APPLIED ENUMERALS OF CUTOFF RESOLVENT ESTIMATES TO THE WAVE EQUATION

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Abstract. We consider solutions to the linear wave equation on non-compact Riemannian manifolds without boundary when the geodesic flow admits a filamentary hyperbolic trapped set. We obtain a polynomial rate of local energy decay with exponent depending only on the dimension.

1. Introduction

In this paper we consider solutions to the linear wave equation on the non-compact Riemannian manifolds with trapping studied by Nonnenmacher-Zworski [NoZw]. Let \((X,g)\) be a Riemannian manifold of odd dimension \(n \geq 3\) without boundary, with (non-negative) Laplace-Beltrami operator \(-\Delta\) acting on functions. The Laplace-Beltrami operator is an unbounded, essentially self-adjoint operator on \(L^2(X)\) with domain \(H^2(X)\).

In order to quote the results of [NoZw] we also need the following analyticity assumption: \(\exists \theta_0 \in [0,\pi]\) such that the \(a_\alpha(x,h)\) are extend holomorphically to

\[
\left\{ r\omega : \omega \in \mathbb{C}^n, \text{ dist } (\omega, S^n) < \epsilon, \ r \in \mathbb{C}, \ |r| \geq R_0, \ \arg r \in [-\epsilon, \theta_0 + \epsilon]\right\}.
\]

As in [NoZw], the analyticity assumption immediately implies

\[
\partial_x^\beta \left( \sum_{|\alpha| \leq 2} a_\alpha(x,h) \xi^\alpha - |\xi|^2 \right) = o(|x|^{-|\beta|}) \langle \xi \rangle^2, \ |x| \to \infty.
\]

We assume also that the classical resolvent \((-\Delta - \lambda^2)^{-1}\) has a holomorphic continuation to a neighbourhood of \(\lambda \in \mathbb{R}\) as a bounded operator \(L^2_{\text{comp}} \to L^2_{\text{loc}}\).
We consider solutions $u$ to the following wave equation on $X \times \mathbb{R}_t$. 

\[
\begin{cases}
(-D_t^2 - \Delta) u(x, t) = 0, & (x, t) \in X \times [0, \infty) \\
u(x, 0) = u_0 \in H^1(X) \cap C^\infty_c(X), \\
D_t u(x, 0) = u_1 \in L^2(X) \cap C^\infty_c(X),
\end{cases}
\tag{1.1}
\]

For $u$ satisfying (1.1) and $\chi \in C^\infty_c(X)$, we define the local energy, $E_\chi(t)$, to be

\[
E_\chi(t) = \frac{1}{2} \left( \| \chi \partial_t u \|^2_{L^2(X)} + \| \chi u \|^2_{H^1(X)} \right).
\]

Local energy for solutions to the wave equation has been well studied in various settings. Morawetz [Mor], Morawetz-Phillips [MoPh], and Morawetz-Ralston-Strauss [MRS] study the wave equation in non-trapping exterior domains in $\mathbb{R}^n$, showing the local energy decays exponentially in odd dimensions $n \geq 3$, and polynomially in even dimensions. This has been generalized to cases with non-trapping potentials [Val] and compact non-trapping perturbations of Euclidean space [Vod].

In the case of elliptic trapped rays, it is known that (see [Ral]) exponential decay of the local energy is generally not possible. Ikawa [Ika1, Ika2] shows in dimension 3 there is exponential local energy decay with a loss in derivatives in the presence of trapped rays between convex obstacles, provided the obstacles are sufficiently small and far apart. In the case $X$ is Euclidean outside a compact set, $\partial X \neq \emptyset$, and with no assumptions on trapping, Burq shows in [Bur1] that $E_\chi(t)$ decays at least logarithmically with some loss in derivatives. The author shows in [Chr3] that if there is one hyperbolic trapped orbit with no other trapping, then the local energy decays exponentially with a loss in derivative (including the case $\partial X = \emptyset$).

The main result of this paper is that if there is a hyperbolic trapped set which is sufficiently “thin”, then the local energy decays at least polynomially, with an exponent depending on the dimension $n$.

