ON THE SPECTRAL DENSITY OF A CLASS OF CHAOTIC TIME SERIES

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Abstract. The purpose of this paper is to show explicitly the spectral density function of the stationary stochastic process determined by a certain class of two-dimensional maps \( F_\alpha \) defined below (\( \alpha \) is a parameter in \( (0, 1) \)), the random variable \( \phi(x, y) = x \) and the invariant probability \( \nu \) described below.

We first define the transformation \( T_\alpha : [0, 1] \to [0, 1] \) given by
\[
T_\alpha(x) = \begin{cases} 
\frac{x}{\alpha} & \text{if } 0 \leq x < \alpha \\
\frac{\alpha(x - \alpha)}{1 - \alpha} & \text{if } \alpha \leq x \leq 1
\end{cases}
\]
where \( \alpha \in (0, 1) \) is a constant. The map \( T_\alpha \) describes a model for a particle (or the probability of a certain kind of element in a given population) that moves around, in discrete time, in the interval \([0, 1]\).

The results presented here can be stated either for \( T_\alpha \) or for \( F_\alpha \) but we prefer the latter. The results for \( T_\alpha \) can be obtained from the more general setting described by \( F_\alpha \).

The map \( F_\alpha \) is defined from \( K = ([0, 1] \times (0, \alpha)) \cup ([0, \alpha] \times [\alpha, 1]) \subset \mathbb{R}^2 \) to itself and is given by \( F_\alpha(x, y) = (T_\alpha(x), G_\alpha(x, y)) \) for \((x, y) \in K\), where
\[
G_\alpha(x, y) = \begin{cases} 
\alpha y & \text{if } 0 \leq x < \alpha \\
\alpha + \frac{1 - \alpha}{\alpha} y & \text{if } \alpha \leq x \leq 1
\end{cases}
\]

The spectral density function of the stationary process with probability \( \nu \) (invariant for \( F_\alpha \) and absolutely continuous with respect to the Lebesgue measure)
\[
Z_t = X_t + \xi_t = \phi(F_\alpha^n(X_0, Y_0)) + \xi_t \text{ for } t \in \mathbb{Z}
\]
where \((X_0, Y_0) \in \mathbb{R}^2\) and \( \{\xi_t\}_{t \in \mathbb{Z}} \) is a white noise process, is given explicitly (Theorem 1) by
\[
fZ(\lambda) = fX(\lambda) + \frac{\sigma^2}{2\pi \text{ var } (X_t)}[\gamma \{\exp(i\lambda)\} + \gamma \{\exp(-i\lambda)\} - C(0)] + \frac{\sigma^2}{2\pi}
\]
for all \( \lambda \in [0, 2\pi] \), where var \((X_t) = (\alpha^2 - \alpha + 1)(\alpha^2 - 5\alpha + 5)\{12(2 - \alpha)^2\}^{-1} \) \( \gamma \) is given by Equation (2.10) of Proposition 5 and \( C(0) = (1 + \alpha^2 - \alpha^2)(3(2 - \alpha))^{-1} \). We also estimate the parameter \( \alpha \) based on a time series.

Keywords. Spectral density; chaotic time series; dynamical systems.

1. INTRODUCTION

We shall present a complete spectral analysis of the stationary stochastic process
\[
Z_t = X_t + \xi_t \\
= \phi(F'_a(X_0, Y_0)) + \xi_t \\
= \phi(F_a(X_{t-1}, Y_{t-1})) + \xi_t \text{ for } t \in \mathbb{Z}
\]  
(1.1)

where \( \phi(x, y) = x \) is a random variable, \( \xi_t \) is a white noise process and \( F_a \) is the transformation defined below.

The map \( F_a \) is defined from \( K = ([0, 1] \times (0, \alpha)) \cup ([0, \alpha] \times [\alpha, 1]) \subset \mathbb{R}^2 \) to itself and is given by \( F_a(x, y) = (T_a(x), G_a(x, y)) \) where the transformation \( T_a: [0, 1] \rightarrow [0, 1] \) is defined by

\[
T_a(x) = \begin{cases} 
  \frac{x}{\alpha} & \text{if } 0 \leq x < \alpha \\
  \frac{\alpha(x - \alpha)}{1 - \alpha} & \text{if } \alpha \leq x \leq 1
\end{cases}
\]  
(1.2)

with \( \alpha \in (0, 1) \) a constant, and

\[
G_a(x, y) = \begin{cases} 
  \alpha y & \text{if } 0 \leq x < \alpha \\
  \alpha + \frac{1 - \alpha}{\alpha} y & \text{if } \alpha \leq x \leq 1
\end{cases}
\]  
(1.3)

The stationary process (1.1) will be considered with respect to a certain stationary (or invariant) measure \( \nu \) that will be defined in Section 3.

