LINEAR SPECTRAL TRANSFORMATIONS FOR MULTIVARIATE ORTHOGONAL POLYNOMIALS AND MULTISPECTRAL TODA HIERARCHIES

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ABSTRACT. Linear spectral transformations of orthogonal polynomials in the real line, and in particular Geronimus transformations, are extended to orthogonal polynomials depending on several real variables. Multivariate Christoffel–Geronimus–Uvarov formulæ for the perturbed orthogonal polynomials and their quasi-tau matrices are found for each perturbation of the original linear functional. These expressions are given in terms of quasi-determinants of bordered truncated block matrices and the 1D Christoffel–Geronimus–Uvarov formulæ in terms of quotient of determinants of combinations of the original orthogonal polynomials and their Cauchy transforms, are recovered. A new multispectral Toda hierarchy of nonlinear partial differential equations, for which the multivariate orthogonal polynomials are reductions, is proposed. This new integrable hierarchy is associated with non-standard multivariate biorthogonality. Wave and Baker functions, linear equations, Lax and Zakharov–Shabat equations, KP type equations, appropriate reductions, Darboux/linear spectral transformations, and bilinear equations involving linear spectral transformations are presented. Finally, the paper includes an Appendix devoted to multivariate Uvarov transformations. Particular attention is paid to 0D-Uvarov perturbations and also to the 1D-Uvarov perturbations, which require of the theory of Fredholm integral equations.

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1. INTRODUCTION

The aim of this paper is twofold, in the first place we discuss an extension of the linear spectral transformation given in [81] for orthogonal polynomials in the real line (OPRL) to several real variables; i.e., to complex multivariate orthogonal polynomials in real variables (MVOPR). Secondly, to generalize the Toda hierarchy introduced in [13] in the context of MVOPR, to a more general case, that we have named multispectral Toda hierarchy. For this new integrable hierarchy, which has the MVOPR as a particular reduction, we find the multivariate linear spectral transformations.

1.1. Historical background and state of the art. Elwin Christoffel, when discussing Gaussian quadrature rules in [21], found explicit formulæ —in terms of kernel polynomials— for several masses of this type, of the Stieltjes function orthogonal polynomials for several reasons, is in particular remarkable its close relation with Padé approximations. That is, he worked out in §1 the linear spectral transformation of a moment linear functional is given as a composition of Christoffel and Geronimus transformations [81].

The Stieltjes function \( F(x) := \sum_{n=0}^{\infty} \frac{(\delta(x) \cdot x^n)}{\delta(x-a)} \) of a linear functional \( u \in (\mathbb{R}[x])' \) is relevant in the theory of orthogonal polynomials for several reasons, is in particular remarkable its close relation with Padé approximation theory, see [18, 48]. Alexei Zhedanov studied in [81] the following rational spectral transformations of the Stieltjes function

\[
F(x) \mapsto \tilde{F}(x) = \frac{A(x)F(x) + B(x)}{C(x)F(x) + D(x)},
\]

as a natural extension of the above mentioned three canonical transformations. Here \( A(x), B(x), C(x) \) and \( D(x) \) are polynomials such that \( \tilde{F}(x) = \sum_{n=0}^{\infty} \frac{(\tilde{a}_n x^n)}{\tilde{\delta}(x)} \) is a new Stieltjes function. Linear spectral transformations correspond to the choice \( c(x) = 0 \), of which particular cases are the canonical Christoffel transformations \( \tilde{F}(x) = (x-a)F(x) - F_0 \) and the canonical Geronimus transformation of \( \tilde{F}(x) = \frac{F(x)+F_0}{x-a} \). Every linear spectral transformation of a moment functional is given as a composition of Christoffel and Geronimus transformations [81].
These transformations are referred generically as Darboux transformations, a name coined in the context of integrable systems in [55]. Gaston Darboux, when studying the Sturm–Liouville theory in [23], explicitly treated these transformations, which he obtained by a simplification of a geometrical transformation founded previously by Théodore Moutard [63]. In the OPRL framework, such a factorization of Jacobi matrices has been studied in [19, 80], and also played a key role in the study of bispectrality [46, 45]. In the differential geometry context, see [32], the Christoffel, Geronimus, Uvarov and linear spectral transformations are related to geometrical transformations like the Laplace, Lévy, adjoint Lévy and the fundamental Jonas transformations.

Regarding orthogonal polynomials in several variables we refer the reader to the excellent monographs [31, 78]. Milch [60] and Karlin and McGregor [49] considered multivariate Hahn and Krawtchouk polynomials in relation with growth birth and death processes. Since 1975 substantial developments have been achieved, let us mention the spectral properties of these multivariate Hahn and Krawtchouk polynomials, see [43]. Orthogonal polynomials and cubature formulae on the unit ball, the standard simplex, and the unit sphere were studied in [79] finding a strong connections between both themes. The common zeros of multivariate orthogonal polynomials were discussed in [77] where relations with higher dimensional quadrature problems were found. A description of orthogonal polynomials on the bicircle and polycircle and their relation to bounded analytic functions on the polydisk is given in [50], here a Christoffel–Darboux like formula, related in this bivariate case with stable polynomials, and Bernstein–Szegő measures are used, allowing for a new proof of Ando theorem in operator theory. Bivariate orthogonal polynomials linked to a moment functional satisfying the two-variable Pearson type differential equation and an extension of some of the characterizations of the classical orthogonal polynomials in one variable was discussed in [35]; in the paper [36] an analysis of a bilinear form obtained by adding a Dirac mass to a positive definite moment functional in several variables is given.

Darboux transformations for multivariate orthogonal polynomials were first studied in [13, 14] in the context of a Toda hierarchy. These transformations are the multidimensional extensions of the Christoffel transformations. In [14] we presented for the first time a multivariate extension of the classical 1D Christoffel formula, in terms of quasi-determinants [40, 39, 65], and poised sets [65, 14]. Also in this general multidimensional framework we have studied in [15] multivariate Laurent polynomials orthogonal with respect to a measure supported in the unit torus, finding in this case the corresponding Christoffel formula. In [7] linear relations between two families of multivariate orthogonal polynomials were studied. Despite that [7] does not deal with Geronimus formulae, it deals with linear connections among two families of orthogonal polynomials, a first step towards a connection formulae for the multivariate Geronimus transformation.

Sato [68, 69] and Date, Jimbo, Kashiwara and Miwa [24, 26, 25] introduced geometrical tools, like the infinite-dimensional Grassmannian and infinite dimensional Lie groups and Lie algebras, which have become essential, in the description of integrable hierarchies. We also mention [64], were the factorization problems, dressing procedure, and linear systems where shown to be the keys for integrability. Multicomponent versions of the integrable Toda equations [73, 74, 72] played a prominent role in the connection with orthogonal polynomials and differential geometry. In [16, 17, 47, 57, 58] multicomponent versions of the KP hierarchy were analyzed, while in [56, 59] we can find a study of the multi-component Toda lattice hierarchy, block Hankel/Toeplitz reductions, discrete flows, additional symmetries and dispersionless limits. In [6, 9] the relation of the multicomponent KP–Toda with mixed multiple orthogonal polynomials was discussed.

Adler and van Moerbeke showed the prominent role played by the Gauss–Borel factorization problem for understanding the strong bonds between orthogonal polynomials and integrable systems. In particular, their studies on the 2D Toda hierarchy –what they called the discrete KP hierarchy– neatly established the deep connection among standard orthogonality of polynomials and integrability of nonlinear equations of Toda type, see [1, 2, 3, 4, 5] and also [34]. Let us also mention that multicomponent Toda systems or non-Abelian versions of Toda equations with matrix orthogonal polynomials was studied, for example, in [61, 11] (on the real line) and in [62, 10] (on the unit circle).
The approach to linear spectral transformations and Toda hierarchies used in this paper, which is based on the Gauss–Borel factorization problem, has been used before in different contexts. We have connected integrable systems with orthogonal polynomials of diverse types:

1. As already mentioned, mixed multiple orthogonal polynomials and multicomponent Toda was analyzed in [9].
2. Matrix orthogonal Laurent polynomials on the circle and CMV orderings were considered [12].
3. The Christoffel transformation has been recently discussed for matrix orthogonal polynomials in the real line [3].

1.2. Results and layout of the paper. First, we complete this introduction with some background material from [13]. Then, in §2 we discuss the Geronimus type transformation for multivariate orthogonal polynomials. We introduce the resolvents and find the connection formulae. The multivariate extension of the Geronimus determinantal formula depends on the introduction of a semi-infinite matrix $R$, that for the 1D case is encoded in the Cauchy transforms of the OPRL, the second kind functions. However, no such connection exists in this more general scenario, and the multivariate Cauchy transform of the MVOPR does not provide the necessary aid for finding the multivariate formula for Geronimus transformations (aid which is provided by the semi-infinite matrix $R$). Then, we end the section by discussing the 1D reduction and recovering the Geronimus results [44]. A similar approach can be found in §3 for the linear spectral for which we present a multivariate quasi-determinantal Christoffel–Geronimus–Uvarov formula [81], and we give a brief discussion of the existence of poised sets.

In [13] we considered semi-infinite matrices having the adequate symmetries, that we call multi-Hankel, so that a multivariate moment functional or moment semi-infinite matrix appeared. In section 4 we are ready to abandon this more comfortable MVOPR situation and explore different scenarios by assuming that $G$ could be arbitrary, as far it is Gaussian factorizable. We are dealing with perturbations of non-standard multivariate biorthogonality. We first give the general setting for this integrable hierarchy, that we have named multi-spectral Toda lattice hierarchy, finding the corresponding Lax and Zakharov–Shabat equations and the role played by the Baker and adjoint Baker functions. Some reductions, like the multi-Hankel that leads to dynamic MVOPR, and extensions of it are presented. We also consider the action of the discussed multivariate linear spectral transformations and find the Christoffel–Geronimus–Uvarov formula in this broader scenario. To end the paper, we find generalized bilinear equations that involve linear spectral transformations.

We have also included an appendix to discuss multivariate Uvarov transformations. For the 0D-Uvarov transformation, which can be considered an immediate extension of the results of Uvarov [25], connection formulas are found. The general situation is discussed in terms of jets, we then particularize to mass perturbation, which for the OPRL case appears in [25] and in the multivariate case in [27], and to a dipole perturbation. The more appealing 1D-Uvarov perturbation is also discussed, and a connection formula is given in terms of a solution of an integral Fredholm equation.

1.3. Preliminary material. Following [14], a brief account of complex multivariate orthogonal polynomials in a D-dimensional real space (MVOPR) is given. Cholesky factorization of a semi-infinite moment matrix will be keystone to built such objects. Consider D independent real variables $x = (x_1, x_2, \ldots, x_D) \in \Omega \subseteq \mathbb{R}^D$, and the corresponding ring of complex multivariate polynomials $\mathbb{C}[x] \equiv \mathbb{C}[x_1, \ldots, x_D]$. Given a multi-index $\alpha = (\alpha_1, \ldots, \alpha_D) \in \mathbb{Z}_+^D$ of non-negative integers write $x^\alpha = x_1^{\alpha_1} \cdots x_D^{\alpha_D}$ and say that the length of $\alpha$ is $|\alpha| := \sum_{a=1}^{D} \alpha_a$. This length induces a total ordering of monomials: $x^\alpha < x^{\alpha'} \iff |\alpha| < |\alpha'|$. For each non-negative integer $k \in \mathbb{Z}_+$ introduce the set

$$[k] := \{ \alpha \in \mathbb{Z}_+^D : |\alpha| = k \},$$

built up with those vectors in the lattice $\mathbb{Z}_+^D$ with a given length $k$. The graded lexicographic order for $\alpha_1, \alpha_2 \in [k]$ is

$$\alpha_1 > \alpha_2 \iff \exists p \in \mathbb{Z}_+ \text{ with } p < D \text{ such that } \alpha_{1_1} = \alpha_{2_1}, \ldots, \alpha_{1_p} = \alpha_{2_p} \text{ and } \alpha_{1_{p+1}} < \alpha_{2_{p+1}},$$
and if \( \alpha^{(k)} \in [k] \) and \( \alpha^{(1)} \in [l] \), with \( k < l \) then \( \alpha^{(k)} < \alpha^{(1)} \). Given the set of integer vectors of length \( k \) use the lexicographic order and write
\[
[k] = \{ \alpha_1^{(k)}, \alpha_2^{(k)}, \ldots, \alpha_{||k||}^{(k)} \} \quad \text{with} \quad \alpha_\alpha > \alpha_\beta.
\]
Here \( ||k|| \) is the cardinality of the set \( k \), i.e., the number of elements in the set. This is the dimension of the linear space of homogenous multivariate polynomials of total degree \( k \). Either counting weak compositions or multisets one obtains the multi-choose number, \( ||k|| = \binom{D+k}{k} = \binom{D+k-1}{k-1} \). The dimension of the linear space \( \mathbb{C}_k[x_1, \ldots, x_D] \) of multivariate polynomials of degree less or equal to \( k \) is
\[
N_k = 1 + ||2|| + \cdots + ||k|| = \binom{D+k}{D}.
\]
The vector of monomials
\[
\chi := \begin{pmatrix} x_{[0]} \\ x_{[1]} \\ \vdots \\ x_{[k]} \\ \vdots \end{pmatrix}
\]
will be useful. Observe that for \( k = 1 \) we have that the vectors \( \alpha_\alpha^{(1)} = e_a \) for \( a \in \{1, \ldots, D\} \) form the canonical basis of \( \mathbb{R}^D \), and for any \( j \in [k] \) we have \( \alpha_j = \sum_{a=1}^{D} \alpha_j^a e_a \). For the sake of simplicity unless needed we will drop off the super-index and write \( \alpha_j \) instead of \( \alpha_j^{(k)} \), as it is understood that \( ||\alpha_j|| = k \).

The dual space of the symmetric tensor powers is isomorphic to the set of symmetric multilinear functionals on \( \mathbb{C}^D \). (Sym\(^k\)(\( \mathbb{C}^D \))\(^* \) \( \cong S(\text{Sym}^k(\mathbb{C}^D), \mathbb{C}) \). Hence, homogeneous polynomials of a given total degree can be identified with symmetric tensor powers. Each multi-index \( \alpha \in [k] \) can be thought as a weak \( D \)-composition of \( k \) (or weak composition in \( D \) parts), \( k = x_1 + \cdots + x_D \). Notice that these weak compositions may be considered as multisets and that, given a linear basis \( \{e_a\}_{a=1}^{D} \) of \( \mathbb{C}^D \) one has the linear basis \( \{e_{a_1} \circ \cdots \circ e_{a_k}\}_{a_1 \cdots a_k \in \mathbb{Z}_+^k} \) for the symmetric power \( \text{Sym}^k(\mathbb{C}^D) \), where the multisets \( 1 \leq a_1 \leq \cdots \leq a_k \leq D \) have been used. In particular, the vectors of this basis \( e_{a_1} \circ \cdots \circ e_{a_k} \) and its duals \( e_{a_1}^* \circ \cdots \circ e_{a_k}^* \) are in bijection with monomials of the form \( x_{a_1} \circ \cdots \circ x_{a_k} \). The lexicographic order can be applied to \( \text{Sym}^k(\mathbb{C}^D) \), then a linear basis of \( \text{Sym}^k(\mathbb{C}^D) \) is the ordered set
\[
B_c = \{ e_{a_1}, \ldots, e_{a_{||k||}} \} \quad \text{with} \quad e_{a_1} := \alpha_1^{\circ a_1} \circ \cdots \circ e_{a_k}^{\circ a_k} \quad \text{so that} \quad \chi_{[k]}[x] = \sum_{i=1}^{||k||} x_{a_i} e_{a_i} \).
\]
For more information see [22, 33, 76].

Consider semi-infinite matrices \( A \) with a block or partitioned structure induced by the graded reversed lexicographic order
\[
\Lambda_0 = \begin{pmatrix} A_{[0],[0]} & A_{[0],[1]} & \cdots \\ A_{[1],[0]} & A_{[1],[1]} & \cdots \\ \vdots & \vdots & \ddots \end{pmatrix}, \quad \Lambda_{[k],[t]} = \begin{pmatrix} A_{\alpha^{(k)},\alpha^{(t)}} & \cdots & A_{\alpha^{(k)},\alpha^{(l)}} \\ \vdots & \ddots & \vdots \\ A_{\alpha^{(k)},\alpha^{(t)}} & \cdots & A_{\alpha^{(k)},\alpha^{(l)}} \end{pmatrix} \in \mathbb{C}^{||k|| \times ||l||}.
\]
Use the notation \( 0_{||k||,||l||} \in \mathbb{C}^{||k|| \times ||l||} \) for the rectangular zero matrix, \( 0_{||k||} \in \mathbb{C}^{||k||} \) for the zero vector, and \( I_{||k||} \in \mathbb{C}^{||k|| \times ||k||} \) for the identity matrix. For the sake of simplicity just write \( 0 \) or \( I \) for the zero or identity matrices, and assume that the sizes of these matrices are the ones indicated by their position in the partitioned matrix.

The vector space of complex multivariate polynomials \( \mathbb{C}_k[x] \) in \( D \) real variables of degree less or equal to \( k \) with the norm \( \| \sum_{|\alpha| \leq k} P_{\alpha} x^{\alpha} \|_n := \sum_{|\alpha| \leq k} |P_{\alpha}| \), gives a nesting of Banach spaces \( \mathbb{C}_n[x] \subset \mathbb{C}_{n+1}[x] \) whose inductive limit gives a topology to the space \( \mathbb{C}[x] \). The elements of the algebraic dual \( u \in (\mathbb{C}[x])^* \), which are called linear functionals, are linear maps \( u : \mathbb{C}[x] \to \mathbb{C} \); the notation \( P(x) \mapsto \langle u, P(x) \rangle \) will be used. Two polynomials \( P(x), Q(x) \in \mathbb{C}[x] \) are said orthogonal with respect to \( u \) if \( \langle u, P(x)Q(x) \rangle = 0 \). The
topological dual $(\mathbb{C}[x])'$ has the dual weak topology characterized by the semi-norms $\{ \| \cdot \|_p : p(x) \in \mathbb{C}[x]' \}$. This family of seminorms is equivalent to the family of seminorms given by $\| u \|^{(k)} := \sup_{|\alpha| = k} |\langle u, x^\alpha \rangle|$. Moreover, the topological dual $(\mathbb{C}[x])'$ is a Fréchet space and $(\mathbb{C}[x])' = (\mathbb{C}[x])^*$ and every linear functional is continuous. Linear functionals can be multiplied by polynomials $\langle Qu, P(x) \rangle := \langle u, Q(x)P(x) \rangle$, $\forall P(x) \in \mathbb{C}[x]$. It can be also shown, that in this case the space of generalized functions $(\mathbb{C}[x])'$ coincide with the space of formal series $\mathbb{C}[x]$. For more information regarding linear functional’s approach to orthogonal polynomials see [51, 52] and [67, 66].

However, we need to deal with generalized functions with a support and the linear functionals we have discussed so far are not suitable for that. We proceed to discuss several possibilities to overcome this problem. The space of distributions is a space of generalized functions when the fundamental functions space is the complex valued smooth functions of compact support $\mathcal{D} := C^\infty_0(\mathbb{R}^D)$, the space of test functions, see [20, 41, 42]. Now, there is a clear meaning for the set of zeroes of a distribution $u \in \mathcal{D}'$, $u$ is zero on a domain $\Omega \subset \mathbb{R}^D$ if for any fundamental function $f(x)$ with support in $\Omega$ we have $\langle u, f \rangle = 0$. The complement, which is closed, is the support $\text{supp } u$ of the distribution $u$. Distributions of compact support, $u \in \mathcal{E}'$, are generalized functions with fundamental functions space is the topological space of complex valued smooth functions $\mathcal{E} = C^\infty(\mathbb{R}^D)$. Thus, as $\mathbb{C}[x] \subset \mathcal{E}$ we have $\mathcal{E}' \subset (\mathbb{C}[x])' \cap \mathcal{D}'$. These distributions of compact support is a first example of an appropriate framework for the consideration of polynomials and supports simultaneously. More general setting appears within the space of tempered distributions $\mathcal{S}'$—which are distributions, $\mathcal{S}' \subset \mathcal{D}'$. Now, the fundamental functions space is given by the Schwartz space $\mathcal{S}$ of complex valued fast decreasing functions, see [20, 41, 42]. Then, we can consider the space of fundamental functions of smooth functions of slow growth $\mathcal{O}_M \subset \mathcal{E}$, whose elements are smooth functions having all its derivatives bounded by a polynomial of certain degree. As $\mathbb{C}[x], \mathcal{S} \subseteq \mathcal{O}_M$, for the corresponding set of generalized functions we find that $\mathcal{O}'_M \subset (\mathbb{C}[x])' \cap \mathcal{S}'$. Thus, these distributions give a second suitable framework. Finally, for a third suitable framework we need to introduce bounded distributions. Let us consider as space of fundamental functions, the linear space $\mathcal{B}$ of bounded smooth functions, i.e., with all its derivatives in $L^\infty(\mathbb{R}^D)$, being the corresponding space of generalized functions $\mathcal{B}'$ the bounded distributions (not to be confused with compact support). Notice that, as $\mathcal{D} \subset \mathcal{B}$ we have that bounded distributions are distributions $\mathcal{B}' \subset \mathcal{D}'$. Then, we consider the space of fast decreasing distributions $\mathcal{O}'_\mathcal{C}$ given by those distributions $u \in \mathcal{D}'$ such that for each positive integer $k$, we have $(\sqrt{1 + (x_1)^2 + \cdots + (x_D)^2})^k u \in \mathcal{B}'$ is a bounded distribution. Any polynomial $P(x) \in \mathbb{C}[x]$, with $\deg P = k$, can be written as $P(x) = (\sqrt{1 + (x_1)^2 + \cdots + (x_D)^2})^k F(x), \quad F(x) = \frac{P(x)}{(\sqrt{1 + (x_1)^2 + \cdots + (x_D)^2})^k} \in \mathcal{B}$.

