

MONIC MONOMIAL REPRESENTATIONS I: GORENSTEIN-PROJECTIVE MODULES

XIU-HUA LUO, PU ZHANG∗

Abstract. For a k-algebra A, a quiver Q, and an ideal I of kQ generated by monomial relations, let Λ := A ⊗_k kQ/I. We introduce the monic representations of (Q, I) over A. We give properties of the structural maps of monic representations, and prove that the category mon(Q, I, A) of the monic representations of (Q, I) over A is a resolving subcategory of rep(Q, I, A). We introduce the condition (G). The main result claims that a Λ-module is Gorenstein-projective if and only if it is a monic module satisfying (G). As consequences, the monic Λ-modules are exactly the projective Λ-modules if and only if A is semisimple; and they are exactly the Gorenstein-projective Λ-modules if and only if A is selfinjective, and if and only if mon(Q, I, A) is Frobenius.

Key words and phrases. monic representations, Gorenstein-projective modules

1. Introduction and preliminaries

1.1. The representation category rep(Q, I, k) of a bounded quiver (Q, I) over field k ([28], [4]), and the morphism category ([3]) of k-algebra A, can be reviewed as the representation category rep(Q, I, A) of (Q, I) over A, or equivalently, the module category Λ-mod with Λ := A ⊗_k kQ/I. If I = 0, this viewpoint induces the notion of monic Λ-modules ([25]), and we get the monomorphism category mon(Q, A) consisting of monic representations of Q over A.

In fact, there is a long history of studying mon(Q, A), initiated by G. Birkhoff [7]. When Q is of type An, it is the submodule category ([29]-[31], [26]) or the filtered chain category ([1], [33], [34]). In particular, C.M. Ringel and M. Schmidmeier have established the Auslander-Reiten theory of the submodule category ([30]); and D. Simson have studied the representation type ([33]). A reciprocity of monomorphism operator and perpendicular operator is given ([35]); and by D. Kussin, H. Lenzing and H. Meltzer ([23], [24]) and X.W. Chen [10], the monomorphism category is related to the singularity theory.

However, for I ≠ 0, how to define monic representations of (Q, I) over A is a problem. In this paper we consider the case that I is generated by monomial relations. Definition 2.1 is inspired by the property of projective kQ/I-modules. This fits our aim of describing Gorenstein-projective Λ-module via monic representations of (Q, I) over A. If A is not semisimple, then the monic representations of (Q, I) over A are much more than the projective Λ-modules. We give properties of the structural maps X_p of a monic representation X (Theorem 2.3). The category mon(Q, I, A) of the monic representations of (Q, I) over A is proved to be a resolving subcategory of rep(Q, I, A) (Theorem 3.1).

1.2. On the other hand, Gorenstein-projective modules (M. Auslander and M. Bridger [2]; E.E. Enochs and O.M.G. Jenda [13]) enjoy more pleasant stable properties than projective modules (cf. [8], [6], [32], [18]). They form a main ingredient in the relative homological algebra (cf. [14], [5], [17], [11], [20]), and are widely used in the representation theory and algebraic geometry (cf. [8], [22], [9], [16], [6], [12], [19]). An important feature is that the category GP(A) of Gorenstein-projective modules is Frobenius, and hence
its stable category $GP(A)$ is triangulated. If $A$ is Gorenstein then the singularity category of $A$ is triangle equivalent to $GP(A)$ ([8]), and hence has the Auslander-Reiten triangles ([15]).

Thus we need explicitly construct all the Gorenstein-projective modules of an algebra $\Lambda$. This turns out to be closely related to the monomorphism category (cf. [35], [19], [25], [36]). However, if we view an algebra as $\Lambda := A \otimes_k kQ/I$, this relation is only known at the case of $I = 0$ ([25]).

We introduce the condition (G). The main result Theorem 4.1 claims that a $\Lambda$-module is Gorenstein-projective if and only if it is a monic representation of $(Q, I)\Lambda$-modules via quivers. This generalizes the corresponding result in [25] for $\Lambda$-modules, respectively, of two algebras via a bimodule ([36]). So we get

$$\{\text{projective } \Lambda\text{-module}\} \subseteq \{\text{Gorenstein-projective } \Lambda\text{-module}\} = \{\text{monic } \Lambda\text{-module satisfying (G)}\} \subseteq \{\text{monic } \Lambda\text{-module}\}.$$ 

As applications of Theorem 4.1, the first inclusion is an equality if and only if $A$ is semisimple; and the last inclusion is an equality if and only if $A$ is selfinjective, and if and only if $\text{mon}(Q, I, A) = \text{Frob}(Q, I, A)$.

1.3. Throughout $A$ is a finite-dimensional algebra over field $k$, $A$-$mod$ the category of finitely generated left $A$-modules, $Q$ a finite acyclic quiver, $I$ an ideal of path algebra $kQ$ generated by monomial relations, and $\Lambda := A \otimes_k kQ/I$. All tensors $\otimes$ are over $k$. Let $Q_0$ and $Q_1$ be the set of vertices and the set of arrows, respectively, $s(\alpha)$ and $e(\alpha)$ the starting and the ending vertex of arrow $\alpha$, respectively. Path $p$ is a sequence $\alpha_1 \cdots \alpha_l$ of arrows with each $e(\alpha_i) = s(\alpha_{i+1})$; and it is non-zero if $p \notin I$. Connection of paths is from right to left. Vertex $v$ is a path of length 0, denoted by $e_v$, and $P(v)$ is the indecomposable projective $kQ/I$-module $(kQ/I)e_v$. Let $P$ be the set of paths, $s(p)$, $e(p)$ and $l(p)$ the starting vertex, the ending vertex and the length of $p \in P$, respectively. Label $Q_0$ as $1, \cdots, n$, such that $j > i$ if $\alpha : j \rightarrow i$ is in $Q_1$. So $n$ is a source and 1 is a sink. If $p = \alpha q \in P$ with $\alpha \in Q_1$, then $\alpha$ is the last arrow of $p$, denoted by $la(p) := \alpha$. Let $\rho := \{\rho_1, \cdots, \rho_l\}$ is the minimal set of generators of $I$. For $i, j \in Q_0$, we put

$$A(\rightarrow i) := \{\alpha \in Q_1 \mid e(\alpha) = i\}$$

$$P(\rightarrow i) := \{ p \in P \mid e(p) = i, \ l(p) \geq 1, \ p \notin I \}$$

$$A(j \rightarrow i) := \{\alpha \in Q_1 \mid s(\alpha) = j, \ e(\alpha) = i\}$$

$$P(j \rightarrow i) := \{ p \in P \mid s(p) = j, \ e(p) = i, \ l(p) \geq 1, \ p \notin I \}.$$ 

For a non-zero path $p$ with $l(p) \geq 1$, we put

$$K_p := \{q \in P(\rightarrow s(p)) \mid pq \in I\}. \tag{1.1}$$

1.4. A representation $X$ of $Q$ over $A$ is a datum $(X_i, X_\alpha, i \in Q_0, \ \alpha \in Q_1)$, where each $X_i$ is an $A$-module and each $X_\alpha : X_{s(\alpha)} \rightarrow X_{e(\alpha)}$ is an $A$-map. A morphism $f : X \rightarrow Y$ is a datum $(f_i, i \in Q_0)$, where $f_i : X_i \rightarrow Y_i$ is an $A$-map, such that for each arrow $\alpha : j \rightarrow i$ we have

$$Y_\alpha f_j = f_i X_\alpha. \tag{1.2}$$

We call $X_i$ and $f_i$ the $i$-th branch of $X$ and the $i$-th branch of $f$, respectively. We also write $X$ as $(X_i, \ X_\alpha, \ \alpha \in Q_1)$, and a morphism as $(f_i, \ \alpha \in Q_1)$. For path $p = \alpha_1 \cdots \alpha_l$ with each $\alpha_i \in Q_1$, let $X_p$ denote the $A$-map $X_{\alpha_l} \cdots X_{\alpha_1} : X_{s(\alpha_l)} \rightarrow X_{e(\alpha_l)}$.

A representation $X$ of $(Q, I)$ over $A$ is a representation $X$ of $Q$ over $A$, such that $X_{\rho_i} = 0$ for each $\rho_i \in \rho$. Denote by $\text{rep}(Q, I, A)$ the category of finite-dimensional representations of $(Q, I)$ over $A$. A
morphism $f$ is a monomorphism (resp., an epimorphism, an isomorphism) if and only if each $f_i$ is injective (resp., surjective, an isomorphism). A sequence $0 \to X \xrightarrow{f} Y \xrightarrow{g} Z \to 0$ of morphisms in $\text{rep}(Q, I, A)$ is exact if and only if $0 \to X_i \xrightarrow{f_i} Y_i \xrightarrow{g_i} Z_i \to 0$ is exact in $A$-mod for each $i \in Q_0$.

**Lemma 1.1.** ([4, p.57], [28, p.44]) *We have an equivalence $A\text{-mod} \cong \text{rep}(Q, I, A)$ of categories.*

Throughout we will identify a $\Lambda$-module with a representation of $(Q, I)$ over $A$.

As an object of $\text{rep}(Q, I, k)$ we have $P(v) = (e_i(kQ/I)e_v, P(v)_\alpha, i \in Q_0, \alpha \in Q_1)$, where $P(v)_\alpha : e_{s(\alpha)}(kQ/I)e_v \to e_{t(\alpha)}(kQ/I)e_v$ is the $k$-map sending path $w$ to $\alpha w$. Consider functors $- \otimes P(v) : A\text{-mod} \to \text{rep}(Q, I, A)$ and $- \mapsto \text{rep}(Q, I, A) \to A\text{-mod}$ (by taking the $i$-th branch).

**Lemma 1.2.** (1) $(- \otimes P(v), -\mapsto)$ is an adjoint pair.

(2) The indecomposable projective $\Lambda$-modules are exactly $P \otimes P(v)$, where $P$ are indecomposable projective $A$-modules. In particular, branches of projective $\Lambda$-modules are projective $A$-modules.

**Proof.** Let $X = (X_i, X_\alpha, i \in Q_0, \alpha \in Q_1) \in A\text{-mod}$, $M \in A\text{-mod}$, and $f = (f_i, i \in Q_0) \in \text{Hom}_A(M \otimes P(v), X)$. Then $f_i \in \text{Hom}_A(M, X_i)$. Since $M \otimes P(v) = (M \otimes e_i(kQ/I)e_v, \text{id}_M \otimes P(v)_\alpha, i \in Q_0, \alpha \in Q_1)$, it follows from (1.2) that for $i \neq v$ we have

$$f_i = \begin{cases} 
0, & \text{if there are no non-zero paths from } v \text{ to } i; \\
n \otimes p \mapsto X_p f_v(m), & \text{if there is a non-zero path } p \text{ from } v \text{ to } i.
\end{cases}$$

By (1.3) we see that $f \mapsto f_v$ gives an injective map $\text{Hom}_A(M \otimes P(v), X) \to \text{Hom}_A(M, X_v)$. It is also surjective, since for any given $f_v \in \text{Hom}_A(M, X_v)$, $f = (f_i, i \in Q_0) : M \otimes P(v) \to X$ is indeed a morphism in $\text{rep}(Q, I, A)$, where $f_i$ is given by (1.3).

(2) is clear.

1.5. A complete $A$-projective resolution is an exact sequence of finitely generated projective $A$-modules $P^\bullet = \cdots \to P^{-1} \to P^0 \xrightarrow{d^0} P^1 \to \cdots$, such that $\text{Hom}_A(P^\bullet, A)$ is also exact. An $A$-module $M$ is Gorenstein-projective, if there is a complete $A$-projective resolution $P^\bullet$ such that $M \cong \text{Ker } d^0$ ([13]). Let $P(A)$ be the full subcategory of $A$-mod of projective modules, and $\mathcal{GP}(A)$ the full subcategory of $A$-mod consisting of Gorenstein-projective modules. Then $\mathcal{GP}(A) \subseteq \perp_A := \{ X \in A\text{-mod} \mid \text{Ext}^1_A(X, A) = 0, \forall i \geq 1 \}$; and $\mathcal{GP}(A) = A$-mod if and only if $A$ is self-injective. If $\text{gl.dim } A < \infty$ then $\mathcal{GP}(A) = P(A)$ (but the converse is not true) and if $A$ is a Gorenstein algebra (i.e., $\text{inj.dim } A < \infty$ and $\text{inj.dim } A < \infty$) then $\mathcal{GP}(A)$ is contravariantly finite in $A$-mod and $\mathcal{GP}(A) = \perp_A ([14, 11.5.4, 11.5.3])$ (but the converse is not true). Note that $\mathcal{GP}(A)$ is a resolving subcategory of $A$-mod, i.e., $\mathcal{GP}(A) \supseteq P(A)$, $\mathcal{GP}(A)$ is closed under extensions, the kernels of epimorphisms, and direct summands ([17]), and that $\mathcal{GP}(A)$ is a Frobenius category with relative projective-injective objects being projective $A$-modules ([6]).

