FACE NUMBERS OF CUBICAL BARYCENTRIC SUBDIVISIONS

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Abstract. The cubical barycentric subdivision $\text{sd}_c(K)$ of a cubical complex $K$ is introduced as an analogue of the barycentric subdivision of a simplicial complex. Explicit formulas for the short and long cubical $h$-vector of $\text{sd}_c(K)$ are given, in terms of those of $K$. It is deduced that symmetry and nonnegativity of these $h$-vectors, as well as real rootedness of the short cubical $h$-polynomial, are preserved under cubical barycentric subdivision. The asymptotic behavior of the short and long cubical $h$-vectors of successive cubical barycentric subdivisions of $K$ is also determined.

1. Introduction

The present work is partly motivated by [2]. In that article, Brenti and Welker study the transformation of the $h$-vector of a simplicial complex $\Delta$ under barycentric subdivision. They express the entries of the $h$-vector of the barycentric subdivision of $\Delta$ as nonnegative integer linear combinations of those of the $h$-vector of $\Delta$. In particular, they show that symmetry and nonnegativity of the $h$-vector are preserved under barycentric subdivision. Moreover, they prove that if the $h$-vector of $\Delta$ is nonnegative, then the $h$-polynomial of the barycentric subdivision of $\Delta$ has only real roots.

It is natural to inquire whether similar results hold for non-simplicial complexes. We study the transformation of the short and the long cubical $h$-vector of a cubical complex $K$ under a natural cubical analogue of simplicial barycentric subdivision, which we call cubical barycentric subdivision. The cubical barycentric subdivision $\text{sd}_c(K)$ is a cubical complex which subdivides $K$, so that the poset of nonempty faces of $\text{sd}_c(K)$ is isomorphic to the set of closed intervals in the poset of nonempty faces of $K$, partially ordered by inclusion (see Section 2.3 for a precise definition). Our main result expresses the entries of the short and long cubical $h$-vector of $\text{sd}_c(K)$ explicitly as nonnegative linear combinations of the entries of the corresponding $h$-vector of $K$ (Theorems 3.2 and 4.1). From these expressions we deduce that symmetry and nonnegativity of the short and long cubical $h$-vector, as well as real rootedness of the short cubical $h$-polynomial, are preserved under cubical barycentric subdivision. We also study the asymptotic behavior of the short and long cubical $h$-polynomials under successive cubical barycentric subdivisions (Corollaries 3.8 and 4.5).

Acknowledgement. I would like to thank Christos Athanasiadis for suggesting this problem and for his extensive comments on earlier versions of this paper and Volkmar Welker for useful discussions.

Supported by the Cyprus State Scholarship Foundation.
2. PRELIMINARIES

This section reviews basic definitions concerning cubical complexes and their $f$-vectors and $h$-vectors and introduces the concept of cubical barycentric subdivision for such complexes. For background and any undefined terminology on partially ordered sets and on polytopes and polyhedral complexes we refer the reader to [4, Chapter 3] and to [3, 5], respectively.

2.1. Cubical complexes. We denote by $C_d$ the standard $d$-dimensional cube $[0,1]^d \subseteq \mathbb{R}^d$. Any polytope which is combinatorially isomorphic to $C_d$ is said to be a combinatorial $d$-cube. A cubical complex is a finite collection $K$ of combinatorial cubes in $\mathbb{R}^n$, such that (i) every face of an element of $K$ also belongs to $K$ and (ii) the intersection of any two elements of $K$ is a face of both. The elements of $K$ are called faces.

We denote by $\mathcal{F}(K)$ the face poset of $K$, meaning the set of faces of $K$, partially ordered by inclusion. This poset is a meet-semilattice. The empty face is the minimum element of $\mathcal{F}(K)$ and the vertices of $K$ are its atoms. The maximal elements of $\mathcal{F}(K)$ are called facets. The dimension of $K$, denoted by $\text{dim}(K)$, is defined as the maximum dimension of a face.

