Automorphism group of principal bundles, Levi reduction and invariant connections

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Abstract
Let $M$ be a compact connected complex manifold and $G$ a connected reductive complex affine algebraic group. Let $E_G$ be a holomorphic principal $G$–bundle over $M$ and $T \subset G$ a torus containing the connected component of the center of $G$. Let $N$ (respectively, $C$) be the normalizer (respectively, centralizer) of $T$ in $G$, and let $W$ be the Weyl group $N/C$ for $T$. We prove that there is a natural bijective correspondence between the following two:

1. Torus subbundles $T$ of $\text{Ad}(E_G)$ such that for some (hence every) $x \in M$, the fiber $T_x$ lies in the conjugacy class of tori in $\text{Ad}(E_G)$ determined by $T$.
2. Quadruples of the form $(E_W, \phi, E_C', \tau)$, where $\phi : E_W \longrightarrow M$ is a principal $W$–bundle, $\phi^* E_G \supset E_C' \longrightarrow E_W$ is a holomorphic reduction in structure group of $\phi^* E_G$ to $C$, and

$$\tau : E_C' \times N \longrightarrow E_C'$$

is a holomorphic action of $N$ on $E_C'$ extending the natural action of $C$ on $E_C'$, such that the composition $\psi \circ \tau$ coincides with the composition of the quotient map $E_C' \times N \longrightarrow (E_C'/C) \times (N/C) = (E_C \times N)/(C \times C)$ with the natural map $(E_C'/C) \times (N/C) \longrightarrow E_W$.

The composition of maps $E_C' \psi : E_W \phi : E_W \longrightarrow M$ defines a principal $N$–bundle on $M$. This principal $N$–bundle $E_N$ is a reduction in structure group of $E_G$ to $N$. Given a complex connection $\nabla$ on $E_G$, we give a necessary and sufficient condition for $\nabla$ to be induced by a connection on $E_N$. This criterion relates Hermitian–Einstein connections on $E_G$ and $E_C'$ in a very precise manner.
Keywords Principal bundle · Torus bundle · Levi reduction · Adjoint bundle · Hermitian-Einstein connection

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1 Introduction

Let $M$ be a compact connected complex manifold. Take a holomorphic vector bundle $E$ on $M$. In [3] the following question was addressed: When is the vector bundle $E$ the direct image of a vector bundle over an étale cover of $M$? The main result of [3] described all possible way $E$ is realized as the direct image of a vector bundle over an étale cover of $M$. The main result of [3] says that they are parametrized by the subbundles of the adjoint bundle $\text{Ad}(E) \to M$ whose fibers are tori. To explain this with more details, given any triple $(Y, \beta, F)$, where $\beta: Y \to M$ is an étale covering ($Y$ need not be connected) and $F$ is a holomorphic vector bundle on $Y$ such that $E = \beta^* F$, we construct a torus subbundle of $\text{Ad}(E)$; this subbundle is in fact the invertible part of $\beta^* \mathcal{O}_Y \subset \text{End}(\beta^* F)$. Conversely, given a subbundle of $\text{Ad}(E)$ with the typical fiber being a torus, we construct a triple $(Y, \beta, F)$ of the above form such that $E = \beta^* F$. In [4], these results were generalized to the context of parabolic (orbifold) vector bundles over any Riemann surface; see [7] for a somewhat related question.

Our aim here is to formulate and address the question in the context of principal bundles. Since direct image of a principal bundle does not quite make sense, a reformulation is warranted.

Let $G$ be a connected reductive complex affine algebraic group. Fix a complex torus $T \subset G$ that contains the connected component, containing the identity element, of the center of $G$. Denote the normalizer (respectively, centralizer) of $T$ in $G$ by $N$ (respectively, $C$). This $C$ is a Levi factor of a parabolic subgroup of $G$. The quotient $N/C$ is a finite group, which we shall denote by $W$.

We prove the following (see Theorem 2.5):

**Theorem 1.1** Take a holomorphic principal $G$–bundle $E_G$ over $M$. There is a natural bijective correspondence between the following two:

1. Torus subbundles $T$ of $\text{Ad}(E_G)$ such that for some (hence every) $x \in M$, the fiber $T_x$ lies in the conjugacy class of tori in $\text{Ad}(E_G)$ determined by $T$. 

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(2) Quadruples of the form \((E_W, \phi, E'_C, \tau)\), where \(\phi : E_W \to M\) is a principal \(W\)-bundle, \(\phi^*E_G \supset E'_C\), \(\psi : E'_C \to E_W\) is a holomorphic reduction in structure group of the principal \(G\)-bundle \(\phi^*E_G\) to the subgroup \(C\), and

\[\tau : E'_C \times N \to E'_C\]

is a holomorphic action of \(N\) on \(E'_C\) extending the natural action of \(C\) on the principal \(C\)-bundle \(E'_C\), such that the diagram of maps

\[\begin{array}{ccc}
E'_C \times N & \xrightarrow{\tau} & E'_C \\
\downarrow & & \downarrow \psi \\
E_W \times W := (E'_C/C) \times (N/C) & \to & E_W
\end{array}\]

is commutative, where \(E'_C \times N \to (E'_C/C) \times (N/C) = (E'_C \times N)/(C \times C)\) is the quotient map, and \(\psi\) is the natural projection.

If we set \(G = \text{GL}(r, \mathbb{C})\) in Theorem 1.1, then the above mentioned result of [3] is obtained.

Consider the composition of maps \(E'_C \xrightarrow{\psi} E_W \xrightarrow{\phi} M\) in (2). The action of \(N\) on \(E'_C\) and this composition of maps together produce a holomorphic principal \(N\)-bundle over \(M\). This principal \(N\)-bundle over \(M\) will be denoted by \(E_N\). Since \(\phi^*E_G = E_W \times_M E_G\), we have a natural projection \(\phi^*E_G \to E_G\). Now using the composition of maps

\[E_N = E'_C \to \phi^*E_G \to E_G,\]  

(1.1)

the principal \(N\)-bundle \(E_N\) is a holomorphic reduction in structure group of the principal \(G\)-bundle \(E_G\) to the subgroup \(N \subset G\). Therefore, a complex connection on \(E_N\) induces a complex connection on \(E_G\).

Given a complex connection on \(E_G\), it is natural to ask whether it is induced by a complex connection on \(E_N\).

We prove the following criterion for it (see Theorem 3.1 and Remark 3.3):

**Theorem 1.2** Given a complex connection \(D_0\) on \(E_G\), let \(D_1\) be the complex connection on the associated adjoint bundle \(\text{Ad}(E_G)\) induced by \(D_0\). The connection \(D_0\) on \(E_G\) is induced by a complex connection on the principal \(N\)-bundle \(E_N\) if and only if the induced connection \(D_1\) on \(\text{Ad}(E_G)\) preserves the corresponding torus subbundle \(T \subset \text{Ad}(E_G)\) in (1) of Theorem 1.1.

When the connection \(D_1\) on \(\text{Ad}(E_G)\) induced by \(D_0\) preserves the torus subbundle \(T \subset \text{Ad}(E_G)\), the connection \(D_0\) is holomorphic if and only if the connection on \(E_N\) inducing \(D_0\) is holomorphic.

A complex connection on \(E_N\) defines a complex connection on the principal \(C\)-bundle \(E'_C\) on \(E_W\), because \(E_N = E'_C\) (see (1.1)) and the map \(\phi\) is étale.

Now assume that \(M\) is Kähler; fix a Kähler form \(\omega\) on \(M\) in order to define degree of a torsionfree coherent analytic sheaf on \(M\). This enables us to define stable and polystable principal \(G\)-bundles on \(M\). Fix a maximal compact subgroup \(K_G \subset G\) to define the Hermitian–Einstein equation for principal \(G\)-bundles. So \(K_C := C \cap K_G\) is a maximal compact subgroup of \(C\). A holomorphic principal \(G\)-bundle on \(M\) admits a Hermitian–Einstein connection if and only if it is polystable [1,8,14,19]. The pulled back form \(\phi^*\omega\) on \(E_W\) is Kähler. However, \(E_W\) need not be connected. Polystable bundles and Hermitian–Einstein connections on bundles over \(E_W\) are defined in a suitable way.
We prove the following (see Proposition 3.4):

**Proposition 1.3** Take $E_G$ and $E'_C$ as in Theorem 1.1. Assume that the principal $G$–bundle $E_G$ on $M$ is polystable. Let $\nabla$ be the Hermitian–Einstein connection on $E_G$. Then the following two hold:

1. The principal $C$–bundle $E'_C$ on $E_W$ is polystable.
2. The Hermitian–Einstein connection $\phi^*\nabla$ on $\phi^*E_G$ preserves the reduction $E'_C$ of structure group of the principal $G$–bundle $\phi^*E_G$ to the subgroup $C \subset G$. Furthermore, the connection on $E'_C$ given by $\phi^*\nabla$ is Hermitian–Einstein.

Let $z(\mathfrak{t})$ denote the center of the Lie algebra of $K_G$. Let $z(\mathfrak{k}_C)$ be the center of the Lie algebra of the maximal compact subgroup $K_C = K_G \cap C$ of $C$. We have $z(\mathfrak{t}) \subset z(\mathfrak{k}_C)$.

We also prove the following (see Proposition 3.5):

**Proposition 1.4** Take $E_G$ and $E'_C$ as in Theorem 1.1. Assume that the principal $C$–bundle $E'_C$ over $E_W$ is polystable. Let $\nabla$ be the Hermitian–Einstein connection on $E'_C$. Assume that the element of $z(\mathfrak{k}_C)$ given by the curvature of $\nabla$ lies in the subspace $z(\mathfrak{t})$. Then the following two hold:

1. The principal $G$–bundle $E_G$ on $M$ is polystable.
2. The Hermitian–Einstein connection on $E_G$ is given by $\nabla$.

As indicated in Sect. 5, all the above results extend to the equivariant setup. This means that the results of [4] extend to the context of complex reductive affine algebraic groups.

