Quantum key distribution based on orthogonal states

Hao Shu

College of Mathematics, South China University of Technology, Guangzhou, 510641, P. R. China

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Quantum key distribution (QKD) is one of the most significant areas in quantum information theory. For nearly four decades, substantial QKD protocols and cryptographic methods are developed. In early years, the security of protocols are depend on switching different bases, which, in fact, are based on non-orthogonal states. The most famous example is the BB84 protocol. Later, other techniques were developed for orthogonal states cryptography. Representations of such protocols including the GV protocol and order-rearrangement protocols. It might be harder to implement protocols based on orthogonal states since they require extra techniques to obtain security. In this paper, we give two QKD protocols based on orthogonal states. One of them needs not to employ order-rearrangement techniques while the other needs. We give analyses of their security and efficiency. Also, anti-noisy discussions would be given. Our protocols are highly efficient when considering consumptions of both qubits and classical bits while they are robust over several noisy channels. Moreover, the requirements of maximally entangled states maybe less than previous protocols and so the efficiency of measurements may be increased.

Keywords: Quantum key distribution; order-rearrangement; orthogonal states; noise; qubit.

I. INTRODUCTION

In information theory, cryptography is always one of the most important fields. Unfortunately, the most used cryptosystem, the RSA system, is not secure in quantum era[1]. The essential reason why the RSA system is not secure is that its security is depend on low capacity of classical computations. To obtain the unconditional security, one has to find a cryptographic scheme of which the security is only depend on physical laws. Nowadays, the only classical cryptography which is proven to be secure is encoding messages with an one-time pad. On the other hand, transmitting an one-time pad by classical channels could be totally insecure since classical messages can be cloned without being detected. However, quantum effects can provide possibilities of transmitting an one-time pad, of which the security is only depend on physical laws and can be proven mathematically. Such a task is called a quantum key distribution (QKD).

In nearly four decades, great developments have been made since the first QKD protocol was proposed in 1984[2], which obtaining the security by switching two mutual unbiased bases and of which the security has been proven[3]. After the BB84 protocol, several BB84-like protocols were proposed, such as Ekert’s protocol[4], BBM92 protocol[5], six-states protocol[6] and so on[7][8]. These protocols obtain the security by employing non-orthogonal states. It was not until 1995 that the first cryptographic protocol based on orthogonal states was published[9]. The idea of the protocol is sending states with a time delay such that the eavesdropper can never get entire states without being detected. Another technique can be employed to implement a QKD protocol based on orthogonal states is the order-rearrangement. Protocols employing this technique can be found in [9][10][11][12][13]. Other protocols including[14][15][16][17][18][19][20].

On the other hand, besides designing QKD protocols, there is another problem. In practical, Channels employed to implement a QKD protocol are always noisy. Therefore, one has to analyse the robustness of a protocol over noisy channels. Previous works including[21][22][23][24] for collective noises, [25][26][27][28] for Pauli noises and [29][30][31][32][33][34] for amplitude damping (AD) and phase damping (PD) noises. There are also other researches on noises[35][36][37][38][39][40], for examples.

In this paper, two QKD protocols based on orthogonal states are proposed in section II while efficient analyses are given in section III, comparing with several protocols. The discussions of the security are given in section IV and implementing protocols over noisy environments would be argued in section V. The last section, section VI, is devoted to conclusions. Comparing with certain previous protocols, our protocols, on one hand, are highly efficient when considering consumptions of both qubits and classical bits, and on the other hand, are robust over several noisy channels. Moreover, the requirements of maximally entangled states maybe less and so the efficiency of measurements may be increased.

II. TWO PROTOCOLS

Let us propose the protocols in this section, which are stated as follow.

Protocol I:

Step 1: Alice and Bob agree to encode 00, 11, 01, 10 by states $|00\rangle_{AB}, |11\rangle_{AB}, |\varphi\rangle_{AB} = \frac{1}{\sqrt{2}}(|01\rangle_{AB} - |10\rangle_{AB})$, $|\varphi^{'})_{AB} = \frac{1}{\sqrt{2}}(|01\rangle_{AB} + |10\rangle_{AB})$. 

| States | Alice | Bob |
|--------|-------|------|
| 00     | 00    | 00   |
| 11     | 11    | 11   |
| 01     | 01    | 01   |
| 10     | 10    | 10   |
| $\varphi$ | 00    | 11   |
| $\varphi'$ | 10    | 01   |

...
Step 2: To share a N 2-bits key string, Alice creates a string of 2N states chosen randomly in $S = \{|00\rangle_{AB}, |11\rangle_{AB}, |\varphi\rangle_{AB}, |\varphi'\rangle_{AB}\}$ which is only known by her.

Step 3: Alice creates $N/2$ decoy states, all be $|+\rangle = 1/\sqrt{2}|0 + 1\rangle$ and inserts them in the states string in step 2 randomly. Now Alice has a states string with 2.5N states and she records the positions of decoy states by a 2.5N-bits string $r = r_1r_2...r_{2.5N}$. In more details, $r_i = 1$ if the i-th state is a decoy state and $r_i = 0$, otherwise.

