GENETIC PROGRAMMING WITH LOCAL SCORING

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Abstract. We present new techniques for synthesizing programs through sequences of mutations. Among these are (1) a method of local scoring assigning a score to each expression in a program, allowing us to more precisely identify buggy code, (2) suppose-expressions which act as an intermediate step to evolving if-conditionals, and (3) cyclic evolution in which we evolve programs through phases of expansion and reduction. To demonstrate their merits, we provide a basic proof-of-concept implementation which we show evolves correct code for several functions manipulating integers and lists, including some that are intractable by means of existing Genetic Programming techniques.

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1. Introduction

From Darwinian evolution, a process of local optimization, emerged highly complex organisms. An organism is, in a sense, a program encoded by DNA determining physical actions based on observed data. It is therefore natural to ask whether it would also be possible to evolve through natural selection computer code that solves a given problem. This is the task at the heart of Genetic Programming. This paper is aimed at describing a hill-climbing algorithm for evolving programs, taking an approach that is quite different from established Genetic Programming methods. As a starting point, let us first sketch an extremely simple hill-climbing algorithm for evolving programs for a programming language $\mathcal{L}$. For that algorithm, we will need

1. an initial program $P_0 \in \mathcal{L}$

Date: November 11, 2022.

1For example, this could be a randomly generated program or a program that we are trying to debug.
a scoring function $S : \mathcal{L} \rightarrow [0, 1]$ assessing how well a program is performing at solving a given task with 1 meaning perfect,² and

(3) for each program $P \in \mathcal{L}$, a probability distribution $M(P)$ on $\mathcal{L}$ of “mutations” of $P$.³

Given this, the (non-deterministic) algorithm produces a sequence of programs $P_0, P_1, \ldots \in \mathcal{L}$ in which $P_0$ is the initial program and for $k \geq 1$,

$$P_k := \begin{cases} P'_k, & \text{if } S(P'_k) > S(P_{k-1}); \\ P_{k-1}, & \text{otherwise}; \end{cases}$$

where $P'_k$ is randomly sampled from $M(P_{k-1})$. The algorithm terminates with solution $P_k$ when $S(P_k) = 1$, which may or may not happen eventually. Put simply, the algorithm attempts to improve a program one small step at a time until it arrives at a satisfying solution, quite similar to how a human might approach writing or debugging a program.

This paper describes some inadequacies of this primitive idea and how to overcome them, culminating in a feasible algorithm for evolving programs along these lines. Our investigation is structured as follows.

(1) In Section 2, we shall see how assign a score to each expression in a program as opposed to the entire program at once. In this way, we get a more precise understanding of which parts of a program are buggy. Later, these “local scores” are leveraged to choose mutations so that the expressions that are the most buggy are the most likely to be mutated.

(2) In Section 3, we introduce `suppose`-expressions. These are expressions that are waiting to be promoted into `if`-conditionals. This two-stage process gives an efficient and precise way to evolve conditions for `if`-conditionals.

(3) In Section 4, we discuss greedy criteria for when a mutation should be regarded successful. Specifically, we will mutate one expression at a time, and the mutation is regarded successful if the score of the new expression is strictly greater than the score of the old expression.

(4) In Section 5, we describe an evolutionary process which consists of cycles. Each cycle is divided into four stages: stretching, mutation, rewinding, and compression. This design allows us to probe new program structures while avoiding endless program expansion.

(5) In Section 6, we present a basic proof-of-concept implementation of these ideas and show that it is able to generate correct code for several small problems. On one well-known problem, the even-parity problem, we also compare our implementation to some found in the literature and find that our methods outperforms the best of the considered implementations by more than an order of magnitude.

(6) In Section 7, we discuss future theoretical and practical directions, including integration with Deep Learning techniques.

²$S$ will typically work by running the program on a pre-defined list of inputs and then comparing the actual outputs to the desired outputs, assigning a score based on how many of these cases are correct.

³A good choice of the distribution $M(P)$ should minimize the expected score loss $L(P) := S(P) - E[S(M(P))]$. In particular, likely mutations should ideally only affect the program behavior slightly. However, small changes in code often have major effect on the behavior of the program, so typically $L > 0$. 
1.1. **Functional Programming as a setting for Genetic Programming.** This paper is focused on functional programs, although some of its techniques are applicable to other paradigms as well. The lack of side effects lends itself to Genetic Programming as it means that changing a part of the code has a more predictable effect on the program behavior: we need not worry about how a state change could affect the behavior of expressions elsewhere in the program.

2. **Local Scoring**

When evolving programs, we need to be able to quantify how well a program is performing. As we discussed in the introduction, one could do that by assigning the program a score based on how well it adheres to its test cases. However, with such an approach to scoring, if a test case fails, we gain little insight into what parts of the code could be the root cause of the failure. If we were able to trace back such failure to specific parts of the code, we could focus on mutating those parts.

In this section, we describe a simple method for assigning each expression a score. We assume that every type is an Algebraic Data Type so that values can be viewed as trees; each node corresponds to a value constructor and its children correspond to its arguments. Then, the idea is to augment each node in the value tree with a *trace*, which is a list of expressions that could affect the value at that node. To state this method precisely, we exhibit a rudimentary programming language implementing this idea.

2.1. A programming language with traced values. As the setting for this paper, we consider a simplistic, functional programming language whose values are “traced”. We first describe its syntax:

\[
\begin{align*}
  f &::= g_1 \mid g_2 \mid \cdots & \text{function} \\
  c &::= \text{True} \mid \text{False} \mid d_1 \mid d_2 \mid \cdots & \text{value constructor} \\
  x &::= y_1 \mid y_2 \mid \cdots & \text{variable} \\
  t &::= 1 \mid 2 \mid \cdots & \text{expression tag} \\
  E &::= x \mid c(e, \ldots, e) \mid f(e, \ldots, e) \\
  &\quad| \text{case } e \text{ of } c(x, \ldots, x) \rightarrow e; \cdots; c(x, \ldots, x) \rightarrow e \text{ end} \\
  &\quad| \text{if } e \text{ then } e \text{ else } e \text{ end} \\
  &\quad| \text{suppose } e \text{ then } e \text{ end} \\
  e &::= E^{(t)} & \text{tagged expression} \\
  D &::= f(x, \ldots, x) = e & \text{function definition} \\
  P &::= D \mid D; P & \text{program}
\end{align*}
\]

We stipulate that

1. in a program \(P\), each function \(g_i\) may be declared at most once,
2. in a program \(P\), each participating expression has a unique tag, that is, among all the bodies of functions and all their subexpressions, each tag \(i\) appears at most once,
3. cases in a *case*-expression should be disjoint, that is, the constructors appearing on the left-hand side of the arms should be pairwise distinct,
(4) A variable can be bound at most once in each case of a case-expression, that is, the left-hand side of the arm takes the form \( c(x_1, \ldots, x_n) \) where \( x_1, \ldots, x_n \) are pairwise distinct variables, and

(5) Variables bound by a case of a case-expression cannot overshadow variables that are already bound.

We now describe the typing rules. We use a simple nominal type system with types

\[ \tau ::= \text{Bool} \mid \tau_1 \mid \tau_2 \mid \cdots. \]

A typing context \( \Gamma \) is a collection of judgments of the forms

1. A variable typing: \( x : \tau \),
2. A function signature: \( f : (\tau_1, \ldots, \tau) \rightarrow \tau \), or
3. A type definition: \( \tau ::= c(\tau_1, \ldots, \tau) \mid \cdots \mid c(\tau_1, \ldots, \tau) \).

