Bessel convolutions on matrix cones

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Abstract

In this paper we introduce probability-preserving convolution algebras on cones of positive semidefinite matrices over one of the division algebras \( \mathbb{F} = \mathbb{R}, \mathbb{C} \) or \( \mathbb{H} \) which interpolate the convolution algebras of radial bounded Borel measures on a matrix space \( M_{p,q}(\mathbb{F}) \) with \( p \geq q \). Radiality in this context means invariance under the action of the unitary group \( U_p(\mathbb{F}) \) from the left. We obtain a continuous series of commutative hypergroups whose characters are given by Bessel functions of matrix argument. Our results generalize well-known structures in the rank one case, namely the Bessel-Kingman hypergroups on the positive real line, to a higher rank setting. In a second part of the paper, we study structures depending only on the matrix spectra. Under the mapping \( r \mapsto \text{spec}(r) \), the convolutions on the underlying matrix cone induce a continuous series of hypergroup convolutions on a Weyl chamber of type \( B_q \). The characters are now Dunkl-type Bessel functions. These convolution algebras on the Weyl chamber naturally extend the harmonic analysis for Cartan motion groups associated with the Grassmann manifolds \( U(p,q)/(U_p \times U_q) \) over \( \mathbb{F} \).

1. Introduction

We start with the basic guiding example which corresponds to the rank-one case of our subsequent constructions. For a natural number \( p \geq 2 \), consider the set \( M_{b}^{\text{rad}}(\mathbb{R}^p) \) of regular bounded Borel measures on \( \mathbb{R}^p \) which are radial, i.e. invariant under orthogonal transformations. \( M_{b}^{\text{rad}}(\mathbb{R}^p) \) is a commutative Banach algebra with the usual convolution of measures. Transferring this structure to \( \mathbb{R}^+ = [0, \infty) \) via the mapping \( x \mapsto |x| = (x_1^2 + \ldots + x_p^2)^{1/2} \), one obtains a commutative Banach algebra of Borel measures on \( \mathbb{R}^+ \). Calculation in polar coordinates shows that its convolution \( *_p \) is determined on point measures by

\[
\delta_r *_p \delta_s(f) = c_p \int_0^\pi f \left( \sqrt{r^2 + s^2 - 2rs \cos \theta} \right) \sin^{p-2} \theta \, d\theta, \quad r, s \in \mathbb{R}^+, \, f \in C(\mathbb{R}^+) 
\]

with a normalization constant \( c_p > 0 \). Now, the above assignment defines a probability measure \( \delta_r *_p \delta_s \) not only for integer \( p \) but for all real \( p > 1 \), and it extends uniquely to a bilinear and weakly continuous convolution on the space \( M_b(\mathbb{R}^+) \) of regular bounded Borel measures on \( \mathbb{R}^+ \) which is commutative, associative, and probability-preserving (\([\text{KI}]\)). The interesting point about this family of convolutions is that analytic properties which are valid for integer indices \( p \), due to their origin in radial analysis on \( \mathbb{R}^p \), remain true for general indices where no longer any group structure is present. For example, consider the normalized Bessel functions.
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\[ j_\alpha(z) = {}_0F_1(\alpha + 1; -z^2/4) \] with index \( \alpha = p/2 - 1 \).

If \( p \) is a natural number, then

\[ j_\alpha(|x|) = \int_{S_{p-1}} e^{-i|\xi||x|} d\sigma(\xi), \quad x \in \mathbb{R}^p \]

\((d\sigma)\) denotes the normalized Lebesgue surface measure on the unit sphere \( S_{p-1} \), and the homomorphism property of the exponential function entails the product formula

\[ j_\alpha(r)j_\alpha(s) = \delta_r \ast_p \delta_s(j_\alpha). \]

This formula, however, extends to arbitrary indices \( p > 1 \), see [W]. The space \( \mathbb{R}_+ \) together with the convolution \( \ast_p \) is a prominent example of a commutative hypergroup, called the Bessel-Kingman hypergroup (BKH). A hypergroup is a locally compact Hausdorff space \( X \) together with a probability preserving convolution of measures on \( X \) which generalizes the measure algebra of a locally compact group; in particular, one requires a unit element and the existence of an involution on \( X \) generalizing the group inverse in a suitable way; for details we refer to [Le] and Section 2.3 below. There is a rich harmonic analysis for commutative hypergroups extending the analysis on locally compact abelian groups. In particular, there is a Haar measure, a dual space, and a Fourier transform satisfying a Plancherel theorem. In our example \( (\mathbb{R}_+, \ast_p) \) the dual space consists of the Bessel functions

\[ \{ \varphi_s(r) = j_\alpha(rs), s \in \mathbb{R}_+ \}, \quad \alpha = \frac{p}{2} - 1, \]

and the hypergroup Fourier transform is given by a Hankel transform,

\[ \hat{f}^p(s) = \frac{2^{-p/2}}{\Gamma(p/2)} \int_{\mathbb{R}_+} f(r)\varphi_s(r)r^{p-1}dr. \]

This extends the fact that the Fourier transform of a radial function \( F(x) = f(|x|) \in L^1(\mathbb{R}^p) \) is again radial and given by a Hankel transform of \( f \) with integral index \( p \).

In the present paper, we generalize the Bessel convolutions described above to a higher rank setting, where the space of “radii” is realized as a cone of positive semidefinite matrices. More precisely, we construct convolution algebras on such cones which interpolate radial convolution algebras on spaces of non-squared matrices. The setting is as follows: For \( q \in \mathbb{N} \) and a natural number \( p \geq q \) consider the space \( M_{p,q} = M_{p,q}(\mathbb{F}) \) of \( p \times q \) matrices over one of the division algebras \( \mathbb{F} = \mathbb{R}, \mathbb{C} \) or the quaternions \( \mathbb{H} \). It has the structure of a Euclidean vector space with scalar product \( (x|y) = \text{Re}tr(x^*y) \) where \( x^* = \overline{x}^t \). A function (or measure) on \( M_{p,q} \) is called radial if it is invariant under the action of the unitary group \( U_p = U_p(\mathbb{F}) \) on \( M_{p,q} \) by left multiplication,

\[ U_p \times M_{p,q} \to M_{p,q}, \quad (u, x) \mapsto ux. \quad (1.1) \]

Observe that this action is via orthogonal transformations, and that \( x \) and \( y \) are contained in the same \( U_p \)-orbit if and only if \( x^*x = y^*y \). Thus the space of \( U_p \)-orbits is naturally parametrized by the cone \( \Pi_q = \Pi_q(\mathbb{F}) \) of positive semidefinite \( q \times q \)-matrices over \( \mathbb{F} \). If \( q = 1 \) and \( \mathbb{F} = \mathbb{R} \), then \( \Pi \) coincides with the nonnegative real line \( \mathbb{R}_+ \) and radiality is equivalent to rotational invariance in \( \mathbb{R}^p \). Generalizing the classical case, there is a radial harmonic analysis on \( M_{p,q} \) which is based on polar coordinates with \( \Pi_q \) as radial part and the Stiefel manifold

\[ \Sigma_{p,q} = \{ x \in M_{p,q} : x^*x = I_q \} \cong U_p/U_{p-q} \]

as a transversal manifold. In [FQ], this is developed to some extent within the general framework.
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of analysis on symmetric cones. Indeed, the open cone
\[ \Omega_q = \{ r \in \Pi_q : r \text{ strictly positive definite} \} \]
is a symmetric cone within the space \( H_q = \{ x \in M_q(\mathbb{F}) : x = x^* \} \) of Hermitian \( q \times q \) matrices over \( \mathbb{F} \) which carries a natural Euclidean Jordan algebra structure of rank \( q \); see Section 2.1 for details.

For each integer \( q \geq p \), we interpret radial analysis on \( M_{p,q} \) in the concise context of a commutative “orbit hypergroup” convolution on the cone \( \Pi_q \) which is derived from the orbit structure w.r.t. the action of \( U_p \) on \( M_{p,q} \). Similar to the rank-one case, the characters of this hypergroup, i.e. the multiplicative functions which make up the dual, are obtained by taking the means of the characters on \( M_{p,q} \) – usual exponential functions – over the Stiefel manifold. This implies that they are given in terms of Bessel functions \( J_\mu \) on the cone \( \Pi_q \) with (half) integer index \( \mu = pd/2 \), where \( d = \dim_\mathbb{F} \mathbb{F} \). The Bessel function of index \( \mu \) is a hypergeometric series of the form
\[
J_\mu(x) = \sum_{\lambda \geq 0} \frac{(-1)^{|\lambda|}}{(|\mu|)_\lambda |\lambda|!} \cdot Z_\lambda(x), \quad x \in H_q;
\]
here the summation goes over all partitions \( \{ \lambda \in \mathbb{N}_0^q : \lambda_1 \geq \ldots \geq \lambda_q \} \), \( (|\lambda|)_\lambda \) is a generalized Pochhammer symbol, and the \( Z_\lambda \) are renormalized versions of the so-called spherical polynomials associated with the underlying cone. They are homogeneous of degree \( |\lambda| \) and satisfy \((\text{trr})^k = \sum_{|\lambda|=k} Z_\lambda(x) \) for all \( k \in \mathbb{N}_0 \); c.f. Section 2.1. The multiplicativity of the Bessel functions \( J_\mu \) with respect to the orbit convolution on \( \Pi_q \) expresses itself in a positive product formula. Under the technical condition \( p \geq 2q \), this product formula can be written in a way which allows analytic continuation with respect to the index \( \mu \). We thus obtain a positive product formula for all Bessel functions with index \( \mu \geq d(q-1/2) \), and an associated continuous series of hypergroup structures \(*_\mu\) on the cone \( \Pi_q \) whose dual is given by the functions
\[
\{ r \mapsto \varphi_s(r) = J_\mu \left( \frac{1}{4} rs^2 r \right), \quad s \in \Pi_q \}.
\]
Actually, each hypergroup \((\Pi_q, *_\mu)\) is self-dual in a natural way; the neutral element is 0 and the involution is the identity mapping. For matrix cones, the Hankel transform of \( F_1 \) can now be identified with the \( L^2\)-Fourier transform on the underlying hypergroup; but in addition to the results of \( \mathbb{F} \), it is also (and primarily) defined as a Fourier transform on \( L^1\)-convolution algebra.

Before continuing with structural aspects, let us spend some words on Bessel functions of matrix argument. They trace back to ideas of Bochner and the fundamental work of Herz \( \mathbb{H} \) and Constantine \( \mathbb{C} \) in the real case. Much of the interest in these functions is motivated by questions in number theory and multivariate statistics. In particular, they occur naturally in relation with non-central Wishart distributions, which generalize non-central \( \chi^2 \)-distributions to the higher rank case, see \( \mathbb{C} \) and \( \mathbb{M} \). Nowadays, Bessel functions of matrix argument are imbedded into rich theories of multivariable special functions. First, they can be considered the \( _0F_1 \) class among general \( pF_q \)-hypergeometric functions of matrix argument, where hypergeometric series are defined in terms of the spherical polynomials; see \( \mathbb{G} \) as well as \( \mathbb{J} \) for an introduction. Second, all this can be done in the general setting of abstract Jordan algebras and symmetric cones, see \( \mathbb{E} \). In any case, the spherical polynomials depend only on the eigenvalues of their argument. Considered as functions of the spectra, they can be identified with Jack polynomials of a certain index depending on the underlying cone; this was first observed
by Macdonald [M2]. There is a natural theory of hypergeometric expansions in terms of Jack polynomials (see [Ka]) which encompasses the theory on symmetric cones. Finally, hypergeometric expansions of such kind are intimately related to the modern theory of hypergeometric functions associated with root systems as developed by Heckman, Opdam, Dunkl and others. As functions of the spectra, Gaussian hypergeometric functions on a symmetric cone can be identified with hypergeometric functions associated with a root system of type $BC$ with a specific choice of parameters ([BO]). Similarly, Bessel functions on a symmetric cone can be considered as a subclass of the Bessel functions associated with reduced root systems of type $B$ in the sense of [O], which play a fundamental role in the theory of rational Dunkl operators [D1], [D2]. This connection goes essentially back to [BF] and is made precise in Section 4.3 below.

In the second part of this paper, we consider structures which depend only on the spectra of the matrices from the underlying cone $\Pi_q$. This amounts to assume invariance under the action of the unitary group $U_q = U_q(\mathbb{F})$ by conjugation,

$$r \mapsto uru^{-1}, \quad u \in U_q.$$ 

The orbits under this action are naturally parametrized by the set $\Xi_q$ of possible spectra of matrices from $\Pi_q$, the eigenvalues being ordered by size:

$$\Xi_q = \{ \xi = (\xi_1, \ldots, \xi_q) \in \mathbb{R}^q : \xi_1 \geq \ldots \geq \xi_q \geq 0 \}.$$ 

$\Xi_q$ is a closed Weyl chamber for the reflection group $B_q$ which acts on $\mathbb{R}^q$ by permutations and sign changes of the coordinates. Via the canonical mapping from $\Pi_q$ onto $\Xi_q$ which assigns to each matrix its spectrum, the continuous series of hypergroup structures $(\Pi_q, \ast_{\mu})$ with $\mu \geq d(q-1/2)$ induces a series of commutative hypergroup structures $\circ_{\mu}$ on the chamber $\Xi_q$. The transfer is established by means of so-called orbital mappings. The convolution, Haar measure and dual space of each hypergroup on the chamber are made explicit. In particular, the Haar measure of $(\Xi_q, \circ_{\mu})$ is (up to a constant factor) given by $h_{\mu}(\xi)d\xi$ where

$$h_{\mu}(\xi) = \prod_{i=1}^{q} \xi_i^{2\gamma+1} \prod_{i<j}(\xi_i^2 - \xi_j^2)^d, \quad \gamma = \mu - \frac{d}{2}(q-1) - 1.$$ 

The hypergroup characters turn out to be certain Dunkl-type Bessel functions

$$\xi \mapsto J_{k}(\xi, i\eta), \quad \eta \in \Xi_q$$

associated with the $B_q$-root system $\{ \pm e_i, \pm e_i \pm e_j \} \subset \mathbb{R}^q$. Here $k$ is a parameter on the root system which is constant on each subset of roots corresponding to a conjugacy class of reflections; in our situation it is given by $k = (k_1, k_2)$ with

$$k_1 = \mu - \frac{d}{2}(q-1) - \frac{1}{2} \quad \text{on} \quad \pm e_i; \quad k_2 = \frac{d}{2} \quad \text{on} \quad \pm e_i \pm e_j.$$ 

The hypergroup convolution on the Weyl chamber matches the generalized Dunkl translation ([R1]) for Weyl group invariant functions, and we have an interpretation of the Dunkl transform as a hypergroup Fourier transform.

In the geometric cases $\mu = pd/2$, the support of the probability measure $\delta_\xi \circ_{\mu} \delta_\eta$ on $\Xi_q$ describes the set of possible singular spectra of sums $x + y$ with matrices $x, y \in M_{p,q}$ having given singular spectra $\xi$ and $\eta$. Further, the characters are just the bounded spherical functions of the Euclidean type Riemannian symmetric space $(U_p \times U_q) \ltimes M_{p,q} / (U_p \times U_q)$ associated with the Grassmann manifold $U(p,q)/U_p \times U_q$. For general $\mu$ they are characterized, within the theory
of rational Dunkl operators, as the unique analytic solution of a so-called Bessel system; see [O].

It is conjectured that for arbitrary root systems and non-negative multiplicities, the associated
Dunkl-type Bessel functions satisfy a positive product formula and can be characterized as
the characters of a commutative hypergroup structure on the underlying Weyl chamber. The
three continuous series \((d = 1, 2, 4)\) for \(B_q\) obtained in this paper are, to our knowledge, the
first affirmative examples beyond the group cases associated with Cartan motion groups of
reductive symmetric spaces. Some background, together with further partial results, is given in [R2].

