UNIMODULAR ROWS OVER AFFINE ALGEBRAS OVER ALGEBRAIC CLOSURE OF A FINE FIELD

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ABSTRACT. In this article, we prove that if $R$ is an affine algebra of dimension $d \geq 4$ over $\mathbb{F}_p$ and $1/(d-1)! \in R$, then any unimodular row over $R$ of length $d$ can be mapped to a factorial row by elementary transformations.

1. INTRODUCTION

All the rings are assumed to be commutative noetherian with unity 1.

In [15], R.G. Swan and J. Towber showed that if $(a^2, b, c) \in Um_3(R)$ then it can be completed to an invertible matrix over $R$. This result of Swan and Towber was generalised by Suslin in [13] who showed that if $(a_d^0, a_1, \ldots, a_r) \in Um_{r+1}(R)$ then it can be completed to an invertible matrix. Rao also studied the unimodular rows over $R[X]$ where $R$ is a local ring in [7], [8]. In [10], author studied the unimodular rows over finitely generated rings over $\mathbb{Z}$ and proved that if $R$ is a finitely generated ring over $\mathbb{Z}$ of dimension $d, d \geq 2, \frac{1}{d!} \in R$, then any unimodular row over $R[X]$ of length $d + 1$ can be mapped to a factorial row by elementary transformations.

In [3, Theorem 7.5], Fasel, Rao and Swan proved that if $R$ is a normal affine algebra of dimension $d \geq 4$ over $\mathbb{F}_p$ and $1/(d-1)! \in R$, then any unimodular row over $R$ of length $d$ can be mapped to a factorial row by elementary transformations. In this article we remove their normality condition. In particular, we prove:

Theorem 1.1. Let $R$ be an affine algebra of dimension $d \geq 4$ over $\mathbb{F}_p$ and $1/(d-1)! \in R$. Let $v = (v_1, \ldots, v_d) \in Um_d(R)$. Then there exists $\varepsilon \in E_d(R)$ such that

$$(v_1, \ldots, v_d)\varepsilon = (w_1, w_2, \ldots, w_d)^{(d-1)!}$$

for some $(w_1, \ldots, w_d) \in Um_d(R)$. In particular, $Um_d(R) = e_1SL_d(R)$.

In this article, we consider the action of symplectic matrices on unimodular rows and prove:

Theorem 1.2. Let $R$ be an affine algebra of dimension $d$ over $\mathbb{F}_p$ and $d \equiv 2 \pmod{4}$. Let $v \in Um_d(R)$. Assume that $SL_{d+1}(R) \cap E(R) = E_{d+1}(R)$. Then $v$ is completable to a symplectic matrix.

We study the problem of injective stability for affine algebras over $\mathbb{F}_p$ and prove:
Theorem 1.3. Let $R$ be an affine algebra of dimension $d \geq 4$ over $\overline{\mathbb{F}}_p$. Let $\alpha \in \text{SL}_d(R) \cap \text{E}_{d+1}(R)$. Then $\alpha$ is isotopic to the identity matrix.

2. Preliminaries

In this section, we give some basic definitions and collect some known results which will be used later in the article.

2.1. Basic definitions and notations.

Definition 2.1. A row $v = (a_0, \ldots, a_r) \in R^{r+1}$ is said to be unimodular if there is a $w = (b_0, \ldots, b_r) \in R^{r+1}$ with $\langle v, w \rangle = \Sigma_{i=0}^r a_i b_i = 1$ and $Um_{r+1}(R)$ will denote the set of unimodular rows (over $R$) of length $r + 1$.

The elementary linear group $E_{r+1}(R)$ acts on the rows of length $r + 1$ by right multiplication. Moreover, this action takes unimodular rows to unimodular rows: $\frac{Um_{r+1}(R)}{E_{r+1}(R)}$ will denote the set of orbits of this action; and we shall denote by $[v]$ the equivalence class of a row $v$ under this equivalence relation.