The typing rules are:

| TrueType | FalseType | VarType |
|----------|-----------|---------|
| \( \Gamma \vdash \text{True}^{(t)} : \text{Bool} \) | \( \Gamma \vdash \text{False}^{(t)} : \text{Bool} \) | \( x : a \in \Gamma \) |
| \( \Gamma \vdash f(e_1, \ldots, e_n)^{(t)} : b \) | \( \Gamma \vdash x^{(t)} : a \) | |

| FunctionType |
|-------------|
| \( \Gamma \vdash f(e_1, \ldots, e_n) : b \) |

| ConstructorType |
|-----------------|
| \( \Gamma \vdash e_1 : a_1, \ldots, e_n : a_n \) |
| \( a ::= \cdots \mid c(a_1, \ldots, a_n) \mid \cdots \in \Gamma \) |
| \( \Gamma \vdash c(e_1, \ldots, e_n)^{(t)} : a \) |

| CaseType |
|---------|
| \( \Gamma \vdash \text{case } d \text{ of } c_1(x_1,1, \ldots, x_1,n_1) \rightarrow e_1; \cdots ; c_m(x_m,1, \ldots, x_m,n_m) \rightarrow e_n \end{t} : b \) |

| IfType | SupposeType |
|--------|-------------|
| \( \Gamma \vdash \text{if } c \text{ then } e_1 \text{ else } e_2 \end{t} : a \) | \( \Gamma \vdash \text{suppose } c \text{ then } e \end{t} : a \) |

We shall often write just \( P \) when we mean a well-typed program \((P, \Gamma)\), letting the typing context \( \Gamma \) containing all the type definitions and function signatures be implicit.

At last, we explain the semantics of our programming language. The values of the programming language are

\[ v ::= c^{[\pm t, \ldots, \pm t]}(v_1, \ldots, v) \]

where \( \pm ::= + \mid - \). Conceptually, each value constructor is tagged with a trace, a list of expressions that could affect the value. Its trace need not contain the traces of the arguments \((v_1, \ldots, v)\) whose traces in turn consist of expressions that could affect the arguments but not the parent value constructor. A sign \( \pm \) should be thought of as measuring the “influence” of an expression: \(+\) means that if the value is bad then the expression should be punished, and \(-\) means that if the value is bad then the expression should be rewarded. The relevance of \(-\) will be revealed in Section 3.
An evaluation context $\sigma$ is a collection of bindings of the form

$$B ::= x = v \mid f(x, \ldots, x) = e$$

such that each variable $x$ and each function $f$ appears at most once in the left-hand sides. For a program $P$, one extracts an evaluation context $\sigma(P)$ by collecting all its function definitions.

We use the notation $\sigma \vdash e \downarrow v$ to denote that an expression $e$ evaluates to a value $v$ in the evaluation context $\sigma$. Furthermore, we make use of the notation

$$(c_{[t_{k+1}, \ldots, t_n]}(v_1, \ldots, v_k))^{[t_0, \ldots, t_k]} := c_{[t_0, t_1, \ldots, t_n]}(v_1, \ldots, v_k)$$

where $t_i$ are signed tags. We shall sometimes elide the + in +$t$ for notational convenience. The rules for evaluation are as follows.

| Var          | $\sigma, x = v \vdash x^{(t)} \downarrow v^{[t]}$ |
|--------------|-----------------------------------------------|
| Cons         | $\sigma \vdash e_1 \downarrow v_1, \ldots, e_n \downarrow v_n$ |
|              | $\sigma \vdash c(e_1, \ldots, e_n)^{(t)} \downarrow v^{[t]}(v_1, \ldots, v_n)$ |

| Call         | $\sigma \vdash e_1 \downarrow v_1, \ldots, e_n \downarrow v_n \quad x_1 = v_1, \ldots, x_n = v_n \vdash e \downarrow v$ |
|--------------|--------------------------------------------------|
|              | $\sigma, f(x_1, \ldots, x_n) = c(e_1, \ldots, e_n)^{(t)} \downarrow v^{[t]}$ |

| Case         | $\sigma \vdash \text{case } d \text{ of } \cdots ; c(x_1, \ldots, x_n) = e; \cdots \text{ end}^{(t_0)} \downarrow v^{[t_0, t_1, \ldots, t_k]}$ |
|--------------|-----------------------------------------------|
|              | $\text{IfTrue}$                                |
|              | $\sigma \vdash c \downarrow \text{True}^{[t_1, \ldots, t_n]}, e_1 \downarrow v$ |
|              | $\sigma \vdash \text{if } c \text{ then } e_1 \text{ else } e_2 \text{ end}^{(t_0)} \downarrow v^{[t_0, t_1, \ldots, t_n]}$ |
|              | $\text{IfFalse}$                               |
|              | $\sigma \vdash c \downarrow \text{False}^{[t_1, \ldots, t_n]}, e_2 \downarrow v$ |
|              | $\sigma \vdash \text{if } c \text{ then } e_1 \text{ else } e_2 \text{ end}^{(t_0)} \downarrow v^{[t_0, t_1, \ldots, t_n]}$ |

We postpone stating the semantics of suppose-expressions to Section 3.

Note that whenever an expression is evaluated, the value it produces is tagged with a unique identifier for that expression. The rules are defined in such a way that each value ends up carrying traces of expressions that could affect the value.

To illustrate, let us elaborate a bit on the Case rule. The rule says that what a case-expression evaluates to may be affected by

1. the case-expression $t_0$ itself since, trivially, replacing the entire case-expression could change its value,
2. those expressions $t_1, \ldots, t_n$ that could affect the entirety, even the value constructor, of the value attained by $d$ that we are casing on, and
3. those expressions that could affect the value attained by the body $e$ of the chosen arm.

The rules IfTrue and IfFalse are quite similar.\(^4\)

The reason that we include if-conditions in spite of having case-expressions is that they simplify our discussion of suppose-expressions in Section 3.
2.2. Extracting local scores. We shall need to consider values with other kinds of tags. Define
\[ s^u := c^u (v^u, \ldots, v^u). \]
Of particular use are untagged values \( s^1 \), traced values \( v = v^{[x:t, \ldots, x:t]} \), and score-tagged values \( s^{[0,1]} \).

Fix a base typing context \( \Gamma_{\text{types}} \) containing just the type definitions we shall use. To assign scores to expressions in a program \((P, \Gamma)\) with \( \Gamma_{\text{types}} \subseteq \Gamma \), we will need a test criterion consisting of the data

1. a name and signature \( f : (a_1, \ldots, a_n) \to b \) for the function we are testing, where \( a_1, \ldots, a_n, b \) are types defined in \( \Gamma_{\text{types}} \);
2. untagged input values \( (v_{1,k} : a_1, \ldots, v_{n,k} : a_n) \) for \( k = 1, \ldots, N \), and
3. for each \( k = 1, \ldots, N \), an assessment function

\[ s_{(v_{1,k}, \ldots, v_{n,k})} : \{w \in v^1 \mid w : b\} \to \{w \in v^{[0,1]} \mid w : b\} \]

which takes a \( b \)-typed value and sets the tag at each node to a score (i.e., does not change the underlying untagged value) such that all the tags in \( s_{(v_{1,k}, \ldots, v_{n,k})}(w) \) are 1 if and only if \( w \) is the desired result of \( f(v_{1,k}, \ldots, v_{n,k}) \).

We call the function specified in (1) the primary function.

For example, suppose \( P \) consisted of just one function definition \( \text{swap}(x, y) = \cdots \) and \( \Gamma \) consisted of the type definition \( \text{Bool}2 := \text{Pair}(\text{Bool}, \text{Bool}) \) and the function signature \( \text{swap} : \text{Bool2} \to \text{Bool2} \). If we were to assess how well \( \text{swap} \) swaps the boolean values in the pair like \((x, y)\), the test criterion could consist of

1. the name and signature \( \text{swap} : \text{Bool2} \to \text{Bool2} \),
2. the inputs \( \text{Pair}^1(\text{True}^1, \text{True}^1), \text{Pair}^1(\text{False}^1, \text{True}^1), \text{Pair}^1(\text{True}^1, \text{False}^1), \text{Pair}^1(\text{False}^1, \text{False}^1) \), and
3. the assessment functions

\[ s_{\text{Pair}}(a_1^1, a_2^1) : \text{Pair}^1(b_1^1, b_2^1) \to \text{Pair}^1(\delta(b_1, a_2) + \delta(b_2, a_1)/2, b_1^1, b_2^1) \]

where \( \delta(x, y) \) is 1 if \( x = y \) and 0 otherwise.

