Determination of the phase of an electromagnetic field via incoherent detection of fluorescence

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We show that the phase of a field can be determined by incoherent detection of the population of one state of a two-level system if the Rabi frequency is comparable to the Bohr frequency so that the rotating wave approximation is inappropriate. This implies that a process employing the measurement of population is not a square-law detector in this limit. We discuss how the sensitivity of the degree of excitation to the phase of the field may pose severe constraints on precise rotations of quantum bits involving low-frequency transitions. We present a scheme for observing this effect in an atomic beam, despite the spread in the interaction time.

It is well known that the amplitude of an atomic state is necessarily complex. Whenever a measurement is made, the square of the absolute value of the amplitude is the quantity we generally measure. The electric or magnetic field generated by an oscillator, on the other hand, is real, composed of the sum of two complex components. In describing semiclassically the atom-field interaction involving such a field, one often side-steps this difference by making the so-called the rotating wave approximation (RWA), under which only one of the two complex components is kept, and the counter-rotating part is ignored. Under this approximation, an atom interacting with a field enables one to measure only the intensity, and not the phase of the driving field. This is the reason why most detectors are so-called square-law detectors.

In this article, we show how a single atom by itself can detect the absolute phase of a driving field, by making use of the interference between the co- and counter-rotating parts of the excitation, while the Rabi frequency is not negligible compared to the transition frequency. This detection is performed by measuring incoherently the population of either of the two states of a two level atom. This implies that a process employing the measurement of population is not a square-law detector in this limit. We discuss how the sensitivity of the degree of excitation to the phase of the field may enable phase teleportation using a pair of entangled atoms, but poses severe constraints on precise rotations of quantum bits involving low-frequency transitions. We also present a scheme for observing this effect in an atomic beam, despite the spread in the interaction time.

We consider an ideal two-level system where a ground state \( |0\rangle \) is coupled to a higher energy state \( |1\rangle \). We also assume that the \( 0 \leftrightarrow 1 \) transitions are magnetic dipolar, with a transition frequency \( \omega \), and the magnetic field is of the form \( B = B_0 \cos(\omega t + \phi) \). We now summarize briefly two-level dynamics without the RWA. In the dipole approximation, the Hamiltonian can be written as:

\[
\hat{H} = \epsilon (\sigma_0 - \sigma_z)/2 + g(t) \sigma_x
\]

where \( g(t) = -g_0 [\exp(i\omega t + i\phi) + c.c.] / 2 \), \( \sigma_i \) are Pauli matrices, and \( \epsilon = \omega \) corresponds to resonant excitation. The state vector is written as:

\[
|\xi(t)\rangle = \begin{pmatrix}
C_0(t) \\
C_1(t)
\end{pmatrix}.
\]

We perform a rotating wave transformation by operating on \( |\xi(t)\rangle \) with the unitary operator \( \hat{Q} \), where:

\[
\hat{Q} = (\sigma_0 + \sigma_z)/2 + \exp(i\omega t + i\phi)(\sigma_0 - \sigma_z)/2.
\]

The Schrödinger equation then takes the form (setting \( \hbar = 1 \)): \( \dot{\xi} = -i\hat{H}|\xi(t)\rangle \) where the effective Hamiltonian is given by:

\[
\hat{H} = \alpha(t)\sigma_+ + \alpha^*(t)\sigma_-,
\]

with \( \alpha(t) = -(g_0/2) [\exp(-i2\omega t - i2\phi) + 1] \), and in the rotating frame the state vector is:

\[
|\tilde{\xi}(t)\rangle \equiv \hat{Q}|\xi(t)\rangle = \begin{pmatrix}
\tilde{C}_0(t) \\
\tilde{C}_1(t)
\end{pmatrix}.
\]

Now, one may choose to make the rotating wave approximation (RWA), corresponding to dropping the fast oscillating term in \( \alpha(t) \). This corresponds to ignoring effects (such as the Bloch-Siegert shift ) of the order of \( g_0/\omega \), which can easily be observable in experiment if \( g_0 \) is large enough. On the other hand, by choosing \( g_0 \) to be small, one can make the RWA for any value of \( \omega \). We explore both regimes in this paper. As such, we find the general results without the RWA.

