q - Magnetism at roots of unity

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Abstract

We study the thermodynamic properties of a family of integrable 1D spin chain hamiltonians associated with quantum groups at roots of unity. These hamiltonians depend for each primitive root of unit on a parameter $s$ which plays the role of a continuous spin. The model exhibits ferrimagnetism even though the interaction involved is between nearest neighbors. The latter phenomenon is interpreted as a genuine quantum group effect with no “classical” analog. The discussion of conformal properties is given.
After Heisenberg [1], spin is the key word for understanding the magnetic properties of metals. In 1 spatial dimension we have many exactly solvable models, which can be treated by means of Bethe ansatz technique [2]. These models can be used to deepen our intuition on such non trivial subject as magnetism. Quantum groups [3] provide the mathematical ground for studying integrable one dimensional spin chains. Moreover, the different integrable generalizations of the original $S = 1/2$ Heisenberg model are associated in one to one fashion with the different irreps of $U_q(SL(2))$ where the deformation parameter $q$ is related to the anisotropy of the chain.

Heisenberg’s ideas of magnetism can be extended naturally in the context of quantum groups, in a sense that the rotational group $SU(2)$ is replaced by $U_q(SL(2))$. For generic $q$ this replacement is not essential, just because the finite dimensional irreps of $SU(2)$ and the ones of its quantum deformation are the same. If we want to get a typical signal of the effect of defining the spin variables by finite dimensional irreps of $U_q(SL(2))$, “$q$ - magnetism”, we need to work in the very special regime, a $q$ root of unity, where we have finite dimensional irreps of $U_q(SL(2))$ without “classical” ($q = 1$) analog [4].

In this letter we start a systematic study of the magnetic properties of 1 dimensional spin chains using non regular finite dimensional irreps of $U_q(SL(2))$ at roots of unity. The main new phenomena we find, concerning the magnetic properties, is ferrimagnetism, i.e. a disordered ground state with non vanishing magnetization. This kind of behavior is known in systems possessing complex topology of interaction [5], while here the appearance of this phenomena is directly tied with the special irreps used to define the spin variables of the chain.

The quantum group $U_q(SL(2))$ with $q = \epsilon$, $\epsilon^N = 1$ is generated by the operators $E$, $F$ and $K = \epsilon^{2s}$. The peculiar thing about $\epsilon$ being a root of unity is that $E^{N'}$, $F^{N'}$ and $K^{N'}$ are central elements (where $N' = N$ if $N$ is odd and $N' = N/2$ if $N$ is even). These central elements, together with the Casimir, label the irreps of $U_q(SL(2))$. Regular irreps, which are the $q$-deformations of the usual integer and half-integers spin representations, satisfy $E^{N'} = F^{N'} = 0$ and $K^{N'} = \pm 1$. Nilpotent irreps of $U_q(SL(2))$ are a slight generalization of the regular ones, in a sense, that the generator $K$ takes on the generic value $\epsilon^{2s}$, where $s$ is our “continuous” spin. The dimension of these nilpotent irreps is always $N'$.

The “nilpotent” - spin chain hamiltonian is defined in the standard way as:

$$H(s) = -iI \frac{\partial \ln t(u, s)}{\partial u} \bigg|_{u=0}$$  \hspace{1cm} (1)

where $I$ is an overall coupling constant and $t(u, s)$ is the transfer matrix defined by the quantum $R$-matrix $R^{ss'}(u)$ intertwining two nilpotent irreps of $U_q(S\hat{L}(2))$ [6].