Step 4: Alice sends partite B of the states string to Bob.

Step 5: After receiving the particles, Bob publicly announces this fact.

Step 6: After Alice receives Bob’s receipt, she sends partite A of the states string to Bob together with the string $r$.

Step 7: Bob receives the states string and the string $r$. He then measures decoy states via basis $\{1\rangle = 1/\sqrt{2}|0 + 1\rangle$ and other states via basis $S = \{|00\rangle_{AB}, |11\rangle_{AB}, |\varphi\rangle_{AB}, |\varphi'\rangle_{AB}\}$.

Step 8: Alice and Bob run checking procedures as follow. Bob publishes all his outcomes on decoy states and the outcomes on half of other states (let us called them checking states) chosen randomly. Alice checks whether the checking states are agreed with what she created and calculates the error rate.

Step 9: If the error rate is acceptable, Alice and Bob agree a secret key by the outcomes of remaining N states, which are not employed as checking states.

Step 10: Alice and Bob repeat the above procedure until they share a sufficiently long secret key and run error correcting procedures or private amplified procedures if needed.

Protocol II:

Step 1: Alice and Bob agree to encode 00, 11, 01, 10 by states $|00\rangle_{AB}, |11\rangle_{AB}, |\varphi\rangle_{AB}, |\varphi'\rangle_{AB} = 1/\sqrt{2}|01 + 01\rangle_{AB}, |\varphi'\rangle_{AB}$, respectively, in $C^2 \otimes C^2$.

Step 2: To share a N 2-bits key string, Alice creates a string of 2N states chosen randomly in $S = \{|00\rangle_{AB}, |11\rangle_{AB}, |\varphi\rangle_{AB}, |\varphi'\rangle_{AB}\}$ which is only known by her.

Step 3: Alice divides the 2N states into N blocks such that each block includes two adjacent states. Alice chooses a random N-bits string $s = s_1, s_2, ..., s_N$ and exchanges the order of B partite of the first state and the A partite of the second state in the i-th block if $s_i = 1$.

Step 4: Alice sends all partite of the states string to Bob.

Step 5: After receiving the particles, Bob publicly announces this fact.

Step 6: After Alice receives Bob’s receipt, she sends the string $s$ to Bob.

Step 7: Now, Bob has the states string and the string $s$. He then reorders the states by informations of the string $s$, recovering them and then measures via basis $S$.

Step 8: Alice and Bob run a checking procedure as follow. Bob publishes the positions and outcomes on half of states (let us called them checking states) chosen randomly. Alice checks whether the checking states are agreed with what she created and calculates the error rate.

Step 9: If the error rate is acceptable, Alice and Bob agree a secret key by the outcomes of remaining N states, which are not employed as checking states.

Step 10: Alice and Bob repeat the above procedure until they share a sufficiently long secret key and run error correcting procedures or private amplified procedures if needed.

III. ANALYSES OF THE PROTOCOLS

Before given a secure proof, let us give analyses of the protocols.

A. Efficiency

In protocol I, to generate a N 2-bits key string, Alice and Bob consume 2N states in $C^2 \otimes C^2$ and $N$ single qubit states. Totally, they consume 4.5N qubits. And on the other hand, Alice and Bob consume 2.5N classical bits for publishing the string $r$, 2N classical bits for publishing the positions of checking states and 2N classical bits for the outcomes of N checking states. In more details, for example, Bob sends a string $b = b_1b_2,...,b_{2N}$ to Alice and $b_i = 0$ means that the i-th state is not a checking state, and $b_i = 1$, otherwise. And for the N checking states, Bob sends a string $c = c_1c_2,...,c_N$ to Alice and $c_j = 0, 1, 2, 3$ means that the outcome of the j-th state is $|0\rangle, |1\rangle, |\varphi\rangle, |\varphi'\rangle$, respectively. Of course, they have to consume another three classical bits including Bob declaring his receipt in step 5. Alice and Bob publishing whether their errors checking procedures are passed. The total classical bits needed (for classical communications) in such a protocol are nearly 6.5N bits. Equivalently, they consume nearly 2.25N qubits and 3.25N classical bits for a N-bits key string.

In protocol II, to generate a N 2-bits key string, Alice and Bob consume 2N states in $C^2 \otimes C^2$. Totally, they consume 4N qubits. And on the other hand, Alice and Bob consume N classical bits for publishing the string $s$, 2N classical bits for publishing the positions of checking states and 2N classical bits for the outcomes of N checking states. In more details, for example, Bob sends a string $d = d_1d_2,...,d_{2N}$ to Alice and $d_i = 0$ means that the i-th state is not a checking state, and $d_i = 1$, otherwise. And for the N checking states, Bob sends a
string \( e = e_1 e_2, \ldots, e_N \) to Alice and \( e_j = 0, 1, 2, 3 \) means that the outcome of the \( j \)-th state is \( |00\rangle, |11\rangle, |\varphi\rangle, |\varphi'\rangle \), respectively. Of course, they have to consume another two classical bits including Bob declaring his receipt in step 5, Alice publishing whether the errors checking procedure is passed. The total classical bits (for classical communications) needed in such a protocol are nearly \( 5N \) bits. Equivalently, they consume nearly \( 2N \) qubits and \( 2.5N \) classical bits for a \( N \)-bits key string.

