Thermalization of gluons with Bose-Einstein condensation

Zhe Xu *,1 Kai Zhou,1,2 Pengfei Zhuang,1 and Carsten Greiner2

1Department of Physics, Tsinghua University and Collaborative Innovation Center of Quantum Matter, Beijing 100084, China
2Institut für Theoretische Physik, Johann Wolfgang Goethe-Universität Frankfurt, Max-von-Laue-Strasse 1, 60438 Frankfurt am Main, Germany

We study the thermalization of gluons far from thermal equilibrium in relativistic kinetic theory. The initial distribution of gluons is assumed to resemble that in the early stage of ultrarelativistic heavy ion collisions. Only elastic scatterings in a static, non-expanding gluonic matter are considered. At first we show that the occurrence of condensation in the limit of vanishing particle mass requires a general constraint for the scattering matrix element. Then the thermalization of gluons with Bose-Einstein condensation is demonstrated in a transport calculation. We see a continuously increasing over-population of low energy gluons, followed by a decrease to the equilibrium distribution, when the condensation occurs. The times of the completion of the gluon condensation and of the entropy production are calculated. These times scale inversely with the energy density.

A new state of matter composed of quarks and gluons, the quark-gluon plasma (QGP), has been produced in experiments of ultrarelativistic heavy ion collisions at the Relativistic Heavy Ion Collider (RHIC) and at the Large Hadron Collider (LHC) [1,3]. On the one hand, the evolution of the QGP is well described by relativistic hydrodynamics. The extractions from the flow measurements yield a small number of the shear viscosity over the entropy density (η/s) of the QGP [4], which indicates that the QGP is a strongly coupled QCD matter. On the other hand, the initially produced quarks and gluons are far from thermal equilibrium. How these partons thermalize toward QGP within a short time of about 1 fm/c is still an open question, although different approaches [4,5] have been developed to study this issue.

Recently a new idea of the formation of Bose-Einstein condensates in ultrarelativistic heavy ion collisions [7] raises again the interest in thermalization of quarks and gluons produced in such reactions. For high colliding energy the color glass condensate formed in the collision gives an over-occupied initial condition of gluons, when compared with the thermal Bose-Einstein distribution with a temperature obtained by assuming a sudden equilibration with energy and gluon number conservation. This over-occupation, if it happens in the real case, may lead to the formation of a Bose-Einstein condensate, which affects the thermalization process. Detailed studies based on the kinetic transport theory have shown the onset of Bose-Einstein condensation [8,9] from an over-occupied initial condition for a static, non-expanding gluon system. In this letter we evolve the system beyond the onset toward the full thermalization with a complete Bose-Einstein condensation. It is an important step forward to understand the parton thermalization in ultrarelativistic heavy ion experiments.

In this work the space-time evolution of gluons is described in a kinetic transport model BAMPS [2] that solves the Boltzmann equation for gluons. We consider a static, non-expanding gluon system that is initially out of equilibrium and assume gluon number conservation by including only gluonic elastic scatterings in the collision term of the Boltzmann equation. Once the initial distribution of gluons is given, the gluon system will finally evolve to the thermal equilibrium distribution

\[ f_{eq}(\mathbf{p}) = \frac{1}{\mathcal{Z}_{BE}} \left( 1 + (2\pi)^3 n_c \delta^{(3)}(\mathbf{p}) \right), \]

which is the solution of the Boltzmann equation in the long time limit. The temperature \( T_{BE} \) and the density of the gluon condensate at equilibrium \( n_c^{eq} \) are obtained by using the assumed gluon number and energy conservation. We will show that our numerical solution agrees with Eq. (1), and will determine the time scale of the full thermalization for various initial conditions from weak to strong over-occupation.

We note that although particle number changing processes may prevent the Bose-Einstein condensation, studies within field theories including particle production and annihilation processes showed the emergence of a condensate for an intermediate time window [9]. Also, results from kinetic transport calculations indicate an acceleration of the onset of gluon condensation due to gluon bremsstrahlung processes [11].

