On the Diagonal Stability of $k$-Positive Linear Systems

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Abstract—We consider $k$-positive linear systems, that is, systems that map the set of vectors with up to $k-1$ sign variations to itself. For $k=1$, this reduces to positive linear systems. It is well-known that stable positive linear time invariant (LTI) systems admit a diagonal Lyapunov function. This property has many important implications. A natural question is whether stable $k$-positive systems also admit a diagonal Lyapunov function. This paper shows that, in general, the answer is no. However, for both continuous-time and discrete-time $n$-dimensional systems that are $(n-1)$-positive we provide a sufficient condition for diagonal stability.

I. INTRODUCTION

Lyapunov functions are a powerful tool for stability analysis and control synthesis. For linear time invariant (LTI) systems, stability is equivalent to the existence of a quadratic Lyapunov function (QLF), that can be obtained constructively based on the eigenvectors of the Hamiltonian matrix [15].

The LTI is called diagonally stable (DS) if it is possible to find a QLF with a diagonal matrix. Diagonal stability of LTI systems has attracted considerable attention in the systems and control community (see e.g. the monograph [10]), as the existence of a diagonal Lyapunov function (DLF) has important implications to certain nonlinear systems associated with the LTI. The existence of a DLF can also facilitate analysis and control community (see e.g. the monograph [10]), as the existence of a diagonal Lyapunov function has important implications to certain nonlinear systems associated with the LTI. The existence of a DLF can also facilitate analysis and control community (see e.g. the monograph [10]), as the existence of a diagonal Lyapunov function has important implications to certain nonlinear systems associated with the LTI.

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In general, stable LTIs do not admit a DLF [3]. However, it is well-known that positive LTIs admit a DLF (see, e.g., [14]). Recently, the notion of positive linear systems was generalized to $k$-positive linear systems [18], [1]. For $k=1$, this reduces to positive linear systems. This naturally raises the question of whether $k$-positive LTIs also admit a DLF.

This paper studies the diagonal stability of both discrete-time (DT) and continuous-time (CT) LTIs. Our main results include the following. We first show that in general $k$-positive systems with $k>1$ are not DS. However, for $n$-dimensional LTIs that are $(n-1)$-positive we derive a sufficient condition for DS.

The remainder of this paper is organized as follows. The next section reviews known results that will be used later on. This includes a short description of $k$-positive systems and also a review of conditions for DS of LTIs. Sections III and IV detail our main results.

II. PRELIMINARIES

We use standard notation. The non-negative orthant is denoted by $\mathbb{R}_+^n := \{ x \in \mathbb{R}^n \mid x_i \geq 0, \ i = 1, \ldots, n \}$. For a vector $y \in \mathbb{R}^n \setminus \{ 0 \}$, let $s^-(y)$ denote the number of sign variations in $y$ after deleting all its zero entries, with $s^-(0)$ defined as zero. For $y \in \mathbb{R}^n$, let $s^+(y)$ denote the maximal possible number of sign variations in $y$ after each zero entry is replaced by either $+1$ or $-1$. For example, for $n = 5$ and $y = [1, -2, 0, 0, \pi]^T$, we have $s^-(y) = 2$ and $s^+(y) = 4$. Obviously,

$0 \leq s^-(y) \leq s^+(y) \leq n-1$ for all $y \in \mathbb{R}^n$.

Let $[1, n] := \{ 1, \ldots, n \}$. For any $k \in [1, n]$, define the sets:

\begin{align*}
P^-_k &:= \{ z \in \mathbb{R}^n : s^-(z) \leq k-1 \}, \\
P^+_k &:= \{ z \in \mathbb{R}^n : s^+(z) \leq k-1 \}.
\end{align*}

(1)

For example, $P^+_1 = \mathbb{R}^n_+ \cup (-\mathbb{R}^n_+)$, and $P^+_1$ is the interior of $P^+_1$, denoted $\text{int}(P^+_1)$. It is not difficult to show that $P^+_k$ is closed, and that $P^+_k = \text{int}(P^+_k)$ for any $k \in [1, n-1]$ [1].

A linear dynamical system is called $k$-positive if its flow maps $P^+_k$ to $P^+_k$, and strongly $k$-positive if its flow maps $P^+_k \setminus \{ 0 \}$ to $P^+_k$. These systems were introduced and analyzed in the recent papers [1], [18]. For the case $k = 1$, this reduces to the definition of a positive and a strongly positive system [8].

A set $K \subset \mathbb{R}^n$ is called a cone if $x \in K$ implies that $cx \in K$ for any $c > 0$. The dual cone is

$K^* := \{ y \in \mathbb{R}^n \mid y^T x \geq 0 \ \text{for all} \ x \in K \}$.

A cone $K$ is called proper if it is convex, closed, and pointed (i.e. $K \cap (-K) = \{ 0 \}$). For example, $\mathbb{R}^n_+$ is a proper cone.

For a matrix $X \in \mathbb{R}^{n \times m}$, we write $X \geq 0$ [$X > 0$] if all its entries are non-negative [positive]. We say that $X$ is non-negative [positive] if $X \geq 0$ [$X > 0$]. A matrix $A \in \mathbb{R}^{n \times n}$ is called Metzler if all its off-diagonal entries are non-negative. For a symmetric matrix $P \in \mathbb{R}^{n \times n}$, we use the notation $P > 0$ to indicate that $P$ is positive-definite, i.e. $x^T P x > 0$ for all $x \in \mathbb{R}^n \setminus \{ 0 \}$.