### B. Comparing with previous protocols

Note that in the ordinary BB84 protocol\(^2\), to agree a \( N \)-bits key string, Alice and Bob have to consume \( 4N \) qubits, \( 4N \) classical bits for Bob informing Alice which basis he chooses to measure the states, \( 4N \) classical bits for Alice informing Bob which states are checked, \( 2N \) classical bits for Alice informing Bob which states are checking states and \( N \) classical bits for Alice informing Bob the outcomes on checking states. Totally, they consume nearly \( 4N \) qubits and \( 11N \) classical bits.

In the modified BB84 protocol\(^3\) with a Hadamard gate, it consumes \( 2N \) qubits and \( 2N \) classical bits for publishing operations on states and \( 2N \) classical bits for publishing positions of checking states and another \( N \) classical bits for publishing outcomes on checking states. Totally, they consume \( 2N \) qubits and nearly \( 5N \) classical bits for a \( N \)-bits key string.

In previous protocols based on order-rearrangement of orthogonal states\(^4\), \( |01\rangle \), \( |11\rangle \), \( |12\rangle \), \( |13\rangle \), the consumptions of qubits and classical bits are not less than protocol II above. For example, in \( |01\rangle \), the consumption of states is equal to protocol II but has to employ more entangled states and a full Bell measurement, which is difficult to implement, for the reason that the efficiency of a Bell measurement is low, in experimental, nowadays. Protocol II needs not to measure a full set of Bell states, which might be more efficient when implementing. And on the other hand, the previous protocol requires a four-states rearrangement in four cases while protocol II only requires a two-states rearrangement in two cases, and so could be more realistic.

In addition, our protocols could be more robust over noisy channels than previous protocols. For examples, BB84 protocol and the previous order-rearrangement protocol might not be implemented over collective dephasing(CD) noisy channels. However, our protocols above can immune CD noises. The discussions of noises would be given in section V.

### IV. SECURITY

Let us discuss the security of the protocols. Assume that there is an eavesdropper, says Eve, who wants to steal the secret key of Alice and Bob. We would assume that Alice and Bob hold authenticated classical channels which might not be private, and quantum channels without any further assumption. That means that Eve might eavesdrop the classical communications but with no abilities to forge messages or pretend to be one of the legitimated parties, while she can do anything under physical laws in quantum channels. The security means that Eve can not get enough informations on the secret key or she would create errors which are detectable by checking procedures with high probabilities. We also assume that Eve provides a collective attack, that is Eve attacks every state Alice sent by the same method.

For Eve, she mainly has three kinds of attacks. She might intercept a state sent by Alice and implement one of the three actions. Firstly, she might take the state herself and send another state created by her to Bob (let us called this a substituted attack), instead. Secondly, she might measure the state and resend it to Bob (let us called this a measure-resend attack). Thirdly, she might add an auxiliary party and send another state created by her to Bob (let us called this a purified attack).

#### A. purified attack

Let us analyse purified attacks firstly. For such attacks, the decoy states in the protocol I (step 3) can be aborted, and so the efficiency can be increased.

1. **Eve purifies via single qubits**

If Eve purifies states sent by Alice via single qubits, the security of the protocols (both protocol I and protocol II) can correspond to previous protocols, such as \( |17\rangle \) or \( |0\rangle \). For example, if Eve purifies states sent by Alice via basis \( \{|i\rangle\}_{i=0,1} \), which would change states \( |00\rangle, |11\rangle, |\varphi\rangle, |\varphi'\rangle \) into \( |00\rangle_{AB}|00\rangle_{EE'}, |11\rangle_{AB}|11\rangle_{EE'}, |\varphi\rangle_{pAEE'} = \frac{1}{\sqrt{2}}(0101 - 1010)_{ABEE'}, |\varphi'\rangle_{pAEE'} = \frac{1}{\sqrt{2}}(0101 + 1010)_{ABEE'} \), respectively, where \( E \) and \( E' \) are partite of Eve’s. However, Alice and Bob can detect Eve on checking states \( |\varphi\rangle \) or \( |\varphi'\rangle \). In more details, \( |\varphi\rangle_{pAEE'} = \frac{1}{\sqrt{2}}(0101 - 1010)_{ABEE'} = \frac{1}{\sqrt{2}}(|\varphi\rangle|\varphi\rangle + |\varphi'\rangle|\varphi\rangle)_{ABEE'}. \) When Bob measuring the state via basis \( S \) on partite \( A \) and \( B \), he will get an outcome \( |\varphi\rangle \) or \( |\varphi'\rangle \) with equal probabilities. The calculation of \( |\varphi'\rangle \) is essentially same. The error rate created by Eve and detectable by Alice and Bob is now \( \frac{1}{2} \) for Alice and Bob choose an entangled state for checking with probability \( \frac{1}{2} \) and get an error outcome with probability \( \frac{1}{2} \), if so.
basis \( \{|+\rangle = \frac{1}{\sqrt{2}}(0+1), \{-\rangle = \frac{1}{\sqrt{2}}(0-1) \} \), then an error occurs with probability \( \frac{1}{2} \) for every checking state and so the error rate is \( \frac{1}{2} \).

