Stability region and critical delay

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Abstract

The location of roots of the characteristic equation of a linear delay differential equation (DDE) determines the stability of the linear DDE. However, by its transcendency, there is no general criterion on the contained parameters for the stability. Here we mainly concentrate on the study of a simple transcendental equation (\(z + a - we^{-\tau z} = 0\)) with coefficients of real \(a\) and complex \(w\) and a delay parameter \(\tau > 0\) to tackle this transcendency brought by delay. The consideration is twofold: (i) to give the stability region in the parameter space for Eq. (*) by using the critical delay and (ii) to compare this with a graphical method (so-called the method of D-partitions) by combining with the delay sequence obtained by conditions for purely imaginary roots. By (i), we obtain another proof of Hayes’ and Sakata’s results, which reveals the nature of imaginary \(w\) case in Eq. (*). By (ii), we propose a method combining the analytic one and geometric one. This combination is important because it will be helpful in studying characteristic equations having higher-dimensional parameters.

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1 Introduction

Retarded functional differential equations (RFDEs) give a mathematical formulation of a class of delay differential equations (DDEs). It is well-known that for the exponential stability of the trivial solution of a given linear RFDE, it is necessary and sufficient that all the roots of the corresponding characteristic equation have negative real parts (refs. Hale and Verduyn Lunel [27] and Diekmann et al. [20]). Such a characteristic equation is transcendental in general, and it has infinitely many roots in principle in the complex number plane $\mathbb{C}$. For this reason, it is difficult to obtain the condition on the contained parameters for which all the roots are located in the left half of the complex plane $\mathbb{C}$.

Many authors have elaborated on the study of the transcendental equations obtained as characteristic equations of linear RFDEs, where various studies exist depending on the nature of the time-delay structure and on the form of differential equations (e.g., higher-order equations, systems of equations, or neutral equations). We refer the reader
to Stépán [54] as a general reference of the stability problem of linear RFDEs. See e.g., [9], [19], [33], [46], [55], [24], and [59] for recent studies. See also [25] for linear stability analysis of partial differential equations with time-delay. However, the understanding of a simple transcendental equation of the form

\[ z + a - w e^{-\tau z} = 0 \tag{*} \]

with complex coefficients \(a\) and \(w\) and with a delay parameter \(\tau > 0\) has not yet completed. Here Eq. (*) is obtained as the characteristic equation of a scalar linear DDE

\[ \dot{x}(t) = -ax(t) + wx(t - \tau) \quad (t \in \mathbb{R}, x(t) \in \mathbb{C}) \tag{1.1} \]

by assuming a complex exponential solution \(x(t) = e^{zt}\). In this paper, we will call the region in the parameter space for which all the roots of Eq. (*) have negative real parts the stability region. Usually, delay parameters have special natures different from those which usual control parameters have. For this reason, we separate delay parameters from the parameter space in which stability region is considered.

The purpose of this paper is to provide a unified perspective to obtain the stability region of Eq. (*) with real \(a\) and complex \(w\), where the existence and the expression of the critical delay \(\tau_c(a, w)\) are essential. Here the critical delay \(\tau_c(a, w)\) means the threshold \(\tau\)-value which divides the positive real number line into two parts so that all the roots of Eq. (*) have negative real parts for \(\tau \in (0, \tau_c(a, w))\), and Eq. (*) has a root with positive real part for \(\tau \in (\tau_c(a, w), \infty)\). In [39], it has been essentially shown that the critical delay \(\tau_c(a, w)\) is given by

\[ \tau_c(a, w) = \frac{1}{\sqrt{|w|^2 - a^2}} \left[ |\text{Arg}(w)| - \arccos \left( \frac{a}{|w|} \right) \right] \tag{1.2} \]

for real \(a\) and complex \(w\) satisfying \(w \neq 0\) and \(\Re(w) < a < |w|\).

Here \(\Re(w)\) denotes the real part of \(w\), \(\text{Arg}(w) \in (-\pi, \pi]\) is the principal value of the argument of nonzero \(w\), and

\[ \arccos : [-1, 1] \to [0, \pi] \]

is the inverse function of \(\cos : [0, \pi] \to [-1, 1]\). In the following three paragraphs, we briefly review some studies related to Eq. (*)

The case of imaginary \(a\) is a source of many interesting dynamics (e.g., see [60], [30], and [22]). We note that a necessary and sufficient condition on \(a, w,\) and \(\tau\) for which all the roots of Eq. (*) have negative real parts is obtained in [43], and it has been applied to the stabilization of an unstable equilibrium of autonomous ordinary differential equations by the delayed feedback control proposed by Pyragas [47] (cf. [30]). Here the choice of imaginary \(a\) is essential because there is no stabilization in Eq. (*) as increasing the delay parameter \(\tau\) from 0 if \(a\) is real. However, the detailed stability region in \((a, w)\)-space, which is a real 4-dimensional space because \(a\) and \(w\) are complex, has not been obtained (cf. [11]).

The case of real \(a\) and \(w\) is studied by Hayes [28]. Unlike the complex coefficients case, the complete picture of the stability region in \((a, w)\)-plane has been obtained (refs.
In this case, it is important that \((a, w)-\text{plane}\) is 2-dimensional. The case of real \(a\) and complex \(w\) is studied by Sakata [49]. Although the picture of the stability region is depicted, the statement and the proof are complicated, which makes it difficult to understand the nature of this situation. We note that \((a, w)-\text{space}\) for real \(a\) and complex \(w\) is 3-dimensional, but the consideration can be reduced to regions in \((a, |w|)-\text{plane}\) by fixing the argument of \(w\).

It should be pointed out that there are incorrect results in the literature. Borrowing one of the results and the arguments discussed by Braddock and van den Driessche [10], Belair [3, Theorem 2.6] has discussed the stability region of Eq. (1.5) in the complex \(w\)-plane by letting \(a := 1\). As is already pointed out by Takada, Hori, and Hara [56], [3, Theorem 2.6] is incorrect. We note that the corresponding result in [10] is also incorrect.

In this paper, we will show that the critical delay function

\[
(a, w) \mapsto \tau_c(a, w) \in (0, \infty) \tag{1.3}
\]

has enough power to deduce the stability region of Eq. (1.5) in \((a, w)\)-space by solving the inequality

\[
\tau_c(a, w) > \tau \tag{1.4}
\]

with respect to \((a, w)\) for the case that \(a\) is real and \(w\) is complex. We call the corresponding method the \textit{method by critical delay}. The idea of solving inequality (1.4) is to use the following expression of the critical delay:

\[
\tau_c(a, w) = \frac{1}{\sqrt{|w|^2 - a^2}} \left[ |\text{Arg}(w)| - \text{arccot} \left( \frac{a}{\sqrt{|w|^2 - a^2}} \right) \right]. \tag{1.5}
\]

Here \(\cot \theta := \cos \theta / \sin \theta\) for \(\theta \notin \pi \mathbb{Z}\), and \(\text{arccot} : \mathbb{R} \to (0, \pi)\) denotes the inverse function of \(\cot |(0, \pi) : (0, \pi) \to \mathbb{R}\).

To see the effectiveness of the method by critical delay, we first concentrate our consideration on the real \(a, w\) case and give another proof of Hayes’ result by using the critical delay function. From expression (1.5), it will be turned out that solving an inequality

\[
C(\theta) := \theta \cot \theta < r
\]

with respect to \(\theta \in (0, \pi)\) is essential, and by the monotonicity of the function \(C : (0, \pi) \to (-\infty, 1)\), the condition is simply expressed by the inverse function \(C^{-1} : (-\infty, 1) \to (0, \pi)\). This may be elementary but will give an insight into the analysis of real \(a\) and imaginary \(w\) case. We next move to the consideration of the real \(a\) and imaginary \(w\) case and give another proof of Sakata’s result by using the critical delay function. Expression (1.5) also leads us to solve an inequality

\[
C(\theta; \varphi) := \theta \cot(\theta - \varphi) < r \tag{1.6}
\]

for \(\theta \in (0, \varphi)\), and it plays an essential role to obtain the stability region in \((a, |w|)-\text{plane}\). Here \(\varphi \in (0, \pi)\) corresponds to the absolute value of the principal value of the argument \(\text{Arg}(w)\) of \(w \in \mathbb{C} \setminus \mathbb{R}\), and the real parameter \(r\) will be given appropriately. We note that the real \(w\) case can be considered as a limiting case of \(\varphi \uparrow \pi\).

The above mentioned approach for the proof of Sakata’s result is simple but needs elaborative calculations to obtain the behavior of the function \(C(\cdot; \varphi)\). In the literature,
there is another method to obtain the stability regions of transcendental equations, which is the so called method of D-partitions (refs. El'sgol'ts and Norkin [21], Kolmanovskii and Nosov [35]). Basically, this is a method to obtain hyper-surfaces in the parameter space by considering the condition on the parameters under which a given transcendental equation has a root $i\Omega$ for some real number $\Omega$. Here $i$ is the imaginary unit, and the real number $\Omega$ corresponds to the angular frequency of the corresponding periodic solutions (i.e., $T = 2\pi/|\Omega|$ for the period $T > 0$ of the periodic solution$^1$).

In this paper, we will also compare the above mentioned another proof of Sakata’s result based on the method by critical delay with the method of D-partitions. A direct calculation will show that for each fixed $\text{Arg}(w)$, $a$ and $|w|$ are parametrized by the angular frequency $\Omega$. Here we have two distinct points from the real $w$ case: (i) The property that $i\Omega$ is a root of Eq. (2) does not necessarily imply that its complex conjugate $-i\Omega$ is a root of Eq. (2). Therefore, it is insufficient to only consider the case $\Omega > 0$. (ii) $|w|$ should be kept positive.

The main ingredient in this paper for the study of the stability region of Eq. (2) via the method of D-partitions is to connect the curves parametrized by angular frequency with the sequence composed of the $\tau$-values for which Eq. (2) has purely imaginary roots. These $\tau$-values are essentially obtained in [39], but there is an ambiguity because two sequences composed of $\tau$-values are given in [39]. The above mentioned connection gives a “one-to-one correspondence” between the curves and the $\tau$-values of the sequence. Furthermore, it naturally produce an “ordering” of the curves via the ordering of the $\tau$-values.

This paper is organized as follows. In Section 2, we give a detailed explanation of the method by critical delay for Eq. (2) with $a \in \mathbb{R}$ and $w \in \mathbb{C}$. Here the domain of definition of critical delay function (1.3) is also determined. In Section 3 to give a general insight into the method by critical delay, we study the dependence of real parts of roots of Eq. (2) on the delay parameter $\tau > 0$ for the case of $a \in \mathbb{C}$ and $w \in \mathbb{C} \setminus \{0\}$. In Subsection 3.3 we examine the crossing direction of a purely imaginary root $i\Omega_0$ for some $\Omega_0 \in \mathbb{R} \setminus \{0\}$ with respect to the delay parameter $\tau > 0$. Then the obtained result will reveal that its crossing direction is determined by the sign of

$$\Omega_0(\Omega_0 + \Im(a)),$$

where $\Im(a)$ is the imaginary part of $a$. The above value is not necessarily positive if $\Im(a) \neq 0$. Therefore, the imaginary $a$ case will be more complicated. For this reason, we assume that $a$ is real in the later sections. In Section 4, we will find conditions on $\tau$ for which Eq. (2) has a purely imaginary root to know what happens for general parameter values. Sections 5 and 6 are devoted to the investigation of the stability region of Eq. (2) by the method by critical delay, where another proof of Hayes’ and Sakata’s results are given. In Section 7, we discuss the comparison between the method by critical delay and the method of D-partitions. In Section 8, we will discuss the imaginary $a$ case and the case of multiple delays to contribute to possible future researches. Appendix A gives a proof of the key theorem in this paper via the Lambert $W$ function.

**Notations**

Let $i$ denote the imaginary unit. For a complex number $z$, its complex conjugate is denoted by $\bar{z}$. When $z \neq 0$, let $\text{Arg}(z)$ be the principal value of the argument, i.e., $\text{Arg}(z) \in (-\pi, \pi]\n$
and \( z = |z|e^{i\text{Arg}(z)} \). Throughout this paper, we will use the following notation.

**Notation 1.** For each \( a, w \in \mathbb{C} \), let \( T(a, w) \) denote the set of all \( \tau > 0 \) for which all the roots of Eq. (\( * \))
\[
z + a - we^{-\tau z} = 0
\]
have negative real parts.

When \( w = 0 \), we have
\[
T(a, 0) = \begin{cases} (0, \infty) & (\Re(a) > 0), \\ \emptyset & (\Re(a) \leq 0) \end{cases}
\]
because Eq. (\( * \)) becomes \( z + a = 0 \).

### 2 Method by critical delay for real \( a \) and complex \( w \)

In this section, we study Eq. (\( * \))
\[
z + a - we^{-\tau z} = 0
\]
for given \( a \in \mathbb{R} \), \( w \in \mathbb{C} \), and \( \tau > 0 \). The basics of our consideration is the following theorem.

**Theorem 2.1** (cf. [39]). Suppose \( a \in \mathbb{R} \) and \( w \in \mathbb{C} \). Then the following statements hold:

(I) \( T(a, w) = (0, \infty) \) if and only if \( a \geq |w| \) and \( a > \Re(w) \).

(II) \( T(a, w) \) is a nonempty proper subset of \( (0, \infty) \) if and only if \( w \neq 0 \) and \( \Re(w) < a < |w| \).

In this case, \( T(a, w) = (0, \tau_c(a, w)) \) holds, where \( \tau_c(a, w) > 0 \) is expressed by
\[
\tau_c(a, w) = \frac{1}{\sqrt{|w|^2 - a^2}} \left[ |\text{Arg}(w)| - \arccos \left( \frac{a}{|w|} \right) \right]
\]
or
\[
\tau_c(a, w) = \frac{1}{\sqrt{|w|^2 - a^2}} \left[ |\text{Arg}(w)| - \arccot \left( \frac{a}{\sqrt{|w|^2 - a^2}} \right) \right].
\]

(III) \( T(a, w) \) is empty if and only if \( a \leq \Re(w) \).

**Remark 1.** The condition \( a \geq |w| \) and \( a > \Re(w) \) in (I) is equivalent to
\[
a \in \begin{cases} (|w|, \infty) & (\text{Arg}(w) = 0), \\ [|w|, \infty) & (\text{otherwise}). \end{cases}
\]

Therefore, it is also equivalent to \( a \geq |w| \) and \( a \neq w \).
Theorem 2.1 except (1.5) is obtained by Matsunaga [39, Theorem 2] for the imaginary $w$ case with a different expression of the critical delay. See the latter discussion for the comparison. We note that expression (1.5) is obtained by using an identity

$$\arccos(x) = \text{arccot} \left( \frac{x}{\sqrt{1-x^2}} \right) \quad (x \in (-1, 1)),$$

which is obtained from

$$\cot(\arccos x) = \frac{x}{\sqrt{1-x^2}} \quad (x \in (-1, 1)).$$

We will call the subsets corresponding to the cases (I), (II), and (III) in Theorem 2.1 the delay-independent stability region, delay-dependent stability or instability region, and delay-independent instability region, respectively. See Fig. 1 for an illustration of the each regions in $(a, |w|)$-plane obtained by Theorem 2.1 when $\text{Arg}(w) = 0$ and $\text{Arg}(w) = 3\pi/4$, respectively. Here we are considering $\text{Arg}(w)$ as a parameter.

### 2.1 Comparison with Matsunaga’s expression of critical delay

In [39], the critical delay is given by

$$\frac{\text{sgn}(b)}{\sqrt{b^2 - a^2}} \left[ \arccos \left( -\frac{a}{b} \right) - |\theta| \right], \quad (2.1)$$

where the parameter $w$ corresponds to $-be^{i\theta}$ with $b \in \mathbb{R} \setminus \{0\}$ and $-\pi/2 \leq \theta \leq \pi/2$, and $\text{sgn}(b) := b/|b|$. The value (2.1) is indeed equal to the right-hand side of (1.2) as the following lemma shows.

