Generalization Properties of Doubly Online Learning Algorithms

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Abstract

Doubly online learning algorithms are scalable kernel methods that perform very well in practice. However, their generalization properties are not well understood and their analysis is challenging since the corresponding learning sequence may not be in the hypothesis space induced by the kernel. In this paper, we provide an in-depth theoretical analysis for different variants of doubly online learning algorithms within the setting of nonparametric regression in a reproducing kernel Hilbert space and considering the square loss. Particularly, we derive convergence results on the generalization error for the studied algorithms either with or without an explicit penalty term. To the best of our knowledge, the derived results for the unregularized variants are the first of this kind, while the results for the regularized variants improve those in the literature. The novelties in our proof are a sample error bound that requires controlling the trace norm of a cumulative operator, and a refined analysis of bounding initial error.

1 Introduction

In nonparametric regression, we are given a set of samples of the form \(\{(x_i, y_i)\}_{i=1}^T\), where each \(x_i \in \mathbb{R}^d\) is an input, \(y_i\) is a real-valued output, and the samples are drawn i.i.d. from an unknown distribution on \(\mathbb{R}^d \times \mathbb{R}\). The goal is to learn a function which can be used to predict future outputs based on the inputs.

Kernel methods \([16, 4, 19]\) are a popular nonparametric technique based on choosing a hypothesis space to be a reproducing kernel Hilbert space (RKHS). Online learning algorithms \([7, 2]\) (often called stochastic gradient methods \([12, 10]\) in convex optimization) are among the most efficient and fast learning algorithms. At each iteration, they compute a gradient estimate with respect to a new sample point and then updates the current solution by subtracting the scaled gradient estimate. In general, the computational complexities for training are \(O(T + Td)\) in space and \(O(T^2d)\) in time, due to the nonlinearity of kernel methods. In recent years, different types of online learning algorithms, either with or without an explicit penalty term, have been proposed and analyzed, see e.g. \([2, 21, 23, 15, 20, 13, 6, 9]\) and references therein.

In classic online learning algorithms, all sampling points need be stored for testing. Thus, the implementation of the algorithm may be difficult in learning problems with
high-dimensional inputs and large datasets. To tackle such a challenge, an alternative stochastic method, called doubly online learning algorithm was proposed in [5]. The new algorithm is based on the random feature approach proposed in [11]. The latter result is based on Bochner’s theorem and shows that most shift-invariant kernel functions can be expressed as an inner product of some suitable random features. Thus the kernel function at each iteration in the original online learning algorithm can be estimated (or replaced) by a random feature. As a result, the new algorithm allows us to avoid keeping all the sample points since it only requires generating the random features and recovers past random resampling them using specific random seeds [5]. The computational complexities of the algorithm are \(O(T)\) (independent of the dimension of the data) in space and \(O(T^2d)\) in time. Numerical experiments given in [5], show that the algorithm is fast and comparable with state-of-the-art algorithms. Convergence results with respect to the solution of regularized expected risk minimization were derived in [5] for doubly online learning algorithms with regularization, considering general Lipschitz and smooth losses.

In this paper, we study generalization properties of doubly online learning algorithms in the framework of nonparametric regression with the square loss. Our contributions are theoretical. First, for the first time, we prove generalization error bounds for doubly online learning algorithms without regularization, either using a fixed constant step-size or a decaying step-size. Compared with the regularized version studied in [5], doubly online learning algorithms without regularization do not involve the model selection of regularization parameters, and thus it may have some computational advantages in practice. Secondly, we also prove generalization error bounds for doubly online learning algorithms with regularization. Compared with the results in [5], our convergence rates are faster and do not require the bounded assumptions on the gradient estimates as in [5], see the discussion section for details. The key ingredients to our proof are an error decomposition and an induction argument, which enables us to derive total error bounds provided that the initial (or approximation) and sample errors can be bounded. The initial and sample errors are bounded using properties from integral operators and functional analysis. The difficulty in the analysis is the estimation of the sample error, since the sequence generated by the algorithm may not be in the hypothesis space. The novelty in our proof is the estimation of the sample error involving upper bounding a trace norm of an operator, and a refined analysis of bounding the initial error.

The rest of the paper is organized as follows. In the next section, we introduce the learning setting we consider and the doubly online learning algorithms. In Section 3 we present the main results on generalization properties for the studied algorithms and give some simple discussions. Sections 4 to 7 are devoted to the proofs of all the main results.

2 Learning Setting and Doubly Online Learning

Learning a function from a given finite number of instances through efficient and practical algorithms is the basic goal of learning theory. Let the input space \(X\) be a closed subset of Euclidean space \(\mathbb{R}^d\), the output space \(Y \subset \mathbb{R}\), and \(Z = X \times Y\). Let \(\rho\) be a fixed
Borel probability measure on $Z$, with its induced marginal measure on $X$ and conditional measure on $Y$ given $x \in X$ denoted by $\rho_X(\cdot)$ and $\rho(\cdot|x)$ respectively. In statistical learning theory, the probability measure $\rho$ is unknown, but only a set of sample points $z = \{z_i = (x_i, y_i)\}_{i=1}^T$ of size $T \in \mathbb{N}$ is given. Here, we assume that the sample points are independently and identically drawn from the distribution $\rho$.

The quality of a function $f : X \to Y$ can be measured in terms of the expected risk with the square loss defined as

$$\mathcal{E}(f) = \int_Z (f(z) - y)^2 d\rho(z).$$

In this case, the function minimizing the expected risk over all measurable functions is the regression function given by

$$f_\rho(x) = \int_Y y d\rho(y|x), \quad x \in X.$$  \hfill (2.2)

For any $f \in L^2_\rho$, it is easy to prove that

$$\mathcal{E}(f) - \mathcal{E}(f_\rho) = \|f - f_\rho\|_\rho^2.$$  \hfill (2.3)

Here, $L^2_\rho$ is the Hilbert space of square integral functions with respect to $\rho_X$, with its induced norm given by $\|f\|_\rho = \|f\|_{L^2_\rho} = (\int_X |f(x)|^2 d\rho_X(x))^{1/2}$. Throughout this paper we assume that $\int_Y y^2 d\rho < \infty$. Thus, using (2.3) with $f = 0$, $\mathcal{E}(f_\rho) + \|f_\rho\|_\rho^2$ is finite.

Kernel methods is based on choosing a hypothesis space as a reproducing kernel Hilbert space (RKHS). Recall that a reproducing kernel $K$ is a symmetric function $K : X \times X \to \mathbb{R}$ such that $(K(u_i, u_j))_{i,j=1}^t$ is positive semidefinite for any finite set of points $\{u_i\}_{i=1}^t$ in $X$. The kernel $K$ defines a RKHS $(\mathcal{H}_K, \|\cdot\|_K)$ as the completion of the linear span of the set $\{K_x(\cdot) := K(x, \cdot) : x \in X\}$ with respect to the inner product $(K_x, K_u)_K := K(x, u)$.

Online learning is an important class of efficient algorithms to perform learning tasks. Over the past few decades, several variants of online learning algorithms have been studied, many of which take the form of

$$h_{t+1} = (1 - \lambda)h_t - \eta_t (h_t(x_t) - y_t)K_{x_t}, \quad t = 1, \ldots, T,$$  \hfill (2.4)

and generalization properties have been derived. Here $\{\eta_t > 0\}$ is a step-size sequence, and $\lambda$ can be chosen as a positive constant depending on the sample size $\lambda(T) > 0$ \cite{21, 20}, or to be zero \cite{23, 15, 9}. In general, the computational complexities of the algorithm are $O(T + Td)$ in space and $O(T^2d)$ in time.

According to Bochner’s theorem, a continuous kernel $K(x, x') = k(x - x')$ on $\mathbb{R}^d$ is positive definite if and only if $k(\delta)$ is the Fourier transform of a non-negative measure. Thus, most shift-invariant kernel functions can be expressed as an integration of some random features. A basic example for the Gaussian kernel is detailed as follows.

**Example 2.1 (Random Fourier Features \cite{11}).** Let the Gaussian kernel

$$K(x, x') = e^{-\frac{|x-x'|^2}{2\sigma^2}},$$
for some $\sigma > 0$. Then according to Fourier inversion theorem, and by a simple calculation, one can prove that

$$K(x, x') = \frac{\sigma^d}{(\sqrt{2\pi})^{d+2}} \int_{\mathbb{R}^d} \int_0^{2\pi} \sqrt{2 \omega^\top x + b} \sqrt{2 \omega^\top x' + b} e^{-\frac{\sigma^2 |\omega|^2}{2}} d\omega db.$$

Replacing $K_{x_i}$ in (2.3) by an unbiased estimate with respect to a random feature, we get the doubly online learning algorithm\footnote{Note that [4] studied the algorithm with a general convex loss function. Specializing to the square loss leads to the algorithm (2.6).}. Let $\mu$ be another probability measure on a measurable set $V$, and $\phi : V \times X \to \mathbb{R}$ a square-integrable (with respect to $\mu \otimes \rho_X$) function. Assume that the kernel $K$ can be written as \footnote{Note that [5] studied the algorithm with a general convex loss function. Specializing to the square loss leads to the algorithm (2.6).}

$$K(x, x') = \int_V \phi(v, x)\phi(v, x')d\mu = \langle \phi(\cdot, x), \phi(\cdot, x') \rangle_{L^2_\mu}, \quad \forall x, x' \in X. \quad (2.5)$$

Let $\nu_1, \ldots, \nu_T$ be $T$ elements in $V$, i.i.d. according to the distribution $\mu$. The doubly online learning algorithm associated with random features $\{\phi_{\nu_t}\}$ is defined by $f_1 = 0$ and

$$f_{t+1} = (1 - \eta_t \lambda) f_t - \eta_t (f_t(x_t) - y_t) \phi_{\nu_t}(x_t)\phi_{\nu_t}, \quad t = 1, \ldots, T. \quad (2.6)$$

The computational complexities of the algorithm are $O(T)$ (independent of the dimension of the data) in space and $O(T^2d)$ in time.

In this paper, we study the generalization properties of Algorithm (2.6), either with a fixed constant step-size $\{\eta_t = \eta\}$, or a decaying step-size $\{\eta_t = \eta t^{-\theta}\}$, $\theta \in (0, 1)$, where $\lambda \geq 0$. Under basic assumptions in the standard learning theory and with appropriate choices of parameters, we shall prove upper bounds for the excess expected risks, i.e., $\mathbb{E}\|f_T - f_\rho\|^2$.  

