Nonlocal pseudospin dynamics in a quantum Ising chain

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Abstract

The existence of topological zero modes in nontrivial phase of quantum Ising chain results in not only the Kramers-like degeneracy spectrum, but also dynamic response for non-Hermitian perturbation in the ordered phase (2021 Phys. Rev. Lett. 126 116 401). In this work, we investigate the possible response of the degeneracy spectrum for Hermitian perturbations. We provide a single-particle description of the model in the ordered phase, associating with an internal degree of freedom characterized as a pseudospin. The effective magnetic field, arising from both local and nonlocal perturbations in terms of string operators, acts on the pseudospin. We show that the action of string operator can be realized via a quench under the local perturbations. As an application, any ground states and excited states for the Hamiltonian with perturbation can be selected to identify the quantum phase, by adding the other perturbations to trigger a quench and measuring the Loschmidt echo.

1. Introduction

Identifying the quantum phase diagram of a physical system is of vital importance in both condensed matter physics and quantum information science. In the past few decades, a large number of theoretical and experimental studies in this field have emerged [1–12]. The transverse field Ising chain [13–15], which is one of the paradigmatic model to explore quantum phase transition (QPT) and quantum information science, plays a key role in this realm. Experimentally, the transverse field Ising chain has direct realizations [2, 8], and is also achievable with ultracold atoms in optical lattices [10, 16]. In this model, the integrability has possibly attracted the attention of the researchers the most. In spite of the simplicity, it possesses all the basic elements of the QPT. The competition between the nearest neighbor interaction and the external magnetic field leads to an ordered phase and a disordered phase, which are separated by a quantum critical point, accompanied by a spontaneous symmetry breaking.

Methodologically, triggering the quantum quench dynamics [17–21] by suddenly changing system parameters is frequently used to study the QPT. After the quench, the system undergoes nonequilibrium dynamics [17], and one can measure the Loschmidt echo (LE) [22–27] to quantify the deviation of the evolved state from the initial state. Generally speaking, since the evolved state contains information of both the initial state and the postquench Hamiltonian, the behavior of LE can reflect the physical properties of the system. Besides the application in QPT, it is interesting to employ the quench protocol to manipulating the spin degrees of freedom in quantum information science. Experimentally, the observation of quench dynamics in a quantum Ising chain have been realized in a trapped-ion quantum simulator [20], where the ion chain is initialized in a ground state of the prequench Hamiltonian with zero magnetic field and suddenly apply a large magnetic field.

In this paper, we investigate the possible response of the degeneracy spectrum of the transverse field Ising chain to Hermitian perturbations, which is more convenient for experimental implementation than a non-Hermitian one [28]. A non-Hermitian Hamiltonian, which effectively describes the dynamics at short time for an open quantum system [29–31], is more complicated to be considered in experiment, since the form of non-Hermitian term depends on the coupling to the environment. Also, at a longer time, the effect of decoherence must be considered. On the other hand, most of the studies about quench dynamics focused on the ground state
and some simple state, such as saturated ferromagnetic state, as the initial states. We tried to extend the initial states to any excited states, by utilizing the identical split of all energy levels of the system under certain perturbation. We focus on the model with open boundary condition. It is shown that [28] the Kramers-like degeneracy spectrum in the ordered phase is related to the topological zero modes of the Kitaev chain [32], thus the degeneracy is topologically protected, in other words, robust against certain local perturbations. However, it is not easy to manipulate such an inner degree of freedom, as well as breaking the degeneracy with arbitrary local perturbation. Our motivations are to provide a quench protocol to identify the topology-related degeneracy spectrum and the quantum phases, and seek possible action to manipulate the inner degree of freedom for this model with Hermitian perturbations that result in the identical split in the spectrum. To this end, we provide a single-particle description of the model in the ordered phase, associating with an internal degree of freedom characterized as a pseudospin. The effective magnetic field acts on the pseudospin, which arises from both local and nonlocal perturbations in terms of string operators. As an application, a quantum state living in the two-fold degenerate subspace of the system in the ordered phase can be considered as a single qubit, and with the effective magnetic field, the time evolution operator can achieve two quantum gates—phase gate and Hadamard gate. To our knowledge, there is no direct experimental realization of the string operators, although some multi-site interacting terms have been discussed in other theoretical spin models [33, 34]. Thus, we provide a scheme to realize the actions of the string operators discussed in this paper. We simplify the perturbations and show that the string operator action can be realized via a quench under the local perturbations. Any ground state or excited state for the Hamiltonian with perturbation can be selected for identifying the quantum phase, by adding the other perturbations to trigger a quench and measuring the LE. A possible application in the thermal state is also discussed. It is worth noting that most of the studies about QPT and quench dynamics focused on the ground state of the system. In addition, numerical simulations for a finite-size system are provided to support our results.