Therefore, given a fast decreasing distribution $u \in \mathcal{O}'_\mathcal{C}$ we may consider $\langle u, P(x) \rangle = \langle \left( \sqrt{1 + (x_1)^2 + \cdots + (x_D)^2} \right)^k u, F(x) \rangle$ which makes sense as $(\sqrt{1 + (x_1)^2 + \cdots + (x_D)^2})^k u \in \mathcal{B}', F(x) \in \mathcal{B}$. Thus, $\mathcal{O}'_\mathcal{C} \subset (\mathbb{C}[x])' \cap \mathcal{D}'$. Moreover it can be proven that $\mathcal{O}'_M \subset \mathcal{O}'_\mathcal{C}$, see [51]. Summarizing this discussion, we have found three generalized function spaces suitable for the discussion of polynomials and supports simultaneously:

$\mathcal{E}' \subset \mathcal{O}'_M \subset \mathcal{O}'_\mathcal{C} \subset ((\mathbb{C}[x])' \cap \mathcal{D}')$.

**Definition 1.1.** Associated with the linear functional $u \in (\mathbb{C}[x])'$ define the following moment matrix $G := \langle u, \chi(x) (\chi(x))^T \rangle$.

In block form can be written as

$$G := \begin{pmatrix} G_{[0],[0]} & G_{[0],[1]} & \cdots \\ G_{[1],[0]} & G_{[1],[1]} & \cdots \\ \vdots & \vdots & \ddots \end{pmatrix}.$$
Truncated moment matrices are given by
\[ G^{[1]} := \begin{pmatrix} G_{[0],[0]} & \cdots & G_{[0],[1]} \\ \vdots & \ddots & \vdots \\ G_{[1-1],[0]} & \cdots & G_{[1-1],[1]} \end{pmatrix}. \]

Notice that from the above definition we know that

**Proposition 1.1.** The moment matrix is a symmetric matrix, \( G = G^\top \).

This result implies that a Gauss–Borel factorization of it, in terms of lower unitriangular and upper triangular matrices, is a Cholesky factorization.

In terms of quasi-determinants, see [38, 65], we have

**Proposition 1.2.** If the last quasi-determinants \( \Theta_* (G^{[k+1]}), k \in \{0,1,\ldots\} \), of the truncated moment matrices are invertible the Cholesky factorization

\[
(1.1) \quad G = S^{-1}HS^{-\top},
\]

with

\[
S^{-1} = \begin{pmatrix} \mathbb{I} & 0 & 0 & \cdots \\ (S^{-1})_{[1],[0]} & \mathbb{I} & 0 & \cdots \\ \vdots & (S^{-1})_{[2],[1]} & \mathbb{I} & \vdots \\ \end{pmatrix}, \quad H = \begin{pmatrix} H_{[0]} & 0 & 0 & \cdots \\ 0 & H_{[1]} & 0 & \cdots \\ \vdots & \vdots & \ddots & \vdots \end{pmatrix},
\]

and Hermitian quasi-tau matrices \( H_{[k]} = (H_{[k]})^\top \), can be performed. Moreover, the rectangular blocks can be expressed in terms of last quasi-determinants of truncations of the moment matrix

\[
H_{[k]} = \Theta_* (G^{[k+1]}), \quad (S^{-1})_{[k],[1]} = \Theta_* (G^{[k+1]}_k) \Theta_* (G^{[k+1]})^{-1}.
\]

**Definition 1.2.** The monic MVOPR associated to the linear functional \( u \) are

\[
(1.2) \quad P(x) = S \chi(x) = \begin{pmatrix} P_{[0]}(x) \\ P_{[1]}(x) \\ \vdots \end{pmatrix}, \quad P_{[k]}(x) = \sum_{\ell=0}^{k} S_{[k],[\ell]} \chi_{[\ell]}(x) = \begin{pmatrix} P_{\alpha_1^{(k)}}(x) \\ \vdots \\ P_{\alpha_{[k]}^{(k)}}(x) \end{pmatrix}, \quad P_{\alpha_{[k]}^{(k)}}(x) = \sum_{l=0}^{k} \sum_{j=1}^{[l]} S_{\alpha_1^{(k)},[j]}^{(l)} x^{\alpha_j^{(l)}}.
\]

Observe that \( P_{[k]}(x) = \chi_{[k]}(x) + \beta_{[k]} \chi_{[k-1]}(x) + \cdots \) is a vector constructed with the polynomials \( P_{\alpha_i}(x) \) of degree \( k \), each of which has only one monomial of degree \( k \); i.e., we can write \( P_{\alpha_i}(x) = x^{\alpha_i} + Q_{\alpha_i}(x) \), with \( \deg Q_{\alpha_i} < k \). Here \( \beta \) is a semi-infinite matrix with all its elements being zero but for its first subdiagonal \( \beta = \text{subdiag}_1(\beta_{[1]}, \beta_{[2]}, \ldots) \) with coefficients given by \( \tilde{\beta}_{[k]} := S_{[k],[k-1]} \).

**Proposition 1.3** (Orthogonality relations). The MVOPR satisfy

\[
\langle u, P_{[k]}(x) (P_{[l]}(x))^\top \rangle = \delta_{k,l} H_{[k]}.
\]

which implies

\[
(1.3) \quad \langle u, P_{[k]}(x) (P_{[l]}(x))^\top \rangle = \langle u, P_{[k]}(x) (\chi_{[l]}(x))^\top \rangle = 0, \quad l = 0, 1, \ldots, k - 1,
\]

\[
(1.4) \quad \langle u, P_{[k]}(x) (P_{[k]}(x))^\top \rangle = \langle u, P_{[k]}(x) (\chi_{[k]}(x))^\top \rangle = H_{[k]}.
\]

Therefore, the following orthogonality conditions

\[
\langle u, P_{\alpha_i}(x) P_{\alpha_j}(x) \rangle = \langle u, P_{\alpha_i}(x) x^{\alpha_j} \rangle = 0,
\]

are fulfilled for \( l \in \{0,1,\ldots,k-1\}, i \in \{1,\ldots,[k]\} \) and \( j \in \{1,\ldots,[l]\} \), with the normalization conditions

\[
\langle u, P_{\alpha_i}(x) P_{\alpha_j}(x) \rangle = \langle u, P_{\alpha_i}(x) x^{\alpha_i} \rangle = H_{\alpha_i,\alpha_j}, \quad i, j \in \{1,\ldots,[k]\}.
\]
Definition 1.3. The spectral matrices are given by

\[
\Lambda_a = 
\begin{pmatrix}
0 & (\Lambda_a)_{[0],[1]} & 0 & 0 & \cdots \\
0 & 0 & (\Lambda_a)_{[1],[2]} & 0 & \cdots \\
0 & 0 & 0 & (\Lambda_a)_{[2],[3]} & \cdots \\
\vdots & \vdots & \vdots & \vdots & \ddots 
\end{pmatrix}, \quad a \in \{1, \ldots, D\},
\]

where the entries in the first block superdiagonal are

\[
(\Lambda_a)_{\alpha_i^{(k)}, \alpha_j^{(k+1)}} = \delta_{\alpha_i^{(k)} + e_a, \alpha_j^{(k+1)}}, \quad a \in \{1, \ldots, D\}, \quad i \in \{1, \ldots, |k|\}, \quad j \in \{1, \ldots, |k + 1|\},
\]

and the associated vector

\[
\Lambda := (\Lambda_1, \ldots, \Lambda_D)^T.
\]

Finally, we introduce the Jacobi matrices

\[
J_a := S\Lambda_a S^{-1}, \quad a \in \{1, \ldots, D\},
\]

and the Jacobi vector

\[
J = (J_1, \ldots, J_D)^T.
\]

Proposition 1.4. (1) The spectral matrices commute among them

\[
\Lambda_a \Lambda_b = \Lambda_b \Lambda_a, \quad a, b \in \{1, \ldots, D\}.
\]

(2) The spectral properties

\[
\Lambda_a \chi(x) = x_a \chi(x), \quad a \in \{1, \ldots, D\}
\]

hold.

(3) The moment matrix G satisfies

\[
\Lambda_a G = G (\Lambda_a)^T, \quad a \in \{1, \ldots, D\}.
\]

(4) The Jacobi matrices \( J_a \) are block tridiagonal and satisfy

\[
J_a H = H J_a^T, \quad a \in \{1, \ldots, D\}.
\]

Definition 1.4. The Christoffel–Darboux kernel is

\[
K_n(x, y) := \sum_{m=0}^{n} (P_{[m]}(x))^T (H_{[m]})^{-1} P_{[m]}(y)
\]

In terms of the Christoffel–Darboux kernel and a linear functional \( u \in \mathcal{O}'_M \), we define the operator acting on \( \mathcal{O}_M \) as follows

\[
S_n(f)(x) := \langle u, f(y) K_n(y, x) \rangle .
\]

Proposition 1.5. (1) If \( P(x) = \sum_{j \geq 0} c_{[j]} P_{[j]}(x) \in \mathbb{C}[x] \subset \mathcal{O}_M \) is an arbitrary multivariate polynomial of degree \( n \), we have

\[
S_n(P)(x) = \sum_{m=0}^{n} c_{[m]} P_{[m]}(x).
\]

(2) For any vector \( n \in \mathbb{C}^D \), the following Christoffel–Darboux formula is fulfilled

\[
(n \cdot (x - y)) K_n(x, y) = (P_{[n+1]}(x))^\dagger \left( (n \cdot \Lambda)_{[n],[n+1]} \right)^T (H_{[n]})^{-1} P_{[n]}(y) - (P_{[n]}(x))^\dagger (H_{[n]})^{-1} (n \cdot \Lambda)_{[n],[n+1]} P_{[n+1]}(y).
\]
2. Geronomus transformations

In this section a Geronomus transformation for MVOPR is discussed, if we understand the Christoffel transformation as the perturbation by the multiplication by a polynomial, its right inverse, the Geronomus transformation, might be thought as the perturbation obtained by dividing by a polynomial. We also need a discrete part concentrated at the zeroes of the polynomial denominator, now an algebraic hypersurface.

2.1. Geronomus transformations in the multivariate scenario. Given a polynomial \( \Omega_2(\mathbf{x}) \in \mathbb{C}[\mathbf{x}] \) we may consider its principal ideal

\[
(\Omega_2) := \{ \Omega_2(\mathbf{x})P(\mathbf{x}) : P(\mathbf{x}) \in \mathbb{C}[\mathbf{x}] \}.
\]

This ideal is closely related to the algebraic hypersurface in \( \mathbb{C}^D \) of its zero set

\[
Z(\Omega_2) := \{ \mathbf{x} \in \mathbb{C}^D : P(\mathbf{x}) = 0 \}.
\]

The kernel of a linear functional \( v \in (\mathbb{R}[\mathbf{x}])' \) is defined by

\[
\text{Ker}(v) := \{ P(\mathbf{x}) \in \mathbb{C}[\mathbf{x}] : \langle v, P(\mathbf{x}) \rangle = 0 \}.
\]

We know that \( \mathbb{C}[\mathbf{x}] \) acts on \( (\mathbb{C}[\mathbf{x}])' \) by left multiplication, but for the transformations we are dealing with we also need the notion of division by polynomials.

**Definition 2.1.** Given fastly decreasing generalized function \( u \in \mathcal{O}_{\mathbf{c}}' \) and a polynomial \( \Omega_2(\mathbf{x}) \in \mathbb{C}[\mathbf{x}] \), such that \( Z(\Omega_2) \cap \text{supp}(u) = \emptyset \), the set of all the linear functionals \( \check{u} \in (\mathbb{C}[\mathbf{x}])' \) such that

\[
\Omega_2 \check{u} = u,
\]

is called its Geronomus transformation.

Notice that there is not a unique linear functional \( \check{u} \in (\mathbb{C}[\mathbf{x}])' \) satisfying such a requirement. Indeed, suppose that a solution is found and denote it by \( \frac{u}{\Omega_2} \), then all possible perturbations \( \check{u} \) verifying (2.1) will have the form

\[
\check{u} = \frac{u}{\Omega_2} + v,
\]

where the linear functional \( v \in (\mathbb{C}[\mathbf{x}])' \) is such that \( (\Omega_2) \subseteq \text{Ker}(v) \); i.e.,

\[
\Omega_2 v = 0.
\]

For example, given a positive Borel measure \( d\mu(\mathbf{x}) \) and the associated linear functional

\[
\langle u, P(\mathbf{x}) \rangle = \int P(\mathbf{x}) \, d\mu(\mathbf{x}),
\]

we can choose \( \frac{u}{\Omega_2} \in (\mathbb{C}[\mathbf{x}])' \) as the following linear functional

\[
\left\langle \frac{u}{\Omega_2}, P(\mathbf{x}) \right\rangle = \int P(\mathbf{x}) \frac{d\mu(\mathbf{x})}{\Omega_2(\mathbf{x})},
\]

which makes sense if \( Z(\Omega_2) \cap \text{supp}(d\mu) = \emptyset \). Any multivariate polynomial has a unique, up to constants, factorization in terms of prime polynomials

\[
\Omega_2(\mathbf{x}) = \prod_{i=1}^{N} (\Omega_{2,i}(\mathbf{x}))^{d_i},
\]

where \( \Omega_{2,i} \) are prime polynomials for \( i \in \{1, \ldots D\} \) and the multiplicities \( \{d_1, \ldots, d_N\} \) are positive integers such that \( m_2 = \deg \Omega_2 = d_1 \deg \Omega_{2,1} + \cdots + d_2 \deg \Omega_{2,N} \). Let us consider for each prime factor \( \Omega_{2,i}, i \in \)
\{1, \ldots, N\} a set of measures \( \{ d \xi_{i, \alpha} \}_{\alpha \in \mathbb{Z}^+_{d_i}} \) with \( \text{supp} \left( d \xi_{i, \alpha} \right) \subseteq Z(\Omega_{2,1}) \). Then, a linear functional \( v \) of the form

\begin{equation}
\langle v, P(x) \rangle = \sum_{i=1}^{N} \sum_{\alpha \in \mathbb{Z}^+_{d_i}} \int_{Z(\Omega_{2,1})} \frac{\partial^\alpha}{\partial x^\alpha} P(x) d \xi_{i, \alpha},
\end{equation}

is such that \( \Omega_2 \subseteq \text{Ker}(v) \).

In the \( D = 1 \) context, where up to constants \( \Omega_2(x) = (x - q_1)^{d_1} \cdots (x - q_N)^{d_N} \), with different roots \( \{q_1, \ldots, q_N\} \), and multiplicities \( \{d_1, \ldots, d_N\} \) such that \( d_1 + \cdots + d_N = m_2 \), the most general form of \( v \) is, in terms of the Dirac linear functional \( \delta \) and its derivatives, given by

\begin{equation}
v = \sum_{i=1}^{N} \sum_{j=0}^{d_i-1} \zeta_i^{(j)} \delta^{(j)}(x - q_i), \quad \zeta_i^{(j)} \in \mathbb{R}.
\end{equation}

Observe that for multiplicities greater than 1 we have linear functionals of higher order and therefore not linked to measures, which are linear functionals of order zero.

From hereon we assume that both linear functionals \( u \) and \( \tilde{u} \) give rise to well defined families of MVOPR, equivalently that all their moment matrix block minors are nonzero \( \det G^{[k]} \neq 0, \det \tilde{G}^{[k]} \neq 0, \forall k \in \{1, 2, \ldots\} \).

**Proposition 2.1.** The moment matrices \( \tilde{G} \) and \( G \), of the perturbed linear functional \( \tilde{u} \) and unperturbed linear functional \( u \), respectively, satisfy

\begin{equation}
\Omega_2(\lambda) \tilde{G} = \tilde{G} \Omega_2(\lambda^T) = G.
\end{equation}

**Proof.** It is a direct consequence of the spectral property \( \Omega_2(\lambda) \chi(x) = \Omega_2(x) \chi(x) \), that is deduced from (1.6). Indeed,

\begin{align*}
\Omega_2(\lambda) \langle \tilde{u}, \chi(x) \rangle \langle \chi(x) \rangle^T &= \langle \tilde{u}, \Omega_2(x) \chi(x) \rangle \langle \chi(x) \rangle^T \\
&= \langle \Omega_2 \tilde{u}, \chi(x) \rangle \langle \chi(x) \rangle^T \\
&= \langle u, \chi(x) \rangle \langle \chi(x) \rangle^T \quad \text{use (2.1).}
\end{align*}

Let us notice that for a given semi-infinite matrix \( G \) there is not a unique \( \tilde{G} \) satisfying (2.5). In fact, observe that given any generalized function \( v \) of the form (2.5) and any semi-infinite block vector \( \zeta = (\zeta_0, \zeta_1, \ldots)^T, \zeta_i \in \mathbb{R} \), we have

\( \Omega_2(\lambda) \langle v, \chi(x) \rangle \zeta^T = 0. \)

and if \( \tilde{G} \) satisfies (2.5) so does \( \tilde{G} + \langle v, \chi(x) \rangle \zeta^T \).

2.2. **Resolvents and connection formulæ.**

**Definition 2.2.** The resolvent matrices are

\( \omega_1 := \tilde{S} S^{-1}, \quad (\omega_2)^T := S \Omega_2(\lambda)(\tilde{S})^{-1}, \)

given in terms of the lower unitriangular block semi-infinite matrices \( S \) and \( \tilde{S} \) of the Cholesky factorizations of the moment matrices \( G = S^{-1} \bar{H} (S^{-1})^T \) and \( \tilde{G} = (\tilde{S})^{-1} (\bar{H}) (\tilde{S})^{-1}, \) respectively.

**Proposition 2.2.** We have that

\begin{equation}
\tilde{H} \omega_2 = \omega_1 H.
\end{equation}

**Proof.** It follows from the Cholesky factorization of \( G \) and \( \tilde{G} \) and from (2.1). \( \square \)
We now decompose the perturbing multidimensional polynomial $Q_2$ in its homogeneous parts $Q_2(x) = \sum_{n=0}^{m_2} Q_2^{(n)}(x)$ where $Q_2^{(n)}(x)$ are homogeneous polynomials of degree $n$, i.e., $Q_2^{(n)}(sx) = s^n Q_2^{(n)}(x)$, for all $s \in \mathbb{R}$.

**Proposition 2.3.** In terms of block subdiagonals the adjoint resolvent $\omega_1$ can be expressed as follows

$$\omega_1 = \begin{cases} \mathcal{H} \Omega_2^{(m_2)} (\Lambda^\top) \mathcal{H}^{-1} \\ \text{m}_2\text{-th subdiagonal} \end{cases} + \begin{cases} \mathcal{H} \left( Q_2^{(m_2-1)} (\Lambda^\top) + Q_2^{(m_2)} (\Lambda^\top) \beta^\top \beta \right) \mathcal{H} \mathcal{H}^{-1} \\ \left( \text{m}_2 - 1 \right)\text{-th subdiagonal} \end{cases} + \begin{cases} \mathcal{H} \mathcal{H}^{-1} \\ \text{diagonal} \end{cases}$$

**Proof.** The resolvent $\omega_1$ is a block lower unitriangular semi-infinite matrix and the adjoint resolvent $\omega_2$ has all its superdiagonals but for the first $m$ equal to zero. The result follows from (2.6). \qed

Incidentally, and not essential for further developments in this paper, we have the following two Propositions regarding Jacobi matrices

**Proposition 2.4.** The following UL and LU factorizations

$$Q_2(J) = (\omega_2)^\top \omega_1,$$  
$$Q_2(\bar{J}) = \omega_1 (\omega_2)^\top,$$

hold.

**Proof.** Both follow from Proposition 2.1 and the Cholesky factorization which imply

$$Q_2(\Lambda)(\bar{S})^{-1} \bar{H}(\bar{S}^{-1})^\top = S^{-1}H(S^{-1})^\top,$$

and a proper cleaning does the job. \qed

From the first equation in the previous Proposition we get

**Proposition 2.5.** The block truncations $(Q_2(\bar{J}))^{[k]}$ admit a LU factorization

$$(Q_2(\bar{J}))^{[k]} = \omega_1^{[k]} (\omega_2^{[k]})^\top$$

in terms of the corresponding truncations of resolvents.

**Proposition 2.6.** We have

$$\det((Q_2(\bar{J}))^{[k]}) = \prod_{l=0}^{k-1} \det H_{[l]}$$

and therefore $(Q_2(\bar{J}))^{[k]}$ is a regular matrix.

**Proof.** To prove this result just use Propositions 2.4 and 2.3 and the assumption that the minors of the moment matrix and the perturbed moment matrix are not zero. \qed

The next connection relations will be relevant for the finding of the Gerominus formulæ

**Proposition 2.7 (Connection formulæ).** The followings relations are fulfilled

$$\omega_2^{\top} \bar{P}(x) = Q_2(x) P(x),$$

$$\omega_1 P(x) = \bar{P}(x).$$
2.3. **The multivariate Geronimus formula.** To extend to multidimensions the Geronimus determinantal expressions for the Geronimus transformations \[44\] we need a new object. In the 1D case it is enough to use the Cauchy transforms of the OPRL, so closely related to the Stieljes functions. However, in this multivariate scenario we have not been able to use the corresponding multivariate Cauchy transforms, see \[13\], precisely because of complications motivated by the multidimensionality. Instead, we have been able to use an alternative path by introducing a semi-infinite matrix $R$ that in the 1D case, using a partial fraction expansion, can be expressed in terms of the mentioned Cauchy transforms and Geronimus type combinations. This new element is essential in the finding of a new multivariate Geronimus quasi-determinantal formula.

**Definition 2.3.** We introduce the semi-infinite block matrices

$$R := \langle \tilde{u}, P(x)(\chi(x))^T \rangle.$$

**Proposition 2.8.** The formula

$$R = \rho + \theta,$$

$$\rho := \langle u, \frac{P(x)(\chi(x))^T}{Q_2(x)} \rangle,$$

$$\theta := \langle v, P(x)(\chi(x))^T \rangle,$$

holds.

