Small gauge transformations and universal geometry in heterotic theories

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Abstract

We explore the observation that small gauge transformations in heterotic supergravity are related to a choice of connection on the moduli space. We start by describing the action of small gauge transformations in heterotic supergravity. We show a convenient gauge fixing is holomorphic gauge together with residual gauge fixing involving the holomorphic top form. This gauge fixing, combined with the equations of motion, allows us to determine the Hodge decomposition and so a map between parameters and field deformations. At large radius, we give a prescription for how to compute the $\alpha'$--corrections to the Hodge decomposition and apply this to the moduli space metric. We then show this gauge fixing is related to a choice of holomorphic structure and Lee form on the universal bundle. The field strength for this connection is related to second order deformation theory and we point out it is generically the case that higher order deformations do not commute.

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1. Introduction

In a recent series of papers [1–3] we developed a theory of derivatives for describing the moduli space \( M \) of heterotic vacua realising \( N = 1 \) supersymmetry in a \( R^{1,3} \) spacetime, and used it to compute various properties such as the natural Kähler metric on \( M \). We work at large radius, in which the supergravity approximation is valid and so the heterotic flux \( H \) is subleading in \( \alpha' \). These theories are defined by a complex 3-fold \( X \) with \( c_1(X) = 0 \) and a hermitian form \( \omega \) as well as a holomorphic vector bundle \( E \to X \) with a connection \( A \) satisfying the Hermitian–Yang–Mills equation and a well-defined three-form \( H \). The anomaly relation yields a modified Bianchi identity for \( H \).

\[
dH = -\frac{\alpha'}{4} \left( \text{Tr} (F^2) - \text{Tr} (R^2) \right),
\]

while the supersymmetry relation implies the manifold is non-Kähler\(^1\)

\[
H = d^c \omega, \quad d^c \omega = \frac{1}{3!} J^m J^n J^p (d \omega)_{mnp},
\]

where \( J^m = J_n^m dx^n \) is a 1-form constructed out of the complex structure. For a fixed topology, these equation have parameters which describe families of heterotic vacua. The parameters are also coordinates for the manifold \( M \). As in [4] we take a constant dilaton, which as argued there, is a consequence of demanding a well-defined supergravity approximation.

To warm-up, consider \( X \) to be fixed so that \( M \) is the moduli space of connections \( A \). This is closely related to the context of the Kobayashi–Hitchin correspondence [5]. It is also the context of a heterotic theory in the \( \alpha' \to 0 \) limit. The connection \( A \) transforms under gauge symmetries

\[
A \to \Phi A = \Phi A \Phi^{-1} - (d \Phi) \Phi^{-1}.
\]

We wish to study deformations \( \delta A \). The background gauge principle implies two types of transformation properties for \( \delta A \):

\[
\delta A \to \Phi \delta A = \Phi \delta A \Phi^{-1}, \quad \delta A \to \delta A + d_A \phi, \quad \phi \ll 1.
\]

The former is a classical symmetry of the background, while the latter is a symmetry of the quantum theory, and we refer to as a small gauge transformation.

To make progress in understanding the moduli space \( M \), we need a relation between variations of parameters and deformations of fields called the Kodaira–Spencer map — see [6] for details — in such a way that the transformation laws of \( \delta A \) are respected. A solution implementing the first symmetry property was described in [2]. One introduces a

\(^1\)In the above equations \( R \) is the curvature two-form evaluated with the Hull connection \( \Theta^H = \Theta^L + \frac{1}{4} H \).

We denote by \( x^m \) the real coordinates of \( X \) and its complex coordinates by \((x^m, x^\overline{m})\). The coordinates along \( M \) are denoted by \( y^a \), and complex coordinates by \((y^a, y^\overline{a})\). In the following we will generally omit the wedge product symbol ‘\( \wedge \)’ between forms, unless doing so would lead to ambiguity.
connection $\Lambda = \Lambda_a \, \text{d}y^a$, so that it has legs along the moduli space $M$ and also transforms in a manner parallel to $A$

$$\Lambda \to \Phi \Lambda_a = \Phi \Lambda_a \Phi^{-1} - (\partial_a \Phi) \Phi^{-1}. \quad (1.5)$$

The deformation and its relation to parameters is given by $\delta A = \delta y^a \mathcal{D}_a A$ where $\mathcal{D}_a A$ contains two connections $A$ and $\Lambda$

$$\mathcal{D}_a A = \partial_a A - d_A \Lambda_a , \quad \text{where} \quad d_A \Lambda_a = d\Lambda_a + [A, \Lambda_a]. \quad (1.6)$$

It is easy to check that under (1.3)–(1.5) the covariant derivative $\mathcal{D}_a A$ transforms homogeneously as required. The small gauge transformation property of (1.6) was not studied in [1–3] in any detail. In this paper we explore this in more detail.

At the standard embedding the moduli space $M$ is reduced to that of Calabi-Yau manifolds and a closed $B$-field. This moduli space exhibits special geometry. One way to state the property of special geometry is that there exists canonical choice of coordinates on the moduli space deriving from the periods of the Calabi-Yau. Deformations can be written as partial derivatives with respect to these canonical coordinates of the background fields. Yukawa couplings are partial derivatives of the prepotential, a certain holomorphic function. In heterotic theories we do not have the analogue of the periods and there is no obvious choice of canonical coordinates. This means we need to more covariant formalism, and this is why, at root, we need to introduce covariant derivatives and the associated universal geometry [1]. The choice of covariant derivative corresponds to a choice of connection on $M$. This gives us a geometric way to study all the underlying gauge symmetries of the heterotic theories.

Indeed, there is a close connection with small gauge transformations and the theory of universal geometry developed in [1]. We start from the observation that under a small deformation $\Lambda_a \to \Lambda_a - \phi_a$, we have

$$\mathcal{D}_a A \to \Phi \mathcal{D}_a A = \mathcal{D}_a A + d_A \phi_a ,$$

which is exactly a small gauge transformation. So if we fix small gauge transformations, we must have fixed $\Lambda$ and its field strength, at least to some degree. In the space $\mathcal{A}$ of all connections, background gauge transformations move a basepoint $A \in \mathcal{A} \to \Phi A \in \mathcal{A}$. The points $\Phi A$ form the gauge orbit. The tangent space at $A \in \mathcal{A}$ has an orthogonal decomposition into a vertical subspace tangential to gauge orbits and a horizontal subspace spanned by $\{ \mathcal{D}_a A \}$, which are precisely the deformations of $A$. The definition of the horizontal subspace is determined by the connection $\Lambda$, and that choice is preserved under the action of background gauge transformations. The choice of $\Lambda$ therefore should be of no physical consequence in the spacetime theory. This is illustrated in Figure 1.

\[\text{The manifold } M \text{ is a subspace of the quotient } \mathcal{A}/G \text{ where } G \text{ is the action of the Lie group in (1.3)}.\]
Consider $\mathfrak{X} = X \times M$ and on it a connection $\mathfrak{A} = A + \Lambda$ for a universal bundle $\mathcal{U}$, with a field strength $\mathcal{F} = d\Lambda + \Lambda^2$. Under decomposition of legs, there is $F$ with legs purely along $X$; a mixed term $F_{a m} d x^m = \mathcal{D}_a A$, which is exactly the covariant derivative; and finally there is a third term $F_{a b}$, whose legs lie purely along $M$. In light of the previous paragraph, $F_{a b}$ influences the map between parameters $y^a$ and deformations $\delta A$. At first order in deformation theory we point out that taking the connection to depend holomorphically
on parameters $\mathcal{D}_\pi A = 0$ is actually a gauge fixing, we call holomorphic gauge. At second order in deformations, we show there is an independent gauge fixing that needs to be done and that holomorphic gauge amounts to $\mathcal{F}_{\alpha \beta} = 0$. Hence, holomorphic gauge to second order in deformation theory corresponds to the universal bundle being holomorphic with $\mathcal{F}^{(0,2)} = 0$.

In this paper we extend this observation to the heterotic vacuum. The manifold $X$ is not fixed and depends on parameters. We show this is forced upon us by the equations of motion and so we must consider a fibration $\mathcal{X}$ whose fibres are the manifolds $X$ and base space the moduli space $M$. There are other gauge symmetries to consider: diffeomorphisms of $X$ and transformations of the $B$-field. The former imply $X$ is fibered over $M$, and in [1] we studied this fibration labelled $\mathcal{X}$. The latter imply the presence of a 2-gerbe $B$ on $\mathcal{X}$. Consequently, we need to introduce additional connections on $M$ and their role will be to define a relation between parameters $y^a$ and deformations of fields. The connection $c^a_m dy^a \otimes \partial_m$ is associated to diffeomorphisms of $X$. Deformations of $c^a$ result in small diffeomorphisms of tensor fields on $X$. Small gerbe transformations are realised by deformations of an object $B_{mn}dy^a dx^m$.

A convenient gauge fixing is holomorphic gauge in which not just $\mathcal{D}_\pi A = 0$ but analogous conditions apply for deformations of complex structure and the complexified hermitian form. There is a residual gauge freedom which can be gauge fixed by the condition $\delta \Omega^{(3,0)} = k \Omega$, where $\Omega$ is the holomorphic top–form; equivalently the $(2,1)$–forms describing deformations of complex structure are $\partial$–closed in this gauge. We then apply the equations of motion in this gauge. This allows us to determine the Hodge decomposition, showing that holomorphic gauge is not the same as harmonic gauge in general. The equations are highly coupled. We give a prescription for solving these equations perturbatively in $\alpha'$ and apply this to the moduli space metric.
2. Small gauge transformations and heterotic moduli

We derive the action of small diffeomorphisms, gauge transformations and small gerbe transformations. Our derivation starts with small diffeomorphisms and from the equations of motion we are able to deduce the action of small gauge and small gerbes transformations on $\delta A$ and $\delta B$ respectively.

Suppose on $X$ we have a tensor field $T_{m_1...m_p}$ which can undergo a diffeomorphism

$$T_{n_1...n_k}(\tilde{x}) = T_{m_1...m_k}(x),$$

where $\tilde{x} = x + \varepsilon$ with $\varepsilon^m$ small

$$T_{m_1...m_k}(\tilde{x}) \simeq T_{m_1...m_k}(x) + \varepsilon^n \partial_n T_{m_1...m_k}(x) + \cdots + (\partial_m \varepsilon^n) T_{m_1...n}(x) + \cdots + (\partial_m \varepsilon^p) T_{m_1...p}(x) = T_{m_1...m_k}(x) + (\mathcal{L}_\varepsilon T)_{m_1...m_k}(x),$$

where $\mathcal{L}_\varepsilon$ is the Lie derivative taken with respect to the vector $\varepsilon^m$. We have worked to first order in $\varepsilon$. When studying deformations $\delta T$ of the tensor, the small diffeomorphisms are regarded as unphysical and so in the physical theory we identify

$$\delta T \sim \delta T + \mathcal{L}_\varepsilon T, \quad (2.3)$$

Appendix §B carefully explains the gauge fixing of diffeomorphisms in the study of the moduli space of Calabi-Yau manifolds in three different ways. We now describe how this works for fields of heterotic to first order in deformations.

2.1. Complex structure $J$

On $X$ there is an integrable complex structure $J$ and this facilitates introducing holomorphic coordinates $x^m = (x^\mu, \bar{x}^\nu)$. Deformations that modify complex structure can be expressed in terms of the undeformed complex structure as

$$\delta J = \delta J^\mu_\nu \, dx^\nu \otimes \partial_\mu + \delta J^\nu_\nu \, dx^\nu \otimes \partial_\mu .$$

To first order in deformation theory, demanding the Nijenhuis tensor be preserved. When decomposed into type we get

$$\overline{\partial}(\delta J^\mu) = 0 , \quad \partial(\delta J^\mu) = 0 ,$$

where the notation here is as in the introduction $\delta J^\mu = \delta J^\mu_\nu \, dx^\nu$. Small diffeomorphisms induce an identification $\delta J \sim \delta J + \mathcal{L}_\varepsilon J$ where because $\delta J$ is a real tensor, the vector $\varepsilon$ must also be real. Decomposing into complex type this equation becomes

$$\delta J^\mu \sim \delta J^\mu + 2i \overline{\partial} \varepsilon^\mu \quad \text{and} \quad \delta J^\nu \sim \delta J^\nu - 2i \partial \varepsilon^\nu .$$

Hence, $\delta J^\mu \, dx^\nu \in H^{(0,1)}(X, \mathcal{T}_X^{(1,0)})$. We expand in a basis for the cohomology group and this defines the Kodaira–Spencer map between tangent vectors and field variations

$$\delta J^\mu_\nu = \delta y^a (2i \Delta \alpha^\mu_\nu) .$$
2.2. Holomorphic $(3,0)$-form $\Omega$

Consider the d-closed holomorphic $(3,0)$–form $\Omega$. A first order deformation $\delta \Omega$ obeys three equations
\[
\partial \delta \Omega^{(3,0)} = 0 , \quad \overline{\partial} \delta \Omega^{(3,0)} + \partial \delta \Omega^{(2,1)} = 0 , \quad \overline{\partial} \delta \Omega^{(2,1)} = 0 .
\]

These equations are solved by
\[
\delta \Omega^{(3,0)} = \delta y^a \left( k_a \Omega + \partial \xi^{(2,0)}_a \right) , \quad \delta \Omega^{(2,1)} = \delta y^a \chi_a , \quad \partial \chi_a = - \overline{\partial} \delta \xi^{(2,0)}_a ,
\]
where $\xi^{(2,0)}_a$ are arbitrary $(2,0)$–forms, the $k_a$ are constant over $X$ and $\chi_a = \Delta_a^\mu \Omega_\mu$ are $\overline{\partial}$–closed $(2,1)$–forms.

At this point it is convenient to describe a two–parameter family of connections on $T_X$ given as follows
\[
\Theta^{(\epsilon,\rho)}_{\mu \nu} = \Theta^\text{LC}_{\mu \nu} + \frac{\epsilon - \rho}{2} H_{\mu \nu} , \quad \Theta^{(\epsilon,\rho)}_{\mu \sigma} = 0 ,
\]
where $\Theta^\text{LC}$ is the Levi–Civita connection. The Bismut connection is given by $\Theta^{B} = \Theta^{(-1,0)}$, the Hull connection by $\Theta^{H} = \Theta^{(1,0)}$ and the Chern connection by $\Theta^{\text{Ch}} = \Theta^{(0,-1)}$.

Recall that we also assume a $g_s$–perturbative string background so that we can trust the $\alpha'$–expansion. It would be interesting to understanding deviations from this assumption but doing so would likely need technology beyond supergravity. Supersymmetry implies that $\nabla^B_\mu \Omega = \nabla^B_\mu \Omega = 0$. From this it follows that
\[
H_{\mu \nu} = 0 , \quad \partial_\mu \| \Omega \|^2 = 0 .
\]

Using $H = d^c \omega$, this means geometrically the Lee form $\frac{1}{2} \omega^{mn} (d \omega)_{mn}$ vanishes. The Lee form is a 1–form measuring the non–primitive part of $d \omega$. Such manifolds are called balanced manifolds.

A small diffeomorphism acts as on the pure part as\footnote{Due to $H_{\mu \nu} = 0$ divergences of this vector with respect to Levi–Civita and Bismut coincide $\nabla^{\text{LC}}_{\nu} \varepsilon^\nu = \nabla^{\text{B}}_{\nu} \varepsilon^\nu$. Furthermore, with the choice of $\varepsilon^\nu$ we find
\[
\nabla^{\text{LC}}_{\nu} \varepsilon^\nu = - \frac{1}{3! \| \Omega \|^2} \Omega^{\rho \tau} \left( \partial \xi^{(2,0)}_a \right)_{\nu \rho \tau} .
\]}
\[
\delta \Omega^{(3,0)} \rightarrow \delta \Omega^{(3,0)} + \partial (\varepsilon^\nu \Omega_\nu) = \delta \Omega^{(3,0)} + (\nabla^{\text{LC}}_{\nu} \varepsilon^\nu) \Omega .
\]
2.3. The hermitian form $\omega$

There is a compatible hermitian form $\omega$. A real deformation of $\omega$ can be written as

$$\delta \omega^{(2,0)} = \delta y^a \Delta_a \omega_{\mu} \pi^\mu, \quad \delta \omega^{(1,1)} = \delta y^a (\partial_a \omega)^{(1,1)}, \quad \delta \omega^{(0,2)} = \delta y^a \Delta_a \omega_{\mu}, \quad \omega_m = \omega_{mn} dx^n.$$ 

It is subject to small diffeomorphisms

$$\delta \omega \sim \delta \omega + L_{\epsilon_a} \omega = \delta \omega + \epsilon_a^m (d\omega)_m + d(\epsilon_a^m \omega_m).$$

If $d\omega = 0$ the manifold is Kähler and small diffeomorphisms generate $d$–exact shifts of $\delta \omega$. In heterotic theories this is not the case.

2.4. The gauge field $F$

The gauge field $A$ and its field strength $F = dA + A^2$ also transform under small diffeomorphisms. However this case is complicated by gauge symmetries in which $\Phi F = F \Phi^{-1}$ and $A$ transforms according to (1.3). So in relating $F(x + \epsilon)$ and $F(x)$ we need a covariant generalisation of a Lie derivative, (2.2), which is

$$F(\tilde{x}) \simeq F(x) + \epsilon^m (dA)_m + d(\epsilon^m A_m). \quad (2.5)$$

Using the Bianchi identity we find an identification

$$\delta F(x) \sim \delta F(x) + dA(\epsilon^m F_m). \quad (2.6)$$

We write $\mathcal{A} = A^{(0,1)}$ so that $A = A - A^\dagger$. Holomorphy of $\mathcal{E}$ means $F^{(0,2)} = \overline{\partial}_A^2 = 0$. Taking the $(0,2)$ part of (2.6) and solving for $\delta \mathcal{A}$ gives

$$\delta \mathcal{A} \sim \delta \mathcal{A} + \epsilon^\mu F_\mu + \overline{\partial}_A \phi,$$

where $\phi$ is a section of $\text{End} \mathcal{E}$. In principle the last term could be closed but not exact. However, this would represent a change in moduli space coordinates (see below). This term is interpreted as an independent gauge symmetry, small gauge transformations, and we see we arrived at it for free by studying small diffeomorphisms.

We can expand the deformation in terms of covariant derivatives with respect to parameters

$$\delta \mathcal{A} = \delta y^a \mathcal{D}_a \mathcal{A}, \quad \text{where} \quad \Phi \mathcal{D}_a \mathcal{A} = \Phi \mathcal{D}_a \mathcal{A} \Phi^{-1}.$$ 

The covariant derivatives both ensure the transformation law (1.4) is satisfied and are representation of the Kodaira–Spencer map relating tangent vectors on the moduli space to deformations of fields [2].

The fluctuation $\mathcal{D}_a \mathcal{A}$ satisfies the Atiyah equation

$$\overline{\partial}_A (\mathcal{D}_a \mathcal{A}) = \Delta_a \mu F_\mu. \quad (2.7)$$
In terms of covariant derivatives, the gauge symmetry action becomes

$$D_a A \sim D_a A + \varepsilon_a \mu F_\mu + \overline{\partial} A \phi_a ,$$  

and it is a symmetry of (2.7) provided $\Delta_{a} \mu \sim \Delta_{a} \mu + \partial \varepsilon_a \mu$.

In the literature a bundle modulus is typically associated to a fluctuation $D_a A \in H^1(X, \text{End } E)$. From (2.7) these correspond to $\Delta_a = 0$. While it may obvious to some readers, we note this is only true in a particular gauge. A more invariant statement is that a bundle modulus satisfies

$$\partial A (D_a A) = (\partial A \kappa_a \mu) F_\mu ,$$

for any $\kappa_a \mu$. We will see in the heterotic theory that in fact fluctuations $D_a A$ are coupled to deformations of the complex structure and hermitian structure of $X$.

### 2.5. The three-form $H$

Consider the three-form

$$H = d B - \frac{\alpha}{4} \left( \text{CS}[A] - \text{CS}[\Theta] \right) , \quad \text{CS}[A] = \text{Tr} \left( A d A + \frac{2}{3} A^3 \right) ,$$

defined so that it satisfies the Bianchi identity (1.1). Here $\Theta$ is the gauge potential for frame transformations, which we suppress for now. Under background gauge transformations

$$\Phi B = B - \frac{\alpha}{4} \left\{ \text{Tr}(A Y) - U \right\} , \quad \frac{1}{3} \text{Tr}(Y^3) = d U ,$$

where $Y = \Phi^{-1} d \Phi$ and $d U = \frac{1}{3} \text{Tr} Y^3$.