The organization of the paper is as follows: In Section 2, background on symmetric cones and
Bessel functions on cones, as well as some hypergroup analysis are provided. Section 3 is devoted
to the study of Bessel convolutions on matrix cones. In Subsection 3.1., orbit convolutions
derived from matrix spaces \(M_{p,q}\) are considered. In 3.3., the corresponding product formula for
the involved Bessel functions is analytically extended with respect to the index, and in 3.4. the
associated series of hypergroup convolutions on the cone are studied. In particular, their Haar
measure and the dual are determined. In 3.5. we analyse an interesting critical index. Section 4
is devoted to the induced convolution algebras on a Weyl chamber of type \(B\). They are derived
in 4.1. from the convolutions on the matrix cones and are then, in the two final subsections,
put into relation to rational Dunkl theory.

2. Preliminaries

In this introductory chapter we provide some relevant background on symmetric cones, in
particular matrix cones, and about Bessel functions on such cones. In the main part of the
paper, we shall introduce orbit convolutions and their “interpolations” on matrix cones within
the framework of hypergroup theory. As a preparation, a short account on the relevant notions
and facts is included in the present section. For a general background on hypergroups, the reader
is referred to the fundamental article [Je] (where the notion ”conv” is being used instead of
”hypergroup”), or to the monograph [BH]. An excellent reference for analysis on symmetric
cones is the book [FK], for special functions on matrix cones see also [GR1] as well as the
classical papers of Herz, James and Constantine, [H], [Ja], [Co].

2.1 Analysis on symmetric cones

Let \(\mathbb{F}\) be one of the division algebras \(\mathbb{F} = \mathbb{R}, \mathbb{C}\) or the quaternions \(\mathbb{H}\). We denote by \(t \mapsto \overline{t}\)
the usual conjugation in \(\mathbb{F}\) and by \(\Re t = \frac{1}{2}(t + \overline{t})\) the real part of \(t \in \mathbb{F}\). Consider the set of
Hermitian \(q \times q\)-matrices over \(\mathbb{F}\),

\[
H_q = H_q(\mathbb{F}) = \{x \in M_q(\mathbb{F}) : x = x^*\}; \quad x^* = \overline{\text{tr}}.
\]

We regard \(H_q\) as a Euclidean vector space with scalar product \((x|y) = \Re \text{tr}(xy)\), where \(\text{tr}\) denotes the trace on \(M_q(\mathbb{F})\). The dimension of \(H_q\) over \(\mathbb{R}\) is

\[
n = q + \frac{d}{2}q(q - 1), \quad d = \dim_{\mathbb{R}} \mathbb{F}.
\]

With the above scalar product and the Jordan product \(x \circ y = \frac{1}{2}(xy + yx)\), the matrix space
\(H_q\) becomes a Euclidean (equivalently, a formally real) Jordan algebra with unit \(I = I_q\), the
unit matrix. The rank of \(H_q\), i.e. the number of elements of each Jordan frame in \(H_q\), is \(q\).

The set \(\Omega_q = \Omega_q(\mathbb{F})\) of positive definite matrices from \(H_q\) is a symmetric cone. Recall that a
symmetric cone $\Omega$ is a proper, non-empty convex cone in a finite-dimensional Euclidean vector space which is self-dual and homogeneous in the sense that its group of linear automorphisms acts transitively. Let $G$ denote the connected component of this group. Then $K = G \cap O(V)$ is a maximal compact subgroup of $G$ and $\Omega \cong G/K$, a Riemannian symmetric space. The matrix cones are realized as $\Omega_q(\mathbb{F}) \cong GL_q(\mathbb{F})/U_q(\mathbb{F})$. Hereby $GL_q(\mathbb{F})$ acts via $v \mapsto gp^t$, which reduces to conjugation when restricted to the unitary subgroup $U_q(\mathbb{F})$. Our main interest will be in the closure of $\Omega_q$ relative to $H_q$ which coincides with the set of positive semidefinite matrices over $\mathbb{F}$,

$$\Pi_q = \Pi_q(\mathbb{F}) = \{x^*x : x \in H_q\} = \{x^2 : x \in H_q\}.$$  

For a general Euclidean Jordan algebra $V$, the interior $\Omega$ of the set $\{x^2 : x \in V\}$ is a symmetric cone, and each symmetric cone in a finite-dimensional Euclidean vector space $V$ can be realized in such a way. For details see Section III.3 of [FR]. The simple Euclidean Jordan algebras correspond to the irreducible symmetric cones and are classified. Up to isomorphism, there are the above series $H_q(\mathbb{F})$ with $\mathbb{F} = \mathbb{R}, \mathbb{C}, \mathbb{H}$, the exceptional Jordan algebra $H_3(\mathbb{O})$, as well as one infinite series of rank-2 algebras corresponding to the Lorentz cones

$$\Lambda_n = \{(x', x_n) \in \mathbb{R} \times \mathbb{R}^{n-1} : x_n^2 - |x'|^2 > 0, x_n > 0\}.$$  

Let $V$ be a simple Euclidean Jordan algebra of rank $q$. We recall that for each $x \in V$ there is a Jordan frame $\{e_1, \ldots, e_q\}$ such that $x = \sum_{i=1}^q \xi_i e_i$ with real numbers $\xi_i$. Up to ordering, the $\xi_i$ are uniquely determined and are called the eigenvalues of $x$. Notice that $x \in \Omega$ iff all its eigenvalues are positive. The trace and determinant of $x$ are defined by

$$\text{tr} x = \sum_{i=1}^q \xi_i, \quad \Delta(x) = \prod_{i=1}^q \xi_i.$$  

In the Jordan algebras $H_q(\mathbb{F})$, the function $\Delta$ coincides with the usual determinant det if $\mathbb{F} = \mathbb{R}$ or $\mathbb{C}$ while for $\mathbb{F} = \mathbb{H}$ we have $\Delta(x) = (\det x)^{1/2}$ when $x$ is being considered as a complex $2 \times 2$-matrix in the standard way.

The basic functions for the harmonic analysis on a symmetric cone, and at the same time the basic constituents for all kinds of hypergeometric functions on it, are the so-called spherical polynomials. To write them down, we have to introduce some more notation:

Consider a fixed Jordan frame $\{e_1, \ldots, e_q\}$ of $V$. For $1 \leq j \leq q$ and $x \in V$ we denote by $\Delta_j(x)$ the principal minors of $\Delta(x)$ with respect to this frame; in particular $\Delta_q(x) = \Delta$ and each $\Delta_j$ is a polynomial function on $V$ which is positive on the cone $\Omega$. Recall that a $q$-tuple $\lambda = (\lambda_1, \ldots, \lambda_q) \in \mathbb{N}_0^q$ is called a partition if $\lambda_1 \geq \lambda_2 \geq \cdots \geq \lambda_q > 0$. The weight $|\lambda|$ of $\lambda$ is defined as $|\lambda| = \lambda_1 + \ldots + \lambda_q$. Following a standard convention, we write $\lambda \geq 0$ in order to indicate that $\lambda$ is a partition. For a partition $\lambda$, the power function $\Delta_\lambda$ on $V$ is defined by

$$\Delta_\lambda(x) = \Delta_1(x)^{\lambda_1} \Delta_2(x)^{\lambda_2} \cdots \Delta_q(x)^{\lambda_q},$$  

It is a homogeneous polynomial of degree $|\lambda|$ and positive on $\Omega$. In particular, if $x = \sum_{i=1}^q \xi_i e_i$ with $\xi_1, \ldots, \xi_q \in \mathbb{R}$ then $\Delta_j(x) = \xi_1 \cdots \xi_j$ and $\Delta_\lambda(x) = \xi_1^{\lambda_1} \cdots \xi_q^{\lambda_q}$. The spherical polynomials of $\Omega$ are indexed by partitions and defined by

$$\Phi_\lambda(x) = \int_K \Delta_\lambda(kx) dk, \quad \lambda \geq 0, \ x \in V$$  

where $dk$ is the normalized Haar measure on $K$. Notice that $\Phi_\lambda(e) = 1$. $\Phi_\lambda$ is $K$-invariant and therefore depends only on the eigenvalues of its argument. The $\Phi_\lambda$ are just the polynomial spherical functions of $\Omega \cong G/K$. For the matrix cones over $\mathbb{R}$ they are known as zonal
polynomials, for those over \( C \) they coincide with the Schur polynomials. Actually, for each symmetric cone the spherical polynomials are given in terms of Jack polynomials \( [\text{Sta}] \). This has been first observed by Macdonald \( [\text{M2}] \) (see also \( [\text{F}] \) and the notes in \( [\text{FK}] \), Chapt. XI). More precisely, let us consider the renormalized polynomials

\[
Z_\lambda(x) := d_\lambda \frac{|\lambda|!}{(n/q)_\lambda} \cdot \Phi_\lambda(x)
\]

where \( d_\lambda \) denotes the dimension of the vector space of polynomials on \( V \subseteq C \) which is generated by the elements \( z \mapsto \Delta_\lambda(g^{-1}z) \), \( g \in G \). (c.f. Sect.XI.5. of \( [\text{FK}] \)). Let further \( C^\alpha_\lambda \) denote the Jack polynomials of index \( \alpha > 0 \) \( [\text{Sta}] \), normalized such that

\[
(\xi_1 + \ldots + \xi_q)^k = \sum_{\lambda \geq 0, |\lambda| = k} C^\alpha_\lambda(\xi) \quad \forall k \in \mathbb{N}_0
\]

(c.f. \( [\text{Ka}] \)). Then for \( x \in V \) with eigenvalues \( \xi = (\xi_1, \ldots, \xi_q) \) we have

\[
Z_\lambda(x) = C^\alpha_\lambda(\xi)
\]

where \( \alpha = 2/d \) with \( d = \dim_\mathbb{R} F \) for the matrix cones over \( F \) and \( d = n-2 \) for the Lorentz cone \( \Lambda_n \).

In normalization constants, we shall also need the gamma function of a symmetric cone,

\[
\Gamma_\Omega(z) = \int_{\Omega} e^{-trx} \Delta(x) z^{-n/q} dx
\]

which converges absolutely for \( z \in \mathbb{C} \) with \( \Re z > \frac{d}{2}(q-1) = \frac{n}{q} - 1 \) and can be written in terms of the classical gamma function as

\[
\Gamma_\Omega(z) = (2\pi)^{(n-q)/2} \prod_{j=1}^{q} \Gamma\left(z - \frac{d}{2}(j-1)\right)
\]

see \( [\text{FK}] \), Chapt. VII.1.

2.2 Bessel functions on a symmetric cone

Hypergeometric expansions in terms of spherical polynomials have a long history in multivariate statistics, tracing back to the work of Herz \( [\text{H}] \), James \( [\text{Ja}] \) and Constantine \( [\text{Co}] \). They are important in the study of Wishart distributions and for questions related with total positivity \( ([\text{GR2}] \). In view of the above connection between spherical polynomials and Jack polynomials it is natural to treat these classes of functions in the more general framework of multivariable hypergeometric functions based on Jack polynomial expansions (see \( [\text{Ka}] \)). In our context, only hypergeometric functions of type \( {}_0F_1 \) (which are Bessel functions) will be relevant.

Let \( \alpha > 0 \) be a fixed parameter. For partitions \( \lambda = (\lambda_1, \ldots, \lambda_q) \) we introduce the generalized Pochhammer symbol

\[
(c)_\lambda^\alpha = \prod_{j=1}^{q} \left(c - \frac{1}{\alpha}(j-1)\right)_{\lambda_j}, \quad c \in \mathbb{C}.
\]

For an index \( \mu \in \mathbb{C} \) satisfying \( (\mu)_\lambda^\alpha \neq 0 \) for all \( \lambda \geq 0 \) (in particular for \( \mu \) with \( \Re \mu > \frac{d}{2}(q-1) \)), the generalized hypergeometric function \( {}_0F_1^\alpha(\mu; \cdot) \) on \( \mathbb{C}^q \) is defined by

\[
{}_0F_1^\alpha(\mu; \xi) = \sum_{\lambda \geq 0} \frac{1}{(\mu)_\lambda^\alpha |\lambda|!} \cdot C^\alpha_\lambda(\xi).
\]
It is known (Kn) that this series converges absolutely for all \( \xi \in \mathbb{C}^q \). Similarly, a \(_0F_1\)-hypergeometric function of two arguments is defined by

\[
_0F_1^\alpha(\mu; \eta) = \sum_{\lambda \geq 0} \frac{1}{(\mu)_\lambda^\alpha |\lambda|!} \cdot \frac{C_\lambda^\alpha(\xi)C_\lambda^\alpha(\eta)}{C_\lambda^\alpha(1)}, \quad \alpha = (1, \ldots, 1).
\]

Now suppose that \( \alpha = 2/d \) where \( d \) is the dimension constant of a simple Euclidean Jordan algebra \( V \) of rank \( q \) corresponding to the symmetric cone \( \Omega \). Then the Bessel function \(_0F_1^\alpha(\mu; \cdot)\) essentially coincides with the Bessel function \( \mathcal{J}_\mu \) associated with \( \Omega \) in the sense of [FK]. Indeed, the latter is defined by

\[
\mathcal{J}_\mu(x) = \sum_{\lambda \geq 0} \frac{(-1)^{|\lambda|}}{(\mu)_\lambda^\alpha |\lambda|!} Z_\lambda(x) \quad \text{for} \quad x \in V; \quad \alpha = 2/d \tag{2.2}
\]

Thus for \( x \in V \) with eigenvalues \( \xi = (\xi_1, \ldots, \xi_q) \) we have

\[
\mathcal{J}_\mu(x) = _0F_1^{2/d}(\mu; -\xi). \tag{2.3}
\]

In Section 4.1 we shall also work with Bessel functions of two arguments \( x, y \in V \),

\[
\mathcal{J}_\mu(x, y) := \sum_{\lambda \geq 0} \frac{(-1)^{|\lambda|}}{(\mu)_\lambda^\alpha |\lambda|!} Z_\lambda(x)Z_\lambda(y) \quad \text{where} \quad e \text{ is the unit of } V.
\]

where \( e \) is the unit of \( V \). For \( x, y \in V \) with eigenvalues \( \xi = (\xi_1, \ldots, \xi_q) \) and \( \eta = (\eta_1, \ldots, \eta_q) \) respectively, we thus have

\[
\mathcal{J}_\mu(x, y) = _0F_1^{2/d}(\mu; i\xi, i\eta). \tag{2.4}
\]

The following estimate is especially useful for large indices \( \mu \).

**Lemma 2.1.** For \( \alpha > 0 \) let \(_0F_1^\alpha(\mu; \cdot)\) denote the associated generalized Bessel function of index \( \mu \) on \( \mathbb{C}^q \). Suppose that \( \Re \mu > \frac{1}{\alpha}(q - 1) \). Then for \( \xi = (\xi_1, \ldots, \xi_q) \in \mathbb{C}^q \),

\[
|_0F_1^\alpha(\mu; \xi)| \leq e^{(\xi_1 + \ldots + |\xi_q|).}
\]

**Proof.** For \( n \in \mathbb{N}_0 \) and \( z \in \mathbb{C} \) with \( \Re z > 0 \) we have

\[
\left| \frac{\Gamma(z)}{\Gamma(z + n)} \right| \leq \frac{\Gamma(\Re z)}{\Gamma(\Re z + n)} \leq 1.
\]

Therefore \( |(\mu)_\lambda^\alpha| \geq |(\Re \mu)_\lambda^\alpha| \geq 1 \). Moreover, it is known that the coefficients of the Jack polynomial \( C_\lambda^\alpha \) in its monomial expansion are all non-negative (see KnS). This implies that

\[
|C_\lambda^\alpha(\xi)| \leq C_\lambda^\alpha(|\xi_1|, \ldots, |\xi_q|).
\]

Using relation (2.2), we therefore obtain

\[
|_0F_1^\alpha(\mu; \xi)| \leq \sum_{\lambda \geq 0} \frac{1}{|\lambda|!} C_\lambda^\alpha(|\xi_1|, \ldots, |\xi_q|) = e^{(\xi_1 + \ldots + |\xi_q|).
\]

**Corollary 2.2.** Let \( \mathcal{J}_\mu \) denote the Bessel function of index \( \mu \) associated with the symmetric cone \( \Omega \) inside the Jordan algebra \( V \) of rank \( q \). Suppose that \( \Re \mu > \frac{1}{\alpha}(q - 1) \). Then for \( x \in V \) with eigenvalues \( \xi_1, \ldots, \xi_q \),

\[
|\mathcal{J}_\mu(x)| \leq e^{(\xi_1 + \ldots + |\xi_q|).
\]
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We mention at this point that for \( \Re \mu > d(q - 1) + 1 \) and \( a \in \Omega \), the Bessel function on \( \Omega \) has the absolutely convergent integral representation (an inverse Laplace integral)

\[
J_\mu(r) = \frac{\Gamma_\Omega(\mu)}{(2\pi i)^n} \int e^{trw} e^{-(w^{-1}|r|)} \Delta(w)^{-\mu} dw \quad (r \in \Omega)
\]

see Prop. XV.2.2. of [FK]. This integral was originally used by Herz [H] to define Bessel functions on the cones of positive definite matrices over \( \mathbb{R} \); see also [FT] and [Di] for arbitrary symmetric cones. Corollary 2.2 slightly improves the order estimate for \( J_\mu \) given in [H], p.486 for \( \mathbb{F} = \mathbb{R} \).

