Closing gaps in problems related to Hamilton cycles in random graphs and hypergraphs

Asaf Ferber *

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Abstract

We show how to adjust a very nice coupling argument due to McDiarmid in order to prove/reprove in a novel way results concerning Hamilton cycles in various models of random graph and hypergraphs. In particular, we firstly show that for $k \geq 3$, if $pn^{k-1}/\log n$ tends to infinity, then a random $k$-uniform hypergraph on $n$ vertices, with edge probability $p$, with high probability (w.h.p.) contains a loose Hamilton cycle, provided that $(k-1)n$. This generalizes results of Frieze, Dudek and Frieze, and reproves a result of Dudek, Frieze, Loh and Speiss. Secondly, we show that there exists $K > 0$ such for every $p \geq (K \log n)/n$ the following holds: Let $G_{n,p}$ be a random graph on $n$ vertices with edge probability $p$, and suppose that its edges are being colored with $n$ colors uniformly at random. Then, w.h.p the resulting graph contains a Hamilton cycle with which all the colors appear (a rainbow Hamilton cycle). Bal and Frieze proved the latter statement for graphs on an even number of vertices, where for odd $n$ their $p$ was $\omega((\log n)/n)$. Lastly, we show that for $p = (1 + o(1))(\log n)/n$, if we randomly color the edge set of a random directed graph $D_{n,p}$ with $(1 + o(1))n$ colors, then w.h.p. one can find a rainbow Hamilton cycle where all the edges are directed in the same way.

1 Introduction

In this paper we show how to adjust a very nice coupling argument due to McDiarmid [7] in order to prove/reprove problems related to the existence of Hamilton cycles in various random graphs/hypergraphs models. The first problem we consider is related to the existence of a loose Hamilton cycle in a random $k$-uniform Hypergraph.

A $k$-uniform hypergraph is a pair $H = (V, E)$, where $V$ is the set of vertices and $E \subseteq \binom{V}{k}$ is the set of edges. In the special case where $k = 2$ we simply refer to it as a graph and denote it by $G = (V, E)$. The random $k$-uniform hypergraph $H_{n,p}^{(k)}$ is defined by adding each possible edge with probability $p$ independently at random, where for the case $k = 2$ we denote it by $G_{n,p}$ (the usual binomial random graph). We define a loose Hamilton cycle as a cyclic ordering of $V$ for which the edges consist of $k$ consecutive vertices, and for each two consecutive edges $e_i$ and $e_{i+1}$ we have $|e_i \cap e_{i+1}| = 1$ (where we consider $n+1 = 1$). It is easy to verify that if $n$ is not divisible by $k-1$ then such a cycle cannot exist.

*Department of Mathematics, Yale University and Department of Mathematics, MIT. Emails: asaf.ferber@yale.edu and ferbera@mit.edu.
Frieze 4 and Dudek and Frieze 2 showed that for \( p = \omega(\log n/n) \), the random \( k \)-uniform hypergraph \( H_{n,p}^{(k)} \) w.h.p. (with high probability) contains a loose Hamilton cycle in \( H_{n,p}^{(k)} \) whenever \( 2(k-1)/n \). Formally, they showed:

**Theorem 1.1.** The following hold:

(a) (Frieze) Suppose that \( k = 3 \). Then there exists a constant \( c > 0 \) such that for \( p \geq (c \log n)/n \) the following holds

\[
\lim_{4|n| \to \infty} \Pr\left[H_{n,p}^{(3)} \text{ contains a loose Hamilton cycle}\right] = 1.
\]

(b) (Dudek and Frieze) Suppose that \( k \geq 4 \) and that \( p n^{k-1}/\log n \) tends to infinity. Then

\[
\lim_{2(k-1)|n| \to \infty} \Pr\left[H_{n,p}^{(k)} \text{ contains a loose Hamilton cycle}\right] = 1.
\]

The assumption \( 2(k-1)|n \) is clearly artificial, and indeed, in [3] Dudek, Frieze, Loh and Speiss removed it and showed analog statement to [1] where there the only restriction on \( n \) is to be divisible by \( k-1 \) (which is optimal).

