On slip pulses at a sheared frictional viscoelastic/ non deformable interface

C. Caroli

Groupe de Physique des Solides, 2 place Jussieu, 75251 Paris cedex 05, France.

(November 13, 2018)

Abstract

We study the possibility for a semi-infinite block of linear viscoelastic material, in homogeneous frictional contact with a non-deformable one, to slide under shear via a periodic set of “self-healing pulses”, i.e. a set of drifting slip regions separated by stick ones. We show that, contrary to existing experimental indications, such a mode of frictional sliding is impossible for an interface obeying a simple local Coulomb law of solid friction. We then discuss possible physical improvements of the friction model which might open the possibility of such dynamics, among which slip weakening of the friction coefficient, and stress the interest of developing systematic experimental investigations of this question.

46.35.+z, 46.50.+a, 62.20.Mk, 81.40.Pq, 83.50.Fc
I. INTRODUCTION

A few recent qualitative observations \[1\], \[2\] on the frictional motion of sheared gels sliding along smooth glass surfaces point towards the existence of inhomogeneous modes of frictional sliding. Namely, in some limited range of values of -small- shearing rates, sliding seems to occur via propagation of a quasi-periodic pattern of sliding zones of finite width, separated by non moving regions, where the interface sticks. These “slip pulses” drift at velocities $c \simeq \text{mm/sec}$, while the remote average (pulling) velocity lies in the $1 - 10\mu\text{m/sec}$ range. Their width is typically tens of micrometers. Analogous observations have been made by Brune et al \[3\] on a sliding rubber foam, and by Mouwakeh et al. \[4\] on the elastomer polyurethane.

The topology of such sliding modes is reminiscent of that of Schallamach waves \[5\], which have been documented \[6\] in the case of some very compliant transparent rubbers sliding on smooth glass. They consist of quasi periodic zones, of width typically $l \simeq 100\mu\text{m}$, with space periods roughly $\sim 10l$ \[7\], where the rubber buckles, so that the two surfaces get separated by a distance comparable with $l$. These separation waves have drift velocities $\sim \text{mm/sec}$, for remote velocities $\sim \mu\text{m/sec}$.

However, the slip pulses in gels do not seem to be associated with any interface separation. In this respect, they are more comparable with the so-called “self-healing slip pulses”, on which the attention of mechanicians has been focussing recently \[8\], following the suggestion by Heaton \[9\] that some major seismic events may have occurred, not by quasi simultaneous sliding of the whole rupture zone, but via fast propagation of localized sliding zones of small extent.

These observations all point towards a common question about the nature of frictional sliding, which can be schematized as follows. Consider two very thick blocks of solid materials with dissimilar elastic properties, in frictional contact along a planar interface of infinite lateral extent (Figure 1). This system bears a remote homogeneous compressive stress $\tau_{22}^*$, normal to the interface. Assume that, under the remote shear stress $\tau_{12}^*$, the upper block (I)
slides towards \( x_1 > 0 \) at a remote point velocity \( v_0 \) with respect to the lower one (II). Such motion can of course occur in a homogeneous mode, where stresses are uniform. Along the (homogeneous) interface, the friction law, which we assume to obey the Amontons-Coulomb proportionality between shear and normal loads, imposes that

\[
\tau_{12}^* = -f_d(v_0)\tau_{22}^*
\]  

(1)

If such is the case, given \( \tau_{22}^* \) and \( v_0 \), the remote shear sliding stress is fixed. In the Coulomb approximation, where fine variations of the dynamic friction coefficient are ignored, \( f_d \) reduces to a constant.

The question then arises of whether or not this homogeneous sliding mode is stable with respect to small non homogeneous perturbations of the stress and strain fields localized in the surface region. In other words, do deformation waves exist along a sliding frictional interface? If so, are they damped, or amplified, or neutral? This question has been studied extensively, for dissimilar linear elastic materials, with Coulomb friction, by several authors, in particular Weertmann [10], Adams [11], and Martins et al [12], whose results are synthetized in a recent article by Ranjith and Rice [8]. They find that, when \( \mu_d \neq 0 \) and when such interface waves exist [13], the corresponding sliding velocity field along the interface has, for a mode of wavelength \( k \), the form :

\[
v(x_1) = v_k \exp \left[ ik (x_1 - ct) + a \right]
\]  

(2)

Given the elastic moduli, the drift velocity \( c \) and amplification coefficient \( a \) are real positive constants. That is, waves drifting along (resp. against) the direction of \( v_0 \) are amplified (resp. damped). Homogeneous sliding is thus linearly unstable against perturbations of all wavelengths.

These results are derived under the assumption that the interface is sliding everywhere. As amplification proceeds, the sliding velocity necessarily vanishes at some points. This suggests that sliding might occur via a periodic set of self healing slip pulses, separated by stick regions. A family of such pulses has been built by Adams [11] for dissimilar elastic solids.
Their drift velocity \( c \), which depends on the values of elastic moduli, is, roughly speaking, on the order of a sound velocity. So, their dynamics is controlled by inertia. However, such self-sustaining (stationary) dynamical patterns are singular in the following sense. Since perturbations of all wavelengths, however small, are amplified, any initially localized perturbation gives rise to diverging oscillations at arbitrary small time: Adams’s pulses have zero measure attractors. This so-called “ill-posedness” most likely signals that the Coulomb friction law misses some of the physical processes which control the fast dynamics of fracture at frictional interfaces between elastic materials, i.e. their high frequency response - a problem which is currently under study [8].