2.2. Face enumeration. Let $K$ be a $(d-1)$-dimensional cubical complex. We denote by $f_i(K)$ the number of $i$-dimensional faces of $K$. The $f$-vector of $K$ is defined as

$$f(K) = (f_0(K), f_1(K), \ldots, f_{d-1}(K)).$$

The polynomial

$$f_K(x) = \sum_{i=0}^{d-1} f_i(K) x^i$$

is called the $f$-polynomial of $K$. The short cubical $h$-polynomial of $K$ is defined in [1] by the equation

$$h^{(sc)}_K(x) = \sum_{i=0}^{d-1} h^{(sc)}_i(K) x^i = \sum_{j=0}^{d-1} f_j(K)(2x)^j(1-x)^{d-1-j}.$$  

The vector $h^{(sc)}(K) = (h^{(sc)}_0(K), h^{(sc)}_1(K), \ldots, h^{(sc)}_{d-1}(K))$ of coefficients of this polynomial is called the short cubical $h$-vector of $K$. The polynomials $f_K(x)$ and $h^{(sc)}_K(x)$ are related by the equations

$$h^{(sc)}_K(x) = (1-x)^{d-1} f_K\left(\frac{2x}{1-x}\right)$$

and

$$2^{d-1} f_K(x) = (x+2)^{d-1} h^{(sc)}_K\left(\frac{x}{x+2}\right).$$

As a result, the entries of $f(K)$ can be expressed in terms of those of $h^{(sc)}(K)$ and vice versa by the equations

$$f_j(K) = 2^{-j} \sum_{i=0}^{j} \binom{d-1-i}{d-1-j} h^{(sc)}_i(K)$$

for $j = 0, \ldots, d-1$. For $j = d-1$ we have

$$f_{d-1}(K) = h^{(sc)}_{d-1}(K).$$
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and

\[ h^{(sc)}_i(K) = \sum_{j=0}^{i} \binom{d-1-j}{d-1-i} (-1)^{i-j} 2^j f_j(K). \]

The (long) cubical h-vector

\[ h^{(c)}(K) = (h^{(c)}_0(K), h^{(c)}_1(K), \ldots, h^{(c)}_d(K)) \]

of \( K \) is defined in [1] by the recursive formula

\[ h^{(sc)}_i(K) = h^{(c)}_i(K) + h^{(c)}_{i+1}(K), \quad \text{for } 0 \leq i \leq d - 1 \]

and the initial condition \( h^{(c)}_0(K) = 2^{d-1} \). The polynomial

\[ h^{(c)}_K(x) = \sum_{i=0}^{d} h^{(c)}_i(K) x^i \]

is called the (long) cubical h-polynomial of \( K \). The short and long cubical h-polynomials of \( K \) are related by the equation

\[ (1 + x) h^{(c)}_K(x) = 2^{d-1} + xh^{(sc)}_K(x) + 2^{d-1}(-x)^{d+1}\tilde{\chi}(K), \]

where

\[ \tilde{\chi}(K) = -1 + \sum_{i=0}^{d-1} (-1)^i f_i(K) = -1 + f_K(-1) \]

is the reduced Euler characteristic of \( K \). The entries of \( h^{(c)}(K) \) can be expressed in terms of those of \( h^{(sc)}(K) \) by the equation

\[ h^{(c)}_i(K) = \sum_{j=0}^{i-1} (-1)^{i+j-1} h^{(sc)}_j(K) + (-1)^i 2^{d-1}, \quad 1 \leq i \leq d. \]

The cubical h-vectors of \( K \) can also be expressed in terms of the simplicial h-vectors of the links of the vertices of \( K \) (these links are simplicial complexes); see [1, Theorem 9].

