TOPOLOGICAL RANK DOES NOT INCREASE BY
NATURAL EXTENSION OF CANTOR MINIMALS

TAKASHI SHIMOMURA

Abstract. Downarowicz and Maass (2008) have defined the topological rank for all Cantor minimal homeomorphisms. On the other hand, Gambaudo and Martens (2006) have expressed all Cantor minimal continuous surjections as the inverse limits of certain graph coverings. Using the aforementioned results, we previously extended the notion of topological rank to all Cantor minimal continuous surjections. In this paper, we show that taking natural extensions of Cantor minimal continuous surjections does not increase their topological ranks. Further, we apply the result to the minimal symbolic case.

1. Introduction

In [DM08], Downarowicz and Maass presented a remarkable result, i.e., a Cantor minimal system of finite topological rank $K > 1$ is expansive. They used properly ordered Bratteli diagrams and adopted a noteworthy technique. In [BKM09], Bezuglyi, Kwiatkowski, and Medynets extended the result to non-minimal aperiodic homeomorphic cases. In this paper, a zero-dimensional system implies a pair $(X, f)$ of a compact zero-dimensional metrizable space $X$ and a continuous surjective map $f : X \to X$. A zero-dimensional system is a Cantor system if $X$ does not contain any isolated point. In [S14], we showed that every zero-dimensional system is expressed as an inverse limit of a sequence of covers of finite directed graphs. In this paper, instead of the term “sequence of graph covers,” we use the term “graph covering” or just “covering” for short. In [GM06], Gambaudo and Martens had already represented general Cantor minimal continuous surjections by a type of graph covering. In a previous paper [S15], we extended the definition of topological rank to Cantor minimal continuous surjections by applying the Gambaudo–Martens type of graph covering, and we showed that a Cantor minimal continuous surjection of finite topological rank $K > 1$ has a natural extension that is expansive. We also showed that the two topological ranks are equal to each other in the case of homeomorphic Cantor

Date: May 14, 2018.

2010 Mathematics Subject Classification. Primary 37B05, 37B10, 54H20.

Key words and phrases. graph covering, zero-dimensional, Bratteli diagram, topological rank, minimal, subshift.
minimal systems. In addition, we presented some related results. In this paper, we show that taking natural extensions of Cantor minimal continuous surjections does not increase their topological ranks. Further, we apply the result to the minimal symbolic case.

2. Preliminaries

Let \( \mathbb{Z} \) denote the set of all integers; \( \mathbb{N} \), the set of all non-negative integers; and \( \mathbb{N}^+ \), the set of all positive integers. In this section, to prepare graph coverings of the Gambaudo–Martens type, we repeat the construction of general graph coverings for general zero-dimensional systems originally given in [S14, §3]. For \( m \leq n \), we denote \( r_{n, m} =: [n, n + 1, \ldots, m] \). A pair \( G = (V, E) \) consisting of a finite set \( V \) and a relation \( E \subseteq V \times V \) on \( V \) can be considered as a directed graph with vertices \( V \) and an edge from \( u \) to \( v \) when \( (u, v) \in E \). Unlike the case of Bratteli diagrams, which are well known and defined in §3, multiple edges from a vertex \( u \) to \( v \) are not permitted. Here, we note that the expression \( (V, E) \) is also used to represent a Bratteli diagram. If we write “graph \( G \),” “graph \( G = (V, E) \),” or “surjective directed graph \( G = (V, E) \),” we imply a finite directed graph. When the expression \( (V, E) \) represents a Bratteli diagram, we explicitly write “Bratteli diagram \( (V, E) \).”

Notation 2.1. In this paper, we assume that a finite directed graph \( G \) is a surjective relation, i.e., for every vertex \( v \in V \), there exist edges \( (u_1, v), (v, u_2) \in E \).

For directed graphs \( G_i = (V_i, E_i) \) with \( i = 1, 2 \), a map \( \varphi : V_1 \to V_2 \) is said to be a graph homomorphism if for every edge \( (u, v) \in E_1 \), it follows that \( (\varphi(u), \varphi(v)) \in E_2 \). In this case, we write \( \varphi : G_1 \to G_2 \). For a graph homomorphism \( \varphi : G_1 \to G_2 \), we say that \( \varphi \) is edge-surjective if \( \varphi(E_1) = E_2 \). Suppose that a graph homomorphism \( \varphi : G_1 \to G_2 \) satisfies the following condition:

\[
(u, v), (u, v') \in E_1 \text{ implies that } \varphi(v) = \varphi(v').
\]

In this case, \( \varphi \) is said to be +directional. Suppose that a graph homomorphism \( \varphi \) satisfies both of the following conditions:

\[
(u, v), (u, v') \in E_1 \text{ implies that } \varphi(v) = \varphi(v') \text{ and } (u, v), (u', v) \in E_1 \text{ implies that } \varphi(u) = \varphi(u').
\]

Then, \( \varphi \) is said to be bidirectional.

