ATTRACTORS FOR SINGULARLY PERTURBED HYPERBOLIC EQUATIONS ON UNBOUNDED DOMAINS

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Abstract. For an arbitrary unbounded domain $\Omega \subset \mathbb{R}^3$ and for $\varepsilon > 0$, we consider the damped hyperbolic equations

$$
\varepsilon u_{tt} + u_t + \beta(x)u - \sum_{ij}(a_{ij}(x)u_{x_j})_{x_i} = f(x, u), \quad x \in \Omega, \ t \in [0, \infty[,
$$

$$(H_\varepsilon)$$

$$u(x, t) = 0, \quad x \in \partial \Omega, \ t \in [0, \infty[. $$

and their singular limit as $\varepsilon \to 0$, i.e. the parabolic equation

$$
\varepsilon u_{tt} + u_t + \beta(x)u - \sum_{ij}(a_{ij}(x)u_{x_j})_{x_i} = f(x, u), \quad x \in \Omega, \ t \in [0, \infty[,
$$

$$(P)$$

$$u(x, t) = 0, \quad x \in \partial \Omega, \ t \in [0, \infty[. $$

Under suitable assumptions, $(H_\varepsilon)$ possesses a compact global attractor $A_\varepsilon$ in the phase space $H_0^1(\Omega) \times L^2(\Omega)$, while $(P)$ possesses a compact global attractor $\mathcal{A}_0$ in the phase space $H_0^1(\Omega)$, which can be embedded into a compact set $\mathcal{A}_0 \subset H_0^1(\Omega) \times L^2(\Omega)$. We show that, as $\varepsilon \to 0$, the family $(A_\varepsilon)_{\varepsilon \in [0, \infty[}$ is upper semicontinuous with respect to the topology of $H_0^1(\Omega) \times H^{-1}(\Omega)$. We thus extend a well known result by Hale and Raugel in three directions: first, we allow $f$ to have critical growth; second, we let $\Omega$ be unbounded; last, we do not make any smoothness assumption on $\partial \Omega$, $\beta(\cdot)$, $a_{ij}(\cdot)$ and $f(\cdot, u)$.

1. Introduction

In their paper [13] Hale and Raugel considered the damped hyperbolic equations

$$
\varepsilon u_{tt} + u_t - \Delta u = f(u) + g(x), \quad x \in \Omega, \ t \in [0, \infty[,
$$

$$u(x, t) = 0, \quad x \in \partial \Omega, \ t \in [0, \infty[. $$

and their singular limit as $\varepsilon \to 0$, i.e. the parabolic equation

$$
u_t - \Delta u = f(u) + g(x), \quad x \in \Omega, \ t \in [0, \infty[,
$$

$$u(x, t) = 0, \quad x \in \partial \Omega, \ t \in [0, \infty[. $$

In [13] the set $\Omega$ is a bounded smooth domain or a convex polyhedron, $\varepsilon$ is a positive constant, $g \in L^2(\Omega)$ and $f$ is a $C^2$ function of subcritical growth such that

$$
\limsup_{|u| \to \infty} \frac{f(u)}{u} \leq 0.
$$
Under these assumptions, for any fixed \( \varepsilon > 0 \) the corresponding hyperbolic equation generates a global semiflow which possesses a compact global attractor \( \mathcal{A}_\varepsilon \) in the phase space \( H^1_0(\Omega) \times L^2(\Omega) \) (see [2,8,12]). Moreover, the limiting parabolic equation generates a global semiflow which possesses a compact global attractor \( \widetilde{\mathcal{A}}_0 \) in the phase space \( H^1_0(\Omega) \) (see [5,12]). Due to the smoothing effect of parabolic equations, it turns out that \( \mathcal{A}_0 \) is actually a compact subset of \( H^2(\Omega) \). Hence one can define the set

\[
\mathcal{A}_0 = \{(u, \Delta u + f(u) + g) \mid u \in \mathcal{A}_0\},
\]

which is a compact subset of \( H^1_0(\Omega) \times L^2(\Omega) \). Hale and Raugel proved that the family \( \{(\mathcal{A}_\varepsilon)_{\varepsilon \in [0,\infty]} \) is upper semicontinuous with respect to the topology of \( H^1_0(\Omega) \times L^2(\Omega) \), i.e.

\[
\lim_{\varepsilon \to 0^+} \sup_{y \in \mathcal{A}_\varepsilon} \inf_{z \in \mathcal{A}_0} |y - z|_{H^1_0 \times L^2} = 0.
\]

In this paper we extend the result of Hale and Raugel in three directions: firstly, we allow \( f \) to have critical growth; secondly, we let \( \Omega \) be unbounded; thirdly, we replace \( f(u) + g(x) \) by \( f(x, u) \) and \( -\Delta \) by \( \beta(x)u - \sum_{i,j}(a_{ij}(x)u_{x_i})_{x_j} \), without any smoothness assumption on \( \partial \Omega \), \( \beta(\cdot) \), \( a_{ij}(\cdot) \) and \( f(\cdot, u) \).

In [13] the proof of the main result relies on some uniform \( (H^2 \times H^1) \)-estimates for the attractors \( \mathcal{A}_\varepsilon \), combined with the compactness of the Sobolev embedding \( H^1_0(\Omega) \subset L^2(\Omega) \). The uniform \( (H^2 \times H^1) \)-estimates are obtained through a bootstrapping argument originally due to Haraux [14]. Such argument works only if \( f \) is subcritical, and if \( \Omega \) is such that the domain of the \( L^2(\Omega) \)-realization of \( -\Delta \) is \( H^2(\Omega) \cap H^1_0(\Omega) \) (e.g. if \( \Omega \) is a convex polyhedron).

A different bootstrapping argument was proposed by Grasselli and Pata in [10,11]. Their argument also works in the critical case, and is based on certain a-priori estimates that can be obtained “within an appropriate Galerkin approximation scheme”. Here, “appropriate” means “on a basis of eigenfunctions of \(-\Delta\)”. Therefore, their approach cannot be used in the case of an unbounded domain \( \Omega \). More recently, in [15] Pata and Zelik obtained \((H^2 \times H^1)\)-estimates for \( \mathcal{A}_\varepsilon \) without using bootstrapping arguments, but again their a-priori estimates are obtained “within an appropriate Galerkin approximation scheme”. We point out that also in [10,11,15] \( \Omega \) must have the property that the domain of the \( L^2(\Omega) \)-realization of \(-\Delta\) is \( H^2(\Omega) \cap H^1_0(\Omega) \). Moreover, the Nemitski operator associated with \( f \) must be Lipschitz continuous from \( H^2(\Omega) \cap H^1_0(\Omega) \) to \( H^2(\Omega) \) in [15] and from \( D((-\Delta)^{(\alpha+1)/2}) \) to \( D((-\Delta)\alpha/2) \) for all \( 0 \leq \alpha \leq 1 \) in [10,11]. Therefore, if one wants to replace \( f(u) + g(x) \) by \( f(x, u) \), one needs to impose severe smoothness conditions on \( f(x, u) \) with respect to the space variable \( x \).

If \( \Omega \) is unbounded, the embedding \( H^1_0(\Omega) \subset L^2(\Omega) \) is no longer compact, and this poses some additional difficulties even for the existence proof of the attractors \( \mathcal{A}_\varepsilon \). In [6,7], Feireisl circumvented these difficulties by decomposing any solution \( u(t, x) \) into the sum \( u_1(t, x) + u_2(t, x) \) of two functions, such that \( u_1(t, \cdot) \) is asymptotically small, and \( u_2(t, \cdot) \) has a compact support which propagates with speed \( 1/\varepsilon^2 \). As \( \varepsilon \to 0 \), the speed of propagation tends to infinity, and, indeed, the estimates obtained
by Feireisl are not uniform with respect to $\varepsilon$. It is therefore apparent that, if one wants to pass to the limit as $\varepsilon \to 0$, a different approach is needed.

In our previous paper [16] we proved the existence of compact global attractors for damped hyperbolic equations in unbounded domains using the method of tail-estimates (introduced by Wang in [19] for parabolic equations), combined with an argument due to Ball [3] and elaborated by Raugel in [18]. Here we exploit the same techniques to establish an upper semicontinuity result similar to that of Hale and Raugel, when $\Omega$ is an unbounded domain and $f$ is critical. Our arguments do not rely on $(H^2 \times H^1)$-estimates for the attractors $A_\varepsilon$. Therefore they also apply to the case of an open set $\Omega$ for which the domain of the $L^2(\Omega)$-realization of $-\Delta$ is not $H^2(\Omega) \cap H^1_0(\Omega)$ (e.g. if $\Omega$ is the exterior of a convex polyhedron).

Before we describe in detail our assumptions and our results, we need to introduce some notation. In this paper, $N = 3$ and $\Omega$ is an arbitrary open subset of $\mathbb{R}^N$, bounded or not. For $a$ and $b \in \mathbb{Z}$ we write $[a..b]$ to denote the set of all $m \in \mathbb{Z}$ with $a \leq m \leq b$. Given a subset $S$ of $\mathbb{R}^N$ and a function $\nu: S \to \mathbb{R}$ we denote by $\tilde{\nu}: \mathbb{R}^N \to \mathbb{R}$ the trivial extension of $\nu$ defined by $\tilde{\nu}(x) = 0$ for $x \in \mathbb{R}^N \setminus S$. Given a function $g: \Omega \times \mathbb{R} \to \mathbb{R}$, we denote by $\hat{g}$ the Nemitski operator which associates with every function $u: \Omega \to \mathbb{R}$ the function $\hat{g}(u): \Omega \to \mathbb{R}$ defined by

$$\hat{g}(u)(x) = g(x, u(x)), \quad x \in \Omega.$$ 

Unless specified otherwise, given $k \in \mathbb{N}$ and functions $g, h: \Omega \to \mathbb{R}^k$ we write

$$\langle g, h \rangle := \int_\Omega \sum_{m=1}^k g_m(x) h_m(x) \, dx,$$

whenever the integral on the right-hand side makes sense.

