Clifford Algebroids and Nonholonomic Spinor Deformations of Taub–NUT Spacetimes

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February 16, 2005

Abstract

In this paper we examine a new class of five dimensional (5D) exact solutions in extra dimension gravity possessing Lie algebroid symmetry. The constructions provide a motivation for the theory of Clifford nonholonomic algebroids elaborated in Ref. [1]. Such Einstein–Dirac spacetimes are parametrized by generic off–diagonal metrics and nonholonomic frames (vielbeins) with associated nonlinear connection structure. They describe self–consistent propagations of (3D) Dirac wave packets in 5D nonholonomically deformed Taub NUT spacetimes and have two physically distinct properties: First, the metrics are with polarizations of constants which may serve as indirect signals for the presence of higher dimensions and/or nontrivial torsions and nonholonomic gravitational configurations. Second, such Einstein–Dirac solutions are characterized by new type of symmetries defined as generalizations of the Lie algebra structure constants to nonholonomic Lie algebroid and/or Clifford algebroid structure functions.

Keywords: Lie algebroids, Clifford algebroids, nonholonomic frames, nonlinear connections, exact solutions, Einstein–Dirac equations, extra dimension gravity.

PACS Classification: 02.40.-k, 04.20.Gz, 04.50.+h, 04.90.+e
2000 AMS Sub.Clas.: 15A66, 17B99, 53A40, 81R25, 83C20, 83E15

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1 Introduction

Recently we studied a new class of solutions in string gravity (with nontrivial limits to the Einstein’s gravity) possessing Lie algebroid symmetry \[2\] and describing black holes in solitonic backgrounds, see Ref. \[3\] for details on algebroid theory and related bibliography. These solutions were generated using the method of anholonomic frames with associated nonlinear connection (N–connection) structure \[4\]. The technique can be extended for spinor variables and Einstein–Dirac spaces and motivates the concept of Clifford algebroids \[1\]. In this paper we apply such spinor and Lie algebroid methods in five dimensional (5D) gravity in order to construct metrics defining nonholonomic deformations of the Taub NUT metric to certain Einstein–Dirac algebroids defined by exact solutions of the Einstein–Dirac equations with Lie algebroid symmetry. \[1\]

Various classes of 4D metrics induced from 5D Kaluza–Klein theory and brane/string gravity are involved in many modern studies of physics related to gravitational instantons and monopoles and to the geodesic geometry of higher dimension spacetimes \[5\]; the constructions where generalized to generic off–diagonal and locally anisotropic gravitational configurations in Refs. \[6\].
Exact solutions play a special role in classical and quantum gravity providing a testing ground for fundamental concepts and general methods which can be studied by approximation methods. Moreover, any approximations are usually based on some exact solutions. An exact solution of the Einstein equations is characterized by corresponding symmetries and boundary (asymptotic) conditions. For instance, in modern astrophysics the asymptotically Minkowski/de Sitter metrics, with spherical or cylindrical symmetry, are of special interests. In cosmology, additionally to the spherically symmetric solutions, one considers anisotropic models defining spacetimes with Lie group symmetry. Nevertheless, following various fundamental purposes in quantum gravity and string theory, it is important to investigate more general spacetime models described by generic off–diagonal metrics and nonholonomic structures, deformed (non) commutative symmetries and non–perturbative gravitational vacuum configurations (solitons, instantons, monopoles, spinor and pp–waves,...). In this line, the approach to constructing and investigating classes of solutions with Lie algebroid symmetry distinguishes a new direction in mathematical gravity and string theory. Such spacetimes preserve a number of important features of manifolds with group symmetry but possess a more reach geometric structure combining the properties of bundle spaces and various type of nonlinear symmetries, singular maps and nonholonomic constraints.

There are certain applications of the algebroid theory in geometric mechanics [7] and, recently, there were elaborated some algebroid approaches in the theory of gauge fields, gravity and strings and noncommutative geometry [3, 9, 11, 2]. Perhaps, one can reflect on groupoid and algebroid program of geometrizations of physics considered as a modern versions of the Felix Klein’s ”Erlanger Program” (1887) when instead of groups and algebras one considers, respectively, groupoids and algebroids. In a more general context, containing spinors and Clifford algebras, one has to extend the constructions to the so called C–space, Clifford space, i.e. the space of Clifford numbers, or Clifford aggregates (see details and references in [10]), but provided with additional geometric structures, for instance, of Clifford nonholonomic algebroid [2], in order to describe certain classes of nonlinear gravitational interactions with generalized symmetries.

Motivated by the mentioned developments and prospects, in this paper we study some explicit examples of exact solutions in gravity when the Lie algebroid and Clifford algebroid structures are modelled as Einstein–Dirac configurations defined by nonholonomic deformations of Taub NUT metrics. The Hawking’s [5] suggestion that the Euclidean Taub-NUT metric might give rise to the gravitational analogue of the Yang–Mills instanton holds true for generalization to ”algebroid instantons”. Nevertheless, in this case
the solutions have some anisotropically polarized constants being of higher dimension gravitational vacuum polarization origin. These nonholonomically deformed Taub-NUT metrics also satisfy the vacuum Einstein’s equations with zero cosmological constant when the spherical symmetry is deformed, for instance, into an ellipsoidal/ toroidal configuration or transformed into locally anisotropic wormhole metrics, see details in Refs. [6]. Such algebroid Taub-NUT metrics can be used for generation of deformations of the space part of the line element defining Lie algebroid modifications of the Kaluza-Klein monopole solutions proposed by Gross and Perry and Sorkin [11].

The Schrödinger quantum modes and the Dirac equation in the Euclidean Taub-NUT geometry were analyzed using algebraic and analytical methods [12, 13, 14]. One of the purposes of this paper is to prove that the approach can be developed in order to include into consideration algebroid Taub-NUT backgrounds, in the context of the generalization of gauge-invariant theories of the Dirac field. In a more explicit form, in the present work, we develop an algebroid $SO(4,1)$ gauge like theory of the Dirac fermions considered for spacetimes with generic off–diagonal metrics, for instance, in the external field of the Kaluza-Klein monopole [13] which is deformed to Lie algebroid configurations. Our aim is also to emphasize some new features of the Einstein theory in higher dimension spacetime when the locally anisotropic properties, induced by anholonomic constraints and extra dimension gravity, are characterized by Clifford algebroid symmetries. We construct new classes of exact solutions of the Einstein–Dirac equations defining 3D soliton–spinor configurations propagating self–consistently in a 5D Lie algebroid Taub NUT spacetime and analyze certain physical properties of such geometries.

We note that in this paper the 5D spacetime is modelled as a direct time extension of a 4D Riemannian space provided with a corresponding spinor structure, i.e. our spinor constructions are not defined by some Clifford algebra associated to a 5D bilinear form but, for simplicity, they are considered to be extended from a spinor geometry defined for a 4D Riemannian space.

We start in Section 2 with an introduction into the theory of (gravitational) nonholonomic Lie algebroids. In Section 3 we study 5D metrics characterized by Lie algebroid symmetries and nonholonomic constraints. Two examples of such exact solutions are analyzed. We also point the possibility to model corrections to the Newton low by extra–dimension polarizations and nonholonomic frames which is very similar to warped geometries. In Section 4, we consider the Dirac equations on gravitational algebroids. Then, we generate new solutions of the 5D Einstein – Dirac equations constructed as generalizations of algebroid Taub NUT vacuum metrics to configurations with Dirac spinor energy–momentum source, i.e. to Clifford algebroids. Finally, in Section 5, we discuss and conclude the work. The Appendix contains
some ansatz formulas for Einstein equations with nonholonomic variables and their solutions.

2 Spacetimes with Lie N–Algebroid Symmetry

Let us consider a 5D spacetime $V$ with a conventional splitting of dimension, $\dim V = n + m = 5$, where $n \geq 2$, which is a (pseudo) Riemannian manifold, or a more general one with nontrivial torsion (i.e. a Riemann–Cartan manifold) of necessary smooth class. The splitting can be defined in global, coordinate free form, as a locally non–integrable (nonholonomic) distribution globalized for every point $u \in V$ resulting in a Whitney type sum

$$TV = hV \oplus vV$$

stating a decomposition of the tangent bundle $TV$ into certain conventional horizontal (h) and vertical (v) subspaces. We call a such manifold $V$ to be N–anholonomic being provided with a nonlinear connection (in brief, N–connection) structure $N$ defined by (1), see details in Refs. [1, 2].

We state the typical notations for abstract (coordinate) indices and geometrical objects defined with respect to an arbitrary local basis (system of reference, vielbein, or funfbein for 5D spacetimes) $e_{\alpha} = (e_i, v_a)$ on $V$. The small Greek indices $\alpha, \beta, \gamma, ...$ run values 1, 2, ..., $n + m$ and $i, j, k, ...$ and $a, b, c, ...$ respectively label the geometrical objects on the base and typical ”fiber” and run, correspondingly, the values 1, 2, ..., $n$ and 1, 2, ..., $m$. The dual base is denoted by $e^{\alpha} = (e^i, v^a)$. The local coordinates of a point $u \in V$ are written $u = (x, u)$, or $u^\alpha = (x^i, u^a)$, where $x^i$ are local $h$–coordinates, with respect to $e_i$, and $u^a$ are local $v$–coordinates with respect to the basis $v_a$.

A N–connection $N$ is also given by its coefficients,

$$N = \Lambda^\alpha_i(u)dx^i \otimes \frac{\partial}{\partial u^\alpha} = N^b_i(u)e^i \otimes v_b,$$

where there are underlined the indices defined with respect to the local coordinate basis

$$e_{\underline{\alpha}} = \partial_{\underline{\alpha}} = \partial/\partial u^\underline{\alpha} = (e_{\underline{i}} = \partial_{\underline{i}} = \partial/\partial x^i, v_{\underline{a}} = \partial_{\underline{a}} = \partial/\partial u^a)$$

2A manifold is called nonholonomic if it is provided with a nonintegrable distribution; in literature, one uses the equivalent term 'anholonomic'. In this papers, we consider a special case of nonholonomic manifolds when the anholonomy is defined only by the N–connection structure.
and its dual 
\[ e_\alpha^\alpha = du^\alpha = (e^i = dx^i, e^\alpha = du^\alpha). \]

The class of linear connections is parametrized by linear dependencies on \( u^\alpha \), i.e. \( N^\alpha_i(x, u) = \Gamma^\alpha_i_j(x) u^j \).