**Lemma 2.2.** Let $w = -be^{i\theta}$ for $b \in \mathbb{R} \setminus \{0\}$ and $\theta \in [-\pi/2, \pi/2]$. Then

$$\frac{\text{sgn}(b)}{\sqrt{b^2 - a^2}} \left[ \arccos \left( -\frac{a}{b} \right) - |\theta| \right] = \frac{1}{\sqrt{|w|^2 - a^2}} \left[ |\text{Arg}(w)| - \arccos \left( \frac{a}{|w|} \right) \right]$$

holds.
Proof. The proof is divided into the following two cases.

- Case 1: \( b > 0 \). Since \(|w|e^{i \text{Arg}(w)} = be^{i(\theta + \pi)}\), we have

\[
|w| = b \quad \text{and} \quad \text{Arg}(w) = \begin{cases} 
\theta - \pi & (0 < \theta \leq \pi/2), \\
\theta + \pi & (-\pi/2 \leq \theta \leq 0),
\end{cases}
\]

where \( \text{Arg}(w) < 0 \) for \( 0 < \theta \leq \pi/2 \) and \( \text{Arg}(w) > 0 \) for \( -\pi/2 \leq \theta \leq 0 \). Therefore,

\[
-|\theta| = \begin{cases} 
-\text{Arg}(w) - \pi & (0 < \theta \leq \pi/2), \\
\text{Arg}(w) - \pi & (-\pi/2 \leq \theta \leq 0)
\end{cases} = |\text{Arg}(w)| - \pi.
\]

By using this and an identity

\[
\arccos(x) + \arccos(-x) = \pi \quad (x \in [-1, 1])
\]

we obtain

\[
\arccos\left(-\frac{a}{b}\right) - |\theta| = |\text{Arg}(w)| - \arccos\left(\frac{a}{|w|}\right).
\]

- Case 2: \( b < 0 \). Since \(|w| = -b\) and \( \text{Arg}(w) = \theta \), we have

\[
\arccos\left(-\frac{a}{b}\right) - |\theta| = -\left[|\text{Arg}(w)| - \arccos\left(\frac{a}{|w|}\right)\right].
\]

This completes the proof.

2.2 Critical delay and its domain of definition for real \( a \) and \( w \)

The following result is a direct consequence of Theorem 2.1. Therefore, the proof can be omitted. See [6, Theorem 2.3] for the result with \( a = 1 \) and \( w \leq 0 \). See [7, Theorem] for the result with general \( a, w \in \mathbb{R} \).

**Theorem 2.3** ([7], cf. [15], [6]). Suppose \( a, w \in \mathbb{R} \). Then the following statements hold:

(I) \( T(a, w) = (0, \infty) \) if and only if \( a > 0 \) and \( -a \leq w < a \).

(II) \( T(a, w) \) is a nonempty proper subset of \( (0, \infty) \) if and only if

\[
w < -|a|.
\]

In this case, \( T(a, w) = (0, \tau_c(a, w)) \) holds, where \( \tau_c(a, w) > 0 \) is expressed by

\[
\tau_c(a, w) = \frac{1}{\sqrt{w^2 - a^2}} \arccos\left(\frac{a}{w}\right) = \frac{1}{\sqrt{w^2 - a^2}} \arccot\left(\frac{a}{\sqrt{w^2 - a^2}}\right).
\]

(III) \( T(a, w) \) is empty if and only if \( w \geq a \).
Figure 2: Decomposition of \((a, w)\)-plane obtained by Theorem 2.3

See Fig. 2 for the picture of \((a, w)\)-plane decomposed by the nature of \(T(a, w)\) given in Theorem 2.3. It is a special case of Fig. 1. The above subset
\[
\{(a, w) \in \mathbb{R}^2 : w < -|a|\}
\]
is the domain of definition of critical delay function (1.3) \((a, w) \mapsto \tau_c(a, w) \in (0, \infty)\) when \(w\) varies in the real number line. We note that the expressions of \(\tau_c(a, w)\) above are obtained by identities (2.2) and
\[
\arccot(x) + \arccot(-x) = \pi \quad (x \in \mathbb{R})
\]
since \(w < 0\).

Remark 2. When \(a > w\) and Eq. (4) is not asymptotically stable independent of delay, Cooke and Grossman [15, Section 2] discussed the existence of the critical delay value \(\tau_0 > 0\) satisfying the following properties: (1) For all \(0 < \tau < \tau_0\), all the roots have negative real parts, (2) For all \(\tau > \tau_0\), there exists a root whose real part is positive. However, the explicit expression of \(\tau_0\) is not given in [15].

2.3 Method by critical delay and stability region

For the sake of clarity, we introduce the following notation.

Notation 2. Let
\[
D_c := \{(a, w) \in \mathbb{R} \times (\mathbb{C} \setminus \{0\}) : \Re(w) < a < |w|\},
\]
be a linear cone which is the domain of definition of critical delay function (1.3). It corresponds to the delay-dependent stability or instability region (II).

Here a nonempty subset \(C\) (of a linear topological space over \(\mathbb{R}\)) is called a linear cone if for every \(v \in C\) and every \(s > 0\), \(sv \in C\) holds.

Theorem 2.1 shows that all the roots of Eq. (1) have negative real parts if and only if \(a\) and \(w\) satisfy one of the following properties:

(I) \(a \geq |w|\) and \(a > \Re(w)\).

(II) \((a, w) \in D_c\) and inequality (1.4) \(\tau_c(a, w) > \tau\) holds.
From expressions (1.2) or (1.5) of the critical delay, we have
\[ \tau_c(sa, sw) = \frac{1}{s} \tau_c(a, w) \] (2.4)
for every \( s > 0 \) and every \((a, w) \in D_c\). Therefore, for each fixed \( \tau > 0 \), inequality (1.4) is equivalent to \( \tau_c(\tau a, \tau w) > 1 \). This means that all the roots of Eq. (2) have negative real parts if and only if \( a \) and \( w \) satisfy one of the following properties:

(I) \( a \geq |w| \) and \( a > \Re(w) \).

(II) \((a, w) \in D_c \) and an inequality \( \tau_c(\tau a, \tau w) > 1 \) holds.

We call the above method to find the stability region the method by critical delay.

3 Dependence of roots with respect to delay parameter

Let \( a \in \mathbb{C} \) and \( w \in \mathbb{C} \setminus \{0\} \) be given. In this section, we study the dependence of zeros of the function \( h(\cdot; a, w, s) : \mathbb{C} \to \mathbb{C} \) defined by

\[ h(z; a, w, s) := z + a - we^{-sz} \] (3.1)
on the real parameter \( s \in \mathbb{R} \).

3.1 Argument by Rouché’s theorem

For each fixed \( s_0 \in \mathbb{R} \), we have

\[ |h(z; a, w, s) - h(z; a, w, s_0)| = |w| |e^{-sz} - e^{-s_0z}| = |w| |e^{-s_0z}| e^{-(s-s_0)z} - 1| \]

for all \( z \in \mathbb{C} \) and all \( s \in \mathbb{R} \). Let \( D \) be a given open disk of the complex plane \( \mathbb{C} \) so that \( h(\cdot; a, w, s_0) \) does not have any zeros on the boundary \( \partial D \) of \( D \). We note that such an open disk can be always chosen because \( h(\cdot; a, w, s_0) \) is not constant. We have

\[ \sup_{z \in \partial D} |e^{-s_0z}| \leq M \]

for some \( M > 0 \). Furthermore, the term \( |e^{-(s-s_0)z} - 1| \) converges to 0 as \( s \to s_0 \) uniformly in \( z \in \partial D \). This shows that

\[ \sup_{z \in \partial D} |h(z; a, w, s) - h(z; a, w, s_0)| < \inf_{z \in \partial D} |h(z; a, w, s_0)| \]

holds for all sufficiently small \( |s - s_0| \) because \( h(\cdot; a, w, s_0) \) has no zeros on \( \partial D \). Therefore, by applying Rouché’s theorem, there exists a \( \delta > 0 \) such that for all \( s \in (s_0 - \delta, s_0 + \delta) \), \( h(\cdot; a, w, s) \) and \( h(\cdot; a, w, s_0) \) have the same number of zeros on \( D \) counted with multiplicity.

By combining the above argument and Theorem 2.1, the following lemma is obtained.

Lemma 3.1. Let \( a \in \mathbb{R} \) and \( w \in \mathbb{C} \setminus \{0\} \) be fixed so that \((a, w) \in D_c\). Then Eq. (2) with \( \tau = \tau_c(a, w) \) does not have any roots with positive real parts and has a root on the imaginary axis.
Proof. By combining the above argument and Theorem 2.1, it holds that Eq. (8) with \( \tau = \tau_c(a, w) \) does not have any roots with positive real parts. To show the existence of a root on the imaginary axis, we use the following estimate: For every zero \( z \) of \( h(\cdot; a, w, s) \),

\[
|z| = |-a + we^{-sz}| \leq |a| + |w|e^{-s\Re(z)}.
\]  

(3.2)

We now suppose \( s \) is nonnegative. From (3.2), all the zeros of \( h(\cdot; a, w, s) = 0 \) whose real parts are greater than or equal to \( \sigma \in \mathbb{R} \) should be located in the closed disk with center at 0 and radius \( |a| + |w|e^{-s\sigma} \).

This shows that the number of such zeros is finite counted with multiplicity. Therefore, the combination of the argument by Rouché’s theorem and Theorem 2.1 also shows that Eq. (8) with \( \tau = \tau_c(a, w) \) has a root on the imaginary axis.

3.2 Asymptotics of real parts of roots as delay tends to zero

Inequality (3.2) also holds when \( a \) is complex and shows that the real part \( \Re(z) \) satisfies

\[
|\Re(z)| \leq |a| + |w|e^{-s\Re(z)}
\]

(3.3)

because of \( |\Re(z)| \leq |z| \). To analyze inequality (3.3), we will use the real branches of the Lambert \( W \) function. Here the Lambert \( W \) function is the inverse function of a complex function \( \mathbb{C} \ni z \mapsto ze^z \in \mathbb{C} \) in the sense of set-valued function, i.e., \( W(\zeta) \) is defined by

\[ W(\zeta) := \{ z \in \mathbb{C} : ze^z = \zeta \} \]

for any \( \zeta \in \mathbb{C} \). We refer the reader to Corless et al. [17] as a review paper on the Lambert \( W \) function. See [13] for a short survey of the Lambert \( W \) function. For a discussion on the naming of the Lambert \( W \) function, e.g., see [29]. There is an attempt to generalize the Lambert \( W \) function (see [42]).

The Lambert \( W \) function is strongly related to Eq. (3). Indeed, the set of all roots of Eq. (3) is equal to

\[ \frac{1}{\tau}W(\tau we^{\tau a}) - a := \left\{ z - a : z \in \frac{1}{\tau}W(\tau we^{\tau a}) \right\} \]

(3.4)

because Eq. (3) can be transformed into

\[ \tau(z + a) \cdot e^{\tau(z + a)} = \tau we^{\tau a} . \]

We note that it is common to use the Lambert \( W \) function for numerical investigations of the location of the roots of Eq. (3) (e.g., see [1], [30], [31], [52], [22], [57], [61], [9], and [55]).

As is mentioned above, we use the real branches of the Lambert \( W \) function for the study of the asymptotics of real parts of roots of Eq. (3) as \( \tau \downarrow 0 \). Since the function \( \mathbb{R} \ni x \mapsto xe^x \in \mathbb{R} \) is strictly monotonically increasing on \( [-1, \infty) \) and strictly monotonically decreasing on \( (-\infty, -1] \), the Lambert \( W \) function has two real branches \( W_0 : [-1/e, \infty) \to \mathbb{R} \) and \( W_{-1} : [-1/e, 0) \to \mathbb{R} \). These graphs are depicted in Fig. 3.

The following is the analytic result showing the asymptotics of real parts of roots.
Theorem 3.2. Let $a \in \mathbb{C}$ and $w \in \mathbb{C} \setminus \{0\}$ be fixed. For each sufficiently small $\tau > 0$, let $\Sigma_0^+(\tau)$, $\Sigma_0^-(\tau)$, and $\Sigma_{-1}(\tau)$ be positive numbers given by

$$
\Sigma_0^+(\tau) := \frac{1}{\tau}W_0(\tau|w|e^{-\tau|a|}) + |a|,
$$

$$
\Sigma_0^-(\tau) := \frac{1}{\tau}W_0(-\tau|w|e^{\tau|a|}) - |a|,
$$

and

$$
\Sigma_{-1}(\tau) := \frac{1}{\tau}W_{-1}(-\tau|w|e^{\tau|a|}) - |a|.
$$

Then for every sufficiently small $\tau > 0$, any root of Eq. (3.3) satisfies

$$
\Re(z) \leq \Sigma_{-1}(\tau) \quad \text{or} \quad \Sigma_0^-(\tau) \leq \Re(z) \leq \Sigma_0^+(\tau),
$$

where

$$
\lim_{\tau \downarrow 0} \Sigma_0^+(\tau) = \pm(|a| + |w|) \quad \text{and} \quad \lim_{\tau \downarrow 0} \Sigma_{-1}(\tau) = -\infty
$$

hold.

Proof. Let $s$ be positive. We transform inequality (3.3) as follows.

- On inequality $\Re(z) - |a| \leq |w|e^{-s\Re(z)}$. By multiplying both sides of the inequality by $se^{(\Re(z)-|a|)} > 0$, it becomes

$$
s(\Re(z) - |a|)e^{s(\Re(z)-|a|)} \leq s|w|e^{-s|a|}.
$$

Since the right-hand side is positive, the above inequality can be solved as

$$
s(\Re(z) - |a|) \leq W_0(s|w|e^{-s|a|}),
$$

i.e., $\Re(z) \leq \Sigma_0^+(s)$. 

Figure 3: Real branches $W_0$ (solid) and $W_{-1}$ (dashed) of the Lambert $W$ function

\[\]
• On inequality $\Re(z) + |a| \geq -|w|e^{-s\Re(z)}$: By multiplying both sides of the inequality by $se^{s(\Re(z) + |a|)} > 0$, it becomes

$$s(\Re(z) + |a|)e^{s(\Re(z) + |a|)} \geq -s|w|e^{s|a|}.$$ 

We may assume

$$0 < s|w|e^{s|a|} < \frac{1}{e}$$

by choosing $s > 0$ sufficiently small. Since $-s|w|e^{s|a|} \in (-1/e, 0)$, the above inequality can be solved as

$$s(\Re(z) + |a|) \geq W_0(-s|w|e^{s|a|}),$$

or

$$s(\Re(z) + |a|) \leq W_{-1}(-s|w|e^{s|a|}),$$

i.e., $\Re(z) \geq \Sigma^0_0(s)$ or $\Re(z) \leq \Sigma_{-1}(s)$.

By combining the above inequalities, we obtain the condition on $\Re(z)$. The limit relations follow by $W'_0(0) = 1$ and $\lim_{y \to 0} W_{-1}(y) = -\infty$. This completes the proof.

### 3.3 Crossing direction of a purely imaginary root

We investigate the crossing direction of a purely imaginary root of $h(\cdot; a, w, s)$ with respect to the real parameter $s \in \mathbb{R}$. It has been considered by Cooke and Grossman [15, Section 2] (for real $a, w$ case) and Matsunaga [39, Lemma 3] (for real $a$ and complex $w$ case). We note that $h(0; a, w, s) = 0$ for some $s$ if and only if $a = w$. In this case,

$$h(0; a, a, s) = 0$$

for all $s \in \mathbb{R}$, and therefore, only the trivial branch is obtained as a function of $s$. This is a consequence of the argument by Rouché’s theorem.

