Lower bounds on complexity of geometric 3-orbifolds

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Abstract

We establish a lower bound on the complexity orientable locally orientable geometric 3-orbifolds in terms of Delzant’s T-invariants of their orbifold-fundamental groups, generalizing previously known bounds for complexity of 3-manifolds.

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Introduction

The aim of this note is to provide lower bounds on the so-called complexity of 3-orbifolds in terms of relative T-invariant of their orbifold fundamental groups. More precisely, we consider orientable locally orientable geometric 3-orbifolds and provide two types of bounds: one in terms of the T-invariant relative to a finite set of subgroups (which however is not uniquely defined) and another in terms of the T-invariant relative to an elementary family of (finite) subgroups, which is uniquely defined by the orbifold under consideration. The bounds thus obtained generalize the bound on the complexity of 3-manifolds in terms of the T-invariants of their fundamental groups, that was already established in [4].

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1 Main definitions

In this section we briefly recall the notions of orbifold, its complexity, and the $T$-invariant of a group.

**Local structure of orbifolds** For the general theory of orbifolds we refer the reader to [10], to the recent [1], and to the extensive bibliography of the latter (the essential definitions, as well as an overview of results concerning geometrization of orbifolds, can also be found in the survey [3]). We just recall that an orbifold of dimension $n$ is a topological space with a singular smooth structure, locally modelled on a quotient of $\mathbb{R}^n$ under the action of a finite group of diffeomorphisms. We will confine ourselves to the case of compact orientable locally orientable 3-orbifolds. Such an orbifold $X$ is given by a compact support 3-manifold $|X|$ together with a singular set $S(X)$. Given the assumption of local orientability, $S(X)$ is a finite collection of circles and univalent graphs tamely embedded in $|X|$, where the univalent vertices are given by the intersection of $S(X)$ with $\partial|X|$ and each component of $S(X)$ minus the vertices is endowed with an order in $\{p \in \mathbb{N} : p \geq 2\}$, with the restriction that the three germs of edges incident to each vertex should have orders $(2, 2, p)$, for arbitrary $p$, or $(2, 3, p)$, for $p \in \{3, 4, 5\}$.

**Uniformizable $n$-orbifolds** An orbifold-covering is a map between orbifolds locally modelled on a map of the form $\mathbb{R}^n/\Delta \rightarrow \mathbb{R}^n/\Gamma$, naturally defined whenever $\Delta < \Gamma < \text{Diff}^+_{\text{orb}}(\mathbb{R}^n)$. If $M$ is a manifold and $\Gamma$ is a group acting properly discontinuously on $M$ then $M/\Gamma$ possesses a naturally defined structure of an orbifold such that the natural projection $M \rightarrow M/\Gamma$ is an orbifold-covering map. Any orbifold obtained as $M/\Gamma$ is said to be uniformizable. We note that in the case $n = 2$ the set of non-uniformizable 2-orbifolds (also called bad 2-orbifolds) admits the following easy description:

**Lemma 1.1.** The only non-uniformizable closed 2-orbifolds are $(S^2; p)$, the 2-sphere with one cone point of order $p$, and $(S^2; p, q)$, the 2-sphere with cone points of orders $p \neq q$.

**Spherical and discal orbifolds** We now introduce notation and terminology for several particular types of 2- and 3-orbifolds (the notation we employ follows [8]). We define $D^3_o$ to be $D^3$, the ordinary discal 3-orbifold, $D^3_o(p)$ to be $D^3$ with singular set a trivially embedded arc with arbitrary order $p$,
and $D^3(p, q, r)$ to be $D^3$ with singular set a trivially embedded triod with edges of admissible orders $p$, $q$, $r$. We call $D^3_c(p)$ and $D^3_v(p, q, r)$ respectively cyclic discal and vertex discal 3-orbifolds; we will occasionally suppress the indication of orders and write $D^3_c$ and $D^3_v$ to denote a generic cyclic or vertex discal 3-orbifold.

We also define the ordinary, cyclic, and vertex spherical 2-orbifolds, denoted respectively by $S^2_o$, $S^2_c(p)$, and $S^2_v(p, q, r)$, as the 2-orbifolds bounding the corresponding discal 3-orbifolds $D^3_o$, $D^3_c(p)$, and $D^3_v(p, q, r)$. Finally, we define the ordinary, cyclic, and vertex spherical 3-orbifolds, denoted respectively by $S^3_o$, $S^3_c(p)$, and $S^3_v(p, q, r)$, as the 3-orbifolds obtained by mirroring the corresponding discal 3-orbifolds $D^3_o$, $D^3_c(p)$, and $D^3_v(p, q, r)$ in their boundary. We will also use the notation $S^2_\ast$ and $S^3_\ast$ to indicate a spherical 2- or 3-orbifold with generic orders.