The graph of the map \( T_a \) is shown in Figure 1. The action of the piecewise diffeomorphism \( F_a \) is presented in Figure 2. The transformation \( F_a \) is a modification of the well-known Baker transformation.

The map \( T_a \) describes a model for a particle that moves around in the interval \([0, 1]\). If the particle is at position \( x \), then after a unit of time it jumps to \( T_a(x) \) and so on. According to the model considered here suppose the spatial position of the particle is \( T_a(x) = X_t, \quad t \in \mathbb{N} \), in the interval \([0, 1]\). If the particle \( X_t \) is in the interval \([0, \alpha]\), it has a uniformly spread possibility of jumping to any point \( X_{t+1} \) in \([0, 1]\). However, if it is in the interval \([\alpha, 1]\) it has a uniformly spread possibility of jumping to any point \( X_{t+1} \) in the interval \([0, \alpha]\).

We are primarily interested in the map \( T_a \), but for defining the spectral density in the classical form (see Brockwell and Davis, 1987)

\[
f_X(\lambda) = \frac{1}{2\pi} \sum_{k=-\infty}^{\infty} \exp(-ik\lambda) \rho_X(k) \geq 0 \quad 0 \leq \lambda < 2\pi
\]

we need to consider \( \rho_X(k) \) for negative \( k \) and hence we need a bijective map. Therefore, we have to consider \( F_a \), the natural extension of \( T_a \) (Bogomolny and Carioli, 1993).

The piecewise diffeomorphism \( F_a \) leaves invariant (see the definition in Lopes and Lopes, 1995) an ergodic probability \( \nu \) on \( K \subset \mathbb{R}^2 \), absolutely continuous with respect to the Lebesgue measure, that will be described in Section 3.
FIGURE 1.

FIGURE 2.
Choosing a point \((x_0, y_0)\) at random, according to the Lebesgue measure (or according to \(\nu\)), the spectral properties of the process \(Z_t\) will be analyzed. More precisely, we shall present explicitly the analytical expression of the spectral density function of such a stochastic process (see Section 3). We refer the reader to Lasota and Mackey (1994) and to Lopes and Lopes (1995) for general definitions and more detailed explanations for the context of the class of problems considered here.

In Grossmann and Thomae (1977) a recursive relation for the autocorrelation coefficients is presented for a different class of maps: tent maps. These results allow one to obtain in that case an explicit closed form for the autocorrelation coefficients and an explicit expression for the spectral density (see also Lopes and Lopes, 1995). The results obtained here cannot be obtained from Grossmann and Thomae (1977) because of the different structure of branches and the fact that the invariant measure is not a Lebesgue measure anymore. A generalization of the result of Grossmann and Thomae (1977) is presented in Lopes et al. (1997).

In Section 2 we present the basic results for a map \(T_\alpha\) which are used in Section 3 for obtaining results for the map \(F_\alpha\).

The main result of this paper is the expression (3.1) for the spectral density function of the stochastic process \(X_t = \phi(F_\alpha^t)\) (in terms of radial limits in the unit complex disk) presented in Theorem 1 in Section 3. For more general results see Lopes et al. (1997).

The explicit expression of the spectral density function (as obtained here) of a stochastic process allows us to analyze the efficiency of a given numerical method for estimating the spectrum, based on the closeness of the estimation obtained from the method to the true spectral density function.

We also estimate the parameter \(\alpha\) at the end of Section 2.

We refer the reader to Lopes et al. (1997) for results about the periodogram of times series obtained from expanding maps.

2. THE AUTOCORRELATION FUNCTION

Before considering the transformation \(F_\alpha\) we need to consider the transformation \(T_\alpha\).