Slip pulses in gels or rubbers, on the contrary, are slow dynamical objects whose velocities, comparable with those of Schallamach waves, are much too low for inertia to be relevant. Their dynamics is certainly controlled by the dissipation associated with the viscoelasticity of these materials.

We therefore concentrate, in this article, on the following question. Let block (I) be an incompressible linear viscoelastic material with, for simplicity, a single viscous relaxation time. It slides slowly on a smooth non-deformable material, and interface friction obeys a simple local Coulomb law. Under such conditions, are non inertial periodic slip pulses, stationary in the drifting frame, a possible mode of motion?

In Section II we formulate the corresponding mathematical problem, and derive the form of its analytical solutions. We show in Section III that none of these is compatible with the stick conditions to be satisfied in the non moving parts of the interface. Hence, in this as well as in the inertial regime, a Coulomb law with a constant dynamical friction coefficient is incompatible with the existence of such modes of motion. We discuss in Section IV possible physical tracks towards improvements of the simple Coulomb model, which might be relevant to the problem of inhomogeneous sliding, and stress the interest of corresponding experimental investigations.
II. GENERAL FORMULATION

We follow closely the approach of Adams [11] and of Comninou and Dundurs [14] restricted to the case where (see Figure 1) block (II) \((x_2 < 0)\) is non deformable. Block (I) is submitted to the uniform remote stresses \(\tau_{22}^* < 0\) and \(\tau_{12}^*\). It is infinitely extended along \(x_1\), and made of an incompressible material, with a linear viscoelastic shear response described by:

\[
\tau_{12}(t) = \int_{-\infty}^{t} dt' \mu(t - t') \dot{u}_{12}(t')
\]

\(\tau_{12}\) and \(u_{12}\) are the shear stress and deformation, confined to the \((x_1, x_2)\) plane, and we model the time dependent shear modulus as a single time Kelvin one, namely:

\[
\mu(t) = \mu_\infty + (\mu_0 - \mu_\infty) e^{-t/\tau}
\]

We will moreover assume that the relaxed modulus \(\mu_\infty\) is much smaller than the short time one, \(\mu_0\). To fix ideas, for compliant rubbers, values of \(\mu_\infty/\mu_0 \lesssim 10^{-3}\) are typical.

We want to study dynamic patterns, where (I) slides towards \(x_1 > 0\) with the uniform remote velocity \(v_0\), with space period \(\lambda = 2\pi/k\), and drift velocity \(c\) in the frame of (II). The corresponding form of the displacements \(u_1, u_2\) reads:

\[
u_1 = v_0 t + \sum_{m \geq 1} D_{m1}(x_2) e^{imk(x_1 - ct)}
\]

\[
u_2 = D_{02} + \sum_{m \geq 1} D_{m2}(x_2) e^{imk(x_1 - ct)}
\]

Solving the wave propagation equation together with the condition of non separation at the interface: \(u_2(x_1, x_2 = 0, t) = 0\), one obtains straightforwardly (see Appendix A), in the incompressible limit (Poisson coefficient \(\nu = 1/2\)) for the interfacial sliding velocity \(v_s(\eta) = \partial u_1/\partial t |_{x_2=0}\):

\[
v_s(\eta) = v_0 + cRe \sum_{m \geq 1} B_m e^{im\eta}
\]
where $\eta = k (x_1 - ct)$ while the interfacial shear and vertical stresses read:

$$\tau_{12} (\eta) = \tau_{12}^* + \text{Re} \sum_{m \geq 1} -iB_m \mu_m (1 + \sigma_m) e^{im\eta}$$  

(8)

$$\tau_{22} (\eta) = \tau_{22}^* + \text{Re} \sum_{m \geq 1} B_m \mu_m (1 - \sigma_m) e^{im\eta}$$  

(9)

with

$$\sigma_m = \left[ 1 - \frac{\rho c^2}{\mu_m} \right]^{1/2} \quad \text{Re} \sigma_m > 0$$  

(10)

We restrict our attention to the very slow modes observed experimentally, for which, whatever the frequency $(mck)$, $\rho c^2 << \mu_m$, so that

$$\sigma_m \simeq 1 - \frac{\rho c^2}{2\mu_m}$$  

(11)

$\rho$ is the mass density of material (I), $\mu_m$ its (complex) elastic modulus at frequency $(mck)$

$$\mu_m = \hat{\mu} (mck) = \left[ -i\omega \int_0^\infty dt \mu (t) e^{i\omega t} \right]_{\omega=mck}$$  

(12)

The unknown coefficients $B_m$ must be determined from the second boundary condition along the interface, which we want to describe a set of slipping regions of length $2l$ separated by sticking ones.

We assume friction to be described by a simple local Coulomb law, with a constant dynamic friction coefficient $f$ equal to the static one, that is:

(i) Slip regions : $-\alpha + 2p\pi < \eta < \alpha + 2p\pi$

$$\tau_s (\eta) = \tau_{12} (\eta) + f \tau_{22} (\eta) = 0 \quad v_s > 0$$  

(13)

(ii) Stick regions : $-\alpha + 2p\pi < \eta < -\alpha + 2(p + 1)\pi$

$$v_s = 0 \quad f \tau_{22} < \tau_{12} < -f \tau_{22}$$  

(14)

This description of interface friction calls for a few comments. Indeed, it assumes tacitly - as is common in contact mechanics [15] - that one can legitimately define a local and
space-independent friction coefficient. Since solid friction results from the average effect of disipative flips of bistable pinned elastic units, this can be true only on a scale much larger than (i) the size $b$ of the basic unit, and (ii) the scale $L$ of interface inhomogeneities. The detailed analysis [16] of the Rice-Ruina phenomenological law of dynamic friction [17] has shown that $b$ is of nanometric order. So, our assumption is justified for interfaces with homogeneous intimate contact. Such is indeed the case for the gels or very compliant rubbers which we have in mind here, as long as elastic deformations vary on scales much larger than nanometers – which sets a lower limit on the size of Dugdale-Barenblatt-like fracture head regions.

