On Topological Terms in the $O(3)$ Nonlinear Sigma Model

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Abstract. Topological terms in the $O(3)$ nonlinear sigma model in $(1 + 1)$ and $(2 + 1)$ dimensions are re-examined based on the description of the $SU(2)$-valued field $g$. We first show that the topological soliton term in $(1 + 1)$ dimensions arises from the unitary representations of the group characterizing the global structure of the symmetry inherent in the description, in a manner analogous to the appearance of the $\theta$-term in Yang-Mills theory in $(3 + 1)$ dimensions. We then present a detailed argument as to why the conventional Hopf term, which is the topological counterpart in $(2+1)$ dimensions and has been widely used to realize fractional spin and statistics, is ill-defined unless the soliton charge vanishes. We show how this restriction can be lifted by means of a procedure proposed recently, and provide its physical interpretation as well.

PACS codes: 11.10.Kk; 11.27.+d; 03.65.-w

Keywords: Hopf term; Fractional spin; Solitons; Non-linear sigma model

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1. Introduction

The $O(3)$ nonlinear sigma model (NSM) is a model ubiquitous in physics, being used in fields ranging from condensed matter physics to high energy physics. It describes physical systems that undergo a spontaneous breakdown of a global symmetry $O(3)$ by means of a vector field $\mathbf{n}(\mathbf{x}, t)$ with unit length $|\mathbf{n}|^2 = \sum_{\alpha=1}^{3} (n^\alpha)^2 = 1$. The dynamics of the model is governed by the Lagrangian,

$$L_0(\mathbf{n}) = \frac{1}{2\lambda^2} (\partial_\mu \mathbf{n})^2, \quad (1.1)$$

where $\lambda$ is a coupling constant and $\mu$ runs from 0 to the spacetime dimension $d + 1$. It is customary to assume that the field approaches to a constant vector $\mathbf{n}_0$ at spatial infinity,

$$\mathbf{n}(\mathbf{x}, t) \rightarrow \mathbf{n}_0 \quad \text{as} \quad |\mathbf{x}| \rightarrow \infty. \quad (1.2)$$

Due to this boundary condition (1.2), at a fixed time $t$ the field $\mathbf{n}(\mathbf{x}, t)$ can be regarded as a map from the (effective) space $S^d$ to the target $S^2$ with the fixed value $\mathbf{n}_0$ assigned to the image of spatial infinity. In other words, the configuration space $C_d$ of the model is given by the set of based maps from $S^d$ to $S^2$, i.e., $C_d = \text{Map}_0(S^d, S^2)$.

It has been widely known that the $O(3)$ NSM admits topological terms which can be added to (1.1). The best known topological term is the soliton term in $(1+1)$-dimensions,

$$L_{\text{soliton}} = \frac{\hbar \theta}{8\pi} \epsilon_{abc} n^a \partial_\mu n^b \partial_\nu n^c, \quad (1.3)$$

which is formed out of the volume element of the target space $S^2$. The presence of $\hbar$, along with the angle parameter $\theta$, signals the fact that the term is of quantum origin. On the other hand, in $(2+1)$ dimensions one has the Hopf term, which has been used to bestow fractional spin and statistics upon instanton (skyrmion) excitations [1, 2, 3, 4]. In terms of the field strength $F_{\mu\nu} := -\epsilon^{abc} n^a \partial_\mu n^b \partial_\nu n^c$ and the connection $A_\lambda$ given as a solution to $F_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu$, the conventional Hopf term used in the literature takes the form of the Chern-Simons term,

$$L_{\text{Hopf}} = -\frac{\hbar \theta}{32\pi^2} \epsilon^{\mu\nu\lambda} F_{\mu\nu} A_\lambda, \quad (1.4)$$

where again $\theta$ is an angle parameter.

The two topological terms mentioned above share the origin in that they arise from the same topological structure characterized by the fundamental group of the respective configuration space,

$$\pi_1(C_1) = \pi_2(S^2) = \mathbb{Z}, \quad \pi_1(C_2) = \pi_3(S^2) = \mathbb{Z}, \quad (1.5)$$
which are derived from the identities for homotopy groups $\pi_k(\text{Map}_0(S^n, S^m)) = \pi_{k+n}(S^m)$ valid for non-negative integers $k$ (see, e.g., [5]). However, the two terms are not quite the same in status because of the difference in the structure of connectedness,

$$\pi_0(C_1) = \pi_1(S^2) = 0, \quad \pi_0(C_2) = \pi_2(S^2) = \mathbb{Z}. \quad (1.6)$$

The disconnectedness of the space $C_2$ suggests that there are solitons/antisolitons which may hamper the topological term (1.4) to be defined firmly. In fact, this has been pointed out earlier in [5, 6] where it was shown that the Hopf term in the conventional form (1.4) is well-defined only in the vanishing soliton number sector.

The prime aim of the present paper is to examine closely the Hopf term in the (2+1)-dimensional NSM, and thereby show that the procedure proposed in [5, 6] to extend the applicable domain of the term to the whole configuration space is viable mathematically and also natural from physical point of view. Our argument is based primarily on the adjoint orbit parameterization (AOP) of the model where one uses a group ($SU(2)$)-valued field $g$ instead of the vector field $n$ in the original description. The AOP has been introduced in [7, 8] to describe the NSM in the general framework of $G/H$ coset models, but it turns out to be advantageous over the original description especially in treating topological quantities, which is a property crucial for implementing the procedure proposed.