2. Eve purifies via two-qubits state

Now assume that Eve attacks by purifying states via S. This attack is not suitable for protocol I since in protocol I, Eve can only obtain one partite of states in the same time, if she tries to employ such an attack. Let us analyse

\[
\begin{align*}
&b_{\varphi \varphi'} = |\varphi\rangle_{12}|\varphi\rangle_{34}, b_{\varphi' \varphi} = |\varphi'\rangle_{12}|\varphi\rangle_{34}, b_{00} = |00\rangle_{12}|00\rangle_{34}, b_{11} = |11\rangle_{12}|11\rangle_{34}, \\
&b_{\varphi'0} = |\varphi'\rangle_{12}|00\rangle_{34}, b_{\varphi0'} = |00\rangle_{12}|\varphi'\rangle_{34}, b_{23} = |\varphi'\rangle_{12}|11\rangle_{34}, b_{1\varphi} = |11\rangle_{12}|\varphi\rangle_{34}, \\
&b_{\varphi1} = |\varphi'\rangle_{12}|11\rangle_{34}, b_{1\varphi'} = |11\rangle_{12}|\varphi'\rangle_{34}, b_{01} = |00\rangle_{12}|11\rangle_{34}, b_{10} = |11\rangle_{12}|00\rangle_{34}.
\end{align*}
\]

(1)

with equal probabilities. Let us calculate the error rate for each case. Assume that \( |b\rangle_{xy} \) becomes \( |b\rangle_{xy}p \) after being purified via S on partite 1, 3 and partite 2, 4, respectively. And let Eve’s partite be \( E_1, E_2, E_3, E_4 \).

\[
\begin{align*}
b_{\varphi' \varphi'} &= |\varphi'\rangle_{12}|\varphi'\rangle_{34} = \frac{1}{2}(1010 + 0110 + 1001 + 1010)_{1234} = \frac{1}{2}(0011 + 0110 + 1001 + 1100)_{1234} \\
&= \frac{1}{2}(0011 + \varphi'\varphi' - \varphi\varphi + 1100)_{1234},
\end{align*}
\]

(2)

\[
\begin{align*}
b_{\varphi p \varphi' p} &= \frac{1}{2}(0010011 + \varphi'\varphi'\varphi'\varphi' - \varphi\varphi\varphi\varphi + 11001100)_{1234 E_1 E_2 E_3 E_4} \\
&= \frac{1}{4}(0011)|\varphi'\rangle_{-\varphi\rangle_{34}} + |1100)|\varphi'\rangle_{-\varphi\rangle_{34}} + |\varphi\varphi\rangle_{0011 + 1100 - \varphi'\varphi' - \varphi\varphi} \\
&\quad + |\varphi'\varphi'\rangle_{0011 + 1100 + \varphi'\varphi' + \varphi\varphi} + |\varphi'\varphi'\rangle_{0011 - 1100} + |\varphi\varphi\rangle_{0011 - 1100})_{1234 E_1 E_2 E_3 E_4}.
\end{align*}
\]

The probability of getting the correct outcome by measuring via S on partite 1, 2 is \( \frac{7}{10} \). Similarly, error rates for \( b_{\varphi p \varphi p}, b_{\varphi p' \varphi p} \) and \( b_{\varphi p' \varphi p} \) are all

\[
\begin{align*}
b_{\varphi 0} &= |\varphi\rangle_{12}|00\rangle_{34} = \frac{1}{\sqrt{2}}(1000 - 1000)_{1234} = \frac{1}{\sqrt{2}}(0010 - 1000)_{1234} = \frac{1}{2}(00\varphi' - 00\varphi - \varphi'00 + \varphi00)_{1234}. \\
b_{\varphi 0' p} &= \frac{1}{2}(00\varphi'00\varphi - 00\varphi00\varphi - \varphi'00\varphi'00 + \varphi00\varphi00)_{1234 E_1 E_2 E_3 E_4} \\
&= \frac{1}{4}(00)|\varphi + \varphi'|)|00\rangle_{34} + |00\rangle_{34})|\varphi' - \varphi\rangle_{34} + |\varphi\varphi\rangle_{0010 + 00\varphi + \varphi00 + \varphi00} \\
&\quad + |\varphi\varphi\rangle_{0010 + 00\varphi + \varphi00 - \varphi00})_{1234 E_1 E_2 E_3 E_4}.
\end{align*}
\]

(3)

The probability of getting the correct outcome (that is \( |\varphi\rangle \)) by measuring via S on partite 1, 2 is \( \frac{3}{4} \) and so the error rate is \( \frac{3}{4} \). Similarly, error rates for \( b_{\varphi 0 p}, b_{\varphi 1 p} \) and \( b_{\varphi 1 p} \) are all \( \frac{3}{4} \). And on the other hand, the probability
of getting the correct outcome by measuring via $S$ on partite $3$, $4$ (that is $|00\rangle$) is $\frac{1}{2}$ and so the error rate is $\frac{1}{2}$.

