Device-independent certification of tensor products of quantum states using single-copy self-testing protocols

Ivan Šupić, 1 Daniel Cavalcanti, 2 and Joseph Bowles 2

1 Département de Physique Appliquée, Université de Genève, 1211 Genève, Switzerland
2 ICFO-Institut de Ciencies Fotòniques, The Barcelona Institute of Science and Technology, 08860 Castelldefels (Barcelona), Spain

(Dated: October 1, 2019)

Self-testing protocols are methods to determine the presence of shared entangled states in a device-independent scenario, where no assumptions on the measurements involved in the protocol are made. A particular type of self-testing protocol, called parallel self-testing, can certify the presence of copies of a state, however such protocols typically suffer from the problem of requiring a number of measurements that increases with respect to the number of copies one aims to certify. Here we propose a procedure to transform single-copy self-testing protocols into a procedure that certifies the tensor product of an arbitrary number of (not necessarily equal) quantum states, without increasing the number of parties or measurement choices. Moreover, we prove that self-testing protocols that certify a state and rank-one measurements can always be parallelized to certify many copies of the state. Our results have immediate applications for unbounded randomness expansion.

Introduction — Bell nonlocality describes measurement correlations which are rigidly incompatible with the notion of local determinism [Bel64, BCP+14]. Namely, all local deterministic theories satisfy bounds—called Bell inequalities—which limit the strength of the correlations between measurement outcomes of two spatially distant and non-communicating systems. Interestingly, it is possible to violate such Bell inequalities in quantum experiments [HBD+15, SMSC+15, GVW+15]. Such violating correlations, called nonlocal, are closely related to quantum resources such as entanglement and measurement incompatibility, essential for the development of modern day quantum technologies.

Bell nonlocality also plays a crucial role in so-called device-independent protocols. It turns out that the violation of a Bell inequality is a function of the observed correlations alone, regardless of the underlying physical realization. Thus, the sole observation of a Bell inequality violation witnesses the presence of both entanglement and incompatibility without having knowledge or making any assumptions about the underlying experimental implementation. Such an assumption-free verification is named device-independent and has a special significance in cryptographic scenarios [ABG+07, PAM+10].

The maximal violation of some Bell inequalities can even imply the precise form of the underlying state and measurements. This can be seen as a device-independent tomography, and has received the name of self-testing [ŠB19, MY04]. The simplest example of such phenomenon is the maximal violation of the Clauser-Horn-Shimony-Holt (Bell) inequality [CHSH69], which can be used to self-test the maximally entangled pair of qubits and mutually unbiased local measurements [PR92, Tsi93, SW87].

There exist nonlocal correlations that self-test several copies of a quantum state, a process called parallel self-testing. These self-testing protocols have immediate applications in situations where high amounts of entanglement is needed, such as randomness expansion [CY14], parallel quantum key distribution [JMS17, Vid17], delegated quantum computing [RUV13, CGJV17], and universal entanglement certification [BŠCA18a]. One drawback of the first parallel self-testing protocols is that they require a number of local measurements that increases exponentially with the number of copies one wants to certify [WBMS16, Col17, CN16, McK17, BŠCA18b]. This fact increases the time-cost and the randomness consumption of the protocol (relevant for cryptographic applications). More recently, techniques to reduce the number of local measurements to poly(log(n)) and log(n) were found [NV17, NV18, CRSV18, OV16, BŠCA18b]. There exist protocols to self-test entangled states of arbitrary dimension with a constant number of three or four inputs per party [YN13, CGS17]. These protocols, when applied to copies of quantum states require making joint measurements between the local subsystems of each copy, making them challenging from an experimental perspective.

In this letter, we show a procedure to combine different self-testing protocols into a protocol that self-tests tensor products of quantum states, without increasing the number of required measurements. The combined protocol has the advantage of not requiring joint measurements among the copies. As a key application, we show a way of self-testing n copies of the two-qubit maximally entangled state using only two measurements per party (see Figure 1, right). This is the first self-testing protocol using the minimum number of local measurements possible for certifying an unbounded amount of entanglement. This procedure can therefore be used to convert one random bit into an arbitrary number of private random bits.

Main idea of the method— For simplicity, we explain how our method can transform the self-testing based on the Clauser-Horn-Shimony-Holt (CHSH) game into a self-test for copies of a two-qubit maximally entangled state. The scenario consists of two space-like separated parties, Alice and Bob, making local measurements on a shared quantum system. Alice and Bob apply one of two measurements each, labeled $x = 0, 1$ and $y = 0, 1$ respectively, and their goal is to obtain outputs $a = 0, 1$ and $b = 0, 1$ such that $a \oplus b = x \cdot y$ where $\oplus$ is addition modulo 2. Their score is defined as

$$\omega = \frac{1}{4} \sum_{x,y} P(a \oplus b = x \cdot y),$$

i.e. the probability of satisfying the winning condition averaged over a uniform choice of $x, y$. If Alice and Bob make use of classical resources (or
separable states), the best score they can achieve is $\omega \leq \frac{3}{2}$, which is equivalent to satisfying the CHSH Bell inequality

$$\langle A_0 B_0 \rangle + \langle A_0 B_1 \rangle + \langle A_1 B_0 \rangle - \langle A_1 B_1 \rangle \leq 2,$$

where $\langle A_{\lambda} B_{\gamma} \rangle = \sum_{x,y} (-1)^{ab} P(a,b|x,y)$. On the other hand, if they share a maximally entangled pair of qubits $|\phi^+\rangle = \frac{1}{\sqrt{2}} (|00\rangle + |11\rangle)$, Alice’s measurements correspond to the Pauli observables $A_0 = \sigma_Z$ and $A_1 = \sigma_X$ and Bob’s measurements to $B_0 = (\sigma_Z + \sigma_X)/\sqrt{2} = \sigma_+ + \sigma_- + \sigma_\pm$, and they can achieve the score $\omega = (1+1/\sqrt{2})/2 \approx 0.8536$. This strategy violates the CHSH Bell inequality to a value of $2\sqrt{2}$, and is the largest violation possible in quantum theory. This maximum value is also known to self-test the state $|\phi^+\rangle$. Thus, up to a possible local change of basis and extra unused degrees of freedom, the state $|\phi^+\rangle$ is the only state that achieves this value (see later for a formal definition of self-testing).

The CHSH inequality can also be used to self-test $n$ copies of $|\phi^+\rangle^\otimes n$ via parallel self-testing [Col17, McK17]. One way to achieve this is as follows. Alice and Bob in each round receive $n$-bit inputs $x = (x_1, x_2, \ldots, x_n)$ and $y = (y_1, y_2, \ldots, y_n)$, and return $n$-bit outputs $a = (a_1, a_2, \ldots, a_n)$ and $b = (b_1, b_2, \ldots, b_n)$. To self-test the state, Alice measures $\sigma_Z$ or $\sigma_X$ on her local qubit $i$ depending on $x_i = 0, 1$, and Bob does similarly, measuring $\sigma_+ \text{ or } \sigma_- \text{ depending on } y_i = 0, 1$ and returning outcome $b_i$ (see Fig. 1b). The resulting correlations maximally violate $n$ CHSH inequalities in parallel and imply Alice and Bob share a tensor product of $n$ EPR pairs.

We consider, in turn, a scenario in which each party has only two choices of measurements $x = 0, 1$ and $y = 0, 1$ (Fig. 1c). If $x = 0$ Alice measures all of her qubits in the $\sigma_Z$-basis, whereas if $x = 1$ she measures all of them in the $\sigma_X$-basis. In both cases she returns an output consisting of $n$ bits $a = (a_1, a_2, \ldots, a_n)$ corresponding to the outcomes of each of the $n$ measurements. Bob proceeds similarly, measuring all his qubits in $\sigma_+ \text{ and } \sigma_- \text{ bases and returning } b = (b_1, b_2, \ldots, b_n)$. For any pair $(a_i, b_i)$, the condition $P(a_i \oplus b_i = x \cdot y) = \omega_q$ is satisfied, and one might naively think that this information alone is enough to conclude that the state is $|\phi^+\rangle^\otimes n$. This would be a mistake however, as the following counter-example shows. Suppose Alice and Bob measure a single copy of $|\phi^+\rangle$, obtaining outputs $a, b$ such that $P(a \oplus b = x \cdot y) = \omega_q$. Then, they set $a_i = a$ and $b_i = b$ for all $i$, leading to $P(a_i \oplus b_i = x \cdot y) = \omega_q$ for all $i$. Thus, one can achieve $P(a_i \oplus b_i = x \cdot y) = \omega_q$ for all $i$ with only a single copy of $|\phi^+\rangle$.