**Notation** $\mathbb{N}$ denotes the set of positive integers. $(a)_+ = \max(a, 0)$ for any $a \in \mathbb{R}$. For $t \in \mathbb{N}$, the set $\{1, 2, \ldots, t\}$ is denoted by $[t]$. We will use the following conventional notations $0^0 = 1$, $0/0 = 1$, $\prod_{j=t+1}^t a_j = 1$ and $\sum_{j=t+1}^t a_j = 0$ for any sequence of real numbers $\{a_j\}_{j \in \mathbb{N}}$. For any operator $L : H \to H$, on a Hilbert space $H$, $I$ denotes the identity operator on $H$ and $\Pi_{t+1}^T(L) = \prod_{k=t+1}^T (I - \eta_k L)$ when $t \in [T - 1]$ and $\Pi_{t+1}^T(L) = I$. For a given bounded operator $L : \mathcal{L}^2_\rho \to \mathcal{L}^2_\rho$, $\|L\|$ denotes the operator norm of $L$, i.e., $\|L\| = \sup_{f \in \mathcal{L}^2_\rho, \|f\|=1} \|Lf\|_\rho$. For two positive sequences $\{a_i\}_i$ and $\{b_i\}_i$, $a_i \leq O(b_i)$ (or $a_i \lesssim b_i$) stands for $a_i \leq C b_i$ for some positive constant $C$ (independent of $i$) for all $i$. The indicator function of a subset $A$ is denoted by $1_A$.  

## 3 Generalization Properties for Doubly Online Learning

In this section, after introducing some basic assumptions, we state our main results, following with simple discussions.

### 3.1 Assumptions

We first make the following basic assumption, with respect to the RKHS and its associated kernel as well as the underlying features.
Assumption 1. $\mathcal{H}_K$ is separable and $K$ is measurable. Furthermore, there exists a positive constant $\kappa \geq 1$, such that $K(x, x) \leq \kappa^2$ and $\phi_v(x)\phi_v(x') \leq \kappa^2$ almost surely with respect to $\mu \otimes \rho_X$.

The bounded assumptions on the kernel function and random features are fairly common. For example, when $K(\cdot, \cdot)$ is a Gaussian kernel $K(x, x') = e^{-\|x-x'\|^2/2}$, we have $\kappa^2 = 1$.

To present our next assumption, we need to introduce the integral operator $L_K : L^2_\rho \to L^2_\rho$, defined as

$$L_K(f) = \int_X f(x) K_x d\rho_X(x). \quad (3.1)$$

Under Assumption 1, the operator $L_K$ is known to be symmetric, positive definite and trace class. Thus, its power $L_\kappa$ is well defined for $\kappa > 0$. Particularly, we know that

$$\|L_\kappa^\frac{1}{2} g\|_\rho, \text{ for all } g \in \mathcal{H}_K.$$ 

We make the following assumption on the regularity of the regression function.

Assumption 2. There exists $\kappa > 0$ and $R > 0$, such that $\|L_\kappa^\frac{1}{2} f_\rho\|_\rho \leq R$.

The above assumption is very standard in nonparametric regression. It characterizes how big is the subspace the target function $f_\rho$ lies in. Particularly, the bigger the $\kappa$ is, the more stringent is the assumption and the smaller is the subspace, since $L_\kappa^\frac{1}{2}(L^2_\rho) \subseteq L_\kappa^\frac{1}{2}(L^2_\rho)$ when $\kappa_1 \geq \kappa_2$. Moreover, when $\kappa = 0$, we are making no assumption as $\|f_\rho\|_\rho < \infty$ holds trivially, while for $\kappa = 1/2$, we are requiring $f_\rho \in \mathcal{H}_K$.

Finally, the last assumption is related to the capacity of the RKHS.

Assumption 3. For some $\gamma \in [0, 1]$ and $c_\gamma > 0$, $L_K$ satisfies

$$\text{tr}(L_K(L_K+\lambda I)^{-1}) \leq c_\gamma \lambda^{-\gamma}, \text{ for all } \lambda > 0. \quad (3.3)$$

The left hand-side of (3.3) is called as the effective dimension or the degrees of freedom. It can be related to covering/entropy number conditions, see [18, 19] for further details. Assumption 3 is always true for $\gamma = 1$ and $c_\gamma = \kappa^2$, since $L_K$ is a trace class operator which implies the eigenvalues of $L_K$, denoted as $\sigma_i$, satisfy $\text{tr}(L_K) = \sum_i \sigma_i \leq \kappa^2$. The case $\gamma = 1$ is referred to as the capacity independent setting. Assumption 3 with $\gamma \in [0, 1]$ allows to derive better error rates. It is satisfied, e.g., if the eigenvalues of $L_K$ satisfy a polynomial decaying condition $\sigma_i \sim i^{-1/\gamma}$, or with $\gamma = 0$ if $L_K$ is finite rank.

3.2 Main Results

We are now ready to present our main results, whose proofs are postponed to Section 7. Our first and main result provides generalization error bounds for the studied algorithms with $\lambda = 0$ and a constant (but depending on $T$) step-size.
Theorem 3.1. Under Assumptions[1][2][3] Let \( \{f_t\}_{t \in [T]} \) be generated by \( \{2, 0\} \) with \( \lambda = 0 \), \( \eta_t = \eta(T) \) for all \( t \in [T] \) such that
\[
0 < \eta(T)^{\gamma+1} T^{\gamma} \ln(2T) \leq \frac{1}{4\kappa^2(c_\gamma + \kappa^2)}.
\]

Then,
\[
\mathbb{E}[\|f_{T+1} - f_\rho\|^2] \leq O((\eta(T)T)^{-2\zeta} + \eta(T)^{\gamma+1} T^{\gamma} \ln T). \tag{3.5}
\]

According to (3.4), to derive a convergence result from the above theorem, one can choose \( \eta(T) = \eta_1 T^{-\alpha} \), with \( \frac{2\gamma}{\gamma+2} < \alpha < 1 \) for some appropriate \( \eta_1 \). The error bound (3.5) is composed of two terms, which arise from estimating the initial and sample errors respectively in our proof, and are controlled by \( \eta(T) \) directly. A bigger \( \eta(T) \) may lead to a smaller initial error but may enlarge the sample error, while a smaller \( \eta(T) \) may reduce the sample error but may enlarge the initial error. Solving this trade-off leads to the best rate obtainable from the above theorem, which is stated next.

Corollary 3.2. Under Assumptions[1][2][3] let \( \{f_t\}_{t \in [T]} \) be generated by \( \{2, 0\} \) with \( \lambda = 0 \) and
\[
\eta_t = \frac{\zeta}{4\kappa^2(c_\gamma + \kappa^2)(\zeta + 1)} T^{-\frac{\gamma+1}{\gamma+2}}, \quad \forall t \in [T]. \tag{3.6}
\]

Then,
\[
\mathbb{E}[\|f_{T+1} - f_\rho\|^2] \leq O(T^{-\frac{2\zeta}{\gamma+1}} \ln T). \tag{3.7}
\]

The above corollary asserts that with an appropriate fixed step-size, the doubly online learning algorithm without regularization achieves generalization error bounds of order \( O(T^{-\frac{2\zeta}{\gamma+1}} \ln T) \).

As mentioned before, Assumption[3] is always satisfied with \( c_\gamma = \kappa^2 \) and \( \gamma = 1 \), which is called as the capacity independent case. Setting \( c_\gamma = \kappa^2 \) and \( \gamma = 1 \) in Corollary 3.2 we have the following results in the capacity independent cases.

Corollary 3.3. Under Assumptions[1][2] let \( \{f_t\}_{t \in [T]} \) be generated by \( \{2, 0\} \) with \( \lambda = 0 \) and
\[
\eta_t = \frac{\zeta}{8\kappa^4(\zeta + 1)} T^{-\frac{2\zeta+1}{2\zeta+2}}, \quad \forall t \in [T].
\]

Then,
\[
\mathbb{E}[\|f_{T+1} - f_\rho\|^2] \leq O(T^{-\frac{2\zeta}{2\zeta+1}} \ln T).
\]

The above corollary can be further simplified as follows if we consider the special case \( f_\rho \in \mathcal{H}_K \), i.e, Assumption[2] with \( \zeta = 1/2 \).

Corollary 3.4. Under Assumption[1] let \( f_\rho \in \mathcal{H}_K \) and \( \{f_t\}_{t \in [T]} \) be generated by \( \{2, 0\} \) with \( \lambda = 0 \) and \( \eta_t = 1/(24\kappa^4 \sqrt{T^2}) \), \( \forall t \in [T] \). Then,
\[
\mathbb{E}[\|f_{T+1} - f_\rho\|^2] \leq O(T^{-\frac{2}{2\zeta+1}} \ln T).
\]

\( ^2 \)Here, the constant \( C \) in the right-hand side of (3.5) is depending only on \( R, \zeta, \|f_\rho\|, \mathcal{E}(f_\rho), \kappa^2, c_\gamma \), and will be given explicitly in the proof.
Theorem 3.1 and its corollaries provide generalization error bounds for the studied algorithm without regularization in the fixed step-size setting. In the next theorem, we give generalization error bounds for the studied algorithm (2.4) without regularization in a decaying step-size setting.

**Theorem 3.5.** Under Assumptions 1, 2 and 3, let $\gamma \neq 1$, $\lambda = 0$ and $\eta_t = \eta_1 t^{-\theta}$ for all $t \in \mathbb{N}$ with $\frac{\gamma}{\gamma+1} < \theta < 1$ and $\eta_1$ such that

$$0 < \eta_1 \leq \frac{1}{4\kappa^2(2^{2\theta}c_\gamma + \kappa^2)c_{\theta,\gamma}},$$

(3.8)

where

$$c_{\theta,\gamma} = \max_{t \in [T]} \left\{ t^{\gamma - \theta(\gamma+1) + (2\theta - 1)\ln 2t} \right\}.$$

Then, for any $t \in [T]$,

$$\mathbb{E}[\|f_{t+1} - f_\rho\|^2] \leq O(t^{2\zeta(\theta - 1) + \gamma - \theta(\gamma+1) + (2\theta - 1)\ln t}).$$

(3.10)

Similarly, there is a trade-off problem in the error bounds of the above theorem. Balancing the last two terms of the error bounds, we get the following corollary.

**Corollary 3.6.** Under Assumptions 1, 2 and 3, let $\gamma \neq 1$, $\lambda = 0$ and $\eta_t = \eta_1 t^{-\theta}$ for all $t \in [T]$.

a) If $2\zeta < 1 - \gamma$, then by selecting $\theta = \frac{2\zeta + \gamma}{2\zeta + \gamma + 1}$ and $\eta_1 = \frac{\zeta}{4\kappa^2(2c_\gamma + \kappa^2)}$,

$$\mathbb{E}[\|f_{t+1} - f_\rho\|^2] \leq O(t^{\frac{2\zeta}{2\zeta + \gamma + 1}\ln t}).$$

(3.11)

b) If $2\zeta \geq 1 - \gamma$, then by selecting $\theta = 1/2$ and $\eta_1 = \frac{1 - \gamma}{6\kappa^2(2c_\gamma + \kappa^2)}$,

$$\mathbb{E}[\|f_{t+1} - f_\rho\|^2] \leq O(t^{\frac{2\zeta}{2\zeta + \gamma + 1}}\ln t).$$

(3.12)

Corollary 3.6 asserts that with an appropriate choice of the decaying exponent for the step-size, the doubly online learning algorithm without regularization has a generalization error bound of order $O(T^{-\frac{2\zeta}{\gamma+1}\ln T})$ when $2\zeta < 1 - \gamma$, or of order $O(T^{\frac{2\zeta}{\gamma+1}}\ln T)$ when $2\zeta \geq 1 - \gamma$. Comparing Corollary 3.2 with Corollary 3.6, the latter has a slow convergence rate when $2\zeta \geq 1 - \gamma$. This suggests that the fixed step-size setting may be more favourable.