The eigenvalues of $A \in \mathbb{R}^{n \times n}$ are denoted by $\lambda_i(A), i = 1, \ldots, n$, ordered such that

$|\lambda_1(A)| \geq |\lambda_2(A)| \geq \cdots \geq |\lambda_n(A)| \geq 0.$

(2)

The spectral radius of $A$ is $\rho(A) = |\lambda_1(A)|$. 
A. Invariant cones and diagonal stability

The next result details the spectral properties of matrices that map a proper cone to itself.

**Lemma 1**: [5, Thm. 3.2] If \( A \in \mathbb{R}^{n \times n} \) leaves a proper cone \( K \subset \mathbb{R}^n \) invariant, i.e., \( AK \subset K \), then (a)

1. The spectral radius of \( A \) is an eigenvalue;
2. If \( \lambda \) is an eigenvalue of \( A \) such that \( |\lambda| = \rho(A) \), then \( \deg \lambda \leq \deg \rho(A) \);
3. \( K \) contains a dominant eigenvector of \( A \), i.e., an eigenvector corresponding to \( \rho(A) \);
4. \( K^* \) contains a dominant eigenvector of \( A^T \).

The next result provides a sufficient spectral condition guaranteeing that a matrix maps some proper cone to itself.

**Lemma 2**: [5, Thm. 3.5] If \( \rho(A) \) is an eigenvalue of \( A \in \mathbb{R}^{n \times n} \), and \( \deg \lambda \leq \deg \rho(A) \) for every eigenvalue \( A \) such that \( |\lambda| = \rho(A) \), then \( A \) leaves a proper cone invariant.

If \( A \in \mathbb{R}^{n \times n} \) is non-negative, then \( x(j + 1) = Ax(j) \) is called a **positive DT-LTI** system. If \( A \in \mathbb{R}^{n \times n} \) is Metzler, then \( z(t) = Ax(t) \) is called a **positive CT-LTI** system. It is well-known that the flow of positive LTI systems leaves the proper cone \( \mathbb{R}_+^n \) invariant [16]. The following results show that stable positive LTI systems are DS.

**Lemma 3**: (see e.g., [14, Prop. 2]) If \( A \in \mathbb{R}^{n \times n} \) with \( A \geq 0 \) then the following statements are equivalent: (a)

1. The matrix \( A \) is Schur, i.e., \( \rho(A) < 1 \);
2. There exists \( \xi > 0 \) such that \( A\xi < \xi \);
3. There exists \( z > 0 \) such that \( A^T z < z \);
4. There exists \( z > 0 \) such that \( P \succ 0 \) and \( A^T P A < P \);
5. The matrix \((I - A)\) is nonsingular and \((I - A)^{-1} \succ 0 \).

**Remark 1**: Let \( A \in \mathbb{R}^{n \times n} \) be non-negative and Schur. Pick \( x, y \in \mathbb{R}^n \) with \( x, y \succ 0 \). Then \( \xi := (I - A)^{-1} x, z := (I - A^T)^{-1} y, P := \text{diag}(z_1, \ldots, z_n) \) satisfy conditions (b), (c), and (d) in Lemma 3 respectively. This provides a constructive procedure to obtain a DLF for positive DT-LTI systems. Furthermore, note that if \( A \in \mathbb{R}^{n \times n} \) is Schur and \( A \leq 0 \), then \(( -A)\) is Schur and non-negative. In this case, Lemma 3 ensures that there exists a diagonal matrix \( P \in \mathbb{R}^{n \times n} \) such that \( P \succ 0 \) and \( A^T P A \not\prec P \).

**Lemma 4**: (see e.g., [14, Prop. 1]) If \( A \in \mathbb{R}^{n \times n} \) is Metzler then the following statements are equivalent: (a)

1. The matrix \( A \) is Hurwitz;
2. There exists \( \xi \in \mathbb{R}^n \) such that \( \xi > 0 \) and \( A\xi \ll \xi \);
3. There exists \( z \in \mathbb{R}^n \) such that \( z > 0 \) and \( A^T z \ll 0 \);
4. There is a diagonal matrix \( P \in \mathbb{R}^{n \times n} \) such that \( P > 0 \) and \( PA + A^T P \prec 0 \);
5. \( A \) is nonsingular and \( A^{-1} \ll 0 \).

**Remark 2**: Let \( A \in \mathbb{R}^{n \times n} \) be Metzler and Hurwitz. Fix \( x, y \in \mathbb{R}^n \) with \( x, y \gg 0 \). Then \( \xi := -A^{-1} x, z := -A^{-T} y, P := \text{diag}(z_1, \ldots, z_n) \) satisfy conditions (b), (c), and (d) in Lemma 4 respectively.

The analysis of \( k \)-positive systems is based on compound matrices. The next subsection reviews the \( k \)th-order multiplicative and additive compound of a matrix.

B. Compound Matrices

For an integer \( n \geq 1 \) and \( k \in [1, n] \), let \( Q_{k,n} \) denote the ordered set of all strictly increasing sequences of \( k \) integers chosen from \([1, n]\). We denote the \( r \) := \( \binom{n}{k} \) elements of \( Q_{k,n} \) by \( k_1, \ldots, k_r \), with the \( k_i \)-s ordered lexicographically. For example, \( Q_{2,3} = \{k_1, k_2, k_3\} \), with \( k_1 = \{1, 2\}, k_2 = \{1, 3\}, \) and \( k_3 = \{2, 3\} \).

