Discrete Differential Geometry and Lattice Field Theory

Miguel Lorente
Departamento de Física, Universidad de Oviedo, 33007 Oviedo, Spain

Abstract

We develop a difference calculus analogous to the differential geometry by translating the forms and exterior derivatives to similar expressions with difference operators, and apply the results to fields theory on the lattice [Ref. 1]. Our approach has the advantage with respect to other attempts [Ref. 2-6] that the Lorentz invariance is automatically preserved as it can be seen explicitely in the Maxwell, Klein-Gordon and Dirac equations on the lattice.

1 A difference calculus of several independent variables

Given a function of one independent variable the forward and backward differences are defined as

\[ \Delta f(x) \equiv f(x + \Delta x) - f(x) \quad \text{and} \quad \nabla f(x) \equiv f(x) - f(x - \Delta x) \]

Similarly, we can define the forward and backward promediate operator

\[ \tilde{\Delta} f(x) \equiv \frac{1}{2} \{ f(x + \Delta x) + f(x) \} \quad \text{and} \quad \tilde{\nabla} f(x) \equiv \frac{1}{2} \{ f(x - \Delta x) + f(x) \} \]

Hence the difference or promediate of the product of two functions follows:

\[ \nabla \{ f(x)g(x) \} = \Delta f(x)\tilde{\Delta} g(x) + \tilde{\Delta} f(x)\Delta g(x) \quad \text{(1.1)} \]

\[ \tilde{\Delta} \{ f(x)g(x) \} = \tilde{\Delta} f(x)\tilde{\Delta} g(x) + \frac{1}{4}\Delta f(x)\Delta g(x) \quad \text{(1.2)} \]

This calculus can be enlarged to functions of several independent variables. We use the following definitions:

\[ \Delta_x f(x, y) \equiv f(x + \Delta x, y) - f(x, y) \]
\[ \Delta_y f(x, y) \equiv f(x, y + \Delta y) - f(x, y) \]
\[ \tilde{\Delta}_x f(x, y) \equiv \frac{1}{2} \{ f(x + \Delta x, y) + f(x, y) \} \]
\[ \tilde{\Delta}_y f(x, y) \equiv \frac{1}{2} \{ f(x, y + \Delta y) + f(x, y) \} \]
\[ \Delta f(x, y) \equiv f(x + \Delta x, y + \Delta y) - f(x, y) \]
\[ \tilde{\Delta} f(x, y) \equiv \frac{1}{2} \{ f(x + \Delta x, y + \Delta y) + f(x, y) \} \]
These definitons can be easily generalized to more independent variables but for the sake of brevity we restrict ourselves to two independent variables. From the last definitions it can be proved the following identities:

\[
\Delta f(x, y) = \Delta_x \Delta_y f(x, y) + \Delta_x \Delta_y f(x, y) \quad (1.3)
\]
\[
\bar{\Delta} f(x, y) = \bar{\Delta}_x \bar{\Delta}_y f(x, y) + \frac{1}{4} \Delta_x \Delta_y f(x, y) \quad (1.4)
\]

We can also construct the difference calculus for composite functions. For the sake of simplicity we restrict ourselves to functions of two dependent variables and two independent ones, \( f(u(x, y), v(x, y)) \).

We define:

\[
\Delta_u f = f(u + \Delta u, v) - f(u, v)
\]
\[
\Delta_v f = f(u, v + \Delta v) - f(u, v)
\]
\[
\bar{\Delta}_u f = \frac{1}{2} \{ f(u + \Delta u, v) + f(u, v) \}
\]
\[
\bar{\Delta}_v f = \frac{1}{2} \{ f(u, v + \Delta v) + f(u, v) \}
\]
\[
\Delta_x f = f(u(x + \Delta x, y), v(x + \Delta x, y)) - f(u(x, y), v(x, y))
\]
\[
\Delta_y f = f(u(x, y + \Delta y), v(x, y + \Delta y)) - f(u(x, y), v(x, y))
\]
\[
\bar{\Delta}_x f = \frac{1}{2} \{ f(u(x + \Delta x, y), v(x + \Delta x, y)) + f(u(x, y), v(x, y)) \}
\]
\[
\bar{\Delta}_y f = \frac{1}{2} \{ f(u(x, y + \Delta y), v(x, y + \Delta y)) + f(u(x, y), v(x, y)) \}
\]

from which the following identities can be proved:

\[
\Delta f = \Delta_u \Delta_v f + \bar{\Delta}_u \Delta_v f = \Delta_x \Delta_y f + \bar{\Delta}_x \Delta_y f
\]
\[
\bar{\Delta} f = \bar{\Delta}_u \bar{\Delta}_v f + \frac{1}{4} \Delta_u \Delta_v f = \bar{\Delta}_x \bar{\Delta}_y f + \frac{1}{4} \Delta_x \Delta_y f
\]