2.3. Cubical barycentric subdivision. Let \( K \) be a cubical complex. The cubical barycentric subdivision \( \text{sd}_c(K) \) of \( K \) is the polyhedral complex defined as follows: Vertices of \( \text{sd}_c(K) \) are the barycenters of the nonempty faces of \( K \). Furthermore, to each closed interval \([F, G]\) in the poset \( \mathcal{F}(K) \setminus \{\emptyset\} \) of nonempty faces of \( K \) corresponds a face of \( \text{sd}_c(K) \). This face is equal to the convex hull of the barycenters of all elements of \([F, G]\); see Figure 1 for an example.

![Figure 1: A cubical complex and its cubical barycentric subdivision](image-url)
The face poset of \( sd_c(K) \) is isomorphic to the set of closed intervals in the poset \( \mathcal{F}(K) \setminus \{ \emptyset \} \) of nonempty faces of \( K \), partially ordered by inclusion, with a minimum element adjoined. This fact implies that \( sd_c(K) \) is also a cubical complex; its dimension is equal to that of \( K \).

3. The short cubical \( h \)-vector

This section studies the short cubical \( h \)-polynomial of \( sd_c(K) \) with respect to its nonnegativity, symmetry and real rootedness. The asymptotic behavior of the short cubical \( h \)-polynomial of the \( n \)-th iterated cubical barycentric subdivision of \( K \), as \( n \) approaches infinity, is also explored. Here and in the following section, \( K \) denotes a \( (d - 1) \)-dimensional cubical complex and \( sd_c(K) \) stands for its cubical barycentric subdivision.

To express the short cubical \( h \)-vector of \( sd_c(K) \) in terms of that of \( K \), we first need to determine the relationship between the corresponding \( f \)-vectors.

**Proposition 3.1.** The \( f \)-vectors of \( K \) and \( sd_c(K) \) are related as follows:

\[
 f_i(sd_c(K)) = 2^i \sum_{j=i}^{d-1} \binom{j}{i} f_j(K), \quad i = 0, \ldots, d - 1.
\]

**Proof.** We recall that there is a one to one correspondence between the set of \( i \)-dimensional faces of \( sd_c(K) \) and the set of closed intervals \([x, y]\) of rank \( i \) in the face poset of \( K \). To count these intervals, we note that there are \( f_j(K) \) ways to choose a face \( y \) of \( K \) of given dimension \( j \). Since every such face is a combinatorial \( j \)-cube, there are \( 2^i \binom{j}{i} \) ways to choose a face \( x \) of \( y \) of codimension \( i \). As a result, there are \( 2^i \binom{j}{i} f_j(K) \) intervals \([x, y]\) of rank \( i \) in the face poset of \( K \) with \( \dim(y) = j \). Summing over all \( j \geq i \), we obtain (9). \( \square \)

**Theorem 3.2.** The short cubical \( h \)-vectors of \( K \) and \( sd_c(K) \) are related as follows:

\[
 h_i^{(sc)}(sd_c(K)) = \sum_{j=0}^{d-1} B(d, i, j) h_j^{(sc)}(K), \quad i = 0, \ldots, d - 1,
\]

where the coefficients \( B(d, i, j) \) are nonnegative rational numbers, determined by the generating function

\[
 \sum_{i=0}^{d-1} B(d, i, j) x^i = \frac{1}{2^{d-1}} (3x + 1)^j (x + 3)^{d-1-j}.
\]

**Proof.** Equations (9) can be rewritten as

\[
 f_{sd_c(K)}(x) = f_K(1 + 2x),
\]

where \( f_K(x) \) and \( f_{sd_c(K)}(x) \) are the \( f \)-polynomials of \( K \) and \( sd_c(K) \), respectively. Using (2) and (3), we can further rewrite (12) as

\[
 2^{d-1} h_{sd_c(K)}^{(sc)}(x) = (x + 3)^{d-1} h_K^{(sc)}(x + 1) \left( \frac{3x + 1}{x + 3} \right),
\]

so that

\[
 h_{sd_c(K)}^{(sc)}(x) = \frac{1}{2^{d-1}} \sum_{j=0}^{d-1} h_j^{(sc)}(K)(3x + 1)^j (x + 3)^{d-1-j}.
\]
The result follows by equating the coefficients of \( x^i \) in the two hand sides of this equation. \( \square \)

Recall that a vector \((a_0, a_1, \ldots, a_{n-1}) \in \mathbb{R}^n\) is said to be nonnegative (respectively, symmetric) if \( a_i \geq 0 \) (respectively, \( a_i = a_{n-1-i} \)) holds for \( 0 \leq i \leq n-1 \).

