Quantum optical properties of the radiation field in the Dicke model

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Abstract. We study the optical signatures of the quantum critical behaviour associated with the quantum phase transition exhibited by the Dicke model in the thermodynamic limit. We obtain an effective Hamiltonian for the radiation field, which resembles a degenerate parametric amplifier and which reproduces the Dicke’s model critical behaviour. We identify the state of the radiation field in the sub-radiant and super-radiant phases, and show that the optical squeezing and photon statistics present striking behaviour in the vicinity of the phase transition. Our results are both of theoretical interest and of relevance to recent experimental proposals that hope to realize the quantum phase transition in the Dicke model.

1. Introduction
Quantum Phase Transitions (QPTs) are well known to be drastic changes in the physical properties of a system at zero temperature due to the variation of an external parameter that drives the phase transition. There are several theoretical models which serve as tools for the study of such fascinating quantum phenomena, being the Dicke model one of particular interest as it describes collective phenomena in a light-matter system [1]. Not least does the quantum phase transition in the Dicke model provide an excellent example of collective phenomena, but the presence of the interaction between the radiation and matter subsystems and the occurrence of the phase transition in the radiation field, provides an interesting setting for the study of quantum phase transitions in contrast to the more usual magnetic systems which serve as the principle prototypes for such quantum phenomena. In recent years, there have been several attempts to identify further properties of a system exhibiting interesting behaviour at or near a phase transition, and which therefore serve as potential indicators of critical phenomena. Much of the previous work that studies phase transition phenomena in the Dicke model has focused on the matter subsystem in both the method of solution and the results [1–3], consequently only the macroscopic occupation of the radiation field is usually considered so as to indicate the actual occurrence of the phase transition. However, it is an obvious question to ask how the radiation itself field behaves as we vary the driving parameter of the QPT using standard measures from quantum optics. In this paper, we shift focus away from the matter system by constructing an effective Hamiltonian in the bosonic operators of the radiation field, following a similar approach to that taken in Ref. [3]. As such we are able to describe the Dicke model entirely in terms of the radiation field operators. We rederive the quantum critical point and scaling for the phase transition and continue to describe the system in the two different phases. Finally, we go on to consider the macroscopic occupation, the optical squeezing and photon statistics of the radiation field as we drive the system through the QPT. Our results show that the photon statistics present striking behaviour in the vicinity of the phase transition and help motivate the exploration of such exotic phenomena.
quantum phenomena in other matter-radiation settings such as atom-cavity, nanostructure-cavity, and nanostructure-photonic-band-gap systems [4–6].

2. Effective Hamiltonian for the radiation field

We consider here the Dicke model which describes the interaction between a single-mode photon field and $N$ non-interacting two-level systems [1, 3]:

\[ H = a^\dagger a + \epsilon J_z + \frac{\lambda}{2\sqrt{N}} (a + a^\dagger) J_x, \]  

where $J_z = \frac{1}{2} \sum_{i=1}^{N} \sigma_z^i$ and $J_x = \frac{1}{2} \sum_{i=1}^{N} (\sigma_+^i + \sigma_-^i)$ are the collective angular momentum operators, and the operators $a, a^\dagger$ and $\sigma_-^i, \sigma_+^i$ correspond to the photon field and two-level atom $i$ respectively. It is well known that this model exhibits a phase transition at both zero and finite temperature [1, 3]. We are interested here in describing the QPT from the perspective of the radiation field. As such we proceed to eliminate the matter degrees of freedom and write a size-consistent effective Hamiltonian for the photon field using the cumulant method of Polatsek and Becker [7]. We consider $H = H_0 + H_I$ with:

\[ H_0 = H_b = \epsilon J_z \]
\[ H_I = H_a + H_ab = a^\dagger a + \frac{2\lambda}{\sqrt{N}} \left( a^\dagger + a \right) J_x, \]

where $H_a$ denotes the Hamiltonian of the subsystem $a$ (photon field), $H_b$ that of subsystem $b$ (matter) and $H_ab$ represents the interaction between the two subsystems. An effective Hamiltonian for the subset $a$ is given by:

\[ H_a^{\text{eff}} = -\frac{1}{\beta} \left( e^{-\beta (H_I + L_0)} - 1 \right)_b. \]

