There is only one time

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Abstract: We draw a picture of physical systems that allows us to recognize what is this thing called "time" by requiring consistency not only with our notion of time but also with the way time enters the fundamental laws of Physics, independently of one using a classical or a quantum description. Elements of the picture are two non-interacting and yet entangled quantum systems, one of which acting as a clock, and the other one doomed to evolve. The setting is based on the so called "Page and Wootters (PaW) mechanism" [1], and updates [2–4], with tools from Lie-Group [5] and large-N quantum approaches [6–10]. The overall scheme is quantum, but the theoretical framework allows us to take the classical limit, either of the clock only, or of the clock and the evolving system altogether; we thus derive the Schrödinger equation in the first case, and the Hamilton equations of motion in the second one. Suggestions about possible links with general relativity and gravity are also put forward.

I. INTRODUCTION

The notion of time is deeply rooted into our perception of reality, which is why, for centuries, time has entered Physics as a fundamental ingredient that is not to be questioned. Then, general relativity (GR) and quantum mechanics (QM) intervened in opposite directions: GR gave time the same status of position, while QM made time a parameter, external to the theory and not recognizable as an observable. While the introduction of "spacetime" in GR appears as an elegant intuition, fully consistent with classical physics, the fact that time cannot be treated as any other observable in QM is disturbing. As a consequence, discussions about the role of time in QM have been developed, leading to different proposals on how to overcome what seems a serious inconsistency of the theory. Reporting upon these discussions goes beyond the scope of this paper; therefore, in what follows we will only refer to the proposal that provides our starting point. This was introduced by D. N. Page and W. K. Wootters in 1983 [1] to formalize the idea that the expression "at a certain time \( t' \)" should be understood as "conditioned to a clock being in a state labeled by a certain value \( t' \)." This proposal, to which we will refer as the "Page and Wootters (PaW) mechanism", is based upon three assumptions: (i) the clock does not interact with the system to which it provides the parameter \( t \), but (ii) it is entangled with it; moreover, (iii) clock and system together are in an eigenstate of the total Hamiltonian (with eigenvalue that can be set equal to zero, for the sake of simplicity and without loss of generality). The Paw mechanism has been extensively used, and its assumptions scrutinized, in the recent literature, both from the theoretical and the experimental viewpoint [2–4, 11–21].

Most discussions about time in QM are aimed at understanding what is the status of time in the quantum description, as if there were no problem as far as one stays classical. However, if one believes that there do not exist quantum systems and classical ones, but rather that some quantum systems behave in a way that, under certain conditions, is efficiently described by the laws of classical physics, than there must be just one time. In other terms, the procedure used to identify what time is in QM must have a well defined classical limit, fully consistent with classical physics and the way time enters the classical equation of motion. In this work we construct such a procedure, and demonstrate that it consistently produces not only the Schrödinger equation for quantum systems, but also the Hamilton equations of motion (e.o.m.) for classical ones, with the parameter playing the role of time being the same in both cases. We tackle the quantum-to-classical crossover via the large-N approach based on Generalized Coherent States (GCS) from Refs. [6–10], where it is demonstrated that the theory describing a quantum system for which GCS can be constructed flows into a well defined classical theory if few specific conditions upon its GCS hold in the \( N \to \infty \) limit (\( N \) quantifies the number of microscopic quantum components, sometimes referred to as the number of degrees of freedom or dynamical variables, in the literature). By "classical limit" we will hereafter mean the large-N limit with the above conditions on GCS enforced.
II. ENTANGLED, AND YET NON-INTERACTING

We consider a composite quantum system $\Psi = C + \Gamma$, with $C$ the clock and $\Gamma$ the evolving system; we assume that $\Psi$ is isolated, with Hamiltonian $\hat{H}$, and in a pure state $\ket{\Psi}$ which is entangled w.r.t. the partition $C$ and $\Gamma$; as in Ref. [2], the double bracket indicates states in $\mathcal{H}_\Psi = \mathcal{H}_C \otimes \mathcal{H}_\Gamma$, with $\mathcal{H}_\Gamma$ the Hilbert space of $\Gamma$. Referring to the PaW mechanism, we assume that

$$\hat{H} \ket{\Psi} = 0 ,$$

and take $C$ and $\Gamma$ non-interacting, i.e.

$$\hat{H} = \hat{H}_C \otimes \mathbb{1}_\Gamma - \mathbb{1}_C \otimes \hat{H}_\Gamma,$$  \hspace{1cm} (2)

where the irrelevant minus sign in front of the term acting on $\Gamma$ is our choice for the sake of a lighter notation. In view of dealing with a parameter that must be continuous to represent time, we resort to a parametric representation (see supplementary material) of $\ket{\Psi}$ with GCS for the clock [22–24], and write

$$\ket{\Psi} = \int_{\mathcal{M}_C} d\mu(\Omega) \chi(\Omega) \ket{\Omega} \otimes \phi(\Omega) , $$

where $\ket{\Omega}$ are the GCS defined via the group-theoretical construction [25, 26] for the Lie group $G_C$ associated with the algebra $\mathfrak{g}_C$ to which the Hamiltonian $\hat{H}_C$ belongs, and $\chi(\Omega)$ can be chosen real without loss of generality. The $M$-tuples $\Omega = (\Omega_1, \Omega_2, ..., \Omega_M)$, with $\Omega_m \in \mathbb{C} \forall m$, identify points on $\mathcal{M}_C$, which is a $2M$-dimensional manifold with a simplectic structure, and $M$ related to the dimension of $\mathfrak{g}_C$. The measure $d\mu(\Omega)$ is invariant w.r.t. the elements of $G_C$ and ensures that GCS form a complete set upon $\mathcal{H}_C$, thus providing a resolution of the identity. The positive function $\chi^2(\Omega)$ is a normalized probability distribution on $\mathcal{M}_C$, and $\phi(\Omega) \in \mathcal{H}_\Omega$ is normalized, and hence describes a physical state of $\Gamma$, parametrically dependent on $\Omega$. Notice that the $\Omega$-dependence of $\ket{\phi(\Omega)}$ survives iff $\ket{\Psi}$ is entangled.

There is a certain degree of freedom in the group-theoretic construction of GCS (see for instance Tables I and II in Ref. [27]), due to the possibility of choosing different set of generators for $\mathfrak{g}_C$, i.e. different Cartan basis, and an arbitrary state $\ket{G}$ from which to start the construction, so called reference state. As for the generators of semisimple algebras, we remind that the Cartan decomposition classifies them into diagonal, $\{ \hat{D}_s \}$, and raising operators, $\{ \hat{R}_m, \hat{R}_{-m} \}$, according to $[\hat{D}_s, \hat{D}_s] = 0, [\hat{D}_s, \hat{R}_m] = d_{sm} \hat{R}_m, [\hat{R}_m, \hat{R}_{-m}] = \sum_s d_{sm} \hat{D}_s$, and $[\hat{R}_m, \hat{R}_{m'}] = c_{mm'} \hat{R}_{m+m'}$, where the coefficients $\{d_{sm}\}, \{c_{mm'}\}$ are the so called structure constants. By way of example, for the semisimple algebra $\mathfrak{su}(2)$ and $\mathfrak{su}(1, 1)$, that define the spin and pseudo-spin coherent states, respectively, it is $M = 1$, with $\hat{R}_1 = \hat{S}^-$ and $\hat{K}^-$. When spin squeezing is considered, it is $M = 2$, with $\hat{R}_2 = (\hat{S}^-)^2$. The non-semisimple algebra $\mathfrak{h}_4$ that defines the harmonic-oscillator coherent states has a similar decomposition into diagonal ($\hat{a} \hat{a}^\dagger, \hat{a}$) and creation/annihilation ($\hat{a}^\dagger, \hat{a}$) operators, leading to the same results hereafter derived for the semisimple case, as shown in the supplemental material.

