Two problems of binomial sums involving harmonic numbers

Nadia N. Li¹* and Wenchang Chu¹,²

*Correspondence: lina20190606@outlook.com
¹School of Mathematics and Statistics, Zhoukou Normal University, Henan, China
Full list of author information is available at the end of the article

Abstract

Two open problems recently proposed by Xi and Luo (Adv. Differ. Equ. 2021:38, 2021) are resolved by evaluating explicitly three binomial sums involving harmonic numbers, that are realized mainly by utilizing the generating function method and symmetric functions.

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1 Introduction and outline

Denote by \( \mathbb{N} \) the set of natural numbers with \( \mathbb{N}_0 = \mathbb{N} \cup \{0\} \). For an indeterminate \( x \), define the rising and falling factorials by

\[
(x)_0 = 1 \quad \text{and} \quad (x)_n = x(x + 1) \cdots (x + n - 1) \quad \text{for } n \in \mathbb{N},
\]

\[
(x)_n = x(x - 1) \cdots (x - n + 1) \quad \text{for } n \in \mathbb{N}.
\]

The harmonic numbers of higher order are given by

\[
H^{(\lambda)}_0 = 1 \quad \text{and} \quad H^{(\lambda)}_n = \sum_{k=1}^{n} \frac{1}{k^\lambda} \quad \text{for } n, \lambda \in \mathbb{N}.
\]

In order to reduce lengthy expressions, we shall employ the notations of elementary and complete symmetric functions. For a finite set \( S \) of real numbers, we define these functions by \( \Phi_n(x|S) = \Psi_n(x|S) \equiv 1 \) and

\[
\Phi_n(x|S) = \sum_{\sum_{k=0}^{n} k_y = n \atop 0 \leq k_y \leq 1} \prod_{y \in S} \frac{1}{(x + \alpha)^{k_y}} \quad \text{for } n \in \mathbb{N},
\]

\[
\Psi_n(x|S) = \sum_{\sum_{k=0}^{n} k_y = n \atop 0 \leq k_y \leq n} \prod_{y \in S} \frac{1}{(x + \alpha)^{k_y}} \quad \text{for } n \in \mathbb{N}.
\]
We shall also need the signless Stirling numbers of the first kind (see [6]) which are determined by the connection coefficient of expanding the shifted factorials into monomials

\[(y)_n = \sum_{k=0}^{n} \left[ \begin{array}{c} n \\ k \end{array} \right] y^k. \tag{3}\]

There exist numerous summation formulae involving harmonic numbers (cf. [1–3, 7, 8]). In a recent paper [9], Xi and Luo proposed the following two open problems.

**Problem I** Let \(x\) be an indeterminate. For \(m, n \in \mathbb{N}_0\) with \(m > n\), how to calculate the combinatorial sums

\[\sum_{k=0}^{n} (-1)^k \binom{n}{k} \binom{m+k}{k} \quad \text{and} \quad \sum_{k=0}^{n} (-1)^k \binom{n}{k} \binom{m+k}{k} \frac{x}{x+k} ?\]

**Problem II** Let \(x\) be an indeterminate. For \(m, n, \lambda, \rho \in \mathbb{N}_0\), what are the combinatorial sums

\[\sum_{k=0}^{n} (-1)^k \binom{n}{k} \binom{m+k}{k} \left(\frac{x}{x+k}\right) \text{ and } \sum_{k=0}^{n} (-1)^k \binom{n}{k} \binom{m+k}{k} \left(\text{H}_\rho^{(\lambda)} - \text{H}_k^{(\lambda)}\right) ?\]

The first binomial sum in Problem I can easily be evaluated by the Chu–Vandermonde convolution formula as follows:

\[\sum_{k=0}^{n} (-1)^k \binom{n}{k} \binom{m+k}{k} = \sum_{k=0}^{n} \binom{n}{n-k} (-m-1) = \binom{n-m-1}{n} = (-1)^m \binom{m}{n}. \]

As the primary motivation, the aim of the present paper is to resolve these problems and evaluate the remaining three sums explicitly in the following theorems.

