ROOT SYSTEM OF A PERTURBATION OF A SELFADJOINT OPERATOR WITH DISCRETE SPECTRUM

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Abstract. We analyze the perturbations $T + B$ of a selfadjoint operator $T$ in a Hilbert space $H$ with discrete spectrum $\{t_k\}$, $T \phi_k = t_k \phi_k$, as an extension of our constructions in [1] where $T$ was a harmonic oscillator operator. In particular, if $t_{k+1} - t_k \geq c k^{\alpha - 1}$, $\alpha > 1/2$ and $\|B \phi_k\| = o(k^{\alpha - 1})$ then the system of root vectors of $T + B$, eventually eigenvectors of geometric multiplicity 1, is an unconditional basis in $H$.

1. Statement of main results

Let $H$ be a separable Hilbert space. Consider an operator $T$ with domain $\text{dom} T$ whose spectrum consists of a countable set of eigenvalues $\tau = \{t_k\}_{k=1}^{\infty}$ with corresponding eigenvectors $\{\phi_k\}$,

$$T \phi_k = t_k \phi_k,$$

which form an orthonormal basis in $H$. Let us also assume that $t_{k+1} - t_k > 0$ and that for some fixed $p \in \mathbb{Z}^+$, $d > 0$

$$t_{k+p} - t_k > d \quad \forall k \in \mathbb{Z}_+.$$  \hfill (1.1)

Define $\triangle t_k = t_{k+1} - t_k$. Then (1.1) says $\sum \triangle t_k + \triangle t_{k+1} + \ldots + \triangle t_{k+p-1} > d \quad \forall k$. Hence, for any $k \in \mathbb{Z}_+$, there exists $\gamma(k) \in \{0, 1, \ldots, p-1\}$ such that

1. $\triangle t_k + \gamma(k) \geq d/p$ and

2. $\triangle t_{k+j} < d/p$ \quad $\forall j < \gamma(k)$.

Let $j_1 = 1$ and $j_k = j_{k-1} + \gamma(j_{k-1})$ for $k > 1$ and define $T_k = t_{j_k}$. Define the intervals

$$F_1 = [T_1 - \frac{d}{2p}, T_2 + \frac{d}{2p}], \quad F_k = [T_k + \frac{d}{2p}, T_{k+1} + \frac{d}{2p}], \quad k > 1.$$  \hfill (1.2)

It follows that

$$\tau \subset \bigcup_{k=1}^{\infty} F_k \quad \text{and} \quad \#(\tau \cap F_k) \leq p \quad \forall k.$$  \hfill (1.3)

Set

$$\Pi_k = \{a + ib : a \in F_k, |b| \leq \frac{d}{2p}\}, \quad \Gamma_k = \partial \Pi_k$$

and for $z \notin \text{Sp} T$,

$$R^0(z) = (z - T)^{-1}.$$  \hfill (1.3)

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With
\[ P_k^0 = \frac{1}{2\pi i} \int_{\Gamma_k} R^0(z) dz \]
we have a resolution of the identity \( \sum_{k=1}^{\infty} P_k^0 \).

Consider the perturbed operator \( L = T + B \) with \( B \) closed and \( \text{dom}B \supseteq \text{dom}T \). Set \( \beta = \{ \beta_k = \| B\phi_k \|^2 \} \). In Proposition 1 we will use the condition
\[ \limsup \beta_k < \left( \frac{d}{2p} \right)^2 \left( \frac{1}{8p(1 + \pi^2/3)} \right). \]
This condition implies the existence of integers \( M, N \) such that
\[ \beta_k \leq \left( \frac{d}{2p} \right)^2 \left( \frac{1}{8p(1 + \pi^2/3)} \right) \forall k \geq M, \]
and
\[ \| \beta \|_{\infty} \leq \frac{d^2}{16p^2} \left( 2p \sum_{j=N+1}^{\infty} 1/j^2 \right)^{-1}. \]

Define \( h \) to be a positive constant which satisfies
\[ \sum_{j=0}^{\infty} \frac{1}{h^2 + \left( \frac{d}{2p} \right)^2} \leq \frac{1}{8p \| \beta \|_{\infty}} \]
and set
\[ \Pi_0 = \{ a + ib : -h \leq a \leq T_{M+N+1} + \frac{d}{2p}, |b| \leq h \}, \quad \Gamma_0 = \partial \Pi_0, \]
\[ R(z) = (z - L)^{-1} \quad \forall z \notin \text{Sp}L. \]

**Proposition 1.** Suppose the conditions (1.1) and (1.4) hold and that \( M, N \) satisfy (1.5) - (1.6); \( K = M + N \). Then, with the notation (1.2) - (1.3), \( \text{Sp}L \) is discrete and contained in \( \Pi_0 \cup \bigcup_{j=K+1}^{\infty} \Pi_j \).

This proposition implies that the following operators are well-defined
\[ S_K = \frac{1}{2\pi i} \int_{\partial \Pi_0} R(z) dz, \]
\[ P_k = \frac{1}{2\pi i} \int_{\partial \Gamma_k} R(z) dz \quad \text{for} \ k \geq K + 1. \]

**Proposition 2.** Under the conditions of Proposition 1,
\[ \text{dim}S_K = \sum_{j=1}^{K} \text{dim}P_j^0 \leq pK, \]
\[ \text{dim}P_j = \text{dim}P_j^0 \leq p \quad \text{for all} \ j \geq K + 1 \quad \text{and} \]
\[ \| R(z) \|^2 \leq \left( \frac{d}{p} \right)^2 \quad \forall z \notin \Pi_0 \cup \bigcup_{j=K+1}^{\infty} \Pi_j. \]
Theorem 3. Suppose the condition (1.1) holds and \( \|B\phi_k\| \to 0 \) as \( k \to \infty \). Then there is a bounded operator \( W \) such that \( WP_kW^{-1} = P_k^0 \), \( \dim P_k^0 \leq p \) for all \( k > K \) and \( WS_kW^{-1} = \sum_{k=1}^{K} P_k^0 \). Hence, \( \{S_K, P_{K+1}, P_{K+2}, \ldots\} \) is a Riesz system of projections.

Basically this statement is proven in [9, Thm. 2] where the condition (1.4) is weaker (see (1.2) there) but the dimension of the projectors \( \{P_k\} \) in the Riesz system are bounded by \( 2p \), not by \( p \). Our alternative approach –as in [1]– is based on the boundedness of the discrete Hilbert transform and its adjustments.

We will also consider the case in which the sequence of eigenvalues satisfies the growth condition

\[
(1.10) \quad t_{k+1} - t_k \geq \kappa k^{\alpha - 1} \quad \forall k \in \mathbb{N}
\]

where \( \alpha \in (0, \infty) \setminus \{1\} \).

Define

\[
(1.11) \quad v = 2^{\frac{1}{1-\alpha}}
\]

and put

\[
V_0 = [0, v) \cap \mathbb{N}, \quad V_k = [v^k, v^{k+1}) \cap \mathbb{N} \quad \forall k \in \mathbb{N}.
\]

Consider a closed operator \( B \) with \( \text{dom}B \supseteq \text{dom}T \) and

\[
(1.12) \quad \|B\phi_k\| = c_k k^{\alpha - 1} \quad \text{with} \quad \lim_{k \to \infty} c_k = 0.
\]

(See the remark in Section 7.1). Set \( L = T + B \) and \( c_\infty = \sup |c_k| \).