Here the index $c$ denotes cumulant averaging [8] and the thermal average is carried out with respect to the matter degrees of freedom, i.e. $\langle A \rangle_b = \text{Tr}_b \left( e^{-\beta J_z} A \right) / \text{Tr}_b \left( e^{-\beta J_z} \right)$. The Liouvillian superoperator $L_0$ is defined by $L_0 A = [H_b, A]$. The cumulants are then expanded in a series as follows:

\[ \langle e^{-\beta (H_I + L_0)} - 1 \rangle_b^c = \sum_{\nu = 1}^{\infty} \frac{(-\beta)^\nu}{\nu!} \langle (H_I + L_0)^\nu \rangle^c. \]

We are able to calculate the first two cumulants exactly as:

\[ \langle (H_I + L_0)^c \rangle = a^\dagger a, \quad \langle (H_I + L_0)^2 \rangle^c = \lambda^2 (a + a^\dagger)^2. \]

We then calculate higher order cumulants and find that, if only considering terms linear and quadratic in $a$ and $a^\dagger$, we may write a general expression for every even and odd cumulant as:

\[ \langle (H_I + L_0)^{2n} \rangle^c = \lambda^2 e^{2n-2} (a + a^\dagger)^2 \]
\[ \langle (H_I + L_0)^{2n+1} \rangle^c = \lambda^2 e^{2n+1-2} (a + a^\dagger)^2 \tanh \left( \frac{\beta \epsilon}{2} \right). \]

It is worthy of note that if we include higher order terms in $a$ and $a^\dagger$, terms are introduced into the effective Hamiltonian that provide infinite contributions to the energy in the $\beta \to \infty$. We will show that terms to second order are all that is required to find the salient features of the model at zero-temperature. The effective Hamiltonian becomes

\[ H_a^{\text{eff}} = a^\dagger a - \frac{\lambda^2}{\epsilon} \tanh \left( \frac{\beta \epsilon}{2} \right) (a + a^\dagger)^2. \]
In the zero limit $T \to 0$ we have

$$H^\text{eff}_{a(T=0)} = \omega \left( a^\dagger a + \frac{1}{2} \right) + \gamma \left( a^{\dagger 2} + a^2 \right) - \frac{1}{2}$$

(2)

with $\omega = 1 - \frac{2\lambda^2}{c}$ and $\gamma = -\frac{\lambda^2}{c}$. The effective Hamiltonian given in equation 2 indicates that the optical properties of the Dicke model can be mapped onto an optical degenerate parametric process in which a classical field interacts with a non-linear medium. Equation 2 belongs to a widely studied class of squeezing Hamiltonians with SU(1,1) symmetry which have been shown to exhibit a ground state phase transitions [9]. This mapping provides a familiar setting in which we may consider the optical signatures of the phase transition in the Dicke model and suggests an alternative method for experimental verification and investigation of the Dicke QPT.

3. The radiation field and the critical behaviour

We proceed to demonstrate the validity of the effective Hamiltonian derived in the previous section by correctly reproducing the salient features of the Dicke model, and then go on to discuss new results.