From Eqs.\ref{eq:6a} and \ref{eq:6b}, one gets two coupled differential equations:

\[
\ddot{\tilde{C}}_0(t) = -(g_0/2) [1 + \exp(-i2\omega t - i2\phi)] \tilde{C}_1(t)
\]

\[
\ddot{\tilde{C}}_1(t) = -(g_0/2) [1 + \exp(+i2\omega t + i2\phi)] \tilde{C}_0(t).
\]
We assume \(|C_0(t)|^2 = 1\) is the initial condition, and proceed further to find an approximate analytical solution of Eq. 3. Given the periodic nature of the effective Hamiltonian, the general solution to Eq. 3 can be written in the form:

\[
|\tilde{\xi}(t)| = \sum_{n=-\infty}^{\infty} \left( a_n \right) \exp(n(-i2\omega t - i2\phi)).
\]  
(7)

Inserting Eq. 3 in Eq. 4, and equating coefficients with same frequencies, one gets for all \(n\):

\[
\begin{align*}
\dot{a}_n &= i2n\omega a_n + i\eta_g(b_n + b_{n-1})/2, \\
\dot{b}_n &= i2n\omega b_n + i\eta_g(a_n + a_{n+1})/2.
\end{align*}
\]  
(8a)

Here, the coupling between \(a_0\) and \(b_0\) is the conventional one present when the RWA is made. The couplings to the nearest neighbors, \(a_{\pm 1}\) and \(b_{\pm 1}\), are detuned by an amount \(2\omega\), and so on. To the lowest order in \((g_o/\omega)\), we can ignore terms with \(|n| > 1\), thus yielding a truncated set of six equations:

\[
\begin{align*}
\dot{a}_0 &= i\eta_g(b_0 + b_{-1})/2, \\
\dot{b}_0 &= i\eta_g(a_0 + a_{-1})/2, \\
\dot{a}_1 &= i2\omega a_1 + i\eta_g(b_1 + b_0)/2, \\
\dot{b}_1 &= i2\omega b_1 + i\eta_g(a_1 + a_0)/2, \\
\dot{a}_{-1} &= -i2\omega a_{-1} + i\eta_g b_{-1}/2, \\
\dot{b}_{-1} &= -i2\omega b_{-1} + i\eta_g(a_{-1} + a_0)/2.
\end{align*}
\]  
(9a)

We consider \(g_0\) to have a time-dependence of the form \(g_0(t) = g_0 M \left[1 - \exp(-t/\tau_{sw})\right]\), where the switching time constant \(\tau_{sw}\) is large compared to other characteristic time scales such as \(1/\omega\) and \(1/\eta_0 M\). Under this condition, one can solve these equations by employing the method of adiabatic elimination, which is valid to first order in \(\eta \equiv (g_0/4\omega)\). Note that \(\eta\) is also a function of time, and can be expressed as \(\eta(t) = \eta_0 \left[1 - \exp(-t/\tau_{sw})\right]\), where \(\eta_0 \equiv (g_0 M/4\omega)\). To solve the set of equations above, we consider first Eqs. 9 and 10. In order to simplify these two equations further, one needs to diagonalize the interaction between \(a_{-1}\) and \(b_{-1}\). Define \(\mu_- \equiv (a_{-1} - b_{-1})\) and \(\mu_+ \equiv (a_{-1} + b_{-1})\), which now can be used to re-express these two equations in a symmetric form as:

\[
\begin{align*}
\dot{\mu}_- &= -i(2\omega + g_0/2)\mu_- - i\eta g a_0/2, \\
\dot{\mu}_+ &= -i(2\omega - g_0/2)\mu_+ + i\eta g a_0/2.
\end{align*}
\]  
(10a)