The remainder of this paper is organized as follows. In section 2, we present the transverse field Ising chain and its symmetries. In section 3, we introduce the pseudospin description for the model in the ordered phase and investigate the response of the degenerate spectrum to perturbations. In section 4, we discuss the simplification of the perturbations, the realization of the nonlocal operator, and the possible applications in the QPT and thermal state with numerical results of LEs. Finally, we summarize our results in section 5.

2. Model and symmetries

In this section, we present the Hamiltonian and a brief review on its basic properties, based on which we perform our investigations in this work. The model considered is the transverse field Ising chain with open boundary condition, defined by the Hamiltonian

\[ H_0 = -J \sum_{j=1}^{N-1} \sigma_j^x \sigma_{j+1}^x + g \sum_{j=1}^{N} \sigma_j^z, \]  

(1)

where \( \sigma_j^\alpha (\alpha = x, y, z) \) are the Pauli operators on site \( j \) and parameter \( g (g > 0) \) is the transverse field strength. For simplicity, the following discussion assumes that \( J = 1 \). It can be checked that the model respects two symmetries. The first one is the parity symmetry, that is, the parity operator \( p = \prod_{j=1}^{N} (-\sigma_j^z) \) is commutative with the Hamiltonian. The second one is a little subtle and is crucial to our main conclusion [28, 35]. The model with periodic boundary condition is exactly solvable and has been well studied [13]. At zero temperature, the QPT at \( g = 1 \) separates an ordered phase of the system \( (g < 1) \) from a disordered phase \( (g > 1) \). However, when we consider the model with open boundary condition, it possesses an exclusive symmetry in the ordered phase \( g < 1 \) in thermodynamic limit. Defining a nonlocal operator

\[ D = \frac{1}{2} \sqrt{1 - g^2} \sum_{j=1}^{N} g^{j-1} \prod_{l<j} (-\sigma_l^z) \sigma_j^z - i \prod_{l<j} (-\sigma_l^z) \sigma_{N-j+1}^z, \]  

(2)

(where \( i = \sqrt{-1} \)), we have \( D \) and \( D^\dagger \) commute with the Hamiltonian [28], which is referred to as edge-spin symmetry, since it is the outcome of the edge operator of the Kitaev chain [32]. Operator \( D \) is a fermion operator, obeying the relations \( \{D, D^\dagger\} = 1 \) and \( D^2 = (D^\dagger)^2 = 0 \). It should be noted that it is contingent on the following conditions: \( g < 1 \), a large \( N \) limit, and open boundary. In addition, operator \( D \) is non-universal and \( g \)-dependent.

From these symmetries, we have following implications: the complete eigenstates \( \{ \left| \psi^+_{m} \right>, \left| \psi^-_{m} \right> \} \) of \( H_0 \) with eigenenergy \( \epsilon_{+m}, H_0 \left| \psi^+_{m} \right> = \epsilon_{+m} \left| \psi^+_{m} \right>, \) span two invariant subspaces for any value of \( g \) where \( \pm \) denotes the eigenvalues of parity operator \( p \). Importantly, within the region \( g < 1 \), the edge-spin symmetry guarantees the existence of eigenstates degeneracy \( \epsilon_{+m} = \epsilon_{-m} = \epsilon_m \), referred to as Kramers-like degeneracy. Accordingly, we also

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**References:**

1. [28] J. Phys. Commun. 6 (2022) 095006 K L Zhang and Z Song
have the relations
\[ i(D^\dagger - D)(\ket{\psi_n^+} \pm \ket{\psi_n^-}) = \pm i(\ket{\psi_n^+} \mp \ket{\psi_n^-}), \]
and
\[ p(\ket{\psi_n^+} \pm \ket{\psi_n^-}) = \ket{\psi_n^+} \mp \ket{\psi_n^-}, \]
which play an important role in the quench dynamics, as demonstrated in the following section.