**Theorem 1.** Suppose $(X, g)$ satisfies the assumptions of the introduction, $\dim X = n \geq 3$ is odd, and $(X, g)$ admits a compact hyperbolic fractal trapped set, $K_E$, in the energy level $E > 0$ with topological pressure $P_E(1/2) < 0$. Assume there is no other trapping and $(-\Delta - \lambda^2)^{-1}$ admits a holomorphic continuation to a strip around $\mathbb{R} \subset \mathbb{C}$. Then for each $\epsilon > 0$ and $s > 0$, there is a constant $C > 0$, depending on $\epsilon$, $s$, and the support of $u_0$ and $u_1$, such that

\[
E_\chi(t) \leq C \left( \frac{\log(2 + t)}{t} \right)^{\frac{s}{2}} \left( \| u_0 \|^2_{H^{1+s}(X)} + \| u_1 \|^2_{H^s(X)} \right).
\tag{1.2}
\]

**Remark 1.1.** It is expected that Theorem 1 is not optimal, and in fact an exponential or sub-exponential estimate holds. Similar to in [Chr3], we expect applications to the nonlinear wave equation, although there are certain technical difficulties to overcome.

The proof of Theorem 1 is a consequence of an adaptation of [Bur1] Théorème 1 to this setting and the following resolvent estimates.

**Theorem 2.** Suppose $(X, g)$ satisfies the assumptions of Theorem 1. Then for any $\chi \in C^\infty_c(X)$ and any $\epsilon > 0$ there is a constant $C = C_{\chi, \epsilon} > 0$ such that

\[
\| \chi (-\Delta - \lambda^2)^{-1} \chi \|_{L^2(X) \to L^2(X)} \leq C \frac{\log(1 + \langle \lambda \rangle)}{\langle \lambda \rangle},
\]

where $\langle \lambda \rangle$ denotes the topological pressure of $K_E$.

for
\[ \lambda \in \left\{ \lambda : \left|\text{Im} \lambda \right| \leq \begin{cases} C, & |\text{Re} \lambda|^{\frac{3n}{2} - \epsilon}, \\ C', & |\text{Re} \lambda| \geq C \end{cases} \right\}. \]

**Remark 1.2.** The proof of Theorem 1 depends more on the neighbourhood in which the resolvent estimates hold than on the estimates themselves. Given a complex neighbourhood of the real axis, any polynomial cutoff resolvent estimate will give the same local energy decay rate. Theorem 2 represents a gain over the estimates in [NoZw, Theorem 5] in the sense that the estimate holds in a complex neighbourhood of \( \mathbb{R} \), rather than just on \( \mathbb{R} \).

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## 2. Proof of Theorem 2

To prove Theorem 2, we use the results of Nonnenmacher-Zworski [NoZw] to prove a high energy estimate for the resolvent with complex absorbing potential, then use the holomorphic continuation to bound the cutoff resolvent by a constant for low energies. If we consider the problem

\[
(-\Delta - \lambda^2)u = f,
\]

and restrict our attention to values \( |\lambda| \geq C \) for some constant \( C > 0 \), we can transform equation (2.1) into a semiclassical problem for fixed energy by setting

\[ \lambda = \sqrt{z}/h \]

for \( z \sim 1 \) and \( 0 < h \leq h_0 \). Then (2.1) becomes

\[ (P - z)u = h^2 f, \]

where

\[ P = -h^2 \Delta \]

is the semiclassical Laplacian.

The following Proposition is the high energy resolvent estimate from [NoZw] with the improvement that the estimate holds in a larger neighbourhood of \( \mathbb{R} \subset \mathbb{C} \).