## 2 Torus subbundle and Levi reduction in structure group of a principal bundle

### 2.1 A Levi reduction from a torus subbundle

Let $G$ be a connected complex reductive affine algebraic group. A parabolic subgroup of $G$ is a Zariski closed connected subgroup $P \subset G$ such that the quotient variety $G/P$ is projective. The unipotent radical of a parabolic subgroup $P \subset G$ is denoted by $\text{Ru}(P)$. The quotient $P/\text{Ru}(P)$ is a reductive affine complex algebraic group. A connected reductive complex algebraic subgroup $L(P) \subset P$ is called a Levi factor of $P$ if the composition of maps

$$L(P) \hookrightarrow P \twoheadrightarrow P/\text{Ru}(P)$$

is an isomorphism [6, p. 158, § 11.22]. There are Levi factors of $P$; any two Levi factors of $P$ differ by the inner automorphism of $P$ produced by an element of $\text{Ru}(P)$ [6, p. 158, § 11.23], [11, § 30.2, p. 184].

Fix a Borel subgroup $B_G \subset G$, and also fix a maximal torus $T_G \subset B_G$. Given any parabolic subgroup $P \subset G$, there is some element $g_0 \in G$ such that we have $B_G \subset g_0^{-1}P g_0$ [11, p. 134, Theorem 21.3]. Henceforth, whenever we consider a parabolic subgroup of $G$, we would assume that $P \supset B_G$. The connected component of the center of $G$ containing the identity element will be denoted by $Z_0(G)$; this $Z_0(G)$ is isomorphic to a product of copies of the multiplicative group $\mathbb{C}^* := \mathbb{C} \setminus \{0\}$.

Let $M$ be a compact connected complex manifold. Let

$$p_0 : E_G \longrightarrow M$$  \hspace{1cm} (2.1)
be a holomorphic principal $G$–bundle over $M$; this means that $E_G$ is a holomorphic fiber bundle over $M$ equipped with a holomorphic right-action of $G$

$$q_0 : E_G \times G \longrightarrow E_G$$

such that the above projection $p_0$ is $G$–invariant and, furthermore, the resulting map to the fiber product

$$E_G \times G \longrightarrow E_G \times_M E_G, \ (y, z) \mapsto (y, q_0(y, z))$$

is a biholomorphism. For notational convenience, for any $(y, z) \in E_G \times G$, the point $q_0(y, z) \in E_G$ will be denoted by $yz$. Given a holomorphic principal $G$–bundle $E_G$ as above, consider the quotient of $E_G \times G$, where two points $(y, z)$ and $(y_1, z_1)$ of $E_G \times G$ are identified if there is some $g_0 \in G$ such that $y_1 = yg_0$ and $z_1 = g_0^{-1}zg_0$. Let

$$f_0 : E_G \times G \longrightarrow \text{Ad}(E_G) := (E_G \times G)/\sim$$

be this quotient. Each fiber of the projection

$$p : \text{Ad}(E_G) \longrightarrow M, \ (y, z) \longmapsto p_0(y), \ (y, z) \in E_G \times G$$

is a group isomorphic to $G$, where the group operation is given by $(y, z) \cdot (y', z') = (y, zz')$ (it is straightforward to check that the group operation is well-defined, meaning it is independent of $y$); note that the map $(y, z) \longmapsto z$ is an isomorphism of $\text{Ad}(E_G)_x := p_0^{-1}(x)$ with $G$. For any $x \in M$, the above isomorphism between $\text{Ad}(E_G)_x$ and $G$ depends on the choice of the point $y \in p_0^{-1}(x)$. However, for two different choices of $y$, the corresponding isomorphisms differ by an inner automorphism of $G$. In other words, $\text{Ad}(E_G)_x$ and $G$ are identified uniquely up to an inner automorphism. This $\text{Ad}(E_G)$ is called the adjoint bundle for $E_G$.

The adjoint action of any $z \in G$ on $G$ fixes the subgroup $Z_0(G)$ pointwise. Therefore, $M \times Z_0(G) \longrightarrow M$ is a subbundle of $\text{Ad}(E_G)$.

Let

$$\mathbb{T} \subset \text{Ad}(E_G) \xrightarrow{p} M$$

be a holomorphic sub-fiber bundle such that

- $M \times Z_0(G) \subset \mathbb{T}$, and
- for every point $x \in M$, the fiber

$$\mathbb{T}_x := \mathbb{T} \cap \text{Ad}(E_G)_x \subset \text{Ad}(E_G)_x$$

is a torus (it need not be a maximal torus of $\text{Ad}(E_G)_x$).

Take any point $x \in M$. Since $\text{Ad}(E_G)_x$ is identified with $G$ up to an inner automorphism, the torus $\mathbb{T}_x \subset \text{Ad}(E_G)_x$ determines a conjugacy class of tori in $G$. From the rigidity of tori in $G$, it follows that this conjugacy class of tori in $G$ is independent of the choice of the point $x \in M$. Note that any torus in $G$ is conjugate to a sub-torus of $T_G$, and the space of sub-tori in $T_G$ is a discrete (countable) set. Fix a torus

$$T \subset T_G \subset G$$

in the conjugacy class determined by $\mathbb{T}$. Note that we have

$$Z_0(G) \subset T,$$

because $M \times Z_0(G) \subset \mathbb{T}$. 
Let
\[ N := N_G(T) \subset G \] (2.5)
be the normalizer in \( G \) of the subgroup \( T \) in (2.4). The connected component of \( N \)
\[ C := N_0 \subset N \] (2.6)
containing the identity element is in fact the centralizer of \( T \) in \( G \). This component \( C \) of \( N \) is a Levi factor of a parabolic subgroup of \( G \) [18, § 3] (see also [17, p. 110, Theorem 6.4.7(i)]). The quotient \( N/C \) is the Weyl group associated to \( T \subset G \); it is a finite group. The normalizer of \( C \) in \( G \) actually coincides with \( N \).

Now let
\[ E_N \subset E_G \] (2.7)
be the unique largest subset such that
\[ E_N \times T \subset f_0^{-1}(\mathbb{T}) \subset E_G, \] (2.8)
where \( f_0, \mathbb{T} \) and \( T \) are as in (2.2), (2.3) and (2.4) respectively. Let
\[ p_0 : E_N \longrightarrow M \] (2.9)
be the restriction of the projection in (2.1); this repetition of notation should not cause any confusion. Clearly, \( E_N \) is a complex manifold, and the projection \( p_0 \) in (2.9) is holomorphic because the projection in (2.1) is holomorphic.

We will prove that \( E_N \) is a holomorphic principal \( N \)--bundle over \( M \), where \( N \) is constructed in (2.5). For this first note that for any \( y \in E_N \) and \( g_0 \in G \), we have
\[ f_0(yg_0, t) = f_0(y, g_0t_g_0^{-1}) \] (2.10)
[see the construction of \( f_0 \) in (2.2)]. Using (2.10) we will show that \( yg_0 \in E_N \) if and only if \( g_0 \in N \), which would imply that \( E_N \) is a holomorphic principal \( N \)--bundle over \( M \). To see that \( yg_0 \in E_N \) for all \( g_0 \in N \), first note that if \( g_0 \in N \), then \( g_0t_g_0^{-1} \in T \) whenever \( t \in T \). Therefore, in view of (2.10), the given condition that \( \{y\} \times T \subset f_0^{-1}(\mathbb{T}) \) immediately implies that \( \{yg_0\} \times T \subset f_0^{-1}(\mathbb{T}) \) if \( g_0 \in N \). So we have \( yg_0 \in E_N \) for all \( g_0 \in N \).

To prove the converse, note that the given condition that \( \{y\} \times T \subset f_0^{-1}(\mathbb{T}) \) implies that \( f_0(\{y\} \times T) = \mathbb{T}_{p_0(y)} \), because \( T \) is identified with \( \mathbb{T}_{p_0(y)} \) up to an inner automorphism. Therefore, if \( yg_0 \in E_N \), then from (2.10) it follows that \( g_0t_g_0^{-1} \in T \) for all \( t \in T \). This implies that \( g_0 \in N \) if \( yg_0 \in E_N \).

From (2.7), we conclude that the principal \( N \)--bundle \( E_N \) is a holomorphic reduction in structure group of \( E_G \) to the subgroup \( N \) in (2.5). Equivalently, \( E_G \) coincides with the holomorphic principal \( G \)--bundle on \( M \) obtained by extending the structure group of the holomorphic principal \( N \)--bundle \( E_N \) using the inclusion of \( N \) in \( G \).

For notational convenience, the Weyl group \( N/C \) in (2.6) will be denoted by \( W \). Let
\[ E_W := E_N/C = E_N \times^N W \xrightarrow{\phi} M \] (2.11)
be the principal \( W \)--bundle over \( M \) obtained by extending the structure group of the principal \( N \)--bundle \( E_N \) using the quotient map \( N \longrightarrow N/C = W \). So \( \phi \) in (2.11) is an étale Galois covering with Galois group \( \text{Gal}(\phi) = W \). This \( E_W \) need not be connected.

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Lemma 2.1 The pulled back principal $N$–bundle $\phi^*E_N \longrightarrow E_W$ has a tautological reduction in structure group to the subgroup $C \subset N$ in (2.6), where $\phi$ is the projection in (2.11).

Proof Consider the quotient map $E_N \longrightarrow E_N/C = E_W$. This makes $E_N$ a holomorphic principal $C$–bundle over $E_W$. We will denote by $E'_C$ this holomorphic principal $C$–bundle over $E_W$. The identity map $E_N = E'_C \longrightarrow E_N$ is $C$–equivariant. For any $z \in E_W$, the $C$–equivariant map $(E'_C)_z \longrightarrow (E_N)_{\phi(z)}$, obtained by restricting the above identity map of $E_N$, is evidently an embedding. Therefore, $E'_C$ is a holomorphic reduction in structure group of the principal $N$–bundle $\phi^*E_N$ to the subgroup $C \subset N$. ☐

Since $E_N$ is a holomorphic reduction in structure group of $E_G$ to $N$, Lemma 2.1 implies that $E'_C$ is also a holomorphic reduction in structure group of $\phi^*E_G$ to the subgroup $C \subset G$.

Let $\psi : E'_C = E_N \longrightarrow E_W := E_N/C$ (2.12) be the quotient map.

We note that the group $N$ acts holomorphically on $E'_C$, because $E'_C = E_N$; for this action $\tau : E'_C \times N \longrightarrow E'_C$ of $N$ on $E'_C$, the following diagram of maps is evidently commutative:

$$
\begin{array}{ccc}
E'_C \times N & \xrightarrow{\tau} & E'_C \\
\downarrow & & \downarrow \psi \\
E_W \times W := (E'_C/C) \times (N/C) & \longrightarrow & E_W
\end{array}
$$

(2.13)

where $E'_C \times N \longrightarrow (E'_C/C) \times (N/C) = (E'_C \times N)/(C \times C)$ is the natural quotient map, and $E_W \times W \longrightarrow E_W$ is the action of $W$ on the principal $W$–bundle $E_W$, while $\psi$ is the map in (2.12).

