HIGHER ORDER CONVERGENCE FOR A CLASS OF SET DIFFERENTIAL EQUATIONS WITH INITIAL CONDITIONS

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ABSTRACT. In this paper, we obtain some rapid convergence results for a class of set differential equations with initial conditions. By introducing the partial derivative of set valued function and the m-hyperconvex/hyperconcave functions (m ≥ 1), and using the comparison principle and quasilinearization, we derive two monotone iterative sequences of approximate solutions of such equations that involve the sum of two functions, and these approximate solutions converge uniformly to the unique solution with higher order.

1. Introduction. In recent years, set differential equations (SDEs) which are defined in semilinear metric space have aroused many researcher’s attention as an independent subject area because of its advantages. In particular, the theory of ordinary differential equations and ordinary differential systems can be regarded as the special cases of SDEs. Some basic results have been established on existence, uniqueness, continuous dependence of solutions and the corresponding stability properties [20, 21, 3, 16, 29, 14, 4, 30, 5, 28]. Meanwhile, some generalized results have also been established for various types of SDEs, such as set functional differential equations [1, 2], impulsive set differential equations [7, 25, 24, 11], set differential equations involving causal operators [8, 12], set differential equations in Banach spaces [22, 23], set dynamic equations on time scales [15], to name a few.

It is well known that the convergence of solutions plays an important role in the development of qualitative theory. The method of upper and lower solutions coupled with the monotone iterative technique has been widely used in the treatment of nonlinear differential equations (see the monograph [18]). With this method we can prove that the monotone sequences are convergent uniformly. This fruitful method has been extended to set differential equations in a general setup which includes various known results, and the reader is referred to the papers of Devi and Vatsala [9], Devi [10], Dhaigude and Naidu [13], McRae, Devi and Drici [26] for details. In [10], the authors introduced a partial ordering in the metric space \( (K_c(\mathbb{R}^n), D) \) and developed a monotone iterative method for set differential equations when the forcing function is the sum of a convex and a concave function.

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From the practical point of view, the higher order convergence of sequences of approximate solutions is also very important. Quasilinearization provides an approach for obtaining approximate solutions of nonlinear differential equations, and the convergence is quadratic which is useful in practice, as detailed in the monograph of Lakshmikantham and Vatsala [19]. However, to the best of our knowledge, there are few results on higher order convergence for set differential equations (see [31]). Motivated by [17, 27], the aim of this paper is to extend the generalized quasilinearization to set differential equations involving the sum of two functions. For the purpose, by introducing the definition of partial derivative of set valued function, $m$-hyperconvex and $m$-hyperconcave function, and applying the comparison principle, we construct two monotone convergence sequences and prove that the coupled solutions converge, with high convergence order, to the unique solution of the set differential equations with initial conditions.

2. Preliminaries. Firstly, we give some preliminaries to be used in this paper.

Let $K(\mathbb{R}^n)$ ($K_c(\mathbb{R}^n)$) denote all nonempty compact subsets of $\mathbb{R}^n$ (all nonempty compact convex subsets of $\mathbb{R}^n$). The Hausdorff metric is defined by

$$D[A, B] = \max\{\sup_{x \in B} d(x, A), \sup_{y \in A} d(y, B)\},$$

where $d(x, A) = \inf\{d(x, y) : y \in A\}$, $A$ and $B$ are bounded sets in $\mathbb{R}^n$. The Hausdorff metric satisfies the following properties

$$D[A + C, B + C] = D[A, B] \text{ and } D[A, B] = D[B, A],$$
$$D[\lambda A, \lambda B] = \lambda D[A, B],$$
$$D[A, B] \leq D[A, C] + D[C, B],$$

for all $A, B, C \in K_c(\mathbb{R}^n)$ and $\lambda \in \mathbb{R}^+$.  

**Definition 2.1.** (See [20]) Let $I$ be any interval in $\mathbb{R}^+$. The set valued mapping $U : I \to K_c(\mathbb{R}^n)$ is Hukuhara differentiable at a point $t_0 \in I$, if there exists $D_HU(t_0) \in K_c(\mathbb{R}^n)$ such that the limits

$$\lim_{h \to 0^+} \frac{U(t_0 + h) - U(t_0)}{h} \quad \text{and} \quad \lim_{h \to 0^+} \frac{U(t_0) - U(t_0 - h)}{h}$$

both exist and are equal to $D_HU(t_0)$, where $h > 0$ is sufficiently small and $I = [0, T]$, $T > 0$ is a constant.

In order to obtain the conclusion of higher order convergence, we give the following definitions of partial derivative of set valued function.

**Definition 2.2.** (See [6]) Let a set valued function $F : I \times K_c(\mathbb{R}^n) \to K_c(\mathbb{R}^n)$ be given. Let $e_k = (e_k^1, \ldots, e_k^n)$ be the vector such that $e_k^j = 0, k \neq j$ and $e_k^k = 1$, $U_0 \in K_c(\mathbb{R}^n)$ and $h > 0$ sufficiently small. Then the function $F$ is Hukuhara partial differentiable at $U_0$, if there exists $D_{H_{U_0}}F(t, U_0) \in 2^{K_c(\mathbb{R}^n)}$ such that the limits

$$\lim_{h \to 0^+} \frac{F(t, U_0 + he_k) - F(t, U_0)}{h} \quad \text{and} \quad \lim_{h \to 0^+} \frac{F(t, U_0) - F(t, U_0 - he_k)}{h}$$

both exist in the topology of $(K_c(\mathbb{R}^n), D)$ and are equal to $D_{H_{U_0}}F(t, U_0)$. Clearly implicit in the definition of $D_{H_{U_0}}F(t, U_0)$ is the existence of the Hukuhara differences

$$F(t, U_0 + he_k) - F(t, U_0) \quad \text{and} \quad F(t, U_0) - F(t, U_0 - he_k).$$
Since the $K_c(\mathbb{R}^n)$ are compact, convex subsets in $\mathbb{R}^n$, we can identify $D_{H^c_0} F(t, U_0)$ in the following suitable form:

$$D_{H^c_0} F(t, U_0) = \left[ \frac{\partial F(t, U_0)}{\partial u_0} : u_0 \in U_0 \in K_c(\mathbb{R}^n) \right] \in 2^{K_c(\mathbb{R}^n)},$$

where

$$\frac{\partial F(t, U_0)}{\partial u_0} = \left( \frac{\partial F(t, U_0)}{\partial u_0^1}, \ldots, \frac{\partial F(t, U_0)}{\partial u_0^n} \right)$$

such that $\frac{\partial F(t, U_0)}{\partial u_i} \in K_c(\mathbb{R}^n)$ for each $i$.

