The Role of Symmetry in Mathematics

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Abstract

Over the past few decades the notion of symmetry has played a major role in physics and in the philosophy of physics. Philosophers have used symmetry to discuss the ontology and seeming objectivity of the laws of physics. We introduce several notions of symmetry in mathematics and explain how they can also be used in resolving different problems in the philosophy of mathematics. We use symmetry to discuss the objectivity of mathematics, the role of mathematical objects, the unreasonable effectiveness of mathematics and the relationship of mathematics to physics.

1 Introduction

Different philosophical conceptions of the nature of mathematics are designed to account for our intuitions about the ontology, epistemology, and semantics of mathematics. The account we provide in this paper gives a novel and unified account of what mathematics is, why we are certain of mathematics, and why we see the semantics of mathematics in line with the semantics of scientific discourse. We show how we can address these foundational questions by thinking about mathematics as satisfying a certain set of symmetries. It will also provide us with an assessment of the relationship between mathematics and science. We do this by providing a naturalistic conception of mathematics that is also sensitive to contemporary mathematical practice.

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Our account relies on the notions of symmetry that have lately held deep interest for physicists and philosophers of science. Researchers begin by inquiring into what it means for laws of physics to satisfy symmetries. Typically, a law satisfies a particular symmetry if it holds despite a change in some parameter of the set of phenomena covered by the law. An obvious symmetry is invariance with respect to space which dictates that if the location at which an experiment is performed is changed the results of the experiment will nonetheless be the same. Another symmetry, invariance with respect to time, mandates that the results of an experiment will stay the same even if the time that an experiment is performed changes. Section 2 elaborates and discusses other more abstract forms of symmetry.

Einstein changed physics forever by taking these ideas in a novel direction. He showed that rather than looking for symmetries that given laws satisfy, physicists should use symmetries to construct the laws of nature. This makes symmetries the defining property of the laws instead of an accidental feature. These ideas were taken very seriously by particle physicists. Their search for forces and particles are essentially searches for various types of symmetries. Philosophers have used these ideas to guide several topics in the philosophy of science. In particular, the symmetries explain the seeming objectivity of the laws of physics. Since the laws of physics are invariant with respect to space, the laws will be the same wherever they are examined. Since the laws of physics are invariant with respect to time, they appear timeless and universal. Some philosophers go further and even question whether the laws of physics actually exist. They posit that rather than there being laws of physics in nature, the physicist is really selecting those families of phenomena that seem to be universal and calling them laws of nature.

Our goal here is to show that this is true for mathematics as well. The notion of symmetry we focus on we call symmetry of semantics. Mathematical statements are about abstract or concrete entities of a certain type. Symmetry of semantics says that if the entities are appropriately exchanged for other entities, then the truth value of the mathematical statement remains the same. To a model theorist or a logician this notion is related to validity. Our point is that this type of validity is a form of symmetry. It allows some aspect of the mathematical statement to vary while leaving its truth value intact.

As with the symmetries in physics, the symmetries in mathematics help us deal with different issues in the philosophy of mathematics. We discuss the seeming objectivity of mathematics, foundations of mathematics, unification of different mathematical fields, the creation of new mathematical fields, and several other topics. We also discuss the very practice of “doing mathematics” from the point of view of symmetry. We discuss the philosophical consequences of the symmetries in mathematics and also discuss their relationship with the symmetries of physics. This gives us a novel explanation of the unreasonable effectiveness of mathematics in the natural sciences as well as the general relationship between mathematics and the hard sciences. To foreshadow, we show that the symmetries of physics are, in essence, a subset of the symmetries of mathematics. Numerous consequences are derived from this view.
Category theory is a branch of mathematics that deals with structures and functions that can be seen as changing or warping those structures. It is with this ability to warp structures that the types of symmetries we are interested in become evident. In particular, there is a result in higher algebra and category theory that reconstructs an ideal structure from the symmetries in the category of representations/algebras of the ideal structure. This is, to a certain extent, analogous to what physicists do when they reconstruct the ideal law of physics from the symmetries of the phenomena they find in the real world. We hint at this analogy in the final section of this paper.

Our paper proceeds as follows: in Section 2 we briefly describe the prominent and evolving role that symmetry plays in the development of physics. In section 3 we discuss the kinds of symmetries found in mathematics. Section 4 uses these symmetry considerations to address some important philosophical questions while staying true to mathematical practice. Section 5 illustrates this with a brief mathematical discussion of the importance of the role of symmetries in category theory.

2 The role of symmetry in physics

A reasonable history of physics can be given in terms of the ever-expanding place for symmetry in understanding the physical world. We briefly outline this expanding history from its origins in the classification of crystals to its role in explaining and defining all the “laws of nature.”

“Symmetry” was initially employed in science as it is in everyday language. Bilateral symmetry, for example, is the property an object has if it would look the same when the left and the right sides are swapped. In general an object has symmetry if it appears the same when viewed from different perspectives. A cube thus has six-sided symmetry while a sphere is perfectly symmetrical because it looks the same from any of its infinite positions.

Pierre Curie formulated one of the earliest symmetry rules about nature. He showed that if a cause has a certain symmetry, then the effect will have a corresponding symmetry. Although Curie was mostly concerned with crystals and the forces that create them, the “Curie Symmetry Principle” has been employed throughout physics.

Physicists have generalized the term “symmetry” from descriptions of objects to descriptions of laws of nature. A law of nature exhibits symmetry when we can transform the phenomena under the scope of the law in certain ways and still make use of the same law to get a correct result. We say that such a law is “invariant” with respect to those transformations.

An early noticed example of a symmetry exhibited by a law of nature is the fact that the results of an experiment remain correct when the location of an experiment is changed. A ball can be dropped in Pisa or in Princeton and the time needed for the ball to hit the ground will be the same (all other relevant conditions held fixed).

\[^2\]For philosophical introductions to symmetry see the Introduction in [BC03], [BC07], [BC08] and [Ban12]. For a popular introduction to the physical issues see [LH04].
factors being equal). We thus say that the law of gravity is invariant with respect to location. This fact about locations of experiments was so obvious and taken for granted that scientists did not notice or articulate it as a type of symmetry for some time.

Similarly, the laws of nature are invariant with respect to time. A physical process can be studied today or tomorrow and the results will be the same. The orientation of an experiment is also irrelevant and we will get identical results (again, \textit{ceteris paribus}) regardless of which way the experiment is facing.

