Variational Bayes (VB) applied to latent Dirichlet allocation (LDA) is the original inference mechanism for LDA. Many variants of VB for LDA, as well as for VB in general have been developed since LDA's inception in 2013, but standard VB is still widely applied to LDA. Variational message passing (VMP) is the message passing equivalent of VB and is a useful tool for constructing a variational inference solution for a large variety of conjugate exponential graphical models (there is also a non-conjugate variant available for other models). In this article we present the VMP equations for LDA and also provide a brief discussion of the equations. We hope that this will assist others when deriving variational inference solutions to other similar graphical models.

Keywords Latent Dirichlet Allocation, Variational, Graphical Model, Message Passing, VMP, derivation

1 Introduction

The LDA graphical model[1], shown in Figure 1 has become more popular in the last two decades due to increase in availability of compute resources, the large volumes of data now available for consumption [2] as well as the advances in NLP. LDA can be used to extract latent topics of any type from a wide range of inputs but is most commonly known for its ability to extract latent semantic topics from text corpora. Borrowing an elegant statement from Using Variational Inference and MapReduce to Scale Topic [3], LDA "is appealing for noisy data because it requires no annotation and discovers, without any supervision, the thematic trends in a corpus". Some typical use cases of other applications of LDA are,

- banking and finance: clustering banking clients based on their transactions and identification of insurance fraud [4].
- genomics: using multilocus genotype data to learn about population structure and assign individuals to populations [5], classification of gene expression in healthy and diseased tissues [6] and prediction of the functional effects of genetic variation [7].
- image processing: clustering of images and scene recognition [8][9].
- medical: feature extraction for rare and emerging diseases [10] and medical data set clustering [11][12].

For a more comprehensive list, see the review article, Latent Dirichlet allocation (LDA) and topic modeling: models, applications, a survey [13].

Exact inference is intractable for many useful graphical models such as LDA [1][14][461][15]. A range of approximate techniques can be used to overcome this difficulty. Particle based approaches (such as Markov chain Monte Carlo [14][462]) are computationally expensive [16] and due to the availability of larger data sources, faster approaches that are as accurate [17][18] such as variational Bayes (VB) [19][19] have gained popularity.
Figure 1: Plate model of LDA system as a Bayes net. Each node in the graph is assigned a name based on the random variable that is to the left of the conditioning bar for its initial belief.

2 Variational Bayes as an approximate inference technique for LDA

Although collapsed Gibbs sampling (a type of MCMC technique) is often the inference technique of choice for LDA, the original article where LDA was introduced by David Blei [1], used VB. Note that this is standard variational Bayes and not the online version which was only introduced for LDA in 2011 [20] or the stochastic variant introduced in 2013 [21]. Both of these later methods are beneficial when there are larger amounts of data and do not improve LDA performance significantly. For improved performance and scalability, the structured stochastic variational inference algorithm from 2015 [22] is preferred (the independence assumptions are weakened resulting in a more accurate depiction of the posterior interdependencies). Standard VB, however, is still a popular technique for LDA due to its simplicity and many Python implementations.

Variational message passing (VMP) [23, 24] is the message passing equivalent of VB (the standard version) and is a useful tool for constructing a variational inference solution for a large variety of conjugate exponential graphical models. A non-conjugate variant of VMP, NCVMP, is also available for certain other model [25]. An advantage of VMP is that it can speed up the process of deriving a variational solution to a new graphical model [15]. Although the algorithm is simple, it can be confusing to someone new to the field unless applied to an example model that is similar in form to the model that they wish to apply it to.

In this article we present the VMP equations for LDA and also provide a brief discussion of the equations. The aim of this article is to assist others when deriving variational inference solutions to other similar graphical models, and also invite others to publish their VMP and NCVMP derivations to other interesting graphical models.

3 The exponential family (EF) of distributions

The VMP algorithm is limited to distributions in the exponential family. The exponential family is the only family of distributions with finite-sized sufficient statistics. Many useful distributions, including the Dirichlet and categorical distributions, fall into this family (incidentally, all of the distributions in LDA do too).

For a random vector, \(x\) with parameters \(\eta\), distributions within the exponential family can be written in the mathematically convenient form,

\[
p(x; \eta) = \frac{1}{Z(\eta)} h(x) \exp \{ \eta^T T(x) \},
\]

where \(\eta\) is called the natural parameters, and \(T(x)\) is the sufficient statistics vector (since with a sufficiently large sample, the probability of \(x\) under \(\eta\) only depends on \(x\) through \(T(x)\)). The partition function, \(Z(\eta)\), normalises the distribution to unity volume i.e.

\[
Z(\eta) = \int h(x) \exp \{ \eta^T T(x) \} \, dx.
\]

We can also formulate Equation (1) as,

\[
p(x; \eta) = h(x) \exp \{ \eta^T T(x) - A(\eta) \} \quad \text{with} \quad A(\eta) \triangleq \log Z(\eta).
\]
We call \(A(\eta)\) the log-partition or cumulant function since it can be used to find the moments of a distribution. Below we show the first cumulant (the mean) of exponential family distributions which will be used later,

\[
\nabla_\eta A(\eta) = \nabla_\eta \ln \left[ \int h(x) \exp \{ \eta^T T(x) \} \, dx \right]
\]

\[
= \int h(x) \exp \{ \eta^T T(x) \} \, dx \nabla_\eta \left[ \int h(x) \exp \{ \eta^T T(x) \} \, dx \right]
\]

\[
= \int \frac{1}{Z(\eta)} h(x) \exp \{ \eta^T T(x) \} T(x) \, dx
\]

\[
= \int p(x) T(x) \, dx
\]

\[
= \langle T(x) \rangle_{p(x)}
\]

(4)

This shows that we can find the first expected moment of a distribution in the exponential family by taking the derivative of its log partition function. This will always be the same as finding the expected value of the sufficient statistics vector.

