ON THE EXIT FROM A FINITE INTERVAL FOR THE RISK PROCESSES WITH STOCHASTIC PREMIUMS

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In this article the almost semi-continuous step-process $\xi(t)$ is considered. The conditional characteristic functions of the jumps of $\xi(t)$ have the form $E\left[e^{i\alpha \xi_k} / \xi_k > 0\right] = c(c - i\alpha)^{-1}$. For such processes the boundary functionals connected with the exit from the finite interval are investigated.

The problems on the exit from the finite interval for the process $\xi(t)$ ($t \geq 0, \xi(0) = 0$) with stationary independent increments were considered by many authors (see, for example [1, ch. IV, § 2]). In [1] the joint distributions of extrema and the distributions of the values of the process up to the exit from the interval were expressed in terms of rather complicate series of the “convolutions” of

$$
\Gamma^\pm(s, x, y) = E\left[e^{-s \tau^\pm(\pm x), \gamma^\pm(\pm x) \leq y}\right],
$$

where

$$
\tau^\pm(\pm x) = \inf\{t > 0 : \pm \xi(t) > x\}, \gamma^\pm(\pm x) = \pm \xi(\tau^\pm(\pm x)) \mp x, x > 0.
$$

Simpler relations for the Wiener processes are established in [1, p. 463] and in [2, § 27]. In [3] - [6], the mentioned problems are investigated for the semi-continuous processes $\xi(t)$ ($\xi(t)$ have jumps of one sign). For these processes, in [7] - [8] the density of distribution of $\xi(t)$ up to the exit from the interval was represented in terms of the resolvent functions $R_s(x)$ (introduced by V.S. Korolyuk in [3]).

We’ll consider the compound Poisson process

$$
\xi(t) = \sum_{k \leq \nu(t)} \xi_k,
$$

where $\nu(t)$ is the Poisson process with the rate $\lambda > 0$. The distributions of $\xi_k$ satisfy the next condition ($F(x)$ is the cumulative distribution function)

$$
P\{\xi_k < x\} = qF(x)I\{x \leq 0\} + (1 - pe^{-cx})I\{x > 0\}, c > 0, p + q = 1. \tag{1}
$$

The process $\xi(t)$ is the almost upper semi-continuous piecewise constant process. We can represent $\xi(t)$ as the claim surplus process $\xi(t) = C(t) - S(t)$ with the stochastic premium function

$$
C(t) = \sum_{k \leq \nu_1(t)} \eta_k, \eta_k > 0, E e^{i\alpha \eta_k} = \frac{c}{c - i\alpha}, c > 0,
$$

and with the process of claims $S(t) = \sum_{k \leq \nu_2(t)} \xi_k, \xi_k > 0$. $\nu_1(t), \nu_2(t)$ - are independent Poisson processes with the rates $\lambda_1, \lambda_2 > 0, \lambda_1 + \lambda_2 = \lambda$ (for details see [8]).

Note, that $C(t) \to 0$ and $\xi(t) \to -S(t)$ as $c \to \infty$, where $-S(t)$ is the non-increasing process.
Let $C_c(t)$ be the process with the cumulant
\[
\psi_c(\alpha) = \lambda_c \left( \frac{c}{e - i\alpha} - 1 \right), \quad \lambda_c = ac, \ a > 0,
\]
then $\psi_c(\alpha) \xrightarrow{c \to \infty} i\alpha a$, consequently $C_c(t) \xrightarrow{c \to \infty} at$, and $\xi_c(t) = C_c(t) - S(t) \to \xi_0(t) = at - S(t)$, where the limit process $\xi_0(t)$ is the classical upper semi-continuous risk process with the non-stochastic premium function $C(t) = at$.

Let $\theta_s$ be the exponentially distributed random variable ($P\{\theta_s > t\} = e^{-st}; \ s, t > 0$). Then the randomly stopped process $\xi(\theta_s)$ have the characteristic function (c.f.)
\[
\varphi(s, \alpha) = E e^{i\alpha \xi(\theta_s)} = \frac{s}{s - \psi(\alpha)},
\]
where
\[
\psi(\alpha) = \lambda p(c(c - i\alpha)^{-1} - 1) + \lambda q(\phi(\alpha) - 1), \quad \phi(\alpha) = \int_{-\infty}^{\alpha} e^{i\alpha x} dF(x).
\]
(2)

Let us denote the first exit time from the interval $(x - T, x)$, $0 < x < T, T > 0$:
\[
\tau(x, T) = \inf \{t > 0: \xi(t) \notin (x - T, x)\},
\]
and the events
\[
A_+(x) = \{\omega : \xi(\tau(x, T)) \geq x\}, \quad A_-(x) = \{\omega : \xi(\tau(x, T)) \leq x - T\}.
\]
Then
\[
\tau(x, T) = \begin{cases} 
\tau^+(x, T) = \tau^+(x), & \omega \in A_+(x); \\
\tau^-(x, T) = \tau^-(x - T), & \omega \in A_-(x).
\end{cases}
\]
Overshoots at the moments of the exit from the interval we denote by the following relations:
\[
\gamma^+(x) = x - T - \xi(\tau^-(x, T)), \quad \gamma^-(x) = \xi(\tau^+(x, T)) - x.
\]

