A Dirac-type equation for spacelike neutrinos

Tsao Chang
Center for Space Plasma and Aeronomy Research
University of Alabama in Huntsville
Huntsville, AL 35899
Email: changt@cspar.uah.edu

Based on experimental evidences supporting the hypothesis that neutrinos might be spacelike particles, a new Dirac-type equation is proposed and a spin-$\frac{1}{2}$ spacelike quantum theory is developed. The new Dirac-type equation provides a solution for the puzzle of negative mass-square of neutrinos. This equation can be written in two spinor equations coupled together via nonzero mass while respecting maximum parity violation, and it reduces to one Weyl equation in the massless limit. Some peculiar features of spacelike neutrino are discussed in this theoretical framework.

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I. INTRODUCTION

A model has recently been presented to fit the cosmic ray spectrum at $E \approx 1 - 4$ PeV \cite{1} using the hypothesis that the electron neutrino is a tachyon. This model yields a value for $m^2(\nu_e) \approx -3$ eV$^2$, which is consistent with the results from recent measurements in tritium beta decay experiments \cite{2-4}. Moreover, the muon neutrino also exhibits a negative mass-square \cite{5}. However, up to now, there is no satisfactory relativistic quantum theory to describe neutrinos as spin-$\frac{1}{2}$ tachyons.

The negative value of the neutrino mass-square simply means:

$$E^2/c^2 - p^2 = m^2_c < 0$$

The right-hand side in Eq.(1) can be rewritten as $(-m^2_sc^2)$, then $m_s$ has a positive value. The subscript $s$ means spacelike particle, i.e. tachyon.

Based on special relativity and known as re-interpretation rule, tachyon as a hypothetical particle was proposed by Bilaniuk et al. in the sixties \cite{6-8}. For tachyons, the relation of momentum and energy is shown in Eq.(1). The negative value on the right-hand side of Eq.(1) means that $p^2$ is greater than $(E/c)^2$. The velocity of a tachyons, $u_s$, is greater than speed of light. The momentum and energy in terms of $u_s$ are as follows:

$$p = \frac{m_su_s}{\sqrt{u^2_s/c^2 - 1}}, \quad E = \frac{m_sc^2}{\sqrt{u^2_s/c^2 - 1}}$$

Any physical reference system is built by timelike particles (such as atoms, molecules etc.), which requires that a reference frame must move slower than light. On the other hand, once a tachyon is created in an interaction, its speed is always greater than the speed of light. The neutrino is the most likely candidate for a tachyon because it has left-handed spin in any reference frame \cite{9,10}. However, anti-neutrino always has right-handed spin. Considering the measured mass-square is negative for the muon neutrino, Chodos, Hauser and Kostelecky \cite{9} suggested in 1985 that the muon neutrino might be a tachyon. They also suggested that one could test the tachyonic neutrino in high energy region using a strange feature of tachyon: $E_u$ could be negative in some reference frames \cite{11,12}. This feature has been further studied by Ehrlich \cite{1}. Therefore, it is required to construct a spacelike quantum theory for neutrinos.
The first step in this direction is usually to introduce an imaginary mass, but these efforts could not reach the point of constructing a consistent quantum theory. Some early investigations of Dirac-type equations for tachyons are listed in Ref. [13,14]. An alternative approach was investigated by Chodos et al. [9]. They examined the possibility that muon neutrino might be tachyonic fermion. A form of the lagrangian density for tachyonic neutrinos was proposed. Although they did not obtain a satisfactory quantum theory for tachyonic fermions, they suggested that more theoretical work would be needed to determine a physically acceptable theory.

