Complex Riccati equations as a link between different approaches for the description of dissipative and irreversible systems

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Abstract. Quantum mechanics is essentially described in terms of complex quantities like wave functions. The interesting point is that phase and amplitude of the complex wave function are not independent of each other, but coupled by some kind of conservation law. This coupling exists in time-independent quantum mechanics and has a counterpart in its time-dependent form. It can be traced back to a reformulation of quantum mechanics in terms of nonlinear real Ermakov equations or equivalent complex nonlinear Riccati equations, where the quadratic term in the latter equation explains the origin of the phase-amplitude coupling. Since realistic physical systems are always in contact with some kind of environment this aspect is also taken into account. In this context, different approaches for describing open quantum systems, particularly effective ones, are discussed and compared. Certain kinds of nonlinear modifications of the Schrödinger equation are discussed as well as their interrelations and their relations to linear approaches via non-unitary transformations. The modifications of the aforementioned Ermakov and Riccati equations when environmental effects are included can be determined in the time-dependent case. From formal similarities conclusions can be drawn how the equations of time-independent quantum mechanics can be modified to also include the environmental aspects.

1. Introduction
In his talk “Square root of minus one, complex phase and Erwin Schrödinger” [1] presented 25 years ago at a conference in London celebrating the centenary of the birth of Schrödinger, the Nobel Laureate Chen Ning Yang claimed that with quantum mechanics, for the first time, the imaginary unit enters physics in a fundamental way and “complex numbers became a conceptual element of the very foundation of physics”. He continued that the very meaning of the fundamental equations of matrix mechanics (Heisenberg’s commutation relation) and of wave mechanics (the time-dependent Schrödinger equation (TDSE)) “would be totally destroyed if one tries to get rid of i by writing them in terms of real and imaginary parts”.

I totally agree with this statement but want to go even a step further. The reason it is not sufficient to simply write these complex equations in terms of two real ones will be explained in section 2 using Madelung’s hydrodynamic formulation [2] of Schrödinger’s theory where the complex wave function is written in polar form. This shows that phase and amplitude (or real and imaginary parts), and hence the corresponding equations of motion, are not independent of each other but uniquely coupled. For the time-independent (TI)SE, it will be shown that
this coupling is connected with a conservation law that leads to a nonlinear (NL) formulation of quantum mechanics in terms of a (real) so-called Ermakov equation [3]. This can be rewritten in the form a complex NL Riccati equation where the origin of the coupling becomes obvious. Formally, a similar situation is also found in the case of the TDSE, at least for quadratic Hamiltonians with Gaussian wave packet (WP) solutions.

Since all realistic physical systems are in contact with some kind of environment, which introduces phenomena like dissipation and irreversibility, it will be discussed in section 3 how these aspects can be included in our investigation. Several of the many different approaches to take the environment into account will be mentioned. Our main emphasis will be on models where only the effect of the environment on the system of interest, but not the details of the environmental degrees of freedom, is considered. In this context, modifications of the TDSE leading to explicitly TD Hamiltonians or NL modifications of the SE will be discussed; in particular, a NLSE with complex logarithmic nonlinearity that avoids the shortcomings of other approaches.

In section 4 this logarithmic NLSE will be used to show the modifications of the Ermakov and Riccati equations that were obtained in the TD case when a dissipative environment is taken into account. The formal similarities between the TD and TI situation in the conservative case give a hint as to how the TISE can be modified in order to also include the environmental effects. This leads to a deeper understanding of the connections between several of the NL approaches.

The relations between the NL approaches and those based on explicitly TD Hamiltonians will be shown in section 5. The transition between two of these explicitly TD models, via a unitary transformation, bears similarity with the relation between two of the NL approaches. The connections between the complex Riccati equations that exist in all these approaches will be specified and reduced to one equation that is relevant for a description in physical terms.

Finally, the results are summarized in section 6 where the interrelations between the approaches discussed are also presented schematically and further perspectives given.

2. Phases, amplitudes and complex Riccati equations in time-independent and time-dependent quantum mechanics

In classical mechanics the deterministic particle aspect is expressed by the equations of motion for the trajectory. In the most general, the Hamiltonian form, the integration of these equations becomes trivial, if the Hamiltonian function is canonically transformed into one that is identically zero. The corresponding generating function is the action

$$S(r, t),$$

leading to the Hamilton–Jacobi equation

$$\frac{\partial}{\partial t} S + H = \frac{\partial}{\partial t} S + \frac{1}{2m} (\nabla S)^2 + V = 0$$

with the momentum

$$p = \nabla S.$$ This equation was Schrödinger’s starting point for the “derivation” of his wave equation [4].

It should be mentioned that all the classical equations of motion (i.e., also the Newtonian and Lagrangian form) are (for conservative forces) invariant under time reversal.

On the classical level, also a probabilistic description of a mechanical system exists in terms of a continuity equation for a probability density $$\varrho_\Gamma(q, p, t)$$ in the 6N dimensional phase space ($$\Gamma$$ space, for an N-particle system),

$$\frac{\partial}{\partial t} \varrho_\Gamma + \nabla_\Gamma (\varrho_\Gamma \mathbf{v}_\Gamma) = 0 .$$

with coordinate $$q = (q_1, ..., q_{3N})$$, momentum $$p = (p_1, ..., p_{3N})$$ and $$\nabla_\Gamma = \left( \frac{\partial}{\partial q}, \frac{\partial}{\partial p} \right), \mathbf{v}_\Gamma = (q, p),$$ where overdots denote time-derivatives. For Hamiltonian systems $$\nabla_\Gamma \mathbf{v}_\Gamma = 0$$ is valid and Eq.
(2) reduces to the Liouville equation
\[
\frac{\partial}{\partial t} \varrho + v \nabla (\varrho \Gamma) = \frac{\partial}{\partial t} \varrho \Gamma + \{H, \varrho \Gamma\}_F = 0 ,
\]
whith \(\{...,\}_F\) representing the Poisson brackets. Also Eqs. (2) and (3) are invariant under time-reversal.

Equations (1) and (2) appear in modified form also in quantum mechanics where the modifications are characteristic for the complex nature of the quantum mechanical wave function and reflect the coupling of these two equations and hence the coupling of the deterministic particle-like aspect and the probabilistic wave-like aspect in quantum mechanics. This becomes obvious using Madelung’s hydrodynamic formulation [2], where the wave function is written in polar form as
\[
\Psi(r, t) = \varrho^{1/2}(r, t) \exp \left( \frac{i}{\hbar} S(r, t) \right)
\]
with the square root of the probability density \(\varrho = \Psi^* \Psi\) as amplitude and \(\frac{\hbar}{i} S\) as phase (r is the position vector in three dimensions).

Inserting this form into the TDSE
\[
i\hbar \frac{\partial}{\partial t} \Psi(r, t) = \left( -\frac{\hbar^2}{2m} \Delta + V \right) \Psi(r, t) = H_L \Psi(r, t)
\]
(with \(\Delta = \text{Laplace operator}\) leads to a modified Hamilton–Jacobi equation for the phase,
\[
\frac{\partial}{\partial t} S + \frac{1}{2m} (\nabla S)^2 + V - \frac{\hbar^2}{2m} \frac{\varrho^{1/2}}{\varrho^{1/2}} = 0 ,
\]
and a continuity equation (in position space),
\[
\frac{\partial}{\partial t} \varrho + \frac{1}{m} \nabla (\varrho \nabla S) = 0 ,
\]
for the amplitude.

In Eq. (6) that determines the evolution of the phase occurs an additional term, the last one on the rhs, that depends on the amplitude \(\varrho^{1/2}\) of the wave function and is often deceptively called “quantum potential” \((V_{qu})\) although it is essentially related to the kinetic energy of the quantum system (see below). This equation reflects the classical (deterministic) trajectory aspect of the system. However, via \(\varrho\), also a probabilistic element enters.

The probability density fulfills Eq. (7) that determines the evolution of the amplitude of the wave function, but contains the (gradient of the) phase in the probability current \(j = \varrho (\frac{\hbar}{m} \nabla S)\), where the gradient of the action \(S\) is, as usual, associated with the momentum of the system. So, obviously the Hamilton–Jacobi equation (6) and the continuity equation (7) (or, phase and amplitude of the wave function \(\Psi\)) are coupled. This coupling is not arbitrary but related to a conservation law which will be shown in the following for the TI case.

