Field analogue of the Ruijsenaars-Schneider model

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ABSTRACT: We suggest a field extension of the classical elliptic Ruijsenaars-Schneider model. The model is defined in two different ways which lead to the same result. The first one is via the trace of a chain product of $L$-matrices which allows one to introduce the Hamiltonian of the model and to show that the model is gauge equivalent to a classical elliptic spin chain. In this way, one obtains a lattice field analogue of the Ruijsenaars-Schneider model with continuous time. The second method is based on investigation of general elliptic families of solutions to the 2D Toda equation. We derive equations of motion for their poles, which turn out to be difference equations in space with a lattice spacing $\eta$, together with a zero curvature representation for them. We also show that the equations of motion are Hamiltonian. The obtained system of equations can be naturally regarded as a field generalization of the Ruijsenaars-Schneider system. Its lattice version coincides with the model introduced via the first method. The limit $\eta \to 0$ is shown to give the field extension of the Calogero-Moser model known in the literature. The fully discrete version of this construction is also discussed.

KEYWORDS: Integrable Field Theories, Integrable Hierarchies, Lattice Integrable Models, Quantum Groups

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1 Introduction

Our main purpose in this paper is to introduce a (1+1)-dimensional field theory generalization of the elliptic N-body Ruijsenaars-Schneider model [1, 2] which is usually regarded as a relativistic extension of the Calogero-Moser system. This is done in two different ways, so
the paper consists of two main parts. In the first part (sections 2–4) we define a discrete
space classical Ruijsenaars-Schneider chain starting from the classical homogeneous elliptic
GL\(_N\) spin chain on \(n\) sites. Assuming periodic boundary conditions in discrete space, it is
a finite-dimensional integrable system of classical mechanics. By construction, it is gauge
equivalent to the elliptic spin chain (or lattice version of the generalized Landau-Lifshitz
model) with some special choice of the classical spin matrix at each site. In the second
part (sections 5 and 6) we introduce a (1+1)-dimensional field analogue of the Ruijsenaars-
Schneider model with continuous space variable whose natural finite-dimensional reduction
turns out to be equivalent to the Ruijsenaars-Schneider chain introduced in the first part.
Our method is based on investigation of general elliptic solutions (called elliptic families)
to the difference version of the 2D Toda equation. We derive equations of motion for
their poles together with a zero-curvature representation for them and show that they are
Hamiltonian. The limit to differential equations in space rather than difference is shown to
give the field extension of the Calogero-Moser model introduced in [3] via analyzing elliptic
families of solutions to the Kadomtsev-Petviashvili equation.

Regarding the motivation of our work, we should say that the field analog of the
Ruijsenaars-Schneider system is a new integrable model which is valuable by itself. Its
role in modern mathematical physics can be compared with the role of the field extension
of the Calogero-Moser system, which is known to be gauge equivalent to continuous 1+1
integrable Landau-Lifshitz model [4, 5].

Below we describe the contents of both parts of the paper in more details.

The classical homogeneous elliptic GL\(_N\) spin chain on \(n\) sites is a widely known integrable
system. It is also called an integrable GL\(_N\)-generalization of the lattice Landau-Lifshitz
equation [6–10]. It is defined via the (classical) monodromy matrix depending on a spectral
parameter \(z\) as a product of the Lax matrices at each site:

\[
T(z) = L^1(z) L^2(z) \ldots L^n(z), \quad L^i(z) \in \text{Mat}(N, \mathbb{C}).
\] (1.1)

Each Lax matrix depends on a set of dynamical variables (coordinates in the phase space),
which are combined into a matrix \(S^i \in \text{Mat}(N, \mathbb{C})\), so that \(L^i(z) = L^i(z, S^i)\). The trace
of the monodromy matrix \(t(z) = \text{tr}T(z)\) is a generating function of Hamiltonians. They
are in involution with respect to the classical quadratic \(r\)-matrix structure (with the
Belavin-Drinfeld elliptic classical \(r\)-matrix [11] and \(c\) be an arbitrary constant)

\[
\{L^i(z), L^j(w)\} = \frac{1}{c} \delta^{ij} [L^i(z)L^j(w) - r_{12}(z-w)],
\] (1.2)

which is equivalent to \(n\) copies of the classical generalized Sklyanin algebras at each site.
In this paper we use a modified description of the classical Sklyanin’s elliptic Lax matrix.
Namely, following [12] we define \(L^i(z, S^i)\) as

\[
L^i(z, S^i) = \text{tr}_2(R^i_{12}(z) S^i_2), \quad S^i_2 = 1_N \otimes S^i \in \text{Mat}(N, \mathbb{C})^{\otimes 2},
\] (1.3)

where \(R^i_{12}(z) \in \text{Mat}(N, \mathbb{C})^{\otimes 2}\) is the quantum Baxter-Belavin elliptic \(R\)-matrix [13, 14] and
we use the standard convention on numbering the spaces where the matrices act. Let us
stress that although $R_{12}^0(z)$ is quantum, the Lax matrix (1.3) is classical. The parameter $\eta$ usually plays the role of the Planck constant since in the classical limit $\eta \to 0$ we have $R_{12}^0(z) = \eta^{-1} 1_N \otimes 1_N + r_{12}(z) + \ldots$, where $r_{12}(z)$ is the classical $r$-matrix entering (1.2). At the same time in (1.3) $\eta$ is regarded as the relativistic deformation parameter, similarly to what happens in the Ruijsenaars-Schneider model. In fact, the explicit dependence on $\eta$ can be removed by some simple re-definitions. However, we keep it since the form (1.3) has the following important property [18] (see also [15, 19–23]). In the case when $S^i$ is a rank 1 matrix ($S^i = \xi^i \otimes \psi^i, \xi^i, \psi^i \in \mathbb{C}^N$), the Lax matrix can be represented in the factorized form

$$L^i(z, S^i) = g(z + N\eta, q^i) e^{P^i/c} g^{-1}(z, q^i) \in \text{Mat}(N, \mathbb{C}), \quad P^i = \text{diag}(p^i_1, \ldots, p^i_N)$$

(1.4)

(up to a scalar factor), where $q^i$ denotes a set of $N$ coordinate variables $q^i = \{q^i_1, \ldots, q^i_N\}$ and the explicit form of the matrix $g(z, q^i)$ is given below in the main text. It is known as the intertwining matrix entering the IRF-Vertex correspondence [24–27]. The factorization (1.4) provides an explicit parametrization of the matrix $S^i$ through the canonical variables $p^i_k, q^i_k, i = 1, \ldots, n, k = 1, \ldots, N$, thus providing the classical analogue for representation of the generalized Sklyanin algebra by difference operators. Moreover, the gauge transformed Lax matrix

$$g^{-1}(z, q^i)L^i(z, S^i)g(z, q^i) = g^{-1}(z, q^0)g(z + N\eta, q^i)e^{P^i/c} := L^{RS}(z, p^i, q^i)$$

(1.5)

is equal to the Lax matrix of the classical $N$-body elliptic Ruijsenaars-Schneider model with momenta $p^i$, coordinates of particles $q^i$ and the relativistic deformation parameter $\eta$. The “velocity of light” $c$ enters (1.5) as a normalization factor. On the spin chain side it is the constant in the r.h.s. of (1.2).

We restrict ourselves to the case when all matrices of dynamical variables $S^i$ are of rank one. Then, taking into account (1.4)–(1.5), we represent the transfer matrix $t(z)$ in the form

$$t(z) = \text{tr}\left(\tilde{L}^1(z)\tilde{L}^2(z)\ldots \tilde{L}^n(z)\right),$$

(1.6)

where

$$\tilde{L}^i(z) = g^{-1}(z, q^0-1)g(z + N\eta, q^i)e^{P^i/c}, \quad i = 1, \ldots, n \quad \text{and} \quad q^0 = q^n.$$  

(1.7)

The Lax matrices $\tilde{L}^i(z)$ can be found explicitly. Then we derive equations of motion generated by a special Hamiltonian flow. It is the one which has continuous limit to the (1+1)-dimensional field theory (the generalized Landau-Lifshitz equation) for the elliptic spin chain.

More precisely, we prove that the transfer-matrix (1.6) provides the Hamiltonian

$$H = c \sum_{k=1}^n \log h_{k-1,k}, \quad h_{k-1,k} = \sum_{j=1}^N b^k_j, \quad b^k_j = \prod_{l=1}^N \frac{\vartheta(q^k_j - q^{k-1}_l - \eta)}{\vartheta(-\eta) \prod_{k \neq j} \vartheta(q^k_j - q^k_i)} e^{\vartheta/c},$$

(1.8)

}\footnote{An explanation of the presence of quantum $R$-matrix in a classical model comes from associative Yang-Baxter equation $R^0_{12}R^0_{23} = R^0_{13}R^0_{12} + R^{0-k}_{23} R^0_{12} (R^0_{ab} = R^0_{ba}(x_a - x_b))$, which is fulfilled by the quantum Baxter-Belavin elliptic $R$-matrix in the fundamental representation of $\text{GL}_N$. This equation (and its degenerations) provides the classical Lax equation for the Lax matrix (1.3). In this respect the associative Yang-Baxter equation unifies classical and quantum integrable structures. See [15–17] and references therein.}
where $\vartheta(z)$ is the odd Jacobi theta-function (A.8), and $\bar{q}^j_k = q^j_k - \sum_{i=1}^N q^i_k/N$ are coordinates “in the center of masses frame” at each site. This Hamiltonian generates equations of motion (in the Newtonian form)

\[
\frac{\ddot{q}^k_i}{\dot{q}^k_i} = -\sum_{l=1}^N \dot{q}^l_{i+1} E_1(q^k_i - q^l_{i+1} + \eta) - \sum_{l=1}^N \dot{q}^l_{i-1} E_1(q^k_i - q^l_{i-1} - \eta) + 2 \sum_{l \neq i}^N \dot{q}^l_k E_1(q^k_i - q^l_k) + \sum_{m,l=1}^N \dot{q}^m_l \dot{q}^k_{l+1} E_1(q^m_l - q^k_{l+1} + \eta) - \sum_{m,l=1}^N q^m_k \dot{q}^k_{l+1} E_1(q^m_k - q^k_{l+1} + \eta),
\]

where $E_1(z)$ is the logarithmic derivative of the odd Jacobi theta-function (A.18). With some simple normalization factor the Lax matrix (1.7) turns into

\[
L^{ij}_{kl}(z) = \phi(z, \bar{q}^{k-1}_l - \bar{q}^k_j + \eta) \dot{q}^k_j,
\]

where $\phi(z, q)$ is the Kronecker elliptic function (A.14). Equations of motion (1.9) are equivalently written in the form of the semi-discrete zero curvature (Zakharov-Shabat) equation

\[
\frac{d}{dt} L^k_i(z) = L^k_i(z) M^{ik}_i(z) - M^{ik-1}_i(z) L^k_i(z).
\]

with $M$-matrices

\[
M^{ik}_{ij}(z) = -(1 - \delta_{ij}) \phi(z, q^k_q - q^k_j) \dot{q}^k_j + \delta_{ij} \sum_{m,l=1}^N \dot{q}^{k+1}_m \dot{q}^k_l E_1(q^k_m - q^{k+1}_l + \eta) + \delta_{ij} \left(-E_1(z) \dot{q}^k_i \sum_{m, m \neq l}^N \dot{q}^m_l E_1(q^k_i - q^m_l) - \sum_{m=1}^N \dot{q}^{k+1}_m E_1(q^k_i - q^{k+1}_m + \eta) \right). \tag{1.12}
\]

At this stage the model is discrete and finite-dimensional. To proceed to field generalization we use another approach. We will see that the 1+1 version corresponds to straightforward field extension of the described Ruijsenaars-Schneider chain.

The idea of the other approach is to exploit the close connection between elliptic solutions to nonlinear integrable equations and many-body systems. The investigation of dynamics of poles of singular solutions to nonlinear integrable equations was initiated in the seminal paper [28], where elliptic and rational solutions to the Korteweg-de Vries and Boussinesq equations were studied. As it was proved later in [29, 30], poles of rational solutions to the Kadomtsev-Petviashvili (KP) equation as functions of the second hierarchical time $t_2$ move as particles of the integrable Calogero-Moser system [31–34]. The method suggested by Krichever [35] for elliptic solutions of the KP equation consists in substituting the solution not in the KP equation itself but in the auxiliary linear problem for it (this implies a suitable pole ansatz for the wave function). This method allows one to obtain the equations of motion together with their Lax representation.

Dynamics of poles of elliptic solutions to the 2D Toda lattice and modified KP (mKP) equations was studied in [36], see also [37]. It was proved that the poles move as particles of}

\[2\text{Notice that } q^k_i - \bar{q}^k_i = \bar{q}^k_i - q^k_i.\]
the integrable Ruijsenaars-Schneider many-body system which is a relativistic generalization of the Calogero-Moser system.

In the paper [3] elliptic families of solutions to the KP equation were studied. In this more general case the solution is assumed to be an elliptic function not of $x = t_1$, as it was assumed before, but of a general linear combination of higher times of the KP hierarchy. It was shown that poles of such solutions as functions of $x = t_1$ and $t_2$ move according to equations of motion of the field generalization of the Calogero-Moser system. In this paper we extend this result to elliptic families of solutions to the 2D Toda system. We derive equations of motion for such solutions. These equations of motion can be naturally thought of as a field generalization of the Ruijsenaars-Schneider system. In the limit when the parameter $\eta$ having the meaning of the inverse velocity of light tends to 0, the obtained equations of motion become those dealt with in the paper [3]. We also consider elliptic families of solutions to the fully difference integrable version of the 2D Toda hierarchy. We derive equations of motion of the field generalization of the Calogero-Moser system.

It was shown that poles of such solutions as functions of $x$ assumed before, but of a general linear combination of higher times of the KP hierarchy. The vectors $\Gamma$ is a finite number of points $\lambda_i = \lambda_i(x, t)$. Therefore, for elliptic families we have:

$$\Theta \left( V_0 x/\eta + \sum_{k \geq 1} V_k t_k + W \lambda + Z \right) = f(x, t) e^{\gamma_1 \lambda + \gamma_2 \lambda^2} \prod_{i=1}^N \sigma(\lambda - \lambda_i(x, t)), \quad (1.14)$$

with a function $f(x, t)$ and some constants $\gamma_1, \gamma_2$. Here $\sigma(\lambda)$ is the Weierstrass $\sigma$-function defined in the appendix A. The zeros $\lambda_i$ of the tau-function are poles of the elliptic solutions.

We show that the equations of motion of the poles $\lambda_i = \lambda_i(x, t)$, where $t = t_1$, are given by

$$\ddot{\lambda}_i(x) + \sum_{k=1}^N \left( \dot{\lambda}_i(x) \dot{\lambda}_k(x - \eta) \zeta(\lambda_i(x) - \lambda_k(x - \eta)) + \dot{\lambda}_i(x) \dot{\lambda}_k(x + \eta) \zeta(\lambda_i(x) - \lambda_k(x + \eta)) \right)$$

$$+ \sum_{k \neq i} (c(x - \eta, t) - c(x, t)) \dot{\lambda}_i(x) = 0. \quad (1.15)$$
Here
\[
c(x, t) = \frac{1}{\beta} \sum_{i,k=1}^{N} \dot{\lambda}_i(x) \dot{\lambda}_k(x + \eta) \zeta(\lambda_i(x) - \lambda_k(x + \eta)), \quad \beta = \sum_{i=1}^{N} \dot{\lambda}_i(x), \tag{1.16}
\]
and \(\zeta(\lambda)\) is the Weierstrass \(\zeta\)-function (a close relative of the function \(E_1\) in (1.9)).

Equations (1.15) are represented in the zero-curvature form
\[
\dot{L}(x) + L(x)M(x) - M(x + \eta)L(x) = 0, \tag{1.17}
\]
and the Lax pair is obtained explicitly in section 5.3. Then, the equations (1.15) can be naturally restricted to the lattice by setting \(\lambda_k^i = \lambda_i(x_0 + k\eta)\) and rewritten as
\[
\ddot{\lambda}_i^k + \sum_{j=1}^{N} \left( \dot{\lambda}_i^k \dot{\lambda}_{j}^{k-1} \zeta(\lambda_i^k - \lambda_j^{k-1}) + \dot{\lambda}_i^k \dot{\lambda}_{j}^{k+1} \zeta(\lambda_i^k - \lambda_j^{k+1}) \right)
- 2 \sum_{j: j \neq i}^{N} \dot{\lambda}_i^k \dot{\lambda}_{j}^{k} \zeta(\lambda_i^k - \lambda_j^{k}) + (c^{k-1}(t) - c^k(t)) \dot{\lambda}_i^k = 0, \tag{1.18}
\]
with
\[
c^k(t) = \frac{1}{\beta} \sum_{i,j=1}^{N} \dot{\lambda}_i^k \dot{\lambda}_{j}^{k+1} \zeta(\lambda_i^k - \lambda_j^{k+1}), \tag{1.19}
\]
in which form they can be shown to be equivalent to equations (1.9). Details of the equivalence between (1.15)–(1.19) and (1.9)–(1.12) are given in section 5.4.

In section 5.6 we also describe the \(\eta \to 0\) limit to the (1+1)-dimensional Calogero-Moser field theory discussed in [3]. The fully discrete version of the equations (1.18) is obtained in section 6. In appendix A the necessary definitions and properties of elliptic functions are given. In appendix B we describe properties of the elliptic \(R\)-matrix which are used in the derivation of the Ruijsenaars-Schneider spin chain. In appendix C, using the factorization formulae for the Lax matrix, we obtain the explicit change of variables between the Ruijsenaars-Schneider model and the relativistic top.

2 Ruijsenaars-Schneider model in the form of relativistic top

In this section we recall the necessary preliminaries related to the Ruijsenaars-Schneider model and relativistic top. From the point of view of the next sections this case corresponds to the “Ruijsenaars-Schneider chain” on one site.

2.1 Classical Ruijsenaars-Schneider model

The standard Hamiltonian and equations of motion. The elliptic Ruijsenaars-Schneider model is defined by the Lax matrix [2]
\[
L_{ij}^{\text{RS}}(z) = \phi(z, q_{ij} + \eta) b_j, \quad i, j = 1, \ldots, N, \tag{2.1}
\]
where \( \phi(z, q) \) is the Kronecker function defined in (A.14) and

\[
    b_j = \prod_{k:k \neq j}^{N} \frac{\vartheta(q_j - q_k - \eta)}{\vartheta(q_j - q_k)} e^{p_j/c}, \quad c = \text{const} \in \mathbb{C}.
\]

Here \( \vartheta(z) \) is the odd Jacobi theta-function (A.8). Note that original definition of \( b_j \) in [2] is different from (2.2). This is due to a freedom in the definition of (2.1)–(2.2) coming from the canonical map

\[
p_j \to p_j + c_1 \log \prod_{k:k \neq j}^{N} \frac{\vartheta(q_j - q_k + \eta)}{\vartheta(q_j - q_k - \eta)}
\]

with arbitrary constant \( c_1 \).