Classical laws of motion formulated by Galileo and Newton display more sophisticated symmetries called “Galilean invariance” or “Galilean relativity.” These symmetries show that the laws of motion are invariant with respect to an observation in an inertial frame of reference — the laws of motion remain unchanged if an object is observed while stationary or moving in a uniform, constant velocity. Galileo elegantly illustrates these invariances describing experiments that can be performed inside a closed ship (Gal53). Mathematicians have formulated this invariance by studying different transformations of reference frames. A “Galilean transformation” is thus a change from one frame of reference to another that differs by a constant velocity: \( x \rightarrow x + vt \), where \( v \) is a constant velocity. Mathematicians realized that all these Galilean transformations form a group which they called the “Galilean group.” In broad terms then, the laws of classical physics are invariant with respect to the Galilean group.

To see what it means for laws of physics to be invariant with respect to Galilean transformations consider the following. Imagine a passenger in a car traveling a steady 50 miles per hour along a straight line (neither accelerating nor decelerating). A passenger is throwing a ball up and catching it when it comes down. To the passenger (and any other observer in the moving car) the ball is going straight up and straight down. However to an observer standing still on the sidewalk, the ball leaves the passenger’s hand and is caught by the passenger’s hand but it does not go straight up and down. Rather the ball travels along a parabola because the ball goes up and down while the car is also moving forward. The two observers are not observing different laws of physics in action. The same law of physics applies to them both. But when the stationary observer sees the ball leaving the passenger’s hand, it does not only have a vertical component. Since the car is also moving forward at 50 miles per hour the ball’s motion has a horizontal component too. The two observers see the phenomena from different perspectives but the results of the laws must be the same. Each observer must be able to use the same law of physics to calculate where and when the ball will land despite the fact that they make different observations. Thus, the law is invariant with respect to the ability to swap the two perspectives and still get the same answer. The law is symmetric. One can view this with the aid of the following commutative diagram:
The top part shows the ability of the two perspectives to be swapped. Each observer can calculate the ball’s trajectory and both of them must come to the same conclusion about the location of the ball when it lands.

Another way of expressing this is to say that observers cannot determine whether they are moving at a constant velocity or standing still just by looking at the ball. The laws of physics cannot be used to differentiate between them because the laws operate identically from either perspective.

One of the most significant changes in the role of symmetry in physics was Einstein’s formulation of the Special Theory of Relativity (STR). When considering the Maxwell equations that describe electromagnetic waves Einstein realized that regardless of the velocity of the frame of reference, the speed of light will always appear to be traveling at the same rate. Einstein went further with this insight and devised the laws of STR by postulating an invariance: the laws are the same even when the frame of reference is moving close to the speed of light. He found the equations by first assuming the symmetry. Einstein’s radical insight was to use symmetry considerations to formulate laws of physics.

Einstein’s revolutionary step is worth dwelling upon. Before him, physicists took symmetry to be a property of the laws of physics: the laws happened to exhibit symmetries. It was only with Einstein and STR that symmetries were used to characterize relevant physical laws. The symmetries became a priori constraints on a physical theory. Symmetry in physics thereby went from being an a posteriori sufficient condition for being a law of nature to an a priori necessary condition. After Einstein, physicists made observations and picked out those phenomena that remained invariant when the frame of reference was moving close to the speed of light and subsumed them under a law of nature. In this sense, the physicist acts as a sieve, capturing the invariant phenomena, describing them under a law of physics, and letting the other phenomena go.

The General Theory of Relativity (GTR) advanced the relevance of symmetry further by incorporating changes in acceleration. Starting with the advent of GTR Einstein postulated that the laws of nature be understood as invariant even when acceleration is taken into account, i.e. if the observer is accelerating.

In 1918 symmetry became even more relevant to (the philosophy of) physics when Emmy Noether proved a celebrated theorem that connected symmetry to the conservation laws that permeate physics. The theorem states that for every continuous symmetry of the laws of physics, there must exist a related
conservation law. Furthermore, for every conservation law, there must exist a related continuous symmetry. For example, the fact that the laws of physics are invariant with respect to space corresponds to conservation of linear momentum. The law says that within a closed system the total linear momentum will not change and the law is “mandated” by the symmetry of space. Time invariance corresponds to conservation of energy. Orientation invariance corresponds to conservation of angular momentum, etc (see e.g. [Fey67] Ch. 4, [Wei92] Ch VI, and [Ste06] for discussion). Noether’s theorem had a profound effect on the workings of physics. Whereas physics formerly first looked for conservation laws, it now looked for different types of symmetries and derived the conservation laws from them. Increasingly, symmetries became the defining factor in physics.

Currently, particle physics is one of the more interesting fields where symmetries are sought and found. The field originally postulated three symmetries: a parity invariance with respect to space reflection that lets us swap right and left, a translation from going one way in time to the other, and the charge replacement of a particle with a corresponding anti-particle. Particle physics continues the effort to find more and more abstract symmetries such as gauge symmetry. The idea is to allow the laws of physics to remain the same no matter how the phenomena are described.

The physicist Victor Stenger unites the many different types of symmetries under what he calls “point of view invariance.” That is, all the laws of physics must remain the same regardless of how they are viewed. Stenger ([Ste06]) demonstrates how much of modern physics can be recast as laws that satisfy point of view invariance. We can visualize this with a generalization of the previous commutative diagram.

**Point of View Symmetry**

The top part shows the ability of the two perspectives to be swapped. Each perspective can be used to calculate the process of the physical phenomena and both must get the same result.

Symmetry also plays a role in more speculative areas of physics. Our best way forward beyond the standard model are attempts to unify all interactions in nature. One of them, supersymmetry, postulates that there is a symmetry that relates matter to forces in nature. Supersymmetry requires us to postulate the existence of a partner matter particle for every known particle that carries a force, and a force particle for every matter particle. The idea here is that the laws of physics are invariant if we swap all the matter for all the force. None of
the partner particles have yet been discovered, but because they are mandated by the symmetries it is what scientists are looking for.

Symmetry, as we have described it, is only part of the story. In numerous cases a law of physics actually violates a symmetry law and breaks into several different laws via a mechanism known as “symmetry breaking.” These broken symmetries are as conceptually important as the symmetries themselves. The way a symmetry breaks determines certain constants of nature. But the question of why a symmetry should break in one way and not another is not presently understood. Researchers are at a loss when they leave the constraints set by symmetry.