The three–form $H$ is well-defined on $X$. A variation of it is

$$\partial_a H = d B_a - \frac{\alpha}{2} \text{Tr} (D_a AF) ,$$

where $B_a$ is defined as

$$B_a = D_a B + \frac{\alpha}{4} \text{Tr} (A D_a A) - d B_a ,$$

and the field $B_a$ is a 1–form on $X$. The definition of $D_a B$ and the symmetry properties of $B_a$ become clearer in universal geometry. The transformation property of $D_a B$ under background transformations mimics that of $B$ in (2.10) and is discussed in [2]:

$$\Phi D_a B = D_a B - \frac{\alpha}{4} \left( \text{Tr} (D_a A Y) - 3 U_a \right) , \quad d U_a = 0 .$$

---

4At this point we can explain the last term being exact. Suppose we write $D_a A \sim \tilde{D}_a A = D_a A + \varepsilon_a F_\mu + \gamma_a$ where $\overline{\partial} A \gamma_a = 0$. Suppose it is non–trivial in cohomology and can be expanded in a basis $\gamma_a = \gamma_a b D_b A$, where $D_b A$ are basis elements for the cohomology group and $\gamma_a b$ is a parameter dependent matrix. Then,

$$D_a A \sim \tilde{D}_a A = (\delta_a b + \gamma_a b) D_b A + \varepsilon_a \mu F_\mu + \overline{\partial} A \phi_a .$$

This is a transformation law for a change of parameters and can be absorbed by a redefinition of the $\delta y^a$.
We infer the transformation law for $B_a$ by using (2.11) and the transformation law for the gauge field (2.8) to find
\[
\partial_a H \sim d\left(\tilde{B}_a - \frac{\alpha'}{2} \text{Tr}(\phi_a F)\right) - \frac{\alpha'}{2} \text{Tr}\left((\mathcal{D}_a A + \varepsilon_a^m F_m) F\right),
\]
where $\tilde{B}_a$ is to be determined. On the other hand, (2.3) implies
\[
\partial_a H \sim \partial_a H - \frac{\alpha'}{2} \varepsilon_a^m \text{Tr}(F_m F) + d(\varepsilon_a^m H_m).
\]
Comparing the last two equations gives
\[
B_a \sim \tilde{B}_a = B_a + \varepsilon_a^m H_m + \frac{\alpha'}{2} \text{Tr}(\phi_a F) + db_a,
\]
where $b_a$ is a real 1–form. This equation holds up to a gauge invariant $d$–closed term. We have taken this to be exact $db_a$. As explained previously, any non-trivial element of $H^2(X, \mathbb{R})$ can be absorbed by a redefinition of the parameter space coordinates. In the $\alpha' \to 0$ limit, fluctuations of the $B$–field admit a symmetry $\delta B = \delta y^a \mathcal{D}_a B$. Just as we could infer the small transformation law $\mathcal{D}_a A \sim \mathcal{D}_a A + d_A \phi_a$ by considering the infinitesimal limit $\Phi = 1 - \phi$ in (1.3), we could apply the same limit to (2.10) to read off
\[
\mathcal{D}_a B \sim \mathcal{D}_a B + \frac{\alpha'}{4} \text{Tr}(A d \phi_a).
\]
For future reference, this means the following combination transforms as
\[
\mathcal{D}_a B + \frac{\alpha'}{4} \text{Tr}(A \mathcal{D}_a A) \sim \mathcal{D}_a B + \frac{\alpha'}{4} \text{Tr}(A \mathcal{D}_a A) + \frac{\alpha'}{2} \text{Tr}(\phi_a F) - d\left(\frac{\alpha'}{2} \text{Tr}(A \phi_a)\right).
\]
We will return to this in the next section.

2.6. Summary of small transformations on fields

We now put all of this together. First, it is convenient to form the complexified combinations
\[
Z_a = B_a + i \mathcal{D}_a \omega, \quad \overline{Z}_a = B_a - i \mathcal{D}_a \omega,
\]
where we denote $\mathcal{D}_a \omega^{(p,q)} = (\partial_a \omega)^{(p,q)}$ while on a real form $\mathcal{D}_a \omega = \partial_a \omega$.

We record for posterity, the action of a combined small diffeomorphism, gerbe and gauge transformation on heterotic moduli fields:
\[
\Delta_a^\mu \sim \Delta_a^\mu + \overline{\partial} \varepsilon_a^\mu, \quad \mathcal{D}_a A \sim \mathcal{D}_a A + \varepsilon_a^\mu F_\mu + \overline{\partial}_A \phi_a,
\]
\[
Z_a \sim Z_a + \varepsilon_a^m (H + \text{id} \omega)_m + \frac{\alpha'}{2} \text{Tr}(F \phi_a) + d(b_a + i \varepsilon_a^m \omega_m),
\]
\[
\overline{Z}_a \sim \overline{Z}_a + \varepsilon_a^m (H - \text{id} \omega)_m + \frac{\alpha'}{2} \text{Tr}(F \phi_a) + d(b_a - i \varepsilon_a^m \omega_m).
\]
2.7. The equations of motion

One equation of motion is that the complex structure is integrable. A second is the \text{Atiyah equation (2.7).} We have already checked that the small transformations (2.17) are a symmetry of these equations, see below (2.8). We now check the remaining equations of motion.

We start with \( H = d^c \omega \). Using \( d^c \omega = J^m \partial_m \omega - (dJ^m) \omega_m \) we have

\[
\begin{align*}
( \partial_a d^c \omega )^{(0,3)} &= -i \overline{\partial} (\partial_a \omega )^{(0,2)}, \\
( \partial_a d^c \omega )^{(1,2)} &= 2i \Delta_a \mu (\partial \omega )_\mu - i \partial (\partial_a \omega )^{(0,2)} - i \overline{\partial} (\partial_a \omega )^{(1,1)},
\end{align*}
\]

(2.18)

Projecting \( \partial_a H \) in (2.11) onto type we find

\[
\begin{align*}
\overline{\partial} Z_a^{(0,2)} &= 0, \\
\partial Z_a^{(0,2)} + \overline{\partial} Z_a^{(1,1)} &= 2i \Delta_a \mu (\partial \omega )_\mu + \frac{\alpha'}{2} \text{Tr} (D_a AF).
\end{align*}
\]

(2.19)

There are two other equations given by complex conjugation. It is now straightforward to see that (2.17) is a symmetry of (2.19) provided the \text{Bianchi identity (1.1)} holds.

For later convenience, we note that a solution to the first equation of (2.19) with \( h^{(0,2)} = 0 \) is given by \( Z_a^{(0,2)} = \overline{\partial} \beta_a^{(0,1)} \) for some complex \((0,1)-\text{form} \beta_a^{(0,1)}\). The second equation then becomes

\[
\overline{\partial} (Z_a^{(1,1)} - \partial \beta_a^{(0,1)}) = 2i \Delta_a \mu (\partial \omega )_\mu + \frac{\alpha'}{2} \text{Tr} (D_a AF).
\]

(2.20)

In \( g_s \)-\text{perturbative heterotic vacua}, the dilaton is a constant in the supergravity limit. The conformally balanced condition becomes \( d(\omega^2) \). It has a variation

\[
d(\partial_a \omega \omega) = 0.
\]

(2.21)

The action of (2.17) on \( \partial_a \omega \) is a Lie derivative \( \partial_a \omega \sim \partial_a \omega + L_{\epsilon} \omega \). This is a symmetry of (2.21). To see this, we use two properties of the operator \( L \), namely Leibnitz rule and \([L, d] = 0\). The calculation goes as follows

\[
d(L_{\epsilon} \omega \omega) = (L_{\epsilon} d\omega \omega + (L_{\epsilon} \omega) d\omega = L_{\epsilon} \omega (\omega d\omega) = \frac{1}{2} L_{\epsilon} \omega d(\omega^2) = 0.
\]

(2.22)

Finally, the HYM equation \( \omega^2 F = 0 \) has a variation

\[
\omega^2 (d_A D_a A) + 2 F \omega \partial_a \omega = 0.
\]

(2.23)

Note that

\[
(d^2 \phi_a) \omega^2 = [F, \phi_a] \omega^2 = [F \omega^2, \phi_a] = 0.
\]

(2.24)

Hence, under (2.17) this is invariant because

\[
\left( d_A (d_A \phi_a + \epsilon_a^m) F_m \right) \omega^2 + 2 F \omega L_{\epsilon} \omega = L_{\epsilon} (F \omega^2) = 0,
\]

(2.25)

where the Lie derivative acting on some gauge group \( G \)-\text{charged object} \( \xi \) is defined

\[
L_{\epsilon} \xi = \epsilon^m (d_A \xi)_m + d_A (\epsilon^m \xi_m).
\]

(2.26)

This confirms is what, of course, obvious: the equations of motion are invariant under gauge transformations. But it also serves as a useful consistency check that (2.17) is the correct transformation law.
3. Gauge fixing

We now fix the gauge freedom. Our gauge fixing is holomorphic gauge together a condition on \( \delta \Omega \). That is, given some choice of complex structure on the moduli space \( M \), we set \( \mathcal{D}_\tau A = 0 \) and \( Z_\tau = 0 \). It is important to note that taking a holomorphic variation of equations of motion secretly assumes this choice of gauge, a point that can lead to confusion in the literature. There is a residual gauge freedom which we use to fix \( \delta \Omega^{(3,0)} \) to be harmonic; this also implies \( \partial \chi_\alpha = 0 \) and \( \nabla_\mu \Delta_\alpha^\mu = 0 \) for an appropriate choice of \( \nabla_\mu \).

3.1. A warm–up

Consider a deformation of the Hermitian Yang–Mills equation 
\[
\omega^2 F = 0 ,
\]
where \( F = dA + A^2 \) and \( A = A - A^\dagger \). A variation of this equation on a fixed manifold \( X \) results in 
\[
\overline{\partial}_A^\dagger (\delta A) = -\partial_A^\dagger (\delta A^\dagger) .
\]
This provides no constraints on \( \delta A \), but notice it does depend on both \( \delta A \) and \( \delta A^\dagger \) and so is a real equation. Suppose we can take a holomorphic variation of this equation in which \( \delta A^\dagger = 0 \). Then we end up with a constraint 
\[
\overline{\partial}_A^\dagger (\delta A) = 0 .
\]
It naively looks as though the HYM equation fixed us to harmonic gauge. However, this conclusion is incorrect. The gauge fixing occurred earlier on, in the assumption that we could take a holomorphic variation. Indeed, under a small gauge transformation \( \delta A^\dagger \sim \delta A^\dagger + \partial_A^\dagger \phi^i \), and so it is only true that \( \delta A^\dagger = 0 \) in a particular gauge. We call this holomorphic gauge. So the correct statement is that in holomorphic gauge the Hermitian Yang–Mills equation further constrains \( \delta A \) to be in harmonic gauge. The aim of this section is to systematically study the heterotic equations of motion and gauge fixing, and then describe how these uniquely fix the physical degrees of freedom.

As a toy example to warm–up consider a \( d = 4 \) supersymmetric \( U(1) \) gauge theory with \( N + 1 \) chiral multiplets whose bosons are denoted \( \phi^i \) can carry charge \(+1\) under the gauge symmetry. The scalar potential is \( V = \frac{1}{2} D^2 \) where the D-term is \( D = \phi^i \overline{\phi}_i - r \) with \( r \) the Fayet–Iliopoulos (FI) parameter. We have set the coupling constant to unity. The fields \( \phi^i \) can be interpreted as complex coordinates on \( \mathbb{C}^{N+1} \) with the flat hermitian metric used in lowering the index \( \phi^i \). The space of classical vacua, therefore, corresponds to the single D–term vanishing modulo gauge transformations and is
\[
\mathbb{P}^N = \mathbb{C}^{N+1}/\!\!/U(1) ,
\]
where the \( \!\!/ \) denotes the symplectic quotient: the gauge quotient \( \phi^i \sim \phi^i e^{i\lambda} \), \( \lambda \in \mathbb{R} \) and the moment map \( \phi^i \overline{\phi}_i - r = 0 \) imposed simultaneously.\(^5\)

\(^5\)It is well–known we can equally view this moduli space as a holomorphic quotient \( (\mathbb{C}^{N+1} - 0)/\mathbb{C}^* \).
Given a point $\phi^i \in \mathbb{P}^N$ we can study deformations $\phi^i \to \phi^i + \delta \phi^i$. Deformations $\delta \phi^i$ inherit a gauge symmetry $\delta \phi^i \sim \delta \phi^i + i \delta \lambda$ for some small gauge parameter $\delta \lambda$ which is real. These are a simple example of small gauge transformations which we discuss in this paper. The D-terms impose an equation of motion on the fluctuations

$$\delta \phi^i \phi_i + \phi^i \delta \phi_i = 2 \text{Re} (\delta \phi^i \phi_i) = 0 .$$

The D-terms do not fix the $U(1)$ gauge symmetry. This is expected — after all the equations of motion are gauge invariant. Instead a gauge fixing is an additional condition such as $\text{Im} (\delta \phi^i \phi_i) = \xi$ for some $\xi \in \mathbb{R}$. The D–term together with the gauge fixing allow us to determine the physical fluctuations. It is not hard to see they are exactly tangent vectors to $\mathbb{P}^N$ about a point. In the heterotic analysis the equations of motion, including the HYM and balanced equation, together with gauge fixing will allow us to determine physical degrees of freedom in the same sort of way.

### 3.2. Holomorphy and holomorphic gauge

In §2, expressions were written in real coordinates with real parameters. However, a supersymmetric theory always comes with a complex parameter space $M$. Hence, its tangent space is complexified (see Appendix §A) and we gain much computational power by introducing holomorphy. Loosely speaking, holomorphy in this context means we need to decide a map between field deformations and holomorphic tangent vectors $\delta y^\alpha$. Typically, this relation is stated after (implicitly) gauge fixing. For this reason, we first write down the relations without fixing a gauge. This will amount to certain antiholomorphic combinations being exact. We will then show how to fix the gauge freedom in §2 can be used to goto ‘holomorphic gauge’ in which we recover the usual expressions given in the literature.

Let us illustrate this with integrable complex structure deformations. These obey $\partial \Delta^a = 0$. Substituting $\delta y^\alpha \Delta^a = \delta y^\alpha \Delta^a + \delta y^\beta \Delta^a$ into this equation, we take holomorphy to mean

$$\Delta^\alpha = \partial \kappa^\alpha .$$

In other words, the non–trivial elements of $H^1(X, T_X)$ are associated to holomorphic tangent vectors, but we have not set the antiholomorphic deformations to zero. This is because under a small diffeomorphism $\Delta^a \sim \Delta^a + \partial \epsilon^a$ and so putting to $\partial \kappa^\alpha = 0$ would amount to partially gauge fixing small diffeomorphism. This becomes more involved once the remaining fields of heterotic are involved. Let us write out how this works.

Consider the gauge field. Writing $\delta A = \delta y^\alpha D_\alpha A + \delta y^\alpha D_\pi A$ and substituting into the Atiyah equation (2.7) the antiholomorphic deformation satisfies $\partial A D_\pi A = \partial A (\kappa_\pi \mu F_\mu)$ and holomorphy means we take the solution to be exact

$$D_\pi A = \kappa_\pi \mu F_\mu + \partial A \Phi_\pi ,$$

This amounts to complexifying the gauge symmetry $U(1) \to \mathbb{C}^*$ and forgetting about the D–terms. In a $d = 4$ supersymmetric field theory we can study either the real compact gauge group or the complexified gauge group. Whether this holds in string field theory is far from clear. That being so, we work with the real compact gauge group in heterotic theories to guarantee self–consistency.
for some section $\Phi_\pi$ of End $\mathcal{E}$. What if the solution to this equation were not to be exact?
The answer is that any non–trivial elements can be absorbed by a change in complex structure of the moduli space [2] (see footnote below (2.8)).

The complexified hermitian form $Z_\mu$ is included through the second equation of (2.19)

$$\bar{\partial} \left( Z_{\pi}^{(1,1)} - \partial \beta_\pi^{(0,1)} - 2i \kappa_\pi^\mu (\partial \omega)_\mu - \frac{\alpha^1}{2} \text{Tr} \left( \Phi_\pi F \right) \right) = 0.$$  

Holomorphy here amounts to the solution of this equation being exact. As above, if we were to take this solution to be not exact, then any non–trivial elements of $H^{(1,1)}(X, \mathbb{C})$ on the right hand side can be absorbed by a change in complex structure of the moduli space (see [2]). That being so, we have

$$Z_{\pi}^{(1,1)} = 2i \kappa_\pi^\mu (\partial \omega)_\mu + \frac{\alpha^1}{2} \text{Tr} \left( \Phi_\pi F \right) + \partial \beta_\pi^{(0,1)} + \bar{\partial} \xi_\pi^{(1,0)}.$$  

Hence, without reference to a choice of gauge, holomorphicity for the moduli fields is defined as

$$\Delta_\pi^\mu = \bar{\partial} \kappa_\pi^\mu, \quad \mathcal{D}_\pi A = \kappa_\pi^\mu F_\mu + \bar{\partial}_A \Phi_\pi,$$

$$Z_{\pi}^{(1,1)} = 2i \kappa_\pi^\mu (\partial \omega)_\mu + \frac{\alpha^1}{2} \text{Tr} \left( \Phi_\pi F \right) + \partial \beta_\pi^{(0,1)} + \bar{\partial} \xi_\pi^{(1,0)}, \quad Z_{\pi}^{(0,2)} = \bar{\partial} \beta_\pi^{(0,1)}.$$  

The fields in (3.2) transform under a small transformation as

$$\Delta_\pi^\mu \sim \Delta_\pi^\mu + \bar{\partial} \varepsilon_\pi^\mu, \quad \mathcal{D}_\pi A \sim \mathcal{D}_\pi A + \varepsilon_\pi^\mu F_\mu + \bar{\partial}_A \phi_\pi,$$

$$Z_{\pi}^{(1,1)} \sim Z_{\pi}^{(1,1)} + 2i \varepsilon_\pi^\mu (\partial \omega)_\mu + \frac{\alpha^1}{2} \text{Tr} \left( \phi_\pi F \right) + \partial \left( b_\pi^{(0,1)} + i \varepsilon_\pi^\mu \omega_\mu \right) + \bar{\partial} \left( b_\pi^{(1,0)} + i \varepsilon_\pi^\mu \omega_\mu \right),$$

where we use (2.8) with $\delta \phi = \delta y^a \phi_a + \delta y^\tau \phi_\pi$. We can equivalently express this as

$$\kappa_\pi^\mu \sim \kappa_\pi^\mu + \varepsilon_\pi^\mu, \quad \Phi_\pi \sim \Phi_\pi + \phi_\pi, \quad \beta_\pi^{(0,1)} \sim \beta_\pi^{(0,1)} + b_\pi^{(0,1)} + i \varepsilon_\pi^\nu \omega_\nu, \quad \xi_\pi^{(1,0)} \sim \xi_\pi^{(1,0)} + b_\pi^{(1,0)} + i \varepsilon_\pi^\mu \omega_\mu,$$

where in fact the last two equations are correct up to $\bar{\partial}$– and $\partial$–closed term respectively.

We need to gauge fix. Holomorphic gauge amounts to $\varepsilon_\pi^\mu = -\kappa_\pi^\mu$ and $\phi_\pi = -\Phi_\pi$ so that so that $\Delta_\pi^\mu = 0$ and $\mathcal{D}_\pi A = 0$ respectively. Furthermore, demand $b, \varepsilon$ solve

$$b_\pi^{(0,1)} - i \kappa_\pi^\nu \omega_\nu = -\beta_\pi^{(0,1)} + \bar{\partial} \psi_\pi, \quad b_\pi^{(1,0)} + i \varepsilon_\pi^\nu \omega_\nu = -\xi_\pi^{(1,0)} + \bar{\partial} \psi_\pi,$$

so that $Z_{\pi}^{(0,2)} = 0$ and $Z_{\pi}^{(1,1)} = 0$ respectively. There is a residual gauge freedom parameterised by $\psi_\pi$ a complex $(0, 1)$–form on the moduli space $M$. Finally, we also set

$$b_\alpha^{(0,1)} + i \varepsilon_\alpha^\nu \omega_\nu = -\beta_\alpha^{(0,1)} + \bar{\partial} \psi_\alpha,$$

where $\psi_\alpha dy^a$ is a complex $(1, 0)$–form on $M$ and is independent from $\psi_\pi dy^\pi$ in (3.3). This amounts to $Z_{\alpha}^{(0,2)} = 0$. The fact $\Delta_\pi^\mu = 0$ implies $(\mathcal{D}_\alpha \omega)^{(2,0)} = 0$ and so together with
\[ Z^{(0,2)} = 0 \] we see that \( B^{(2,0)}_\alpha = \mathcal{D}_\alpha \omega^{(2,0)} = 0 \). We can summarise the gauge fixing as

\[ \Delta \pi^\mu = 0 , \quad \mathcal{D}_\pi \mathcal{A} = 0 , \quad \overline{Z}^{(2,0)} = 0 , \quad Z^{(1,1)} = 0 , \quad Z^{(0,2)} = \overline{Z}^{(0,2)} = 0 . \quad (3.5) \]

We have included \( \overline{Z}^{(2,0)} \) as part of holomorphy; this will be justified in the next section.

Two of the equations \( Z^{(1,1)} = Z^{(0,2)} = 0 \) imply \( B^{(0,2)}_\alpha = -i \mathcal{D}_\alpha \omega^{(0,2)} \) and \( B^{(1,1)}_\alpha = i \mathcal{D}_\alpha \omega^{(1,1)} \) respectively. Hence, the physical degrees of freedom are

\[ \mathcal{D}_\alpha \mathcal{A} , \quad \Delta \epsilon^\mu , \quad \mathcal{D}_\alpha \omega^{(1,1)} , \quad \mathcal{D}_\alpha \omega^{(0,2)} , \quad (3.6) \]

though these are related by the equations of motion that we describe below. In the following we will interchangably use \( Z^{(1,1)}_\alpha \) and \( \overline{Z}^{(0,2)}_\alpha \) for the last pair respectively; they are equal up to a numerical factor.

In the physical theory \( \delta J, \delta A, \partial_\tau \omega \) and \( B_\alpha \) are real, and so we understand that we always pair holomorphic and antiholomorphic deformations in such a way to give a real deformation.