Definition 2.2. A graph homomorphism \( \varphi : G_1 \to G_2 \) is called a cover if it is a +directional edge-surjective graph homomorphism.
For a sequence $G_1 \xleftarrow{\varphi_2} G_2 \xleftarrow{\varphi_3} \cdots$ of graph homomorphisms and $m > n$, we write $\varphi_{m,n} := \varphi_{n+1} \circ \varphi_{n+2} \circ \cdots \circ \varphi_m$. Then, $\varphi_{m,n}$ is a graph homomorphism. If all $\varphi_i$ $(i \in \mathbb{N}^+)$ are edge-surjective, then every $\varphi_{m,n}$ is edge-surjective. If all $\varphi_i$ $(i \in \mathbb{N}^+)$ are covers, every $\varphi_{m,n}$ is a cover. Let $G_0 := \{ \{ v_0 \} \} \subset V_0$. For a sequence of graph covers $G_1 \xleftarrow{\varphi_2} G_2 \xleftarrow{\varphi_3} \cdots$, we attach the singleton graph $G_0$ at the head. We refer to a sequence of graph covers $G_0 \xleftarrow{\varphi_1} G_1 \xleftarrow{\varphi_2} G_2 \xleftarrow{\varphi_3} \cdots$ as a graph covering or just a covering. Let us express the directed graphs as $G_i := (V_i, E_i)$ for $i \in \mathbb{N}$. Define

$$V_0 := \{ (x_0, x_1, x_2, \ldots) \in \prod_{i=0}^{\infty} V_i \mid x_i = \varphi_{i+1}(x_{i+1}) \text{ for all } i \in \mathbb{N} \}$$

and

$$E_0 := \{ (x, y) \in V_0 \times V_0 \mid (x_i, y_i) \in E_i \text{ for all } i \in \mathbb{N} \},$$

each equipped with the product topology. Further, each $V_i$ is equipped with the discrete topology.

**Notation 2.3.** Let $X = V_0$, and let us define a map $f : X \to X$ by $f(x) = y$ if and only if $(x, y) \in E_0$. For each $n \in \mathbb{N}$, the projection from $V_n$ to $V_n$ is denoted by $\varphi_{n,n}$. For $v \in V_n$, we denote a clopen set $U(v) := \varphi_{n,n}^{-1}(v)$. For a subset $V \subset V_n$, we denote a clopen set $U(V) := \bigcup_{v \in V} U(v)$.

We can state the following:

**Theorem 2.4** (S14 Theorem 3.9 and Lemma 3.5). Let $G$ be a covering $G_0 \xleftarrow{\varphi_1} G_1 \xleftarrow{\varphi_2} G_2 \xleftarrow{\varphi_3} \cdots$. Let $X = V_0$ and let us define $f : X \to X$ as above. Then, $f$ is a continuous surjective mapping and $(X, f)$ is a zero-dimensional system. Conversely, every zero-dimensional system can be written in this manner. Furthermore, if all $\varphi_n$ are bidirectional, then this zero-dimensional system is a homeomorphism and every compact zero-dimensional homeomorphism is written in this manner.

We write $(V_G, E_G)$ as $G_\infty$. Take a subsequence $n_0 = 0 < n_1 < n_2 < \cdots$. Then, we can get essentially the same covering

$$G_0 \xleftarrow{\varphi_{n_1,0}} G_{n_1} \xleftarrow{\varphi_{n_2,n_1}} G_{n_2} \cdots.$$ 

It is evident that the new covering produces a naturally topologically conjugate zero-dimensional system. Following the terminology in the theory of Bratteli–Vershik systems, we refer to this procedure as **telescoping**.

**Notation 2.5.** Let $G = (V, E)$ be a surjective directed graph. A sequence of vertices $(v_0, v_1, \ldots, v_l)$ of $G$ is said to be a walk of length $l$ if $(v_i, v_{i+1}) \in E$ for all $0 \leq i < l$. We denote $l(w) := l$. We say that a walk $w = (v_0, v_1, \ldots, v_l)$ is a path if $v_i$ $(0 \leq i \leq l)$ are mutually distinct. A walk $c = (v_0, v_1, \ldots, v_l)$ is said to be a cycle of period $l$ if $v_0 = v_l$, and a cycle $c = (v_0, v_1, \ldots, v_l)$
is a circuit of period $l$ if $v_i \ (0 \leq i < l)$ are mutually distinct. Further, a circuit $c$ and a path $p$ are considered to be subgraphs of $G$ with period $l(c)$ and length $l(p)$, respectively. Let $\mathcal{C}(G)$ be the set of all circuits of $G$. For a walk $w = (v_0, v_1, \ldots, v_l)$, we define $V(w) := \{ v_i \mid 0 \leq i \leq l \}$ and $E(w) := \{ (v_i, v_{i+1}) \mid 0 \leq i < l \}$. For a subgraph $G'$ of $G$, we define $V(G')$ and $E(G')$ in the same manner; in particular, $V(G) = V$ and $E(G) = E$.

Next, we introduce a proposition that describes a condition of minimality of the inverse limit of a graph covering.

**Proposition 2.6.** Let $G_0 \overset{\varphi_1}{\rightarrow} G_1 \overset{\varphi_2}{\rightarrow} G_2 \overset{\varphi_3}{\rightarrow} \cdots$ be a covering. Then, the resulting zero-dimensional system $G_\infty$ is minimal if and only if for all $n \geq 0$, there exists an $m > n$ such that every $c \in \mathcal{C}(G_m)$ satisfies $V(\varphi_m,n(c)) = V(G_n)$.