We choose the Cartan basis so that $\hat{H}_C$ depends linearly on one of its diagonal operators only, say $\hat{H}_C = \varsigma \hat{D}_1 + K$, where $K$ is a real arbitrary constant and $\varsigma^2 = \pm 1$ such that $\epsilon := \varsigma \epsilon_1$ is real and positive for some $\epsilon$, which ensures $\hat{H}_C$ is hermitian. For the sake of a lighter notation, we also normalize the raising and diagonal operators so that $\varsigma^2 \sum_s d_{sm}^2 \rightarrow 2$. As for the reference $\ket{G}$, we set it as the minimal weight state, $\hat{R}_m \ket{G} = 0 \forall m$, which is easily seen to be an eigenstate of the diagonal operators, $\hat{D}_s \ket{G} = g_s \ket{G}$. In particular, hence, it is $\hat{H}_C \ket{G} = \epsilon_0 \ket{G}$, with $\epsilon_0 := \varsigma g_1 + K$, and we will hereafter take $K$ so that $\epsilon_0 = 0$.

Once the Cartan basis and the reference state are chosen, GCS are generated via

$$\ket{\Omega} = e^{\Omega \hat{R}^\dagger-\Omega^\ast \hat{R}} \ket{G} ,$$

where $\hat{R} := (\hat{R}_1, \hat{R}_2, ..., \hat{R}_M)$; notice that the index $m$ runs from $1$ to $M$ both in $\Omega_m$ and in $\hat{R}_m$, by definition. GCS as from Eq. (4) are normalized and non-orthogonal, and expectation values of operators upon them, $(\bra{\Omega} \hat{O} \ket{\Omega})$, are often dubbed symbols, indicated by $O(\Omega)$. For more technical details on this section, we refer the reader to the supplemental material.

III. A QUANTUM CLOCK FOR A QUANTUM SYSTEM

We consider the set of GCS defined by $\Omega_\ell = (0, 0, ..., 0), \ell$ chosen at will amongst those for which $\epsilon$ is real and positive. Given that $\Omega_\ell \in \mathbb{C}$, we will hereafter use

$$\lambda := \Omega_\ell = \varrho e^{-i \varphi} ,$$

with $\varrho \in [0, \infty)$ and $\varphi \in (-\infty, \infty)$. Using the BCH formulas proper to $\mathfrak{g}_C$, and the definition (4), it can be easily shown that

$$\ket{\lambda} := \ket{\Omega_\ell} = N_\varrho e^{\lambda \hat{R}_\ell} \ket{G} ,$$

with $\Lambda = \left| \text{tan}(\varsigma \varrho) \right| e^{-i \varphi}$ and $N_\varrho$ a normalization factor that does not depend on $\varphi$. Furthermore, from the Cartan commutation rule $[\hat{D}_{\delta}, \hat{R}_{\ell}] = d_{\delta \ell} \hat{R}_{\ell}$ it follows $[\hat{H}_C, e^{\Lambda \hat{R}_\ell}] = \epsilon \Lambda^\ast \hat{R}_\ell e^{\Lambda \hat{R}_\ell}$, leading to

$$\langle \lambda | \hat{H}_C | \Omega \rangle = \langle G | N_\varrho e^{\Lambda \hat{R}_\ell} \hat{H}_C | \Omega \rangle =$$

$$= i \epsilon \frac{d}{d \varphi} \langle \lambda | \Omega \rangle ,$$

Once defined the partial inner product $\langle \cdot | \cdot \rangle : \mathcal{H}_C \otimes \mathcal{H}_\Gamma \rightarrow \mathcal{H}_\Gamma$, such that $\langle \gamma | \{ \xi \} \rangle = \langle \gamma | \{ \xi \} \{ \psi \} \rangle \forall \xi \in \mathcal{H}_C$ and $\forall \gamma \in \mathcal{H}_\Gamma$, we project the constraint (1) in the form

$$\langle \lambda | \hat{H} | \Psi \rangle = 0 .$$


with \( \hat{H} \) and \(|\Psi\rangle\) as in Eq. (2) and (3), and find, by virtue of the result (7),

\[
\frac{ie}{d\varphi} |\Phi_\varphi(\varphi)\rangle = \hat{H}_\varphi |\Phi_\varphi(\varphi)\rangle ,
\]

(9)

where

\[
|\Phi_\varphi(\varphi)\rangle := \langle \lambda |\Psi\rangle = \int_{\mathcal{M}_C} d\mu(\Omega) \chi(\Omega) \langle \lambda |\Omega\rangle |\phi(\Omega)\rangle
\]

(10)

is an un-normalized element of \( \mathcal{H}_\varphi \), and we have introduced a notation that highlights the different meaning that the dependence on \( \varphi \) will have in what follows, w.r.t. that on \( \varphi \). Reminding that \( \epsilon \) is real and positive, Eq. (9) has the same form of the Shrödinger equation, with the real parameter

\[
\frac{\hbar}{\epsilon} \varphi
\]

(11)

playing the role of time, as found resorting to other parametric representations [1, 2, 20, 22]. However, Eq. (9) is not the Schrödinger equation, as \(|\Phi_\varphi(\varphi)\rangle\) is not normalized. This is most often considered an amendable fault, as from Eq. (9) it follows \( \frac{d}{d\varphi} \langle \Phi_\varphi(\varphi) |\Phi_\varphi(\varphi)\rangle = 0 \) meaning that, should \(|\Phi_\varphi(\varphi)\rangle\) have a non-vanishing and finite norm, Eq. (9) would also hold for its normalized sibling. Before considering this point, let us collect some more clues on the meaning of \( \varphi \) and \( \varphi \).

Getting back to the operator \( \hat{R}_\varphi \) introduced at the beginning of this section, one can define [28, 29] the so-called "phase-operator" \( \hat{\phi} \), via

\[
\hat{R}_\varphi = (\hat{R}_\varphi \hat{R}_\varphi^\dagger)^{1/2} e^{-i\hat{\phi}} .
\]

(12)

From the commutation rules between elements of the Cartan basis, reminding that \( \hat{H}_C = \zeta \hat{D}_1 + K \) and \( \epsilon = \zeta \delta_{1\ell} \in \mathbb{R}^7 \), it follows

\[
[\hat{H}_C, \sin \hat{\phi}] = i \epsilon \cos \hat{\phi} ,
\]

(13)

and hence (see for instance Ref. [30])

\[
\Delta \hat{H}_C \Delta \sin \hat{\phi} \geq \left| \frac{\epsilon}{2} \cos \hat{\phi} \right| ,
\]

(14)

with \( \Delta \hat{B} := (\langle \hat{B}^2 \rangle - \langle \hat{B} \rangle^2)^{1/2} \) for any hermitian operator \( \hat{B} \). Noticing that Eqs. (1)-(2) imply a relation between \( \hat{H}_C \) and the energy of the system, while Eqs. (5) and (12) relate \( \phi \) with \( \varphi \), one might say that the inequality (14) is the ancestor of the time-energy uncertainty relation for \( \Gamma \), after setting \( \varphi \ll 1 \) and the parameter (11) precisely as time, a statement that is made clear in the next section.