Given a matrix \( A \in \mathbb{R}^{n \times n} \) and \( k_1, k_2 \in Q_{k,n} \), let \( A[k_1, k_2] \) denote the submatrix of \( A \) consisting of the rows indexed by \( k_i \) and columns indexed by \( k_j \). Thus, \( A[k_1, k_2] \in \mathbb{R}^{k \times k} \). Let \( A[k_1, k_2] := \text{det}(A[k_1, k_2]) \), i.e., the \( k \)-minors of \( A \) determined by the rows in \( k_i \) and the columns in \( k_j \).

The \( k \)th multiplicative compound (**MC**) of \( A \) is the matrix \( A(k) \in \mathbb{R}^{n \times r} \), whose entries, written in lexicographic order, are \( A[k_1, k_2] \). For example, for \( n = 3 \) and \( k = 2 \),

\[
A(2) = \begin{bmatrix} A([1, 2, 3]); A([1, 2, 3]); A([1, 2, 3]); A([1, 2, 3]) & A([1, 2, 3]); A([1, 2, 3]); A([1, 2, 3]); A([1, 2, 3]) \\
A([1, 2, 3]); A([1, 2, 3]); A([1, 2, 3]); A([1, 2, 3]) & A([1, 2, 3]); A([1, 2, 3]); A([1, 2, 3]); A([1, 2, 3]) \\
A([1, 2, 3]); A([1, 2, 3]); A([1, 2, 3]); A([1, 2, 3]) & A([1, 2, 3]); A([1, 2, 3]); A([1, 2, 3]); A([1, 2, 3]) \end{bmatrix}
\]

The MC satisfies the following properties.

**Lemma 5**: Let \( A, B \in \mathbb{R}^{n \times n} \) and pick \( k \in \mathbb{S}_n \). Then (i)

1. \((AB)(k) = A(k)B(k)\);
2. If \( A \) is nonsingular then \((A^{-1})(k) = (A(k))^{-1}\);
3. \((A^T)(k) = (A(k))^T\);
4. If \( A \) exists then \((A)^{(k)} = (A(k))^{1}\);
5. If \( A \) is Schur, then \( A(k) \) is Schur;
6. If \( A \) is a diagonal matrix, then \( A(k) \) is a diagonal matrix.
7. If \( A \succ 0 \), then \( A(k) \succ 0 \).

For the sake of completeness, we provide a proof for some of these properties.

**Proof**: Property (1) follows from the Cauchy-Binet formula (see e.g. [7, Thm. 1.1.1]). Using (1) yields

\[
I(k) = (AA^{-1})(k) = A(k)(A^{-1})(k),
\]

and

\[
I(k) = (A^{-1}A)(k) = (A^{-1})(k)A(k).
\]

Since \( I(k) \) is the \( r \times r \) identity matrix, this proves (2). Also, (1) implies that

\[
A(k) = (A^T A)(k) = (A^T)(k)(A)(k)
\]

which proves (3). Properties (3) and (6) follow from the definition of the MC.

If \( A \in \mathbb{R}^{n \times n} \) has eigenvalues \( \lambda_1, \ldots, \lambda_n \), then every eigenvalue of \( A(k) \) is a product of \( k \) of the \( \lambda_i \)-s (see e.g. [13]). Therefore, if \( \rho(A) < 1 \), then \( \rho(A(k)) < 1 \). This proves (5).

To prove (7), note that if \( A \succ 0 \) then \( A \) is symmetric, and all the eigenvalues of \( A \) are positive. Property (3) implies that \( A(k) \) is symmetric. Every eigenvalue of \( A(k) \) is the product of \( k \) positive numbers so it is positive. Thus, \( A(k) \succ 0 \).

**Remark 3**: If the eigenvalues \( \tilde{\lambda}_1, \ldots, \tilde{\lambda}_r \) of \( A(k) \) are ordered lexicographically, that is,

\[
\tilde{\lambda}_q = \prod_{s \in k_q} \lambda_s(A), \quad q = 1, \ldots, r,
\]
where $Q_{k,n} = \{\kappa_1, \ldots, \kappa_r\}$, then the ordering defined in (2) implies that
\[ |\lambda_1| \geq |\lambda_2| \geq \cdots \geq |\lambda_r|. \]
For example, if $n = 3$ and $k = 2$, then $\lambda_1 = \lambda_1(A)\lambda_2(A)$, $\lambda_2 = \lambda_2(A)\lambda_3(A)$, and $\lambda_3 = \lambda_3(A)\lambda_3(A)$. Hence, $|\lambda_1(A)| \geq |\lambda_2(A)| \geq |\lambda_3(A)|$ implies that $|\lambda_1| \geq |\lambda_2| \geq |\lambda_3|$.

Given $A \in \mathbb{R}^{n \times n}$, the $k$th additive compound (AC) of $A$ is the $r \times r$ matrix defined by
\[ A^{(k)} := \frac{d}{d\varepsilon} (I + \varepsilon A)^{(k)}|_{\varepsilon=0} \tag{4} \]
(see e.g., [12]). By definition, $A^{(1)} = A$ and $A^{(n)} = \det(A)$, so (4) yields $A^{[1]} = A$ and $A^{[n]} = \text{tr}(A)$. Note that (4) implies that for any $\varepsilon \in \mathbb{R}$,
\[ (I + \varepsilon A)^{(k)} = I + \varepsilon A^{[k]} + o(\varepsilon). \tag{5} \]

The AC satisfies the following properties.