Note, however, that the situation is different when dealing with multicontact Greenwood-like interfaces [18]. These prevail with stiff materials, such as metals, glasses or rocks, which are not polished down to nanometric roughness. Then, the small scale cutoff is provided by the average distance between contacting asperities, commonly lying in the 100$\mu$m/sec range. This, in our opinion, should be kept in mind when attempting, for such interfaces, to regularize the above mentioned ill-posedness problem, since, on space scales $\lesssim L$ pinning strength fluctuations become non negligible.

Taking condition (11) into account we set, following Comninou and Dundurs [14], for the periodic function $v_s(\eta)$, in $-\pi \leq \eta \leq \pi$:

\begin{align}
    v_s(\eta) &= 0 & \alpha < |\eta| < \pi & \text{(stick)} \\
    v_s(\eta) &= v(\eta) & -\alpha < \eta < \alpha & \text{(slip)}
\end{align}

Hence, from equation (8):

\begin{align}
    B_m &= \frac{1}{\pi c} \int_{-\alpha}^{\alpha} d\xi \, v(\xi) e^{-im\xi} & (m \geq 1) \\
    v_0 &= \frac{1}{2\pi} \int_{-\alpha}^{\alpha} d\xi \, v(\xi)
\end{align}

Using equations (15), (16) together with (8), (9), and with the help of the relation
\[ \sum_{m \geq 1} e^{imx} = -\frac{1}{2} + \sum_{n=-\infty}^{\infty} \delta(x - 2n\pi) + \frac{i}{2} PV \left[ \cot g \frac{x}{2} \right] \]  

(19)

(where \( PV \) designates the Cauchy principal value), one finally gets, for the interfacial “sliding stress” \( \tau_s \)

\[ \tau_s(\eta) = \tau_s^* + \frac{f\rho c^2}{2} [V(\eta) - V_0] - \frac{1}{\pi} \int_{-\alpha}^{\alpha} d\xi V(\xi) \int_{0}^{\infty} dt \mu(t) \frac{d}{dt} \left[ \cot g \frac{\eta - \xi + ckt}{2} \right] \]  

(20)

where we have set \( V(\eta) = v(\eta)/c \), and \( \int f \, dx \equiv PV \int f \, dx \).

Integrating by parts the last term in the r.h.s. of Eq. (16), the condition Eq. (11) for frictional sliding within the slip pulses provides us with the integral equation to be satisfied by the interfacial reduced velocity field in \((-\alpha < \eta < \alpha\) ), namely :

\[ \tau_s^* + \frac{f\rho c^2}{2} [V(\eta) - V_0] + \frac{\mu_0}{\pi} \int_{-\alpha}^{\alpha} d\xi V(\xi) \cot g \frac{\eta - \xi}{2} + \frac{1}{\pi} \int_{-\alpha}^{\alpha} d\xi V(\xi) \int_{0}^{\infty} dt \frac{d\mu}{dt} \left[ \cot g \frac{\eta - \xi + ckt}{2} \right] = 0 \]

(21)

with:

\[ \int_{-\alpha}^{\alpha} d\xi V(\xi) = 2\pi V_0 \]  

(22)

Once Eqs. (17) are solved for \( V(\eta) \), interfacial stresses in the stick regions should be calculated from Eq. (16), Eq. (11) then providing the final condition for slip pulses to exist.

Expression (21) separates explicitly the instantaneous elastic shear effects (3\(^{rd}\) term) from the contribution of viscoelastic relaxation (4\(^{th}\) term). The second term, which derives from the perturbation of the normal stress \( \tau_{22} \), is, for our very slow pulses, smaller than the integral ones by a factor \( (c/c_s)^2 \), where \( c_s \) is some sound velocity. We will therefore neglect it from now on.

The cotg form of the elastic kernels results from imposing space periodicity to the patterns. Eq. (21) can be rewritten in a form more standard in fracture mechanics by setting, in \((-\pi < \eta < \pi\)):

\[ u = \tan \frac{\eta}{2} \quad y = \tan \frac{\xi}{2} \quad \Phi(u) = \frac{V(u)}{1 + u^2} \quad a = \tan \frac{\alpha}{2} = \tan \frac{kl}{2} \]

(23)
Some elementary algebra then leads, in \((-a < u < a)\), to:

\[
\frac{\tau_s}{1+u^2} = \frac{2\mu_0}{\pi} \int_{-a}^{a} dy \frac{\Phi(y)}{u-y} + \int_{-a}^{a} dy \Phi(y)k(u,y) - \frac{2V_0 u - \tau_s^*}{1+u^2} = 0 \tag{24}
\]

with:

\[
\int_{-a}^{a} dy \Phi(y) = \pi V_0 \tag{25}
\]

and we have set:

\[
k(u,y) = \frac{2}{\pi} \int_{0}^{\infty} dt \mu'(t) \frac{1 + ytg(\frac{ckt}{2})}{u - y + (1 + uy)tg(\frac{ckt}{2})} \tag{26}
\]

The singular integral equation \((24)\) belongs to a class which was studied extensively by Mushkelishvili [19]. In his terminology, the first two terms on the l.h.s. constitute the “dominant” part. The viscoelastic kernel \(k(u,y)\) satisfies the regularity condition:

\[
\lim_{u \to y} [(u - y)k(u,y)] = 0.
\]

This entails that its plays no role in the strength of the singularities of the solutions – i.e., as is intuitively reasonable, these are ruled by the instantaneous elastic response of the deformable medium.