For the special case $N = 3$ and $s = 1$ the hamiltonian (1) coincides with the Fateev-Zamolodchikov hamiltonian [7] with the anisotropy fixed by $q = e^{2\pi i/3}$. The hermiticity condition on the hamiltonian $H(s)$ are given by:
\[ v_s \sin \gamma \varepsilon \sin \gamma (2s - k + 1) > 0 \quad k = 1, 2, ..., N' - 1 \] (2)

where \( q = e^{i\gamma} \) and \( v_s = \pm 1 \) is the spin parity. Equation (2) is equivalent to the condition \( E^\dagger = v_s F \) for the corresponding nilpotent s-irrep. The hermiticity regions that follow from (2) (for \( \varepsilon = e^{2\pi i/N} \)) are:

\[
\begin{align*}
N \text{ even:} & \quad \frac{1}{2} - \frac{1}{p_0} + \frac{1}{4} - \frac{v_s}{4} < s < \frac{1}{2} + \frac{1}{4} - \frac{v_s}{4} \\
N \text{ odd:} & \quad \frac{1}{2} - \frac{3}{4p_0} + \frac{1}{4} - \frac{v_s}{4} < s < \frac{1}{2} - \frac{1}{4p_0} + \frac{1}{4} - \frac{v_s}{4}
\end{align*}
\]

(3)

where \( p_0 = \frac{N}{2} \). In what follows, we shall consider \( N > 4 \) (\( N \) even) and \( N > 3 \) (\( N \) odd). In the trivial case \( N = 4 \), the hamiltonian (1) is essentially that of \( XX \)-model in magnetic field. The case \( N = 3 \) [8] requires special treatment which will be given elsewhere [9].

Notice that for \( N \) even the middle point of both spin intervals \( (v_s = \pm 1) \) corresponds to a regular integer or half-integer spin. For \( N \) odd only the interval of negative parity contains such a point. For all these middle points \( s_0 \)'s the corresponding hamiltonians \( H(s_0) \) are identical to a higher spin \( XXZ \) models with anisotropy \( \gamma = \frac{2\pi}{N} \). It is interesting to observe that \( 2s_0 + 1 \) is not a Takahashi number [10]. Apparently for that reason, Kirillov and Reshetikhin [11] do not consider this case in their, otherwise, general analysis. On the other hand, Babujian and Tsvelick [12] have considered one of these points \( (s = \frac{N - 2}{4} \) for \( N \) even). However, we do not believe that their results are correct at this point.

The hamiltonian (1) can be diagonalized by means of the standard Bethe ansatz [6]. The Bethe ansatz equations read in our case:

\[
\begin{bmatrix} \text{sh} \frac{\gamma}{2}(\lambda_j + 2is) \\ \text{sh} \frac{\gamma}{2}(\lambda_j - 2is) \end{bmatrix}^L = -\prod_{k=1}^M \frac{\text{sh} \frac{\gamma}{2}(\lambda_j - \lambda_k + 2is)}{\text{sh} \frac{\gamma}{2}(\lambda_j - \lambda_k - 2is)}
\]

(4)

with the energy eigenvalues given by:

\[
E_M = -\sum_{k=1}^M \frac{I \sin 2\gamma s}{\text{sh} \frac{\gamma}{2}(\lambda_k + 2is)[\text{sh} \frac{\gamma}{2}(\lambda_k - 2is)]}
\]

(5)

where \( s \) is our “generic” spin, subject only to hermiticity requirements (3).

To solve (4) we will use the String Hypothesis (SH) [13]:

\[
\lambda_l^0 = \lambda_n^0 + i[n + 1 - 2l + \frac{\pi}{2\gamma}(1 - v_s v_n)]
\]

(6)

where \( l = 1, ..., n \) and \( \lambda_n^0 \) is the real valued center of the string of length \( n \) and parity \( v_n = \pm 1 \). It can be proven that the allowed strings are determined by the Takahashi condition [10]:

\[
v_n \sin \gamma (n - l) \sin \gamma l > 0 \quad l = 1, 2, ..., n - 1
\]

(7)
whenever the hermiticity condition (2) holds true. Strictly speaking, SH is legitimate only if the number of BA roots is much smaller than the number of sites. However, it has been shown [14] that the SH can be safely used for the nonzero magnetic field or temperature.

