ON THE QUADRATIC WIENER FUNCTIONAL ASSOCIATED WITH THE MALLIAVIN DERIVATIVE OF THE SQUARE NORM OF BROWNIAN SAMPLE PATH ON INTERVAL

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Abstract
Exact expressions of the stochastic oscillatory integrals with phase function $\int_0^T (\int_t^T w(s)ds)^2 dt$, $\{w(t)\}_{t \geq 0}$ being the $1$-dimensional Brownian motion, are given. As an application, the density function of the distribution of the half of the Wiener functional is given.

1 Introduction and statement of result

The study of quadratic Wiener functionals, i.e., elements in the space of Wiener chaos of order 2, goes back to Cameron-Martin [1, 2] and Lévy [8]. While a stochastic oscillatory integral with quadratic Wiener functional as phase function has a general representation via Carleman-Fredholm determinant ([3, 6, 10]), in our knowledge, a few examples, where the integrals are represented with more concrete functions like the ones used by Cameron-Martin and Lévy, are available. See [1, 2, 8, 6, 10] and references therein. In this paper, we study a new quadratic Wiener functional which admits a concrete expression of stochastic oscillatory integral, and apply the expression to compute the density function of the Wiener functional.

Let $T > 0$, $W$ be the space of all $\mathbb{R}$-valued continuous functions $w$ on $[0, T]$ with $w(0) = 0$, and $P$ be the Wiener measure on $W$. The Wiener functional investigated in this paper is

$$q(w) = \int_0^T \left( \int_t^T w(s)ds \right)^2 dt, \quad w \in W.$$ 

The functional $q$ interests us because it is a key ingredient in the study of asymptotic theory on $W$. Namely, recall the Wiener functional

$$q_0(w) = \int_0^T w(t)^2 dt, \quad w \in W.$$ 

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which was studied first by Cameron-Martin [1, 2, 8]. As is well-known ([15]), the stochastic oscillatory integral
\[ \int_\mathcal{W} \exp(\zeta q_0/2)\delta_y(w(T))dP, \]
where \( \delta_y(w(T)) \) is Watanabe’s pull back of the Dirac measure \( \delta_y \) concentrated at \( y \in \mathbb{R} \) via \( w(T) \), relates to the fundamental solution to the heat equation associated with the Schrödinger operator \( (1/2)\{(d/dx)^2 + \zeta x^2\} \), which describes the quantum mechanics of harmonic oscillator. If we denote by \( \mathcal{H} \) the Cameron-Martin subspace of \( \mathcal{W} \) (the subspace of all absolutely continuous \( h \in \mathcal{W} \) with square integrable derivative \( \dot{h} \)) and set \( \langle h, g \rangle_{\mathcal{H}} = \int_0^T \dot{h}(t)\dot{g}(t)dt \) and \( \|h\|_{\mathcal{H}}^2 = \langle h, h \rangle_{\mathcal{H}} \) for \( h, g \in \mathcal{H} \), then it is straightforward to see that
\[ q = \frac{1}{4}\|\nabla q_0\|_{\mathcal{H}}^2, \]
where \( \nabla \) denotes the Malliavin gradient. Thus \( q \) determines the stationary points of \( q_0 \). It should be noted that, in the context of the Malliavin calculus, the set of stationary points of \( q_0 \), i.e. the set \( \{\nabla q_0 = 0\} = \{q = 0\} \) is determined uniquely up to equivalence of quasi-surely exceptional sets. On account of the stationary phase method on finite dimensional spaces (cf.[4]), \( q \) would play an important role in the study of asymptotic behavior of the stochastic oscillatory integral \( \int_\mathcal{W} \exp(\zeta q_0)\psi dP \) with amplitude function \( \psi \) (cf. [9, 11, 12], in particular [13, 14]).