Definition 2.2. A unimodular row $v \in Um_n(R)$ is said to be completable if there exists a matrix $\alpha \in GL_n(R)$ such that $v = e_1 \alpha$.

Notation 2.3. Let $\psi_1 = \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix}$, $\psi_n = \psi_{n-1} \perp \psi_1$ for $n > 1$.

Notation 2.4. Let $\sigma$ be the permutation of the natural numbers given by $\sigma(2i) = 2i - 1$ and $\sigma(2i - 1) = 2i$.

Definition 2.5. $Sp_{2m}(R)$: the group of all $2m \times 2m$ matrices $\{ \alpha \in GL_{2m}(R) \mid \alpha^t \psi_m \alpha = \psi_m \}$.

Definition 2.6. $ESp_{2m}(R)$: We define for $1 \leq i \neq j \leq 2m$, $z \in R$,

$$se_{ij}(z) = \begin{cases} I_{2m} + zE_{ij}, & \text{if } i = \sigma(j); \\ I_{2m} + zE_{ij} - (-1)^{i+j}zE_{\sigma(j)\sigma(i)}, & \text{if } i \neq \sigma(j). \end{cases}$$

It is easy to verify that all these matrices belong to $Sp_{2m}(R)$. We call them the elementary symplectic matrices over $R$. The subgroup generated by them is called the elementary symplectic group and is denoted by $ESp_{2m}(R)$.

Definition 2.7. Let $I$ be an ideal of a ring $R$. A unimodular row $v \in Um_n(R)$ which is congruent to $e_1$ modulo $I$ is called unimodular relative to ideal $I$. Set of unimodular rows relative to ideal $I$ will be denoted by $Um_n(R, I)$.

Let $I$ be an ideal of a ring $R$. We shall denote by $GL_n(R, I)$ the kernel of the canonical mapping $GL_n(R) \longrightarrow GL_n(\frac{R}{I})$. Let $SL_n(R, I)$ denotes the subgroup of $GL_n(R, I)$ of elements of determinant 1.

Let $I$ be an ideal of a ring $R$. We shall denote by $Sp_{2m}(R, I)$ the kernel of the canonical mapping $Sp_{2m}(R) \longrightarrow Sp_{2m}(\frac{R}{I})$.

Definition 2.8. Let $I$ be an ideal of $R$. The elementary group $E_n(I)$ is the subgroup of $E_n(R)$ generated as a group by the elements $e_{ij}(x)$, $x \in I$, $1 \leq i \neq j \leq n$.

The relative elementary group $E_n(R, I)$ is the normal closure of $E_n(I)$ in $E_n(R)$.
Theorem 2.1. Let $I$ be an ideal of $R$. The elementary symplectic group $ESp_{2m}(I)$ is the subgroup of $ESp_{2m}(R)$ generated as a group by the elements $e_{ij}(x), x \in I, 1 \leq i \neq j \leq 2m$.

The relative elementary symplectic group $ESp_{2m}(R, I)$ is the normal closure of $ESp_{2m}(I)$ in $ESp_{2m}(R)$.

2.2. The Suslin matrices. First recall the Suslin matrix $S_r(v, w)$. These were defined by Suslin in [13, Section 5]. We recall his inductive process: Let $v = (a_0, v_1), w = (b_0, w_1)$, where $v_1, w_1 \in M_{1,r}(R)$. Set $S_0(v, w) = a_0$ and set

$$S_r(v, w) = \begin{bmatrix} a_0 I_{2^{r-1}} & S_{r-1}(v_1, w_1) \\ -S_{r-1}(w_1, v_1)^T & b_0 I_{2^{r-1}} \end{bmatrix}.$$

The process is reversible and given a Suslin matrix $S_r(v, w)$ one can recover the associated rows $v, w$, i.e. the pair $(v, w)$.