### 3.3. Residual gauge symmetries

Residual transformations must preserve holomorphic gauge. Firstly, there are no solutions to \( \partial \varepsilon^{\alpha \mu} = 0 \) provided \( h^{(0,2)} = 0 \) as can be seen by contracting with \( \Omega_{\mu} \). So any residual diffeomorphisms must have form \( \varepsilon^\alpha_\mu \). Secondly, holomorphy completely fixes \( \phi_\alpha \). To see this note that we need \( \partial \mathcal{A} \phi_{\pi} = 0 \) and stability implies \( H^0(X, \text{End} \mathcal{E}) \) is trivial and so \( \phi_{\pi} \) is a constant on \( X \). We take the connection \( \Lambda_\alpha \) to be antihermitian so that \( \phi_\alpha = -(\phi_{\pi})^\dagger \).

Therefore small gauge transformations for the bundle are fixed. With those conditions in mind, we now explore the remaining gauge freedoms.

We still have a residual gauge freedom deriving from (3.3) and (3.4) of the form

\[ b^{(0,1)}_\pi = \overline{\partial} \psi_\pi , \quad b^{(1,0)}_\pi + i \varepsilon^{\pi}_{\tau} \omega_\pi = \partial \psi_\pi , \quad b^{(1,0)}_\pi - i \varepsilon^{\pi}_{\tau} \omega_\pi = \overline{\partial} \psi_\pi , \quad (3.7) \]

where \( \overline{\psi}_\pi = (\psi_\alpha)^\ast \). Define \( \varepsilon^{\pi}_{\tau} = i \varepsilon^{\mu}_{\tau} \omega_\mu \), we can invert these equations

\[ b^{(1,0)}_\pi = \frac{1}{2} \partial (\psi_\pi + \overline{\psi}_\pi) , \quad \varepsilon^{(1,0)}_\pi = \frac{1}{2} \partial (\psi_\pi - \overline{\psi}_\pi) , \quad b^{(0,1)}_\pi = \overline{\partial} \psi_\pi , \quad \varepsilon^{(0,1)}_\pi = 0 . \quad (3.8) \]

Referring back to \( \S \ 2.2 \), in holomorphic gauge \( \delta \Omega \) is a holomorphic function of parameters and coordinates. Under a small diffeomorphism \( \delta \Omega^{(3,0)} = \delta \Omega^{(3,0)} + (\nabla_{\xi}^{\ast c} \varepsilon^\nu) \Omega \). Without loss of generality we can write an arbitrary holomorphic vector as

\[ \varepsilon^\nu_\alpha = -\frac{1}{2\|\Omega\|^2} \overline{\Omega}^{\nu \rho \sigma}_{\pi} (\xi_\alpha + \partial \xi^{(1,0)}_\alpha)_{\rho \sigma} , \quad (3.9) \]

where \( \partial \xi^{(1,0)}_\alpha \) is a local representative of a closed 2-form. Then, note that \( \overline{\mathcal{D}}^{(0,1)}_c = -g^{\mu \nu} \partial_\mu \varepsilon_\pi = -\nabla_\nu \varepsilon^\nu \). Due to \( H^{\mu \nu} \varepsilon^\nu = 0 \) we are free to use any of the Levi–Civita, Bismut, Hull or Chern connections in that expression and that is why the label is left blank.

\[ \text{If we were considering a complexified gauge group then } A \text{ is no longer antihermitian, and so } \delta \mathcal{A} \text{ and } \delta \mathcal{A}^\dagger \text{ are independent degrees of freedom. That being so, } \phi_\alpha \text{ is independent from } \phi_{\pi} \text{ and so } \mathcal{D}_{\pi} \mathcal{A} = 0 \text{ gauge fixes } \phi_{\pi} \text{ but does nothing to } \phi_\alpha . \]
Hence,
\[ \nabla_\nu \varepsilon_\alpha^\nu = -\frac{1}{3! ||\Omega||^2} \nabla_{\nu \rho \sigma} (\partial \xi_\alpha)_{\nu \rho \sigma}, \]
and we kill the \( \partial \xi^{(2,0)} \) in (2.4). Comparing with (3.8), the existence of the gauge transformation amounts to solving a Poisson equation
\[ \Delta \bar{\psi}_\alpha - \psi_\alpha = -\frac{i}{3! ||\Omega||^2} \nabla_{\nu \rho \sigma} (\partial \xi_\alpha)_{\nu \rho \sigma}. \]

As discussed in Appendix §B.1 the obstruction to solving this Poisson equation is that the source be orthogonal to zero modes. On \( X \) these are constants and so we see orthogonality is always satisfied:
\[ \frac{i}{3! ||\Omega||^2} \int_X \text{vol} \nabla_{\nu \rho \sigma} (\partial \xi_\alpha)_{\nu \rho \sigma} \sim i \int_X \partial \xi^{(2,0)} \Omega = 0. \]

Consequently, we can find a \( \bar{\psi}_\alpha - \psi_\alpha \) so that \( \delta \Omega^{(3,0)} = (\delta k) \Omega \) is harmonic for some parameter dependent constant \( \delta k \). The closed form \( \partial \xi^{(1,0)}_\alpha \) corresponds to \( \nabla_{\nu} \varepsilon_\alpha^\nu = 0 \). However, from (3.8) we see that such gauge transformations correspond to zero modes on \( X \) viz. \( \Box \bar{\psi}_\alpha - \bar{\psi}_\alpha = 0 \). The only zero modes on \( X \) are constants, which for reasons explained in Appendix §B.1, must be zero and so \( \psi_\alpha = \bar{\psi}_\alpha \) and \( \partial \xi^{(0,1)} \) must vanish. The only residual freedom, therefore, is contained with \( b_\alpha = d\psi_\alpha \). Such gauge transformations do not affect any of the fields because the pertinent transformation law is \( B_\alpha \sim B_\alpha + db_\alpha \) which clearly leaves \( B_\alpha \) invariant. Hence, we have completely fixed the gauge, the aim of this section.

A final comment is that in this gauge
\[ \partial \chi_\alpha = 0. \]

Using \( \chi_\alpha = \Delta_\alpha \mu \Omega_\mu \), the identity \( (\partial \Delta_\alpha \mu) \Omega_\mu = \partial_\mu \Delta_\alpha \mu \Omega \) and the fact \( \Omega = \frac{1}{3! f} \epsilon_{\mu \nu \rho} d\pi^{\mu \nu \rho} \) with \( |f|^2 = ||\Omega||^2 \sqrt{g} \) we have
\[ \partial (\Delta_\alpha \mu \Omega_\mu) = - (\partial_\mu \Delta_\alpha \mu + \Delta_\alpha \mu \partial_\mu \log \sqrt{g}) \Omega = - (\nabla_{\mu}^{H/Ch} \Delta_\alpha \mu) \Omega = 0, \]
where in the last equality \( \nabla_{\mu}^{H/Ch} \) are a family of connections with \( \epsilon - \rho = 1 \). This, in particular, includes the Hull and Chern connections. Hence, in this gauge we find
\[ \nabla_{\mu}^{H/Ch} \Delta_\alpha \mu = 0. \]

Under a small diffeomorphism this condition is preserved. Seeing this is an interesting exercise as it requires the condition \( \partial_\mu \partial_\nu \log \sqrt{g} = 0 \), a consequence of \( |f|^2 = \sqrt{g} ||\Omega||^2 \), the norm \( ||\Omega||^2 \) being constant and \( f \) being holomorphic. As the metric is not Kähler this condition is not the same as Ricci flatness.

We can rephrase this gauge fixing in terms of \( \mathfrak{D}_\alpha \omega^{(0,2)} \) as follows. This will be useful in the next subsection. First, the coderivative as given in Appendix C can be used to write
\[ \overline{\delta} \Delta_\alpha \mu = - \nabla_{\alpha}^{Ch} \omega^{\mu \nu} \Delta_\alpha \mu = - \nabla_{\nu}^{Ch} \Delta_\alpha (\mu \nu) + \nabla_{\nu}^{Ch} \Delta_\alpha [\mu \nu]. \]

Second, combine this equation with (3.10) and the relation \( \mathfrak{D}_\alpha \omega^{(0,2)} = 2i \Delta_\alpha \omega^{[\mu \nu]} \) to give
\[ \overline{\delta} \Delta_\alpha \mu = ig^{\mu \nu} \nabla_{\nu}^{Ch} \omega^{(0,2)} \Delta_\alpha \mu = 2i \Delta_\alpha \omega^{(0,2)} \Delta_\alpha \mu. \]
4. The Hodge decomposition and the moduli space metric

We now show how gauge fixing together with all of the equations of motion, including the hermitian Yang–Mills equation and balanced equation, allow us to solve for the terms in the Hodge decomposition of the heterotic moduli in (3.6).

Mathematically we are determining the Kodaira–Spencer map which associates parameters with field deformations. We do not a priori assume any structure about the moduli space, for example, we do not label parameters as complex structure or bundle moduli. We find that even at first order in $\alpha^\prime$, the moduli fields are incredibly coupled and there is not invariant distinction between moduli.

When the background is Calabi-Yau at large radius, so that supergravity is guaranteed to be a good approximation, we show how one may go about solving these coupled differential equations for the field deformations in (3.6). As an application of this work, we give a prescription for how to compute the moduli space metric derived in [1, 2].

4.1. The equations of motion

The physical degrees of freedom in (3.6) are $\mathcal{D}_\alpha\omega^{(0,2)}$, $\mathcal{D}_\alpha\omega^{(1,1)}$, $\mathcal{D}_\alpha A$ and $\Delta_\alpha$. The equations of motion from the Nijenhuis tensor, Atiyah equation and (2.19) are

$$\partial\Delta_\alpha^\mu = 0 ,$$

$$\partial_\mu \mathcal{D}_\alpha A = \Delta_\alpha^\mu F_\mu ,$$

$$\partial Z^{(1,1)}_\alpha = 2i \Delta_\alpha^\mu (\partial\omega)_\mu + \frac{\alpha^\prime}{2} \text{Tr} (\mathcal{D}_\alpha A F) - \alpha^\prime \left( \nabla_\nu \Delta_\alpha^\mu + i \nabla^\nu \mathcal{D}_\alpha \omega^{(0,1)}_\nu \right) R^\nu_\mu ,$$

(4.1)

where $R^\mu_\nu = R^\rho_\nu_\rho_\mu \text{d}x^\rho$ is the Riemann tensor. We have now included the contribution of the spin connection $\mathcal{D}_\alpha \Theta$. This depends, to zeroth order in $\alpha^\prime$, on the metric moduli with the relation derived in [1]:

$$\mathcal{D}_\alpha \Theta^{\nu\sigma} = \nabla_\sigma \Delta_\alpha^\nu + i \nabla^\nu \mathcal{D}_\alpha \omega^{(0,1)}_\sigma ,$$

(4.2)

which together with the symmetry $\mathcal{D}_\alpha \Theta^{\nu\sigma} = -g^{\nu\lambda} \mathcal{D}_\alpha \Theta^{\rho\lambda} g_{\rho\sigma}$ allows us to write

$$\text{Tr} (\mathcal{D}_\alpha \Theta^{(0,1)} R) = 2 \mathcal{D}_\alpha \Theta^{\nu\sigma} R^\nu_\mu = 2 \left( \nabla_\nu \Delta_\alpha^\nu + i \nabla^\nu \mathcal{D}_\alpha \omega^{(0,1)}_\nu \right) R^\nu_\mu ,$$

(4.3)

where we have used $R^\mu_\nu = -R^\nu_\mu$. Note that in [1] we checked that the spin connection is holomorphic $\mathcal{D}_\alpha \Theta_\mu = 0$ and that this holds provided we are in the gauge fixing discussed here, at least to this order in $\alpha^\prime$.

A motivation for this paper is that the equations (4.1), in particular the last one, are exactly captured by the universal bundle in [1]. We explore this further in §5, and understand this result as being the statement that the universal bundle being holomorphic corresponds to field deformations of heterotic string being in holomorphic gauge as constructed in the previous subsection. It is, mathematically speaking, natural for the universal bundle to be holomorphic and justifies a posteri our choice of holomorphic gauge in §3.2 in studying heterotic supergravity.
The balanced condition means \( d(\omega^2) = 0 \). Taking a real deformation and decomposing into type there is one non-trivial equation

\[
\partial(D_\alpha \omega^{(0,2)} \omega) + \overline{\partial}(D_\alpha \omega^{(1,1)} \omega) = 0 ,
\]

with the remaining equation given by complex conjugation. In the gauge in which \( \delta \Omega^{(3,0)} \) is harmonic, it follows that \( \delta \log \sqrt{g} = \omega^{\mu \nu} \delta \omega_{\mu \nu} \) is a function of parameters only. This contraction appears in the Hodge dual of \( \delta \omega^{(1,1)} \), see (C.10). As a reminder, in holomorphic gauge \( D_\alpha \omega^{(2,0)} = 0 \) and \( \overline{\partial} \xi^{(1,1)}_\alpha = 0 \) meaning \( \mathcal{Z}^{(1,1)}_\alpha = 2i D_\alpha \omega^{(1,1)} \).

By definition of the adjoint operators and a result in (C.10), we find two conditions derive from the balanced equation

\[
\overline{\partial}^\dagger D_\alpha \omega^{(1,1)} = 0 , \quad \overline{\partial}^\dagger D_\alpha \omega^{(0,2)} = \partial^\dagger D_\alpha \omega^{(1,1)} .
\] (4.4)

In virtue of the first equation, the Hodge decomposition for \( D_\alpha \omega^{(1,1)} \) with respect to the \( \overline{\partial} \)-operator is

\[
D_\alpha \omega^{(1,1)} = D_\alpha \omega^{(1,1)}_{\text{harm}} + \overline{\partial}^\dagger \xi^{(1,2)}_\alpha ,
\] (4.5)

that is, there is no \( \overline{\partial} \)-exact term. The first term is \( \overline{\partial} \)-harmonic and is completely determined via Hodge theory of \( X \) up to a parameter dependent matrix. This matrix can be accounted for by a holomorphic change of coordinates on the moduli space. The \( \overline{\partial} \)-coexact bit is determined by the moduli equation (4.1). To see this note that \( \mathcal{Z}^{(1,1)}_\alpha = 2i D_\alpha \omega^{(1,1)} \) and so

\[
\overline{\partial} \overline{\partial}^\dagger \xi^{(1,2)}_\alpha = \Delta_\alpha^\mu (\partial_\omega)_\mu - \frac{i \alpha^i}{4} \text{Tr}(D_\alpha AF) + \frac{i \alpha^i}{2} \left( \nabla_\nu \Delta_\alpha^\mu + i \nabla^\nu D_\alpha \omega^{0,1}_\nu \right) R^\nu_\mu .
\]

The Hodge decomposition of the \((1,2)\)-form is

\[
\xi^{(1,2)}_\alpha = \xi^{(1,2)}_{\text{harm}} + \overline{\partial} \rho^{(1,1)}_\alpha + \overline{\partial}^\dagger \lambda^{(1,3)}_\alpha .
\]

The terms \( \xi^{(1,2)}_{\text{harm}} \) and \( \overline{\partial}^\dagger \lambda^{(1,3)}_\alpha \) do not contribute to \( D_\alpha \omega^{(1,1)} \) and so are not physical degree of freedom. The remaining term is determined by substituting into the previous equation, and inverting the Laplacian

\[
\xi^{(1,2)}_\alpha = \square^{-1} \left( \Delta_\alpha^\mu (\partial_\omega)_\mu - \frac{i \alpha^i}{4} \text{Tr}(D_\alpha AF) \right) + \frac{i \alpha^i}{2} \left( \nabla_\nu \Delta_\alpha^\mu + i \nabla^\mu D_\alpha \omega^{0,1}_\nu \right) R^\nu_\mu + \overline{\partial}^\dagger \text{–closed} .
\] (4.6)

The term in the bracket is not a zero mode of Laplacian by assumption and we have not written a co–closed term as it does not contribute to (4.5). We will come back to this equation in a moment.

Returning to (4.4), the second equation can be combined with (3.11) to gain further information about \( \Delta_\alpha \). Start with the left hand side

\[
\overline{\partial}^\dagger D_\alpha \omega^{(0,2)} = - \star \partial(D_\alpha \omega^{(0,2)} \omega) = - \star \left( \partial D_\alpha \omega^{(0,2)} \right) \omega - \star \left( D_\alpha \omega^{(0,2)} \partial \omega \right) ,
\]

and make use of (C.12) to express this as

\[
\overline{\partial}^\dagger D_\alpha \omega^{(0,2)} = - \nabla^\text{Ch} \pi(D_\alpha \omega^{(0,1)}_\pi) - \frac{i}{2} D_\alpha \omega^{0,1}_\pi \partial \omega^\pi \pi .
\]
The second equation in (4.4) can now be written as

$$\nabla^\text{Ch}_\nu (\mathcal{D}_\alpha \omega^{(0,1)}_\nu) = -\partial^\dagger \mathcal{D}_\alpha \omega^{(1,1)} + \frac{i}{2} \mathcal{D}_\alpha \omega_{\mu\nu} (\partial \omega)^{\mu\nu}.$$  (4.7)

The gauge fixing in (3.11) becomes

$$\overline{\partial}^\dagger \Delta^{\alpha}_{\mu} = -i (\partial^\dagger \mathcal{D}_\alpha \omega^{(1,1)}_\nu) g^{\mu\nu} + \frac{1}{2} \mathcal{D}_\alpha \omega_{\mu\nu} (\partial \omega)^{\mu\nu}.$$  (4.8)

The Hodge decomposition of $\Delta_\alpha$ with respect to the $\partial$ operator is

$$\Delta_\alpha = \Delta_\alpha^{\text{harm}} + \partial \kappa_\alpha, \quad \kappa_\alpha = \kappa_\alpha^{\mu} \partial_\mu.$$  (4.9)

The first term is determined by Hodge theory as was the case for the hermitian form. We have used that $\Delta_\alpha$ is integrable $\partial \Delta_\alpha = 0$ and so there is no $\overline{\partial}$-coexact term. Substituting into (4.8) and using that the tangent bundle is stable $H^0(\overline{\partial}, \mathcal{T}_X) = 0$ to invert the Laplacian, we can solve for the vector

$$\kappa_\alpha^{\mu} = \square^{-1} \left( -i (\partial^\dagger \mathcal{D}_\alpha \omega^{(1,1)}_\nu) g^{\mu\nu} + \frac{1}{2} \mathcal{D}_\alpha \omega_{\mu\nu} (\partial \omega)^{\mu\nu} \right).$$  (4.10)

We will return to this equation shortly.

Finally, consider the Hermitian–Yang–Mills (HYM) equation. The Donaldson–Uhlenbeck–Yau (DUY) theorem asserts that given the bundle is stable, we find a unique connection $A$ such that

$$\omega^2 F = 0.$$  

Suppose this is the case. Under a deformation of the gauge field, with $X$ fixed, we end up

$$\overline{\partial}_A^\dagger \mathcal{D}_\alpha A = -\partial^\dagger_A \mathcal{D}_\alpha A^\dagger,$$

which is an equation of motion of the fluctuations.

Suppose parameters also vary the manifold $X$. In that case a holomorphic variation of the HYM is

$$i (\mathcal{D}_\alpha \omega_{\mu\nu}) F^{\mu\nu} + \overline{\partial}_A \mathcal{D}_\alpha A = 0,$$  (4.11)

where we use (C.10). Write a Hodge decomposition

$$\mathcal{D}_\alpha A = \mathcal{D}_\alpha A^{\text{harm}} + \overline{\partial}_A \Phi_\alpha + \overline{\partial}_A^{\dagger} \Psi^{(0,2)}_\alpha,$$  (4.12)

for some $\text{End} E$–valued $(0,2)$–form $\Psi_\alpha$ and scalar $\Phi_\alpha$. The harmonic representative is determined purely by Hodge theory. The remaining pieces are determined by the Atiyah equation and the HYM equation as we now show.

Firstly, the Atiyah equation gives

$$\overline{\partial}_A \overline{\partial}_A^{\dagger} \Psi^{(0,2)}_\alpha = \Delta_\alpha^{\mu} F_\mu.$$  (4.13)

As we have done previously, the Hodge decomposition of $\Psi^{(0,2)}_\alpha = \overline{\partial}_A \Xi^{(0,1)}_\alpha + \overline{\partial}^{\dagger} (\cdots)$, where the second term does not contribute to $\mathcal{D}_\alpha A$ and so is not written explicitly. The
Atiyah equation becomes a Laplacian which we can invert for unobstructed deformations \( \overline{\partial}_A \Xi^{(0,1)} = \square_A^{-1} (\Delta_\alpha \omega F_\mu) \). Hence, we determine the last term in (4.12)

\[
\Psi_\alpha^{(0,2)} = \square_A^{-1} (\Delta_\alpha \omega F_\mu) + \overline{\partial}_A^j - \text{closed} ,
\]

where we have not written the \( \overline{\partial}_A^j \text{–closed} \) term as it does not contribute to \( \mathcal{D}_A \).

Secondly, because \( (\partial^\dagger A)^2 = 0 \) we find

\[
\square_A \Phi_\alpha = -i(\mathcal{D}_\alpha \omega F_\mu) , \quad (4.15)
\]

The bundle satisfies the HYM and so is stable, thence \( \square_A \) has trivial kernel. This means we can invert the Laplacian for \( \Phi_\alpha \).