2.1. Nonlinear formulations and invariants in time-independent quantum mechanics

In 1994, G. Reinisch [5] presented a NL formulation of the TISE. Since in this case \(\frac{\partial}{\partial t} \varrho = 0\) and \(\frac{\partial}{\partial t} S = -E\) are valid, the continuity equation (6) (we now use the notation \(\varrho^{1/2} = |\Psi| = a\) turns into
\[
\nabla (a^2 \nabla S) = 0
\]
and the modified Hamilton–Jacobi equation into

\[ -\frac{\hbar^2}{2m} \Delta a + (V - E) a = -\frac{1}{2m} (\nabla S)^2 a. \]  

(9)

Equation (8) is certainly fulfilled for \( \nabla S = 0 \), turning (9) into the usual TISE for the real wave function \( a = |\Psi| \) with position-independent phase \( S(\mathbf{r}) \). (NB: the kinetic energy term divided by \( a, -\frac{\hbar^2}{2m} \frac{\Delta a}{a} \), is just identical to \( V_{\text{qu}} \! \) !)

However, Eq. (8) can also be fulfilled for \( \nabla S \neq 0 \) if only the conservation law

\[ \nabla S = \frac{C}{a^2} \]  

(10)

with constant (or, at least, position-independent) \( C \) is fulfilled. This relation now shows explicitly the coupling between phase and amplitude of the wave function. Inserting (10) into the rhs of Eq. (9) changes this into the real NL Ermakov equation

\[ \Delta a + 2m \frac{\hbar^2}{\hbar^2} (E - V) a = \left( \frac{1}{\hbar} \nabla S \right)^2 a = \left( \frac{C}{\hbar} \right)^2 \frac{1}{a^3} \]  

(11)

with inverse cubic nonlinearity (this equation had been studied by V. Ermakov [3] long before quantum mechanics was developed and we will come back to it in the TD case). Knowing the solution of (11), the phase (up to an integration constant) can be determined via (10) and thus the complex wave function can be obtained.

For a better understanding of the origin of the coupling expressed by Eq. (10) it can be used that the real NL Ermakov equation (11) is equivalent to a complex NL Riccati equation. In our case, this quadratic NL equation reads

\[ \nabla \left( \frac{\nabla \Psi}{\Psi} \right) + \left( \frac{\nabla \Psi}{\Psi} \right)^2 + 2m \frac{\hbar^2}{\hbar^2} (E - V) = 0 \]  

(12)

with the complex variable \( \left( \frac{\nabla \Psi}{\Psi} \right) = \frac{\nabla a}{a} + \frac{\imath}{\hbar} \nabla S \).

It is straightforward to show that Eq. (12) can be linearized to yield the usual TISE,

\[ -\frac{\hbar^2}{2m} \Delta \Psi + V \Psi = E \Psi, \]  

(13)

but inserting the definition of the complex variable into Eq.(12) and separating real and imaginary parts shows that the imaginary part leads to the conservation law (10), whereas the real part leads to the Ermakov equation (11). The origin of the coupling between phase and amplitude is obviously the quadratic nonlinearity since it introduces a term proportional to \( (\nabla S)^2 \) into the real part (Eq. (11)) and is responsible for the product term proportional to \( (\nabla a)(\nabla S) \) in the imaginary part, leading to Eq. (10).

The same type of NL Riccati and Ermakov equations also occurs in the TD case, as will be shown in the following.

2.2. Nonlinear formulations and invariants in time-dependent quantum mechanics

The TDSE (5) always possesses for at most quadratic Hamiltonians exact analytic solutions in the form of Gaussian WPs that can be written (in the following in one dimension with position variable \( x \)) as

\[ \Psi(x, t) = N(t) \exp \left\{ i \left[ y(t) \dot{x}^2 + \frac{1}{\hbar} < \dot{x} > + K(t) \right] \right\}, \]  

(14)
where the maximum of the WP is at $x = \eta(t)$ which is identical to the mean value of the position, $\langle x \rangle = \int_{-\infty}^{+\infty} dx \Psi^* x \Psi = \eta$. $\tilde{x} = x - \eta$ and the width of the WP is related with the imaginary part of the complex quantity $y(t)$ via $y_I = \frac{1}{\bar{\alpha}} \left( \langle \tilde{x}^2 \rangle - \langle x \rangle^2 \right)$. ($N$ is a, possibly TD, normalization factor, $\langle p \rangle = m\dot{\eta}$ the classical momentum and $K$ a purely TD function that will not be relevant for the following.) Inserting WP $(14)$ into the SE $(5)$ leads to the equations of motion for $\eta(t)$ and $y(t)$. Particularly for the harmonic oscillator (HO) (i.e., $V = \frac{m}{2} \omega^2 x^2$ with possibly TD $\omega(t)$), one obtains the usual Newtonian equation of motion

$$\ddot{\eta} + \omega^2(t) \eta = 0 \quad (15)$$

for the maximum and a complex Riccati equation, but now for the TD variable $\left( \frac{2\hbar}{m} y \right)$,

$$\frac{2\hbar}{m} \dot{y} + \left( \frac{2\hbar}{m} y \right)^2 + \omega^2(t) = 0 \quad (16)$$

Introducing a new variable $\alpha_L(t)$ via

$$\frac{2\hbar}{m} y_I = \hbar \frac{1}{2m \langle \tilde{x}^2 \rangle_L} = \frac{1}{\alpha_L^2} \quad (17)$$

allows one to determine the real part $\frac{2\hbar}{m} y_R$ from the imaginary part of Eq. $(16)$ as

$$\frac{2\hbar}{m} y_R = \frac{\dot{\alpha}_L}{\alpha_L} \quad (18)$$

Inserting this into the real part of $(16)$ yields again an Ermakov equation,

$$\ddot{\alpha}_L + \omega^2(t) \alpha_L = \frac{1}{\alpha_L^2} \quad (19)$$

Eliminating $\omega^2(t)$ from Eqs.$(15)$ and $(19)$ leads to the dynamical invariant

$$I_L = \frac{1}{2} \left[ (\dot{\eta} \alpha_L - \eta \dot{\alpha}_L)^2 + \left( \eta \frac{\dot{\alpha}_L}{\alpha_L} \right)^2 \right] = \text{constant} \quad (20)$$

that also exists for TD $\omega(t)$, whereas the corresponding Hamiltonian is no longer an invariant (for further details see, e.g., [3] [6] [7] [8] [9]).

Also the NL Riccati equation $(16)$ can be linearized via the ansatz $\left( \frac{2\hbar}{m} y \right) = \frac{1}{\alpha} \lambda$ with complex $\lambda(t)$ to yield the complex Newtonian equation

$$\ddot{\lambda} + \omega^2(t) \lambda = 0 \quad (21)$$

With the polar form $\lambda = \alpha e^{i\varphi}$ the complex variable $\frac{2\hbar}{m} y$ can be written as $\frac{1}{\alpha} = \frac{1}{\alpha} \alpha + i \varphi$. Inserting this into Eq. $(16)$ shows that the imaginary part leads to the conservation law

$$\dot{\varphi} = \frac{1}{\alpha^2} \quad (22)$$

which represents a kind of conservation of angular momentum for the motion of $\lambda(t)$ in the complex plane [10]. From the real part of Eq. $(16)$ one finally obtains the Ermakov equation $(19)$ with $\alpha \equiv \alpha_L$. 

5
In figure 1, the time-dependent and space-dependent cases are compared to show the formal similarity. The complex Riccati equations are given, as well as their linearized forms, leading to second-order differential equations for complex variables. These variables are given in polar form and the relations between phase and amplitude of these variables are expressed in terms of conservation laws. These laws are equivalent to the imaginary part of the respective Riccati equations if the polar form of the complex variables is inserted. The real part of the Riccati equations is then identical with the NL Ermakov equations. The essential difference between the two cases is that temporal and spatial variables and the respective derivatives are interchanged, as well as $\omega^2(t)$ and $\frac{2m}{h^2}(E - V(x))$.

