Factorization of the 3d superconformal index

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Abstract: We prove that 3d superconformal index for general $\mathcal{N} = 2$ $U(N)$ gauge group with fundamentals and anti-fundamentals with/without Chern-Simons terms is factorized into vortex and anti-vortex partition function. We show that for simple cases, 3d vortex partition function coincides with a suitable topological open string partition function. We provide much more elegant derivation at the index level for $\mathcal{N} = 2$ Seiberg-like dualities of unitary gauge groups with fundamental matters and $\mathcal{N} = 4$ mirror symmetry.
1. Introduction

Recently, there has been renewed interest in nonperturbative dualities between three dimensional theories such as mirror symmetry and Seiberg-like dualities. This is explained in part by the availability of sophisticated tools such as the partition function on $S^3$ and the superconformal index. Using these tools, one can give impressive evidence for various 3d dualities. Some of works in this area are [1]-[19].
It turns out that the partition function has another interesting property, i.e., it is factorized into vortex and anti-vortex partition function [20]. Schematically

$$Z(z, \bar{z}) = Z_{\text{vortex}}(z)Z_{\text{antivortex}}(\bar{z}) = |Z_{\text{vortex}}(z)|^2$$ (1.1)

where $z$ traces the vortex number while $\bar{z}$ traces the anti-vortex number. This is reminiscent of the conformal blocks of the 2-dimensional conformal field theories. The above factorization was shown to hold for abelian gauge theories. Thus it is more desirable to show this factorization for the general nonabelian cases. And it would be an interesting question that the similar holds for 3d superconformal index. In fact, it is recently shown that similar factorization holds for 2-dimensional $\mathcal{N} = 2$ supersymmetric partition function in terms of vortex and anti-vortex partition function[21, 22]. Since 3d index is the partition function defined on $S^1 \times S^2$, the two sphere partition function is recovered from the 3d index by taking the radius of $S^1$ to be small. Thus we expect that the factorization should hold for 3d superconformal index as well.

The purpose of this paper is to show explicitly that such factorization indeed occurs for 3d superconformal index. More explicitly we show that for $U(N)$ gauge theories with $N_f$ fundamental and $\tilde{N}_f$ fundamentals and show that the index is factorized into vortex and anti-vortex partition function on $R^2 \times S^1$ whenever $\max(N_f, \tilde{N}_f) \geq N$. This condition is the condition of the existence of the vortex solutions of the underlying field theories. This is done by explicit residue evaluation of the associated matrix integral of the index, similar to 2d case.

The factorized form of the index has a number of merits and we just explore a few of them in this paper, relegating the full explorations elsewhere. The first one is that we have the explicit expressions of the index after the matrix integral. Obviously since we have the explicit expressions for the index, it would be much more convenient to explore the various dualities. Previously the index is expanded in power series of the conformal dimension of the gauge invariant BPS operators. In this way, one can check various dualities by working out the index of the both sides to some orders in operator dimensions. Though it certainly gives impressive evidences, in this way the full analytic proof cannot be achieved. We will show that explicit factorized formulae of the index reveal much more transparent structures of the dualities. We will see this by working out the index of the dual pairs of Aharony duality with unitary gauge group. The proof of the equality of the index is reduced to show the nontrivial identity of the combinatorical character.

Furthermore in 2d case, the vortex partition function has the direct connection to the topological open string amplitude. We expect that similar holds for 3d vortex partition function since 2d vortex partition function is so called the homological limit of 3d vortex partition function. We show that vortex partition function is the same as topological open string partition function for simple cases but certainly has the
obvious generalizations for much more numerous examples. This is also resonant with the recent proposal by Iqbal and Vafa [24] that the integrand of the 3d superconformal index is given by the square of the topological open string amplitude. It would be interesting to explore the precise relation between the 3d vortex partition function and the open topological string. 

The content of the paper is as follows. In section 2, we summarize the basic structures of the superconformal index. We carefully study the $U(1)$ gauge theory with a fundamental chiral multiplet with Chern-Simons (CS) level $-1/2$ following [23], find subtleties such as the relative phase of the different monopole sector, in the usual index computation, which will be useful for later computation. In section 3, we firstly work out the factorization for $U(1)$ gauge theory without CS terms, which is technically simpler. Then we summarize the factorization of the general cases, deferring the full proof to the appendix. We also work out the explicit examples of the factorization and show the associated vortex partition function admits topological open string interpretation. Furthermore we show that in some of the examples vortex partition function can be understood as 3d defect of the 5d field theory. In section 4, we apply the factorized index to understand the $\mathcal{N} = 2$ Seiberg-like dualities for unitary gauge group, known as Aharony duality. Factorized index reveals much more clearly such duality should hold at the index level. We briefly touch upon the $\mathcal{N} = 4$ Seiberg-like dualities and mirror symmetry and postpone the further explorations elsewhere.

As this work is close to end, we receive the related paper by [25]. As far as we understand, they do not give the general formulae for the factorized index as we do.

2. 3d superconformal index

2.1 Summary of the 3d superconformal index

Let us discuss the superconformal index for $\mathcal{N} = 2 \ d = 3$ superconformal field theories (SCFT). The bosonic subgroup of the 3d $\mathcal{N} = 2$ superconformal group is $SO(2, 3) \times SO(2)$. There are three Cartan elements denoted by $\epsilon, j_3$ and $R$ which come from three factors $SO(2)_\epsilon \times SO(3)_{j_3} \times SO(2)_R$ in the bosonic subgroup, respectively. The superconformal index for an $\mathcal{N} = 2 \ d = 3$ SCFT is defined as follows [26]:

$$I(x, t) = \text{Tr}(-1)^F \exp(-\beta'\{Q,S\}) x^{\epsilon+j_3} \prod_a t_a^{\epsilon_a}$$  

(2.1)

where $Q$ is a supercharge with quantum numbers $\epsilon = \frac{1}{2}, j_3 = -\frac{1}{2}$ and $R = 1$, and $S = Q^\dagger$. The trace is taken over the Hilbert space in the SCFT on $\mathbb{R} \times S^2$ (or equivalently over the space of local gauge-invariant operators on $\mathbb{R}^3$). The operators $S$ and $Q$ satisfy the following anti-commutation relation:

$$\{Q, S\} = \epsilon - R - j_3 := \Delta.$$  

(2.2)
As usual, only BPS states satisfying the bound $\Delta = 0$ contribute to the index, and therefore the index is independent of the parameter $\beta'$. If we have additional conserved charges $f_\alpha$ commuting with the chosen supercharges $(Q,S)$, we can turn on the associated chemical potentials $t_\alpha$, and then the index counts the number of BPS states weighted by their quantum numbers.

The superconformal index is exactly calculable using the localization technique [27, 28]. It can be written in the following form:

$$I(x,t) = \sum_{m\in\mathbb{Z}} \int da \frac{1}{|W_m|} e^{-S^{(0)}_{CS}(a,m)} e^{t_\alpha(a,m)} \prod_a t_\alpha^{q_\alpha(a,m)} \exp \left[ \sum_{n=1}^{\infty} \frac{1}{n} f_{\text{tot}}(e^{ina}, t^n, x^n) \right]. \quad (2.3)$$

The origin of this formula is as follows. To compute the trace over the Hilbert space on $S^2 \times \mathbb{R}$, we use path-integral on $S^2 \times S^1$ with suitable boundary conditions on the fields. The path-integral is evaluated using localization, which means that we have to sum or integrate over all BPS saddle points. The saddle points are spherically symmetric configurations on $S^2 \times S^1$ which are labeled by magnetic fluxes on $S^2$ and holonomy along $S^1$. The magnetic fluxes are denoted by $\{m\}$ and take values in the cocharacter lattice of $G$ (i.e. in $\text{Hom}(U(1), T)$, where $T$ is the maximal torus of $G$), while the eigenvalues of the holonomy are denoted $\{a\}$ and take values in $T$. $S^{(0)}_{CS}(a,m)$ is the classical action for the (monopole-holonomy) configuration on $S^2 \times S^1$, $\epsilon_0(m)$ is the Casimir energy of the vacuum state on $S^2$ with magnetic flux $m$, $q_\alpha(m)$ is the $f_\alpha$-charge of the vacuum state, and $b_\alpha(a,m)$ represents the contribution coming from the electric charge of the vacuum state. The last factor comes from taking the trace over a Fock space built on a particular vacuum state. $|W_m|$ is the order of the Weyl group of the part of $G$ which is left unbroken by the magnetic fluxes $m$. These ingredients in the formula for the index are given by the following explicit expressions:

$$S^{(0)}_{CS}(a,m) = i \sum_{\rho \in R_F} k_\rho(m) \rho(a), \quad (2.4)$$

$$b_\alpha(a,m) = -\frac{1}{2} \sum_\Phi \sum_{\rho \in R_\Phi} |\rho(m)| \rho(a),$$

$$q_\alpha(m) = -\frac{1}{2} \sum_\Phi \sum_{\rho \in R_\Phi} |\rho(m)| f_\alpha(\Phi),$$

$$\epsilon_0(m) = \frac{1}{2} \sum_\Phi (1 - \Delta_\Phi) \sum_{\rho \in R_\Phi} |\rho(m)| - \frac{1}{2} \sum_{\alpha \in G} |\alpha(m)|,$$
\[ f_{\text{tot}}(x, t, e^{ia}) = f_{\text{vector}}(x, e^{ia}) + f_{\text{chiral}}(x, t, e^{ia}), \]
\[ f_{\text{vector}}(x, e^{ia}) = - \sum_{a \in G} e^{ia(a) \cdot |a(m)|}, \]
\[ f_{\text{chiral}}(x, t, e^{ia}) = \sum_{\Phi} \sum_{\rho \in R_{\Phi}} \left[ e^{i\rho(a) \cdot t_a \cdot x|^{\rho(m)| + \Delta_{\Phi}}} - e^{-i\rho(a) \cdot t_a - f_a(\Phi) \cdot x|^{\rho(m)| + 2 - \Delta_{\Phi}}} \right] \]

where \( \sum_{\rho \in R_{\Phi}} \sum_{\Phi} \cdot \sum_{\rho \in R_{\Phi}} \) and \( \sum_{\alpha \in G} \) represent summations over all fundamental weights of \( G \), all chiral multiplets, all weights of the representation \( R_{\Phi} \), and all roots of \( G \), respectively.

We will find it convenient to rewrite the integrand in (2.3) as a product of contributions from the different multiplets. First, note that the single particle index \( f \) enters via the so-called Plethystic exponential:
\[ \exp \left( \sum_{n=1}^{\infty} \frac{1}{n} f(x^n, t^n, z^n = e^{ina}, m) \right) \] (2.5)
while we define \( z_j = e^{ia_j} \). It will be convenient to rewrite this using the \( q \)-product, defined for \( n \) finite or infinite:
\[ (z; q)_n = \prod_{j=0}^{n-1} (1 - zq^j). \] (2.6)
Specifically, consider a single chiral field \( \Phi \), whose single particle index is given by\(^1\):
\[ \sum_{\rho \in R_{\Phi}} \left( e^{i\rho(a) \cdot t_a \cdot f_a(\Phi) \cdot x|^{\rho(m)| + \Delta_{\Phi}}} - e^{-i\rho(a) \cdot t_a - f_a(\Phi) \cdot x|^{\rho(m)| + 2 - \Delta_{\Phi}}} \right). \] (2.7)
Then we can write the Plethystic exponential of this as follows:
\[ \prod_{\rho \in R_{\Phi}} \exp \left[ \sum_{n=1}^{\infty} \frac{1}{n} \left( e^{i\rho(a) \cdot t_a \cdot n \cdot f_a(\Phi) \cdot x|^{\rho(m)| + n \Delta_{\Phi}}} - e^{-i\rho(a) \cdot t_a - n \cdot f_a(\Phi) \cdot x|^{\rho(m)| + 2n - n \Delta_{\Phi}}} \right) \right]. \] (2.8)
By rewriting the denominator as a geometric series and interchanging the order of summations, one finds that this becomes:
\[ \prod_{\rho \in R_{\Phi}} \frac{(e^{-i\rho(a) \cdot t_a - f_a(\Phi) \cdot x|^{\rho(m)| + 2 - \Delta_{\Phi}}; x^2)_0)_{\infty}}{(e^{i\rho(a) \cdot t_a \cdot f_a(\Phi) \cdot x|^{\rho(m)| + \Delta_{\Phi}}; x^2)_0)_{\infty}}. \] (2.9)
The full index will involve a product of such factors over all the chiral fields in the theory, as well as the contribution from the gauge multiplet. It is given by:
\[ I(x, t) = \sum_{m \in \mathbb{Z}} \oint \prod_{j} \frac{dz_j}{2\pi iz_j} \frac{1}{|W_m|} e^{-S_{CS}(m,a)} Z_{\text{gauge}}(x, z, m) \prod_{\Phi} Z_{\Phi}(x, t, z, m) \] (2.10)
\(^1\)Note that \( a \) in \( \rho(a) \) and the subscript \( a \) in \( t_a \) or \( f_a \) denote the different objects.
where:

\[ Z_{\text{gauge}}(x, z = e^{ia}, m) = \prod_{\alpha \in \text{ad}(G)} x^{-|\alpha(m)|} \left( 1 - e^{i\alpha(a) x |\alpha(m)|} \right), \]

\[ Z_{\Phi}(x, t, z, m) = \prod_{\rho \in \mathbb{R}} \left( x^{(1-\Delta\Phi)} e^{-i\rho(a) \sum_a t_a - f_a(\Phi)} \right)^{\rho(m)/2} \left( \frac{e^{-i\rho(a) t_a - f_a(\Phi) x |\rho(m)| + 2 - \Delta\Phi; x^2_{\infty}}}{(e^{i\rho(a) t_a f_a(\Phi) x |\rho(m)| + \Delta\Phi; x^2_{\infty}}} \right). \]

Note that by shifting \( t_a \rightarrow t_a x^{\alpha_a} \), one can change the value of the R-charge \( \Delta\Phi \). Hence \( \Delta\Phi \) remains the free parameter for generic cases.

We are mainly interested in this ordinary index and work out the factorization. However two important generalizations are worthy of mention, which will be useful in comparison with the result of [23] in the following subsection. The first one is the notion of the generalized index. When we turn on the chemical potential \( t_a \), this can be regarded as turning on a Wilson line for a fixed background gauge field. The generalized index is obtained when we turn on the nontrivial magnetic flux \( n_a \) for the corresponding background gauge field. Only the contribution to the chiral multiplets are changed and this is given by the replacement \( \rho(m) \rightarrow \rho(m) + \sum_a f_a(\Phi)n_a \)

\[ Z_{\Phi}(x, t, z, m) = \prod_{\rho \in \mathbb{R}} \left( x^{(1-\Delta\Phi)} e^{-i\rho(a) \sum_a t_a - f_a(\Phi)} \right)^{\rho(m)/2 + \sum_a f_a(\Phi)n_a/2} \left( \frac{e^{-i\rho(a) t_a - f_a(\Phi) x |\rho(m)| + 2 - \Delta\Phi; x^2_{\infty}}}{(e^{i\rho(a) t_a f_a(\Phi) x |\rho(m)| + \Delta\Phi; x^2_{\infty}}} \right). \]

(2.11)

Here \( n_a \) should take integer value as does \( m_j \).

For every \( U(N) \) gauge group, we can define another abelian symmetry \( U(1)_T \) whose conserved current is \( *F \) of overall \( U(1) \) factor. To couple this topological current to background gauge field we introduce \( BF \) term \( \int A_{BG} \wedge \text{tr} dA + \cdots \) and terms needed for supersymmetric completion. This introduces to the index

\[ x^n w^{\sum_j m_j} \]

(2.12)

where \( n \) is the new discrete parameter representing the topological charge of \( U(1)_T \) while \( w \) is the chemical potential for \( U(1)_T \).