The main task of our paper is the finding the next moment generating functions (m.g.f.) of the functionals connected with the exit from the interval.

\[
Q(T, s, x) = E e^{-s\tau(x, T)}, \quad Q^T(s, x) = E \left[ e^{-s\tau^+(x, T) - s\tau^-(x, T)}, A_+\right], \quad Q_T(s, x) = E \left[ e^{-s\tau^-(x, T) - s\tau^+(x, T)}, A_-\right],
\]
\[
V^+(s, \alpha, x, T) = E \left[ e^{i\alpha \gamma^+(x, T)} - e^{i\alpha \gamma^-(x, T)}, A_+\right], \quad V_+(s, \alpha, x, T) = E \left[ e^{i\alpha \gamma^-(x, T)} - e^{i\alpha \gamma^+(x, T)}, A_-\right],
\]
\[
V(s, \alpha, x, T) = E \left[ e^{i\alpha \xi(\theta_s), \tau(x, T) > \theta_s}\right],
\]

Let us denote the extrema $\xi^+(t) = \sup_{0 \leq s \leq t} (\inf) \xi(s), \ xi^-= \sup_{0 \leq s < \infty} (\inf) \xi(s)$, the joint distribution of
\[
\{\xi(\theta_s), \xi^+(\theta_s), \xi^-(\theta_s)\}:
\]
\[
H_s(T, x, y) = P \{\xi(\theta_s) < y, \xi^+(\theta_s) < x, \xi^-(\theta_s) > x - T\} = P \{\xi(\theta_s) < y, \tau(x, T) > \theta_s\},
\]
and
\[
P_\pm(s, x) = P \{\xi^\pm(\theta_s) < x\}, \ x \geq 0, \ p_\pm(s) = P \{\xi^\pm(\theta_s) = 0\}, \ q_\pm(s) = 1 - p_\pm(s);
\]
\[\varphi_{\pm}(s, \alpha) = \pm \int_{0}^{\pm\infty} e^{i\alpha x} dP_{\pm}(s, x),\]

\[T^{\pm}(s, x) = E \left[ e^{-s\tau^{\pm}(x)}, \tau^{\pm}(x) < \infty \right], \ x \geq 0.\]

**Lemma 1.** For the process \(\xi(t)\) with cumulant (2) the main factorization identity is represented by relations

\[\varphi(s, \alpha) = \varphi_{+}(s, \alpha)\varphi_{-}(s, \alpha), \ \Re \alpha = 0; \quad (3)\]

\[\varphi_{+}(s, \alpha) = \frac{p_{+}(s)(c - i\alpha)}{\rho_{+}(s) - i\alpha}, \quad (4)\]

where \(\rho_{+}(s) = \sigma(s)\) is the positive root of Lundberg's equation \(\psi(-ir) = s, \ s > 0.\)

\[P \{\xi^{+}(\theta_{s}) > x\} = T^{+}(s, x) = q_{+}(s)e^{-\sigma(s)x}, \ x > 0. \quad (5)\]

If \(m > 0:\)

\[\lim_{s \to 0} \rho_{+}(s)s^{-1} = \rho'_{+}(0) = m^{-1}, \ \lim_{s \to 0} P_{-}(s, x) = P \{\xi^{-} < x\}, \ x < 0. \quad (6)\]

If \(m < 0:\)

\[\lim_{s \to 0} \rho_{+}(s) = \rho_{+} > 0; \ \lim_{s \to 0} s^{-1} P \{\xi^{-}(\theta_{s}) > x\} = E \rho^{-}(x), \ x < 0. \quad (7)\]

If \(\sigma_{1}^{2} = D\xi(1) < \infty\) and \(m = \lambda \left( p_{c} - 1 - \frac{\rho F(0)}{\psi(0)} \right) = 0 \left( \bar{F}(0) = \int_{-\infty}^{0} F(x)dx \right),\)

\[\lim_{s \to 0} \rho_{+}(s)s^{-1/2} = \frac{\sqrt{\sigma}}{\sigma_{1}}; \ \lim_{s \to 0} s^{-1/2} P'_{-}(s, x) = f_{0}(x), \ x < 0,
\]

\[f_{0}(x) = k_{0} \frac{\partial}{\partial x} \left( \int_{0}^{\infty} P \{\bar{\xi}_{0}(t) < x\} dt \right) = -k_{0} \frac{\partial}{\partial x} E \tau_{0}(x), \ x < 0; \quad (8)\]

where \(k_{0} = c\sigma_{1} \left( \sqrt{2} \right)^{-1}, \ \tau_{0}(x) = \inf \left\{ t > 0 : \bar{\xi}_{0}(t) < x \right\}, \ x < 0, \ \bar{\xi}_{0}(t) is the decreasing process with the spectral measure

\[\Pi_{0}(dx) = \lambda q(cF(x)dx + dF(x)), \ x < 0.\]