Recent excitement over the discovery of the Higgs boson reveals a triumph of the role of symmetry in physics. Scientists postulated that there was a symmetry in place at the time of the big bang, and it was only when this symmetry was “broken” via the “Higgs mechanism” that it was possible for mass to exist. By discovering the Higgs boson physics was able to provide the mechanism by which mass was produced out of the perfect symmetry of the initial state of the universe. The Higgs mechanism was postulated only on the strength of the presumed symmetries. The recent discovery of the Higgs boson as the culmination of an extensive research program has further vindicated the methodology of postulating symmetries to discover fundamental properties of the universe.

Physics also respects another symmetry, which as far as we know has not been articulated as such. The symmetry we refer to is similar to the symmetry of time and place that was obvious for millennia but not articulated until the last century. Namely, a law of physics is applicable to a class of physical objects such that one can exchange one physical object of the appropriate type for another of that type with the law remaining the same. Consider classical mechanics. The laws for classical mechanics work for all medium sized objects not moving close to the speed of light. In other words, if a law works for an apple, the law will also work for a moon. Quantities like size and distance must be accounted for, but when a law is stated in its correct form, all the different possibilities for the physical entities are clear, and the law works for all of them. We shall call this invariance for a law of nature, symmetry of applicability, i.e. a law is invariant with respect to exchanging the objects to which the law is applied. We shall see later that this is very similar to a type of symmetry that is central to mathematics.

To sum up our main point, philosophically the change in the role of symmetry has been revolutionary. Physicists have realized that symmetry is the defining property of laws of physics. In the past, the “motto” was that a law of physics respects symmetries. In contrast, the view since Einstein is: That which respects symmetries is a law of physics. In other words, when looking at the physical phenomena, the physicists picks out those those that satisfy certain symmetries and declares those classes of phenomena to be operating under a law of physics. Stenger summarizes this view ⁴See [Ban88] for a related discussion of the discovery of the Ω⁻ particle.
as follows “...the laws of physics are simply restrictions on the ways physicists may draw the models they use to represent the behavior of matter” ([Ste06]: 8). They are restricted because they must respect symmetries. From this perspective, a physicist observing phenomena is not passively taking in the laws of physics. Rather the observer plays an active role. She looks at all phenomena and picks out those that satisfy the requisite symmetries.

This account explains the seeming objectivity of the laws of physics. In order for a set of phenomena to fall under a single law of physics, it must hold in different places, at different times, be the same from different perspectives, etc. If it does not have this “universality,” then it cannot be a law of physics. Since, by definition, laws of nature have these invariances, they appear independent of human perspective. Symmetry thus became fundamental to the philosophical question of the ontology of laws of physics.

3 The role of symmetry in mathematics

Consider the following three examples:

(1) Many millennia ago someone noticed that if five oranges are combined with seven oranges there will be twelve oranges in total. It was also noticed that when five apples are combined with seven apples there is a total of twelve apples. That is, if we substitute apples for oranges the rule remains true. In a leap of abstraction, a primitive mathematician formulated a rule that in effect says $5 + 7 = 12$. This last short abstract statement holds for any objects that can be exchanged for oranges. The symbols represent any abstract or real entities such as oranges, apples, or manifolds. A similar commutative diagram can be used to illustrate this.

\[\begin{array}{c}
\text{statement about oranges} \\
\downarrow \\
\text{exchange} \\
\downarrow \\
\text{evaluate} \\
\downarrow \\
\text{truth value}
\end{array}\]

\[\begin{array}{c}
\text{statement about apples} \\
\downarrow \\
\text{evaluate} \\
\downarrow \\
\text{truth value}
\end{array}\]

\textbf{Symmetry of Fruit Exchange/}
\textbf{Symmetry of Applicability}

The top part shows the ability to swap apples for oranges. Each statement can be evaluated and must produce the same truth value.

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Whether or not there are laws of nature at all or whether they should be eliminated in favor of symmetries in a matter of considerable controversy among philosophers of science. See van Fraassen ([Fra89]) and Earman ([Ear04]) for stronger and weaker versions of eliminationist views on this issue. Our account is agnostic about this.
Ancient Egyptians studied different shapes in order to measure the earth so that they can have their taxes and inheritance properly assessed. Their drawings on papyrus represented shapes that could be used to divide up the fields on the banks of the Nile. Archimedes could make the same shapes with sticks in the sand. Drawings can accurately describe properties of these shapes regardless of what they represent: plots of land, Mondrian paintings, shipping containers, or whatever.

In modern times, mathematicians talk about numerous geometrical or topological theorems such as the Jordan Curve Theorem. This statement says that any non-self-intersecting (simple) closed continuous curve (like an oval) in the plane splits the plane into two regions, an “inside” and an “outside.” If you exchange one curve for another you will change the two regions. The curve could represent a children’s maze or a complicated biological drawing and the Jordan Curve Theorem remains the same.

One of the central theorems in algebra is Hilbert’s Nullstellensatz. This says that there is a relationship between ideals in polynomial rings and algebraic sets. The point is that for every ideal, there is a related algebraic set and vice versa. In symbols:

$$I(V(J)) = \sqrt{J}$$

for every ideal $J$. If you swap one ideal for another ideal, you get a different algebraic set. If you change the algebraic set, you get a different ideal. This theorem relates the domains of algebra and geometry and is the foundation of algebraic geometry.

In these examples we made use of ways of changing the semantics (referent) of mathematical statements. We swapped oranges for apples, changed shapes, transformed curves, and switched ideals. Our central claim is that this ability to alter what a mathematical statement denotes is a fundamental property of mathematics. Of course not all transforms are permitted. If we swap some of the oranges for some of the apples, for example, we will not necessarily get the same true mathematical statement. If we substitute a simple closed curve for a non-simple closed curve (like a figure 8), the Jordan Curve Theorem will not hold true. Such transformations are not legal. We can only change what the statement means in a structured way. Call this structured changing that is permitted a uniform transformation. Our main point is that this uniform transformation and the fact that statements remain true under such a transformation is a type of symmetry. Recall, a symmetry allows us to change or transform an object or “law” and still keep some vital property invariant. If a mathematical statement is true, and we uniformly transform the referent of the statement, the statement remains true. Mathematical statements are invariant with respect to uniform transformations. We call the property of mathematical statements that allows it to be invariant under a change of referent symmetry of semantics. The truth value of the mathematical statement remains the same despite the change of semantic content.

Symmetry of semantics can be illustrated with the following familiar dia-
The top part shows the ability to swap connotations of mathematical statements with any two elements of the domain of discourse. Each statement can be evaluated and must arrive at the same truth value.