For each \( k \in \mathbb{N} \) define

\[
\Pi_k = \{a + ib : t_k - (\kappa/2)(k - 1)^{\alpha - 1} \leq a \leq t_k + (\kappa/2)k^{\alpha - 1}, \quad |b| \leq (\kappa/2)k^{\alpha - 1}\}
\]

and \( \Lambda_k = \partial \Pi_k \) so that

\[
(1.14) \quad |\Lambda_k| \leq 4\kappa k^{\alpha - 1} \quad \text{and} \quad \{t_k\} \subset \bigcup_{j=1}^{\infty} \Pi_j
\]

Now select \( N \) large enough so that

\[
(1.15) \quad v^{N\alpha} > c_\infty^2 \left( \frac{1}{1 - \frac{1}{v}} \right) \quad \text{and} \quad c_j^2 \leq (1/4) \left( \frac{16}{\pi^2} \left( 1 + \frac{2\pi^2}{3} + \frac{4}{1 - \frac{1}{v}} \right)^{-1} \quad \forall j \in V_j, \quad j \geq N/2. \right)
\]
Finally set
\[ \ell = \sup \{ \cup_{j \leq N} V_j \}, \]
(1.17)
\[ Y = \left( 4c_\infty^2 v \sum_{j=1}^N (\sqrt{v/2})^{2j} \right)^{1/2} \]
and
(1.18)
\[ \Pi_0 = \{ a + ib : -Y \leq a \leq t_\ell + (\kappa/2)\ell^{\alpha-1}, \quad |b| \leq Y \} \]
and define
(1.19)
\[ R^0(z) = (z - T)^{-1}, \quad R(z) = (z - L)^{-1}, \]
(1.20)
\[ Q_j^0 = \frac{1}{2\pi i} \int_{\Lambda_j} R^0(z)dz \quad \forall j \in \mathbb{N}, \quad Q_j = \frac{1}{2\pi i} \int_{\Lambda_j} R(z)dz \quad \forall j > \ell \]
and
(1.21)
\[ U_\ell = \frac{1}{2\pi i} \int_{\Lambda_0} R(z)dz. \]

**Proposition 4.** Suppose the conditions (1.10) and (1.12) hold with \( \alpha \in (0, \infty) \setminus \{1\} \). Then \( SpL \) is discrete and eventually simple. Furthermore, with the notation (1.17)-(1.20), we have:
\[ SpL \subset \Pi_0 \cup (\cup_{j=\ell+1}^{\infty} \Pi_j), \]
\[ \dim U_\ell = \sum_{j=1}^{\ell} \dim Q_j^0, \quad \text{and} \quad \dim Q_j = \dim Q_j^0 = 1 \quad \forall j > \ell. \]

**Proposition 5.** Fix \( n \in \mathbb{N} \) with \( n > \ell \). Then for each \( z \in \Lambda_n \) we have:
\[ \| R(z) \| \leq \begin{cases} \kappa^{1-\alpha} & \text{if} \quad 1/2 < \alpha < 1, \\ \kappa(n-1)^{1-\alpha} & \text{if} \quad 1 < \alpha < \infty. \end{cases} \]

**Theorem 6.** Let \( \alpha \in (1/2, \infty) \setminus \{1\} \) and suppose the condition (1.10) and (1.12) hold. Then there is a bounded operator \( W \) such that \( WU_\ell W^{-1} = \sum_{k=1}^{\ell} Q_k^0 \) and \( WQ_k W^{-1} = Q_k^0 \) for all \( k > \ell \). Hence, \( \{U_\ell, Q_{\ell+1}, Q_{\ell+2}, \ldots\} \) is a Riesz system of projections.

Let us notice that Propositions 4-5 and Theorem 6 can be reformulated in an proper way for \( \alpha = 1 \). This would necessitate additional notation. We refer the reader to our previous paper [1] where the case \( \alpha = 1 \) is formulated and proven in detail.

2. **Technical preliminaries**

Define \( B(\ell^2(\mathbb{N})) \) to be the space of all bounded linear operators on \( \ell^2(\mathbb{N}) \). Given a strictly increasing sequence of real numbers \( a = (a_k) \) define the
Lemma 9. Suppose $a_{k+1} - a_k > \delta$ $\forall k$. Then $G_a \in B(\ell^2(\mathbb{N}))$.

Proof. Write $\mathbb{R} = \cup_{k \in \mathbb{Z}} I_k$, $I_k = [(k - 1/2)\delta/2, (k + 1/2)\delta/2]$. Then by (2.2), $\#(I_k \cap a) = 0$ or 1. Enumerate $\{j (\frac{\delta}{2}) : \#(I_j \cap a) = 1\}$ in increasing order and call the sequence $\tilde{a}$. It follows from Lemma 7 that the GDHT $G_{\tilde{a}} \in B(\ell^2(\mathbb{N}))$ with $\|G_{\tilde{a}}\| \leq (2/\delta)\|G\|$. Thus, with $A := G_a - G_{\tilde{a}}$ it suffices to show $\|A\| \leq \frac{2\pi^2}{3\delta}$.

Consider the matrix entries $A_{k,k} = 0$ $\forall k$, and for $j \neq k$,

$$|A_{j,k}| = \frac{1}{a_j - a_k} - \frac{1}{\tilde{a}_j - \tilde{a}_k} = \frac{(\tilde{a}_j - a_j) - (\tilde{a}_k - a_k)}{(a_j - a_k)(\tilde{a}_j - \tilde{a}_k)}.$$

By (2.2) we have $|a_j - a_k| > |j - k|\delta$, $|\tilde{a}_j - \tilde{a}_k| \geq |j - k|\delta/2$, and $|\tilde{a}_j - a_j|, |\tilde{a}_k - a_k| < \delta/2$. Hence,

$$|A_{j,k}| \leq \frac{\delta/2 + \delta/2}{(|j - k|(\delta/2))(|j - k|\delta)} = \frac{2/\delta}{|j - k|}\frac{1}{|j - k|^2}.$$

So by Lemma 8, $\|A\| \leq \frac{2\pi^2}{3\delta}$. \hfill \Box
Lemma 10. Suppose $a_k$ is a strictly increasing sequence with $a_k \uparrow \infty$ and $(z_k)$ is a complex sequence satisfying

\begin{align}
|\text{Im}z_k| &< \delta, \\
\text{Re}z_k &\in (a_{k-1} + \delta, a_{k+1} - \delta) \quad \text{and} \\
|\text{Re}z_k - a_k| &< \Delta \quad \forall k.
\end{align}

Then the operator $Z_a$ defined by

$$(Z_a \xi)(n) = \sum_{k \neq n} \frac{\xi_k}{a_k - z_n}$$

is bounded in $\ell^2$ with

$$\|Z_a\| \leq \frac{1}{2\delta} \|G\| + \frac{2\delta \pi^2}{3\Delta^2}.$$  

Proof. By (2.3), $a_{k+1} - a_k > 2\delta \quad \forall k$. Hence, by Lemma 9, $\|G_a\| \leq \frac{1}{2\delta} \|G\|$. Now, set $A = G_a - Z_a$. It suffices to show $\|A\| \leq \frac{2\delta \pi^2}{3\Delta^2}$. Consider the matrix elements $A_{j,k} = 0 \quad \forall k$ and for $j \neq k$,

$$|A_{j,k}| = \left| \frac{1}{a_k - a_n} - \frac{1}{a_k - z_n} \right| = \left| \frac{a_n - z_n}{(a_k - a_n)(a_k - z_n)} \right| \leq \frac{2\delta}{\Delta |k - n| \Delta |k - n|} = \frac{2\delta}{\Delta^2 |k - n|^2}.$$ 

