KERNELS OF VECTOR-VALUED TOEPLITZ OPERATORS

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Abstract. Let $S$ be the shift operator on the Hardy space $H^2$ and let $S^*$ be its adjoint. A closed subspace $F$ of $H^2$ is said to be nearly $S^*$-invariant if every element $f \in F$ with $f(0) = 0$ satisfies $S^* f \in F$. In particular, the kernels of Toeplitz operators are nearly $S^*$-invariant subspaces. Hitt gave the description of these subspaces. They are of the form $F = g(H^2 \ominus uH^2)$ with $g \in H^2$ and $u$ inner, $u(0) = 0$. A very particular fact is that the operator of multiplication by $g$ acts as an isometry on $H^2 \ominus uH^2$. Sarason obtained a characterization of the functions $g$ which act isometrically on $H^2 \ominus uH^2$. Hayashi obtained the link between the symbol $\varphi$ of a Toeplitz operator and the functions $g$ and $u$ to ensure that a given subspace $F = gK_u$ is the kernel of $T_\varphi$. Chalendar, Chevrot and Partington studied the nearly $S^*$-invariant subspaces for vector-valued functions. In this paper, we investigate the generalization of Sarason’s and Hayashi’s results in the vector-valued context.

1. Introduction

To begin this section, we present the scalar results of Hitt, Sarason and Hayashi which will be generalized throughout this paper.

We denote by $H^2$ the classical Hardy space of analytic functions on the unit disc $D$, and by $H^2(\mathbb{C}^m)$ the $\mathbb{C}^m$-vector-valued Hardy space consisting of $m$ copies of $H^2$. The shift $S$ is the operator of multiplication by the variable $z$ and $S^*$ is its adjoint. The (closed) $S^*$-invariant subspaces of $H^2$ are called model subspaces. They are of the form $K_u = H^2 \ominus uH^2$, where $u$ is an inner function.

For $\varphi \in L^\infty$, the Toeplitz operator with symbol $\varphi$ is defined by $T_\varphi f := p_+(\varphi f)$, where $p_+$ is the orthogonal projection from $L^2$ onto $H^2$.

Hitt [Hit88] introduced the nearly $S^*$-invariant subspaces:

Definition 1.1. A closed subspace $F$ of $H^2$ is said to be a nearly $S^*$-invariant subspace if every element $f \in F$ with $f(0) = 0$ satisfies $S^* f \in F$.

In particular, the kernel of a Toeplitz operator is a nearly $S^*$-invariant subspace. Hitt obtained the complete description of this spaces:

Theorem 1.2 (Hitt, 1988). Let $F$ be a non-trivial nearly $S^*$-invariant subspace. Let $g$ be the unique unit-norm function in $F$, positive at the origin, that is orthogonal to $F \cap zH^2$. Then there exists an inner function $u$ vanishing at zero such that, for
all \( f \in \mathcal{F} \), there exists a unique \( f_0 \in K_u \) and \( f = gf_0 \). Furthermore, \( \| f \|_2 = \| f_0 \|_2 \).

In other words, multiplication by \( g \) acts isometrically on \( K_u \).

Two questions arise.

(1) The first one was already posed by Sarason in [Sar88] where he made this remark: "The latter theorem leaves mysterious the relation between the function \( g \) and the space \( K_u \). Given a function \( g \) of unit norm in \( H^2 \), what are the \( S^*\)-invariant subspaces \( K_u \) that can arise with \( g \) in Hitt’s theorem?"

(2) Which nearly \( S^*\)-invariant subspaces are kernels of Toeplitz operators.

Sarason obtained the following answer to the first question:

**Theorem 1.3** (Sarason, 1988). Let \( g \) be an outer function of unit norm, and \( u \) an inner function with \( u(0) = 0 \). We define two analytic functions on the disc:

\[
\begin{align*}
f(z) &= \frac{1}{2\pi} \int_0^{2\pi} \frac{e^{i\theta} + z}{e^{i\theta} - z} |g(e^{i\theta})|^2 \, d\theta \\
b(z) &= \frac{f(z) - 1}{f(z) + 1}.
\end{align*}
\]

Then the following statements are equivalent:

(1) multiplication by \( g \) acts isometrically from \( K_u \) to \( \mathcal{F} \);
(2) \( bH^2 \subset uH^2 \) (i.e. \( b = ub_0 \));
(3) \( K_u \subset (1 - T_b T_b^*)^{1/2} H^2 \).

The answer to the second question is given by Hayashi in [Hay85, Hay86, Hay90] and Sarason found an alternative proof in [Sar94a]. This answer is expressed in terms of exposed points of the unit ball of \( H^1 \), also called rigid functions.

Before stating Hayashi’s result, we need some definitions. With the previous notation, let \( \mathcal{F} = gK_u \) be a nearly \( S^*\)-invariant space and let \( b \) be the function associated to \( g \) as in Theorem 1.3. Because \( \log(1 - |b|^2) \) is integrable, we can build an outer function \( a \) such that \( |a|^2 + |b|^2 = 1 \) a.e. on \( \mathbb{T} \). Then \((b, a)\) is called a corona pair (or pair) associated to \( g \). Thanks to Theorem 1.3, \( b = ub_0 \). If \( \mathcal{F} \) is the kernel of a Toeplitz operator, then \((b_0, a)\) is a corona pair associated to the outer function \( g_0 := a/(1 - b_0) \). Some pairs, called special pairs, verify an additional property which will be precisely defined in section 5. Admitting this, we can reformulate Hayashi’s result as follows (see also [Sar94a]):

**Theorem 1.4** (Hayashi, 1985). The subspace \( \mathcal{F} = gK_u \) is the kernel of a Toeplitz operator if and only if the pair \((b_0, a)\) is special and \( g_0^2 \) is rigid.

We would like to generalize the previous theorems to vector-valued functions. The paper is organized as follows. In section 2 we define the vector- or matrix-valued objects: we recall the inner-outer matricial factorization, we comment on the generalization of Theorem 1.2, and we recall the definition of de Branges–Rovnyak spaces, the vector-valued analogue of \( \mathcal{H}(b) := (1 - T_b T_b^*)^{1/2} H^2 \) appearing in Theorem 1.3.

In section 3 we transcribe Sarason’s approach to the vectorial case. We build the analogue of the functions \( b \) and \( u \). Thanks to de Branges–Rovnyak spaces, we obtain the matricial version of Theorem 1.3. The matrices do not commute, so we need to modify the original scalar proof given by Sarason. An example illustrates this kind of problem.

In section 4 we would like to describe the kernels of Toeplitz operators. We begin with some examples. This allows us to illustrate the difficulties due to the
dimension, and to establish some notation. We then investigate the descriptions of kernels of Toeplitz operators of finite dimension.

Finally, in section 3, we obtain the full description of the kernels of Toeplitz operators. We establish the desired generalization of Hayashi’s Theorem.

2. HARDY SPACES OF VECTOR-VALUED FUNCTIONS

2.1. Inner-outer factorization. As usual with Hardy spaces, we identify a function with its radial limits.

Let \( F, G \) be two subspaces of \( \mathbb{C}^m \) of dimension \( r \). Nikolskii, in [Nik02] page 14, calls \( \Theta \in H^\infty(F \to G) \) an inner function if its boundary values \( \Theta(\xi) \) are surjective isometries for a.e. \( \xi \in \mathbb{T} \).

It will be more convenient to say that \( \Theta \in H^\infty(\mathbb{C}^m \to \mathbb{C}^m) \) is an inner function if its boundary values \( \Theta(\xi) \) are partial isometries for a.e. \( \xi \in \mathbb{T} \), with kernel and range independent of \( \xi \) a.e. in \( \mathbb{T} \). In other words, an inner function is a square-matrix-valued function such that there exist two subspaces \( F, G \) of \( \mathbb{C}^m \) with the same dimension \( r \) for which \( \Theta|_{F} \in H^\infty(F \to G) \) is an inner function in the sense of Nikolskii. The rank of \( \Theta(\xi) \) is equal to \( r \) for a.e. \( \xi \in \mathbb{T} \).

Here are two examples of inner functions of rank 2. The first one will be discussed later (see Theorem 2.7). Let \( \theta \) be an inner scalar function and \( a, b \in K_{a, \theta} \) verifying \( |a|^2 + |b|^2 = 1 \) a.e. on \( \mathbb{T} \). Define \( \varphi \in H^\infty(\mathbb{C}^2 \to \mathbb{C}^2) \) and \( \Theta \in H^\infty(\mathbb{C}^3 \to \mathbb{C}^3) \) by the following formulae:

\[
F = \begin{pmatrix} \mathbb{C} & \mathbb{C} \\ \mathbb{C} & 0 \end{pmatrix}, \quad G := \begin{pmatrix} \mathbb{C} \\ 0 \end{pmatrix}, \quad \varphi := \begin{pmatrix} a & -b \\ \theta \bar{b} & \theta \bar{a} \end{pmatrix} \quad \text{and} \quad \Theta := \begin{pmatrix} a & 0 & -b \\ \theta \bar{b} & 0 & \theta \bar{a} \\ 0 & 0 & 0 \end{pmatrix}.
\]

Both \( \varphi, \Theta \) are inner of rank 2. Note that \( \varphi \in H^\infty(\mathbb{C}^m \to \mathbb{C}^m) \) is inner of rank \( m \) if and only if \( \det \varphi \) is inner.

Recall the Beurling–Lax theorem [Lax59]: If a closed subspace \( M \subset H^2(\mathbb{C}^m) \) is invariant by the shift, then there exists an inner function \( \Theta \) such that \( M = \Theta H^2(\mathbb{C}^m) \). This description is unique up to multiplication by an unitary matrix.

Next, we recall the notion of outer vector-valued function. The outer scalar functions are cyclic vectors for the shift. For \( g \in H^2(\mathbb{C}^m) \), we define \( \mathcal{G} \), the smallest \( S \)-invariant subspace containing \( g \), by \( \mathcal{G} := \text{span}(S^k g : k \in \mathbb{N}) \). Thanks to Beurling–Lax theorem, there exists \( \Theta \), inner with of 1, such that \( \mathcal{G} = \Theta H^2(\mathbb{C}^m) \). We say that \( g \) is outer if \( \Theta \) is a constant matrix. Then \( \mathcal{G} = H^2(\Theta(0)\mathbb{C}^m) \). It will be useful to write \( \mathcal{G} := \Theta(0)\mathbb{C}^m \). Finally, the function \( g \) is a cyclic vector for \( S \) in \( H^2(\mathcal{G}) \).