Following reference [19], there are four families of solutions of equation \((24)\), each of which is associated with one of the basic functions, characteristic of the dominant part:

\[
Z_{\epsilon_t,\epsilon_h}(y) = (y + a)^{\epsilon_t/2}(a - y)^{\epsilon_h/2} \tag{27}
\]

where the indices \((\epsilon_t,\epsilon_h) = (+1, -1)\) control the convergent or divergent behavior of \(Z\) at the tail and head edges of the slip zone.

One can then transform, for each family, Eq.(24) into an equivalent non-singular Fredholm integral equation. It moreover turns out that, when we specialize to the single relaxation time model for \(\mu(t)\) (Eq.(4)), analytical expressions for the solutions of these equations can be obtained explicitly, thus allowing us to draw explicit conclusions about their existence.

In view of the heaviness of the (otherwise straightforward) algebra involved, we will exemplify the method in full detail only for one of the families, namely the \((+-)\) one.
III. THE FOUR FAMILIES OF SOLUTIONS: V-FIELDS AND EXISTENCE CONDITIONS

A. The \((+-)\) family

The corresponding basic function

\[
Z_{+-}(y) = \sqrt{(y + a) / (a - y)}
\]  

(28)

The implementation of Mushkelishvili's method is performed in Appendix B. For \(\mu(t)\) as specified by Eq.(4), the non singular equation equivalent to Eq.(24) reads:

\[
\Phi_{+-}(u) = H_{+-}(u) + \frac{\Delta \mu}{\mu_0} \frac{2\exp(2\tan^{-1}u/ck\tau)}{ck\tau(1 + u^2)} \int_u^a dz \Phi_{+-}(z) \exp\left(-2\tan^{-1}z/ck\tau\right)
\]

(29)

with:

\[
H_{+-}(u) = -\frac{Z_{+-}(u)}{2\mu_0} \frac{C^*u + D^*}{1 + u^2} + \frac{\Delta \mu}{\mu_0} \frac{W_{+-}}{ck\tau} \left[ \frac{Z_{+-}(u)}{\pi} G(u) + \frac{\beta}{1 + u^2} \exp(2\tan^{-1}u/ck\tau) \right]
\]

(30)

\[
W_{+-} = 2 \int_{-a}^a dz \Phi(z) \exp\left(-\frac{2\tan^{-1}z}{ck\tau}\right)
\]

(31)

\[
G(u) = -\beta \int_{\Psi}^{\infty} du \frac{e^{2\tan^{-1}u/ck\tau}}{(u - \Psi)(1 + \Psi^2)} \sqrt{\Psi - a / (\Psi + a)} - (1 + \beta) \int_{-\infty}^{-\Psi} du \frac{e^{2\tan^{-1}u/ck\tau}}{(u - \Psi)(1 + \Psi^2)} \sqrt{-\Psi + a / -\Psi - a}
\]

(32)

\[
\beta = (e^{2\pi/ck\tau} - 1)^{-1} \quad \Delta \mu = \mu_0 - \mu_{\infty}
\]

(33)

\[
C^* = 2V_0\mu_\infty \cos \frac{\alpha}{2} - \tau_s^* \sin \frac{\alpha}{2}
\]

(34)

\[
D^* = -2V_0\mu_\infty \sin \frac{\alpha}{2} - \tau_s^* \cos \frac{\alpha}{2}
\]

(35)

Expression(32) for \(G(u)\) is valid for solutions whose slip zone length \(2l\) satisfies the condition

\[ 2l < \frac{\lambda}{2} \]

We assume this to hold, in accordance with experimental observations, which indicate values of \(2l/\lambda < < 1\).
Eq. (29) is a first order differential equation for the function \( \int_u^a dz \Phi_+(z) \exp[-2\tan^{-1}z/ck\tau] \), which is straightforwardly solved into:

\[
\Phi_+(u) = H_+(u) + \frac{2}{ck\tau} \frac{\Delta \mu}{\mu_0} \frac{1}{1+u^2} \int_u^a dz \ H_+(z) \exp \left[ \frac{\mu_{\infty}}{\mu_0} \frac{2(tan^{-1}u - tan^{-1}z)}{ck\tau} \right]
\] (36)

This defines a family of slip velocity fields, each of which is labelled by the four dimensionless parameters \( V_0 = v_0/c, \tau_s^*/\mu_0, l/\lambda = a/2\pi, c\tau/\lambda \). Two of the physical parameters, \( v_0 \) and \( \tau_s^* \), are “external” : in an experiment, one imposes in general an average sliding velocity – hence \( v_0 \) is fixed – and measures \( \tau_s^* \). \( l, c, \lambda \), are the internal parameters of the family. This defines a problem of dynamical selection, namely : if sliding patterns exist, are \( l, c, \lambda \), and hence \( \tau_s \), uniquely defined when \( v_0 \) is fixed, or not? In order to clear up this important question, it is necessary to list the relations between them - or, alternately, the conditions to be satisfied by \( \Phi_+ \) as given by Eq.(36). These are:

(i) Two consistency conditions, expressing that the remote velocity and stresses are simply the \( k=0 \) components of the corresponding fields. This is expressed by relation (25) and by an analogous equation for \( \tau_s \):

\[
\tau_s^* = \frac{1}{\pi} \int_{-\infty}^{+\infty} du \ \frac{\tau_s(u)}{1+u^2}
\] (37)

where \( \tau_s(u) \) is related to \( \Phi_+(u) \) by the first of equalities (24).