**Corollary 3.3.** If \( K \) has nonnegative short cubical \( h \)-vector, then so does \( \text{sd}_c(K) \).

**Proof.** Equation (11) implies that the coefficients \( B(d, i, j) \) are nonnegative for all \( 0 \leq i, j \leq d-1 \). Thus the statement follows from (10). \( \square \)

**Corollary 3.4.** If \( K \) has symmetric short cubical \( h \)-vector, then so does \( \text{sd}_c(K) \).

**Proof.** Replacing \( x \) by \( 1/x \) in (11) and multiplying by \( x^{d-1} \), we find that
\[
B(d, i, j) = B(d, d-1-i, d-1-j).
\]
Assume that \( h^{(sc)}(K) \) is symmetric, so that \( h^{(sc)}_{d-1-j}(K) = h^{(sc)}_j(K) \) holds for \( 0 \leq j \leq d-1 \). Then
\[
h^{(sc)}_{d-1-i}(\text{sd}_c(K)) = \sum_{j=0}^{d-1} B(d, d-1-i, j) h^{(sc)}_j(K)
= \sum_{k=0}^{d-1} B(d, d-1-i, d-1-k) h^{(sc)}_{d-1-k}
= \sum_{k=0}^{d-1} B(d, i, k) h^{(sc)}_k = h^{(sc)}_i(\text{sd}_c(K))
\]
and hence \( h^{(sc)}(\text{sd}_c(K)) \) is also symmetric. \( \square \)

**Corollary 3.5.** The short cubical \( h \)-polynomial of \( K \) has only real roots if and only if the same holds for \( \text{sd}_c(K) \).

**Proof.** This statement follows easily from (13). The details are left to the reader. \( \square \)

**Example 3.6.** For the boundary complex \( K \) of the 3-dimensional cube we have \( h^{(sc)}_K(x) = 8(1 + x + x^2) \). This polynomial has positive coefficients and two non-real complex roots. By Corollaries 3.3 and 3.4, so does the short cubical \( h \)-polynomial of \( \text{sd}_c(K) \). This situation is in contrast with what holds for barycentric subdivisions of simplicial complexes (see [2, Theorem 2]).

We denote by \( \text{sd}_c^n(K) \) the \( n \)th iterated cubical barycentric subdivision of \( K \), i.e. \( \text{sd}_c^0(K) = K \) and \( \text{sd}_c^n(K) = \text{sd}_c(\text{sd}_c^{n-1}(K)) \) for \( n \geq 1 \). The short cubical \( h \)-polynomial of \( \text{sd}_c^n(K) \) has the following simple expression, in terms of the short cubical \( h \)-polynomial of \( K \).

**Proposition 3.7.** The short cubical \( h \)-polynomials of \( K \) and \( \text{sd}_c^n(K) \) are related as follows:
\[
(14) \quad h^{(sc)}_{\text{sd}_c^n(K)}(x) = \left( \frac{(2^n - 1)x + 2^n + 1}{2} \right)^{d-1} h^{(sc)}_K \left( \frac{(2^n + 1)x + 2^n - 1}{(2^n - 1)x + 2^n + 1} \right).
\]
Proof. This follows from (13) by induction on $n$. □

The following statement implies that all complex roots of the short cubical $h$-polynomial of $sd^n_c(K)$ converge to $-1$ as $n \to \infty$.