**Proposition 2.1.** Suppose \( W \in C^\infty(X; [0, 1]), W \geq 0 \) satisfies

\[ \text{supp} W \subset X \setminus B(0, R_1), \quad W \equiv 1 \text{ on } X \setminus B(0, R_2), \]

for \( R_2 > R_1 \) sufficiently large, and

\[ \| (P - iW - z)^{-1} \|_{L^2 \to L^2} \leq C_N \left( 1 + \log(1/h) + \frac{h^N}{\text{Im} z} \right), \]

for \( z \in [E - \delta, E + \delta] + i(-ch, ch) \). Then for each \( \epsilon > 0 \) and each \( \chi \in C_0^\infty(X) \), there is a constant \( C = C_{\epsilon, \chi} > 0 \) such that

\[ \| \chi (P - z)^{-1} \chi \|_{L^2 \to L^2} \leq C \frac{\log(1/h)}{h}, \]

for \( z \in [E - c_1 h, E + c_1 h] + i(-c_2 h^{3n/2 + 1 + \epsilon}, c_2 h^{3n/2 + 1 + \epsilon}) \).
We first improve Lemma 9.2 in order to get cutoff resolvent estimates with the absorbing potential in a polynomial neighbourhood of the real axis. The proof of the following lemma is an adaptation of the “three-lines” theorem from complex analysis and borrows techniques from Chr1, BuZw, NoZw and the references cited therein.

**Lemma 2.2.** Suppose $F(z)$ is holomorphic on
\[ \Omega = [-1, 1] + i(-c_-, c_+), \]
and satisfies
\[
\log |F(z)| \leq M, \quad z \in \Omega,
\]
\[
|F(z)| \leq \alpha + \frac{\gamma}{\Im z}, \quad z \in \Omega \cap \{\Im z > 0\}.
\]
Then if $\gamma \leq \epsilon M^{-3/2}$ for $\epsilon > 0$ sufficiently small, there exists a constant $C = C_\epsilon > 0$ such that
\[
|F(z)| \leq C\alpha, \quad z \in [-1/2, 1/2] + i(-M^{-3/2}, M^{-3/2}).
\]

**Proof.** Choose $\psi(x) \in C_c^\infty([-1, 1]), \psi \equiv 1$ on $[-1/2, 1/2]$, and set
\[
\varphi(z) = \beta^{-1/2} \int e^{-(x-\beta y)/\beta} \psi(x) dx,
\]
where $0 < \beta < 1$ and $c > 0$ will be chosen later. The function $\varphi(z)$ enjoys the following properties:

(a) $\varphi(z)$ is holomorphic in $\Omega$,
(b) $|\varphi(z)| \leq C$ on $\Omega \cap \{\Im z \leq \beta^{1/2}\}$,
(c) $|\varphi(z)| \geq C^{-1}$ on $\{\Re z \leq 1/2\} \cap \{\Im z \leq \beta\}$ if $c > 0$ is chosen appropriately,
(d) $|\varphi(z)| \leq Ce^{-C/\beta}$ for $z \in \{\pm 1\} + i(-\beta^{1/2}, \beta^{1/2})$.

Now for $a \in \mathbb{R}$ to be determined, set
\[
g(z) = e^{iaz} \varphi(z) F(z).
\]
For $\delta_- > 0$ to be determined, let
\[
\Omega' := \Omega \cap \{-\delta_- \leq \Im z \leq \delta_+\}.
\]
We have the following bounds for $g(z)$ on the boundary of $\Omega'$:
\[
\log |g(z)| \leq \begin{cases} 
-C/\beta + M - a \Im z, & \text{Re } z = \pm 1, \text{ if } |\Im z| \leq \beta^{1/2}; \\
C + M + a\delta_-, & \text{Im } z = -\delta_- \geq -\beta^{1/2}; \\
C + \log(\alpha + \gamma/\delta_+) - a\delta_+, & \text{Im } z = \delta_+ \leq \beta^{1/2}.
\end{cases}
\]
We want to choose $a, \beta$, and $\delta_\pm$ to optimize these inequalities. Choosing $a = -2M/\delta_-$ yields
\[
\log |g(z)| \leq C - M \quad \text{for } \Im z = -\delta_-,
\]
and choosing $\delta_+ = |2/a|$ yields
\[
\log |g(z)| \leq C + \log(\alpha + \gamma/\delta_+) + 2, \quad \text{for } \Im z = \delta_+.
\]
Finally, choosing $\beta = C'/M$ for an appropriate $C' > 0$ yields
\[
\log |g(z)| \leq -C^{-1} M \quad \text{for } \Re z = \pm 1, \quad \text{Im } z \leq \max\{\delta_+, \delta_-\},
\]
and taking $\delta_- = C''M^{-1/2}, \delta_+ = C''M^{-3/2}$ gives
\[
\log |g(z)| \leq C'' + \log(\alpha + \gamma/\delta_+) \quad \text{on } \partial \Omega'.
\]
In order to conclude the stated inequality on $F(z)$, we need to invert $e^{-iaz}\varphi(z)$, which, from the definition of $a$ and the properties of $\varphi$ stated above, is possible for $z \in [-1/2, 1/2] + i(-M^{-3/2}, M^{-3/2})$. Then for $z$ in this range and $\gamma$ satisfying $\gamma \leq \epsilon M^{-3/2}$, 