(ii) The interfacial stress field must also satisfy the stick inequality Eq.(14). One easily checks, with the help of Eq.(24), that a divergence of \( \Phi \) at an edge, \( u = \pm a \), of the slip zone results in a diverging \( \tau_s \) at the corresponding stick zone edge, and therefore in the violation of the stick condition. \( Z_+ \) (Eq.(28)) diverges at the slip head \( u = a \). For \( u \to a \):

\[
\Phi_+(u) = \frac{Z_+(u)}{2\mu_0} \left[ -\frac{C * a + D*}{1+a^2} + \frac{2\Delta \mu}{\pi} \frac{W_+}{ck\tau} G(a) \right] + \Re(u)
\] (38)

with \( \lim_{u \to a} \Re(u) = 0 \).

A necessary condition for \( \Phi_+ \) to be acceptable is that the coefficient of \( Z_+ \) in Eq.(38) vanishes, i.e., using Eqs.(29,35) :
\[ \tau_s^* \cos \frac{\alpha}{2} = -\frac{2}{\pi} \Delta \mu \frac{W_{+-} G(a)}{ck\tau} \]  

(39)

So, for solutions of the \((+-)\) class, the five pattern parameters are linked by three relations. That is, for a given \(v_0\), this class of patterns, if they exist, form a one parameter family. We will comment further on this conclusion in section IV.

Let us now come back to the “regularization condition” Eq. (39). From conditions (15) and (16), the interfacial sliding stress \(\tau_s\) must be non positive everywhere. Hence, its \(u\)-average \(\tau_s^*\), must be strictly negative.

On the other hand, the \(v_s\), and thus the \(\Phi\) field must, by Eq. (13), be positive everywhere in the slip zone. Then definition (31) entails that \(W_{+-} > 0\). Finally, using Eq. (32), one gets:

\[ G(a) = -\int_a^\infty d\Psi \frac{\text{sh}\left[ \frac{ck\tau}{\pi} \left( \frac{\Psi}{2} - \tan^{-1}\Psi \right) \right]}{(1 + \Psi^2)\sqrt{\Psi^2 - a^2} \text{sh} \left( \frac{\pi}{ck\tau} \right)} < 0 \]  

(40)

Therefore, condition (39) can never be satisfied. No solution of type \((+-)\) exists. In other words, viscous relaxation and pulse-pulse interaction effects can never be sufficient to cancel the square-root singularity due to the instantaneous elastic response.

B. The \((-+)\) and \((--)\) families

For the \((-+)\) family, whose \(Z\) function diverges at the slip zone tail only, the analysis parallels completely the above one, the \(\Phi_{+-}\) fields obey a set of equations with exactly the same structure as that of Eqs. (29-32), differing only in the detailed algebraic expressions of \(G(u), C^*, D^*\). The regularization condition analogous to Eq. (39), now to be imposed at \(u \to -a\), is again immediately shown to have no solutions.

\(Z_{--}\) diverges at both slip edges, hence two regularization conditions, and one shows similarly that the condition obtained from their difference cannot be satisfied. Note, however, that the counting argument tells us that, if \((--)\) patterns could exist, one only of the five parameters would be free, i.e. there could exist at most one dynamical pattern at a given sliding velocity.
C. The (++) family

As \( Z_{++} \) vanishes at both slip zone edges, no regularity condition has to be imposed \[20\]. (+++) solutions, if any, form a two parameter family. The analysis of Appendix B again leads to an expression of \( \Phi_{++} \) with the same structure as Eqs. \[29-32\]. One can then write explicitly the self consistency equation \[25\] for \( V_\theta \). We will skip here the corresponding tedious but straightforward algebra, and only quote the final form of Eq.\[25\], which can be written as:

\[
V_\theta \left[ 1 + O \left( \frac{\mu_\infty}{\mu_0} \right) \right] = \frac{\tau^*_s}{2\mu_0} \frac{1}{ck\tau} \left( 1 - \cos \frac{\alpha}{2} \right)^2 \cos \frac{\alpha}{2}
\]  

(41)

For the systems we are interested in, as already mentioned, \( \mu_\infty/\mu_0 << 1 \). Then, again, under the stick restriction which imposes that \( \tau^*_s < 0 \), condition \[29\] cannot be fulfilled.

IV. DISCUSSION

The above analysis leads us to a strong statement, which seems to contradict existing qualitative observations. Namely, an interface with Coulomb friction between a viscoelastic and a non-deformable material cannot sustain slow sliding via a periodic set of alternating non inertial slip pulses and stick regions.

We believe that the reason for this contradiction must be traced to the fact that the Coulomb model of friction which we have assumed misses some physical elements which probably play a crucial role in the dynamics of patterns with fracture-like singularities. This may appear more clearly when one notices that this model, which describes the interface as infinitely rigid below the friction threshold, then, once this is reached, sliding under constant stress, is the exact 2D equivalent of the Hill model of bulk ideal plasticity \[21\], well known to generate numerous artefact instabilities due to its highly singular character.

Clearly, the main weakness of the Coulomb model lies in its overschematic description of the transition between stick and slip i.e., for our patterns, in the details of interface boundary conditions at the edges of a slip zone. In the case of the analogous mode-I problem – the
Griffith crack – it is well known that discontinuous boundary conditions (vanishing normal stresses and displacements in, respectively, the cracked and uncracked regions of the crack plane) miss a major physical ingredient, namely the finite range of atomic decohesion and its associated energy cost. Taking this into account regularizes the stress field at the fracture head, by smearing its square root singularity over the Dugdale-Barenblatt cohesive zone [22].