We extend this construction to define the outer matrix-valued functions. Let \( g_1, \ldots, g_r \in H^2(\mathbb{C}^m) \), with \( r \leq m \), be a independent family of vector-valued functions. Let \( G \in H^2(\mathbb{C}^r \to \mathbb{C}^m) \) be the rectangular matrix-valued functions where the columns are \( (g_t)_{t \leq r} \). In this case we write \( G = [g_1, \ldots, g_r] \). It is said to be outer if \( \mathcal{G} := \text{span}(S^k g_t : \ell \leq r, k \in \mathbb{N}) = \Theta H^2(\mathbb{C}^m) \), where \( \Theta \) is a constant partial isometry of rank \( r \). Then, we will write \( \mathcal{G} = \Theta(0)\mathbb{C}^m, \dim G = r, \) and \( \mathcal{G} = H^2(\mathcal{G}) \). Due to the rank theorem, there exists an unitary mapping \( \Theta_0 : \mathbb{C}^r \to \mathcal{G} \). To \( G \), we associate \( \tilde{G} \in H^2(\mathbb{C}^r \to \mathbb{C}^r) \) such that \( G := \Theta_0 \tilde{G} \). This allows us to translate the properties of square-matrix-valued functions to rectangular ones.

For more details about inner-outer factorization of square matrix-valued functions with determinant different from zero, see [KK97]. In particular the Definition
5.3 in [KK97] of Beurling left outer function coincides with that of outer given above. The Smirnov–Nevanlinna class $\mathcal{N}^+(\mathbb{C}^m \to \mathbb{C}^m)$ of square matrix-valued functions is the set of all matrices with entries in the scalar Smirnov–Nevanlinna class. The Definition 3.1 in [KK97] of outer function in $\mathcal{N}^+(\mathbb{C}^m \to \mathbb{C}^m)$ is that $E$ is outer if $\det E$ is outer in $\mathcal{N}^+$. The authors shows that all definitions of outer functions are equivalent in $H^2(\mathbb{C}^m \to \mathbb{C}^m)$. Theorem 5.4 in [KK97] says that, given a function $F$ in $\mathcal{N}^+(\mathbb{C}^m \to \mathbb{C}^m)$, $\det F(z) \neq 0$, there exist functions $F_i$ inner and $F_o$ outer (resp. $F'_i, F'_o$), unique up to a unitary matrix, such that $F = F_i F_o$ (resp. $F = F'_i F'_o$). Furthermore, Theorem 3.1 of [KK97] will be useful later: Let $E \in \mathcal{N}^+(\mathbb{C}^m \to \mathbb{C}^m)$ an outer square-matrix-valued function. Then $\det(z) \neq 0$ for all $z \in \mathbb{D}$ and $E^{-1} \in \mathcal{N}^+(\mathbb{C}^m \to \mathbb{C}^m)$.

2.2. Nearly $S^*$-invariant subspaces of $H^2(\mathbb{C}^m)$. The next result is the description of the nearly $S^*$-invariant subspaces of $H^2(\mathbb{C}^m)$. For more details, see [CCP].

**Theorem 2.1.** Let $\mathcal{F} \subset H^2(\mathbb{C}^m)$ be a non-trivial nearly $S^*$-invariant subspace. Let $(g_1, \ldots, g_r)$ be an orthonormal basis of

$$W := \mathcal{F} \cap \{ \mathcal{F} \cap zH^2(\mathbb{C}^m) \}^\perp.$$

Then $r := \dim W \leq m$ and there exist an integer $r'$, $1 \leq r' \leq r$, and $U \in H^\infty(\mathbb{C}^r \to \mathbb{C}^r)$ inner, rank $U = r'$, such that

$$\mathcal{F} = \{ g_1, \ldots, g_r \} \left( H^2(\mathbb{C}^r) \ominus U H^2(\mathbb{C}^r) \right) = GK_U.$$

For all $f \in \mathcal{F}$, there exists an unique $f_0 \in K_U$ such that $f = G f_0$. Furthermore, $\| f_0 \|_{H^2(\mathbb{C}^r)} = \| f \|_{H^2(\mathbb{C}^m)}$.

Because the columns of $G$ form an orthonormal basis of $W$, the norm of $G \in H^2(\mathbb{C}^r \to \mathbb{C}^m)$ is 1. For any $h \in H^2(\mathbb{C}^r)$, we define $TGh$ to be the Fourier projection of the $L^1(\mathbb{C}^m)$ function $Gh$ on $H^2(\mathbb{C}^m)$. It is an unbounded operator, but, as in the scalar case, it is an isometry on $K_U$.

2.3. De Branges–Rovnyak spaces. Now, we will recall the definition and the main properties of de Branges–Rovnyak spaces. For more details, see the first chapter of [Sar94b]. Let $H_1$ and $H$ be two Hilbert spaces and $B \in \mathcal{L}(H_1, H)$ be a bounded operator. We define $\mathcal{M}(B)$ to be the range space $BH_1$ with the inner product that makes $B$ a coisometry on $H$:

$$\forall f, g \in H_1 \cap (\ker B)^\perp, \quad \langle Bf, Bg \rangle_{\mathcal{M}(B)} := \langle f, g \rangle_{H_1}.$$

For a contraction $B$, the inclusion is a contraction from $\mathcal{M}(B)$ to $H$. The complementary space $\mathcal{H}(B)$ is defined to be $\mathcal{M}((Id_H - BB^*)^{1/2})$. In the particular case where $B$ is the multiplication by an inner function $b$, then $\mathcal{M}(B) = BH^2(\mathbb{C}^m)$ and $\mathcal{H}(B) = K_B$. In this case, the inner products of $\mathcal{M}(B)$ and $\mathcal{H}(B)$ coincide with the $H^2$ inner product and these two spaces are really complementary spaces in the $H^2$ sense. In this article, $H$ and $H_1$ will be Hardy spaces like $H^2(\mathbb{C}^m)$ or closed subspaces of $H^2(\mathbb{C}^m)$ isometrically equivalent to $H^2(\mathbb{C}^r)$, and $B$ will be the multiplication by a matrix $B$ in the unit ball of $H^\infty(\mathbb{C}^r \to \mathbb{C}^m)$.

The reproducing kernels in $H^2(\mathbb{C}^m)$ are $k_\lambda u := \frac{1}{1 - \lambda z} u$ for $\lambda \in \mathbb{D}$ and $u \in \mathbb{C}^m$. Thus, for all $f \in H^2(\mathbb{C}^m)$, the reproducing kernels verify

$$\langle f, k_\lambda u \rangle_2 = \langle f(\lambda), u \rangle_{\mathbb{C}^m}.$$
Because the inclusion from $\mathcal{H}(B)$ to $H^2$ is contractive, de Branges–Rovnyak spaces have kernel functions, and a simple calculation shows that
\[ k^B_\lambda u := \frac{I_{d_r} - B(z)B(\lambda)^*}{1 - \lambda z} u \quad \text{and} \quad \langle f, k^B_\lambda u \rangle_{\mathcal{H}(B)} = \langle f(\lambda), u \rangle_{\mathcal{C}^m}. \]
Given a symbol $B$, we write $M(B)$ (resp. $\mathcal{H}(B)$) instead of $M(T_B)$ (resp. $\mathcal{H}(T_B)$).

3. Toeplitz operators acting as an isometry on a model space

In this section, we verify that the tools used by Sarason [Sar88] can be applied to matrix-valued functions.

3.1. A matricial intertwining. Let $(g_\ell)_{\ell \leq r}$ be an orthogonal basis of $W$ and let $G \in H^2(\mathbb{C}^r \rightarrow \mathbb{C}^m)$ be the matrix-valued function $[g_1, \ldots, g_r]$.
We denote by $H^2(\mathbb{C}^m, \mu_G)$ the Hardy space of vector-valued functions with the norm
\[ \|q\|^2_{H^2(\mathbb{C}^m, \mu_G)} := \frac{1}{2\pi} \int_0^{2\pi} \|G(e^{i\theta})q\|^2_{\mathbb{F}^m} d\theta. \]
Remember that $\mathcal{G} = \text{span}(S^k g_\ell : \ell \leq r, k \geq 0)$. Let $f = Gq$ be in $H^2(\mathbb{C}^m)$. This forces $q$ to be in $H^2(\mathbb{C}^m, \mu_G)$.
We would like to build from $G$ the functions $F$ and $B$, the analogues of those appearing in Theorem 1.3. After, we will show that $T_{I_{d_r} - B}T_{B^*}$ is an isometry from $\mathcal{G}$ to $\mathcal{H}(B)$, or equivalently, $Gq$ is an isometry from $H^2(\mathbb{C}^m, \mu_G)$ to $\mathcal{H}(B)$. As a consequence, we will obtain the following equality, the key of the proof of the generalization of Theorem 1.3:
\[ I_{d_r} - T_B T_{B^*} = (T_{I_{d_r} - B} T_{G^*}) (T_{I_{d_r} - B} T_{G^*})^*. \]
We begin by defining $F$ the analytic function on the disc by
\[ \forall z \in \mathbb{D}, \quad F(z) := \frac{1}{2\pi} \int_0^{2\pi} \frac{e^{i\theta} + z}{e^{i\theta} - z} G(e^{i\theta})^* G(e^{i\theta}) d\theta. \]
Note that, if $G = UG'$, where $U$ is inner of rank $m$, then the functions $F$ and $F'$ are the same. Because the $(g_\ell)_{\ell \leq r}$ form an orthogonal basis of $W$, the coefficient $F(0)_{i,j}$ is $(g_i, g_j)_{H^2}$. So, $F(0) = I_{d_r}$. For $z_0 \in \mathbb{D}$, let $u \in \mathbb{C}^r$ be an eigenvector of the matrix $F(z_0)$. Then $\Re \langle F(z_0)u, u \rangle$ is a Poisson integral, so the real parts of the eigenvalues of $F(z_0)$ are $\|G(z_0)u\|^2 \geq 0$. This implies that the moduli of the eigenvalues of $F(z_0) + I_{d_r}$ are greater than 1, so $F(z_0) + I_{d_r}$ is invertible.
Next, we define $B$, the matrix-valued Herglotz integral of $\mu_G$, by
\[ B(z) := (F(z) + I_{d_r})^{-1} (F(z) - I_{d_r}). \]
Because $F(0) = I_{d_r}$, the function $B$ vanishes in zero. For all $u \in \mathbb{C}^r$,\[ \|F(z) \pm I_{d_r})u\|^2 = \|F(z)u\|^2 + \|u\|^2 \geq 2 \Re \langle F(z)u, u \rangle. \]
Then, because $\Re \langle F(z)u, u \rangle \geq 0$,
\[ \|F(z) + I_{d_r})u\|^2 \geq \|(F(z) - I_{d_r})u\|^2 \]
and $B$ lies in the unit ball of $H^\infty(\mathbb{C}^r \rightarrow \mathbb{C}^r$). We can therefore consider $\mathcal{H}(B)$.