Adiabatic following then yields (again, to lowest order in \(\eta\)) \(\mu_- \approx -\eta_0 a_0 \) and \(\mu_+ \approx \eta_0 a_0\), which in turn yields \(a_{-1} \approx 0\) and \(b_{-1} \approx \eta_0 a_0\). In the same manner, we can solve equations 8c and 8d, yielding:

\[
\begin{align*}
\dot{a}_1 &= i2\omega a_1 + i\eta_g(b_1 + b_0)/2, \\
\dot{b}_1 &= i2\omega b_1 + i\eta_g(a_1 + a_0)/2, \\
\dot{a}_{-1} &= -i2\omega a_{-1} + i\eta_g b_{-1}/2, \\
\dot{b}_{-1} &= -i2\omega b_{-1} + i\eta_g(a_{-1} + a_0)/2. \\
\end{align*}
\]  
(10b)

Note that the amplitudes of \(a_{-1}\) and \(b_1\) are vanishing (each proportional to \(\eta^2\)) to lowest order in \(\eta\), and thereby justifying our truncation of the infinite set of relations in Eq. 3. It is easy to show now:

\[
\begin{align*}
\dot{a}_0 &= i\eta_g b_0/2 + i\Delta(t) a_0/2, \\
\dot{b}_0 &= i\eta_g a_0/2 - i\Delta(t)b_0/2,
\end{align*}
\]  
(11a)

where \(\Delta(t) = g_0^2(t)/4\omega\) is essentially the Bloch-Siegert shift. Eq. 11 can be thought of as a two-level system excited by a field detuned by \(\Delta\). For simplicity, we assume that this detuning is dynamically compensated for by adjusting the driving frequency \(\omega\). This assumption does not affect the essence of the results to follow, since the resulting correction to \(\eta\) is negligible. With the initial condition of all the population in \(|0\rangle\) at \(t = 0\), the only non-vanishing (to lowest order in \(\eta\) ) terms in the solution of Eq. 11 are:

\[
\begin{align*}
a_0(t) &\approx \cos\left(\gamma_0^0(t)/2\right), \\
b_0(t) &\approx i\sin\left(\gamma_0^0(t)/2\right), \\
a_1(t) &\approx -i\eta_0\sin\left(\gamma_0^0(t)/2\right), \\
b_{-1}(t) &\approx \eta_0\cos\left(\gamma_0^0(t)/2\right), \\
g_0^0(t) &= 1/t \int_0^t g_0(t) dt = g_0 \left[1 - (t/\tau_{sw})^{-1}\exp(-t/\tau_{sw})\right].
\end{align*}
\]  

We have verified this solution via numerical integration of Eq. 11 as shown later. Inserting this solution in Eq. 12, and reversing the rotating wave transformation, we get the following expressions for the components of Eq. 2:

\[
\begin{align*}
C_0(t) &= \cos\left(\gamma_0^0(t)/2\right) - 2\eta_0\sin\left(\gamma_0^0(t)/2\right), \\
C_1(t) &= i\exp(-i(\omega t + \phi))\sin\left(\gamma_0^0(t)/2\right) + 2\eta_0\cos\left(\gamma_0^0(t)/2\right),
\end{align*}
\]  
(12a)

where we have defined \(\Sigma \equiv (i/2)\exp(-i(2\omega t + 2\phi))\), and \(\eta_0 = (g_0 M/4\omega)\). To lowest order in \(\eta\), this solution is normalized at all times. Note that if one wants to carry this excitation on an ensemble of atoms using \(\pi/2\) pulse and measure the population of the state \(|1\rangle\) after the excitation terminates (at \(t = \tau\) when \(g^0(t)/\tau = \pi/2\) ), the result would be a output signal given by:

\[
|C_1(1, \tau, \phi)|^2 = \frac{1}{2} \left[1 + 2\eta_0\sin(2\omega\tau + 2\phi)\right],
\]  
(13)

which contains information of both the amplitude and the phase of the driving field. This is our main result.