In analogy with this, and based upon the nature of the stress-strain characteristics of overconsolidated clays, Palmer and Rice [23] proposed a model for sliding along a concentrated slip surface in which the sliding shear stress is assumed to decrease with relative displacement as shown on Figure 2. This enabled them to analyze the “mode-II fracture” problem of shear band propagation in such materials.

Let us assume for the moment that we can modify our Coulomb model in a similar manner. That is, let us assume that the shear stress in the sliding state is given by:

$$\tau_{12} = -f\tau_{22} + \delta\tau_0[\delta(x_1, t)]$$  \hspace{1cm} (42)

where $\delta\tau_0$ is maximum for $\delta = 0$ and has a small range $\delta_0 << 2l$. We define $\delta(x_1, t)$ as the displacement at the interface point $x_1$ from its position when it was in the preceding stick zone, i.e. up till the head of the slip zone under consideration reached it. So:

$$\delta(x_1, t) = u_1(x_1, t) - u_1(x_1, t - \frac{a - x_1}{c}) = \int_{\eta}^{\alpha} d\eta' V(\eta') \equiv \delta(\eta)$$  \hspace{1cm} (43)

In Eq.(21) for the velocity field, $\tau^*_s$ must now be substituted by $\tau^*_s - \delta\tau_0[\delta(\eta)]$.

In order to fix ideas, let us concentrate on $(+-)$ solutions. Repeating the analysis of Appendix B leads again to expression (36), with:

$$H_{+-}(u) \rightarrow H_{+-}(u) - \frac{Z_{+-}(u)}{2\mu_0}I(a)$$  \hspace{1cm} (44)

$$I(a) = -\frac{1}{\pi} \int_{-a}^{a} dy \frac{\delta\tau_0[\delta(y)]}{(1 + y^2)(a - y)Z_{+-}(y)}$$  \hspace{1cm} (45)

The regularity condition then becomes:
\[ \tau_s^* \cos \frac{\alpha}{2} + \frac{2\Delta \mu W_{+}G(a)}{\pi ck\tau} = \frac{I(a)}{\pi} \]  

(46)

The l.h.s. of Eq.(46) must, as shown above, be negative. From Eq.(45), \( I(a) < 0 \). So, the introduction of a “cost for incipient sliding” via \( \delta \tau_0 \) is sufficient to lift the incompatibility which we found to hold for the Coulomb friction model. Clearly, the same formal result applies for the \((-+)\) and \((-)\) classes. That is, the localized incremental stress \( \delta \tau_0 \) plays a role comparable to that of the cohesive stress in mode-I fracture, namely it smoothes out the stress singularity by spreading it over a zone of incipient sliding of small but finite extent.

Once this formal remark has been made, one should however come back to the possible physical interpretation of such a modification of the friction model. A decrease in the frictional stress with the slip distance is likely to be associated with a change upon sliding of the internal structure of the nanometer-thick adhesive interfacial junction. Moreover, in order for the peaked structure of \( \tau_{12}(\delta) \) to reproduce itself at each successive slip zone head, the structure of the junction must relax non negligibly on the duration \( \Delta \tau_{st} = (\lambda - 2l)/c \) of a stick (typically, \( \Delta \tau_{st} \) lies in the range of seconds).

Such a scenario is plausible for junctions composed of long molecules – either because a molecular layer of lubricant is present or because the junction is formed by molecular tails from the sliding material itself. Then, sliding is likely to give rise to a slip weakening of friction associated with molecular elongation and restrengthening by structural relaxation during stick. These are precisely the physical ingredients invoked to explain the hysteretic frictional dynamics observed in a number of boundary lubrication experiments \cite{24}, \cite{25}, \cite{26}.

However, inclusion of slip-weakening of dynamical friction is not the only possible improvement on the Coulomb model susceptible to allow for slip pulses. Indeed, a series of recent works by Langer et al \cite{27}, \cite{28} on the viscoplasticity of amorphous solids point very convincingly towards the crucial importance of a realistic description – of the rate and state type – of the gradual cross-over of the mechanical shear response from mainly elastic to mainly dissipative. As already pointed out, solid friction along a continuous interface is
nothing but 2D interfacial viscoplasticity, to which the bulk analysis should be transposable. Hence the need for the elaboration of a phenomenology which can bridge realistically between static and dynamic solid friction. Work in that direction, based upon experimental studies of dynamic interfacial shear response, is presently in progress.

This discussion naturally leads us to emphasize the need for the development of systematic experimental studies of interfacial slip pulses, and the interest which they present. The main questions to be elucidated are:

(i) the precise conditions for frictional sliding to occur in this mode. This includes systematic characterization of the bulk viscoelasticity of systems which do exhibit this behavior, and qualification of the relevant range of driving velocities $v_0$.

(ii) the $v_0$-dependence of the apparent friction coefficient $\tau_{12}^* / |\tau_{22}^*|$, and the question of pattern selection. Namely, is the slip pattern unique for a given $v_0$, or, for example, does it depend on the lateral size of the sliding block? In other words, does injection at the back free edge of the slider play a crucial role in the selection of the pattern wavelength $\lambda$, or not?

Further elucidation of these questions would also be of value to shed further light on the still largely open question of the physics of shear interfacial fracture.

ACKNOWLEDGMENTS

It is a pleasure to thank J.R. Rice for a number of illuminating discussions about this and related subjects. I am indebted to T.Baumberger for drawing my attention to this question, and for permanent exchange during the course of this work, and to O. Ronsin and B. Velicky for fruitful discussions.

