Dense semigroups of triangular matrices

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April 11, 2018

Abstract

Let $\mathbb{K} = \mathbb{R}$ or $\mathbb{C}$, and $T_n(\mathbb{K})$ be the set of $n \times n$ lower triangular matrices with entries in $\mathbb{K}$. We show that $T_n(\mathbb{K})$ has dense subsemigroups that are generated by $n + 1$ matrices.

2010 Mathematics Subject Classification: Primary 47D03; Secondary 20H20.
Key words and phrases: Dense subsemigroups, Triangular matrices.

1 Introduction

Dense subgroups of Lie groups have been studied by many mathematicians, dating back to Auerbach, who proved that every compact semisimple Lie group has a 2-generator dense subgroup [3]. Kuranishi [9] proved that if $G$ is a semisimple Lie group and $a, b$ are two elements near the identity, then the subgroup generated by $a$ and $b$ is dense in $G$ if and only if $\log a$ and $\log b$ generate the Lie algebra $\mathfrak{g}$. More recently, Abels and Vinberg [2] showed that every connected semisimple Lie group with finite center has 2-generator dense subsemigroups. In the real case, Breuillard and Gelander [4] proved that every dense subgroup of a connected semisimple real Lie group has a 2-generator dense subgroup.

Dense subgroups of linear spaces in particular have also been studied. Wang [10] has shown that every dense subgroup of the group of orientation
preserving Möbius transformations on the $n$-dimensional unit sphere has a dense subgroup generated by at most $n$ elements, where $n \geq 2$. A similar statement was proved by Cao in [5] for $U(n, 1)$. Finally, examples of 2-generator dense subsemigroups of $n \times n$ matrices in both real and complex cases can be constructed [8].

In connection to hypercyclicity, the following results have inspired the results of this paper. Feldman proved that there exists a dense subsemigroup of $n \times n$ diagonal matrices generated by $n + 1$ diagonal matrices [6]. Abels and Manoussos in [1] have shown that the minimum number of generators of a triangular non-diagonalizable abelian subsemigroup with a dense or somewhere dense orbit is $n + 1$ in the real case and $n + 2$ in the complex case.

The following theorem is the main result of this article.

**Theorem 1.** There exist $(n + 1)$-generator dense subsemigroups of $n \times n$ triangular matrices in both real and complex cases.

Note that $n + 1$ is the least number of generators of a dense subsemigroup of diagonal matrices, and so it is also the least number of generators of a dense subsemigroup of lower triangular matrices.

## 2 Proof of the main theorem

We use Feldman’s construction [6] of a hypercyclic $(n + 1)$-tuple of $n \times n$ diagonal matrices to construct an $(n + 1)$-generator dense subsemigroup of lower triangular matrices. A set of $n \times n$ commuting matrices $(A_1, \ldots, A_k)$ with entries in $\mathbb{K} = \mathbb{R}$ or $\mathbb{C}$ is called a hypercyclic $k$-tuple, if there exists a vector $X \in \mathbb{K}^n$ such that the set

$$\{A_1^{m_1} \cdots A_k^{m_k} X : m_1, \ldots, m_k \in \mathbb{N}\}$$

is dense in $\mathbb{K}^n$. It is straightforward to show that if $(D_0, \ldots, D_n)$ is a hypercyclic $(n + 1)$-tuple of diagonal matrices, then the semigroup generated by $D_0, \ldots, D_n$, denoted by $\langle D_0, \ldots, D_n \rangle$ is dense in the set of all $n \times n$ diagonal matrices. In addition, even though Feldman’s theorem is stated for $\mathbb{C}$, but the real case can be concluded easily from the complex case.

In the sequel, all of the statements hold for matrices with entries in $\mathbb{K} = \mathbb{R}$ or $\mathbb{C}$. Therefore, we might drop the reference to the field under consideration.

We begin with the following lemmas.
Lemma 2. Let $A$ be a lower triangular $n \times n$ matrix with diagonal entries $A_{ii} = a_i$, $1 \leq i \leq n$, such that

$$0 < |a_1| < \cdots < |a_n|.$$  

Then there exists $\lambda > 0$, which depends only on $A$, such that for all $1 \leq i, j \leq n$ and all $k \geq 1$, we have

$$|A^k|_{ij} \leq \lambda |a_i|^k. \quad (1)$$

The proof of Lemma 2 can be found in [7].