The assessment functions are defined in such a way that the resulting tag to the \text{Pair} constructor measures how close the value is to the desired value \( \text{Pair}(a_2^1, a_1^1) \), and the resulting tag to the boolean in each coordinate measures how close the respective coordinate is to the desired value. Notice how the assessment functions give perfect scores when the behavior is as desired:

\[ s_{\text{Pair}}(a_1^1, a_2^1) (\text{Pair}^1(a_2^1, a_1^1)) = \text{Pair}^1(a_2^1, a_1^1). \]

Let us return to the general setup (1)-(3) and explain how to compute a score for each expression in the program \( P \). We assume that \( P \) contains such an \( f \) and that \( f \) always terminates. The process for obtaining expression-level scores is as follows.

\[ \text{Max Vistrup} \]
(a) For each $k = 1, \ldots, N$, evaluate the function call $f(\text{tr}(v_1,k), \ldots, \text{tr}(v_n,k))$ in $P$ and call its (traced) value $w_k$. Here, $\text{tr}(v)$ is the traced value obtained from the untagged value $v$ by setting each tag to the empty trace $[]$.

(b) For each $k = 1, \ldots, N$, compute $w'_k := s(v_1,k, \ldots, v_n,k)(\text{untag}(w_k))$ where $\text{untag}(w_k)$ is the untagged value obtained from the traced value $w_k$ by replacing each tag with $(.)$.

(c) For each node in $w_k$, extract the positively charged tags $+t_1, \ldots, +t_p$ from its trace, the negatively charged tags $-u_1, \ldots, -u_q$ from its trace, and the tag $s \in [0,1]$ of the corresponding node in $w'_k$; then collect all the tuples $(t_i, s)$ for $i = 1, \ldots, p$ and $(u_i, 1-s)$ for $i = 1, \ldots, q$ into a list $\ell$.

(d) An expression in $P$ with the tag $t$ is assigned the score, 
$$SP(t) := \frac{\sum_{(t,s) \in \ell} s}{\sum_{(t,s) \in \ell} 1}$$ 
averaging over all the scores in $\ell$ associated to the tag $t$.

Note how a positively charged tag $+t$ in some trace in $w_k$ contribute to the average defining $SP(t)$ with the score produced by $s(v_1,k, \ldots, v_n,k)$ corresponding to the trace whereas a negatively charged tag $-t$ contribute with the inversion of that score. The net effect is that an expression with tag $t$ that appears in a trace positively charged (resp. negatively charged) is incentivized to maximize (resp. minimize) the score corresponding to that trace if we are to maximize the score $SP(t)$.

For the rest of the paper, we shall fix a test criterion (with respect to the fixed $\Gamma_{\text{types}}$). Moreover, from now on, we will only consider programs $P$ that contain the type definitions $\Gamma_{\text{types}}$ and a function definition for the primary function (1). In particular, for each such program $P$, we get a scoring function $SP$. If $e$ is an expression in $P$ with expression tag $t$, we write $SP(e) := SP(t)$. Also, when the context resolves ambiguity, we shall omit the subscript $P$, writing $S$ instead of $SP$.

### 3. SUPPOSE-EXPRESSIONS

Let us informally consider the problem of evolving code for the minimum function $\min : \mathbb{Z} \times \mathbb{Z} \rightarrow \mathbb{Z}$. We may assess a candidate implementation based on some example arguments $A \subseteq \mathbb{Z} \times \mathbb{Z}$ ranging over small numbers. Start with the initial program

$$(P_0) \quad \min(n, m) = 0.$$ 

This is not faring particularly well: perhaps, there are a few $(n, m) \in A$ with $0 = n \leq m$ or $0 = m \leq n$ but they make up an increasingly small fraction as $|A|$ grows. If we try to mutate $P_0$ by replacing the body of $\min(n, m)$ by various randomly chosen expressions and testing it on $A$, it is not hard to find the improved program

$$(P_1) \quad \min(n, m) = n,$$ 

which performs a lot better: it is correct whenever $n \leq m$ which is roughly $\frac{1}{2}$ of the time. However, at this point, we stagnate. The next program we want is

$$(P_2) \quad \min(n, m) = \text{if } n \leq m \text{ then } n \text{ else } m \text{ end},$$ 

but this is quite a leap from $P_1$ and would therefore take a long time to discover through picking out random expressions. More complex examples are even less tractable. How do we overcome this problem of evolving if-conditionals?
suppose-expressions allow us to separate the problem of evolving the condition from the problem of evolving the else-branch. In the above informal example, small intermediate steps between $P_1$ and $P_2$ could be

\[
\begin{align*}
\text{min}(n, m) &= n \\
\rightarrow \text{min}(n, m) &= \text{suppose True then } n \text{ end} \\
\rightarrow \text{min}(n, m) &= \text{suppose } n \leq m \text{ then } n \text{ end} \\
\rightarrow \text{min}(n, m) &= \text{if } n \leq m \text{ then } n \text{ else } n \text{ end} \\
\rightarrow \text{min}(n, m) &= \text{if } n \leq m \text{ then } n \text{ else } m \text{ end},
\end{align*}
\]

Conceptually, suppose-expressions encourage the condition, initially constant True, to evolve to be True whenever the body is correct and False otherwise. When the condition for the suppose-expression is good enough, we will promote the suppose-expression into an if-conditional:

\[
\text{suppose } c \text{ then } e \text{ end } \rightarrow \text{if } c \text{ then } e \text{ else } e \text{ end}.
\]

This evolutionary strategy is established in Section 5.

With this discussion in mind, let us now state the semantics of suppose-expressions. We use the notation established in Section 2. Define $+t := -t$ and $-t := +t$. The rules are then as follows.

\[
\begin{align*}
\text{SupposeTrue} & \quad \sigma \vdash c \downarrow \text{True}^{[t_1, \ldots, t_n]}, e \downarrow v \\
& \quad \sigma \vdash \text{suppose } c \text{ then } e \text{ end}^{(t_0)} \downarrow v^{[t_0, t_1, \ldots, t_n]} \\
\text{SupposeFalse} & \quad \sigma \vdash c \downarrow \text{False}^{[t_1, \ldots, t_n]}, e \downarrow v \\
& \quad \sigma \vdash \text{suppose } c \text{ then } e \text{ end}^{(t_0)} \downarrow v^{[t_0, -t_1, \ldots, -t_n]}.
\end{align*}
\]

Thus, if we disregard traces, suppose-expressions just evaluate their bodies. However, if we follow the Local Scoring model set out in Section 2, we see that if $c$ evaluates to True, the better $e$ performs, the higher score is assigned to $c$, whereas if $c$ evaluates to False, the better $e$ performs, the lower score is assigned to $c$.

We will need the following lemma in Section 5.

**Lemma 3.1.** Let $P$ be a program. Suppose $P$ contains an expression suppose True$^{(t)}$ then $e$ end$^{(u)}$. Then $S(t) = S(u)$.

**Proof.** Recall the algorithm specified in Section 2. Note that the steps (a)-(c) are the same no matter whether we are computing $S(t)$ or $S(u)$. From the rule SUPPOSETRUE, we see that during step (a), $+u$ (resp. $-u$) will always appear coupled with $+t$ (resp. $-t$) in traces. It follows that $\ell$ has exactly $m$ pairs $(u, s)$ if and only if it has exactly $m$ pairs $(t, s)$. Therefore,

\[
S(t) = \sum_{(t, s) \in \ell} s = \sum_{(u, s) \in \ell} s = S(u),
\]

completing the proof. \(\square\)

4. Some mutation principles

We discuss here some simple principles for how to mutate programs by leveraging Local Scoring. These will be used in combination with the methods presented in Section 5.
4.1. Mutually exclusive mutation. Changing a single expression has a high chance of breaking the program, but changing multiple expressions has an even higher chance of doing so. To deal with this fragility, we shall change just one expression in each mutation step.