APPENDIX A:

We briefly sketch here the derivation of eqs. (7-9) for a purely elastic system. Let $\lambda, \mu$ be its Lame coefficients, related to the Young modulus $E$ and to the Poisson ratio $\nu$ by:
\[ \mu = \frac{E}{2(1 + \nu)} \quad \lambda + \mu = \frac{E}{2(1 + \nu)(1 - 2\nu)} \quad (A1) \]

The elastic displacement \( \mathbf{u} = (u_1, u_2) \) obeys the Lame equation:

\[ \rho \ddot{\mathbf{u}} = (\lambda + \mu) \nabla \cdot \mathbf{u} + \mu \Delta \mathbf{u} \quad (A2) \]

with \( \rho \) the mass density. The stresses are given by:

\[ \frac{\tau_{ij}}{\mu} = \frac{\partial u_i}{\partial x_j} + \frac{\partial u_j}{\partial x_i} + (\beta^2 - 2) \delta_{ik} \nabla \cdot \mathbf{u} \quad (A3) \]

where:

\[ \beta^2 = \frac{2(1 - \nu)}{1 - 2\nu} \quad (A4) \]

One then sets:

\[ u_i(x_1, x_2, t) = u_i^*(x_2) + \sum_m U_{im}(x_2) e^{im(x_1-ct)} \quad (A5) \]

with \( u^* \) the displacement field corresponding to uniform sliding under the homogenous stresses \( \tau^* \). Solving equation (A2) together with the condition of non separation at the interface \( u_2 |_{x_2=0} = 0 \), one gets:

\[ u_1(x_1, x_2, t) - u_1^*(x_2) = Re \sum_{m \geq 1} A_m \left[ -\frac{k^2}{s_+ s_-} e^{-ms_+ x_2} + e^{-ms_- x_2} \right] e^{im(x_1-ct)} \quad (A6) \]

\[ u_2(x_1, x_2, t) - u_2^* = Re \sum_{m \geq 1} A_m \frac{ik}{s_-} \left[ -e^{-ms_+ x_2} + e^{-ms_- x_2} \right] e^{im(x_1-ct)} \quad (A7) \]

with:

\[ s_+^2 = k^2 \left( 1 - \frac{\rho c^2}{\lambda + 2\mu} \right) \quad s_-^2 = k^2 \left( 1 - \frac{\rho c^2}{\mu} \right) \quad Re s_+ > 0 \quad (A8) \]

Then, with the help of Eq.(A3), and in the incompressible limit \( \lambda \rightarrow \infty \), one obtains the expression for the interface stresses and the interface sliding velocity:

\[ \tau_{12} |_{x_2=0} = \tau_{12}^* + \mu Re \sum_{m \geq 1} mk A_m \left( \frac{k}{s_- - \frac{s_-}{k}} \right) e^{im(x_1-ct)} \quad (A9) \]
\( \tau_{22} |_{x=0} = \tau_{22}^* + \mu \text{Re} \sum_{m \geq 1} \text{im} k A_m \left[ -2 + \frac{k}{s_-} + \frac{s_-}{k} \right] e^{\text{im} k (x_1 - ct)} \) \tag{A10}

\( v_s = v_0 - c \text{Re} \sum_{m \geq 1} \text{im} k A_m \left[ 1 - \frac{k}{s_-} \right] e^{\text{im} k (x_1 - ct)} \) \tag{A11}

Setting:

\( B_m = \text{im} k A_m \left( \frac{k}{s_-} - 1 \right) \) \tag{A12}

and substituting \( s_- \) by \( s_{m-} = k \sigma_m \) (Eq.\( (A3) \)) appropriate to the viscoelastic system directly yields expressions \( (7)-(11) \).

**APPENDIX B:**

Following [19], the singular integral equation (24), valid in \(-a < u < a\):

\[
\frac{2\mu_0}{\pi} \int_{-a}^{a} dy \frac{\Phi(y)}{u-y} + \int_{-a}^{a} dy \Phi(y) k(u, y) = F(u)
\] \tag{B1}

where \( k \) is given by Eq.\( (26) \) and:

\[
F(u) = \frac{2V_0 u - \tau_s^*}{1 + u^2}
\] \tag{B2}

is equivalent, for the (\(+\)-) family of solutions, to:

\[
\Phi + K \ast k \ast \Phi = K \ast F
\] \tag{B3}

with

\[
[K \ast f](u) = -\frac{Z_{+-}(u)}{2\mu_0 \pi} \int_{-a}^{a} dy \frac{f(y)}{Z_{+-}(y)(u-y)}
\] \tag{B4}

Integrating in the complex \( y \)-plane along the contour shown on Figure 3, one finds:

\[
K \ast F = -\frac{Z_{+-}(u)}{2\mu_0} \frac{C^* u + D^*}{1 + u^2}
\] \tag{B5}

where \( C^*, D^* \) are given by equations \( (34), (35) \).
On the other hand:

\[
(K \ast k \ast \Phi) (u) = - \frac{Z_+ (u)}{2 \mu_0 \pi} \int_{-a}^{a} dz \Phi (z) J \to (y, z)
\]  (B6)

\[
J \to (y, z) = \frac{2}{\pi} \int_{-a}^{a} dy \frac{dy}{Z_+ (u) (u - y)} \int_{0}^{\infty} ds \frac{\mu' (s)}{y - \frac{z - T (s)}{1 + z T (s)}}
\]  (B7)

with:

\[
T (s) = \tan \frac{ck s}{2}
\]  (B8)