In [12], Suslin proves that if $\langle v, w \rangle = v \cdot w^T = 1$, then by row and column operations one can reduce $S_r(v, w)$ to a matrix $\beta_r(v, w)$ of size $r + 1$ whose first row is $(a_0, a_1, a_2^2, \ldots, a_r^r)$. This in particular proves that rows of such type can be completed to an invertible matrix of determinant 1. We call $\beta_r(v, w)$ to be a compressed Suslin matrix.

2.3. Some assorted results. We state two results from [16, Corollary 17.3, Corollary 18.1, Theorem 18.2].

Theorem 2.1. Let $R$ be an affine $C$-algebra of dimension $d \geq 2$, where $C$ is either a subfield $F$ of $\mathbb{F}_p$ or $C = \mathbb{Z}$. Then

- If $d = 2$, then $E_3(R)$ acts transitively on $Um_3(R)$.
- If $d \geq 3$, then $sr(R) \leq d$. As a consequence, $E_{d+1}(R)$ acts transitively on $Um_{d+1}(R)$.

We will state the part of Swan’s Bertini needed in our proof [14, Theorems 1.3, 1.4] (see also [5, Theorem 2.3]).

Theorem 2.2. Let $R$ be a geometrically reduced affine ring of dimension $d$ over an infinite field $k$ and $(a, a_1, \ldots, a_r) \in Um_{r+1}(R)$. Then there exist $b_1, \ldots, b_r \in R$ such that if $a' = a + a_1 b_1 + \ldots + a_r b_r$, then $R/(a')$ is a reduced affine algebra of dimension $d - 1$ which is smooth at all smooth points of $R$.

We note a result of Fasel, Rao and Swan in [14, Theorems 7.5].

Theorem 2.3. Let $R$ be a smooth affine algebra of dimension $d \geq 3$ over an algebraically closed field $k$ and $(\gcd(d-1), \text{char}(k)) = 1$. Let $v = (v_1, \ldots, v_d) \in Um_d(R)$. Then there exists $\varepsilon \in E_d(R)$ such that

$$v\varepsilon = (w_1, w_2, \ldots, w_{d}^{(d-1)!})$$

for some $(w_1, w_2, \ldots, w_d) \in Um_d(R)$.

Proof of the above theorem implicitly shows the following:

Remark 2.4. Let $R$ be a smooth affine algebra of dimension $d \geq 3$ over an algebraically closed field $k$ and $\gcd(2m, \text{char}(k)) = 1$. Let $v = (v_1, \ldots, v_d) \in Um_d(R)$. Then there exists $\varepsilon \in E_d(R)$ such that

$$v\varepsilon = (w_1, w_2, \ldots, w_d^{2m})$$

for some $(w_1, w_2, \ldots, w_d) \in Um_d(R)$. 

Next we note a result of Vaserestein [16 Lemma 5.5].

**Lemma 2.5.** Let $R$ be a ring. Then for any $m \geq 1$, $E_{2m}(R)e_1 = (Sp_{2m}(R) \cap E_{2m}(R))e_1$.

**Remark 2.6.** It was observed in [2, Lemma 2.13] that Vaserestein’s proof actually shows that $E_{2m}(R)e_1 = ESP_{2m}(R)e_1$.

We recall a result of Basu, Chattopadhyay and Rao [1, Lemma 2.13].

**Lemma 2.7.** Let $R$ be a ring of dimension $d \geq 1$ and $S_\gamma(v,w) \in \text{SL}_2(R/I)$ and $r \geq d$ for $v, w \in \text{Um}_{r+1}(R,I)$ satisfying $\langle v, w \rangle = 1$. Let us assume that $r \equiv 1 \pmod{4}$. Then there exists $\varepsilon, J, \in ESP_\gamma(R,I)$ such that $S_\gamma(v,w)\varepsilon J = (I_{2r-k} \perp \gamma)\varepsilon$ for some $\gamma \in Sp_k(R,I)$ and $\varepsilon \in E_{2r}(R,I)$. Here $k = d + 1$ if $d$ is odd and $k = d$ if $d$ is even.