A physical realization of this result can be appreciated best by considering an experimental arrangement of the type illustrated in Fig. 4. Here, a single group of atoms (e.g., atoms held in a dipole force trap) are subjected to a resonant, oscillating magnetic field, which is turned on adiabatically with a switching time-constant \(\tau_{sw}\), starting at \(t = 0\). After an interaction time of \(\tau\), the population of the excited state \(|1\rangle\) is determined instantaneously (i.e., with a time-constant much faster than \(1/\omega\) and \(1/g_0 M \)) by coupling this state to an optically excited state \(|2\rangle\) with a laser, and monitoring the resulting fluorescence. Such a measurement would correspond to the expression in Eq. 13. Defining the phase of the field at \(t = \tau\) to be \(\phi_\tau \equiv \omega\tau + \phi\), the expression of Eq. 13 can be rewritten as:

\[
|C_1(\tau)|^2 = \frac{1}{2} \left[1 + 2\eta\sin(2\omega\tau + 2\phi)\right].
\]  

In Fig. 4(a) we have shown the evolution of the excited state population \(|C_1(\tau)|^2\) as a function of interaction time \(\tau\), using the analytical expression of Eq. 13(b).
FIG. 1. Schematic illustration of an experimental arrangement for measuring the phase dependence of the population of the excited state |1⟩: (a) The microwave field couples the ground state (|0⟩) to the excited state (|1⟩). A third level, |2⟩, which can be coupled to |1⟩ optically, is used to measure the population of |1⟩ via fluorescence detection. (b) The microwave field is turned on adiabatically with a switching time-constant τ_sw, and the fluorescence is monitored after a total interaction time of τ.

FIG. 2. Illustration of the Bloch-Siegert Oscillation (BSO): (a) The population of state |1⟩, as a function of the interaction time τ, showing the BSO superimposed on the conventional Rabi oscillation. (b) The BSO oscillation (amplified scale) by itself, produced by subtracting the Rabi oscillation from the plot in (a). (c) The time-dependence of the Rabi frequency. Inset: BSO as a function of the absolute phase of the field.

Under the RWA, this curve would represent the conventional Rabi oscillation. However, we notice here some additional oscillations, which is magnified and shown separately in figure 2(b), produced by subtracting the conventional Rabi oscillation (sin²(g(t)/2)) from figure 2(a). That is, figure 2(b) corresponds to what we call the Bloch-Siegert Oscillation (BSO), given by \( \eta \sin(g'_0(\tau)\tau)\sin(2\phi_\tau) \). The dashed curve (c) shows the time-dependence of the Rabi frequency. These analytical results agree very closely to the results which are obtained via direct numerical integration of Eq.[3] Note that the BSO is at twice the frequency of the driving field, and its amplitude is enveloped by a function that vanishes when all the atoms are in a single state.

Consider next a situation where the interaction time τ, is fixed so that we are at the peak of the BSO envelope (which corresponds to a (π/2) pulse for the Rabi oscillation). We further assume that τ is long enough so that \( g_0(\tau) \approx g_{0M} \). The experiment is now repeated many times, with a different value of \( \phi \) each time. The corresponding population of |1⟩ is given by \( \eta_0 \sin(2\phi_\tau) = \eta_0 \sin(2(\omega \tau + \phi)) \), and is plotted as a function of \( \phi \) in the inset of figure. This dependence of the population of |1⟩ on the initial phase \( \phi \) (and, therefore, on the final phase \( \phi_\tau \)) makes it possible to measure these quantities.

As indicated above, these effects can be observed in an experiment where, for example, a stationary collection of atoms are excited repeatedly by a microwave field. However, a more robust process for observing this effect can be realized using an atomic beam. For illustration, consider first a situation where the atoms are emitted in regular intervals Δt, and propagate in the z direction with a fixed velocity \( v \). We can describe such a source by the line density of the number of atoms at a position \( z \) and at an instant \( t: M(z, t) = m \sum_{l=0}^{\infty} \delta[z - v(t - l\Delta t)] \).