Theorems 3.1 and 3.5 provide generalization error bounds for doubly online learning algorithms without regularization. In the next theorem, we give generalization error bounds for doubly online learning algorithms with regularization.

**Theorem 3.7.** Under Assumptions 1, 2 and 3, let $\zeta \leq 1$, $\gamma \neq 1$, $\lambda = T^{\theta - 1 + \epsilon}$, $\eta_t = \eta_1 t^{-\theta}$ for all $t \in \mathbb{N}$, with $\frac{\gamma}{\gamma+1} < \theta < 1$, $0 < \epsilon \leq 1 - \theta$, and $\eta_1$ such that (3.3). Then,

$$\mathbb{E}_{\{z_j, v_j\}_{j=1}^T}[\|f_{T+1} - f_\rho\|^2] \leq O(T^{2\zeta(\theta - 1 + \epsilon) + T^{\gamma - \theta(\gamma+1)}\ln T}).$$

(3.13)

Balancing the two terms from the error bounds in the above theorem to optimize the bounds, we can get the following results.
Corollary 3.8. Under Assumptions 4, 2 and 3, let $\zeta \leq 1$, $\gamma \neq 1$. For all $t \in [T]$, let $\eta_t = \frac{1}{\sqrt{3(\zeta_0 + \kappa_0)(1 + \zeta_0)}} t^{-\frac{3}{2(\zeta_0 + 1)}}$ and $\lambda = T^{-\frac{1}{4(\zeta_0 + 1)}}$, with $0 < \epsilon \leq \frac{2}{2^{\zeta_0 + 1}}$. Then

$$E[\|f_{T+1} - f_0\|^2_2] \leq O(T^{-\frac{2}{2^\zeta + 1}} + \ln T). \quad (3.14)$$

The above corollary asserts that for some appropriate choices on the regularized parameter $\lambda$ and the decaying exponent $\theta$ of the step-size, doubly online learning algorithm with regularization achieves generalization error bounds of order $O(T^{-\frac{2}{2^\zeta + 1}} + \ln T)$, where $\epsilon$ can be arbitrarily close to zero. The convergence rate from Corollary 3.8 is essentially the same as that from Corollary 3.2 for $\zeta \leq 1$. For the case $\zeta \geq 1$, the best obtainable rate from Corollary 3.8 for the studied algorithm is of order $O(T^{-\frac{1}{2^\zeta}} \ln T)$. This is the so-called saturation effect in learning theory. Such a saturation effect also occurs in the analysis for kernel ridge regression.

In general, the random feature $\phi_v$ does not belong to $H_K$. However, there are cases where $\phi_v \in H_K$. In this case, the error bounds stated in Theorems 3.3, 3.5 and 3.7 can be further improved. For the sake of brevity, we only provide convergence results of the studied algorithm in the cases of fixed step-size setting and $\lambda = 0$. Similar results for the other cases can be easily derived and we left them to the interested readers.

Theorem 3.9. Under Assumptions 7 and 8, assume $\phi_v \in H_K$ and there exists some $c_\mu > 0$ such that $\|\phi_v\|_{H_K} \leq c_\mu$, for all $v \in V$. Let $\lambda = 0$ and $\eta_t = \eta(T) = \frac{2^\zeta}{\kappa(\zeta_0 + 4)(2^\zeta + 1)}$ for all $t \in [T]$. Then,

$$E[\|f_{T+1} - \phi_v\|^2_2] \leq O(T^{-\frac{2}{2^\zeta + 1}} \ln T). \quad (3.15)$$

The rate in the above theorem is order $O(T^{-\frac{2}{2^\zeta + 1}} \ln T)$, which is always faster than that of Corollary 3.2. Moreover, it is optimal up to a logarithmic factor in the capacity independent case, i.e., without considering Assumption 3.

Discussions We compare our results with those in [5]. A regularized version of doubly online learning algorithms with a convex loss function was studied in [5]. When the loss function is the square loss, the algorithm in [5] is exactly Algorithm (2.0). Theorem 6] asserts that with high probability, the learning sequence generated by (2.0) with $\lambda > 0$ and $\eta_t \simeq \frac{1}{T}$, satisfies

$$\mathcal{E}(f_{T+1}) - \mathcal{E}(f_\lambda) \lesssim \lambda^{-2}T^{-\frac{1}{2}} \log T, \quad (3.16)$$

provided that $\|f_t\|_\infty \lesssim 1$. Here $f_\lambda$ is the solution of the regularized risk minimization

$$\min_{f \in H_K} \mathcal{E}(f) + \lambda \|f\|^2_K.$$

Combining (3.15) with the fact that [17] under Assumption 2 with $\zeta \leq 1$,

$$\mathcal{E}(f_\lambda) - \mathcal{E}(f_\rho) \lesssim \lambda^{2\zeta},$$

one has

$$\mathcal{E}(f_{T+1}) - \mathcal{E}(f_\rho) \lesssim \lambda^{-2}T^{-\frac{1}{2}} \log T + \lambda^{2\zeta}.$$
The optimal obtainable error bounds is achieved by setting \( \lambda^* \simeq T^{-\frac{\zeta}{4}} \), in which case,
\[
E(f_{T+1}) - E(f_\rho) \lesssim T^{-\frac{\zeta}{4}} \log T.
\]
Comparing the above result with Corollaries 3.3 and 3.8, the error bounds (or order \( O(T^{-\frac{\zeta}{4}}) \) in the capacity independent case) from Corollaries 3.3 and 3.8 are better, while they does not require the bounded assumption \( \|f_t\|_{\infty} \lesssim 1 \).

We discuss some issues that might be considered in the future. First, our generalization error bounds are in expectation, and it would be interesting to derive high-probability error bounds in the future. Second, the rates in our results are not optimal and they should be further improved in the future by using a more involved technique. Finally, in this paper, we only consider simple stochastic gradient methods (SGM) with last iterates. It would be interesting to extend our analysis to different variants of SGM, such as the fully online learning [22, 20], SGM with mini-batches [3], the stochastic average gradient [14], averaging SGM [6], and multi-pass SGM [8] in the future.

4 Error Decomposition

The rest of this paper is devoted to proving our main results. To this end, we need some preliminary analysis and a key error decomposition.

For notational simplicity, we denote \( L_K + \lambda I \) by \( L_{K,\lambda} \) for any \( \lambda \geq 0 \), and set the residual vector
\[
r_t = f_t - f_\rho, \quad \forall t \in \mathbb{N}.
\]
Since \( \{f_t\}_t \) is generated by (2.6), subtracting \( f_\rho \) from both sides of (2.6), by direct computations, one can easily prove that
\[
r_{t+1} = (I - \eta_t L_{K,\lambda})r_t - \eta_t M_t - \eta_t \lambda f_\rho, \tag{4.1}
\]
where we denote
\[
M_t = L_K(f_t - f_\rho) - (f_t(x_t) - y_t)\phi_v(x_t)\phi_v.
\]
For notational simplicity, we denote \( \Pi_{k+1}^t(L_{K,\lambda}) = \prod_{j=k+1}^t (I - \eta_j L_{K,\lambda}) \) for \( k \in [t - 1] \) and \( \Pi_{t+1}^t(L_{K,\lambda}) = I \). Using the iterated relationship (4.1) multiple times, we can prove the following error decomposition.

**Proposition 4.1.** For any \( t \in [T] \), we have the following error decomposition
\[
E_{(z_j,v_j)} f_{j=1}^t \|r_{t+1}\|_{\rho}^2 = \|S_1(t)\|_{\rho}^2 + E_{(z_j,v_j)} f_{j=1}^t \|S_2(t)\|_{\rho}^2, \tag{4.3}
\]
where
\[
S_1(t) = \Pi_1^t(L_{K,\lambda}) f_\rho + \lambda \sum_{k=1}^t \eta_k \Pi_{k+1}^t(L_{K,\lambda}) f_\rho, \tag{4.4}
\]
and
\[
S_2(t) = \sum_{k=1}^t \eta_k \Pi_{k+1}^t(L_{K,\lambda}) M_k. \tag{4.5}
\]
that $M_k$ from (2.6), we know that for any $r_{t+1} = -\left(\Pi_1^t(L_{K,\lambda})f_\rho + \lambda \sum_{k=1}^{t} \eta_k \Pi_{k+1}(L_{K,\lambda})f_\rho\right) + \sum_{k=1}^{t} \eta_k \Pi_{k+1}^t(L_{K,\lambda})M_k$, which is exactly

$$r_{t+1} = -S_1(t) + S_2(t). \quad (4.6)$$

In the rest of the proof, we will write $S_i(t)$ as $S_i$ ($i = 1, 2$) for short, and use the notation $E$ for $E_{\{z_j,v_j\}_j}^{\t}$. Following from (4.6), we get

$$E[\|r_{t+1}\|^2_\rho] = E[\|S_1 + S_2\|^2] = \|S_1\|^2 + E[\|S_2\|^2] - 2E[\langle S_1, S_2 \rangle].$$

From (2.6), we know that for any $k \in [T]$, $f_{k+1}$ is depending only on $z_1, z_2, \cdots, z_k$ and $v_1, v_2, \cdots, v_k$. Also, note that the family $\{z_k, v_k\}_{k=1}^{t}$ is independent. Thus, we can prove that $M_k$ has the following vanishing property:

$$E_{z_k,v_k}[M_k] =
\begin{align*}
&= L_K(f_k - f_\rho) - E_{x_k}[f_k(x_k) - E_{y_k|x_k}[y_k]]E_{v_k}[\phi_{v_k}(x_k)\phi_{v_k}] \\
&= L_K(f_k - f_\rho) - E_{x_k}[f_k(x_k) - E_{y_k|x_k}[y_k]]K_{x_k} \\
&= L_K(f_k - f_\rho) - E_{x_k}[f_k(x_k) - f_\rho(x_k)]K_{x_k} \\
&= 0. \quad (4.7)
\end{align*}$$

Therefore,

$$E[\langle S_1, S_2 \rangle] = \sum_{k=1}^{t} \eta_k \langle S_1, \Pi_{k+1}^t(L_{K,\lambda})E[M_k] \rangle = 0.$$

The proof is complete. \qed

The error decomposition (4.3) is fairly common in analyzing standard online learning algorithms [23]. The term $\|S_1(t)\|^2_\rho$ is related to an initial error, which is deterministic and will be estimated in the next section. The term $E_{\{z_j,v_j\}_j}^{\t}[\|S_2(t)\|^2_\rho]$ is a sample error depending on the sample, which will be estimated in Section 5.

