The Mathematical Model of Transverse Vibrations of the Three-Layer Plate

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Abstract. The article in a flat setting investigated the antisymmetric oscillations of a three-layer plate, which is infinite in plan. It is believed that the plate is not symmetrical in thickness. Based on the exact solutions of the equations of the linear theory of elasticity in transformations, a theory of unsteady transverse vibrations of a three-layer plate is developed. The oscillation equations are derived with respect to two auxiliary functions, which are the main parts of the longitudinal and transverse displacements of the points of some “intermediate” surface of the middle layer. The distance of this surface to the coordinate plane of the plate is arbitrary. All components of the stress tensors and displacement vectors at the points of the layers are expressed, like the vibration equations, through the introduced auxiliary functions. The problem of harmonic antisymmetric vibrations of an elastic three-layer plate is solved.

1. Introduction
Multilayer plates and shells are widely used in various fields of technology. The sphere of usage of the three-layer plates is extremely wide. It includes such areas as construction, aircraft manufacturing, instrumentation, etc. Therefore, the calculation of such plates for the action of various dynamic loads is widely used in the design and operation of engineering structures, often working in extreme conditions on the effects of explosive, seismic and other loads [1].
Three-layer structures, the appearance of which was dictated by needs, primarily in the military field, fully met the requirements for a combination of mechanical characteristics of materials and functional requirements for engineering structures. The conditions for the intensive use of three-layer and multilayer structures contributed to the need to develop effective methods for calculating such elements [2]. Therefore, simultaneously with the beginning of the use of multilayer structures, theories of their calculation began to develop.
Superiority in constructing a theory for calculating multilayer plates belongs to S. G. Lehnitsky, who created the theory, later called the “zig-zag” theory [3]. The use and refinement of the zig-zag theory continues to this day, as evidenced by numerous studies [4]. The work [5] is devoted to the development of a new theory of shear deformation for multilayer and composite plates. The displacement field is approximated using the product of trigonometric and exponential functions. The theory takes into account an adequate distribution of transverse shear deformations over the plate thickness and tangential boundary conditions without stresses on the boundary surface. A similar representation of the shear strain using the exponential function in the displacement field is the subject of [6]. It presents a new model of a multilayer composite structure, which describes the thickness distribution of shear
stress taking into account the conditions of the free boundary on the upper and lower surfaces using an exponential function.

The number of published studies on the development of new models and theories of unsteady oscillations, homogeneous and multilayer plates and shells subjected to external dynamic loads is small compared with the case of statics [7]. One of the directions for solving this problem is devoted several scientific works [8,9]. These works are based on the method of general solutions in transformations that are used to satisfy the conditions specified on the surfaces of plates and shells. Many scientific works devoted to the creation of models can be cited where deformation is taken into account by attracting certain functions of the hyperbolic [10], power [11] and mixed hyperbolic with irrational [12] types. The new theory of higher-order shear and normal deformation [13] adjoins here for the analysis of bending and free vibrations of multilayer plates with functionally gradient isotropic front plates. A n-order model [14] has also been developed for shear deformation for calculating a functionally graded and composite multilayer plate. The logical continuation of these studies are included several scientific investigations such as [15, 16], in which new mathematical models of stationary oscillations of plates and shells are proposed, based on the exact solution of the corresponding three-dimensional viscoelastic problems.

The methods developed in these works for developing the theory of oscillations of plates and shells were then used to study layered structures [17, 18]. An analyses of a large number of published works on the vibrations of homogeneous and layered plates, taking into account the viscoelastic properties of the material, shows that bending vibrations are the most studied. These studies are still ongoing [19-21]. At the same time, the study of studies on unsteady oscillations of three-layer plates, taking into account the new requirements of modern technology, allows us to conclude that there are still many problems that are far from being solved and therefore, new studies in this direction are relevant.

2. Methods

Consider a three-layer, infinite in plan, elastic plate. We assume that the plate consists of two bearing layers with thicknesses \(h_1\) and \(h_2\), and the middle layer with thicknesses \(2h_0\) (Figure 1). In the case when the space between the bearing layers is filled with lighter, i.e. less rigid material, the middle layer is called aggregate. During deriving the equations of vibration, we assume that both the plate as a whole and each of its layers separately strictly obey the mathematical linear theory of elasticity and are described in exact formulations by its three-dimensional equations.