It follows from Lemma 8 that $\|A\| \leq \frac{2\delta \pi^2}{3\Delta^2}$. \qed

Define $\ell^2(H)$ with the norm

$$\|\xi\|_{\ell^2(H)} = \sum_{j=1}^{\infty} \|\xi_j\|_{\ell^2}^2, \quad \xi = (\xi_k), \xi_k \in H.$$ 

Lemma 11. Suppose $a$, $z$, and $Z_a$ are as in Lemma 10. Consider the operator $Z_a^V$ in $\ell^2(H)$

$$(Z_a^V \xi)(n) = \sum_{k \neq n} \frac{\xi_k}{a_k - z_n}.$$ 

Then $\|Z_a^V\|_{\ell^2(H)} = \|Z_a\|_{\ell^2} \leq \frac{1}{2\delta} \|G\| + \frac{2\delta \pi^2}{3\Delta^2}$. 

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$$(Z_a^V \xi)(n) = \sum_{k \neq n} \frac{\xi_k}{a_k - z_n}.$$ 

Then $\|Z_a^V\|_{\ell^2(H)} = \|Z_a\|_{\ell^2} \leq \frac{1}{2\delta} \|G\| + \frac{2\delta \pi^2}{3\Delta^2}$.
Proof. Suppose \( \xi = (\xi_k) \in \ell^2(H) \) with \( \xi_j = \sum_{k=1}^{\infty} \xi_j^{(k)} \phi_k \in H \). Then

\[
\|Z_a^V \xi\|^2 = \sum_{n=1}^{\infty} \| (Z_a^V \xi)(n) \|^2 = \sum_{n=1}^{\infty} \sum_{k=1}^{\infty} \xi_n^{(k)} \sum_{j \neq n} a_j - z_n \quad \text{and} \quad \|Z_a^V \xi\|^2 \leq \|Z_a\|^2 \sum_{n=1}^{\infty} \xi_n^{(k)}^2 H
\]

Then

\[
\|Z_a^V \xi\|^2 = \sum_{n=1}^{\infty} \| (Z_a^V \xi)(n) \|^2 = \sum_{n=1}^{\infty} \sum_{k=1}^{\infty} \xi_n^{(k)} \sum_{j \neq n} a_j - z_n \quad \text{and} \quad \|Z_a^V \xi\|^2 \leq \|Z_a\|^2 \sum_{n=1}^{\infty} \xi_n^{(k)}^2 H
\]

We now move to a series of lemmas which will be used in the proofs of Proposition 4 and Theorem 6. The proofs of these lemmas for values of \( 0 < \alpha < 1 \) and \( \alpha > 1 \) follow a similar pattern so we only present proofs for values of \( \alpha < 1 \).

**Lemma 12.** Suppose \( \alpha > 0 \), \( \alpha \neq 1 \), \( \{t_k\}_{1}^{\infty} \) satisfies (1.10), \( m \in V_M \) and \( n-1 \in V_N \) with \( M < N - 2 \). Then \( t_n - t_m \geq c(1 - 1/v) v^\alpha N \) with \( v \in [1,1] \).

**Proof.** We have

\[
t_n - t_m = (t_n - t_{n-1}) + (t_{n-1} - t_{n-2}) + \ldots + (t_{m+1} - t_m)
\]

Suppose first \( \alpha \in (0,1) \). By the mean value theorem if \( a < b \) we have

\[
\alpha a^{\alpha - 1} > b^\alpha - a^\alpha \geq \alpha b^\alpha - 1.
\]

Hence,

\[
c[(n-1)^{\alpha - 1} + (n-2)^{\alpha - 1} + \ldots + m^{\alpha - 1}] \geq (c/\alpha)[n^\alpha - m^\alpha]
\]

\[
\geq (c/\alpha)[v^\alpha N - v^{(N-1)\alpha}] \geq (c/\alpha)\alpha v^N (\alpha - 1)[v^N - v^{(N-1)N}]
\]

\[
= c(1 - 1/v) v^\alpha N.
\]

A similar argument can be used for \( \alpha > 1 \). We omit the details. \( \square \)

The following lemma generalizes the boundedness of the discrete Hilbert transform. It is a basic tool in our proof of Theorem 6. In fact, our proof of Theorem 6 only works for values of \( \alpha > 1/2 \) because the following lemma does not hold for \( \alpha \leq 1/2 \) (see Remark 14).

**Lemma 13.** Suppose \( \alpha \in (1/2, \infty) \) and \( \{t_k\}_{1}^{\infty} \) satisfies (1.10). Then there is a constant \( \tilde{C} > 0 \) depending only on \( \alpha \) such that for any \( (b_m) \in \ell^2(N) \) we
have
\[
(2.5) \quad \sum_{n=1}^{\infty} \left| \sum_{m \neq n} \frac{m^{\alpha-1}b_m}{t_m - t_n} \right|^2 \leq \tilde{C}\|b\|^2.
\]

Remark 14. Of course, (2.5) does not hold if \( \sum 1/t_n^2 = \infty \) (even for \( b = e_1 \), i.e., \( b(1) = 1, b(m) = 0, m > 1 \)) so \( t_n = n^a, 0 < a \leq 1/2 \) or sequences with the growth condition (1.10) with \( a \leq 1/2 \) could not be analyzed with some analog of Lemma [13] or Lemma [15].

Proof. Let \( b \in \ell^2(\mathbb{N}) \). First suppose \( 1/2 < \alpha < 1 \) so that \( v \) from (1.11) can be written as \( v = 2^{2(1+\delta)} \) with \( \delta > 0 \). Set \( \gamma = \frac{1+2\delta}{2} \). By Cauchy’s inequality we have
\[
(2.6) \quad \sum_{n=1}^{\infty} \left| \sum_{m \neq n} \frac{m^{\alpha-1}b_m}{t_m - t_n} \right|^2 = \sum_{N=1}^{\infty} \sum_{n \in V_N} \left| \sum_{M=1}^{\infty} \sum_{m \in V_M} \frac{m^{\alpha-1}b_m}{t_m - t_n} \right|^2 \\
\leq \left( \frac{2}{1 - \gamma^{-1/2}} \right) \sum_{N=1}^{\infty} \sum_{n \in V_N} \left| \sum_{M=1}^{\infty} \gamma^{|N-M|} \sum_{m \in V_M} \frac{m^{\alpha-1}b_m}{t_m - t_n} \right|^2 \\
= \left( \frac{2}{1 - \gamma^{-1/2}} \right) (S_1 + S_2)
\]
where
\[
S_1 = \sum_{N=1}^{\infty} \sum_{n \in V_N} \left| \sum_{M=1}^{\infty} \gamma^{|N-M|} \sum_{m \in V_M} \frac{m^{\alpha-1}b_m}{t_m - t_n} \right|^2 \\
S_2 = \sum_{N=1}^{\infty} \sum_{n \in V_N} \left| \sum_{M=1}^{\infty} \gamma^{|N-M|} \sum_{m \in V_M} \frac{m^{\alpha-1}b_m}{t_m - t_n} \right|^2.
\]