2. Monic monomial representations

**Definition 2.1.** A representation $X = (X_i, X_\alpha, i \in Q_0, \alpha \in Q_1)$ of $(Q, I)$ over $A$ is a monic (monomial) representation, or a monic (monomial) $\Lambda$-module, provided that $X$ satisfies the conditions:

(m1) For each $i \in Q_0$, the sum $\sum_{\alpha \in A(\rightarrow i)} \text{Im } X_\alpha$ is a direct sum $\bigoplus_{\alpha \in A(\rightarrow i)} \text{Im } X_\alpha$;

(m2) For each $\alpha \in Q_1$, there holds $\text{Ker } X_\alpha = \sum_{q \in K_\alpha} \text{Im } X_q$, where $K_\alpha$ is as in (2.1).

Easy to see that (m1) and (m2) are independent. If $K_\alpha = \emptyset$, i.e., $\alpha$ is not the last arrow of all $\rho_i \in \rho$, then $\sum_{q \in K_\alpha} \text{Im } X_q$ is understood to be zero. We keep this convention throughout. In this case
(m2) says that $X_\alpha$ is injective (this is the case if $s(\alpha)$ is a source). In general, (m2) exactly says that
\[ \bigoplus_{q \in K_{\alpha}} X_{s(q)} \xrightarrow{(X_{\alpha})_{q \in K_{\alpha}}} X_{s(\alpha)} \xrightarrow{X_{e(\alpha)}} X_i \] is an exact sequence.

Denote by $\text{mon}(Q, I, A)$ the full subcategory of $\text{rep}(Q, I, A)$ consisting of monic representations, which is called the monomorphism category of $(Q, I)$ over $A$. Since $\text{mon}(Q, I, A)$ is closed under the direct summands, it is a Krull-Schmidt category ([28]).

**Example 2.2.** (1) If $I = 0$, then (m1) and (m2) exactly says that $(X_\alpha)_{\alpha \in A(\rightarrow i)} : \bigoplus_{\alpha \in A(\rightarrow i)} X_{s(\alpha)} \to X_i$ is injective for each $i \in Q_0$ ([25]).

(2) Indecomposable projective $kQ/I$-module $P(v) = (e_i(kQ/I)e_v, P(v)_\alpha, i \in Q_0, \alpha \in Q_1) \in \text{rep}(Q, I, k)$ is monic for each $v \in Q_0$, and this gives all the indecomposable monic $kQ/I$-modules (see Corollary [12.3]).

(3) Let $M$ be an arbitrary $A$-module. Then $M \otimes P(v) = (M \otimes e_i(kQ/I)e_v, \text{Id}_M \otimes P(v)_\alpha, i \in Q_0, \alpha \in Q_1)$ is a monic $\Lambda$-module for each $v \in Q_0$. Thus projective $\Lambda$-modules are monic (cf. Lemma [12.2]). So, monic $\Lambda$-modules are much more than just projective $\Lambda$-modules, if $A$ is not semisimple.

(4) Let $A := k[x]/(x^2)$, and $\Lambda := A \otimes kQ/I$ with
\[ Q := \begin{array}{ccc}
4 & \gamma & 3 \\
\beta_1 & 0 & 2 \\
\beta_2 & 0 & 1
\end{array} \]
and $I := \langle \beta_1, \beta_2 \rangle$. Note that $A$ has only two indecomposable modules $A$ and $k := k\bar{x}$ (up to isomorphisms), with $\text{Hom}_A(A, k) = k\bar{x}$, $\text{Hom}_A(k, A) = k\sigma$. $\text{Hom}_A(A, A) = k\text{Id}_A \oplus k\bar{x}$, where $0 \to k \bar{x} \to A \to k \to 0$ is the canonical exact sequence. Then
\[ X := \begin{array}{ccc}
k \xrightarrow{x_{\gamma} = \sigma} A \xrightarrow{x_{\beta_1} = \left(\begin{array}{cc}1 & 0 \\ 0 & 1 \end{array}\right)} A \oplus k \xrightarrow{x_{\beta_2} = \left(\begin{array}{cc}1 & 0 \\ 0 & 1 \end{array}\right)} k \oplus k
\end{array} \]
is a monic $\Lambda$-module.

The following result gives properties of the structural $A$-maps of monic monomial representations.

**Theorem 2.3.** Let $X = (X_i, X_\alpha, i \in Q_0, \alpha \in Q_1)$ be a monic $\Lambda$-module, and $p$ a non-zero path with $l(p) \geq 1$. Then
\begin{enumerate}
\item $\text{Ker} X_p = \sum_{q \in K_p} \text{Im} X_q$, where $K_p$ is as in (1.1). Thus, if $s(p)$ is a source then $X_p$ is injective.
\item $\text{Ker} X_p = \bigoplus_{\beta \in B_1} \text{Im} X_\beta \oplus \bigoplus_{\beta \in B_2} \text{Ker} X_{\beta_p}$, where
\[ B_1 := \{ \beta \in A(\rightarrow s(p)) \mid \beta = \text{la}(q) \text{ for some } q \in P(\rightarrow s(p)), \ p\beta \notin I \}, \]
\[ B_2 := \{ \beta \in A(\rightarrow s(p)) \mid \beta = \text{la}(q) \text{ for some } q \in P(\rightarrow s(p)), \ p\beta \notin I, \ pq \notin I \} \]
\[(\text{If } B_1 = \emptyset = B_2, \text{ then it says that } X_p \text{ is injective}).
\item Let $j$ and $i$ be distinct vertices of $Q$. Then $\sum_{q \in P(j \rightarrow i)} \text{Im} X_q = \bigoplus_{q \in P(j \rightarrow i)} \text{Im} X_q$.
\end{enumerate}

**Proof.** (1) Use induction on the length $l(p)$. If $l(p) = 1$, then the assertion is (m2). Suppose that $l(p) \geq 2$ with $p = p'\alpha$ and $\alpha \in Q_1$. Clearly $\sum_{q \in K_p} \text{Im} X_q \subseteq \text{Ker} X_p$. Let $x \in \text{Ker} X_p$. Then $X_p(x) = X_{p'}X_\alpha(x) = 0$, i.e., $X_\alpha(x) \in \text{Ker} X_{p'}$. Since $l(p') < l(p)$, by the induction $X_\alpha(x) \in \text{Ker} X_{p'} = \sum_{q \in K_{p'}} \text{Im} X_q$. For each path
If \( q \in K_{p'} := \{ q \in P(\rightarrow s(p')) \mid p'q \in I \} \), we consider the last arrow \( \beta \) of \( q \) with \( q = \beta q' \), where \( q' \) is a path, possibly of length 0. Graphically we have

\[
\begin{array}{c}
s(\alpha) = s(p) \\
\downarrow \alpha \\
s(q') \xrightarrow{\beta} s(\beta) \xrightarrow{\alpha} s(\alpha) = s(p') \xrightarrow{\beta} s(p).
\end{array}
\]

We divide all these paths \( q \) into two classes via \( p' \beta \in I \), or \( p' \beta \notin I \). Put

\[
B_1 := \{ \beta \in \mathcal{A}(\rightarrow s(p')) \mid \beta = \text{la}(q) \text{ for some } q \in P(\rightarrow s(p')), p' \beta \in I \},
\]

\[
B_2 := \{ \beta \in \mathcal{A}(\rightarrow s(p')) \mid \beta = \text{la}(q) \text{ for some } q \in P(\rightarrow s(p')), p' \beta \notin I, p'q \in I \}.
\]

Then \( B_1 \cap B_2 = \emptyset \). For those paths \( q = \beta q' \in K_{p'} \) such that \( \beta \in B_1 \) (this contains the case \( l(q') = 0 \)), we use the inclusion \( \text{Im} \ X_q \subseteq \text{Im} \ X_\beta \). Since for \( \beta \in B_1 \) we have \( p' \beta \in I \), it follows that

\[
\sum_{q \in K_{p'}} \text{Im} \ X_q = \sum_{\beta \in B_1} \text{Im} \ X_\beta \overset{(m1)}{=} \bigoplus_{\beta \in B_1} \text{Im} \ X_\beta.
\]

Then by \((m1)\) we have

\[
X_\alpha(x) \in \text{Ker} \ X_{p'} = ( \bigoplus_{\beta \in B_1} \text{Im} \ X_\beta ) \oplus ( \bigoplus_{\beta \in B_2} \text{Im} \ X_\beta ) = \sum_{q' \in \mathcal{P}(\rightarrow s(\alpha))} \text{Im} \ X_{q'}.
\]

Since \( p'\alpha = p \notin I \), we have \( \alpha \notin B_1 \). It follows from \((*)\) that either \( X_\alpha(x) = 0 \) (if \( \alpha \notin B_2 \)), or (if \( \alpha \in B_2 \))

\[
X_\alpha(x) \in X_\alpha \sum_{q' \in \mathcal{P}(\rightarrow s(\alpha))} \text{Im} \ X_{q'}.
\]

In the both cases there are \( x_{q'} \in X_{s(q')} \) such that

\[
x = \sum_{q' \in \mathcal{P}(\rightarrow s(\alpha))} X_{q'}(x_{q'}) \in \text{Ker} \ X_\alpha.
\]

Since \( \text{Ker} \ X_\alpha = \sum_{q \in \mathcal{P}(\rightarrow s(\alpha))} \text{Im} \ X_q \) by \((m2)\), it follows that \( x \in \sum_{q \in \mathcal{P}(\rightarrow s(\alpha))} \text{Im} \ X_q \), i.e., the assertion \((1)\) holds.

\((2)\) in fact has been proved above (see the equality in \((*)\)).

\(3)\) We first prove the following

**Claim:** If \( \sum_{q \in \mathcal{P}(j \rightarrow s(p))} X_q(x_q) \in \text{Ker} \ X_p \) with each \( x_q \in X_{s(q)} \), then \( X_p X_q(x_q) = 0 \) for each \( q \).

Set \( l_p := 0 \) if \( s(p) \) is a source; otherwise \( l_p \) is defined to be the maximal length of all the paths \( q \) of length \( \geq 1 \) to \( s(p) \). We stress that \( l_p \neq \max \{ l(q) \mid q \in \mathcal{P}(\rightarrow s(p)) \} \) in general, since \( q \in \mathcal{P}(\rightarrow s(p)) \) means that \( q \) is a non-zero path (considering all the paths rather than only non-zero paths here will make the argument below easier). Use induction on \( l_p \). If \( l_p = 0 \) then the assertion trivially holds.

Suppose that \( l_p \geq 1 \). Put \( x := \sum_{q \in \mathcal{P}(j \rightarrow s(p))} X_q(x_q) \). Note that the index set \( \{ q \in \mathcal{P}(j \rightarrow s(p)) \mid pq \notin I \} \) is a disjoint union of

\[
C_1 := \{ \beta \in \mathcal{A}(j \rightarrow s(p)) \mid p\beta \notin I \}
\]

and

\[
C_2 := \{ \beta \in \mathcal{A}(j \rightarrow s(p)) \mid p\beta \in I \}.
\]

Then, \( X_q(x_q) = 0 \) for each \( q \in C_2 \).

