Analyzing and constructing general nonspreading wave packets

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We show the method for constructing nonspreading wave packets whose shape and motion can be general. We analyze the time evolution of nonspreading wave packets by decomposing the Hamiltonian into two parts. Of the two, one changes the instantaneous state, the other does not. Through this decomposition, the time evolution operator is shown to be effectively a spatial shifting operator. This explains why nonspreading wave packets can be nonspreading. And we show that the part of the Hamiltonian which changes the instantaneous state governs the motion of the nonspreading wave packets.

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I. INTRODUCTION

Nonspreading wave packets (NSWP) are quantum packets whose probability density function, $|\Psi(x,t)|^2$, do not change their form while propagating. There are stationary wave packets which correspond to the energy eigenstates of a Hamiltonian with a static potential energy. These energy eigenstates may be viewed as stationary NSWP, or the trivial NSWP. These stationary NSWP are, in fact, important for constructing propagating NSWP with shapes in a general form. We will discuss constructing general NSWP in Section II.

The first nontrivial nonspreading wave packet was found by Schrödinger in 1926. This packet is normalizable and is of the form as a shifted ground state of a harmonic oscillator. The motion of the packet behaves just like a classical particle which is being shifted from its equilibrium position and executing a simple harmonic motion. After Schrödinger, Senitzky in 1954 extended the result showing that shifted higher energy eigenstates are also nonspreading. Senitzky also showed that the expectation value of the energy of a nonspreading wave packet is $E_n + E_{cl}$, where $E_n$ is the quantum mechanical energy of a wave packet when it is stationary, and $E_{cl}$ the classical energy of the particle. Thus, though the motion of a nonspreading wave packet is the same as a classical particle, the energy content of the packet is different from that of a classical particle. We may view $E_n$ as the structural energy of a stationary wave packet besides its classical kinetic energy $E_{cl}$. The extension of nonspreading packets from a shifted ground state to a shifted higher energy eigenstates were also found by Roy and Singh, Yan, and Mentrup et al., etc.

Much later than Schrödinger, Berry and Balazs in 1979 found another interesting nonspreading Airy packet in free space. Airy packets, with a form described by Airy function $Ai(x)$, though are bounded functions, however, are not square integrable. Especially, an Airy packet does not move in a uniform velocity in free space; instead, it accelerates. Thus, in contrast to Schrödinger’s packets, the motion of an Airy wave packet is different from that of a classical particle. This is due to that Airy packet is not normalizable, and therefore does not really describe a particle. Nonspreading Airy packets even exist in time-dependent uniform force.

There are a lot of investigation on the interesting optical Airy beams. The Airy optical beams was first observed by Christodoulides. And electron Airy beams were also constructed by Noa. The phenomenon of nonspreading wave packets is an interesting subject. The essential ingredients of a nonspreading wave packet are: its shape function $f(x)$, its motion $d(t)$, and its phase function $\phi(x,t)$. In what follows, we will show that stationary energy eigenstates can be used to generate propagating NSWP with the same shapes.

In Section II, we discuss the method for constructing NSWP from stationary states. In Section III, we explain why NSWP can be nonspreading from the point of view of time evolution. In Section IV, we use our method to reproduce Berry, Balazs, Schrödinger and Senitzky’s NSWP. In section V, we generalize the formalism to three dimensions.

II. CONSTRUCTING NSWP FROM ARBITRARY STATIC POTENTIAL ENERGY FUNCTION $V(x)$

In this section, we discuss how to construct NSWP from an arbitrary static potential energy function, $V(x)$. The time evolved wave function $\Psi(x,t)$ of an NSWP with a trajectory described by $x = d(t)$ should be of the form as
\[ \Psi(x, t) = f(q) e^{i \phi(x, t)} , \]  
\[ q = x - d(t) , \]  
(1)  

where \( f(x) \) and \( \phi(x, t) \) are real functions, and \( d(t) \) can be arbitrarily designed. The function \( d(t) \) describes the motion of the packet, and \( d(t) \) is the group velocity of the packet. We have \( |\Psi(x, t)|^2 = |f(q)|^2 \), so that \( |\Psi(x, t)|^2 \) has the same form all the time. For the previous NSWPs obtained by Schrödinger and Senitzky, \( f(x) \) is the eigenfunction of a Hamiltonian with \( V(x) = (1/2)kx^2 \). And \( V(x) = Ax \) for NSWPs obtained by Berry and Balazs; if \( A = 0 \), it means a free space. To construct NSWPs for a more general case, we let the shape function \( f(x) \) be determined from the time independent Schrödinger equation

\[ \left[ -\frac{\hbar^2}{2m} \partial_x \partial_x + V(x) \right] f(x) = E_f f(x) . \]  
(3)

where \( V(x) \) is an arbitrary static potential energy.

