Additive renormalization of the specific heat of O(n) symmetric systems in three-loop order

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Abstract

We present three-loop formulas for the additive renormalization constant $A(u, \epsilon)$ and associated renormalization group function $B(u)$ for the specific heat of the O(n) symmetric $\phi^4$ model. Using this result, we obtain also the amplitude function above $T_C$ within the minimally renormalized theory at fixed $d = 3$. At the fixed point, the three-loop correction to $B(u)$ turns out to be small (about 3% for $n = 2$). We note that a correction of this size may become important at the level of accuracy expected in future experiments.

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Field-theoretic renormalization group (RG) calculations based on (Borel) resummations of several orders of perturbation theory have yielded accurate predictions for the critical exponents [1] and for many of the universal amplitude ratios [2] of O(n) symmetric systems. For $n > 1$, however, similar predictions for amplitude ratios involving quantities defined below $T_C$ are not available since the relevant perturbation series have not yet been extended to sufficiently high order for resummations to be effective. It has been pointed out recently [3], that such higher order calculations for the amplitude functions of the specific heat and superfluid density would be needed for a fully quantitative test of universality along the
λ-line of 4He. Another quantity which enters the formulas for the amplitude ratios and which has, so far, been computed only in low order is the renormalization group function $B(u)$ associated with the additive part of the renormalization of the specific heat. Like the amplitude functions, this function has additional relevance for the analysis of experimental data in the nonasymptotic region \[4–7\].

On the basis of specific heat measurements taken in earth orbit, Lipa et al. [8] have shown that the rounding of data near the λ-transition of 4He due to gravity-induced pressure gradients [9], can be avoided for reduced temperatures as small as $t \simeq 10^{-9}$. Previously, these effects restricted the range of useful data to temperatures $10^{-6} \lesssim t \lesssim 10^{-2}$ implying the need for theoretical constraints in the analysis [10]. An unconstrained fit to the data in Ref. [8] yielded the critical exponent value $\alpha = -0.01285(38)$ and an estimate $A^+/A^- = 1.054(1)$ for the ratio of leading amplitudes. The uncertainty in this value of $\alpha$ is smaller than that of the “best” RG prediction [11] $\alpha = -0.016(6)$ by about an order of magnitude—a clear call for greater theoretical accuracy. Further experiments in reduced gravity have been planned [12].

The purpose of this note is to examine the three-loop approximation to $B(u)$ from the standpoint of the minimally renormalized $\phi^4$ theory in fixed dimension [5,13–16]. This function is expected to deviate from its leading order approximation $B(u) \simeq n/2$ by a small amount of $O(\eta)$ [5,14,17], where $\eta$ is the exponent describing the decay of spatial correlations at $T = T_C$. While the role played by $B(u)$ is expected to be a minor one relative to the exponents and amplitude functions [4,14,16], it is nevertheless of interest to know whether a correction of this size would become significant at the higher level of accuracy expected in reduced gravity experiments.

The renormalization of the specific heat within the minimal subtraction scheme at fixed dimension has been described in detail in Ref. [15]. For definitions and notation, see also Refs. [3,13,10]. In three-loop order, the additive renormalization $A(u, \epsilon)$ and RG function $B(u)$ are given by
\[ A(u, \epsilon) = -2n \frac{1}{\epsilon} - 8n(n + 2) \frac{u}{\epsilon^2} + a \frac{u^2}{\epsilon^3} + O(u^3), \]  

\[ a = -32n(n + 2) \left[ (n + 4) - \frac{5}{3} \epsilon + \frac{1}{8} \epsilon^2 \right], \]  

\[ B(u) = \frac{n}{2} \left[ 1 + 6(n + 2)u^2 + O(u^3) \right], \]

which we have obtained within the “massless” theory (that is, for \( k \neq 0 \) and \( T = T_C \)) since the pole terms are more readily evaluated there. The relevant vacuum diagrams with two \( \phi^2 \) insertions are shown in Fig. 1; their contributions near \( d = 4 \) are

\[ I_A = 2nA_d k^{-\epsilon} \frac{1}{\epsilon} \left[ 1 + \frac{\epsilon}{2} + \left[ 2 - \zeta(2) \right] \frac{\epsilon^2}{4} + O(\epsilon^3) \right], \]  