2-suborbifolds and irreducible 3-orbifolds We say that a 2-orbifold $\Sigma$ is a suborbifold of a 3-orbifold $X$ if $|\Sigma|$ is embedded in $|X|$ so that $|\Sigma|$ meets $S(X)$ transversely (in particular, it does not meet the vertices), and $S(\Sigma)$ is given precisely by $|\Sigma| \cap S(X)$, with matching orders.

A spherical 2-suborbifold $\Sigma$ of a 3-orbifold $X$ is called essential if it does not bound in $X$ a discal 3-orbifold. A 3-orbifold $X$ is called irreducible if it does not contain any non-uniformizable 2-suborbifold and every spherical 2-suborbifold of $X$ is inessential (in particular, it is separating).

Geometric 3-orbifolds An orbifold is said to be geometric if it admits a complete geometric structure in the sense of orbifolds. Namely, let $\tilde{X}$ be a (simply) connected manifold, and let $G$ be a group acting effectively and transitively on $X$ such that all point stabilizers are compact. An orbifold admits a geometric structure modelled on $(\tilde{X}, G)$ if its maximal orbifold atlas contains an atlas such that all the folding groups consist of appropriate restrictions of the elements of $G$ and all the charts are again restrictions of elements of $G$.

In particular, it can be shown that any geometric 3-orbifold $X$ is isomorphic to an orbifold of the form $\tilde{X}/\Gamma$, where $\tilde{X}$ is a simply connected manifold and $\Gamma$ acts properly discontinuously on $\tilde{X}$, see for instance [10]. The group $\Gamma$ (viewed up to isomorphism) is called the orbifold-fundamental group of $X$ (a more direct definition can be found, for instance, in [9]).

We note, although we won’t need this, that the analogue of geometrization
conjecture for orbifolds was announced by Thurston in 1982 and that the complete proof of the conjecture can be found in [2].

**Simple and special polyhedra** From now on we will employ the piecewise linear viewpoint, which is equivalent to the smooth one in dimensions 2 and 3. A *simple polyhedron* is a compact polyhedron $P$ such that the link of each point of $P$ can be embedded in the space given by a circle with three radii. In particular, $P$ has dimension at most 2. A point of a simple polyhedron is called a *vertex* if its link is given precisely by a circle with three radii; a regular neighbourhood of a vertex is shown in Fig. 1 on the far right. It follows from the figure that the vertices are isolated, hence finite in number (the case when there are no vertices is also allowed). The complexity $c(P)$ of $P$ is the number of vertices that $P$ contains.

Two more restrictive types of polyhedra will be used below. A simple polyhedron $P$ is called *almost-special* if the link of each point of $P$ is given by a circle with zero or two or three radii. The three possible types of neighbourhoods of points of $P$ are shown in Fig. 1. The points of the second and the third type are called *singular*, and the set of all singular points of $P$ is denoted by $S(P)$. We say that $P$ is special if it is almost-special, $S(P)$ contains no circle component, and $P \setminus S(P)$ consists of open 2-discs.

**Spines of 3-orbifolds and complexity** The notions we define in this paragraph were introduced in [8] and generalize the corresponding notions for 3-manifolds. Let $X$ be a closed orientable locally orientable 3-orbifold, and let $P$ be a simple polyhedron contained in $|X|$. We say that $P$ is a *spine* of $X$ if the following conditions are satisfied:

- The intersections between $P$ and $S(X)$ occur only in surface points of $P$ and non-vertex points of $S(X)$, and they are transverse;
If $U(P)$ is an open regular neighbourhood of $P$ in $|X|$ then each component of $X \setminus U(P)$ is isomorphic to one of the discal 3-orbifolds $\mathbb{D}^3_\ast$. We will normally identify $X \setminus U(P)$ with $X \setminus P$, making a distinction only when it matters; it is easy to check that this choice does not cause any ambiguity.

If $P$ is a spine of $X$ as above, we define the following function (that depends not only on $P$ as an abstract polyhedron but also on its position in $|X|$ relative to $S(X)$):

$$c(P, S(X)) = c(P) + \sum\{p - 1 : x \in P \cap S(X), \text{ the order of } S(X) \text{ at } x \text{ is } p\}.$$

Then the complexity $c(X)$ of the 3-orbifold $X$ is defined as the minimum of $c(P, S(X))$ taken over all spines $P$ of $X$. We note that $c$ is always well-defined because every orbifold has simple spines [8, Section 2].