Lemma 3.1. For all $u, v \in \mathbb{C}^m$ we have:
\[ \langle G_k u, G_k v \rangle_{H^2} = \langle k^B_e (I_{d_r} - B(u)^*)^{-1}u, k^B_e (I_{d_r} - B(z)^*)^{-1}v \rangle_{\mathcal{H}(B)}. \]
Finally, we interpret $\langle Gk_wu, Gk_zv \rangle_{H^2}$ in terms of $F$:

$$
\langle Gk_wu, Gk_zv \rangle_{H^2} = \frac{1}{2\pi} \int_0^{2\pi} \frac{1}{(1 - \bar{w}e^{i\theta})(1 - ze^{-i\theta})} \overline{(G(e^{i\theta})u, G(e^{i\theta})v)} \, d\theta \\
= \frac{1}{2\pi(1 - \bar{w}z)} \int \frac{1}{2} \left[ \frac{e^{-i\theta} + \bar{w}}{e^{-i\theta} - \bar{w}} + \frac{e^{i\theta} + z}{e^{i\theta} - z} \right] \overline{(G(e^{i\theta})u, G(e^{i\theta})v)} \, d\theta \\
= \frac{1}{2(1 - \bar{w}z)} \langle (F(w)^* + F(z))u, v \rangle_{C^m}.
$$

Because $(Id_r + B(z))$ and $(Id_r - B(z))^{-1}$ commute,

$$
F(w)^* + F(z) = (Id_r - B(z))^{-1} (Id_r + B(z)) + (Id_r + B(w)^*)(Id_r - B(w)^*)^{-1} \\
= 2(Id_r - B(z))^{-1} [Id_r - B(z)B(w)^*] (Id_r - B(w)^*)^{-1}.
$$

Finally, we interpret $\langle Gk_wu, Gk_zv \rangle_{H^2}$ in terms of inner product of $k_w^B(z)$:

$$
\langle Gk_wu, Gk_zv \rangle_{H^2} = \frac{1}{2(1 - \bar{w}z)} \langle (F(w)^* + F(z))u, v \rangle_{C^m} \\
= \frac{1}{1 - \bar{w}z} \langle (Id_r - B(z))^{-1} [Id_r - B(z)B(w)^*] (Id_r - B(w)^*)^{-1}u, v \rangle_{C^m} \\
= \langle (Id_r - B(z))^{-1}k_w^B(z)(Id_r - B(w)^*)^{-1}u, v \rangle_{C^m} \\
= \langle k_w^B(Id_r - B(w)^*)^{-1}u, k_w^B(Id_r - B(z)^*)^{-1}v \rangle_{H(B)}.
$$

$\square$

The following lemma is useful in connection with de Branges–Rovnyak spaces ([Sar94b] I-5):

**Lemma 3.2** (Douglas's criterion). Let $H$, $H_1$ and $H_2$ be Hilbert spaces, and let $A : H_1 \to H$, $B : H_2 \to H$ be contractions. We define $\mathcal{M}(A) := AH_1$ and $\mathcal{M}(B) := BH_2$. Then $\mathcal{M}(A) = \mathcal{M}(B)$ is equivalent to $AA^* = BB^*$.

Remember that $\mathcal{G} = \text{span}(S^k g_\ell : \ell \leq r, k \in \mathbb{N}) = \text{span}(Gk_wu : u \in C^r, w \in \mathbb{D})$ and that $\Theta$ is an inner function such that $\mathcal{G} = \Theta H^2(C^m)$. (When $G$ is outer, $\Theta$ is a constant unitary-matrix)

**Lemma 3.3.**

1. For all $u \in C^m$, $T_{Id_r - B}T_{G^*}$ maps $Gk_wu$ to $k_w^B(Id_r - B(w)^*)^{-1}u$.
2. If $G$ is outer, then $T_{Id_r - B}T_{G^*}$ is an isometry from $\mathcal{G}$ onto $\mathcal{H}(B)$.
3. If $G = G_i G_o$, with $G_i$ inner and $G_o$ outer, then $T_{Id_r - B}T_{G^*}$ is a coisometry of $\mathcal{G}$ to $\mathcal{H}(B)$ with null space $K_{G_i} \cap \mathcal{G}$.
4. Define $\mathcal{M}(T_{Id_r - B}T_{G^*}) := T_{Id_r - B}T_{G^*}G$. Equipped with the inner product

$$
\langle T_{Id_r - B}T_{G^*}h_1, T_{Id_r - B}T_{G^*}h_2 \rangle := \langle h_1, h_2 \rangle_{2} \quad \forall h_1, h_2 \in \mathcal{G} \cup (\ker T_{Id_r - B}T_{G^*})^\perp,
$$

$\mathcal{M}(T_{Id_r - B}T_{G^*})$ coincides with the de Branges–Rovnyak space $\mathcal{H}(B)$.  **
Proof. (1) We begin by computing the range of $Gk_wu$ by $T_{Id_r-B}T_{G^*}$:

\[
\langle (T_{Id_r-B}T_{G^*})Gk_wu, k_z v \rangle_{H^2} = \langle (Id_r - B(z))Gk_wG(z)k_v(z)u, v \rangle_{\mathbb{C}^m} = \\
\langle T_{G^*}Gk_wu, k_z(Id_r - B^*)v \rangle_{H^2} = \\
\langle Gk_wu, Gk_z(Id_r - B(z)^*)v \rangle_{H^2} = \\
\langle k_w^B(Id_r - B(w)^{-1})u, k_z^B(Id_r - B(z)^{-1})v \rangle_{\mathcal{H}(B)} = \\
\langle k_w^B(Id_r - B(w)^{-1})u, k_zv \rangle_{H^2}.
\]

Therefore, $(T_{Id_r-B}T_{G^*})$ sends $Gk_wu$ to $k_w^B(Id_r - B(w)^{-1})u$.

(2) The inner product of two functions in $\mathcal{G}$ is equal to the inner product of their images in $\mathcal{H}(B)$:

\[
\langle T_{Id_r-B}T_{G^*}Gk_wu, T_{Id_r-B}T_{G^*}Gk_zv \rangle_{H^2} = \langle k_w^B(Id_r - B(w)^*)^{-1}u, k_z^B(Id_r - B(z)^{-1})v \rangle_{\mathcal{H}(B)} = \\
\langle Gk_wu, Gk_zv \rangle_{H^2}.
\]

Because $G$ is outer, the functions $Gk_wu$ span $\mathcal{G}$, reduced to $H^2(\mathbb{G})$, and the result follows.

(3) The definition of $B$ does not depend on $G$, so $T_{Id_r-B}T_{G^*} = T_{Id_r-B}T_{G^*}|_{\mathcal{G}}$. But $T_{Id_r-B}T_{G^*}$ sends $G_{\mathcal{G}}$ isometrically to $\mathcal{H}(B)$, which is dense in $\mathcal{G}$, so we get the result by continuation. Moreover, $\ker T_{Id_r-B}T_{G^*}|_{\mathcal{G}} = \ker T_{G^*}|_{\mathcal{G}}$, so $G = G_{\mathcal{G}}$.

(4) The last sentence allows us to identify the two de Branges–Rovnyak spaces:

\[
\mathcal{M}(T_{Id_r-B}T_{G^*}) := T_{Id_r-B}T_{G^*}\mathcal{G} = \mathcal{H}(B) = \mathcal{M}(\mathcal{H}(B) = \mathcal{M}\left((Id_r - TB_{B^*})^{1/2}\right). 
\]

\[\square\]

Theorem 3.4. As operators on $H^2(C^r)$, we have $(T_{Id_r-B}T_{G^*})(T_{Id_r-B}T_{G^*})^* = Id_r - TB_{B^*}$.

Proof. By the Douglas criterion, Lemma 3.2 implies that $\mathcal{M}(T_{Id_r-B}T_{G^*}) = \mathcal{H}(B)$ is equivalent to $(T_{Id_r-B}T_{G^*})(T_{Id_r-B}T_{G^*})^* = Id_r - TB_{B^*}$ as operators on $H^2(C^r)$.

3.2. A matricial version of Sarason’s theorem. With the previous notation,

Theorem 3.5. Let $G = [g_1, \ldots, g_r] \in H^2(C^r \to \mathbb{C}^m)$ and let $U \in \mathbb{H}^\infty(C^r \to C^r)$ be inner of rank $r$ vanishing at zero. Then

$T_G|_{K_0}$ is an isometry $\iff T_B^*K_U = \{0\} \iff BH^2(C^r) \subset UH^2(C^r)$.

The proof follows Sarason’s ideas, with modifications to bypass the fact that $T_B^*K_U$ might not be a subspace of $K_U$.

Proof. The last equivalence is obvious. Suppose that $T_B^*K_U = \{0\}$. Then $T_Gh = T_GT_{Id_r-B}h$ for all $h \in K_U$. Thanks to Lemma 3.2, $(T_{Id_r-B}T_{G^*})(T_{Id_r-B}T_{G^*})^* = Id_r - TB_{B^*}$, and we compute the norm of $\|T_Gh\|_{H^2}^2$:

\[
\|T_Gh\|_{H^2}^2 = \|T_GT_{Id_r-B}h\|_{H^2}^2 = \langle (T_{Id_r-B}T_{G^*})(T_{Id_r-B}T_{G^*})^* h, h \rangle = \\
\langle (Id_r - TB_{B^*})h, h \rangle = \langle h, h \rangle_{H^2} = \|h\|_{H^2}^2.
\]

Thus, $T_G$ acts as an isometry on $K_U$. 

Conversely, suppose that $T_G|_{K_U}$ is an isometry. Let $h \in K_U$. Lemma 3.3 and Theorem 3.3 assert that $T_{id,-B}T_G \cdot G \to \mathcal{H}(B)$ is a coisometry, and $(T_{id,-B}T_G \cdot G) (T_{id,-B}T_G \cdot G)^* = Id_B - T_B T_B^*$ on $H^2(\mathbb{C}_r)$. It follows that

$$||T_G T_{id,-B}h||^2_B = \langle (Id_B - T_B T_B^*)h, h \rangle_B^2,$$

whose development is:

$$||T_G h||^2 - \langle T_G T_B h, T_G h \rangle - \langle T_G h, T_G T_B h \rangle + ||T_G T_B h||^2 = ||h||^2 - ||T_B h||^2.$$

Using the hypothesis $||T_G h|| = ||h||$, we get

$$\langle T_G T_B h, T_G h \rangle + \langle T_G h, T_G T_B h \rangle = ||T_B h||^2 + ||T_G T_B h||^2. \tag{3.1}$$

Now, with Sarason’s trick, we will show that $T_G T_B h = 0$, for $h \in K_U$. Remember that $U(0) = 0$, so $\mathbb{C}^r \subset K_U$ and because $B(0) = 0$, we get $T_B v = 0$ for all $v \in \mathbb{C}^m$. With $c \in \mathbb{C}$ and $v \in \mathbb{C}^m$, $h + cv$ stays in $K_U$ and $T_B (h + cv) = T_B h$.

Replacing $h$ by $h + cv$ in the equality 3.1 we have:

$$\langle T_G T_B h, T_G (h + cv) \rangle + \langle T_G (h + cv), T_G T_B h \rangle = ||T_B h||^2 + ||T_G T_B h||^2.$$

This is equivalent to

$$2 \Re \langle c(T_G T_B h, T_G v) \rangle = ||T_B h||^2 + ||T_G T_B h||^2 - 2 \Re \langle T_G T_B h, T_G h \rangle. \tag{3.2}$$

This holds for all $c \in \mathbb{C}$, so necessarily

$$\Re \langle T_G T_B h, T_G h \rangle = 0 \text{ and } ||T_B h||^2 + ||T_G T_B h||^2 - 2 \Re \langle T_G T_B^* h, h \rangle = 0.$$

The first equality holds for all $v \in \mathbb{C}^m$, so $T_G T_B h(0) = 0$. Replacing $h$ by $S^k h$, which stays in $K_U$, we deduce that $T_G T_B T_B^* S^k h(0) = 0$ and so $T_G T_B^* T_B = 0$. This implies that $T_B h \in \ker T_G$ or $T_G T_B h \in \ker T_G^*$. We denote $f = T_G T_B h$.