(i) Ground state energy, quantum critical point and scaling. Figure 1 shows the ground state energy in the Dicke model as obtained by a numerical calculation of both the effective Hamiltonian, equation (2) and the full Dicke Hamiltonian. The results demonstrate that the ground state of the Dicke model is well described by the effective Hamiltonian. In order to describe the critical behaviour, we begin by performing a Bogoliubov transformation from the operators $a$, $a^\dagger$ to squeezed $b$-bosons, such that $b^\dagger = ua + va^\dagger$ and $|u|^2 - |v|^2 = 1$:

$$b = \frac{a + aa^\dagger}{\sqrt{1 - \alpha^2}}, \quad b^\dagger = \frac{a^\dagger + \alpha a}{\sqrt{1 - \alpha^2}}.$$

and substitute for $a$, $a^\dagger$ in equation (2). Ignoring the constant additive of $\frac{1}{2}$, as we are interested here in the energy gap between the ground and first excited states, and with the use of the bosonic commutation relation $[b, b^\dagger] = 1$ we obtain

$$H^\text{eff}_{a(T=0)} = \frac{1}{1 - \alpha^2} \left\{ \left( \gamma \alpha^2 - \omega \alpha + \gamma \right) \left( b^{\dagger 2} + b^2 \right) + \left( \omega \alpha^2 - 4\gamma \alpha + \omega \right) \left( b^b b + \frac{1}{2} \right) \right\}.$$

We choose $\alpha$ such that the coefficient of the $(b^{\dagger 2} + b^2)$ term is zero and apply the so-called resonance condition, $\epsilon = 1$. We find

$$\alpha = -\frac{1 - 2\lambda^2 \pm \sqrt{1 - 4\lambda^2}}{2\lambda^2}$$

and the effective Hamiltonian, having arbitrarily chosen one of the two equivalent solutions, becomes $H^\text{eff}_{a(T=0)} = -\sqrt{1 - 4\lambda^2} \left( b^{\dagger} b + \frac{1}{2} \right)$. This effective Hamiltonian is clearly that of a simple harmonic oscillator in the $b$-bosons and as such it may be diagonalised by the number states of the $b$-boson operators, i.e. $E_n = -\sqrt{1 - 4\lambda^2} \left( n + \frac{1}{2} \right)$. Therefore the energy gap between the ground and first excited states is $\Delta E = \sqrt{1 - 4\lambda^2}$ and as such we find a quantum critical point at $\lambda = \lambda_c = 0.5$. In addition, the energy gap scales $\propto \sqrt{\lambda_c - \lambda}$ as $\lambda \to \lambda_c$ in agreement with previous results [1]. The sub-radiant phase, with $\lambda < \lambda_c$, is clearly well described by the above Bogoliubov transformation and the effective Hamiltonian is simply mapped to a simple harmonic oscillator. In the super-radiant phase, with $\lambda > \lambda_c$, the effective Hamiltonian in equation 2 resembles an inverted oscillator, as we will show later. For each phase we will calculate in addition to the photon number, the optical squeezing which
Figure 1. Ground state energy as a function of the light-matter coupling $\lambda$, as calculated with the size-consistent Hamiltonian of equation 2 (solid line) and the full Dicke model Hamiltonian (circles).

Figure 2. Analytical (circles) and numerical (solid line) calculation of the photon occupation number plotted as a function of $\lambda$. The inset shows the saturation of $N$ for large values of $\lambda$.

describes the fluctuations in the field quadratures, and the Mandel parameter $Q$ which characterizes the photon statistics [10].

(ii) Sub-radiant phase. We are able to find analytical expressions for $N$, $Q$ and the optical squeezing by exploiting the $SU(1, 1)$ symmetry of the effective Hamiltonian [9]. The Lie algebra of $SU(1, 1)$ [9] is generated by introducing the three operators $K_0, K_\pm$:

$$K_0 = \frac{1}{2} \left( a^\dagger a + \frac{1}{2} \right), \quad K_+ = \frac{1}{2} a^\dagger a^\dagger, \quad K_- = \frac{1}{2} aa.$$

