THE ATIYAH-BOTT LEFSCHETZ FORMULA APPLIED TO THE BASED LoOPS ON SU(2)

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Abstract. The Atiyah-Bott Lefschetz Formula is a well-known formula for computing the equivariant index of an elliptic operator on a compact smooth manifold. We provide an analogue of this formula for the based loop group \( \Omega SU(2) \) with respect to the natural \((T \times S^1)\)-action. From this result we also derive an effective formula for computing characters of certain Demazure modules.

1. Introduction

Let \( G \) be a compact, simply connected Lie group. Denote by \( \Omega G \) the set of all smooth loops based at \( e \in G \), i.e. \( \Omega G = \{ f \in C^\infty(S^1, G) \mid f(1) = e \} \). We then consider the subgroup of polynomial loops \( \Omega_{poly} G = \{ f \in \Omega G \mid f(t) = \sum_{k=-N}^{N} a_k t^k \} \), the based loops with finite Fourier expansion. Let \( G_C \) be the complexification of \( G \). It is known that \( \Omega_{poly} G \) is homotopy equivalent to the affine Grassmannian of \( G_C \) defined by \( \text{Gr} = G_C((t))/G_C[[t]] \), where \( G_C((t)) \) and \( G_C[[t]] \) denote Laurent series with values in \( G_C \) and power series with values in \( G_C \) respectively.

\( \Omega G \) is an infinite-dimensional symplectic manifold equipped with the 2-form

\[
\omega(X(t), Y(t)) = \frac{1}{2\pi} \int_0^{2\pi} \langle X(t), Y'(t) \rangle dt
\]

There are two natural Lie groups acting on \( \Omega G \): First the maximal torus \( T \subset G \) acts by pointwise conjugation \( k \cdot \theta(t) = k \theta(t) k^{-1} \) for \( k \in T \), second the circle group \( S^1 \) acts by rotating the loops. However, since the loops must be based, we instead have a "based rotation" given by \( \theta \cdot \gamma(t) = \gamma(t + \theta) \gamma(\theta)^{-1} \) for \( \theta \in S^1 \). These two actions commute with each other. (See [2] for more details)

This \((T \times S^1)\)-action is in fact Hamiltonian with respect to \( \omega \), and the corresponding moment map is given by \( \mu = (E, \rho) : \Omega G \to \text{Lie}(T \times S^1)^* \) with

\[
E(\gamma) = \frac{1}{4\pi} \int_0^{2\pi} \| \gamma(t)^{-1} \gamma'(t) \|^2 dt
\]

\[
\rho(\gamma) = \frac{1}{2\pi} \int_0^{2\pi} \text{pr}_{\text{Lie}(T)}(\gamma(t)^{-1} \gamma'(t)) dt
\]

The moment map and symplectic form restrict to the polynomial loops, and for the remainder of the paper we will refer to \( \Omega_{poly} G \) as \( \Omega G \) for simplicity.

In the 1960s, it was conjectured that the \( G \)-equivariant index of a transversally elliptic operator can be computed purely in data terms around the \( G \)-fixed points. Such a formula was proven by Atiyah and...
Bott \[1\] and is given in Section \[5\] As we have mentioned previously, $\Omega G$ is homotopy equivalent to $Gr$, so we may also consider $\Omega G$ as an infinite dimensional Grassmannian. As such, it exhibits properties similar to those of flag manifolds and Grassmannians. One such property is the Bruhat decomposition, which says that $Gr$ is an (infinite) disjoint union of affine spaces called \textbf{Bruhat cells}. This property implies that $Gr$ has a filtration by finite-dimensional (possibly singular) spaces formed by finite unions of Bruhat cells, these spaces are known as \textbf{Schubert varieties}.

The Atiyah-Bott formula however, applies only to smooth manifolds. To this end, we use a well-known desingularization of Schubert varieties called \textbf{Bott-Samelson manifolds}. It was shown by Kumar \[6\] that:

1. The $(T \times S^1)$-equivariant indices of the prequantum line bundles on the Schubert varieties and Bott-Samelson manifolds are isomorphic.
2. The equivariant index of the prequantum line bundle on $\Omega G \cong Gr$ is the inverse limit of the equivariant indices of the Schubert varieties.

Armed with these two facts, we may then compute positive-energy representations of $\Omega G$ in the case of $G = SU(2)$ as a limit of equivariant indices of Bott-Samelson manifolds, on which we may employ the Atiyah-Bott fixed point formula. Kumar proved that the character formula for the Bott-Samelson was the same as the character formula for the Schubert varieties. So we prove a character formula for the Bott-Samelson manifolds, and take a limit as the dimension of the Bott-Samelson manifolds goes to infinity.

As a corollary, we obtain an \textbf{effective} character formula for Demazure modules of the affine Lie algebra $\widehat{\mathfrak{sl}}_2$ (i.e. a formula containing explicit weights). The character of Demazure modules was previously only known as a composition of Demazure operators. We also obtain an affine analogue of the Kostant multiplicity formula for semisimple Lie algebras.

2. The \textbf{Affine Weyl Group}

In this section we follow the notes of Magyar \[8\]. First we recall that the Weyl group associated to type $A_{n-1}$ is isomorphic to the symmetric group $S_n$. Given the finite Weyl group $W$ associated to type $A_{n-1}$, the affine Weyl group $\overline{W}$ is the semidirect product $W \rtimes Q$, where $Q$ denotes the coroot lattice of $SU(n)$. Another way of expressing this is that each element in $\overline{W}$ is given by the product of a reflection $r_\alpha$ and a translation $t^\beta$, where $\alpha$ is a root and $\beta \in Q$.