### 3.1 Maximum likelihood estimation of distributions in the exponential family

For data \(D = \{x_1, x_2, \ldots, x_N\}\), we define the likelihood as,

\[
p(x_1, x_2, \ldots, x_n | \eta) = \prod_n p(x_n; \eta)
\]

\[
= \prod_n h(x_n) \left( \frac{1}{Z(\eta)} \right)^N \exp \left\{ \eta^T \sum_n T(x_n) \right\}
\]

(5)

The log likelihood can therefore be written as,

\[
l(D; \eta) = \ln \left( \prod_n h(x_n) \right) - NA(\eta) + \eta^T \sum_n T(x_n).
\]

(6)

This is a concave function of \(\eta\) and therefore is unimodal. To find the Maximum Likelyhood (ML) estimate we can therefore set the gradient w.r.t. \(\eta\) equal to zero,

\[
\nabla_\eta l(X; \eta) = 0 - \nabla_\eta NA(\eta) + \sum_n T(x_n)
\]

\[
\therefore \nabla_\eta A(\eta) = \frac{1}{N} \sum_n T(x_n).
\]

(7)

(8)

Reflecting on Equation (4) we see that,

\[
\langle T(x) \rangle_{p(x)} = \frac{1}{N} \sum_n T(x_n),
\]

(9)

since the ML estimate must equal the theoretical expected sufficient statistics.

### 4 The graphical model describing latent Dirichlet allocation (LDA)

Latent Dirichlet Allocation (LDA) is a hierarchical graphical model that can be represented by the directed graphical model shown in Figure 1.

It is comprised of the following random variables / vectors:
• topic-document random vectors, $\theta_m$: we have $M$ of these; one per document. Each $\theta_m$ is K-dimensional with
$$\theta_{m,k}$$
indicating the probability of topic $k$ in document $m$ with $\{\theta_{m,k} \in \mathbb{R} | \theta_{m,k} \in (0, 1)\}$.
• topic-document random variables, $Z_{m,n}$: we have one of these for each word in each document and therefore
$$\prod_{m=1}^M N_m$$
RVs within the graph (with $N_m$ words in document $m$). These RVs can also take on one of $K$ values and describe the probability of each word deriving from the various topics.
• Word-topic random variables, $W_{m,n}$: we once again have one of these for each word in each document. Each of these random variables can take on one of $V$ words in the vocabulary (each $v$ represents a word in the vocabulary).
• word-topic random vectors $\phi_k$: we have $K$ of these (which we can see as a single set), one per topic where
$$\phi_k = \{\phi_{1|Z=k}, \phi_{2|Z=k}, ..., \phi_{V|Z=k}\}$$
Each vector, $\phi_k$, is V-dimensional with $\phi_{k,v}$ indicating the probability of word $v$ in topic $k$ with $\{\phi_{k,v} \in \mathbb{R} | \phi_{k,v} \in (0, 1)\}$.

The graphical model shown in Figure 1 allows us to visually identify the conditional dependencies in the LDA model. Arrows in the graph indicate the direction of dependence. From 1, we can see that the $i$th word in document $m$ is $W_{m,n}$. This word depends on the topic $Z_{m,n}$ present in the document, which, selects the Dirichlet random vector $\phi_k$ describing the words present in each topic. Based on the graph we can identify the following distributions,

• topic-document Dirichlet distributions are used to inform our belief about the topics present within each document, as well as how confident we are in this belief. They have a belief of $\Psi_{\theta_m}(\theta_m; \alpha_m) = p(\theta_m | \alpha_m, \ldots, \alpha_m, K)$.
• topic-document distributions: each distribution informs the local belief about the topic proportions for each word in each document. They have an initial belief of $\Psi_{Z_{m,n}}(Z_{m,n}, \theta_m) = p(Z_{m,n} | \theta_m)$ and should the VMP approach be used, categorical form is used throughout message passing.
• word-topic conditional distributions are conditioned on the RV $Z_{m,n}$, which means that if we could observe the topic, we are left with either a categorical or Polya distribution. These distributions effectively tell us what the topic proportions are per word within a document, for each topic. They initially have a belief of $\Psi_{W_{m,n}}(W_{m,n}, Z_{m,n}, \Phi) = p(W_{m,n} | Z_{m,n}, \Phi)$ (and this remains so for VMP).
• word-topic Dirichlet distributions (a Dirichlet set) inform our belief that a word should be assigned to a topic as well as our confidence in this. These have a belief of $\Psi_{\phi_k}(\phi_k) = p(\phi_k | \beta_k)$. We can see the group of all the word-topic Dirichlet distributions in our graph as a set of Dirichlet distributions, denoted by $\Psi_{\phi}(\Phi) = p(\Phi | B)$.

In the next section we will introduce variation inference, a type of approximate inference which we will later apply to the LDA graphical model to calculate its posterior distributions based on priors we will chose.