Summarizing, we have so far collected results that point to \( \hbar \varphi / \epsilon \) as "the time" for the evolving system, but the overall picture is not that provided by QM, where the quantum character of the clock is totally absent; this is the reason why we take our next step.

### IV. A CLASSICAL CLOCK FOR A QUANTUM SYSTEM

We now assume that the quantum theory describing \( C \) satisfies the conditions ensuring it flows into a well defined classical theory when the clock becomes macroscopic, according to the large-\( N \) quantum approach based on GCS, as briefly described in the Introduction. In particular, we use that GCS are the only quantum states that survive the quantum-to-classical crossover, insofar doing becoming orthogonal

\[
\lim_{N \to \infty} |\Omega| |\Omega'\rangle \to \delta(|\Omega - \Omega'|) ,
\]

(15)

and defining the classical states identified by the corresponding points \( \Omega \) on the classical phase-space \( \mathcal{M} \). As for the observables, the only ones that stay meaningful throughout the crossover must obey

\[
\lim_{N \to \infty} \frac{\langle \Omega |\hat{A}| \Omega'\rangle}{\langle \Omega |\Omega'\rangle} < \infty ,
\]

(16)

so as to transform into well defined functions on the classical phase-space. Using Eq. (15) one can easily show that \( \langle \Phi_\varphi(\varphi) |\Phi_\varphi(\varphi)\rangle \to \chi^2(\lambda) \) in the classical limit for the clock; moreover, it is \( \chi^2(\lambda) = \chi^2(\varphi) \) due to Eq. (9). Therefore, reminding that \( \chi^2(\varphi) \) is a normalized probability distribution, any \( \varphi \) for which \( \chi^2(\varphi) \neq 0 \) defines a physical state

\[
|\phi(\varphi)\rangle := \frac{|\Phi_\varphi(\varphi)\rangle}{\sqrt{\chi^2(\varphi)}} ,
\]

(17)

whose dependence on \( \varphi \) is ruled by

\[
\frac{ie}{d\varphi} |\phi(\varphi)\rangle = \hat{H}_\varphi |\phi(\varphi)\rangle ,
\]

(18)

which is the Schrödinger equation with \( t = \hbar \varphi / \epsilon \). In fact, the above result is a derivation of the Schrödinger equation akin to that suggested in the original work by Page and Wootters [1], with state-normalization ensured by construction, for a classical clock. We notice, though, that as a byproduct of having specifically addressed the normalization issue, the state (17) has a further dependence on the real parameter \( \varphi \). In order to understand its meaning as far as the evolving system is concerned, we get back to the constraint (1) and its projection upon a GCS \(|\lambda\rangle\) of the clock, Eq. (8), with \(|\Psi\rangle\) as in Eq. (3).

Considering that \( \langle \lambda |\Omega\rangle \) is finite for finite \( N \), we write

\[
0 = \langle \lambda |\hat{H}|\Psi\rangle =
\]

\[
\int_{\mathcal{M}} d\mu(\Omega) \chi(\Omega) \langle \lambda |\Omega\rangle \left( \frac{\langle \lambda |\hat{H}_C|\lambda\rangle}{\langle \lambda |\lambda\rangle} - \hat{H}_\varphi \right) |\phi(\Omega)\rangle
\]

(19)

that becomes, in the classical limit for \( C \) where Eqs. (15) and (16) hold and for any \( \varphi \) such that \( \chi^2(\varphi) \neq 0 \),

\[
\hat{H}_\varphi |\phi_\varphi(\varphi)\rangle = E_\varphi(\varphi) |\phi_\varphi(\varphi)\rangle ,
\]

(20)

with

\[
E_\varphi(\varphi) = \langle \lambda |\hat{H}_C|\lambda\rangle ;
\]

(21)
the r.h.s. of the above equation, which is the symbol of $\hat{H}_c$ on $|\lambda\rangle$, can be calculated and reads (see supplemental material)

$$H_c(\varphi) := \langle \lambda | \hat{H}_c | \lambda \rangle = \frac{\epsilon}{2} b^2 (\cos(2\varphi) - 1), \quad \text{(22)}$$

with $\epsilon^2 b^2 = \sum_s g_s d_s$. It is relevant that Eq. (22) follows from algebraic properties, and therefore holds in general, regardless of the details of the theory that describes the clock. Furthermore, $H_c(\varphi)$ does not depend on $\varphi$, which justifies the use of the notation $E_\Gamma(\varphi)$ in Eq. (21) and allows one to consistently relate Eq. (20) with the stationary Schrödinger equation for $\Gamma$, with $\varphi$ the parameter that sets its energy.

An uncertainty relation

Let us now consider what happens when making measurements on the clock. We know that GCS are the only quantum states that survive the quantum-to-classical crossover according to $|\Omega\rangle \to \Omega$, as described above and thoroughly discussed in the literature [24, 31–34]. This means that performing a quantum measurement upon a system whose behaviour can be effectively described as if it were classical, is tantamount to select one GCS $|\Omega\rangle$ to be the ancestor of the observed classical state or, which is the same, say that the combined effect of a measurement and the classical limit is to make $\chi^2(\Omega)$ become a Dirac-$\delta$ around the point $\Omega$ on $\mathcal{M}_c$ that identifies the observed classical state. Let us now take such state to be one of the GCS $|\lambda\rangle$, consistently with the task of making measurements of observables that characterize it as a clock, such as $H_c$ or $\sin \varphi$ in Eq. (13). When taking the classical limit of the clock, it can be demonstrated [28, 29] that

$$\langle \lambda | \sin \varphi | \lambda \rangle \to \sin \varphi, \quad \langle \lambda | \cos \varphi | \lambda \rangle \to \cos \varphi; \quad \text{(23)}$$

this result, together with the definition $\Delta E_\Gamma(\varphi) := \Delta H_c(\varphi)$ (that follows from Eqs. (20)-(21)) and a small-$\varphi$ approximation, provides

$$\Delta E_\Gamma(\varphi) \Delta \varphi \geq \frac{\epsilon}{2}, \quad \text{(24)}$$

which we recognize, once the parameter $\hbar \varphi/\epsilon$ is identified with time, as a proper energy-time uncertainty relation for $\Gamma$. We will further comment upon this result in the concluding section.

Collecting all the clues so far obtained, we conclude this section maintaining that the parameter (11) is what we call "time" in QM, a statement that we express as

$$t_{\text{QM}} = \frac{\hbar}{\epsilon} \varphi, \quad \text{(25)}$$

where the apex QM indicates that this is the parameter that enters the quantum description of evolving systems.

This is not the end of the story, though, because it is now necessary to demonstrate that when the system $\Gamma$ undergoes the quantum-to-classical crossover, the above results lead to the Hamilton e.o.m., with the parameter $\hbar \varphi/\epsilon$ still playing the role of time. To this purpose, in the next section we take the classical limit also for the evolving system, thus moving into a completely classical setting.