2.2 A torus subbundle from a Levi reduction

We will now describe a reverse of the construction done in Sect. 2.1.

As before, let $E_G$ be a holomorphic principal $G$–bundle over $X$. Take a torus $T \subset T_G \subset B_G$ such that $Z_0(G) \subset T$. As before, the normalizer (respectively, centralizer) of $T$ in $G$ will be denoted by $N$ (respectively, $C$). Denote the Weyl group $N/C$ by $W$.

Let $\phi : E_W \longrightarrow W$ be a principal $W$–bundle. Assume that the holomorphic principal $G$–bundle $\phi^*E_G \longrightarrow E_W$ has a holomorphic reduction in structure group to the subgroup $C \subset G$

$$
E'_C \subset \phi^*E_G
$$

(2.14)

such that

(1) the principal $C$–bundle $E'_C$ is equipped with a holomorphic action of the complex group $N$

$$
\tau : E'_C \times N \longrightarrow E'_C
$$

that extends the natural action of the subgroup $C$ on the principal $C$–bundle $E'_C$, and
(2) the diagram of maps

\[
\begin{array}{ccc}
E'_C \times N & \xrightarrow{\tau} & E'_C \\
\downarrow & & \downarrow \\
E_W \times W := (E'_C/C) \times (N/C) & \xrightarrow{\psi} & E_W \\
\end{array}
\]

(2.15)
is commutative, where \(E'_C \times N \longrightarrow (E'_C/C) \times (N/C) = (E'_C \times N)/(C \times C)\) is the natural quotient map, and \(\psi\) is the projection of \(E'_C\) to \(E_W\), so \(\psi\) coincides with the restriction to \(E'_C\) of the pullback \(\phi^* p_0\), where \(p_0\) is the projection in (2.1); note that (2.15) is similar to (2.13).

**Lemma 2.2** The action of \(W\) on the principal \(W\)-bundle \(E_W\) has a canonical lift to an action of \(W\) on the adjoint bundle \(\text{Ad}(E'_C) = E'_C \times^C C \longrightarrow E_W\) for \(E'_C\). This action of \(W\) on \(\text{Ad}(E'_C)\) preserves the group structure of the fibers of \(\text{Ad}(E'_C)\).

**Proof** The adjoint action of \(N\) on itself preserves \(C\) because it is the connected component of \(N\) containing the identity element. We recall that \(\text{Ad}(E'_C)\) is the quotient of \(E'_C \times C\) where two elements \((y, z), (y_1, z_1) \in E'_C \times C\) are identified if there is an element \(c \in C\) such that \(y_1 = yc\) and \(z_1 = c^{-1}zc\).

Consider the following action of \(N\) on \(E'_C \times C\): the action of any \(n \in N\) sends any \((y, z) \in E'_C \times C\) to \((\tau(y, n), n^{-1}zn)\), where \(\tau\) is the action of \(N\) in (2.15) that extends the action of \(C\) on \(E'_C\). So for any \(c \in C\), the action of \(n\) sends \((yc, c^{-1}zc)\) to \((\tau(yc, n), n^{-1}c^{-1}zcn) = (\tau(y, cn), (cn)^{-1}zcn)\) (recall that the restriction of the action \(\tau\) to \(C \subset N\) coincides with the natural action of \(C\) on \(E'_C\)). The image of \((yc, c^{-1}zc)\) (respectively, \((\tau(y, cn), (cn)^{-1}zcn)\)) in the quotient space \(\text{Ad}(E'_C)\) of \(E'_C \times C\) coincides with the image of \((y, z)\) (respectively, \((yc, c^{-1}zc)\)). Therefore, the above action of \(N\) on \(E'_C \times C\) produces an action of \(N\) on the quotient manifold \(\text{Ad}(E'_C)\). Next note that if \(n \in C\), then the image of \((\tau(y, n), n^{-1}zn)\) in \(\text{Ad}(E'_C)\) coincides with the image of \((y, z)\) in \(\text{Ad}(E'_C)\). Consequently, the above action of \(N\) on \(\text{Ad}(E'_C)\) produces an action of \(W = N/C\) on \(\text{Ad}(E'_C)\).

The above action of \(W\) on \(\text{Ad}(E'_C)\) preserves the group structure of the fibers of \(\text{Ad}(E'_C)\), because the adjoint action of a group on itself preserves the group structure. \(\square\)

Consider the action \(\tau\) of \(N\) on \(E'_C\) in (2.15). From (2.15) it follows that this action and the composition of maps \(E'_C \xrightarrow{\psi} E_W \xrightarrow{\phi} M\) together define a holomorphic principal \(N\)-bundle over \(M\). This holomorphic principal \(N\)-bundle over \(M\) will be denoted by \(E_N\).

**Lemma 2.3** The holomorphic principal \(G\)-bundle over \(M\), obtained by extending the structure group of the above defined principal \(N\)-bundle \(E_N\) using the inclusion of \(N\) in \(G\), is identified with \(E_G\).

**Proof** Let \(E_N \longrightarrow E_G\) be the composition of the natural map \(\phi^* E_G \longrightarrow E_G\) with the inclusion \(E_N = E'_C \hookrightarrow \phi^* E_G\) in (2.14). This map \(E_N \longrightarrow E_G\) is clearly \(N\)-equivariant. This implies that \(E_N\) is a reduction in structure group of the principal \(G\)-bundle \(E_G\) to the subgroup \(N\). In other words, \(E_G\) is identified with the holomorphic principal \(G\)-bundle over \(M\) obtained by extending the structure group of the principal \(N\)-bundle \(E_N\) using the inclusion of \(N\) in \(G\). \(\square\)

Let

\[\text{Ad}(E'_C) \supset T' \longrightarrow E_W\] (2.16)
be the bundle of connected components of centers containing identity element; so for any \( y \in E_W \), the fiber \((\mathbb{T}')_y\) of \( \mathbb{T} \) is the connected component of the center of the group \( \text{Ad}(E'_C)_y \) containing the identity element. Note that \( \mathbb{T}' \) is identified with the trivial bundle \( E_W \times T \rightarrow E_W \), because \( T \) is the connected component, containing the identity element, of the center of \( C \); the identification between \( E_W \times T \) and \( \mathbb{T}' \) sends any \((y, z) \in E_W \times T \) to the equivalence class of \((y', z) \) in the quotient \( \text{Ad}(E'_C) \) of \( E'_C \times C \), where \( y' \) is any element of the fiber \((E_C')_y \) (this equivalence class does not depend on the choice of the point \( y' \) in \((E_C')_y \)). The action of \( W \) on \( \text{Ad}(E'_C) \) in Lemma 2.2 preserves \( \mathbb{T}' \) because the action of \( W \) preserves the group structure of the fibers of \( \text{Ad}(E'_C) \) (hence the centers of fibers are preserved implying that their connected components containing identity element are also preserved).

Consequently, we have a torus subbundle

\[
\mathbb{T} := \mathbb{T}'/W \subset \text{Ad}(E'_C)/W \subset \text{Ad}(\phi^*E_G)/W = \text{Ad}(E_G).
\]  

(2.17)

The construction of \( \mathbb{T} \) in (2.17) from \( E'_C \) in (2.14) is the inverse of the construction of \( E'_C \) in Lemma 2.1 from \( \mathbb{T} \) in (2.3). More precisely, starting with \( \mathbb{T} \) in (2.3), construct \( E'_C \) as in Lemma 2.1. Now set \( E'_C \) in (2.14) to be this holomorphic principal \( C \)–bundle constructed from \( \mathbb{T} \) in (2.3). Then the torus bundle \( \mathbb{T} \) constructed in (2.17) coincides with the torus bundle in (2.3) that we started with.

Conversely, start with \( E'_C \) in (2.14) and construct \( \mathbb{T} \) as in (2.17). Setting this to be the torus bundle in (2.3), the principal \( C \)–bundle constructed in Lemma 2.1 coincides with \( E'_C \) in (2.14) that we started with.

**Remark 2.4** Although the principal \( C \)–bundle \( E'_C \) in Lemma 2.2 does not, in general, descend to \( M \), note that the adjoint bundle \( \text{Ad}(E'_C) \), being \( W \) equivariant (see Lemma 2.2), descends to \( M \) as a subbundle of the adjoint bundle \( \text{Ad}(E_N) \) of \( E_N \) in Lemma 2.3. For every point \( x \in M \), the fiber of \( \text{Ad}(E'_C)/W \) over \( x \) is the connected component of \( \text{Ad}(E_N)_x \) containing the identity element. Recall that for any \( z \in E_W \), the fiber \( \mathbb{T}'_x \) in (2.16) is the connected component of the center of the group \( \text{Ad}(E'_C)_z \) containing the identity element. Consequently, for any \( x \in X \), the fiber \( \mathbb{T}'_x \) in (2.17) is the connected component, containing the identity element, of the center of the fiber of \( \text{Ad}(E'_C)/W \) over \( x \).

Combining the results of Sects. 2.1 and 2.2, we have the following:

**Theorem 2.5** Let \( E_G \) be a holomorphic principal \( G \)–bundle over \( M \) and \( T \subset G \) a torus containing \( Z_0(G) \). The normalizer (respectively, centralizer) of \( T \) in \( G \) will be denoted by \( N \) (respectively, \( C \)), while the Weyl group \( N/C \) will be denoted by \( W \). There is a natural bijective correspondence between the following two:

1. Torus subbundles \( \mathbb{T} \) of \( \text{Ad}(E_G) \) such that for some (hence every) \( x \in M \), the fiber \( \mathbb{T}_x \) lies in the conjugacy class of tori in \( \text{Ad}(E_G) \) determined by \( T \).
2. Quadruples of the form \((E_W, \phi, E'_C, \tau)\), where \( \phi : E_W \rightarrow M \) is a principal \( W \)–bundle, \( \phi^*E_G \supset E'_C \xrightarrow{\psi} E_W \) is a holomorphic reduction in structure group of \( \phi^*E_G \) to \( C \), and

\[
\tau : E'_C \times N \rightarrow E'_C
\]

is a holomorphic action of \( N \) on \( E'_C \) extending the natural action of \( C \) on \( E'_C \), such that the diagram of maps

\[
\begin{array}{ccc}
E'_C \times N & \xrightarrow{\tau} & E'_C \\
\downarrow & & \downarrow \psi \\
E_W \times W := (E'_C/C) \times (N/C) & \rightarrow & E_W
\end{array}
\]
is commutative, where $E_C' \times N \longrightarrow (E_C' / C) \times (N / C) = (E_C' \times N) / (C \times C)$ is the quotient map.