### 2.2 Comparison to DGG

In the paper by Dimofte, Gaiotto and Gukov [23] (DGG), the simplet mirror pair of \( \mathcal{N} = 2 \) theory was considered and along with it revealed some subtleties in the index computation. The claim is that the theory of one free chiral multiplet with global \( U(1) \) symmetry at CS level \( \frac{1}{2} \) is mirror to \( U(1) \) gauge theory at CS level \( -\frac{1}{2} \),
coupled to a single fundamental chiral multiplet. According to DGG, for the free chiral theory the index is given by

$$I_\Delta(m; q, \zeta) = \left(-q^2\right)^{\frac{1}{2}(m+|m|)} \zeta^{-\frac{1}{2}(m+|m|)} \prod_{r=0}^{\infty} \frac{1 - q^{r+\frac{1}{2}|m|+1}\zeta^{-1}}{1 - q^{r+\frac{1}{2}|m|}\zeta}. \quad (2.13)$$

Note that we use the zero $R$-charge for the free chiral but value of $R$-charge can be altered by shifting $\zeta \to \zeta x^\alpha$ for a suitable $\alpha$. The index of $U(1)$ theory is [23]

$$I_{U(1)}(m'; q, \zeta') = \sum_{m \in \mathbb{Z}} \oint \frac{dz}{2\pi i z} \zeta^m \zeta'^m \left(-q^2\right)^{-\frac{1}{2}(m-|m|)} z^{-\frac{1}{2}(m-|m|)} \prod_{r=0}^{\infty} \frac{1 - q^{r+\frac{1}{2}|m|+1}\zeta^{-1}}{1 - q^{r+\frac{1}{2}|m|}\zeta}. \quad (2.14)$$

It is proved that $I_\Delta(m; q, \zeta) = I_{U(1)}(m; q, \zeta)$.

In order to compare it to our index, let us slightly change the variables as follows:

$$I_{U(1)}(m'; x^2, w) = \sum_{m \in \mathbb{Z}} \oint \frac{dz}{2\pi i z} w^{m} z^{m'} \left(-x\right)^{-\frac{1}{2}(m-|m|)} z^{-\frac{1}{2}(m-|m|)} \prod_{k=0}^{\infty} \frac{1 - z^{-1}x^{-2k+2k+2|m|}}{1 - zx^{-2k+2k+2|m|}}. \quad (2.15)$$

Note that $U(1)$ gauge theory has topological $U(1)$ global symmetry whose current is given by $*F$ and $w$ corresponds to its chemical potential. Under the mirror map, the global symmetry of chiral theory is mapped to the topological symmetry. Hence $\zeta$ is mapped to $w$. The expression appearing at DGG is slightly different from the standard expression one obtains following the prescription specified at the previous subsection or at [28]. For $U(1)$ with CS level $-1/2$, the index is given by

$$I(x, w, m') = \sum_{m \in \mathbb{Z}} \oint \frac{dz}{2\pi i z} w^{m} z^{m'} x^{-1/2} \left(-x\right)^{\frac{1}{2}(m-|m|)} \prod_{k=0}^{\infty} \frac{1 - z^{-1}x^{-2k+2k+2|m|}}{1 - zx^{-2k+2k+2|m|}}. \quad (2.16)$$

The term $x^{-|m|/2}$ comes from the zero point energy contribution. At first DGG expression appears to change the zero point energy for positive and negative flux sector. However the computation in [28] shows that the one-loop determinant is symmetric under $m \to -m$ hence the zero point energy should be symmetric under $m \to -m$, which comes from the suitable regularization of one-loop determinant. The resolution is that if we assign different $R$-charge in the free theory by $\zeta \to \zeta q^\alpha$ we modify the $U(1)$ theory by $w \to wx^{2\alpha}$. Using this freedom, if one shifts $w \to wx^{-1/2}$ one obtains DGG eq. (2.15) from the standard computation eq. (2.16). On the other hand, in the $U(1)$ theory there’s no freedom to change the assigned $R$-charge of the charged chiral field and we assign zero $R$-charge for the scalar of the chiral multiplet. One might worry that this $R$-charge can violate the unitarity of the SCFT. However, the chiral field itself is not a gauge invariant operator. Furthermore all of the gauge

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2We use a convention of the opposite sign for the CS level to DGG.

3The factor $\left(-1\right)^{\frac{1}{2}(m-|m|)}$ will be explained in the next paragraph.
invariant operators of the theory are captured by the index of the free chiral theory due to the mirror symmetry. Thus the assigned zero $R$-charge does not lead to any inconsistency. Furthermore one can show that the standard index of the $U(1)$ theory eq. (2.16) is equal to the free chiral theory with the canonical $R$-charge $1/2$, i.e.,

$$
I_{\Delta} (m; q, \zeta) = \left(q^{1/2}\right)^{1/2 (m+|m|)} (-\zeta)^{-1/2 (m+|m|)} \prod_{r=0}^{\infty} \frac{1 - q^{r+1/2} |m| + \frac{1}{2} \zeta^{-1}}{1 - q^{r+1/2} |m| + \frac{1}{2} \zeta} 
$$

(2.17)

with $m = m'$, $q = x^2$ and $\zeta = w$. Thus if we use the standard index computation we have the duality between $U(1)$ theory with CS level $-1/2$ with one charged chiral with zero $R$-charge and the free chiral with CS level $1/2$ with the standard $R$-charge assignment.

On the other hand DGG assigns subtle relative phase factor $(-1)^{-\kappa m'/2}$ between positive and negative flux sector. This phase factor cannot be derived from the usual index computation since it concerns on the relative phase of the different flux sector. In DGG, this relative phase factor have been checked extensively so we include this phase in later computations. It turns out that this phase is crucial for the factorization of the indices.

For reference, for $U(1)$ theory with CS level $\kappa$ with $N_f$ fundamental chiral and $\tilde{N}_f$ anti-fundamental chiral, the flavor symmetry is $U(1)_A \times SU(N_f) \times SU(\tilde{N}_f)$. The index we will use is as follows:

$$
I(x, t, \tilde{t}, w, \kappa) = \sum_{m \in \mathbb{Z}} \oint_{\mathbb{T}^2} \frac{dz}{2\pi i z} w^m x^{1/2 (N_f + \tilde{N}_f)} |m| (-z)^{-\kappa m - \frac{1}{2} (N_f - \tilde{N}_f)} |m| \tau^{-\frac{1}{2} (N_f + \tilde{N}_f)} |m| 
$$

(2.18)

where $\tau, t_a, \tilde{t}_a$ are the fugacities for $U(1)_A$, Cartans of $SU(N_f), SU(\tilde{N}_f)$ respectively. Note that we include the additional phase $(-1)^{-\kappa m - \frac{1}{2} (N_f - \tilde{N}_f)} |m|$ to the original index. Similar factor will be included for non-abelian cases as well.

3. Factorization

3.1 Abelian without CS terms

We first consider the factorization for the abelian case without CS terms. Similar but slightly more complicated derivation works for $U(N)$ theory with fundamentals and anti-fundamentals in the presence of Chern-Simons terms. The general derivation is relegated to the appendix. The superconformal index for $U(1)$ gauge theory is given
by

\[ I(x, t, w) = \sum_{m \in \mathbb{Z}} \int \frac{dz}{2\pi i z} w^m \prod_{\Phi} Z_{\Phi}(x, t, z, m) \quad (3.1) \]

If one considers \( N_f \) fundamental and \( \tilde{N}_f \) antifundamental chiral multiplets,

\[
\prod_{\Phi} Z_{\Phi}(x, t, \tilde{t}, \tau, z, m) = x^{(1-\Delta_{\Phi})(N_f+\tilde{N}_f)|m|/2} (-z)^{(N_f-\tilde{N}_f)|m|/2} x^{-\Delta_{\Phi}|m|/2} \prod_{a=1}^{N_f} \left( 1 - \frac{z^{-1} t_a^{-1} x^{-|m|+2-\Delta_{\Phi}+2k}}{1 - z t_a x^{|m|+\Delta_{\Phi}+2k}} \right) \prod_{a=1}^{\tilde{N}_f} \left( 1 - \frac{z^{-1} \tilde{t}_a^{-1} x^{-|m|+2-\Delta_{\Phi}}}{1 - z^{-1} \tilde{t}_a x^{|m|+\Delta_{\Phi}+2k}} \right) \quad (3.2)
\]

where \( t_a \) and \( \tilde{t}_a \) correspond to fugacities for \( SU(N_f) \times SU(\tilde{N}_f) \); \( \tau \) is a fugacity for \( U(1)_A \) as in the previous section. We will set \( \Delta_{\Phi} = 0 \), which can be restored by deforming \( \tau \to \tau x^{\Delta_{\Phi}} \). Note that the infinite product in the expression only makes sense for \( |x| < 1 \). Assuming \( |t_a\tau|, |\tilde{t}_a\tau| < 1 \), which can be extended by analytic continuation after integration, poles from the antifundamental part lie inside the integration contour. In addition, the integrand also has a pole at the origin, which makes the integration difficult, for \( N_f \geq \tilde{N}_f \). Fortunately for \( N_f > \tilde{N}_f \) one could change the integration variable \( z \to 1/z \) to exclude the pole at the origin and would take poles from the fundamental part, which are now inside the contour, instead of the poles from the antifundamental part. For \( N_f = \tilde{N}_f \) one should take account of the pole at the origin.

Firstly we deal with the \( N_f > \tilde{N}_f \) case. Changing the variable \( z \to 1/z \) is equivalent to summing residues at poles outside the contour, which come from the fundamental part: \( z = t_b^{-1} \tau^{-1} x^{-|m|+2l} \) for \( b = 1, \ldots, N_f \) and \( l = 0, 1, \ldots \). After performing the contour integration the index is given by

\[
I_{N_f>\tilde{N}_f}^{N_f}(x, t, \tilde{t}, \tau, w)
\]

\[
= \sum_{m \in \mathbb{Z}} \sum_{b=1}^{N_f} (\sum_{l=0}^{\infty} (1 - (N_f-\tilde{N}_f)|m|/2) w_m t_b^{(N_f-\tilde{N}_f)|m|/2} x^{-\tilde{N}_f|m|/2} x^{(N_f+\tilde{N}_f)|m|/2} + (N_f-\tilde{N}_f)(|m|+2|m|/2)|m|/2)
\]

\[
\left( \prod_{a=1}^{N_f} \prod_{k=0}^{\infty} \frac{1 - t_a^{-1} x^{-|m|+2l+2k}}{1 - t_a x^{-2l+2k}} \right) \left( \prod_{a=1}^{\tilde{N}_f} \prod_{k=0}^{\infty} \frac{1 - \tilde{t}_a^{-1} \tau^{-2} x^{-2l+2k}}{1 - \tilde{t}_a \tau x^{-2l+2k}} \right) \right) \quad (3.3)
\]

Let us rewrite the terms \( \prod_{a=1,k=0}^{N_f,\infty} 1 - t_a^{-1} x^{-2l+2k} \) and \( \prod_{a=1,k=0}^{\tilde{N}_f,\infty} 1 - \tilde{t}_a^{-1} \tau^{-2} x^{-2l+2k} \)
as follows:

\[
\prod_{a=1,k=0}^{N_f,\infty} 1 - t_b^{-1} t_a x^{-2l+2k} \quad (\text{3.4})
\]

\[
\prod_{a=1,k=0}^{N_f,l-1} 1 - t_b^{-1} t_a x^{-2l+2k} \quad (\text{3.5})
\]

and

\[
\prod_{a=1,k=0}^{N_f,\infty} 1 - t_b^{-1} t_a^{-1} x^{-2l+2k}
\]

Using these one can rewrite the index as follows:

\[
I_{N_f}^{N_f} (x, t, \bar{t}, \tau, w)
\]

\[
= \sum_{m=0}^{N_f} \sum_{b=1}^{\infty} \sum_{l=0}^{(N_f-\bar{N}_f)|m|/2} (-1)^{-(N_f-\bar{N}_f)|m|/2} t_b^{(N_f-\bar{N}_f)|m|/2} \tau^{-N_f m} x^{(N_f+\bar{N}_f)|m|/2 + (N_f-\bar{N}_f)(|m|+2)|m|/2} / \sum_{m=0}^{N_f} \sum_{b=1}^{\infty} \sum_{l=0}^{(N_f-\bar{N}_f)|m|/2} (-1)^{-(N_f-\bar{N}_f)|m|/2} t_b^{(N_f-\bar{N}_f)|m|/2} \tau^{-N_f m} x^{(N_f+\bar{N}_f)|m|/2 + (N_f-\bar{N}_f)(|m|+2)|m|/2}...\]

In order to proceed further one should rearrange the summations. Thanks to the symmetry \(|m| + l \leftrightarrow l\) one can rearrange the summations as \(\sum_{m \in \mathbb{Z}} \sum_{l=0}^{\infty} = \sum_{n=0}^{\infty} \sum_{n=0}^{\infty} \)
where \( n \equiv l + \frac{|m|}{2} + \frac{m}{2}, \bar{n} \equiv l + \frac{|m|}{2} - \frac{m}{2} \). Finally the index can be written in the factorized form:

\[
I^{N_f, \bar{N}_f}(x, t, \bar{t}, \tau, w) = \sum_{b=1}^{N_f} \sum_{n=0}^{\bar{N}_f} \sum_{\bar{n}=0}^{N_f} \left( -1 \right)^{-(N_f-\bar{N}_f)(n+\bar{n})/2} \sum_{(\tau)}(N_f-\bar{N}_f)(n+\bar{n})/2 - \bar{N}_f(n+\bar{n})/2(2(N_f-\bar{N}_f)(n+\bar{n})/2)
\]

\[
\times \left( \prod_{k=0}^{\bar{N}_f} \frac{1}{\prod_{b=1}^{N_f} 1 - t_b - \bar{t}_a \tau x^{2k}} \right) \left( \prod_{k=0}^{N_f} \frac{1}{\prod_{b=1}^{\bar{N}_f} 1 - t_b - \bar{t}_a \tau x^{2k}} \right)
\]

\[
\times \left( \prod_{k=0}^{\bar{N}_f} \frac{1}{\prod_{b=1}^{N_f} 1 - t_b - \bar{t}_a \tau x^{2k}} \right) \left( \prod_{k=0}^{N_f} \frac{1}{\prod_{b=1}^{\bar{N}_f} 1 - t_b - \bar{t}_a \tau x^{2k}} \right)
\]

\[
= \sum_{b=1}^{N_f} \left[ \prod_{k=0}^{\bar{N}_f} \left( \prod_{b=1}^{N_f} 1 - t_b - \bar{t}_a \tau x^{2k} \right) \left( \prod_{k=0}^{N_f} \frac{1}{\prod_{b=1}^{\bar{N}_f} 1 - t_b - \bar{t}_a \tau x^{2k}} \right) \right]
\]

More concisely,

\[
I^{N_f, \bar{N}_f}(x = e^{-\gamma}, t = e^{iM}, \bar{t} = e^{i\bar{M}}, \tau = e^{i\mu}, w)
\]

\[
= \sum_{b=1}^{N_f} \left[ \prod_{k=0}^{\bar{N}_f} \left( \prod_{b=1}^{N_f} 1 - t_b - \bar{t}_a \tau x^{2k} \right) \left( \prod_{k=0}^{N_f} \frac{1}{\prod_{b=1}^{\bar{N}_f} 1 - t_b - \bar{t}_a \tau x^{2k}} \right) \right]
\]

where we identified some parameters as follows: \( x = e^{-\gamma}, t_a = e^{iM_a}, \bar{t}_a = e^{i\bar{M}_a} \) and \( \tau = e^{i\mu} \). The second last and last lines correspond to the \( \mathcal{N} = 2 \) vortex and antivortex partition functions on \( \mathbb{R}^2 \times S^1 \) as appearing in [20, 29]. In [29], vortex quantum mechanics is considered and the vortex zero sector is not handled. Thus we cannot directly compare the prefactor corresponding to the one-loop determinant. Perturbative part was checked in the other example [20] by matching to the 2d result. One can easily check that prefactor of eq. (3.8) also factorizes and gives rise to the one-loop determinant which matches the known result.
The index for \( N_f < \tilde{N}_f \) is simply obtained by interchanging \( t_a \leftrightarrow \tilde{t}_a \):

\[
I^{N_f < \tilde{N}_f}(x = e^{-\gamma}, t = e^{iM}, \tilde{t} = e^{i\tilde{M}}, \tau = e^{i\mu}, \omega)
= \sum_{b=1}^{\tilde{N}_f} \prod_{k=0}^{\infty} \left( \sum_{a=1}^{N_f} \frac{1 - \tilde{t}_b t_a^{-1} x^{2k}}{1 - \tilde{t}_b t_a^{-2} x^{2k}} \right) \prod_{a=1}^{N_f} \frac{1 - \tilde{t}_b^{-1} t_a^{-1} x^{-2} x^{2k}}{1 - \tilde{t}_b^{-1} t_a^{-2} x^{2k}} \right]
\times \sum_{n=0}^{\infty} \left( -1 \right)^{\left( \tilde{N}_f - N_f \right) n/2} (-w)^n \prod_{k=0}^{\infty} \left( \prod_{a=1}^{N_f} \frac{2 \sinh \frac{-iM_a - iM_b + 2 \mu + 2 \gamma}{2}}{\sinh (k - n) \prod_{a=1}^{N_f} 2 \sinh \frac{iM_a - iM_b + 2 \gamma (1 + k)}{2}} \right) \prod_{a=1}^{N_f} 2 \sinh \frac{-iM_a - iM_b + 2 \mu + 2 \gamma}{2}
\times \sum_{\tilde{n}=0}^{\infty} \left( -1 \right)^{\left( \tilde{N}_f - N_f \right) \tilde{n}/2} (-w)^{-\tilde{n}} \prod_{k=0}^{\infty} \left( \prod_{a=1}^{N_f} \frac{2 \sinh \frac{-iM_a - iM_b + 2 \mu + 2 \gamma}{2}}{\sinh (\tilde{k} - \tilde{n}) \prod_{a=1}^{N_f} 2 \sinh \frac{iM_a - iM_b + 2 \gamma (1 + k)}{2}} \right) \prod_{a=1}^{N_f} 2 \sinh \frac{-iM_a - iM_b + 2 \mu + 2 \gamma}{2}

(3.9)

For \( N_f = \tilde{N}_f \) the integrand also has poles both at the origin and at infinity. The residue at the origin is given by

\[
\text{Res}(\ldots, 0) = e^{N_f |m|} x^{-N_f |m|} \lim_{z \to 0} \prod_{a=1}^{N_f} \prod_{k=0}^{\infty} \frac{z - t_a^{-1} \tau^{-1} x^{|m|} + 2 + 2k}{1 - z \tilde{t}_a \tau x^{|m|} + 2k} \frac{1 - z \tilde{t}_a^{-1} \tau^{-1} x^{-|m|} + 2 + 2k}{z - \tilde{t}_a^{-1} \tau x^{-|m|} + 2k}
\]

(3.10)

assuming \(|t_a^{-1} \tau^{-1} x^2| < 1\), which doesn’t spoil the original range of parameters that we already assumed. Since the residues at the other poles are the same as those for \( N_f \neq \tilde{N}_f \), both results for \( N_f > \tilde{N}_f \) and \( N_f < \tilde{N}_f \) even work for \( N_f = \tilde{N}_f \). One can also check the result is reduced to the known result of 2d partition function of \( \mathcal{N} = 2 \) theories in the small radius limit of \( S^3 \) [21, 22]. The same is true of non-abelian cases, which we will summarize in the next subsection.