We also present here a path-integral derivation of the soliton term (1.3) — which is well-defined for any configurations in the (1+1)-dimensional NSM — by a method similar to that used to induce the $\theta$-term in Yang-Mills theory in (3+1) dimensions [9]. This will highlight the analogy of the two terms as a quantum mechanically induced topological term, and at the same time elucidate the usefulness of the AOP for the NSM.

We begin our discussion by reviewing the Hamiltonian formulation of the model in $(d+1)$ dimensions using the AOP in section 2. We then show in section 3 that in the (1+1)-dimensional NSM the soliton term arises from the unitary representations of the fundamental group associated with the gauge symmetry inherent to the AOP. In section 4, the Hopf term in the (2+1)-dimensional NSM is examined in detail in the original description, and then in section 5 it is re-examined in the AOP to present the procedure of extension and its physical interpretation. Section 6 is devoted to our conclusion and discussions, including the possibility of the Hopf term being a topological ‘Wess-Zumino term’ in (1+1) dimensions.

### 2. Hamiltonian formulation in the AOP

The AOP is a description of the NSM identifying the target space of the field $n(x, t)$ as a nontrivial adjoint orbit of a Lie group $G$. For our $O(3)$ NSM, we take $G = SU(2)$ and
use \( \{ T_a = \sigma_a/2i; a = 1, 2, 3 \} \) for a basis of the Lie algebra \( \mathfrak{su}(2) \). With an \( SU(2) \)-valued field \( g(\mathbf{x}, t) \) we then consider

\[
g T_3 g^{-1} = n^a T_a, \tag{2.1}
\]

where the l.h.s. forms the adjoint orbit passing through \( T_3 \), which is then identified with the target \( S^2 \) of \( \mathfrak{n}(\mathbf{x}, t) \) satisfying \( |\mathfrak{n}| = 1 \). Note that the AOP possesses inherent ambiguity associated with the subgroup \( U(1) \subset SU(2) \) generated by the element \( T_3 \). Indeed, \( \mathfrak{n}(\mathbf{x}, t) \) is unchanged under the transformations

\[
g(\mathbf{x}, t) \rightarrow g(\mathbf{x}, t) h(\mathbf{x}, t), \tag{2.2}
\]

for any (smooth) function \( h(\mathbf{x}, t) \in U(1) \). On account of this ambiguity, the boundary condition for \( g(\mathbf{x}, t) \) which corresponds to (1.2) for \( \mathfrak{n}(\mathbf{x}, t) \) becomes

\[
g(\mathbf{x}, t) \rightarrow g_0 k(\mathbf{x}, t) \quad \text{as} \quad |\mathbf{x}| \rightarrow \infty, \tag{2.3}
\]

where \( g_0 \in SU(2) \) is a constant element fulfilling \( g_0 T_3 g_0^{-1} = n_0^a T_a \) and \( k(\mathbf{x}, t) \in U(1) \) is an arbitrary function defined at spatial infinity. The boundary condition (2.3) implies that, at a fixed time, unlike \( \mathfrak{n} \) which can be regarded as a map \( S^d \rightarrow S^2 \), the map \( g \) used in the AOP can only be treated as a map from a \( d \)-dimensional disc to the group, \( D^d \rightarrow SU(2) \), with \( \partial D^d \simeq S^{d-1} \) being identified with spatial infinity. Under the AOP the original Lagrangian (1.1) becomes\(^3\)

\[
\mathcal{L}_0(g) = \frac{1}{2\lambda^2} \text{Tr} (g^{-1} \partial_\mu g|_\tau)^2. \tag{2.4}
\]

The Lagrangian (2.4) has a (trivial) local \( U(1) \) symmetry under (2.2) due to the ambiguity of the AOP mentioned above.

In order to construct the Hamiltonian formulation of the NSM in \( (d + 1) \)-dimensional spacetime, we adopt Dirac’s procedure for constrained systems and introduce a set of \( SU(2) \) variables \( \xi^a(\mathbf{x}) \), \( a = 1, 2, 3 \), to parametrize \( g(\mathbf{x}) = g(\xi^a(\mathbf{x})) \). From their conjugate momenta \( \pi_a = \partial \mathcal{L}(g)/\partial \dot{\xi}^a \) and the matrix \( N^a_b T_b := g^{-1}(\partial g/\partial \xi^a) \), we define

\[
J^a = (N^{-1})^a_b \pi_b = \begin{cases} \frac{1}{\lambda^2} N^c_a \dot{\xi}^c, & a = 1, 2, \\ 0, & a = 3, \end{cases} \tag{2.5}
\]

with \( (N^{-1})^a_b \) being the inverse of \( N^a_b \), that is, \( (N^{-1})^a_b N^b_c = N^a_b (N^{-1})^b_c = \delta^a_c \). The point to be noted is that \( J(\mathbf{x}) = J^a(\mathbf{x}) T_a \) is independent of the parameterization \( \xi^a \) and, together