4.2. The Hodge decomposition in holomorphic gauge

We collate the results (4.5), (4.9), and (4.12) together in one place

\[
\mathcal{D}_\alpha \omega^{(1,1)} = \mathcal{D}_\alpha \omega^{(1,1)} \text{harm} + \overline{\partial}_A^j \xi^{(1,2)} , \quad \Delta_\alpha = \Delta_\alpha \text{harm} + \overline{\partial}_A \kappa_\alpha , \quad \mathcal{D}_\alpha A = \mathcal{D}_\alpha A \text{harm} + \overline{\partial}_A \Phi_\alpha + \overline{\partial}_A^j \Psi^{(0,2)} .
\]

The field \( \mathcal{D}_\alpha \omega^{(0,2)} = \Delta_\alpha \omega F_\mu \) obeys the following equation \( \overline{\partial}_A^j \Delta_\alpha \omega = i g^{\mu \rho} \nabla^\rho \mathcal{P}(\mathcal{D}_\alpha \omega F_\mu) \).

The gauge fixing and equations of motion allow us to determine the exact and co-exact terms. Collecting (4.6), (4.10), (4.14), (4.15) together in one place:

\[
\xi^{(1,2)} = \square_A^{-1} (\Delta_\alpha \omega F_\mu) - \frac{i \alpha^\lambda}{4} \text{Tr}(\mathcal{D}_\alpha A F) + \frac{i \alpha^\lambda}{2} (\nabla^\nu \Delta_\alpha \omega + \nabla^\mu \mathcal{D}_\alpha \omega^{(0,1)} \mathcal{R}_\mu^\nu) + \overline{\partial}_A^j - \text{closed} , \quad (4.17a)
\]

\[
\kappa_\alpha^\mu = \square_A^{-1} (\overline{\partial}_A^j \mathcal{D}_\alpha \omega^{(1,1)}) + \frac{1}{2} (\mathcal{D}_\alpha \omega F_\mu) (\partial^\omega)^{\nu \mu} , \quad (4.17b)
\]

\[
\Psi^{(0,2)}_\alpha = \square_A^{-1} (\Delta_\alpha \omega F_\mu) + \overline{\partial}_A^j - \text{closed} , \quad \Phi_\alpha = \square_A^{-1} (\overline{\partial}_A^j \mathcal{D}_\alpha \omega F_\mu) . \quad (4.17c)
\]

We have assumed the Laplacian is invertible: that is the sources have no zero modes of the Laplacian. The conditions for this to be the case are written in the previous subsection, we repeat here. Consider the first equation of (4.17c). We have \( \Delta_\alpha \omega F_\mu \) is \( \overline{\partial}_A \text{–closed} \). The Laplacian is invertible if and only if \( \Delta_\alpha \omega F_\mu \) is \( \overline{\partial}_A \text{–exact} \), which is of course, exactly the Atiyah condition (2.7). Similarly, the source in (4.17a) needs to be \( \overline{\partial} \text{–exact} \) which is exactly the last line of (4.1). On the other Laplacians for \( \kappa_\alpha \) and \( \Phi_\alpha \) are always invertible provided the bundles are stable. So we really view (4.17a)–(4.17c) as being supplemented by (4.1).

We have gauge fixed. Hence, the \( \overline{\partial}_A \) and \( \overline{\partial} \text{–exact} \) pieces in (4.16) are physical. We could try to discard them by a gauge transformation, but as we have fixed the gauge completely, this would inevitably lead to a condition such as \( \mathcal{D}_\alpha A \dagger \neq 0 \) or \( \partial \chi_\alpha \neq 0 \). Such terms were set to zero in deriving the equations of above. This is just as in electromagnetism: in say
Coulomb gauge, the potential \( A \) is physical as the gauge has been fixed, and we no longer have the freedom to perform gauge transformations.

Turning to (4.17a)–(4.17c), supplemented by (4.1), these are highly coupled equations, and finding a direct solution is very hard.\(^7\) These equations are another, rather explicit, echo of the intuition that there is no clear decoupling of parameters between \( X \) and \( \mathcal{E} \). One approach is to proceed perturbatively in \( \alpha' \). Expand fields in \( \alpha' \) with a square bracket denoting the order

\[
\mathcal{D}_\alpha \omega = [\mathcal{D}_\alpha \omega]_0 + \alpha' [\mathcal{D}_\alpha \omega]_1 + \ldots , \quad \xi_\alpha = [\xi_\alpha]_0 + \alpha' [\xi_\alpha]_1 + \ldots
\]

At zeroth order in \( \alpha' \), as we are assuming a large radius limit so that SUGRA embeds into string theory, we have \([d \omega]_0 = 0\). The right hand side of the first equation in (4.17a) is therefore \( \mathcal{O}(\alpha') \) and so \([\mathcal{D}_\alpha \omega^{(1,2)}]_0 = 0\) to this order. Hence, \([\mathcal{D}_\alpha \omega^{(1,1)}]_0\) is harmonic with respect to the zeroth order Calabi-Yau metric \([g_\mu \nu]_0\). Also, \([\mathcal{D}_\alpha \omega^{(0,2)}]_0 = 0\) as follows by the calculation in appendix §B.3. Using equivalence of Kähler Laplacians, the right hand side of (4.17b) also vanishes and so \([\Delta_\alpha]_0\) is harmonic with respect to \([g_\mu \nu]_0\). Finally, the sources in (4.17c) completely determines \([\mathcal{D}_\alpha \mathcal{A}]_0\). In summary, \([\mathcal{D}_\alpha \omega]_0\), \([\Delta_\alpha]_0\) are both harmonic with respect to the Calabi-Yau metric while \([\mathcal{D}_\alpha \mathcal{A}]_0\) is harmonic at zeroth order only when \([\Delta_\alpha]_0 = [\mathcal{D}_\alpha \omega^{(1,1)}]_0 = 0\).

We now do the first order in \( \alpha' \) analysis. Due to \( \partial \omega = \mathcal{O}(\alpha') \) the right hand side of the first equation in (4.17a) is determined by \([\Delta_\alpha]_0\) and \([\mathcal{D}_\alpha \mathcal{A}]_0\). Hence, \([\mathcal{D}_\alpha \omega]_1\) can be computed by inverting the Laplacian and using the \( \alpha' \)-corrected harmonic representative \( \mathcal{D}_\alpha \omega^{(1,1)}_{\text{harm}} \). We have determined \([\mathcal{D}_\alpha \omega^{(1,1)}]_1\). This can then be used in (4.17b). Neglecting the term \((\mathcal{D}_\alpha \omega_{\mu \nu})(\partial \omega)_{\mu \nu} = \mathcal{O}(\alpha'^2)\), we can solve for \([\kappa_\alpha]_1\) and determine \([\Delta_\alpha]_1\). Finally, substitute \([\mathcal{D}_\alpha \omega^{(1,1)}]_1\) and \([\Delta_\alpha]_1\) into (4.17c) and solve for \([\mathcal{D}_\alpha \mathcal{A}]_1\). We are finished to \( \mathcal{O}(\alpha') \).

### 4.3. Comments

We pause to make some comments on results so far. First, we now see how quantum corrections modify the field deformations: they do so by the exact and co-exact pieces. These pieces are determined by contractions of zeroth order harmonic representatives with the background fields. The harmonic representatives are a linear combination of elements of \( H^1(X, \mathcal{T}_X^{0,1}) \oplus H^1(X, \mathcal{T}_X^{1,0}) \oplus H^1(X, \text{End} V) \). The role of Atiyah equation and anomaly cancellation is to determine which linear combinations to take. For example, we need conditions like

\[
[\Delta_\alpha]_0 [F_\mu]_0 + [\Delta_\alpha]_1 [F_\mu]_1 + [\Delta_\alpha]_0 [F_\mu]_1 = \mathcal{D}_{\mathcal{A}} - \text{exact} ,
\]

\[
[\Delta_\alpha]_0 [(\partial \omega)_\mu]_1 - \frac{i}{4} \left[ \text{Tr} (\mathcal{D}_\alpha \mathcal{A} F) - 2 (\nabla_\nu \Delta_\alpha + i \nabla_\mu \mathcal{D}_\alpha \omega^{(0,1)}_\nu) \mathcal{B}_\mu \right]_0 = \mathcal{D}_{\mathcal{A}} - \text{exact} . \tag{4.18}
\]

The elements that satisfy these equations form a vector space which is related to a first cohomology of an appropriate bundle, see [7–11] building on the work of [12] who describe the zeroth order in \( \alpha' \) deformations at the level of a non–linear sigma models, taking into account gauge symmetries. However, in constructing these bundles it is convenient to

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\(^7\)We would like to thank Mario García–Fernandez for discussion on this point.
treat deformations of the spin connection $\mathcal{D}_\alpha \Theta$ as independent degrees of freedom. Doing so gives rise to additional but unphysical directions in the fibres of the bundle and one needs to presumably perform an additional quotient to get the physical degrees of freedom. In (4.18) we have not treated $\mathcal{D}_\alpha \Theta$ as an independent degree of freedom. Instead we have used (4.2) relating $\mathcal{D}_\alpha \Theta$ to the metric moduli. This should amount to a quotient of the cohomology but seeing this remains future work.

Second, the analysis performed here is complementary to finding how the background fields are corrected perturbatively in $\alpha'$, see for example [4, 13–15]. In these works the gauge-fixing for the corrections to the background is discussed. Recall that we assume a perturbative string background in which the 10–dimensional dilaton is a constant, see [4] for more details.

Third, the field $\Delta_\alpha$ describes a holomorphic deformation of complex structure. Note that it is perturbatively corrected in $\alpha'$. This is unlike the non–renormalisation that occurs in some type II theories.

Fourth, it is unlikely that one can keep $X$ fixed and vary the bundle at first order in $\alpha'$. To see this, suppose we try to hold $X$ fixed. Then $[\Delta_\alpha]_0 = \mathcal{D}_\alpha \omega_0 = 0$ and $[\mathcal{D}_\alpha A]_0$ are harmonic. At first order in $\alpha'$, (4.17a) implies that $[\mathcal{D}_\alpha \omega^{(1,1)}]_1$ is turned on and sources (4.17b)–(4.17c), unless the following cancellation happens

$$[\text{Tr} (\mathcal{D}_\alpha A^{\text{harm}} F)]_0 = 0 ,$$

which is a much stronger condition than (4.18). This phenomenon arose because the conformally balanced and HYM equation, together with gauge fixing holomorphic gauge, coupled these degrees of freedom together.

Fifth, the fixing of small gauge transformations is a necessary part of the map between parameters and physical field deformations. Our choice of holomorphic gauge is natural for a supersymmetric theory in which the moduli space is complex. Whether there exists choice of gauge in which (4.17c) further simplify is far from clear.

### 4.4. An application to the moduli space metric

We can apply these results to the Kähler moduli space metric derived in [1, 2]

$$g^\mathfrak{z}_{\alpha \bar{\beta}} = \frac{1}{V} \int_X \left( \Delta_{\alpha} \star \Delta_{\bar{\beta}} g_{\mu \nu} + \mathcal{D}_\alpha \omega^{(1,1)} \star \mathcal{D}_{\bar{\beta}} \omega^{(1,1)} + \frac{\alpha'}{4} \text{Tr} (\mathcal{D}_\alpha A \star \mathcal{D}_{\bar{\beta}} A) \right)$$

$$+ \frac{\alpha'}{2V} \int_X \text{vol} \left( \Delta_{\alpha \mu \nu} \Delta_{\bar{\beta} \rho \sigma} + \mathcal{D}_\alpha \omega_{\mu \rho} \mathcal{D}_{\bar{\beta}} \omega_{\sigma \nu} \right) R^\mathfrak{z}_{\mu \nu \rho \sigma} + O(\alpha'^2) .$$

This metric is a series of inner products together with an $\alpha'$–correction involving a contraction with the Riemann tensor. The term involving the Riemann tensor appears because of (4.2) as mentioned above.

A nice thing that happens is that the exact and co–exact terms in (4.16) either cancel or contribute in a way that is higher order in $\alpha'$. This is due to harmonic, exact and
co–exact forms being mutually orthogonal. The $\alpha'$–corrections to the harmonic forms are also orthogonal. For example, writing

$$\Delta^{\text{harm}} = [\Delta^{\text{harm}}]_0 + \alpha' [\Delta^{\text{harm}}]_1.$$  

We may as well take $[\Delta^{\text{harm}}]_1$ to be orthogonal to $[\Delta^{\text{harm}}]_0$ as is done in say [4]. Any contribution which is “parallel” is just a change of basis and represents a vacuous change. Hence,

$$\int_X \Delta^{\mu} * \Delta^\nu g_{\mu\nu} = \int_X [\Delta^{\mu}]_0 * [\Delta^\nu]_0 g_{\mu\nu} + O(\alpha'^2).$$

A similar result applies for the other inner products. Perhaps what was not obvious before the analysis in this section was whether there could be corrections to $\mathcal{D}_\alpha \omega$ or $\Delta_\alpha$ which would contribute to the metric at $O(\alpha')$. We now see the answer is no; the $\alpha'$–corrections come either from $*$ or are written explicitly as above. It would be interesting to see how if this structure persists at next order in $\alpha'$.

We can give a concrete prescription for computing the metric

$$g^{\text{q}}_{\alpha\beta} = \frac{1}{V} \int_X \left( [\Delta^{\mu}]_0 * [\Delta^\nu]_0 g_{\mu\nu} + [\mathcal{D}_\alpha \omega^{(1,1)}]_0 * [\mathcal{D}_\omega \omega^{(1,1)}]_0 + \frac{\alpha'}{4} \text{Tr} \left( [\mathcal{D}_\alpha A]_0 * [\mathcal{D}_\omega A]_0 \right) \right)$$

$$+ \frac{\alpha'}{2V} \int_X \left[ \text{vol} \left( \Delta^{\mu} \mathcal{D}_\omega \mathcal{D}_\rho \mathcal{D}_\sigma + \mathcal{D}_\alpha \omega^{\mu\rho} \mathcal{D}_\omega \omega^{\sigma} \right) F_{\mu\rho\sigma} \right]_0 + O(\alpha'^2).$$  

The terms $[\Delta^{\mu}]_0$ and $[\mathcal{D}_\alpha \omega^{(1,1)}]_0$ are the zeroth order harmonic forms. The only $\alpha'$–correction in the first two terms comes from background metric $g_{\mu\nu}$, and their computation are discussed in [4, 14, 15].

We are to view this as the metric on the quantum corrected cohomology ring corresponding to heterotic moduli. For models connected to the standard embedding this ring is computed using gauged linear sigma models in [16–18], while a mathematics discussion is in [19, 20]; see [21–23] for some reviews. Note that this is physical string theory metric, in which the spin connection is determined in terms of the remaining moduli. The lowest order contribution was determined in [1] and from a different point of view in [4].
5. First order universal geometry and gauge transformations

In the previous section we gauge-fixed and used the equations of motion to isolate the physical degrees of freedom. In this section, we put this into the context of the universal geometry constructed in [1, 2]. We will find that a small transformations correspond to a first order deformation of a connection on the moduli space. This means that the the choice of connection on $M$ corresponds to a choice of gauge. This makes sense: neither the connection on $M$ nor the choice of gauge are physical quantities and so should not enter the physical theory.

So far we have not been too detailed about parameter space derivatives. They are needed for general covariance of the theory as symmetries can depend on coordinates of $X$ but also on parameters.

5.1. Why universal geometry?

In the classic works on complex structure moduli space for CY manifolds, such as [24, 25], the authors benefited from the existence of special coordinates called periods of $\Omega$. In those coordinates $\Omega$ takes a simple form

$$\Omega(x, z) = z^a \alpha_a(x) - \mathcal{G}_a(z) \beta^a(x),$$

(5.1)

where $\alpha_a, \beta^a$ are a symplectic basis for $H^3(X)$. Within this choice of special coordinates it is sensible to compute deformations through a partial derivative $\partial_a$. This is because the tensorial part of $\Omega$ does not depend on parameters: the forms $\alpha_a$ and $\beta^a$ are “constant” across $M$. This feature remains true as long as diffeomorphisms and symplectic rotations do not depend on parameters.

The same story goes for the field $B + i \omega$, the so-called Kähler cone. There exist special coordinates on $M$ in which this depends linearly on parameters

$$(B + i \omega)(x, t) = t^a \omega_a(x),$$

(5.2)

with $\omega_a$ a choice of “constant” harmonic $(1, 1)$-forms.

The existence of coordinates on $M$ of this type is a remarkable feature. One might wonder, given the intricate analysis of the previous section, if similar coordinates exist for heterotic moduli. If one wants to investigate without assuming that special coordinates like above exist, a more covariant formalism is needed. Universal geometry allows to do that.

The caveat – or the gain, according to one’s view – of universal geometry is that one needs to study connections on the moduli space and their associated curvatures, as well as higher order tensors. In particular, we cannot really assume that deformations commute $[\mathcal{D}_a, \mathcal{D}_b] = 0$ as a starting point. We will see in the next section how important is the role of gauge fixing: curvatures have their own small gauge transformations and these affect commutativity.
5.2. Small gauge transformations on a fixed manifold \( X \)

We begin with one of the simplest cases: the moduli space of bundle deformations on a fixed manifold \( X \). In that case, we consider the action of only small gauge transformations. In the notation of [1, 2], we denote \( \mathcal{X} = X \times M \) and \( \mathcal{A} = A + \Lambda \), where \( \Lambda = \Lambda_a \, dy^a \), is the connection for the universal bundle \( \mathcal{U} \to \mathcal{X} \). This discussion at the end of the last section makes it clear that this is only really valid in the \( \alpha' \to 0 \) limit of a heterotic theory.

The covariant parameter derivative of \( A \) on a fixed manifold \( X \) is

\[
\mathcal{D}_a A = \partial_a A - d_A \Lambda_a ,
\]

(5.3)

and as observed in [1, 2] this coincides with a component of the field strength \( \mathcal{F}_{am} \, dx^m = \mathcal{D}_a A \). For the moment we consider real coordinates and forms; holomorphy will follow shortly. This derivative was constructed in [1, 2] to be manifestly covariant under background gauge transformations. However, as we discovered in the previous section, there is an action of small gauge transformations

\[
\mathcal{D}_a A \to \mathcal{D}_a A + d_A \phi_a ,
\]

(5.4)

for some \( \text{End} \mathcal{E} \)-valued \( \phi_a \). The key observation is that by comparing (5.3)-(5.4) we see that a small gauge transformation precisely corresponds to a deformation of the connection \( \Lambda_a \to \Lambda_a - \phi_a \).

(5.5)

This deformation, in general, modifies the field strength \( \mathcal{F}_{ab} = \partial_a \Lambda_b - \partial_b \Lambda_a + [\Lambda_a, \Lambda_b] \) as

\[
\mathcal{F}_{ab} \to \mathcal{F}_{ab} - \mathcal{D}_a \phi_b + \mathcal{D}_b \phi_a ,
\]

\[
\mathcal{D}_a \phi_b = \partial_a \phi_b + [\Lambda_a, \phi_b] .
\]

(5.6)

The geometric picture is the following. Consider the space of all connections \( \mathcal{A} \) and pick a base point \( A \). The space \( \mathcal{A} \) can be thought of as a fibration over \( M \) [5], whose fibres are the gauge orbits generated by (1.3). Deformations of the basepoint \( A \) along the gauge orbit are regarded as trivial, that is \( \delta A \sim \delta A + d_A \phi \). Non–trivial deformations are defined by introducing a connection, which is a 1-form on the base \( M \) of the fibration. The Kodaira–Spencer map identifies gauge inequivalent deformations of connections \( \delta A \) with the tangent space \( T_M \) [26]. In fact, the mixed component \( \mathcal{F}_{am} \) is an explicit realisation of the Kodaira–Spencer map \( T_y M \to \Omega^1(X, \text{End} \mathcal{E}) \). This is the same in spirit as the Donaldson map used in Donaldson theory. This mixed component is precisely the covariant derivative in (5.3). We now see that the ambiguity in the choice of deformation \( \delta A \) in \( \mathcal{A} \) corresponds to a choice of connection on \( M \). Viewed in the language of \( \mathcal{A} \) therefor, it is not surprising that we have discovered that small gauge transformations correspond to deformations of the connection \( \Lambda_a \, dy^a \) on the moduli space \( M \).

We now introduce holomorphy. In [1, 2] we took this to mean that the bundle \( \mathcal{U} \to \mathcal{X} \) is itself a holomorphic bundle, and so \( \mathcal{F}^{(0,2)} = 0 \). The mixed component of this equation is \( \mathcal{D}_\pi \mathcal{A} = 0 \). We now know this is actually part of holomorphic gauge. This is not surprising: asserting \( \mathcal{U} \) is a holomorphic bundle involves a choice of connection \( \Lambda_\pi \) and the space of
such connections is related to gauge fixing. Indeed, if we deform $\Lambda_\alpha \rightarrow \Lambda_\alpha - \phi_\alpha$, then we see that holomorphy is preserved if

$$ \overline{\partial}_A \phi_\alpha = 0, \quad \mathcal{D}_\alpha \phi_\alpha = \mathcal{D}_\beta \phi_\alpha. $$

The first equation is discussed in the first paragraph of (3.3), and the only possible solutions are parameter dependent constants. In particular, it means that $\phi_\alpha$ must commute with the algebra $\mathfrak{g}$ as alluded to above. The second equation we will interpret in the next section. It is a further constraint on the parameter dependence of gauge transformations $\phi_\alpha$, which are associated with gauge fixing second order deformations. We note that $\phi_\alpha \sim \phi_\alpha + \mathcal{D}_\alpha \psi$ is a symmetry of this equation.