### 3.2 Factorization: summary of nonabelian cases

Now we summarize the factorized index formula for non-abelian cases in the presence of CS terms. The superconformal index in the presence of nonzero Chern-Simons term is written as

\[
I(x, t, w, \kappa) = \sum_{\bar{m} \in \mathbb{Z}^N / S_N} \oint dz_j \prod_{j} \frac{1}{2\pi i z_j} |W_m| w^{\sum_j m_j e^{-S_{CS}(a, m)} Z_{gauges}(x, z, m)} \prod_{\Phi} Z_{\Phi}(x, t, z, m)
\]

(3.11)

where \( S_{CS}^{(0)}(a, m) = \frac{i}{2} \sum_{\rho \in R_F} \kappa \rho(m) \rho(a) \). The Chern-Simons term with level \( \kappa \) induces the classical action term in the path integral. It leads to the pole at \( z_i = 0 \) or \( z_i = \infty \) according to the sign of \( \kappa m \). As shown in the appendix, one can show that the
residues at these poles are zero. The contour integral over the holonomy variables of the gauge group can be written as

\[
I^{N_f,N_f}(x, t, \tilde{t}, w, \kappa) = \frac{1}{N!(N_f-N)!} \sum_{\sigma(t)} \prod_{i=1}^{N} \frac{2 \sinh \frac{i M_i - i M_j}{2}}{\prod_{a=1(\neq j)}^{N_f}} \left[ \prod_{a=1}^{N_f} 1 - t_j t_a^{-1} x_a^{2k+2} \right] \left[ \prod_{a=1}^{N_f} \frac{1 - t_j t_a^{-1} x_a^{2k+2}}{1 - t_j t_a^{-1} x_a^{2k+2}} \right]
\]

where \( n = \sum_j n_j, \tilde{n} = \sum_j \tilde{n}_j \) and \( \sigma(t) \) denotes the permutation of \( t_a \)'s. Here the perturbative and vortex contributions are given by

\[
I_{\{n_j\}}(x, t, \tilde{t}, \tau, \kappa) = (-1)^{-\kappa n - (N_f-N) n/2} e^{i \pi \sum_j (M_j n_j + m_j n_j + \eta n_j^2)} \prod_{j=1}^{N_f} 2 \sinh \frac{i M_i - i M_j}{2} \prod_{a=1(\neq j)}^{N_f} \frac{1 - t_j t_a^{-1} x_a^{2k+2}}{1 - t_j t_a^{-1} x_a^{2k+2}} \]

where the prefactor depending on \( \kappa \) in the vortex part appears due to the nonzero Chern-Simons term.

### 3.3 Factorization of mirror of one free chiral

Let us consider a \( U(1) \) theory with a single chiral multiplet in the fundamental representation. If one also turns on the level \( -\frac{1}{2} \) CS interaction and the fixed background magnetic flux \( m' \) corresponding to the topological global symmetry \( U(1)_T \), the index is given by

\[
I(x, w', m') = \sum_{m \in \mathbb{Z}} \oint \frac{dz}{2\pi i z} w^m z^{m'} x^{\frac{1}{2}|m|} (-z)^{\frac{1}{2}(m-|m|)} \prod_{k=0}^{\infty} \frac{1 - z^{-1} x^{m+2\Delta_+ + 2k}}{1 - z x^{m+2\Delta_+ + 2k}}. \tag{3.14}
\]

This can be written as

\[
I(x, w, m') = \sum_{m \in \mathbb{Z}} \oint \frac{dz}{2\pi i z} w^m z^{m'} (-x)^{-\frac{1}{2}(m-|m|)} z^{\frac{1}{2}(m-|m|)} \prod_{k=0}^{\infty} \frac{1 - z^{-1} x^{m+2\Delta_+ + 2k}}{1 - z x^{m+2\Delta_+ + 2k}} \tag{3.15}
\]

where we redefined \( w' x^{\frac{1}{2}} \to w \). A factor \( x^{\frac{1}{2}} \) is additionally absorbed to \( w \) for later convenience.

One may consider its mirror description, a single free chiral theory with the level \( \frac{1}{2} \) CS interaction. As introduced in the previous section the index for the mirror description is given by [23]

\[
\mathcal{I}_\Delta(m; \eta, \zeta) = (-q \zeta) \prod_{r=0}^{\infty} 1 - q^{r+\frac{1}{2}|m+1|} \zeta^{-1} \tag{3.16}
\]
where \( m \) and \( \zeta \) are magnetic flux and the Wilson line of the fixed background \( U(1) \) vector field coupling to the conserved current of the \( U(1) \) flavor symmetry. Again the parameters are identified with ours as follows:

\[
m = m', \quad q = x^2, \quad \zeta = w.
\] (3.17)

It was argued in [23] that the index (3.15) agrees with (3.16). Here we revisit the index agreement using the factorized form of the index. The factorized form of (3.15) is given as follows:

\[
I(x = e^{-\gamma}, w, m') = \sum_{n=0}^{\infty} \left[ \frac{w^n x^{-m'n} - \frac{n(n+1)}{2}}{2} \left( \prod_{k=1}^{n} \sinh \gamma k \right) \right] \times \sum_{\bar{n}=0}^{\infty} \left[ (-w)^{-\bar{n}} x^{-m'\bar{n} + \frac{n(n+1)}{2}} \left( \prod_{k=1}^{\bar{n}} \sinh \gamma k \right) \right]^{-1}.
\] (3.18)

As before the first summation corresponds to the vortex partition function while the second summation corresponds to the antivortex partition function. One may check that the vortex partition function can be written as a Plethystic exponential:

\[
Z_{\text{vortex}}(x, w, m') \equiv \sum_{n=0}^{\infty} \left[ \frac{w^n x^{-m'n} - \frac{n(n+1)}{2}}{2} \left( \prod_{k=1}^{n} \sinh \gamma k \right) \right] = \exp \left[ \frac{\sum_{n=1}^{\infty} 1}{n} \frac{w^n x^{-m'n}}{1 - x^{2n}} \right].
\] (3.19)

Likewise, the antivortex partition function also has the Plethystic exponential form:

\[
Z_{\text{anti}}(x, w, m') \equiv \sum_{\bar{n}=0}^{\infty} \left[ (-w)^{-\bar{n}} x^{-m'\bar{n} + \frac{n(n+1)}{2}} \left( \prod_{k=1}^{\bar{n}} \sinh \gamma k \right) \right]^{-1} = \exp \left[ -\frac{\sum_{n=1}^{\infty} 1}{n} \frac{x^{-n(m'+2)}}{1 - x^{2n}} \right].
\] (3.20)

On the other hand, it was pointed out in [23] that the free chiral index (2.15) has a more concise form as follows:

\[
I_{\Delta}(m; q, \zeta) = \prod_{r=0}^{\infty} \frac{1 - q^{r-\frac{1}{2}m+1} \zeta^{-1}}{1 - q^{r-\frac{1}{2}m} \zeta}.
\] (3.21)

\(^4\)Compared with the general formula, the power in \( x \) has \( x^{-\frac{n(n+1)}{2}} \) while the general formula appearing at the appendix has \( x^{-\frac{n}{2}} \). The reason is that (3.15) matches with the free theory with zero \( R \)-charge for the free chiral while the standard factorized formula matches with the free chiral with canonical \( R \)-charge. Two expressions are related by the shift \( w \rightarrow wx^{\frac{1}{2}} \).
One can see that the denominator, which comes from the scalar, is exactly the vortex partition function while the numerator, which comes from the fermion, is the antivortex partition function:

\[
Z_{vortex}(q^{\frac{1}{2}}, \zeta, m) = \exp \left[ \sum_{n=1}^{\infty} \frac{1}{n} \frac{\zeta^n q^{-\frac{1}{2} mn}}{1 - q^n} \right] = \prod_{r=0}^{\infty} \frac{1}{1 - q^{r - \frac{1}{2} m} \zeta},
\]

\[
Z_{anti}(q^{\frac{1}{2}}, \zeta, m) = \exp \left[ - \sum_{n=1}^{\infty} \frac{1}{n} \frac{\zeta^{-n} q^{(-\frac{1}{2} m + 1)n}}{1 - q^n} \right] = \prod_{r=1}^{\infty} 1 - q^{r - \frac{1}{2} m + 1} \zeta^{-1}.
\]

### 3.4 Relation to topological open string amplitude

The form of the vortex partition function has the close relation to the topological open string amplitude. As the first example we consider the vortex partition function for \(U(1)\) gauge theory with Chern-Simons level \(-\frac{1}{2}\) with a single chiral multiplet. As already explained at the previous subsection, the vortex partition function is given by

\[
\sum_{n=0}^{\infty} w^n x^{\frac{n(n+1)}{2}} \prod_{k=1}^{n} (2\sinh \gamma k)^{-1} = \sum_{n=0}^{\infty} w^n (1 - e^{-2\gamma})(1 - e^{-4\gamma}) \cdots (1 - e^{-2n\gamma})
\]

with \(x = e^{-\gamma}\). Now consider the topological open string for a Lagrangian brane in \(C^3\) as explained in [30, 24]⁵:

\[
Z_{brane}(z, t, q) = \sum_{\mu} s_{\mu}(z) C_{00\mu}(t, q)
\]

\[
= \sum_{n=0}^{\infty} \frac{t^{\frac{n}{2}} z^n}{(1 - q)(1 - q^2) \cdots (1 - q^n)} = \prod_{i=0}^{\infty} \frac{1}{1 - q^i t^\frac{1}{2} z}.
\]

This coincides with the vortex partition function if we identify \(z = w, t = 1, q = e^{-2\gamma}\). To compare with the index of the free chiral field with the canonical \(R\)-charge we need the shift \(z \to z \sqrt{q}\). Then

\[
|Z_{brane}|^2 = \frac{Z_{brane}(z, q)}{Z_{brane}(\bar{z}, q)} = \prod_{n=1}^{\infty} \frac{1 - z q^{n+\frac{1}{2}}}{1 - \bar{z} q^{n+\frac{1}{2}}}
\]

which coincides with the free chiral index as explained in [24]. To compare with the free chiral index of arbitrary \(R\)-charge or its mirror dual, one simply change the open string modulus \(z \to z^\alpha q^\beta\) for a suitable \(\alpha\). Note that it’s crucial to have Chern-Simons term to match the vortex partition function with the topological open string amplitude.

⁵We use the refined vertex formalism to write down the topological string partition function. For the notation, please refer to [31]. \(s_{\mu}\) in the formula denotes the Schur function.
For this simple example, we generalize the matching between the homological vortex partition function of two dimensions and the topological open string partition function to the full 3d K-theoretic vortex partition function. In \cite{LZ}, many more examples of the matching between the 2d vortex partition function and the topological open string were found. We expect that this surely lifts to the matching between the 3d vortex partition function and the topological open string. Furthermore in the homological version, 2d vortex theory is realized as the surface operator of 4d gauge theories. We expect that this lifts to the 3d defect operator in 5d superconformal field theories. We will work out a simple example in the next subsection.

As a next example, we can consider $U(1)$ gauge theory with one fundamental and one antifundamental chiral theory. As will be shown in the subsection 4.1, the superconformal index of the theory is given by

$$I_{N_c=N_f=1} = \prod_{l=0}^{\infty} \frac{1 - \tau^{-2} x^{2l+2}}{1 - \tau^{2} x^{2l}} \times Z_{N_c=N_f=1}^{\text{vortex}} \times Z_{N_c=N_f=1}^{\text{anti}}. \quad (3.26)$$

Here $N_c$ denotes the rank of the gauge group while $N_f = \tilde{N}_f$ denotes the number of fundamental and antifundamental multiplets. The vortex partition function is given by

$$Z_{\text{vortex}}^{N_c=N_f=1} = \exp \left[ \sum_{n=1}^{\infty} n x^n \frac{\tau^n - \tau^{-n}}{1 - x^{2n}} \right]. \quad (3.27)$$

Note that the vortex part as well as perturbative part is given by the free chiral index. Hence this $U(1)$ theory can again be written in terms of topological open string amplitude.

If one considers the more general $U(1)$ non-chiral theory with $N_f = \tilde{N}_f = N$, the index can be written as

$$I_{N_f=\tilde{N}_f=N} = \sum_{b=1}^{N} \left( Z_{1-\text{loop}}^{b} Z_{\text{vortex}}^{b} \right) \times \left( Z_{1-\text{loop,anti}}^{b} Z_{\text{anti}}^{b} \right). \quad (3.28)$$

where

$$Z_{1-\text{loop}}^{b} = \prod_{k=1}^{\infty} \prod_{a=1(\not= b)}^{N} \frac{1 - t_b t_a^{-1} x^{2k}}{1 - t_a^{2k} x^{2(k-1)}}, \quad (3.29)$$

$$Z_{1-\text{loop,anti}}^{b} = \prod_{k=1}^{\infty} \prod_{a=1(\not= b)}^{N} \frac{1 - t^{-1} t_a^{-1} x^{-2k}}{1 - t_a^{2k} x^{2(k-1)}}, \quad (3.30)$$

$$Z_{\text{vortex}}^{b} = \sum_{n=0}^{\infty} \left[ u^{-N} x^{n} \right] x^n \prod_{k=1}^{N} \frac{1 - t_b t_a^{-1} x^{2(k-1)}}{1 - x^{2k}} \prod_{a=1(\not= b)}^{N} \frac{1 - t_a^{-1} x^{2(k-1)}}{1 - t_a x^{2k}}, \quad (3.31)$$

$$Z_{\text{anti}}^{b} = \sum_{n=0}^{\infty} \left[ u^{-1} \tau^{-N} x^{n} \right] x^n \prod_{k=1}^{N} \frac{1 - t_b t_a^{-1} x^{2(k-1)}}{1 - x^{2k}} \prod_{a=1(\not= b)}^{N} \frac{1 - t_a x^{2(k-1)}}{1 - t_a^{-1} x^{2k}}. \quad (3.32)$$

\footnote{This relation is parallel to 4d Nekrasov partition function and its 5d version.}
The vortex partition function (3.31) is the same as that of [20]. In fact, with identifications

\[ t_b t_a^{-1} = e^{-2\pi b D_{a b}}, \]
\[ t_b^2 t_a^2 \tau^2 = e^{-2\pi b C_{a b}}, \]
\[ x^2 = q, \]
\[ w \tau^{-N} x^N = z, \]

one can see that

\[ Z_{\text{b vortex}} = Z_V^{(b)}, \quad Z_{1\text{-loop}}^b = Z_{1\text{-loop}}^{(b)}, \]

where \( Z^{(b)} \)'s are partition functions for the non-chiral theory given in [20].