\(^3\) Conventions: The trace is normalized as \( \text{Tr} T_a T_b = \delta^{ab} \), and the symbol \( |\tau\rangle \) denotes \( X|\tau\rangle = \sum_{a=1}^2 X^a T_a \) for \( X = \sum_{a=1}^3 X^a T_a \). Our antisymmetric tensor \( \epsilon^{\mu\nu} \) has the sign \( \epsilon^{01} = 1 \), and we use \( \dot{u} := \partial u/\partial t \) and \( u' := \partial u/\partial x \) (for \( d = 1 \)).
with \( g(x) \), parametrizes the phase space of the model, forming the following fundamental Poisson bracket,

\[
\{ J^a(x), J^b(y) \} = -\epsilon^{abc} J^c(x) \delta^{(d)}(x - y),
\]

\[
\{ J^a(x), g(y) \} = g(x) T_3 \delta^{(d)}(x - y),
\]

\[
\{ g(x), g(y) \} = 0.
\]

The Legendre transform of the Lagrangian (2.4) then leads to the Hamiltonian

\[
H_0(g, J) = \int d^d x H_0
\]

with

\[
H_0(g, J) = \lambda^2 \frac{\lambda^2}{2} \text{Tr}(J|_r)^2 + \frac{1}{2\lambda^2} \text{Tr} (g^{-1} \partial_i g|_r)^2.
\]

Note that our primary constraint (in Dirac’s notation),

\[
\phi(g, J) \approx 0, \quad \text{where} \quad \phi(g, J) := J^3(x),
\]

generates the infinitesimal right transformation associated with the gauge transformation (2.2). Since the constraint (2.8) commutes with the Hamiltonian (2.7) under the Poisson bracket (2.6), no further (secondary) constraints arise.

3. The soliton term

We now specialize to the (1+1)-dimensional case and show that the soliton term (1.3) arises as a result of quantization. To this end, we first observe that for \( d = 1 \) the spatial infinity consists of two points \( x = \pm \infty \) and we can always eliminate the arbitrary function \( k(x, t) \) in (2.3) by a gauge transformation (2.2). Thus, instead of (2.3) we may consider the simplified boundary condition, \( g(x, t) \to g_0 \) as \( |x| \to \infty \), without loss of generality. Under this condition, at a fixed time the function \( h(x, t) \) in (2.2) becomes a map from the (effective) space \( S^1 \) to the target \( U(1) \simeq S^1 \). Thus, those gauge transformations can be classified by the winding number,

\[
w(h) = \frac{1}{4\pi} \int_{S^1} dx \text{Tr} T_3(h^{-1}h')
\]

of the map, and a representative map possessing the winding number \( n \in \mathbb{Z} \) is given by

\[
h_n(x; L) := e^{4n\pi x T_3/L}, \quad \text{where} \quad L \text{ is the length of the space } S^1.
\]

Gauge transformations with zero winding number can be generated by infinitesimal transformations and are called ‘small gauge transformations,’ whereas those with non-zero winding number are called ‘large gauge transformations.’ Let us next see the consequences of the invariance of the theory under these transformations in quantum theory.
Upon quantizing the model in Schrödinger picture, state vectors are represented by wave functionals $\Phi[g(x)]$ where the argument $g(x)$ denotes a configuration at a fixed time. Observables on phase space are now regarded as self-adjoint operators (although we use the same notation as before) and, in particular, the canonical momenta conjugate to $\xi^a(x)$ are realized by functional derivatives $\pi^a(x) = -i\hbar \delta/\delta \xi^a(x)$. The gauge symmetry of the theory is ensured by requiring that under gauge transformations (2.2) physical functionals be invariant

$$\Phi_{\text{phys}}[g(x) h(x)] = e^{iF[h(x)]} \Phi_{\text{phys}}[g(x)],$$

(3.2)

up to a phase $e^{iF}$ given by some functional $F[h(x)]$. For small gauge transformations this (with $e^{iF} = 1$) follows from the condition that implements the first class constraint (2.8) in the quantum theory,

$$J_3(x) \Phi_{\text{phys}}[g(x)] = (N-1)^3 \sum_b \delta_{\xi^b} \Phi_{\text{phys}}[g(x)] = 0,$$

(3.3)

which is the analogue of the Gauss’ law in gauge theory.

To find out the phase factor $e^{iF[h]}$ acquired for large gauge transformations, let us note that any map $h(x)$ having the winding number $w(h) = n$ can be decomposed as $h(x) = h_n(x; L)\tilde{h}(x)$ using the representative map $h_n(x; L)$ and $\tilde{h}(x)$ that has zero winding number. It then follows from (3.3) that the factor $e^{iF[h]}$ depends only on the winding number of $h(x)$, and it is given by a unitary representation of the additive group $\mathbb{Z}$, namely, we have

$$\Phi_{\text{phys}}[g(x) h(x)] = e^{i\theta w(h)} \Phi_{\text{phys}}[g(x)],$$

(3.4)

with an arbitrary angle parameter $\theta \in [0,2\pi)$. The energy eigenstates are obtained by solving the Schrödinger equation, $H_0(g, J) \Phi_{\text{phys}} = E \Phi_{\text{phys}}$.