V. A CLASSICAL CLOCK FOR A CLASSICAL SYSTEM

Let us now consider what happens when the system $\Gamma$ becomes macroscopic in a way that makes its behaviour amenable to the laws of classical physics. As in the previous section, the problem is tackled in terms of GCS in the large-$N$ limit. Therefore, besides the GCS for the clock $\{|\Omega\rangle\}$ defined in Sec. II, here we also use the GCS for the system, i.e. those relative to the Lie algebra $g_k$ proper to the quantum theory that describes $\Gamma$. These will be indicated by $\{|\gamma\rangle\}$, where $\gamma = (\gamma_1, \gamma_2, ..., \gamma_J)$ with $\gamma_j \in \mathbb{C}$ and $J$ related to the dimension of $g_k$. Each $|\gamma\rangle$ univocally identifies one point on the manifold $\mathcal{M}_s$, whose (real) dimension is $2J$.

Using the resolution of the identity upon $H_c$ and $\mathcal{H}_c$ in terms of the GCS $\{|\Omega\rangle\}$ and $\{|\gamma\rangle\}$, respectively, we write the state $|\Psi\rangle$ of the overall system as

$$|\Psi\rangle = \int_{\mathcal{M}_c} d\mu(\Omega) \int_{\mathcal{M}_s} d\mu(\gamma) \beta(\Omega, \gamma) |\Omega\rangle \otimes |\gamma\rangle, \quad \text{(26)}$$

where

$$\beta(\Omega, \gamma) := \langle \Omega | \langle \gamma | \Psi \rangle \rangle = \chi(\Omega) \langle \gamma | \phi(\Omega) \rangle \quad \text{(27)}$$

is a function on $\mathcal{M}_c \times \mathcal{M}_s$, whose square modulus, $\chi^2(\Omega) \langle \gamma | \phi(\Omega) \rangle^2$ is the conditional probability for $\Gamma$ to be in the state $|\gamma\rangle$ when $C$ is in the state $|\Omega\rangle$, given that the global system $\Psi$ is in the pure state $|\Psi\rangle$. In other terms, $\beta(\Omega, \gamma)$ is different from zero only on those pairs $(\Omega, \gamma) \in \mathcal{M}_c \times \mathcal{M}_s$ that define states $|\Omega\rangle \otimes |\gamma\rangle \in H_c \otimes \mathcal{H}_c$ which are present in the decomposition of $|\Psi\rangle$ in terms of GCS, Eq. (26).

Projecting the constraint (1) upon one specific state $|\Omega\rangle \otimes |\gamma\rangle$, we write

$$0 = \langle \Omega | \langle \gamma | \hat{H} | \Psi \rangle \rangle = \int_{\mathcal{M}_c} d\mu(\Omega) \int_{\mathcal{M}_s} d\mu(\gamma) \beta(\Omega, \gamma) \langle \Omega | \hat{H}_c | \Omega \rangle \langle \gamma | \gamma \rangle \times \left[ \langle \Omega | \hat{H}_c | \Omega \rangle - \langle \gamma | \hat{H}_c | \gamma \rangle \right] \quad \text{(28)}$$

that becomes, in the classical limit for $C$ and $\Gamma$, i.e. assuming Eqs. (15) and (16) hold not only for the GCS and the Hamiltonian of the clock but also for those of the system,

$$H_c(\Omega) = H_c(\gamma) \quad \text{(29)}$$

for $(\Omega, \gamma)$ such that $\beta(\Omega, \gamma) \neq 0$, meaning that the configurations $(\Omega, \gamma)$ into which the original quantum state $|\Psi\rangle$ can flow when clock and system behave according to the rules of classical physics, must obey Eq. (29). In particular, if one considers the configurations amongst those for which $\beta(\Omega, \gamma) \neq 0$ that have $\Omega = (0,0, ..., \Omega_t, ..., 0)$, corresponding to the GCS $|\lambda\rangle$ introduced in Sec. III and identified by the complex variable $\lambda = q e^{-i\varphi}$, these will belong to a submanifold $(\mathcal{U}_c \subset \mathbb{C}) \times (\mathcal{U}_T \subset \mathcal{M}_c)$ such that a map $F: \mathcal{U}_c \to \mathcal{U}_T$ exists, defined by

$$\lambda \in \mathcal{U}_c \mapsto u \in \mathcal{U}_T : H_T(u = F(\lambda)) = H_c(\varphi). \quad \text{(30)}$$
As the explicit form of $F$ is arbitrary, we fix it as follows. We consider that $\mathcal{M}_c$ has a symplectic structure, which means that it exists a Darboux chart
\[
D: \gamma \in \mathcal{M}_c \rightarrow (q, p) := ((q_1, p_1), (q_2, p_2), \ldots, (q_J, p_J)) \in \mathbb{R}^{2J},
\]
such that $\{q_i, p_j\}_c = \hbar^{-1} \delta_{i,j}$ with $\hbar = \text{const.}$, where $\{\cdot, \cdot\}_c$ are Poisson brackets on $\mathcal{M}_c$.

That relates the parametrization of GCS via $J$-dimensional complex vectors $\{\gamma\}$ that one obtains via $J$ pairs of real, canonically conjugated, variables $(q_j, p_j)$. For these pairs, referring to Ref. [27], we choose
\[
q_j - i\kappa^2 p_j = v_{\gamma} \sqrt{2\kappa} \sin(\varpi \varphi)e^{-i\varphi}
\]
with $\varpi \in \mathbb{R}$ constant unit vector, i.e. $\sum_j v_j^2 = 1$. As far as condition (30) is fulfilled, other choices are possible, without affecting the overall scheme and the subsequent results. Once $F$ is given, the so-called "pullback-by-$F^*$" map, sometimes indicated by $F^*$, is also defined, according to $F^*: \omega^{(k)} \rightarrow \omega^{(k)}_{\gamma} = F^*(\omega^{(k)}_c)$, where $\omega^{(k)}_{\gamma}$ are $k$-forms on $U_{\gamma}(\gamma)$. In particular, for $k = 0$, i.e. when considering functions, it is $(F^* f_{\gamma})(\lambda) = f_{\gamma}(u) = F(\lambda)$. Applying $F^*$ on the symplectic 2-form defining the standard Poisson brackets in (31), we obtain the Poisson brackets on $\mathcal{M}_c$, that read (see supplemental material)
\[
\{f_{\gamma}, g_{\gamma}\}_c = \frac{1}{\hbar \delta^{2s}} \delta^{(2s, \varphi)} \left( \frac{\partial f_{\gamma}}{\partial q} \frac{\partial g_{\gamma}}{\partial \varphi} - \frac{\partial g_{\gamma}}{\partial q} \frac{\partial f_{\gamma}}{\partial \varphi} \right)
\]
\[
(33)
\]
\[\forall f_{\gamma}, g_{\gamma} \text{ generic functions on } \mathcal{M}_c. \]