### 2.3 Hypergroups

We start with some notation: For a locally compact Hausdorff space \( X \), let \( M_b(X) \) denote the Banach space of all bounded regular (complex) Borel measures on \( X \) with total variation norm, and \( M^1(X) \subset M_b(X) \) the set of all probability measures. With \( \delta_x \) we denote the point measure in \( x \in X \). We use the notions \( C(X), C_b(X) \) and \( C_c(X) \) for the spaces of continuous complex-valued functions on \( X \), those which are bounded, and those having compact support respectively. Further, \( C_0(X) \) is the set of functions from \( C(X) \) which vanish at infinity.

**Definition 2.3.** A hypergroup \((X, \ast)\) is a locally compact Hausdorff space \( X \) with a bilinear and associative convolution \(* \) on \( M_b(X) \) with the following properties:

1. The map \((\mu, \nu) \mapsto \mu \ast \nu\) is weakly continuous, i.e. w.r.t. the topology induced by \( C_b(X) \).
2. For all \( x, y \in X \), the product \( \delta_x \ast \delta_y \) of point measures is a compactly supported probability measure on \( X \).
3. The mapping \((x, y) \mapsto \text{supp}(\delta_x \ast \delta_y)\) from \( X \times X \) into the space of nonempty compact subsets of \( X \) is continuous with respect to the Michael topology (see [Je]).
4. There is a neutral element \( e \in X \), satisfying \( \delta_e \ast \delta_x = \delta_x \ast \delta_e = \delta_x \) for all \( x \in X \).
5. There is a continuous involutive automorphism \( x \mapsto \overline{x} \) on \( X \) such that \( \delta_x \ast \delta_{\overline{x}} = (\delta_y \ast \delta_x)^{-} \) and \( x = y \iff e \in \text{supp}(\delta_x \ast \delta_y) \). (For \( \mu \in M_b(X) \), the measure \( \mu^{-} \) is given by \( \mu^{-}(A) = \mu(\overline{A}) \) for Borel sets \( A \subseteq X \)).

A hypergroup \((X, \ast)\) is called commutative if its convolution is commutative.

For a hypergroup \((X, \ast)\), the space \((M_b(X), \ast)\) is a Banach algebra with unit \( \delta_e \). Notice that by the density of the finitely supported measures in \( M_b(X) \), the convolution is uniquely determined as soon as it is given for point measures.

Of course every locally compact group is a hypergroup with the usual group convolution. In our context, only commutative hypergroups will be relevant. Of particular interest will be the following

**Example 2.4.** ([Je], Chapt. 8) Let \((G, +)\) be a locally compact abelian group and \( K \) a compact subgroup of \( \text{Aut} \, G \). Then the space \( G^K = \{ K.x : x \in G \} \) of \( K \)-orbits in \( G \) is a locally compact Hausdorff space with the quotient topology and becomes a commutative hypergroup with the (natural) definition

\[
(\delta_{K.x} \ast \delta_{K.y})(f) = \int_K f(K.(x + ky))dk, \quad f \in C_c(G^K).
\]

\((G^K, \ast)\) is called a orbit hypergroup; its neutral element is \( K.0 = 0 \) and the involution is \( (K.x)^{-} = K.(-x) \).


In the following, we collect some further ingredients underlying the harmonic analysis on a commutative hypergroup \( X \). There exists (up to normalization) a unique Haar measure \( \omega \) on \( X \), i.e., a positive Radon measure satisfying

\[
\int_X f(x * y) d\omega(y) = \int_X f(y) d\omega(y) \quad \forall \, x \in X, \, f \in C_c(X)
\]

where we use the notation

\[
f(x * y) := \int_X f(\delta_x * \delta_y).
\]

Similar to the dual of a locally compact abelian group, one defines the dual space of \( X \), i.e., a positive Radon measure satisfying

\[
\hat{\varphi}(x) := \int_X \varphi(x) d\mu(x), \quad \varphi \in \hat{X},
\]

and on \( L^1(X, \omega) \) by \( \hat{f} := \hat{f_\omega} \). The Fourier transform is injective, and there exists a unique positive Radon measure \( \pi \) on \( \hat{X} \), called the Plancherel measure of \((X, \ast)\), such that \( f \mapsto \hat{f} \) extends to an isometric isomorphism from \( L^2(X, \omega) \) onto \( L^2(\hat{X}, \pi) \). While the Haar measure \( \omega \) has full support, the support of \( \pi \) may be a proper subset of \( \hat{X} \) only.

**Example 2.5.** [Continuation of example 2.4] For a Haar measure \( m \) on the group \( G \), the image measure of \( m \) under the canonical map \( \pi : G \to G^K \) provides a Haar measure on the orbit hypergroup \( G^K \). Further, it is easily seen that the functions

\[
\varphi_\alpha(K,x) := \int_K \alpha(k.x) dk, \quad \alpha \in \hat{G},
\]

belong to the dual of \( G^K \). Actually, we have

**Lemma 2.6.**

1. \( \hat{G^K} = \{ \varphi_\alpha : \alpha \in \hat{G} \} \).

2. \( \varphi_\alpha = \varphi_{\alpha'} \iff \alpha \text{ and } \alpha' \text{ are contained in the same orbit under the dual action of } K \text{ on } \hat{G}, \)

   given by \((k,\alpha')(x) = \alpha(k^{-1}.x)\)

**Proof.** It is known that \( \hat{G^K} \) can be identified with the extremal points of the set

\[
\Xi = \{ \beta \in C(G) : \beta \text{ positive definite and } K \text{-invariant with } \beta(0) = 1 \};
\]

the identification being given by \( \beta \mapsto \varphi_\beta, \varphi_\beta(K,x) = \beta(x) \text{ for } x \in G, \text{ see } \text{Ro}. \) By Bochner’s theorem, each \( \hat{\beta} \in \Xi \) is of the form \( \hat{\beta}(x) = \int_G \alpha(x) d\mu(\alpha) \) with some \( K \)-invariant probability measure \( \mu \) on \( \hat{G} \). In this way, the extremal points of \( \Xi \) correspond to the measures of the form

\[
\mu_\alpha = \int_K \delta_{k,\alpha} dk \quad \alpha \in \hat{G}.
\]

Further, \( \mu_\alpha = \mu_{\alpha'} \iff \alpha \text{ and } \alpha' \text{ belong to the same } K\text{-orbit in } \hat{G}. \)

\[\square\]

A commutative hypergroup \((X, \ast)\) is called self-dual if there exists a homeomorphism \( \Psi : X \to \hat{X} \) such that

\[
\Psi(x)(z)\Psi(y)(z) = \int_X \Psi(w)(z) d(\delta_x * \delta_y)(w) \quad \forall \, x, y, z \in X.
\]
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In that case, \( \hat{X} \) is a dual hypergroup in the sense of [Je], 12.4, where the convolution product \( \delta_{\Psi(x)} * \delta_{\Psi(y)} \) is just the image measure of \( \delta_x * \delta_y \) under \( \Psi \). It is easily checked that the image measure \( \Psi(\omega) \) of the Haar measure \( \omega \) on \( X \) is a Haar measure on \( \hat{X} \). Thus by Theorem 12.4A of [Je], \( \Psi(\omega) \) coincides up to a multiplicative constant with the Plancherel measure \( \pi \) of \( X \).

3. Convolution structures associated with Bessel functions on a matrix cone

3.1 Orbit hypergroups on a matrix cone

For natural numbers \( p \geq q \), consider the matrix space \( M_{p,q} = M_{p,q}(\mathbb{F}) \) of \( p \times q \)-matrices over \( \mathbb{F} \). We regard \( M_{p,q} \) as a real vector space, equipped with the Euclidean scalar product \( \langle x | y \rangle := \text{Re} \text{tr}(x^*y) \) and norm \( \| x \| = \sqrt{\text{tr}(x^*x)} \). Here \( \text{Re} t = \frac{1}{2}(t + \overline{t}) \) is the real part of \( t \in \mathbb{F} \) and \( \text{tr} \) denotes the trace in \( M_q(\mathbb{F}) \). In the square case \( p = q, \| \cdot \| \) is just the Frobenius norm. Let us consider the action of the unitary group \( U_p \) on \( M_{p,q} \) by left multiplication,

\[ U_p \times M_{p,q} \rightarrow M_{p,q}, \quad (u, x) \mapsto ux. \]

The orbit space \( M_{p,q}^{U_p} \) for this action can be identified with the space \( \Pi_q = \Pi_q(\mathbb{F}) \) of positive semidefinite \( q \times q \) matrices over \( \mathbb{F} \) via

\[ U_p x \mapsto \sqrt{x^*x} =: |x|. \]

Here for \( r \in \Pi_q \), \( \sqrt{r} \) denotes the unique positive semidefinite square root of \( r \). It is easy to see that the above bijection becomes a homeomorphism when \( M_{p,q}^{U_p} \) is equipped with the quotient topology and \( \Pi_q \) with the subspace topology induced from \( M_q \). (Indeed, both the canonical map \( x \mapsto U_p x \) from \( M_{p,q} \) onto \( M_{p,q}^{U_p} \) and the mapping \( x \mapsto \sqrt{x^*x} \), \( M_{p,q} \rightarrow \Pi_q \) are open and continuous.) Notice that the Stiefel manifold

\[ \Sigma_{p,q} = \{ x \in M_{p,q} : x^*x = I_q \} \]

is the orbit of the block matrix

\[ \sigma_0 := \begin{pmatrix} I_q \\ 0 \end{pmatrix} \in M_{p,q}. \]

Before calculating the orbit hypergroup convolution for the above action, let us recall the basic aspects of radial analysis in \( M_{p,q} \) as developed in [FT]. As in the introduction, radiality here means invariance under the above action of \( U_p \); a function \( F \) on \( M_{p,q} \) is radial iff it is of the form \( F(x) = f(|x|) \) for some \( f : \Pi_q \rightarrow \mathbb{C} \). Suitable polar coordinates in \( M_{p,q} \) are defined as follows: Let \( d\sigma \) denote the \( U_p \)-invariant measure on \( \Sigma_{p,q} \), normalized according to \( \int_{\Sigma_{p,q}} d\sigma = 1 \), and let

\[ M_{p,q}' = \{ x \in M_{p,q} : \Delta(x^*x) \neq 0 \} \]

which is open and dense in \( M_{p,q} \). Then the mapping

\[ \Omega_q \times \Sigma_{p,q} \rightarrow M_{p,q}', \quad (r, \sigma) \mapsto \sigma \sqrt{r} \]

is a diffeomorphism, and for integrable functions \( f : M_{p,q} \rightarrow \mathbb{C} \) one has

\[ \int_{M_{p,q}} f(x) dx = C_{p,q} \int_{\Omega_q} \int_{\Sigma_{p,q}} f(\sigma \sqrt{r}) \Delta(r)^\gamma dr d\sigma \]

with

\[ C_{p,q} = \frac{\pi^{dpq/2}}{\Gamma_{\Omega_q}(\frac{dp}{2})} \quad \text{and} \quad \gamma = \frac{dp}{2} - \frac{n}{q}, \quad (3.1) \]
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As before, \( n \) is the real dimension of \( H_q \). Our notion slightly differs from that of [FK] (and the monograph [FT]); it is adapted to the left action \( (\cdot) \) of \( U_p \) on \( M_{p,q} \) while the notion of [FT] would require to have \( U_p \) acting from the right on \( M_{q,p} \). As Lebesgue measure on \( M_{p,q} \) is \( U_p \)-invariant, the Fourier transform of a radial function in \( L^1(M_{p,q}) \) is again radial, and also the convolution of two radial functions is radial; both can be calculated by use of polar coordinates, c.f. [FT]. We shall come back to this shortly. Following Example 2.4, we obtain the orbit hypergroup convolution

\[
(\delta_r \ast \delta_s)(f) = \int_{U_p} f(|\sigma_0 r + u\sigma_0 s|) du, \quad r, s \in \Pi_q.
\]

The image measure of \( du \) under the mapping \( U_p \to \Sigma_{p,q}, u \mapsto u\sigma_0 \) is \( U_p \)-invariant and therefore coincides with the \( U_p \)-invariant measure \( d\sigma \). Hence,

\[
(\delta_r \ast \delta_s)(f) = 
\int_{\Sigma_{p,q}} f(|\sigma_0 r + \sigma s|) d\sigma = 
\int_{\Sigma_{p,q}} f\left(\sqrt{r^2 + s^2 + r\bar{s}s + (s\bar{r}r)}\right) d\sigma
\]

(3.2)

where \( \bar{s} = \sigma_0^t \sigma \) is the \( q \times q \)-matrix whose rows are given by the first \( q \) rows of \( \sigma \). Actually, this convolution depends on \( p \) (and \( q \)), which we suppress for the moment. The neutral element of the orbit hypergroup \( (\Pi_q, \ast) \) is 0, and the involution is the identity mapping (because \( x \in M_{p,q} \) and \(-x\) are in the same \( U_p\)-orbit). Further, according to Lemma 2.6, the dual space of \( (\Pi_q, \ast) \) consists of the functions \( \varphi_s, s \in \Pi_q \) with

\[
\varphi_s(r) = 
\int_{U_p} e^{-i(r\sigma_0 r + \mu s |r\sigma_0 s|)} du = 
\int_{\Sigma_{p,q}} e^{-i(r\sigma|s\sigma r)} d\sigma.
\]

These are Bessel functions. Indeed, according to Propos. XVI.2.3. of [FK] we have for \( x \in M_{p,q} \) the identity

\[
\int_{\Sigma_{p,q}} e^{-i(r\sigma|s\sigma)} d\sigma = \mathcal{J}_\mu\left(\frac{1}{4}s|x|\right), \quad \mu = \frac{pd}{2}
\]

(3.4)

where \( \mathcal{J}_\mu \) is the Bessel function of index \( \mu \) associated with the symmetric cone \( \Omega_q \) as in Section 2.2.2. Thus

\[
\varphi_s(r) = \mathcal{J}_\mu\left(\frac{1}{4}rs^2 r\right).
\]

We mention at this point that up to a constant factor, \( \varphi_s(r) \) coincides with the Bessel function \( J(r^2, s^2) \) in [FK]. As \( \mathcal{J}_\mu(x) \) depends only on the eigenvalues of \( x \) and as the matrices \( rs^2 r \) and \( sr^2 s \) have the same eigenvalues, we see that

\[
\varphi_s(r) = \varphi_r(s) \quad \forall r, s \in \Pi_q.
\]

This implies that the hypergroup \( (\Pi_q, \ast) \) is self-dual via \( r \mapsto \varphi_r \). In order to stress the dependence of its convolution on \( p \) (or equivalently, on the index \( \mu = pd/2 \)), we denote it by \( *_\mu \) from now on and write \( \Pi_{q,\mu} \) for the orbit hypergroup \( (\Pi_q, *_\mu) \).