Specifically, what types of transformations are uniform transformations? First, the entities we swap must be part of a certain class of elements. Every mathematical statement defines a class of entities which we call its domain of discourse. This domain contains the entities for which the uniform transformation can occur. When a mathematician says “For any integer $n$ . . .”, “Take a Hausdorff space . . .”, or “Let $C$ be a cocommutative coassociative coalgebra with an involution . . .” she is defining a domain of discourse. Furthermore, any statement that is true for some element in that domain of discourse is true for any other. A uniform transformation is one in which one element in the domain is substituted for another. Notice that the domain of discourse for a statement can consist of many classes of entities. Each statement might have $n$-tuples of entities, like an algebraically closed field, a polynomial ring and an ideal of that ring. Every mathematical statement has an associated domain of discourse which defines the entities that we can uniformly transform.

Different domains of discourse are indicative of different branches of mathematics. Logic deals with the classes of propositions while topology deals with various subclasses of topological spaces. The theorems of algebraic topology deal with domains of discourses within topological spaces and algebraic structures. One can (perhaps naively) say that the difference between applied mathematics and pure mathematics is that, in general, the domains of discourse for applied mathematical statements are usually concrete entities while the domains of discourse for pure mathematical statements are generally abstract entities.

With the concept of domains of discourse in mind one can see how variables are so central to mathematical discourse and why mathematicians from Felix Klein to Tarski, Whitehead, Frege, Russell, and Peano all touted their import for mathematics. Variables are placeholders that tell how to uniformly transform referents in statements. Essentially, a variable indicates the type of object that is being operated on within the theory and the way to change its
value within the statement. For example in the statement

\[ a \times (b + c) = (a \times b) + (a \times c) \]

which expresses the fact that multiplication distributes over addition, the \( a \) shows up twice on the right side of the equation. If we substitute something for \( a \) on the left, then, in order to keep the statement true, that substitution will have to be made twice on the right side. In contrast to \( a \), the \( b \) and \( c \) each occur once on both sides of the equation. Again, the variables show us how to uniformly transform the entities.

The values of the variables, for us, are mathematical objects. They are any entities in a domain of discourse defined by a mathematical statement. So oranges, apples, and stick drawings in the sand are mathematical objects. As long as we can transform those objects into other objects within the same domain of discourse they are mathematical objects. We can transform seven oranges into the elements of the set \( 7 = \{0, 1, 2, 3, 4, 5, 6\} \) and give equal status to each of them as mathematical objects. Mathematicians prefer to use 7 because of the generality it connotes. But this is misleading. Seven oranges are just as good at representing that number in any mathematical statement. Any statement about the number seven can be made with a transformation of the elements from the set of seven oranges. The mathematical statements made by applied mathematicians are no less true than the statements made by pure mathematicians. Concrete models of mathematical theories are just as good as abstract models.

Symmetry of semantics should look familiar to logicians and model theorists as the definition of validity. A logical formula is valid if it is true under every interpretation. That is, it must be true for any object in the domain of discourse. Thus, symmetry of semantics is not a radical idea. Rather, the novelty is viewing validity as a type of symmetry. We shall see that this symmetry is as fundamental to mathematics as many symmetries are to physics.

Now that we see how mathematics satisfies symmetry of semantics, let us return to the analogy with physics. Rather than understanding mathematical statements as satisfying symmetry of semantics, we argue that it is that which satisfies these symmetries that we call mathematics. As with physics, in the past whereas we used to understand that:

A mathematical statement satisfies symmetry of semantics.

we now claim that:

A statement that satisfies symmetry of semantics is a mathematical statement.

In other words, given the many expressible statements a mathematician finds, her job is to choose and organize those that satisfy symmetry of semantics. In contrast, if a statement is true in one instance but false in another instance, then it is not mathematics. In the same way that the physicist acts as a “sieve” and

\(^5\)Frege’s influence on this definition should be evident. A finite number for Frege consists of the equivalence class of the finite sets where two sets are equivalent if there is an isomorphism from one set to another. When we talk of the equivalence class 5 we are ignorant of which set of the equivalence class we are discussing. Are we talking about 5 apples or 5 cars?
chooses those phenomena that satisfy the required symmetries to codify into physical law, so too the mathematician chooses those statements that satisfy symmetry of semantics and dubs it mathematics.

Many statements in general do not satisfy symmetry of semantics; statements containing vague words, for example. Even some mathematical-sounding statements such as “If x is like y and y is like z, then x is like z” simply fail in most cases because “like” is not exact enough to be part of mathematics.

One may object to this view by saying that allowing mathematics to be whatever satisfies symmetry of semantics is too inclusive. Many general statements not traditionally thought of as mathematical also satisfy symmetry of semantics. For example “all women are mortal” is a general non-mathematical statement whereby any woman can be exchanged with any other in the connotation and the statement still satisfies symmetry of semantics. We agree that symmetry of semantics can be found such general statements, though we do not see this as an objection. We are addressing the properties of mathematics. Many branches of science, like applied mathematics, make use of general statements with domains of discourse that do not contain “traditional” mathematical objects. One would be hard pressed to find a good dividing line between theoretical physics and mathematics, theoretical computer science and mathematics, etc. Many statements in both pure and applied sciences do satisfy symmetry of semantics and to the extent that a branch of science is mathematical we expect it to have symmetry of semantics.

“We stress again, that many general or universal statements are in fact mathematics. From our perspective if the statement is strong enough so that it is true for every element of the implied domain of discourse, then it is mathematics. Nor do we shirk away from this definition. Mathematics and its many subdisciplines discuss many different types of objects. No one would say that mathematics is only about numbers and shapes. It is also about propositions, fluid flows, connections on vector bundles, chemical bonds, towels, apples, oranges, etc. Mathematics is about anything where one can reason in an exact manner in such a way that no element in its domain of discourse is exceptional. One might object that by this criterion nothing is outside of our definition of mathematics. This is false. Consider the following statements “all spoons are silverware” and “all silverware is metal”, therefore “all spoons are metal.” This is a general statement that is not part of mathematics. While it is part of everyday speech — and would be considered generally true — it is not mathematics. Some spoons are not silverware. There are plastic spoons that are neither silverware nor metal. These statements are not exact enough to be part of mathematics. If it was more exact, then the statements would in fact be part of a logic discussion and fall under the dominion of mathematics.”