2.1. Formulation of the problem.

Given the unlimited size of the plate, further we will assume that it is under conditions of plane deformation. Therefore, we will consider the plate in a system of rectangular coordinates \(Oxz\) and direct the axis \(Ox\) along the midline of the cross section, and the axis \(Oz\) up, perpendicular to the axis \(Ox\). We call the supporting layers of the plate the first and second (in accordance with their thicknesses \(h_1\) and \(h_2\)) layers, and the middle layer is zero. Therefore, we will consider the plate in a system of rectangular coordinates and direct the axis along the midline of the cross section, and the axis up, perpen-
Transverse vibrations of the plate are excited under boundary conditions described by Hooke's linear law. In the case of plane deformation, by introducing the components of displacement vectors according to the formulas

\[ U_m = \frac{\partial \varphi_m}{\partial x} - \frac{\partial \psi_m}{\partial z}, \quad W_m = \frac{\partial \varphi_m}{\partial z} + \frac{\partial \psi_m}{\partial x}, \quad (m = 0, 1, 2) \]  

(1)

the equations of motion of the points of layers in a Cartesian system can easily be reduced to wave equations

\[ (\lambda_m + 2\mu_m)(\Delta \varphi_m) = \rho_m \varphi_m; \quad \mu_m(\Delta \psi_m) = \rho_m \psi_m. \]  

(2)

Here \( \lambda_m, \mu_m, \rho_m \) \((m = 0, 1, 2)\)-elastic coefficients (Lame) and bulk density of the layers; \( \varphi_m \) and \( \psi_m \)-some potential functions to be determined; \( \Delta \)-two-dimensional Laplace operator.

Transverse vibrations of the plate are excited under boundary conditions

\[ \sigma_{zz}^{(0)} = \sigma_{zz}^{(i)}, \quad \tau_{xz}^{(0)} = \tau_{xz}^{(i)}, \quad \tau_{yz}^{(0)} = 0 \]

and

\[ u_0 = u_i, \quad w_0 = w_i, \quad i = \{1, n\} \text{ \(z = h_0\)}, \quad \{2, n\} \text{ \(z = -h_0\)}. \]

(4)

The initial conditions of the problem are considered zero.

When defining the displacement components in the form (1), the stresses are given by the expressions

\[ \sigma_{xx}^{(m)} = \lambda_m (\Delta \varphi_m) + 2\mu_m \left( \frac{\partial^2 \varphi_m}{\partial x^2} - \frac{\partial^2 \psi_m}{\partial z^2} \right), \quad \sigma_{zz}^{(m)} = \lambda_m (\Delta \varphi_m) + 2\mu_m \left( \frac{\partial^2 \varphi_m}{\partial z^2} + \frac{\partial^2 \psi_m}{\partial x^2} \right), \]

\[ \sigma_{xz}^{(m)} = \mu_m \left( \frac{\partial^2 \varphi_m}{\partial x\partial z} + \frac{\partial^2 \psi_m}{\partial x\partial z} \right). \]

2.2. The equations of oscillation of a three-layer plate

For solving assigned task, the functions of external influences from (3) can be represented as [20]

\[ f_1(x, t) = \int_{\alpha}^{\infty} \frac{\sin kx}{\cos kx} \int_{\alpha}^{\infty} f_1(k, p)e^{ipt} dp, \quad f_2(x, t) = \int_{\alpha}^{\infty} \frac{\cos kx}{\sin kx} \int_{\alpha}^{\infty} f_2(k, p)e^{ipt} dp, \]

(5)

where \( f_1(k, p), f_2(k, p) \)-functions, regular at \( \text{Re } p \geq 0 \), having a finite number of poles, taking arbitrary values within a certain area \( \Omega(k, p) \), containing the spacing of the imaginary axis \( (-i\omega_0, i\omega_0) \), decreasing when not slower at \( p \to \mp i\infty \), then \( |p| \approx \infty, \text{где } n_0 \gg 1 \), and such like \( \Omega(k, p) \) outside their values are negligible.

