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A quasimartingale characterization of $p$-stable type Banach spaces.

Florian HECHNER $^1$ and Bernard HEINKEL$^2$

$^1$Corresponding author
IRMA, University of Strasbourg and CNRS,
7 rue René-Descartes, 67084 STRASBOURG CEDEX
hechner@math.unistra.fr

$^2$IRMA, University of Strasbourg and CNRS,
7 rue René-Descartes, 67084 STRASBOURG CEDEX
heinkel@math.unistra.fr

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Abstract

We characterize Banach spaces $B$ of stable-type $p$ ($1 < p < 2$) by
the property that for every sequence $(X_i)$ of $B$-valued random variables,
independent, centered and fulfilling some integrability assumption, the
sequence \( \left( \frac{X_1 + \cdots + X_n}{n^{1/p}} \right) \) is a quasimartingale.

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1 Introduction

“How to characterize the regularity of a Banach space $(\mathcal{B}, \| \cdot \|)$ by the fact
that a kind of classical strong law of large numbers (SLLN) holds for $\mathcal{B}$-valued random variables (r.v.)?” is a well know problem. Two kinds of regular Banach
spaces – the spaces of Rademacher type $p$ and the spaces of stable type $p$ –
have been nicely characterized in that way (see chapter 9 in [3]).

Here our goal will be to show that the SLLN characterization of spaces of
stable type $p$ ($1 < p < 2$) can be made more precise in supposing that the
“normalized sums” obeying the SLLN have even a quasimartingale behaviour.

To begin with, we recall some definitions.

In the sequel, $(\mathcal{B}, \| \cdot \|)$ will be a real separable Banach space, equipped with
its Borel $\sigma$-field $\mathcal{B}$. A $\mathcal{B}$-valued r.v. $X$ is a measurable function defined on
a probability space \((\Omega, \mathcal{T}, P)\) with values in \((\mathcal{B}, \mathcal{F})\). Such a r.v. is said to be (strongly) integrable if \(E\|X\| < +\infty\) and \(\forall f \in \mathcal{F}, E[f(X)] = 0\); this is denoted by \(E(X) = 0\).

Let \(p \geq 1\) be given. The weak-\(\ell_p\) norm of a sequence \(a := (a_1, \ldots, a_n)\) of real numbers is defined as follows:

\[
∥a∥_{p,∞} := \sup_{t>0} (t^p \text{Card}(i : |a_i| > t))^{1/p} = \sup_{k=1} a_k^* k^{1/p},
\]

where \((a_1^*, \ldots, a_n^*)\) denotes the non-increasing rearrangement of the sequence \((|a_1|, \ldots, |a_n|)\).

Let now \((\varepsilon_k)\) be a sequence of independent Rademacher random variables (that is \(P(\varepsilon_k = 1) = P(\varepsilon_k = -1) = \frac{1}{2}\)).

Rademacher type \(p\) spaces and stable type \(p\) spaces are defined as follows:

**Definition 1.**

1. Let \(1 < p < 2\). The Banach space \((\mathcal{B}, ∥·∥)\) is of Rademacher type \(p\) if there exists a constant \(c(p) > 0\) such that for every finite sequence \((x_i)\) in \(\mathcal{B}\):

\[
E \left( \left\| \sum_{i=1}^n \varepsilon_i x_i \right\|^p \right)^{\frac{1}{p}} \leq c(p) \left( \sum_{i=1}^n ∥x_i∥^p \right)^{\frac{1}{p}}.
\]

2. Let \(1 \leq p \leq 2\). The Banach space \((\mathcal{B}, ∥·∥)\) is of stable type \(p\) if there exists a constant \(C(p) > 0\) such that for every finite sequence \((x_i)\) in \(\mathcal{B}\):

\[
E \left( \left\| \sum_{i=1}^n \varepsilon_i x_i \right\| \right) \leq C(p) ∥(∥x_1∥, \ldots, ∥x_n∥)∥_{p,∞}.
\]

**Remark 2.**

1. If \((\mathcal{B}, ∥·∥)\) is of stable type \(p\), then there exists \(q > p\), such that \((\mathcal{B}, ∥·∥)\) is also of stable type \(q\).

2. If \((\mathcal{B}, ∥·∥)\) is of stable type \(p\), it is also of Rademacher type \(p\).

3. The above definition of stable type is not the classical one (which involves standard stable r.v.), but an equivalent statement (see for instance [3], proposition 9.12) which will be used in our proofs.