Then, $f \in \ker T_G^* \cap \mathcal{G}$ and there exists $q \in H^2(\mathbb{C}^m, \mu_G)$ such that $f = G q$. The norm of $f$ is $||f||^2 = \langle G^* G q, q \rangle = \langle T_G, f, q \rangle = 0$. Finally, $T_B K_U \subset \ker T_G$.

The second equality of (3.2) implies the following equivalences:

$$||T_B h||^2 + ||T_G T_B h||^2 - 2 \Re \langle T_G T_B^* h, h \rangle = 0 \iff ||T_B h||^2 - ||T_G h||^2 + ||T_G T_B h||^2 - 2 \Re \langle T_G T_B h, h \rangle = 0 \iff ||T_G T_{id,-B} h||^2 = 0 \iff ||T_G T_{id,-B} h||^2 = ||T_B h||^2.$$

But we know that $T_B K_U \subset \ker T_G$, so $||T_G T_{id,-B} h||^2 = ||T_B h||^2$ and $||T_B h||^2 = 0$. So we get $T_B K_U = \{0\}$ as desired.

\[\square\]

**Corollary 3.6.** The operator $T_{id,-B} T_G^*$ acts on $\mathcal{F}$ as division by $G$.

**Proof.** Let $Gh \in \mathcal{F}$. Then, thanks to the last theorem, $T_B h = 0$. So,

$$T_{id,-B} T_G^* Gh = T_{id,-B} T_G T_{id,-B} h$$

and Lemma 3.4 implies that $(Id - T_B T_{id,-B}) h = h$. \[\square\]

In the original proof, Sarason uses the fact that scalar model spaces $K_u$ (or more generally de Branges spaces \cite{Sar94} II-7) are stable under the action of $T_b$ for every symbol $b \in H^\infty$. This does not hold for matrix symbols. The inclusion $T_B K_U \subset K_U$ means that $BUH^2(\mathbb{C}^r) \subset UH^2(\mathbb{C}^r)$, and so $U^* BU \in H^\infty(\mathbb{C}^r \to \mathbb{C}^r)$. This is obvious if $B$ and $U$ commute.
In this section, we will construct an example in $H^\infty(\mathbb{C}^2 \to \mathbb{C}^2)$ where $T_B \cdot K_U$ is not contained in $K_U$. The following characterization of $(2 \times 2)$-matrix-valued inner functions is due to Garcia, in [Gar06].

**Theorem 3.7.** Let $U \in H^\infty(\mathbb{C}^2 \to \mathbb{C}^2)$. Then $U$ is inner if and only if $U$ is of the form:

$$U = \begin{pmatrix} a & -b \\ \theta b & \theta a \end{pmatrix}$$

where $\theta := \det U$ is inner, and $a, b \in K_{z\theta}$ verify $|a|^2 + |b|^2 = 1$ a.e. on $T$.

Garcia gives an interesting example of an inner function by taking $a := (1 + \theta)/2$, $b := -i(1 - \theta)/2$ and

$$V := \frac{1}{2} \begin{pmatrix} (1 + \theta) & i(1 - \theta) \\ -i(1 - \theta) & (1 + \theta) \end{pmatrix}.$$  

Taking $\theta = z$, for example, we notice that the entries are outer scalar functions.

We can look for a nearly diagonal terms $\theta b$, and $b$ with

$$G := \begin{pmatrix} b_1 & b_2 \\ b_3 & b_4 \end{pmatrix} \in H^\infty(\mathbb{C}^2 \to \mathbb{C}^2).$$

A calculation shows that $U^*BU$ is equal to:

$$\begin{pmatrix} b_1 + b_4 + \text{Re}(\theta)(b_1 - b_4) + i\text{Im}(\theta)(b_3 + b_2) & b_2 - b_3 + \text{Re}(\theta)(b_3 + b_2) + i\text{Im}(\theta)(b_1 + b_4) \\ b_3 - b_2 + \text{Re}(\theta)(b_3 - b_2) + i\text{Im}(\theta)(b_1 + b_4) & -b_1 + b_4 + \text{Re}(\theta)(b_1 + b_4) + i\text{Im}(\theta)(b_3 - b_2) \end{pmatrix}.$$  

If we suppose that $b_4 = -b_1$ and $b_3 = -b_2$, then

$$U^*BU = \begin{pmatrix} \text{Re}(\theta)b_1 & b_2 + \text{Im}(\theta)b_1 \\ -2b_2 + \text{Im}(\theta)b_1 & -\text{Re}(\theta)b_1 \end{pmatrix},$$

which is not in $H^\infty(\mathbb{C}^2 \to \mathbb{C}^2)$ and so $T_B \cdot K_U \not\subset K_U$.

**Remark 3.8.** In [Sar88], Sarason establishes an alternative proof of Theorem 2.1 using Corollary 3.1. This approach could be generalized to the vector-valued case.

4. **Kernel of Toeplitz operators**

4.1. **Some examples.** For a nearly $S^*$-invariant subspace $F = GK_U$, we recall that $W = F \cap (F \cap zH^2(\mathbb{C}^{m}))^\perp$, and $r := \dim W \leq m$. If $m = 2$, then we have two ways to build $F$ with $\dim F = 2$.

**Example 1.** If $r = 2$, $G = [g_1, g_2]$ and $U = zI_{d_2}$. Let $F$ be the space

$$\left\{ \begin{pmatrix} a(1 + z)^{1/2} \\ b(1 - z)^{1/2} \end{pmatrix} ; (a, b) \in \mathbb{C}^2 \right\}.$$  

Because $f(0) = 0$ implies $f = 0$, it is $S^*$-nearly invariant. We see that $W = F$ and $F = T_G \mathbb{C}^2$ with

$$G(z) = \frac{1}{2} \begin{pmatrix} (1 + z)^{1/2} & 0 \\ 0 & (1 - z)^{1/2} \end{pmatrix}.$$  

The functions $\frac{1}{2}(1 + z)^{1/2}$ and $\frac{1}{2}(1 - z)^{1/2}$ are outer in $H^2$ and $G$ is outer, because its determinant is outer (see section 2.1). Moreover, $G = H^2(\mathbb{C}^2)$.

Is $F$ the kernel of a Toeplitz operator $T_\varphi$? We will build $\varphi \in L^\infty(\mathbb{C}^2 \to \mathbb{C}^2)$ as the following. Remark that $G(e^{it})^*G^{-1}(e^{it})$ is diagonal. The diagonal terms are $e^{-\frac{1}{2}it}$ and $-e^{\frac{1}{2}it}$. So, $G^*G^{-1}$ lies in $L^\infty(\mathbb{C}^2 \to \mathbb{C}^2)$ and then

$$T_{\varphi}(z) = \begin{pmatrix} z^{-\frac{i}{2}} & 0 \\ 0 & -z^{-\frac{i}{2}} \end{pmatrix} \text{ a.e. } z \in T.$$  

$$T_{\varphi}(z)G(z)^{-1} = p_+ \begin{pmatrix} z^{-\frac{i}{2}} & 0 \\ 0 & -z^{-\frac{i}{2}} \end{pmatrix} \text{ a.e. } z \in T.$$  


Every $f$ in $\mathcal{F}$ is of the form $f = Ge$, with $e \in \mathbb{C}^2$, and $T_{zG^*G^{-1}}f = p_+(zG^*e) = 0$. So $\mathcal{F}$ is the kernel of $T_{zG^*G^{-1}}$.

**Example 2.** We modify the previous example to get a nearly $S^*$-invariant subspace which is not the kernel of a Toeplitz operator. Let $\mathcal{F}$ and $G$ be defined by

$$\mathcal{F} = \left\{ \left( \begin{array}{c} a(1 + z) \\ b(1 - z) \end{array} \right) : (a, b) \in \mathbb{C}^2 \right\}$$

and $G(z) = \frac{1}{\sqrt{2}} \begin{pmatrix} 1 + z & 0 \\ 0 & 1 - z \end{pmatrix}$.

We will show that if it is the kernel of a Toeplitz operator, it is also the kernel of the Toeplitz operator with symbol $\varphi(e^{it}) := G^*(e^{it})U^*(e^{it})G^{-1}(e^{it})$. This symbol is a diagonal matrix. The diagonal terms are $e^{-2it}$ and $-1$ a.e. $e^{it} \in \mathbb{T}$. But

$$\ker T_\varphi = \left\{ \left( \begin{array}{c} a + bz \\ 0 \end{array} \right) : (a, b) \in \mathbb{C}^2 \right\} \neq \mathcal{F},$$

and so $\mathcal{F}$ fails to be the kernel of a Toeplitz operator.

**Example 3.** Let $r = 1$, $G = [g_1]$ and $\dim K_U = 2$. Let $\mathcal{F}$ be defined by

$$\mathcal{F} := \left\{ \left( \begin{array}{c} a + bz \\ 0 \end{array} \right) : (a, b) \in \mathbb{C}^2 \right\}.$$

This space is the nearly $S^*$-invariant $\mathcal{F} = GK_U$ with $G(z) = \begin{pmatrix} 1 \\ 0 \end{pmatrix}$ and $U(z) = z^2$.

With the notation defined in section [2.1] we have

$$G := \text{span}(S^*G) = \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}, \quad \Theta_0 = \begin{pmatrix} 1 \\ 0 \end{pmatrix}, \quad \text{and } G = \begin{pmatrix} 1 \\ 0 \end{pmatrix}.$$

Because $\mathcal{G} = H^2(\mathbb{G})$, the function $G$ is outer. As we saw before, $\tilde{G}$ is the $(1 \times 1)$-square matrix such that $G = \Theta_0 \tilde{G}$. Here, $\tilde{G} = 1$. Furthermore, $\mathcal{F} = \ker T_\varphi$ where

$$\varphi := \begin{pmatrix} \tilde{G}^*U^*G^{-1} & 0 \\ 0 & 1 \end{pmatrix} = \begin{pmatrix} \tilde{z}^2 & 0 \\ 0 & 1 \end{pmatrix}.$$
Let \( L : H^1(\mathbb{C}^m \to \mathbb{C}^m) \to \mathbb{C} \)

\[
H \mapsto \text{tr} \left( \frac{1}{2\pi} \int_0^{2\pi} H(e^{it})A_F(e^{it}) \, dt \right),
\]

where \( \text{tr} \) denote the trace. It is easy to verify that \( L(F) = 1 \) and that \( F \) is rigid if and only if \( L \) is unique.