3. Nonlocal pseudospin and Loschmidt echo

It is not surprising that a Hermitian perturbation can lift the degeneracy. However, it should lead to a fascinating dynamic phenomenon if an identical split in each level in the spectrum is obtained, which enable the same oscillatory dynamics of the excited states as that of the ground state. Moreover, hybridizing two robust degenerate states on demand is a central task of quantum information processing, since these states are immune to weak local perturbations.

We first focus on the ordered quantum phase $0 < g < 1$, considering a perturbed Hamiltonian
\[ H = H_0 + H', \]
with $H'$ being the combination of three types of actions,
\[ H' = \kappa_x (D^\dagger + D) + i \kappa_y (D^\dagger - D) + \kappa_z p. \]

We note that the perturbation is nonlocal, containing the string operators $\prod_{l} (-\sigma_i^z)$. The perturbed Hamiltonian in equation (3) can be changed to the fermionic form by Jordan-Wigner transformation [36]. In this representation $H'$ only contains the local terms, however, this is impractical in experimental aspect since $H'$ break the fermionic parity of the system [32, 37, 38]. In contrast, the parity symmetry of a spin system does not need to be conserved.

Any pair of degenerate eigenstates ($\ket{\psi_n^+}$, $\ket{\psi_n^-}$) with energy $\varepsilon_n$ spans a diagonal block in the form
\[ B \cdot \sigma = \begin{pmatrix} \kappa_x & \kappa_y \\ \kappa_y & \kappa_x \end{pmatrix}, \]
with parameter vector $B = (\kappa_x, \kappa_y, \kappa_z)$ and Pauli matrix $\sigma = (\sigma_x, \sigma_y, \sigma_z)$. Therefore, under the basis ($\ket{\psi_1^+}$, $\ket{\psi_1^-}$, $\ket{\psi_2^+}$, $\ket{\psi_2^-}$, $\ket{\psi_3^-}$), we get an equivalent Hamiltonian for $H$
\[ H_{eq} = \bigoplus_{n=1}^{Z_n} (B \cdot \sigma + \varepsilon_n I_2), \]
where $I_2$ denotes the $2 \times 2$ identity matrix. In this single-particle description with pseudo spin, it is obvious that the present of $B$ splits the degeneracy of the energy levels $\varepsilon_n$. This is not contradictory with the previous claim about the degeneracy of the energy spectrum is robust. The degeneracy is robust against the random variations on the uniform distribution of parameters $(J, g)$ in $H_0$. This can be proved by the robustness of the edge-spin symmetry: it can be checked that the commutation relation $[D, H_0] = 0$ still hold for the system with local perturbation on system parameters in large $N$ limit [39], leading to the degeneracy that is robust against local perturbation. This means that when site-dependent $(J_n, g)$ are disordered and the field $B$ is zero, a qubit is stable in the degenerate subspace, without the influence of dynamics phase factor.

