Stability analysis of the solutions of kinetic equations for modelling of gas flows at arbitrary heat ratio

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Abstract.
In the presented paper the stability of the solutions of kinetic equations, which formed the models of ideal compressible gases at arbitrary heat ratio, is investigated. Models are widely used in applications of the lattice Boltzmann method, presented in the literature. The stability analysis is based on the considering of linear approximation and on the transition to the system of ordinary differential equations with complex coefficients. After the computations the stability criteria of the models are formulated for the cases of fixed values of relaxation time. The stability analysis is an important stage of the modelling process, because it may realize some help in choosing of the proper model for using in practical simulations.

1. Introduction
Lattice Boltzmann method (LBM) in the last decades has established itself as a powerful tool for computational modelling of complex processes in gases and fluids [1, 2]. The most advantages of LBM are the explicit type of its computational schemes, local nature and wide possibilities for parallel realization of its computational algorithms. The most of applications of LBM deal with the incompressible and semi-compressible fluids and gases. But some practical problems, such as a shock wave propagation, Kelvin – Helmholtz and Richtmyer – Meshkov instabilities, detonation, and combustion, required to consider the models of highly compressible gases [3].

First LB models for the simulation of compressible flows were proposed in the first years of LBM development. Alexander et al in [4] proposed a model, where the possibility to use of a flexible sound speed is introduced. So the Mach number can be varied. Yan et al in [5] proposed LB model for simulation of the compressible Euler equations. In this model the case of three energy levels takes into account. Sun and Hsu in [6, 7] proposed a so-called adaptive LBM, where the particle velocities may vary with Mach number and internal energy. Kataoka and Tsutahara in [8, 9] proposed the models for the simulation of the compressible Euler and Navier – Stokes equations, which may be discretized by various approaches, such as finite-difference or finite-volume methods, so they may use variable time and space steps, so as adaptive meshes. Watari and Tsutahara in [10, 11] constructed so-called multispeed models for simulation of an ideal gas with arbitrary heat ratio. In 2013 Gan et al proposed a model for the simulation of compressible ideal gases with arbitrary heat ratio, which is based on the representation of an equilibrium distribution function using a momentum matrix. In [13] this model is applied to the simulation of high-speed compressible viscous flows with a boundary layer. Despite the variety
of the proposed models, the problem of its theoretical investigation and comparison of models is still unsolved.

The presented paper is devoted to the stability analysis of the models, used in LBM for simulation of highly compressible flows. The asymptotic stability of the solutions of the system of kinetic equations is investigated after its linearization. The numerical procedure for the stability analysis is proposed and realized. The stability criteria are formulated and some recommendations for the use of the models are formulated.

The paper has a following structure. In Section 2 the models used in LBM for the simulation of compressible gases are presented. In Section 3 the problem of the stability investigation is stated. In Section 4 the results of the analysis are discussed. Some concluding remarks are made in Section 5.

2. Lattice Boltzmann models for compressible flows at arbitrary heat ratio

Let us consider the ideal gas with the following equation-of-state:

\[ p = (\gamma - 1)\rho e, \]

where \( p \) is the pressure, \( \rho \) is a density, \( e \) is the internal energy, \( \gamma \) is the heat ratio. Full energy \( \varepsilon \) is computed from the values of \( e \) by following formulae:

\[ \varepsilon = \rho e + \frac{\rho u^2}{2}, \]

where \( u = |\mathbf{u}| \), where \( \mathbf{u} \) is the gas velocity.

In the presented papers three models of compressible gas, widely used in LBM, are considered. Only the case of two-dimensional flows is considered.

2.1. Model proposed by Kataoka and Tsutahara [8]

The model is based on the following system of kinetic equations:

\[ \frac{\partial f_i}{\partial t} + V_i \cdot \nabla f_i = -\frac{1}{\lambda} (f_i - f_i^{(eq)}), \]  \hspace{1cm} (1)

where \( t \) is a time, \( x_\alpha \) are the space variables, which formed the radius-vector \( \mathbf{r} \), \( f_i = f_i(t, \mathbf{r}) \) are distribution functions of particles with the velocities \( \mathbf{V}_i \), \( \lambda \) is the relaxation time, \( f_i^{(eq)} \) are the equilibrium distribution functions.

The macroscopic variables are computed from the values of \( f_i \) by the following formulas:

\[ \rho = \sum_{j=1}^{n} f_j, \quad \rho \mathbf{u} = \sum_{j=1}^{n} f_j \mathbf{V}_j, \quad \rho \left( e + \frac{u^2}{2} \right) = \frac{1}{2} \sum_{j=1}^{n} f_j (V_j^2 + \eta_j^2), \] \hspace{1cm} (2)

where \( \eta_j^2/2 \) are the energy levels of the extra degrees of freedom.