Suppose $h(z_0; a, w, s_0) = 0$. Since the derivative of $h(\cdot; a, w, s)$ is given by

$$h'(z; a, w, s) = 1 + swe^{-sz},$$

we have $h'(z_0; a, w, s_0) = 1 + s_0(z_0 + a)$.

**Theorem 3.3** (cf. [15], [39]). Let $a \in \mathbb{C}$ and $w \in \mathbb{C}\{0\}$ be given. Suppose $h(i\Omega_0; a, w, s_0) = 0$ holds for some $\Omega_0 \in \mathbb{R} \setminus \{0\}$ and $s_0 \in \mathbb{R}$. We assume that $h'(i\Omega_0; a, w, s_0) \neq 0$, i.e.,

$$i\Omega_0 + a \neq -\frac{1}{s_0}$$

holds. Then there exist a $\delta > 0$ and a unique continuously differentiable function

$$z(\cdot): (s_0 - \delta, s_0 + \delta) \rightarrow \mathbb{C}$$

\footnote{By using $W_0$, the condition can be expressed as

$$s < \frac{1}{|a|} W_0\left(\frac{|a|}{|w|e}\right)$$

when $a \neq 0$.}
such that $z(s_0) = i\Omega_0$ and $h(z(s); a, w, s) = 0$ holds for all $|s - s_0| < \delta$. The derivative of $z(\cdot)$ at $s_0$ is calculated as

$$
z'(s_0) = \frac{\Omega_0(\Omega_0 + i\Im(a))}{(1 + s_0\Re(a))^2 + |s_0(\Omega_0 + i\Im(a))|^2} - i\Omega_0 s_0\left[\frac{(\Omega_0 + i\Im(a))^2 + \Re(a)^2 + \Re(a)}{(1 + s_0\Re(a))^2 + |s_0(\Omega_0 + i\Im(a))|^2}\right].$$

**Proof.** The existence of the function $z(\cdot)$ is a consequence of the implicit function theorem. Therefore, we only need to calculate the derivative $z'(s_0)$. Since

$$\frac{\partial h}{\partial z} = 1 + sw e^{-sz} \quad \text{and} \quad \frac{\partial h}{\partial s} = zw e^{-sz},$$

$z(\cdot)$ satisfies a differential equation

$$z'(s) = \frac{z(s)(z(s) + a)}{1 + s(z(s) + a)}$$

by the implicit differentiation. Here $we^{-sz} = z(s) + a$ is used. We note that the right-hand side of the above differential equation does not depend on $w$ explicitly. By letting $s := s_0$, we have

$$z'(s_0) = \frac{\Omega_0(\Omega_0 + i\Im(a)) - i\Omega_0 \Re(a)}{1 + s_0\Re(a) + is_0(\Omega_0 + i\Im(a))}.$$

We omit the further calculation.

**Remark 3.** When $s_0 = 0$, $h'(z_0; a, w, s_0) = 1$ holds. When $s_0 \neq 0$, the condition $h'(z_0; a, w, s_0) \neq 0$ is equivalent to

$$z_0 \neq -\frac{1}{s_0} - a,$$

which is automatically satisfied when $z_0$ is purely imaginary and $a$ is real.

**Remark 4.** Theorem [3,3] shows that the crossing direction of purely imaginary zero with respect to the real parameter $s$ is determined by the sign of the real number

$$\Omega_0(\Omega_0 + i\Im(a)).$$

It makes clear the difference between the real $a$ case and the imaginary $a$ case. This should be compared with [16, Proposition 1] and [8, Section 3].

### 4 Purely imaginary roots

Let $a \in \mathbb{R}$ and $w \in \mathbb{C} \setminus \{0\}$. In this section, we find the $\tau$-values for which Eq. (1) has a purely imaginary roots.

#### 4.1 Angular frequency equations

For some $\Omega \in \mathbb{R}$, if $i\Omega$ is a root of Eq. (1) if and only if $i\Omega + a - we^{-i\tau\Omega} = 0$, i.e.,

$$\begin{cases} a - |w|\cos(\text{Arg}(w) - \tau\Omega) = 0, \\
\Omega - |w|\sin(\text{Arg}(w) - \tau\Omega) = 0. \end{cases}$$
When $\Omega$ is considered to be an unknown variable, we call (4.1) the *angular frequency equations*. In this consideration, it is natural to assume that the delay parameter $\tau$ is also one of the unknown variables. Then it is expected that one can solve (4.1) with respect to $(\Omega, \tau)$ for each given $(a, w) = (a, |w|, \text{Arg}(w))$.

From angular frequency eqs. (4.1), $\Omega$ necessarily satisfies
\[ |w|^2 = a^2 + \Omega^2 \]
by the trigonometric identity $\cos^2(\cdot) + \sin^2(\cdot) \equiv 1$. This also imposes that $|w|^2 - a^2 \geq 0$, i.e.,
\[ |a| \leq |w| \quad \text{and} \quad \Omega = \pm \sqrt{|w|^2 - a^2}. \]

When $|a| = |w|$, the following statements hold:

- Suppose $a = |w|$. Then Eq. (4.1) has a root on the imaginary axis if and only if $\text{Arg}(w) = 0$.
- Suppose $a = -|w|$. Then Eq. (4.1) has a root on the imaginary axis if and only if $\text{Arg}(w) = \pi$.

In the above cases, 0 is the only root on the imaginary axis. This consideration shows that nontrivial situations will occur only when $w \neq 0$ and $|a| < |w|$. We note that $D_c$ is contained in this region, and the delay-independent instability region (III) intersects with this region.

### 4.2 Angular frequencies

To study nontrivial situations, we introduce the following notation.

**Notation 3.** Suppose $a \in \mathbb{R}$, $w \in \mathbb{C} \setminus \{0\}$, and $|a| < |w|$. Let
\[ \Omega(a, w) := \Omega(a, |w|) := \sqrt{|w|^2 - a^2}. \]

Here the expression of $\Omega(a, w)$ does not depend on the argument $\text{Arg}(w)$ of $w$.

By using this notation, the critical delay $\tau_c(a, w)$ is expressed by
\[ \tau_c(a, w) = \tau_c(a, |w|, \text{Arg}(w)) \]
\[ := \frac{1}{\Omega(a, |w|)} \left[ |\text{Arg}(w)| - \arccos \left( \frac{a}{|w|} \right) \right] \]
for every $(a, w) \in D_c$. We note that $\tau_c(a, w)$ depends on the absolute value of the argument of $w$.

The following lemmas give equivalent forms to angular frequency eqs. (4.1) under the additional conditions of
\[ \text{Arg}(w) - \tau\Omega \in [0, \pi] + 2k\pi \quad \text{or} \quad \text{Arg}(w) - \tau\Omega \in (-\pi, 0) + 2k\pi \]
for some $k \in \mathbb{Z}$. We note that such an integer $k$ uniquely exists by the decomposition
\[ \mathbb{R} = \bigcup_{k \in \mathbb{Z}} \left((\pi, \pi] + 2k\pi \right) \]
(4.2)
of the real number line.
Lemma 4.1. Suppose $a \in \mathbb{R}$, $w \in \mathbb{C} \setminus \{0\}$, and $|a| < |w|$. Let $k \in \mathbb{Z}$ be given. Then $\Omega \in \mathbb{R} \setminus \{0\}$ satisfies angular frequency eqs. (4.1) and $\text{Arg}(w) - \tau\Omega \in [0, \pi] + 2k\pi$ if and only if

$$\Omega = \Omega(a, w) \quad \text{and} \quad \tau\Omega = \text{Arg}(w) - \arccos \left( \frac{a}{|w|} \right) - 2k\pi$$

hold.

Proof. (Only-if-part). By the assumption, we have

$$\text{Arg}(w) - \tau\Omega - 2k\pi = \arccos \left( \frac{a}{|w|} \right).$$

Therefore,

$$\Omega = |w| \sin \left( \arccos \left( \frac{a}{|w|} \right) \right) = \Omega(a, w)$$

holds.

(If-part). We only need to check whether the second equation of Eqs. (4.1) holds. This is verified in view of

$$|w| \sin(\text{Arg}(w) - \tau\Omega) = |w| \sin \left( \arccos \left( \frac{a}{|w|} \right) \right) = \Omega(a, w) = \Omega.$$

This completes the proof. $\square$

In the similar way, the following lemma is obtained. The proof can be omitted.

Lemma 4.2. Suppose $a \in \mathbb{R}$, $w \in \mathbb{C} \setminus \{0\}$, and $|a| < |w|$. Let $k \in \mathbb{Z}$ be given. Then $\Omega \in \mathbb{R} \setminus \{0\}$ satisfies angular frequency eqs. (4.1) and $\text{Arg}(w) - \tau\Omega \in (-\pi, 0) + 2k\pi$ if and only if

$$\Omega = -\Omega(a, w) \quad \text{and} \quad \tau\Omega = \text{Arg}(w) + \arccos \left( \frac{a}{|w|} \right) - 2k\pi$$

hold.

4.3 Sequence of $\tau$-values

In view of Lemmas 4.1 and 4.2 we introduce the following notation.

Notation 4. Suppose $a \in \mathbb{R}$, $w \in \mathbb{C} \setminus \{0\}$, and $|a| < |w|$. For each integer $n \geq 1$, let

$$\tau_n^+(a, w) := \frac{1}{\Omega(a, w)} \left[ \pm \text{Arg}(w) - \arccos \left( \frac{a}{|w|} \right) + 2n\pi \right]$$

and

$$\tau_n^-(a, w) := \frac{1}{\Omega(a, w)} \left[ \pm \text{Arg}(w) - \text{arccot} \left( \frac{a}{\Omega(a, w)} \right) + 2n\pi \right],$$

where $\tau_n^\pm(a, w)$ are always positive because of $\arccos(a/|w|) < \pi$.

Remark 5. We have

$$\tau_n^+(a, w) - \tau_n^-(a, w) = \frac{2 \text{Arg}(w)}{\Omega(a, w)}$$

(4.3)

for all $n \geq 1$. 

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The following theorem gives conditions on $\tau$ under which Eq. \((\ast)\) has a purely imaginary root for each given $a$ and $w$.

**Theorem 4.3** (cf. [39]). Suppose $a \in \mathbb{R}$, $w \in \mathbb{C} \setminus \{0\}$, and $|a| < |w|$. Then Eq. \((\ast)\) has a root on the imaginary axis if and only if one of the following conditions is satisfied:

(i) $(a, w) \in D_c$ and $\tau = \tau_c(a, w)$.

(ii) $\tau = \tau_n^+(a, w)$ for some integer $n \geq 1$.

(iii) $\tau = \tau_n^-(a, w)$ for some integer $n \geq 1$.

Furthermore, the following statements hold:

- When (i) with $\text{Arg}(w) > 0$ or (ii) holds, Eq. \((\ast)\) has only the root $i\Omega(a, w)$ on the imaginary axis.

- When (i) with $\text{Arg}(w) < 0$ or (iii) holds, Eq. \((\ast)\) has only the root $-i\Omega(a, w)$ on the imaginary axis.

**Proof.** It holds that Eq. \((\ast)\) has a root on the imaginary axis if and only if there exists $\Omega \in \mathbb{R} \setminus \{0\}$ satisfying angular frequency eqs. (4.1). From (4.2), the consideration is divided into the following two cases.

- **Case 1:** $\text{Arg}(w) - \tau \in [0, \pi] + 2k\pi$ for some $k \in \mathbb{Z}$. In this case, angular frequency eqs. (4.1) is reduced to
  \[
  \Omega = \Omega(a, w) \quad \text{and} \quad \tau\Omega = \text{Arg}(w) - \arccos \left( \frac{a}{|w|} \right) - 2k\pi
  \]
  from Lemma 4.1. The positivity of $\tau\Omega$ imposes the following conditions:
  - $k = 0$, $\text{Arg}(w) > 0$, and $a > \Re(w)$. In this case, $\tau = \tau_c(a, w)$.
  - $k \leq -1$. In this case, $\tau = \tau_k^+(a, w)$.

- **Case 2:** $\text{Arg}(w) - \tau \in (-\pi, 0) + 2k\pi$ for some $k \in \mathbb{Z}$. In this case, angular frequency eqs. (4.1) is reduced to
  \[
  \Omega = -\Omega(a, w) \quad \text{and} \quad \tau\Omega = \text{Arg}(w) + \arccos \left( \frac{a}{|w|} \right) - 2k\pi
  \]
  from Lemma 4.2. The negativity of $\tau\Omega$ imposes the following conditions:
  - $k = 0$, $\text{Arg}(w) < 0$, and $a > \Re(w)$. In this case, $\tau = \tau_c(a, w)$.
  - $k \geq 1$. In this case, $\tau = \tau_k^-(a, w)$.

This completes the proof. \hfill \square

It is sufficient to consider the case $\text{Arg}(w) \in [0, \pi]$ (i.e., the case $\Im(w) \geq 0$) because we obtain

\[
\bar{z} + a - \bar{w}e^{-\tau\bar{z}} = 0
\]

by taking the complex conjugate. However, we do not adopt this assumption because it does not bring us any simplification as the statement and the proof show.
For each given \((a,w)\) satisfying \(|a| < |w|\), Theorem 4.3 gives conditions on \(\tau\) for which Eq. \((\ast)\) has a purely imaginary root. Here the corresponding angular frequency is uniquely determined as a function of \((a,w)\) and does not explicitly depend on \(\tau\).

**Remark 6.** In the sense of asymptotic stability of the trivial solution, DDE \((\ref{eq:1})\) and the DDE

\[ \dot{x}(t) = -\tau a + \tau wx(t - 1) \]

are equivalent. However, the angular frequency of the latter equation explicitly depends on the parameter \(\tau\) through the coefficients \(\tau a\) and \(\tau w\). This is natural because the latter DDE is obtained by the change of time variables \(t \rightarrow \tau t\) depending on the parameter \(\tau\).

### 4.3.1 Comparison with Matsunaga’s sequences

Matsunaga \([39, \text{Lemmas 1 and 2}]\) has obtained the similar results, where the case \(0 < \theta \leq \pi/2\) is only considered and the proof is divided into the cases \(b > 0\) (\([39, \text{Lemma 1}]\)) and \(b < 0\) (\([39, \text{Lemma 2}]\)). We recall that the parameter \(w = -be^{i\theta}\) \((b \in \mathbb{R} \setminus \{0\}, \theta \in [-\pi/2, \pi/2])\) is used in \([39]\). In these results, there are sequences \((\tau^+_n)_{n=0}^{\infty}\) for the case \(b > 0\) and \((\sigma^+_n)_{n=0}^{\infty}\) for the case \(b < 0\) for which Eq. \((\ast)\) has a nonzero root on the imaginary axis. These are given by

\[
\tau^+_n := \frac{1}{\sqrt{b^2 - a^2}} \left[ \pm \theta + \arccos \left( -\frac{a}{b} \right) + 2n\pi \right],
\]

and

\[
\sigma^+_n := \frac{1}{\sqrt{b^2 - a^2}} \left[ \theta - \arccos \left( -\frac{a}{b} \right) + 2n\pi \right],
\]

\[
\sigma^-_n := \frac{1}{\sqrt{b^2 - a^2}} \left[ -\theta - \arccos \left( -\frac{a}{b} \right) + (2n + 2)\pi \right].
\]

It seems that there are two sequences for the existence of a nonzero root on the imaginary axis. However, Theorem 4.3 shows that this is not the situation, which is clarified in the following lemmas.