Consider the role of gauge symmetries and the structure group of $\mathcal{U}$. The connection $A$ transforms under a gauge transformation

$$ A \rightarrow \Phi_A = \Phi(A - Y)\Phi_A^{-1}, \quad Y = \Phi_A^{-1}d\Phi_A, $$

where $\Phi_A$ is an element of the structure group of the universal bundle $\mathcal{U}$. If we fix $y \in M$ then the universal bundle $\mathcal{U}$ is such that it reduces to $\mathcal{E} \rightarrow X$ with structure algebra $\mathfrak{g}$. We take the structure algebra of $\mathcal{U}$ to be $\mathfrak{g} = \mathfrak{g} \oplus \mathfrak{g}_b$, where we are leaving $\mathfrak{g}_b$ arbitrary apart from it being a real, semi-simple and compact Lie algebra and $[\mathfrak{g}_b, \mathfrak{g}_b] = \mathfrak{g}$. The connection $A$ is taken to be valued in $\mathfrak{g}$, while $\Lambda$ is valued in $\mathfrak{g}_b$. We also demand that $\mathcal{D}_a A \in \Omega^1(X, \text{End} \mathcal{E})$ and so is in particular valued in $\text{ad}_g$.

We see that (5.7) decomposes into two separate transformation laws

$$ A \rightarrow \Phi_A = \Phi(A - Y)\Phi_A^{-1} , \quad \Lambda \rightarrow \Phi_\Lambda = \Phi(\Lambda_a - Y_a)\Phi_\Lambda^{-1} , \quad Y_a = \Phi_\Lambda^{-1}d_\Lambda \Phi_A, $$

where $\Phi$ is exponentiation of the Lie algebras $\mathfrak{g}$. If we take the limit $\Phi = 1 - \phi$ and $\Phi_U = 1 - \phi_U$, then we can gain some intuition for small gauge transformations. The transformation law reduces to

$$ A \rightarrow A + d_A \phi, \quad \Lambda_a \rightarrow \Lambda_a + \mathcal{D}_a \phi_U. $$

We have already studied the deformations in the first equation. So let us for now restrict ourselves to those of the form $d_A \phi = d_A \phi_U = 0$. We take this to mean that $\phi, \phi_U$ depend only on parameters and commute with $\mathfrak{g}$. This means they are $\mathfrak{g}_b$-valued. If we denote $\phi_a = \mathcal{D}_a \phi_U$, then we see that $\mathcal{F}$ transforms as

$$ F \rightarrow F, \quad \mathcal{D}_a A \rightarrow \mathcal{D}_a A, \quad \mathcal{F}_{ab} \rightarrow \mathcal{F}_{ab} + [\mathcal{D}_a, \mathcal{D}_b] \phi_U = \mathcal{F}_{ab} + [\mathcal{F}_{ab}, \phi_U]. $$

where we use $d_A(\mathcal{D}_a \phi_U) = \mathcal{D}_a(d_A \phi_U) - [\mathcal{D}_a A, \phi_U] = 0$. Hence, there are deformations of $\Lambda_a$ that do not modify $A$ nor $\mathcal{D}_a A$; these are like gauge-for-gauge transformations.

An curious aside: recall the space of deformations of holomorphic bundles $\mathcal{E} \rightarrow X$ is a moduli space $M$. We see here that the space of deformations of $\mathcal{U}$ with $\delta A = 0$ reduces to the space of deformations of holomorphic $\mathfrak{g}_b$–bundles on $M$. We leave this for further work.
5.3. Small gerbes

Just as \( F_a = D_a A \) came from the mixed term of a field strength, the form \( B_a \) in (2.12) is most elegantly defined as most easily follows by considering the mixed component of \( H^a \):

\[
B_a = H_a = D_a B + \frac{\alpha^2}{4} \text{Tr} (A D_a A) - d B_a, \quad D_a B = \partial_a B - \frac{\alpha^3}{4} \text{Tr} (\Lambda_a d A). \tag{5.8}
\]

This gives an interpretation to the 1–form \( B_a \) as the mixed component \( B_{am} \) of the universal \( B \)–field. Under background gauge transformations \( B \sim B + \frac{\alpha^2}{2} \text{Tr} \gamma A + \frac{\alpha^3}{4} U \), and applying this to \( B_a \) and \( B_a \) we find exactly the transformation law (2.13).

We immediately see that small gerbe transformations \( B_a \rightarrow B_a + \partial b_a \) are realised by a deformation

\[
B_a \rightarrow B_a - b_a, \tag{5.9}
\]

and so we view \( B_a \) in an analogous fashion to \( \Lambda_a \) for the gauge transformations: it is a ‘gerbe–connection’ on the moduli space \( M \). The tangibility \([2, 1]\) of \( H^a \) is also deformed

\[
H_{ab} \rightarrow H_{ab} - D_a b_b + D_b b_a. \tag{5.10}
\]

If we compare this with the gauge transformation law for the the field strength \( F_{ab} \) in (5.6), then we see that \( B_a \) is naturally viewed as a connection for gerbe transformations and \( H_{ab} \) is its related field strength.

In the previous section we deduced that small gauge transformations, parameterised by \( \phi_a \), result in a transformation law for \( B_a \) in (2.15) and that for \( D_a B \) in (2.16). The realisation of this in universal geometry is more subtle. While a deformation \( \Lambda_a \rightarrow \Lambda_a - \phi_a \) gives the correct law for \( D_a A \), we have

\[
D_a B \rightarrow D_a B + \frac{\alpha^3}{4} \text{Tr} (\phi_a d A),
\]

which does not agree with (2.15). We are missing the fact that \( B_a \) is to be simultaneously deformed. This is a manifestation of the fact the \( B \)–field is charged under the gauge symmetry. The correct deformation turns out to be

\[
\Lambda_a \rightarrow \Lambda_a - \phi_a, \quad \text{and} \quad B_a \rightarrow B_a - \frac{\alpha^3}{4} \text{Tr} (A \phi_a). \tag{5.11}
\]

It is an exercise to see that \( H^a \) transforms in the same manner as (2.15).

To reconcile with (2.16) one could redefine \( D_a B \) as follows

\[
B_a = D_a B + \frac{\alpha^3}{4} \text{Tr} (A D_a A) - d \left( B_a + \frac{\alpha^3}{4} (\text{Tr} \Lambda_a A) \right), \quad D_a B = \partial_a B - \frac{\alpha^3}{4} \text{Tr} (A d \Lambda_a). \tag{5.12}
\]

If we do so, we see that under (5.11) the derivative \( D_a B \) has the correct transformation law. We note that this definition of the covariant derivative of the \( B \)-field differs from [2] by a \( d \)–exact term. The symmetry property (2.16) gave us no choice in this matter. Physically, there is no difference as \( D_a B \) is only ever defined up to \( d \)–closed forms, a manifestation of its parent \( dB \).
Higher tangibilities of the three-form $H$ also transform. For example,

$$\mathcal{H}_{ab} \to \mathcal{H}_{ab} + \frac{\alpha^4}{2} \text{Tr} (\phi_a \mathcal{D}_b A - \phi_b \mathcal{D}_a A),$$

where in analogy with (5.11) we defined the rule

$$\mathcal{B}_{ab} \to \mathcal{B}_{ab} - \frac{\alpha^4}{4} \text{Tr} (\phi_a \Lambda_b - \phi_b \Lambda_a).$$

### 5.4. Small diffeomorphisms

We now suppose that $X$ and the bundle $\mathcal{E}$ can simultaneously deform. As in [1] we interpret diffeomorphisms as a gauge symmetry. This necessitates introducing an invariant basis of forms $e^m, dy^a$ and vectors $\partial_m, e_a$ where

$$e^m = dx^m + c_a^m dy^a, \quad e_a = \partial_a - c_a^m \partial_m,$$

which factorise the hermitian form on $X$ into a block diagonal form

$$\omega = \frac{1}{2} \omega_{mn} e^m e^n + \frac{1}{2} \omega_{ab} dy^a dy^b. \quad (5.12)$$

The symbol $c_a^m$ transforms like a connection under diffeomorphisms and its role will be analogous to $\Lambda_a$. As described in [1] we need to express quantities in the ‘$e$’–basis, and if there is any ambiguity will denote the result by a superscript $\sharp$ symbol. A key example is the gauge field $A$. When expressed in the ‘$e$’–basis it has a transformed component

$$A = A_m e^m + A^a_{\sharp} dy^a,$$

$$A^a_{\sharp} = \Lambda_a - A_m c^m_a.$$  

Consider a form $\eta$ without any additional gauge symmetries and whose legs lie purely along the manifold $X$. Then a result derived in [1] is that its deformation $\delta \eta$ is a Lie derivative with respect to the vectors $e_a$:

$$\delta \eta = \delta y^a \mathcal{L}_{e_a} \eta. \quad (5.13)$$

Due to the fibration structure of $\mathcal{X}$, the Lie derivative functions as a directional direction, $\mathcal{D}_a = \mathcal{L}_{e_a} \eta$ and it is helpful to write out the expression explicitly

$$\mathcal{D}_a \eta = \frac{1}{q!} \left( e_a (\eta_{m_1 \cdots m_q}) - c_a^{n_1 m_1} \eta_{m_2 \cdots m_q} - c_a^{n_2 m_1} \eta_{m_1 m_2 \cdots m_q} - \cdots \right) e^{m_1 \cdots m_q}. \quad (5.14)$$

If we have a tensor of the form

$$\xi = \xi_{n_1 \cdots n_s}^{m_1 \cdots m_r} e^{n_1} \otimes \cdots \otimes e^{n_s} \otimes \partial_{m_1} \otimes \cdots \otimes \partial_{m_r},$$

then its derivative is defined as

$$\mathcal{D}_a \xi_{n_1 \cdots n_s}^{m_1 \cdots m_r} = e_a (\xi_{n_1 \cdots n_s}^{m_1 \cdots m_r}) + c_{a m_1}^{m_1} \xi_{n_1 \cdots n_s}^{m_2 \cdots m_r} + \cdots + c_{a m_r}^{m_r} \xi_{n_1 \cdots n_s}^{m_1 \cdots m_r-1} - c_{a}^{k m_1} \xi_{n_1 \cdots n_s}^{m_1 \cdots m_r} - \cdots - c_{a}^{k n_s} \xi_{n_1 \cdots n_s-1}^{m_1 \cdots m_r}. \quad (5.15)$$

As we introduce additional gauge symmetries and associated connections we will generalise this covariant derivative. For ease of notation, we will use the same symbol except where a possible ambiguity may arise.

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A small diffeomorphism, $\delta \eta \rightarrow \delta \eta + \mathcal{L}_\varepsilon \eta$ is realised by a deformation of the connection $c_a$:

$$c_a \rightarrow c_a - \varepsilon_a, \quad \varepsilon_a = \varepsilon^m_a \partial_m. \tag{5.16}$$

It is elementary to see that

$$\mathcal{D}_a \eta \rightarrow \mathcal{D}_a \eta + \mathcal{L}_\varepsilon \eta. \tag{5.17}$$

The connection $c_a$ has a curvature tensor defined as the Lie bracket of two vertical vectors $S_{ab}^m = [e_a, e_b]^m$. Under the deformation (5.16) we find

$$S_{ab}^m \rightarrow S_{ab}^m + \mathcal{D}_a \varepsilon_b^m - \mathcal{D}_b \varepsilon_a^m. \tag{5.18}$$

As derived in [1], this curvature tensor appears in forms on $X$ with two or more legs along the moduli space. For example, in complex coordinates

$$(d c \omega)_{\alpha \beta} = -i S_{\alpha \beta}^\mu \omega_\mu, \quad (d c \omega)_{\alpha} = i S_{\alpha}^\mu \omega_\mu, \quad (d c \omega)_{\alpha \beta} = -i S_{\alpha \beta}^\mu \omega_\mu + i S_{\alpha \beta}^\mu \omega_\mu.$$ 

If we start with $S = 0$ and demand it is preserved then

$$\mathcal{D}_a \varepsilon_b^m - \mathcal{D}_b \varepsilon_a^m = 0. \tag{5.19}$$

Locally, this amounts to $\varepsilon_a^m = \mathcal{D}_a v^m$ for some vertical vector field $v^m \partial_m$. See [1] for the definition of $\mathcal{D}_a$ acting on mixed component tensors.

The connection $c_a$ determines tangibility, and so is also modified by deformations. In the following discussion it is convenient to compare the deformed frame, denoted by a tilde with the undeformed frame. The basis of 1-forms deforms as

$$e^m \rightarrow \tilde{e}^m = e^m - dy^a \varepsilon_a^m. \tag{5.20}$$

This has interesting consequences. Consider the field strength $F$. Firstly

$$F = \frac{1}{2} F_{mn} e^m e^n + \mathcal{F}_{an} dy^a e^n + \frac{1}{2} \mathcal{F}_{ab} dy^{ab} = \frac{1}{2} F_{mn} \tilde{e}^m \tilde{e}^n + \mathcal{F}_{an} dy^a \tilde{e}^n + \frac{1}{2} \mathcal{F}_{ab} dy^{ab} \tag{5.21}$$

Comparing with $F$ expressed in terms of its tilded components:

$$\tilde{F} = \frac{1}{2} F_{mn} \varepsilon_a^m \varepsilon_a^n + \tilde{\mathcal{F}}_{an} dy^a \varepsilon_a^n + \frac{1}{2} \tilde{\mathcal{F}}_{ab} dy^{ab},$$

and identifying $\mathcal{F}_{an} = \mathcal{D}_a A_n$ and $\mathcal{F}_{an} = \tilde{\mathcal{D}}_a A_n$ we find a transformation law

$$\tilde{\mathcal{D}}_a A = \mathcal{D}_a A + \varepsilon_a^m F_m, \quad \tilde{\mathcal{F}}_{ab} = \mathcal{F}_{ab} - \varepsilon_a^n \mathcal{D}_a A_n + \varepsilon_b^n \mathcal{D}_a A_n.$$

The second equation implies the curvature on $M$ is also deformed in a non–trivial way. The first equation is exactly the transformation law under a small diffeomorphisms as derived in supergravity. When applied to $\mathcal{D}_a A$ and combined with $A^\alpha_a \rightarrow A^\alpha_a - \phi_a$ — note that we use $A^\alpha$ and not $A$ as we are in the ‘$e$’–basis — we recover exactly (2.8)

$$\tilde{\mathcal{D}}_a A = \mathcal{D}_a A + \varepsilon^\alpha_a \mathcal{F}_\mu + \tilde{\mathcal{D}} A \phi_a.$$
In the same way, we determine that field strength \( H \) transforms and from this we deduce the transformation law for \( H_a = B_a \) under small diffeomorphisms

\[
\tilde{B}_a = B_a + \varepsilon_a^m H_m .
\]

Again this is in exact agreement with supergravity.

There is a complex structure on \( \mathcal{X} \)

\[
J = J_m^n e^n \otimes \partial_n + J^a_b dy^a \otimes e_b . \tag{5.21}
\]

The manifold \( \mathcal{X} \) being complex means the indices must be pure \( c = 0 \).

(5.22)

It is interesting to study how \( J \) transforms under the change of ‘\( e \)–basis’ in (5.19):

\[
\tilde{J} = J_m^n e^n \otimes \partial_n + J^a_b dy^a \otimes e_b + (\varepsilon_a^m J_m^n - J^a_b \varepsilon_b^n) dy^a \partial_n . \tag{5.23}
\]

At first we see the structure is no longer block diagonal, which is not surprising considering with have deformed \( c_a \). However, as the complex structure is integrable, the Frobenius theorem tells us we can find a compatible set of holomorphic coordinates and in these coordinates we see the off–diagonal terms vanish when \( \varepsilon_a^\pi = 0 \). So deformations that preserve integrability of \( J \) are identified with ones that preserve its block diagonal nature. This has an echo in supergravity. In the last section, we saw that part of holomorphic gauge is

\[
\Delta_\mu = 0 .
\]

Small diffeomorphisms that preserve holomorphic gauge are of the form \( \varepsilon_\mu^\alpha = 0 \). So we learn that holomorphic gauge amounts to \( J \) being integrable and block diagonal.

Furthermore, in [1] it was shown that \( \Delta_\alpha^\mu \mu = -\partial_\alpha c_\mu^\mu \). Hence, under a deformation \( c_\alpha^\mu \rightarrow c_\alpha^\mu - \varepsilon_\alpha^\mu \) we have exactly the correct transformation law

\[
\Delta_\alpha^\mu \mu \rightarrow \tilde{\Delta}_\alpha^\mu \mu = \Delta_\alpha^\mu \mu + \partial_\alpha \varepsilon_\mu^\mu . \tag{5.24}
\]

Before we proceed we pause to introduce some technical details, which while maybe mildly painful, but nonetheless are necessary to account for the diffeomorphism symmetries correctly. Recall a form with \( p \) legs along \( M \) and \( q \) legs along \( X \) is said to have tangibility \([p,q]\). We denote the set of such forms by \( \Omega^{[p,q]}(\mathcal{X}) \). The diffeomorphic invariant way to distinguish legs in this way is through the ‘\( e \)–basis’. This means however, we need to introduce a generalisation of the de Rahm operator that acts like the de Rahm operator on \( X \), but annihilates the bases \( e^m \). Neither of the operators \( e^m \partial_n \) or \( dx^m \partial_m \) achieve this. So with a slight abuse of notation\(^8\), we redefine \( d \) and \( \mathcal{D} \) to act as

\[
d f = (\partial_m f) e^m , \quad d e^m = 0 , \quad d (dy^a) = 0 ,
\]

\[
\mathcal{D} f = e_a(f) dy^a , \quad \mathcal{D} e^m = -e^m_{a,n} dy^a e^n , \quad \mathcal{D} (dy^a) = 0 . \tag{5.25}
\]

\(^8\)In [1] we used the Iceland ‘deth’ symbol \( \mathcal{D} \) for derivatives of corpus while here we denote this by the usual \( d \)–operator; here we use \( \mathcal{D} \) which was denoted \( \mathcal{D}^\sharp \) in [1]. We hope by simplifying the number of derivative symbols used, the paper will be easier to digest.
It is straightforward to check that
\[ d^2 = 0 \quad \text{and} \quad \{d, \mathcal{D}\} = 0. \tag{5.26} \]

Note that \( d \) now maps a \([p, q]\)-form to a \([p, q+1]\)-form and \( \mathcal{D} \) maps a \([p, q]\)-form to a \([p+1, q]\) form. There are strong similarities with complex manifolds. A form of definite holomorphic type, say \((r, s)\), is acted upon by the \( \partial \)- and \( \bar{\partial} \)-operators that map this form to \((r+1, s)\) and \((r, s+1)\). In the limit where \( c_a \to 0 \), which includes the case of a fixed manifold \( X \), the operators above reduce to what has been used in previous sections.

The de Rham operator \( d \) acting on a vertical form \( \eta \) decomposes
\[ d\eta = d\eta + \mathcal{D}\eta - \frac{1}{2(q-1)!} \eta_{m_1 \cdots m_{q-1} n} S_{bc} n^b d^c y^e e^{m_1 \cdots m_{q-1}}. \tag{5.27} \]

The term involving \( S_{ab} = [e_a, e_b] \) measures the lack of integrability of tangibility. In general a tangibility \([p, q]\) form with \( q \geq 1 \) is mapped to a form with three different tangibilities divided up into branches as shown
\[
\Omega^{[p,q]}(X) \xrightarrow{d} \Omega^{[p+1,q]}(X) \xrightarrow{\mathcal{D}} \Omega^{[p+2,q-1]}(X) \xrightarrow{s} \Omega^{[p+2,q-1]}(X).
\]

The hermitian form is given in (5.12) and its deformation, derived from (5.13), is given by \( \mathcal{D}_a \omega = L_{e_a} \omega \). Just as \( F \) includes \( \mathcal{D}_a A \) as its mixed component, the deformation of \( \omega \) is a mixed component of \( d\omega \):
\[ d\omega = \frac{1}{3!} (d\omega)_{mnp} e^{mnp} + \frac{1}{2} \mathcal{D}_a \omega_{mn} d^b y^c e^{mn} - \frac{1}{2} S_{bc} n^b \omega_{nm} d^c y^e e^{m} , \tag{5.28} \]

where
\[ (d\omega)_{mnp} = \partial_m \omega_{np} + \partial_n \omega_{pm} + \partial_p \omega_{mn} , \]

and \( \mathcal{D}_a \omega \) is of the form (5.14) with \( q = 2 \). The transformation law under \( c_a \to \tilde{c}_a = c_a - \varepsilon_a \) follows directly from (5.14):
\[ \mathcal{D}_a \omega \to \tilde{\mathcal{D}}_a \omega = \mathcal{D}_a \omega + \varepsilon_a^m (d\omega)_m + d(\varepsilon_a^m \omega_m) . \tag{5.29} \]

where the notation here means
\[ (d\omega)_m = \frac{1}{2} (\partial_m \omega_{np} + \partial_n \omega_{pm} + \partial_p \omega_{mn}) e^{np} , \quad d(\varepsilon_a^m \omega_m) = \partial_p (\varepsilon_a^m \omega_{mn}) e^{pm} . \]

This is exactly the transformation law of a small diffeomorphism. It is also possible to deduce this transformation law by studying how \( d\omega \) transforms under the change of frame.
5.5. Holomorphy

This section opened with a discussion of the role of holomorphy for the gauge field. In universal geometry this is the fact $F^{(0,2)} = 0$ and results in gauge fixing $A$ to holomorphic gauge. We now extend this to other universal fields.