On the other hand, $q_j$ and $p_j$ are by all means functions on $\mathcal{M}_c$, as seen in Eq. (32); therefore, using Eq. (33) with $g_{\gamma} = H_{\gamma}(\varphi)$ from Eq. (22) we evaluate $\{q_j, H_{\gamma}\}_c$ and $\{p_j, H_{\gamma}\}_c$, and find (see supplemental material) $\{q_j, H_{\gamma}\}_c = \frac{\epsilon dq_j}{\hbar d\varphi}$, and $\{p_j, H_{\gamma}\}_c = \frac{\epsilon dp_j}{\hbar d\varphi}$. Finally, using $\{f_{\gamma}(\lambda), g_{\gamma}(\lambda)\}_c = \{f_{\gamma}(u), g_{\gamma}(u)\}_c$, we obtain
\[
\begin{align*}
\{q_j, H_{\gamma}\}_c &= \frac{\epsilon dq_j}{\hbar d\varphi} \\
\{p_j, H_{\gamma}\}_c &= \frac{\epsilon dp_j}{\hbar d\varphi}
\end{align*}
\]
\[
(34)
\]
i.e. the Hamilton e.o.m. ruling the dynamics of classical systems, once time is recognized as the parameter
\[
\epsilon^{CL} = \frac{\hbar}{\epsilon \varphi},
\]
\[
(35)
\]
where the apex CL indicates that this is the parameter that enters the classical description of evolving systems. Getting back to Eq. (25) and setting the arbitrary constant $\hbar$ in the Poisson-brackets of the Darboux chart (31) equal to $\hbar$, we finally obtain
\[
\epsilon^{QM} = \epsilon^{CL} = \frac{\hbar}{\epsilon \varphi}.
\]
\[
(36)
\]
This last equation, together with the derivation in one same framework of both the quantum-mechanical Schrödinger equation (18) and the classical Hamilton e.o.m. (34), represents the main result of this work, which is discussed in the next and last section.

VI. DISCUSSION, CONCLUSIONS, AND FURTHER DEVELOPMENTS

In the last decades we have learnt that when quantum macroscopic systems can be effectively studied as if they were classical (which is what should be meant by "classical"), their geometrical properties follow from the algebraic structure of the quantum theory originally describing them (see for instance the way a specific phase-space emerges as symplectic manifold involved in the GCS construction for one assigned quantum Lie-algebra). This is by itself quite a breakthrough, as it allows to establish a dialogue between classical and quantum physics without resorting to disjointed interventions such as quantization or, in the opposite direction, non-unitary state-reduction.

When considering more than one system, things become ever more interesting. In fact, when a quantum system interacts with a classical environment (be it a magnetic field, or a thermal bath, or some macroscopic environment), the pure states of the former acquire a parametric dependence that testifies the existence of the latter, and gives rise to geometrical effects such as the quantum Berry-phase [35-38]. Awe comes, though, as these effects emerge even without interaction, as far as the systems are entangled and some physical constraint is enforced, such as Eq. (1) in the PaW mechanism. Indeed, this is how states of a quantum system come to depend on time according to the Schrödinger equation, as also shown in this work. In such setting, coordinates of points in manifolds and elements of Hilbert spaces (e.g. $\varphi$, and $|\psi(\varphi)\rangle$ in this work) relate to each other via rules, such as the Schrödinger equation or the time-energy uncertainty relation, whose generality is that of the physical principles. To this respect we like to comment upon two of our results: First we notice that the energy of the system $\Gamma$, i.e. $E_{\Gamma}(\varphi)$ in Eq. (20), does not depend on time, i.e. on $\varphi$, consistently with the fact that the Hamiltonian of an isolated system cannot depend on time. Then we underline that, as in Refs. [4, 20], the inequality (24) does not follow from the non-commutativity between $\hat{H}_{\ell}$ and some other operator acting on $\hat{H}_{\ell}$: it is rather an indirect consequence of the inequality (13), which regards operators acting on the clock, plus the constraint (1) and the possibility, given by the use of GCS, of describing the clock as a classical object without wiping out one of its most relevant quantum feature, namely its being entangled with the evolving system.

What is most remarkable, though, is that a genuinely quantum feature such as entanglement survives even in a completely classical setting, there continuing to cause the emergence of such a fundamental ingredient of our everyday life as time, which is what we have here demonstrated by deriving the Hamilton e.o.m (34). In fact, our results in the fully classical setting unravel another tangle of classical physics, namely
the relation between phase-space and space-time. This relation emerges from the fact that when the global system is in the pure state $|Ψ⟩$, the only configurations that survive its classical limit are those identified by points $(Ω, γ) ∈ M_C × M_r$, where the probability $|β(Ω, γ)|^2$ is different from zero. Therefore, while the phase-space of $Γ$ is the 2$J$ dimensional simplectic manifold $M_r$ defined by the GCS $[γ]$ introduced in Sec. V, its space-time is the $(J + 1)$-dimensional real hypersurface defined by Eqs. (29) and (32), whose points (i.e. events) are identified by the coordinates $(hϕ/ε; q)$, with $ϕ = \arg λ ∈ ℝ$ from Eq. (5) and $q = q(γ) ∈ ℝ^J$ from the Darboux chart (31), such that $β(q; ϕ, q, P)$ is different from zero for some $q$ (i.e. energy of the clock) and $P$ (i.e. momentum of the system). Notice that, if $C$ and $Γ$ were not entangled, i.e. $|Ψ⟩ = |C⟩ ⊗ |Γ⟩$, it would be $β(q; ϕ, q, P) = χ(Ω)(⟨q, P|Γ⟩)$, with no relation between instants of time $ϕ$ and position in space $q$, i.e. with no causal relation between events. In other terms, as emerged in different contexts (see for instance Ref. [39]) not only quantum entanglement is what makes physical systems to evolve, but also provides their spacetime with a causal structure.

Despite effects of entanglement without interaction being already phenomenal, we think that taking possible interactions into account will lead to substantial developments of this work. One might first consider adding a quantum environment with which $Γ$ starts interacting while being already entangled with the clock. This should describe the dynamics of the density operator of $Γ$, and show how, and under what conditions, the Liouville-VonNeumann equation emerges, with clues about the non-unitary evolution of non-isolated systems. The presence of multiple clocks, possibly interacting amongst themselves, also seems an intriguing enrichment, particularly in view of some recent works by other authors [19, 21, 40]. However, the most compelling follow-up of this work, in our opinion, is that of relating the picture it proposes with that provided by relativity. In fact, we expect relativistic quantum mechanics and quantum-field-theory to find their place in the hybrid setting of Sec. IV, where studying how the expectation values of operators on $ℋ_{c}$ get to depend on $(q, ϕ)$ via the parametric dependence of the states $|ϕ_q(ϕ)⟩$, might help understanding some unclear aspects of the way special relativity encounters quantum mechanics. Moreover, having connected the classical formalism that set the scene for general relativity and gravity with a full quantum description, we think we have ideal tools for breaking through some of the obstacles that make quantum gravity so difficult to process. In particular, we believe that studying the probability distribution $|β(λ; q, p)|^2$ in relation to the original Lie algebras $g_c$ and $g_r$, and/or the specific form of the quantum Hamiltonian $H$ may provide a link between the geodesic principle and the Schrödinger equation; furthermore, taking into account a possible interaction between evolving system and clock, as suggested in Ref. [19], or between different clocks, as in Ref. [21], might explain spacetime deformation, and hence gravity, from a quantum viewpoint. Work in this direction is in progress, particularly referring to the case of Schwartzschild black-holes [23] and Hawking radiation [41].

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**VIII. SUPPLEMENTARY MATERIAL**

### A. GCS: General Coherent States

Generalized Coherent States (GCS) are an extension of the field coherent states firstly introduced by R. Glauber in 1963 [42]. The group-theoretic construction was derived ten years later by A. Perelomov [25] and R. Gilmore [26], independently. GCS are normalized elements of Hilbert spaces which are in one-to-one correspondence with the points of a smooth manifold, that has all the properties requested to a classical phase-space. In the following, we briefly introduce GCS according to the procedure described by Gilmore and coworkers in Ref. [27].

