Preliminary result on stochastic system control theory for aperiodic sample-data systems

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Abstract: In this paper, we obtain some preliminary results on stochastic control theory for time-varying linear systems both continuous and discrete, and further apply to aperiod sample-data linear systems. The Itô’s lemma is utilized in this proposed theory, and deduced that the stability of a linear time-varying system is determined by the eigenvalues expectation of system matrix, which coincidences with the stable conditions for time-invariant system, i.e. Hurwitz for continuous systems or inside the unit circle for discrete systems. The control method for aperiod time-invariant sample-data system is also derived. It is shown that the stable condition is determined by the expectation of the sample-interval but the up-bound and the aperiod interval can be arbitrarily large even infinity. To verify the efficiency of our theory, serval experiments are demonstrated in the final of the paper.

Key Words: Aperiod control;stochastic control system; sample-data system; time-varying system; Itô’s lemma

1 Introduction

Recently, with the increasing demand for networked control in industrial systems, the network controllers and sensors are communicating in a decentralized environment. Due to the existence of communication interference, delay and blocking phenomena, the control sample period becomes non-static, for which the normal discrete control theory cannot be applicable. In order to solve above problems, a new emerging theory on aperiodic sampling system has drawn the attention of scholars, and many control methods are designed and proposed, such as Lyapunov function construction method and event-trigger or self-triggered control strategy. However, the construction of this theory is still facing many significant challenges, it is of great significance to analyze and study the stability and control methods of aperiodic sampling control systems from a systematic perspective.

Aperiod sampling system design goal is that the controller should ensure that the system under non-periodic sampling conditions of the closed-loop system robust stability. Due to the uncertainty and time-varying nature of the control sampling interval of aperiodic sampling system, the stability of the system depends on the characteristics of the controlled object and the sampling period, so it can not be directly studied using the traditional discrete control system theory. At this stage, the research on this kind of system generally includes three kinds of research methods:

The first kind is based on the method of delay modeling, by adding the sampling variation as the control delay into the system modeling, the non-periodic sampling system model is a continuous-time system with a form of input with delay control based on the Lyapunov-krasovskii theory. Such methods can investigate uncertainties with time-varying state equations and system stability conditions under nonlinear structures, but it is often difficult to give controller design methods.

The second type is based on the discrete time modeling method, and the system is modeled as a stable system with unknown sampling delay. Such methods are usually analyzed by selecting the nominal sampling period and the exponential term caused by the variation of sampling period is modeled as the uncertain structure error of the system time-varying, making the system state transition matrix an additive uncertain function and then analyzed Lyapunov stability conditions.

The third type is founded on stochastic theory. Serval stochastic condition and control methods have been obtained according to the stochastic stability theory of continues and discrete systems with support of the Itô’s differential equation theory. Firstly, the stochastic stability theory of continues and discrete systems are achieved, after which the non-periodic sampling system is modeled by the discrete time method, and the stochastic stability is obtained immediately. At last some injecting control method is proposed according to the stochastic stability theory.
2 Problem formulation

Investigate the following aperiodic sampling linear system
\[ \dot{x}(t) = Ax(t) + Bu(t) \]  
(1)

Where \( x \in \mathbb{R}^n \) is the system state, \( u \in \mathbb{R}^m \) is the system input, and the system matrix \( A \) is the Hurwitz matrix of proper dimensionality.

The system performs the sampling control according to the following discrete time series
\[ 0 = t_0 < t_1 < \cdots < t_k < \cdots \]

Control input in each sampling interval to maintain regular segments,
\[ u(t) = Kx(t_k), \quad \forall t \in [t_k, t_{k+1}) \]  
(2)

Define each sampling interval \( T_k \) as
\[ T_k = t_{k+1} - t_k \]

And suppose that \( T_k \) is unsteady and satisfies only the boundedness, that is:
\[ 0 < T_{\text{min}} \leq T_k \leq T_{\text{max}} < \infty, \quad \forall k \]

The system has the following features under the control of \( T \)
\[ x(t_{k+1}) = \Phi(T_k)x(t_k) \]
\[ \Phi(T_k) = e^{A_T} + \int_0^{T_k} e^{A(T_k - \tau)} B d\tau K \]  
(3)

We can further present equation \( T \) with the compact matrix multiply form as in \( 8 \)
\[ x(t_{k+1}) = \begin{bmatrix} I & 0 \end{bmatrix} \exp \left( \begin{bmatrix} A & B \\ 0 & 0 \end{bmatrix} T_k \right) \begin{bmatrix} I \\ K \end{bmatrix} x(t_k) \]  
(4)

If \( T_{\text{min}} = T_{\text{max}} \), ie \( T_k \), the system becomes a periodic sampling control system whose stability \( \Phi(T_k) \) The root position of the decision. But if \( T_k \) is unsteady, the stability condition of the aperiodic sampling control system is \( \forall T_k \in [T_{\text{min}}, T_{\text{max}}] \) is located in the unit circle, which belongs to the solution of infinite-dimensional equations.