The aim of this paper is to show

**Theorem 1.** (i) For sufficiently small \( \lambda > 0 \), the following identities hold.

\[ \int_\mathcal{W} \exp(\lambda q/2) dP = \left\{ \frac{1}{\cosh(\lambda^{1/4}T) \cos(\lambda^{1/4}T)} \right\}^{1/2}, \]

\[ \int_\mathcal{W} \exp(\lambda q/2)\delta_0(w(T))dP = \frac{\lambda^{1/8}}{\sqrt{\pi}\sinh(\lambda^{1/4}T) \cos(\lambda^{1/4}T) + \sin(\lambda^{1/4}T) \cos(\lambda^{1/4}T)}}^{1/2}. \]

(ii) Define \( \theta(u; x) \) and \( p_T(x) \) for \( u \in [0, \pi/2] \) and \( x \geq 0 \) by

\[ \theta(u; x) = \sum_{k=-\infty}^{\infty} (-1)^k \left\{ u + (2k + 1)\pi \right\}^3 e^{-x(u+(2k+1)\pi)^4/T^4}, \]

\[ p_T(x) = \frac{4}{\pi T^4} \int_0^{\pi/2} \frac{\theta(u; x)}{\sqrt{\cos u}} du. \]

Then \( p_T \) is the density function of the distribution of \( q/2 \) on \( \mathbb{R} \);

\[ P(q/2 \in dx) = p_T(x)\chi_{[0,\infty)}(x)dx, \]

where \( \chi_{[0,\infty)} \) denotes the indicator function of \( [0,\infty) \).

The assertion (i) of Theorem 1 will be shown in Section 2 and (ii) will be proved in Section 3.
2 Proof of Theorem 1 (i)

In this section, we shall show the identities (1) and (2). The proof is broken into several steps, each being a lemma. We first show

Lemma 1. Define the Hilbert-Schmidt operator $A : \mathcal{H} \to \mathcal{H}$ by

$$Ah(t) = \int_0^t ds \int_0^T du \int_0^u dv \int_0^v da \ h(a), \quad h \in \mathcal{H}, \ t \in [0, T].$$

Then it holds that

$$q = QA + \frac{T^4}{6}, \quad (4)$$

where $QA = (\nabla^*)^2 A$, $\nabla^*$ being the adjoint operator of the Malliavin gradient $\nabla$. Moreover, $A$ is of trace class and $\text{tr} A = T^4/6$. In particular, $q = QA + \text{tr} A$.

Proof. Due to the integration by parts on $[0, T]$, it is easily seen that

$$\langle \nabla^2 q, h \otimes k \rangle_{\mathcal{H}^{\otimes 2}} = 2 \int_0^T \left( \int_0^T h(s) ds \right) \left( \int_0^T k(s) ds \right) dt = 2 \langle Ah, k \rangle_{\mathcal{H}} \quad (5)$$

for $h, k \in \mathcal{H}$, where $\mathcal{H}^{\otimes 2}$ denotes the Hilbert space of all Hilbert-Schmidt operators on $\mathcal{H}$, and $\langle \cdot, \cdot \rangle_{\mathcal{H}^{\otimes 2}}$ does its inner product. Hence

$$\nabla^2 q = 2A. \quad (6)$$

Let $\mathcal{C}_2$ be the space of Wiener chaos of order 2. Since

$$w(s)w(u) - s = w(s)^2 - s + w(s)\{w(u) - w(s)\} \in \mathcal{C}_2 \quad \text{for } u \geq s,$$

we have that

$$q - \frac{T^4}{6} = 2 \int_0^T \int_1^T \int_0^s (w(s)w(u) - s) dudsdt \in \mathcal{C}_2.$$

From this and (6), we can conclude the identity (4).

Let $\{h_n\}_{n=1}^\infty$ be an orthonormal basis of $\mathcal{H}$, and define $k_t \in \mathcal{H}$, $t \in [0, T]$, by

$$k_t(s) = \int_0^s (T - \max\{t, u\}) du, \quad s \in [0, T].$$

Since $\int_0^T h_n(s) ds = \langle k_t, h_n \rangle_{\mathcal{H}}$, due to (5), we obtain that

$$\sum_{n=1}^{\infty} \langle Ah_n, h_n \rangle_{\mathcal{H}} = \int_0^T \sum_{n=1}^{\infty} \langle k_t, h_n \rangle_{\mathcal{H}}^2 dt = \int_0^T \|k_t\|^2_{\mathcal{H}} dt = \frac{T^4}{6}. \quad (7)$$