The generators include the standard generators for the finite Weyl group: $s_i = r_{\alpha_i}$ for $i = 1, \cdots, n - 1$. There is an extra generator in the affine Weyl group given by $s_0 = r_{\beta} t^{-\theta}$, where $\theta$ is the highest root. For these generators, the multiplication is given by

$$(r_{\alpha_{i_1}} t^{\beta_{i_1}})(r_{\alpha_{i_2}} t^{\beta_{i_2}}) = r_{\alpha_{i_1}} r_{\alpha_{i_2}} t^{\alpha_{i_1} + r_{\alpha_{i_2}}(\alpha_{i_1})}. $$

\textbf{Example 2.1.} $SU(2)$ has a coroot lattice isomorphic to $\mathbb{Z}$ and its Weyl group is the trivial group $e$, so the Weyl group for affine $\mathfrak{sl}_2$ is the semidirect product $e \times Q \cong \mathbb{Z}$.  

3. \textbf{Bott-Samelson Manifolds}

First we state the important Bruhat decomposition. Let $B$ denote the Iwahori subgroup $\{ f \in G_C[[t]] \mid f(0) \in B \}$ where $B$ is a the standard subgroup of upper triangular matrices of $G$. Let $\ell = \text{rank } G$, then we have the following definition

\textbf{Definition 3.1} (Kumar, \[6\] Definition 6.1.18, p. 186). For any subset $Y \subset \{ 1, \cdots, \ell \}$ define the \textbf{standard parabolic subgroup} $P_Y$ as $P_Y = B W_Y B$ where $W_Y$ is the subgroup of the affine Weyl group generated by $\{ s_i \}_{i \in Y}$.

We consider the case where $Y = \{ 1, \cdots, \ell \}$ and we denote the corresponding subgroup $P_Y$ simply by $P$. Of course in this case, the group $W_Y$ is simply the finite Weyl group $W$.

\textbf{Theorem 3.2} (Bruhat Decomposition). $Gr = \bigsqcup_{w \in \overline{W}} B w P / P$, each component $B w P / P$ is isomorphic to an affine space of dimension $\ell(w)$.

For any reduced word $w$, we may take its lower order ideal $I_w := \{ v \in \overline{W} \mid v \leq w \}$. Then the disjoint union $\bigsqcup_{w \in I_w} B w P / P$ is known as the Schubert variety associated to $w$. It is denoted by $X_w$.

In general $X_w$ may have singularities. In fact in the case of the finite-dimensional flag manifold $G_C / B$, $X_w$ is
smooth if and only if \( w \), seen as an element of \( S_n \), avoids (1324) and (2143) (see [12] for example). Therefore in order to use Atiyah and Bott’s formula, we must use the desingularization given by Bott-Samelson manifolds.

In this paper we will use Magyar’s topological description of Bott-Samelson manifolds (as opposed to the description using lattices in \( \mathsf{Gr} \)). This approach allows us to use well-known coordinates on the Bott-Samelson manifolds to compute fixed point data.

For each reduced word \( w \), denote by \( M_w \) the Bott-Samelson associated to \( w \). It is a desingularization of \( X_w \).

3.1. **Magyar’s Topological Description.** For this paper it suffices to give a description in type A. Let \( \alpha = e_i - e_j \), be a root and \( k \) be an integer. Define \( K_{\alpha_i-e_i,k} \) as the group generated by the maximal torus \( T \in K \) and matrices with a copy of \( \mathsf{SU}(2) \) in the \((ij)\)-block, in the form

\[
\begin{pmatrix}
a & -bt^k \\
bt^{-k} & \varpi
\end{pmatrix}
\]

In type A, the root \( e_i - e_{i+1} \) corresponds to the simple root \( \alpha_i \) for \( i = 1, \ldots, n-1 \).

Let \( \alpha_0 = (\sum_{i} \alpha_i, 1) \) and \( \alpha_i = (\alpha_i, 0) \) for \( i \geq 1 \) denote the affine simple root and the finite simple roots respectively. We now denote by \( K_{(i)} \) the group \( K_{\alpha_i} \).

**Definition 3.3.** Let \( w = s_i_1 s_i_2 \cdots s_i_n \) be a reduced word, the Bott-Samelson manifold associated to \( w \) is given by the group \( M_w := (K_{(i_1)} \times K_{(i_2)} \times \cdots \times K_{(i_n)}) / T^n \) where \( K_{(i_1)} \times K_{(i_2)} \times \cdots \times K_{(i_n)} \) has the following action of \( T^n \):

\[
(x_1, \ldots, x_n) \cdot (t_1, \ldots, t_n) = (x_1 t_1, t_1^{-1} x_2 t_2, \ldots, t_{n-1}^{-1} x_n t_n)
\]

3.2. **Coordinates on \( M_w \).** For a Bott-Samelson manifold \( M_w \) there is a birational multiplication map \( \eta : M_w \to X_w \) to the corresponding Schubert variety. If we consider the Schubert variety as a subset of the loop group, then the Schubert variety inherits the \((T \times S^1)\)-action, where \( T \) acts by conjugation and \( S^1 \) acts by rotation.

We want an action on \( M_w \) that is equivariant with respect to \( \eta \). The action that does the job is

\[
(3.1) \quad t \cdot (k_1, \ldots, k_n) = (tk_1, k_2, \ldots, k_{n-1}, k_n)
\]

\[
(3.2) \quad \theta \cdot (k_1, \ldots, k_n) = (\theta \cdot k_1, \theta \cdot k_2, \ldots, \theta \cdot k_{n-1}, \theta \cdot k_n)
\]

We observe that \( M_w \) is defined by the equivalence relation

\[
(3.3) \quad (k_1, \ldots, k_n) = (k_1 t_1, t_1^{-1} k_2 t_2, \ldots, t_{n-1}^{-1} k_n t_n)
\]

Hence the fixed points are \( (k_1, \ldots, k_n) \) such that \( k_i = \bar{s}_{i_i} \) or 1. Therefore we have a total of \( 2^n \) isolated fixed points.