which satisfy the commutation relations $[K_0, K_\pm] = \pm K_\pm, [K_-, K_+] = 2K_0$. We may then rewrite the effective Hamiltonian in equation (2) as $H_{a(T=0)}^{eff} = 2\omega K_0 + \frac{1}{2} \gamma (K_+ + K_-)$ with $\omega = 1 - \frac{2\lambda^2}{\epsilon}$ as before but with $\gamma = -\frac{4\lambda^2}{\epsilon}$. To calculate expectation values of this Hamiltonian we follow Ref. [9] and introduce the $SU(1, 1)$ coherent states $|\xi, k\rangle = \exp \left\{ zK_+ - z^* K_- \right\} |0, k\rangle$ where $z = -\theta(2)e^{-i\phi}$ and $\xi = -\tanh(\theta/2)e^{-i\phi}$. $\theta$ and $\phi$ are group parameters with ranges $(-\infty, \infty)$ and $[0, 2\pi]$ respectively. The state $|0, k\rangle$ with $k = \frac{1}{4}$ is the vacuum squeezed state. $N$, $Q$, and the squeezing can be calculated in terms of expectation values of the three generators. For example,

$$\langle \xi, k | K_0 | \xi, k \rangle = k \cosh \theta, \quad \langle \xi, k | K_\pm | \xi, k \rangle = -k \sinh \theta e^{\pm i\phi}.$$

By considering the classical system dynamics as in Ref. [9] we write equations of motion for $\theta$ and $\phi$ as

$$\dot{\theta} = -(k \sinh \theta)^{-1} \frac{\partial \mathcal{H}}{\partial \phi}, \quad \dot{\phi} = (k \sinh \theta)^{-1} \frac{\partial \mathcal{H}}{\partial \theta},$$

where $\mathcal{H} = \langle \xi, k | H_{a(T=0)}^{eff} | \xi, k \rangle$. So we obtain:

$$\mathcal{H} = 2\omega k \cosh \theta - \gamma k \sinh \theta \cos \phi - \frac{1}{2},$$

$$\dot{\theta} = -\gamma \sin \phi,$$

$$\dot{\phi} = 2\omega - \gamma \coth \theta \cos \phi.$$
The stationary points of this system are defined by the conditions $\dot{\theta} = \dot{\phi} = 0$. We find $\phi = n\pi$ with $n = 0, 1, 2$, and if we choose $n$ odd we get $\theta = \arccoth \left( \frac{1 - 2\lambda^2}{2\lambda^2} \right)$. Again we see that no real solution exists above the critical point $\lambda_c = 0.5$. This is due to the energy $\mathcal{E}$ becoming unbounded as we pass from the sub-radiant into the super-radiant phase and we will return to this point later on. In the sub-radiant phase the photon number $N = \langle \xi, k | a^\dagger a | \xi, k \rangle$ can be written as

$$N = 2(K_0) - \frac{1}{2} = 2k \cosh \theta - \frac{1}{2}.$$ 

For a vacuum squeezed state, the above expression reproduces the numerical results obtained for $N$ as a function of the coupling parameter $\lambda$ through out the sub-radiant phase, as it can be seen in figure 2. This implies that in the sub-radiant phase the field is effectively in a squeezed state and this is confirmed from a consideration of the optical squeezing. The squeezing is defined in terms of the quadrature operators $X_1 = \frac{1}{2}(a + a^\dagger)$ and $X_2 = \frac{1}{2\sqrt{\lambda}}(a - a^\dagger)$ for which $[X_1, X_2] = i/2$, which leads to $(\Delta X_1)^2(\Delta X_2)^2 \geq \frac{1}{16}$ where $(\Delta X_1)^2 = \langle X_1^2 \rangle - \langle X_1 \rangle^2$. Squeezing exist if $(\Delta X_1)^2 \leq \frac{1}{4}$. We find that

$$\langle \Delta X_{1,2} \rangle^2 = k (\cosh \theta \mp \cos \phi \sinh \theta)$$

For $k = \frac{1}{2}$ and $\phi = n\pi$ ($n = 0, 1, 2$, as before) $(\Delta X_1)^2 \leq \frac{1}{4}$ throughout the sub-radiant phase (see figure 3(b)), and the ground state is a minimum uncertainty squeezed state, i.e. $(\Delta X_1)^2(\Delta X_2)^2 = \frac{1}{16}$.

