ON THE DECYCLING NUMBER OF 4-REGULAR RANDOM GRAPHS

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Abstract

The decycling number \( \phi(G) \) of a graph \( G \) is the smallest number of vertices which can be removed from \( G \) so that the resulting graph has no cycles. Bau, Wormald and Zhou conjectured in [1] that with probability tending to one the decycling number of the random 4-regular graph \( G_4(n) \) on \( n \) vertices is equal to \( \lceil (n+1)/3 \rceil \). In this paper we show that this conjecture holds asymptotically, i.e. asymptotically almost surely \( \lim_{n \to \infty} \phi(G_4(n))/n = 1/3 \).

1 Introduction

For a finite graph \( G \) with vertex set \( V(G) \) a subset \( S \subseteq V(G) \) is said to be a decycling set (sometimes also called feedback vertex set) if \( G \setminus S \) is acyclic. The minimum cardinality of a decycling set of \( G \) is said to be the decycling number of \( G \), denoted by \( \phi(G) \) in this paper.

A graph without cycles is called a forest. A subforest \( F \) of a graph \( G \) is a subgraph of \( G \) that is a forest. An induced subgraph \( H \) of a graph \( G \) is a subgraph of \( G \), in which for every two vertices \( u, v \) of \( H \), \( uv \) is an edge in \( H \) if and only if \( uv \) is an edge in \( G \). It is an easy observation that one may rephrase the problem of finding an upper bound for \( \phi(G) \) as finding a lower bound for the maximum number of vertices in an induced subforest of \( G \). The reformulated version has a long history; if not even earlier, it was already considered by Kirchhoff in 1847 in its work on spanning trees, see [12].

From a computational point of view, the decision problem has been shown to be NP-complete in [11], and even for special families of graphs such as bipartite graphs, perfect graphs or planar graphs it remains NP-complete, see [21]. For certain subclasses of graphs such as cubic graphs, however, polynomial-time algorithms are known, see [14].

In a \( d \)-regular graph \( G \) with \( n \) vertices every decycling set of minimal cardinality is incident with at most \( d\phi(G) \) edges and its removal leaves a forest with at most \( n - \phi(G) - 1 \) edges. Therefore we must have that

\[
\frac{dn}{2} - d\phi(G) \leq n - \phi(G) - 1,
\]

or equivalently,

\[
\phi(G) \geq \left\lceil \frac{(d/2 - 1)n + 1}{d - 1} \right\rceil.
\]

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In particular for any 4-regular graph $G$ we have

$$\phi(G) \geq \left\lceil \frac{n+1}{3} \right\rceil. \tag{2}$$

However, the gap between this lower bound and the value of $\phi(G)$ can be linear in $n$. For example, Bondy, Hopkins and Staton constructed in [5] a family of cubic graphs with $\phi(G) = \left\lceil \frac{3n+2}{8} \right\rceil$, while the bound (1) gives $\phi(G) \geq \left\lceil \frac{n+2}{4} \right\rceil$ for $d = 3$.

In this paper we study random regular graphs. For every $d \geq 1$ denote by $G_d(n)$ the set of all $d$–regular graphs (we allow graphs to have loops and multiple edges; graphs without loops and without multiple edges are called simple). Denote also by $\mathcal{G}_d(n)$ the random $d$–regular graph with $n$ vertices following the uniform distribution over the set $\mathcal{G}_d(n)$.

For a sequence of probability spaces $(\Omega_n, \mathcal{F}_n, \mathbb{P}_n)_{n \geq 1}$ and a sequence of events $(A_n)_{n \geq 1}$, where $A_n \in \mathcal{F}_n$ for every $n \geq 1$, we say that $(A_n)_{n \geq 1}$ happens asymptotically almost surely or a.a.s., if $\lim_{n \to +\infty} \mathbb{P}_n(A_n) = 1$.

The sequence of events $(A_n)_{n \geq 1}$ itself is said to be asymptotically almost sure or again a.a.s.

In [1] it was shown that a.a.s. $\phi(G_3(n)) = \left\lceil \frac{n+2}{4} \right\rceil$. In the same paper, the authors obtained lower and upper bounds on $\phi(G_d(n))$ for every $d \geq 4$, which hold with probability tending to one. The technique used for the upper bounds was the differential equations method, which was introduced by Wormald in [10].

For $d = 4$ this approach led to the asymptotically almost sure bound $\phi(G_4(n))/n \leq 0.3787$. The authors then corrected in a subsequent version a slight mistake in the original proof and obtained the slightly weaker a.a.s. bound $\phi(G_4(n))/n \leq 0.3955$. Later, Hoppen and Wormald showed in [8] that for every 4-regular graph $G$ with sufficiently large girth, $\phi(G)/n \leq 0.3955$. Salez then used in [13] the differential equation method together with contraction of vertices of degree two and an elaborate strategy of exposing information of neighbors to improve the upper bound: he showed that a.a.s. $\phi(G_4(n))/n \leq 0.3408$.

Schmidt, Pfister and Zdeborová analyzed in [16] the so called CoreHD algorithm (originally introduced by Zdeborová, Zhang and Zhou in [22]), thereby obtaining that a.a.s. $\phi(G_4(n))/n \leq 0.3462$. They also gave heuristics how to further improve this algorithm to obtain that a.a.s. $\phi(G_4(n))/n \leq 0.3376$.

In [1] it was conjectured that $\phi(G_4(n))/n \leq 0.3376$. This was later proved by [2] using the cavity method. Our main theorem shows that the conjecture holds asymptotically (up to $o(n)$). More precisely, we show the following theorem:

**Theorem 1.1.** Asymptotically almost surely, $$\lim_{n \to +\infty} \frac{\phi(G_4(n))}{n}$$ exists and is equal to $\frac{1}{3}$.

**Notation.** We use standard graph theory notations. For a graph $G = (V, E)$ we call order (resp. size) of $G$ the cardinality $|V|$ (respectively $|E|$) of its vertex (resp. edge) set. A leaf is a vertex of degree exactly one. A path of length $k$ is a graph with $k + 1$ vertices $\{v_1, v_2, \ldots, v_{k+1}\}$ and $k$ edges $v_i v_{i+1}$, $1 \leq i \leq k$. Since we work with multigraphs, notice that a path is uniquely defined by its edges, but not by its vertices in general. For a connected subset of vertices $S \subseteq V$ (i.e. a set such that each pair of vertices of $S$ is connected by a path in $G$) we denote by $cc(S)$ the connected component of $S$, i.e., the subgraph of $G$ induced by the subset of vertices that can be reached by a path starting in $S$. The graph $G$ is $k$-vertex connected or simply $k$-connected if the deletion of any $k - 1$ vertices of $G$ does not disconnect $G$. Maximal 2-connected subgraphs of any given graph are edge-disjoint and organized into 2-connected components or blocks in a tree-like manner, see for example ([9], Section 3.1). Indeed, there is at most one shortest path between two different 2-connected components: if there were two such paths, one could enlarge some of the existing 2-connected components. The reduced 2-core $\overline{C_2}(H)$ of a graph $H$ is defined as the union of its 2-connected components. We also define the 2-core $\overline{C_2}(H)$ of $H$ as the unique subgraph of $H$ maximal with respect to inclusion obtained by connecting some of the components of the reduced 2-core by paths, see Figure [1].
Outline of the proof.

First we prove that a.a.s. there exists an independent set \( A \subseteq V(G_4(n)) \) of linear size for which the graph \( B = G_4(n) \setminus A \) remains connected. We then show that with high probability we can keep increasing the size of the set \( A \) step by step while maintaining \( B = G_4(n) \setminus A \) connected by performing exactly one of the following modifications at each step:

1. Find a vertex \( v \) of degree four in \( B \), for which \( B \setminus \{v\} \) remains connected, and add it to \( A \).

2. Find two vertices \( u, v \in B \), both of degree three in \( B \) and sharing a common neighbor \( w \) in \( A \), for which \( (B \setminus \{u, v\}) \cup \{w\} \) remains connected. Then, add both \( u \) and \( v \) to \( A \) and take \( w \) out of \( A \).

In order to show that we do not run out of available vertices in \( B \) with the desired properties, we carefully analyze the process in the configuration model which is defined in detail in the next section. We first find the distribution of the random graph \( B \) among all connected graphs with a given number of vertices and edges. Next, we define and make use of the notions of reduced 2-core and 2-core of a given graph. We will need them as tools to analyze the structure of the graph \( B \) via its 2-connected components. This will provide us with a way of producing a uniform graph from a careful extension of the 2-core of \( B \). In fact, the 2-core of a graph will be uniquely defined. Therefore the distribution of the 2-core of \( B \) will be be obtained by counting the connected graphs (with the respective number of vertices and edges) having this particular 2-core. This will consequently allow us to reconstruct the distribution of \( G_d(n) \) from the distribution of the 2-core of \( B \).

In the course of the proof we show that each 2-connected component of the reduced 2-core of \( B \) with a given degree sequence is distributed uniformly among all 2-connected graphs with this degree sequence. Using this observation, for each 2-connected component of \( B \) we find with high probability either one vertex \( v \) of \( B \) or one pair of vertices \( u, v \) of \( B \) with a common neighbor \( w \) in \( A \) that can be used to enlarge \( A \) according to the modifications 1 and 2 respectively and then move on to the next step. We then justify that with high probability such a strategy will allow us to obtain an independent set of size \((1/3 + \varepsilon)n\) for every \( \varepsilon > 0 \).

Note that this detour over 2-cores and reduced 2-cores is needed to obtain structural information of the random graphs \( B \). Without such an approach, it is not clear that, for example, there exists a way to sample directly the 2-connected components of \( B \) so that finally we would be able to reconstruct the uniform distribution on \( G_d \).