Let \( G \) be a compact Lie group, for any simple root \( \alpha \in \mathfrak{g}^* \) there is an \( \mathsf{SU}(2) \)-subgroup associated to \( \alpha \). Let \( \Psi_\alpha \) denote the embedding of that \( \mathsf{SU}(2) \)-subgroup into \( G \). Then we have the following description of the affine coordinates on the Bott-Samelson manifold (see [7]):

**Proposition 3.4.** There are affine charts around each fixed point \( k = (k_1, \ldots, k_n) \in M_w \) given by

\[
(3.4) \quad \Phi_{(k_1, \ldots, k_n)} : (z_1, \ldots, z_n) \mapsto (\Psi_{\alpha_1}(m_{z_1}) \cdot k_1, \ldots, \Psi_{\alpha_n}(m_{z_n}) \cdot k_n)
\]

where

\[
m_{z_i} := \frac{1}{\sqrt{1 + |z_i|^2}} \begin{pmatrix} 1 & -z_i \\ z_i & 1 \end{pmatrix}
\]

if \( k_i = e \) and

\[
m_{z_i} := \frac{i}{\sqrt{1 + |z_i|^2}} \begin{pmatrix} z_i & 1 \\ 1 & -z_i \end{pmatrix}
\]

if \( k_i = \bar{s}_{i_i} \).
Even though these coordinates are not holomorphic, we can still treat $M_w$ as a real manifold of dimension $2n$ and use the real coordinates $x_i, y_i$ for each $z_i$. Each Bott Samelson manifold $M_w$ has $2^{l(w)}$ fixed points, indexed by what are called galleries. We now give the following definition due to Häringich [4]:

**Definition 3.5.** Let $w = s_{i_1} \cdots s_{i_n}$ be a reduced word. A gallery associated to $w$ is an element in the set $\{0, 1\}^n$. We denote a gallery $\gamma$ by an $n$-tuple $(\gamma_1, \cdots, \gamma_n)$. We define the realization of a gallery $\gamma$ by the word $s_{i_1}^{\gamma_1} \cdots s_{i_n}^{\gamma_n}$; it is denoted by $u(\gamma)$.

**Example 3.6.** Let $w = s_1 s_2 s_1$, then there are $2^3 = 8$ possible galleries associated to $w$. If we take the gallery $\gamma = (0, 1, 1)$, then its realization is $u(\gamma) = s_1^0 s_2^1 s_1^1 s_2^1 = s_2 s_1$. Note that two galleries may have the same realization, e.g. $u((0, 0, 0))$ and $u((1, 0, 1))$ are both equal to the identity word $e$.

**Proposition 3.7.** Let $w$ be a reduced word. The fixed points of $M_w$ under the $(T \times S^1)$-action are in bijection with the set of galleries associated to $w$. In particular $M_w$ has $2^{l(w)}$ fixed points.

### 3.3. Euler class of tangent spaces at fixed points.

**Corollary 3.8.** The equivariant Euler class of the tangent space of $M_w$ at the fixed point $k = (k_1, \cdots, k_n)$ is given by

$$\left( \prod_{j=1}^{n} \left( \prod_{k=1}^{j} s_{i_k}^{e_k} \right) \cdot (-\alpha_j) \right) X^n,$$

where $e_j = 1$ if $k_j = 1$ and $e_j = 0$ if $k_j = 0$.

**Proof.** Differentiating the chart $\Phi_{\{k_1, \cdots, k_n\}}$ at 0, we get that the tangent space decomposes into

$$\text{span}\{F_{s_{i_1}^1 \alpha_{i_1}}, \cdots, F_{s_{i_n}^k \alpha_{i_n}}\}$$

where $\{F_{\alpha_{i_1}}, F_{\alpha_{i_1}} H_{\alpha_{i_1}}\}$ is the $\mathfrak{sl}_2$-triple associated to $\alpha_{i_1}$.

### 4. Geometric Quantization

#### 4.1. Prequantum line bundles.

A prequantum line bundle over a symplectic manifold $(M, \omega)$ is a Hermitian line bundle with connection $(L, \langle , \rangle, \nabla)$, such that the curvature $\Omega$ of the connection is cohomologous to $\omega$. Let $\Phi$ denote the moment map of a Hamiltonian $G$-action on $M$, then for all $X \in \mathfrak{g}$ we define the quantum operator $H_X := \nabla_{X^\#} + i\langle \Phi, X \rangle$, where $X^\#$ is the fundamental vector field associated to $X$. The map $X \mapsto H_X$ is a Lie algebra homomorphism, and each $H_X$ acts on the space of sections of $L$. Therefore we can consider the sections of $L$ as a representation of $G$ (through the exponential map). Note that if $p \in M$ is a fixed point, then $\nabla_{X^\#} = 0$ for all $X \in \mathfrak{g}$. So the character of $\mathfrak{g} = \exp(X) = \exp(i\langle \Phi, X \rangle)$.

If in addition $M$ is Kahler, then we can use the Kahler polarization on $M$ to get a representation of $G$ on the space of holomorphic sections.

#### 4.2. Line bundles on $\Omega G$.

Let $\mathcal{L}(\lambda)|_{w}$ denote the prequantum line bundle on $\Omega G$ associated to the dominant weight $\lambda$. This bundle can be restricted to Schubert varieties $X_w$ and then pulled back to the Bott-Samelson manifold $M_w$ to get a prequantum line bundle $\eta^* \mathcal{L}(\lambda)|_{X_w}$ on $M_w$.

**Proposition 4.1** (Kumar). The line bundles on $\Omega G$ are parametrized by $k \mathfrak{w}_G$, $k \in \mathbb{Z}$ where $\mathfrak{w}_G$ is the 0-th fundamental weight of $\mathfrak{g}$. Therefore $\text{Pic}(\Omega G) \cong \mathbb{Z}$.