Using the “Takahashi zone” terminology, we have for the allowed strings \((n_j, v_j)\):

\[
\begin{align*}
N \text{ even} : & \quad \left\{ \begin{array}{l}
0 - \text{zone} \quad n_j = j, v_{n_j} = +1 \quad 1 \leq j \leq \nu - 1 \\
1 - \text{zone} \quad n_\nu = 1, v_\nu = -1 \quad j = \nu
\end{array} \right.
\end{align*}
\]

\[
\begin{align*}
N \text{ odd} : & \quad \left\{ \begin{array}{l}
0 - \text{zone} \quad n_j = j, v_j = +1 \quad 1 \leq j \leq \nu - 1 \\
1 - \text{zone} \quad n_\nu = 1, v_\nu = -1 \quad j = \nu \\
2 - \text{zone} \quad n_{\nu+2} = \nu, \quad v_{\nu+2} = +1 \quad j = \nu + 2
\end{array} \right.
\end{align*}
\]

where \(\nu = \frac{N}{2}\) for \(N\) even and \(\nu = \frac{N-1}{2}\) for \(N\) odd.

In the thermodynamic limit equations (4) become:

\[
\tilde{a}_j = (-1)^r_j (\rho_j + \rho_j^h) + \sum_k T_{jk} \ast \rho_k
\]

where \(\rho_j(\rho_j^h)\) is the density of \(j\)-strings (\(j\)-holes) and \((-1)^r_j\) is the sign of \(\tilde{a}_j(\lambda)\) and “\(\ast\)” stands for convolution. The Fourier transforms of the functions which appear in (9) are given by:

\[
\hat{T}_{jk} = g(\omega; |n_j - n_k|, v_{n_j}v_{n_k}) + g(\omega; |n_j + n_k|, v_{n_j}v_{n_k})
\]

\[
\hat{\tilde{a}}_j = \sum_{l=0}^{n_j-1} g(\omega; 2s + 1 - n_j + 2l, v_s v_{n_j})
\]

\[
g(\omega; n; v) = -\frac{sh2p_0\omega((n \frac{p_0}{2} + 1 - v)}{shp_0\omega}; \quad ((x)) \text{ is Dedekind function}
\]

Following Yang and Yang [15] we minimize the free energy \(F = E - TS\) to obtain:

\[
\frac{F}{L} = -T \sum_j \int_{-\infty}^{+\infty} d\lambda |\tilde{a}_j(\lambda)| ln(1 + n_j^{-1})
\]

\[
lnn_j = -\frac{4p_0I}{T} \tilde{a}_j + \sum_k (-1)^r_k T_{jk} \ast ln(1 + n_k^{-1})
\]
\[ \eta_j = e^{\frac{\epsilon_j(\lambda)}{T}} = \frac{\rho_j^b(\lambda)}{\rho_j(\lambda)} \]

In the \( T = 0 \) limit we get the following results for the ground state and the spectrum of excitations (see table 1).

| \( N \) | \( I \) | Ground state strings | Positive energy strings | Zero energy strings |
|--------|--------|-----------------------|------------------------|-------------------|
| even   | > 0    | \( \nu - 1 \)         | \( \nu \)              | the rest          |
| even   | < 0    | \( \nu \)             | the rest               | none              |
| odd    | > 0    | \( \nu + 2 \)         | \( \nu, \nu + 1 \)     | the rest          |
| odd    | < 0    | \( \nu, \nu + 1 \)    | the rest               | none              |

Table 1.

The entries in the table above refer to the label \( j \) of the strings \((n_j, v_j)\).

We observe that this spectrum of the strings at zero temperature is independent of the value of the spin \( s \), as long as it belongs to the hermiticity regions (3). A comparison of the spectrum given above with that of ref. [11] shows that they are quite different. Interestingly enough, there is only one kind of string filling the Dirac sea (except for the case of \( N \) odd and \( I < 0 \)). This will be important when we discuss the conformal properties of our models.

The \( T \to \infty \) limit of equations (11) provides a justification of the SH. In fact, we get

\[ \lim_{T \to \infty} \frac{F_T}{L} = -\ln N', \]

which implies that the total number of states is correctly given by \((N')^L\).