Thus $A$ is of trace class and $\text{tr} A = T^4/6$. \hfill $\square$

We next recall the following assertion achieved in [5, 7].
Lemma 2. Let $U : \mathcal{H} \rightarrow \mathcal{H}$ be a Hilbert-Schmidt operator admitting a decomposition $U = U_V + U_F$ with a Volterra operator $U_V : \mathcal{H} \rightarrow \mathcal{H}$ and a bounded operator $U_F : \mathcal{H} \rightarrow \mathcal{H}$ possessing the finite-dimensional range $R(U_F)$.

(i) For sufficiently small $\lambda \in \mathbb{R}$, it holds that

$$
\int_{\mathcal{W}} \exp(\lambda Q_U/2) dP = \{\det(I - \lambda U_F(I - \lambda U_V)^{-1})\}^{-1/2} e^{-(\lambda/2) \text{tr} U_F}.
$$

(ii) Let $E$ be a subspace of $R(U_F)$ and $\{\eta_1, \ldots, \eta_d\}$ be a basis of $E$. Define the Wiener functional $\eta : \mathcal{W} \rightarrow \mathbb{R}^d$ by $\eta = (\nabla^* \eta_1, \ldots, \nabla^* \eta_d)$. Then, for sufficiently small $\lambda \in \mathbb{R}$, it holds that

$$
\int_{\mathcal{W}} \exp(\lambda Q_U/2) \delta_0(\eta) dP = \frac{1}{\sqrt{(2\pi)^d \det C(\eta)}} \{\det(I - \lambda U_F(I - \lambda U_V)^{-1})\}^{-1/2} e^{-(\lambda/2) \text{tr} U_F},
$$

where $U_F^2 = -\pi_E U_V + (I - \pi_E) U_F$, $\pi_E : \mathcal{H} \rightarrow \mathcal{H}$ being the orthogonal projection onto $E$, and $C(\eta) = (\langle \eta_i, \eta_j \rangle_H)_{1 \leq i, j \leq d}$.

Proof. The essential part of the proof can be found in [5, 7]. For the completeness, we give the proof.

Due to the splitting property of the Wiener measure, it holds that

$$
\int_{\mathcal{W}} \exp(\lambda Q_U/2) dP = \{\det_2(I - \lambda U)\}^{-1/2},
$$

where $\det_2$ denotes the Carleman-Fredholm determinant. For example, see [3, 7]. Observe that, for Hilbert-Schmidt operators $C, D : \mathcal{H} \rightarrow \mathcal{H}$ such that $C$ is of trace class, it holds that

$$
\det_2(I + C)(I + D) = \det_2(I + C) \det_2(I + D) e^{-\text{tr} C(I + D)}.
$$

Since $\det_2(I - \lambda U_V) = 1$, substituting $C = -\lambda U_F(I - \lambda U_V)^{-1}$ and $D = -\lambda U_V$ into (9), we obtain that

$$
\det_2(I - \lambda U) = \det_2(I - \lambda U_F(I - \lambda U_V)^{-1}) e^{\lambda \text{tr} U_F}.
$$

Thus (7) has been shown.

Put $U_0 = (I - \pi_E) U(I - \pi_E)$ and $U_1 = \pi_E U \pi_E$. Then it holds ([7, 12]) that

$$
\int_{\mathcal{W}} \exp(\lambda Q_U/2) \delta_0(\eta) dP = \frac{1}{\sqrt{(2\pi)^d \det C(\eta)}} \{\det_2(I - \lambda U_0)\}^{-1/2} e^{-(\lambda/2) \text{tr} U_1}.
$$

Setting $U^2 = (I - \pi_E) U$, and substituting $C = -\lambda U_F^2(I - \lambda U_V)^{-1}$ and $D = -\lambda U_V$ into (9), we see that

$$
\det_2(I - \lambda U_0) = \det_2(I - \lambda U^2) = \det_2(I - \lambda U_F^2(I - \lambda U_V)^{-1}) e^{\lambda \text{tr} U_1^2}.
$$

