The Lie Group Structure of the $\eta - \xi$ Space-time and its Physical Significance

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Abstract

The $\eta - \xi$ space-time is suggested by Gui for the quantum field theory in 1988. This paper consists of two parts. The first part is devoted to the discussion of the global properties of the $\eta - \xi$ space-time. The result contains a proof which asserts that the $\eta - \xi$ space-time is homeomorphic to $\mathbb{C}^* \times \mathbb{C}^* \times \mathbb{C}^2$ by means of two explicit maps, which shows that the $\eta - \xi$ space-time allows a Lie group structure. Thus some transformation groups, one of which is isomorphic to the Lorentz group in two dimensions, can be found. The other part of the paper is the discussion about the embedding of some subspaces in the $\eta - \xi$ space-time. In particular, it is pointed out that the Euclidean space-time and the Minkowskian space-time are linked in a way in the $\eta - \xi$ space-time such that the tilde field appears naturally. In addition some formulae in the $\eta - \xi$ space-time reappear in a more natural way.

Keywords $\eta - \xi$ space-time, Lie group, Imaginary-time theory, Real-time theory, Transformation group

1 Introduction

The $\eta - \xi$ space-time (Gui's space-time) is proposed by Gui for the quantum field theory in [1], [2], and [3]. In his recent paper [4] the idea and progress on this new space-time are reviewed and some problems to be solved are given as well. There are two problems among that encouraged by [4], the symmetry problem and tilde field problem, in which we are interested.

In the paper the fact that the $\eta - \xi$ space-time is homeomorphic to $\mathbb{C}^* \times \mathbb{C}^* \times \mathbb{C}^2$ by means of two explicit maps, one of which gives the $\eta - \xi$ space-time the Lie group structure directly, is showed. From these results some excellent formulae in the theory of $\eta - \xi$ space-time reappear in a more natural way. Next it comes out that the Euclidean space-time $S^1 \times \mathbb{R}^3$ for the imaginary-time theory and the Minkowskian space-time $\mathbb{R}^4$ for the real-time theory join as two subspaces of the $\eta - \xi$ space-time. This property leads the tilde field to appear naturally.

The $\eta - \xi$ space-time is denoted by $V$ which is defined in our way by $V = \mathbb{C}^4 - C$ where an element in $\mathbb{C}^3$ is denoted by four complex numbers $(\eta, \xi, y, z)$, and $C$ is the algebraic set determined by the equation
with the metric $g$ which is defined by

$$ds^2 = \frac{1}{\alpha^2(\xi^2 - \eta^2)}(-d\eta^2 + d\xi^2) + dy^2 + dz^2$$

where $\alpha \neq 0$, is a real constant number. There is no loss in generality by assuming that $\alpha = 1$ in this paper.

In order to define the metric $g$ on the whole space-time we delete the algebraic set $C$ from $C^4$.

Let $V_0 = \mathbb{C}^2 - C_0$, where $C_0$ is the algebraic set determined by the equation

$$\xi^2 - \eta^2 = 0$$

with the metric $g_0$ which is defined by

$$ds^2 = \frac{1}{\xi^2 - \eta^2}(-d\eta^2 + d\xi^2)$$

It is obvious that $V = V_0 \times \mathbb{C}^2$, thus it suffices to research the topology of $V_0$.

We have need of a little knowledge of Lie groups, for these see [5] or [6].

2 The structure of $V_0$

In order to give $V_0$ a group structure we consider the subgroup $T$ of $GL(2, \mathbb{C})$, that is

$$T = \{ \begin{pmatrix} \gamma & 0 \\ 0 & \delta \end{pmatrix} | \gamma \delta \neq 0 \}$$

It is obviously homeomorphic to $\mathbb{C}^* \times \mathbb{C}^*$. Then we have a map

$$V_0 \rightarrow T$$

$$(\eta, \xi) \rightarrow \begin{pmatrix} \xi - \eta & 0 \\ 0 & \xi + \eta \end{pmatrix}$$

It is a homeomorphism map. If identify $\eta, \xi$ in $V_0$ with the element

$$\begin{pmatrix} \xi & \eta \\ \eta & \xi \end{pmatrix} \in GL(2, \mathbb{C})$$

then $V_0$ can be considered as a subgroup, which is a conjugate of $T$, of $GL(2, \mathbb{C})$.