The next step is to determine a potential energy, denoted by \( V_{\text{nswp}}(x, t) \), in which NSWPs described in Eqs.\((1), (2)\) can be constructed. The time dependent Schrödinger equation for such potential energy is

\[ i\hbar \partial_t \Psi(x, t) = H \Psi(x, t) = \left[ -\frac{\hbar^2}{2m} \partial_x \partial_x + V_{\text{nswp}}(x, t) \right] \Psi(x, t) , \]  
(4)

We can determine \( V_{\text{nswp}}(x, t) \) from Eq.\((4)\). We first rewrite Eq.\((4)\) as

\[ i\hbar \partial_t \Psi(x, t) + \frac{\hbar^2}{2m} \partial_x \partial_x \Psi(x, t) = V_{\text{nswp}}(x, t) \Psi(x, t) , \]  
(5)

Substituting the \( \Psi(x, t) \) in Eqs.\((1), (2)\) to the left side of Eq.\((5)\) and comparing it with the right side, we can then determine the \( V_{\text{nswp}}(x, t) \). For the first term of the left side of Eq.\((5)\), we have

\[ i\hbar \partial_t \Psi(x, t) = \left[ \dot{d}(t) P - \dot{d}(t) \hbar \partial_x \phi(x, t) - \hbar \partial_t \phi(x, t) \right] \Psi(x, t) , \]  
(6)

where, \( P = -i\hbar \partial_t / \partial x \) is the momentum operator. For the second term of the left side of Eq.\((5)\), we have

\[ \partial_x \partial_x \Psi(x, t) = e^{i \phi(x, t)} \partial_q \partial_q f(q) + 2i \partial_x \phi(x, t) \partial_x \Psi(x, t) + \left( \partial_x \phi(x, t) \right)^2 \Psi(x, t) . \]  
(7)

The term \( \partial_q \partial_q f(q) \) above can be replaced by \( V(q) \) and \( E_f \). This is due to that, from Eq.\((3)\), we have the following formula

\[ \left[ -\frac{\hbar^2}{2m} \partial_q \partial_q + V(q) \right] f(q) = E_f f(q) . \]  
(8)

Substituting Eq.\((8)\) into Eq.\((7)\), we obtain

\[ \frac{\hbar^2}{2m} \partial_x \partial_x \Psi(x, t) = \left[ V(q) - E_f - \frac{\hbar^2}{m} \partial_x \phi(x, t) P + \frac{\hbar^2}{2m} (\partial_x \phi(x, t))^2 \right] \Psi(x, t) . \]  
(9)

Adding Eqs.\((6)\) and \((9)\) and comparing it with Eqs.\((5)\), we then obtain the form of \( V_{\text{nswp}}(x, t) \). We have

\[ V_{\text{nswp}}(x, t) = \left[ \dot{d}(t) - \frac{\hbar}{m} \partial_x \phi(x, t) \right] P - \dot{d}(t) \hbar \partial_x \phi(x, t) - \hbar \partial_t \phi(x, t) + V(q) - E_f + \frac{\hbar^2}{2m} (\partial_x \phi(x, t))^2 . \]  
(10)
This is the general form of $V_{\text{nswp}}(x, t)$. We consider this general form elsewhere for the case that EM interactions are involved. At present, we do not consider a potential energy that is $P$ dependent. Then, the term containing $P$ in the right side of Eq. (10) should be zero. This then yields

$$
\dot{d}(t) = \frac{\hbar}{m} \partial_x \phi(x, t).
$$

(11)

From Eq. (11), $\phi(x, t)$ can be easily determined, we have

$$
\phi(x, t) = \phi_1(t)x + \phi_0(t).
$$

(12)

$$
\phi_1(t) = \frac{m \dot{d}(t)}{\hbar}.
$$

(13)