Since $\text{tr} U_1^2 + \text{tr} U_1 = \text{tr} U_F$, we obtain (8).
It is not known if, by just watching specific shape of quadratic Wiener functional, one can tell that the associated Hilbert-Schmidt operator admits a decomposition as a sum of a Volterra operator and a bounded operator with finite dimensional range. However, in our situation, we know a priori that the operator \( A \) admits such a decomposition. Namely, the Hilbert-Schmidt operator \( B \) associated with \( q_0 \) admits such a decomposition ([7]). Being equal to the square of \( B \) (see Remark 1 below), so does \( A \). The following lemma gives the concrete expression of the decomposition of \( A \).

**Lemma 3.** Define \( I, A_V, A_F : \mathcal{H} \to \mathcal{H} \) by

\[
Ih(t) = \int_0^t h(s)ds, \quad t \in [0, T],
\]

\[
A_V h = I^\dagger h, \quad A_F h = \left\{ \frac{T^2}{2} I h(T) - T^3 h(T) \right\} \eta_1 - \frac{1}{6} I h(T) \eta_2, \quad h \in \mathcal{H},
\]

where \( \eta_j(t) = t^{2j-1}, t \in [0, T], j = 1, 2 \). Then (i) \( A = A_V + A_F \), (ii) \( A_V \) is a Volterra operator, (iii) \( R(A_F) = \{ an_1 + bn_2 \mid a, b \in \mathbb{R} \} \), (iv) \( \text{tr} A_F = \text{tr} A \), and (v) for \( \lambda > 0 \), it holds that

\[
(I - \lambda A_V)^{-1} h(t) = \frac{1}{2} \int_0^t h(s) \{ \cosh(\lambda^{1/4}(t-s)) + \cos(\lambda^{1/4}(t-s)) \} ds,
\]

\( h \in \mathcal{H}, t \in [0, T] \). (10)

**Proof.** The assertions (i) and (ii) follow from the very definitions of \( A \) and \( A_V \). The assertion (iv) is an immediate consequence of these and Lemma 1. By the definition of \( A_F \), the inclusion \( R(A_F) \subset \{ an_1 + bn_2 \mid a, b \in \mathbb{R} \} \) is obvious. To see the converse inclusion, it suffices to notice that \( A_F \eta_1 = (5T^4/24) \eta_1 - (T^2/12) \eta_2 \) and \( A_F \eta_2 = (7T^6/60) \eta_1 - (T^4/24) \eta_2 \). Thus (iii) has been verified.

To see (v), let \( (I - \lambda A_V)g = h \) and \( f = I^\dagger g \). It then holds that \( f^{(4)} - \lambda f = h \), where \( f^{(n)} = (d/dt)^n f \). This leads us to the ordinary differential equation;

\[
\frac{d}{dt} \begin{pmatrix} f \\ f^{(1)} \\ f^{(2)} \\ f^{(3)} \end{pmatrix} = \begin{pmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ \lambda & 0 & 0 & 0 \end{pmatrix} \begin{pmatrix} f \\ f^{(1)} \\ f^{(2)} \\ f^{(3)} \end{pmatrix} + \begin{pmatrix} 0 \\ 0 \\ 0 \\ h \end{pmatrix}, \quad \begin{pmatrix} f(0) \\ f^{(1)}(0) \\ f^{(2)}(0) \\ f^{(3)}(0) \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \\ 0 \end{pmatrix}.
\]

It is then easily seen that

\[
f^{(3)}(t) = \frac{1}{2} \int_0^t h(s) \{ \cosh(\lambda^{1/4}(t-s)) + \cos(\lambda^{1/4}(t-s)) \} ds.
\]

Since \( g = f^{(4)} \), this implies the identity (10). \( \square \)