**Theorem 1** Let \(x\) be an indeterminate. Then for \(m, n \in \mathbb{N}_0\), the following algebraic identity holds:

\[\frac{n!}{(x)_{n+1}} \frac{m-x}{m} = \sum_{k=0}^{n} (-1)^{k+1} \binom{n}{k} \binom{m+k}{k} x^k + \sum_{k=1}^{m-n} (-1)^{m+k} \binom{m}{n+k} \frac{(x+n+1)_{k-1}}{(n+1)_k}. \]

We remark that when \(m > n\), this theorem evaluates the second sum in Problem I by determining the polynomial part of the rational function explicitly as in the last line, which vanishes for \(m \leq n\), instead.
**Theorem 2** Let \( x \) be an indeterminate. Then for \( m, n, \lambda \in \mathbb{N}_0 \), the following algebraic identity holds:

\[
\sum_{k=0}^{n} \binom{n}{k} \binom{m+k}{k} \left( \frac{(-1)^k}{x+k} \right) = \frac{n!}{(x)_{n+1}} \left( \frac{m-x}{m} \right) \sum_{k=1}^{\lambda} \frac{\Phi_{k-1}(x[1,m])}{(k-1)!} \Psi_{k-1}(x[0,n]) \\
+ \sum_{k=1}^{m-n} \frac{(-1)^{n+k+\lambda}}{(\lambda-1)!} \binom{m}{n+k} \frac{(x+n+1)_{k-1}}{(n+1)_k} \Phi_{k-1}(x+n[1,k-1]).
\]

**Theorem 3** Let \( x \) be an indeterminate. Then for \( m, n, \lambda, \rho \in \mathbb{N}_0 \), the following algebraic identity holds:

\[
\sum_{k=0}^{n} (-1)^k \binom{n}{k} \binom{m+k}{k} \left[ H^{(\lambda)}_{\rho+k} - H^{(\lambda)}_k \right] = \frac{n!}{m!} \sum_{j=1}^{\lambda} \sum_{i=1}^{m} \sum_{k=1}^{\rho} \frac{(-1)^{i+j}}{(j)_{n+1}} \frac{\Psi_{j-k}(j[0,n])}{(k-1)!} \frac{m-j}{m} \Phi_{k-1}(-j[1,m]) \\
+ \sum_{k=1}^{\lambda} \sum_{j=m+1}^{\rho} \frac{\Psi_{j-k}(j[0,n])}{(j)_{n+1}} \frac{m-j}{m} \Phi_{k-1}(-j[1,m]) \\
+ \sum_{k=1}^{m-n} \sum_{j=1}^{\rho} \frac{(-1)^{n+k+j}}{(\lambda-1)!} \binom{m}{n+k} \frac{(j+n+1)_{k-1}}{(n+1)_k} \Phi_{k-1}(j+n[1,k-1]).
\]

The rest paper will be organized as follows. In the next section, we shall prove Theorem 1 by determining explicitly the polynomial part of a rational function when its numerator degree is greater than that of the denominator. Then Theorems 2 and 3 will be shown in Sect. 3 by establishing two analytical formulae of the derivatives of higher order for a polynomial function of the rising factorial and its reciprocal. The informed reader will notice that by employing symmetric functions \( \Phi \) and \( \Psi \), several involved expressions become simpler than those appearing in [9], where the Bell polynomials were employed.

### 2 Proof of Theorem 1

Observe that the rational function below can be decomposed into partial fractions

\[
\frac{n!}{(x)_{n+1}} \left( \frac{m-x}{m} \right) = P^m_n(x) + \sum_{k=0}^{n} \frac{A_k}{x+k},
\]

where \( P^m_n(x) \) is a polynomial of degree \( m - n - 1 \) in \( x \) which reduces to zero when \( m \leq n \), and the coefficients \( A_k \) are determined by the limits

\[
A_k = \lim_{x \to -k} (x+k) \left\{ \frac{n!}{(x)_{n+1}} \left( \frac{m-x}{m} \right) \right\} = (-1)^k \binom{n}{k} \binom{m+k}{m}.
\]
Therefore, we have found the equality
\[
\frac{n!}{(x)_{n+1}} \left( \frac{m-x}{m} \right) = P_n^m(x) + \sum_{k=0}^{n} \frac{(-1)^k}{x+k} \binom{n}{k} \frac{m+k}{k}.
\]  
(4)

By scaling down \(m\) and then making use of
\[
\frac{m-x}{m} = \frac{m+k - k+x}{m},
\]
we can rewrite the last equality as
\[
\frac{n!}{(x)_{n+1}} \left( \frac{m-x}{m} \right) = \frac{m-x}{m} \left( \frac{m-1-x}{m} \right) = \frac{m-x}{m} \left[ P_{n-1}^{m-1}(x) + \sum_{k=0}^{n} \frac{(-1)^k}{x+k} \binom{n}{k} \left( \frac{m-1+k}{k} \right) \right] = \frac{m-x}{m} P_{n-1}^{m-1}(x) + \sum_{k=0}^{n} \frac{(-1)^k}{x+k} \binom{n}{k} \left( \frac{m+k}{k} \right) - \sum_{k=0}^{n} \frac{(-1)^k}{m} \binom{n}{k} \left( \frac{m-1+k}{k} \right).
\]