**Lemma 6:** Let $A, B \in \mathbb{R}^{n \times n}$ and pick $k \in [1, n]$. Then
(i) \[ (A + B)^{(k)} = A^{(k)} + B^{(k)}; \]
(ii) For any nonsingular matrix $\Gamma \in \mathbb{R}^{n \times n}$, $(\Gamma A\Gamma^{-1})^{(k)} = \Gamma^{(k)} A^{(k)} (\Gamma^{-1})^{(k)}$;
(iii) $(A^T)^{(k)} = (A^{(k)})^T$;
(iv) If $A$ is Hurwitz, then $A^{[k]}$ is Hurwitz;
(v) If $A$ is a diagonal matrix, then $A^{[k]}$ is a diagonal matrix;
(vi) If $A \succ 0$ [or $A \prec 0$], then $A^{[k]} \succ 0$ [or $A^{[k]} \prec 0$].

For the sake of completeness, we provide a proof.

**Proof:** By Property (1) in Lemma 5,
\[ (I + \varepsilon A)^{(k)} = (I + \varepsilon I + \varepsilon A + \varepsilon B + \varepsilon o(\varepsilon))^{(k)} \tag{6} \]
and combing this with (5) implies that $(A + B)^{(k)} = A^{(k)} + B^{(k)}$.

By (4) and Lemma 5,
\[ (\Gamma A\Gamma^{-1})^{(k)} = \frac{d}{d\varepsilon} (I + \varepsilon \Gamma A\Gamma^{-1})^{(k)}|_{\varepsilon=0} = \frac{d}{d\varepsilon} (I + \varepsilon A)^{(k)}|_{\varepsilon=0} = \frac{d}{d\varepsilon} (\Gamma^{(k)} (I + \varepsilon A)^{(k)} (\Gamma^{-1})^{(k)})|_{\varepsilon=0} = \Gamma^{(k)} A^{(k)} (\Gamma^{-1})^{(k)} \tag{7} \]
Properties (3) and (5) follow from (4) and Lemma 5.

The proofs of (4) and (6) follow from the fact that every eigenvalue of $A^{[k]}$ is a sum of $k$ of the $\lambda_i(A)$s [13].

C. Necessary and Sufficient Conditions for $k$-Positivity

A matrix $A \in \mathbb{R}^{n \times n}$ is called [strictly] sign-regular of order $k$ (denoted by $[S]SR_k$) if either $A^{(k)} \leq 0$ [$A^{(k)} \leq 0$] or $A^{(k)} \geq 0$ [$A^{(k)} \geq 0$]. In other words, all minors of order $k$ of $A$ have the same [strict] sign. To refer to the common sign of the entries of $A^{(k)}$, we use the signature $\epsilon_k \in \{-1, 1\}$.

For example, if $A^{(k)}$ is $SSR_k$ [$SR_k$] with signature $\epsilon_k = 1$ then all the $k$-minors of $A$ are positive [non-negative].

The next result provides a necessary and sufficient condition for a nonsingular matrix to map $P_k^+$ to itself.

**Theorem 1:** [4] Let $T \in \mathbb{R}^{n \times n}$ be a nonsingular matrix and pick $k \in [1, n]$. Then
(a) \[ TP_k^+ \subset P_k^+ \] if and only if $T$ is $SR_k$;
(b) \[ T(P_k^+ \setminus \{0\}) \subset P_k^+ \] if and only if $T$ is $SSR_k$.

For example, for $k = 1$ this implies that $T(\mathbb{R}_+^n \cup (-\mathbb{R}_+^n)) \subseteq (\mathbb{R}_+^n \cup (-\mathbb{R}_+^n))$ if the entries of $T$ are all non-negative or all non-positive, and that $T(\mathbb{R}_+^n \cup (-\mathbb{R}_+^n)) \subseteq \text{int}(\mathbb{R}_+^n \cup (-\mathbb{R}_+^n))$ if the entries of $T$ are all positive or all negative.

Based on Thm. 1 the next two results give necessary and sufficient conditions for an LTI system to be $k$-positive.

**Theorem 2:** Let $A \in \mathbb{R}^{n \times n}$ be a nonsingular matrix and pick $k \in [1, n]$. Then, the DT-LTI system
\[ x(j + 1) = Ax(j) \tag{7} \]
is [strongly] $k$-positive if and only if $A^{[k]}$ is $[S]SR_k$.

**Theorem 3:** [18] Let $A \in \mathbb{R}^{n \times n}$ and pick $k \in [1, n]$. Then, the CT-LTI system
\[ \dot{x}(t) = Ax(t) \tag{8} \]
is $k$-positive if and only if $A^{[k]}$ is Metzler. Additionally, $A^{[k]}$ is strongly $k$-positive if $A^{[k]}$ is Metzler and irreducible.

**Remark 4:** As shown in [18], if $A^{[k]}$ is Metzler, then the state-transition matrix $\Phi(t, t_0) := \exp(A(t - t_0))$ of $\dot{x} = Ax$ is $SR_k$ for all $t \geq t_0$. Since $\Phi(t) = \Phi(t, t_0)x(t_0)$, Thm. 1 implies that if $x(t_0) \in P_k^+$, then $x(t) \in P_k^+$ for all $t \geq t_0$.

For $k = 1$ we have $A^{[k]} = A$, so this reduces to the requirement that $A$ is Metzler. For $k > 1$ the requirement that $A^{[k]}$ is Metzler can also be stated in terms of the entries of $A$.