### 2.3 A tautological connection on a torus bundle

We return to the setup of Sect. 2.1. As in (2.3), let

$$\mathbb{T} \subset \text{Ad}(E_G) \xrightarrow{p} M$$

be a holomorphic sub-fiber bundle containing $M \times Z_0(G)$ such that for every point $x \in M$, the fiber

$$T_x := \mathbb{T} \cap p^{-1}(x) \subset \text{Ad}(E_G)_x$$

is a torus.

A flat connection on the fiber bundle $\mathbb{T}$ is said to be compatible with the group structure of the fibers of $\mathbb{T}$ if for any two locally defined flat sections $s$ and $t$ of $\mathbb{T}$, defined over an open subset $U \subset M$, the section $s \cdot t$ of $\mathbb{T}|_U$ is again flat.

**Proposition 2.6** There is a tautological flat holomorphic connection on $\mathbb{T}$ which is compatible with the group structure of the fibers of $\mathbb{T}$.

**Proof** Consider the étale Galois covering $\phi : E_W \longrightarrow M$ is (2.11). Let

$$E_C' \subset \phi^* E_G$$

be the holomorphic reduction in structure group constructed in Lemma 2.1. The adjoint action of $C$ on the torus $T$ in (2.4) is trivial, because $T$ is contained in the center of $C$. Therefore, $T$ defines a trivial subbundle

$$E_W \times T \subset \text{Ad}(E_C') \subset \text{Ad}(\phi^* E_N) = \phi^* \text{Ad}(E_N) \subset \phi^* \text{Ad}(E_G) = \text{Ad}(\phi^* E_G),$$

(2.18)

where $E_N$ is the principal $N$–bundle constructed in (2.7). Note that $E_W \times T \subset \text{Ad}(E_C')$ is the bundle consisting of connected components of the centers, containing the identity element, of the fibers of $\text{Ad}(E_C')$, and $\text{Ad}(E_C')$ is the connected component of $\text{Ad}(\phi^* E_N) = \phi^* \text{Ad}(E_N)$ containing the section given by the identity elements of the fibers.

Let $D_0$ denote the trivial (flat) connection on the trivial bundle $E_W \times T \longrightarrow E_W$ in (2.18). The tautological action of the Galois group $W = \text{Gal}(\phi)$ on $\phi^* \text{Ad}(E_N)$ evidently preserves the subbundle $E_W \times T \subset \phi^* \text{Ad}(E_N)$. The resulting action of $W$ on $E_W \times T$ clearly preserves the above trivial connection $D_0$ on $E_W \times T \longrightarrow E_W$.

Since the connection $D_0$ is preserved by the action of $W = \text{Gal}(\phi)$, it produces a flat holomorphic connection on the subbundle $(E_W \times T) / W$

$$\text{Ad}(E_N) = (\phi^* \text{Ad}(E_N)) / W \supset (E_W \times T) / W \longrightarrow E_W / W = M$$

of $\text{Ad}(E_N)$. But this subbundle $(E_W \times T) / W$ is identified with $\mathbb{T}$, because $E_W \times T \subset \phi^* \text{Ad}(E_G)$ in (2.18) coincides with $\phi^* \mathbb{T} \subset \phi^* \text{Ad}(E_G)$ (see (2.17)); recall that $\mathbb{T}'$ in (2.17) is identified with $E_W \times T$.

\(\square\) Springer
3 Torus subbundles and connection

3.1 Connections on a principal bundle

Let $H$ be a complex Lie group. The Lie algebra of $H$ will be denoted by $\mathfrak{h}$. Let

$$\beta : E_H \longrightarrow M$$

be a holomorphic principal $H$–bundle on $M$. The holomorphic tangent bundles of $M$ and $E_H$ will be denoted by $TM$ and $TE_H$ respectively. The quotient

$$\text{At}(E_H) := (TE_H)/H \longrightarrow E_H/H = M$$

is a holomorphic vector bundle which is called the Atiyah bundle for $E_H$. The differential $d\beta : TE_H \longrightarrow \beta^*TM$ for the above projection $\beta$, being $H$–invariant, produces a surjective homomorphism

$$d'\beta : \text{At}(E_H) \longrightarrow TM.$$ 

The kernel of $d'\beta$ is the relative tangent bundle for $\beta$ and it is identified with the adjoint vector bundle $\text{ad}(E_H) = \ker(d\beta)/H$. We recall that $\text{ad}(E_H)$ is the quotient of $E_H \times \mathfrak{h}$, where two points $(y, v)$ and $(y_1, v_1)$ of $E_H \times \mathfrak{h}$ are identified if there is some $h_0 \in H$ such that $y_1 = y h_0$ and $v_1 = \text{Ad}(h_0^{-1})(v)$. Therefore, $\text{ad}(E_H) \longrightarrow M$ is the Lie algebra bundle for the bundle $\text{Ad}(E_H)$ of Lie groups on $M$. We have the short exact sequence of holomorphic vector bundles on $M$

$$0 \longrightarrow \text{ad}(E_H) \longrightarrow \text{At}(E_H) \xrightarrow{d'\beta} TM \longrightarrow 0,$$

which is known as the Atiyah exact sequence. A complex connection on $E_H$ is a $C^\infty$ homomorphism of vector bundles

$$D_0 : TM \longrightarrow \text{At}(E_H)$$

such that $(d'\beta) \circ D_0 = \text{Id}_TM$ (see [2]). A holomorphic connection on $E_H$ is a holomorphic homomorphism of vector bundles

$$D_0 : TM \longrightarrow \text{At}(E_H)$$

such that $(d'\beta) \circ D_0 = \text{Id}_TM$.

Let $H'$ be a complex Lie group and $\eta : H \longrightarrow H'$ a holomorphic homomorphism of Lie groups. Let

$$E_{H'} := E_H \times^\eta H \longrightarrow M$$

be the holomorphic principal $H'$–bundle over $M$ obtained by extending the structure group of the holomorphic principal $H$–bundle $E_H$ using the above homomorphism $\eta$. Recall that $E_{H'}$ is the quotient of $E_H \times H'$ where any two points $(y, z)$ and $(y_1, z_1)$ of $E_H \times H'$ are identified if there is an element $h \in H$ such that $y_1 = y h$ and $z_1 = \eta(h^{-1})z$. So sending any $y \in E_H$ to the equivalence class of $(y, e)$, where $e \in H'$ is the identity element, we get a holomorphic map

$$\eta_0 : E_H \longrightarrow E_{H'}$$

which satisfies the equation

$$\eta_0(y h) = \eta_0(y) \eta(h) \quad (3.1)$$
for all \( y \in E_H \) and \( h \in H \). The action of \( H \) on \( E_H \) produces an action of \( H \) on the tangent bundle \( TE_H \); using \( \eta \), the action of \( H' \) on the tangent bundle \( TE_{H'} \) produces an action of \( H \) on \( TE_{H'} \). From (3.1), it follows immediately that the differential \( d\eta_0 \) of the map \( \eta_0 \) is \( H \)-equivariant. Therefore, \( d\eta_0 \) induces a homomorphism

\[ \eta_1 : \text{At}(E_H) = (TE_H)/H \rightarrow (TE_{H'}/H') = \text{At}(E_{H'}) . \]

This homomorphism \( \eta_1 \) satisfies the equation \( \eta_1(\text{ad}(E_H)) \subset \text{ad}(E_{H'}) \); in other words, we have a commutative diagram of holomorphic homomorphisms of vector bundles

\[
\begin{array}{cccccc}
0 & \rightarrow & \text{ad}(E_H) & \rightarrow & \text{At}(E_H) & \xrightarrow{d'g} & TM & \rightarrow & 0 \\
\downarrow & & \eta_1 & & \downarrow & & \parallel & & \\
0 & \rightarrow & \text{ad}(E_{H'}) & \rightarrow & \text{At}(E_{H'}) & \rightarrow & TM & \rightarrow & 0
\end{array}
\]

where the horizontal sequences are the Atiyah exact sequences. Therefore, if \( D_0 : TM \rightarrow \text{At}(E_H) \) is a complex connection on \( E_H \), then

\[ \eta_1 \circ D_0 : TM \rightarrow \text{At}(E_{H'}) \]

is a complex connection on \( E_{H'} \); this \( \eta_1 \circ D_0 \) is called the connection on \( E_{H'} \) induced by the connection \( D_0 \) on \( E_H \). If the connection \( D_0 \) is holomorphic, then the induced connection \( \eta_1 \circ D_0 \) is clearly holomorphic also.

### 3.2 Criterion for induced connection

We continue with the setup of Sect. 2.1. Take a torus subbundle

\[ \mathbb{T} \subset \text{Ad}(E_G) \xrightarrow{p} M \]

as in (2.3). Fix \( T \) as in (2.4), and construct \( N \) as in (2.5). Let \( E_N \subset E_G \) be the holomorphic reduction in structure group to \( N \) constructed in (2.7).

Let \( D_0 \) be a complex connection on \( E_G \). Our aim in this section is to establish a necessary and sufficient condition for \( D_0 \) to be induced by a complex connection on \( E_N \).

The connection \( D_0 \) on \( E_G \) induces a complex connection on every holomorphic fiber bundle \( E_G(F) \) associated to \( E_G \) for a holomorphic action of \( G \) on a complex manifold \( F \). In particular, \( D_0 \) induces a complex connection on the bundle \( \text{Ad}(E_G) \) associated to \( E_G \) for the adjoint action of \( G \) on itself. Let \( D_1 \) denote the complex connection on \( \text{Ad}(E_G) \) induced by \( D_0 \).

**Theorem 3.1** The connection \( D_0 \) on \( E_G \) is induced by a complex connection on the principal \( N \)-bundle \( E_N \) if and only if the induced connection \( D_1 \) on \( \text{Ad}(E_G) \) preserves the torus subbundle \( \mathbb{T} \subset \text{Ad}(E_G) \).