Evaluating the last sum by means of the Chu–Vandemonde formula and then comparing the resultant expression with (4), we get the following recurrence relation:
\[
P_n^m(x) = \frac{m-x}{m} P_{n-1}^{m-1}(x) - \frac{(-1)^n}{m} \binom{m-1}{n}.
\]  
(5)

In order to find an explicit expression for \(P_n^m(x)\), let \(Q_n := P_n^{m+n}(x)\). Then the equality corresponding to (5) becomes
\[
Q_m = \frac{m+n-x}{m+n} Q_{m-1} - \frac{(-1)^n}{m+n} \binom{m+n-1}{n}.
\]  
(6)

It is routine to figure out the initial values \(Q_0 = 0\) and \(Q_1 = \frac{(-1)^{n+1}}{n+1}\). Then we can manipulate the generating function
\[
Q(y) := \sum_{m=1}^{\infty} Q_n y^{m+n}
\]
\[
= \sum_{m=1}^{\infty} \left( 1 - \frac{x}{m+n} \right) Q_{m-1} y^{m+n} - \sum_{m=1}^{\infty} \frac{(-1)^n}{m+n} \binom{m+n-1}{n} y^{m+n}.
\]

By differentiating the last equation with respect to \(y\),
\[
Q'(y) = \frac{d}{dy} \left[ yQ(y) \right] - xQ(y) - \sum_{m=1}^{\infty} \frac{(-1)^n}{m+n} \binom{m+n-1}{n} y^{m+n-1},
\]
and then evaluating the binomial series on the right, we find, after some simplification, that $Q(y)$ satisfies the following differential equation:

$$(1 - y)Q'(y) - (1 - x)Q(y) = \frac{y^n}{(y - 1)^{x+1}}. \quad (7)$$

It is trivial to check that the corresponding homogeneous equation

$$\frac{Q'(y)}{Q(y)} = \frac{1 - x}{1 - y}$$

has the binomial solution

$$Q(y) = \Omega(1 - y)^{x-1} \quad (8)$$

where $\Omega$ is an arbitrary constant. When $\Omega := \Omega(y)$ is considered as a function of $y$, substituting the above solution into (7) gives rise to

$$\Omega'(y) = (-1)^{n+1}y^n(1 - y)^{-x-n-1}.$$

Therefore, we have the integral representation

$$\Omega(y) = (-1)^{n+1} \int_0^y T^n(1 - T)^{-x-n-1} dT.$$

Define for simplicity

$$J_n := \int_0^y T^n(1 - T)^{-x-n-1} dT \quad \text{with} \quad J_0 = \frac{(1 - y)^{-x} - 1}{x}.$$

According to integration by parts, we can calculate $J_n$ as follows:

$$J_n = \int_0^y T^n(1 - T)^{-x-n-1} dT = \frac{y^n}{x + n}(1 - y)^{-x-n} - \frac{n}{x + n}J_{n-1}$$

$$= \frac{y^n}{x + n}(1 - y)^{-x-n} - \frac{ny^{n-1}}{(x + n)2}(1 - y)^{1-x-n} + \frac{(n)2}{(x + n)2}J_{n-2}.$$

By means of the induction principle, we can show that

$$J_n = \sum_{k=0}^{n-1} \frac{(-1)^k (n)_k}{(x + n)k+1} y^{n-k}(1 - y)^{k-x-n} + \frac{(-1)^n (n)_n}{(x + n)n} J_0$$

$$= \sum_{k=0}^{n} \frac{(-1)^k (n)_k}{(x + n)k+1} y^{n-k}(1 - y)^{k-x-n} + \frac{(-1)^{n+1}n!}{(x)_{n+1}},$$

which is equivalent to the expression

$$\Omega(y) = \frac{n!}{(x)_{n+1}} - \sum_{k=0}^{n} \frac{(-1)^{n+k} (n)_k}{(x + n)k+1} y^{n-k}(1 - y)^{k-x-n}.$$
Substituting this into (8), we obtain the explicit generating function

\[ Q(y) = \frac{n!}{(x)_{n+1}}(1-y)^{x-1} - \sum_{k=0}^{n} \frac{(-1)^{n-k} (n)_k}{(x+n)_{k+1}} y^{n-k} (1-y)^{k-1-n}. \]