We can define also the difference operators

\[
\Delta_{ux} f \equiv f(u + \Delta u, v) - f(u, v)
\]
\[
\Delta_{uy} f \equiv f(u, v + \Delta v) - f(u, v)
\]
\[
\Delta_{vx} f \equiv f(u + \Delta x, v) - f(u, v)
\]
\[
\Delta_{vy} f \equiv f(u, v + \Delta y) - f(u, v)
\]

and similarly for the promediate operator

\[
\bar{\Delta}_{ux} f = \frac{1}{2} \{ f(u + \Delta u, v) + f(u, v) \}
\]
\[
\bar{\Delta}_{uy} f = \frac{1}{2} \{ f(u + \Delta u, v) + f(u, v) \}
\]
\[
\bar{\Delta}_{vx} f = \frac{1}{2} \{ f(u, v + \Delta x) + f(u, v) \}
\]
\[
\bar{\Delta}_{vy} f = \frac{1}{2} \{ f(u, v + \Delta y) + f(u, v) \}
\]
From which we deduce the following identities:

\[ \Delta_x f = \Delta_u \Delta_v f + \Delta_u \Delta_v f \]  
(1.5)

\[ \Delta_y f = \Delta_u \Delta_v f + \Delta_u \Delta_v f \]  
(1.6)

\[ \bar{\Delta}_x f = \bar{\Delta}_u \bar{\Delta}_v f + \frac{1}{4} \Delta_{uu} \Delta_{vv} f \]  
(1.7)

\[ \bar{\Delta}_y f = \bar{\Delta}_u \bar{\Delta}_v f + \frac{1}{4} \Delta_{uu} \Delta_{vv} f \]  
(1.8)

These formulas can easily be applied to vector-valued functions:

\[ \bar{u} = (u_1(x), u_2(x), \ldots, u_n(x)) = \bar{u}(x) \]

and its “tangent vector”

\[ \frac{\Delta \bar{u}}{\Delta x} = \left( \frac{\Delta u_1}{\Delta x}, \frac{\Delta u_2}{\Delta x}, \ldots, \frac{\Delta u_n}{\Delta x} \right) \equiv \bar{v}(x) \]

An immediate application is the four-position and four-velocity vectors in special relativity:

\[ x^\mu(\tau) \equiv (x^1(\tau), x^2(\tau), x^3(\tau), x^4(\tau)) \]

\[ V^\mu(\tau) \equiv \left( \frac{\Delta x^1}{\Delta \tau}, \frac{\Delta x^2}{\Delta \tau}, \frac{\Delta x^3}{\Delta \tau}, \frac{\Delta x^4}{\Delta \tau} \right) \]

These vector-valued vector can be expressed as

\[ \bar{u} = u^a \bar{e}_a \]

for a given set of orthonormal vectors \( \bar{e}_a \)

### 2 Discrete differential forms

Given a vectorial space \( V^n \) over \( \mathbb{Z} \) we can define a real-valued linear function over \( \mathbb{Z} \)

\[ f(u) \equiv \langle \omega, u \rangle \quad u \in V^n \]  
(2.1)

The forms \( \omega \) constitute a vectorial linear space (dual space) \( *V^n \), and can be expanded in terms of a basis \( \omega^\alpha \)

\[ \omega = \sigma_\alpha \omega^\alpha \]

The basis \( e_\beta \) of \( V^n \) and \( \omega^\alpha \) of \( *V^n \) can be contracted in the following way

\[ \langle \omega^\alpha, e_\beta \rangle \delta^\alpha_\beta \]  
(2.2)

hence

\[ \langle \omega, e_a \rangle = \sigma_\alpha , \quad \langle \omega^\beta, u \rangle = u^\beta , \quad \langle \omega, u \rangle = \sigma_\alpha u^\alpha \]  
(2.3)
If we take $\omega^\beta = \Delta x^\beta$ as coordinate basis for the linear forms we can construct discrete differential forms (a discrete version of the continuous differential forms).