**Lemma 7:** [18] Let $A \in \mathbb{R}^{n \times n}$ with $n > 2$. \begin{enumerate} \item Pick $k \in \{2, \ldots, n - 2\}$. Then $A^{[k]}$ is Metzler if and only if the entries of $A$ satisfy the following three conditions: \begin{enumerate} \item $\sum_{i,j=1}^{n-k} a_{ij} (1 - 1^{k+1} + a_{11}) \geq 0$.
\item $a_{ij} \geq 0$ for all $i, j$ with $|i - j| = 1$, and
\item $a_{ij} = 0$ for all $i, j$ with $|i - j| < n - 1$.
\end{enumerate} \item $A^{[n-1]}$ is Metzler if and only if all $a_{ij} \geq 0$ for all $i, j$ such that $|i - j| = 1, 3, 5, \ldots$, and $a_{ij} \leq 0$ for all $i, j$ such that $|i - j| = 2, 4, 6, \ldots$. \end{enumerate}

Note that these requirements are sign-pattern requirements. Note also that there is no constraint on the diagonal entries of $A$.

We will show below that in general $k$-positive LTI systems are not DS. To prove this, we briefly review necessary conditions for diagonal stability.

D. Necessary Conditions for Diagonal Stability

We begin with the CT case. Recall that $A(\kappa_i | \kappa_j)$ is called a principal minor if $\kappa_i = \kappa_j$.

**Theorem 4:** [3, Theorem 2] Let $A \in \mathbb{R}^{n \times n}$. If there exists a diagonal positive definite matrix $P \in \mathbb{R}^{n \times n}$ such that $PA + A^T P < 0$, then all the principal minors of $(-A)$ are positive.
To state an analogous result for the DT case, we require the following lemma.

**Lemma 8**: [17, Theorem 3] A matrix $A_\lambda \in \mathbb{R}^{n \times n}$ is Schur iff the matrix $A_{\lambda} := (A_\lambda + I)(A_\lambda - I)^{-1}$ is Hurwitz. Furthermore, a matrix $W \in \mathbb{R}^{n \times n}$ satisfies $WA_\lambda A_\lambda^T W < 0$ iff $A_\lambda^T W A_\lambda < W$.

Combining Thm. 4 and Lemma 8 yields the following result.

**Theorem 5**: Let $A \in \mathbb{R}^{n \times n}$. If there exists a diagonal positive definite matrix $P \in \mathbb{R}^{n \times n}$ such that $A^T P A < P$, then all the principal minors of $-(A + I)(A - I)^{-1}$ are positive.

The next sections describe our main results.

**III. k-positivity does not imply Diagonal Stability**

Since $1$-positivity is just positivity and stable positive LTI systems are DS, a natural question is: do stable $k$-positive systems admit a DLF? We now show that in general the answer is no.

Consider the DT-LTI system with $A = \begin{bmatrix} \frac{-1}{4} & \frac{3}{4} \\ \frac{1}{2} & \frac{-1}{2} \end{bmatrix}$. The eigenvalues of $A$ are $\lambda_1(A) = -0.7260 + 0.4949i$, $\lambda_2(A) = -0.7260 - 0.4949i$, and $\lambda_3(A) = 0.1602$. Note that $\rho(A) = 0.8768 < 1$, so $A$ is Schur. A calculation yields

$$A^{(2)} = \frac{1}{49} \begin{bmatrix} 14 & 19 & 13 \\ 10 & 1 & 3 \\ 22 & 33 & 11 \end{bmatrix},$$

so $A$ is SSR$_2$ with $e_2 = 1$. Let $B := -(A + I)(A - I)^{-1}$. Then

$$B = \frac{1}{461} \begin{bmatrix} 204 & -119 & 140 \\ -182 & 183 & -378 \\ 497 & -21 & 323 \end{bmatrix}. \quad (9)$$

Note that

$$B(\{1, 3\} \{1, 3\}) = \det \left( \frac{1}{461} \begin{bmatrix} 204 & 140 \\ 497 & 323 \end{bmatrix} \right) < 0.$$  

Thm. 5 implies that the stable and 2-positive DT-LTI system does not admit a DLF.

A similar result holds for CT-LTI systems. Consider $A = \begin{bmatrix} -21 & 11 & -14 \\ 18 & -19 & 37 \\ -49 & 21 & -33 \end{bmatrix}$. The eigenvalues of $A$ are $\lambda_1(A) = -72.6785$, $\lambda_2(A) = -0.1608 + 5.8827i$, $\lambda_3(A) = -0.1608 - 5.8827i$, so $A$ is Hurwitz. A calculation yields $A^{[2]} = \begin{bmatrix} 40 & 37 & 14 \\ 21 & -54 & 11 \\ 49 & 18 & -52 \end{bmatrix}$, which is Metzler and irreducible (and also Hurwitz). Hence, this system is strongly 2-positive. Let $B := -A$. Then

$$B(\{2, 3\} \{2, 3\}) = \det \left( \begin{bmatrix} 19 & -37 \\ -21 & 33 \end{bmatrix} \right) < 0.$$  

Thm. 4 implies that this stable and strongly 2-positive CT-LTI is not DS.

Summarizing, stable $k$-positive systems are in general not DS.

**IV. Diagonal Stability of $(n - 1)$-Positive LTI Systems**

The examples in Section III are of systems of dimension $n = 3$ that are strongly 2-positive yet are not DS. This section provides a sufficient condition ensuring that an $(n - 1)$-positive LTI is DS.