In fact we have

$$\begin{pmatrix} \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} \\ -\frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} \end{pmatrix}^{-1} \begin{pmatrix} \xi & \eta \\ \eta & \xi \end{pmatrix} \begin{pmatrix} \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} \\ -\frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} \end{pmatrix} = \begin{pmatrix} \xi - \eta & 0 \\ 0 & \xi + \eta \end{pmatrix}.$$
It is easily checked that \((0, 1)\) is the identity, the inverse element of \((\eta, \xi)\) is 
\[
(\frac{-\eta}{\xi^2 - \eta^2}, \frac{\xi}{\xi^2 - \eta^2})
\]
and the multiplication is associative and commutative. In addition, there is a representation

\[
\Phi : V_0 \rightarrow GL(2, \mathbb{C})
\]

\[
\Phi(\eta, \xi) = \begin{pmatrix} \xi & \eta \\ \eta & \xi \end{pmatrix}
\]

In fact, it is the above identification. In addition we have also

\[
(\eta, \xi) \ast (\eta', \xi') = (\xi' \eta + \eta \xi', \xi' \xi + \eta' \eta)
\]

As a group, \(V_0\) has the left translation, that is

\[
L_{(\gamma, \delta)} : V_0 \rightarrow V_0
\]

\[
L_{(\gamma, \delta)}(\eta, \xi) = (\gamma, \delta) \ast (\eta, \xi)
\]

for a fixed element \((\gamma, \delta) \in V_0\). It is easy to show that the left translations 
preserve the metric \(g_0\). It follows that the set of the left translations forms a 
transformation group, which is isomorphic to \(V_0\). of \(V_0\).

Let

\[
G_1 = \{(\gamma, \delta) \in V_0| \delta^2 - \gamma^2 = 1\} \quad \text{and} \quad G_2 = \{(\gamma, \delta) \in G_1| \gamma, \delta \in \mathbb{R}\}
\]

it is easy to show that \(G_1\) is a subgroup of the group \(V_0\) and \(G_2\) is a subgroup 
of \(G_1\). Since \(\Phi(G_2)\) is the Lorentz group in two dimensions, \(\eta - \xi\) space-time 
has a transformation group which is isomorphic to the Lorentz group in two 
dimensions.

As a commutative Lie group, \(V_0\) has its Lie algebra \(V_0\) with the trivial bracket 
, which is isomorphic to \(\mathbb{C}^2\) by taking a basis, for example

\[
\frac{\partial}{\partial \eta}, \frac{\partial}{\partial \xi}
\]

This fact allows us to define the exponential map

\[
\exp : \mathbb{C}^2 \rightarrow V_0
\]

\[
\exp(z_0, z_1) = \Phi^{-1} \exp\left(\begin{pmatrix} z_1 & z_0 \\ z_0 & z_1 \end{pmatrix}\right)
\]

\[
= \Phi^{-1}(\exp\left(\begin{pmatrix} z_1 & 0 \\ 0 & z_1 \end{pmatrix}\right) \exp\left(\begin{pmatrix} 0 & z_0 \\ z_0 & 0 \end{pmatrix}\right))
\]

\[
= \Phi^{-1}\begin{pmatrix} e^{z_1} \cosh z_0 & e^{z_1} \sinh z_0 \\ e^{z_1} \sinh z_0 & e^{z_1} \cosh z_0 \end{pmatrix}
\]

\[
= (e^{z_1} \sinh z_0, e^{z_1} \cosh z_0)
\]

This expression and that of \([1]\) are coincide exactly. Then we find out

\[
\frac{1}{\xi^2 - \eta^2}(-d\eta^2 + d\xi^2) = -d\xi_0^2 + d\xi_1^2
\]

(2.1)
Since $V_0$ is a commutative Lie group, the map $\exp$ is a homomorphism from the additive group of $\mathbb{C}^2$ to $V_0$ with the composition $\ast$. Thus the translation “$(a, b)\ast$” in $\mathbb{C}^2$ corresponds to the translation “$(\gamma, \delta)\ast$” in $V_0$, where $(\gamma, \delta) = \exp(a, b)$. It follows from the connectedness of $V_0$ that $Q = \exp : \mathbb{C}^2 \to V_0$ is the universal covering map of $V_0$. Thus there is another homeomorphism, if write $\mathbb{C}^2 = \mathbb{R}_0 \times i\mathbb{R}_0 \times \mathbb{R}_1 \times i\mathbb{R}_1$,

$$Q : (\mathbb{R}_0 \times S^1_0) \times (\mathbb{R}_1 \times S^1_1) \to V_0$$

$$Q(u_0, e^{i\nu_0}, u_1, e^{i\nu_1}) = \tilde{Q}(u_0 + i\nu_0, u_1 + i\nu_1) \quad 0 \leq \nu_0, \nu_1 < 2\pi \quad (2.2)$$

and a map $\Pi : \mathbb{C}^2 \to (\mathbb{R}_0 \times S^1_0) \times (\mathbb{R}_1 \times S^1_1)$ such that $\tilde{Q} = Q \circ \Pi$, the composition of two maps.