**Proof** First assume that there is a complex connection \( D_2 \) on the principal \( N \)-bundle \( E_N \) such that the connection on \( E_G \) induced by \( D_2 \) coincides with \( D_0 \). Let \( D_3 \) be the complex connection on \( \text{Ad}(E_N) \) induced by \( D_2 \). Let \( \mathcal{G} \subset \text{Ad}(E_N) \) be the sub-fiber bundle whose fiber over any \( x \in M \) is the connected component of \( \text{Ad}(E_N)_x \) containing the identity element. So \( \mathcal{G} \) coincides with \( \text{Ad}(E'_G)/W \) in Remark 2.4. The connection \( D_3 \) on \( \text{Ad}(E_N) \) clearly preserves \( \mathcal{G} \). Let \( D_4 \) denote the complex connection on \( \mathcal{G} \) given by \( D_3 \).

Since \( D_0 \) is induced by \( D_2 \), it follows that the connection \( D_1 \) on \( \text{Ad}(E_G) \) preserves the torus subbundle \( \mathbb{T} \) if and only if the connection \( D_4 \) on \( \mathcal{G} \) preserves \( \mathbb{T} \).
We recall that the torus subbundle $\mathbb{T} \subset G$ is the bundle of connected components of centers, containing the identity element, of the fibers of $G$ (see Remark 2.4). From this it can be deduced that the connection on $D_4$ on $G$ preserves $\mathbb{T}$. To prove this, it is convenient to switch to the Lie algebra bundles from the Lie group bundles, because it is easier the work with connections on vector bundles.

Let $\tilde{T}$ (respectively, $\tilde{G}$) be the Lie algebra bundle over $M$ corresponding to the Lie group bundle $T$ (respectively, $G$). Note that $\tilde{G}$ coincides with $\text{ad}(E_N)$, because $N/C$ is a finite group so $\text{Lie}(N) = \text{Lie}(C)$. Since $\mathbb{T} \subset G$ is the bundle of connected components of centers, containing the identity element, of the fibers of $G$, it follows immediately that for any $x \in M$, the fiber $\tilde{T}_x$ is the center of the Lie algebra $\tilde{G}_x$. Let $D'_4$ be the complex connection on the vector bundle $\tilde{G}$ given by the connection $D_4$ on $G$; note that $D'_4$ coincides with the connection on $\text{ad}(E_N)$ induced by the connection $D_2$ on $E_N$. If $s$ and $t$ are locally defined holomorphic sections of $\tilde{G}$ defined over an open subset $U \subset M$, then we have

$$D'_4([s, t]) = [D'_4(s), t] + [s, D'_4(t)],$$

because $D'_4$ is compatible with the Lie algebra structure of the fibers of $\tilde{G}$. Now if $s$ is a section of the subbundle $\tilde{T}$, then

$$[s, t] = 0 = [s, D'_4(t)].$$

Therefore, we conclude that $[D'_4(s), t] = 0$ if $s$ is a section of $\tilde{T}|_U$. Hence $D'_4(s)$ is a $C^\infty$ section of $\tilde{T}|_U \otimes \Omega^1_U$ if $s$ is a holomorphic section of $\tilde{T}|_U$. Consequently, the connection $D'_4$ on $\tilde{G}$ preserves the subbundle $\tilde{T}$. Hence the connection $D_4$ on $G$ preserves $\mathbb{T}$.

To prove the converse, assume that the connection $D_0$ on $E_G$ has the following property: the connection $D_1$ on $\text{Ad}(E_G)$ induced by $D_0$ preserves the torus subbundle $T \subset \text{Ad}(E_G)$.

Recall that $\text{ad}(E_G)$ is the Lie algebra bundle on $M$ corresponding to the bundle $\text{Ad}(E_G)$ of groups. Let $D'_1$ be the complex connection on the vector bundle $\text{ad}(E_G)$ induced by $D_1$. As before, $\tilde{T}$ denotes the Lie algebra bundle on $M$ corresponding to $T$, so $\tilde{T}$ is an abelian subalgebra bundle of $\text{ad}(E_G)$. The given condition that the connection $D_1$ on $\text{Ad}(E_G)$ preserves $T$ immediately implies that the connection $D'_1$ preserves the subbundle $\tilde{T} \subset \text{ad}(E_G)$.

If $s$ and $t$ are locally defined holomorphic sections of $\text{ad}(E_G)$ defined over an open subset $U \subset M$, then we have

$$D'_1([s, t]) = [D'_1(s), t] + [s, D'_1(t)];$$

because $D'_1$ is compatible with the Lie algebra structure of the fibers of $\text{ad}(E_G)$. Now if $s$ is a section of the subbundle $\tilde{T}|_U$, and $t$ is a section of $\tilde{G}|_U$ (defined earlier), then we have

$$[s, t] = 0 = [D'_1(s), t];$$

indeed, $[s, t] = 0$ because $\tilde{T}$ is the bundle the centers of $\tilde{G}$, and $[D'_1(s), t] = 0$ because $D'_1$ preserves $\tilde{T}$. Therefore, we have

$$[s, D'_1(t)] = 0$$

if $s$ is a holomorphic section of the subbundle $\tilde{T}|_U$ and $t$ is a holomorphic section of $\tilde{G}|_U$. From this it follows that the subbundle $\tilde{G} \subset \text{ad}(E_G)$ is preserved by the connection $D'_1$ on $\text{ad}(E_G)$. Indeed, as noted before, $\tilde{G} = \text{ad}(E_N) \subset \text{ad}(E_G)$, because the Lie algebras $\text{Lie}(N)$ and $\text{Lie}(C)$ coincide (the quotient $N/C$ is a finite group). Hence for any $x \in M$, the subalgebra
\[ \widehat{G}_x \subset \text{ad}(E_G)_x \]

is the centralizer of \( \widehat{T}_x \).

The normalizer of \text{Lie}(C) in \text{Lie}(G) is \text{Lie}(C) itself; this is because the normalizer of \( C \) in \( G \) is \( N \), and \text{Lie}(C) = \text{Lie}(N). Therefore, from the above observation that \( \widehat{\mathcal{G}} = \text{ad}(E_N) \) is preserved by the connection \( D_1' \) on \text{ad}(E_G) it follows that the connection \( D_0 \) on \( E_G \) preserves \( E_N \). In other words, the connection \( D_0 \) on \( E_G \) is induced by a connection on \( E_N \). This completes the proof. \( \square \)

**Proposition 3.2** Let \( D_0 \) be a complex connection on \( E_G \) such that the induced connection \( D_1 \) on \text{Ad}(E_G) \) preserves the torus subbundle \( T \subset \text{Ad}(E_G) \). Then the connection on \( T \) given by \( D_1 \) coincides with the tautological connection on \( T \) in Proposition 2.6.

**Proof** From Theorem 3.1, we know that there is a complex connection \( D_2 \) on the principal \( N \)-bundle \( E_N \) that induces \( D_0 \). Consider the pulled back connection \( \phi^* \) \( D_2 \) on \( \phi^* E_N \), where \( \phi \) is the projection from \( E_W \) in (2.11). This connection \( \phi^* D_2 \) gives a connection on the principal \( C \)-bundle \( E'_C \) in Lemma 2.1, because \( E'_C \) is a union of some connected component of \( \phi^* E_N \). Let \( D'' \) be the complex connection on \text{Ad}(E'_C) \) induced by this connection on \( E'_C \) given by \( \phi^* D_2 \).

The adjoint action of \( C \) on \( T \) is trivial because \( T \) is contained in the center of \( T \). Therefore,

- the subbundle

\[ E_W \times T \subset \text{Ad}(E'_C) \]

in (2.18) is preserved by the connection \( D'' \) on \text{Ad}(E'_C), and

- the connection on \( E_W \times T \) given by \( D'' \) coincides with the trivial connection of the trivial bundle.

Now from the construction of the tautological connection on \( T \) in Proposition 2.6 it follows that it coincides with the connection on \( T \) given by \( D_1 \). \( \square \)

**Remark 3.3** We use the notation in Theorem 3.1. Let \( D_0 \) be a complex connection on \( E_G \) such that the induced connection \( D_1 \) on \text{Ad}(E_G) \) preserves the torus subbundle \( T \subset \text{Ad}(E_G) \). Let \( D_2 \) be the complex connection on \( E_N \) inducing \( D_0 \). If the connection \( D_2 \) is holomorphic, then \( D \) is holomorphic, because holomorphic connections induce holomorphic connection, as noted in Sect. 3.1. Since \( N \) is a subgroup of \( G \), and the principal \( G \)-bundle \( E_G \) is the extension of structure group of the principal \( N \)-bundle \( E_N \) for the inclusion of \( N \) in \( G \), it follows that the Atiyah bundle \( \text{At}(E_N) \) is a holomorphic subbundle of \( \text{At}(E_G) \). Therefore, if the connection \( D \) is holomorphic then \( D_2 \) is also holomorphic.

### 3.3 Hermitian–Einstein connection and Levi reduction

In the subsection, we assume that \( M \) is Kähler and it is equipped with a Kähler form \( \omega \). For any torsionfree coherent analytic sheaf \( W \) on \( X \), define

\[
\text{degree}(W) := \int_M c_1(\det W) \wedge \omega^{d-1},
\]

where \( d = \dim_G M \) and the determinant line bundle \( \det W \) is constructed as in [12, Ch. V, § 6]. A torsionfree coherent analytic sheaf \( W \) is called \textit{stable} (respectively, \textit{semistable}) if every coherent analytic subsheaf \( F \subset W \) with \( 0 < \text{rank}(F) < \text{rank}(W) \), we have

\[
\frac{\text{degree}(F)}{\text{rank}(F)} < \frac{\text{degree}(W)}{\text{rank}(W)} \quad \text{(respectively, } \frac{\text{degree}(F)}{\text{rank}(F)} \leq \frac{\text{degree}(W)}{\text{rank}(W)} \text{)}
\]

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(see [12, p. 168]). Also, \( W \) is called \textit{polystable} if it is semistable and direct sum stable sheaves.

We shall now recall a generalization of these notions to the context of principal bundles [13,14].