For convenience we may wish to use states which are invariant under all gauge transformations. This can be accomplished by introducing

$$K(g) = \frac{1}{4\pi} \int dx \ Tr T_3(g^{-1}g'),$$

(3.5)

which transforms under gauge transformations (2.2) as $K(gh) = K(g) + w(h)$. Then the desired states which are invariant even under large gauge transformations can be obtained by

$$\Psi[g(x)] := e^{i\theta K} \Phi_{\text{phys}}[g(x)].$$

(3.6)

Noting that $J_a e^{i\theta K} = e^{i\theta K}(J_a + \frac{\hbar \theta}{4\pi} \epsilon^{3ab} Tr T_b(g^{-1}g'))$, we see that the invariant states $\Psi[g(x)]$ obey the Schrödinger equation in the form $H_\theta(g, J) \Psi = E \Psi$, where $H_\theta(g, J) = \int dx \mathcal{H}_\theta$ is the modified Hamiltonian with

$$\mathcal{H}_\theta(g, J) = \frac{\lambda^2}{2} \left( J^a - \frac{\hbar \theta}{4\pi} \epsilon^{3ab} Tr T_b(g^{-1}g') \right)^2 + \frac{1}{2\lambda^2} \ Tr (g^{-1}g'|_c)^2.$$

(3.7)
One can put the above formulation of quantum theory in the path-integral formalism by means of the standard Faddeev-Popov prescription (see, e.g., [10]), where one introduces a gauge fixing condition $\chi(g, J) \approx 0$ corresponding to the constraint $\phi(g, J) \approx 0$ in (2.8) and considers the partition function in phase space,

$$Z = \int \mathcal{D}g \mathcal{D}J \delta(\phi) \delta(\chi) \left| \det \{\phi, \chi\} \right| \exp \left[ \frac{i}{\hbar} \int d^2x \left( \text{Tr} J (g^{-1} \dot{g}) - \mathcal{H}_\theta \right) \right],$$

with $\mathcal{D}g \mathcal{D}J = \prod_{x, t} \left[ \text{Tr} (g^{-1} dg)^3 \prod_a dJ^a \right]$. Choosing the gauge fixing condition to be $J$-independent $\chi(g, J) = \chi(g)$, and noting that $\{\phi, \chi\}$ gives the infinitesimal gauge transformation $\delta \chi$, one can carry out the $J$-integrations to get the configuration space path-integral,

$$Z = \int \mathcal{D}g \delta(\chi) \left| \det \delta \chi \right| \exp \left[ \frac{i}{\hbar} \int d^2x \left( \mathcal{L}_0(g) - \frac{\hbar \theta}{8\pi} \epsilon^{\mu\nu} \partial_\mu \text{Tr} T_3 (g^{-1} \partial_\nu g) \right) \right].$$

The first term $\mathcal{L}_0(g)$ in the exponent of the path-integral is the Lagrangian (2.4), whereas the second term is just the soliton term $L_{\text{soliton}}$ in (1.3) as can be readily confirmed upon using (2.1). It is also obvious that the path-integral measure $\mathcal{D}g \delta(\chi) \left| \det \delta \chi \right|$ must be the same as the measure $\mathcal{D}n$ for the field $n(x, t)$. Indeed, we can see this explicitly if we employ the Euler angle decomposition $g = e^{\alpha T_3} e^{\beta T_2} e^{\gamma T_3}$ for which the measure reads $\mathcal{D}g = \prod \sin \beta \ d\alpha \ d\beta \ d\gamma = \mathcal{D}g \mathcal{D}n$, and whereby choose $\chi(g) = \gamma$ which has $\delta \chi = \text{const}$. We therefore arrive at

$$Z = \int \mathcal{D}n \exp \left[ \frac{i}{\hbar} \int d^2x \left( \frac{1}{2\lambda^2} (\partial_\mu n)^2 + \frac{\hbar \theta}{8\pi} \epsilon^{\mu\nu} \epsilon_{abc} n^a \partial_\mu n^b \partial_\nu n^c \right) \right],$$

which shows that the soliton term (1.3) is induced upon quantization in the $O(3)$ NSM in $(1 + 1)$ dimensions.

4. The Hopf term in $n(x, t)$

The Hopf term (1.4) has been widely used in the physics literature especially in the context of fractional spin and statistics in $(2 + 1)$ dimensions. However, unlike the soliton term in $(1 + 1)$ dimensions, the Hopf term is not well-defined mathematically for generic configurations, and hence it requires a careful consideration before it is used. To examine the problem in detail, we first recall the Hopf invariant used in mathematics (see, e.g., [11]). Let $f$ be a map $S^3 \to S^2$ and $F$ be a generator of the de Rham cohomology $H^2_{\text{DR}}(S^2) = \mathbb{R}$.