### 3 The embedding of subspaces

Restricting $Q$ to $\{t\} \times S^1_0 \times \mathbb{R}_1 \times \{1\}$ for a fixed $t$, we have an embedding

$$Q_{t,t} : S^1_0 \times \mathbb{R}_1 \to V_0$$

$$Q_{t,t}(e^{i\tau}, x_1) = Q(t, e^{i\tau}, x_1, 1)$$

$$= (e^{x_1} \sinh(t + i\tau), e^{x_1} \cosh(t + i\tau)) \quad (3.1)$$

In particular, if $t = 0, Q_{I,0}(e^{i\tau}, x_1) = (i e^{x_1} \sin \tau, e^{x_1} \cos \tau)$.

If $\tau$ is fixed but $t$ is variable we have another embedding

$$Q_{R,\tau} : \mathbb{R}_0 \times \{e^{i\tau}\} \times \mathbb{R}_1 \to V_0$$

$$Q_{R,\tau}(t, x_1) = \tilde{Q}(t + i\tau, x_1) = (e^{x_1} \sinh(t + i\tau), e^{x_1} \cosh(t + i\tau)) \quad (3.2)$$

Let $(\sinh i\tau', \cosh i\tau') \in V_0$, then

$$(\sinh i\tau', \cosh i\tau') \ast (e^{x_1} \sinh(t + i\tau), e^{x_1} \cosh(t + i\tau))$$

$$= (e^{x_1} \sinh(t + i(\tau + \tau')), e^{x_1} \cosh(t + i(\tau + \tau')))$$

It follows that $Q_{R,\tau+\tau'}(\mathbb{R}_0 \times \{e^{i(\tau+\tau')}\} \times \mathbb{R}_1)$ can be obtained from $Q_{R,\tau}(\mathbb{R}_0 \times \{e^{i\tau}\} \times \mathbb{R}_1)$ by translation along $(\sinh i\tau', \cosh i\tau')$. In particular, after the translation along $(0, -1)$, we have

$$Q_{R,\tau+\tau}(t, x_1) = (-e^{x_1} \sinh(t + i\tau), -e^{x_1} \cosh(t + i\tau))$$

$$= -Q_{R,\tau}(t, x_1)$$

For example, when $\tau = 0$, we have

$$Q_{R,0}(t, x_1) = (e^{x_1} \sinh t, e^{x_1} \cosh t)$$

$$Q_{R,\pi}(t, x_1) = (-e^{x_1} \sinh t, -e^{x_1} \cosh t)$$
The subspaces $Q_{R,0}(\mathbb{R}_0 \times \{1\} \times \mathbb{R}_1) = V_I$ and $Q_{R,\pi}(\mathbb{R}_0 \times \{-1\} \times \mathbb{R}_1) = V_{II}$, as is known, are called the universe $I$ and the universe $II$.

Next the line elements are calculated. From (2.1), (2.2) and (3.1), we have

$$ds^2 = dv^2 + dx_1^2$$

(3.3)

This is the Euclidean line element in $Q_{I,t}(S^1_0 \times \mathbb{R}_1)$. From (2.1) and (3.2) we have

$$ds^2 = -dt^2 + dx_1^2$$

(3.4)

This is the Minkowskian line element in $Q_{R,\tau}(\mathbb{R}_0 \times \{e^{i\tau}\} \times \mathbb{R}_1)$.

Let $v = (a, b) \in \mathbb{C}^2$, the Lie algebra of $V_0$, we have the one-parameter subgroup

$$\rho_v(\mu) = \exp \mu v = (e^{i\mu b} \sinh \mu a, e^{i\mu b} \cosh \mu a).$$

It induces a one-parameter transformation group

$$\rho_v(\mu) * (\eta, \xi) = e^{i\mu b}(\eta \cosh \mu a + \xi \sinh \mu a, \xi \cosh \mu a + \eta \sinh \mu a)$$

and its generator

$$(\eta, \xi) \mapsto (b \eta + a \xi, b \xi + a \eta)$$

(3.5)

Because of metric-preserving the generator is a Killing field.