**A. Discrete-Time Case**

To prove the main results below we require an auxiliary result that uses the ordering in Remark 3 to derive an expression for the largest eigenvalue of a matrix in terms of the eigenvalues of its MC.

**Lemma 9**: Pick $A \in \mathbb{R}^{n \times n}$ and denote its eigenvectors by $\lambda_i$, $i = 1, \ldots, n$, such that

$$|\lambda_1| \geq |\lambda_2| \geq \cdots \geq |\lambda_n|.$$  

Pick $k \in [1, n - 1]$. Let $r := \binom{n}{k}$ and denote the eigenvalues of $A^{(k)}$, ordered lexicographically, by $\bar{\lambda}_1, \ldots, \bar{\lambda}_r$. Define

$$\ell := \frac{\prod_{i=1}^{(n-1)} \bar{\lambda}_i}{\left( \prod_{i=1}^r \bar{\lambda}_i \right)^{r-1}} \quad (11)$$

Then $\ell = \lambda_1^{(n-1)}$.

**Example 1**: For $k = 1$ we have $\bar{\lambda}_i = \lambda_i$, $i = 1, \ldots, n$, and (11) yields $\ell = \lambda_1$. For $k = n - 1$ Eq. (11) yields

$$\ell = \frac{\bar{\lambda}_1 \cdots \bar{\lambda}_{n-1}}{(\bar{\lambda}_n)^{n-1}} = \frac{\lambda_1 \cdots \lambda_{n-1}}{(\lambda_n)^{n-1}}.$$  

It is clear that $\bar{\lambda}_1 \cdots \bar{\lambda}_n = \prod_{i=1}^{n} \lambda_i^{(n-1)}$, and thus,

$$\ell = \frac{(\lambda_1 \cdots \lambda_{n})^{n-1}}{(\lambda_2 \cdots \lambda_n)^{n-1}} = (\lambda_1)^{n-1}.$$  

As another example consider the case $n = 4$ and $k = 2$. Then $r = \binom{4}{2} = 6$, and $\bar{\lambda}_1 = \lambda_1 \lambda_2$, $\bar{\lambda}_2 = \lambda_1 \lambda_3$, $\bar{\lambda}_3 = \lambda_1 \lambda_4$, $\lambda_4 = \lambda_2 \lambda_3$, $\lambda_5 = \lambda_2 \lambda_4$, $\lambda_6 = \lambda_3 \lambda_4$, so (11) yields

$$\ell = \frac{\lambda_1 \lambda_2 \lambda_3}{(\lambda_4 \lambda_5 \lambda_6)^{1/2}} = \lambda_1^2.$$  

**Proof of Lemma 9**: It follows from the lexicographic ordering of the $\bar{\lambda}_i$s that

$$\prod_{i=1}^{(n-1)} \bar{\lambda}_i = \lambda_1^{(n-1)} \prod_{i=2}^n \lambda_i^{(n-2)}, \quad (12)$$

and

$$\prod_{i=1}^r \bar{\lambda}_i = \lambda_1^{(n-1)} \prod_{i=2}^n \lambda_i^{(n-2)}. \quad (13)$$
Substituting this in (11) gives
\[
\ell = \lambda_1^{(n-1)} \prod_{k=2}^{n} \lambda_{k}^{(n-k)} = \lambda_1^{(n-1)},
\] (14)
where the second equality follows from the identity \( \binom{n-2}{k-2} \binom{n-2}{k-2} = \binom{n-2}{k-2} \).

Suppose that \( A \in \mathbb{R}^{n \times n} \) is \( SR_n \), that is, either \( A^{(k)} \geq 0 \) or \( A^{(k)} \leq 0 \). Suppose also that \( A \) is Schur. Then Lemma 5 implies that \( A^{(k)} \) is Schur. By Lemma 3 there exists a diagonal and positive definite matrix \( D \in \mathbb{R}^{r \times r} \), where \( r := \binom{n}{k} \), such that \( (A^{(k)})^T DA^{(k)} \prec D \).

It turns out that in order to extend this to a DLF for \( A \), we require that the equation
\[
P^{(k)} = D
\] (15)
admits a diagonal and positive definite solution \( P \in \mathbb{R}^{n \times n} \).

The next example demonstrates that such a solution does not always exist.

**Example 2:** Consider the case \( n = 4 \) and \( k = 2 \), so \( r = \binom{4}{2} = 6 \). Let \( D \in \mathbb{R}^{6 \times 6} \) be a diagonal and positive-definite matrix. Then (15) becomes
\[
p_1 p_2 = d_1, \quad p_1 p_3 = d_2, \quad p_1 p_4 = d_3,
p_2 p_3 = d_4, \quad p_2 p_4 = d_5, \quad p_3 p_4 = d_6.
\]
This is a set of 6 equations in the four unknowns \( p_1, \ldots, p_4 \), so in general it does not admit a solution.

The next result shows that (15) does admit a solution for the case \( k = n - 1 \).

**Lemma 10:** If \( D \in \mathbb{R}^{n \times n} \) is diagonal and positive definite then there exists a diagonal and positive definite matrix \( P \in \mathbb{R}^{n \times n} \) such that \( P^{(n-1)} = D \).