A simple scheme to mutate a program $P$ is to first pick a random expression. The probability for picking a given expression $e$ should be proportional to $1 - S(e)$, so that we are more likely to mutate what is broken and we never mutate what works perfectly\footnote{In particular, under this scheme, dead code is never mutated.} This expression is then replaced by a new, randomly generated expression of the same type (with expression tags disjoint from the expression tags used in $P$), giving a new program $P'$, the mutant of $P$.

4.2. Greedy Principle of Mutation. Suppose we use a hill-climbing algorithm which has as state a program $P$ that it repeatedly tries to mutate until it finds a good mutant, a successor, which then replaces $P$, repeating this process until the program solves some task perfectly. How do we determine if a given mutant $P'$ should replace $P$ as its successor?

If we go purely by some kind of global score based on the behavior of the program, as in Section\footnote{In particular, under this scheme, dead code is never mutated.} we fail to evolve the conditions for $\textit{suppose}$-expressions changing which has no exterior effect on the program behavior. We therefore propose another criterion. Recall that we mutate a program $P$ to a program $P'$ by replacing some expression $e$ by a new expression $e'$. The Greedy Principle of Mutation stipulates that $P'$ should be the successor to $P$ if and only if $S_{P'}(e') > S_P(e)$. The principle is “greedy” since the mutation can (and usually does) change the scores of expressions that are not $e$ and yet we only focus on how the score of $e$ changes.

Notice how the condition is $S_{P'}(e') > S_P(e)$ as opposed to $S_{P'}(e') \geq S_P(e)$. This is a deliberate choice. It means that we prefer doing no mutation to mutating: we only accept mutations that strictly improve the expression they change. We shall refer to this property as mutation conservativity.

5. Cyclic Evolution

We have described $\textit{suppose}$-expressions but we have yet to explain how $\textit{suppose}$ should be introduced during evolution and when these should be promoted into if-conditionals. When mutating a program $P$, we would like to sometimes replace an expression

\[ e \xrightarrow{\textit{suppose} \ True \ then} e \textit{end} \]

after which mutations can begin to improve $\text{True}$ to a more meaningful condition. If we were to just follow the simplistic scheme set out in Section\footnote{In particular, under this scheme, dead code is never mutated.} such replacement might be tried but the resulting program will never succeed the existing program because the score of the replacing expression is the same as the score of the original expression $e$. One could circumvent this issue by making an exception to mutation conservativity, making this specific type of mutation always succeed. This, however, is not a very good solution to the problem at hand: the program could then grow unbounded as we add more and more redundant $\textit{suppose}$-expressions.

What we propose instead is to divide the evolution into phases of growth, phases of mutation, and phases of reduction; similar to how a person who is trying to gain muscle may divide their eating habits into phases of bulking and phases of cutting.
Specifically, instead of proceeding one mutation at a time, the basic unit of the algorithm will be a cycle consisting of four phases:

1. **Stretching** in which a program is replaced by an equivalent, more redundant program by introducing redundant *suppose*-expressions, redundant if-conditionals, redundant *case*-expressions, and redundant new functions.
2. **Mutation** in which a number small changes are made to the program, replacing the program whenever a mutation satisfying the Greedy Principle of Mutation is found.
3. **Rewinding** in which we try to undo the stretches done in (a).
4. **Compression** in which we try to rewrite the program to an equivalent, smaller form.

The algorithm we propose evolves programs by repeatedly applying cycles until a program performing perfectly (in accord with the test criterion) is found.

We describe basic ways to implement each phase. We emphasize that each can be extended and refined in many ways, some of which will be discussed in Section 7.

5.1. **Stretching.** The stretching phase makes the program more redundant, setting the scene for the mutation phase to turn that redundancy into useful structure. The stretching phase consists of \( N \) stretches for some fixed constant \( N > 0 \).

Each stretch picks out a random expression in the program with the probability of choosing an expression \( e \) being proportional to \( 1 - S(e) \). The stretched program \( P' \) is constructed from \( P \) by picking randomly among the following possibilities:

1. replacing \( e \leadsto \text{suppose True then } e \text{ end} \),
2. if \( e = \text{suppose } c \text{ then } e' \text{ end} \) and \( S(c) > S(e') \), replacing \( e \leadsto \text{if } c \text{ then } e' \text{ else } e' \text{ end} \),
3. picking a variable \( x : \tau \) bound in the context of \( e \) and replacing
   
   \[ e \leadsto \text{case } x \text{ of } \]
   
   \[ c_1(x_1, \ldots, x_n) \rightarrow e[c_1(x_1, \ldots, x_n)]; \]
   
   \[ \vdots \]
   
   \[ c_k(x_1, \ldots, x_n) \rightarrow e[c_k(x_1, \ldots, x_n)] \]
   
   \[ \text{end}, \]

   where (a) \( x_1, \ldots \) are variables that do not participate in \( e \), (b) \( c_1, \ldots, c_k \) are the value constructors for \( \tau \):

   \[ \tau := c_1(\cdots(n_1 \text{ arguments})) | \cdots | c_k(\cdots(n_k \text{ arguments})) \],

   and (c) \( e[x/e'] \) denotes \( e \) with all occurrences of \( x \) replaced by \( e' \), or

4. for \( x_1 : a_1, \ldots, x_n : a_n \) the variables bound in the context of \( e : b \), adding a new function

   \[ f : (b, a_1, \ldots, a_n) \rightarrow b \]
   
   \[ f(y, x_1, \ldots, x_n) = y \]

   and replacing \( e \leadsto f(e, x_1, \ldots, x_n) \).

In each case, the replacing expression should be tagged with some tags disjoint from those used in \( P \), but we elide this for notational simplicity.

Let us elaborate on the condition \( S(c) > S(e') \) in stretch (2). The justification for this condition comes from Lemma 3.1 which precludes stretch (2) from happening when \( c = \text{True} \). In this way, when a stretch (1) introduces a *suppose*-expression,
the condition must first improve (which can happen in the mutation phase) before a stretch (2) can promote it into an if-conditional. As such, we do not end up immediately creating a lot of redundant if- conditionals.

5.2. Mutation. The mutation phase consists of \( N_{\text{mutate}} \) mutations, successful or not, where \( N_{\text{mutate}} > 0 \) is some fixed constant. A mutation changes the program in the way described Section 4.3. A mutation is considered successful if it satisfies the Greedy Principle of Mutation (see Section 4.2) in which case the mutant program replaces the program and the search continues.

5.3. Rewinding. The purpose of stretching was to create new program structures that the mutation phase could make use of. However, not all stretches create useful program structures. If we return to our informal example of Section 3 with evolving the min function, we could have the stretch

\[
\min(n, m) = \begin{cases} n & \text{if } n \leq m \\ m & \text{else} \end{cases}
\]

But that would not be of much use because in the else-branch \( n \) is never the right expression. So, unless the last \( n \) is mutated, the condition for the suppose-expression will just converge to False. We must be able to deal effectively with such “useless stretches” to stop the program that is being evolved from accumulating redundancy.

After the mutation phase has completed, we propose to clean up the stretches by “rewinding” them. Specifically, for each stretching operation \( s \), we define a corresponding rewinding operation \( \overline{s} \), which is a transformation of programs that undoes (is left-inverse to) \( s \). Then, if stretches \( s_1, \ldots, s_{N_{\text{stretch}}} \) were made in the stretching phase in that order, the rewinding phase applies rewinding operations \( \overline{s}_{N_{\text{stretch}}}, \ldots, \overline{s}_1 \) in that order (notice the reversal of order).