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4 Since the decomposition is possible only locally in $SU(2)$, for a more rigorous treatment one needs to introduce a set of patches to cover the $SU(2)$. The obstruction for a naive global gauge fixing is also evident in that in terms of $g$ the soliton term is a total divergence while in terms of $n$ it is not.
Since $H^2_{\text{DR}}(S^3) = 0$ the pullback $f^*F$ of $F$ under the map $f$ (i.e., $F$ regarded as a 2-form on $S^3$) admits the form $f^*F = dA$ with some 1-form $A$ on $S^3$. The Hopf invariant associated to the map $f$ is then given by

$$H(f) := -\frac{1}{16\pi^2} \int_{S^3} A \wedge dA,$$  \hspace{1cm} (4.1)

where the normalization is chosen such that $H(f) \in \mathbb{Z}$ when $F$ is normalized as $\int_{S^2} F \in 4\pi \mathbb{Z}$. Note that the topological invariant $H(f)$ in (4.1) is independent of the choice of $A$. In other words, despite that there exists an ambiguity in $A$ (under $U(1)$ gauge transformations $A \to A - d\Lambda$) and hence in the integrand in (4.1), the integral is still uniquely determined. With an appropriately normalized $F$ the Hopf invariant $H(f)$ becomes an integer characterizing the map $f$.

Evidently, if our vector field $n$ can be regarded as a map $S^3 \to S^2$ with $S^3$ our (effective) spacetime, then by identifying $n$ with the above $f$ and also $\frac{1}{2} F_{\mu\nu} dx^\mu \wedge dx^\nu$ with the 2-form $f^*F$, the integral of the Hopf term (1.4) over the spacetime $S^3$ becomes identical to (4.1) up to an overall constant and, therefore, it is well-defined. However, the boundary condition in space (1.2) implies that we shall be considering the NSM in a spacetime of the form $M = S^2 \times I$ where $I$ is a time interval, say, $I = [0, T]$, or $M = S^2 \times S^1$ if an additional periodic boundary condition in time is imposed. Thus in general our map is given by $n : M \to S^2$ with these $M$, for which we have $H^2_{\text{DR}}(M) \neq 0$. Accordingly, there is no guarantee to find such (globally defined) $A$ satisfying $n^*F = dA$ to a given map $n$. In other words, the naïve integral,

$$I(A; M) := -\frac{1}{16\pi^2} \int_M A \wedge dA,$$  \hspace{1cm} (4.2)

cannot be used for providing the topological term we want. At this point one may think that a solution is to regard $A$ as a $U(1)$ connection over the base space $M$, that is, to find $A$ given locally on patches which are introduced to trivialize the fibre bundle. That this does not work can be seen as follows.

Suppose $M$ is covered by two local patches, $M_1$ and $M_2$, on which we have a 1-form $A_1$ and $A_2$, respectively, satisfying $n^*F = dA_a$ for $a = 1, 2$. One then may consider, instead of (4.2), the sum of integrals$^5$

$$I(A_1; M_1) + I(A_2; M_2) - I(A_1; M_1 \cap M_2).$$  \hspace{1cm} (4.3)

$^5$ One can also use $A_2$ instead of $A_1$ in the last term, and this creates another ambiguity in defining the topological term.
This is, however, not invariant under gauge transformations $A_a \rightarrow A_a - d\Lambda_a$ performed separately on the patches with $\Lambda_1$ and $\Lambda_2$ chosen independently, because its variation, $\int_{\partial M_1} \Lambda_1 dA_1$ plus contributions from other two terms, does not vanish for generic $\Lambda_a$. (The gauge noninvariance of the conventional Hopf term has been pointed out earlier in [12].) Actually, this is a problem with the Chern-Simons term in gauge theory (rather than with the Hopf term in the NSM), where it is assumed that the connection $A$ is globally defined [13], which is enough if one is interested in the perturbative analysis of the theory. The real problem with the Hopf term is that in the $O(3)$ NSM this assumption excludes solitons or anti-solitons unless the total charge vanishes, which are considered to be responsible for physical phenomena of our interest such as fractional spin. Indeed, for the spacetime $M = S^2 \times I$ (or $M = S^2 \times S^1$) the assumption amounts to the requirement that the 2-form $n^*F$ be a trivial element of $H^2_{DR}(M)$ and, therefore, we find that the soliton charge,

$$Q(n) := -\frac{1}{4\pi} \int_{S^2} n^*F,$$  

becomes $Q(n) = -(1/4\pi) \int_{S^2} dA = 0$, that is, the map $n$ must belong to the sector where the soliton charge vanishes.

Conversely, it is possible to show that the restriction to the sector $Q(n) = 0$ is sufficient for the topological term to be well-defined in the NSM. For this, we first note that, for a generic spacetime $M$, the condition $H^2_{DR}(M) = 0$ is not still enough for ensuring the integral (4.2) to be well-defined, because the gauge invariance of the integral (4.2) requires that the 1-form $A$ must in general be gauge-fixed on $\partial M$ if $\partial M \neq 0$. In quantum theory, however, such a gauge fixing is not necessary with our space time $M = S^2 \times I$. The reason for this is that, as is well-known, in quantum theory we can always place periodic boundary conditions in time, and this allows us to put the time period $I$ into $S^1$ and thereby remove the boundary of the spacetime. We then notice that if the aforementioned restriction to the $Q(n) = 0$ sector is made in the NSM, we can deform any map $n(x)$ at $t = 0$ and $T$ continuously to the constant map $n(x) = n_0$ without changing the integral in (4.2) (since we have $\partial(S^2 \times S^1) = 0$). This procedure allows us to regard $n$ as a map $S^3 \rightarrow S^2$ by shrinking the space $S^2$ to a point at the both ends of time and, consequently,