**Proof.** Relations (3) - (7) were proved in \([7]\) - \([8]\). If \(m = 0 \left( p = c\rho F(0) \right),\) then

\[\varphi(s, \alpha) = \frac{s(c - i\alpha)}{s(c - i\alpha) - i\alpha \lambda \left( p - qF(\alpha)(c - i\alpha) \right)}, \ \bar{F}(\alpha) = \int_{-\infty}^{0} e^{i\alpha x} F(x)dx.\]

On the basis of factorization identity (3) as \(s \to 0,\) we get

\[\frac{1}{\sqrt{s}} \varphi_{-}(s, \alpha) = \frac{\rho_{+}(s) - i\alpha}{\rho_{+}(s)(c - i\alpha) - i\alpha \lambda \left( p - qF(\alpha)(c - i\alpha) \right)} \to \bar{f}_{0}(\alpha), \quad (9)\]

\[\bar{f}_{0}(\alpha) = \frac{c\sigma_{1}}{\sqrt{2} - \lambda q} \left[ \left( \bar{F}(\alpha) - \bar{F}(0) \right) c + \varphi(\alpha) - 1 \right] = \frac{c\sigma_{1}}{\sqrt{2} - \lambda q} \left[ \frac{1}{\sqrt{2} - \lambda q} \right], \quad (10)\]

\[\bar{\psi}_{0}(\alpha) = \int_{-\infty}^{0} (e^{i\alpha x} - 1) \Pi_{0}(dx), \ \Pi_{0}(dx) = \lambda q(cF(x)dx + dF(x)), \ x < 0.\]

Let's denote

\[\varphi_{0}(s, \alpha) = E e^{i\alpha \bar{\xi}_{0}(\theta_{s})} = \frac{s}{s - \bar{\psi}_{0}(\alpha)}, \quad (11)\]
where $\tilde{\xi}_0(t)$ is the decreasing process with the cumulant $\tilde{\psi}_0(\alpha)$. Since
\[
\frac{c\sigma_1}{\sqrt{2}} \rho_0(s,\alpha) s^{-1} \to \tilde{f}_0(\alpha) = \int_{-\infty}^0 e^{i\alpha x} f_0(x) dx, \ s \to 0,
\]
we get that
\[
f_0(x) = k_0 \frac{\partial}{\partial x} \left( \int_0^\infty P \{ \tilde{\xi}_0(t) < x \} dt \right),
\]
or
\[
-f_0(x) = k_0 \frac{\partial}{\partial x} \int_0^\infty P \{ \tau_0(x) > t \} dt = k_0 \frac{\partial}{\partial x} E \tau_0(x), \ x < 0.
\]

Let’s introduce the set of boundary functions on the interval $I \subset (-\infty, \infty)$:
\[
\mathfrak{L}(I) = \left\{ G(x) : \int_I |G(x)| dx < \infty \right\},
\]
and the set of integral transforms:
\[
\mathfrak{R}(I) = \left\{ g^0(\alpha) : g^0(\alpha) = C + \int_I e^{i\alpha x} G(x) dx \right\}.
\]

Let’s denote the projection operations on $\mathfrak{R}((-\infty, \infty))$ by the next relations
\[
[g^0(\alpha)]_I = \int_I e^{i\alpha x} G(x) dx, \quad [g^0(\alpha)]_0 = C + \int_I e^{i\alpha x} G(x) dx,
\]
\[
[g^0(\alpha)]_- = [g^0(\alpha)]_{(-\infty,0)}, \quad [g^0(\alpha)]_+ = [g^0(\alpha)]_{(0,\infty)}.
\]

The main results of our paper are included in the next two assertions.