The introduction of the perturbation term $H'$ allows two possible applications through time evolution. First, the operation on the robust degenerate state can be realized. We stress that this is not an outcome of any Hermitian perturbations lifting degeneracy. It needs three independent Hermitian perturbations to realize the full operation on the robust degenerate state. In the degenerate subspace with index $n$, an arbitrary state $\ket{\psi_n} = \alpha \ket{\psi_n^+} + \beta \ket{\psi_n^-}$ acts as a single qubit, where $\alpha$ and $\beta$ are complex numbers encoding quantum information. An arbitrary unitary operation on this state can be realized by the time evolution operator
\[ U(t) = e^{-i\kappa_x t} e^{-iB \cdot \sigma}, \]
\[ = e^{-i\kappa_x t} \left[ \cos \left(\frac{B \cdot \sigma}{2B} t\right) - i \frac{B \cdot \sigma}{2B} \sin \left(\frac{B \cdot \sigma}{2B} t\right) \right], \]
by choosing appropriate parameters $B = (\kappa_x, \kappa_y, \kappa_z)$ and evolved time $t$. For example, when choosing $B = (0, 0, \kappa_z)$, it realizes the action of the phase gate
In this section, we analyze the perturbation term and try to propose a practical scheme to demonstrate the dynamic detection of the phase diagram. We note that \( p \) is a typical string operator, which is a challenge to realize.
in experiment. We first consider the possible realization of the perturbation in practice. The perturbation term $H'$ in equation (6) commutes to the unperturbed Hamiltonian $H_0$, then the whole Hamiltonian $H_0 + H'$ is exactly solvable. However, the operators in $H'$ are $g$ dependent and need to be deliberately designed in practice.

4.1. Simplified perturbations

First, we consider a simplification of the perturbation in equation (6), that is, only the dominant terms of $H'$ are taken into account:

$$H'_0 = \kappa_x \sigma_1^x - \kappa_y \prod_{i=1}^{N-1} (-\sigma_1^z) \sigma_N^y + \kappa_z \prod_{i=1}^{N} (-\sigma_1^z)$$

$$= \kappa_x \sigma_1^x - i\kappa_y g \sigma_N^y + \kappa_z b,$$

which is equal to a small $g$ limit of $H'$. We find that the local operator $\sigma_1^x$ and the nonlocal operator $g$ are two elemental actions of the perturbations. The advantages of considering this perturbation are two folds: (i) $H'_0$ is independent of the system parameter $g$, allowing us to implement the LE detection when the system parameters are unknown. (ii) The form of $H'_0$ is simpler and thus is more possible for experimental implementation. To see the effects of the perturbation $H'_0$ on the spectrum of $H_0$, in figure 1, we present the spectrum of the low-lying eigenstates of the Hamiltonian $H = H_0 + H'_0$ for different parameters $(\kappa_x, \kappa_y, \kappa_z)$ and $g$. We can see that in the ordered phase, the perturbations with different $(\kappa_x, \kappa_y, \kappa_z)$ all lead to almost equal level splitting for a fixed $g$ and as $g$ varying, which suggests that perturbation $H'_0$ has the same effect as that in equation (6). As a local perturbation, $\sigma_1^x$ can lift the degeneracy [see figure 1(b)], which has the same effect as the nonlocal case in figures 1(c) and (d). We also note that perturbations $\kappa_x \sigma_1^x$ and $-i\kappa_y g \sigma_N^y$ both break the parity symmetry, while $\kappa_z b$ preserves it [see the red (even parity) and black (odd parity) lines in figures 1(a) and (d)].

In contrast, we investigate the effect of another type of perturbation, containing operator $\sigma_N^y$. Without loss of generality, we consider the case with Hamiltonian

$$H = H_0 + \sum_{j=1}^{N} \gamma_j \sigma_j^y,$$

where $\gamma_j$ is an arbitrary set of real numbers. Applying a set of local transformation [35]

$$\tau_j^x = \sigma_j^x,$$

$$\tau_j^y = \eta_j \sigma_j^y - \eta_j \sigma_j^z,$$

$$\tau_j^z = \eta_j \sigma_j^z + \eta_j \sigma_j^y,$$

Figure 1. Spectrum of the low-lying eigenstates for Hamiltonian $H = H_0 + H'_0$ as a function of $g$ with parameters $(\kappa_x, \kappa_y, \kappa_z)$ (a) $(0, 0, 0)$, (b) $(0, 0, 1)$, (c) $(0, 0, 1)$, and (d) $(0, 0, -0.1)$, obtained numerically through exact diagonalization. The red and black colors of the lines in (a) and (d) denote even and odd parities of the corresponding eigenstates, respectively, while the eigenstates of spectrum (b) and (c) are not the eigenstates of parity operator $p$. The spacing of the energy splits equal to $2\sqrt{\kappa_x^2 + \kappa_y^2 + \kappa_z^2}$ approximately. Here $E_0$ is the ground-state energy. System parameters are $N = 10$ and $J = 1$. 