In fact, if \( q \in C_2 \), then \( q \notin C_1 \), so by \((m2)\) we have

\[
\sum_{q \in C_2} \text{Im} \ X_q \subseteq \sum_{q \in \mathcal{P}(\rightarrow s(\alpha))} \text{Im} \ X_{q'} = X_\alpha \text{Im} \ X_{s(q)}.
\]

By \((m1)\), it follows that

\[
\sum_{q \in C_2} \text{Im} \ X_q \subseteq X_\alpha \text{Im} \ X_{s(q)} \subseteq \sum_{q \in \mathcal{P}(\rightarrow s(\alpha))} \text{Im} \ X_{q'}.
\]

Therefore, \( x_q = 0 \) for each \( q \in C_2 \).
and

\[ \{ q \in P(j \to s(p)) \mid pq \notin I, \ l(q) \geq 2 \}. \]

Denote by \( C_2 \) the set of the last arrows of \( q \in \{ q \in P(j \to s(p)) \mid pq \notin I, \ l(q) \geq 2 \}. \) Since any arrow \( \beta \) in \( C_2 \) does not starts from \( p \), it follows that \( C_1 \cap C_2 = \emptyset \). Thus \( x = x_1 + x_2 \) with

\[
x_1 := \sum_{\beta \in C_1} X_\beta(x_\beta), \quad x_2 = \sum_{\beta \in C_2} X_\beta(\sum_{q \in P(j \to s(\beta))} X_q(x_q)).
\]

On the other hand, by (2) we have \( \text{Ker} \ X_\beta = ( \bigoplus \text{Im} \ X_\beta ) \oplus ( \bigoplus \ X_\beta(\text{Ker} \ X_{\beta'}) \), where

\[
B_1 := \{ \beta \in A(\to s(p)) \mid \beta = \text{la}(q) \text{ for some } q \in P(\to s(p)), \ p \beta \in I \},
\]

\[
B_2 := \{ \beta \in A(\to s(p)) \mid \beta = \text{la}(q) \text{ for some } q \in P(\to s(p)), \ p \beta \notin I, \ pq \notin I \}
\]

Thus \( x = y_1 + y_2 \) with \( y_1 \in \sum_{\beta \in B_1} \text{Im} \ X_\beta \) and \( y_2 \in \sum_{\beta \in B_2} \ X_\beta(\text{Ker} \ X_{\beta'}). \) Note that

\[
C_1 \cap C_2 = \emptyset, \quad B_1 \cap B_2 = \emptyset, \quad B_1 \cap C_1 = \emptyset, \quad B_1 \cap C_2 = \emptyset.
\]

By \( x_1 + x_2 = y_1 + y_2 \) and (m1) we see that \( y_1 = 0 \). So \( x = x_1 + x_2 \in \sum_{\beta \in B_2} X_\beta(\text{Ker} \ X_{\beta'}). \) We discuss each summand \( X_q(x_q) \) of

\[
x = \sum_{q \in P(j \to s(p))} X_q(x_q) = \sum_{\beta \in C_1} X_\beta(x_\beta) + \sum_{\beta \in C_2} X_\beta(\sum_{q \in P(j \to s(\beta))} X_w(x_w))
\]

in several cases. Our aim is to prove that \( X_pX_q(x_q) = 0 \) for each summand \( X_q(x_q) \) of \( x \).

Case 1. If \( \beta \in C_1 \setminus B_2 \), then by (m1) we see that the summand \( X_\beta(x_\beta) \) of \( x \) is 0, and hence \( X_pX_\beta(x_\beta) = 0 \).

Case 2. If \( \beta \in C_1 \cap B_2 \), then by (m1) we see that \( X_\beta(x_\beta) = X_\beta(x_\beta') \) with \( x_\beta' \in \text{Ker} \ X_{\beta'} \), it follows that

\[
X_pX_\beta(x_\beta) = X_pX_\beta(x_\beta') = X_{\beta'}(x_\beta') = 0.
\]

Case 3. If \( \beta \in C_2 \setminus B_2 \), then by (m1) we see that the summand \( X_\beta(\sum_{q \in P(j \to s(\beta))} X_w(x_w)) \) of \( x \) is 0, and hence

\[
\sum_{w \in P(j \to s(\beta))} X_w(x_w) \in \text{Ker} \ X_\beta \subseteq \text{Ker} \ X_{\beta'}.
\]

Since \( l_{\beta'} < l_p \), by induction \( X_{\beta'}X_w(x_w) = 0 \), i.e., \( X_{\beta'}X_w(x_w) = 0 \).

Case 4. If \( \beta \in C_2 \cap B_2 \), then by (m1) we see that \( X_\beta(\sum_{q \in P(j \to s(\beta))} X_w(x_w)) = X_\beta(x_\beta') \) with \( x_\beta' \in \text{Ker} \ X_{\beta'} \). It follows that

\[
X_pX_\beta(\sum_{q \in P(j \to s(\beta))} X_w(x_w)) = X_pX_\beta(x_\beta') = X_{\beta'}(x_\beta') = 0,
\]

i.e., \( \sum_{w \in P(j \to s(\beta))} X_w(x_w) \in \text{Ker} X_{\beta'} \). Since \( l_{\beta'} < l_p \), by induction \( X_{\beta'}X_w(x_w) = 0 \), i.e., \( X_pX_{\beta'}X_w(x_w) = 0 \).

All together Claim is proved.
Now, let $\sum_{p \in \mathcal{P}(j \to i)} x_p(x_p) = 0$ with each $x_p \in X_j$. Then we have

$$0 = \sum_{p \in \mathcal{P}(j \to i)} X_p(x_p) = \sum_{\alpha \in \mathcal{A}(j \to i)} X_\alpha(x_\alpha) + \sum_{p \in \mathcal{P}(j \to i) \setminus \mathcal{A}(j \to i)} X_p(x_p)$$

$$= \sum_{\alpha \in \mathcal{A}(j \to i)} X_\alpha(x_\alpha) + \sum_{\beta \in \mathcal{A}(s(\beta) \to i)} X_\beta(\sum_{\beta \notin I} X_q(x_{\beta q})).$$

Since all those arrows $\beta$ do not start from $j$, by (m1) we have $X_\alpha(x_\alpha) = 0$ for $\alpha \in \mathcal{A}(j \to i)$ and $X_\beta(\sum_{\beta \notin I} X_q(x_{\beta q})) = 0$ for all the $\beta$'s above. Thus $\sum_{\beta \in \mathcal{A}(s(\beta) \to i)} X_q(x_{\beta q}) \in \text{Ker} \beta$. It follows from Claim that each $X_\beta X_q(x_{\beta q}) = 0$. This proves (3). 

3. Resolvability

A full subcategory $\mathcal{X}$ of $\Lambda$-mod is resolving if $\mathcal{X}$ contains all the projective $\Lambda$-modules, $\mathcal{X}$ is closed under extensions, the kernels of epimorphisms, and direct summands ([2]).

**Theorem 3.1.** The monomorphism category $\text{mon}(Q, I, A)$ is a resolving subcategory of $\Lambda$-mod.

Thus, $\text{mon}(Q, I, A)$ has enough projective objects, which are exactly projective $\Lambda$-modules. In particular, each branch of projective objects of $\text{mon}(Q, I, A)$ is a projective $\Lambda$-module.

**Proof.** By Example 2.2(3) $\text{mon}(Q, I, A)$ contains $\mathcal{P}(\Lambda)$. Clearly $\text{mon}(Q, I, A)$ is closed under direct summands. Let $0 \to X \xrightarrow{f} Y \xrightarrow{g} Z \to 0$ be an exact sequence in $\text{rep}(Q, I, A)$ with $X, Z \in \text{mon}(Q, I, A)$.

**Claim 1:** $Y$ satisfies (m1), i.e., $\sum_{\alpha \in \mathcal{A}(\to i)} \text{Im} Y_\alpha = \bigoplus_{\alpha \in \mathcal{A}(\to i)} \text{Im} Y_\alpha$ for each $\alpha \in Q_0$.

The argument is a diagram chasing. By (1.2) we have a commutative diagram with exact rows

$$\begin{array}{cccccc}
0 & \rightarrow & \bigoplus_{\alpha \in \mathcal{A}(\to i)} X_{s(\alpha)} & \xrightarrow{\oplus f_{s(\alpha)}} & \bigoplus_{\alpha \in \mathcal{A}(\to i)} Y_{s(\alpha)} & \xrightarrow{\oplus g_{s(\alpha)}} & \bigoplus_{\alpha \in \mathcal{A}(\to i)} Z_{s(\alpha)} & \rightarrow & 0 \\
0 & \xrightarrow{X_i} & \xrightarrow{f_i} & \xrightarrow{Y_i} & \xrightarrow{g_i} & \xrightarrow{Z_i} & 0.
\end{array}$$

Note that $\sum_{\alpha \in \mathcal{A}(\to i)} \text{Im} Y_\alpha = \text{Im}(Y_{s(\alpha)})_{\alpha \in \mathcal{A}(\to i)}$. Let $\sum_{\alpha \in \mathcal{A}(\to i)} Y_{s(\alpha)}(y_\alpha) = 0$ for some $y_\alpha \in Y_{s(\alpha)}$. By the right square above we have

$$\sum_{\alpha \in \mathcal{A}(\to i)} g_{s(\alpha)}(y_\alpha) = \sum_{\alpha \in \mathcal{A}(\to i)} Z_{s(\alpha)}(y_\alpha) = 0.$$

Since $Z$ satisfies (m1), we have $Z_{s(\alpha)}(y_\alpha) = 0$ for each $\alpha$. By (m2) on $Z$ we get $g_{s(\alpha)}(y_\alpha) = \sum_{q \in K_{s(\alpha)}} Z_{q}(z_q)$ for some $z_q \in Z_{q}$. Since $g_{s(\alpha)} : Y_{s(\alpha)} \rightarrow Z_{s(\alpha)}$ is surjective, there exists $y_q \in Y_{s(\alpha)}$ such that $z_q = g_{s(\alpha)}(y_q)$. So by (1.2) we have (note that $e(q) = s(\alpha)$)

$$g_{s(\alpha)}(y_\alpha) = \sum_{q \in K_{s(\alpha)}} Z_{q}(g_{s(\alpha)}(y_q)) = \sum_{q \in K_{s(\alpha)}} g_{s(\alpha)}(Y_q(y_q))$$

for each $\alpha \in \mathcal{A}(\to i)$, i.e., $y_\alpha - \sum_{q \in K_{s(\alpha)}} Y_q(y_q) \in \text{Ker} g_{s(\alpha)} = \text{Im} f_{s(\alpha)}$. Hence

$$y_\alpha - \sum_{q \in K_{s(\alpha)}} Y_q(y_q) = f_{s(\alpha)}(x_\alpha)$$

for some $x_\alpha \in X_{s(\alpha)}$. Applying $Y_\alpha$ and by $Y_{\alpha} Y_q = 0$ for each $q \in K_{s(\alpha)}$, we have

$$Y_{\alpha}(y_\alpha) = \text{Im}(Y_{s(\alpha)}(x_\alpha)) = f_{s(\alpha)}(X_{s(\alpha)}(x_\alpha)).$$
By $\sum_{\alpha \in \mathcal{A}(-i)} Y_{\alpha}(y_{\alpha}) = 0$ we have $\sum_{\alpha \in \mathcal{A}(-i)} f_i(X_{\alpha}(x_{\alpha})) = 0$, and hence $\sum_{\alpha \in \mathcal{A}(-i)} X_{\alpha}(x_{\alpha}) = 0$ since $f_i$ is injective. Since $X$ satisfies (m1), we have $X_{\alpha}(x_{\alpha}) = 0$ for each $\alpha \in \mathcal{A}(\rightarrow i)$, and hence $Y_{\alpha}(y_{\alpha}) = f_i(X_{\alpha}(x_{\alpha})) = 0$. This proves Claim 1.

Claim 2: $Y$ satisfies (m2), i.e., $\ker Y_{\alpha} = \sum_{q \in \mathcal{K}_{\alpha}} \text{Im} Y_q$ for each $\alpha \in Q_1$.

