COHERENT STATES AND GEOMETRIC PHASES IN CALOGERO-SUTHERLAND MODEL

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Abstract

Exact coherent states in the Calogero-Sutherland models (of time-dependent parameters) which describe identical harmonic oscillators interacting through inverse-square potentials are constructed, in terms of the classical solutions of a harmonic oscillator. For quasi-periodic coherent states of the time-periodic systems, geometric phases are evaluated. For the $A_{N-1}$ Calogero-Sutherland model, the phase is calculated for a general coherent state. The phases for other models are also considered.

*Supported in part by grant No. R02-2000-00040 and by grant No. R04-2000-00002 from the Korea Science & Engineering Foundation.
1. Introduction

The Calogero-Sutherland (CS) models \cite{1, 2} which describe identical $N$-body harmonic oscillators interacting through inverse square potentials in one-dimension have long been of great interest. These models are closely related to the random matrix model \cite{3, 4} and have been found relevant for the descriptions of various physical phenomena \cite{5}. While the generalization of the model on a circle without confining harmonic potential (the Sutherland model) \cite{6, 7} has been of great interest \cite{5}, we also mention that generalization has been considered in various directions \cite{8, 9}. The underlying algebraic structure of the models has been analyzed by the exchange operator formalism \cite{10}, and the symmetric polynomials in the wave functions of the models have been studied through the quantum inverse scattering method \cite{11, 12}.

The close relationship between the CS model and $N$-body non-interacting harmonic oscillators has been noticed from the inception of the model, and one of the prominent features of a harmonic oscillator system is the existence of coherent states \cite{13, 14}; A coherent state whose center of the probability distribution function moves along a classical solution can be given by applying a displacement operator to an eigenstate of a constant Hamiltonian system, while a generalized coherent state (squeezed state) whose width are described by the classical solutions can be obtained through a unitary squeeze transformation. For a CS model with harmonic potential, a (squeezed-type) coherent state has been shown to exist \cite{15, 16}. Recently, it has been further shown that, the coherent states of the CS models (of time-dependent parameters) can be derived from the eigenstates of
the model (of constant parameters) through the (squeeze- and displacement-type) unitary transformations [17], as in the harmonic oscillator [18]. For the coherent states of the CS model of $N$-body system, the center and width of the particle number density function are described by the classical solutions of a harmonic oscillator [15, 17].

If a system is described by a periodic Hamiltonian and a wave function is quasi-periodic (periodic up to a global phase factor) under the time-evolution with (an integral multiple of) the Hamiltonian’s period, a geometric phase, the geometric part of a change in the phase of a wave function, can be defined [19, 20]. These geometric phases have attracted great interest both theoretically and experimentally [21]. For the CS models, the condition for the quasi-periodicity of a coherent state is exactly same to that in the harmonic oscillator system [17, 22], and steps forward the calculation of a geometric phase has been done for a coherent state corresponding to the ground state [23, 24].

In this article, the coherent states and geometric phases of the CS models with harmonic potential will be systematically studied. It will be shown that, thanks to the orthogonality and recurrence relations among the coherent states (or the eigenstates), the geometric phases for the coherent states can be evaluated in terms of the classical solutions of a harmonic oscillator, without explicit knowledge of the normalization of the wave functions. For the $A_N$ CS model where the interaction is written in terms of the differences between the positions of two particles, both of squeeze- and displacement-type unitary transformations can be applied to give the coherent states, and the geometric phase will be found as a sum of the contributions from the motion of the center and from the squeezing of the coherent state. If the symmetric polynomial in the coherent state is
trivially 1, the contribution from the squeezing is proportional to the energy eigenvalue.

The geometric phases for coherent states of the $B_N$ model and another model will also be considered.

In the next section, the known facts about $A_N$ model will be recapitulated and the coherent states will be found. In section 3, the geometric phases will be evaluated for a general coherent state of the $A_N$ model. In section 4, the coherent states and geometric phases of the $B_N$ and another model will be evaluated. A summary will be given in the last section.