## 5 Estimating Initial Error

In this section, we will upper bound the initial error, namely, the first term of the right-hand side of (4.3). To this end, we introduce the following two lemmas.

**Lemma 5.1.** Let $\lambda \geq 0$, $\zeta \geq 0$, and $\eta_k$ be such that $0 \leq \eta_k(\lambda + \kappa^2) \leq 1$ for all $k \in \mathbb{N}$. Then for all $t \in \mathbb{N}$,

$$\lambda \left| \sum_{k=1}^{t} \eta_k \Pi_{k+1}^t(L_{K,\lambda})L_{K,\lambda}^{\zeta} \right| \leq \lambda^{\min(\zeta, 1)} \kappa^{2(\zeta-1)} + 1_{\{\lambda > 0\}}. \quad (5.1)$$

**Proof.** (5.1) holds trivially for the case $\lambda = 0$. Now, we consider the case $\lambda > 0$. Recall that $L_K$ is a self-adjoint, compact, and positive operator on $L_\rho^2$. According to the
Letting $c_i = \lambda + \sigma_i$ for each $i$, we have
\[
(\lambda + \sigma_i) \sum_{k=1}^{t} \eta_k \prod_{j=k+1}^{t} (1 - \eta_j (\lambda + \sigma_i))
\]
\[
= \sum_{k=1}^{t} (1 - (1 - \eta_k c_i)) \prod_{j=k+1}^{t} (1 - \eta_j c_i)
\]
\[
= \sum_{k=1}^{t} \left( \prod_{j=k+1}^{t} (1 - \eta_j c_i) - \prod_{j=k}^{t} (1 - \eta_j c_i) \right)
\]
\[
= \left( 1 - \prod_{j=1}^{t} (1 - \eta_j c_i) \right) \leq 1.
\]
Therefore, we get
\[
\lambda \left\| \sum_{k=1}^{t} \eta_k \Pi_{k+1}^{(L_{K\lambda})} L_{K}^{\zeta} \right\| \leq \sup_i \frac{\lambda \sigma_i^{\zeta}}{\lambda + \sigma_i}.
\]
When $\zeta \in [0, 1]$, we have
\[
\frac{\lambda \sigma_i^{\zeta}}{\lambda + \sigma_i} = \lambda^{\zeta} \left( \frac{\lambda}{\lambda + \sigma_i} \right)^{1-\zeta} \left( \frac{\sigma_i}{\lambda + \sigma_i} \right)^{\zeta} \leq \lambda^{\zeta}.
\]
When $\zeta > 1$,
\[
\frac{\lambda \sigma_i^{\zeta}}{\lambda + \sigma_i} \leq \lambda \sigma_i^{\zeta-1} \leq \lambda \kappa^{2(\zeta-1)}.
\]
From the above analysis, we can get (5.1). The proof is complete. \qed

**Lemma 5.2.** Under the assumptions of Lemma [5.1] we have for $t \in \mathbb{N}$ and any non-negative integer $k \leq t - 1$,
\[
\| \Pi_{k+1}^{L_{K\lambda}} L_{K}^{\zeta} \| \leq \exp \left\{ -\lambda \sum_{j=k+1}^{t} \eta_j \right\} \left( \frac{\zeta}{\sum_{j=k+1}^{t} \eta_j} \right)^{\zeta}.
\]
(5.2)

The above lemma is essentially proved in [23, 20]. For completeness, we provide a proof in the appendix.

Now, we can upper bound the initial error as follows.

**Proposition 5.3.** Under Assumption [2] let $\eta_t = \eta_1 t^{-\theta}$ for all $t \in \mathbb{N}$, with $\eta_1 > 0$ such that $\eta_1 (\lambda + \kappa^2) \leq 1$ and $\theta \in [0, 1]$. Then, for any $t \in \mathbb{N}$,
\[
\| S_1(t) \|_\rho \leq R \kappa^{2(\zeta-1)} + \lambda^{\min(\zeta, 1)} + \left( \frac{\zeta}{\eta_1} \right)^{\zeta} R \cdot \exp \left\{ -\lambda \eta_1 \frac{t^{1-\theta}}{2} \right\} t^{(\theta-1)} \left\{ t^{-\eta_1 \lambda} \{\ln(t+1)\}^{-\zeta} \right\} \text{, when } \theta \neq 1,
\]
\[
\text{and}
\]
\[
\| S_1(t) \|_\rho \leq 2 \| f_\rho \|_\rho.
\]
(5.3)
Proof. Note that $S_1(t)$ is given by (4.4). Thus, we have
\[
\|S_1(t)\|_{\rho} \leq \left\| \lambda \sum_{k=1}^{t} \eta_k \Pi_{k+1}^t(L_{K,\lambda}) f_\rho \right\|_{\rho} + \|\Pi_1^t(L_{K,\lambda}) f_\rho\|_{\rho}.
\] (5.5)

With Assumption 2 we can write $f_\rho = L_{\rho} g_0$ for some $\|g_0\|_{\rho} \leq R$. We thus derive
\[
\|S_1(t)\|_{\rho} \leq R \lambda \sum_{k=1}^{t} \eta_k \Pi_{k+1}^t(L_{K,\lambda}) L_{\rho}^k + R\|\Pi_1^t(L_{K,\lambda}) L_{\rho}^k\|.
\]

Note that $\eta_k = \eta_1 k^\theta$ with $\eta_1 > 0$ satisfying $\eta_1 (\lambda + \kappa^2) \leq 1$ and $\theta \in [0,1]$ implies that $0 \leq \eta_k (\lambda + \kappa^2) \leq 1$ for all $k \in \mathbb{N}$. Thus, we can use (5.1) and (5.2) to bound the last two terms of (5.6) and get that
\[
\|S_1(t)\|_{\rho} \leq R \kappa^{2(\zeta - 1) + \lambda \min(\zeta,1)} + R \exp \left\{ -\lambda \sum_{k=1}^{t} \eta_k \right\} \left( \frac{\zeta}{\sum_{k=1}^{t} \eta_k} \right)^\zeta.
\] (5.6)

Observe that
\[
\sum_{k=1}^{t} k^{-\theta} \geq \sum_{k=1}^{t} \int_{k}^{k+1} x^{-\theta} dx = \frac{(t+1)^{1-\theta} - 1}{1-\theta}, \quad \text{when } \theta \in [0,1),
\]
and that by the mean value theorem,
\[
\frac{(t+1)^{1-\theta} - 1}{1-\theta} \geq \frac{t(1-\theta)(t+1)^{-\theta}}{1-\theta} \geq \frac{t^{1-\theta}}{2}.
\] (5.7)

We thus have
\[
\sum_{k=1}^{t} \eta_k = \eta_1 \sum_{k=1}^{t} k^{-\theta} \geq \left\{ \begin{array}{ll}
\eta_1 t^{1-\theta}/2, & \text{when } \theta \in [0,1),
\eta_1 \ln(t+1), & \text{when } \theta = 1,
\end{array} \right.
\] (5.8)
and consequently,
\[
\exp \left\{ -\lambda \sum_{k=1}^{t} \eta_k \right\} \left( \frac{\zeta}{\sum_{k=1}^{t} \eta_k} \right)^\zeta \\
\leq \left( \frac{\zeta}{\eta_1} \right)^\zeta \left\{ \begin{array}{ll}
\exp \left\{ -\lambda \eta_1 t^{1-\theta}/2 \right\} t^{(\theta-1)\zeta}, & \text{when } \theta \in [0,1),
\eta_1 \lambda \left( \ln(t+1) \right)^{-\zeta}, & \text{when } \theta = 1.
\end{array} \right.
\]

Putting the above inequality into (5.6), we get the desired bound (5.6).

Besides, by (5.5), we also have
\[
\|S_1(t)\|_{\rho} \leq \|\Pi_1^t(L_{K,\lambda})\| \|f_\rho\|_{\rho} + \left\| \lambda \sum_{k=1}^{t} \eta_k \Pi_{k+1}^t(L_{K,\lambda}) \right\| \|f_\rho\|_{\rho}.
\] (5.9)

Since
\[
\|\Pi_1^t(L_{K,\lambda})\| = \sup \prod_{j=1}^{t} (1 - \eta_k(\lambda + \sigma_i)) \leq 1,
\]
and by setting $\zeta = 0$ in (5.1),
\[
\left\| \lambda \sum_{k=1}^{t} \eta_k \Pi_{k+1}^t(L_{K,\lambda}) \right\| \leq 1_{(\lambda > 0)}.
\]

Introducing the last two inequalities into (5.9), we get the desired bound (5.3). The proof is complete. □
6 Bounding Sample Error

In this section, we will upper bound the sample error, i.e., the last term of (4.3). We first introduce the following decomposition.

Proposition 6.1. Under Assumption 7 let \( \{ f_{i+1} \}_{i=1}^{T} \) be generated by Algorithm (2.6), \( S_2(t) \) be given by (4.3), with \( M_t \) given by (4.2). Then for any \( t \in \mathbb{N} \),

\[
E_{\{z_j,v_j\}_{j=1}^{t}} \left\{ \| S_2(t) \|_{\rho}^2 \right\} \leq \kappa^2 \sum_{k=1}^{t} \eta_k^2 E_{v_k} \left[ \| \Pi_{k+1}^{t} (L_{K,}\lambda) \phi_{v_k} \|_{\rho}^2 \right] E_{\{z_j,v_j\}_{j=1}^{t}} \left[ E(f_k) \right]. \tag{6.1}
\]

**Proof.** As in the proof of Proposition 4.1, we will use the notation \( E \) for \( E_{\{z_j,v_j\}_{j=1}^{t}} \). Note that \( S_2(t) \) is given by (4.3). Thus, we have

\[
E[\| S_2(t) \|_{\rho}^2] = \sum_{k,l=1}^{t} \eta_k \eta_l E[\Pi_{k+1}^{l} (L_{K,}\lambda) M_k, \Pi_{l+1}^{k} (L_{K,}\lambda) M_l]_{\rho}.
\]