**Proof.** From [S16a] (a),(d),(e) of Theorem 3.5, the conclusion is obvious. □

3. Bratteli–Vershik systems

**Definition 3.1.** A Bratteli diagram is an infinite directed graph $(V, E)$, where $V$ is the vertex set and $E$ is the edge set. These sets are partitioned into non-empty disjoint finite sets $V = V_0 \cup V_1 \cup V_2 \cup \cdots$ and $E = E_1 \cup E_2 \cup \cdots$, where $V_0 = \{ v_0 \}$ is a one-point set. Each $E_n$ is a set of edges from $V_{n-1}$ to $V_n$. Therefore, there exist two maps $r, s : E \rightarrow V$ such that $r : E_n \rightarrow V_n$ and $s : E_n \rightarrow V_{n-1}$ for $n \geq 1$, i.e., the range map and the source map, respectively. Moreover, $s^{-1}(v) \neq \emptyset$ for all $v \in V$ and $r^{-1}(v) \neq \emptyset$ for all $v \in V \setminus V_0$. We say that $u \in V_{n-1}$ is connected to $v \in V_n$ if there exists an edge $e \in E_n$ such that $s(e) = u$ and $r(e) = v$. Unlike the case of graph coverings, multiple edges between $u$ and $v$ are permitted. The rank $K$ of a Bratteli diagram is defined as $K := \liminf_{n \rightarrow \infty} \#V_n$, where $\#V_n$ is the number of elements in $V_n$.

Let $(V, E)$ be a Bratteli diagram and $m < n$ be non-negative integers. We define

$$E_{m,n} := \{ p \mid p \text{ is a path from a } u \in V_m \text{ to a } v \in V_n \}.$$

Then, we can construct a new Bratteli diagram $(V', E')$ as follows:

$$V' := V_0 \cup V_1 \cup \cdots \cup V_m \cup V_n \cup V_{n+1} \cup \cdots$$

$$E' := E_1 \cup E_2 \cup \cdots \cup E_m \cup E_{m,n} \cup E_{n+1} \cup \cdots.$$

The source map and the range map are also defined naturally. This procedure is called telescoping.

**Definition 3.2.** A Bratteli diagram is called simple if, after (at most countably many) telescopings, we get that for all $n \geq 0$, all pairs of vertices $u \in V_n$ and $v \in V_{n+1}$ are joined by at least one edge.
Definition 3.3. Let \((V, E)\) be a Bratteli diagram such that \(V = V_0 \cup V_1 \cup V_2 \cup \cdots\) and \(E = E_1 \cup E_2 \cup \cdots\) are the partitions, where \(V_0 = \{ v_0 \}\) is a one-point set. Let \(r, s : E \to V\) be the range map and the source map, respectively. We say that \((V, E, \leq)\) is an ordered Bratteli diagram if the partial order \(\leq\) is defined on \(E\) such that \(e, e' \in E\) are comparable if and only if \(r(e) = r(e')\). In other words, we have a linear order on each set \(r^{-1}(v)\) with \(v \in V \setminus V_0\). The edges \(r^{-1}(v)\) are numbered from 1 to \(\#(r^{-1}(v))\).

Let \(n > 0\) and \(e = (e_n, e_{n+1}, e_{n+2}, \ldots), e' = (e'_n, e'_{n+1}, e'_{n+2}, \ldots)\) be cofinal paths from the vertices of \(V_{n-1}\), which might be different. We obtain the lexicographic order \(e < e'\) as follows:

\[
\text{if } k \geq n \text{ is the largest number such that } e_k \neq e'_k, \text{ then } e_k < e'_k.
\]

Definition 3.4. Let \((V, E, \leq)\) be an ordered Bratteli diagram. Let \(E_{\max}\) and \(E_{\min}\) denote the sets of maximal and minimal edges, respectively. A path is maximal (resp. minimal) if all the edges constituting the path are elements of \(E_{\max}\) (resp. \(E_{\min}\)).

Definition 3.5. An ordered Bratteli diagram is properly ordered if it is simple and if it has a unique maximal path and a unique minimal path, denoted respectively by \(x_{\max}\) and \(x_{\min}\).

Definition 3.6 (Vershik map). Let \((V, E, \leq)\) be a properly ordered Bratteli diagram. Let

\[
E_{0,\infty} := \{(e_1, e_2, \ldots) \mid r(e_i) = s(e_{i+1}) \text{ for all } i \geq 1\},
\]

with the subspace topology of the product space \(\prod_{i=1}^{\infty} E_i\). We can define a Vershik map \(\phi : E_{0,\infty} \to E_{0,\infty}\) as follows:

If \(e = (e_1, e_2, \ldots) \neq x_{\max}\), then there exists the least \(n \geq 1\) such that \(e_n\) is not maximal in \(r^{-1}(r(e_n))\). Then, we can select the least \(f_n > e_n\) in \(r^{-1}(r(e_n))\). Let \(v_{n-1} = s(f_n)\). Then, it is easy to obtain the unique least path \((f_1, f_2, \ldots, f_{n-1})\) from \(v_0\) to \(v_{n-1}\). We define

\[
\phi(e) := (f_1, f_2, \ldots, f_{n-1}, f_n, e_{n+1}, e_{n+2}, \ldots).
\]

Further, we define \(\phi(x_{\max}) = x_{\min}\). The map \(\phi : E_{0,\infty} \to E_{0,\infty}\) is called the Vershik map.