**Lemma 4.** The identity (1) holds.

**Proof.** Let \( \eta_1, \eta_2 \in \mathcal{H} \) be as described in Lemma 3, and put \( f_j = (I - \lambda A_V)^{-1} \eta_j, j = 1, 2 \). By virtue of Lemma 3, we have that

\[
I f_1(t) = \frac{\lambda^{-1/2}}{2} \{ \cosh(\lambda^{1/4}t) - \cos(\lambda^{1/4}t) \},
\]

\[
I^3 f_1(t) = \frac{\lambda^{-1}}{2} \{ \cosh(\lambda^{1/4}t) + \cos(\lambda^{1/4}t) - 2 \},
\]

\[
I^3 f_2(t) = 3\lambda^{-1} \{ \cosh(\lambda^{1/4}t) + \cos(\lambda^{1/4}t) - 2 \},
\]

\[
I^3 f_2(t) = 3\lambda^{-3/2} \{ \cosh(\lambda^{1/4}t) - \cos(\lambda^{1/4}t) \} - 3\lambda^{-1}t^2.
\]
Hence, if we set \( \alpha_\lambda = \cosh(\lambda^{1/4} T) \) and \( \beta_\lambda = \cos(\lambda^{1/4} T) \), then

\[
(I - \lambda A_F)(I - \lambda A_V)^{-1} \eta_1
= \left\{ -\frac{T^2 \lambda^{1/2}}{4}(\alpha_\lambda - \beta_\lambda) + \frac{1}{2}(\alpha_\lambda + \beta_\lambda) \right\} \eta_1 + \frac{\lambda^{1/2}}{12}(\alpha_\lambda - \beta_\lambda)\eta_2,
\]

\[
(I - \lambda A_F)(I - \lambda A_V)^{-1} \eta_2
= \left\{ -\frac{3T^2}{2}(\alpha_\lambda + \beta_\lambda) + 3\lambda^{-1/2}(\alpha_\lambda - \beta_\lambda) \right\} \eta_1 + \frac{1}{2}(\alpha_\lambda + \beta_\lambda)\eta_2.
\]

Thus, by virtue of (iii), it holds that

\[
\det(I - \lambda A_F)(I - \lambda A_V)^{-1}
= \det \left( \begin{array} {cc}
-\frac{T^2 \lambda^{1/2}}{4}(\alpha_\lambda - \beta_\lambda) + \frac{1}{2}(\alpha_\lambda + \beta_\lambda) & \frac{\lambda^{1/2}}{12}(\alpha_\lambda - \beta_\lambda) \\
-\frac{3T^2}{2}(\alpha_\lambda + \beta_\lambda) + 3\lambda^{-1/2}(\alpha_\lambda - \beta_\lambda) & \frac{1}{2}(\alpha_\lambda + \beta_\lambda)
\end{array} \right) = \alpha_\lambda \beta_\lambda.
\]

This implies the identity (1), because Lemmas 1, 2, and 3 yield that

\[
\int_W \exp(\lambda q/2) dP = \{\det(I - \lambda A_F)(I - \lambda A_V)^{-1}\}^{-1/2}.
\]

Lemma 5. The identity (2) holds.

Proof. Let \( \eta_j, j = 1, 2 \), be as in Lemma 3 (iii), and \( E = \{c\eta_1 | c \in \mathbb{R}\} \). Define \( A_F^k \) as described in Lemma 2 with \( U = A, U_V = A_V \), and \( U_F = A_F \). Since \( \pi_E h = (h(T)/T)\eta_1 \) for any \( h \in H \), we have that

\[
A_F^k h = \left\{ -\frac{1}{T}I^1 h(T) + \frac{T^2}{6}I^2 h(T) \right\} \eta_1 - \frac{1}{6}I h(T) \eta_2.
\]

Let \( f_1, f_2 \) be as in the proof of Lemma 4. Then we see that

\[
I^1 f_1(t) = \frac{\lambda^{-5/4}}{2} (\sinh(\lambda^{1/4} t) + \sin(\lambda^{1/4} t)) - \lambda^{-1} t,
\]

\[
I^1 f_2(t) = 3\lambda^{-7/4} \{\sinh(\lambda^{1/4} t) - \sin(\lambda^{1/4} t)\} - \lambda^{-1} t^3.
\]