We begin with complex structure $J$. Holomorphic gauge in supergravity is the condition $\Delta \alpha^\mu = 0$. As described above this amounts to the connection $c_a$ being pure in indices, or equivalently, the fibration $X$ being complex.

As for the moduli space of Calabi-Yau manifolds the deformations of $B$ and $\omega$ are combined and holomorphy is gauge fixed to be

$$Z^{(2,0)}_{\alpha} = 0, \quad Z^{(1,1)}_{\alpha} = 0, \quad Z^{(0,2)}_{\alpha} = Z^{(0,2)}_{\alpha} = 0.$$

For the hermitian form $\omega$ and $B$–field, holomorphy is captured by

$$\| - d\omega = 0.$$

To see this we need only consider the tangibility $[1, 2]$ component, which has one leg along the moduli space $M$ and the remainder along $X$. Then, note that $\frac{1}{2} H_{\pi mn} e^{mn} = B_\pi$ and $\frac{1}{2} (d^c \omega)_{\pi mn} e^{mn} = -i D_\pi \omega^{(1,1)} + i D_\pi \omega^{(2,0)}$. Hence, we find

$$\Delta \alpha^\mu = 0, \quad D_\alpha A = 0, \quad Z^{(2,0)}_{\alpha} = 0, \quad Z^{(1,1)}_{\alpha} = 0, \quad Z^{(0,2)}_{\alpha} = Z^{(0,2)}_{\alpha} = 0.$$

As promised this is precisely the hermitian moduli part of holomorphic gauge summarised in (3.5).

In §3.3 we observed there is a residual gauge freedom which can be used to set $\delta \Omega^{(3,0)}$ to be harmonic. Equivalently, $\omega^{\mu\nu} D_\alpha \omega_{\mu\nu}$ is a constant on $X$. There is a geometric echo of this gauge fixing in the universal geometry. We can consider the Lee form for the universal geometry

$$\frac{1}{2} \omega^{mn} (d\omega)_{mn} + \frac{1}{2} \omega^{ab} (d\omega)_{ab} = \frac{1}{2} d\gamma^p \omega^{mn} D_a \omega_{mn} + \frac{1}{2} \omega^{mn} (d\omega)_{mn} - \frac{1}{2} \omega^{ab} (S_{ab}^m \omega_m),$$

The gauge fixing implies the first term, which has a leg along the moduli space, is a constant over manifold $X$; it depends only on parameters. The second and third terms have legs along the manifold. The first term is the usual Lee form and the second term is an extrinsic contribution coming from the embedding of $X$ in $\mathcal{X}$. For heterotic theories with constant dilaton the second term vanishes, and if we do as in [1] and set $S_{ab} = 0$ then the final term also vanishes. So the residual gauge fixing of §3.3 amounts to the Lee form on the universal bundle being closed. It would be interesting to explore this geometric condition in the case where the dilaton is not constant.
6. A taste of second order universal geometry

The study of higher order deformations is necessary if one wants to fully understand the moduli space. They can capture quantities such as the metric $g^+_ab$ or the Yukawa couplings $\kappa_{abc}$, and will be crucial to any heterotic generalisation of special geometry. Furthermore, the work of [27] suggests they govern the underlying algebra of heterotic moduli spaces.

Our aim here is to give a taste of what is to be learnt at second order with an example which is the study of holomorphic bundles on a fixed manifold, a situation which occurs at the standard embedding or at zeroth order in heterotic. We will find that studying this moduli space in the context of universal geometry will actually allow us to learn something useful about the role of curvature $F_{ab}$ on the moduli space. We will not achieve the ultimate aim of describing the theory of higher order covariant derivatives for heterotic theories. This we hope to publish soon. Indeed, even at first order when one fleshes out the details of gauge fixing, this is already extremely complicated as demonstrated in §4.2.

We study the moduli of holomorphic stable vector bundles $E \to X$ over a fixed complex manifold with a balanced metric $d(\omega^2) = 0$. The structure group of $E$ is a compact real group $G$, with Lie algebra $\mathfrak{g}$. This example is discussed in §5.2. It is relevant to the mathematics of the Kobayashi–Hitchin correspondence, see e.g. [5]. It is also relevant to heterotic theories to lowest order in $\alpha'$. To next order in $\alpha'$, the example becomes more complicated. As we have shown in section §4.2 to first order in $\alpha'$ the heterotic equations of motion demand the complex structure and hermitian structure of $X$ also vary with a change in the bundle.

The manifold $X$ is fixed so we do not worry about $c_a$ or $S_{ab}$. Therefore, we consider a universal bundle $U \to M \times X$. We are ignoring $\alpha'$–corrections, which as discussed in §4.2 will force $X$ to vary, and so introduce a connection $c_a$. In this limit, the bundle is equipped with a connection:

$$A = \Lambda_a dy^a + A, \quad F = \frac{1}{2} F_{ab} dy^{ab} + dy^a F_a + F, \quad F_a = \mathcal{D}_a A,$$

with Bianchi identity

$$\mathcal{D}_a F = d_A (\mathcal{D}_a A) \quad [\mathcal{D}_a, \mathcal{D}_b] A = -d_A F_{ab}.\quad \quad \quad (6.1)$$

A deformation of the gauge field can be written in real coordinates as

$$\delta A = \delta y^a \mathcal{D}_a A + (\delta y^a \otimes \delta y^b) \mathcal{D}_a \mathcal{D}_b A + \cdots.$$

Recall the first order analysis in §3.2–§4.2. Holomorphy on a fixed manifold means that $\mathcal{D}_{\pi} A = \overline{\partial}_A \Phi_\pi$ and fixing to holomorphic gauge means $\mathcal{D}_{\pi} A = 0$. This completely fixes the gauge at first order. The deformation above becomes

$$\delta A = \delta y^a \mathcal{D}_a A + (\delta y^a \otimes \delta y^b) \mathcal{D}_a \mathcal{D}_b A + (\delta y^\tau \otimes \delta y^\rho) \mathcal{D}_\pi \mathcal{D}_\beta A + \cdots.$$

The commutators imply $[\mathcal{D}_\beta, \mathcal{D}_\alpha] A = \overline{\partial}_A F_{\alpha\beta}$ and $[\overline{\partial}_\tau, \mathcal{D}_\alpha] A = \overline{\partial}_A F_{\alpha\tau}$ and we immediately see that deformations do not necessarily commute. However, a key point is that the choice of connection $\Lambda_a$ is closely related to gauge fixing. What we show here that under the
assumption $X$ is fixed we can find a gauge in which $F_{\alpha\beta} = 0$. But the term $F_{\alpha\beta}$ is not necessarily zero. So deformations of the gauge field do not necessarily commute.

To see this we start by defining holomorphy at second order in same way as first order:

$$D_\beta D_\alpha A = \partial A \Phi_{\alpha\beta}, \quad D_\alpha D_\beta A = \bar{\partial} A \Phi_{\alpha\beta}.$$  \hspace{1cm} (6.3)

As $D_\beta D_\alpha A$ is $\bar{\partial} A$-exact and we find two additional equations

$$D_\beta D_\alpha A = \bar{\partial} A \Phi_{\alpha\beta}, \quad F_{\alpha\beta} = -\Phi_{\alpha\beta} + \Phi_{\beta\alpha}.$$  \hspace{1cm} (6.4)

That $F$ is antihermitian implies $\Phi^{\dagger}_{\alpha\beta} + \Phi_{\alpha\beta} = \Phi^{\dagger}_{\beta\alpha} + \Phi_{\beta\alpha}$.

Finally, there is a purely holomorphic derivative. We write down a Hodge decomposition

$$D_\alpha D_\beta A = D_\alpha D_\beta A^{\text{harm}} + \partial A \Phi_{\alpha\beta} + \partial^{\dagger} A \Psi_{\alpha\beta},$$  \hspace{1cm} (6.5)

where the harmonic term will be proportional to the first order variations $D_\alpha A$ through some parameter-dependent coefficients.

Recall, the action of small gauge transformations is described by $\Lambda_a \rightarrow \Lambda_a + \phi_a$. The field $\phi_a$ evaluated at $y_0 \in M$ gauge fixes the first order derivative $D_\alpha A$, which as described previously, is holomorphic gauge $D_\alpha A = 0$. Given this gauge fixing, the second order derivatives transform under $D_\alpha \phi_b|_{y_0} = \partial_\alpha \phi_b + [\Lambda_a, \phi_b]|_{y_0}$ as follows

$$D_\beta D_\alpha A \sim D_\beta D_\alpha A + \bar{\partial} A (D_\beta \phi_a).$$  \hspace{1cm} (6.6)

Holomorphic gauge at second order is given by setting $D_\pi \phi_{\pi\gamma} = -\Phi_{\pi\gamma}$ and $D_\alpha \phi_{\alpha\gamma} = -\Phi_{\alpha\gamma}$ so that

$$D_\pi D_\alpha A = 0, \quad D_\beta D_\pi A = 0.$$  \hspace{1cm} (6.7)

Note that antihermiticity requires $D_\pi \phi_{\alpha\beta} = -(D_\beta \phi_{\pi\gamma})^{\dagger} = \Phi^{\dagger}_{\beta\alpha}$. This, together with the fact that there are no $\bar{\partial} A$-closed scalars, completely fixes the gauge freedom at this order.

From the commutator of the first equation in (6.6) we gather that $F_{\alpha\beta} = 0$ and the universal bundle is holomorphic. In passing we observe that this is only true after gauge-fixing second order deformations to holomorphic gauge. There is not enough freedom, however, to get rid of $F_{\alpha\beta}$. A perhaps useful intuition for this is a simple counting. Second derivatives are given by four independent quantities

$$D_\alpha D_\beta A, \quad D_\pi D_\beta A, \quad D_\alpha D_\pi A, \quad D_\pi D_\beta A,$$  \hspace{1cm} (6.8)

while small gauge transformations amount to two independent quantities

$$D_\alpha \phi_\beta, \quad D_\alpha \phi_{\alpha\beta}.$$  \hspace{1cm} (6.9)

Hence, we can only gauge-fix only two among (6.7), which is what we did in (6.6).

In this gauge we have

$$D_\beta D_\alpha A = \bar{\partial} A F_{\alpha\beta}, \quad F_{\alpha\beta} = \Phi_{\beta\alpha} + \Phi^{\dagger}_{\beta\alpha} = \Phi_{\alpha\beta} + \Phi^{\dagger}_{\alpha\beta}.$$  \hspace{1cm} (6.10)
We now turn to the equations of motion. A real variation of the Bianchi identity (6.2) is

\[ \mathcal{D}_a \mathcal{D}_b F = d_A \mathcal{D}_a \mathcal{D}_b A + \{ \mathcal{D}_a A, \mathcal{D}_b A \} . \]  

(6.10)

The manifold \( X \) has fixed complex structure. This implies \( \mathcal{D}_a \mathcal{D}_b F^{(0,2)} = 0 \) and projecting the previous equation onto type we find

\[ \overline{\partial}_A (\mathcal{D}_a \mathcal{D}_b A) + \{ \mathcal{D}_a A, \mathcal{D}_b A \} = 0 . \]  

(6.11)

When indices of mixed holomorphic type are considered, we do not learn anything new from this. When \( a, b \) are holomorphic, we recognise the second order Maurer–Cartan equation

\[ \overline{\partial}_A (\mathcal{D}_a \mathcal{D}_b A) + \{ \mathcal{D}_a A, \mathcal{D}_b A \} = 0 . \]  

(6.12)

From this we can read the coefficients \( \Psi_{\alpha\beta} \) in the Hodge decomposition (6.4)

\[ \Box_A \Psi_{\alpha\beta} = -\{ \mathcal{D}_a A, \mathcal{D}_b A \} . \]  

(6.13)

Unobstructedness of deformations means that the bracket does not contain zero-modes and this is the condition that allows to invert the Laplacian and solve for \( \Psi_{\alpha\beta} \).

There is another equation to consider, the HYM equation \( \omega^2 F = 0 \). A second variation of this, while keeping \( X \) fixed, gives

\[ \omega^2 \left( d_A \mathcal{D}_a \mathcal{D}_b A + \{ \mathcal{D}_a A, \mathcal{D}_b A \} \right) = 0 . \]  

(6.14)

In holomorphic gauge (6.6) this gives two equations

\[ \overline{\partial}_A^\dagger \mathcal{D}_a \mathcal{D}_b A = 0 , \quad \overline{\partial}_A^\dagger \mathcal{D}_\beta \mathcal{D}_\alpha A = -i \omega^{\mu\nu} \{ \mathcal{D}_\alpha A, \mathcal{D}_\beta A^\dagger \}_{\mu\nu} . \]  

(6.15)

Substituting (6.4) and (6.9) inside these and using stability to invert the Laplacian, we finally obtain

\[ \Phi_{\alpha\beta} = 0 , \quad \mathcal{F}_{\alpha\beta} = \Box_A^{-1} \left( -i \omega^{\mu\nu} \{ \mathcal{D}_\alpha A, \mathcal{D}_\beta A^\dagger \}_{\mu\nu} \right) . \]  

(6.16)

The non-primitive part of the bracket of mixed variations acts as a source for \( \mathcal{F}_{\alpha\beta} \). Therefore, deformations of stable bundles over a fixed manifold do not necessarily commute.

It would be very interesting to apply this second order analysis with the beautiful work in [27] who described ‘holomorphic’ data in terms of a functional of \( SU(3) \) structures. In that work the starting point at first order in deformations implicitly assumes holomorphic gauge (together with the residual gauge constraint \( \delta \Omega^{(3,0)} \) is harmonic) and at second order assumes holomorphic derivatives commute. The latter condition may or may not be true. Determining this requires a complete analysis of gauge fixing second order derivatives and determining if the curvature tensor \( S_{ab}, H_{ab}, F_{ab} \) are non–zero. Doing so is therefore both interesting and important.
7. Conclusions

We started with the humble ambition of writing down the action of small gauge transformations together with clarifying gauge fixing in heterotic theories. We achieved this goal.

Along the way we clarified certain issues. We related the choice of gauge fixing to a choice of connection on the moduli space. We found holomorphic gauge corresponds to the universal bundle being holomorphic. We reiterated that taking a holomorphic deformation of fields is itself a gauge fixing, holomorphic gauge. There is a residual gauge freedom which we showed can be used to gauge fix the deformation of the holomorphic top form to be harmonic $\delta \Omega^{(3,0)} = k \Omega$ with $k$ a constant. This gauge fixing implies $\partial \chi_\alpha = 0$.

We checked that the equations of motion are properly invariant under gauge transformations and they do not fix the gauge. They can then be used to write explicit expressions for terms in the Hodge decomposition of all the fields, which as we have gauge fixed, are the physical degrees of freedom. The role of quantum corrections is to generate exact and co–exact terms. The coupled nature of the equations makes it very hard to disentangle the heterotic moduli as ‘bundle’ or ‘complex structure’ or ‘hermitian moduli’. For example, in heterotic theories one cannot deform the bundle without also deforming the manifold. Nonetheless we give a prescription for how to compute through the mess applying it to the moduli space metric.

We showed the choice of gauge is related to a choice of connection on the moduli space $M$ corresponds to choice of small gauge. For example, holomorphic gauge in spacetime corresponds to the universal bundle being holomorphic $F^{(0,2)} = 0$. At second order deformations do not obviously commute and gauge fixing is related to the field strengths $F_{ab}, S_{ab}$ and $H_{ab}$ on the moduli space. In a toy example, we found gauge fixing allowed us to eliminate some, but not all of these field strengths. Consequently, bundle deformations do not commute.

There are many interesting questions. We hope to publish a full analysis of second order universal geometry in the near future. This has important implications for any ‘heterotic special geometry’ such as relations between the Yukawa couplings and the moduli space metrics. Is it possible to formulate a description of the moduli space in a fully gauge invariant way? The discussion in this paper and the literature makes use of a nice choice of gauge. Is there a more gauge invariant formulation of these results in which we do not need to gauge fix? Or can we show our results do not depend on our choice of gauge?

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A. Complexified tangent spaces and gauge groups

In this short appendix we write down some well-known mathematical facts, largely drawn from [28], about complexification of vector spaces and groups, and relate these facts to gauge transformations.

Let $X$ be a differentiable manifold of dimension $m$. Let $g, h \in \mathcal{F}(X)$ be two real-valued smooth functions. Then, $f = g + ih : X \rightarrow \mathbb{C}$ is a complex valued function and is an element of the complexified set of smooth functions of $X$ denoted as $\mathcal{F}(X)^C$. Its complex conjugate is $\overline{f} = g - ih$ and it is real if and only if $\overline{f} = f$.

Let $\mathcal{V}$ be a real vector space with real dimension $m$. Its complexification $\mathcal{V}^C$ is a vector space of complex dimension $m$ whose elements consist of $V + iW$ with $V, W \in \mathcal{V}$, and multiplication by a scalar is enhanced from real scalars to complex scalar quantities. $\mathcal{V} \subset \mathcal{V}^C$ with the identification of $V + i0 \in \mathcal{V}^C$ and $V \in \mathcal{V}$. Such vectors are said to be real. The complex conjugate of $Z = V + iW$ is denoted $\overline{Z} = V - iW$ and $Z$ is real if and only if $Z = \overline{Z}$. A linear operator $e$ acts on $\mathcal{V}^C$ is the obvious way: $e(V + iW) = e(V) + ie(W)$. If $e \rightarrow \mathbb{R}$ is a linear function, so that $e \in \mathcal{V}^*$, then its complexification acts on $\mathcal{V}^C$ as $e : \mathcal{V}^C \rightarrow \mathbb{C}$. The quantity $e$ is said to be real if $e(V + iW) = e(V - iW)$. Tensors are complexified in the obvious way: $T = T_1 + iT_2$ where $T_1, T_2$ are tensors of the same type. A tensor is real if $T = T$.

Let $e_m$ be a basis for $\mathcal{V}$. If we regard these as complex vectors the same basis $\{e_m\}$ becomes a basis for $\mathcal{V}^C$. That is, if $V = V^m e_m$ and $W = W^m e_m$ are both real vectors then $V + iW = (V^m + iW^m)e_m \in \mathcal{V}^C$, where $\overline{e_m} = e_m$. Hence, $\dim_{\mathbb{R}} \mathcal{V} = \dim_{\mathbb{C}} \mathcal{V}^C$.

The tangent space $\mathcal{T}_pX = \{V^m \partial_m \mid V^m \in \mathbb{R}\}$ is an example of a real vector space. Its complexification is therefore $\mathcal{T}_pX^C = \{(V^m + iW^m)\partial_m \mid V^m, W^m \in \mathbb{R}\}$. As above, the basis elements for $\mathcal{T}_pX$ are regarded as basis elements for the complexified tangent space. The co-tangent space is complexified in the obvious way $\mathcal{T}^*_pX = \{\omega + i\eta \mid \omega, \eta \in \mathcal{T}^*_pX\}$. Tensors are similarly complexified, and the extension to the tangent bundle $\mathcal{T}_X^C$ and co-tangent bundle $\mathcal{T}^*_X^C$ follows.

If $X$ has $\dim_{\mathbb{R}} X = 2k$ and admits an integrable complex structure then we can find a split of the coordinates so that $\frac{\partial}{\partial x^\mu}, \ldots, \frac{\partial}{\partial x^k}, \frac{\partial}{\partial y^\mu}, \ldots, \frac{\partial}{\partial y^k}$ are a basis for $\mathcal{T}_pX$ so that $J_\mu = \frac{\partial}{\partial y^\mu}$ and $J_\mu = -\frac{\partial}{\partial y^\mu}$. These vectors are also a basis for the complexified tangent space $\mathcal{T}_pX^C$, so that $\dim_{\mathbb{C}} \mathcal{T}_pX^C = 2k$. Furthermore, we can redefine the basis so that
\[
\frac{\partial}{\partial z^\mu} = \frac{1}{2} \left( \frac{\partial}{\partial x^\mu} - i \frac{\partial}{\partial y^\mu} \right), \quad \frac{\partial}{\partial \overline{z}^\mu} = \frac{1}{2} \left( \frac{\partial}{\partial x^\mu} + i \frac{\partial}{\partial y^\mu} \right), \quad \mu, \overline{\mu} = 1, \ldots, k,
\]
is also basis for $\mathcal{T}_pX^C$ but one in which $J$ is constant and diagonal. Note that $\frac{\partial}{\partial z^\mu} = \frac{\partial}{\partial z^\mu}$. With this choice of basis we have a canonical split of the vector fields
\[
\mathcal{T}_X^C = \mathcal{T}_X^{(1,0)} + \mathcal{T}_X^{(0,1)}.
\]
and so a vector $V \in \mathcal{T}_X^C$ is
\[
V^\mu \partial_\mu + W^\overline{\mu} \partial_{\overline{\mu}}, \quad V^\mu, W^\overline{\mu} \in \mathbb{C}, \quad \mu, \overline{\mu} = 1, \ldots, k.
\]
This is real if and only $\overline{V^\mu} = W^\overline{\mu}$. We often write this condition as $\overline{V^\mu} = V^\overline{\mu}$ or as $\overline{V^{(1,0)}} = V^{(0,1)}$. Note that $\dim_{\mathbb{C}} \mathcal{T}_X^{(1,0)} = \dim_{\mathbb{C}} \mathcal{T}_X^{(0,1)} = k$.  