Where $\phi_0(t)$ is an arbitrary function depending only on time. We see that $\phi(x, t)$ is simply linear in $x$. We previous had also obtained this result in another way [9]. Substituting Eqs. (12)-(13) into Eq. (10), we obtain

$$
V_{\text{nswp}}(x, t) = V(q) - m \dot{d}(t)x + G(t),
$$

(14)

$$
G(t) \equiv -E_f - \frac{m}{2} \dot{d}(t)^2 - \hbar \dot{\phi}_0(t).
$$

(15)

We see that $V_{\text{nswp}}(x, t)$ contains the original potential energy $V(x)$ but with the argument $x$ replaced by $q$. The function $G(t)$ is a function depending only on time. Since $G(t)$ depends only on time, it has no influence on dynamics. We may simply set $G(t) = 0$. However, it is better to keep this term for the sake of later convenience. Once we have chosen a functional form for $G(t)$, we can then determine $\phi_0(t)$ from eq.(15). The result is shown in Eq. (22).

We thus have achieved the determination of the potential energy $V_{\text{nswp}}(x, t)$ that supports the NSWPs. We conclude our results as the following:

$$
\hbar \frac{\partial \Psi(x, t)}{\partial t} = H \Psi(x, t).
$$

(16)

$$
H = -\frac{\hbar^2}{2m} \partial_x \partial_x + V_{\text{nswp}}(x, t).
$$

(17)

$$
V_{\text{nswp}}(x, t) = V(q) - m \dot{d}(t)x + G(t).
$$

(18)

This Hamiltonian system, Eqs. (16)-(18), supports NSWPs for which the wave packets are of the form as:

$$
\Psi(x, t) = f(q) e^{i\phi(x, t)}, \quad q = x - \dot{d}(t).
$$

(19)

$$
\left[ -\frac{\hbar^2}{2m} \partial_x \partial_x + V(x) \right] f(x) = E_f f(x).
$$

(20)

$$
\phi(x, t) = \frac{m \dot{d}(t)}{\hbar}x + \phi_0(t),
$$

(21)

$$
\phi_0(t) = -\frac{1}{\hbar} \int_0^t \left( E_f + G(t) + \frac{m}{2} \dot{d}(t)^2 \right) dt.
$$

(22)

Where we simply set $\phi_0(0) = 0$. We easily check that the $\Psi(x, t)$ given in Eqs. (19)-(22) satisfies the Schrödinger equation described in Eqs. (16)-(18). This then completes the construction of NSWPs. We start from an arbitrary static potential function $V(x)$, then we obtain the potential energy $V_{\text{nswp}}(x, t)$ which supports NSWPs.

From Eq. (13), we note that there is a moving potential energy $V(q)$ which moves along with the packet. This seems reasonable, if we are having a propagating nonspreading wave packet. There is also a potential energy term $-m \dot{x} \dot{d}(t)$. This term can be understood from the classical mechanics. As, classically, a particle with a trajectory $d(t)$, then the force needed for that motion is $F = m \dot{d}(t)$, which corresponds to a potential energy term as $-m \dot{d}(t)x$.

As an example for what we have constructed, we consider a time independent potential such as

$$
V(x) = \lambda x^4.
$$

(23)
The eigenfunctions, $f_{\lambda}(x)$, obtained from this potential energy offer the shape of NSWPs. The NSWP-constructing potential is:

$$V_{\text{nswp}}(x,t) = \lambda(x - d(t))^4 - mx\dot{d}(t) + G(t),$$

where $d(t)$ can be arbitrarily designed. This potential energy $V_{\text{nswp}}(x,t)$ then provides NSWPs with their shapes being of the form as $f_{\lambda}(x)$. The wave functions $\Psi(x,t)$, described in Eqs. (19)-(22), then are the NSWPs moving with a trajectory $x = d(t)$.