In fact, by (1.2) we have a commutative diagram with exact rows

\[
\begin{array}{ccccccccc}
0 & \rightarrow & \bigoplus_{q \in \mathcal{K}_{\alpha}} X_{s(q)} & \oplus f_{s(q)} & Y_{s(q)} & \oplus g_{s(q)} & \bigoplus_{q \in \mathcal{K}_{\alpha}} Z_{s(q)} & \rightarrow & 0 \\
& & \downarrow & & \downarrow & & \downarrow & & \\
0 & \rightarrow & X_{s(\alpha)} & & Y_{s(\alpha)} & & Z_{s(\alpha)} & \rightarrow & 0 \\
\downarrow (X_q)_{q \in \mathcal{K}_{\alpha}} & & \downarrow f_{s(\alpha)} & & \downarrow (Y_q)_{q \in \mathcal{K}_{\alpha}} \oplus g_{s(\alpha)} & & \downarrow (Z_q)_{q \in \mathcal{K}_{\alpha}} & & \\
0 & \rightarrow & X_{e(\alpha)} & & Y_{e(\alpha)} & & Z_{e(\alpha)} & \rightarrow & 0 \\
& & \downarrow & & \downarrow & & \downarrow & & \\
0 & & 0 & & 0 & & 0 & & 0
\end{array}
\]

Viewing columns as complexes such that $X_{s(\alpha)}$ is at position zero, the diagram above gives a short exact sequence of complexes, and hence it induces a long exact sequence of cohomologies. The assumption that $X$ and $Z$ satisfy (m2) means that the 0-th cohomologies of the complexes on the left and on the right are zero. It follows that the 0-th cohomologies of the complex at the middle is also zero, i.e., $Y$ satisfies (m2). This proves Claim 2, and hence that $\text{mon}(Q, I, A)$ is closed under extensions.

Now we prove that $\text{mon}(Q, I, A)$ is closed under the kernels of epimorphisms. Let $0 \rightarrow X \xrightarrow{f} Y \xrightarrow{g} Z \rightarrow 0$ be an exact sequence in $\text{rep}(Q, I, A)$ with $Y, Z \in \text{mon}(Q, I, A)$. We need to prove that $X$ satisfies (m1) and (m2). Let $\sum_{\alpha \in \mathcal{A}(\rightarrow i)} X_{\alpha}(x_{\alpha}) = 0$ for some $x_{\alpha} \in X_{s(\alpha)}$. By the left square in (3.1) we have

$\sum_{\alpha \in \mathcal{A}(\rightarrow i)} Y_{\alpha}(Y_{\alpha}(y_{\alpha})) = 0$. Since $Y$ satisfies (m1), we have $Y_{\alpha}(y_{\alpha}) = 0$ for each $\alpha$. By (1.2) this is exactly $f_i X_{\alpha}(x_{\alpha}) = 0$. Since $f_i$ is injective, we have $X_{\alpha}(x_{\alpha}) = 0$ for each $\alpha \in \mathcal{A}(\rightarrow i)$. This proves that $X$ satisfies (m1). It remains to prove that $X$ satisfies (m2), i.e., $\ker X_{\alpha} = \sum_{q \in \mathcal{K}_{\alpha}} \text{Im} X_q$ for each $\alpha \in E$. This follows from the following sub-lemma.

Finally, by Example 2.2(3) projective $\Lambda$-modules are monic. From this and the fact that $\text{mon}(Q, I, A)$ is closed under the kernels of epimorphisms we see that $\text{mon}(Q, I, A)$ has enough projective objects, which are exactly projective $\Lambda$-modules. \[\square\]

Sub-Lemma 3.2. Let $0 \rightarrow X \xrightarrow{f} Y \xrightarrow{g} Z \rightarrow 0$ be an exact sequence in $\text{rep}(Q, I, A)$ with $Y$ and $Z$ in $\text{mon}(Q, I, A)$. Then $\ker X_p = \sum_{q \in \mathcal{K}_p} \text{Im} X_q$ for each non-zero path $p$, where $K_p$ is as in (1.1).

Proof. Define $l_p$ as in the proof of Theorem 2.2(3). Use induction on $l_p$. If $l_p := 0$, then $s(p)$ is a source and hence $\ker Y_p = 0$ by Theorem 2.2(1). So $\ker X_p = 0$, i.e., the assertion holds.
Suppose that the assertion holds for all the non-zero paths with \( l_p < m \) (\( m \geq 1 \)). We prove the assertion for non-zero path \( p \) with \( l_p = m \). Let \( x \in \text{Ker} \, X_p \). By the commutative diagram with exact rows

\[
\begin{array}{c}
0 \rightarrow X_{s(p)} \xrightarrow{f_s(p)} Y_{s(p)} \xrightarrow{g_s(p)} Z_{s(p)} \rightarrow 0 \\
0 \rightarrow X_{e(p)} \xrightarrow{f_e(p)} Y_{e(p)} \xrightarrow{g_e(p)} Z_{e(p)} \rightarrow 0
\end{array}
\]

we have \( f_s(p)(x) \in \text{Ker} \, Y_p \). By Theorem 2 we know that \( f_s(p)(x) \) is of the form

\[
f_s(p)(x) = \sum_{\beta \in B_1} Y_\beta(y_\beta) + \sum_{\beta \in B_2} Y_\beta(y_\beta)
\]

where \( y_\beta \in Y_s(\beta) \) for each \( \beta \in B_1 \cup B_2 \), moreover \( y_\beta \in \text{Ker} \, Y_{p,\beta} \) for each \( \beta \in B_2 \). By (1.2) we have

\[
\sum_{\beta \in B_1 \cup B_2} Z_\beta g_s(\beta)(y_\beta) = \sum_{\beta \in B_1 \cup B_2} g_s(p)Y_\beta(y_\beta) = g_s(p)f_s(p)(x) = 0.
\]

Since \( B_1 \cap B_2 = \emptyset \) and \( Z \) satisfies (m1), we have \( Z_\beta g_s(\beta)(y_\beta) = 0 \) for each \( \beta \in B_1 \cup B_2 \). Then by (m2) on \( Z \) we have \( g_s(\beta)(y_\beta) \in \text{Ker} \, Z_\beta = \sum_{q \in K_\beta} \text{Im} \, Z_q \), so there are some \( z_q \in Z_q \) such that \( g_s(\beta)(y_\beta) = \sum_{q \in K_\beta} Z_q(z_q) \).

Since \( g_q(\beta) \) is surjective, there are \( y_q \in Y_q(\beta) \) such that \( z_q = g_q(\beta)(y_q) \). Thus

\[
g_s(\beta)(y_\beta) = \sum_{q \in K_\beta} Z_q(g_q(\beta)(y_q)) = \sum_{q \in K_\beta} g_q(\beta)(Y_q(y_q)),
\]

and hence \( y_\beta = \sum_{q \in K_\beta} Y_q(y_q) \in \text{Ker} \, g_s(\beta) = \text{Im} \, f_s(\beta) \), i.e., \( y_\beta = -\sum_{q \in K_\beta} Y_q(y_q) = g_s(\beta)(x_\beta) \) for some \( x_\beta \in X_s(\beta) \).

Thus for each \( \beta \in B_1 \cup B_2 \) we have \( Y_\beta(y_\beta) = Y_\beta f_s(\beta)(x_\beta) \).

Now for each \( \beta \in B_2 \), since \( y_\beta \in \text{Ker} \, Y_{p,\beta} \), we have

\[
f_e(p)X_\beta X_\beta(x_\beta) = Y_\beta f_s(p)X_\beta(x_\beta) = Y_\beta Y_\beta f_s(\beta)(x_\beta) = Y_\beta Y_\beta(y_\beta) = 0,
\]

and hence \( x_\beta \in \text{Ker} \, X_{p,\beta} \) for each \( \beta \in B_2 \). Since \( l_{p,\beta} < l_p \), by induction \( \text{Ker} \, X_{p,\beta} = \sum_{q \in K_{p,\beta}} \text{Im} \, X_q \). So

\[
f_s(p)(x) = \sum_{\beta \in B_1} Y_\beta f_s(\beta)(x_\beta) + \sum_{\beta \in B_2} Y_\beta f_s(\beta)(x_\beta)
\]

\[
= \sum_{\beta \in B_1} f_s(p)X_\beta(x_\beta) + \sum_{\beta \in B_2} f_s(p)X_\beta(x_\beta).
\]

Since \( f_s(p) \) is injective, it follows that

\[
x = \sum_{\beta \in B_1} X_\beta(x_\beta) + \sum_{\beta \in B_2} X_\beta(x_\beta) \subseteq \sum_{\beta \in B_1} \text{Im} \, X_\beta + \sum_{\beta \in B_2} \text{Im} \, X_\beta \subseteq \sum_{q \in K_\beta} \text{Im} \, X_q.
\]

This completes the proof.

It seems to be more difficult to get all the injective objects of \( \text{mon}(Q, I, A) \). We can only give some indecomposable injective objects of \( \text{mon}(Q, I, A) \). This is needed in the next section.

**Proposition 3.3.** Let \( N \) be an indecomposable injective \( A \)-module and \( P(v) \) the indecomposable projective \( kQ/I \)-module at \( v \in Q_0 \). Then \( N \otimes P(v) \) is an indecomposable injective object of \( \text{mon}(Q, I, A) \).

**Proof.** By Lemma 3 \( N \otimes P(v) \) is an indecomposable object of \( \text{mon}(Q, I, A) \). Put \( L := D(A_A) \otimes kQ/I \), where \( D := \text{Hom}_k(-, k) \). It suffices to prove that \( L \) is an injective object of \( \text{mon}(Q, I, A) \). Use induction on \( |Q_0| \). Write \( L \) as \( L = (L_i, L_\alpha, i \in Q_0, \alpha \in Q_1) \). Let \( Q'' \) be the quiver by deleting sink vertex 1 from
Q, and \( I'' := \langle \rho_1 \mid e(\rho_i) \neq 1 \rangle \). For a \( A \)-module \( X \), let \( X'' \) be the \((A \otimes kQ''/I'')\)-module by deleting the first branch \( X_1 \) from \( X \). For a \( A \)-map \( f \), we similarly define \( f'' \).

Let \( 0 \to X \xrightarrow{f} Y \xrightarrow{g} Z \to 0 \) be an exact sequence and \( h : X \to L \) a morphism, both in \( \text{mon}(Q,I,A) \). Then \( 0 \to X'' \xrightarrow{f''} Y'' \xrightarrow{g''} Z'' \to 0 \) is an exact sequence in \( \text{mon}(Q'',I'',A) \). By induction \( L'' := D(A_A) \otimes kQ/I'' \) is an injective object of \( \text{mon}(Q'',I'',A) \). So there is a morphism \( u'' = \begin{pmatrix} u_1 \\ u_2 \\ \vdots \\ u_n \end{pmatrix} : Y'' \to L'' \) in \( \text{rep}(Q'',I'',A) \) such that \( h'' = u'' f'' \). Thus, it suffices to prove the following

**Claim:** there is an \( A \)-map \( u_1 : Y_1 \to L_1 \) such that \( u = \begin{pmatrix} u_1 \\ \vdots \\ u_n \end{pmatrix} : Y \to L \) is a morphism in \( \text{rep}(Q,I,A) \), and that \( h_1 = u_1 f_1 \).

Since \( L_1 \) is a direct of copies of \( D(A_A) \), it is an injective \( A \)-module. So there is an \( A \)-map \( u'_1 : Y_1 \to L_1 \) such that \( h_1 = u'_1 f_1 \). Consider the \( A \)-map \( (L_\alpha u_\alpha - u'_1 Y_\alpha)_{\alpha \in A(\to 1)} : \bigoplus_{\alpha \in A(\to 1)} Y_\alpha \to L_1 \) and the exact sequence of \( A \)-modules

\[
0 \to \bigoplus_{\alpha \in A(\to 1)} X_\alpha \xrightarrow{f_{\alpha(\to 1)}} \bigoplus_{\alpha \in A(\to 1)} Y_\alpha \xrightarrow{g_{\alpha(\to 1)}} \bigoplus_{\alpha \in A(\to 1)} Z_\alpha \to 0.
\]

Since \( f \) and \( h \) are morphisms, we have

\[
(L_\alpha u_\alpha - u'_1 Y_\alpha)_{\alpha \in A(\to 1)} \circ g_{\alpha(\to 1)} = (L_\alpha u_\alpha f_{\alpha(\to 1)} - u'_1 Y_\alpha f_{\alpha(\to 1)})_{\alpha \in A(\to 1)}
\]

\[
= (L_\alpha u_\alpha f_{\alpha(\to 1)} - u'_1 f_1 X_\alpha)_{\alpha \in A(\to 1)} = (L_\alpha h_{\alpha(\to 1)} - h_1 X_\alpha)_{\alpha \in A(\to 1)} = 0.
\]

It follows that \( (L_\alpha u_\alpha - u'_1 Y_\alpha)_{\alpha \in A(\to 1)} \) factors through \( g_{\alpha(\to 1)} \), i.e., there is an \( A \)-map \( v : \bigoplus_{\alpha \in A(\to 1)} Z_\alpha \to L_1 \) such that

\[
(L_\alpha u_\alpha - u'_1 Y_\alpha)_{\alpha \in A(\to 1)} = v \circ g_{\alpha(\to 1)}.
\]

Put \( E \) to be the set of the last arrows of \( \rho_i \in \rho \). Write \( A(\to 1) \) as a disjoint union \( A(\to 1) = A_1 \cup A_2 \), where \( A_1 := A(\to 1) \setminus E \) and \( A_2 := A(\to 1) \cap E \). For each \( \alpha \in A_2 \), put \( Z^{\alpha} := (Z_q)_{q \in K_{\alpha}} = \bigoplus_{q \in K_{\alpha}} Z_\alpha(q) \to Z_{\alpha(q)} \). Similarly we have \( Y^{\alpha} \) and \( L^{\alpha} \). Clearly we have

\[
Z_{\alpha} Z^{\alpha} = 0, \quad Y_{\alpha} Y^{\alpha} = 0, \quad L_{\alpha} L^{\alpha} = 0.
\]

By (m2) we have \( \text{Im} Z^{\alpha} = \ker Z_{\alpha}, \quad \text{Im} Y^{\alpha} = \ker Y_{\alpha} \).