(17)
with the factors $\eta_j^+ = 1/\sqrt{1 + \gamma_j^2}$ and $\eta_j^- = \gamma_j/\sqrt{1 + \gamma_j^2}$, we have

$$H = -\sum_{j=1}^{N-1} \tau_j^+ \tau_{j+1}^- + \sum_{j=1}^{N} \sqrt{\gamma_j^2 + \gamma_{j+1}^2} \tau_j^-,$$

(20)

which is still a transverse field Ising chain since the new spin operators still satisfy the Lie algebra commutation relations

$$[\tau_j^\mu, \tau_k^\nu] = \sum_{\lambda=x,y,z} 2i \epsilon^{\mu\nu\lambda} \tau_j^\lambda.$$

(21)

Then weak perturbation $\kappa \sigma_j^z \gamma_j$ cannot lift the degeneracy of $H_0$, since small derivation from uniform $g$ does not affect the topological zero modes [32].

4.2. Realization of string operator action

Second, we consider to realize the action of string operator $p$ by the time evolution under a time-dependent local Hamiltonian

$$H_p(t) = g(t) \sum_{j=1}^{N} \sigma_j^z,$$

(22)

which describes the action of an extra time-dependent transverse field. Here the coefficient is defined as

$$g(t) = \begin{cases} \frac{\pi}{2\Delta}, & 0 < t \leq \Delta \\ 0, & \text{otherwise}. \end{cases}$$

(23)

After time $\Delta$, the effect of $H_p(t)$ on the degenerate state $|\psi_0\rangle = \alpha |\psi^+_0\rangle + \beta |\psi^-_0\rangle$ can be expressed as the time evolution operator

$$U(\Delta) = \exp \left[ -i \int_0^\Delta H_p(t) \, dt \right] = \prod_{j=1}^{N} \exp \left[ -i \int_0^\Delta \frac{\pi}{2\Delta} \sigma_j^z \, dt \right] = i^N \prod_{j=1}^{N} (-\sigma_j^z).$$

(24)

It indicates that the time evolution operator takes the role of the operator $p$, i.e.,

$$U(\Delta)(\alpha |\psi^+_0\rangle + \beta |\psi^-_0\rangle) = i^N (\alpha |\psi^+_0\rangle - \beta |\psi^-_0\rangle).$$

(25)

To verify this result, we perform numerical simulation for a quench process defined as

$$H_{\text{pre}} = H_0 + \kappa \sigma_1^z,$$

$$H_{\text{pos}} = H_0 + H_p.$$

(26)

The purpose of performing the prequench is to lift the degeneracy in the ordered phase, so that the eigenstates of the system become certain. The initial state is taken as the ground state of the prequench Hamiltonian. According to the previous results, it can be approximately expressed in the following form

$$|\Phi(0)\rangle = \sin \frac{\theta}{2} |\psi^+_g\rangle - \cos \frac{\theta}{2} e^{i\varphi} |\psi^-_g\rangle.$$

(27)

Then in the ordered phase, the expected final state is

$$|\Phi(\Delta)\rangle = U(\Delta) |\Phi(0)\rangle = i^N \left( \sin \frac{\theta}{2} |\psi^+_g\rangle + \cos \frac{\theta}{2} e^{i\varphi} |\psi^-_g\rangle \right).$$

(28)

It is expected that the LE obeys $L(\Delta) = \cos^2 \theta = \kappa_0^2 / |B|^2 = 0$. While in the disordered phase, the ground state of $H_0$ is non-degenerate and is separated from the excited state by an energy gap, then it is expected that $L(\Delta) \approx 1$. This scheme realizes the action of string operator $p$, i.e., the action of the first order term of time evolution operator $e^{-i\kappa g \sigma_1^z}$. The numerical results of LEs obtained by exact diagonalization with system parameters $g = 0.5$ and 1.5 are presented in figure 2(a), which are in accord with our analysis. Similarly, the actions of other string operators such as $\prod_{j=1}^{N} (-\sigma_j^z) \sigma_j^\alpha$ ($\alpha = x, y, z$), can be realized by this scheme.
4.3. Quantum phase transition