A \( \tilde{N} \)–connection structure induces a system of preferred vielbeins on \( V \):

Let us consider a 'vielbein' transform 
\[ e_\alpha = e_\alpha^\alpha(u) e_\alpha \] and 
\[ e^\alpha = e^\alpha_\beta(u) e^\alpha \]

given respectively by a nondegenerated matrix \( e_\beta^\alpha(u) \) and its inverse \( e^\alpha_\beta(u) \). Such matrices respectively parametrize maps from a local coordinate frame \( e_\alpha \) and co–frame \( e^\alpha \) to any general frame \( e_\alpha = (e_i, v^a) \) and co–frame \( e^\alpha = (e^i, v^a) \). If we consider a subclass of matrix transforms (2) linearly depending on \( N^\alpha_i(x, u) \), with the coefficients

\[ e_\alpha^\beta(u) = \begin{bmatrix} e_i^i(u) & N_i^b(u) e_\alpha^b(u) \\ 0 & e_\alpha^\alpha(u) \end{bmatrix} \] (3)

and

\[ e^\alpha_\beta(u) = \begin{bmatrix} e^i_i(u) & -N^b_k(u) e^k_i(u) \\ 0 & e^\alpha_\alpha(u) \end{bmatrix}, \] (4)

we generate \( N \)–adapted frames

\[ e_\alpha = (e_i = \frac{\partial}{\partial x^i} - N^b_i v_b, v_b) \] (5)

and their dual coframes

\[ e^\alpha = (e^i, v^b = v^b + N^b_i d x^i), \] (6)

for any \( v_b = e_b^b \partial_b \) satisfying the condition \( v^c | v^b = \delta^b_c \). In a particular case, we can take \( v_b = \partial_b \). The operators \([5]\) and \([6]\) are the so–called "N–elongated" partial derivatives and differentials which define a \( N \)–adapted differential calculus on \( N \)–anholonomic manifolds.\(^3\)

The Lie algebroids structure on \( N \)–anholonomic manifolds (equivalently, Lie \( N \)–algebroids) is defined by the corresponding sets of functions \( \tilde{\rho}_i^a(x, u) \)

\(^3\)we use 'boldface' symbols in order to emphasize that the geometrical objects are defined in \( N \)–adapted form, with invariant \( h \)– and \( v \)–components, on a manifold provided with \( N \)–connection structure
and $C^f_{ab}(x, u)$, see details in Refs. [1, 2]. For such Lie N–algebroids, the structure relations satisfy the conditions

$$\hat{\rho}(v_b) = \hat{\rho}_b(x, u) e_i,$$

$$[v_d, v_b] = C^f_{db}(x, u) v_f$$

and the structure equations of the Lie N–algebroid are written

$$\sum_{\text{cyclic}(a,b,c)} \left( \hat{\rho}_d e_j(C^f_{be}) + C^f_{ag} C^g_{be} - C^f_{be} \hat{\rho}_a Q^{f'}_{f'bej} \right) = 0,$$

for $Q^{f'bej} = e_i^b e_j^e e_{f'}^j e_{f'}^j e_i^a e_{f'}^b$ with the values $e_i^b$ and $e_{f'}^j$ defined by the N–connection. The anchor is defined as a map $\hat{\rho} : V \rightarrow hV$ and the Lie bracket structure $C^f_{ab}$ is considered on the spaces of sections $\text{Sec}(vV)$. For trivial N–connections, we can put $N^a_i = 0$ and obtain the usual Lie algebroid constructions with $e_i \rho^i_b \rightarrow \partial_i \rho^i_b$ when the structure functions $\rho^i_b(x)$ and $C^f_{ab}(x)$ do not depend on $v$–variables $u^a$. We can say that a Lie N–algebroid geometry is modelled on a spacetime $V$ provided with nontrivial N–connection $N$ and Lie N–algebroid structures $\hat{\rho}_b$ and $C^f_{ab}$ subjected to the conditions (7) – (9).

Such spacetimes are generalizations of the (pseudo) Riemannian manifolds with Lie group symmetry.

The curvature of a N–connection $\Omega \equiv -N_v$ is introduced as the Nijenhuis tensor

$$N_v(X, Y) \equiv [vX, vY] + vv[X, Y] - v[vX, Y] - v[X, vY]$$

for any vector fields $X$ and $Y$ on $V$ associated to the vertical projection "v" defined by this N–connection, i. e.

$$\Omega = \frac{1}{2} \Omega^b_{ij} e^i \wedge e^j \otimes v_b$$

with the coefficients

$$\Omega^b_{ij} = e_j N^a_i - e_i N^a_j + N^b_{ij} v_b \left( N^a_i \right) - N^b_{ji} v_b \left( N^a_i \right).$$

The vielbeins (5) satisfy certain nonholonomy (equivalently, anholonomy) relations

$$[e_\alpha, e_\beta] = W^\gamma_{\alpha\beta} e_\gamma$$

with nontrivial anholonomy coefficients

$$W^a_{jk} = \Omega^a_{jk}(x, u), \quad W^b_{ie} = v_e N^b_i(x, u) \quad \text{and} \quad W^v_{ae} = C^v_{ae}(x, u)$$

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reflecting the fact that the Lie algebroid is N–anholonomic.

A metric $g$ on $V$ can be written in N–adapted form,

$$g = g_{\alpha\beta}(u) e^\alpha \otimes e^\beta = g_{ij}(u) e^i \otimes e^j + h_{cb}(u) v^c \otimes v^b,$$  \hspace{1cm} (11)

where $g_{ij} = g(e_i, e_j)$ and $h_{cb} = g(v_c, v_b)$ and $e_\nu = (e_i, v_b)$ and $e^\mu = (e^i, v^b)$ are, respectively, just the vielbeins (5) and (6). We can define the anchored map for the ”contravariant” $v$–part of (11),

$$h_{cb}(u) v^c \otimes v^c \rightarrow h_{cb}(u) \tilde{\rho}^c_i \tilde{\rho}^b_j e^i \otimes e^j,$$  \hspace{1cm} (12)

modelling a h–metric $N h_{ij} = g_{ij}$.

A distinguished connection (d–connection) $D = \{\Gamma_{\beta\gamma}\}$ on $V$ is a linear connection conserving under parallelism the Whitney sum (1). This mean that a d–connection $D$ may be represented by h- and $v$–components in the form $\Gamma_{\beta\gamma} = (L^i_{jk}, \tilde{L}^a_{bb}, B^i_{jc}, \tilde{B}^a_{bc})$, stated with respect to N–elongated frames (6) and (5), defining a N–adapted splitting into h– and v–covariant derivatives, $D = hD + vD$, where $hD = (L, \tilde{L})$ and $vD = (B, \tilde{B})$.

A distinguished tensor (in brief, d–tensor; for instance, a d–metric (11)) formalism and d–covariant differential and integral calculus can be elaborated \[1, 2, 4, 6\] for spaces provided with general N–connection, d–connection and d–metric structure by using the mentioned type of N–elongate operators. The simplest way to perform a d–tensor covariant calculus is to use N–adapted differential forms with the coefficients defined with respect to (6) and (5), for instance, $\Gamma_{\beta\gamma} = \Gamma_{\beta\gamma}^i e^i$.

The torsion $T^\alpha$ and curvature $R_{\beta\gamma}^\alpha$ are defined by standard formulas but for N–adapted differential forms: we have respectively

$$T^\alpha = De^\alpha = de^\alpha + \Gamma^\alpha_{\beta\gamma} \wedge e^\beta$$  \hspace{1cm} (13)

and

$$R_{\beta\gamma}^\alpha = D\Gamma^\alpha_{\beta\gamma} = d\Gamma^\alpha_{\beta\gamma} - \Gamma^\gamma_{\beta\delta} \wedge \Gamma^\alpha_{\delta\gamma},$$  \hspace{1cm} (14)

see Refs. \[1, 2\] for explicit formulas $T^\alpha = \{T^\alpha_{\beta\gamma}\}$ and $R_{\beta\gamma}^\alpha = \{R_{\beta\gamma\alpha}^\alpha\}$ for the coefficients computed with respect to N–adapted frames (6) and (5).

The Ricci d–tensor $R_{\beta\gamma}$ can be computed by contracting the corresponding indices

$$R_{\beta\gamma} = R_{\beta\gamma\alpha}^\alpha$$

and scalar curvature is

$$\tilde{R} = g^{\beta\gamma} R_{\beta\gamma}.$$
The Einstein equations are written in the form

\[ G_{\beta\gamma} = R_{\beta\gamma} - \frac{1}{2} g_{\beta\gamma} \hat{R} = \Upsilon_{\alpha\beta} \]  

for a general source, \( \Upsilon_{\alpha\beta} \), of matter fields and possible extra dimension corrections.

A Riemann–Cartan algebroid (in brief, RC–algebroid) is a Lie algebroid \( \mathcal{A} = (V, [\cdot, \cdot], \rho) \) associated to a N–anholonomic spacetime \( V \) provided with a N–connection \( N \), symmetric metric \( g(u) \) and linear connection \( \Gamma(u) \) structures resulting in a metric compatible covariant derivative \( D \), when \( Dg = 0 \), but, in general, with non–vanishing torsion. In this work, we shall investigate some classes of metrics \( g(u) \) and linear connections \( \Gamma(u) \) modelling RC–algebroids as exact solutions of the 5D Einstein–Dirac equations.