**Proof.** Just write $\tilde{u} = \frac{u}{Q_2} + v$, with $(Q_2) \subseteq \text{Ker} v$. \qed

**Proposition 2.9.** If the linear functional $u$ is of order zero with an associated Borel measure $d \mu(x)$ we can write

$$\rho = \int P(x)(\chi(x))^T \frac{d \mu(x)}{Q_2(x)},$$

and if $Q_2(x) = (Q_{2,1}(x))^{d_1} \cdots (Q_{2,N}(x))^{d_N}$ is a prime factorization, and $v$ is taken as in (2.3) we can write

$$\theta = \sum_{i=1}^{N} \sum_{\alpha \in \mathbb{Z}^D, |\alpha| < d_i} \int_{Z(Q_{2,i})} \frac{\partial^{\alpha}(P(x)(\chi(x))^T)}{\partial x^{\alpha}} \frac{d \xi_{i,\alpha}(x)}{d \xi_{i,\alpha}(x)}.$$

**Proposition 2.10.** The following relations

$$(\omega_1 R)_{[k],l} = 0, \quad l \in \{0, 1, \ldots, k-1\},$$

$$(\omega_1 R)_{[k],[k]} = H_{[k]},$$

hold true.

**Proof.** A direct computation leads to the result. Indeed,

$$\omega_1 R = \langle \tilde{u}, \omega_1 P(x)(\chi(x))^T \rangle = \langle \tilde{u}, \tilde{P}(x)(\chi(x))^T \rangle$$

recall (2.7)

and the orthogonality equations (1.3) and (1.4) give the desired conclusion. \qed

**Proposition 2.11.** (1) The truncations $R^{[k]}$ are nonsingular for all $k \in \mathbb{Z}_+$. 

(2) The adjoint resolvent entries satisfy

$$(\omega_1)_{[k],[0]}, \ldots, (\omega_1)_{[k],[k-1]} = -(R_{[k],[0]}^{[k]}, \ldots, R_{[k],[k-1]}^{[k]}) (R^{[k]})^{-1}.$$
(3) We can express each entry of the adjoint resolvent as

\[
(\omega_1)_{[k],[l]} = -(R_{[k],[0]}, \ldots, R_{[k],[k-1]})^{-1} \begin{pmatrix}
0_{[0],[1]} \\
\vdots \\
0_{[l-1],[1]} \\
\mathbb{I}_{[1]} \\
0_{[l+1],[1]} \\
\vdots \\
0_{[k],[1]}
\end{pmatrix}, \quad l \in \{0, 1, \ldots, k-1\}.
\]

**Proof.**

(1) We can write

\[
R^{[k+1]} = S^{[k+1]} G^{[k+1]}
\]

so that

\[
\det R^{[k+1]} = \prod_{l=0}^{k} \det H_{[l]} \neq 0.
\]

(2) From Propositions 2.3 and 2.10 we deduce

\[
(\omega_1)_{[k],[0]} R_{[0],[1]} + \cdots + (\omega_1)_{[k],[k-1]} R_{[k-1],[1]} = -R_{[k],[1]}, \quad l \in \{0, 1, \ldots, k-1\}.
\]

Therefore, we get

\[
(\omega_1)_{[k],[0]}, \ldots, (\omega_1)_{[k],[k-1]} R^{[k]} = -(R_{[k],[0]}, \ldots, R_{[k],[k-1]}),
\]

from where (2.8) follows. 

\[
\square
\]

**Theorem 2.1.** We can express the new MVOPR, \( \hat{P}_{[k]}(x) \), and the quasi-tau matrices \( \hat{H}_{[k]} \) in terms of the non-perturbed ones as follows

\[
\hat{P}_{[k]}(x) = \Theta_s \left( \begin{array}{cccc}
R_{[0],[0]} & \cdots & R_{[k],[k-1]} & P_{[0]}(x) \\
\vdots & \ddots & \vdots & \vdots \\
R_{[k],[0]} & \cdots & R_{[k],[k-1]} & P_{[k]}(x)
\end{array} \right),
\]

\[
\hat{H}_{[k]} = \Theta_s (R^{[k+1]}).
\]

**Proof.** From (2.7) we deduce

\[
\hat{P}_{[k]}(x) = (\omega_1)_{[k],[0]} P_{[0]}(x) + \cdots + (\omega_1)_{[k],[k-1]} P_{[k-1]}(x) + P_{[k]}(x)
\]

and Proposition 2.11 implies

\[
\hat{P}_{[k]}(x) = P_{[k]}(x) - (R_{[k],[0]}, \ldots, R_{[k],[k-1]}) (R^{[k]})^{-1} \begin{pmatrix}
P_{[0]}(x) \\
\vdots \\
P_{[k-1]}(x)
\end{pmatrix}
\]

and, consequently, (2.11) follows.

From Proposition 2.10 we get

\[
(\omega_1)_{[k],[0]} R_{[0],[k]} + \cdots + (\omega_1)_{[k],[k-1]} R_{[k-1],[k]} + R_{[k],[k]} = \hat{H}_{[k]},
\]

now recall (2.8) to deduce

\[
\hat{H}_{[k]} = R_{[k],[k]} - (R_{[k],[0]}, \ldots, R_{[k],[k-1]}) (R^{[k]})^{-1} \begin{pmatrix}
R_{[0],[k]} \\
\vdots \\
R_{[k-1],[k]}
\end{pmatrix},
\]

so that (2.12) is proven. Let us mention that it also follows from (2.10). 

\[
\square
\]
The previous relations involve a growing number of terms as $k$ increases. However, for $k \geq m_2$ this changes.

**Definition 2.4.**

1. If $k > m_2$, take an ordered set of multi-indices
   
   $\mathcal{M}_k := \{ \mathbf{\beta}_i \in (\mathbb{Z}_+)^D : |\mathbf{\beta}_i| < k \}^{R_{k,m_2}}$

   with cardinal given by
   
   $r_{k,m_2} := |\mathcal{M}_k| = N_{k-1} - N_{k-m_2-1} = ||k-m_2|| + \cdots + ||k-1||$.

2. Associated with this set consider the truncations
   
   $R^{[\mathcal{M}_k]} := \begin{pmatrix}
   R_{[k-m_2],[\beta_1]} \cdots R_{[k-m_2],[\beta_{r_{k,m_2}}} \\
   \vdots \quad \vdots \\
   R_{[k-1],[\beta_1]} \cdots R_{[k-1],[\beta_{r_{k,m_2}}]} 
   \end{pmatrix}$

   $R_{\mathcal{M}_k} := \{ R_{[k],[\beta_1]}, \ldots, R_{[k],[\beta_{r_{k,m_2}}]} \}$

3. Then, the set $\mathcal{M}_k$ is said to be poised if the corresponding truncation is not singular
   
   $\det R^{[\mathcal{M}_k]} = 0$.

**Proposition 2.12.** Poised sets do exist.

**Proof.** We need to ensure that among all subsets $\mathcal{M}_k$ of multi-indices of length less than $k$ there is at least one such that $\det R^{[\mathcal{M}_k]} \neq 0$. We proceed by contradiction. If we assume that there is no such set the matrix

$$
\begin{pmatrix}
R_{[k-m_2],[0]} \cdots R_{[k-m_2],[k-1]} \\
\vdots \quad \vdots \\
R_{[k-1],[0]} \cdots R_{[k-1],[k-1]} 
\end{pmatrix}
$$

is not full rank and, consequently, $R_{[k]}$ will be singular, which is in contradiction with our assumptions. 

**Proposition 2.13.** For $k \geq m_2$ and a poised set of multi-indices $\mathcal{M}_k$, we have

$$
((\omega_1)_{[k],[k-m_2]}, \ldots, (\omega_1)_{[k],[k-1]}) = -R_{\mathcal{M}_k}(R^{[\mathcal{M}_k]})^{-1}.
$$

**Proof.** Observe that Propositions 2.3 and 2.10 imply

$$(\omega_1)_{[k],[k-m_2]} R_{[k-m_2],[1]} + \cdots + (\omega_1)_{[k],[k-1]} R_{[k-1],[1]} = -R_{[k],[1]},$$

for $l \in \{0, 1, \ldots, k-1\}$. Hence, we deduce

$$
((\omega_1)_{[k],[k-m_2+1]}, \ldots, (\omega_1)_{[k],[k]}) R^{[\mathcal{M}_k]} = -R_{\mathcal{M}_k},
$$

from where the result follows.

**Theorem 2.2** (Multivariate Gerominus formulæ). For $k \geq m_2$ and a given a poised set of multi-indices $\mathcal{M}_k$ we can write

$$
\hat{\mathcal{P}}_{[k]}(x) = \Theta_{\ast} \begin{pmatrix}
R_{[k-m_2],[\beta_1]} \cdots R_{[k-m_2],[\beta_{r_{k,m_2}}} \cdot P_{[k-m_2]}(x) \\
\vdots \quad \vdots \\
R_{[k],[\beta_1]} \cdots R_{[k],[\beta_{r_{k,m_2}}]} \cdot P_{[k]}(x)
\end{pmatrix}.
$$
In this case, for the quasi-tau matrices we have the following two expressions

\[
\begin{align*}
\mathcal{H}_{[k]}(\mathcal{Q}_2(\mathcal{A}))_{[k-m_2],[k]}^\top &= \Theta_* \begin{pmatrix} R_{[k-m_2],\beta_1} \cdots R_{[k-m_2],\beta_{r_{k,m_2}}} H_{[k-m_2]} \\ R_{[k-m_2+1],\beta_1} \cdots R_{[k-m_2+1],\beta_{r_{k,m_2}}} 0_{[k-m_2+1],[k-m_2]} \\ \vdots & \vdots \\ R_{[k],\beta_1} \cdots R_{[k],\beta_{r_{k,m_2}}} 0_{[k],[k-m_2]} \end{pmatrix}, \\
\mathcal{H}_{[k]} &= \Theta_* \begin{pmatrix} R_{[k-m_2],\beta_1} \cdots R_{[k-m_2],\beta_{r_{k,m_2}}} R_{[k-m_2],[k]} \\ R_{[k-m_2+1],\beta_1} \cdots R_{[k-m_2+1],\beta_{r_{k,m_2}}} R_{[k-m_2+1],[k]} \\ \vdots & \vdots \\ R_{[k],\beta_1} \cdots R_{[k],\beta_{r_{k,m_2}}} R_{[k],[k]} \end{pmatrix},
\end{align*}
\]

Proof. When \( k \geq m_2 \) we can use (2.7)

\[
\mathcal{P}_{[k]}(x) = (\omega_1)_{[k],[k-m_2]} P_{[k-m_2]}(x) + \cdots + (\omega_1)_{[k],[k-1]} P_{[k-1]}(x) + P_{[k]}(x),
\]

and Proposition 2.13 leads to 2.14. From Proposition 2.3 we get

\[
(\omega_1)_{[k],[k-m_2]} = \mathcal{H}_{[k]}(\mathcal{Q}_2(\mathcal{A}))_{[k-m_2],[k]}^\top (H_{[k-m_2]})^{-1},
\]

while Proposition 2.13 tells us that

\[
(\omega_1)_{[k],[k-m_2]} = -R_{\mathcal{M}_k}(R_{[\mathcal{M}_k]})^{-1} \begin{pmatrix} I_{[k-m_2]} \\ 0_{[k-m_2+1],[k-m_2]} \\ \vdots \\ 0_{[k],[k-m_2]} \end{pmatrix},
\]

and, consequently, (2.15) is proven. Then, to prove (2.16) just recall Proposition 2.10 and write

\[
\mathcal{H}_{[k]} = \begin{pmatrix} (\omega_1)_{[k],[k-m_2]} & \cdots & (\omega_1)_{[k],[k-1]} \end{pmatrix} \begin{pmatrix} R_{[k-m_2],[k]} \\ \vdots \\ R_{[k-1],[k]} \end{pmatrix} + R_{[k],[k]},
\]

and use Proposition 2.13 to conclude

\[
\mathcal{H}_{[k]} = R_{[k],[k]} - R_{\mathcal{M}_k}(R_{[\mathcal{M}_k]})^{-1} \begin{pmatrix} R_{[k-m_2],[k]} \\ \vdots \\ R_{[k-1],[k]} \end{pmatrix}.
\]

\[
\square
\]

2.4. Recovering the 1D Geronimus formula. Let us assume that \( D = 1 \), then \(|k| = 1\) and \( N_{k-1} = k \) and for \( k \geq m_2 \) we have \( r_{k,m_2} = m_2 \), so we can choose the indices as \([0,1,\ldots,m_2-1]\) (there are other possibilities but let us suppose that it is poised) as they all are less than \( k \). Let us assume that \( Q_2(x) = (x - q_1) \cdots (x - q_{m_2}) \), has \( m_2 \) simple zeroes \( \{q_1,\ldots,q_{m_2}\} \), and let us consider the Cauchy transforms \( C_k(x) \) of the orthogonal polynomials \( P_k(x) \) of the original measure \( d\mu(x) \) given by

\[
C_k(x) := \int \frac{P_k(y)}{y-x} \, d\mu(y).
\]

The point is that the two set of numbers \( \{C_k(q_1),\ldots,C_k(q_{m_2})\} \) and \( \{\rho_{0,k},\rho_{k,1},\ldots,\rho_{k,m_2-1}\} \) are linked by the Vandermonde matrix

\[
\mathcal{V} = \begin{pmatrix} 1 & \cdots & 1 \\ q_1 & \cdots & q_{m_2} \\ \vdots & \vdots & \vdots \\ q_1^{m_2-1} & \cdots & q_{m_2}^{m_2-1} \end{pmatrix}.
\]
and the diagonal matrix

\[ \mathcal{D} := \text{diag} \left( \prod_{i \in \{1, \ldots, m_2\}, i \neq 1} (q_1 - q_i), \ldots, \prod_{i \in \{1, \ldots, m_2\}, i \neq m_2} (q_{m_2} - q_i) \right), \]

by the formula

\[ (\rho_{k,0}, \ldots, \rho_{k,m_2-1}) = (C_k(q_1), \ldots, C_k(q_{m_2})) \mathcal{D}^{-1} \mathcal{V}^\top. \tag{2.17} \]

This relation can be obtained from the identity

\[ \frac{(x - q_1) \cdots (x - q_i) \cdots (x - q_{m_2})}{(x - q_1) \cdots (x - q_{m_2})} = \frac{1}{x - q_i}, \]

where by \((x - q_i)\) we mean that this factor has been deleted from the product, by expanding the numerator —according to Vieta’s formulae— in terms of elementary symmetric polynomials of the roots, \(e_j(q_1, \ldots, q_{m_2}), j \in \{0, 1, \ldots, m_2\}\). Moreover, we have the following formulae

\[ \left( \begin{array}{ccc} \rho_{k-m_2,0} & \cdots & \rho_{k-m_2,m_2-1} \\ \vdots & \ddots & \vdots \\ \rho_{k-1,0} & \cdots & \rho_{k-1,m_2-1} \end{array} \right) = \left( \begin{array}{ccc} C_{k-m_2}(q_{1}) & \cdots & C_{k-m_2}(q_{m_2}) \\ \vdots & \ddots & \vdots \\ C_{k-1}(q_{1}) & \cdots & C_{k-1}(q_{m_2}) \end{array} \right) \mathcal{D}^{-1} \mathcal{V}^\top. \]

Regarding the \(\theta_{k,n}\) terms we must recall that a general form of \(d_\nu\) in the 1D scenario is given in (2.4), from where one concludes that

\[ \left( \begin{array}{ccc} \theta_{k-m_2,0} & \cdots & \theta_{k-m_2,m_2-1} \\ \vdots & \ddots & \vdots \\ \theta_{k-1,0} & \cdots & \theta_{k-1,m_2-1} \end{array} \right) = \left( \begin{array}{ccc} P_{k-m_2}(q_{1}) & \cdots & P_{k-m_2}(q_{m_2}) \\ \vdots & \ddots & \vdots \\ P_{k-1}(q_{1}) & \cdots & P_{k-1}(q_{m_2}) \end{array} \right) \mathcal{V}^\top \]

where

\[ \tilde{\zeta} = \text{diag}(\zeta_1, \ldots, \zeta_{m_2}). \]

Hence, if

\[ \xi_j := \zeta_j \prod_{i \in \{1, \ldots, m_2\}, i \neq j} (q_j - q_i), \quad \phi_1(x, \xi) := C_1(x) + \xi P_1(x), \]

we get

\[ R^{M_k} = \left( \begin{array}{ccc} \phi_{k-m_2}(q_1, \xi_1) & \cdots & \phi_{k-m_2}(q_{m_2}, \xi_{m_2}) \\ \vdots & \ddots & \vdots \\ \phi_{k-1}(q_1, \xi_1) & \cdots & \phi_{k-1}(q_{m_2}, \xi_{m_2}) \end{array} \right) \mathcal{D}^{-1} \mathcal{V}^\top, \]

\[ R_{M_k} = (\phi_k(q_1, \xi_1), \ldots, \phi_k(q_{m_2}, \xi_{m_2})) \mathcal{D}^{-1} \mathcal{V}^\top. \]

Therefore,

\[ R_{M_k} (R^{[R_k]})^{-1} = (\phi_k(q_1, \xi_1), \ldots, \phi_k(q_{m_2}, \xi_{m_2})) \left( \begin{array}{ccc} \phi_{k-m_2}(q_1, \xi_1) & \cdots & \phi_{k-m_2}(q_{m_2}, \xi_{m_2}) \\ \vdots & \ddots & \vdots \\ \phi_{k-1}(q_1, \xi_1) & \cdots & \phi_{k-1}(q_{m_2}, \xi_{m_2}) \end{array} \right)^{-1}. \]
We finally get for, $k \geq m_2$, the perturbed polynomials the Geronimus formula \[14\]

\[
\hat{P}_k(x) = \Theta_s \left( \begin{array}{c}
\phi_{k-m_2}(q_1, \xi_1) \cdots \phi_{k-m_2}(q_{m_2}, \xi_{m_2}) \ p_{k-m_2}(x) \\
\vdots \\
\phi_k(q_1, \xi_1) \cdots \phi_k(q_{m_2}, \xi_{m_2}) \ p_k(x) \\
\phi_{k-m_2}(q_1, \xi_1) \cdots \phi_{k-m_2}(q_{m_2}, \xi_{m_2}) \ p_{k-m_2}(x) \\
\vdots \\
\phi_k(q_1, \xi_1) \cdots \phi_k(q_{m_2}, \xi_{m_2}) \ p_k(x) \\
\phi_{k-1}(q_1, \xi_1) \cdots \phi_{k-1}(q_{m_2}, \xi_{m_2}) \\
\end{array} \right) = \Theta_s \left( \begin{array}{c}
\phi_{k-m_2}(q_1, \xi_1) \cdots \phi_{k-m_2}(q_{m_2}, \xi_{m_2}) \ H_{k-m_2} \\
\phi_{k-m_2+1}(q_1, \xi_1) \cdots \phi_{k-m_2+1}(q_{m_2}, \xi_{m_2}) \ 0 \\
\vdots \\
\phi_k(q_1, \xi_1) \cdots \phi_k(q_{m_2}, \xi_{m_2}) \ 0 \\
\phi_{k-m_2+1}(q_1, \xi_1) \cdots \phi_{k-m_2+1}(q_{m_2}, \xi_{m_2}) \\
\vdots \\
\phi_k(q_1, \xi_1) \cdots \phi_k(q_{m_2}, \xi_{m_2}) \\
\phi_{k-1}(q_1, \xi_1) \cdots \phi_{k-1}(q_{m_2}, \xi_{m_2}) \\
\end{array} \right) = (-1)^{m_2+1} \Theta_s \left( \begin{array}{c}
\phi_{k-m_2}(q_1, \xi_1) \cdots \phi_{k-m_2}(q_{m_2}, \xi_{m_2}) \\
\phi_{k-m_2+1}(q_1, \xi_1) \cdots \phi_{k-m_2+1}(q_{m_2}, \xi_{m_2}) \\
\vdots \\
\phi_k(q_1, \xi_1) \cdots \phi_k(q_{m_2}, \xi_{m_2}) \\
\phi_{k-1}(q_1, \xi_1) \cdots \phi_{k-1}(q_{m_2}, \xi_{m_2}) \\
\end{array} \right) \ H_{k-m_2}.
\]

and the perturbed squared norms

\[
\hat{H}_k = \Theta_s \left( \begin{array}{c}
\phi_{k-m_2}(q_1, \xi_1) \cdots \phi_{k-m_2}(q_{m_2}, \xi_{m_2}) \\
\phi_{k-m_2+1}(q_1, \xi_1) \cdots \phi_{k-m_2+1}(q_{m_2}, \xi_{m_2}) \\
\vdots \\
\phi_k(q_1, \xi_1) \cdots \phi_k(q_{m_2}, \xi_{m_2}) \\
\phi_{k-1}(q_1, \xi_1) \cdots \phi_{k-1}(q_{m_2}, \xi_{m_2}) \\
\end{array} \right) \ H_{k-m_2}.
\]

### 3. Linear spectral transformations

Once we have discussed the multivariate Geronimus transformation we are ready to consider the more general linear spectral transform, that might be thought as the multiplication by a rational function, plus an extra contribution living in the zeroes of the polynomial in the denominator. Uvarov perturbations are treated in the Appendix.

**Definition 3.1.** For a given generalized function $u \in \mathcal{O}^*_u$, let us consider two coprime polynomials $Q_1(x), Q_2(x) \in \mathbb{C}[x]$, i.e., with no common prime factors, degrees $\deg Q_1 = m_1$ and $\deg Q_2 = m_2$, and such that $Z(Q_2) \cap \text{supp}(u) = \emptyset$. Then, the set of linear functionals $\hat{u}$ such that

\[
Q_2 \hat{u} = Q_1 u.
\]

is called a linear spectral transformation.