Organization of the paper. In Section 2 we present preliminary results from graph theory and discrete probability. Section 3 is dedicated to the proof of Theorem 1. In Section 4 we conclude with a short outlook.

2 Preliminaries

2.1 Graph theoretic preliminaries

We make use of the following elementary results and observations from graph theory. First, we use the following result from [13]:

**Theorem 2.1.** Let \( G \) be a connected graph with \( n \) vertices and minimal degree \( \delta \).

- If \( \delta \geq 3 \), then \( G \) contains a spanning tree with at least \( \frac{n}{4} + 2 \) leaves.

- If \( \delta \geq 4 \), then \( G \) contains a spanning tree with at least \( \frac{2n + 8}{5} \) leaves.

**Observation 2.2.** Every subforest \( F \) of a connected graph \( G \) can be extended to a spanning tree of \( G \).
Proof. List the edges of $E(G) \setminus E(F) = \{e_1, e_2, \ldots, e_s\}$ with $s = |E(G) \setminus E(F)|$. We construct a sequence of forests $(F_i)_{0 \leq i \leq s}$ by exploring this list. We define $F_0 = F$. Then $F_i$ is equal to $F_{i-1}$, if $F_{i-1} \cup e_i$ contains a cycle, and $F_{i-1} \cup e_i$ otherwise.

The final subgraph $F_s$ is therefore maximal acyclic subgraph of $G$ with respect to inclusion, so a spanning tree of $G$.

Next, for a graph $G$ with $v_G$ vertices, $e_G$ edges and $c_G$ connected components, we define its total excess $\rho(G) = e_G + c_G - v_G$.

Observation 2.3. $\rho(G) \geq 0$ for every graph $G$. Moreover, $\rho(G) = 0$ exactly when $G$ is a forest.

Proof. Let $G$ have $v_G$ vertices, $e_G$ edges and $c_G$ connected components as above. We prove the observation by induction on $c_G$.

In the base case $c_G = 1$ we have that $G$ is connected. Then, since $e_G \geq v_G - 1$, $\rho(G) \geq 0$ and equality holds exactly when $G$ is a tree.

Suppose that the statement is true for every graph $G$ with $c_G \leq c$. Let $G'$ be a graph with $c_G + 1$ connected components. Then, by connecting two vertices of $G'$ from different connected components by an edge $e$ we obtain a graph $G''$ with $c_{G''} = c$ and $e_{G''} = e_{G'} + 1$. Thus, by the induction hypothesis the parameter $\rho(G'') = \rho(G')$ is non-negative and it is zero if and only if $G''$ is a forest. As the edge $e$ connects two different connected components of $G'$, we conclude that $G''$ is a forest exactly when $G'$ is a forest itself. The induction is finished.

Observation 2.4. For a graph $G$ with $\rho(G) \geq 1$, one may find a vertex $v \in V(G)$ for which $\rho(G \setminus v) \leq \rho(G) - 1$.

Proof. Since $\rho(G) \geq 1$, by Observation 2.3 $G$ contains a cycle. Let $v$ be a vertex in this cycle and denote by $d$ its degree. By deleting $v$ we reduce the number of vertices of $G$ by one, we reduce the number of edges by $d$, and we increase the number of connected components by at most $d - 2$. Overall this gives $\rho(G \setminus v) \leq (e_G - d) + (c_G + d - 2) - (v_G - 1) = \rho(G) - 1$, which proves the claim.

The following observation is well known. We include proof for the sake of completeness.

Observation 2.5. The 2-core of a graph $H$ can be obtained by consecutive deletions of vertices of degree at most one.

Proof. Let $e$ be an edge of $H$. If $e$ participates in a cycle, it is never deleted by the above procedure. Since every cycle is in some 2-connected component of $H$, it is included in $\overline{C_2}(H)$. Otherwise, $e$ connects two different connected components in $H \setminus e$. If they both contain cycles, $e$ is on the unique path between two
edge-disjoint 2-connected components, in which case it cannot be deleted by the above procedure: indeed, edges of 2-connected components are not deleted by the previous argument, and hence all vertices on the paths between any two of them (if any) always have degree at least two. At the same time, $e$ participates in $C_2(H)$. If at least one of these connected components of $H \setminus e$ is a tree, then after several steps $e$ will be deleted. It remains to notice that in this case $e$ is not on a path between any two 2-connected components of $H$, so it does not participate in $C_2(H)$ as well.

For connected graphs we have the following corollary:

**Corollary 2.6.** The 2-core of a connected graph is connected.

**Proof.** This follows from Observation 2.5 and the fact that after deleting a vertex of degree one the graph remains connected. \hfill \Box

Let $d \geq 3$ be an integer. For a connected graph $H$ of maximal degree at most $d$ we observe the following.

**Observation 2.7.** There are at least $\frac{2(\rho(H) - 1)}{d - 2}$ vertices in $C_2(H)$ of degree at least three.

**Proof.** Since $H$ is connected, $\rho(H) = e_H - v_H + 1$. By Observation 2.5, the 2-core of $H$ can be constructed from $H$ by consecutively deleting vertices of degree one. This operation does not change the value of $\rho$. Hence we have that $\rho(C_2(H)) = e_{C_2(H)} - v_{C_2(H)} + 1 = e_H - v_H + 1$.

For a graph $G$ denote by $d_i(G)$ the number of vertices of degree $i$. By the handshaking lemma

$$\sum_{i=1}^{d} id_i(G) = 2e_G.$$  

Applying this for $G = C_2(H)$ we get that

$$\sum_{i=3}^{d} (i - 2)d_i(C_2(H)) + 2v_{C_2(H)} = 2e_{C_2(H)},$$

and therefore

$$(d - 2) \sum_{i=3}^{d} d_i(C_2(H)) \geq 2(e_{C_2(H)} - v_{C_2(H)}) = 2(\rho(C_2(H)) - 1) = 2(\rho(H) - 1).$$

This proves the claim. \hfill \Box

## 2.2 Probabilistic preliminaries

We first recall the following version of Chernoff’s bound (see for example [10]).

**Theorem 2.8.** Let $X \sim Bin(n, p)$ be a binomial random variable with $E(X) = np = \mu$. For every $0 < \delta < 1$,

$$\mathbb{P}(X \leq (1 - \delta)\mu) \leq \exp \left( - \frac{\delta^2 \mu}{2} \right). \quad (3)$$

In the next lemma, $U_1$ and $U_2$ are urns with sizes $n_1$ and $n_2$. We are interested in the regime when $n_1$ and $n_2$ go to infinity, and moreover, we assume that there is a constant $c > 0$ for which $\frac{n_2}{n_1} \leq c$. Let $s = s(n_1) \to \infty$ and $s = o(n_1)$. We throw $s$ balls $b_1, b_2, \ldots, b_s$ one after the other into $U_1$ or $U_2$. For every ball, the probability to be thrown into $U_j$ (for $j \in \{1, 2\}$) is proportional to the free space left in $U_j$ at the time of throwing the ball. Denote the set of balls in urn $U_j$ after the $i$-th step by $B(U_j(i))$. We have the following lemma:
Lemma 2.9. There is \( s_0 = s_0(c,n_1) \in \mathbb{N} \) such that for any fixed \( s \geq s_0 \) one has that \( |B(U_1(s))| \geq \frac{s}{2(2c+1)} \)

with probability at least \( 1 - \exp \left( - \frac{s}{16c + 8} \right) \).

Proof. We introduce a new stochastic process, which will be stochastically dominated by \((\mathbb{1}_{b_i \in U_1})_{1 \leq i \leq s}\). Prepare \( s \) pairs of urns \((U^1_i, U^2_i)_{1 \leq i \leq s}\) with \( \forall i \leq s, |U^1_i| = n_1 - (i-1) \) and \( |U^2_i| = n_2 \) and one ball \( b^i \) for each of the \( s \) pairs.

Now, let \((X_i)_{1 \leq i \leq s}\) be random variables distributed uniformly in \([0, 1]\). At the \( i \)-th step, throw the ball \( b_i \) into \( U_1 \) if and only if

\[
X_i \leq \frac{n_1 - |B(U_1(i-1))|}{n_1 + n_2 - (i-1)}
\]

and throw the ball \( b^i \) into \( U^1_i \) if and only if

\[
X_i \leq \frac{|U^1_i|}{n_1 + n_2 - (i-1)} = \frac{n_1 - (i-1)}{n_1 + n_2 - (i-1)}.
\]

Note that both thresholds correspond exactly to the quotient between the free space left in \( U_1 \) at step \( i \) (resp. in \( U^1_i \)) divided by the total free space left in \( U_1 \cup U_2 \) at step \( i \) (resp. in \( U^1_i \cup U^2_i \)).

Since

\[
\forall i \leq k-1, \quad \frac{n_1 - (i-1)}{n_1 + n_2 - (i-1)} \leq \frac{n_1 - |B(U_1(i-1))|}{n_1 + n_2 - (i-1)},
\]

we conclude that for every \( 1 \leq i \leq s \), the event that \( b^i \) is thrown into \( U^1_i \) in the auxiliary process is included in the event that \( b_i \) goes to \( U_1 \) in the original process. Hence, in order to prove the statement, it suffices to prove the result for the auxiliary process.

Clearly, in the auxiliary process, the probability that the ball \( b^i \) is thrown into \( U^1_i \) is minimal for \( i = s \), for which it is \( \frac{n_1 - (s-1)}{n_1 + n_2 - (s-1)} \). Moreover, for every \( n_1 \) large enough we have that \( \frac{n_2}{2c} \leq n_1 - s + 1 \). For any such \( n_1 \) we have

\[
\frac{n_1 - (s-1)}{n_1 + n_2 - (s-1)} \geq \frac{n_2/2c}{n_2/2c + n_2} = \frac{1}{2c+1}.
\]

Thus, in turn, the collection of (independent) random variables \((\mathbb{1}_{b \in U_1})_{1 \leq i \leq s}\) stochastically dominates a collection of independent Bernoulli random variables \((Y_i)_{1 \leq i \leq s}\) with parameter \( \frac{1}{2c+1} \).