Moreover, exposed points are extreme points (in the sense of convexity). The extreme points of the unit ball of \( H^1(\mathbb{C}^m \to \mathbb{C}^m) \) are the outer functions with norm 1. For more results see [Cam77] and [BW03]. Before stating the next lemma, we define precisely the norm \( \| \cdot \| \) which we shall use. Let \( (e_k)_k \) be the canonical basis of \( \mathbb{C}^m \). Then \( \| \cdot \| \) is the matricial norm defined by

\[
\| A \|^2 := \sum_{k=1}^{m} (Ae_k, Ae_k)_{\mathbb{C}^m}.
\]

**Lemma 4.2.** Let \( G \in H^2(\mathbb{C}^m \to \mathbb{C}^m) \) and \( F = T_G C^m \) be a nearly \( S^* \)-invariant subspace of \( H^2(\mathbb{C}^m \to \mathbb{C}^m) \). If \( F \) is the kernel of a Toeplitz operator, then \( G^2 \) is rigid and \( F = \ker T_{\bar{z}G^*G^{-1}} \).

**Proof.** Let \( \varphi \) be the symbol of the Toeplitz operator for which \( F \) is the kernel. Because \( G \in \ker T_{\varphi} \), there exists \( H \in H^2(\mathbb{C}^m \to \mathbb{C}^m) \) such that \( \varphi G = \bar{z}H^* \). We begin by showing that \( H \) is outer. Let \( H = VH_\alpha \) the inner-outer decomposition of \( H \). Then \( \varphi G = \bar{z}H^*_\alpha V^* \), so \( \varphi G V = \bar{z}H^*_\alpha \). Finally, \( GV \in \ker T_{\varphi} \), but \( \ker T_{\varphi} = GC^m \), so \( V \) is constant.

We would like to write \( T_G \) as the product of two Toeplitz operators, such that the first one is injective and the symbol of the second one depends only on \( G \). Because \( G \in H^2(\mathbb{T}, C^m \to C^m) \) is outer, \( \det G(z) \) is a outer scalar function in \( N^+ \) and for all \( z \in \mathbb{D} \), the inverse \( G(z)^{-1} \) has a sense. The function \( G^{-1} \) lives in \( N^+ \). Now, we consider the polar decomposition of \( G(z) = A_G(z)R_G(z) \) with \( R_G(z) = (G(z)^*G(z))^{1/2} \) positive-definite hermitian and \( A_G(z) \) unitary. We have

\[
G(z)^{-1} = A^*(z)R(z)^{-1} \quad \text{and} \quad G(z)^* = A^*(z)R(z).
\]

Because \( G(z) \) is outer, \( G = H^2(\mathbb{C}^m) \). We can say that \( GH^\infty(\mathbb{C}^m) \) is dense in \( G \). Let \( f = Gf' \in GH^\infty(\mathbb{C}^m) \). We define \( \psi \in L^\infty(\mathbb{T}, C^m \to C^m) \) by

\[
\psi = \bar{z}H^*(G^{-1})^*G^*G^{-1},
\]

and then \( T_G f = T_{\bar{z}G^*G^{-1}} T_G \) is exactly \( T_{(G^{-1})^*} T_{G^*G^{-1}} f \). We extend by continuity from \( GH^\infty(\mathbb{C}^m) \) to \( H^2(\mathbb{C}^m) \).

Because \( G \) and \( H \) are outer, \( G^{-1} \) and \( G^{-1}H \) are outer in \( N^+(\mathbb{C}^n \to \mathbb{C}^m) \). But \( \varphi G = \bar{z}H^* \), so we have \( \varphi(z) = \bar{z}H^*(z)G(z)^{-1} \). So, for almost every \( e^{it} \in \mathbb{T} \), the
norm $\| G(e^{it})^{-1} H(e^{it}) \|$ is given by:
\[
\| G(e^{it})^{-1} H(e^{it}) \| = \| H(e^{it}) H(e^{it})^s - H(e^{it}) G(e^{it})^{-1} \|
= \| A_H(e^{it})^2 e^{it} \varphi(e^{it}) \|
= \| \varphi(e^{it}) \|.
\]

This proves that $G^{-1} H \in H^\infty(\mathbb{C}^m \to \mathbb{C}^m)$ is outer with the same norm as $\varphi$. So, the kernel of the Toeplitz operator with symbol $G^{-1} H$ is trivial. By construction, $\ker T_{z^* G^{-1}} = \mathcal{F}$.

We shall prove that $G^2$ is rigid.

Let $J$ be a function with the same argument as $G^2$. We can suppose that it is outer, because if not then we can build an outer function with the same argument. Indeed, if $J = J_1 J_2$, then $-(\text{Id} + J_1)^2(\text{Id} - J_1^*)^2 = 2\text{Id} - J_1^* - J_1^2$ is positive and $J_1^*(\text{Id} + J_1)^2 J$ or $-J_1^*(\text{Id} - J_1)^2 J$ is outer with the same argument as $G^2$.

Let $J_1 = J^{1/2} = A_F(z) R_{J_2}(z)^{1/2}$. The hypothesis on $J$ implies that $R_J \neq R_{G^2}$, so it is the same with $F_1$ and $G$. Because $J_1^* J_1^{-1} = A_G^2 G^* G^{-1}$ for a.e. $e^{it} \in \mathbb{T}$, then for all $u \in \mathbb{C}^m$, $T_{z^* G^{-1}} F_1 u = 0$. But $\ker T_{z^* G^{-1}} = G \mathbb{C}^m$, so $J_1 u \in G \mathbb{C}^m$. The contradiction follows. By hypothesis, $F_1$ is not a multiple of $G$ by a constant matrix.

The following consequence is an interesting characterization of the rigid functions. It is the matricial analogue of a result of Sarason ([Sar94b], chapter X page 70) used to show that $1 + z$ is rigid.

**Proposition 4.3.** If $F \in H^2(\mathbb{C}^m \to \mathbb{C}^m)$ is outer, then $F^2$ is rigid if and only if the Toeplitz operator $T_{F^* F^{-1}}$ has a trivial kernel.

**Proof.** Because $\ker S_*= \mathbb{C}^m$ and $T_{z^* F^* F^{-1}} = S^* T_{F^* F^{-1}}$, then $T_{F^* F^{-1}}$ is trivial if and only if $\dim \ker T_{z^* G^{-1}} = m$. But, thanks to the previous lemma, this is true only if $F^2$ is rigid.

Here is the matricial analogue of Lemma 1 of [Sar94a] page 161 for an outer square matrix $G$.

**Lemma 4.4.** Let $G \in H^2(\mathbb{C}^m \to \mathbb{C}^m)$ be outer and let $U \in H^\infty(\mathbb{C}^m \to \mathbb{C}^m)$ be inner vanishing at zero. Let $\mathcal{F} = T_G K_U$ be the kernel of a Toeplitz operator. Then it is the kernel of $T_{G^* U^* G^{-1}}$.

**Proof.** This proof follows Sarason’s. The fact that $\mathcal{F}$ is the kernel of a Toeplitz $T_\varphi$ allows us to show that $G^* U^* G^{-1}$ defines a symbol in $L^\infty(\mathbb{T}, \mathbb{C}^m \to \mathbb{C}^m)$. We begin the proof by building an outer function with the same norm as $\varphi$, then we can suppose that $\varphi$ takes values in the set of norm-1 matrices. Let $(e_k)_{1 \leq k \leq m}$ the canonical basis of $\mathbb{C}^m$.

Because $p_+ (\varphi g_k) = 0$, there exists $h_k \in H^2(\mathbb{C}^m)$ such that $\varphi G e_k = \overline{h_k}$ for $k \in \{1, \ldots, m\}$. The norm $\| \varphi(e^{it}) G(e^{it}) \|^2$ is equal to $\sum_{k=1}^m (h_k, h_k)$, so log $\| \varphi(e^{it}) G(e^{it}) \|$ = $\frac{1}{2} \log (\sum_{k=1}^m (h_k, h_k))$.

But the function $e^{it} \mapsto \sum_{k=1}^m (h_k(e^{it}), h_k(e^{it}))$ is in $H^1$, so it is log-integrable, just as $e^{it} \mapsto \| \varphi(e^{it}) G(e^{it}) \|$. We deduce from $\| \varphi(e^{it}) G(e^{it}) \| \leq \| \varphi(e^{it}) \| \| G(e^{it}) \|$. 

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that $e^{it} \rightarrow \|\varphi(e^{it})\|$ is log-integrable. Via the Poisson kernel (cf \cite{KK97}), we build $\psi \in H^\infty(\mathbb{C}^m \to \mathbb{C}^m)$ outer with the same norm:

$$
\psi(z) := \exp\left(\frac{1}{2\pi} \int_\mathbb{T} \frac{e^{it} + z}{e^{it} - z} \log \left(\varphi^*(e^{it})\varphi(e^{it})\right)^{1/2} \, dt\right).
$$

Because $\psi$ is outer, so is its determinant and the matrix $\psi(z)$ is invertible for $z \in \mathbb{D}$ and the inverse is in $\mathcal{N}^+$. Then the radial limits exist almost everywhere and $\psi^{-1}\varphi \in L^\infty(\mathbb{T}, \mathbb{C}^m \to \mathbb{C}^m)$. The values $\psi^{-1}\varphi(e^{it})\varphi(e^{it})$ are matrices with norm 1 for a.e. $e^{it} \in \mathbb{T}$.

So, there exists $\chi \in L^\infty(\mathbb{T}, \mathbb{C}^m \to \mathbb{C}^m)$ which takes values in the set of norm-1 matrices such that $\varphi = \psi^*\chi$. But, ker $T_{\psi^*}$ is trivial, so ker $T_{\varphi}$ = ker $T_\chi$ and even if we replace $\varphi$ by $\chi$, we can suppose that $\varphi$ takes values that are matrix of norm 1.

Because $p_+(\varphi G) = 0$, there exists $H \in H^2(\mathbb{C}^m \to \mathbb{C}^m)$ such that $\varphi G = H^*$ and $H(0) = 0$.

We note $H = H_1H_o$, and we will show that $H_1 = VU$ with $V$ a unitary constant matrix.

Now, we prove that $U$ divides $H_1$. For all $h \in K_U$, $Gh \in \ker T_\varphi$ implies that $\varphi Gh$, which is equal to $H_o^*H_1^*h$, lies in $zH^2(\mathbb{C}^m)$. Because $H_o$ is outer, $H_o^*h \in zH^2(\mathbb{C}^m)$ and then $h$ is orthogonal to $H_o^*H_1^*h$.

The converse inclusion holds: Let $h \in K_H$, be bounded. Then, there exists $H = H_1H_o \in H^2(\mathbb{C}^m \to \mathbb{C}^m)$ such that $T_\varphi Gh = p_+(H_o^*H_1^*h)$. Then for all $k \in H^\infty(\mathbb{C}^m)$, we have

$$
\langle T_\varphi Gh, k \rangle = \langle H_o^*H_1^*h, k \rangle = \langle h, H_oH_1k \rangle = 0.
$$

Because the bounded functions are dense in $K_H$, we obtain that $K_H \subset K_U$, and so $H_1 = VU$ with $V$ unitary constant.

As in the proof of Lemma 1.2, we write $\varphi = H_o^*U^*G^{-1}$. Then $p_+(\varphi f) = p_+(H_o^*(G^{-1})^*G^*U^*G^{-1} f)$, and $T_\varphi = T_{(G^{-1}H_o)}T_{G^*U^*G^{-1}}$.

