ON FOURIER MULTIPLIERS IN FUNCTION SPACES WITH PARTIAL HÖLDER CONDITION AND THEIR APPLICATION TO THE LINEARIZED CAHN-HILLIARD EQUATION WITH DYNAMIC BOUNDARY CONDITIONS

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Abstract. We give relatively simple sufficient conditions on a Fourier multiplier so that it maps functions with the Hölder property with respect to a part of the variables to functions with the Hölder property with respect to all variables. By using these sufficient conditions we prove solvability in Hölder classes of the initial-boundary value problems for the linearized Cahn-Hilliard equation with dynamic boundary conditions of two types. In addition, Schauder’s estimates are derived for the solutions corresponding to the problem under study.

1. Introduction. The topic of Fourier multipliers in anisotropic Hölder spaces has been addressed in several papers, most of them dealing with Besov spaces. Note that (non-integer) Hölder spaces are a particular case of Besov spaces, see [1], [2], [12], [23], [36], [83]. All these papers contain various sufficient conditions for the boundedness of a Fourier multiplier in Besov spaces, including $B_{\infty,\infty}^s$. Our sufficient conditions are different from those in the pointed out papers. Without going into a deep comparison, we note that our sufficient conditions (and the conditions from [57]) are close to [36]. Moreover, the conditions from [36] are more precise. At the same time the paper [36] deals with isotropic case. Our paper, instead, deals with an anisotropic smoothness and with the smoothness with respect only to a part of the variables. In general, the author is not aware of any papers on multipliers in functional spaces with partial regularity.

In this paper we present simple theorems on Fourier multipliers, which we hope can be used to obtain Schauder’s estimates also for some other PDE. The motivation for our paper was the paper by O.A.Ladyzhenskaya [57]. (see also [58]). The original idea presented in [57] and [58] is taken from [46], Theorem 7.9.6. This idea is classical and it is based on the Littlewood-Paley decomposition. Paper [57] gives some simple sufficient conditions on a Fourier multiplier to provide bounded mapping with this multiplier in anisotropic Hölder spaces. These conditions can be easily verified in

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many particular problems for partial differential equations as it was demonstrated in [57] for the Stokes system.

Note that the sufficient conditions from [36], [57] can be relatively easy verified in the case when a multiplier is an anisotropic-homogeneous function of degree zero (or it is close to such a function in some sense). A fundamentally important fact in [57] is that the anisotropy of a Hölder space, where the action of a multiplier takes place, must coincide with the anisotropy of the multiplier itself (see Theorem 1.1 below).

Though the results of [36], [57], are applicable to a broad class of problems for partial differential equations (as it was pointed out in [57]), they are still not applicable to many problems, where the anisotropy of a Hölder space does not coincide with the anisotropy of a multiplier. They are not applicable also to situations, where one should consider a Hölder space of functions with the Hölder conditions with respect to only a part of independent variables. Among such problems we first mention “nonclassical” statements connected to the so called “Newton polygon” - see, for example, [20], [21], [22], [35], [42]. The book [22] contains detailed analysis of the symbols relevant to such problems with respect to its action in $L_p$ and Triebel-Lizorkin spaces.

A particular subclass of such problems is the class of problems for parabolic equations with highest derivatives in boundary conditions including the Wentzel conditions. This subclass includes also problems with dynamic boundary conditions- [19], [21], [26], [34], [37], [67], [69], [82]. As it is well known, such problems for parabolic and elliptic equations are not included in the standard general theory of parabolic boundary value problems - see, for example, [3], [47], [48], [63], [75].

In this paper we apply results on Fourier multipliers in spaces of functions with partial Hölder condition to the initial-boundary value problems for linearized Cahn-Hilliard equation with dynamic boundary conditions of two types. To our knowledge, such problems were previously under investigation only in spaces of functions with integrable derivatives - see, for example, [19], [21], [26], [34], [37], [67], [69], [82]. Note that in our case the anisotropy of a multiplier does not coincide with the anisotropy of the corresponding Hölder space. In this situation we consider smoothness of functions with respect to each variable separately. Besides, we do not use the notion of Newton’s polygon explicitly. We rather consider spaces of function with different smoothness inside the domain of their definition and on the boundary of the domain. In addition we use the framework of the anisotropic spaces $C^{1_1,\ldots,1_N}$ as they are defined in (4).

Note also that problems with dynamic boundary conditions arise as a linearization of many well-known free boundary problems such as the Stefan problem, the two-phase filtration problem for two compressible fluids (the parabolic version of the Muskat-Verigin problem), the Hele-Shaw problem, the classical evolutionary Muskat-Verigin problem for elliptic equations - see, for example, [3], [18], [27]-[29], [31]-[33], [45], [55], [64], [65], [67], [70]-[72], [80], [81], [84], [85]. In the latter two cases, we are dealing with problems in which the unknown function $u(x,t)$ satisfies a condition with the derivative with respect to time $u_t$. Analysis of these problems leads to consideration of the smoothness of certain potentials, in which the density satisfies the Hölder condition only in the variables $x$. This is consistent with the study of the respective multipliers in spaces of functions satisfying the Hölder
condition only in the variable \( x \). Note that this difficulty arises in any study of evolution problems associated with elliptic equations.

Besides, the studying of smoothness of solutions to some problems with respect to only a part of independent variables (including obtaining corresponding Schauder’s estimates) has it’s own history and it is an important direction of investigations. In particular, we deal with such situations when the semigroup of operators with parameter \( t > 0 \) and with a generator defined on some Hölder space - see, for example, \([24],[25],[30],[43],[44],[50]-[54],[59]-[63],[73],[77],[78]\). In all such cases we can use a theorem about multipliers in spaces of functions with smoothness with respect to only a part of independent variables. This permits to consider the smoothness with respect to each variable separately.

Let us introduce now some notation and formulate an assertion, which is a simple consequence of Theorem 2.1 and lemmas 2.1, 2.2 from \([57]\).

Let for a natural number \( N \)

\[ \gamma \in (0,1), \alpha = (\alpha_1, \alpha_2, \ldots, \alpha_N), \quad \alpha_1 = 1, \alpha_k \in (0,1], k = \overline{2,N}. \]  

Denote by \( C^{\gamma \alpha}(R^N) \) the space of continuous in \( R^N \) functions \( u(x) \) with the finite norm

\[ \|u\|_{C^{\gamma \alpha}(R^N)} \equiv |u|_{R^N}^{(\gamma \alpha)} = |u|_{R^N}^{(0)} + \sum_{i=1}^{N} (u)_{x_i,R^N}^{(\gamma \alpha_i)}, \]

where

\[ |u|_{R^N}^{(0)} = \sup_{x \in R^N} |u(x)|, \quad (u)_{x_i,R^N}^{(\gamma \alpha)} = \sup_{x \in R^N, h > 0} \frac{|u(x_1, \ldots, x_i + h, \ldots, x_N) - u(x)|}{h^{\gamma \alpha}}, \]

Along with the spaces \( C^{\gamma \alpha}(R^N) \) with the exponents \( \gamma \alpha_i < 1 \) we consider also spaces \( C^{\vec{\alpha}}(R^N) \), where \( \vec{l} = (l_1, l_2, \ldots, l_N), l_i \) are arbitrary positive non-integers. The norm in such spaces is defined by

\[ \|u\|_{C^{\vec{\alpha}}(R^N)} \equiv |u|_{R^N}^{(\vec{\alpha})} = |u|_{R^N}^{(0)} + \sum_{i=1}^{N} (u)_{x_i,R^N}^{(l_i)}, \]

where \( l_i \) is the integer part of the number \( l_i \), \( D_{x_i}^{[l_i]}u \) is the derivative of order \( [l_i] \) with respect to the variable \( x_i \) of a function \( u \). Seminorm (5) can be equivalently defined by \((38],[74],[79])\)

\[ (u)_{x_i,R^N}^{(l_i)} = \sup_{x \in R^N, h > 0} \frac{|D_{x_i}^{[l_i]}u(x_1, \ldots, x_i + h, \ldots, x_N) - D_{x_i}^{[l_i]}u(x)|}{h^{l_i}}, \]

where \( \Delta_{h,x_i}u(x) = u(x_1, \ldots, x_i + h, \ldots, x_N) - u(x) \) is the difference of a function \( u(x) \) with respect to the variable \( x_i \) and with the step \( h \), \( \Delta_{h,x_i}^k u(x) = \Delta_{h,x_i} \left( \Delta_{h,x_i}^{k-1} u(x) \right) = \left( \Delta_{h,x_i} \right)^k u(x) \) is the difference of power \( k \), \( k > l_i \) is arbitrary but fixed. In fact, functions from the space \( C^{\vec{\alpha}}(R^N) \) have also mixed derivatives up to definite orders and all derivatives are Hölder continuous with respect to all variables with some exponents.
in accordance with the ratios between the exponents \( l_i \). Namely, if \( \vec{k} = (k_1, ..., k_N) \) with nonnegative integers \( k_i, k_i \leq [l_i] \), and

\[
\omega = 1 - \sum_{i=1}^{N} \frac{k_i}{l_i} > 0,
\]

then

\[
D_x^\vec{\alpha} u(x) \in C^\vec{\alpha}(R^N), \quad \| D_x^\vec{\alpha} u(x) \|_{C^\vec{\alpha}(R^N)} \leq C \| u \|_{C^\vec{\alpha}(R^N)},
\]

(7)

where \( \vec{d} = (d_1, ..., d_N) \) and \( d_i = \omega l_i \).