The formation of a Bose-Einstein condensate is the consequence of quantum statistics of bosons and has been observed in experiments using ultracold atoms [12]. The Boltzmann equation including Bose statistics reads

\[
\left( \frac{\partial}{\partial t} + \frac{\mathbf{p}_i}{E_i} \frac{\partial}{\partial \mathbf{r}_i} \right) \mathbf{f}_1 = \frac{1}{2E_1} \int d\Gamma_2 \frac{1}{\nu} \int d\Gamma_3 d\Gamma_4 |M_{34\rightarrow12}|^2 \times \left[ f_3 f_4 (1 + f_1)(1 + f_2) - f_1 f_2 (1 + f_3)(1 + f_4) \right] (2\pi)^4 \times \delta^{(4)}(p_3 + p_4 - p_1 - p_2),
\]

where \( f_i = f_i(\mathbf{r}, \mathbf{p}_i, t) \) and \( d\Gamma_i = d^3p_i/(2E_i)/(2\pi)^3, i = 1, 2, 3, 4 \). \( \nu \) equals 2 when the particle 3 and 4 are identical particles. Otherwise \( \nu \) equals 1. \( (1 + f_1)(1 + f_2) \) and \( (1 + f_3)(1 + f_4) \) are the Bose factors, with which the distribution (1) is a solution of the Boltzmann equation (2). The growth of the condensate density is governed

*E-mail: xuzhe@mail.tsinghua.edu.cn
by interactions between condensate and noncondensate particles with the Boltzmann approach, while the time evolution of the macroscopic wave function of the condensate is not considered.

Separating \( f_i \) into two parts for the gas (noncondensate) and condensate, \( f_i = f_i^\text{gas} + f_i^\text{cond} \) with \( f_i^\text{cond} = (2\pi)^3n_e\delta^{(3)}(p_i) \), we obtain the equations for \( f_i^\text{gas} \) and \( f_i^\text{cond} \), respectively. Integrating Eq. (2) for \( f_i^\text{cond} \) over \( d^3p_i/(2\pi)^3 \) gives the condensation rate

\[
\frac{dn_c}{dt} = \frac{n_c}{64\pi^3} \int dE_3dE_4\left[ f_3f_4 - f_3(1 + f_3 + f_4) \right] \times E \left[ \frac{M_{34-12}^2}{s} \right]_{s=2mE}, \quad (3)
\]

where \( E = E_3 + E_4 = E_2 \) is the total energy, \( s \) is the invariant mass of the process, and \( m \) denotes the particle mass at rest. Here we have assumed that \( f \) only depends on the absolute value of \( p \). Thus, \( f_2 \) equals \( f \) at \( p_2 = |p_3 + p_4| = \sqrt{E^2 - s} \). The two terms on the right hand side of Eq. (3) correspond to kinetic processes for the condensation and the evaporation, respectively. The energy-momentum conservation in these processes leads to the constraint \( s = 2mE_2 \). Equation (3) is exactly the same as that derived in [13], if a factor of 1/2 is taken due to the double counting of identical particles.

For vanishing rest mass, i.e., in the limit \( s \to 0 \), the ratio \( |M_{34-12}^2|/s \) must have a nonzero value, in order to make a condensate. This is a general condition for particle condensation in such extreme case. For elastic scatterings of massless gluons in leading order of perturbative QCD (pQCD) a regularized matrix element squared with the small angle approximation, \( |M_{gg\to gg}|^2 \approx s^2/(t - m^2) \), is often used [14], where \( s, t \) are the Mandelstam variables, \( m \) the screening mass. We see that this form cannot lead to a condensation. Instead we take [15]

\[
|M_{gg\to gg}|^2 \approx 144\pi^2\alpha_s^2 \frac{s^2}{t(t - m^2)} \quad (4)
\]
in the following calculations. Due to (4) \( \sigma_{gg\to gg} \) is divergent for scatterings involving noncondensate particles. The divergence is logarithmic and is regularized by a upper cutoff of \( t \). We note that scatterings with \( t \) approaching to zero do not contribute to thermalization.

To solve the Boltzmann equation (2) we modify the currently used partonic transport model BAMPS (Boltzmann Approach of Multiparton Scatterings) [2] to include the Bose factors. A new scheme [16] is employed to simulate collisions stochastically between test particles. Instead of a collision probability we define a differential collision probability

\[
\frac{dP_{22}}{d\Omega} = \frac{\nu_{\text{rel}}}{N_{\text{test}}} \frac{d\sigma_{22}}{d\Omega}(1 + f_3)(1 + f_4) \frac{\Delta t}{\Delta V}, \quad (5)
\]