To conclude, we need to show that $G^{-1}H_o$ lies in $H^\infty(\mathbb{C}^m \to \mathbb{C}^m)$. The two functions $H_o$ and $G$ are outer, so $H_o^{-1}G$ is in the Nevanlinna-Smirnov class. To end the proof, we observe that $\|G^{-1}H_o\| = \|\varphi\|$, which was done at the end of the proof of Lemma 1.2. So ker $T_{(G^{-1}H_o)}$ is trivial, and ker $T_{\varphi} = \ker T_{G^*U^*G^{-1}}$.

**4.3. The case $r < m$.** Let $G \in H^2(\mathbb{C}^r \to \mathbb{C}^m)$ be an outer function. With the notation of section 2.1, we consider the space $\mathcal{G} = H^2(\mathbb{G})$, the unitary mapping $\Theta_0 : \mathbb{C}^r \to \mathbb{G}$ and $G = \Theta_0\hat{G}$ with $\hat{G} \in H^2(\mathbb{C}^r \to \mathbb{C}^r)$ outer.

Let $\Theta_1 : \mathbb{C}^{m-r} \to \mathbb{G}^\perp$ be an unitary mapping. Then we decompose $H^2(\mathbb{C}^m)$ as follows:

$$
H^2(\mathbb{C}^m) = \mathcal{G} \oplus \mathcal{G}^\perp = H^2(\mathbb{G}) \oplus H^2(\mathbb{G}^\perp) = \begin{pmatrix} \Theta_0 & 0 \\ 0 & \Theta_1 \end{pmatrix} \begin{pmatrix} H^2(\mathbb{C}^r) \\ H^2(\mathbb{C}^{m-r}) \end{pmatrix}.
$$

We denote $\Theta$ the $(m \times m)$-unitary matrix with diagonal $\Theta_0, \Theta_1$.

**Lemma 4.5.** Let $\mathcal{F} = T_G\mathbb{C}^r$ the kernel of a Toeplitz operator. We suppose that $\dim \mathcal{F} = r$. Then $\hat{G} \hat{G} \in H^1(\mathbb{C}^r \to \mathbb{C}^r)$ is a rigid function and $\mathcal{F}$ is the kernel of the Toeplitz operator with symbol

$$
\phi := \Theta \begin{pmatrix} z\hat{G}^* & 0 \\ 0 & \text{Id}_{m-r} \end{pmatrix} \Theta^*.
$$
Proof. Let $\phi$ in $L^\infty(T, \mathbb{C}^m \to \mathbb{C}^m)$ be defined as in (12).

If $f \in G^1$, then $\Theta^* f \in H^2\{(0_C^r) \oplus H^2(\mathbb{C}^{m-r})\}$ and $T_\phi f = f$, so ker $T_\phi|_{G^1} = \{0\}$.

If $f \in G$, then $\Theta^* f \in H^2(\mathbb{C}^r \oplus \{0_{m-r}\})$.

We denote by $p_r$ the orthogonal projection from $\mathbb{C}^m$ to $\mathbb{C}^r \oplus \{0_{m-r}\}$. Let $\varphi = p_r \Theta^* \phi p_r$, then $T_\varphi$ is a Toeplitz operator on $H^2(\mathbb{C}^r)$. Write $\tilde{F} = \tilde{G}\mathbb{C}^r$. We verify that $\tilde{F} = p_r(\Theta^* \mathbb{G}^r) = p_r \Theta F$ is nearly $S^r$-invariant of dimension $r$ and that ker $T_\varphi = \tilde{F}$. Then, we apply Lemma 4.5 to $\tilde{F}$ in $H^2(\mathbb{C}^r)$, which implies that $G^2$ is rigid and $\tilde{F} = \text{ker} T_{\varphi'|G\perp}$.

To conclude,

$$\phi = \Theta \begin{pmatrix} \tilde{G}^* U^* \tilde{G}^{-1} & 0 \\ 0 & Id_{m-r} \end{pmatrix} \Theta^*$$

satisfies $T_\phi = F$ and $G^2$ is rigid.

\[\square\]

**Lemma 4.6.** Let $F = T_G K_U$ be the kernel of a Toeplitz operator. Then $G^2 \in H^1(\mathbb{C}^r \to \mathbb{C}^r)$ is a rigid function and $F$ is the kernel of the Toeplitz operator with symbol

$$\phi := \Theta \begin{pmatrix} \tilde{G}^* U^* \tilde{G}^{-1} & 0 \\ 0 & Id_{m-r} \end{pmatrix} \Theta^*.$$ 

The proof uses the Lemmas 4.3 and 4.5. The naive idea is to apply Lemma 4.3 to $F$, considered as a subspace of $\mathbb{C} = H^2(\mathbb{G})$. Then we have dim $W = r = \text{dim} \ G$. Unfortunately, the range of $T_\phi|_{G}$ is a priori not in $G$. We need to find a new symbol $\phi'$ such that $T_{\phi'|G} = G$ and $T_{\phi'|G\perp} = G\perp$ with $F = \text{ker} T_{\phi'|G}$.

Proof. We begin by building a nearly $S^r$-invariant subspace $F'$ which will be the kernel of a certain Toeplitz operator with symbol $\phi'$, such that $W'$ is of dimension $m$. It will be more convenient to write $F = T_{G_0} K_{U_0} = \text{ker} T_{\phi_0}$.

By hypothesis, $G_0 \in H^2(\mathbb{C}^r)$ is outer and its columns form an orthonormal basis of $W$. Let $(e_{r+k})_{k=1,\ldots,m-r}$ be an orthonormal basis of $G\perp$. Because $G_0 = H^2(\mathbb{G})$, $G_1 := \frac{1}{\sqrt{2}}[(1 + z)^{1/2} e_k]_{r+1 \leq k \leq m}$ is an outer matrix and $\tilde{G}_1 = \Theta_1^* G_1$ is square. This choice of $\frac{1}{\sqrt{2}}(1 + z)$, which is a rigid scalar function, implies that $G_1^2$ is rigid in $H^1(\mathbb{C}^{m-r} \to \mathbb{C}^{m-r})$. Because $\tilde{G}$, the diagonal matrix with blocks $G_0, G_1$ is outer, $G := [G_0, G_1] \in H^2(\mathbb{C}^m \to \mathbb{C}^m)$ is outer.

Let $U_1 := z I_{m-r}$. It is inner and $U_1(0) = 0$ and $U'$, the matrix with diagonal blocks $U_0$ and $U_1$, is of rank $m$. Finally, we can consider $F' = T_{G_1} K_{U'} = T_{G_0} K_{U_0} \perp T_{G_1} K_{U_1}$. But $T_{G_0} K_{U_0} \subset G$ and $T_{G_1} K_{U_1} \subset G\perp$, so $W' = W_0 \oplus W_1$. We remark that $W_1 = G_1 \mathbb{C}^m-r$ and dim $W' = m$.

It remains to show that $F'$ is the kernel of a Toeplitz operator with symbol $\varphi'$. First of all, because $G_1^2$ is rigid, Lemma 1.5 applied to $T_{G_1} K_{U'}$, asserts that this space is the kernel of the Toeplitz with symbol $\phi_1 := \Theta \begin{pmatrix} \tilde{G}_1^* & 0 \\ 0 & \tilde{G}_1^* G_1 \end{pmatrix} \Theta^*.$

By hypothesis, $\mathbb{G}$ is the kernel of $T_{\phi_0}$. Let $\varphi' := \phi_0 |_{\mathbb{G}} + \phi_1 |_{\mathbb{G}\perp}$, where $\phi_0$ is the matrix $\Theta \begin{pmatrix} I_r & 0 \\ 0 & 0 \end{pmatrix} \Theta^*$ of projection from $H^2(\mathbb{C}^m)$ to $\mathbb{G} = H^2(\mathbb{G})$. It is clear that $\varphi' \in L^\infty(T, \mathbb{C}^m \to \mathbb{C}^m)$ and that $F' = \text{ker} T_{\varphi'}$.

Now, we can apply Lemma 4.3 to $F'$, the kernel of $T_{\varphi'}$. So, $G_1^2$ must be rigid. But it is a block diagonal matrix, so $G_1^2$ must be rigid. Moreover, $F'$ is the kernel
of $T'_\phi$ with $\phi' := G^*U^*G^{-1} = \Theta \tilde{G}^*U^*\tilde{G}^{-1}\Theta^*$. Finally, the symbol $\phi'$ is given by

$$
\phi' = \Theta \left( \begin{array}{cc} \tilde{G}^*_0U^*_0\tilde{G}^{-1}_0 & 0 \\
0 & Id_{m-r} \end{array} \right) \Theta^*.
$$

The diagonal structure of $\Theta^*\phi'\Theta$ implies that the range of $T_{\phi'}|_{\mathcal{G}_0}$ is contained in $\mathcal{G}_0$.

To conclude, we will show that $\mathcal{F} = \ker T_{\phi_0}$ where the symbol $\phi_0$ is:

$$
\phi_0 = \Theta \left( \begin{array}{cc} \tilde{G}^*_0U^*_0\tilde{G}^{-1}_0 & 0 \\
0 & Id_{m-r} \end{array} \right) \Theta^* \in L^\infty(\mathbb{T}, \mathbb{C}^m \to \mathbb{C}^m).
$$

Applying Lemma 4.4 in $\mathcal{G}_0 = H^2(\mathcal{G}_0)$ with $\dim \mathcal{G}_0 = r = \dim W$, we have $\mathcal{F}$ is the kernel of $T_{\tilde{G}^*_0\phi_0\tilde{G}^{-1}_0}$, so it is the kernel of the Toeplitz operator with symbol $\tilde{G}^*_0U^*_0\tilde{G}^{-1}_0$. It remains to complete the symbol on $\mathcal{G}_0^\perp$. Then $\mathcal{F}$ is the kernel of the Toeplitz operator with symbol

$$(4.3) \quad \phi := \Theta \left( \begin{array}{cc} \tilde{G}^*_0U^*_0\tilde{G}^{-1}_0 & 0 \\
0 & Id_{m-r} \end{array} \right) \Theta^*.$$

\[\square\]

Let $\mathcal{F}$ be a nearly $S^*$-invariant subspace of the form $T_GK_U$ where $G$ is outer. We can summarize the section by saying that $\mathcal{F}$ is the kernel of a Toeplitz operator if and only if $\tilde{G}^2$ is rigid. Furthermore, the formula (4.3) expresses explicitly a symbol depending only on $G$ and $U$.

5. A MATRICIAL VERSION OF HAYASHI’S THEOREM

The purpose of this section is to obtain a matricial version of Hayashi’s theorem (Hay86). This section treats mainly the case $r = m$. We need some results about de Branges–Rovnyak spaces to state the theorem.

The first chapter of Sarason’s book (Sar94b) is general enough for the matricial case. Sarason specializes to the scalar case in the next chapters. The ideas come from the third and fourth chapters. In this section, we adapt some of them to the matricial case. The main problem is the lack of commutativity.