Extracting the coefficient of \( y^{m+n} \) across the last equation yields

\[ Q_m = \left[ y^{m+n} \right] Q(y) = \left( \frac{m+n-x}{m+n} \right) \frac{n!}{(x)_{n+1}} - \sum_{k=0}^{n} \frac{(-1)^{n-k} (n)_k}{(x+n)_{k+1}} \left( \frac{m+n}{n-k} \right) \left( \frac{m+n}{n-k} \right). \]

By reformulating the last sum with respect to \( k \) as

\[ \sum_{k=0}^{n} \frac{n!}{(x)_{n+1}} \sum_{k=0}^{n} \left( \frac{m+n}{m+k} \right) \left( \frac{m+n}{n-k} \right) \]

we find finally the binomial expression

\[ Q_m = \frac{n!}{(x)_{n+1}} \sum_{k=1}^{m} \left( \frac{m+n}{m-k} \right) \left( \frac{m+n}{n+k} \right) = \sum_{k=1}^{m} (-1)^{n+k} \left( \frac{m+n}{n+k} \right) \left( \frac{m+n}{n+k} \right). \]

This gives consequently the desired formula stated in Theorem 1:

\[ P_m^{m+n}(x) = Q_{m-n}(x) = \sum_{k=1}^{m-n} (-1)^{n+k} \left( \frac{m+n}{n+k} \right) \left( \frac{x+n+1}{n+1} \right). \]

### 3 Proofs of Theorems 2 and 3

For the derivative operator \( \mathcal{D} \) with respect to \( x \), we have the following analytical formulae of higher order derivatives:

\[ \mathcal{D}^n \prod_{a \in S} (x + \alpha) = \Phi_n(x | S) \prod_{a \in S} (x + \alpha), \quad (9) \]

\[ \mathcal{D}^n \prod_{a \in S} \frac{1}{x + \alpha} = \Psi_n(x | S) \frac{n!(-1)^n}{\prod_{a \in S} (x + \alpha)}. \quad (10) \]

The first one in (9) can be evaluated easily by induction on \( n \). In order to prove the second one in (10), define

\[ R(x) := \prod_{a \in S} \frac{1}{x + \alpha} \quad \text{and} \quad \mathcal{D}^n R(x) = R(x) G_n. \quad (11) \]
where the function $G_n$ remains to be determined with the initial values

$$G_0 = 1 \quad \text{and} \quad G_1 = -\Psi_1(x|S).$$

Then by making use of the Leibniz rule, we have

$$D^{\lambda+1}R(x) = -D^\lambda \left[ R(x)\Psi_1(x|S) \right]$$

$$= -\sum_{k=0}^\lambda \binom{\lambda}{k} D^{\lambda-k}R(x)D^k \Psi_1(x|S)$$

$$= -R(x) \sum_{k=0}^\lambda \binom{\lambda}{k} G_{\lambda-k} D^k \Psi_1(x|S),$$

which leads us to the binomial recursion

$$G_{\lambda+1} = -\sum_{k=0}^\lambda \binom{\lambda}{k} G_{\lambda-k} D^k \Psi_1(x|S). \quad (12)$$

In order to find an explicit expression for $G_\lambda$, we examine the exponential generating function defined by

$$G(y) := \sum_{\lambda=0}^\infty \frac{y^\lambda}{\lambda!} G_\lambda.$$

According to (12), its derivative with respect to $y$ can be expressed as

$$G'(y) = \sum_{\lambda=0}^\infty \frac{y^\lambda}{\lambda!} G_{\lambda+1} = -\sum_{\lambda=0}^\infty \frac{y^\lambda}{\lambda!} \sum_{k=0}^\lambda \binom{\lambda}{k} G_{\lambda-k} D^k \Psi_1(x|S)$$

$$= -\sum_{k=0}^\infty \frac{y^k}{k!} D^k \Psi_1(x|S) \sum_{\lambda=k}^\infty \frac{y^{\lambda-k}}{(\lambda-k)!} G_{\lambda-k}.$$ 