One thus needs to consider more information than just the CHSH score of each copy. One possibility is to consider the marginal statistics, since in the parallel $n$-copy strategy one has $P(a|x) = 1/2^n$ whereas in the single-copy strategy the local output bits $a_i$ are always perfectly correlated (with a similar situation for Bob). However this will not work, since by using $n$ bits of pre-shared randomness $(\lambda_1, \ldots, \lambda_n)$ and post-processing their outputs according to $a_i = a + \lambda_i$ and $b_i = b + \lambda_i$, the local output bits of the single-copy strategy can be decorrelated without affecting the CHSH scores. Notice however that there is still a crucial difference between the two strategies that remains: in each round of the single-copy strategy, if the first pair satisfies $a_1 + b_1 = x \cdot y$, then all other pairs also satisfy $a_i + b_i = x \cdot y$. This is not the case for the $n$-copy strategy, where each pair has a probability $\omega_q$ to satisfy the condition in each round, independent of the other pairs. It is precisely this difference that we use as an inspiration to design our self-testing protocol.

**Lifting self-testing protocols**—Before stating our main result, let us define self-testing in a precise way. Consider a general bipartite Bell scenario, where $x, y \in \{0, 1, \ldots, m-1\}$ and $a, b \in \{0, 1, \ldots, o-1\}$. Alice and Bob share the state $\rho_{AB} = \text{tr}_P [\rho_A |\psi\rangle \langle \psi|_{\ABP}^{\rho}]$, with $|\psi\rangle_{\ABP}^{\rho}$ being any purification of $\rho_{AB}$. The probabilities of obtaining outputs $a$ and $b$, when
the inputs are \( x \) and \( y \), respectively, are given by
\[
p(a, b | x, y) = \langle \psi | M_{a|x} \otimes N_{b|y} \otimes I_P | \psi \rangle,
\]
where \( \{M_{a|x}\} \) and \( \{N_{b|y}\} \) are Alice’s and Bob’s local projective measurement operators. We say that the probabilities \( \{p(a, b | x, y)\} \) self-test the reference experiment \( \{\langle \psi' \rangle_{A'B'}, M'_{a|x}, N'_{b|y}\} \) if observing them in an experiment implies the existence of a local unitary transformation \( U = U_{AA'} \otimes V_{BB'} \otimes I_P \) such that
\[
U |\langle \psi' \rangle_{ABP} \otimes |00\rangle_{A'B'}\rangle = |\xi\rangle_{ABP} \otimes |\psi\rangle_{A'B'};
\]
\[
U |M_{a|x} \otimes N_{b|y} \otimes I_P |\psi\rangle_{ABP} \otimes |00\rangle_{A'B'}\rangle = |\xi\rangle_{ABP} \otimes \left(M'_{a|x} \otimes N'_{b|y} |\psi'\rangle_{A'B'}\right).
\]
These equations state that, up to ancillary degrees of freedom and local basis transformations, the state \( |\psi\rangle \) is equivalent to \( |\psi'\rangle \), and the measurements \( M_{a|x} \) and \( N_{b|y} \) act on \( |\psi\rangle \) in the same way as \( M'_{a|x} \) and \( N'_{b|y} \) act on \( |\psi'\rangle \). The state \( |\xi\rangle \) is usually called the junk state.

Our main result is as follows

**Theorem 1.** Consider a Bell expression \( \mathcal{I} \) and \( \{p(a, b | x, y)\} \) for a scenario where Alice and Bob have \( m \) inputs and \( o \) outputs each, such that the value \( \mathcal{I}(\{p(a, b | x, y)\}) = \beta \) self-tests the reference experiment \( \mathcal{R} = \{\langle \psi' \rangle, M_{a|x}, N_{b|y}\} \), where \( M'_{a|x}, N'_{b|y} \) are rank-one projectors and \( p(a, b | x, y) > 0 \) \( \forall \ a, b \) for each combination \( (x, y) \) appearing in \( \mathcal{I} \). Consider the scenario of Fig. 1b with \( m \) inputs and \( o \) outputs per party. Then, there exists a collection of \( n \) non-linear Bell expressions \( J^i \) \( (i = 1, \ldots, n) \) for this scenario that self-test the reference experiment \( R_n = \{\langle \psi' \rangle \otimes \otimes_{i=1}^n M_{a|x}, \otimes_{i=1}^n N_{b|y}\} \).

The nonlinear Bell expressions \( J^i \) in Theorem 1 are constructed as follows. Define \( J^i = \mathcal{I}(\{p(a_1, b_1 | x, y, a_{i-1}, b_{i-1})\}) \) and the conditional Bell expressions for the pair \( i > 1 \) as
\[
J^i_{a_{i-1}b_{i-1}} = \mathcal{I}(\{p(a_1, b_1 | x, y, a_{i-1}, b_{i-1})\}),
\]
where \( a_{i-1} = (a_1, a_2, \ldots, a_{i-1}) \) and \( b_{i-1} = (b_1, b_2, \ldots, b_{i-1}) \). \( J^i_{a_{i-1}b_{i-1}} \) gives the value of \( \mathcal{I} \) for the pair \( i \) conditioned on observing the particular values \( a_{i-1} = (a_1, a_2, \ldots, a_{i-1}) \) and \( b_{i-1} = (b_1, b_2, \ldots, b_{i-1}) \). Note that in order for these conditional Bell expressions to be well defined we require that \( p(a_{i-1}, b_{i-1} | x, y) > 0 \) \( \forall \ a_{i-1}, b_{i-1} \) for each combination \( (x, y) \) appearing in \( \mathcal{I} \). This is automatically the case for the reference experiment \( R_n \) due to the properties of \( R \). The Bell expression \( J^i \) is defined as
\[
J^i = \frac{1}{o^{2(i-1)}} \sum_{a_{i-1}b_{i-1}} T^i_{a_{i-1}b_{i-1}},
\]
for \( i > 1 \) and \( J^1 = I^1 \). The observation that \( J^i = \beta \forall i = 1, \ldots, n \) self tests \( R_n \). The proof is inductive (given a self-test of \( k \) copies, the value of \( J^{k+1} \) self-tests an additional copy) and is given in the Appendix A.

A direct consequence of Theorem 1 is the possibility of self-testing \( n \) copies of the two-qubit maximally entangled state \( |\phi^+\rangle \) (itself a maximally entangled state of local dimension \( 2^n \)) with only two measurement settings per party via the CHSH Bell inequality. More precisely

**Corollary 1.** Let \( M_{a|x}, N_{b|y} (x, y \in \{0, 1\}, a, b \in \{\pm 1\}) \) be the local measurements that lead to the maximal violation of the CHSH Bell inequality when applied to \( |\phi^+\rangle \).
Then the correlations obtained by performing the experiment \( R = \{|\phi^+\rangle, M_{a|x}, N_{b|y}\} \) in the parallel scheme of Fig. 1c self-test the reference experiment \( \{|\phi^+\rangle \otimes \otimes_{i=1}^n M_{a|x}, \otimes_{i=1}^n N_{b|y}\} \).

In particular, this means that an unbounded amount of entanglement can be certified in a device-independent manner with the minimum number of local measurements possible.

Although Theorem 1 holds only for the case of perfect statistics, one can investigate the robustness to noise of Corollary 1 for the case \( n = 2 \), via the technique proposed in [BNS+15, YVB+14]. The precise noise model we consider is one in which each copy of the state is subject to the same level of white noise. That is, the observed correlations are generated using the same measurement strategy on the state \( \rho_{\nu} \otimes \rho_{\nu} \) where
\[
\rho_{\nu} = \nu |\phi^+\rangle \langle \phi^+| + (1 - \nu) \mathbb{I}/4.
\]

In [BNS+15, YVB+14] a semi-definite program is given that calculates a value \( f \), such that for any state \( \rho_{AB} \) leading to the observed correlations, there exists a local transformation mapping \( \rho_{AB} \) to a state that has fidelity at least \( f \) with the reference state (in this case \( |\phi^+\rangle \otimes |\phi^+\rangle \)). In the case of perfect self-testing (3), one has \( f = 1 \), which then decreases as a function of the noise parameter. Fig. 2 shows the values of \( f \) obtained as a function of \( \nu \) for this noise model.

**Extensions** – Although Theorem 1 is defined for bipartite Bell inequalities, and equal number of inputs for Alice and Bob, it can be generalized to more general scenarios. In what follows we discuss some possible extensions.

1. **Parallel self-testing protocols from full statistics** – While Theorem 1 refers to the self-tests based on the maximal violation of some Bell inequality, it is worth noting that similar claims can be made for self-testing protocols based on the observation of a particular set of correlations. The main requirement is that the reference correlations cannot involve any probability equal to zero. An example of such protocol is given in the Appendix B.