### 5. The Atiyah-Bott fixed point theorem

Now we state a very important formula by Atiyah and Bott [1]:

**Theorem 5.1.** Assume that $M$ is manifold with a Hamiltonian $G$-action and that the fixed point set $M^G$ is finite. Then for any $G$-equivariant vector bundle $V$ on $M$, we get

$$\sum_i (-1)^i \text{tr}(H^i(M, V)) = \sum_{p \in M^G} \frac{\text{tr}(V|_p)}{\det(1 - T^*_p M)}$$

Our goal is to derive a formula for the based loop group $\Omega SU(2)$ in the following form:

$$\sum_i (-1)^i \text{tr}(H^i(\Omega SU(2), V)) = \sum_{w \in \mathcal{W}} \text{tr}(V|_w) \cdot R_w(\Omega SU(2)).$$
Here \( R_w(\Omega SU(2)) \) are certain rational functions depending on \( w \in \overline{W} \). Since the fixed points of the \((T \times S^1)\)-action are indexed by the affine Weyl group elements, this formula is in the exact same form as the Atiyah-Bott formula, except that it contains an infinite sum on the RHS. Note that written in this form, the analogous rational function for compact smooth \( G \)-manifolds would be \( R_p(M) = \det(1 - T_p^*M)^{-1} \). In this case the character of the tangent cone \( \text{ch}(\text{gr}(O_{p,M})) \) is exactly equal to \( \det(1 - T_p^*M) = R_p(M)^{-1} \). For a singular space however, the numerator of the character of the tangent cone will not necessarily be 1.

We run into a small technical difficulty: the based loop group has a filtration by Schubert varieties, which are in general very singular. Therefore we cannot apply this theorem directly. However, we can apply this result to the Bott-Samelson manifolds, which are desingularizations of Schubert varieties. We are allowed to do so due to the following result:

**Proposition 5.2** (Grossberg-Karshon, Kumar).  \[ \mathbb{H}^p(M_w, \eta^*\mathcal{L}(\lambda)|_{X_w}) \cong \mathbb{H}^p(X_w, \mathcal{L}_\lambda|_{X_w}) \]

Furthermore, Kumar proved that

**Theorem 5.3** ([6] Theorem 8.1.25 p. 271). \[ \lim_{w \to \overline{W}} \mathbb{H}^0(M_w, \eta^*\mathcal{L}(\lambda)|_{X_w})^* \cong \Lambda_{\lambda} \text{ where } \Lambda_{\lambda} \text{ is the irreducible representation of the Kac-Moody group with highest weight } \lambda \text{. Here elements of } \overline{W} \text{ are ordered by the Bruhat order.} \]

Therefore it suffices to apply the Atiyah-Bott fixed point theorem to the Bott-Samelson manifolds to obtain the character of the virtual line bundle \( \sum_i (-1)^i \mathbb{H}^i(\mathcal{L}_\lambda|_{X_w}) \). Then we will take a limit over words \( w \) in the affine Weyl group to obtain \( \sum_i (-1)^i \mathbb{H}^i(\Omega SU(2), \mathcal{L}_\lambda) \) in the following sections.

### 5.1. Application to the Bott-Samelson Manifolds

To summarize our setup so far, we now have:

- A smooth presymplectic manifold \((M_w, \eta^*(\omega|_{X_w}))\)
- A Hamiltonian \((T \times S^1)\)-action on \(M_w\) with moment map \( \mu \circ \eta : M_w \to \text{Lie}(T \times S^1)^* \), and \( 2^n \) fixed points parametrized by galleries \( \{ \gamma \in \Gamma_w \} \)
- A prequantum line bundle \( \eta^*\mathcal{L}(\lambda)|_{X_w} \) on \( M_w \)

Applying the Atiyah-Bott formula yields

\[
\sum_i (-1)^i \text{tr}(\mathbb{H}^i(M_w, \eta^*\mathcal{L}(\lambda)|_{X_w})) = \sum_{p \in M_w^{T \times S^1}} \frac{\text{tr}(\eta^*\mathcal{L}(\lambda)|_{X_p})}{\det(1 - T_p^*M_w)}
\]

\[
= \sum_{\gamma \in \Gamma_w} \frac{\text{tr}(\eta^*\mathcal{L}(\lambda)|_{X_{\gamma}})}{\det(1 - T_\gamma^*M_w)} \quad \text{(the fixed points are galleries associated to } w \text{)}
\]

\[
= \sum_{\gamma \in \Gamma_w} \frac{e^{-i\mu(\gamma)}}{\det(1 - T_\gamma^*M_w)} \quad \text{(by Section 4.1)}
\]

\[
= \sum_{\nu \leq w} \sum_{\gamma : \gamma(\nu) = \nu} \frac{1}{\det(1 - T_\gamma^*M_w)}
\]

\[
= \sum_{\nu \leq w} \frac{e^{i(\nu - \lambda)} \prod_{\gamma : \gamma(\nu) = \nu} (1 - e^{s^1_{\gamma_1} \alpha_{\nu_1}})(1 - e^{s^2_{\gamma_1} \alpha_{\nu_2}}) \cdots (1 - e^{s^n_{\gamma_1} \alpha_{\nu_n}})}{\prod_{\nu \leq w} \frac{1}{\det(1 - T_\gamma^*M_w)}}
\]

Here the last equality is due to the fact that \( \det(1 - T_\gamma^*M_w) = \prod\limits_{\alpha} (1 - e^{-\alpha}) \), and \( \alpha \) ranges over all isotropy weights of \( T_\gamma^*M_w \). These were computed in the section on Bott-Samelson manifolds. Note that by Proposition 5.2 the last line is also equal to \( \sum_i (-1)^i \text{tr}(\mathbb{H}^i(X_w, \mathcal{L}(\lambda)|_{X_w})) \).