Define further the space \( H^{\gamma_\alpha}(R^N) = C^{\gamma_\alpha}(R^N) \cap L_2(R^N) \) as the space of functions \( u(x) \) from \( C^{\gamma_\alpha}(R^N) \) with the finite norm

\[
\| u \|_{H^{\gamma_\alpha}(R^N)} \equiv \| u \|_{L_2(R^N)} + \sum_{i=1}^{N} (u^{(\gamma_\alpha)})_{x_i,R^N}.
\]

Analogously define the space \( \tilde{H}^{\vec{l}}(R^N) \) with arbitrary positive non-integer \( l_i \) with the norm

\[
\| u \|_{\tilde{H}^{\vec{l}}(R^N)} \equiv \| u \|_{L_2(R^N)} + \sum_{i=1}^{N} (u^{(l_i)})_{x_i,R^N}.
\]

(9)

It was shown in [57] that \( \| u \|^{(0)}_{R^N} \leq C \| u \|_{H^{\gamma_\alpha}(R^N)} \) and so

\[
\| u \|^{(\gamma_\alpha)}_{R^N} \leq C \| u \|_{H^{\gamma_\alpha}(R^N)},
\]

(10)

where here and below we denote by \( C, \nu \) all absolute constants or constants depending on fixed data only.

It is important to note here that the results of [57] were proved for considerably more narrow spaces than \( H^{\gamma_\alpha}(R^N) \) in our definition. In [57] the space \( H^{\gamma_\alpha}(R^N) \) means closer of functions from \( C^{\gamma_\alpha}(R^N) \) with compact supports in norm (8). But we will show below that pretty simple reasonings permit to extend the results of [57] to the whole spaces \( H^{\gamma_\alpha}(R^N) \) in our definition - see the end part of the proof of Theorem 2.1.

Let a function \( \tilde{m}(\xi), \xi \in R^N \), be defined in \( R^N \) and let it be measurable and bounded. Define the operator \( M : H^{\gamma_\alpha}(R^N) \to L_2(R^N) \) according to the formula

\[
Mu \equiv F^{-1}(\tilde{m}(\xi)\tilde{u}(\xi)),
\]

(11)

where for a function \( u(x) \in L_1(R^N) \),

\[
\tilde{u}(\xi) \equiv F(u) = \int_{R^N} e^{-ix\xi} u(x) dx
\]

(12)

is the Fourier transform of \( u(x) \) and we extend the Fourier transform on the space \( L_2(R^N) \). Denote by \( F^{-1}\tilde{u}(\xi) \) the inverse Fourier transform of the function \( \tilde{u}(\xi) \). Since \( u(x) \in L_2(R^N) \) and the function \( \tilde{m}(\xi) \) is bounded, the operator \( M \) is well defined. We call the function \( \tilde{m}(\xi) \) a Fourier multiplier.

Let the set of the variables \( (\xi_1, ..., \xi_N) = \xi \) is represented as a union of \( r \) subsets of length \( N_i \), \( i = 1, ..., r \) so that

\[
N_1 + ... + N_r = N, \quad \xi = (y_1, ..., y_r), \quad y_1 = (\xi_1, ..., \xi_{N_1}), ..., y_r = (\xi_{N_1} + ... + N_{r-1} + 1, ..., \xi_N).
\]

Let further \( \omega_i, i = 1, ..., r, \) are multi-indices of length \( N_i \)

\[
\omega_1 = (\omega_{1,1}, ..., \omega_{1,N_1}), ..., \omega_r = (\omega_{r,1}, ..., \omega_{r,N_r}), \quad \omega_{i,k} \in N \cup \{0\},
\]
and the symbol $D_y^{\omega_i} \tilde{u}(\xi)$ means a derivative of a function $\tilde{u}(\xi)$ of order $|\omega_i| = \omega_{i,1} + \ldots + \omega_{i,N_i}$ with respect to the group of variables $y_i = (\xi_{k_1}, \ldots, \xi_{k_{N_i}})$, that is $D_y^{\omega_i} \tilde{u}(\xi) = D_{\xi_{k_1}}^{\omega_{i,1}} \ldots D_{\xi_{k_{N_i}}}^{\omega_{i,N_i}} \tilde{u}(\xi)$. Let also $p \in (1, 2]$ and positive integers $s_i$, $i = 1, r$, satisfy the inequalities
\[ s_i > \frac{N_i}{p}, \quad i = 1, r. \] (13)

Denote for $\nu > 0$
\[ B_{\nu} = \{ \xi \in \mathbb{R}^N : \nu \leq |\xi| \leq \nu^{-1} \}. \]

Suppose that for some $\nu > 0$ the function $\tilde{m}(\xi)$ satisfies with some $\mu > 0$ uniformly in $\lambda > 0$ the condition
\[ \sum_{|\omega_i| \leq s_i} \left\| D_y^{\omega_i} D_{\xi_1}^{\omega_1} \ldots D_{\xi_N}^{\omega_N} \tilde{m}(\lambda \frac{\xi_1}{\nu}, \ldots, \lambda \frac{\xi_N}{\nu}) \right\|_{L_p(B_{\nu})} \leq \mu. \] (14)

**Theorem 1.1.** ( [57] : T.2.1, L.2.1, L.2.2, T.2.2, T.2.3) If the function $\tilde{m}(\xi)$ satisfies condition (14) then the operator $M$ defined in (11) is a linear bounded operator from the space $\mathcal{H}^{\gamma, \alpha}(\mathbb{R}^N)$ to itself and
\[ \|Mu\|_{\mathcal{H}^{\gamma, \alpha}(\mathbb{R}^N)} \leq C(N, \gamma, \alpha, p, \nu, s_i) \mu \|u\|_{\mathcal{H}^{\gamma, \alpha}(\mathbb{R}^N)}, \] (15)
\[ \sum_{i=1}^{N} \langle \psi, (\gamma \mu) \rangle_{x_i, R_N} \leq C(N, \gamma, \alpha, p, \nu, s_i) \mu \sum_{i=1}^{N} \langle \psi, (\gamma \mu) \rangle_{x_i, R_N}. \] (16)

Condition (14) can be easily verified very often in applications to differential equations. It is the case when the function $\tilde{m}(\xi)$ is anisotropic homogeneous of degree zero, that is when $\tilde{m}(\lambda \frac{\xi_1}{\nu}, \ldots, \lambda \frac{\xi_N}{\nu}) = \tilde{m}(\xi)$. Note also that condition (14) contains derivatives of the function $\tilde{m}(\xi)$ with respect to variables $y_i$ only up to the order $s_i$. The case $r = 1$, $N_1 = N$, $p = 2$ is contained in Lemma 2.1 in [57] and Lemma 2.2 in [57] contains the case $r = N$, $N_1 = 1$, $s_i = 1$. The general case completely analogous (see the proofs of lemmas 2.2-2.4 below).

Note that besides [1], [2], [12], [23], [36], [83] some other theorems about Fourier multipliers in H"older spaces can be found in [39]-41, [66], [76].

The further content of the paper is as follows. In Section 2, we prove a theorem about Fourier multipliers in spaces of functions with the H"older condition with respect to only a part of the independent variables. In this section we also give comparatively simple sufficient conditions for the theorem. As a conclusion of the section we demonstrate applications of the theorem about Fourier multipliers by two very simple but interesting in our opinion examples for the Laplace equation and for the heat equation. In section 3, we apply the results of Section 2 to an investigation of the model problems for the linearized Cahn-Hilliard equation with dynamic boundary conditions of two types. On the basis of these results, in section 4 we formulate solvability theorems and estimates for the solutions of boundary value problems for the linearized Cahn-Hilliard equation with dynamic boundary conditions. We only describe the scheme of the proof because it is identical to the scheme of [75].

**2. Theorems about Fourier multipliers in spaces of functions with H"older condition with respect to a part of variables.** In this section we prove a theorem about Fourier multipliers, which is a generalization of Theorem 2.1 from [57]. The scheme of the proof is a modification of the corresponding schemes from [46], [57].
Define an anisotropic “distance” \( \rho \) in space \( R^N \) between points \( x \) and \( y \) according to the formula

\[
\rho(x - y) = \sum_{k=1}^{N-2} |x_k - y_k| + |x_{N-1} - y_{N-1}|^\alpha + |x_N - y_N|^\beta, \quad \alpha \in (0, 1], \beta > 0. \tag{17}
\]

Let us stress that the exponent \( \beta > 0 \) is an arbitrary positive number. Choose a function \( \omega(\rho) : [0, +\infty) \to [0, 1] \) from the class \( C^\infty \) such that \( \omega \equiv 1 \) on the interval \([1/2, 2]\) and \( \omega \equiv 0 \) on the set \([0, 1/4] \cup [4, +\infty) \). Denote

\[
\chi : R^N \to [0, 1], \quad \chi(\xi) \equiv \omega(\rho(\xi)), \quad \xi \in R^N.
\]

Let a function \( \tilde{m}(\xi) \in C(R^N \setminus \{0\}) \) be bounded. For \( x \in R^N \) and for an integer \( j \in Z \) denote

\[
A_j x \equiv (2^j x_1, ..., 2^j x_{N-2}, 2^{\frac{j}{2}} x_{N-1}, 2^{\frac{j}{2}} x_N), \quad a_j = \det A_j = 2^{j(N-2) + \frac{j}{2} + \frac{j}{2}}. \tag{18}
\]