When \( k \neq l \), without loss of generality, we can assume that \( k > l \). Recall that \( M_t \) is given by (4.2), and \( M_t \) is depending only on \( \{ z_j, v_j \}_{j=1}^{t} \). Thus, we have

\[
E(\Pi_{k+1}^{l} (L_{K,}\lambda) M_k, \Pi_{l+1}^{k} (L_{K,}\lambda) M_l)_{\rho} = E_{\{z_j,v_j\}_{j=1}^{T-1}}(\Pi_{k+1}^{l} (L_{K,}\lambda) E_{z_k,v_k}[M_k], \Pi_{l+1}^{k} (L_{K,}\lambda) M_l)_{\rho} = 0,
\]

where we have used the vanishing property (4.2) for the last equality. We thus get

\[
E[\| S_2(t) \|_{\rho}^2] = \sum_{k=1}^{t} \eta_k^2 E[\Pi_{k+1}^{t} (L_{K,}\lambda) M_k]_{\rho}^2.
\]

Using the fact that for any random variable \( \xi \in L_{\rho}^2 \), \( E[\xi - E[\xi]]_{\rho}^2 = E[\xi]_{\rho}^2 - \| E[\xi] \|_{\rho}^2 \leq E[\| \xi \|_{\rho}^2] \), with \( \xi = \Pi_{k+1}^{l} (L_{K,\lambda})(f_k(x_k) - y_k) \phi_{v_k}(x_k) \phi_{v_k} \), we get

\[
E[\| S_2(t) \|_{\rho}^2] \leq \sum_{k=1}^{t} \eta_k^2 E \left\{ (f_k(x_k) - y_k)^2 \phi_{v_k}(x_k) \| \Pi_{k+1}^{t} (L_{K,}\lambda) \phi_{v_k} \|_{\rho}^2 \right\}.
\]

By Assumption \( \Pi, \phi_{v_k}^2(x_k) \leq \kappa^2 \) almost surely. And note that \( f_k \) is depending only on \( \{ z_j, v_j \}_{j=1}^{k-1} \). We thus can relax the above inequality as (6.1). The proof is complete. \( \square \)

Based on the above proposition and using an inducted argument, one can prove the following result.

Proposition 6.2. Instate the assumptions and notations of Proposition 6.1. Assume that (5.4) and that

\[
\kappa^2 \max_{t \in [T]} \left\{ \sum_{k=1}^{t} \eta_k^2 E_{v_k} \left[ \| \Pi_{k+1}^{t} (L_{K,}\lambda) \phi_{v_k} \|_{\rho}^2 \right] \right\} \leq 1/2. \tag{6.2}
\]

Then, for any \( t \in [T] \),

\[
E_{\{z_j,v_j\}_{j=1}^{t}} \| \rho_{t+1} \|_{\rho}^2 \leq \| S_1(t) \|_{\rho}^2 + \left( 8 \| f_{\rho} \|_{\rho}^2 + 2E(f_{\rho}) \right) \kappa^2 \sum_{k=1}^{t} \eta_k^2 E_{v_k} \left[ \| \Pi_{k+1}^{t} (L_{K,}\lambda) \phi_{v_k} \|_{\rho}^2 \right]. \tag{6.3}
\]

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Proof. By Proposition 6.1, we have (6.1). Plugging with (2.3), we get
\[
E_{\{z_j, v_j\}_{j=1}^t} \{ ||S_2(t)||^2_{\rho} \}
\leq \left( \sup_{k \in [t]} E_{\{z_j, v_j\}_{j=1}^{k-1}} \{ ||r_k||^2_{\rho} + \mathcal{E}(f_{\rho}) \} \right) \kappa^2 \sum_{k=1}^t \eta_k^2 E_{v_k} [||\Pi_{k+1}^t(L_{K,\lambda})\phi_{v_k}||^2_{\rho}].
\]
Introducing the above into the error decomposition (4.3), we have
\[
E_{\{z_j, v_j\}_{j=1}^t} \{ ||r_{t+1}||^2_{\rho} \}
\leq ||S_1(t)||^2_{\rho} + \left( \sup_{k \in [t]} E_{\{z_j, v_j\}_{j=1}^{k-1}} \{ ||r_k||^2_{\rho} + \mathcal{E}(f_{\rho}) \} \right) \kappa^2 \sum_{k=1}^t \eta_k^2 E_{v_k} [||\Pi_{k+1}^t(L_{K,\lambda})\phi_{v_k}||^2_{\rho}].
\] (6.4)

Letting \( t = 1 \) in the above inequality, with \( r_1 = f_1 - f_\rho = -f_{\rho} \),
\[
E_{z_1, v_1} [||r_2||^2_{\rho}] \leq ||S_1(1)||^2_{\rho} + (||f_\rho||^2_{\rho} + \mathcal{E}(f_{\rho})) \kappa^2 \eta_1^2 E_{v_1} [||\phi_{v_1}||^2_{\rho}].
\]
This verifies (6.3) for \( t = 1 \). Now for any fixed \( t \in [T] \) with \( t \geq 2 \), assume that (6.3) holds for each \( t' \in [t-1] \). In this case,
\[
\sup_{k \in [t]} E_{\{z_j, v_j\}_{j=1}^{k-1}} [ ||r_k||^2_{\rho} ]
\leq \sup_{k \in [T]} [ ||S_1(k)||^2_{\rho} + (8 ||f_{\rho}||^2_{\rho} + 2\mathcal{E}(f_{\rho})) \sup_{k \in [T]} \left\{ \kappa^2 \sum_{j=1}^k \eta_j^2 E_{v_j} [||\Pi_{j+1}^k(L_{K,\lambda})\phi_{v_j}||^2_{\rho}] \right\}]
\leq 4 ||f_{\rho}||^2_{\rho} + (8 ||f_{\rho}||^2_{\rho} + 2\mathcal{E}(f_{\rho})) / 2 = 8 ||f_{\rho}||^2_{\rho} + \mathcal{E}(f_{\rho}),
\]
where for the last inequality, we used (6.1) and (6.2). Therefore, using (6.4), we get
\[
E_{\{z_j, v_j\}_{j=1}^t} [ ||r_{t+1}||^2_{\rho} ] \leq ||S_1(t)||^2_{\rho} + (8 ||f_{\rho}||^2_{\rho} + 2\mathcal{E}(f_{\rho})) \kappa^2 \sum_{k=1}^t \eta_k^2 E_{v_k} [||\Pi_{k+1}^t(L_{K,\lambda})\phi_{v_k}||^2_{\rho}],
\]
which verifies the case \( t \). Thus, by an inducted argument, we prove the desired results.

From the above result, we see that the sample error is upper bounded in terms of
\[
\sum_{k=1}^t \eta_k^2 E_{v_k} [||\Pi_{k+1}^t(L_{K,\lambda})\phi_{v_k}||^2_{\rho}],
\]
provided that (6.2) holds. We thus only need to estimate \( \sum_{k=1}^t \eta_k^2 E_{v_k} [||\Pi_{k+1}^t(L_{K,\lambda})\phi_{v_k}||^2_{\rho}] \) in the rest of this section, considering two different cases.

### 6.1 General Case

In this subsection, we estimate the term related to sample error in the general cases. The trace of any trace operator \( L : \mathcal{L}_\rho^2 \rightarrow \mathcal{L}_\rho^2 \) is denoted by tr(\( L \)). We first introduce the following three lemmas.

**Lemma 6.3.** We have for any \( t \in \mathbb{N} \) and any \( k \in [t] \),
\[
E_{v_k} [||\Pi_{k+1}^t(L_{K,\lambda})\phi_{v_k}||^2_{\rho}] = \text{tr}(\Pi_{k+1}^t(L_{K,\lambda})^2 L_K).
\] (6.5)
Proof. By Proposition 6.1, we have (6.5). Since
\[ \mathbb{E}_{\nu_k} \left[ \| \Pi_{k+1}^t (L_{K,\lambda}) \phi_{\nu_k} \|^2_{\rho} \right] = \mathbb{E}_{\nu_k} tr(\Pi_{k+1}^t (L_{K,\lambda}) \phi_{\nu_k} \otimes \phi_{\nu_k} \Pi_{k+1}^t (L_{K,\lambda})) = tr(\Pi_{k+1}^t (L_{K,\lambda}) \mathbb{E}_{\nu_k} [\phi_{\nu_k} \otimes \phi_{\nu_k}] \Pi_{k+1}^t (L_{K,\lambda})), \]
and for any \( f, g \in L^2_{\rho}, \)
\[ \langle \mathbb{E}_{\nu}[\phi_{\nu} \otimes \phi_{\nu}] f, g \rangle_{\rho} = \mathbb{E}_{\nu}[\phi_{\nu}, f]_{\rho} \langle \phi_{\nu}, g \rangle_{\rho} = \int_X \int_X f(x)g(t)\phi_{\nu}(x)\phi_{\nu}(t)d\rho_X(x)d\rho_X(t)d\mu(v) = \int_X \int_X f(x)g(t)K(x,t)d\rho_X(x)d\rho(t) = (LKF, g)_{\rho}, \]
where for the third equality, we used (6.6). Therefore,
\[ \mathbb{E}_{\nu_k} \left[ \| \Pi_{k+1}^t (L_{K,\lambda}) \phi_{\nu_k} \|^2_{\rho} \right] = tr(\Pi_{k+1}^t (L_{K,\lambda}) L_K \Pi_{k+1}^t (L_{K,\lambda})), \]
which leads to the desired result (6.5). The proof is complete.