Denote by \( X = \sum_{i=1}^{s} Y_i \) and note that \( \mathbb{E}(X) = \frac{s}{2c+1} \). By Theorem 2.8 applied with \( \delta = \frac{1}{2} \), we get that

\[
\mathbb{P} \left( X \leq \frac{s}{2(2c+1)} \right) \leq \exp \left( - \frac{s}{16c + 8} \right).
\]

Since the process \((\mathbb{1}_{b \in U_1})_{1 \leq i \leq s}\) stochastically dominates \((Y_i)_{1 \leq i \leq s}, |B(U_1(s))|\) stochastically dominates \( X \). The statement follows. \( \square \)

We now introduce the probability space we will be working with till the end of this paper - the configuration model introduced in [2] and further developed by Bollobás in [4] and by Wormald in [17]. We are given \( dn \) points, with \( dn \) being even, indexed by \((P_{i,j})_{1 \leq i \leq d, 1 \leq j \leq n}\) and regrouped into \( n \) buckets according to their second index. The probability space we work with is the space of perfect matchings of these \( dn \) points equipped with the uniform probability. We call configuration a perfect matching of \((P_{i,j})_{1 \leq i \leq d, 1 \leq j \leq n}\). We also call partial configuration a matching of \((P_{i,j})_{1 \leq i \leq d, 1 \leq j \leq n}\), which is not necessarily perfect. We now reconstruct the random \( d \)-regular graph model as follows. We identify the \( d \)-point buckets with the vertices of our random graph. By abuse of terminology, we use both buckets and vertices in the sequel to refer to the same objects by the above identification. An edge in the random
regular graph between two (not necessarily different) vertices \( v \) and \( v' \) corresponds to an edge of the configuration between a point \( P \) in the bucket \( v \) and a point \( P' \) in the bucket \( v' \). It is well known that this model is contiguous to the uniform distribution on random regular graphs, see [2]. More precisely, two sequences of probability measures \( (P_n) \) and \( (Q_n) \) are contiguous, if for every sequence of measurable properties \( A_n \), \( \lim_{n \to \infty} P_n(A_n) = 0 \iff \lim_{n \to \infty} Q_n(A_n) = 0 \). Indeed, it can also be easily verified that the probability to obtain a simple graph in the configuration model tends to a positive constant with \( n \) for every fixed \( d > 0 \): the limit is given by \( \exp(-d^2 - 1)/4 \), see [10]. Hence, any property holding a.a.s. in the configuration model also holds a.a.s. for a simple random regular graph. Therefore by abuse of language we consider until the end of this paper that the random \( d \)-regular graph is generated from the configuration model.

Denote by \( X^k_d \) the number of cycles of length \( k \) in the random \( d \)-regular graph, and denote by \( Y^k_d \) the maximal number of edge-disjoint cycles of length \( k \). Throughout the paper fix \( \beta \in (0, 1/2) \) (its value will be given in the end).

**Lemma 2.10.** For every fixed \( \ell \geq 1 \) and \( d \geq 3 \) there are positive constants \( c = c(\ell,d) \) and \( C = C(\ell,d) \) such that for every \( n \geq 1 \) and \( k \) such that \( 1 \leq k \leq n^\beta \), we have

\[
\Pr(Y^k_d(n) \geq k) \leq C \exp(-ck).
\]

**Proof.** By (20), Theorem 2.6) and [3] the number of cycles of length \( \ell \) in a random \( d \)-regular graph converges in distribution to a Poisson random variable and in particular the probability of not having any cycle of length \( \ell \) converges to a positive constant. Thus there are \( n_0 = n_0(\ell,d) \in \mathbb{N} \) and \( c' = c'(\ell,d) \in (0,1) \) such that for every \( n \geq n_0 \), the probability of not having an \( \ell \)-cycle is at least \( c' \). Using this we prove first that the probability that the greedy algorithm (that is, the algorithm taking the first cycle of length \( \ell \) in an arbitrary order) finds \( k \) edge-disjoint cycles of length \( \ell \) is at most \( \exp(-c'k) \).

Indeed, we have that

\[
\Pr(\text{The greedy algorithm finds } k \text{ edge-disjoint } \ell - \text{cycles } \Gamma_1, \Gamma_2, \ldots, \Gamma_k) \\
= \Pr(\Gamma_1) \prod_{i=1}^{k-1} \Pr(\text{The greedy algorithm finds an } \ell - \text{cycle } \Gamma_{i+1} \text{ disjoint from } \bigcup_{j=1}^{i} E(\Gamma_j) \mid \Gamma_1, \Gamma_2, \ldots, \Gamma_i).
\]

Now, we remark that for every \( i \) we have that, conditionally on \( \Gamma_1, \Gamma_2, \ldots, \Gamma_i \), the random graph \( G_d \setminus \{E(\Gamma_1), E(\Gamma_2), \ldots, E(\Gamma_i)\} \) has a uniform distribution among all graphs with this prescribed degree sequence. Indeed, recall that the choice of the \( \ell \)-cycles \( \Gamma_1, \Gamma_2, \ldots, \Gamma_i \) consists in the choice of vertices (or buckets) that participate in them, the choice of points in these buckets that serve as endvertices of the edges in \( \bigcup_{j=1}^{i} E(\Gamma_j) \), and the choice of matching between these points. In other words, it comes down to revealing the partial configuration \( \Gamma_1, \Gamma_2, \ldots, \Gamma_i \). Thus, the (random) partial configuration on the points not used in the construction of \( \Gamma_1, \Gamma_2, \ldots, \Gamma_i \) remains uniform over all possible partial configurations on this set of points.

Thus, since the probability to have an \( \ell \)-cycle in the graph \( G_d \setminus \{E(\Gamma_1), E(\Gamma_2), \ldots, E(\Gamma_i)\} \) is dominated by the one in \( G_d \) (fewer edges left) we have that for every \( i \leq k - 1 \),

\[
\Pr(\text{The greedy algorithm finds an } \ell - \text{cycle } \Gamma_{i+1} \text{ edge-disjoint from } \bigcup_{j=1}^{i} E(\Gamma_j) \mid \Gamma_1, \Gamma_2, \ldots, \Gamma_i) \leq 1 - c'.
\]

This proves that

\[
\Pr(\text{The greedy algorithm finds } k \text{ edge-disjoint } \ell - \text{cycles}) \leq (1 - c')^k.
\]

It remains to notice that the largest number \( k \) of edge-disjoint \( \ell \)-cycles \( \Gamma_1, \Gamma_2, \ldots, \Gamma_k \) found by the greedy algorithm is at least a \( \frac{1}{\ell} \)-fraction of the maximal number of edge-disjoint \( \ell \)-cycles \( \Gamma'_1, \Gamma'_2, \ldots, \Gamma'_k \) in \( G_d \).
independently of the sequence of local decisions made by the algorithm. Indeed, each of \( \Gamma_1', \Gamma_2', \ldots, \Gamma_{k'} \) should contain at least one edge from \( \bigcup_{i=1}^{k} E(\Gamma_i) \), so \( k' \geq k' \). We conclude that

\[
P(Y^d_{\ell} \geq k') \leq (1 - c')^{-\ell}.
\]  

(4)

Since (4) holds for every \( n \geq n_0 \), one can choose for example \( c = \frac{\ln((1 - c')^{-1})}{\ell} \) and \( C = \exp(cn_0) \), and the lemma follows.

From Lemma 2.10 we obtain immediately the following corollary:

**Corollary 2.11.** In a random 4-regular (respectively 3-regular) graph, the probability of \( X_2^4 \) (respectively \( X_3^4 \)) being at least \( k \) is at most \( C \exp(-ck) \) for some constants \( c, C > 0 \). The same holds for the number of loops \( X_1^4 \) (respectively \( X_3^4 \)).

**Proof.** Observe that the total number of cycles of length 2 is at most six times the maximal number of edge-disjoint cycles of length 2. Observe also that all cycles of length 1 are edge-disjoint. The corollary follows by Lemma 2.10.

A *smoothing* of a vertex of degree two in a graph consists in deleting the vertex and then joining its two neighbors by an edge. In particular, if the two neighbors are already connected by one or more edges, the number of edges between them increases by one after the smoothing. A *contraction of an edge* in a graph consists in deleting the edge and identifying its endvertices. Remark that smoothing of a vertex \( v \) of degree two is equivalent to contracting any of the edges incident with \( v \).

**Lemma 2.12.** Let \( G' \) be a uniform random graph with \( d_2' \) vertices of degree two and \( d_3' \) vertices of degree three, where \( d_3' \) is even. Then, by sampling \( G' \) and smoothing all vertices of degree two, we generate the uniform random 3-regular graph \( G_3(d_3') \) with \( d_3' \) vertices.

**Proof.** Let \((Q_{i,j})_{1 \leq i \leq 3, 1 \leq j \leq d'_2} \cup (Q_{i,j})_{1 \leq i \leq 2, d'_2+1 \leq j \leq d'_2+d'_3} \) be the points of the matching at the origin of the configuration model for \( G' \). Let also \((R_{i,j})_{1 \leq i \leq 3, 1 \leq j \leq d'_2} \) be the points of the matching at the origin of the configuration model for the random 3-regular graph \( G_3(d'_3) \). We present a coupling between the probability space of the matchings of \((Q_{i,j})_{1 \leq i \leq 3, 1 \leq j \leq d'_2} \cup (Q_{i,j})_{1 \leq i \leq 2, d'_2+1 \leq j \leq d'_2+d'_3} \) and of \((R_{i,j})_{1 \leq i \leq 3, 1 \leq j \leq d'_2} \). We perform the following algorithm generating the graphs \( G' \) and \( G_3(d'_3) \) at the same time.