Again, there is not a unique $\hat{u}$ satisfying this condition. In fact, assume we have found such linear functional that we denote as $\frac{Q_1}{Q_2} u$, then all possible perturbations $\hat{u}$ verifying (2.1) will have the form

\[
\hat{u} = \frac{Q_1}{Q_2} u + v
\]

where, as for the Geronimus transformation, the linear functional $v \in (\mathbb{C}[x])'$ is such that $(Q_2) \subseteq \text{Ker}(v)$; i.e.,

\[
Q_2 v = 0.
\]

**Proposition 3.1.** A linear spectral transformation $u \mapsto \hat{u}$ can be obtained by performing first a Geronimus transformation and then a Christoffel transformation:

\[
u \mapsto \tilde{u} \mapsto \hat{u},
\]
where
\[ Q_2 \hat{u} = u, \quad \hat{u} = Q_1 \hat{u}. \]

For example, for a given a positive Borel measure \( d \mu(x) \) with associated zero order linear functional
\[ \langle u, P(x) \rangle = \int P(x) \, d \mu(x), \]
we can choose \( \frac{Q_1}{Q_2} u \in (\mathbb{C}[x])' \) as the following linear functional
\[ \left\langle \frac{Q_1}{Q_2} u, P(x) \right\rangle = \int P(x) \frac{Q_1(x)}{Q_2(x)} \, d \mu(x), \]
which makes sense if \( Z(Q_2) \cap \text{supp}(d \mu) = \emptyset. \)

**Proposition 3.2.** If \( \hat{G} \) is the moment matrix of the perturbed linear functional \( \hat{u} \) we have
\[ Q_2(\Lambda) \hat{G} = \hat{G} Q_2(\Lambda^T) = Q_1(\Lambda) G = G Q_1(\Lambda^T). \]

*Proof.* It is proven as follows
\[
Q_2(\Lambda) \langle \hat{u}, \chi(x)(\chi(x))^T \rangle = \langle \hat{u}, Q_2(x)(\chi(x))^{(\chi(x))^T} \rangle \\
= \langle Q_2 \hat{u}, \chi(x)(\chi(x))^T \rangle \\
= \langle Q_1 u, \chi(x)(\chi(x))^T \rangle \\
= \langle u, Q_1(x)(\chi(x))^T \rangle \\
= Q_1(\Lambda) \langle u, \chi(x)(\chi(x))^T \rangle \\
\]
\( \square \)

### 3.1. Resolvents and connection formulæ.

**Definition 3.2.** The resolvent matrices are
\[ \omega_1 := \hat{S} Q_1(\Lambda) S^{-1}, \quad (\omega_2)^T := S Q_2(\Lambda) \hat{S}^{-1}, \]
given in terms of the lower unitriangular matrices \( S \) and \( \hat{S} \) of the Cholesky factorizations of the moment matrices \( G = S^{-1} H (S^{-1})^T \) and \( \hat{G} = (\hat{S})^{-1} (H)(\hat{S})^{-1} \).

**Proposition 3.3.** The resolvent matrices satisfy
\[ \hat{H} \omega_2 = \omega_1 H. \]

*Proof.* It follows from the Cholesky factorization of \( G \) and \( \hat{G} \) and from Proposition 3.2 \( \square \)

**Proposition 3.4.** The resolvent matrices \( \omega_1 \) and \( \omega_2 \) are block banded matrices. All their block superdiagonals above the \( m_1 \)-th and all their subdiagonals below \( m_2 \)-th are zero. In particular, the \( m_1 \)-th block superdiagonal of \( \omega_1 \) is \( Q_1^{(m_1)}(\Lambda) \) while its \( m_2 \)-th block subdiagonal is \( \hat{H}(Q_2^{(m_2)}(\Lambda^T)) H^{-1}. \)

*Proof.* From Definition 3.2 we deduce that both \( \omega_1 \) or \( (\omega_2)^T \) are semi-infinite matrices with all its block superdiagonals outside the block diagonal band going from the \( m_1 \)-th superdiagonal to \( m_2 \)-th subdiagonal being zero, and with the \( m_1 \) or \( m_2 \) superdiagonal equal to \( Q_1^{(m_1)}(\Lambda) \) and \( Q_2^{(m_2)}(\Lambda) \), respectively. Consequently, if (3.3) is taken into account we deduce the band block structure. \( \square \)

Incidentally, and as a byproduct let us notice

**Proposition 3.5.** The following factorizations hold
\[ Q_1(J) Q_2(J) = Q_2(J) Q_1(J) = (\omega_2)^T \omega_1, \]
\[ Q_1(J) Q_2(J) = Q_2(J) Q_1(J) = \omega_1 (\omega_2)^T. \]
The truncations satisfy
\[
\det \left[ (Q_1(J))^{[k]} \right] = \det \left( (\omega_1)^{[k]} \right), \quad \det \left( Q_2(J) \right)^{[k]} = \det \left( (\omega_2)^{[k]} \right).
\]

**Proof.** In the one hand, Definitions 1.3 and 3.2 imply
\[
Q_1(J) = SS^{-1}\omega_1, \quad Q_2(J) = \omega_2^TSS^{-1},
\]
\[
Q_1(\tilde{J}) = \omega_1SS^{-1}, \quad Q_2(\tilde{J}) = \tilde{S}^{-1}\omega_2^T,
\]
from where we conclude the factorizations (3.4). \(\square\)

**Proposition 3.6 (Connection formulae).** The following relations are fulfilled
\[
(\omega_2)^TP(x) = Q_2(x)P(x),
\]
\[
\omega_1P(x) = Q_1(x)\tilde{P}(x).
\]

**Proof.** It follows from (1.2) and Definition 3.2. \(\square\)

3.2. **The multivariate Christoffel–Geronimus–Uvarov formula.** We are ready to deduce a multivariate extension of the Christoffel–Geronimus–Uvarov formula for linear spectral transformations, [21,44,75,81].

**Definition 3.3.** We introduce the semi-infinite block matrices
\[
R := \langle \tilde{u}, P(x)(\chi(x))^T \rangle.
\]

**Proposition 3.7.** The formula
\[
R = \rho + \theta,
\]
\[
\rho := \left\langle u, \frac{P(x)(\chi(x))^T}{Q_2(x)} \right\rangle, \quad \theta := \left\langle v, \frac{P(x)(\chi(x))^T}{Q_1(x)} \right\rangle,
\]
holds.

**Proof.** Just write \(\tilde{u} = \frac{u}{\partial Q_2} + \frac{\nu}{\partial Q_1}\), with \((Q_2) \subset \text{Ker} \nu\), and \(\tilde{u} = \frac{Q_2}{Q_1} \cdot u + v\). \(\square\)

As in the Geronimus situation

**Proposition 3.8.** When the linear functional \(u\) is of order zero with associated Borel measure \(d\mu(x)\) we have
\[
\rho = \int \frac{P(x)(\chi(x))^T \cdot d\mu(x)}{Q_2(x)}
\]
and for a given prime factorization \(Q_2 = (Q_{2,1})^{d_1} \cdots (Q_{2,N})^{d_N}\) and \(v\) taken as in (2.3) we can write
\[
\theta = \sum_{i=1}^{N} \sum_{\alpha \in \mathbb{Z}_+^d} \int_{Z(Q_{2,i})} \frac{\partial^\alpha}{\partial x^\alpha} \left( \frac{P(x)(\chi(x))^T}{Q_1(x)} \right) d\xi_{i,\alpha}(x).
\]

**Proposition 3.9.** The following relations
\[
(\omega_1R)^{[k]} = 0, \quad l < k,
\]
\[
(\omega_1R)^{[k]}_{[k], [l]} = \hat{H}_{k},
\]
hold for the linear spectral type transformation.

**Proof.** Just follow the proof of Proposition 2.10 \(\square\)

**Definition 3.4.** For \(m_1 > 0\) we consider a set of different multi-indices \(M_k = \{ \beta_i : |\beta_i| < k \}_{i=1}^{N_2m_2}\), with cardinal given by
\[
r_{2|k,m_2} := |M_k| = \begin{cases} N_{k-1} - \sum_{l=0}^{k-1} |l|, & k < m_2 \\ N_{k-1} - N_{k-m_2-1} - \sum_{l=0}^{k-1} |l|, & k \geq m_2. \end{cases}
\]
We also consider a set of different nodes \( \mathcal{N}_k = \{ p_i \}_{i=1}^{r_1[k,m_1]} \), in the algebraic hypersurface \( Z(\Omega_1) \) of zeroes of \( \Omega_1 \), where

\[
r_{1,k,m_1} := |\mathcal{N}_k| = N_{k+m_1-1} - N_{k-1} = |k| + \cdots + |k + m_1 - 1|.
\]

Finally, we introduce the set \( S_k := \mathcal{N}_k \cup \mathcal{N}_k \), the union of the sets of multi-indices and nodes with cardinal given by

\[
r_{k,m} := |S_k| = r_{1,k,m_1} + r_{2,k,m_2} = \begin{cases} N_{k+m_1-1}, & k < m_2, \\ N_{k+m_1-1} - N_{k-m_2-1}, & k \geq m_2. \end{cases}
\]

**Definition 3.5.** When \( k < m_2 \) a set of nodes is poised if

\[
\begin{vmatrix}
R_{[0],[0]} & \cdots & R_{[0],[k-1]} & P_{[0]}(p_1) & \cdots & P_{[0]}(p_{r_{1,k,m_1}}) \\
\vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\
R_{[k+m_1-1],[k-1]} & \cdots & R_{[k+m_1-1],[k-1]} & P_{[k+m_1-1]}(p_1) & \cdots & P_{[k+m_1-1]}(p_{r_{1,k,m_1}})
\end{vmatrix} \neq 0.
\]

For \( k \geq m_2 \), we say that the set \( S_k \) of nodes and multi-indices is poised if

\[
\begin{vmatrix}
R_{[k-m_2],[\beta_1]} & \cdots & R_{[k-m_2],[\beta_{r_{2,k,m_2}}]} & P_{[k-m_2]}(p_1) & \cdots & P_{[k-m_2]}(p_{r_{1,k,m_1}}) \\
\vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\
R_{[k+m_1],[\beta_1]} & \cdots & R_{[k+m_1],[\beta_{r_{2,k,m_2}}]} & P_{[k+m_1]}(p_1) & \cdots & P_{[k+m_1]}(p_{r_{1,k,m_1}})
\end{vmatrix} \neq 0.
\]

**Theorem 3.1** (Christoffel–Geronimus–Uvarov formula for multivariate linear spectral transformations). Given a poised set \( S_k \), of multi-indices and nodes, the perturbed orthogonal polynomials, generated by the linear spectral transformation given in Definition 3.1 can be expressed, for each \( k \in \mathbb{Z}_+ \), as

\[
\hat{P}_{[k]}(x) = \frac{(\Omega_1(\Lambda))_{[k],[k+m_1]}}{\Omega_1(x)} \times \Theta_*(\begin{vmatrix}
R_{[0],[0]} & \cdots & R_{[0],[k-1]} & P_{[0]}(p_1) & \cdots & P_{[0]}(p_{r_{1,k,m_1}}) & P_{[0]}(x) \\
\vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\
R_{[k+m_1],[0]} & \cdots & R_{[k+m_1],[k-1]} & P_{[k+m_1]}(p_1) & \cdots & P_{[k+m_1]}(p_{r_{1,k,m_1}}) & P_{[k+m_1]}(x)
\end{vmatrix}),
\]

and

\[
\hat{H}_{[k]} = (\Omega_1(\Lambda))_{[k],[k+m_1]} \times \Theta_*(\begin{vmatrix}
R_{[0],[0]} & \cdots & R_{[0],[k-1]} & P_{[0]}(p_1) & \cdots & P_{[0]}(p_{r_{1,k,m_1}}) & R_{[0],[k]} \\
\vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\
R_{[k+m_1],[0]} & \cdots & R_{[k+m_1],[k-1]} & P_{[k+m_1]}(p_1) & \cdots & P_{[k+m_1]}(p_{r_{1,k,m_1}}) & R_{[k+m_1],[k]}
\end{vmatrix}).
\]

When \( k \geq m_2 \), we also have for the perturbed MVOPR

\[
\hat{P}_{[k]}(x) = \frac{(\Omega_1(\Lambda))_{[k],[k+m_1]}}{\Omega_1(x)} \times \Theta_*(\begin{vmatrix}
R_{[k-m_2],[\beta_1]} & \cdots & R_{[k-m_2],[\beta_{r_{2,k,m_2}}]} & P_{[k-m_2]}(p_1) & \cdots & P_{[k-m_2]}(p_{r_{1,k,m_1}}) & P_{[k-m_2]}(x) \\
\vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\
R_{[k+m_1],[\beta_1]} & \cdots & R_{[k+m_1],[\beta_{r_{2,k,m_2}}]} & P_{[k+m_1]}(p_1) & \cdots & P_{[k+m_1]}(p_{r_{1,k,m_1}}) & P_{[k+m_1]}(x)
\end{vmatrix}),
\]
The quasi-tau matrices are subject to

\[
\hat{H}_1 \left( \left( Q_2(A) \right)_{[k-m_2],[k]} \right) = \left( Q_1(A) \right)_{[k],[k+m_1]}
\]

Second, we analyze the consequences of (3.6) and (3.5). In the one hand, from (3.6) we have for \( t < k \)

\[
(\omega_1)_{[k],[0]} R_{[0],[1]} + \cdots + (\omega_1)_{[k],[k+m_1]} R_{[k+m_1-1],[1]} = -\left( Q_1(A) \right)_{[k],[k+m_1]} R_{[k+m_1],[1]}.
\]

Moreover, when \( k \geq m_2 \) and \( 1 < k \), it is also true that

\[
(\omega_1)_{[k],[k-m_2]} R_{[k-m_2],[1]} + \cdots + (\omega_1)_{[k],[k+m_1]} R_{[k+m_1-1],[1]} = -\left( Q_1(A) \right)_{[k],[k+m_1]} R_{[k+m_1],[1]}.
\]

On the other hand, from (3.5), given a zero \( p \) of \( Q_1(x) \) we can write

\[
(\omega_1)_{[k],[0]} P_{[0],[p]} + \cdots + (\omega_1)_{[k],[k+m_1]} P_{[k+m_1-1],[p]} = -\left( Q_1(A) \right)_{[k],[k+m_1]} P_{[k+m_1],[p]},
\]

and when \( k \geq m_2 \) it can be written as follows

\[
(\omega_1)_{[k],[k-m_2]} P_{[0],[p]} + \cdots + (\omega_1)_{[k],[k+m_1]} P_{[k+m_1-1],[p]} = -\left( Q_1(A) \right)_{[k],[k+m_1]} P_{[k+m_1],[p]}.
\]

Regarding the sizes of the resolvent matrices involved let us remark

\[
\left( (\omega_1)_{[k],[0]}, \ldots, (\omega_1)_{[k],[k+m_1-1]} \right) \in \mathbb{R}^{[k] \times (N_{k+m_1-1})},
\]

\[
\left( (\omega_1)_{[k],[k-m_2]}, \ldots, (\omega_1)_{[k],[k+m_1-1]} \right) \in \mathbb{R}^{[k] \times (N_{k+m_1-1} - N_{k-m_2-1})}, \quad k \geq m_2.
\]

Thus, for \( k < m_2 \) we can write

\[
(3.8) \quad \left( (\omega_1)_{[k],[0]}, \ldots, (\omega_1)_{[k],[k+m_1-1]} \right) = -\left( Q_1(A) \right)_{[k],[k+m_1]} \left( R_{[k+m_1],[0]}, \ldots, R_{[k+m_1],[k-1]}, P_{[k+m_1],[p_1]}, \ldots, P_{[k+m_1],[p_{r_1,k+m_1}]} \right) -1
\]

\[
\times \left( \begin{array}{cccc}
R_{[0],[0]} & \cdots & R_{[0],[k-1]} & P_{[k],[p_1]} & \cdots & P_{[k],[p_{r_1,k+m_1}]} \\
\vdots & \ddots & \vdots & \vdots & \ddots & \vdots \\
R_{[k+m_1-1],[0]} & \cdots & R_{[k+m_1-1],[k-1]} & P_{[k+m_1-1],[p_1]} & \cdots & P_{[k+m_1-1],[p_{r_1,k+m_1}]} \end{array} \right),
\]

Proof. First, we reckon that

\[
(\omega_1)_{[k],[k+m_1]} = \left( Q_1(A) \right)_{[k],[k+m_1]}.
\]
while for \( k \geq m_2 \)

\[
(3.9) \quad ((\omega_1)_{[k],[k-m_2]}, \ldots, (\omega_1)_{[k],[k+m_1-1]} = \\
- (Q_1(A))_{[k],[k+m_1]} \begin{pmatrix}
R_{[k-m_2],\beta_1} & \ldots & R_{[k-m_2],\beta_{2|k-m_2}}, P_{[k-m_2]}(p_1), \ldots, P_{[k+m_1]}(p_{\nu(k,k_m)}) \\
\vdots & \ldots & \vdots & \ddots & \vdots \\
R_{[k+m_1-1],\beta_1} & \ldots & R_{[k+m_1-1],\beta_{2|k-m_2}}, P_{[k+m_1-1]}(p_1), \ldots, P_{[k+m_1-1]}(p_{\nu(k,k_m)})
\end{pmatrix}^{-1},
\]

and similarly for \( k < m_2 \). Now, recalling the connection formula (3.5) we derive the stated result.

Proposition 3.4 implies

\[
(\omega_1)_{[k],[k-m_2]} = \hat{H}_{[k]} \left( (Q_2(A))_{[k-m_2],[k]} \right) ^\top (H_{[k-m_2]})^{-1},
\]

i.e., the first quasi-determinantal expression for \( \hat{H}_{[k]} \) is proven.

Finally, from (3.7) we get

\[
(\omega_1)_{[k],[k-m_2]} \mathcal{R}_{[k-m_2],[k]} + \cdots + (\omega_1)_{[k],[k+m_1]} \mathcal{R}_{[k+m_1-1],[k]} + (Q_1(A))_{[k],[k+m_1]} \mathcal{R}_{[k+m_1],[k]} = \hat{H}_{[k]},
\]

and (3.9) we get the second quasi-determinantal expression for \( \hat{H}_{[k]} \).

For the finding of a multivariate Christoffel formula for Christoffel transformations we need the conco-ourse of poised sets, and the existence of them deeply depends on the algebraic hypersurface of the zeros \( \Omega_1(x) \) of the perturbing polynomial \( \Omega_1(x) \), see [14]. In fact, for a factorization in terms of irreducible polynomials, \( \Omega_1(x) = \prod_{i=1}^N (\Omega_{1,i}(x))^{d_i} \), with \( d_1 = \cdots = d_N = 1 \) we require the poised set to belong only to the mentioned algebraic hypersurfaces and not to any other of lower degree. Moreover, if any of the multiplicities \( d_1, \ldots, d_N \) is bigger than 1 we need to introduce multi-Wronskians expressions. For the Geronimus case this is not necessary as we have Wronskians already encoded in the linear functional \( v \) and, consequently, in \( R \). However, the linear spectral transformations is a composition of Geronimus and Christoffel transformations. Therefore, we have a similar situation as that described in [14]. In fact, to have poised sets the requirements discussed in that paper are necessary. Thus, the formulae given make sense only when all multiplicities of the irreducible factors of \( \Omega_1 \) are 1. Otherwise, a multi-Wronskian generalization is needed [14].

### 3.3. The 1D case: recovering the 1D Christoffel–Geronimus–Uvarov formula.

In the scalar case \( D = 1 \) we take two polynomials with simple roots

\[
\Omega_1(x) = (x - p_1) \cdots (x - p_{m_1}), \quad \Omega_2(x) = (x - q_1) \cdots (x - q_{m_2}).
\]

Then, we have \( r_{1|k,m_1} = m_1 \) and \( r_{2|k,m_2} = m_2 \) and we can take the \( m_2 \) indexes (not \textit{multi} as we have \( D = 1 \)) as \( \beta = 0, 1, \ldots, m_2 - 1 \) (we have more possibilities). Moreover, we have

\[
\begin{pmatrix}
\rho_{k-m_2,0} & \cdots & \rho_{k-m_2,m_2-1} \\
\vdots & \ddots & \vdots \\
\rho_{k+m_1-1,0} & \cdots & \rho_{k+m_1-1,m_2-1}
\end{pmatrix} = \begin{pmatrix}
P_{k-m_2}(q_1) & \cdots & P_{k-m_2}(q_{m_2}) \\
\vdots & \ddots & \vdots \\
P_{k+m_1-1}(q_1) & \cdots & P_{k+m_1-1}(q_{m_2})
\end{pmatrix} D^{-1}v^T,
\]

For the \( \theta_{k,n} \) terms we must recall that the general form of \( d \nu \) in the 1D scenario is given in (2.4), and obtain

\[
\begin{pmatrix}
\theta_{k-m_2,0} & \cdots & \theta_{k-m_2,m_2} \\
\vdots & \ddots & \vdots \\
\theta_{k,0} & \cdots & \theta_{k,m_2}
\end{pmatrix} = \begin{pmatrix}
P_{k-m_2}(q_1) & \cdots & P_{k-m_2}(q_{m_2}) \\
\vdots & \ddots & \vdots \\
P_{k}(q_1) & \cdots & P_{k}(q_{m_2})
\end{pmatrix} \xi D^{-1}v^T,
\]

\[
(\theta_{k+m_1,0}, \ldots, \theta_{k+m_1,m_2-1}) = (P_{k+m_1}(q_1), \ldots, P_{k+m_1}(q_{m_2})) \xi D^{-1}v^T,
\]
where
\[ \xi_j := \frac{\xi_j}{Q_1(q_j)} \prod_{i \in \{1, \ldots, m_2\}, i \neq j} (q_j - q_i), \]
and consider
\[ \phi_1(x, \xi) := C_1(x) + \xi P_1(x). \]
Consequently, we have the perturbed polynomials determinantal expressions
\[
\hat{P}_k(x) = \frac{1}{Q_1(x)} \Theta_x \begin{pmatrix}
\phi_{k-m_2}(q_1, \xi_1) & \ldots & \phi_{k-m_2}(q_{m_2}, \xi_{m_2}) & P_{k-m_2}(p_1) & \ldots & P_{k-m_2}(p_{m_1}) & P_{k-m_2}(x) \\
\vdots & & \vdots & \vdots & & \vdots & \vdots \\
\phi_{k+m_1}(q_1, \xi_1) & \ldots & \phi_{k+m_1}(q_{m_2}, \xi_{m_2}) & P_{k+m_1}(p_1) & \ldots & P_{k+m_1}(p_{m_1}) & P_{k+m_1}(x)
\end{pmatrix}
\]
which coincides with formulæ (3.19) and (3.20) in [81]. Notice that, it also coincide with the Uvarov’s formulæ in §1.3 but not specifically with multivariate polynomials in mind.