The Hamiltonian

\[
    H^{RS} = c \frac{\text{tr} L^{RS}(z)}{\phi(z, \eta)} = c \sum_{j=1}^{N} b_j(p, q)
\]

with the canonical Poisson brackets

\[
\{p_i, q_j\} = \delta_{ij}, \quad \{p_i, p_j\} = \{q_i, q_j\} = 0
\]

provides the following equations of motion:

\[
    \dot{q}_j = \{H^{RS}, q_j\} = \partial_{p_j} H^{RS} = b_j = \prod_{k:k \neq j}^{N} \frac{\vartheta(q_j - q_k - \eta)}{\vartheta(q_j - q_k)} e^{p_j/c}.
\]

We see that the Hamiltonian is proportional to the sum of velocities:

\[
\frac{1}{c} H^{RS} = \sum_{j=1}^{N} \dot{q}_j
\]

and the Lax matrix (2.1) takes the form

\[
    L^{RS}_{ij}(z) = \phi(z, q_{ij} + \eta) \dot{q}_j.
\]

The Hamiltonian equations for momenta are as follows:

\[
\frac{1}{c} \dot{p}_i = \frac{1}{c} \{H^{RS}, p_i\} = - \frac{1}{c} \partial_{q_i} H^{RS}
\]

\[
= \sum_{l:l \neq i}^{N} (\dot{q}_l + \dot{q}_i) E_1(q_{il}) - \dot{q}_l E_1(q_{il} - \eta) - \dot{q}_i E_1(q_{il} + \eta),
\]

where \( q_{ij} = q_i - q_j \), and \( E_1(w) = \vartheta'(w)/\vartheta(w) \) (A.17). By differentiating both parts of (2.6) with respect to time we get

\[
\frac{\dot{q}_i}{q_i} = \frac{1}{c} \dot{p}_i + \sum_{l:l \neq i}^{N} (\dot{q}_l - \dot{q}_i)(E_1(q_{il} - \eta) - E_1(q_{il})).
\]
Plugging (2.9) into (2.10) we get the well known equations of motion of the elliptic Ruijseaars-Schneider model in the Newtonian form:

\[ \ddot{q}_i = \sum_{k,k \neq i}^N q_i \dot{q}_k (2E_1(q_{ik}) - E_1(q_{ik} + \eta) - E_1(q_{ik} - \eta)) , \quad i = 1, \ldots, N . \quad (2.11) \]

Equations of motion (2.11) are equivalent to the Lax equation

\[ \dot{L}^{RS}(z) \equiv \{ H^{\text{RS}}, L^{RS}(z) \} = [L^{RS}(z), M^{\text{RS}}(z)] \quad (2.12) \]

with the M-matrix

\[ M^{\text{RS}}_{ij}(z) = -(1 - \delta_{ij}) \phi(z, q_i - q_j) \dot{q}_j - \delta_{ij} \left( \dot{q}_i (E_1(z) + E_1(\eta)) + \sum_{k:k \neq i}^N \dot{q}_k (E_1(q_{ik} + \eta) - E_1(q_{ik})) \right) . \quad (2.13) \]

This follows from a direct calculation with the help of (A.20) and (A.19).

**Logarithm of the Hamiltonian.** Alternatively, one can use the following Hamiltonian:

\[ H' = c \log H^{\text{RS}} = c \log \sum_{j=1}^N b_j , \quad (2.14) \]

Then

\[ \dot{q}_j = \frac{\partial H'}{\partial p_j} = \frac{b_j}{H^{\text{RS}}} , \quad \frac{1}{c} \dot{p}_i = -\frac{1}{c} \frac{\partial H'}{\partial q_i} = -\frac{1}{H^{\text{RS}}} \frac{\partial H^{\text{RS}}}{\partial q_i} , \quad (2.15) \]

so that (cf. (2.7))

\[ \sum_{j=1}^N \dot{q}_j = 1 . \quad (2.16) \]

The Lax matrix (2.1) becomes now

\[ L_{ij}^{\text{RS}}(z) = \phi(z, q_{ij} + \eta) \dot{q}_j H^{\text{RS}} \quad (2.17) \]

instead of (2.8) but this makes no difference since \( H^{\text{RS}} \) is a conserved quantity. It is easy to see that the equations of motion in the Newtonian form (2.11) remain the same with the Hamiltonian (2.14). We will use the Hamiltonian description similar to the presented above (with logarithm of the Hamiltonian) in the Ruijseaars chain.

### 2.2 Classical relativistic top

Let us consider the elliptic GL\(_N\) spin chain on a single site. It is an integrable system called relativistic top [12]. The Lax matrix is as follows:

\[ \mathcal{L}^\alpha(z) = \sum_{a \in \mathbb{Z}_N \times \mathbb{Z}_N} T_a S_a \varphi_a(z, \omega_a + \eta) , \quad (2.18) \]

where \( S_a \) are dynamical variables (classical spins) numbered by the index \( a = (a_1, a_2) \in \mathbb{Z}_N \times \mathbb{Z}_N \) in the special matrix basis \( T_a \in \text{Mat}(N, \mathbb{C}) \) (B.3), which is often used for elliptic
quantum $R$-matrices. The set of functions $\varphi_a(z, \omega_a + \eta)$ and the quantities $\omega_a$ are given in (A.13). The dynamical variables are combined into a matrix $S \in \text{Mat}(N, \mathbb{C})$:

$$S = \sum_{i,j=1}^{N} S_{ij} E_{ij} = \sum_{a_1,a_2=0}^{N-1} S_a T_a,$$

(2.19)

where $E_{ij}$ are the usual matrix units. Using the property (B.5) let us rewrite the Lax matrix (2.18) in terms of the Baxter-Belavin elliptic $R$-matrix $R_{12}^\eta(z)$ [13, 40] in the form (B.10):

$$R_{12}^\eta(z) = \frac{1}{N} \sum_{a \in \mathbb{Z}_N \times \mathbb{Z}_N} T_a \otimes T_{-a} \varphi_a(z, \omega_a + \eta) \in \text{Mat}(N, \mathbb{C})^{\otimes 2}.$$

(2.20)

Alternative equivalent forms are given in the appendix B. In terms of the $R$-matrix, the Lax matrix (2.18) acquires the following compact form:

$$L^\eta(z) = \text{tr}_2(R_{12}^\eta(z)S_2), \quad S_2 = 1_N \otimes S,$$

(2.21)

where the trace is over the second tensor component.

We emphasis that the $R$-matrix is quantum, while the Lax matrix is classical. The parameter $\eta$ plays the role of the Planck constant in the $R$-matrix since the classical $r$-matrix comes from (2.20) in the classical limit:

$$R_{12}^\eta(z) = \frac{1}{N\eta} + r_{12}(z) + O(\eta),$$

(2.22)

$$r_{12}(z) = \frac{1}{N} 1_N \otimes 1_N E_1(z) + \frac{1}{N} \sum_{a \in \mathbb{Z}_N^2, a \neq 0} T_a \otimes T_{-a} \varphi_a(z, \omega_a) \in \text{Mat}(N, \mathbb{C})^{\otimes 2}.$$  

(2.23)

At the same time $\eta$ plays the role of the relativistic deformation parameter in the relativistic top model similarly to the Ruijsenaars-Schneider model. In the standard approach [6, 8] the parameter $\eta$ is absent in the Lax matrix and so it does not enter the classical (Poisson) Sklyanin algebra. Below we explain how this parameter can be eliminated by some redefinitions. However, it is important for us to keep it in the Lax matrix because in this form the latter has a nice property of factorization.

### Classical Sklyanin algebra

The classical quadratic $r$-matrix Poisson structure is:

$$\{L_1^\eta(z), L_2^\eta(w)\} = \frac{1}{c} [L_1^\eta(z) L_2^\eta(w), r_{12}(z-w)],$$

(2.24)

where the classical $r$-matrix is given by (2.23). Plugging the Lax matrix (2.18) into (2.24) and using identity (A.22) one gets

$$\{S_\alpha, S_\beta\} = \frac{1}{c} \sum_{\xi \in \mathbb{Z}_N^2, \xi \neq 0} \kappa_{\alpha-\beta, \xi} S_{\alpha-\xi} S_{\beta+\xi} \left( E_1(\omega_\xi) - E_1(\omega_{\alpha-\beta-\xi}) + E_1(\omega_{\alpha-\xi} + \eta) - E_1(\omega_{\beta+\xi} + \eta) \right),$$

(2.25)

which is the classical Sklyanin algebra. The constants $\kappa_{\alpha, \beta}$ are defined in (B.4).

---

3The classical $r$-matrix is used for construction of the non-relativistic top similarly to (2.21). See [4, 16].

4The coefficient $1/c$ in (2.24) is introduced here in order to match the relation with the Ruijsenaars-Schneider model.
Eliminating the parameter $\eta$. Let us comment on the form of the classical Lax matrix $\mathcal{L}_\eta(z)$ (2.18), (2.21). Usually [6, 8] the classical Lax matrix of the top is written as

$$\mathcal{L}(z, \tilde{S}) = 1_N \tilde{S}_0 + \sum_{a \in \mathbb{Z}_N^2, a \neq 0} T_\alpha \tilde{S}_a \varphi_a(z, \omega_a).$$

(2.26)

It is known to satisfy (2.24), which provides the classical Sklyanin Poisson algebra for $N^2$ generators $\tilde{S}_a$. Writing (2.24), we assume that it is also fulfilled for $\mathcal{L}_\eta(z)$. It happens for the following reason [12, 41]. First of all, this can be verified by a direct calculation, so that the classical Sklyanin algebra for $S_a$ contains additional parameter $\eta$. However, this dependence is artificial. Using the relation

$$\frac{\varphi_a(z - \eta, \omega_a + \eta)}{\varphi(z - \eta, \eta)} = \frac{\varphi_a(z, \omega_a)}{\varphi_a(\eta, \omega_a)},$$

(2.27)

one easily obtains

$$\frac{1}{\varphi(z - \eta, \eta)} \mathcal{L}_\eta(z - \eta, S) = \mathcal{L}(z, \tilde{S})$$

(2.28)

if

$$S = \mathcal{L}(\eta, \tilde{S}).$$

(2.29)

Using (2.26) in the basis $T_\alpha$ (B.3) we may write (2.29) explicitly:

$$S_0 = \tilde{S}_0, \quad S_\alpha = \tilde{S}_\alpha \varphi_\alpha(\eta, \omega_\alpha) \text{ for } \alpha \neq 0.$$  

(2.30)

Let us remark that a similar phenomenon with the same change of variables take place in quantum Sklyanin algebra generated by exchange relation $\tilde{R}^h_{12}(z - w)\tilde{\mathcal{L}}^h_1(z)\tilde{\mathcal{L}}^h_2(w) = \tilde{\mathcal{L}}^h_2(w)\tilde{\mathcal{L}}^h_1(z)\tilde{R}^h_{12}(z - w)$. Then it contains two parameters $h$ and $\eta$, but the latter can be removed by (2.29) or fixed somehow. For example, in the case $h = \eta$ the Sklyanin algebra has representation $\tilde{S}_a = T_{-a}$ since the exchange relation turns into the Yang-Baxter equation in this case. So that the second parameter is artificial.

We see that the two Lax matrices $\mathcal{L}_\eta(z)$ and (2.26) are related by the explicit change of variables (2.29) or (2.30), and the shift of the spectral parameter $z \to z - \eta$ does not effect (2.24) because $r_{12}(z - w)$ depends on the difference of spectral parameters. In what follows we need the explicit dependence on $\eta$ in $\mathcal{L}_\eta(z)$ for establishing its relation to the Ruijsenaars-Schneider model. For this purpose we will consider $S$ to be a rank one matrix (this is not true for $\tilde{S}$). The possibility to fix the rank of matrix $S$ follows from equations of motion (see (2.35) below). The eigenvalues of $S$ are conservation laws on these equations.

Lax pair. The Lax equation follows from (2.24) in the following way. Since $S = \text{Res}_{w=0} \mathcal{L}^h(w)$, then the residue at $w = 0$ of both parts of (2.24) yields

$$\{\mathcal{L}^h_1(z), S_2\} = \left[\mathcal{L}^h_1(z)S_2, \frac{1}{c} r_{12}(z)\right].$$

(2.31)

Taking trace of both parts of (2.31) in the second tensor component, we get the Lax equation

$$\dot{\mathcal{L}}^\eta(z) = \{H^{\text{top}}, \mathcal{L}^\eta_1(z)\} = [\mathcal{L}^\eta_1(z), M(z)], \quad M(z) = -\text{tr}_2 \left(r_{12}(z)S_2\right),$$

(2.32)
where the Hamiltonian is
\[ H^{\text{top}} = c \text{ tr } S = c \frac{\text{tr} L^\eta(z)}{\phi(z, \eta)}. \]  
(2.33)

More precisely,
\[ M(z) = - S_0 1_N E_1(z) - \sum_{\alpha \in \mathbb{Z}^2 \times 2N, \alpha \neq 0} T_\alpha S_\alpha \varphi_\alpha(z, \omega_\alpha). \]  
(2.34)

Equations of motion take the form:
\[ \dot{S} = [S, J^\eta(S)], \]  
(2.35)
\[ J^\eta(S) = 1_N S_0 E_1(\eta) + \sum_{\alpha \in \mathbb{Z}^2 \times 2N, \alpha \neq 0} T_\alpha S_\alpha J^\eta_\alpha, \quad J^\eta_\alpha = E_1(\eta + \omega_\alpha) - E_1(\omega_\alpha). \]  
(2.36)

They follow from the Lax equation (2.32) under the substitution (2.18), (2.34) and usage of (B.4) and the identity (A.20).

2.3 Factorization of Lax matrices and relation between the models

Following [24, 25, 27], we introduce the intertwining matrix
\[ g(z, q) = \Xi(z, q) \left( \frac{d^0}{} \right)^{-1} \]  
(2.37)
with
\[ \Xi_{ij}(z, q) = \theta \left[ \frac{1}{2} - \frac{i}{N} \right] \left( z - Nq_j + \sum_{m=1}^{N} q_m \mid N \tau \right), \]  
(2.38)
and the diagonal matrix
\[ d^0_{ij}(z, q) = \delta_{ij} d^0_j = \delta_{ij} \prod_{k: k \neq j} \theta(q_j - q_k), \]  
(2.39)
where the theta function with characteristics is defined in (A.7). The matrix (2.37) is the intertwining matrix entering the relations of the IRF-Vertex correspondence. Its properties are described in the appendix B.

Factorization formula. It was observed in [18] (at quantum level) that the Lax matrix (2.1)–(2.2) can be represented in the factorized form
\[ L^{RS}_{ij}(z) = \frac{\theta''(0)}{\theta'(\eta)} \sum_{k=1}^{N} g_{ik}^{-1}(z, q) g_{kj}(z + N \eta, q) e^{p_j/c}, \]  
(2.40)
or
\[ L^{RS}(z) = \frac{\theta''(0)}{\theta'(\eta)} g^{-1}(z, q) g(z + N \eta, q) e^{P/c}, \quad P = \text{ diag}(p_1, \ldots, p_N). \]  
(2.41)

Moreover, the gauge transformed Lax matrix
\[ L^\eta(z) = g(z, q) L^{RS}(z) g^{-1}(z, q) = \frac{\theta''(0)}{\theta'(\eta)} g(z + N \eta, q) e^{P/c} g^{-1}(z, q) \]  
(2.42)
is the Lax matrix of type (2.18) since it has the same quasi-periodic properties (see (3.3) below) and a simple pole at $z = 0$. In contrast to (2.18) a special choice of the residue $S$ is assumed in (2.42). It is a rank one matrix. We may fix the rank since the eigenvalues of $S$ are conserved on the dynamics given by (2.35). Likewise the spinless Calogero-Moser model is related to the coadjoint orbit of minimal dimension. Relation (2.42) can be viewed as the classical version of the IRF-Vertex relation (B.15). It provides the change $S = S(p, q, \eta, c)$ from canonical variables to spin variables, which will be discussed in detail in the next subsection. Let us compute the residue of both parts of (2.42) at $z = 0$. For this purpose we need the properties of the matrix $g(z, q)$ (B.18)–(B.21). In particular, it is degenerated at $z = 0$, and the residue $\tilde{g}(0, q) = \text{Res}_{z=0} g^{-1}(z, q)$ is a rank one matrix. In this way we get

\[ S = \xi \otimes \psi: \quad \xi = \frac{\partial'(0)}{\partial(\eta)} g(N\eta, q) e^{P/c} \tilde{g}(0, q) \]  

(2.43)

or

\[ S = \xi \otimes \psi, \quad \xi = \frac{\partial'(0)}{\partial(\eta)} g(N\eta) e^{P/c} \rho, \quad \psi = \frac{1}{N} \rho^T \tilde{g}(0). \]  

(2.44)

with $\rho$ from (B.19) and $\tilde{g}(0, q)$ from (B.20).

**Factorization from IRF-Vertex relations.** Notice that on one hand we deal with the Lax matrix $L^\eta(z)$ in the form (2.21), and on the other hand we use its factorized form (2.42) for a special choice of $S$. A connection between these two representations come from the IRF-Vertex relation, which includes both $R$-matrix and the matrix $g(z, q)$. We review it in appendix B. The easiest way is to use the identity (B.24) in $\text{Mat}(N, \mathbb{C}) \otimes \mathbb{C}$, which includes a special matrix $O_{12} \in \text{Mat}(N, \mathbb{C}) \otimes \mathbb{C}$ (B.25) with the property (B.26). Following [23], multiply both parts of (B.24) by $S_2 = 1_N \otimes S$ with $S$-matrix presented in the form (2.44)

\[ \frac{\partial'(0)}{\partial(\eta)} g_2(N\eta) e^{P_2/c} \tilde{g}_2(0, q) R_{12}^\eta(z) = \frac{\partial'(0)}{\partial(\eta)} g_2(N\eta) e^{P_2/c} g_1(z + N\eta, q) O_{12} g_2^{-1}(N\eta, q) g_1^{-1}(z, q). \]  

(2.45)

Next, compute the trace over the second tensor component of both parts (2.45). The property (B.26) simplifies the r.h.s. of (2.45) since $\text{tr}_2(O_{12} e^{P_2/c}) = e^{P/c}$, and therefore

\[ L^\eta(z, S) = \text{tr}_2(R_{12}^\eta(z) S_2) = \frac{\partial'(0)}{\partial(\eta)} g(z + N\eta, q) e^{P/c} g^{-1}(z, q). \]  

(2.46)

In what follows we also need degeneration of the factorized form. By comparing (2.40) and (2.1) in the $\eta \to 0$ limit we get

\[ \left( g^{-1}(z) g'(z) \right)_{ij} = \frac{1}{N} \delta_{ij} \left( E_1(z) - \sum_{k: k \neq i}^N E_1(q_{ik}) \right) + \frac{1}{N} (1 - \delta_{ij}) \phi(z, q_{ij}). \]  

(2.47)

In section 4 we also use degeneration of (2.45) coming from (B.27) to derive the accompany $M$-matrix.
Explicit change of variables. The explicit change of variables $S_a = S_a(p, q, \eta, c)$ can be found in [18] (see also [21, 22]) in the elliptic case. (The trigonometric and rational cases were addressed in [12].) In appendix C we derive this formula in the elliptic case. In our notation it takes the form

$$S_a = \frac{(-1)^{a_1 + a_2}}{N} e^{\pi i a_2 \omega_a} \sum_{m=1}^{N} e_{p_m/c} e^{2 \pi i a_2 (\eta - \bar{q}_m)} \frac{\vartheta(\eta + \omega_a)}{\vartheta(\eta)} \prod_{l: l \neq m} \frac{\vartheta(q_m - q_l - \eta - \omega_a)}{\vartheta(q_m - q_l)}, \quad (2.48)$$

where $\bar{q}_m$ is the coordinate in the center of masses frame. The classical Sklyanin generators $S_a$ are dynamical variables in the relativistic top model described above, and (2.48) provides its relation to the Ruijsenaars-Schneider model in the special case $\text{rk}(S) = 1$ generated by the gauge equivalence (2.42). Put it differently, (2.48) is a classical analogue of the representation of the generalized Sklyanin algebra by difference operators (in the classical limit the shift operators are substituted by exponents of momenta).

Therefore, we have the following statement. The set of functions $S_a = S_a(p, q, \eta, c)$ satisfy the classical Sklyanin algebra Poisson brackets (2.25) computed by means of the canonical Poisson brackets (2.5). The Lax matrix (2.46) satisfy the classical exchange relations (2.24). The proof of a similar statement was proposed in [22] by a direct gauge transformation relating the $r$-matrix structure (2.24) with the dynamical one known for the Ruijsenaars-Schneider model [20].

We also claim that the matrix of dynamical variables $S$ with components (2.48) is represented in the form (2.44). The proof and explicit expressions for $\xi$ and $\psi$ are given in appendix C.

### 3 Classical GL$_N$ elliptic spin chain

Here we review properties of the Lax matrices for elliptic classical spin chains and describe the Hamiltonian flow which is then used for constructing the Ruijsenaars-Schneider chain in the next section.