It is similar to define the $k$th order partial derivative of mapping $F$.

**Definition 2.3.** Let a mapping $D_{H^c_0} F : I \times K_c(\mathbb{R}^n) \to (2^{k-1})^{K_c(\mathbb{R}^n)}$ be given. Let $e_k = (e_k^1, \ldots, e_k^n)$ be the vector such that $e_k^j = 0, k \neq j$ and $e_k^k = 1, U_0 \in K_c(\mathbb{R}^n)$ and $h > 0$ sufficiently small. Then the function $F$ is Hukuhara $k$th partial differentiable at $U_0$, if there exists $D_{H^c_0} F(t, U_0) \in (2^{k})^{K_c(\mathbb{R}^n)}$ such that the limits

$$\lim_{h \to 0^+} \frac{D_{H^c_0} F(t, U_0 + he_k) - D_{H^c_0} F(t, U_0)}{h}$$

and

$$\lim_{h \to 0^+} \frac{D_{H^c_0} F(t, U_0) - D_{H^c_0} F(t, U_0 - he_k)}{h}$$

both exist in the topology of $((2^{k-1})^{K_c(\mathbb{R}^n)}, D)$ and are equal to $D_{H^c_0} F(t, U_0)$. Clearly implicit in the definition of $D_{H^c_0} F(t, U_0)$ is the existence of the Hukuhara differences

$$D_{H^c_0} F(t, U_0 + he_k) - D_{H^c_0} F(t, U_0) \quad \text{and} \quad D_{H^c_0} F(t, U_0) - D_{H^c_0} F(t, U_0 - he_k).$$

Similarly, we can identify $D_{H^c_k} F(t, U_0)$ in the following suitable form:

$$D_{H^c_k} F(t, U_0) = \left[ \frac{\partial D_{H^c_0} F(t, U_0)}{\partial u_0} : u_0 \in U_0 \in K_c(\mathbb{R}^n) \right] \in (2^{k})^{K_c(\mathbb{R}^n)},$$

where

$$\frac{\partial D_{H^c_0} F(t, U_0)}{\partial u_0} = \left( \frac{\partial D_{H^c_0} F(t, U_0)}{\partial u_0^1}, \ldots, \frac{\partial D_{H^c_0} F(t, U_0)}{\partial u_0^n} \right)$$

such that $\frac{\partial D_{H^c_0} F(t, U_0)}{\partial u_i} \in (2^{k-1})^{K_c(\mathbb{R}^n)}$ for each $i$.

**Definition 2.4.** Let $D_{H^c_k} F(t, U_0) \in (2^{k})^{K_c(\mathbb{R}^n)}, P \in K_c(\mathbb{R}^n)$. We define $D_{H^c_k} F(t, U_0)^{P^{2k-1}}$ in the following form:

$$D_{H^c_k} F(t, U_0)^{P^{2k-1}} = \left[ \frac{P^T \cdots P^T [D_{H^c_k} F(t, U_0) P]^T}{k} \right] \cdots \left[ P^T \cdots P^T [D_{H^c_k} F(t, U_0) P]^T \right] \cdots \left[ P^T \cdots P^T [D_{H^c_k} F(t, U_0) P]^T \right].$$

Similarly, we define $D_{H^c_k} F(t, U_0)^{P^{2k}}$ in the following form:

$$D_{H^c_k} F(t, U_0)^{P^{2k}} = \left[ \frac{P^T \cdots P^T [D_{H^c_k} F(t, U_0) P]^T}{k} \right] \cdots \left[ P^T \cdots P^T [D_{H^c_k} F(t, U_0) P]^T \right] \cdots \left[ P^T \cdots P^T [D_{H^c_k} F(t, U_0) P]^T \right].$$
With these preliminaries, we consider the following set differential equations
\[ D_H U = F(t, U) + G(t, U), \quad U(0) = U_0 \in K_c(\mathbb{R}^n), \]  
where \( F, G \in C[I \times K_c(\mathbb{R}^n), K_c(\mathbb{R}^n)] \) and \( D_H U \) is the Hukuhara derivative of \( U \).  

The mapping \( U \in C^1[I, K_c(\mathbb{R}^n)] \) is said to be a solution of \( (1) \) on \( I \), if it satisfies \( (1) \) on \( I \). Since \( U(t) \) is continuously differentiable, we have
\[ U(t) = U_0 + \int_0^t D_H U(s) ds, \quad t \in I. \]

We therefore associate with the IVP \( (1) \) the following integral equation
\[ U(t) = U_0 + \int_0^t \left[ F(s, U(s)) + G(s, U(s)) \right] ds, \quad t \in I, \]
where the integral is the Hukuhara integral which is defined as
\[ \int F(s) ds = \left\{ \int f(s) ds : f \text{ is any continuous selector of } F \right\}. \]

It is obvious that \( U(t) \) is a solution of IVP \( (1) \) on \( I \) i f it satisfies \( (2) \) on \( I \).

The following properties are useful in proving our results.  
If \( I \to K_c(\mathbb{R}^n) \) is integrable, we have
\[ \int_0^{t_2} F(s) ds = \int_0^{t_1} F(s) ds + \int_{t_1}^{t_2} F(s) ds, \quad 0 \leq t_1 \leq t_2 \leq T, \]
and
\[ \int_0^T \lambda F(s) ds = \lambda \int_0^T F(s) ds, \quad \lambda \in \mathbb{R}^+. \]

If \( F, G : I \to K_c(\mathbb{R}^n) \) are integrable, then \( D[F(\cdot), G(\cdot)] : I \to \mathbb{R} \) is integrable and
\[ D \left[ \int_0^t F(s) ds, \int_0^t G(s) ds \right] \leq \int_0^t D[F(s), G(s)] ds. \]

We denote by \( K \) the subfamily of \( K_c(\mathbb{R}^n) \) consisting of sets \( Z \in K_c(\mathbb{R}^n) \) such that any \( z \in Z \) is a nonnegative vector of \( n \)-components satisfying \( z_i \geq 0 \) for \( i = 1, 2, \ldots, n \). Furthermore, we introduce a partial ordering in \( K_c(\mathbb{R}^n) \) as follows.

**Definition 2.5.** For any \( U \) and \( V \in K_c(\mathbb{R}^n) \), if there exists a \( Z \in K_c(\mathbb{R}^n) \) such that \( Z \in K \) and \( U = V + Z \), then we say that \( U \geq V \). Similarly, we can define \( U \leq V \).

In order to obtain the main results of this paper, we follow the procedure in the proof of [20, 19] and give the following lemmas and definition.