A particular example of this discrete form is the total difference operator $(1,3)$ of a function of several discrete variables written in the following way:

$$
\Delta f(x, y) = \left( \frac{\Delta_y f}{\Delta x} \right) \Delta x + \left( \frac{\Delta_x f}{\Delta y} \right) \Delta y
$$

(2.4)

For these discrete forms we can define the exterior product of two form $\sigma$ and $\delta$

$$
\rho \wedge \sigma = -\sigma \wedge \rho
$$

which is linear in both arguments.

For the coordinate basis we also have

$$
\Delta x \wedge \Delta y = -\Delta y \wedge \Delta x
$$

With the help of this exterior product we can construct a second order discrete differential form or 2-form, namely

$$
\rho \wedge \sigma = -\rho_{\alpha} \Delta x^\alpha \wedge \sigma_\alpha \Delta x^\beta = \frac{1}{2} (\rho_{\alpha} \sigma_\beta - \rho_\beta \sigma_\alpha) \Delta x^\alpha \wedge \Delta x^\beta \equiv \sigma_{\alpha\rho} \Delta x^\alpha \wedge \Delta x^\beta
$$

(2.5)

where $\sigma_{\alpha\rho}$ is an antisymmetric tensor. Similarly we can define a discrete $p$-form in a $n$-dimensional space ($p < n$)

$$
\sigma = \frac{1}{p!} \sigma_{i_1 i_2 \ldots i_p} \Delta x^{i_1} \wedge \Delta x^{i_2} \ldots \wedge \Delta x^{i_p}
$$

where $\sigma_{i_1 i_2 \ldots i_p}$ is a totally antisymmetric tensor.

The dual of a $p$-form in a $n$-dimensional space is the $(n-p)$-form $^\ast \alpha$ with components

$$
(^\ast \alpha)_{k_1 k_2 \ldots k_{n-p}} = \frac{1}{p!} \alpha^{i_1 i_2 \ldots i_p} \varepsilon_{i_1 i_2 \ldots i_p k_1 \ldots k_{n-p}}
$$

where $\varepsilon$ is the $n$-dimensional Levy-Civitá totally antisymmetric tensor ($\varepsilon_{123\ldots} \equiv 1$).

We give now some examples:

**Energy-momentum 1-form**

$$
P = -E \Delta t + P_x \Delta x + P_y \Delta y + P_z \Delta z
$$

(2.6)

where $(P_x, P_y, P_z, iE) \equiv P_n$ is the four-momentum.

**Vector potential 1-form**

$$
A = A_\mu \Delta x^\mu = A_x \Delta x + A_y \Delta y + A_z \Delta z + A_t \Delta t
$$

(2.7)

where $A_\mu = (A_x, A_y, A_z, A_t)$ is the four-potential.
Charge-current 1-form

\[ J = J_\mu \Delta x^\mu = J_x \Delta x + J_y \Delta y + J_z \Delta z - \rho \Delta t \]  

where \( J_\mu \) is the density current four-vector.

Faraday 2-form

\[ F = E_x \Delta x \wedge \Delta t + E_y \Delta y \wedge \Delta t + E_z \Delta z \wedge \Delta t + B_x \Delta y \wedge \Delta z + B_y \Delta z \wedge \Delta x + B_z \Delta x \wedge \Delta y = \frac{1}{2} F_{\mu \nu} \Delta x^\mu \wedge \Delta x^\nu \]  

with \((B_x, B_y, B_z) \equiv \vec{B}\) and \((E_x, E_y, E_z) \equiv \vec{E}\) the magnetic and electric field, respectively.