**Theorem 1.** For the process $\xi(t)$ with cumulant (2) $Q^T(s, x)$ has the next form $(0 < x < T)$
\[
Q^T(s, x) = q_+(s) e^{-\rho_+(s)x} \int_{x-T}^0 e^{\rho_+(s)y} dP_-(s, y) \times
\]
\[
\times \left[ e^{-\rho_+(s)T} \int_{-\infty}^{-T} e^{c(T+y)} dP_-(s, y) + \int_{-T}^0 e^{\rho_+(s)y} dP_-(s, y) \right]^{-1}. \quad (9)
\]

**Theorem 2.** For the process $\xi(t)$ with cumulant (2) the joint distributions of \{$\tau^+ (x, T), \gamma_+^T(x)$\} and \{$\tau^+ (x, T), \xi(\tau^+ (x, T))$\} are determined by the next relations
\[
\begin{cases}
V^+(s, \alpha, x, T) = \frac{c}{c - i\alpha} Q^T(s, x), \ 0 < x < T, \\
V_+(s, \alpha, x, T) = e^{i\alpha x} V^+(s, \alpha, x, T) = \frac{c}{c - i\alpha} Q^T(s, x).
\end{cases} \quad (10)
\]

The ch.f. of $\xi(\theta_s)$ before the exit time from the interval has the form
\[
V(s, \alpha, x, T) = \varphi_+(s, \alpha) [\varphi_-(s, \alpha) (1 - V_+(s, \alpha, x, T))]_{[x-T, \infty)}
\]
\[
= \varphi_+(s, \alpha) [\varphi_-(s, \alpha) (1 - c e^{i\alpha(x - \alpha)} Q^T(s, x))]_{[x-T, \infty)}, \quad (11)
\]
the corresponding distribution has the next density \((x - T < z < x, \ z \neq 0)\)

\[
\begin{align*}
\frac{\partial}{\partial z} H_s(T, x, z) &= h_s(T, x, z) \\
&= \left( p_+(s) P_+(s, z) - q_+(s) \rho_+(s) \int_{z}^{x} e^{\rho_+(s)(y-z)} d\mu(y) \right) I\{\{z < 0\} + \\
&\quad + \rho_+(s) Q^T(s, x) \int_{z-x}^{x} e^{\rho_+(s)(y-x)} d\mu(y),
\end{align*}
\]

and the next atomic probability

\[
P\{\xi(\tau_0) = 0, \tau(x, T) > \tau_0\} = P\{\xi(\tau_0) = 0\} = p_-(s) p_+(s) = \frac{s}{s + \lambda}.
\]

**Proof.** First, let us prove Theorem 1. From the stochastic relations for \(\tau_+^+(x, T), \gamma_T^+(x) (\xi = \xi_1 \text{ have the cumulative distribution function } F_1(x), \zeta - \text{the moment of the first jump of } \xi(t)):\n
\[
\tau_+^+(x, T) = \begin{cases}
\zeta, \ z > x,
\zeta + \tau_+^+(x - \xi, T), x - T < \xi < x,
\end{cases}
\]

\[
\gamma_T^+(x) = \begin{cases}
\xi - x, \ z > x,
\gamma_T(x - \xi), x - T < \xi < x,
\end{cases}
\]

we have the next equation for \(V^+(s, \alpha, x, T)\)

\[
(s + \lambda) V^+(s, \alpha, x) = \left( \frac{\lambda p_c}{c - i\alpha} e^{-\alpha x} + \lambda \int_{x-T}^{x} V^+(s, \alpha, x-z) dF_1(z) \right), \ 0 < x < T. \tag{13}
\]

If \(\alpha = 0\), then from (13) we obtain the equation for \(Q^T(s, x)\)

\[
(s + \lambda) Q_T^+(s, x) = \lambda p c e^{-\alpha x} + \lambda \int_{x-T}^{x} Q_T^+(s, x-z) dF_1(z), \ 0 < x < T. \tag{14}
\]

Since \(P(A_+(x)) = 1\) for \(x < 0\), then we have the next boundary conditions

\[
Q^T(s, x) = \begin{cases}
0, \ x > T, \\
1, \ x < 0.
\end{cases}
\]

After the replacement

\[
\mathcal{Q}^T(s, x) = 1 - Q^T(s, x),
\]

relation (14) yields the equation for \(\mathcal{Q}^T(s, x) \ (0 < x < T)\)

\[
(s + \lambda) \mathcal{Q}^T(s, x) = s + \lambda F(x - T) + \lambda \int_{0}^{T} \mathcal{Q}^T(s, z) F_1(x - z) dF_1(z),
\]

which after prolonging for \(x > 0\) has the form:

\[
(s + \lambda) \mathcal{Q}^T(s, x) = s C(x) + \lambda \int_{-\infty}^{\infty} \mathcal{Q}^T(s, z) F_1(x - z) dz + C_T^+(s, x), \tag{15}
\]

\[
C(x) = I\{x > 0\}, \ C_T^+(s, x) = e^{-\alpha x}, \ x > 0,
\]

\[
\mathcal{C}_T(s) = \lambda p [e^{cT} - e^{\mathcal{C}_T^+(s)}], \ \mathcal{Q}^*_T(s) = e^{cT} \mathcal{Q}^T(s, x) dx. \tag{16}
\]
Let's introduce the function \( C_\epsilon(x) = e^{-\epsilon x}C(x), x > 0 \), and consider instead of (15) the equation for \( Y_\epsilon(T, s, x) \) (\( \epsilon > 0 \)):  
\[
(s + \lambda)Y_\epsilon(T, s, x) = sC_\epsilon(x) + \lambda \int_{-\infty}^{\infty} Y_\epsilon(T, s, x - z)dF_1(z) + C_\epsilon^T(s, x), \quad x > 0. \tag{17}
\]