In section 3, we built from $G$ an outer function $B$ such that $\mathcal{H}(B)$ is unitarily equivalent to $H^2(\mathbb{C}^m, \mu_G)$. We saw that $B$ is in the unit ball of $H^\infty(\mathbb{C}^m \to \mathbb{C}^m)$ and a little investigation shows that $\log(I - B^*B) \in L^1(\mathbb{T}, \mathbb{C}^m \to \mathbb{C}^m)$. (Let $A$ be the outer function $2\tilde{G}(F + Id)^{-1}$, then for all $z \in \mathbb{D}$, $A^*(z)A(z) + B^*(z)B(z) = Id$, and the conclusion follows.)

In this section, we need to reverse the construction, beginning with a $B$ in the unit ball of $H^\infty(\mathbb{C}^m \to \mathbb{C}^m)$ verifying $\log(I - B(e^{it})^*B(e^{it})) \in L^1(\mathbb{T}, \mathbb{C}^m \to \mathbb{C}^m)$. (Such functions are non-extreme points of the unit ball of $H^\infty(\mathbb{C}^m \to \mathbb{C}^m)$, see Treil’s theorem in Nik02 page 85).

We define $A \in \mathcal{N}^+(\mathbb{C}^m \to \mathbb{C}^m)$ to be the Poisson integral

$$
A(z) := \exp \left( \frac{1}{2\pi} \int_0^{2\pi} \frac{e^{it} + z}{e^{it} - z} \log(I - B(e^{it})^*B(e^{it})) \, dt \right).
$$

By definition, $A$ is outer and $A(e^{it})^*A(e^{it}) + B(e^{it})^*B(e^{it}) = Id$ a.e. on $\mathbb{T}$. Because $B$ is bounded, so is $A$. 

Proposition 5.2.

Let \( H \) to \( f \in \mathcal{T} \) defined in section 3 is equal to absolutely continuous, then \( (\text{absolutely continuous}, G \) basis and products \( \langle \text{polynomials in the coefficients of } Id \rangle \) is \( B \). Finally, the hypothesis \( \mu \) is positive, so we can define the Herglotz integral corresponding to \( B \) analogous to \( \text{corresponding to } \). So, for \( u \in C^m \), we have \( \langle \text{Re } ((I + B)(I - B)^{-1}) u, u \rangle_{C^m} = \| A(I - B)^{-1} u \|_{C^m}^2 \) which is positive, so we can define the Herglotz integral corresponding to \( B \) analogous to \( \). Let \( \mu \in M(\mathbb{T}) \) be the positive-definite-valued measure defined by \( \forall z \in \mathbb{D}, \ F(z) := (I + B(z))(I - B(z))^{-1} = : iV + \int_{\mathbb{T}} \frac{e^{i\theta} + z}{e^{i\theta} - z} d\mu(e^{i\theta}). \) Then, the Radon-Nikodym derivative of the absolutely continuous component of \( \mu \) is \( R^2_{(I - B)^{-1} A} \in L^1(\mathbb{T}, C^m \rightarrow C^m) \). We define \( G := (I - B)^{-1} A \). This function is outer, because its determinant is outer. It is the quotient of two outer functions in \( H^\infty \), so \( G \in \mathcal{N}^+ \) (compute the inverse of \( (I - B) \) with the formula of the comatrix, whose coefficients are polynomials in the coefficients of \( I - B \), so are outer functions). The fact that \( R^2_{(I - B)^{-1} A} \in L^1(\mathbb{T}, C^m \rightarrow C^m) \) implies that \( G \in H^2(C^m \rightarrow C^m) \). If \( \mu \) is absolutely continuous, then \( F(z) := (I + B(z))(I - B(z))^{-1} = iV + \int_{\mathbb{T}} \frac{e^{i\theta} + z}{e^{i\theta} - z} G^*(e^{i\theta}) G(e^{i\theta}) \frac{d\theta}{2\pi}. \) Finally, the hypothesis \( B(0) = 0 \) implies \( V = 0 \). Let \( (e_k)_{1 \leq k \leq m} \) be the canonical basis of \( C^m \). The coefficients of \( F(0) = Id = \int_{\mathbb{T}} G^*(e^{i\theta}) G(e^{i\theta}) \frac{d\theta}{2\pi} \) are the inner products \( \langle G_{e_k}, G_{e_m} \rangle_{H^2(C^m)} \), so \( G \) is a matrix which columns form an orthogonal basis and \( G \) is of unit norm in \( H^2(C^m) \).

In this context, we say that \( (B, A) \) is a corona pair, or a pair. When \( \mu \) is absolutely continuous, \( (B, A) \) is said to be a special pair. Then, the measure \( \mu_G \) defined in section 3 is equal to \( \mu \) and \( Gf \in H^2(C^m) \) is equivalent to \( f \in H^2(C^m, \mu) \).

Recall that \( G := (I - B)^{-1} A \) is outer, so thanks to Lemma 3.3.2, the operator \( T_{Id-B} T_G : H^2(C^m) \rightarrow \mathcal{H}(B) \) is an isometry. To be a special pair means that for all \( f \in H^2(C^m, \mu), T_{Id-B} T_G \cdot Gf = f \), the operator \( T_{Id-B} T_G \cdot Gf = f \) represents the division by \( G \). So, we can reformulate Theorem 3.3.

Corollary 5.1. Let \( (B, A) \) be a pair. Then \( T_{Id-B} T_G \cdot Gf = f \) is an isometry from \( H^2(C^m) \) to \( \mathcal{H}(B) \). The pair is special if and only if \( T_{Id-B} T_G \cdot Gf = f \) is surjective.

The next Proposition is the analogue of the Proposition 6 in [Sat94a].

Proposition 5.2.

1. Let \( (B, A) \) be a pair. Then, \( AH^2(C^m) \subset \mathcal{H}(B) \).
2. If the pair \( (B, A) \) is special and \( G = (I - B)^{-1} A \), then \( AH^2(C^m) \) is dense in \( \mathcal{H}(B) \) if and only if \( G^2 \) is rigid.
3. If \( AH^2(C^m) \) is dense in \( \mathcal{H}(B) \) (relatively to \( \| \cdot \|_{\mathcal{H}(B)} \)), then \( (B, A) \) is special.

Consequently, if \( AH^2(C^m) \) is dense in \( \mathcal{H}(B) \), then \( G^2 \) is rigid.
Proof. (1) Using the property of Toeplitz operators, we obtain easily:
\[ \forall f \in H^2(C^m, \mu), T_{Id-B}T_G \cdot T_{G^{-1}} f = T_{Id-B}G f = (Id - B)G f = Af. \]

So, \( T_{Id-B}T_G \cdot T_{G^{-1}} \) sends the range of \( T_{G^{-1}} \) in \( AH^2(C^m) \). But, if \((B, A)\) is special pair, then the range of \( T_{Id-B}T_G \cdot T_{G^{-1}} \) is \( \mathcal{H}(B) \). So, \( AH^2(C^m) \) is contained in all \( \mathcal{H}(B) \).

(2) If \((B, A)\) is special, then \( T_{Id-B}T_G \cdot T_{G^{-1}} \) is a surjective isometry from \( H^2(C^m) \) to \( \mathcal{H}(B) \). Because \( A \) is outer, \( AH^2(C^m) \) is dense in \( H^2(C^m) \), but \( AH^2(C^m) \) is in \( \mathcal{H}(B) \) so \( AH^2(C^m) \) is dense in \( \mathcal{H}(B) \). Because \( T_{Id-B}T_G \cdot T_{G^{-1}} = T_A \), and \( T_{Id-B}T_G \cdot T_{G^{-1}} \) is an isometry, \( AH^2(C^m) \) is dense equivalent to the range of \( T_{G^{-1}} \) is dense in \( H^2(C^m) \). This means that the kernel of \( T_{G^{-1}} \) is reduced to zero. We conclude by Proposition [Sar94b] IV-3 that \( G^2 \) is a rigid function.

(3) We need to prove that the range of \( T_{Id-B}T_G \cdot T_{G^{-1}} \) is \( \mathcal{H}(B) \). If \( AH^2(C^m) \) is dense in \( \mathcal{H}(B) \), then the range of \( T_{Id-B}T_G \cdot T_{G^{-1}} \) is dense in \( \mathcal{H}(B) \). But \( T_{Id-B}T_G \cdot T_{G^{-1}} \) is a special pair. We conclude that \((B, A)\) is a special pair. \( \square \)

In the scalar case, for a function \( b \in H^\infty \) verifying \( \log(1 - |b|^2) \in L^1 \), the polynomials are dense in \( \mathcal{H}(b) \) ([Sar94b] II-4). We verify the matrix analogue, namely that \( \{pu, p \in Pol_+, u \in C^m \} \) is dense in \( \mathcal{H}(B) \).

Because \( T_A T_{A^*} \leq T_A T_A \), and \( T_{A^*} T_A = Id - T_{B^*} T_B \), and \( T_B T_{B^*} \leq T_{B^*} T_B \), Douglas’s Lemma [Sar94b] 5.2 implies that \( \mathcal{M}(A) \subset \mathcal{M}(A^*) = \mathcal{M}(B^*) \subset \mathcal{H}(B) \).

Let \( p \) be a polynomial and \( u \) be a vector in \( C^m \). The range \( T_{A^*} pu \) is of the form \( qu \), where \( q \) is a polynomial with the same degree as \( p \). Because \( \{pu, p \in Pol_+, u \in C^m \} \) is dense in \( H^2(C^m) \), so it is in \( \mathcal{M}(T_{A^*}) = \mathcal{H}(B^*) \). To complete the proof, we need to show that \( \mathcal{M}(A^*) \) is dense in \( \mathcal{H}(B) \).

The link between \( \mathcal{H}(B) \) and \( \mathcal{H}(B^*) \) is a corollary of Douglas’s Lemma (see [Sar94b] I-8): \( h \) is in \( \mathcal{H}(B) \) is equivalent to \( T_{B^*} h \in \mathcal{H}(B^*) \).

Because \( A \) is outer, \( ker T_{A^*} = \{0\} \), so there exists an unique \( h^+ \) such that \( T_{B^*} h = T_A h^+ \). Moreover, the following formula (see [Sar94b] IV-1) holds in the matrix case:

\[ \forall h_1, h_2 \in \mathcal{H}(B), \quad \langle h_1, h_2 \rangle_B = \langle h_1, h_2 \rangle_2 + \langle h_1^+, h_2^+ \rangle_2. \]

Let \( h \) be in \( \mathcal{H}(B) \) orthogonal to \( \mathcal{M}(A^*) \) (for the inner product of \( \mathcal{H}(B) \)). Then, for all \( n \geq 0 \), \( \langle T_{A^*} S^n h, h \rangle_{\mathcal{H}(B)} = 0 \), and by unicity, \( (T_{A^*} S^n h)^+ = T_A S^n h^+ \).