In accordance with the accepted representations for the function of external influences \( \varphi_m(x, z, t) \) and \( \psi_m(x, z, t) \), we also represent potential functions in the form (3.1), substitution of which in (2) gives the ordinary Bessel differential equations with respect to the functions transformed by (5) \( \tilde{\varphi}_m(z, k, p) \) and \( \tilde{\psi}_m(z, k, p) \) [21]

\[ \frac{d^2 \tilde{\varphi}_m}{dz^2} - \alpha_m^2 \tilde{\varphi}_m = 0, \quad \frac{d^2 \tilde{\psi}_m}{dz^2} - \beta_m^2 \tilde{\psi}_m = 0, \quad (m = 0, 1, 2) \]

(6)

where \( \alpha_m^2 = k^2 + \rho_m \rho^2 (\lambda_m + 2\mu_m) \); \( \beta_m^2 = k^2 + \rho_m \rho^2 \mu_m \); \( \text{arg } \alpha = \text{arg } \beta = 0 \), at \( \rho > 0 \).
The solutions of equations (6) in the case of transverse vibrations of the plate, taking into account the antisymmetric effects in the boundary conditions (3), will be

$$\bar{\varphi}_m(z, k, p) = A_m^s(k, p)sh(a_mz), \quad \varphi_m(z, k, p) = B_m^s(k, p)ch(b_mz), \quad (m = 0, 1, 2)$$

(7)

Having expressed the transformed displacements $\bar{U}_m$ and $\bar{W}_m$ through solutions (3,3), we expand the hyperbolic functions in them in power series. Further, as the desired functions, we take the main parts of the transformed displacements $\bar{U}_0$ and $\bar{W}_0$ such a surface of the middle layer of the plate, distance from the surface $z = 0$, which is equal to $\xi = 0$, where $\xi = const$ number which is an satisfying inequality $-h_s < \xi < h_s$. For this, in the decompositions mentioned, we restrict ourselves to the zero approximation and accept $z = \xi$ and $m = 0$. Then introducing the notation $\bar{U}_0(0)$ and $\bar{W}_0(0)$ for the main parts of the transformed displacements we obtain following

$$(\beta_0^2 - k^2)B_0^s = k\bar{W}_0(0) + \frac{1}{\xi} \bar{U}_0(0); \quad \alpha_s(\beta_0^2 - k^2)A_0^s = \beta_0^2\bar{W}_0(0) - \frac{k}{\xi} \bar{U}_0(0).$$

(8)

Substituting (8) into the decompositions of the transformed displacements for $m = 0$, we obtain

$$\bar{U}_0 = \sum_{n=0}^{\infty} q_0 Q_0^m \left( \frac{k}{\xi} U_0^m - \beta_0^2 W_0^m \right) - \frac{\beta_0^2}{\xi} U_{00}^m \left( \frac{\xi}{2\nu} + 1 \right) e^{-\xi},$$

$$\bar{W}_0 = \sum_{n=0}^{\infty} q_0 Q_0^m \left( \frac{k}{\xi} U_0^m - \beta_0^2 W_0^m \right) + \frac{\beta_0^2}{\xi} W_{00}^m \left( \frac{\xi}{2\nu} + 1 \right) e^{-\xi},$$

where

$$Q_0^m = (\alpha_0^2 - \beta_0^2) \left( \alpha_0^2 - \beta_0^2 \right), \quad Q_0^0 = 0, \quad Q_0^1 = 1, \quad Q_0^2 = \alpha_0^2 + \beta_0^2, \ldots, m = 0, 1, 2; q_0 = 1 - L_m M_m.7$$

Substituting solutions (7) into contact conditions (4), at $z = h_0$ we obtain the equations

$$kA_0^s sh(\alpha_0 h_0 - \beta_0 B_0^s) sh(\beta_0 h_0) = kA_0^s sh(\alpha_0 h_0 - \beta_0 B_0^s) sh(\beta_0 h_0),$$

$$\alpha_0 \alpha_0^2 ch(\alpha_0 h_0 - \beta_0 B_0^s) ch(\beta_0 h_0),$$

which make up a system of two algebraic equations for two unknowns $A_1^s$ and $B_1^s$. Having solved this system, we will have

$$A_1^s = \frac{1}{(\beta_0^2 - k^2)\Delta_1^s} \left[ \frac{\beta_0^2}{\alpha_0^2} \Delta_1^s + k \Delta_1^s \right] \bar{W}_0^m - \frac{1}{\xi} \left[ \frac{k}{\alpha_0^2} \Delta_1^s + k \Delta_1^s \right] \bar{U}_0^m,$$