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6. This is easily realized in the path-integral for a particle, where one needs to define a \textit{relative} phase to a given pair of paths, rather than to define an \textit{absolute} phase to each path, in order to provide the transition amplitude. The relative phase between a pair of two paths obeying the same boundary conditions at $t = 0$ and $T$ can be regarded as the phase attached to the path given by connecting the two paths (with the time of one of the paths reversed). This amounts to defining a phase to an arbitrary loop which is a path possessing the same initial and final points.
the integral in (4.2), which is just the Hopf invariant (4.1), can be used as a topological term in the model.\textsuperscript{7}

5. The Hopf term in \(g(x,t)\)

The foregoing argument shows that there are basically three obstacles for the naïve integral \(I(A;M)\) in (4.2) to be well-defined as the topological term associated with the Hopf invariant (4.1). These obstacles are related to the conditions that the term be (i) well-defined as an integral over the spacetime \(M\), (ii) a topological invariant, and (iii) gauge invariant. As we shall see shortly, in the AOP the situation concerning these conditions turns out to be quite different, and we shall exploit it in order to define the topological term to any configurations, fulfilling all the three conditions.

To this end, we first note that the corresponding \(F\) becomes then the 2-form on the target of the map \(g: D^2 \times I \to SU(2) \simeq S^3\), and it is given by
\[
F = -\text{Tr}T_3 (g^{-1} dg)^2.
\]
Since \(H^2_{DR}(D^2 \times I) = 0\) we always have a 1-form \(A\) such that the pullback of \(F\) under \(g\) becomes \(g^* F = dA\) — an obvious solution is \(A = \text{Tr}T_3 (g^{-1} dg)\). Thus we now have \(A\) local in the variable \(g\) in contrast to the previous \(A\) which is nonlocal in \(n\). Accordingly, in the AOP the integral (4.2) turns into the local expression \([7]\),
\[
I(A;M) = I(g) := \frac{1}{48\pi^2} \int_M \text{Tr}(g^{-1}dg)^3,
\]
where now \(\bar{M} = D^2 \times I\), and this shows that condition (i) is fulfilled. In passing we mention that, on account of the boundary condition (2.3), the AOP version of the soliton charge (4.4) reads
\[
Q(g) := -\frac{1}{4\pi} \int_{D^2} g^* F = -\frac{1}{4\pi} \int_{\partial D^2} \text{Tr}T_3 (g^{-1} dg),
\]
which gives the winding number of the map \(k : \partial D^2 \simeq S^1 \to U(1)\) at space infinity.

Let us now observe that, when the spacetime \(\bar{M}\) for \(g\) can be regarded as \(D^2 \times S^1\) as can be done in the path-integral, the integral (5.1) becomes a topological invariant irrespective of the soliton sector we are in. Indeed, under an arbitrary variation \(g \to g + \delta g\) we obtain
\[
I(g + \delta g) - I(g) = \frac{1}{16\pi^2} \int_{\partial \bar{M}} \text{Tr}g^{-1}\delta g (g^{-1} dg)^2,
\]
which vanishes for any \(g\) which are assumed to be of the form (2.3) at \(\partial \bar{M} = \partial D^2 \times S^1\) and hence \((g^{-1} dg)^2|_{\partial \bar{M}} = (k^{-1} dk)^2|_{\partial \bar{M}} = 0\). However, the problem is that \(I(g)\) is not invariant

\textsuperscript{7} In the literature [2, 3, 4, 14] the term (4.2) is considered even for sectors with nonzero soliton numbers under the Coulomb gauge. However, as has been pointed out in [6], the fractional spin evaluated directly from the term becomes gauge dependent and physically unacceptable.
under time-dependent gauge transformations (2.2) possessing nontrivial winding numbers along $S^1$. This can be confirmed directly by observing

$$I(gh) - I(g) = -\frac{1}{16\pi^2} \int_{\partial M} \text{Tr}(k^{-1} dk) \wedge (dh h^{-1}) .$$

(5.4)

To evaluate the r.h.s. of (5.4), let $L$ be the length of the boundary $\partial D^2 \simeq S^1$ and regard the domain of the integral $\partial M = S^1 \times S^1$ as the rectangle $I^2 = [0, L] \times [0, T]$ in which periodic boundary conditions are imposed on the maps $k$ and $h$. Introducing the coordinates $(x, t) \in I^2$, we put

$$k(x, t) = e^{\xi(x, t) T_3} h_m(x; L) h_l(t; T) , \quad h(x, t) = e^{\eta(x, t) T_3} h_n(t; T) ,$$

(5.5)

where $\xi(x, t)$ and $\eta(x, t)$ are periodic functions in $I^2$, $h_m(x; L)$, $h_l(t; T)$, $h_n(t; T)$ are the representative maps defined earlier with $m, n, l \in \mathbb{Z}$. Note that the integer $m$ in $k(x, t)$ equals (minus) the soliton number $-Q(g)$, whereas for $h(x, t)$ no such integer appears since we have $Q(h) = 0$ on account of the fact that for gauge transformations $h$ must be given on $D^2$ which is contractible. Substituting (5.5) into (5.4) we find

$$I(gh) - I(g) = -\left( \frac{1}{16\pi^2} \right) \left( \frac{4n\pi}{T} \right) \left( \frac{4m\pi}{L} \right) \int_{I^2} dx \wedge dt = n Q(g) ,$$

(5.6)

that is, condition (iii) is not fulfilled unless $Q(g) = 0$.