According to the conclusions in sections 3 and 4.1, it is expected that the dynamics behavior of LE can be utilized to identify different quantum phases when we consider a simplified version of the perturbation in equation (17). It should lead to the similar oscillatory behavior of LE described in equation (16), since the non-degenerate eigenstates are not sensitive to the perturbation of $H'_0$. These features allow us to observe significantly different dynamical behaviors in different quantum phases when the initial state is chosen as any eigenstate of $H\text{pre}$.

In the following, we consider the numerical simulation of quench process under the Hamiltonian

$$H\text{pre} = H_0 + H_0^x(\kappa_x = 0.05, \kappa_y = 0, \kappa_z = 0),$$

$$H\text{pos} = H_0 + H_0^x(\kappa_x = 0.05, \kappa_y = 0.1, \kappa_z = 0),$$

(29)

where $H_0^x(\kappa_x, \kappa_y, \kappa_z)$ is defined in equation (17). Here $\kappa_x$, $\kappa_y$, and $\kappa_z$ should be small compared to the energy scale of the system, so that $H_0^x$ can be considered as a perturbation. We choose nonzero $\kappa_x$, $\kappa_y$, and zero $\kappa_z$ in order to compare with the analytical result in equation (16). Similar results can be obtained for other $B$ values, as long as the parameter vectors $B = (\kappa_x, \kappa_y, \kappa_z)$ for the prequench and posquench Hamiltonians are not parallel or antiparallel to each other. The initial state is prepared as the ground state of $H\text{pre}$. In figure 2(b), we presented the LEs for different $g$, which are calculated by exact diagonalization. We can see that the results are in accord with our predictions for both phases. For small $g$, the minimum value of LE is the same as that of equation (16), which is estimated under the complex version of quench term in equation (6).

This verifies that the proposed quench protocol can be utilized to identify different quantum phase of the transverse field Ising chain. To compare with the phase diagram in the thermodynamic limit, where the ordered phase and the disordered phase are separated by the critical point $g_c = 1$ [13], we introduce the average LE in the time interval $[0, T]$, which captures the change of the dynamics characteristic near the critical point, and can be used to infer the behavior of LE for a system with larger $N$. The average LE is defined as

$$L(g) = \frac{1}{T} \int_0^T L(t) dt,$$

(30)

the value of which in the ordered phase can be estimated from equation (16) in long-time limit, that is

$$L(g \ll 1) = \frac{2\kappa^2 + \kappa^2_y}{2(\kappa^2_x + \kappa^2_y)}.$$

(31)

While in the disordered phase, it is expected that $L(g > T) = 1$. The numerical results of average LEs for different $g$ and $N$ are presented in figure 2(c), which are obtained by exact diagonalization. We can see that when the system size is larger, the average LE is closer to the ideal values (gray dashed line) that are expected in the thermodynamic limit.

4.4. Thermal state

Now we discuss a possibility of applying the quench protocol to the thermal state when $g < 1$. In the previous section, we have known that for the Ising chain with parameters $g < 1$, the robust degeneracy occurs not only in the ground states, but in all energy levels [see figure 1(a)]. In general, a thermal state of system $H_0$ with temperature $\beta^{-1}$ can be written as $\rho_0 = e^{-\beta H_0}/Z e^{-\beta E_0}$, which preserves no quantum information. However, the robust degeneracy of the spectrum may enable a thermal state to preserve the quantum information in each degenerate subspace, in the case that the thermalization is induced by local perturbation from the environment.
Consider such a state as an initial state, with density matrix
\[
\rho = \frac{\sum_{n=1}^{2^{N-1}} e^{-i\theta_n} |\Phi_n\rangle \langle \Phi_n|}{\sum_{n=1}^{2^{N-1}} e^{-i\theta_n}},
\] (32)
where
\[
|\Phi_n\rangle = \sin \frac{\theta_n}{2} |\psi_e^n\rangle - \cos \frac{\theta_n}{2} e^{i\varphi_n} |\psi_o^n\rangle.
\] (33)