By Lemma [12] and another application of Cauchy’s inequality
\[
(2.7) \quad S_1 \leq \left[ \sum_{N=1}^{\infty} \sum_{n \in V_N} \sum_{M=1}^{\infty} \gamma^{|N-M|} \sum_{m \in V_M} \frac{m^{2(\alpha-1)}(t_m - t_n)^2}{(t_m - t_n)^2} \right] \sum_{m \in V_M} b_m^2 \\
\leq \sum_{M=1}^{\infty} \sum_{m \in V_M} b_m^2 \sum_{N=1}^{\infty} \sum_{M=1}^{\infty} \gamma^{|N-M|} \frac{\|V_M\| \cdot \|V_N\| 2^{-2M}}{(v/2)^{2\max(M,N)}} \\
\leq \sum_{M=1}^{\infty} \sum_{m \in V_M} b_m^2 \sum_{N=1}^{\infty} \sum_{M=1}^{\infty} \gamma^{|N-M|} v^{M+N+2} 2^{-2M} (2/v)^{2\max(M,N)}.
\]
We will show that the following is uniformly bounded in $M$

$$\sum_{N=1}^{\infty} \gamma |N-M| v^{-|M-N|} 2^{2\max(M,N)-2M}.$$ 

We have

$$\sum_{N=1}^{\infty} \gamma |N-M| v^{-|M-N|} 2^{2\max(M,N)-2M} = \sum_{N=1}^{\infty} \gamma |N-M| 2^{-2(1+\delta)|M-N|} 2^{2\max(M,N)} \leq \sum_{N=1}^{\infty} \gamma |N-M| (2-2\delta)^{|M-N|} \leq \frac{2}{1 - \frac{3}{2\delta}}.$$ 

Combining this bound with (2.7), we conclude

$$(2.8) \quad S_1 \leq \left( \frac{2}{1 - \frac{3}{2\delta}} \right) \|b\|^2.$$ 

Now

$$S_2 \leq \sum_{N=1}^{\infty} \sum_{n \in V_N} \sum_{|M-N| \leq 1} \gamma |M-N| \left( \sum_{m \in V_M} \frac{m^{2(\alpha-1)}}{(t_m - t_n)^2} \right) \left( \sum_{m \in V_M} |b_m|^2 \right)$$

$$\leq \sum_{N=1}^{\infty} \sum_{|M-N| \leq 1} \left( \# F_M \cdot \# F_N \right) \gamma |M-N| 2^{-2M} (v/2)^{-2\max(M,N)} \left( \sum_{m \in V_M} |b_m|^2 \right)$$

$$\leq \sum_{N=1}^{\infty} \sum_{|M-N| \leq 1} v^{M+N+2} \gamma |M-N| v^{-2\max(M,N)} 2^{-2M+2\max(M,N)} \left( \sum_{m \in V_M} |b_m|^2 \right)$$

$$\leq 16 v^4 \sum_{N=1}^{\infty} \sum_{|M-N| \leq 1} \left( \sum_{m \in V_M} |b_m|^2 \right) \leq 3 \cdot 16 \cdot v^4 \gamma \|b\|^2.$$ 

Combining these bounds with (2.6) and (2.8) we have

$$\sum_{n=1}^{\infty} \sum_{m \neq n} \frac{m^{\alpha-1}b_m}{t_m - t_n}^2 \leq \left( \frac{2}{1 - \gamma^{-1/2}} \right) \left( \left( \frac{2}{1 - \frac{3}{2\delta}} \right) + 3 \cdot 16v^4\gamma \right) \|b\|^2.$$ 

So (2.5) is proved for $1/2 < \alpha < 1$.

The proof for $\alpha > 1$ is similar. We omit the details. \qed

The following lemma can be proven in the same manner as Lemma 13. We omit the details.
Lemma 15. Suppose $\alpha \in (1/2, \infty) \setminus \{1\}$ and $\{t_k\}_{k=1}^\infty$ satisfies (1.10). Let $\{z_k\}$ be a sequence such that $|z_k - t_k| \leq (c/2)k^{\alpha-1}\forall k \in \mathbb{N}$. Then there is a constant $C > 0$ depending only on $\alpha$ such that for any $(b_m) \in \ell^2(\mathbb{N})$ we have

$$\sum_{n=1}^\infty \left| \sum_{m \neq n} m^{\alpha-1}b_m \right|^{2} \leq C\|b\|^2.$$ 

The following vector-valued version of Lemma 15 can be proven in the same manner as Lemma 11. We omit the details.

Lemma 16. Suppose $\alpha \in (1/2, \infty) \setminus \{1\}$ and $\{t_k\}_{k=1}^\infty$ satisfies (1.10). Let $\{z_k\}$ be a sequence such that $|z_k - t_k| \leq (c/2)k^{\alpha-1}\forall k \in \mathbb{N}$. Then there is a constant $C > 0$ depending only on $\alpha$ such that for any $(b_m) \in \ell^2(H)$ we have

$$\sum_{n=1}^\infty \left| \sum_{m \neq n} m^{\alpha-1}b_m \right|^{2} \leq C\|b\|^2.$$ 

3. Proof of Proposition 1 and Proposition 2

Proof. Let $z \notin \bigcup_0^\infty \Pi_k$. To show $z \notin \text{Sp}L$ it suffices to show $\|BR^0(z)\| \leq 1/2$ since then $R(z) = R^0(z)(I - BR^0(z))^{-1}$ is well defined. To this end let $f \in H$ with $\|f\|^2 = 1$, $f = \sum f_k\phi_k$.

We have

$$\|BR^0(z)f\|^2 = \|BR^0(z)\sum f_k\phi_k\|^2 = \|\sum \frac{f_k\phi_k}{z - t_k}\|^2 \leq \|f\|^2 \sum \frac{\beta_k}{|z - t_k|^2} = \sum \frac{\beta_k}{|z - t_k|^2}.$$ 

Consider first the case $\text{Re}z \in F_m$, for some $m \geq M + N + 1$. We have

$$\sum_{j \in F_m} \frac{\beta_j}{|z - t_k|^2} = \sum \frac{\beta_j}{(\text{Re}z - t_k)^2 + \text{Im}z^2} \leq \left(\frac{2p}{d}\right)^2 \left(\sum_{j \in F_m} \beta_j + \sum_{J=1}^N + \sum_{J=N+1}^\infty \frac{1}{J^2} \sum_{j \in F_{m+J}} \beta_j\right).$$

Because $m \geq M + N + 1$, $m \pm N \geq M$ whenever $J < N$. So, by (1.3) we have

$$\sum_{j \in F_m} \beta_j + \sum_{J=1}^N \frac{1}{J^2} \sum_{j \in F_{m+J}} \beta_j \leq p \left(1 + 2 \sum_{J=1}^N \frac{1}{J^2}\right) \left(\frac{1}{8p(1 + \pi^2/3)}\right) \left(\frac{d}{2p}\right)^2$$

and by (1.4) we have

$$\sum_{J=N+1}^\infty \frac{1}{J^2} \sum_{j \in F_{m+J}} \beta_j \leq 2p \left(\sum_{J=N+1}^\infty \frac{1}{J^2}\right) \left(\frac{d^2}{16p^2}\right) \left(\frac{p}{J=N+1} \frac{1}{J^2}\right).$$
Combining (3.1) with (3.2-3.3) we conclude that
\[ \|BR^0(z)f\|^2 \leq 1/4 \]
whenever \( \text{Re}z \in F_m \) for some \( m \geq M + N + 1 \).