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B. Gauge fixing small diffeomorphisms on Calabi–Yau manifolds

Suppose the manifold $X$ is Kähler so that $d\omega = 0$ and $h^{(0,2)} = h^{(0,1)} = 0$. On a Kähler manifold, there is a relation between the Laplacians

$$\square_{\rho} = \square_{\bar{\sigma}} = \frac{1}{2} d_{\rho}.$$  \hfill (B.1)

B.1. Real deformations

Consider a real deformation. By this we mean that

$$\delta = \delta y^\alpha \partial_{\alpha} + \delta y^\bar{\beta} \partial_{\bar{\beta}}$$

so that when applied to the Kähler form, the deformation is manifestly real ($\delta \omega^* = \delta \omega$).

Using (B.1), $h^{0,2} = h^{2,0} = 0$ and the Hodge decomposition being unique, we can Hodge decompose the equation of motion $d\delta \omega = 0$ with respect to the $d$-operator

$$\delta \omega = \gamma + d\beta ,$$

where $\gamma$ is $d$–harmonic $(1,1)$–form.

Small diffeomorphisms act as

$$\delta \omega \sim \delta \omega + d(\varepsilon^m \omega_m) .$$

We can partially fix this freedom by setting $\varepsilon^m \omega_m = -\beta + d\psi$, where $\psi$ is an arbitrary function on $X$. In this gauge $\delta \omega = \gamma^{(1,1)}$ is harmonic. There is a residual gauge freedom parameterised by $\psi$.

There is also the holomorphic $(3,0)$ form $\Omega$. Its deformation decomposed into type is

$$\delta \Omega = \delta \Omega^{(3,0)} + \delta \Omega^{(2,1)}.$$  \hfill (B.2)

The Hodge decomposition of $\delta \Omega^{(3,0)}$ with respect to the $\partial$–operator is

$$\delta \Omega^{(3,0)} = k \Omega + \partial \zeta^{(2,0)} = (k + k_\zeta(x))\Omega ,$$  \hfill (B.3)

where in the first equality we use $h^{(3,0)} = 1$ and so $\Omega$ is the unique harmonic $(3,0)$–form and $k$ is a parameter dependent constant. In the second equality we have shown the $\partial$–exact term is proportional to $\Omega$, except where the coefficient $k_\zeta(x) = \frac{1}{3!|\Omega|^2}(\partial \zeta)^{\mu\nu\rho}\Omega^{\mu\nu\rho}\Omega$ is not a constant. The equation of motion $d\delta \Omega = 0$ implies

$$\partial \chi = \overline{\partial} \partial \zeta^{(2,0)} , \quad \overline{\partial} \chi = 0 ,$$

where $\chi = \delta \Omega^{(2,1)}$.

We can also use $\Omega = \frac{1}{2} f(x,y)\epsilon_{\mu\nu\rho} dx^{\mu\nu\rho}$ where $f = e^{\phi}|f|$ and $|f|^2 = \|\Omega\|^2 \sqrt{g}$. The pure part of the deformation is

$$\delta \Omega^{(3,0)} = \left(i \delta \phi + \frac{1}{2} (\delta \log \|\Omega\|^2 + \omega^{\mu\sigma} \delta \omega_{\mu\sigma})\right) \Omega ,$$

where $\delta \phi$ is a globally defined function on $X$. Supersymmetry implies $\|\Omega\|$ is a constant.

Using the fact the Levi–Civita connection on a Kähler manifold is metric and hermitian so that $\nabla_{\rho} \omega = 0$ and torsionless

$$\partial_{\rho} \left(\omega^{\mu\sigma} \delta \omega_{\mu\sigma}\right) = (\nabla_{\rho} \omega^{\mu\sigma}) \delta \omega_{\mu\sigma} + \omega^{\mu\sigma} (\nabla_{\rho} \delta \omega_{\mu\sigma})$$

$$= -i \nabla^{\sigma} \delta \omega_{\rho\sigma} - i g^{\mu\sigma} (\partial \delta \omega^{(1,1)})_{\rho\mu\sigma} .$$  \hfill (B.4)
With \( \delta \omega \) harmonic, it is both \( \partial \)-closed and \( \overline{\partial} \)-coclosed and so both terms vanish. Hence \( \omega^{\mu \nu} \delta \omega_{\mu \nu} \) is a constant.

We still have the residual gauge freedom parameterised by \( \varepsilon^m \omega_m = d\psi \). While \( \delta \omega \) is invariant, \( \delta \Omega^{(3,0)} \) transforms

\[
\delta \Omega^{(3,0)} \sim \delta \Omega^{(3,0)} - i(\nabla_\mu \nabla^\mu \psi)\Omega ,
\]

where we use \( \partial (\varepsilon^m \Omega_m) = (\nabla_\mu \varepsilon^\mu) \Omega \) and \( \varepsilon^\mu = -i \nabla^\mu \psi \). This implies an action on \( \delta \phi \) given by

\[
\delta \phi \sim \delta \phi - \nabla^\mu \nabla_\mu \psi .
\]

At this point, we pause to recount some properties of the Laplacian acting on a scalar \( \Psi \) with a source \( \Phi \) on the compact Kähler manifold \( X \):

\[
\Box \partial \Psi = \nabla^\mu \nabla_\mu \Psi = \Phi .
\]  

We divide \( \Psi \) into a formal sum of zero modes \( \Box \partial \Psi_0 = 0 \) and non-zero modes \( \Box \partial \Psi_i = \lambda_i \Psi_i \), where \( \lambda_i \neq 0 \):

\[
\Psi = \Psi_0 + \sum_i \Psi_i .
\]

The obstruction to solving the equation (B.7) is that the source \( \Phi \) be orthogonal to the zero modes \( \int_X \Psi_0 \Phi = 0 \). On \( X \) the zero modes are constants. Hence, pick a \( \Psi_0 \neq 0 \) and evaluate the following integral in two different ways

\[
\int \Psi_0 \nabla^\mu \nabla_\mu \Psi_i = \lambda_i \Psi_0 \int \Psi_i = - \int \nabla^\mu \Psi_0 \nabla_\mu \Psi_i = 0 ,
\]

we see that the non-zero modes have vanishing integral \( \int_X \Psi_i = 0 \). If \( \Phi = k \) for some constant \( k \) then does (B.7) have a solution? The integral of a total divergence on a compact manifold vanishes

\[
\int_X \nabla^\mu \nabla_\mu \Psi = k V = 0 ,
\]

where \( V \) is the volume of \( X \) and so we see that \( k = 0 \).

We now apply these lessons to the gauge fixing (B.6). We divide up the source \( \delta \phi \) into zero modes \( \delta \phi_0 \) and non-zero modes \( \delta \phi_{nz} \). The facts above indicate that \( \int_X \delta \phi_{nz} = 0 \) and so we can find a \( \psi \) so that \( \nabla^\mu \nabla_\mu \psi = \delta \phi_{nz} \). However, we cannot find a \( \psi \) in which \( \nabla^\mu \nabla_\mu \psi = \delta \phi_0 \) for \( \delta \phi_0 \) a non-zero constant. Hence, by a choice of gauge, we can kill the non-zero modes of \( \delta \phi \) but not the zero modes, which are constants.  

With the choice of gauge, we have \( \delta \omega \) being harmonic and \( \delta \Omega^{(3,0)} = k \Omega \) for some parameter-dependent constant \( k \). The equation \( d \delta \Omega = 0 \) gives

\[
\partial \chi = \partial \overline{\partial} \xi^{(2,0)} = 0 , \quad \overline{\partial} \chi = 0 ,
\]

and we have \( \nabla_\mu \Delta^\mu_\alpha = 0 \).

---

9 We have completely fixed the gauge freedom as \( \nabla^\mu \nabla_\mu \psi = k \) has solution only for \( k = 0 \), in which case \( \psi \) are constants and acts on the fields only through its derivatives.
B.2. Holomorphic deformations

Let us now repeat this analysis in which \( \delta h = \delta y^\alpha \partial_\alpha \) is a holomorphic deformation. This means \( \delta h \omega \) is no longer real. The equation \( d \delta h \omega = 0 \) gives three parts

\[
\overline{\partial} \delta h \omega^{(0,2)} = 0, \quad \partial \delta h \omega^{(0,2)} + \overline{\partial} \delta h \omega^{(1,1)} = 0, \quad \partial \delta h \omega^{(1,1)} = 0.
\]

The first equation is solved by

\[
\delta h \omega^{(0,2)} = \overline{\partial} \xi^{(0,1)}.
\]

The second equation becomes \( \overline{\partial}(\delta h \omega^{(1,1)} - \partial \xi^{(0,1)}) = 0 \) which has a solution

\[
\delta h \omega^{(1,1)} = \gamma^{(1,1)} + \partial \xi^{(0,1)} + \overline{\partial} \beta^{(1,0)},
\]

where \( \gamma^{(1,1)} \) is \( \overline{\partial} \)-harmonic. The final equation gives \( \overline{\partial} \partial \beta^{(1,0)} = 0 \) and after using \( h^{1,0} = h^{2,0} = 0 \) we have \( \beta^{(1,0)} = \partial \Phi \). Hence,

\[
\delta h \omega^{(1,1)} = \gamma^{(1,1)} + \partial(\xi^{(0,1)} - \overline{\partial} \Phi).
\]

The gauge symmetries are small holomorphic diffeomorphisms, in which \( \varepsilon^\nu = 0 \) and \( \varepsilon^\mu \) is free. These preserve \( \delta h \omega^{(2,0)} = 0 \). The action on the remaining parts is

\[
\delta h \omega^{(0,2)} \sim \delta h \omega^{(0,2)} + i \overline{\partial} \varepsilon^{(0,1)}, \quad \delta h \omega^{(1,1)} \sim \delta h \omega^{(1,1)} + i \partial \varepsilon^{(0,1)}.
\]

We can choose a gauge in which \( \delta h \omega^{(0,2)} = 0 \) by setting

\[
i \varepsilon^{(0,1)} = -\xi^{(0,1)} + \overline{\partial} \psi.
\]

The remaining term \( \psi \) can be fixed by demanding \( \delta h \omega^{(1,1)} \) be harmonic

\[
\psi = \Phi.
\]

While \( \delta h \omega \) is harmonic it is not necessarily real.

The holomorphic top form \( \delta h \Omega \) has a decomposition as before in (B.2)–(B.3). The utility of \( \delta \) being a holomorphic deformation is that \( \delta h \Omega = 0 \) and so

\[
\delta h \Omega = (\delta h \log f) \Omega = (\delta h \log |f|^2) \Omega = (\delta h \log \|\Omega\|^2 + \omega^{\mu\nu} \delta h \omega_{\mu\nu}) \Omega.
\]

As before, the Levi–Civita connection satisfies \( \nabla_{\rho} \omega_{\mu\nu} = 0 \) and \( \partial \)-closure of \( \delta h \omega^{(1,1)} \) implies \( \nabla_{\rho} \delta h \omega_{\mu\nu} = \nabla_{\mu} \delta h \omega_{\rho\nu} \). Hence,

\[
\partial_{\rho}(\omega^{\mu\sigma} \delta h \omega_{\mu\sigma}) = \omega^{\mu\sigma} \nabla_{\mu} \delta h \omega_{\rho\sigma} = -i \nabla_{\rho} \delta h \omega_{\mu\sigma}.
\]

In the gauge in which \( \delta h \omega^{(1,1)} \) is harmonic it follows that \( \omega^{\mu\sigma} \delta h \omega_{\mu\sigma} \) is a constant. Hence, \( \delta h \Omega^{(3,0)} = k \Omega \) for some parameter–dependent constant \( k \) and \( \partial \xi^{(2,0)} = 0 \). This is the same gauge as the previous subsection.
B.3. A third way

To satisfy ourselves the previous calculation is correct, we repeat the calculation this time starting from a holomorphic variation of $\Omega$. The decomposition of $\delta \Omega^{(3,0)}$ with respect to the $\partial$–operator gives

$$\delta \Omega^{(3,0)} = k \Omega + \partial \xi^{(2,0)},$$

where $\partial \xi^{(2,0)} = k \xi \Omega$. A gauge transformation acts as

$$\delta \Omega \rightarrow \delta \Omega + d(\varepsilon^\nu \Omega_\nu) = \delta \Omega + (\nabla_\nu \varepsilon^\nu)\Omega.$$

We choose $\varepsilon^\nu = -\frac{1}{2 \|\Omega\|^2} \Omega^{\rho\sigma}(\xi^{\rho\sigma} + (\partial \zeta^{(1,0)})_{\rho\sigma})$, where $\zeta^{(1,0)}$ is an arbitrary one form. With this choice $\nabla_\nu \varepsilon^\nu = -\frac{1}{3! \|\Omega\|^2} \Omega^{\rho\sigma}(\partial \xi^{(2,0)})_{\nu\rho\sigma}$ and $\delta \Omega^{(3,0)} = k \Omega$ for some constant $k$. We see that $\zeta^{(1,0)}$ is a residual gauge freedom that does not affect $\delta \Omega$.

The variation of the Kähler form is holomorphic, and so the equation of motion $d \delta \omega = 0$ implies

$$\overline{\partial} \delta \omega^{(0,2)} = 0, \quad \partial \delta \omega^{(0,2)} + \overline{\partial} \delta \omega^{(1,1)} = 0, \quad \partial \delta \omega^{(1,1)} = 0.$$

The solution to these equations is

$$\delta \omega^{(0,2)} = \overline{\partial} \alpha^{(0,1)}, \quad \delta \omega^{(1,1)} = \gamma^{(1,1)} + \partial \alpha^{(0,1)}, \quad \delta \omega^{(2,0)} = 0,$$

where $\gamma^{(1,1)}$ is a harmonic $(1,1)$–form. There is a condition on $\alpha^{(0,1)}$ that derives from the gauge fixing of $\delta \Omega$. To see this, we first turn to a calculation used several times above

$$\partial(\omega^{\mu\nu} \delta \omega^{\mu\nu}) = i \overline{\partial} \delta \omega^{(1,1)} = 0, \quad \overline{\partial}(\omega^{\mu\nu} \delta \omega^{\mu\nu}) = -ig^{\mu\nu}(\overline{\partial} \delta \omega^{(1,1)})_{\nu\rho} + i \partial \delta \omega^{(1,1)} = 0,$$

The vanishing of these equation is the gauge fixing $\delta \Omega^{(3,0)} = k \Omega$. Consider the first equation:

$$\overline{\partial} \delta \omega^{(1,1)} = (\nabla^\nu \nabla_\mu \alpha_{\nu \rho}) dx^\rho = \partial(\nabla^\nu \alpha_{\nu \rho}) = 0.$$

There are no holomorphic functions on $X$ and so this implies $\nabla^\nu \alpha_{\nu \rho}$ is a constant. In fact, the constant must vanish, as can be seen by integrating it over the manifold and using that a total divergance on a compact manifold has vanishing integral. Hence,

$$\overline{\partial} \alpha^{(0,1)} = 0.$$

On the other hand, we write this in terms of the Hodge dual

$$\overline{\partial}^\dagger \alpha^{(0,1)} = -\ast \partial \ast \alpha^{(0,1)} = -\ast \partial(\alpha_{\nu \rho} \Omega^{\nu \rho}) \frac{i \overline{\Omega}}{\|\Omega\|^2} = 0,$$

The fact this vanishes implies $\alpha_{\nu \rho} \Omega^{\nu \rho} = \partial \beta^{(1,0)}$ for some $(1,0)$–form. This equation is invertible

$$\alpha^{(0,1)} = \frac{1}{2 \|\Omega\|^2} \Omega^{\nu\rho}(\partial \beta^{(1,0)})_{\nu\rho}.$$

We finally, see that $\alpha^{(0,1)}$ takes the same form as the residual gauge freedom described above, and that by setting $\zeta^{(1,0)} = -\beta^{(1,0)}$ we fix $\delta \omega^{(1,1)} = \gamma^{(1,1)}$ to be harmonic.

We have now illustrated how to gauge fix to harmonic gauge in three separate ways, and this serves as a useful guide for gauge fixing the heterotic system.
B.4. Conformally balanced condition

A Calabi–Yau manifold is in particular a balanced manifold and satisfies

\[ d\omega^2 = 0. \]

A holomorphic variation gives a pair of independent equations

\[ \partial(\delta_h\omega^{(0,2)}) + \overline{\partial}(\delta_h\omega^{(1,1)}) = 0, \quad \partial(\delta_h\omega^{(1,1)}) = 0. \]

Under a small diffeomorphism (B.8) these equations are preserved provided we use \( d\omega = 0 \). Hence, a holomorphic variation of the balanced condition on a Kähler manifold is not a gauge fixing.

Using the following Hodge duals

\[ \ast \delta_h\omega^{(1,1)} = (\omega^{\mu\nu}\delta_h\omega_{\mu\nu})\omega^2 - \delta_h\omega^{(1,1)}\omega, \quad \ast \delta_h\omega^{(0,2)} = \delta_h\omega^{(0,2)}\omega, \]

we get

\[ \partial \ast \delta_h\omega^{(1,1)} = \overline{\partial} \ast \delta_h\omega^{(0,2)} + i \overline{\partial}(\omega^{\mu\nu}\delta_h\omega_{\mu\nu}), \quad \overline{\partial} \ast \delta_h\omega^{(1,1)} = i \partial(\omega^{\mu\nu}\delta_h\omega_{\mu\nu}). \]

C. Hodge duals and contractions

We establish a set of notations and conventions, as well enumerate several useful relations.

Real Riemannian manifolds

We start with a \( D \)-dimensional compact Riemannian manifold \( X \) with metric \( g \).

Inner products for forms

The pointwise inner product on (exterior powers of) the cotangent space uses the inverse metric \( g^{-1} = g^{mn}\partial_m \otimes \partial_n \). Given \( k \)-forms \( \eta, \xi \) it is defined as

\[ g^{-1}(\eta, \xi) = \frac{1}{k!} \eta_{m_1...m_k} g^{m_1n_1}...g^{m_kn_k} \xi_{n_1...n_k} = \frac{1}{k!} \eta^{n_1...n_k} \xi_{m_1...m_k}, \quad \text{(C.1)} \]

which is a real function over \( X \), whose integral defines an inner product for forms

\[ (\cdot, \cdot) : \Omega^k(X) \times \Omega^k(X) \to \mathbb{R}, \]

with

\[ (\eta, \xi) = \frac{1}{V} \int_X \text{vol} g^{-1}(\eta, \xi) = \frac{1}{V k!} \int_X \text{vol} \eta^{m_1...m_k} \xi_{m_1...m_k}. \]

This is positive definite and with the definition above the zero-form 1 and the \( D \)-form \( \text{vol} \) have unit norm. Recall, the Riemannian volume form is

\[ \text{vol} = \frac{1}{D!} \sqrt{g} \epsilon_{m_1...m_D} dx^{m_1...m_D}, \]

where \( \epsilon_{m_1...m_D} \) is the constant antisymmetric symbol.
Contraction of forms

Given two forms $\eta_k$ and $\xi_l$, where the subscript denotes their degree and $k \leq l$, we can form their contraction, or interior product. The symbol that denotes this operation is $\lrcorner$: $\Omega^k(X) \times \Omega^l(X) \to \Omega^{l-k}(X)$,

and acts on forms as follows

\[
\eta_k \lrcorner \xi_l = \frac{1}{k!(l-k)!} \eta^{m_1...m_k} \xi_{m_1...m_k n_1...n_l-k} \, dx^{n_1...n_l-k} = \frac{1}{k!} \eta^{m_1...m_k} \xi_{m_1...m_k} .
\]

The inner product (C.1) becomes then just a special case of contraction

\[
g^{-1}(\eta, \xi) = \eta \lrcorner \xi .
\]

An interesting feature of this operator is that it is the adjoint of the wedge product

\[
(\sigma_{l-k} \lrcorner \eta_k, \xi_l) = (\eta_k, \sigma_{l-k} \wedge \xi_l) .
\]

Hodge star operator

The Hodge dual operator $\star : \Omega^k(X) \to \Omega^{D-k}(X)$, defined as

\[
\star \eta = \sqrt{g} \frac{k!(D-k)!}{k!} \eta^{m_1...m_k} \epsilon_{m_1...m_k n_1...n_{D-k}} \, dx^{n_1...n_{D-k}} .
\]

Hence, $\eta \star \xi = (\eta \lrcorner \xi) \text{vol}$.

The $\star$ operator satisfies the identities

\[
\star^2 \eta_k = (-)^{k(D-k)} \eta_k ,
\]

\[
g^{-1}(\eta, \xi) = g^{-1}(\star \eta, \star \xi) .
\]

The first line means that $\star$ is invertible with possible eigenvalues $\pm 1$ or $\pm i$ according to the degree of the form and the number of dimensions. The second property means $\star$ is an isometry.