Denote further \( \tilde{m}_j(\xi) = \tilde{m}(\xi) \chi(A^{-1}_j \xi) \) and denote by \( m_j(x) \) the inverse Fourier transform of the function \( \tilde{m}_j(\xi) \). Denote also

\[
n_j(x) = a_j^{-1} m_j(A_j^{-1} x). \tag{19}
\]

For convenience we also denote for \( x \in R^N \) the variables \( x' = (x_1, ..., x_{N-2}, x_{N-1}) \), \( x'' = (x_1, ..., x_{N-2}) \). Let with some \( \mu > 0 \) the following conditions are satisfied

\[
\tilde{m}(\xi)|_{\xi'=0} = \tilde{m}(0, ..., 0, \xi_N) \equiv 0, \quad \xi_N \in R^1, \tag{20}
\]

\[
\int_{R^N} (1 + |x''|^\gamma + |x_{N-1}|^\alpha |x_1|)|n_j(x)|dx \leq \mu, \quad j \in Z. \tag{21}
\]

Let, finally, we have a function \( u(x) \in L_2(R^N) \) with the property

\[
\|u\|_{L_2(R^N)} + \|u\|_{x''', R_N} + \|u\|_{x_{N-1}', R_N} < \infty. \tag{22}
\]

Note that such a function may be unbounded - see the example before Theorem 2.7. Denote

\[
v(x) = m(x) * u(x) \equiv F^{-1}(\tilde{m}(\xi) \tilde{u}(\xi)).
\]

**Theorem 2.1.** Under conditions (20) - (22) the function \( v(x) \) satisfies Hölder conditions with respect to all variables and the following estimate is valid

\[
\|v\|_{x''', R_N} + \|v\|_{x_{N-1}', R_N} + \|v\|_{x_{N-1}, R_N} \leq C \mu \left( \|u\|_{x''', R_N} + \|u\|_{x_{N-1}', R_N} + \|u\|_{x_{N-1}, R_N} \right). \tag{23}
\]

**Proof.** We will try to retain the notation from [57], where it is possible.

Assume first that the function \( u(x) \) has compact support and belongs to the class \( C^\infty \) so that it’s Fourier transform is smooth and decays at infinity at any power rate.

Let \( \psi \in C^\infty((0, 0)), \) \( 0 \leq \psi \leq 1, \) \( \psi \equiv 1 \) on \([0, 1]\) and \( \psi \equiv 0 \) on \([2, \infty) \). Denote \( \varphi(\rho) = \psi(\rho) - \psi(2\rho) \) for \( \rho \in (0, \infty) \). The function \( \varphi(\rho) \) possess the properties:

\[
\varphi(\rho) \equiv 0 \text{ on } [0, 1/2] \text{ and } \varphi(\rho) \equiv 0 \text{ on } [2, \infty). \]

Denote further

\[
\varphi_j(\rho) = \varphi\left( \frac{\rho}{2^j} \right), \quad \varphi_j : [0, \infty) \to [0, 1], \quad j \in Z.
\]

This set of the functions satisfies by the definition

\[
\sum_{j=-\infty}^{\infty} \varphi_j(\rho) = 1, \quad \rho \in (0, \infty). \tag{24}
\]
Define functions \( \Phi \) and \( \bar{\Phi}_j : R^N \rightarrow [0, 1] \) according to the formulas (\( \rho \) is from (17))
\[
\Phi(x) = \rho \circ \rho(x) = \rho(\rho(x)),
\]
(25)
\[
\bar{\Phi}_j(x) = \varphi \circ \rho(x) = \varphi(\rho(x)) = \varphi \left( \frac{\rho(x)}{2^j} \right).
\]
(26)
\[
\varphi \left( \rho \left( \frac{\xi''_n}{2^j}, \frac{\xi_{N-1}}{2^{j/\alpha}}, \frac{\xi_N}{2^{j/\beta}} \right) \right) = \Phi \left( \frac{\xi''_n}{2^j}, \frac{\xi_{N-1}}{2^{j/\alpha}}, \frac{\xi_N}{2^{j/\beta}} \right).
\]
(27)
By the definition of the functions \( \bar{\Phi}_j \) and in view of (24)
\[
\sum_{j=-\infty}^{\infty} \bar{\Phi}_j(x) = 1, \quad \xi \in R^N \setminus \{0\}.
\]
(28)
We use this equality to represent the function \( \bar{u}(x) = F(u(x)) \) as
\[
\bar{u}(x) = \sum_{j=-\infty}^{\infty} \bar{u}_j(x), \quad \bar{u}_j(x) = \bar{u}(x) \bar{\Phi}_j(x), \quad \xi \in R^N \setminus \{0\}.
\]
(29)
Denote also \( \Phi(x) = F^{-1}(\Phi(x)), \quad \Phi_j(x) = F^{-1}(\bar{\Phi}_j(x)). \)
(30)
In view of (27)
\[
\Phi_j(x) = F^{-1}(\bar{\Phi}_j(x)) = \frac{1}{(2\pi)^N} \int_{R^N} \overline{e^{i\xi \cdot x}} \Phi \left( \frac{\xi''_n}{2^j}, \frac{\xi_{N-1}}{2^{j/\alpha}}, \frac{\xi_N}{2^{j/\beta}} \right) d\xi.
\]
Making in the last integral the change of the variables \( \xi'' = 2^j \eta'', \xi_{N-1} = 2^{j/\alpha} \eta_{N-1}, \xi_N = 2^{j/\beta} \eta_N, \) we obtain
\[
\Phi_j(x) = 2^{(N-2)j + \frac{j}{\alpha} + \frac{j}{\beta}} \Phi \left( 2^{j/\alpha} x_{N-1}, 2^{j/\beta} x_N \right) = a_j \Phi(A_j x).
\]
(31)
In view of the above definition of the function \( \chi_j(x) \) and in view of the definition of the functions \( \bar{\Phi}(x), \bar{\Phi}_j(x) \) we have for all \( \xi \in R^N \)
\[
\bar{\Phi}(x) = \bar{\Phi}(x) \chi_j(x), \quad \bar{\Phi}_j(x) = \bar{\Phi}_j(x) \chi \left( \frac{\xi''_n}{2^j}, \frac{\xi_{N-1}}{2^{j/\alpha}}, \frac{\xi_N}{2^{j/\beta}} \right).
\]
Consequently,
\[
\bar{u}_j(x) = \bar{u}(x) \bar{\Phi}_j(x) = \bar{u}_j(x) \chi \left( \frac{\xi''_n}{2^j}, \frac{\xi_{N-1}}{2^{j/\alpha}}, \frac{\xi_N}{2^{j/\beta}} \right) \equiv \bar{u}_j(x) \chi_j(x),
\]
(32)
where
\[
\chi_j(x) \equiv \chi \left( \frac{\xi''_n}{2^j}, \frac{\xi_{N-1}}{2^{j/\alpha}}, \frac{\xi_N}{2^{j/\beta}} \right).
\]
(33)
Denote in the sense of distributions
\[
v(x) = m(x) * u(x),
\]
that is
\[
\bar{v}(x) = \bar{m}(x) \bar{u}(x) = \sum_{j=-\infty}^{\infty} \bar{m}(x) \bar{u}_j(x) = \sum_{j=-\infty}^{\infty} \bar{m}_j(x) \bar{u}_j(x) \equiv \sum_{j=-\infty}^{\infty} \bar{v}_j(x),
\]
(35)
where
\[
\bar{m}_j(x) \equiv \bar{m}(x) \chi_j(x), \quad \bar{v}_j(x) = \bar{m}_j(x) \bar{u}_j(x).
\]
(36)
From (35) we have the equality

\[ v(x) = \sum_{-\infty}^{\infty} v_j(x). \] (37)

Since the function \( u(x) \in C^\infty(R^N) \cap L_2(R^N) \) and \( \tilde{m}(\xi) \) is bounded, the function \( v(x) \) in the left hand side of (37) belongs to the class \( L_2(R^N) \). The series in the right side of (37) converges to it in the sense of tempered distributions, although the functions \( v_j(x) \) are of class \( C^\infty \) because the functions \( \tilde{v}_j(\xi) \) have compact supports.

(In fact, the series in (28) converges pointwise and its terms are non-negative. Since the partial sums do not exceed 1, the series in (35), and hence the series in (37), converges in the space \( L_2(R^N) \). However, the convergence in the sense of the distributions is sufficient for us.)

Consider the function \( v_j(x) = F^{-1}(\tilde{v}_j(\xi)) \). Using its definition and (18), we represent this function in the form

\[ v_j(x) = u(x) * \Phi_j(x) * m_j(x) = \int_{R^N} u(x-y)dy \int_{R^N} m_j(y-z)\Phi_j(z)dz = \]

\[ = \int_{R^N} u(x-y)dy \int_{R^N} m_j(y-z)a_j \Phi(A_jz)dz. \] (38)

Making in the last integral the change of the variables \( k = A_jz, \ dk = a_jdz \), we obtain

\[ v_j(x) = \int_{R^N} u(x-y)dy \int_{R^N} m_j(y-A^{-1}_j k)\Phi(k)dk. \]

Making now the change of the variables \( y = A^{-1}_j z \), we arrive at the expression

\[ v_j(x) = \int_{R^N} u(x-A^{-1}_j z)dz \int_{R^N} [a^{-1}_j m_j(A^{-1}_j(z-k))] \Phi(k)dk \equiv \int_{R^N} u(x-A^{-1}_j z)\theta_j(z)dz, \] (39)

where

\[ \theta_j(z) = n_j(z) * \Phi(z) = \int_{R^N} n_j(z-k)\Phi(k)dk, \] (40)

and the function \( n_j(z) \) is defined in (19).