As examined in [20] one can also check that the vortex partition function \( Z_{\text{b vortex}}^b \) is exactly the same as the open topological string partition function on the Lagrangian brane placed at the \( b \)-th gauge leg of the toric diagram in Figure 1, i.e., \( \alpha_b \in \{1^n|n = 0, 1 \cdots \} \). The corresponding topological partition function is given by [20, 32]

\[ Z_{\text{top}}^b = \sum_n A_n^{(b)} z^n, \]

\[ A_n^{(b)} = \frac{K_{\cdots 1^n \cdots}}{K_{\cdots \cdots}} = \frac{1}{\prod_{k=1}^n (1 - q^k)} \prod_{a > b} \prod_{k=1}^n (1 - Q_{\alpha_b \beta_a} q^{k-1}) \prod_{a < b} \prod_{k=1}^n (1 - Q_{\alpha_a \alpha_b} q^{(k-1)}) \]

(3.37)

where the Kähler parameters are defined by:

\[ Q_{\alpha_a \alpha_a'} = \prod_{k=a}^{a'-1} M_k F_k, \]
\[ Q_{\alpha_a \beta_a'} = Q_{\alpha_a \alpha_a'} M_a', \]
\[ Q_{\beta_a \alpha_a'} = Q_{\alpha_a \alpha_a'} M_a^{-1}, \]

(3.37)
with $a < a'$. For a fixed $b$, if we identify the parameters as follows:

$$z \prod_{a<b} M_a^{-1} = w T^{-N} x^N,$$

$$Q_{\alpha,\beta} = t_b t_a \tau^2, \quad a \geq b$$

$$Q_{\beta,\alpha}^{-1} = t_b t_a \tau^2, \quad a < b$$

$$Q_{\alpha,\alpha} = t_b t_a^{-1} x^2, \quad a > b$$

$$Q_{\alpha,\alpha}^{-1} = t_b t_a^{-1} x^2, \quad a < b$$

one immediately sees that the topological partition function $Z_{\text{top}}^b$ is the same as the vortex partition function for abelian theories that we obtained, $Z_{\text{vortex}}^b$.

### 3.5 The partition function for $U(1)^N$ quiver theories and closed topological string

More interestingly the closed string geometry on the strip geometry considered in [20], for which $\alpha_b$ is now the trivial representation, has the close relation to the 5d partition function on $S^1 \times S^4$. In this case the 5d gauge theory defined on the strip geometry is $U(1)^N$ quiver theory.

If we consider first the closed string amplitude on the strip geometry it can be worked out using the refined topological vertex. In [33] a similar geometry given in Figure 2 was examined where the leftmost leg and the rightmost leg are identified. If we disconnect that leg we again obtain the strip geometry we are interested in.
The closed string amplitude for Figure 2 is given by

\[
Z_{L}^{\text{inst}} \left( \tilde{Q}; q, t \right) = \sum_{\{\lambda_{2a}\}}^{N} \prod_{s \in \lambda_{2a}} \tilde{Q}_{2a}^{[\lambda_{2a}]} \prod_{s \in \lambda_{2a}} \left( 1 - \tilde{Q}_{2a-1} q^{\ell_{2a-2}(s) a_{2a-2}(s)+1} \right) \prod_{s \in \lambda_{2a}} \left( 1 - \tilde{Q}_{2a-1} q^{-\ell_{2a}(s)-1 a_{2a}-2(s)} \right)
\]

where \( \tilde{Q}_{\alpha} = Q_{\alpha} \left( \frac{q}{\bar{q}} \right)^{-1} \). As in the other examples of geometric engineering, this closed topological string amplitude leads to the Nekrasov instanton partition function. In 5-dimensions full instanton partition function was not worked out for theories with bifundamental fields. However Nekrasov partition function of such quiver in four-dimension was worked out in [34]. One can see that this can be obtained from the closed string amplitude. The unrefined version of the amplitude is obtained by setting \( t = q \):

\[
Z_{L}^{\text{inst}} \left( \tilde{Q}_{\alpha}; q, q \right) = \sum_{\{\lambda_{2a}\}}^{N} \prod_{s \in \lambda_{2a}} \left( 1 - \tilde{Q}_{2a+1} q^{\ell_{2a}(s)+a_{2a+2}(s)+1} \right) \prod_{s \in \lambda_{2a}} \left( 1 - \tilde{Q}_{2a-1} q^{-\ell_{2a}(s)-a_{2a-2}(s)-1} \right)
\]

\[
= \sum_{\{\lambda_{2a}\}}^{N} \prod_{s \in \lambda_{2a}} \left( \tilde{Q}_{2a}^{1/2} \tilde{Q}_{2a+1}^{1/2} \tilde{Q}_{2a-1}^{1/2} \right)^{[\lambda_{2a}]} q^{\sum_{s \in \lambda_{2a}} a_{2a+2}(s)-a_{2a-2}(s)/2}
\]

\[
\times \prod_{s \in \lambda_{2a}} \sinh \frac{\beta}{2} [h_{h_{2a},2a+2}(s) + M_{2a+1}] \sinh \frac{\beta}{2} [h_{h_{2a},2a-2}(s) - M_{2a-1}] \]

where we defined that \( \tilde{Q}_{\alpha} \equiv e^{-\beta M_{\alpha}} \), \( q \equiv e^{-\beta h} \). \( \beta M_{\alpha} \) and \( \beta h \) here are the five-dimensional chemical potentials while \( M_{\alpha} \) and \( h \) are the four-dimensional parameters. \( h_{\alpha,\beta}(s) \) is the hook length defined by \( h_{\alpha,\beta}(s) = \ell_{\alpha}(s) + a_{\beta}(s) + 1 \). In order to obtain the four-dimensional partition function, one would take \( \beta \to 0 \):

\[
Z_{L}^{\text{inst}} \left( \tilde{Q}_{\alpha}; q, q \right) \bigg|_{\beta \to 0} = \sum_{\{\lambda_{2a}\}}^{N} \prod_{s \in \lambda_{2a}} \frac{[h_{h_{2a},2a+2}(s) + M_{2a+1}] [h_{h_{2a},2a-2}(s) - M_{2a-1}]}{[h_{h_{2a},2a}(s)]^{2}}
\]

which is the same as the partition function for quiver theories given in [34] with identifications \( M_{2a+1} = a_{2a} - a_{2a+2} + m \) where \( m \) denotes the mass of the bifundamentals.\(^8\) Thus it is quite reasonable that the above topological string amplitude

\(^7\)If we consider the 2d partition \( \lambda = \{ \lambda_{1} \geq \lambda_{2} \cdots \} \), this can be represented by a Young diagram. We draw the \( \lambda_{1} \) boxes on the leftmost column and \( \lambda_{2} \) boxes on the next-leftmost column and so on. For an element \( s = (i, j) \in \lambda \), \( a(s) \) denotes the boxes on the right and \( l(s) \) denotes the boxes on top, i.e., \( a(i,j) = \lambda_{j} - i \), \( l(i,j) = \lambda_{i} - j \). For more details, refer to [31].

\(^8\)The hook length \( h_{\alpha,\beta}(s) \) is denoted by \( \ell_{\alpha}(s) \) in [34].
gives the 5d Nekrasov partition function for $U(1)^N$ quiver theories. One can cut the leftmost leg and the the rightmost leg, which are identified, by taking $Q_{2N} \to 0$. This give rise to the closed string amplitude for the strip geometry we original considered. The amplitude is given by

\[
Z_{5d}^{inst}(\tilde{Q}; q, t) \big|_{Q_{2N} \to 0} = \sum_{\{\lambda_{2a}\}}^{N-1} \prod_{a=1}^{\lambda_{2a}} \prod_{s \in \lambda_2} \left(1 - \tilde{Q}_1 q^{-\ell_1(s)-1} t^{-a_1(s)}\right) \prod_{s \in \lambda_2 N-2} \left(1 - \tilde{Q}_2N-1 q^{-\ell_2N-2(s)-1} t^{-a_2(s)+1}\right) \\
\times \prod_{\alpha=2}^{N-1} \prod_{s \in \lambda_{2a-2}} \left(1 - \tilde{Q}_2 a_{\alpha-2} q^{-\ell_2 a_{\alpha-2}(s)-1} t^{-a_2(s)+1}\right) \prod_{s \in \lambda_{2a}} \left(1 - \tilde{Q}_2 a_{\alpha} q^{-\ell_2 a_{\alpha}(s)-1} t^{-a_2(s)}\right) \prod_{\alpha=1}^{N-1} \prod_{s \in \lambda_{2a}} \left(1 - q^{\ell_2 a_{\alpha}(s)} t^{a_2(s)+1}\right) \prod_{s \in \lambda_{2a}} \left(1 - q^{-\ell_2 a_{\alpha}(s)-1} t^{-a_2(s)}\right)
\]

(3.43)

where $a_0(s = (i, j)) = -i$.

One might wonder since abelian theory is trivial in 5d so that its nonperturbative part is also trivial. However abelian theories can have small instantons and it’s quite subtle how to include them. For example if we consider 5d $U(1)$ theory and if we define its nonperturbative completion to give the Nekrasov partition function, 5d partition function of $U(1)$ theory on $S^{5}$ gives the index of single M5 brane in 6d [35].

The general structure of the 5d index worked out at [36] has the structure

\[
\int da PE(f_{mat}(x, y, e^{ia}, t) + f_{vec}(x, y, e^{ia})) |I_{inst}(x, y, e^{ia}, t, q)|^2
\]

(3.44)

where $da$ is the Haar measure for the gauge group, $PE$ denotes Plethystic exponential, which gives the one-loop determinant and $I_{inst}$ is Nekrasov instanton partition function. $x, y$ is the chemical potential for Cartans of Lorentz symmetry $SU(2)_1 \times SU(2)_2 \subset SO(5)$, $x = e^{-\gamma_1}, y = e^{-\gamma_2}$ and $t$ is the usual chemical potential for the flavor symmetry. Here $SU(2)_1$ is also twisted with $SU(2)_R$ R-symmetry. Finally $q$ is introduced to track the instanton number. Thus for $U(1)^N$ quiver 5d partition function has the same form where $I_{inst}$ is now identified with closed string amplitude. This is consistent with the recent proposal by [24].

In addition, the perturbative part is also factorized and the whole index can be written as

\[
I = \int da |I_{pert}(x, y, e^{ia}, t) I_{inst}(x, y, e^{ia}, t, q)|^2.
\]

(3.45)

Now let’s check if perturbative part matches. In the refined vertex formalism, the preturbative part is automatically built in. In our case, it is given by [33]

\[
Z_{pert} = \exp \left(-\sum_{n=1}^{\infty} \frac{1}{n} \left(\sum_{\alpha} \tilde{Q}_{2\alpha-1}^{\alpha} \right) \left(\frac{1}{t^{n+\frac{1}{2}} - t^{-\frac{1}{2}}} \right) \left(\frac{1}{q^{n+\frac{1}{2}} - q^{-\frac{1}{2}}} \right)\right)
\]

(3.46)
This should match the one-loop determinant of the bifundamental fields. The general expression for the one-loop determinant for the matter fields are given by

\[ f_{\text{mat}}(x, y, a) = \frac{x}{(1 - xy)(1 - x/y)} \sum_w (e^{-i\vec{w} \cdot \vec{a}} + e^{i\vec{w} \cdot \vec{a}}) \]  
(3.47)

where \( w \) is the weight of the representation. We suppress the chemical potential of the flavor symmetry. We can see that it has the explicit factorized structure and we can just look for \( e^{-i\vec{w} \cdot \vec{a}} \) part to compare with the topological string expression. For bifundamentals, we have

\[ f_{\text{pert}} = \frac{x}{(1 - xy)(1 - x/y)} (e^{-i(\alpha_1 - \alpha_2)} + e^{-i(\alpha_2 - \alpha_3)} + \ldots + e^{-i(\alpha_N - \alpha_1)}). \]  
(3.48)

This coincides with the corresponding topological string expression if we identify

\[ x = \sqrt{tq}, \quad y = \sqrt{q/t}, \quad \tilde{Q}_{2k-1} = e^{-i(\alpha_k - \alpha_{k-1})}. \]  
(3.49)

Furthermore since the open string amplitude was obtained by introducing the Lagrangian brane in the strip geometry, and this leads to the 3d index of the nontrivial SCFT, it is natural to expect that introducing Lagrangian brane corresponds to introducing the surface operator in 5d \( U(1)^N \) quiver theory. This is the T-dual of the Hanany-Witten set up for the surface operator in 4-dimension so we expect this is the 3d defect of the 5d theory.

This lead to an interesting lesson that apparently trivial 5d theory\(^9\) can have nontrivial defect operator, which corresponds to nontrivial 3d SCFT. Furthermore we saw that the partition function of 5d theory with the defect operator matches the closed+open string amplitude since the the vertex partition function appearing at [20], is normalized by the closed string partition function. The vortex partition function has the structure

\[ Z^b_{\text{vortex}} = \sum_n z^n \frac{K(1^n)}{K(0)} \]  
(3.50)

where \( K(1^n) \) is the string partition function with the insertion of the brane with the representation \( (1^n) \) while \( K(0) \) denotes the string partition function with the trivial representation, i.e., the closed string partition function.

4. \( \mathcal{N} = 2 \) Seiberg-like dualities

4.1 Simple cases

In this section we consider Seiberg-like (or Aharony duality) for three dimensional \( U(N) \) gauge theories with \( \mathcal{N} = 2 \) supersymmetries proposed in [37]. The duality

\(^9\)One way to see the 5d index computation of the this theory is to regard it as a twisted partition function on \( S^1 \times S^4 \).
relates two gauge theories which we call the “original” theory and the “dual” theory. Two dual theories have different gauge groups and matter contents but they flow to the same theory in the infrared.

The original theory is a $U(N)$ gauge theory which consists of $N_f$ fundamental chiral multiplets $Q_a$ and $N_f$ anti-fundamental chiral multiplets $\bar{Q}^a$ as well as $U(N)$ vector multiplets. This theory has no superpotential. On the other hand, the dual theory is a $U(N_f - N)$ gauge theory with $N_f$ pairs of fundamental $q^a$ and anti-fundamental $\bar{q}^a$ chiral multiplets. In addition, the dual theory contains gauge singlet chiral multiplets, $M_{ab}$ and $V^\pm$, and they couple to the charged matters through the superpotential, $W = q^a M_{ab} \bar{q}^b + V^+_\pm \bar{V}^-_\pm$. Here $\bar{V}^\pm_\pm$ are chiral superfields corresponding to monopole operators which parametrize the Coulomb branch of the dual theory. The global symmetry of both theories is $SU(N_f) \times SU(N_f)$ flavor symmetry acting on the fundamental and anti-fundamental matters, respectively, times $U(1)_A \times U(1)_T$, where $U(1)_A$ is an axial symmetry rotating fundamental and anti-fundamental matters by the same phase and $U(1)_T$ is a topological symmetry whose current is given by $\ast \text{Tr} F$.

Under the duality, mesonic operators $Q_a \bar{Q}^b$ and monopole operators with topological charges $\pm 1$ of the original theory are mapped to singlet fields $M_{ab}$ and $V^\pm_\pm$ of the dual theory, respectively. This duality map together with the superpotential $W$ determines global charge assignment of chiral fields of the dual theory.

The superconformal indices for several dual pairs have been computed. The indices are expanded by conformal dimensions of BPS operators and show agreement between BPS spectra of two dual theories at some leading orders. Here we present factorized representations of superconformal indices for simple cases that shows 3d Seiberg-like dualities in a clearer way.