$$|F(z)| \leq C\alpha(1 + \epsilon) \leq C'\alpha,$$

as claimed. □

Now to prove Proposition 2.1, as in [NoZw], we apply Lemma 2.2 to $F(\zeta) = \langle (P - iW - h\zeta)^{-1}f, g \rangle_{L^2}$, for $f, g \in L^2$. For $M$ we use the well-known estimate 

$$\|((P - iW - z)^{-1}\|_{L^2 \rightarrow L^2} \leq C e^{Ch^{-n-\epsilon}}, \quad \text{Im} z \geq -h/C,$$

and take $M = C h^{-n-\epsilon}$. For the other parameters, we take $\gamma = h^N, \quad \alpha = c_0 + \log(1/h)$.

Rescaling, we conclude 

$$\|((P - iW - z)^{-1}\| \leq C \frac{\log(1/h)}{h}$$

in the stated region. Then we apply the remainder of the proof [NoZw, Theorem 5]. □

3. Proof of Theorem 1

In this section we adapt the proof of [Bur1, Théorème 1] to the case where one has better resolvent estimates. We first present a general theorem on semigroups (see [Bur1, Théorème 3] and [Leb]).

Let $H$ be a Hilbert space, $B(\xi)$ a meromorphic family of unbounded linear operators on $H$, holomorphic for $\text{Im} \xi < 0$. Assume for $\text{Im} \xi \leq 0$, 

$$\text{Im} (B(\xi)u, u)_H \geq 0.$$

Let $\text{Dom}(B) = \text{Dom}(1 - iB(-i))$ denote the domain of $B$. Assume for $\text{Im} \xi < 0$, $\xi - B(\xi)$ is bijective and bounded with respect to the natural norm on $\text{Dom}(B)$, 

$$\|u\|^2_{\text{Dom}(B)} = \|u\|^2_H + \|B(-i)u\|^2_H,$$

and 

$$\|((\xi - B(\xi))^{-1}\|_{H \rightarrow H} \leq C |\text{Im} \xi|^{-1}.$$ 

Assume that $B(\xi) \in \mathcal{S}^1(\mathbb{R}; \mathcal{L}(\text{Dom}(B), H))$. That is, $B(\xi)$ is a symbol with respect to $\xi$ and assume that, as operators on $\text{Dom}(B)$, 

$$B(D_s) e^{i\xi s} = e^{i\xi s} B(\xi + D_s) = e^{i\xi s} B(\xi),$$

since members of $\text{Dom}(B)$ do not depend on $s$. We assume $B$ satisfies the identity 

$$B(D_t) \psi(t)U(t) = \psi(t)B(\psi'/i\psi + D_t)U(t) = \psi(t)(B(D_t) + A(t))U(t),$$

for $\psi(t) \in C^\infty(\mathbb{R})$, and $U \in C^\infty(\mathbb{R}; \text{Dom}(B))$. Here, $A(t)$ is a linear operator, bounded on $H$ and has compact support contained in $\text{supp} \psi'$. 
By the Hille-Yosida Theorem, for every \( k \in \mathbb{N} \) and \( s \geq 0 \), we can construct the operators
\[
e^{isB(D_x)}(1 - iB(-i))^k,
\]
where \( e^{isB(D_x)} \) satisfies the evolution equation
\[
\begin{cases}
(D_x - B(D_x))e^{isB(D_x)} = 0, \\
e^{isB(D_x)}|_{s=0} = \text{id}.
\end{cases}
\]