In order to construct GCS, three inputs are necessary:  
1) a Lie-algebra \( g \), or the related Lie-group \( G \),  
2) a Hilbert space \( \mathcal{H} \) which is the carrier space of an irreducible representation of \( g \), and  
3) a normalized element \( | \Omega \rangle \) of \( \mathcal{H} \).

Referring to a specific system for which GCSs are to be constructed, the inputs are as follows: \( H \) is the Hilbert space of the system; \( \mathfrak{g} \) is the Lie-algebra whose representation via operators on \( \mathcal{H} \) contains the Hamiltonians of the system, meaning that the representation of the related Lie-Group \( G \) contains all its propagators, which is why \( \mathfrak{g} \) is often dubbed *dynamical* group. The normalized element \( | \Omega \rangle \) of \( \mathcal{H} \) is a physically accessible state of the system, usually called *reference* state. For the sake of clarity we will hereafter identify \( g \) and \( \mathcal{H} \) with their respective representations on \( H \). Once the inputs are given, the procedure returns three outputs:  
1) the subgroup \( \mathcal{F} \subset G \) whose elements leave \( | \Omega \rangle \) unchanged apart from an irrelevant overall phase, and the associated coset \( G/\mathcal{F} \), such that every \( g \in G \) can be written as a unique decomposition of two group-elements, one belonging to \( \mathcal{F} \) and the other to \( G/\mathcal{F} \), i.e. \( g = \Omega \tilde{\mathbf{f}} \) with \( \tilde{\mathbf{f}} \in \mathcal{F} \) and \( \Omega \in G/\mathcal{F} \);  
2) the GCS  
   \[ |\Omega\rangle := \Omega |G\rangle, \quad \forall \Omega \in G/\mathcal{F}; \]  
3) a measure \( d\mu(\Omega) \) on \( G/\mathcal{F} \) which is invariant under the action of the elements of \( G \), and therefore dubbed *invariant*
where such a resolution of the identity upon $\mathcal{H}$ is provided
\[ \int_{\mathcal{G}/\mathcal{F}} d\mu(\Omega) \langle \Omega | \Omega \rangle = \frac{1}{2} \mathcal{H}. \] (38)

The GCS are normalized, $\langle \Omega | \Omega \rangle = \langle G | \hat{g}^{-1} \hat{g} | G \rangle = \langle G | G \rangle = 1, \forall \hat{g} \in \mathcal{G},$ but non-orthogonal,
\[ \langle \Omega | \Omega' \rangle = \langle G | \hat{g}^{-1} \hat{g}' | G \rangle = \langle G | \hat{g}^{-1} \hat{g} | G \rangle e^{i\theta} = \langle \Omega | \hat{g}' | G \rangle e^{i\theta} \neq 0, \]
\[ \forall \hat{g}, \hat{g}', \hat{g}'' \in \mathcal{G}, \text{ and } \Omega, \Omega' \in \mathcal{G}/\mathcal{F}. \]

For this reason they are said to provide an "overcomplete" set of states for $\mathcal{H},$ where "complete" refers to Eq. (38), while "over" means that they are too many for being all orthogonal to each other.

As for the reference state $\langle \Omega \rangle,$ a common, yet not mandatory, choice is that of taking it as an extremal state; for instance, one can choose $|G\rangle$ as the minimal-weight state such that $\hat{R}_m |G\rangle = 0 \forall m,$ with $\hat{R}_m$ defined below.

Getting an explicit expression for the operators $\hat{\Omega},$ and hence of the GCS via Eq. (37), requires a characterization of the algebra. In particular, if $\mathfrak{g}$ is semisimple, one can consider its Cartan decomposition, that classifies the generators as diagonal, $\{\hat{D}_\delta\},$ or raising, $\{\hat{R}_m, \hat{R}_{-m}\},$ operators, according to
\[ \hat{[D}_\delta, \hat{D}_\theta] = 0, \quad \hat{[D}_\delta, \hat{R}_m] = \delta_{dm} \hat{R}_m, \]
\[ \hat{[R}_m, \hat{R}_{-m}] = \sum s \delta_{sm} \hat{D}_s, \quad \hat{[R}_m, \hat{R}_{m'}] = c_{m'm} \hat{R}_{m+m'}. \] (39)

where $\{d_{sm}\}, \{c_{m'm}\}$ are the so called structure constants, while $m, m'$ and $\delta, \theta$ go from 1 to some upper value $M$ and $D,$ respectively, that depend on the algebra itself (in the case of $\mathfrak{su}(2),$ for instance, it is $M = D = 1,$ and if spin-squeezing is also considered, it is $M = 2$ and $D = 1$). In any irreducible representation of $\mathfrak{g}$ it is possible to choose the raising operators such that $\hat{R}_m = \hat{R}_m \forall m,$ and, consistently, hermitian or anti-hermitian diagonal operators $\hat{D}_s = +(-) \delta \hat{D}_s \forall \delta,$ depending on the structure constants $\{d_{sm}\}$ being real or imaginary. The diagonal operators have the reference state amongst their eigenstates, i.e., $\hat{D}_\delta |G\rangle = g_\delta |G\rangle \forall \delta.$ Since the Cartan decomposition is available, it can be shown that the elements of $\mathcal{G}/\mathcal{F}$ in the definition (37) take the form
\[ \hat{\Omega} = \exp \left( \sum_m \Omega_m \hat{R}_m - \Omega'_m \hat{R}_m \right), \] (40)

where the coefficients $\Omega_m \in \mathbb{C}$ are coordinates of one point $\Omega$ of the differentiable manifold $\mathcal{M},$ which is associated to $\mathcal{G}/\mathcal{F}$ via the quotient manifold theorem [43]. Using a complex projective representation of $\mathcal{G}/\mathcal{F},$ GCS can also be written as
\[ |\Omega\rangle = N(|\eta(\Omega)|) e^{\sum_m \eta_m \hat{R}_m} |G\rangle \] (41)

where the normalization constant $N(|\eta(\Omega)|)$ and the relation between the $\eta_m$-coordinates and the $\Omega_m$ ones can be obtained via the BCH formulas.