Now consider the case \( \text{Re}z < T_{M+N+1} + d/2p \), i.e. \( \text{Re}z \notin F_m \) \( \forall m \geq M + N + 1 \). Then \( z \notin \cup_{j=1}^{\infty} \Pi_j \) implies \( |z - t_k|^2 \geq h^2 + \text{Re}(|z| - t_k)^2 \).

Thus
\[ \sum_{k=1}^{\infty} \frac{\beta_k}{|z - t_k|^2} \leq \sum_{k=1}^{\infty} \frac{\beta_k}{h^2 + (|\text{Re}z| - t_k)^2} = \sum_{j=1}^{\infty} \frac{\beta_k}{h^2 + (|\text{Re}z| - t_k)^2} \leq 2 \sum_{k=1}^{\infty} \frac{p\|\beta\|_\infty}{h^2 + (\frac{d}{2p})^2} \leq \frac{2p\|\beta\|_\infty}{8p} = 1/4. \]

So we have shown that \( \|R^0(z)B\|^2 \leq 1/4 \) for all \( z \notin \Pi_0 \cup \cup_{k=K+1}^{\infty} \Pi_k \).

Thus, \( \text{SpL} \subset \Pi_0 \cup \cup_{k=K+1}^{\infty} \Pi_k \). Also, \( \|R(z)\| = \|R^0(z)(I - BR^0(z))^{-1}\| \leq \|R^0(z)\|(1/2) \leq d/p \) for all \( z \notin \Pi_0 \cup \cup_{k=K+1}^{\infty} \Pi_k \).

A standard argument (see [3]) shows that
\[ \text{Trace} \frac{1}{2\pi i} \int_{\Gamma_n} (z - T - tB)^{-1} dz, \quad 0 \leq t \leq 1, \]
is a continuous integer-valued scalar function so it is constant and (1.7 - 1.8) hold.

4. PROOF OF THEOREM 3

We first reproduce Lemma 4.17(a) from [4]. See also [3].

**Lemma 17.** Let \( \{Q^0_k\}_{j \in \mathbb{Z}_+} \) be a complete family of orthogonal projections in a Hilbert space \( X \) and let \( \{Q_k\}_{j \in \mathbb{Z}_+} \) be a family of (not necessarily orthogonal) projections such that \( Q_jQ_k = \delta_{j,k}Q_j \). Assume that
\[ \text{dim}(Q^0_0) = \text{dim}(Q_0) = m < \infty \]
\[ \sum_{j=1}^{\infty} \|Q^0_j(Q_j - Q^0_j)u\|^2 \leq c_0\|u\|^2, \quad \text{for every} \quad u \in X \]
where \( c_0 \) is a constant smaller than 1. Then there is a bounded operator \( W : X \to X \) with bounded inverse such that \( Q_j = W^{-1}Q^0_jW \) for \( j \in \mathbb{Z}_+ \).

We are now ready to prove Theorem 3

**Proof.** By Lemma 17 it suffices to show \( \exists N_* \in \mathbb{N} \) such that for all \( f \in H \) with \( \|f\| = 1 \),
\[ \sum_{n \geq N_*} \|P^0_n(P_n - P^0_n)f\|^2 \leq 1/2. \]
Fix \( n > K \) (with \( K \) from Proposition 1) and \( f = \sum f_k \phi_k \in H \) with \( \|f\| = 1 \). Then
\[
P_n - P_n^0 = \frac{1}{2\pi i} \int_{\Gamma_n} (R(z) - R^0(z)) \, dz = \frac{1}{2\pi i} \int_{\Gamma_n} R(z)BR^0(z) \, dz.
\]
So by Proposition 2, inequality (1.9),
\[
\| (P_n - P_n^0)f \|_2^2 = \frac{1}{2\pi} \| \int_{\Gamma_n} R(z)BR^0(z) f \, dz \|_2^2
\leq \frac{1}{2\pi} \left[ \int_{\Gamma_n} \|R(z)BR^0(z)f\| \right]^2 \leq C \left[ \int_{\Gamma_n} \|BR^0(z)f\| \right]^2
\]
where \( C = \frac{1}{2\pi} \left( \frac{d}{p} \right)^2 \).

For \( n \geq K \) define \( z_n^* \in \Gamma_n \) to be a point where the maximum of the sum
\[
\| \sum_{k=1}^{\infty} \frac{f_k B \phi_k}{z - t_k} \|
\]
is attained. Note that \( (z_n^*) \) depends on \( f \). Since \( |\Gamma_k| \leq 3C \quad \forall k \geq 1 \), we have by (4.1)
\[
\| (P_n - P_n^0)f \|_2^2 \leq C |\Gamma_n|^2 \left\| \sum_{k=1}^{\infty} \frac{f_k B \phi_k}{z - t_k} \right\|_2
\leq C(3d)^2 \left\| \sum_{k=1}^{\infty} \frac{f_k B \phi_k}{z - t_k} \right\|_2.
\]

Suppose, for now, that \( p = 1 \) so that \( \#(\tau \cap F_k) \leq 1 \quad \forall k \geq 1 \). We will show that given any \( \epsilon > 0 \) if we choose \( N_1 \) as in (4.5) below, then
\[
\sum_{n \geq N_1} \left\| \sum_{k=1}^{\infty} \frac{f_k B \phi_k}{z_n^* - t_k} \right\|^2 < \epsilon \quad \forall \|f\| = 1.
\]
Note that \( z_n^* \in \Gamma_n \) depends on \( f \).

Recall that \( G \) is the cannonical discrete Hilbert transform and set
\[
C_1 = 4 \left( \frac{d\|G\|}{p} + \frac{\pi^2}{3dp} \right).
\]
Select \( M_1 \) large enough so that
\[
\|B \phi_k\|^2 \leq \epsilon/C_1 \quad \forall k \geq M_1
\]
and \( N_1 \) large enough so that whenever \( w \in \Gamma_n \quad \forall n \geq K \)
\[
\sum_{n=N_1}^{\infty} \left| w - t_{M_1} \right|^{-2} \leq \frac{\epsilon}{4\|\beta\|_{M_1}} \quad \forall m \leq M_1.
\]
Then
\[ \left\| \sum_{k=1}^{\infty} \frac{f_k B \phi_k}{z_n^* - t_k} \right\|^2 \leq 2 \left( \left\| \sum_{k \leq M_1} \frac{f_k B \phi_k}{z_n^* - t_k} \right\|^2 + \left\| \sum_{k > M_1} \frac{f_k B \phi_k}{z_n^* - t_k} \right\|^2 \right). \]

By Cauchy's inequality we have
\[ \left\| \sum_{k \leq M_1} \frac{f_k B \phi_k}{z_n^* - t_k} \right\|^2 \leq \left( \sum_{k \leq M_1} |f_k|^2 \|B \phi_k\|^2 \right) \left( \sum_{k \leq M_1} |z_n^* - t_k|^{-2} \right). \]

So by (4.5)
\[ (4.6) \quad 2 \sum_{n \geq N_1} \left\| \sum_{k \leq M_1} \frac{f_k B \phi_k}{z_n^* - t_k} \right\|^2 \leq \|B \phi_k\|^2 C_1 \left( \sum_{n \geq N_1} \frac{z_n^* - t_{M_1}}{2} \right) < \epsilon/2. \]

It follows from Lemma 11 and (4.4) that
\[ (4.7) \quad 2 \sum_{n \geq N_1} \left\| \sum_{k > M_1} \frac{f_k B \phi_k}{z_n^* - t_k} \right\|^2 \leq 2 \|B \phi_k\|^2 C_1 \left( \sum_{n \geq N_1} \frac{z_n^* - t_{M_1}}{2} \right) < \epsilon/2. \]

Hence, combining (4.6) and (4.7) we have proven (4.3).