On RC–algebroids, the Levi–Civita linear connection \( \nabla = \{ \nabla \Gamma^a \} \), by definition, satisfying the metricity and zero torsion conditions, is not adapted to the global splitting (11) and can not applied for elaborating N–adapted and algebroid constructions, see details and discussion in Refs. [11, 12]. Nevertheless, there is a preferred canonical \( d \)–connection structure \( \hat{\Gamma} \) constructed only from the metric and N–connection coefficients \( [g_{ij}, h_{ab}, N^a_i] \) and satisfying the metricity conditions \( \hat{D}g = 0 \) and \( \hat{T}^i_{jk} = 0 \) and \( \hat{T}^a_{bc} = 0 \). In explicit form, the \( h–v \)–components of the canonical \( d \)–connection \( \hat{\Gamma}^\gamma_{\alpha\beta} = (\hat{L}^i_{jk}, \hat{L}^a_{bk}, \hat{B}^i_{jc}, \hat{B}^a_{bc}) \), are given by formulas

\[
\hat{L}^i_{jk} = \frac{1}{2} g^{ir} \left[ e_k(g_{jr}) + e_j(g_{kr}) - e_r(g_{jk}) \right],
\hat{L}^a_{bk} = v_b(N^a_k) + \frac{1}{2} h^{ac} \left[ e_k(h_{bc}) - h_{dc} v_b(N^d_k) - h_{db} v_c(N^d_k) \right],
\hat{B}^i_{jc} = \frac{1}{2} g^{ik} v_c(g_{jk}),
\hat{B}^a_{bc} = \frac{1}{2} h^{ad} \left[ v_c(h_{bd}) + v_b(h_{cd}) - v_d(h_{bc}) \right].
\]

The formulas (10), (13) and (14) are defined on the nonholonomic spacetime \( V \) and contain the partial derivative operator \( v_c = \partial/\partial u^c \). We can emphasize the Lie N–algebroid structure by working with “boldface” operators \( v_c \rightarrow \nabla v_c = \hat{\rho}^i_c(x,u) e_i \) (see formulas (7) and (13)). A such “anchoring” of formulas defines canonical maps for \( d \)–metrics, anholonomic frames, \( d \)–connections and \( d \)–torsions from \( V \) to \( Sec(vV) \). By anchoring the N–elongated differential operators, we can define and compute (substituting \( v_c \) by \( \hat{\rho}^i_c e_i \) into (16)) the canonical \( d \)–connection \( \hat{\Gamma}^\gamma_{\alpha\beta} \) on \( Sec(vV) \) stating a canonical map \( \hat{\Gamma}^\gamma_{\alpha\beta} \rightarrow \rho \hat{\Gamma}^\gamma_{\alpha\beta} \).
3 Algebroid Taub NUT spaces

The standard Kaluza-Klein monopole was constructed by embedding the Taub-NUT gravitational instanton into 5D theory, adding the time coordinate in a trivial way [11]. There were investigated locally anisotropic variants of such solutions [6] when anisotropies are modelled by effective polarizations of the induced magnetic field. The aim of this Section is to analyze nonholonomic deformations of the Taub–NUT solutions when the metrics possess Lie N–algebroid symmetry.

3.1 Background metrics and deformations to gravitational algebroids

The Taub NUT solution of the 5D vacuum Einstein equations ($R_{\alpha\beta} = 0$, for the Levi–Civita connection) is expressed by the line element

$$ds^2_{(5D)} = dt^2 + ds^2_{(4D)},$$

$$ds^2_{(4D)} = -Q^{-1}(dr^2 + r^2d\theta^2 + \sin^2\theta d\phi^2) - Q(dx^4 + A_i dx^i)^2$$

where

$$Q^{-1} = 1 + \frac{m_0}{r}, m_0 = const.$$ (18)

The functions $A_i$ are static ones associated to the electromagnetic potential

$$A_r = 0, A_\theta = 0, A_\phi = 4m_0 (1 - \cos \theta)$$

resulting into "pure" magnetic field

$$\vec{B} = \text{rot} \vec{A} = m_0 \frac{\vec{r}}{r^3}$$

of a Euclidean instanton; $\vec{r}$ is the spherical coordinate’s unity vector.

The metric (17) defines a spacetime with global symmetry of the group $G_s = SO(3) \otimes U_4(1) \otimes T_t(1)$ since the line element is invariant under the global rotations of the Cartesian space coordinates and $x^4$ and $t$ translations of the Abelian groups $U_4(1)$ and $T_t(1)$ respectively. We note that the $U_4(1)$ symmetry eliminates the so called NUT singularity if $x^4$ has the period $4\pi m_0$. The mentioned group symmetries can be deformed in Lie N–algebroid one by corresponding nonholonomic frame transforms. Let us consider this procedure in details:

We introduce a new 5th coordinate,

$$x^4 \rightarrow \varsigma = x^4 - \int \mu^{-1}(\theta, \phi) d\xi(\theta, \phi),$$

where $\mu^{-1}(\theta, \phi)$ is a function that satisfies certain conditions that ensure the nonholonomic deformation of the metric.
when
\[ d\xi + 4m_0(1 - \cos \theta) d\theta = dx^4 + 4m_0(1 - \cos \theta) d\varphi. \]
This holds, for instance, for
\[ d\xi = \mu(\theta, \varphi) d(\zeta - y^4) = \frac{\partial \xi}{\partial \theta} d\theta + \frac{\partial \xi}{\partial \varphi} d\varphi, \]
when
\[ \frac{\partial \xi}{\partial \theta} = 4m_0(1 - \cos \theta) \mu \quad \text{and} \quad \frac{\partial \xi}{\partial \varphi} = -4m_0(1 - \cos \theta) \mu \]
with
\[ \mu = (1 - \cos \theta)^{-2} \exp[\theta - \varphi]. \]
The changing of coordinate \( \ref{20} \) describes a reorientation of the 5th coordinate in a such way as we could have only one nonvanishing component of the electromagnetic potential
\[ A_\theta = 4m_0(1 - \cos \theta). \]

The next step, we consider an auxiliary 5D metric\(^4\)

\[
\begin{align*}
\delta s^2_{(5D)} &= dt^2 + d\hat{s}^2_{(4D)}, \\
d\hat{s}^2_{(4D)} &= -dr^2 - r^2 d\theta^2 - r^2 \sin^2 \theta d\varphi^2 - Q^2(d\zeta + A_\theta d\theta)^2,
\end{align*}
\]
where
\[ d\hat{s}^2_{(4D)} \rightarrow d\hat{s}^2_{(4D)} = Q^2 d\hat{s}^2_{(4D)}. \]
This metric will be transform into some exact solutions after corresponding N–anholonomic transforms.

Let us consider a 5D ansatz of type \( \ref{11} \)
\[
\begin{align*}
\delta s^2 &= (dx^1)^2 + g_2(x^k)(dx^2)^2 + g_3(x^k)(dx^3)^2 \\
&
+ h_4(x^k, v)(e^4)^2 + h_5(x^k, v)(e^5)^2, \\
e^4 &= dy^4 + w_i(x^k, v) dx^i \quad \text{and} \quad e^5 = dy^5 + n_i(x^k, v) dx^i
\end{align*}
\]
with the time like coordinate \( x^1 = t \) and "anisotropic" dependence on coordinate \( y^4 = v \) and running of indices like \( i, j, ... = 1, 2, 3 \) and \( a, b, ... = 4, 5 \).
The set of coordinates \( x^k = (x^2, x^3) \) and \( y^5 \) can be any parametrization of the space line and the 5th dimension coordinates. This way, the complete

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\(^4\)It is not a solution of the Einstein equations, but its corresponding deformations will generate a class of exact solutions
set of local coordinates is stated in the form \( u = \{ u^\alpha = (t, x^2, x^3, y^4, y^5) \} \). The \( N \)-connection coefficients are parametrized in the form \( N_i^4 = w_i(x^k, v) \) and \( N_i^5 = n_i(x^k, v) \). We also write

\[
\eta^\alpha = (1, \eta^\alpha_i, \eta^\alpha_a) \quad \text{and} \quad q^\alpha = (1, q^\alpha_i, q^\alpha_a) \]

where \( \eta^\alpha = (1, \eta_i, \eta_a) \) are called 'polarization' functions. The values \( q^\alpha = (1, q_i, q_a) \) are just the coefficients of the metric (21) if \( \eta^\alpha \to 1 \) and \( w_i, n_i \to 0 \).

Our aim is to find certain nontrivial values \( \eta^\alpha \) and \( w_i, n_i \) when the metric (22) defines a solution with Lie \( N \)-algebroid symmetry, i.e. there are satisfied some conditions of type (12),

\[
g^{ij}(u) = h^{ab}(u) \tilde{\rho}^i_c(u) \tilde{\rho}^j_b(u). \]

Such anchor conditions, for effectively diagonalized metrics (with respect to \( N \)-adapted frames), must be satisfied both for \( \eta^\alpha = 1 \) and nontrivial values of \( \eta^\alpha \), i.e.

\[
g^i = h^4 (\tilde{\rho}^i_4)^2 + h^5 (\tilde{\rho}^i_5)^2, \quad \text{for} \ \eta^\alpha \neq 1; \\
q^i = q^4 (\tilde{\rho}^i_4)^2 + q^5 (\tilde{\rho}^i_5)^2, \quad \text{for} \ \eta^\alpha = 1,
\]

where \( g^i = 1/g_i, h^a = 1/h_a, \eta^a = 1/\eta_a \) and \( q^\alpha = 1/q_\alpha \).