As a first result in this paper, we give a very short proof for the result of Dudek, Frieze, Loh and Speiss while weakening (a) a bit. Formally, we prove the following theorem:

**Theorem 1.2.** The following hold:

(a) Suppose that \( k = 3 \). Then for every \( \varepsilon > 0 \) there exists a constant \( c > 0 \) such that for \( p \geq (c \log n)/n \) the following holds

\[
\lim_{2|n| \to \infty} \Pr\left[H_{n,p}^{(3)} \text{ contains a loose Hamilton cycle}\right] \geq 1 - \varepsilon.
\]

(b) Suppose that \( k \geq 4 \) and that \( p n^{k-1}/\log n \) tends to infinity. Then

\[
\lim_{(k-1)|n| \to \infty} \Pr\left[H_{n,p}^{(k)} \text{ contains a loose Hamilton cycle}\right] = 1.
\]

Another problem we handle with is the problem of finding a rainbow Hamilton cycle in a randomly edge-colored random graph. For an integer \( c \), let us denote by \( G_{n,p}^c \) the random graph \( G_{n,p} \), where each of its edges is being colored, uniformly at random with a color from \([c]\). A Hamilton cycle in \( G_{n,p}^c \) is called rainbow if all its edges receive distinct colors. Clearly, a rainbow Hamilton cycle can not exists whenever \( c < n \). Bal and Frieze 11 showed that for some constant \( K > 0 \), if \( p \geq (K \log n)/n \), the \( G_{n,p}^c \) w.h.p. contains a rainbow Hamilton cycle, provided that \( n \) is even. For the odd case, they proved similar statement but for \( p = \omega((\log n)/n) \). We overcome this and show the following:

**Theorem 1.3.** There exists a constant \( K > 0 \) such that \( G_{n,p}^n \) w.h.p. contains a rainbow Hamilton cycle.

It is well known (see e.g. [6]) that a Hamilton cycle appear (w.h.p.) in \( G_{n,p} \) for \( p \approx (\log n)/n \). Therefore, one would expect to prove an analog for Theorem 1.3 in this range of \( p \). However, it is easy to see that in this range, while randomly color the edges of \( G_{n,p} \) with \( n \) colors, w.h.p. not all the colors appear. Frieze and Loh 5 proved that for \( p = (1+\varepsilon)(\log n)/n \) and for \( c = n + \Theta(n/\log \log n) \),
a graph $G^c_{n,p}$ w.h.p. contains a rainbow Hamilton cycle. It is thus natural to consider the same problem for a randomly edge-colored directed random graph, denoted by $D^c_{n,p}$ (we allow edges to go in both directions). Note that in directed graphs we require to have a directed Hamilton cycle, which is a Hamilton cycle with all arcs pointing to the same direction.

The following theorem will follow quite immediately:

**Theorem 1.4.** Let $p = (1 + \varepsilon)(\log n)/n$ and let $c = n + \Theta(n/\log \log n)$. Then $D^c_{n,p}$ w.h.p. contains a rainbow Hamilton cycle.

Our proof is based on a very nice coupling argument due to McDiarmid [7] and on Theorem 1.1.

## 2 Auxiliary results

In this section we present some variants of a very nice argument by McDiarmid [7]. For the convenience of the reader we add a proof for one of them, and the rest will be left as easy exercises. Before stating our lemmas, let us define the directed random $k$-uniform hypergraph $D^{(k)}_{n,p}$ in the following way: Each ordered $k$-tuple $(x_1, \ldots, x_k)$ consisting of $k$ distinct elements of $[n]$ appears as an arc with probability $p$, independently at random. In the special case where $k = 2$ we simply write $D_{n,p}$. A directed loose Hamilton cycle is a loose Hamilton cycle where consecutive vertices are now arcs of $D^{(k)}_{n,p}$ and the last vertex of every arc is the first of the consecutive one. In the following lemma we show that the probability for $D^{(k)}_{n,p}$ to have a directed loose Hamilton cycle is lower bounded by the probability for $H^{(k)}_{n,p}$ to have one.