Hence, if we put \( \sigma_\lambda = \sinh(\lambda^{1/4} T) \) and \( \tau_\lambda = \sin(\lambda^{1/4} T) \), then

\[
(I - \lambda A_F^k)(I - \lambda A_V)^{-1} \eta_1
= \left\{ \frac{\lambda^{-1/4}}{2T}(\sigma_\lambda + \tau_\lambda) - \frac{T^2 \lambda^{1/2}}{12}(\alpha_\lambda - \beta_\lambda) \right\} \eta_1 + \frac{\lambda^{1/2}}{12}(\alpha_\lambda - \beta_\lambda)\eta_2,
\]

\[
(I - \lambda A_F^k)(I - \lambda A_V)^{-1} \eta_2
= \left\{ \frac{\lambda^{-3/4}}{T}(\sigma_\lambda - \tau_\lambda) - \frac{T^2}{2}(\alpha_\lambda + \beta_\lambda) \right\} \eta_1 + \frac{1}{2}(\alpha_\lambda + \beta_\lambda)\eta_2.
\]

Since \( R(A_F^k) \subset R(A_F) \), by Lemma 3 (ii), this yields that

\[
\det(I - \lambda A_F^k)(I - \lambda A_V)^{-1}
= \det \left( \begin{array} {cc}
\frac{\lambda^{-1/4}}{2T}(\sigma_\lambda + \tau_\lambda) - \frac{T^2 \lambda^{1/2}}{12}(\alpha_\lambda - \beta_\lambda) & \frac{\lambda^{1/2}}{12}(\alpha_\lambda - \beta_\lambda) \\
\frac{\lambda^{-3/4}}{T}(\sigma_\lambda - \tau_\lambda) - \frac{T^2}{2}(\alpha_\lambda + \beta_\lambda) & \frac{1}{2}(\alpha_\lambda + \beta_\lambda)
\end{array} \right)
= \lambda^{-1/4} \left\{ \sigma_\lambda \beta_\lambda + \tau_\lambda \alpha_\lambda \right\}.
\]
The identity (2) follows from this, because Lemmas 1, 2, and 3 imply that
\[
\int^W \exp(\lambda q/2) \delta_0(w(T)) dP = \int^W \exp(\lambda Q_A/2) \delta_0(\nabla^* \eta) dP e^{(\lambda/2) \text{tr} A} \\
= \frac{1}{\sqrt{2\pi T}} \left\{ \det(I - \lambda A^2(I - \lambda A)^{-1}) \right\}^{-1/2}.
\]

**Remark 1.** It may be interesting to see that (1) is also shown by using the infinite product expression. Namely, define \(B : \mathcal{H} \rightarrow \mathcal{H}\) by
\[
Bh(t) = \int_0^t \int_s^T h(u) du \, ds, \quad h \in \mathcal{H}, \; t \in [0, T].
\]
Then there exists an orthonormal basis \(\{h_n\}_{n=0}^{\infty}\) of \(\mathcal{H}\) so that
\[
B = \sum_{n=0}^{\infty} \left( \frac{T}{(n + \frac{1}{2})\pi} \right)^2 h_n \otimes h_n.
\]
See [10]. Since \(A = B^2\), it holds that
\[
A = \sum_{n=0}^{\infty} \left( \frac{T}{(n + \frac{1}{2})\pi} \right)^4 h_n \otimes h_n. \tag{11}
\]
In conjunction with Lemma 1, this implies that
\[
q = Q_A + \text{tr} A = \sum_{n=0}^{\infty} \left( \frac{T}{(n + \frac{1}{2})\pi} \right)^4 (\nabla^* h_n)^2.
\]
Due to the splitting property of the Wiener measure, we then obtain that
\[
\int^W \exp(\lambda q/2) dP = \left( \prod_{n=0}^{\infty} \left\{ 1 - \lambda \left( \frac{T}{(n + \frac{1}{2})\pi} \right)^4 \right\} \right)^{-1/2} \\
= \left( \prod_{n=0}^{\infty} \left\{ 1 + \lambda^{1/2} \left( \frac{T}{(n + \frac{1}{2})\pi} \right)^2 \right\} \prod_{n=0}^{\infty} \left\{ 1 - \lambda^{1/2} \left( \frac{T}{(n + \frac{1}{2})\pi} \right)^2 \right\} \right)^{-1/2}.
\]

Due to the infinite product expressions of \(\cosh x\) and \(\cos x\), this implies (1).