5 Variational Message passing

Variational Bayes (VB) is a framework that approximates the full posterior distribution over a model’s parameters and latent variables in an iterative Expectation Maximization (EM) like manner 19 once variational equations for a graphical model have been derived.

Variational message passing (VMP) presents an alternative to deriving these equations for each model 23 24 and then using VB for the optimization. The formulation of optimization in terms of local computations is required to translate VB into its message passing variant, VMP.

5.1 The generic VMP algorithm

For conjugate-exponential models (those where conjugacy exists between all parent-child relationships and where all distributions are in the exponential family) we can perform variational inference by performing local computations and message passing.

Based on the derivation in 23, these local computations depend only on the variables within the Markov blanket of a node. The nodes in the Markov blanket (shown in Figure 2) are nodes that are either parents, children or co-parents of the node. The co-parents of a node are the parents of its children (excluding the node itself).

For Bayesian networks, the joint distribution can be expressed in terms of the conditional distributions at each node, $i$:

$$p(x) = \prod_i p(x_i | pa_i)$$

(10)
where \( pa_i \) are the parents of node \( i \) and \( x_i \) the variable(s) associated with node \( i \) (these can be hidden or visible).

\[
\text{pa}_{x_j}
\]

\[
\text{ch}_{x_j}
\]

Figure 2: Figure showing the Markov blanket of a node \((x_j)\). The nodes in the Markov blanket are the shaded nodes and defined by the set of parents (\(\text{pa}\)), children (\(\text{ch}\)) and co-parents (\(\text{cp}\)) of a node. The variational update equation for a node \(x_j\) depends only on expectations over variables appearing in the Markov blanket of that node [24].

In this section we present the general VMP message passing update equations and use a small example (the graphical model shown in Figure 3) to explain the broader principles involved in VMP. As with the directed graphical model in Figure 1, we assign names to each node in the graph based on the random variable left of the conditioning bar at initialisation.

\[
\theta \rightarrow x \rightarrow y
\]

Figure 3: Figure showing the child and parent nodes in the example we use to present the message passing equations. In our example we use \( x \) as the parent node and \( y \) as the child node, where the node names correspond to the random vectors.

We can write the node \( x \) (the parent of \( y \)) in exponential family form as,

\[
p(x; \eta) = \frac{1}{Z(\eta)} h(x) \exp \left\{ \eta^T T(x) \right\},
\]

\[
= h(x) \exp \left\{ \eta^T T(x) - A(\eta) \right\} \quad \text{with} \quad A(\eta) \triangleq \ln Z(\eta),
\]

\( \eta \) being the natural parameters, \( T(x) \), the sufficient statistics vector and \( A(\eta) \) the log-partition function.

If we limit ourselves to distributions in this family, the prior and posterior distributions have the same form (as shown in by Winn in Section 1.4.3 of his Dissertation [23]). When we perform inference, we therefore only need to update the values of the parameters and do not change the functional form [23].

5.1.1 Message to a child node

Continuing with the graphical model in Figure 3 the parent to child node message (for parent node \( x \) and child node \( y \)) is simply the expectation of the sufficient statistics vector [24],

\[
\mu^{x \rightarrow y} = \langle T(x) \rangle_{p(x)}.
\]
We can calculate this by using the derivative of the log partition function as seen in Equation 4,
\[ \langle T(x) \rangle = \nabla_{\eta} A(\eta) \] 
(14)
The parent to child message is therefore,
\[ \mu_{x \rightarrow y} = \nabla_{\eta} A(\eta) \] 
(15)
This message containing the expected values of the natural parameters of \( x \), now becomes the new natural parameters of \( y \). We can therefore write our child node’s distribution as \( p(y | \langle T(x) \rangle) \) after receiving a message from node \( x \).

5.1.2 Message to a parent node

Because we limit ourselves to conjugate-exponential models, the exponential form of the child distribution can always be re-arranged to match that of the parent distribution. This is due to the multi-linear properties of conjugate-exponential models [23].

We define the re-arranged version of the sufficient statistics (which are in the correct functional form to send a child to parent message) as \( \varphi(y) \). To read more about this, refer to Winn’s work [23, 24]. A child to parent message can therefore be written as,
\[ \mu_{y \rightarrow x} = \langle \varphi(y) \rangle_{p(y)} \cdot \] 
(16)
Note that if any node, \( a \), is observed then the messages are as defined above but with \( \langle \varphi(a) \rangle \) replaced by \( \varphi(a) \) (if we know the true values, we use them).

When the parent node has received all its required messages, we can update its posterior distribution. \( q_x' \) by finding its updated natural parameter vector \( \eta' \),
\[ \eta' = \{ \mu_{x_i \rightarrow x_j} \} x_i \in pa_{x_j} + \sum_{k \in ch_{x_j}} \mu_{x_k \rightarrow x_j} \]. 
(17)
Updating the parent node \( x \) from the graphical model in Figure 3 will result in the update equation,
\[ \eta' = \mu_{\theta \rightarrow x} + \mu_{y \rightarrow x} \cdot \] 
(18)
The full algorithm for VMP is summarised in Algorithm 1. We can see how this aligns with Equation??.

**Algorithm 1** Variational Message Passing (VMP) [23]

**Initialization:**
Initialize each factor distribution \( q_j \) by initializing the corresponding moment vector \( \langle T_j(x_j) \rangle \) with random vector \( x_j \).