The microwave field is assumed to be spatially varying, corresponding to a Rabi frequency that vanishes for \( z < z_0 \). For \( z \geq z_0 \), it is given by: \( g_0(z) = g_{0M} [1 - \exp(-z/z_{sw})] \) where \( z_{sw} = vt_{sw} \) represents the distance over which the field is switched on. Under these assumptions, and in the limit where \( \Delta t \rightarrow 0 \) (corresponding to a continuous, monovelocity atomic beam), it is easy to show that the normalized population \( S \) of |1⟩ measured at a position \( z = z_0 + v\tau \) (where \( \tau \) is the interaction duration corresponding to a π/2 pulse) as a function of time is given simply by \( [1/2 + \eta_0 \sin(2(\omega t + 2\phi))] \), assuming that the microwave field at this position is \( B = B_0 \cos(\omega t + \phi) \). Note that \( (\omega t + \phi) \) represents the absolute phase of the field as seen by the atoms when they arrive at this position. Thus, measurement of \( S \) directly reveals the absolute phase of the field, modulo 2π. Alternatively, one can mix \( S \) with another signal \( F \) corresponding to the second harmonic of a phase-shifted version of the microwave field (\( F = F_0 \cos[2(\omega t + \phi - \pi/2 - \theta)] \)), where \( \theta \) is a controlled, variable phase shift, and observe the ac component of this signal, which will be proportional to \( \cos(\theta) \).

Consider next the more easily realizable situation where the atomic beam is produced by an effusive oven. Such a beam is typically characterized by a normalized velocity distribution \( f(v) = 2v^3 u^{-4} \exp(-v^2/u^2) \), where \( u = \sqrt{2kT/m} \) is the most probable velocity, \( K \) is the Boltzman constant, \( T \) is the temperature and \( m \) is the mass of the atom. The atoms that contribute to \( S \)
come from all different velocity groups. As such, each group experiences a different initial phase (i.e., phase at \( z = z_0 \)), and one might think that this would cause the signal to wash out. However, notice that \( S \) corresponds not to the initial phase, but rather to the phase at the observation point. Since the atoms contributing to \( S \) all have, by definition, arrived at this point at the same time, the signal will have the same time dependence, independent of the velocity group. Explicitly, the expression for \( S \) now becomes:

\[
S(t) = \int_{v=0}^{\infty} dv f(v)\left[ |\sin(g_0'(\tau_v)\tau/2)|^2 + \eta \sin(g_0'(\tau_v)\tau_v) \sin(2\omega(t+\phi)) \right].
\]  

(14)

where \( \tau_v = \tau u/v \) is the effective interaction time for the atoms with velocity \( v \), and is a constant, corresponding to the interaction time for the most probable velocity \( u \). The terms inside the integral in Eq.14 are independent of \( t \) and \( \phi \), and the effect of this averaging is simply to reduce the amplitude of the oscillatory signal observed.

When a quantum bit (qubit) is represented by two non-degenerate states of a massive particle, it is necessary to apply a field at a frequency matching the energy difference between these states, in order to produce an arbitrary rotation of the qubit. In order to minimize the decoherence rate of such a qubit, one often chooses to use low energy spin transitions. In general, one is interested in performing these transitions as fast as possible. As such, it is desirable to use a strong Rabi frequency.

The ratio of the Rabi frequency to the qubit transition frequency is therefore not necessarily very small. One example of such a situation is already seen to occur in qubits based on trapped ions, for example [8–10]. Under this condition, one can not ignore the effect of the counter-rotating term. However, as we have shown here, the degree of excitation (e.g., the amplitude of the excited state) depends not only on the product of the Rabi frequency and the duration of the excitation, but also on the phase of the field at the time the interaction stops. Thus, one has to keep track of the phase of the excitation field at the location of the qubit [11,12]. In principle, the phase-tracking approach embodied in this paper itself can be used for this purpose. Alternatively, one has to limit the strength of the Rabi frequency to a level dictated by the precision required of the particular qubit operation involved. We note that this effect is present for both direct radio-frequency excitation, as well as for indirect Raman excitation, which are functionally equivalent [13,14]. Finally, we point out that by making use of distant entanglement, this mechanism may enable teleportation of the phase of a field that is encoded in the atomic state amplitude, for potential applications to remote frequency locking [16,19].

In conclusion, we have shown that when a two-level atomic system is driven by a strong periodic field, the Rabi oscillation is accompanied by another oscillation at twice the transition frequency, and this oscillation carries the information about the absolute phase of the driving field. One can detect this phase by simply measuring only the population of the excited state. We have also shown how this effect may be observed using an atomic beam even if it has a substantial velocity spread. Finally, we have shown how this effect has to be taken into account in qubit operations.

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