1. Choose an arbitrary point \( Q_{i',j'} \) with \( 1 \leq i' \leq 3, 1 \leq j' \leq d'_3 \) (if it exists, if not, go to point 5.) that has not been matched yet. Prepare to match the point \( R_{i',j'} \).
2. Match \( Q_{i',j'} \) with some unmatched point \( Q = Q_{i'',j''} \) among \((Q_{i,j})_{1 \leq i \leq 3, 1 \leq j \leq d'_2} \cup (Q_{i,j})_{1 \leq i \leq 2, d'_2+1 \leq j \leq d'_2+d'_3} \).
3. If \( j'' \leq d'_3 \), match \( R_{i'',j''} \) and \( R_{i',j'} \). Then, return to 1.
4. If \( j'' \geq d'_3 + 1 \), then keep the point \( R_{i',j'} \) waiting and perform 2. with \( Q_{3-i'',j''} \) instead of \( Q_{i'',j''} \).
5. Match all points among \((Q_{i,j})_{1 \leq i \leq 2, d'_2+1 \leq j \leq d'_2+d'_3} \) that remain unmatched uniformly at random.

**Corollary 2.13.** Sampling uniformly a random graph with \( d_2' \) vertices of degree two and \( d_3' \) vertices of degree three, where \( d_3' \) is even, conditionally on that graph being 2-connected, and then applying smoothing on the vertices of degree two is equivalent to sampling uniformly a random 2-connected 3-regular graph.

**Proof.** Smoothing a vertex of degree two in a 2-connected graph preserves the 2-connectivity. Moreover, the graph is 2-connected after the smoothing if and only if it was 2-connected before the smoothing, see for example ([6], Section 3.1). Iterating these observations for a sequence of smoothings we obtain that the initially sampled graph is 2-connected if and only if the final 3-regular graph is 2-connected. It remains to apply Lemma 2.12 conditionally on \( G' \) being 2-connected.

8
3 Proof of Theorem 1.1

In this section we show that the lower bound given in (2) is tight up to lower order terms, thus proving Theorem 1.1. We start with a lemma, which shows that for both $d \in \{3, 4\}$ we are a.a.s. able to find a linear size independent set $A$ in $G_d(n)$ while at the same time maintaining the subgraph $G_d(n) \setminus A$ of $G_d(n)$ connected. Recall that $\beta$ is a fixed real number in the interval $\left(0, \frac{1}{4}\right)$.

**Lemma 3.1.** Fix an integer $n \geq 5$. Fix $d \in \{3, 4\}$ and $k \leq n^\beta$ non-negative integers. There are constants $C, c > 0$ such that, in a random $d$-regular graph $G_d = G_d(n)$ on $n$ vertices conditioned to be connected (or 2-connected or 3-connected), with probability at least $1 - C \exp(-ck)$ one can find an independent set $A \subseteq V(G_d)$ of size $\frac{n}{20} - \frac{16}{5}k$, for which $G_d \setminus A$ remains a connected graph.

**Proof.** From Theorem 2.1 we know that connected simple graphs with $n$ vertices and minimal degree three have spanning trees with at least $\frac{n}{4}$ leaves. Our goal now will be to modify our random graph so that the modification is a simple connected graph of minimal degree 3.

We deal with 3-regular and 4-regular graphs at the same time. By Corollary 2.11 we get by a union bound that there are constants $C', c' > 0$ such that $\max(X_d^1, X_d^2)$ is at most $k$ with probability at least $1 - 2C' \exp(-c'k)$ for both $d = 3$ and $d = 4$. Thus, by deleting loops and identifying parallel edges, with probability at least $1 - 2C' \exp(-c'k)$ we obtain a graph in which at most $3k$ of the vertices are of degree less than three. Equivalently, the set of $d$-regular graphs with $\max(X_d^1, X_d^2) > k$ is at most a $2C' \exp(-c'k)$-portion of all $d$-regular graphs. Since by ([4], Section 7.6) for all $d \geq 3$ the number of $d$-regular connected (or 2-connected or 3-connected) graphs is of the same order as the number of $d$-regular graphs we conclude that the probability of a random connected (respectively 2-connected or 3-connected) $d$-regular graph to have $\max(X_d^1, X_d^2) > k$ is at most $C'' \exp(-c''k)$ for some constant $C'' > 0$.

Define $G_d'$ to be the simple graph, obtained from $G_d$ by deleting loops and identifying parallel edges. Connect every vertex of degree less than three in $G_d'$ to some other vertices of $G_d'$ to form a new simple graph $G_d''$ of minimal degree at least three. We call these newly constructed edges false edges. Remark that we need at most $4k$ false edges with probability at least $1 - C'' \exp(-c''k)$ (indeed, every loop gives rise to two false edges and every set of $j$ multiple edges gives rise to $2(j - 1)$ false edges). It follows from Theorem 2.1 that $G_d''$ has a spanning tree with at least $\frac{n}{4}$ leaves. Take an arbitrary spanning tree with at least that many leaves. The deletion of all false edges of this tree forms a forest $F_d \subset G_d$ with at most $4k + 1$ connected components. Moreover, $F_d$ has at least $\frac{n}{4} - 8k$ leaves since the deletion of an edge in a forest can decrease the number of leaves by at most two. Since we work under the event that $G_d$ is connected (respectively 2-connected or 3-connected), the graph $G_d'$ is connected (respectively 2-connected or 3-connected) and $F_d$ can be thus extended to a spanning tree of $G_d'$ by Observation 2.2. Since connecting two connected components of $F_d$ by an edge can decrease the number of leaves in $F_d$ by at most two each time, the number of leaves in this last spanning tree is at least $\frac{n}{4} - 8k - 2 \times 4k = \frac{n}{4} - 16k$.

We remark that, first, deleting an arbitrary set of leaves of this spanning tree does not disconnect $G_d'$ as it does not disconnect the spanning tree, and second, every vertex of $G_d'$ has degree at most four. Thus we can greedily choose at least one fifth of all leaves of the spanning tree to form an independent set. The lemma is proved.

**Remark.** From the proof of Lemma 3.1 with $k = n^\beta$ and the fact that asymptotically almost every $d$-regular graph is connected for every $d \geq 3$ (see for example [18]) we conclude that a.a.s. we have that a random $d$-regular graph possesses a spanning tree with at least $\frac{n}{4} - o(n)$ leaves and so an independent set of at least $\frac{n}{20} - o(n)$ vertices leaving the graph induced by the remaining vertices connected.

Recall that every graph $G$ can be decomposed into 2-connected components forming the reduced 2-core of $G$, denoted $C_2(G)$, paths connecting some of the 2-connected components, which together with the
reduced 2-core form the 2-core of $G$, denoted $\overline{C}_2(G)$, and subtrees of $G$, each containing at most one vertex from $\overline{C}_2(G)$. We remark that by Observation 2.3 the number of connected components of the 2-core of a graph is equal to the number of connected components of the graph that contain a cycle. In particular, the 2-core of a connected graph is empty, if the graph is a tree, and it is connected otherwise.

Let $E_k$ be the event "The random regular graph $G_4 = G_4(n)$ contains an independent set $A_k$ of size $k$ and $V(G_4) \setminus A_k$ induces a connected graph". Fix $\varepsilon > 0$ and denote by $m = m(n) = \frac{n}{\varepsilon} - \frac{2\varepsilon}{3}n^3$ and $M = M(n) = \left(\frac{2\varepsilon}{3} - \varepsilon\right)n - 1$ (or rather the integer parts of these numbers, we omit from now on floor and ceiling functions for readability). By the remark after Lemma 3.1 we have that $E_m$ happens a.a.s. Our goal will be to prove the following lemma.

**Lemma 3.2.** For every $k \in [m, M)$, $\mathbb{P}(E_{k+1}|E_k) \geq 1 - C_\varepsilon \exp(-c_\varepsilon n^{3-\sqrt{8}})$, where $c_\varepsilon, C_\varepsilon > 0$ are constants depending only on $\varepsilon$.

We show how one can derive the proof of Theorem 1.1 from Lemma 3.2.

**Proof of Theorem 1.1 assuming Lemma 3.2.** We have that

$$\mathbb{P}(E_{M+1}) \geq \mathbb{P}(E_M)|\mathbb{P}(E_{M+1}|E_M) \geq \cdots \geq \mathbb{P}(E_m) \prod_{k=m}^{M} \mathbb{P}(E_{k+1}|E_k).$$

Now, by Lemma 3.1 and the following remark,

$$\mathbb{P}(E_m = E_{m(n)}) \xrightarrow{n \to +\infty} 1,$$

and since

$$\prod_{k=m}^{M} \mathbb{P}(E_{k+1}|E_k) \geq \left(1 - C_\varepsilon \exp(-c_\varepsilon n^{3-\sqrt{8}})\right)^{M-m} \geq \left(1 - C_\varepsilon \exp(-c_\varepsilon n^{3-\sqrt{8}})\right)^n \xrightarrow{n \to +\infty} 1,$$

the probability of $E_{M+1}$ tends to one.

Now, under the event $E_{M+1}$, consider the graph $B_{M+1} = G_4 \setminus A_{M+1}$. It is connected and has $2n - 4(M + 1) = (\frac{2}{3} + 4\varepsilon)n$ edges and $n - (M + 1) = (\frac{2}{3} + \varepsilon)n$ vertices. Thus $\rho(B_{M+1}) = 3n + 1$. By Observation 2.3 and Observation 2.4 one can obtain a forest from $B_{M+1}$ by deleting at most $3\varepsilon n - 1$ vertices from $B_{M+1}$. Overall, this proves that on the event $E_{M+1}$ the decycling number of $G_4$ is at most $M + 1 + 3\varepsilon n + 1 = (\frac{1}{3} + 2\varepsilon)n + 1$. Thus we get that a.a.s.

$$\lim sup_{n \to +\infty} \frac{\phi(G_4(n))}{n} \leq \frac{1}{3} + 2\varepsilon.$$

Since this holds for arbitrary $\varepsilon > 0$, together with [2] this shows that the limit exists and is equal to $1/3$. The theorem is proved.