### 3 Proof of Theorem 1 (ii)

In this section, we shall show Theorem 1 (ii). We first describe how we realize \(\{\cosh z \cos z\}^{1/2}\) for complex number \(z\). Represent \(z \in \mathbb{C}\) as \(z = re^{i\theta}\) with \(r \geq 0\) and \(-\frac{3}{2}\pi \leq \theta < \frac{3}{2}\pi\) to define \(\sqrt{z} = r^{1/2}e^{i\theta/2}\), where \(i^2 = -1\). The
Riemann surface of the 2-valued function \( z^{1/2} \) is realized by switching \( \sqrt{z} \) and \( -\sqrt{z} \) on the half line consisting of \( i\xi, \xi < 0 \). Set
\[
G(z) = \begin{cases} 
\sqrt{\cos z}, & \text{if a) } |\text{Re } z| < \frac{\pi}{2}, \text{ or} \\
\text{b) } \text{Im } z > 0, -\frac{3\pi}{2} + 4k\pi \leq \text{Re } z < \frac{\pi}{2} + 4k\pi (k \in \mathbb{Z}), \text{ or} \\
\text{c) } \text{Im } z < 0, -\frac{3\pi}{2} + 4k\pi \leq \text{Re } z < \frac{\pi}{2} + 4k\pi (k \in \mathbb{Z}), \\
-\sqrt{\cos z}, & \text{if a) } \text{Im } z > 0, \frac{5\pi}{2} + 4k\pi \leq \text{Re } z < \frac{5\pi}{2} + 4k\pi (k \in \mathbb{Z}), \text{ or} \\
\text{b) } \text{Im } z < 0, \frac{3\pi}{2} + 4k\pi \leq \text{Re } z < \frac{7\pi}{2} + 4k\pi (k \in \mathbb{Z}).
\end{cases}
\]

Then \( G \) is holomorphic on \( \mathbb{C} \setminus \{ \xi | \xi \in \mathbb{R}, |\xi| \geq \pi/2 \} \), and realizes \( \{ \cos z \}^{1/2} \). Hence \( G(z)G(iz) \) is holomorphic on \( D_0 \equiv \mathbb{C} \setminus \{ \xi, i\xi | \xi \in \mathbb{R}, |\xi| \geq \pi/2 \} \) and does not vanish in \( D_0 \). Recalling that \( \cosh z = \cos(iz) \), we write \( \{ \cosh(z \cos z) \}^{1/2} \) for \( G(z)G(iz) \).

We next extend the identity (1) holomorphically. Since there exists \( \delta > 0 \) such that \( \exp(\delta q/2) \) is integrable with respect to \( P \) and \( q \geq 0 \), the mapping
\[
\{ z \in \mathbb{C} | \text{Re } z < \delta \} \ni z \mapsto \int_{\mathbb{W}} \exp(zq/2)dP
\]
is holomorphic. \( \{ \cosh(zT \cos(zT)) \}^{-1/2} \) being holomorphic in \( D_0 \), we can find a domain \( D \subset \mathbb{C} \) such that
\[
D \supset \left\{ r e^{i\theta} \bigg| r \geq 0, \theta \in \bigcup_{k=0}^{3} \left[ \frac{\pi}{8} + \frac{k\pi}{2}, \frac{3\pi}{8} + \frac{k\pi}{2} \right] \right\}; \quad \text{and} \quad \int_{\mathbb{W}} \exp(z^4q/2)dP = \frac{1}{\{ \cosh(zT) \cos(zT) \}^{1/2}} \quad \text{for every } z \in D. \tag{12}
\]