On the basis of a previously introduced theorem [HPS92, Theorem 4.7], we can find a correspondence that a properly ordered Bratteli diagram brings about the Vershik map that is a minimal homeomorphic zero-dimensional system. Conversely, a minimal homeomorphic zero-dimensional system is represented as the Vershik map of a properly ordered Bratteli diagram. In [DM08], Downarowicz and Maass introduced the topological rank for Cantor minimal homeomorphisms.
Definition 3.7. Let \((X, f)\) be a Cantor minimal homeomorphism. Then, the topological rank of \((X, f)\) is \(1 \leq K \leq \infty\) if it has a Bratteli–Vershik representation with a Bratteli diagram of rank \(K\), and \(K\) is the minimum of such numbers.

4. Covering of Gambaudo–Martens Type

In this section, we introduce a covering of the Gambaudo–Martens type and define the topological rank for all Cantor minimal continuous surjections. Then, we prepare the proof of our main result. In [GM06 Theorem 2.5], Gambaudo and Martens showed that every Cantor minimal system is an inverse limit of a special type of graph covering. In our context, their construction of a graph covering is as follows. Let \(G_0 \leftarrow^\varphi_1 G_1 \leftarrow^\varphi_2 G_2 \leftarrow^\varphi_3 \cdots\) be a graph covering. As usual, we assume that \(G_0\) is a singleton graph \(\{(v_0), (v_0, v_0)\}\). We shall construct graphs \(G_n\) with an \(n \geq 1\) such that there exist a unique vertex \(v_{n,0}\) and a finite number of circuits \(c_{n,i} (1 \leq i \leq r_n)\) that start and end at \(v_{n,0}\). Roughly, if two circuits meet at a vertex, then the remaining circuits merge until they reach the end.

Definition 4.1. We say that a covering \(G_0 \leftarrow^\varphi_1 G_1 \leftarrow^\varphi_2 G_2 \leftarrow^\varphi_3 \cdots\) is of the Gambaudo–Martens type if for each \(n > 0\), there exist a vertex \(v_{n,0}\), a finite number of circuits \(c_{n,i} (1 \leq i \leq r_n)\), and a covering map \(\varphi_n\) such that

(a) \(c_{n,i}\) can be written as \((v_{n,0} = v_{n,i,0}, v_{n,i,1}, v_{n,i,2}, \ldots, v_{n,i,l(n,i)} = v_{n,0})\) with \(l(n,i) \geq 1\),

(b) \(\bigcup_{i=1}^r E(c_{n,i}) = E(G_n)\),

(c) if \(v_{n,i,j} = v_{n,i',j'}\) with \(j, j' \geq 1\), then \(v_{n,i,j+k} = v_{n,i',j'+k}\) for \(k = 0, 1, 2, \ldots\), until \(j+k = l(n,i)\) and \(j'+k = l(n,i')\) at the same time,

(d) \(\varphi_n(v_{n,0}) = v_{n-1,0}\) for all \(n \geq 1\), and

(e) \(\varphi_n(v_{n,i,1}) = v_{n-1, i,1}\) for all \(n \geq 1\) and \(1 \leq i \leq r_n\).

We say that a covering of this type is a GM-covering for short. We denote \(\mathcal{C}_n := \mathcal{C}(G_n) = \{c_{n,i} \mid 1 \leq i \leq r_n\}\). A GM-covering is said to be simple if for all \(n > 0\), there exists an \(m > n\) such that for each \(1 \leq i \leq r_m\), \(E(\varphi_{m,n}(c_{m,i})) = E(G_n)\). By Proposition 2.6, this condition makes the resulting zero-dimensional system minimal. If we want to avoid the case in which the resulting zero-dimensional system has an isolated point, we have to add the following condition: for every \(n \geq 1\) and every vertex \(v\) of \(G_n\), there exist an \(m > n\) and distinct vertices \(u_1, u_2\) of \(G_m\) such that \(\varphi_{m,n}(u_1) = \varphi_{m,n}(u_2) = v\). The rank of a GM-covering is the integer \(1 \leq K \leq \infty\) defined by \(K := \liminf_{n \to \infty} r_n\).

Remark 4.2. For \(n \geq 0\) and \(1 \leq i \leq r_n\), we can write

\[\varphi_n(c_{n,i}) = c_{n-1,a(n,i,1)c_{n-1,a(n,i,2)} \cdots c_{n-1,a(n,i,k(n,i))}};\]
such that $a(n, i, 1) = 1$ for all $i$ with $1 \leq i \leq r_n$.

**Notation 4.3.** By telescoping, we can add the following condition to a simple GM-covering: for every $n \geq 1$ and every $i$ ($1 \leq i \leq r_n$), $E(\varphi_n(c_{n,i})) = E(G_{n-1})$. Hereafter, if we say that a GM-covering is simple, we assume that this condition is satisfied.

**Theorem 4.4** (Gambaudo and Martens, [GM06]). A zero-dimensional system is minimal (not necessarily homeomorphic) if and only if it is represented as the inverse limit of a simple GM-covering.