**Lemma 4.4.** Let \(w = -be^{i\theta}\) for \(b \in \mathbb{R} \setminus \{0\}\) and \(\theta \in [-\pi/2, \pi/2]\). Suppose \(b > 0\) and \(\theta > 0\). Then for all \(n \geq 0\),

\[
\tau^+_n = \tau^+_{n+1}(a, w) \quad \text{and} \quad \tau^-_n = \begin{cases} 
\tau_c(a, w) & (n = 0, \Re(w) < a < |w|), \\
\tau^-_n(a, w) & (n \geq 1)
\end{cases}
\]

hold.

**Proof.** By the assumption, \(|w| = b\) and \(\text{Arg}(w) = \theta - \pi < 0\) from the proof of Lemma 2.2. Therefore, we have

\[
\theta + \arccos \left( -\frac{a}{b} \right) = (\text{Arg}(w) + \pi) + \left[ \pi - \arccos \left( \frac{a}{|w|} \right) \right]
\]

and

\[
-\theta + \arccos \left( -\frac{a}{b} \right) = -(\text{Arg}(w) + \pi) + \left[ \pi - \arccos \left( \frac{a}{|w|} \right) \right]
\]

from identity \((\ref{eq:2})\). This shows the conclusion.
Lemma 4.5. Let \( w = -b e^{i\theta} \) for \( b \in \mathbb{R} \setminus \{0\} \) and \( \theta \in [-\pi/2, \pi/2] \). Suppose \( b < 0 \) and \( \theta > 0 \). Then for all \( n \geq 0 \),

\[
\sigma^+_n = \begin{cases} 
\tau_c(a, w) & (n = 0, \Re(w) < a < |w|), \\
\tau^+_n(a, w) & (n \geq 1)
\end{cases}
\]

and \( \sigma^-_n = \tau^-_{n+1}(a, w) \) hold.

Proof. By the assumption, we have \( |w| = -b \) and \( \text{Arg}(w) = \theta > 0 \). Then the expressions are obtained by simply using these relations. \( \square \)

Remark 7. The positivity of \( \tau_0^- \) (for the case that \( b > 0 \) and \( \theta \in (0, \pi/2) \)) and \( \sigma_0^+ \) (for the case \( b < 0 \) and \( \theta \in (0, \pi/2) \)) must be checked because this is essential for the expression of \( \tau_c(a, w) \). However, this has not been performed in [39].

4.4 Ordering of \( \tau \)-values

Here we consider the ordering of \( \tau_c(a, w) \) and \( \tau_n^\pm(a, w) \) for \( n \geq 1 \). The cases \( \text{Arg}(w) = 0 \) or \( \text{Arg}(w) = \pi \) are special.

Lemma 4.6. Suppose \( a \in \mathbb{R}, \ w \in \mathbb{R} \setminus \{0\}, \) and \( |a| < |w| \). Then the following statements hold:

1. If \( w > 0 \), then \( \tau_c(a, w) \) is not defined and we have

\[
\tau^+_n(a, w) = \tau^-_n(a, w) = \frac{1}{\Omega(a, w)} \left[ -\arccos \left( \frac{a}{w} \right) + 2n\pi \right]
\]

for all integer \( n \geq 1 \).

2. If \( w < 0 \), then \( \tau_c(a, w) \) is defined. Furthermore, we have

\[
\tau_c(a, w) = \tau^-_1(a, w) = \frac{1}{\Omega(a, w)} \arccos \left( \frac{a}{|w|} \right)
\]

and

\[
\tau^+_n(a, w) = \tau^-_{n+1}(a, w) = \frac{1}{\Omega(a, w)} \left[ \arccos \left( \frac{a}{|w|} \right) + 2n\pi \right]
\]

for all integer \( n \geq 1 \).

Proof. 1. Since \( \text{Arg}(w) = 0 \), Theorem [2.3] shows that \( \tau_c(a, w) \) is not defined. The expressions of \( \tau_n^\pm(a, w) \) are obvious.

2. Since \( \text{Arg}(w) = \pi \), \( \tau_c(a, w) \) is defined from Theorem [2.3]. Furthermore, we have

\[
\tau^-_1(a, w) = \tau_c(a, w) = \frac{1}{\Omega(a, w)} \left[ \pi - \arccos \left( \frac{a}{|w|} \right) \right]
\]

and

\[
\tau^-_{n+1}(a, w) = \tau^+_n(a, w) = \frac{1}{\Omega(a, w)} \left[ \pi - \arccos \left( \frac{a}{|w|} \right) + 2n\pi \right].
\]

In view of identity [2.22], the expressions are obtained. \( \square \)
Lemma 4.7. Suppose $a \in \mathbb{R}, w \in \mathbb{C} \setminus \{0\}, |a| < |w|$, and $\text{Arg}(w) \in (0, \pi)$. Then for all $n \geq 1$,
$$
\tau_n^-(a, w) < \tau_n^+(a, w) < \tau_{n+1}^-(a, w)
$$
holds. Furthermore, if $a > \Re(w)$, then
$$
0 < \tau_c(a, w) < \tau_1^-(a, w)
$$
holds.

Proof. The inequality $\tau_n^-(a, w) < \tau_n^+(a, w)$ is a consequence of (4.3). The inequality $\tau_n^+(a, w) < \tau_{n+1}^-(a, w)$ follows by
$$
\tau_{n+1}^- - \tau_n^+ = \frac{2(\pi - \text{Arg}(w))}{\Omega(a, w)} > 0.
$$
The inequality $\tau_c(a, w) < \tau_1^-(a, w)$ for the case $a > \Re(w)$ follows by the same reasoning.

Lemma 4.8. Suppose $a \in \mathbb{R}, w \in \mathbb{C} \setminus \{0\}, |a| < |w|$, and $\text{Arg}(w) \in (-\pi, 0)$. Then for all $n \geq 1$,
$$
\tau_n^+(a, w) < \tau_n^-(a, w) < \tau_{n+1}^+(a, w)
$$
holds. Furthermore, if $a > \Re(w)$, then
$$
0 < \tau_c(a, w) < \tau_1^+(a, w)
$$
holds.

Proof. The inequality $\tau_n^+(a, w) < \tau_n^-(a, w)$ is a consequence of (4.3). The inequality $\tau_n^-(a, w) < \tau_{n+1}^+(a, w)$ follows by
$$
\tau_{n+1}^- - \tau_n^+ = \frac{2(\pi + \text{Arg}(w))}{\Omega(a, w)} > 0.
$$
The inequality $\tau_c(a, w) < \tau_1^+(a, w)$ for the case $a > \Re(w)$ follows by the same reasoning.

5 Method by critical delay and stability region for real $a$ and $w$

In this section, we study Eq. (2) with real $a, w$ and find the stability region via the method by critical delay. As is used in Introduction, we will use the following notation.

Notation 5. Let $C: (0, \pi) \to \mathbb{R}$ be the function defined by
$$
C(\theta) := \theta \cot \theta = \theta \cdot \frac{\cos \theta}{\sin \theta}
$$
for all $\theta \in (0, \pi)$. 
Since\[ C'(\theta) = \frac{\cos \theta}{\sin \theta} - \frac{\theta}{\sin^2 \theta} = \frac{\sin 2\theta - 2\theta}{2\sin^2 \theta} < 0 \]
for \( \theta \in (0, \pi) \) and
\[
\lim_{\theta \downarrow 0} \frac{\theta}{\sin \theta} \cdot \cos \theta = 1,
\]
the function \( C: (0, \pi) \rightarrow \mathbb{R} \) is strictly monotonically decreasing and satisfies
\[
\lim_{\theta \downarrow 0} C(\theta) = 1 \text{ and } \lim_{\theta \uparrow \pi} C(\theta) = -\infty.
\]
See Fig. 4 for the graph of the function \( C: (0, \pi) \rightarrow \mathbb{R} \).

The above properties show that the function \( C \) gives a one-to-one correspondence between open intervals \((0, \pi)\) and \((-\infty, 1)\). Therefore, the inverse function \( C^{-1}: (-\infty, 1) \rightarrow \mathbb{R} \) is strictly monotonically decreasing and satisfies
\[
\lim_{r \rightarrow -\infty} C^{-1}(r) = \pi \text{ and } \lim_{r \uparrow 1} C^{-1}(r) = 0. \quad (5.1)
\]

### 5.1 Inequality on critical delay and stability region

The following function will naturally appear for solving the inequality \( \tau_c(a, w) > 1 \).

**Definition 5.1.** Let \( R: (-\infty, 1) \rightarrow \mathbb{R} \) be the function defined by
\[
R(r) := \frac{C^{-1}(r)}{\sin C^{-1}(r)}
\]
for all \( r < 1 \). We also have
\[
R(r) = \frac{r}{\cos C^{-1}(r)}
\]
when \( r \neq 0 \) because \( C^{-1}(r) \cot C^{-1}(r) = r \).

We note that \( R(r) \) can be considered as a function of \( C^{-1}(r) \). The following lemma gives qualitative properties of the function \( R \).
Lemma 5.2. The function $R: (-\infty, 1) \to \mathbb{R}$ is strictly monotonically decreasing and satisfies $\lim_{r \to -\infty} R(r) = \infty$ and $\lim_{r \uparrow 1} R(r) = 1$. Furthermore,

$$\lim_{r \to -\infty} \frac{R(r)}{|r|} = 1$$

holds.

Proof. Since

$$\frac{d}{d\theta} \sin \theta = \frac{\sin \theta - \theta \cos \theta}{\sin^2 \theta} = 1 - \frac{\theta \cot \theta}{\sin \theta} > 0$$

for $\theta \in (0, \pi)$, the function $(0, \pi) \ni \theta \mapsto \theta/\sin \theta \in \mathbb{R}$ is strictly monotonically increasing. Therefore, it holds that the function $R$ is strictly monotonically decreasing because the function $C^{-1}: (-\infty, 1) \to \mathbb{R}$ is strictly monotonically decreasing. The limits are consequences of (5.1) and $R(r) = r/cos C^{-1}(r)$ for $r \neq 0$. \qed

Theorem 5.3. Let $a, w \in \mathbb{R}$ be given so that $w < -|a|$. Then $\tau_c(a, w) > 1$ if and only if

$$a > -1 \quad \text{and} \quad -R(-a) < w.$$

hold.

Proof. The inequality $\tau_c(a, w) > 1$ is equivalent to

$$\sqrt{w^2 - a^2} < \arccot \left( -\frac{a}{\sqrt{w^2 - a^2}} \right)$$

by the expression of $\tau_c(a, w)$. Let $X(a, w) := \sqrt{w^2 - a^2}$. Since $\cot |(0, \pi): (0, \pi) \to \mathbb{R}$ is strictly monotonically decreasing, the above inequality can be solved as

$$a > -1 \quad \text{and} \quad X(a, w) < C^{-1}(-a).$$

By solving the last inequality with respect to $w$, we obtain

$$w^2 < a^2 + C^{-1}(-a)^2$$

$$= C^{-1}(-a)^2 (\cot^2 C^{-1}(-a) + 1),$$

which is equivalent to $-R(-a) < w$ because of the negativity of $w$. \qed

From Theorem 5.3, the region in $(a, w)$-plane obtained by the inequality $\tau_c(a, w) > 1$ is expressed by the function $R: (-\infty, 1) \to \mathbb{R}$, whose qualitative properties have been revealed by Lemma 5.2.

The following stability condition on $a$, $w$, and $\tau$ is obtained as a corollary of Theorems 2.3 and 5.3. The result is due to Hayes [28, Theorem 1].

Corollary 5.4 ([28], refs. [27], [20]). Suppose $a, w \in \mathbb{R}$. Then all the roots of Eq. (11) have negative real parts if and only if

$$a > -\frac{1}{\tau} \quad \text{and} \quad -\frac{1}{\tau} R(-\tau a) < w < a$$

hold.
Proof. From Theorem 2.3, all the roots of Eq. (1) have negative real parts if and only if one of the following conditions is satisfied:

- $a > 0$ and $-a \leq w < a$.
- $w < -|a|$ and $\tau_c(a, w) > \tau$.

Here the second condition becomes

$$\tau a > -1 \text{ and } -R(-\tau a) < \tau w < -\tau |a|$$

from equality (2.4) and Theorem 5.3. By combining this and the first condition, we obtain the conclusion from Lemma 5.2.

Remark 8. In general, it is not apparent how to derive the expression of the critical delay from Corollary 5.4. By following the argument of the proof of Theorem 5.3 in reverse, we obtain the following equivalences: For $w < -|a|$, 

$$\tau a > -1 \text{ and } \frac{1}{\tau} R(-\tau a) < w$$

$\iff \tau a > -1 \text{ and } \tau \sqrt{w^2 - a^2} < C^{-1}(-\tau a)$

$\iff 0 < \tau \sqrt{w^2 - a^2} < \pi \text{ and } \tau \sqrt{w^2 - a^2} \cot\left(\tau \sqrt{w^2 - a^2}\right) > -\tau a$

$\iff \tau \sqrt{w^2 - a^2} < \arccot\left(\frac{a}{\sqrt{w^2 - a^2}}\right)$.

This shows

$$\tau_c(a, w) = \frac{1}{\sqrt{w^2 - a^2}} \arccot\left(\frac{a}{\sqrt{w^2 - a^2}}\right).$$

5.2 Parametrization of stability boundary curve

Since the function $C^{-1}: (-\infty, 1) \to \mathbb{R}$ gives a one-to-one correspondence between the open intervals $(-\infty, 1)$ and $(0, \pi)$, the curve

$$\left\{ \left( a, \frac{1}{\tau} R(-\tau a) \right) : a > -\frac{1}{\tau} \right\}$$

in $(a, w)$-plane is parametrized by

$$a = -\frac{1}{\tau} \theta \cot \theta \text{ and } w = -\frac{\theta}{\tau \sin \theta} \quad (5.2)$$

for $\theta \in (0, \pi)$ in view of

$$-\tau a = C^{-1}(-\tau a) \cot C^{-1}(-\tau a).$$

The stability boundary curves are depicted in Fig. 5 for the cases of $\tau = 1$ and $\tau = 1/3$.

The picture is well-known in the literature (see [27, Figure 5.1 in Chapter 5] and [20, Figure XI.1 in Chapter XI]). See also [39, Comment after Theorem A].
5.2.1 Comparison with a study via Pontryagin’s results

For simplicity, let $\tau = 1$. In [27, Theorem A.5], the necessary and sufficient condition for which all the roots of Eq. (5.2) have negative real parts is given as follows via Pontryagin’s results (see [27, Theorems A.3 and A.4]):

$$a > -1, \quad a - w > 0, \quad \text{and} \quad w > -C^{-1}(-a) \sin C^{-1}(-a) + a \cos C^{-1}(-a).$$

However, the parametrization (5.2) is not directly obtained by this expression. The above condition is same as that given in Corollary 5.4 in view of

$$-C^{-1}(-a) \sin C^{-1}(-a) + a \cos C^{-1}(-a)$$

$$= -\frac{1}{\sin C^{-1}(-a)} \left( C^{-1}(-a) \sin^2 C^{-1}(-a) - a \sin C^{-1}(-a) \cos C^{-1}(-a) \right)$$

$$= -R(-a),$$

where $-a \sin C^{-1}(-a) = C^{-1}(-a) \cos C^{-1}(-a)$ is used.

The above procedure can be understood as a process eliminating the parameter $a$, which is explained as follows. Suppose $a, w \in \mathbb{R}$. By substituting $z = i\Omega$ ($\Omega \in \mathbb{R} \setminus \{0\}$) in Eq. (3), we have

$$w = (i\Omega + a)(\cos \tau \Omega + i \sin \tau \Omega)$$

$$= a \cos \tau \Omega - \Omega \sin \tau \Omega + i(a \sin \tau \Omega + \Omega \cos \tau \Omega).$$

Here we are using the equivalent expression $(z + a)e^{\tau z} - w = 0$ for Eq. (3). Then the assumption $\Im(w) = 0$ leads to $a \sin \tau \Omega + \Omega \cos \tau \Omega = 0$, i.e.,

$$\Omega \cot \tau \Omega = -a$$

when $a \neq 0$. Therefore, $w$ is expressed by

$$w = a \cos \tau \Omega - \Omega \sin \tau \Omega = -\frac{\Omega}{\sin \tau \Omega}$$

in the same way as above. The above consideration is related to the method of D-partitions. See Section 7 for the detail.