By straightforward computations (see explicit formulas and details in Refs. [1, 2, 4, 6]) we can check the nontrivial components of the Einstein tensor \( \hat{G}^\alpha_\beta \) for the canonical \( \hat{d} \)-connection \( \hat{\Gamma}^\gamma_\alpha_\beta \) (16) satisfy the conditions

\[
\hat{G}^1_1 = -(\hat{R}^2_2 + \hat{R}^4_4), \quad \hat{G}^2_2 = \hat{G}^3_3 = -\hat{R}^4_4(x^2, x^3, v), \quad \hat{G}^4_4 = \hat{G}^5_5 = -\hat{R}^2_2(x^2, x^3).
\]

This means that the Einstein equations (15) for the ansatz (22) are compatible for nonvanishing sources and if and only if the nontrivial components of the source, with respect to the frames (5) and (6), are any functions of type

\[
\hat{\Upsilon}^2_2 = \hat{\Upsilon}^3_3 = \Upsilon_2(x^2, x^3, v), \quad \hat{\Upsilon}^4_4 = \hat{\Upsilon}^5_5 = \Upsilon_4(x^2, x^3) \quad \text{and} \quad \hat{\Upsilon}^1_1 = \Upsilon_2 + \Upsilon_4.
\]

Parametrizations of sources in the form (25) can be satisfied for quite general distributions of matter, torsion and dilatonic fields in string gravity or other gravity models. In this paper, we shall consider that there are given certain values \( \Upsilon_2 \) and \( \Upsilon_4 \) which vanish in the vacuum cases or can be induced by certain packages of spinor waves.

A very general class of exact solutions of the Einstein equations (15), with nontrivial sources of type (25) parametrized by the metric ansatz (22),
see Appendix (for simplicity, those formulas derived for the conditions (59)), is described by off–diagonal metrics of type

\[\delta s^2 = (dx^1)^2 - g_k(x^i) (dx^k)^2 - h_{[0]}^2(x^i) \left[f^* (x^i, v)\right]^2 \left|\varsigma_T(x^i, v)\right| (e^4)^2 - \left[f (x^i, v) - f_0 (x^i)\right]^2 (e^5)^2,\]

\[e^4 = dv + w_k (x^i, v) dx^k, \quad e^5 = du^5 + n_k (x^i, v) dx^k, \quad (26)\]

where the coefficients \(g_k(x^i)\) are constrained to be a solution of the 2D equation

\[g_{3}^{*} - \frac{g_{2}^{*}g_{3}^{*}}{2g_{2}} - \frac{(g_{3}^{*})^2}{2g_{3}} + g_{2}^{*} - \frac{g_{2}g_{3}^{*}}{2g_{2}} - \frac{(g_{2}^{*})^2}{2g_{2}} = 2g_{2}g_{3} \Upsilon_{4}(x^2, x^3) \quad (27)\]

for a given source \(\Upsilon_{4}(x^3)\) where, for instance, \(g_{3}^{*} = \partial g_{3}/\partial x^2\) and \(g_{2}^{*} = \partial g_{2}/\partial x^3\) and \(f^* = \partial f/\partial v\). It is always possible to find solutions of this equation, defining 2D Riemannian metrics, which are conformally flat, at least in non–explicit form. Hereafter we shall consider that \(g_k(x^i)\) are certain defined functions. The rest of functions from (26) can be computed in the form:

\[\varsigma_T(x^i, v) = 1 - \frac{1}{12} \int \Upsilon_{2}(x^k, v) \frac{\partial}{\partial v} [f(x^i, v) - f_0(x^i)]^3 dv; \quad (28)\]

the N–connection coefficients \(N_i^4 = w_i(x^k, v)\) and \(N_i^5 = n_i(x^k, v)\) are

\[w_i = - \frac{\partial \varsigma_T(x^k, v)}{\varsigma_T^*(x^k, v)} \quad (29)\]

and

\[n_k = n_{k[4]}(x^i) + n_{k[2]}(x^i) \int \varsigma_T(x^i, v) \frac{\partial}{\partial v} [f(x^i, v) - f_0(x^i)]^{-3} dv. \quad (30)\]

The set of functions (27)–(30) defines a class of exact solution of the 5D Einstein equations depending on arbitrary nontrivial functions \(f(x^i, v)\) (with \(f^* \neq 0\), \(h_{0}^*(x^i)\), \(\varsigma_4[0](x^i)\), \(n_{k[4]}(x^i)\) and \(n_{k[2]}(x^i)\), and sources \(\Upsilon_{2}(x^k, v)\) and \(\Upsilon_{4}(x^3)\)) which have to be defined from certain boundary conditions and physical considerations.\(^5\) It is not difficult to see that the metric (26) defines

\(^5\)Any metric (26) with \(h_{0}^* \neq 0\) and \(h_{5}^* \neq 0\) has the property to be generated by a function of four variables \(f(x^i, v)\) with emphasized dependence on the anisotropic coordinate \(v\), because \(f^* \neq \partial_v f \neq 0\) and by arbitrary sources \(\Upsilon_{2}(x^k, v), \Upsilon_{4}(x^3)\). The rest of arbitrary functions not depending on \(v\) have been obtained in result of integration of partial differential equations.
a Lie N–anholonomic algebroid. This follows from a parametrization of type (23),
\[ g_i = q_i \eta_i, \quad -h^2_{00} \left( x^i \right)^2 \left| \varsigma \right| = q_4 \eta_4, \quad - \left[ f - f_0 \right]^2 = q_5 \eta_5, \]
which allows to define the polarization functions \( \eta_\alpha \) and compute the non-trivial anchor coefficients (24).

The Lie N–algebroid structure is finally defined for such classes of metrics if the structure functions \( C^d_{ab}(x, u) = \) are chosen to satisfy the algebraic relations (9) with given values for \( \hat{\rho}_i \) (24) and defined N–elongated operators \( e_i \). In result, the second equation in (9) will be satisfied as a consequence of the first one. This restricts the classes of possible v–frames, \( v_b = e_b(x, u) \partial / \partial u_b \), where \( e_b(x, u) \) must solve the algebraic equations (3). We conclude, that the Lie N–algebroid symmetry imposes certain algebraic constraints on the coefficients of vielbein deformations generating the off–diagonal solutions.

The sourceless case with vanishing \( \Upsilon_2 \) and \( \Upsilon_4 \) can be distinguished in the form: Any off–diagonal metric (26) with \( h_0^2(x^i) = h_0^2 = \text{const}, \ w_i = 0 \) and \( n_k \) computed as in (30) but for \( \varsigma_\Upsilon = 1 \) defines a vacuum solution of 5D Einstein equations for the canonical d–connection (16). By imposing additional constraints on arbitrary functions from \( N^5_i = n_i \) and \( N^5_i = w_i \), we can select just those off–diagonal gravitational configurations when the Levi–Civita connection and the canonical d–connections are related to the same class of solutions of the vacuum Einstein equations, see details in Ref. [1, 2, 4, 6]. With the aim to analyze the gravitational algebroids in general form, in this paper, we shall consider nontrivial torsion configurations induced by nonholonomic frames.

### 3.2 Two examples of Taub NUT algebroids

We outline two classes of exact solutions of 5D vacuum Einstein equations on Lie N–algebroids which, in Section 4.2, will be extended to configurations with spinor matter field source.

#### 3.2.1 Static gravitational algebroids with angular polarization

A stationary ansatz for of type (26) with explicit dependence on the "anisotropic" angular coordinate \( v = \varphi \) is taken in the form

\[
\begin{align*}
\delta s^2 &= dt^2 - \delta s^2_{(4D)}, \\
\delta s^2_{(4D)} &= -dr^2 - r^2 d\theta^2 - r^2 \sin^2 \theta \eta_1(r, \theta, \varphi) \delta \varphi^2 - Q^2(r) \eta_5(r, \theta, \varphi) \delta \varsigma^2, \\
\delta \varphi &= d\varphi + w_1(r, \theta, \varphi) dt + w_2(r, \theta, \varphi) dr + w_3(r, \theta, \varphi) d\theta, \\
\delta \varsigma &= d\varsigma + n_1(r, \theta, \varphi) dt + n_2(r, \theta, \varphi) dr + n_3(r, \theta, \varphi) d\theta, 
\end{align*}
\]
where $q_2 = -1$ and $q_3 = -r^2$ solves the equation (27) for $\Upsilon_4 = 0$ and $\eta_{4,5}(r, \theta, \varphi)$ define classes of exact vacuum solutions with $\Upsilon_2 = 0$. For stationary metrics, one does not consider dependencies of the metric coefficients on the time coordinate.

The formulas (28) and (29), for $\Upsilon_2 = 0$, allow respectively to take $\zeta = 1$ and $w_i = 0$. We also consider $n_1 = 0$ and $n_3 = 0$ which can be obtained from (30) by stating zero the corresponding integration functions, i.e. $n_{1[1,2]}(x^i) = 0$ and $n_{3[1,2]}(x^i) = 0$. The formulas (31) give

$$q_2 = -1, \eta_2 = 1; q_3 = -r^2, \eta_3 = 1; q_4 = -r^2 \sin^2 \theta, q_5 = -Q^2(r)$$ (32)

where $\eta_4(r, \theta, \varphi)$ and $\eta_5(r, \varphi)$ satisfy the conditions

$$h_{[0]}^2(r) \left[ f^*(r, \varphi) \right]^2 = r^2 \sin^2 \theta \eta_4(r, \theta, \varphi), \quad [f(r, \varphi) - f_0(r)]^2 = Q^2(r) \eta_5(r, \varphi).$$ (33)

Such conditions are satisfied for any set of functions $f(r, \varphi), f_0(r), h_{[0]}(r, \theta), \eta_4(r, \theta, \varphi)$ and $\eta_5(r, \varphi)$ having limits $\eta_{4,5} \to 1$ for $\varphi \to 0$.