**Iteration:**
1. For each node \( x_j \) in turn:
   (a) Retrieve messages from all parent and child nodes, as defined in Equation 14 and Equation 16. This requires child nodes to retrieve messages from the co-parents of \( x_j \) (Figure 2).
   (b) Compute updated natural parameter vector \( \eta'_j \) using equation Equation 17.
   (c) Compute updated moment vector \( \langle T_j(x_j) \rangle \) given the new setting of the parameter vector.
2. Calculate the new value of the lower bound ELBO(\( q \)) (optional).

**Termination:**
If the increase in the bound is negligible or a specified number of iterations has been reached, stop. Otherwise repeat from step 1.

The full algorithm as given by Winn [23] is shown in Algorithm 1.
6 The VMP algorithm for LDA

Because LDA has a simple, known structure per document, it is sensible to construct a fixed message passing schedule; this is not always the case for graphical models, where in some cases we rather base the schedule on message priority (using divergence measures to prioritise messages based on the impact they will make).

6.1 Message passing schedule

This message passing schedule is generic and is used for all three algorithms namely VMP, ALBU and VMP+. In the case of ALBU where we do updates, this is considered the message being sent.

Algorithm 2 Loopy message passing for LDA

For each epoch:

For each document $m$:

Do a forward sweep for word $n$ in this document:

- send message from $\Psi_{\Phi}(\Phi; B)$ to $\Psi_{W,m,n}(W_{m,n}, Z_{m,n}, \Phi)$
- observe word $W_{m,n} = v$
- send message from $\Psi_{W,m,n}(W_{m,n}, Z_{m,n}, \Phi)$ to $\Psi_{Z,m,n}(Z_{m,n}, \theta_m)$
- send message from the $\Psi_{Z,m,n}(Z_{m,n}, \theta_m)$ to $\Psi_{\theta_m}(\theta_m; \alpha_m)$

Do a backward sweep for word $n$ in this document:

- send message from $\Psi_{\theta_m}(\theta_m; \alpha_m)$ to $\Psi_{Z,m,n}(Z_{m,n}, \theta_m)$
- send message from $\Psi_{Z,m,n}(Z_{m,n}, \theta_m)$ to the $\Psi_{W,m,n}(W_{m,n}, Z_{m,n}, \Phi)$

For each word in each document:

- send message from the $\Psi_{W,m,n}(W_{m,n}, Z_{m,n}, \Phi)$ to $\Psi_{\Phi}(\Phi)$

Now that we have defined our inference algorithms as well as the message passing scheme that we will use for the LDA graphical model, we can now present VMP for LDA. Here we describe the distribution at each node and also derive the child to parent and parent to child messages (where applicable) for the LDA graph. We keep the messages dependent only on the current round of message passing which can be considered to be one epoch.

6.2 The topic-document Dirichlet, $\Psi_{\theta_m}$

In exponential family form we can write each topic-document Dirichlet as,

$$\ln \text{Dir}(\theta_m; \alpha_m) = \begin{bmatrix} \alpha_{m,1} - 1 \\ \alpha_{m,2} - 1 \\ \vdots \\ \alpha_{m,K} - 1 \end{bmatrix}^T \begin{bmatrix} \ln \theta_{m,1} \\ \ln \theta_{m,2} \\ \vdots \\ \ln \theta_{m,K} \end{bmatrix} - \ln \Gamma(\sum_k \alpha_{m,k}) \prod_k \Gamma(\alpha_{m,k}) \quad (19)$$

For each $m$ we can identify the natural parameters as,

$$\eta = \begin{bmatrix} \alpha_1 - 1 \\ \alpha_2 - 1 \\ \vdots \\ \alpha_K - 1 \end{bmatrix} \quad (20)$$

and the sufficient statistics as,

$$T(\theta) = \begin{bmatrix} \ln \theta_1 \\ \ln \theta_2 \\ \vdots \\ \ln \theta_K \end{bmatrix} \quad (21)$$

Message to a child node, $\Psi_{Z,m,n}$ The parent to child node message (for parent node $\theta_m$ and child node $Z_{m,n}$) is simply the expectation of the sufficient statistics vector $[24]$,

$$\mu_{\theta_m \rightarrow Z_{w,n}} = \langle T(\theta_m) \rangle_{p(\theta_m)} \quad (22)$$
Using Equation 14, we can calculate this expectation using the derivative of the log partition function. It is known that Winn [23, p128],

\[
< \ln(\theta_k) >_{p(\theta_k)} = \Psi(\alpha_k) - \Psi(\sum_k \alpha_k),
\]

(23)

where \(\Psi\) is the digamma function. The parent to child message from each topic-document Dirichlet distribution is therefore,

\[
\mu_{\theta_m \rightarrow Z_{m,n}} = \begin{bmatrix}
\psi(\alpha_{m,1}) - \psi(\sum_k \alpha_{m,k}) \\
\vdots \\
\psi(\alpha_{m,K}) - \psi(\sum_k \alpha_{m,k})
\end{bmatrix}
\]

(24)

We can now insert these expected sufficient statistics into the child distribution. The natural parameter vector of the topic-document distributions is now,

\[
\eta_{Z_{m,n}} = \begin{bmatrix}
\ln \theta'_{m,1} \\
\vdots \\
\ln \theta'_{m,K}
\end{bmatrix}
\]

(25)

with \(\theta'\) being the updated topic proportions. Note the conjugacy between the Dirichlet and categorical distributions: this allows us to simply update the natural parameters [2, 23].