### 6. The Localization Theorem in Equivariant K-theory

The following two subsections are due to Kumar. (See [6] Chapters 11 and 12)
6.1. **Kostant-Kumar nil-Hecke ring.** Let \( G \) be a Kac-Moody group and \( T \) be a maximal torus. Then denote by \( A[T] \) the group algebra of the character group of \( T \). Denote by \( Q[T] \) the fraction field of \( A[T] \) and let \( \overline{Q[T]} \) be the completion of \( A[T] \): the set of formal infinite sums \( \sum_{e^h \in X(T)} n_{\lambda} e^h \) (\( n_{\lambda} \in \mathbb{Z} \)) with coefficients in \( Q[T] \). Then form the \( Q[T] \) -vector space \( \overline{Q[T]}_W \) with basis \( \{\delta_w\}_{w \in W} \). The space \( \overline{Q[T]}_W \) is an associative ring with multiplication given by

\[
\left( \sum_v q_v \delta_v \right) \cdot \left( \sum_w q_w \delta_w \right) = \sum_w q_w (\nu q_w) \delta_{vw}
\]

and identity \( \delta_e \) (see [6], p. 450).

6.2. **A special element.** Let \( s_1 \) be an element in \( \overline{W} \) which is a simple reflection. Then there is a special element \( T_{s_1} = \frac{1}{1-e^{\alpha_1}} \delta_{s_1} + \frac{1}{1-e^{-\alpha_1}} \delta_e \in Q[T]_W \). For a general word \( w \in W \), we define \( T_w = T_{s_1} \cdots T_{s_{\text{in}}} \). Then we can calculate the coefficients of \( T_w \) with respect to the \( Q[T] \)-basis. Write \( T_w = \sum_{v \leq W} b_{w,v} \delta_v \), then we have

\[
b_{w,v} = \sum \left( (1 - e^{-s_1^{\epsilon_1} \alpha_1}) \cdots (1 - e^{-s_1^{\epsilon_n} \alpha_n}) \right)^{-1}
\]

where the sum is over all \( n \)-tuples \( (\epsilon_1, \cdots, \epsilon_n) \in \{0,1\}^n \) satisfying \( s_1^{\epsilon_1} \cdots s_n^{\epsilon_n} = v \).

**Example 6.1.** We would like to compute \( b_{s_1 s_0, s_1} \). The only subword of \( s_1 s_0 \) equalling \( s_1 \) is \( s_1 s_0^0 \). So

\[
b_{s_1 s_0, s_1} = \frac{1}{1-e^{-s_1 \alpha_1}} \cdot \frac{1}{1-e^{-s_0 \alpha_0}} = \frac{1}{1-e^{\alpha_1}} \cdot \frac{1}{1-e^{-(\alpha_0+2\alpha_1)}}
\]

We also define an involution on \( Q[T]_W \) by \( e^\lambda \mapsto e^{-\lambda} \).

6.3. **Character of the tangent cone.** Let \( Y = G/B \) denote the affine flag variety. Its fixed points under the \( T \) action by left multiplication are parametrized by \( v \in \overline{W} \). Kumar proved the following statement about the \( T \)-characters of tangent cones in the Schubert varieties \( X_w \) around these points.

**Theorem 6.2.** (Kumar, [6] Theorem 12.1.3, p. 451)

\[
\text{ch}(\text{gr}(\mathcal{O}_v, X_w)) = \overline{b_{w,v}}
\]

However we are working with the case of the affine Grassmannian \( G = G/P \) where \( P \) is the maximal parabolic containing all the finite simple roots. Here the \( T \)-fixed points are parametrized by minimal length cosets in \( \overline{W}/W \). So we make a slight modification to the formula above.

Recall the application of the Atiyah-Bott fixed point formula to the Bott-Samelson manifold \( M_w \) given in section [5.1], the last line contained expressions which may be written in the form of \( b_{w,v} \) for certain words \( w, v \). Indeed we may rewrite it as

\[
\sum_{v \leq W} e^{i(v,\lambda)} \sum_{\gamma : W(\gamma) = \nu} \frac{1}{(1 - e^{s_1^{\epsilon_1} \alpha_1}) \cdots (1 - e^{s_1^{\epsilon_n} \alpha_n})} = \sum_{v \leq W} e^{i(v,\lambda)} \cdot b_{w,v} \cdot \sum_{v' \in W, vW = v'W} \overline{b_{w,v'}}.
\]

The last equality follows because the fixed points in the Schubert variety are indexed by cosets \( vW \) and galleries whose realizations belong to the same cosets are mapped to the same value under \( \mu \).

By virtue of Proposition [5.2] we now have a fixed point formula for the Schubert varieties as well:
Proposition 6.3.\[ \sum_i (-1)^i \text{tr}(H^i(X_{w_i}, L(\lambda)|X_{w_i})) = \sum \text{cosets } vW \sum_{v' \in W} \overline{b_{w,v'}} \]

In the language of Section 4, \( R_v(X_w) = \sum_{v' \in W, vW = v'W} \overline{b_{w,v'}} \). In the next section we will derive an effective formula for \( R_v(X_w) \) for the Schubert varieties of \( \Omega SU(2) \).

7. Calculations for \( G = SU(2) \)

We can consider the space of polynomial based loops in \( SU(2) \) as the affine Grassmannian \( G/P \) where \( P \) is the maximal parabolic subgroup corresponding to the set \( I = \{ 1 \} \). For the reduced Weyl group \( W/W_1 \) (see Section 2 for definitions), it is totally ordered w.r.t Bruhat order, with an ascending chain \( s_0 \leq s_1 s_0 \leq s_0 s_1 s_0 \leq ... \). Then we denote the length \( n \) element of this chain by \( w_n \). In this way, the \( T \times S^1 \)-fixed points of \( \Omega SU(2) \) are indexed by the integers.

Moreover \( \Omega SU(2) \) has a filtration by Schubert varieties, and in this case they correspond exactly to the elements of the ascending chain described above, i.e. \( X_{s_0} \subset X_{s_1 s_0} \subset X_{s_0 s_1 s_0} \subset ... \subset \Omega SU(2) \). For more details, see [9].