Now suppose \( p > 1. \) Reindex the sequences \( t_k, f_k, \) and \( \phi_k \) in such a way that for all \( k \geq 1 \)
\[ \tau \cap F_k = \{ t_k^{(1)} \leq t_k^{(2)} \leq \ldots \leq t_k^{(J_k)} \}, \quad J_k \leq p. \]

Then for \( K \geq k, \)
\[ C \sum_{n \geq K} \left\| \sum_{k=1}^{\infty} \frac{f_k B \phi_k}{z_n^* - t_k} \right\|^2 = C \sum_{n \geq K} \left\| \sum_{j=1}^{J_k} f_k^{(j)} B \phi_k^{(j)} \right\|^2 \leq 2^p C \sum_{j=1}^{p} \sum_{n \geq K} \left\| \sum_{k=1}^{\infty} \frac{f_k^{(j)} B \phi_k^{(j)}}{z_n^* - t_k^{(j)}} \right\|^2. \]

Note that if \( J_k < p \) some terms in the series are taken to be 0. For each \( j \leq p \) the sequence \( t_k^{(j)} \leq t_k^{(2)} \leq \ldots \) satisfies (1.1) with \( p = 1. \) So by taking \( \epsilon = 1/(2 \cdot 2^p C) \) and applying (4.3) for each \( j \leq p \) the Theorem is proven by (4.2).

5. Proof of Proposition 4 and Proposition 5

Suppose that \( z \notin \Pi_0 \cup \bigcup_{j=1}^{\infty} \Pi_j. \) We will show that \( \|R^0(z)B\|^2 \leq 1/2. \) It follows that
\[ (5.1) \quad R(z) = (I - R^0(z)B)^{-1} R^0(z) \]
is well defined. Let $f \in H$ with $\|f\| = 1$, $f = \sum f_k \phi_k$. Then

$$\| R^0(z)Bf \|^2 = \| \sum f_j B\phi_j \|^2 \leq \left( \sum |f_j| \|B\phi_j\| \right)^2 \leq \sum \|B\phi_j\|^2 |z - t_j|^2.$$

Suppose first that $\text{Re} \ z > t_\ell + (c_\ell/2)\ell^{\alpha-1}$ so that $\text{Re} \ z \in [v^{\tilde{N}}, v^{\tilde{N}+1})$ for some $\tilde{N} > N$. Then

$$\sum_{j=1}^\infty \frac{c_j^2 j^{2(\alpha-1)}}{|z - t_j|^2} = S_1 + S_2$$

with

$$S_1 = \sum_{J=1}^{\tilde{N} - N/2} \sum_{j \in V_J} \frac{c_j^2 j^{2(\alpha-1)}}{|z - t_j|^2}, \quad S_2 = \sum_{J=\tilde{N} - N/2}^\infty \sum_{j \in V_J} \frac{c_j^2 j^{2(\alpha-1)}}{|z - t_j|^2}.$$

We have

$$S_1 \leq c_\infty^2 \sum_{J=1}^{\tilde{N} - N/2} \sum_{j \in V_J} \frac{j^{2(\alpha-1)}}{|z - t_j|^2}.$$

If $0 < \alpha < 1$, then for each $J \leq \tilde{N} - 1 - N/2$,

$$\sum_{j \in V_J} \frac{j^{2(\alpha-1)}}{|z - t_j|^2} \leq \# V_J \frac{2^{-2J}}{v^{2(N-1)\alpha}} \leq \frac{2^{-2J}(v/2)^2(v/2)^{-2\tilde{N}}}{v^{N+1}2^{-2J}(v/2)^2(2/v)^{2\tilde{N}}} \leq v^{J+1}2^{-2J}(v/2)^2(2/v)^{2\tilde{N}} \leq (2/v)^2(\tilde{N}-J)^{-2}v^{-J} \leq v(2/v)^N v^{-J}.$$

It follows from (1.15) that

$$S_1 \leq c_\infty^2 (2/v)^N \sum_{J=1}^{\tilde{N} - N/2} v^{-J} \leq c_\infty^2 (2/v)^N \left( \frac{1}{1 - 1/v} \right) < 1/4.$$
So it follows from (1.15) that

\[
S_1 \leq c_\infty^2 v^{(2v)^{-N}} \sum_{J=1}^{\tilde{N}-N/2} v^{-J} \leq c_\infty^2 (2v)^{-N} \left( \frac{1}{1-1/v} \right) < 1/4.
\]

Now

\[
S_2 = \sum_{J=\tilde{N}-N/2}^{\infty} \sum_{j \in V_J} \frac{c_J^2 2^{(\alpha-1)}}{|z - t_j|^2} \leq \sup_{J \geq \tilde{N}-N/2} \left[ \left( \sum_{J=\tilde{N}-N/2}^{\infty} + \sum_{J=\tilde{N}-1,\tilde{N},\tilde{N}+1} \right) \sum_{j \in V_J} \frac{j^{2(\alpha-1)}}{|z - t_j|^2} \right].
\]

Let \( \text{Re} z \in (t_k, t_{k+1}], k \in \tilde{V}_{\tilde{N}-1} \cup \tilde{V}_{\tilde{N}} \cup \tilde{V}_{\tilde{N}+1} \). Then

\[
|z - t_k|^2 \geq (v/2)^2 k^{2(\alpha-1)} \geq (v/2)^2 2^{2(\tilde{N}-1)}
\]

and for \( j \neq k, j \in \tilde{V}_{\tilde{N}-1} \cup \tilde{V}_{\tilde{N}} \cup \tilde{V}_{\tilde{N}+1} \) we have

\[
|z - t_j|^2 \geq (\kappa/2)^2 |j - k|^{2(\tilde{N}-1)}.
\]

Thus,

\[
\sum_{J=\tilde{N}-1,\tilde{N},\tilde{N}+1} \sum_{j \in V_J} \frac{j^{2(\alpha-1)}}{|z - t_j|^2} \leq \frac{2^{2(\tilde{N}+1)}}{(\kappa/2)^2 2^{2(\tilde{N}-1)}} + \sum_{J=\tilde{N}-1,\tilde{N},\tilde{N}+1} \sum_{j \neq k} \frac{2^{2(\tilde{N}+1)}}{(v/2)^2 |j - k|^{2(\tilde{N}-1)}} \leq \frac{16}{c^2} \left( 1 + 2 \sum_{j=1}^{\infty} j^{-2} \right) = \frac{16}{\kappa^2} \left( 1 + 2 \pi^2 / 3 \right).\]

Furthermore, for \( 0 < \alpha < 1 \)

\[
\sum_{J=\tilde{N}-N/2}^{\infty} \sum_{j \in V_J} \frac{j^{2(\alpha-1)}}{|z - t_j|^2} \leq \sum_{J=\tilde{N}-N/2}^{\infty} \sum_{j \neq \tilde{N}-1,\tilde{N},\tilde{N}+1} \frac{(\# F_J) 2^{-2J}}{(v/2)^2 (J-1)^2} \leq 4 \sum_{J=\tilde{N}-M}^{\infty} \frac{v^{J+1}}{v^{2J-2}} \leq 4v \sum_{J=1}^{\infty} v^{-J} \leq 4 \left( \frac{1}{1-1/v} \right).
\]