Next, we define a total order on the set $\Delta = \{(i, j) : 1 \leq j \leq i \leq n\}$ as follows:

$$(i, j) \preceq (r, s) \iff i - j < r - s \lor (i - j = r - s \land i \leq r).$$

Let $\mathcal{T}_{(r, s)}$ be the set of $n \times n$ lower triangular matrices $T$ such that $T_{ij} = 0$ for all $(i, j) \prec (r, s)$. Clearly, if $(r, s) \preceq (t, u)$, then $\mathcal{T}_{(u, v)} \subseteq \mathcal{T}_{(r, s)}$. Moreover, each $\mathcal{T}_{(r, s)}$ is closed under matrix multiplication, $(r, s) \in \Delta$.

Lemma 3. Let $D = (a_1, \ldots, a_n)$ be a diagonal matrix such that

$$0 < |a_1| < \cdots < |a_n|. \quad (2)$$

Let $(r, s) \in \Delta$ and $T \in \mathcal{T}_{(r, s)}$. If $A = D + T$, then

$$(A^k)_{rs} = \left(\frac{a_r^k - a_s^k}{a_r - a_s}\right) T_{rs}. \quad (3)$$

Proof. For $T_1, T_2 \in \mathcal{T}_{(r, s)}$ and diagonal matrices $D_1$ and $D_2$, one has

$$[(D_1 + T_1)(D_2 + T_2)]_{rs} = \sum_{l=1}^{n} (D_1 + T_1)_{rl}(D_2 + T_2)_{ls}$$

$$= \sum_{l=s}^{r} (D_1 + T_1)_{rl}(D_2 + T_2)_{ls}$$

$$= (D_1 + T_1)_{rs}(D_2 + T_2)_{ss} + (D_1 + T_1)_{rr}(D_2 + T_2)_{rs}$$

$$= (T_1)_{rs}(D_1)_{ss} + (D_2)_{rr}(T_2)_{rs}, \quad (4)$$

since for $s < l < r$, we have $(r, l) \preceq (r, s)$, and so $(D_1 + T_1)_{rl} = (T_1)_{rl} = 0$. The equation (3) then follows from (1) and induction on $k$. \qed
Lemma 4. Let \( \langle D_\alpha : \alpha \in J \cup \{0\} \rangle \) be a dense subsemigroup of diagonal matrices, where \( J \) is a finite index set. Suppose that the diagonal matrix \( D_0 = (a_1, \ldots, a_n) \) is such that (2) holds. Let \( T \) be a lower triangular matrix with \( T_{ii} = 0 \) for all \( 1 \leq i \leq n \). Then the closure of the semigroup generated by \( \langle D_0 + T, D_\alpha : \alpha \in J \rangle \) contains all diagonal matrices.

Proof. We prove that if \( T \in \mathcal{T}_{(r,s)} \), where \( (r, s) \succeq (2, 1) \), then the closure of \( \langle D_0 + T, D_\alpha : \alpha \in J \rangle \) includes \( D_0 + T' \) for some \( T' \in \mathcal{T}_{(r,s)} \), where \( (r, s)' \) is the successor of \( (r, s) \) in the ordered set \( (\Delta, \preceq) \). Equivalently, we show that there exists \( T' \in \mathcal{T}_{(r,s)} \) such that \( D_0 + T' \) belongs to the closure of \( \langle D_0 + T, D_\alpha : \alpha \in J \rangle \) and in addition \( (T')_{rs} = 0 \). It will then follow from a finite induction on the ordered set \( (\Delta, \preceq) \), starting with \( (2, 1) \in \Delta \) and ending in \( (n, 1) \in \Delta \), that \( D_0 \) belongs to the closure of \( \langle D_0 + T, D_\alpha : \alpha \in J \rangle \), and the claim follows from the assumption that \( \langle D_\alpha : \alpha \in J \cup \{0\} \rangle \) is dense in the set of diagonal matrices.

Let \( A = D_0 + T \). By our density assumption, for each \( \alpha \in J \cup \{0\} \), there exists a sequence of exponents \( \{d_\alpha^k\}_{k=1}^\infty \) such that \( \lim_{k \to \infty} d_\alpha^k = \infty \) and

\[
\left( \prod_{\alpha \in J} D_\alpha^{d_\alpha^k} \right) D_0^{d_0^k} \to B,
\]

as \( k \to \infty \), where \( B \) is an arbitrary diagonal matrix. Let

\[
M_k = \left( \prod_{\alpha \in J} D_\alpha^{d_\alpha^k} \right) A^{d_0^k}.
\]

It follows from (5) that for all \( 1 \leq i \leq n \), we have

\[
\lim_{k \to \infty} (M_k)_{ii} = B_{ii}.
\]