If $I \subset \mathbb{R}$, $Y$ and $X$ are normed spaces with $Y \subset X$ and if $u: I \to Y$ is a function which is differentiable as a function into $X$ then we denote its $X$-valued derivative by $\partial(u; X)$. Similarly, if $X$ is a Banach space and $u: I \to X$ is integrable as a function into $X$, then we denote its $X$-valued integral by $\int_I (u(t); X) \, dt$.

**Assumption 1.1.**

1. $a_0, a_1 \in ]0, \infty[ \text{ are constants and } a_{ij}: \Omega \to \mathbb{R}, \ i, j \in [1..N] \text{ are functions in } L^\infty(\Omega) \text{ such that } a_{ij} = a_{ji}, \ i, j \in [1..N], \text{ and for every } \xi \in \mathbb{R}^N \text{ and a.e. } x \in \Omega, \ a_0|\xi|^2 \leq \sum_{i,j=1}^N a_{ij}(x) \xi_i \xi_j \leq a_1|\xi|^2. \ A(x) := (a_{ij}(x))_{i,j=1}^N, \ x \in \Omega.$

2. $\beta: \Omega \to \mathbb{R}$ is a measurable function with the property that

   (i) for every $\tau \in ]0, \infty[$ there is a $C_\tau \in [0, \infty[$ with $||\beta||^{1/2} u_{L^2}^2 \leq \tau |u|_{H^1}^2 + C_\tau |u|_{L^2}^2$ for all $u \in H^1_0(\Omega)$;
   
   (ii) $\lambda_1 := \inf \{ \langle A \nabla u, \nabla u \rangle + \langle \beta u, u \rangle \mid u \in H^1_0(\Omega), \ |u|_{L^2} = 1 \} > 0$.

**Assumption 1.2.**

1. $f: \Omega \times \mathbb{R} \to \mathbb{R}$ is such that, for every $u \in \mathbb{R}$, $f(\cdot, u)$ is (Lebesgue-)measurable, $f(\cdot, 0) \in L^2(\Omega)$ and for a.e. $x \in \Omega$, $f(x, \cdot)$ is of class $C^2$ and such that
\[ \partial_t f(\cdot, 0) \in L^\infty(\Omega) \text{ and } |\partial_{uu} f(x, u)| \leq C(1 + |u|) \text{ for some constant } C \in [0, \infty[, \text{ every } u \in \mathbb{R} \text{ and a.e. } x \in \Omega; \]

\[ (2) \ f(x, u)u - \overline{\nu} F(x, u) \leq c(x) \text{ and } F(x, u) \leq c(x) \text{ for a.e. } x \in \Omega \text{ and every } u \in \mathbb{R}. \]

Here, \( c \in L^2(\Omega) \) is a given function, \( \overline{\nu} \in [0, \infty[ \) is a constant and \( F: \Omega \times \mathbb{R} \to \mathbb{R} \) is defined, for \((x, u) \in \mathbb{R}\), by

\[
F(x, u) = \int_0^u f(x, s) \, ds,
\]

whenever \( f(x, \cdot): \mathbb{R} \to \mathbb{R} \) is continuous and \( F(x, u) = 0 \) otherwise.

Note that Assumptions 1.1 and 1.2 imply the hypotheses of [16].

Let \( D(B_\varepsilon) \) be the set of all \((u, v) \in H^1_0(\Omega) \times L^2(\Omega)\) such that \( v \in H^1_0(\Omega) \) and \(-\beta u + \sum_{ij} (a_{ij} u_{x_j}) x_i \) (in the distributional sense) lies in \( L^2(\Omega)\). It turns out that the operator

\[
B_\varepsilon(u, v) = (-v, (1/\varepsilon) v + (1/\varepsilon) \beta u - (1/\varepsilon) \sum_{ij} (a_{ij} u_{x_j}) x_i), \quad (u, v) \in D(B_\varepsilon)
\]

is the generator of a \((C_0)\)-semigroup \( e^{-B_\varepsilon t}, t \in [0, \infty[ \) on \( H^1_0(\Omega) \times L^2(\Omega) \). Moreover, the Nemitski operator \( \hat{f} \) is a Lipschitzian map of \( H^1_0(\Omega) \) to \( L^2(\Omega) \). Results in [4] then imply that the hyperbolic boundary value problem

\[
\varepsilon u_{tt} + u_t + \beta(x) u - \sum_{ij} (a_{ij} (x) u_{x_j}) x_i = f(x, u), \quad x \in \Omega, \ t \in [0, \infty[, \\
u(x, t) = 0, \quad x \in \partial\Omega, \ t \in [0, \infty[ 
\]

with Cauchy data at \( t = 0 \) has a unique (mild) solution \( z(t) = (u(t), v(t)) \) in \( H^1_0(\Omega) \times L^2(\Omega) \), given by the “variation-of-constants” formula

\[
z(t) = e^{-B_\varepsilon t} z(0) + \int_0^t e^{-B_\varepsilon (t-s)} (0, (1/\varepsilon) \hat{f}(u(s))) \, ds.
\]

For \( \varepsilon \in [0, \infty[ \) we define \( \pi_\varepsilon \) to be the local semiflow on \( H^1_0(\Omega) \times L^2(\Omega) \) generated by the (mild) solutions of this hyperbolic boundary value problem. We can summarize the results of [16] in the following:

**Theorem 1.3.** Under Assumptions 1.1 and 1.2, \( \pi_\varepsilon \) is a global semiflow and it has a global attractor \( A_\varepsilon \).

Analogously, consider the parabolic boundary value problem

\[
u_t + \beta(x) u - \sum_{ij} (a_{ij} (x) u_{x_j}) x_i = f(x, u), \quad x \in \Omega, \ t \in [0, \infty[, \\
u(x, t) = 0, \quad x \in \partial\Omega, \ t \in [0, \infty[ 
\]
with Cauchy data at $t = 0$. Letting $A$ denote the sectorial operator on $L^2(\Omega)$ defined by the differential operator $u \mapsto \beta u - \sum_{ij}(a_{ij}u_{x_j})_{x_i}$, we have that $D(A)$ is the set of all $u \in H^2_0(\Omega)$ such that the distribution $\beta u - \sum_{ij}(a_{ij}u_{x_j})_{x_i}$ lies in $L^2(\Omega)$. Again, the Cauchy problem has a unique (mild) solution $u(t)$ in $H^1_0(\Omega)$, given by the “variation-of-constants” formula

$$u(t) = e^{-At}u(0) + \int_0^t e^{-A(t-s)}\hat{f}(u(s))\,ds.$$

Let $\tilde{\pi}$ be the local semiflow on $H^1_0(\Omega)$ generated by the (mild) solutions of this parabolic boundary value problem. Results in [17] imply that $\tilde{\pi}$ is a global semiflow and has a global attractor $\tilde{\mathcal{A}}$ (see also [1]). Moreover, it is proved in [17] that $\mathcal{A} \subset D(A)$ and $\tilde{\mathcal{A}}$ is compact in $D(A)$ endowed with the graph norm.

Let $\Gamma: D(A) \to H^1_0(\Omega) \times L^2(\Omega)$ be defined by $\Gamma(u) = (u, Au + \hat{f}(u))$. Set $A_0 := \Gamma(\mathcal{A})$. Then we have the following main result of this paper:

**Theorem 1.4.** The family $(A_\varepsilon)_{\varepsilon \in [0, \infty[}$ is upper semicontinuous at $\varepsilon = 0$ with respect to the topology of $H^1_0(\Omega) \times H^{-1}(\Omega)$, i.e.

$$\lim_{\varepsilon \to 0^+} \sup_{y \in A_\varepsilon} \inf_{z \in A_0} |y - z|_{H^1_0 \times H^{-1}} = 0.$$

Actually a stronger result is established in Theorem 3.9 below.

## 2. Preliminaries

In this section we collect a few preliminary results. We begin with an abstract lemma established in [17]:

**Lemma 2.1.** Suppose $(Y, \langle \cdot, \cdot \rangle_Y)$ and $(X, \langle \cdot, \cdot \rangle_X)$ are (real or complex) Hilbert spaces such that $Y \subset X$, $Y$ is dense in $(X, \langle \cdot, \cdot \rangle_X)$ and the inclusion $(Y, \langle \cdot, \cdot \rangle_Y) \to (X, \langle \cdot, \cdot \rangle_X)$ is continuous. Then for every $u \in X$ there exists a unique $w_u \in Y$ such that

$$\langle v, w_u \rangle_Y = \langle v, u \rangle_X \text{ for all } v \in Y.$$

The map $B: X \to X$, $u \mapsto w_u$ is linear, symmetric and positive. Let $B^{1/2}$ be a square root of $B$, i.e. $B^{1/2}: X \to X$ linear, symmetric and $B^{1/2} \circ B^{1/2} = B$. Then $B$ and $B^{1/2}$ are injective and $R(B)$ is dense in $Y$. Set $X^{1/2} = X^{1/2}_B = R(B^{1/2})$ and $B^{-1/2}: X^{1/2} \to X$ be the inverse of $B^{1/2}$. On $X^{1/2}$ the assignment $\langle u, v \rangle_{1/2} := \langle B^{-1/2}u, B^{-1/2}v \rangle_X$ is a complete scalar product. We have $Y = X^{1/2}$ and $\langle \cdot, \cdot \rangle_Y = \langle \cdot, \cdot \rangle_{1/2}$.