The velocities \( \mathbf{V}_i \) are presented by following formulas:

\[ \mathbf{V}_i = \begin{cases} (0, 0), & i = 1, \\ V_1 \left( \cos \left( \frac{\pi i}{2} \right), \sin \left( \frac{\pi i}{2} \right) \right), & i = 2, 3, 4, 5, \\ V_2 \left( \cos \left( \frac{\pi i}{2} + \frac{1}{4} \right), \sin \left( \frac{\pi i}{2} + \frac{1}{4} \right) \right), & i = 6, 7, 8, 9, \end{cases} \]

the values of \( \eta_i \) are presented as:

\[ \eta_i = \begin{cases} \eta_0, & i = 1, \\ 0, & i = 2, 9, \end{cases} \]
where \(V_1, V_2 (V_2 \neq V_1)\) and \(\eta_0\) are given nonzero constants, \(V_1 = V\), where \(V - l/\delta t, l\) is a mean free path, \(\delta t\) is a mean free time. The equilibrium distribution functions are presented as:

\[
f^{(eq)}_i = \rho(A_i + B_i u_\alpha V_{i\alpha} + D_i u_\alpha u_\beta V_{i\alpha \beta}), \quad i = 1, 9,
\]

where the expressions for \(A_i, B_i\) and \(D_i\) are presented in [8]. So the model is based on the nine nonlinear differential equations.

2.2. Model of Watari [11]

In [11] the following system of kinetic equations is proposed:

\[
\frac{\partial f_{kl}}{\partial t} + V_{kl\alpha} \frac{\partial f_{kl}}{\partial x_\alpha} = -\frac{1}{\lambda} (f_{kl} - f^{(eq)}_{kl}), \quad (3)
\]

where \(f_{kl}\) are the distribution functions of the particles with velocities \(V_{kl}\). The subscript \(k\) indicate a group of particle velocities, where the translational speed is \(V_k\) and \(l\) indicate the particle direction. The macrovariables are computed by formulas (2), but the sums are written on two indexes \(k\) and \(l\). The expressions for \(f^{(eq)}_{kl}\) are presented in [11].

The vectors \(V_{kl}\) are presented by the same formulas, as in the case of Kataoka and Tsutahara model, but the eight velocities \(V_k, k = 1, 8\) take place in every direction \(l = 1, 8\), and one zero velocity. Parameters \(\eta_k\) are presented as: \(\eta_0 = 0, \eta_i \neq 0, i = 1, 4, \eta_i = 0, i = 5, 8\). So this model consists of 65 nonlinear differential equations, which formed the system (3).

2.3. Model of Gan et al [12]

The model is based on the system (1), but the velocity set without the zero velocity is used:

\[
V_i = \begin{cases} 
V(\pm1, 0), & i = 1, 4, \\
V(\pm1, \pm1), & i = 5, 8, \\
2V(\pm1, 0), & i = 9, 12, \\
2V(\pm1, \pm1), & i = 13, 16.
\end{cases}
\]

The equilibrium distribution functions formed the vector \(f^{(eq)}\), which may be considered as a solution of the following linear system:

\[
Cf^{(eq)} = M. \quad (4)
\]

System (4) is formed by an extension of the system of expressions for macrovariables (2) on the case of 16 moments of equilibrium distribution functions. In (4) \(M\) is a vector of moments and \(C\) is a square matrix. Expressions for \(M\) and \(C\) are presented at [12]. Vector \(f^{(eq)}\) is easily obtained by the inversion of \(C\): \(f^{(eq)} = C^{-1}M\). The model consist of 16 nonlinear differential equations.

3. Stability analysis

Let us consider the procedure of the stability investigation of the solutions of initial problems for the systems of kinetic equations. The whole procedure will be considered for the case of system (1), written in dimensionless variables.

The stationary flow regime on the unbounded domain is considered. The regime is characterized by the constant values of macrovariables: \(u = \bar{U}, \rho = \bar{\rho}, e = \bar{e}\), where \(U\) is the dimensionless constant vector, \(\bar{\rho}\) and \(\bar{e}\) are the dimensionless constants. This flow regime
corresponds to the following stationary solution of a system (1): \( \tilde{f}_i = f_i^{(eq)}(\tilde{\rho}, \tilde{e}, \tilde{U}) \), which is considered as an unperturbed solution of a system (1).

Let us represent the solution of (1) in the following form:

\[
f_i(t, r) = \tilde{f}_i + f_i'(t, r),
\]

(5)

where \( f_i' \) are the perturbations of the unperturbed solutions \( \tilde{f}_i \). After the substitution of (5) into (1) and linearization of the obtained system, the following linear system is obtained:

\[
\frac{\partial f_i'}{\partial t} + V_{i\alpha} \frac{\partial f_i'}{\partial x_\alpha} = -\frac{1}{\tau} \left( f_i' - \sum_{s=1}^{n} A_{is} f_s' \right),
\]

(6)

where \( \tau \) is a dimensionless relaxation time, \( A_{is} = \frac{\partial f_i^{(eq)}}{\partial f_{si}}(\tilde{f}) \) are the components of the Jacobian matrix.