Similarly, error rates of $b_{0_1p'}$, $b_{1_p2'}$ and $b_{1_p2}$ are all $\frac{1}{2}$.

$$b_{01} = |00\rangle_{12}|11\rangle_{34} = |0011\rangle_{1234} = |0101\rangle_{1324} = \frac{1}{2}(|\varphi + \varphi' + \varphi'\varphi + \varphi'\varphi'\rangle_{1234}$$

$$b_{01p} = \frac{1}{2}(|\varphi\varphi\varphi + \varphi'\varphi'\varphi' + \varphi'\varphi'\varphi + \varphi'\varphi'\varphi'\rangle_{1234}E_1E_2E_3E_4$$

$$=\frac{1}{4}(|0011\rangle|\varphi + \varphi'|\varphi + \varphi'| + |1100\rangle|\varphi' - \varphi|\varphi' - \varphi|$$

$$+ |\varphi|\varphi'\varphi - \varphi'\varphi'\varphi - \varphi'\varphi'\varphi + \varphi'\varphi'\varphi' + \varphi'\varphi'\varphi'\rangle_{1234}E_1E_2E_3E_4.$$

The probability of getting the correct outcome (that is $|00\rangle$) by measuring via $S$ on partite $1$, $2$ is $\frac{1}{2}$ and so the error rate is $\frac{1}{2}$. Similarly, the error rate of $b_{10p}$ is $\frac{3}{4}$.

$$b_{00} = |00\rangle_{12}|00\rangle_{34} = |0000\rangle_{1234} = |0000\rangle_{1324}$$

$$b_{00p} = |00000000\rangle_{1234}E_1E_2E_3E_4 = |00000000\rangle_{1234}E_1E_2E_3E_4.$$

The probability of getting the correct outcome (that is $|00\rangle$) by measuring via $S$ on partite $1$, $2$ is $\frac{1}{2}$ and so the error rate is $0$. Similarly, the error rate of $b_{11p}$ is $0$.

Now, the average error rate for Eve guessing the order wrong is $\frac{1}{2}(4\times\frac{1}{2} + 4\times\frac{1}{2} + 4\times\frac{1}{2} + 2\times\frac{1}{2} + 2\times0) = \frac{9}{2}$, and so the whole error rate is $\frac{9}{4}$, which is larger than $\frac{1}{2}$. the error rate of attacking the BB84 protocol by purification.

**B. Substituted attack**

Let us state the substituted attack. Step 3 of protocol I is not needed in such case and so the efficiency or security can be increased. Eve might take the state sent by Alice herself, measuring it or keeping it until she obtaining more informations. However, Alice and Bob will not continue the procedure until Bob receiving a state. Thus, Eve has to send another state to Bob, instead. Eve might get enough informations after stealing the A partite in protocol I or after Alice publishes the order string in protocol II. She can measure the state via basis $S$ and so know exactly what Alice sent. But Bob can detect the attack, for the reason Eve might substitute a product state by an entangled state or substitute an entangled state by a product state. In both cases, Bob’s measuring outcome might be incorrect.

In more details, in protocol I, if Eve steals the B partite of the state and sends one partite of her state, instead. Since, at this time, Eve is not able to discriminate whether the state is separated or entangled, she can not send a state to Bob and transform it into the state she needs after stealing the A partite, since local transformations (local unitary operations) of a state can not generate or break entanglement. For example, assume that Eve sends a partite of a product state, says partite $E_1$ of $|0\rangle_{E_1}|0\rangle_{E_2}$ to Bob and keeps the state sent by Alice in step 4. If after stealing partite $A$ of the state and finding that the state is entangled, for example, be $|\varphi\rangle$, she can not prevent her being detectable. Since now no matter what state she sends to Bob, Bob will get a product state and there is at least with probability $\frac{1}{2}$ he will obtain an error outcome when measuring via basis $S$. The same argument is suitable if Eve sends a maximally entangled state, instead. In such case, if she finds that Alice sent a product state, she can do nothing to decrease the error rate of Bob less than $\frac{1}{2}$. Hence, the error rate is not less than $\frac{1}{2}$, for Eve wrongly guessing the state sent by Alice is entangled or separated and then Bob’s measuring outcome is incorrect, if so.

These arguments are also hold for protocol II. Since Eve can not know the order of the states before sending her states to Bob, she might send a product state instead of a entangled state or conversely. If so, the error rate on Bob’s checking procedure will not less than $\frac{1}{2}$ and the whole error rate will not less than $\frac{1}{4}$.