6.2.1 The topic-document categorical, \(\Psi_{Z_{m,n}}\)

In exponential form we can represent the topic-document categorical distribution for a specific word in a specific document as,

\[
\ln \text{Cat}(Z_{m,n}|\theta_m) = \begin{bmatrix}
\ln \theta_{m,1} \\
\ln \theta_{m,2} \\
\vdots \\
\ln \theta_{m,K}
\end{bmatrix}^\top \begin{bmatrix}
\mathbb{I}_{Z_{m,n} = 1} \\
\mathbb{I}_{Z_{m,n} = 2} \\
\vdots \\
\mathbb{I}_{Z_{m,n} = K}
\end{bmatrix} - \ln(\sum_k \theta_{m,k}),
\]

(26)

\[
\therefore A(\theta_m) = \ln(\sum_k \theta_{m,k}),
\]

(27)

where the function \(\mathbb{I}[Z = k]\) is known as the Iverson bracket:

\[
\mathbb{I}[\text{[Proposal]}] = \begin{cases}
1 & \text{[Proposal] is true} \\
0 & \text{otherwise}
\end{cases}
\]

Message to a child node, \(\Psi_{W_{m,n}}\) The messages from the topic-document categorical distributions to the conditional word-topic categorical distributions are also parent to child messages. Based on Equation 13 the update equation is,

\[
\mu_{Z_{m,n} \rightarrow W_{m,n}} = \langle T(Z_{m,n}) \rangle_{p(Z_{m,n})}
\]

(28)

Equation 27 defines the log-partition function that can be used to calculate this moment using Equation 14. To do this we need to re-parameterise the natural parameter vector from Equation 26. This is shown below for a single word in a single topic for each \(k\),

\[
\eta_k = \ln \theta'_k, \\
\therefore \theta'_k = e^{\eta_k},
\]

\[
\therefore A(\eta) = \ln(\sum_j e^{\eta_j}).
\]

From this we can calculate the expected sufficient statistics using Equation 4 for a specific topic \(k\):
Equation 4 for a specific topic \( k \):

\[
\frac{\delta}{\delta \eta_k} A(\eta) = \sum_k e^{\eta_k} = \sum_k \theta_{m,k} = \theta^*_{m,k}
\]

where the \( \theta^* \)'s are normalised. 

Each \( \theta^*_{m,k} \) is a normalized topic proportion for a single topic. We can write the full parent to child message for a word within a topic as,

\[
\mu_{p2c}^{Z_{m,n} \rightarrow W_{m,n}} = \begin{bmatrix}
\theta^*_{m,1} \\
\vdots \\
\theta^*_{m,K}
\end{bmatrix},
\]

with,

\[
\theta_{m,k} \sum_k \theta_{m,k}
\]

These updated topic proportion are then used to update the word-topic conditional categorical distributions.

**Message to a parent node, \( \Psi_{\theta_m} \)** The message from a topic-document categorical to a topic-document Dirichlet is a child to parent message. Assuming that each topic-document categorical has received its message from the corresponding word-topic node \( W_{m,n} \), we can re-arrange Equation 26 i.t.o. \( \theta_m \) as follows,

\[
\ln p(Z_{m,n}|\theta_m) = \left[ \begin{bmatrix}
Z_{m,n} = 1 \\
Z_{m,n} = 2 \\
\vdots \\
Z_{m,n} = K
\end{bmatrix} \right]^T \begin{bmatrix}
\ln \theta_{m,1} \\
\ln \theta_{m,2} \\
\vdots \\
\ln \theta_{m,K}
\end{bmatrix} - \ln \left( \sum_k \theta_{m,k} \right).
\]

If we have received a message from the node to the right of the topic-document node, the natural parameter vector of the topic-document distribution will have been modified and we can call it \( \theta'_m \).

Because the message sent to the topic-document node is a child to parent message (from the respective word-topic node), this message is added to the natural parameter vector such that

\[
\ln \tilde{\theta}_m = \ln \theta_m + \ln \theta'_m = \ln(\theta_m p_{m,n}),
\]

Re-normalizing so that \( \sum_k \theta'_{m,k} = 1 \) gives,

\[
\ln \theta'_{m,k} = \ln \left( \frac{\theta_m p_{m,n}}{\sum_k \theta_m p_{m,n}} \right),
\]

where \( \{p_{m,n,1}, \ldots, p_{m,n,K}\} \) are the topic probabilities for a specific word in a specific document as given by the message from \( W_{m,n} \). In the case where no message has been received from \( W_{m,n} \), then \( \ln \theta'_m = \ln \theta_m \). In LDA we will only ever update the topic-document Dirichlet from the topic-document categorical after receiving a message from the word-topic side of the graph (except at initialisation, when this message will be uninformative).
We can now define the child to parent message towards the word-topic Dirichlet to be,

\[ \mu_{Z_{m,n} \rightarrow \theta_m}^{c \rightarrow p} = \langle \phi(Z_{m,n}) \rangle_{p(Z_{m,n})} \]

\[ = \left\langle \begin{bmatrix} Z_{m,n} = 1 \\ Z_{m,n} = 2 \\ \vdots \\ Z_{m,n} = K \end{bmatrix} \right\rangle_{p(Z_{m,n})} \]

\[ = \begin{bmatrix} \theta_{m,1} \\ \theta_{m,2} \\ \vdots \\ \theta_{m,K} \end{bmatrix} \]

\[ = \begin{bmatrix} \frac{\theta_{m,1} p_{m,n,1}}{\sum_k \theta_{m,1} p_{m,n,k}} \\ \frac{\theta_{m,2} p_{m,n,2}}{\sum_k \theta_{m,2} p_{m,n,k}} \\ \vdots \\ \frac{\theta_{m,K} p_{m,n,K}}{\sum_k \theta_{m,K} p_{m,n,k}} \end{bmatrix} \]