We deal with the classical version [6, 8] of the generalized elliptic (anisotropic) homogeneous spin chain on $n$ sites associated with GL$_N$. It is described by the elliptic Baxter-Belavin $R$-matrix [13, 40]. The generating function of the Hamiltonians is given by trace $t(z)$ of the monodromy matrix $T(z)$:

$$t(z) = \text{tr} T(z), \quad T(z) = L^1(z) L^2(z) \ldots L^n(z), \quad (3.1)$$

where $L^i(z)$ is the classical Sklyanin’s Lax matrix (2.18) on the $i$-th site of the chain. It is fixed by the quasi-periodic properties on the lattice $\mathbb{Z} \oplus \tau \mathbb{Z}$ in the complex plane (defining the elliptic curve $\Sigma_\tau = \mathbb{C}/(\mathbb{Z} \oplus \tau \mathbb{Z})$) and the residue at a simple pole $z = 0$:

$$\text{Res}_{z=0} L^i(z) = S^i = \sum_{k,j=1}^{N} E_{kj} S^i_{kj} \in \text{Mat}(N, \mathbb{C}) \quad (3.2)$$

(it is the only pole in the fundamental domain). Here $E_{kj}$ is the standard matrix basis in Mat($N, \mathbb{C}$) (matrix units) and $S^i_{kj}$ are the classical Sklyanin’s generators at $i$-th site.

$^5$The latter means that the variables $S^i$ satisfy the quadratic Poisson brackets (2.25) for any fixed $i$. 

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- 13 –
monodromy properties are as follows:
\[ L^i(z + 1) = Q_1^{-1} L^i(z) Q_1, \quad L^i(z + \tau) = \exp(-2\pi i \eta) Q_2^{-1} L^i(z) Q_2, \] (3.3)
where \( Q_{1,2} \in \text{Mat}(N, \mathbb{C}) \) are finite-dimensional representations for generators of the Heisenberg group given by (B.1). More explicitly,
\[ L^i(z) = \sum_{a \in \mathbb{Z}_N \times \mathbb{Z}_N} S^i_a T_a \varphi_a(z, \omega_a + \eta), \] (3.4)
where \( T_a \) is the special basis (B.3) in \( \text{Mat}(N, \mathbb{C}) \) constructed by means of the matrices \( Q_1, Q_2 \). Similarly to (2.21), we can write the Lax matrices (3.4) in the compact form:
\[ L^i(z) = \text{tr}_2(R_{12}^i(z)S^i_2), \quad S^i_2 = 1_{N \times N} \otimes S^i \in \text{Mat}(N, \mathbb{C}) \otimes \mathbb{C}. \] (3.5)

Consider the lattice version of the generalized Landau-Lifshitz model, i.e. the classical elliptic spin chain [6]. It is defined by the monodromy matrix \( T(z) \) (3.1) with the Lax matrices \( L^i(z) \) (3.4) or (3.5). The Poisson structure is given by \( n \) copies of (2.24):
\[ \{ L^i_1(z), L^j_2(w) \} = \frac{1}{c} \delta^{ij} [L^i_1(z) L^j_2(w), r_{12}(z - w)]. \] (3.6)

In order to have a local Hamiltonian (when only neighbouring sites interact), the residues \( S^i \)
\[ S^i = \text{Res}_{z=0} L^i(z), \quad i = 1, \ldots, n \] (3.7)
should be rank one matrices:
\[ S^i = \xi^i \otimes \psi^i, \] (3.8)
where \( \xi^i \in \mathbb{C}^N \) are column-vectors and \( \psi^i \in \mathbb{C}^N \) are row-vectors. Then the local Hamiltonian is defined as follows. Let us compute the coefficient of \( t(z) \) (3.1) in front of \( 1/z^n \). It equals
\[ \exp(H/c) = \text{Res}_{z=0} z^{n-1} t(z) = \text{tr}(S^1 S^2 \ldots S^n). \] (3.9)

Plugging (3.8) into (3.9) and taking its logarithm, we get
\[ H = c \log \text{tr}(S^1 S^2 \ldots S^n) = c \sum_{k=1}^n \log h_{k,k+1}, \] (3.10)
\[ h_{k,k+1} = (\psi^k, \xi^{k+1}) = \sum_{l=1}^N \psi^k_l \xi^{k+1}_l, \] (3.11)
where \( \xi^{n+1} = \xi^1 \) and the notation \( (\psi^k, \xi^{k+1}) \) means the standard scalar product. To get equations of motion, consider
\[ \text{tr}_2\{ L^k_1(z), T_2(w) \} \overset{(3.6)}{=} -L^k(z) M^k(z, w) + M^{k-1}(z, w) L^k(z), \] (3.12)
where
\[ M^k(z, w) = -\frac{1}{c} \text{tr}_2\left( L^1_2(w) \ldots L^k_2(w) r_{12}(z - w) L^{k+1}_2(w) \ldots L^n_2(w) \right). \] (3.13)
By taking the coefficient of the \( n \)th order pole at \( w = 0 \) in (3.12)–(3.13) and dividing both parts of (3.12) by \( \exp(H/c) \) (3.9), we see that the Lax matrices \( L^k(z) \) satisfy a set of the semi-discrete Zakharov-Shabat equations\(^6\)

\[
\dot{L}^k(z) = \{H, L^k(z)\} = L^k(z)M^k(z) - M^{k-1}(z)L^k(z), \tag{3.14}
\]

where

\[
M^k(z) = -\text{tr}_2(r_{12}(z)S_{2}^{k+1,k}), \quad \text{Res}_{z=0}M^k(z) = -S^{k+1,k}, \tag{3.15}
\]

and

\[
S^{k+1,k} = \frac{\xi^{k+1} \otimes \psi^k}{h_{k,k+1}}. \tag{3.16}
\]

The second order pole at \( z = 0 \) in the r.h.s. of (3.14) is cancelled out since \( S^kS^{k+1,k} = S^k \), i.e.

\[
\frac{1}{h_{k,k+1}}(\xi^k \otimes \psi^k)(\xi^{k+1} \otimes \psi^k) - \frac{1}{h_{k-1,k}}(\xi^k \otimes \psi^{k-1})(\xi^k \otimes \psi^k) = 0. \tag{3.17}
\]

Similarly to (2.34) we have the following explicit expression for \( M^k(z) \):

\[
M^k(z) = -S_0^{k+1,k}1_NE_1(z) - \sum_{\alpha \in \mathbb{Z}^{2}, \alpha \neq 0} T_\alpha S^{k+1,k}_\alpha \varphi_\alpha(z, \omega_\alpha). \tag{3.18}
\]

The equations of motion are of the form

\[
\dot{S}^k = S^kJ^n(S^{k+1,k}) - S^{k+1,k}J^n(S^k), \tag{3.19}
\]

where \( J^n \) is the linear map (2.36).

4 The Ruijsenaars-Schneider chain

This section is organized as follows. First, we define the lattice finite-dimensional analogue of the Ruijsenaars-Schneider model (the Ruijsenaars spin chain) and find its Lax matrix. In subsection 4.2 the Hamiltonian and equations of motion are derived similarly to those for the elliptic spin chain described in the previous section. In subsection 4.3, using a set of IRF-Vertex type relations, we compute the \( M \)-matrices entering the semi-discrete zero curvature (Zakharov-Shabat) equations. Finally, we explain how the obtained Lax pair can be modified in order to have a form similar to the ordinary Ruijsenaars-Schneider model.

4.1 Classical \( L \)-matrix

Let us parameterize all the \( L \)-matrices of the elliptic spin chain in (3.1) by \( n \) sets of canonical variables \( p^k_i, q^k_j, i, j = 1, \ldots, N, k = 1, \ldots, n \)

\[
\{p^k_i, q^j_l\} = \delta^{kl} \delta_{ij}. \tag{4.1}
\]
as in (2.42), so that

\[ L^k(z) = \frac{\vartheta'(0)}{\vartheta'(\eta)} g(z + N\eta, q^k) e^{P_k/c} g^{-1}(z, q^k), \quad P^k = \text{diag}(p_1^k, \ldots, p_N^k) \tag{4.2} \]

and

\[ S^k = S^k(p^k, q^k) = \xi^k \otimes \psi^k, \quad k = 1, \ldots, n \tag{4.3} \]

with

\[ \xi^k = \xi^k(p^k, q^k) = \frac{\vartheta'(0)}{\vartheta'(\eta)} g(N\eta, q^k) e^{P^k/c} \rho, \quad \psi^k = \psi^k(q^k) = \frac{1}{N} \rho^T \tilde{g}(0, q^k), \tag{4.4} \]

where \( \rho \) is the column vector (B.19). Plugging the Lax matrices being written in the factorized form (4.2) into the monodromy matrix (3.1), we get

\[ T(z) = \left( \frac{\vartheta'(0)}{\vartheta'(\eta)} \right)^n g(z + N\eta, q^1) e^{P^1/c} g^{-1}(z, q^1) g(z + N\eta, q^2) e^{P^2/c} g^{-1}(z, q^2) \ldots, \tag{4.5} \]

and, therefore, the transfer-matrix \( t(z) \) (3.1) can be equivalently rewritten in the form

\[ t(z) = \text{tr} \left( \tilde{L}^1(z) \tilde{L}^2(z) \ldots \tilde{L}^n(z) \right) \tag{4.6} \]

(by identifying \( q^0 = q^n \)), where

\[ \tilde{L}^k(z) = g^{-1}(z, q^{k-1}) L^k(z) g(z, q^k) \tag{4.7} \]

or (from (4.2))

\[ \tilde{L}^k(z) = \frac{\vartheta'(0)}{\vartheta'(\eta)} g^{-1}(z, q^{k-1}) g(z + N\eta, q^k) e^{P^k/c}. \tag{4.8} \]

To obtain explicit an expression for \( \tilde{L}^k(z) \), we need to compute the matrix \( g^{-1}(z, q^{k-1}) g(z + N\eta, q^k) \):\(^{2.37}\)

\[ g^{-1}(z, q^{k-1}) g(z + N\eta, q^k) = d^0(q^k) \Xi^{-1}(z, q^k) \Xi(z + N\eta, q^k) \left( d^0(q^k) \right)^{-1}, \tag{4.9} \]

where we have introduced the notation

\[ q_i^k = q_i^k - \frac{1}{N} \sum_{j=1}^{N} q_j^k, \tag{4.10} \]

i.e. each \( g \)-matrix depends on the coordinates in the center of masses frame (see the definitions (2.37)–(2.39) and notice that \( d^0(q^k) = d^0(\tilde{q}^k) \)). It is necessary for the following reason. The Lax matrix should have a pole at some fixed point \((z = 0)\), and the latter comes from the inverse of \( g(z) \). The pole at \( z = 0 \) then appears from \( \det \Xi(z, q) \) (B.18). Finally, the theta function \( \vartheta(z) \) in (B.18) comes from (2.38) in the following way: \( \vartheta\left( \frac{1}{N} \sum_{k=1}^{N} (z - N\tilde{q}_k) \right) = \vartheta(z) \), where we used \( \sum_{k=1}^{N} \tilde{q}_k = 0 \).
Coming back to the calculation (4.9), we use the following formula proved in [18]:

\[
\left(-\vartheta'(0)\Xi^{-1}(z,q^{k-1})\Xi(z+N\eta,q^{k})\right)_{ij} = \phi(z,q^{k-1}_i - q^{k-1}_j + \eta) \frac{\prod_{l=1}^{N} \vartheta(q^{k}_j - q^{k-1}_l - \eta)}{\prod_{l \neq j} \vartheta(q^{k}_j - q^{k-1}_l)}.
\] (4.11)

Plugging also the matrices $d^0$ (2.39) into (4.9), we get

\[
\left(-\vartheta'(0)g^{-1}(z,q^{k-1})g(z+N\eta,q^{k})\right)_{ij} = \phi(z,q^{k-1}_i - q^{k-1}_j + \eta) \frac{\prod_{l=1}^{N} \vartheta(q^{k}_j - q^{k-1}_l - \eta)}{\prod_{l \neq j} \vartheta(q^{k}_j - q^{k-1}_l)} .
\] (4.12)

Note that under the identification $\bar{q}^{k-1} := \bar{q}^k$ the upper product in the r.h.s. acquires the factor $\vartheta(-\eta)$. Dividing by it the both sides, we reproduce the Lax matrix of the Ruijsenaars-Schneider model (2.1)–(2.2) or (2.41).

Finally, for the $L$-matrices (4.8) entering the transfer matrix (4.6) we have:

\[
\tilde{L}^k_{ij}(z) = \phi(z,\bar{q}^{k-1}_i - \bar{q}^k_j + \eta) \frac{\prod_{l=1}^{N} \vartheta(q^k_j - q^{k-1}_l - \eta)}{\vartheta(-\eta) \prod_{l \neq j} \vartheta(q^k_j - q^{k-1}_l)} e^{\rho_j^k/c}.
\] (4.13)

4.2 Hamiltonian and equations of motion

**The Hamiltonian.** The Hamiltonian can be obtained from $t(z)$ (4.6) in the same way as in the spin chain case (see (3.9)–(3.11)). For this purpose compute the residue of $\tilde{L}^k_{ij}(z)$:

\[
\operatorname{Res}_{z=0} \tilde{L}^k(z) = \rho^T \otimes b^k,
\] (4.14)

where $\rho$ is taken from (B.19) and $b^k$ is a row-vector, so that

\[
\operatorname{Res}_{z=0} \tilde{L}^k_{ij}(z) = b^k_j,
\] (4.15)

Then

\[
\exp(H/c) = \operatorname{Res}_{z=0} z^{-1}t(z) = \text{tr} \left((\rho^T \otimes b^1)(\rho^T \otimes b^2) \ldots (\rho^T \otimes b^n)\right).
\] (4.16)

Finally, the Hamiltonian is of the form

\[
H = c \sum_{k=1}^{n} \log h_{k,k+1}, \quad h_{k,k+1} = (\rho^T, b^{k+1})
\] (4.17)

and

\[
h_{k-1,k} = (\rho^T, b^k) = \sum_{j=1}^{N} b^k_j = \sum_{j=1}^{N} \frac{\prod_{l=1}^{N} \vartheta(q^k_j - q^{k-1}_l - \eta)}{\vartheta(-\eta) \prod_{l \neq j} \vartheta(q^k_j - q^{k-1}_l)} e^{\rho_j^k/c}.
\] (4.18)

\[– 17 –\]
By construction, the trace $t(z)$ (4.6) coincides with the one for the elliptic spin chain (3.1) under the substitution (4.2)–(4.4). To see this, we mention that the terms $h_{k,k+1}$ entering (3.10)–(3.11) and those from (4.17)–(4.18) are equal to each other:

$$h_{k-1,k} = (\rho^T, b^k) = (\psi^{k-1}, \xi^k)$$  \hspace{1cm} (4.19)

for $\xi^k$ and $\psi^k$ defined in (4.4). In order to verify (4.19), one should compare the trace of the residue of $\tilde{L}^k(z)$ computed from (4.8) and (4.13).

**The Hamiltonian equations of motion.** Let us proceed to the equations of motion. From (4.17)–(4.18) we have

$$\dot{q}^k_i = \frac{\partial H}{\partial p_i^k} = \frac{h^k_i}{h_{k-1,k}}.$$  \hspace{1cm} (4.20)

The latter yields

$$\sum_{i=1}^N \dot{q}^k_i = 1 \quad \text{for all } k.$$  \hspace{1cm} (4.21)

With (4.20) the Lax matrix (4.13) takes the form:

$$\tilde{L}^k_{ij}(z) = \phi(z, q^{k-1}_i - q^k_i + \eta) b^k_j, \quad b^k_j = h_{k-1,k} q^k_j.$$  \hspace{1cm} (4.22)

Next,

$$\frac{1}{c} \dot{p}^k_i = -\frac{1}{c} \frac{\partial H}{\partial q^k_i} = -\frac{1}{h_{k-1,k}} \frac{\partial}{\partial q^k_i} h_{k-1,k} - \frac{1}{h_{k,k+1}} \frac{\partial}{\partial q^k_i} h_{k,k+1}.$$  \hspace{1cm} (4.23)

Its r.h.s. is evaluated from the explicit expression (4.18):

$$\frac{1}{h_{k-1,k}} \frac{\partial}{\partial q^k_i} h_{k-1,k} = \dot{q}^k_i \sum_{i=1}^N E_1(q^k_i - q^{k-1}_i - \eta) - \dot{q}^k_i \sum_{i \neq i} E_1(q^k_i - q^k_i) + \sum_{i \neq i} q^k_i E_1(q^k_i - q^k_i)$$

$$- \frac{1}{N} \sum_{i=1}^N \dot{q}^k_i \sum_{m=1}^N E_1(q^k_i - q^k_m - \eta),$$  \hspace{1cm} (4.24)

$$\frac{1}{h_{k,k+1}} \frac{\partial}{\partial q^k_i} h_{k,k+1} = -\sum_{i=1}^N q^{k+1}_i E_1(q^{k+1}_i - q^k_i - \eta) + \frac{1}{N} \sum_{i=1}^N \dot{q}^{k+1}_i \sum_{m=1}^N E_1(q^{k+1}_i - q^k_m - \eta),$$  \hspace{1cm} (4.25)

where the last terms (double sums) come from dependence on the center of masses coordinates (4.10). Summing up (4.24) and (4.25), we get the following equation for momenta (4.23):

$$\frac{1}{c} \dot{p}^k_i = -\dot{q}^k_i \sum_{i=1}^N E_1(q^k_i - q^{k-1}_i - \eta) - \sum_{i=1}^N q^{k+1}_i E_1(q^k_i - q^{k+1}_i + \eta) + \sum_{i \neq i} (q^k_i + q^k_i) E_1(q^k_i - q^k_i)$$

$$+ \frac{1}{N} \sum_{i=1}^N \dot{q}^k_i \sum_{m=1}^N E_1(q^k_i - q^{k+1}_i - \eta) - \frac{1}{N} \sum_{i=1}^N \dot{q}^{k+1}_i \sum_{m=1}^N E_1(q^{k+1}_i - q^k_m - \eta).$$  \hspace{1cm} (4.26)

The second line of this equation is independent of the index $i$. It has appeared from the dependence of $\dot{q}^k_i$ on the center of masses coordinates $\sum_i q^k_i$ at each $(k$-th) site.
The Newtonian form. Let us represent the Hamiltonian equations of motion in the Newtonian form. By differentiating both parts of (4.20) with respect to the time $t$, we get

$$
\dot{q}_i^k = \frac{b_k^i}{h_{k-1,k}} - \frac{\dot{h}_{k-1,k}}{h_{k-1,k}} \dot{q}_i^k = \dot{q}_i^k \left( \frac{b_k^i}{b_i^k} - \frac{\dot{h}_{k-1,k}}{h_{k-1,k}} \right),
$$
(4.27)

where

$$
\partial_t \log h_{k-1,k} = \frac{\dot{h}_{k-1,k}}{h_{k-1,k}} = \frac{1}{h_{k-1,k}} \sum_{l=1}^N \dot{b}_l^k = \sum_{l=1}^N q_l^k \frac{\dot{b}_l^k}{b_l^k}.
$$
(4.28)

We see that we need to compute $\dot{b}_l^k/b_i^k$. From the definition of $b_l^k$ (4.15) we have

$$
\dot{b}_l^k = \sum_{i=1}^N (q_i^k - \check{q}_i^{k-1}) E_1(q_i^k - q_l^{k-1} - \eta) - \sum_{i \neq l}^N (q_l^k - q_l^k) E_1(q_l^k - q_i^k) + \frac{1}{c} \partial_t \hat{q}_l^k.
$$
(4.29)