Consider the \( A \)-map \( \bigoplus_{\alpha \in A_2} Z^{\alpha} : \bigoplus_{q \in K_{\alpha}} (\bigoplus_{\alpha \in A_2} Z_\alpha(q)) \to \bigoplus_{\alpha \in A_2} Z_{\alpha(q)} \), also the \( A \)-map

\[
\begin{pmatrix} 0 \\ \alpha \in A_2 \end{pmatrix} Z^{\alpha} : \bigoplus_{q \in K_{\alpha}} (\bigoplus_{\alpha \in A_2} Z_\alpha(q)) \to \bigoplus_{\alpha \in A_1} (\bigoplus_{\alpha \in A_2} Z_{\alpha(q)}) \oplus \bigoplus_{\alpha \in A_2} Z_\alpha \to \bigoplus_{\alpha \in A(\to 1)} Z_{\alpha(q)}.
\]

Note that

\[
\text{Im} \left( \begin{pmatrix} 0 \\ \alpha \in A_2 \end{pmatrix} Z^{\alpha} \right) = \bigoplus_{\alpha \in A_2} \ker Z_{\alpha}, \quad \text{Im} \left( \begin{pmatrix} 0 \\ \alpha \in A_2 \end{pmatrix} Y^{\alpha} \right) = \bigoplus_{\alpha \in A_2} \ker Y_{\alpha}.
\]

Thus

\[
\begin{pmatrix} 0 \\ \alpha \in A_2 \end{pmatrix} Z^{\alpha} = \sigma_2 \circ \bigoplus_{\alpha \in A_2} Z_{\alpha}^{\alpha},
\]

(3.4)
where \( \bigoplus \tilde{Z}^\alpha : \bigoplus (\bigoplus \alpha \in A \bigoplus Z_{s(q)}) \to \bigoplus \text{Ker} \ Z_{\alpha} \), and \( \sigma_2 : \bigoplus \text{Ker} \ Z_{\alpha} \hookrightarrow \bigoplus Z_{s(\alpha)} \) is the embedding. By (1.2) we have the commutative diagram

\[
\begin{array}{ccc}
\bigoplus (\bigoplus \alpha \in A \bigoplus Y_{s(q)}) & \xrightarrow{0} & \bigoplus Y_{s(\alpha)} \\
\bigoplus (\bigoplus \alpha \in A \bigoplus \alpha \in A \bigoplus g_{s(q)}) \downarrow & & \downarrow \bigoplus g_{s(\alpha)} \\
\bigoplus (\bigoplus \alpha \in A \bigoplus \alpha \in A \bigoplus \alpha \in A \bigoplus \alpha \in A \bigoplus Z_{s(q)}) & \xrightarrow{0} & \bigoplus Z_{s(\alpha)} \\
\end{array}
\]

(3.5)

where the vertical maps are surjective. For each \( \alpha \in A_2 \) we have the exact sequence

\[
0 \to \text{Ker} \ Z_{\alpha} \to Z_{s(\alpha)} \xrightarrow{\tilde{Z}^\alpha} \text{Im} \ Z_{\alpha} \to 0
\]

induced by \( Z_{\alpha} : Z_{s(\alpha)} \to Z_1 \), and hence we have exact sequence

\[
0 \to \bigoplus \text{Ker} \ Z_{\alpha} \to \bigoplus Z_{s(\alpha)} \xrightarrow{\pi} \bigoplus \text{Im} \ Z_{\alpha} \to 0.
\]

Thus we have exact sequence

\[
0 \to \bigoplus \text{Ker} \ Z_{\alpha} \xrightarrow{\sigma_2} \bigoplus Z_{s(\alpha)} \xrightarrow{\pi} \bigoplus \text{Im} \ Z_{\alpha} \to 0,
\]

where \( \pi := \left( \begin{array}{cccc}
\text{id} & \bigoplus \alpha \in A_1 Z_{s(\alpha)} & 0 \\
0 & \bigoplus \alpha \in A_2 \tilde{Z}^\alpha \\
\end{array} \right) \). Composing \( v : \bigoplus Z_{s(\alpha)} \to L_1 \) on the right with \( \sigma_2 \circ \bigoplus \tilde{Z}^\alpha \circ \bigoplus g_{s(q)} \), by (3.4), (3.5), (3.2) and (3.3) we have

\[
v \circ \sigma_2 \circ \bigoplus \tilde{Z}^\alpha \circ \bigoplus g_{s(q)} = v \circ \left( \bigoplus \alpha \in A_2 Z_{s(\alpha)} \right) \circ \left( \bigoplus \alpha \in A_2 \alpha \in A \bigoplus \alpha \in A \bigoplus Y_{s(q)} \right)
\]

\[
= v \circ \left( \bigoplus \alpha \in A \bigoplus Y_{s(q)} \right) = (L_{\alpha} u_{s(\alpha)} - u_1 Y_{\alpha})_{\alpha \in A \to 1} \circ \left( \bigoplus \alpha \in A Y_{s(q)} \right)
\]

\[
= (L_{\alpha} u_{s(\alpha)} - u_1 Y_{\alpha})_{\alpha \in A_2} \bigoplus \alpha \in A \bigoplus Y_{s(q)} = (L_{\alpha} u_{s(\alpha)} Y_{s(\alpha)} - u_1 Y_{\alpha} Y_{s(\alpha)})_{\alpha \in A_2}
\]

\[
= (L_{\alpha} u_{s(\alpha)} Y_{s(\alpha)})_{\alpha \in A_2} = (L_{\alpha} L^{\alpha} \bigoplus \alpha \in A \bigoplus u_{s(q)})_{\alpha \in A_2} = 0,
\]

where we have used the following commutative diagram

\[
\begin{array}{ccc}
\bigoplus \alpha \in A \bigoplus Y_{s(q)} & \xrightarrow{Y_{s(\alpha)}} & \bigoplus \alpha \in A \bigoplus Y_{s(\alpha)} \\
\bigoplus \alpha \in A \bigoplus u_{s(q)} \downarrow & & \downarrow \bigoplus u_{s(\alpha)} \\
\bigoplus \alpha \in A \bigoplus L_{s(q)} & \xrightarrow{L^{\alpha}} & \bigoplus \alpha \in A \bigoplus L_{s(\alpha)}
\end{array}
\]

Since \( \bigoplus \tilde{Z}^\alpha \circ \bigoplus g_{s(q)} \) is surjective, we get \( v \circ \sigma_2 = 0 \), thus \( v \) factors through \( \pi \). Hence there is an \( A \)-map

\[
t : \left( \bigoplus \alpha \in A \bigoplus Z_{s(\alpha)} \right) \oplus \left( \bigoplus \alpha \in A \bigoplus \text{Im} \ Z_{\alpha} \right) \to L_1
\]
such that $v = t \pi$. Since $L_1$ is an injective $A$-module, there is an $A$-map $w : Z_1 \rightarrow L_1$ such that the diagram
\[
\begin{array}{ccc}
\bigoplus_{\alpha \in A_1} Z_{s(\alpha)} & \oplus & \bigoplus_{\alpha \in A_2} \text{Im } Z_{\alpha} \\
\downarrow t & & \downarrow w \\
L_1 & \rightarrow & Z_1
\end{array}
\]
commutes, where $\sigma$ is the embedding (note that $(Z_{\alpha})_{\alpha \in A_1}$ is injective by (m2), and hence $((Z_{\alpha})_{\alpha \in A_1}, \sigma)$ injective by (m1)). All together we have
\[
v = t \pi = w((Z_{\alpha})_{\alpha \in A_1}, \sigma)\pi = w((Z_{\alpha})_{\alpha \in A_1}, \sigma)
\begin{pmatrix}
\text{id} \\
0
\end{pmatrix}
\bigoplus_{\alpha \in A_2} Z_{\alpha}
\]
(3.6)

Now put $u_1 := u_1' + wg_1 : Y_1 \rightarrow L_1$, where $g_1 : Y_1 \rightarrow Z_1$ is the 1-st branch of $g : Y \rightarrow Z$. Then $u_1f_1 = u_1'f_1 + wg_1f_1 = u_1'f_1 = h_1$. It remains to prove that $u = \begin{pmatrix}
u_1' \\
u_2 \\
\vdots \\
u_n
\end{pmatrix} : Y \rightarrow L$ is a $\Lambda$-map, i.e., for each arrow $\alpha : s(\alpha) \rightarrow 1$ the diagram
\[
\begin{array}{ccc}
Y_{s(\alpha)} & \xrightarrow{u_{s(\alpha)}} & L_{s(\alpha)} \\
\downarrow Y_{\alpha} & & \downarrow L_{\alpha} \\
L_1 & \xrightarrow{u_1} & L_1
\end{array}
\]
commutes. In fact by (3.2) and (3.6) we have
\[
(L_{\alpha}u_{s(\alpha)} - u_1'Y_{\alpha})_{\alpha \in A(-r)} = v \circ \bigoplus g_{s(\alpha)}
\]
\[
= w \circ (Z_{\alpha})_{\alpha \in A(-r)} \circ \bigoplus g_{s(\alpha)} = w \circ (Z_{\alpha}g_{s(\alpha)})_{\alpha \in A(-r)}
\]
\[
= w \circ (g_1Y_{\alpha})_{\alpha \in A(-r)} = (wg_1Y_{\alpha})_{\alpha \in A(-r)}.
\]
This means that $L_{\alpha}u_{s(\alpha)} - u_1'Y_{\alpha} = wg_1Y_{\alpha}$, i.e., $L_{\alpha}u_{s(\alpha)} = u_1'Y_{\alpha}$.

This proves Claim, and hence completes the proof. 

4. Main result

Let $A$ be a finite-dimensional algebra over field $k$, $Q$ a finite acyclic quiver, $I$ an ideal of $kQ$ generated by monomial relations, and $\Lambda := A \otimes kQ/I$. Note that $\Lambda$ is not necessarily a Gorenstein algebra. Our aim is to characterize the Gorenstein-projective $\Lambda$-modules.

**Theorem 4.1.** Let $X = (X_1, X_{\alpha}, i \in Q_0, \alpha \in Q_1)$ be an arbitrary $\Lambda$-module. Then $X$ is Gorenstein-projective if and only if $X$ is a monic $\Lambda$-module satisfying condition (G), where
\[
(G) \quad \text{For each } i \in Q_0, X_i \text{ and } X_i/(\bigoplus_{\alpha \in A(-r)} \text{Im } X_{\alpha}) \text{ are Gorenstein-projective } A\text{-modules.}
\]

Note that for a monic $\Lambda$-module $X$, by (m1) we have $X_i/(\bigoplus_{\alpha \in A(-r)} \text{Im } X_{\alpha}) = X_i/(\bigoplus_{\alpha \in A(-r)} \text{Im } X_{\alpha})$.

4.1. Before a proof we give some applications. If $G$ is an indecomposable Gorenstein-projective $A$-module, then $G \otimes P(v)$ is an indecomposable monic $A$-module for each $v \in Q_0$ (cf. Lemma 1.2 and Example 2.2(3)). It is easy to see that $G \otimes P(v)$ satisfies condition (G). Thus by Theorem 4.1 we have

**Corollary 4.2.** Let $G$ be an indecomposable Gorenstein-projective $A$-module. Then $G \otimes P(v)$ is an indecomposable Gorenstein-projective $\Lambda$-module for each $v \in Q_0$. 