For each topic-document Dirichlet, for one branch in the graph (one word, \( n \), in a single document \( m \)) and for each topic, \( k \), we have the update,

\[ \alpha_m' = \frac{\theta_{m,k} p_{m,n,k}}{\sum_k \theta_{m,k} p_{m,n,k}} + \alpha_{m,k}^{\text{prior}} \]

once the respective message has been received from its \( W_{m,n} \). Over all words in the document we therefore have,

\[ \alpha_m' = \sum_n \frac{\theta_{m,k} p_{m,n,k}}{\sum_k \theta_{m,k} p_{m,n,k}} + \alpha_{m,k}^{\text{prior}} \]

We will later be able to update this equation with the actual values that \( p_{m,n,k} \) represent (they are yet to be derived).

6.2.2 The word conditional categorical, \( \Psi_{W_{m,n}} \)

The \( n \)'th word of a document is described by one of \( K \) topi-word distributions. We call this a conditional categorical distribution; for a topic \( k \) this reduces to a single categorical. For all \( K \) we can write,

\[ \ln \text{Cat}(W_{m,n}|Z_{m,n}, \Phi) = \left[ \sum_k \frac{[Z_{m,n} = k]}{\sum_k [Z_{m,n} = k]} \ln \phi_1|Z_{m,n} = k \right]^T \left[ \begin{bmatrix} W_{m,n} = 1 \\ W_{m,n} = 2 \\ \vdots \\ W_{m,n} = V \end{bmatrix} \right] \]

\[ - \sum_k \frac{[Z_{m,n} = k]}{\sum_k [Z_{m,n} = k]} \ln \left( \sum_v \phi_v|Z_{m,n} = k \right) \]

where the vocabulary over all words range from 1 to \( V \).

Message to a Dirichlet parent node, \( \Psi_{\Phi} \) To send child to parent messages from the word-topic conditional categorical distributions to word-topic Dirichlet distribution, re-paramitarisation is required. When we consider only one topic, \( k \), we can think of there being a single message to be sent from the categorical representing the word proportions for this topic to the respective Dirichlet distribution, \( \phi_k \). Figure 4 shows how a conditional categorical can be viewed as a set of categorical distributions, each conditioned on a unique \( k \).
Figure 4: This figure shows the notation used for working with a conditional categorical distribution, \( W_{m,n} \). In (a) we show the full conditional categorical and in (b) we show the separate categorical distributions, each conditioned on a specific topic.

After parameterising i.t.o. \( \phi_{m,n|Z_{m,n}=k} \) we get,

\[
\ln p(W_{m,n}|Z_{m,n}, \phi_k) = \begin{bmatrix} \prod_{Z_{m,n}=k}[W_{m,n}=1] \\ \vdots \\ \prod_{Z_{m,n}=k}[W_{m,n}=v] \\ \vdots \\ \prod_{Z_{m,n}=k}[W_{m,n}=V] \end{bmatrix}^T \begin{bmatrix} \ln \phi_1|Z_{m,n}=k \\ \vdots \\ \ln \phi_v|Z_{m,n}=k \\ \vdots \\ \ln \phi_V|Z_{m,n}=k \end{bmatrix} + \text{terms involving } \ln \phi_{Z_{m,n}=\neq k}.
\]

where \( \phi_{v|Z_{m,n}=k} \) are the topic proportions for word \( v \) in topic \( k \). The messages from one of these categorical distributions \( p(W_{m,n}|Z_{m,n}=k, \phi_k) \) to \( \phi_k \) can then be written as, \( m,n|Z_{m,n}=k \to \phi_k \)

\[
m_{W_{s} \to \phi_k} = \langle \varphi(W_{m,n}|Z_{m,n}=k, Z_{m,n}=k) \rangle_{p(W_{m,n}|Z_{m,n}=k, Z_{m,n}=k)} = \langle \prod_{Z_{m,n}=k}[W_{m,n}=v] \rangle_{p(W_{m,n}|Z_{m,n}=k, Z_{m,n}=k)} = \langle \prod_{Z_{m,n}=k}[1] \rangle_{p(Z_{m,n}=k)} \text{ because } W_{m,n} \text{ is observed}
\]

Note that because \( W_{m,n} \) is observed, the values in the vector for all entries except for where \( W_{m,n} = v \) is zero. To update \( \phi_k \) we simply add all the incoming message to the respective \( \beta_k \) values. For a specific word in the vocabulary, \( v \), this would be,

\[
\beta'_{k,v} = \sum_m \sum_n \theta^*_{m,k} + \beta_{k,v}^{\text{prior}}
\]