Note that we can remove “bar” from velocities $\dot{q}$ in the first sum since

$$
\check{q}_i^k - \check{q}_j^m = \hat{q}_i^k - \hat{q}_j^m
$$
(4.30)

for any values of indices due to (4.21). Using (4.30) and plugging (4.26) into (4.29), we get

$$
\frac{\dot{b}_l^k}{b_l^k} = -\sum_{i=1}^N q_i^{k+1} E_1(q_i^k - \check{q}_l^{k+1} + \eta) - \sum_{i=1}^N q_i^{k-1} E_1(q_i^k - \check{q}_l^{k-1} - \eta) + 2 \sum_{i \neq l}^N q_l^k E_1(q_l^k - q_i^k)
$$

$$
+ \frac{1}{N} \sum_{i=1}^N \sum_{m=1}^N E_1(q_i^k - \check{q}_m^{k-1} - \eta) - \frac{1}{N} \sum_{l=1}^N \sum_{m=1}^N \sum_{l=1}^N \sum_{m=1}^N E_1(q_l^{k+1} - q_m^k - \eta) - \partial_t \log h_{k-1,k}.
$$
(4.31)

Therefore, from (4.27) we obtain the following result:

$$
\check{q}_i^k - \check{q}_j^m = -\sum_{i=1}^N q_i^{k+1} E_1(q_i^k - \check{q}_l^{k+1} + \eta) - \sum_{i=1}^N q_i^{k-1} E_1(q_i^k - \check{q}_l^{k-1} - \eta) + 2 \sum_{i \neq l}^N q_l^k E_1(q_l^k - q_i^k)
$$

$$
+ \frac{1}{N} \sum_{i=1}^N \sum_{m=1}^N E_1(q_i^k - \check{q}_m^{k-1} - \eta) - \frac{1}{N} \sum_{l=1}^N \sum_{m=1}^N \sum_{l=1}^N \sum_{m=1}^N E_1(q_l^{k+1} - q_m^k - \eta) - \partial_t \log h_{k-1,k}.
$$
(4.32)

The last term $\partial_t \log h_{k-1,k}$ can be found using (4.28) and (4.31). We have:

$$
\partial_t \log h_{k-1,k} = -\sum_{m,l=1}^N q_m^l q_i^{k+1} E_1(q_m^l - q_i^{k+1} + \eta) + \sum_{m,l=1}^N q_m^l q_m^{k-1} E_1(q_m^{k-1} - q_i^k + \eta)
$$

$$
+ \frac{1}{N} \sum_{l=1}^N \sum_{m=1}^N E_1(q_i^k - \check{q}_m^{k-1} - \eta) - \frac{1}{N} \sum_{l=1}^N \sum_{m=1}^N \sum_{l=1}^N \sum_{m=1}^N E_1(q_l^{k+1} - q_m^k - \eta),
$$
(4.33)

where for the last line we also used (4.21). Note also that the latter expression can be represented in the form

$$
\partial_t \log h_{k-1,k} = e^{k-1} - e^k,
$$
(4.34)
As is seen above, the Ruijsenaars-Schneider chain is the gauge transformed elliptic spin chain together with the change of variables (4.13) and the Lax matrices (4.13) and the Hamiltonians (4.17) and (3.11) coincide. From the relation (4.7) between the Lax matrices, we follow the strategy used in [23] to reproduce the Ruijsenaars-Schneider chain together with the change of variables (4.3)–(4.4). With this identification, the semi-discrete Zakharov-Shabat representation for the Ruijsenaars-Schneider chain (4.35)–(4.41) through the canonical variables (4.38)–(4.41) from the IRF-Vertex relations. First, let us express $M^{k}(z)$ (3.15)–(3.16) through the canonical variables

$$M^{k}(z) = - \frac{1}{\hbar_{k,k+1}} \frac{\partial' (0)}{\partial (\eta)} \frac{1}{N} \text{tr}_{2} \left( \left( \xi^{k+1} (p^{k+1}, q^{k+1}) \otimes \psi^{k}(q^{k}) \right) r_{12}(z) \right).$$

Plugging $\xi^{k+1}$ and $\psi^{k}$ from (4.4), we have

$$M^{k}(z) = - \frac{1}{h_{k,k+1}} \frac{\partial' (0)}{\partial (\eta)} \frac{1}{N} \text{tr}_{2} \left( (\rho \otimes \rho^T)_{2} g_{2}(0, q^{k}) r_{12}(z) g_{2}(N \eta, q^{k+1}) e^{P_{k+1}/c} \right).$$

Next, we substitute $g(0, q^{k}) r_{12}(z)$ from (B.27), where all matrices in the r.h.s. depend on $q^{k}$. Using $(\rho \otimes \rho^T)_{12} = N \mathcal{O}_{12}$, we obtain:

$$M^{k}(z) = - \frac{1}{h_{k,k+1}} \frac{\partial' (0)}{\partial (\eta)} \frac{1}{N} \text{tr}_{2} \left( g_{1}(z, q^{k}) \mathcal{O}_{12} g_{2}(0, q^{k}) g_{1}^{-1}(z, q^{k}) \right) g_{2}(N \eta, q^{k+1}) e^{P_{k+1}/c}.$$
Then the gauged transformed $M$-matrix (4.38) takes the form

$$
\hat{M}^k(z) = -g^{-1}(z) g'(z) G - F + g^{-1}(z, q^k) \dot{g}(z, q^k),
$$

(4.42)

where

$$
G = \frac{1}{h_{k,k+1}} \text{tr}_2 \left( \mathcal{O}_{12} \frac{\vartheta' (0)}{\vartheta (\eta)} \tilde{g}_2(0, q^k) g_2(N \eta, q^{k+1}) e^{P_{k+1} / c} \right)
$$

(4.43)

and

$$
F = \frac{1}{h_{k,k+1}} \text{tr}_2 \left( \mathcal{O}_{12} \frac{\vartheta' (0)}{\vartheta (\eta)} A_2(q^k) g_2(N \eta, q^{k+1}) e^{P_{k+1} / c} \right).
$$

(4.44)

Let us compute the matrices $F$ and $G$. Consider the residue of $\hat{L}^{k+1}(z)$ at $z = 0$. On the one hand it comes from (4.8), and on the other hand it can be found from (4.14):

$$
\hat{L}^{k+1}[0] = \text{Res}_{z=0} \hat{L}^{k+1}(z) = \frac{\vartheta' (0)}{\vartheta (\eta)} \tilde{g}(0, q^k) g(N \eta, q^{k+1}) e^{P_{k+1} / c} = \rho \otimes b^{k+1}.
$$

(4.45)

Plugging it into (4.43) and taking also into account (4.19), we see that $G$ is the identity matrix:

$$
G = 1_N.
$$

(4.46)

In order to compute the matrix $F$, consider the $z^0$-term in the expansion of $\hat{L}^{k+1}(z)$ near $z = 0$. Using the factorized form (4.8) and the expansion (B.20), we obtain:

$$
\hat{L}^{k+1}[0] = \frac{\vartheta' (0)}{\vartheta (\eta)} \tilde{g}(0, q^k) g'(N \eta, q^{k+1}) e^{P_{k+1} / c} + \frac{\vartheta' (0)}{\vartheta (\eta)} A(q^k) g(N \eta, q^{k+1}) e^{P_{k+1} / c}.
$$

(4.47)

The first term in the r.h.s. of (4.47) is obtained by differentiating both sides of (4.45) with respect to $\eta$:

$$
\partial_\eta \hat{L}^{k+1}[0] = -E_1(\eta) \hat{L}^{k+1}[0] + N \frac{\vartheta' (0)}{\vartheta (\eta)} \tilde{g}(0, q^k) g'(N \eta, q^{k+1}) e^{P_{k+1} / c},
$$

(4.48)

so that

$$
\frac{\vartheta' (0)}{\vartheta (\eta)} A(q^k) g(N \eta, q^{k+1}) e^{P_{k+1} / c} = \hat{L}^{k+1}[0] - \frac{1}{N} \left( \partial_\eta \hat{L}^{k+1}[0] + E_1(\eta) \hat{L}^{k+1}[0] \right).
$$

(4.49)

Expressions $\hat{L}^{k+1}[0]$ and $\hat{L}^{k+1}[0]$ are known explicitly from (4.13) and (A.17):

$$
\hat{L}^{k+1}[0] = b_j^{k+1}, \quad \hat{L}^{k+1}[0] = b_j^{k+1} E_1(q_i^k - q_j^{k+1} + \eta),
$$

$$
\left( \partial_\eta \hat{L}^{k+1}[0] + E_1(\eta) \hat{L}^{k+1}[0] \right)_{ij} = -b_j^{k+1} \sum_{l=1}^N E_1(q_j^{k+1} - q_l^k - \eta).
$$

(4.50)

Plugging all this into (4.49) and dividing both parts by $h_{k,k+1}$, we obtain (using also the definition (4.20)):

$$
\frac{1}{h_{k,k+1}} \frac{\vartheta' (0)}{\vartheta (\eta)} \left( A(q^k) g(N \eta, q^{k+1}) e^{P_{k+1} / c} \right)_{ij}
$$

$$
= q_j^{k+1} E_1(q_i^k - q_j^{k+1} + \eta) + \frac{1}{N} q_j^{k+1} \sum_{l=1}^N E_1(q_j^{k+1} - q_l^k - \eta).
$$

(4.51)
Using the property (B.26), we find the (diagonal) matrix $F$ (4.44):

$$F_{ij} = \delta_{ij} \sum_{m=1}^{N} q_{m}^{k+1} E_{1}(\bar{q}_{m}^{k} - \bar{q}_{m}^{k+1} + \eta) + \delta_{ij} \frac{1}{N} \sum_{l,m=1}^{N} q_{m}^{k+1} E_{1}(\bar{q}_{m}^{k+1} - \bar{q}_{l}^{k} - \eta).$$

(4.52)

To get the final answer for $\dot{M}_{ij}^{k}(z)$ (4.42), let us simplify its last term $g^{-1}(z, \eta)\dot{g}(z, \eta)$. Using its definition (2.37)–(2.39), we have:

$$g^{-1}(z)\dot{g}(z) = g^{-1}(z)g'(z) \left(-N \text{diag}(\dot{\phi}) + 1_N \sum_{k=1}^{N} \dot{\phi}_k \right) - \dot{d}(d^{-1})^{-1}. \tag{4.53}$$

Substitute (4.53) and (4.46) into (4.42). Due to (4.21) the term $-g^{-1}g'(z)G$ is canceled with the one proportional to $\sum_k \dot{\phi}_k$ in (4.53):

$$\dot{M}_{ij}^{k}(z) = -F - Ng^{-1}(z)g'(z)\text{diag}(\dot{\phi}) - \dot{d}(d^{-1})^{-1}. \tag{4.54}$$

The quantity $g^{-1}(z)g'(z)$ is known from (2.47) and

$$\left(\dot{d}(d^{-1})^{-1}\right)_{ii} = \sum_{m:l,m\neq i}^{N} (q_{i}^{k} - q_{m}^{k}) E_{1}(q_{i}^{k} - q_{m}^{k}). \tag{4.55}$$

Therefore,

$$\dot{M}_{ij}^{k}(z) = -(1 - \delta_{ij})\phi(z, q_{i}^{k} - q_{j}^{k}) \dot{q}_{j}^{k} - \delta_{ij} E_{1}(z)\dot{q}_{i}^{k}$$

$$+ \delta_{ij} \left( \sum_{m:l,m\neq i}^{N} q_{m}^{k} E_{1}(q_{i}^{k} - q_{m}^{k}) - \sum_{m=1}^{N} q_{m}^{k+1} E_{1}(\bar{q}_{i}^{k} - \bar{q}_{m}^{k+1} + \eta) - \frac{1}{N} \sum_{l,m=1}^{N} \delta_{ij} \sum_{m=1}^{N} q_{m}^{k+1} E_{1}(\bar{q}_{m}^{k+1} - \bar{q}_{l}^{k} - \eta) \right). \tag{4.56}$$

To summarize, we have proved that the semi-discrete Zakharov-Shabat equation (4.37) holds for the matrices (4.22) and (4.56) on the equations of motion of the Ruijsenaars-Schneider chain (4.20), (4.26) or (4.31)–(4.33). This can be also verified by direct substitution using identities (A.20) and (A.19) similarly to verification of the Lax pair for the Ruijsenaars-Schneider model (2.1), (2.13).

**Modified Lax pair.** All Lax matrices (4.22) can be simultaneously divided by $h_{k-1,k}$. Then the resultant Lax matrix depend on the velocities (4.20). From the point of view of the ordinary Ruijsenaars-Schneider model it is similar to transition to the logarithm of Hamiltonian (2.15). Consider

$$L_{ij}^{k}(z) = \tilde{L}_{ij}^{k}(z) \frac{1}{h_{k-1,k}} = \phi(z, q_{i}^{k-1} - q_{j}^{k} + \eta)\dot{q}_{j}^{k}. \tag{4.57}$$

This can be done since the transfer matrix is divided by conserved quantity (4.16):

$$t'(z) = \text{tr}\left( L_{1}(z) L_{2}(z) \ldots L_{m}(z) \right) = \frac{t(z)}{h_{1,2}h_{2,3} \ldots h_{n,1}} = t(z) \exp(-H/c), \tag{4.58}$$

(4.40)
The Lax equation for $L^{jk}(z)$ is of the form:

$$\frac{d}{dt}L^{jk}(z) = L^{jk}(z)\hat{M}^k(z) - \hat{M}^{k-1}(z)L^{jk}(z) - L^{jk}(z)\partial_t \log h_{k-1,k},$$  \hspace{1cm} (4.59)

Using (4.34) the last term in (4.59) can be removed by redefining $\hat{M}^k(z)$:

$$M^{jk}(z) = \hat{M}^{k}(z) + c^k 1_N.$$  \hspace{1cm} (4.60)

Then

$$\frac{d}{dt}L^{jk}(z) = L^{jk}(z)M^{jk}(z) - M^{jk-1}(z)L^{jk}(z).$$  \hspace{1cm} (4.61)

Explicit form of the $M$-matrix (4.60) is as follows:

$$M^{jk}_{ij}(z) = -(1 - \delta_{ij})\phi(z, q^k_j - q^k_i) \hat{q}^k_j - \delta_{ij}E_1(z) q^k_i + \delta_{ij} \left( \sum_{m ; m \neq i}^N \hat{q}^k_m E_1(z) \hat{q}^k_j - \hat{q}^k_m \right) + \delta_{ij} \left( \sum_{m = 1}^N \hat{q}^{k+1}_m E_1(z) \hat{q}^k_j - \hat{q}^{k+1}_m + \eta \right) \right).$$  \hspace{1cm} (4.62)

To summarize, the Lax pair (4.57) and (4.62) satisfy the semi-discrete zero curvature equation (4.61) and provide equations of motion of the Ruijsenaars-Schneider chain (4.36).

5 Field analogue of the elliptic Ruijsenaars-Schneider system from elliptic families of solutions to the 2D Toda lattice

In this section we derive equations of motion for poles of general elliptic solutions (which we call elliptic families) to the 2D Toda lattice hierarchy and show that they are Hamiltonian and equivalent to (4.36) under some simple substitutions and redefinitions.

5.1 The 2D Toda lattice hierarchy

Following [42], we briefly review the 2D Toda lattice hierarchy. The sets of independent variables are two infinite sets of continuous time variables $\mathbf{t} = \{ t_1, t_2, t_3, \ldots \}$, $\mathbf{t} = \{ \bar{t}_1, \bar{t}_2, \bar{t}_3, \ldots \}$ and a discrete integer-valued variable $n$ which is sometimes denoted as $t_0$. The main objects are two pseudo-difference Lax operators

$$\mathbf{L} = e^{\partial_n} + \sum_{k \geq 0} U_{k,n} e^{-k\partial_n}, \quad \bar{\mathbf{L}} = a_n e^{-\partial_n} + \sum_{k \geq 0} \bar{U}_{k,n} e^{k\partial_n},$$  \hspace{1cm} (5.1)

where $e^{\partial_n}$ is the shift operator acting as $e^{\pm \partial_n} f(n) = f(n \pm 1)$ and the coefficient functions are functions of $\mathbf{t}$, $\mathbf{\bar{t}}$. The equations of the hierarchy are differential equations for the functions $a_n, U_{k,n}, \bar{U}_{k,n}$. They are encoded in the Lax equations

$$\partial_{t_m} \mathbf{L} = [\mathcal{B}_m, \mathbf{L}], \quad \partial_{\bar{t}_m} \bar{\mathbf{L}} = [\mathcal{B}_m, \bar{\mathbf{L}}] \quad \mathcal{B}_m = (\mathbf{L}^m)_{\geq 0},$$  \hspace{1cm} (5.2)

$$\partial_{\bar{t}_m} \mathbf{L} = [\bar{\mathcal{B}}_m, \mathbf{L}], \quad \partial_{t_m} \bar{\mathbf{L}} = [\bar{\mathcal{B}}_m, \bar{\mathbf{L}}] \quad \bar{\mathcal{B}}_m = (\bar{\mathbf{L}}^m)_{< 0},$$  \hspace{1cm} (5.3)
where we use the notation
\[
\left( \sum_{k \in \mathbb{Z}} U_{k,n} e^{k \partial_n} \right) \geq 0 = \sum_{k \geq 0} U_{k,n} e^{k \partial_n}, \quad \left( \sum_{k \in \mathbb{Z}} U_{k,n} e^{k \partial_n} \right) < 0 = \sum_{k < 0} U_{k,n} e^{k \partial_n}.
\]

For example, \( B_1 = e^{\partial_n} + b_n, \) \( \bar{B}_1 = a_n e^{-\partial_n}, \) where we have denoted \( U_{0,n} = b_n. \) It can be shown that the zero curvature (Zakharov-Shabat) equations
\[
\begin{align*}
\partial_t B_m - \partial_t B_n + [B_m, B_n] &= 0, \quad (5.4) \\
\partial_{\bar{t}} B_m - \partial_t \bar{B}_n + [B_m, \bar{B}_n] &= 0, \quad (5.5) \\
\partial_{\bar{t}} \bar{B}_m - \partial_{\bar{t}} \bar{B}_n + [\bar{B}_m, \bar{B}_n] &= 0. \quad (5.6)
\end{align*}
\]
provide an equivalent formulation of the hierarchy.

The 2D Toda equation is the first member of the hierarchy. It is obtained from (5.5) at \( m = n = 1 \) which is equivalent to the system of equations
\[
\begin{align*}
\partial_t \log a_n &= b_n - b_{n-1} \\
\partial_{\bar{t}} b_n &= a_n - a_{n+1}.
\end{align*}
\]
Excluding \( b_n \) from this system, we get the differential equation for \( a_n: \)
\[
\partial_t \partial_{\bar{t}} \log a_n = 2a_n - a_{n+1} - a_{n-1}. \quad (5.7)
\]
It is one of the forms of the 2D Toda equation. In terms of the function \( \varphi_n \) introduced through the relation \( a_n = e^{\varphi_n - \varphi_{n-1}} \) it acquires the familiar form
\[
\partial_t \partial_{\bar{t}} \varphi_n = e^{\varphi_n - \varphi_{n-1}} - e^{\varphi_{n+1} - \varphi_n}. \quad (5.8)
\]

The universal dependent variable of the hierarchy is the tau-function \( \tau_n = \tau_n(t, \bar{t}). \) The change of the dependent variables from \( a_n, b_n \) to the tau-function,
\[
a_n = \frac{\tau_{n+1} \tau_{n-1}}{\tau_n^2}, \quad b_n = \partial_{\bar{t}} \log \frac{\tau_{n+1}}{\tau_n},
\]
brings the 2D Toda equation to the form \([43]\)
\[
\partial_t \partial_{\bar{t}} \log \tau_n = -\frac{\tau_{n+1} \tau_{n-1}}{\tau_n^2}. \quad (5.10)
\]

At fixed \( n \) and \( \bar{t} \) the equations of the 2D Toda lattice hierarchy are reduced to the Kadomtsev-Petviashvili (KP) hierarchy with the independent variables \( \{t_1, t_2, t_3, \ldots\} \), with the KP equation (the first member of the hierarchy) being satisfied by
\[
u_n = \partial^2_{t_1} \log \tau_n. \quad (5.11)
\]

An important class of solutions to the 2D Toda lattice hierarchy is the algebraic-geometrical solutions constructed from a smooth algebraic curve \( \Gamma \) of genus \( g \) with some extra data. The tau-function for such solutions is given by \([38, 39]\)
\[
\tau_n(t, \bar{t}) = e^{Q(n, t, \bar{t})} \Theta \left( V_0 n + \sum_{k \geq 1} V_k t_k + \sum_{k \geq 1} \bar{V}_k \bar{t}_k + Z \right), \quad (5.12)
\]

- 24 -
where $Q$ is a quadratic form of its variables and $\Theta$ is the Riemann theta-function with the Riemann’s matrix being the period matrix of holomorphic differentials on the curve $\Gamma$. Components of the $g$-dimensional vectors $V_k, \bar{V}_k$ are $b$-periods of certain normalized meromorphic differentials on $\Gamma$ with poles at two marked points $P_0, P_\infty \in \Gamma$ (see [38, 39] for details).