Calculate now the derivatives \( v_{x_i}, \ i = 1, 2, ..., N \). For this we use properties of a convolution and analogously to (38) represent \( v_j(x) \) as

\[ v_j(x) = \int_{R^N} a_j \Phi(A_j(x-y))dy \int_{R^N} u(y-z)m_j(z)dz. \]

Let first \( i = N - 1 \). Then

\[ (v_j(x))_{x_{N-1}} = 2^{\frac{N}{2}} \int_{R^N} a_j \Phi^{(i)}(A_j(x-y))dy \int_{R^N} u(y-z)m_j(z)dz, \]

where

\[ \Phi^{(i)}(z) = \frac{\partial \Phi}{\partial z_i}(z), \ i = 1, 2, ..., N. \] (41)
Using properties of a convolution and making a series of changes of variables with the matrices $A_j$ and $A_j^{-1}$, we obtain completely analogously to (39)

$$(v_j(x))_{z_{N-1}} = 2\pi \int_{\mathbb{R}^N} u(x - A_j^{-1}z)(\theta_j)_{z_{N-1}}(z)dz$$

(42)

where

$$(\theta_j)_{z_{N-1}}(z) = \int_{\mathbb{R}^N} n_j(z - k)\Phi_{k_{N-1}}(k)dk.$$  

Completely analogously, for $1 \leq i \leq N - 2$

$$(v_j(x))_{z_i} = 2^i \int_{\mathbb{R}^N} u(x - A_j^{-1}z)(\theta_j)_{z_i}(z)dz,$$

and for $i = N$

$$(v_j(x))_{z_N} = 2\pi \int_{\mathbb{R}^N} u(x - A_j^{-1}z)(\theta_j)_{z_N}(z)dz,$$

and also in more general case

$$D^k_{\mathbb{R}^N} (v_j(x)) = 2\pi \int_{\mathbb{R}^N} u(x - A_j^{-1}z)D^k_{\mathbb{R}^N} (\theta_j)_{z_N}(z)dz, \quad k = 0, 1, 2, \ldots.$$

Now note that for almost all $z_N$

$$f_j(z_N) \equiv \int_{\mathbb{R}^{N-1}} \theta_j(z', z_N)dz' = 0, \quad f_j^{(i)}(z_N) \equiv \int_{\mathbb{R}^{N-1}} (\theta_j)_{z_i}(z', z_N)dz' = 0,

\int_{\mathbb{R}^{N-1}} D^k_{\mathbb{R}^N} (\theta_j)(z', z_N)dz' \equiv 0.$$  

(44)

We show the first of these relations as the second and the third are completely similar. It suffices to show that the Fourier transform of $f_j(z_N)$ with respect to $z_N$ is identically equal to zero

$$F_N(f_j) = \overline{f}_j(\xi_N) = \int_{\mathbb{R}^1} e^{-iz\xi}dz_N \int_{\mathbb{R}^{N-1}} \theta_j(z', z_N)dz' =

= \int_{\mathbb{R}^{N-1}} dz' \int_{\mathbb{R}^1} e^{-iz\xi}\theta_j(z', z_N)dz_N = \int_{\mathbb{R}^{N-1}} F_N(\theta_j)(z', \xi_N)dz'.$$

Since the integral with respect to $z'$ of the function $F_N(\theta_j)(z', \xi_N)$ is equal to the value at $\xi' = 0$ of its Fourier transform with respect to the same variables $z'$,

$$\int_{\mathbb{R}^{N-1}} F_N(\theta_j)(z', \xi_N)dz' = \left[ \int_{\mathbb{R}^{N-1}} e^{-iz'\xi'} F_N(\theta_j)(z', \xi_N)dz' \right]_{\xi' = 0} = \overline{\theta}_j(\xi', \xi_N)_{\xi' = 0} = \overline{\theta}_j(0, \xi_N) \equiv 0.$$

The last identity follows from the fact that

$$\overline{m}_j(\xi) = \int_{\mathbb{R}^N} e^{-iz\xi}m_j(A_j^{-1}x)dx = \overline{m}_j(A_j\xi),$$  

(45)
and consequently
\[ \tilde{\theta}_j(\xi) = \tilde{m}_j(\xi)\tilde{\Phi}(\xi) = \tilde{m}_j(A_j\xi)\tilde{\Phi}(\xi) = \tilde{m}(A_j\xi)\chi_j(\xi)\tilde{\Phi}(\xi). \]

Therefore in view of (20) we have \( \tilde{\theta}_j(0,\xi_N) \equiv 0 \). Thus the first relation (41) is proved. The second and the third are similar.

Let us obtain now the estimates
\[
\begin{align*}
\int_{R^N} |z''| \gamma |\theta_j(z)| \, dz & \leq C\mu, \\
\int_{R^N} |z_{N-1}''| \alpha \gamma |\theta_j(z)| \, dz & \leq C\mu, \quad j \in Z, \quad (46) \\
\int_{R^N} |z''| \gamma \left|\frac{\partial}{\partial z_k}(\theta_j(z))\right| \, dz & \leq C\mu, \\
\int_{R^N} |z_{N-1}''| \alpha \gamma \left|\frac{\partial}{\partial z_k}(\theta_j(z))\right| \, dz & \leq C\mu, \quad j \in Z, k = 1, N, \quad (47)
\end{align*}
\]

where \( \mu \) is from condition (21). We obtain only the first inequality (46) because the rest is quite similar. Indeed, if \( y'' \in R^{N-2} \) then we use the inequality \( |z''| \gamma \leq C (|y''| \gamma + |z'' - y''| \gamma) \) and in view of the definition of \( \theta_j \) in (40) we obtain
\[
\begin{align*}
\int_{R^N} |z''| \gamma |\theta_j(z)| \, dz & \leq \int_{R^N} |z''| \gamma dz \int_{R^N} |n_j(y)||\Phi(z-y)| \, dy \\
& \leq C \int_{R^N} dz \int_{R^N} |y''| |n_j(y)| |\Phi(z-y)| \, dy + C \int_{R^N} dz \int_{R^N} |n_j(y)||z'' - y''| |\Phi(z-y)| \, dy \\
& \leq \int_{R^N} (1 + |y''| \gamma) |n_j(y)| \, dy \leq C\mu,
\end{align*}
\]
as it follows from properties of the function \( \Phi(x) \) and from (21).

Let us now estimate the Hölder constant of the function \( v(x) \). For this we estimate \( |v_j(x)| \) and \( |(v_j)_x(x)|. \) From (39) and (44) it follows that
\[
\begin{align*}
v_j(x) &= \int_{R^1} dz_N \int_{R^{N-1}} u(x - A_j^{-1}z)\theta_j(z)dz' = \\
&= \int_{R^N} dz_N \int_{R^{N-1}} \left[u(x - A_j^{-1}z) - u(x'',x_{N-1},x_N - \frac{z_N}{2j^\beta})\right]\theta_j(z)dz' = \\
&= \int_{R^N} \left[u(x'' - \frac{z''}{2j},x_{N-1} - \frac{z_{N-1}}{2j^{1/\alpha}},x_N - \frac{z_N}{2j^\beta}) - u(x'',x_{N-1},x_N - \frac{z_N}{2j^\beta})\right]\theta_j(z)dz.
\end{align*}
\]

From this, in view of the fact that \( u(x) \) satisfies the Hölder condition with respect to the first \( N - 1 \) variables and in view of estimate (46), it follows that
\[
|v_j(x)| \leq C2^{-\gamma j} \langle u \rangle^{\gamma,\alpha} \int_{R^N} (|z''| \gamma + |z_{N-1}''| \alpha \gamma) |\theta_j(z)| \, dz \leq C\mu \langle u \rangle^{\gamma,\alpha} 2^{-\gamma j}.
\]

(50)
And for \( i = 1, N - 2 \) we similarly obtain

\[
(v_j)_{x_i}(x) = 2^j \int_{R^N} \left[ u(x'' - \frac{z''}{2^j}, x_{N-1} - \frac{z_{N-1}}{2^j/\alpha}, x_N - \frac{z_N}{2^j/\beta}) - u(x'', x_{N-1}, x_N - \frac{z_N}{2^j/\beta}) \right] (\theta_j)_{x_i}(z) dz.
\]

Therefore we have similarly to (50)

\[
|(v_j)_{x_i}(x)| \leq C \mu \langle u \rangle \gamma, \alpha \gamma 2^j - \gamma j, \quad i = 1, N - 2.
\]

Likewise for \( i = N - 1 \) and for \( i = N \)

\[
|(v_j)_{x_{N-1}}(x)| \leq C \mu \langle u \rangle \gamma, \alpha \gamma 2^j - \gamma j, \quad j = 1, 2, ..., \quad (52)
\]

and more generally for \( i = N \)

\[
|D_x^{k}(v_j)(x)| \leq C \mu \langle u \rangle \gamma, \alpha \gamma 2^j - \gamma j, \quad k = 1, 2, ..., \quad (54)
\]

Let now \( x, z \in R^N \), \( z \) is fixed and let \( y = x + z \). Consider first the case when \( \beta \gamma < 1 \).