Another symmetry that mathematics has we call symmetry of syntax. This says that any mathematical object can be described (syntax) in many different ways. For example we can write 6 as $2 \times 3$ or $2 + 2 + 2$ or $54/9$. The number $\pi$

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6Mark Steiner (Ste05) treats those fields as applications of mathematics.
can be expressed as $\pi = C/d$, $\pi = 2i \log \frac{1-i}{1+i}$, or the continued fraction

$$\pi = 3 + \cfrac{1}{7 + \cfrac{1}{15 + \cfrac{1}{1 + \cfrac{1}{292 + \cfrac{1}{1 + \cfrac{1}{1 + \ddots}}}}}.$$  

Similarly we can talk about a “non-self-intersecting continuous loop,” “a simple closed curve,” or “a Jordan curve” and mean the same thing. The point is that the results of the mathematics will be the same regardless of the syntax we use. Mathematicians often aim to use the simplest syntax possible, so they may write “6” or $\pi$ instead of some equivalent statement, but ultimately the choice is one of convenience, as long as each option is expressing the same thing.

These are not the only symmetries that mathematical statements satisfy. Some symmetries are taken for granted to such an extent that even mentioning them seems strange. For example, mathematical truths are invariant with respect to time and space: if they are true now then they will also be true tomorrow, if they are true in Manhattan they are true on Mars. It is similarly irrelevant who asserts a theorem or in what language a theorem is stated, or if it stated at all.

## 4 Some Philosophical Consequences

So far we have described various symmetries that pertain to mathematics. Let us now see what this contributes to our understanding of the problems in the philosophy of science and mathematics. We will consider several subjects starting with issues about foundations of mathematics.

**Foundations.** There are different aspects of the foundations of mathematics: Ontological foundations describe the constituent nature of mathematical objects. Epistemological foundations are tasked with explaining why mathematical statements are so convincing as compared to other areas of knowledge ([Mar95], [Azz05]) and why it appears objective. Methodological foundations describe (the) methodology common throughout mathematics.

We will clarify our foundational accounts, starting with epistemology. The epistemic confidence we have in our mathematics has its origins in an *a prioristic* concept of symmetry instantiated in mathematics. We are certain about mathematical results because we have decided *a priori* that the mathematically tractable entities we deal with are those entities that are amenable to uniform transformations. If they are not amenable to such transformations, then we have no reason to be certain they will behave the way we want and so we exclude them from mathematics. The fact that we already decided *a priori* how mathematics will work allows us to be certain that our results will turn out the way we expect.

But if certainty in mathematics comes from symmetry considerations and our symmetry considerations are the same as those in science, shouldn’t our
science provide the same certainty as mathematics? Why are we still less certain about physics than we are about mathematics? The reason is that science depends on capturing phenomena under a given symmetry. If we are not aware of some phenomenon or do not know how it is captured by a symmetry, our law will be incomplete or imperfect. Thus the anomalies that Newton was unaware of or could not handle were not subsumed in his system. Einstein found phenomena that were outside Newton’s domain of objects that he could swap in a universal transformation and had to describe an even “larger” symmetry that could accommodate them. As long as there are unexplored phenomena (like the perturbations in the perihelion of Mercury or near light speed objects) our certainty about science will be lower because we will not know the extent of the domains of discourse that we can uniformly transform or the relevant symmetry of applicability.

Aside for assuring certainty, any epistemology of mathematics should also explain the apparent objectivity of mathematics. Mathematical discoveries are sometimes made simultaneously by individuals working independently and the facts of mathematics are true in all places, times, and perspectives. This objectivity has led many to believe in the independent reality of mathematics and its objects just as it leads one to believe in the reality of the objects studies by physics. But, we need not make the leap from objectivity to realism. The recognition that symmetry is at the center of mathematical epistemology reveals that the objectivity of mathematics is an artifact of the way mathematics has been set up. By selecting only those statements that are invariant with regard to what a statement is referring to, the mathematician ensures that the statement is objective and universal. Analogous to Kant who saw mathematical objectivity as a form of intuitions about space and time, we see symmetry as the precondition under which mathematics is done. Symmetry undergirds both physics and mathematics. Since mathematical statements can by definition refer to so many different entities and since mathematics draws on so fundamental a principle, the entities appear to exist independently.

Our approach to the ontological foundations of mathematics should thus be clear. We have a metaphysically simple account. Any object that can be manipulated in a uniform transformation is a mathematical object. We need not appeal to the existence of abstract entities or structures to describe the nature of mathematical entities. We countenance anything from points to peppers as mathematical objects.

Does this thereby commit a kind of category error by calling a pepper a mathematical object? If not, have we just declared by fiat that, trivially, mathematical objects exist as we have identified them with ordinary objects?

We have argued that a mathematical object is any object that is amenable to mathematical treatment. Both seven apples and seven can occupy a domain of discourse. Occupying a domain of discourse is the only relevant criteria for the referent of mathematical statements. That is why it is not odd to assert that “an apple and an apple are two apples” and we call that a mathematical statement just as we might call “1+1 = 2” a mathematical statement. Moreover, because mathematical discourse appears so much like the discourse of ordinary
languages, it is a philosophical desideratum that we preserve a uniform semantics of our mathematical and ordinary languages ([Ben73]). “2 is bigger than 1” has the same structure and truth conditions as the “Empire State Building is bigger than the Chrysler building” and we expect that reflected in our semantics. And this is exactly what is done when we allow apples, squares, cars, and numbers to be mathematical objects. All can be swapped as part of a uniform transformation so it is not surprising that our theory of reference for looks the same for both natural language and mathematics.

But are we not deriving our ontology from our methodology? Just because we know what is amenable to mathematical treatment does not mean we know what a mathematical object is. Shouldn’t we still have to articulate a theory of what 7 is? Perhaps. But our ontology is neutral about that question. For 7 to exist means nothing above the fact that it can be swapped in a uniform transformation with other objects (or “objects”) in a way that preserves symmetry of semantics. That is a core property of mathematics. Once we account for the properties of mathematics in all contexts in which the property applies, demanding an account of what mathematics really deals with, is outside the scope of what mathematics could and should aim for. This is a mathematical analogue of what philosophers of science call empirical adequacy. Looking for the underlying homunculus or soul or constituent parts of the numbers, manifolds, sets, infinitesimals, or whatever, is all we need.