In order to extend the domain of $I(g)$ to nonvanishing soliton number sectors, a possible procedure proposed in [6] is that one imposes a (partial) gauge fixing condition on $g(x, t)$ such that at spatial infinity it become time-independent, i.e.,

$$\frac{\partial}{\partial t} g(x, t) \big|_{\partial D^2} = 0 .$$

(5.7)

Indeed, this excludes those gauge transformations by $h(x, t)$ having $n \neq 0$ on $\partial D^2$, and hence the r.h.s. of (5.6) vanishes irrespective of the soliton charge $Q(g)$. Thus we see that, upon imposing (5.7), the integral $I(g)$ defines a topological invariant and, at the same time, it is gauge invariant for any $g$. It remains, therefore, to find the meaning of the topological invariant so defined.

For this, let us consider the configuration,

$$\bar{g}(x, t) := g_0 g^{-1}(x, 0) g(x, t) .$$

(5.8)

Using the additivity of the soliton number, $Q(g_1 g_2) = Q(g_1) + Q(g_1)$ for any $g_1, g_2$ obeying (2.3), we find that $Q(\bar{g}) = 0$. Thus we are allowed to regard $\bar{g}$ as one converted from $g$
to the vanishing soliton number sector, for which the integral \( I(\bar{g}) \) in (5.1) gives the Hopf invariant without imposing (5.7). On the other hand, it can be readily shown [6] that, once the condition (5.7) is imposed, the conversion does not change the integral, \( I(g) = I(\bar{g}) \). It thus follows that the topological invariant defined above for any \( g \) with (5.7) is nothing but the Hopf invariant for the converted configuration \( \bar{g} \).

Note that conversion to the vanishing soliton number sector is far from unique. However, since the integral (5.1) is a topological invariant, any conversion of the form,

\[
\bar{g}(x, t) := g_A(x) g(x, t),
\]  

(5.9)
gives the same value for \( I(\bar{g}) \) as long as the static configuration \( g_A \) has the soliton number opposite to that of \( g \) so that \( Q(\bar{g}) = 0 \) (and we may also require \( g_A|_{\partial D^2} = 1 \) so that \( \bar{g} \) still obeys (2.3)). This observation allows for the following physical interpretation to the topological invariant we just assigned to \( g \). Suppose that the original configuration \( g(x, t) \) has soliton number \( n \), and that it is localized in some finite domain in space. Choose then \( g_A(x) \) such that it has soliton number \( (-n) \) and also localized in some other domain which do not intersect with the domain of \( g \). Then what the conversion (5.9) is doing is that it places a static anti-soliton \( g_A \) somewhere far from the soliton \( g \). Roughly speaking, the Hopf invariant counts the number of twists made by the configuration during the
time interval $[0, T]$, and for $\bar{g}$ these twists are performed by the soliton part. The static anti-soliton does not play a role in this, except that it provides a ground in which the Hopf invariant becomes well-defined by nullifying the soliton charge (see Fig. 1). In fact, the physical picture originally used by Wilczek and Zee [1] to discuss fractional spin (for the case $n = 1$) is the initial soliton/anti-soliton pair creation and annihilation at the final time, which is the case where the conversion is made according to (5.8) for a soliton $g(x, t)$ which stays far from the static anti-soliton $g_A(x) = g^{-1}(x, 0)$ except when the creation and annihilation take place at $t = 0$ and $t = T$, respectively. Our procedure presented here provides a mathematical ground for the picture, in the light of finding a well-defined topological invariant corresponding to the Hopf invariant. We also point out that it is possible to carry out the procedure in the original description in terms of $n$, but it becomes more involved than the one given here which exploits fully the advantage of the group properties of the AOP.

6. Conclusion and discussions

In this paper we discussed two types of topological terms in the $O(3)$ NSM, one is the soliton term in $(1 + 1)$ dimensions and the other is the Hopf term in $(1 + 1)$ dimensions. In contrast to the soliton term which is well-defined and can be derived from the unitary representation of the fundamental group of the configuration space $\mathcal{C}_1$, the Hopf term used in the literature is ill-defined and, against the general expectation, it cannot serve to produce fractional spin and statistics in its conventional form. We argued that the conversion procedure, which has been proposed earlier to make the Hopf term well-defined and is equivalent to a partial gauge fixing in the AOP, is natural both mathematically and physically.