**Proof.** See [GM06] Theorem 2.5. \hfill $\square$

As an analogue of topological rank for Cantor minimal homeomorphisms, we say that a minimal zero-dimensional system has **topological rank** $K$ if there exists a simple GM-covering of rank $K$, and $K$ is the minimum of such numbers (see [DM08]). In [DM08], it was shown that a Cantor minimal homeomorphism with finite topological rank $K > 1$ is expansive, i.e., topologically conjugate to a minimal two-sided subshift. The remainder of this section is devoted to preparing the statement of our main result and its proof. Suppose that a simple GM-covering $G_0 \xleftarrow{\partial_1} G_1 \xleftarrow{\partial_2} G_2 \xleftarrow{\partial_3} \cdots$ produces a minimal zero-dimensional system $G_\infty$. We write $G_\infty = (X, f)$. We assume that $(X, f)$ is not a single periodic orbit. Then, because of minimality, $(X, f)$ is a Cantor system and has no periodic orbits. Therefore, the minimal length of the circuits of $G_n$ becomes infinity, i.e., we get $l(n, i) \to \infty$ uniformly as $n \to \infty$.

**Notation 4.5.** For $(X, f)$, we construct the natural extension $(\hat{X}_f, \sigma)$ as follows:

- $\hat{X}_f := \{ \ldots, x_{-1}, x_0, x_1, x_2, \ldots \} \in X^\mathbb{Z}$ if $f(x_i) = x_{i+1}$ for all $i \in \mathbb{Z}$;
- for $\hat{x} = (\ldots, x_{-1}, x_0, x_1, x_2, \ldots) \in \hat{X}_f$, $\sigma$ shifts $\hat{x}$ to the left, i.e., $(\sigma(\hat{x}))_i = x_{i+1}$ for all $i \in \mathbb{Z}$.

It is easy to check that if $(X, f)$ is minimal, then $(\hat{X}_f, \sigma)$ is minimal. For an $\hat{x} \in \hat{X}_f$ and an $i \in \mathbb{Z}$, we denote $\hat{x}(i) := x_i$. Then, $(\sigma(\hat{x}))(i) = \hat{x}(i + 1)$ for all $i \in \mathbb{Z}$. We use many notations and concepts from [DM08].

For every $\hat{x} \in \hat{X}_f$ and $i \in \mathbb{Z}$, there exists a unique $u_{n,i} \in V(G_n)$ such that $x_i \in U(u_{n,i})$. Therefore, for each $n \geq 0$, a unique sequence $\hat{x}|_n = (\ldots, u_{n-2}, u_{n-1}, u_{n,0}, u_{n,1}, \ldots)$ of vertices of $G_n$ is defined such that $x_i \in U(u_{n,i})$ for all $i \in \mathbb{Z}$. We write $\hat{x}|_n(i) := u_{n,i}$ for all $i \in \mathbb{Z}$. Although the vertex $u_{n,i}$ is uniquely determined for each $\hat{x}$, $n \geq 0$ and $i \in \mathbb{Z}$, the circuit $c_{n,t}$ with $u_{n,i} \in V(c_{n,t})$ may not be unique. Nevertheless, if $x_i \in U(v_{n,0})$ for some
$\begin{array}{c|c|c|c|c|c|c|}
\hat{x}_{n,1} & \hat{x}_{n,3} & \ldots & \hat{x}_{n,15} & \hat{x}_{n,1} & \hat{x}_{n,2} & \hat{x}_{n,13} \\
\hat{x}_{n+1,1} & \hat{x}_{n+1,3} & \ldots & \hat{x}_{n+1,15} & \hat{x}_{n+1,1} & \hat{x}_{n+1,2} & \hat{x}_{n+1,13}
\end{array}$

**Figure 1.** $n$th and $(n+1)$th rows of a linked array system with cuts.

$i \in \mathbb{Z}$, then there exists a unique $t$ $(1 \leq t \leq r_n)$ such that $x_{i+1} \in U(v_{n,t,1})$; therefore, $x_{i+j} \in U(v_{n,t,j})$ for all $0 \leq j \leq l(n,t)$.

**Notation 4.6.** We write this $t$ as $t(\hat{x}, n, i)$, and $c_{n,t}$ as $c(\hat{x}, n, i)$, for all $n \geq 0$ and $i \in \mathbb{Z}$.