We say that a spine $P$ of $X$ is minimal if $c(P, S(X)) = c(X)$ and every proper subpolyhedron of $P$ which is also a spine of $X$ is actually homeomorphic to $P$. We cite the following property of such spines (actually established in [8] under some slightly relaxed assumptions), that will be important for us:

**Lemma 1.2.** [8, Lemma 2.1] Let $P$ be a minimal spine of an orbifold $X$, and let $\alpha$ be a connected component of $S(X)$ minus the vertices. Then $\alpha \cap P$ consists of precisely one point.

**Orbifolds with minimal special spines** In addition to the orbifold $S^3_v$ described above, we also need to consider certain orbifolds $(\mathbb{P}^3, F_p)$ and $(L_{3,1}, F_p)$. In both cases $F_p$ is a circle of order $p$, given by a non-singular fibre of the natural Seifert fibration; since we will have to include the case of manifolds $S^3_v, \mathbb{P}^3, L_{3,1}$, we stipulate that if $K$ is a knot then $K_p$ denotes $K$ equipped with the cone order $p$ if $p \geq 2$, and it denotes the empty set if $p = 1$. With this stipulation we can write $S^3_0 = S^3_v(1)$.

In the above notation, it was established in [8, Proposition 2.4] that the complexity of any of $S^3_v(p)$, $(\mathbb{P}^3, F_p)$, $(L_{3,1}, F_p)$ is equal to $p - 1$; moreover, any minimal spine of any of these orbifolds is not special.

On the other hand, we have the following result, also established in [8].

**Theorem 1.3.** [8, Theorem 2.6] If $X$ is an irreducible 3-orbifold and not $S^3_v(p)$, $(\mathbb{P}^3, F_p)$, $(L_{3,1}, F_p)$, or $S^3_v(p, q, r)$, then any minimal spine of $X$ is special.
Duality  We define a triangulation of a 3-orbifold $X$ to be a triangulation of the manifold $|X|$ such that $S(X)$ is a subset of the 1-skeleton. Note that by a triangulation of a 3-manifold $M$ we mean a realization of $M$ as a gluing of a finite number of tetrahedra along a complete system of simplicial pairings of the lateral faces and that, in particular, we allow multiple and self-adjacencies of the tetrahedra. Recall now that there is a well-known duality between the set of such triangulations of $M$ and the set of special polyhedra $P$ embedded in $M$ in such a way that $M \setminus P$ is a union of open 3-discs; indeed, any $P$ as above induces a natural cellularization of $M$ the dual of which is a triangulation, and vice versa. This duality extends, with reservations, also to the case of 3-orbifolds.

**Proposition 1.4.** [8, Proposition 2.8] Let $X$ be a 3-orbifold as in Theorem 1.3. Then dual to any minimal spine of $X$ there is a triangulation (in the sense of orbifolds).

**T-invariant relative a set of subgroups** Finally, we need the following definition introduced in [4].

**Definition 1.5.** Let $G$ be a group, and let $(C_1, \ldots, C_n)$ be a family of its subgroups. Then the T-invariant of the pair $(G; C_1, \ldots, C_n)$ is defined by the following condition: $T(G; C_1, \ldots, C_n) \leq k$ if and only if there exists a simply connected simplicial polyhedron $\Pi$ of dimension 2 such that $G$ acts on $\Pi$, this action is simplicial and possesses the following properties:

- the number of faces of $\Pi$ modulo the action of $G$ is no more than $k$;
- the stabilizers of vertices of $\Pi$ are conjugate to $C_1, \ldots, C_n$ and each of these groups fixes some vertex of $\Pi$.

## 2 Lower bounds

In this section we establish two types of lower bounds on the complexity of 3-orbifolds that are mentioned in the introduction.

**Bounds with respect to a finite family** Consider a geometric orbifold $X$. Denote by $\tilde{X}$ the universal orbifold covering of $X$ and by $\pi$ the covering of $X$ by $\tilde{X}$ induced by the action of $\pi_1^{\text{orb}}(X)$ on $\tilde{X}$. To proceed, we recall the following easy facts:
• Let \( x \in \tilde{X} \) be a point such that \( \pi(x) \in S(X) \), the singular set of \( X \). Then the stabilizer \( St(x) \) of \( x \) in \( \pi_1^{\text{orb}}(X) \) is a finite subgroup.