Next we move on to compute entropy \( S \):

\[ \frac{S}{L} = \sum_j \int d\lambda \rho_j(\lambda) [(1 + \eta_j)\ln(1 + \eta_j) - \eta_j\ln \eta_j] \]  

(12)

Making use of equations (11), we obtain in low temperature limit:

\[ \frac{S}{L} = \begin{cases} 
\frac{2T \pi}{6\nu} [3 - \frac{6}{\nu + 2}]; & I > 0 \\
\frac{2T \pi}{6\nu} \frac{\nu}{\nu + 1}; & I < 0 
\end{cases} \]  

(13)

\[ \frac{S}{L} = \begin{cases} 
\frac{2T \pi}{6\nu} [3 - \frac{6}{\nu + 2}]; & I > 0 \\
\frac{4T}{\pi} \left[ \frac{1}{\nu} L(\frac{\nu}{2\nu + 1}) + \frac{1}{\nu} L(\frac{\nu + 1}{2\nu + 1}) \right]; & I < 0 
\end{cases} \]

where \( v^s, v^s_1 \) and \( v^s_2 \) are speeds of sound:

\[ v^s = \frac{N}{2} |I|, \quad v^s_1 = \frac{N |I|}{\nu - 1}, \quad v^s_2 = N |I| \]  

(14)
and \( L(x) \) is the dilogarithmic Roger’s function [16]. Notice that for \( N \) odd and \( I < 0 \) we have two different speeds of sound. For the remaining cases there is only one speed of sound so that the underlying CFT has a central extension \( c \) given by:

\[
\frac{\partial S}{\partial T} \equiv - \frac{\partial^2 F}{\partial T^2} = \frac{\pi c}{3v^s}
\]  

From equation (13) we get:

\[ I > 0, \quad c = \frac{3s_{\text{eff}}}{s_{\text{eff}}+1} \]
\[ I < 0, \quad c = 1 \quad \text{for} \ N \ \text{even} \]

where

\[ s_{\text{eff}} = \begin{cases} \frac{N-2}{4}, & N \ \text{even} \\ \frac{N-1}{4}, & N \ \text{odd} \end{cases} \]

When \( N \) is odd and \( I < 0 \) there are two different strings filling the ground state and two different speeds of sound. This fact indicates that rotational invariance is broken which, in turn, implies that we do not have a full conformal invariance. This situation has already been discussed in literature [17], where a broken CFT (in the sense given above) can be viewed as a sum of two independent CFT’s. In our case, we have not been able to identify any of the broken pieces with reasonable CFT.

Finally, we present our results for the magnetization of the ground state at \( T = 0 \) which is defined as:

\[
M = \frac{s^2}{L} = s - \sum_{j \in \text{Ground state}} n_j \int \rho_j(\lambda) d\lambda
\]

The results are collected in table 2:

| \( N \) | \( I \) | \( M \) |
|------|------|------|
| even | > 0  | \( M = \frac{N}{2}[s - \left(\frac{N}{2} - 1\right)^{3-v_s}] \) |
| even | < 0  | \( M = \frac{N}{2}\left(\frac{1-v_s}{3-v_s}\right) \) |
| odd  | > 0  | \( M = N\left[s - \left(\frac{N-1}{2}\right)^{3-v_s}\right] \) |
| odd  | < 0  | \( M = -N\left[s + 1 - \left(\frac{N+1}{2}\right)^{3-v_s}\right] \) |

Table 2.

From table 2, we see that for generic \( s \) (subject to hermiticity condition (3)) the ground state exhibits ferrimagnetic behavior. More precisely, when spin \( s \) takes on values different from integer or half-integer then magnetization is non null.

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To summarize: \( q \) being a root of unity made it possible to depart from regular representations and this, in turn, led to the new phenomenon of ferrimagnetism for a system governed by local (nearest neighbors interaction) Hamiltonian. In our future publications we hope to report on our study of magnetic properties of the model as well as on the further analysis of conformal properties and to present our study of scattering matrices along with quantum numbers of low lying excitations. The details of the results presented here will be given elsewhere [9].

Acknowledgments.
We are grateful to L. Nirenberg for the prompt typing of this manuscript.

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