Let the transformation \(T_\alpha: [0, 1] \rightarrow [0, 1]\) be as in (1.2), where \(\alpha \in (0, 1)\) is a constant. The derivative of \(T_\alpha(x)\) at \(x\) is \(a = 1/\alpha\) if \(0 \leq x < 1/\alpha\) and \(b = \alpha/(1 - \alpha)\) if \(1/\alpha < x \leq 1\).

A piecewise monotonic differentiable transformation \(T\) is expanding if there exists \(\beta > 1\) such that \(T'(x) > \beta\) for all \(x\) where \(T\) is differentiable.

One observes that \(a\) is always greater than 1; however, \(b \leq 1 \iff \alpha \leq 1/2\) and \(b > 1 \iff \alpha > 1/2\). The transformation \(T_\alpha\) is an expanding map (see Robinson, 1995) when \(\alpha > 1/2\). It is easy to show that, when \(\alpha < 1/2\), \(T_\alpha^2\) is an expanding map.

We are interested here in finding an invariant measure \(\mu\) absolutely
continuous with respect to the Lebesgue measure (see Lasota and Yorke, 1973; Parry and Pollicott, 1990; Robinson, 1995) and also in analyzing the autocorrelation function associated with the stationary stochastic process \( (T_\alpha^t, \mu) \).

The general existence of absolutely continuous invariant measures for expanding maps is known from the Lasota–Yorke theorem (see Lasota and Yorke, 1973). We need explicit expressions and we have obtained this density in a closed form as in (2.1) below.

The invariant measure for the transformation \( T_\alpha \) is of the form (see Lopes et al., 1995)

\[
g(x) = \begin{cases} 
\frac{1}{\alpha(2 - \alpha)} & \text{if } 0 \leq x < \alpha \\
1 & \text{if } \alpha \leq x \leq 1.
\end{cases}
\]  

Consider in the sequel the following notation:

\[
c = \frac{1}{\alpha(2 - \alpha)}, \quad d = \frac{1}{2 - \alpha}.
\]  

When \( T \) (or \( T^2 \)) is an expanding map the measure \( \mu \) is ergodic (see Parry and Pollicott, 1990). Hence, the measure \( \mu \) given by expression (2.1) is an ergodic measure (applying the last statement to \( T_\alpha \) or \( T^2_\alpha \)).

Analogously to Lopes and Lopes (1995), we consider \( F_\alpha : K \rightarrow K \) \((K \text{ will be defined later})\) the natural extension of \( T_\alpha \) (see Bogomolny and Carioli, 1993). The one-dimensional measure \( \mu \) can be extended to a two-dimensional measure \( \nu \) (defined later in Section 3) invariant for \( F_\alpha \). We shall give an explicit expression for the spectral density function of the stationary stochastic process \( \phi(F_\alpha^t) = X_t, \) the random variable \( \phi(x, y) = x \) and the measure \( \nu \).

Consider now the stationary stochastic process given by (1.1), where \( \{\xi_t\}_{t \in \mathbb{Z}} \) is a noise process. For simplicity of exposition we consider \( \xi_t \sim N(0, \sigma_\xi^2) \) for any \( t \in \mathbb{Z} \), i.e. a Gaussian white noise process. The noise \( \xi_t \) has no dynamic characteristic in the model considered here and could, in fact, be eliminated from the model in (1.1). The model with noise is nevertheless more general and we will leave the noise in the model.

We assume that \( \{(X_t, Y_t)\}_{t \in \mathbb{Z}} \) and \( \{\xi_t\}_{t \in \mathbb{Z}} \) are uncorrelated processes. We define the autocorrelation function of order \( k \) of the process \( \{X_t\}_{t \in \mathbb{Z}} \) by

\[
\rho_X(k) = \frac{\text{cov}(X_t, X_{t+k})}{\text{var}(X_t) \text{var}(X_{t+k})^{1/2}} = \frac{E[X_t \phi\{F_\alpha^k(X_t, Y_t)\}]}{\text{var}(X_t) \text{var} [\phi\{F_\alpha^k(X_t, Y_t)\}]^{1/2}} = \frac{E\{xT_\alpha^k(x)\} - \{E(x)\}^2}{\text{var}(x)},
\]  

\( \rho_X(k) \) is the autocorrelation function of order \( k \) of the process \( \{X_t\}_{t \in \mathbb{Z}} \).
Our goal is to derive the spectral density function of the process \( \{X_t\}_{t \in \mathbb{Z}} \)
\[
f_X(\lambda) = \frac{1}{2\pi} \sum_{k=-\infty}^{\infty} \exp(-ik\lambda) \rho_X(k) \quad \text{for any } \lambda \in [0, 2\pi).
\]
Hence, we need to derive the autocorrelation coefficients \( \rho_X(k), k \in \mathbb{Z} \), defined in (2.3).