3. Completion to a Special Linear Matrix

In this section we prove that unimodular rows of length $d$ over $R$ can be elementarily mapped to a factorial row when $R$ is an affine algebra of dimension $d \geq 4$ over $\overline{F}_p$ and $1/(d-1)! \in R$. We also prove this result in the relative case.

**Theorem 3.1.** Let $R$ be an affine algebra of dimension $d \geq 4$ over $\overline{F}_p$ and $1/(d-1)! \in R$. Let $v = (v_1, \ldots, v_d) \in \text{Um}_d(R)$. Then there exists $\varepsilon \in E_d(R)$ such that

$$(v_1, \ldots, v_d)\varepsilon = (w_1, w_2, \ldots, w_d^{(d-1)!})$$

for some $(w_1, \ldots, w_d) \in \text{Um}_d(R)$. In particular, $\text{Um}_d(R) = e_1 \text{SL}_d(R)$.

**Proof:** In view of [7 Remark 1.4.3], we may assume that $R$ is reduced. Let $J$ be the ideal defining singular locus of $R$. Since $R$ is a reduced ring, $\text{ht}(J) \geq 1$. Let $\overline{R} = R/J$. Thus $\dim(\overline{R}) \leq d - 1$ and $\overline{\varpi} = (\overline{\varpi}_1, \ldots, \overline{\varpi}_d) \in \text{Um}_d(\overline{R})$. In view of Theorem 2.1 we have $\dim(\overline{R}) \leq d - 1$ such that $(\overline{\varpi}_1, \ldots, \overline{\varpi}_d) \in \text{Um}_{d-1}(\overline{R})$, where $\overline{\varpi}_i = \varpi_i + k_i v_1, 2 \leq i \leq d$. Note that $[(\varpi_1, \varpi_2, \ldots, \varpi_d)] = [(\varpi_1, \varpi_2, \ldots, \varpi_d)]$.

Since $(\overline{\varpi}_2, \ldots, \overline{\varpi}_d) \in \text{Um}_{d-1}(\overline{R})$, we can elementarily transform $(\overline{\varpi}_1, \overline{\varpi}_2, \ldots, \overline{\varpi}_d)$ to $(1, 0, \ldots, 0)$. Thus we may assume that $(v_1, \ldots, v_d) \equiv e_1$ mod $J$. Now in view of the Theorem 2.2 we add multiples of $v_2, \ldots, v_d$ to $v_1$ to transform it to $v'_1$ such that $R/(v'_1)$ is smooth at the smooth points of $R$. Since $v'_1 \equiv 1$ mod $J$, $R/(v'_1)$ is smooth of dimension $d - 1$.

Thus in view of the Remark 2.4 there exists $\varpi \in E_{d-1}(\overline{R})$ such that $(\varpi_2, \ldots, \varpi_d)\varpi = (w_2, \ldots, w_d^{(d-1)!})$. Let $\varepsilon$ be a lift of $\varpi$ and upon making appropriate elementary transformations we get

$$(v_1, v_2, \ldots, v_d)\varepsilon = (w_1, w_2, \ldots, w_d^{(d-1)!}).$$

**Lemma 3.2.** Let $R$ be an affine algebra of dimension $d \geq 4$ over $\overline{F}_p$ and let $v \in \text{Um}_d(R,I)$ for some ideal $I \subset R$. Let $J$ denotes the ideal defining singular locus of $R$. Then there exists $\varepsilon \in E_d(R,I)$ such that $v\varepsilon = (u_1, u_2, \ldots, u_d)$ with $u_1 \equiv 1$ mod $(I \cap J)$. 