In the locally isotropic limit of the solution for $n_2(r, \theta, \varphi)$, when $\varphi \to 0$, we obtain the particular magnetic configuration contained in the metric (21) if we impose the condition that

$$n_{2[0]} + n_{2[1]} \lim_{\varphi \to 0} \int \frac{\partial}{\partial \varphi} \left[ f(r, \theta, \varphi) - f_0(r, \theta) \right]^{-3} d\varphi = A_\theta = 4m_0 (1 - \cos \theta),$$

which defines only one function from two unknown values $n_{2[0]}(r, \theta)$ and $n_{2[1]}(r, \theta)$. This has a corresponding physical explanation. From the usual Kaluza–Klein procedure, we induce the 4D gravitational field (metric) and 4D electromagnetic field (potentials $A_i$), which satisfy the Maxwell equations in 4D pseudo–Riemannian spacetime. For the case of spherical, locally isotropic, symmetries the Maxwell equations can be written for vacuum magnetic fields without any polarizations. When we introduce into consideration anholonomic constraints and non–spherical symmetries, the effective magnetic field could be effectively polarized by higher dimension gravitational field or vacuum nonlinear gravitational interactions. In the simplest case, we can put $n_{2[0]}(r, \theta) = 0$ and $n_{2[1]}(r, \theta) = 4m_0 (1 - \cos \theta)$ and write

$$n_2(r, \theta, \varphi) = 4m(r, \theta, \varphi) (1 - \cos \theta)$$

where the gravitationally anisotropically polarized mass is defined

$$m(r, \theta, \varphi) = \int \frac{\partial}{\partial \varphi} \left[ f(r, \theta, \varphi) - f_0(r, \theta) \right]^{-3} d\varphi,$$
with
\[ \lim_{\phi \to 0} m(r, \theta, \varphi) = m_0. \]

This class of 5D vacuum gravitational static algebroid solutions can be represented in the form
\[
\begin{align*}
\delta s^2 &= dt^2 - \delta s^2_{(4D)}, \\
\delta s^2_{(4D)} &= -dr^2 - r^2 d\theta^2 - r^2 \sin^2 \theta \, \eta_4(r, \theta, \varphi) d\varphi^2 - Q^2(r) \, \eta_5(r, \theta, \varphi) \delta \varsigma^2, \\
\delta \xi &= d\varsigma + 4m(r, \theta, \varphi) (1 - \cos \theta) \, dr.
\end{align*}
\]

Considering \( \eta_1 = 1 \) and \( q_1 = 1 \) and introducing the gravitational polarizations \( \eta_\alpha \) for the data (32) and (33), we can compute the anchor coefficients by solving the algebraic equations (24) and define the Lie N–algebroid structure with any \( C^\alpha_{bc} \) satisfying the conditions (9). Such d–metrics are similar to the Taub NUT vacuum metric (17) and its 4D conformally transformed partner (21) with that difference that the coefficients are polarized by nonholonomic constraints. Their Lie algebroid symmetry is a nonholonomic defformations of the global symmetry defined by the group \( G_s = SO(3) \otimes U_4(1) \otimes T_l(1) \). We can treat (34) as a static algebroid Taub NUT solution.

### 3.2.2 Solutions with extra–dimension induced polarization

Another class of solutions is constructed if we consider a d–metric of the type (26) with explicit dependence on the “anisotropic” angular coordinate \( v = \varsigma \) is taken in the form
\[
\begin{align*}
\delta s^2 &= dt^2 - \delta s^2_{(4D)}, \\
\delta s^2_{(4D)} &= -dr^2 - r^2 d\theta^2 - Q^2(r) \, \eta_4(t, r, \theta, \varsigma) d\varphi^2 - r^2 \sin^2 \theta \, \eta_5(t, r, \theta, \varsigma) \delta \varsigma^2, \\
\delta \varsigma &= d\varsigma + w_1(t, r, \theta, \varsigma) dt + w_2(t, r, \theta, \varsigma) dr + w_3(t, r, \theta, \varsigma) d\theta, \\
\delta \varphi &= d\varphi + n_1(t, r, \theta, \varsigma) dt + n_2(t, r, \theta, \varsigma) dr + n_3(t, r, \theta, \varsigma) d\theta.
\end{align*}
\]

The equation (27) is satisfied for \( \Upsilon_4 = 0 \), but \( \eta_{4,5}(t, r, \theta, \varsigma) \) are defined for a class of exact solutions with nonzero source, \( \Upsilon_2(r, \theta) \neq 0 \), in equation (53) from Appendix.

For simplicity, we can consider stationary solutions when the functions \( \eta_{4,5} \) and \( w_i, n_i \) do not depend on time variable \( t \) and consider functions of type \( f(r, \theta, \varsigma) \), \( f_0(r, \theta) \) and any integrable \( \Upsilon_2(r, \theta) \). The solutions (28), (29) and (30) are respectively written in the form
\[
\varsigma_T(r, \theta, \varsigma) = 1 - \frac{1}{12} \int \Upsilon_2(r, \theta, \varsigma) \frac{\partial}{\partial \varsigma} [f(r, \theta, \varsigma) - f_0(r, \theta)]^3 d\varsigma;
\]
the N–connection coefficients \( N_i^4 = w_i(x^k, v) \) and \( N_i^5 = n_i(x^k, v) \) are

\[
\begin{align*}
  w_1 &= 0, \quad w_2 = -\frac{\partial \zeta_T (r, \theta, \varsigma)}{\partial r} \left( \frac{\partial \zeta_T (\theta, \varsigma)}{\partial \varsigma} \right)^{-1}, \\
  w_3 &= -\frac{\partial \zeta_T (r, \theta, \varsigma)}{\partial \theta} \left( \frac{\partial \zeta_T (\theta, \varsigma)}{\partial \varsigma} \right)^{-1},
\end{align*}
\]

and

\[
n_k = n_{k[1]} (r, \theta) + n_{k[2]} (r, \theta) \int \zeta_T (r, \theta, \varsigma) \frac{\partial}{\partial \varsigma} [f (r, \theta, \varsigma) - f_0 (r, \theta)]^{-3} d\varsigma.
\]

We have to impose the limits

\[
w_2 \to 0 \text{ and } w_3 \to A_\theta = 4m_0 (1 - \cos \theta) \text{ for } \varsigma \to 0,
\]

this can be obtained by a corresponding class of functions \( \Upsilon_2 (\theta, \varsigma) \) and \( f (\theta, \varsigma), f_0 (\theta) \) such that

\[
\lim_{\varsigma \to 0} [f (r, \theta, \varsigma) - f_0 (r, \theta)]^2 = r^2 \sin^2 \theta \quad \text{for } \lim_{\varsigma \to 0} [f^* (r, \theta, \varsigma)]^2 = Q^2 (r)
\]

and \( n_k \to 0 \) for \( \varsigma \to 0 \) in order to get (21).

Expressing

\[
|h_4| = h_0^2 (r, \theta) [f^* (r, \theta, \varsigma)]^2 |\zeta_T (r, \theta, \varsigma)| \quad \text{and} \quad |h_5| = [f (r, \theta, \varsigma) - f_0 (r, \theta)]^2,
\]

\[
\eta_4 = [f^* (r, \theta, \varsigma)]^2 \quad \text{and} \quad \eta_5 = [1 - f (r, \theta, \varsigma) / f_0 (r, \theta)]^2 \quad \text{(35)}
\]

and parametrizing the integration functions to have

\[
\zeta_T = 1 - \frac{1}{12} \int \Upsilon_2 (r, \theta, \varsigma) d [f (r, \theta, \varsigma) - f_0 (r, \theta)]^3, \quad \text{(36)}
\]

\[
\begin{align*}
  w_1 &= 0, \quad w_2 = |f^*|^{-1} \frac{\partial}{\partial r} [f - f_0], \quad w_3 = |f^*|^{-1} \frac{\partial}{\partial \theta} [f - f_0], \\
  n_k &= n_{k[1]} (r, \theta) + n_{k[2]} (r, \theta) (f - f_0)^{-2},
\end{align*}
\]

with the limits \( f \to 0, \ f^* \to 1 \) and

\[
|f^*|^{-1} \frac{\partial}{\partial \theta} [f - f_0] \to 4m_0 (1 - \cos \theta)
\]

for \( \varsigma \to 0 \) and \( h_0^2 (r, \theta) = Q^2 (r) = -q_4 \) and \( (f_0)^2 = r^2 \sin^2 \theta = -q_5 \). We have \( n_k \to 0 \) with \( f \to 0 \) for \( \varsigma \to 0 \) if \( n_{k[1]} (r, \theta) = -n_{k[2]} (r, \theta) \). We put \( n_{1[1]} (r, \theta), n_{1[2]} (r, \theta) = 0 \) in order to get static solutions.
The data for a such solution are concluded for the metric
\[
\delta s^2 = dt^2 - \delta s_{(4D)}^2, \\
\delta s_{(4D)}^2 = -dr^2 - r^2 d\theta^2 - Q^2(r) [f^*(r, \theta, \varsigma)]^2 |\varsigma^\tau(r, \theta, \varsigma)| \delta \varsigma^2 - [r \sin \theta - f(r, \theta, \varsigma)]^2 \delta \varphi^2,
\]
\[
\delta \varsigma = d\varsigma + [f^*]^{-1} \left[ \frac{\partial}{\partial r} |f - f_0| dr + \frac{\partial}{\partial r} |f - f_0| d\theta \right],
\]
\[
\delta \varphi = d\varphi + [1 - (f - f_0)^{-2}] \left[ n_{2[1]}(r, \theta) dr + n_{2[2]}(r, \theta) d\theta \right].
\]

generated by functions \(f(r, \theta, \varsigma), f_0(r, \theta)\) and \(n_{2[1]}(r, \theta), n_{2[2]}(r, \theta)\) satisfying the above stated conditions. In this case we have \(\eta_1 = 1\) and \(q_1 = 1\) : introducing the gravitational polarizations \(\eta_\alpha\) for the data (31) and (37), we can compute the anchor coefficients by solving the algebraic equations (24) and define the Lie N–algebroid structure with any \(C^a_{bc}\) satisfying the conditions (9). We note that this metric can not be transformed into a vacuum because the (36) with \(w_3 \neq 0\) are possible for \(\varsigma^\tau \neq 1\) and \(\Upsilon_2 \neq 0\). Such type of gravitational algebroid configurations can be generated from the Taub NUT solution by nonholonomic transforms only by nontrivial matter sources or by any matter like corrections from extra dimension, for instance, in string theory. The polarization functions depend explicitly on extra coordinate.