**Table 1**: Comparison between the space-dependent and the time-dependent Riccati equations, their linearized forms and the corresponding Ermakov equations for the conservative case.

| Time          | Space          |
|---------------|----------------|
| NL complex    |                |
| $\frac{d}{dt} \left( \frac{\dot{\lambda}}{\lambda} \right) + \left( \frac{\dot{\lambda}}{\lambda} \right)^2 + \omega^2(t) = 0$ | $\nabla \left( \frac{\nabla \Psi}{\Psi} \right) + \left( \frac{\nabla \Psi}{\Psi} \right)^2 + \frac{2m}{h^2}(E - V(x)) = 0$ |
| linearized    |                |
| $\dot{\lambda} + \omega^2 \lambda = 0$ | $\Delta \Psi + \frac{2m}{h^2}(E - V)\Psi = 0$ |
| complex variable |                |
| $\dot{\lambda} = a e^{i\phi}$ | $\Psi = a e^{is/h}$ |
| conservation law |                |
| $\dot{\phi} = \frac{1}{a^2}$ | $\nabla S = \frac{C}{a^2}$ |
| NL real       |                |
| $\ddot{a} + \omega^2 = \dot{\phi}^2 a = \frac{1}{a^2}$ | $\Delta a + \frac{2m}{h^2}(E - V) = \frac{1}{h^2} (\nabla S)^2, a = \left( \frac{C}{h^2} \right)^2 \frac{\partial^2}{\partial x^2}$ |

**Figure 1**: Comparison between the space-dependent and the time-dependent Riccati equations, their linearized forms and the corresponding Ermakov equations for the conservative case.

### 3. Irreversibility, dissipation and effective NLSEs

Since realistic physical systems are always in contact with some kind of environment and this coupling usually introduces the phenomena irreversibility and dissipation (but not necessarily always both simultaneously!), the questions arise of how this can be taken into account in the formalism of classical (Hamiltonian) mechanics and, especially, in a quantum mechanical context.

In the trajectory picture, dissipation can be included by adding a phenomenological friction force that is proportional to velocity (or momentum) to the Newtonian equation of motion, thus turning it into an irreversible evolution equation, i.e.,

$$m\ddot{x} = m\dot{v} = -m\gamma v - \frac{\partial}{\partial x} V$$

which is the Langevin equation without stochastic contribution ($\gamma$ is the friction coefficient). This equation is used to describe Brownian motion from a trajectory point of view. An equivalent description of this phenomenon can also be given in terms of (classical) probability distributions $\varrho_{cl}$ via the Fokker–Planck equations that contain irreversible diffusion terms. Particularly in position space, this can be written in the form of the Smoluchowski equation for $\varrho_{cl}(x,t)$,

$$\frac{\partial}{\partial t} \varrho_{cl} + \frac{\partial}{\partial x} \left( \frac{F(x)}{m\gamma} \varrho_{cl} \right) - \frac{kT}{m\gamma} \frac{\partial^2}{\partial x^2} \varrho_{cl} = 0,$$  

(24)
with the conservative force $F(x)$, Boltzmann’s constant $k$ and temperature $T$. Comparison with Einstein’s theory of Brownian motion [11] shows that the coefficient of the diffusion term fulfils the Einstein relation $D = \frac{kT}{m\gamma}$.

However, Eqs. (23) and (24) do not fit consistently into the Lagrange/Hamilton formalism that is invariant under canonical transformations and provides a basis for quantization. So, these phenomenological equations do not yet provide the answer to the questions posed at the beginning of this Section.

There are numerous approaches in the literature to answer these questions in different ways; most of them with some problematic aspects. In the following, several of them and their interrelations will be discussed, particularly those which give rise to Riccati and Ermakov equations similar to the ones investigated in Section 2, so that a comparison between the conservative and dissipative case is possible.

The models use different approaches to reach their goal, namely:

1. Modification of the classical Lagrange/Hamilton formalism with subsequent canonical quantization (e.g., Caldirola–Kanai [12] [13], expanding coordinate system [14];
2. system-plus-reservoir approach (e.g., Bateman [15], Caldeira–Leggett [16] [17] [18], Kossakowski–Lindblad [19] [20];
3. non-unitary time-evolution (e.g., Gisin [21] [22], Beretta [23]) [24];
4. addition of friction terms to the SE according to Ehrenfest’s equation for mean values, leading to NLSEs (e.g., Kostin [25], Cho [26], Albrecht [27], Süssmann [28], Hasse [29]);
5. addition of an irreversible term on the level of the probability density, e.g., via a Smoluchowski equation, also leading to NLSEs (e.g., Schuch–Chung–Hartmann [30] [31] [32], Doebner–Goldin [33] [34]).

3.1. Modified Lagrange/Hamilton formalism

The addition of a so-called “friction potential” to the classical Lagrangian or Hamiltonian in order to obtain the above-mentioned friction force is not consistent with the conventional Lagrange or Hamilton formalism. Therefore, Caldirola [12] and Kanai [13] used a different approach by multiplying the classical Lagrangian by an exponential factor, leading to

$$\hat{L}_{\text{CK}} = \left[ \frac{m}{2} \ddot{x}^2 - V(x) \right] e^{\gamma t}$$ (25)

from where, via the Euler–Lagrange equation, the friction force in the equation of motion arises

$$\ddot{x} + \gamma \dot{x} + \frac{1}{m} \frac{\partial}{\partial x} V = 0.$$ (26)

Defining the canonical momentum in the usual way as

$$\frac{\partial}{\partial \dot{x}} \hat{L}_{\text{CK}} = \hat{p} = m \dot{x} e^{\gamma t} = p e^{\gamma t}$$ (27)

allows to construct the corresponding Hamiltonian function

$$\hat{H}_{\text{CK}} = \frac{1}{2m} e^{-\gamma t} \hat{p}^2 + e^{\gamma t} V(x)$$ (28)

that, via the Hamiltonian equations of motion, also provides the desired friction term. It is important to note that the transition from the physical variables $(x, p)$ to the canonical ones, $\dot{x} = x, \dot{p} = e^{\gamma t} p$, is a non-canonical transformation! Applying canonical quantization to
\[ \hat{H}_{CK}, \text{i.e., replacing } \hat{p} \text{ by } \frac{\hbar}{i} \frac{\partial}{\partial x}, \text{ provides the corresponding linear, but explicitly TD Hamiltonian} \]

\[ i\hbar \frac{\partial}{\partial t} \hat{\Psi}_{CK}(x, t) = \left\{ e^{-\gamma t} \left( -\frac{\hbar^2}{2m} \frac{\partial^2}{\partial x^2} \right) + e^{\gamma t} V \right\} \hat{\Psi}_{CK}(x, t). \tag{29} \]

However, calculating the uncertainty product of position and (physical) momentum using the solutions of (29) leads to an exponential decay and therefore an apparent violation of the uncertainty product. The reason for this, and a possible solution of this problem, will be given in section 5.

Another approach that does not face this problem and also starts from a modified Lagrange/Hamilton formalism is likewise based on a non-canonical transformation of the variables; however, this time also changes the position variable by transition to an exponentially-expanding coordinate system. The new (canonical) variables \((Q, P)\) are related to the physical ones \((x, p)\) via

\[ Q = e^{\gamma t/2} x , \quad P = m \dot{Q} = me^{\gamma t/2}(\dot{x} + \frac{\gamma}{2} x) , \tag{30} \]

leading to the Hamiltonian function

\[ \hat{H}_{exp} = \frac{1}{2m}P^2 + \frac{m}{2} \Omega^2 Q^2 = \text{const.} \tag{31} \]

with \(\Omega^2 = \left( \omega^2 - \frac{\gamma^2}{4} \right). \) \(\hat{H}_{exp}\) is a constant of motion, provides the correct equation of motion including the friction force (when expressed in the physical variables) and can be canonically quantized via \(P \rightarrow \hbar \frac{\partial}{\partial Q}\) (for further details, see [14]). The relation between this approach and the one of Caldirola–Kanai will be elaborated in section 5.