2. **Combining self-testing protocols** – Theorem 1 gives a recipe to build a new protocol self-testing the state \( |\psi_i\rangle \) starting from a given protocol to self-test the state \( |\psi_i\rangle \). In fact, our construction can also be used to generate self-testing protocols for a tensor product of different states \( \otimes_{i=1}^n |\psi_i\rangle \), provided that for each \( i \) there exists a protocol that self-tests the state \( |\psi_i\rangle \). Assume that every \( |\psi_i\rangle \) is self-tested through the maximal value of a Bell expression \( \mathcal{I}_i \). The protocol for testing \( \otimes_{i=1}^n |\psi_i\rangle \) consists of using a different Bell expression \( \mathcal{I}_i \) for each \( i \) when defining the different conditional Bell expressions (4). An important constraint is that individual self-tests must be compatible, i.e. they must be characterised by the same number of inputs. A possible combined protocol is the self-test of a tensor product of different partially entangled pairs.
of qubits through the use of different tilted CHSH inequalities [BP15]. For more details see Appendix C.

(3) All self-testing protocols can be parallelized – In this section we discuss conditions for parallel self-testing without aiming to keep the number of inputs constant. As mentioned in the introduction, there exist several parallel self-tests with the total number of inputs scaling exponentially with the number of copies to be self-tested. Such self-tests are built for (partially) entangled pairs of qubits based on the (tilted) CHSH inequality [Col17, McK17], maximally entangled pairs of qubits based on the magic square game [Col17, CN16] or magic pentagram game [KM18] and GHZ states based on the Mermin inequality [BKM19]. One interesting question is thus whether any self-testing protocol for a state $|\psi\rangle$ can be ‘parallelized’ to self-test the tensor product $|\psi\rangle^\otimes n$ (without caring about the total number of inputs). We are able to give a positive answer to even a more general problem of self-testing tensor product $\otimes_{i=1}^n |\psi_i\rangle$ given that there are self-tests for the individual states $|\psi_i\rangle$.

**Theorem 2.** Consider a set of $n$ bipartite Bell expressions $\{I_i\}$ characterised by $n_1$ inputs and $a_i$ outputs, respectively, such that the value $I_i(\{p(a_i, b_i|x_i, y_i)\}) = \beta_i$ self-tests the reference experiment $R_{\beta_i} = \{\psi_i\}, M_{a_i|x_i}, N_{b_i|y_i}\},$ where $M_{a_i|x_i}, N_{b_i|y_i}$ are rank-one projective. For each $i$. Then, the correlations obtained by performing the $R_{\beta_i}$’s in parallel as in Fig. 1b self-test the reference experiment $R_{\beta_i} =$

$\{\otimes_{i=1}^n |\psi_i\rangle, \otimes_j M_{a_j|x_j}, \otimes_i N_{b_i|y_i}\}.$

Note that this theorem does not make any constraints on the reference probabilities appearing in the individual Bell expressions. The result is proven by constructing a single Bell expression whose maximal value self-tests $R_{\beta_i}$. For each pair $i$ define the Bell value conditioned on particular choice of other inputs $x_j, y_j, j \neq i$:

$$I^i_{x(i), y(i)} = \sum a_{(i)} b_{(i)} I_i(\rho(x, y))$$

where $a_{(i)} = \{a_1, \ldots, a_{i-1}, a_{i+1}, \ldots, a_n\}$ and similarly for $b_{(i)}$, $x_{(i)}$ and $y_{(i)}$. The Bell expression $I^i_{x(i), y(i)}$ averaged over all other inputs $x_{(i)}$ and $y_{(i)}$ is

$$J^i = \frac{1}{\prod_{j \neq i} M_{a_j|x_j}} \sum x_{(i)} y_{(i)} I^i_{x(i), y(i)}.$$

If $J^i$ achieves its maximal value, it means that for for every $x_i$ and $y_i$ $T^i_{x(i), y(i)}$ also achieves the maximal value - a crucial step towards proving the self-testing statement. The proof of the theorem is given in Appendix D. While all previous results on parallel self-testing also discussed robustness we omit it here for brevity. If one would want to make the above result robust, the standard techniques used, for example in [McK17] or [Col17] can be used.

**Application: unbounded randomness expansion.** – Self-testing is intrinsically related to device-independent randomness expansion. This is because, once we certify that the system is in the state $|\psi\rangle$, we can also conclude that any external system is uncorrelated to it. Thus, an external observer can not predict the outcomes of the measurements applied to the system of interest, i.e. the outcomes are random. In particular, if the state is maximally entangled of local dimension $d$, and the measurements applied to it are rank-one projective, the amount of random bits obtained per round is $\log_2(d)$. Notice, however, that in a Bell test some initial amount of randomness must be consumed in the choice of inputs. Thus, a typically used figure of merit used to certify the efficiency of the randomness generation protocol is the the trade-off between the initial randomness consumed and the final randomness obtained. Our procedure applied to the CHSH inequality and self-testing $n$ copies of a maximally entangled state shows that the best trade-off can be achieved, where only one bit of randomness is used to generate $\log_2(d)$ bits per round.

**Discussion** – In this manuscript we introduced a new procedure useful in parallel self-testing. It allows to certify highly entangled quantum states in a black-box scenario with a constant number of measurements. Such certification schemes are important in protocols for randomness expansion: a small amount of randomness can be expanded to a string of unbounded length. At the heart of our construction lies an interesting insight: independent Bell violations can be used to ensure independence of sources even when the measurement schemes are perfectly correlated. There are several directions for future research on the topic. It would be interesting to explore how tolerant to noise our scheme is. More specifically one might check if the techniques from [NV17, NV18]
can be used to make robustness bounds of our protocol independent on the dimension of the self-tested state while still keeping number of inputs constant. Furthermore, the condition for self-testing can be seen as the maximal violation of a non-linear Bell inequality. One might try to understand if this can be achieved using a single linear Bell inequality.

Note—While working on this project we became aware of the work [SSKA19] exploring self-testing of quantum systems of arbitrary local dimension with minimal number of measurements.

Acknowledgements—We thank Jean-Daniel Bancal for sharing numerical data and to Jed Kaniewski and Yeong-Cherng Liang for useful comments and suggestions. IS acknowledges funding from SNSF (grant DIAQ and BRIDGE project "Self-testing QRNG"). DC acknowledges a Ramon y Cajal fellowship, Spanish MINECO (Severo Ochoa SEV-2015-0522), Fundació Privada Cellex and Generalitat de Catalunya (CERCA Program). JB acknowledges funding from Juan de la Cierva-formación and the AXA chair in quantum information science.

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## Appendices

### Appendix A: Proof of Theorem 1

Before starting the proofs let us introduce some useful notation. The probabilities to observe outcomes \( a = (a_1, \cdots, a_n) \) and \( b = (b_1, \cdots, b_n) \) when the inputs are \( x \) and \( y \) are given by the Born rule:

\[
p(a, b|x, y) = \text{tr}[M_{a|x} \otimes N_{b|y} \rho^{ABP}].
\]

\[ (A1) \]

Let us introduce further auxiliary measurement operators

\[
M_{a|x}^{(i)} = \sum_a M_{a|x} a = a \quad N_{b|y}^{(i)} = \sum_b N_{b|y} b = b.
\]

\[ (A2) \]

With \( M_{a|x}^{(i)} \) we simply denote the reference single qubit measurements. Since we extract the tensor product of the reference state into the ancillary Hilbert space, to ease keeping track of the number of the extracted reference states we denote the Hilbert spaces of ancillary systems with \( A_j \) and \( B_j \) (instead of \( A' \) and \( B' \) in the main text). In order to relax the notation when writing measurement operators we omit the Hilbert space notation. Thus, we employ the following notation: \( M_{a|x}^{(i)} \equiv M_{a|x}^{(i)A} \), \( M_{a|x} \equiv M_{a|x}^A \), \( N_{b|y}^{(i)} \equiv N_{b|y}^{(i)B} \), \( N_{b|y} \equiv N_{b|y}^B \) for physical measurements and \( M_{a|x}^{A_j} \equiv M_{a|x}^{A_j} \), \( N_{b|y}^{A_j} \equiv N_{b|y}^{A_j} \) for reference measurements.