III. THE TIME EVOLUTION AND THE DYNAMICS OF NSWPS

In this section, we discuss how nonspreading wave packets can be nonspreading. We start from the time evolution operator. The infinitesimal time evolution operator is defined as:

$$\Psi(x,t + dt) = U(t,t + dt)\Psi(x,t),$$

$$U(t,t + dt) = e^{-\frac{i}{\hbar}H(t)dt}.$$  

We investigate how the wave function $\Psi(x,t)$ changes with time. For that, we decompose the Hamiltonian $H(t)$ into two parts, $\tilde{H}(t)$ and $H_c(t)$. The first part, $\tilde{H}(t)$, is the one that does not change the state $\Psi(x,t)$. The second part, $H_c(t)$, does change $\Psi(x,t)$. To determine $\tilde{H}(t)$, we substitute Eqs. (11)-(13) into Eq. (9). This yields the following interesting equation

$$\left[-\frac{\hbar^2}{2m}\partial_x\partial_x + V(q) - \dot{d}(t)P\right]\Psi(x,t) = \left[E_f - \frac{m}{2}\dot{d}(t)^2\right]\Psi(x,t).$$  

This is an eigenvalue equation for $\Psi(x,t)$. We write the left side in terms of $\tilde{H}(t)$, and then the equation is written as

$$\tilde{H}(t)\Psi(x,t) = \tilde{E}(t)\Psi(x,t),$$

$$\tilde{H}(t) = -\frac{\hbar^2}{2m}\partial_x\partial_x + V(q) - \dot{d}(t)P,$$

$$\tilde{E}(t) = E_f - \frac{m}{2}\dot{d}(t)^2.$$  

Thus the Hamiltonian $\tilde{H}(t)$ in Eq. (29) is the Hamiltonian that we are looking, as $\tilde{H}(t)$ does not change the state $\Psi(x,t)$. We then decompose the Hamiltonian $H(t)$ into the form as: $H = \tilde{H}(t) + H_c(t)$. Then $H_c(t) = H(t) - \tilde{H}(t)$. This then determines $H_c(t)$ which is the part of the Hamiltonian that changes the state $\Psi(x,t)$. We have therefore obtained the results:

$$H(t) = \tilde{H}(t) + H_c(t)$$

$$H_c(t) = \dot{d}(t)P - m\ddot{d}(t)x + G(t)$$

With this form, then

$$U(t,t + dt)\Psi(x,t)$$

$$= e^{-\frac{i}{\hbar}H_c(t)dt}e^{-\frac{i}{\hbar}\tilde{H}(t)dt}\Psi(x,t)$$

$$= e^{-\frac{i}{\hbar}(\tilde{E}(t) + G(t))dt}e^{\frac{i}{\hbar}m\dot{d}(t)dtx}e^{-\frac{i}{\hbar}\dot{d}(t)dtP}\Psi(x,t)$$

We see that the time evolution operator $U(t,t + dt)$ is effectively a spatial shifting operator: $e^{-i\dot{d}(t)dtP/\hbar}$. It is well known that $e^{i\alpha(t)P/\hbar}$ shifts a spatial function $h(x)$ into $h(x + \alpha(t))$. Thus, we see that the operation of the infinitesimal operator $U(t,t + dt)$ acting on the state $\Psi(x,t)$ transfers the state into a shifted state $\Psi(x + \dot{d}(t)dt,t)$. This explains why nonspreading packet can be nonspreading. The amount of the spatial shift is $dx = \dot{d}(t)dt$. Thus, $\dot{d}(t)$ is the group velocity of the NSWP.
We now discuss formally the operation of $U(t, t + dt)$ on the state $\Psi(x, t)$. From Eq. (35), we need first calculate the operation of $e^{-\frac{i}{\hbar}\tilde{d}(t)dt\hat{P}}$ acting on $\Psi(x, t)$, and then the operation of $e^{\frac{i}{\hbar}m\tilde{d}(t)dt\hat{x}}$, and then the operation of $e^{-\frac{i}{\hbar}(\tilde{E}(t)+\tilde{G}(t))dt}$. For the operation of $e^{-\frac{i}{\hbar}\tilde{d}(t)dt\hat{P}}$ acting on $\Psi(x, t)$, we have

$$e^{-\frac{i}{\hbar}\tilde{d}(t)dt\hat{P}}\Psi(x, t) = e^{-\frac{i}{\hbar}\tilde{d}(t)dt(f(x - d(t)))} e^{i(\phi_1(t)x + \phi_0(t))},$$

$$= f(x - d(t) - d(t)dt) e^{i(\phi_1(t)x - \frac{i}{\hbar}\tilde{d}(t)^2dt + \phi_0(t))},$$

$$= f(x - d(t + dt)) e^{i(\phi_1(t)x - \frac{i}{\hbar}\tilde{d}(t)^2dt + \phi_0(t))}. \quad (36)$$