It follows from (37) that in the sense of distributions

\[
v(x) - v(x + z) = \sum_{j=-\infty}^{\infty} v_j(x) - \sum_{j=-\infty}^{\infty} v_j(x + z) = \sum_{j=-\infty}^{\infty} (v_j(x) - v_j(x + z)) = \sum_{j=-\infty}^{\infty} (v_j(x) - v_j(y)) \quad (55)
\]

and we emphasize that this equality is understood not pointwise for every \( x \in R^N \) but in the sense of distributions. Here \( v(x + z) \) is a shift of the distribution \( v(x) \) and the functions \((v_j(x) - v_j(x + z)) = (v_j(x) - v_j(y))\) are usual smooth functions, although they are summarized in (55) in the sense of distributions. We now show that the series in the right hand side of (55) converges not only in the sense of distributions but also uniformly. This would mean that the equation (55) can be understood pointwise for every \( x \in R^N \) and, moreover, will estimate the difference \(|v(x) - v(x + z)|\). Consider the sum (finite or infinite) \((y=x+z)\)

\[
S(x, z) = \sum_{j=-\infty}^{\infty} |v_j(x) - v_j(y)| = \sum_{j \geq n_0} |v_j(x) - v_j(y)| + \sum_{j \leq n_0} |v_j(x) - v_j(y)| \equiv S_1 + S_2, \quad (56)
\]

where \( n_0 = - \log_2 \rho(x - y) \). To estimate \( S_1 \) we use inequalities (50)

\[
S_1 \leq \sum_{j \geq n_0} 2^j |v_j|^{(0)} \leq C \mu \langle u \rangle \gamma, \alpha \gamma \sum_{j \geq n_0} 2^{-\gamma j} \leq C \mu \langle u \rangle \gamma, \alpha \gamma 2^{-n_0} \sum_{k=0}^{\infty} 2^{-\gamma k} \leq C \mu \langle u \rangle \gamma, \alpha \gamma \rho^{\gamma}(x - y). \quad (57)
\]

To estimate \( S_2 \) we use the mean value theorem for the difference \(|v_j(x) - v_j(y)|\) and estimates (51)-(53) for the corresponding derivatives

\[
S_2 \leq C \sum_{j \leq n_0} \left( \sum_{k=1}^{N} |x_k - y_k| (v_j)_{x_k}^{(0)} \right) \leq C \mu \langle u \rangle \gamma, \alpha \gamma \sum_{j \leq n_0} \left( |x'' - y''| 2^{j-\gamma} + |x_{N-1} - y_{N-1}| 2^{\frac{j}{\beta} - \gamma} + |x_N - y_N| 2^{\frac{j}{\alpha} - \gamma} \right) \leq \quad (58)
\]
From the estimates for where we also used estimate (54). We have

\[ C \text{main difficulty lies in the fact that the space of the author of this article shows that this procedure is not generally known. The} \]

Therefore, this issue is presented below in details.

The author is grateful to those reviewers of his papers who comment on the matter.

Thus by definition (6) the theorem is proved for smooth \( u(x) \) with compact support.

Let now \( \beta \gamma < 1 \). The proof in this case requires only a small change. First, selecting in the previous proof points \( x \) and \( y \) such that \( x_N = y_N \), that is considering the Hölder property of the function \( v(x) \) only with respect to the variables \( x' \), we obtain estimate (58) with \( |x_N - y_N| = 0 \). This proves (59) for such \( x \) and \( y \) and hence this gives the desired smoothness of \( v(x) \) with respect to the variables \( x' \). Now the smoothness property of this function in the variable \( x_N \) should be considered separately. For this purpose, with definition (6) in mind, we need to consider \( k \)-th difference in variable \( x_N \) of the function \( v(x) \). Let \( k \) be a sufficiently large positive integer such that \( k/\beta > \gamma, h > 0 \). Similarly to the previous

\[ |\Delta_{h,x_N}^k v(x)| \leq \sum_{j=-\infty}^\infty |\Delta_{h,x_N}^k v_j(x)| = \sum_{j \geq n_0} |\Delta_{h,x_N}^k v_j(x)| + \sum_{j \leq n_0} |\Delta_{h,x_N}^k v_j(x)| \equiv S_1 + S_2, \]

where \( n_0 = -\log_2 h^\beta \). The sum \( S_1 \) is estimated in exactly the same way as above. This gives

\[ S_1 \leq C \mu \langle u \rangle_{x'}^{(\gamma,\alpha \gamma)} h^{\beta \gamma}. \]

The sum \( S_2 \) is also evaluated as before taking into account the fact that

\[ |\Delta_{h,x_N}^k v_j(x)| \leq C k \left| D_{x_N}^k v_j(x) \right|^{(0)} \leq C \mu \langle u \rangle_{x'}^{(\gamma,\alpha \gamma)} 2^{\frac{\beta}{\beta - \gamma}}. \]

From the estimates for \( S_1 \) and \( S_2 \) it follows that

\[ |\Delta_{h,x_N}^k v(x)| \leq C \mu \langle u \rangle_{x'}^{(\gamma,\alpha \gamma)} h^{\beta \gamma}. \]

Thus by definition (6) the theorem is proved for smooth \( u(x) \) with compact support.

Free ourselves now from the condition of compactness of the support of \( u(x) \) and from the condition \( u(x) \in C^\infty \). This procedure is quite simple. But the experience of the author of this article shows that this procedure is not generally known. The main difficulty lies in the fact that the space \( C^\infty \) is not dense in Hölder spaces. The author is grateful to those reviewers of his papers who comment on the matter. Therefore, this issue is presented below in details.
Free first from the condition for the compactness of the support of \( u(x) \). So let

\[
u(x) \in C^\infty(N) \cap L_2(R^N), \quad \max |u(x)| \leq u_0 < \infty. \tag{60}
\]

Let \( \eta(x) \in C^\infty, \eta(x) \equiv 1 \) for \( |x| \leq 1 \) and \( \eta(x) \equiv 0 \) for \( |x| \geq 2 \). For \( R > 1 \) denote

\[
u_R(x) = u(x)\eta(x/R).\]

The functions \( \nu_R(x) \) have the following properties:

\[
u_R(x) \in C^\infty(R^N); \quad ||u - u_R||_{L_2(R^N)} \to 0, R \to \infty; \quad \max |u_R(x)| \leq C_\eta u_0, \tag{61}
\]

\[
\langle \nu_R \rangle(x^{\gamma,\alpha}) \leq \langle u \rangle(x^{\gamma,\alpha}) \max |\eta(x/R)| + \langle \eta \rangle(x^{\gamma,\alpha}) \max |u| \leq C_\eta \langle u \rangle(x^{\gamma,\alpha}) + C_\eta u_0 R^{-\gamma\alpha}. \tag{62}
\]

Denote

\[
\nu_R(x) = F^{-1}(\tilde{m}(\xi)\tilde{u}_R(\xi)) = m(x) * \nu_R(x).
\]

From the boundedness of \( \tilde{m}(\xi) \) and from (61) it follows that

\[
||u - \nu_R||_{L_2(R^N)} \to 0, R \to \infty. \tag{63}
\]

Besides, since \( \nu_R(x) \) has compact support and \( \nu_R(x) \in C^\infty \), by the above proof we have

\[
\langle \nu_R \rangle(x^{\gamma,\alpha,\beta}) \leq C \langle u \rangle(x^{\gamma,\alpha,\beta}) \leq C \langle u \rangle(x^{\gamma,\alpha}) + C_\eta u_0 R^{-\gamma\alpha}. \tag{64}
\]

From the last estimate, (63), (10), and from the Arzela theorem it follows that \( v_R \to v \) uniformly on compact sets. Let \( h > 0 \) be fixed. From (64) we have for any fixed \( x \in R^N \)

\[
\frac{|v_R(x'',x_{N-1}+h,x_N) - v_R(x'',x_{N-1},x_N)|}{h^{\gamma\alpha}} \leq C_\eta \langle u \rangle(x^{\gamma,\alpha}) + C_\eta u_0 R^{-\gamma\alpha}. \tag{65}
\]

Passing here to the limit as \( R \to \infty \), we get

\[
\frac{|v(x'',x_{N-1}+h,x_N) - v(x'',x_{N-1},x_N)|}{h^{\gamma\alpha}} \leq C_\eta \langle u \rangle(x^{\gamma,\alpha}).
\]

Since \( h \) and \( x \) are arbitrary, this means that

\[
\langle v \rangle(x^{\alpha\gamma},R^N) \leq C_\eta \langle u \rangle(x^{\gamma,\alpha}). \tag{66}
\]

Now similar reasonings with shifts in other variables of the functions \( \nu_R(x) \) give eventually

\[
\langle \nu_R \rangle(x^{\gamma,\alpha,\beta},x_{N-1,x_N}) \leq C \langle u \rangle(x^{\gamma,\alpha}), \tag{67}
\]

that is the theorem is proved under conditions (60).