As the tangent vector of $V_0$, a vector $v$ in $V_0$ has the same causal character as that of the Killing field induced by $v$. In particular, if $v$ is the time-like vector $\frac{\partial}{\partial \tau}$, see Eq.(2.1), then we have a time-like Killing field, see Eq.(3.5),

$$\xi \frac{\partial}{\partial \eta} + \eta \frac{\partial}{\partial \xi}$$

which is exactly what [2] wanted.

In conclusion we point out that

1. $\mathbb{C}^2$ is the universal covering space of $V_0$.
2. $Q_{I,t}(S^1_0 \times \mathbb{R}_1)$ have the Euclidean line element and the universal covering space $\mathbb{R}_0 \times \mathbb{R}_1$ for every $t$.
3. $Q_{R,\tau}(\mathbb{R}_0 \times \{e^{i\tau}\} \times \mathbb{R}_1)$ have the Minkowskian line element, in particular, $Q_{R,0}(\mathbb{R}_0 \times \{1\} \times \mathbb{R}_1) = V_I$ and $Q_{R,\pi}(\mathbb{R}_0 \times \{-1\} \times \mathbb{R}_1) = V_{II}$ are the universe $I$ and the universe $II$, respectively.

4 The imaginary-time and the real-time

Let $\phi$ is a field on $V_0$. The imaginary-time theory considers

$$\phi_0 : S^1_0 \times \mathbb{R}_1 \xrightarrow{Q_{I,t}} V_0 \xrightarrow{\phi} \mathbb{R}$$

and the real-time theory considers

$$\phi_1 : \mathbb{R}_0 \times \{1\} \times \mathbb{R}_1 \xrightarrow{Q_{R,0}} V_0 \xrightarrow{\phi} \mathbb{R}$$

$$\phi_2 : \mathbb{R}_0 \times \{-1\} \times \mathbb{R}_1 \xrightarrow{Q_{R,\pi}} V_0 \xrightarrow{\phi} \mathbb{R}$$
Figure 1: The time path $C'$ in $\mathbb{C} = \mathbb{R}_0 \times i\mathbb{R}_0$

Figure 2: $C'' = \Pi(C')$ in $(S^1_0 \times \mathbb{R}_1) \cup \bigcup_{\tau = 0, \pi} (\mathbb{R}_0 \times \{e^{i\tau}\} \times \mathbb{R}_1)$

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In order to link the imaginary-time and the real-time note that \( Q_{I,t}(S^1_0 \times \mathbb{R}_1) \)
share the space \( \mathbb{R}_1 \) with \( Q_{R,\tau}(\mathbb{R}_0 \times \{e^{i\tau}\} \times \mathbb{R}_1), \tau = 0, \pi, \) in \( V_0 \). Now we look at
the calculation of the real-time thermal Green’s functions by computing a path
integral along the time path \( C' \) in \( C \), where \( C' \) consists of \( C_1, C_2, C_3 \) and \( C_4 \). (Figure 1 , see[ 7 ]). We can identify the “time-plane” \( C \) with \( \mathbb{R}_0 \times i\mathbb{R}_0 \), then
let \( C'' = \Pi(C') \), see Figure 2. It is easy to find
(1) If \( F \to 0 \), then \( C'' \) will be deformable in
\((S^1_0 \times \mathbb{R}_1) \cup \bigcup_{\tau=0,\pi}(\mathbb{R}_0 \times \{e^{i\tau}\} \times \mathbb{R}_1)\)
into a circle and the imaginary-time theory is obtained.
(2) If \( F \to +\infty \), then \( \phi_1 \) and \( \phi_2 \) are obtained, where \( \phi_2 \) is the tilde field.

5 An open problem

Because \( \eta - \xi \) space-time is homotopy equivalent to the torus, it has nontrivial
cohomology groups, precisely, \( H^1(V_0;\mathbb{Z}_2) = \mathbb{Z}_2 \oplus \mathbb{Z}_2, H^2(V_0;\mathbb{Z}) = \mathbb{Z} \). As a result
of these properties, there are nontrivial real and complex line bundles over \( V_0 \).
It seems to me that the physical significance of these facts is unknown.

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