It is worth remarking that protocol I without step 3 can not employ four Bell states instead of $S$. The four Bell states can be transformed into each other via local transformations. For this reason, Eve can steal partite B of states sent by Alice while sending a partite of maximally entangled states of her in step 4. Then she can send the other partite with transformations depending on outcomes of her Bell measurements after stealing partite A.
C. Measure-resend attack

The secure analyses for such attacks are similar to above. If Eve measures via single qubits and resends the states to Bob, then Bob will receive product states. Thus, Bob can detect Eve by outcomes of entangled states sent by Alice. The probability would be at least 1/2 for half of checking states be entangled and with probabilities 1/2 being incorrect for those states. To against such an attack, step 3 of protocol I is not needed, similar to above.

Eve might choose to measure via two-qubits states. This attack can only happen in protocol II, since Eve can never obtain both partite of states in protocol I when employing a measure-resend attack only. In protocol II, since Eve can not know the orders of states until she finishing her measurement and resending states to Bob, she might change correlations of states. For example, let Eve measure states sent by Alice via basis S, the only basis for Eve might gather the secret key without increasing errors. If she guesses the order incorrect, she will measuring her measurement and resending states to Bob, she might change correlations of states. For example, let Eve measure states sent by Alice via basis S, the only basis for Eve might gather the secret key without increasing errors. If she guesses the order incorrect, she will measuring her measurement and resending states to Bob, she might change correlations of states. For example, let Eve measure states sent by Alice via basis S, the only basis for Eve might gather the secret key without increasing errors.

D. Two stages attack

Two stages attacks only suitable for protocol I, since in protocol II, Alice sends states to Bob in only one stage. Let us assume that Eve attacks protocol I with different strategies, on step 4 and step 6, when Alice sending different partite to Bob. If Eve employs a substituted attack in step 4, she might substitute product states instead of entangled states or substitute entangled states instead of product states, which would increase error rates as in substituted attacks only. If Eve employs measure-resend attacks in step 4, she might break entanglement of states, and errors would be occurred as in measure-resend attacks only.

The only case left is that Eve purifies states in step 4, and measures states in step 6. For example, Eve might employ states via basis \(\{|j\rangle|j = 0, 1\}\) in step 4, twice. The states become \(\{|0000\rangle_{ABEE'}, |1111\rangle_{ABEE'}\}\), \(|\varphi'\rangle = \frac{1}{\sqrt{2}}(0111 - 1000)_{ABEE'} = \frac{1}{\sqrt{2}}|\varphi'\rangle - |\varphi\rangle_{ABEE'}\\), \(|\varphi''\rangle = \frac{1}{\sqrt{2}}(0111 + 1000)_{ABEE'} = \frac{1}{\sqrt{2}}|\varphi''\rangle - |\varphi\rangle_{ABEE'}\\) after Eve operates the bit-flip gate on partite E', respectively and corresponding to \(\{|00\rangle_{AB}, |11\rangle_{AB}, |\varphi\rangle_{AB}, |\varphi''\rangle_{AB}\}\). After Eve steals partite A, she measures partite AE via basis S. If the outcome is \(|00\rangle\) or \(|11\rangle\), she knows the state is \(|00\rangle\) or \(|11\rangle\), respectively and she sends partite E' to Bob. If the outcome is \(|\varphi\rangle\), she operates nothing and sends partite E' to Bob while if the outcome is \(|\varphi''\rangle\), she operates the phase-flip gate on partite E' and sends partite E' to Bob. Here, the attack is undetectable without step 3 and Eve obtains all informations on product states. That is why Alice employs step 3, the decoy states. With these states, Eve’s purified attack above can be detected. On such an attack, when Bob measuring decoy states via basis \(\{|+\rangle, |-\rangle\}\), outcomes are incorrect with half of the probability. That is the error rate on decoy states is \(\frac{1}{2}\), which can be employed to detect Eve.

V. IMPLEMENT PROTOCOLS OVER NOISY CHANNELS

We shall give discussions of implementing the protocols over noisy channels including collective dephasing(CD) noises, collective rotation (CR) noises, Pauli noises, amplitude damping(AD) and phase damping(PD) noises and mixtures of them.

A. Collective dephasing (CD)

Collective dephasing noises assume that the whole protocol is implemented in a same-time cycle and so the noises affect each qubit equivalently via \(
\begin{bmatrix}
1 & 0 \\
0 & e^{i\phi}
\end{bmatrix}
\)
under the computational basis, where \(\phi\) is the parameter depending on the noise. Previous results for such noises including \[21\],\[22\] ,\[23\],\[24\] .

For such noises, we shall modify protocol I, step 3, substituting decoy states \(|+\rangle\) by \(|\varphi\rangle\), and protocol I step 7, measuring via S on decoy states, instead. Protocol II needs not to be modified. Now, Protocol I and protocol II completely immune such noises, since all states in the protocols are changed nothing but global phases when affecting by the noises. The consumption of states in protocol I now becomes 2.5N qubits for a N-bits key string while the protocol II remains unchanged.