**Remark 3.**

Let \((\mathcal{B}, ∥·∥)\) be a Banach space and \(1 \leq p \leq 2\). If \(\mathcal{B}\) is of (Rademacher) type \(p\), there exists a constant \(c(p)\) such that for every finite sequence \((X_i)\) with \(∥X_i∥ \in L^p\):

\[
E \left( \left\| \sum_{i=1}^n X_i \right\|^p \right) \leq c(p) \sum_{i=1}^n E[∥X_i∥|^p].
\]

(1)
The stable type $p$ has been characterized in terms of Marcinkiewicz-Zygmund like SLLN:

**Theorem 4** (Maurey-Pisier [5]).
Let $1 < p < 2$. The following two properties are equivalent:
1. $(\mathcal{B}, \| \cdot \|)$ is of stable type $p$.
2. For every bounded sequence $(x_i)$ in $\mathcal{B}$, the sequence \( \left( \frac{1}{n^{1/p}} \sum_{k=1}^{n} \varepsilon_k x_k \right) \) converges a.s. to 0.

**Theorem 5** (Woyczynski [8]).
Let $1 < p < 2$. The following two properties are equivalent:
1. $(\mathcal{B}, \| \cdot \|)$ is of stable type $p$.
2. For every sequence $(X_i)$ of independent, strongly centered $\mathcal{B}$-valued r.v. for which there exists a nonnegative r.v. $\xi$ with $E\xi^p < +\infty$ such that:
   \[ \exists c > 0, \forall t > 0, \forall i \in \mathbb{N}^*, P(\|X_i\| > t) \leq cP(\xi > t), \]
   the sequence $(\frac{S_n}{n^{1/p}})$ converges a.s. to 0, where, as usual, $S_n := X_1 + \cdots + X_n$.

In this paper, we will prove a result in the same spirit as Woyczynski’s result, but in which the SLLN behaviour of the sequence $(\frac{S_n}{n^{1/p}})$ is even a quasimartingale behaviour.

## 2 A quasimartingale characterization of spaces of stable type $p$.

We start this section by defining the quasimartingale behaviour of $(\frac{S_n}{n^{1/p}})$:

**Definition 6.**
Let $(X_k)$ be a sequence of independent, strongly centered $(\mathcal{B}, \| \cdot \|)$-valued r.v. Denote $S_n := X_1 + \cdots + X_n$ and $\mathcal{F}_n := \sigma(X_1, \ldots, X_n)$. Let $p \in [1, 2]$. The sequence $(\frac{S_n}{n^{1/p}}, \mathcal{F}_n)$, or simpler $(\frac{S_n}{n^{1/p}})$, is a quasimartingale if:
\begin{equation}
\sum_{n=1}^{+\infty} E \left\| \frac{S_{n+1}}{(n+1)^{1/p}} - \frac{S_n}{n^{1/p}} \bigg| \mathcal{F}_n \right\| < +\infty. \tag{2}
\end{equation}

**Remark 7.**
Since the r.v. $(X_k)$ are independent and centered, condition (3) is equivalent to:
\[ \sum_{n=1}^{+\infty} E\|S_n\| \frac{1}{n^{1+1/p}} < +\infty. \]
Now we are able to state our characterization of spaces of stable type $p$ ($1 < p < 2$):

**Theorem 8.**

Let $1 < p < 2$. The following two properties are equivalent:

1. $B$ is a stable type $p$ space;

2. For every sequence $(X_n)$ of independent, strongly centered r.v., such that

$$\int_0^{+\infty} g^{1/p}(t) dt < +\infty,$$

where

$$\forall t > 0, \ g(t) := \sup_{n \geq 1} \mathbb{P}(\|X_n\| > t),$$

$(\frac{S_n}{n^{1/p}})$ is a quasimartingale.

**Remark 9.**

1. Condition (3) is not surprising: indeed, in the i.i.d. case, it can be written:

$$\int_0^{+\infty} \mathbb{P}^{1/p}(\|X\| > t) dt < +\infty,$$

which condition is necessary for $(\frac{S_n}{n^{1/p}})$ being a quasimartingale in every Banach space $B$ (see [1], proposition 3).