### 3.2. System-plus-reservoir approach

A more conventional approach is to consider the system of interest together with the environment as a closed Hamiltonian system and to determine the evolution of the system while interacting with the environment. Albeit, using this method means there are still different ways of describing the environment: a) by a single other degree of freedom (see Bateman [15]), b) by a large number of degrees of freedom (e.g., Caldeira–Leggett [16] [17] [18]) c) using an effective description without considering any individual environmental degree of freedom (see, e.g., sections 3.3 - 3.5).

Method (a) was introduced by Bateman applying the Hamiltonian

\[ H_B = \frac{p_x p_y}{m} + \frac{\gamma}{2} (y p_y - x p_x) + m \left( \omega^2 - \frac{\gamma^2}{4} \right) x y. \tag{32} \]

The system with variables \(x\) and \(p_x\) obeys the equation of motion

\[ \ddot{x} + \gamma \dot{x} + \omega^2 x = 0 \tag{33} \]

with the dissipative damping force, whereas the environment is represented by one other system with variables \(y\) and \(p_y\) obeying

\[ \ddot{y} - \gamma \dot{y} + \omega^2 y = 0 \tag{34} \]

with an accelerating force of the same magnitude as the friction force, but with a different sign. The energy lost by the first system is gained by the second and the Hamiltonian \(H_B\) remains
constant. There are attempts in the literature to link this approach to the one by Caldirola–Kanai [35] and a recent achievement was made to connect it directly to the approach using the expanding coordinates [36].

The usual system-plus-reservoir approach, however, takes into account explicitly the interaction between the system and individual environmental degrees of freedom, representing them for instance by a large number of HOs. An ansatz of this kind, e.g., the one by Caldeira and Leggett, can be described by the Hamiltonian \( H_{\text{Tot}} = H_S + H_{SR} + H_R + \) correction term where \( H_S \) and \( H_R \) depend only on the variables of system or reservoir, respectively, whereas \( H_{SR} \) depends on both types of variables. The correction term was added by hand to compensate for a negative shift of \( V(x) \). By averaging over the environmental degrees of freedom, letting the number of environmental degrees go to infinity etc., one finally obtains an equation of motion for the system of interest that contains a dissipative friction force. It has been shown [37] [38] that the Caldirola–Kanai Hamiltonian can be derived from the Caldeira–Leggett ansatz!

However, there is also a problematic aspect of this approach. Hamiltonian \( H_{\text{Tot}} \) is usually applied to derive the corresponding master equation for the density operator \( \rho_{\text{op}} \). But the dissipative term in the equation for the system can lead to violation of the positivity of \( \rho_{\text{op}} \). This problem can be avoided if the dissipative term has a particular formal structure. This requirement is fulfilled by the Kossakowski–Lindblad generators [19] [20] but they provide only a formal mathematical requirement that must be satisfied but give no physical guideline for the explicit choice of the operators in the dissipative term.

A general problem with the approaches using master equations for the density operator is that the computational effort scales at least with \( N^2 \) if \( N \) is the number of relevant degrees of freedom. Therefore, it would be more desirable to have a description in terms of pure states since, in principle, a linear scaling would be possible. In the following, approaches will be presented that assume a quantum system can always be described by a pure state wave function whether it interacts with some kind of environment or not. In these approaches the individual degrees of the environment are not explicitly taken into account only the effect of the environment on the observable properties of the system.

3.3. Non-unitary time-evolution

In his approach Gisin [21] [22] assumed that the time-evolution of an open quantum system should be characterized by a direction of time and therefore it should be non-unitary. This can be reached by adding an imaginary term, e.g., \( iB \) with \( B = B^* \) to the SE; but this in general destroys the normalizability of the wave function. A way of eliminating this shortcoming is to subtract the mean value of this imaginary term, thus regaining normalizability, but for the price of introducing nonlinearity via the mean value. Gisin used the ansatz

\[
\frac{i\hbar}{\partial t} \Psi(t) = \{H_L - i(B - <B>)\} \Psi(t),
\]

which is due to the appearance of \( <B> \) a NLSE. In order to also describe dissipation, Gisin made the assumption \( B = kH_L \); but this provides a degree of energy dissipation that contradicts the known contribution from the classical part, i.e., \( E \neq -2\gamma T \) with \( T \) = kinetic energy.

In a similar approach, Beretta [23] [24] tried to describe non-equilibrium systems (without dissipation) where the imaginary term was supposed to be related to the entropy of the system that should increase during time-evolution. The particular form of his NLSE was

\[
\frac{i\hbar}{\partial t} \Psi = \{H_L + \gamma \frac{\hbar}{2i} (\ln \rho - <\ln \rho>)\} \Psi,
\]
a form that will appear again in another approach in section 3.5.
3.4. Nonlinear “friction potentials”

Whereas the two afore-mentioned approaches were essentially motivated by the irreversibility of the time-evolution of an open quantum system, the ones that will be presented below focus on the dissipative aspect. As mentioned earlier, the equation of motion including a dissipative friction force cannot be obtained by simply adding a term to the classical Lagrangian or Hamiltonian function. The idea of the following approaches was, however, to start already on the quantum mechanical level and add a term that would provide, according to Ehrenfest’s theorem, the correct velocity- (or momentum)-dependent friction force in the equation of motion for the mean-values. A problem arises in this context in that a velocity or momentum defined via 
\[ p = \frac{\hbar}{i} \frac{\partial}{\partial x} \psi \]

is in general a complex quantity due to the fact that \( \psi(x,t) \) is generally complex. For the mean values, this does not really matter since the (imaginary) contribution from the amplitude of the wave function, \( v_+ = \frac{\hbar}{2m} \nabla \ln \psi \), always vanishes on an average, \( \langle v_+ \rangle = 0 \), whereas the (real) contribution from the phase, \( v_- = \frac{\hbar}{2m} \nabla \ln \psi^* \), (on an average) provides the classical velocity, \( \langle v_- \rangle = \dot{\eta} \), (and this is actually the definition of velocity used in the continuity equation for the probability density \( \rho = \psi^* \psi \)).

So the question arises: which definition of velocity is relevant for the construction of a “friction potential” that should generally provide a force proportional to velocity? Is it \( v_+ \), \( v_- \) or \( v_+ + v_- \)?

In his approach, Kostin [25] used the definition of \( v_- \) that provides the classical velocity when averaged, leading to the friction term

\[ W_{\text{Kos}} = \frac{\gamma}{2} \left( \ln \left( \frac{\Psi_{NL}}{\Psi_{NL}^*} \right) - \ln \left( \frac{\Psi_{NL}^*}{\Psi_{NL}} \right) \right) \]  

where the subtraction of the mean-value guarantees that the mean-value of the NL Hamiltonian \( H_{NL} = H_L + W_{\text{Kos}} \) is identical to the mean-value of the kinetic and potential energies, i.e., \( \langle H_{NL} \rangle = \langle H_L \rangle = \langle T \rangle + \langle V \rangle \).

However, this approach suffers certain shortcomings:

a) if \( \psi \) is real, as in the case of the eigenfunctions of the HO, \( \psi = \psi^* \) and \( W_{\text{Kos}} \) vanishes, meaning that the solutions of the undamped HO are also solutions of the frictionally-damped problem;

b) the frequency occurring for the damped HO is not the correct reduced frequency \( \Omega = (\omega^2 - \gamma^2/4)^{1/2} \) known from the classical counterpart;

c) since \( W_{\text{Kos}} \) is real, it does not contribute to the equation of motion for the probability density. Therefore this remains a reversible continuity equation, whereas the averaged equation of motion and the NLSE are irreversible evolution equations; a fact that needs explanation.

An attempt at solving these problems was made by Cho [19] who tried to take into account an additional “diffusion velocity” which is similar to the abovementioned imaginary contribution to the velocity definition in quantum mechanics, \( v_{\text{diff}} = -\frac{\hbar}{2m} \nabla \rho \). However \( v_{\text{diff}} \) is real and, although getting rid of the first of the three shortcomings, the other two, the wrong frequency for the damped HO and the reversible continuity equation for \( \rho(x,t) \), remain.