Maxwell 2-form (dual of Faraday 2-form)

\[ *F = \frac{1}{2} \varepsilon_{\mu \nu \lambda \kappa} F^{\lambda \kappa} \Delta x^\mu \wedge \Delta x^\nu = - B_x \Delta x \wedge \Delta t - B_y \Delta y \wedge \Delta t - B_z \Delta z \wedge \Delta t + E_x \Delta y \wedge \Delta z + E_y \Delta z \wedge \Delta x + E_z \Delta x \wedge \Delta y \]  

3 Exterior calculus

Given a 1-form in a two-dimensional space

\[ \omega = a(x, y) \Delta x + b(x, y) \Delta y \]

we can define the exterior difference, in the similar way as the exterior derivative, namely,

\[ \Delta \omega \equiv \Delta a \wedge \Delta x + \Delta b \wedge \Delta y \]

\[ = \left( \frac{\Delta \Delta a}{\Delta x} \Delta x + \frac{\Delta \Delta b}{\Delta y} \Delta y \right) \wedge \Delta x + \left( \frac{\Delta \Delta b}{\Delta x} \Delta x + \frac{\Delta \Delta a}{\Delta y} \Delta y \right) \wedge \Delta y \]

\[ = \left( \frac{\Delta \Delta a}{\Delta x} \Delta x - \frac{\Delta \Delta b}{\Delta y} \Delta y \right) \Delta x \wedge \Delta y \]  

where in the last expression we have used the properties of the exterior product.

This definition of exterior difference can be easily written for 1-form in \( n \)-dimensional space.

Given a 2-form in a 3-dimensional space,

\[ \omega = a(x, y, z) \Delta y \wedge \Delta z + b(x, y, z) \Delta z \wedge \Delta x + c(x, y, z) \Delta x \wedge \Delta y \]  

we can also define the exterior difference as:

\[ \Delta \omega = \Delta a \wedge \Delta y \wedge \Delta z + \Delta b \wedge \Delta z \wedge \Delta x + \Delta c \wedge \Delta x \wedge \Delta y \]

\[ = \left( \frac{\Delta \Delta a}{\Delta x} \Delta x + \frac{\Delta \Delta b}{\Delta y} \Delta y + \frac{\Delta \Delta c}{\Delta z} \Delta z \right) \Delta x \wedge \Delta y \wedge \Delta z \]
Given a 3-form in a 4-dimensional space
\[ \omega = a \Delta y \wedge \Delta z + \Delta t + b \Delta z \wedge \Delta t \wedge \Delta x + c \Delta t \wedge \Delta x \wedge \Delta y + d \Delta x \wedge \Delta y \wedge \Delta z \]
we can define an exterior difference as before:
\[ \omega = \Delta a \wedge \Delta y \wedge \Delta z \wedge \Delta t + \Delta b \wedge \Delta z \wedge \Delta t \wedge \Delta x + \Delta c \wedge \Delta t \wedge \Delta x \wedge \Delta y + \Delta d \wedge \Delta x \wedge \Delta y \wedge \Delta z \]
\[ = \left( \frac{\Delta_x \Delta_y \Delta_z \Delta_t a}{\Delta z} - \frac{\Delta_x \Delta_y \Delta_z \Delta_t b}{\Delta y} + \frac{\Delta_x \Delta_y \Delta_z \Delta_t c}{\Delta y} - \frac{\Delta_x \Delta_y \Delta_z \Delta_t d}{\Delta t} \right) \Delta x \wedge \Delta y \wedge \Delta z \wedge \Delta t \] (3.4)

The exterior derivative applied to the product of a 0-form (scalar function \( f \)) and a 1-form (\( \omega = a \Delta x + b \Delta y \)) is
\[ \Delta (f \omega) = \tilde{\Delta} f \Delta \omega + \Delta f \wedge \tilde{\Delta} \omega \] (3.5)
where \( \tilde{\Delta} f \) is expressed in (1.4) and \( \tilde{\Delta} \omega = \tilde{\Delta} a \Delta x + \tilde{\Delta} b \Delta y \)

The exterior difference of the product of two 1-forms is easily obtained
\[ \Delta \{ \omega_1 \wedge \omega_2 \} = \Delta \omega_1 \wedge \tilde{\Delta} \omega_2 - \tilde{\Delta} \omega_1 \wedge \Delta \omega_2 \] (3.6)

The exterior difference of the product of a p-form \( \rho \) and a q-form \( \sigma \) is
\[ \Delta \{ \rho \wedge \sigma \} = \Delta \rho \wedge \tilde{\Delta} \sigma + (-1)^p \tilde{\Delta} \rho \wedge \Delta \sigma \] (3.7)

Finally for any p-form \( \omega \) we have
\[ \Delta^2 \omega = \Delta (\Delta \omega) = 0 \] (3.8)