Finally, we analyze the possibility to generate warped (on the extra dimension) gravitational algebroid configurations. In the original Randal–Sundrum scenaria [17] the Newtonian gravitational potential takes the form
\[
V(r) = G_N \frac{m_1 m_2}{r} \left( 1 + \frac{1}{r^2 k^2} \right)
\]
for two interacting point masses \(m_1\) and \(m_2\), where \(G_N\) is the 4D gravitational constant and \(k\) is the “warping” factor. In the papers [18, 6] we concluded that anholonomic coordinates can give rise to “anisotropic” deviations of the Newton potential, like
\[
V(r) = G_N \frac{m_1 m_2}{r} \left( 1 + \frac{e^{-2k y|y|}}{r^2 k^2} \right)
\]
where \(y\) can be a space like or extra dimension coordinate. We can consider such deviations in the Taub NUT algebroids. For instance, we consider instead of \(Q^{-1}\) a function
\[
\tilde{Q}^{-1} = 1 + \frac{m_0}{r} \left( 1 + \frac{1}{r^2 k^2} \right), m_0, k = \text{const.} \quad (38)
\]
It is possible to substitute \( Q \rightarrow \tilde{Q} \) and/or to include a factor of type \((1+e^{-2k_y|b|/r^2k^2})\) into \( \eta_5 \) for the class of solutions (34). In result, one generates warped static gravitational algebroids from a vacuum solution but with nonholonomic (off–diagonal) metric terms which play the role of source from the "isotropic" brane constructions. More similarities with the former brane constructions may be considered for the class of metrics (37) where \( \Upsilon_2 \) can be approximated as a constant tension (source of anisotropy), in general depending on extra dimension, for the second class of gravitational algebroids.

4 Einstein–Dirac Algebroids

The geometry of spinors on Clifford algebroids is elaborated in Ref. [1]. Such spinors can be included in the left or right minimal ideals of Clifford algebras generalized to Clifford spaces (C–spaces, elaborated in details in Refs. [10]), in our case provided with additional N–connections and/or algebroid structure. In this Section, we consider Dirac spinors defined with respect to N–anholonomic frames on 5D gravitational algebroid spaces.

4.1 Dirac equations on gravitational algebroids

For a d–metric (22) with coefficients

\[ g_{\alpha\beta}(u) = (g_{ij}(u), h_{ab}(u)) = (1, g_{\hat{i}}(u), h_{\hat{a}}(u)), \]

where \( \hat{i} = 1, 2; \hat{a} = 0, 1, 2; a = 3, 4 \), defined with respect to an N–adapted basis (5), we can easily define the funfbein (pentad) fields

\[
\begin{align*}
  e^{\mu}_i &= e^i_\mu \delta^{\mu}_i = \{e^i_\mu = e^i_\mu \delta_i, e^a_\mu = e^a_\mu \partial_a\}, \\
  e^{\mu}_a &= e^a_\mu \delta^{\mu}_a = \{e^a_\mu = e^a_\mu \delta^a, e^a_\mu = e^a_\mu \delta^a\}
\end{align*}
\]

satisfying the conditions

\[
\begin{align*}
  g_{ij} &= e^i_\gamma e^j_\gamma g_{\gamma \gamma} \text{ and } h_{ab} = e^a_\alpha e^b_\beta h_{\alpha \beta}, \\
  g_{\hat{i} \hat{i}} &= \text{diag}[1, -1, -1] \text{ and } h_{\hat{a} \hat{a}} = \text{diag}[-1, -1].
\end{align*}
\]

The d–metric (22) is effectively diagonal. This allows to write

\[
\begin{align*}
  e^i_\gamma &= \sqrt{|g_{\gamma \gamma}|} \delta^i_\gamma \text{ and } e^a_\alpha = \sqrt{|h_{\alpha \beta}|} \delta^a_\alpha,
\end{align*}
\]

where \( \delta^i_\gamma \) and \( \delta^a_\alpha \) are Kronecker's symbols.
The Dirac spinor fields on nonholonomically deformed Taub NUT spaces [6],

\[ \Psi (u) = [\Psi^\alpha (u)] = [\psi^I (u), \chi^I (u)], \]

where \( I = 0, 1 \), are defined with respect to the 4D Euclidean tangent subspace belonging the tangent space to the 5D N–anholonomic manifold \( V \). The 4 \times 4 dimensional gamma matrices \( \gamma^{\alpha'} = [\gamma^1, \gamma^2, \gamma^3, \gamma^4] \) are defined in the usual way, in order to satisfy the relation

\[ \{ \gamma^{\alpha'}, \gamma^{\beta'} \} = 2 g^{\alpha' \beta'}, \quad (40) \]

where \( \{ \gamma^{\alpha'}, \gamma^{\beta'} \} \) is a symmetric commutator, \( g^{\alpha' \beta'} = (-1, -1, -1, -1) \), which generates a Clifford algebra distinguished on two holonomic and two anholonomic directions. In order to extend the (40) relations for unprimed indices \( \alpha, \beta \ldots \) we conventionally complete the set of primed gamma matrices with a matrix \( \gamma^0 \), i.e. write \( \gamma^\alpha = [\gamma^0, \gamma^1, \gamma^2, \gamma^3, \gamma^4] \) when

\[ \{ \gamma^\alpha, \gamma^\beta \} = 2 g^{\alpha \beta}. \]

The coefficients of N–anholonomic gamma matrices can be computed with respect to anholonomic bases [5] by using respectively the funfbein coefficients

\[ \tilde{\gamma}^\beta (u) = e^\beta_\alpha (u) \gamma^\alpha. \]

We can also define an equivalent covariant derivation of the Dirac spinor field, \( \vec{\nabla}_\alpha \Psi \), by using pentad decompositions of the d–metric [22],

\[ \vec{\nabla}_\alpha \Psi = \left[ e_\alpha + \frac{1}{4} S_{\alpha \beta \gamma} (u) e_\alpha^\beta (u) \gamma^\beta \gamma^\gamma \right] \Psi, \quad (41) \]

where there are introduced N–elongated partial derivatives and the coefficients

\[ (S_{\alpha \beta \gamma} (u) = (D_{\gamma} e^\alpha_\alpha) e^\beta_\beta e^\gamma_\gamma \]

are transformed into rotation Ricci d–coefficients \( S_{\alpha \beta \gamma} \) which together with the d–covariant derivative \( D_{\gamma} \) are defined by anholonomic pentads and anholonomic transforms of the Christoffel symbols. In the canonical case, we should take the operator of canonical d–connection \( \hat{D}_{\gamma} \) with coefficients (16).

For diagonal d–metrics, the funfbein coefficients can be taken in their turn in diagonal form and the corresponding gamma matrix \( \tilde{\gamma}^\alpha (u) \) for anisotropic curved spaces are proportional to the usual gamma matrix in flat spaces \( \gamma^\gamma. \)
The Dirac equations on Clifford algebroids [6] are written in the simplest form with respect to anholonomic frames,

\[(i\hat{\gamma}^\alpha (u) \nabla_\alpha - \mu)\Psi = 0, \tag{42}\]

where \(\mu\) is the mass constant of the Dirac field. The Dirac equations are the Euler–Lagrange equations for the Lagrangian

\[
\mathcal{L}^{(1/2)}(u) = \sqrt{|g|}\{[\Psi^+ (u) \hat{\gamma}^\alpha (u) \nabla_\alpha \Psi (u) \\
- (\nabla_\alpha \Psi^+ (u))\hat{\gamma}^\alpha (u) \Psi (u)] - \mu \Psi^+ (u) \Psi (u)\}, \tag{43}\]

where by \(\Psi^+ (u)\) we denote the complex conjugation and transposition of the column \(\Psi (u)\). Varying \(\mathcal{L}^{(1/2)}\) on d–metric (43) we obtain the symmetric energy–momentum d–tensor

\[
\Upsilon_{\alpha\beta} (u) = \frac{i}{4}[\Psi^+ (u) \hat{\gamma}_\alpha (u) \nabla_\beta \Psi (u) + \Psi^+ (u) \hat{\gamma}_\beta (u) \nabla_\alpha \Psi (u) \\
- (\nabla_\alpha \Psi^+ (u))\hat{\gamma}_\beta (u) \Psi (u) - (\nabla_\beta \Psi^+ (u))\hat{\gamma}_\alpha (u) \Psi (u)]. \tag{44}\]

By straightforward calculations we can verify that because the conditions \(\hat{D}_\gamma e_\alpha^\alpha = 0\) are satisfied the Ricci rotation coefficients vanishes,

\[
\mathbf{S}_{\alpha\beta\gamma} (u) = 0 \text{ and } \nabla_\alpha \Psi = \delta_\alpha \Psi,
\]

and the N–anholonomic Dirac equations (42) transform into

\[(i\hat{\gamma}^\alpha (u) e_\alpha - \mu)\Psi = 0. \tag{45}\]

Further simplifications are possible for Dirac fields depending only on coordinates \((t, x^2 = r, x^3 = \theta)\), i.e. \(\Psi = \Psi(x^k)\) when the equation (45) transforms into

\[
(i\gamma^1 \partial_t + i\gamma^2 \frac{1}{\sqrt{|g_2|}} \partial_2 + i\gamma^3 \frac{1}{\sqrt{|g_3|}} \partial_3 - \mu)\Psi = 0.
\]

The equation (45) simplifies substantially in \(\zeta\)–coordinates

\[
(t, \zeta^2 = \zeta^2(r, \theta), \zeta^3 = \zeta^3(r, \theta)),
\]

defined as to be satisfied the conditions

\[
\frac{\partial}{\partial \zeta^2} = \frac{1}{\sqrt{|g_2|}} \partial_2 \text{ and } \frac{\partial}{\partial \zeta^3} = \frac{1}{\sqrt{|g_3|}} \partial_3 \tag{46}\]
We can consider a more simple equation

\[
(-i\gamma_1 \frac{\partial}{\partial t} + i\gamma_2 \frac{\partial}{\partial \zeta^2} + i\gamma_3 \frac{\partial}{\partial \zeta^3} - \mu)\Psi(t, \zeta^2, \zeta^3) = 0. \tag{47}
\]

The equation (47) describes the wave function of a Dirac particle of mass \(\mu\) propagating in a three dimensional Minkowski flat plane which is imbedded as an \(N\)–adapted distribution into a 5D Lie \(N\)–algebroid.