We can see that the scaled topic proportions for word \( v \) simply get added to the respective \( \beta_k \) values. The actual values (based on current \( \alpha \) values) for \( \theta^*_{m,k} \) can be computed by substituting Equation 25 into Equation 29 (this normalizes the values) and finally into 42.

\[
\beta'_{k,v} = \sum_m \sum_n \frac{\theta^*_{m,k}}{\sum_k \theta^*_{m,k}} + \beta_{k,v}^{\text{prior}}
\]

\[
= \sum_m \sum_n \frac{\theta^*_{m,k}}{\sum_k \theta^*_{m,k}} + \beta_{k,v}^{\text{prior}}
\]

\[
= \sum_m \sum_n \frac{\exp (\psi(\alpha_{m,k}) - \psi(\sum_k \alpha_{m,k}))}{\sum_k (\exp (\psi(\alpha_{m,k}) - \psi(\sum_j \alpha_{m,j})))} + \beta_{k,v}^{\text{prior}}
\]
**Message to a categorical parent node, Ψ_{Z_{m,n}}** The word-topic conditional categorical is a child of the topic-document categorical; we send a child to parent message for each \( k \). We can re-write Equation 39 i.t.o. \( Z_{m,n} \) this time to give,

\[
\ln p(W_{m,n}|Z_{m,n}, \Phi) = \begin{bmatrix}
\sum_v [W_{m,n} = v] \ln \phi_{v|Z_{m,n}=1} \\
\vdots \\
\sum_v [W_{m,n} = v] \ln \phi_{v|Z_{m,n}=k} \\
\vdots \\
\sum_v [W_{m,n} = v] \ln \phi_{v|Z_{m,n}=K} 
\end{bmatrix}^T \begin{bmatrix}
\mathbb{1}[Z_{m,n} = 1] \\
\vdots \\
\mathbb{1}[Z_{m,n} = k] \\
\vdots \\
\mathbb{1}[Z_{m,n} = K] 
\end{bmatrix}.
\] (46)

After observing the word, \( v \), Equation 46 reduces to a categorical. Using Equation 29, we can then write the the child to parent message as,

\[
\tilde{\mu}^{c2p}_{W_{m,n}\rightarrow Z_{m,n}} = \begin{bmatrix}
\ln \phi_{v|Z_{m,n}=1} \\
\vdots \\
\ln \phi_{v|Z_{m,n}=k} \\
\vdots \\
\ln \phi_{v|Z_{m,n}=K}
\end{bmatrix}
\] (47)

where \( v \) is the observed word as in Equation 41, and the message is un-normalized. This is because we have taken a slice through the word-topic distributions for a specific word and the result is not a true distribution. We normalize the message to obtain the topic proportions (for each word in each document) giving,

\[
\mu^{c2p}_{W_{m,n}\rightarrow Z_{m,n}} = \begin{bmatrix}
\ln \phi^*_v|Z_{m,n}=1 \\
\vdots \\
\ln \phi^*_v|Z_{m,n}=k \\
\vdots \\
\ln \phi^*_v|Z_{m,n}=K
\end{bmatrix}
\] (48)

with

\[
\phi^*_v|Z_{m,n}=k = \frac{\phi_{v|Z_{m,n}=k}}{\sum_k \phi_{v|Z_{m,n}=k}}.\] (49)

To get the updated document topic proportions we update the natural parameter vector by adding these topic weightings to the current document topic proportions,

\[
\eta'_{Z_{m,n}} = \begin{bmatrix}
\ln \theta_{m,1} + \ln \phi^*_v|Z_{m,n}=1 \\
\vdots \\
\ln \theta_{m,K} + \ln \phi^*_v|Z_{m,n}=K
\end{bmatrix}.
\] (50)

Reflecting now on Equation 56 we can see that \( p_{m,n,k} = \phi^*_v|Z_{m,n}=k \) and we can write the message more completely as,

\[
\alpha'_m = \theta_m k \phi^*_v|Z_{m,n}=k + \alpha^\text{prior}_m k.
\] (51)

It is worth noting that the actual values for \( \theta_m k \) are based on the current epoch’s \( \alpha \) values and can be computed by substituting Equation 25 into 42,

\[
\alpha'_m = \exp \left( \psi(\alpha_{m,1}) - \psi(\sum_k \alpha_{m,k}) \right) \phi^*_v|Z_{m,n}=k + \alpha^\text{prior}_m k.
\] (52)

### 6.2.3 The word-topic Dirichlet set, \( \Psi_\Phi \)

For the entire graph, we have: \( \Phi = \{ \phi_1, \ldots, \phi_K \} \). For each topic \( k \) we write,
\[ \ln \text{Dir}(\phi_k; \beta_k) = \begin{bmatrix} \beta_{k,1} - 1 \\ \beta_{k,2} - 1 \\ \vdots \\ \beta_{k,V} - 1 \end{bmatrix}^T \begin{bmatrix} \ln \phi_{k,1} \\ \ln \phi_{k,2} \\ \vdots \\ \ln \phi_{k,V} \end{bmatrix} - \ln \left( \prod_v \Gamma(\beta_{k,v}) \right). \tag{53} \]

**Message to a child node, \( \Psi_{W,m,n} \)** These messages are very similar to the ones on the topic-document side of the graph since they are also parent to child messages from Dirichlet distributions.