We prove the theorem by using mathematical induction. In the first step we prove the base case, \( i.e. \) that the theorem holds when \( n = 1 \). In the second step we prove the so-called inductive step, saying that if the theorem holds for some natural number
The whole theorem is proven by demonstrating the correctness of the base and the inductive step. The validity of the base step holds trivially since the theorem assumes that the Bell inequality under consideration is self-testing. The condition \( L_1 = \beta \) implies the existence of the local unitary \( U_1 = U_{AA_1} \otimes U_{BB_1} \otimes I_P \) such that

\[
U_1|\psi^{ABP}\rangle \otimes |00\rangle^{A_1B_1} = |\xi_i^{ABP}\rangle \otimes |\psi_{x}^{A_1B_1}\rangle,
\]

(A3)

\[
U_1M_{a|x}^{(1)} \otimes N_{b|y}^{(1)}|\psi^{ABP}\rangle \otimes |00\rangle^{A_1B_1} = |\xi_i^{ABP}\rangle \otimes \left(M_{a|x}^{A_1} \otimes N_{b|y}^{B_1}\right)|\psi_{x}^{A_1B_1}\rangle.
\]

(A4)

To start the inductive step assume that the theorem holds for \( i - 1 \), i.e. that \( \sum_{a_{i-1}b_{i-1}} T_{a_{i-1}b_{i-1}}^i = \sigma^{2(i-1)} \beta \) implies there exist the local unitary \( U_i = U_{AA_1 \ldots A_i} \otimes U_{BB_1 \ldots B_i} \otimes I_P \) such that

\[
U_i|\psi^{ABP}\rangle \otimes |00\rangle^{A_1B_1} \ldots \otimes |00\rangle^{A_iB_i} = |\xi_i^{ABP}\rangle \otimes |\psi_{x}^{A_1B_1} \ldots \otimes A_iB_i\rangle,
\]

(A5)

\[
U_iM_{a|x} \otimes N_{b|y}|\psi^{ABP}\rangle \otimes |00\rangle^{A_1B_1} \ldots \otimes |00\rangle^{A_iB_i} = |\xi_i^{ABP}\rangle \otimes \bigotimes_{j=1}^{i} M_{a_j|x}^{A_j} \otimes N_{b_j|y}^{B_j}|\psi_{x}^{A_1B_1} \ldots \otimes A_iB_i\rangle.
\]

(A6)

By summing (A6) over \( b_i \) and using the completeness relation \( \sum b_i N_{b_i|y} = I \) we obtain

\[
U_iM_{a|x}|\psi^{ABP}\rangle \otimes |00\rangle^{A_1B_1} \ldots \otimes |00\rangle^{A_iB_i} = |\xi_i^{ABP}\rangle \otimes \bigotimes_{j=1}^{i} M_{a_j|x}^{A_j} \otimes I_{B_j}|\psi_{x}^{A_1B_1} \ldots \otimes A_iB_i\rangle,
\]

which can be rewritten as

\[
\left(U_{AA_1 \ldots A_i}M_{a|x} \otimes I_{A_1 \ldots A_{i-1}}U_{AA_1 \ldots A_i}^\dagger \right)U_i|\psi^{AB}\rangle \otimes |00\rangle^{A_1B_1} \ldots \otimes |00\rangle^{A_iB_i} =
\]

\[
= |\xi_i^{ABP}\rangle \otimes \bigotimes_{j=1}^{i} M_{a_j|x}^{A_j} \otimes I_{B_j}|\psi_{x}^{A_1B_1} \ldots \otimes A_iB_i\rangle.
\]

(A7)

By comparing (A7) and (A5) we obtain

\[
U_{AA_1 \ldots A_i}M_{a|x} \otimes I^{A_1 \ldots A_{i-1}}U_{AA_1 \ldots A_i}^\dagger = S_{a|x}^{A} \otimes M_{a|x}^{A_{1} \ldots A_{i}}
\]

(A8)

where \( S_{a|x}^{A} = |\xi_i\rangle \) for all \( a_i \) and \( x \). Note that A8 is correct in case \( \text{tr}_{B_1 \ldots B_i} |\psi_{x}\rangle \langle \psi_{x}| \) is full rank. Since \( U_{AA_1 \ldots A_i} \) preserves the identity the condition

\[
\sum_{a_i} S_{a|x}^{A} \otimes M_{a|x}^{A_{1} \ldots A_{i}} = I^{A} \otimes I_{A_{1} \ldots A_{i}}
\]

(A9)

must be satisfied. Since \( \sum_{a_i} M_{a|x}^{A_{1}} = I \), the condition is satisfied if and only if \( S_{a|x}^{A} = I \) for all \( a_i \) and \( x \). This can be seen through the simple reasoning. \( U_{i,A} \) is unitary and thus \( I \geq S_{a|x}^{A} \otimes M_{a|x}^{A_{1} \ldots A_{i}} \geq 0 \). Hence, for arbitrary quantum states \( \rho^{A} \) and \( \tau^{A_1 \ldots A_i} \) it holds \( \text{tr}(S_{a|x}^{A} \rho) = a_i \) and \( \text{tr}(M_{a|x}^{A_{1} \ldots A_{i}} \tau) = b_i \) where \( 0 \leq a_i, b_i \leq 1 \). Eq. (A9) implies \( \sum a_i b_i = 1 \) and the completeness of \( M_{a|x}^{A_{1}} \) implies \( \sum b_i = 1 \). Thus, it also holds \( \sum_i (1-a_i)b_i = 0 \). The sum of nonnegative numbers is equal to zero if and only if each of them is equal to zero. Since the argumentation must hold for all states it implies \( S_{a|x}^{A} = I \) for all \( a_i \) and \( x \). Eq. (A8) reduces to

\[
U_{AA_1 \ldots A_i}M_{a|x} \otimes I^{A_1 \ldots A_{i-1}}U_{AA_1 \ldots A_i}^\dagger = I^{A} \otimes M_{a|x}^{A_{1} \ldots A_{i}}
\]

(A10)

for all \( a_i \) and \( x \). Analogous conclusion can be obtained for Bob’s operators

\[
U_{BB_1 \ldots B_i}N_{b|x} \otimes I^{B_1 \ldots B_{i-1}}U_{BB_1 \ldots B_i}^\dagger = I^{B} \otimes N_{b|x}^{B_{1} \ldots B_{i}}
\]

(A11)

for all \( b_i \) and \( y \). Furthermore, since \( M_{a|x} = M_{a,0|x} + M_{a,1|x} \) and similarly for Bob, eq. (A10) and (A11) imply:

\[
U_{AA_1 \ldots A_i}M_{a|x}^{A} \otimes I^{A_{1} \ldots A_{i-1}}U_{AA_1 \ldots A_i}^\dagger = K_{a,0|x}^{A} \otimes M_{a|x}^{A_{1} \ldots A_{i}}
\]

(A12)

\[
U_{BB_1 \ldots B_i}N_{b|x}^{B} \otimes I^{B_1 \ldots B_{i-1}}U_{BB_1 \ldots B_i}^\dagger = K_{b,0|x}^{B} \otimes N_{b|x}^{B_{1} \ldots B_{i}}
\]

(A13)
where operators $K_{a_i, a_{i+1} | x}$ and $L_{b_i, b_{i+1}}$ satisfy

$$\sum_{a_{i+1}} K_{a_i, a_{i+1} | x} = \mathbb{1}, \quad \sum_{b_{i+1}} L_{b_i, b_{i+1} | y} = \mathbb{1}$$

(A14)

Let us now write down the expression for the conditional Bell value

$$\mathcal{T}^{i+1}_{a_i, b_i} = \sum_{a_{i+1}, b_{i+1} | x y} \frac{b^{xy}_{a_{i+1}, b_{i+1}}}{p(a_i, b_i | x y)} \text{tr}
\left[\left(M'_{a_{i+1} | x} \otimes N'_{b_{i+1} | y}\right) \rho_{AB} \left(M_{a_i | x} \otimes N_{b_i | y}\right) \otimes |00\rangle\langle 00| \right] A_1 B_1 \cdots A_i B_i.$$

(A15)

Given the self-testing statement (A6) it can be rewritten in the following way

$$\mathcal{T}^{i+1}_{a_i, b_i} = \sum_{a_{i+1}, b_{i+1} | x y} \frac{b^{xy}_{a_{i+1}, b_{i+1}}}{p(a_i, b_i | x y)} \text{tr}
\left[\left(M_{a_i | x} \otimes N_{b_i | y}\right) U_i^\dagger \mathcal{K}_{a_i} \mathcal{K}_{i+1} \mathcal{ABP} \mathcal{U}_i \right].$$

(A16)

The unitary operators $U_i$ can be cyclically shifted to obtain

$$\mathcal{T}^{i+1}_{a_i, b_i} = \sum_{a_{i+1}, b_{i+1} | x y} \frac{b^{xy}_{a_{i+1}, b_{i+1}}}{p(a_i, b_i | x y)} \text{tr}
\left[\left(M_{a_i | x} \otimes N_{b_i | y}\right) \otimes \mathbb{1} A_1 B_1 \cdots A_i B_i U_i^\dagger \right. \times
\left. \bigotimes_{j=1}^i \left(\mathcal{K}_{a_i} \mathcal{K}_{i+1} \mathcal{ABP} \mathcal{U}_i \right) \right].$$

(A17)

The expression can be further simplified

$$\mathcal{T}^{i+1}_{a_i, b_i} = \sum_{a_{i+1}, b_{i+1} | x y} \frac{b^{xy}_{a_{i+1}, b_{i+1}}}{p(a_i, b_i | x y)} \text{tr}
\left[\left(M_{a_i | x} \otimes N_{b_i | y}\right) \otimes \mathbb{1} A_1 B_1 \cdots A_i B_i U_i^\dagger \right. \times
\left. \bigotimes_{j=1}^i \left(\mathcal{K}_{a_i} \mathcal{K}_{i+1} \mathcal{ABP} \mathcal{U}_i \right) \right].$$