The Toda type flows discussed in [13] for multivariate moment matrices can be extended further. The integrable hierarchy has the MVOPR as solutions, but this is only a part of its space of solutions, as the MVOPR sector corresponds to a particular choice of \( G \).

4. Extension to a multispectral 2D Toda lattice

We explore the situation described in §1.3 but not specifically with multivariate polynomials in mind. The block structure of the semi-infinite matrices has been described there. In [13] we considered a semi-infinite matrix \( G \) such that \( \Lambda_a G = G(\Lambda_a)^\top \), \( a \in \{1, \ldots, D\} \), a Cholesky factorization
\[ G = S^{-1}H(S)^{-\top}, \]
and flows preserving this structure. In that manner we obtained nonlinear equations for which the MVOPR provided solutions. Then, in [14] we derived a quasi-determinantal Christoffel formula for the multivariate Christoffel transformations for MVOPR. A similar development could be performed here with the more general linear spectral transformations, but we will follow an even more general approach.

The Toda type flows discussed in [13] for multivariate moment matrices can be extended further. The integrable hierarchy has the MVOPR as solutions, but this is only a part of its space of solutions, as the MVOPR sector corresponds to a particular choice of \( G \). In this paper we will analyze this Toda hierarchy, that we name as multispectral 2D Toda hierarchy, in its own, associated as we will see to non standard orthogonality. Therefore, we now consider any possible block Gaussian factorizable semi-infinite matrix
\[ G = (S_1)^{-1}H(S_2)^{-\top} \]
where, $S_1,S_2$ are lower unitriangular block semi-infinite matrices, and $H$ is a diagonal block semi-infinite matrix.

4.1. Non-standard multivariate biorthogonality. Bilinear forms.

**Definition 4.1.** In the linear space of multivariate polynomials $\mathbb{R}[x]$ we consider a bilinear form $\langle \cdot, \cdot \rangle$ whose Gramm semi-infinite matrix is $G$, i.e.

$$\langle P(x), Q(x) \rangle = \sum_{|\alpha| \leq \deg P, |\beta| \leq \deg Q} P_\alpha G_{\alpha,\beta} Q_\beta, \quad G_{\alpha,\beta} = \langle x^\alpha, x^\beta \rangle. \quad (4.1)$$

Whenever the sum $\sum_{\alpha,\beta \in \mathbb{Z}_+^D} P_\alpha G_{\alpha,\beta} Q_\beta$ converges in some sense, the corresponding extension of this bilinear form to the linear space of power series $\mathbb{C}[x]$ can be considered.

In general, the semi-infinite matrix $G$ has no further structure and, consequently, we do not expect it to be symmetric or to be related to a linear functional, for example. We say that we are dealing with a non standard bilinear form. The bilinear form (4.1) induces another bilinear form which is a bilinear map from semi-infinite vectors of polynomials (or power series when possible) into the semi-infinite matrices.

**Definition 4.2.** Given to semi-infinite vectors of polynomials $v(x) = (v_\alpha(x))_{\alpha \in \mathbb{Z}_+^D}$ and $w(x) = (w_\alpha(x))_{\alpha \in \mathbb{Z}_+^D}$, with $v_\alpha, w_\alpha \in \mathbb{C}[x]$ (or $\mathbb{C}[x]$ when possible) we consider the following semi-infinite matrix

$$\langle v(x), (w(x))^\top \rangle = \left(\langle v(x), (w(x))^\top \rangle_\alpha,\beta \right), \quad \langle v(x), (w(x))^\top \rangle_\alpha,\beta := \langle v_\alpha(x), w_\beta(x) \rangle, \quad \alpha,\beta \in \mathbb{Z}_+^D. \quad (4.2)$$

A similar definition holds for a polynomial $p(x) \in \mathbb{C}[x]$, i.e.,

$$\langle v(x), p(x) \rangle := \left(\langle v_\alpha(x), p(x) \rangle_\alpha \right), \quad \langle p(x), (v(x))^\top \rangle := \left(\langle p(x), v_\alpha(x) \rangle_\alpha \right)^\top. \quad (4.3)$$

**Proposition 4.1.** Given three semi-infinite vectors $v^{(i)}(x) = (v^{(i)}_\alpha(x))_{\alpha \in \mathbb{Z}_+^D}$, $i \in \{1,2,3\}$, the formulae

$$\langle v^{(1)}(x), (v^{(2)}(x))^\top \rangle v^{(3)}(z) = \langle v^{(1)}(x), (v^{(2)}(x))^\top v^{(3)}(z) \rangle, \quad (v^{(3)}(z))^\top \langle v^{(1)}(z), (v^{(2)}(x))^\top \rangle = \langle (v^{(3)}(z))^\top v^{(1)}(x), v^{(2)}(x) \rangle \quad (4.2)$$

hold.

Using this non standard bilinear form we can write

$$G = \langle \chi(x), (\chi(x))^\top \rangle. \quad (4.3)$$

When there is a linear form $u \in (\mathbb{C}[x])'$ such that $\langle P(x), Q(x) \rangle = \langle u, P(x)Q(x) \rangle$ we find that $G = \langle u, \chi(x) (\chi(x))^\top \rangle$ is the corresponding moment matrix.

**Proposition 4.2.** For any polynomial $\Omega(x) \in \mathbb{C}[x]$ we have

$$\Omega(A) G = \langle \Omega(x)\chi(x), (\chi(x))^\top \rangle, \quad G(\Omega(A))^\top = \langle (\chi(x))^\top \Omega(x) \rangle. \quad (4.2)$$

**Proof.** Use (4.6). □

4.2. A multispectral 2D Toda hierarchy. In terms of the continuous time parameters sequences $t = \{t_1, t_2\} \subset \mathbb{R}$ given by

$$t_i := \{t_i,\alpha\}_{\alpha \in \mathbb{Z}_+^D}, \quad i \in \{1,2\},$$

we consider the time power series

$$t_i(x) := \sum_{\alpha \in \mathbb{Z}_+^D} t_{i,\alpha} x^\alpha, \quad i \in \{1,2\},$$
the following vacuum wave semi-infinite matrices
\[ W_i^{(0)}(t_i) = \exp \left( \sum_{\alpha \in \mathbb{Z}^D} t_i, \alpha \Lambda^\alpha \right), \quad i \in \{1, 2\}, \]
and the perturbed semi-infinite matrix
\[ (4.4) \quad G(t) = W_1^{(0)}(t_1)G(W_2^{(0)}(t_2))^{-T}. \]

Notice that these flows do respect the multi-Hankel condition, if initially we have \( \Lambda_a G = G(\Lambda_a)^T, \ a \in \{1, \ldots, D\} \), then, for any further time, we will have \( \Lambda_a G(t) = G(t)(\Lambda_a)^T, \ a \in \{1, \ldots, D\} \).

We will assume that the block Gaussian factorization do exist, at least for an open subset of times containing \( t = 0 \)
\[ (4.5) \quad G(t) = (S_1(t))^{-1}H(t)(S_2(t))^{-T}. \]

Then, we consider the semi-infinite vectors of polynomials
\[ (4.6) \quad P_1(t, x) := S_1(t)\chi(x), \quad P_2(t, x) := S_2(t)\chi(x), \]
being its component \( P_{i, \alpha}(t, x), \ i \in \{1, 2\}, \alpha \in \mathbb{Z}_+^D \), a \( t \)-dependent monic multivariate polynomial in \( x \) of degree \( |\alpha| \).

Then, the Gaussian factorization \( (4.5) \) implies the bi-orthogonality condition
\[ \langle P_{1, |k|}(t, x), P_{2, |l|}(t, x) \rangle = \delta_{k, l}H_{|k|}(t). \]
Here we used the bilinear form \( \langle \cdot, \cdot \rangle \) with Gramm matrix \( G(t) \). We also consider the wave matrices
\[ (4.7) \quad W_1(t) := S_1(t)W_1^{(0)}(t_1), \quad W_2(t) := S_2(t)(W_2^{(0)}(t_2))^T, \]
where \( \tilde{S}_2 := H(t)(S_2(t))^{-T}. \)

**Proposition 4.3.** The wave matrices satisfy
\[ (4.8) \quad (W_1(t))^{-1}W_2(t) = G. \]

**Proof.** It follows from the Gauss–Borel factorization \( (4.5) \). \( \square \)

Given a semi-infinite matrix \( \Lambda \) we have unique splitting \( \Lambda = \Lambda_+ + \Lambda_- \) where \( \Lambda_+ \) is an upper triangular block matrix while is \( \Lambda_- \) a strictly lower triangular block matrix. The Gaussian factorization \( (4.8) \) has the following differential consequences

**Proposition 4.4.** The following equations hold
\[ \frac{\partial S_1}{\partial t_{1, \alpha}}(S_1)^{-1} = -\left( S_1\Lambda^\alpha(S_1)^{-1} \right)_-, \quad \frac{\partial S_1}{\partial t_{2, \alpha}}(S_1)^{-1} = \left( \tilde{S}_2(\Lambda^\top)^\alpha(\tilde{S}_2)^{-1} \right)_-, \]
\[ \frac{\partial S_2}{\partial t_{1, \alpha}}(S_2)^{-1} = \left( S_1\Lambda^\alpha(S_1)^{-1} \right)_+, \quad \frac{\partial S_2}{\partial t_{2, \alpha}}(S_2)^{-1} = -\left( \tilde{S}_2(\Lambda^\top)^\alpha(\tilde{S}_2)^{-1} \right)_+. \]

**Proof.** Taking right derivatives of \( (4.8) \) yields
\[ \frac{\partial W_1}{\partial t_{i, \alpha}}(W_1)^{-1} = \frac{\partial W_2}{\partial t_{i, \alpha}}(W_2)^{-1}, \quad i \in \{1, 2\}, \quad j \in \mathbb{Z}_+, \]
where
\[ \frac{\partial W_1}{\partial t_{1, \alpha}}(W_1)^{-1} = \frac{\partial S_1}{\partial t_{1, \alpha}}(S_1)^{-1} + S_1\Lambda^\alpha(S_1)^{-1}, \quad \frac{\partial W_1}{\partial t_{2, \alpha}}(W_1)^{-1} = \frac{\partial S_1}{\partial t_{2, \alpha}}(S_1)^{-1}, \]
\[ \frac{\partial W_2}{\partial t_{1, \alpha}}(W_2)^{-1} = \frac{\partial S_2}{\partial t_{1, \alpha}}(S_2)^{-1}, \quad \frac{\partial W_2}{\partial t_{2, \alpha}}(W_2)^{-1} = \frac{\partial S_2}{\partial t_{2, \alpha}}(S_2)^{-1} + \tilde{S}_2(\Lambda^\top)^\alpha(\tilde{S}_2)^{-1}, \]
and the result follows immediately. \( \square \)

As a consequence, we deduce
Proposition 4.5. The multicomponent 2D Toda lattice equations

\[
\frac{\partial}{\partial t_2,e_b} \left( \frac{\partial H_{[k]}(H_{[k]})^{-1}}{\partial t_1,e_a} \right) + (\Lambda_a)_{[k],[k+1]}H_{[k+1]} \left( (\Lambda_b)_{[k],[k+1]} \right)^\top (H_{[k]})^{-1} \\
- H_{[k]} \left( (\Lambda_b)_{[k-1],[k]} \right)^\top (H_{[k-1]})^{-1} (\Lambda_a)_{[k-1],[k]} = 0
\]

hold.

Proof. From Proposition 4.4 we get

\[
\frac{\partial H_{[k]}(H_{[k]})^{-1}}{\partial t_1,e_a} = \beta_{[k]}(\Lambda_a)_{[k-1],[k]} - (\Lambda_a)_{[k],[k+1]} \beta_{[k+1]}, \quad \frac{\partial \beta_{[k]}}{\partial t_2,e_b} = H_{[k]} \left( (\Lambda_b)_{[k-1],[k]} \right)^\top (H_{[k-1]})^{-1},
\]

where \( \beta_{[k]} \in \mathbb{R}^{[k] \times [k-1]} \), \( k = 1, 2, \ldots \), are the first subdiagonal coefficients in \( S_1 \).

These equations are just the first members of an infinite set of nonlinear partial differential equations, an integrable hierarchy. Its elements are given by

Definition 4.3. The Lax and Zakharov–Shabat matrices are given by

\[
L_{1,a} := S_1 \Lambda_a (S_1)^{-1}, \quad L_{2,a} := \tilde{S}_2 (\Lambda_a)^\top (\tilde{S}_2)^{-1},
\]

\[
B_{1,\alpha} := (L_1^\alpha)_+, \quad B_{2,\alpha} := (L_2^\alpha)_-.
\]

The Baker functions are defined as

\[
\Psi_1(t,z) := W_1(t) \chi(z), \quad \Psi_2(t,z) := W_2(t) \chi^*(z),
\]

and the adjoint Baker functions by

\[
\Psi_1^*(t,z) := (W_1(t))^{-\top} \chi^*(z), \quad \Psi_2^*(t,z) := (W_2(t))^{-\top} \chi(z),
\]

here we switch for \( x \in \mathbb{R}^D \) to \( z \in \mathbb{C} \). We also consider the multivariate Cauchy kernel

\[
\mathcal{C}(z,x) := \frac{1}{\prod_{i=1}^{D} (z_i - x_i)}.
\]

Proposition 4.6. The Lax matrices can be written as

\[
L_{1,a}(t) = W_1(t) \Lambda_a (W_1(t))^{-1}, \quad L_{2,a}(t) = W_2(t) (\Lambda_a)^\top (W_2(t))^{-1},
\]

and satisfy commutativity properties

\[
[L_{1,a}(t), L_{1,b}(t)] = 0, \quad [L_{2,a}(t), L_{2,b}(t)] = 0, \quad a, b \in \{1, \ldots , D\},
\]

and the spectral properties

\[
L_{1,a}(t) \Psi_1(t,x) = x_a \Psi_1(t,x), \quad (L_{2,a}(t))^{-\top} \Psi_2^*(t,x) = x_a \Psi_2^*(t,x), \quad a \in \{1, \ldots , D\}.
\]

The Cauchy kernel satisfies

\[
(\chi(x))^\top \chi^*(z) = \mathcal{C}(z,x), \quad |z_i| > |x_i|, \quad i \in \{1, \ldots , D\}.
\]

Theorem 4.1. The Baker functions can be expressed in terms of the orthogonal polynomials, the multivariate Cauchy kernel and the bilinear form as follows

\[
\Psi_1(t,z) = e^{t_1(x)} P_1(t,z), \quad (4.11)
\]

\[
\Psi_2(t,z) = e^{-t_2(z)} (H(t))^{-\top} P_2(t,z), \quad (4.12)
\]

\[
\Psi_2(t,z) = \langle \Psi_1(t,x), C(z,x) \rangle, \quad (4.13)
\]

\[
|z_i| > |x_i|, \quad i \in \{1, \ldots , D\},
\]

\[
(\Psi_1^*(t,z))^\top = \langle \mathcal{C}(z,x), (\Psi_2^*(t,x))^\top \rangle, \quad (4.14)
\]

\[
|z_i| > |x_i|, \quad i \in \{1, \ldots , D\}.
\]
Proof. Equation (4.11) follows easily
\[
\Psi_1(t, x) = W_1(t) \chi_1(x),
\]
\[
= S(t) W_1(0) (t_1) \chi(x)
\]
\[
= e^{t_1(x)} S(t_1) \chi_1(x)
\]
\[
= e^{t_1(x)} P_1(t, x)
\]
from Definition 4.3
see (4.7)
consequence of (4.6)
directly from (4.6).

To get (4.12) we argue similarly
\[
\Psi_2^2(t, z) = (W_2(t))^{-T} \chi(z),
\]
\[
= H^{-T} S_2(t) (W_2(0) (t_2))^{-1} \chi(z)
\]
\[
= e^{-t_2(z)} H^{-T} S_2(t) \chi(z)
\]
\[
= e^{-t_2(z)} H^{-T} P_2(t, z)
\]
from Definition 4.3
see (4.7)
consequence of (1.6)
follows from (4.6).

To show (4.13) we proceed as follows, assume that \(|z_i| > |x_i|, i \in \{1, \ldots, D\}.
\[
\Psi_2(t, z) = W_2(t) \chi^*(z)
\]
\[
= W_1(t) G \chi^*(z)
\]
\[
= W_1(t) (\chi(x), (\chi(x))^T) \chi^*(z)
\]
\[
= \langle W_1(t) \chi(x), (\chi(x))^T \chi^*(z) \rangle
\]
\[
= \Psi_1(t, x), \mathcal{C}(z, x)
\]
from Definition 4.3
use the factorization (4.8)
introduce the bilinear form expression (4.3)
use properties (4.2)
consequence of (4.10) and Definition 4.3

We now prove (4.14), for \(|z_i| > |x_i|, i \in \{1, \ldots, D\},
\[
\Psi_1^2(t, z) = (W_1(t))^{-T} \chi^*(z)
\]
\[
= (W_2(t))^{-T} G^T \chi^*(z)
\]
\[
= (W_2(t))^{-T} ((\chi^*(z))^T G)^T
\]
\[
= (W_2(t))^{-T} \left( \left( ((\chi^*(z))^T \chi(x), (\chi(x))^T) \right)^T \right)
\]
\[
= (W_2(t))^{-T} \left( \left( \mathcal{C}(z, x), (\chi(x))^T \right) \right)^T
\]
\[
= \left( \left( \mathcal{C}(z, x), (\Psi_2^2(t, x))^T \right) \right)^T
\]
from Definition 4.3 again.

\[\square\]

Proposition 4.7 (The integrable hierarchy). The wave matrices obey the evolutionary linear systems
\[
\frac{\partial W_1}{\partial t_{1, \alpha}} = B_{1, \alpha} W_1,
\]
\[
\frac{\partial W_1}{\partial t_{2, \alpha}} = B_{2, \alpha} W_1,
\]
\[
\frac{\partial W_2}{\partial t_{1, \alpha}} = B_{1, \alpha} W_2,
\]
\[
\frac{\partial W_2}{\partial t_{2, \alpha}} = B_{2, \alpha} W_2,
\]
the Baker and adjoint Baker functions solve the following linear equations
\[
\frac{\partial \Psi_1}{\partial t_{1, \alpha}} = B_{1, \alpha} \Psi_1,
\]
\[
\frac{\partial \Psi_1}{\partial t_{2, \alpha}} = B_{2, \alpha} \Psi_1,
\]
\[
\frac{\partial \Psi_2}{\partial t_{1, \alpha}} = B_{1, \alpha} \Psi_2,
\]
\[
\frac{\partial \Psi_2}{\partial t_{2, \alpha}} = B_{2, \alpha} \Psi_2,
\]
\[
\frac{\partial \Psi_1}{\partial t_{1, \alpha}} = -(B_{1, \alpha})^T \Psi_1,
\]
\[
\frac{\partial \Psi_1}{\partial t_{2, \alpha}} = -(B_{2, \alpha})^T \Psi_1,
\]
\[
\frac{\partial \Psi_2}{\partial t_{1, \alpha}} = -(B_{1, \alpha})^T \Psi_2^*,
\]
\[
\frac{\partial \Psi_2}{\partial t_{2, \alpha}} = -(B_{2, \alpha})^T \Psi_2^*,
Proposition 4.9. The Baker function in terms of the first block subdiagonal \( \beta \) the Lax matrices are subject to the following Lax equations
\[
\frac{\partial L_{i,a}}{\partial t_{j,\alpha}} = [B_{j,\alpha}, L_{i,a}],
\]
and Zakharov–Sabat matrices fulfill the following Zakharov–Shabat equations
\[
\frac{\partial B_{i',\alpha'}}{\partial t_{i,\alpha}} - \frac{\partial B_{i,\alpha}}{\partial t_{i',\alpha'}} + [B_{i,\alpha}, B_{i',\alpha'}] = 0.
\]

Proof. Follows from Proposition 4.4. \( \square \)

In this Proposition, as expected, given two semi-infinite block matrices \( A, B \) the notation \([A, B] = AB - BA\) stands for the usual commutator of matrices.

4.3. KP type hierarchies. In [13] it is shown that the KP type construction appears also in the MVOPR context. Here we show that they admit an extension to this broader scenario not linked to MVOPR of multisinctorial Toda hierarchies.

Definition 4.4. Given two semi-infinite matrices \( Z_1(t) \) and \( Z_2(t) \) we say that
\[
\begin{align*}
&\bullet Z_1(t) \in \mathcal{W}_1^{(0)} \text{ if } Z_1(t)(W_1^{(0)}(t_1))^{-1} \text{ is a block strictly lower triangular matrix.} \\
&\bullet Z_2(t) \in \mathcal{W}_2^{(0)} \text{ if } Z_2(t)(W_2^{(0)}(t_2))^{-T} \text{ is a block upper triangular matrix.}
\end{align*}
\]

Then, we can state the following congruences

Proposition 4.8. Given two semi-infinite matrices \( Z_1(t) \) and \( Z_2(t) \) such that
\[
\begin{align*}
&\bullet Z_1(t) \in \mathcal{W}_1^{(0)}, \\
&\bullet Z_2(t) \in \mathcal{W}_2^{(0)}, \\
&\bullet Z_1(t)G = Z_2(t).
\end{align*}
\]

then
\[
Z_1(t) = 0, \quad Z_2(t) = 0.
\]

Proof. Observe that
\[
Z_2(t) = Z_1(t)G = Z_1(t)(W_1(t))^{-1}W_2(t),
\]
where we have used (4.8). From here we get
\[
Z_1(t)(W_1^{(0)}(t_1))^{-1}(S_1(t))^{-1} = Z_2(t)(W_2^{(0)}(t_2))^{-T}(S_2(t))^{-1},
\]
and, as in the LHS we have a strictly lower triangular block semi-infinite matrix while in the RHS we have an upper triangular block semi-infinite matrix, both sides must vanish and the result follows. \( \square \)

Definition 4.5. When \( A - B \in \mathcal{W}_1^{(0)} \) we write \( A = B + \mathcal{W}_1^{(0)} \) and if \( A - B \in \mathcal{U}_2^{(0)} \) we write \( A = B + \mathcal{U}_2^{(0)} \).