Denote  
\[
ye(T, s, \alpha) = \int_0^\infty e^{i\alpha x}Y_\epsilon(T, s, x)dx, \quad \tilde{C}_\epsilon(\alpha) = \int_0^\infty e^{i\alpha x}C_\epsilon(x)dx,
\]
\[
\tilde{C}_T(s, \alpha) = \int_0^\infty e^{i\alpha x}C_\epsilon^T(s, x)dx.
\]

After integral transform from (17) we obtain the next equation  
\[
(s - \psi(\alpha))Ye(T, s, \alpha) = s\tilde{C}_\epsilon(\alpha) + \tilde{C}_T(s, \alpha) - [ye(\alpha)\varphi(\alpha)]_-
\]
or  
\[
sYe(T, s, \alpha)\varphi^{-1}(s, \alpha) = s\tilde{C}_\epsilon(\alpha) + \tilde{C}_T(s, \alpha) - [ye(\alpha)\varphi(\alpha)]_-. \tag{18}
\]

After using the factorization decomposition (3) and the projection operation \([\cdot]_+\), relation (18) yields  
\[
sYe(T, s, \alpha)\varphi_+(s, \alpha) = \varphi_-(s, \alpha)\left( s\tilde{C}_\epsilon(\alpha) + \tilde{C}_T(s, \alpha) \right)_+. \tag{19}
\]

By inverting of (19), we obtain  
\[
sY_\epsilon(T, s, x) = s\int_0^T B_\epsilon(x - y)dP_+(s, y) + \int_0^x B(s, x - y, T)dP_+(s, y), \tag{20}
\]
\[
B_\epsilon(x) = \int_{-\infty}^x e^{-\epsilon(x-y)}dP_-(s, y) = \int_{-\infty}^0 e^{-\epsilon(x-y)}dP_-(s, y) = e^{-\epsilon x}e^{\epsilon \xi^-(\theta_s)},
\]
\[
B(s, x, T) = \tilde{C}_T(s) \int_{-\infty}^{x-T} e^{-c(x-y)}dP_-(s, y), \quad x > 0.
\]

Taking into account that \( C_\epsilon(x) \to I \{ x > 0 \} \) as \( \epsilon \to 0 \), then \( Y_\epsilon(T, s, x) \to \overline{Q}^T(s, x) \) as \( \epsilon \to 0, 0 < x < T \). So Eq. (20) yields  
\[
\overline{Q}^T(s, x) = sP_+(s, x) + p(s)B(s, x, T) + \int_{x+0}^T B(s, x - z, T)P_+(s, z)dz.
\]

Taking into account that  
\[
q_+(s)p_+(s) \int_0^x \int_{-\infty}^{x-T} e^{-c(z-y)}dP_-(s, y)e^{-\rho_+(s)(x-z)}dz = =q_+(s)p_+(s) \int_{-\infty}^{x-T} e^{-\rho_+(s)(x+y)}dP_-(s, y) \int_{\max(0, y+T)}^x e^{-cq_+(s)z}dz = =p_+(s) \int_{-\infty}^{x-T} e^{cy_-(s)z}dP_-(s, y) + + \int_{-\infty}^{x-T} e^{\rho_+(s)(y+T-x)-cT}dP_-(s, y) - \int_{-\infty}^{x-T} e^{-c(x-y)}dP_-(s, y),
\]

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we have
\[ sQ^T(s, x) = sP_+(s, x) + p_+(s)C_T(s)e^{-\rho_+(s)x} \times \left[ \int_{-T}^{\infty} e^{r y} dP_-(s, y) + \int_{-T}^{x-T} e^{-cT + \rho_+(s)(y+T)} dP_-(s, y) \right]. \]

From the last equation we can find \( C_T(s) \), and \( Q^*_T(T) \), and then get (9).

