcal{L}^{-2}, \mathcal{R}^2 \), respectively. Then,
\[
\mathcal{F}_{\mathcal{L}^{-2}}((\alpha, \gamma), (\beta, \delta)) = -2 \mathcal{F}_{\mathcal{L}}((\alpha, \gamma), (\beta, \delta)) = -2 \frac{i}{\pi} \text{Re Tr} \int_M (\alpha^{(0,1)} \wedge \beta^{(0,1)*})
\]
\[
= -\frac{i}{\pi} \text{Tr} \int_M \alpha \wedge \beta
\]
(Since there is no \( \Phi \)-term in \( \mathcal{L} \), \( \gamma \) and \( \delta \) donot contribute).

\[
\mathcal{F}_{\mathcal{R}^2}((\alpha, \gamma), (\beta, \delta)) = 2 \mathcal{F}_{\mathcal{R}}((\alpha, \gamma), (\beta, \delta))
\]
\[
= 2 \frac{i}{\pi} \text{Re Tr} \int_M \gamma^{(0,1)} \wedge \delta^{(0,1)*}
\]
\[
= -2 \frac{i}{\pi} \text{Re Tr} \int_M \gamma^{(0,1)} \wedge \delta^{(1,0)}
\]
\[
= -\frac{i}{\pi} \text{Re Tr} \int_M (\gamma^{(0,1)tr} \wedge (\delta^{(1,0)tr}))
\]
\[
= -\frac{i}{\pi} \text{Re Tr} \int_M \gamma^{(1,0)} \wedge \delta^{(0,1)}
\]
\[
= 2 \frac{i}{\pi} \text{Re Tr} \int_M \gamma^{(1,0)} \wedge \delta^{(1,0)*}
\]
\[
= -\frac{i}{\pi} \text{Tr} \int_M \gamma \wedge \delta
\]
Note: $\gamma^{(0,1)}$ and $\delta^{(0,1)*}$ contributes because of the term $\Phi^{(0,1)}$ in $\mathcal{R}$. $\alpha$, $\beta$ donot contribute to this curvature because in the definition of $\mathcal{R}$ the gauge equivalence class of $A_0$ is fixed.

It is easy to check that the curvature of $\mathcal{P}$ is

$$\mathcal{F}_{\mathcal{L}^{-2}} + \mathcal{F}_{\mathcal{R}^2} = \frac{i}{\pi} \Omega.$$  

Recall that

$$\mathcal{I}(\alpha^{(0,1)}) = i\alpha^{(0,1)},$$
$$\mathcal{I}(\gamma^{(1,0)}) = i\gamma^{(1,0)},$$
$$\mathcal{I}(\alpha^{(1,0)}) = -i\alpha^{(1,0)},$$
$$\mathcal{I}(\gamma^{(0,1)}) = -i\gamma^{(0,1)}.$$  

Thus w.r.t. $\mathcal{I}$, $A^{(0,1)}$ is holomorphic and $\Phi^{(0,1)}$ is antiholomorphic. But in $\mathcal{P}^{-1} = \mathcal{L}^2 \otimes \mathcal{R}^{-2}$ has the $A^{(0,1)}$-term as it is and the $\Phi^{(0,1)}$-term in the inverse bundle. Thus $\mathcal{P}^{-1}$ is $\mathcal{I}$-holomorphic.

Thus we have the following:

**Theorem 4.2.** $\mathcal{P}^{-1}$ is a holomorphic prequantum line bundle on $\mathcal{M}$ with curvature $-\frac{i}{\pi} \Omega$.

**Polarization:** We can take $\mathcal{I}$-holomorphic sections of $\mathcal{P}^{-1}$ as our Hilbert space.

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