(A19)

where we just used the definition of $M_{a_i | x}$ and $N_{b_i | y}$ to obtain the first equality, the relations (A12) to obtain the second equality and the orthogonality of projective measurements to obtain the third one. The last two equalities come from the property of trace $\text{tr}(A \otimes B) = \text{tr}(A) \text{tr}(B)$ and observing that $p(a_i, b_i | x y) = \prod_{j=1}^i \text{tr} \left[\left(M'_{a_j | x} \otimes N'_{b_j | y}\right) |\psi\rangle\langle \psi'| A_j B_j \right]$. Let us now sum

$$\sum_{a_{i+1}, b_{i+1}} \frac{b^{xy}_{a_{i+1}, b_{i+1}}}{p(a_i, b_i | x y)} \text{tr}
\left[\left(M_{a_i | x} \otimes N_{b_i | y}\right) \otimes \mathbb{1} A_1 B_1 \cdots A_i B_i U_i^\dagger \right. \times
\left. \bigotimes_{j=1}^i \left(\mathcal{K}_{a_i} \mathcal{K}_{i+1} \mathcal{ABP} \mathcal{U}_i \right) \right].$$

(A20)

$$\sum_{a_{i+1}, b_{i+1}} \frac{b^{xy}_{a_{i+1}, b_{i+1}}}{p(a_i, b_i | x y)} \text{tr}
\left[\left(M_{a_i | x} \otimes N_{b_i | y}\right) \otimes \mathbb{1} A_1 B_1 \cdots A_i B_i U_i^\dagger \right. \times
\left. \bigotimes_{j=1}^i \left(\mathcal{K}_{a_i} \mathcal{K}_{i+1} \mathcal{ABP} \mathcal{U}_i \right) \right].$$

(A21)

$$\sum_{a_{i+1}, b_{i+1}} \frac{b^{xy}_{a_{i+1}, b_{i+1}}}{p(a_i, b_i | x y)} \text{tr}
\left[\left(M_{a_i | x} \otimes N_{b_i | y}\right) \otimes \mathbb{1} A_1 B_1 \cdots A_i B_i U_i^\dagger \right. \times
\left. \bigotimes_{j=1}^i \left(\mathcal{K}_{a_i} \mathcal{K}_{i+1} \mathcal{ABP} \mathcal{U}_i \right) \right].$$

(A22)

$$\sum_{a_{i+1}, b_{i+1}} \frac{b^{xy}_{a_{i+1}, b_{i+1}}}{p(a_i, b_i | x y)} \text{tr}
\left[\left(M_{a_i | x} \otimes N_{b_i | y}\right) \otimes \mathbb{1} A_1 B_1 \cdots A_i B_i U_i^\dagger \right. \times
\left. \bigotimes_{j=1}^i \left(\mathcal{K}_{a_i} \mathcal{K}_{i+1} \mathcal{ABP} \mathcal{U}_i \right) \right].$$

(A23)
different conditional Bell values

\[
\sum_{a,b_i} T^{i+1}_{a,b_i} = \sum_{a_i,b_i+1} b^{xy}_{a_i+1,b_i} \text{tr} \left[ (K_{a_i,a_{i+1}x} \otimes L_{b_i,b_i+1|y}) |\xi_i\rangle \langle \xi_i|^{\text{ABP}} \right]
\]

\[
= o^2 \sum_{a_i+1,b_i+1xy} b^{xy}_{a_i+1,b_i+1} \text{tr} \left[ (K^{(i+1)}_{a_i+1x} \otimes L^{(i+1)}_{b_i+1|y}) |\xi_i\rangle \langle \xi_i|^{\text{ABP}} \right]
\]

where we introduced new operators

\[
o^{(i+1)} K^{(i+1)}_{a_i+1x} = \sum_{a_i} K_{a_i,a_{i+1}x}, \quad o^{(i+1)} L^{(i+1)}_{b_i+1|y} = \sum_{b_i} L_{b_i,b_{i+1}|y}.
\]

Note that \(K^{(i+1)}_{a_{i+1}x}\) and \(L^{(i+1)}_{b_{i+1}|y}\) are positive and satisfy the completeness relations \(\sum_{a_i} K^{(i+1)}_{a_{i+1}x} = \mathbb{1}\) and \(\sum_{b_i} L^{(i+1)}_{b_{i+1}|y} = \mathbb{1}\) (see eq. (A14) ). Hence they represent valid quantum measurements. The condition from the theorem imposes \(\sum_{a,b_i} T^{i+1}_{a,b_i} = o^2 \beta\), or equivalently

\[
\sum_{a_i+1,b_i+1xy} b^{xy}_{a_i+1,b_i+1} \text{tr} \left[ (K^{(i+1)}_{a_i+1x} \otimes L^{(i+1)}_{b_i+1|y}) |\xi_i\rangle \langle \xi_i|^{\text{ABP}} \right] = \beta.
\]

Since the Bell inequality is self-testing the reference experiment the eq. (A26) implies the existence of the local unitary transformation \(U'_{i+1} = U'_{AA_{i+1}} \otimes U'_{BB_{i+1}} \otimes \mathbb{1}_P\) such that

\[
U'_{i+1} \left[ |\psi\rangle^{\text{ABP}} \otimes |00\rangle_{A_i+1B_{i+1}} \right] = |\xi_{i+1}\rangle^{\text{ABP}} \cdot |\psi\rangle^{A_i+1B_{i+1}}
\]

\[
U'_{i+1} \left[ K^{(i+1)}_{a_{i+1}x} \otimes L^{(i+1)}_{b_{i+1}|y} |\xi_{i+1}\rangle^{\text{ABP}} \otimes |00\rangle_{A_i+1B_{i+1}} \right] = |\xi_{i+1}\rangle^{\text{ABP}} \otimes \left( M'_{a_i|x} \otimes N'_{b_i|y} \otimes |\psi\rangle^{A_{i+1}B_{i+1}} \right)
\]

Combining eqs (A27)-(A28) with (A3)-(A4) leads to the parallel self-testing of \(\otimes_{j=1}^{i+1} |\psi\rangle\):

\[
U'_{i+1} \left[ U_i \left[ |\psi\rangle^{\text{ABP}} \otimes |00\rangle_{A_1B_1} \right] \otimes |00\rangle_{A_i+1B_{i+1}} \right] = |\xi_{i+1}\rangle^{\text{ABP}} \otimes \bigotimes_{j=1}^{i+1} |\psi\rangle^{A_jB_j}
\]

\[
U'_{i+1} \left[ U_i \left[ M_{a_i|x} \otimes N_{b_i|y} |\psi\rangle^{\text{ABP}} \otimes |00\rangle_{A_1B_1} \right] \otimes |00\rangle_{A_{i+1}B_{i+1}} \right] = |\xi_{i+1}\rangle^{\text{ABP}} \otimes \left( M'_{a_i|x} \otimes N'_{b_i|y} \otimes |\psi\rangle^{A_{i+1}B_{i+1}} \right)
\]

The eq. (A30) is not yet the one present in the theorem. Let us consider the expression from the theorem

\[
U'_{i+1} \left[ U_i \left[ M_{a_{i+1}|x} \otimes N_{b_{i+1}|y} |\psi\rangle^{\text{ABP}} \otimes |00\rangle_{A_1B_1} \right] \otimes |00\rangle_{A_{i+1}B_{i+1}} \right] =
\]

\[
= U'_{i+1} \left[ \left( K^{(i+1)}_{a_i+1x} \otimes L^{(i+1)}_{b_i+1|y} |\xi_{i+1}\rangle^{\text{ABP}} \right) \otimes \left( M'_{a_i|x} \otimes N'_{b_i|y} \otimes |\psi\rangle^{A_{i+1}B_{i+1}} \right) \right]
\]

Eq. (A28) implies \(U'_{AA_{i+1}} \left[ K^{(i+1)}_{a_{i+1}x} \otimes \mathbb{1}^{A_{i+1}} \right] U'_{AA_{i+1}} = \mathbb{1}^{A} \otimes M'_{a_{i+1}x}^{A_{i+1}}\). Given eq. (A12), this implies

\(U'_{AA_{i+1}} \left[ K^{(i+1)}_{a_{i+1}x} \otimes \mathbb{1}^{A_{i+1}} \right] U'_{AA_{i+1}} = K^{A}_{a_{i+1}x} \otimes M'_{a_{i+1}x}^{A_{i+1}}\). The eq. (A30) can be rewritten as

\[
U'_{i+1} \circ U_i \left[ M_{a_i|x} \otimes N_{b_i|y} |\psi\rangle^{\text{ABP}} \otimes |00\rangle_{A_1B_1} \right] = U'_{i+1} \circ U_i \left[ \left( \sum_{a_{i+1}} M_{a_{i+1}|x} \otimes \sum_{b_{i+1}} N_{b_{i+1}|y} |\psi\rangle^{\text{ABP}} \right) \otimes |00\rangle_{A_1B_1} \right]
\]

\[
= U'_{i+1} \left[ \left( \sum_{a_{i+1}} K_{a_{i+1}x} \otimes \sum_{b_{i+1}} L_{b_{i+1}|y} |\xi_{i+1}\rangle^{\text{ABP}} \right) \otimes \left( M'_{a_i|x} \otimes N'_{b_i|y} \otimes |\psi\rangle^{A_{i+1}B_{i+1}} \right) \right]
\]

\[
= \sum_{a_{i+1},b_{i+1}} \left( K_{a_{i+1}x} \otimes L_{b_{i+1}|y} |\xi_{i+1}\rangle^{\text{ABP}} \right) \otimes \left( M'_{a_i|x} \otimes N'_{b_i|y} \otimes |\psi\rangle^{A_{i+1}B_{i+1}} \right)
\]
This equation is equivalent to (A30) if and only if $\bar{0}$ isometry. We omit the self-testing proof here and direct reader’s attention to \[WWS16\]. The isometry used in the proof is the Swap for $n$ to the Theorem 1 can be applied to build a self-testing protocol for a tensor product of measurement choices, certifying inequality violation.