An open dense subset \( U \) of \( M \) will be called \textit{big} if the complement \( M \setminus U \) is a complex analytic subspace of \( M \) of complex codimension at least two. For a holomorphic line bundle \( L \) on a big open subset \( \iota : U \hookrightarrow M \), the degree of \( L \) is defined to the degree of the direct image \( \iota_*L \).

A character \( \chi \) of a parabolic subgroup \( P \subset G \) is called \textit{anti-dominant} if the holomorphic line bundle on \( G/P \) associated to \( \chi \) is nef. Moreover, if the associated line bundle is ample, then \( \chi \) is called \textit{strictly anti-dominant}.

Let \( E_G \) be a holomorphic principal \( G \)-bundle over \( M \), where \( G \), as before, is a connected reductive affine complex algebraic group. It is called \textit{stable} (respectively, \textit{semistable}) if for all triples of the form \( (P, E_P, \chi) \), where

- \( P \subset G \) is proper (not necessarily maximal) parabolic subgroup,
- \( E_P \subset E_G \) is a holomorphic reduction in structure group of \( E_G \) to \( P \) over a big open subset \( U \subset M \), and
- \( \chi \) is a strictly anti-dominant character of \( P \) which is trivial on the center of \( G \),

the following holds:

\[
\text{degree}(E_P(\chi)) > 0
\]

(respectively, \( \text{degree}(E_P(\chi)) \geq 0 \)), where \( E_P(\chi) \) is the holomorphic line bundle on \( U \) associated to the principal \( P \)-bundle \( E_P \) for the character \( \chi \). (See [1,13,14].)

Let \( P \) be a parabolic subgroup of \( G \) and \( E_P \subset E_G \) a holomorphic reduction in structure group over \( M \) of \( E_G \) to the subgroup \( P \). Such a reduction in structure group is called \textit{admissible} if for every character \( \chi \) of \( P \) trivial on the center of \( G \), the associated holomorphic line bundle \( E_P(\chi) \) on \( M \) is of degree zero.

A holomorphic principal \( G \)-bundle \( E_G \) on \( M \) is called \textit{polystable} if either \( E_G \) is stable or there is parabolic subgroup \( P \subset G \) and a holomorphic reduction in structure group \( E_{L(P)} \subset E_G \) over \( M \) to a Levi factor \( L(P) \) of \( P \), such that

- the principal \( L(P) \)-bundle \( E_{L(P)} \) is stable, and
- the reduction in structure group of \( E_G \) to \( P \) given by the extension of the structure group of \( E_{L(P)} \) to \( P \), using the inclusion of \( L(P) \) in \( P \), is admissible.

(See [1,14].)

Fix a maximal compact subgroup

\[
K_G \subset G.
\]  

(3.2)

Let \( E_{K_G} \subset E_G \) be a \( C^\infty \) reduction in structure group over \( M \) of \( E_G \) to the subgroup \( K_G \). Then there is a unique \( C^\infty \) connection \( \nabla \) on \( E_{K_G} \) such that the connection on \( E_G \) induced by \( \nabla \) is a complex connection [2, pp. 191–192, Proposition 5].

Let \( z(\mathfrak{k}) \) denote the center of the Lie algebra of \( K_G \). A \( C^\infty \) reduction in structure group over \( M \)

\[
E_{K_G} \subset E_G
\]

is called a Hermitian–Einstein reduction if the corresponding connection \( \nabla \) has the property that the curvature \( K(\nabla) \) of \( \nabla \) satisfies the equation

\[
K(\nabla) \wedge \omega^{d-1} = c\omega^d
\]  

(3.3)
for some $c \in z(\mathfrak{t})$, where $d = \dim_{\mathbb{C}} M$. If $E_{K_G}$ is a Hermitian–Einstein reduction, then the connection on $E_G$ induced by the corresponding connection $\nabla$ on $E_{K_G}$ is called a Hermitian–Einstein connection.

A holomorphic principal $G$–bundle $E_G$ on $M$ admits a Hermitian–Einstein connection if and only if $E_G$ is polystable, and furthermore, if $E_G$ is polystable, then it has a unique Hermitian–Einstein connection. When $M$ is a complex projective manifold and $G = \text{GL}(r, \mathbb{C})$, this was proved in [8,9]; when $M$ is Kähler and $G = \text{GL}(r, \mathbb{C})$, this was proved in [19]; when $M$ is a complex projective manifold and $G$ is an arbitrary complex reductive group, this was proved in [14]; when $M$ is Kähler and $G$ is an arbitrary complex reductive group, this was proved in [1].

Now consider the setup of Theorem 2.5; let $T$, $C$, $N$ and $W$ be as in Theorem 2.5.

Let $E_G$ be a holomorphic principal $G$–bundle over $M$. Let $\mathbb{T}$ be a torus subbundles of $\text{Ad}(E_G)$ such that for some (hence every) $x \in M$, the fiber $\mathbb{T}_x$ lies in the conjugacy class of tori in $\text{Ad}(E_G)$ determined by $T$. Let $(E_W, \phi, E'_C, \tau)$ be the corresponding quadruple in Theorem 2.5.

Equip $E_W$ with the Kähler form $\phi^*\omega$. Note that $E_W$ need not be connected. Take a holomorphic principal $H$–bundle $E_H$ on $E_W$, where $H$ is a connected complex reductive affine algebraic group (we do not use the notation $G$ because it may create confusion as $G$ is used above). We will call $E_H$ to be polystable if the following two conditions hold:

1. the restriction of $E_H$ to each connected component of $E_W$ is polystable, and
2. for each character $\chi$ of $H$, the associated holomorphic line bundle $E_H(\chi)$ on $E_W$ has the property that degrees of its restriction to the connected components of $E_W$ coincide.

(If $E_W$ is connected then this condition is vacuously satisfied.)

Fix a maximal compact subgroup $K_H \subset H$. Let $z(\mathfrak{t}_h)$ denote the center of the Lie algebra of $K_H$. A $C^\infty$ reduction in structure group over $E_W$

$$E_{K_H} \subset E_H$$

will be called a Hermitian–Einstein reduction if the corresponding connection $\nabla$ has the property that there is an element $c \in z(\mathfrak{t}_h)$ such that the curvature $\mathcal{K}(\nabla)$ of $\nabla$ satisfies the equation

$$\mathcal{K}(\nabla) \wedge (\phi^*\omega)^{d-1} = c(\phi^*\omega)^d. \quad (3.4)$$

If $E_{K_H}$ is a Hermitian–Einstein reduction, then the connection on $E_H$ induced by the corresponding connection $\nabla$ on $E_{K_H}$ will be called a Hermitian–Einstein connection.

If $E_H$ is polystable, then the Hermitian–Einstein connections on the restrictions of $E_H$ to the connected components of $E_W$ together produce a Hermitian–Einstein connection on $E_H$. The Hermitian–Einstein connection for the restriction of $E_H$ to each component of $E_W$ produces an element of $z(\mathfrak{t}_h)$ (the element $c$ in (3.3)). The second condition in the definition of polystability for $E_H$ ensures that this element of $z(\mathfrak{t}_h)$ is independent of the component of $E_W$; the elements of $z(\mathfrak{t}_h)$ for different connected components of $E_W$ coincide. Furthermore, the Hermitian–Einstein connection on $E_H$ is unique. Conversely, if $E_H$ admits a Hermitian–Einstein connection, then the restriction of $E_H$ to each connected components of $E_W$ is polystable. Since the element of $z(\mathfrak{t}_h)$ for the Hermitian–Einstein connection [the element $c$ in (3.4)] does not depend on the component of $E_W$, the second condition in the definition of polystability is satisfied for $E_H$.

The intersection $K_C := C \cap K_G$ is a maximal compact subgroup of $C$. It will be used for defining the Hermitian–Einstein equation for holomorphic principal $C$–bundles on $E_W$. 

\[ Springer \]
Proposition 3.4 Assume that the principal G–bundle $E_G$ on $M$ is polystable. Let $\nabla$ be the Hermitian–Einstein connection on $E_G$. Then the following two hold:

1. The principal C–bundle $E_C'$ on $E_W$ is polystable.
2. The Hermitian–Einstein connection $\phi^*\nabla$ on $\phi^*E_G$ preserves the reduction $E_C'$ of structure group of $\phi^*E_G$ to C. Furthermore, the connection on $E_C'$ given by $\phi^*\nabla$ is Hermitian–Einstein.

Proof Recall that $T$ is contained in the center of $C$. So the natural action of $C$ on the principal $C$–bundle $E_C'$ commutes with the action of $T \subset C$ on $E_C'$. So $T$ acts on the total space of $E_C'$ preserving its principal C–bundle structure. Recall that $T'$ in (2.16) is identified with $E_W \times T$. The above action of $T$ on $E_C'$ produces the identification of $E_W \times T$ with $T'$. Since $\phi^*E_G$ is identified with the principal $G$–bundle on $E_W$ obtained by extending the structure group $E_C'$ to the group $G$, the action of $T$ on $E_C'$ produces an action of $T$ on $\phi^*E_G$. Indeed, $\phi^*E_G$ is the quotient of $E_C' \times G$ where two elements $(y, z)$ and $(y', z')$ of $E_C' \times G$ are identified if there is a $g \in C$ such that $y' = yg$ and $z' = g^{-1}z$. Consider the diagonal action of $T$ on $E_C' \times G$ given by the above action of $T$ on $E_C'$ and the trivial action of $T$ on $G$. This diagonal action of $T$ on $E_C' \times G$ descends to an action of $T$ on the quotient space $\phi^*E_G$.

Consider the Hermitian–Einstein connection $\phi^*\nabla$ on $\phi^*E_G$. From the uniqueness of a Hermitian–Einstein connection, it follows immediately that the above action of $T$ on $\phi^*E_G$ preserves the Hermitian–Einstein connection $\phi^*\nabla$. Since $T = T'/W = (E_W \times T)/W$ [see (2.17)], this implies that the subbundle $T \subset \text{Ad}(E_G)$ is preserved by the connection on $\text{Ad}(E_G)$ induced by $\nabla$. Hence from Theorem 3.1 if follows that the connection $\nabla$ is induced by a complex connection $\nabla'$ on the principal $N$–bundle $E_N$.