Let $k(0) \in \mathbb{Z}$ such that $x_{k(0)} \in U(v_{n,0})$, and let $k(1) > k(0)$ be the least $k > k(0)$ such that $x_k \in U(v_{n,0})$. Then, we combine the interval $u_{n,k(0)}, u_{n,k(0)+1}, \ldots, u_{n,k(1)-1}$ with the unique circuit $c(\hat{x}, n, i)$ with $k(0) \leq i < k(1)$. Thus, we obtain a sequence of $c_{n,8}$, and we denote it as $\hat{x}[n]$. We write $\hat{x}[n](i) = c(\hat{x}, n, i)$ for all $n \geq 0$ and $i \in \mathbb{Z}$. To mark the beginning of a circuit, it is sufficient to change $c(\hat{x}, n, i)$ to $\hat{c}(\hat{x}, n, i)$ for each $i \in \mathbb{Z}$ with $u_{n,i} = v_{n,0}$. Nevertheless, as in [DM08], for each sequence $\hat{x}[n]$ of circuits of $G_n$ ($n > 0$), instead of changing the symbol, we make an $n$-cut at position $i \in \mathbb{Z}$ just before $i \in \mathbb{Z}$ with $\hat{x}[n](i) = v_{n,0}$, i.e., if there exists an $n$-cut at position $i \in \mathbb{Z}$, then $c(\hat{x}, n, i-1)$ and $c(\hat{x}, n, i)$ are separated by the cut (see Figure 1). Note that we can recover the sequence of vertices of $G_n$ from $\hat{x}[n]$. An $\hat{x}[0]$ becomes just a sequence of $e_0 := (v_0, v_0)$. For an interval $[n, m]$ with $m > n$, the combination of rows $\hat{x}|_n^m$ with $n \leq n' \leq m$ is denoted as $\hat{x}|_{[n,m]}$, and the combination of rows $\hat{x}|^m$ with $n \leq n' \leq m$ is denoted as $\hat{x}|_{[n,m]}$. The array system of $\hat{x}$ is the infinite combination $\hat{x}|_{[0,\infty)}$ of all rows $\hat{x}|_n^m$ $0 \leq n < \infty$. The linked array system of $\hat{x}$ is the infinite combination $\hat{x}|_{[0,\infty)}$ of all rows $\hat{x}[n]$ ($0 \leq n < \infty$) (see Figure 2). Note that from the information of $\hat{x}|_{[0,\infty)}$, we can recover $\hat{x}|_{[0,\infty)}$ and also identify $\hat{x}$ itself. If each circuit of $G_n$ is considered to be just an alphabet, then for $n \geq 0$ and $I < J$, we can consider a finite sequence of circuits of $G_n$,

$$
\hat{x}[n](I), \hat{x}[n](I+1), \ldots, \hat{x}[n](J),
$$

even if the completion of the circuits is cut off at the right or left end in the above sequence.

**Definition 4.7.** Let $\hat{X}_f := \{ \hat{x}|_{[0,\infty)} \mid \hat{x} \in \hat{X}_f \}$ be a set of sequences of symbols that are vertices of $G_n$ $(0 \leq n < \infty)$. The topology is generated by cylinders such that for $\hat{x} \in \hat{X}_f$ and $N, I > 0$,

$$
\hat{C}(\hat{x}, N, I) := \{ \hat{y}|_{[0,\infty)} \mid \hat{y} \in \hat{X}_f, \hat{y}|_{[0,N]}(i) = \hat{x}|_{[0,N]}(i) \text{ for all } i \text{ with } -I \leq i \leq I \}.
$$

The shift map $\sigma : \hat{X}_f \rightarrow \hat{X}_f$ is defined as above. Then, $(\hat{X}_f, \sigma)$ is a zero-dimensional system, and we refer to it as an array system of $(\hat{X}_f, \sigma)$. Let
For $m$ is also defined. The set $X_n$ determine a set of circuits arranged in a square form as in Figure 3. Circuits by $\phi_c$ determine a set of circuits arranged in a square form as in Figure 3. Let $\phi_c : X_n \to X_n$ denote a set of circuits arranged in a square form as in Figure 3.

**Remark 4.8.** Clearly, $(\hat{X}_f, \sigma)$ is topologically conjugate to $(\hat{X}_f, \sigma)$. Moreover, because $(\hat{X}_f, \sigma)$ has a continuous factor map to $(\hat{X}_f, \sigma)$ and it is bijective as described above, $(\hat{X}_f, \sigma)$ is also topologically conjugate to $(\hat{X}_f, \sigma)$.

The row $\hat{x}[n]$ is precisely separated into circuits by the cuts. Note that for $m > n$, if there exists an $m$-cut at position $k$, then there exists an $n$-cut at position $k$. For each circuit $c_{n,i}$, we can determine a series of circuits by $\varphi_n(c_{n,i}) = c_{n-1,1}c_{n-1,a(n,i,1)}c_{n-1,a(n,i,2)} \cdots c_{n-1,a(n,i,k(n,i))}$. Furthermore, each $c_{n-1,a(n,i,j)}$ determines a series of circuits by the map $\varphi_{n-1}$. Thus, we can determine a set of circuits arranged in a square form as in Figure 3. Following [DM08], this form is said to be the $n$-symbol and denoted by $c_{n,i}$.

For $m < n$, the projection $c_{n,i}[m]$ that is a finite sequence of circuits of $G_m$ is also defined. The set $X_n := \{ \hat{x}[n] \mid \hat{x} \in \hat{X}_f \}$ is a two-sided subshift of the finite set $\mathcal{C}_n \cup \{ \hat{c}_{n,1}, \hat{c}_{n,2}, \ldots, \hat{c}_{n,r_n} \}$. The factoring map is denoted by $\pi_n : \hat{X}_f \to X_n$, and the shift map is denoted by $\sigma_n : X_n \to X_n$. We simply write $\sigma = \sigma_n$ for all $n$ if there is no confusion.