At first, we estimate the dynamics of thermal state \(\rho\) under the quenched Hamiltonian
\[
H_{\text{pos}} = H_0 + \kappa_s (D^\dagger + D).
\] (34)
We choose this quenched Hamiltonian since the simplified version of the perturbation term is the simplest. Similar results can be obtained for the quench provided in equation (6). In fact, we have
\[
| \langle \Phi_n| e^{-iH_{\text{wet}}t} |\Phi_n\rangle |^2 = \cos^2 (\kappa_s t) + \sin^2 (\kappa_s t) \sin^2 \theta \cos^2 2\theta
\] (35)
which is a periodic function of time. We consider two types of distribution of \(\{\theta_n, \varphi_n\}\), which are encoded with different informations. (i) Random distribution: \(\{\theta_n\}\) is taken as a random sample that is uniformly distributed over the interval \([0, \pi]\) and \(\{\varphi_n\} = \{2\theta_n\}\). Then the LE can be estimated by ignoring the Boltzmann factor in high temperature and large \(N\) limits, that is
\[
L(t) = \frac{1}{2\pi} \sum_{n=1}^{2^{N-1}} | \langle \Phi_n| e^{-iH_{\text{wet}}t} |\Phi_n\rangle |^2 \\
\approx \frac{1}{\pi} \int_0^\pi d\theta [\cos^2 (\kappa_s t) + \sin^2 (\kappa_s t) \sin^2 \theta \cos^2 2\theta] \\
= \frac{1}{4} + \frac{3}{4} \cos^2 (\kappa_s t).
\] (36)
(ii) Fixed parity: the thermal state consists of the levels with the same parity, that is, \(\theta_n = \pi\) and \(\{\varphi_n\}\) is taken as a random sample in \([0, 2\pi]\). Then we have
\[
L(t) \approx \frac{1}{2\pi} \int_0^{2\pi} d\varphi \cos^2 (\kappa_s t) \\
= \cos^2 (\kappa_s t).
\] (37)
For both cases, \(L(t)\) are periodic functions but with different amplitudes, and note that the minimal value for the latter is zero. When the posquench Hamiltonian is fixed, the difference of the amplitudes is originate from the initial states with parameters \(\theta_n\) and \(\varphi_n\), which may preserve different quantum informations, and reflect the coherence between the states with even and odd parity. When the initial state contains only one component of parity, the amplitude is 1.

In practice, based on the above analysis, one can consider the following quenched Hamiltonian instead of equation (34)
\[
H_{\text{pos}} = H_0 + \kappa_s \sigma_1^z,
\] (38)
and the definition of LE for density matrix is
\[
L(t) = \langle \text{Tr} \sqrt{\rho(0) \rho(t) \sqrt{\rho(0)}} \rangle^2,
\] (39)
which is also known as the Uhlmann fidelity [40, 41], characterizing the similarity between the initial state \(\rho(0)\) and evolved state \(\rho(t) = e^{-iH_{\text{wet}}t}\rho(0)e^{iH_{\text{wet}}t}\). The numerical results for the two types of random initial states described above are presented in figure 3(a). As a comparison, the numerical result for initial thermal state with canonical ensemble distribution \(\rho_0 = e^{-\beta H_0}/\text{Tr}e^{-\beta H_0}\) is also given. We can see that for initial states \(\rho\) of two cases, the LEs are close to the results in equations (36) and (37), although the definitions of LEs and the forms of the quench Hamiltonian are different. For the initial thermal state \(\rho_0\), the dynamics is not sensitive to the perturbation in equation (38). We can see that the amplitudes reflect the coherence between the states with even and odd parity. The amplitude is zero for the canonical ensemble distribution \(\rho_0\).