A more pragmatic approach was taken in constructing a most general “friction potential” \( W_{\text{Gen}} \) as a product of position and momentum and their mean values, requiring that

\[ \langle \frac{\partial}{\partial x} W \rangle = \gamma \langle p \rangle \]  

is always fulfilled [29]. The corresponding “friction potential” can be given as

\[ W_{\text{Gen}} = \gamma \langle p \rangle (x - \langle x \rangle) + \frac{\gamma}{2} C [(x - \langle x \rangle), (p - \langle p \rangle)]_+ \]  

(with \( [ , ]_+ = \text{anticommutator} \)), that fulfils the requirement for any value of \( C \).
There are several particular choices for $C$ in the literature:

a) $C = 1$: Süssmann [28] used this ansatz to obtain

$$W_{\text{Sü}} = \frac{\gamma}{2} [(x - < x >), p]_+$$

which, for complex $\Psi$ due to the operator $p = \frac{\hbar}{i} \frac{\partial}{\partial x}$, is generally a complex quantity. This, however, still provides the wrong frequency for the damped HO. More concerning the contribution of $W_{\text{Sü}}$ to the continuity equation can be seen in the next section.

b) $C = 0$: Albrecht’s choice [27] leads to

$$W_{\text{Alb}} = \gamma < p > (x - < x >), \quad (41)$$

which is a real quantity, therefore still leading to a reversible continuity equation for $\psi$ and also providing an undamped frequency for the damped HO.

c) $C = \pm 1/2$: The correct frequency $\Omega = (\omega^2 - \gamma^2/4)^{1/2}$ for the damped HO can be obtained for $C = \pm 1/2$, a choice taken by Hasse [29] where $C = -1/2$ leads to unphysical results (wrong sign in the exponent of the Gaussian WP). So, for $C = +1/2$, Hasse obtained

$$W_{\text{Has}} = \frac{\gamma}{2} [(x - < x >), (p+ < p >)]_+$$

which is a complex quantity, thus also contributing an (irreversible) term to the equation for $\psi(x,t)$. Only the mean value of $W_{\text{Has}}$ does not vanish, $< W_{\text{Has}} > \neq 0$, so that $< H_{NL} > \neq < T > + < V >$.

Comparing Hasse’s approach with the two afore-mentioned ones, one finds

$$W_{\text{Has}} = \frac{1}{2} (W_{\text{Sü}} + W_{\text{Alb}}) \quad (43)$$

3.5. Broken time-symmetry for the probability density

Approaches via the averaged equation of motion and using condition (38) contain a lot of arbitrariness. Equation (38) only makes statements about the derivative of the “friction potential” (and even worse, only about the mean value of this quantity) and compares it with the mean-value of velocity (which is itself ambiguous, as shown above). We therefore tried (see [30] [31] [32]) an approach that is not based on the dissipative aspect of the trajectory picture but on the irreversible aspect of the probability picture. For this purpose, we add a time-reversal symmetry-breaking diffusion term to the continuity equation for $\psi$, thus turning it, as previously discussed, into an irreversible Smoluchowski equation,

$$\frac{\partial}{\partial t} \psi + \frac{\partial}{\partial x} (\psi v) - D \frac{\partial^2}{\partial x^2} \psi = 0 \quad (44)$$

but now for the quantum mechanical $\psi(x,t)$ and with the $v_-$ mentioned earlier. For the continuity equation, Madelung [39] and Mrowka [40] had shown that with a bilinear ansatz for the probability current density, $j = \rho \left( \frac{1}{m} \frac{\partial}{\partial x} S \right) = \psi \frac{\partial}{\partial x} \psi^* - \frac{\hbar}{2mi} \left( \Psi^* \frac{\partial}{\partial x} \Psi - \Psi \frac{\partial}{\partial x} \Psi^* \right)$, this can be separated into the TDSE and its complex conjugate where the separation “constant” is proportional to the potential $V$. However, due to the diffusion term in (44), $\Psi$- and $\Psi^*$-dependent terms can no longer be separated because of the last term on the rhs of

$$-D \frac{\partial^2 \psi}{\partial x^2} = -D \left\{ \Psi^* \frac{\partial^2}{\partial x^2} \Psi + \Psi \frac{\partial^2}{\partial x^2} \Psi^* + 2 \left( \frac{\partial}{\partial x} \Psi \right) \left( \frac{\partial}{\partial x} \Psi^* \right) \right\} \quad (45)$$
For separation one needs that the condition

$$-D \frac{\partial^2 \rho}{\partial \tau^2} = F_1(\Psi) + F_2(\Psi^*)$$

has to be fulfilled.

This can be reached by introducing the additional separation condition

$$-D \frac{\partial^2 \rho}{\partial \tau^2} = \gamma (\ln \rho - <\ln \rho>) ,$$

where $\ln \rho = \ln \Psi + \ln \Psi^*$. This leads to an additional complex logarithmic term [23] in the SE for $\Psi_{NL}(x,t)$,

$$i\hbar \frac{\partial}{\partial t} \Psi_{NL}(x,t) = \{H_L + \frac{\gamma}{i} (\ln \Psi_{NL} - <\ln \Psi_{NL}>)\} \Psi_{NL}(x,t) = \{H_L + W_{SCH}\} \Psi_{NL}(x,t) ,$$

where $H_L$ is the usual linear Hamiltonian. The additional NL term $W_{SCH}$ can be written as real and imaginary contributions in the form

$$W_{SCH} = W_R + iW_I = \frac{\gamma \hbar}{2i} \left( \ln \frac{\Psi_{NL}}{\Psi_{NL}^*} - <\ln \frac{\Psi_{NL}}{\Psi_{NL}^*}> \right) + \frac{\gamma \hbar}{2i} (\ln \rho_{NL} - <\ln \rho_{NL}>)$$

where the real part only depends on the phase of the wave function and provides the friction force in the averaged equation of motion. The imaginary part does not contribute to dissipation but introduces irreversibility into the evolution of the wave function. It corresponds to the diffusion term in the Smoluchowski equation, but still allows for normalizability due to the subtraction of the mean value of $\ln \rho$. Comparison with the afore-mentioned approaches shows that the real part is just identical to Kostin’s term and the imaginary part corresponds to Beretta’s term (for pure states).

The imaginary part breaks the time-reversal symmetry on the level of the probability density, introduces a non-unitary time evolution and turns the Hamiltonian into a non-Hermitian one while still guaranteeing normalizable wave functions and real energy mean values since its mean value vanishes.

Also from the real part of $W_{SCH}$ no additional term to the energy mean value occurs, so this is still given by the mean value of the operators of kinetic and potential energies. This real part is however not arbitrary but is uniquely determined by the separation condition and provides the correct dissipative friction forces in the equation of motion for the mean values. Besides, the ratio of energy dissipation (for the classical contribution) is in agreement with the classical counterpart and arises because the mean values are calculated with $\Psi_{NL}$ instead of $\Psi_L$ (the solution of Eq. (5)).

The real part, by itself, would provide dissipation but retain a unitary time-evolution of the wave function, whereas the imaginary part, on its own, would provide irreversibility via a non-unitary time-evolution but no dissipation. Consequently only the combination of real and imaginary parts provides all the desired properties of the quantum system under consideration. The reason for this is the coupling of phase and amplitude of the wave function since $W_R$ depends on the phase and $W_I$ on the amplitude.

We should mention another approach by Doebner and Goldin based on the same Smoluchowski equation for $\rho(x,t)$ that confirms the additional diffusion term on group-theoretical grounds [24]. The authors do not try to separate this term but add half of it to
the SE and the other half to its complex conjugate, leading to a NLSE where an arbitrary real, possibly NL, term \( R[\Psi, \Psi^*] \) can be added, i.e.,

\[
i\hbar \frac{\partial}{\partial t} \Psi = \{ H_L + i \frac{\hbar}{2} D \frac{\partial^2 \varrho}{\partial^2 \varrho} + R[\Psi, \Psi^*] \} \Psi .
\]  

(50)

Without \( R[\Psi, \Psi^*] \), the additional term leads to an irreversible non-unitary time-evolution, but not necessarily to dissipation; this depends essentially on \( R \). For Gaussian WPs, the diffusion term is identical to the \( \ln \varrho \)-term of \( W_{SCH} \). These authors also discuss the possibility of linearizing NLSEs via NL gauge transformations [25], what has been compared with an alternative method for linearization (see section 5) using non-unitary transformations [26].