Let’s note, that \( Q^T(s, x) \to \mathcal{P}_+(s, x) \), as \( T \to \infty \) and \( Q^T(s, x) \to 0 \), as \( c \to \infty \). If we consider, instead of \( \xi(t) \), the process \( \xi_c(t) = C_c(t) - S(t) \), then relation (9) yields
\[ Q^T_c(s, x) = q^c_+(s)E \left[ e^{\rho^c_+(s)(\xi_c^+(\theta_s) + T - x)}, \xi_c^-(\theta_s) + T - x > 0 \right] \times \left( E \left[ e^{c^c(\xi_c^-(\theta_s) + T)}, \xi_c^-(\theta_s) + T < 0 \right] + E \left[ e^{\rho^c_+(s)(\xi_c^-(\theta_s) + T)}, \xi_c^-(\theta_s) + T > 0 \right] \right)^{-1}. \]

Taking into account that for \( x > 0 \): \( P \{ \xi_+^c(\theta_s) > x \} = q^c_+(s)e^{-c^c_+(s)x} \to e^{-\rho^c_0(s)x} \), where \( \rho^c_0(s) \) is the positive solution of the equation
\[ \psi^I_0(-ir) := ar - \lambda_2 \left( \int_{-\infty}^{0} e^{rz} dF(x) - 1 \right) = 0, \]
we get \( Q^T_c(s, x) \to Q^T_c(s, x) \) as \( c \to \infty \). If we denote
\[ \xi^0_+(t) = \sup_{0 \leq u \leq t} (\inf) \xi^0(u), \]
then
\[ Q^T(s, x) = E \left[ e^{\rho^0_0(s)(\xi^0_+^+(\theta_s) + T - x)}, \xi^0_+^-(\theta_s) + T - x > 0 \right] \left( E \left[ e^{\rho^0_0(s)(\xi^0_+^-(\theta_s) + T)}, \xi^0_+^-(\theta_s) + T > 0 \right] \right)^{-1} \]
\[ = \int_0^{T-x} e^{\rho^0_0(s)(T-x-y)} dP \left\{ -\xi^0_+(\theta_s) < y \right\} \left( \int_0^T e^{\rho^0_0(s)(T-y)} dP \left\{ -\xi^0_+(\theta_s) < y \right\} \right)^{-1} \]
\[ = R_s(T - x)R_s^{-1}(T), \]
where the last relation is the well-known formula (see [3]) for the upper semi-continuous processes.

Proof. Consider the proof of the second theorem. The first relation in (10) follows from equations (13) and (14). The second relation follows from the first one. The first equality in (11) was proved in [9]. After inverting (11), we get
\[ h_s(T, x, z) = p_+(s) \frac{\partial}{\partial z} P_-(s, z) I \{ z < 0 \} + q_+(s) \rho_+(s) \int_{x-T}^{\min\{z, 0\}} e^{-\rho_+(s)(z-y)} dP_-(s, y) - Q^T(s, x) \left[ p_+(s) \frac{\partial}{\partial z} P \left\{ \xi_-(\theta_s) + \theta'_c + x \leq z \right\} + \right. \]
\[ + q_+(s) \rho_+(s) \int_{x-T}^z e^{-\rho_+(s)(z-y)} dP \left\{ \xi_-(\theta_s) + \theta'_c + x < z \right\} \right]. \]

Using the integral transform of (21) with respect to the distribution of \( \theta'_c \) we get formula (12).

**Corollary 1.** For the joint distribution \( \{ \tau^-(x, T), \xi(x, \tau^-(x, T)) \} \) we have
\[ sE \left[ e^{-s\xi(x, \tau^-(x, T))}, \xi(x, \tau^-(x, T)) < z, A_-(x) \right] = \int_{x-T}^z \Pi_-(z-y) dH_s(T, x, y), z \leq x - T, \]
where $H_s(T,x,y)$ is determined by its density (12) and $\Pi_-(x) = \int_{-\infty}^{x} \Pi(dy), \ x < 0.$

The probability of the lack of exit (non-exit) from the interval $(x - T, x)$ has the form

$$P\{\tau(x,T) > \theta_s\} = P\{\xi^-(\theta_s) > x - T\} - Q^T(s,x) \left[ \int_{-\infty}^{-T} e^{c(z+T)} dP_-(s,z) + P\{\xi^-(\theta_s) > -T\} \right].$$

(23)

The m.g.f. for $\tau(x,T)$ and $\tau^-(x,T)$ are determined in the following way

$$\begin{align*}
Q(T,s,x) &= 1 - P\{\tau(x,T) > \theta_s\}, \ 0 < x < T, \\
Q_T(s,x) &= Q(T,s,x) - Q^T(s,x), \ 0 < x < T.
\end{align*}$$

(24)

Proof. Formula (22) follows from [6, Theorem 7.3]. By substitution (12) in (22), we obtain the relation in terms of $Q^T(s,x)$ and the truncated distribution of $\xi^-(\theta_s) + \theta_s^c$. Taking into account that

$$P\{\tau(x,T) > \theta_s\} = \int_{x-T}^{x} dH_s(T,x,z) =$$

$$= P\{\xi^-(\theta_s) > x - T\} - q_+(s) \int_{x-T}^{0} e^{p_+(s)(y-(x-T))} dP_-(s,z) +$$

$$+ Q^T(s,x) \left[ \int_{-\infty}^{0} e^{p_+(s)(z+T)} dP_-(s,z) - P\{\xi^-(\theta_s) > -T\} \right],$$

and using formula (9), we obtain (23) after some simple transformations. Substituting (23) into the first relation of (24) we find the m.g.f. of $\tau(x,T)$, and then we can get the m.g.f. of $\tau^-(x,T)$ (see the second relation in (24)).