Now suppose \( \chi_j, j = 1, 2 \) are bounded operators \( H \to H \), and \( \chi_1(\xi - B(\xi))^{-1}\chi_2 \) continues holomorphically to the region
\[
\Omega = \left\{ \xi \in \mathbb{C} : |\text{Im}\xi| \leq \begin{cases} C, & |\text{Re}\xi| \leq C \\ P(|\text{Re}\xi|), & |\text{Re}\xi| \geq C \end{cases} \right\},
\]
where \( P(|\text{Re}\xi|) > 0 \) and is monotone decreasing (or constant) as \( |\text{Re}\xi| \to \infty \). Assume
\[
\|\chi_1(\xi - B(\xi))^{-1}\chi_2\|_{H \to H} \leq G(|\text{Re}\xi|)
\]
for \( \xi \in \Omega \), where \( G(|\text{Re}\xi|) = \mathcal{O}(|\text{Re}\xi|^N) \) for some \( N \geq 0 \). We further assume that the propagator \( e^{isB(D_x)} \) “acts finitely locally,” in the sense that for \( s \in [0, 1] \),
\[
\tilde{\chi}_2 := e^{isB(D_x)}\chi_2
\]
is also a bounded operator on \( H \), and \( \chi_1(\xi - B(\xi))^{-1}\tilde{\chi}_2 \) continues holomorphically to \( \Omega \) and satisfies the estimate (3.1) with \( G \) replaced by \( CG \) for a constant \( C > 0 \).

**Theorem 3.** Suppose \( B(\xi) \) satisfies all the assumptions above, and let \( k \in \mathbb{N} \), \( k > N + 1 \). Then for any \( F(t) \to 0, \) monotone increasing, satisfying
\[
F(t)^{k+1} \leq \exp(tP(F(t))),
\]
we have
\[
\|\chi_1 e^{itB(D_x)}(1 - iB(-i))^k\tilde{\chi}_2\|_{H \to H} \leq CF(t)^{-k}. \tag{3.3}
\]

As in [Bur1], Theorem \( \Delta \) follows from Theorem 3 by setting
\[
B = \begin{pmatrix} 0 & -i\Delta \\ -i\text{id} & 0 \end{pmatrix},
\]
the Hilbert space \( H = H^1(X) \times L^2(X) \), and \( \chi_j \in C_c^\infty(X) \) for \( j = 1, 2 \). The commutator \([\chi_2, B]\) is compactly supported and bounded on \( H \), so if \( \tilde{\chi}_2 \in C_c^\infty(X) \) is supported on a slightly larger set than \( \chi_2 \), we have
\[
\|\chi_1 e^{itB}\chi_2\|_{\text{Dom}(B^k) \to H} = \|\chi_1 e^{itB}\chi_2(1 - iB)^{-k}\|_{H \to H} \leq C\|\chi_1 e^{itB}(1 - iB)^{-k}\tilde{\chi}_2\|_{H \to H}.
\]

Taking \( k = 2, \ P(t) = t^{-3n/2-\epsilon/2} \), and
\[
F(t) = \left( \frac{t}{\log t} \right)^{2/(3n+\epsilon)},
\]
yields (1.2) for \( s \geq k \). We observe the spaces \( H^{1+s} \times H^s \) are complex interpolation spaces, hence interpolating with the trivial estimate
\[
E_\chi(t) \leq \|u_0\|^2_{H^1} + \|u_1\|^2_{L^2},
\]
yields (1.2) for $s \geq 0$.

\[ \square \]

**Remark 3.1.** Evidently, if we have polynomial resolvent bounds in a fixed strip around the real axis, we have exponential local energy decay for the wave equation with a loss in derivatives. Further, if $H = L^2(X)$ for $X$ a compact manifold, this theorem may be applied to the damped wave operator with $\chi_1 = \chi_2 = 1$ to conclude there is exponential energy decay with loss in derivatives for solutions to the damped wave equation if there is a polynomial bound on the inverse of the damped wave operator in a strip. This corrects a mistake in the proof of [Chr1, Theorem 5].