Above, we have used Eq. (13). We see that under the operation of $e^{-\frac{i}{\hbar}\tilde{d}(t)dt\hat{P}}$, the original $f(x - d(t))$ is shifted to $f(x - d(t + dt))$, and brings in an extra phase factor $e^{-\frac{i}{\hbar}\tilde{d}(t)^2dt}$. Next, we discuss the operation of the phase factor $e^{im\tilde{d}(t)dx/\hbar}$ acting on both sides of Eq. (36). From the right side of Eq. (36), we see that this phase factor can be absorbed to the phase factor $e^{i\phi_1(t)x}$. As

$$e^{\frac{i}{\hbar}m\tilde{d}(t)dx}e^{i\phi_1(t)x} = e^{\frac{i}{\hbar}m(d(t)+\tilde{d}(t)dt)x}$$

$$= e^{\frac{i}{\hbar}m(d(t)+d(t)dt)x}$$

$$= e^{i\phi_1(t)+d(t)dt}x. \quad (37)$$

We then have the following result obtained form the two successive operations:

$$e^{\frac{i}{\hbar}m\tilde{d}(t)dx}e^{-\frac{i}{\hbar}\tilde{d}(t)dt\hat{P}}\Psi(x, t) = f(x - d(t + dt)) e^{i(\phi_1(t)+d(t)dt)x - \frac{i}{\hbar}\tilde{d}(t)^2dt + \phi_0(t))} \quad (38)$$

Finally, we discuss the operation of the phase factor $e^{-i(\tilde{E}(t)+\tilde{G}(t))dt/\hbar}$ acting on both sides of Eq. (38). We also easily see that this phase factor can be absorbed into the phase factor $e^{i\phi_0(t)}$. As, from the right side of Eq. (38), we see we need calculate the following formula:

$$e^{-\frac{i}{\hbar}(\tilde{E}(t)+\tilde{G}(t))dt}e^{(\frac{i}{\hbar}\tilde{d}(t)^2dt + \phi_0(t))} = e^{-\frac{i}{\hbar}(\tilde{E}(t)+\tilde{G}(t)+m\tilde{d}(t)^2)dt}e^{i\phi_0(t)}$$

$$= e^{\frac{i}{\hbar}(\phi_0(t)+\phi_0(t)dt)}$$

$$= e^{i\phi_0(t)+d(t)dt}. \quad (39)$$

Above, we have used Eqs. (22), (30). In all, from all of these successive operations, we have then obtained that

$$\Psi(x, t + dt) = U(t, t + dt)\Psi(x, t)$$

$$= e^{-\frac{i}{\hbar}(\tilde{E}(t)+\tilde{G}(t))dt} e^{\frac{i}{\hbar}m\tilde{d}(t)dt}\hat{x} e^{-\frac{i}{\hbar}\tilde{d}(t)dt\hat{x}}(x - d(t)) e^{i(\phi_1(t)x + \phi_0(t))} \quad (40)$$

$$f(x - d(t + dt)) e^{i(\phi_1(t)+d(t)dt)x + \phi_0(t))dt})} \quad (41)$$

The action of the infinitesimal time evolution operator $U(t, t + dt)$ on the state $\Psi(x, t)$ indeed remains the functional form of an NSWP.

As NSWP move like classical particles, the dynamics of the motion should then be governed by an effective Hamiltonian. From Eq. (31), we see that the Hamiltonian is decomposed into two parts. The part of the Hamiltonian, $H_c(t)$, is the part of the Hamiltonian which involves the action of shifting the state $\Psi(x, t)$. The motion of NSWP is thus effectively determined by $H_c(t)$ only. And, therefore, if we take an NSWP as a classical particle, then we expect that $H_c(t)$ would play classically the same role as a Hamiltonian that governs the motion of a particle. To verify that, we check the Hamilton equations obtained from the Hamiltonian $H_c(t)$. We have

$$\dot{x} = \frac{\partial H_c}{\partial \hat{P}} = \dot{\hat{d}}(t), \quad (42)$$

$$\dot{\hat{P}} = \frac{\partial H_c}{\partial x} = m\dot{\hat{d}}(t). \quad (43)$$