Let us first consider the $U(1)$ gauge theory with $N_f = 1$ flavor which would give the simplest duality model. The proposed dual theory is the $U(0)$ theory, i.e. non-gauge theory, with chiral multiplets $M$ and $V_\pm$. After vortex-antivortex factorization, the superconformal index of the original theory is given by

$$I^{N=N_f=1} = \prod_{l=0}^{\infty} \frac{1 - \tau^{-2} x^{2l+2}}{1 - \tau^2 x^{2l}} \times Z^{N=N_f=1}_{\text{vortex}} \times Z^{N=N_f=1}_{\text{anti}}. \quad (4.1)$$

The vortex index is the sum over all vortex number $n$’s. After some calculation it can be written as a Plethystic exponential form

$$Z^{N=N_f=1}_{\text{vortex}} = \sum_{n=0}^{\infty} (-w)^n \prod_{k=1}^{n} \frac{\tau^{-1} x^{-k} - \tau x^{k-1}}{x^{-(k-1)-n} - x^{k-1-n}} = \exp \left[ \sum_{n=1}^{\infty} \frac{1}{n} w^n \frac{(\tau^n - \tau^{-n}) x^n}{1 - x^{2n}} \right] \quad (4.2)$$

We checked the last identity up to the order $\mathcal{O}(w^9)$. The anti-vortex index is easily obtained from the vortex index by replacing $w$ to $w^{-1}$. Moreover it turns out that the
superconformal index of $N = N_f = 1$ theory can be rewritten as a simple Plethystic exponential form

$$I^{N=N_f=1} = \exp \left( \sum_{n=1}^{\infty} \frac{1}{n} f(x^n, \tau^n, w^n) \right) ,$$  \hspace{1cm} (4.3)

$$f(x, \tau, w) = \frac{\tau x^{2\Delta_Q} - \tau^{-2} x^{2-2\Delta_Q}}{1-x^2} + \frac{\tau^{-1} x^{1+\Delta_Q} - \tau x^{1+\Delta_Q}}{1-x^2} (w + w^{-1}) ,$$

where we restored $R$-charge $\Delta_Q$ of the chiral boson $Q$ of the original theory. Amazingly this form of the index is exactly the same as the superconformal index of the dual theory. The function $f$ is identical to the single letter index in the dual theory. As the chiral field $M$ of the dual theory is identified with the meson operator $\bar{Q}Q$ of the original theory, its $R$-charge and $U(1)_A$ charge are $2\Delta_Q$ and $+2$ respectively, and therefore the letter index of $M$ is given by the first term of $f$. The second term of $f$ comes from the letter contribution of dual chiral multiplets $V_{\pm}$ which is mapped to monopole operators with $U(1)_T$ charges $\pm 1$. In general, zero point energies and $U(1)_A$ charges of monopole operators with GNO charge $(\pm 1, 0, \ldots , 0)$ for $N = N_f$ theories are

$$\epsilon_0 = N_f (1-\Delta_Q) - (N-1) = 1 - N\Delta_Q ,$$  \hspace{1cm} (4.4)

$$b_{U(1)_A} = -N_f = -N$$

from (2.4). One can then see that the single letter index of $V_{\pm}$ for $N = N_f = 1$ case agrees with the second term of $f$.

This theory is known to be mirror-dual to the XYZ theory [38]. The chiral fields $M, V_{\pm}$ in the dual theory correspond to $X, Y, Z$ fields of the superpotential $W = -MV_{+}V_{-}$, so that they should have $R$-charges $\Delta_M = \Delta_V = \frac{2}{3}$. As shown in [39], the $R$-charge of the original chiral field is determined to be $\Delta_Q = \frac{1}{3}$ in IR, and therefore one can see from (4.3) that the dual chiral fields have the correct $R$-charges in the IR fixed point.

More generally, one can express the superconformal indice for $U(N)$ gauge theories with $N_f = N$ fundamental and anti-fundamental matters in duality manifest forms using the factorization. The dual theory is a $U(0)$ theory with chiral multiplet $M_a^b$ and $V_{\pm}$. The vortex indices for $N = N_f$ theories reduce to Plethystic exponential forms

$$Z_{\text{vortex}}^{N=N_f} = \sum_{n=0}^{\infty} \sum_{i,j} \prod_{n=1}^{N} \prod_{k=1}^{N} l_i^{-1/2} l_j^{-1/2} x^{-(k-1)} - l_i^{-1/2} l_j^{-1/2} x^{k-1} \tau x^{k-1/2} \tau x^{k-1/2} w^{-n} \left( \sum_{n=1}^{\infty} \frac{1}{n} w^n \right) ,$$

\hspace{1cm} (4.5)

We explicitly checked the last identity for some low values of $n$ and $N$. Together with the antivortex partition function, this can be interpreted as the multi-particle index
for singlet chiral fields \( V_\pm \) of the dual theory. All of the \( t \) dependence are cancelled out, which is expected since \( V_\pm \) are the flavor singlets. Restoring \( R \)-charge by shifting \( \tau \to \tau x^{\Delta Q} \), one can check the chiral field \( V_\pm \) has correct \( R \)-charge, \( 1 - N\Delta Q \), and \( U(1)_A \) charge, \( -N \). Then the superconformal index after combining the perturbative part can also be rewritten as duality manifest form

\[
I^{N=N_f} = \prod_{i,j} \prod_{l=0}^{\infty} \frac{1 - t_l^{-1}t_j^{-1} - t_l^{-1}}{1 - t_l^{-1}t_j^{-1} - t_l^{-1}t_j^{-1}}^2 \times Z_{vortex}^{N=N_f} \times Z_{anti}^{N=N_f}
\]

\[
= \exp \left[ \sum_{n=1}^{\infty} \frac{1}{n} f_{N=N_f}(x^n, t_1^n, t_2^n, x^n, w^n) \right], \tag{4.6}
\]

\[
f_{N=N_f} = \sum_{i,j}^{N} t_l^{-1} t_j^{-1} x^{2\Delta Q} - t_l^{-1} t_j^{-1} x^{-2-2\Delta Q} \frac{1}{1 - x^2} + \frac{\tau - N x^{1-N\Delta Q} - \tau N x^{1+N\Delta Q}}{1 - x^2}(w + w^{-1})
\]

This precisely agrees with superconformal index of the dual theory with \( N \times N \) chiral fields \( M_{ij} \) and two chiral fields \( V_\pm \). When \( N > 1 \), the dual theories flow to free theories. One can check it first for \( N = 2 \), where the \( Z \)-extrimization of [39] determines \( R \)-charge of the original chiral fields as \( \Delta_Q = \frac{1}{2} \). Then \( R \)-charges of the dual chiral fields are fixed to be \( \Delta_M = \Delta_V = \frac{1}{2} \) and so the dual theory is obviously free. For \( N > 2 \), it seems to be impossible to have free dual theory by adjusting the original \( R \)-charge \( \Delta_Q \). However, as we see from the index formula (4.6), the index of the IR conformal theory is written as that of non-interacting free fields and therefore IR degrees of freedom can carry new \( U(1) \) charges for accidental symmetry which emerges only at the IR fixed point. The UV \( R \)-symmetry then mixes with this extra \( U(1) \) symmetry so that the dual chiral fields \( M, V_\pm \) become free fields in IR.

Now we consider further generalization to \( N_f > N \) theories. Unlike the previous cases which are mostly free theories, dual theories are now interacting gauge theories. Let us first consider \( U(1) \) gauge theory with \( N_f = 2 \) pairs of fundamental and anti-fundamental matters. The dual theory is also \( U(1) \) gauge theory with \( N_f = 2 \) flavors, but has additional \( 2 \times 2 \) chiral fields \( M_{ab} \) and two chiral fields \( V_\pm \). The superconformal index of the original theory is

\[
I^{(1,2)} = \sum_{\sigma(t)} \prod_{a=1}^{2} \left[ 1 - t_1^{-1} x^{2l+2} - t_2^{-1} x^{2l+2} \right] \times Z_{vortex}^{(1,2)}(x, \sigma(t), \hat{t}, \tau, w) \times Z_{anti}^{(1,2)}(x, \sigma(t), \hat{t}, \tau, w^{-1}),
\]

\[
Z_{pert}^{(1,2)} = \prod_{l=0}^{\infty} \left[ 1 - t_1^{-1} t_2^{-l} x^{2l+2} \right] \times \left[ 1 - t_2^{-1} t_1^{-l} x^{2l+2} \right],
\]

\[
Z_{vortex}^{(1,2)} = \sum_{n=0}^{\infty} (-w)^n \prod_{k=1}^{n} \prod_{a=1}^{2} \left( t_1^{-1} t_a^{-1} x^{-(k-1-n)} - x^{-k-1-n} \right) \left( t_1^{-1} t_2^{-1} x^{-(k-1-n)} - t_2^{-1} t_1^{-1} x^{-(k-1-n)} \right) \tag{4.7}
\]

and \( Z_{anti}^{(1,2)} = Z_{vortex}^{(1,2)}(w \to w^{-1}) \) where \( I^{(N,N_f)} \) denotes the index of the original theory with \( U(N) \) gauge group and \( N_f \) flavors, and \( \sigma(t) \) runs over permutations of \( \{t_1, t_2\} \).
In fact this index also has the duality manifest expression. The perturbative part \(Z_{\text{pert}}^{(1,2)}\) with exchange of \(t_a\)'s can be rewritten as

\[
Z_{\text{pert}}^{(1,2)}(t_1 \leftrightarrow t_2) = \prod_{l=0}^{\infty} \prod_{a,b}^{2} \frac{1 - t_a^{-1} t_b \tau^{-2} x^{2l+2} + 2 \Delta_Q}{1 - t_a^{-1} t_b \tau^{-1} x^{2l+2} + 2(1 - \Delta_Q)} \times \frac{1 - t_a^{-1} t_b^{-1} \tau^{-2} x^{2l+2} - 2 \Delta_Q}{1 - t_a^{-1} t_b^{-1} \tau^{-1} x^{2l+2} + 2(1 - \Delta_Q)}
\]

where \(\tilde{Z}_{\text{pert}}^{(1,2)} \equiv Z_{\text{pert}}^{(1,2)}(t \to t^{-1}, \tilde{t} \to \tilde{t}^{-1}, \tau \to \tau^{-1}, \Delta_Q \to 1 - \Delta_Q)\). We shall identify \(\tilde{Z}_{\text{pert}}^{(1,2)}\) to the perturbative part of charged chiral fields \(q^a, \tilde{q}_a\) in the dual theory. Also the second infinity product term in the second line of (4.8) will be identified with the index contribution of the meson field \(M_a^b\) of the dual theory. Similarly, we define the dual vortex index as \(Z_{\text{vortex}}^{(1,2)} \equiv Z_{\text{vortex}}^{(1,2)}(t \to t^{-1}, \tilde{t} \to \tilde{t}^{-1}, \tau \to \tau^{-1}, \Delta_Q \to 1 - \Delta_Q)\) and find that

\[
Z_{\text{vortex}}^{(1,2)}(t_1, t_2) = \tilde{Z}_{\text{vortex}}^{(1,2)}(t_2, t_1) \times \exp \left[ \sum_{n=1}^{\infty} \frac{1}{n} w^n \frac{\tau^{-2n, x^{2n}(1 - \Delta_Q) - \tau^{2n, x^{2n} \Delta_Q}}}{1 - x^{2n}} \right]
\]

We checked this identity by expanding both sides with vortex number up to \(O(w^6)\). The Plethystic exponential term on the right hand side corresponds to the index contribution from chiral fields \(V_\pm\) of the dual theory. Finally, collecting all the result, the original index becomes

\[
I^{(1,2)} = \tilde{I}^{(1,2)} \times \exp \left[ \sum_{n=1}^{\infty} \frac{1}{n} f^{(1,2)}(x^n, t^n, \tilde{t}^n, \tau^n, w^n) \right]
\]

where

\[
\tilde{I}^{(1,2)} = \sum_{\sigma(t)} \tilde{Z}_{\text{pert}}^{(1,2)}(\sigma(t)) \times \tilde{Z}_{\text{vortex}}^{(1,2)}(\sigma(t)) \times \tilde{Z}_{\text{anti}}^{(1,2)}(\sigma(t)),
\]

\[
f^{(1,2)} = \sum_{a,b}^{2} \frac{t_a \tilde{t}_b^{-1} \tau^{-2} x^{2\Delta_Q} - t_a^{-1} \tilde{t}_b^{-1} \tau^{-2} x^{2-2\Delta_Q}}{1 - x^2} \times \frac{\tau^{-2} x^{2(1 - \Delta_Q) - \tau^2 x^2 \Delta_Q}}{1 - x^2}(w + w^{-1})
\]

This is exactly the same as the superconformal index of the dual theory, which is a \(U(1)\) gauge theory with charged chiral multiplets \(q^a, \tilde{q}_a\), singlet chiral multiplets \(M_a^b\) and \(V_\pm\). The superpotential \(W\) implies that \(R\)-charges for \(q, \tilde{q}\) are \(1 - \Delta_Q\) and other charges are opposite to \(Q, \tilde{Q}\) of the original theory. Thus the index \(\tilde{I}^{(1,2)}\) encodes the contributions from the chiral multiplets \(q, \tilde{q}\). One can also check that the single letter index \(f^{(1,2)}\) represents the letter indices for \(M_a^b\) and \(V_\pm\) with correct \(R\)-charge and global charges.
4.2 General cases

One can generalize the \( N = 1, N_f = 2 \) example in the previous subsection to general \( N,N_f \) in the same way. The index contribution of the singlet matters \( M_a^b, V_x \) is straightforward, and the contribution of \( q, \bar{q} \) is obtained by replacing \( N \rightarrow N - N_f, t, \tilde{t} \rightarrow t^{-1}, \tilde{t}^{-1}, \tau \rightarrow \tau^{-1} \) in the original index. Thus, the superconformal index for the dual theory is given by

\[
I(x = e^{-\tau}, t = e^{iM}, \tilde{t} = e^{i\bar{M}}, \tau = e^{\mu}, w) = \prod_{b=1}^{\infty} \left( \prod_{a,b=1}^{N_f} \frac{1-t_a^{-1} t^{-1}_b \tau^{-2} x^{2+2k}}{1-t_a t^{-1}_b \tau^2 x^{2k}} \right) \left( \prod_{k=1}^{N_f} \frac{1-w^{-1} t^{-1}_a \tau^{-1} x^{-1} N_f + N + 2k}{1-w^{-1} t^{-1}_a \tau^{-1} x^{-1} N_f - N + 1 + 2k} \right) \times \sum_{1 \leq b_1 < \cdots < b_{N_f-N} \leq N_f} \left( \prod_{j=1}^{N_f-N} \right) - 2 \sinh \left( \frac{iM_{b_j} - iM_{b_j'}}{2} \right) \right)
\]

where \( \{b_j\}^c = \{1, \ldots, N_f\} - \{b_1, \ldots, b_{N_f-N}\} \). The second line comes from the singlet matters \( M_a^b \) and \( V_x \). The fourth line is the perturbative part coming from \( q^a \) and \( \bar{q}_a \), which will be called \( Z_{\text{pert}} \). The last two lines are vortex and antivortex parts, which will be called \( Z_{\text{vortex}} \) and \( Z_{\text{anti}} \).