Within this subsection we will write \( t_{i,(a_1,a_2,...,a_p)} \) to denote \( t_{i,a} \) with \( \alpha = e_{a_1} + \cdots + e_{a_p} \). We introduce the diagonal block matrices \( V_{a,b} = \text{diag}((V_{a,b})_{[0]}, (V_{a,b})_{[1]}, (V_{a,b})_{[2]}, \ldots) \)
\[
(4.15) \quad V_{a,b} := \frac{\partial \beta_1}{\partial t_{1,a}} \Lambda_{b}, \quad (V_{a,b})_{[k]} = \frac{\partial \beta_{1,[k]}}{\partial t_{1,a}} (\Lambda_{b})_{[k-1],[k]}, \quad U_{a,b} := -V_{a,b} - V_{b,a}.
\]
in terms of the first block subdiagonal \( \beta_1 \) of \( S_1 \).

Proposition 4.9. The Baker function \( \Psi_1 \) satisfies
\[
(4.16) \quad \frac{\partial \Psi_1}{\partial t_{1,(a,b)}} = \frac{\partial^2 \Psi_1}{\partial t_{1,a} \partial t_{2,b}} + U_{a,b} \Psi_1.
\]
Proof. In the one hand,
\[
\frac{\partial W_1}{\partial t_{1,(a,b)}} = \left( \frac{\partial S_1}{\partial t_{1,(a,b)}} + S_1 \Lambda_a \Lambda_b \right) W_1^{(0)}(t_1)
\]
\[
\frac{\partial^2 W_1}{\partial t_{1,a} \partial t_{1,b}} = \left( \frac{\partial^2 S_1}{\partial t_{1,a} \partial t_{1,b}} + \frac{\partial S_1}{\partial t_{1,a}} \Lambda_b + \frac{\partial S_1}{\partial t_{1,b}} \Lambda_a + S_1 \Lambda_a \Lambda_b \right) W_1^{(0)}(t_1)
\]
so that
\[
\left( \frac{\partial}{\partial t_{1,(a,b)}} - \frac{\partial^2}{\partial t_{1,a} \partial t_{1,b}} \right) (W_1) = -\left( \frac{\partial \beta_1}{\partial t_{1,a}} \Lambda_b + \frac{\partial \beta_1}{\partial t_{1,b}} \Lambda_a \right) W_1^{(0)}(t_1) + \partial W_1^{(0)}
\]
and, consequently,
\[
\left( \frac{\partial}{\partial t_{1,(a,b)}} - \frac{\partial^2}{\partial t_{1,a} \partial t_{1,b}} + \frac{\partial \beta_1}{\partial t_{1,a}} \Lambda_b + \frac{\partial \beta_1}{\partial t_{1,b}} \Lambda_a \right) (W_1) = \partial W_1^{(0)}.
\]

On the other hand,
\[
\frac{\partial W_2}{\partial t_{1,(a,b)}} = \frac{\partial S_2}{\partial t_{1,(a,b)}} W_2^{(0)}(t_2), \quad \frac{\partial^2 W_2}{\partial t_{1,a} \partial t_{1,b}} = \frac{\partial^2 S_2}{\partial t_{1,a} \partial t_{1,b}} W_2^{(0)}(t_2)
\]
Now, we apply Proposition 4.8 with
\[
Z_i = \left( \frac{\partial}{\partial t_{1,(a,b)}} - \frac{\partial^2}{\partial t_{1,a} \partial t_{1,b}} - U_{a,b} \right) (W_i), \quad i = 1, 2,
\]
to get the result.

Proceeding similarly we can reproduce the results of [13] for this more general case. The proofs are essentially as are there with slight modifications as just shown in the above developments. Associated with the third order times \(t_{1,(a,b,c)}\) we introduce the following block diagonal matrices
\[
V_{a,b,c} = \text{diag}((V_{a,b,c})_{[0]}, (V_{a,b,c})_{[1]}, (V_{a,b,c})_{[2]}, \ldots)
\]
with
\[
V_{a,b,c} := \frac{\partial \beta_{1}^{(2)}}{\partial \Lambda_b \Lambda_c} - \frac{\partial \beta_1}{\partial t_{1,a}} \Lambda_b \Lambda_c,
\]
\[
(V_{a,b,c})_{[k]} = \left( \frac{\partial \beta_1^{(2)}}{\partial \Lambda_b} (\Lambda_b)_{[k-2],[k-1]} - \frac{\partial \beta_1}{\partial \Lambda_b} (\Lambda_b)_{[k-1],[k]} \right) (\Lambda_c)_{[k-1],[k]},
\]
The Baker functions \(\Psi_1\) satisfies the third order linear differential equations
\[
\frac{\partial \Psi_1}{\partial t_{1,(a,b,c)}} = \frac{\partial^3 \Psi_1}{\partial t_{1,a} \partial t_{1,b} \partial t_{1,c}} - V_{a,b} \frac{\partial \Psi}{\partial t_c} - V_{c,a} \frac{\partial \Psi}{\partial t_{1,b}} - V_{b,c} \frac{\partial \Psi}{\partial t_{1,a}}
\]
\[
- \left( \frac{\partial V_{a,b}}{\partial t_{1,c}} \frac{\partial \Psi}{\partial t_{1,a}} + \frac{\partial V_{b,c}}{\partial t_{1,a}} \frac{\partial \Psi}{\partial t_{1,b}} + \frac{\partial V_{c,a}}{\partial t_{1,b}} \frac{\partial \Psi}{\partial t_{1,a}} + \frac{\partial V_{a,b}}{\partial t_{1,c}} + \frac{\partial V_{b,c}}{\partial t_{1,a}} + \frac{\partial V_{c,a}}{\partial t_{1,b}} \right) \Psi_1,
\]
and a matrix type KP system of equations for $\beta_{1,[k]}$ and $\beta_{1,[k]}^{(2)}$ emerges \cite{13}. For example, if we denote $t_{1,a}^{(3)} = t_{3,(a,a,a)}$ and $t_{1,a}^{(2)} = t_{1,(a,a)}$ we get the nonlinear partial differential system

$$0 = \frac{\partial}{\partial t_{1,a}} \left[ \frac{1}{2} \frac{\partial^2 \beta_1}{\partial t_{1,a}^2} - \frac{1}{2} \frac{\partial^2 \beta_1}{\partial t_{1,a}^2} + \frac{1}{2} \frac{\partial^2 \beta_1}{\partial t_{1,a}^2} \right] \Lambda a,$$

$$0 = 3 \frac{\partial}{\partial t_{1,a}} \left[ \frac{1}{2} \frac{\partial^2 \beta_1}{\partial t_{1,a}^2} - \frac{1}{2} \frac{\partial^2 \beta_1}{\partial t_{1,a}^2} + \frac{1}{2} \frac{\partial^2 \beta_1}{\partial t_{1,a}^2} \right] \Lambda a,$$

$$4.4. \text{Reductions.} \text{ We explore superficially some possibilities for reductions.}$$

**Definition 4.6.** *Given two polynomials $Q_1(x), Q_2(x) \in \mathbb{C}[x]$ a semi-infinite matrix $G$ is said $(Q_1, Q_2)$-invariant if

$$Q_1(A)G = GQ_2(A^\top)$$

We will use the notation

$$L_1 := (L_{1,1}, \ldots, L_{1,D})^\top, \hspace{1cm} L_2 := (L_{2,1}, \ldots, L_{2,D})^\top.$$

Observe that according to Proposition 4.2 this reduction implies for the associated bilinear forms

$$\langle Q_1(x)\chi(x), (\chi(x))^\top \rangle = \langle \chi(x), (\chi(x))^\top Q_2(x) \rangle.$$

**Proposition 4.10.** *Given two polynomials $Q_1(x), Q_2(x) \in \mathbb{C}[x]$, with powers written as

$$(Q_1(x))^n = \sum_{\alpha \in \mathbb{Z}_+^D} Q_{1,\alpha}^n x^\alpha, \hspace{1cm} (Q_2(x))^n = \sum_{\alpha \in \mathbb{Z}_+^D} Q_{2,\alpha}^n x^\alpha$$

and a $(Q_1, Q_2)$-invariant initial condition $G$ we find that

1. The Lax semi-infinite matrices satisfy

$$Q_1(L_1) = Q_2(L_2).$$

2. For $n \in \{1, 2, \ldots\}$ the wave matrices satisfy

$$\sum_{\alpha \in \mathbb{Z}_+^D} Q_{1,\alpha}^n \frac{\partial W_1}{\partial t_{1,\alpha}} + \sum_{\alpha \in \mathbb{Z}_+^D} Q_{2,\alpha}^n \frac{\partial W_1}{\partial t_{2,\alpha}} = W_1(Q_1(A))^n,$$

$$\sum_{\alpha \in \mathbb{Z}_+^D} Q_{1,\alpha}^n \frac{\partial W_2}{\partial t_{1,\alpha}} + \sum_{\alpha \in \mathbb{Z}_+^D} Q_{2,\alpha}^n \frac{\partial W_2}{\partial t_{2,\alpha}} = W_2(Q_2(A^\top))^n,$$

and the Lax matrices fulfill the invariance conditions

$$\sum_{\alpha \in \mathbb{Z}_+^D} Q_{1,\alpha}^n \frac{\partial L_1}{\partial t_{1,\alpha}} + \sum_{\alpha \in \mathbb{Z}_+^D} Q_{2,\alpha}^n \frac{\partial L_1}{\partial t_{2,\alpha}} = 0,$$

$$\sum_{\alpha \in \mathbb{Z}_+^D} Q_{1,\alpha}^n \frac{\partial L_2}{\partial t_{1,\alpha}} + \sum_{\alpha \in \mathbb{Z}_+^D} Q_{2,\alpha}^n \frac{\partial L_2}{\partial t_{2,\alpha}} = 0.$$

**Proof.** (1) Use (4.8), (4.9) and (4.17) for (4.18).
(2) Observe that
\[ \sum_{\alpha \in \mathbb{Z}_1^D} Q^n_{1,\alpha} B_{1,\alpha} = \left( (Q_1(L_1))^n \right) \uparrow , \quad \sum_{\alpha \in \mathbb{Z}_1^D} Q^n_{2,\alpha} B_{2,\alpha} = \left( (Q_1(L_2))^n \right) \downarrow \]
and consequently,
\[ \sum_{\alpha \in \mathbb{Z}_1^D} Q^n_{1,\alpha} B_{1,\alpha} + \sum_{\alpha \in \mathbb{Z}_1^D} Q^n_{2,\alpha} B_{2,\alpha} = (Q_1(L_1))^n = (Q_2(L_2))^n, \]
and systems (4.19) and (4.20) follow from Proposition 4.7.

An illustration of these type of the reductions is the case studied in previous sections involving multivariate orthogonal polynomials to a given generalized function $u \in (\mathbb{C}[x])'$ with $G = \langle u, \chi \chi^T \rangle$. As we know this implies $\Lambda \alpha G = G(\Lambda \alpha)^T$, $\alpha \in \{1, \ldots, D\}$, so that $L_{1,\alpha} = S_1 \Lambda S_1^{-1} = S_2 \Lambda^T S_2^{-1} = L_{2,\alpha}$, $\alpha \in \{1, \ldots, D\}$. Moreover, these conditions imply an invariance property under the flows introduced, as we have that
\[ \left( \partial_{1,\alpha} + \partial_{2,\alpha} \right) W_1 = W_1 \Lambda^\alpha, \quad \left( \partial_{1,\alpha} + \partial_{2,\alpha} \right) W_2 = W_2 (\Lambda^T) \alpha, \]
\[ \left( \partial_{1,\alpha} + \partial_{2,\alpha} \right) L_{1,\alpha} = 0, \quad \left( \partial_{1,\alpha} + \partial_{2,\alpha} \right) L_{2,\alpha} = 0. \]

4.5. The linear spectral transformation for the multispectral 2D Toda hierarchy. We extend the linear spectral transform for MVOPR to the more general framework of the multispectral Toda lattice just discussed. As a main result in Theorem 4.2 we get quasi-determinantal expressions for the transformed Baker function $(\hat{\Psi}_1|k|)(t)$ and the quasi-tau matrices $\hat{H}_{1|k|}(t)$.

**Definition 4.7.** Given two coprime polynomials $Q_1(x)$ and $Q_2(x)$, $\deg Q_i = m_i$, we consider an initial condition $G$ and a perturbed one $\hat{G}$ such that
\[ \hat{G} Q_2(\Lambda^T) = Q_1(\Lambda) G. \]
We can achieve the perturbed semi-infinite matrix $\hat{G}$ in two steps, using an intermediate matrix $\tilde{G}$. First, we perform a Geronimus type transformation
\[ \hat{G} Q_2(\Lambda^T) = G \]
and second, a Christoffel type transformation
\[ \hat{G} = \Omega_1(\Lambda) \hat{G}. \]

**Proposition 4.11.** Under the evolution prescribed in (4.4) if (4.21), (4.22) and (4.23) we have
\[ \hat{G}(t) Q_2(\Lambda^T) = Q_1(\Lambda) G(t), \quad \hat{G}(t) Q_2(\Lambda^T) = G(t), \quad \hat{G}(t) = Q_1(\Lambda) \hat{G}(t). \]

**Proof.** We just check the first as the others follow in an analogous manner:
\[ \hat{G}(t) Q_2(\Lambda^T) = W_{1}^{(0)}(t_1) \hat{G}(W_{2}^{(0)}(t_2))^{-T} Q_2(\Lambda^T) = W_{1}^{(0)}(t_1) \hat{G} Q_2(\Lambda^T) (W_{2}^{(0)}(t_2))^{-T} = W_{1}^{(0)}(t_1) Q_1(\Lambda) G(W_{2}^{(0)}(t_2))^{-T} = Q_1(\Lambda) G(t). \]
In terms of bilinear forms (4.22) reads
\[ \langle \chi(x), (\chi(x))^\top Q_2(x) \rangle^* = \langle \chi(x), (\chi(x))^\top \rangle \]
so that assuming we can divide by polynomials inside these bilinear forms a solution to (4.22)
\[ (4.24) \]
\[ \tilde{G} = \langle \chi(x), (\chi(x))^\top Q_2(x) \rangle + \langle v, \chi(x)(\chi(x))^\top \rangle \]
where \( v \in (\mathbb{C}[x])' \) and \( (Q_2(x)) \subset \text{Ker}(v) \). In fact, a more general case will be
\[ \tilde{G} = \langle \chi(x), (\chi(x))^\top Q_2(x) \rangle + \langle v, A\chi(x)(\chi(x))^\top \rangle \]
where \( A \) is a semi-infinite matrix with rows having only a finite number of non vanishing coefficients.

**Definition 4.8.** We introduce the resolvents
\[ \omega_1(t) := \hat{S}_1(t)Q_1(A)(S_1(t))^{-1}, \quad \omega_2(t) := \left( S_2(t)Q_2(A)(\hat{S}_2(t))^{-1} \right)^\top. \]

**Proposition 4.12.** The resolvent matrices satisfy
\[ (4.25) \]
\[ \tilde{H}(t)\omega_2(t) = \omega_1(t)H(t). \]
The resolvents \( \omega_1(t), \omega_2(t) \) are block banded matrices, having different from zero only the first \( m_1 \) block superdiagonals and the first \( m_2 \) block subdiagonals.

**Proof.** From the LU factorization we get
\[ (\hat{S}_1(t))^{-1}\hat{H}(t)(\hat{S}_2(t))^{-\top}Q_2(A^\top) = Q_1(A)(S_1(t))^{-1}H(t)(S_2(t))^{-\top}, \]
so that
\[ \tilde{H}(t)\left( S_2(t)Q_2(A)(\hat{S}_2(t))^{-1} \right)^\top = \hat{S}_1(t)Q_1(A)(S_1(t))^{-1}H(t). \]

\[ \square \]

In this more general scenario Proposition 3.4 still holds for these new resolvents, not connected in principle with any linear functional. We have

**Proposition 4.13** (Connection formulas). We have
\[ \omega_1(t)P_1(t,x) = Q_1(x)\hat{P}_1(t,x), \]
\[ (\omega_2(t))^\top \hat{P}_2(t,x) = Q_2(x)P_2(t,x). \]

**Definition 4.9.** We introduce the semi-infinite matrix
\[ (4.26) \]
\[ R(t) := S_1(t)\tilde{G}(t) \]

**Proposition 4.14.** The matrix \( R(t) \) can be expressed as follows
\[ (4.27) \]
\[ R(t) = \left\langle P_1(t,x), \frac{(\chi(x))^\top}{Q_2(x)} \right\rangle + \left\langle v, P_1(t,x)(\chi(x))^\top \right\rangle. \]

**Proof.** Recall (4.24) and (4.26).\[ \square \]

**Proposition 4.15.** We have the following relations
\[ (\omega_1(t)R(t))_{[k],l} = 0, \quad l = 0, 1, \ldots, k - 1 \]
\[ (\omega_1(t)R(t))_{[k],[k]} = \tilde{H}_{[k]}(t) \]
Proof. Just follow the next chain of equalities

\[
\omega_1(t) R(t) = \hat{S}_1(t) Q_1(A)(S_1(t))^{-1} S_1(t) \hat{G}(t)
\]
\[= \hat{S}_1(t) Q_1(A) \hat{G}(t)\]
\[= \hat{S}_1(t) \hat{G}(t)\]
\[= \hat{H}(t)(\hat{S}_2(t))^{-T}\]  \hspace{1cm} \text{from (4.11)}

(4.28)

and the matrix \(\omega_1 R\) is an upper triangular block matrix with \(\hat{H}\) as its block diagonal. \(\square\)

Proceeding as we did for (3.8) and (3.9) we can deduce analogous equations in this new context. For \(k < m_2\) we can write

\[
\big( (\omega_1)_{[k],[0]}(t), \ldots, (\omega_1)_{[k],[k+m_1-1]}(t) \big) = \\
- (Q_1(A))_{[k],[k+m_1]} \big( R_{[k+m_1],[0]}(t), \ldots, R_{[k+m_1],[k-1]}(t), \psi_{1,[k+m_1]}(t, p_1), \ldots, \psi_{[k+m_1]}(t, p_{r_1}) \big) \\
\times \begin{pmatrix}
R_{[k],[0]}(t) & \cdots & R_{[0],[k-1]}(t) & \psi_{1,[k]}(t, p_1) & \cdots & \psi_{1,[k]}(t, p_{r_1}) \\
\vdots & \ddots & \vdots & \vdots & \ddots & \vdots \\
R_{[k+m_1-1],[0]}(t) & \cdots & R_{[k+m_1-1],[k-1]}(t) & \psi_{1,[k+m_1-1]}(t, p_1) & \cdots & \psi_{1,[k+m_1-1]}(t, p_{r_1})
\end{pmatrix}^{-1},
\]

while for \(k \geq m_2\)

\[
\big( (\omega_1)_{[k],[k-m_2]}(t), \ldots, (\omega_1)_{[k],[k+m_1-1]}(t) \big) = \\
- (Q_1(A))_{[k],[k+m_1]} \big( R_{[k+m_1],[0]}(t), \ldots, R_{[k+m_1],[k-1]}(t), \psi_{1,[k+m_1]}(t, p_1), \ldots, \psi_{1,[k+m_1]}(t, p_{r_1}) \big) \\
\times \begin{pmatrix}
R_{[k-m_2],[0]}(t) & \cdots & R_{[0],[k-m_2]}(t) & \psi_{1,[k-m_2]}(t, p_1) & \cdots & \psi_{1,[k-m_2]}(t, p_{r_1}) \\
\vdots & \ddots & \vdots & \vdots & \ddots & \vdots \\
R_{[k+m_1-1],[0]}(t) & \cdots & R_{[k+m_1-1],[k-1]}(t) & \psi_{1,[k+m_1-1]}(t, p_1) & \cdots & \psi_{1,[k+m_1-1]}(t, p_{r_1})
\end{pmatrix}^{-1},
\]

We also have

\[
\big( \omega_1(t) \big)_{[k],[k+m_1]} = (Q_1(A))_{[k],[k+m_1]}.
\]

Then, we extend Definitions 3.4 and 3.5 to this new scenario, and find a version of Theorem 3.1 in terms of the Baker functions