This shows that indeed the Hamiltonian $H_c(t)$ governs the motion of NSWP. We should note that it is $H_c(t)$, not the total Hamiltonian $H(t)$, that governs the motion of NSWP. This then explains why NSWP can be accelerated in free space, as the corresponding $H_c(t)$, in the case of a free space, is not a constant, it is linear in $x$, and therefore accelerates a wave packet. We discuss this in more detail in Section IV.

In general, the function $V_{\text{nswp}}(x, t)$ is time dependent. It would be interesting to investigate whether it is possible for static potential energy systems to supply NSWP. In fact, this is possible, and these potential energies are just those explored by Schrödinger and Berry and Balaz. Below, we show that.
IV. DERIVING PREVIOUS KNOWN NSWPS

A. Berry and Balazs’s NSWPs (1)

In what follows, we show that we can derive the known examples of NSWPs from the method introduced in Sec. II. For the first case, we consider:

\[ V(x) = Ax, \]

where A is a real constant. This potential energy leads to the eigenvalue equation:

\[ -\frac{\hbar^2}{2m}\partial_x^2 f(x) + Af(x) = Ef(x). \]  \hspace{1cm} (44)

The solution of \( f(x) \) is expressed in terms of Airy function \( Ai(x) \) as the following

\[ f(x) = Ai\left(\frac{2Am}{\hbar^2}\right)^{1/3}\left(x - \frac{Ef}{A}\right). \]  \hspace{1cm} (45)

\( f(x) \) will then be used as the shape function of the NSWPs. The potential energy \( V_{\text{nswp}}(x,t) \) which provides NSWPs can be obtained from (18). We have

\[ V_{\text{nswp}}(x,t) = A(x - d(t)) - m\ddot{d}(t)x + G(t) \]  \hspace{1cm} (46)

\[ = \left(A - m\ddot{d}(t)\right)x + \left(G(t) - A\dot{d}(t)\right), \]  \hspace{1cm} (47)

where \( d(t) \) describes the motion of the packet. If we consider the case that \( V_{\text{nswp}}(x,t) \) is time independent, then the values for \( A \) and \( G(t) \) can be chosen as follows:

\[ A - m\ddot{d}(t) = 0, \]  \hspace{1cm} (48)

\[ G(t) - A\dot{d}(t) = 0. \]  \hspace{1cm} (49)

This then means that

\[ V_{\text{nswp}}(x,t) = 0. \]  \hspace{1cm} (50)

Eq. (50) describes a free space. We thus have the interesting result: a free space system can offer NSWPs. This was found by Berry and Balaz in 1979. Eq. (48) shows that the motion is with a constant acceleration, as we have

\[ d(t) = \frac{1}{2}(A/m)t^2, \]

where we set \( d(0) = 0 \). And then we have \( G(t) = A^2t^2/(2m) \) from Eq.(49). From Eqs.(21)-(22), we also obtain

\[ \phi(x,t) = Atx + \phi_0(t), \]

\[ \phi_0(t) = -\left(E_f t + \frac{A^2 t^3}{3m}\right), \]  \hspace{1cm} (51)

where we set \( \phi_0(0) = 0 \). From Eq. (19), the full wave function of the NSWP is therefore

\[ \Psi(x,t) = Ai\left[\left(\frac{2Am}{\hbar^2}\right)^{1/3}\left(x - \frac{Ef}{A} - \frac{At^2}{2m}\right)\right] e^{\frac{i}{\hbar}\left[Ai(x - \frac{4At^2}{3m}) - E_f t\right]} \]  \hspace{1cm} (52)

If we set \( E_f = 0 \) and \( A = B^3/(2m) \). Then we have

\[ \Psi(x,t) = Ai\left[\left(\frac{B^3}{\hbar^2}\right)^{1/3}\left(x - \frac{B^3 t^2}{4m^2}\right)\right] e^{\frac{i}{\hbar}\left[\frac{3B^3}{4m^2}(x - \frac{B^3 t^2}{4m^2})\right]} \]  \hspace{1cm} (53)