By abuse of the notation, we shall denote \( \phi(x) = x \) and \( \phi(x, y) = x \) by the same letter \( \phi \).

In an analogous way to Lopes and Lopes (1995, Section 2), we show that for positive \( k \) the autocorrelation functions of order \( k \) of the dynamical systems \((F_\alpha(x, y), \phi(x, y), \nu)\) and \((T_\alpha(x), \phi(x), \mu)\) are the same. For negative values of \( k \) the autocorrelation function of order \( k \) of \( F_\alpha \) is equal to the corresponding autocorrelation function \( -k \) for \((F_\alpha, \phi, \nu)\) (or for \((T_\alpha, \phi, \mu)\)). These properties will be described in Section 3.

There is no meaning for the autocorrelation function of \( T_\alpha \) at negative lag \( k \) because \( T_\alpha \) is not an invertible map.

First we need three technical propositions involving the transformation \( T_\alpha \). The proofs of Propositions 1 to 5 will not be presented here (see Lopes et al., 1995). Proposition 1 gives a characterization of the \( k \)th iterate of the transformation \( T_\alpha(x) \) by a recursive formula.

**Proposition 1.** The \( k \)th iterate of the transformation \( T_\alpha(x) \) defined by expression (1.2) is given by
\[
T_\alpha^k(x) = \begin{cases} 
T_\alpha^{k-1}\left(\frac{x}{\alpha}\right) & \text{if } 0 \leq x < \alpha \\
T_\alpha^{k-2}\left(\frac{x-\alpha}{1-\alpha}\right) & \text{if } \alpha \leq x \leq 1
\end{cases}
\] (2.4)
for any integer \( k \leq 2 \).

We need Proposition 1 to prove Proposition 2 which will be used in Proposition 3.

**Proposition 2.** The integral \( A(k) = \int_{0}^{1} T_\alpha^k(x)dx \) satisfies the recursive equation
\[
A(k) = \alpha A(k-1) + (1-\alpha)A(k-2)
\] (2.5)
for any integer \( k \geq 2 \), with initial values \( A(0) = 1/2 \) and \( A(1) = (2-\alpha)/2 \).

The next proposition shows a three-term relation that will be used in Proposition 4.

**Proposition 3.** The integral \( B(k) = \int_{0}^{1} xT_\alpha^k(x)dx \) satisfies the recursive equation
\[ B(k) = \alpha^2 B(k - 1) + (1 - \alpha)^2 B(k - 2) + \alpha(1 - \alpha)A(k - 2) \]  

(2.6)

for any integer \( k \geq 2 \), with initial values \( B(0) = 1/3 \) and \( B(1) = (1 + \alpha)(2 - \alpha)\alpha/6 \).

From Propositions 2 and 3 we shall derive recursively the autocorrelation function \( \rho_X(k) \), \( k \geq 0 \), of the process \( \{X_t\}_{t \in \mathbb{N}} \).

**Proposition 4.** The autocorrelation function of order \( k \) of the process \( \{X_t\}_{t \in \mathbb{N}} \) defined in expression (1.1) is given by

\[ \rho_X(k) = \frac{E\{X_t T_A^k(X_t)\} - \{(1 + \alpha - \alpha^2)/2(2 - \alpha)\}^2}{(\alpha^2 - \alpha + 1)(\alpha^2 - 5\alpha + 5)/12(2 - \alpha)^2} \]  

(2.7)

where \( E\{X_t T_A^k(X_t)\} \), denoted by \( C(k) \), is given by the three-term relation

\[ C(k) = (\alpha^2 c B(k - 1) + (1 - \alpha^2) B(k - 2) + \alpha(1 - \alpha) A(k - 2) \]  

(2.8)

for any integer \( k \geq 2 \), with \( A(k) \) given by (2.5), \( B(k) \) given by (2.6) and the constants \( c \) and \( d \) defined in expression (2.2). Moreover, the initial values \( C(0) \) and \( C(1) \) are given by

\[ C(0) = \frac{1 + \alpha^2 - \alpha^3}{3(2 - \alpha)} \quad \quad C(1) = \frac{\alpha(4 - \alpha - \alpha^2)}{6(2 - \alpha)}. \]