In the following, the NLSE (48) with complex logarithmic nonlinearity will be used to show the influence of a dissipative environment on the TDSE and the corresponding Riccati and Ermakov equations. This will allow one to draw conclusions for corresponding modifications in the time-independent case which will show closer connections between the approaches presented in sections 3.4 and 3.5.

4. Modifications of the complex Riccati and nonlinear Ermakov equations

4.1. Modifications due to dissipation and irreversibility in the time-dependent case

Inserting WP (14) into the NLSE (48) yields, for \( \eta(t) = <x>_N \), and hence for the motion of the WP maximum, the Newtonian equation including the friction force,

\[
\ddot{\eta} + \gamma \dot{\eta} + \omega^2 \eta = 0
\]

(51)

and for the complex variable \( (2\hbar m y)_N \) the modified Ricatti equation

\[
\left( \frac{2\hbar m y}{m} \right)_N + \gamma \left( \frac{2\hbar m y}{m} \right)_N + \left( \frac{2\hbar m y}{m} \right)_N^2 + \omega^2(t) = 0
\]

(52)

with an additional linear term depending on the friction coefficient \( \gamma \).

The relation between the imaginary part of this variable and the position uncertainty or Ermakov variable, respectively, remains unchanged as in the conservative case,

\[
\left( \frac{2\hbar m y_I}{m} \right)_N = \frac{\hbar}{2m <\dot{x}^2>_N} = \frac{1}{\alpha^2_N},
\]

(53)

but the real part differs by a contribution from the friction coefficient, i.e.,

\[
\left( \frac{2\hbar m y_R}{m} \right)_N = \frac{\dot{\alpha}_N}{\alpha_N} - \gamma_2.
\]

(54)

Inserting (53) and (54) into the Riccati equation (52) turns this into the Ermakov equation

\[
\ddot{\alpha}_N + \left( \omega^2 - \frac{\gamma^2}{4} \right) \alpha_N = \frac{1}{\alpha^2_N},
\]

(55)

i.e., in comparison with the conservative case, only \( \omega^2 \) has been replaced by \( \Omega^2 = \left( \omega^2 - \frac{\gamma^2}{4} \right) \). Again, from Eqs. (51) and (55) a dynamical invariant can be obtained that now has the form [44] [45] [46]

\[
I_N = \frac{1}{2} e^{\gamma t} \left[ \dot{\eta} \alpha_N - \left( \dot{\alpha}_N - \frac{\gamma}{2} \alpha_N \right) \eta^2 + \left( \frac{\eta}{\alpha_N} \right)^2 \right].
\]

(56)
Riccati equation (52) can also be linearized, now using the ansatz \( \frac{2h_{\text{m}}}{m} y \) to yield the Newtonian equation with linear friction term

\[
\ddot{\lambda} + \gamma \dot{\lambda} + \omega^2 \lambda = 0 ,
\]

for the complex variable \( \dot{\lambda}(t) = \lambda e^{-\gamma t/2} \). Inserting the polar form into Eq. (52) leads to the unchanged conservation law

\[
\dot{\phi} = \frac{1}{\alpha_{NL}}
\]

as in the conservative case and the modified Ermakov equation (55).

Now we compare Eqs. (51), (52) and (57) with the corresponding Eqs. (15), (16) and (21) in the conservative case. In the linear second-order differential equations for \( \eta \) and \( \lambda \), a linear term with first derivative has been added while, in the first-order Riccati equation, an additional term linear in \( \frac{2h_{\text{m}}}{m} y \) appears. All the additional terms depend on the coefficient of the friction force. But now the question arises: is it possible to modify the corresponding equations in the time-independent case (see Fig. 1) in a similar way to include the effect of a dissipative environment into the TISE? The answer will be given in the next subsection.

4.2. Modifications due to dissipation and irreversibility in the space-dependent case

The space-dependent Riccati equation corresponding to Eq. (16) was Eq. (12) that could be linearized to yield the TISE (13). The modifications from Eqs. (16) and (21) to (52) and (57) in the time-dependent case would mean, in the space-dependent situation, the addition of a term \( \Gamma \left( \nabla \Psi \right) \) to Eq. (12) or a term \( \Gamma \nabla \Psi \) to Eq. (13), respectively, where the coefficient \( \Gamma \) should somehow be related to the friction force but, in general, could be a complex function of \( \mathbf{r} \) and \( t \), \( \Gamma(\mathbf{r},t) \).

Since \( \Psi \) is complex, a contribution from the imaginary part of the additional term \( \Gamma \nabla \Psi \) to the continuity equation for \( \varrho = \Psi^* \Psi \) should arise. What should be the specific form of this term? To answer this question, the following assumptions shall be made:

1. The SE (5) can be obtained from the continuity equation (7) (according to Madelung [39] and Mrowka [40]) via separation. The log NLSE (48) can be obtained from the Smoluchowski equation (44) with additional diffusion term (and condition (47), as shown in section 3.5) also via separation. Therefore, the additional term \( \Gamma \nabla \Psi \) in the modified TISE should also originate from an additional term in the continuity equation plus separation. That is, a term

\[
\Gamma \nabla \Psi = \Gamma (\Psi^* \nabla \Psi + \Psi \nabla \Psi^*)
\]

should occur in the evolution equation for \( \varrho \). The contribution from this term would enter the SE with an imaginary coefficient.

2. The additional term in the equation for \( \varrho \), at least for cases with Gaussian WP solutions (i.e., \( V = 0 \) and HO) discussed in this paper, should have the same effect as the diffusion term or the \( \ln \varrho \)-term, i.e.,

\[
-D \Delta_x \varrho = \gamma (\ln \varrho - < \ln \varrho >) \varrho = \frac{\gamma}{2} \left( 1 - \frac{x^2}{< x^2 >} \right) \varrho
\]

Comparison shows that

\[
\nabla_x \left( \frac{\gamma}{2} \varrho \right) = \frac{\gamma}{2} \left( 1 - \frac{x^2}{< x^2 >} \right) \varrho
\]
yields the desired result! (In our one-dimensional case $\nabla x$ means $\frac{\partial}{\partial x}$ and the subscript $x$ will be omitted in the following.)

After separation this contributes an additional term $W_{TI} = \frac{\gamma}{2} \left( \frac{1}{2} + \tilde{x} \frac{\nabla \Psi}{\Psi} \right)$ in the SE that already looks familiar when rewritten as

$$W_{TI} \Psi = \left( \frac{\gamma}{4} \frac{\hbar}{i} + \frac{\gamma}{2} \frac{\hbar}{i} \nabla \right) \Psi = \frac{\gamma}{4} \left[ \tilde{x}, p_{op} \right] \Psi = \frac{1}{2} W_{S\text{"us}} \Psi . \quad (62)$$

However:

a) $< W_{TI} > = \frac{\gamma}{4} < \tilde{x} > < p > = < W_{\text{Has}} > \neq 0$, but this could be compensated for by a normalization factor; and

b) $< -\nabla W_{TI} > = - \frac{\gamma}{2} < p >$, i.e., half of the friction force is missing.

Considering for instance the TISE for the HO with real wave functions, this would not actually cause a problem since, in this case, $< p > = 0$. However, if one would also take into account cases where the phase of the wave function matters, a further real contribution $\frac{\gamma}{2} \tilde{x} < p > = \frac{1}{2} W_{\text{Alb}}$ would be needed, in addition to $W_{TI}$. Where could the missing terms originate from?