For a reduced word \( w = s_{i_1} \cdots s_{i_n} \) define \( S_w = \prod_{j=1}^{n} e^{s_{i_1} \cdots s_{i_{j-1}}(\alpha_{i_j})} \). Denote \( S_{w_n} \) by \( S_n \), and similarly denote \( R_{w_n}, X_{w_n} \) by \( R_n, X_n \) respectively.

Definition 7.1. We define the following restricted partitions:
\[
P_{n,k}(m) := \text{the number of integer partitions of } m \text{ into at most } n \text{ parts, all of which are at most } k.
\]

\[
p_{n,k}(m) := \text{the number of integer partitions of } m \text{ into exactly } n \text{ parts, all of which are at most } k.
\]

Example 7.2. \( p_{3,2}(4) = 2 \) since we have \( 2+2, 2+1+1, p_{3,2}(4) = 1 \) since we only have the partition \( 2+1+1 \).

We also denote the number of all integer partitions of \( m \) by \( P(m) \).

We take a moment here to give a reminder on the roots of the affine Lie algebra \( \hat{sl}_2 \). In contrast to semisimple Lie algebras, affine Lie algebras have both real and imaginary roots (roots which have nonpositive length). Let \( \delta = \alpha_0 + \alpha_1 \) denote the smallest positive imaginary root of \( \hat{sl}_2 \). The real roots of \( \hat{sl}_2 \) are given by the set \( \Delta_{re} = \{ \alpha_1 + n\delta \mid n \in \mathbb{Z} \} \) and the positive real roots form the set \( \Delta_{re} = \{ \alpha_1 + n\delta \mid n \in \mathbb{N} \cup \{0\} \} \). For more details on roots of affine Lie algebras, see [7].

Theorem 7.3. The rational functions in the localization formula for the affine Grassmannian \( \Omega G \) are given by
\[
R_m(\Omega G) = \frac{(-1)^m S_m(1 - e^{w_1})}{\prod_{\alpha \in \Delta_{re}^+} (1 - e^{\alpha})} \sum_{n=0}^{\infty} P(n) e^{n\delta}
\]
where \( \Delta_{re}^+ \) denotes the positive real roots of the affine Lie algebra \( \hat{sl}_2 \).

Theorem 7.4. (Second version) The rational functions in the localization formula for the affine Grassmannian \( \Omega G \) are given by
\[
R_m(\Omega G) = \frac{(-1)^m S_m(1 - e^{w_1})}{\prod_{\alpha \in \Delta^+} (1 - e^{\alpha})}
\]
where \( \Delta^+ \) denotes the positive roots of the affine Lie algebra \( \hat{sl}_2 \).
The sum in the numerator in Theorem 7.3 is simply the infinite product \( \prod_{n=0}^{\infty} (1 - e^{n\delta})^{-1} \), and each factor encodes an imaginary root, complementing the infinite product over all positive real roots in Theorem 7.3.

Let us do a few sample computations.

\[
R_1(X_3) = \frac{(-1)S_1(1 - e^{w_1 \cdot \alpha_1})(1 + e^{\delta} + e^{2\delta})}{(1 - e^{\alpha_1})(1 - e^{\alpha_2})(1 - e^{\alpha_3})(1 - e^{\alpha_0 + \delta})(1 - e^{\alpha_0 + 2\delta})}
\]

\[
R_2(X_4) = \frac{S_2(1 - e^{w_2 \cdot \alpha_1})(1 + e^{\delta} + e^{2\delta} + e^{3\delta})}{(1 - e^{\alpha_0})(1 - e^{\alpha_1})(1 - e^{\alpha_1 + \delta})(1 - e^{\alpha_1 + 2\delta})(1 - e^{\alpha_1 + 3\delta})}
\]

\[
R_0(X_4) = \frac{(1 - e^{\alpha_0})(1 + e^{\delta} + e^{2\delta})(1 + e^{3\delta})}{(1 - e^{\alpha_0})(1 - e^{\alpha_0 + \delta})(1 - e^{\alpha_0 + 2\delta})(1 - e^{\alpha_0 + 3\delta})}
\]

\[
R_{id}(X_3) = \frac{e^{2\delta}(1 + e^{\delta})}{(1 - e^{\alpha_0 + \delta})(1 - e^{\alpha_0 + 2\delta})(1 - e^{\alpha_0 + 3\delta})}
\]

To prove the theorem we first prove a finite-dimensional version, let \( \overline{m} \) denote \( m \mod 2 \) for any integer \( m \).

**Lemma 7.5.** The rational functions in the localization formula for the Schubert varieties are given by

\[
R_m(X_n) = \frac{(-1)^m S_m(1 - e^{w_m \cdot \alpha_1})}{\prod_{i=0}^{m + \left\lfloor \frac{m}{2} \right\rfloor} 1 - e^{\alpha_{m+i+1}\delta}} \sum_{j=0}^{\left\lfloor \frac{m}{2} \right\rfloor} \prod_{i=0}^{\left\lfloor \frac{m}{2} \right\rfloor} 1 - e^{\alpha_{m+i}\delta} \left( \prod_{j=0}^{\left\lfloor \frac{m}{2} \right\rfloor} 1 - e^{\alpha_{m+1+j}\delta} \right)
\]

To do this we need an identity on restricted partitions:

**Proposition 7.6.** \( P_{a,b}(j) + P_{a+1,b-1}(j-a-1) = P_{a+1,b}(j) \)

**Proof.** Notice that by definition,

\[
P_{a+1,b}(j) = \sum_{n=0}^{a+1} P_{n,b}(j) = \sum_{n=0}^{a} P_{n,b}(j) + P_{a+1,b}(j) = P_{a,b}(j) + P_{a+1,b}(j)
\]

therefore it suffices to prove that \( P_{a+1,b-1}(j-a-1) = P_{a+1,b}(j) \). First we establish some bounds on \( j \):

If \( j > (a+1)b \), then both sides equal 0. Also if \( j < a+1 \), then both sides equal 0. So we assume that \( a+1 \leq j \leq (a+1)b \).