Free at last from the condition \( u(x) \in C^\infty \) and from the boundedness of \( u(x) \). Let now \( u(x) \) satisfies only conditions (22). Denote by \( \omega_\delta(x) \) some smoothing kernel with the parameter \( \delta > 0 \) and with the support of dimension \( C\delta \). Denote \( u_\delta(x) = u(x) * \omega_\delta \). We have the following well-known properties of such convolution

\[
\langle u_\delta \rangle(x^{\gamma,\alpha},x_{N-1},x_N) \leq C \langle u \rangle(x^{\gamma,\alpha}), \quad ||u_\delta - u||_{L_2(R^N)} \to 0, \delta \to 0, \quad |u_\delta(x)| \leq C_\delta. \tag{68}
\]

and in particular \( u_\delta \) satisfies (60). Hence, by the above proof,

\[
\langle u_\delta \rangle(x^{\gamma,\alpha,\beta},x_{N-1},x_N) \leq C \langle u \rangle(x^{\gamma,\alpha,\beta}) \leq C \langle u \rangle(x^{\gamma,\alpha}), \tag{69}
\]

where \( u_\delta = m * u_\delta \) and

\[
\langle u \rangle(x^{\gamma,\alpha,\beta},x_{N-1},x_N) \equiv \langle u \rangle(x^{\gamma},R^N) + \langle u \rangle(x^{\alpha\gamma},x_{N-1},R^N) + \langle u \rangle(x^{\beta\gamma},x_N,R^N). \tag{70}
\]

Besides, from the boundedness of \( \tilde{m}(\xi) \), the definition of \( v_\delta \), and from the second relation in (66) it follows that (\( v = m * u \))

\[
||v_\delta - v||_{L_2(R^N)} \leq C||u_\delta - u||_{L_2(R^N)} \to 0, \delta \to 0. \tag{71}
\]
From (67), (68), (10), and from the Arzela theorem it follows now that the sequence \( v_\delta \) converges (at least for a subsequence) uniformly on compact sets \( v_\delta \rightarrow v, \delta \rightarrow 0 \).

Let now, as above, \( h > 0 \) be fixed. Taking into account (67), consider as above the relation
\[
|v_\delta(x''', x_{N-1} + h, x_N) - v_\delta(x''', x_{N-1}, x_N)| \leq (v_\delta(x''', x_{N-1}, x_N) \leq C(u(x''', x_{N-1}), \gamma, \alpha). \]

Now fixing in this relation a point \( x \in \mathbb{R}^N \) and passing to the limit as \( \delta \rightarrow 0 \), we obtain in view of the uniform convergence
\[
|v(x''', x_{N-1} + h, x_N) - v(x''', x_{N-1}, x_N)| \leq C(u(x''', x_{N-1}), \gamma, \alpha), \]
which means
\[
(v(x'''))_{x_{N-1}} \leq C(u(x''', x_{N-1}), \gamma, \alpha). \]

Similar arguments with respect to other variables of the function \( v(x) \) complete the proof of the theorem. \( \square \)

Following the idea of [57] and similar to the conditions of Theorem 1.1, we give simple sufficient conditions on \( \tilde{m}(\xi) \) to have condition (21). Note first that after the change of the variables \( y = A_j^{-1} x \) we obtain
\[
\tilde{m}_j(\xi) = \int_{\mathbb{R}^N} e^{ix\xi} a_j^{-1} m_j(A_j^{-1} x) dx = \int_{\mathbb{R}^N} e^{i(y,A_j)\xi} m_j(y) dy = \tilde{m}_j(A_j\xi) = \tilde{m}(A_j\xi) \chi(\xi). \quad (69)
\]
Denote for \( \lambda > 0 \) \( A_j \xi = (\lambda \xi''', \lambda^{\frac{3}{2}} \xi_{N-1}, \lambda^{\frac{1}{2}} \xi_N) \) and denote \( B_0 = \{ \xi \in \mathbb{R}^N : 1/8 \leq \rho(\xi) \leq 8 \} \). All sufficient conditions to have (21), which we state below, are linked with the property of the Fourier transform
\[-ix_k f(x) = \tilde{f}_k(\xi), \]
as well as with the well-known Hausdorff-Young inequality
\[
\|f(x)\|_{L_{p'}(\mathbb{R}^N)} \leq C_{N,p} \|\tilde{f}(\xi)\|_{L_p(\mathbb{R}^N)}, \quad p \in (1, 2], \quad p' = \frac{p}{p-1}. \quad (70)
\]

**Lemma 2.2.** Let uniformly in \( \lambda > 0 \)
\[
\tilde{m}(A_j \xi) \in W^s_p(B_0), \quad p \in (1, 2], \quad s > \frac{N}{p} + \gamma.
\]
Then conditions (21) are satisfied and
\[
\mu \leq \sup_\lambda C \|\tilde{m}(A_j \xi)\|_{W^s_p(B_0)}. \quad (71)
\]

**Proof.** (compare [57]).
In view of (69) for \( r > N/p \)
\[
\int_{\mathbb{R}^N} (1 + |x''|^\gamma + |x_{N-1}|^\alpha) |n_j(x)| dx \leq C \int_{\mathbb{R}^N} (1 + x^2)^\frac{\gamma x}{2} |n_j(x)| (1 + x^2)^{-\frac{\gamma x}{2}} dx \leq \]
\[
\leq C \left( \int_{\mathbb{R}^N} (1 + x^2)^\frac{\gamma x}{2} |n_j(x)| \right)^{\frac{1}{p'}} dx \left( \int_{\mathbb{R}^N} (1 + x^2)^{-\frac{2p}{2}} dx \right)^{\frac{1}{p}} \leq
\]
\[ \leq C \left[ \int_{\mathbb{R}^N} \left( \sum_{|\omega| = 0}^{\gamma+r} |D_{\xi}^\omega \bar{n}_j(\xi)|^p \right) d\xi \right]^{\frac{1}{p}} \leq C \| \bar{m}(A_j \xi) \|_{W^{r+r}(B_0)}. \]

The lemma follows. \[ \square \]

The same idea that was used in the proof of Lemma 2.2 can be used by groups of the variables. That is, for example, we obtain with \( r > (N - 1)/p \)

\[ \int_{\mathbb{R}^N} (1 + |x'|^{\gamma} + |x_{N-1}|^{\alpha}) |n_j(x)| \, dx \leq C \int_{\mathbb{R}^N} (1 + (x')^2)^{\frac{r}{2}} |n_j(x)| \, dx \]

\[ = C \int_{\mathbb{R}^N} (1 + (x')^2)^{\frac{r}{2}} (1 + i x_N) |n_j(x)| \left[ (1 + (x')^2)^{-\frac{r}{2}} (1 + i x_N)^{-1} \right] \, dx \]

\[ \leq C \left\{ \int_{\mathbb{R}^N} \left[ (1 + (x')^2)^{\frac{r}{2}} (1 + i x_N) |n_j(x)| \right]^{\frac{p}{r'}} \, dx \right\}^{\frac{1}{p'}} \left\{ \int_{\mathbb{R}^N} (1 + (x')^2)^{-\frac{r}{2}} (1 + i x_N)^{-p} \, dx \right\}^{\frac{1}{p}} \]

\[ \leq C \sum_{|\omega'| \leq \gamma + r, \omega_N \in \{0,1\}} \left\| D_{\xi}^{(\omega', \omega_N)} \bar{m}(A_j \xi) \right\|_{L_p(B_0)}, \]

where the sum in the last expression is considered in all multi-indices \( \omega = (\omega', \omega_N) \) such that \( |\omega'| \leq \gamma + r \) with \( r > (N - 1)/p \) and \( \omega_N \in \{0,1\} \).

Thus the following lemma holds.

**Lemma 2.3.** Let for any \( \lambda > 0 \) and for some \( p \in (1,2] \) with \( s > (N - 1)/p \) we have

\[ M_1 \equiv \sup_{\lambda > 0} \sum_{|\omega'| \leq \gamma + s, \omega_N \in \{0,1\}} \left\| D_{\xi}^{(\omega', \omega_N)} \bar{m}(A_j \xi) \right\|_{L_p(B_0)} < \infty. \]

Then condition (21) is satisfied and \( \mu \leq CM_1. \)

Formulate for example yet another assertion, which can be proved exactly the same way as the previous two lemmas taking into account that \( (1 + |x'|^{\gamma} + |x_{N-1}|^{\alpha}) \)

\[ \leq \prod_{k=1}^{N-1} (1 + |ix_k|) \] and using the multiplication and division by \( \prod_{k=1}^{N-1} (1 + ix_k) \).

**Lemma 2.4.** Suppose that for some \( p \in (1,2] \) the following condition is satisfied

\[ M_2 \equiv \sup_{\lambda > 0} \sum_{\omega} \left\| D_{\xi}^{(\alpha, \omega)} \bar{m}(A_j \xi) \right\|_{L_p(B_0)} < \infty, \]

where the sum is taken over all multi-indices \( \omega = (\omega_1, ..., \omega_{N-1}, \omega_N) \) such that \( \omega_1, ..., \omega_{N-1} \in \{0,1\}, \omega_N \in \{0,1\} \).

Then condition (21) is satisfied and \( \mu \leq CM_2. \)

The fact that the multiplier \( \bar{m}(\xi) \) in Theorem 2.1 uses the smoothness of the density \( u(x) \) for all variable \( x' \) except for one variable \( x_N \) is insignificant and was considered only for the simplicity. Directly from the proof of Theorem 2.1 it follows that completely analogous to this proof the following assertion can be proved.