The Mandel Q-parameter in the sub-radiant phase is is given by [10]

$$Q = \frac{\langle a^\dagger a a^\dagger a \rangle - \langle a^\dagger a \rangle^2}{\langle a^\dagger a \rangle} - 1 = \frac{4(K_0^4) - 4(K_0)^2}{2(K_0) - \frac{1}{2}} - 1 = k ((1 + 2k) \cosh(2\theta) + 2k - 1) - 4k^2 \cosh^2 \theta \frac{2k \cosh \theta - \frac{1}{2}}{2k \cosh \theta - \frac{1}{2}} - 1.$$ 

This expression for $Q$, with $k = \frac{1}{2}$ is plotted in figure 4, which shows a good agreement with the numerical results for the sub-radiant phase.

(iii) Super-radiant phase. To describe the behaviour of the system in the super-radiant phase we introduce the canonical position and momentum operators

$$\hat{x} = \frac{1}{\sqrt{2\omega}}(a + a^\dagger), \quad \hat{p} = \frac{1}{i} \sqrt{\frac{\omega}{2}}(a - a^\dagger)$$

with commutation relation $[\hat{x}, \hat{p}] = i$. In terms of these operators, the effective Hamiltonian of equation (2) reads

$$H_{a_{T=0}}^{eff} = \frac{\hat{p}^2}{2\bar{m}} + \frac{1}{2}\bar{m}\bar{\omega}^2 \hat{x}^2, \quad (3)$$

with

$$\bar{m} = \frac{1}{1 - \frac{2\lambda}{\omega}}, \quad \bar{\omega}^2 = \omega^2 \left( 1 + \frac{2\gamma}{\omega} \right) \left( 1 - \frac{2\gamma}{\omega} \right)$$

and where we have again ignored the constant factor of $\frac{1}{2}$. The coefficients of $\hat{x}^2$ and $\hat{p}^2$ are plotted against $\lambda$ in figure 5. The figure shows that in the sub-radiant phase, with $\lambda < 0.5$, the system is equivalent to a harmonic oscillator and as such is square-integrable. In the super-radiant phase, with $\lambda > 0.5$, the system becomes first an inverted harmonic oscillator first in momentum and then in position. The properties of the inverted potential harmonic oscillator have been well documented elsewhere [11]. In particular, it is interesting to note the inverted potential harmonic oscillator has been used as a model of instability in relation to quantum chaos [12]. This would confirm that the system undergoes a transition from quasi-integrable to quantum chaotic at the critical point as first shown by Emary et al. [13].
Although the system cannot be described simply in an analytic form in the super-radiant phase, except in the limit $\lambda \to \infty$, the quantities in which we are interested can be calculated from numerical simulations. In addition the numerical simulations suggest that, with the exception of the energy, each quantity of interest rapidly converges to a single value as $\lambda$ is increased. Only a small number of photons are required to obtain extremely accurate results. The scaling remains intact as we increase the system size and only the magnitude of the energy is found to increase significantly, as would be expected.

Figures 1, 2, and 3 all show numerical results for both the sub- and super-radiant phases and analytical results for the sub-radiant phase on the same graphs. The occupation number $N$ is presented in figure 2. It shows that $N$ remains small but finite as we increase $\lambda$ until we reach the critical point and the occupation becomes macroscopic – the system ceases to be sub-radiant and enters a super-radiant phase. The behaviour depicted in figure 2 compares very well with the properties predicted for the of the output photon flux in a recent proposal that hopes to realize the Dicke model [4]. As $\lambda$ is increased further $N$ converges to a single value as it is shown in figure 2(inset).