II. A NEW DIRAC-TYPE EQUATION

In this paper, we will start with a different approach for deriving a new Dirac-type equation for spacelike neutrinos. In order to avoid introducing imaginary mass, Eq. (1) can be rewritten as

\[ E = (c^2p^2 - m_s^2c^4)^{1/2} \]  

where \( m_s \) is called proper mass. For instance, \( m_s(\nu_e) = 1.6 \text{ eV} \), if taking \( m^2(\nu_e) = -2.5 \text{ eV}^2 \) [15]. To follow Dirac’s approach [16], the Hamiltonian must be first order in the momentum operator \( \hat{p} \):

\[ \hat{E} = c\bar{\alpha} \cdot \hat{p} + \beta_s m_s c^2 \]  

with \( (\hat{E} = i\hbar \partial/\partial t, \hat{p} = -i\hbar \nabla) \). \( \bar{\alpha} = (\alpha_1, \alpha_2, \alpha_3) \) and \( \beta_s \) are 4×4 matrix, which are defined as

\[ \alpha_i = \begin{pmatrix} 0 & \sigma_i \\ \sigma_i & 0 \end{pmatrix}, \quad \beta_s = \begin{pmatrix} 0 & I \\ -I & 0 \end{pmatrix} \]  

where \( \sigma_i \) is 2×2 Pauli matrix, \( I \) is 2×2 unit matrix. Notice that \( \beta_s \) is a new matrix, which is different from the one in the traditional Dirac equation. We will discuss the property of \( \beta_s \) in a later section.

When we take the square of both sides of Eq. (4), and consider the following relations:

\[ \alpha_i\alpha_j + \alpha_j\alpha_i = 2\delta_{ij} \]
\[ \alpha_i\beta_s + \beta_s\alpha_i = 0 \]
\[ \beta_s^2 = -1 \]  

where
then the relation in Eq. (1) or Eq. (3) is reproduced. Since Eq. (3) is related to Eq. (2), this means $\beta_s$ is the right choice to describe neutrinos as tachyons. Notice that the relation between the matrix $\beta_s$ and the traditional matrix $\beta$ is as follows:

$$ \beta_s = \beta \gamma_5, \quad \text{where} \quad \beta = \begin{pmatrix} I & 0 \\ 0 & -I \end{pmatrix}, \quad \gamma_5 = \begin{pmatrix} 0 & I \\ I & 0 \end{pmatrix} $$ (7)

We now study the spin-$\frac{1}{2}$ the property of neutrino (or anti-neutrino) as a tachyonic fermion.

Denote the wave function as

$$ \Psi = \begin{pmatrix} \varphi(x,t) \\ \chi(x,t) \end{pmatrix} \quad \text{with} \quad \varphi = \begin{pmatrix} \varphi_1 \\ \varphi_2 \end{pmatrix}, \quad \chi = \begin{pmatrix} \chi_1 \\ \chi_2 \end{pmatrix} $$

the complete form of the new Dirac-type equation, Eq. (4), becomes

$$ \hat{E}\Psi = c(\vec{\alpha} \cdot \hat{p})\Psi + \beta_s m_s c^2 \Psi $$ (8)

It can also be rewritten as a pair of two-component equations:

$$ i\hbar \frac{\partial \varphi}{\partial t} = -i c \hbar \vec{\sigma} \cdot \nabla \chi + m_s c^2 \chi $$

$$ i\hbar \frac{\partial \chi}{\partial t} = -i c \hbar \vec{\sigma} \cdot \nabla \varphi - m_s c^2 \varphi $$ (9)

From the equation (8), the continuity equation is derived:

$$ \frac{\partial \rho}{\partial t} + \nabla \cdot \vec{j} = 0 $$ (10)

and we have Eq. (10) can be rewritten as

$$ \rho = \Psi^\dagger \gamma_5 \Psi, \quad \vec{j} = c(\Psi^\dagger \gamma_5 \vec{\alpha} \Psi) $$ (11)

where $\rho$ and $\vec{j}$ are probability density and current; $\Psi^\dagger$ is the Hermitian adjoint of $\Psi$.