For a stretch \( s \) out of the \( N_{\text{stretch}} \) stretches done in the stretching phase, we will now describe its corresponding rewinding operation \( \overline{s} \). \( \overline{s} \) depends on which type (1)-(4) of stretch \( s \) is:

1) If \( s \) is a stretch of type (1), \( \overline{s} \) is the following program transformation. Let \( e_1, \ldots, e_n \) be the suppose-expressions that originated from \( s \). Let \( c_1, \ldots, c_n \) be their respective condition expressions and \( e_1', \ldots, e_n' \) their respective bodies. For \( i = 0, \ldots, n \), replace \( e_i \mapsto e'_i \) if \( S(c_i) \leq S(e_i) \).

2) If \( s \) is a stretch of type (2), \( \overline{s} \) is the following program transformation. Let \( e_1, \ldots, e_n \) be the if- conditionals that originated from \( s \). Let \( c_1, \ldots, c_n \) be their respective condition expressions and \( e_1', \ldots, e_n' \) their respective True-branches. For \( i = 0, \ldots, n \), replace \( e_i \mapsto \text{suppose } e_i \text{ then } e'_i \text{ end} \) if the latter expression attains a score in the program with this replacement made at least the score \( S(e_i) \).

3) If \( s \) is a stretch of type (3), \( \overline{s} \) is the following program transformation. Let \( x \) be the variable chosen in \( s \). Let \( e_1, \ldots, e_n \) be the expressions case \( x \) of \( \cdots \) that originated from \( s \). Let \( c_1(x_1, \ldots, x_{n_1}), \ldots, c_k(x_1, \ldots, x_{n_k}) \) be the cases that appear in each of those case-expressions. For \( i = 1, \ldots, n \), do the following.

   a) Look for a \( 1 \leq j \leq k \) such that (i) \( n_j > 0 \), (ii) the expression \( e'_{i,j} := e_{i,j} | c_j(x_1, \ldots, x_{n_j}) = x_i] \) where \( e_{i,j} \) is the body of the case-expression \( e_i \) to the case \( c_j \), does not reference the variables \( x_1, \ldots, x_{n_j} \), and (iii)
after replacing \( e_i \) with \( e'_{i,j} \), the new \( e'_{i,j} \) attains a score at least the
score \( S(e_i) \). If such a \( j \) exists, replace \( e_i \rightleftharpoons e'_{i,j} \) in the program.

(b) If not, look instead for a \( 1 \leq j \leq k \) such that (i) \( n_j = 0 \) and (ii) after
replacing \( e_i \) with \( e_{i,j} \), the new \( e_{i,j} \) attains a score at least the score
\( S(e_i) \). If such a \( j \) exists, replace \( e_i \rightleftharpoons e_{i,j} \) in the program.

(c) If such a \( j \) could not be found either, do nothing.

(4) If \( s \) is a stretch of type (4), \( \pi \) is the following program transformation.
Let \( e_1, \ldots, e_n \) be the call-expressions that originated from \( s \). Let \( f \) be
the name of the function they are calling. Suppose its definition in the
program is \( f(x_1, \ldots, x_k) = e \). If \( e \) contains a subexpression \( f(\cdots) \), do
nothing. Otherwise, do the following. For each \( i = 1, \ldots, n \), replace
\( e_i = f(a_1, \ldots, a_k) \rightleftharpoons e[x_1/a_1] \cdots [x_k/a_k] \). If after this transformation the
expression \( f(\cdots) \) appears nowhere in the program, furthermore remove the
function definition \( f(x_1, \ldots, x_k) = e \) from the program.

As before, we elide tags in notation, implicitly tagging expressions with unique tags.
And again, \( e[a/b] \) denotes the expression \( e \) with all occurrences of \( a \) replaced by \( b \).

Note that in each of these rewinding operations, we talk of possibly multiple
expressions \( e_1, \ldots, e_n \). This is because an expression created by one stretch may
be duplicated by another (stretches of type (2) or (3)). It is therefore necessary to
track what expressions originate in what stretches. This can be done by tagging
each expression by some extra bookkeeping data of which we shall not deal with
the details.

The condition \( S(e_i) \leq S(e_i) \) in (1) is again justified by Lemma 3.1 In particular,
the condition is only fulfilled if the condition \( c_i \) is at least as bad as the initial
condition \textbf{True}. In effect, we throw away a \textbf{suppose} if its condition did not manage
to improve at all during the mutation phase.

5.4. Compression. The final phase is compression, replacing the program by an
equivalent, shorter form. There are indeed many ways to approach this problem of
removing excess; we present one which we found works reasonably well in practice.

As input for the algorithm, we need to run the program on the examples in
the test criterion that we fixed in Section 2.2 (see step (a)) and collect for each
equation \( e \) an input–output table: the set with an element \((\sigma, \text{untag}(v))\) for each
evaluation \( \sigma \vdash e \downarrow v \) done.

We say that an expression \( e \) is \textit{linear} if either \( e = x^{(t)} \) for \( x \) a variable or
\( e = c(e_1, \ldots, e_n)^{(t)} \) for \( c \) a value constructor and \( e_1, \ldots, e_n \) linear expressions.

Recall that we defined a program to consist of a list of function declarations
\( f(x_1, \ldots, x_n) = e \). The compression algorithm first applies the following procedure
to each of the bodies \( e \).

(1) If \( e = c(e_1, \ldots, e_n)^{(t)} \) for \( c \) a value constructor, apply the procedure recursively
to \( e_1, \ldots, e_n \).

(2) If \( e = \textbf{suppose} c \textbf{ then } b \textbf{ end}^{(t)} \), apply the procedure recursively to \( c \) and
\( b \); and if after doing so \( e \) became \( e' = \textbf{suppose } c' \textbf{ then } b' \textbf{ end}^{(t)} \) where \( c' \)
admits no variable \( x^{(u)} \) as a subexpression, further replace \( e' \rightleftharpoons b' \).

(3) If neither of the above cases apply, try to find a linear expression \( \ell \) with
the behavior specified by the input–output table of \( e \). (The algorithm for
finding \( \ell \) is postponed to Section 5.4.1) If such \( \ell \) exists, pick one and
replace \( e \rightsquigarrow \ell \) (in which we tag \( \ell \) with expression tags disjoint from those used in \( P \)), and we’re done. If not, continue to the next steps.

(4) If \( e = \text{if } c \text{ then } a \text{ else } b \end{f}, \) apply the procedure recursively to \( c, a, \) and \( b. \) Suppose that after doing so \( e \) became \( e' = \text{if } c' \text{ then } a' \text{ else } b' \end{f}. \) If \( c' = \text{True}(u) \) for some \( u, \) further replace \( e' \rightsquigarrow a'. \) Or, if \( c' = \text{False}(a) \) for some \( u, \) further replace \( e' \rightsquigarrow b'. \)

(5) If \( e = f(e_1, \ldots, e_n)\end{f} \) for \( f \) a function, apply the procedure recursively to \( e_1, \ldots, e_n. \)

After this, the algorithm removes from the program function definitions \( g(\cdots) = \cdots \) for functions \( g \) that are not reachable from the primary function \( f \) in the call graph.

The resulting program is the result of the compression phase.

Note that, in each step the \( w_k \) used in the definition of \( S \) in Section 2.2 remain unchanged if we disregard traces. This is the sense in which the original program is equivalent to the compressed program, that is, they are equivalent on known input.