Now let $A$ be the sectorial operator on $L^2(\Omega)$ defined by the differential operator $u \mapsto \beta u - \sum_{ij}(a_{ij}u_{x_j})_{x_i}$. Then $A$ generates a family $X^\alpha = X^\alpha_A$, $\alpha \in \mathbb{R}$, of fractional power spaces with $X^{-\alpha}$ being the dual of $X^\alpha$ for $\alpha \in ]0, \infty[$. We write

$$H_\alpha = X^{\alpha/2}, \quad \alpha \in \mathbb{R}.$$
For $\alpha \in \mathbb{R}$ the operator $A$ induces an operator $A_{\alpha}: H_{\alpha} \to H_{\alpha-2}$. In particular, $H_0 = L^2(\Omega)$ and $A = A_2$.

Note that, thanks to Assumption 1.1, the scalar product

$$\langle u, v \rangle_{H_0^1} = \langle A\nabla u, \nabla v \rangle + \langle \beta u, v \rangle, \quad u, v \in H_0^1(\Omega)$$

on $H_0^1(\Omega)$ is equivalent to the usual scalar product on $H_0^1(\Omega)$. Moreover,

$$\langle u, v \rangle_{H_0^1} = \langle A_2u, v \rangle, \quad u \in D(A_2), v \in H_0^1(\Omega).$$

**Corollary 2.2.** $H_1 = H_0^1(\Omega)$ with equivalent norms. Consequently $H_{-1} = H^{-1}(\Omega)$ with equivalent norms.

**Proof.** Set $(X, \langle \cdot, \cdot \rangle_X) = (L^2(\Omega), \langle \cdot, \cdot \rangle)$ and $(Y, \langle \cdot, \cdot \rangle_Y) = (H_0^1(\Omega), \langle \cdot, \cdot \rangle_{H_0^1})$. Then $Y$ is dense in $X$ and the inclusion $Y \to X$ is continuous. Let $B_2: X \to X$ be the inverse of $A_2$. Then for all $u \in X, B_2u \in Y$ and for all $v \in Y$

$$\langle v, u \rangle_X = \langle v, B_2u \rangle_Y.$$

Thus $B_2 = B$ where $B$ is as in Lemma 2.1. Now the lemma implies the corollary. □

**Corollary 2.3.** The linear operator $A_1: H_1 \to X := H_{-1}$ is self-adjoint hence sectorial on $X$. Let $X_0^\alpha, \alpha \in [0, \infty]$, be the family of fractional powers generated by $A_1$. Then $X^{1/2} = L^2(\Omega)$ with equivalent norms.

**Proof.** Set $(X, \langle \cdot, \cdot \rangle_X) = (H_{-1}, \langle \cdot, \cdot \rangle_{H_{-1}})$ and $(Y, \langle \cdot, \cdot \rangle_Y) = (H_0, \langle \cdot, \cdot \rangle_{H_0})$. Then $Y$ is dense in $X$ and the inclusion $Y \to X$ is continuous. Let $B_1: X \to X$ be the inverse of $A_1$. Then for all $u \in X, B_1u \in Y$ and for all $v \in Y$

$$\langle v, u \rangle_X = \langle B_1v, B_1u \rangle_{H_1} = \langle v, B_1u \rangle_Y.$$

Thus $B_1 = B$ where $B$ is as in Lemma 2.1. Now the lemma implies the corollary. □

We end this section by quoting a result proved in [16], which can be used to rigorously justify formal differentiation of various functionals along (mild) solutions of semilinear evolution equations.

**Theorem 2.4.** Let $Z$ be a Banach space and $B: D(B) \subset Z \to Z$ the infinitesimal generator of a $(C_0)$-semigroup of linear operators $e^{-Bt}$ on $Z$, $t \in [0, \infty[$. Let $U$ be open in $Z$, $Y$ be a normed space and $V: U \to Y$ be a function which, as a map from $Z$ to $Y$, is continuous at each point of $U$ and Fréchet differentiable at each point of $U \cap D(B)$. Moreover, let $W: U \times Z \to Y$ be a function which, as a map from $Z \times Z$ to $Y$, is continuous and such that $D^\tau V(z)(Bz + w) = W(z, w)$ for $z \in U \cap D(B)$ and $w \in Z$. Let $\tau \in [0, \infty[$ and $I := [0, \tau]$. Let $\bar{z} \in U$, $g: I \to Z$ be continuous and $z$ be a map from $I$ to $U$ such that

$$z(t) = e^{-Bt}\bar{z} + \int_0^t e^{-B(t-s)}g(s)\, ds, \quad t \in I.$$

Then the map $V \circ z: I \to Y$ is differentiable and

$$(V \circ z)'(t) = W(z(t), g(t)), \quad t \in I.$$
3. Proof of the main result

In order to establish our main result we need uniform estimates for the attractors $A_{\varepsilon}$ in $H_0^1(\Omega) \times L^2(\Omega)$.

**Lemma 3.1.** Let $f$ be as in Assumption 1.2. Then there is a constant $C \in [0, \infty[$ such that for all $u, v \in \mathbb{R}$ and for a.e. $x \in \Omega$,

$$|\partial_u f(x, u)| \leq C(1 + |u|^2),$$

$$|\partial_u f(x, v) - \partial_u f(x, u)| \leq C(1 + |u| + |v - u|)|v - u|$$

and

$$|f(x, v) - f(x, u) - \partial_u f(x, u)(v - u)| \leq C(1 + |u| + |v - u|)|v - u|^2.$$

**Proof.** For all $u, v \in \mathbb{R}$ and a.e. $x \in \Omega$ we have

$$\partial_u f(x, v) - \partial_u f(x, u) = \int_0^1 \partial_{uu} f(x, u + s(v - u))(v - u) \, ds$$

and

$$f(x, v) - f(x, u) - \partial_u f(x, u)(v - u) = (v - u)^2 \int_0^1 \theta \left[ \int_0^1 \partial_{uu} f(x, u + r\theta(v - u)) \, dr \right] \, d\theta$$

This easily implies the assertions of the lemma. \qed

**Proposition 3.2.** Let $f$ and $F$ be as in Assumption 1.2. Then, for every measurable function $v: \Omega \to \mathbb{R}$, both $\hat{f}(v)$ and $\hat{F}(v)$ are measurable and for all measurable functions $u, h: \Omega \to \mathbb{R}$

(3.1) \quad $|\hat{f}(u)|_{L^2} \leq |\hat{f}(0)|_{L^2} + C(|u|_{L^2} + |u|_{L^6}^3),$

(3.2) \quad $|\hat{f}(u + h) - \hat{f}(u)|_{L^2} \leq C|h|_{L^2} + C(|u|_{L^6}^2 + |h|_{L^6})|h|_{L^6},$

(3.3) \quad $|\hat{F}(u)|_{L^1} \leq C(|u|_{L^2}^2 + |u|_{L^6}^4/4 + |u|_{L^2} |\hat{f}(0)|_{L^2},$

(3.4) \quad $|\hat{F}(u + h) - \hat{F}(u)|_{L^1} \leq (|\hat{f}(0)|_{L^2} + C(|u|_{L^2} + |h|_{L^2}) + 4C(|u|_{L^6}^2 + |h|_{L^6}) |h|_{L^2},$

and

(3.5) \quad $|\hat{F}(u + h) - \hat{F}(u) - \hat{f}(u)h|_{L^1} \leq (C|h|_{L^2} + C(|u|_{L^6}^2 + |h|_{L^6})|h|_{L^6})|h|_{L^2}.$

Finally, for every $r \in [3, \infty[$ there is a constant $C(r) \in [0, \infty[$ such that for all $u, h \in H_0^1(\Omega)$

(3.6) \quad $|\hat{f}(u + h) - \hat{f}(u)|_{H^{-1}} \leq C(r)|h|_{L^2} + C(r) |u|_{L^6}^2 + |h|_{L^6}^2 |h|_{L^2}.$

**Proof.** Lemma 3.1 implies that $f$ satisfies the hypotheses of [16, Proposition 3.11], to which the reader is referred for details. \qed

For $s \in [2, 6]$ we denote by $C_s \in [0, \infty[$ an imbedding constant of the inclusion induced map from $H_1$ to $L^s(\Omega)$. 