\qed
Proof: Let \( v = (v_1, v_2, \ldots, v_d) \in \text{Um}_d(R, I) \) and \( \overline{R} = R/J \). We have \( \eta = (\overline{v_1}, \overline{v_2}, \ldots, \overline{v_d}) \in \text{Um}_d(\overline{R}, \overline{T}) \). In view of the Theorem 2.1, \( \text{sr}(\overline{R}) \leq d - 1 \). Thus upon adding multiples of \( \overline{v_d} \) to \( \overline{v_1}, \overline{v_2}, \ldots, \overline{v_{d-1}} \), we may assume that \( (\overline{v_1}, \overline{v_2}, \ldots, \overline{v_d}) \in \text{Um}_{d-1}(\overline{R}, \overline{T}) \). Let \( \overline{w} \in \text{Um}_{d-1}(\overline{R}, \overline{T}) \) be such that \( \sum_{i=1}^{d-1} \overline{w_i} = 1 \). Let

\[
\varpi = \begin{bmatrix} T & 0 & \cdots & (1 - \overline{v_d})\overline{w_1} \\ 0 & T & \cdots & (1 - \overline{v_d})\overline{w_2} \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & T \end{bmatrix} \in \text{E}_d(\overline{R}).
\]

Then \( (\overline{v_1}, \overline{v_2}, \ldots, \overline{v_d})\varpi = (\overline{v_1}, \overline{v_2}, \ldots, \overline{v_{d-1}}, 1) \). There exists \( \overline{y_2} \in \text{E}_d(\overline{R}, \overline{T}) \) such that \( (\overline{v_1}, \overline{v_2}, \ldots, \overline{v_{d-1}}, 1)\overline{y_2} = (1, 0, \ldots, 0, 1) \). Now \( (1, 0, \ldots, 0, 1) \equiv 1 \mod \overline{T} \). Since \( 1 - (1 - \overline{v_d}\overline{w_1}) = \overline{0} \mod \overline{T} \), there exists \( \overline{y_3} \in \text{E}_d(\overline{R}, \overline{T}) \) such that \( \overline{y_3} = (1, 0, \ldots, 0, 1) - (1 - \overline{v_d}\overline{w_1})\overline{y_2} = (1, 0, \ldots, 0, 1) \). Let \( \varpi = \overline{v_1}\varpi^-1\overline{y_3} \in \text{E}_d(\overline{R}, \overline{T}) \). Let \( \varepsilon \in \text{E}_d(R, I) \) be a lift of \( \varpi \). Then we have \( v\varepsilon = (u_1, u_2, \ldots, u_d) \) with \( u_1 \equiv 1 \mod (I \cap J) \).

Theorem 3.3. Let \( R \) be an affine algebra of dimension \( d \geq 4 \) over \( \mathbb{F}_p \) and let \( v \in \text{Um}_d(R, I) \) for some ideal \( I \subset R \). Then there exists \( \varepsilon \in \text{E}_d(R, I) \) such that

\[
v\varepsilon = (w_1, \ldots, w_d^{(d-1)!})
\]

for some \( (w_1, \ldots, w_d) \in \text{Um}_d(R, I) \).

Proof: Let \( v = (v_1, \ldots, v_d) \in \text{Um}_d(R, I) \) and \( J \) be the ideal defining singular locus of \( R \). In view of the Lemma 3.2, we may assume that \( v_1 \equiv 1 \mod (I \cap J) \). Let \( v_1 = 1 - \lambda \) for some \( \lambda \in I \cap J \). Note that \( (v_1, \ldots, v_d) \sim (v_1, \lambda v_2, \ldots, \lambda v_d) \). Now in view of the Theorem 2.2, we add multiples of \( \lambda v_2, \ldots, \lambda v_d \) to \( v_1 \), to transform it to \( v' \) such that \( R/(v') \) is smooth outside the singular set of \( R \). Since \( v' \equiv 1 \mod J \), \( R/(v') \) is a smooth affine algebra of dimension \( d - 1 \).