$$B_1^s = \frac{1}{(\beta_0^2 - k^2)\Delta_1^s} \left[ \frac{\beta_0^2}{\alpha_0^2} \Delta_1^s + k \Delta_1^s \right] \bar{W}_0^m - \frac{1}{\xi} \left[ \frac{k}{\alpha_0^2} \Delta_1^s + k \Delta_1^s \right] \bar{U}_0^m,$$

where

$$\Delta_1^s = \alpha_0 \alpha_0^2 sh(\beta_0 h_0) ch(\alpha_0 h_0) k^2 sh(\alpha_0 h_0) ch(\beta_0 h_0) \left[ \alpha_0 \alpha_0^2 ch(\alpha_0 h_0) ch(\beta_0 h_0) \right],$$

$$\Delta_1^s = k \alpha_0 \alpha_0^2 sh(\alpha_0 h_0) ch(\alpha_0 h_0) k^2 sh(\alpha_0 h_0) ch(\beta_0 h_0) \left[ \alpha_0 \alpha_0^2 ch(\alpha_0 h_0) ch(\beta_0 h_0) \right].$$

On the other hand, substituting solutions (7) into contact conditions (4) for $z = -h_0$, we find $A_2^s$, $B_2^s$. We represent the stresses $\sigma_{xz}^{(m)}$, as well as (5).Then, boundary conditions (3) can be written as

$$\sigma_{xz}^{(m)}(2k \alpha_0 A_0^2 ch(\alpha_0 (h_0 + h_1)) - \beta_0^2 + k^2) B_0^2 ch(\beta_0 (h_0 + h_1)) = \bar{f}(k, p),$$

$$-k \beta_0^2 B_0^2 sh(\alpha_0 (h_0 + h_1)) + k \beta_0^2 A_0^2 sh(\alpha_0 (h_0 + h_1)) = \bar{f}(k, p).$$

(9)

Substituting the expressions of constants $A_1^s$, $B_1^s$ into (9) and expanding the trigonometric functions in series, on the left-hand sides of (9) by the degrees of the coordinate $z$ and by reversing the system of equations obtained in this way, we will have the general equations of the transverse vibrations of a three-layer plate. These equations have infinitely high orders in derivatives. We assume that the truncation conditions for the infinite series indicated are satisfied and will be limited to zero or first approximations in the expansions. As a result, we obtain approximate equations of vibration of
a three-layer plate for solving applied problems in which we pass to dimensionless variables by the formulas

\[ b_i t = t' l, \quad U_0(t) = U' l, \quad W_0(t) = W'_h h_0, \quad z = z' h_0, \quad x = x' l, \quad \xi = \xi' h_0, \quad h_1 = h'_1 h_0, \quad h_2 = h'_2 h_0 \]

we obtain the equations:

\[
\frac{W_0(\xi)}{l^2} = \left( \frac{\partial^2}{\partial \xi^2} - \frac{\partial^2}{\partial x^2} \right) \frac{\partial^2 W_0(\xi)}{\partial x^2} - \frac{h'_0}{6\xi^2 l^2} \left[ \left( 2 - \frac{b'_0}{a'_0} \right) \frac{\partial^2 \xi}{\partial t^2} + \left( 1 + 2q_0 \right) \frac{\partial^2 \xi}{\partial x^2} + \frac{6\xi^2}{h'_0} \frac{\partial^2 \xi}{\partial x^2} + 8q_0 \left( 1 + h'_1 \right) \frac{\partial^4 \xi}{\partial x^4} \right] \frac{\partial U_0(\xi)}{\partial \xi} =
\]

\[
= \frac{\partial^2 f_1^{(2)}}{\partial t^2} + \frac{4h'_0 b'_2 h'_0}{3l^2} q_1 \left( 1 + h'_1 \right) \frac{\partial^4 f_1^{(1)}}{\partial x^4} + \left( 1 + h'_1 \right) \frac{\partial^2 f_1^{(1)}}{\partial t^2},
\]

\[
\frac{1}{l^2} \left[ \left( 1 - 2q_0 \right) \frac{\partial^2 \xi}{\partial t^2} - \frac{\partial^2 \xi}{\partial x^2} \right] 2 \frac{l^2}{h'_0} \frac{\partial^2 \xi}{\partial t^2} + \frac{8b'_0 q_2 \left( 1 + h'_1 \right)}{3h'_0} \frac{\partial^4 \xi}{\partial x^4} \right] U_0(\xi) +
\]

\[
= \frac{2l}{h'_0} \frac{\partial^2 f_2^{(1)}}{\partial t^2} + q_1 h_2 \left[ 2 + h'_2 \left( \frac{\partial^2 \xi}{\partial t^2} - \frac{b'_2}{b'_0} \frac{\partial^2 \xi}{\partial x^2} \right) \right] \frac{\partial U_0(\xi)}{\partial \xi}.
\]

Where \( a_0 \) - velocity of longitudinal waves in the material of the middle layer; \( b_0, b_1, b_2 \) - shear wave velocities in layer materials; \( l \) - plate length.