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Proposition 3.3. Let \( f \) be as in Assumption 1.2, \( I \subset \mathbb{R} \) be an interval, \( u \) be a continuous map from \( I \) to \( H_1 \) such that \( u \) is continuously differentiable into \( H_0 \) with \( v := \partial(u; H_0) \). Then the composite map \( \hat{f} \circ u : I \to H_0 \) is defined, \( \hat{f} \circ u \) is continuously differentiable into \( H_{-1} \) and \( g := \partial(\hat{f} \circ u; H_{-1}) = (\partial_u \hat{f} \circ u) \cdot v \). Moreover, for every \( t \in I \),

\[
|g(t)|_{H_{-1}} \leq C(C_2 + C_6|u(t)|_{L^2})|v(t)|_{L^2} \leq C(C_2 + C_6C_3|u(t)|_{H_1})|v(t)|_{L^2}.
\]

Proof. It follows from Proposition 3.2 that for every \( w \in H_1, \hat{f}(w) \in H_0 \). Thus \( \hat{f} \circ u \) is defined as a function from \( I \) to \( H_0 \). Moreover, for every \( t \in I \) and \( \zeta \in H_1 \), the function \( \partial_u \hat{f}(u(t)) \cdot v(t) \cdot \zeta : \Omega \to \mathbb{R} \) is measurable and so by Lemma 3.1 and Hölder's inequality

\[
|\partial_u \hat{f}(u(t)) \cdot v(t) \cdot \zeta|_{L^1} \leq C|v(t)|_{L^2}|\zeta|_{L^2} + C|u(t)|^2|v(t)|_{L^2}|\zeta|_{L^6}.
\]

It follows that for every \( t \in \mathbb{R}, g(t) = \partial_u \hat{f}(u(t)) \cdot v(t) \in H_{-1} \) and (3.7) is satisfied. Moreover, for \( s, t \in I \),

\[
|\partial_u \hat{f}(u(t)) \cdot v(t) - \partial_u \hat{f}(u(s)) \cdot v(s)|_{H_{-1}} \leq \sup_{\zeta \in H_1, |\zeta|_{H_1} \leq 1} |\partial_u \hat{f}(u(t)) \cdot v(t) \cdot \zeta - \partial_u \hat{f}(u(s)) \cdot v(s) \cdot \zeta|_{L^1} \leq \sup_{\zeta \in H_1, |\zeta|_{H_1} \leq 1} T_1(t)(\zeta) + \sup_{\zeta \in H_1, |\zeta|_{H_1} \leq 1} T_2(t)(\zeta),
\]

where

\[
T_1(t)(\zeta) = |(\partial_u \hat{f}(u(t)) - \partial_u \hat{f}(u(s))) \cdot v(t) \cdot \zeta|_{L^1}
\]

and

\[
T_2(t)(\zeta) = |\partial_u \hat{f}(u(s)) \cdot (v(t) \cdot \zeta - v(s) \cdot \zeta)|_{L^1}.
\]

By Lemma 3.1 we obtain, for all \( \zeta \in H_1 \) with \( |\zeta|_{1} \leq 1 \),

\[
T_1(t)(\zeta) \leq C|1 + |u(s)|| + |u(t) - u(s)|| \cdot |u(t) - u(s)| \cdot |\zeta|_{L^2}|v(t)|_{L^2}
\]

\[
\leq C|u(t) - u(s)|_{L^3}|v(t)|_{L^2}|\zeta|_{L^6} + C|u(s)|_{L^6}|u(t) - u(s)|_{L^6}|v(t)|_{L^2}|\zeta|_{L^6}
\]

\[
+ C|u(t) - u(s)|_{L^6}|u(t) - u(s)|_{L^6}|v(t)|_{L^2}|\zeta|_{L^6} \leq CC_3C_6|u(t) - u(s)|_{H_1}|v(t)|_{L^2} + CC_3^3|u(s)|_{H_1}|u(t) - u(s)|_{H_1}|v(t)|_{L^2}
\]

\[
+ CC_6^2|u(t) - u(s)|_{H_1}^2|v(t)|_{L^2}.
\]

and

\[
T_2(t)(\zeta) \leq C|1 + |u(s)||^2 \cdot |\zeta|_{L^2}|v(t) - v(s)|_{L^2}
\]

\[
\leq C(|\zeta|_{L^2} + ||u(s)||^2 |\zeta|_{L^6}|v(t) - v(s)|_{L^2}
\]

\[
\leq C(C_2 + C_6^3|u(s)|_{H_1}^2)|v(t) - v(s)|_{L^2}.
\]
Since $u$ is continuous into $H_1$ and $v$ is continuous into $H_0 = L^2(\Omega)$ it follows that
\[
\sup_{\zeta \in H_1, |\zeta|_{H_1} \leq 1} T_1(t)(\zeta) + \sup_{\zeta \in H_1, |\zeta|_{H_1} \leq 1} T_2(t)(\zeta) \to 0 \text{ as } t \to s
\]
so the map $(\partial_u \hat{f} \circ u) \cdot v$ is continuous into $H_{-1}$.

Now, for $s, t \in I$, $t \neq s$,
\[
(t-s)^{-1}|(\hat{f} \circ u)(t) - (\hat{f} \circ u)(s) - \partial_u \hat{f}(u(s)) \cdot v(s)|_{H_{-1}}
\]
\[
= \sup_{\zeta \in H_1, |\zeta|_{H_1} \leq 1} (t-s)^{-1}|(\hat{f} \circ u)(t) \cdot \zeta - (\hat{f} \circ u)(s) \cdot \zeta - \partial_u \hat{f}(u(s)) \cdot v(s) \cdot \zeta|_{L^1}
\]
\[
\leq (t-s)^{-1} \sup_{\zeta \in H_1, |\zeta|_{H_1} \leq 1} T_3(t)(\zeta) + (t-s)^{-1} \sup_{\zeta \in H_1, |\zeta|_{H_1} \leq 1} T_4(t)(\zeta)
\]
where
\[
T_3(t)(\zeta) = |g_{t,\zeta}|_{L^1}
\]
with $g_{t,\zeta} = (\hat{f} \circ u)(t) \cdot \zeta - (\hat{f} \circ u)(s) \cdot \zeta - \partial_u \hat{f}(u(s)) \cdot (u(t) - u(s)) \cdot \zeta$ and
\[
T_4(t)(\zeta) = |\partial_u \hat{f}(u(s)) \cdot (u(t) - u(s) - v(s)) \cdot \zeta|_{L^1}.
\]

Now, by Lemma 3.1, for all $\zeta \in H_1$ with $|\zeta|_{H_1} \leq 1$ and for a.e. $x \in \Omega$
\[
|g_{t,\zeta}(x)| \leq C(1 + |u(s)(x)| + |u(t)(x) - u(s)(x)||u(t)(x) - u(s)(x)|^2)|\zeta(x)|
\]
so
\[
T_3(t)(\zeta) \leq C(|u(t) - u(s)|_{L^2}|u(t) - u(s)|_{L^2}|\zeta|_{L^6})
\]
\[
+ C(|u(s)|_{L^6}|u(t) - u(s)|_{L^6}|u(t) - u(s)|_{L^2}|\zeta|_{L^6})
\]
\[
+ C(|u(t) - u(s)|_{L^6}|u(t) - u(s)|_{L^6}|u(t) - u(s)|_{L^2}|\zeta|_{L^6})
\]
\[
\leq CC_6(C_3|u(t) - u(s)|_{H_1}|u(t) - u(s)|_{L^2})
\]
\[
+ CC_6(C_0^2|u(s)|_{H_1}|u(t) - u(s)|_{H_1}|u(t) - u(s)|_{L^2})
\]
\[
+ CC_6(C_0^2|u(t) - u(s)|_{H_1}^2|u(t) - u(s)|_{L^2}).
\]

Since $u$ is continuous into $H_1$ and locally Lipschitzian into $H_0 = L^2(\Omega)$ it follows from (3.8) that
\[
(t-s)^{-1} \sup_{\zeta \in H_1, |\zeta|_{H_1} \leq 1} T_3(t)(\zeta) \to 0 \text{ as } t \to s.
\]

We also have
\[
T_4(t)(\zeta) \leq C(|1 + |u(s)|^2|\zeta|_{L_2}|u(t) - u(s) - v(s)|_{L^2})
\]
\[
\leq (C|\zeta|_{L_2} + C|u(s)|^2|L^3|\zeta|_{L^6}|u(t) - u(s) - v(s)|_{L^2}
\]
\[
\leq C(C_2 + CC_0^3|u(s)|_{H_1}^2|u(t) - u(s) - v(s)|_{L^2}
\]
Since $(t-s)^{-1}|u(t) - u(s) - v(s)|_{L^2} \to 0$ as $t \to s$ it follows that
\[
(t-s)^{-1} \sup_{\zeta \in H_1, |\zeta|_{H_1} \leq 1} T_4(t)(\zeta) \to 0 \text{ as } t \to s.
\]

It follows that $\hat{f} \circ u$, as a map into $H_{-1}$, is differentiable at $s$ and $\partial_u(\hat{f} \circ u; H_{-1})(s) = (\partial_u f \circ u)(s) \cdot v(s)$. The proposition is proved. \qed
**Proposition 3.4.** Let $\varepsilon \in ]0, \infty[ \,$ be arbitrary. Define the function $\tilde{V} = \tilde{V}_\varepsilon: H_1 \times H_0 \to \mathbb{R}$ by

$$
\tilde{V}(u, v) = (1/2)\langle u, u \rangle_{H_1} + (1/2)\varepsilon \langle v, v \rangle - \int_{\Omega} F(x, u(x)) \, dx, \quad (u, v) \in H_1 \times H_0.
$$