Then \( P_{a+1,b-1}(j-a-1) \) counts the number of Young diagrams with at most \( a+1 \) rows and at most \( b-1 \) columns, with \( j-a-1 = j-(a+1) \) total blocks. We call this set of Young diagrams \( Y_{a+1,b-1}(j-(a+1)) \).

On the other hand \( P_{a+1,b}(j) \) counts the number of Young diagrams with exactly \( a+1 \) rows and at most \( b-1 \) columns, with \( j \) total blocks, call this set of Young diagrams \( y_{a+1,b-1}(j) \). Then we have a map

\[
Y_{a+1,b-1}(j-(a+1)) \to y_{a+1,b-1}(j)
\]

\[
(\lambda_1, \cdots, \lambda_{a+1}) \mapsto (\lambda_1 + 1, \cdots, \lambda_{a+1} + 1)
\]

which is in fact a bijection. Therefore \( P_{a+1,b-1}(j-a-1) = P_{a+1,b}(j) \).

**Proposition 7.7.** \( P_{a+1,b-1}(j) + P_{a,b}(j-b) = P_{a+1,b}(j) \)

**Proof.** By the correspondence to Young diagrams, we see that \( P_{a,b}(j) = P_{b,a}(j) \) for all \( a, b, j \) just by conjugation. Then the identity follows from Proposition 7.6 by setting \( a = b - 1, b = a + 1 \).

**Proof of lemma 7.5.**

The proof of the lemma is by induction on \( n \).

For \( n = 1 \), \( X_1 = M_1 = \begin{pmatrix} a & -bt \\ bt & a \end{pmatrix} \) where \( a, b \in \mathbb{C} \). This space is diffeomorphic to \( \mathbb{CP}^1 \) and the only fixed
points of \( X_1 \) are the north and south poles whose moment map images are \( \text{id} \) and \( s_0 \) respectively (i.e. the correspond to the Weyl group elements \( \text{id} \) and \( s_0 \)). Using formula (6.2) we get that \( R_0(X_1) = R_{\text{id}}(X_1) = \frac{1}{1-e^{-\alpha}} \) and \( R_1(X_1) = R_{s_0}(X_1) = \frac{1}{1-e^{-2\alpha}} \). So the lemma is true for \( n = 1 \).

Now assume that the lemma is true for \( n = k \), now we wish to compute the rational functions \( R_m(X_{k+1}) \) for all \( m \leq k + 1 \). Notice that \( \omega_{k+1} = s_\alpha \cdot \omega_k \) for some simple root \( \alpha \), and we recall that \( R_m(X_{k+1}) \) is a sum of the rational functions \( \sum_{mW = m'W} P_{k_{1,m}} \). Notice that if \( w = s_i \cdot w' \) then we have the identity

\[
\mathbf{b}_{w,v} = (1 - e^{\alpha_i})^{-1} (\mathbf{b}_{w',v} - e^{-\alpha_i} s_i \cdot \mathbf{b}_{w',s_i \cdot v}),
\]

since the subwords of \( w \) that equal \( v \) are either: (1) \( \text{id} \) concatenated with the subwords of \( w' \) that form \( v \), or (2) \( s_i \) concatenated with the subwords of \( w' \) that form \( s_i \cdot v \).

And because \( s_i \) turns \( \alpha_i \) into a negative root, we "polarize" the second term with a factor of \(-e^{\alpha_i}\). Therefore putting together the last two formulas we get \( R_m(X_{k+1}) = (1 - e^{\alpha_i})^{-1} (R_m(X_k) - e^{\alpha_i} s_i \cdot R_{s_i \cdot w_m}(X_k)) \).

Notice that when \( m = 0 \) and \( k + 1 \) is even (i.e. the fixed point is the identity coset and \( s_i = s_1 \)), then \( s_i \) and \( e \) belong to the same coset, so in this case \( R_0(X_{k+1}) = (1 - e^{\alpha_i})^{-1} (1 - e^{\alpha_1} s_1) R_0(X_k) \), which amounts to applying the Demazure operator \( D_{s_1} \) to \( R_0(X_k) \).

There are two cases, either \( s_i \cdot w_m = w_{m-1} \) or \( s_i \cdot w_m = w_{m+1} \).

Case 1: \( s_i \cdot w_m = w_{m-1} \) which implies that \( k - m \) is odd.

In this case we have

1. \( \left[ \frac{k-m}{2} \right] + 1 = \left[ \frac{k-m}{2} \right] = \left[ \frac{k-(m-1)}{2} \right] = \left[ \frac{k-(m-1)}{2} \right] \)
2. \( s_i = s_{\alpha_{k-1}} \)
3. \( s_i \cdot (\alpha_k + j \delta) = s_i \cdot \alpha + j \delta \)
4. \( s_i \cdot S_{m-1} = -e^{-\alpha_i} \cdot S_m \)

Now by the induction hypothesis we can compute \( R_m(X_{k+1}) \) readily as

\[
R_m(X_{k+1}) = (1 - e^{\alpha_i})^{-1} (R_m(X_k) - e^{-\alpha_i} s_i \cdot R_{s_i \cdot w_m}(X_k)).
\]

Before carrying out the induction step, we simplify notation by letting

\[
A = \sum_{j=0}^{\left\lfloor \frac{k-m}{2} \right\rfloor} \left( \begin{array}{c} k-m \cr j \end{array} \right) e^{j \delta} - \sum_{j=0}^{\left\lfloor \frac{k-(m-1)}{2} \right\rfloor} \left( \begin{array}{c} k-(m-1) \cr j \end{array} \right) e^{j \delta}
\]

and

\[
B = \sum_{j=0}^{\left\lfloor \frac{k-m}{2} \right\rfloor} \left( \begin{array}{c} k-m \cr j \end{array} \right) e^{j \delta} - \sum_{j=0}^{\left\lfloor \frac{k-(m-1)}{2} \right\rfloor} \left( \begin{array}{c} k-(m-1) \cr j \end{array} \right) e^{j \delta}
\]

We now give a detailed computation of \( R_m(X_k) - e^{-\alpha_i} s_i \cdot R_{s_i \cdot w_m}(X_k) \):
Given in terms of iterated Demazure operators. In this section we let Demazure modules.