By (11) and Lemma 1, as an easy application of the Malliavin calculus, we see that the distribution of \( q/2 \) on \( \mathbb{R} \) admits a smooth density function \( p_T(x) \) ([14, Lemma 3.1]). Since \( q \geq 0 \), \( p_T(x) = 0 \) for \( x \leq 0 \). Hence, in what follows, we always assume that \( x > 0 \). By the inverse Fourier transformation, we have that
\[
p_T(x) = \frac{1}{2\pi} \int_{\mathbb{R}} e^{-ixt} I(t)dt, \quad \text{where } I(t) = \int_{\mathbb{W}} \exp(itq/2)dP. \tag{13}
\]

For \( R > 0 \), let \( \Gamma_+(R) \) (resp. \( \Gamma_-(R) \)) be the directed line segment in \( \mathbb{C} \) starting at the origin and ending at \( Re^{ix/8} \) (resp. \( Re^{-ix/8} \)). Then, parameterizing \( \Gamma_\pm(R) \) by \( t^{1/4}e^{\pm ix/8}, t \in [0, R^4] \), we have that
\[
\int_{\Gamma_{\pm}(R)} f(z^4)z^3dz = \pm i \int_0^{R^4} f(\pm it)dt
\]
for any piecewise continuous function \( f \) on \( i\mathbb{R} \), where and in the sequel, the symbol \( \pm \) takes \( + \) or \( - \) simultaneously. Plugging this into (13), and then substituting (12), we obtain that
\[
2\pi p_T(x) = \lim_{R \to \infty} \left\{ 4i \int_{\Gamma_-(R)} \frac{z^3e^{-xz^4}}{\{ \cosh(zT) \cos(zT) \}^{1/2}}dz \\
- 4i \int_{\Gamma_+(R)} \frac{z^3e^{-xz^4}}{\{ \cosh(zT) \cos(zT) \}^{1/2}}dz \right\}. \tag{14}
\]
Thanks to the estimation that
\[ |\cosh(u + iv)\cos(u + iv)|^2 \geq \sinh^2 u \max\{\cos^2 u, \sinh^2 v\}, \]
it is a routine exercise of complex analysis to show that
\[
\lim_{R \to \infty} \int_{\Gamma_{\pm}(R)} \frac{z^3 e^{-zx^4}}{(\cosh(zT) \cos(zT))^{1/2}} \, dz = \int_0^\infty \frac{u^3 e^{-ux^4}}{\lim_{h \to 0} \{\cosh(uT \pm ih) \cos(uT \pm ih)\}^{1/2}} \, du. \tag{15}
\]
Moreover, by the definition of \{\cosh z \cos z\}^{1/2}, we have that
\[
\lim_{h \to 0} \{\cosh(uT \pm ih) \cos(uT \pm ih)\}^{1/2} = \begin{cases} \\
\sqrt{\cosh(uT) \cos(uT)}, & \text{if } -\pi - (\pm \frac{\pi}{2}) + 4k\pi \leq uT < \pi - (\pm \frac{\pi}{2}) + 4k\pi, \\
-\sqrt{\cosh(uT) \cos(uT)}, & \text{if } \pi - (\pm \frac{\pi}{2}) + 4k\pi \leq uT < 3\pi - (\pm \frac{\pi}{2}) + 4k\pi,
\end{cases}
\]
Substitute this and (15) into (14) to see that
\[
2\pi p_T(x) = 8i \sum_{k=0}^{\infty} \int_{((\pi/2)+2k\pi)/T}^{(3\pi/2)+2k\pi)/T} \frac{(-1)^k u^3 e^{-ux^4}}{\sqrt{\cosh(uT) \cos(uT)}} \, du.
\]
This implies Theorem 1 (ii), because
\[
\int_{((\pi/2)+2k\pi)/T}^{(3\pi/2)+2k\pi)/T} \frac{u^3 e^{-ux^4}}{\sqrt{\cosh(uT) \cos(uT)}} \, du = \frac{1}{iT^4} \int_0^{\pi/2} \frac{v + (2k + 1)\pi}{\sqrt{\cosh\{v + (2k + 1)\pi\} \cos v}} \, dv
- \frac{1}{iT^4} \int_0^{\pi/2} \frac{v - (2k + 1)\pi}{\sqrt{\cosh\{v - (2k + 1)\pi\} \cos v}} \, dv.
\]

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