Now, we can show that \( h \) is null. For all \( n \geq 0 \), we have
\[
0 = \langle h, T_{A^*} S^n h \rangle_{\mathcal{H}(B)} = \langle h, T_{A^*} S^n h \rangle_2 + \langle h^+, T_A S^n h^+ \rangle_2
\]
\[
= \frac{1}{2\pi} \int_\mathbb{T} (\langle A(e^{it}) h(e^{it}), h(e^{it}) \rangle_C^m + \langle A(e^{it}) h^+(e^{it}), h^+(e^{it}) \rangle_C^m) e^{int} dt.
\]

The scalar function \( \phi : e^{it} \rightarrow \langle A(e^{it}) h(e^{it}), h(e^{it}) \rangle_C^m + \langle A(e^{it}) h^+(e^{it}), h^+(e^{it}) \rangle_C^m \) lies in \( H_0^1 \). The same calculation with \( \langle T_{A^*} S^n h, h \rangle_{\mathcal{H}(B)} = 0 \) implies that \( \phi \in H_0^1 \), so \( \phi \) is constant equal to zero, and we obtain the density of \( \{pu, p \in Pol_+, u \in C^m \} \) in \( \mathcal{H}(B) \).

**Corollary 5.3.** Let \((B, A)\) be a special pair. If \( G^2 \) is rigid, and \( U \in H^\infty(C^m \rightarrow C^m) \) is an inner function of rank \( m \), then \((UB, A)\) is special and \( (I - UB)^{-1} A \) is rigid.

A consequence of \((I - UB)^{-1} A \) being rigid is that \((I - UB)^{-1} A \) is outer.
Proof. Because $U$ is inner and $(B, A)$ is a pair, $(BU, A)$ is a pair too.

Once again, Douglas’s Lemma 3.2 allows us to show that $\mathcal{H}(B) \subset \mathcal{H}(UB)$. We have $T_{UB} = T_U T_B \leq T_B$, so is $I - T_B T_B = I - T_{UBT_B} U$. Then, $\mathcal{H}(B) \subset \mathcal{H}(UB)$.

The polynomials are dense in $\mathcal{H}(B)$ and in $\mathcal{H}(UB)$, so $\mathcal{H}(B)$ is dense in $\mathcal{H}(UB)$. Thanks to Proposition 5.2, $G^2$ rigid and $(B, A)$ special implies that $AH^2(\mathbb{C}^m)$ is dense in $\mathcal{H}(B)$, so in $\mathcal{H}(UB)$. Proposition 5.2 again, tell us that the pair $(UB, A)$ is a special. As a consequence, the corresponding function $((I - UB)^{-1}A)^2$ is rigid.

□

Lemma 5.4. Let $(B, A)$ be a pair and let $U \in H^\infty(\mathbb{C}^m \to \mathbb{C}^m)$ be an inner function of rank $m$ verifying $B = UB_0$. Let $A' := (I - B_0 U)G$. Then $T_{Id-B} T_{G^{-1}}$ maps the range of the operator $T_{G^{-1}U G}$ onto $UA'H^2(\mathbb{C}^m)$.

Proof. By construction, it is clear that $A'$ is outer in $H^\infty(\mathbb{C}^m \to \mathbb{C}^m)$. Because $B = UB_0$, we obtain the equality

$$(Id - B)UG = (Id - UB_0)UG = U(Id - B_0 U)G = UA'.$$

So the range of $T_{Id-B} T_{G^{-1}U G}$ is $UA'H^2(\mathbb{C}^m)$.

□

Now, we state the matricial analogue of Hayashi’s theorem.

Theorem 5.5. Let $F = GK_U$ be a nearly $S^*$-invariant subspace of $H^2(\mathbb{C}^m)$, where $G \in H^2(\mathbb{C}^m \to \mathbb{C}^m)$ is outer and $U \in H^\infty(\mathbb{C}^m \to \mathbb{C}^m)$ is inner, verifying $U(0) = 0$ and rank $U = m$. We write

$$A := (Id - UB_0)G, \quad A' := (Id - B_0 U)G \quad \text{and} \quad G' := (Id - B_0)A'.$$

Then, $F$ is the kernel of a Toeplitz operator if and only if $B = UB_0$, the pair $(B_0, A')$ is special and $G'^2$ is rigid.

Proof. Let $F = GK_U$ be the kernel of a Toeplitz operator. Then, Theorem 3.2 implies that $B = UB_0$, and Lemma 4.4 gives that $F = \ker T_{G^{-1}U G}$ and so

$$H^2(\mathbb{C}^m) = F \oplus \perp T_{G^{-1}U G} H^2(\mathbb{C}^m).$$

The section I-10 of [Sar94b], valid in the matricial case, gives the following orthogonal decomposition:

$$\mathcal{H}(B) = K_U \oplus_{\mathcal{H}(B_0)} U \mathcal{H}(B_0).$$

The operator $T_{Id-B} T_{G^{-1}}$ is an isometry from $H^2(\mathbb{C}^m)$ to $\mathcal{H}(B)$. Lemma 5.4 tells us that it maps the range of $T_{G^{-1}U G}$ onto $UT_{A'} H^2(\mathbb{C}^m)$. Moreover, it maps $F$ on $K_U$. So, we get the following diagram:

$$
\begin{array}{ccc}
T_{Id-B} T_{G^{-1}} & \downarrow & \downarrow \\
H^2(\mathbb{C}^m) & \oplus \perp & T_{G^{-1}U G} H^2(\mathbb{C}^m) \\
\mathcal{H}(B) & = & K_U \oplus_{\mathcal{H}(B_0)} U \mathcal{H}(B_0).
\end{array}
$$

It follows that $UA'H^2(\mathbb{C}^m)$ is dense in $U \mathcal{H}(B_0)$. On $\mathcal{H}(B)$, $T_U$ is an isometry, so $A'H^2(\mathbb{C}^m)$ is dense in $\mathcal{H}(B_0)$. We conclude this implication using Proposition 5.2. The fact that $A'H^2(\mathbb{C}^m)$ is dense in $\mathcal{H}(B_0)$ implies that $(B_0, A')$ is special and $G'^2$ is rigid.
Conversely, we can reverse the reasoning. The pair \((B_0, A')\) is special and \(G^2_0\) is rigid imply that \(A' H^2(\mathbb{C}^m)\) is dense in \(H(B_0)\) and so \(U A' H^2(\mathbb{C}^m)\) is dense in \(U H(B_0)\). The diagram holds to be true, so

\[
T_{Id-B}T_G \cdot T_G^{-1} U G H^2(\mathbb{C}^m) = T_{Id-B}T_G \cdot (U H(B_0)) = K_U.
\]

It follows that \(\mathcal{F} = G K_U = T_G^{-1} U G H^2(\mathbb{C}^m)\) is the kernel of the Toeplitz operator \(T_G \cdot U \cdot G^{-1}\).

As mentioned in Sarason’s article [Sar94], the proof contains a recipe for constructing a non-trivial proper subspace \(\mathcal{F} \subset H^2(\mathbb{C}^m)\) which is the kernel of a Toeplitz operator. We repeat the process.

We begin with the particular case \(r = m\): Take an outer function \(G_0 \in H^2(\mathbb{C}^m \to \mathbb{C}^m)\) such that \(G^2_0\) is rigid and with an inner function \(U \in H^\infty(\mathbb{C}^m \to \mathbb{C}^m)\) vanishing at zero. The pair associated to \(G_0\) is \((B_0, A')\). Let \(B = U B_0\), \(G = (Id - B_0 U)^{-1} A'\) and \(A = (Id - U B_0) G\). Thanks to Proposition 5.2, the pair \((B, A)\) is special and \(G^2\) is rigid. Then \(\mathcal{F} = G K_U\) is a nearly \(S^*\)-invariant subspace which is the kernel of the Toeplitz operator with symbol \(G^* U^* G^{-1}\). With this construction, \(\dim W\) is equal to \(m\).

We can adapt the general case \(r < m\) from the particular case \(r = m\). With the previous notation, \(\mathcal{F} = G K_U\), with \(G \in H^2(\mathbb{C}^r \to \mathbb{C}^m)\) outer, so \(\mathcal{G} = H^2(\mathbb{G})\) where \(\mathbb{G}\) is a subspace of \(\mathbb{C}^m\) of dimension \(r\). Working in \(H^2(\mathbb{G})\) allow us to apply the previous theorem and the unitary matrix \(\Theta_0\) to come back in \(H^2(\mathbb{C}^m)\).

\section*{References}

*BW03* P. Beneker and J. Wiegerinck. The boundary of the unit ball in \(H^1\)-type spaces. In *Function spaces (Edwardsville, IL, 2002)*, volume 328 of *Contemp. Math.*, pages 59–84. Amer. Math. Soc., Providence, RI, 2003.

*Cam77* M. Cambern. Invariant subspaces and extremum problems in spaces of vector-valued functions. *J. Math. Anal. Appl.*, 57(2):290–297, 1977.

*CCP* I. Chalendar, N. Chevrot, and J.R. Partington. Nearly invariant subspaces for backward shifts on vector-valued Hardy spaces. *Journal of Operator Theory*, to appear, #1767.

*Gar06* S. R. Garcia. Conjugation and Clark operators. In *Recent advances in operator-related function theory*, volume 393 of *Contemp. Math.*, pages 67–111. Amer. Math. Soc., Providence, RI, 2006.

*Hay85* E. Hayashi. The solution sets of extremal problems in \(H^1\). *Proc. Amer. Math. Soc.*, 93(4):690–696, 1985.

*Hay86* E. Hayashi. The kernel of a Toeplitz operator. *Integral Equations Operator Theory*, 9(4):588–591, 1986.

*Hay90�* E. Hayashi. Classification of nearly invariant subspaces of the backward shift. *Proc. Amer. Math. Soc.*, 110(2):441–448, 1990.

*Hit88* D. Hitt. Invariant subspaces of \(H^2\) of an annulus. *Pacific J. Math.*, 134(1):101–120, 1988.

*KK97* V. E. Katsnelson and B. Kirstein. On the theory of matrix-valued functions belonging to the Smirnov class. In *Topics in interpolation theory (Leipzig, 1994)*, volume 95 of *Oper. Theory Adv. Appl.*, pages 299–350. Birkhäuser, Basel, 1997. Also available as http://www.citebase.org/abstract?id=oai:arXiv.org:0706.1901.

*Lax59* P. D. Lax. Translation invariant spaces. *Acta Math.*, 101:163–178, 1959.

*Nik02* N. K. Nikolski. *Operators, functions, and systems: an easy reading. Vol. 1*, volume 92 of *Mathematical Surveys and Monographs*. American Mathematical Society, Providence, RI, 2002.

*Sar88* D. Sarason. Nearly invariant subspaces of the backward shift. In *Contributions to operator theory and its applications (Mesa, AZ, 1987)*, volume 35 of *Oper. Theory Adv. Appl.*, pages 481–493. Birkhäuser, Basel, 1988.
[Sar94a] D. Sarason. Kernels of Toeplitz operators. In Toeplitz operators and related topics (Santa Cruz, CA, 1992), volume 71 of Oper. Theory Adv. Appl., pages 153–164. Birkhäuser, Basel, 1994.

[Sar94b] D. Sarason. Sub-Hardy Hilbert spaces in the unit disk. University of Arkansas Lecture Notes in the Mathematical Sciences, 10. John Wiley & Sons Inc., New York, 1994.

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