On the basis of formulas (6) - (8) we can get the next statement about the limit behavior of $Q^T(s,x)$ and $h_s(T,x,z)$, as $s \to 0$.

Corollary 2. Function $h'_0(T,x,z) = \lim_{s \to 0} s^{-1} h_s(T,x,z) \ (x-T < z < x, \ z \neq 0, \ 0 < x < T)$ according to the sign of $m$ have the next forms:

if $m > 0$

$$h'_0(T,x,z) = \frac{1}{m} \left( e^{-c \frac{\partial}{\partial z}} P\{\xi^- < z\} - P\{\xi^- > z\} \right) I\{z < 0\} + \frac{1}{m} Q^T(x) P\{\xi^- > z - x\};$$

(25)

if $m < 0$

$$h'_0(T,x,z) = \left( -p_+ \frac{\partial}{\partial z} E\tau^-(z) + q_+ \rho_+ \int_{z}^{0} e^{p_+(y-z)} dE\tau^-(y) \right) I\{z < 0\} -$$

$$- Q^T(x) \rho_+ \int_{z-x}^{0} e^{p_+(y-(z-x))} dE\tau^-(y);$$

(26)

if $m = 0$

$$h'_0(T,x,z) = \left( -\frac{\partial}{\partial z} E\tau_0(z) - c \lambda^{-1} + c \int_{z}^{0} \frac{\partial}{\partial y} E\tau_0(y) dy \right) I\{z < 0\} +$$

$$+ cQ^T(x) \left( \lambda^{-1} - \int_{z-x}^{0} \frac{\partial}{\partial y} E\tau_0(y) dy \right).$$

(27)

The ruin probability

$$Q^T(x) = \lim_{s \to 0} Q^T(s,x)$$

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(according to the sign of \( m \)) is determined from (9) in the following way

\[
Q^T(x) = \begin{cases} 
\int_{x-T}^{0} dP \{ \xi > y \} \times \\
\times \left[ e^{c(T+y)}dP \{ \xi > y \} + \int_{-T}^{0} dP \{ \xi > y \} \right]^{-1}, m > 0, \\
q_+ e^{-\rho_+x} \left( \frac{1}{\lambda p_+} - \int_{x-T}^{0} e^{\rho_+y} \frac{\partial}{\partial y} \tau^{-y}(dy) \right) \times \\
\left[ \frac{1}{\lambda p_+} - e^{-\rho_+} \int_{-\infty}^{0} e^{c(T+y)} \frac{\partial}{\partial y} \tau^{-y}(dy) - \int_{-T}^{0} e^{\rho_+y} \frac{\partial}{\partial y} \tau^{-y}(dy) \right]^{-1}, m < 0, \\
\left( \lambda^{-1} - \int_{x-T}^{0} \frac{\partial}{\partial y} \tau_0(y)dy \right) \times \\
\left[ \lambda^{-1} - \int_{-\infty}^{0} e^{c(T+y)} \frac{\partial}{\partial y} \tau_0(y)dy - \int_{-T}^{0} \frac{\partial}{\partial y} \tau_0(y)dy \right]^{-1}, m = 0.
\end{cases}
\]

The distribution of \( \xi(\tau^{-}(x,T)) \) has the next form:

\[
P \{ \xi(\tau^{-}(x,T)) < z, A_-(x) \} = \frac{1}{\lambda} \Pi_-(z) + \int_{x-T}^{0} \Pi_-(z-y) h_0'(T, x, y)dy + \\
+ \int_{0+}^{x} \Pi_-(z-y) h_0'(T, x, y)dy, z < x - T.
\]

Corollary 3. For the process \( \xi(t) \) with the cumulant function

\[
\psi(\alpha) = \lambda p(c - i\alpha)^{-1} - 1) + \lambda q(b + i\alpha)^{-1} - 1)
\]

\( Q^T(x) \) is represented in the following way \((0 < x < T)\)

\[
Q^T(x) = \begin{cases} 
\left( 1 - q_- e^{\rho_-(x-T)} \right) \left( 1 - q_- e^{c + \rho_-} - e^{\rho_-} \right)^{-1}, m > 0, \\
q_+ e^{-\rho_+x} \left( 1 - b(p_+ + b)^{-1} e^{\rho_+(x-T)} \right) \left( 1 - b(p_+ + b)^{-1} q_+ e^{-\rho_+} \right)^{-1}, m < 0, \\
c(1 + b(T - x)) \frac{1}{b + c + bcT}, m = 0.
\end{cases}
\]