Let $z: \mathbb{R} \to H_1 \times H_0$, $z(t) = (z_1(t), z_2(t))$, $t \in \mathbb{R}$, be a solution of $\pi_\varepsilon$. Then $\tilde{V} \circ z$ is differentiable and

$$(\tilde{V} \circ z)'(t) = -\|z_2(t)\|_{L^2}^2, \quad t \in \mathbb{R}.$$

**Proof.** This is an application of Theorem 2.4 (for the details see [16, Proposition 4.1]). \qed

**Proposition 3.5.** Let $\varepsilon \in ]0, \infty[ \,$ be arbitrary. Define the function $V = V_\varepsilon: H_0 \times H_{-1} \to \mathbb{R}$ by

$$
V(v, w) = (1/2)\langle v, v \rangle + (1/2)\varepsilon \langle w, w \rangle_{H_{-1}}, \quad (v, w) \in H_0 \times H_{-1}.
$$

Let $z: \mathbb{R} \to H_1 \times H_0$, $z(t) = (z_1(t), z_2(t))$, $t \in \mathbb{R}$, be a solution of $\pi_\varepsilon$. Then $z = (z_1, z_2)$ is differentiable as a map into $H_0 \times H_{-1}$ with $z_2 = \partial(z_1; H_0)$. Let

$u = z_1, \quad v = z_2, \quad w = \partial(v; H_{-1})$ and $g = (\partial_uf \circ u) \cdot v$. Then the function $\alpha: \mathbb{R} \to \mathbb{R}, \quad t \mapsto V(v(t), w(t))$ is differentiable and for every $t \in \mathbb{R}$

$$
\alpha'(t) = -\langle w(t), w(t) \rangle_{H_{-1}} + \langle g(t), w(t) \rangle_{H_{-1}}.
$$

**Proof.** For $\varepsilon \in ]0, \infty[ \,$ and $\kappa \in \mathbb{R}$ let $B_{\varepsilon, \kappa}: H_\kappa \times H_{\kappa-1} \to H_{\kappa-1} \times H_{\kappa-2}$ be defined by

$$
B_{\varepsilon, \kappa}(z) = (-z_2, (1/\varepsilon)(z_2 + A_{\kappa}z_1)), \quad z = (z_1, z_2) \in H_\kappa \times H_{\kappa-1}.
$$

It follows that $-B_{\varepsilon, \kappa}$ is $m$-dissipative on $H_{\kappa-1} \times H_{\kappa-2}$ (cf [16, proof of Proposition 3.6]). Moreover, if $z: \mathbb{R} \to H_1 \times H_0$ is a solution of $\pi_\varepsilon$, then

$$
\begin{align*}
  z(t) &= e^{-B_{\varepsilon, \kappa}(t-t_0)}z(t_0) + \int_{t_0}^t e^{-B_{\varepsilon, \kappa}(t-s)}(0, (1/\varepsilon)\tilde{f}(z_1(s))) ; H_1 \times H_0) \, ds \\
  &= e^{-B_{\varepsilon, 1}(t-t_0)}z(t_0) + \int_{t_0}^t e^{-B_{\varepsilon, 1}(t-s)}(0, (1/\varepsilon)\tilde{f}(z_1(s))) ; H_0 \times H_{-1}) \, ds,
\end{align*}
$$

$t, t_0 \in \mathbb{R}, t_0 \leq t$.

Since $z(t_0) \in D(B_{\varepsilon, 1})$ and $t \mapsto (0, (1/\varepsilon)\tilde{f}(z_1(s)))$ is continuous into $D(B_{\varepsilon, 1})$ it follows from [9, proof of Theorem II.1.3 (i)] that $z = (u, v)$ is differentiable as a map into $H_0 \times H_{-1}$ with $v = \partial(u; H_0)$. Now, in $H_{-1}$,

$$
w = \partial(v; H_{-1}) = (1/\varepsilon)(v - A_1 \circ u + \tilde{f} \circ u) = (1/\varepsilon)(v - A_0 \circ u + \tilde{f} \circ u).$$
It follows from Proposition 3.3 that \( w \) is differentiable into \( H_{-2} \) and
\[
\partial(w; H_{-2}) = (1/\varepsilon)(w - A_0 \circ v + g).
\]
Again [9, proof of Theorem II.1.3 (i)] implies that
\[
(v, w)(t) = e^{-B_{\varepsilon, -1}(t-t_0)}(v, w)(t_0) + \int_{t_0}^{t} (e^{-B_{\varepsilon, -1}(t-s)}(0, (1/\varepsilon)g(s)); H_{-2} \times H_{-3}) \, ds
\]
\[
= e^{-B_{\varepsilon, 1}(t-t_0)}(v, w)(t_0) + \int_{t_0}^{t} (e^{-B_{\varepsilon, 1}(t-s)}(0, (1/\varepsilon)g(s)); H_{0} \times H_{-1}) \, ds,
\]
\( t, t_0 \in \mathbb{R}, t_0 \leq t \).

Now note that the function \( V = V_\varepsilon \) is Fréchet differentiable and
\[
DV(v, w)(\bar{v}, \bar{w}) = \langle v, \bar{v} \rangle_{H_0} + \varepsilon \langle w, \bar{w} \rangle_{H_{-1}}.
\]
Thus for \((u, v) \in D(-B_{\varepsilon, 1}) = H_1 \times H_0 \) and \((\bar{v}, \bar{w}) \in H_0 \times H_{-1}\)
\[
DV(v, w)(-B_{\varepsilon, 1}(v, w) + (\bar{v}, \bar{w})) = \langle v, w + \bar{v} \rangle_{H_0}
\]
\[
+ \varepsilon \langle w, -(1/\varepsilon)w - (1/\varepsilon)A_1v + \bar{w} \rangle_{H_{-1}} = \langle v, \bar{v} \rangle_{H_0} - \langle w, w \rangle_{H_{-1}} + \varepsilon \langle w, \bar{w} \rangle_{H_{-1}}.
\]
Here we have used the fact that
\[
\langle w, A_1v \rangle_{H_{-1}} = \langle A_1^{-1}w, A_1^{-1}A_1v \rangle_{H_1} = \langle A_1^{-1}w, v \rangle_{H_1} = \langle w, v \rangle_{H_0}
\]
as \( A_1^{-1}w = A_2^{-1}w \in H_2 \). Defining \( W: (H_0 \times H_{-1}) \times (H_0 \times H_{-1}) \to \mathbb{R} \) by
\[
W((v, w), (\bar{v}, \bar{w})) = \langle v, \bar{v} \rangle_{H_0} - \langle w, w \rangle_{H_{-1}} + \varepsilon \langle w, \bar{w} \rangle_{H_{-1}}
\]
we see that \( W \) is continuous. Now it follows from (3.9) and Theorem 2.4 that \( \alpha = V_\varepsilon \circ (v, w) \) is differentiable and
\[
\alpha'(t) = -\langle w(t), w(t) \rangle_{H_{-1}} + \langle w(t), g(t) \rangle_{H_{-1}}, \quad t \in \mathbb{R}.
\]
The proof is complete. \( \square \)

**Proposition 3.6.** Let \( \varepsilon_0 \in ]0, \infty[ \) be arbitrary. Then for every \( r \in ]0, \infty[ \) there is a constant \( C(r, \varepsilon_0) \in ]0, \infty[ \) such that whenever \( \varepsilon \in ]0, \varepsilon_0] \) and \( z = (u, v): \mathbb{R} \to H_1 \times H_0 \) is a solution of \( \pi_\varepsilon \) with \( \sup_{t \in \mathbb{R}} (|u(t)|_{H_1}^2 + \varepsilon |v(t)|_{H_0}^2) \leq r \) and \( w := \partial(v; H_{-1}) \), then
\[
\sup_{t \in \mathbb{R}} (|v(t)|_{H_0}^2 + \varepsilon |w(t)|_{H_{-1}}^2) \leq C(r, \varepsilon_0).
\]

**Proof.** By \( C_1(r) \in ]0, \infty[ \), resp. \( C_1(r, \varepsilon_0) \in ]0, \infty[ \) we denote various constants depending only on \( r \), resp. on \( r \) and \( \varepsilon_0 \), but independent of \( \varepsilon \in ]0, \varepsilon_0] \) and the choice of a solution \( z \) of \( \pi_\varepsilon \) with \( \sup_{t \in \mathbb{R}} (|u(t)|_{H_1}^2 + \varepsilon |v(t)|_{H_0}^2) \leq r \).
Let \( \varepsilon \in [0, \varepsilon_0] \) be arbitrary, \( \alpha(t) = V_\varepsilon(v(t), w(t)), t \in \mathbb{R} \), and \( g = (\partial_u \hat{f} \circ u) \cdot v \).