The solution \(\Psi = \Psi(t, \zeta^2, \zeta^3)\) of (47) is searched in the form

\[
\Psi = \begin{cases} 
\Psi^{(+)}(\zeta) = \exp[-i(k_1 t + k_2 \zeta^2 + k_3 \zeta^3)]\varphi^1(k) & \text{for positive energy;} \\
\Psi^{(-)}(\zeta) = \exp[i(k_1 t + k_2 \zeta^2 + k_3 \zeta^3)]\chi^1(k) & \text{for negative energy,}
\end{cases}
\]

with the condition that \(k_1\) is identified with the positive energy and \(\varphi^1(k)\) and \(\chi^1(k)\) are constant bispinors. To satisfy the Klein–Gordon equation we must have

\[k^2 = (k_1)^2 - (k_2)^2 - (k_3)^2 = \mu^2.\]

The Dirac equations implies

\[(\sigma^i k_i - \mu)\varphi^1(k) \text{ and } (\sigma^i k_i + \mu)\chi^1(k),\]

where \(\sigma^i (i = 1, 2, 3)\) are Pauli matrices corresponding to a realization of gamma matrices as to a form of splitting to usual Pauli equations for the bispinors \(\varphi^1(k)\) and \(\chi^1(k)\).

In the rest frame for the horizontal plane parametrized by coordinates \(\zeta = \{t, \zeta^2, \zeta^3\}\) there are four independent solutions of the Dirac equations,

\[
\varphi^1_{(1)}(\mu, 0) = \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}, \quad \varphi^1_{(2)}(\mu, 0) = \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix},
\]

\[
\chi^1_{(1)}(\mu, 0) = \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix}, \quad \chi^1_{(2)}(\mu, 0) = \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix}.
\]

We consider wave packets of type (for simplicity, we can use only superpositions of positive energy solutions)

\[
\Psi^{(+)}(\zeta) = \int \frac{d^3 p}{2\pi^3} \frac{\mu}{\sqrt{\mu^2 + (k^2)^2}} \sum_{[\alpha]=1,2,3} \frac{b(p, [\alpha])\varphi^{[\alpha]}(k) \exp [-ik_1\zeta]}{b^{[\alpha]}(p, [\alpha])}, \tag{48}
\]

22
when the coefficients $b(p, [\alpha])$ define a current (the group velocity)

$$J^2 \equiv \left\langle \frac{p^2}{\sqrt{\mu^2 + (k^2)^2}} \right\rangle = \sum_{[\alpha]=1,2,3} \int \frac{d^3p}{2\pi^3} \frac{\mu |b(p, [\alpha])|^2 p^2}{\sqrt{\mu^2 + (k^2)^2}}$$

with $|p^2| \sim \mu$ and the energy–momentum d–tensor (44) has nontrivial coefficients

$$\Upsilon^1_1 = 2\Upsilon(\zeta^2, \zeta^3) = k_1 \Psi^+ \gamma_2 \Psi, \quad \Upsilon^2_2 = -k_2 \Psi^+ \gamma_2 \Psi, \quad \Upsilon^3_3 = -k_3 \Psi^+ \gamma_3 \Psi$$

where the holonomic coordinates can be reexpressed $\zeta^i = \zeta^i(x^i)$. We must take two or more waves in the packet and choose such coefficients $b(p, [\alpha])$, satisfying corresponding algebraic equations, in order to get from (49) the equalities

$$\Upsilon^2_2 = \Upsilon^3_3 = \Upsilon(\zeta^2, \zeta^3) = \Upsilon(x^2, x^3),$$

required by the conditions (44).

Finally, in this Section, we note that the ansatz for the 5D metric (22) and 4D spinor fields depending on 3D h–coordinates $\zeta = \{t, \zeta^2, \zeta^3\}$ reduce the Dirac equations to the usual ones projected on a flat 3D spacetime. This configuration is $N$–adapted, because all coefficients are computed with respect to $N$–adapted frames. The spinor source $\Upsilon(x^2, x^3)$ induces a corresponding Clifford algebroid configuration.

### 4.2 Algebroid Taub NUT — Dirac Fields

In this subsection, we construct two new classes of solutions of the Einstein–Dirac fields generalizing the Taub NUT metrics defined by data (34) and (37) to be respective solutions of the Einstein equations, see (52)–(55) in the Appendix, with a nonvanishing diagonal energy momentum d–tensor

$$\Upsilon^\alpha_{\beta} = \{2\Upsilon(r, \theta), \Upsilon(r, \theta), \Upsilon(r, \theta), 0, 0\}$$

for a Dirac wave packet satisfying the conditions (49) and (50).

#### 4.2.1 Clifford algebroids with angular polarizations

The vacuum d–metric (34) was constructed by taking $\varsigma_\Upsilon = 1$ for $\Upsilon_2 = 0$ (28). For a nontrivial Dirac spinor source $\Upsilon_2 = \Upsilon_2^2 = \Upsilon(r, \theta)$ (50), we compute a nontrivial matter polarization $\varsigma_\Upsilon$ defined in general form by introducing this source in formula (28). This results in nonzero values of $w_i$, defined by $\varsigma_\Upsilon$ in formulas (29) and modified $n_i$ because $\varsigma_\Upsilon$ is also present in the formulas (30). The integration functions in $w_i$ and $n_i$ can be any
ones subjected to the conditions that \( w_1 = 0, n_1 = 0, w_{2,3} \to 0, n_3 \to 0 \) but \( n_2 \to 4m \) \((r, \theta, \varphi) (1 - \cos \theta)\) for \( \Upsilon \to 0 \). For a such source, the d–metric (54) transforms into the form

\[
\delta s^2 = dt^2 - \delta s^2_{(4D)},
\]

\[
\delta s^2_{(4D)} = -dr^2 - r^2 d\theta^2 - r^2 \sin^2 \theta \, \eta_1(r, \theta, \varphi) |\delta \varphi^2 - Q^2(r) \eta_5(r, \theta, \varphi) \delta \zeta^2;\]

\[
\delta \varphi = d\varphi + w_2(r, \theta, \varphi) dr + w_3(r, \theta, \varphi) d\theta,
\]

\[
\delta \zeta = d\zeta + n_2(r, \theta, \varphi) dr + n_3(r, \theta, \varphi) d\theta,
\]

where

\[
\zeta_\Upsilon (r, \theta, \varphi) = 1 - \frac{1}{12} \int \Upsilon(r, \theta, \varphi) \frac{\partial}{\partial \varphi} [f(r, \theta, \varphi) - f_0(r, \theta)]^3 d\varphi.
\]

The nontrivial N–connection coefficients are computed

\[
w_2 = -\frac{\partial \zeta_\Upsilon (r, \theta, \varphi)}{\partial \varphi \zeta_\Upsilon (r, \theta, \varphi)}, w_3 = -\frac{\partial \zeta_\Upsilon (r, \theta, \varphi)}{\partial \varphi \zeta_\Upsilon (r, \theta, \varphi)},
\]

for \( \partial_r = \partial / \partial r, \partial_\theta = \partial / \partial \theta, \partial_\varphi = \partial / \partial \varphi \), and

\[
n_{2,3} (r, \theta, \varphi) = n_{2,3[1]} (r, \theta) + n_{2,3[2]} (r, \theta) \times
\]

\[
\int \zeta_\Upsilon (r, \theta, \varphi) \frac{\partial}{\partial \varphi} [f(r, \theta, \varphi) - f_0(r, \theta)]^3 d\varphi.
\]

The data (33) are also modified because of \( \zeta_\Upsilon \)

\[
h^2_{[0]} (r) |f^*(r, \varphi)|^2 |\zeta_\Upsilon (r, \theta, \varphi)| = r^2 \sin^2 \theta \eta_4(r, \theta, \varphi).
\]

This results in a modifications of the Lie algebroid anchor structure functions because \( \zeta_\Upsilon \)–modified \( \eta_4 \) and \( h_4 \) (see formula (23)) change the solution of the algebraic equations for \( \hat{\varphi}_a \) (24). So, the anchor structure is also deformed if a vacuum gravitational algebroid is deformed to a such type of Einstein–Dirac algebroid.

### 4.2.2 Clifford algebroids with extra dimension polarizations

The solution (37) was constructed for a general matter source \( \Upsilon_2(r, \theta, \varsigma) \). The spinor source (50) can be considered a particular case when the matter energy–momentum tensor does not depend on extra dimension coordinate \( \varsigma \), i.e. \( \Upsilon_2 = \Upsilon(r, \theta) \), which results in a particular type of polarization \( \zeta_\Upsilon (r, \theta, \varsigma) \) computed just for a such \( \Upsilon_2(r, \theta) \), see data (36). So, the d–metric (37)
describes the solutions of the Einstein–Dirac equations as particular cases characterized by a proper anchor configuration because this type of $\zeta_\tau(r, \theta, \varsigma)$ is related to (35) and (24) defining the algebroid configuration. This type of Clifford algebroids are also static but with coefficients depending explicitly on extra dimension coordinate $\varsigma$. Such $d$-metrics are determined by the Dirac spinor field and do not have limits to vacuum configurations.

Finally, we emphasize that all types of solutions considered in this work can be generalized to stationary configurations by introducing certain coefficients $w_1 dt$ and/or $n_1 dt$ in the off–diagonal part. Such terms have to be computed by corresponding formulas (62) and (63) with nontrivial integration functions depending on $h$–coordinates.

5 Discussion and Conclusions

We have constructed a new class of nonholonomically deformed Taub NUT spacetimes possessing Lie algebroid symmetry. Such static gravitational algebroid configurations were generalized for nontrivial sources of Dirac spinor fields, i.e. the metrics were extended to define exact solutions of the Einstein–Dirac equations. They consist explicit examples of nonholonomic manifolds provided with distributions defining nonlinear connections when the Lie algebra symmetry was deformed to a Lie algebroid or Clifford algebroid symmetry.