From (2.7) we obtain the quantity \( C(k) \) recursively but not in a closed form. This is not enough for obtaining the spectral density function of the process \( \{X_t\}_{t \in \mathbb{Z}} \) explicitly. We can, equivalently, describe the quantities \( A(k), B(k) \) and \( C(k) \) by the following power series:

\[ \varphi(z) = \sum_{k=0} A(k)z^k \quad \psi(z) = \sum_{k=0} B(k)z^k \quad \gamma(z) = \sum_{k=0} C(k)z^k. \]  

(2.9)

The previous proposition (the three-term relation (2.8)) implies a relation between the functions \( \varphi(z), \psi(z) \) and \( \gamma(z) \) defined above (see Proposition 5) and this property will permit us to obtain \( \gamma(z) \). From Proposition 5 we can derive the spectral density function in a closed form (see Section 3).

**Proposition 5.** The power series for \( A(k), B(k) \) and \( C(k) \) as in expression (2.9) are given, respectively, by

\[ \varphi(z) = \frac{1 + \alpha z(1 - \alpha)}{2(1 - \alpha)z + 1}(1 - z) \]

\[ \psi(z) = \frac{2 - \alpha z(\alpha^2 + \alpha - 2) + 6\alpha(1 - \alpha)z^2 \varphi(z)}{6(1 - \alpha^2z - (1 - \alpha)^2z^2)} \]

\[ \gamma(z) = \frac{2\alpha^2(1 - \alpha) + 2 + \alpha z(2 - \alpha - \alpha^2)}{6(2 - \alpha)} \]
A few points should be made about estimation of the parameter \( \alpha \) from a times series with \( N \) observations. Note that the estimation of \( \alpha \) for the stationary stochastic process (1.1) can be obtained from Proposition 4. This follows from the fact that

\[
\frac{\alpha(4 - \alpha - \alpha^2)}{6(2 - \alpha)} = C(1) = \int_0^1 x T^0_\alpha(x) d\mu(x).
\]

By considering a times series \( \{Z_t\}_{t=1}^N \) and by using Birkhoff’s ergodic theorem we can estimate \( \alpha \) by solving the equation

\[
\hat{C}(1) = \frac{1}{N} \sum_{t=1}^{N-1} Z_t Z_{t+1} = \frac{\hat{\alpha}(4 - \hat{\alpha} - \hat{\alpha}^2)}{6(2 - \hat{\alpha})}
\]

in the variable \( \hat{\alpha} \).

3. THE SPECTRAL DENSITY FUNCTION

In this section we shall use the results for \( T_\alpha \) (obtained in the previous section) for the map \( F_\alpha \).

We observe that \( F_\alpha \) is a homeomorphism of \( K \) and \( F^n_\alpha \) is of the form

\[
F^n_\alpha(x, y) = (T^n_\alpha(x), G_{\alpha,n}(x, y)).
\]

That is, the action of \( F_\alpha \) in the first variable is just the action of \( T_\alpha \).

Now we shall define the \( F_\alpha \)-invariant measure \( \nu \) on \( K \), absolutely continuous with respect to the Lebesgue measure \( dxdy \).

For sets of the form \( A_1 \times A_2 \) where \( A_1 \subset (0, \alpha) \) and \( A_2 \subset (\alpha, 1) \) or \( A_1 \subset (\alpha, 1) \) and \( A_2 \subset (0, \alpha) \), we define \( \nu(A_1 \times A_2) = (2 - \alpha)\mu(A_1)\mu(A_2) \).

For sets of the form \( A_1 \times A_2 \) where \( A_1 \subset (0, \alpha) \) and \( A_2 \subset (0, \alpha) \), we define \( \nu(A_1 \times A_2) = (2 - \alpha)\alpha\mu(A_1)\mu(A_2) \).