We first need a lemma.

**Lemma 3.2.** For $k > N + 1$, the propagator satisfies the following identity on $H$:

\[
e^{itB(D_t)} \frac{(1 - iB(-i))^k}{(1 - iB(-i))^k} = \frac{1}{2\pi i} \int_{Im \xi = -\frac{1}{2}} e^{it\xi}(1 - i\xi)^{-k}(\xi - B(\xi))^{-1}d\xi.\]

**Proof.** We write $I_k$ for the right hand side and observe both the left hand side and $I_k$ satisfy the evolution equation

\[(D_t - B(D_t))w = 0.\]

To calculate $I_k(0)$, we deform the contour to see

\[I_k(0) = \frac{1}{2\pi i} \left( \int_{Im \xi = -C} - \int_{\partial B(-i, \epsilon)} \right)(1 - i\xi)^{-k}(\xi - B(\xi))^{-1}d\xi.\]

Letting $C \to \infty$, the first integral vanishes. Thus we need to calculate the second integral. For $k = 1$, this is the residue formula, while for $k > 1$ the formula follows by induction and the continuity of $B(\xi)$ as $\epsilon \to 0$.

Thus the left hand side and $I_k$ have the same initial conditions, and the lemma is proved. \[ \square \]

**Proof of Theorem 3.** Now, as in [Bur1], we introduce a cutoff in time to make the equation inhomogeneous, then analyze the integral separately for low and high frequencies in $\xi$. In order to maintain smoothness, we convolve with a Gaussian. For an initial condition $u_0 \in H$, let $V(t) = e^{itB(D_t)}\chi_2 u_0$, and consider $U(t) = \psi(t)(1 - iB(-i))^{-k}V(t)$ for $\psi(t) \in C^\infty(\mathbb{R})$ satisfying $\psi \equiv 0$ for $t \leq 1/3$, $\psi \equiv 1$ for $t \geq 2/3$, and $\psi' \geq 0$. We observe by the sub-unitarity of $e^{itB(D_t)}$ for $t \geq 0$,

\[ ||U(t)|| \leq C||V(t)|| \leq C' ||u_0||,\]

where for the remainder of the proof, $|| \cdot || = || \cdot ||_H$ unless otherwise specified.

The family $U(t)$ satisfies

\[(D_t - B(D_t))U = \tilde{A}(t)(1 - iB(-i))^{-k}V(t),\]

where $\tilde{A}$ is a bounded operator on $H$ with support contained in $[1/3, 2/3]$. As $U(0) = 0$, Duhamel’s formula yields

\[ U(t) = \int_0^t e^{i(t-s)B(D_t)}\tilde{A}(s)(1 - iB(-i))^{-k}V(s)ds,\]
and by Lemma 3.2
\[ U(t) = \int_{s=0}^{t} \int_{\Im \xi = -1/2} e^{i(t-s)\xi} \hat{A}(s)(1 - i\xi)^{-k}(\xi - B(\xi))^{-1} V(s)d\xi ds. \]

For a function \( F(t) > 0 \), monotone increasing in \( t \) to be selected later, we will cut off frequencies in \( |\xi| \) above and below \( F(t)^2 \). We convolve with a Gaussian to smooth this out:
\[
U(t) = \int_{s=0}^{t} \int_{\Im \xi = -1/2} e^{i(t-s)\xi} \hat{A}(s)(1 - i\xi)^{-k}(\xi - B(\xi))^{-1} \cdot (c_0/\pi)^{1/2} e^{-c_0(\lambda - \xi/F(t))^2} V(s)d\lambda d\xi ds \\
= \int_{0}^{t} \int_{\Im \xi = -1/2} \left( \int_{|\lambda| \leq F(t)} + \int_{|\lambda| \geq F(t)} \right) (\cdot) d\lambda d\xi ds \\
=: I_1 + I_2.
\]