Let \( v' = 1 - \eta \) for some \( \eta \in I \cap J \). Note that in \( \overline{R} = R/(v') \), we have \( \overline{\eta} = \overline{1} \). In view of the Theorem 2.2, there exists \( \varepsilon \in \text{E}_{d-1}(\overline{R}) = \text{E}_{d-1}(\overline{\eta}) \) such that \( \lambda v_2, \ldots, \lambda v_d \varepsilon = (\overline{w_1}, \ldots, \overline{w_d}) \). Let \( \varepsilon \in \text{E}_{d-1}(\overline{\eta}) \subset \text{E}_{d-1}(I) \) be a lift of \( \varepsilon \). Therefore \( \lambda v_2, \ldots, \lambda v_d \varepsilon = (w_2, \ldots, w_d^{(d-1)!}) \mod (I v') \). Thus, we have

\[
(v_1, \ldots, v_d) \sim (v'_1, \lambda v_2, \ldots, \lambda v_d) \in \text{E}_d(R, I)
\]

\[
\sim (w_1, w_2, \ldots, w_d^{(d-1)!}), \text{ where } w_1 = v'_1.
\]

Now we consider the problem of injective stability and prove that stably elementary matrices of size \( d \) are isotopic to the identity. To prove this we need the following proposition.

Proposition 3.4. Let \( R \) be an affine algebra of dimension \( d \geq 4 \) over \( \mathbb{F}_p \). Let \( v \in \text{Um}_d(R) \) be such that \( v \equiv e_1 \mod (s) \) for some \( s \in R \). Then there exists \( \sigma \in \text{SL}_d(R) \) with \( \sigma \equiv I_d \mod (s) \) such that \( v = e_1 \sigma \).

Proof: Let \( S = R[X]/(X^2 - sX) \). Then \( S \) is an affine algebra of dimension \( d \geq 4 \) over \( \mathbb{F}_p \). Since \( v \equiv e_1 \mod (s) \), let \( v = e_1 + sw \) for some \( w \in R^d \). Let \( u(X) = e_1 + Xw \).
Claim: \( u(X) \in \text{Um}_d(S) \).

Proof of the claim: Since \( v = e_1 + sw \), there exist \( w'_1, \ldots, w'_d \in R \) such that
\[
(1 + sw_1)w'_1 + sw_2w'_2 + \cdots + sw_dw'_d = 1.
\]
Suppose to the contrary that \( u(X) \notin \text{Um}_d(S) \). Thus \( \langle 1 + Xw_1, Xw_2, \ldots, Xw_d \rangle \subset p \) for some \( p \in \text{Spec}(S) \). Therefore
\[
(1 + Xw_1)w'_1 + \cdots + Xw_dw'_d = w'_1 + X(w_1w'_1 + \cdots w_dw'_d)
= 1 + (X - s)(w_1w'_1 + \cdots w_dw'_d) \in p.
\]
Since \( X(X - s) = 0 \) in \( S \), \( X \in p \). As \( 1 + Xw_1 \in p, 1 \in p \) which is not possible. Thus \( u(X) \in \text{Um}_d(S) \).

Now \( u(s) = v \) and \( u(0) = e_1 \). By Theorem 3.3, there exists \( \alpha(X) \in \text{SL}_d(S) \) such that \( u(X) = e_1\alpha(X) \).

Upon taking \( \sigma = \alpha(0)^{-1}\alpha(s) \), we have \( v = e_1\sigma \) and \( \sigma \equiv I_d \) mod \( s \). \( \square \)

Theorem 3.5. Let \( R \) be an affine algebra of dimension \( d \geq 4 \) over \( \overline{\mathbb{F}}_p \). Let \( \alpha \in \text{SL}_d(R) \cap \text{E}_{d+1}(R) \). Then \( \alpha \) is isotopic to the identity matrix.