• For every two points \( x, y \in \tilde{X} \) such that \( \pi(x), \pi(y) \) belong to the same connected component of \( S(X) \) minus the vertices, the orders of subgroups \( St(x) \) and \( St(y) \) are equal to each other and to \( p \), the cyclic order associated to that component.

Denote by \( c_X \) the number of circular components in \( S(X) \) minus vertices and by \( v_X \) the number of vertices in \( S(X) \).

**Theorem 2.1.** Let \( X \) be a closed orientable locally orientable geometric 3-orbifold with non-empty singular set such that \( X \) has a minimal special spine. Then there is a finite family of subgroups \( C_1, \ldots, C_{c_X+v_X} \leq \pi_1^{\text{orb}}(X) \) such that

\[
c(X) \geq \frac{1}{2} T(\pi_1^{\text{orb}}(X); C_1, \ldots, C_{c_X+v_X}) + \sum_{\alpha \subseteq S(X)} (p - 1),
\]

where the latter sum is taken over all connected components \( \alpha \) of \( S(X) \) minus the vertices, and \( p \) is the order associated to \( \alpha \).

**Remark 2.2.** As follows from Theorem 1.3, most geometric 3-orbifolds admit minimal special spines.

**Proof.** Let \( P \) be a minimal special spine of \( X \). It follows from Lemma 1.2 that

\[
c(X) = \#V(P) + \sum_{\alpha \subseteq S(X)} (p - 1),
\]

where \( \alpha, p \) are as in the statement of the theorem. So we actually need to estimate \( \#V(P) \).

Let \( \tau \) be any triangulation of \( X \) defined in Proposition 1.4 (i.e., dual to \( P \) and such that \( S(X) \) is contained in the 1-skeleton of \( \tau \)). This triangulation lifts to a triangulation \( \tilde{\tau} \) of \( \tilde{X} \). Denote by \( Q \) the 2-skeleton of \( \tau \) and by \( \tilde{Q} \) the 2-skeleton of \( \tilde{\tau} \). Then obviously \( \pi_1^{\text{orb}}(X) \) acts on \( \tilde{Q} \), and \( \tilde{Q}/\pi_1^{\text{orb}}(X) = Q \).

Since \( S(X) \) is contained in the 1-skeleton of \( \tau \) and therefore of \( Q \), its pre-image \( \pi^{-1}(S(X)) \) is contained in the 1-skeleton of \( \tilde{Q} \). Hence the action of \( \pi_1^{\text{orb}}(X) \) is simplicial, and all points with nontrivial stabilizers are contained in the 1-skeleton of \( \tilde{Q} \).

Notice that among the stabilizers of vertices of \( \tilde{Q} \) there is only a finite number of non-conjugate subgroups. Indeed, if \( \pi(u) = \pi(v) \) then \( St(u) \) is
conjugate to $St(v)$, and $Q$ contains only a finite number of vertices. We
claim that, if $S(X)$ is non-empty then the number $\#V(Q)$ of vertices of $Q$ is $c_X + v_X$.

Since triangulation $\tau$ is dual to $P$, this number is exactly the number of
connected components of $X \setminus P$. Obviously, this number is at least $c_X + v_X$
because each vertex of $S(X)$ must lie in some ball of $X \setminus P$, and that ball
cannot contain any other vertices of $S(X)$ or have non-empty intersection
with any circular components. Also, every circular component must intersect
some ball, and it will be the only connected component of $S(X)$ intersecting
that ball.

On the other hand, suppose that $S(X)$ is non-empty and the complement
$X \setminus P$ contains an ordinary discal 3-orbifold $B$. Then there is a 2-cell $c$ of $P$
separating $B$ from another discal 3-orbifold $B'$. Obviously, $c$ does not contain
any points of $S(X)$. So we can puncture $\tau$ into discal 3-orbifold $P'$ of $\tau$. Notice that $c$ must contain
some vertices, so $\#V(P') < \#V(P)$ and hence $c(P', S(X)) < c(P, S(X))$. This
contradicts the minimality of $P$. Hence every component of $X \setminus P$ has
non-empty singular set.

Finally, by Lemma 1.2 each circular component $\alpha$ of $S(X)$ is contained
in the closure of precisely one component of $X \setminus P$, which approaches the
cell intersected by $\alpha$ from both sides. By the same lemma each non-closed
edge of $S(X)$ intersects exactly two (with multiplicity) components of $X \setminus P$, namely the ones containing its ends. This and the absence of ordinary discal
3-orbifolds implies that any component of $X \setminus P$ contains either a vertex of
$S(X)$ or a portion of a circular component. Hence the number of components
of $X \setminus P$ is indeed $c_X + v_X$.