**Analysis of \( I_1 \):** From the resolvent and propagator continuation properties, the integrand in \( I_1 \) is holomorphic in \( \{ \Im \xi < 0 \} \cup \Omega \). Observe if \( |\Re \xi| \geq F(t)^2 \), then the integrand is rapidly decaying, hence we can deform the contour in \( \xi \) to
\[
\Gamma = \{ \xi \in \mathbb{C} : \Im \xi = \begin{cases} C, & \Re \xi \leq C, \\ P(\Re \xi), & \Re \xi \geq C. \end{cases} \}
\]

We further break \( I_1 \) into integrals where \( \Re \xi \) is larger than or smaller than \( F^2(t) \):
\[
I_1 = \int_{0}^{t} \left( \int_{\Gamma \cap \{ |\Re \xi| \leq AF(t)^2 \}} + \int_{\Gamma \cap \{ |\Re \xi| \geq AF(t)^2 \}} \right) \int_{|\lambda| \leq F(t)} (\cdot) d\lambda d\xi ds \\
=: J_1 + J_2.
\]

For \( J_1 \), if \( t \geq 2 \), since \( P(\Re \xi) \) is monotone decreasing, we have
\[
\Im \xi \geq P(AF(t)^2),
\]
and on the support of \( \hat{A} \), we have \( t - s \geq t - 1 \). Hence
\[
\|\chi_1 J_1\| \leq C \int_{\Gamma \cap \{ |\Re \xi| \leq AF(t)^2 \}} \int_{|\lambda| \leq F(t)} e^{-(t-1)P(AF(t)^2)} (\xi)^{-k} G(\Re \xi) \cdot \left| e^{-c_0(\lambda - \xi/F(t))^2} \right| d\lambda d\xi \|u_0\| \\
\leq CAF(t)^2e^{-tP(AF(t)^2)}\|u_0\|.
\]

For \( J_2 \), we observe that for \( A \) large enough and \( |\Re \xi| \geq AF(t)^2 \),
\[
\Re (\lambda - \xi/F(t))^2 \geq C^{-1}(\lambda^2 + (\Re \xi)^2/F(t)^2).
\]

Hence,
\[
\|\chi_1 J_2\| \leq C \int_{\Gamma \cap \{ |\Re \xi| \geq AF(t)^2 \}} \int_{|\lambda| \leq F(t)} (\xi)^{-k} G(\Re \xi) \cdot \left| e^{-c_0(\lambda - \xi/F(t))^2} \right| d\lambda d\xi \|u_0\| \\
\leq C \int_{|\eta| \geq F(t)} F(t)e^{-c_1\eta^2} d\eta \|u_0\| \\
\leq CF(t)e^{-c_2F(t)}\|u_0\|.
\]
Analysis of $I_2$: Set

$$J(\tau) = \int_{s=0}^{1} \int_{|\lambda| \geq F(t)} A(s)e^{i(\tau - s)\xi} (1 - i\xi)^{-k}(\xi - B(\xi))^{-1}$$

\[
\cdot (c_0/\pi) \frac{1}{2} e^{-c_0(\lambda - \xi/F(t))^2} V(s) d\lambda d\xi ds,
\]

which for $\tau \geq 1$ is equal to $U(\tau)$. Observe

$$(D_\tau - B(D_\tau))J(\tau) = \int_{s=0}^{1} \int_{|\lambda| \geq F(t)} A(s)e^{i(\tau - s)\xi} (1 - i\xi)^{-k}$$

\[
\cdot (c_0/\pi) \frac{1}{2} e^{-c_0(\lambda - \xi/F(t))^2} V(s) d\lambda d\xi ds
\]