**Lemma 2.6.** Assume that \( V \in C^1[I, K_c(\mathbb{R}^n)] \), \( D_H V \leq AV + \sigma \), where \( A = (a_{ij}) \) is an \( n \times n \) matrix satisfying \( a_{ij} \geq 0, \ i \neq j, \ \sigma \in C[I, K_c(\mathbb{R}^n)] \). Then
\[ V(t) \leq V(0) e^{At} + \int_0^t e^{A(t-s)} \sigma(s) ds, \quad t \in I. \]

**Definition 2.7.** Let \( V, W \in C^1[I, K_c(\mathbb{R}^n)] \). Then \( V, W \) are said to be
\( (A_1) \): coupled lower and upper solutions of type I of \( (1) \) if \( D_H V \leq F(t, V) + G(t, W), \ V(0) \leq U_0, \ \text{and} \ D_H W \geq F(t, W) + G(t, V), \ W(0) \geq U_0, \ t \in I; \)
\( (A_2) \): coupled lower and upper solutions of type II of \( (1) \) if \( D_H V \leq F(t, W) + G(t, V), \ V(0) \leq U_0, \ \text{and} \ D_H W \geq F(t, V) + G(t, W), \ W(0) \geq U_0, \ t \in I. \)
Lemma 2.8. Assume that \( F, G \in C[I, K_c(\mathbb{R}^n)] \), and one of the following conditions holds:

\((K_1)\): \( V, W \) are coupled lower and upper solutions of type I of (1) and
\[ F(t, X_1) - F(t, X_2) \leq L(X_1 - X_2), \quad G(t, X_1) - G(t, X_2) \geq -L(X_1 - X_2), \]
where \( X_1 \geq X_2 \) and \( L \geq 0 \) is a constant.

\((K_2)\): \( V, W \) are coupled lower and upper solutions of type II of (1) and
\[ F(t, X_1) - F(t, X_2) \geq -L(X_1 - X_2), \quad G(t, X_1) - G(t, X_2) \leq L(X_1 - X_2), \]
where \( X_1 \geq X_2 \) and \( L \geq 0 \) is a constant.

Then \( V(0) \leq W(0) \) implies \( V(t) \leq W(t) \) on \( I \).

Lemma 2.9. Assume that \( F, G \in C[\Omega, K_c(\mathbb{R}^n)] \), where \( \Omega = \{(t, U) : V(t) \leq U(t) \leq W(t), \ t \in I\} \), and one of the following conditions holds:

\((I_1)\): \( V, W \) are coupled lower and upper solutions of type I of (1) such that \( V(t) \leq W(t) \) on \( I \), and \( G(t, U) \) is monotone nonincreasing in \( U \) for \( t \in I \).

\((I_2)\): \( V, W \) are coupled lower and upper solutions of type II of (1) such that \( V(t) \leq W(t) \) on \( I \), and \( F(t, U) \) is monotone nonincreasing in \( U \) for \( t \in I \).

Then there exists a solution \( U(t) \) of (1) satisfying \( V(t) \leq U(t) \leq W(t) \) on \( I \).

3. Main results. Let \( D_{H^k} F(t, U) \) denote the \( k \)th partial derivative of \( F \) with respect to \( U \), \( \theta \) denote the null set, and \( \|U\| \) denote the maximum norm of \( U(t) \) on \( I \).

We firstly introduce the following definition.

Definition 3.1. A set valued function \( F \) is called \( m \)-hyperconvex, if \( D_{H^m+1} F \geq \theta \); \( F \) is called \( m \)-hyperconcave if the inequality is reversed, where \( m \geq 1 \) is a constant.

Assume that \( F(t, U) \) is hyperconvex in \( U \) of order \( m - 1 \), then we have the following inequalities depending on whether \( m \) is even or odd.

(i) \( m = 2k \)
\[ F(t, \eta) \geq \sum_{i=0}^{2k-1} \frac{D_{H^i} F(t, \xi)(\eta - \xi)^i}{i!}; \quad \sum_{i=0}^{2k-2} \frac{D_{H^i} F(t, \xi)(\eta - \xi)^i}{i!} + \frac{D_{H^{2k-1}} F(t, \eta)(\eta - \xi)^{2k-1}}{(2k-1)!}; \quad (6) \]

\[ F(t, \eta) \leq \sum_{i=0}^{2k} \frac{D_{H^i} F(t, \xi)(\eta - \xi)^i}{i!}; \quad \sum_{i=0}^{2k-1} \frac{D_{H^i} F(t, \xi)(\eta - \xi)^i}{i!} + \frac{D_{H^{2k}} F(t, \eta)(\eta - \xi)^{2k}}{(2k)!}; \quad (7) \]

(ii) \( m = 2k + 1 \)
\[ F(t, \eta) \geq \sum_{i=0}^{2k} \frac{D_{H^i} F(t, \xi)(\eta - \xi)^i}{i!}; \quad \eta \geq \xi, \quad \eta \leq \xi; \quad \eta \geq \xi, \quad \eta \leq \xi; \quad \eta \geq \xi, \quad \eta \leq \xi; \quad \eta \geq \xi, \quad \eta \leq \xi. \]

\[ F(t, \eta) \leq \sum_{i=0}^{2k-1} \frac{D_{H^i} F(t, \xi)(\eta - \xi)^i}{i!} + \frac{D_{H^{2k}} F(t, \eta)(\eta - \xi)^{2k}}{(2k)!}; \quad \eta \geq \xi, \quad \eta \leq \xi. \]

\[ F(t, \eta) \geq \sum_{i=0}^{2k} \frac{D_{H^i} F(t, \xi)(\eta - \xi)^i}{i!} + \frac{D_{H^{2k}} F(t, \eta)(\eta - \xi)^{2k}}{(2k)!}; \quad \eta \leq \xi. \]
Similarly, when $G(t, U)$ is hyperconcave in $U$ of order $m - 1$, then we have the following inequalities depending on whether $m$ is even or odd.