Note that one might show that correlations (B1)-(B2) maximally violate some Bell inequality and the procedure corresponding to this can be used to build a self-testing protocol for a tensor product of $n$ maximally entangled qubit pairs. The reference measurement observables are

\[
A'_i = \sigma_z, \quad A'_i = \sigma_x, \\
B'_i = \cos \gamma \sigma_z + \sin \gamma \sigma_x, \quad B'_i = \cos \delta \sigma_z - \sin \delta \sigma_x.
\]

We omit the self-testing proof here and direct reader’s attention to \[WWS16\]. The isometry used in the proof is the Swap isometry $U$ and physical experiment reproducing correlations (B1)-(B2) satisfies the following equations

\[
U \left( |\psi\rangle^{\text{ABP}} \otimes |00\rangle^{A_iB_i} \right) = |\xi\rangle^{\text{ABP}} \otimes |\phi^+\rangle^{A_iB_i},
\]

\[
U \left( (M_{a|z} \otimes N_{b|y}) |\psi\rangle^{\text{ABP}} \otimes |00\rangle^{A_iB_i} \right) = |\xi\rangle^{\text{ABP}} \otimes (M'_{a|z} \otimes N'_{b|y}) |\phi^+\rangle^{A_iB_i}.
\]

Note that one might show that correlations (B1)-(B2) maximally violate some Bell inequality and the procedure corresponding to the Theorem 1 can be applied to build a self-testing protocol for a tensor product of $n$ maximally entangled qubit pairs. However, we still find it useful to show how to deal with a self-testing protocol based on the reproduction of the whole set of correlations. In (2, 2, 2) case one can use standard methods to find the Bell inequality maximally violated by some extremal point of the set of quantum correlations (for example NPA hierarchy methods), but in more complicated scenarios this might not be an easy task. Furthermore, most of the known self-testing protocols for multipartite states are not based on the maximal Bell inequality violation.

In what follows we show how to use the above given self-test to build the another self-test, using the same number of measurement choices, certifying $|\phi^+\rangle \otimes |\phi^+\rangle$. Let us introduce the following notation

\[
A_x^{(k)} = \sum_a (-1)^{a_k} M_{a|x}, \quad B_y^{(k)} = \sum_b (-1)^{b_k} N_{a|y}, \quad \text{for } k = 1, 2.
\]

The condition for self-testing in this case is reproduction of the following correlations:

\[
\langle \psi | A_0^{(1)} \otimes B_0^{(1)} | \psi \rangle = \cos \gamma, \quad \langle \psi | A_0^{(1)} \otimes B_1^{(1)} | \psi \rangle = -\cos \delta, \quad \text{for } a, b
\]

\[
\langle \psi | A_1^{(1)} \otimes B_0^{(1)} | \psi \rangle = \sin \gamma, \quad \langle \psi | A_1^{(1)} \otimes B_1^{(1)} | \psi \rangle = \sin \delta.
\]

We have proved the inductive step, and with it completed the theorem proof.

Appendix B: Example: parallel self-testing beyond Bell inequalities

In this section we give example of lifting the self-testing protocol which is based not on the maximal violation of a Bell inequality but reproduction of the whole set of correlations. For the sake of simplicity we chose the self-testing protocol in the simplest (2, 2, 2) scenario. The self-testing correlations are

\[
\begin{align*}
\langle \psi | A_0 \otimes B_0 | \psi \rangle &= \cos \gamma, \\
\langle \psi | A_0 \otimes B_1 | \psi \rangle &= -\cos \delta, \\
\langle \psi | A_1 \otimes B_0 | \psi \rangle &= \sin \gamma, \\
\langle \psi | A_1 \otimes B_1 | \psi \rangle &= \sin \delta,
\end{align*}
\]

for $\gamma \neq \delta$ and $\gamma, \delta \in (0, \pi/4]$. $A_i$ and $B_j$ are observables defined as $A_i = M_{a|i} - M_{1|i}$ and $B_i = N_{0|i} - N_{1|i}$. In \[WWS16\] it is proven that this set of correlations self-tests the maximally entangled pair of qubits. The reference measurement observables are

\[
A'_i = \sigma_z, \quad A'_i = \sigma_x, \\
B'_i = \cos \gamma \sigma_z + \sin \gamma \sigma_x, \quad B'_i = \cos \delta \sigma_z - \sin \delta \sigma_x.
\]

We omit the self-testing proof here and direct reader’s attention to \[WWS16\]. The isometry used in the proof is the Swap isometry $U$ and physical experiment reproducing correlations (B1)-(B2) satisfies the following equations

\[
U \left( |\psi\rangle^{\text{ABP}} \otimes |00\rangle^{A_iB_i} \right) = |\xi\rangle^{\text{ABP}} \otimes |\phi^+\rangle^{A_iB_i},
\]

\[
U \left( (M_{a|z} \otimes N_{b|y}) |\psi\rangle^{\text{ABP}} \otimes |00\rangle^{A_iB_i} \right) = |\xi\rangle^{\text{ABP}} \otimes (M'_{a|z} \otimes N'_{b|y}) |\phi^+\rangle^{A_iB_i}.
\]

Note that one might show that correlations (B1)-(B2) maximally violate some Bell inequality and the procedure corresponding to the Theorem 1 can be applied to build a self-testing protocol for a tensor product of $n$ maximally entangled qubit pairs. However, we still find it useful to show how to deal with a self-testing protocol based on the reproduction of the whole set of correlations. In (2, 2, 2) case one can use standard methods to find the Bell inequality maximally violated by some extremal point of the set of quantum correlations (for example NPA hierarchy methods), but in more complicated scenarios this might not be an easy task. Furthermore, most of the known self-testing protocols for multipartite states are not based on the maximal Bell inequality violation.

In what follows we show how to use the above given self-test to build the another self-test, using the same number of measurement choices, certifying $|\phi^+\rangle \otimes |\phi^+\rangle$. Let us introduce the following notation

\[
A_x^{(k)} = \sum_a (-1)^{a_k} M_{a|x}, \quad B_y^{(k)} = \sum_b (-1)^{b_k} N_{b|y}, \quad \text{for } k = 1, 2.
\]
In terms just of the observed probabilities the set of conditions (B6)-(B11) can be written as

\[ p(a_1 = b_1|00) - p(a_1 \neq b_1|00) = \cos \gamma, \quad p(a_1 = b_1|01) - p(a_1 \neq b_1|01) = -\cos \delta, \]  
\[ p(a_1 = b_1|10) - p(a_1 \neq b_1|10) = \sin \gamma, \quad p(a_1 = b_1|11) - p(a_1 \neq b_1|11) = \cos \delta, \]  
\[ (p(a_2 = b_2|x = 0, y = 0, a_1 = a, b_1 = b) - p(a_2 \neq b_2|x = 0, y = 0, a_1 = a, b_1 = b)) = \cos \gamma \]  
\[ (p(a_2 = b_2|x = 0, y = 1, a_1 = a, b_1 = b) - p(a_2 \neq b_2|x = 0, y = 1, a_1 = a, b_1 = b)) = -\cos \delta \]  
\[ (p(a_2 = b_2|x = 1, y = 0, a_1 = a, b_1 = b) - p(a_2 \neq b_2|x = 1, y = 0, a_1 = a, b_1 = b)) = \sin \gamma \]  
\[ (p(a_2 = b_2|x = 1, y = 1, a_1 = a, b_1 = b) - p(a_2 \neq b_2|x = 1, y = 1, a_1 = a, b_1 = b)) = \sin \delta \]  