Proof: Since \( \alpha \in \text{SL}_d(R) \cap \text{E}_{d+1}(R) \), there exists a \( \beta(T) \in \text{E}_{d+1}(R[T]) \) such that
\[
\beta(0) = I_{d+1} \quad \text{and} \quad \beta(1) = 1 \perp \alpha.
\]
Let \( A = R[T], s = T^2 - T \) and \( v = e_1\beta(T) \). By Proposition 3.4, there exists \( \gamma(T) \in \text{SL}_{d+1}(R[T], (s)) \) such that \( v = e_1\gamma(T) \). Therefore \( e_1\beta(T)\gamma(T)^{-1} = e_1 \). Thus
\[
\beta(T)\gamma(T)^{-1} = (1 \perp \eta(T))\prod_{i=2}^{d+1} E_{i1}(f_i(T)),
\]
where \( \eta(T) \in \text{SL}_d(R[T]) \) and \( f_i(T) \in R[T] \) with the property that \( f_i(0) = f_i(1) = 0 \) for \( 2 \leq i \leq d+1 \). Now,
\[
\beta(0)\gamma(0)^{-1} = I_{d+1} = 1 \perp \eta(0).
\]
Since \( \beta(1) = 1 \perp \alpha, \gamma(1) = I_{d+1} \), we have
\[
\beta(1)\gamma(1)^{-1} = (1 \perp \alpha)I_{d+1} = 1 \perp \eta(1).
\]
Thus \( \eta(T) \) is an isotopy from \( \alpha \) to \( I_d \). \( \square \)

4. Completion to a symplectic matrix

In this section, we prove that symplectic matrices of size \( d \) acts transitively on unimodular rows of length \( d \) when \( d \equiv 2 \) (mod 4).

Lemma 4.1. Let \( R \) be an affine algebra of dimension \( d \) over \( \overline{\mathbb{F}}_p \) and \( d \equiv 2 \) (mod 4). Let \( v \in \text{Um}_d(R) \). Assume that \( \text{SL}_{d+1}(R) \cap \text{E}(R) = \text{E}_{d+1}(R) \). Then \( v \) is completable to a symplectic matrix.

Proof: Let \( d = 2 \) and \( v = (v_1, v_2) \in \text{Um}_2(R) \). Let \( w_1, w_2 \in R \) be such that \( v_1w_1 + v_2w_2 = 1 \). Consider
\[
\alpha = \begin{bmatrix} v_1 & v_2 \\ -v_2 & w_1 \end{bmatrix} \in \text{SL}_2(R) = \text{Sp}_2(R).
\]
Thus \( v \) is completable to a symplectic matrix.
Thus we may assume that \( d \geq 6 \). In view of Theorem 3.1 there exists \( \varepsilon \in \text{E}_d(R) \) such that \( v \varepsilon = \chi_d(v_1) \) for some \( v_1 \in \text{Um}_d(R) \). By Remark 2.6 there exists \( \varepsilon_1 \in \text{ESp}_d(R) \) such that \( v_1 \varepsilon_1 = \chi_d(v_1) \). Let \( w_1 \in \text{Um}_d(R) \) be such that \( \langle v_1, w_1 \rangle = 1 \). By Lemma 2.7 we have \( S_{d-1} \varepsilon, w_1 \rangle \in \text{ESp}_{d-1}(R) \) such that

\[
S_{d-1}(v, w)\varepsilon, w_1 \rangle = (I_{2d-1} - d \perp \gamma)\varepsilon_2
\]

for some \( \gamma \in \text{Sp}_d(R) \) and \( \varepsilon_2 \in \text{E}_{2d-1}(R) \). Therefore \( S_{d-1}(v, w) \) and \( \gamma \) are stably elementary equivalent. One knows that \( S_{d-1}(v_1, w_1) \) and \( \beta_{d-1}(v_1, w_1) \) are stably elementary equivalent, we have \( \beta_{d-1}(v_1, w_1)\gamma^{-1} \in \text{SL}_d(R) \cap \text{E}_{d+1}(R) \). By [4] Corollary 3.9], first row of an 1-stably elementary matrix is elementarily completeable. Thus \( e_1\beta_{d-1}(v_1, w_1)\gamma^{-1} = e_1\delta' \) for some \( \delta' \in \text{E}_d(R) \). Now by Remark 2.6 \( e_1\delta' = e_1\delta \) for some \( \delta \in \text{ESp}_d(R) \). Therefore

\[
e_1\beta_{d-1}(v_1, w_1)\gamma^{-1} = e_1\delta.
\]