Figure 3 present the variances in the two quadrature operators (figures 3(a) and (b)), and the The Mandel parameter, $Q$, in figure 4. The variances in the quadratures indicate that the radiation field is in an ideal squeezed state in the sub-radiant phase, but there is a discontinuity in $(\Delta X_2)^2$ at the quantum critical point and the radiation field ceases to be in ideal squeezed state. As $\lambda$ is increased further the radiation field ceases to be in a squeezed state at all. Such discontinuity in $(\Delta X_2)^2$ is associated with a sudden change in the photon statistics as indicated by the Mandel parameter depicted in figure 4. In the
sub-radiant phase we have $Q > 0$ indicating that the photon statistics are super-poissonian. However, at the quantum critical point, $Q$ is divergent and above $\lambda_c$, $Q < 0$, indicating that the statistics have become sub-poissonian. As $\lambda$ is increased further $Q \rightarrow -1$ and as such $Q$ is taking on the most negative value of $Q$ possible, that of a Fock state $|10\rangle$.

In summary, we have reproduce the ground state energy, quantum critical point and scaling for the Dicke model by deriving an effective Hamiltonian for the radiation field. This is in stark contrast to previous work where the methodology tends to focus on the matter subsystem. We consider the effective Hamiltonian in the two different phases, sub-radiant and super-radiant, and demonstrate that the system undergoes a transition from quasi-integrable to quantum chaotic at the critical point. We also show that there are singularities in the $Q$-parameter and in the variance of the quadrature operator $X^2$ at the critical point and that the phase transition is accompanied by a change in the photon statistics. Finally, we have identified the state of the radiation field in both phases of the Dicke model at zero temperature as the matter-radiation field coupling $\lambda$ is varied.

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[1] Dicke R H 1954 Phys. Rev. 170 379; Jarrett T C, Lee C F and Johnson N F 2006 Phys. Rev. B 74 121301(R); Lambert N, Emary C, and Brandes T 2004 Phys. Rev. Lett. 92 073602; Dusuel S and Vidal J 2004 Phys. Rev. Lett. 93 237204; Emary C and Brandes T 2003 Phys. Rev. Lett. 90 044101; Lee C F and Johnson N F 2004 Phys. Rev. Lett. 93 083001

[2] Brandes T 2005 Phys. Rep. 408 315

[3] Reslen J, Quiroga L and Johnson N F 2005 Europhys. Lett. 69 8; Europhys. Lett. 72 153

[4] Dimer F, Estienne B, Parkins A S and Carmichael H J 2006 Preprint quant-ph/0607115

[5] Yoshih T, Scherer A, Hendrickson J, Khitrova G, Gibbs H M, Rupper G, Ell C, Shchekin O B and Deppe D G 2004 Nature 432 200; Wallraff A, Schuster D I, Blais A, Frunzio L, Huang R-S, Majer J, Kumar S, Girvin S M, Schoelkopf R J 2004 Nature 431 162; Guthöhllein G R, Keller M, Hayasaka K, Lange W, Walther W 2001 Nature 414 49

[6] Olaya-Castro A and Johnson N F 2005 Handbook of Theoretical and Computational Nanotechnology vol 10, ed M Rieth and W Schommers (California:American Scientific Publishers) (Preprint quant-ph/0406133)

[7] Polatsek G and Becker K W 1997 Phys. Rev. B 55 16096

[8] Kubo R 1962 J. Phys. Soc. Jpn. 17 1100

[9] Gerry C C and Kiefer J 1990 Phys. Rev. A 41 27; Gerry C C and Silverman S 1982 J. Math. Phys. 23 1995; Gerry C C 1982 Phys. Lett. 119B 381

[10] Scully M O and Zubairy M S 1997 Quantum Optics (Cambridge: Cambridge University Press)

[11] Barton G 1986 Ann. Phys. 166 322

[12] Miller P A and Sarkar S 1998 Phys. Rev. E 58 4217

[13] Emary C and Brandes T 2003 Phys. Rev. E 67 66203