B. Collective rotation (CR)

Collective rotation noises assume that the whole protocol is implemented in a same-time cycle and so the noises affect each qubit equivalently via \(
\begin{bmatrix}
cos\theta & sin\theta \\
sin\theta & -cos\theta
\end{bmatrix}
\)
under the computational basis, where \(\theta\) is the parameter depending on the noise and evolving upon time. Previous results for such noises including \[21\],\[22\],\[23\],\[24\] .

We shall modify protocol I and II both. Firstly, we substitute S by \(S' = \{|\varphi\rangle_{AB}, |\varphi''\rangle_{AB} = \frac{1}{\sqrt{2}}|00\rangle + |11\rangle\}_{AB}\) in both protocol I and protocol II. Then in protocol I, decoy states in step 3 are substituted by \(|\varphi\rangle\) of B. And in
step 7, Bob measures via $S'$, instead. The reason for employing $S'$ instead of $S$ is that states in $S'$ are unchanged when affecting by CR while states in $S$ might not. The consumption of states now becomes $5N$ qubits for protocol I and $4N$ qubits for protocol II for a $N$-bits key string.

C. Pauli noises

Pauli noises act on each qubit via Pauli operators, $I = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$, $Z = \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}$, $X = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}$, $ZX = \begin{bmatrix} 0 & 1 \\ 1 & -1 \end{bmatrix}$. Under the computational basis, with probabilities $p$, $p$, we have the following:

$$|++⟩ = \frac{1}{\sqrt{2}}(|00⟩ + |11⟩), \quad |−−⟩ = \frac{1}{\sqrt{2}}(|00⟩ - |11⟩).$$

$1. One Pauli channel$

In one Pauli channel, states suffer two of the four Pauli operators and one of which is $I$.

Let us assume that states suffer $I$ with a probability $p$ and $Z$ with a probability $1-p$.

As for protocol I, to deal with the noise, Alice might prefer to send states supplied by auxiliary partite. She sends two partite together and assumes that they suffer the same effect. In more details, Alice employs states $|00⟩|++⟩⟨++|00⟩$, $|11⟩|−−⟩⟨−−|11⟩$, $|φ⟩|−−⟩⟨−−|φ⟩$, $|φ⟩|++⟩⟨++|φ⟩$, respectively. Alice always sends partite $A$, $A'$ together and $B$, $B'$ together and assumes that partite $A$, $A'$ always suffer the same effect while partite $B$, $B'$ always suffer the same effect. After Bob receives states (step 4 and step 6), he measures partite $A$, $A'$ and $B$, $B'$, respectively, via basis $\{+,−\}$ for $A$, $B'$ and basis $\{|0⟩,|1⟩\}$ for $A'$, $B'$. If the outcomes on $A'$, $A''$ are $|+⟩$ and $|0⟩$, he does nothing, while he transforms partite $A$ via the operator $Z$, $X$, $ZX$ if the outcomes are $|−⟩$ and $|0⟩$, $|+⟩$ and $|1⟩$, $|−⟩$ and $|1⟩$ on $A'$ and $A''$, respectively. The same manipulations are also applied for partite $B$, $B'$ and $B''$. Then he continues the procedure (step 7). Hence, the auxiliary partite are employed for detecting whether states are influenced by $Z$, $X$, $ZX$ and if so, Bob corrects them via $Z$, $X$, $ZX$, respectively.

As for protocol II, Alice only needs to employ exactly one auxiliary partite for each block while qubits of the auxiliary partite are always be $|0⟩$. Bob operates via $X$ on the ordinary state when getting the outcome $|1⟩$ by measuring via basis $\{|0⟩,|1⟩\}$ on the auxiliary partite of the state while he does nothing, otherwise. This is because operator $Z$ affects nothing for protocol II and so they only need to confirm whether operator $X$ is applied, that is whether states are affected by $X$ or $ZX$. Other procedures are similar to above.

D. Phase damping (PD) and Amplitude damping (AD)

In two Pauli channels, $p$, $p$, $p$, $p$ might all be non-zero, and so the final states might be completely mixed. We follow the above method to deal with such noises.

For protocol I, Alice employs states $|00⟩|++⟩⟨++|00⟩$, $|11⟩|−−⟩⟨−−|11⟩$, $|φ⟩|−−⟩⟨−−|φ⟩$, $|φ⟩|++⟩⟨++|φ⟩$, instead of $|00⟩|AB⟩|AB⟩|AB⟩|AB⟩|AB⟩$, $|11⟩|AB⟩|AB⟩|AB⟩|AB⟩|AB⟩$, respectively, and decoy states become $|++⟩|BB⟩|BB⟩|BB⟩|BB⟩|BB⟩$, instead of $|++⟩|AB⟩|AB⟩|AB⟩|AB⟩|AB⟩$. Alice always sends partite $A$, $A'$, $A''$ together and $B$, $B'$, $B''$ together and assumes that partite $A$, $A'$, $A''$ always suffer the same effect while partite $B$, $B'$, $B''$ always suffer the same effect. After Bob receives states (step 4 and step 6), he measures partite $A$, $A'$, $A''$ and $B'$, $B''$, respectively, via basis $\{+,−\}$ for $A$, $B'$ and basis $\{|0⟩,|1⟩\}$ for $A''$, $B''$. If the outcomes on $A'$, $A''$ are $|+⟩$ and $|0⟩$, he does nothing, while he transforms partite $A$ via the operator $Z$, $X$, $ZX$ if the outcomes are $|−⟩$ and $|0⟩$, $|+⟩$ and $|1⟩$, $|−⟩$ and $|1⟩$ on $A'$ and $A''$, respectively. The same manipulations are also applied for partite $B$, $B'$ and $B''$. Then he continues the procedure (step 7). Hence, the auxiliary partite are employed for detecting whether states are influenced by $Z$, $X$, $ZX$ and if so, Bob corrects them via $Z$, $X$, $ZX$, respectively.