**Corollary 3.8.** We have

$$
\frac{1}{2n(n-1)}h^{(sc)}_{sd^n_c(K)}(x) \to f_{d-1}(K)(x+1)^{d-1}
$$

coefficientwise, as $n \to \infty$. In particular, the short cubical $h$-polynomial of $sd^n_c(K)$ has positive and unimodal coefficients for all large $n$.

**Proof.** The first statement follows from (13) and the fact that $h^{(sc)}_K(1) = 2^{d-1} f_{d-1}(K)$ by straightforward computations. The second statement follows from the first and well-known unimodality properties of binomial coefficients. □

**Remark 3.9.** We do not know of an example of cubical complex $K$ for which $h^{(sc)}(K)$ is nonnegative and $h^{(sc)}(sd_c(K))$ is not unimodal.

## 4. The cubical $h$-vector

This section proves results on the cubical $h$-vector of $sd_c(K)$ analogous to those on the short cubical $h$-vector in Section 3.

**Theorem 4.1.** The cubical $h$-vectors of $K$ and $sd_c(K)$ are related as follows:

$$
(15) \quad h^{(c)}_i(sd_c(K)) = \sum_{j=0}^d C(d, i, j)h^{(c)}_j(K),
$$

where the coefficients $C(d, i, j)$ are nonnegative rational numbers, determined by the generating function

$$
(16) \quad \sum_{j=0}^d \sum_{i=0}^d C(d, i, j)x^iy^j = \frac{1}{1+x}(1+x^{d+1}y^d) + \frac{xy}{2^{d-3}} \frac{(x+3)^{d-1}-(3x+1)^{d-1}y^{d-1}}{x+3-(3x+1)y} + \frac{x}{2^{d-1}(1+x)}((x+3)^{d-1}+(3x+1)^{d-1}y^d).
$$

**Proof.** Since $h^{(c)}_0(K) = h^{(c)}_0(sd_c(K)) = 2^{d-1}$, equation (15) is valid for $i = 0$ if we set $C(d, 0, 0) = 1$ and $C(d, 0, j) = 0$ for $1 \leq j \leq d$. This agrees with (16), since the right-hand side reduces to the constant polynomial with value 1 for $x = 0$. Using (3), (8) and (10), we find that for $1 \leq i \leq d$

$$
\begin{align*}
\sum_{k=0}^{i-1} & (-1)^{i+k-1} h^{(sc)}_k(sd_c(K)) + (-1)^i 2^{d-1} = \sum_{k=0}^{i-1} (-1)^{i+k-1} \sum_{j=0}^{d-1} B(d, k, j) h^{(sc)}_j(K) + (-1)^i 2^{d-1} \\
& = \sum_{k=0}^{i-1} (-1)^{i+k-1} \sum_{j=0}^{d-1} B(d, k, j) h^{(sc)}_j(K) + h^{(c)}_{j+1}(K) + (-1)^i h^{(c)}_0(K) \\
& = \sum_{j=0}^d C(d, i, j)h^{(c)}_j(K),
\end{align*}
$$
where
\begin{equation}
(17) \quad C(d, i, 0) = \sum_{k=0}^{i-1} (-1)^{i+k-1} B(d, k, 0) + (-1)^i
\end{equation}
and
\begin{equation}
(18) \quad C(d, i, j) = \sum_{k=0}^{i-1} (-1)^{i+k-1}(B(d, k, j) + B(d, k, j - 1))
\end{equation}
for $1 \leq j \leq d$, under the convention that $B(d, k, d) = 0$ for all $k$. Using (11), we compute that for $1 \leq j \leq d - 1$
\[
\sum_{i=0}^{d} C(d, i, j)x^i = \sum_{i=0}^{d} \sum_{k=0}^{i-1} (-1)^{i+k-1}(B(d, k, j) + B(d, k, j - 1)) x^i
\]
\[
= \sum_{k=0}^{d-1} (-1)^{k-1}(B(d, k, j) + B(d, k, j - 1))(-x)^{k+1} \frac{1 - (-x)^{d-k}}{1 + x}
\]
\[
= \frac{1}{1 + x} \left( x \sum_{k=0}^{d-1} B(d, k, j)x^k + (-x)^{d+1} \sum_{k=0}^{d-1} B(d, k, j)(-1)^k
\right.
\]
\[
+ x \sum_{k=0}^{d-1} B(d, k, j - 1)x^k + (-x)^{d+1} \sum_{k=0}^{d-1} B(d, k, j - 1)(-1)^k
\left.
\right)
\]
\[
= \frac{1}{2^{d-1}(1 + x)}(x(3x + 1)^j(x + 3)^{d-1-j} + (-x)^{d+1}(-2)^j2^{d-1-j}
\]
\[
+ x(3x + 1)^{j-1}(x + 3)^{d-j} + (-x)^{d+1}(-2)^{j-1}2^{d-j})
\]
and hence
\begin{equation}
(19) \quad \sum_{i=0}^{d} C(d, i, j)x^i = \frac{x(3x + 1)^{j-1}(x + 3)^{d-1-j}}{2^{d-3}}.
\end{equation}