To answer this question, we need to revisit Madelung’s hydrodynamic formulation. From the continuity equation (7), rewritten in the notation used in section 2.1., i.e., $\rho = a^2$,

$$\frac{\partial}{\partial t} a^2 + \frac{1}{m} \nabla \left( a^2 \nabla S \right) = 0 \quad (63)$$

in the TI case with $\frac{\partial}{\partial t} a^2 = 0$, the conservation law $\nabla S = C/a^2$, was derived. Taking into account the additional term (61), this is obviously no longer valid. However, introducing a modified action function $S'$ via

$$\frac{\partial}{\partial t} a^2 + \frac{1}{m} \nabla \left( a^2 \nabla S + m \frac{\gamma}{2} \tilde{x} \right) =$$

$$\frac{\partial}{\partial t} a^2 + \frac{1}{m} \nabla \left( a^2 \nabla S' \right) = 0 \quad (64)$$

with

$$S' = S + m \frac{\gamma}{4} \tilde{x}^2 + f(t) \quad (65)$$

allows us to find, for $\frac{\partial}{\partial t} a^2 = 0$, again a conservation law of the form

$$\nabla S' = \frac{C}{a^2} . \quad (66)$$

Separation of Eq. (64) introduces the term $W_{TI}$ into the modified TISE, what leads to an additional contribution of the phase in the Hamilton–Jacobi-type equation (6),

$$\frac{\partial}{\partial t} S + \frac{1}{2m} \left( \nabla S \right)^2 + V - \frac{\hbar^2}{2m} \frac{\Delta a}{a} + \frac{\gamma}{2} \tilde{x} (\nabla S) = 0 , \quad (67)$$

i.e., the last term on the lhs.

To obtain a form that is consistent with the modified action in Eq. (64), $S$ will be replaced by $S'$ according to (65). Specifically, the time-derivative of the action supplies the additional terms,

$$\frac{\partial}{\partial t} S = \frac{\partial}{\partial t} S' + \frac{\gamma}{2} < p > \tilde{x} - \frac{\partial}{\partial t} f(t) , \quad (68)$$
leading to Eq. (67) expressed in terms of $S'$ as

$$\frac{\partial}{\partial t} S' + \frac{1}{2m} (\nabla S')^2 + \left[ V - \frac{m \gamma}{2} \frac{\hat{x}^2}{4} + \frac{\gamma}{2} < p > \hat{x} - \frac{\partial}{\partial t} f \right] - \frac{h^2}{2m} \frac{\Delta a}{a} = 0 . \quad (69)$$

With $\frac{\partial}{\partial t} S' = -E'$ and $\frac{\partial}{\partial t} f = \frac{\gamma}{4} < \hat{x}, \hat{p} >$ and applying (66), Eq. (69) can be rewritten as the Ermakov equation

$$\Delta a + \frac{2m}{\hbar^2} \left[ \dot{E}_R - \dot{V} \right] a = \left( \frac{1}{\hbar} \nabla S' \right)^2 a = \left( \frac{C}{\hbar} \right)^2 \frac{1}{a^4} \quad (70)$$

with $\dot{E}_R = E' - \frac{\gamma}{4} < \hat{x}, \hat{p} > - \frac{\gamma}{4} < \hat{x}, \hat{p} >$ and $\dot{V} = V - \frac{m \gamma}{4} \hat{x}^2$.

Including the two additional terms originating from the time-dependence of $S$ expressed in terms of $S'$, the full irreversible and dissipative term $W_{TI}$ now reads

$$\hat{W}_{TI} = \frac{\gamma}{2} [\hat{x}, \hat{p}]_+ + \frac{\gamma}{2} < p > \hat{x} - \frac{\gamma}{4} < [\hat{x}, \hat{p}]_+ > = \hat{W}_{Has} - < \hat{W}_{Has} > = \hat{W}_{SCH} , \quad (71)$$

where the first additional (real) term just provides the missing contribution $\frac{1}{2} \hat{W}_{Alb}$ that is necessary to obtain the correct friction force in the equation of motion for the mean values and the second term is equal to the mean-value of Hasse’s friction term and must be subtracted so that $< \hat{W}_{TI} > = 0$. For the Gaussian WPs $\hat{W}_{TI} \Psi_{WP} = \hat{W}_{SCH} \Psi_{WP}$ is valid. The modified TISE including these two additional terms can be rewritten as

$$E' \Psi = \left\{ - \frac{\hbar^2}{2m} \Delta_x + \frac{\gamma}{2} \frac{\hbar}{i} \nabla_x + \left( V + \frac{\gamma}{4} < p > - \frac{\gamma}{4} < [\hat{x}, \hat{p}]_+ > + i \hbar \frac{\gamma}{4} \right) \right\} \Psi \quad (72)$$

or in the form of the corresponding modified complex Riccati equation

$$\nabla \left( \frac{\nabla \Psi}{\Psi} \right) + i \frac{\gamma}{\hbar} \left( \frac{\nabla \Psi}{\Psi} \right) + \left( \frac{\nabla \Psi}{\Psi} \right)^2 + \frac{2m}{\hbar^2} (\hat{E} - V) = 0 \quad (73)$$

with the complex quantity

$$\hat{E} = E' - \frac{\gamma}{2} < p > \hat{x} + \frac{\gamma}{4} < [\hat{x}, \hat{p}]_+ > - i \hbar \frac{\gamma}{4} . \quad (74)$$

In figure 2, the time-dependent and space-dependent cases are again compared but now including the environmental effect. Whereas in the time-dependent situation the coefficient of the additional linear term in the Riccati equation and the first-derivative term in its linearized form do not explicitly depend on time $t$ (only an implicit time-dependence via $\gamma = \gamma(t)$ is possible), in the space-dependent case the coefficient depends linearly on the spatial variable and is, in contrast with the TD case, purely imaginary. In the TD case, the conservation law remains unchanged while in the space-dependent case the phase is changed in a way that corresponds to a unitary transformation of the wave function. On the other hand, the amplitude of the complex variable in the space-dependent situation remains unchanged whereas in the TD case, the amplitude is multiplied with an exponential damping factor. Nevertheless, the resulting Ermakov equations have the same structure in both cases. So, it seems that a change of the phase, corresponding to a unitary transformation, can have the same effect as a change of the amplitude of a complex variable which would correspond to a non-unitary transformation. The role of non-unitary transformations in the context of our dissipative quantum systems will be discussed in the next section.
Figure 2: Comparison between the space-dependent and the time-dependent Riccati equations, their linearized forms and the corresponding Ermakov equations for the dissipative case.

5. Connections between nonlinear and linear dissipative approaches via non-unitary transformations

The connection between the explicitly TD approach of Caldirola–Kanai and the logarithmic NLSE (48) can be established by referring to Schrödinger’s first communication on wave mechanics [1] which is based on the Hamilton–Jacobi form of classical mechanics. Writing the Hamilton–Jacobi equation in terms of $S_c$ as

$$\frac{\partial}{\partial t} S_c + H \left( x, \frac{\partial}{\partial x} S_c, t \right) = 0$$

(with the momentum $p_c = \frac{\partial}{\partial x} S_c$) Schrödinger arrived at the Hamiltonian operator $H_L = -\frac{\hbar^2}{2m} \frac{\partial^2}{\partial x^2} + V(x)$ via a variational ansatz after he introduced the wave function $\Psi(x,t)$ via $S_c = \frac{\hbar}{i} \ln \Psi$, where the subscript c (added by us) indicates that this action is a complex quantity, since $\Psi$ is generally a complex function (a fact that Schrödinger did not like at all in the beginning [47]).

Reversing Schrödinger’s procedure, but starting now with Eq. (48) (divided by $\Psi$, which causes no problems for Gaussian WPs) and, using the definition of $S_c$, one arrives at

$$\left( \frac{\partial}{\partial t} + \gamma \right) S_c + H = -\gamma < S_c > .$$

This is, of course, as little rigorous as Schrödinger’s first attempt was, however it follows his idea of connecting the classical Hamilton–Jacobi theory with a wave (mechanical) equation. The purely TD term $-\gamma < S_c >$ is necessary mainly for normalization purposes (and can therefore be absorbed by the normalization coefficient) and shall be neglected in the following.

Multiplying the remaining Eq. (76) by $e^{\gamma t}$ and using the definitions

$$\dot{S}_c = e^{\gamma t} S_c \quad \text{and} \quad \dot{H} = e^{\gamma t} H$$

Figure 2: Comparison between the space-dependent and the time-dependent Riccati equations, their linearized forms and the corresponding Ermakov equations for the dissipative case.
it can be rewritten as a Hamilton–Jacobi equation

$$\frac{\partial}{\partial t} \hat{S}_c + \hat{H} = 0 .$$  \hfill (78)

From the definition of the action function, it follows that the wave function $\hat{\Psi}(x, t)$ in the transformed (canonical) system is connected with the wave function $\Psi(x, t)$ in the physical system via the non-unitary relation

$$\ln \hat{\Psi} = e^{\gamma t} \ln \Psi .$$  \hfill (79)

Consequently, the (complex) momenta in the two systems are connected via

$$\hat{p}_c = \frac{\hbar}{i} \frac{\partial}{\partial x} \ln \hat{\Psi} = e^{\gamma t} \frac{\hbar}{i} \frac{\partial}{\partial x} \ln \Psi_{NL} = e^{\gamma t} p_c ,$$  \hfill (80)

which is equivalent to the connection between the canonical and the kinetic momentum in the Caldirola–Kanai approach. The non–canonical connection between the classical variables $(x, p)$ and $(\hat{x}, \hat{p})$ (see section 3.1) corresponds to the non–unitary transformation between $\Psi$ and $\hat{\Psi}$.