As shown above, sample-data systems can be regarded as an time-varying discrete system, which can be further treated as some time-varying sampled continuous system. Therefore the stability of sample-data systems can be determined by the deduced time-varying discrete or continuous system. In the following paper, some novel theory is firstly proposed on the stability of time-varying continuous systems, which is subsequently derived to discrete systems.

3 Stability of linear time-varying systems

The linear time-varying systems with state feedback control system has following dynamic,
\[ \dot{x}(t) = A(t)x(t) \]  
(5)

where \( x \) is state, \( A(t) \) is system matrix.

To analysis stability of the system \( 5 \), we assume that:

Assumption 1 \( A(t) \) is Borel measurable on \( t \in \mathbb{R} \).

Theorem 1 The linear time-varying continuous systems with dynamic \( 5 \) satisfying assumption 7 is stable if the expectation \( E[A(t)] \) is Hurwitz.

Proof According to the It\'o’s lemma, the stochastic differential equation \( 5 \) has unique result under assumption 11
\[ x(t) = e^{A(t)x(t)} \]  
(6)

The expectation of the states is obtained by
\[ E[x(t)] = E\left[ e^{A(t)x(t)} \right] \]  
(7)

According to the assumptions that \( E[A(t)] \) is Hurwitz, the state expectation \( E[x(t)] \) is asymptotic to zero, i.e.
\[ \lim_{t \to \infty} E[x(t)] = 0 \]  
(8)

Further, the expectation of second moment can be obtained as
\[ E[x^T x] = E\left[ e^{A(t)x(t)} \right]^T e^{A(t)x(t)} \]  
(9)

Apparently, since \( A(t) \) is Hurwitz, \( A(t) + A^T(t) \) is also Hurwitz. Assume the eigenvalues of \( A(t) \) is \( \lambda_1, \lambda_2, \ldots, \lambda_n \) whose real part is less than zero, such that \( 2\lambda_1, 2\lambda_2, \ldots, 2\lambda_n \) is the eigenvalues of \( A(t) + A^T(t) \) which all place on the left side of S-space. Therefore, the limit of second moment expectation is also zero,
\[ \lim_{t \to \infty} E[x^T(t)x(t)] = 0 \]  
(10)

With the definition 1, the sufficiency of the stable condition has been achieved.

4 Stabilization control for sample-data systems

In this section, we firstly propose the stochastic stability theorem on discrete systems which can be regarded as proposition of theorem continuous systems. With model \( 3 \), the stable condition and control method is applicable for sample-data systems.

Investigate following discrete system
\[ x(k+1) = F(k)x(k) \]  
(11)

where \( x \) is state, \( k \) is discrete time instance, \( F \) is state transfer function. In this paper, we further assume that \( F(k) \) is nonsingular.

Remark 1 The nonsingular assumption on state transfer function is generally valid when \( F \) is the discretization of some continuous system matrix, e.g. sample-data systems.

Theorem 2 The discrete system \( 11 \) is stochastic stable if the eigenvalues of state transfer function expectation \( F(k) \) are inside of the unit circle, i.e. the norms of any eigenvalue less than 1.
Proof: With assumption 2, we can always found some matrix $A_k$ such that

$$F(k) = e^{A_k}$$  \hspace{1cm} (12)

Construct a piecewise constant function $A(t)$ as

$$A(t) = A_k, \ k \leq t \leq k+1 \text{ for } \forall k \in N \hspace{1cm} (13)$$

which apparently is borel measurable on $t$.

Let function (13) to be the system matrix for linear dynamic (5).

$$\dot{x}(t) = A(t)x(t)$$  \hspace{1cm} (14)

From the Itô’s lemma, the state trajectory satisfies

$$x(t) = e^{A(t)}x(s), \text{ for } k \leq t \leq k+1, \ k-1 \leq s \leq k$$  \hspace{1cm} (15)

Subsequently, that state value (15) is always coincidence with discrete system (11) at time instance $t = 1, 2, \ldots, k, \ldots$.

$$x(t) = x(k) \text{ for } k \leq t \leq k+1$$

According to Theorem 1, if the expectation of $A(t)$ is Hurwitz, the continuous system (14) is stable, which means that the state of the discrete system (11) is also asymptotic to zero. Therefore the discrete system is stable if the expectation of $A(t)$ is hurwitz which also means that the eigenvalues of state transfer function expectation $F(k)$ should be inside of the unit circle by the relation (12).