(i) $m = 2k$

\[
G(t, \eta) \leq \sum_{i=0}^{2k-1} \frac{D_{H^2}^i G(t, \xi)(\eta - \xi)^i}{i!}, \tag{12}
\]

\[
G(t, \eta) \geq \sum_{i=0}^{2k-2} \frac{D_{H^2}^i G(t, \xi)(\eta - \xi)^i}{i!} + \frac{D_{H^2}^{2k-1} G(t, \eta)(\eta - \xi)^{2k-1}}{(2k-1)!}; \tag{13}
\]

(ii) $m = 2k + 1$

\[
G(t, \eta) \leq \sum_{i=0}^{2k} \frac{D_{H^2}^i G(t, \xi)(\eta - \xi)^i}{i!}, \quad \eta \geq \xi, \tag{14}
\]

\[
G(t, \eta) \geq \sum_{i=0}^{2k} \frac{D_{H^2}^i G(t, \xi)(\eta - \xi)^i}{i!}, \quad \eta \leq \xi, \tag{15}
\]

\[
G(t, \eta) \geq \sum_{i=0}^{2k-1} \frac{D_{H^2}^i G(t, \xi)(\eta - \xi)^i}{i!} + \frac{D_{H^2}^{2k} G(t, \eta)(\eta - \xi)^{2k}}{(2k)!}, \quad \eta \geq \xi, \tag{16}
\]

\[
G(t, \eta) \leq \sum_{i=0}^{2k-1} \frac{D_{H^2}^i G(t, \xi)(\eta - \xi)^i}{i!} + \frac{D_{H^2}^{2k} G(t, \eta)(\eta - \xi)^{2k}}{(2k)!}, \quad \eta \leq \xi. \tag{17}
\]

**Theorem 3.2.** For the IVP (1), assume that

(H$_1$): $V, W \in C^1[I, K_c(\mathbb{R}^n)]$ are coupled lower and upper solutions of type I of (1), and $V(t) \leq W(t)$,

(H$_2$): $F(t, U)$ is hyperconvex in $U$ of order $2k - 1$, $G(t, U)$ is hyperconcave in $U$ of order $2k - 1, k \geq 1$, where $F, G \in C^{2k}[\Omega, K_c(\mathbb{R}^n)]$, i.e. $D_{H^2}^{2k} F(t, U) \geq \theta, D_{H^2}^{2k} G(t, U) \leq \theta$,

(H$_3$): $D_{H^2} G(t, U) \leq \min_{i!} |D_{H^2} G(t, U)| \frac{(W - V)^{2k-1}}{(2k-2)!} \leq \theta$ on $\Omega$,

(H$_4$): $F$ and $G$ map bounded sets into bounded sets in $K_c(\mathbb{R}^n)$.

Then there exist monotone sequences $\{V_n\}, \{W_n\}$ with $V = V_0, W = W_0, n \geq 0$, which converge uniformly and monotonically to the unique solution of (1) and the convergence is of order $2k$.

**Proof.** In view of assumption $D_{H^2}^{2k} F \geq \theta$ and $D_{H^2}^{2k} G \leq \theta$, we obtain the inequalities (6), (7), (12) and (13). Consider the following coupled IVPs:

\[
D_H X(t, V, W; X, Y)
\]

\[
\equiv \sum_{i=0}^{2k-1} \frac{D_{H^2}^i F(t, V)(X - V)^i}{i!} + \sum_{i=0}^{2k-2} \frac{D_{H^2}^i G(t, W)(Y - W)^i}{i!}
\]

\[
+ \frac{D_{H^2}^{2k-1} G(t, V)(Y - W)^{2k-1}}{(2k-1)!}, \quad X(0) = U_0; \tag{18}
\]
where \( V \exists \) a solution \( Y \) and we can use Lemma 2.9 to associate with (20), (21) and conclude that there exists a solution of (19) with 
\[ D_H Y = N(t, V, W; Y, X) \]

\[
\equiv \sum_{i=0}^{2k-2} \frac{D_{H^2} F(t, W)(Y - W)^i}{i!} + \frac{D_{H_2^{2k-1}} F(t, V)(Y - W)^{2k-1}}{(2k-1)!} \\
+ \sum_{i=0}^{2k-1} \frac{D_{H^2} G(t, V)(X - V)^i}{i!}, \quad Y(0) = U_0, 
\]

where \( V(0) \leq U_0 \leq W(0) \).

The assumption \((H_1)\) and inequalities (6), (13) imply

\[ D_H V \geq F(t, V) + G(t, V) = M(t, V, W; V, W), \quad V(0) \leq U_0; \quad (20) \]

\[ D_H W \geq F(t, W) + G(t, V) \]

\[
\geq \sum_{i=0}^{2k-1} \frac{D_{H^2} F(t, V)(W - V)^i}{i!} + \sum_{i=0}^{2k-2} \frac{D_{H_2^{2k-1}} G(t, W)(V - W)^i}{i!} \\
+ \frac{D_{H_2^{2k-1}} G(t, V)(V - W)^{2k-1}}{(2k-1)!} \\
= M(t, V, W; W, V), \quad W(0) \geq U_0. 
\]

By the assumption \((H_3)\) and the Taylor series expansion of \( M(t, V, W; X, Y) \), we obtain

\[ D_H V \leq F(t, V) + G(t, V) \]

\[
\leq \sum_{i=0}^{2k-2} \frac{D_{H^2} F(t, W)(V - W)^i}{i!} + \frac{D_{H_2^{2k-1}} F(t, V)(V - W)^{2k-1}}{(2k-1)!} \\
+ \sum_{i=0}^{2k-1} \frac{D_{H^2} G(t, V)(W - V)^i}{i!}, \\
= N(t, V, W; V, W), \quad V(0) \leq U_0; 
\]

\[ D_H W \geq F(t, W) + G(t, V) = N(t, V, W; W, V), \quad W(0) \geq U_0. \quad (23) \]

By the assumption \((H_3)\) and the Taylor series expansion of \( N(t, V, W; Y, X) \), we get

\[ D_H X N(t, V, W; Y, X) = D_H X G(t, X) - \frac{D_{H_2^{2k}} G(t, \eta)(X - V)^{2k-1}}{(2k-1)!} \leq \theta, \]

where \( V \leq \eta \leq X \). Thus \( N(t, V, W; Y, X) \) is nonincreasing in \( X \) and we can use Lemma 2.9 to associate with (22), (23) and conclude that there exists a solution \( W_1 \) of (19) with \( V = V_0, W = W_0 \) such that \( V_0 \leq W_0 \leq W_0 \) on \( I \).
Furthermore, in view of the inequalities (6), (13), \( V_0 \leq W_1 \leq W_0 \) and the condition \((H_2)\), we have

\[
D_H V_1 = M(t, V_0, W_0; V_1, W_1) \\
\leq \sum_{i=0}^{2k-1} \frac{D_{H_v}^i F(t, V_0)(V_1 - V_0)^i}{i!} + \sum_{i=0}^{2k-2} \frac{D_{H_w}^i G(t, W_0)(W_1 - W_0)^i}{i!} \\
+ \frac{D_{H_v}^{2k-1} G(t, W_1)(W_1 - W_0)^{2k-1}}{(2k-1)!} \\
\leq F(t, V_1) + G(t, W_1), \quad V_1(0) = U_0.
\]

By (7), (12), \( V_0 \leq W_1 \leq W_0 \) and the condition \((H_2)\), we have

\[
D_H W_1 = N(t, V_0, W_0; W_1, V_1) \\
\geq \sum_{i=0}^{2k-2} \frac{D_{H_w}^i F(t, W_0)(W_1 - W_0)^i}{i!} + \frac{D_{H_v}^{2k-1} F(t, W_1)(W_1 - W_0)^{2k-1}}{(2k-1)!} \\
+ \sum_{i=0}^{2k-1} \frac{D_{H_v}^i G(t, V_0)(V_1 - V_0)^i}{i!} \\
\geq F(t, W_1) + G(t, V_1), \quad W_1(0) = U_0.
\]

By the \((K_1)\) of Lemma 2.8, we get \( V_1 \leq W_1 \) on \( I \). Hence we can prove \( V_0 \leq V_1 \leq W_1 \leq W_0 \) on \( I \).