In what follows we identify the set $V(G_4)$ with $[n]$. Fix $k \in [m, M]$.

By definition we have that on the event $E_k$ there exists a random subset $A_k$ of $[n]$ of size $k$ such that $A_k$ is an independent set of $G_4$ and $G_4 \setminus A_k$ is a connected graph. We denote by $F_k$ the event "$[k]$ is an independent set and $[n] \setminus [k]$ induces a connected graph". We show that working under $F_k$ instead of $E_k$ does not make a difference. Let $i_1 \leq i_2 \leq \cdots \leq i_k$ be the (random set of) vertices in $A_k$ and $j_1 \leq j_2 \leq \cdots \leq j_{n-k}$ be the (random set of) vertices in $V(G_4) \setminus A_k$. Let also $\sigma_k$ be a (random) permutation defined by

$$\sigma_k(l) = \begin{cases} i_l & \text{if } l \leq k \\ j_{l-k} & \text{otherwise.} \end{cases}$$
Moreover, for a graph $G$ on the vertex set $[n]$ and a permutation $\pi \in S^n$, define

$$\pi(G) = ([n], \{(\pi(i), \pi(j)) \mid (i, j) \in E(G)\}).$$

We recall that in our setting graphs can have parallel edges and therefore $E(G)$ is a multiset.

**Observation 3.3.** For any 4-regular graph $G$

$$P(G_4 = G \mid F_k) = P(G_4 = \sigma_k(G) \mid E_k).$$

**Proof.** Conditionally under $F_k$ one has a uniform probability measure on the family of random 4-regular graphs with $[k]$ being an independent set and $[n] \setminus [k]$ inducing a connected graph. Conditionally under $E_k$ one has a uniform probability measure on the family of random 4-regular graphs with $A_k$ being an independent set and $[n] \setminus A_k$ inducing a connected graph. Since $\sigma_k$ is a bijection between these two families, the claim holds. \qed

**Lemma 3.4.** For every $k \in [m, M]$, $P(E_{k+1}|F_k) \geq 1 - C_\varepsilon \exp(-c_\varepsilon n^3 - \sqrt{\varepsilon})$, where $c_\varepsilon, C_\varepsilon > 0$ are constants depending only on $\varepsilon$.

We now give the proof of Lemma 3.2 assuming Lemma 3.3.

**Proof of Lemma 3.2** By Lemma 3.4 we have

$$P(E_{k+1}|E_k) = P(\{A_{k+1} \text{ is an independent set in } G_4 \text{ and } [n] \setminus A_{k+1} \text{ induces a connected graph} \mid E_k\) 

$$= P(\{\sigma_k^{-1}(A_{k+1}) \text{ is an independent set in } G_4 \text{ and } [n] \setminus \sigma_k^{-1}(A_{k+1}) \text{ induces a connected graph} \mid F_k\) 

$$= P(E_{k+1}|F_k).$$

The proof of Lemma 3.2 is finished. \qed

Our goal until the end of this section will be to prove Lemma 3.4.

By restricting our attention to the (random) subgraph $B_k$ of $G_4$ induced by the vertices $[n] \setminus [k]$, we obtain a probability distribution $U_k$ induced by $P(\cdot | F_k)$ on the set of connected graphs of maximal degree at most four with $n - k$ vertices and $2n - 4k$ edges. Recall that we work in the configuration model and therefore the event $\{B_k = G\}$ contains not only purely graph theoretical information about $B_k$, but it also indicates a partial configuration on a subset of $(P_{i,j})_{1 \leq i \leq 4, 1 \leq j \leq n}$.

**Observation 3.5.** $U_k$ is a uniform distribution on the set of partial configurations associated to connected graphs of maximal degree at most four with $n - k$ vertices and $2n - 4k$ edges.

**Proof.** Let $G'$ be a partial configuration satisfying the above constraints. By construction of the configuration model there are $4k$ points in the buckets of $V(G')$ and $4k$ points in the buckets in $[k]$ to be matched. Since we work under $F_k$, the above sets of $4k$ points should form the two parts of a bipartite graph. The number of the above bipartite graphs is $(4k)!$, and hence we have

$$P(B_k = G' \mid F_k) = \frac{(4k)!}{P(F_k)(4n - 1)!!},$$

where $(4n - 1)!!$ is the total number of configurations on $(P_{i,j})_{1 \leq i \leq 4, 1 \leq j \leq d}$. Since the right hand side of (5) does not depend on $G'$, the observation is proved. \qed
By Observation 3.3 the probability to sample one particular 2-core of \(B_k\) is proportional to the number of configurations inducing this 2-core. Thus, the probability distribution \(U_k\) in turn induces another probability distribution \(\mathcal{P}_k\) (not necessarily uniform) on the set of 2-cores of connected graphs of degree at most four with \(n-k\) vertices and \(2n-4k\) edges. In other words, \(\mathcal{P}_k\) is obtained by counting since it arises from a uniform distribution. We look for a way to recover the uniform probability on the set of configurations on \((P_{i,j})_{1 \leq i < j \leq n}\) conditionally under \(F_k\) via sampling a 2-core \(\overline{C_k}\) (from now on \(C_2 := C_2(B_k)\) and \(C_2 := C_2(B_k)\) are seen as subgraphs of \(B_k\)) according to the probability distribution \(\mathcal{P}_k\). Observe that, by definition of \(\mathcal{P}_k\), it is sufficient to find a way to recover the uniform matching distribution conditioned under both \(F_k\) and the sampled 2-core \(\overline{C_k}\). Recall that the choice of the 2-core consists in the choice of partial configuration i.e. vertices (or buckets) that participate in it, the choice of points in these buckets that serve as endvertices of the edges, present in the 2-core, and the choice of the matching between these points.

Having already sampled a 2-core \(\overline{C_k}\) from \(\mathcal{P}_k\), we have at our disposal a certain (random) number \(s\) of vertices in \(B_k\) to attach to \(\overline{C_k}\) and \(k\) vertices in \(A_k\). Now, in each of the \(s\) vertices (buckets) of \(B_k\) to attach, we choose uniformly at random one of its four points and call it the \(-pointer\) of this vertex. This point will be incident to the edge connecting the vertex to \(A\) for every \(i\) for every \(v\) and is not a pointer is called \(\text{non-pointer}\).

Our goal, as already pointed out, will be to recover the uniform matching distribution between the points that remain unmatched in \(\overline{C_k}\) under the event \(F_k\). Let us try to understand how this reformulates in terms of pointers and non-pointers. Every \(-pointer\) has to be matched to an unmatched point in \(B_k\).

This point should necessarily be a non-pointer by the fact that \(-pointers\) serve to attach their respective vertices to the 2-core \(\overline{C_k}\) via a path in \(B_k\). Also, we know that no two \(-pointers\) are matched to each other for the same reason, and more generally that there are no vertices \(v_1, v_2, \ldots, v_\ell \in B_k \setminus \overline{C_k}\) such that for every \(i \in \{1, 2, \ldots, \ell\}\), the pointer of \(v_i\) is matched to a non-pointer of \(v_{i+1}\) (indices are taken modulo \(\ell\)). One can verify without much effort that these are the only restrictions imposed by the conditioning.

We thus wish to be able to sample a uniform matching between pointers and non-pointers with the restriction that there are no vertices \(v_1, v_2, \ldots, v_\ell \in B_k \setminus \overline{C_k}\) such that for every \(i \in \{1, 2, \ldots, \ell\}\), the pointer of \(v_i\) is matched to a non-pointer of \(v_{i+1}\) (indices are taken modulo \(\ell\)). We remind the reader that for a graph \(G\) and a connected subset of vertices \(S\) of \(G\), we denote by \(cc(S)\) the connected component of \(S\) in \(G\). By abuse of notation for a subgraph \(G'\) of \(G\) contained in a single connected component of \(G\) we sometimes denote \(cc(G') = cc(V(G'))\). This notation depends on the original graph \(G\), which will be specified unless no ambiguity is possible. In order to recover the uniform matching distribution between the points that remain unmatched in \(\overline{C_k}\) under the event \(F_k\), we perform the following procedure:

**Algorithm 1.**

1. If there is no vertex in \(B_k\) not yet in the connected component of \(\overline{C_k}\) in \(B_k\), go to point 2. Otherwise, choose a vertex \(v\) of \(B_k\), which is not yet in \(cc(\overline{C_k})\), uniformly at random and attach its pointer to a uniformly chosen unmatched non-pointer. This results in a new edge \(vu\) in \(B_k\).
   
   (a) If \(u\) is in \(cc(\overline{C_k})\), return to point 1.
   
   (b) If \(u\) is not in \(cc(\overline{C_k})\) and its pointer is unmatched, attach its pointer to a uniformly chosen unmatched non-pointer. This forms an edge \(uw\). Redo the case analysis ((a), (b) or (c)) with \(w\) instead of \(u\).
   
   (c) If \(u\) is not in \(cc(\overline{C_k})\) and its pointer is already matched, stop the procedure and reinitialize the process.

2. Choose a vertex in \(A_k\) and attach consecutively each of its pointers to an unmatched non-pointer in \(B_k\) uniformly at random. Repeat this until all vertices in \(A_k\) have been processed. □
Notice that this algorithm is reinitialized only when a cycle in $B_k$ not in the connected component of $\overline{C_2}$ appears.