With a little algebra one can show that the following expression holds:

\[
\prod_{a,b=1}^{\infty} \left( \prod_{k=0}^{\infty} \left( \prod_{a,b=1}^{N_f} \frac{1-t_a^{-1} t^{-1}_b \tau^{-2} x^{2+2k}}{1-t_a t^{-1}_b \tau^2 x^{2k}} \right) \left( \prod_{k=1}^{N_f} \frac{1-w^{-1} t^{-1}_a \tau^{-1} x^{-1} N_f + N + 2k}{1-w^{-1} t^{-1}_a \tau^{-1} x^{-1} N_f - N + 1 + 2k} \right) \times \sum_{1 \leq b_1 < \cdots < b_{N_f-N} \leq N_f} \left( \prod_{j=1}^{N_f-N} \right) - 2 \sinh \left( \frac{iM_{b_j} - iM_{b_j'}}{2} \right) \right)
\]

\[
= \left( \prod_{a,b=1}^{\infty} \frac{1-t_a^{-1} t^{-1}_b \tau^{-2} x^{2+2k}}{1-t_a t^{-1}_b \tau^2 x^{2k}} \right) \left( \prod_{a,b=1}^{\infty} \frac{1-w^{-1} t^{-1}_a \tau^{-1} x^{-1} N_f + N + 2k}{1-w^{-1} t^{-1}_a \tau^{-1} x^{-1} N_f - N + 1 + 2k} \right) \times \left( \prod_{a,b=1}^{\infty} \frac{1-t_a^{-1} t^{-1}_b \tau^{-2} x^{2+2k}}{1-t_a t^{-1}_b \tau^2 x^{2k}} \right) \left( \prod_{a,b=1}^{\infty} \frac{1-w^{-1} t^{-1}_a \tau^{-1} x^{-1} N_f + N + 2k}{1-w^{-1} t^{-1}_a \tau^{-1} x^{-1} N_f - N + 1 + 2k} \right) \times \sum_{1 \leq b_1 < \cdots < b_{N_f-N} \leq N_f} \left( \prod_{j=1}^{N_f-N} \right) - 2 \sinh \left( \frac{iM_{b_j} - iM_{b_j'}}{2} \right) \right)
\]

for an arbitrary subset \( \{b_j\} \subset \{1, \ldots, N_f\} \). It teaches us that we can write the
perturbative part of the original index as follows:

\[
Z^{(b_j)}_{\text{pert}}(x, t, \tilde{t}, \tau) = \tilde{Z}^{(b_j)c}_{\text{pert}}(x, t, \tilde{t}, \tau) \times \left( \prod_{a,b=1}^{N_f} \prod_{k=0}^{\infty} \frac{1 - t_a^{-1} t_b^{-1} \tau^{-2} x^{2+2k}}{1 - t_a^{-1} t_b \tau^2 x^{2k}} \right)
\]

\[
= \tilde{Z}^{(b_j)c}_{\text{pert}}(x, t, \tilde{t}, \tau) \times \exp \left[ \sum_{n=1}^{\infty} \frac{1}{n} f_M(x^n, \tilde{t}^n, \tilde{\tau}^n, \tau^n) \right], \quad (4.14)
\]

\[
f_M(x, t, \tilde{t}, \tau) = \sum_{a,b=1}^{N_f} t_a^{-1} t_b^{-1} \tau^2 - t_a^{-1} t_b^{-1} \tau^{-2} x^2.
\]

(4.15)

One would note that \( f_M \) is exactly the letter index for the Mesons \( M_a^b \). Now every term of the remaining part has a nonzero power of \( w \) except 1. In addition, \( Z_{\text{vertex}} \) only has positive powers of \( w \) while \( Z_{\text{anti}} \) has negative powers of \( w \). Therefore, we conjecture that the following identities hold:

\[
Z^{(b_j)}_{\text{vertex}}(x, t, \tilde{t}, \tau, w) = \tilde{Z}^{(b_j)c}_{\text{vertex}}(x, t, \tilde{t}, \tau, w) \times \left( \prod_{k=0}^{\infty} \frac{1 - w^{-1} \tau^{-N_f} x^{1-N_f+N+2k}}{1 - w^{-1} \tau^{-N_f} x^{1-N_f+N+1+2k}} \right)
\]

\[
= \tilde{Z}^{(b_j)c}_{\text{vertex}}(x, t, \tilde{t}, \tau, w) \times \exp \left[ \sum_{n=1}^{\infty} \frac{1}{n} f_+(x^n, \tilde{t}^n, \tilde{\tau}^n, \tau^n, w^n) \right], \quad (4.16)
\]

\[
Z^{(b_j)}_{\text{anti}}(x, t, \tilde{t}, \tau, w) = \tilde{Z}^{(b_j)c}_{\text{anti}}(x, t, \tilde{t}, \tau, w) \times \left( \prod_{k=0}^{\infty} \frac{1 - w^{-1} \tau^{-N_f} x^{1-N_f+N+2k}}{1 - w^{-1} \tau^{-N_f} x^{1-N_f+N+1+2k}} \right)
\]

\[
= \tilde{Z}^{(b_j)c}_{\text{anti}}(x, t, \tilde{t}, \tau, w) \times \exp \left[ \sum_{n=1}^{\infty} \frac{1}{n} f_-(x^n, \tilde{t}^n, \tilde{\tau}^n, \tau^n, w^n) \right], \quad (4.17)
\]

\[
f_\pm(x, t, \tilde{t}, \tau, w) = w^\pm \frac{\tau^{-N_f} x^{1-N_f+N+1} - \tau^{N_f} x^{1-N_f+N}}{1 - x^2},
\]

(4.18)

which is the generalization of the numerically tested identities for special \( N, N_f \) above. We also check validity of these formulae by extensive numerical computation. Note that \( f_+ + f_- = f_{V_+} + f_{V_-} \) where \( f_{V_\pm} \) are the letter indices for the singlets \( V_\pm \), which have nonzero topological charges:

\[
f_{V_\pm}(x, t, \tilde{t}, \tau, w) = \frac{w^\pm \tau^{-N_f} x^{N_f-N+1} - w^\mp \tau^{N_f} x^{1-N_f+N}}{1 - x^2}.
\]

(4.19)

In fact, \( f_+ \) gets the contribution from the scalar of \( V_+ \) and the fermion of \( V_- \) while \( f_- \) gets the contribution from the fermion of \( V_+ \) and the scalar of \( V_- \). For both the perturbative part and the vortex part, the contribution with certain choice of \( \{b_j\}_{\text{orig}} \) for the original theory is exactly the same as the contribution with the complementary
choice of \( \{b_j\}_{\text{dual}} = \{b_j\}_{\text{orig}} \). Summing over all possible choices of \( \{b_j\} \), the total indices for both theories are thus the same for any \( N \) and \( N_f \geq N \). Note that the perturbative contribution of \( Q_a \) and \( \tilde{Q}^a \) maps to that of \( q^a, \tilde{q}_a \) and the contribution of \( M_a \) while the vortex and antivortex contributions of \( Q_a, \tilde{Q}^a \) map to those of \( q^a, \tilde{q}_a \) and the contribution of \( V_\pm \).

### 4.3 \( \mathcal{N} = 4 \) Seiberg-like duality and mirror symmetry

\( \mathcal{N} = 4 \) Seiberg-like dualities were proposed in [4, 29] based on brane configuration of Type IIB string theory. Under the duality, a \( U(N) \) gauge theory with \( N_f \) fundamental hypermultiplets is conjectured to be dual to another \( U(N_f - N) \) theory with \( N_f \) hypers in the infrared.

In this section we consider the simplest example of \( \mathcal{N} = 4 \) Seiberg-like duality. At low energy, an \( \mathcal{N} = 4 \) \( U(1) \) gauge theory with a fundamental hypermultiplet and a free theory of one hypermultiplet flow to the same theory. This is also the simplest example of the mirror symmetry. The free hypermultiplet is so called the twisted hypermultiplet in the context of mirror symmetry. As two theories are simple enough, we can easily compare two superconformal indices of them and check this duality conjecture. In the case at hand, the \( U(1) \) gauge multiplet of the original theory couples to one fundamental and anti-fundamental chiral matters while the adjoint chiral matter is decoupled from it. So there is a similarity between this \( U(1) \) theory and \( \mathcal{N} = 2 \) \( U(1) \) gauge theory with \( N_f = \tilde{N}_f = 1 \) chiral matters up to the decoupled adjoint chiral. In fact, once we assign the correct global charges to \( \mathcal{N} = 2 \) fields, it is easy to write the superconformal index of \( \mathcal{N} = 4 \) \( U(1) \) gauge theory using \( \mathcal{N} = 2 \) result. Then the superconformal index for \( U(1) \) theory with \( N_f = 1 \) fundamental hyper after factorization becomes

\[
I_{N=4}^{\mathcal{N}=4} = I_{N=2}^{\mathcal{N}=2}(\Delta_Q = \frac{1}{2}, \tau = y^{1/2}) \times \exp \left[ \sum_{n=1}^{\infty} \frac{1}{n} \frac{y^{-n} - y^n}{1 - x^{2n}} \right] 
\]

\[
= \exp \left[ \sum_{n=1}^{\infty} \frac{1}{n} \frac{y^{-n/2} x^{n/2} - y^{n/2} x^{-n/2}}{1 - x^{2n}} (w^n + w^{-n}) \right] 
\]  

\[ (4.20) \]

Here \( I_{N=2}^{\mathcal{N}=2} \) is the index of (4.3) for \( \mathcal{N} = 2 \) theory, and we set R-charge of bosonic fields to be \( \Delta_Q = \frac{1}{2} \) and introduced the chemical potential \( y \) for the off-diagonal \( U(1)_A \) of \( SU(2)_L \times SU(2)_R = SO(4) \) R-symmetry. The exponential term on the right hand side of the first line is from the adjoint chiral multiplet. The final expression is written as the Plethystic exponential of one free hypermultiplet that agrees with the duality proposal. In the dual theory \( w \) is interpreted as the \( U(1) \) flavor chemical potential. Note that this also perfectly matches with mirror symmetry. Under the mirror symmetry the monopole operator of the \( U(1) \) is mapped to the twisted free hypermultiplet. Note that \( w \) at eq. (4.20) is the vortex number, which is nothing
but the monopole charge. In the mirror side this is mapped to the charge of the flavor symmetry of the free hypermultiplet. The detailed exploration of the mirror symmetry and $\mathcal{N} = 4$ Seiberg-like duality in terms of the factorization will appear elsewhere.

5. Concluding remarks

There would be manifold generalizations one can pursue related to the current work. The first one is the direct proof of the factorization using the localization. For the 2d partition function, it is explicitly worked out in [22]. Certainly it is more desirable to more general gauge groups and general matters, which will have applications for Seiberg-like dualities for classical groups with two index matters. This was explored in [40].

For simple cases, we already saw the vortex partition function coincides with the corresponding topological open string amplitude. Such pattern will hold for more general cases and it would be desirable to work out explicitly. In [30], the 2d vortex arises as the surface operator of the 4d supersymmetric gauge theory and we expect that this will be lifted to the 3d defect to the 5d SCFT. The 3d SCFT realized as the IR limit of the 3d gauge theory flows to the 2d CFT upon the dimensional reduction. Thus many of the properties of 2d CFTs such as conformal blocks and $tt^*$ equations will be lifted to the corresponding 3d CFTs, which is interesting to explore. In the same spirit, the relation between 3d mirror symmetry and 2d mirror symmetry would be worked out in similar way. 2d mirror symmetry in the nonabelian gauge group setup is explored recently[41, 42] and it would be interesting to find its relation to 3d mirror symmetry.

Finally the proof of the duality such as Seiberg-like duality, mirror symmetry will be greatly simplified with the factorized form of the index and it is worth attempting analytic proof of the index equality for dual pairs.

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A. Factorization: nonabelian cases

In this section we are going to derive the factorized form of the superconformal index for a $U(N)$ gauge theory with $N_f$ fundamental and $\tilde{N}_f$ antifundamental flavors. At first the superconformal index is given by

$$I(x, t, w, \kappa) = \sum_{\vec{m} \in \mathbb{Z}^n/S_N} \oint \frac{dz_j}{2\pi i z_j} \frac{1}{|W_m|} \prod_j m_j e^{-S_{CS}(a,m)} Z_{gauge}(x, z, m) \prod_{\Phi} Z_{\Phi}(x, t, z, m)$$  \hspace{1cm} (A.1)

where

$$e^{-S_{CS}(a,m)} = e^{-i Tr_{CS}(a+\pi)m},$$  \hspace{1cm} (A.2)

$$Z_{gauge}(x, z) = e^{i a, m} = \prod_{\alpha \in ad(G)} x^{-|\alpha(m)|/2} \left(1 - e^{i a, x|x|\alpha(m)}\right),$$  \hspace{1cm} (A.3)

$$Z_{\Phi}(x, t, z) = e^{i a, m} = \prod_{\rho \in R_{\Phi}} \left[x^{(1-\Delta_{\Phi})} e^{-i \rho(a+\pi)} \prod_a t_a^{f_a(\Phi)}\right]^{(|\rho(m)|/2)} \frac{\left(e^{-i \rho(a)} \prod_a t_a^{f_a(\Phi)} x|x|\rho(m)+2-\Delta_{\Phi}, x^2\right)_{\infty}}{\left(e^{-i \rho(a)} \prod_a t_a^{f_a(\Phi)} x|x|\rho(m)+2-\Delta_{\Phi}, x^2\right)_{\infty}}.$$  \hspace{1cm} (A.4)

$(a; q)_n$ is the $q$-Pochammers symbol defined by

$$(a; q)_n = \prod_{k=0}^{n-1} (1 - aq^k), \hspace{1cm} |q| < 1.$$  \hspace{1cm} (A.5)

We have included the nontrivial phase shift $a_j \rightarrow a_j + \pi$ for nonzero magnetic flux vacua. If one considers a $U(N)$ gauge theory with $N_f$ fundamental and $\tilde{N}_f$ antifundamental chiral multiplets,

$$e^{-S_{CS}(a,m)} = \prod_{j=1}^{N} (-z_j)^{-m_j},$$  \hspace{1cm} (A.6)

$$Z_{gauge}(x, z, m) = \prod_{i,j=1}^{N} x^{-|m_i-m_j|/2} \left(1 - z_i z_j^{-1} x|x|^{m_i-m_j}\right),$$  \hspace{1cm} (A.7)

$$\prod_{\Phi} Z_{\Phi}(x, t, z, m)$$

$$= x^{(1-\Delta_{\Phi})N_f + N_{\tilde{f}}} \sum_{|m_j|/2} \left[\prod_{j=1}^{N} (-z_j)^{-N_f - N_{\tilde{f}}|m_j|/2}\right] \tau^{-N_f - N_{\tilde{f}} \sum|m_j|/2}$$

\begin{equation}
\times \prod_{j=1}^{N_f} \prod_{k=0}^{\infty} \left[\prod_{a=1}^{N_f} \frac{1 - z_j^{-1} t_a^{-1} x^{-1} x|m_j|+2-\Delta_{\Phi}+2k}{1 - z_j t_a x|m_j|+\Delta_{\Phi}+2k}\right] \prod_{a=1}^{N_{\tilde{f}}} \frac{1 - z_j^{-1} t_a^{-1} x|m_j|+2-\Delta_{\Phi}+2k}{1 - z_j t_a x|m_j|+\Delta_{\Phi}+2k} \tag{A.8}
\end{equation}
where \( t_a \) and \( \tilde{t}_a \) correspond to fugacities for \( SU(N_f) \times SU(\tilde{N}_f) \); \( \tau \) is a fugacity for \( U(1)_A \). One expects that \( \kappa + (N_f + \tilde{N}_f)/2 \) should be an integer due to the quantization of the effective CS level. In addition, we will set \( \Delta \Phi = 0 \), which can be restored by deforming \( \tau \to \tau x^{\Delta \Phi} \). The infinite product only makes sense for \( |x| < 1 \). Thus, if we assume \( |t_a \tau|, |\tilde{t}_a \tau| < 1 \), which can be extended by analytic continuation after integration, poles from the antifundamental part lie inside the integration contour. Indeed, the integrand also has a pole at the origin, which makes the integration difficult, for \( N_f \geq \tilde{N}_f \). Fortunately for \( N_f > \tilde{N}_f \) one could change the integration variables \( z_j \to 1/z_j \) to exclude the pole at the origin and would take poles from the fundamental part, which are now inside the contour, instead of the poles from the antifundamental part. For \( N_f = \tilde{N}_f \), however, one should take account of the pole at the origin.