When one considers algebraic-geometrical solutions, it is natural to treat $t_0 = n$ as a continuous rather than discrete variable. Namely, let us introduce the continuous variable $x = x_0 + \eta n$, where $\eta$ is a constant (a lattice spacing), then the Toda equation becomes a difference equation in $x$:

$$\partial_t \partial_{\bar{t}} \log a(x) = 2a(x) - a(x + \eta) - a(x - \eta).$$

It is equivalent to the zero curvature equation $\partial_{\bar{t}} B_1 - \partial_t \bar{B}_1 + [B_1, \bar{B}_1] = 0$ for the difference operators

$$B_1 = e^{\eta \partial_x} + b(x), \quad \bar{B}_1 = a(x) e^{-\eta \partial_x},$$

which is the compatibility condition of the linear problems

$$\partial_t \psi(x) = \psi(x + \eta) + b(x) \psi(x), \quad \partial_{\bar{t}} \psi(x) = a(x) \psi(x - \eta)$$

for a wave function $\psi$.

### 5.2 Elliptic families

Let us fix the variables $\tilde{t}$ and consider the dependent variables as functions of $x, t$. A general solution to the 2D Toda and KP equations is known to be of the form

$$a(x, t) = \frac{\tau(x + \eta, t) \tau(x - \eta, t)}{\tau^2(x, t)},$$

$$b(x, t) = \partial_1 \log \frac{\tau(x + \eta, t)}{\tau(x, t)},$$

$$u(x, t) = \partial_{\bar{t}}^2 \log \tau(x, t).$$

We are going to consider solutions that are elliptic functions with respect to some variable $t_k$ or a linear combination $\lambda = \beta_0 x + \sum_k \beta_k t_k$. We call them elliptic families. The elliptic families form a subclass of algebraic-geometrical solutions. As it was already mentioned in Introduction, an algebraic-geometrical solution is elliptic with respect to some direction if there exists a $g$-dimensional vector $W$ such that it spans an elliptic curve $\mathcal{E}$ embedded in the Jacobian of the curve $\Gamma$: 

$$\tau(x, t, \lambda) = e^{Q(x, t)} \Theta \left( V_0 x/\eta + \sum_{k \geq 1} V_k t_k + W \lambda + Z \right).$$

This is a nontrivial transcendental constraint. The space of corresponding algebraic curves has codimension $g - 1$ in the moduli space of all the curves. If such a vector $W$ exists,
then the theta-divisor intersects the shifted elliptic curve $E + V_0 x/\eta + \sum_k V_k t_k$ at a finite number of points $\lambda_i = \lambda_i(x, t)$. Therefore, for elliptic families we have:

$$\Theta \left( V_0 x/\eta + \sum_{k \geq 1} V_k t_k + W \lambda + Z \right) = f(x, t) e^{\gamma_1 \lambda + \gamma_2 \lambda^2} \prod_{i=1}^{N} \sigma(\lambda - \lambda_i(x, t)).$$

(5.18)

Here $\gamma_1, \gamma_2$ are constants and $\sigma(\lambda)$ is the Weierstrass $\sigma$-function defined in the appendix. The form of the exponential factor in the right hand side of (5.18) follows from monodromy properties of the theta-function. The function $f(x, t)$ does not depend on $\lambda$. In the tau-function, it is modified by the factor $e^{Q(x,t)}$ which also does not depend on $\lambda$. Having in mind the discrete version, we will also denote $\lambda^k = \lambda(x)$ for $x = x_0 + k \eta$.

In what follows we denote $t_1 = t$. From (5.18) we conclude that if $b(x, t, \lambda)$ is an elliptic family of solutions to the 2D Toda equation, then it has the form

$$b(x, t, \lambda) = \sum_{i=1}^{N} \left( \lambda_i(x) \zeta(\lambda - \lambda_i(x)) - \dot{\lambda}_i(x + \eta) \zeta(\lambda - \lambda_i(x + \eta)) \right) + c(x, t),$$

(5.19)

where dot means the $t$-derivative and $c(x, t)$ is some function. Since $a(x, t, \lambda), b(x, t, \lambda)$ and $u(x, t, \lambda)$ given by (5.16) are elliptic functions of $\lambda$, one should have

$$\sum_{i=1}^{N} \left( \lambda_i(x + \eta) + \dot{\lambda}_i(x - \eta) - 2\dot{\lambda}_i(x) \right) = 0,$$

(5.20)

$$\sum_{i=1}^{N} \dot{\lambda}_i(x + \eta) = \sum_{i=1}^{N} \dot{\lambda}_i(x),$$

(5.21)

$$\sum_{i=1}^{N} \ddot{\lambda}_i(x) = 0.$$  

(5.22)

From (5.20), (5.21), (5.22) it follows that

$$\sum_{i=1}^{N} \lambda_i(x) = \alpha x + \beta t + \alpha_0, \quad \dot{\alpha} = \dot{\beta} = 0.$$  

(5.23)

Here $\alpha_0, \alpha, \beta$ are unspecified $\eta$-periodic functions of $x$. We can say that the requirement of ellipticity implies that the “center of masses” of the points $\lambda_i$ moves linearly in time.

A meromorphic function $f(\lambda)$ is called a double-Bloch function if it satisfies the following monodromy properties:

$$f(\lambda + 2 \omega_\alpha) = B_\alpha f(\lambda), \quad \alpha = 1, 2.$$  

(5.24)

The complex constants $B_\alpha$ are called Bloch multipliers. Our goal is to find $b(x, t, \lambda)$ such that the equation

$$\partial_t \psi(x) - \psi(x + \eta) - b(x) \psi(x) = 0$$  

(5.25)

has sufficiently many double-Bloch solutions. The existence of such solutions turn out to be a very restrictive condition. The double-Bloch functions with simple poles $\lambda_i$ can be represented in the form

$$\psi(x) = \sum_{i=1}^{N} c_i(x) \Phi(\lambda - \lambda_i(x), z),$$  

(5.26)
where \( c_i \) are residues at the poles \( \lambda_i \) and the function \( \Phi(\lambda, z) \) is defined in (A.2). The variable \( z \) has the meaning of the spectral parameter. In what follows we often suppress the second argument of \( \Phi \) writing simply \( \Phi(\lambda, z) := \Phi(\lambda) \).

### 5.3 Equations of motion of the field analogue of the elliptic Ruijsenaars-Schneider system

Our strategy is similar to that of the work [3]. We are going to substitute (5.19), (5.26) into (5.25) and cancel all the poles which are at \( \lambda = \lambda_i(x) \) and \( \lambda = \lambda_i(x + \eta) \). The substitution gives:

\[
\sum_{i=1}^{N} c_i(x)\Phi(\lambda - \lambda_i(x)) - \sum_{i=1}^{N} c_i(x)\dot{\lambda}_i(x)\Phi'(\lambda - \lambda_i(x)) - \sum_{i=1}^{N} c_i(x + \eta)\Phi(\lambda - \lambda_i(x + \eta))
- \sum_{i=1}^{N} \left( (\dot{\lambda}_i(x)\zeta(\lambda - \lambda_i(x)) - \dot{\lambda}_i(x + \eta)\zeta(\lambda - \lambda_i(x + \eta)) \right) \sum_{j=1}^{N} c_j(x)\Phi(\lambda - \lambda_j(x))
- c(x, t) \sum_{i=1}^{N} c_i(x)\Phi(\lambda - \lambda_i(x)) = 0.
\]

The cancellation of poles yields the following system of equations:

\[
c_i(x + \eta) = \dot{\lambda}_i(x + \eta) \sum_{j=1}^{N} c_j(x)\Phi(\lambda_i(x + \eta) - \lambda_j(x)), \tag{5.27}
\]

\[
\dot{c}_i(x) = \dot{\lambda}_i(x) \sum_{j\neq i} c_j(x)\Phi(\lambda_i(x) - \lambda_j(x)) + c_i(x) \sum_{j\neq i} \dot{\lambda}_j(x)\zeta(\lambda_i(x) - \lambda_j(x))
- c_i(x) \sum_{j=1}^{N} \dot{\lambda}_j(x + \eta)\zeta(\lambda_i(x) - \lambda_j(x + \eta)) + c_i(x)c(x, t). \tag{5.28}
\]

Introducing the matrices \( L, M \) as

\[
L_{ij}(x, z) = \dot{\lambda}_i(x + \eta)\Phi(\lambda_i(x + \eta) - \lambda_j(x), z), \tag{5.29}
\]

\[
M_{ij}(x, z) = (1 - \delta_{ij}) \dot{\lambda}_i(x)\Phi(\lambda_i(x) - \lambda_j(x), z) \tag{5.30}
+ \delta_{ij} \left( \sum_{k\neq i} \dot{\lambda}_k(x)\zeta(\lambda_i(x) - \lambda_k(x)) - \sum_{k=1}^{N} \dot{\lambda}_k(x + \eta)\zeta(\lambda_i(x) - \lambda_k(x + \eta)) + c(x, t) \right),
\]

one can write the system (5.27), (5.28) in the form

\[
c_i(x + \eta) = \sum_{j=1}^{N} L_{ij}(x)c_j(x), \quad \dot{c}_i(x) = \sum_{j=1}^{N} M_{ij}(x)c_j(x). \tag{5.31}
\]

Let us also introduce the matrices

\[
A_{ij}^+(x) = \Phi(\lambda_i(x + \eta) - \lambda_j(x)), A_{ij}^0(x) = (1 - \delta_{ij})\Phi(\lambda_i(x) - \lambda_j(x)) \tag{5.32}
\]
and diagonal matrices
\[ \Lambda_{ij}(x) = \delta_{ij}\lambda_i(x), \]
\[ D^0_{ij}(x) = \delta_{ij} \sum_{k:k\neq i} \dot{\lambda}_k(x)\zeta(\lambda_i(x) - \lambda_k(x)), \]
\[ D^+_{ij}(x) = \delta_{ij} \sum_{k:k\neq i} \dot{\lambda}_k(x \pm \eta)\zeta(\lambda_i(x) - \lambda_k(x \pm \eta)). \]

In terms of these matrices, the matrices \( L \) and \( M \) read:
\[ L(x) = \dot{\Lambda}(x + \eta)A^+(x), \quad M(x) = \dot{\Lambda}(x)A^0(x) + D^0(x) - D^+(x) + c(x,t)I, \]
where \( I \) is the unity matrix. The compatibility condition of the overdetermined system \((5.27), (5.28)\) is the semi-discrete zero curvature (Zakharov-Shabat) equation
\[ R(x) := \dot{L}(x) + L(x)M(x) - M(x + \eta)L(x) = 0. \]

The matrices \( L, M \) here depend on the spectral parameter \( z \). We have:
\[ R(x) = \dot{\Lambda}(x + \eta)A^+(x) + \dot{\Lambda}(x + \eta)\left( S(x) + A^+(x)(D^0(x) - D^+(x)) - (D^0(x + \eta) - D^+(x + \eta))A^+(x) + (c(x,t) - c(x + \eta,t))A^+(x) \right), \]
where
\[ S(x) = \dot{A}^+(x) + A^+(x)\dot{\Lambda}(x)A^0(x) - A^0(x + \eta)\dot{\Lambda}(x + \eta)A^+(x). \]

Using \((A.5), (A.6)\), we calculate:
\[ \dot{\Lambda}^+_i(x) = (\dot{\lambda}_i(x + \eta) - \dot{\lambda}_j(x))\Phi(\lambda_i(x + \eta) - \lambda_j(x)) \]
\[ \times \left( \zeta(\lambda_i(x + \eta) - \lambda_j(x) + \mu) - \zeta(\lambda_i(x + \eta) - \lambda_j(x)) - \zeta(\mu) \right), \]
and
\[ \left( A^+(x)\dot{\Lambda}(x)A^0(x) - A^0(x + \eta)\dot{\Lambda}(x + \eta)A^+(x) \right)_{ij} \]
\[ = \sum_{k:k\neq j} \Phi(\lambda_i(x + \eta) - \lambda_k(x))\Phi(\lambda_k(x) - \lambda_j(x))\dot{\lambda}_k(x) \]
\[ - \sum_{k:k\neq i} \Phi(\lambda_i(x + \eta) - \lambda_k(x + \eta))\Phi(\lambda_k(x + \eta) - \lambda_j(x))\dot{\lambda}_k(x + \eta) \]
\[ = \Phi(\lambda_i(x + \eta) - \lambda_j(x)) \left( \sum_{k:k\neq j} \dot{\lambda}_k(x + \eta)\zeta(\lambda_i(x + \eta) - \lambda_k(x)) - \sum_{k:k\neq j} \dot{\lambda}_k(x)\zeta(\lambda_j(x) - \lambda_k(x)) \right) \]
\[ - \Phi(\lambda_i(x + \eta) - \lambda_j(x)) \left( \sum_{k:k\neq i} \dot{\lambda}_k(x + \eta)\zeta(\lambda_i(x + \eta) - \lambda_k(x + \eta)) \right. \]
\[ \left. - \sum_{k:k\neq i} \dot{\lambda}_k(x + \eta)\zeta(\lambda_j(x) - \lambda_k(x + \eta)) \right) \]
\[ + (\dot{\lambda}_i(x + \eta) - \dot{\lambda}_j(x))\Phi(\lambda_i(x + \eta) - \lambda_j(x))(\zeta(\mu) - \zeta(\lambda_i(x + \eta) - \lambda_j(x) + \mu)). \]
In the calculation, the condition (5.21) was taken into account. Therefore, we have:

\[ S_{ij}(x) = \Phi(\lambda_i(x+\eta) - \lambda_j(x))(D_{ii}^+(x+\eta) + D_{jj}^+(x) - D_{ii}^0(x+\eta)) \]  
\hspace{1cm} \text{(5.36)}

and, combining everything together, we obtain the matrix identity

\[ R(x) = \left( \dot{\Lambda}(x+\eta)\dot{\Lambda}^{-1}(x+\eta) + D^-(x+\eta) + D^+(x+\eta) - 2D^0(x+\eta) + (c(x,t) - c(x+\eta,t))I \right)L(x), \]
\hspace{1cm} \text{(5.37)}

from which we see that the compatibility condition is equivalent to vanishing of the diagonal matrix in front of \( L(x) \):

\[ \dot{\Lambda}(x+\eta)\dot{\Lambda}^{-1}(x+\eta) + D^-(x+\eta) + D^+(x+\eta) - 2D^0(x+\eta) + (c(x,t) - c(x+\eta,t))I = 0. \]  
\hspace{1cm} \text{(5.38)}

This results in the equations of motion

\[ \ddot{\lambda}_i(x) + \sum_{k=1}^{N} \left( \dot{\lambda}_i(x)\dot{\lambda}_k(x-\eta)\zeta(\lambda_i(x) - \lambda_k(x-\eta)) + \dot{\lambda}_i(x)\dot{\lambda}_k(x+\eta)\zeta(\lambda_i(x) - \lambda_k(x+\eta)) \right) 
- 2 \sum_{k,k\neq i}^{N} \dot{\lambda}_i(x)\dot{\lambda}_k(x)\zeta(\lambda_i(x) - \lambda_k(x)) + (c(x-\eta,t) - c(x,t))\dot{\lambda}_i(x) = 0. \]  
\hspace{1cm} \text{(5.39)}

Equations (5.39) resemble the Ruijsenaars-Schneider equations of motion (2.11) and provide their field generalization. If \( \lambda_i(x) = x_i(t) + x \), then equations (5.39) become the equations of motion for the elliptic Ruijsenaars-Schneider system with coordinates of particles \( x_i \) (with \( c(x,t) = c(x+\eta,t) \)).

The condition (5.23) allows us to find the explicit form of the function \( c(x,t) \). Summing equations (5.39) over \( i = 1, \ldots, N \) and using (5.23), we get

\[ c(x,t) = \left( \sum_{i=1}^{N} \dot{\lambda}_i(x) \right)^{-1} \sum_{i,k=1}^{N} \dot{\lambda}_i(x)\dot{\lambda}_k(x+\eta)\zeta(\lambda_i(x) - \lambda_k(x+\eta)) \]  
\hspace{1cm} \text{(5.40)}

(up to an arbitrary function of \( t \) and an \( \eta \)-periodic function of \( x \) which do not affect the equations of motion).

Let us also present the lattice version of equations (5.39) which are obtained from them after the substitution \( \lambda_i^k = \lambda_i(k\eta + x_0) \):

\[ \ddot{\lambda}_i^k + \sum_{j=1}^{N} \left( \dot{\lambda}_i^k\dot{\lambda}_j^{k-1}\zeta(\lambda_i^k - \lambda_j^{k-1}) + \dot{\lambda}_i^k\dot{\lambda}_j^{k+1}\zeta(\lambda_i^k - \lambda_j^{k+1}) \right) 
- 2 \sum_{j,j\neq i}^{N} \dot{\lambda}_i^k\dot{\lambda}_j^k\zeta(\lambda_i^k - \lambda_j^k) + (c^{k-1}(t) - c^k(t))\dot{\lambda}_i^k = 0 \]  
\hspace{1cm} \text{(5.41)}

with

\[ c^k(t) = \frac{1}{\beta} \sum_{i,j=1}^{N} \lambda_i^k\lambda_j^{k+1}\zeta(\lambda_i^k - \lambda_j^{k+1}). \]  
\hspace{1cm} \text{(5.42)}

Here \( \beta = \sum_{i=1}^{N} \dot{\lambda}_i^k \) may be regarded as a constant since \( \beta = \sum_{i=1}^{N} \dot{\lambda}_i(x) \) is an \( \eta \)-periodic function of \( x \) (see (5.23)).
5.4 Equivalence to the equations of sections 4.2 and 5.3

It is not difficult to see that with the conditions (5.20)–(5.22) equations (4.36), (5.42) with $\beta = 1$ are equivalent to (4.36). Indeed, identifying $\lambda_k^j = q_k^j$ and passing to the center of masses frame, we see that

\[
\lambda_k^j - \lambda_{k}^{j-1} = q_k^j - q_{k-1}^j + \alpha \eta/N, \\
\lambda_k^j - \lambda_{k+1}^j = q_k^j - q_{k+1}^j - \alpha \eta/N,
\]

where $\alpha$ is defined in (5.23), so that the arguments of the $\zeta$- and $E_1$-functions in (4.36) and (5.41) coincide if $\alpha = -N$. Next, if one chooses the periods to be $2\omega_1 = 1$, $2\omega_2 = \tau$, the $\zeta(z)$- and $E_1(z)$-functions differ by a term linear in $z$ (see (A.23)). The corresponding contribution to $c^k(t)$ (5.42) is

\[
\frac{1}{\beta} \sum_{i=1}^{N} \lambda_i^k \lambda_i^k \sum_{j=1}^{N} \lambda_j^{k+1} - \frac{1}{\beta} \sum_{i=1}^{N} \lambda_i^{k+1} \lambda_i^k \sum_{j=1}^{N} \lambda_j^{k} = \sum_{i=1}^{N} \lambda_i^k \lambda_i^k - \sum_{i=1}^{N} \lambda_i^{k+1} \lambda_i^{k+1} \tag{5.43}
\]

since $\sum_{j=1}^{N} \lambda_j^k = \beta$. Writing equations (5.41) as

\[
\frac{\dot{\lambda}_k^j}{\lambda_k^k} + \sum_{j=1}^{N} \left( \dot{\lambda}_j^{k-1} \zeta(\lambda_k^j - \lambda_j^{k-1}) + \dot{\lambda}_j^{k+1} \zeta(\lambda_k^j - \lambda_j^{k+1}) \right) \\
- 2 \sum_{j:j \neq i} \dot{\lambda}_j^k (\lambda_k^j - \lambda_j^k) + c^{k-1}(t) - c^k(t) = 0, \tag{5.44}
\]

we find the corresponding contribution from the sums over $j$ to be

\[
\sum_{j=1}^{N} \dot{\lambda}_j^{k-1} (\lambda_k^j - \lambda_j^{k-1}) + \sum_{j=1}^{N} \dot{\lambda}_j^{k+1} (\lambda_k^j - \lambda_j^{k+1}) - 2 \sum_{j=1}^{N} \dot{\lambda}_j^k (\lambda_k^j - \lambda_j^k) \\
= \lambda_k^k \sum_{j=1}^{N} (\dot{\lambda}_j^{k-1} + \dot{\lambda}_j^{k+1} - 2 \dot{\lambda}_j^k) - \sum_{j=1}^{N} (\dot{\lambda}_j^{k-1} \lambda_j^{k-1} + \dot{\lambda}_j^{k+1} \lambda_j^{k+1} - 2 \dot{\lambda}_j^k \lambda_j^k), \tag{5.45}
\]

so this contribution cancels with the one coming from $c^k(t)$ (5.46).