2.3. The stress-strain state of the plate

Along with the vibration equations, formulas are derived for all components of the stress tensors and displacement vectors at the points of all three layers of the plate. For example, the expressions for the displacement \( W_0 \), as well as the stresses \( \sigma_{zz}^{(0)} \) at the points of the middle layer, corresponding to the degrees of the oscillation equations (10) have the form

\[
W_0 = \left\{ 1 + \frac{z^2}{2} \left( 1 - q_0 \right) \xi + \frac{z^4}{24} \left[ \frac{1 - q_0}{b'_0} \frac{\partial^4 \xi}{\partial t^4} - \left( \frac{1 - 2q_0}{b'_0} + \frac{1 - q_0}{a'_0} \right) \frac{\partial^4 \xi}{\partial x^4} + \left( 1 - 2q_0 \right) \frac{\partial^4 \xi}{\partial x^4} \right] \right\} W_0(\xi) -
\]

\[
- \frac{1}{\xi} \left[ \frac{z^2}{2} q_0 \xi + \frac{z^4}{24} q_2 \left[ \frac{1}{b'_0} + \frac{1}{a'_0} \right] \frac{\partial^2 \xi}{\partial t^2} - 2 \frac{\partial^2 \xi}{\partial x^2} \right] \frac{\partial U_0(\xi)}{\partial \xi}, \quad (11)
\]

\[
\sigma_{zz}^{(0)} = \mu_0 \left[ 1 + \frac{b'_2}{b'_0} \frac{\partial^2 \xi}{\partial t^2} - \frac{\partial^2 \xi}{\partial x^2} \right] W_0(\xi) + \frac{1}{\xi} \left[ 1 + \frac{z^2}{2} \left( \frac{\partial^2 \xi}{\partial t^2} - \left( 1 - 2q_0 \right) \frac{\partial^2 \xi}{\partial x^2} \right) \right] U_0(\xi).
\]

Here \( \xi = \frac{1}{b'_0} \frac{\partial^2 \xi}{\partial t^2} - \frac{\partial^2 \xi}{\partial x^2} \).

Expressions for stresses have similar forms. \( \sigma_{zz}^{(0)} \) and \( \sigma_{zz}^{(0)} \). The derivation of formulas for displacements and stresses at the points of the upper and lower bearing layers is also not difficult, but they are more cumbersome. Therefore, for example, we give these formulas in the operator form for displacements \( U_1 \) and \( W_1 \), as well as for stress \( \sigma_{zz}^{(1)} \), which have the form

\[
W_i = L_\xi \left[ N_i W_0^{(0)} + N_2 \frac{\partial W_0^{(0)}}{\partial x} \right], \quad U_i = L_\xi \left[ M_i \frac{\partial W_0^{(0)}}{\partial x} + M_s U_0^{(0)} \right], \quad \sigma_{zz}^{(1)} = \mu_1 L_\xi \left( R_1 \frac{\partial W_0^{(0)}}{\partial x} + R_1 U_0^{(0)} \right), \quad (12)
\]

here \( L_\xi, L_n, L_\sigma \) - fourth-order linear differential operators; through \( L_\xi \) the inverse operator is indicated \( L_\xi ; N_i, M_i, R_1 \) - also linear differential operators of no higher than fourth order.
The above expressions for the stress and displacement components (11) and (12) make it possible to determine the stress-strain state of an arbitrary point of a three-layer plate from the main parts $W^{(0)}_0$ and $U^{(0)}_0$ from the results of solving differential equations (10).