Note: Although $\Psi$ and $\hat{\Psi}$ both depend explicitly on the same variables, $x$ and $t$, the two wave functions are analytically different functions of $x$ and $t$ and have different physical meanings due to the non-unitary transformation (79). Ignoring this fact leads to unphysical results like violation of the uncertainty principle (for further details, see [48]).

Expressing $\hat{H}$ in terms of the canonical momentum $\hat{p}_c$ and following Schrödinger’s quantization procedure, finally yields the modified SE (29) of the Caldirola–Kanai approach.

In section 3.1 it was already mentioned that a canonical description of the damped HO is also possible in terms of an exponentially-expanding system where the transformation (30) between the canonical and the physical variables is also a non-canonical transformation. Applying this canonical formalism, the equations of motion, after back-transformation into the physical variables, also provide the equation of motion (26) including the friction force. Whereas this approach and the one by Caldirola–Kanai are both connected with the physical level via a non-canonical transformation, the connection between both of them on the canonical level is given by the canonical transformation

$$Q = e^{\gamma t/2} \hat{x} = e^{\gamma t/2} x ,$$

$$P = e^{-\gamma t/2} \hat{p} + m \frac{\gamma}{2} e^{\gamma t/2} \hat{x} .$$  \hfill (81)

Since $\hat{H}_{CK}$ is explicitly TD, while $\hat{H}_{exp}$ is a constant of motion, the transformation between them must involve an explicitly TD generating function $\hat{F}_2$, i.e.,

$$\hat{H}_{exp} = \hat{H}_{CK} + \frac{\partial}{\partial t} \hat{F}_2$$  \hfill (82)

with

$$\hat{F}_2(\hat{x}, P, t) = \hat{x} P e^{\gamma t/2} - m \frac{\gamma}{4} \hat{x}^2 e^{\gamma t} .$$  \hfill (83)

Consequently, also the action functions in both approaches are different and related by

$$\hat{S}_{exp} = \hat{S}_{CK} + m \frac{\gamma}{4} \hat{x}^2 e^{\gamma t}$$  \hfill (84)

or with $\hat{S}_{CK} = S e^{\gamma t}$ and $\hat{x} = x$

$$\hat{S}_{exp} = \left( S + m \frac{\gamma}{4} x^2 \right) e^{\gamma t} = S' e^{\gamma t} .$$  \hfill (85)
The change from $S$ to $S'$ is the same that occurred in section 4.2 when the modified conservation law (66), $\nabla S' = C/a^2$, was obtained after the irreversible term (61) had been added to the continuity equation.

With Schrödinger’s definition $S = \hbar \ln \Psi$, the canonical transformation corresponds to a phase-factor $\frac{\hbar}{\pi} x^2$ in the wave function. So, the transition between the NL physical level and the formal (linear) canonical level is given classically by a non-canonical transformation (corresponding to the non-unitary transformation (79) of the wave function). Whilst, on the canonical level, the transition between the two approaches discussed is possible via a canonical transformation (corresponding to a unitary transformation of the wave functions involving the above-mentioned phase factor). These relations are shown again schematically in fig. 3.

Finally a short comparison will be made with the canonical approach of Bateman that was introduced at the beginning of section 3.2. Earlier attempts [25] were made to link this approach to the one of Caldirola–Kanai by imposing some constraints. A recent attempt [26] takes advantage of the fact that the Bateman Hamiltonian $H_B$, Eq. (32), as well as the Hamiltonian for the expanding system $\hat{H}_{\text{exp}}$, Eq. (31), represent the constant initial energy of system plus reservoir. Therefore, it is possible to transform the Bateman Hamiltonian exactly into $\hat{H}_{\text{exp}}$ by expressing the variables of $H_B$ in terms of the physical variables $x$ and $\dot{x}$. Since the (canonical) transformation between $\hat{H}_{\text{exp}}$ and $\hat{H}_{\text{CK}}$ is known (see above), the exact link between $H_B$ and $\hat{H}_{\text{CK}}$ can also be obtained (further details will be published elsewhere).

**Figure 3:** Connections between the physical and canonical descriptions of dissipative systems.

### 6. Conclusions

A fundamental difference between classical mechanics and quantum mechanics is the occurrence of complex quantities in quantum mechanics, not just for computational convenience but as a conceptual element of the foundations of physics. As a first example, Madelung’s polar form of the complex wave function was used where the phase reflects the trajectory aspect of the system, whereas the amplitude takes into account the probabilistic aspect. Utilizing a NL formulation of the TISE, in terms of a Ermakov equation, it has been shown that phase and amplitude are not independent of each other but coupled via some kind of conservation law. The real NL Ermakov equation is equivalent to a complex NL Riccati equation where the quadratic term of this equation shows the origin of the afore-mentioned coupling. Formally, the same situation was found in the TDSE for cases where exact Gaussian WP solutions exist. A comparison of
the respective quantities can be found in figure 4.

Several approaches were presented to effectively include the phenomena of irreversibility and dissipation into the systems mentioned above and their interrelations discussed. Specifically, a NL modification of the TDSE using a complex logarithmic nonlinearity was studied in detail and the changes in comparison with the corresponding conservative systems were analyzed. The modifications of the complex Riccati equation and its linearized form were of particular interest because, from there, the corresponding modifications for the TISE were concluded. This notably led to an additional first-derivate term with imaginary coefficient and a modified definition of the action in the phase of the wave function in order to obtain a similar conservation law and phase-amplitude coupling as in the conservative case (see also fig. 4).

**Figure 4:** Interrelations between the space-dependent and time-dependent cases for conservative and dissipative systems.

This modification of the TISE allowed for a closer comparison between our logarithmic NLSE and the approaches by Süssmann, Albrecht and Hasse. On a level where operators and wave functions have the same physical meanings as in conventional quantum mechanics, our approach and that of Hasse’s differ only by a phase factor. However, there are also linear approaches based on modified classical Hamiltonian functions (involving non-canonical transformations!) that attempt to describe the same dissipative systems but seem to provide, in certain cases, unphysical results. Analyzing the relation between our approach and the one of Caldirola–Kanai showed that this is mainly due to a non-unitary transformation between the log-NLSE using the conventional definitions of operators and wave functions, and the modified TDSEs using the canonical formalism but applying it to operators and wave functions that have different meanings from the conventional ones. Taking these facts into account, a consistent transition is possible between these two levels of description (see fig. 5).

Another canonical approach is related to the one of Caldirola–Kanai via a unitary transformation using an expanding coordinate system leading to a Hamiltonian that is a constant of motion. The corresponding change of the phase of the wave function is similar to the one that occurred when dissipation was introduced into the TISE ($S \rightarrow S'$). Finally, this constant Hamiltonian can be used to obtain an exact link between the NL and canonical approaches and the one by Bateman using one explicit, additional environmental system to mimic the surrounding heat bath.
Figure 5: Interrelations between different physical and canonical descriptions of dissipative systems.

In conclusion, it was possible to establish connections between our logarithmic NL approach and the other approaches discussed in this paper (see fig. 5) based mainly on the complex Riccati equations that occur in this context.

Finally, it should be noted that the same type of Ermakov equation can be formally obtained either by changing the amplitude of the complex quantity that is used to linearize the Riccati equation (see lhs of fig. 2), or by changing the phase of the complex quantity (see rhs of fig. 2). In terms of quantum mechanical wave functions, this would mean that a non-unitary transformation could be replaced by a unitary one and vice-versa. This will be investigated in further detail elsewhere.

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