The $L-M$ pair (4.22), (4.56) discussed in sections 4.2, 4.3 is also equivalent to the $L-M$ pair (5.29), (5.30). Indeed, one can straightforwardly check that with the identification of the $L$- and $M$-matrices

\[
\hat{L}_{ij}^k(z) = -h_{k-1,k} e^{E_1(z)(\lambda_k^j - \lambda_j^k)} L_{j_1}^{k-1}(-z), \tag{5.46}
\]

\[
\hat{M}_{ij}^k(z) = e^{E_1(z)(\lambda_k^j - \lambda_j^k)} M_{j_1}^k(-z) - \delta_{ij} (E_1(z) \dot{\lambda}_j^k + \bar{c}^k). \tag{5.47}
\]

the semi-discrete Zakharov-Shabat equations (4.37) and (5.35) become equivalent. In the right hand sides of (5.46), (5.47) the matrices $L^k$, $M^k$ are given by (5.29), (5.30) under
the identification $x = k\eta + x_0$: $L^k = L(k\eta + x_0)$, $M^k = M(k\eta + x_0)$. For the modified Lax pair (4.57), (4.62) the relations (5.46), (5.47) slightly simplify:

\begin{align}
L^k_{ij}(z) &= -e^{E_1(z)(\lambda^k_i - \lambda^k_j)} L^{k-1}_{ji}(-z), \\
M^k_{ij}(z) &= e^{E_1(z)(\lambda^k_i - \lambda^k_j)} M^k_{ji}(-z) - \delta_{ij} E_1(z) \dot{\lambda}^k_i.
\end{align}

### 5.5 Hamiltonian structure

Let us show that the equations (5.41) with $c^k(t)$ given by (5.42) are Hamiltonian. We fix the canonical Poisson brackets

\begin{align}
\{p^k_i, p^l_j\} = \{\lambda^k_i, \lambda^l_j\} = 0, \quad \{\lambda^k_i, p^l_j\} = \delta_{ij}\delta_{kl}.
\end{align}

The Hamiltonian is

\begin{align}
H = \frac{\beta}{\eta} \sum_{k=1}^n \log H_k
\end{align}

with

\begin{align}
H_k = \sum_{i=1}^N e^{\eta p^k_i} \prod_{j=1}^N \frac{\sigma(\lambda^k_i - \lambda^{k-1}_j)}{\prod_{j:j \neq i}^N \sigma(\lambda^k_i - \lambda^k_j)}.
\end{align}

The first set of Hamiltonian equations is

\begin{align}
\dot{\lambda}_i^k = \frac{\partial H}{\partial p_i^k} = \beta e^{\eta p^k_i} \frac{\prod_{j=1}^N \sigma(\lambda_i^k - \lambda_j^{k-1})}{\prod_{j:j \neq i}^N \sigma(\lambda_i^k - \lambda_j^k)}.
\end{align}

Taking the time derivative of (logarithm of) this equation, we get

\begin{align}
\eta \dot{p}_i^k = \dot{\lambda}_i^k \frac{\lambda_i^k}{\dot{\lambda}_i^k} - \sum_{j=1}^N (\dot{\lambda}_i^k - \dot{\lambda}_j^{k-1}) \zeta(\lambda_i^k - \lambda_j^{k-1}) + \sum_{j:j \neq i}^N (\dot{\lambda}_i^k - \dot{\lambda}_j^k) \zeta(\lambda_i^k - \lambda_j^k) + \partial_t \log H_k.
\end{align}

The second set of Hamiltonian equations is

\begin{align}
\dot{p}_i^k(x) = -\frac{\partial H}{\partial \lambda_i^k}.
\end{align}

The variation of the Hamiltonian is

\begin{align}
\eta \delta H = \beta \sum_{k=1}^n \frac{\delta H_k}{H_k}
= \sum_{k=1}^n \sum_{i,j=1}^N \dot{\lambda}_i^k \zeta(\lambda_i^k - \lambda_i^{k-1})(\delta \lambda_i^k - \delta \lambda_i^{k-1}) - \sum_{k=1}^n \sum_{i,j:i \neq j}^N \dot{\lambda}_i^k \zeta(\lambda_i^k - \lambda_j^k)(\delta \lambda_i^k - \delta \lambda_j^k).
\end{align}
The Hamiltonian is
\[ H = \sum_{k=1}^{N} \lambda_k^k (\lambda_k^k - \lambda_k^{k-1}) \delta \lambda_k^k + \sum_{k=1}^{N} \lambda_k^{k+1} (\lambda_k^{k+1} - \lambda_k^{k+1}) \delta \lambda_k^k \]
\[ - \sum_{k=1}^{N} \lambda_k^k (\lambda_k^k - \lambda_k^{k+1}) \delta \lambda_k^k. \]

From here we see that
\[ \eta \delta H = - \sum_{l=1}^{N} \lambda_l^k (\lambda_l^k - \lambda_l^{k-1}) - \sum_{l=1}^{N} \lambda_l^{k+1} (\lambda_l^{k+1} - \lambda_l^{k+1}) + \sum_{l:i \neq l} (\lambda_l^k + \lambda_l^k) (\lambda_l^k - \lambda_l^k). \]

Comparing with (5.54), we obtain:
\[ \frac{\lambda_l^k}{\lambda_l^k} + \sum_{l=1}^{N} \lambda_l^{k+1} (\lambda_l^{k+1} - \lambda_l^{k+1}) + \sum_{l=1}^{N} \lambda_l^{k+1} (\lambda_l^{k+1} - \lambda_l^{k+1}) - 2 \sum_{l:i \neq l} \lambda_l^k (\lambda_l^k - \lambda_l^k) + \partial_t \log H_k = 0. \]

The calculation of \( \partial_t \log H_k = \dot{H}_k/H_k \) is straightforward using (5.52) and (5.56). The result is
\[ \partial_t \log H_k = c_k^{k-1}(t) - c_k^k(t), \]

where \( c_k(t) \) is given by (5.42). Therefore, the equations of motion (5.41) are reproduced.

### 5.6 The limit to 1+1 Calogero-Moser field theory

In this section we show that the \( \eta \to 0 \) limit of the field Ruijsenaars-Schneider model yields the field Calogero-Moser model as it appears in [3].

Instead of the lattice version (5.51) it is convenient to work with the equivalent \( x \)-dependent Hamiltonian density
\[ H(x) = \frac{\beta}{\eta} \log \left( \sum_{i=1}^{N} e^{\eta p_i} \sigma(\lambda_i(x) - \lambda_i(x - \eta)) \prod_{l:i \neq l} \frac{\sigma(\lambda_l(x) - \lambda_l(x - \eta))}{\sigma(\lambda_l(x) - \lambda_l(x))} \right), \]

and the canonical Poisson brackets
\[ \{ p_i(x), p_j(y) \} = \{ \lambda_i(x), \lambda_j(y) \} = 0, \quad \{ \lambda_i(x), p_j(y) \} = \delta_{ij} \delta(x - y). \]

The Hamiltonian is \( \int H(x) dx \). The Hamiltonian (5.51) is a straightforward lattice version of it. The calculation which is completely parallel to the one performed in section 5.5 shows that this Hamiltonian generates the equations of motion (5.39), (5.40) (with the only change that the derivatives with respect to the canonical variables \( \lambda_i^k, p_i^k \) on the lattice become variational derivatives with respect to the canonical variables \( \lambda_i(x), p_i(x) \)).

We are interested in the \( \eta \)-expansion of (5.58) as \( \eta \to 0 \). We have:
\[ H(x) = \frac{\beta}{\eta} \log \left[ \eta \sum_{i=1}^{N} \left( 1 + \eta p_i + \frac{1}{2} \eta^2 p_i^2 + O(\eta^3) \right) \left( \lambda_i^k - \frac{1}{2} \eta \lambda_i^k + \frac{1}{6} \eta^2 \lambda_i^k \right) \right. \]
\[ \times \exp \left( \sum_{j:j \neq i} \left( \eta \lambda_j^k \delta(\lambda_i - \lambda_j) - \frac{1}{2} \eta^2 \lambda_j^k \delta \lambda_i^k \right) \right), \]
where prime denotes the $x$-derivative. Equation (5.23) implies that

$$
\sum_{i=1}^{N} \lambda_i'(x) = \alpha. \tag{5.60}
$$

Since $\alpha$ is in general an $\eta$-periodic function of $x$, in the limit $\eta \to 0$ it is natural to assume that $\alpha$ is a constant. In the limit $\eta \to 0$ the vector from the marked point $P_0$ to the marked point $P_\infty$ on the Riemann surface $\Gamma$ becomes the tangent vector at $P_\infty$. This means that the $x$-flow tends to the $t_1$-flow, and so the limit of $\alpha$ as $\eta \to 0$ is equal to $\beta$. Therefore, the first few terms of the $\eta$-expansion of $H(x)$ are

$$
H(x) = \text{const} + (1 + O(\eta))H_1^{\text{CM}}(x) - \frac{\eta}{2} H_2^{\text{CM}}(x) + O(\eta^2), \tag{5.61}
$$

where

$$
H_1^{\text{CM}}(x) = \sum_{i=1}^{N} \tilde{p}_i \lambda_i' \tag{5.62}
$$
is the first Hamiltonian density of the field Calogero-Moser model (a field analogue of the total momentum) and

$$
H_2^{\text{CM}}(x) = -\sum_{i=1}^{N} \tilde{p}_i^2 \lambda_i' - \frac{1}{4} \sum_{i=1}^{N} \lambda_i'' \lambda_i' - \frac{1}{3} \sum_{i=1}^{N} \lambda_i''' + \frac{1}{\beta} \left( \sum_{i} \tilde{p}_i \lambda_i' \right)^2 - \frac{1}{2} \sum_{i,j:i\neq j} \left( \lambda_i'' \lambda_j' - \lambda_j'' \lambda_i' \right) \zeta(\lambda_i - \lambda_j) + \frac{1}{2} \sum_{i,j:i\neq j} \left( \lambda_i' \lambda_j'^2 + \lambda_j' \lambda_i'^2 \right) \varphi(\lambda_i - \lambda_j) \tag{5.63}
$$
is the second (standard) Hamiltonian density. The Hamiltonian of the model is

$$
H_2^{\text{CM}} = \int H_2^{\text{CM}}(x) dx. \tag{5.64}
$$

Up to a total derivative and the canonical transformation

$$
p_i \rightarrow \tilde{p}_i = p_i - \frac{\lambda_i''}{2 \lambda_i'} + \sum_{j:j\neq i} \lambda_j' \zeta(\lambda_i - \lambda_j) \tag{5.65}
$$
the Hamiltonian density (5.63) coincides with the Hamiltonian density for the field Calogero-Moser model presented in [3].

The fact that the transformation (5.65) is canonical can be verified straightforwardly. The only nontrivial calculation that is required is to show that $\{\tilde{p}_i(x), \tilde{p}_j(y)\} = 0$. This can be done using the identities

$$
f(x) \delta''(x - y) - f(y) \delta''(y - x) = -\left( f'(x) \delta'(x - y) - f'(y) \delta'(y - x) \right), \tag{5.66}
$$

$$
f(x) \delta'(x - y) + f(y) \delta'(y - x) = -f'(x) \delta(x - y) \tag{5.67}
$$
for the delta-function and its derivatives.
6 Fully discrete version

The fully discrete (or difference) version of the above construction can be obtained by considering elliptic families of solutions to the Hirota bilinear difference equation \[44\] for the tau-function \(\tau^{l,m}(x)\), where \(l, m\) are discrete times:
\[
\tau^{l,m}(x+\eta)\tau^{l+1,m+1}(x) - \kappa \tau^{l,m+1}(x+\eta)\tau^{l+1,m}(x) + (\kappa - 1)\tau^{l+1,m}(x+\eta)\tau^{l,m+1}(x) = 0. \tag{6.1}
\]
Here \(\kappa\) is a parameter. This equation is known to provide an integrable time discretization of the 2D Toda equation. One of the auxiliary linear problems for the equation (6.1) is \[45\]
\[
\psi^{m+1}(x) = \psi^m(x + \eta) - \kappa \frac{\tau^m(x)\tau^{m+1}(x + \eta)}{\tau^m(x + \eta)\tau^m(x)} \psi^m(x), \tag{6.2}
\]
where the index \(l\) is supposed to be fixed.

The elliptic families of solutions with elliptic parameter \(\lambda\) are given by
\[
\tau^{l,m}(x) = \rho^{l,m}(x)e^{c_1\lambda + c_2\lambda^2} \prod_{j=1}^N \sigma(\lambda - \lambda_j^{l,m}(x)), \tag{6.3}
\]
where \(\rho^{l,m}(x)\) is some function which does not depend on \(\lambda\) and \(c_1, c_2\) are constants. If the constraint
\[
\sum_{j=1}^N \left(\mu_j^{m+1}(x + \eta) - \mu_j^{m+1}(x)\right) = \sum_{j=1}^N \left(\mu_j^m(x + \eta) - \mu_j^m(x)\right) \tag{6.4}
\]
is satisfied, then the coefficient in front of the second term in the right hand side of (6.2) is an elliptic function of \(\lambda\) and we can find double-Bloch solutions of the form
\[
\psi^m(x) = \sum_{i=1}^N c_i^m(x)\Phi(\lambda - \lambda_i^m(x), z). \tag{6.5}
\]

The substitution of (6.5) into (6.2) yields:
\[
\sum_{i=1}^N c_i^{m+1}(x)\Phi(\lambda - \lambda_i^{m+1}(x)) - \sum_{i=1}^N c_i^m(x + \eta)\Phi(\lambda - \lambda_i^m(x + \eta))
+ \kappa_m(x) \prod_{j=1}^N \sigma(\lambda - \lambda_j^m(x))\sigma(\lambda - \lambda_j^{m+1}(x + \eta)) \prod_{j=1}^N c_j^m(x)\Phi(\lambda - \lambda_j^m(x)) = 0,
\]
where
\[
\kappa_m(x) = \frac{\kappa \rho^m(x)\rho^{m+1}(x + \eta)}{\rho^{m+1}(x)\rho^m(x + \eta)}.
\]

It is enough to cancel poles in the left hand side at \(\lambda = \lambda_j^{m+1}(x)\) and \(\lambda = \lambda_j^m(x + \eta)\). A direct calculation shows that the conditions of cancellation of the poles read
\[
c_i^m(x + \eta) = f_i^m(x) \sum_{j=1}^N c_j^m(x)\Phi(\lambda_i^m(x + \eta) - \lambda_j^m(x)), \tag{6.6}
\]
\[
c_i^{m+1}(x) = g_i^m(x) \sum_{j=1}^N c_j^m(x)\Phi(\lambda_i^{m+1}(x) - \lambda_j^m(x)), \tag{6.7}
\]
where

\[ f_i^m(x) = \kappa_m(x) \frac{\prod_{j=1}^{N} \sigma(\lambda_i^m(x + \eta) - \lambda_j^m(x)) \sigma(\lambda_i^m(x + \eta) - \lambda_j^{m+1}(x + \eta))}{\prod_{j=1}^{N} \sigma(\lambda_i^m(x + \eta) - \lambda_j^m(x)) \prod_{j:j \neq i} \sigma(\lambda_i^m(x + \eta) - \lambda_j^{m+1}(x + \eta))}. \quad (6.8) \]

\[ g_i^m(x) = -\kappa_m(x) \frac{\prod_{j=1}^{N} \sigma(\lambda_i^{m+1}(x) - \lambda_j^m(x)) \sigma(\lambda_i^{m+1}(x) - \lambda_j^{m+1}(x + \eta))}{\prod_{j=1}^{N} \sigma(\lambda_i^{m+1}(x) - \lambda_j^m(x + \eta)) \prod_{j:j \neq i} \sigma(\lambda_i^{m+1}(x) - \lambda_j^{m+1}(x))}. \quad (6.9) \]

Note that

\[ \sum_{i=1}^{N} (f_i^m(x) - g_i^m(x)) = 0 \quad (6.10) \]

as the sum of residues of the elliptic function

\[ \varphi(\lambda) = \prod_{j=1}^{N} \sigma(\lambda - \lambda_j^m(x)) \sigma(\lambda - \lambda_j^{m+1}(x + \eta)) \]

\[ \times \prod_{j=1}^{N} \sigma(\lambda - \lambda_j^{m+1}(x + \eta)) \sigma(\lambda - \lambda_j^m(x)). \]

Introduce the matrices \( L^m(x), M^m(x) \):

\[ L_{ij}^m(x, z) = f_i^m(x) \Phi(\lambda_i^m(x + \eta) - \lambda_j^m(x), z), \quad (6.11) \]

\[ M_{ij}^m(x, z) = g_i^m(x) \Phi(\lambda_i^{m+1}(x) - \lambda_j^m(x), z). \quad (6.12) \]

They depend on the spectral parameter \( z \). In terms of these matrices, the equations (6.6), (6.7) read

\[ c_i^m(x + \eta) = \sum_{j=1}^{N} L_{ij}^m(x)c_j^m(x), \quad c_i^{m+1}(x) = \sum_{j=1}^{N} M_{ij}^m(x)c_j^m(x). \quad (6.13) \]

The compatibility condition of the linear problems (6.6), (6.7) has the form of the fully discrete zero curvature equation

\[ R^m(x) := L^{m+1}(x)M^m(x) - M^m(x + \eta)L^m(x) = 0. \quad (6.14) \]

We have:

\[ R_{ij}^m(x) = f_i^{m+1}(x) \sum_{k=1}^{N} g_k^m(x) \Phi(\lambda_i^{m+1}(x + \eta) - \lambda_k^{m+1}(x)) \Phi(\lambda_k^{m+1}(x) - \lambda_i^m(x)) + g_i^m(x + \eta) \sum_{k=1}^{N} f_k^m(x) \Phi(\lambda_i^{m+1}(x + \eta) - \lambda_k^m(x)) \Phi(\lambda_k^m(x + \eta) - \lambda_i^m(x)). \quad (6.15) \]

Cancellation of the leading singularity at \( z = 0 \) leads to the condition

\[ f_i^{m+1}(x) \sum_{k=1}^{N} g_k^m(x) - g_i^m(x + \eta) \sum_{k=1}^{N} f_k^m(x) = 0 \]

\[ \text{JHEP07(2022)023} \]
Taking into account (6.10) we see that it is equivalent to

\[ f_i^{m+1}(x) = g_i^m(x + \eta). \]  

(6.16)

Now we are going to prove that if (6.16) is satisfied, then \( R_{ij}^m(x) = 0 \), so the zero curvature equation is fulfilled. The proof is along the lines of ref. \[45\]. Using the identity (A.6), we rewrite (6.15) as

\[
R_{ij}^m(x) = \Phi(\lambda_i^{m+1}(x + \eta) - \lambda_j^m(x)) f_i^{m+1}(x) \sum_{k=1}^N g_k^m(x)
\times \left( \zeta(\lambda_i^{m+1}(x + \eta) - \lambda_k^{m+1}(x)) + \zeta(\lambda_k^{m+1}(x) - \lambda_j^m(x)) + \zeta(\mu) - \zeta(\lambda_i^{m+1}(x + \eta) - \lambda_j^m(x)) \right)
- \Phi(\lambda_i^{m+1}(x + \eta) - \lambda_j^m(x)) g_i^m(x + \eta) \sum_{k=1}^N f_k^m(x)
\times \left( \zeta(\lambda_i^{m+1}(x + \eta) - \lambda_k^m(x + \eta)) + \zeta(\lambda_k^m(x + \eta) - \lambda_j^m(x)) + \zeta(\mu) - \zeta(\lambda_i^{m+1}(x + \eta) - \lambda_j^m(x)) \right)
\]

Using (6.10) and (6.16), we can represent it in the form

\[ R_{ij}^m(x) = g_i^m(x + \eta) \Phi(\lambda_i^{m+1}(x + \eta) - \lambda_j^m(x)) G_{ij}^m(x), \]  

(6.17)

where

\[
G_{ij}^m(x) = \sum_{k=1}^N \left( g_k^m(x) \zeta(\lambda_i^{m+1}(x + \eta) - \lambda_k^{m+1}(x)) + g_k^m(x) \zeta(\lambda_k^{m+1}(x) - \lambda_j^m(x)) - f_k^m(x) \zeta(\lambda_i^{m+1}(x + \eta) - \lambda_k^m(x + \eta)) - f_k^m(x) \zeta(\lambda_k^m(x + \eta) - \lambda_j^m(x)) \right).
\]

But \( G_{ij}^m(x) \) is the sum of residues of the elliptic function

\[
F(\lambda) = \left( \zeta(\lambda_i^{m+1}(x + \eta) - \lambda) + \zeta(\lambda - \lambda_j^m(x)) \right) \prod_{j=1}^N \frac{\sigma(\lambda - \lambda_j^m(x)) \sigma(\lambda - \lambda_j^{m+1}(x + \eta))}{\sigma(\lambda - \lambda_j^m(x + \eta)) \sigma(\lambda - \lambda_j^{m+1}(x + \eta))}
\]

and, therefore, \( G_{ij}^m(x) = 0 \).