Now suppose that \( V_n, W_n \) are the solutions of IVPs (18) and (19) respectively, such that \( V = V_{n-1}, W = W_{n-1}, V_{n-1} \leq V_n \leq W_n \leq W_{n-1}, \) and

\[
D_H V_n \leq F(t, V_n) + G(t, W_n), \quad D_H W_n \geq F(t, W_n) + G(t, V_n).
\]

We intend to prove that \( V_n \leq V_{n+1} \leq W_{n+1} \leq W_n \), in which \( V_{n+1}, W_{n+1} \) are the solutions respectively of IVPs (18) and (19) such that \( V = V_n, W = W_n \).

Applying (6), (13) and (26) imply

\[
D_H V_n \leq F(t, V_n) + G(t, W_n) = M(t, V_n, W_n; V_n, W_n), \quad V_n(0) \leq U_0; \quad (27)
\]

\[
D_H W_n \geq F(t, W_n) + G(t, V_n) \\
\geq \sum_{i=0}^{2k-1} \frac{D_{H_v}^i F(t, V_n)(W_n - V_n)^i}{i!} + \sum_{i=0}^{2k-2} \frac{D_{H_w}^i G(t, W_n)(V_n - W_n)^i}{i!} \\
+ \frac{D_{H_v}^{2k-1} G(t, V_n)(V_n - W_n)^{2k-1}}{(2k-1)!} \\
= M(t, V_n, W_n; V_n, W_n), \quad W_n(0) \geq U_0.
\]

These show that \( V_n \) and \( W_n \) are lower and upper solutions of (18) respectively. Taking account of the fact that \( M(t, V, W; X, Y) \) is nonincreasing in \( Y \) and Lemma 2.9, we can conclude that there exists a solution \( V_{n+1} \) of (18) such that \( V_n \leq V_{n+1} \leq W_n \) by applying (27) and (28).
Similarly, applying (7), (12) and (26), we have
\[ D_H V_n \leq F(t, V_n) + G(t, W_n) \]
\[ \leq \sum_{i=0}^{2k-2} \frac{D_{H^i} F(t, W_n)(V_n - W_n)^i}{i!} + \frac{D_{H^{2k-1}} F(t, V_n)(V_n - W_n)^{2k-1}}{(2k-1)!} \]
\[ + \sum_{i=0}^{2k-1} \frac{D_{H^i} G(t, V_n)(W_n - V_n)^i}{i!}, \]
\[ = N(t, V_n, W_n; V_n, W_n), \quad V_n(0) \leq U_0; \]
\[ D_H W_n \geq F(t, W_n) + G(t, V_n) = N(t, V_n, W_n; W_n, V_n), \quad W_n(0) \geq U_0. \]  

(29) 

(30) 

These show that \( V_n, W_n \) are also lower and upper solutions of (19) respectively. Taking account of the fact that \( N(t, V, W; Y, X) \) is nonincreasing in \( X \) and Lemma 2.9, we can conclude that there exists a solution \( W_{n+1} \) of (19) such that \( V_n \leq W_{n+1} \leq W_n \) by applying (29) and (30).

From (6), (13) and the condition \( (H_2) \), \( V_n \leq W_{n+1} \leq W_n \), we get
\[ D_H V_{n+1} = M(t, V_n, W_n; V_{n+1}, W_{n+1}) \]
\[ \leq \sum_{i=0}^{2k-1} \frac{D_{H^i} F(t, V_n)(V_{n+1} - V_n)^i}{i!} + \frac{2k-2}{2k-1} \frac{D_{H^{2k-1}} G(t, W_n)(W_{n+1} - W_n)^{2k-1}}{(2k-1)!} \]
\[ + \sum_{i=0}^{2k-1} \frac{D_{H^i} G(t, V_n)(W_n - V_n)^i}{i!}, \quad V_{n+1}(0) = U_0. \]  

(31) 

By (7), (12) and the condition \( (H_2) \), \( V_n \leq W_{n+1} \leq W_n \), we obtain
\[ D_H W_{n+1} = N(t, V_n, W_n; W_{n+1}, V_{n+1}) \]
\[ \geq \sum_{i=0}^{2k-2} \frac{D_{H^i} F(t, W_n)(W_{n+1} - W_n)^i}{i!} + \frac{2k-1}{2k-1} \frac{D_{H^{2k-1}} G(t, V_n)(V_{n+1} - V_n)^i}{i!} \]
\[ + \sum_{i=0}^{2k-1} \frac{D_{H^i} G(t, W_n)(W_n - V_n)^i}{i!}, \quad W_{n+1}(0) = U_0. \]  

(32) 

Therefore, we have \( V_{n+1} \leq W_{n+1} \) by applying the \( (K_1) \) of Lemma 2.8 on \( I \). This proves \( V_n \leq V_{n+1} \leq W_{n+1} \leq W_n \) in \( I \). Thus through induction, we obtain
\[ V_0 \leq V_1 \leq \cdots \leq V_n \leq W_n \leq \cdots \leq W_1 \leq W_0. \]

According to (26), \( V_n \) and \( W_n \) are coupled lower and upper solutions of type I of (1). We know \( D_{H^i} G(t, U) \leq \theta \) on \( \Omega \) from the assumption \( (H_3) \). It then follows from the Lemma 2.9 that \( U(t) \) is a solution of (1) on \( I \) satisfying \( V_n(t) \leq U(t) \leq W_n(t) \). Hence, we can prove that \( V_n \leq U \leq W_n \) on \( I \) for all \( n \), namely
\[ V_0 \leq V_1 \leq \cdots \leq V_n \leq U \leq W_n \leq \cdots \leq W_1 \leq W_0. \]