**Observation 3.6.** Every configuration inducing a 4-regular graph satisfying $F_k$ and with 2-core $\overline{C_2}$ of $B_k$ appears as outcome with the same probability.

Proof. Every possible outcome of the algorithm is equally probable by construction. Moreover, the set of outcomes coincides with the set of configurations obeying $F_k$ and with 2-core $\overline{C_2}$ of $B_k$ due to the analysis made just before Algorithm 1. □

**Observation 3.7.** The number of vertices of $\overline{C_2}$ of degree three or four is at least $3\varepsilon n$.

Proof. This follows from Observation 2.7 and from the choice of $k \leq M = M(n) = (\frac{1}{3} - \varepsilon)n - 1$: indeed, one has that $B_k$ is of maximal degree four and $\rho(B_k) = 2n - 4k + 1 - (n - k) = n - 3k + 1 \geq 3\varepsilon n + 1$. □

We denote by $C_2$ the reduced 2-core of $B_k$ induced by $\overline{C_2}$. Let us denote by $S(B_k)$ the (random) set of vertices of $C_2$ that are either

- of degree four in $C_2$ and are not cutvertices in their connected component in $C_2$ or
- of degree three both in $\overline{C_2}$ and in $C_2$.

One may observe that $S(B_k)$ contains exactly the vertices of $C_2$ of degree at least three in $C_2$ that do not disconnect $\overline{C_2}$ when being deleted. If there is a vertex $v$ of degree four in $C_2$ then one could increase the independent set $A_k$ by just adding $v$ to $A_k$. As it was in $C_2$, its connected component in $C_2$ will remain connected. Moreover, it was not adjacent to any vertex in $\overline{C_2} \setminus C_2$ as it is already of degree four in $C_2$, so $\overline{C_2} \setminus v$ remains connected and therefore $B_k \setminus v$ remains connected as well.

From now on we restrict ourselves by conditioning on the (bad) event that there is no vertex of degree four in $S(B_k)$. That is, we condition on the event that all vertices of $S(B_k)$ are of degree three in both $\overline{C_2}$ and in $C_2$. Our goal then will be to find two vertices $u$ and $v$ in $S(B_k)$, which are connected by an edge to the same vertex $w$ in $A_k$ and such that $B_k \setminus \{u, v\}$ remains connected. This will allow us to send $u$ and $v$ to $A_k$ and $w$ to $B_k$, thus increasing the size of $A_k$ and leaving the graph in $B_k$ connected at the same time.

Let $\omega = \omega(n)$ throughout the paper be a function tending to infinity with $n$ sufficiently slowly (the growth of $\omega$ is given implicitly below). The following lemma shows that, since the total number of "small" cycles is a.a.s. sublinear, the number of 2-connected components is a.a.s. sublinear as well:

**Lemma 3.8.** There is a constant $c > 0$ such that $C_2$ contains at most $\frac{n}{\omega(n)}$ 2-connected components with probability at least $1 - \exp(-cn^\beta)$.

Proof. We argue by contradiction. Suppose that there is some $\varepsilon' > 0$ such that for every constant $c' > 0$ there are infinitely many positive integers $n$ such that the number of 2-connected components of $C_2$ is at least $\varepsilon'n$ with probability at least $\exp(-c'n^\beta)$. For every $n$ with the property mentioned above we conclude that with probability at least $\exp(-c'n^\beta)$ the number of 2-connected components of $C_2$ of size at most $\frac{4}{\varepsilon'}$ is at least $\frac{\varepsilon'n}{2}$. Indeed, the contrary would mean that there are less than

$$\frac{2n}{\varepsilon'} + \frac{\varepsilon'n}{2} = \varepsilon'n$$

2-connected components, contrary to our assumption. Here we use the fact that different 2-connected components are edge-disjoint.
Every 2-connected component of $C_2$ of size at most $\frac{4}{\varepsilon'}$ contains a cycle of length at most $\frac{4}{\varepsilon'}$ and cycles in different 2-connected components of $C_2$ are edge-disjoint. Thus, with probability at least $\exp(-c'n^\beta)$ there are at least $\frac{\varepsilon' n}{2}$ edge-disjoint cycles of length at most $\frac{4}{\varepsilon'}$.

On the other hand, by Lemma 2.10 we know that for every positive integer $\ell \leq \frac{4}{\varepsilon'}$ there are constants $c_\ell, C_\ell > 0$ for which $P(Y_4^\ell \geq k) \leq C_\ell \exp(-c_\ell k)$ for every $1 \leq k \leq n^\beta$. Define $c = \min_{1 \leq \ell \leq 4} c_\ell$ and $C = \max_{1 \leq \ell \leq 4} C_\ell$. Then, by choosing $\varepsilon' < 2$ we have that

$$P\left(\sum_{i=1}^{4} Y_4^i \geq \frac{\varepsilon' n^\beta}{2}\right) \leq P\left(\max_{1 \leq i \leq 4} Y_4^i \geq \frac{\varepsilon' n^\beta}{2} \right) \leq C \exp\left(-\frac{c\varepsilon'^2}{8} n^\beta\right).$$

But then by choosing $\varepsilon' < \frac{c\varepsilon'^2}{8}$ we obtain that for infinitely many $n$ we have

$$\exp(-c'n^\beta) \leq P\left(\sum_{i=1}^{4} Y_4^i \geq \frac{\varepsilon' n^\beta}{2}\right) \leq \exp\left(-\frac{c\varepsilon'^2}{8} n^\beta\right).$$

This is a contradiction. The lemma is proved.

**Lemma 3.9.** There are at most $2\left(\frac{n}{\omega(n)} - 1\right)$ vertices in $C_2$ that are either of different degree in $C_2$ and in $\overline{C_2}$ or cutvertices in $\overline{C_2}$ with probability at least $1 - \exp(-cn^\beta)$ for some constant $c > 0$.

**Proof.** By Lemma 3.8 there exists some constant $c > 0$ such that with probability at least $1 - \exp(-cn^\beta)$, $C_2$ contains at most $\frac{n}{\omega(n)}$ 2-connected components. Recall that in order to obtain $\overline{C_2}$ from $C_2$, the 2-connected components of $C_2$ are connected to each other by paths in a (unique) tree-like way. In other words, by identifying the 2-connected components of $C_2$ and vertices and the paths between them (possibly of length zero) with edges, we obtain a tree. The number of edges of this tree is equal to the number of 2-connected components in $C_2$ minus one, and hence only twice as many vertices of $C_2$ can be of different degree in $C_2$ and in $\overline{C_2}$ or cutvertices in $\overline{C_2}$. Thus, conditionally under having at most $\frac{n}{\omega(n)}$ 2-connected components, at most $2\left(\frac{n}{\omega(n)} - 1\right)$ vertices can be either of different degree in $C_2$ and in $\overline{C_2}$ or cutvertices in $\overline{C_2}$. The lemma follows.

**Lemma 3.10.** There is a constant $c > 0$ such that $S(B_k)$ contains at least $3\varepsilon n - 2\left(\frac{n}{\omega(n)} - 1\right)$ vertices with probability at least $1 - \exp(-cn^\beta)$.

**Proof.** By Lemma 3.9 with probability at least $1 - \exp(-cn^\beta)$, there are at most $2\left(\frac{n}{\omega(n)} - 1\right)$ vertices of $C_2$, which are either of different degree in $C_2$ and in $\overline{C_2}$ or cutvertices in $\overline{C_2}$. Since by Observation 3.7 there are at least $3\varepsilon n$ vertices of degree three or four in $\overline{C_2}$, we conclude that with probability at least $1 - \exp(-cn^\beta)$ there are at least $3\varepsilon n - 2\left(\frac{n}{\omega(n)} - 1\right)$ vertices in $S(B_k)$. 

14
Lemma 3.11. There is a constant $c = c(\varepsilon) > 0$ such that for every $n \in \mathbb{N}$ large enough, with probability at least $c$, Algorithm 1 is never reinitialized.

Proof. Recall that we condition under the event that $S(B_k)$ contains only vertices of degree three. By Lemma 3.10 there is a constant $c' > 0$ such that $|S(B_k)| \geq 3\varepsilon n - 2\left(\frac{n}{\omega(n)} - 1\right)$ with probability at least $1 - \exp(-c'n^2)$. Since the event $|S(B_k)| \geq 3\varepsilon n - 2\left(\frac{n}{\omega(n)} - 1\right)$ holds a.a.s., we may and do condition on it.