7.1. Loop Groups

weight convention in their book standard convention is with highest weights, as in Kumar [6]. However, Pressley and Segal use the lowest weights. However, we note that this is mostly a matter of convention (using lowest weights vs. highest weights). The

compute the character of $E^\lambda$ of the dual representations given by the pullback bundles on the Bott-Samelson manifolds (see Theorem 5).

Now setting $\lambda = \sum_{i=1}^{m} \alpha_i$ be a dominant weight, and let $w$ be a fixed point of $\Omega SU(2)$. The restricted partitions become partitions outright. And we get the rational functions $R_w(\Omega SU(2))$ for each fixed point of $\Omega SU(2)$.

Remark: Strictly speaking, we should have taken the complex conjugate of the quantities in Theorems 7.3 and 7.4. This is because the character of the positive energy representation was equal to the inverse limit of the dual representations given by the pullback bundles on the Bott-Samelson manifolds (see Theorem 5). However, we note that this is mostly a matter of convention (using lowest weights vs. highest weights). The standard convention is with highest weights, as in Kumar [6]. However, Pressley and Segal use the lowest weight convention in their book Loop Groups (see [10]).

7.1. Demazure modules. A consequence of Lemma 7.5 is that it allows us to write an effective character formula for Demazure modules associated to the Schubert varieties. Previously, these characters were only given in terms of iterated Demazure operators. In this section we let $g$ be a Kac-Moody algebra and $b$ a Borel subalgebra.

Definition 7.8. Let $\lambda$ be a dominant weight, and let $V(\lambda)$ be the irreducible representation with highest-weight $\lambda$. For any $w \in W$, form $E_w(\lambda) = b \cdot v_w(\lambda)$, it is a submodule of $V(\lambda)$ called the Demazure submodule of $V(\lambda)$ associated to $w$.

The Demazure submodule of $V(\lambda)$ associated to $w$ is isomorphic to the restriction of the prequantum line bundle $L_\lambda$ to the Schubert variety $X_w$ (see Kumar [6]). Therefore Lemma 7.5 allows us to effectively compute the character of $E_w(\lambda)$ for $\lambda = \sum_{i=1}^{m} \alpha_i$ and for any $w \in W$.

Remark: Previously, characters of Demazure modules were computed using the Demazure character formula. This formula allows one to write the character $ch E_w$ as iterated applications of certain Demazure operators to the weight $e^w$, i.e. $ch E_w = D_{s_{i_1}} \cdots D_{s_{i_n}} e^w$ where $w = s_{i_1} \cdots s_{i_n}$ is a decomposition of the
reduced word \( w \). These Demazure operators act on the group algebra \( A[T] \) (see Section 6.1). For a simple reflection \( s_i \), the Demazure operator is defined by
\[
D_{s_i}(e^\lambda) = \frac{e^\lambda - e^{s_i\lambda - s_i}}{1 - e^{-\alpha_i}}.
\]

8. A Kostant Multiplicity Function

Each factor \((1 - e^{\alpha})\) in the denominator can be written as \(\sum_{n=0}^{\infty} e^{n\alpha}\), so we can compute the multiplicities of the character in the last section as a sum of partition functions, defined below

**Definition 8.1.** The partition function of a weight \( \mu \in \mathfrak{t}^* \) is defined as \( N(\mu) = \# \) of solutions of the equation
\[
\sum_{\alpha \in \Delta^+} n_\alpha \alpha = \mu \quad \text{where each } n_\alpha \in \mathbb{N} \cup \{0\}.
\]
Also define \( N_\beta(\mu) = \# \) of solutions of the equation
\[
\sum_{\alpha \in \Delta^+ \setminus \{\beta\}} n_\alpha \alpha = \mu \quad \text{where each } n_\alpha \in \mathbb{N} \cup \{0\}.
\]

For a reduced word \( w = s_{i_1} \cdots s_{i_n} \) we also define \( S_w = \sum_{j=1}^{n} s_{i_1} \cdots s_{i_{j-1}}(\alpha_{i_j}) \). Denote \( S_m := S_{w_m} \) in the Weyl group of affine \( \mathfrak{sl}_2 \).

Then by reading off the rational functions derived in the last section, we obtain the following Kostant multiplicity formula for \( \Omega SU(2) \):

**Proposition 8.2.** Let \( \lambda = k\mathfrak{w}_0, k \in \mathbb{Z}_{>0} \), then the multiplicity of the weight \( \alpha \) in the irreducible representation \( L_\lambda \) is given by
\[
m(\alpha, L_\lambda) = \sum_{m=0}^{\infty} (-1)^m \left( N(\alpha - (w_m \cdot \lambda + S_m)) - N(\alpha - (w_m \cdot (\lambda + \alpha_1) + S_m)) \right)
\]

An immediate corollary is the following identity:

**Corollary 8.3.**
\[
N(\alpha - (w_m \cdot \lambda + S_m)) - N(\alpha - (w_m \cdot (\lambda + \alpha_1) + S_m)) = N_{w_m \cdot \alpha_1}(\alpha - (w_m \cdot \lambda + S_m))
\]

**Remark:** The sum given above will always be finite. This is because only finitely many terms will be partitions of positive weights.

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