The above discussion is summarized in the following lemma.
Lemma 5.5. Suppose $\theta \notin \pi \mathbb{Z}$ and $r \in \mathbb{R}$. Then $\theta \cot \theta = r$ implies

$$\frac{\theta}{\sin \theta} = r \cos \theta + \theta \sin \theta.$$ 

Furthermore, if $\cos \theta \neq 0$, then the converse also holds.

Proof. We only need to show the converse under the assumption of $\cos \theta \neq 0$. Then the equation is equivalent to

$$\frac{\theta}{\sin \theta} (1 - \sin^2 \theta) = r \cos \theta.$$ 

By dividing the both sides by $\cos \theta$, we obtain $\theta \cot \theta = r$. \hfill $\square$

5.3 Remark on Hayes’ result

Hayes [28] considered a transcendental equation

$$s = ce^s$$

for an unknown $s \in \mathbb{C}$ and a given constant $c \in \mathbb{R}$ to investigate Eq. (1) with $\tau = 1$ for real $a$ and $w$. Since this is equivalent to $(-s)e^{-s} = -c$, the set of all roots of the above equation coincides with $-W(-c)$. Hayes [28] did not use the concept of the Lambert $W$ function, but the study is considered to be the investigation of the principal complex branch $W_0(\zeta)$ for real $\zeta$. Based on this approach, the following result has been obtained by Hayes [28, Lemma 2].

Theorem 5.6 ([28]). Let $\zeta \in \mathbb{R}$ and $\sigma \in \mathbb{R}$ be given. Then $\Re(z) < \sigma$ for all $z \in W(\zeta)$ if and only if

$$\sigma > -1 \quad \text{and} \quad -R(-\sigma)e^\sigma < \zeta < \sigma e^\sigma$$

hold.

This is logically equivalent to Corollary 5.4 by the discussions in Appendix A.

6 Method by critical delay and stability region for real $a$ and imaginary $w$

In this section, we study Eq. (1) with real $a$ and complex $w$ and find the stability region via the method by critical delay.

We will use the following notation.

Notation 6. For each $\varphi \in (0, \pi)$, let $C(\cdot; \varphi) : [0, \varphi) \to \mathbb{R}$ be the function defined by

$$C(\theta; \varphi) := \theta \cot(\theta - \varphi) = \theta \cdot \frac{\cos(\theta - \varphi)}{\sin(\theta - \varphi)}$$

for all $\theta \in [0, \varphi)$. 

6.1 Property of the function $C(\cdot; \varphi)$

We first study the case $\varphi \in (0, \pi/2]$.

**Lemma 6.1.** Let $\varphi \in (0, \pi/2]$ be given. Then the function $C(\cdot; \varphi) : [0, \varphi) \to \mathbb{R}$ is strictly monotonically decreasing and satisfies $\lim_{\theta \uparrow \varphi} C(\theta; \varphi) = -\infty$.

**Proof.** We have

$$\frac{d}{d\theta} C(\theta; \varphi) = \frac{\cos(\theta - \varphi) - \theta}{\sin(\theta - \varphi)} - \frac{\theta}{\sin^2(\theta - \varphi)} = \frac{\sin(2\theta - 2\varphi) - 2\theta}{2\sin^2(\theta - \varphi)}$$

for all $\theta \in [0, \varphi)$. Let

$$f(\theta) := \sin(2\theta - 2\varphi) - 2\theta \quad (\theta \in [0, \varphi)).$$

Then its derivative is

$$f'(\theta) = 2(\cos(2\theta - 2\varphi) - 1) < 0.$$ 

Since $f(0) = -\sin(2\varphi) \leq 0$, $f(\theta) < 0$ holds for all $\theta \in (0, \varphi)$. Therefore, the monotonicity is obtained. The limit is a consequence of $\lim_{\theta \uparrow 0} \cot \theta = -\infty$. \qed

Lemma 6.1 shows that for each given $\varphi \in (0, \pi/2]$, the function $C(\cdot; \varphi) : [0, \varphi) \to \mathbb{R}$ has its inverse function $C^{-1}(\cdot; \varphi) : (0, \infty, 0] \to \mathbb{R}$ which is strictly monotonically decreasing and satisfies

$$\lim_{r \to -\infty} C^{-1}(r; \varphi) = \varphi \quad \text{and} \quad C^{-1}(0; \varphi) = 0. \quad (6.1)$$

To study the case $\varphi \in (\pi/2, \pi)$, we need to introduce the following notation.

**Definition 6.2.** For each $\varphi \in (\pi/2, \pi)$, let $\theta = S(\varphi)$ be the unique solution of

$$\sin(2\theta - 2\varphi) = 2\theta$$

in $(0, \varphi)$, or equivalently, $S(\varphi) \in (0, \varphi)$ satisfies

$$\sin(S(\varphi) - \varphi) \cos(S(\varphi) - \varphi) = S(\varphi).$$

We note that the above $S(\varphi)$ coincides with $\phi^*$ used in [49, Lemma 2]. The above unique existence of $S(\varphi)$ is ensured by the following lemma.

**Lemma 6.3.** Let $\varphi \in (\pi/2, \pi)$ be given. Then the following statements hold:

- $\sin(2\theta - 2\varphi) - 2\theta > 0$ for all $\theta \in [0, S(\varphi))$,

- $\sin(2S(\varphi) - 2\varphi) = 2S(\varphi)$, and

- $\sin(2\theta - 2\varphi) - 2\theta < 0$ for all $\theta \in (S(\varphi), \varphi)$.
Proof. Let \( f(\theta) := \sin(2\theta - 2\phi) - 2\theta \) for all \( \theta \in [0, \phi) \). Then

\[
f'(\theta) = 2(\cos(2\theta - 2\phi) - 1) < 0.
\]

Since

\[
f(0) = -\sin(2\phi) > 0 \quad \text{and} \quad \lim_{\theta \to \phi} f(\theta) = -2\phi < 0,
\]

there exists a unique \( \theta_\phi \in (0, \phi) \) such that \( f(\theta_\phi) = 0 \) by the intermediate value theorem. By the monotonicity, \( \theta_\phi = S(\phi) \) holds. Then \( f(\theta) > 0 \) for all \( \theta \in [0, S(\phi)) \) and \( f(\theta) < 0 \) for all \( \theta \in (S(\phi), \phi) \).

By using the value \( S(\phi) \) for \( \phi \in (\pi/2, \pi) \), we obtain the following lemma.

**Lemma 6.4.** Let \( \phi \in (\pi/2, \pi) \) be given. Then the function \( C(\cdot; \phi) : [0, \phi) \to \mathbb{R} \) is strictly monotonically increasing on \( [0, S(\phi)] \) and is strictly monotonically decreasing on \( [S(\phi), \phi) \). Furthermore, the function attains its maximum

\[
M(\phi) := \cos^2(S(\phi) - \phi)
\]

at \( S(\phi) \) and \( \lim_{\theta \uparrow \phi} C(\theta; \phi) = -\infty \) holds.

**Proof.** By the proof of Lemma 6.1 we have

\[
\frac{d}{d\theta} C(\theta; \phi) = \frac{\sin(2\theta - 2\phi) - 2\theta}{2\sin^2(\theta - \phi)}
\]

for all \( \theta \in [0, \phi) \). Therefore, the monotonicity properties stated in Lemma 6.3 follow by Lemma 6.3. The maximum can be calculated by using the relation \( S(\phi) = \sin(S(\phi) - \phi) \cos(S(\phi) - \phi) \). The limit is a consequence of \( \lim_{\theta \to \phi} \cot \theta = -\infty \).

**Remark 9.** Since \( C(\phi - (\pi/2); \phi) = 0 \), we have

\[
S(\phi) < \phi - \frac{\pi}{2}
\]

for all \( \phi \in (\pi/2, \pi) \).

See Fig. 6 for the graphs of the function \( C(\cdot; \phi) : [0, \phi) \to \mathbb{R} \) for each cases of \( \phi \in (0, \pi/2] \) and \( \phi \in (\pi/2, \pi) \).

Based on Lemma 6.4 we introduce the following.

**Definition 6.5.** For each given \( \phi \in (\pi/2, \pi) \), let \( C_1(\cdot; \phi) \) and \( C_2(\cdot; \phi) \) be the restrictions of the function \( C(\cdot; \phi) : [0, \phi) \to \mathbb{R} \) to the intervals \([0, S(\phi))\) and \([S(\phi), \phi)\), respectively.

Then Lemma 6.4 shows that the inverse functions \( C_1^{-1}(\cdot; \phi) : [0, M(\phi)] \to \mathbb{R} \) and \( C_2^{-1}(\cdot; \phi) : (-\infty, M(\phi)] \to \mathbb{R} \) have the following properties:

- \( C_1^{-1}(\cdot; \phi) \) is strictly monotonically increasing and satisfies

  \[
  C_1^{-1}(0; \phi) = 0 \quad \text{and} \quad C_1^{-1}(M(\phi); \phi) = S(\phi).
  \]  

- \( C_2^{-1}(\cdot; \phi) \) is strictly monotonically decreasing and satisfies

  \[
  \lim_{r \to -\infty} C_2^{-1}(r; \phi) = \phi \quad \text{and} \quad C_2^{-1}(M(\phi); \phi) = S(\phi).
  \]
\[ \theta = \varphi \]

(a) \( \varphi \in (0, \pi/2] \)

(b) \( \varphi \in (\pi/2, \pi) \). The function attains its maximum \( M(\varphi) = \cos^2(S(\varphi) - \varphi) \) at \( \theta = S(\varphi) \).

Figure 6: Graphs of \( C(\theta; \varphi) = \theta \cot(\theta - \varphi) \) for \( \varphi \in (0, \pi/2] \) and \( \varphi \in (\pi/2, \pi) \)

6.2 Inequality on critical delay and stability region for the case \( |\text{Arg}(w)| \in (0, \pi/2] \)

We use the function introduced below.

**Definition 6.6.** For each \( \varphi \in (0, \pi/2] \), let \( R(\cdot; \varphi): (-\infty, 0] \rightarrow \mathbb{R} \) be the function defined by

\[
R(r; \varphi) := -\frac{C^{-1}(r; \varphi)}{\sin(C^{-1}(r; \varphi) - \varphi)} = -\frac{r}{\cos(C^{-1}(r; \varphi) - \varphi)}. \tag{6.4}
\]

Here we have

\[-\frac{\pi}{2} \leq -\varphi < C^{-1}(r; \varphi) - \varphi < 0\]

for all \( r < 0 \).

The following lemma gives qualitative properties of the function \( R(\cdot; \varphi): (-\infty, 0] \rightarrow \mathbb{R} \) for each given \( \varphi \in (0, \pi/2] \).

**Lemma 6.7.** Let \( \varphi \in (0, \pi/2] \) be given. Then the function \( R(\cdot; \varphi): (-\infty, 0] \rightarrow \mathbb{R} \) is strictly monotonically decreasing and satisfies \( \lim_{r \to -\infty} R(r; \varphi) = \infty \) and \( R(0; \varphi) = 0 \).

Furthermore,

\[
\lim_{r \to -\infty} \frac{R(r; \varphi)}{|r|} = 1 \quad \text{and} \quad \lim_{r \uparrow 0} \frac{R(r; \varphi)}{|r|} = \frac{1}{\cos \varphi}
\]

hold. Here we are interpreting that \( 1/\cos \varphi = \infty \) when \( \varphi = \pi/2 \).

**Proof.** We have

\[
\frac{d}{d\theta} \left( -\frac{\theta}{\sin(\theta - \varphi)} \right) = \frac{\sin(\theta - \varphi) - \theta \cos(\theta - \varphi)}{\sin^2(\theta - \varphi)}
\]

\[
= \frac{1 - \theta \cot(\theta - \varphi)}{\sin(\theta - \varphi)}
\]

\[> 0\]
for all $\theta \in (0, \varphi)$ because $\theta \cot(\theta - \varphi) < 0$ and $\sin(\theta - \varphi) < 0$ hold. Therefore, the monotonicity property of the function $R(r; \varphi)$ follows by the monotonicity property of the function $C^{-1}(r; \varphi)$. The remaining properties are direct consequences of (6.4) and (6.1), where

$$R(r; \varphi) = \frac{1}{|r|} \cos(C^{-1}(r; \varphi) - \varphi)$$

holds for $r < 0$. This completes the proof.

**Theorem 6.8.** Let $a \in \mathbb{R}$ and $w \in \mathbb{C} \setminus \mathbb{R}$ be given so that $(a, w) \in D_c$. Suppose $\varphi := |\text{Arg}(w)| \in (0, \pi/2]$. Then $\tau_c(a, w) > 1$ if and only if

$$a > 0 \text{ and } |w| < R(-a; \varphi)$$

hold.

**Proof.** The inequality $\tau_c(a, w) > 1$ becomes

$$\sqrt{|w|^2 - a^2 - \varphi + \pi - \arccot \left(\frac{a}{\sqrt{|w|^2 - a^2}}\right)} = \arccot \left(\frac{a}{\sqrt{|w|^2 - a^2}}\right)$$

by the expression of $\tau_c(a, w)$ and identity (2.3). Let $X(a, w) := \sqrt{|w|^2 - a^2}$. Since $\cot(0, \pi) : (0, \pi) \to \mathbb{R}$ is strictly monotonically decreasing, the above inequality can be solved as

$$a > 0 \text{ and } X(a, w) < C^{-1}(-a; \varphi).$$

By solving the last inequality with respect to $|w|$, we obtain

$$|w|^2 < a^2 + C^{-1}(-a; \varphi)^2$$

$$= C^{-1}(-a; \varphi)^2 [\cot^2(C^{-1}(-a; \varphi) - \varphi) + 1],$$

which is equivalent to $|w| < R(-a; \varphi)$.

**Corollary 6.9 (19).** Suppose $a \in \mathbb{R}$, $w \in \mathbb{C} \setminus \mathbb{R}$, and $\varphi := |\text{Arg}(w)| \in (0, \pi/2]$. Then all the roots of Eq. (16) have negative real parts if and only if

$$a > 0 \text{ and } |w| < \frac{1}{\tau} R(-\tau a; \varphi)$$

hold.

**Proof.** From Theorem 2.1, all the roots of Eq. (16) have negative real parts if and only if one of the following conditions is satisfied:

- $a \geq |w|$ and $a > \Re(w)$.
- $(a, w) \in D_c$ and $\tau_c(a, w) > \tau$.

Here $a > \Re(w)$ is automatically satisfied under the condition $a \geq |w|$, and the second condition becomes

$$\tau \Re(w) < \tau a < \tau |w|, \ \tau a > 0, \text{ and } \tau |w| < R(-\tau a; \varphi)$$

from equality (2.4) and Theorem 6.8. By combining this and the first condition, we obtain the conclusion from Lemma 6.7 (see also Fig. 1).
Since the function \( C^{-1}(\cdot; \varphi) \) gives a one-to-one correspondence between the open intervals \((-\infty, 0)\) and \((0, \varphi)\), the curve
\[
\left\{ \left( a, \frac{1}{\tau} R(-\tau a; \varphi) \right) : a > 0 \right\}
\]
in \((a,|w|)\)-plane is parametrized by
\[
a = -\frac{1}{\tau} \theta \cot(\theta - \varphi) \quad \text{and} \quad |w| = -\frac{\theta}{\tau \sin(\theta - \varphi)} \quad (6.5)
\]
for \( \theta \in (0, \varphi) \) in view of
\[-\tau a = C^{-1}(-\tau a; \varphi) \cot (C^{-1}(-\tau a; \varphi) - \varphi).\]
See Fig. 7 for the picture of boundary curves when \( \varphi = \pi/2 \) and \( \varphi = \pi/4 \).

