COMMENT

Comment on “Boson-fermion model beyond the mean-field approximation”

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Abstract. In a recent paper [Alexandrov A S 1996 J. Phys.: Condens. Matter 8 6923–32; cond-mat/9603111], it has been suggested that there is no Cooper pairing in boson-fermion models of superconductivity. We show that this conjecture is based on an inconsistent approximation that violates an exact identity. Quite generally, the divergence of the fermion t-matrix (the Thouless criterion) is accompanied by the condensation of a boson mode.

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In a recent publication \[1\], Alexandrov has argued that there is no Cooper pairing in boson-fermion models \[2, 3\]. It was suggested that the Thouless criterion for the onset of long-range order is inconsistent with the requirement that the physical energy of a boson be non-negative.

We would like to point out in this respect that the proof is based on two approximations and therefore is not exact. (a) The fermion t-matrix is approximated by a ladder series. (b) The boson self-energy is approximated by a single two-fermion bubble. Below we will show that there exists an exact identity relating these two quantities. Therefore, choosing an approximation for the t-matrix fixes the form of the approximate boson propagator. Alexandrov’s choice (b) violates that identity and makes his approximation inconsistent (in the sense of Baym and Kadanoff \[4\]).

Although Equation 7 in \[1\], which determines $T_c$, is derived from the linearized gap equation, an alternative (but fully equivalent) derivation uses the divergence of the fermion t-matrix at zero energy and momentum as a criterion for the transition \[5\]. Alexandrov adopts an effective short-range fermion interaction in the form $V = v^2 D_0(0,0) + V_c$ (the first two diagrams in Figure 1), where $D_0(q, \Omega_n)$ is the free boson propagator, $v$ is the boson-fermion coupling, and $V_c$ is a short-range fermion repulsion. In the ladder approximation, one obtains the usual result for the t-matrix:

$$T(0,0) = \frac{V}{1 - VB},$$

where, in Alexandrov’s notation, $B = -N^{-1} \int dp \ G(p)G(-p)$ is a short-hand for a bubble with two fermion lines. The divergence of the t-matrix then requires that

$$1 - [v^2 D_0(0,0) + V_c]B = 0,$$

which is precisely Equation 7 of Alexandrov.

Turning to the boson self-energy, we supplement Alexandrov’s single diagram (a bubble with two fermion lines) by a ladder series to account for Coulomb repulsion $V_c$ between the intermediate fermions (Figure 2). This choice will be justified below. Upon including these diagrams, the boson propagator at $q = 0$ and $\Omega_n = 0$ is given by the
equation
\[ D^{-1}(0, 0) = D_0^{-1}(0, 0) - \frac{v^2 B}{1 - V_0 B}, \] (3)

At the condensation temperature, bosons with zero momentum have zero energy, i.e., \( D(0, 0) = \infty \), and we recover Equation 2.

\[ \text{Figure 2.} \text{ Boson self-energy in the ladder approximation. Only the first graph is included by Alexandrov.} \]

Evidently, when the boson self-energy is properly modified, the Thouless criterion no longer contradicts the sum rule (6) in [1]. Instead, we find that the t-matrix diverges exactly at the condensation temperature of the bosons. This remarkable conclusion is substantiated by an exact identity presented in Figure 3. It indicates that the divergence of \( T(0, 0) \) must be accompanied by that of \( D(0, 0) \).

The proof of the identity uses the equations of motion for a non-self-interacting boson field \( \phi(x, t) \) with a Hamiltonian
\[ H_B = \int dx \phi^\dagger(x, t) \mathcal{H}_B(x) \phi(x, t), \] (4)
where \( \mathcal{H}_B(x) \) is a linear operator acting on \( x \). The boson field is coupled to a fermion field \( \psi_\sigma(x, t) \) by the interaction Hamiltonian
\[ H_{BF} = v \int dx [\phi^\dagger(x, t) \psi_\uparrow(x, t) \psi_\downarrow(x, t) + \text{H.c.}]. \] (5)

In what follows, \( x \) denotes spatial coordinates and the (complex) time \( t \) varies along a straight line between 0 and \( \tau = -i\hbar/k_B T \).

Introduce time-ordered boson (\( D \)), two-fermion (\( G_2 \)) and mixed (\( M \)) propagators
\[ D(x, t; x', t') = -i\langle T[\phi(x, t) \phi^\dagger(x', t')] \rangle, \] (6)
\[ M(x, y, t; x', t') = -i\langle T[\psi_\uparrow(x, t) \psi_\downarrow(y, t) \phi^\dagger(x', t')] \rangle, \] (7)
\[ G_2(x, y, t; x', y', t') = -i\langle T[\psi_\uparrow(x, t) \psi_\downarrow(y, t) \psi_\downarrow(y', t') \psi_\uparrow(x', t')] \rangle. \] (8)

By using equations of motions for \( \phi \) and \( \phi^\dagger \), we obtain
\[ [i\partial/\partial t - \mathcal{H}_B(x)] D(x, t; x', t') = \delta(x - x') \delta(t - t') + vM(x, x, t; x', t'), \] (9)
\[ [-i\partial/\partial t' - \mathcal{H}_B(x')] M(x, x, t; x', t') = vG_2(x, x, t; x', x', t'). \] (10)

These differential relations can be integrated with the aid of the free \((v = 0)\) boson propagator \( D_0(x, t; x', t') \). The following identity is then obtained for the corresponding
Figure 3. An identity for the exact boson propagator (thick dashed line). Thin dashed line: free boson propagator.

Fourier coefficients:

\[ D(x; x'|n) = D_0(x; x'|n) \]
\[ + v^2 \int dx_1 \int dx_2 \ D_0(x; x_1|n)G_2(x_1, x_1; x_2, x_2|n)D_0(x_2; x'|n). \]  

(11)

\[ D(x; x'|n) \] is defined in a standard way,

\[ D(x; t; x'; t') = \frac{1}{\tau} \sum_{n=-\infty}^{\infty} D(x; x'|n) \exp \left[ -i\Omega_n(t - t') \right], \]

(12)

where \( \Omega_n = 2\pi n/\tau \). Finally, we isolate the disconnected part of \( G_2 (GG) \) and express the connected part in terms of the proper vertex \((GGTGG)\), as shown in Figure 3.

We are now in position to justify our choice of Feynman diagrams for the approximate boson self-energy (Figure 2). To this end, we can determine the dressed boson propagator directly from the identity of Figure 3, by using the approximate expression for the t-matrix (Figure 1) as the input. In this way, the exact result (11) is built into the approximate theory from the start. The resulting boson propagator is shown in Figure 4. Higher-order graphs replaced with the dots are generated by similarly omitted graphs of Figure 1. The diagrams with only two free boson lines determine the boson self-energy, which by inspection coincides with our choice (Figure 2).

In conclusion, we have shown that Alexandrov’s conjecture (no Cooper pairing in boson-fermion models) is incorrect. A careful revision of his approach reveals an exact identity that expresses equivalence of the Thouless criterion to Bose condensation in a wide class of boson-fermion models. Being non-self-consistent, the approximation made by Alexandrov violates this identity and therefore leads to a contradiction.

References

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Figure 4. The boson propagator obtained in a self-consistent approximation (by combining Figures 1 and 3). The diagrams are grouped according to the number of free boson lines. The boxed set of graphs (with the external boson lines amputated) gives the boson self-energy.