Lemma 6.4. Under Assumptions 1 and 3, let \( \lambda \geq 0, \) and \( \eta_k \in \mathbb{R}^+ \) be such that \( \eta_k(\lambda + \kappa^2) \leq 1 \) for all \( t \in \mathbb{N}. \) Then for any \( k, t \in \mathbb{N} \) with \( k \leq t - 1, \) we have
\[ tr(\Pi_{k+1}^t (L_{K,\lambda})^2 L_K) \leq 2c_\gamma \exp \left\{ -2\lambda \sum_{j=k+1}^{t} \eta_j \right\} \left( \frac{2c}{2e \sum_{j=k+1}^{t} \eta_j} \right)^{\gamma-1}. \] (6.6)
Proof. Recall that \( L_K \) is a self-adjoint, compact and positive operator on \( L^2_{\rho}, \) and \( L_K \)
has only non-negative singular values \( \{\sigma_i\}_{i=1}^\infty \) such that \( \kappa^2 \geq \sigma_1 \geq \sigma_2 \geq \cdots \geq 0. \) Fix \( k \in [t-1]. \) For any \( \lambda_0 > 0, \)
\[ tr(\Pi_{k+1}^t (L_{K,\lambda})^2 L_K) = tr((L_K + \lambda_0)\Pi_{k+1}^t (L_{K,\lambda})^2 L_K (L_K + \lambda_0)^{-1}) \leq \| (L_K + \lambda_0)\Pi_{k+1}^t (L_{K,\lambda})^2 \| tr(L_K (L_K + \lambda_0)^{-1}) \leq \| (L_K + \lambda_0)\Pi_{k+1}^t (L_{K,\lambda})^2 \| c_\gamma \lambda_0^{-\gamma}, \]
where for the last inequality we used Assumption 3. Note that by Lemma 5.2, we have
\[ \| \Pi_{k+1}^t (L_{K,\lambda})^2 L_K \| = \| \Pi_{k+1}^t (L_{K,\lambda}) L_K^{1/2} \|^2 \leq \exp \left\{ -2\lambda \sum_{j=k+1}^{t} \eta_j \right\} \left( \frac{1}{2e \sum_{j=k+1}^{t} \eta_j} \right), \]
and
\[ \| \Pi_{k+1}^t (L_{K,\lambda})^2 \| \leq \exp \left\{ -2\lambda \sum_{j=k+1}^{t} \eta_j \right\}. \]
Therefore, we get
\[ tr(\Pi_{k+1}^t (L_{K,\lambda})^2 L_K) \leq c_\gamma \exp \left\{ -2\lambda \sum_{j=k+1}^{t} \eta_j \right\} \left( \frac{1}{2e \sum_{j=k+1}^{t} \eta_j} + \lambda_0 \right) \lambda_0^{-\gamma}. \]
Choosing \( \lambda_0 = \frac{1}{2e \sum_{j=k+1}^{t} \eta_j}, \) we can get the desired result. The proof is complete. \( \square \)
Lemma 6.5. Let $c \geq 0$, $\gamma \in [0, 1]$ and $\theta \in [0, 1]$. Then for any $t \in \mathbb{N}$ with $t \geq 2$,
\[
\sum_{k=1}^{t-1} k^{-\theta} \exp \left\{-c \sum_{j=k+1}^{t} j^{-\theta} \right\} \left( \sum_{j=k+1}^{t} j^{-\theta} \right)^{\gamma-1} \leq 2^{2^{\theta} - 2 \theta} t^{-\theta} \ln(2t) \left( \exp \left\{-ct^{-\theta} / 2 \right\} t^{(2^{\theta} - 1) + \min \left\{ 1, \left( t^{1-\theta} c \right)^{-\gamma} \right\} \right). \tag{6.7}
\]

Proof. For any $j \in [k+1, t]$, we have $j^{-\theta} \geq t^{-\theta}$. Thus,
\[
\sum_{k=1}^{t-1} k^{-\theta} \exp \left\{-c \sum_{j=k+1}^{t} j^{-\theta} \right\} \left( \sum_{j=k+1}^{t} j^{-\theta} \right)^{\gamma-1} \leq \sum_{k=1}^{t-1} k^{-\theta} \exp \left\{-c(t-k)t^{-\theta} \right\} \left( (t-k)t^{-\theta} \right)^{\gamma-1}.
\]

For $k \leq (t-1)/2$, we have $(t-k)t^{-\theta} \geq t^{1-\theta}/2$. Therefore,
\[
\sum_{k \leq (t-1)/2} k^{-\theta} \exp \left\{-c \sum_{j=k+1}^{t} j^{-\theta} \right\} \left( \sum_{j=k+1}^{t} j^{-\theta} \right)^{\gamma-1} \leq \exp \left\{-ct^{1-\theta} / 2 \right\} 2^{1-\gamma} t^{(1-\theta)(\gamma-1)} \sum_{k \leq (t-1)/2} k^{-\theta}.
\]

Applying
\[
\sum_{k=1}^{t} k^{-\theta'} \leq t^{\max(1-\theta', 0)} \sum_{k=1}^{t} k^{-1} \leq t^{(1-\theta')} + \ln(t) \tag{6.8}
\]
to bound $\sum_{k \leq (t-1)/2} k^{-2\theta}$, we get
\[
\sum_{k \leq (t-1)/2} j^{-\theta} \exp \left\{-c \sum_{j=k+1}^{t} j^{-\theta} \right\} \left( \sum_{j=k+1}^{t} j^{-\theta} \right)^{\gamma-1} \leq 2^{1-\gamma - (1-2\theta)} \exp \left\{-ct^{1-\theta} / 2 \right\} t^{(1-\theta)(\gamma-1) + (1-2\theta)} + \ln(t) / 2 \leq 2^{\min(1-2\theta, -\gamma)} \exp \left\{-ct^{1-\theta} / 2 \right\} t^{\gamma-\theta(\gamma+1)} t^{(2^{\theta} - 1) + \ln(2t)}.
\]

For $t/2 \leq k \leq t-1$, $k^{-\theta} \leq 2^{2\theta} t^{-2\theta}$. We thus have
\[
\sum_{k \geq t/2} k^{-2\theta} \exp \left\{-c \sum_{j=k+1}^{t} j^{-\theta} \right\} \left( (t-k)t^{-\theta} \right)^{\gamma-1} \leq 2^{2^{\theta} - 2\theta} \sum_{k \geq t/2} k^{-2\theta} \exp \left\{-c \sum_{j=k+1}^{t} j^{-\theta} \right\} \left( (t-k)t^{-\theta} \right)^{\gamma-1} \leq 2^{2^{\theta} t^{-\theta} (\gamma+1)} \sum_{1 \leq k \leq t/2} \exp \left\{-ct^{-\theta} k \right\} k^{\gamma-1}.
\]

On the one hand, for any $c \geq 0$ and $\gamma \geq 0$, by (6.8),
\[
\sum_{1 \leq k \leq t/2} \exp \left\{-ct^{-\theta} k \right\} k^{\gamma-1} \leq \sum_{1 \leq k \leq t/2} k^{\gamma-1} \leq 2^{-\gamma} t^{\gamma} \ln(2t) \leq 2^{-\gamma} t^{\gamma} \ln(2t).
\]

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On the other hand, when $c > 0$ and $\gamma > 0$, we subsequently apply (39) (see the appendix) with $x = r^{-\theta}k$, $\zeta = \gamma$, and (6.8) to get
\[
\sum_{1 \leq k \leq t/2} \exp \left\{ -ct^{-\theta}k \right\} k^{\gamma-1} \leq \left( \frac{t}{ec} \right)^{\gamma} t^{\theta \gamma} \sum_{1 \leq k \leq t/2} k^{-1} \leq 2^{2\gamma-1} c^{\gamma} t^{\theta \gamma} \ln(2t).
\]
Note that the above inequality also holds for $c = 0$, $\gamma \geq 0$, or $c > 0$, $\gamma = 0$, as we used the conventional notations that $(0/0)^{0} = 1$ and $(1/0) = \infty$. Consequently, we derive
\[
\sum_{k \geq t/2} \eta_{k} k^{-2\theta} \exp \left\{ -c(t-k)t^{-\theta} \right\} \left( (t-k)t^{-\theta} \right)^{\gamma-1} \leq 2^{2\gamma-1} t^{\gamma-\theta(\gamma+1)} \ln(2t) \min \left( t^{\gamma}, c^{\gamma} t^{\theta \gamma} \right) = 2^{2\gamma-1} t^{\gamma-\theta(\gamma+1)} \ln(2t) \min \left( 1, \left( t^{1-\theta} c \right)^{-\gamma} \right).
\]
From the above analysis, one can conclude the proof. 

We now can estimate the term related to sample error as follows.

**Proposition 6.6.** Under Assumptions 7 and 3, let $\eta_k = \eta_k k^{-\theta}$ for all $k \in \mathbb{N}$, with $0 < \eta_1 \leq 1/(\lambda + \kappa^2)$, $\theta \in [0, 1]$ and $\lambda \geq 0$. Then, for all $t \in \mathbb{N},$
\[
\sum_{k=1}^{t} \frac{\eta_k^2 \mathbb{E}_{v_k} \left[ \| \Pi_{k+1}^{t} (L_{K,\lambda}) \phi_{v_k} \|_{\rho} \right]^2}{\| \phi_{v_k} \|_{\rho}^2} \leq (2^{2\theta+\gamma-1} c_\gamma + \kappa^2) F_{\eta_1, \theta, \gamma}(t),
\]
where
\[
F_{\eta_1, \theta, \gamma}(t) = \eta_1^{\gamma+1} t^{\gamma-\theta(\gamma+1)} \ln(2t) \left( \exp \left\{ -\lambda t^{1-\theta} \right\} \right) \left( t^{(2\theta-1)+1} + 1 \right).
\]

**Proof.** Following from Lemma 6.3 and $\text{tr}(L_K) \leq \kappa^2$, we have
\[
\sum_{k=1}^{t} \frac{\eta_k^2 \mathbb{E}_{v_k} \left[ \| \Pi_{k+1}^{t} (L_{K,\lambda}) \phi_{v_k} \|_{\rho} \right]^2}{\| \phi_{v_k} \|_{\rho}^2} \leq \sum_{k=1}^{t-1} \eta_k^2 \text{tr}(\Pi_{k+1}^{t} (L_{K,\lambda})^2 L_K) + \eta_t^2 \kappa^2.
\]
Note that $\eta_j = \eta_j j^{-\theta}$ implies $\eta_j (\lambda + \sigma_j) \leq \eta_1 (\lambda + \kappa^2) \leq 1$ for all $j \in \mathbb{N}$. Applying Lemma 6.4 we get
\[
\sum_{k=1}^{t-1} \eta_k^2 \mathbb{E}_{v_k} \left[ \| \Pi_{k+1}^{t} (L_{K,\lambda}) \phi_{v_k} \|_{\rho} \right]^2 \
\leq 2c_\gamma \sum_{k=1}^{t-1} \eta_k^2 \exp \left\{ -2\lambda \sum_{j=k+1}^{t} \eta_j \right\} \left( 2e \sum_{j=k+1}^{t} \eta_j \right)^{\gamma-1} \
\leq 2^{2\gamma-1} c_\gamma \eta_1^{\gamma+1} \sum_{k=1}^{t-1} k^{-2\theta} \exp \left\{ -2\eta_1 \lambda \sum_{j=k+1}^{t} j^{-\theta} \right\} \left( \sum_{j=k+1}^{t} j^{-\theta} \right)^{\gamma-1}.
\]
Using Lemma 6.5 with $c = 2\eta_1 \lambda$, by a direct computation, we get
\[
\sum_{k=1}^{t-1} \eta_k^2 \mathbb{E}_{v_k} \left[ \| \Pi_{k+1}^{t} (L_{K,\lambda}) \phi_{v_k} \|_{\rho} \right]^2 \leq 2^{2\theta+\gamma-1} c_\gamma F_{\eta_1, \theta, \gamma}(t).
\]
Introducing the above inequality into (6.11), and by using the fact that since $\eta_1 \leq 1/(\lambda + \kappa^2)$, $\kappa^2 \geq 1$ and $\gamma, \theta \in [0, 1],$

$$\eta_1^2 t^{-20} = \eta_1^{1+\gamma} t^{-\theta(\gamma+1)} \eta_1^{1-\gamma} t^{(\theta-1)\gamma-\theta} \leq \eta_1^{1+\gamma} t^{\gamma-\theta(\gamma+1)} \leq \mathcal{F}_{\eta_1, \theta, \gamma}(t),$$

one can get the desired result. The proof is finished. \hfill \Box

### 6.2 Special Case

In this subsection, we estimate the term related to sample error in the special case, $f_\rho \in \mathcal{H}_K$. 