Let a function \( \bar{m}(\xi) \in C(\mathbb{R}^N \setminus \{0\}) \) be bounded. Let \( x \in \mathbb{R}^N \), let \( K \in (0,N) \) be an integer, \( x = (x^{(1)}, x^{(2)}), x^{(1)} = (x_1, ..., x_K), x^{(2)} = (x_{K+1}, ..., x_N) \) and similarly \( \xi = (\xi^{(1)}, \xi^{(2)}), \xi^{(1)} = (\xi_1, ..., \xi_K), \xi^{(2)} = (\xi_{K+1}, ..., \xi_N) \). Let \( \alpha = (\alpha_1, ..., \alpha_K), \beta = (\beta_K, ..., \beta_N), \alpha_k \in (0,1], \beta_k > 0 \), and \( \gamma \in (0,1). \)
Denote for \( x \in \mathbb{R}^N \) and for an integer \( j \in \mathbb{Z} \)
\[
A_j x = (2^{\frac{j}{m}} x_1, ..., 2^{\frac{j}{m}} x_K, 2^{\frac{j}{m}+1} x_{K+1}, 2^{\frac{j}{m}} x_N), \quad a_j = \det A_j.
\] Denote as above \( \tilde{m}_j(\xi) = \tilde{m}(\xi) \chi(A_j^{-1} \xi), \) and let \( m_j(x) \) be the inverse Fourier transform of the function \( \tilde{m}_j(\xi), \)
\[
n_j(x) = a_j^{-1} m_j(A_j^{-1} x).
\] Let with some \( \mu > 0 \) the following conditions are satisfied
\[
\tilde{m}(\xi)|_{\xi^{(1)}=0} = \tilde{m}(0, \xi^{(2)}) \equiv 0, \quad \xi^{(2)} \in \mathbb{R}^{N-K},
\]
\[
\int_{\mathbb{R}^N} (1 + \sum_{k=1}^K |x_k|^{\alpha_k}) |n_j(x)| dx \leq \mu, \quad j \in \mathbb{Z}.
\] Suppose finally that a function \( u(x) \in L_2(\mathbb{R}^N) \) and satisfies the Hölder condition with respect to a part of the variables
\[
\langle u \rangle^{(\alpha\gamma)}_{x^{(1)}, \mathbb{R}^N} = \sum_{k=1}^K \langle u \rangle^{(\alpha_k\gamma)}_{x_k, \mathbb{R}^N} < \infty.
\] Denote as above
\[
v(x) \equiv Mu \equiv m(x) * u(x) \equiv F^{-1}(\tilde{m}(\xi)\tilde{u}(\xi)).
\]

**Theorem 2.5.** Let conditions (74), (75) are satisfied. Then the function \( v(x) \) satisfies the Hölder condition with respect to all variables and
\[
\langle v \rangle^{(\alpha\gamma,\beta\gamma)}_{x^{(1)}, x^{(2)}, \mathbb{R}^N} \leq C_\mu \langle u \rangle^{(\alpha\gamma)}_{x^{(1)}, \mathbb{R}^N},
\]
where
\[
\langle v \rangle^{(\alpha\gamma,\beta\gamma)}_{x^{(1)}, x^{(2)}, \mathbb{R}^N} = \sum_{k=1}^K \langle u \rangle^{(\alpha_k\gamma)}_{x_k, \mathbb{R}^N} + \sum_{k=K+1}^N \langle u \rangle^{(\beta_k\gamma)}_{x_k, \mathbb{R}^N}.
\]
Completely analogous to the proof of Lemmas 2.2, 2.3 a sufficient condition for inequalities (75) can be obtained. Similarly to Lemma 2.2 we have the following assertion.

For \( \lambda > 0 \) denote \( A_\lambda \xi = (\lambda^{\frac{1}{m}} \xi_1, ..., \lambda^{\frac{1}{m}} \xi_K, \lambda^{\frac{1}{m}+1} \xi_{K+1}, ..., \lambda^{\frac{1}{m}} \xi_N) \) and denote \( B_0 = \{ \xi \in \mathbb{R}^N : 1/8 \leq \rho(\xi) \leq 8 \} \), where \( \rho(\xi) = \sum_{k=1}^K |\xi_k|^{\alpha_k} + \sum_{k=K+1}^N |\xi_k|^{\beta_k}. \)

**Lemma 2.6.** Let uniformly in \( \lambda > 0 \)
\[
\tilde{m}(A_\lambda \xi) \in W_p^s(B_0), \quad p \in (1,2], \quad s > \frac{N}{p} + \gamma.
\]
Then conditions (75) are satisfied and
\[
\mu \leq \sup_{\lambda} C \| \tilde{m}(A_\lambda \xi) \|_{W_p^s(B_0)}.
\]

We also have a more general assertion similar to Theorem 1.1. Denote the spaces
\[
\mathcal{H}_{x^{(1)}}^{\alpha\gamma}(\mathbb{R}^N) \supset C_{x^{(1)}}^{\alpha\gamma}(\mathbb{R}^N) \cap L_2(\mathbb{R}^N), \quad \mathcal{H}_{x^{(1)}, x^{(2)}}^{\alpha\gamma,\beta\gamma}(\mathbb{R}^N) = C_{x^{(1)}, x^{(2)}}^{\alpha\gamma,\beta\gamma}(\mathbb{R}^N) \cap L_2(\mathbb{R}^N),
\]
with the norms
\[
\| u \|_{\mathcal{H}_{x^{(1)}}^{\alpha\gamma}(\mathbb{R}^N)} \equiv \| u \|_{L_2(\mathbb{R}^N)} + \langle u \rangle^{(\alpha\gamma)}_{x^{(1)}, \mathbb{R}^N},
\]
\[
\| u \|_{\mathcal{H}_{x^{(1)}, x^{(2)}}^{\alpha\gamma,\beta\gamma}(\mathbb{R}^N)} \equiv \| u \|_{L_2(\mathbb{R}^N)} + \langle u \rangle^{(\alpha\gamma,\beta\gamma)}_{x^{(1)}, x^{(2)}, \mathbb{R}^N}.
\]
It is worth noting that the space $H_{x_1}^{\alpha,\gamma}$ with partial regularity contains also unbounded functions because the relation (10) is not valid for this space. As an example of such a function can serve the function of two variables ($\alpha=1$)

$$u(x^{(1)}, x^{(2)}) = |x^{(2)}| - \delta e^{-|x^{(1)}|} |x^{(2)}|^{\delta}$$

with $\delta \in (0, 1), 2\delta + \delta/\gamma < 1$. Analogously to Lemmas 2.2, 2.3 we have the following assertion.

**Theorem 2.7.** Let condition (74) be satisfied. Let further in the notation of Theorem 1.1 instead of condition (13) the following condition be satisfied

$$s_i > \frac{N_i}{p} + \gamma, \quad i = 1, r, \quad p \in (1, 2].$$

Let also similar to (14) the following condition be satisfied

$$\sup_{\lambda > 0} \sum_{|\omega_i| \leq s_i} \| D_{y_1}^{\omega_1} D_{y_2}^{\omega_2} \ldots D_{y_r}^{\omega_r} \tilde{m}(A\lambda \xi) \|_{L_p(B_\nu)} \leq \mu.$$  

Then the operator $M$ is a bounded linear operator from $H_{x_1}^{\alpha,\gamma}$ ($R^N$) to $H_{x_1}^{\alpha,\beta,\gamma}$ ($R^N$) and

$$\|Mu\|_{H_{x_1}^{\alpha,\beta,\gamma}} (R^N) \leq C \mu \|u\|_{H_{x_1}^{\alpha,\gamma}} (R^N),$$

$$\langle Mu \rangle_{H_{x_1}^{\alpha,\beta,\gamma}, R^N} \leq C \mu \langle u \rangle_{H_{x_1}^{\alpha,\gamma}, R^N},$$

In the following section, we will demonstrate an application of the proven statements about multipliers to some initial-boundary value problems for the linearized Cahn-Hilliard equation with dynamic boundary conditions of two types. Here we give simple examples of applications of Theorems 2.1, 2.7.

**Example 1.** Let a function $u(x)$ has compact support in $R^N$ and satisfies the Poisson equation

$$\Delta u(x) = f(x),$$

where a function $f(x)$ has compact support in $R^N$ and satisfies the Hölder condition for some single variable, for example, $x_1$ with an exponent $\gamma \in (0, 1)$

$$\langle f \rangle_{x_1}^{(\gamma)} = \sup_{h > 0} \frac{|f(x + h \xi) - f(x)|}{h^{\gamma}} < \infty.$$  

Consider all the second derivatives of $u(x)$ containing the derivative with respect to $x_1$. It is well known that in terms of Fourier transform we have the equality

$$\tilde{\partial^2 u}/\partial x_1^2 (\xi) = C \frac{\xi_k \xi_1}{\xi_1^2} \tilde{f}(\xi), \quad k = 1, N.$$  

Since the function $\tilde{m}(\xi) = \frac{\xi_k \xi_1}{\xi_1^2}$ has the property $\tilde{m}(\lambda \xi) = \tilde{m}(\xi)$ for any positive $\lambda$ and is smooth away from the origin, it is easy to verify the conditions of Theorem 2.7. Consequently,

$$\langle \tilde{\partial^2 u}/\partial x_1^2 \rangle_{x_1}^{(\gamma)} \leq C \langle f \rangle_{x_1}^{(\gamma)}, \quad k = 1, N,$$

where the Hölder constant in the left hand side of this inequality is taken with respect to all variables, not only with respect to $x_1$. 
Note that in estimate (83) only the second derivatives containing the derivative with respect to \( x_1 \) are present. This fact is essential as the following example shows. Let \( \eta(x) \in C_0^\infty(R^3) \). Consider the function
\[
u_1(x) = u_1(x_1, x_2, x_3) = (x_2^2 - x_3^2 + x_2 x_3) \ln(x_2^2 + x_3^2) \eta(x_1, x_2, x_3).
\]
It is immediately verified that the function with compact support \( u_1(x) \) satisfies equation (81) with right-hand side satisfying (82). However, it’s second derivatives that do not contain the derivative with respect to \( x_1 \) do not satisfy Hölder condition and they are unbounded in a neighborhood of the origin.