### 6.3 Inequality on critical delay and stability region for the case \(|\text{Arg}(w)| \in (\pi/2, \pi)\)

We use the functions introduced below.

**Definition 6.10.** For each \( \varphi \in (\pi/2, \pi) \), let \( R_1(\cdot; \varphi) : [0, M(\varphi)] \to \mathbb{R} \) be the function defined by
\[
R_1(r; \varphi) := -\frac{C_1^{-1}(r; \varphi)}{\sin(C_1^{-1}(r; \varphi) - \varphi)} = -\frac{r}{\cos(C_1^{-1}(r; \varphi) - \varphi)} \quad (6.6)
\]
Here we have
\[-\pi < -\varphi < C_1^{-1}(r; \varphi) - \varphi < S(\varphi) - \varphi < -\frac{\pi}{2}\]
for all \( 0 < r < M(\varphi) \).

**Definition 6.11.** For each \( \varphi \in (\pi/2, \pi) \), let \( R_2(\cdot; \varphi) : (-\infty, M(\varphi)] \to \mathbb{R} \) be the function defined by
\[
R_2(r; \varphi) := -\frac{C_2^{-1}(r; \varphi)}{\sin(C_2^{-1}(r; \varphi) - \varphi)} = -\frac{r}{\cos(C_2^{-1}(r; \varphi) - \varphi)} \quad (6.7)
\]
Here we have

\[-\pi < S(\varphi) - \varphi < C_2^{-1}(r; \varphi) - \varphi < -\frac{\pi}{2}\]

for all \(0 < r < M(\varphi)\) and

\[-\frac{\pi}{2} \leq C_2^{-1}(r; \varphi) - \varphi < 0\]

for all \(r \leq 0\).

The following lemmas give qualitative properties of the functions \(R_1(\cdot; \varphi)\) and \(R_2(\cdot; \varphi)\).

The proofs are similar to that of Lemma 6.7 but with (6.2) and (6.3).

**Lemma 6.12.** Let \(\varphi \in (\pi/2, \pi)\) be given. Then the function \(R_1(\cdot; \varphi)\) is strictly monotonically increasing and satisfies

\(R_1(0; \varphi) = 0\) and \(R_1(M(\varphi); \varphi) = \sqrt{M(\varphi)}\).

Furthermore,

\[\lim_{r \to 0} \frac{R_1(r; \varphi)}{r} = \frac{1}{|\cos \varphi|} \text{ and } R_1(r; \varphi) > \frac{r}{|\cos \varphi|}\]

hold.

**Proof.** The monotonicity property follows by the similar way to the proof of Lemma 6.7. All the remaining properties are consequences of (6.2) and (6.3) in view of \(\sqrt{M(\varphi)} = -\cos(S(\varphi) - \varphi)\).

The proof of the following lemma is similar to that of Lemma 6.12. Therefore, it can be omitted.

**Lemma 6.13.** Let \(\varphi \in (\pi/2, \pi)\) be given. Then the function \(R_2(\cdot; \varphi)\) is strictly monotonically decreasing and satisfies

\(\lim_{r \to -\infty} R_2(r; \varphi) = \infty\) and \(R_2(M(\varphi); \varphi) = \sqrt{M(\varphi)}\).

Furthermore,

\[\lim_{r \to -\infty} \frac{R_2(r; \varphi)}{|r|} = 1\]

holds.

**Theorem 6.14.** Let \(a \in \mathbb{R}\) and \(w \in \mathbb{C} \setminus \mathbb{R}\) be given so that \((a, w) \in D_c\). Suppose \(\varphi := |\text{Arg}(w)| \in (\pi/2, \pi)\). Then \(\tau_c(a, w) > 1\) if and only if one of the following conditions is satisfied:

(i) \(a \geq 0\) and \(|w| < R_2(-a; \varphi)\).

(ii) \(-M(\varphi) < a < 0\) and \(R_1(-a; \varphi) < |w| < R_2(-a; \varphi)\).

**Proof.** In the same way as the proof of Theorem 6.8, we obtain the inequality

\[X(a, w) - \varphi + \pi < \arccot \left(\frac{a}{X(a, w)}\right),\]

where \(X(a, w) := \sqrt{|w|^2 - a^2}\). From Lemma 6.11 this can be solved as
(i) \( a \geq 0 \) and \( X(a, w) < C_2^{-1}(-a; \varphi) \), or

(ii) \(-M(\varphi) < a < 0 \) and \( C_1^{-1}(-a; \varphi) < X(a, w) < C_2^{-1}(-a; \varphi) \).

\( \bullet \) Case (i): By solving \( X(a, w) < C_2^{-1}(-a; \varphi) \) with respect to \(|w|\), we obtain

\[
|w| < R_2(-a; \varphi)
\]

in the similar way to the proof of Theorem 6.8.

\( \bullet \) Case (ii): By solving \( C_2^{-1}(-a; \varphi) < X(a, w) < C_2^{-1}(-a; \varphi) \) with respect to \(|w|\), we obtain

\[
R_1(-a; \varphi) < |w| < R_2(-a; \varphi)
\]

in the similar way to the proof of Theorem 6.8.

This completes the proof.

The following is a consequence of Theorem 6.14. It is proved in the similar way to Corollary 6.9, and therefore, the proof can be omitted.

**Corollary 6.15** ([49]). Suppose \( a \in \mathbb{R}, w \in \mathbb{C}\setminus\mathbb{R}, \) and \( \varphi := |\text{Arg}(w)| \in (\pi/2, \pi) \). Then all the roots of Eq. (1) have negative real parts if and only if one of the following conditions is satisfied:

(i)

\[
a \geq 0 \quad \text{and} \quad |w| < \frac{1}{\tau}R_2(-\tau a; \varphi).
\]

(ii)

\[
-\frac{1}{\tau}M(\varphi) < a < 0 \quad \text{and} \quad \frac{1}{\tau}R_1(-\tau a; \varphi) < |w| < \frac{1}{\tau}R_2(-\tau a; \varphi).
\]

Finally, we discuss the parametrization of the boundary curve. Since

\( \bullet \) the function \( C_2^{-1}(\cdot; \varphi) \) gives a one-to-one correspondence between \((\infty, M(\varphi))\) and \((S(\varphi), \varphi)\),

\( \bullet \) the function \( C_1^{-1}(\cdot; \varphi) \) gives a one-to-one correspondence between \((0, M(\varphi))\) and \((0, S(\varphi))\),

the curves

\[
\begin{align*}
\left\{ \left( a, \frac{1}{\tau}R_2(-\tau a; \varphi) \right) : a > -\frac{1}{\tau}M(\varphi) \right\}, \\
\left\{ \left( a, \frac{1}{\tau}R_1(-\tau a; \varphi) \right) : -\frac{1}{\tau}M(\varphi) < a < 0 \right\}
\end{align*}
\]

in \((a, |w|)\)-plane are parametrized by \([0, \varphi]\) for \( \theta \in (S(\varphi), \varphi) \) and for \( \theta \in (0, S(\varphi)) \), respectively. By taking \( \theta = (0, \varphi) \), we also obtain the parametrization of the joined curve. See Fig. 8 for the picture of boundary curves when \( \varphi = 9\pi/10 \) and \( \varphi = 3\pi/4 \).
6.4 Remarks

Expression of critical delay  By following the arguments of the proofs of Theorems 6.8 and 6.14 in reverse, one can obtain the expression of the critical delay from Corollaries 6.9 and 6.15 in the same reasoning as Remark 8.

On Sakata’s result  Corollaries 6.9 and 6.15 are due to Sakata [49, Theorem]. The visualization of boundary curves was also obtained by Sakata [49] without the parametrization and the parameter range.

On parameter range given by Matsunaga  Matsunaga [39, Theorem B] gave a re-statement of Sakata’s result with the following parametrization of the boundary curve:

\[ a = -\frac{1}{\tau} \theta \cot(\theta - |\psi|) \quad \text{and} \quad b = \frac{\theta}{\tau \sin(\theta - |\psi|)} \]

for \( \theta \in (|\psi| - \pi, |\psi|) \). Here \( w = -be^{i\psi} \) for \( b \in \mathbb{R} \setminus \{0\} \) and \( \psi \in [-\pi/2, \pi/2] \). We now compare this with the parametrization (6.3) with the parameter range \( \theta \in (0, \varphi) \).

- Case 1: \( b > 0 \). By the proof of Lemma 2.2, we have

\[ b = |w| \quad \text{and} \quad -|\psi| = \varphi - \pi. \]

Therefore, the above parametrization becomes

\[ a = -\frac{1}{\tau} \theta \cot(\theta + \varphi) \quad \text{and} \quad |w| = -\frac{\theta}{\tau \sin(\theta + \varphi)} \]

for \( \theta \in (-\varphi, -\varphi + \pi) \).

- Case 2: \( b < 0 \). By the proof of Lemma 2.2, we have

\[ b = -|w| \quad \text{and} \quad |\psi| = \varphi. \]

Therefore, the above parametrization becomes

\[ a = -\frac{1}{\tau} \theta \cot(\theta - \varphi) \quad \text{and} \quad |w| = -\frac{\theta}{\tau \sin(\theta - \varphi)} \]

for \( \theta \in (\varphi - \pi, \varphi) \).
We note that the parametrization in Case 1 is also expressed by that in Case 2 because the appearing functions are even. However, the parameter range \((\varphi - \pi, \varphi)\) is not consistent with \((0, \varphi)\).

7 Comparison with the method of D-partitions

In this section, we apply the method of D-partitions to Eq. (∗) for the case of real \(a\) and complex \(w\) and to compare this with the results obtained in Sections 5 and 6.

7.1 Method of D-partitions

In this subsection, we briefly summarize the method of D-partitions. We consider a transcendental equation having \(n\)-th real parameters \((p_1, \ldots, p_n)\). As is mentioned in Introduction, the delay parameters are not included in \((p_1, \ldots, p_n)\) for the purpose of obtaining the stability region. In other words, the delay parameters are fixed in the following consideration of the method of D-partitions.

Assuming the situation that the transcendental equation has a root \(i\Omega\) on the imaginary axis, we have the two constraints

\[ f_1(p_1, \ldots, p_n, \Omega) = 0 \text{ and } f_2(p_1, \ldots, p_n, \Omega) = 0 \]

which are obtained by the real and imaginary parts of the left-hand side of the considering transcendental equation. Here it is assumed that the domain of the function \(f = (f_1, f_2)\) is open in the extended parameter space (i.e., \((p_1, \ldots, p_n, \Omega)\)-space), and \(f\) is sufficiently smooth in its domain. Then the regular level set theorem states that the set of all solutions \((p_1, \ldots, p_n, \Omega)\) satisfying the above constraints is an \((n-1)\)-dimensional smooth submanifold embedded in the extended parameter space if \(0\) is a regular value of the function \(f\). Furthermore, if one can choose indices \(1 \leq i < j \leq n\) so that the Jacobian determinant

\[ \left| \frac{\partial(f_1, f_2)}{\partial(p_i, p_j)} \right| \]

is nonzero at some point, then by applying the implicit function theorem, the solution set are locally represented by the graph of some functions

\[ p_i = p_i((p_k)_{k \neq i, j}, \Omega) \text{ and } p_j = p_j((p_k)_{k \neq i, j}, \Omega). \]

If this can be possible globally, then one obtain hyper-surfaces in the parameter \((p_1, \ldots, p_n)\)-space by removing the angular frequency \(\Omega\) from the extended parameter space.

We refer the reader to [21, Chapter III.3], [35, Subsection 3.2 in Chapter 2], and [20, Sections XI.1 and XI.2 in Chapter XI] for the details of the method of D-partitions including the analysis of Eq. (3) for the case that \(a\) and \(w\) are real numbers. See [18] for a note stressing the importance of converting the parameters \(a\) and \(w\) to original model parameters in the method of D-partitions. See also [5], [34] for applications of the method of D-partitions to differential equations with distributed delay and [19] to a neutral delay differential equation.

7.2 Conditions for roots on the imaginary axis with real \(a\)

Suppose \(a \in \mathbb{R}\) and \(w \in \mathbb{C}\setminus\{0\}\). The proof of Theorem 2.1 in [39] relies on the investigation of the condition on \(a\), \(w\), and \(\tau\) for which Eq. (3) has a purely imaginary root (i.e., a nonzero root on the imaginary axis). We note that Eq. (3) has a root 0 if and only if \(w = a\).
7.2.1 An interpretation via implicit function theorem

Angular frequency eqs. (4.1)

\[
\begin{align*}
a - |w| \cos(\text{Arg}(w) - \tau \Omega) &= 0, \\
\Omega - |w| \sin(\text{Arg}(w) - \tau \Omega) &= 0
\end{align*}
\]

can be considered as a system of equations with respect to the five variables \(a \in \mathbb{R}, \rho := |w| > 0, \psi := \text{Arg}(w) \in (-\pi, \pi], \tau > 0, \text{ and } \Omega \in \mathbb{R}\). We now give an interpretation on Lemmas 4.1 and 4.2 from the viewpoint of the implicit function theorem.

Let

\[
\begin{align*}
f(a, \rho, \psi, \tau, \Omega) := (f_1(a, \rho, \psi, \tau, \Omega), f_2(a, \rho, \psi, \tau, \Omega)) \\
&:= (a - \rho \cos(\psi - \tau \Omega), \Omega - \rho \sin(\psi - \tau \Omega)).
\end{align*}
\]

We study the solution set of the equation \(f(a, \rho, \psi, \tau, \Omega) = 0\). The Jacobian determinant \(\left| \frac{\partial(f_1, f_2)}{\partial(\tau, \Omega)} \right|\) is calculated as

\[
\begin{align*}
&= -\rho \Omega \sin(\psi - \tau \Omega)[1 + \rho \tau \cos(\psi - \tau \Omega)] - \{-\rho \tau \sin(\psi - \tau \Omega) \cdot \rho \Omega \cos(\psi - \tau \Omega)\} \\
&= -\rho \Omega \sin(\psi - \tau \Omega).
\end{align*}
\]

Therefore, by restricting the domain of definition of the function \(f\) to the subset satisfying \(\Omega \sin(\psi - \tau \Omega) \neq 0\), both of \(\tau\) and \(\Omega\) can be written as functions of \((a, \rho, \psi)\).

The independency of \(\Omega\) from \(\psi\) is also derived by calculating the partial derivative \(\partial \Omega / \partial \psi\) as follows: By partially differentiating \(f(a, \rho, \psi, \tau, \Omega) = 0\) with respect to \(\psi\), we have

\[
\begin{align*}
&\rho \sin(\psi - \tau \Omega) \cdot \left[ 1 - \frac{\partial(\tau \Omega)}{\partial \psi} \right] = 0, \\
&\frac{\partial \Omega}{\partial \psi} - \rho \cos(\psi - \tau \Omega) \cdot \left[ 1 - \frac{\partial(\tau \Omega)}{\partial \psi} \right] = 0.
\end{align*}
\]

Therefore, we necessarily have \(\partial \Omega / \partial \psi = 0\) if \(\sin(\psi - \tau \Omega) \neq 0\).

7.3 Curves parametrized by angular frequency

Suppose \(a \in \mathbb{R}\) and \(w \in \mathbb{C} \setminus \{0\}\). For each given \(\Omega \in \mathbb{R}\) and \(\tau > 0\), we will find a condition on \(a\) and \(w\) for which Eq. (4.1) has a root \(\text{i}\Omega\). Since Eq. (4.1) has a root 0 if and only if \(a = w\), it is sufficient to find a purely imaginary root of Eq. (4.1).

We use the following notation.