This is the result derived by Berry and Balaz. Thus, starting from a static potential \( V(x) = Ax \), we show that a system with potential energy \( V_{\text{nswp}}(x,t) = 0 \) can supply NSWPs. The shape of the packet is of the form as Airy function and moves in a constant acceleration.
We can also explain why Airy packets can accelerate in a free space. The reason is because the motion of an Airy packet is governed not by the total Hamiltonian, but by the Hamiltonian, \( H_c(t) \). Referring to Eq. (32), we have

\[
H_c(t) = \frac{A t}{m} P - A x + \frac{A t^2}{2m}.
\]

We see that \( H_c(t) \) is not a constant in free space. And, in fact, \( H_c(t) \) contains a term as \( A x \) which results a constant force \( F = A \). It is this force that accelerates the Airy packet. Thus, we should keep in mind the important role played by the Hamiltonian \( H_c(t) \) when analyzing the time evolution of a quantum packet.

**B. Berry and Balazs’s NSWPs (2)**

Eq. (47) shows that we may extend from the case of a free space to a more general system to construct NSWPs. We note that we may choose the values for \( A \) and \( G(t) \) in Eq. (47) as

\[
A - m \ddot{d}(t) = F(t), \tag{55}
\]

\[
G(t) - A d(t) = 0, \tag{56}
\]

where \( F(t) \) is an arbitrary function. Substituting Eq. (55) and (56) into (47), we obtain

\[
V_{\text{nswp}}(x, t) = -F(t)x. \tag{57}
\]

We then show that a system with a potential energy \( V_{\text{nswp}}(x, t) = -F(t)x \) can also supply NSWPs. This result was also found by Berry and Balazs in 1979. We note that, in this case, \( V_{\text{nswp}}(x, t) \) may be time dependent. The function \( d(t) \) can be solved from Eq. (55), and \( G(t) \) can be determined from Eq. (56). The phase function \( \phi(x, t) \) can also be determined from Eqs. (20)-(21). We have the result:

\[
\Psi(x, t) = Ai\left(\frac{2Am}{\hbar^2}\right)^{1/3} \left( x - \frac{E_f}{A} - d(t) \right) e^{i\left(\phi_1(t)x + \phi_0(t)\right)}, \tag{58}
\]

where

\[
d(t) = \frac{A t^2}{2m} + \frac{1}{m} \int_0^t \int_0^\tau F(x)dx d\tau. \tag{59}
\]

\[
\phi_1(t) = \frac{At}{\hbar} + \frac{1}{\hbar} \int_0^t F(\tau)d\tau. \tag{60}
\]

\[
\phi_0(t) = -\frac{E_f t}{\hbar} - \frac{A^2 t^3}{3m\hbar} - \frac{1}{2m\hbar} \int_0^t \left[ \int_0^\tau F(\eta)d\eta \right]^2 d\tau - \frac{A}{m\hbar} \left[ \int_0^\tau \int_0^\eta F(\zeta)d\zeta d\eta d\tau + \int_0^\tau \int_0^\eta \int_0^\zeta F(\xi)d\xi d\eta d\tau \right]. \tag{61}
\]

If we set \( E_f = 0 \) and \( A = B^3/(2m) \), we then reproduce Berry and Balazs’s result for the case \( V_{\text{nswp}}(x, t) = -F(t)x \).

From Eq. (47), we see that there is no other way to construct NSWPs with Airy function as the shape function. We have also derived this result in Ref. [10].

**C. Schrödinger and Senitzky’s NSWPs**

We now consider constructing NSWPs in the system of simple harmonic oscillator (SHO). We start from the potential energy as
\[ V(x) = \frac{1}{2}m\omega^2 x^2. \] (62)

This potential energy leads to the well known eigenvalue equation.