**Proof:** The proof is constructive. The equation \( P^{(n-1)} = D \) can be written as
\[
\prod_{s \in \kappa_a} p_s = d_q > 0, \quad q = 1, \ldots, n
\] (16)
where \( \kappa_1, \ldots, \kappa_n \in Q_{n-1,n} \). For any \( s \in [1, n] \), let \( j = j(s) \) be the single element in \( [1, n] \setminus \kappa_s \). A lengthy but straightforward computation shows that then the solution of (16) is
\[
p_s = \frac{1}{d_{j(s)}} \prod_{q \in \kappa_s} d_q^{1/q}.
\] (17)
Note that this implies that \( p_s > 0 \) for any \( s \).

**Example 3:** Consider the case \( n = 4 \). Then \( P^{(3)} = D \) is equivalent to the equations
\[
p_1 p_2 p_3 = d_1, \quad p_1 p_2 p_4 = d_2, \quad p_1 p_3 p_4 = d_3, \quad p_2 p_3 p_4 = d_4.
\]
Eq. (17) yields
\[
p_1 = (d_1 d_3 d_4)^{1/3} / d_1^{1/3}, \quad p_2 = (d_1 d_2 d_4)^{1/3} / d_2^{1/3}, \quad p_3 = (d_1 d_2 d_3)^{1/3} / d_3^{1/3}, \quad p_4 = (d_2 d_3 d_4)^{1/3} / d_4^{1/3},
\]
and it is easy to verify that this is indeed a solution.

The next result provides a condition guaranteeing that \( P^{(n-1)} \) can be “uplifted” to a DLF for the original system.

**Proposition 1:** Suppose that \( A \in \mathbb{R}^{n \times n} \) is \( SR_{n-1} \) and Schur. Then (i)

1) There exists a diagonal positive definite matrix \( P \in \mathbb{R}^{n \times n} \) such that
\[
(A^{(n-1)})^T P^{(n-1)} A^{(n-1)} \prec P^{(n-1)}.
\] (18)

2) \( A^T P A \prec P \) iff the lexicographically ordered eigenvectors \( \lambda_i(M) \), \( i = 1, \ldots, n \), of the matrix
\[
M := (P^{(n-1)})^{-\frac{1}{2}} (A^{(n-1)})^T P^{(n-1)} A^{(n-1)} (P^{(n-1)})^{-\frac{1}{2}}
\]
satisfy
\[
\left| \prod_{i=1}^{n-1} \lambda_i(M) \right| < 1.
\] (19)

**Proof:** (i) Since \( A \) is \( SR_{n-1} \) and Schur, \( A^{(n-1)} \in \mathbb{R}^{n \times n} \) is non-negative (or nonpositive) and Schur. Lemma 3 implies that there exists a diagonal positive definite matrix \( D \in \mathbb{R}^{n \times n} \) such that
\[
D^{-\frac{1}{2}} (A^{(n-1)})^T D A^{(n-1)} D^{-\frac{1}{2}} \prec I.
\] (20)
By Lemma 10 there exists a diagonal and positive definite matrix \( P \in \mathbb{R}^{n \times n} \) such that \( P^{(n-1)} = D \). Therefore, (20) can be rewritten as \( M \prec I \) and this proves (18).

(ii) By Lemma 5
\[
M = (P^{-\frac{1}{2}} (n-1) (A^T (n-1) P^{(n-1)} A^{(n-1)} (P^{-\frac{1}{2}} (n-1)))
=(P^{-\frac{1}{2}} (A^T P A P^{-\frac{1}{2}} (n-1))
By Lemma 9, condition (19) is equivalent to \( \rho(P^{-\frac{1}{2}} (A^T P A P^{-\frac{1}{2}} (n-1)) \prec I \), that is, \( A^T P A \prec P \).

**Example 4:** Consider the DT-LTI system \( x(j + 1) = Ax(j) \) with
\[
A = \begin{bmatrix}
\frac{1}{2} & \frac{1}{2} & 0 \\
\frac{1}{2} & \frac{1}{2} & 0 \\
0 & 0 & \frac{1}{2}
\end{bmatrix}.
\] (21)
The eigenvalues of \( A \) are \( \lambda_1(A) = -0.6376 + 0.4374i \), \( \lambda_2(A) = -0.6376 - 0.4374i \), and \( \lambda_3(A) = 0.1503 \). Note that \( A \) is Schur, and \( \rho(A) = |\lambda_1(A)| = 0.7732 \) is not an eigenvalue. Hence, Lemma 11 implies that \( A \) does not leave a proper cone invariant. A calculation shows that \( A \) exists a diagonal \( D > 0 \) such that \( D^{-\frac{1}{2}} (A^2)^T D A^2 D^{-\frac{1}{2}} \prec I \).

According to Remark 1 one such \( D \) can be obtained as \( D = \text{diag} \left( \frac{21}{2}, \frac{13}{2}, \frac{7}{2} \right) \). Based on Lemma 10 the diagonal and positive definite matrix \( P \in \mathbb{R}^{3 \times 3} \) such that \( P^{(2)} = D \) is \( P = \text{diag} \left( \frac{887}{1176}, \frac{184}{509}, \frac{147}{184} \right) \). Let \( M := (P^{(2)})^{-\frac{1}{2}} (A^2)^T P^{(2)} A^2 (P^{(2)})^{-\frac{1}{2}} \). It can be verified that in this case (19) holds. Thus, Prop. 1 asserts that \( A^T P A \prec P \). Indeed, the eigenvalues of \( A^T P A - P \) are \(-0.9659, -0.4473, -0.2727 \), so \( A^T P A - P \) is negative definite.