2. Property (4) implies that $\mathbb{E}\|X\|^p < +\infty$ (see [1], remark 4).

3. There exists a small class of r.v. $X$ such that $\mathbb{E}\|X\|^p < +\infty$ and for which (3) does not hold (see example 1 in [1]).

So – comparing theorems 5 and 8 – one sees that the price to pay for getting a quasimartingale behaviour for $(\frac{S_n}{n^{1/p}})$ instead of a simple a.s. convergence to 0, is to sharpen a little bit the hypothesis $\mathbb{E}\xi^p < +\infty$ of theorem 5.

**Proof.**

In the sequel, $c_k$ will denote positive constants which precise value does not matter.

Let us show the implication $1 \Rightarrow 2$.

First consider the special case where there exists $M > 0$ such that: $\forall t > M, \ g(t) = 0$. Then $\forall k, \ \|X_k\| \leq M$ a.s. The space $B$ being of Rademacher
type $q$ for some $q > p$ by comments 1) and 2) following definition 1, and using relation (3), one has :

$$
\frac{\mathbb{E}\|S_n\|}{n^{1+1/p}} \leq \frac{c(q)}{n^{1+1/p}} \left( \sum_{k=1}^{n} \mathbb{E}\|X_k\|^q \right)^{\frac{1}{q}} \leq \frac{c(q)M^{\frac{1}{q}}}{n^{1+\frac{1}{p}}}
$$

and the series having general terms $\frac{\mathbb{E}\|S_n\|}{n^{1+\frac{1}{p}}}$ converges.

From now we suppose that $\forall t > 0, g(t) > 0$.

By a classical symmetrization argument it suffices to consider the case of symmetrically distributed r.v. $(X_k)$. For showing that under condition (3) the series having general term $\frac{\mathbb{E}\|S_n\|}{n^{1+\frac{1}{p}}}$ converges, one will split each r.v. $X_1, \ldots, X_n$ involved in the sum $S_n$ into two parts $U_{n,k}$ and $V_{n,k}$ by truncating $X_k$ at a suitable level $v_n$.

For defining $v_n$, one first notices that, $g$ being decreasing, one has :

$$
\sup_{t>0} t^p g(t) \leq \left( \int_0^{+\infty} g^\frac{1}{p}(x)dx \right)^p < +\infty.
$$

Furthermore, multiplying the $X_k$ by a suitable constant if necessary, one can suppose without loss of generality that :

$$
\sup_{t>0} t^p g(t) \leq 1. \quad (5)
$$

Now define $v_n := \inf \{ t > 0 | g(t) \leq \frac{1}{n} \}$.

It follows from the definition of $g$ and (5) that :

$$
v_n \leq n^{1/p} \quad \text{and} \quad g(v_n) \leq \frac{1}{n}.
$$

For every $n \in \mathbb{N}^*$ and $k = 1, \ldots, n$, one considers the following centered r.v. (by symmetry) :

$$
U_{n,k} := X_k 1_{\{\|X_k\| \leq v_n\}} \quad \text{and} \quad V_{n,k} := X_k 1_{\{\|X_k\| > v_n\}},
$$

and the associated sums :

$$
A_n := \sum_{k=1}^{n} \frac{U_{n,k}}{n^{1+\frac{1}{p}}} \quad \text{and} \quad B_n := \sum_{k=1}^{n} \frac{V_{n,k}}{n^{1+\frac{1}{p}}}.
$$

For showing that the series having general terms $\frac{\mathbb{E}\|S_n\|}{n^{1+\frac{1}{p}}}$ converges, one will show that :

$$
\sum_{n=1}^{+\infty} \mathbb{E}\|A_n\| < +\infty \quad (6)
$$
and

\[ \sum_{n=1}^{+\infty} \mathbb{E} \| B_n \| < +\infty. \quad (7) \]

We first prove (6).

By remark 1, following definition 1, there exists \( q > p \) such that \((\mathcal{B}, \| \cdot \|)\) is also \( q \)-stable.