A connection on $E_N$ gives a connection on the principal $C$–bundle $E_C'$; recall that the total spaces of $E_N$ and $E_C'$ coincide. The connection on $E_C'$ given by the above connection $\nabla'$ on the principal $N$–bundle $E_N$ will be denoted by $\nabla"$. Since the connection $\nabla'$ on $E_N$ induces the connection $\nabla$ on $E_G$, from the definition of $\nabla"$ it follows that the connection $\phi^*\nabla'$ on $\phi^*E_G$ is induced by $\nabla"$. As the connection $\phi^*\nabla$ is Hermitian–Einstein, it follows immediately that the inducing connection $\nabla"$ is also Hermitian–Einstein.

Since $\nabla"$ is a Hermitian–Einstein connection on $E_C'$, the principal C–bundle $E_C'$ is polystable.

We will now prove a converse of Proposition 3.4. Consider the maximal compact subgroup $K_C := C \cap K_G$ of $C$. Let $z(t_c)$ be the center of the Lie algebra of $K_C$. Note that $z(t) \subset z(t_c)$, because the connected component of the center of $K_G$, containing the identity element, is contained in $C$.

Proposition 3.5 Assume that the principal C–bundle $E_C'$ over $E_W$ is polystable. Let $\nabla$ be the Hermitian–Einstein connection on $E_C'$. Assume that the element of $z(t_c)$ given by the curvature of $\nabla$ [the element $c$ in (3.4)] lies in the subspace $z(t)$. Then the following two hold:

1. The principal G–bundle $E_G$ on $M$ is polystable.
2. The Hermitian–Einstein connection on $E_G$ is given by $\nabla$.

Proof Since $\phi^*E_G$ is the extension of structure group of $E_C'$ to $G$ using the inclusion of $C$ in $G$, a connection on $E_C'$ induces a connection on $\phi^*E_G$. Let $\nabla'$ be the connection of $\phi^*E_G$ induced by the connection $\nabla$ on $E_C'$. Since $\nabla$ satisfies the Hermitian–Einstein equation with the element of $z(t_c)$, given by the curvature of $\nabla$, lying in the subspace $z(t)$, it follows immediately that $\nabla'$ satisfies the Hermitian–Einstein equation for $\phi^*E_G$. 

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Consider the natural action of the Galois group \( \text{Gal}(\phi) = W \) on the pulled back bundle \( \phi^*EG \). From the uniqueness of the Hermitian–Einstein connection on \( \phi^*EG \) it follows immediately that this action of \( W \) on \( \phi^*EG \) preserves the Hermitian–Einstein connection \( \nabla' \). Hence \( \nabla' \) descends to a connection on \( EG \); this connection on \( EG \) given by \( \nabla'' \) will be denoted by \( \nabla'' \). Since \( \nabla' \) satisfies the Hermitian–Einstein equation, it follows immediately that \( \nabla'' \) also satisfies the Hermitian–Einstein equation.

Since \( EG \) admits a Hermitian–Einstein connection, namely \( \nabla'' \), the principal \( G \)–bundle \( EG \) is polystable. \( \Box \)

4 Higgs bundles and Levi reduction

In this section, we work with the setup of Sect. 2.1. Take a torus bundle

\[ T \subset \text{Ad}(EG) \xrightarrow{p} M \]

as in (2.3). Fix \( T \) as in (2.4), and construct \( N \) as in (2.5). Let \( E_N \subset E_G \) be the holomorphic reduction in structure group to \( N \) constructed in (2.7).

Let \( \iota : \text{ad}(E_N) \longrightarrow \text{ad}(E_G) \) be the inclusion map. Take a holomorphic vector bundle \( V \) on \( M \).

Let

\[(\iota \otimes \text{Id}_V)_* : H^0(M, \text{ad}(E_N) \otimes V) \longrightarrow H^0(M, \text{ad}(E_G) \otimes V)\]

be the natural homomorphism. For any \( x \in M \), the adjoint action of \( \text{Ad}(E_G)_x \) on \( \text{ad}(E_G)_x \) and the trivial action of \( \text{Ad}(E_G)_x \) on \( V_x \) together produce a diagonal action of \( \text{Ad}(E_G)_x \) on \( (\text{ad}(E_G) \otimes V)_x \). A section

\[ \theta \in H^0(M, \text{ad}(E_G) \otimes V) \]

is said to be fixed by \( T \) if for every \( x \in M \) and \( g \in T_x \), the above action of \( g \) on \( (\text{ad}(E_G) \otimes V)_x \) fixes \( \theta(x) \).

**Lemma 4.1** A section

\[ \theta \in H^0(M, \text{ad}(E_G) \otimes V) \]

lies in the image of the above homomorphism \((\iota \otimes \text{Id}_V)_*\) if and only if \( \theta \) is fixed by \( T \).

**Proof** For any \( x \in M \), the fixed point locus \((\text{ad}(E_G)_x)^T_x\) for the adjoint action of \( T_x \) on \( \text{ad}(E_G)_x \) is \( \text{ad}(E_N)_x \). The lemma follows from this fact. \( \Box \)

4.1 \( G \)-Higgs bundles

In this subsection, we set \( V = \Omega^1_M \). The Lie algebra structure of fibers of \( \text{ad}(E_G) \) and the natural projection \( \bigotimes^2 \Omega^1_M \longrightarrow \Omega^2_M \) together define a homomorphism

\[ (\text{ad}(E_G) \otimes \Omega^1_M) \otimes (\text{ad}(E_G) \otimes \Omega^1_M) \longrightarrow \text{ad}(E_G) \otimes \Omega^2_M \]

which we shall denote by \( \bigwedge \).

We recall that a Higgs field on \( E_G \) is a holomorphic section

\[ \theta \in H^0(M, \text{ad}(E_G) \otimes \Omega^1_M) \]
such that \( \theta \wedge \theta = 0 \) \([10,15,16]\). A \( G \)–Higgs bundle on \( M \) is a pair of the form \((E_G, \theta)\), where \( E_G \) is a holomorphic principal \( G \)–bundle over \( M \) and \( \theta \) is a Higgs field on \( E_G \).

As in Sect. 3.3, we assume that \( M \) is Kähler and it is equipped with a Kähler form \( \omega \).

A \( G \)–Higgs bundle \((E_G, \theta)\) is called stable (respectively, semistable) if for all triples of the form \((P, E_P, \chi)\), where

- \( P \subset G \) is proper (not necessarily maximal) parabolic subgroup,
- \( E_P \subset E_G \) is a holomorphic reduction in structure group of \( E_G \) to \( P \) over a big open subset \( U \subset M \) such that
  \[
  \theta|_U \in H^0(U, \text{ad}(E_P) \otimes \Omega^1_U) \subset \text{ad}(E_G) \otimes \Omega^1_U,
  \]
  and
- \( \chi \) is a strictly anti-dominant character of \( P \) which is trivial on the center of \( G \),

the following holds:

\[
\text{degree}(E_P(\chi)) > 0
\]
(respectively, \( \text{degree}(E_P(\chi)) \geq 0 \)), where \( E_P(\chi) \) is the holomorphic line bundle over \( U \) associated to the principal \( P \)–bundle \( E_P \) for the character \( \chi \). (See \([5,16]\).)

A \( G \)–Higgs bundle \((E_G, \theta)\) over \( M \) is called polystable if either \((E_G, \theta)\) is stable or there is parabolic subgroup \( P \subset G \) and a holomorphic reduction in structure group \( E_L(P) \subset E_G \) over \( M \) to a Levi factor \( L(P) \) of \( P \), such that

- \( \theta \) in \( H^0(M, \text{ad}(E_L(P)) \otimes \Omega^1_M) \),
- the \( L(P) \)–Higgs bundle \((E_L(P), \theta)\) is stable, and
- the reduction in structure group of \( E_G \) to \( P \) given by the extension of the structure group \( E_L(P) \) to \( P \), using the inclusion of \( L(P) \) in \( P \), is admissible.

Fix a maximal compact subgroup \( K_G \) as in (3.2). Let \((E_G, \theta)\) be a \( G \)–Higgs bundle on \( M \). A \( C^\infty \) reduction in structure group over \( M \)
\[
E_{K_G} \subset E_G
\]
is called a Hermitian–Yang–Mills reduction if the corresponding connection \( \nabla \) on \( E_{K_G} \) has the property that the curvature \( \mathcal{K}(\nabla) \) of \( \nabla \) satisfies the equation
\[
(\mathcal{K}(\nabla) + [\theta, \theta^*]) \wedge \omega^{d-1} = c\omega^d,
\] (4.1)
where \( c \) and \( d \) are as in (3.3), and \( \theta^* \) is the adjoint of \( \theta \) (the \( \mathbb{R} \)–vector space \( \text{Lie}(G) \) has the decomposition \( \text{Lie}(G) = \text{Lie}(K_G) \oplus \mathfrak{p} \); the adjoint is the conjugate linear automorphism of the vector space \( \text{Lie}(G) \) that acts on \( \text{Lie}(K_G) \) as multiplication by \(-1\) and acts on \( \mathfrak{p} \) as the identity map). If \( E_{K_G} \) is a Hermitian–Yang–Mills reduction, then the connection on \( E_G \) induced by the corresponding connection \( \nabla \) on \( E_{K_G} \) is called a Hermitian–Yang–Mills connection \([5,15,16]\).

A \( G \)–Higgs bundle \((E_G, \theta)\) admits a Hermitian–Yang–Mills connection if and only if \((E_G, \theta)\) is polystable, and furthermore, if \((E_G, \theta)\) is polystable, then it has a unique Hermitian–Yang–Mills connection \([5,10,15]\).

Let \( T, C, N \) and \( W \) be as in Theorem 2.5. Let \((E_G, \theta)\) be a \( G \)–Higgs bundle on \( M \). Let \( \mathbb{T} \) be a torus subbundles of \( \text{Ad}(E_G) \) such that

- for some (hence every) \( x \in M \), the fiber \( \mathbb{T}_x \) lies in the conjugacy class of tori in \( \text{Ad}(E_G) \) determined by \( T \), and
- for every \( x \in M \), the action of \( \mathbb{T}_x \) on \( \text{ad}(E_G) \otimes \Omega^1_M \) fixes the element \( \theta(x) \).
Let \((E_W, \phi, E'_C, \tau)\) be the corresponding quadruple in Theorem 2.5. Equip \(E_W\) with the Kähler form \(\phi^*\omega\).

The notions of Higgs bundle, polystability and Hermitian–Yang–Mills connection extend to \(E_W\) as before.