\[ -\frac{\hbar^2}{2m} \frac{\partial^2}{\partial x^2} \psi_n(x) + \frac{1}{2}m\omega^2 x^2 \psi_n(x) = E_n \psi_n(x), \] (63)

\[ E_n = (n + \frac{1}{2})\hbar\omega. \] (64)

where \( \psi_n(x) \) is the \( n \)th eigenfunction of SHO. This eigenfunction \( \psi_n(x) \) will then be used as the shape function for NSWPs. The corresponding potential energy \( V_{nswp}(x,t) \) can be obtained from (18). This yields

\[ V_{nswp}(x,t) = \frac{1}{2}m\omega^2 (x - d(t))^2 - m\ddot{d}(t)x + G(t) \] (65)

\[ = \frac{1}{2}m\omega^2 x^2 - m\left(\omega^2 d(t) + \ddot{d}(t)\right)x + \left(G(t) + \frac{1}{2}m\omega^2 d(t)^2\right), \] (66)

To obtain Schrödinger’s result, the \( V_{nswp}(x,t) \) should be time independent. We may then set

\[ \omega^2 d(t) + \ddot{d}(t) = 0, \] (67)

\[ G(t) + \frac{1}{2}m\omega^2 d(t)^2 = 0. \] (68)

This means that

\[ V_{nswp}(x,t) = \frac{1}{2}m\omega^2 x^2. \] (69)

This then shows that a simple harmonic oscillator system can offer NSWPs. This was found by Schrödinger in 1926 and Senitzky in 1954. The functions \( d(t) \) and \( G(t) \) can be determined from Eqs. (67), (68). We then have the NSWPs described in the following:

\[ \Psi(x,t) = \psi_n(x - d(t))e^{i[\phi_1(t)x + \phi_0(t)]}, \] (70)

where

\[ d(t) = A \sin(\omega t), \] (71)

\[ \phi_1(t) = \frac{1}{\hbar}mA\omega \cos(\omega t), \] (72)

\[ \phi_0(t) = -\frac{m\omega x_0^2}{4\hbar} \sin(2\omega t) - \frac{E_n t}{\hbar}. \] (73)

Above, we set \( d(0) = 0 \), and \( \phi_0(0) = 0 \). The NSWPs in the SHO system execute a simple harmonic motion. We have therefore reproduced Schrödinger and Senitzky’s results.

From Eq. (66), we see that we can not arrange \( V_{nswp}(x,t) \) into a form as \( m\tilde{\omega}(t)^2 x^2/2 \), where \( \tilde{\omega}(t) \) is a time dependent frequency. Thus, a harmonic oscillator system with a time dependent frequency \( \tilde{\omega}(t) \) can not support a nonspreading wave packet, as stated in Yan’s paper [4].

V. THE GENERALIZATION TO THREE DIMENSIONS

Our result can easily be generalized to three dimensions. The same, we start from a static potential \( V(\vec{r}) \), then we have the eigenvalue equation:

\[ -\frac{\hbar^2}{2m} \nabla \cdot \nabla f(\vec{r}) + V(\vec{r})f(\vec{r}) = E f(\vec{r}). \] (74)
The corresponding NSWP-construction potential energy $V_{nswp}(\vec{r}, t)$ is of the form as:

$$V_{nswp}(\vec{r}, t) = V(\vec{r} - \vec{d}(t)) - m\vec{r} \cdot \ddot{\vec{d}}(t) + G(t),$$  

(75)

$$G(t) \equiv -E_f - \frac{m}{2} \dot{\vec{d}}(t)^2 - \hbar \phi_0(t).$$  

(76)

The wave functions of the NSWPs constructed from $V_{nswp}(\vec{r}, t)$ are described as follows

$$\Psi(\vec{r}, t) = f(\vec{r} - \vec{d}(t)) e^{i\phi(\vec{r}, t)},$$  

(77)

$$\phi(\vec{r}, t) = \frac{m}{\hbar} \dot{\vec{d}}(t) \cdot \vec{r} + \phi_0(t),$$  

(78)

$$\phi_0(t) = -\frac{1}{\hbar} \int_0^t \left( E_f + G(t) + \frac{m}{2} \dot{\vec{d}}(t)^2 \right) dt.$$  

(79)

We easily check that the $\Psi(\vec{r}, t)$ given in Eqs. (77)-(79) satisfies the Schrödinger equation

$$i\hbar \frac{\partial \Psi(\vec{r}, t)}{\partial t} = -\frac{\hbar^2}{2m} \vec{\nabla} \cdot \vec{\nabla} \Psi(\vec{r}, t) + V_{nswp}(\vec{r}, t)\Psi(\vec{r}, t).$$  

(80)

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