In Example 2.2(3) we have known that projective \( \Lambda \)-modules are monic.

**Corollary 4.3.** Monic \( \Lambda \)-modules are exactly projective \( \Lambda \)-modules if and only if \( A \) is a semisimple algebra.

In particular, monic \( kQ/I \)-modules are exactly projective \( kQ/I \)-modules.

**Proof.** It suffices to prove that \( A \) is semisimple if and only if any monic \( \Lambda \)-module is projective. Without loss of generality we may assume that \( A \) is connected (i.e., \( A \) can not be a product of two non-zero algebras).

Suppose that \( A \) is semisimple. By Wedderburn Theorem \( A \cong M_n(R) \), the matrix algebra over a division \( k \)-algebra \( R \), hence \( A \otimes kQ/I \cong M_n(R \otimes kQ/I) \). Since \( M_n(R \otimes kQ/I) \) is Morita equivalent to \( R \otimes kQ/I \cong RQ/I \), and \( \text{gl.dim.}RQ/I < \infty \) (as in the case of \( R = k \)), we have \( \text{gl.dim.}(A \otimes kQ/I) < \infty \). Now let \( M \) be a monic \( \Lambda \)-module. Then by Theorem 4.1 \( M \) is a Gorenstein-projective \( \Lambda \)-module (since in this case any \( \Lambda \)-module is projective, and hence condition (G) holds automatically). While by [14, 10.2.3] Gorenstein-projective modules over an algebra of finite global dimension must be projective, it follows that \( M \) is a projective \( \Lambda \)-module.

Conversely, assume that any monic \( \Lambda \)-module is projective. Let \( M \) be an arbitrary \( \Lambda \)-module. Consider \( \Lambda \)-module \( X = M \otimes P(1) \), where \( P(1) \) is the simple projective \( kQ/I \)-module at sink vertex 1. Then \( X \) is a monic \( \Lambda \)-module, and hence a projective \( \Lambda \)-module by assumption. Thus 1-st branch \( M \) of \( X \) is a projective \( \Lambda \)-module (cf. Lemma 1.2(2)). This proves that \( A \) is semisimple.

For a Frobenius category we refer to [27] and [21, Appendix A].

**Corollary 4.4.** The following are equivalent:

1. \( A \) is a self-injective algebra;
2. \( \mathcal{GP}(A \otimes kQ/I) = \text{mon}(Q, I, A) \);
3. \( \text{mon}(Q, I, A) \) is a Frobenius category.

**Proof.** \((i) \implies (ii)\): If \( A \) is self-injective, then every \( \Lambda \)-module is Gorenstein-projective, and hence \( (ii) \) follows from Theorem 4.1. The implication \((ii) \implies (iii)\) is well-known.

\((iii) \implies (i)\): Taking sink vertex 1 of \( Q \), by Proposition 3.3 \( D(A_A) \otimes P(1) \) is an injective object of \( \text{mon}(Q, I, A) \), where \( D := \text{Hom}_k(\langle - \rangle, k) \), hence by assumption it is a projective object of \( \text{mon}(Q, I, A) \). While each branch of a projective object of \( \text{mon}(Q, I, A) \) is a projective \( \Lambda \)-module (cf. Theorem 5.1), it follows that the 1-st branch \( D(A_A) \) of \( D(A_A) \otimes P(1) \) is a projective \( \Lambda \)-module, i.e., \( A \) is self-injective. 

Let \( D^b(\Lambda) \) be the bounded derived category of \( \Lambda \), and \( K^b(\mathcal{P}(\Lambda)) \) the bounded homotopy category of \( \mathcal{P}(\Lambda) \). By definition the singularity category \( D^b_{sg}(\Lambda) \) of \( \Lambda \) is the Verdier quotient \( D^b(\Lambda)/K^b(\mathcal{P}(\Lambda)) \). If \( \Lambda \) is Gorenstein, then there is a triangle-equivalence \( D^b_{sg}(\Lambda) \cong \mathcal{GP}(\Lambda) \) ([8, 4.4.1]; [16, 4.6]). Note that if \( A \) is Gorenstein, then \( \Lambda = A \otimes_k kQ/I \) is also Gorenstein. By Corollary 1.1 we have

**Corollary 4.5.** Let \( A \) be a self-injective algebra, \( Q, I \) and \( \Lambda = A \otimes kQ/I \) as usual. Then there is a triangle-equivalence \( D^b_{sg}(\Lambda) \cong \text{Mon}(Q, I, A) \).

4.2. For proving Theorem 4.1 we first show that the condition (G) can be simplified.

**Lemma 4.6.** Let \( X = (X_i, X_\alpha, \ i \in Q_0, \ \alpha \in Q_1) \) be a monic \( \Lambda \)-module such that \( X_i/(\sum_{\alpha \in A(\rightarrow i)} \text{Im}X_\alpha) \) is Gorenstein-projective for each \( i \in Q_0 \). Then

1. For each non-zero path \( p \), \( \text{Im}X_p \) is Gorenstein-projective.
2. \( X \) satisfies condition (G).
Lemma 4.7. Let \( A \in \mathcal{A} \), \( \mathcal{A} := \text{proj.dim}_A M < \infty \) and proj.dim\(M_B < \infty \), and \( T := (\Lambda A B) \) the triangular matrix ring. We assume that \( T \) is an Artin algebra ([4], p.72), and consider finitely generated \( T \)-modules. Recall that a \( T \)-module can be identified with a triple \((X, Y, f)\), where \( X \in \mathcal{A}_0\), \( Y \in \mathcal{B}_0\), and \( f : M \otimes B Y \to X \) is an \( A \)-map. An \( A \)-map \((X, Y, f) \to (X', Y', f')\) can be identified with a pair \((f, f')\), where \( f \in \text{Hom}_A(X, X')\), \( g \in \text{Hom}_B(Y, Y')\), such that \( f \circ g = f' \circ (\text{id} \otimes g) \). The following description of \( \mathcal{G}(T) \) is a special case of [49, Thm. 1.4],

\[
\alpha \in A \\Rightarrow \mathcal{G}(T) = \mathcal{A} \}
\]

and hence it is Gorenstein-projective by induction (since \( l_{p\beta} < l_p \)). This proves (1).

(2) We need to show that each \( X_i \), Gorenstein-projective. This follows from the exact sequence \( 0 \to \bigoplus_{\alpha \in A} \text{Im}X_{i\alpha} \to X_i \to \bigoplus_{\alpha \in A} \text{Im}X_{i\alpha} \to 0 \), the assumption and (1), again using the fact that \( \mathcal{G}(A) \) is closed under extensions.

\[
\sum_{\alpha \in A} \text{Im}X_{i\alpha} \to \text{Im}X_i \to \bigoplus_{\alpha \in A} \text{Im}X_{i\alpha} \to 0
\]

Proof. (1) Note that \( \text{Im}X_p \cong X_{s(p)}/\text{Ker}X_p \). Use induction on \( l_p := \max\{l(q) \mid q \in \mathcal{P}_s = 1(q) \} \). If \( l_p = 0 \), then \( s(p) \) is a source and \( X_p \) is injective, and hence \( \text{Im}X_p \cong X_{s(p)} \); while in this case \( \sum_{\alpha \in A} \text{Im}X_{i\alpha} = 0 \) and hence by the assumption \( X_{s(p)} \) is Gorenstein-projective.

Suppose \( l_p \geq 1 \). Since \( \mathcal{G}(A) \) is closed under extensions, by the assumption and the exact sequence

\[
0 \to \bigoplus_{\alpha \in A} \text{Im}X_{i\alpha} \to \text{Im}X_p \to \bigoplus_{\alpha \in A} \text{Im}X_{i\alpha} \to 0
\]

it suffices to prove that the left hand side term is Gorenstein-projective.

By Theorem 4.1 (2) \( \text{Ker}X_p = (\bigoplus \text{Im}X_{i\alpha}) \oplus (\bigoplus \text{Im}X_{i\beta}(\text{Ker}X_{p\beta})) \), where \( B_1 := \{ \beta \in A \mid p\beta \notin I \} \) and \( B_2 := \{ \beta \in A \mid \beta = 1 + q \} \) for some \( q \in \mathcal{P}_s \). Put \( \Omega := \{ \beta \in A \mid p\beta \notin I \} \) and \( \text{Ker}X_p \subseteq \bigoplus \text{Im}X_{i\beta} \). It follows that

\[
\bigoplus_{\alpha \in A} \text{Im}X_{i\alpha} \oplus \bigoplus_{\beta \in B_1} \text{Im}X_{i\beta} \oplus \bigoplus_{\beta \in B_2} \text{Im}X_{i\beta}(\text{Ker}X_{p\beta}) \]

If \( \alpha \in \Omega \) then \( l_\alpha < l_p \), and hence by induction \( \text{Im}X_{i\alpha} \) is Gorenstein-projective. If \( \beta \in B_2 \) then

\[
\frac{\text{Im}X_{i\beta}}{\text{Ker}X_{i\beta} \oplus \text{Im}X_{i\beta}} \cong \frac{\text{Im}X_{i\beta}}{\text{Ker}X_{i\beta} \oplus \text{Im}X_{i\beta}}
\]

and hence it is Gorenstein-projective by induction (since \( l_{p\beta} < l_p \)). This proves (1).

4.3. Let \( A \) and \( B \) be rings, \( M \) an \( A-B \)-bimodule, and \( T := (\Lambda A B) \) the triangular matrix ring. We assume that \( T \) is an Artin algebra ([4], p.72), and consider finitely generated \( T \)-modules. Recall that a \( T \)-module can be identified with a triple \((X, Y, f)\), where \( X \in \mathcal{A}_0\), \( Y \in \mathcal{B}_0\), and \( f : M \otimes B Y \to X \) is an \( A \)-map. An \( A \)-map \((X, Y, f) \to (X', Y', f')\) can be identified with a pair \((f, f')\), where \( f \in \text{Hom}_A(X, X')\), \( g \in \text{Hom}_B(Y, Y')\), such that \( f \circ g = f' \circ (\text{id} \otimes g) \). The following description of \( \mathcal{G}(T) \) is a special case of [49, Thm. 1.4].

Lemma 4.7. Let \( M \) be an \( A-B \)-bimodule with proj.dim\(A M < \infty \) and proj.dim\(M_B < \infty \), and \( T := (\Lambda A B) \). Then \((X, Y, f) \in \mathcal{G}(T) \) if and only if \( f : M \otimes B Y \to X \) is an injective \( A \)-map, \( \text{Coker} \phi \in \mathcal{G}(A) \), and \( Y \in \mathcal{G}(B) \).
In order to apply Lemma 4.7 we put

\[ \rho' := \{ \rho_i \in \rho \mid s(\rho_i) \neq n \}; \quad I' := \{ \rho_i \mid \rho_i \in \rho' \}; \]

\[ Q' := \text{the quiver obtained from } Q \text{ by deleting vertex } n; \]

\[ P(n) := \text{the indecomposable projective } kQ/I\text{-module at } n; \]

\[ \Lambda' := A \otimes kQ'/I'; \quad M := A \otimes \text{projdim } P(n). \]

Then \( M \) is a \( \Lambda' \)-\( A \)-bimodule. The point is that \( \Lambda = A \otimes kQ/I \) is of the form \( \Lambda = \left( \frac{\Lambda'}{\Lambda} \right) \). Since \( \text{gl.dim.} kQ'/I' < \infty \), we have \( \text{projdim}_{kQ'/I'} \text{projdim}(n) < \infty \), and hence \( \text{projdim}_A M < \infty \). Clearly, as a right \( A \)-module \( M \) is projective. For applying Lemma 4.7 in this setting-up, we have \( \Lambda \)-module \( \tilde{X} = (X_i, X_\alpha, \ i \in Q_0, \ \alpha \in Q_1) \) as \( X = (X'_{X_n})_{\phi} \) where \( X' = (X_i, X_\alpha, \ i \in Q_0, \ \alpha \in Q_1') \) is a \( \Lambda' \)-module, \( X_n \) is an \( A \)-module, and \( \phi : M \otimes_A X_n \rightarrow X' \) is a \( \Lambda' \)-map. The explicit expression of \( \phi \) is given in the proof of Lemma 4.9.