Considering a plane wave along the $z$ axis for a right-handed particle, the helicity $H = (\vec{\sigma} \cdot \hat{p})/|\hat{p}| = 1$, then Eq. (8) or (9) yields the following relation:

$$ \chi = \frac{cp - m_s c^2}{E} \varphi $$ (12)
III. COVARIANT FORM AND EXPLICIT SOLUTIONS

The new Dirac-type equation (8) can be written in a covariant form:

\[ i\hbar \gamma^\mu \partial_\mu \Psi - m_s c \gamma_5 \Psi = 0 \]  

Here the standard convention for the Dirac matrices are used:

\[ \gamma^0 = \beta = \begin{pmatrix} I & 0 \\ 0 & -I \end{pmatrix}, \quad \gamma^i = \begin{pmatrix} 0 & \sigma_i \\ -\sigma_i & 0 \end{pmatrix} \]  

For a free particle with momentum \( \vec{p} \) in the z direction, the plane wave can be represented by

\[ \Psi(z,t) = \psi_\sigma \exp\left[ \frac{i}{\hbar} (pz - Et) \right] \]  

where \( \psi_\sigma \) is a four-component bispinor. Substituting this bispinor into the wave equation (8) or (13), the explicit form of two bispinors with positive-energy states are listed as follows:

\[ \psi_1 = \psi_{\uparrow(+)} = N \begin{pmatrix} 1 \\ 0 \\ A \\ 0 \end{pmatrix}, \quad \psi_2 = \psi_{\downarrow(+)} = N \begin{pmatrix} 0 \\ -A \\ 0 \\ 1 \end{pmatrix} \]  

and other two bispinors with the negative-energy states are:

\[ \psi_3 = \psi_{\uparrow(-)} = N \begin{pmatrix} 1 \\ 0 \\ -A \\ 0 \end{pmatrix}, \quad \psi_4 = \psi_{\downarrow(-)} = N \begin{pmatrix} 0 \\ A \\ 0 \\ 1 \end{pmatrix} \]  

where the component \( A \) and the normalization factor \( N \) are

\[ A = \frac{cp - m_s c^2}{|E|}, \quad N = \sqrt{\frac{|p + m_s c|}{2m_s c}} \]  

For \( \psi_1 = \psi_{\uparrow(+)} \), the conserved current in Eq.(11) becomes:

\[ \rho = \Psi_1 \gamma_5 \Psi_1 = \frac{|E|}{m_s c^2}, \quad j = \frac{p}{m_s} \]
Let $\bar{\Psi} = \Psi^\dagger \beta$, we can obtain the following scalars:

$$\bar{\Psi}_1 \Psi_1 = \bar{\Psi}_3 \Psi_3 = 1$$  
$$\bar{\Psi}_2 \Psi_2 = \bar{\Psi}_4 \Psi_4 = -1$$  

(20a) \hspace{1cm} (20b)

In addition, the pseudo scalar for each spinor satisfies:

$$\bar{\Psi} \gamma_5 \Psi = 0$$  

(21)

IV. PARITY VIOLATION FOR NEUTRINOS

In order to compare the new Dirac-type equation Eq.(7) with the two component Weyl equation in the massless limit, we now consider a linear combination of $\varphi$ and $\chi$:

$$\xi = \frac{1}{\sqrt{2}} (\varphi + \chi), \quad \eta = \frac{1}{\sqrt{2}} (\varphi - \chi)$$  

(22)

where $\xi(\vec{x}, t)$ and $\eta(\vec{x}, t)$ are two-component spinor functions. In terms of $\xi$ and $\eta$, Eq.(10) becomes

$$\rho = \xi^\dagger \xi - \eta^\dagger \eta, \quad \vec{j} = c (\xi^\dagger \vec{\sigma} \xi + \eta^\dagger \vec{\sigma} \eta)$$  

(23)

Moreover, Eq.(8) can be rewritten in Weyl representation:

$$i \hbar \frac{\partial \xi}{\partial t} = -ic \vec{\sigma} \cdot \nabla \xi - m_s c^2 \eta$$

$$i \hbar \frac{\partial \eta}{\partial t} = ic \vec{\sigma} \cdot \nabla \eta + m_s c^2 \xi$$  

(24)

In the above equations, both $\xi$ and $\eta$ are coupled via the mass term $m_s$.