Finally, let us write down equations (6.16) explicitly:

\[
\prod_{j=1}^N \frac{\sigma(\lambda_i^m(x) - \lambda_j^m(x - \eta)) \sigma(\lambda_i^m(x) - \lambda_j^{m+1}(x)) \sigma(\lambda_i^m(x) - \lambda_j^{m-1}(x + \eta)) \sigma(\lambda_i^m(x) - \lambda_j^{m+1}(x + \eta))}{\sigma(\lambda_i^m(x) - \lambda_j^m(x + \eta)) \sigma(\lambda_i^m(x) - \lambda_j^{m-1}(x)) \sigma(\lambda_i^m(x) - \lambda_j^{m+1}(x - \eta)) \sigma(\lambda_i^m(x) - \lambda_j^{m-1}(x - \eta))} = \frac{\kappa_m(x - \eta)}{\kappa_{m-1}(x)}.
\]  

(6.18)

Note that

\[
\frac{\kappa_m(x - \eta)}{\kappa_{m-1}(x)} = \frac{\rho^{m-1}(x) \rho^m(x + \eta) \rho^m(x - \eta)}{\rho^{m-1}(x + \eta) \rho^m(x - \eta) \rho^{m+1}(x)}.
\]  

(6.19)

This is the field analog of the doubly discrete Ruijsenaars-Schneider system [46]. Similar equations were obtained in [47].
7 Conclusion

In this paper we have introduced two integrable models one of which is a natural lattice version of the other. The first one is a finite-dimensional system which we call the Ruijsenaars-Schneider chain. We show that it is gauge equivalent to a special case of the homogenous classical elliptic (XYZ) spin chain when residues of all Lax matrices in the chain are of rank one. The second one is the field analogue of the Ruijsenaars-Schneider model with continuous space and time variables. The definition of this model is the main result of the paper. This is a (1+1)-dimensional model which admits a semi-discrete zero curvature (Zakharov-Shabat) representation for elliptic Lax pair with spectral parameter. This model is obtained through a multi-pole ansatz for general elliptic solutions (elliptic families) of the 2D Toda lattice hierarchy. Then we show that a natural space discretization of this model coincides with the Ruijsenaars-Schneider chain.

The fully discrete version of the model (i.e. discrete in both space and time) is also introduced. It is based on studying elliptic families of solutions to the Hirota bilinear difference equation [44] which is known to provide the integrable discretization of the 2D Toda equation. The corresponding equations of motion for poles of the elliptic solutions are very similar to those obtained in [47] from a general ansatz for elliptic $L$-$M$ pair.

We also discuss a limit of the model which coincides with the field analogue of the Calogero-Moser system introduced in [48] and reproduced in [3] as a dynamical system for poles of elliptic families of solutions to the Kadomtsev-Petviashvili equation. Note that by construction the Ruijsenaars-Schneider chain is gauge equivalent to the elliptic spin chain. A similar gauge equivalence exists between the (1+1)-dimensional Calogero-Moser field theory and the continuous Landau-Lifshitz equation [4, 5]. The exact relation between the obtained (1+1)-dimensional field analogue of the Ruijsenaars-Schneider model and the (semi-discrete) equations of the Landau-Lifshitz type will be discussed elsewhere.

A Elliptic functions

Weierstrass $\sigma$-, $\zeta$- and $\wp$-functions. The $\sigma$-function with quasi-periods $2\omega_1$, $2\omega_2$ such that $\text{Im}(\omega_2/\omega_1) > 0$ is defined by the infinite product

$$\sigma(x) = \sigma(x|\omega_1,\omega_2) = x \prod_{s \neq 0} \left(1 - \frac{x}{s}\right)^{1/4} e^{\frac{s^2}{2} + \frac{x^2}{2s}}, \quad s = 2\omega_1m_1 + 2\omega_2m_2 \quad \text{with integer } m_1, m_2. \quad (A.1)$$

It is connected with the Weierstrass $\zeta$- and $\wp$-functions by the formulas $\zeta(x) = \sigma'(x)/\sigma(x)$, $\wp(x) = -\zeta'(x) = -\partial_x^2 \log \sigma(x)$. We also need the function $\Phi = \Phi(x,z)$ defined as

$$\Phi(x,z) = \frac{\sigma(x+z)}{\sigma(z)\sigma(x)} e^{-\zeta(z)x}. \quad (A.2)$$

It has a simple pole at $x = 0$ with residue 1 and the expansion

$$\Phi(x,z) = \frac{1}{x} - \frac{1}{2} \wp(z)x + \ldots, \quad x \to 0. \quad (A.3)$$
The quasiperiodicity properties of the function $\Phi$ are
\[
\Phi(x + 2\omega_\alpha, z) = e^{2(\zeta(\omega_\alpha)z - \zeta(z)\omega_\alpha)}\Phi(x, z).
\] (A.4)

In the main text we often suppress the second argument of $\Phi$ writing simply $\Phi(x, z) = \Phi(x)$. We will also need the $x$-derivative $\Phi'(x, z) = \partial_x \Phi(x, z)$. The function $\Phi$ satisfies the following identities:
\[
\Phi'(x) = \Phi(x)\left(\zeta(x + z) - \zeta(x) - \zeta(z)\right),
\] (A.5)
\[
\Phi(x)\Phi(y) = \Phi(x + y)\left(\zeta(x) + \zeta(y) + \zeta(z) - \zeta(x + y + z)\right)
\] (A.6)

which are used in the main text.

**Theta-functions.** The theta-functions with characteristics $a,b$ are defined as follows:
\[
\theta \left[ \frac{a}{b} \right] (z | \tau) = \sum_{j \in \mathbb{Z}} \exp \left( 2\pi i (j + a) \frac{2\tau}{2} + 2\pi i (j + a)(z + b) \right).
\] (A.7)

In our paper, we consider the case of rational characteristics $a,b \in \frac{1}{N} \mathbb{Z}$. In particular, the odd theta function used in the paper ($\theta_1(z)$ in the Jacobi notation) is
\[
\vartheta(z) = \theta \left[ \frac{1}{2} \right] (z | \tau).
\] (A.8)

The following quasi-periodicity properties hold. For $a, b, a' \in (1/N)\mathbb{Z}$
\[
\theta \left[ \frac{a}{b} \right] (z + 1 | \tau) = e(a) \theta \left[ \frac{a}{b} \right] (z | \tau),
\] (A.9)
\[
\theta \left[ \frac{a + 1}{b} \right] (z | \tau) = \theta \left[ \frac{a}{b} \right] (z | \tau),
\] (A.10)
\[
\theta \left[ \frac{a}{b} \right] (z + a' | \tau) = e \left( -a'^2 \frac{\tau}{2} - a'(z + b) \right) \theta \left[ \frac{a + a'}{b} \right] (z | \tau),
\] (A.11)

where we denote
\[
e(x) := \exp(2\pi i x)
\] (A.12)

for brevity.

**The Kronecker function and the function $E_1$.** We also use the following set of $N^2$ functions:
\[
\varphi_a(z, \omega_a + \eta) = e(a_2z/N) \phi(z, \omega_a + \eta), \quad \omega_a = \frac{a_1 + a_2 \tau}{N},
\] (A.13)
where $a = (a_1, a_2) \in \mathbb{Z}_N \times \mathbb{Z}_N$ and
\[
\phi(z, u) = \frac{\vartheta'(0)\vartheta(z + u)}{\vartheta(z)\vartheta(u)}
\] (A.14)
is the Kronecker function. It has a simple pole at $z = 0$ with residue 1:

\[
\text{Res}_{z=0}\phi(z, u) = 1.
\] (A.15)

The quasi-periodicity properties are as follows:

\[
\phi(z + 1, u) = \phi(z, u), \quad \phi(z + \tau, u) = e(-u)\phi(z, u).
\] (A.16)

The expansion of $\phi(z, u)$ near $z = 0$ has the form

\[
\phi(z, u) = \frac{1}{z} + E_1(u) + \frac{E_1^2(u) - \varphi(u)}{2} + O(z^2),
\] (A.17)

where

\[
E_1(u) = \frac{\vartheta'(u)}{\vartheta(u)}.
\] (A.18)

It follows from the definition (A.14) that

\[
\partial_z \phi(z, u) = (E_1(z + u) - E_1(z))\phi(z, u),
\]

\[
\partial_u \phi(z, u) = (E_1(z + u) - E_1(u))\phi(z, u).
\] (A.19)

We also use a set of widely known addition formulae:

\[
\phi(z, u_1)\phi(z, u_2) = \phi(z, u_1 + u_2)\left(E_1(z) + E_1(u_1) + E_1(u_2) - E_1(z + u_1 + u_2)\right),
\] (A.20)

\[
\phi(z_1, u_1)\phi(z_2, u_2) = \phi(z_1, u_1 + u_2)\phi(z_2 - z_1, u_2) + \phi(z_2, u_1 + u_2)\phi(z_1 - z_2, u_1)
\] (A.21)

and

\[
\phi(z, u_1 - v)\phi(w, u_2 + v)\phi(z - w, v) - \phi(z, u_2 + v)\phi(w, u_1 - v)\phi(z - w, u_1 - u_1 - v)
\]

\[
= \phi(z, u_1)\phi(w, u_2)\left(E_1(v) - E_1(u_1 - u_2 - v) + E_1(u_1 - v) - E_1(u_2 + v)\right).
\] (A.22)

**Relation to the Weierstrass functions.** The above definitions of the Weierstrass functions (A.1)–(A.3) are easily related to those given in terms of theta-functions (A.14)–(A.17) if we choose the periods to be $2\omega_1 = 1$, $2\omega_2 = \tau$:

\[
\zeta(z) = E_1(z) + 2\eta_0 z,
\]

\[
\eta_0 = -\frac{1}{6} \frac{\vartheta''''(0)}{\vartheta'(0)},
\] (A.23)

\[
\sigma(z) = \frac{\vartheta(z)}{\vartheta'(0)} e^{\eta_0 z^2},
\] (A.24)

\[
\Phi(z, u) = \phi(z, u) e^{-zE_1(u)}.
\] (A.25)

Under the substitution (A.25) the identities (A.5)–(A.6) are transformed into (A.19) and (A.20) respectively. The Weierstrass $\wp$-function appearing in (A.17) is

\[
\wp(u) = -\partial_u^2 \log \vartheta(u) + \frac{1}{3} \frac{\vartheta''''(0)}{\vartheta'(0)}.
\] (A.26)
Some relations for theta-functions with rational characteristics. Using definition (A.7), one can rewrite the set of functions (A.13) in a slightly different form. Set

$$\theta_\alpha(z, \tau) = \theta \left[ \frac{\alpha_1 + 1/2}{N} \right] (z, \tau), \quad \alpha \in \mathbb{Z}_N \times \mathbb{Z}_N$$  \hspace{1cm} (A.27)

Then for any \( \alpha \) we have

$$\frac{\theta_\alpha(z + \eta, \tau)}{\theta_\alpha(\eta, \tau)} = e(\alpha_2 z/N) \frac{\vartheta(z + \eta + \omega_\alpha)}{\vartheta(\eta + \omega_\alpha)}$$  \hspace{1cm} (A.28)

and, therefore,

$$\varphi_\alpha(z, \eta + \omega_\alpha) = \vartheta'(0) \frac{\theta_\alpha(z + \eta, \tau)}{\theta_\alpha(\eta, \tau)}.$$  \hspace{1cm} (A.29)

Introduce also

$$\theta(j) = \theta \left[ \frac{1}{2} - \frac{j}{N} \right] (z, N\tau), \quad j \in \mathbb{Z}_N.$$  \hspace{1cm} (A.30)

Then for \(-\vartheta(z)\) we have

$$\theta \left[ \frac{1}{2} - \frac{j}{N} \right] (z, \tau) = C(\tau) \prod_{j=0}^{N-1} \theta(j)(z), \quad C(\tau) = \frac{\vartheta'(0, \tau) \frac{1}{N}}{\prod_{j=1}^{N-1} \theta(j)(0)},$$  \hspace{1cm} (A.31)

so that the following relation holds:

$$\vartheta'(0, \tau) \prod_{j=0}^{N-1} \theta(j)(z) \frac{1}{\prod_{j=1}^{N-1} \theta(j)(0)} = -\vartheta'(0, N\tau).$$  \hspace{1cm} (A.32)

Consider the matrix

$$X_{ij}(x_j) = \theta \left[ \frac{1}{2} - \frac{j}{N} \right] (N x_j | N\tau).$$  \hspace{1cm} (A.33)

Then the following determinant of the Vandermonde type formula holds [18]:

$$\det X = C_N(\tau) \vartheta \left( \sum_{k=1}^{N} x_k \prod_{i<j} \vartheta(x_j - x_i), \quad C_N(\tau) = \frac{(-1)^N}{\eta(\tau)^{(N-1)(N-2)/2}};$$  \hspace{1cm} (A.34)

where \( \eta(\tau) \) is the Dedekind eta-function:

$$\eta(\tau) = e^{\frac{\pi i \tau}{12}} \prod_{k=1}^{\infty} \left( 1 - e^{2\pi i k} \right) = \left( \frac{\vartheta'(0)}{2\pi} \right)^{1/3}.$$  \hspace{1cm} (A.35)

Finite Fourier transformation on \( \mathbb{Z}_N \). For any \( m \in \mathbb{Z} \) and \( N \in \mathbb{Z}_+ \)

$$e^{2\pi i m \eta} \phi(N\eta, z + m\tau | N\tau) = \frac{1}{N} \sum_{k=0}^{N-1} e^{-2\pi i m \frac{k}{N}} \phi \left( z, \eta + \frac{k}{N} \tau \right),$$  \hspace{1cm} (A.36)

$$\phi(z, \eta | \tau) = \sum_{k=0}^{N-1} e^{2\pi i z k} \phi(Nz, \eta + k\tau | N\tau).$$  \hspace{1cm} (A.37)
B Elliptic $R$-matrix and its properties

Matrix basis. Consider the pair of $N \times N$ matrices

$$(Q_1)_{kl} = \delta_{kl} \exp \left( \frac{2\pi i}{N} k \right), \quad (Q_2)_{kl} = \delta_{k-l+1=0 \mod N}. \quad (B.1)$$

They satisfy the properties

$$Q_2^{a_2} Q_1^{a_1} = \exp \left( \frac{2\pi i}{N} a_1 a_2 \right) Q_1^{a_1} Q_2^{a_2}, \quad a_{1,2} \in \mathbb{Z}; \quad Q_1^N = Q_2^N = 1_{N \times N}, \quad (B.2)$$

so that these matrices represent the generators of the Heisenberg group. Let us construct a special basis in $\text{Mat}(N, \mathbb{C})$ in terms of (B.1) in the following way:

$$T_a = T_{a_1 a_2} = \exp \left( \frac{\pi i}{N} a_1 a_2 \right) Q_1^{a_1} Q_2^{a_2}, \quad a = (a_1, a_2) \in \mathbb{Z}_N \times \mathbb{Z}_N. \quad (B.3)$$

In particular, $T_0 = T_{(0,0)} = 1_N$. For the product we have

$$T_a T_\beta = \kappa_{\alpha,\beta} T_{\alpha + \beta}, \quad \kappa_{\alpha,\beta} = \exp \left( \frac{\pi i}{N} (\beta_1 a_2 - \beta_2 a_1) \right), \quad \alpha + \beta = (\alpha_1 + \beta_1, \alpha_2 + \beta_2) \quad (B.4)$$

Also

$$\text{tr}(T_a T_\beta) = N \delta_{\alpha+\beta,(0,0)}. \quad (B.5)$$

Let us perform the transformation relating the standard matrix basis $E_{ij}$ in $\text{Mat}(N, \mathbb{C})$, given by $(E_{ij})_{kl} = \delta_{ik} \delta_{jl}$, with (B.3). For the pair of matrices $Q_{1,2}$ (B.1) and integer numbers $a_1, a_2$ we have

$$Q_1^{a_1} = \sum_{k=1}^{N} E_{kk} e \left( \frac{ka_1}{N} \right), \quad Q_2^{a_2} = \sum_{k=1}^{N} E_{k-a_2,k}, \quad (B.6)$$

where in the last sum we assume the value of index $k - a_2$ modulo $N$. Then for the basis matrix $T_a$ (B.3) one gets

$$T_a = e \left( -\frac{a_1 a_2}{2N} \right) \sum_{k=1}^{N} E_{k-a_2,k} e \left( \frac{ka_1}{N} \right). \quad (B.7)$$

For an arbitrary matrix $B = \sum_{i,j=1}^{N} E_{ij} B_{ij} \in \text{Mat}(N, \mathbb{C})$ its components $B_a = B_{(a_1, a_2)}$ in the basis $T_a$ can be found using (B.5) and (B.7):

$$B_a = \frac{1}{N} \text{tr}(B T_{-a}) = \frac{1}{N} e \left( -\frac{a_1 a_2}{2N} \right) \sum_{k=1}^{N} B_{k,k-a_2} e \left( -\frac{a_1 k}{N} \right). \quad (B.8)$$

Similarly, given a set of components $B_{(a_1, a_2)}$, $a_{1,2} \in \mathbb{Z}_N$ for a matrix $B \in \text{Mat}(N, \mathbb{C})$ in the basis $\{T_a\}$ we have the following expression for its components $B_{ij}$ in the standard basis:

$$B_{ij} = \begin{cases} \sum_{a_1=0}^{N-1} B_{(a_1, j-i)} e \left( \frac{a_1(j+i)}{2N} \right), & j \geq i, \\ \sum_{a_1=0}^{N-1} B_{(a_1, j-i+N)} e \left( \frac{a_1(j+i-N)}{2N} \right), & j < i. \end{cases} \quad (B.9)$$
The Baxter-Belavin $R$-matrix. We use the Baxter-Belavin elliptic quantum $R$-matrix in the following form:

$$R_{12}^h(z) = \frac{1}{N} \sum_{a \in \mathbb{Z}_N \times \mathbb{Z}_N} T_a \otimes T_{-a} \varphi_a(z, \omega_a + h) \in \text{Mat}(N, \mathbb{C})^{\otimes 2}.$$ \hfill (B.10)

It is equivalently written in the standard basis as follows [40]:

$$R_{12}^h(z) = \sum_{i,j,k,l=1}^N R_{ij,kl} E_{ij} \otimes E_{kl},$$ \hfill (B.11)

$$R_{ij,kl} = -\vartheta'(0, N\tau) \frac{\theta^{(i-k)}(z + Nh)}{\theta^{(j-k)}(z) \theta^{(i-j)}(Nh)} \delta_{i+k=j+l \mod N}.$$ \hfill (B.12)

It is also convenient to write it in terms of the Kronecker function (A.14). For this purpose we need the identity

$$-\vartheta'(0, N\tau) \frac{\theta^{(a+b)}(z + u)}{\theta^{(a)}(z) \theta^{(b)}(u)} = e\left(\frac{abr - au - bz}{N}\right) \phi(z - a\tau, u - br | N\tau)$$

$$= e\left(-\frac{u}{N}\right) \varphi_{(0, -\frac{b}{N})}(z - a\tau, u - br | N\tau).$$ \hfill (B.13)

Then

$$R_{ij,kl} = \delta_{i+k=j+l \mod N} e\left(\frac{(k-j)(j-i)\tau + (k-j)Nh + (j-i)z}{N}\right) \phi(z + (k-j)\tau, Nh + (j-i)\tau | N\tau).$$ \hfill (B.14)

Equivalence between different representations can be shown by using the relation between the bases $E_{ij}$, $T_a$ and the Fourier formulae (A.36)–(A.37).