By a similar argument, if \( \alpha > 1 \) we also have

\[
\sum_{J=\tilde{N}-N/2}^{\infty} \sum_{j \in V_J} \frac{j^{2(\alpha-1)}}{|z - t_j|^2} \leq 4 \left( \frac{1}{1-1/v} \right).
\]
Hence, by (1.16)

\[ S_2 \leq \sup_{\substack{j \in V_j \\ J \geq N-N/2}} \left( \frac{(16/\kappa^2)(1 + 2\pi^2/3)}{1 - 1/v} \right) < 1/4. \]

Now suppose that Rez \( \leq t_\ell + (\kappa/2)\ell^{\alpha-1} \) so that \( \text{dist}(z, \tau)^2 \geq Y^2 \). Suppose \( 0 < \alpha < 1 \). Then

\[
\sum_{j=1}^{\infty} c_j^2 |z - t_j|^2 \left( \alpha - 1 \right) \leq c_{\infty} \sum_{j=1}^{N} \frac{(\#V_j)2^{-2J}}{Y^2} + \sup_{j \in V_j} c_j^2 \sum_{J=1}^{\infty} \frac{(\#V_J)2^{-2J}}{\nu^{2\alpha(J-1)}} \]

\[
\leq c_{\infty} \nu \sum_{j=1}^{N} (\sqrt{\nu}/2)^{2J} + \sup_{j \in V_j} \sum_{J \geq N+1} (2/\nu)^2 \nu^{1-J} < 1/2
\]

by (1.16) and (1.17). We have shown (5.2) \( \| (I - R^0(z)B)^{-1} \| \leq 2 \)

so that by (5.1) \( R(z) \) is well-defined.

By a similar argument for \( \alpha > 1 \) we also have

\[
\sum_{j=1}^{\infty} c_j^2 |z - t_j|^2 \leq 1/2.
\]

We omit the details.

Now, definition (1.13) implies that for \( z \in \Lambda_n \),

\[
\| R^0(z) \| \leq \begin{cases} 
(\kappa/2)n^{1-\alpha} & \text{if } 1/2 < \alpha \leq 1, \\
(\kappa/2)(n-1)^{1-\alpha} & \text{if } 1 < \alpha < \infty.
\end{cases}
\]

Hence, inequality (1.22) follows from (5.1) and (5.2) together with (1.14) and (5.3).

6. PROOF OF THEOREM 6

Proof. For the case of \( \alpha = 1 \), this theorem is proven in the paper [1]. Henceforth, we assume \( \alpha \neq 1 \). By Lemma [17] it suffices to show there exists an integer \( N_\ast \) such that

\[
\sum_{n \geq N_\ast} \| Q_n^0(Q_n - Q_n^0)f \|^2 \leq 1/2.
\]

Fix \( n > N \) and \( f = \sum f_k \phi_k \in H \) with \( \| f \| = 1 \). Then

\[
Q_n - Q_n^0 = \frac{1}{2\pi i} \int_{\Lambda_n} (R(z) - R^0(z))dz = \frac{1}{2\pi i} \int_{\Lambda_n} R(z)BR^0(z)dz.
\]
Hence,
\[
\|(Q_n - Q_0^n)\|^2 = \frac{1}{2\pi} \int_{\Lambda_n} R(z)BR^0(z)f dz^2 \\
\leq \frac{1}{2\pi} \left[ \int_{\Lambda_n} \|R(z)BR^0(z)f\| dz \right]^2 \\
= \frac{1}{2\pi} \left[ \int_{\Lambda_n} \left\| \sum_{k=1}^\infty f_k R(z)B\phi_k \frac{z}{z - t_k} \right\| dz \right]^2.
\]

Now define \( z_n^* \in \Lambda_n \) to be a point at which the following sum attains its maximum,
\[
\left\| \sum_{k=1}^\infty f_k B\phi_k \frac{z}{z - t_k} \right\| \quad z \in \Lambda_n.
\]

Combining (1.14) with (1.22) yields
\[
|\Lambda_n|^2 \|R(z)\|^2 \leq 16\kappa^2. 
\]

So,
\[
\|(Q_n - Q_0^n)\|^2 \leq \frac{16\kappa^2}{2\pi} \left[ \left\| \sum_{k=1}^\infty f_k B\phi_k \frac{z_n^*}{z_n^* - t_k} \right\| \right]^2.
\]

Recall the constant \( C \) from Lemma 16. Condition (1.12) implies that there exists an absolute constant \( N_* \) such that
\[
\|B\phi_k\| \leq \frac{2\pi}{32\kappa^2 C} k^{\alpha-1} \quad \forall k \geq N_*.
\]

Thus
\[
\sum_{n \geq N_*} \|Q_n^0(Q_n - Q_0^n)f\|^2 \leq \sum_{n \geq N_*} \|(Q_n - Q_0^n)f\|^2 \\
\leq \frac{16\kappa^2}{2\pi} \sum_{n \geq N_*} \left[ \left\| \sum_{k=1}^\infty f_k B\phi_k \frac{z_n^*}{z_n^* - t_k} \right\| \right]^2.
\]

Finally, combining (6.3) with Lemma 16 and (6.2) yields (6.1) and the proof is complete. \( \Box \)

7. Further remarks

7.1. Our statement of Theorem 6 required the condition
\[
\lim_{k \to \infty} c_k = 0
\]
where \( \{c_k\} \) is defined in (1.12). With a careful accounting of quantities appearing in the proof of Theorem 6 we could have written a constant \( c^* \) such that the condition (7.1) could be replaced by the weaker condition
\[
\limsup c_k \leq c^*.
\]
However, condition (7.1) or (7.2) could not be weakened in a significant way: an assumption $\limsup c_k < \infty$ would not guarantee the statement of Theorem 6. A counterexample in the case $\alpha = 1$ is given in [1], Section 6.3.

Now we’ll adjust the constructions of [1] to get an operator $B$, with

$$\sup_m \{ \| B\phi_k \| (t_{2m} - t_{2m-1})^{-1}, \quad k = 2m - 1, 2m \} = 1/2$$

such that the perturbation $L = T + B$ has a discrete spectrum, all points of $\text{Sp}(T + B)$ are simple eigenvalues, the system $\{ \psi_k \}$ of eigenvectors of $L$ is complete, but it is not a basis in $H$. If $t_n = n^\alpha, \quad 0 < \alpha < \infty$, then (7.3) guarantees that $c_\infty \leq 1/2$.

Special 2-dimensional blocks play an important role in this construction. Put

$$b = \begin{bmatrix} 0 & s \\ -s & 0 \end{bmatrix}, \quad 0 < s < 1, \quad s^2 + h^2 = 1, \quad 0 < h << 1.$$ (7.4)

This choice is a slight adjustment of a 2-dimensional block (64) in [1]. It simplifies elementary calculations of the $\text{Angle}(g^+, g^-)$, etc., for example. Such a block (7.4) could be used to get the same counterexample in Section 6.3, [1] instead of (64) there. Of course, $\|b\| = s$ in $\mathbb{C}^2$ in the Euclidean norm.