The chain of biunivocal relations
\[ \Omega \in \mathcal{G}/\mathcal{F} \Leftrightarrow \Omega \in \mathcal{M} \Leftrightarrow |\Omega\rangle \in \mathcal{H}. \] (42)

is one of the most distinctive feature of the group-theoretic construction, as it establishes that any GCS is univocally associated to a point on $\mathcal{M},$ and viceversa. As a consequence, the invariant measure $d\mu(\Omega)$ induces a measure $d\mu(\Omega)$ upon $\mathcal{M}.$ In fact, it can be demonstrated [27] that $\mathcal{M}$ is endowed with a natural metric that can be expressed in the $\eta_m$-coordinates as
\[ ds^2 = \sum_{m,m'} g_{m'm} d\eta_m d\eta^*_m, \quad \text{where } g_{m'm} := \frac{\partial^2 \log \langle \Omega | \Omega \rangle}{\partial \eta_m \partial \eta^*_m}, \] (43)

with $|\Omega\rangle := |\Omega\rangle / N$ in (41). After $ds^2$ one can define a canonical form on $\mathcal{M},$ i.e. the above mentioned measure on $\mathcal{M},$ via
\[ d\mu(\Omega) \propto \text{const} \times \det(\eta) \prod_m d\eta_m d\eta^*_m. \] (44)

The manifold $\mathcal{M}$ is also equipped with a symplectic structure that allows one to identify it as a phase-space. In particular, the symplectic form on $\mathcal{M}$ has the coordinate representation
\[ \omega = -i \sum_{m,m'} g_{m'm} d\eta_m \wedge d\eta^*_m, \] (45)

can be used to define the Poisson brackets
\[ \{f, g\}_{PB} := i \sum_{m,m'} g^{m'm} \left( \frac{\partial f}{\partial \eta_m} \frac{\partial g}{\partial \eta^*_m} - \frac{\partial f}{\partial \eta^*_m} \frac{\partial g}{\partial \eta_m} \right), \] (46)

with $\sum_n g_{nm'n} g^{nm'} = \delta_m$. In the case of non-semisimple algebras, such as $\mathfrak{h}_4$ and $\mathfrak{h}_6$ for the harmonic and squeezed-harmonic oscillator, respectively, where a Cartan decomposition (39) is not available, analogous decompositions exist, and the same procedure can be adopted. This is explicitly done for $\mathfrak{h}_4$ at the end of this material, where we show that the results are the same as those obtained in the semisimple case.

### B. PRECS: Parametric Representation with Environmental Coherent States

Parametric representations of composite systems can be built whenever a resolution of the identity upon the Hilbert space of one of the subsystems is available. In Ref. [2], for instance, the representation is introduced via $\int dx \langle x | \right) = \hat{1}_C,$ where $|x\rangle$ are the eigenstates of the position operator for one of two subsystems, and the integral is over the real axes. Our choice, which is pivotal to get to our final result, is based on the fact that parametric representations with GCS inherit from the group theoretic construction some properties that are essential in order to follow the quantum-to-classical crossover and formally define a classical limit of a quantum theory, according to the large-$N$ quantum approach.
where \( \{|\gamma\rangle\} \) and \( \{|\xi\rangle\} \) are orthonormal bases for \( \mathcal{H}_f \) and \( \mathcal{H}_s \), respectively. Inserting the above mentioned resolution of the identity upon \( \mathcal{H}_c \), for which we choose the one provided by GCS, Eq. (38), one gets
\[
\langle \Psi \rangle = \int_{\mathcal{M}} d\mu(\Omega) \chi(\Omega) |\Omega\rangle \otimes |\phi(\Omega)\rangle ,
\]
(48)
where \( \chi(\Omega) \) is a function that can be chosen real, being defined via \( \chi^2(\Omega) := \sum_\gamma |\langle \gamma | \chi(\Omega) \xi \rangle|^2 \). The element \( |\phi(\Omega)\rangle \) of \( \mathcal{H}_f \) is normalized, and hence describe a pure state of \( \Gamma \). Due to the normalization of \( |\Psi\rangle \), it is
\[
\int_{\mathcal{M}} d\mu(\Omega) \chi^2(\Omega) = 1 ,
\]
(49)
meaning that \( \chi^2(\Omega) \) can be interpreted as a probability distribution on \( \mathcal{M} \). The above expressions have a clear physical interpretation: reminding that each point \( \Omega \in \mathcal{M} \) is in one-to-one correspondence with a GCS \( |\Omega\rangle \in \mathcal{H}_c \), we can say that \( |\phi(\Omega)\rangle \) is the state of \( \Gamma \) conditioned to \( C \) being in the GCS \( |\Omega\rangle \), a circumstance that occurs with probability \( \chi^2(\Omega) \) when \( \Psi \) is in the pure state \( |\Psi\rangle \). This interpretation is consistent with the following relations \[44\]
\[
\chi^2(\Omega) = \langle \Omega | \rho_C | \Omega \rangle ,
\]
and
\[
\rho_{\Gamma} = \int_{\mathcal{M}} d\mu(\Omega) \chi^2(\Omega) |\phi(\Omega)\rangle \langle \phi(\Omega) | ,
\]
(51)
where \( \rho_{\Gamma(C)} := \text{Tr}_{\mathcal{H}(C)} \langle \Psi | \Psi \rangle \). Notice that the diagonal-like form (51) of \( \rho_{\Gamma} \) is not generally granted for parametric representations such that the identity resolution is in terms of non-orthogonal states, as in the GCS case. In fact, it is the specific overcompleteness of GCS that ensures Eq. (51) to hold.

Finally, it is important to remind that since the parametric representations allow one to use pure states \( |\phi(\Omega)\rangle \) to describe \( \Gamma \), this should by no means be intended as if \( \Gamma \) were in a pure state. In fact, due to the parametric dependence of \( |\phi(\Omega)\rangle \) on \( \Omega \), the density operator \( \rho_{\Gamma} \) in Eq. (51) is not a projector, reflecting that \( C \) and \( \Gamma \) are entangled, as far as the form (47) of \( |\Psi\rangle \) stays general. To this respect, it is easily verified that when \( |\Psi\rangle \) is separable the above parametric dependence dies out.

### C. Derivation of the symbol of \( \hat{H}_C \)

In this part we express the symbol \( \langle \lambda | \hat{H}_C | \lambda \rangle \) of the clock-Hamiltonian, as introduced in Sec. IV of the paper, in terms of the complex parameter \( \lambda := \Omega_l = \varphi e^{-i\varphi} \) that defines the GCS \( |\lambda\rangle \) via \( |\lambda\rangle \equiv e^{i\varphi} |G \rangle \) with \( |G\rangle \) the clock reference state satisfying \( \hat{R}_m |G\rangle = 0 \forall m \), \( \hat{D}_s |G\rangle = g_s |G\rangle \), and \( \hat{W} := W_l := \Omega_l R_f - \Omega_f R_l \). Recalling that \( H_C = \hat{\varphi} \hat{D}_1 + K \) with \( K = -\varphi g_1 \) and \( \varphi^2 = \pm 1 \) such that \( \varepsilon := \varphi g_1 \) is real and positive, we write
\[
\langle \lambda | \hat{H}_C | \lambda \rangle = K + \varepsilon \langle \lambda | e^{-i\varphi} \hat{D}_1 e^{i\varphi} |G\rangle \\
= K + \varepsilon \langle \lambda | \hat{D}_1 + (\hat{W}, \hat{D}_1) + \frac{1}{2i} [\hat{W}, [\hat{W}, \hat{D}_1]] \\
+ \frac{1}{3i} [\hat{W}, [\hat{W}, [\hat{W}, \hat{D}_1]]] \\
+ \frac{1}{4i} [\hat{W}, [\hat{W}, [\hat{W}, [\hat{W}, \hat{D}_1]]]] + \ldots |G\rangle \\
= K + \varepsilon \langle \lambda | \hat{D}_1 + (\hat{W}, \hat{D}_1) + \sum_{\delta} \left[ \frac{1}{2i} (-2d_{1e}d_{\delta}g^2 \hat{D}_s) \\
+ \frac{1}{3i} (-2d_{1e}d_{\delta}g^2 \hat{D}_s) \\
+ \frac{1}{4i} \sum_{\delta} (-2d_{1e}d_{\delta}g^2) (-2d_{\delta}d_{\delta}g^2 \hat{D}_s) + \ldots |G\rangle \\
= \varepsilon d_{1e} \sum_{\delta} g_{\delta}d_{\delta} \left( \sum_{n=0}^{\infty} \frac{(-1)^n}{(2n)!} (\sqrt{2g})^{2n} \right) - 1 := H(C) ,
\]
(52)
where \( a^2 = \sum_\delta g_{\delta}^2 \) and \( \varphi^2 = \sum_\delta g_{\delta}d_{\delta} \). For the sake of a lighter notation, in what follows and in the main work we set \( \varphi^2 = 2 \), which means that the raising and diagonal operators are multiplied by \( \sqrt{2\varphi^2}a \), and their eigenvectors are rescaled accordingly. We thus finally get
\[
\langle \lambda | \hat{H}_C | \lambda \rangle = \frac{\varepsilon b^2}{2} (\cos(2\sqrt{2g}) - 1) .
\]
(53)