A Haar measure \( \omega_\mu \) on the hypergroup \( \Pi_{q,\mu} \) is obtained by taking the image measure of the (normalized) Lebesgue measure \( (2\pi)^{-pqd/2}dx \) on \( M_{p,q} \) under the mapping \( x \mapsto |x| \). Using again polar coordinates, we obtain

\[
\omega_\mu(f) = \frac{2^{-\mu q}}{\Gamma_{\Omega_q}(\mu)} \int_{\Omega_q} f(\sqrt{r})\Delta(r)^n dr, \quad f \in C_c(\Pi_q).
\]

Fourier transform and convolution of radial functions on \( M_{p,q} \) are calculated in our notion a follows: Suppose \( F, G \in L^1(M_{p,q}) \) are radial with \( F(x) = f(|x|) \) and \( G(x) = g(|x|) \). Then the
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The Fourier transform of $F$ is

$$\hat{F}(\lambda) = \frac{1}{(2\pi)^{dpq/2}} \int_{M_{p,q}} F(x) e^{-i(\lambda|x|)} dx = \int_{\Omega_q} f(r) \varphi_{|\lambda|}(r) d\omega_{\mu}(r) = \hat{f}(|\lambda|).$$

The convolution of $F$ and $G$ is given by $F * G(x) = H(|x|)$ with

$$H(r) = (2\pi)^{-dpq/2} \int_{M_{p,q}} F(\sigma_0 r - y) G(y) dy = \frac{2^{-\mu q}}{\Gamma_{\Omega_q}(\mu)} \int_{\Omega_q} \left( \int_{\Sigma_{p,q}} f(\sigma_0 r - \sigma \sqrt{s}) \right) G(\sqrt{s}) \Delta(s)^\mu ds$$

$$= \int_{\Omega_q} \delta_r * \tilde{\delta}_s(f) g(s) d\omega_{\mu}(s) = (f * g)(r).$$

The multiplicativity of the characters $\varphi_s$ implies a positive product formula for the Bessel functions $J_\mu$ with index $\mu = pd/2$, $p \geq q$ an integer:

$$J_\mu(r^2) J_\mu(s^2) = \int_{\Sigma_{p,q}} J_\mu(r^2 + s^2 + r \bar{s} + s \bar{r}^* r) d\sigma \quad \forall r, s \in \Omega_q. \quad (3.5)$$

We shall generalize this formula to Bessel functions of arbitrary index $\mu \in \mathbb{C}$ with $\Re \mu > d(q - 1/2)$. For real indices we shall obtain a positive product formula, which leads to a continuous family of hypergroup structures on $\Omega_q$ beyond those which have a realization as orbit hypergroups as above. The decisive observation towards this aim is that only the reduced matrix $\tilde{\sigma}$ occurs in the integrands of (3.5) and (3.2). In the following section, we shall introduce coordinates on the Stiefel manifold which are adapted to this situation.

### 3.2 Split coordinates on the Stiefel manifold

Let $k \in \mathbb{N}$ with $p - k \geq q$. We decompose $\sigma \in \Sigma_{p,q}$ as $\sigma = \begin{pmatrix} v \\ w \end{pmatrix}$ with $v \in M_{k,q}$ and $w \in M_{p-k,q}$. For fixed $q$, put

$$D_k = \{ v \in M_{k,q} : v^* v < I \}$$

where the notion $x < y$ for $x, y \in M_q(\mathbb{F})$ means that $y - x$ is (strictly) positive-definite.

**Proposition 3.1.** The mapping

$$\Phi : D_k \times \Sigma_{p-k,q} \to \Sigma_{p,q}, \quad (v, \sigma') \mapsto \begin{pmatrix} v \\ \sigma' \sqrt{1 - v^* v} \end{pmatrix}$$

is a diffeomorphism onto a dense and open subset $\tilde{\Sigma}_{p,q}$ of $\Sigma_{p,q}$. Let $d\sigma$ and $d\sigma'$ denote the normalized Riemannian volume elements on $\Sigma_{p,q}$ and $\Sigma_{p-k,q}$ respectively, and let $\eta := \frac{\delta}{2}(p - k) - \frac{n}{q}$. Then on $\tilde{\Sigma}_{p,q}$,

$$d\sigma = c_{k,p} \cdot \Delta(I - v^* v)^\eta d\sigma' dv$$

where

$$c_{k,p} = \left( \int_{D_k} \Delta(I - v^* v)^\eta dv \right)^{-1} = \pi^{-dkq/2} \frac{\Gamma_{\Omega_q}(\frac{dp}{2})}{\Gamma_{\Omega_q}(\frac{d(p-k)}{2})}.$$

In the important special case $p \geq 2q, k = q$ with $\mathbb{F} = \mathbb{R}$, this result goes back to [II]. For the reader’s convenience we nevertheless supply a concise proof along a different approach.
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Proof. It is easily checked that \( \Phi : D_k \times \Sigma_{p-k,q} \to \Sigma_{p,q} \) is \( C^1 \) and injective, and that its image \( \Sigma_{p,q} \) is dense in \( \Sigma_{p,q} \). Let \( f \in C(\Sigma_{p,q}) \) and extend it to \( F \in L^1(M_{p,q}) \) by \( F(\sigma \sqrt{r}) := f(\sigma) \) if \( \frac{1}{2} I < r < \frac{2}{3} I \) and \( F = 0 \) else. Then

\[
\int_{M_{p,q}} F(x)dx = C \int_{\Sigma_{p,q}} f d\sigma \quad \text{with} \quad C = C_{p,q} \cdot \int_{\frac{1}{2} I < r < \frac{2}{3} I} \Delta(r) \gamma dr.
\]

(3.6)

On the other hand, we write \( x \in M_{p,q} \) in block form as \( x = \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} \) with \( x_1 \in M_{k,q} \) and \( x_2 \in M_{p-k,q} \). Then

\[
\int_{M_{p,q}} F(x)dx = C_{p-k,q} \int_{M_{k,q}} \int_{I < r < \frac{2}{3} I} \int_{\Sigma_{p-k,q}} F \left( x_1 \sigma_2 \sqrt{r - x_1^2 x_1} \right) \Delta(r_2) \gamma dr_2 d\sigma_2 dx_1
\]

where we used polar coordinates \( x_2 = \sigma_2 \sqrt{r} \) in \( M_{p-k,q} \). Consider now the successive transforms \( r_2 \mapsto x_1^2 x_1 + r_2 =: r \) with \( x_1 \) fixed and \( x_1 \mapsto v = x_1 r^{-1/2} \) with \( r \) fixed. We have \( dr_2 = dr \) and, by Lemma 2 of [FT], \( dx_1 = \Delta(r) dr_{2} dv_{2} dv_{1} \). Thus in a first step,

\[
\int_{M_{p,q}} F(x)dx = C_{p-k,q} \int_{M_{k,q}} \int_{I < r < \frac{2}{3} I} \int_{\Sigma_{p-k,q}} F \left( \sigma_2 \sqrt{r - x_1^2 x_1} \right) \Delta(r_2) \gamma \Delta(I - v^* v) \gamma dv_{2} dv_{1} dx_1
\]

where \( F \left( \sigma_2 \sqrt{r - x_1^2 x_1} \right) = f \left( \frac{v}{\sigma_2 w} \right) \) with \( w = r - r v^* v \). As \( w^* w = I - v^* v \), it follows that \( w = u \sqrt{I - v^* v} \) with some \( u \in U_q \). The invariance of the Lebesgue measure on \( M_{p-k,q} \) under the action of \( U_q \) by right multiplication easily implies that \( d\sigma_2 \) is invariant under this action of \( U_q \) on \( \Sigma_{p-k,q} \). In view of the identity \( \eta + kd/2 = \gamma \), the above integral therefore becomes

\[
C_{p-k,q} \int_{M_{k,q}} \int_{I < r < \frac{2}{3} I} \int_{D_k} \int_{\Sigma_{p-k,q}} f \left( \sigma_2 \sqrt{r - v^* v} \right) \Delta(r) \gamma \Delta(I - v^* v) \gamma d\sigma_2 dv_{2} dv_{1} dx_1
\]

\[
= C \cdot \frac{C_{p-k,q}}{C_{p,q}} \int_{D_k} \int_{\Sigma_{p-k,q}} f \left( \sigma_2 \sqrt{I - v^* v} \right) \Delta(I - v^* v) \gamma \gamma d\sigma_2 dv_{1}.
\]

Together with (3.6), this gives the stated Jacobian (which does not vanish on \( D_k \times \Sigma_{p-k,q} \)) and proves the claimed diffeomorphism property of \( \Phi \).

If \( p \geq 2q \), then the above proposition with \( k = q \) leads to a nice integration formula for functions \( f(\sigma) \) on \( \Sigma_{p,q} \) which depend only on the first \( q \) rows of \( \sigma \). For abbreviation, we put

\[ q := d(q - \frac{1}{2}) + 1 \]

and for \( \mu \in \mathbb{C} \) with \( \Re \mu > \rho - 1 \),

\[ \kappa_\mu := \int_{D_q} \Delta(I - v^* v) \mu - \rho dv. \]

The explicit value of \( \kappa_\mu \) is obtained by using polar coordinates and Thm. VII.1.7. of [FK] about beta integrals on symmetric cones,

\[ \kappa_\mu = C_{q,q} \int_{r < 1} \Delta(I - r) \mu - \theta \Delta(r)^{d/2 - 1} dr = \pi \Omega_q^d / \Gamma_{\Omega_q}(\mu). \]

Corollary 3.2. Let \( p \geq 2q \). Then for \( f \in C(\Sigma_{p,q}) \) of the form

\[ f(\sigma) = \tilde{f}(\tilde{\sigma}), \quad \tilde{\sigma} = \sigma_0^* \sigma \]
one has
\[ \int_{\Sigma_{\mu, q}} f d\sigma = \frac{1}{\kappa_{pd/2}} \int_{D_q} \tilde{f}(v) \Delta(I - v^*v)^{pd/2 - \vartheta} dv. \]

The dependence on \( p \) now occurs only in the density, not in the domain of integration.

### 3.3 A product formula for Bessel functions with continuous index

Remember that the integrand in the convolution formula (3.2) and the product formula (3.5) depend only on the reduced matrix \( \tilde{\sigma} = \sigma_0^\vartheta \sigma \). By Corollary 3.2 we obtain

**Proposition 3.3.** Suppose that \( p \geq 2q \) and let \( \mu = pd/2 \).

1. The convolution (3.2) (i.e. \( *_{\mu} \)) on \( \Pi_q \) can be written as
\[
(\delta_r *_{\mu} \delta_s)(f) = \frac{1}{\kappa_{\mu}} \int_{D_q} f(\sqrt{r^2 + s^2 + 2rvs + sv^*r}) \Delta(I - v^*v)^{\mu - \vartheta} dv.
\]
2. The Bessel function \( J_\mu \) satisfies the product formula
\[
J_\mu(r^2)J_\mu(s^2) = \frac{1}{\kappa_{\mu}} \int_{D_q} J_\mu(r^2 + s^2 + 2rvs + sv^*r) \Delta(I - v^*v)^{\mu - \vartheta} dv \quad \forall r, s \in \Pi_q.
\]

We are now going to extend the integral formulas of Proposition 3.3 to arbitrary indices \( \mu \) within the half plane \( \{ \mu \in \mathbb{C} : \Re \mu > q - 1 = d(q - 1/2) \} \). We will use the standard technique (c.f. [Ste] for the rank one case), namely analytic continuation with respect to \( \mu \). The argumentation will be based on a classical theorem of Carlson:

**Theorem 3.4** [1], p.186. Let \( f(z) \) be holomorphic in a neighbourhood of \( \{ z \in \mathbb{C} : \Re z \geq 0 \} \) satisfying \( f(z) = O(e^{cz^2}) \) on \( \Re z \geq 0 \) for some \( c < \pi \). If \( f(z) = 0 \) for all \( z \in \mathbb{N}_0 \), then \( f \) is identically zero.

The following theorem is the main result of this section.

**Theorem 3.5.** Let \( \mu \in \mathbb{C} \) with \( \Re \mu > q - 1 = d(q - 1/2) \). Then the Bessel function \( J_\mu \) satisfies the product formula
\[
J_\mu(r^2)J_\mu(s^2) = \frac{1}{\kappa_{\mu}} \int_{D_q} J_\mu(r^2 + s^2 + 2rvs + sv^*r) \Delta(I - v^*v)^{\mu - \vartheta} dv \quad \forall r, s \in \Pi_q. \tag{3.7}
\]

**Remark.** It is easy to obtain from (3.7) the more general identity
\[
J_\mu(x^*x)J_\mu(y^*y) = \frac{1}{\kappa_{\mu}} \int_{D_q} J_\mu(x^*x + y^*y + x^*vy + y^*v^*x) \Delta(I - v^*v)^{\mu - \vartheta} dv \tag{3.8}
\]
for arbitrary \( x, y \in M_q \). For this, recall that every \( x \in M_q \) has a polar decomposition \( x = u|x| \) with \( |x| = \sqrt{x^*x} \in \Pi_q \) and a unitary matrix \( u \in U_q \). Choose \( r = |x| = u^*x \) and \( s = |y| = w^*y \) with \( u, w \in U_q \) in (3.7). Then
\[
J_\mu(x^*x)J_\mu(y^*y) = \frac{1}{\kappa_{\mu}} \int_{D_q} J_\mu(x^*x + y^*y + x^*uyw^*y + y^*uw^*u^*x) \Delta(I - v^*v)^{\mu - \vartheta} dv.
\]
Under the coordinate transform \( v \mapsto uvw^* = : \tilde{v} \) we have \( dv = d\tilde{v}, \Delta(I - v^*v) = \Delta(I - \tilde{v}^*\tilde{v}) \) and \( v \in D_q \Leftrightarrow \tilde{v} \in D_q \). This implies (3.8).
Proof of Theorem 3.5. Let $W := \{\mu \in \mathbb{C} : \Re\mu > q - 1\}$. By the asymptotic properties of the usual gamma function we have

$$\kappa_{\mu} \sim \left( \frac{2}{\mu} \right)^{dq^2/2} \text{ uniformly in } W \text{ as } \mu \to \infty.$$  \hfill (3.9)

Consider now the claimed product formula (3.7). Its left-hand side is holomorphic and, according to Corollary 2.2, also uniformly bounded in $W$ as a function of $\mu$. In order to estimate the right-hand side of (3.7), note that the argument $r^2 + s^2 + rv + sv^*r$ is positive semidefinite for all $v \in D_q$. Moreover,

$$\text{tr}(r^2 + s^2 + rv + sv^*r) = \|r\|^2 + \|s\|^2 + 2(r|vs) \leq \|r\|^2 + 2\|r\|\|vs\| + \|s\|^2 \leq (\|r\| + \|s\|)^2$$  \hfill (3.10)

where again $\| \cdot \|$ is the Frobenius norm on $M_q$. For the last estimate, it was used that $I - v^*v$ is positive-definite and therefore

$$\|s\|^2 - \|vs\|^2 = \text{tr}(s(I - v^*v)s) \geq 0.$$  

Thus by Corollary 2.2

$$\left| J_\mu(r^2 + s^2 + rv + sv^*r) \right| \leq e^{(\|r\| + \|s\|)^2} \quad \forall v \in D_q.$$  

This easily implies that the right-hand side in (3.7) is also holomorphic as a function of $\mu$ in $W$, and can be estimated according to

$$\left| \frac{1}{\kappa_{\mu}} \int_{D_q} J_\mu(r^2 + s^2 + rv + sv^*r) \Delta(I - v^*v)^{\mu - \eta} dv \right| \leq C \cdot \frac{1}{|\kappa_{\mu}|} \int_{D_q} \Delta(I - v^*v)^{\Re\mu - \eta} dv = C \cdot \frac{\kappa_{\Re\mu}}{|\kappa_{\mu}|}$$  \hfill (3.11)

with a constant $C > 0$ independent of $\mu$. In view of (3.9), the last expression is of the form

$$O(\|\mu\|^{dq^2/2}) \quad \text{as } \mu \to \infty \text{ in } W.$$  

Now define $f(z) := g((z + 2q)^{1/2})$ where

$$g(\mu) := J_\mu(r^2)J_\mu(s^2) - \frac{1}{\kappa_{\mu}} \int_{D_q} J_\mu(r^2 + s^2 + rv + sv^*r) \Delta(I - v^*v)^{\mu - \eta} dv, \mu \in W.$$  

Then $f(z) = 0$ for all $z \in \mathbb{N}_0$ by Proposition 3.3. The above considerations further show that $f$ is holomorphic and of polynomial growth on $\Re z > -1$. With Carlson’s theorem, validity of (3.7) follows as claimed.  