\[ I_B = -8n(n + 2)u_0 A_d^2 k^{-2\epsilon} \frac{1}{\epsilon^2} \left[ 1 + \epsilon + \left[ 5 - 2\zeta(2) \right] \frac{\epsilon^2}{4} + O(\epsilon^3) \right], \]  

\[ I_C = 32n(n + 2)^2 u_0^2 A_d^3 k^{-3\epsilon} \frac{1}{\epsilon^3} \left[ 1 + \frac{3}{2} + \left[ 9 - 3\zeta(2) \right] \frac{\epsilon^2}{4} + O(\epsilon^3) \right], \]  

\[ I_D = -\frac{16}{3} n(n + 2)u_0^2 A_d^3 k^{-3\epsilon} \frac{1}{\epsilon^3} \left[ 1 + \frac{15}{4} \epsilon + O(\epsilon^2) \right], \]  

\[ I_E = 64n(n + 2)u_0^2 A_d^3 k^{-3\epsilon} \frac{1}{\epsilon^3} \left[ 1 + 2\epsilon + \left[ 13 - 3\zeta(2) \right] \frac{\epsilon^2}{4} + O(\epsilon^3) \right]. \]

To obtain \( I_E \), we have used Eq. (2.20) of Ref. [18]. The geometric factor \( A_d = 2^{2-d} \pi^{-d/2} \Gamma(3-\frac{d}{2})/(d-2) \) is left unexpanded \[5\]. In three dimensions, our formula for the amplitude function above \( T_C \) reads

\[ F_+(1, u, 3) = -n - 2n(n + 2)u + bu^2 + O(u^3), \]  

\[ b = -4n(n + 2) \left[ n + 4 \ln \frac{4}{3} - \frac{7}{27} \right]. \]

For \( n = 2 \), the \( O(u^2) \) term in Eq. (2) is roughly 3% of the leading term and, since \( \eta \simeq 0.04 \) \[13\], is consistent with the \( O(\eta) \) estimate of Ref. [17] for the net contribution of all higher order terms. It has been suggested \[3,14,17\] that the terms beyond leading order should contribute less than 1% to the function \( B(u) \) and yet, although this contribution is expected to be small, it is not at all clear that it should be so small. One should bear in mind that low order perturbative expressions, such as Eq. (2), cannot by themselves be regarded as
reliable in a purely quantitative sense and that it is usually difficult to anticipate which (low) order of perturbation theory will provide the “best” approximation in any given situation. Indeed, this is the motivation behind the resummations of higher order series that have so far yielded accurate predictions for the exponents and amplitude ratios.

With the above caveat, therefore, let us consider the \( O(u^2) \) term in Eq. (2) to be \( O(\eta) \) and examine its effect on the amplitude ratios [13]

\[
\frac{A^+}{A^-} = \left[ \frac{2Q^*_+}{Q^*_-} \right]^{\frac{\alpha}{\alpha F^*_+ + 4\nu B^*}} \frac{\alpha F^*_+ + 4\nu B^*}{\alpha F^*_+ + 4\nu B^*}, \tag{9}
\]

\[
R_T^{\xi} = \left[ \frac{Q^-_+}{4} \right]^{2/3} (\alpha F^*_+ + 4\nu B^*)^{1/3} \frac{(4\pi)^{2/3}}{G^*}, \tag{10}
\]

where \( Q_\pm, F_\mp \) and \( G \) are the amplitude functions for the correlation lengths, the specific heat below \( T_C \) and the superfluid density, respectively; the asterisk denotes fixed point values. We make use of the Borel summation results given in Refs. [13,14,16] and of the relations \( Q^*_+ = 2\nu P^*_+ \) and \( Q^*_- = 3 - 2Q^*_+ \) where \( P_+ \) is the amplitude function for the quantity \((\partial r_0/\partial \xi^{-2})_{\nu_0} [13]\). In the absence of Borel results for \( n > 1 \) below \( T_C \), we use the most reliable low order approximations for \( F_\mp \) and \( G \), which turn out to be given already in one-loop order [3]. We also set \( u^* = 0.0405, \alpha = 0.11 \ (n = 1) \) and \( u^* = 0.0362, \alpha = -0.013 \ (n = 2) \) and fix \( \nu \) according to \( \nu = (2 - \alpha)/3 \).