**B. Continuous-Time Case**

The next result is the CT analogue of Prop. 1.

**Proposition 2:** Suppose that \( A \in \mathbb{R}^{n \times n} \) is Hurwitz and that \( A^{[n-1]} \) is Metzler. Then (i)
1) There exists a diagonal positive definite matrix \( P \in \mathbb{R}^{n \times n} \) such that
\[
P^{(n-1)} A^{[n-1]} + (A^{[n-1]})^T P^{(n-1)} < 0.
\] (22)

2) The matrix \( A \) satisfies \( PA + A^T P < 0 \) iff the eigenvalues \( 
\lambda_i(H), i = 1, \ldots, n, \) of the matrix
\[
H := (P^{(n-1)})^\frac{1}{2} A^{[n-1]} (P^{(n-1)})^{-\frac{1}{2}} + (P^{(n-1)})^{-\frac{1}{2}} (A^{[n-1]})^T (P^{(n-1)})^{\frac{1}{2}}
\]
satisfy
\[
\sum_{i=1}^{n} (\lambda_i(H) - (n-2)\lambda_j(H)) < 0, \quad j = 1, \ldots, n.
\] (23)

**Proof:** (i) Since \( A \) is Hurwitz, Lemma \( \text{[8]} \) implies that \( A^{[n-1]} \) is also Hurwitz. Additionally, \( A^{[n-1]} \) is Metzler. Hence, Lemma \( \text{[8]} \) implies that there exists a diagonal and positive matrix \( D \in \mathbb{R}^{n \times n} \) such that
\[
DA^{[n-1]} + (A^{[n-1]})^T D < 0,
\] (24)
and Lemma \( \text{[10]} \) yields \( \text{(22)} \).

(ii) Using the properties of compound matrices gives
\[
H = (P^{\frac{1}{2}} AP^{-\frac{1}{2}} + P^{-\frac{1}{2}} A^T P^{\frac{1}{2}})^{[n-1]}.
\] (25)

The relation between the eigenvalues of a matrix and its \((n-1)\) AC yields
\[
\lambda_{n+1-j}(P^{\frac{1}{2}} AP^{-\frac{1}{2}} + P^{-\frac{1}{2}} A^T P^{\frac{1}{2}}) = \frac{1}{n-1} \sum_{i \neq j} (\lambda_i(H) - (n-2)\lambda_j(H)),
\]
for all \( j \in [1,n] \). Therefore, \( \text{(23)} \) is equivalent to \( \lambda_1(P^{\frac{1}{2}} AP^{-\frac{1}{2}} + P^{-\frac{1}{2}} A^T P^{\frac{1}{2}}) < 0, i = 1, \ldots, n \), that is, to \( PA + A^T P < 0 \).

**Example 5:** Consider the CT-LTI \( \text{[8]} \) with
\[
A = \begin{bmatrix}
-3 & 2 & -1 \\
4 & -8 & 5 \\
-7 & 9 & -10
\end{bmatrix}.
\]
The eigenvalues of \( A \) are \( \lambda_1(A) = -16.8789 \), \( \lambda_2(A) = -2.0604 + 0.4449i \), and \( \lambda_3(A) = -2.0604 - 0.4449i \). Note that \( A \) is Hurwitz. A calculation yields
\[
A^{[2]} = \begin{bmatrix}
-11 & 5 & 1 \\
9 & -13 & 2 \\
7 & 4 & -18
\end{bmatrix},
\]
which is Metzler and Hurwitz. Hence, this system is 2-positive. By Lemma \( \text{[3]} \) there exists a diagonal \( D > 0 \) such that \( DA + A^T D < 0 \). According to Remark \( \text{[2]} \) one such \( D \) can be obtained as \( D = \text{diag}(\frac{16}{5}, \frac{1}{5}, \frac{45}{16}) \). Based on Lemma \( \text{[10]} \) the diagonal and positive definite matrix \( P \in \mathbb{R}^{n \times n} \) such that \( P^{(2)} = D \) is computed as \( P = \text{diag}(\sqrt{\frac{16}{5}}, \sqrt{\frac{1}{5}}, \sqrt{\frac{45}{16}}) \). Let \( H := (P^{(2)})^\frac{1}{2} A^{[2]} (P^{(2)})^{-\frac{1}{2}} + (P^{(2)})^{-\frac{1}{2}} (A^{[2]})^T (P^{(2)})^{\frac{1}{2}} \). The eigenvalues of \( H \) are \( \lambda_1(H) = -38.6238 \), \( \lambda_2(H) = -37.1888 \), and \( \lambda_3(H) = -8.1874 \). Note that \( \text{(23)} \) holds in this case. Therefore, Prop. \( \text{[2]} \) asserts that \( PA + A^T P < 0 \). Indeed, the eigenvalues of \( PA + A^T P \) are \(-26.0950, -5.7023, \) and \(-2.8298 \), so \( PA + A^T P < 0 \).

**V. Conclusion**

\( k \)-positive LTIs are a generalization of positive LTIs. Since positive LTIs are DS, a natural question is whether this holds for \( k \)-positive LTIs as well. We showed that the answer is in general no. Stability of the LTI and \( k \)-positivity implies that a certain \( r \)-dimensional “lifted LTI”, with \( r := \binom{n}{k} \), is stable and positive, and thus admits a DLF. But this cannot be used in general to generate a DLF for the original LTI. One exception, however, is the case \( k = n - 1 \), for which we derived a condition guaranteeing the existence of a DLF for the original LTI.

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