The proof goes along the same line as the proof of Theorem 1. Equations (B6)-(B7) imply the existence of the isometry

\[ U_1 = U_{AA_1} \otimes U_{BB_1} \otimes \Phi_P \]  

such that

\[ U_1 \left( |\psi\rangle^{ABP} \otimes |00\rangle^{A_1B_1} \right) = |\xi_1\rangle^{ABP} \otimes |\phi^+\rangle^{A_1B_1}, \]  
\[ U_1 \left( (M_{a|x}^{(1)} \otimes N_{b|y}^{(1)})^{ABP} \otimes |00\rangle^{A_1B_1} \right) = |\xi_1\rangle^{ABP} \otimes (M_{a|x}^{(1)} \otimes N_{b|y}^{(1)})^{A_1B_1}. \]  

These two equations imply the following set of equations

\[ U_{AA_1} \left( M_{a|x}^{(1)} \otimes \Phi_A^{(1)} \right) U_{AA_1}^\dagger = \Phi_A^{(1)} \otimes M_{a|x}^{(1)}, \quad U_{BB_1} \left( N_{b|y}^{(1)} \otimes \Phi_B^{(1)} \right) U_{BB_1}^\dagger = \Phi_B^{(1)} \otimes N_{b|y}^{(1)}, \]  
\[ U_{AA_1} \left( M_{a|x}^{(1)} \otimes \Phi_A^{(1)} \right) U_{AA_1}^\dagger = K_{a_1,a_2|x} \otimes M_{a_1|x}^{(1)}, \quad U_{BB_1} \left( N_{b|y}^{(1)} \otimes \Phi_B^{(1)} \right) U_{BB_1}^\dagger = L_{b_1,b_2|y} \otimes N_{b_1|y}^{(1)}, \]  

where the operators \( K_{a_1,a_2|x}, L_{b_1,b_2|y} \) are positive semidefinite and satisfy

\[ \sum_{a_2} K_{a_1,a_2|x} = \mathbb{I}, \quad \sum_{b_2} L_{b_1,b_2|y} = \mathbb{I} \]  

Given all these equations the first expression from (B8) can be rewritten as

\[ \frac{1}{p(a_1 = a, b_1 = b|00)} \text{tr} \left( (A_0^{(2)} \otimes B_0^{(2)})(M_{a|0}^{(1)} \otimes N_{b|0}^{(1)})^{ABP} \langle \psi | \phi^{ABP} \rangle \right) = \]  
\[ = \frac{1}{p(a_1 = a, b_1 = b|00)} \text{tr} \left( U \left( A_0^{(2)} \otimes B_0^{(2)} \right) U^\dagger |\xi_1\rangle^{ABP} \otimes (M_{a|0}^{(1)} \otimes N_{b|0}^{(1)})^{A_1B_1} \right) \]  
\[ = \frac{1}{p_1(a_1 = a, b_1 = b|00)} \sum_{a_2,b_2} (-1)^{a_2+b_2} \text{tr} \left( \left( K_{a_2,a_2|0} \otimes L_{b_2,b_2|0} \right) |\xi_1\rangle^{ABP} \otimes (M_{a_1|0}^{(1)} \otimes N_{b_1|0}^{(1)})^{A_1B_1} \right) \]  
\[ = \frac{1}{p_1(a_1 = a, b_1 = b|00)} \sum_{a_2,b_2} (-1)^{a_2+b_2} \text{tr} \left( \left( K_{a_2,a_2|0} \otimes L_{b_2,b_2|0} \right) |\xi_1\rangle^{ABP} \right) \text{tr} \left( (M_{a_1|0}^{(1)} \otimes N_{b_1|0}^{(1)})^{A_1B_1} \right) \]  
\[ = \sum_{a_2,b_2} (-1)^{a_2+b_2} \text{tr} \left( \left( K_{a_2,a_2|0} \otimes L_{b_2,b_2|0} \right) |\xi_1\rangle^{ABP} \right) = \cos \gamma. \]  

This equation holds for all \( a, b \in 0, 1 \). Anologous equations can be obtained starting from the other three relations from (B8)-(B10):

\[ \sum_{a_2,b_2} (-1)^{a_2+b_2} \text{tr} \left( \left( K_{a_2,a_2|0} \otimes L_{b_2,b_2|1} \right) |\xi_1\rangle^{ABP} \right) = -\cos \delta \]  
\[ \sum_{a_2,b_2} (-1)^{a_2+b_2} \text{tr} \left( \left( K_{a_2,a_2|1} \otimes L_{b_2,b_2|0} \right) |\xi_1\rangle^{ABP} \right) = \sin \gamma \]  
\[ \sum_{a_2,b_2} (-1)^{a_2+b_2} \text{tr} \left( \left( K_{a_2,a_2|1} \otimes L_{b_2,b_2|1} \right) |\xi_1\rangle^{ABP} \right) = \sin \delta. \]  

Let us now introduce new operators

\[ 2\tilde{A}_x = \sum_{a_1,a_2} (-1)^{a_2} K_{a_1,a_2|x}, \quad 2\tilde{B}_y = \sum_{b_1,b_2} (-1)^{b_2} L_{b_1,b_2|y} \]  

(B29)
Let us introduce the generalized conditional Bell value: 

\[ \beta_{U_{a}S} \]

corresponding to the scenarios with different inputs and/or output size. The general scenario is as follows: the protocol 

\[ U_{2} = U_{AA_{2}} \otimes U_{BB_{2}} \otimes I_{P} \]

such that

\[ U_{2} \left[ |\xi_{1}^{ABP} \otimes |00\rangle^{A_{2}B_{2}} \right] = |\xi_{1}^{ABP} \otimes |\phi^{+}\rangle^{A_{2}B_{2}}. \]

Combining this equation with (B19) we obtain

\[ U_{2} \circ U_{1} \left[ |\psi_{i}^{ABP} \otimes |0000\rangle^{A_{1}A_{2}B_{1}B_{2}} \right] = |\xi_{i}^{ABP} \otimes |\phi^{+}\rangle^{A_{1}B_{1}} \otimes |\phi^{+}\rangle^{A_{2}B_{2}}. \]

The self-testing proof goes along the same lines as the proof for Theorem 1. The only difference is that for every 

\[ \sum_{a,b} \beta_{i} \equiv \sum_{a,b} \sum_{x,y=0} b_{a,b}^{xy} p(ab|xy) \leq \beta_{i} \]

Let us introduce the generalized conditional Bell value:

\[ \mathcal{T}_{i+1}^{a_{i},b_{i}} = \sum_{a,b} \sum_{x,y=0} b_{a,b}^{xy} p(ab|xy) \left( \mathcal{M}_{i+1}^{a_{i}|x} \otimes \mathcal{N}_{i+1}^{a_{i}|x} \right) \left( \mathcal{M}_{b_{i}|y}^{a_{i}|x} \otimes \mathcal{N}_{b_{i}|y}^{a_{i}|x} \right) \]

The conditions for self-testing \( \otimes_{i} |\psi_{i}^{i}\rangle \) by using only \( m \) different inputs per party are the following

- \( \mathcal{T}_{i} = \beta_{i} \)
- \( \sum_{a,b} \sum_{x,y=0} b_{a,b}^{xy} p(ab|xy) \leq \beta_{i+1} \)

The self-testing proof goes along the same lines as the proof for Theorem 1. The only difference is that for every \( i \) the second condition corresponds to the maximal violation of \( \mathcal{T}_{i} \) by the junk state appearing in the self-testing statement for \( i - 1 \).

Appendix C: Combining self-testing protocols to test a tensor product of different quantum states

In the main text we introduced a notion of compatible self-testing protocols, as those using the same number of inputs to self-tests the corresponding reference states. Here we outline how one can build a self-testing protocol certifying a tensor product of \( n \) different states, which can independently self-tested by using compatible self-testing protocols.

Namely, the aim is to self-test a state of the form \( \otimes_{i} |\psi_{i}^{i}\rangle \), knowing that every \( |\psi_{i}^{i}\rangle \) can be self-tested through observing the maximal violation \( \beta_{i} \) of the inequality

\[ \mathcal{T}_{i} = \sum_{a_{i},b_{i}} b_{a_{i},b_{i}}^{xy} p(a_{i},b_{i}|xy) \leq \beta_{i} \]

The conditions for self-testing \( \otimes_{i} |\psi_{i}^{i}\rangle \) by using only \( m \) different inputs per party are the following

- \( \mathcal{T}_{i} = \beta_{i} \)
- \( \sum_{a,b} \sum_{x,y=0} b_{a,b}^{xy} p(ab|xy) \leq \beta_{i+1} \)

The self-testing proof goes along the same lines as the proof for Theorem 1. The only difference is that for every \( i \) the second condition corresponds to the maximal violation of \( \mathcal{T}_{i} \) by the junk state appearing in the self-testing statement for \( i - 1 \).