We have the following generalization of Proposition 3.4:

**Proposition 4.2** Assume that the \(G\)–Higgs bundle \((E_G, \theta)\) is polystable. Let \(\nabla\) be the Hermitian–Yang–Mills connection on \(E_G\). Then the following three hold:

1. \(\theta\) defines a Higgs on the principal \(C\)–bundle \(E'_C\) over \(E_W\).
2. The \(C\)–Higgs bundle \((E'_C, \theta)\) is polystable.
3. The Hermitian–Yang–Mills connection \(\phi^*\nabla\) on \(\phi^*E_G\) preserves the reduction \(E'_C\) of structure group of \(\phi^*E_G\) to the subgroup \(C\). Furthermore, the connection on \(E'_C\) given by \(\phi^*\nabla\) is Hermitian–Yang–Mills connection for the \(C\)–Higgs bundle \((E'_C, \theta)\).

**Proof** In view of Lemma 4.1, and the uniqueness of the Hermitian–Yang–Mills connection on a polystable bundle, the proof works along the same line as the proof of Proposition 3.4. We omit the details. \(\square\)

The following is a generalization of Proposition 3.5:

**Proposition 4.3** Assume that the \(C\)–Higgs bundle \((E'_C, \theta)\) is polystable. Let \(\nabla\) be the Hermitian–Yang–Mills connection on \(E'_C\). Assume that the element of \(z(\mathfrak{g})\) given by the Hermitian–Yang–Mills equation for \((E'_C, \theta)\) lies in \(z(\mathfrak{k})\). Then the following two hold:

1. The \(G\)–Higgs bundle \((E_G, \theta)\) on \(M\) is polystable.
2. The Hermitian–Yang–Mills connection on \(E_G\) is given by \(\nabla\).

**Proof** The proof is similar to the proof of Proposition 3.5. \(\square\)

### 5 Torus for equivariant bundles

Let \(\Gamma\) be a group and

\[
\varphi_0 : \Gamma \times M \longrightarrow M
\]

a left–action such that the self–map \(x \mapsto \varphi_0(\gamma, x)\) of \(M\) is a biholomorphism for every \(\gamma \in \Gamma\). An *equivariant* holomorphic principal \(G\)–bundle on \(M\) is a pair \((E_G, \varphi)\), where \(p_0 : E_G \longrightarrow M\) is a holomorphic principal \(G\)–bundle and

\[
\varphi : \Gamma \times E_G \longrightarrow E_G
\]

is an action of \(\Gamma\) on \(E_G\) such that

- for every \(\gamma \in \Gamma\), the self–map \(y \mapsto (\gamma, y)\) of \(E_G\) is a biholomorphism,
- \(p_0 \circ \varphi = \varphi_0 \circ (\text{Id}_\Gamma \times p_0)\), and
- the actions \(G\) and \(\Gamma\) on \(E_G\) commute.

Let \((E_G, \varphi)\) be an equivariant holomorphic principal \(G\)–bundle on \(M\). For any complex manifold \(F\) equipped with a holomorphic action of \(G\), consider the diagonal action of \(\Gamma\) on \(E_G \times F\) given by the action \(\varphi\) on \(E_G\) and the trivial action of \(\Gamma\) on \(F\). This diagonal action of \(\Gamma\) on \(E_G \times F\) produces an action of \(\Gamma\) on the quotient space of \(E_G \times F\) defining the fiber.
bundle over $M$ associated to $E_G$ for $F$. In particular, $\Gamma$ acts on $\text{Ad}(E_G)$; this action of $\Gamma$ on $\text{Ad}(E_G)$ preserves the group structure of the fibers of $\text{Ad}(E_G)$.

As in (2.3), let

$$T \subset \text{Ad}(E_G) \xrightarrow{p} M$$

be a holomorphic sub-fiber bundle such that

- the above action of $\Gamma$ on $\text{Ad}(E_G)$ preserves $T$,
- $M \times Z_0(G) \subset T$, and
- for every point $x \in M$, the fiber

$$T_x := T \cap \text{Ad}(E_G)_x \subset \text{Ad}(E_G)_x$$

is a torus (it need not be a maximal torus of $\text{Ad}(E_G)_x$).

Fix $T$ as in (2.4). Then the principal $N$–bundle $E_N \subset E_G$ in (2.7) is evidently preserved by the action $\varphi$ of $\Gamma$ on $E_G$. The resulting action of $\Gamma$ on $E_N$ produces an action of $\Gamma$ on the quotient $E_W$ of $E_N$ in (2.11). The projection $\phi$ in (2.11) clearly intertwines the action of $\Gamma$ on $E_W$ and $M$.

Consider the diagonal action $\Gamma$ on $E_W \times E_G$ constructed using the actions of $\Gamma$ on $E_W$ and $E_G$. Recall the $\phi^*E_G$ is the submanifold of $E_W \times E_G$ consisting of all $(x, y) \in E_W \times E_G$ such that $\phi(x) = p_0(y)$, where $p_0$ and $\phi$ are the maps in (2.1) and (2.11) respectively. This submanifold of $E_W \times E_G$ is preserved by the diagonal action of $\Gamma$. Therefore, this action of $\Gamma$ on $E_W \times E_G$ produces an action of $\Gamma$ on $\phi^*E_G$. For this action of $\Gamma$ on $\phi^*E_G$, the projection $\phi^*E_G \longrightarrow E_W$ is clearly $\Gamma$–equivariant. Also, the natural map $\phi^*E_G \longrightarrow E_G$ is also $\Gamma$–equivariant. The diagonal action of $\Gamma$ on $E_W \times E_N$ similarly produces an action of $\Gamma$ on $\phi^*E_N$, where $E_N$ is constructed in (2.7). Note that the inclusion map $\phi^*E_N \hookrightarrow \phi^*E_G$ (which is the pullback of the inclusion map in (2.7)) is $\Gamma$–equivariant. Since the principal $N$–bundle $E_N \subset E_G$ is preserved by the action $\varphi$ of $\Gamma$ on $E_G$, the principal $C$–bundle $E_C \subset \phi^*E_N \subset \phi^*E_G$ constructed in Lemma 2.1 is preserved by the above action of $\Gamma$ on $\phi^*E_G$.

Conversely, fix a torus $T \subset G$ a containing $Z_0(G)$. The normalizer (respectively, centralizer) of $T$ in $G$ will be denoted by $N$ (respectively, $C$), while the Weyl group $N/C$ will be denoted by $W$. Let

$$\phi : E_W \longrightarrow M$$

be a principal $W$–bundle such that $E_W$ is equipped with an action of $\Gamma$ satisfying the following conditions:

- the map $\phi$ is $\Gamma$–equivariant, and
- the actions of $\Gamma$ and $W$ on $E_W$ commute.

Let $(E_G, \varphi)$ be an equivariant holomorphic principal $G$–bundle on $M$. As before, the diagonal action of $\Gamma$ on $E_W \times E_G$ produces an action of $\Gamma$ on $\phi^*E_G$. Let

$$E_C' \subset \phi^*E_G$$

be a holomorphic reduction in structure group of the principal $G$–bundle $\phi^*E_G$ to $C$ such that the action of $\Gamma$ on $\phi^*E_G$ preserves $E_C'$. Assume that we are further given a holomorphic action of the complex Lie group $N$ on $E_C'$

$$\tau : E_C' \times N \longrightarrow E_C'$$

such that
the restriction of the map $\tau$ to $E'_{\mathbb{C}} \times C$ is the natural action of $C$ on the principal $C$–bundle $E'_{\mathbb{C}}$,

- the actions of $N$ and $\Gamma$ on $E'_{\mathbb{C}}$ commute, and

- the diagram of maps

$$
\begin{array}{ccc}
E'_{\mathbb{C}} \times N & \overset{\tau}{\longrightarrow} & E'_{\mathbb{C}} \\
\downarrow & & \downarrow \psi \\
E_W \times W := (E'_{\mathbb{C}}/C) \times (N/C) & \longrightarrow & E_W
\end{array}
$$

is commutative, where $E'_{\mathbb{C}} \times N \longrightarrow (E'_{\mathbb{C}}/C) \times (N/C) = (E'_{\mathbb{C}} \times N)/(C \times C)$ is the quotient map, and $\psi$ is the natural projection of the principal $C$–bundle.

Then the action of $\Gamma$ on $\text{Ad}(E_G)$, given by the action $\varphi$ of $\Gamma$ on $E_G$, preserves the torus subbundle $\mathcal{T} \subset \text{Ad}(E_G)$ constructed in (2.17).

Therefore, Theorem 2.5 has the following generalization:

**Theorem 5.1** Let $(E_G, \varphi)$ be an equivariant holomorphic principal $G$–bundle on $M$ and $T \subset G$ a torus containing $Z_0(G)$. The normalizer (respectively, centralizer) of $T$ in $G$ will be denoted by $N$ (respectively, $C$), while the Weyl group $N/C$ will be denoted by $W$. There is a natural bijective correspondence between the following two:

1. $\Gamma$–invariant torus subbundles $\mathcal{T}$ of $\text{Ad}(E_G)$ such that for some (hence any) $x \in M$, the fiber $\mathcal{T}_x$ lies in the conjugacy class determined by $T$.

2. Quadruples of the form $(E_W, \varphi, E'_{\mathbb{C}}, \tau)$, where $\varphi : E_W \longrightarrow M$ is a $\Gamma$–equivariant principal $W$–bundle, $E'_{\mathbb{C}} \subset \varphi^*E_G$ is a $\Gamma$–invariant holomorphic reduction in structure group of $\varphi^*E_G$ to $C$, and $\tau : E'_{\mathbb{C}} \times N \longrightarrow E'_{\mathbb{C}}$ is a holomorphic action of $N$ on $E'_{\mathbb{C}}$ extending the natural action of $C$ on the principal $C$–bundle $E'_{\mathbb{C}}$, such that the actions of $N$ and $\Gamma$ on $E'_{\mathbb{C}}$ commute, and the diagram of maps

$$
\begin{array}{ccc}
E'_{\mathbb{C}} \times N & \overset{\tau}{\longrightarrow} & E'_{\mathbb{C}} \\
\downarrow & & \downarrow \\
E_W \times W := (E'_{\mathbb{C}}/C) \times (N/C) & \longrightarrow & E_W
\end{array}
$$

is commutative.

All the results in Sects. 3 and 4 also extend to the equivariant setup.

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