The solution of (6) may be presented in the following form:

\[
f_i'(t, r) = F_i(t) \exp(jkr),
\]

(7)

where \( j^2 = -1 \) and \( k \) is the wave vector.

After the substitution of (7) into (6), the following system of linear ordinary differential equations with complex coefficients is obtained:

\[
\dot{F}_i(t) = \sum_{s=1}^{n} G_{is} F_s(t),
\]

(8)

where the components \( G_{ik} \) are presented as:

\[
G_{is} = \begin{cases} 
-jk_{i\alpha} V_{i\alpha} - \frac{1}{\tau} + \frac{1}{\tau} A_{is}, & s = i, \\
\frac{1}{\tau} A_{is}, & s \neq i.
\end{cases}
\]

The solutions of (8) are asymptotically stable if and only if the following inequalities take place for real parts of eigenvalues of matrix \( G \): \( \text{Re}(\lambda_s) < 0 \), \( s = 1, n \).

Due to the complex asymmetric structure of matrix \( G \), the eigenvalue problem may be solved only numerically. In the presented investigation the procedures of Matlab are used. The stability analysis is realized by the fixation of values of \( \tilde{\rho}, \tilde{e} \) and \( V \), and the values of \( \tau, k \) and \( U \) (where \( U = |\tilde{U}| \)) are varied. The stability is characterized by the following function: \( \chi = \max_{s} \text{Re}(\lambda_s) \). If at some parameter values this function is larger than zero, it is concluded, that at these values the solution of (1) is unstable.

The main purpose of the realized analysis is to determine the value of the relation \( U/V \), i.e. the relation of the gas velocity to particle velocity, at which the instability occurs, so in this way the boundary of the stability domain is defined.

4. Results of the analysis

After the computations, the surfaces, which represents the 3D plots of \( \chi \) as the function of \( k \) (the situation of \( k_x = k_y = k \) is considered) and \( U/V \) are constructed, and critical values of \( U/V \) are obtained at various values of \( \tau \). It is interesting to investigate the case of values of \( \tau \) close to zero, due to its corresponding to the case of high Reynolds number and less viscosity [14].

At fig. 1 the plots of plots of \( \chi \) are presented for the case of \( \tau = 10^{-3} \). As it can be seen, at axis \( U/V \) the value corresponds to the transition of \( \chi \) from negative to positive values exists for all the models. For the model proposed in [8], the instability occurs in the following value: \( U/V \approx 0.7122 \), for the model, proposed in [11] \( U/V \approx 3.1919 \) and for the model, proposed in [12] \( U/V \approx 2.3612 \). As it can be seen, the highest value of this relation takes place in the case of the model, proposed by Watari [11]. It must be noted, that model consists of 65 nonlinear equations, but it is a price which must be paid to the stability.
Figure 1. Plots of $\chi$: a) case of model proposed in [8]; b) case of the model proposed in [10]; c) case of the model proposed in [12]

5. Conclusion
In the presented paper the stability of the solutions of kinetic equations, which formed the models of ideal compressible gases at arbitrary heat ratio, is investigated. Models are widely used in applications of LBM, presented in literature. The stability analysis is based on the considering of linear approximation and on the transition to the system of ordinary differential equations with complex coefficients. After the computations the stability criteria of the models are formulated for the cases of fixed values of relaxation time. The stability analysis is an important stage of the modelling process, because it may realize some help in choosing of the proper model for using in practical simulations.

References
[1] Chen S and Doolen G 1998 Ann. Rev. of Fluid Mech. 30 329
[2] Huang H, Sukop M and Lu X-Y 2015 Multiphase lattice Boltzmann method: theory and applications (Oxford: Wiley)
[3] Xu A et al 2012 Front. of Phys. 7 582
[4] Alexander F et al 1992 Phys. Rev. A 46 1967
[5] Yan G, Chen Y and Hu S 1999 Phys. Rev. E 59 454
[6] Sun C 1998 Phys. Rev. E 58 7283
[7] Sun C and Hsu A 2003 Phys. Rev. E 68 016303
[8] Kataoka T and Tsutahara M 2004 Phys. Rev. E 69 056702
[9] Kataoka T and Tsutahara M 2004 Phys. Rev. E 69 035701(R)
[10] Watari M and Tsutahara M 2004 Phys. Rev. E 70 016703
[11] Watari M 2007 Phys. A 382 502
[12] Gan Y et al 2013 Europhys. Lett. 103 24003
[13] Qiu R et al 2017 Appl. Math. Mod. 48 567
[14] Sofonea V and Sekerka R 2003 J. of Comp. Phys. 184 422