\( \nu_{\text{rel}} = s/(2E_1E_2) \) is the relative velocity of the incoming particles, \( N_{\text{test}} \) the number of test particles per real particle, \( \Delta V \) the volume element around the collision point, \( \Delta t \) the time step in the calculation, and \( d\sigma_{22}/d\Omega \) the differential cross section of elastic gluon scatterings. At first a solid angle \( \Omega \) is sampled according to \( d\sigma_{22}/d\Omega \) for each particle pairs considered. In case of the occurrence of a collision the momenta \( p_3, p_4 \) of the outgoing particles are thus determined. Kinematic constraints ensure exact energy-momentum conservation in each collisions. We also obtain \( f_3 \) and \( f_4 \) from the extracted \( f \) at \( p_3 \) and \( p_4 \), respectively. Secondly a random number between zero and the value of a normalized reference function \( dF/d\Omega \) at \( \Omega \) will decide whether a collision will occur: if this random number is smaller than the differential collision probability \( dP_{22}/d\Omega \) at \( \Omega \), a collision occurs; otherwise not.

Figure 1 shows the energy distributions of noncondensate gluons at various times. \( f_0 = 2.0 \) is chosen in the example. From the upper panel of Fig. 1 we observe a continuous increase at low energy and at energy higher than \( Q_s \), whereas between some intermediate energy and

\[
T_{BE} = \left( 15f_0/4 \right)^{1/4} Q_s/\pi, \quad n_c|\eta_c|/n = 1 - 19.43\pi^{-3} f_0^{-1/4}. \quad (7)
\]

There is a critical value of \( f_0^c = 0.154 \), below which \( n_c|\eta_c|/n \) becomes negative and no Bose-Einstein condensation occurs. We set \( Q_s = 1 \text{ GeV} \) and vary \( f_0 \) from 0.4 to 2.0 to change the number and energy density. Gluons are homogeneously distributed in a box, for which the side length is set to be 3 fm. A periodic boundary condition is taken. The box is divided by cubic cells with equal volume \( \Delta V \). The side length of a cell is 0.125 fm. Large \( \alpha_s \)-gluon is produced, the two incoming gluons should have parallel momenta, which is impossible in numerical calculations. We define a cutoff energy \( \epsilon \). Gluons with energy smaller than \( \epsilon \) are regarded as condensate gluons. Since the production rate of the condensate particles should not depend on the particular numerical implementation, \( \epsilon \) has to be chosen sufficiently small to avoid numerical artifacts. The details and numerical confirmation of the new scheme will be shown in a forthcoming paper [17].

The momentum distribution of initial gluons is assumed to be

\[
f_{\text{init}}(p) = f_0\theta(Q_s - |p|), \quad (6)
\]

which resembles that in the early stage of ultrarelativistic heavy ion collisions [7]. The initial momentum distribution has been simplified to be isotropic, although the new method introduced [17] can be applied for anisotropic momentum distributions. From (6) we obtain the temperature at equilibrium and the condensate fraction of the total density

\[
T_{BE} = \left( 15f_0/4 \right)^{1/4} Q_s/\pi, \quad n_c|\eta_c|/n = 1 - 19.43\pi^{-3} f_0^{-1/4}. \quad (7)
\]
where \( Q_s \) is close to zero, \( \mu \) is the potential for the starting time of the gluon condensation, \( \tau_c \). At this moment \( n_c \) probably has a nonzero value, because we have made the approximation to regard particles with energy smaller than \( \epsilon \) as condensate gluons. In the calculations we set \( \epsilon = 0.0025 \text{ GeV} \).

We note that both \( \tau_c \) and \( n_c \) at \( \tau_c \) should be determined from a theory describing the phase transition to a Bose-Einstein condensate \([11]\), which is beyond the scope of the Boltzmann approach. However, we will see these values will not affect the dynamic behavior of the growth of the condensate appreciably. In the calculation for \( f_0 = 2.0 \) the condensation starts at \( \tau_c = 0.376 \text{ fm/c} \). The energy distribution at \( \tau_c \) is far from thermal equilibrium. This particular feature differs from the production of Bose-Einstein condensates in experiments using ultracold atoms \([12]\), where the system is close to thermal equilibrium. From the upper panel of Fig. 4 we also see that until 0.5 fm/c the condensation is too weak to reduce the increasing overpopulation at low energies. \( T_{\text{eff}} \) increases to 10 GeV at 0.5 fm/c, which is much larger than the equilibrium temperature, \( T_{\text{BE}} = 0.53 \text{ GeV} \).