For comparing Eq. (24) with the well known Weyl equation, we take a limit, $m_s = 0$, then the first equation in Eq. (24) reduces to

$$\frac{\partial \xi_\varphi}{\partial t} = -c \vec{\sigma} \cdot \nabla \xi_\varphi$$  

(25)

In addition, the second equation in Eq. (24) vanishes because $\varphi = \chi$ when $m_s = 0$. 
Eq. (25) is the two-component Weyl equation for describing antineutrinos $\bar{\nu}$, which is related to the maximum parity violation discovered in 1956 by Lee and Yang \cite{18,19}. They pointed out that no experiment had shown parity to be a good symmetry for weak interactions. Now we see that, in terms of Eq. (24), once if neutrino has some mass, no matter how small it is, two equations should be coupled together via the mass term while still respecting maximum parity violation.

Indeed, the Weyl equation (25) is only valid for antineutrinos since a neutrino always has left-handed spin, which is opposite to antineutrino. For this purpose, we now introduce a transformation:

$$\vec{\alpha} \rightarrow -\vec{\alpha}$$ \hspace{1cm} (26)

It is easily seen that the anticommutation relations in Eq. (6) remain unchanged under this transformation. In terms of Eq. (26), Eq. (4) becomes

$$\hat{E}\Psi_{\nu} = -c(\vec{\alpha} \cdot \hat{p})\Psi_{\nu} + \beta_{s}m_{s}c^{2}\Psi_{\nu}$$ \hspace{1cm} (27)

Furthermore, $\vec{\sigma}$ should be replaced by $(-\vec{\sigma})$ from Eq. (5-9) and Eq. (24) for describing a neutrino. Therefore, the two-component Weyl equation for massless neutrino becomes:

$$\frac{\partial \xi_{\nu}}{\partial t} = c\vec{\sigma} \cdot \nabla \xi_{\nu}$$ \hspace{1cm} (28)

In fact, the transformation (26) is associated with the CPT theorem. Some related discussions can be found in Ref.\cite{20,21}.

V. REMARKS

In this paper, the hypothesis that neutrinos might be tachyons is further investigated. A spin-$\frac{1}{2}$ spacelike quantum theory is developed on the basis of the new Dirac-type equation. It provides a solution for the puzzle of negative mass-square of neutrinos.

Spacelike neutrinos have many peculiar features, which are very different from all other particles. For instance, neutrinos only have weak interactions with other particles. Neutrino has left-handed spin in any reference frame. On the other hand, anti-neutrino always has right-handed spin. This means that the speed of neutrinos must be equal to or greater than the speed of light. Otherwise, the spin direction of neutrino would be changed in some reference frames. Moreover, the
energy of a tachyonic neutrino (or anti-neutrino), $E_\nu$, could be negative in some reference frames. We will discuss the subject of the negative energy in another paper.

The electron neutrino and the muon neutrino may have different non-zero proper masses. If taking the data from Ref.[15], then we obtain $m_s(\nu_e) = 1.6$ eV and $m_s(\nu_\mu) = 0.13$ MeV. In this way, we can get a natural explanation why the numbers of e-lepton and $\mu$-lepton are conserved respectively.

Comparing with the electron mass, the mass term of the e-neutrino in Eq.(13) is approximately close to zero. Moreover, from Eq.(21), $\bar{\Psi}\gamma_5\Psi = 0$ for Spacelike neutrinos. It means that the mass term in Eq.(13) may be negligible in most cases. In fact, the momentum of a neutrino is much greater than $m_s c$ in most measurements. For instance, let $p_s = 10 m_s c = 16 eV/c$, Eq.(2) yields the speed of e-neutrino: $u_s = \frac{1}{1.005}c$. Therefore, spacelike neutrinos behave just like the massless neutrinos. Besides, we have the coefficient $A \simeq 1$ in Eq.(18). This similarity may also play role at the level of quantum field theory and SU(2) gauge theory.

According to special relativity [22], if there is a spacelike particle, it might travel backward in time. However, a re-interpretation rule has been introduced since the sixties [6-8]. Another approach is to introduce a kinematic time under a non-standard form of the Lorentz transformation [23-27]. Therefore, special relativity can be extended to the spacelike region, and tachyons are allowed without causality violation.

Generally speaking, the above spin-$\frac{1}{2}$ spacelike quantum theory provides a theoretical framework to study the hypothesis that neutrinos are tachyonic fermions. More measurements on the cosmic ray at the spectrum knee and more accurate tritium beta decay experiments are needed to further test the above theory.

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