**Lemma 2.1.** Let $k \geq 3$. Then, for every $p := p(n) \in (0,1)$ we have

$$
\Pr \left[ D^{(k)}_{n,p} \text{ contains a directed loose Hamilton cycle} \right] \geq \Pr \left[ H^{(k)}_{n,p} \text{ contains a loose Hamilton cycle} \right].
$$

**Proof.** (McDiarmid) Let us define the following sequence of random directed hypergraphs $\Gamma_0, \Gamma_1, \ldots, \Gamma_N$, where $N = \binom{n}{k}$ in the following way: Let $e_1, \ldots, e_N$ be an arbitrary enumeration of all the (unordered) $k$-tuples contained in $[n]$. For each $e_i$ one can define $k!$ different orientations. Now, in $\Gamma_i$, for every $j \leq i$ and for each of the $k!$ possible orderings of $e_j$, we add the corresponding arc with probability $p$, independently at random. For every $j > i$, we include all possible orderings of $e_j$ or none with probability $p$, independently at random. Note that $\Gamma_0$ is $H^{(k)}_{n,p}$ while $\Gamma_N$ is $D^{(k)}_{n,p}$. Therefore, in order to complete the proof it is enough to show that

$$
\Pr \left[ \Gamma_i \text{ contains a directed loose Ham. cycle} \right] \geq \Pr \left[ \Gamma_{i-1} \text{ contains a directed loose Ham. cycle} \right].
$$

To this end, assume we exposed all arcs but those coming from $e_i$. There are three possible scenarios:

(a) $\Gamma_{i-1}$ contains a directed loose Hamilton cycle without considering $e_i$, or

(b) $\Gamma_{i-1}$ does not contain a directed loose Hamilton cycle even if we add all possible orderings of $e_i$, or

(c) $\Gamma_{i-1}$ contains a directed loose Hamilton cycle using at least one of the orderings of $e_i$. 

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Note that in (a) and (b) there is nothing to prove. In case (c), the probability for $\Gamma_{i-1}$ to have a directed loose Hamilton cycle is $p$, where the probability for $\Gamma_i$ to have such a cycle is at least $p$. This completes the proof of the lemma.

In the second lemma, we show that given an integer $c$, one can lower bound the probability of $D_{n,p}^c$ to have a rainbow directed Hamilton cycle by the probability of $G_{n,p}^c$ to have such a cycle.

**Lemma 2.2.** Let $c$ be a positive integer. Then, for every $p := p(n) \in (0, 1)$ we have

$$\Pr[D_{n,p}^c \text{ contains a rainbow directed Hamilton cycle}] \geq \Pr[G_{n,p}^c \text{ contains a rainbow Hamilton cycle}].$$

Note that by combining the result of Bal and Frieze [1] with Lemma 2.2 we immediately obtain the following corollary:

**Corollary 2.3.** There exists a constant $K > 0$ such that for every $p \geq (K \log n)/n$ we have

$$\Pr[D_{n,p}^n \text{ contains a rainbow Hamilton cycle}] = 1,$$

provided that $n$ is even.

### 3 Proofs of our main results

In this section we prove Theorems 1.2, 1.3 and 1.4. We start with proving Theorem 1.2.

**Proof of Theorem 1.2** Suppose that $(k - 1)|n$ and that $2(k - 1)$ does not divide $n$. Let $f^2(n)$ be a function that tends arbitrarily slowly to infinity and suppose that $p = \frac{f^2(n) \log n}{n^{k-1}}$. Note that by deleting the orderings of a $D_{n,q}^{(k)}$, using a similar argument as in a multi-round exposure (we refer the reader to [6] for more details), we obtain a $H_{n,s}^{(k)}$ where $(1 - q)^{kl} = 1 - s$ (one can just think about $D_{n,q}^{(k)}$ as an undirected hypergraph such that for every $e \in \binom{[k]}{k}$ there are $k!$ independent trials to decide whether to add it).

Now, let us choose $q$ in such a way that $(1-p/2)(1-q)^{kf(n)} = 1-p$, and observe that $q \geq \frac{p}{2k!f(n)} = \omega \left( \log n/n^{k-1} \right)$. We generate $H_{n,p}^{(k)}$ in a multi-round exposure and present it as a union $\bigcup_{i=0}^{f(n)} H_i$, where $H_0$ is $H_{n,p}^{(k)}$ and $H_i$ is $D_{n,q}^{(k)}$ (which, as stated above, is like $H_{n,s}^{(k)}$ with $(1-q)^{kl} = 1-s$) for each $1 \leq i \leq f(n)$ (of course, ignoring the orientations). In addition, all the $H_i$’s are considered to be independent.