Remarks. 1. We may as well establish a Bochner type integral representation for the Bessel function $J_\mu$ with $\Re\mu > q - 1$ by analytic continuation. Indeed, for $\mu = \frac{q^2}{2}$ with an integer $p \geq q$, we obtain from (3.1) that for all $x \in M_q$,

$$J_\mu(x^*x) = \int_{\Sigma_{p,q}} e^{-2i(\sigma|\sigma x)} d\sigma = \int_{\Sigma_{p,q}} e^{-2i(\tilde{\sigma}|x)} d\sigma.$$  

If $p \geq 2q$ then according to Corollary 3.2 this can be written as

$$J_\mu(x^*x) = \frac{1}{\kappa_{\mu}} \int_{D_q} e^{-2i(v|x)} \Delta(I - v^*v)^{\mu - \eta} dv.$$  \hfill (3.12)

Analytic continuation with respect to $\mu$ as above shows that (3.12) remains valid for all $\mu \in \mathbb{C}$ with $\Re\mu > q - 1$. From this identity, it follows by the Riemann-Lebesgue Lemma for the
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additive matrix group \((M_q, +)\) that \(J_\mu \in C_0(\Pi_q)\). In case \(\mathbb{F} = \mathbb{R}\), the integral representation \([3.12]\) for \(\Re\mu > q - 1\) goes back to \([1]\), where it was proven by a different method.

2. In the rank one case \(q = 1\) we have

\[ J_\mu \left(\frac{r^2}{4} \right) = j_{\mu - 1}(r) \quad (r \in \mathbb{R}_+) \]

with the one variable Bessel functions

\[ j_\alpha(z) = \, {}_0F_1(\alpha + 1; -\frac{z^2}{4}) = \sum_{n=0}^{\infty} \frac{(-1)^n \Gamma(\alpha + 1)}{n! \Gamma(\alpha + n + 1)} \left(\frac{z}{2}\right)^{2n}. \]

Formulas \([3.12]\) and \([3.7]\) reduce to the well-known Mehler formula

\[ j_\alpha(r) = c_\alpha \int_{-1}^{1} e^{-irt} (1 - t^2)^{\alpha - 1/2} dt \]

and the product formula

\[ j_\alpha(r) j_\alpha(s) = c_\alpha \int_{-1}^{1} j_\alpha(\sqrt{r^2 + s^2 + 2rst})(1 - t^2)^{\alpha - 1/2} dt, \]

which are both valid for \(\alpha \in \mathbb{C}\) with \(\Re\alpha > -1/2\).

We finish this section with an alternative form of product formula \([3.7]\) and the integral representation \([3.12]\) for the Bessel functions \(J_\mu\). Let

\[ B := \{ z \in M_{1,q}(\mathbb{F}) : |z| < 1 \} \]

denote the unit ball in \(M_{1,q}(\mathbb{F}) \cong \mathbb{F}^q\) with respect to the standard norm \(|z| = (\sum_{j=1}^{q} \overline{z_j} z_j)^{1/2}\).

It is easily checked that

\[ \Delta(I - z^* z) = 1 - |z|^2. \]

**Lemma 3.6.** The mapping

\[ P(y_1, \ldots, y_q) := \begin{pmatrix} y_1 \\ y_2 (I - y_1^* y_1)^{1/2} \\ \vdots \\ y_q (I - y_{q-1}^* y_{q-1})^{1/2} \cdots (I - y_1^* y_1)^{1/2} \end{pmatrix}; \quad y_1, \ldots, y_q \in B \]

establishes a diffeomorphism from \(B^q\) onto \(D_q\) with the Jacobi determinant

\[ |\det dP(y)| = \prod_{j=1}^{q-1} (1 - |y_j|^2)^{d(q-j)/2}. \]

**Proof.** For \(k\) with \(2 \leq k \leq q\), the mapping

\[ B \times D_{k-1} \rightarrow D_k, \quad (z, w) \mapsto \left( \frac{z}{w \sqrt{1 - z^* z}} \right) \]

is obviously a diffeomorphism. By Lemma 2 of \([17]\) and Fubini’s theorem, its Jacobi determinant is given by \((1 - |z|^2)^{d(k-1)/2}\). \((q - 1)\)-fold iteration of this decomposition yields the assertion. \(\square\)
It is easily checked that for \( v = P(y) \in D_q \),
\[
\Delta(I - v^*v) = \prod_{j=1}^q \Delta(I - y_j^*y_j) = \prod_{j=1}^q (1 - |y_j|^2).
\]

This implies

**Corollary 3.7.** Let \( \zeta \in \mathbb{C} \) with \( \Re \zeta > -1 \). Then the image measure of \( \Delta(I - v^*v)\zeta dv \) under \( P^{-1} : D_q \to B^d \) is given by \( \prod_{j=1}^q (1 - |y_j|^2)^{\zeta+d(q-j)/2}dy \).

Application to the product formula (3.7) and the integral representation (3.12) for the Bessel function \( J_\mu \) gives

**Corollary 3.8.** Let \( \mu \in \mathbb{C} \) with \( \Re \mu > \varrho - 1 \). Then for all \( r, s \in \Pi_q \),
\[
J_\mu(r^2)J_\mu(s^2) = \frac{1}{\kappa_\mu} \int_{B^d} J_\mu(r^2 + s^2 + rP(y)s + sP(y)^*r) \prod_{j=1}^q (1 - |y_j|^2)^{\mu-\varrho+d(q-j)/2}dy;
\]
\[
J_\mu(r^2) = \frac{1}{\kappa_\mu} \int_{B^d} e^{-2(P(y)r)} \prod_{j=1}^q (1 - |y_j|^2)^{\mu-\varrho+d(q-j)/2}dy.
\]

### 3.4 Bessel convolutions on the cone \( \Pi_q \)

For real \( \mu > \varrho - 1 \) the measure \( \kappa_\mu^{-1} : \Delta(I - v^*v)^{\mu-\varrho} \) in product formula (3.7) is a probability measure on \( D_q \). We shall see that (3.7) leads to a hypergroup convolution on \( \Pi_q \) which is of the same form as those of Proposition 3.3. This will give us a continuous series of commutative hypergroup structures on \( \Pi_q \) which interpolate those occurring as orbit hypergroups for the indices \( \mu = pd/2, p \geq 2q \) an integer.

**Theorem 3.9.** Fix an index \( \mu \in \mathbb{R} \) with \( \mu > \varrho - 1 \).

(a) The assignment
\[
(\delta_r *_\mu \delta_s)(f) := \frac{1}{\kappa_\mu} \int_{D_q} f(\sqrt{r^2 + s^2 + rvs + sv^*r}) \Delta(I - v^*v)^{\mu-\varrho} dv; \quad f \in C(\Pi_q)
\]
defines a commutative hypergroup structure \( \Pi_{q,\mu} = (\Pi_q, *_\mu) \) with neutral element 0 and the identity mapping as involution. The support of \( \delta_r *_\mu \delta_s \) satisfies
\[
supp(\delta_r *_\mu \delta_s) \subseteq \{ t \in \Pi_q : ||t|| \leq ||r|| + ||s|| \}.
\]

(b) A Haar measure of the hypergroup \( \Pi_{q,\mu} \) is given by
\[
\omega_\mu(f) = \frac{2^{-q\mu}}{\Gamma_{\Omega_q}(\mu)} \int_{\Omega_q} f(\sqrt{r}) \Delta(r)^\gamma dr
\]
with \( \gamma = \mu - \frac{q}{2}(q - 1) - 1 = \mu - \frac{n}{q} \).

**Remark.** The specific normalization of the Haar measure is motivated by Theorem 3.11 below.

**Proof.** Ad (a). Clearly \( \delta_r *_\mu \delta_s \) is a probability measure on \( \Pi_q \) and
\[
supp(\delta_r *_\mu \delta_s) = \{ \sqrt{r^2 + s^2 + rvs + sv^*v} : v \in \mathcal{D}_q \},
\]
which does not depend on \( \mu \). The stated support inclusion is immediate from estimate (3.10). This shows property (2) in the definition of a hypergroup. (3) is clear because it is known to be
true for indices $\mu = pd/2$ which lead to an orbit hypergroup structure. Property (4) is obvious, and (5) with $\mathfrak{r} = r$ is again true in general because it is true in the orbit hypergroup cases. For the proof of (1) it suffices to show that for each $f \in C_b(\Pi_q)$ the mapping $(r, s) \mapsto f(r * \mu s)$ is continuous. But this is clear from the continuity of the map $(r, s, v) \mapsto f(\sqrt{r^2 + s^2 + rv + sv^2})$ on $\Pi_q^2 \times D_q$. Commutativity of the convolution is clear, and for the proof of associativity it again suffices to consider point measures. So let $r, s, t \in \Pi_q$ and $f \in C_b(\Pi_q)$. Then 

$$
\delta_r *_\mu (\delta_s *_\mu \delta_t)(f) = \frac{1}{\kappa_\mu^2} \int_{D_q} \int_{D_q} f(H(r, s, t; v, w)) \Delta(I - v^* v)^{\mu - q} \Delta(I - w^* w)^{\mu - q} dv dw =: I(\mu)
$$

with a certain $\Pi_q$-valued argument $H$ that is independent of the index $\mu$. Similarly,

$$(\delta_r *_\mu \delta_s) *_\mu \delta_t(f) = I'(\mu)$$

with some $\mu$-independent argument $H'$ instead of $H$. The integrals $I(\mu)$ and $I'(\mu)$ are well-defined and holomorphic in $\{\mu \in \mathbb{C} : \Re \mu > q - 1\}$. Further, we know that $I(\mu) = I'(\mu)$ for all $\mu = pd/2$ with an integer $p \geq 2q$. By analytic continuation as in the proof of Theorem 3.5 (use Bessel convolutions) again) we obtain validity of this relation for all $\mu$ with $\Re \mu > q - 1$.

Ad (b). We have to prove that

$$
\int_{\Omega_q} f(s *_\mu \sqrt{r}) \Delta(r)^\gamma dr = \int_{\Omega_q} f(\sqrt{r}) \Delta(r)^\gamma dr
$$

for all $f \in C_c(\Pi_q)$ and $s \in \Pi_q$. Equivalently,

$$
\int_{\Omega_q} \int_{D_q} f(\sqrt{s^2 + r^2 + sv\sqrt{r} + \sqrt{rv^*}}) \Delta(I - v^* v)^{\mu - q} \Delta(\sqrt{r})^{\mu - q} dr dv
$$

$$
= \int_{\Omega_q} f(\sqrt{r}) \Delta(\sqrt{r})^{\mu - q} dr.
$$

(3.13)

We know from Section 3.1 that this relation is true if $\mu = pd/2$ with an integer $p \geq 2q$. Moreover, the stated result is well known in case $d = q = 1$ for all $\mu$ in question; actually, in these cases $*_{\mu}$ is just the convolution of a Bessel-Kingman hypergroup (see Section 3.5 of [BH]). So we may assume that $dq \geq 2$. Let again $W = \{\mu \in \mathbb{C} : \Re \mu > q - 1\}$. Then $\Re \mu - \frac{q}{2} > 0$ for all $\mu \in W$, and for fixed $f$ both sides of (3.13) are well-defined and holomorphic as functions of $\mu$ in $W$. We shall again carry out analytic continuation with respect to $\mu$, based on Carlson’s theorem. We proceed in two steps.

Step 1. For $R > 0$ let $B_R := \{r \in \Pi_q : \|r\| \leq R\}$. Choose $R > 0$ with $R^q \leq e^\pi/d$. The explicit formula for the convolution $*_{\mu}$ shows that there exist constants $S_0, R_0 > 0$ such that for $f \in C_c(\Pi_q)$ with $\operatorname{supp} f \subseteq B_{R_0}$ and for $s \in \Pi_q$ with $\|s\| \leq S_0$ we have

$$
f(s *_{\mu} \sqrt{r}) = 0 \text{ if } \|r\| \geq R.
$$

Suppose that $|f| \leq C$ on $\Pi_q$ and notice that $\Delta(r) \leq \|r\|^q$. The left-hand side of (3.13) may then be estimated according to

$$
|LHS| \leq C \int_{B_R} \Delta(r)^{2q - \frac{q}{2}} dr \cdot \int_{D_q} \Delta(I - v^* v)^{2q - q} dv
$$

$$
\leq C \cdot \kappa_{2q} \cdot \operatorname{vol}(B_R) \cdot (R^q)^{2q - \frac{q}{2}}
$$

(3.14)

Due to the asymptotics (3.9), $\mu \mapsto \kappa_{2q}$ is bounded on $W$. Thus by our initial assumption on
Theorem make up the complete dual: The right-hand side of (3.13) coincides with the left-hand side for $s = 0$ and is therefore of (at most) the same order as a function of $\mu$. As in the proof of Theorem (3.5), we have to substitute $\mu = (z + 2q)^{\frac{d}{2}}$ to obtain holomorphic functions in $z$ on $\Re z > -1$ which coincide for all $z \in \mathbb{N}_0$ and are of order $O(e^{\pi|z|/2})$ in $\Re z \geq 0$. Carlson’s theorem now implies the assertion under the above restrictions on $f$ and $s$.

Step 2. Let $f \in C_c(\Pi_q)$ and $s \in \Pi_q$ be arbitrary. For $\delta > 0$ define $f_\delta(r) := f(\delta r)$. We shall use that the convolution $\ast \mu$ is homothetic, i.e.

$$f_\delta(r \ast \mu s) = f(\delta r \ast \mu \delta s).$$

Fix constants $S_0, R_0$ as in step 1 and choose $\delta > 1$ such that $\text{supp} f_\delta \subseteq B_{R_0}$ and $\| \frac{s}{\delta} \| \leq S_0$. Then

$$I(s) := \int_{\Omega_q} f(s \ast \mu \sqrt{r}) \Delta(r)^\gamma dr = \int_{\Omega_q} f_\delta \left( \frac{s}{\delta} \ast \mu \sqrt{r} \right) \Delta(r)^\gamma dr$$

$$= \delta^{2q\gamma + 2n} \int_{\Omega_q} f_\delta \left( \frac{s}{\delta} \ast \mu \sqrt{t} \right) \Delta(t)^\gamma dt.$$

As $f_\delta$ and $\frac{s}{\delta}$ satisfy the conditions of step 1, we conclude that $I(s) = I(0)$, which finishes the proof of part (b).

Our next aim is to determine the dual and the Plancherel measure of the hypergroup $\Pi_{q,\mu}$ with $\mu > q - 1$. For $s \in \Pi_q$, we define

$$\varphi_s(r) = \varphi_s^\mu(r) := J_{\mu} \left( \frac{1}{4} sr^2 s \right), \quad r \in \Pi_q.$$

Notice that

$$\varphi_s(r) = \varphi_r(s)$$

(3.15)

because $J_{\mu}$ depends only on the eigenvalues of its argument. Moreover, we have

Lemma 3.10. Let $\mu > q - 1$. Then for each $s \in \Pi_q$, $\varphi_s = \varphi_s^\mu$ belongs to $C_b(\Pi_q)$ with $\| \varphi_s \|_\infty = \varphi_s(0) = 1$. If $s \in \Omega_q$, then even $\varphi_s \in C_0(\Pi_q)$.

Proof. The first assertion is immediate from the Bochner-type integral representation (3.22) for $J_{\mu}$. Now suppose $s \in \Omega_q$. Then $r \to \infty$ in $\Pi_q$ implies that $r^2 \to \infty$ and also $sr^2 s \to \infty$, because $s$ is invertible. The second assertion thus follows from the fact that $J_{\mu}$ vanishes at infinity.

From this Lemma together with the product formula (3.28) it is immediate that each $\varphi_s$ with $s \in \Pi_q$ belongs to $\hat{\Pi}_{q,\mu}$. The results of Section 3.1 suggest that these Bessel functions actually make up the complete dual:

Theorem 3.11. (1) The dual space of $\Pi_{q,\mu}$ with $\mu > q - 1$ is given by

$$\hat{\Pi}_{q,\mu} = \{ \varphi_s = \varphi_s^\mu : s \in \Pi_q \}.$$

(2) The hypergroup $\Pi_{q,\mu}$ is self-dual via the homeomorphism $\Psi : \Pi_{q,\mu} \to \hat{\Pi}_{q,\mu}$, $s \mapsto \varphi_s$. Under this identification, the Plancherel measure $\pi_{\mu}$ on $\hat{\Pi}_{q,\mu}$ coincides with the Haar measure $\omega_\mu$.