**Notation 7.** Let \(\tau > 0\) be given. For each \(\Omega \in \mathbb{R} \setminus \{0\}\) and \(\psi \in (-\pi, \pi]\) satisfying \(\tau \Omega - \psi \notin \pi \mathbb{Z}\), let

\[
a(\Omega, \psi; \tau) := -\Omega \cot(\tau \Omega - \psi) \quad \text{and} \quad \rho(\Omega, \psi; \tau) := -\frac{\Omega}{\sin(\tau \Omega - \psi)}.
\]

For each \(\tau > 0\), \(a(\Omega, \psi; \tau)\) and \(\rho(\Omega, \psi; \tau)\) are expressed as

\[
a(\Omega, \psi; \tau) = \frac{1}{\tau} a(\tau \Omega, \psi; 1) \quad \text{and} \quad \rho(\Omega, \psi; \tau) = \frac{1}{\tau} \rho(\tau \Omega, \psi; 1). \quad (7.1)
\]
From (4.1), Eq. (∗) has a root $i\Omega$ ($\Omega \neq 0$) if and only if $\sin(\text{Arg}(w) - \tau\Omega) \neq 0$, i.e.,

$$|w| = \frac{\Omega}{\sin(\text{Arg}(w) - \tau\Omega)} = \rho(\Omega, \text{Arg}(w); \tau),$$

and

$$a = |w|\cos(\text{Arg}(w) - \tau\Omega) = a(\Omega, \text{Arg}(w); \tau).$$

This means that by varying $\Omega \in \mathbb{R} \setminus \{0\}$ so that $\tau\Omega - \text{Arg}(w) \notin \pi\mathbb{Z}$ for each given $\tau > 0$ and each fixed $\text{Arg}(w)$, we obtain the parametrization of curves in $(a, |w|)$-plane on which Eq. (∗) has purely imaginary roots. This is the method of D-partitions in our case.

We introduce the following notation.

**Notation 8.** For each integer $k \neq 0$ and $\psi \in (-\pi, \pi]$, let

$$I_k(\psi) := \begin{cases} (-\pi, 0) + \psi + 2k\pi & (k \geq 1), \\ (0, \pi) + \psi + 2k\pi & (k \leq -1). \end{cases}$$

Let

$$I_c(\psi) := \begin{cases} (0, \psi) & (\psi \geq 0), \\ (\psi, 0) & (\psi \leq 0). \end{cases}$$

Here we are interpreting that $I_c(\psi)$ is empty when $\psi = 0$.

**Lemma 7.1.** Let $\psi \in (-\pi, \pi]$ and $\tau > 0$ be given. Suppose $\Omega \in \mathbb{R} \setminus \{0\}$. Then $\rho(\Omega, \psi; \tau) > 0$ if and only if

$$\tau\Omega \in I_c(\psi) \cup \bigcup_{k \in \mathbb{Z} \setminus \{0\}} I_k(\psi)$$

holds.

**Proof.** From (7.1), it is sufficient to consider the case $\tau = 1$. We also consider the case $\psi > 0$. The proof is divided into the following two cases.

- **Case 1: $\Omega > 0$.** In this case, the positivity of $\rho(\Omega, \psi; 1)$ is equivalent to $\sin(\Omega - \psi) < 0$. This is equivalent to

$$\Omega - \psi \in (-\pi, 0) + 2k\pi$$

for some $k \in \mathbb{Z}$. Since $\Omega - \psi > -\psi \geq -\pi$, $k$ necessarily satisfies $k \geq 0$. We note that the condition for the case $k = 0$ becomes $\Omega \in (0, \psi) = I_c(\psi)$ because of $\psi - \pi \leq 0$.

- **Case 2: $\Omega < 0$.** The same reasoning imposes $\sin(\Omega - \psi) > 0$, i.e.,

$$\Omega - \psi \in (0, \pi) + 2k\pi$$

for some $k \in \mathbb{Z}$. Since $\Omega - \psi < -\psi < 0$, $k$ necessarily satisfies $k \leq -1$.

The similar proof is valid for the case $\psi < 0$, and the proof for the case $\psi = 0$ is more simpler than these cases. This completes the proof.

In view of Lemma [7.1] we introduce the following notation.
**Notation 9.** For each $\psi \in (-\pi, \pi]$ and each $\tau > 0$, let

$$
\Gamma_*(\psi; \tau) := \left\{ (a(\Omega, \psi; \tau), \rho(\Omega, \psi; \tau)e^{i\psi}) : \tau \Omega \in I_*(\psi) \right\},
$$

$$
\tilde{\Gamma}_*(\psi; \tau) := \left\{ (a(\Omega, \psi; \tau), \rho(\Omega, \psi; \tau)) : \tau \Omega \in I_*(\psi) \right\},
$$

where the symbol $\ast$ denotes $c$ or some nonzero integer $k$.

For each $\ast \in \{c\} \cup (\mathbb{Z} \setminus \{0\})$, we consider $\Gamma_*(\psi; \tau)$ as a parametrized curve with the parametrization given by

$$
\frac{1}{\tau} I_*(\psi) \ni \Omega \mapsto (a(\Omega, \psi; \tau), \rho(\Omega, \psi; \tau)e^{i\psi}) \in \mathbb{R} \times (\mathbb{C} \setminus \{0\}).
$$

(7.2)

We note that

$$
\Gamma_*(\psi; \tau) = \frac{1}{\tau} \Gamma_*(\psi; 1)
$$

(7.3)

holds from (7.1).

By using the above notation, under the assumption of $\text{Arg}(w) = \psi$, the set of all $(a, w)$ for which Eq. (9) has purely imaginary roots is represented by

$$
\Gamma_c(\psi; \tau) \cup \bigcup_{k \in \mathbb{Z} \setminus \{0\}} \Gamma_k(\psi; \tau).
$$

Here we briefly study the location of each curves. For all $\Omega$ satisfying $\tau \Omega \in I_c(\psi) \cup \bigcup_{k \in \mathbb{Z} \setminus \{0\}} I_k(\psi)$, the inequalities

$$
-\rho(\Omega, \psi; \tau) < a(\Omega, \psi; \tau) < \rho(\Omega, \psi; \tau)
$$

are obtained. This can be seen by dividing all the terms by $\Omega / \sin(\tau \Omega - \psi) < 0$ because the resulting inequalities are

$$
-1 < -\cos(\tau \Omega - \psi) < 1.
$$

The above inequalities are also obtained by

$$
\rho(\Omega, \psi; \tau)^2 - a(\Omega, \psi; \tau)^2 = \Omega^2 \left[ \frac{1}{\sin^2(\tau \Omega - \psi)} - \cot^2(\tau \Omega - \psi) \right] = \Omega^2 > 0
$$

(7.4)

under the assumption of $\rho(\Omega, \psi; \tau) > 0$. Therefore, all the curves $\Gamma_*(\psi; \tau)$ are contained in a linear cone

$$
\{(a, w) \in \mathbb{R} \times (\mathbb{C} \setminus \{0\}) : |a| < |w| \}
$$

in $(a, w)$-space.

Calculation (7.4) also shows that each curve $\Gamma_*(\psi; \tau)$ does not have a self-intersection, namely, parametrization (7.2) gives a one-to-one correspondence between $(1/\tau) I_*(\psi)$ and $\Gamma_*(\psi; \tau)$. We note that this is a natural consequence from Theorem 4.3.

We next study the curve $\Gamma_c(\psi; \tau)$ in more detail. Here the following remark is useful.

**Remark 10.** $\tau \Omega \in I_c(\psi)$ can be written as $\tau |\Omega| \in (0, |\psi|)$. By combining this and

$$
a(\Omega, \psi; \tau) = a(-\Omega, -\psi; \tau) \quad \text{and} \quad \rho(\Omega, \psi; \tau) = \rho(-\Omega, -\psi; \tau),
$$

\(\Gamma_c(\psi; \tau) = \Gamma_c(|\psi|; \tau)\) holds.
The above remark shows that we only have to consider the case \( \psi > 0 \) to study \( \Gamma_c(\psi; \tau) \).

**Lemma 7.2.** Let \( \psi \in (-\pi, \pi] \) and \( \tau > 0 \) be given. Then for all \( \Omega \) satisfying \( \tau \Omega \in I_c(\psi) \),

\[
a(\Omega, \psi; \tau) > \rho(\Omega, \psi; \tau) \cos \psi
\]

holds.

**Proof.** We only have to consider the case \( \psi > 0 \). From (7.1), it is sufficient to consider the case \( \tau = 1 \). Since \( \rho(\Omega, \psi; 1) > 0 \), the inequality is equivalent to \( \cos(\Omega - \psi) > \cos \psi \). Since \( \Omega \in (0, \psi) \), i.e., \( -\psi < \Omega - \psi < 0 \), \( \cos(\Omega - \psi) > \cos \psi \) is equivalent to

\[-\Omega + \psi < \psi,
\]

which trivially holds. \( \square \)

The above lemma means that for any \( \psi \in (-\pi, \pi] \) and any \( \tau > 0 \), the curve \( \Gamma_c(\psi; \tau) \) is contained in the subset \( D_c \), which is a linear cone and the domain of definition of the critical delay function.

### 7.4 “One-to-one correspondence” and “ordering”

In this subsection, we treat a special “ordering” for the curves \( \Gamma^*_c(\psi; \tau) \).

**Notation 10.** Let \( C \) be a linear cone (in a linear topological space over \( \mathbb{R} \)) and \( \Gamma, \Gamma' \subset C \) be nonempty subsets. We write \( \Gamma \prec \Gamma' \) if the following condition is satisfied: For every \( v \in C \), there exists a unique pair \( (s, s') \) of positive numbers such that \( s < s' \), \( sv \in \Gamma \) and \( s'v \in \Gamma' \).

The above concept should be compared with [34, page 334].

We first consider correspondences between the curves \( \Gamma^*_c(\psi; \tau) \) and \( \tau \)-values, which will be useful for determining the \( \prec \)-ordering of the family of curves \( (\Gamma^*_c(\psi; \tau))_{\tau \in (c) \cup (\mathbb{Z} \setminus \{0\})} \).

**Lemma 7.3.** Let \( \psi \in (-\pi, \pi] \) and \( \tau > 0 \) be given. Let

\[
a := a(\Omega, \psi; \tau) \quad \text{and} \quad w := \rho(\Omega, \psi; \tau) e^{i\psi}
\]

for some \( \tau \Omega \in I_c(\psi) \cup \bigcup_{k \in \mathbb{Z} \setminus \{0\}} I_k(\psi) \). Then the following equivalences hold:

1. \( \tau = \tau_0(a, w) \) if and only if \( \tau \Omega \in I_c(\psi) \).
2. For each integer \( n \geq 1 \), \( \tau = \tau_0^+(a, w) \) if and only if \( \tau \Omega \in I_n(\psi) \).
3. For each integer \( n \geq 1 \), \( \tau = \tau_0^-(a, w) \) if and only if \( \tau \Omega \in I_{-n}(\psi) \).

**Proof.** We prove the statement 1 when \( \psi \neq 0 \).

(Only-if-part). By definition, we have

\[
\tau = \frac{1}{\Omega(a, w)} \left[ |\psi| - \arccos \left( \frac{a}{|w|} \right) \right],
\]

where \( \Omega(a, w) = |\Omega| \) holds from (7.4). This shows \( 0 < \tau |\Omega| < |\psi| \), which is equivalent to \( \tau \Omega \in I_c(\psi) \).
(If-part). Since
\[ \frac{a}{|w|} = \cos(\tau \Omega - \psi) = \cos(\tau |\Omega| - |\psi|), \]
we have \( \arccos(a/|w|) = -\tau|\Omega| + |\psi| \) because \( \tau|\Omega| \in \mathcal{I}_c(|\psi|) \). By using this and (7.4), we obtain
\[ \tau_c(a, w) = \frac{1}{\Omega(a, w)} \left[ |\psi| - \arccos \left( \frac{a}{|w|} \right) \right] = \frac{1}{\Omega} \left[ |\psi| - ( -\tau|\Omega| + |\psi|) \right] = \tau. \]

The above argument shows that \( \tau = \tau_c(a, w) \) is impossible when \( \psi = 0 \). Therefore, this completes the proof of the statement 1. The proofs of the statements 2 and 3 are similar in view of the expressions of \( \tau_{c}^\pm(a, w) \). Therefore, they can be omitted.

We note that the above equivalences are natural consequences from Theorem 4.3. The following theorems are related to Lemma 7.3.

**Theorem 7.4.** Let \( \psi \in (-\pi, \pi] \) and \( \tau > 0 \) be given. Then for every \((a, w) \in D_c \) and every \( s > 0 \), \((sa, sw) \in \Gamma_c(\psi; \tau) \) if and only if
\[ s = \frac{1}{\tau} \tau_c(a, w) \quad \text{and} \quad \text{Arg}(w) = \psi \]
hold.

**Proof.** We only have to consider the case \( \psi \neq 0 \).

(Only-if-part). \((sa, sw) \in \Gamma_c(\psi; \tau) \) is equivalent to
\[ sa = a(\Omega, \psi; \tau) \quad \text{and} \quad sw = \rho(\Omega, \psi; \tau) e^{i\psi} \]
for some \( \Omega \in (1/\tau) \mathcal{I}_c(\psi) \). Then by applying Lemma 7.3 we necessarily have \( \tau = \tau_c(sa, sw) \), where
\[ \tau_c(sa, sw) = \frac{1}{\Omega(sa, sw)} \left[ |\text{Arg}(w)| - \arccos \left( \frac{a}{|w|} \right) \right] = \frac{1}{s} \tau_c(a, w). \]

Therefore, \( s = \tau_c(a, w)/\tau \) is obtained.

(If-part). Let
\[ \Omega := \Omega(sa, sw) = s\Omega(a, w). \]
Then we have
\[ \tau = \frac{1}{s} \tau_c(a, w) = \frac{1}{s\Omega(a, w)} \left[ |\psi| - \arccos \left( \frac{a}{|w|} \right) \right], \]
which implies
\[ \tau|\Omega| = |\psi| - \arccos \left( \frac{a}{|w|} \right) \in (0, |\psi|) = \mathcal{I}_c(|\psi|). \]

By using
\[ \sin \left( \arccos \left( \frac{a}{|w|} \right) \right) = \frac{1}{|w|} \Omega(a, w), \]
we obtain
\[ a(\Omega, \psi; \tau) = \frac{a/|w|}{\Omega(a, w)/|w|} = sa, \]
\[ \rho(\Omega, \psi; \tau) = \frac{s\Omega(a, w)}{\Omega(a, w)/|w|} = s|w|. \]

Therefore, \((sa, sw) \in \Gamma_c(\psi; \tau) \) is concluded.

This completes the proof. \( \square \)
The following is a corollary of Theorem 7.4.

**Corollary 7.5.** For each $\psi \in (-\pi, \pi]$ and each $\tau > 0$, we have
\[
\{(a, w) \in D_c : \text{Arg}(w) = \psi \text{ and } \tau_c(a, w) > \tau\} = \bigcup_{s > 1} \frac{1}{s} \Gamma_c(\psi; \tau).
\]
Consequently,
\[
\{(a, w) \in D_c : \tau_c(a, w) > \tau\} = \bigcup_{\psi \in (-\pi, \pi]} \bigcup_{s > 1} \frac{1}{s} \Gamma_c(\psi; \tau).
\]
holds.

**Proof.** ($\subset$). Let $(a, w) \in D_c$ be chosen so that $\text{Arg}(w) = \psi$ and $\tau_c(a, w) > \tau$. Let
\[
s := \frac{1}{\tau} \tau_c(a, w).
\]
Then $s > 1$ holds by the assumption. Applying Theorem 7.4, we have $(sa, sw) \in \Gamma_c(\psi; \tau)$. Therefore,
\[
(a, w) \in \bigcup_{s' > 1} \frac{1}{s'} \Gamma_c(\psi; \tau)
\]
holds.