2. The coherent states from eigenstates

In this section we will consider the $A_{N-1}$ CS model. The model of unit mass and angular frequency is described by the Hamiltonian

$$H^s_A = \sum_{i=1}^{N} \left( \frac{p_i^2}{2} + \frac{x_i^2}{2} \right) + \sum_{i>j=1}^{N} \frac{\hbar^2 \lambda (\lambda - 1)}{(x_i - x_j)^2}. \quad (1)$$

By defining $y_N$ and $r$ as

$$y_N = \frac{1}{\sqrt{N}} (x_1 + x_2 + \cdots + x_N), \quad r = \left( \sum_{i=1}^{N} x_i^2 \right)^{1/2}, \quad (2)$$

the (unnormalized) eigenstate can be written as \[25\]

$$\phi_{m,n}(x_1, x_2, \cdots, x_N) = \exp \left( -\frac{y_N^2}{2\hbar} \right) H_m \left( \frac{y_N}{\sqrt{\hbar}} \right) \phi^C_n$$

$$= \left( \prod_{i>j=1}^{N} (x_i - x_j)^\lambda \right) \exp \left( -\frac{r^2}{2\hbar} \right) H_m \left( \frac{y_N}{\sqrt{\hbar}} \right) L_n^b \left( \frac{1}{\hbar} (r^2 - y_N^2) \right), \quad (4)$$

where

$$b = \frac{1}{2} (N - 3) + \frac{1}{2} \lambda N (N - 1). \quad (5)$$
In Eq. (4), $H_m$ and $L^n_m$ denote the Hermite and the associated Laguerre polynomial, respectively, and the energy eigenvalue of $\phi_{m,n}$ with positive integers $m, n$ is

$$E_{m,n} = \hbar (m + 2n) + \frac{\hbar}{2}[N + \lambda N(N - 1)].$$

By defining a new coordinate system $\{y_i | i = 1, 2, \cdots N\}$ which satisfies the linear relation $\bar{y} = O\bar{x}$ with an orthogonal matrix $O$, the $H^s_A$ can be written as

$$H^s_A = \left(\frac{y^2_N}{2} + \frac{y^2_N}{2}\right) + H_{C,s},$$

while $H_{C,s}$ depends only on $y_1, y_2, \cdots, y_{N-1}$. In the work by Calogero [1], $\phi_n^C(y_1, y_2, \cdots, y_{N-1})$ has shown to be an eigenstate of $H_{C,s}$ [7].

The system described by the Hamiltonian

$$H_A = \frac{1}{2} \sum_{i=1}^{N} \left( \frac{p_i^2}{M(t)} + M(t)w^2(t)x_i^2 \right) + \frac{\hbar^2 \lambda(\lambda - 1)}{M(t)} \sum_{i>j=1}^{N} \frac{1}{(x_i - x_j)^2},$$

is related to the system of $H^s_A$ through the unitary transformations [7]. If the two linearly independent solutions of the equation of motion

$$\frac{d}{dt}(M\dot{x}) + M(t)w^2(t)x = 0$$

are denoted as $u(t)$ and $v(t)$, and $\rho(t)$ is defined by $\rho(t) = \sqrt{u^2 + v^2}$, with a time-constant $\Omega (\equiv \lambda [\dot{u}(t)u(t) - \ddot{u}(t)v(t)])$, the wave function satisfying the Schrödinger equation

$$i\hbar \frac{\partial}{\partial t} \psi_{m,n} = H_A \psi_{m,n}$$

is given from $\phi^s_{m,n}$ as

$$\psi_{m,n}(t; \bar{x}) = \left(\frac{u - iv}{\rho(t)}\right)^{E_{m,n}/\hbar} U_f U_N \phi^s_{m,n}(x_1, x_2, \cdots, x_N)$$

$$= e^{i(\delta_f + M\dot{u}f\sqrt{\lambda} y_N)/\hbar} \left(\frac{u - iv}{\rho}\right)^{m+2n} \left(\frac{u + iv}{\sqrt{\Omega}}\right)^{-(N + \lambda N(N - 1))/2}$$
\[ \times \exp \left[ -\frac{1}{2\hbar} \left( \frac{\Omega}{\rho^2} - iM\dot{\rho} \right) \left( r^2 - 2\sqrt{N}y_N u_f + N u_f^2 \right) \right] \times H_m \left( \sqrt{\frac{\Omega y_N - \sqrt{N} u_f}{\hbar}} \right) L_n^b \left( \frac{\Omega}{\hbar \rho^2} (r^2 - y_N^2) \right), \] 

(11)

while the overdot denotes differentiation with respect to \( t \). In Eq. (11), \( u_f \) is a linear combination of \( u(t) \) and \( v(t) \), and \( \delta_f \) is defined through the relation \( \dot{\delta}_f = \frac{1}{2} M (w^2 u_f^2 - \dot{u}_f^2) \).