Using (3.7) we see that

\[
|g(t)|_{H_{-1}} \leq C(1 + C_6 C_3^2 r^2)|v(t)|_{H_0}, \quad t \in \mathbb{R}.
\]

Proposition 3.5 implies that

\[
\alpha'(t) \leq -|w(t)|_{H_{-1}}^2 + (1/2)|g(t)|_{H_{-1}}^2 + (1/2)|w(t)|_{H_{-1}}^2
\]

\[
\leq -(1/2)|w(t)|_{H_{-1}}^2 + (1/2)C^2(1 + C_6 C_3^2 r^2)^2|v(t)|_{H_0}^2, \quad t \in \mathbb{R}.
\]

Thus we obtain, for every \( k \in ]0, \infty[ \),

\[
\alpha'(t) + k\alpha(t) \leq (1/2) + (k\varepsilon/2))|w(t)|_{H_{-1}}^2
\]

\[
+ ((1/2)C^2(1 + C_6 C_3^2 r^2)^2 + (k/2))|v(t)|_{H_0}^2, \quad t \in \mathbb{R}.
\]

Choose \( k = k(\varepsilon_0) \in ]0, \infty[ \) such that \( -(1/2) + (k\varepsilon_0/2) < 0 \). Hence we obtain

\[
\alpha'(t) + k\alpha(t) \leq C_1(r, \varepsilon_0)|v(t)|_{H_0}^2 \quad t \in \mathbb{R}.
\]

Using Propositions 3.4 and 3.2 we see that

\[
\int_{t_0}^t |v(s)|_{H_0}^2 \leq C_2(r, \varepsilon_0), \quad t, t_0 \in \mathbb{R}, t_0 \leq t.
\]

It follows that

\[
\alpha(t) = e^{-k(t-t_0)} \alpha(t_0) + C_1(r, \varepsilon_0) \int_{t_0}^t e^{-k(t-s)}|v(s)|_{H_0}^2 ds
\]

\[
\leq e^{-k(t-t_0)} \alpha(t_0) + C_3(r, \varepsilon_0), \quad t, t_0 \in \mathbb{R}, t_0 \leq t.
\]

Using the definition of \( \alpha \) we thus obtain from (3.12)

\[
(1/2)|v(t)|_{H_0}^2 + (1/2)\varepsilon|w(t)|_{H_{-1}}^2 \leq e^{-k(t-t_0)}((1/2)|v(t_0)|_{H_0}^2 + (1/2)\varepsilon|w(t_0)|_{H_{-1}}^2)
\]

\[
+ C_3(r, \varepsilon_0), \quad t, t_0 \in \mathbb{R}, t_0 \leq t.
\]

Since for \( t \in \mathbb{R} \), \( \varepsilon w(t) = -v(t) - A_1 u(t) + \hat{f}(u(t)) \) in \( H_{-1} \), it follows that

\[
\varepsilon|w(t)|_{H_{-1}} \leq |v(t)|_{H_{-1}} + |u(t)|_{H_1} + |\hat{f}(u(t))|_{H_{-1}}
\]

\[
\leq |v(t)|_{H_{-1}} + C_5(r) \leq C_6(r)\varepsilon^{-1/2} + C_5(r), \quad t \in \mathbb{R}.
\]

Thus

\[
\varepsilon|w(t_0)|_{H_{-1}}^2 \leq (1/\varepsilon)(C_6(r)\varepsilon^{-1/2} + C_5(r))^2.
\]
Furthermore,

\begin{equation}
|v(t_0)|^2_{H_0} \leq r/\varepsilon.
\end{equation}

Inserting (3.14) and (3.15) into (3.13) and letting \( t_0 \to -\infty \) we thus see that

\[ |v(t)|^2_{H_0} + \varepsilon|w(t)|^2_{H_{-1}} \leq 2C_3(r, \varepsilon_0), \quad t \in \mathbb{R}. \]

This completes the proof. \( \square \)

Fix a \( C^\infty \)-function \( \overline{\vartheta}: \mathbb{R} \to [0, 1] \) with \( \overline{\vartheta}(s) = 0 \) for \( s \in ]-\infty, 1] \) and \( \overline{\vartheta}(s) = 1 \) for \( s \in [2, \infty[. \) Let

\[ \vartheta := \overline{\vartheta}^2. \]

For \( k \in \mathbb{N} \) let the functions \( \overline{\vartheta}_k: \mathbb{R}^N \to \mathbb{R} \) and \( \vartheta_k: \mathbb{R}^N \to \mathbb{R} \) be defined by

\[ \overline{\vartheta}_k(x) = \overline{\vartheta}(|x|^2/k^2) \quad \text{and} \quad \vartheta_k(x) = \vartheta(|x|^2/k^2), \quad x \in \mathbb{R}^N. \]

The following theorem (actually a rephrasing of Theorem 4.4 in [16]) provides the "tail-estimates" mentioned in the Introduction:

**Theorem 3.7.** Let Assumptions 1.1 and 1.2 be satisfied. Let \( \varepsilon_0 > 0 \) be fixed. Choose \( \delta \) and \( \nu \in ]0, \infty[ \) with

\[ \nu \leq \min(1, \pi/2), \quad \lambda_1 - \delta > 0 \quad \text{and} \quad 1 - 2\delta\varepsilon_0 \geq 0. \]

Under these hypotheses, there is a constant \( c' \in ]0, \infty[ \) and for every \( R \in ]0, \infty[ \) there are constants \( M' = M'(R), \) \( c_k = c_k(R) \in ]0, \infty[, \) \( k \in \mathbb{N} \) with \( c_k \to 0 \) for \( k \to \infty, \) such that for every \( \tau_0 \in ]0, \infty[, \) every \( \varepsilon, 0 < \varepsilon \leq \varepsilon_0 \) and every solution \( z(\cdot) \) of \( \pi_\varepsilon \) on \( I = [0, \tau_0] \) with \( |z(0)|_Z \leq R \)

\[ |z_1(t)|^2_{H_1} + \varepsilon|z_2(t)|^2_{H_0} \leq c' + M'e^{-2\delta \nu t}, \quad t \in I. \]

If \( |z(t)|_Z \leq R \) for \( t \in I, \) then

\[ |\partial_k z_1(t)|^2_{H_1} + \varepsilon|\partial_k z_2(t)|^2_{H_0} \leq c_k + M'e^{-2\delta \nu t}, \quad k \in \mathbb{N}, t \in I. \]

Now we can prove the following fundamental result:

**Theorem 3.8.** Let \( (\varepsilon_n)_n \) be a sequence of positive numbers converging to 0. For each \( n \in \mathbb{N} \) let \( z_n = (u_n, v_n): \mathbb{R} \to H_1 \times H_0 \) be a solution of \( \pi_{\varepsilon_n} \) such that

\[ \sup_{n \in \mathbb{N}} \sup_{t \in \mathbb{R}} (|u_n(t)|^2_{H_1} + \varepsilon_n |v_n(t)|^2_{H_0}) \leq r < \infty. \]

Then, for every \( \alpha \in ]0, 1], \) a subsequence of \( (z_n)_n \) converges in \( H_1 \times H_{-\alpha}, \) uniformly on compact subsets of \( \mathbb{R}, \) to a function \( z: \mathbb{R} \to H_1 \times H_0 \) with \( z = (u, v), \) where \( u \) is a solution of \( \bar{\pi} \) and \( v = \partial(u; H_0). \)
Proof. We may assume that $\varepsilon_n \in [0, \varepsilon_0]$ for some $\varepsilon_0 \in ]0, \infty[$ and all $n \in \mathbb{N}$. Write $u_n = z_{n,1}$ and $v_n = z_{n,2}$, and $n \in \mathbb{N}$. We claim that for every $t \in \mathbb{R}$, the set $\{ u_n(t) | n \in \mathbb{N} \}$ is relatively compact in $H_0$. Let $\vartheta_k, k \in \mathbb{N}$, be as above. Then, choosing $k \in \mathbb{N}$ large enough and using Theorem 3.7 we can make $\sup_{n \in \mathbb{N}} |\vartheta_k u_n(t)|_{H_1}$ as small as we wish. Therefore, by a Kuratowski measure of noncompactness argument, we only have to prove that for every $k \in \mathbb{N}$, the set $S_k = \{ (1 - \vartheta_k)u_n(t) | n \in \mathbb{N} \}$ is relatively compact in $H_0$. Let $U$ be the ball in $\mathbb{R}^N$ with radius $2k$ centered at zero. Then $(1 - \vartheta_k)(U) \subset C^1_0(U)$, so $(1 - \vartheta_k)\tilde{u}_n(t)U \in H^1_0(U)$ for $n \in \mathbb{N}$. Since $H^1_0(U)$ imbeds compactly in $L^2(U)$ and $(1 - \vartheta_k)\tilde{u}_n(t)|(\mathbb{R}^N \setminus U) \equiv 0$, it follows that, indeed, $S_k$ is relatively compact in $H_0$. This proves our claim.

Since, by Proposition 3.6, for each $n \in \mathbb{N}$, $u_n$ is differentiable into $H_0$ and $v_n = \partial(u_n; H_0)$ is bounded in $H_0$ uniformly $t \in \mathbb{R}$ and $n \in \mathbb{N}$, we may assume, using the above claim and Arzelà-Ascoli theorem, and taking subsequences if necessary, that $(u_n)_n$ converges in $H_0$, uniformly on compact subsets of $\mathbb{R}$, to a continuous function $u: \mathbb{R} \to H_0$. Moreover, since, for each $t \in \mathbb{R}$, $(u_n(t))_n$ has a subsequence that is weakly convergent in $H_1$, we see that $u$ takes its values in $H_1$. Let $w_n = \partial(v; H_{-1})$, $n \in \mathbb{N}$.