The stability of aperiodic sample-data systems (3) with discrete time model (4) can be directly obtained with Theorem 2. Let

$$F(k) = \begin{bmatrix} I & 0 \end{bmatrix} \exp \left( \begin{bmatrix} A & B \\ 0 & 0 \end{bmatrix} T_k \right) \begin{bmatrix} I \\ K \end{bmatrix}$$

The expectation of $F(k)$ is derived as

$$E[F(k)] = \begin{bmatrix} I & 0 \end{bmatrix} \exp \left( \begin{bmatrix} E[AT_k] & E[BT_k] \\ 0 & 0 \end{bmatrix} \right) \begin{bmatrix} I \\ K \end{bmatrix} \hspace{1cm} (16)$$

The system stability is determined by the eigenvalues of above expectation (16).

For an aperiodic sample-data system with constant dynamic, i.e., $A(t), B(t)$ are constant matrixes, we can obtain following aperiodic sample-data system stabilization control method.

**Theorem 3** The constant dynamic sample-data system with aperiodic sampling interval $T_k \in [T_{\min}, T_{\max}]$ can be stabilized by finding some constant state feedback gain $K$ such that the following inequality established.

$$\|\text{eig} \begin{bmatrix} I & 0 \end{bmatrix} \exp \left( \begin{bmatrix} A & B \\ 0 & 0 \end{bmatrix} E[T_k] \right) \begin{bmatrix} I \\ K \end{bmatrix} \| < 1$$  \hspace{1cm} (17)

where $\| \cdot \|$ represents 2-norm $\text{eig}(\cdot)$ represents any eigenvalue of the matrix inside the bracket.

Proof: Apparently, omitted.

It is noticeable that since both $A$ and $B$ are constant matrix, only the expectation of the sampling intervals are relevant to the stability. If we have the knowledge of the probability distribution of sampling intervals, the feedback control can be easily determined, without consideration on the range of intervals. Especially, for some particular distribution, the interval up-bound can be arbitrarily large even to infinity, when it may cause the periodic sample-data system unstable with such sample rate.

**5 Example**

To test the validation and the efficiency of our theory and control method, several numerical simulation result is presented in this section. Consider the system in reference [12] described by (5) as an instance

$$\dot{x} = \begin{bmatrix} 1.8 & -1.3 \\ 0 & 1.2 \end{bmatrix} x + \begin{bmatrix} 1 \\ -1.2 \end{bmatrix} u$$

The sampling interval satisfies $T_k \in [0.5, 1.1]$ and obeys continuous uniform distribution.

The state feedback control obtained in (12) is

$$u = \begin{bmatrix} -1.3132 & 1.1614 \end{bmatrix} x$$  \hspace{1cm} (18)

Since sampling interval obeys uniform distribution, the expectation of $T_k$ is $E[T_k] = 0.8$. From Theorem 3 with the state feedback $K = \begin{bmatrix} k_1 & k_2 \end{bmatrix}$ the state transfer function expectation is

$$\begin{bmatrix} I & 0 \end{bmatrix} \exp \left( 0.8 \begin{bmatrix} A & B \\ 0 & 0 \end{bmatrix} \right) \begin{bmatrix} I \\ K \end{bmatrix} = \begin{bmatrix} 4.22 + 2.95 k_1 & -3.49 + 2.95 k_2 \\ -1.61 k_1 & 2.61 - 1.61 k_2 \end{bmatrix}$$  \hspace{1cm} (19)

To ensure the stability, the eigenvalue 2-norm should be less than 1, which concludes that if $K$ should satisfy

$$-11.5232 < 2.0843k_1 - 6.80k_2 < -10.0232$$

Apparently, there are infinity combinations of such $k_1, k_2$. Besides, the feedback (18) in ref [12] is an instance.

By utilizing the feedback (18) in ref. [12], the system still has stable margin to bear even larger up-bound of sampling intervals, which can be easily obtained by Theorem 3 as

$$T_k \in [0.5, 1.568]$$  \hspace{1cm} (20)

under continuous uniform distribution.

In the simulation, the initial value is set as $x_0 = \begin{bmatrix} 1 & 0.8 \end{bmatrix} T$, and sampling intervals obey the new bound (20). The simulation result is presented in following figure.

**6 Conclusion**

We have discussed some Preliminary results on the stability of time-varying systems, and further applied to aperiodic sample-data systems stabilization control method. The time-varying system is modeled as stochastic system, and analyzed
with Itô’s differential theory. We found that the stable condition is determined by the expectation of eigenvalues of the system matrix, real parts less than zero for continuous systems and norms less than 1 for discrete systems. Aperiodic sample-data systems are regarded as time-varying discrete system in this paper, and the stable condition is achieved straightly. For linear time-invariant aperiodic sample-data systems, the stable condition is determined by the expectation of sampling interval but the bonds, by which the stabilization control method has been proposed. The experiment results show the validation of our theory in which the sample-interval upper bound is extremely large than existing method.

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