In particular, the set of vertices of $\tilde{Q}$ is decomposed into $c_X + v_X$ classes
of vertices such that two vertices are in the same class if and only if they
project to the same vertex of $Q$. The stabilizers of vertices in the same
class are conjugate, and we choose exactly one of them. Denote the chosen
subgroups by $C_1, \ldots, C_{c_X+v_X}$. Notice that $\tilde{Q}$ and subgroups $C_1, \ldots, C_{c_X+v_X}$
satisfy the second condition in the definition of the T-invariant.

Since $\tilde{X} \setminus \tilde{Q}$ consists of open balls, $\tilde{Q}$ is simply-connected. Therefore we
have that

$$\#2\text{-faces of } Q \geq T(\pi_1^{\text{orb}}(X); C_1, \ldots, C_{c_X+v_X}).$$

Since $\tau$ is dual to $P$, we have that the number of 2-faces of $P$, and
therefore of $Q$, is equal to the number of edges of $P$. Since $P$ is special, its
singular graph has valence 4. Therefore the number of edges of $P$ is twice the number of vertices of $P$. Summarizing, we have:

$$
\#V(P) \geq \frac{1}{2} T(\pi_1^{\text{orb}}(X); C_1, \ldots, C_{cX+vX}).
$$

This proves the theorem.

**Bounds with respect to an elementary family of subgroups** The subgroups $C_1, \ldots, C_{cX+vX}$ that appear in the statement of Theorem 2.1 are not well-defined by $X$ only; for instance, those among them that are stabilizers of points projecting to vertices of $S(X)$ are well-defined up to conjugacy only. However, the set of the orders of all these subgroups are uniquely defined by the orbifold $X$. Thus, we can obtain a lower bound on the orbifold complexity in terms depending only on $X$, by employing the following definition introduced in [6].

**Definition 2.3.** Let $G$ be a fixed group. A family $\mathcal{C}$ of subgroups of $G$ is called elementary if the following four conditions hold:

(a) if $C \in \mathcal{C}$ then any subgroup of $C$ and any subgroup conjugated to $C$ is in $\mathcal{C}$;

(b) any infinite subgroup in $\mathcal{C}$ is included into a unique maximal subgroup in $\mathcal{C}$, and any increasing union of finite subgroups in $\mathcal{C}$ is in $\mathcal{C}$;

(c) if $C \in \mathcal{C}$ acts on a tree then it fixes a point of that tree or a point of its boundary, or it preserves a pair of points of the boundary possibly permuting them;

(d) if $C \in \mathcal{C}$ is infinite, maximal in $\mathcal{C}$, and $gCg^{-1} = C$ then $g \in \mathcal{C}$.

In particular, the family of all finite subgroups of order not greater than $p$, where $p$ is a fixed number, is an elementary family. Denote it by $\mathcal{C}_p(G)$. We have the following definition, from [6] as well.

**Definition 2.4.** The $T$-invariant $T_p(G)$ relative to $\mathcal{C}_p(G)$ of a group $G$ is defined as $T(G; C_1, \ldots, C_n)$, where the infimum is taken over all finite sets of subgroups from $\mathcal{C}_p(G)$.

**Remark 2.5.** The definition given in [6] actually differs from the one given above, in the following respects:
1) In the former, it is required that $G$ act on $\Pi$ from Definition 1.5 without edge inversions, and that
2) the stabilizers of all edges and faces also be from $C_p$.

However, we observe that in our situation (i.e. that of orbifold-fundamental groups of orientable locally orientable 3-orbifolds) these requirements are automatically satisfied. Indeed, the stabilizer of each edge $\alpha$ is contained in the intersection of stabilizers of its interior points. All those points project to the same connected component of $S(X) \setminus V(S(X))$, the set of vertices. Hence the orders of their stabilizers are all equal to $p$, the local order associated to that connected component. Hence the order of the stabilizer of $\alpha$ is no more than $p$, so it belongs to the family $C_{p(X)}$. Also, since $\pi^{-1}(S(X))$ is contained in the 1-skeleton of $\tilde{Q}$, all faces have trivial stabilizers. Finally, since $X$ is locally orientable, there are no inversions of edges.

Denote now by $p(X)$, where $X$ is an orbifold, the maximum of its local orders. We now have the following result.

**Corollary 2.6.** Let $X$ be an orbifold as in Theorem 2.1. Then

$$c(X) \geq \frac{1}{2} T_{p(X)}(\pi_{1}^{\text{orb}}(X)) + \sum_{\alpha \subset S(X)} (p - 1).$$

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