For every $n \in \mathbb{N}$ and every $t \in \mathbb{R}$,

\begin{equation}
\varepsilon_n w_n(t) = -v_n(t) - A_0 u_n(t) + f(u_n(t))
\end{equation}

in $H_{-1}$. Now, uniformly for $t$ lying in compact subsets of $\mathbb{R}$, $\hat{f}(u_n(t)) \to \hat{f}(u(t))$ in $H_{-1}$ (by Proposition 3.2), $A_0 u_n(t) \to A_0 u(t)$ in $H_{-2}$ and $\varepsilon_n w_n(t) \to 0$ in $H_{-1}$ (by Proposition 3.6). It follows from (3.16) that, uniformly for $t$ in compact subsets of $\mathbb{R}$, $v_n(t) \to v(t)$ in $H_{-2}$, where $v: \mathbb{R} \to H_{-2}$ is a continuous map such that, for every $t \in \mathbb{R}$,

$$v(t) = -A_0 u(t) + \hat{f}(u(t))$$

in $H_{-2}$. It follows that $u$ is differentiable into $H_{-2}$ and $v = \partial(u; H_{-2})$. Then $u$ is differentiable into $H_{-3}$ and, for all $t \in \mathbb{R}$,

$$\partial(u; H_{-3})(t) = -A_{-1} u(t) + \hat{f}(u(t))$$

in $H_{-3}$. Since $\hat{f} \circ u$ is continuous into $D(A_{-1}) = H_{-1}$ it follows that

\begin{equation}
\hat{u}(t) = e^{-A_{-1}(t-t_0)} u(t_0) + \int_{t_0}^{t} (e^{-A_{-1}(t-s)} \hat{f}(u(s)); H_{-3}) \, ds
\end{equation}

$$= e^{-A_1(t-t_0)} u(t_0) + \int_{t_0}^{t} (e^{-A_1(t-s)} \hat{f}(u(s)); H_{-1}) \, ds, \quad t, t_0 \in \mathbb{R}, \, t_0 \leq t.$$

We claim that $u$ is a solution of $\tilde{\pi}$. To this end let $t_0 \in \mathbb{R}$ be arbitrary. Let $\tilde{u}: [0, \infty[ \to H_1$ be the solution of $\tilde{\pi}$ with $\tilde{u}(0) = u(t_0)$ ($\tilde{u}$ exists by results in [17]). We must show that $\tilde{u}(s) = u(s + t_0)$ for all $s \in [0, \infty[$. If not, then there is a $s_0 \geq 0$ with $\tilde{u}(s_0) = u(s_0 + t_0)$ and $\tilde{u}(s_0) \neq u(s_0 + t_0)$ for all $n \in \mathbb{N}$, where $(s_n)_n$ is a
sequence with \( s_n \to s_0^+ \) as \( n \to \infty \). By Corollary 2.3 there is a constant \( C \in [0, \infty[ \) such that
\[
|e^{-A_1 t} w|_{H_0} \leq C t^{-1/2} |w|_{H_{-1}}, \quad w \in H_{-1}, \quad t \in [0, \infty[.
\]
Moreover, by Proposition 3.2, for every \( b \in ]0, \infty[ \) there is an \( L(b) \in ]0, \infty[ \) such that for all \( w_i \in H_1, |w_i|_{H_1} \leq b, i = 1, 2, \)
\[
|\hat{f}(w_2) - \hat{f}(w_1)|_{H_{-1}} \leq L(b)|w_2 - w_1|_{H_0}.
\]
There is an \( \bar{s} \in [s_0, \infty[ \) such that whenever \( s \in [s_0, \bar{s}] \) then \( |u(s + t_0)|_{H_1} < r + 1 \) and \( |\bar{u}(s)|_{H_1} < r + 1 \). Let \( L = L(b) \) where \( b = r + 1 \). Choosing \( \bar{s} \) smaller, if necessary, we can assume that
\[
(3.18) \quad C L(\bar{s} - s_0)^{1/2}/2 < 1.
\]
It follows that, for each \( s \in [s_0, \bar{s}] \),
\[
u(s + t_0) - \bar{u}(s) = \int_{s_0}^s e^{-A_1 (s-r)} [\hat{f}(u(r + t_0)) - \hat{f}(\bar{u}(r))] \, dr
\]
so
\[
|u(s + t_0) - \bar{u}(s)|_{H_0} \leq C \int_{s_0}^s (s-r)^{-1/2} L [ |u(r + t_0) - \bar{u}(r)|_{H_0}] \, dr
\]
\[
\leq C L(\bar{s} - s_0)^{1/2}/2 \sup_{r \in [s_0, \bar{s}]} |u(r + t_0) - \bar{u}(r)|_{H_0}.
\]
In view of (3.18), we obtain that \( u(s + t_0) = \bar{u}(s) \) for \( s \in [s_0, \bar{s}] \), a contradiction, which proves our claim.

We now claim that \( u_n(t) \to u(t) \) in \( H_1 \), uniformly for \( t \) lying in compact subsets of \( \mathbb{R} \). If this claim is not true, then there is a strictly increasing sequence \( (n_k)_n \) in \( \mathbb{N} \) and a sequence \( (t_k)_k \) in \( \mathbb{R} \) converging to some \( t_\infty \in \mathbb{R} \) such that
\[
(3.19) \quad \inf_{k \in \mathbb{N}} |u_{n_k}(t_k) - u(t_\infty)|_{H_1} > 0.
\]
For \( \varepsilon \in ]0, \infty[ \) define the function \( F_{\varepsilon} : H_1 \times H_0 \to \mathbb{R} \) by
\[
F_{\varepsilon}(z) = (1/2)\varepsilon \langle \delta z_1 + z_2, \delta z_1 + z_2 \rangle + (1/2) \langle A \nabla z_1, \nabla z_1 \rangle
\]
\[
+ (1/2) \langle (\beta - \delta + \delta^2 \varepsilon) z_1, z_1 \rangle - \int_\Omega F(x, z_1(x)) \, dx
\]
where \( \delta \in ]0, \infty[ \) is such that \( \lambda_1 - \delta > 0 \) and \( 1 - 2\delta \varepsilon_0 > 0 \). Note that
\[
|u|^2 = \langle A \nabla u, \nabla u \rangle + \langle (\beta - \delta) u, u \rangle, \quad u \in H_1
\]
defines a norm on \( H_1 \) equivalent to the usual norm on \( H_1 \). Let \( \varepsilon \in ]0, \varepsilon_0[ \) and \( \zeta = (\zeta_1, \zeta_2) : ]0, \infty[ \to Z \) be an arbitrary solution of \( \pi_\varepsilon \). Using Theorem 2.4 (cf [16,
Proposition 4.1]) one can see that the function $F_{\varepsilon} \circ \zeta$ is continuously differentiable and for every $t \in [0, \infty[$

\[(F_{\varepsilon} \circ \zeta)'(t) + 2\delta F_{\varepsilon}(\zeta(t)) = \int_{\Omega} (2\delta \varepsilon - 1)(\delta \zeta_1(t)(x) + \zeta_2(t)(x))^2 \, dx\]

\[(3.20)\]

Moreover, define $F_0: H_1 \to \mathbb{R}$ by

\[F_0(u) = (1/2) \langle A\nabla u, \nabla u \rangle + (1/2)((\beta - \delta)u, u) - \int_{\Omega} F(x, u(x)) \, dx, \quad u \in H_1.\]

Every solution $\xi: \mathbb{R} \to H_1$ of $\pi$ is differentiable into $H_1$ so the function $F_0 \circ \xi$ is differentiable and a simple computation shows that for $t \in \mathbb{R}$,

\[(F_0 \circ \xi)'(t) + 2\delta (F_0 \circ \xi)(t) = -\langle \delta \xi(t) + \eta(t), \delta \xi(t) + \eta(t) \rangle\]

\[+ \int_{t}^{t} [\delta \xi(t)(x)f(x, \xi(t)(x)) - 2\delta F(x, \xi(t)(x))] \, dx\]

where $\eta(t) = -A_1 \xi(t) + \hat{f}(\xi(t)), t \in \mathbb{R}$.

Fix $l \in \mathbb{N}$ and, for $k \in \mathbb{N}$, let $\zeta(t) = z_{nk}(t_k - l + t)$ and $\zeta(t) = (u(t_{\infty} - l + t), v(t_{\infty} - l + t)$) for $t \in [0, \infty[$. Then (3.20) and (3.21) imply that

\[F_{\varepsilon_{nk}}(z_{nk}(t_k)) = e^{-2\delta l} F_{\varepsilon_{nk}}(z_{nk}(t_k - l))\]

\[\quad + (2\delta \varepsilon_{nk} - 1) \int_{0}^{l} e^{-2\delta(l-s)} \left( \int_{\Omega} (\delta \zeta_{1,1}(s)(x) + \zeta_{1,2}(s)(x))^2 \, dx \right) \, ds\]

\[+ \int_{0}^{l} e^{-2\delta(l-s)} \left( \int_{\Omega} \delta \zeta_{1,1}(s)(x)f(x, \zeta_{1,1}(s)(x)) \, dx - 2\delta \int_{\Omega} F(x, \zeta_{1,1}(s)(x)) \, dx \right) \, ds.\]

and

\[F_{0}(u(t_{\infty})) = e^{-2\delta l} F_{0}(u(t_{\infty} - l))\]

\[\quad - \int_{0}^{l} e^{-2\delta(l-s)} \left( \int_{\Omega} (\delta \zeta_1(s)(x) + \zeta_2(s)(x))^2 \, dx \right) \, ds\]

\[+ \int_{0}^{l} e^{-2\delta(l-s)} \left( \int_{\Omega} \delta \zeta_1(s)(x)f(x, \zeta_1(s)(x)) \, dx - 2\delta \int_{\Omega} F(x, \zeta_1(s)(x)) \, dx \right) \, ds.\]