Once we constructed the Hopf term as a topological invariant associated with the first homotopy group (1.5), we may ask if it can also be used as a ‘Wess-Zumino term’ in the $(1 + 1)$-dimensional NSM. This possibility arises from the fact that the second homotopy group of the configuration space $\mathcal{C}_1$ reads

$$\pi_2(\mathcal{C}_1) = \pi_3(S^2) = \mathbb{Z}, \quad (6.1)$$

which suggests that the NSM may admit an associated topological term analogous to the Wess-Zumino term in the Wess-Zumino-Novikov-Witten model [15] which has the same second homotopy group. In fact, the usual Wess-Zumino term is given precisely by the integral (5.1) with $M = D^3$ whose boundary $\partial M \simeq S^2$ is identified with the $(1 + 1)$-dimensional spacetime. Since in the NSM the second homotopy group is related to the
Hopf fibration, we may expect that the topological term is again the Hopf term we have just made well-defined.

To examine this possibility, for definiteness we take our spacetime to be $S^2$, which is always possible since by (1.6) there is no obstacle to deform any configurations to a constant one. We however observe that taking the extrapolated manifold $M = D^3$ for our Wess-Zumino term (4.2) is not possible, because a loop in $C_1$ given by the spacetime map $n(x, t)$ cannot always be deformed to a point due to (1.5), unlike in the Wess-Zumino-Novikov-Witten model where it can. Instead, we may consider $M = S^2 \times I$ parameterized by $(x, t, \sigma)$ with the extrapolation parameter $\sigma \in [0, 1]$ in such a way that the map $n(x, t, 1) := n(x, t)$ be extended to $n(x, t, 0)$ given by some fixed configuration possessing the same soliton number as $n(x, t)$. This does not render the integral (4.2) well-defined either, since the integral changes under gauge transformations for $A$ yielding a gauge dependent integral over the spacetime $S^2$. The situation is unaltered even if we take any three-dimensional manifold for $M$ as long as it contains $S^2$ in the boundary $\partial M$, which is a quality requisite to a Wess-Zumino term in $(1 + 1)$ dimensions. We also note that, despite the similarity of the present problem with that of defining the Hopf term in $(2 + 1)$ dimensions, the procedure we adopted in the AOP cannot be employed here, because one cannot use the periodic boundary condition in the direction of the extrapolating parameter $\sigma$, as done in the previous case where the role of $\sigma$ is played by the time $t$.

Finally, we mention that the construction of gauge invariant topological terms as a Wess-Zumino term has also been discussed elsewhere [16] in the context of coset models, where the list of cohomology generators of a symmetric space $G/H$ is exhausted. However, this amounts to finding a local gauge invariant integrand such as $F \wedge F$ or $F \wedge *F$, rather than directly seeking for a gauge invariant integral as we did above without assuming the locality and gauge invariance of its integrand. The conclusion of the non-existence of such a term, however, remains the same.
References

[1] F. Wilczek and A. Zee, \textit{Phys. Rev. Lett.} \textbf{51} (1983) 25; Y.S. Wu and A. Zee, \textit{Phys. Lett.} \textbf{147B} (1984) 325.

[2] M.J. Bowick, D. Karabali and L.C.R. Wijewardhana, \textit{Nucl. Phys.} \textbf{B371} (1986) 417; D. Karabali, \textit{Int. J. Mod. Phys.} \textbf{A6} (1991) 1369.

[3] G.W. Semenoff and P. Sodano, \textit{Nucl. Phys.} \textbf{B328} (1989) 753.

[4] S. Forte, \textit{Rev. Mod. Phys.} \textbf{64} (1992) 193.

[5] H. Kobayashi, I. Tsutsui and S. Tanimura, \textit{Nucl. Phys.} \textbf{B514} (1998) 667.

[6] M. Kimura, H. Kobayashi and I. Tsutsui, \textit{Nucl. Phys.} \textbf{B527} (1998) 624.

[7] A.P. Balachandran, G. Marmo, B.S. Skagerstam and A. Stern, “Classical Topology and Quantum States”, World Scientific, Singapore, 1991.

[8] A.P. Balachandran, A. Stern and G. Trahern, \textit{Phys. Rev.} \textbf{D19} (1979) 2416.

[9] R. Jackiw, in “Current Algebras and Anomalies”, World Scientific, Singapore, 1985.

[10] M. Henneaux and C. Teitelboim, “Quantization of Gauge Systems”, Princeton University Press, New Jersey, 1992.

[11] R. Bott and L.W. Tu, “Differential Forms in Algebraic Topology”, Springer-Verlag, New York, 1982.

[12] H. Otsu, H. Sato, \textit{Prog. Theor. Phys.} \textbf{91} (1994) 1199; \textit{Z. Phys.} \textbf{C64} (1994) 177.

[13] S. Deser, R. Jackiw and S. Templeton, \textit{Phys. Rev. Lett.} \textbf{48} (1982) 975; \textit{Ann. Phys.} \textbf{140} (1982) 372.

[14] E. D’Hoker, \textit{Phys. Lett.} \textbf{357B} (1995) 539.

[15] E. Witten, \textit{Commun. Math. Phys.} \textbf{92} (1984) 455.

[16] E. D’Hoker and S. Weinberg, \textit{Phys. Rev.} \textbf{D50} (1994) 605; E. D’Hoker, \textit{Nucl. Phys.} \textbf{B451} (1995) 725.