For the proof of part (1) we need the following
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Lemma 3.12. The set \( A := \{ \varphi_s : s \in \Pi_q \} \) is closed in \( \hat{\Pi}_{q,\mu} \) with respect to the topology of locally uniform convergence.

Proof. Let \((\varphi_{s_j})_{j \in \mathbb{N}}\) be a sequence in \( A \) converging to \( \alpha \in \hat{\Pi}_{q,\mu} \) locally uniformly. If the sequence \((s_j) \subset \Pi_q\) is bounded, then after passing to a subsequence we may assume that \( s_j \to s \in \Pi_q \) as \( j \to \infty \). Then \( \varphi_{s_j} \to \varphi_s \) and therefore \( \alpha = \varphi_s \in A \). If else the original sequence \((s_j)\) is unbounded, then after passing to a subsequence we may assume that \( s_j \to \infty \). Thus by Lemma 3.10 \( \varphi_{s_j}(r) = \varphi_r(s_j) \to 0 \) for all \( r \in \Omega_q \) as \( j \to \infty \). This implies \( \alpha = 0 \) which contradicts the convention \( 0 \notin \hat{\Pi}_{q,\mu} \).

Proof of Theorem 3.11. In a first step, we establish that \( \Pi_{q,\mu} \) has subexponential growth in the sense of [VI], i.e. for each compact subset \( K \subset \Pi_q \) and each \( c > 1 \), the Haar measure satisfies \( \omega_\mu(K^m) = O(c^m) \). Here \( K^m \) denotes the \( m \)-fold convolution power of \( K \), the convolution product of subsets \( A,B \) of a hypergroup \((X,\ast)\) being defined by \( A \ast B = \bigcup_{x \in A, y \in B} \text{supp}(\delta_x \ast \delta_y) \). Once subexponential growth is known, Theorem 2.17 of [V1] will imply that the support of the Plancherel measure \( \pi_\mu \) coincides with the complete dual \( \hat{\Pi}_{q,\mu} \). For the proof of subexponential growth, it suffices to consider the balls \( B_R = \{ r \in \Pi_q : ||r|| \leq R \} \). From the support properties of \( \ast_\mu \) we see that \( B_R^m \subseteq B_{mR} \). Moreover,

\[
\omega_\mu(B_R) = C \cdot \int_{\|x\| \leq R} \Delta(r)^{\gamma} dr = O(R^{2\gamma+2n}) = O(R^{2\mu}) \quad \text{as } R \to \infty.
\]

Thus for fixed \( R > 0 \), we obtain \( \omega_\mu(B_R^m) = O(m^{2\mu}) \), and the assertion follows.

In a second step, we determine \( \pi_\mu \). The decisive ingredient will be known results about the Hankel transform on a symmetric cone. For \( s, r \in \Omega_q \), define

\[
H_\mu(s, r) = \frac{1}{\Gamma_\Omega_q(\mu)} \mathcal{J}_\mu(\sqrt{s} \sqrt{r}).
\]

Suppose that \( \mu > d(q - 1) + 1 \). Then according to Theorem XV.4.1. of [FK], the Hankel transform

\[
U_\mu F(s) := \int_{\Omega_q} H_\mu(s, r) F(r) \Delta(r)^{\gamma} dr
\]

defines an isometric and involutive isomorphism of \( L^2(\Omega_q, \Delta(r)^{\gamma} dr) \). The argumentation of [H] and [FT], Section 5 shows that this statement actually extends to all \( \mu \in \mathbb{R} \) with \( \mu > \frac{d}{2}(q-1) \), i.e. \( \gamma > -1 \). Let \( f \in L^2(\Pi_q, \omega_\mu) \). Then \( F(r) := f(\sqrt{r}) \) belongs to \( L^2(\Omega_q, \Delta(r)^{\gamma} dr) \) and a short calculation shows that

\[
\hat{f}(\varphi_s) = \int_{\Pi_q} \varphi_s(r) f(r) d\omega_\mu(r) = 2^{-q\mu} U_\mu F(\frac{r^2}{4}) \quad \forall s \in \Omega_q.
\]

Moreover, by the isometry of \( U_\mu \) we readily obtain

\[
\int_{\Omega_q} |\hat{f}(\varphi_s)|^2 d\omega_\mu(s) = \int_{\Omega_q} |f(s)|^2 d\omega_\mu(s).
\]

This shows that the Plancherel measure \( \pi_\mu \) associated with \( \omega_\mu \) is given by

\[
\pi_\mu(g) = \int_{\Pi_q} g(\varphi_s) d\omega_\mu(s), \quad g \in C_c(\hat{\Pi}_{q,\mu}).
\]

Hence the support of \( \pi_\mu \), which we already know to coincide with \( \hat{\Pi}_{q,\mu} \), also coincides with the closure of the set \( \{ \varphi_s : s \in \Pi_q \} \) in \( \hat{\Pi}_{q,\mu} \) with respect to the topology of locally uniform
convergence. The proof of part (1) is therefore accomplished by Lemma 3.12. For part (2), it remains to verify that $\Psi$ is a homeomorphism. Continuity and surjectivity are clear. For injectivity, suppose $\varphi_s = \varphi_r$. Then in view of (3.15) we have $\tilde{\delta}_s = \tilde{\delta}_r$, and the injectivity of the Fourier transform of measures on the hypergroup $\Pi_{q,\mu}$ implies $s = r$. To check continuity of $\Psi^{-1}$ suppose that $\varphi_{s_i} \to \varphi_s$ locally uniformly. Then $\tilde{\delta}_{s_i} \to \tilde{\delta}_s$ locally uniformly on $\widehat{\Pi}_{q,\mu}$. Levy's continuity theorem (Thm. 4.2.2. in [BH]) implies that $\delta_{s_i} \to \delta_s$ weakly, and hence $s_i \to s$.

### 3.5 The limit case $\mu = q - 1$.

Using Corollary 3.7 we see that the convolution $*_{\mu}$ with $\mu > q - 1$ can be written in the alternative form

$$(\delta_r *_{\mu} \delta_s)(f) = \frac{1}{\kappa_{\mu}} \int_{B^q} f \left( \sqrt{r^2 + s^2 + rP(y)s + sP(y)r} \right) \prod_{j=1}^{q} (1 - |y_j|^2)^{\mu-\frac{d}{2}(q-j)} dy.$$ (3.16)

We shall use this representation to determine the limit of the convolution $*_{\mu}$ as $\mu \downarrow q - 1$, where it assumes a degenerate form. As $p - 1 = pd/2$ with $p = 2q - 1$, it is natural to expect that the resulting limit convolution coincides with the orbit hypergroup convolution $*_{\mu}$ on $\Pi_q$ derived from $M_{q,2q-1}$. In the following, $d\sigma$ denotes the normalized surface measure on the unit sphere $S = \{z \in M_{1,q} : |z| = 1\}$.

The coordinate transform $P : B^q \to D_q$ of Lemma 3.6 is assumed to be continuously extended to $\overline{B}^q$.

**Proposition 3.13.** As $\mu \downarrow q - 1$, the convolution product $\delta_r *_{\mu} \delta_s$ converges weakly to the probability measure $\delta_r \ast \delta_s$ on $\Pi_q$ given by

$$(\delta_r \ast \delta_s)(f) = \tilde{\kappa} \int_{B^{q-1}} \int_{S} f \left( \sqrt{r^2 + s^2 + rP(y)s + sP(y)r} \right) \prod_{j=1}^{q-1} (1 - |y_j|^2)^{\mu-\frac{d}{2}(q-j)} |d\sigma_1 \ldots d\sigma_{q-1}|.$$ (3.16)

with a normalization constant $\tilde{\kappa} > 0$. The product $\ast$ defines a commutative hypergroup structure on $\Pi_q$ which coincides with the orbit hypergroup $\Pi_{q,2q-1}$ derived from $M_{q,2q-1}$ as in Section 3.1. In particular,

$$(\delta_r \ast \delta_s)(f) = (\delta_r \ast_{\mu-1} \delta_s)(f) = \int_{\Sigma_{q,2q-1}} f \left( \sqrt{r^2 + s^2 + r\sigma s + (r\sigma)^2} \right) d\sigma$$

and the additional statements of Theorem 3.7 and Theorem 3.11 extend to the case $\mu = q - 1$.

**Proof.** For $\mu > q - 1$ consider the probability measure

$$p_\mu := c_{\mu}^{-1} (1 - |y|^2)^{\mu-\rho} 1_B(y) dy$$
on $M_{1,q}$, where $c_{\mu} := \int_B (1 - |y|^2)^{\mu-\rho} dy$ and $1_B$ denotes the characteristic function of the ball $B$. It is easily checked that $p_\mu$ tends weakly to the normalized surface measure $d\sigma$ as $\mu \to q - 1$. Indeed, let $f \in C(M_{1,q})$ and put $F(\tau) := \int_S f(\tau y) d\sigma(y)$, $\tau \geq 0$. Then with $c_\mu' = \int_0^1 (1 - \tau^2)^{\mu-\rho} d\tau$,

$$\int_B f dp_\mu = \frac{1}{c_\mu'} \int_0^1 F(\tau)(1 - \tau^2)^{\mu-\rho} d\tau \to F(1) \text{ as } \mu \to q - 1.$$ 

This proves that $\delta_r *_{\mu} \delta_s \to \delta_r \ast \delta_s$ weakly. It is clear that $\text{supp}(\delta_r \ast \delta_s) \subseteq \text{supp}(\delta_r *_{\mu} \delta_s)$ for $\mu > q - 1$ which implies the same support inclusion as in Theorem 3.8 (a). In the limit $\mu \to q - 1$
we further obtain that the Bessel functions
\[ \varphi_s(r) = J_{q-1}(\frac{1}{4} sr^2 s), \quad s, r \in \Pi_q \]
satisfy the product formula
\[ \varphi_s(r)\varphi_s(t) = \int_{\Pi_q} \varphi_s(\tau) d(\delta_r \ast_s \delta_t)(\tau) \quad \forall s \in \Pi_q. \]
On the other hand, consider the orbit hypergroup \( X_{q-1} = (\Pi_q, \ast_{q-1}) \) derived from \( M_{p,q} \) with \( p = 2q - 1 \). Its dual space consists exactly of the Bessel functions \( \varphi_s(r) = J_{q-1}(\frac{1}{4} rs^2 r) \) as above. In particular,
\[ \varphi_s(r)\varphi_s(t) = \int_{\Pi_q} \varphi_s(\tau) d(\delta_r \ast_{q-1} \delta_t)(\tau) \quad \forall s \in \Pi_q. \]
The injectivity of the Fourier transform on the hypergroup \( X_{q-1} \) now implies that \( \delta_r \ast_{q-1} \delta_t = \delta_r \ast_s \delta_t \) for all \( r, t \).

**Remarks.**
1. We conjecture that the formulas of Corollary 3.8 permit degenerate extensions to successively larger index ranges. As soon as the exponent in one of the iterated integrals becomes critical, the corresponding integral over \( B \) should be replaced by an integral over \( S \). More precisely, we conjecture that within the range
\[ \{ \mu \in \mathbb{R} : \rho - \frac{d}{2}(q - k) - 1 < \mu \leq \rho - \frac{d}{2}(q - k - 1) - 1 \}, \quad k = q - 1, \ldots, 1 \]
the following product formula is valid:
\[
\mathcal{J}_\mu(r^2)\mathcal{J}_\mu(s^2) = \frac{1}{\kappa_{\mu,j}} \int_{S^{q-k}} \int_{B^k} \mathcal{J}_\mu(r^2 + s^2 + rP(y)s + sP(y)^* r) \prod_{j=1}^{k} (1 - |y_j|^2)^{\mu - 4}\cdot dy_1 \ldots dy_k d\sigma(y_{k+1}) \ldots d\sigma(y_q).
\]
Also, there should be analogous integral representations for \( \mathcal{J}_\mu \) within the above ranges of \( \mu \).

2. Further properties of the hypergroups \( \Pi_{q,\mu} \), concerning their automorphism groups as well as stochastic aspects (such as limit theorems for random walks on matrix cones associated with Bessel convolutions) will be the subject of a forthcoming paper joint with M. Voit.

**4. Hypergroups associated with rational Dunkl operators of type B**

In the analysis of the previous sections, one may be interested in questions which depend only on the spectra of the matrices from the underlying cone \( \Pi_q = \Pi_q(\mathbb{R}) \). This amounts to considering functions and measures on \( \Pi_q \) which are invariant under unitary conjugation. For \( x \in H_q \) we denote by \( \sigma(x) = (\xi_1, \ldots, \xi_q) \in \mathbb{R}^q \) the set of eigenvalues of \( x \) ordered by size, i.e. \( \xi_1 \geq \ldots \geq \xi_q \). The unitary group \( U_q \) acts on \( \Pi_q \) via conjugation, \( (u, r) \mapsto uru^{-1} \), and the orbits under this action are parametrized by the set \( \Xi_q \) of possible spectra \( \sigma(r) \) of matrices \( r \in \Pi_q \),
\[
\Xi_q = \{ \xi = (\xi_1, \ldots, \xi_q) \in \mathbb{R}^q : \xi_1 \geq \ldots \geq \xi_q \geq 0 \}.
\]
We mention that \( U_q \) is the maximal compact subgroup of the automorphism group of \( \Omega_q \); indeed the latter coincides with \( GL_q = GL_q(\mathbb{R}) \), acting on \( H_q \) via \( (g, x) \mapsto gxg^* \). The set \( \Xi_q \) is a closed Weyl chamber of the hyperoctahedral group \( B_q = S_q \times \mathbb{Z}_2^q \) which acts on \( \mathbb{R}^q \) by permutations of the basis vectors and sign changes. In Section 3, we saw that \( \Pi_q \) carries a
continuously parametrized family of commutative hypergroup structures $\ast_\mu$ with $\mu \geq q - 1$, as well as additional orbit hypergroup structures for $\mu = pd/2$, $p \geq q$ an integer. In the following section we are going to show that under the above action of $U_q$ on $\Pi_q$ each convolution $\ast_\mu$ induces a commutative orbit hypergroup convolution $\circ_\mu$ on $\Xi_q$, similar as orbit hypergroups were obtained from (commutative) groups in Section 2.3. In Section 4.2 we shall then identify the characters of $(\Xi_q, \circ_\mu)$ with multivariable Bessel functions of Dunkl type which are associated with the root system of type $B_q$. In effect, we thus obtain a continuous series of commutative hypergroup structures on the chamber $\Xi_q$ whose characters are given by Dunkl-type Bessel functions.

4.1 Convolutions on the spectra of positive definite matrices

In the situation described above, the canonical mapping

$$\pi : \Pi_q \to \Xi_q, \quad r \mapsto \sigma(r)$$

is continuous, surjective and open with respect to the standard topologies on both sets. Therefore the map $\Pi_{U_q}^q \to \Xi_q, U_q r \mapsto \sigma(r)$ becomes a homeomorphism when $\Pi_q$ is equipped with the quotient topology. In the following, $du$ denotes the normalized Haar measure on $U_q$ and $\xi \in \Xi_q$ is always identified with the diagonal matrix diag$(\xi_1, \ldots, \xi_q) \in \Pi_q$ without mentioning. Moreover, we introduce the index set

$$\mathcal{M}_q := \left\{ \frac{pd}{2}, p = q, q + 1, \ldots \right\} \cup \{ \rho - 1, \infty \}.$$

**Theorem 4.1.** (1) For each $\mu \in \mathcal{M}_q$ the chamber $\Xi_q$ carries a commutative hypergroup structure with convolution

$$(\delta_\xi \circ_\mu \delta_\eta)(f) := \int_{U_q} (f \circ \pi)(\xi \ast_\mu \eta u^{-1})du, \quad f \in C(\Xi_q).$$

The neutral element of the hypergroup $\Xi_{q,\mu} := (\Xi_q, \circ_\mu)$ is $0 \in \Xi_q$ and the involution is given by the identity mapping.