Here we are dealing with the \( N_f > \tilde{N}_f \) case first. Changing the variables \( z_j \to 1/z_j \) is equivalent to summing residues at poles outside the contour, which come from the fundamental part: \( z_j = t_{b_j}^{-1} \tau^{-1} x^{-|m_j|} - 2l_j \) for \( b_j = 1, \ldots, N_f \) and \( l_j = 0, 1, \ldots \). After performing the contour integration the index is given by

\[
P^{N_f > \tilde{N}_f}(x, t, \tilde{t}, \tau, w, \kappa) = \sum_{\tilde{m} \in \mathbb{Z}^N/SU} \sum_{b_1, \ldots, b_N = 1}^{N_f} \sum_{\tilde{m}} \frac{1}{|W_{m,j}|} (-1)^{-\kappa \sum m_j - (N_f - \tilde{N}_f) |m_j|/2} \sum m_j \left( \prod_{j=1}^{N_f} c_{m_j + (N_f - \tilde{N}_f) |m_j|/2} \right) \right)
\]

\[
\prod_{i,j = 1}^{N_f} 1 - t_{b_j}^{-1} t_{b_j} x^{m_i - m_j - |m_i| - |m_j| - 2l_i + 2l_j} \right)
\]

\[
\left[ \prod_{j=1}^{N_f} \prod_{a=1, k=0}^{\infty} 1 - t_{b_j}^{-1} t_{a}^{-1} x^{2|m_j| + 2l_j + 2k} \right] \left[ \prod_{a=1, k=0}^{\infty} 1 - t_{b_j}^{-1} t_{a}^{-1} x^{2l_j + 2k} \right] \right)
\]
and $\prod_{a=1}^{N_f} \prod_{k=0}^{\infty} 1 - t_{b_j}^{-1} \tilde{t}_a^{-1} \tau^{-2} x^{-2l_j + 2 + 2k}$. They can be rewritten as follows:

$$
\prod_{i,j=1}^{N} \prod_{(i \neq j)} \left( 1 - t_{b_i}^{-1} t_{b_j} x^{m_i - m_j} - |m_i| + |m_j| - 2l_i + 2l_j \right)
= \prod_{i<j}^{N} \left( \left( 1 - t_{b_i}^{-1} t_{b_j} x^{m_i - m_j} - |m_i| + |m_j| - 2l_i + 2l_j \right) \left( 1 - t_{b_i}^{-1} t_{b_j} x^{m_i - m_j} + |m_i| + |m_j| + 2l_i - 2l_j \right) \right)
= \prod_{i<j}^{N} \left( -x^{m_i - m_j} \right) \left( \frac{1}{t_{b_i}^{-1} t_{b_j} x^{l_i + |m_i|/2 - m_j/2} - (l_i + |m_i|/2 - m_j/2) - t_{b_i}^{-1} t_{b_j} x^{-l_i + |m_i|/2 - m_j/2} + (l_i + |m_i|/2 - m_j/2)} \right)
\times \left( \frac{1}{t_{b_i}^{-1} t_{b_j} x^{l_j + |m_j|/2 + m_i/2} - (l_j + |m_j|/2 + m_i/2) - t_{b_i}^{-1} t_{b_j} x^{-l_j + |m_j|/2 + m_i/2} + (l_j + |m_j|/2 + m_i/2)} \right),
$$
\tag{A.10}

$$
\prod_{a=1}^{N_f} \prod_{k=0}^{\infty} 1 - t_{b_j}^{-1} t_{a} x^{-2l_j + 2k}
= \left( \prod_{a=1}^{N_f} \prod_{k=0}^{l_j-1} 1 - t_{b_j}^{-1} t_{a} x^{-2l_j + 2k} \right) \left( \prod_{a=1}^{N_f} \prod_{k=0}^{l_j-1} 1 - t_{b_j}^{-1} t_{a} x^{2k} \right)
= \left( \prod_{a=1}^{N_f} \prod_{k=0}^{l_j-1} -t_{b_j}^{-1} t_{a} x^{-2l_j + 2k} \right) \left( \prod_{a=1}^{N_f} \prod_{k=0}^{l_j-1} 1 - t_{b_j}^{-1} t_{a} x^{2k} \right)
= \left( \prod_{a=1}^{N_f} \prod_{k=0}^{l_j-1} 1 - t_{b_j}^{-1} t_{a} x^{-2l_j + 2k} \right) \left( \prod_{a=1}^{N_f} \prod_{k=0}^{l_j-1} 1 - t_{b_j}^{-1} t_{a} x^{2k} \right),
$$
\tag{A.11}

$$
\prod_{a=1}^{N_f} \prod_{k=0}^{\infty} 1 - t_{b_j}^{-1} \tilde{t}_a^{-1} \tau^{-2} x^{-2l_j + 2 + 2k}
= \left( \prod_{a=1}^{N_f} \prod_{k=0}^{l_j-1} 1 - t_{b_j}^{-1} \tilde{t}_a^{-1} \tau^{-2} x^{-2l_j + 2k} \right) \left( \prod_{a=1}^{N_f} \prod_{k=0}^{\infty} 1 - t_{b_j}^{-1} \tilde{t}_a^{-1} \tau^{-2} x^{2k} \right)
= \left( \prod_{a=1}^{N_f} \prod_{k=0}^{l_j-1} t_{b_j}^{-1} \tilde{t}_a^{-1} \tau^{-2} x^{2k} \right) \left( \prod_{a=1}^{N_f} \prod_{k=0}^{\infty} 1 - t_{b_j}^{-1} \tilde{t}_a^{-1} \tau^{-2} x^{2k} \right),
$$
\tag{A.12}

We have dropped the absolute value symbol of $|m_i - m_j|$ by assuming $m_1 \geq \cdots \geq m_N$. 

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Using these one can rewrite the index as follows:

\[
\begin{align*}
I^{N_f>\tilde{N}_f}(x, t, \tilde{t}, \tau, w, k) &= \sum_{m_1} \sum_{j=1}^{N_f} \sum_{l=0}^{\infty} \frac{1}{|W_m|} \left( -1 - \kappa \sum m_j - (N_f - \tilde{N}_f) \sum |m_j|^2 + \sum m_j \left( \prod_{j=1}^{N_f} \tilde{b}_{\bar{m}j} + (N_f - \tilde{N}_f)|m_j|/2 \right) \right) \\
\sum_{m_1} \sum_{j=1}^{N_f} \sum_{l=0}^{\infty} \frac{1}{|W_m|} \left( -1 - \kappa \sum m_j - (N_f - \tilde{N}_f) \sum |m_j|^2 + \sum m_j \left( \prod_{j=1}^{N_f} \tilde{b}_{\bar{m}j} + (N_f - \tilde{N}_f)|m_j|/2 \right) \right) \\
\sum_{x_1} \sum_{j=1}^{N_f} \sum_{l=0}^{\infty} \frac{1}{|W_m|} \left( -1 - \kappa \sum m_j - (N_f - \tilde{N}_f) \sum |m_j|^2 + \sum m_j \left( \prod_{j=1}^{N_f} \tilde{b}_{\bar{m}j} + (N_f - \tilde{N}_f)|m_j|/2 \right) \right) \\
\sum_{x_1} \sum_{j=1}^{N_f} \sum_{l=0}^{\infty} \frac{1}{|W_m|} \left( -1 - \kappa \sum m_j - (N_f - \tilde{N}_f) \sum |m_j|^2 + \sum m_j \left( \prod_{j=1}^{N_f} \tilde{b}_{\bar{m}j} + (N_f - \tilde{N}_f)|m_j|/2 \right) \right)
\end{align*}
\]

In order to proceed further one should rearrange the summations. First the summation \( \sum_{m_1} \) is replaced by \( \sum_{m_1} \frac{S_{m_1}}{N_f} \). Thanks to the symmetry \( |m_j| + l_j \leftrightarrow l_j \), one can rearrange the summations as \( \sum_{m_1} \sum_{l_0}^{\infty} = \sum_{n_0}^{\infty} \sum_{n_0}^{\infty} \) where \( n_j = \)
\[ I_{N_{\tau}}^{N_f}(x, t, \tilde{t}, \tau, w, \kappa) \]
\[ = \frac{(-1)^{N(N-1)/2}}{N!} \sum_{b_1, \ldots, b_N=1}^{N_f} \sum_{\tilde{n}=0}^{\infty} \sum_{n=0}^{\infty} \]
\[ (-1)^{-\kappa} \sum (n_j - \tilde{n}_j) - (N_f - \tilde{N}_f) \sum (n_j + \tilde{n}_j)/2, \sum (n_j - \tilde{n}_j) \left( \prod_{j=1}^{N_f} t_{b_j}^{\kappa(n_j - \tilde{n}_j)} + (N_f - \tilde{N}_f)(n_j + \tilde{n}_j)/2 \right) \]
\[ \tau^{\kappa} \sum (n_j - \tilde{n}_j) - (N_f - \tilde{N}_f) \sum (n_j + \tilde{n}_j)/2, (N_f - \tilde{N}_f) \sum (n_j + \tilde{n}_j)/2 + (N_f - \tilde{N}_f) \sum (n_j^2 + \tilde{n}_j^2)/2 \]
\[ \left[ \prod_{i<j}^{N_f} \left( t_{b_i}^{1/2} t_{b_j}^{-1/2} x_i \tilde{x}_j - t_{b_i}^{-1/2} t_{b_j}^{1/2} \tilde{x}_i \tilde{x}_j \right) \right] \]
\[ \times \left[ \prod_{j=1}^{N_f} \left( \prod_{k=0}^{\tilde{n}_f} \prod_{a=1}^{\tilde{N}_f} t_{a}^{1/2} - t_{a}^{-1/2} x_{a}^2 \right) \right] \]
\[ \times \left[ \prod_{k=0}^{\infty} \prod_{a=1}^{N_f} \left( \prod_{a=1}^{\tilde{N}_f} 1 - t_{a}^{-1/2} \tilde{t}_{a} x_{a}^2 \right) \right] \]
\[ = \frac{(-1)^{N(N-1)/2}}{N!} \sum_{b_1, \ldots, b_N=1}^{N_f} \left\{ \prod_{j=1}^{N_f} \prod_{k=0}^{\infty} \left( \prod_{a=1}^{\tilde{n}_f} 1 - t_{a}^{-1/2} \tilde{t}_{a} x_{a}^2 \right) \right\} \]
\[ \times \sum_{\tilde{n}=0}^{\infty} \left[ (-1)^{-\kappa} \sum (n_j - \tilde{n}_j) - (N_f - \tilde{N}_f) \sum (n_j + \tilde{n}_j)/2, \sum (n_j - \tilde{n}_j) \left( \prod_{j=1}^{N_f} t_{b_j}^{\kappa(n_j - \tilde{n}_j)} + (N_f - \tilde{N}_f)(n_j + \tilde{n}_j)/2 \right) \right] \]
\[ \times \left[ \prod_{i<j}^{N_f} \left( t_{b_i}^{1/2} t_{b_j}^{-1/2} x_i \tilde{x}_j - t_{b_i}^{-1/2} t_{b_j}^{1/2} \tilde{x}_i \tilde{x}_j \right) \right] \]
\[ \times \left[ \prod_{k=0}^{\infty} \prod_{a=1}^{N_f} \left( \prod_{a=1}^{\tilde{n}_f} 1 - t_{a}^{-1/2} \tilde{t}_{a} x_{a}^2 \right) \right] \]
\[ = \frac{(-1)^{N(N-1)/2}}{N!} \sum_{b_1, \ldots, b_N=1}^{N_f} \left\{ \prod_{j=1}^{N_f} \prod_{k=0}^{\infty} \left( \prod_{a=1}^{\tilde{n}_f} 1 - t_{a}^{-1/2} \tilde{t}_{a} x_{a}^2 \right) \right\} . \] (A.14)
More concisely,

\[
I^{N_f \to \tilde{N}_f}(x = e^{-\gamma}, t = e^{iM}, \tilde{t} = e^{i\tilde{M}}, \tau = e^{i\mu}, w, \kappa) = (-1)^{N(N-1)/2} \sum_{b_1, \ldots, b_{N_f} = 1}^{N_f} \left\{ \prod_{j=1}^{N} \left( \prod_{a=1}^{N_f} \frac{N_j^{(a)} - 1 - t_b_j t_a^{-1} x^{2+2k}}{N_j^{(a)} - 1 - t_b_j \tilde{t}_a r x^{2+2k}} \right) \left( \prod_{a=1}^{N_f} \frac{\tilde{N}_j^{(a)} \tau^{-1} x^{-2+2k}}{\tilde{N}_j^{(a)} \tau^{-1} x^{-2+2k}} \right) \right\} \]

\times \sum_{\hat{n} = 0}^{\infty} \left[ (-1)^{-\kappa} \sum_{n_f}(N_f - \tilde{N}_f) \sum_{n_f / w} n_f \prod_{j=1}^{N_f} \left( \prod_{b_j} t^{b_j} \right) \tau^{-\kappa} \sum_{n_f} x^{-\kappa} \sum_{n_f} \right]

\[
\frac{1}{2} \left( \prod_{i < j}^{N} t \sinh \frac{i M_{b_i} - i M_{b_j} - 2 \gamma(n_i - n_j)}{2} \right) \prod_{j=1}^{N_f} \left( \prod_{k=0}^{N_f} \frac{\tilde{N}_j^{(k)} - 1 - t_b_j \tilde{t}_a r x^{2+2k}}{\tilde{N}_j^{(k)} - 1 - t_b_j \tilde{t}_a r x^{2+2k}} \right) \left( \prod_{a=1}^{N_f} \frac{\tilde{N}_j^{(a)} \tau^{-1} x^{-2+2k}}{\tilde{N}_j^{(a)} \tau^{-1} x^{-2+2k}} \right) \left( \prod_{a=1}^{N_f} \frac{\tilde{N}_j^{(a)} \tau^{-1} x^{-2+2k}}{\tilde{N}_j^{(a)} \tau^{-1} x^{-2+2k}} \right) \right]

(A.15)

where we identified some parameters as follows: \( x = e^{-\gamma}, t = e^{iM}, \tilde{t} = e^{i\tilde{M}}, \tau = e^{i\mu} \). If \( b_i = b_j \) for \( i \neq j \) the index vanishes because it has antisymmetric contributions of \( n_i \) and \( n_j \). Together with the flavor symmetry it implies that one can arrange \( b_j \) in ascending order: \( b_1 < \cdots < b_N \) and thus replace \( \frac{1}{N!} \sum_{b_1, \ldots, b_{N_f} = 1}^{N_f} \) by \( \sum_{1 \leq b_1 < \cdots < b_N \leq N_f} \). With some tricks described in the next section the index can be written as

\[
I^{N_f \to \tilde{N}_f}(x = e^{-\gamma}, t = e^{iM}, \tilde{t} = e^{i\tilde{M}}, \tau = e^{i\mu}, w, \kappa) = \sum_{1 \leq b_1 < \cdots < b_N \leq N_f} \left\{ \prod_{i < j}^{N} t \sinh \frac{i M_{b_i} - i M_{b_j} - 2 \gamma(n_i - n_j)}{2} \right\} \prod_{j=1}^{N_f} \left( \prod_{k=0}^{N_f} \frac{\tilde{N}_j^{(k)} - 1 - t_b_j \tilde{t}_a r x^{2+2k}}{\tilde{N}_j^{(k)} - 1 - t_b_j \tilde{t}_a r x^{2+2k}} \right) \left( \prod_{a=1}^{N_f} \frac{\tilde{N}_j^{(a)} \tau^{-1} x^{-2+2k}}{\tilde{N}_j^{(a)} \tau^{-1} x^{-2+2k}} \right) \right\} \quad (A.16)

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Each of two summations corresponds to the $N = 2$ vortex and antivortex partition functions on $\mathbb{R}^2 \times S^1$ respectively. Note that $(-1)^{\ldots}$ always give rise to a real valued factor because $\kappa + N_f/2 + \bar{N}_f$ is an integer.

The index for $N_f < \bar{N}_f$ is simply obtained by interchanging $t_a \leftrightarrow \bar{t}_a$ and $\kappa \to -\kappa$:

\[
I_{N_f < \bar{N}_f}(x = e^{-\gamma}, t = e^{iM}, \bar{t} = e^{i\bar{M}}, \tau = e^{i\mu}, w, \kappa) = \sum_{1 \leq b_1 < \ldots < b_N \leq \bar{N}_f} \left\{ \left( \prod_{i,j=1}^{N} \frac{2 \sinh \frac{t_{b_{ij}} - i \bar{t}_{b_{ij}}}{2}}{\prod_{i=1}^{N} \sinh \frac{1}{2} \left( \prod_{i=1}^{N} t_{b_{ij}} - i \bar{t}_{b_{ij}} + 2\gamma_{-1} t_{b_{ij}} + 2\gamma_{-1} \bar{t}_{b_{ij}} + 2(1+k) \right) \right) \right\} \times \sum_{\tilde{n} = 0}^{\infty} \left( \prod_{j=1}^{N} \frac{1}{\prod_{i=1}^{N} \sinh \left( \frac{1}{2} \prod_{i=1}^{N} t_{b_{ij}} - i \bar{t}_{b_{ij}} + 2\gamma_{-1} t_{b_{ij}} + 2\gamma_{-1} \bar{t}_{b_{ij}} + 2(1+k) \right) \right) \right\}
\]

For $N_f = \bar{N}_f$ the integrand also has poles either at the origin or at the infinity depending on the sign of $N + \kappa m_j$. For $N + \kappa m_j > 0$ the residue at the origin is given by

\[
\text{Res}(\ldots, 0) = x^{-\sum_{i \neq j} |m_i - m_j|/2 + N_f \sum |m_j|} x^{-N_f \sum |m_j|} \lim_{j \to 0} \left( \prod_{j=1}^{N} \frac{1}{(N + \kappa m_j - 1)!} \frac{\partial^{N + \kappa m_j - 1}}{\partial z_j^{N + \kappa m_j - 1}} \right)
\]

\[
\left[ \prod_{j=1}^{N} \left( 1 - z_j x^{m_j} \right) \right] \left[ \prod_{a=1}^{N_f} \frac{1}{1 - z_j x^{m_j} + 2k} \right] \frac{1}{1 + z_j x^{m_j} + 2k}
\]

Let us first consider the $N = 1, \kappa = 0$ case. In that case one has a vanishing infinite product:

\[
\sim \prod_{k=0}^{\infty} t_a^{-1} \bar{t}_a^{-1} x^{-2} x^{2k} \to 0
\]

assuming $|t_a^{-1} \bar{t}_a^{-1} x^{-2} x^{2k}| < 1$, which doesn’t spoil the original range of parameters that we already assumed at start. For general $N$ and $\kappa$, there are $N + \kappa m_j - 1$
differentiations. When each of them acts on the above infinite product, an additional factor arises. Nevertheless, one still has a vanishing infinite product because there are only the finite number of such additional factors, which are not singular. In the same manner the residue at the infinity also vanishes. Therefore, since the residues at the other poles are the same as those for \( N_f \neq \tilde{N}_f \), both results for \( N_f > \tilde{N}_f \) and \( N_f < \tilde{N}_f \) even work for \( N_f = \tilde{N}_f \).