In Eq. (10), \( U_N \) and \( U_f \) are defined as

\[
U_N = \left( \frac{\Omega}{\rho^2} \right)^{N/4} \prod_{i=1}^{N} \left( \exp \left[ \frac{i}{\hbar} M \dot{x}_i^2 \right] \exp \left[ -\frac{1}{2} \ln \left( \frac{\rho^2}{\Omega} \right) x_i \frac{\partial}{\partial x_i} \right] \right),
\]

(12)

\[
U_f = e^{\frac{i}{\hbar} N \delta_f} \prod_{i=1}^{N} \left( \exp \left[ \frac{i}{\hbar} M \dot{u}_f x_i \right] \exp \left[ -\frac{i}{\hbar} u_f p_i \right] \right).
\]

(13)

If \( u, v \) are taken as \( \cos t, \sin t \), respectively, the \( \psi_{m,n} \) describes a coherent state of the system of Hamiltonian in Eq. (1) [17].

A general eigenstate of the Hamiltonian in Eq. (1) is written as \[1, 11, 12\]

\[
\phi_{m,n,\kappa}(x_1, x_2, \ldots, x_N) = \phi_{m,n,\kappa} \pi \kappa(x_1, x_2, \ldots, x_N),
\]

(14)

where a partition, \( \kappa \), is an integer vector \( (\kappa_1, \kappa_2, \ldots, \kappa_N) \). With the weight \( k = |\kappa| = \sum_{i=1}^{N} \kappa_i \), the energy eigenvalue of \( \phi_{m,n,\kappa} \) is given as \( E_{m,n} + \hbar k \) (\( \equiv E_{m,n,k} \)). It has been shown that \( \pi \kappa \) is a homogeneous, translational invariant, symmetric polynomial of degree \( k \). Even though we do not know the explicit form of \( \pi \kappa \), from these properties, the coherent states \( \psi_{m,n,\kappa} \) corresponding to \( \phi_{m,n,\kappa} \) can be written as

\[
\psi_{m,n,\kappa} = \left( \frac{u + iv}{\Omega} \right)^{-k} \psi_{m,n,\kappa} \kappa.
\]

(15)
3. The geometric phases for the $A_{N-1}$ model

In this section, we will study the geometric phase of the $A_{N-1}$ model through the definition given by Aharonov and Anandan [19] which is a generalization of Berry’s phase by removing the restriction to adiabatic evolution [21]. Since the geometric phase is defined for a periodic system, we will assume the periodicity of $M(t)$ and $w^2(t)$ under the time-evolution with period $\tau$

\[
M(t + \tau) = M(t), \quad w^2(t + \tau) = w^2(t). \quad (16)
\]

For the quasi-periodicity of the coherent states given in the previous section, as for the harmonic oscillator system, $\rho(t)$ and $u_f(t)$ must be periodic under the time-evolution of an integral multiple of $\tau$. As analyzed by one of the authors [22], for the cases that the two linearly independent homogeneous solutions of Eq. (9) are bounded all over the time, there exists a choice of classical solutions for periodic $\rho$ with the period $\tau'$, where $\tau'$ is $\tau$ or $2\tau$ depending on the model. If $u(t)$ and $v(t)$ are not periodic with period $\tau'$, for the periodicity, $u_f$ will be taken as 0. For a normalized quasi-periodic wave function $\psi$ of a $\tilde{H}$ system with the global phase change $\chi$ such that

\[
\psi(t + \tau'; \vec{x}) = e^{i\chi}\psi(t; \vec{x}), \quad (17)
\]

the geometric phase $\gamma$ is given as [19]

\[
\gamma = \chi + \frac{1}{\hbar} \int_0^{\tau'} <\psi | \tilde{H} | \psi > \\
= \chi + i \int_0^{\tau'} <\psi | \frac{\partial}{\partial t} | \psi >. \quad (18)
\]
The global phase change $\chi_{m,n}$ of $\psi_{m,n}$ under the time-evolution $\tau'$ can be found as

$$\chi_{m,n}(\tau') = -\left( m + 2n + \frac{N + \lambda N(N - 1)}{2} \right) \int_{0}^{\tau'} \frac{\Omega}{M \rho^2} dt. \quad (19)$$