We call a vertex of $B_k \setminus cc(C_2)$ a starting vertex if its pointer is matched before any of its non-pointers and let a chain starting at $v$ be a path (recall that a path is defined by its edges since we work with multigraphs) in $B_k \setminus cc(C_2)$ beginning with a starting vertex $v$ and ending either with a vertex of $C_2$ or with a vertex of $B_k \setminus C_2$ whose pointer was already matched. See Figure 2. Let $v_1$ be the first vertex in $B_k \setminus C_2$ chosen in point 1 of Algorithm 1. The probability that the non-pointer chosen to be matched with the pointer of $v_1$ belongs to a vertex in $C_2$ is at least a constant $c'' = c''(\varepsilon) > 0$, and hence the probability of not attaching it to $C_2$ is at most $1 - c'' < 1$. In this case (point 1 of Algorithm 1) the probability for the next vertex $v_2$ not to be attached to $C_2$ is also at most $1 - c''$. Hence, the probability to have a chain of $t$ consecutive vertices $v_1, v_2, \ldots, v_t$ of $B_k \setminus C_2$ not attached to $C_2$ is at most $(1 - c'')^t$. In this case the probability to form a cycle within this chain is

$$\Pr(\exists t \mid v_1, \ldots, v_t \in B_k \setminus C_2 \text{ is a chain and } (v_1, v_2, \ldots, v_t) \text{ contains a cycle}) \leq \sum_{t \in \mathbb{N}} \Pr(v_1, \ldots, v_t \in B_k \setminus C_2 \text{ is a chain and } (v_1, v_2, \ldots, v_t) \text{ contains a cycle}) \leq \sum_{t \in \mathbb{N}} \Pr(v_1, \ldots, v_t \in B_k \setminus C_2 \text{ is a chain}) \Pr((v_1, v_2, \ldots, v_t) \text{ contains a cycle } \mid v_1, \ldots, v_t \in B_k \setminus C_2 \text{ is a chain}) \leq \sum_{t \in \mathbb{N}} (1 - c'')^t \frac{2t + 1}{\varepsilon n},$$

where the last inequality holds since, on the one hand, there are exactly $2t + 1$ unmatched non-pointers of $v_1, \ldots, v_t$, and on the other hand, at each step the total number of unmatched non-pointers is asymptotically at least $\varepsilon n$ by Lemma 3.10. Indeed, by our conditioning $S(B_k)$ contains only vertices of degree three in $C_2$ and each of them contains an unmatched non-pointer. We conclude that there is a constant $C' = C''(\varepsilon) > 0$ for which the probability that $v_1$ remains disconnected from $C_2$ in $B_k$ is at most.
\[
\sum_{t \in \mathbb{N}} (1 - e^{-n})^{t \frac{2t + 1}{\varepsilon n}} \leq \frac{C'}{n}.
\]

If one was successful to connect \(v_1\) (and all vertices in its chain) to \(\overline{C_2}\) in \(B_k\), for every vertex that remains to be treated by Algorithm 1 the probability to attach it directly to the connected component \(cc(\overline{C_2})\) is at least as high as for \(v_1\). Indeed, each of the already attached vertices has at least two additional non-pointers that could serve to form an edge leading to the connected component \(cc(\overline{C_2})\) and the number of non-pointers in buckets that are not yet in \(cc(\overline{C_2})\) decreased.

Clearly there is no cycle in \(B_k \setminus \overline{C_2}\) if no chain contains a cycle. We partition the set \(T\) of starting vertices into \(s \leq 2C' + 1\) groups \(V_1, V_2, \ldots, V_s\) of at most \(\frac{n}{2C'}\) vertices so that \(V_1\) contains the first \(\frac{n}{2C'}\) starting vertices processed by Algorithm 1, \(V_2\) contains the next \(\frac{n}{2C'}\) starting vertices processed by Algorithm 1, etc. By (6), taking a union bound over the vertices of \(\overline{V}\) vertices in \(V_i\), the probability that all vertices of the \(i\)-th group \(V_i\) end up in \(cc(\overline{C_2})\) is at least \(\frac{n}{2C'} \times \frac{C'}{n} = \frac{1}{2}\). Hence, the probability to attach all starting vertices to \(cc(\overline{C_2})\) is

\[
\mathbb{P}(\forall w \in T, \text{ the chain starting at } w \text{ ends in } cc(\overline{C_2})) = \prod_{i=1}^{\delta} \mathbb{P}(\forall w \in V_i, \text{ the chain starting at } w \text{ ends in } cc(\overline{C_2}) \mid \forall w \in \bigcup_{j=1}^{i-1} V_j, \text{ the chain starting at } w \text{ ends in } cc(\overline{C_2})) \geq \frac{1}{2^s} > \frac{1}{2^{2C' + 1}}.
\]

The lemma follows.

Recall that \(\beta\) is a fixed number in the interval \((0, \frac{1}{2})\). Our goal will be to find a subset \(S'(B_k)\) of \(S(B_k)\) as large as possible with the property that the removal of any pair of vertices in \(S'(B_k)\) does not disconnect \(B_k\) (recall that we condition under the event that there is no vertex of degree four in \(S(B_k)\)). To do this we first construct a superset \(S''(B_k)\) of \(S'(B_k)\) according to the following strategy. For any 2-connected component \(C'_2\) of \(C_2\) such that \(S(B_k) \cap C'_2 \neq \emptyset\), if \(C'_2\) contains at most \(n^3\beta\) vertices of \(S(B_k)\), choose one vertex of \(C'_2\) in \(S(B_k)\) and add it to \(S''(B_k)\). If on the contrary \(C'_2\) contains more than \(n^3\beta\) vertices of \(S(B_k)\), we use the following result:

**Lemma 3.12.** For a 2-connected component \(C'_2\) of \(C_2\) with \(d_3' \geq n^3\beta\) vertices of degree three, with probability at least \(1 - \exp(-cn^{\beta^2})\) we can find a subset of size \(\frac{d_3'}{20} - \frac{16}{5}d_3^\beta\) of the above set of vertices, no pair of which disconnects \(C'_2\).

**Lemma 3.12** will be proved below. Notice that no vertex can be of degree three in two different 2-connected components of \(C_2\) since different 2-connected components are edge-disjoint. Therefore the sets of vertices given by Lemma 3.12 corresponding to different 2-connected components are disjoint as well. Hence, in total Lemma 3.12 produces a set \(S''(B_k)\) of size at least

\[
|\{\text{connected components of } C_2 \text{ with order at most } n^\beta\}| + \sum_{C'_2 : |C'_2| \geq n^\beta} \frac{|C'_2|}{20} - \frac{16}{5}|C'_2|^\beta
\]

vertices with probability at least

\[
(1 - \exp(-cn^{\beta^2}))^{n^{1-\beta}} = 1 - (1 + o(1))n^{1-\beta} \exp(-cn^{\beta^2}),
\]

since the total number of connected components of \(C_2\) of order at least \(n^\beta\) is at most \(n^{1-\beta}\). Now, let \(S'(B_k) = S(B_k) \cap S''(B_k)\).
Lemma 3.14. There is a constant $c > 0$ such that, with probability at least $1 - (1 + o(1))n^{1-\beta} \exp(-cn^{\beta^2})$, $|S'(B_k)| \geq \varepsilon n^{1-\beta}$.

Proof. Recall that by Lemma 3.10, $|S(B_k)| \geq 3\varepsilon n - 2\left(\frac{n}{\omega(n)} - 1\right)$ with probability at least $1 - \exp(-cn^\beta)$. Assuming $n$ sufficiently large, if at least half of the vertices of $S(B_k)$ are in components of order at most $n^\beta$, the statement is clear since $\beta > \beta^2$. Otherwise, by Lemma 3.12 there is a constant $c > 0$ such that with probability at least $1 - (1 + o(1))n^{1-\beta} \exp(-cn^{\beta^2})$, $S''(B_k)$ contains at least $\frac{3\varepsilon n}{20} - o(n)$ vertices. Also, by Lemma 3.13 there is a constant $c' > 0$ such that, with probability at least $1 - \exp(-c'n^{\beta})$, there are at most $2\left(\frac{n}{\omega(n)} - 1\right)$ vertices in $S''(B_k)$ not in $S(B_k)$. Thus, with probability at least $1 - (1 + o(1))n^{1-\beta} \exp(-cn^{\beta^2}) - \exp(-c'n^{\beta}) = 1 - (1 + o(1))n^{1-\beta} \exp(-cn^{\beta^2})$, $|S'(B_k)| = |S''(B_k) \cap S(B_k)| \geq \frac{3\varepsilon n}{40} - o(n) - 2\left(\frac{n}{\omega(n)} - 1\right) = \frac{3\varepsilon n}{40} - o(n)$, and the statement follows.

We come back to the proof of Lemma 3.12. To do this, we prove that conditionally on the number of vertices of $C_2'$ and on the degree sequence of $C_2'$ the distribution of $C_2'$ is uniform among all 2-connected graphs with the same vertex degrees.

Lemma 3.14. Let $G'$ and $G''$ be two connected graphs without vertices of degree one with $\alpha \leq n - k$ vertices and $\beta = n - 3k + \alpha$ edges. The number of extensions of $G'$, respectively $G''$, to a 4-regular graph respecting $F_k$ and such that $C_2 = G'$, respectively $C_2 = G''$, is the same.

Proof. We have that by the choice of $\alpha$ and $\beta$ there are $n - k - \alpha$ $B$-pointers to match (one for each vertex in $B_k \setminus G'$ or $B_k \setminus G''$, respectively) and $4\alpha - 2\beta = 2\alpha + 6k - 2n$ non-pointers in the buckets of $C_2$ for both $G'$ and $G''$. Since the probability to form a cycle disconnected from $C_2$ depends only on the number of non-pointers in $C_2$ (recall point II of Algorithm 1), the probability to extend properly $C_2 = G'$ to $G_4$ under $F_k$ is equal to the probability to extend properly $C_2 = G''$ to $G_4$ under $F_k$. In other words, the number of extensions of $G'$ and $G''$ is the same.

Recall that we condition under the event of having no vertex of degree 4 in any 2-connected component of $B_k$. Under this conditioning we have:

Lemma 3.15. Let $C_2'$ be a 2-connected component of $C_2$. The distribution of $C_2'$ conditionally on $C_2'$ having $d_2'$ vertices of degree two and $d_3'$ vertices of degree three is uniform.

Proof. The claim holds if we prove that for any graph $G'_1$ and any 2-connected graph $G'_2$ with $d_2'$ vertices of degree two and $d_3'$ vertices of degree three, the number of ways to recover $G_4$ from $C_2' = G'_2$ and $G_4 \setminus C_2' = G'_1$ depends only on the choice of $G'_1$ and not on $G'_2$. By Lemma 3.14 it is sufficient to prove that the number of extensions of $C_2' = G'_2$ and any graph $\overline{C_2} \setminus C_2' = \tilde{G}_2$ to a 2-core of $B_k$ with fixed number of vertices and edges is the same for different 2-connected graphs $G'_2$ and depends only on $\tilde{G}_2$. Indeed, by the law of total probability partitioning over all graphs defined on the set of vertices $V(\overline{C_2} \setminus C_2')$ we get

$$\mathbb{P}(C_2' = G_2') = \sum_{\tilde{G}_2} \mathbb{P}(\overline{C_2} \setminus C_2' = \tilde{G}_2) \mathbb{P}(C_2' = G_2' \mid \overline{C_2} \setminus C_2' = \tilde{G}_2).$$

Of course, there could be graphs $\tilde{G}_2$ for which the probability of extension is zero, but the terms in the above sum that correspond to them will be zero.