We should also clarify another aspect of our mathematical ontology. Prominent accounts of mathematical foundationalism claim that all mathematics can be built out of simpler mathematical stuff. For example some accounts claim that all mathematical objects can be built out of sets. But if as we claim, mathematics is not solely built out of real abstract entities like sets, what is the appeal of such reductionist accounts of mathematics? Those kinds of ontological reductions treat “foundation” in connection with mathematics as the ability to show that large parts (or even all) of mathematics can be phrased in some system, and that system is “primitive.” Commonly, since many parts of mathematics can be reduced to set theory and logic, and sets provide a convenient domain of discourse in everyday settings, set theory is taken to be a good candidate for such a foundational system.

This is presumably analogous to the conception of fundamental physics that seeks out the particles in which we, in theory, can express our fundamental ontological statements. This search for fundamental laws or fundamental particles is an important part of contemporary physics. But as we have shown, physics has largely abandoned the idea that programs that search for particles are the starting points for scientific research and theory. Instead we have the presumption that invariances are fundamental, which in turn allows for the discovery of fundamental particles. Thus we really only understand the workings of physics when we look at invariances, not particles. Particles may exist, but knowing about them does not give us insight into the nature of the rest of foundational physics. Programs regarding the foundations of mathematics began by confusing reduction and invariance in the same way, by looking at mathematical “particles” and not the invariances. We see mathematics as more similar to
science than is usually supposed, and thus apply the rules we would apply to science.

When we look for the starting points of mathematics we do not look for its fundamental pieces. One reason for this is that mathematics admits a variety of types of objects that have no reasonable expectation of reducing one to another (and in many cases it would anyway be unclear what reduces to what). Mathematics, in other words, has no fundamental parts, nor can we necessarily find one kind of object with which to phrase all the others. The vocabulary of uniform transformations (i.e. the vocabulary of mathematical methodology) on the other hand is the only way to talk about both abstract and concrete objects.

But still, can’t sets be the foundation? Doesn’t set theory also allow for the discussion of all kinds of objects? The reason we do not use sets as a foundation is that although typical mathematics can be reduced to sets, sets do not exhibit the correct kind of expressive power or display the right kind of symmetries in mathematics to be the fundamental “ground”. Sets only display the symmetry of mathematical objects. That is, set theory shows that all mathematical objects are the same in one way: they can all be “reduced” to the same thing. Since all mathematical objects are the same there are ways in which they can be treated similarly (akin to showing that all (non-fundamental) physical objects reduce to fundamental particles, it fails to deal with of all the other symmetries in nature). As an analogy, recall Frege’s definition of a finite number. In it, the equivalence class representing the number 7 happens to contain \( \{0, 1, 2, 3, 4, 5, 6\} \). But it also has \( \{T, U, P, W, Q, Y, R\} \). Insisting that sets are fundamental to mathematics is akin to insisting that every time we use a set with seven (7) elements we use \( \{0, 1, 2, 3, 4, 5, 6\} \). We can do that but that choice is arbitrary. What is key here is the whole equivalence class and the isomorphisms between the sets.

In an influential paper [Ben65] Paul Benacerraf describes two hypothetical children who are taught about the natural numbers in different ways. Ernie learns that the natural numbers 1, 2, 3... are identified with the sets \( \{\emptyset\}, \{\emptyset, \emptyset\}, \{\emptyset, \{\emptyset\}\} \).... Johnny learns to identify the natural numbers with the sets \( \{\emptyset\}, \{\{\emptyset\}\}, \{\{\{\emptyset\}\}\} \).... The moral of the hypothetical pedagogy is supposed to be that set theory, for example, cannot actually make sense of the myriad of “fundamental mathematical properties” because there are an infinite number such of set-theoretical reductions and there is no single set that corresponds to each number. But on our account this very concern is a symptom of confusing reduction and invariance, not a problem with a view of numbers. Benacerraf’s problem in other words, is exactly our point. The fact that we can swap \( \{\emptyset, \{\emptyset\}\} \) for \( \{\{\emptyset\}\} \) in a uniform transformation shows that set theory itself exhibits symmetry of semantics. However it says nothing about how sets or set theory are foundational. The fact that we can swap one set for another and understand both as 3 is not only unsurprising on our account but expected, because set theory is just another branch of mathematics that exhibits symmetry of semantics, like all the others. It is symmetry of semantics that is truly fundamental, not set theory.

Set theory does not show anything fundamental about numbers because it does not account for how we actually take mathematics to exhibit invariances.
Namely, we take mathematical objects to be invariant in a way that the objects stay the same under a wide range of rule transformations, not just object transformations. Therefore set theory initially appears intuitively like a ground for mathematics, but nonetheless fails, because set theory can do one thing that we expect of a physical reduction, namely exhibit something analogous to an ontological-type reduction of some mathematical objects. But the reduction is inadequate as it cannot capture what is really important (what is really mathematical) about mathematics.

As in physics, the fundamental nature of the objects is important, but they are ultimately derived from the symmetry considerations. Understanding what mathematics is tells us what mathematical objects are. As in physics, methodology generates ontology and not the reverse. Traditionally however this would be problematic because when we think about methodological foundations we think of the Euclidean model where mathematics requires an axiomatic system plus a system of deduction that allows for the generation or construction of the rest of mathematics (or its contemporary model theoretic version). But as Azzouni ([Azz05]) has argued, such a methodological foundation is untenable both because of Gödel’s problem of establishing proper axioms and because terms like “constitute” or “construct” are used metaphorically in mathematics and not literally, as there is no clear ontological hierarchy in mathematics. Symmetry on the other hand is a deeper methodological foundation neither countenancing an axiomatic nor ontological hierarchy, and thereby skirting these kinds of concerns.

Therefore, on this way of looking at mathematics, we need not see any branch of mathematics as ontologically fundamental. The ontology is secondary to what we take to be the methodological underpinnings of mathematics - the search for symmetries. We will return to methodology in the section on mathematical practice below.

**Wigner’s mystery and naturalism.** A philosophical naturalist’s interest in the philosophy of mathematics is the alignment of the ontology, epistemology, and especially methodology of mathematics with those of science. The account we have given is naturalistic as it has mathematics relying on the same *a priori* role of symmetry as fundamental physics. They both take up the idea that the starting point of inquiry are the postulated symmetries, not the “smallest pieces.”

This treatment of mathematics and science explains away the problem of the unreasonable effectiveness of mathematics in the natural sciences. Wigner ([Wig60]) (on one interpretation) has articulated his amazement at the fact that the physical science we discover is shockingly related to the mathematics we need to understand it. Often, science needs to articulate a physical concept and it turns to mathematics; the mathematics was there. Mark Steiner ([Ste92]: 154) sees one version of the problem as stemming from the apparent mismatch of methodologies. How can problems emerging from physics be articulated,
and even solved, using methods that were designed for a completely unrelated purpose?