Similar computations yield
\begin{equation}
(20) \quad \sum_{i=0}^{d} C(d, i, 0)x^i = \frac{1}{1 + x} \left( \frac{x(x + 3)^{d-1}}{2^{d-1}} + 1 \right)
\end{equation}
and
\begin{equation}
(21) \quad \sum_{i=0}^{d} C(d, i, d)x^i = \frac{1}{1 + x} \left( \frac{x(3x + 1)^{d-1}}{2^{d-1}} + x^{d+1} \right).
\end{equation}

From these equations we can infer that the $C(d, i, j)$ are nonnegative rational numbers for all $0 \leq i, j \leq d$. Multiplying the equations (19), (20) and (21) by $y^j$ for $1 \leq j \leq d - 1$, $j = 0$ and $j = d$, respectively, and summing over all $j$ results in (16). \hfill \Box

**Corollary 4.2.** If $K$ has nonnegative cubical $h$-vector, then so does $sd_c(K)$. \hfill \Box

**Corollary 4.3.** If $K$ has symmetric cubical $h$-vector, then so does $sd_c(K)$. 
Proof. As in the proof of Corollary 3.4 it suffices to show that \( C(d, d - i, d - j) = C(d, i, j) \) for all \( 0 \leq i, j \leq d \). This follows from (16) by replacing \( x \) and \( y \) by \( 1/x \) and \( 1/y \), respectively, and multiplying by \( x^d y^d \).

□

Remark 4.4. We do not know of an example of cubical complex \( K \) with nonnegative cubical \( h \)-vector for which \( h(\chi(K)) \) is not real-rooted.

Corollary 4.5. For \( d \geq 2 \), we have

\[
\frac{1}{2^{d(d-1)}} h_{sd_c^2(K)}(x) \to f_{d-1}(K) x (x+1)^{d-2}
\]

coefficientwise, as \( n \to \infty \). In particular, if \((-1)^{d-1} \chi(K) \geq 0\), then the cubical \( h \)-polynomial of \( sd_c^n(K) \) has nonnegative and unimodal coefficients for all large \( n \).

Proof. Since \( sd_c(K) \) is a subdivision of \( K \), we have \( \chi(sd_c(K)) = \chi(K) \) (this equality also follows from (12) by setting \( x = -1 \)). Thus, by applying (17) to \( sd_c^n(K) \), we get

\[
(1 + x) h_{sd_c^2(K)}(x) = 2^{d-1} + x h_{sd_c^2(K)}(x) + 2^{d-1}(-x)^{d+1} \chi(K).
\]

Note that if \((-1)^{d-1} \chi(K) \geq 0\), then \( h_{sd_c^2(K)}(x) = (-2)^{d-1} \chi(K) \geq 0 \). The result follows from (23) and Corollary 3.8.

□

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