2.4. Harmonic vibrations of a three-layer plate

As an example, we consider the problem of antisymmetric (transverse) harmonic vibrations of a three-layer plate based on the obtained approximate equations of oscillation. It should be considered that the plate surfaces are free from external loads. Then the right-hand sides of the oscillation equations (10) will be equal to zero. The solution of differential equations (10) with zero right-hand sides will be sought in the form

$$W^{(0)}_0 = \overline{W}_0 e^{a r - k z}, \quad U^{(0)}_0 = \overline{U}_0 e^{a r - k z},$$

(13)

where $\omega$ - circular frequency; $k$ – wave number. Substituting (13) into the oscillation equations, we have a system of two homogeneous algebraic equations with respect to $\overline{W}_0$ and $\overline{U}_0$

$$a_{11}\overline{W}_0 + a_{12}\overline{U}_0 = 0, \quad a_{21}\overline{W}_0 + a_{22}\overline{U}_0 = 0,$$

(14)

where

$$a_{11} = \frac{b_0^2 h_0^3}{b_1^3 l^3} \omega^4 - \frac{b_0^2 h_0^3}{b_1^3 l^3} \omega^2 k^2,$$

$$a_{22} = \frac{1}{2 \xi} \left[ \frac{b_0^2 h_0^3}{2 b_2^3 l^2} \left( \frac{b_0^2 h_0^3}{b_0^3 a_0^2} (1 + 2 q_0) \omega^2 k^2 - \frac{4 h_0^3}{3 l^2} q_2 (1 + h_2) k^4 + \frac{b_0^2 h_0^3}{b_2} \omega^2 k^2 \right) \right],$$

$$a_{12} = -\frac{k}{\xi} \left[ \frac{b_0^2 h_0^3}{2 b_2^3 l^2} \left( \frac{b_0^2 h_0^3}{b_0^3 a_0^2} (1 + 2 q_0) \omega^2 k^2 - \frac{4 h_0^3}{3 l^2} q_2 (1 + h_2) k^4 - \frac{b_0^2 h_0^3}{b_2} \omega^2 k^2 \right) \right],$$

$$a_{21} = -\frac{k}{\xi} \left[ \frac{4 h_0^3}{3 l^2} q_2 (1 + h_2) k^4 + \frac{b_0^2 h_0^3}{b_2^3 l^4} \omega^2 + \frac{b_0^2 h_0^3}{2 b_2^3 l^4} (1 - 2 q_0) \omega^4 - \frac{b_0^2 h_0^3}{b_2} \omega^2 k^2 \right].$$

From (14) follows the frequency equation

$$a_{11} \cdot a_{22} - a_{12} \cdot a_{21} = 0.$$  

(15)

The last equation (15) was solved numerically using the “Maple 17” application package. In this case, the calculations were performed for steel and aluminum bearing layers of the plate. The values of their physical and mechanical material parameters are as follows: steel - $E=2.0\cdot10^{11}$Па; $\nu=0.25$; $\rho=7850$ $\text{kg/m}^3$; aluminum - $E=0.7\cdot10^{11}$Па; $\nu=0.35$; $\rho=2750$ $\text{kg/m}^3$. The following materials and their physical and mechanical parameters are taken as a filler: polymer - $E_0=5.5\cdot10^{10}$Па; $\nu=0.4$; $\rho=1700$ $\text{kg/m}^3$; fiberglass - $E_0=1.8\cdot10^{10}$Па; $\nu=0.35$; $\rho=1400$ $\text{kg/m}^3$; wood plastic - $E_0=1.2\cdot10^{10}$Па; $\nu=0.35$; $\rho=1200$ $\text{kg/m}^3$ and tantalite - $E_0=0.4\cdot10^{10}$Па; $\nu=0.35$; $\rho=1300$ $\text{kg/m}^3$. The geometric characteristics of the three-layer plate are as follows: thickness of the outer layers $h_1 = h_2 = 0.001$ м; aggregate thickness - $h_0 = 0.03$; 0.05; 0.1 м.

3. Results and Discussions

The results of the calculations are shown in Figures 2-5 as the dependences of the lowest frequency $\omega$ on the wave number $k$. Figure 2 shows the curves of the dependence $\omega \sim k$ of a three-layer plate with the same thicknesses of steel bearing layers (a plate of a symmetrical structure is considered) equal to $h_1 = h_2 = 0.001$ м. At the same time, three values of the thickness of the aggregate polymer are considered for which $h_0 = 0.03$; 0.05; 0.1 м. It is easy to see that for all cases the thickness dependences $\omega \sim k$ are directly proportional. From the graphs it follows that for a fixed value of the wave number, an increase in the thickness of the middle layer of the plate leads to an increase in the oscillation fre-
quency. For example, the frequency values corresponding to the values $h_0 = 0.03; 0.05; 0.1$ m at differ from the value at $h_0 = 0.03$ by 61% and 178%, respectively. With an increase in the wave number, i.e. with the transition to a higher frequency region, these differences increase more and more.