Now suppose that the r.v. \( X_k \) are defined on a probability space \((\Omega, \mathcal{T}, \mathbb{P})\) and consider \((\varepsilon_k)\) a sequence of independent Rademacher r.v. defined on another probability space \((\Omega', \mathcal{T}', \mathbb{P}')\). By symmetry, one has:

\[
\mathbb{E} \| A_n \| = \int_{\Omega} \left( \int_{\Omega'} \frac{1}{n^{1+\frac{1}{p}}} \left\| \sum_{k=1}^{n} \varepsilon_k(\omega') U_{n,k}(\omega) \right\| d\mathbb{P}'(\omega') \right) d\mathbb{P}(\omega).
\]

By application of the definition of the stable type \( q \), one obtains:

\[
\mathbb{E} \| A_n \| \leq \frac{C(q)}{n^{1+\frac{1}{p}}} \mathbb{E} (\| (\| U_{n,1} \|, \ldots, \| U_{n,n} \|) \|_{q,\infty}).
\]

For bounding the tails of the weak-\( \ell_p \) norm of a sequence of positive, independent r.v., we will use the following classical result due to Marcus Pisier [4]:

**Lemma 10.**

For positive valued, independent r.v. \( \xi_1, \ldots, \xi_n \), one has:

\[
\forall q \geq 1, \forall u > 0, \mathbb{P} (\| (\xi_1, \ldots, \xi_n) \|_{q,\infty} > u) \leq \frac{2e}{u^q} \Delta (\xi_1, \ldots, \xi_n), \quad (8)
\]

where \( \Delta (\xi_1, \ldots, \xi_n) = \sup_{t > 0} \left( t^q \sum_{k=1}^{n} \mathbb{P} (\xi_k > t) \right) \).

For simplicity, denote \( \Delta_n \) the quantity \( \Delta (\| U_{n,1} \|, \ldots, \| U_{n,n} \|) \), and notice that by application of lemma [4]:

\[
\mathbb{E} (\| (\| U_{n,1} \|, \ldots, \| U_{n,n} \|) \|_{q,\infty}) = \int_{0}^{+\infty} \mathbb{P} (\| (\| U_{n,1} \|, \ldots, \| U_{n,n} \|) \|_{q,\infty} > u) du
\]

\[
\leq \Delta_n^{\frac{1}{q}} + \int_{\Delta_n^{\frac{1}{q}}}^{+\infty} \frac{2e}{u^q} \Delta_n du
\]

\[
\leq c_2 \Delta_n^{\frac{1}{q}}
\]

So the proof of (6) reduces to the following lemma:

**Lemma 11.**

\[
\sum_{n=1}^{+\infty} \frac{\Delta_n^{\frac{1}{q}}}{n^{1+\frac{1}{p}}} < +\infty
\]
Proof of lemma 11:

One first notices that:

$$\Delta_n \leq \sup_{t \leq v_n} t^q \sum_{k=1}^n \mathbb{P}(\|X_k\| > t) \leq \sup_{t \leq v_n} n t^q g(t) \leq n \left( \int_0^{v_n} g^\frac{1}{q}(u) du \right)^q,$$

the last inequality following from the fact that $g$ is decreasing.

For concluding the proof of lemma 11 it remains to check that the series with general term $a_n := \frac{1}{n^{\frac{1}{p} - \frac{1}{q}}} \int_0^{v_n} g^\frac{1}{q}(u) du$ converges.

First observe that:

$$\sum_{n=1}^{+\infty} a_n \leq \sum_{n=1}^{+\infty} \frac{1}{n^{\frac{1}{p} - \frac{1}{q}}} \sum_{j=0}^{v_{j+1}} g^\frac{1}{q}(u) du,$$

where $v_0 := 0$, and then exchange the summations in $n$ and $j$:

$$\sum_{n=1}^{+\infty} a_n \leq c_3 \left( v_1 + \sum_{j=1}^{+\infty} \frac{1}{j^{\frac{1}{p}}} \left( \int_{v_j}^{v_{j+1}} g^\frac{1}{q}(u) du \right) \right),$$

so, by the definition of $v_{j+1}$:

$$\sum_{n=1}^{+\infty} a_n \leq c_4 \left( v_1 + \int_0^{+\infty} g^\frac{1}{q}(u) du \right),$$

which concludes the proof of lemma 11.

Now we are going to prove (7).