IRF-Vertex correspondence. The matrix (2.37) participates in the IRF-Vertex relation

$$g_2(z_2, q) g_1(z_1, q + Nh^{(2)}) R_{12}^h(h, z_1 - z_2| q) = R_{12}^h(h, z_1 - z_2) g_1(z_1, q) g_2(z_2, q + Nh^{(1)})$$ \hfill (B.15)

between the (vertex type) Baxter-Belavin $R$-matrix (2.20) and the (IRF-type) Felder’s dynamical $R$-matrix [49]:

$$R_{12}^h(h, z_1 - z_2| q) = \sum_{i,j;i\neq j}^N E_{ii} \otimes E_{jj} \phi(Nh, -q_{ij})$$

$$+ \sum_{i,j;i\neq j}^N E_{ij} \otimes E_{ji} \phi(z_1 - z_2, q_{ij}) + \phi(Nh, z_1 - z_2) \sum_{i=1}^N E_{ii} \otimes E_{ii}.$$ \hfill (B.16)

The shift of argument $g_1(z_1, q + Nh^{(2)})$ in (B.15) is understood as

$$g_1(z_1, q + Nh^{(2)}) = P_2^{Nh} g_1(z_1, q) P_2^{-Nh}, \quad P_2^h = \sum_{k=1}^N 1_{N \times N} \otimes E_{kk} \exp\left(\frac{h}{\theta} \frac{\partial}{\partial \theta_{kk}}\right).$$ \hfill (B.17)
Properties of the intertwining matrix. The matrix $g(z,q)$ is degenerated at $z = 0$ due to (A.34):

$$\det \Xi(z,q) = C_N(\tau) \vartheta(z) \prod_{i<j} \vartheta(q_i - q_j),$$

and the factor $\vartheta(z)$ comes from the fact that the sum of coordinates (in the center of masses frame) equals zero.

The matrix $g(z)$ (2.37) satisfies the following properties (see [23] for a review):

1. The matrix $g(z,q)$ is degenerated at $z = 0$ (B.18).

2. The matrix $g(0,q)$ has one-dimensional kernel generated by the vector-column $\rho$:

$$g(0,q)\rho = 0, \quad \rho = (1, 1, \ldots, 1)^T \in \mathbb{C}^N.$$

Properties of this type were described in [40]. Their proof can be also found in [4].

Let us consider $g^{-1}(z,q)$ near $z = 0$:

$$g^{-1}(z,q) = \frac{1}{z} g(0,q) + A(q) + O(z), \quad \tilde{g}(0,q) = \text{Res}_{z=0} g^{-1}(z,q).$$

Then the matrix $\tilde{g}(0)$ is of rank one\(^7\)

$$\tilde{g}(0) = \rho \otimes v, \quad v = \frac{1}{N} \rho^T \tilde{g}(0,q) \in \mathbb{C}^N.$$ 

Below we derive an explicit expression for the inverse of the matrix $g(z,q)$.

IRF-Vertex correspondence for semidynamical $R$-matrix. In [50] the following (semidynamical) $R$-matrix was used for quantization of the Ruijsenaars-Schneider model:

$$R^\text{ACF}_{12}(h, z_1, z_2 | q) = \sum_{i,j: i \neq j} E_{ii} \otimes E_{jj} \phi(Nh, -q_{ij}) + \sum_{i\neq j} E_{ij} \otimes E_{ji} \phi(z_1 - z_2, -q_{ij})$$

$$- \sum_{i, j: i \neq j} E_{ij} \otimes E_{jj} \phi(z_1 + Nh, -q_{ij}) + \sum_{i, j: i \neq j} E_{jj} \otimes E_{ij} \phi(z_2, -q_{ij})$$

$$+ \left( E_1(Nh) + E_1(z_1 - z_2) + E_1(z_2) - E_1(z_1 + Nh) \right) \sum_{i=1}^N E_{ii} \otimes E_{ii},$$

where $E_1$ is defined in (A.17). This $R$-matrix satisfies the quantum Yang-Baxter equation with shifted spectral parameters. Following [51] let us write down its relation to the Baxter-Belavin $R$-matrix (2.20) in the form of type (B.15):

$$R^h_{12}(z_1 - z_2) = g_1(z_1 + Nh, q) g_2(z_2, q) R^\text{ACF}_{12}(h, z_1, z_2 | q) g_2^{-1}(z_2 + Nh, q) g_1^{-1}(z_1, q).$$

\(^7\)Locally, in some basis $g(z,q)$ is represented in the form $\text{diag}(z, 1, \ldots, 1)$. Therefore, $\tilde{g}(0,q)$ has $N-1$ zero eigenvalues.
Multiplying both sides by \( g_2^{-1}(z_2, q) \) and evaluating residue at \( z_2 = 0 \), we get the following useful formula:\(^8\)
\[
\tilde{g}_2(0, q) R_{12}^b(z) = g_1(z + N\hbar, q) \mathcal{O}_{12} \tilde{g}_2^{-1}(N\hbar, q) g_1^{-1}(z, q), \tag{B.24}
\]
where \( \tilde{g}(0) \) is given by (B.20) and
\[
\mathcal{O}_{12} = \sum_{i,j=1}^N E_{ii} \otimes E_{ji}. \tag{B.25}
\]
For an arbitrary matrix \( T = \sum_{i,j} E_{ij} T_{ij} \in \text{Mat}(N, \mathbb{C}) \) we have
\[
\text{tr}_2 (\mathcal{O}_{12} T_2) = \sum_{i=1}^N E_{ii} \sum_{j=1}^N T_{ij}. \tag{B.26}
\]
Besides (B.24), we use its degeneration (see the classical limit (2.22) below) \( \hbar \to 0: \(^9\)
\[
\tilde{g}_2(0, q) r_{12}(z) = g'_1(z) \mathcal{O}_{12} \tilde{g}_2(0) g_1^{-1}(z) + g_1(z) \mathcal{O}_{12} A_2 g_1^{-1}(z), \tag{B.27}
\]
where \( A \) comes from the expansion (B.20) and \( g'(z) \) is the derivative of \( g(z) \) with respect to \( z \).

\section*{C Explicit change of variables}

\textbf{Change of variables.} Here we show how to obtain (2.48) using the factorization formula (2.40) for the Ruijsenaars-Schneider Lax matrix (2.1). Let us compute the \( a = (a_1, a_2) \)-component of the Lax matrix (2.42)
\[
\mathcal{L}_{ij}^a(z) = \frac{\partial^2 (0)}{\partial \eta} \sum_{m=1}^N \Xi_{im}(z + N\eta, q) e^{p_m/c} \Xi_{mj}^{-1}(z, q). \tag{C.1}
\]
Plugging it into (B.8), we get
\[
\mathcal{L}^a_{\alpha}(z) = \frac{1}{N} e \left( -\frac{a_1 a_2}{2N} \right) \frac{\partial^2 (0)}{\partial \eta} \sum_{k,m=1}^N \Xi_{km}(z + N\eta, q) e^{p_m/c} \Xi_{m,k+a_2}^{-1}(z, q) e \left( -\frac{a_1 k}{N} \right). \tag{C.2}
\]
From (2.18) we know that \( \mathcal{L}^a_{\alpha}(z) = S_a \varphi_{\alpha}(z, \omega_a + \eta) \). Therefore, we could find \( S_a \) from \( \mathcal{L}^a_{\alpha}(z) \), which we are going to compute. Let us represent (C.2) in the form
\[
\mathcal{L}^a_{\alpha}(z) = \sum_{m=1}^N e^{p_m/c} \mathcal{L}^a_{\alpha,m}(z), \tag{C.3}
\]
\[
\mathcal{L}^a_{2,m}(z) = \frac{1}{N} e \left( -\frac{a_1 a_2}{2N} \right) \frac{\partial^2 (0)}{\partial \eta} \sum_{k=1}^N \Xi_{km}(z + N\eta, q) \Xi_{m,k+a_2}^{-1}(z, q) e \left( -\frac{a_1 k}{N} \right). \tag{C.4}
\]
\(^8\)Note that in the \( N = 1 \) case (B.24) boils down to the definition (A.14) of the Kronecker function. A similarity of the quantum \( R \)-matrix (2.20) with the Kronecker function underlies the so-called associative Yang-Baxter equation. See [51] and references therein.

\(^9\)Relation (B.27) appears in the \( \hbar^2 \) order, while in \( \hbar^{-1} \) order one has \( \tilde{g}_2(0) = g_1(z) \mathcal{O}_{12} g_1^{-1}(z) \tilde{g}_2(0) \), which is true due to the property (B.21).
Our aim now is to evaluate the latter expression. For this purpose we need the properties (A.9)–(A.11). Using explicit form of the matrix Ξ (2.38), it easy to see from (A.9) that

\[
Ξ_{km}(z + Nη + a_1, q) = (-1)^{a_1}e\left(-\frac{a_1k}{N}\right)Ξ_{km}(z + Nη, q).
\]

Therefore,

\[
\mathcal{L}^\eta_{a,m}(z) = \frac{1}{N}e\left(-\frac{a_1a_2}{2N}\right)(-1)^{a_1}\frac{\vartheta'(0)}{\vartheta(\eta)}\sum_{k=1}^{N}Ξ_{km}(z + Nη + a_1, q)Ξ^{-1}_{m,k+a_2}(z, q).
\]

Next, add and subtract \(a_2τ\) to the argument of \(Ξ_{km}(z + Nη + a_1, q)\). Then using (A.11) with \(a' = -a_2/N\) we obtain

\[
Ξ_{km}(z + Nη + a_1, q) = \vartheta\left[\frac{1}{2} - \frac{k}{N}\right]\left(z - N\bar{q}_m + Nη + a_1 + a_2τ - \frac{a_2}{N}Nτ \mid Nτ\right)
\]

\[
= e\left(-\frac{a_2^2}{2N}τ + \frac{a_2}{N}\right)(z - N\bar{q}_m + Nη + a_1 + a_2τ + \frac{N}{2}))Ξ_{k+a_2,m}(z + N(η + ω_a), q),
\]

where the notation \(ω_a\) (A.13) is used. Plugging it into (C.6), we arrive at

\[
\mathcal{L}^\eta_{a;m}(z) = \frac{1}{N}e\left(\frac{a_1a_2}{2N} + \frac{a_2^2}{2N}\right)(-1)^{a_1+a_2}e(2\eta - \bar{q}_m)e\left(z\frac{a_2}{N}\right)
\]

\[
\times \frac{\vartheta'(0)}{\vartheta(\eta)}\sum_{k=1}^{N}Ξ^{-1}_{m,k+a_2}(z, q)Ξ_{k+a_2,m}(z + N(η + ω_a), q).
\]

Finally, we use (2.40) for the Ruijsenaars-Schneider Lax matrix (2.1)–(2.2). Namely, we need the \(i = j = m\) diagonal element of (2.1) with \(η\) being replaced by \(η + ω_a\) (except for the common factor \(\vartheta'(0)/\vartheta(\eta)\)). This yields

\[
\mathcal{L}^\eta_{a,m}(z) = \frac{1}{N}e\left(\frac{a_2}{2}\right)(-1)^{a_1+a_2}
\]

\[
\times e(2\eta - \bar{q}_m)\varphi_a(z, ω_a + η)\frac{\vartheta(\eta + ω_a)}{\vartheta(\eta)}\prod_{l\neq m}^{N} \frac{\vartheta(q_m - q_l - η - ω_a)}{\vartheta(q_m - q_l)}.
\]

Returning back to (C.3) and canceling \(φ_a(z, ω_a + η)\), we find the final answer

\[
S_a = \frac{(-1)^{a_1+a_2}}{N}e\left(\frac{a_2}{2}\right)\prod_{m=1}^{N}e^{p_m/\epsilon}e(2\eta - \bar{q}_m)\frac{\vartheta(\eta + ω_a)}{\vartheta(\eta)}\prod_{l\neq m}^{N} \frac{\vartheta(q_m - q_l - η - ω_a)}{\vartheta(q_m - q_l)}.
\]

\[
\text{Inverse of the matrix } Ξ(z, q). \quad \text{Consider the set of matrices with components (C.9) in the basis } T_a:
\]

\[
\mathcal{L}^\eta_{in}(z) = \sum_a \mathcal{L}^\eta_{a,m}(z)T_a ∈ \text{Mat}(N, C), \quad m = 1, \ldots, N.
\]

It follows from its initial definition (C.1), (C.3) that the matrix elements in the standard basis are of the form:

\[
\mathcal{L}^\eta_{ij;m}(z) = \frac{\vartheta'(0)}{\vartheta(\eta)}Ξ_{im}(z + Nη, q)Ξ^{-1}_{mj}(z, q).
\]
Therefore,
\[
\Xi_{mj}^{-1}(z, q) = \frac{\theta(\eta)}{\theta'(0)} \frac{L_{ij,m}^{\eta}(z)}{\Xi_{im}(z + N\eta, q)},
\]  
(C.13)
To get an explicit expression, we need to compute the matrices \(L_{ij,m}^{\eta}(z)\), \(m = 1, \ldots, N\) in the standard basis. For this purpose substitute (C.9) into (B.9) with \(B_a = L_{a,m}^{\eta}(z)\). Both cases in the r.h.s. of (B.9) provide the same answer (the latter is verified directly using the transformation properties (A.9)–(A.11) for the theta-function (A.8)):
\[
\frac{\theta(\eta)}{\theta'(0)} \frac{L_{ij,m}^{\eta}(z)}{\Xi_{im}(z + N\eta, q)} = \frac{1}{N} \sum_{a_1=0}^{N-1} e \left( \frac{a_1}{2N} (i + j) \right) e \left( \frac{j - i}{2} \omega(a_1, j-i) \right) (-1)^{a_1+j-i} \times e \left( (j-i)(\eta - \bar{q}_m) \right) e \left( z \frac{j-i}{N} \frac{\theta(z + \eta + \omega(a_1, j-i))}{\theta(z)} \prod_{l: l \neq m} \frac{\theta(q_m - q_l - \eta - \omega(a_1, j-i))}{\theta(q_m - q_l)} \right),
\]  
(C.14)
where \(\omega(a_1, j-i) = \frac{a_1+(j-i)\tau}{N}\). Dividing this expression by \(\Xi_{im}(z + N\eta, q)\) we obtain \(\Xi_{mj}^{-1}(z, q)\) (C.13). Notice that by construction the r.h.s. of (C.13) is independent of \(\eta\), so we put \(\eta = 0\) in the final answer since all entering functions are regular in \(\eta\). Also, the r.h.s. of (C.13) is independent of index \(i\). We fix it as \(i = N\). Finally, using \(\Xi_{Nm}(z + N\eta, q) = -\theta(z + \frac{N-1}{2} + N\eta - N\bar{q}_m|N\tau)\) we obtain
\[
\Xi_{ij}^{-1}(z, q) = \frac{(-1)^{j+1}}{N \theta \left( z + \frac{N-1}{2} - N\bar{q}_i|N\tau \right)} \times \sum_{a_1=0}^{N-1} e \left( \frac{a_1}{2N} j + \frac{a_1 + j\tau}{2N} \right) e(-j\bar{q}_i) e \left( z \frac{j}{N} \frac{\theta(z + \frac{a_1 + j\tau}{N})}{\theta(z)} \prod_{l: l \neq i} \frac{\theta(q_i - q_l - \frac{a_1 + j\tau}{N})}{\theta(q_i - q_l)} \right).
\]  
(C.15)
Equivalently, for the matrix \(g(z, q)\) (2.37) we have
\[
g_{ij}^{-1}(z, q) = \frac{(-1)^{j+1}}{N \theta \left( z + \frac{N-1}{2} - N\bar{q}_i|N\tau \right)} \sum_{a_1=0}^{N-1} e \left( \frac{a_1}{2N} j + \frac{a_1 + j\tau}{2N} \right) e(-j\bar{q}_i) e \left( z \frac{j}{N} \frac{\theta(z + \frac{a_1 + j\tau}{N})}{\theta(z)} \prod_{l: l \neq i} \frac{\theta(q_i - q_l - \frac{a_1 + j\tau}{N})}{\theta(q_i - q_l)} \right),
\]  
(C.16)
Derivation of \(S = \xi \otimes \psi\). The matrix \(\tilde{g}(0, q)\) (B.20) is easily calculated from (C.16):
\[
\tilde{g}_{ij}(0, q) = \frac{(-1)^j}{N \theta \left( \frac{N-1}{2} - N\bar{q}_i|N\tau \right)} \times e \left( \frac{j\tau}{2N} \right) e(-j\bar{q}_i) \frac{1}{\theta'(0)} \sum_{a_1=0}^{N-1} e \left( \frac{a_1}{N} j \right) \prod_{l=1}^{N} \frac{\theta(q_i - q_l - \frac{a_1 + j\tau}{N})}{\theta(q_i - q_l)}.
\]  
(C.17)
Note that due to (B.21) the r.h.s. of (C.17) is independent of the index \(i\), so that the functions
\[
f_j(x, q) = \frac{e(-jx)}{\theta \left( \frac{N-1}{2} - Nx|N\tau \right)} \sum_{a_1=0}^{N-1} e \left( \frac{a_1}{N} j \right) \prod_{l=1}^{N} \frac{\theta(x - q_l - \frac{a_1 + j\tau}{N})}{\theta(x - q_l)}.
\]  
(C.18)
obey the property
\[ f_j(q) = f_j(q_i, q) = f_j(q_k, q) \quad \text{for all } i, j, k. \] (C.19)

In this notation
\[ \tilde{g}_{ij}(0, q) = \frac{(-1)^j}{N\vartheta'(0)} e\left(\frac{j^2}{2N}\right) f_j(q). \] (C.20)

Finally, using (2.43)–(2.44), we find the change of variables (C.10) in the form
\[ S = \xi \otimes \psi \]
with
\[ \xi_i = \frac{\vartheta''(0)}{\vartheta'(\eta)} \sum_{k=1}^N g_{ik}(N\eta) e^{\frac{p_k}{c}}, \quad \psi_j = \frac{(-1)^j}{N\vartheta'(0)} e\left(\frac{j^2}{2N}\right) f_j(q). \] (C.21)

The normalization can be chosen in a different way since (C.21) is defined up to \( \xi_i \rightarrow \lambda \xi_i, \psi_j \rightarrow \psi_j/\lambda \). Let us also mention that in the rational case \( \psi_j \) are elementary symmetric functions of coordinates (see [12]), so that (C.21) provides its elliptic analogue.

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