After 0.5 fm/c the overpopulation at low energies begins to disappear due to the growth of the gluon condensate, as clearly seen in the lower panel of Fig. 4. We find that the distribution at low energies has a profile \( f \sim 1/p \) from \( \tau_c \). The freed energy during the condensation flows to particles with high energies, where the occupation number increases continuously. The energy distribution relaxes to the equilibrium form. We find a perfect agreement with the equilibrium distribution at 3.4 fm/c, which indicates the completion of the gluon condensation, see Fig. 2. We notice that the distribution at 1.5 fm/c is already very close to the equilibrium one.

The growth of the gluon condensate in time is shown in the upper panel of Fig. 2. At early times of the condensation the gluon condensate grows exponentially, which can qualitatively be understood by Eq. (3). At these times the production processes are dominant. When the evaporation processes begin to balance the production processes, the growth of the gluon condensate slows down, and then has a relaxation form.

Numerical calculations are performed for three different \( f_0 \). Larger \( f_0 \) leads to larger particle and energy densities and larger fraction of the density of the gluon condensate to the total gluon density, see Eq. (4). To study the scaling behavior, the density of the gluon condensate is divided by its final value at equilibrium, while the time is multiplied by \( f_0 \). We see an approximate scaling, which shows that the larger the density (larger \( f_0 \)), the faster is the completion of the gluon condensation, and the faster is the thermal equilibration. The scaling behavior of the starting time of the gluon condensation, \( \tau_c \), can be understood by approximating the collision term in the Boltzmann equation (1), \( f_3 f_4 (1 + f_1)(1 + f_2) - f_1 f_2 (1 + f_3)(1 + f_4) \approx f_3 f_4 (f_1 + f_2) - f_1 f_2 (f_3 + f_4) \). It is a good approximation for colliding particles with small energy, where \( f \) is large. If \( f \) at low energies is proportional to \( f_0 \), \( f \approx f_0 f_N(p, t) \), which seems reasonable at least at early times due to the given initial condition \([4]\), and if this relation holds long for large \( f_0 \), then the right hand side.

FIG. 1: (Color online) The energy distributions of noncondensate gluons at various times. \( f_0 = 2 \) is chosen.
The time evolution of the density of the gluon condensate (upper panel) and the entropy production (lower panel). Calculations are performed for various $f_0$.

The lower panel of Fig. 2 shows the entropy production $s(t) - s(t_0 = 0)$ divided by its value at equilibrium. We see a two step production separated by $\tau_c$. This shows that the formation of the gluon condensate is essential for the time scale of thermalization. From Fig. 2 we also see that the completion of the condensation is later than that of entropy production. The full thermalization is reached at $f_0 t_\text{f0} \approx 3 \text{ fm/c}$. In addition, the entropy production does not scale with $f_0 t_\text{f0}$ before $\tau_c$, because the momentum distribution at low energies has a negligible contribution to the entropy and the argument for $\tau_c$-scaling does not apply to the early entropy production. Instead, we find that the early entropy production is independent of $f_0$. The reason lies in the almost same collision rate at the early times, because $\sigma_{gg} \sim 1/m_D^2$ and $m_D^2$ is roughly proportional to $f_0$.

In this letter we demonstrated for the first time the Bose-Einstein condensation of gluons within kinetic theory. Our calculations showed that for the color glass condensate initial conditions, which are far from thermal equilibrium, the gluon condensation is essential for the thermalization process: The times for the occurrence of the condensation, the completion of the condensation and the full thermalization scale inversely with the energy density. For instance, the time for full thermalization is about $1.5 \text{ fm/c}$ for energy density $53 \text{ GeV/fm}^3 (f_0 = 2)$. We also find that the earlier the condensation occurs, the earlier does the second step of the entropy production start. All these are important new findings to get further forward with the understanding of thermalization for systems far from thermal equilibrium.

Some discussions are in order. At first, each gluon may possess a thermal mass, which leads to a nonzero energy of a gluon condensate. This makes the detection of the gluon condensate possible. Photons and dileptons produced in ultrarelativistic heavy ion collisions might provide some measurable hints. Secondly, the inclusion of quark-antiquark annihilations [15], or/and gluon bremsstrahlung processes [11], and the back reactions will violate the gluon number conservation. Whether a gluon condensation will still occur, is an interesting question. Moreover, one has to consider the expansion of quark gluon matter, in order to obtain more realistic results. All these are subjects of future works.

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