We have

$$\begin{bmatrix} 0 & 0 \\ 0 & 2 \end{bmatrix} + b = \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix} + c, \quad c = \begin{bmatrix} -1 & s \\ -s & 1 \end{bmatrix}$$

and

$$cg^\pm = \pm hg^\pm \quad \text{where}$$

$$g^\pm = (1, G^\pm 1), \quad G = \sqrt{\frac{1 + h}{1 - h}}.$$ (7.5)

If $\alpha = \text{Angle}(g^+, g^-)$ then

$$(\cos \alpha)^2 = \frac{(g^+, g^-)^2}{\|g^+\|^2 \cdot \|g^-\|^2} = 1 - h^2 = s^2.$$ (7.6)

So $\sin \alpha = h$.

If $f = \Phi_0(f)u_0 + \Phi_1(f)u_1$ is the standard basis decomposition in $\mathbb{C}^2$ then

$$\|\Phi_0\| = \|\Phi_1\| = 1/\sin \alpha = 1/h.$$ (7.7)

Now we define $B = \{ b(m) \}$ where $b(m)$ are 2-dimensional blocks

$$\frac{1}{2}(t_{2m} - t_{2m-1}) \begin{bmatrix} 0 & s \\ -s & 0 \end{bmatrix}, \quad s = s(m), \quad \text{say} \quad s(m)^2 + (1/m)^2 = 1,$$

on $\mathbb{C}^2 = E_m := \text{Span}\{ \phi_{2m-1}; \phi_{2m} \}.$
Then $E_m$ are invariant subspaces of $T + B$ and, (compare to [1], Lemma 14),

$$(T + B)_m = \frac{1}{2}(t_{2m} + t_{2m-1}) + \frac{1}{2}(t_{2m} - t_{2m-1}) \begin{bmatrix} -1 & s \\ -s & 1 \end{bmatrix}$$

and

$$(T + B)_m \psi_m = \left( \frac{1}{2}(t_{2m} + t_{2m-1}) \pm \frac{1}{2}(t_{2m} - t_{2m-1})h \right), \quad h = 1/m,$

where

$$\psi_m = g^\pm(m) = \phi_{2m-1} + G^\pm \phi_{2m}.$$

We omit further details. With the explicit formulas given it is easy to see that (7.5) with $h = 1/m$ guarantees that $\{\psi_m\}_1^\infty$ is not a basis.

7.2. As an application of Theorem 5, consider the differential operator $T$ on $L^2(\mathbb{R})$ defined by

$$(7.6) \quad Ty = -y'' + |x|^{\beta}y, \quad \text{with } \beta > 1.$$  

The spectrum of $T$ consists of an infinite set of eigenvalues

$$\text{Spec}T = \{\lambda_0 \leq \lambda_1 \leq \lambda_2 \leq \ldots\} \quad \text{with } \lim_{n \to \infty} \lambda_n = \infty.$$ 

The growth of the sequence of eigenvalues is described by the formula

$$(7.7) \quad \lim_{n \to \infty} \left[ 2 \int_0^{\lambda_n^{1/\beta}} (\lambda_n - |x|^{\beta})^{1/2}dx - (n + 1/2)\pi \right] = 0.$$ 

For a proof, see the last section of [10]. It follows from (7.7) by a change of variables that

$$(7.8) \quad \lim_{n \to \infty} \left[ 2\lambda_n^{2/\beta} \Omega_\beta - (n + 1/2)\pi \right] = 0 \quad \text{with } \Omega_\beta = 2 \int_0^1 (1 - x^{\beta})^{1/2}dx.$$ 

Subtracting the $n$th term from the $n + 1$st term in (7.8) we derive

$$(7.9) \quad \lim_{n \to \infty} \left[ \lambda_{n+1}^{2+\beta} - \lambda_n^{2+\beta} \right] = \pi/\Omega_\beta.$$ 

From (7.9) it is straightforward to show that there exist constants $C > 0, N \in \mathbb{N}$ (depending on $\beta$) such that

$$(7.10) \quad \lambda_{n+1} - \lambda_n \geq Cn^{\alpha - 1} \quad \forall n > N, \quad \alpha = \frac{2\beta}{\beta + 2}.$$ 

Let us mention the papers [7], [8] where the eigenvalues for the eigenproblem $-y'' + q(z)y = \lambda y$ are analyzed for polynomial $q(z)$.

Denote the eigenfunction corresponding to $\lambda_n$ by $\phi_n$ and define

$$L(p; \alpha) = \{b : b(x)(1 + |x|^2)^{-\alpha/2} \in L^p(\mathbb{R})\}.$$
We have the following bound on the behavior of \( \phi_n \)

\[
|\phi_n(x)| \leq \frac{K \exp(Q(x))}{|\lambda_n - |x|^{\beta}|^{1/4} + \lambda_n^{-6\beta}}
\]

with

\[
Q(x) = \begin{cases}
- \int_{\lambda_n^{1/\beta}}^x (\lambda_n - |t|^{\beta})^{1/2} dt, & x > \lambda_n^{1/\beta} \\
0, & |x| \leq \lambda_n^{1/\beta} \\
\int_{\lambda_n^{1/\beta}}^{-x} (\lambda_n - |t|^{\beta})^{1/2} dt, & x < -\lambda_n^{1/\beta}.
\end{cases}
\]

For the case \( \beta = 2 \), this inequality is proven in [2], a few changes to this proof boost it to cover \( \beta > 1 \). We omit the details. Such constructions for Schrödinger operators with turning points are discussed in [6, Ch 8.11].

By an argument like that given for Lemma 8 in [1] it follows from (7.11) that if \( b \in L(p; \alpha) \) then \( \|b\phi_n\|_2 \leq Cn^{\frac{2\beta}{p}} \) where

\[
(7.12) \quad \xi = \max\left\{ \frac{1}{3\beta} \left( 1 - \beta + 3\alpha + (\beta - 1)/p \right); \frac{1}{\beta} \left( \alpha - \beta/4 + 1/2 - 1/p \right) \right\}
\]

\[
= \begin{cases}
\frac{1}{3\beta} \left( 1 - \beta + 3\alpha + (\beta - 1)/p \right), & 2 \leq p < 4 \\
\frac{1}{\beta} \left( \alpha - \beta/4 + 1/2 - 1/p \right), & 4 < p.
\end{cases}
\]

In the exceptional case \( p = 4 \) we have

\[
(7.13) \quad \|b\phi_n\|_2 \leq Cn^{\frac{2\beta}{p} + \frac{1-\beta}{\beta+1}} \log(n+2)
\]

The following Proposition follows from (7.10), (7.12), (7.13) and Theorem [6]. Our statement of Theorem [6] does not include \( \alpha = 1 \) (and therefore \( \beta = 2 \)); a proper formulation and proof of this Theorem for \( \alpha = 1 \) can be found in [1].

**Proposition 18.** Let \( T \in (7.6) \), \( b \in L(p, \alpha) \), and define the operator \( B \) on \( L^2(\mathbb{R}) \) by \( Bf = b(x)f(x) \).

Suppose that

\[
(7.14) \quad \begin{cases}
\beta - 1 < p(-4 + 5\beta/2 - 3\alpha) & \text{if } 2 \leq p < 4 \\
2 > p(3 - 3\beta/2 + 2\alpha) & \text{if } 4 \leq p.
\end{cases}
\]

Then the system of eigen and associated functions for the operator \( T + B \) is an unconditional basis.

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