Next, we wish to briefly recall the construction of the array system in [DM08]. Let $(V, E, \geq)$ be a properly ordered Bratteli diagram with the Vershik map $\phi : E_{0,\infty} \to E_{0,\infty}$. Let $x \in E_{0,\infty}$. We write $\phi^n(x) = (e_1, e_2, \ldots)$ for all $i \in \mathbb{Z}$. Then, we can construct a sequence $v_{n,i} = s(e_{n+1,i})$ for all $n \geq 0$ and $i \in \mathbb{Z}$. For each $n \geq 0$, we denote $v_x[n] := (v_{n,i})_{i \in \mathbb{Z}}$ and

**Figure 2.** The first 4 rows of a linked array system.

| $c_0$ | $c_1$ | $c_2$ | $c_3$ |
|-------|-------|-------|-------|
| $c_{1,1}$ | $c_{1,2}$ | $c_{1,3}$ | $c_{1,4}$ |
| $c_{2,1}$ | $c_{2,2}$ | $c_{2,3}$ | $c_{2,4}$ |
| $c_{3,1}$ | $c_{3,2}$ | $c_{3,3}$ | $c_{3,4}$ |
| $c_{4,1}$ | $c_{4,2}$ | $c_{4,3}$ | $c_{4,4}$ |

**Figure 3.** The 2-symbol corresponding to the circuit $c_{2,1}$ of Figure 2.
the combination of these lines as $v_x := v_x[0,\infty)$. For each $n \geq 0$, we make an $n$-cut by the following argument. For each $v \in V_n$, we define $P_v := \{(e_1,e_2,\ldots,e_{n-1}) \mid r(e_{n-1}) = v\}$. We make an $n$-cut just before $(e_1,e_2,\ldots,e_{n-1},\ldots) \in E_{0,\infty}$ such that $(e_1,e_2,\ldots,e_{n-1}) \in P_v$ is minimal. Thus, as in the case of the previous argument of a GM-covering, $v_x[n]$ gets $n$-cuts. We define $Y(V,E,\geq) := \{v_x \mid x \in E_{0,\infty}\}$. Let $Y = Y(V,E,\geq)$ and let $\sigma : Y \to Y$ be the left shift. Then, $(E_{0,\infty},\phi)$ is naturally topologically conjugate to $(Y,\sigma)$.

5. Main Theorem.

In this section, we state our main result and prove the theorem.

**Theorem 5.1** (Main Result). Let $(X,f)$ be a minimal (not necessarily homeomorphic) zero-dimensional system with topological rank $K \geq 1$. Then, its natural extension $(\tilde{X}_f,\sigma)$ has topological rank $\leq K$.

**Proof.** By the assumption, there exists a simple GM-covering $G : G_0 \xleftarrow{\varphi_1} G_1 \xleftarrow{\varphi_2} G_2 \xleftarrow{\varphi_3} \cdots$ of rank $K$ such that $G_0$ is topologically conjugate to $(X,f)$. Therefore, we assume that $G_0 = (X,f)$. In §4, we defined a linked array system $(\tilde{X}_f,\sigma)$. We have stated that

$$\varphi_n(c_{n,i}) = c_{n-1,1}c_{n-1,a(n,i),2}\cdots c_{n-1,a(n,i,k(n,i))}$$

for each $1 \leq i \leq r_n$.

By telescoping, we can assume that $k(n,i) > 2$ for all $1 \leq i \leq r_n$ and $a(n,i,2)$ is independent of $i$. We write $a(n) := a(n,i,2)$. Thus, we can write $\varphi_n(c_{n,i}) = c_{n-1,1}c_{n-1,a(n,i)}d_{n-1,i}$. From here, we make another symbolic linked array system $(Y,\sigma)$. Later, we check whether the symbolic linked array system is actually linked with a properly ordered Bratteli diagram. To make another array system, let $s_n = l(c_{n-1,1})$ for all $n \geq 2$. We denote $S(n) := \sum_{i=2}^{n} s(i)$. Let $\tilde{x} \in \tilde{X}_f$. For each $n \geq 1$, we make another sequence $\tilde{y}[n](i) := \tilde{x}[n](i+S(n))$, i.e., we make different slides for each line $\tilde{x}[n]$ ($n \geq 1$). Let $\tilde{y}_{\tilde{x}} := \tilde{y}[0,\infty)$. The set $Y := \{\tilde{y}_{\tilde{x}} \mid \tilde{x} \in \tilde{X}_f\}$ is a subspace of $\prod_{n \geq 1} \mathcal{C}_n$ with the product topology. We denote the map $\phi : \tilde{X}_f \to Y$ by $\phi(\tilde{x}) = \tilde{y}_{\tilde{x}}$. Evidently, $\phi$ is a bijection. Obviously, $\phi$ is continuous and a homeomorphism. Let $\sigma : Y \to Y$ be the left shift. Then, it is easy to see that $\sigma \circ \phi = \phi \circ \sigma$. Therefore, $(Y,\sigma)$ is topologically conjugate to $(\tilde{X}_f,\sigma)$. It is easy to check that after the slides, the cuts do not have conflicts in different levels, i.e., for $n > m \geq 0$, if an $n$-cut occurred at position $i$, then an $m$-cut has to occur at position $i$ (see Figure 4). For each $n \geq 1$, originally, each $c_{n,i}$ is projected to $c_{n-1,1}c_{n-1,a(n)}d_{n-1,i}$ by the graph map $\varphi_n$. After the slides, $c_{n,i}$ in the $n$th level is projected to $c_{n-1,a(n)}d_{n-1,i}c_{n-1,1}$ (see Figure 4). We write $\varphi'_n(c_{n,i}) := \varphi_n(c_{n,i})$. 