Some examples:
From the Faraday 2-form we write down one set of Maxwell difference equations
\[ \Delta \mathbf{F} = \Delta (\Delta \mathbf{A}) = 0 \]
\[ = \left( \frac{\Delta_x \Delta_y \Delta_z \Delta_t B_x}{\Delta z} + \frac{\Delta_x \Delta_y \Delta_z \Delta_t B_y}{\Delta y} + \frac{\Delta_x \Delta_y \Delta_z \Delta_t B_z}{\Delta y} \right) \Delta x \wedge \Delta y \wedge \Delta z \]
\[ = \left( \frac{\Delta_x \Delta_y \Delta_z \Delta_t E_z}{\Delta t} + \frac{\Delta_x \Delta_y \Delta_z \Delta_t E_y}{\Delta y} - \frac{\Delta_x \Delta_y \Delta_z \Delta_t E_y}{\Delta z} \right) \Delta t \wedge \Delta y \wedge \Delta z \]
\[ = \left( \frac{\Delta_x \Delta_y \Delta_z \Delta_t E_x}{\Delta t} + \frac{\Delta_x \Delta_y \Delta_z \Delta_t E_y}{\Delta z} - \frac{\Delta_x \Delta_y \Delta_z \Delta_t E_y}{\Delta x} \right) \Delta t \wedge \Delta z \wedge \Delta x \]
\[ = \left( \frac{\Delta_x \Delta_y \Delta_z \Delta_t E_x}{\Delta t} + \frac{\Delta_x \Delta_y \Delta_z \Delta_t E_y}{\Delta x} - \frac{\Delta_x \Delta_y \Delta_z \Delta_t E_y}{\Delta y} \right) \Delta t \wedge \Delta x \wedge \Delta y \] (3.9)
from the Maxwell 2-dual form $\ast F$ we get the other set of Maxwell equations:

\[ \Delta^* F = 4\pi^* J \]

\[
\begin{align*}
\left( \frac{\Delta_x \Delta_y \Delta_z \Delta_t F_x}{\Delta x} + \frac{\Delta_x \Delta_y \Delta_z \Delta_t F_y}{\Delta y} + \frac{\Delta_x \Delta_y \Delta_z \Delta_t F_z}{\Delta z} \right) & \Delta x \land \Delta y \land \Delta z \\
+ \left( \frac{\Delta_x \Delta_y \Delta_z \Delta_t F_y}{\Delta y} - \frac{\Delta_x \Delta_y \Delta_z \Delta_t B_z}{\Delta z} + \frac{\Delta_x \Delta_y \Delta_z \Delta_t B_y}{\Delta y} \right) & \Delta t \land \Delta y \land \Delta z \\
+ \left( \frac{\Delta_x \Delta_y \Delta_z \Delta_t E_y}{\Delta y} - \frac{\Delta_x \Delta_y \Delta_z \Delta_t B_x}{\Delta x} + \frac{\Delta_x \Delta_y \Delta_z \Delta_t B_z}{\Delta x} \right) & \Delta t \land \Delta z \land \Delta x \\
+ \left( \frac{\Delta_x \Delta_y \Delta_z \Delta_t E_z}{\Delta z} - \frac{\Delta_x \Delta_y \Delta_z \Delta_t B_y}{\Delta y} + \frac{\Delta_x \Delta_y \Delta_z \Delta_t B_y}{\Delta y} \right) & \Delta t \land \Delta x \land \Delta y \\
= 4\pi \left( \rho \Delta x \land \Delta y \land \Delta z - J_z \Delta t \land \Delta y \land \Delta z \\
- J_y \Delta t \land \Delta z \land \Delta x - J_x \Delta t \land \Delta x \land \Delta y \right) \\
\end{align*}
\]

(3.10)

Taking the exterior derivative of the last equation we get another example of $\Delta^2 = 0$.

\[
\left( \frac{\Delta_x \Delta_y \Delta_z \Delta_t \rho}{\Delta t} + \frac{\Delta_x \Delta_y \Delta_z \Delta_t J_x}{\Delta x} + \frac{\Delta_x \Delta_y \Delta_z \Delta_t J_y}{\Delta y} + \frac{\Delta_x \Delta_y \Delta_z \Delta_t J_z}{\Delta z} \right) \cdot \Delta t \land \Delta x \land \Delta y \land \Delta z = 0
\]

(3.11)

Note that the coefficient of the difference form is the discrete version of the continuity equation.