First notice the following chain of inequalities:

$$\mathbb{E}\|B_n\| \leq \sum_{k=1}^n \mathbb{E}\|V_{n,k}\| \leq \sum_{k=1}^n \int_0^{+\infty} \mathbb{P}(\|V_{n,k}\| > u) du \leq \sum_{k=1}^n \frac{v_n}{n^{\frac{1}{p} + \frac{1}{q}}} \mathbb{P}(\|X_k\| > v_n) + \frac{n}{n^{\frac{1}{p} + \frac{1}{q}}} \int_{v_n}^{+\infty} g(u) du \leq \frac{n v_n}{n^{\frac{1}{p} + \frac{1}{q}}} g(v_n) \leq \frac{v_n}{n^{\frac{1}{p} + \frac{1}{q}}} + \frac{1}{n^{\frac{1}{p}}} \int_{v_n}^{+\infty} g(u) du.$$

Therefore, for proving (7), it suffices to check that condition (3) implies the convergence of the two series with general terms $\frac{v_n}{n^{\frac{1}{p} + \frac{1}{q}}}$ and $\frac{1}{n^{\frac{1}{p}}} \int_{v_n}^{+\infty} g(u) du$.

Lemma 12.

If (3) is fulfilled, then $\sum_{n=1}^{+\infty} \frac{v_n}{n^{\frac{1}{p} + \frac{1}{q}}} < +\infty$.  

7
Proof of lemma 12:
For \( j \in \mathbb{N}^* \), one denotes \( t_j := \frac{v_j + v_{j+1}}{2} \). Then:
\[
\int_0^{+\infty} g^{\frac{1}{p}}(u) du \geq \sum_{j=1}^{+\infty} \int_{t_j}^{v_j} g^{\frac{1}{p}}(t) dt \geq \sum_{j=1}^{+\infty} \frac{1}{(j+1)^{\frac{1}{p}}} \frac{v_{j+1} - v_j}{2}.
\] (9)

Now observe that:
\[
\sum_{j=1}^{n-1} \frac{1}{(j+1)^{\frac{1}{p}}}(v_{j+1} - v_j) = -\frac{v_1}{2^p} + \sum_{j=2}^{n-1} v_j \left( \frac{1}{j^{\frac{1}{p}}} - \frac{1}{(j+1)^{\frac{1}{p}}} \right) + \frac{v_n}{n^{\frac{1}{p}}}.
\] (10)

As:
\[
\frac{v_n}{2^p} g^{\frac{1}{p}}(v_n) \leq \int_{v_n}^{+\infty} g^{\frac{1}{p}}(u) du,
\]
one gets \( \lim_{n \to +\infty} v_n g^{\frac{1}{p}}(v_n) = 0 \) and also \( \lim_{n \to +\infty} \frac{\alpha_n}{n^{\frac{1}{p}}} = 0 \).

As \( \frac{1}{j^{\frac{1}{p}}} - \frac{1}{(j+1)^{\frac{1}{p}}} \geq \frac{c_5}{j^{1+\frac{1}{p}}} \), it follows from (9) and (10) that the series having general term \( \frac{\alpha_n}{n^{\frac{1}{p}}} \) converges.

For completing the proof of the implication 1 \( \implies \) 2 of theorem 8, it remains to check:

**Lemma 13.**
Under (3), one has \( \sum_{n=1}^{+\infty} \frac{1}{n^{\frac{1}{p}}} \int_{v_n}^{v_{n+1}} g(u) du < +\infty \).

Proof of lemma 13.
Let us write:
\[
\alpha := \sum_{n=1}^{+\infty} \frac{1}{n^{\frac{1}{p}}} \int_{v_n}^{v_{n+1}} g(u) du = \sum_{n=1}^{+\infty} \frac{1}{n^{\frac{1}{p}}} \sum_{j=n}^{+\infty} \int_{v_j}^{v_{j+1}} g(u) du.
\]

By exchanging the summations in \( n \) and \( j \), one gets:
\[
\alpha = \sum_{j=1}^{+\infty} \left( \int_{v_j}^{v_{j+1}} g(u) du \right) \sum_{n=1}^{+\infty} \frac{1}{n^{\frac{1}{p}}} \leq c_6 \sum_{j=1}^{+\infty} \left( \int_{v_j}^{v_{j+1}} g^{\frac{1}{p}}(u) du \right) \frac{j^{1+\frac{1}{p}}}{j^{\frac{1}{p}}}
\]
\[
\leq c_7 \int_0^{+\infty} g^{\frac{1}{p}}(u) du.
\]

Let us now show the converse implication 2 \( \implies \) 1 of theorem 8.
We first show a general property which is of independent interest:

**Proposition 14.**
Let \((Y_n)\) be a sequence of independent strongly centered r.v. with values in a
general Banach space \((B, \| \cdot \|)\). Denote by \(T_n\) the sum \(Y_1 + \ldots + Y_n\) and by \(G_n\) the \(\sigma\)-field \(\sigma(Y_1, \ldots, Y_n)\). If \(\left( \frac{T_n}{n^p}, G_n \right)\) is a quasimartingale, then \(\left( \frac{T_n}{n^p} \right)\) converges a.s. to 0.