If \( \xi(t) \) is a symmetric process \((p = q = 1/2, b = c)\), then

\[
Q^T(x) = \frac{1 + c(T - x)}{2 + cT}, Q_T(x) = \frac{1 + cx}{2 + cT}, \quad (0 < x < T).
\]

Proof. Let’s note that the process with cumulant (30) is the almost upper and lower semi-continuous process. Then in addition to relations (4) - (5) we have that

\[
\varphi_-(s, \alpha) = \frac{p_-(s)(b + i\alpha)}{\rho_-(s) + i\alpha},
\]

where \(-\rho_-(s) = -bp_-(s)\) is the negative root of the equation \(\psi(-ir) = s, s > 0,\)

\[
P \{ \xi^- (\theta_s) < x \} = T^- (s, x) = q_-(s)e^{\rho_-(s)x}, x < 0.
\]
If \( m > 0 \), then
\[
\Pr\{\xi^-(\theta_s) < x\} \xrightarrow{s \to 0} \Pr\{\xi^- < x\} = q_- e^{bp_+ x}, \quad x < 0, \quad p_-(s) \xrightarrow{s \to 0} p_- > 0. \tag{34}
\]
Taking into account that \( p_+(s)p_-(s) = s(s + \lambda)^{-1} \), we have, for \( m < 0 \), \( q'_-(s) = -p'_-(s) \to -(\lambda p_+)^{-1} \) as \( s \to 0 \). Hence,
\[
E \tau^-(x) = -\frac{\partial}{\partial s} T^-(s,x)|_{s=0} = \frac{1 - bx}{\lambda p_+}, \quad x < 0. \tag{35}
\]
If \( m = 0 \), then for \( \bar{\xi}_0(t) \), we have \( \Pi_0(dx) = \lambda_0 b e^{b x} dx, \quad x < 0, \quad \lambda_0 = \lambda q(c + b)b^{-1} \), moreover
\[
\bar{\xi}^-_0(t) = \xi_0(t), \quad p^0_-(s) = \Pr\{\bar{\xi}_0(\theta_s) = 0\} = \frac{s}{s + \lambda_0}.
\]
Hence, the m.g.f. of \( \tau_0(x) \) has the form
\[
T^0_0(s,x) = E e^{-s \tau_0(x)} = q^0_-(s)e^{bp^0_+(s)x}, \quad x < 0.
\]
Since \( (p^0_+)'(s) = -(q^0_-)'(s) \to \lambda_0^{-1} \) as \( s \to 0 \), we get
\[
E \tau_0(x) = -\frac{\partial}{\partial s} T^0_0(s,x)|_{s=0} = \frac{1 - bx}{\lambda_0}, \quad x < 0. \tag{36}
\]
Substituting formulas (34) - (36) into the corresponding relations of (28) we get (31). \( \square \)

**Remark.** We should note that it is easy to get the representation of the m.g.f. of the functionals related to the exit from the interval for the almost lower semi-continuous process \( \eta(t) \) (with the parameter \( b > 0 \), by considering that \( \xi(t) = -\eta(t) \)). Particularly,
\[
Q_T(s,x) = q_-(s) \int_0^x e^{p_-(s)(x-y)} dP_+(s,y) \times
\]
\[
\times \left[ \int_T^\infty e^{b(T-y)} dP_+(s,y) + \int_0^T e^{p_-(s)(T-y)} dP_+(s,y) \right]^{-1}. \tag{37}
\]

Let \( \xi(t) \) be the almost upper semi-continuous piecewise constant process. Then \( \xi_1(t) = at + \xi(t), \quad a < 0 \) is the almost upper semi-continuous piecewise linear process. For the process \( \xi_1(t) \) on the basis of the stochastic relations for \( \tau^+(x,T) \):
\[
\tau^+(x,T) = \begin{cases} 
\xi, \quad \xi + a \zeta > x, \\
\xi + \tau^+(x-\xi-a\zeta), \quad x - T < \xi + a \zeta < x,
\end{cases}
\]
we have the next integro-differential equation for \( Q_T(s,x) \)
\[
a \frac{\partial}{\partial x} Q_T(s,x) = \lambda \int_{x-T}^x Q_T(s,x-z)dF_1(z) - (s + \lambda)Q_T(s,x) + \lambda p e^{-cx}, \quad 0 < x < T. \tag{38}
\]
Introducing the function \( \overline{Q}_T(s,x) = 1 - Q_T(s,x) \), and following the reasoning analogous to that for the piecewise constant process \( \xi(t) \) we can get the representation of the functionals related to the exit from the interval \( (x - T, x) \) for the piecewise linear processes.

The two boundary problems for the integer - valued random-walks are considered in [10] and for the process with stationary independent increments are treated in [11].
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