5.4.1. An algorithm for finding linear expressions.

We are presented with the problem of listing the linear expressions \( \ell \) satisfying a given specification. Let us be more precise. We are given a type \( \tau, \) a typing context \( \Gamma \) containing a definition of \( \tau, \) and an input–output table \( T = \{(\sigma_1, v_1), \ldots, (\sigma_n, v_n)\} \) such that for each \( i = 1, \ldots, n, \)

(i) \( \sigma_i \) is an evaluation context and \( v_i \) an untagged value of type \( \tau \) and (ii) all variables found in \( \sigma_i \) are found in \( \Gamma \) and vice versa. The task is then to compute the (untagged) linear expressions \( \ell \) that evaluate to \( v_i \) in context \( \sigma_i \) for every \( i = 1, \ldots, n. \)

If \( n = 0 \) (that is, the input–output table is empty), the problem is equivalent to finding all linear expressions of type \( \tau. \) This possibly infinite list can easily be computed by a breadth-first search applied to the recursive definition of linear expressions given near the beginning of Section 5.4.

Now, suppose \( n > 0. \) For convenience, we shall rename the variables in \( \Gamma \) so that they are called \( x_1, \ldots, x_n. \) Recall the notation for tagged values established in Section 2.2. An \( \ell \)-value is a tagged value \( \in v^\mathcal{P}(\{a, 1, \ldots, n\}) \) where \( \mathcal{P} \) denotes the power set operator. Moreover, given an \( \ell \)-value \( w = c^A(w_1, \ldots, w_k), \) define

\[
\ell(w) := \{x_i^{|i|} \mid i \in \mathbb{N} \cap A\} \cup \begin{cases} \{c^{|i|}(w_1^{|i|}, \ldots, w_k^{|i|}) \mid w_i^{|i|} \in \ell(w_i)\}, & \text{if } c \in A; \\ \emptyset, & \text{otherwise.} \end{cases}
\]

**Lemma 5.1.** \( \ell(w) \) is a finite set.

**Proof.** Proceed by induction over \( w: \) assume that the lemma is true when \( w \) is replaced by any of the \( w_i. \) Since \( A \) is finite, the induction step follows from the formula defining \( \ell(w). \)

To an input–output pair \((\sigma, v)\) where \( v = c^{|i|}(v_1, \ldots, v_k), \) we associate an \( \ell \)-value

\[
w(\sigma, v) := c^{|\sigma| \cup \{i \mid (x_i = v) \in \sigma\}}(w(\sigma, v_1), \ldots, w(\sigma, v_k)).
\]

**Lemma 5.2.** \( \ell(w(\sigma, v)) \) is the set of linear expressions that evaluates to \( v \) in the evaluation context \( \sigma. \)

**Proof.** Proceed by induction on \( v: \) assume that the lemma is true when \( v \) is replaced by any of the \( v_i. \) The linear expressions evaluating to \( v \) in the context \( \sigma \) are the

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8 according to semantics like those given in Section 2.1 but ignoring traces
variables $x_i$ such that $(x_i = v) \in \sigma$ and expressions $c(e_1, \ldots, e_k)$ where each $e_i$ is a linear expression evaluating to $v_i$ in the context $\sigma$. The induction step follows from unfolding definitions.

Given $\ell$-values $w_1 = c_1^A(w_{1,1}, \ldots, w_{1,p})$ and $w_2 = c_2^B(w_{2,1}, \ldots, w_{2,q})$, define

$$w_1 * w_2 := \begin{cases}
    c_1^{A \cap B}(w_{1,1} * w_{2,1}, \ldots, w_{1,p} * w_{2,p}), & \text{if } c_1 = c_2; \\
    c_1^{A \cap B \setminus \{\circ\}}(w_{1,1}, \ldots, w_{1,p}), & \text{otherwise}.
\end{cases}$$

**Lemma 5.3.** $\ell(w_1 * w_2) = \ell(w_1) \cap \ell(w_2)$.

**Proof.** Proceed by induction on $w_1$: assume that the lemma is true when $w_1$ is replaced by any of the $w_{1,i}$.

**Case:** $c_1 = c_2$. Using the induction hypothesis,

$$\ell(w_1 * w_2) = \ell(c_1^{A \cap B}(w_{1,1} * w_{2,1}, \ldots, w_{1,p} * w_{2,p}))$$

$$= \{x_i^0 \mid i \in \mathbb{N} \cap A \cap B\}$$

$$\cup \begin{cases}
    \{c_1^0(w_{1,i}^0, \ldots, w_{1,i}^p) \mid w_{1,i}^p \in \ell(w_{1,i} * w_{2,i})\}, & \text{if } \circ \in A \cap B; \\
    \emptyset, & \text{otherwise}.
\end{cases}$$

$$= \{x_i^0 \mid i \in \mathbb{N} \cap A \cap B\}$$

$$\cup \begin{cases}
    \{c_1^0(w_{1,i}^0, \ldots, w_{1,i}^p) \mid w_{1,i}^p \in \ell(w_{1,i} * w_{2,i})\}, & \text{if } \circ \in A \cap B; \\
    \emptyset, & \text{otherwise}.
\end{cases}$$

$$= \{x_i^0 \mid i \in \mathbb{N} \cap A\} \cup \begin{cases}
    \{c_1^0(w_{1,i}^0, \ldots, w_{1,i}^p) \mid w_{1,i}^p \in \ell(w_{1,i})\}, & \text{if } \circ \in A; \\
    \emptyset, & \text{otherwise}.
\end{cases}$$

$$\cap \begin{cases}
    \{c_1^0(w_{1,i}^0, \ldots, w_{1,i}^p) \mid w_{1,i}^p \in \ell(w_{2,i})\}, & \text{if } \circ \in B; \\
    \emptyset, & \text{otherwise}.
\end{cases}$$

$$= \ell(w_1) \cap \ell(w_2).$$

**Case:** $c_1 \neq c_2$. Unfolding definitions,

$$\ell(w_1 * w_2) = \ell(c_1^{A \cap B \setminus \{\circ\}}(w_{1,1}, \ldots, w_{1,p}))$$

$$= \{x_i^0 \mid i \in \mathbb{N} \cap A \cap B\} \cup \emptyset$$

$$= \{x_i^0 \mid i \in \mathbb{N} \cap A\} \cup \begin{cases}
    \{c_1^0(w_{1,i}^0, \ldots, w_{1,i}^p) \mid w_{1,i}^p \in \ell(w_{1,i})\}, & \text{if } \circ \in A; \\
    \emptyset, & \text{otherwise}.
\end{cases}$$

$$\cap \begin{cases}
    \{c_1^0(w_{1,i}^0, \ldots, w_{1,i}^p) \mid w_{1,i}^p \in \ell(w_{2,i})\}, & \text{if } \circ \in B; \\
    \emptyset, & \text{otherwise}.
\end{cases}$$

$$= \ell(w_1) \cap \ell(w_2).$$

We can now describe the algorithm for finding linear expressions. Combining the lemmata above, we find that the set

$$\ell(w(\sigma_1, v_1) * \cdots * w(\sigma_n, v_n))$$
contains exactly the linear expressions coforming to the input–output table \( T \). Since
the definitions of \( \ell \), \( w \), and \( * \) deal with finite data in finite numbers of steps, we can
compute this set in its entirety by unfolding definitions, thus giving an algorithm.

6. A proof-of-concept implementation

We provide a proof-of-concept implementation which can be found at
\( \text{https://gitlab.com/maxvi/ssgp} \). Excluding comments and blank lines, it
consists of a little less than 2700 lines of Haskell code. Great care was taken
to document the API and every piece of code, amounting to over 1300 lines
of comments. Various QuickCheck tests are also included. We emphasize that
the implementation is a minimum viable product; for instance, it is completely
unoptimized (see Section 7.2).

The implementation deviates from a naïve implementation of the theoretical
ideas described above in a number of ways, many of which we shall cover in this
section. It produces code written in what is, sparing suppos expressions, a subset
of Haskell.