Clearly the sequences \( \{V_n\}, \{W_n\} \) are uniformly bounded on \( I \). To prove that \( \{V_n\}, \{W_n\} \) are equicontinuous, from the condition \( (H_4) \) and properties of Hausdorff
metric, consider for any \( s < t \), where \( t, s \in I \),

\[
D[V_n(t), V_n(s)] = D[U_0 + \int_0^t E(\xi) d\xi, U_0 + \int_0^s E(\xi) d\xi] = D[\int_s^t E(\xi) d\xi, \theta]
\]

\[
\leq \int_s^t D[E(\xi, \theta)] d\xi \leq M|t - s|,
\]

where

\[
E(\xi) = \sum_{i=0}^{2k-1} \frac{D_{H^i} F(\xi, V_{n-1})(V_n - V_{n-1})^i}{i!} + \sum_{i=0}^{2k-2} \frac{D_{H^i} G(\xi, W_{n-1})(W_n - W_{n-1})^i}{i!}
\]

\[
+ \frac{D_{H^{2k-1}} G(\xi, V_{n-1})(W_n - W_{n-1})^{2k-1}}{(2k - 1)!},
\]

and \( M > 0 \) is a constant. Hence \( \{V_n\} \) is equicontinuous on \( I \). Applying Ascoli-Arzela’s Theorem, \( \{V_n\} \) gives a subsequence which converges uniformly. Furthermore, \( \{V_n\} \) is a monotone nondecreasing sequence, then the sequence \( \{V_n\} \) converges uniformly to \( \rho \). Similar arguments apply to the sequence \( \{W_n\} \) which converges uniformly to \( r \). Taking the limit as \( n \to \infty \), we have

\[
\lim_{n \to \infty} V_n = \rho \leq U = r = \lim_{n \to \infty} W_n,
\]

where \( (\rho, r) \) are the coupled minimal and maximal solutions of (1) respectively.

Next, we shall prove that \( \rho \geq r \). In fact, from the following IVPs

\[
D_H V_{n+1} = M(t, V_n, W_n; V_{n+1}, W_{n+1}), \quad V_{n+1}(0) = U_0,
\]

\[
D_H W_{n+1} = N(t, V_n, W_n; W_{n+1}, V_{n+1}), \quad W_{n+1}(0) = U_0,
\]

we get

\[
D_H \rho = F(t, \rho) + G(t, r), \quad \rho(0) = U_0,
\]

\[
D_H r = F(t, r) + G(t, \rho), \quad r(0) = U_0.
\]

Now setting \( \Phi = \rho - r \), we obtain

\[
D_H \Phi = F(t, \rho) + G(t, r) - F(t, r) - G(t, \rho)
\]

\[
\geq -L_1(\rho - r) - L_2(r - \rho) = L(\rho - r),
\]

\[
\Phi(0) = 0,
\]

where \( L = -L_1 + L_2, L_1 \geq 0, L_2 \geq 0 \). Hence we can conclude that \( r \leq \rho \) on \( I \).

This shows \( r = \rho = U \), therefore \( \{V_n\} \) and \( \{W_n\} \) converge uniformly to the unique solution of (1).

Finally, we shall show that the convergence of \( \{V_n\} \) and \( \{W_n\} \) is of order \( 2k \) on \( I \). Let

\[
P_n = U - V_n, \quad P_n(0) = \theta,
\]

\[
Q_n = W_n - U, \quad Q_n(0) = \theta,
\]

where \( U \) is a unique solution of (1) for \( t \in I \) with \( P_n \geq \theta, Q_n \geq \theta \). Using the definitions of \( V_n, W_n \), the Taylor series expansion and the mean value theorem associate with \( (H_2) \), we get

\[
D_H P_{n+1} = F(t, U) + G(t, U) - \left[ \frac{D_{H^{2k-1}} G(t, V_n) (W_{n+1} - W_n)^{2k-1}}{(2k - 1)!} \right]
\]

\[
+ \sum_{i=0}^{2k-2} \frac{D_{H^i} G(t, W_n) (W_{n+1} - W_n)^i}{i!} + \sum_{i=0}^{2k-1} \frac{D_{H^i} F(t, V_n) (V_{n+1} - V_n)^i}{i!}
\]

\[
\leq 0,
\]

\[
\Rightarrow P_{n+1} \leq P_n.
\]

Hence \( \{P_n\} \) is a monotone nondecreasing sequence, then the sequence \( \{P_n\} \) converges uniformly to the unique solution of (1).
\begin{align*}
&= F(t, U) + G(t, U) - \left[\frac{D_{H^{2k-1}} G(t, V_n)(W_{n+1} - W_n)^{2k-1}}{(2k-1)!}\right] \\
&+ G(t, W_{n+1}) - \frac{D_{H^{2k-1}} G(t, \xi_2)(W_{n+1} - W_n)^{2k-1}}{(2k-1)!} \\
&+ F(t, V_{n+1}) - \frac{D_{H^{2k}} F(t, \xi_1)(V_{n+1} - V_n)^{2k}}{(2k)!} \\
&= D_{H_U} F(t, \eta_1)(U - V_{n+1}) + \frac{D_{H^{2k}} F(t, \xi_1)(V_{n+1} - V_n)^{2k}}{(2k)!} \\
&+ D_{H_U} G(t, \eta_2)(U - W_{n+1}) + \frac{D_{H^{2k}} G(t, \eta_3)(\xi_2 - V_n)(W_{n+1} - W_n)^{2k-1}}{(2k-1)!} \\
&\leq D_{H_U} F(t, \eta_1)(U - V_{n+1}) + D_{H_U} G(t, \eta_2)(U - W_{n+1}) \\
&+ \frac{D_{H^{2k}} F(t, \xi_1)(U - V_n)^{2k}}{(2k)!} \\
&+ \frac{D_{H^{2k}} G(t, \eta_3)(U - W_n)^{2k-1}[|W_n - U| + (U - V_n)]}{(2k-1)!} \\
&= D_{H_U} F(t, \eta_1) P_{n+1} - D_{H_U} G(t, \eta_2) Q_{n+1} + \frac{D_{H^{2k}} F(t, \xi_1) P_{2k}^n}{(2k)!} \\
&- \frac{D_{H^{2k}} G(t, \eta_3) Q_{n+1}^{2k-1}(P_n + Q_n)}{(2k-1)!} \\
&\leq M_1 P_{n+1} - N_1 Q_{n+1} + M_2 P_{2k}^n - N_2 Q_{n+1}^{2k-1}(P_n + Q_n),
\end{align*}

where \( V_n \leq \xi_1 \leq V_{n+1}, \ W_{n+1} \leq \xi_2 \leq W_n, \ V_{n+1} \leq \eta_1 \leq U, \ U \leq \eta_2 \leq W_{n+1}, \) and \( V_n \leq \eta_3 \leq \xi_2, \ M_1, \ M_2, \ N_1, \ N_2 \) are \( n \times n \) matrices and inverse matrices exist and