Proof of proposition 14:

As noticed earlier, if \(\left( \frac{T_n}{n^p} \right)\) is a quasimartingale, then :

\[
\sum_{n=1}^{+\infty} \frac{\mathbb{E}\|T_n\|}{n^{1+p}} < +\infty.
\]  

(11)

By Jensen’s inequality :

\[
\forall N \in \mathbb{N}^*, \sum_{n=N}^{+\infty} \frac{\mathbb{E}\|T_n\|}{n^{1+p}} \geq \mathbb{E}\|T_N\| \sum_{n=N}^{+\infty} \frac{1}{n^{1+p}} \geq c_8 \frac{\mathbb{E}\|T_N\|}{N^p},
\]

so by (11),

\[
\lim_{n \to +\infty} \frac{\mathbb{E}\|T_n\|}{n^p} = 0.
\]  

(12)

By the conditional version of Jensen’s inequality :

\[
\forall n \in \mathbb{N}^*, \mathbb{E}(\|T_{n+1}\| | G_n) \geq \|T_n\|,
\]

(13)

so :

\[
\sum_{n=1}^{N} \mathbb{E} \left( \frac{\|T_{n+1}\| - \|T_n\|}{(n+1)^{1+p}} \bigg| G_n \right) \leq \sum_{n=1}^{N} \mathbb{E} \left( \frac{\|T_{n+1}\| - \|T_n\|}{(n+1)^{1+p}} \bigg| G_n \right) + c_9 \sum_{n=1}^{N} \frac{\mathbb{E}\|T_n\|}{n^{1+p}}
\]

and by (13) :

\[
\sum_{n=1}^{N} \mathbb{E} \left( \frac{\|T_{n+1}\| - \|T_n\|}{(n+1)^{1+p}} \bigg| G_n \right) \leq \mathbb{E} \left( \frac{\|T_{N+1}\|}{(N+1)^{1+p}} \right) + c_{10} \sum_{n=1}^{N} \frac{\mathbb{E}\|T_n\|}{n^{1+p}}
\]

Finally, by (11) and (12), the sequence \(\left( \frac{T_n}{n^p} \right)\) is a positive quasimartingale. Therefore, thanks to theorem 9.4 in \(\), it converges a.s. to a limit, which, by (12) is necessary 0.

This concludes the proof of proposition 14.

Let us now come back to the proof of the implication \(2 \implies 1\) of theorem 8.
Let \((\varepsilon_k)\) be a sequence of independent Rademacher r.v. and \((x_k)\) be a bounded sequence of elements in \(\mathcal{B}\). Defining \(M := \sup \|x_k\|\), \(X_k := \varepsilon_k x_k\), one gets:

\[
\forall t > M, \; g(t) = \sup_k \mathbb{P}(\|X_k\| > t) = 0,
\]

so condition (3) holds. Therefore \(\frac{1}{n^p} \sum_{k=1}^n \varepsilon_k x_k\) is a quasimartingale, which by proposition 14 converges a.s. to 0. The \(p\)-stability of the space \((\mathcal{B}, \| \cdot \|)\) then follows from theorem 4.

3 What happens when \(p = 1\)?

It is natural to wonder if the spaces of stable type 1 (see [2] for the definition of stable type 1) also admit a “quasimartingale characterization”. In fact it is the case, by theorem 6 in [2], which can be reformulated as follows:

**Theorem 15.** Let \(\mathcal{B}\) be a Banach space. The following two properties are equivalent:

1. \(\mathcal{B}\) is of stable type 1.
2. For every sequence \((X_n)\) of independent, strongly centered r.v., such that

\[
\int_0^{+\infty} g(t) \, dt < +\infty,
\]

where

\[
\forall t > 0, \; g(t) := \sup_{n \geq 1} \mathbb{P}(\|X_n\| \ln(1 + \|X_n\|) > t),
\]

\((\frac{S_n}{n})\) is a quasimartingale.

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