**Proposition 6.7.** Under Assumption 4, let $\eta_k = \eta_1 k^{-\theta}$ for all $k \in \mathbb{N}$, with $0 < \eta_1 \leq 1/(\lambda + \kappa^2)$, $\theta \in [0, 1]$ and $\lambda \geq 0$. Assume that $\phi_v \in \mathcal{H}_K$ and there exists some $c_\mu > 0$ such that $\|\phi_v\|_{\mathcal{H}_K} \leq c_\mu$, for all $v \in V$. Then, for all $t \in \mathbb{N},$

$$\sum_{k=1}^{t} \eta_k^2 \mathbb{E}_{v_k} \left[ \|\Pi_{k+1}(L_{K,\gamma}) \phi_{v_k}\|_{\rho}^2 \right] \leq (2^{2\theta - 2} c_\mu^2 + 1) \tilde{F}_{\eta_1, \theta, \lambda}(t),$$

(6.12)

where

$$\tilde{F}_{\eta_1, \theta, \lambda}(t) = \eta_1 t^{-\theta} \ln(2t) \left\{ \exp \left\{ -\lambda \eta_1 t^{1-\theta} \right\} t^{(2\theta - 1)\lambda} + 1 \right\}. \quad (6.13)$$

**Proof.** The proof is similar to the proof of Lemma 6.6. We only include the sketch. By (3.2) and $\|\phi_v\|_{\mathcal{H}_K} \leq c_\mu$, we have for any $k \in [t - 1],$

$$\eta_k^2 \|\Pi_{k+1}(L_{K,\gamma}) \phi_{v_k}\|_{\rho}^2 = \eta_k^2 \|L_{K}^{1/2} \Pi_{k+1}^{1/2}(L_{K,\gamma}) \phi_{v_k}\|_{K}^2 \leq \eta_k^2 \|L_{K}^{1/2} \Pi_{k+1}^{1/2}(L_{K,\gamma})\|_{K}^2 \leq c_\mu^2. \quad (6.14)$$

Using Lemma 5.2 to the above, we get

$$\eta_k^2 \|\Pi_{k+1}(L_{K,\gamma}) \phi_{v_k}\|_{\rho}^2 \leq c_\mu^2 \eta_k^2 \exp \left\{ -2\lambda \sum_{j=k+1}^{t} \eta_j \right\} \left\{ 2\eta_1 \sum_{j=k+1}^{t} \eta_j \right\}^{-1}. \quad (6.15)$$

Summing up over $k \in [t - 1]$, introducing with $\eta_j = \eta_1 j^{-\theta}$ and then using Lemma 6.5 we can reach

$$\sum_{k=1}^{t-1} \eta_k^2 \|\Pi_{k+1}(L_{K,\gamma}) \phi_{v_k}\|_{\rho}^2 \leq 2^{2\theta - 2} c_\mu^2 \tilde{F}_{\eta_1, \theta, \lambda}(t). \quad (6.16)$$

Using the above inequality to (6.11), and noting that $\eta_1^2 \kappa^2 \leq \eta_1 t^{-2\theta}$ since $\eta_1 \leq 1/(\lambda + \kappa^2)$, one can prove the desired result. \hfill \Box

### 7 Deriving Total Error

In this section, we estimate the total errors for the studied algorithms with different choices of parameters.
7.1 \( \lambda = 0, \eta_1 = \eta(T), \theta = 0 \)

**Proof of Theorem 7.1.** By Proposition 6.6, we have (6.10), where \( F_{\eta, \theta, \lambda, \gamma} \) is given by (6.10). Plugging with \( \lambda = 0 \), \( \theta = 0 \), \( \eta_1 = \eta(T) \), and using \( 2^{\gamma-1} \leq 1 \) since \( \gamma \in [0,1] \), we have that for any \( t \in [T], \)

\[
\sum_{k=1}^{t} \eta_k^2 \mathbb{E}_{\nu_k} \left[ \| \Pi_{k+1}^T (L_{K, \lambda}) \phi_{\nu_k} \|_p^2 \right] \leq (c_\gamma + \kappa^2) 2\eta(T) \gamma^+ \ln(2t). \tag{7.1}
\]

Taking the maximum over \( t \in [T] \), and then multiplying both sides by \( \kappa^2 \),

\[
\kappa^2 \max_{t \in [T]} \sum_{k=1}^{t} \eta_k^2 \mathbb{E}_{\nu_k} \left[ \| \Pi_{k+1}^T (L_{K, \lambda}) \phi_{\nu_k} \|_p^2 \right] \leq 2\kappa^2(c_\gamma + \kappa^2) \eta(T) \gamma^+ \ln(2T).
\]

Condition (3.4) ensures the right-hand side of the above is less than \( 1/2 \). This verifies (6.2). Thus, we can apply Proposition 6.2 to get (6.3). Note that by Proposition 5.3, the initial error can be bounded as

\[
\| S_1(t) \|_p^2 \leq R^2 \xi^2 (\eta(T)t)^{-2\xi}. \tag{7.2}
\]

Plugging the above inequality and (7.1) into (6.3), we derive

\[
\mathbb{E}[\| r_{t+1} \|_p^2] \leq R^2 \xi^2 (\eta(T)t)^{-2\xi} + 4(4\| f_\rho \|_p^2 + \mathcal{E}(f_\rho)) \kappa^2(c_\gamma + \kappa^2) \eta(T) \gamma^+ \ln(2t), \tag{7.2}
\]

which leads to the desired result. The proof is complete. \( \square \)

**Proof of Corollary 3.2.** We only need to verify (3.4). Since \( \kappa^2 \geq 1 \) and \( \zeta > 0 \), we know that

\[
\frac{\zeta}{4\kappa^2(c_\gamma + \kappa^2)(\zeta + 1)} \leq 1, \text{ and consequently,}
\]

\[
\left( \frac{\zeta}{4\kappa^2(c_\gamma + \kappa^2)(\zeta + 1)} \right)^{\gamma^+} \leq \frac{\zeta}{4\kappa^2(c_\gamma + \kappa^2)(\zeta + 1)}. \tag{7.3}
\]

Therefore,

\[
\eta(T) \gamma^+ T^\gamma \leq \frac{\zeta}{4\kappa^2(c_\gamma + \kappa^2)(\zeta + 1)} T^{-\frac{\gamma + 2\xi(\zeta + 1)}{\gamma + 2\xi + 1}} T^\gamma = \frac{\zeta}{4\kappa^2(c_\gamma + \kappa^2)(\zeta + 1)} T^{-\frac{2\zeta}{\gamma + 2\xi + 1}}. \tag{7.3}
\]

Rewriting \( T^{-\frac{2\zeta}{\gamma + 2\xi + 1}} \) as

\[
2^{-\frac{2\zeta}{\gamma + 2\xi + 1}} \exp \left\{ -\frac{2\zeta}{\gamma + 2\xi + 1} \ln(2T) \right\},
\]

and then applying (3.9) (from the appendix) with \( c = \frac{2\zeta}{\gamma + 2\xi + 1} \ln(2T), x = 1 \) and \( \zeta = 1 \), we know that

\[
\eta(T) \gamma^+ T^\gamma \leq \frac{\zeta}{4\kappa^2(c_\gamma + \kappa^2)(\zeta + 1)} 2^{-\frac{2\zeta}{\gamma + 2\xi + 1}} \gamma^+ \leq \frac{1}{4\kappa^2(c_\gamma + \kappa^2) \ln(2T)},
\]

which leads to (3.4). Thus, following from the proof of Theorem 3.1 we get (7.2). Plugging with (3.6) and (7.3), we get

\[
\mathbb{E}[\| r_{T+1} \|_p^2] \leq \left( R^2(4\kappa^2(c_\gamma + \kappa^2)(\zeta + 1)) \xi^2 + 4\| f_\rho \|_p^2 + \mathcal{E}(f_\rho) \frac{\zeta}{\zeta + 1} \right) T^{-\frac{2\zeta}{\gamma + 2\xi + 1}} \ln(2T).
\]

The proof is complete. \( \square \)
7.2 \( \lambda = 0, \eta_1 = \text{const}, \theta > 0 \)

**Proof of Theorem 7.2** According to Proposition 6.6, we have (6.9), with \( \mathcal{F}_{\theta, \theta, \lambda, \gamma} \) given by (6.10). When \( \lambda = 0 \),

\[
\mathcal{F}_{\theta, \xi_1, \lambda, \gamma} = \eta_1^{\gamma+1} t^{\gamma - \theta(\gamma+1)} \ln(2t) \left( t^{(2\theta-1)+1} + 1 \right) \leq 2\eta_1^{\gamma+1} t^{\gamma - \theta(\gamma+1)+(2\theta-1)+1} \ln(2t).
\]

Also, \( 2^{2\theta+\gamma-1} \leq 2^\theta \) since \( \gamma \leq 1 \). Therefore,

\[
\sum_{k=1}^{t} \eta_k^2 \mathbb{E}_{\nu_k} \left[ \| \Pi_{k+1}^t \left( L_{K, \lambda} \right) \phi_{\nu_k} \|_\rho \|^2 \right] \leq (2^\theta \eta_1 \gamma + \gamma^2) 2\eta_1^{\gamma+1} t^{\gamma - \theta(\gamma+1)+(2\theta-1)+1} \ln(2t).
\]

Taking the maximum over \( t \in [T] \) on both sides, multiplying both sides by \( \kappa^2 \), and recalling that \( c_{\theta, \gamma} \) is given by (3.9),

\[
\kappa^2 \max_{t \in [T]} \sum_{k=1}^{t} \eta_k^2 \mathbb{E}_{\nu_k} \left[ \| \Pi_{k+1}^t \left( L_{K, \lambda} \right) \phi_{\nu_k} \|_\rho \|^2 \right] \leq 2\kappa^2 (2^\theta \eta_1 \gamma + \gamma^2) \eta_1^{\gamma+1} c_{\theta, \gamma}. \quad (7.4)
\]

Condition (3.8) and \( \kappa^2 \geq 1 \) imply that \( \eta_1 \leq 1 \) and thus \( \eta_1^{\gamma+1} \leq \eta_1 \) since \( \gamma \geq 0 \). Therefore, by (3.8),

\[
2\kappa^2 (2^\theta \eta_1 \gamma + \gamma^2) \eta_1^{\gamma+1} c_{\theta, \gamma} \leq 2\kappa^2 (2^\theta \eta_1 \gamma + \gamma^2) \eta_1 c_{\theta, \gamma} \leq 1/2.
\]