In deriving Eq. (19), the fact that we only consider the periodic $\rho$ and $u_f$ with period $\tau'$ has been used. The expression of $H_A^s$ in Eq. (7) clearly shows that the quantum number $m$ comes from the motion of the center of mass which is described by a free harmonic oscillator. This gives the orthogonality

$$<\psi_{m,n} | \psi_{m',n'}> = \tilde{C}_{m,n,n'} \delta_{m,m'}, \quad (20)$$

with some (unknown) constants $\tilde{C}_{m,n,n'}$. If we only consider a sector of $x_1 < x_2 < \cdots < x_N$, $H_A^s$ is a Hermitian. With the relations in Eqs. (10,20), one can thus find that the coherent states satisfy the orthogonality relation

$$<\psi_{m,n} | \psi_{m',n'}> = C_{m,n} \delta_{m,m'} \delta_{n,n'}. \quad (21)$$

Making use of this orthogonality and the recurrence relations in the Hermite and Laguerre polynomials, one can find the following relation

$$\frac{1}{i} < \psi_{m,n} | \frac{\partial}{\partial t} | \psi_{m,n} > = \frac{d}{dt} \chi_{m,n}(t) + \frac{Nu_f}{\hbar} \frac{d}{dt} (M \dot{u}_f) + i(m + 2n) \frac{\dot{\rho}}{\rho} + \frac{3}{2} \frac{\rho^2}{\Omega} \frac{d}{dt} \left( \frac{\Omega}{\rho^2} - iM \frac{\dot{\rho}}{\rho} \right). \quad (22)$$

Eq. (22) is obtained through a long algebra, while, for a simple case, a similar procedure will be described in detail in the next section. The geometric phase $\gamma_{m,n}$ for $\psi_{m,n}$, therefore, reads as

$$\gamma_{m,n} = \frac{N}{\hbar} \int_{0}^{\tau'} \dot{M} \dot{u}_f dt + \frac{E_{m,n}}{\hbar} \int_{0}^{\tau'} \frac{M \rho^2}{\Omega} dt. \quad (23)$$
For $\psi_{m,n,\kappa}$, the global phase change $\chi_{m,n,\kappa}(\tau')$ is given as
\[
\chi_{m,n,\kappa}(\tau') = \chi_{m,n}(\tau') - k \int_{0}^{\tau'} \frac{\Omega}{M \rho^2} dt = -\frac{E_{m.n,k}}{\hbar} \int_{0}^{\tau'} \frac{\Omega}{M \rho^2} dt \tag{24}
\]
while the geometric phase $\gamma_{m,n,\kappa}$ under the $\tau'$-evolution is same to that of $\psi_{m,n}$, as
\[
\gamma_{m,n,\kappa} = \gamma_{m,n}. \tag{25}
\]

Since the global phase change depends only on energy eigenvalue, the geometric phase can be defined for a superposition of the coherent states corresponding to eigenstates of the same energy eigenvalue. Another eigenstate for the Hamiltonian $H_A^s$ can be found as
\[
\phi_n^s(x_1, x_2, \ldots, x_N) = \prod_{i>j=1}^{N} (x_i - x_j)^\lambda \exp \left(-\frac{r^2}{2\hbar}\right) L_{n-b}^{b+\frac{1}{2}} \left(\frac{r^2}{\hbar}\right), \tag{26}
\]
with the energy eigenvalues $\frac{\hbar}{2}[N + \lambda N(N - 1)] + 2\hbar n = E_{0,n}$. From the identities among Hermite and Laguerre polynomials, this eigenstate and the corresponding coherent state $\psi_n$ for the system of $H_A$ are written in terms of $\phi_n^{s_{2l,n-l}}$ and $\psi_n^{s_{2l,n-l}}$, as
\[
\phi_n^s = \sum_{l=0}^{n} \frac{(-)^l}{2^l l!} \phi_n^{s_{2l,n-l}}, \quad \psi_n^s = \sum_{l=0}^{n} \frac{(-)^l}{2^l l!} \psi_n^{s_{2l,n-l}}. \tag{27}
\]
The geometric phase of $\psi_n$ is given as $\gamma_{0,n}$.