Codifferential

The codifferential is denoted

\[
d^\dagger : \Omega^k(X) \to \Omega^{k-1}(X) ,
\]

and is defined as the adjoint of the de Rham operator. That is,

\[
(d\eta_{k-1}, \xi_k) = (\eta_{k-1}, d^\dagger \xi_k) .
\]
In order to find its explicit expression we need to perform an integration by parts and use the first line in (C.3). Boundary terms will be neglected because we assume $X$ has no boundary. The calculation goes as follows

\[
(d\eta_{k-1}, \xi_k) = \frac{1}{V} \int_X d\eta_{k-1} \star \xi_k = (-)^k \frac{1}{V} \int_X \eta_{k-1} \, d \star \xi_k
\]

\[
= (-)^k \frac{1}{V} \int_X \eta_{k-1} \, (-)^{(D-k+1)(k-1)} \star^2 d \star \xi_k = \frac{1}{V} \int_X \eta_{k-1} \star \left( -(-)^{D(k+1)} \star d \star \xi_k \right),
\]

and we end up with

\[
d^\dagger \xi_k = -(-)^D(k-1) \star d \star \xi_k.
\]

Another expression, very useful in calculations, is written in terms of the LC connection. It can be obtained very quickly by reminding that $d$ has a representation

\[
d = dx^m \nabla^\text{LC}_m,
\]

using property (C.2) and an integration by parts, as follows

\[
(d\eta, \xi) = (dx^m \nabla^\text{LC}_m \eta, \xi) = (\nabla^\text{LC}_m \eta, \xi^m) = (\eta, -\nabla^\text{LC}_m \xi^m).
\]

We gather that

\[
d^\dagger \xi_k = -\nabla^\text{LC}_m \xi^m = -\frac{1}{(k-1)!} \nabla^\text{LC}_n \xi^m_{m_1...m_{k-1}} dx^{m_1...m_{k-1}}.
\]

Proving this identity without using (C.2) involves rather cumbersome calculations.

**Complex manifolds**

We now take $X$ to be complex with complex dimension $N$. We denote holomorphic coordinates $x^\mu, x^{\overline{\nu}}$. The manifold is equipped with a hermitian metric

\[
g = g_{\mu\overline{\nu}} (dx^\mu \otimes dx^{\overline{\nu}} + dx^{\overline{\nu}} \otimes dx^\mu), \quad ds^2 = 2g_{\mu\overline{\nu}} \, dx^\mu dx^{\overline{\nu}} \tag{C.3}
\]

and compatible hermitian form $\omega$

\[
\omega = \omega_{\mu\overline{\nu}} \, dx^\mu dx^{\overline{\nu}} = ig_{\mu\overline{\nu}} \, dx^\mu dx^{\overline{\nu}}.
\]

The pointwise inner product for forms respects hermitianity: given forms $\eta, \xi$ of holomorphic type $(p, q)$ we define

\[
g^{-1}(\eta, \overline{\xi}) = \eta \mathcal{J} \overline{\xi} = \frac{1}{p!q!} \eta_{\overline{\mu}_1...\overline{\mu}_p} \overline{\xi}_{\nu_1...\nu_q} \overline{\xi}_{\overline{\mu}_1...\overline{\mu}_p} \overline{\nu}_1...\overline{\nu}_q.
\]

When integrated over the manifold this gives

\[
(\eta, \overline{\xi}) = \frac{1}{V} \int_X \eta \mathcal{J} \overline{\xi}.
\]

Observe how the Hodge $\star$ operator acts on type

\[
\star : \Omega^{(p, q)}(X) \rightarrow \Omega^{(N-q, N-p)}(X).
\]
The de Rham differential splits into the sum of Dolbeault operators $d = \partial + \bar{\partial}$. Analogously, the codifferential also splits $d^\dagger = \partial^\dagger + \bar{\partial}^\dagger$ where
\begin{equation}
\partial^\dagger : \Omega^{(p,q)}(X) \to \Omega^{(p-1,q)}(X), \quad \bar{\partial}^\dagger = -\star \bar{\partial} \star,
\end{equation}
\begin{equation}
\bar{\partial}^\dagger : \Omega^{(p,q)}(X) \to \Omega^{(p,q-1)}(X), \quad \partial^\dagger = -\star \partial \star.
\end{equation}

Let us discuss the volume form on $X$. The holomorphic coordinates have a writing $x^\mu = u^\mu + i v^\mu$ with $u, v$ real coordinates. In terms of the latter, the volume form is defined as
$$\text{vol} = \sqrt{g} \, du^1 \, dv^1 \ldots du^N \, dv^N.$$ Expressed in terms of the complex coordinates:
$$\text{vol} = \frac{i^{N^2}}{N!} \sqrt{|g|} \epsilon_{\mu_1 \ldots \mu_N} \epsilon_{\nu_1 \ldots \nu_N} \, dx^{\mu_1 \ldots \mu_N} \, dx^{\nu_1 \ldots \nu_N}. \quad (C.5)$$

We can also write this in terms of the holomorphic volume form
$$\text{vol} = \frac{i \Omega^j}{||\Omega||^2}.$$ It coincides with the $N$-th power of the hermitian form
$$\frac{1}{N!} \omega^N = \frac{(-)^{N(N-1)} i^N}{N!} g_{\mu_1 \nu_1} \ldots g_{\mu_N \nu_N} \epsilon^{\mu_1 \ldots \mu_N} \epsilon_{\nu_1 \ldots \nu_N} \, dx^{12 \ldots N} \, dx^{12 \ldots N}$$
$$= \frac{i^{N^2}}{N!} \det g_{\mu \nu} \, dx^{12 \ldots N} \, dx^{12 \ldots N}$$
$$= \frac{i^{N^2}}{N!} \sqrt{|g|} \epsilon_{\mu_1 \ldots \mu_N} \epsilon_{\nu_1 \ldots \nu_N} \, dx^{\mu_1 \ldots \mu_N} \, dx^{\nu_1 \ldots \nu_N}.$$ \textbf{Codifferential for $\Delta_\alpha$}

Consider the space $\Omega^{(0,q)}(X, \mathcal{T}_X^{(1,0)})$. We are mostly interested in $q = 1$ but it is not much harder to work in generality. Elements of this space are
$$\eta^\mu = \frac{1}{q!} \eta_{\nu_1 \ldots \nu_q} \, d\nu_1 \ldots d\nu_q.$$ There is a hermitian metric
$$(\eta^\mu, \xi^\nu) = \int_X \eta^\mu \star \xi^\nu \, g_{\mu \nu}, \quad \xi^\nu = \frac{1}{q!} \xi_{\rho_1 \ldots \rho_q} \, d\rho_1 \ldots d\rho_q, \quad (C.6)$$
where we understand that $\star$ treats $\eta^\mu, \xi^\nu$ as forms. For example, when $q = 0, 1$
$$\star v^\mu = v^\mu \, \text{vol}, \quad \star \Delta^\mu = i \Delta^\mu \frac{\omega^2}{2}.$$ There is also a differential operator
$$\bar{\partial} : \Omega^{(0,q)}(X, \mathcal{T}_X^{(1,0)}) \to \Omega^{(0,q+1)}(X, \mathcal{T}_X^{(1,0)}).$$
that raises the degree of one. We use holomorphic coordinates, and this acts covariantly.

With the help of the metric (C.6) we can define its adjoint

\[ \overline{\partial}^\dagger : \Omega^{(0,q)}(X, T_X^{(1,0)}) \to \Omega^{(0,q-1)}(X, T_X^{(1,0)}) , \]

which satisfies the property

\[ (\eta^\mu, \partial \xi^\nu) = (\overline{\partial}^\dagger \eta^\mu, \xi^\nu) . \]

The calculation uses an integration by parts – in which we neglect boundary terms – and standard properties of \( \star \). We end up with

\[ (\eta^\mu, \partial \xi^\nu) = \frac{1}{V} \int_X \left( - \ast \partial (\ast \eta^\rho g_{\rho\sigma}) g^{\sigma\mu} \right) \ast \xi^\nu g_{\mu\nu} . \]

From this we read the expression for the codifferential

\[ \overline{\partial}^\dagger \eta^\mu = - \ast \partial (\ast \eta^\rho g_{\rho\sigma}) g^{\sigma\mu} . \tag{C.7} \]

For \( q = 0 \) this vanishes trivially. When \( q = 1 \), the case we are most interested in, we find using also the balanced condition

\[ \overline{\partial}^\dagger \Delta^\mu = - \ast \frac{i\omega^2}{2} \left( \partial \Delta^\mu + g^{\mu\sigma} \partial g_{\rho\sigma} \Delta^\rho \right) = - \nabla^\mu \Delta \omega . \tag{C.8} \]

Observe how this is different from the gauge-fixing condition (3.10), due to a different ordering of the indices. The two expressions coincide when \( \Delta_{[\rho\sigma]} = 0 \).

**Hodge dual relations on a three-fold**

We can finally enumerate some Hodge dual relations for various types of forms on \( X \).

One-forms, type \((1,0)\):

\[ \ast \eta^{(1,0)} = -i \eta^{(1,0)} \frac{\omega^2}{2} . \tag{C.9} \]

Two-forms, types \((2,0)\) and \((1,1)\):

\[ \ast \eta^{(2,0)} = \eta^{(2,0)} \omega , \]

\[ \ast \eta^{(1,1)} = -i \eta_{\mu} \frac{\omega^2}{2} - \eta^{(1,1)} \omega = (\omega \ast \eta^{(1,1)}) \frac{\omega^2}{2} - \eta^{(1,1)} \omega , \tag{C.10} \]

Three-forms, types \((3,0)\) and \((2,1)\):

\[ \ast \eta^{(3,0)} = -i \eta^{(3,0)} , \]

\[ \ast \eta^{(2,1)} = i \eta^{(2,1)} - \eta_{\mu} \eta^{(1,0)} \omega = i \eta^{(2,1)} - i (\omega \ast \eta^{(2,1)}) \omega , \tag{C.11} \]

Type \((2,3)\):

\[ \ast \eta^{(2,3)} = \frac{i}{2} \eta_{\mu\nu} \ast \eta^{(0,1)} = \frac{i}{2} \omega \ast \eta^{(2,3)} . \tag{C.12} \]

A useful special case of (C.10) is \( \frac{i}{2} \omega^2 = \ast \omega \). Further relations such as that for \( \eta^{(0,1)} \) can be easily determined using complex conjugation.
D. Connection symbols on $X$

We enumerate some commonly used connection symbols in heterotic theories. We list the components in complex coordinates. We also give expressions for various divergences which are useful for calculations in the paper.

Levi-Civita

\[
\Gamma^{\text{LC}}_{\mu \nu \rho} = \frac{1}{2} g^{\sigma \tau} (\partial_{\mu} g_{\sigma \tau} + \partial_{\rho} g_{\mu \tau}) = g^{\sigma \tau} \partial_{\nu} g_{\mu \tau} - \frac{1}{2} H_{\mu \nu \rho} = g^{\sigma \tau} \partial_{\rho} g_{\mu \tau} + \frac{1}{2} H_{\mu \nu \rho},
\]

\[
\Gamma^{\text{LC}}_{\mu \rho} = 0,
\]

\[
\Gamma^{\text{LC}}_{\mu \nu} = \frac{1}{2} g^{\sigma \tau} (\partial_{\mu} g_{\sigma \tau} - \partial_{\nu} g_{\mu \tau}) = \frac{1}{2} H_{\mu \nu \rho},
\]

\[
\Gamma^{\text{LC}}_{\rho \nu} = \frac{1}{2} g^{\rho \sigma} (\partial_{\sigma} g_{\mu \tau} - \partial_{\tau} g_{\mu \sigma}) = -\frac{1}{2} H_{\mu \nu \rho}.
\]

\[
(D.1)
\]

Bismut

\[
\Gamma^{\text{B}}_{\mu \nu \rho} = g^{\nu \tau} \partial_{\rho} g_{\mu \tau} = g^{\nu \tau} \partial_{\mu} g_{\rho \tau} - H_{\mu \nu \rho},
\]

\[
\Gamma^{\text{B}}_{\mu \rho} = 0,
\]

\[
\Gamma^{\text{B}}_{\nu \rho} = 0,
\]

\[
\Gamma^{\text{B}}_{\rho \nu} = g^{\rho \sigma} (\partial_{\sigma} g_{\mu \tau} - \partial_{\tau} g_{\mu \sigma}) = -H_{\mu \nu \rho}.
\]

\[
(D.2)
\]

Hull

\[
\Gamma^{\text{H}}_{\mu \nu \rho} = g^{\nu \tau} \partial_{\rho} g_{\mu \tau} = g^{\nu \tau} \partial_{\mu} g_{\rho \tau} + H_{\mu \nu \rho},
\]

\[
\Gamma^{\text{H}}_{\mu \rho} = 0,
\]

\[
\Gamma^{\text{H}}_{\nu \rho} = 0,
\]

\[
\Gamma^{\text{H}}_{\rho \nu} = g^{\rho \sigma} (\partial_{\sigma} g_{\mu \tau} - \partial_{\tau} g_{\mu \sigma}) = H_{\mu \nu \rho},
\]

\[
\Gamma^{\text{H}}_{\rho \nu} = 0.
\]

\[
(D.3)
\]

Chern

\[
\Gamma^{\text{Ch}}_{\mu \nu \rho} = g^{\nu \tau} \partial_{\rho} g_{\mu \tau} = g^{\nu \tau} \partial_{\mu} g_{\rho \tau} + H_{\mu \nu \rho},
\]

\[
\Gamma^{\text{Ch}}_{\mu \rho} = 0,
\]

\[
\Gamma^{\text{Ch}}_{\nu \rho} = 0,
\]

\[
\Gamma^{\text{Ch}}_{\rho \nu} = 0.
\]

\[
(D.4)
\]

Divergences

The divergence of a vector $\varepsilon^{\mu}$ taken with respect to a generic connection

\[
\nabla_{\mu} \varepsilon^{\mu} = \partial_{\mu} \varepsilon^{\mu} + \varepsilon^{\mu} \Gamma_{\nu \mu}^{\nu}.
\]

\[
(D.5)
\]

To compute this we need the following contraction

\[
\Gamma_{\mu \nu}^{\text{LC}} = \partial_{\mu} \log \sqrt{g} + \frac{1}{2} H_{\mu \nu},
\]

\[
\Gamma_{\mu \nu}^{\text{B}} = \partial_{\mu} \log \sqrt{g},
\]

\[
\Gamma_{\mu \nu}^{\text{H}} = \Gamma_{\mu \nu}^{\text{Ch}} = \partial_{\mu} \log \sqrt{g} + H_{\mu \nu}.
\]

\[
(D.6)
\]

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The four choices above give

\[ \nabla_{\mu} \epsilon^\mu = \partial_{\mu} \epsilon^\mu + \epsilon^\mu \partial_{\mu} \log \sqrt{g} + \frac{1}{2} \epsilon^\mu H_{\mu \nu} , \]

\[ \nabla_{\mu} \epsilon^\mu = \partial_{\mu} \epsilon^\mu + \epsilon^\mu \partial_{\mu} \log \sqrt{g} , \]  

(D.7)

\[ \nabla_{\mu} ^{\text{Ch}} \epsilon^\mu = \nabla_{\mu} ^{\text{Ch}} \epsilon^\mu = \partial_{\mu} \epsilon^\mu + \epsilon^\mu \partial_{\mu} \log \sqrt{g} + \epsilon^\mu H_{\mu \nu} . \]

As for the vector-valued form \( \Delta \pi^\mu \) we have, for a generic connection

\[ \nabla_{\mu} \Delta \pi^\mu = \partial_{\mu} \Delta \pi^\mu + \Delta \pi^\mu \Gamma^\rho_{\rho \mu} - \Gamma^\rho_{\mu \pi} \Delta \pi^\mu , \]  

(D.8)

and the four choices above give

\[ \nabla_{\mu} \Delta \pi^\mu = \partial_{\mu} \Delta \pi^\mu + \Delta \pi^\mu \partial_{\mu} \log \sqrt{g} + \frac{1}{2} \Delta \pi^\mu H_{\mu \rho} - \frac{1}{2} \Delta \rho^\mu H_{\mu \rho} , \]

\[ \nabla_{\mu} \Delta \pi^\mu = \partial_{\mu} \Delta \pi^\mu + \Delta \pi^\mu \partial_{\mu} \log \sqrt{g} - \Delta \mu^\rho H_{\mu \rho} , \]  

(D.9)

\[ \nabla_{\mu} \Delta \pi^\mu = \nabla_{\mu} \Delta \pi^\mu = \partial_{\mu} \Delta \pi^\mu + \Delta \pi^\mu \partial_{\mu} \log \sqrt{g} + \Delta \pi^\mu H_{\mu \rho} . \]
References

[1] P. Candelas, X. de la Ossa, J. McOrist, and R. Sisca, “The Universal Geometry of Heterotic Vacua,” arXiv:1810.00879 [hep-th].

[2] P. Candelas, X. de la Ossa, and J. McOrist, “A Metric for Heterotic Moduli,” Commun. Math. Phys. 356 no. 2, (2017) 567–612, arXiv:1605.05256 [hep-th].

[3] J. McOrist, “On the Effective Field Theory of Heterotic Vacua,” Lett. Math. Phys. 108 no. 4, (2018) 1031–1081, arXiv:1606.05221 [hep-th].

[4] L. Anguelova, C. Quigley, and S. Sethi, “The Leading Quantum Corrections to Stringy Kahler Potentials,” JHEP 1010 (2010) 065, arXiv:1007.4793 [hep-th].

[5] S. Kobayashi, Differential Geometry of Complex Vector Bundles. Iwanami Shoten, 1987.

[6] K. Kodaira, Complex manifolds and deformation of complex structures. Classics in Mathematics. Springer-Verlag, Berlin, 2005.

[7] X. de la Ossa and E. E. Svanes, “Holomorphic Bundles and the Moduli Space of N=1 Supersymmetric Heterotic Compactifications,” JHEP 10 (2014) 123, arXiv:1402.1725 [hep-th].

[8] X. de la Ossa and E. E. Svanes, “Connections, Field Redefinitions and Heterotic Supergravity,” JHEP 12 (2014) 008, arXiv:1409.3347 [hep-th].

[9] M. Garcia-Fernandez, R. Rubio, and C. Tipler, “Infinitesimal moduli for the Strominger system and Killing spinors in generalized geometry,” arXiv:1503.07562 [math.DG].

[10] L. B. Anderson, J. Gray, and E. Sharpe, “Algebroids, Heterotic Moduli Spaces and the Strominger System,” JHEP 07 (2014) 037, arXiv:1402.1532 [hep-th].

[11] M. Garcia-Fernandez, R. Rubio, and C. Tipler, “Holomorphic string algebroids,” arXiv:1807.10329 [math.AG].

[12] I. V. Melnikov and E. Sharpe, “On marginal deformations of (0,2) non-linear sigma models,” Phys.Lett. B705 (2011) 529–534, arXiv:1110.1886 [hep-th].

[13] E. Witten, “New Issues in Manifolds of SU(3) Holonomy,” Nucl. Phys. B268 (1986) 79.

[14] L. Witten and E. Witten, “Large Radius Expansion of Superstring Compactifications,” Nucl. Phys. B281 (1987) 109–126.

[15] J. Gillard, G. Papadopoulos, and D. Tsimpis, “Anomaly, fluxes and (2,0) heterotic string compactifications,” JHEP 06 (2003) 035, arXiv:hep-th/0304126 [hep-th].

[16] J. McOrist and I. V. Melnikov, “Old issues and linear sigma models,” arXiv:1103.1322 [hep-th].
[17] J. McOrist and I. V. Melnikov, “Summing the Instantons in Half-Twisted Linear Sigma Models,” JHEP 02 (2009) 026, arXiv:0810.0012 [hep-th].

[18] J. McOrist and I. V. Melnikov, “Half-Twisted Correlators from the Coulomb Branch,” JHEP 04 (2008) 071, arXiv:0712.3272 [hep-th].

[19] R. Donagi, J. Guffin, S. Katz, and E. Sharpe, “Physical aspects of quantum sheaf cohomology for deformations of tangent bundles of toric varieties,” Adv. Theor. Math. Phys. 17 no. 6, (2013) 1255–1301, arXiv:1110.3752 [hep-th].

[20] R. Donagi, J. Guffin, S. Katz, and E. Sharpe, “A Mathematical Theory of Quantum Sheaf Cohomology,” Asian J. Math. 18 (2014) 387–418, arXiv:1110.3751 [math.AG].

[21] J. McOrist, “The Revival of (0,2) Linear Sigma Models,” Int. J. Mod. Phys. A26 (2011) 1–41, arXiv:1010.4667 [hep-th].

[22] I. Melnikov, S. Sethi, and E. Sharpe, “Recent Developments in (0,2) Mirror Symmetry,” SIGMA 8 (2012) 068, arXiv:1209.1134 [hep-th].

[23] I. V. Melnikov, “An Introduction to Two-Dimensional Quantum Field Theory with (0,2) Supersymmetry,” Lect. Notes Phys. 951 (2019) pp.–1–482.

[24] A. Strominger, “Special Geometry,” Commun. Math. Phys. 133 (1990) 163–180.

[25] P. Candelas and X. de la Ossa, “Moduli Space of Calabi-Yau Manifolds,” Nucl. Phys. B355 (1991) 455–481.

[26] M. Itoh, “Geometry of anti-self-dual connections and Kuranishi map,” J. Math. Soc. Japan 40 no. 1, (1988) 9–33.

[27] A. Ashmore, X. De La Ossa, R. Minasian, C. Strickland-Constable, and E. E. Svanes, “Finite deformations from a heterotic superpotential: holomorphic Chern–Simons and an $L_{\infty}$ algebra,” arXiv:1806.08367 [hep-th].

[28] M. Nakahara, Geometry, topology and physics. Taylor and Francis, 2003.