A comparison of Figure 2 and Figure 3 shows that under identical conditions the oscillation frequency of a plate with steel bearing layers is always less than that of a plate with aluminum bearing layers. But, the difference is small. For example, at $k = 10$ the specified difference is 0.05, which is a percentage of 4%. At the same time, the oscillation frequency of the plate strongly depends on the filler material, which is visible from a comparison of the graphs in Figure 3 and Figure 4. A plate with a filler with a large value of the elastic modulus (polymer in Figure 3) has a lower oscillation frequency than with a filler with a lower value of the elastic modulus (fiberglass in Figure 4). For example, at $k = 7$ the difference is; for $h_0 = 0.1 - 0.98$ ($\approx 120\%$); for $h_0 = 0.05 - 0.2$ ($\approx 42\%$); for $h_0 = 0.03 - 0.15$ ($\approx 52\%$).

Figure 5 shows the dependency graphs $\omega \sim k$ for the thicknesses of steel bearing layers equal to $h_1 = h_2 = 0.001$ at $h_0 = 0.03$. As a filler, polymer, fiberglass, wood plastic and tantalite are accepted, the values of the physical-mechanical parameters of which are given above. The presented results in Figure 5 confirm the earlier conclusion that a plate with a filler with large values of the elastic modulus and density has a lower oscillation frequency than with a filler with lower values of the modulus of elasticity and density.

![Figure 2](image1.png)

**Figure 2.** According to $\omega$ from $k$ at $h_1 = h_2 = 0.001$ and various $h_0$. The materials of the supporting layers are steel, and the filler is polymer.

![Figure 3](image2.png)

**Figure 3.** According to $\omega$ from $k$ at $h_1 = h_2 = 0.001$ and various $h_0$. The materials of the supporting layers are aluminum, and the filler is polymer.

![Figure 4](image3.png)

**Figure 4.** According to $\omega$ from $k$ at $h_1 = h_2 = 0.001$ and various $h_0$ (the materials of the supporting layers are aluminum, and the filler is fiberglass).

![Figure 5](image4.png)

**Figure 5.** According to $\omega$ from $k$ at $h_1 = h_2 = 0.001$; $h_0 = 0.03$ (the materials of the bearing layers - steel, filling-different (polymer, fiberglass, wood plastic, tantalite)).
The smallest values of the elastic modulus and density are from the given series of filler materials of the PCB, which corresponds to large values of frequency (Figure 5). For example, at $k = 7$ the frequency value for tantalite is 0.86, and for polymer 0.26. In this case, the frequencies of the fiberglass plate are lower (0.44) compared to wood plastic (0.48), despite the fact that the elastic modulus of fiberglass is greater than that of wood plastic. This is because wood plastic is much denser than fiberglass.

4. Conclusions
- the theory of non-stationary transverse vibrations of an elastic three-layer kin-plate is developed based on general solutions in transformations of equations of the theory of elasticity, in a flat setting;
- the developed theory allows us to calculate all the components of the displacement vector and the stress tensor in the sections of the plate as a whole and of all layers through the introduced main parts of the intermediate surface of the middle layer;
- the obtained general equations of vibration make it possible to obtain refined equations of the Tymoshenko type and approximate equations of the Kirchhoff type, which can be applied to solve applied problems of engineering practice;
- from a comparative analysis of the obtained numerical results it follows that the vibration equations and formulas for determining the SSS developed in the work allow a high degree of reliability to determine the frequencies of antisymmetric vibrations of three-layer plates. Moreover, the frequency analysis performed on the basis of the presented model requires minimal computational resources;
- regardless of the thickness of the middle layer, the dependence of the frequency on the wave number is directly proportional. For a fixed value of the wave number, an increase in the thickness of the middle layer of the plate leads to an increase in the vibration frequency, which strongly depends on the filler material. A plate with filler with large values of the elastic modulus and density has a lower oscillation frequency than with filler with lower values of the modulus and density.

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