$$=: K(\tau).$$

Hence

$$J(t) = e^{itB(D_\tau)}J(0) + \int_{0}^{t} e^{i(t-s)B(D_\tau)} K(s) ds.$$

Again, by the subunitarity of the propagator, we need to estimate $\|J(0)\|$ and $\int_{0}^{t} \|K(s)\| ds$. For $s \in [1, t]$, since $k > N + 1$, we can deform the $\xi$-contour in the definition of $K$ to $\text{Im} \xi = F(t)$. Then for this range of $s$,

$$\|K(s)\| \leq C \int_{\eta} e^{-(s-2/3)F(t)} |\eta|^{-k} d\eta \|u_0\|,$$

and hence

$$\int_{1}^{t} \|K(s)\| ds \leq CF(t)^{-1} e^{-F(t)/3}.$$ 

For $J(0)$, we first consider $\lambda \geq F(t)$. Since $k > N + 1$, we can deform the $\xi$-contour to

$$\Gamma' = \Gamma_- \cup \Gamma_+$$

where

$$\Gamma_- = \{ \text{Re} \xi \leq F(t)^2/A, \text{Im} \xi = -1/2 \}$$

$$\cup \{ \text{Re} \xi = F(t)^2/A, -F(t) \leq \text{Im} \leq -1/2 \}$$

and

$$\Gamma_+ = \{ \text{Re} \xi \geq F(t)^2/A, \text{Im} \xi = -F(t) \}.$$ 

If $\xi \in \Gamma_-$, we have

$$\text{Re}(\lambda - \xi/F(t))^2 \geq \lambda^2/C,$$

so

$$\int_{\xi \in \Gamma_-} \int_{\lambda \geq F(t)} (\xi)^{-k} G(|\text{Re} \xi|) \cdot e^{-c_0(\lambda - \xi/F(t))^2} V(s) d\lambda d\xi \leq Ce^{-F(t)^2}.$$ 

For $\xi \in \Gamma_+$, we have

$$|e^{-i\lambda\xi}| = e^{-F(t)/3},$$

so the contribution to $\|J(0)\|$ coming from $\lambda \geq F(t)$ is bounded by

$$C(e^{-F(t)^2} + e^{-F(t)/3}).$$
The contribution to $\|J(0)\|$ coming from $\lambda \leq -F(t)$ is handled similarly to obtain the same bound.

We have yet to estimate $\int_0^1 \|K(s)\| ds$. For this we use Plancherel’s formula to write
\[
\left( \int_0^1 \|K(s)\| ds \right)^2 \leq \int_{-\infty}^{\infty} \|K(s)\|^2 ds
\]
\[
= \int_{-\infty}^{\infty} \left(1 - i\xi\right)^{-k} \tilde{A}V(\xi) \int_{|\lambda| \geq F(t)} e^{-c_0(\lambda - \xi/F(t))^2} d\lambda \left\| \lambda \right\|^2 d\xi.
\]
If we estimate this integral by again considering regions where $|\xi| \leq F(t)^2/A$ and $|\xi| \geq F(t)^2/A$ respectively, we see (3.4) is majorized by
\[
C(F(t)^{-2k} + e^{-F(t)^2/C}) \int_{-\infty}^{\infty} \left\| \tilde{A}V(\xi) \right\|^2 d\xi
\]
\[
= C(F(t)^{-2k} + e^{-F(t)^2/C}) \int_{-\infty}^{\infty} \|\tilde{A}V(s)\|^2 ds
\]
\[
\leq C(F(t)^{-2k} + e^{-F(t)^2/C}) \|u_0\|^2.
\]
Combining all of the above estimates, we have
\[
\|U(t)\| \leq C \max \left\{ \frac{F(t)^{-2k}}{e^{-F(t)^2/C} + e^{-F(t)^2/C}}, \frac{F(t)^{-1}e^{-F(t)^2/C}}{F(t)^{-2}e^{-F(t)^2/C} + F(t)e^{F(t)}} \right\} \|u_0\|.
\]
Relabelling $F(t)^2$ as $F(t)$ throughout and applying the condition (3.3), we recover (3.3).

□

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