4. Other models

For exposing the calculational procedure of geometric phase through orthogonality and recurrence relations, we consider a model described by the Hamiltonian
\[
H_w^s = \sum_{i=1}^{N} \left(\frac{p_i^2}{2} + \frac{x_i^2}{2}\right) + \sum_{i>j=1}^{N} \frac{\hbar^2 \lambda (\lambda - 1)}{(x_i - x_j)^2}
\]
+ \sum_{i=1}^{N} \frac{\alpha(\alpha - 1)N(N-1)}{2w_i^2} - \sum_{i>j=1}^{N} \frac{2N\alpha(\alpha + N\lambda)}{w_iw_j}, \quad (28)

where \( w_i = (\sum_{j=1}^{N} x_j) - Nx_i \). The (unnormalized) ground state of this system is given as

\[
\phi_0^w = \left( \prod_{i>j=1}^{N} (x_i - x_j) \right) \left( \prod_{i=1}^{N} w_i \right)^{\alpha} \exp\left(-\frac{r^2}{2\hbar}\right) \quad (29)
\]

with energy eigenvalue \( E_0^w = \hbar(\lambda \frac{N(N-1)}{2} + \alpha N + \frac{N}{2}) \). For the case of \( N = 3 \), from the fact that \( \sum_{i>j=1}^{3} \frac{1}{w_iw_j} = 0 \), one can find that this system corresponds to the one which has long been known [26, 27]. The case of \( N = 4 \) has also been considered in the literature [27].

One of the excited states with the energy eigenvalue \( E_1^w = E_0^w + 2\hbar \) for the system of \( H_w \) is written as

\[
\phi_1^w = (r^2 - E_0^w)\phi_0^w. \quad (30)
\]

For the system described by the Hamiltonian

\[
H_w = \sum_{i=1}^{N} \frac{1}{2} \frac{p_i^2}{M(t)} + M(t)w_i^2(t)x_i^2 + \frac{1}{M(t)} \sum_{i>j=1}^{N} \frac{\hbar^2\lambda(\lambda - 1)}{(x_i - x_j)^2} + \frac{1}{M(t)} \left[ \sum_{i=1}^{N} \frac{\alpha(\alpha - 1)N(N-1)}{2w_i^2} - \sum_{i>j=1}^{N} \frac{2N\alpha(\alpha + N\lambda)}{w_iw_j} \right], \quad (31)
\]

the coherent states are given as

\[
\psi_0^w = \left( \frac{u + iv}{\sqrt{\Omega}} \right)^{-E_0^w/\hbar} \left( \prod_{i>j=1}^{N} (x_i - x_j) \right)^{\lambda} \left( \prod_{i=1}^{N} w_i \right)^{\alpha} \exp\left[ -\frac{r^2}{2\hbar}\left( \frac{\Omega}{\rho^2} - iMt\frac{\lambda}{\rho} \right) \right], \quad (32)
\]

\[
\psi_1^w = \left( \frac{u - iv}{\rho} \right)^{2} \left( \frac{\Omega}{\rho^2}r^2 - E_0^w \right) \psi_0^w. \quad (33)
\]

Since the interaction can be written in terms of the differences between positions of two particles, the displacement-type unitary transformation can also be applied to give the coherent state; in obtaining Eqs. (32–33), however, only squeeze-type transformation
has been used for simplicity. Furthermore, we only consider the geometric phase for
the coherent state $\psi_0^w$ which is obtained from the ground state. From the fact that the
Hamiltonian is a Hermitian in a sector, the orthogonality relation is given as

$$<\psi_0^w|\psi_1^w> = 0.$$  \hspace{1cm} (34)

With the global phase change $\chi_0^w(\tau')$ which satisfies

$$\psi_0^w(t + \tau') = \exp[i\chi_0^w(\tau')]\psi_0^w(t),$$  \hspace{1cm} (35)

one can find the relation

$$\frac{1}{i}\dot{\psi}_0^w = \chi_0^w(t)\psi_0^w + \frac{i}{2\hbar} \frac{\rho^2}{\Omega} \frac{d}{dt} \left( \frac{\Omega}{\rho^2} - iM \frac{\dot{\rho}}{\rho} \right) \left( \frac{\rho}{u - iv} \right)^2 \psi_1^w + E_0\psi_0^w.$$  \hspace{1cm} (36)