A. Zee, completely independent of our concerns, has re-described the problem as the question of “the unreasonable effectiveness of symmetry considerations in understanding nature.” Though our notions of symmetry differ, he comes closest to articulating the way we approach Wigner’s problem when he writes that “Symmetry and mathematics are closely intertwined. Structures heavy with symmetries would also naturally be rich in mathematics” ([Zee90]: 319).

Understanding the role of symmetry however makes the applicability of mathematics to physics not only unsurprising, but completely expected. Physics discovers some phenomenon and seeks to create a law of nature that subsumes the behavior of that phenomenon. The law must not only encompass the phenomenon but a wide range of phenomena. The range of phenomena that is encompassed defines a set and it is that set which symmetry of applicability operates on. (Recall that symmetry of applicability allows us to exchange one object of a type for another of that type.) So a law must be deliberately designed with symmetry of applicability. Mathematics has a built in ability to express these symmetries because the symmetry of applicability in physics is actually just a subset of the symmetry of semantics. That is, the fact that we can exchange one object for another object when dealing with a physical law is simply a special case of exchanging one object for another object in a mathematical statement that expresses the physical law. There is then nothing surprising about the fact that there is some mathematics that is applicable to physics, as the symmetries of physics are a subset of the symmetries of mathematics. Any symmetry we find in physics should (already) be in mathematics.

For example Newton’s established law regarding the relationship of two bodies is

\[ F = G \frac{m_1 m_2}{r^2}. \]

Symmetry of applicability says that \( m_1 \) can correspond to the mass of an apple or of the moon and the formula still holds. Symmetry of semantics says that \( m_1 \) can be a small number (mass of an apple) or a large number (mass of the moon).

It is for this reason too that it is odd to say that mathematics is indispensable for physics (in the Quine-Putnam sense). Symmetry of applicability (in physics) is a subset of the mathematical symmetry of semantics. So it is not that mathematics is indispensable for our best scientific theories, but rather, they would not be our best scientific theories (or a recognizable scientific theory at all) if they could not be mathematized.

Our treatment demystifies why mathematics is so useful to the natural sciences. As we saw, the laws of physics are invariant with respect to the symmetry of applicability. This means that the laws can apply to many different physical entities. Symmetry of applicability is a type of symmetry of semantics. In detail, symmetry of applicability says that a law of nature can apply to many different physical entities of the same type. Symmetry of semantics says that
a mathematical statement can refer to many different entities in the same domain of discourse. When a physicist is formulating a law of physics, she will, no doubt, use the language of mathematics to express this law because she wants the law to be as broad as possible. Mathematics shares and increases this broadness. The fact that some of the mathematics could have been formulated long before the law of physics is discovered is not so strange. Both the mathematician and the physicists chose their statements to be applicable in many different contexts. So it is not that mathematics is unreasonably effective, but rather that if it were not effective, it would not be mathematics. The “mystery” of the unreasonable usefulness of mathematics melts away and supplies another advantage to heeding the role of symmetry in mathematics.

Mathematical practice: Our considerations about symmetry emerge directly from consideration of different aspects of mathematical practice. It is a sine qua non of any philosophical conception of mathematics that it square with the way mathematics actually works. If mathematicians cannot see their craft in an approach to mathematics, so much the worse for the approach.

The day-to-day job of the mathematician is proving theorems. Contrary to the impression given by the typical finished mathematics paper, mathematicians do not generally posit a theorem and then proceed to prove it from axioms. In reality a mathematician has an intuition and formulates some statement. The mathematician tries to prove this statement but almost inevitably finds a counterexample. A counterexample is a breaking (violation) of the symmetry of semantics; there is some element in the supposed domain of discourse for which the statement fails to be true. The mathematician then proceeds to restrict the domain of discourse so that such counterexamples are avoided. Again our indefatigable mathematician tries to prove the theorem but fails, so she weakens the statement. Iterating these procedures over and over eventually leads to a proven theorem. The final theorem may only vaguely resemble the original statement the mathematician wanted to prove. In some sense rather than saying that the “proof comes to the theorem” we might say that “the theorem meets the proof half way.” The mathematician acting as a “sieve” sorts out those statements that satisfy symmetry of semantics from those that do not, and only those that satisfy this symmetry are reported in the circulated and published paper. This is just another way of saying that the day-to-day work of the practicing mathematician involves looking for symmetries. Lakatos’ “rational reconstruction” ([Lak76]) may be cited as an example of this constant struggle to preserve the symmetry of semantics of the Euler formula.

As in physics, we observe symmetry on various levels. We do not just see symmetries being considered as when mathematicians construct individual proofs, but we see it on the level of the formulation of entire mathematical programs also. When symmetries in physics are discovered it is by relying on the idea that there are substantial domains in which transformations are allowed. It is widely recognized that physics progresses by unification. Unifying
an ever larger amount of allowable phenomena under a single given domain — as when Maxwell unified electrical theory and magnetism or when Newton united terrestrial and planetary mechanics — is how science advances. Similarly in mathematics symmetries are discovered when we find that seemingly different mathematical phenomena are really in the same category as an already known transformation and are thereby subsumed under a larger domain; we discover that a new larger class of entities can be uniformly transformed. In other words, we find that there is a union of different domains of discourses which were previously assumed to be comprised of non-interchangeable entities. The “monstrous moonshine” conjecture is one famous case of such unification. In the late 1970s John McKay noticed a completely unexpected relationship between the seemingly different areas of the “monster group” and modular functions. Legend has it that when McKay first heard that the number 196,884 appears in both areas, he shouted “moonshine” as a term of disbelief. Deeper connections between the monster group and modular functions have since been shown, advancing the respective branches of mathematics.