We therefore get the differential equation

$$G'(y) = -G(y) \sum_{k=0}^\infty \frac{y^k}{k!} D^k \Psi_1(x|S)$$

whose solution is given by the exponential function

$$G(y) = \exp \left\{ -\int_0^y \sum_{k=0}^\infty \frac{y^k}{k!} D^k \Psi_1(x|S) \, dy \right\} = \exp \left\{ -\sum_{k=0}^\infty \frac{y^{k+1}}{(k+1)!} D^k \Psi_1(x|S) \right\}. $$
Evaluating the last sum with respect to $k$ explicitly as
\[
\sum_{k=0}^{\infty} \frac{y^{k+1}}{(k+1)!} \mathcal{D}^{k} \Psi_{1}(x|S) = \sum_{k=0}^{\infty} \frac{(-1)^{k}}{k+1} \sum_{\alpha \in S} \frac{y^{k+1}}{(x+\alpha)^{k+1}}
\]
\[
= \sum_{k=1}^{\infty} \frac{(-1)^{k-1}}{k} \sum_{\alpha \in S} \frac{y^{k}}{(x+\alpha)^{k}}
\]
\[
= \sum_{\alpha \in S} \ln \left(1 + \frac{y}{x + \alpha}\right),
\]
we find the simplified generating function
\[
G(y) = \prod_{\alpha \in S} \left(1 + \frac{y}{x + \alpha}\right)^{-1}.
\] (13)

By extracting the coefficient of $y^n$, we confirm the formula (10) as follows:
\[
G_n = n! \left[y^n \right] G(y) = n! (-1)^n \Psi_n(x|S).
\]

### 3.1 Proof of Theorem 2

This can be done by differentiating $\lambda - 1$ times the equality displayed in Theorem 1. Firstly, it is trivial to have
\[
\mathcal{D}^{\lambda-1} \frac{1}{x+k} = (-1)^{\lambda-1} \frac{\lambda-1)!}{(x+k)^{\lambda}}.
\]

Then by making use of the Leibniz rule, we can compute
\[
\mathcal{D}^{\lambda-1} \left(1-x\right)_m \frac{1}{(x)_{m+1}} = (-1)^m \mathcal{D}^{\lambda-1} \frac{(x-1)_m}{(x)_{m+1}}
\]
\[
= (-1)^m \sum_{k=1}^{\lambda} \frac{(\lambda-1)!}{(k-1)!} \mathcal{D}^{k-1} \frac{1}{(x)_{n+1}}
\]
\[
= \frac{(1-x)_m}{(x)_{n+1}} \sum_{k=1}^{\lambda} \frac{(\lambda-1)!}{(k-1)!} \Phi_{k-1} \left(x]\left[-m,-1\right]\right) \Psi_{\lambda-k} \left(x]\left[0,n\right]\right)
\]
\[
= (-1)^{\lambda-1} \frac{(1-x)_m}{(x)_{n+1}} \sum_{k=1}^{\lambda} \frac{(\lambda-1)!}{(k-1)!} \Phi_{k-1} \left(-x]\left[1,m\right]\right) \Psi_{\lambda-k} \left(x]\left[0,n\right]\right),
\]
where we have invoked two derivative formulae (9) and (10). Finally,
\[
\mathcal{D}^{\lambda-1} \left(x+n+1\right)_{k-1} = (x+n+1)_{k-1} \Phi_{\lambda-1} \left(x]\left[n+1,n+k-1\right]\right)
\]
\[
= (x+n+1)_{k-1} \Phi_{\lambda-1} \left(x+n]\left[1,k-1\right]\right).
\]

Substituting the above three expressions into the equality of Theorem 1 and then making some simplifications, we find the algebraic identity in Theorem 2.
3.2 Proof of Theorem 3

Recalling (3), we can deduce, for the signless Stirling numbers, the symmetric function expression (see [4, Chap. V] and [5, §6.1])

\[ \frac{n+1}{k+1} = \left[ y^k \right] (1 + y)_n = \sum_{1 \leq i_1 < i_2 < \ldots < i_{n-k} \leq n} i_1 i_2 \cdots i_{n-k} = n! \sum_{1 \leq j_1 < j_2 < \ldots < j_k \leq n} \frac{1}{j_1 j_2 \cdots j_k} \].