Once the order of the \(y\) and \(s\)-integrals on the r.h.s. of eq. (B7) has been interchanged, the \(y\)-integration can be performed explicitly. However, care must be exercised when performing this interchange, due to the presence of the two principal values. One uses the following identity, which results from the Poincare-Bertrand theorem [19]:

\[
PV \left( \frac{1}{x''-x'} \right) PV \left( \frac{1}{x''-x} \right) = PV \left( \frac{1}{x-x'} \right) \left[ PV \left( \frac{1}{x''-x} \right) - PV \left( \frac{1}{x''-x'} \right) \right] + \pi^2 \delta (x''-x') \delta (x''-x)
\]  (B9)

One thus obtains:

\[
J \to (y, z) = \int_{0}^{\infty} ds \mu' (s) \int_{-a}^{a} dy \frac{dy}{Z_+ (y) (u - y) \left( y - \frac{z - T (s)}{1 + z T (s)} \right)} + Y
\]  (B10)

\[
Y = -\pi^2 \sum_{p=-\infty}^{\infty} \frac{\mu' (s_p (z, u))}{\partial D / \partial s \left|_{s_p} \right.} Z_+ (u) \theta (s_p (z, u))
\]  (B11)

where the \(s_p\) are the zeros of \(D = u - \frac{z - T (s)}{1 + z T (s)}\), i.e.:

\[
s_p (z, u) = \frac{2}{ck} \left[ \phi (z, u) + p \pi \right]
\]  (B12)

\[
\phi (z, u) = \tan^{-1} \left[ \frac{z - u}{1 + zu} \right] - \frac{\pi}{2} < \phi (z, u) < \frac{\pi}{2}
\]  (B13)

From this one gets finally:
\[ (K \ast k \ast \Phi)(u) = \frac{2}{\mu_0 c k (1 + u^2)} \int_{-a}^{a} dz \Phi(z) \mu'[s_0(z, u)] \left[ \theta(\phi(z, u)) + \beta \right] \]

\[ - \frac{Z_{+-}(u)}{2\mu_0} \int_{-a}^{a} dz \Phi(z) \int_{0}^{\infty} ds \frac{\mu'(s) \theta(\psi^2 - a^2)}{Z_{out}^+(\psi)(u - \psi)} \]  

(B14)

where \( \beta \) is defined in eq.(33), and:

\[ Z_{out}^+(\psi) = \theta(\psi - a) \sqrt{\frac{\psi + a}{\psi - a}} + \theta(-\psi - a) \sqrt{\frac{-\psi - a}{-\psi + a}} \]  

(B15)

Straightforward integration then results in eqs.(29-32).
REFERENCES

‡ “Associé au Centre National de la Recherche Scientifique et aux Universités Paris 6 et Paris 7”.

[1] M.A. Rubio, J. Galeano: Phys. Rev. E50, 1000 (1994)

[2] T. Baumberger, private communication

[3] A. Anooshehpoor, J.N. Brune: PAGEOPH 143, 61 (1994)

[4] M. Mouwakeh, B. Villechaise, M. Godet: Eur. J. Mech. A/Solids 10, 545 (1991)

[5] A. Schallamach: Wear 17, 301 (1971)

[6] M. Barquins, R. Courtel: Wear 32, 133 (1975)

[7] O. Ronsin, private communication

[8] K. Ranjith, J.R. Rice: preprint (1999) and references therein

[9] T.H. Heaton: Phys. Earth and Planet. Inter. 64, 1 (1990)

[10] J. Weertmann: J. Mech. Phys. Solids 11, 197 (1980)

[11] G.G. Adams: J. Appl. Mech. 62, 867 (1995); J. Appl. Mech. 65, 470 (1998)

[12] J.A.C. Martins, J. Guimaraes, L.O. Faria: J. Vib. and Acoustics, Trans. ASME 117, 445 (1995)

[13] The precise conditions for their existence are determined by the value of the friction coefficient and, as is the case for interface elastic waves at a cohesive interface, by the values of the ratios of elastic moduli.

[14] M. Comninou, J. Dundurs: J. Appl. Mech. 44, 222 (1977); J. Appl. Mech. 45, 325 (1978)

[15] K.L. Johnson, ContactMechanics, Cambridge Univ. Press (1994)
In classical mode-I fracture problems such as that of the Griffith crack, such a solution has to satisfy an auxiliary condition which expresses the fact that the perturbation of the stress field vanishes at infinity. There is no such condition for our problem – where it is replaced by condition (11), which must hold everywhere in the stick zones (note that $u \to \pm \infty$ corresponds to their midpoints).

[21] R. Hill: *Theory of Plasticity*, Oxford Univ. Press (1950)

[22] L.D. Landau, E. M. Lifshitz: *Theory of Elasticity*, Pergamon Press (1967)

[23] A.C. Palmer, J.R. Rice: Proc. Roy. Soc. London **A332**, 527 (1973)

[24] H. Yoshizawa, J.N. Israelachvili: J. Chem. Phys. **97**, 11300 (1993)

[25] J.M. Carlson, A.A. Batista: Phys. Rev. E**53**, 4153 (1996)

[26] T. Baumberger, C. Caroli: Eur. Phys. J. B**4**, 13 (1998)

[27] M.L. Falk, J.S. Langer: Phys. Rev. E**57**, 7192 (1998)

[28] J.S. Langer, A.E. Lobkovsky: J. Mech. Phys. Solids **46**, 1521 (1998)
FIGURES

FIG. 1. Schematic representation of the sliding system

FIG. 2. Schematic representation of the slip-weakening friction law.

FIG. 3. Contour for integrals of type $K \star F$. 
\[ \tau_{12} \]

\[ -f \tau_{22} \]

\[ \delta_0 \]
$C_\infty$