Our strategy goes as follows: First, take $H_0 = H_{n,p}^{(k)}$ and pick an arbitrary edge $e^* = \{x_1, \ldots, x_k\}$ (trivially, $H_0$ contains an edge w.h.p.). Now, fix an arbitrary ordering $(x_1, \ldots, x_k)$ of $e^*$ and let $V^* = ([n] \setminus \{x_1, \ldots, x_k\}) \cup \{e^*\}$ (that is, $V^*$ is obtained by deleting all the elements of $e^*$ and adding an auxiliary vertex $e^*$). For each $i \geq 1$, whenever we expose $H_i$ we define an auxiliary $k$-uniform directed random hypergraph $D_i$ on a vertex set $V^*$ in the following way. Every arc $e$ of $H_i$ is being added to $D_i$ if it satisfies one of the following:

- $e \cap e^* = \emptyset$, or
- $e \cap e^* = \{x_1\}$, and $x_1$ is not the first vertex of the arc $e$, or
- $e \cap e^* = \{x_k\}$ and $x_k$ is the first vertex of the arc $e$. 

This completes the proof of the lemma.
Note that indeed, by definition, every \( k \)-tuple of \( V^* \) now appear with probability \( p \), independently at random and that \( |V^*| = n - (k-1) \). Therefore, we clearly have that each of the \( D_i \)'s is an independent \( D_{n-(k-1),q}^n \). Moreover, note that \( 2(k-1)/n \) and that each directed loose Hamilton cycle of \( D_i \) with the special vertex \( e^* \) as a starting/ending vertex of the edges touching it corresponds to a (undirected) loose Hamilton cycle of \( H_{n,p}^n \). To see the latter, suppose that \( e^*v_2 \ldots v_te^* \) is such a cycle in \( D_i \). Now, by definition we have that both \( x_kv_2 \ldots v_k \) and \( v_{t-k+2} \ldots v_te^* \) are arcs of \( H_i \), and therefore, by replacing \( e^* \) with its entries \( x_1 \ldots x_k \), one obtains a loose Hamilton cycle in \( H_i \).

Next, by combining Theorem 1.1 with Lemma 2.1, we observe that w.h.p. \( D_i \) contains a directed loose Hamilton cycle. Note that by symmetry we have that the probability for \( e^* \) to be an endpoint of an edge on the Hamilton cycle is \( 2/k \). Therefore, after exposing all the \( D_i \)'s, the probability to fail in finding such a cycle is \((1 - 2/k)^{f(n)} = o(1)\) as desired. This completes the proof.

Next we prove Theorem 1.3.

**Proof of Theorem 1.3.** Let us assume that \( n \) is odd (since otherwise there is nothing to prove) and that \( K > \) is a sufficiently large constant for our needs. Now, let \( q \) be such that \((1 - p/2)(1 - q)^2 = 1 - p\), and present \( G_{n,p}^n \) as a union \( G_1 \cup G_2 \), where \( G_1 = G_{n,p/2}^n \) and \( G_2 = D_{n,q}^n \) (as in the proof of Theorem 1.2, by ignoring orientations one can see \( D_{n,q}^n \) as \( G_{n,s}^n \) with \( s \) satisfying \((1 - q)^2 = 1 - s\)). Next, let \( e^* = (x, y) \) be an arbitrary edge of \( G_1 \) (trivially, w.h.p. there exists an edge), let \( c_1 \) denote its color, and define an auxiliary edge-colored random directed graph \( D \) as follows. The vertex set of \( D \) is \( V^* = ([n] \setminus \{x, y\}) \cup \{e^*\} \) (that is, we delete \( x \) and \( y \) and add an auxiliary vertex \( e^* \)). The arc set of \( D \) consist of all arcs \( uv \) of \( G_2 \) with colors distinct than \( c_1 \) for which one of the following holds:

- \( \{u, v\} \cap \{x, y\} = \emptyset \), or
- \( v = x \), or
- \( u = y \).

A moment’s thought now reveals that \( D \) is \( D_{n-1,s}^{n-1} \), where \( s = (1 - 1/n)q \), that \( n-1 \) is even, and that a rainbow Hamilton cycle of \( D \) corresponds to a rainbow Hamilton cycle of \( G_{n,p}^n \). Now, since \( s \geq (K' \log n)/n \) for some \( K' \) (we can take it to be arbitrary large), it follows from Corollary 2.3 that w.h.p. \( D \) contains a rainbow Hamilton cycle, and this completes the proof.

Lastly, we prove Theorem 1.4.

**Proof of Theorem 1.4.** The proof is an immediate corollary of the result of Frieze and Loh [5] and Lemma 2.2.

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