This gives rise to the following identity:

\[ \Phi_k(0|[1,n]) = \frac{1}{n!} \left[ \frac{n+1}{k+1} \right] . \quad (14) \]

Let \( \rho \) be a natural number. When \( x \to j \) with \( 1 \leq j \leq \rho \), the limiting case of the equation displayed in Theorem 2 reads as

\[
\sum_{k=0}^{n} \binom{n}{k} \binom{m+k}{k} \frac{(-1)^k}{(j+k)^\lambda} = \frac{n!}{(j)_{n+1}} \sum_{k=1}^{m-n} \frac{\Psi_{\lambda-k}(j[0,n])}{(k-1)!} \lim_{x \to j} \left( \frac{m-x}{m} \right)^{k-1} \Phi_{k-1}(-x|[1,m]) + \sum_{k=1}^{m-n} \frac{(-1)^{n+k+\lambda}}{(k-1)!} \left( \frac{m}{n+k} \right)^{k-1} \frac{j_n+1}{(n+1)_k} \Phi_{k-1}(j+n|[1,k-1]). \quad (15)
\]

When \( j > m \), the limit in the middle line is given directly by letting \( x = j \)

\[ \lim_{x \to j} \left( \frac{m-x}{m} \right) \Phi_{k-1}(-x|[1,m]) = \left( \frac{m-j}{m} \right) \Phi_{k-1}(-j|[1,m]) \]

since the two factors on the right are well defined. Instead, for \( 1 \leq j \leq m \), that limit can be determined as

\[
\lim_{x \to j} \left( \frac{m-x}{m} \right) \Phi_{k-1}(-x|[1,m]) = \frac{(1-x)^{m}}{m!} \Phi_{k-1}(-x|[1,m]) = \frac{(-1)^{j-1}}{j^{(m)}} \sum_{i=1}^{k-1} \Phi_{i-1}(-j|[1,j-1]) \Phi_{k-i-1}(-j|[j+1,m]) = \frac{(-1)^{j-1}}{j^{(m)}} \sum_{i=1}^{k-1} (-1)^i \Phi_{i-1}(0|[1,j-1]) \Phi_{k-i-1}(0|[1,m-j]) = \frac{(-1)^{j-1}}{m!} \sum_{i=1}^{k-1} (-1)^i \left[ \frac{m-j+1}{k-i} \right].
\]
where the last line is justified by (14). Finally summing equation (15) over $j$ from 1 to $\rho$, we obtain the following equality involving harmonic numbers:

\[
\sum_{k=0}^{n} (-1)^j \binom{n}{k} \binom{m+k}{k} \left( H_{\rho+k}^{(2)} - H_k^{(2)} \right) = \\
\frac{n!}{m!} \sum_{j=1}^{m} \frac{(-1)^j}{j} \sum_{k=1}^{\lambda} \frac{\Psi_{k-1}(j)[0,n]}{(k-1)!} \sum_{i=1}^{k-1} \left[ j \frac{m-j+1}{k-i} \right] \\
+ \sum_{j=m+1}^{\rho} \frac{n!}{j!} \sum_{k=1}^{\lambda} \frac{\Psi_{k-1}(j)[0,n]}{(k-1)!} \left( \frac{m-j}{m} \right) \Phi_{k-1}(-j[1,m]) \\
+ \sum_{k=1}^{m+n} \sum_{j=1}^{a} (-1)^{nk+\lambda} \binom{m}{n+k} \frac{j+n+1}{(n+1)k} \Phi_{k-1}(j+n[1,k-1]),
\]

which is equivalent to the formula stated in Theorem 3.

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Author details
1 School of Mathematics and Statistics, Zhoukou Normal University, Henan, China. 2 Department of Mathematics and Physics, University of Salento, Lecce, Italy.

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References
1. Boyadzhiev, K.N.: Series with central binomial coefficients, Catalan numbers, and harmonic numbers. J. Integer Seq. 15, Art#12.1.7 (2012)
2. Chu, W.: Harmonic number identities and Hermite–Padé approximations to the logarithm function. J. Approx. Theory 137, 42–56 (2005)
3. Chu, W.: Partial fraction decompositions and harmonic number identities. J. Comb. Math. Comb. Comput. 60, 139–153 (2007)
4. Comtet, L.: Advanced Combinatorics. Reidel, Dordrecht (1974)
5. Graham, R.L., Knuth, D.E., Patashnik, O.: Concrete Mathematics. Addison-Wesley, Reading (1989)
6. Knuth, D.E.: Two notes on notation. Am. Math. Mon. 99, 403–422 (1992)
7. Sofo, A., Srivastava, H.M.: A family of shifted harmonic sums. Ramanujan J. 37, 89–108 (2015)
8. Wang, X.Y., Chu, W.: Binomial series identities involving generalized harmonic numbers. Integers 20, A98 (2020)
9. Xi, G.W., Luo, Q.M.: Some extensions for the several combinatorial identities. Adv. Differ. Equ. 2021, 38 (2021) 1–8. https://doi.org/10.1186/s13662-020-03171-1