Making use of the definition in Eq. (18) and the relation in Eq. (34), one can find that
the geometric phase of $\psi_0^w$ under the $\tau'$-evolution is given as

$$\gamma_0^w = \frac{E_0^w}{\hbar} \int_0^{\tau'} \frac{M\rho^2}{\Omega} dt.$$  \hspace{1cm} (37)

The $B_N$ CS model is described by the Hamiltonian

$$H_B^\alpha = \sum_{i=1}^{N}\left( \frac{p_i^2}{2} + \frac{y_i^2}{2} \right) + \sum_{i=1}^{N}\frac{\hbar^2 \alpha(\alpha - 1)}{2y_i^2} + \hbar^2 \lambda(\lambda - 1) \sum_{i>j=1}^{N} \frac{1}{(y_i - y_j)^2} + \frac{1}{(y_i + y_j)^2},$$  \hspace{1cm} (38)

whose ground state is known as

$$\psi_0^B = \left( \prod_{i>j=1}^{N} (x_i^2 - x_j^2) \lambda \right) \left( \prod_{i=1}^{N} x_i^\alpha \exp\left( -\frac{x_i^2}{2\hbar} \right) \right)$$  \hspace{1cm} (39)

with energy eigenvalue $E_0^B = \hbar\left( \frac{N}{2} + \lambda N(N - 1) + \alpha N \right)$. By defining $\tilde{b} = \frac{N}{2} + \lambda N(N - 1) + N\alpha - 1$, an excited states of energy eigenvalue $E_n^B = E_0^B + 2\hbar n$ can be found as

$$\phi_n^B = \psi_0^B \tilde{b}^n \frac{\hbar^2}{\hbar}.$$  \hspace{1cm} (40)
Since the interaction term can not be written in terms of differences between the positions of two particles, only squeeze-type unitary transformation can be applied to give coherent states. For the system described by the Hamiltonian

\[ H_B^s = \sum_{i=1}^{N} \frac{1}{2} \left( \frac{p_i^2}{M(t)} + M(t)w^2(t)x_i^2 \right) + \sum_{i=1}^{N} \frac{\hbar^2\alpha(\alpha - 1)}{2M(t)x_i^2} + \frac{\hbar^2\lambda(\lambda - 1)}{M(t)} \sum_{i>j=1}^{N} \left[ \frac{1}{(x_i - x_j)^2} + \frac{1}{(x_i + x_j)^2} \right], \quad (41) \]

the coherent state corresponding to \( \phi^B_n \), therefore, reads

\[ \psi^B_n = \left( \frac{u + iv}{\sqrt{\Omega}} \right)^{-E^B_n/h} \left( \frac{u - iv}{\rho} \right)^2 \left( \prod_{i>j=1}^{N} (x_i^2 - x_j^2)^\lambda \right)^{\frac{N}{\alpha}} \left( \prod_{i=1}^{N} x_i^\alpha \right) \times \exp \left[ -\frac{r^2}{2\hbar} \left( \frac{\Omega}{\rho^2} - iM\dot{\rho}/\rho \right) \right] L_n^b \left( \frac{\Omega r^2}{\hbar \rho^2} \right), \quad (42) \]

and the geometric phase is given as

\[ \gamma^B_n = \frac{E^B_n}{\hbar} \int_0^{\tau'} M\rho^2 dt. \quad (43) \]

5. Concluding remarks

We have studied the coherent states and geometric phases in the CS models. Making use of the orthogonality and recurrence relations among the coherent states, the geometric phases for the coherent states have been evaluated in terms of the classical solutions of a harmonic oscillator.

The geometric phase can be written as a sum of the contributions from the motion of the center of mass and from the motion of squeezing. In the harmonic oscillator system, the contribution from the squeezing is proportional to energy eigenvalue of the
corresponding eigenstate [22], while, if the symmetric polynomial in the wave function is not trivially 1, this is not true in the CS models. We believe that the method developed in this paper could be applied for the calculations of geometric phases in the other CS models with confining harmonic potential.
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