The values of \( A^+/A^- \) and \( R_T^{\xi} \) given in Table I illustrate the size and direction of the effect of including the \( O(u^2) \) term in Eq. (2) when all other quantities in Eqs. (9) and (10) are kept fixed. Also shown in the table is the value for \( A^+/A^- \ (n = 1) \) obtained by Bagnuls et al [19], who use a different renormalization scheme, and the experimental values for \(^4\text{He} \) for \( A^+/A^- \) and \( R_T^{\xi} \) obtained by Lipa et al [8] and by Singsaas and Ahlers [10], respectively. In each case, the effect of the \( O(u^2) \) term in Eq. (2) is comparable to the uncertainties given by the authors of Refs. [8,10,19]. Since the possibility of the exact value of \( B^* \) differing from the leading term by \( \sim 3\% \) cannot be ruled out and since the experimental uncertainties are expected to be substantially smaller in the future, it seems that a higher order calculation,
of the kind indicated by the analysis of Ref. [3] for amplitude functions below $T_C$, may be needed for $B(u)$. This conclusion, of course, presumes the future availability of improved estimates for the critical exponent $\alpha$ and for the amplitude functions below $T_C$.

It may be argued that an additive renormalization is unnecessary if the specific heat is represented in terms of its temperature derivative $\partial C^\pm / \partial t$. In this connection, we recall the relations [15]

$$8A_d^{-1} P \pm f^{(3,0)} = (\epsilon - 2\zeta_r) F_\pm + 4B - \beta_u \frac{\partial F_\pm}{\partial u},$$

(11)

where $f^{(3,0)}$ are the amplitude functions for $\partial C^\pm / \partial t$. These formulas, for example, enable the amplitude ratios to be expressed in terms of $f^{(3,0)}$. In that case, resummation results for $B(u)$ would be useful for an internal check of the theory. However, in view of the unusually large uncertainty associated with the present high order result for $f^{(3,0)}$ [12% compared to $\lesssim 1\%$ typically for other amplitude functions [14,16]], the representation based on additive renormalization may well turn out to be the more reliable in quantitative applications. A higher order calculation of $B(u)$ would be needed to answer this question.

Finally, we note that in Ref. [14], the higher order coefficients in the perturbation series for $F_+$ were approximated by use of Eq. (11) with $B(u) \simeq n/2$. This procedure neglects the leading poles of $A(u, \epsilon)$ beyond two-loop order [for example, the term $\sim 1/8 \epsilon^2$ in the square brackets of Eq. (1)]. Using Eqs. (2) and (11), we find that the resummation results for $F_+^*$ are shifted by about 2%. However, since $F_+$ enters the formulas for the amplitude ratios and for the analysis of experimental data only in the combination $\alpha F_+$, the effect here is entirely negligible.

In summary, we have computed, within the framework of the minimally renormalized $\phi^4$ theory at fixed dimension $d = 3$, the three-loop correction to the additive renormalization of the specific heat, Eq. (1), for systems with $O(n)$ symmetry. We have used this to determine the corresponding RG function $B(u)$ in Eq. (4), and amplitude function $F_+$ above $T_C$ in Eq. (8). While the neglect (within the present scheme) of the leading additive poles in the
specific heat beyond two-loop order is justified in analyses based on low order perturbation
theory, these poles may lead to a small systematic effect at the level of accuracy expected in
future experiments \[8,12\].

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FIG. 1. Diagrams of the massless theory contributing to the specific heat. Their expansions near \( d = 4 \) are given in Eqs. (3)–(7).

TABLE I. Values for the amplitude ratios obtained using \( B(u) \simeq n/2 \) [1 & 2 loop] and Eq. (2) [3 loop]. All other quantities in Eqs. (9) and (10) are held fixed as described in the text. The other theoretical and experimental values are included to illustrate the current level of uncertainty for these quantities.

| \( A^+ / A^- \) | \( (A^-)^{1/3} / k_0 \) |
|-----------------|-------------------|
| \( n = 1 \) 0.527 | \( [1 & 2 \) loop] \( 0.541(1)^a \) |
| \( n = 2 \) 1.056 | \( [1 & 2 \) loop] 0.831 |
| \( \) 1.054 | \( [3 \) loop] 0.840 |
| \( \) 1.054(1) | \( 0.85—0.86^c \) |

\( ^a \)Bagnuls et al [19]
\( ^b \)Lipa et al [8]
\( ^c \)Singsaas and Ahlers [10]