By a direct translation from Lemma 4.7 in this setting-up, we have

**Lemma 4.8.** Let \( X = (X'_{X_n})_{\phi} \) be a \( \Lambda \)-module. Then \( X \in \mathcal{G}P(\Lambda) \) if and only if \( X \) satisfies the conditions:

(i) \( X_n \in \mathcal{G}P(A) \);

(ii) \( \phi : M \otimes_A X_n \rightarrow X' \) is an injective \( \Lambda' \)-map;

(iii) \( \text{Coker } \phi \in \mathcal{G}P(\Lambda') \).

Theorem 4.1 will be proved by applying Lemma 4.8 and using induction on \( |Q_0| \). Thus, in the rest part of this paper we write a \( \Lambda \)-module \( X = (X_i, X_\alpha, \ i \in Q_0, \ \alpha \in Q_1) \) as \( X = (X'_{X_n})_{\phi} \), and keep all the notations \( I', Q', \Lambda', P(n), M, X, X', X_n \) and \( \phi \). For an integer \( m \geq 0 \) and a module \( N \), let \( N^m \) denote the direct sum of \( m \) copies of \( N \) (if \( m = 0 \) then \( N^m := 0 \)).

**Lemma 4.9.** Let \( X = (X'_{X_n})_{\phi} \) be a \( \Lambda \)-module. Then

(1) \( \phi : M \otimes_A X_n \rightarrow X' \) is an injective \( \Lambda' \)-map if and only if \( \sum_{p \in P(n \rightarrow i)} \text{Im } X_p = \bigoplus_{p \in P(n \rightarrow i)} \text{Im } X_p \), and \( X_p \) is injective for all \( i \in Q_0 \) and \( p \in P(n \rightarrow i) \).

(2) If \( X \) is monic then \( \phi : M \otimes_A X_n \rightarrow X' \) is injective.

(3) \( \text{Coker } \phi = (X_i/( \bigoplus_{p \in P(n \rightarrow i)} \text{Im } X_p ), \tilde{X}_n, \ i \in Q_0, \ \alpha \in Q_1) \), where for each \( \alpha : j \rightarrow i \) in \( Q_1 \),

\[ \tilde{X}_n : X_j/( \bigoplus_{q \in P(n \rightarrow j)} \text{Im } X_q ) \rightarrow X_j/( \bigoplus_{q \in P(n \rightarrow j)} \text{Im } X_q ) \]

is the \( A \)-map induced by \( X_\alpha \). Explicitly, \( \tilde{X}_n(\overrightarrow{x}) = X_n(x_j) + \bigoplus_{p \in P(n \rightarrow i)} \text{Im } X_p \), where \( \overrightarrow{x} := x_j + \bigoplus_{q \in P(n \rightarrow j)} \text{Im } X_q \).

**Proof.** (1) For \( i \in Q_0 \), put \( m_i := |P(n \rightarrow i)| \). As a representation of \( (Q', I') \) over \( k \), \( \text{radP}(n) \) can be written as \( \left( \begin{array}{c} \vdots \\ \vdots \\ \vdots \\ \vdots \end{array} \right) \), so we have isomorphisms of \( \Lambda' \)-modules

\[ M \otimes_A X_n \cong (\text{radP}(n) \otimes_k A) \otimes_A X_n \cong \text{radP}(n) \otimes_k X_n \cong \left( \begin{array}{c} X_n^m \\ \vdots \\ X_n^{m_n-1} \end{array} \right). \]

Let \( P(n \rightarrow i) = \{ p_1, \ldots, p_{m_i} \} \). Then \( \phi : M \otimes_A X_n \rightarrow X' \) reads

\[ \phi = \left( \begin{array}{c} \phi_1 \\ \vdots \\ \phi_{m_n-1} \end{array} \right) : \left( \begin{array}{c} X_n^m \\ \vdots \\ X_n^{m_n-1} \end{array} \right) \rightarrow \left( \begin{array}{c} X_1 \\ \vdots \\ X_{n-1} \end{array} \right), \]
Claim 1: Coker \( \phi \) satisfies (m1), i.e.,
\[
\sum_{\alpha \in \mathcal{A}'(-\to i)} X_{\alpha}(x_{\alpha}) \subseteq \bigoplus_{\alpha \in \mathcal{A}'(-\to i)} \text{Im} X_{\alpha} = \bigoplus_{\alpha \in \mathcal{A}'(-\to i)} \text{Im} \tilde{X}_{\alpha} \quad \text{for each } i \in Q'_0.
\]
In fact, let \( \sum_{\alpha \in \mathcal{A}'(-\to i)} X_{\alpha}(x_{\alpha}) = 0 \) with each \( x_{\alpha} \in X_{s(\alpha)} \). Then
\[
\sum_{\alpha \in \mathcal{A}'(-\to i)} X_{\alpha}(x_{\alpha}) \subseteq \bigoplus_{\beta \in \mathcal{A}(-\to i)} \text{Im} X_{\beta} + \sum_{\alpha \in \mathcal{A}(s(\alpha) \to i)} X_{\alpha} \left( \sum_{\gamma \in \mathcal{P}(n \to s(\alpha))} \text{Im} X_{\gamma} \right).
\]
Since \( s(\alpha) \neq n \), by (m1) on \( X \) we have
\[
X_{\alpha}(x_{\alpha}) \subseteq X_{\alpha} \left( \sum_{\gamma \in \mathcal{P}(n \to s(\alpha))} \text{Im} X_{\gamma} \right) \subseteq \bigoplus_{\alpha \in \mathcal{A}'(-\to i)} \text{Im} X_{\alpha}
\]
for each \( \alpha \in \mathcal{A}'(-\to i) \). This means \( \tilde{X}_{\alpha}(x_{\alpha}) = 0 \).

Claim 2: Coker \( \phi \) satisfies (m2), i.e.,
\[
\sum_{q \in \mathcal{P}'(-\to s(\alpha))} \text{Im} \tilde{X}_{q} \subseteq \text{Ker} \tilde{X}_{\alpha} \quad \text{for each } \alpha \in Q'_1.
\]
In fact, clearly \( \sum_{q \in \mathcal{P}'(-\to s(\alpha))} \text{Im} \tilde{X}_{q} \subseteq \text{Ker} \tilde{X}_{\alpha} \). Let \( \tilde{X}_{\alpha}(x) = 0 \) with \( x \in X_{s(\alpha)} \). Then \( X_{\alpha}(x) \in \bigoplus_{p \in \mathcal{P}(n \to s(\alpha))} \text{Im} X_{p} \). By the similar argument as in Claim 1 we have \( X_{\alpha}(x) \in X_{\alpha} \left( \sum_{q \in \mathcal{P}(n \to s(\alpha))} \text{Im} X_{q} \right) \).

By (m2) on \( X \) we have in the quotient
\[
\mathcal{T} \in \text{Ker} X_{\alpha} + \bigoplus_{q \in \mathcal{P}(n \to s(\alpha))} \text{Im} X_{q} = \bigoplus_{q' \in \mathcal{P}'(n \to s(\alpha))} \text{Im} X_{q'} + \bigoplus_{q \in \mathcal{P}(n \to s(\alpha))} \text{Im} X_{q}
\]
and
\[
= \bigoplus_{q' \in \mathcal{P}'(n \to s(\alpha))} \text{Im} X_{q'} + \bigoplus_{q \in \mathcal{P}(n \to s(\alpha))} \text{Im} X_{q}.
\]
It follows that \( \mathcal{T} \in \sum_{q' \in \mathcal{P}'(n \to s(\alpha))} \text{Im} \tilde{X}_{q'} \). This proves Claim 2.

Claim 3: Coker \( \phi \) satisfies (G).
In fact, by Lemma 4.9(2) it suffices to prove \[ X_i/(\bigoplus_{p \in \mathcal{P}(n \to i)} \text{Im} X_p) \] \[ \bigoplus_{\alpha \in \mathcal{A}(\to i)} \text{Im} X_{\alpha} \] is Gorenstein-projective for each \( i \in \mathbb{Q}_0 \).

Since \[ \bigoplus_{p \in \mathcal{P}(n \to i)} \text{Im} X_p \subseteq \sum_{\alpha \in \mathcal{A}(\to i)} \text{Im} X_{\alpha} \], it follows that

\[
\sum_{\alpha \in \mathcal{A}(\to i)} X_{\alpha} = \left( \sum_{\alpha \in \mathcal{A}(\to i)} \text{Im} X_{\alpha} + \bigoplus_{p \in \mathcal{P}(n \to i)} \text{Im} X_p \right) / \left( \bigoplus_{p \in \mathcal{P}(n \to i)} \text{Im} X_p \right) = (\sum_{\alpha \in \mathcal{A}(\to i)} \text{Im} X_{\alpha}) / (\bigoplus_{p \in \mathcal{P}(n \to i)} \text{Im} X_p).
\]

Hence

\[
X_i/(\bigoplus_{p \in \mathcal{P}(n \to i)} \text{Im} X_p) \bigoplus_{\alpha \in \mathcal{A}(\to i)} \text{Im} X_{\alpha} \cong X_i/(\bigoplus_{\alpha \in \mathcal{A}(\to i)} \text{Im} X_{\alpha}) \tag{4.1}
\]

which is Gorenstein-projective by (G) on \( X \).

**4.5. Proof of Theorem 4.1.1** Use induction on \( n = |\mathbb{Q}_0| \).

We first prove the sufficiency. Assume that \( X = (X_i, X_{\alpha}, i \in \mathbb{Q}_0, \alpha \in \mathcal{Q}_1) = (X^\prime)_{\phi} \) is a monic \( \Lambda \)-module satisfying (G). We need to prove that \( X \) is Gorenstein-projective. The assertion clearly holds for \( n = 1 \). Suppose that the assertion holds for \( n - 1 \) (\( n \geq 2 \)). It suffices to prove that \( X \) satisfies the conditions (i), (ii) and (iii) in Lemma 4.9. In fact, the condition (i) is contained in the condition (G); and the condition (ii) follows from Lemma 4.10. By Lemma 4.10, \( \text{Coker} \phi \) is a monic \( \Lambda^\prime \)-module satisfying (G). Since \( |\mathbb{Q}_0| = n - 1 \), it follows from induction that \( \text{Coker} \phi \) is Gorenstein-projective, i.e., the condition (iii) in Lemma 4.9 is also satisfied. This proves that \( X \) is Gorenstein-projective.

Now we prove the necessity. Assume that \( X = (X_i, X_{\alpha}, i \in \mathbb{Q}_0, \alpha \in \mathcal{Q}_1) = (X^\prime)_{\phi} \) is a Gorenstein-projective \( \Lambda \)-module. We need to prove that \( X \) is a monic \( \Lambda \)-module satisfying (G). The assertion clearly holds for \( n = 1 \). Suppose that the assertion holds for \( n - 1 \) (\( n \geq 2 \)). By Lemma 4.8, \( X \) is a Gorenstein-projective \( \mathcal{A} \)-module, \( \phi : M \otimes_{\mathcal{A}} X_n \to X^\prime \) is an injective \( \mathcal{A}^\prime \)-map, and \( \text{Coker} \phi \) is a Gorenstein-projective \( \mathcal{A}^\prime \)-module. Since \( \phi \) is injective, by Lemma 4.9(1) we have

1. For each \( i \in \mathbb{Q}_0 \) there holds \( \sum_{p \in \mathcal{P}(n \to i)} \text{Im} X_p = \bigoplus_{p \in \mathcal{P}(n \to i)} \text{Im} X_p \); and
2. \( X_p \) is an injective \( \mathcal{A} \)-map for \( p \in \mathcal{P}(n \to i) \) and \( i \in \mathbb{Q}_0 \).

Since \( \text{Coker} \phi \) is a Gorenstein-projective \( \mathcal{A}^\prime \)-module and \( |\mathbb{Q}_0| = n - 1 \), by induction \( \text{Coker} \phi \) is a monic \( \Lambda^\prime \)-module satisfying (G). Thus we have

3. For each \( i \in \mathbb{Q}_0 \) there holds \( \sum_{\alpha \in \mathcal{A}(\to i)} X_{\alpha} = \bigoplus_{\alpha \in \mathcal{A}(\to i)} \text{Im} X_{\alpha} \); and
4. For each \( \alpha \in \mathbb{Q}_1 \) there holds \( \text{Ker} X_{\alpha} = \sum_{q \in \mathcal{A}((\to i), \alpha)} \text{Im} X_q \).