From a scalar function we get the wave equations in terms of difference operators, namely,

\[ -\ast \Delta^* \Delta \phi \equiv \Box \phi \]

where $\Box$ is the discrete d’Alambertian operator:

\[
\left\{ -\nabla_x \nabla_y \nabla_z \nabla_t \left( \Delta_x \Delta_y \Delta_z \Delta_t \phi \right) + \nabla_x \nabla_y \nabla_z \nabla_t \left( \Delta_x \Delta_y \Delta_z \Delta_t \phi \right) \\
+ \nabla_x \nabla_y \nabla_z \nabla_t \left( \Delta_x \Delta_y \Delta_z \Delta_t \phi \right) + \nabla_x \nabla_y \nabla_z \nabla_t \left( \Delta_x \Delta_y \Delta_z \Delta_t \phi \right) \right\} \phi (xyzt) = 0
\]

(3.12)

(3.13)

From the vector potential 1-forms $A = A_\mu \Delta x^\mu$ we can construct Faraday 2-form $F = \Delta A$ from which the Maxwell equations are derived

\[ \Delta^* F = \Delta^* \Delta A = 4\pi^* J \]

Taking the dual of this expression we obtain

\[ \ast \Delta^* \Delta A = 4\pi J \]

(3.14)

If we choose the Lorentz condition

\[
\frac{\Delta_x \Delta_y \Delta_z \Delta_t A_x}{\Delta x} + \frac{\Delta_x \Delta_y \Delta_z \Delta_t A_y}{\Delta y} + \frac{\Delta_x \Delta_y \Delta_z \Delta_t A_z}{\Delta z} - \frac{\Delta_x \Delta_y \Delta_z \Delta_t A_t}{\Delta t} = 0
\]

(3.15)

we obtain the wave equation for the vector potential

\[ \Box A_\mu = 4\pi J_\mu \]

(3.16)

where $\Box$ is the d’Alambertian defined in (3.13)
4 Lorentz transformations

In order to compute the transformation of the discrete differential forms we start with
the coordinate-independent nature of 1-form

\[ \omega = \omega_\mu \Delta x^\mu \]  \hspace{1cm} (4.1)

where the \( \Delta x^\mu \) are the space-time intervals in Minkowski space-time. From

\[ \Delta x^{\mu'} = \Lambda^{\mu'}_\mu \Delta x^\mu \]  \hspace{1cm} (4.2)

where \( \Lambda^{\mu'}_\mu \) is a global Lorentz transformation, and from the coordinate-f ree expresion for \( \omega \) we get

\[ \omega^{\mu'} = \omega_\nu \Lambda^{\nu'}_\mu \]  \hspace{1cm} (4.3)

Recall that \( \Lambda^{\nu'}_\nu \Lambda^{\mu'}_\rho = \delta^{\nu'}_\rho \)

From the total difference of a function of several variables \( f(x, y, z, t) \)

\[ \Delta f = \frac{\Delta x \Delta y \Delta z \Delta t f}{\Delta x} \Delta x + \frac{\Delta x \Delta y \Delta z \Delta t f}{\Delta y} \Delta y + \frac{\Delta x \Delta y \Delta z \Delta t f}{\Delta z} \Delta z + \frac{\Delta x \Delta y \Delta z \Delta t f}{\Delta t} \Delta t \]  \hspace{1cm} (4.4)

it follows that the coefficients of the 1-forms, namely,

\[ \left( \frac{\Delta x \Delta y \Delta z \Delta t f}{\Delta x}, \frac{\Delta x \Delta y \Delta z \Delta t f}{\Delta y}, \frac{\Delta x \Delta y \Delta z \Delta t f}{\Delta z}, \frac{\Delta x \Delta y \Delta z \Delta t f}{\Delta t} \right) \]  \hspace{1cm} (4.5)

transform covariantly like the coefficients \( \omega_\mu \) of (4.1). Note that in this case the meaning of \( \Delta x, \Delta y, \Delta z, \Delta t \) in the denominator is different from the meaning in the numerator, because the later \( \Delta x^\mu \) are elements of the exterior products, and the former \( \Delta x \) are small scalars. The different roll of these quantities becomes clear in the continuous limit, where \( \Delta x \to ds \) and \( \frac{\Delta x \Delta y \Delta z \Delta t f}{\Delta x} \to \frac{df}{dx} \)