Phase damping noises have kraus operators $E_0 = \begin{bmatrix} \sqrt{1-p} & 0 \\ 0 & \sqrt{1-p} \end{bmatrix}$, $E_1 = \begin{bmatrix} 0 & \sqrt{p} \\ \sqrt{p} & 0 \end{bmatrix}$, $E_2 = \begin{bmatrix} 0 & 0 \\ 0 & \sqrt{1-p} \end{bmatrix}$, where $p$ is the error rate, which means that a state sent by the channel has a probability $1-p$ remains unchanged and a probability $p$ suffers errors.

Amplitude damping noises have kraus operators $E_0 = \begin{bmatrix} 1 & 0 \\ 0 & \sqrt{1-p} \end{bmatrix}$, $E_1 = \begin{bmatrix} 0 & \sqrt{p} \\ \sqrt{p} & 0 \end{bmatrix}$ where $p$ is the error rate.
which means that a state sent by the channel has a probability 1-p suffers E0 and a probability p suffers E1. Previous works including [29][30][31][32][33][34][40].

For such noises, Alice can employ states

\[ |0101\rangle_{AA'BB'}, \quad |1010\rangle_{AA'BB'}, \]
\[ |\psi\rangle_{AA'BB'} = \frac{1}{\sqrt{2}} (|0110\rangle - |0001\rangle)_{AA'BB'}, \]
\[ |\psi'\rangle_{AA'BB'} = \frac{1}{\sqrt{2}} (|0110\rangle + |0001\rangle)_{AA'BB'} \]

instead of \[ |00\rangle_{AB}, |11\rangle_{AB}, |\phi\rangle_{AB}, |\phi'\rangle_{AB}, \] respectively, and decays states \[ |\phi'\rangle_{BB'} \] instead of \[ +\rangle_B \]. That is Alice employs an auxiliary partite for each partite, setting ordinary states of auxiliary partite to be \[ |1\rangle \], then provides C-NOT gates on auxiliary partite. Alice always sends partite A and A', partite B and B' together to assume C-NOT gates on auxiliary partite. Alice always sends partite A and A', partite B and B' together to assume that partite A and A' always suffer the same effect of noises and so do partite B and B'. For Bob, after receiving states, he firstly descends states which losing some partite. The states left are those states not affected by noises. Bob then provides C-NOT gates on those states then continuous step 7. The above arguments are suitable for both protocol I and protocol II.

E. Mixture of noises

To deal with mixtures of noises, one might consider a technique called decoherence-free subspace [41][42][43][44][45][46][47][48]. Decoherence-free states are invariant under collective noises and our arguments for AD and PD noises shall also be applied. However, there is only one decoherence-free state in a two qubits system which can not be employed for coding. For a four qubits system, that is \[ C^4 \otimes C^4 \], decoherence-free subspace is of 2-dimensions. Thus, coding might be applied in such a system with orthogonal states.

VI. CONCLUSION

In this paper, we give two quantum key distribution protocols based on orthogonal states. Protocol I consumes more but needs not to employ an order-rearrangement technique while protocol II consumes less with an order-rearrangement technique. Both protocols employ the same set of coding states, of which half are maximally entangled and the other half are separated. We demonstrate the advantages of the protocols, comparing with certain previous protocols and we provide secure proofs of them. Arguments of implementing the protocols over noisy channels are also proposed.

Our protocols, on one hand, are highly efficient. In protocol II, all states are employed for key states except those for checking, which are always assumed to be half of the states like in the BB84 protocol while in protocol I, another 12.5 percent of states are employed for decoy states but needs not to employ order-rearrangement techniques. Our protocols need not to employ a full Bell measurement which might be lowly efficient. The states we employ are half maximally entangled and half separated. On the other hand, our protocols can be modified for noisy environments and more robust comparing with several previous protocols. For example, collective daphasing noises affect nothing in protocol II while protocol I only needs to be modified a bit to against such noises.

As for remaining problems, one could consider implementing such protocols over other noisy channels. Another direction could be experimental realizations of the protocols. We also mention that another obstacle on implementing such protocols, including the two protocols, other order-rearrangement protocols or even the BB84 protocol with a Harmard gate, is that one needs to employ a quantum memory. Therefore, discussions of a quantum memory might be significant.

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