One can rewrite the factorized index using the permutation of the chemical potentials \( t_a \):

\[
I_{N_f,\tilde{N}_f}^{N_f,\tilde{N}_f}(x, t, \tilde{t}, w, \kappa) = \frac{1}{N!(N_f - \tilde{N}_f)!} \sum_{\sigma(t)} I_{\text{pert}}^{N_f}(x, \sigma(t), \tilde{t}, \tau) \left[ \prod_{\alpha=1}^{N_f} \frac{1- t_j \tilde{t}_\alpha e^{2k(2k+2)}}{1- t_j \tilde{t}_\alpha} \right] \left[ \prod_{\alpha=1}^{\tilde{N}_f} \frac{1- t_\alpha \tilde{t}_j e^{2k(2k+2)}}{1- t_\alpha \tilde{t}_j} \right] \prod_{\alpha=1}^{\tilde{N}_f} \prod_{n_j=1}^{N_f} (-w)^n I_{(n_j)}^{N_f}(x, \sigma(t), \tilde{t}, \tau, \kappa) \left[ \prod_{\alpha=1}^{\tilde{N}_f} (-w)^{-\bar{n}_j} I_{(\bar{n}_j)}^{\tilde{N}_f}(x, \sigma(t), \tilde{t}, \tau, -\kappa) \right]
\]

where \( n = \sum_j n_j, \bar{n} = \sum_j \bar{n}_j \) and \( \sigma(t) \) denotes the permutation of \( t_a \)'s. Here the perturbative and vortex contributions are given by

\[
I_{(n_j)}^{N_f}(x, \sigma(t), \tilde{t}, \tau, \kappa) = (-1)^{-n-\bar{n}}(N_f - \tilde{N}_f)!/2 \epsilon^{i\kappa} \sum_{j}(M_j n_j + \mu_j + i\gamma_n^2)
\]

\[
\prod_{j=1}^{N_f} \prod_{n_j=1}^{N} \prod_{k=1}^{N} \prod_{i=1}^{\bar{n}_j} 2 \sinh \left( -i M_j - i M_j - 2 \mu + 2 \gamma(k-1) \right) \prod_{a=1}^{\tilde{N}_f} \prod_{n_j=1}^{N_f} \prod_{k=1}^{N} \prod_{i=1}^{\bar{n}_j} 2 \sinh \left( i M_a - i M_a + 2 \gamma(k-n_i) \right) .
\]

One can compare the vortex partition function obtained here with the \( N = 4 \) result in [29]. In order to compare our result to that of [29] one would set \( N_f = \tilde{N}_f \), restore the \( R \)-charge to \( \frac{1}{2} \) and restrict \( b_j = j \in \{1, \ldots, N\} \); for the result in [29], set \( \gamma' = 0 \), \( R = -R = \frac{1}{2} \) and redefine \( 2i \gamma \to \gamma \). A factor \( \left( \prod_{i=1}^{N} \sinh \frac{i M_{a_i} - i M_{a_i} + 2 \gamma(k-n_i)}{2} \right)^{-1} \) corresponds to \(-z_{\text{fund}}^{\text{fund}} \) in [29]; likewise,

\[
\prod_{a=1}^{N_f} \prod_{n_j=1}^{N} \prod_{k=1}^{N} \frac{1}{\sinh \left( \frac{i M_a - i M_a + 2 \gamma(k+1)}{2} \right)} = -z_{\text{fund}}^{N_f - N_f}, \quad (A.22)
\]

\[
\prod_{a=1}^{N_f} \prod_{n_j=1}^{N} \prod_{k=1}^{N} \frac{1}{\sinh \left( \frac{i M_a - i M_a + 2 \gamma(k+1)}{2} \right)} = -z_{\text{anti}}^{N_f}, \quad (A.23)
\]

Mass parameters of each result are identified as follows:

\[
i M_j + i \mu = \mu_j + 2 \gamma + \delta, \quad i M_a + i \mu = \mu_a + \delta, \quad i \tilde{M}_b + i \mu = -\mu_b - \delta \quad (A.24)
\]

where \( j = 0, \ldots, N; a = N + 1, \cdots, N_f \); \( b = 1, \cdots, N_f \) and \( \delta \) is an undetermined parameter coming from the gauge symmetry. Asymmetry between \( i M_j \) and \( i M_a \) might come from the fact that \( N \) flavors have nonzero VEVs while \( N_f - N \) flavors do not.
B. Detailed calculations

In this section we will see an identity that can be used when one derives (A.16) from (A.15). First let us focus on \( \prod_{j=1}^{N} \prod_{k=0}^{n_{j}-1} \prod_{a=1}^{N_f} 2 \sinh \frac{-iM_{b_{j}} + iM_{a} + 2\gamma(1+k)}{2} \) in the denominator of the vortex partition part. It decomposes as

\[
\prod_{j=1}^{N} \prod_{k=0}^{n_{j}-1} \prod_{a=1}^{N_f} 2 \sinh \frac{-iM_{b_{j}} + iM_{a} + 2\gamma(1+k)}{2} = \prod_{j=1}^{n_{j}-1} \left( \prod_{i=1}^{N} 2 \sinh \frac{iM_{b_{i}} - iM_{b_{j}} + 2\gamma(1+k)}{2} \right) \left( \prod_{a=1}^{N_f} 2 \sinh \frac{iM_{a} - iM_{b_{j}} + 2\gamma(1+k)}{2} \right). \quad (B.1)
\]

The former factor can be written as

\[
\prod_{i,j=1}^{N} \prod_{k=0}^{n_{j}-1} 2 \sinh \frac{iM_{b_{i}} - iM_{b_{j}} + 2\gamma(1+k)}{2} = \left( \prod_{j=1}^{N} \prod_{k=0}^{n_{j}-1} 2 \sinh \gamma(1+k) \right) \left[ \prod_{i<j}^{N} \left( \prod_{k=0}^{n_{j}-1} 2 \sinh \frac{iM_{b_{i}} - iM_{b_{j}} + 2\gamma(1+k)}{2} \right) \right] = \left( \prod_{j=1}^{N} \prod_{k=0}^{n_{j}-1} 2 \sinh \gamma(1+k) \right) \left[ \prod_{i<j}^{N} (-1)^{n_{i}} \left( \prod_{k=0}^{n_{j}-1} 2 \sinh \frac{iM_{b_{i}} - iM_{b_{j}} + 2\gamma(1+k)}{2} \right) \left( \prod_{k=0}^{n_{i}-1} 2 \sinh \frac{iM_{b_{j}} - iM_{b_{i}} + 2\gamma(1+k)}{2} \right) \right] = \left( \prod_{j=1}^{N} \prod_{k=0}^{n_{j}-1} 2 \sinh \gamma(1+k) \right) \left( \prod_{i<j}^{N} (-1)^{n_{i}} \prod_{k=0}^{n_{i}+n_{j}-1} 2 \sinh \frac{iM_{b_{i}} - iM_{b_{j}} + 2\gamma(k-n_{i})}{2} \right). \quad (B.2)
\]
Also note
\[
\prod_{i<j} \prod_{k=0}^{2\sinh(2\gamma k - n_i)} iM_{b_i} - iM_{b_j} + 2\gamma(k - n_i)
\]
\[
= \left( \prod_{j=1}^{N} \prod_{k=0}^{2\sinh(\gamma(k - n_j))} \right)
\]
\[
\left[ \prod_{i<j} \left( \prod_{k=0}^{2\sinh(2\gamma(k - n_i))} \right) \right]
\]
\[
= (-1)^{\sum_{j}^{N} n_j} \left( \prod_{j=1}^{N} \prod_{k=0}^{2\sinh(\gamma(1 + k))} \right)
\]

Combining those results one obtains the following identity:
\[
\sum_{\vec{n} = 0}^{\infty} \left[ \prod_{i<j} \left( \prod_{k=0}^{2\sinh(2\gamma(n_i - n_j))} \right) \right]
\]
\[
= \left( \prod_{i<j} \prod_{k=0}^{2\sinh(2\gamma(k - n_i))} \right)
\]
\[
\times \sum_{\vec{n} = 0}^{\infty} \left[ (-w)^{\sum_{j}^{N} n_j} \prod_{j=1}^{N} \prod_{k=0}^{2\sinh(2\gamma(1 + k))} \right]
\]

References

[1] A. Giveon and D. Kutasov, “Seiberg Duality in Chern-Simons Theory,” Nucl. Phys. B812 (2009) 1, [arXiv:0808.0360 [hep-th]].

[2] V. Niarchos, “Seiberg Duality in Chern-Simons Theories with Fundamental and Adjoint Matter,” JHEP 0811 (2008) 001, [arXiv:0808.2771 [hep-th]].

[3] V. Niarchos, “$R$-charges, Chiral Rings and RG Flows in Supersymmetric Chern-Simons-Matter Theories,” JHEP 0905 (2009) 054.
[4] A. Kapustin, B. Willett, I. Yaakov, “Nonperturbative Tests of Three-Dimensional Dualities,” [arXiv:1003.5694 [hep-th]].

[5] A. Kapustin, B. Willett, I. Yaakov, “Tests of Seiberg-like Duality in Three Dimensions,” [arXiv:1012.4021 [hep-th]].

[6] D. Bashkirov, A. Kapustin, “Dualities between N = 8 superconformal field theories in three dimensions,” JHEP 1105, 074 (2011). [arXiv:1103.3548 [hep-th]].

[7] C. Krattenthaler, V. P. Spiridonov, G. S. Vartanov, “Superconformal indices of three-dimensional theories related by mirror symmetry,” JHEP 1106 (2011) 008, [arXiv:1103.4075 [hep-th]].

[8] D. Jafferis and X. Yin, “Duality Appetizer,” [arXiv:1103.5700 [hep-th]].

[9] A. Kapustin, “Seiberg-like duality in three dimensions for orthogonal gauge groups,” [arXiv:1104.0466 [hep-th]].

[10] B. Willett and I. Yaakov, “N=2 Dualities and Z Extremization in Three Dimensions,” [arXiv:1104.0487 [hep-th]].

[11] A. Kapustin, B. Willett, “Generalized Superconformal Index for Three Dimensional Field Theories,” [arXiv:1106.2484 [hep-th]].

[12] D. Bashkirov, “Aharony duality and monopole operators in three dimensions,” [arXiv:1106.4110 [hep-th]].

[13] C. Hwang, H. Kim, K.-J. Park, J. Park, “Index computation for 3d Chern-Simons matter theory: test of Seiberg-like duality,” [arXiv:1107.4942 [hep-th]].

[14] D. Gang, E. Koh, K. Lee, J. Park, “ABCD of 3d $\mathcal{N} = 8$ and 4 Superconformal Field Theories,” [arXiv:1108.3647 [hep-th]].

[15] D. Berenstein, M. Romo, “Monopole operators, moduli spaces and dualities,” [arXiv:1108.4013 [hep-th]].

[16] T. Morita, V. Niarchos, “F-theorem, duality and SUSY breaking in one-adjoint Chern-Simons-Matter theories,” [arXiv:1108.4963 [hep-th]].

[17] F. Benini, C. Closset, S. Cremonesi, ”Comments on 3d Seiberg-like dualities,” [arXiv:1108.5373 [hep-th]].

[18] C. Hwang, K.-J. Park, J. Park, “Evidence for Aharony duality for orthogonal gauge groups,” [arXiv:1109.2828 [hep-th]].

[19] O. Aharony, I. Shamir, “On $O(N_c)$ d=3 N=2 supersymmetric QCD Theories,” [arXiv:1109.5081 [hep-th]].

[20] S. Pasquetti, “Factorisation of N = 2 Theories on the Squashed 3-Sphere,” JHEP 1204, 120 (2012) [arXiv:1111.6905 [hep-th]].
[21] N. Doroud, J. Gomis, B. Floch, S. Lee, “Exact Results in D=2 Supersymmetric Gauge Theories,” [arXiv:1206.3895 [hep-th]].

[22] F. Benini, S. Cremonesi “Partition functions of $N = (2, 2)$ gauge theories on $S^2$ and vortices,” [arXiv:1206.2356 [hep-th]].

[23] T. Dimofte, D. Gaiotto and S. Gukov, “3-Manifolds and 3d Indices,” arXiv:1112.5179 [hep-th].

[24] A. Iqbal, C. Vafa “BPS Degeneracies and Superconformal Index in Diverse Dimensions,” [arXiv:1210.3605 [hep-th]].

[25] C. Beem, T. Dimofte, S. Pasquetti “Holomorphic Blocks in Three Dimensions,” [arXiv:1211.1986 [hep-th]].

[26] J. Bhattacharya and S. Minwalla, “Superconformal Indices for $N = 6$ Chern Simons Theories,” JHEP, 0901 (2009) 014, [arXiv:0806.3251 [hep-th]].

[27] S. Kim, “The complete superconformal index for $N=6$ Chern-Simons theory,” Nucl. Phys. B821 (2009) 241, [arXiv:0903.4172 [hep-th]].

[28] Y. Imamura and S. Yokoyama, “Index for three dimensional superconformal field theories with general $R$-charge assignments,” [arXiv:1101.0557 [hep-th]].

[29] H. Kim, J. Kim, S. Kim, K. Lee, “Vortices and 3 dimensional dualities,” [arXiv:1204.3895 [hep-th]].

[30] T. Dimofte, S. Gukov, L. Hollands “Vortex Counting and Lagrangian 3-manifolds,” [arXiv:1006.0977 [hep-th]].

[31] A. Iqbal, C. Kozcaz, C. Vafa “The Refined Topological Vertex,” JHEP 0910:009,2009 [arXiv:0701156 [hep-th]].

[32] A. Iqbal and A. -K. Kashani-Poor, “The Vertex on a strip,” Adv. Theor. Math. Phys. 10, 317 (2006) [hep-th/0410174].

[33] H. Awata and H. Kanno, “Changing the preferred direction of the refined topological vertex,” arXiv:0903.5383 [hep-th].

[34] F. Fucito, J. F. Morales and R. Poghossian, “Instantons on quivers and orientifolds,” JHEP 0410, 037 (2004) [hep-th/0408090].

[35] H. -C. Kim and S. Kim, “M5-branes from gauge theories on the 5-sphere,” JHEP 1305, 144 (2013) [arXiv:1206.6339 [hep-th]]; G. Lockhart and C. Vafa, “Superconformal Partition Functions and Non-perturbative Topological Strings,” arXiv:1210.5909 [hep-th]; H. -C. Kim, J. Kim and S. Kim, “Instantons on the 5-sphere and M5-branes,” arXiv:1211.0144 [hep-th].

[36] H. Kim, S. Kim, K. Lee “5-dim Superconformal Index with Enhanced En Global Symmetry,” JHEP 1210 (2012) 142 [arXiv:1206.6781 [hep-th]].
[37] O. Aharony, “IR duality in d = 3 N=2 supersymmetric USp(2N(c)) and U(N(c))
gauge theories,” Phys. Lett. B 404, 71 (1997) [hep-th/9703215].

[38] O. Aharony, A. Hanany, K. Intriligator, N. Seiberg, M.J. Strassler “ Aspects of
N=2 Supersymmetric Gauge Theories in Three Dimensions,”
Nucl.Phys.B499:67-99,1997 [arXiv:9703110 [hep-th]].

[39] D. Jafferis, “The Exact Superconformal R-Symmetry Extremizes Z,”
[arXiv:1012.3210 [hep-th]].

[40] A. Kapustin, H. Kim, J. Park, “Dualities for 3d Theories with Tensor Matter,”
[arXiv:1110.2547 [hep-th]]; H. Kim, J. Park, “Aharony Dualities for 3d
Theories with Adjoint Matter,” [arXiv:1302.3645 [hep-th]].

[41] H. Jockers, V. Kumar, J. Lapan, D. Morrison, M. Romo “Two-Sphere Partition
Functions and Gromov-Witten Invariants,” [arXiv:1208.6244 [hep-th]].

[42] Exact Kahler Potential from Gauge Theory and Mirror Symmetry “J. Gomis, S.
Lee,” [arXiv:1210.6022 [hep-th]].