The field of algebraic topology provides another example of such a unification. Researchers realized that there is a certain similarity between taking maps between two topological spaces into account and taking homomorphisms between two groups into account. That is, there is a relationship between topological phenomena and algebraic phenomena. Mathematicians went on to use this similarity to try to classify certain topological structures. Category theory grew out of this unification and essentially became a tool for much more unification. Category theory has been derided as “general abstract nonsense” that is “about nothing.” But precisely because of that, it can be about everything. Hence its language can be used in many different areas of mathematics. The Langlands program is another example of unification. Beautifully described in Edward Frenkel’s Love and Math ([Fre13]), the Langlands program is a way of unifying the seemingly different fields of algebraic number theory and automorphic forms. As with symmetry, such unification advances mathematics by giving mathematicians an opportunity to discover more general theorems with wider applications and allows them to apply techniques from one domain to the other.\[8\]

Another way we see symmetry in mathematical practice is by analogy with symmetry breaking. Physical constants, for example, are what happened when symmetries broke the way they did. Symmetry breaking is also an important part of mathematical practice. Probably the first example of a mathematical broken symmetry was discovered by Pythagoras’s student Hippasus. The Pythagoreans believed that every number is rational. Hippasus showed that the diagonal of a square has length $\sqrt{2}$ and it is not a rational number. The idea that every number is rational was thrown overboard (together with Hippasus). That is, $\sqrt{2}$ was the first element in the domain of discourse known to the Greeks as numbers that showed that the domain must be split or broken.

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\[8\]Philip Kitcher (e.g. [Kit76]) touts the importance of these types of cases for mathematics and uses them in the service of demonstrating the existence of mathematical explanation. Emily Grosholz has studied domain unifications extensively. See e.g. [Gro00].
in two. This discovery begins the long fruitful history of rational and irrational numbers. For another example consider the many problems in computability theory. This area of theoretical computer science was started by Turing and others in the 1930s. In the 1960s researchers realized that although there are many problems that are decidable/solvable by a computer, there are some problems that take an exponentially long time to solve. The Euler Cycle Problem asks to find a cycle in a given graph that hits every edge exactly once. In contrast, the Hamiltonian Cycle Problem asks to find a cycle in a given graph that hits every vertex exactly once. Whereas there is a nice polynomial algorithm to solve the Euler Cycle Problem, there is no known polynomial algorithm for the Hamiltonian Cycle Problem. This breaks the domain of discourse of solvable computer problems into two domains: feasibly solvable computer problems and unfeasibly solvable computer problems. This realization - that the usual methods of solving computational problems fail sometimes, created the entire important field of computational complexity theory.

Thus there are various ways in which symmetry considerations aptly describe mathematical practice in the same way they describe scientific practice, lending credence to the idea that this is the proper way to look at mathematics.

5 Category theory and the symmetries of mathematics

One important branch of mathematics that deals with changing objects in mathematical structures is category theory. In this section we discuss the relationship of the notions of symmetry of mathematics with the central notions of category theory. (Knowledge of category theory is not necessary for what follows. Furthermore, this section can be skipped without loss to the philosophical points we made.)

A category has objects and morphisms. The objects are usually thought of as mathematical structures and the morphisms are functions between the structures that preserve some aspect of the structure. For example, the category of topological spaces has topological spaces as objects and continuous maps between topological spaces as morphisms. The category of groups has groups as objects and homomorphisms between groups as morphisms. For any algebraic structure, there is a category where the objects are those algebraic structures and the morphisms are functions that preserve the structure. In a sense, morphisms are ways of dealing with changing the structure.

One can think of the morphisms in a category as ways of changing some elements in the structure. Let $A$ and $B$ be two structures in a category and let $f : A \to B$ be a morphism in the category. In a sense, $a \in A$ gets placed as $f(a) \in B$. The fact that the morphism has to preserve the structure, means that some of the properties of $f(a)$ has to be shared with $a$. Furthermore, some of the properties of $A$ has to be shared with $B$. The strength of the morphism determines what properties of $A$ are in common with properties of $B$. Is the
morphism an injection? A surjection? If it is an isomorphism, then $A$ and $B$ have the same categorical properties.

One of the central ideas of category theory is that particular constructions are defined by the way morphisms in the category are set up. Most constructions in category theory have “universal properties” that describe the construct using morphisms in the category. This is similar to our emphasis that the central idea of a mathematical structure is what is invariant after changing of the elements of the structure. One of the leaders in category theory, F. William Lawvere, summarizes it as follows: “Thus we seem to have partially demonstrated that even in foundations, not substance but invariant form is the carrier of the relevant mathematical information” ([Law64]).

In this paper we are pushing the notion that one can determine mathematical structures and statements by looking at uniform transformations. There is a very interesting set of ideas in higher algebra and category theory that formalizes this notion of determining structures by looking at uniform transforms. First, some preliminaries. In many places in algebra one looks at an ideal structure and then looks at all the representations / models / algebras of that structure. For example, one can look at

- a monoid and the category of sets which the monoid acts on,
- a group and the category of representations of the group,
- a ring and its category of modules,
- a quantum group and its monoidal category of representations,
- an algebraic theory and its category of algebras,
- etc.

In all these cases one can easily go from the ideal structure to the category of representations. There are times, however, when one can go in the reverse direction. From the category of representations and homomorphisms between representations we can reconstruct the ideal structure. This is similar to the main theme of our paper which is about reconstructing an ideal structure by looking at all the ways objects can be exchanged.

Exactly how such reconstructions are done is beyond the scope of this paper. However, the core idea of the reconstruction theorems are simple and goes back to the basic definition of what a homomorphism of algebraic structure is. Consider some algebraic structure and let $A$ and $B$ be representations/models/algebras of that algebraic structure. If $+$ is a binary operation,
then $f : A \rightarrow B$ is a homomorphism if the following square commutes:

\[
\begin{array}{ccc}
A \times A & \overset{+A}{\longrightarrow} & A \\
\downarrow^{f \times f} & & \downarrow^{f} \\
B \times B & \overset{+B}{\longrightarrow} & B.
\end{array}
\]

The reconstructions rest on the idea that the square can be viewed from a slightly different point of view. The usual motto is that Homomorphisms are functions that respect all the operations.

We suggest:

Operations are functions that respect all the homomorphisms.

That is, we reconstruct the operations by looking at all those functions that always respect the ways of changing what we are dealing with. If you can swap one element for another and the operation still works, then it is a legitimate operation. In a sense the operations are on equal footing as the uniform transformations.

All these ideas can perhaps be traced back to Felix Klein’s Erlangen Program which determines properties of a geometric object by looking at the symmetries of that object. Klein was originally only interested in geometric objects, but mathematicians have taken his ideas in many directions. They look at the automorphism group of many structures, e.g. groups, models of arithmetic, vector spaces, algorithms [Yan], etc. In a sense, some of these ideas can be seen as going back to Galois who determined properties of a structure by looking at the set of symmetries of the structure [10].

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[10] For more about the relationship of category theory and invariance of syntax of semantics see the appendix of [YZ]. See also [Kro97] and [Mar09] for more on the history and philosophy of category theory and it relationship to symmetry.
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