\( D_{H_U} F(t, U) \leq M_1, \quad D_{H^{2k}} F(t, U) \leq (2k)! M_2, \)

\( N_1 \leq D_{H_U} G(t, U) \leq \theta, \quad (2k-1)! N_2 \leq D_{H^{2k}} G(t, U) \leq \theta. \)

Similarly, we have

\[ D_{H} Q_{n+1} = \sum_{i=0}^{2k-1} \frac{D_{H^{2k-i}} F(t, W_n)(W_{n+1} - W_n)^i}{i!} + \frac{D_{H^{2k-1}} F(t, V_n)(W_{n+1} - W_n)^{2k-1}}{(2k-1)!} \]

\[ + \sum_{i=0}^{2k-1} \frac{D_{H^{2k-i}} F(t, V_n)(V_{n+1} - V_n)^i}{i!} - F(t, U) - G(t, U) \]

\[ = F(t, W_{n+1}) - \frac{D_{H^{2k-1}} F(t, \xi_1)^2(W_{n+1} - W_n)^{2k-1}}{(2k-1)!} \]

\[ + \frac{D_{H^{2k-1}} F(t, V_n)(W_{n+1} - W_n)^{2k-1}}{(2k-1)!} + G(t, V_{n+1}) \]

\[ - \frac{D_{H^{2k}} G(t, \xi_2)(V_{n+1} - V_n)^{2k}}{(2k)!} - F(t, U) - G(t, U) \]

\[ = D_{H_U} F(t, \eta_1)(W_{n+1} - U) - \frac{D_{H^{2k}} G(t, \xi_2)(V_{n+1} - V_n)^{2k}}{(2k)!} \]
We can denote (33), (34) as inequality

\[
\leq D_{H_U} F(t, \eta_1^1)(W_{n+1} - U) - D_{H_U} G(t, \eta_2^1)(U - V_{n+1})
\]

\[
- D_{H_W}^2 F(t, \eta_3^2)(U - W_n)^{2k-1}[(W_n - U) + (U - V_n)]
\]

\[
- \frac{D_{H_W}^2 G(t, \xi_2^2)(U - V_n)^{2k}}{(2k-1)!}
\]

\[
= D_{H_U} F(t, \eta_1^1)Q_{n+1} - D_{H_U} G(t, \eta_2^1)P_{n+1} + \frac{D_{H_W}^2 F(t, \eta_3^2)Q_n^{2k-1}(Q_n + P_n)}{(2k-1)!}
\]

\[
- \frac{D_{H_W}^2 G(t, \xi_2^2)P_n^{2k}}{(2k)!}
\]

\[
\leq M_1Q_{n+1} - N_1P_{n+1} + M_3Q_n^{2k-1}(Q_n + P_n) - N_3P_n^{2k},
\]

(34)

where \( W_{n+1} \leq \xi_1^1 \leq W_n, \ V_n \leq \xi_2^1 \leq V_{n+1}, \ U \leq \eta_1^1 \leq W_{n+1}, \ V_{n+1} \leq \eta_2^1 \leq U, \) and \( V_n \leq \eta_3^1 \leq \xi_1^1. \) \( M_1, M_3, N_1, N_3 \) are \( n \times n \) matrixes and inverse matrixes exist and

\[
D_{H_W}^2 F(t, U) \leq (2k-1)!M_3, \quad (2k)!N_3 \leq D_{H_W}^2 G(t, U) \leq \theta.
\]

Suppose that

\[
R_n = \begin{pmatrix} P_n \\ Q_n \end{pmatrix}, \quad A = \begin{pmatrix} M_1 & -N_1 \\ -N_1 & M_3 \end{pmatrix}, \quad B_n = \begin{pmatrix} M_2P_n^{2k} - N_2Q_n^{2k-1}(Q_n + P_n) \\ M_3Q_n^{2k-1}(Q_n + P_n) - N_3P_n^{2k} \end{pmatrix}.
\]

We can denote (33), (34) as inequality

\[
D_H R_{n+1} \leq AR_{n+1} + B_n.
\]

Using Lemma 2.6, treating \( B_n \) as a forcing term, we obtain

\[
\theta \leq R_{n+1} \leq \int_0^t e^{A(t-s)}B_n(s) ds, \quad t \in I.
\]

Then we get

\[
\| R_{n+1}(t) \| \leq A^{-1}e^{AT} \| B_n(t) \|.
\]

There exist some constant matrixes \( C_1, \cdots, C_7, \) such that

\[
\max_{t \in I} |P_{n+1}| \leq \max_{t \in I} |C_1P_n^{2k} + C_2Q_n^{2k-1}(Q_n + P_n)|
\]

\[
\leq C_1 \| P_n \|^{2k} + C_2 \| Q_n \|^{2k-1}(\| Q_n \| + \| P_n \|), \quad (35)
\]

\[
\max_{t \in I} |Q_{n+1}| \leq \max_{t \in I} |C_3P_n^{2k} + C_4Q_n^{2k-1}(Q_n + P_n)|
\]

\[
\leq C_3 \| P_n \|^{2k} + C_4 \| Q_n \|^{2k-1}(\| Q_n \| + \| P_n \|). \quad (36)
\]

Using (35) and (36), we get

\[
\max_{t \in I} |P_{n+1}| + \max_{t \in I} |Q_{n+1}| \leq C_5 \| P_n \|^{2k} + C_6 \| Q_n \|^{2k-1}(\| Q_n \| + \| P_n \|)
\]

\[
\leq C_7 \left( \| P_n \| + \| Q_n \| \right)^{2k},
\]

that is

\[
\| U - V_{n+1} \| + \| W_{n+1} - U \| \leq C_7 \left( \| U - V_n \| + \| W_n - U \| \right)^{2k},
\]
where $C_1 + C_3 = C_5$, $C_2 + C_4 = C_6$, $C_7 = \max\{C_5, C_6\}$ and
\[
\|P_n\|^{2k} = \left( \max_{i \in I} |P^{i|2k}|, \cdots, \max_{i \in I} |P^{n|2k}| \right).
\]