### D. The pullback-by-\( F \) and the Poisson brackets on \( \mathcal{M}_C \)

In this part we will explicitly calculate the Poisson brackets \{\cdot, \cdot\} of induced on \( \mathcal{M}_C \) via the pullback-by-\( F \). We recall that, given the manifolds \( \mathcal{M}_C \) and \( \mathcal{M}_f \) for the clock \( C \) and the evolving system \( \Gamma \) as from the GCS construction, the map \( F : U_C \subset \mathcal{M}_C \rightarrow \hat{U}_R \subset \mathcal{M}_G \), is defined as
\[
\begin{cases}
q_j = \sqrt{2} b \sin(\varphi) \cos(\varphi) v_j , \\
p_j = \sqrt{2} b \sin(\varphi) \sin(\varphi) v_j ,
\end{cases}
\]
(54)
with \( \sum_j v_j^2 = 1 \). We remind that the Poisson brackets are defined on a generic symplectic manifold \( \mathcal{M} \) starting from its symplectic form \( \omega = \frac{1}{2} \sum_{\mu\nu} \omega_{\mu
u} \, dx^\mu \wedge dx^\nu \), via
\( \{ f, g \} = \sum_{\mu \nu} \omega^{\mu \nu} \partial_{x^\mu} f \partial_{x^\nu} g \) with \( \sum_{\sigma} \omega^{\sigma \mu} \omega^{\sigma \nu} = \delta^\mu_\nu \), and \( x^\mu (\mu = 1, \ldots, 2n = \dim M) \), \( f, g \) are some coordinates and generic functions on \( M \), respectively. In fact, the Darboux theorem guarantees that there exist local coordinates \( x^\mu = (q_1, \ldots, q_n, p_1, \ldots, p_n) \) such that \( \omega = \hbar \sum_{j=1}^n dp_j \wedge dq_j \) and \( \{ f, g \} = \hbar^{-1} \sum_{j=1}^n (\partial_{q_j} f \partial_{p_j} g - \partial_{p_j} f \partial_{q_j} g) \), with \( \hbar = \text{const.} \). This said, being \( (q_j, p_j) \) in Eq. (54) Darboux coordinates, i.e. \( \{ q_j, p_j \}_\Gamma = \hbar^{-1} \delta_{ij} \), the symplectic form \( \omega_\Gamma \) on \( U_\Gamma \subset M_\Gamma \) is

\[
\omega_\Gamma = \hbar \sum_j dp_j \wedge dq_j .
\] (55)

We can now calculate the pullback-by-\( F \) of \( \omega_\Gamma \) as

\[
(\omega_\Gamma)^* = \hbar \sum_j \left[ \sqrt{2b} \cos(\zeta \varphi) \sin(\varphi) v_j d\varphi
\right.
+ \frac{\sqrt{2b}}{\zeta} \sin(\zeta \varphi) \cos(\varphi) v_j d\varphi
\right.
\]
\[\left. \wedge \left[ \sqrt{2} b \zeta^2 \cos(\zeta \varphi) \cos(\varphi) v_j^2 d\varphi
\right.
- \sqrt{2} b \zeta \sin(\zeta \varphi) \sin(\varphi) v_j d\varphi
\right.
\]
\[\left. \wedge \left[ \sqrt{2} b \zeta \sin(\zeta \varphi) \sin(\varphi) v_j^2 d\varphi \wedge d\varphi \right.
\right.
\]
\[\left. \left. + b^2 \zeta \sin(2\zeta \varphi) v_j^2 \cos^2 \varphi d\varphi \wedge d\varphi \right]\right].
\] (56)

Finally \( (\omega_\Gamma)^* \) defines Poisson brackets on \( M_\Gamma \) via

\[
\{ f_c, g_c \}_c = \frac{1}{\hbar b^2 \zeta \sin(2\zeta \varphi)} \left( \frac{\partial f_c}{\partial q_c} \frac{\partial g_c}{\partial \varphi} - \frac{\partial f_c}{\partial \varphi} \frac{\partial g_c}{\partial q_c} \right).
\] (57)

We clarify that our choice (54) for the map \( F \) follows from the one suggested in Ref. [27], but other choices are possible.

### E. The Heisenberg algebra \( \mathfrak{h}_4 \)

When the Lie algebra \( \mathfrak{g} \), to which the clock Hamiltonian \( \hat{H}_C \) belongs, is semisimple, the GCS are built starting from the Cartan decomposition. However a similar construction can be put forward for the non-semisimple algebra \( \mathfrak{h}_4 \). The latter is defined by the set \( \{ \hat{n}, \hat{a}, \hat{a}^\dagger \} \) with commutation relations \( [\hat{a}, \hat{a}^\dagger] = 1 \), \( [\hat{a}, \hat{n}] = [\hat{a}^\dagger, \hat{n}] = 0 \). The GCS \( |\alpha\rangle \), usually called harmonic-oscillator coherent states or just coherent states, are in one-to-one correspondence with the points of the complex plane \( \mathbb{C} \) and can be equivalently defined as \( |\alpha\rangle = e^{-\alpha^2} |\alpha\rangle \) with \( \alpha \in \mathbb{C} \).

Finally, when \( \Lambda(\varphi) = \epsilon e^{i\varphi} \), the Schrödinger equation for the evolving system \( \Gamma \) can be obtained, as shown in the main work, implementing the PaW mechanism via the PRECS and considering a fixed GCS \( |\alpha\rangle = e^{i\alpha^2} \) with \( \alpha \in \mathbb{C} \).

Again, the temporal parameter \( t_{\text{QM}} \) turns out to be \( t_{\text{QM}} = (\hbar/\epsilon) \varphi \). Moreover, since it is trivial to show that \( H_C(\alpha) = (\langle \alpha | H_C | \alpha \rangle) = \epsilon \alpha^2 \), the considerations concerning the parameter \( q \) and the stationary Schrödinger equation for \( \Gamma \) still apply. For what concerns the uncertainty relation, a phase-operator can be defined via \( \hat{a} = \hat{n}^{1/2} e^{i\varphi} \). Finally, when \( \Gamma \) becomes macroscopic and presents a completely classical behaviour, its dynamics is ruled by the Hamilton equations according to a temporal parameter \( t_{\text{CL}} = t_{\text{QM}} \). This result can be obtained following the same line of reasoning of the main work and choosing the map \( F \) to be \( \hat{q}_j = \hat{p}_j = v_j \sqrt{2\epsilon} \) with \( \sum_j v_j^2 = 1 \).