**Theorem 4.2** (Christoffel–Geronimus–Uvarov formula for multispectral Toda hierarchy). A linear spectral transformation, as in (4.21), for the multispectral Toda hierarchy has the following effects on the Baker function \(\psi_{1,[k]}(t)\) and the quasi-tau matrices \(H_{[k]}(t)\). Given a poised set \(S_k\) of multi-indices and nodes, we have a perturbed Baker function

\[
\psi_{1,[k]}(t, x) = \frac{(Q_1(A))_{[k],[k+m_1]}}{Q_1(x)}
\times \Theta_*(
\begin{pmatrix}
R_{[0],[0]}(t) & \cdots & R_{[0],[k-1]}(t) & \psi_{1,[0]}(t, p_1) & \cdots & \psi_{1,[0]}(t, p_{r_1}) & \psi_{1,[0]}(t, x) \\
\vdots & \ddots & \vdots & \vdots & \ddots & \vdots & \vdots \\
R_{[k+m_1-1],[0]}(t) & \cdots & R_{[k+m_1-1],[k-1]}(t) & \psi_{1,[k+m_1-1]}(t, p_1) & \cdots & \psi_{1,[k+m_1-1]}(t, p_{r_1}) & \psi_{1,[k+m_1-1]}(t, x)
\end{pmatrix},
\]

and a perturbed quasi-tau matrix

\[
H_{[k]}(t) = (Q_1(A))_{[k],[k+m_1]}
\times \Theta_*(
\begin{pmatrix}
R_{[0],[0]}(t) & \cdots & R_{[0],[k-1]}(t) & \psi_{1,[0]}(t, p_1) & \cdots & \psi_{1,[0]}(t, p_{r_1}) & R_{[0],[k]}(t) \\
\vdots & \ddots & \vdots & \vdots & \ddots & \vdots & \vdots \\
R_{[k+m_1-1],[0]}(t) & \cdots & R_{[k+m_1-1],[k-1]}(t) & \psi_{1,[k+m_1-1]}(t, p_1) & \cdots & \psi_{1,[k+m_1-1]}(t, p_{r_1}) & R_{[k+m_1],[k]}(t)
\end{pmatrix}.
\]
When \( k \geq m_2 \) we have the shorter alternative expressions

\[
\hat{\Psi}_{1,[k]}(t, x) = \frac{(\Omega_1(\mathbf{A}))_{[k],[k+m_1]}}{Q_1(x)} \times \Theta_* \left( \begin{array}{cccc}
R_{[k-m_2],p_1}(t) & \cdots & R_{[k-m_2],p_{r_1[k,m_1]}}(t) & \cdots & R_{[k-m_2],[k]}(t) \\
\vdots & \ddots & \vdots & \ddots & \vdots \\
R_{[k+m_1],p_1}(t) & \cdots & R_{[k+m_1],p_{r_1[k,m_1]}}(t) & \cdots & R_{[k+m_1],[k]}(t)
\end{array} \right),
\]

\[
\hat{H}_{1,[k]}(t) = (\Omega_1(\mathbf{A}))_{[k],[k+m_1]}' = (\Omega_1(\mathbf{A}))_{[k+m_1],[k]}(t)
\times \Theta_* \left( \begin{array}{cccc}
R_{[k-m_2],p_1}(t) & \cdots & R_{[k-m_2],p_{r_1[k,m_1]}}(t) & \cdots & R_{[k-m_2],[k]}(t) \\
\vdots & \ddots & \vdots & \ddots & \vdots \\
R_{[k+m_1],p_1}(t) & \cdots & R_{[k+m_1],p_{r_1[k,m_1]}}(t) & \cdots & R_{[k+m_1],[k]}(t)
\end{array} \right).
\]

and

\[
\hat{H}_{1,[k]}(t) \left((\Omega_2(\mathbf{A}))_{[k-m_2],[k]}\right)' = (\Omega_1(\mathbf{A}))_{[k],[k+m_1]}
\times \Theta_* \left( \begin{array}{cccc}
R_{[k-m_2],p_1}(t) & \cdots & R_{[k-m_2],p_{r_1[k,m_1]}}(t) & \cdots & R_{[k-m_2],[k]}(t) \\
\vdots & \ddots & \vdots & \ddots & \vdots \\
R_{[k+m_1],p_1}(t) & \cdots & R_{[k+m_1],p_{r_1[k,m_1]}}(t) & \cdots & R_{[k+m_1],[k]}(t)
\end{array} \right).
\]

Regarding the Baker function \( \Psi_2 \) and its behavior under a general linear spectral transformation, using (4.13), we have for each component

\[
\hat{\Psi}_{2,[k]}(t, z) = \langle \hat{\Psi}_{1,[k]}(t, x), \mathcal{C}(z, x) \rangle,
\]

and consequently Theorem 4.2 provides quasi-determinantal expression for \( \hat{\Psi}_{2,[k]} \) performing the following replacements

\[
\hat{\Psi}_{1,[1]}(t, x) \rightarrow \left( \frac{\hat{\Psi}_{1,[1]}(t, x)}{\hat{Q}_1(x)} \right) \mathcal{C}(z, x), \quad l \in \{k - m_2, \ldots, k + m_1\}.
\]

Alternative expressions are achieved if the relation (4.28) is recalled. Indeed, it implies

(4.29)

\[
\hat{\Psi}_2(t, z) = \omega_1 R(\mathcal{W}_2(t_2))^{\top} \chi^*(z).
\]

Then, using (4.27) we conclude that the replacements to perform in Theorem 4.2 to find a quasi-determinantal expression for \( \hat{\Psi}_{2,[k]} \) are

\[
\hat{\Psi}_{1,[1]}(t, x) \rightarrow \left( \frac{\hat{\Psi}_{1,[1]}(t, x)}{\hat{Q}_1(x)} \right) \mathcal{C}(z, x) + \langle v, e^{t_2(x)}(z, x) \rangle \mathcal{C}(z, x), \quad l \in \{k - m_2, \ldots, k + m_1\}.
\]

In this general setting \( G \) is not restricted by a Hankel type constraint, thus given a polynomial \( \Omega(x) \in \mathbb{R}[x] \) we have

\[
G \Omega(\mathbf{A}^\top) \neq \Omega(\mathbf{A}) G.
\]

For example, instead of (4.21) we may have considered

\[
\Omega_2(\mathbf{A}) \hat{G} = G \Omega_1(\mathbf{A}^\top).
\]

In this case a transposition formally gives

\[
\hat{G}^\top \Omega_2(\mathbf{A}^\top) = \Omega_1(\mathbf{A}) G^\top,
\]
which can be gotten from (4.21) by the replacement \( G \mapsto G^T \) and \( \hat{G} \mapsto \hat{G}^T \); i.e., at the level of the Gauss–Borel factorization (4.5):

\[
\begin{align*}
S_1 & \mapsto S_2, \\
\hat{S}_1 & \mapsto \hat{S}_2, \\
H & \mapsto H^T, \\
\hat{H} & \mapsto \hat{H}^T, \\
S_2 & \mapsto S_1, \\
\hat{S}_2 & \mapsto \hat{S}_1.
\end{align*}
\]

Thus, previous formulæ hold by replacing \( P_1 \) by \( P_2 \) and transposing the matrices \( H_{|[k]} \) and \( \hat{H}_{|[k]} \).

A quite general transformation, which we will not explore in this paper, corresponds to

\[
\Omega_1^k(A) \hat{G} \Omega_2^k(A^T) = \Omega_1^k(A) G \Omega_2^k(A^T),
\]

for polynomials \( \Omega_1^k(x), \Omega_2^k(x) \in \mathbb{R}[x] \). This transformation is preserved by the integrable flows introduced above; i.e.,

\[
\Omega_1^k(A) \hat{G}(t) \Omega_2^k(A^T) = \Omega_1^k(A) G(t) \Omega_2^k(A^T).
\]

Notice that this transformation for a multi-Hankel reduction \( \Lambda_a G = G(\Lambda_a)^T, a \in \{1, \ldots, D\} \), is just the one considered in previous sections.

### 4.6. Generalized bilinear equations and linear spectral transformations

We are ready to show that the Baker functions at different times and their linear spectral transforms satisfy a bilinear equation as in the KP theory, see [24, 26, 25]. In the standard formulation [24, 26, 25] discrete times appeared in the bilinear equation, which in this case are identified, see for example [30], with the linear spectral transformations. To deduce the bilinear equations we use a similar method as in [4, 56, 59].

We begin with the following observation:

**Proposition 4.16.** Wave matrices \( W_i(t), i \in \{1, 2\} \) and their linear spectral transformed wave matrices \( \hat{W}_i(t') \), \( i \in \{1, 2\} \), according to the coprime polynomials \( \Omega_1(x), \Omega_2(x) \in \mathbb{C}[x], \) fulfill

\[
\hat{W}_1(t')\Omega_1(A)(W_1(t))^{-1} = \hat{W}_2(t')\Omega_2(A^T)(W_2(t))^{-1}.
\]

**Proof.** We have

\[
G = (W_1(t))^{-1}W_2(t), \quad \hat{G} = (\hat{W}_1(t'))^{-1}\hat{W}_2(t').
\]

Hence, using (4.21) we deduce

\[
\Omega_1(A)(W_1(t))^{-1}W_2(t) = (\hat{W}_1(t'))^{-1}\hat{W}_2(t')\Omega_2(A^T).
\]

\[ \Box \]

Now, we need

**Lemma 4.1.** Given two semi-infinite matrices \( U \) and \( V \) we have

\[
UV = \frac{1}{(2\pi i)^D} \oint_{T_D^{(r)}} U\chi(z)(V^T\chi^*(z))^T \, dz_1 \cdots dz_D = \frac{1}{(2\pi i)^D} \oint_{T_D^{(r)}} U\chi^*(z)(V^T\chi(z))^T \, dz_1 \cdots dz_D.
\]

**Proof.** Observe that

\[
\chi(\chi^*)^T = \begin{pmatrix}
Z_{[0],[0]} & \ldots & Z_{[0],[1]} & \ldots \\
Z_{[1],[0]} & \ldots & Z_{[1],[1]} & \ldots \\
\vdots & \ddots & \vdots & \ddots 
\end{pmatrix}, \quad Z_{[k],[\ell]} := \frac{1}{z_1 \cdots z_D} \begin{pmatrix}
z^{k_1-\ell_1} & \ldots & z^{k_1-\ell_{[\ell]}} \\
z^{k_2-\ell_1} & \ldots & z^{k_2-\ell_{[\ell]}} \\
\vdots & \ddots & \vdots \\
z^{k_{[k]}-\ell_1} & \ldots & z^{k_{[k]}-\ell_{[\ell]}}
\end{pmatrix}.
\]

If we now integrate in the polydisk distinguished border \( T_D^{(r)} \) using the Fubini theorem we factor each integral in a product of \( D \) factors, where the \( i \)-th factor is an integral over \( z_i \) on the circle centered at
origin of radius \( r_1 \). This is zero unless the integrand is \( z_i^{-1} \) which occurs only in the principal diagonal. Consequently, we have
\[
\int_{\mathbb{T}^D(r)} \chi(z)\chi^*(z)^\top \, dz_1 \cdots dz_D = \int_{\mathbb{T}^D(r)} \chi^*(z)\chi(z)^\top \, dz_1 \cdots dz_D = (2\pi i)^D \mathbb{I},
\]
and the result follows. □

We notice that \( \Psi_1 \) and \( \Psi_2 \) lead to the computation of finite sums, i.e., polynomials, but \( \Psi_1^* \) and \( \Psi_2 \) involve Laurent series. We will denote by \( \mathcal{P}_{2,\alpha}(t) \) and \( \mathcal{P}_{1,\alpha}(t) \) the domains of convergence of \( \Psi_2(t, z) \) and \( \Psi_1(t, z) \), respectively. Recall that these domains are Reinhardt domains; i.e., if \( \mathcal{D} \subset \mathbb{C}^D \) is the domain of convergence then for any \( c = (c_1, \ldots, c_D) \top \in \mathcal{D} \) we have that \( \mathbb{T}^D([c_1], \ldots, [c_D]) \subset \mathcal{D} \).

**Theorem 4.3** (Generalized bilinear equations). For any pair of times \( t \) and \( t' \), points \( r_1 \in \mathcal{P}_{2,\alpha}(t) \) and \( r_2 \in \mathcal{P}_{1,\alpha}(t') \) in the respective Reinhardt domains and \( D \)-dimensional tori \( \mathbb{T}^D(r_1) \) and \( \mathbb{T}^D(r_2) \), and multi-indices \( \alpha, \alpha' \in \mathbb{Z}_+^D \), the Baker and adjoint Baker functions and their linear spectral transformations satisfy the following bilinear identity
\[
\int_{\mathbb{T}^D(r_1)} \Psi_{1,\alpha}(t', z)\Psi_{1,\alpha}'(t, z)Q_1(z) \, dz_1 \cdots dz_D \int_{\mathbb{T}^D(r_2)} \Psi_{2,\alpha'}(t', z)\Psi_{2,\alpha'}'(t, z)Q_2(z) \, dz_1 \cdots dz_D.
\]

**Proof.** From Definition 4.3 and Lemma 4.1, choosing \( U = \hat{W}_1(t')Q_1(A) \) and \( V = (W_1(t))^{-1} \) we get
\[
\hat{W}_1(t')Q_1(A)(W_1(t))^{-1} = \frac{1}{(2\pi i)^D} \int_{\mathbb{T}^D(r_1)} \Psi_{1,\alpha}(t', z)\Psi_{1,\alpha}'(t, z)Q_1(z) \, dz_1 \cdots dz_D,
\]
and choosing \( U = \hat{W}_2(t') \) and \( V = Q_2(A^\top)(W_2(t))^{-1} \) we get
\[
\hat{W}_2(t')Q_2(A^\top)(W_2(t))^{-1} = \frac{1}{(2\pi i)^D} \int_{\mathbb{T}^D(r_2)} \Psi_{2,\alpha'}(t', z)\Psi_{2,\alpha'}'(t, z)Q_2(z) \, dz_1 \cdots dz_D.
\]
Then, Proposition 4.16 implies the result. □

**Appendix A. Uvarov perturbations**

Uvarov considered in §2 of [25] the addition of a finite number of masses, Dirac deltas, to a given measure in the OPRL situation. In this appendix we discuss some elements of the multivariate extension of this construction. There is an immediate extension when one considers masses, see [27]. A bit more involved case is to consider higher multipoles, i.e., derivatives of the Dirac distributions. In [27] a Sobolev type modification was considered, see for example equation (2.16) in that paper, but this can not be modeled by a perturbation \( \hat{u} = u + v \) of a linear functional \( u \) (or measure in that case, \( u = d\mu \)). All this can be considered as a 0-dimensional additive perturbation. However, more interesting and less trivial extensions are to consider higher dimensional additive perturbations. For example, 1D-Uvarov perturbations, i.e., additive perturbations supported over curves. For the 0D-Uvarov perturbations, as was found in [25], one needs to solve a linear system constructed in terms of the non-perturbed Christoffel–Darboux kernel evaluated at the 0D discrete support of the perturbation. We will see that for the 1D scenario the linear system of the 0D case is replaced by a Fredholm integral equation evaluated at the 1D support of the perturbation.

Our approach to the problem is based on a simple relation among perturbed and non perturbed MVOPR which involves the non perturbed Christoffel–Darboux kernel. Let us consider a generalized function \( u \in \mathbb{C}[x]' \) such that is quasidefinite and consider an additive perturbation of it given by another generalized function \( v \in \mathbb{C}[x]' \)
\[
\hat{u} = u + v.
\]
Proposition A.1. For an additive perturbation we have

\[ \hat{P}_{[n]}(x) = P_{[n]}(x) - \langle v, \hat{P}_{[n]}(y)K_{n-1}(y, x) \rangle, \]

\[ \hat{H}_{[n]} = H_{[n]} + \langle v, \hat{P}_{[n]}(x)(P_{[n]}(x))^T \rangle. \]

Proof. From (1.3) and (1.4) we deduce

\[ \left\langle \hat{u}, \hat{P}_{[n]}(x)(P_{[m]}(x))^T \right\rangle = 0, \quad m \in \{0, 1, \ldots, n-1\}, \]

\[ \left\langle \hat{u}, \hat{P}_{[n]}(x)(P_{[m]}(x))^T \right\rangle = \hat{H}_{[n]}, \]

and, consequently,

\[ \left\langle u, \hat{P}_{[n]}(x)(P_{[m]}(x))^T \right\rangle = -\left\langle v, \hat{P}_{[n]}(x)(P_{[m]}(x))^T \right\rangle, \quad m \in \{0, 1, \ldots, n-1\}. \]

Thus, in terms of the Christoffel–Darboux kernel, see Definition (1.4)

\[ \langle u, \hat{P}_{[n]}(x)K_{n-1}(x, y) \rangle = -\langle v, \hat{P}_{[n]}(x)K_{n-1}(x, y) \rangle. \]

Observe that \( \hat{P}_{[n]}(x) - P_{[n]}(x) \) is a multivariate polynomial of degree \( n - 1 \) and, according to (1.8), we conclude

\[ \hat{P}_{[n]}(x) - P_{[n]}(x) = \langle u, (\hat{P}_{[n]}(y) - P_{[n]}(y))K_{n-1}(y, x) \rangle \]

\[ = \langle u, \hat{P}_{[n]}(y)K_{n-1}(y, x) \rangle \]

\[ = -\langle v, \hat{P}_{[n]}(y)K_{n-1}(y, x) \rangle. \]

Finally, we have

\[ \hat{H}_{[n]} = \left\langle u, \hat{P}_{[n]}(x)(P_{[n]}(x))^T \right\rangle + \left\langle v, \hat{P}_{[n]}(x)(P_{[n]}(x))^T \right\rangle \]

\[ = H_{[n]} + \left\langle v, \hat{P}_{[n]}(x)(P_{[n]}(x))^T \right\rangle. \]

\[ \square \]

A.1. 0D-Uvarov multipolar perturbations. Masses (or charges) and dipoles. Here we discuss the more general additive perturbation with finite discrete support. As we have a finite number of points for the support we say that is a 0 dimensional perturbation. Let us proceed and consider a set of couples \( S = \{x_i, \beta_i\}_{i=1}^q \subset \mathbb{R}^D \times \mathbb{Z}_+^D \) and define the associated generalized function

\[ \nu_S := \sum_{i=1}^q \sum_{\alpha \leq \beta_i} (-1)^{\frac{|\alpha|}{\alpha!}} \hat{\xi}_{i, \alpha} \delta^{(\alpha)}(x - x_i). \]

Here the sum over multi-indices extend to all those multi-indices below a given one. The Dirac delta distribution and its derivatives are given by

\[ \langle \delta^{(\alpha)}(x - x_j), P(x) \rangle := (-1)^{|\alpha|} \frac{\partial^{(|\alpha|)} P(x)}{\partial x^{\alpha}} \bigg|_{x=x_j}, \quad \forall P(x) \in C[x], \]

and we have used the lexicographic order for the set of integer multi-indices. Observe that this is the more general distribution with support on \( \{x_j\}_{i=1}^q = \text{supp}(\nu_S) \). From a physical point of view, the delta functions can be understood as point masses. For higher order derivatives, we have an electromagnetic interpretation, for zero order derivatives we have point charges, and first order derivatives could be understood as dipoles, and in general for \( j \)-th order derivatives we are dealing with \( 2^j \)-multipoles (for \( j = 2 \) we have quadrupoles, for \( j = 3 \) we have octopoles, and so on and so forth).
[Definition A.1.](1) Given a multi-index \( \beta \in \mathbb{Z}_{+}^{D} \) with \( |\beta| = k \in \mathbb{Z}_{+} \) we have a corresponding lexicographic ordered set of multi-indices

\[
\{ \alpha_{1}^{(0)}, \alpha_{1}^{(1)}, \ldots, \alpha_{D}^{(1)}, \alpha_{1}^{(2)}, \ldots, \alpha_{1}^{(k)}, \ldots, \alpha_{m}^{(k)} = \beta \},
\]

where \( m \) denotes the position of the multi-index \( \beta \) among those of length \( k \), see §1.3.

(2) Given a couple \((x, \beta) \in \mathbb{R}^{D} \times \mathbb{Z}_{+}^{D} \) and a polynomial \( P \in \mathbb{C}[x] \) we define the jet

\[
\partial_{P}^{\beta}(x) = \left[ \frac{1}{(\alpha_{1}^{(0)})!} \frac{\partial^{\beta \alpha_{1}^{(0)}}}{\partial x^{\alpha_{1}^{(0)}}}(x), \frac{1}{(\alpha_{1}^{(1)})!} \frac{\partial^{\beta \alpha_{1}^{(1)}}}{\partial x^{\alpha_{1}^{(1)}}}(x), \ldots, \frac{1}{(\alpha_{1}^{(k)})!} \frac{\partial^{\beta \alpha_{1}^{(k)}}}{\partial x^{\alpha_{1}^{(k)}}}(x), \ldots, \frac{1}{\beta!} \frac{\partial^{\beta \alpha_{1}^{(1)}}}{\partial x^{\alpha_{1}^{(1)}}}(x) \right].
\]

This a row vector with \( N_{k-1} + m \) components. Recall that the dimension of the linear space of multivariate polynomials of degree less or equal to \( k \) is \( \mathbb{N}_{k} = \binom{D+k}{D} \).

(3) Given the set, we define a matrix collecting the corresponding jets at each puncture \( x_{i} \)

\[
\partial_{P}(S) = \left[ \partial_{P}^{\beta_{1}}(x_{1}), \ldots, \partial_{P}^{\beta_{q}}(x_{q}) \right].
\]

This a row vector with \( N_{S} := \sum_{i=1}^{q} (N_{k_{i}-1} + m_{i}) \) components.

(4) We consider the block antidiagonal matrices \( \xi^{(i)} \in \mathbb{C}^{(N_{k_{i}-1}+m_{i}) \times (N_{k_{i}-1}+m_{i})}, i \in \{1, \ldots, q\} \), with coefficients

\[
(\xi^{(i)})_{\alpha, \beta} := \xi_{i \alpha + \beta}
\]

and the matrix

\[
\Xi := \text{diag}(\xi^{(1)}, \ldots, \xi^{(q)}) \in \mathbb{C}^{N_{S} \times N_{S}}
\]

(5) The Christoffel–Darboux jet is given in terms of product of truncations

\[
\mathcal{K}_{n-1}(S) = (\partial_{P_{[n]}}(S))^{\top} (H^{[n]})^{-1} P_{[n]}(S) \in \mathbb{C}^{N_{S} \times N_{S}}.
\]

Notice that the truncation \( P^{[n]}(x) \) is a vector of polynomials and, therefore, \( \partial_{P^{[n]}}(S) \) is a \( N_{n-1} \times N_{S} \) complex matrix.