TAKASHI SHIMOMURA
Let $\mathcal{G}$ be the symbolic system obtained by the natural extension of $(\Sigma, \sigma)$. By Theorem 6.1 in [S13], we conclude that the natural extension is expansive. Therefore, the natural extension is not an odometer. Thus, by Theorem 6.1 in [S13], we can conclude that the natural extension has topological rank $L > 1$. Next, let $(\Sigma', \sigma)$ be a two-sided minimal subshift with finite topological rank $K > 1$. By the one-sided factor, we mean the one-sided minimal subshift $(\Sigma^+, \sigma)$ that is made by cutting off negative coordinates. Then, the natural extension of $(\Sigma', \sigma)$ is canonically isomorphic to $(\Sigma, \sigma)$. Thus, if $L$ is the topological rank of $(\Sigma^+, \sigma)$, then our main result concludes $L \geq K$. By Theorem 6.8 in [S16b], we can find a two-sided subshift $(\Sigma', \sigma)$ that is topologically conjugate to $(\Sigma, \sigma)$ such that the one-sided factor $(\Sigma'^+, \sigma)$ has topological rank $K$. Thus, we get the next corollary:

**Corollary 5.2.** Let $(\Sigma, \sigma)$ be a two-sided minimal subshift with topological rank $K$. Then, the one-sided factor has topological rank $\geq K$. Furthermore, there exists a two-sided minimal subshift $(\Sigma', \sigma)$ such that $(\Sigma', \sigma)$ is topologically conjugate to $(\Sigma, \sigma)$ and the one-sided factor has topological rank $K$. 

---

**Figure 4.** In $[n-1, n]$ lines, $\bar{x}$ is slid to $\bar{y}_x$. 

$$
\begin{array}{cccc}
\bar{x}[n-1, n] \\
d_{n-1,3} & c_{n-1,1} & c_{n-1,a(n)} & d_{n-1,i} \\
c_{n,j} & c_{n,i} & d_{n-1,i} & c_{n-1,1} \\
\end{array}
$$

$$
\begin{array}{cccc}
\bar{y}_x[n-1, n] \\
d_{n-1,3} & c_{n-1,1} & c_{n-1,a(n)} & d_{n-1,i} \\
c_{n,j} & c_{n,i} & d_{n-1,i} & c_{n-1,1} \\
\end{array}
$$

c_{n-1,a(n)}d_{n-1,i}c_{n-1,1}$ for each $1 \leq i \leq r_n$. We now construct an ordered Bratteli diagram. Let $V_0 := \{v_0\}$ as usual. For each $n \geq 1$, let $V_n := \mathcal{C}_n$. We can write $\varphi_n(c_{n,i}) = c_{n-1,a(n)}c_{n-1,a(n,i),3}c_{n-1,a(n,i,4)} \cdots c_{n-1,a(n,k(n,i))}c_{n-1,1}$. We make the minimal edge $e_{n,1}$ from $c_{n-1,a(n)}$ to $c_{n,i}$ and all the rest in this order. Thus, the minimal edge connects $c_{n-1,a(n)}$ to $c_{n,i}$ regardless of $i$, and the maximal edge connects $c_{n-1,1}$ to $c_{n,i}$ regardless of $i$. The ordered Bratteli diagram thus constructed is denoted as $(V', E', \geq')$. Evidently, it has rank $K$. It is easy to check that $(V', E', \geq')$ is properly ordered. The simplicity follows from the simplicity of $\mathcal{G}$. The Bratteli–Vershik system thus constructed is identical to the symbolic system $(Y, \sigma)$. This completes the proof. \qed
ACKNOWLEDGMENTS: This work was partially supported by JSPS KAKENHI (Grant Number 16K05185).

REFERENCES

[BKM09] S. Bezuglyi, J. Kwiatkowski, and K. Medynets. Aperiodic substitution systems and their Bratteli diagrams. Ergod. Th. Dynam. Sys. 29 (2009), 37–72.

[DM08] T. Downarowicz and A. Maass. Finite-rank Bratteli–Vershik diagrams are expansive. Ergod. Th. Dynam. Sys. 28 (2008), 739–747.

[GM06] J.-M. Gambaudo and M. Martens. Algebraic topology for minimal Cantor sets. Ann. Henri Poincaré 7 (2006), 423–446.

[HPS92] R. H. Herman, I. F. Putnam, and C. F. Skau. Ordered Bratteli diagrams, dimension groups and topological dynamics. Int. J. Math. 3 (1992), 827–864.

[S14] T. Shimomura. Special homeomorphisms and approximation for Cantor systems. Topol. Appl. 161 (2014), 178–195.

[S15] T. Shimomura. Non-homeomorphic topological rank and expansiveness. Kyushu J. Math. 69 (2015), 413–428.

[S16a] T. Shimomura. Graph covers and ergodicity for zero-dimensional systems. Ergod. Th. Dynam. Sys. 36 (2016), 608–631.

[S16b] T. Shimomura. Zero-dimensional almost 1-1 extensions of odometers from graph coverings. Topol. Appl. 209 (2016), 63–90.

NAGOA UNIVERSITY OF ECONOMICS, UCHIKUBO 61-1, INUYAMA 484-8504, JAPAN
E-mail address: tkshimo@nagoya-ku.ac.jp