(2) A Haar measure on $\Xi_{q,\mu}$ is given by

$$\tilde{\omega}_\mu = \pi(\omega_\mu) = d_\mu h_\mu(\xi)d\xi \quad \text{with} \quad h_\mu(\xi) = \prod_{i=1}^q \xi_i^{2\gamma + 1} \prod_{i < j}(\xi_i^2 - \xi_j^2)^d$$

and a constant $d_\mu > 0$.

**Remarks.** 1. The constant $d_\mu$ will be determined in Section 4.2.

2. In the generic case $\mu > q - 1$ the convolution $\circ_\mu$ can be more explicitly written as

$$(\delta_\xi \circ_\mu \delta_\eta)(f) = \frac{1}{\kappa_\mu} \int D_q \int_{U_q} f(\sigma(\sqrt{\xi^2 + u\eta^2 - 1} + \xi \nu \eta - 1 + u\eta^{-1}v^*\xi)) \Delta(I - v^*v)^{\mu-q}dudv.$$

We start with some preparations for the proof of Theorem 4.1.

**Lemma 4.2.** For $u \in U_q$ consider the automorphism of $\Pi_q$ given by $T_u : r \mapsto uru^{-1}$. The image measure of the convolution product $\delta_r \ast_\mu \delta_s \in M^1(\Pi_q)$ under $T_u$ is given by

$$T_u(\delta_r \ast_\mu \delta_s) = \delta_{uru^{-1}} \ast_\mu \delta_{usu^{-1}}.$$  

**Proof.** Recall that the measure $\Delta(I - v^*v)^{\mu-q}dv$ in the convolution formula of Theorem 3.9 is invariant under unitary conjugations. This yields immediately that

$$(g \circ T_u)(r \ast_\mu s) = g(uru^{-1} \ast_\mu usu^{-1}) \quad \forall g \in C(\Pi_q).$$
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Aside: this lemma just says that the automorphism \( T_u \) of \( \Pi_q \) is actually a hypergroup automorphism for each of the convolutions \( *_{\mu} \). The next lemma is a special case of \[FK\], Theorem VI.2.3.

**Lemma 4.3.** For integrable functions \( g : \Pi_q \rightarrow \mathbb{C} \),
\[
\int_{\Pi_q} g(r) dr = \kappa \int_{\Xi_q} \int_{U_q} g(u\xi u^{-1}) du \prod_{i<j}(\xi_i - \xi_j)^d d\xi
\]
with a normalization constant \( \kappa = \kappa_q > 0 \).

**Proof of Theorem 4.1.** Ad (1). We shall employ the technique of \[Mc\], Section 13 for the transfer of hypergroup structures via orbital mappings. Notice first that the continuous open surjection \( \pi : \Pi_q \rightarrow \Xi_q \) is also proper (because \( U_q \) is compact) and thus provides an orbital mapping from the hypergroup \( \Pi_{q,\mu} \) onto \( \Xi_q \) in the sense of \[Mc\], Section 13. For \( \xi \in \Xi_q \) define
\[
\epsilon_\xi := \int_{U_q} \delta_{u\xi u^{-1}} du
\]
which is a probability measure on \( \Pi_q \) and satisfies \( \text{supp} \epsilon_\xi = \pi^{-1}(\xi) \). We claim that each \( \epsilon_\xi \) is \( \pi \)-consistent in the following sense:
\[
\pi(\epsilon_\xi *_{\mu} \delta_t) = \pi(\epsilon_\xi *_{\mu} \delta_t) \quad \text{for all } s, t \in \Pi_q \text{ with } \pi(s) = \pi(t);
\]
here \( \pi \) is extended to \( M^+(\Pi_q) \) by taking image measures. For the proof of (4.1), suppose that \( s, t \in \Pi_q \) satisfy \( \pi(s) = \pi(t) \). Then for each \( g \in C(\Pi_q) \) which is invariant under \( U_q \)-conjugation we have
\[
\int_{\Pi_q} g d(\epsilon_\xi *_{\mu} \delta_s) = \int_{U_q} g(u\xi u^{-1} *_{\mu} s) du = \int_{U_q} g(\xi *_{\mu} u^{-1} su) du.
\]
Notice that for the second identity Lemma 4.2 has been used. As \( s \) and \( t \) have the same spectra, the last integral does not change when \( s \) is replaced by \( t \). This proves (4.1). The orbital mapping \( \pi \) also satisfies \( \pi(0) = 0 \in \Xi_q \) and \( \pi^{-1}(0) = 0 \in \Pi_q \). We can now apply \[Mc\], Theorem 13.5.A. This shows that \( \Xi_q \) becomes a commutative hypergroup with convolution
\[
\delta_\xi \circ_{\mu} \delta_\eta = \pi(\epsilon_\xi *_{\mu} \epsilon_\eta),
\]
the identity mapping as involution and neutral element \( \pi(0) = 0 \). This proves the assertions of part (1).

Ad (2). According to Theorem 13.3.A of \[Mc\], a Haar measure \( \tilde{\omega}_\mu \) on \( \Xi_{q,\mu} \) is given by the image measure of \( \omega_\mu \) under \( \pi \). Let \( f \in C_c(\Xi_q) \) and put \( g = f \circ \pi \), which is \( U_q \)-invariant. Then
\[
\int_{\Xi_q} f d\tilde{\omega}_\mu = \kappa \int_{\Pi_q} g d\omega_\mu = \kappa' \int_{\Xi_q} g(\sqrt{\xi}) \prod_{i=1}^q \xi_i^d \prod_{i<j}(\xi_i - \xi_j)^d d\xi
\]
where \( \sqrt{\xi} = (\sqrt{\xi_1}, \ldots, \sqrt{\xi_q}) \). Up to a constant factor, the last integral coincides with \( \int_{\Xi_q} g(\xi) h_\mu(\xi) d\xi \).

**Remarks.** 1. Recall that for \( \mu = pd/2 \) with an integer \( p \geq q \) the hypergroup \( \Pi_{q,\mu} \) is just the orbit hypergroup obtained from the multiplication action of the unitary group \( U_p \) on \( M_{p,q} \). In this case, the above hypergroup structure of \( (\Xi_q, \circ_{\mu}) \) can also be described as an orbit hypergroup.
derived directly from $M_{p,q}$, as follows: Consider the action of the group $L := U_p \times U_q$ on $M_{p,q}$ by
\[ x \mapsto uxv^{-1}, \quad (u,v) \in L. \]
The orbits of this action are parametrized by the possible sets of singular values of matrices from $M_{p,q}$. Indeed, let $\sigma_{\text{sing}}(x) = \sigma(\sqrt{x^*x}) = (\xi_1, \ldots, \xi_q) \in \Xi_q$ denote the singular spectrum of $x \in M_{p,q}$, the singular values being ordered by size. We have the equivalences
\[ \sigma_{\text{sing}}(x) = \sigma_{\text{sing}}(y) \iff \sigma(x^*x) = \sigma(y^*y) \]
\[ \iff \exists v \in U_q : y^*y = vx^*xv^{-1} = (xv^{-1})^*(xv^{-1}) \]
\[ \iff \exists (u,v) \in U_p \times U_q : y = uxxv^{-1}. \]
Therefore the orbit space $M_{p,q}^L$ can be identified with the chamber $\Xi_q$ via $U_p \times U_q \mapsto \sigma_{\text{sing}}(x)$, and this is easily checked to be a homeomorphism with respect to the natural topologies on both spaces. Notice that $(uxv^{-1})^*uxv^{-1} = vx^*xv^{-1}$. Hence under the mapping $\Phi : M_{p,q} \to \Pi_q$, $x \mapsto \sqrt{x^*x}$, the above action of $L$ on $M_{p,q}$ induces the conjugation action of $U_q$ on $\Pi_q$. Moreover, the orbit convolution on $\Pi_{q,\mu}$ is defined in such a way that
\[ \delta_r * \delta_s = \Phi(Q_r * Q_s) \quad \text{with} \quad Q_r = \int_{U_p} \delta_{u_r\sigma r} \, du \in M^1(M_{p,q}) \]
where $*$ denotes the usual convolution on the additive group $M_{p,q}$. This shows that for $\mu = pd/2$, the convolution of the hypergroup $\Xi_{q,\mu}$ coincides with the convolution of the orbit hypergroup $M_{p,q}^L \cong \Xi_q$, which is in turn naturally identified with the convolution of the Gelfand pair $(\Pi \times M_{p,q}, L)$.

2. For $\mu = pd/2$ with an integer $p \geq q$, the support of the measure $\delta_\xi \circ_\mu \delta_\eta$ describes the set of possible singular spectra of sums $x + y$ made up by matrices $x, y \in M_{p,q}(\mathbb{F})$ with given singular spectra $\xi$ and $\eta$.

Let us return to general indices $\mu \in \mathcal{M}_q$. In analogy to Lemma 2.6 for orbit hypergroups from groups, we expect that the characters of the hypergroup $\Xi_{q,\mu}$ are all obtained by taking $U_q$-means of the characters of $\Pi_{q,\mu}$. For $\xi \in \Xi_q$, define $\psi_\xi = \psi_\xi^\mu \in C_b(\Xi_q)$ by
\[ \psi_\xi(\eta) := \int_{U_q} \varphi_\xi(u \eta u^{-1}) \, du \]
where $\varphi_\xi(r) = \varphi_\xi^\mu(r) = \mathcal{F}_\mu(\frac{1}{2} r \xi^2 r)$. The $\varphi_\xi$ are the characters of $\Pi_{q,\mu}$ which are parametrized by diagonal matrices. Note that by $U_q$-invariance of $\mathcal{F}_\mu$ we have
\[ \varphi_s(uru^{-1}) = \varphi_{s^{-1}su}(r) \quad \forall r, s \in \Pi_q. \quad (4.2) \]
Hence the mean of $\varphi_s$ equals the mean of $\varphi_\xi$ for $\xi = \sigma(s)$ and $\psi_\xi(\eta) = \psi_\eta(\xi)$ for all $\xi, \eta \in \Xi_q$.

THEOREM 4.4. Let $\mu \in \mathcal{M}_q$.

1. The dual space of the hypergroup $\Xi_{q,\mu} = (\Xi_q, \circ_\mu)$ is given by
\[ \hat{\Xi}_{q,\mu} = \{ \psi_\xi = \psi_\xi^\mu \mid \xi \in \Xi_q \}. \]

2. The hypergroup $\Xi_{q,\mu}$ is self-dual via the homeomorphism $\Xi_{q,\mu} \to \hat{\Xi}_{q,\mu}, \xi \mapsto \psi_\xi$. Under this identification, the Plancherel measure $\pi_\mu$ of $\Xi_{q,\mu}$ coincides with the Haar measure $\tilde{\omega}_\mu$. 

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Proof. Ad (1). It is easily checked that each $\psi_\xi$ is multiplicative w.r.t. $\circ_\mu$ and therefore belongs to $\Xi_{q,\mu}$. Indeed, for $\eta, \zeta \in \Xi_q$ we calculate

$$\psi_\xi(\eta \circ_\mu \zeta) = \int_{U_q} (\psi_\xi \circ \pi)(\eta \ast_\mu u \zeta u^{-1})du = \int_{\Pi_q} \int_{U_q} \int_{U_q} \varphi_\xi(wv\zeta^{-1})d(\delta_{\eta \ast_\mu \delta_{u\zeta^{-1}}})(r)dvdu.$$

By Lemma 1.2 this equals

$$\int_{U_q} \int_{U_q} \varphi_\xi(w\eta^{-1} \ast_\mu \eta \zeta u^{-1})dvdu = \psi_\xi(\eta)\psi_\xi(\zeta).$$

It remains to show that each character of $\Xi_{q,\mu}$ is of the form $\psi_\xi$ with some $\xi \in \Xi_q$. For this, notice first that the hypergroup $\Xi_{q,\mu}$ has subexponential growth, just as $\Pi_{q,\mu}$. Thus according to Theorem 2.17 of [V1], the support of the Plancherel measure $\pi_\mu$ of $\Xi_{q,\mu}$ coincides with the full dual $\widehat{\Xi}_{q,\mu}$. Let $\psi \in \Xi_{q,\mu} = \text{supp} \pi_\mu$. Then by Corollary 6 of [V2] there exists a sequence of functions $f_n \in C_c(\Xi_q)$ such that $f_n \circ_\mu f_n$ converges to $\psi$ locally uniformly. Hence $(f_n \circ_\pi)f_n$ converges to $\psi$ locally uniformly on $\Pi_q$, which implies that $\psi$ is positive definite on the hypergroup $\Pi_{q,\mu}$. Notice that $(\psi \circ \pi)(0) = 1$. Thus by Bochner’s Theorem for commutative hypergroups (Thm. 12.3.B of [JL]) and the self-duality of $\Pi_{q,\mu}$, there exists a probability measure $a \in M^1(\Pi_q)$ such that

$$\psi(\eta) = \int_{\Pi_q} \varphi_\eta(s)da(s) = \int_{\Xi_q} \psi_\xi(\eta)d\tilde{a}(\xi).$$

On the other hand, by Bochner’s Theorem for the hypergroup $\Xi_{q,\mu}$, the character $\psi$ of $\Xi_{q,\mu}$ is an extremal point of the set of positive definite functions $f$ on the hypergroup $\Xi_{q,\mu}$ with the additional property $f(0) = 1$. This implies that $\tilde{a}$ must be a point measure, i.e. $\tilde{a} = \delta_\xi$ for some $\xi \in \Xi_q$. Hence $\psi = \psi_\xi$.

Ad (2). The self-duality of $\Xi_{q,\mu}$ is proven in the same way as that of $\Pi_{q,\mu}$ (Theorem 3.11). To determine the Plancherel measure, let $f \in C_c(\Xi_q)$ and $g := f \circ \pi$. Then the Fourier transforms of $f$ and $g$ w.r.t. the hypergroup structures $\Xi_{q,\mu}$ and $\Pi_{q,\mu}$ are related via

$$\hat{g}(s) = \int_{\Pi_q} g(r)\varphi_\xi(r)d\omega_\mu(r) = \int_{\Xi_q} f(\xi)\psi_\xi(\xi)d\tilde{\omega}_\mu(\xi) = (\hat{f} \circ \pi)(s).$$

By the Plancherel theorem for $\Pi_{q,\mu}$ we readily obtain $\int_{\Xi_q} |f|^2d\tilde{\omega}_\mu = \int_{\Xi_q} |\hat{f}|^2d\tilde{\omega}_\mu$. This finishes the proof.

We would like to write the characters $\psi_\xi$ in a more explicit form. Recall that $\varphi_\xi(r) = \mathcal{J}_\mu(\sqrt{r}su^{-1}s\sqrt{r})$ for $r, s \in \Pi_q$, where $\mathcal{J}_\mu$ is given in terms of the spherical series $\mathcal{J}_s$. The spherical polynomials satisfy the product formula

$$\frac{Z_\lambda(r)Z_\lambda(s)}{Z_\lambda(t)} = \int_{U_q} Z_\lambda(\sqrt{rsu^{-1}}r)du \quad \forall r, s \in \Pi_q,$$

see [EK], Cor. XI.3.2. or [GR], Prop. 5.5. This implies an integral representation for the Bessel
functions $\mathcal{J}_\mu$ of two matrix arguments (recall Section 2.2):

$$\mathcal{J}_\mu(r,s) = \int_{U_q} \mathcal{J}_\mu(\sqrt{r}usu^{-1}\sqrt{r})du, \quad r, s \in \Pi_q.$$  

Thus for $\xi, \eta \in \Xi_q$ we have

$$\psi_\xi(\eta) = \int_{U_q} \mathcal{J}_\mu(1/4\xi u\eta^2u^{-1}\xi)du = \mathcal{J}_\mu(\xi^2/2, \eta^2/2). \quad (4.4)$$

We shall use this representation in order to identify the characters $\psi_\xi$ with Dunkl type Bessel functions for the root system of type $B_q$.

### 4.2 Bessel functions associated with root systems

Bessel functions associated with root systems are an important ingredient in the theory of rational Dunkl operators, which was initiated by C.F. Dunkl in the late 80ies ([D1], [D2]). They are a symmetrized version of the Dunkl kernel, which is the analogue of the usual exponential function in this theory. As a subclass, they include the spherical functions of a Cartan motion group, c.f. [dJ2]. In this section we give a brief account on Dunkl theory and the associated Bessel functions; for a general background, the reader is referred to [DX], [D] and [R3].