Appendix D: Proof of Theorem 2

In this section we give a proof of Theorem 2 which deals with combining two or more incompatible self-tests, i.e. self-tests corresponding to the scenarios with different inputs and/or output size. The general scenario is as follows: the protocol \( S_{i} \) can be used to self-test the state \( |\psi_{i}^{i}\rangle \) in the scenario where each party has \( m_{i} \) inputs denoted with \( x_{i}, y_{i} \) and \( a_{i} \) outputs denoted as \( a_{i}, b_{i} \). If the correlations \( p(a_{i},b_{i}|x_{i},y_{i}) \) satisfy the self-testing conditions given by the protocol \( S_{i} \) for each \( i \) then there is isometry mapping the physical state to the \( \otimes_{i} |\psi_{i}^{i}\rangle \). This can be seen as a generalization of parallel self-testing, where usually the reference state is an \( n \)-fold tensor product of some state.

According to the theorem conditions the state \( |\psi_{i}^{i}\rangle \), for \( i \in \{1, \cdots, n\} \), is self-tested through achieving the maximal violation of the Bell inequality \( \mathcal{T}_{i} \), evaluated as

\[ \mathcal{T}_{i} = \sum_{a_{i},b_{i}|x_{i},y_{i}} b_{a_{i},b_{i}}^{xy} p(a_{i},b_{i}|x_{i},y_{i}) = \beta_{i}. \]

The self-testing scenario is as follows: in every round Alice and Bob receive \( n \) classical inputs each, denoted with \( x = \{x_{1}, \cdots, x_{n}\}, y = \{y_{1}, \cdots, y_{n}\} \), where \( x_{i}, y_{i} \in \{0, 1, \cdots, m_{i} - 1\} \). They return strings \( a = \{a_{1}, \cdots, a_{n}\}, b = \{b_{1}, \cdots, b_{n}\} \).
where \( a_i, b_i \in \{0, 1, \ldots , a_i - 1\} \). The measurement operators are denoted by \( M_{a|x} \) for Alice and \( N_{b|y} \) for Bob. Let us introduce the following notation
\[
M_{a|x} = \prod_{j \neq i} m_i \sum_{a_{(i)} \cdot x_{(i)}} M_{a|x}, \quad N_{b|y} = \prod_{j \neq i} m_i \sum_{b_{(i)} \cdot y_{(i)}} N_{b|y}.
\]

These operators are valid measurement operators, as a sum of positive operators they are positive and they satisfy completeness relations \( \sum_a M_{a|x} = \mathbb{1} \) and \( \sum_b N_{b|y} = \mathbb{1} \), for all \( x_i \) and \( y_j \). Let us define Bell-like expressions
\[
T^i_{(x_i),y_{(i)}} = \sum_{a,b,x,y} b^{a,y}_{a,b} p(ab|x,y)
\]

The Bell violation for the \( i \)-th inequality is a normalized sum of values given in D3:
\[
J^i = \prod_{j \neq i} m_j \sum_{x_{(i)},y_{(i)}} T^i_{(x_{(i)}),y_{(i)}}.
\]

Let us set \( \beta = 1 \). The Bell violation \( J^1 \) can be modelled as
\[
J^1 = \sum_{a_1,b_1,x_1,y_1} b^{x_1,y_1}_{a_1,b_1} \langle \psi | M_{a_1|x_1} \otimes N_{b_1|y_1} | \psi \rangle.
\]

If the inequality \( J^1 \) is maximally violated there exist a local unitary \( U_1 = U_{AA_1} \otimes U_{BB_1} \otimes \mathbb{1}_P \) such that
\[
U_1 \left[ |\psi\rangle A^{BP} \otimes |00\rangle A_1B_1 \right] = |\xi_1\rangle A^{BP} \otimes |\psi_1\rangle A_1B_1.
\]

This implies the following relations
\[
U_{AA_1} \left[ M_{a_1|x_1} \otimes \mathbb{1}_A^{A_1} \right] U_{AA_1}^\dagger = \mathbb{1}^A \otimes M_{a_1|x_1}^{A_1}, \quad U_{BB_1} \left[ N_{b_1|y_1} \otimes \mathbb{1}_B^{B_1} \right] U_{BB_1}^\dagger = \mathbb{1}^B \otimes N_{b_1|y_1}^{B_1}.
\]

where
\[
M_{a_1,a_2|x_1,x_2} = \prod_{m=3}^m m_m \sum_{x_3,\ldots, x_m} M_{a|x}, \quad N_{b_1,b_2|y_1,y_2} = \prod_{m=3}^m m_m \sum_{y_3,\ldots, y_m} N_{b|y}.
\]

In the second step the maximal violation of the inequality \( J^2 \) can be modelled as
\[
J^2 = \frac{1}{m^2_1} \sum_{a_1,a_2,b_1,b_2} b^{x_2,y_2}_{a_2,b_2} \langle \psi | M_{a_1,a_2|x_1,x_2} \otimes N_{b_1,b_2|y_1,y_2} | \psi \rangle.
\]

\[
= \frac{1}{m^2_1} \sum_{a_1,a_2,b_1,b_2} b^{x_2,y_2}_{a_2,b_2} \langle \xi_1 | K_{a_1,a_2|x_1,x_2} \otimes L_{b_1,b_2|y_1,y_2} | \xi_1 \rangle \langle \psi' | M_{a_1|x_1} \otimes N_{b_1|y_1} | \psi' \rangle.
\]

\[
= \frac{1}{m^2_1} \sum_{a_1,b_1,x_1,y_1} \langle \psi' | M_{a_1|x_1} \otimes N_{b_1|y_1} | \psi' \rangle \sum_{a_2,b_2} b^{x_2,y_2}_{a_2,b_2} \langle \xi_1 | K_{a_1,a_2|x_1,x_2} \otimes L_{b_1,b_2|y_1,y_2} | \xi_1 \rangle
\]

\[
= \sum_{a_1,b_1,x_1,y_1} \frac{p(a_1,b_1|x_1,y_1)}{m^2_1} \sum_{a_2,b_2} b^{x_2,y_2}_{a_2,b_2} \langle \xi_1 | K_{a_1,a_2|x_1,x_2} \otimes L_{b_1,b_2|y_1,y_2} | \xi_1 \rangle.
\]
Since numbers \( p(a_1 b_1 | x_1 y_1 ) / m^2 \) are positive and sum to one, and all \( K_{a_1,a_2|x_1,x_2} \) and \( L_{b_1,b_2|y_1,y_2} \) are valid measurement operators the observation \( J^2 = \beta_2 \) implies that

\[
\sum_{a_2,b_2,x_2,y_2} b_{a_2,b_2}^2 \langle \xi_1 | K_{a_1,a_2|x_1,x_2} \otimes L_{b_1,b_2|y_1,y_2} | \xi_1 \rangle = \beta_2
\]  

(D16)

for all \( a_1, b_1, x_1, y_1 \). Let us define operators

\[
K^{(2)}_{a_2|x_2} = \frac{1}{m_1} \sum_{a_1,x_1} K_{a_1,a_2|x_1,x_2}, \quad L^{(2)}_{b_2|y_2} = \frac{1}{m_1} \sum_{b_1,y_1} L_{b_1,b_2|y_1,y_2}
\]

(D17)

These operators are valid measurement operators, and they satisfy

\[
\sum_{a_2,b_2,x_2,y_2} b_{a_2,b_2}^2 \langle \xi_1 | K^{(2)}_{a_2|x_2} \otimes L^{(2)}_{b_2|y_2} | \xi_1 \rangle = \beta_2.
\]

(D18)

The maximal violation \( J^2 \) implies that there exists a local unitary

\[
U_2 \left[ |\xi_1\rangle^{AB} \otimes |00\rangle^{A_2B_2} \right] = |\xi_2\rangle^{AB} \otimes |\psi_2\rangle^{A_2B_2}
\]

(D19)

\[
U_2 \left[ (K^{(2)}_{a_2|x_2} \otimes L^{(2)}_{b_2|y_2} |\xi_1\rangle^{AB} \otimes |00\rangle^{A_2B_2} \right] = |\xi_2\rangle^{AB} \otimes (M'_{a_2|x_2} \otimes N'_{b_2|y_2} |\psi_2\rangle^{A_2B_2}).
\]

(D20)

Combining (D6) and (D19) we get a self-testing statement for a tensor product of two different states

\[
U_2 \circ U_1 \left[ |\psi\rangle^{ABP} \otimes |0000\rangle^{A_1A_2B_1B_2} \right] = |\xi_2\rangle^{ABP} \otimes |\psi_1\rangle^{A_1B_1} \otimes |\psi_2\rangle^{A_2B_2}.
\]

(D21)

The process can be further repeated for \( i = 3 \) to \( i = n \), reaching the final statement

\[
U_n \circ \cdots \circ U_1 \left[ |\psi\rangle^{ABP} \otimes |0000\rangle^{A_1 \cdots A_n B_1 \cdots B_n} \right] = |\xi_n\rangle^{ABP} \otimes |\psi_1\rangle^{A_1B_1} \otimes \cdots \otimes |\psi_n\rangle^{A_nB_n}.
\]

(D22)