Since $\zeta_{1,1}(s) \to \zeta_1(s)$ in $H_0$, uniformly for $s$ lying in compact subsets of $\mathbb{R}$, we obtain from Proposition 3.2 that

\[\int_{0}^{l} e^{-2\delta(l-s)} \left( \int_{\Omega} \delta \zeta_{1,1}(s)(x)f(x, \zeta_{1,1}(s)(x)) \, dx - 2\delta \int_{\Omega} F(x, \zeta_{1,1}(s)(x)) \, dx \right) \, ds\]

\[\to \int_{0}^{l} e^{-2\delta(l-s)} \left( \int_{\Omega} \delta \zeta_1(s)(x)f(x, \zeta_1(s)(x)) \, dx - 2\delta \int_{\Omega} F(x, \zeta_1(s)(x)) \, dx \right) \, ds\]
as \( k \to \infty \). We claim that

\[
\limsup_{k \to \infty} (2\delta \varepsilon_{n_k} - 1) \int_0^l e^{-2\delta(l-s)} \left( \int_\Omega (\delta \zeta_{k,1}(s)(x) + \zeta_{k,2}(s)(x))^2 \, dx \right) \, ds
\]

\[
\leq - \int_0^l e^{-2\delta(l-s)} \left( \int_\Omega (\delta \zeta_1(s)(x) + \zeta_2(s)(x))^2 \, dx \right) \, ds.
\]

(3.25)

In fact, since \( 1 - 2\delta \varepsilon_{n_k} \geq 0 \) for all \( k \in \mathbb{N} \) we have by Fatou's lemma

\[
\limsup_{k \to \infty} (2\delta \varepsilon_{n_k} - 1) \int_0^l e^{-2\delta(l-s)} \left( \int_\Omega (\delta \zeta_{k,1}(s)(x) + \zeta_{k,2}(s)(x))^2 \, dx \right) \, ds
\]

\[
= - \liminf_{k \to \infty} (1 - 2\delta \varepsilon_{n_k}) \int_0^l e^{-2\delta(l-s)} \left( \int_\Omega (\delta \zeta_{k,1}(s)(x) + \zeta_{k,2}(s)(x))^2 \, dx \right) \, ds
\]

\[
= - \liminf_{k \to \infty} \int_0^l e^{-2\delta(l-s)} \left( \int_\Omega (\delta \zeta_{k,1}(s)(x) + \zeta_{k,2}(s)(x))^2 \, dx \right) \, ds
\]

\[
\leq - \int_0^l e^{-2\delta(l-s)} \liminf_{k \to \infty} \left( \int_\Omega (\delta \zeta_{k,1}(s)(x) + \zeta_{k,2}(s)(x))^2 \, dx \right) \, ds.
\]

(3.26)

Let \( s \in [0, l] \) be arbitrary.

Since \( ((\zeta_{k,1}(s), \zeta_{k,2}(s)))_{k} \) converges to \((\zeta_1(s), \zeta_2(s))\) weakly in \( H_1 \times H_0 \) it follows that \( ((\zeta_{k,1}(s), \delta \zeta_{k,1}(s) + \zeta_{k,2}(s)))_{k} \) converges to \((\zeta_1(s), \delta \zeta_1(s) + \zeta_2(s))\) weakly in \( H_1 \times H_0 \). It follows that for every \( v \in L^2(\Omega) \)

\[
\langle v, \delta \zeta_{k,1}(s) + \zeta_{k,2}(s) \rangle \to \langle v, \delta \zeta_1(s) + \zeta_2(s) \rangle \text{ as } k \to \infty.
\]

Taking \( v = (\delta \zeta_1(s) + \delta \zeta_2(s)) \) we thus obtain

\[
|((\delta \zeta_1(s) + \delta \zeta_2(s)))|_{L^2}^2 = \langle (\delta \zeta_1(s) + \delta \zeta_2(s)), (\delta \zeta_1(s) + \delta \zeta_2(s)) \rangle
\]

\[
= \lim_{k \to \infty} \langle (\delta \zeta_1(s) + \delta \zeta_2(s)), (\delta \zeta_{k,1}(s) + \delta \zeta_{k,2}(s)) \rangle
\]

\[
\leq |(\delta \zeta_1(s) + \delta \zeta_2(s))|_{L^2} \liminf_{k \to \infty} |(\delta \zeta_{k,1}(s) + \delta \zeta_{k,2}(s))|_{L^2}
\]

and so

\[
(3.27) \quad \int_\Omega (\delta \zeta_1(s)(x) + \zeta_2(s)(x))^2 \, dx \leq \liminf_{k \to \infty} \int_\Omega (\delta \zeta_{k,1}(s)(x) + \zeta_{k,2}(s)(x))^2 \, dx.
\]

Inequalities (3.27) and (3.26) prove (3.25). Since, by Proposition 3.2,

\[
\int_\Omega F(x, u_{n_k}(t_k)(x)) \, dx \to \int_\Omega F(x, u(t_\infty)(x)) \, dx
\]

we obtain, using Proposition 3.6, that

\[
\limsup_{k \to \infty} \mathcal{F}_{\varepsilon_{n_k}} (z_{n_k}(t_k)) = (1/2) \limsup_{k \to \infty} \|u(t_k)\|^2 - \int_\Omega F(x, u(t_\infty)(x)) \, dx
\]
Moreover, there is a constant $C' \in [0, \infty[$ such that
\[
\sup_{k \in \mathbb{N}} \sup_{t \in \mathbb{R}} |\mathcal{F}_{z_{n_k}}(z_{n_k}(t))| + \sup_{t \in \mathbb{R}} |\mathcal{F}_0(u(t))| \leq C'.
\]
Thus
\[
(1/2) \limsup_{k \to \infty} \|u(t_k)\|^2 - \int_{\Omega} F(x, u(t_\infty)(x)) \, dx \leq e^{-2\delta l} C'
\]
\[
- \int_0^l e^{-2\delta(l-s)} \left( \int_{\Omega} (\delta \zeta_1(s)(x) + \zeta_2(s)(x))^2 \, dx \right) \, ds
\]
\[
+ \int_0^l e^{-2\delta(l-s)} \left( \int_{\Omega} \delta \zeta_1(s)(x)f(x, \zeta_1(s)(x)) \, dx - 2\delta \int_{\Omega} F(x, \zeta_1(s)(x)) \, dx \right) \, ds
\]
\[
= e^{-2\delta l} C' + (1/2)\|u(t_\infty)\|^2 - \int_{\Omega} F(x, u(t_\infty)(x)) \, dx
\]
\[
- e^{-2\delta l} \mathcal{F}_0(u(t_\infty - l)) \leq 2e^{-2\delta l} C' + (1/2)\|u(t_\infty)\|^2 - \int_{\Omega} F(x, u(t_\infty)(x)) \, dx.
\]
Thus for every $l \in \mathbb{N}$
\[
\limsup_{k \to \infty} \|u(t_k)\|^2 \leq 4e^{-2\delta l} C' + \|u(t_\infty)\|^2
\]
so
\[
\limsup_{k \to \infty} \|u(t_k)\| \leq \|u(t_\infty)\|.
\]
Since $(u_{n_k}(t_{n_k}))_k$ converges to $u(t_\infty)$ weakly in $H_1$ we have
\[
\liminf_{k \to \infty} \|u_{n_k}(t_{n_k})\| \geq \|u(t_\infty)\|.
\]
Altogether we obtain
\[
\lim_{k \to \infty} \|u_{n_k}(t_{n_k})\| = \|u(t_\infty)\|.
\]
This implies that $(u_{n_k}(t_{n_k}))_k$ converges to $u(t_\infty)$ strongly in $H_1$, a contradiction to (3.19). Thus, indeed, $u_n(t) \to u(t)$ in $H_1$, uniformly for $t$ lying in compact subsets of $\mathbb{R}$.

Now (3.16) implies that $v_n(t) \to v(t)$ in $H_{-1}$, uniformly for $t$ lying in compact subsets of $\mathbb{R}$. Since $(v_n)_n$ is bounded in $H_0$, interpolation between $H_0$ and $H_{-1}$ (cf. [17]) now implies that $v_n(t) \to v(t)$ in $H_{-\alpha}$, uniformly for $t$ lying in compact subsets of $\mathbb{R}$. The proof is complete. 

Now we obtain the main result of this paper.
Theorem 3.9. For every $\alpha \in [0,1]$ the family $(A_\varepsilon)_{\varepsilon \in [0,\infty]}$ is upper semicontinuous at $\varepsilon = 0$ with respect to the topology of $H_1 \times H_{-\alpha}$, i.e.

$$\lim_{\varepsilon \to 0^+} \sup_{y \in A_\varepsilon} \inf_{z \in A_0} |y - z|_{H_1 \times H_{-\alpha}} = 0.$$ 

Proof. Using the first part of Theorem 3.7, choosing $\varepsilon_0 \in ]0,\infty[$ arbitrarily and $\delta \in ]0,\infty[$ such that $\lambda_1 - \delta > 0$ and $1 - 2\delta \varepsilon_0 > 0$ and noting that the constant $c'$ in that theorem is independent of $\varepsilon \in ]0,\varepsilon_0]$, it follows that for all $\varepsilon \in ]0,\varepsilon_0]$ and all $(u,v) \in A_\varepsilon$,

$$|u|^2_{H_1} + \varepsilon |v|^2_{H_0} \leq 2c'.$$

Now an obvious contradiction argument using Theorem 3.8 completes the proof of our main result. \(\square\)

Remark. Theorem 3.9 and Corollary 2.2 imply Theorem 1.4.

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