($\supset$). Let $(a, w) \in (1/s)\Gamma_c(\psi; \tau)$ for some $s > 1$. This means $(sa, sw) \in \Gamma_c(\psi; \tau)$, which implies
\[
s = \frac{1}{\tau} \tau_c(a, w)
\]
from Theorem 7.4. Therefore, the inequality $\tau_c(a, w) > \tau$ is concluded.

This completes the proof. $\square$

In the similar way to the proof of Theorem 7.4 by using the expressions of $\tau_{n\pm}(a, w)$, the following theorem is obtained. The proof can be omitted.

**Theorem 7.6.** Let $\psi \in (-\pi, \pi]$, $\tau > 0$, and $k \in \mathbb{Z} \setminus \{0\}$ be given. Then for every
\[
(a, w) \in \mathbb{R} \times (\mathbb{C} \setminus \{0\})
\]
satisfying $|a| < |w|$ and every $s > 0$, $(sa, sw) \in \Gamma_k(\psi; \tau)$ if and only if
\[
s = \frac{1}{\tau} \tau_{k\pm}(a, w) \quad \text{and} \quad \text{Arg}(w) = \psi
\]
holds. Here $+$ sign corresponds to the case $k > 0$, and $-$ sign corresponds to the case $k < 0$.

We finally obtain the following $\prec$-ordering results.

**Corollary 7.7.** Let $\psi \in (-\pi, \pi) \setminus \{0\}$ and $\tau > 0$ be given. Then for any distinct pair $(\ast, \ast')$ in $\{c\} \cup (\mathbb{Z} \setminus \{0\})$, the curves $\Gamma_{\ast}(\psi; \tau)$ and $\Gamma_{\ast'}(\psi; \tau)$ do not have an intersection. Furthermore, the following statements hold:

- If $\psi \in (0, \pi)$, then we have
  \[
  \Gamma_{-n}(\psi; \tau) \prec \Gamma_n(\psi; \tau) \prec \Gamma_{-(n+1)}(\psi; \tau) \quad (n \geq 1)
  \]
  and
  \[
  \Gamma_c(\psi; \tau) \prec \Gamma_{-1}(\psi; \tau) \cap D_c.
  \]
• If $\psi \in (-\pi, 0)$, then we have
  \[ \Gamma_n(\psi; \tau) \prec \Gamma_{-n}(\psi; \tau) \prec \Gamma_{n+1}(\psi; \tau) \quad (n \geq 1) \]
  and
  \[ \Gamma_c(\psi; \tau) \prec \Gamma_1(\psi; \tau) \cap D_c. \]

Proof. For each $\ast \in \{ \ast \} \cup (\mathbb{Z} \setminus \{0\})$, let
  \[ \tau_\ast(a, w) := \begin{cases} \tau_\ast(a, w) & (\ast = c), \\ \tau_\ast^+(a, w) & (\ast = k > 0), \\ \tau_\ast^-(a, w) & (\ast = k < 0). \end{cases} \]
Here $(a, w) \in D_c$ when $\ast = c$, and $(a, w)$ satisfies $|a| < |w|$ when $\ast = k \neq 0$. From Lemma [7.3] the existence of intersection of the distinct curves $\Gamma_\ast(\psi; \tau)$ and $\Gamma_\ast'(\psi; \tau)$ necessarily implies
  \[ \tau = \tau_\ast(a, w) = \tau_\ast'(a, w) \]
for some $(a, w)$. However, this is impossible because of Lemmas [4.7] and [4.8]. The $\prec$-ordering results are also consequences of Theorems [7.4], [7.6], Lemmas [4.7] and [4.8] by the definition of $\prec$-ordering. This completes the proof. \hfill \Box

The result for the cases $\psi = 0$ or $\psi = \pi$ are special.

Corollary 7.8. Let $\psi \in \{0, \pi\}$ and $\tau > 0$ be given. Then the following statements hold:

1. If $\psi = 0$, then for all $n \geq 1$,
  \[ \Gamma_n(\psi; \tau) = \Gamma_{-n}(\psi; \tau) \prec \Gamma_{n+1}(\psi; \tau) = \Gamma_{-(n+1)}(\psi; \tau) \]

holds.

2. If $\psi = \pi$, then for all $n \geq 1$,
  \[ \Gamma_n(\psi; \tau) = \Gamma_{-(n+1)}(\psi; \tau) \prec \Gamma_{n+1}(\psi; \tau) = \Gamma_{-(n+2)}(\psi; \tau) \]

holds. Furthermore, $\Gamma_c(\psi; \tau) \prec \Gamma_{-1}(\psi; \tau) \cap D_c$ holds.

The proof is similar to the proof of Corollary [7.7] by using Lemma [4.6]. Therefore, it can be omitted.

8 Discussions

As is discussed in Introduction, the study of the stability region of Eq. (**) for the imaginary case is complicated because the critical delay does not exist in general and stability switches may occur (see [43, Theorems 3.2 and 3.3]). Nevertheless, the method of consideration by using critical delay should work in this situation by solving appropriately obtained inequalities on $\tau > 0$. This is a possible future research.

Another direction of a future research is to study characteristic equations of differential equations with multiple delay parameters under the perspective of critical delay. When
the number of delay parameters is two, we are going to consider a transcendental equation of the form
\[ z + a - w_1 e^{-\tau_1 z} - w_2 e^{-\tau_2 z} = 0, \]
where \( \tau_1, \tau_2 > 0 \) are delay parameters. See [14], [26], [1], [37], [50], [45], [48], [36], [9], and [12] for studies of the above transcendental equation, for example. For studies of the stability condition of transcendental equations with multiple delays in the delay parameter space, e.g., see [26], [24], and [32]. See also [53] for a survey article.

It would be also interesting to develop a method to find the stability region without resorting to the explicit expression of the critical delay. This might be possible because the critical delay is considered to be an implicit function. This consideration would have a similarity with the method by the Lambert \( W \) function in [43] since the Lambert \( W \) function is an inverse function.

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\section*{A Analysis based on Lambert \( W \) function}

In this section, we summarize a route to Theorem 2.1 based on the Lambert \( W \) function.

\subsection*{A.1 General results about Lambert \( W \) function}

Since the set of all roots of Eq. (i) is expressed by (3.31)
\[ \frac{1}{\tau} W(\tau we^{\tau a}) - a, \]
all the roots of Eq. (i) have negative real parts if and only if \( \Re(z) < \tau \Re(a) \) holds for all \( z \in W(\tau we^{\tau a}) \). For this type of threshold condition, the following result is obtained in [43, Lemma 3.1].

\begin{theorem}[43] \label{thm:A.1}
Let \( \zeta \in \mathbb{C} \) and \( \sigma \in \mathbb{R} \) be given. Then \( \Re(z) < \sigma \) holds for all \( z \in W(\zeta) \) if and only if \( \zeta \) and \( \sigma \) satisfy one of the following conditions:
\begin{enumerate}
\item \( \sigma e^\sigma > |\zeta| \).
\item \( -|\zeta| < \sigma e^\sigma \leq |\zeta| \) and
\[ |\text{Arg}(\zeta)| > \arccos \left( \frac{\sigma e^\sigma}{|\zeta|} \right) + \sqrt{(|\zeta|e^{-\sigma})^2 - \sigma^2}. \]
\end{enumerate}
\end{theorem}

\begin{remark}
In [43, Lemma 3.1], the condition \( \zeta \neq 0 \) is presumed. However, since \( W(0) = \{0\} \), the property that \( \Re(z) < \sigma \) holds for all \( z \in W(0) \) is equivalent to \( \sigma > 0 \). Therefore, the case \( \zeta = 0 \) can be included in the statement.

The following is a corollary of Theorem A.1 which is not stated in [43].
\end{remark}
Corollary A.2. Let $\zeta \in \mathbb{C}$ and $\sigma \in \mathbb{R}$ be given. Then $\Re(z) \geq \sigma$ holds for some $z \in W(\zeta)$ if and only if $\zeta$ and $\sigma$ satisfy one of the following conditions:

(iii) $\sigma e^\sigma \leq -|\zeta|$.

(iv) $-|\zeta| < \sigma e^\sigma \leq |\zeta|$ and

$$|\text{Arg}(\zeta)| \leq \arccos\left(\frac{\sigma e^\sigma}{|\zeta|}\right) + \sqrt{|\zeta|e^{-\sigma} - \sigma^2}.$$ 

Proof. From Theorem A.1, $\Re(z) \geq \sigma$ holds for some $z \in W(\zeta)$ if and only if both of the conditions (i) and (ii) in Theorem A.1 does not hold. Here

- the condition (i) in Theorem A.1 does not hold if and only if $\sigma e^\sigma \leq |\zeta|$,
- the condition (ii) in Theorem A.1 does not hold if and only if $\sigma e^\sigma \not\in (-|\zeta|, |\zeta|]$ or (iv) holds.

Therefore, the equivalence is obtained.

A.2 Necessary and sufficient conditions

By applying Theorem A.1 with $\zeta = \tau we^{\tau a}$ and $\sigma = \tau \Re(a)$, the following result is immediately obtained in [43, Theorem 1.2].

Theorem A.3. Suppose $a, w \in \mathbb{C}$. Then all the roots of Eq. (43) have negative real parts, i.e., $\tau \in T(a, w)$, if and only if the parameters $a, w, \tau$ satisfy one of the following conditions:

(i) $\Re(a) > |w|$.

(ii) $-|w| < \Re(a) \leq |w|$ and

$$\arccos\left(\cos(\tau \Im(a) + \text{Arg}(w))\right) > \arccos\left(\frac{\Re(a)}{|w|}\right) + \tau \sqrt{|w|^2 - \Re(a)^2}. \quad (A.1)$$

Eq. (43) with complex $a$ and $w$ has been investigated by many authors (e.g., see [31], [2], [10], [38], [14], [58], [52], [33], [11]). However, as far as we know, the necessary and sufficient condition given in Theorem A.3 has not been obtained before [43].

Remark 12. Let $\sigma \in \mathbb{R}$ be given. By letting $z' := z - \sigma$ in the transcendental equation $ze^z = \zeta$, the equation becomes

$$z' + \sigma - \zeta e^{-\sigma}e^{-z'} = 0, \quad (A.2)$$

where $\Re(z) < \sigma$ if and only if $\Re(z') < 0$. Then by applying Theorem A.3 to Eq. (A.2), the statement of Theorem A.1 is obtained. This means that Theorems A.1 and A.3 are logically equivalent.

Remark 13. When $\Re(a) = |w|$, the right-hand side of inequality (A.1) is equal to 0. Therefore, the condition (ii) in Theorem A.3 becomes

$$\arccos\left(\cos(\tau \Im(a) + \text{Arg}(w))\right) > 0,$$

i.e., $\tau \Im(a) + \text{Arg}(w) \not\in 2\pi\mathbb{Z}$. 

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Inequality (A.1) in the condition (ii) contains the delay parameter \( \tau \). This makes clear that the case of imaginary \( a \) (i.e., \( \Im(a) \neq 0 \)) brings a qualitative change to the condition on \( \tau \) for which all the roots of \((\ast)\) have negative real parts. This fact has been partially known in the literature before [43] (e.g., see [60], [58], [30], [40], and [41]). Here the function \( \arccos(\cos(\cdot)) : \mathbb{R} \to \mathbb{R} \) is the \( 2\pi \)-periodic function satisfying
\[
\arccos(\cos(\theta)) = |\theta|
\]
for all \( \theta \in [-\pi, \pi] \). See [43, Theorems 3.2 and 3.3] for further details.

The following is a consequence of Theorem A.3.

**Theorem A.4.** Suppose \( a \in \mathbb{R} \) and \( w \in \mathbb{C} \). Then all the roots of Eq. \((\ast)\) have negative real parts, i.e., \( \tau \in T(a, w) \), if and only if the parameters \( a, w, \) and \( \tau \) satisfy one of the following conditions:

(i) \( a \geq |w| \) and \( a \neq w \).

(ii) \( -|w| < a < |w| \) and
\[
|\Arg(w)| > \arccos \left( \frac{a}{|w|} \right) + \tau \sqrt{|w|^2 - a^2}. \tag{A.3}
\]

**Proof.** When \( w = 0 \), Eq. \((\ast)\) becomes \( z + a = 0 \). Therefore, \(-a\) is the only root of Eq. \((\ast)\), and the condition \( a > 0 \) is included in the condition (i) in Theorem A.4. We now assume \( w \neq 0 \). In view of
\[
\arccos(\cos(\Arg(w))) = |\Arg(w)|,
\]
we only need to check the case \( a = |w| \). In this case, the condition (ii) in Theorem A.3 becomes \( \Arg(w) \neq 0 \) from Remark 13 and \( \Arg(w) = 0 \) is equivalent to \( a = w \). This completes the proof.

**Remark 14.** By letting \( z' := z + i\Im(a) \), Eq. \((\ast)\) becomes
\[
z' + \Re(a) - w e^{i\Im(a)} e^{-\tau z'} = 0.
\]
Therefore, Theorems A.3 and A.4 are logically equivalent.

A proof of Theorem 2.1 is obtained by using Theorem A.4.

**Proof of Theorem 2.1 based on Theorem A.4.** We give the proofs of the statements (I), (II), and (III).

(I) When \( a \geq |w| \) and \( a \neq w \), it holds that all the roots of Eq. \((\ast)\) have negative real parts from Theorem A.4 independently from \( \tau > 0 \). Therefore, \( T(a, w) = (0, \infty) \) holds. Conversely, we suppose that \( T(a, w) = (0, \infty) \) holds. By using Theorem A.4 again, we have “\( a \geq |w| \) and \( a \neq w \)” or “\(-|w| < a < |w|\)”.

We suppose that the latter condition holds and derive a contradiction. Theorem A.4 shows that inequality (A.3) holds for all \( \tau > 0 \). However, this is impossible because of
\[
\lim_{\tau \to \infty} \tau \sqrt{|w|^2 - a^2} = \infty.
\]
The above argument shows that \( T(a, w) = (0, \infty) \) if and only if \( a \geq |w| \) and \( a \neq w \). Thus, the statement (I) is obtained in view of Remark 1.
(II) When \( w \neq 0 \) and \( \Re(w) < a < |w| \), inequality \((A.3)\) is satisfied if \( 0 < \tau < \tau_c(a, w) \). Therefore, \((0, \tau_c(a, w)) \subset T(a, w)\) holds from Theorem \(A.4\). Conversely, we suppose that \( T(a, w) \) is a nonempty proper subset of \((0, \infty)\). From Theorem \(A.4\) we necessarily have \(-|w| < a < |w|\), which implies \( w \neq 0 \). Theorem \(A.4\) also shows that \( \tau \in T(a, w) \) implies that inequality \((A.3)\) holds. Therefore, we have

\[
|\Arg(w)| > \arccos \left( \frac{a}{|w|} \right),
\]

which implies \( \Re(w) < a \) because \( \cos[0, \pi] \) is strictly monotonically decreasing. Then we have \( T(a, w) \subset (0, \tau_c(a, w)) \). The above argument shows the statement (II).

(III) When \( a \leq \Re(w) \), it holds that \( T(a, w) \) is empty from the statements (I) and (II) in Theorem \(2.1\). When \( a > \Re(w) \), the statement (I) in Theorem \(2.1\) also shows that \( T(a, w) = (0, \infty) \) if \( a \geq |w| \). If \( a < |w| \), \( w \) is necessarily nonzero, and the statement (II) in Theorem \(2.1\) shows that \( T(a, w) \) is a nonempty proper subset of \((0, \infty)\). Therefore, the statement (III) is obtained.

This completes the proof. \(\square\)

Theorem \(A.4\) is also obtained from Theorem \(2.1\). Therefore, these theorems are logically equivalent.

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