**Claim 1:** \( X \) satisfies (m1), i.e., for each \( i \in \mathbb{Q}_0 \) there holds \( \sum_{\alpha \in \mathcal{A}(\to i)} X_{\alpha} = \bigoplus_{\alpha \in \mathcal{A}(\to i)} \text{Im} X_{\alpha} \).
In fact, suppose that \( \sum_{\alpha \in \mathcal{A}(\to i)} X_{\alpha}(x_{\alpha}) = 0 \) with each \( x_{\alpha} \in X_{s(\alpha)} \). Then \( \sum_{\alpha \in \mathcal{A}'(\to i)} \tilde{X}_{\alpha}(\tilde{x}_{\alpha}) = 0 \). By (3) we have \( \tilde{X}_{\alpha}(\tilde{x}_{\alpha}) = 0 \) for each \( \alpha \in \mathcal{A}'(\to i) \), and hence \( \tilde{x}_{\alpha} \in \sum_{q \in \mathcal{P}'(\to s(\alpha))} \text{Im} \tilde{X}_{q} \) by (4). So there exists \( y_{\alpha} \in \sum_{q \in \mathcal{P}'(\to s(\alpha))} \text{Im} X_{q} \) such that \( x_{\alpha} - y_{\alpha} \in \bigoplus_{p \in \mathcal{P}(n \to s(\alpha))} \text{Im} X_{p} \). Thus there exists \( x'_{\alpha} = \sum_{p \in \mathcal{P}(n \to s(\alpha))} X_{p}(x_{p,\alpha}) \) with each \( x_{p,\alpha} \in X_{s(p)} \) such that \( x_{\alpha} - y_{\alpha} = x'_{\alpha} \). Since \( X_{\alpha}X_{q} = 0 \) for all \( \alpha \in I' \), we have \( X_{\alpha}(y_{\alpha}) = 0 \), and hence \( X_{\alpha}(x_{\alpha}) = X_{\alpha}(x'_{\alpha}) \). It follows that
\[
0 = \sum_{\alpha \in \mathcal{A}(\to i)} X_{\alpha}(x_{\alpha}) = \sum_{\alpha \in \mathcal{A}(\to i)} X_{\alpha}(x_{\alpha}) + \sum_{\alpha \in \mathcal{A}'(\to i)} X_{\alpha}(x_{\alpha}) = \sum_{\alpha \in \mathcal{A}(\to i)} X_{\alpha}(x_{\alpha}) + \sum_{\alpha \in \mathcal{A}(\to i)} X_{\alpha}(x_{\alpha}) + \sum_{\alpha \in \mathcal{A}'(\to i)} X_{\alpha}(x_{\alpha}).
\]
By (1) this sum is a direct sum, hence \( X_{\alpha}(x_{\alpha}) = 0 \) for all \( \alpha \in \mathcal{A}(\to i) \), and \( X_{\alpha}X_{\alpha}(x_{\alpha}) = 0 \) for all \( \alpha \in \mathcal{A}'(\to i) \) and \( p \in \mathcal{P}(n \to s(\alpha)) \). Thus all together \( X_{\alpha}(x_{\alpha}) = 0 \) for each \( \alpha \in \mathcal{A}(\to i) \). This proves Claim 1.

**Claim 2:** \( X \) satisfies (m2), i.e., for each \( \alpha \in Q_{1} \) there holds \( \text{Ker} X_{\alpha} = \sum_{q \in \mathcal{P}'(\to s(\alpha))} \text{Im} X_{q} \).

Let \( x \in \text{Ker} X_{\alpha} \). Then \( \bar{x} \in \text{Ker} \tilde{X}_{\alpha} \). By (4) we have
\[
\bar{x} = \sum_{q \in \mathcal{P}'(\to s(\alpha))} X_{q}(x_{q}) + \bigoplus_{p \in \mathcal{P}(n \to s(\alpha))} \text{Im} X_{p}
\]
for some \( x_{q} \in X_{s(q)} \), and hence there is \( y_{p} \in X_{\alpha} \) for each \( p \in \mathcal{P}(n \to s(\alpha)) \), such that
\[
x = \sum_{q \in \mathcal{P}'(\to s(\alpha))} X_{q}(x_{q}) + \sum_{p \in \mathcal{P}(n \to s(\alpha))} X_{p}(y_{p}) = \sum_{q \in \mathcal{P}'(\to s(\alpha))} X_{q}(x_{q}) + \sum_{p \in \mathcal{P}(n \to s(\alpha))} X_{p}(y_{p}) + \sum_{p \in \mathcal{P}(n \to s(\alpha))} X_{p}(y_{p}).
\]
Since \( X_{\alpha}(x_{\alpha}) = 0 \) and \( X_{\alpha}X_{\alpha} = 0 \) for all \( \alpha \in I \), we have \( \sum_{p \in \mathcal{P}(n \to s(\alpha))} X_{\alpha}X_{\alpha}(y_{p}) = 0 \). By (1) we have \( X_{\alpha}X_{\alpha}(y_{p}) = 0 \) for each \( p \in \mathcal{P}(n \to s(\alpha)) \) with \( \alpha \in I \). But by (2) \( X_{\alpha}X_{\alpha} \) is injective, so \( y_{p} = 0 \) for all \( \alpha \in I \). Thus
\[
x = \sum_{q \in \mathcal{P}'(\to s(\alpha))} X_{q}(x_{q}) + \sum_{p \in \mathcal{P}(n \to s(\alpha))} X_{p}(y_{p}) \in \sum_{q \in \mathcal{P}'(\to s(\alpha))} \text{Im} X_{q}.
\]
This proves Claim 2.

By Claims 1 and 2 \( X \) is a monic \( A \)-module. It remains to prove that \( X \) satisfies (G). By Lemma 4.1 it suffices to prove that \( X_{i}/( \bigoplus_{p \in \mathcal{P}(n \to i)} \text{Im} X_{p}) \) is a Gorenstein-projective \( A \)-module for each \( i \in Q_{1} \).

We have claimed that \( \text{Coker} \phi \) satisfies (G), i.e., for each \( i \in Q_{1}^{1} \), \( (X_{i}/( \bigoplus_{p \in \mathcal{P}(n \to i)} \text{Im} X_{p}))/\text{Im} \tilde{X}_{\alpha} \) is a Gorenstein-projective \( A \)-module, which is isomorphic to \( X_{i}/( \bigoplus_{p \in \mathcal{P}(n \to i)} \text{Im} X_{p}) \) by (4.1). This completes the proof of Theorem 4.1.

**References**

[1] D. M. Arnold, Abelian groups and representations of finite partially ordered sets, Canad. Math. Soc. Books in Math., Springer-Verlag, New York, 2000.
[2] M. Auslander, M. Bridger, Stable module theory, Mem. Amer. Math. Soc. 94., Amer. Math. Soc., Providence, R.I., 1969.
[3] M. Auslander, Representation dimension of artin algebras, Queen Mary College Math. Notes, London, 1971.
[4] M. Auslander, I. Reiten, S. O. Smalø, Representation Theory of Artin Algebras, Cambridge Studies in Adv. Math. 36., Cambridge Univ. Press, 1995.
[5] L. L. Avramov, A. Martsinkovsky, Absolute, relative, and Tate cohomology of modules of finite Gorenstein dimension, Proc. London Math. Soc. 85(3)(2002), 393-440.
[6] A. Beligiannis, Cohen-Macaulay modules, (co)torsion pairs and virtually Gorenstein algebras, J. Algebra 288(1)(2005), 137-211.
[7] G. Birkhoff, Subgroups of abelian groups, Proc. Lond. Math. Soc. II. Ser. 38(1934), 385-401.
[8] R.-O. Buchweitz, Maximal Cohen-Macaulay modules and Tate cohomology over Gorenstein rings, Unpublished manuscript, Hamburg (1987), 155pp.
[9] R.-O. Buchweitz, G.-M. Greuel, F.-O. Schreyer, Cohen-Macaulay modules on hypersurface singularities II, Invent. Math. 88(1)(1987), 165-182.
[10] X. W. Chen, The stable monomorphism category of a Frobenius category, Math. Res. Lett. 18(1)(2011), 125-137.
[11] L. W. Christensen, A. Frankild, H. Holm, On Gorenstein projective, injective and flat dimensions-a functorial description with applications, J. Algebra 302(1)(2006), 231-279.
[12] L. W. Christensen, G. Piepmeyer, J. Striuli, R. Takahashi, Finite Gorenstein representation type implies simple singularity, Adv. Math. 218(2008), 1012-1026.
[13] E. E. Enochs, O. M. G. Jenda, Gorenstein injective and projective modules, Math. Z. 220(4)(1995), 611-633.
[14] E. E. Enochs, O. M. G. Jenda, Relative homological algebra, De Gruyter Exp. Math. 30. Walter De Gruyter Co., 2000.
[15] D. Happel, Triangulated categories in representation theory of finite dimensional algebras, London Math. Soc. Lecture Notes Ser. 119, Cambridge Univ. Press, 1988.
[16] D. Happel, On Gorenstein algebras, in: Representation theory of finite groups and finite-dimensional algebras, Prog. Math. 95, 389-404, Birkhauser, Basel, 1991.
[17] H. Holm, Gorenstein homological dimensions, J. Pure Appl. Algebra 189(1-3)(2004), 167-193.
[18] Z. Y. Huang, Proper resolutions and Gorenstein categories, J. Algebra 393(1)(2013), 142-169.
[19] O. Iyama, K. Kato, J. I. Miyachi, Recollement on homotopy categories and Cohen-Macaulay modules, J.K-Theory 8(3)(2011), 507-541.
[20] P. Jørgensen, Existence of Gorenstein projective resolutions and Tate cohomology, J. Eur. Math. Soc. 9(2007), 59-76.
[21] B. Keller, Chain complexes and stable categories, Manuscripta Math. 67 (1990), 379-417.
[22] H. Knörrer, Cohen-Macaulay modules on hypersurface singularities I, Invent. Math. 88(1)(1987), 153-164.
[23] D. Kussin, H. Lenzing, H. Meltzer, Nilpotent operators and weighted projective lines, J. Reine Angew. Math. 685(6)(2010), 33-71.
[24] D. Kussin, H. Lenzing, H. Meltzer, Triangle singularities, ADE-chains, and weighted projective lines, Adv. Math. 237(2013), 194-251.
[25] X. H. Luo, P. Zhang, Monic representations and Gorenstein-projective modules, Pacific J. Math. 264(1)(2013), 163-194.
[26] A. Moore, The Auslander and Ringel-Tachikawa theorem for submodule embeddings, Comm. Algebra 38(2010), 3805-3820.
[27] D. Quillen, Higher algebraic K-theory I, In: Lecture Notes in Math. 341, 85-147, Springer-Verlag, 1973.
[28] C. M. Ringel, Tame algebras and integral quadratic forms, Lecture Notes in Math. 1099, Springer-Verlag, 1984.
[29] C. M. Ringel, M. Schmidmeier, Submodules categories of wild representation type, J. Pure Appl. Algebra 205(2)(2006), 412-422.
[30] C. M. Ringel, M. Schmidmeier, The Auslander-Reiten translation in submodule categories, Trans. Amer. Math. Soc. 360(2)(2008), 691-716.
[31] C. M. Ringel, M. Schmidmeier, Invariant subspaces of nilpotent operators I, J. rein angew. Math. 614 (2007), 1-30.
[32] S. Sather-Wagstaff, T. Sharif, D. White, Stability of Gorenstein categories, J. London Math. Soc. 77(2) (2008), 481-502.
[33] D. Simson, Representation types of the category of subprojective representations of a finite poset over $K[t]/(t^m)$ and a solution of a Birkhoff type problem, J. Algebra 311(2007), 1-30.
[34] D. Simson, Tame-wild dichotomy of Birkhoff type problems for nilpotent linear operators, J. Algebra 424(2015), 254-293.
[35] P. Zhang, Monomorphism categories, cotilting theory, and Gorenstein-projective modules, J. Algebra 339(2011), 180-202.
[36] P. Zhang, Gorenstein-projective modules and symmetric recollements, J. Algebra 388 (2013), 65-80.

Xiu-Hua Luo, xiuhualo2014@163.com
Dept. of Math., Nantong Univ., Nantong 226019, China

Pu Zhang, pzhang@sjtu.edu.cn
Dept. of Math., Shanghai Jiao Tong Univ., Shanghai 200240, China