For the 2-form in Minkowski space

\[ \mathbf{F} = \frac{1}{2} F_{\mu \nu} \Delta x^\mu \wedge \Delta x^\nu \]

we obtain

\[ \mathbf{F} = \frac{1}{2} F_{\mu \nu \rho} \Delta x^\mu \wedge \Delta x^\nu \wedge \Delta x^\rho \]

because of the coordinate independent nature of the Faraday 2-form. From the transformation of \( \Delta x^\mu \) (see (4.2)) and the properties of the exterior product we get

\[ F_{\mu \nu \rho} = F_{\alpha \beta} \Lambda^\alpha_\mu \Lambda^\beta_\nu \]  \hspace{1cm} (4.6)

The same technic can be applied to components of discrete p-forms. The dual of a p-form are also discrete \((n - p)\)-form, therefore their components transform covariantly like totally antisymmetric tensor. For instance the components of Maxwell 2-forms.
\[(^* F)_{\alpha\beta} = \epsilon_{\alpha\beta\mu\nu} F^{\mu\nu} \quad (\epsilon_{1234} = +1)\]

transform covariantly

\[(^* F)_{\alpha\prime\beta\prime} = (^* F)_{\kappa\lambda} \Lambda_{\alpha\prime\beta\prime}^{\kappa\lambda} \quad (4.7)\]

With this definition of the transformation of the components of a discrete \(p\)-form we can prove the covariance of the discrete wave equation, and the covariance of Maxwell equation in discrete form.

5 Application to Klein-Gordon and Dirac wave equation on the lattice

We define the scalar function on the \((3+1)\) dimensional cubic lattice

\[\phi (j_1 \epsilon_1, j_2 \epsilon_2, j_3 \epsilon_3, n \tau) \equiv \phi (\vec{j}, n)\]

where \(\epsilon_1, \epsilon_2, \epsilon_3, \tau\) are small quantities in the space-time directions and \(j_1, j_2, j_3, n\) are integer numbers.

We define the difference operators

\[\delta^+_{\mu} \equiv \frac{1}{\epsilon_\mu} \Delta_{\mu} \prod_{\nu \neq \mu} \bar{\Delta}_\nu, \mu, \nu = 1, 2, 3, 4\]

\[\delta^-_{\mu} \equiv \frac{1}{\epsilon_\mu} \nabla_{\mu} \prod_{\nu \neq \mu} \bar{\nabla}_\nu\]

\[\eta^+ \equiv \prod_{\mu=1}^4 \bar{\Delta}_\mu\]

\[\eta^- \equiv \prod_{\mu=1}^4 \bar{\nabla}_\mu\]

Then the Klein-Gordon wave equations defined on the grid points of the lattice can be read off

\[
\left( \delta^+_{1} \delta^-_{1} + \delta^+_{2} \delta^-_{2} + \delta^+_{3} \delta^-_{3} - \delta^+_{4} \delta^-_{4} - M^2 \eta^+ \eta^- \right) \phi (\vec{j}, n) = 0 \quad (5.1)
\]

It can be verified by direct substitution that the plane wave solution satisfy the difference equation

\[f (\vec{j}, n) \equiv \left( \frac{1 + \frac{1}{2} i \epsilon_1 k_1}{1 - \frac{1}{2} i \epsilon_1 k_1} \right)^{j_1} \left( \frac{1 + \frac{1}{2} i \epsilon_2 k_2}{1 - \frac{1}{2} i \epsilon_2 k_2} \right)^{j_2} \left( \frac{1 + \frac{1}{2} i \epsilon_3 k_3}{1 - \frac{1}{2} i \epsilon_3 k_3} \right)^{j_3} \left( \frac{1 - \frac{1}{2} i \tau \omega}{1 + \frac{1}{2} i \tau \omega} \right)^n \quad (5.2)\]

provided the “dispersion relation” is satisfied

\[\omega^2 - k_1^2 - k_2^2 - k_3^2 = M^2 \quad (5.3)\]
From the last section the Klein-Gordon equation is invariant under finite Lorentz transformations. Imposing boundary conditions on the plane waves we can construct a complete set of orthogonal functions, hence a Fourier analysis can be developed as it has been done in Ref. [1].