Let $G$ be a finite reflection group on $\mathbb{R}^q$ (equipped with the usual Euclidean scalar product $\langle ., . \rangle$), and let $R$ be the reduced root system of $G$. We extend the action of $G$ to $\mathbb{C}^q$ and $\langle ., . \rangle$ to a bilinear form on $\mathbb{C}^q \times \mathbb{C}^q$. A function $k : R \rightarrow \mathbb{C}$ which is invariant under $G$ is called a multiplicity function on $R$. Important special cases of reflection groups are the symmetric group $S_q$, which acts on $\mathbb{R}^q$ by permuting the standard basis vectors $e_i$, and the hyperoctahedral group $B_q = S_q \times \mathbb{Z}_2$ which acts by permutations of the basis vectors and sign changes. The root system of $B_q$ is given by $R = \{e_i, 1 \leq i \leq q\} \cup \{\pm e_i, 1 \leq i < j \leq q\}$, and a multiplicity on it is of the form $k = (k_1, k_2)$ where $k_1$ is the value on the roots $\pm e_i$ and $k_2$ is the value on the roots $\pm e_i \pm e_j$.

For a finite reflection group $G$ and a fixed multiplicity function $k$ on its root system, the associated (rational) Dunkl operators are defined by

$$T_\xi = T_\xi(k) = \partial_\xi + \frac{1}{2} \sum_{\alpha \in R} k_\alpha(\alpha, \xi) \frac{1}{(\alpha, .)}(1 - \sigma_\alpha), \quad \xi \in \mathbb{C}^q;$$

here $\sigma_\alpha$ denotes the reflection in the hyperplane perpendicular to $\alpha$ and the action of $G$ is extended to functions on $\mathbb{C}^q$ via $g.f(\xi) = f(g^{-1}\xi)$. The $T_\xi$ are homogeneous of degree $-1$ on the space $\mathcal{P} = \mathbb{C}[\mathbb{C}^q]$ of polynomial functions on $\mathbb{C}^q$ and they commute: $T_\xi T_\eta = T_\eta T_\xi$ ([D1]). Hence the map $\xi \mapsto T_\xi$ extends uniquely to a linear map $p \mapsto p(T), \mathcal{P} \rightarrow \text{End}(\mathbb{C}^q)$. The Dunkl operators induce a sesquilinear pairing

$$[p, q]_k = (p(T)\overline{q})(0)$$

on $\mathcal{P}$, where $\overline{q}(\xi) := \overline{q(x)}$. In the following we assume that $k$ is non-negative. Then $[p, q]_k$ is actually a scalar product on $\mathcal{P}$, see Prop. 2.4. of [DX]. Moreover, for each fixed $w \in \mathbb{C}^q$, the joint eigenvalue problem

$$T_\xi f = \langle \xi, w \rangle f \quad \forall \xi \in \mathbb{C}^q; \quad f(0) = 1$$

has a unique holomorphic solution $f(z) = E_k(z, w)$ called the Dunkl kernel. It is symmetric in its arguments and satisfies $E_k(\lambda z, w) = E_k(z, \lambda w)$ for all $\lambda \in \mathbb{C}$ as well as $E_k(gz, w) = E_k(z, gw)$.
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for all \( g \in G \). The generalized Bessel function

\[
J_k(z, w) := \frac{1}{|G|} \sum_{g \in G} E_k(z, gw)
\]

is \( G \)-invariant in both arguments. Moreover, \( g(z) = J_k(z, w) \) is the unique holomorphic solution of the "Bessel system"

\[
p(T)g = p(w)g \quad \forall p \in \mathcal{P}^G; \quad g(0) = 1
\]

where \( \mathcal{P}^G \) denotes the subalgebra of \( G \)-invariant polynomials in \( \mathcal{P} \), see [O]. For crystallographic reflection groups and certain values of \( k \), the operators \( p(T) \), when restricted to \( G \)-invariant functions on \( \mathbb{R}^q \), constitute the system of invariant differential operators of a Euclidean-type symmetric space and the Bessel functions \( J_k(\cdot, w) \) can be identified with the associated spherical functions; for details see [dJ2]. The Dunkl kernel \( E_k \) gives rise to an integral transform on \( \mathbb{R}^q \) called the Dunkl transform. Let \( w_k \) denote the weight function

\[
w_k(x) = \prod_{\alpha \in R} \left| \langle \alpha, x \rangle \right|^{2k_\alpha}
\]

on \( \mathbb{R}^q \). The Dunkl transform is the integral transform on \( L^1(\mathbb{R}^q, w_k) \) defined by

\[
f \mapsto \hat{f}^k, \quad \hat{f}^k(\xi) = c_k^{-1} \int_{\mathbb{R}^q} f(x) E_k(-i\xi, x) w_k(x) dx \quad (\xi \in \mathbb{R}^q)
\]

with the constant

\[
c_k := \int_{\mathbb{R}^q} e^{-|x|^2/2} w_k(x) dx.
\]

A thorough study of this transform is given in [dJ1]. It has many properties in common with the usual Fourier transform to which it reduces in case \( k = 0 \). In particular, the Dunkl transform (as normalized above) extends to an isometric isometric isomorphism of \( L^2(\mathbb{R}^q, w_k) \), and \( (T_\eta f)^\wedge_k(\xi) = i\langle \xi, \eta \rangle \hat{f}^k(\xi) \) for differentiable \( f \) of sufficient decay. It is a long-standing open question whether \( L^1(\mathbb{R}^q, w_k) \) can be given the structure of a commutative Banach algebra so that the Dunkl transform becomes the Gelfand transform on its (symmetric) spectrum, similar as for commutative hypergroups. In the rank-one case there is such a convolution, but it is not positivity-preserving. For details and affirmative results in this direction see [R2]. It is however conjectured that for arbitrary \( G \) and \( k \geq 0 \), the Bessel functions \( J_k \) have a positive product formula which leads to a commutative hypergroup structure on a distinguished closed Weyl chamber \( \Xi \) of \( G \), the dual of this hypergroup being made up by the functions \( \xi \mapsto J_k(\xi, \eta), \eta \in \Xi \). In rank one and in all Cartan motion group cases this is true, see [R2]. In the following, we shall confirm this conjecture for three continuous series of multiplicities for root system \( B_q \). Indeed, we shall identify the characters of the hypergroups \( \Xi_{q, \mu} \) in Section 4.1 with Dunkl-type Bessel functions for \( B_q \) and thus obtain hypergroup structures with these Bessel functions as characters.

4.3 Dunkl theory and the convolutions on the Weyl chamber

Let us denote by \( J_k^B \) the Dunkl-type Bessel function associated with the reflection group \( G = B_q \) and multiplicity \( k = (k_1, k_2) \). and by \( [\cdot, \cdot]^B_k \) the associated Dunkl pairing. For \( z = (z_1, \ldots, z_q) \in \mathbb{C}^q \) we put \( z^2 = (z_1^2, \ldots, z_q^2) \). The following key result identifies \( J_k^B \) with a generalized \( 0F_1 \)-hypergeometric function of two arguments (recall the notions of Section 2.2):
Proposition 4.5. Let \( k = (k_1, k_2) \geq 0 \) and \( k_2 > 0 \). Then for all \( z, w \in \mathbb{C}^q \),

\[
J^B_k(z, w) = 0F_1^\alpha \left( \mu; \frac{z^2}{2}, \frac{w^2}{2} \right) \quad \text{with} \quad \alpha = \frac{1}{k_2}, \mu = k_1 + (m - 1)k_2 + \frac{1}{2}.
\]

This result was already mentioned in Section 6 of \([BF]\), but the reasoning there is rather sketchy, and there is an erraneous sign in one of the arguments. We therefore include a proof by different methods.

Proof. The modified Jack polynomials \( p_\lambda(z) = C_\lambda^\alpha(z^2) \), indexed by partitions \( \lambda \geq 0 \), are homogeneous of degree \( 2|\lambda| \) and form a basis of the vector space \( \mathcal{P}^G \) for \( G = B_q \). Thus the Bessel function \( J^B_k \) has a homogeneous expansion of the form

\[
J^B_k(z, w) = \sum_{\lambda \geq 0} a_\lambda(w)p_\lambda(z)
\]

with certain coefficients \( a_\lambda(w) \in \mathbb{C} \). In view of the Bessel system we have

\[
p_\lambda(Tz)J^B_k(z, w)|_{z=0} = p_\lambda(w)J^B_k(0, w) = C_\lambda^\alpha(w^2),
\]

where the superscript \( z \) indicates operation w.r.t. the variable \( z \). On the other hand, the results of \([BF]\) (relation (2.9) and the formula on top of p. 214) imply that the \( p_\lambda \) are orthogonal with respect to \([. , .]^B_k\) with

\[
[p_\lambda, p_\mu]^B_k = 4|\lambda|! |\mu|! (\mu)^\alpha \cdot C_\lambda^\alpha(1) =: M_{\lambda\mu}, \quad \mu = k_1 + (m - 1)k_2 + \frac{1}{2}.
\]

Differentiation of (4.6) (recall that \( \nu \) has real coefficients) now gives

\[
p_\lambda(Tz)J^B_k(z, w)|_{z=0} = \sum_{\nu \geq 0} a_\lambda(w)[p_\lambda, p_\nu]^B_k = M_{\lambda\nu}a_\lambda(w).
\]

Hence \( a_\lambda(w) = M^{-1}_{\lambda\lambda}C_\lambda^\alpha(w^2) \), which implies the assertion.

As a consequence of (4.6), Bessel functions associated with a symmetric cone can now be identified with Dunkl Bessel functions of type \( B_q \) with specific multiplicities:

Corollary 4.6. Let \( \Omega \) be an irreducible symmetric cone inside a Euclidean Jordan algebra of rank \( q \). Then for \( r, s \in \mathbb{R}^q \) with eigenvalues \( \xi = (\xi_1, \ldots, \xi_q) \) and \( \eta = (\eta_1, \ldots, \eta_q) \) respectively, we have

\[
J^B_k(r, s) = J^B_k(\xi, \eta)
\]

where \( k \) is given by \( k = k(\mu, d) = (\mu - \frac{d}{2}(q - 1) - \frac{1}{2}, \frac{d}{2}) \).

Now consider again the hypergroup structures \( \Xi_{q, \mu} = (\Xi, a_\mu) \) on

\[
\Xi_q = \{ \xi = (\xi_1, \ldots, \xi_q) \in \mathbb{R}^q : \xi_1 \geq \ldots \geq \xi_q \geq 0 \}
\]

which is actually a closed Weyl chamber for the reflection group \( B_q \). The consequence of our above identification can be formulated in a twofold way:

Corollary 4.7. The characters of the hypergroup \( \Xi_{q, \mu}, \mu \in \mathcal{M}_q \) are given by

\[
\psi_\mu(\xi) = J^B_k(\xi, \eta), \quad \eta \in \Xi_q,
\]

with the multiplicity \( k = k(\mu, d) \) as in the previous corollary.
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**Corollary 4.8.** Consider the root system of type $B_q$ with a multiplicity $k = (k_1, k_2)$ where $k_2 = \frac{d}{2}$ with $d \in \{1, 2, 4\}$ and $k_1 = \frac{d}{2}(p - q + 1) - \frac{1}{2}$ for integer $p \geq q$ or arbitrary $k_1 \geq \frac{1}{2}(d - 1)$. Then the associated Dunkl-type Bessel functions $\xi \mapsto J_k^B(\xi, \eta)$ are the characters of the hypergroup $(\Xi_B^\circ_{\mu} k)$ on the closed Weyl chamber $\Xi_q$, where $\mu = k_1 + (q - 1)k_2 + \frac{1}{2}$ and the convolution $\circ_{\mu}$ is defined over $F = \mathbb{R}, \mathbb{C}, \mathbb{H}$, depending on the value of $d$. In particular, the Bessel function $J_k^B$ satisfies the positive product formula

$$J_k^B(\xi, z)J_k^B(\eta, z) = \int_{\Xi_q} J_k^B(\zeta, z) d(\delta_\xi \circ_{\mu} \delta_\eta)(\zeta) \quad \forall \xi, \eta \in \Xi_q, z \in \mathbb{C}^q.$$ 

The hypergroup Fourier transform on $\Xi_{q,\mu}$ is given by

$$\hat{f}(\eta) = \int_{\Xi_q} f(\xi) J_k^B(\xi, i\eta) d\tilde{\omega}_\mu(\xi),$$

with $\tilde{\omega}_\mu = d_\mu h_\mu(\xi)d\xi$ as in Theorem 4.4. Notice that $h_\mu$ coincides up to a constant factor with the weight $w_k$ for $k = k(\mu, d)$. As $w_k$ is $B_q$-invariant, we therefore have

$$\hat{f}(\eta) = \text{const} \cdot \hat{F}^k(\eta),$$

where $F$ denotes the $B_q$-invariant extension of $f$ to $\mathbb{R}^q$ and $\hat{F}^k$ its Dunkl transform. Notice that $\hat{F}^k$ is $B_q$-invariant as well. With the Plancherel Theorem for the Dunkl transform at hand, we are now also in a position to determine the normalization constant $d_\mu$ of $\tilde{\omega}_\mu$ as announced previously. Indeed, recall from Theorem 4.4 that the Plancherel measure of the hypergroup $\Xi_{q,\mu}$ coincides with $\tilde{\omega}_\mu$ under the natural identification of $\Xi_{q,\mu}$ with its dual. Using this and the Plancherel theorem for the Dunkl transform, we readily obtain

$$\hat{f} = \hat{F}^k \big|_{\Xi_q}$$

and

$$d_\mu = \left( \int_{\Xi_q} h_\mu(x) e^{-|x|^2/2} dx \right)^{-1}.$$ 

The value of $d_\mu$ can be calculated explicitly; it is a particular case of a Selberg type integral which was evaluated by Macdonald [M1] for the classical root systems.

In the general Dunkl setting, there is a generalized translation on suitable function spaces which replaces the usual group addition to some extent, see [R1], [R2] and the references cited there. On $L^2(\mathbb{R}^q, w_k)$, this translation is defined by

$$\tau_\eta f(\xi) = c_k^{-1} \int_{\mathbb{R}^q} \hat{f}^k(\xi) E_k(i\xi, \zeta) E_k(i\eta, \zeta) w_k(\zeta) d\zeta.$$ 

On has $\tau_\eta : L^2(\mathbb{R}^q, w_k) \rightarrow L^2(\mathbb{R}^q, w_k)$ with $\tau_\eta f = E_k(i\eta, \zeta) \hat{f}^k(\zeta)$. If we restrict to the Weyl group invariant case for $B_q$ with multiplicities as in Corollary 4.8, then this generalized translation just coincides with the translation defined in terms of hypergroup convolution. If, say, $f$ belongs to $L^2(\mathbb{R}^q, w_k)$ and is also continuous and Weyl group invariant, then we have with the notations of Corollary 4.8

$$\tau_\eta f(\xi) = \delta_\xi *_{\mu} \delta_\eta(f) \quad \forall \xi, \eta \in \Xi_q.$$ 

When $\mu = \frac{pd}{2}$ with an integer $p \geq q$, i.e. $k_1 = \frac{d}{2}(p - q + 1) - \frac{1}{2}$, then the Bessel functions $J_k^B(\cdot, z)$ can be identified with the spherical functions of the Cartan motion group associated with the Grassmann manifold $U(p, q)/(U_p \times U_q)$. This follows from the discussion in [R12] (see also [R2]), and is in accordance with Remark 4.4 which implies that for $\mu = \frac{pd}{2}$, the
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The convolution of \( \Xi_{q,\mu} \) coincides with that of biinvariant measures for the Gelfand pair \(((U_p \times U_q) \rtimes M_{p,q}, U_p \times U_q))\). The multiplicative functions coincide with the (elementary) spherical functions of this Gelfand pair. Thus the hypergroups \( \Xi_{q,\mu} \) with \( \mu \geq \rho - 1 \) interpolate the discrete series of convolution algebras derived from the tangent space analysis on Grassmann manifolds.

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