**Theorem 3.3.** For the IVP (1), assume that
\begin{enumerate}
  \item[(H_5):] $V, W \in C^1[I, K_c(\mathbb{R}^n)]$ are coupled lower and upper solutions of type I of (1), and $V(t) \leq W(t)$,
  \item[(H_6):] $F(t, U)$ is hyperconvex in $U$ of order $2k$, $G(t, U)$ is hyperconcave in $U$ of order $2k$, $k \geq 1$, where $F, G \in C^{2k+1}[\Omega, K_c(\mathbb{R}^n)]$, i.e. $D^{2k+1}_U F(t, U) \geq \theta$, $D^{2k+1}_U G(t, U) \leq \theta$,
  \item[(H_7):] $D^{2k+1}_U G(t, U) \leq \min_{\Omega}[D^{2k+1}_U G(t, U)] (W - V)^{2k} / (2k - 1)! \leq \theta$ on $\Omega$,
  \item[(H_8):] $F$ and $G$ map bounded sets into bounded sets in $K_c(\mathbb{R}^n)$.
\end{enumerate}
Then there exist monotone sequences $\{V_n\}$, $\{W_n\}$ with $V = V_0$, $W = W_0$, $n \geq 0$, which converge uniformly and monotonically to the unique solution of (1) and the convergence is of order $2k + 1$.

**Proof.** We associate with the inequalities (8), (9), (14) and (15), and consider the following coupled IVPs:
\begin{align*}
D_H V_n &= M(t, V_{n-1}, W_{n-1}; V_n, W_n) \equiv 
\sum_{i=0}^{2k} \frac{D^{2k}_H(V(t, V_{n-1}) V_{n-1})^i}{i!} 
+ \sum_{i=0}^{2k} \frac{D^{2k}_H G(t, W_{n-1}) (W_n - W_{n-1})^i}{i!}, V_n(0) = U_0.
\end{align*}

\begin{align*}
D_H W_n &= N(t, V_{n-1}, W_{n-1}; V_n, W_n) \equiv 
\sum_{i=0}^{2k} \frac{D^{2k}_H(F(t, W_{n-1}) W_{n-1})^i}{i!} 
+ \sum_{i=0}^{2k} \frac{D^{2k}_H G(t, W_{n-1}) (V_n - V_{n-1})^i}{i!}, W_n(0) = U_0.
\end{align*}

The rest of proof is similar to that of Theorem 3.2, and so we omit it here.

**Theorem 3.4.** For the IVP (1), assume that
\begin{enumerate}
  \item[(H_9):] $V, W \in C^1[I, K_c(\mathbb{R}^n)]$ are coupled lower and upper solutions of type II of (1), and $V(t) \leq W(t)$,
  \item[(H_{10}):] $F(t, U)$ is hyperconvex in $U$ of order $2k - 1$, $G(t, U)$ is hyperconcave in $U$ of order $2k - 1$, $k \geq 1$, where $F, G \in C^{2k}[\Omega, K_c(\mathbb{R}^n)]$, i.e. $D^{2k}_U F(t, U) \geq \theta$, $D^{2k}_U G(t, U) \leq \theta$,
  \item[(H_{11}):] $D^{2k}_U F(t, U) \leq - \max_{\Omega}[D^{2k}_U F(t, U)] (W - V)^{2k-1} / (2k - 2)! \leq \theta$ on $\Omega$,
  \item[(H_{12}):] $F$ and $G$ map bounded sets into bounded sets in $K_c(\mathbb{R}^n)$.
\end{enumerate}
Then there exist monotone sequences $\{V_n\}$, $\{W_n\}$ with $V = V_0$, $W = W_0$, $n \geq 0$ which converge uniformly and monotonically to the unique solution of (1) and the convergence is of order $2k$. 

\qed
The rest of proof is similar to that of Theorem 3.2, and so we omit it here. \qed

**Theorem 3.5.** For the IVP (1), assume that

\begin{align*}
(H_{13}) & : V, W \in C^1([I, \mathbb{K}_c(\mathbb{R}^n)]) are coupled lower and upper solutions of type II of (1), and V(t) \leq W(t), \\
(H_{14}) & : F(t, U) is hyperconvex in U of order 2k, G(t, U) is hyperconcave in U of order 2k, k \geq 1, where F, G \in C^{2k+1}(\Omega, \mathbb{K}_c(\mathbb{R}^n)). i.e., D_{H^2k+1}F(t, U) \geq \theta, \\
& \quad \quad \quad D_{H^2k+1}G(t, U) \leq \theta, \\
(H_{15}) & : D_{H^2k+1}F(t, U) \leq -\max_{11} (D_{H^2k+1}F(t, U))(W - V)^{2k} \leq \theta \quad \text{on} \quad \Omega, \\
(H_{16}) & : F and G map bounded sets into bounded sets in \mathbb{K}_c(\mathbb{R}^n). \end{align*}

Then there exist monotone sequences \{V_n\}, \{W_n\} with \( V = V_0, W = W_0, n \geq 0, \) which converge uniformly and monotonically to the unique solution of (1) and the convergence is of order \( 2k + 1. \)

**Proof.** We associate with the inequalities (10), (11), (16) and (17), and consider the following coupled IVPs:

\begin{align*}
D_H V_n &= M(t, V_{n-1}, W_{n-1}; V_n, W_n) \\
& = \sum_{i=0}^{2k-1} \frac{D_{H^i} F(t, W_{n-1})(W_n - W_{n-1})^i}{i!} + \sum_{i=0}^{2k-2} \frac{D_{H^i} G(t, V_{n-1})(V_n - V_{n-1})^i}{i!} \\
& \quad + \frac{D_{H^{2k}} G(t, W_{n-1})(V_n - W_{n-1})^{2k}}{(2k)!}, \quad V_n(0) = U_0. \\
D_H W_n &= N(t, V_{n-1}, W_{n-1}; V_n, W_n) \\
& = \sum_{i=0}^{2k-2} \frac{D_{H^i} F(t, V_{n-1})(V_n - V_{n-1})^i}{i!} + \frac{D_{H^{2k+1}} G(t, V_{n-1})(V_n - V_{n-1})^{2k}}{(2k+1)!}, \quad W_n(0) = U_0.
\end{align*}
\[ D_H W_n = N(t, V_{n-1}, W_{n-1}; W_n, V_n) \equiv 2k - 1 \sum_{i=0}^{2k-1} \frac{D_{H^i} F(t, V_{n-1})(V_n - V_{n-1})^i}{i!} + \frac{D_{H^{2k}} F(t, W_{n-1})(V_n - V_{n-1})^{2k}}{(2k)!} + \frac{D_{H^{2k}} G(t, W_{n-1})(W_n - W_{n-1})^{2k}}{(2k)!}, \quad W_n(0) = U_0. \]

The rest of proof is similar to that of Theorem 3.2. Thus we omit it here.

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