**Theorem A.1** (0D-Uvarov multipolar perturbation). Given a discrete additive perturbation of the form

\[
\hat{u} = u + \sum_{i=1}^{q} \sum_{\alpha \subseteq P_{i}} (-1)^{|\alpha|} \alpha! \xi_{i \alpha \alpha} \delta(\alpha)(x - x_{i}),
\]

the new MVOPR and quasi-tau matrices are given by the following quasi-determinantal expressions

\[
\hat{P}_{[n]}(x) = \Theta_{S} \left( \begin{array}{c|c} I_{N_{S}} + \Xi \mathcal{K}_{n-1}(S) \frac{\Xi(\partial_{\mathcal{K}_{n-1}(n)}(S))^{\top}}{P_{[n]}(x)} \\ \hline \partial_{P_{[n]}}(S) \end{array} \right),
\]

\[
\hat{H}_{[n]} = \Theta_{S} \left( \begin{array}{c|c} I_{N_{S}} + \Xi \mathcal{K}_{n-1}(S) \frac{-\Xi(\partial_{P_{[n]}}(S))^{\top}}{H_{[n]}(n)} \end{array} \right).
\]

**Proof.** From (A.1) we conclude that

\[
\hat{P}_{[n]}(x) = P_{[n]}(x) - \partial_{P_{[n]}}(S) \Xi(\partial_{\mathcal{K}_{n-1}(n)}(S))^{\top}
\]

and, therefore, we deduce

\[
\partial_{P_{[n]}}(S) = \partial_{P_{[n]}}(S) - \partial_{P_{[n]}}(S) \Xi \mathcal{K}_{n-1}(S)
\]

that is, the unknowns \( \partial_{P_{[n]}}(S) \) satisfy the following linear system

\[
\partial_{P_{[n]}}(S)(I_{N_{S}} + \Xi \mathcal{K}_{n-1}(S)) = \partial_{P_{[n]}}(S).
\]
Let us prove that the quasiddefiniteness of \( u \) implies that \( I_{N_s} + \Xi K_{n-1}(S) \) is not singular, we follow \[23\]. If we assume that \( I_{N_s} + \Xi K_{n-1}(S) \) is singular there must exist a non-zero vector \( C \in C^{N_s} \) such that \( (I_{N_s} + \Xi K_{n-1}(S))C = 0 \) and, consequently, such that \( \tilde{\mathcal{P}}_{[n]}(S)C = 0 \). Now, let us observe that from

\[
I_{N_s} + \Xi K_n(S) = I_{N_s} + \Xi K_{n-1}(S) + \Xi(\mathcal{P}_{[n]}(S))^{\top}(H_{[n]})^{-1}\tilde{\mathcal{P}}_{[n]}(S).
\]

we get \( (I_{N_s} + \Xi K_n(S))C = 0 \) and, consequently, \( \tilde{\mathcal{P}}_{[n+1]}(S)C = 0 \). By induction, we deduce that \( \tilde{\mathcal{P}}_{[m]}(S)C = 0 \), for \( m \in \{n, n+1, \ldots \} \); i.e., the generalized function \( \mathcal{G}(S) := \sum_{i=1}^{q} \sum_{\alpha \leq \beta_i} \frac{(-1)^{\alpha}}{\alpha!} C_{i, \alpha} \delta^\alpha(x - x_i) \) is such that \( \langle \mathcal{G}(S)C, P_{[1]}(x) \rangle = 0 \), for \( m \in \{n, n+1, \ldots \} \). Equivalently, that is to say

\[
\mathcal{G}(S)C \in \left( \{P_\alpha | \alpha| \geq n \} \right)^\perp = \{ \hat{u} \in (C[x])^* : \langle \hat{u}, P_\alpha \rangle = 0, \forall \alpha \in \mathbb{Z}_+^D : |\alpha| \geq n \}.
\]

In the one hand, the orthogonality relations for the MVOPR leads us to conclude that \( \left( \{P_\alpha | \alpha| \geq n \} \right)^\perp = C \{ P_\alpha | |\alpha| < n \} \), with the covectors defined by \( \langle P_\alpha, P_\beta \rangle = \delta_{\alpha, \beta} \); thus, \( \dim \left( \{P_\alpha | |\alpha| \geq n \} \right)^\perp = N_{n-1} \). In the other hand, we notice that \( C \{ x^\alpha u | |\alpha| < n \} \subset \left( \{P_\alpha | |\alpha| \geq n \} \right)^\perp \), but as \( \dim C \{ x^\alpha u | |\alpha| < n \} = N_{n-1} \) we deduce \( \left( \{P_\alpha | |\alpha| \geq n \} \right)^\perp = C \{ x^\alpha u | |\alpha| < n \} \). Consequently, there exists a non-zero polynomial \( Q(x) \in C[x] \), \( \deg Q \leq n - 1 \), such that \( \mathcal{G}(S)C = Qu \). Finally, let us notice that \( \prod_{i=1}^{q} (x - x_i)^{\beta_i} \in \text{Ker} (\mathcal{G}(S)C) \), so that \( \prod_{i=1}^{q} (x - x_i)^{\beta_i} Q(x)u = 0 \), and \( u \) can not be quasiddefinite, in contradiction with the initial assumptions.

Now, as \( I_{N_s} + \Xi K_{n-1}(S) \) is not singular we deduce

\[
\tilde{\mathcal{P}}_{[m]}(S) = \tilde{\mathcal{P}}_{[m]}(S)(I_{N_s} + \Xi K_{n-1}(S))^{-1}.
\]

Thus,

\[
\hat{\mathcal{P}}_{[n]}(x) = P_{[n]}(x) - \tilde{\mathcal{P}}_{[m]}(S)(I_{N_s} + \Xi K_{n-1}(S))^{-1}\Xi (\mathcal{G}_{K_{n-1}(\cdot, x)}(S))^{\top}
\]

and the result follows. From \[A.2\] we have

\[
\hat{H}_{[n]} = H_{[n]} + \tilde{\mathcal{P}}_{[m]}(S)\Xi (\mathcal{P}_{[n]}(S))^{\top}
\]

\[
= H_{[n]} + \tilde{\mathcal{P}}_{[m]}(S)(I_{N_s} + \Xi K_{n-1}(S))^{-1}\Xi (\mathcal{P}_{[n]}(S))^{\top}.
\]

\[\square\]

**Corollary A.1 (0D-Uvarov mass perturbation).** Given the set of pairs \( \{x_i, \xi_i\}_{i=1}^{q} \), positions and masses, and a discrete additive mass perturbation of the form

\[
\hat{u} = u + \sum_{i=1}^{q} \xi_i \delta(x - x_i),
\]

the new MVOPR and quasi-tau matrices are given by the following quasi-determinantal expressions

\[
\hat{P}_{[n]}(x) = \Theta_* \left[ \begin{array}{cccc}
1 + \xi_1 K_{n-1}(x_1, x_1) & K_{n-1}(x_1, x_2) & \cdots & K_{n-1}(x_1, x_q) & \xi_1 K_{n-1}(x_1, x) \\
K_{n-1}(x_2, x_1) & 1 + \xi_2 K_{n-1}(x_2, x_2) & \cdots & K_{n-1}(x_2, x_q) & \xi_2 K_{n-1}(x_2, x) \\
\vdots & \vdots & \ddots & \vdots & \vdots \\
K_{n-1}(x_q, x_1) & K_{n-1}(x_q, x_2) & \cdots & 1 + \xi_q K_{n-1}(x_q, x_q) & \xi_q K_{n-1}(x_q, x) \\
P_{[n]}(x_1) & P_{[n]}(x_2) & \cdots & P_{[n]}(x_q) & P_{[n]}(x)
\end{array} \right],
\]

\[
\hat{H}_{[n]} = \Theta_* \left[ \begin{array}{cccc}
1 + \xi_1 K_{n-1}(x_1, x_1) & K_{n-1}(x_1, x_2) & \cdots & K_{n-1}(x_1, x_q) & -\xi_1 (P_{[n]}(x_1)) \top \\
K_{n-1}(x_2, x_1) & 1 + \xi_2 K_{n-1}(x_2, x_2) & \cdots & K_{n-1}(x_2, x_q) & -\xi_2 (P_{[n]}(x_2)) \top \\
\vdots & \vdots & \ddots & \vdots & \vdots \\
K_{n-1}(x_q, x_1) & K_{n-1}(x_q, x_2) & \cdots & 1 + \xi_q K_{n-1}(x_q, x_q) & -\xi_q (P_{[n]}(x_q)) \top \\
P_{[n]}(x_1) & P_{[n]}(x_2) & \cdots & P_{[n]}(x_q) & H_{[n]}
\end{array} \right].
\]

**Proof.** We take \( S_0 = \{ x_i, \beta_i \}_{i=1}^q \) with \( \beta_i = \alpha_i^{(0)} \), for \( i \in \{1, \ldots, q \} \), i.e., we have no derivatives in the delta functions, we have \( N_S = q \) and

\[
\nabla_x \partial \big|_{\alpha} (\xi^{(i)}) = \xi_i \in \mathbb{C},
\]

\[
\Xi = \text{diag}(\xi_1, \ldots, \xi_q)
\]

\[
\mathcal{K}_{n-1}(S_0) = [p_{[n]}(x_1), \ldots, p_{[n]}(x_q)] \top (H_{[n]})^{-1} [p_{[n]}(x_1), \ldots, p_{[n]}(x_q)] = [k_{n-1}(x_1, x_1)],
\]

\[
\mathcal{K}_{n-1,(x)}(S_0) = [k_{n-1}(x_1, x_1), \ldots, k_{n-1}(x_1, x_q)]]
\]

and the result follows. \( \square \)

This result was discussed by Uvarov in §2 of [25] and its multivariate extension was presented in [27].

We now illustrate the general 0D-Uvarov transformations formulæ by considering the addition of first order derivatives, in physical language the addition of dipoles instead of masses. We consider \( D \)-dimensional gradient operator \( \nabla = \left[ \frac{\partial}{\partial x_1}, \ldots, \frac{\partial}{\partial x_D} \right] \). In terms of it, for each vector \( \xi = (\xi_1, \ldots, \xi_D) \top \in \mathbb{C}^D \) we have the normal derivative \( \nabla_{\xi} = \sum_{a=1}^D \xi_a \frac{\partial}{\partial x_a}. \) Finally, given a function \( K(x, y) : \mathbb{R}^D \times \mathbb{R}^D \to \mathbb{C} \) we denote by

\[
K^{(\nabla, 0)}(x, y) := \left[ \frac{\partial K}{\partial x_1}(x, y), \ldots, \frac{\partial K}{\partial x_D}(x, y) \right] \top,
\]

\[
K^{(0, \nabla)}(x, y) := \left[ \frac{\partial K}{\partial y_1}(x, y), \ldots, \frac{\partial K}{\partial y_D}(x, y) \right] \top,
\]

\[
k^{(\nabla, \nabla)}(x, y) := \left[ \begin{array}{cc}
\frac{\partial^2 K}{\partial x_1 \partial y_1}(x, y) & \frac{\partial^2 K}{\partial x_1 \partial y_D}(x, y) \\
\vdots & \vdots \\
\frac{\partial^2 K}{\partial x_D \partial y_1}(x, y) & \frac{\partial^2 K}{\partial x_D \partial y_D}(x, y)
\end{array} \right],
\]

\[
k^{(\xi, 0)}(x, y) := \sum_{a=1}^D \xi_a \frac{\partial K}{\partial x_a}(x, y),
\]

\[
k^{(\xi, \nabla)}(x, y) := \left[ \sum_{a=1}^D \xi_a \frac{\partial^2 K}{\partial x_a \partial y_1}(x, y), \ldots, \sum_{a=1}^D \xi_a \frac{\partial^2 K}{\partial x_a \partial y_D}(x, y) \right].
\]
Corollary A.2 (0D-Uvarov dipole perturbation). Given couples of vectors (positions and strength of the dipoles) \( S_1 = \{x_i, \xi_i\}_{i=1}^q \) and a corresponding discrete additive dipolar perturbation of the form

\[
\hat{u} = u + \sum_{i=1}^q \nabla \xi_i \delta(x - x_i),
\]

the new MVOPR and quasi-tau matrices are given by the following quasi-determinantal expressions

\[
\hat{P}_{[n]}(x) = \Theta_s,
\]

\[
\begin{bmatrix}
1 + K_{n-1}^{(\xi_j,0)}(x_1, x_1) & K_{n-1}^{(\xi_j,\nabla)}(x_1, x_1) & \cdots & K_{n-1}^{(\xi_j,0)}(x_1, x_q) & K_{n-1}^{(\xi_j,\nabla)}(x_1, x_q) & K_{n-1}^{(\xi_j,0)}(x_1, x_q)

\xi_i K_{n-1}(x_j, x_1) & 1_D + \xi_i K_{n-1}^{(0,\nabla)}(x_j, x_1) & \cdots & \xi_i K_{n-1}(x_j, x_q) & \xi_i K_{n-1}^{(0,\nabla)}(x_j, x_q) & \xi_i K_{n-1}(x_j, x_q)

K_{n-1}^{(\xi_j,0)}(x_q, x_1) & K_{n-1}^{(\xi_j,\nabla)}(x_q, x_1) & \cdots & K_{n-1}^{(\xi_j,0)}(x_q, x_q) & K_{n-1}^{(\xi_j,\nabla)}(x_q, x_q) & K_{n-1}^{(\xi_j,0)}(x_q, x_q)

\xi_q K_{n-1}(x_q, x_1) & \xi_q K_{n-1}^{(0,\nabla)}(x_q, x_1) & \cdots & \xi_q K_{n-1}(x_q, x_q) & \xi_q K_{n-1}^{(0,\nabla)}(x_q, x_q) & \xi_q K_{n-1}(x_q, x_q)

P_{[n]}(x_1) & \nabla P_{[n]}(x_1) & \cdots & P_{[n]}(x_q) & \nabla P_{[n]}(x_q) & P_{[n]}(x)
\end{bmatrix},
\]

\[
\hat{H}_{[n]} = \Theta_s,
\]

\[
\begin{bmatrix}
1 + K_{n-1}^{(\xi_j,0)}(x_1, x_1) & K_{n-1}^{(\xi_j,\nabla)}(x_1, x_1) & \cdots & K_{n-1}^{(\xi_j,0)}(x_1, x_q) & K_{n-1}^{(\xi_j,\nabla)}(x_1, x_q) & K_{n-1}^{(\xi_j,0)}(x_1, x_q)

\xi_i K_{n-1}(x_j, x_1) & 1_D + \xi_i K_{n-1}^{(0,\nabla)}(x_j, x_1) & \cdots & \xi_i K_{n-1}(x_j, x_q) & \xi_i K_{n-1}^{(0,\nabla)}(x_j, x_q) & \xi_i K_{n-1}(x_j, x_q)

K_{n-1}^{(\xi_j,0)}(x_q, x_1) & K_{n-1}^{(\xi_j,\nabla)}(x_q, x_1) & \cdots & K_{n-1}^{(\xi_j,0)}(x_q, x_q) & K_{n-1}^{(\xi_j,\nabla)}(x_q, x_q) & K_{n-1}^{(\xi_j,0)}(x_q, x_q)

\xi_q K_{n-1}(x_q, x_1) & \xi_q K_{n-1}^{(0,\nabla)}(x_q, x_1) & \cdots & \xi_q K_{n-1}(x_q, x_q) & \xi_q K_{n-1}^{(0,\nabla)}(x_q, x_q) & \xi_q K_{n-1}(x_q, x_q)

P_{[n]}(x_1) & \nabla P_{[n]}(x_1) & \cdots & P_{[n]}(x_q) & \nabla P_{[n]}(x_q) & H_{[n]}
\end{bmatrix}
\]

Proof. In this case we take \( S = S_1 = \{x_i, \beta_i\}_{i=1}^q \) with \( \beta_i = \alpha_{D_i}^{(1)} \), for \( i \in \{1, \ldots, q\} \), and choose

\[
\xi_{i, \alpha_i^{(0)}} = 0,
\]

\[
\xi_i = \begin{bmatrix}
\xi_{i, \alpha_i^{(1)}} \\
\vdots \\
\xi_{i, \alpha_D^{(1)}}
\end{bmatrix} \in \mathbb{C}^D.
\]

we have \( N_s = q(D + 1) \) and

\[
\mathcal{J}_{p_{[n]}(S_1)} = \left[ p_{[n]}(x_1), \nabla p_{[n]}(x_1), \ldots, p_{[n]}(x_q), \nabla p_{[n]}(x_q) \right],
\]

\[
\xi^{(i)} = \begin{bmatrix}
0 \\
\xi_i \\
0_D
\end{bmatrix} \in \mathbb{C}^{(D+1) \times (D+1)},
\]

\[
\Xi = \text{diag}(\xi^{(1)}, \ldots, \xi^{(q)}) \in \mathbb{C}^{q(D+1) \times q(D+1)},
\]

\[
\mathcal{K}_{n-1}(S_1) = \begin{bmatrix}
K_{n-1}(x_1, x_1) & K_{n-1}^{(\xi_j,0)}(x_1, x_1) & \cdots & K_{n-1}(x_1, x_q) & K_{n-1}^{(\xi_j,0)}(x_1, x_q) & K_{n-1}(x_1, x_q)

K_{n-1}(x_1, x_1) & K_{n-1}^{(\xi_j,\nabla)}(x_1, x_1) & \cdots & K_{n-1}(x_1, x_q) & K_{n-1}^{(\xi_j,\nabla)}(x_1, x_q) & K_{n-1}(x_1, x_q)

\vdots & \vdots & \ddots & \vdots & \vdots & \vdots

K_{n-1}(x_1, x_1) & K_{n-1}^{(\xi_j,0)}(x_1, x_1) & \cdots & K_{n-1}(x_1, x_q) & K_{n-1}^{(\xi_j,0)}(x_1, x_q) & K_{n-1}(x_1, x_q)

K_{n-1}(x_1, x_1) & K_{n-1}^{(\xi_j,\nabla)}(x_1, x_1) & \cdots & K_{n-1}(x_1, x_q) & K_{n-1}^{(\xi_j,\nabla)}(x_1, x_q) & K_{n-1}(x_1, x_q)
\end{bmatrix}.
\]
Therefore, we compute

\[ I_{N_1} + \Xi \kappa_{n-1}(S_1) \]

and the result follows.

\[ \square \]

A.2. 1D-Uvarov perturbations and Fredholm integral equations. We have discussed 0-dimensional additive perturbations of D-dimensional generalized functions in full generality. However, we reckon that this is a very limited analysis, as in this multivariate context much more general perturbations do exist, as is illustrated by \[ (2.3) \]. We now discuss a very particular example, adding a 1D massive string. For this aim we assume that we have a parametrized curve, i.e. a smooth map from the interval \( I \subset \mathbb{R} \) to \( \mathbb{R}^D \):

\[ \gamma : I \to \mathbb{R}^D, \]

as well as a weight function \( w : I \to \mathbb{C} \). Then, the linear functional \( v \) is

\[ \langle v, P \rangle = \int_I P(\gamma(t))w(t) \, d\, t. \]

Recalling \[ (A.1) \] we can write

\[ (A.3) \quad \hat{P}_{[n]}(x) = P_{[n]}(x) - \int_I \hat{P}_{[n]}(\gamma(s))K_{n-1}(\gamma(s), x)w(s) \, ds. \]

Now, let us remark one of the basic ideas in the proof of Theorem \[ (A.1) \]. First, one uses \[ (A.1) \] and then evaluates on the support of the distribution. In that case, we evaluated again at the points where the delta functions and its derivatives where supported. In this case, we should evaluate it again at the curve \( \gamma \).

**Definition A.2.** We introduce some notation

\[ \hat{\pi}_{[n]}(t) := \hat{P}_{[n]}(\gamma(t)), \quad \pi_{[n]}(t) := P_{[n]}(\gamma(t)), \quad \kappa_{n-1}(t, s) := K_{n-1}(\gamma(t), \gamma(s))w(s). \]

Then, \[ (A.3) \] implies the following integral Fredholm equation

\[ (A.4) \quad \hat{\pi}_{[n]}(t) = \pi_{[n]}(t) - \int_I \hat{\pi}_{[n]}(s)\kappa_{n-1}(s, t) \, ds. \]

This integral equation, having as integral kernel \( \kappa_{n-1}(t, s) \) a separable one, can be solved explicitly. In fact,

**Proposition A.2.** The solution of the separable Fredholm equation \[ (A.4) \] can be expressed as a last quasi-determinant as follows

\[ \hat{\pi}_{[n]}(t) = \Theta_{\ast} \begin{bmatrix} H_{[0]} + \int_I \pi_{[0]}(s)(\pi_{[0]}(s))^\top w(s) \, ds \quad \cdots \quad \int_I \pi_{[0]}(s)(\pi_{[n-1]}(s))^\top w(s) \, ds \\ \int_I \pi_{[1]}(s)(\pi_{[0]}(s))^\top w(s) \, ds \quad \cdots \quad \int_I \pi_{[1]}(s)(\pi_{[n-1]}(s))^\top w(s) \, ds \\ \vdots \\ \int_I \pi_{[n-1]}(s)(\pi_{[0]}(s))^\top w(s) \, ds \quad \cdots \quad H_{[n-1]} + \int_I \pi_{[n-1]}(s)(\pi_{[n-1]}(s))^\top w(s) \, ds \\ \int_I \pi_{[n]}(s)(\pi_{[0]}(s))^\top w(t) \, ds \quad \cdots \quad \int_I \pi_{[n]}(s)(\pi_{[n-1]}(s))^\top w(s) \, ds \end{bmatrix} \begin{bmatrix} \pi_{[0]}(t) \\ \pi_{[1]}(t) \\ \vdots \\ \pi_{[n-1]}(t) \\ \pi_{[n]}(t) \end{bmatrix}. \]

**Proof.** The separability of the kernel means that the kernel \( \kappa_{n-1}(t, s) \) can be written

\[ \kappa_{n-1}(s, t) = \sum_{m=0}^{n-1} (\pi_{[m]}(s))^\top (H_{[m]})^{-1}\pi_{[m]}(t). \]
Now, with the notation
\[ C_{[n],[m]} := \int_I \pi_{[n]}(s)\pi_{[m]}(s)^\top w(s) \, ds \]
we can write the Fredholm equation
\[ \hat{\pi}_{[n]}(t) = \pi_{[n]}(t) - \sum_{m=0}^{n-1} C_{[n],[m]}(H_{[m]})^{-1} \pi_{[m]}(t), \]
that can be introduced in (A.5) to get
\[ C_{[n],[0]}c_{0} + \cdots + C_{[n],[n-1]}c_{n-1} = \left[ A_{[n],[0],0}, \ldots, A_{[n],[n-1],0} \right] (H_{[n]})^{-1} \left[ A_{[0],[0],0} \ldots A_{[0],[n-1],0} \right] = A_{[n],[0],0} \ldots A_{[n],[n-1],0} \]
and the result follows.

**Proposition A.3.** Given the solution \( \hat{\pi}_{[n]}(t) \) to the Fredholm equation (A.4), we find the perturbed MVOPR and squared norms can be expressed
\[ \hat{P}_{[n]}(x) = P_{[n]}(x) - \int_I \hat{\pi}_{[n]}(t)K_{n-1}(\gamma(t), x)w(t) \, dt, \]
\[ \hat{H}_{[n]} = H_{[n]} + \int_I \hat{\pi}_{[n]}(t)\pi_{[n]}(t)^\top w(t) \, dt. \]

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