For each topic, \( k \), the messages sent to all word-topic categorical distributions (one for each word in each topic) will be identical,

\[ \mu_{\phi_k \rightarrow W_{m,n}}^{\phi_k} = \begin{bmatrix} \psi(\beta_{k,1}) - \psi(\sum_v \beta_{k,v}) \\ \vdots \\ \psi(\beta_{k,V}) - \psi(\sum_v \beta_{k,v}) \end{bmatrix}, \tag{54} \]

The additional complexity comes from the fact that the child conditional categorical distributions need to assimilate messages from \( K \) Dirichlet distributions (Figure 4) and not only from one, as in the topic-document side of the graph.

We now perform a similar update to the update seen in Equation 25 except that we have \( K \) messages added instead of only one. For each \( k \) we have,

\[ \eta'_{W_{m,n},\phi_k} = \begin{bmatrix} \psi(\beta_{k,1}) - \psi(\sum_v \beta_{k,v}) \\ \vdots \\ \psi(\beta_{k,V}) - \psi(\sum_v \beta_{k,v}) \end{bmatrix} = \begin{bmatrix} \ln \phi'_{1|Z_{m,n}=k} \\ \vdots \\ \ln \phi'_{V|Z_{m,n}=k} \end{bmatrix}, \tag{55} \]

where \( \phi' \) denotes the updated values. We can now expand Equation 38 to be,

\[ \alpha'_{m,k} = \exp \left( \psi(\alpha_{m,k}) - \psi(\sum_k \alpha_{m,k}) \right) \phi'_{m,n|Z_{m,n}=k} + \alpha_{m,k} \]

\[ = \exp \left( \psi(\alpha_{m,k}) - \psi(\sum_k \alpha_{m,k}) \right) \left( \frac{\phi'_{m,n|Z_{m,n}=k}}{\sum_k \phi'_{m,n|Z_{m,n}=k}} \right) + \alpha_{m,k} \]

\[ = \exp \left( \frac{\psi(\alpha_{m,k}) - \psi(\sum_k \alpha_{m,k})}{\sum_k \psi(\beta_{k,v}) - \psi(\sum_v \beta_{k,v})} \right) + \alpha_{m,k} \tag{56} \]

### 7 Discussion

The child to parent and parent to child formation of these equations results in interesting behaviour for the LDA graphical model. If we consider the two ends of the graphical model where Dirichlet distributions are updated, the updates for the respective parameters (\( \alpha \)’s for the topic-document side, \( \Psi_{\beta_{m,n}} \), shown in Equations 51 and 56 and (\( \beta \)’s for the word-topic side, \( \Psi_{\phi_{m}} \) shown in Equations 42 and 43), seem vastly different.

On the word-topside (the \( K \) Dirichlet distributions to the right in Figure 1) we have a Dirichlet update for each word in each document updating a single word-topic Dirichlet per topic, \( \Psi_{\phi_k} \). These updates are only dependent on the latest \( \alpha \) parameters (after transformation with Digamma functions and normalisation after observing each word). This is due to the fact that the node in the graph where \( W \) is observed (the word conditional categorical, \( \Psi_{W_{m,n}} \)) is always a child node and never a parent. It’s parameters are never update by addition to its natural parameter vector, but only by insertion. This results in the situation for LDA that for this node, we never mix information from both of its parents.

When information flows though the word conditional categorical, \( \Psi_{W_{m,n}} \) from the topic-document side (from the topic-document categorical, \( \Psi_{Z_{m,n}} \)), the belief \( j \) information is not added to the beliefs from the word-topside, \( \Psi_{\phi} \), (except by the transformation of observation and normalisation that happens inside \( \Psi_{\phi} \)). In the reverse direction, when the topic-document categorical, \( \Psi_{Z_{m,n}} \), is updated, it passes on a transformed version of the beliefs from the set of word-topic Dirichlet distributions, \( \Psi_{\phi} \), (once again using the Digamma function to transform the \( \beta \) parameters.

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On the other side of the graph (the Dirichlet distributions, $\Psi_{\theta_m}$, to the left in Figure 1), the Dirichlet update is from a node, the topic-document categorical, that acts as both a child and a parent, depending on the direction of the messages being passed to/from it. The topic-document distribution’s belief, $\Psi_{Z_{m,n}}$, get updated when the message from the word-topic node get added to its natural parameter vector. Its belief therefore will contain a combination of information that was send in the latest round from the topic-document Dirichlet “up-stream” of it, $\Psi_{\theta_m}$, (its parent) and information from the word-topic conditional categorical, “down-stream”, $\Psi_{W_{m,n}}$, (its child). When it updates its parent Dirichlet, $\Psi_{\theta_m}$, (by adding to its natural parameter vector), it uses this combined belief, and therefore contains transformed versions of both $\alpha$ and $\beta$ parameters.

We know from Winn [24] that the VMP algorithm is message passing the analog of VB. This means that these equations can be compared with the LDA update equations (the smoothed version) in Blei’s original LDA article [1]. One can note the similarities in these equations although the formulation is different due to VMP using message passing.

8 Conclusion

In this article, we give a brief overview of variational message passing (VMP) and where it fits into the world of approximate inference. We derive the VMP equations for the LDA graphical model in detail and then provide insight into some of these messages. Based on this work, we invite others to present their update equations for other more complex variants of the LDA model as well as for other non-conjugate models.

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