Let us condition on $\overline{C_2} \setminus C_2'$. Suppose that there are $\alpha_2$ pairs of edges of $\overline{C_2} \setminus C_2'$ coming from another 2-connected component to be attached in pairs to a vertex $v$ of $C_2'$ (recall that this corresponds to $v$ being a cutvertex) and $\alpha_1$ ends of paths in $\overline{C_2} \setminus C_2'$ also waiting to be attached to $C_2'$ (this corresponds to other
Figure 3: Top figure: Possible configuration for $C'_2 \cup (\overline{C_2} \setminus C'_2)$. Middle figure: A possible matching between the unmatched points in $C'_2$ and the unmatched points in $\overline{C_2} \setminus C'_2$. Note that here $C'_2$ is indeed a 2-connected component of $\overline{C_2}$. Bottom figure: An impossible matching between the unmatched points in $C'_2$ and the unmatched points in $\overline{C_2} \setminus C'_2$. Note that here $C'_2$ is not a 2-connected component of $\overline{C_2}$. 
2-connected components connected to $C'_2$ by paths of length at least one). One may verify without much effort that these are the only possibilities for edges between $C'_2$ and $\overline{C_2} \setminus C'_2$, see Figure 3.

Then, each of the $\alpha_2$ pairs of edges have to be attached to vertices of degree two in $C'_2$, which makes $2^{\alpha_2} \frac{d'_2!}{(d'_2 - \alpha_2)!}$ choices, and attaching the remaining $\alpha_1$ ends of paths can be made in $(d'_3 + 2(d'_2 - \alpha_2))!$ ways. Thus, the total number of extensions of $C'_2 \cup (\overline{C_2} \setminus C'_2)$ to $\overline{C_2}$ with the prescribed number of vertices of degree two and three in $C'_2$ is equal to

$$2^{\alpha_2} \frac{d'_2!}{(d'_2 - \alpha_2)!}(d'_3 + 2(d'_2 - \alpha_2))!.$$ 

Since this number depends only on $d'_2, d'_3$ and $\alpha_2$ and not on the choice of graph $G'_2$ with $d'_2$ vertices of degree two and $d'_3$ vertices of degree three, the lemma is proved. \hfill \Box

Now we are ready to prove Lemma 3.12.

**Proof of Lemma 3.12.** By Lemma 3.15 conditionally on the degree sequence of $C'_2$ we know that $C'_2$ is uniformly distributed among the set of 2-connected graphs with the same degree sequence. By Lemma 2.12 smoothing of all vertices of degree two leads to a uniform random 2-connected 3-regular graph $\tilde{C}_2'$ on $d'_3$ vertices, which by Lemma 3.11 (applied with $k = (d'_3)^{3/2}$) contains with probability at least $1 - C \exp(-cd'_3^3) \geq 1 - C \exp(-cn^{3/2})$ a set of at least $\frac{d'_3}{20} - \frac{16}{5} d'_3^{3/2}$ vertices of degree three, no pair of which disconnects $\tilde{C}_2'$. It remains to observe that no pair of vertices in this same set disconnects $C'_2$ as well. \hfill \Box

We can now relate our previous observations with to the urn lemma mentioned in the previous section. Recall once again that we condition under having no vertex of degree 4 in any 2-connected component of $B_k$.

**Corollary 3.16.** There are constants $C, c > 0$ and $n_0 \in \mathbb{N}$ such that, for every $n \geq n_0$, with probability at least $1 - C \exp(-\varepsilon n^{1-\beta}/88) - (1 + o(1)) C n^{1-\beta} \exp(-cn^{3/2})$ there are at least $\frac{\varepsilon n^{1-\beta}}{22}$ vertices of $\mathcal{S}'(B_k)$, which are connected to a vertex in $A_k$.

**Proof.** By Observation 3.13 we have that $|\mathcal{S}'(B_k)| \geq \varepsilon n^{1-\beta}$ with probability at least $1 - (1 + o(1)) n^{1-\beta} \exp(-cn^{3/2})$. We condition under this event. Consider the first $\varepsilon n^{1-\beta}$ vertices of $\mathcal{S}'(B_k)$ and start to attach their non-pointers uniformly at random. For each unmatched non-pointer in such a vertex there are $4k$ pointers in $A_k$ and $p \leq n - 4k$ pointers in $B_k$ except the ones being already matched. Moreover, $k \geq m(n) = \frac{n}{20} - \frac{16}{5} n^{\beta}$, so for every large enough $n$ we have $k \geq \frac{n}{24}$ and so $p \leq \frac{5n}{6}$. This allows us to apply Lemma 2.3 with $U_1 = \{\text{pointers of } A_k\}, U_2 = \{\text{pointers of } B_k\}$ and $s = \varepsilon n^{1-\beta}$ (so we have $\frac{|U_2|}{|U_1|} \leq 5$), where the balls to throw correspond to the first $s$ non-pointers in $\mathcal{S}'(B_k)$ to attach. Hence, by Lemma 2.3 with probability at least $1 - \exp(-\varepsilon n^{1-\beta}/88)$, there will be at least $\frac{\varepsilon n^{1-\beta}}{22}$ non-pointers corresponding to vertices in $\mathcal{S}'(B_k)$ matched with pointers coming from vertices in $A_k$.

On the other hand, by Lemma 3.11 matching uniformly pointers and non-pointers produces (for $n$ large enough) no cycles outside of $cc(\overline{C_2})$ in $B_k$ with probability bounded below by some constant $c' > 0$. By Observation 4.6 this means that with probability at least $c'$ we obtain a 4-regular graph, chosen uniformly among all 4-regular graphs satisfying $F_k$ and with a fixed 2-core $\overline{C_2}$. Conditionally under this event, by the first paragraph of the proof of the lemma we get that with probability at most

$$\exp(-\varepsilon n^{1-\beta}/88) + (1 + o(1)) n^{1-\beta} \exp(-cn^{3/2}).$$
there will be less than \( \frac{\varepsilon n^{1-\beta}}{22} \) non-pointers corresponding to vertices in \( S'(B_k) \) matched with pointers coming from vertices in \( A_k \). Thus, for a uniform 4-regular graph satisfying \( F_k \) and having 2-core \( C_2 \), with probability at most
\[
\frac{1}{e} \left( \exp(-\varepsilon n^{1-\beta}/88) + (1 + o(1))n^{1-\beta} \exp(-cn^{\beta^2}) \right)
\]
there will be less than \( \frac{\varepsilon n^{1-\beta}}{22} \) vertices of \( S'(B_k) \), which are connected to a vertex in \( A_k \). The lemma follows.

**Proof of Lemma 3.4.** By Corollary 3.16 we have that for every large enough \( n \) with probability at least
\[
1 - C \exp(-\varepsilon n^{1-\beta}/88) - (1 + o(1))Cn^{1-\beta} \exp(-cn^{\beta^2})
\]
at least \( \frac{\varepsilon n^{1-\beta}}{22} \) vertices of \( S'(B_k) \) are matched to \( A \)-pointers. Thus, conditioning on this event, the probability that all these non-pointers are connected to \( A \)-pointers in different buckets (vertices) is
\[
\frac{\varepsilon n^{1-\beta}}{22} \prod_{i=0}^{k-1} \frac{4k - 4i}{4k - i} \leq \frac{\varepsilon n^{1-\beta}}{22} \prod_{i=0}^{k-1} \left( 1 - \frac{i}{n} \right).
\]
Using \( \log(1 - x) \leq -x \) we deduce that
\[
\log \prod_{i=0}^{\varepsilon n^{1-\beta}/22} \left( 1 - \frac{i}{n} \right) \leq \frac{1}{n} \sum_{i=0}^{\varepsilon n^{1-\beta}/22-1} i \leq -\frac{\varepsilon^2 n^{1-2\beta}}{484} (1 + o(1)).
\]
Hence, the expressions we need to take into consideration are
\[
1 - C \exp(-\varepsilon n^{1-\beta}/88) - (1 + o(1))Cn^{1-\beta} \exp(-cn^{\beta^2})
\]
(the probability of the event we condition under at the beginning of the proof of the lemma) and
\[
1 - \exp(-cn^{1-2\beta})
\]
(the probability of the event just calculated in the proof of this lemma). By choosing \( \beta = \sqrt{2} - 1 \) we minimize asymptotically the upper bound we have for the probability of any bad event, which is
\[
C \exp(-\varepsilon n^{1-\beta}/88) + (1 + o(1))Cn^{1-\beta} \exp(-cn^{\beta^2}) + \exp(-cn^{1-2\beta}).
\]
This finishes the proof of the lemma.

4 Conclusion and outlook

We showed that random 4-regular graphs can be made acyclic by taking out relatively few vertices. Although our approach does not give a polynomial-time algorithm, it gives an idea how a fast algorithm might possibly work: trying to keep a connected graph \( B \) throughout the process, at each step either take out from \( B \) a vertex of degree four in \( B \) and add it to a stable set \( A \) that grows sequentially or take out two vertices of degree three in \( B \) sharing one common neighbor \( w \) in \( A \), add them to \( A \), take \( w \) out of \( A \) and send it back to \( B \). It would be interesting to see whether an approach similar to ours can be used (both algorithmically and non-algorithmically) to give better bounds for random \( d \)-regular graphs with \( d \geq 5 \).

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