Thus, (6.2) holds. Now, we can apply Proposition 6.2 to obtain (6.3). By Proposition 6.3 with \( \lambda = 0 \), the initial error can be estimated as

\[
\| S_1(t) \|_\rho^2 \leq (\zeta/\eta_1)^{2\kappa^2} R^2 \gamma^{2\kappa(\theta-1)}.
\]

Introducing the above and (7.4) into (6.3), we get

\[
\mathbb{E}[\| r_{t+1} \|_\rho^2] \leq (\zeta/\eta_1)^{2\kappa^2} R^2 \gamma^{2\kappa(\theta-1)+4}(4\mathcal{E}(f_\rho) + ||f_\rho||^2_\rho) \kappa^2 (2^\theta \eta_1 \gamma + \gamma^2) \eta_1^{\gamma+1} t^{\gamma - \theta(\gamma+1)+(2\theta-1)+1} \ln(2t).
\]

The proof is complete. \( \square \)

**Proof of Corollary 7.4** If \( \gamma + 1 < \theta \leq 1/2 \) and \( \gamma \in (0, 1) \), we have \( 2^\theta \leq \sqrt{2} \), and by (6.9),

\[
(2t)^{\gamma - \theta(\gamma+1)} = \exp \{ -(\theta(\gamma+1) - \gamma) \ln(2t) \} \leq \frac{1}{e(\theta(\gamma+1) - \gamma) \ln(2t)},
\]

which implies

\[
c_{\theta, \gamma} = \max_{t \in [T]} \left\{ t^{\gamma - \theta(\gamma+1)} \ln(2t) \right\} \leq \frac{2^{\theta(\gamma+1)-\gamma}}{e(\theta(\gamma+1) - \gamma)} \leq \frac{1}{\sqrt{2}(\theta(\gamma+1) - \gamma)}.
\]

Therefore, (3.8) holds if

\[
0 \leq \eta_1 \leq \frac{\theta(\gamma+1) - \gamma}{3\kappa^2 (2\eta_1 \gamma + \gamma^2)}.
\]

When \( 2\zeta - 1 < \gamma, \theta = \frac{2\zeta + \gamma}{2\zeta + \gamma + 1} \), and \( \eta_1 = \frac{\zeta}{3\kappa^2 (2\eta_1 \gamma + \gamma^2)} \) obviously,

\[
\theta = 1 - \frac{1}{2\zeta + \gamma + 1} < \frac{1}{2}.
\]

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and 
\[
\eta_1 \leq \frac{2\zeta}{3\kappa^2(2c_\gamma + \kappa^2)(2\zeta + \gamma + 1)} = \frac{\theta(\gamma + 1) - \gamma}{3\kappa^2(2c_\gamma + \kappa^2)}.
\]
This proves \((7.5)\) and consequently, \((7.6)\). Following the proof of Theorem \(3.5\) we thus have \((7.5)\), which can be relaxed as
\[
\mathbb{E}[\|r_{t+1}\|_\rho^2] \leq \left\{ (3\kappa^2(2c_\gamma + \kappa^2))^2 R^2 + 2\zeta (4\mathcal{E}(f_\rho) + \|f_\rho\|_\rho^2) \ln(2t) \right\} t^{-\frac{2\zeta}{3\kappa^2(2c_\gamma + \kappa^2)}}.
\]
This leads to the first result of the theorem.

When \(2\zeta \geq 1 - \gamma, \theta = 1/2\) and \(\eta_1 = \frac{1-\gamma}{6\kappa^2(2c_\gamma + \kappa^2)}\). Condition \((7.3)\) is satisfied trivially. Thus, following the proof of Theorem \(3.5\), we thus have \((7.5)\) and consequently,
\[
\mathbb{E}[\|r_{t+1}\|_\rho^2] \leq \left\{ (6\kappa^2(2c_\gamma + \kappa^2)) \zeta/(1-\gamma))^2 R^2 t^{-\zeta} + (4\mathcal{E}(f_\rho) + \|f_\rho\|_\rho^2) t^{-\frac{2\zeta}{2(1-\gamma)}} \ln(2t),
\]
which leads to the second result of the theorem by noting that \(t^{-\zeta} \leq t^{-\frac{2\zeta}{2(1-\gamma)}}\). The proof is complete. 

\(\square\)

7.3 \(\lambda > 0, \eta_1 = \text{const}, \theta > 0\)

\textit{Proof of Theorem 7.7.} Following the proof of Theorem 3.5, we know that the condition (6.2) from Proposition 6.2 is satisfied, and thus it holds that (6.3). Introducing (7.3) and (6.9) into (6.8), with \(t = T\), \(\lambda = T^{\theta-1+\epsilon}\), \(\epsilon \in (0, 1-\theta)\) and \(\zeta \in (0, 1]\), by a direct calculation, we get
\[
\mathbb{E}[\|r_{T+1}\|_\rho^2] \leq R^2 \left( 1 + (\zeta/\eta_1)^2 \right)^2 T^{2\zeta(\theta-1+\epsilon)} + 2\kappa^2 (2\theta c_\gamma + \kappa^2)(4\mathcal{E}(f_\rho) + \|f_\rho\|_\rho^2 + \mathcal{E}(f_\rho)) \mathcal{F}_{\eta_1, \theta, \lambda, \gamma}(T).
\]
Recalling that \(\mathcal{F}_{\eta_1, \theta, \lambda, \gamma}(t)\) is given by (6.10), by \(\lambda = T^{\theta-1+\epsilon}\) and (4.9) (from the appendix),
\[
\exp\{-\eta_1 \lambda T^{1-\theta}\} = \exp\{-\eta_1 T^{\theta}\} \leq \left( \frac{(2\theta - 1 + \epsilon)/\eta_1}{\ln(2T)} \right)^{(2\theta-1+\epsilon)/\epsilon} \leq (\epsilon \eta_1)^{-(2\theta-1+\epsilon)/\epsilon} T^{-(2\theta-1+\epsilon)}.
\]
Thus,
\[
\mathcal{F}_{\eta_1, \theta, \lambda, \gamma}(T) \leq \eta_1^{\gamma+1} T^{-\theta(\gamma+1)} \ln(2T) \left( (\epsilon \eta_1)^{-(2\theta-1+\epsilon)/\epsilon} + 1 \right).
\]
It thus follows from the above analysis that
\[
\mathbb{E}[\|r_{T+1}\|_\rho^2] \leq R^2 \left( 1 + (\zeta/\eta_1)^2 \right)^2 T^{2\zeta(\theta-1+\epsilon)} + 2\kappa^2 (4\mathcal{E}(f_\rho)(2\theta c_\gamma + \kappa^2)\eta_1^{\gamma+1} \left( (\epsilon \eta_1)^{-(2\theta-1+\epsilon)/\epsilon} + 1 \right) T^{-\theta(\gamma+1)} \ln(2T).
\]
The proof is complete. 

\(\square\)

\textit{Proof of Corollary 3.8.} We will use Theorem 3.7 with \(\theta = \frac{2\zeta + \gamma}{2\zeta + 3}\) and \(\epsilon\) replaced by \(\frac{\zeta}{2}\). Obviously, we only need to prove the condition (3.8) is true. By a similar argument as that for (7.0), we know that (3.8) is true if
\[
0 < \eta_1 \leq \frac{\theta(\gamma + 1) - \gamma}{3\kappa^2(2\theta c_\gamma + \kappa^2)}.
\]
This can be verified by noting that \(\theta(\gamma + 1) - \gamma = \frac{2\zeta}{2\zeta + 3} \geq \frac{\zeta}{3+1} \geq \frac{\zeta}{2}\) and \(2\theta = 2^{3/2} \leq 3\). The proof is complete.

\(\square\)
Proof of Theorem 3.9. By Proposition 6.7, we have (6.12). Multiplying both sides by $\kappa^2$, with $\lambda = 0$, $\theta = 0$, and $\eta_1 = \eta(T)$

$$
\kappa^2 \sum_{k=1}^{t} \eta_2^2 \mathbb{E}_{\phi_{v_k}}[\|\Pi_{k+1}^t (L_{K,\lambda}) \phi_{v_k}\|_\rho^2] \leq 2 \kappa^2 (c^2_\mu / 4 + 1) \eta(T) \ln(2t). \tag{7.7}
$$

By (9) (from the appendix),

$$
T^{-\frac{2\zeta}{2\zeta + 1}} = 2^{\frac{2\zeta}{2\zeta + 1}} \exp\left\{-\frac{2\zeta}{2\zeta + 1} \ln(2T)\right\} \leq 2^{\frac{2\zeta}{2\zeta + 1}} \frac{2\zeta + 1}{e2\zeta \ln(2T)} \leq \frac{2\zeta + 1}{2\zeta \ln(2T)}.
$$

Thus,

$$
2 \kappa^2 (c^2_\mu / 4 + 1) \eta(T) \ln(2T) \leq 1/2.
$$

This verifies (6.2), and consequently, we have (6.3). Applying (5.3) and (7.7) into (6.3), we get

$$
\mathbb{E}[\|r_{T+1}\|_\rho^2] \leq \left((\kappa^2 (c^2_\mu / 2 + 2) (2\zeta + 1))^{2\zeta} R^2 + (8\|f_\rho\|_\rho^2 + 2\mathcal{E}(f_\rho))\zeta / (2\zeta + 1)\right) T^{-\frac{2\zeta}{2\zeta + 1}} \ln(2T).
$$

The proof is complete. \hfill \square

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Appendix

Proof of Lemma 5.2. Similar to the proof for Lemma 5.1 we have

\[ \| \Pi_{k+1}(L_{K,\lambda})L^\zeta_K \| = \sup_i \prod_{j=k+1}^t (1 - \eta_j(\lambda + \sigma_i))\sigma_i^\zeta. \]

Using the basic inequality

\[ 1 + x \leq e^x \quad \text{for all } x \geq -1, \quad (\text{.8}) \]

with \( \eta_j(\lambda + \sigma_i) \leq 1 \), we get

\[
\| \Pi_{k+1}(L_{K,\lambda})L^\zeta_K \| \leq \sup_i \exp \left\{ -(\lambda + \sigma_i) \sum_{j=k+1}^t \eta_j \right\} \sigma_i^\zeta \]

\[
\leq \exp \left\{ -\lambda \sum_{j=k+1}^t \eta_j \right\} \sup_{x \geq 0} \exp \left\{ -x \sum_{j=k+1}^t \eta_j \right\} x^\zeta. \]

The maximum of the function \( g(x) = e^{-cx}x^\zeta \) (with \( c > 0 \)) over \( \mathbb{R}_+ \) is achieved at \( x_{\text{max}} = \zeta/c \), and thus

\[
\sup_{x \geq 0} e^{-cx}x^\zeta = \left( \frac{\zeta}{ec} \right)^\zeta. \quad (\text{.9})
\]

Using this inequality, one can get the desired result (5.2). \qed