It is not difficult to see that \( \nu \) is invariant for \( F_\alpha \) and it is absolutely continuous with respect to the Lebesgue measure. The measure \( \nu \) satisfies

\[
\nu\{A \times (0, 1)\} = \mu(A) \quad \text{when} \quad A \subset (0, \alpha), \quad \text{and} \quad \nu\{A \times (0, \alpha)\} = \mu(A) \quad \text{when} \quad A \subset (\alpha, 1).
\]

THEOREM 1. The spectral density function of the process

\[
Z_t = X_t + \xi_t = \phi\{F^n_\alpha(X_0, Y_0)\} + \xi_t \quad \text{for} \quad t \in \mathbb{Z}
\]

is given by

\[
f_Z(\lambda) = \frac{1}{2\pi \text{var}(X_t)} \left[ \gamma\{\exp(i\lambda)\} + \gamma\{\exp(-i\lambda)\} - C(0) \right] + \frac{\sigma_x^2}{2\pi}
\]

for any \( \lambda \in [0, 2\pi) \) (3.1)
where $\text{var}(X_t) = (\alpha^2 - \alpha + 1)(\alpha^2 - 5\alpha + 5)/12(2 - \alpha)^2$, $C(0) = (1 + \alpha^2 - \alpha^3)/3(2 - \alpha)$ and $\gamma(z)$ is given by expression (2.10) of Proposition 5. The point $(X_0, Y_0)$ is chosen randomly according to the measure $\nu$ (or according to the Lebesgue measure).

Proof. The integral of $\nu$ with respect to any function $H$ that depends only on the $x$ variable is such that

$$\int H(x)d\nu(x, y) = \int H(x)d\mu(x). \quad (3.2)$$

We observe that $\phi(x, y) = x$ is a random variable and $F_\alpha$: $K \to K$ defines a stationary stochastic process $X_t = \phi\{F_\alpha^t(X_0, Y_0)\}$ with respect to the invariant probability $\nu$ defined above.

From expression (3.2) and for any positive $t \in \mathbb{N}$,

$$\int \phi\{F_\alpha^t(x, y)\}\phi(x, y)d\nu(x, y) = \int \phi\{T_\alpha^t(x)\}\phi(x)d\mu(x).$$

For any positive $t \in \mathbb{N}$ (i.e. when $-t$ is negative)

$$\int \phi\{F_\alpha^{-t}(x, y)\}\phi(x, y)d\nu(x, y) = \int \phi(x, y)\phi\{F_\alpha^{-t}(x, y)\}d\nu(x, y)$$

because $\nu$ is invariant for $F_\alpha$. Therefore, from (3.2) and for any positive $t \in \mathbb{N}$,

$$\int \phi\{F_\alpha^{-t}(x, y)\}\phi(x, y)d\nu(x, y) = \int \phi\{T_\alpha^{-t}(x)\}\phi(x)d\mu(x). \quad (3.3)$$

The conclusion is that the autocorrelation coefficients $C(t) = C(-t)$, $t \in \mathbb{N}$, of the stochastic process given by the random variable $\phi(x, y) = x$, the transformation $F_\alpha$ and the probability $\nu$ can be obtained from the autocorrelation coefficients obtained previously for the stochastic process given by the random variable $\phi(x) = x$, the transformation $T_\alpha$ and the probability $\mu$.

The spectral density function of the process $\{X_t\}_{t \in \mathbb{Z}}$ is given by

$$f_X(\lambda) = \frac{1}{2\pi} \sum_{k=-\infty}^{\infty} \exp(-ik\lambda) \rho_X(k) \geq 0$$

for all $\lambda \in [0, 2\pi)$ (see Brockwell and Davis, 1987). Therefore, the spectral density function of the process (1.1) is given by (3.1).

From Parry and Polliccott (1990) it is known that $\rho_X(k)$ decays exponentially to zero; hence $f_X(\lambda)$ is an analytic function, for any $\lambda \in [0, 2\pi)$.

Remark. The power series $\gamma(z)$ is an analytic function on the disk $\{z \in C||z|| < 1\}$ and expression (3.1) has the meaning of the radial limit

$$\lim_{r \to 1} r \exp(i\lambda) = \exp(i\lambda) = z.$$
In this sense, the series
\[ \sum_{n \in \mathbb{Z}} \exp(in\lambda) = 2 \Re \left\{ \frac{1}{1 - \exp(i\lambda)} \right\} - 1 = 0 \quad \text{for } \lambda \neq 0 \]
even though the series \( \sum_{n \in \mathbb{Z}} \exp(in\lambda) \) does not converge. We are using this fact in expression (3.1).

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