Since the constructions of the thermal states for cases (i) and (ii) are based on the two-fold degenerate spectrum (see the definition in equation (32), where \(E_n = \epsilon_n^e = \epsilon_n^o\) is the energy for degenerate states \(|\psi_n^e\rangle\)), the definition of the thermal state in case (i) and (ii) is absent in the disordered phase, where the energy levels are not degenerate. Therefore, the quench dynamics in the disordered phase can be discussed only for the initial state of canonical ensemble distribution \(\rho_0\). To investigate the behavior of LE in the disordered phase, numerical calculations are carried out with the initial state \(\rho_0\) for different \(g\), and the results are presented in figure 3(b). As we can see, the LEs are not sensitive to the variation of parameter \(g\), meaning that the phase transition can not be detected for thermal ensembles by the quenched Hamiltonian considered in this paper.
To see the temperature dependence, we carried out the numerical calculations. The results in figure 4(a) show that for case (i) with randomly distributed phase factors, the amplitudes of LEs are slightly different at different inverse temperature $\beta$. In figures 4(b) and (c), we can see that the LEs are both not sensitive to the change of temperature for case (ii) with fixed parity and canonical ensemble distribution $\rho_0$.

The system size $N$ mainly affects the pseudo critical point $g = g_{pc}$ ($g_{pc} \to 1$ when $N \to \infty$) for the transition between the degenerate and non-degenerate regions. Since the discussion on the thermal states focus on the degenerate region, the finite size effect can be avoided by choosing $g \ll 1$.

5. Summary

In summary, we have studied the consequence of the Hermitian nonlocal perturbation term on the transverse field Ising chain. The Hermitian perturbations is more convenient for experimental implementation than the non-Hermitian method. We proposed a pseudospin description for the Hamiltonian with perturbation term. In this description, the perturbation acts as an effective magnetic field, which lift the degenerate spectrum of the Hamiltonian in the ordered phase. The identical split in each level enable the same oscillatory dynamics for the excited states as that for the ground state, which means that the quench protocol can be applied to the ground state, as well as the excited states. We have shown that the string operator action can be realized via a quench process under the local perturbations. As an application, it is demonstrated that any ground states and excited states for the Hamiltonian with perturbation can be selected for identifying the quantum phase, by adding the other perturbation to trigger a quench and measuring the LE. Our method provides another option to identify the quantum phases, while it is failed for the thermal state. It is possible to be improved by seeking for other quench protocol, such as using other postquench Hamiltonian. It can be applied to other model with open boundary condition, as long as the corresponding edge operator exists. Our work, including the numerical results of LEs for a small-size system, provides a possible realization of the nonlocal operation as well as alternative quench protocol to detect the QPT. In addition, the result may shed light on the protocol of quantum information processing based on nonlocal pseudospin as qubit.

Figure 3. (a) Numerical results of LEs for the quench processes under Hamiltonian in equation (38) and three different initial thermal states with $\beta = 1$. Here the legend ‘$\rho_0$ Case (i)’ denotes randomly distributed phase factors; ‘$\rho_0$ Case (ii)’ denotes fixed parity; and ‘$\rho_0$’ is the canonical ensemble distribution. The parameters of the system are $g = 0.4, N = 10$ and $\kappa_x = 0.1$. (b) Numerical results of LEs in different phases for the initial state of canonical ensemble distribution $\rho_0$. Other parameters are $\beta = 1, N = 10$ and $\kappa_x = 0.1$. 

Figure 4. (a) Numerical results of LEs for different inverse temperatures $\beta$ and three different initial thermal states: (a) Case (i) randomly distributed phase factors. (b) Case (ii) with fixed parity. Here the same random distribution of $\{\theta_n \varphi_n\}$ are used for different $\beta$ to produce comparable results. (c) Canonical ensemble distribution $\rho_0$. The parameters of the system are $g = 0.4, N = 10$ and $\kappa_x = 0.1$. 

To see the temperature dependence, we carried out the numerical calculations. The results in figure 4(a) show that for case (i) with randomly distributed phase factors, the amplitudes of LEs are slightly different at different inverse temperature $\beta$. In figures 4(b) and (c), we can see that the LEs are both not sensitive to the change of temperature for case (ii) with fixed parity and canonical ensemble distribution $\rho_0$.
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Data availability statement

All data that support the findings of this study are included within the article (and any supplementary files).

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