The discrete version of the Dirac wave equation can be written as

\[(\gamma_1 \delta^+_1 + \gamma_2 \delta^+_2 + \gamma_3 \delta^+_3 - i \gamma_4 \delta^+_4 + M \eta^+) \psi (\vec{j}, n) = 0 \quad (5.4)\]

where \(\gamma_\mu, \mu = 1, 2, 3, 4\) are the usual Dirac matrices. Applying the operator

\[\left(\gamma_1 \delta^-_1 + \gamma_2 \delta^-_2 + \gamma_3 \delta^-_3 - i \gamma_4 \delta^-_4 - M \eta^-\right)\]

from the left on both sides of (5.4) we recover the Klein-Gordon equation (5.1). Let now construct solutions to (5.4) of the form

\[\psi (\vec{j}, n) = \omega (\vec{k}, E) \ f (\vec{j}, n)\]

where the \(f (\vec{j}, n)\) are given in (5.2).

The four-component spinors \(\omega (\vec{k}, E)\), with momentum \(\vec{k} \equiv (k_1, k_2, k_3)\), must satisfy

\[(i \vec{\gamma} \cdot \vec{k} - \gamma_4 E + M) \omega (\vec{k}, E) = 0 \quad (5.5)\]

as in the continuous case. Multiplying this equation from the left by \((i \vec{\gamma} \cdot \vec{k} - \gamma_4 E - M)\)

we obtain the dispersion relation

\[E^2 - \vec{k}^2 = M^2 \quad (5.6)\]

From (5.5) we obtain the spinors solutions corresponding to positive energy as in the continuous case

\[n_{k\sigma} \equiv \omega_\sigma (\vec{k}, E) = \frac{E_k + M^{1/2}}{2M} \left(\begin{array}{c} \xi_\sigma \\ \frac{\vec{\sigma} \cdot \vec{k}}{E_k + M} \xi_\sigma \end{array}\right) \quad (5.7)\]

with \(E_k \equiv +\sqrt{\vec{k}^2 + M^2}, \sigma = 1, 2\) and

\[\xi_1 = \left(\begin{array}{c} 1 \\ 0 \end{array}\right), \quad \xi_2 = \left(\begin{array}{c} 0 \\ 1 \end{array}\right)\]

Similarly for negative energy spinors.

In order to analyze the Lorentz invariant of the Dirac wave equation on the lattice we take the Lorentz transformations of the difference operators as in (4.3)

\[\delta_{\mu}' = \Lambda_{\mu\nu} \delta_\nu \quad (5.8)\]

\[\delta^{-\mu}' = \Lambda_{\mu\nu} \delta^-_\nu \quad (5.9)\]

The linear transformation law for \(\psi\)
\[ \psi' = S \psi \]
is determined by the requirement that \( \psi' \) satisfy the same equation in the transformed frame as does \( \psi \) in the original frame

\[ \left( \gamma_\mu \delta_\mu^+ + M \right) \psi' = 0 \quad (5.10) \]

With (5.8) we have

\[ \left( \Lambda_{\mu\nu} \gamma_\nu \delta_\mu^+ + M \right) S \psi = 0 \quad (5.11) \]

Multiplying from the left by \( S^{-1} \) we recover the Dirac equation in the original form provided

\[ S^{-1} \gamma_\mu S = \Lambda_{\mu\nu} \gamma_\nu \quad (5.12) \]

A particular solution for the Lorentz transformations of the spinors in the case of rotations is

\[ \psi' = S \psi = (\cos \alpha + i \sin \alpha \gamma_5) \psi \quad (5.13) \]

where \( \alpha \) is the angle of rotation and \( \gamma_5 \equiv \gamma_1 \gamma_2 \gamma_3 \gamma_4 \). In the case of pure Lorentz transformation we have

\[ \psi' = S \psi = \cosh \beta \psi + \sinh \beta \gamma_4 \gamma_2 \overline{\psi} \quad (5.14) \]

with \( \overline{\psi} = \gamma_4 \psi^* \) and \( \beta \) is the usual parameter such that \( \tanh \beta = \frac{v}{c} \).

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