Since \( \mathcal{T}_{(r,s)} \) is closed under multiplication, we have \( M_k - B \in \mathcal{T}_{(r,s)} \). Moreover, it follows from Lemma 2 that for all \( 1 \leq j \leq i \):

\[
|M_k|_{ij} = \left| \left( \prod_{\alpha \in J} D_\alpha^{d_\alpha^k} \right) A^{d_0^k} \right|_{ij} = \prod_{\alpha \in J} D_\alpha^{d_\alpha^k} \cdot \left| A^{d_0^k} \right|_{ij} \leq \lambda \prod_{\alpha \in J} D_\alpha^{d_\alpha^k} \cdot \left| D_0^{d_0^k} \right|_{ii} \leq \lambda \left| \left( \prod_{\alpha \in J} D_\alpha^{d_\alpha^k} \right) D_0^{d_0^k} \right|_{ii} \to \lambda |B_{ii}|,
\]
as \( k \to \infty \). In particular, for each \((i, j)\) with \(1 \leq i, j \leq n\), the sequence \(\{ (M_k)_{ij} \}_{k=1}^{\infty} \) is a bounded sequence, hence, by deriving a common convergent subsequence, we obtain a matrix \(M\) in the closure of \(\langle D_0 + T, D_\alpha : \alpha \in J \rangle\) such that \(M - B \in T_{(r,s)}\).

Next, by Lemma 3, we have

\[
(M_k)_{rs} = \left[ \left( \prod_{\alpha \in J} D_{\alpha}^{d_\alpha} \right) A_0^{d_0} \right]_{rs}
\]

\[
= \left( \prod_{\alpha \in J} D_{\alpha}^{d_\alpha} \right)_{rr} \left( A_0^{d_0} \right)_{rs}
\]

\[
= \left( \prod_{\alpha \in J} D_{\alpha}^{d_\alpha} \right)_{rr} \left( \frac{a_r - a_s}{a_r - a_s} \right) T_{rs}
\]

\[
= \left[ \left( \prod_{\alpha \in J} D_{\alpha}^{d_\alpha} \right) D_0^{d_0} \right]_{rr} \left( \frac{1 - (a_s/a_r)^{d_0}}{a_r - a_s} \right) T_{rs}
\]

\[
\to \frac{B_{rr} T_{rs}}{a_r - a_s},
\]

as \( k \to \infty \), since \( r > s \), and so \(|a_s/a_r| < 1\). It follows that

\[
M_{rs} = \lim_{k \to \infty} (M_k)_{rs} = \frac{B_{rr} T_{rs}}{a_r - a_s}.
\]

So far, we have proved that for any arbitrary diagonal matrix \(B\), there exists \(M = \eta(B)\) in the closure of \(\langle D_0 + T, D_\alpha : \alpha \in J \rangle\) such that \(M - B \in T_{(r,s)}\) and \(M_{rs} = B_{rr} T_{rs} / (a_r - a_s)\). Let \(B_2\) be the diagonal matrix with \((B_2)_{rr} = -1\) and \((B_2)_{ii} = 1\) for all \(i \neq r\). Also, let \(B_1 = B_2 D_0\). Finally, let \(M_1 = \eta(B_1)\) and \(M_2 = \eta(B_2)\). Then for \(T' = M_1 M_2 - D_0\), we have \(T' \in T_{(r,s)}\) and \(D_0 + T' = M_1 M_2\), which belongs to the closure of \(\langle D_0 + T, D_\alpha : \alpha \in J \rangle\). Moreover,

\[
(T')_{rs} = (M_1 M_2)_{rs} = \sum_{i=1}^{n} (M_1)_{ri}(M_2)_{is}
\]

\[
= (M_1)_{rs}(M_2)_{ss} + (M_1)_{rr}(M_2)_{rs}
\]

\[
= \frac{(B_1)_{rr} T_{rs} (B_2)_{ss} + (B_1)_{rr} (B_2)_{rr} T_{rs}}{a_r - a_s}
\]

\[
= \frac{-a_r T_{rs}}{a_r - a_s} + \frac{a_r T_{rs}}{a_r - a_s} = 0,
\]

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Theorem 5. Suppose that the semigroup \( \langle D_0 : \alpha \in J \cup \{0\} \rangle \) of \( n \times n \) diagonal matrices is dense in the set of all \( n \times n \) diagonal matrices with entries in \( K = \mathbb{R} \) or \( \mathbb{C} \), where \( J \) is a finite index set. Suppose that \( D_0 = (a_1, \ldots, a_n) \) is such that the inequalities in (2) hold. Let \( T \) be a matrix such that
\[
T_{ij} \neq 0 \iff 1 \leq j < i \leq n.
\] (6)
Then \( \langle D_0 + T, D_\alpha : \alpha \in J \rangle \) is dense in the set of all \( n \times n \) lower triangular matrices with entries in \( K \).