6.1. Results. We have chosen a few examples, tabulated below, which highlight
some strengths and weaknesses of our implementation.

| Name     | Signature       | Avail. funcs. | Convergence | Avg. iter. |
|----------|-----------------|---------------|-------------|------------|
| sum      | \( \mathbb{Z} \to \mathbb{Z} \) | \( + : (\mathbb{Z}, \mathbb{Z}) \to \mathbb{Z} \) | 5/5 = 100% | 149.4      |
| min      | \( (\mathbb{Z}, \mathbb{Z}) \to \mathbb{Z} \) | \( \leq : (\mathbb{Z}, \mathbb{Z}) \to \text{Bool} \) | 5/5 = 100% | 373.6      |
| min’     | \( \mathbb{Z} \to \mathbb{Z} \)     | \( \leq : (\mathbb{Z}, \mathbb{Z}) \to \text{Bool} \) | 5/5 = 100% | 1076.4     |
| insert   | \( (\mathbb{Z}, \mathbb{Z}) \to \mathbb{Z} \) | \( \leq : (\mathbb{Z}, \mathbb{Z}) \to \text{Bool} \) | 5/5 = 100% | 1485.6     |
| helloworld | \( () \to \mathbb{Z} \)       | \( \leq : (\mathbb{Z}, \mathbb{Z}) \to \text{Bool} \) | 5/5 = 100% | 2611.0     |
| sort     | \( \mathbb{Z} \to \mathbb{Z} \)     | \( \leq : (\mathbb{Z}, \mathbb{Z}) \to \text{Bool} \) | 0/5 = 0%   | NA         |
| reverse  | \( \mathbb{Z} \to \mathbb{Z} \)     | \( \leq : (\mathbb{Z}, \mathbb{Z}) \to \text{Bool} \) | 0/5 = 0%   | NA         |

Figure 6.1. Some benchmarks of our proof-of-concept implementation.

```haskell
insert :: Int -> [Int] -> [Int]
insert x0 (Cons[Int] x1 (Cons[Int] x2 x3)) =
  if f0 x0 x1 (Cons[Int] x2 x3) then
    Cons[Int] x1 (insert x0 (Cons[Int] x2 x3))
  else
    Cons[Int] x0 (Cons[Int] x1 (Cons[Int] x2 x3))
insert x0 Nil[Int] = Cons[Int] x0 Nil[Int]
insert x0 (Cons[Int] x1 Nil[Int]) =
  if f0 x0 x1 Nil[Int] then
    Cons[Int] x1 (Cons[Int] x0 Nil[Int])
  else
    Cons[Int] x0 (Cons[Int] x1 Nil[Int])
f0 :: Int -> Int -> [Int] -> Bool
f0 x0 x1 x2 = leq x1 x0
```

Figure 6.2. Example of code produced by our proof-of-concept implementation.
Each example was run five times. Each run was set to have a maximum of 10 cycles (see Section 5), after which we would declare the evolution to not converge. The constants used were \(N\text{\_stretch} = 3\) and \(N\text{\_mutate} = 300\). The last column records the average number of iterations (stretches, mutations, rewinds, compressions) that occurred before evolution converged to correct code. At each iteration, a new program is proposed and this program is then run on a pre-fixed set of examples through which local scores are assigned.

Every type is an Algebraic Data Type. Integers are encoded by their binary representation. Lists are defined recursively in a left-biased way as \(Z ::= \text{Cons}(Z, Z) \mid \text{Nil}\). There is no standard library; each example can only use the functions specified in the third column, nothing else. If one were to add more useful functions there, one would get better convergence rates.

Let us elaborate on the behavior of some of the tabulated functions. “\(\text{min}'\)” picks out the smallest number from a list, defaulting to 0 if the list is empty. “insert” inserts a number into a list of numbers before the first entry that is greater than or equal to the number. “helloworld” lists the ASCII codes for the string “Hello World”. The rest are self-explanatory.

6.2. Comparison with other works. Koza [Koz94] established the even-parity problem as a benchmark for Genetic Programming. The objective is to evolve code for the function \(\text{Bool} \rightarrow \text{Bool}\) that returns \textbf{True} if the list has even number of \textbf{True}s and \textbf{False} otherwise. Briggs and O’Neill [BO06] use this problem to compare a number of approaches found in literature. We reproduce their benchmarks below, adding a benchmark of our own proof-of-concept implementation.

| Approach               | Citation | Assessments |
|------------------------|----------|-------------|
| GP with Local Scoring  | This paper | 882         |
| Exhaustive enumeration | [BO06]   | 9478        |
| PolyGP                 | [Yu99]   | 14,000      |
| GP with combinators    | [BO06]   | 58,616      |
| GP with iteration      | [Kir01]  | 60,000      |
| Generic GP             | [WL96]   | 220,000     |
| OOGP                   | [AL06]   | 680,000     |
| GP with ADFs           | [Koz94]  | 1,440,000   |

Figure 6.3. Comparison of the smallest number of program assessments needed for 99% convergence rate on the even-parity problem across various approaches. Smaller numbers are better. (No claims are made as to exhaustiveness of the listed approaches.)

The last column records estimates for the smallest number of program assessments needed during evolution for an algorithm to find a correct solution with 99% probability. A program assessment scores a program by running it on a number of examples.

We ran our proof-of-concept implementation 100 times on the even-parity problem. We used a Lenovo ThinkPad X1 Carbon with an Intel Core i7-8550U CPU. It took approximately 11 minutes in total. Every run eventually converged on correct code, with the average number of programs tested before convergence being 398.83.
The best run needed to test just 68 programs and the worst 940 programs. Generalizing from our data, we estimate that if we stop evolution after 882 programs, we arrive at a correct solution with 99% probability. The only built-in function made available from the onset was the binary operation, not $\text{Bool} \rightarrow \text{Bool}$.

6.3. **Local scores and deep recursion.** For scores $S(e)$, we use instead a weighted average in which each summand is weighted by the reciprocal of the call-stack depth. In effect, deep recursive calls have less influence on scores.

6.4. **Continuous scoring.** The Hello World example uses a “continuous” scoring function, meaning that it takes into account how close the string that `helloworld()` produces is to the desired string “Hello World”, instead of just rewarding 1 when correct and 0 when incorrect (in which case evolving the correct code would be virtually impossible). Local scoring means that if `helloworld()` returned the string “Hello Gorld”, we trace the problematic character, G, back to the expression that produced it and then change that expression little by little, inching in on the correct character, W.

6.5. **Timeouts and restrictions to recursion.** A fundamental issue is that the problem of bounding the running time of an arbitrary program in a Turing-complete language is undecidable as was famously shown by Turing in 1936 [Tur36]. To overcome this issue, we imposed restrictions on what is allowed during execution:

1. We impose a limit on how many function calls can be made in one program execution. If exceeded, the local score of every expression that was evaluated during the execution gets punished.

2. We impose restrictions on what forms of recursions are allowed. Specifically, we shall only allow a call $f(v_1, \ldots, v_n)$ if there is an $i$ such that for all calls $f(w_1, \ldots, w_n)$ on the call-stack, $|v_i| < |w_i|$. Here, $|v|$ denotes the “size” of a value $v$, that is, its number of nodes. If this condition is violated, the local score of every expression on the “expression stack” (the expression-level analogue of the call-stack) gets punished.

The restriction (2) alone ensures termination, as the following lemma shows.

**Lemma 6.1.** There is no infinite sequence $a_1, a_2, \ldots \in \mathbb{N}^n$ with the property that each $a_i$ has a coordinate $a_{i,j} \in \mathbb{N}$ such that $a_{i,j} < a_{i',j'}$ for any $i' < i$.