Thus \( \chi_d(v_1) = e_1\beta_{d-1}(v_1, w_1) = e_1\gamma \). Since \( v_1 \in \text{E}_d(v_1) \), we have \( v = e_1\delta \varepsilon^{-1} \) and \( \delta \varepsilon^{-1} \in \text{Sp}_d(R) \).

\[ \Box \]

**Proposition 4.2.** Let \( R \) be an affine algebra of dimension \( d \) over \( \overline{\mathbb{F}}_p \) and \( d \equiv 2 \mod 4 \). Let \( v \in \text{Um}_d(R) \) be such that \( v \equiv e_1 \mod (s) \) for some \( s \in R \). Assume that \( \text{SL}_{d+1}(R) \cap \text{E}(R) = \text{E}_{d+1}(R) \). Then there exists \( \sigma \in \text{Sp}_d(R) \) such that \( \sigma \equiv I_d \mod (s) \) and \( v = e_1\sigma \).

**Proof:** The arguments of this proof are very similar to the proof of Proposition 3.4. \[ \Box \]

The next result shows that a stably elementary symplectic matrix over \( R \) is isotopic to the identity matrix under certain conditions when \( R \) is an affine algebra of dimension \( d \) over \( \overline{\mathbb{F}}_p \).

**Theorem 4.3.** Let \( R \) be an affine algebra of dimension \( d \) over \( \overline{\mathbb{F}}_p \). Assume that \( 1/(d+1)! \in \overline{\mathbb{F}}_p \) and \( d \equiv 1 \mod (4) \). Further assume that \( \text{SL}_{d+1}(R) \cap \text{E}(R) = \text{E}_{d+1}(R) \). Let \( \sigma \in \text{Sp}_{d-1}(R) \cap \text{ESp}_{d+1}(R) \). Then \( \sigma \) is isotopic to the identity.

**Proof:** Since \( \sigma \in \text{Sp}_{d-1}(R) \cap \text{ESp}_{d+1}(R) \), there exists \( \delta(T) \in \text{ESp}_{d+1}(R[T]) \) such that \( \delta(1) = I_2 \perp \sigma, \delta(0) = I_{d+1} \). Let \( v(T) = e_1\alpha(T) \in \text{Um}_{d+1}(R[T], (T^2 - T)) \). By Proposition 4.2 \( v(T) = e_1\alpha(T) \in \text{Sp}_{d+1}(R[T]) \) and \( \alpha(T) \equiv I_2 \mod (T^2 - T) \). Thus \( e_1\delta(T) = e_1\alpha(T) \). Therefore one has \( e_1\delta(T)\alpha(T)\gamma^{-1} = e_1 \). Now note that \( \delta(T)\alpha(T)^{-1} \) is an isotopy of \( I_2 \perp \sigma \) to \( I_{d+1} \). The matrix \( \delta(T)\alpha(T)^{-1} \) looks like

\[
\begin{pmatrix}
1 & 0 & 0 \\
* & 1 & * \\
* & 0 & \eta(T)
\end{pmatrix}
\]

for some \( \eta(T) \in \text{Sp}_{d-1}(R[T], (T^2 - T)) \). Now note that \( \eta(0) = I_{d-1} \) and \( \eta(1) = \sigma \). Thus \( \sigma \) is isotopic to the identity. \[ \Box \]

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