Proof. By Lemma 4, the closure of \( \langle D_0 + T, D_\alpha : \alpha \in J \rangle \) contains all diagonal matrices. Proof of Theorem 5 is by induction on \( n \). The case \( n = 1 \) is trivial. Therefore, suppose the claim is true for \( n - 1 \), where \( n \geq 2 \). Given an \( n \times n \) matrix \( X \), let \( \sigma(X) \) be the upper left \((n-1) \times (n-1)\) block of \( X \). In particular, the semigroup generated by \( \langle \sigma(D_\alpha) : \alpha \in J \cup \{0\} \rangle \) is dense in the set of diagonal \((n-1) \times (n-1)\) matrices. It follows from the inductive hypothesis that \( \langle \sigma(D_0 + T), \sigma(D_\alpha) : \alpha \in J \rangle \) is dense in the set of \((n-1) \times (n-1)\) lower triangular matrices. In particular, for each lower triangular \((n-1) \times (n-1)\) matrix \( R \), there exists a sequence \( L_i = \begin{bmatrix} R_i & 0 \\ V_i & x_i \end{bmatrix} \in \bar{\Lambda}, \ i \geq 1, \) such that \( R_i \to R \) as \( i \to \infty \), and \( x_i \neq 0 \). Here \( \bar{\Lambda} \) is the closure of \( \Lambda = \langle D_0 + T, D_\alpha : \alpha \in J \rangle \). Choose a sequence of nonzero numbers \( \{a_i\}_{i=1}^\infty \) such that \( \lim_{i \to \infty} a_i V_i = 0 \). Then, we have
\[
\begin{bmatrix} I_{n-1} & 0 \\ 0 & a_i \end{bmatrix} L_i \begin{bmatrix} I_{n-1} & 0 \\ 0 & x/(a_i x_i) \end{bmatrix} = \begin{bmatrix} R_i & 0 \\ a_i V_i & x \end{bmatrix} \to \begin{bmatrix} R & 0 \\ 0 & x \end{bmatrix},
\] (7)
as \( i \to \infty \), which implies that:
\[
\begin{bmatrix} R & 0 \\ 0 & x \end{bmatrix} \in \bar{\Lambda},
\] (8)
for all \((n-1) \times (n-1)\) lower triangular matrices and any given \( x \in \mathbb{R} \), since the three matrices on the left side of equation (7) belong to \( \bar{\Lambda} \) (note that all diagonal matrices belong to \( \bar{\Lambda} \) by Lemma 4). Next, we show that every \( n \times n \)
triangular matrix is a product of two matrices of the form \([8]\) with \(D_0 + T\). We write
\[
D_0 + T = \begin{bmatrix} A' & 0 \\ V & a_n \end{bmatrix}.
\]
Let
\[
B = \begin{bmatrix} U & 0 \\ W & z \end{bmatrix}
\]
be any lower triangular matrix such that vector \(W\) has no zero entries, and \(z \neq 0\). Let \(x = z/a_n\) and choose an invertible diagonal matrix \(S\) such that \(xVS = W\). Finally, let \(R = US^{-1}(A')^{-1}\). One has
\[
\begin{bmatrix} R & 0 \\ 0 & x \end{bmatrix} \begin{bmatrix} A' & 0 \\ V & a_n \end{bmatrix} \begin{bmatrix} S & 0 \\ 0 & 1 \end{bmatrix} = \begin{bmatrix} RA'S & 0 \\ xVS & xa_n \end{bmatrix} = \begin{bmatrix} U & 0 \\ W & z \end{bmatrix},
\]
which implies that \(B \in \bar{\Lambda}\). Since \(\bar{\Lambda}\) is closed, the conditions that \(W\) has no zero entries and \(z \neq 0\) can be removed to conclude that \(\Lambda\) contains all lower triangular matrices, and the proof is completed. \(\square\)

We are now ready to state the proof of Theorem 1.

**Proof of Theorem 1.** It follows from [6, Theorem 3.4] and its proof that there exists a dense subsemigroup of diagonal matrices generated by \(n + 1\) diagonal matrices \(D_0, \ldots, D_n\), where the diagonal entries of \(D_0 = (a_1, \ldots, a_n)\) are such that \(|a_i| \neq |a_j|\) for all \(i \neq j\). Via a permutation, we can assume that, in addition, the entries of \(D_0\) satisfy the inequalities in \([2]\). It then follows from Theorem 5 that for any matrix \(T\) that satisfies condition \([4]\), the semigroup generated by \(D_0 + T, D_1, \ldots, D_n\) is dense in the set of \(n \times n\) triangular matrices. \(\square\)

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