There were distinguished two classes of 5D algebroid spacetimes: The first one is stated by solutions with angular local anisotropy and can extended from vacuum configurations to nonlinear gravitational spinor interactions. The second one describes nonholonomic gravitational configurations induced by spinor sources and depends explicitly on extra dimension coordinate. Such metrics do not have vacuum limits if the nonlinear connection and Lie algebroid structure functions are not trivial. All constructed classes of solutions have smooth limits to the usual Taub NUT metric, can be extended to stationary configurations and further deformed to other algebroid or non–algebroid symmetries, for instance, to ellipsoid or toroidal configurations, deformed to wormholes, with, or not, warped factors and/or 3D solitonic gravitational waves like was proven in Refs. [6].

Let us now conclude the properties of such Einstein–Dirac algebroids.

1. They are described by generic off–diagonal metrics and related nonholonomic vielbeins with associated nonlinear connection structure. Such Einstein, Einstein–Cartan and Einstein–Dirac spacetimes are with polarization of constants and metric and connection coefficients. In general, there are nontrivial torsion coefficients, induced by frame non-
holonomy. By imposing certain constrains the constructions can be transformed into 4D (pseudo) Riemannian configurations.

2. The metrics and connections for the solutions possessing algebroid symmetries depends on certain classes of integration functions. This is a typical property for the algebroid approaches to strings in gravity [8]. It follows from general properties of the systems of partial nonlinear equations to which the Einstein equations are reduced for generic off–diagonal ansatz. In the ’simplest’ case of diagonal ansatz depending on one variable (like the Schwarzshild metric) the gravitational field equations transform into a nonlinear second order differential equation. Its general solution depends on integration constants which are physically defined from certain symmetry and boundary conditions in order to get in the limit just the Newton potential for a component of metric. By applying the anholonomic frame method we construct more general classes of solutions derived for partial differential equations. Such solutions contain integration functions depending on two, three or four variables and the corresponding spacetimes possess symmetries characterized not only by structure constants but also by structure functions and nonholonomic distributions.

3. We can restrict a set of vacuum or nonvacuum off–diagonal metrics by imposing additional physical conditions like the Lie algebroid symmetry and fixing certain limits to well known solutions (for instance, to the Taub NUT metric). Nevertheless, even in such cases a certain dependence on some integrability functions is preserved. It can be eliminated only by fixing an exact system of reference with a prescribed type of nonlinear connection and structure anchor and Lie algebra type structure functions. This fixes the type of gravitational polarizations in the vicinity of some points on a nonholonomic manifold. But the general conclusion is that: when we deal with gravitational algebroids, we operate with classes of metrics and connections and corresponding classes of symmetries, i. e. with sets of spacetimes.

4. As explicit examples, we defined and analyzed some Clifford algebroid configurations describing packages of 3D Dirac waves self–consistently propagating in 5D nonholonomic spacetimes. They are distinguished with respect to the so–called N–adapted frames where the symmetry properties can be defined in the simplest way. With respect to local coordinate frames, the nonlinear gravitational–spinor interactions mix into generic off–diagonal metric and vielbein configurations depending on four coordinates.
5. In a more general context, the method can be applied for definition of algebroid solutions with additional noncommutative symmetries, supersymmetric and/or complex variables, quantum deformations in string gravity and for generalized Finsler spaces like we elaborated for ‘non–algebroid’ but also nonholonomic spaces [4, 6, 9, 18].

Acknowledgement: The work is supported by a sabbatical fellowship of the Ministry of Education and Research of Spain.

A Einstein Equations and Anholonomic Variables

We outline some necessary formulas for the non-trivial components of the Einstein equations (15) for the d-metric (22) and canonical d-connection (16), see references [1, 2, 4, 6] for details on proofs and related references. One obtained a system of partial differential equations:

\[ R_2^2 = R_3^3 = -\frac{1}{2g_2g_3} [g_2^{**} - g_2 g_3^{*} \left( \frac{(g_3^*)^2}{2g_3} + g_2^* - \frac{g_2 g_3'}{2g_3} - \frac{(g_2')^2}{2g_2} \right) \] (52)

\[ S_4^4 = S_5^5 = -\frac{1}{2h_4 h_5} \left[ h_5^{**} - h_5^* \left( \ln \sqrt{|h_4 h_5|} \right)^* \right] = -\Upsilon_2(x^2, x^3, v). \] (53)

\[ R_{4i} = -w_i \frac{\beta}{2h_5} - \frac{\alpha_i}{2h_5} = 0, \] (54)

\[ R_{5i} = -h_5 \frac{h_i^{**} + \gamma n_i^*}{2h_4} = 0, \] (55)

where

\[ \alpha_i = \partial_i h_5^* - h_5^* \partial_i \ln \sqrt{|h_4 h_5|}, \beta = h_5^{**} - h_5^* \left( \ln \sqrt{|h_4 h_5|} \right)^*, \] (56)

\[ \gamma = 3h_5^{**}/2h_5 - h_4^*/h_4 \]

\[ h_4^* \neq 0, h_5^* \neq 0, \]

cases with vanishing \( h_4^* \) and/or \( h_5^* \) should be analyzed additionally.

The system of second order nonlinear partial differential equations (52)–(55) can be solved in general form if there are given certain values of functions \( g_2(x^2, x^3) \) (or, inversely, \( g_3(x^2, x^3) \)), \( h_4(x^i, v) \) (or, inversely, \( h_5(x^i, v) \)), \( \omega(x^i, v) \) and of sources \( \Upsilon_2(x^2, x^3, v) \) and \( \Upsilon_4(x^2, x^3) \).

We outline the main steps of constructing exact solutions and for the case \( \Upsilon_4 = 0 \) when the equation (52), equivalently, (27), is solved by the
h–components of d–metric \( g_1 = 1, g_2 = -1 \) and \( g_3 = -(x^2)^2 \). For \( \Upsilon_2 = 0 \), the equation (53) relates two functions \( h_4(x^i, v) \) and \( h_5(x^i, v) \) following two possibilities:

- a) to compute

\[
\sqrt{|h_5|} = h_{5[1]}(x^i) + h_{5[2]}(x^i) \int \sqrt{|h_4(x^i, v)|} \, dv, \quad h_4^*(x^i, v) \neq 0;
\]

\[
= h_{5[1]}(x^i) + h_{5[2]}(x^i) v, \quad h_4^*(x^i, v) = 0,
\]

for some functions \( h_{5[1,2]}(x^i) \) stated by boundary conditions;

- b) or, inversely, to compute \( h_4 \) for a given \( h_5(x^i, v) \), \( h_5^* \neq 0 \),

\[
\sqrt{|h_4|} = h_{[0]}(x^i) (\sqrt{|h_5(x^i, v)|})^*,
\]

with \( h_{[0]}(x^i) \) given by boundary conditions. It is convenient two consider the parametrization

\[
|h_4| = h_{[0]}^2(x^i) \left[f^* (x^i, v)\right]^2 \quad \text{and} \quad |h_5(x^i, v)| = (f(x^i, v) + f_0(x^i))^2
\]

(59)
solving (58). We note that the sourceless equation (53) is satisfied by arbitrary pairs of coefficients \( h_4(x^i, v) \) and \( h_{5[0]}(x^i) \). Solutions with \( \Upsilon_2 \neq 0 \) can be found by ansatz of type

\[
h_5[\Upsilon_2] = h_5, \quad h_4[\Upsilon_2] = \varsigma_4(x^i, v) \ h_4,
\]

(60)

where \( h_4 \) and \( h_5 \) are related by formula (57), or (58). Substituting (60), we obtain

\[
\varsigma_4(x^i, v) = \varsigma_{4[0]}(x^i) - \int \Upsilon_2(x^2, x^3, v) \frac{h_4 h_5}{4h_5^*} \, dv,
\]

(61)

where \( \varsigma_{4[0]}(x^i) \) are arbitrary functions.

The exact solutions of (54) for \( \beta \neq 0 \) are defined from an algebraic equation, \( w_i \beta + \alpha_i = 0 \), where the coefficients \( \beta \) and \( \alpha_i \) are computed as in formulas (56) by using the solutions for (52) and (53). The general solution is

\[
w_k = \partial_k \ln[\sqrt{|h_4 h_5^*|/|h_5^*|}] / \partial_v \ln[\sqrt{|h_4 h_5|/|h_5^*|}],
\]

(62)

with \( \partial_v = \partial / \partial v \) and \( h_5^* \neq 0 \). If \( h_5^* = 0 \), or even \( h_5^* \neq 0 \) but \( \beta = 0 \), the coefficients \( w_k \) could be arbitrary functions on \( (x^i, v) \). For the vacuum Einstein
equations this is a degenerated case imposing the compatibility conditions \( \beta = \alpha_i = 0 \), which are satisfied, for instance, if the \( h_4 \) and \( h_5 \) are related as in the formula (55) but with \( h_{[0]} (x^i) = \text{const.} \).

Having defined \( h_4 \) and \( h_5 \) and computed \( \gamma \) from (56) we can solve the equation (55) by integrating on variable ”v” the equation \( n_{i}^{*} + \gamma n_{i}^{*} = 0 \). The exact solution is

\[
 n_k = n_{k[1]} (x^i) + n_{k[2]} (x^i) \int [h_4 / (\sqrt{|h_5|})^3] dv, \quad h_5^{*} \neq 0;
\]

\[
 = n_{k[1]} (x^i) + n_{k[2]} (x^i) \int h_4 dv, \quad h_5^{*} = 0; \quad (63)
\]

\[
 = n_{k[1]} (x^i) + n_{k[2]} (x^i) \int [1 / (\sqrt{|h_5|})^3] dv, \quad h_4^{*} = 0,
\]

for some functions \( n_{k[1,2]} (x^i) \) stated by boundary conditions.

Summarizing the results for the nondegenerated cases when \( h_4^{*} \neq 0 \) and \( h_5^{*} \neq 0 \) and (for simplicity, for a trivial conformal factor \( \omega \)), we derive an explicit result for 5D exact solutions for gravitational algebroids given by ansatz (26).

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