**Proof.** Assume for contradiction that such sequence $a_*$ exists. The sequence may be partitioned into $n$ subsequences, $p_{1,*}, \ldots, p_{n,*}$, such that for each $i = 1, \ldots, n$, $p_{i,k,i} < p_{i,k',i}$ for any $k' < k$. Since $a_*$ is infinite, one of these subsequences, say $p_{j,*}$, is infinite. But then $p_{j,1,j} > p_{j,2,j} > \cdots$ is an infinite descending sequence in $\mathbb{N}$, which is not possible. $\square$

6.6. **Controlled Recursion.** When a recursive function is broken, it is usually catastrophically broken. This is because errors propagate (and multiply) upwards through recursion. This affects negatively the quality of the local scores which we want to reflect “closeness” to a solution. For example, if we were to define natural numbers inductively using successors, $\mathbb{N} := 0 \mid S(\mathbb{N})$, then the recursive definition

- $0 \oplus m := m$
- $S(n) \oplus m := n \oplus m$
is only one \( S \) away from being the correct definition of addition \((S(n) \oplus m = S(n \oplus m))\), yet in terms of actual behavior it is quite far away: \( n \oplus m = m \). To deal with this issue, we employ a technique we shall call \textit{Controlled Recursion}, in which we answer recursive calls using training data. For example, if we were trying to evolve \( \oplus \) to be addition, we would, when producing the values \( w_1, \ldots, w_N \) in Section 2.2 evaluate the recursive call \( n \oplus m \) in the body of the second equation above as \( n + m \) instead of using the actual code for \( \oplus \). That means that the output only differs by 1 from the desired output, and the score is accordingly better (that is, if we use a continuous scoring function as in Section 6.4) than if we were to answer recursive calls using the actual code. To discourage degenerate self-referential definitions like \( S(n) \oplus m := S(n) \oplus m \), we only answer recursive calls like this that are “smaller” than the calls on the call-stack, a la the restriction (2) we imposed in Section 6.3.

6.7. \textbf{Stretches.} Since the examples did not appear to need it, we disabled the stretch (4) described in Section 5.1. Moreover, for stretch (1), we only introduce \textit{suppose}-expressions at the top of bodies of equations.\footnote{It would probably be better to allow them to be introduced wherever but have a mechanism pushing them up the expression tree (“generalization”) before they are promoted into \textit{if}-conditionals so as to not have to reinvent the same conditions for various subexpressions.}

6.8. \textbf{Limitations.} A significant issue is the performance suboptimality of our proof-of-concept implementation. With optimized code execution, we could perhaps try many times more mutations in the same amount of time, thereby searching much deeper for new mutations. We discuss this further in Section 7.2.

Another striking issue is that of overfitting. For example, overfitting obstructs discovery of correct code for the function \textit{reverse}: \([\mathbb{Z}] \rightarrow [\mathbb{Z}] \) reversing the items of a list. Instead of attacking the problem algorithmically, it generates more and more cases: one for the empty list \([\ ]\), one for \([x_1]\), one for \([x_1, x_2]\), and so on. While it obviously does get closer to a correct solution in terms of behavior, it does not approach a correct solution in terms of code. The issue is that lists are defined recursively in a left-biased way, making operations on the beginning of the list far easier to discover than operations on the end.

A human is able to immediately tell that such code is bad style; perhaps if one taught the same intuition to a neural network, that could be used to stir away from overfitting. See Section 7.3 for more discussion.

Overfitting was not an issue in any of the converging examples from Section 6.1.

7. \textbf{Extensions and future work}

7.1. \textbf{Crossover and parallelism.} In contrast to a lot of the literature (see Section 6.2), we use what is a kind of hill-climbing algorithm: at every point in the algorithm, we have exactly one program. It is typical to instead evolve a population of programs, mutating them and using a procedure called \textit{crossover} to combine two programs producing one or more offspring programs. We remark that this could be added on top of our hill-climbing method, making for highly parallelized Genetic Programming: one could do \( C \) cycles of hill-climbing (see Section 5) for each program in the population \( \{P_1, \ldots, P_n\} \) in parallel, yielding a new population \( \{P'_1, \ldots, P'_n\} \) which is then extracted for crossover combining programs at random and killing off all but the \( n \) best resulting programs. This procedure is repeated
until a solution is found. With this, many hill-climbing processes run in parallel but still “share discoveries” once in a while, avoiding too much duplicated effort.

7.2. **Optimizations.** By far the most significant component performance-wise is that which executes programs. As we were more focused on the theoretical background, no serious attempt at optimizing this was made in our proof-of-concept implementation written in Haskell. Let us discuss some techniques and ideas that could be used in a performant implementation.

1. Use a more performant language such as Rust or C.
2. Currently, we represent code as an Abstract Syntax Tree which is quite inefficient for a number of reasons. For instance, execution requires dereferencing a lot of pointers resulting in many cache misses. A more efficient, flat representation of code, such as a stack machine, could be used instead.
3. Considering that the same program is executed many times ranging over the different test cases, it would make sense to first compile the program to (optimized) machine code. Compilation is typically a heavy process and it would therefore be preferable if only the part of the program that changed was recompiled (incremental recompilation).
4. Since we are running the same (or similar) code again and again and do possibly overlapping recursion, it would be beneficial to make heavy use of memoization, trading CPU cycles for increased memory usage.
5. If the programs were instead executed with respect to a lazy evaluation strategy, programs that contain bloat would execute much faster. For example, calling a function and passing its return value to another function in an argument that is unused would then not have any effect on performance. In some instances, this would result in an exponential speedup: in our experiments, it was a common occurrence that correct but extremely slow programs would evolve quickly with the majority of evolution then spent on reducing the exponential running time that is due to unused evaluations, as such exponential running time hinders getting perfect scores on some of the bigger test cases due to the timeouts described in Section 6.5.
6. When testing a mutation $e \rightarrow e'$, it is only necessary to recompute a $w_i$ (see Section 2.2) if it contains the tag of $e$ in one of its traces, yet we recompute all.
7. A significant amount of time is spent on testing out bad mutations, most of which are scoring extremely poorly. One could use heuristics for quickly discarding large swaths of bad mutations with only few false positives. Local Scoring is particularly apt for this. For example, if one was to test a mutation $e \rightsquigarrow e'$ of an expression $e$, one could first compute the score of $e'$ with respect to a small, randomly chosen subset of $w_1, \ldots, w_N$ (but only selecting necessary $w_i$'s as in (6)), and then preemptively discard the mutation if this “approximate score” is significantly smaller than $S(e)$.
8. While our proof-of-concept implementation does ensure that the exact same program is not tried multiple times, one could employ more advanced means of ensuring that equivalent mutations are not attempted multiple times. One would need to specify algorithms that are able to tell if a mutation on

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10 Or perhaps, instead of $S(e)$, it would be better to use the corresponding approximate score computed for $e$. This would necessitate storing not just the local scores but also all summands of the averages, so that averages of subsets can be computed at a later point.
a program $P$ corresponds to a mutation on a program $P'$ equivalent to $P$ by a number of stretches (see Section 5.1). Perhaps a hash function could be devised for this purpose.

7.3. Integration with Deep Learning methods. We are choosing new code to try out essentially at random instead of doing so intelligently and drawing from an intuitive understanding of what changes are needed in what situations. Recently, there has been major breakthroughs on the subject of using Deep Learning methods to generate code (e.g., [CTJ+21] and [LCC+22]). An interesting direction would be to hybridize the approaches, combining the overview of Local Scoring with the intuition of Deep Learning. Let us speculate a bit on what that could look like.

1. One could incorporate into the scores an assessment of the “plausibility” of code, as determined by a Deep Learning model, rewarding code that looks like human-written code. This could help guide the search in directions of known patterns, combating issues like overfitting and getting stuck in local minima.

2. What kind of mutation or stretch to do could be chosen using a model that is trained (on a fixed training set of code) to predict what is most likely to cause improvements. Deep Learning-based code generation could be used to propose candidates for the replacing expressions that we use when mutating instead of generating these at random.

3. Additionally, since we use mutation conservativity (see Section 4.2), there will typically be long periods of attempting mutations before a successful mutation is found. One could collect data from these failed mutations. An idea is to then use Deep Learning to analyze these attempted mutations and their resulting local scores in order to get a sense of which mutations do well, which could then be leveraged to pick out qualified guesses for new mutations making use of problem-specific knowledge. Perhaps, one could even train a model during evolution on this data to predict what mutations are likely to be successful.

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