This example shows that condition (20) on a multiplier can not be generally dropped. Although the author does not known how close it is to the sharp condition.

**Example 2.** It is interesting, in our opinion, to consider the following simple example for a parabolic equation. Let a function \( u(x, t) \) with compact support in \( R^N \times R^1 \) satisfies the heat equation
\[
\frac{\partial u}{\partial t} - \Delta u = f(x, t),
\]
where the right hand side \( f(x, t) \) with compact support satisfies the Hölder condition with respect to the variable \( t \) only with an exponent greater than 1/2, that is
\[
\langle f(x, t) \rangle_{\gamma, t, R^N \times R^1} < \infty, \quad \gamma \in \left(\frac{1}{2}, 1\right).
\]
Making in (84) the Fourier transform and denoting the variable of the Fourier transform with respect to \( t \) by \( \xi_0 \), we obtain
\[
\frac{\tilde{\partial u}}{\partial t} = C \frac{i \xi_0}{i \xi_0 + \xi^2} \tilde{f}(\xi, \xi_0).
\]
Then it follows from Theorem 2.7 that
\[
\left\langle \frac{\partial u}{\partial t} \right\rangle_{\gamma, t, R^N \times R^1} + \left\langle \frac{\partial u}{\partial t} \right\rangle_{2\gamma, x, R^N \times R^1} \leq C \langle f(x, t) \rangle_{\gamma, t, R^N \times R^1}.
\]
In particular, since \( 2\gamma \in (1, 2) \), the derivative \( \frac{\partial u}{\partial t} \) has derivatives with respect to \( x_i \)
\[
\sum_{i=1}^N \left\langle \frac{\partial^2 u}{\partial x_i \partial x_i} \right\rangle_{2\gamma - 1, x, R^N \times R^1} \leq C \langle f(x, t) \rangle_{\gamma, t, R^N \times R^1}.
\]
Note again that, as in the previous example, the second derivatives with respect to the variables \( x_i \) can be unbounded in general, for example, \( u(x_1, x_2, t) = (x_1^2 - x_2^2 + x_2 x_1) \ln(x_1^2 + x_2^2) \eta(x_1, x_2) \psi(t) \), \( \eta \in C_0^\infty(R^2) \), \( \psi \in C_0^\infty(R^1) \).

**Remark 1.** Instead of conditions (74) in Theorem 2.7 on the multiplier \( \tilde{m}(\xi) = \tilde{m}(\xi^{(1)}, \xi^{(2)}) \) one can impose a more general condition. Namely, we can require that the function \( \tilde{m}(\xi)|_{\xi^{(1)}=0} = \tilde{m}(0, \xi^{(2)}) \) is a bounded multiplier from \( L_0(R^N) \cap L_2(R^N) \) into itself, and the function \( \tilde{m}_d(\xi) \equiv \tilde{m}(\xi) - \tilde{m}(0, \xi^{(2)}) \) satisfies condition (75). In this case it is immediately verified that the multiplier \( \tilde{m}(\xi) = \tilde{m}(0, \xi^{(2)}) + \tilde{m}_d(\xi) \) is a bounded multiplier from the space \( H^{N/2}_{x^{(1)}}(R^N) \) into itself without increasing the smoothness in the variables \( x^{(2)} \). Thus we obtain the following corollary from Theorem 2.7.

**Corollary 1.** If in Theorem 2.7 instead of conditions (74) the condition
\[
\tilde{m}(0, \xi^{(2)}) \equiv \text{const}
\]
is satisfied, or, more generally, the function $\hat{m}(\xi)|_{\xi_1=0} = \hat{m}(0,\xi^{(2)})$ is a bounded multiplier from $L_{\infty}(\mathbb{R}^N) \cap L_2(\mathbb{R}^N)$ into itself and the function $\hat{m}_0(\xi) \equiv \hat{m}(\xi) - \hat{m}(0,\xi^{(2)})$ satisfies condition (75), then the operator $M$ is a bounded linear mapping from the space $H^\gamma_{\xi_1}(\mathbb{R}^N)$ to $H^\gamma_{\xi_1}(\mathbb{R}^N)$ and

$$
\|Mu\|_{H^\gamma_{\xi_1}(\mathbb{R}^N)} \leq C\mu \|u\|_{H^\gamma_{\xi_1}(\mathbb{R}^N)}. 
$$

To prove this corollary, it suffices to note that operators given by Fourier multipliers are translationally invariant. In particular, denoting by $M^*$ the operator with the multiplier $\hat{m}(0,\xi^{(2)})$, we have

$$(M^* u)(x + h\varepsilon^n) = M^* (u(x + h\varepsilon^n)).$$

Thus

$$
\frac{(M^* u)(x + h\varepsilon^n) - (M^* u)(x)}{h^\gamma} = M^* \left( \frac{u(x + h\varepsilon^n) - u(x)}{h^\gamma} \right),
$$

which leads to the assertion of the corollary.

Completing this section, we formulate a lemma concerning properties of the Fourier transform, which we need in the future. This lemma is a special case of the well-known theorems about the properties of the Fourier transforms of distributions with supports in cones. The proof is essentially contained in more general form in [49], Ch.2.

**Lemma 2.8.** Let a function $\hat{f}(i\xi_0,\xi)$, $\xi_0 \in \mathbb{R}^1$, $\xi \in \mathbb{R}^N$, be defined on $\mathbb{R}^{N+1}$ and can be extended to a function $f(i\xi_0 + a,\xi)$ in the domain $a \geq 0$ in such a way that the function $\hat{f}(i\xi_0 + a,\xi)$ has the properties:

1) it is continuous in this domain;
2) $\hat{f}(i\xi_0 + a,\xi)$ is analytic with respect to the variable $p = i\xi_0 + a$ in the domain $a > 0$;
3) it satisfies in $a \geq 0$ and with some exponents $m_1$, $m_2$ the following estimate

$$
|\hat{f}(p,\xi)| \leq C(1 + |p|^{m_1})(1 + |\xi|^{m_2}).
$$

Then the inverse Fourier transform in the sense of distributions

$$
f(t, x) = (2\pi)^{-(N+1)} \int_{\mathbb{R}^{N+1}} e^{it\xi_0 + iz\xi} \hat{f}(i\xi_0,\xi) d\xi d\xi_0
$$

possesses the property

$$
f(t, x) \equiv 0, \quad t < 0.
$$

The proof of this lemma consists of the representation of the function $\hat{f}(p,\xi)$ in the form

$$
\hat{f}(p,\xi) = (p + \xi^2)^M \hat{f}(p,\xi) \equiv (p + \xi^2)^M \tilde{F}(p,\xi),
$$

where $M > m_1 m_2$ is a sufficiently large natural number so that the function $\tilde{F}(p,\xi)$ decreases sufficiently fast at infinity. Then it is well known that the inverse Laplace-Fourier transform $F(t, x)$ from $\tilde{F}(p,\xi)$ (the inverse Laplace transform with respect to the variable $p$) is identically equal to zero for $t < 0$. At the same time $f(t, x) = C(\partial/\partial t - \Delta)^M F(t, x)$, which implies the lemma.
3. Model problems in a half-space for the linearized Cahn-Hilliard equation with dynamic boundary conditions. In this section we consider the Schauder’s estimates for initial boundary value problems in a half-space for the linearized Cahn-Hilliard equation with dynamic boundary conditions. These problems are not included in the well-known general theory of parabolic initial-boundary value problems (see [3], [47], [48], [63], [75]). However, we significantly use the results of [75].

Recall spaces of smooth functions from (4), (5) we use below. Let $\Omega$ be a domain in $\mathbb{R}^N$, which can be unbounded. Denote by $\Omega_T = \Omega \times (0, T)$, where $T > 0$ or $T = +\infty$. We use Banach spaces $C^{l_1,l_2}(\Omega_T)$ with elements $u(x,t)$, $x \in \Omega$, $t \in [0,T]$, $l_1, l_2 > 0$ are non-integer. These spaces are defined, for example, in [74] and consist of functions with smoothness with respect to the variables $x$ up to the order $l_1$ and with smoothness with respect to the variable $t$ up to the order $l_2$, i.e., having a finite norm

$$|u|^{(l_1,l_2)}_{\Omega_T} \equiv \|u\|_{C^{l_1,l_2}(\Omega_T)} \equiv |u|^{(0)}_{\Omega_T} + \sum_{|\alpha|=|l_1|} \langle D^\alpha_x u \rangle^{(l_1-|\alpha|)}_{x,\Omega_T} + \langle D^\alpha_t u \rangle^{(l_2-|\alpha|)}_{t,\Omega_T}. \quad (86)$$

Here $\alpha = (\alpha_1, ..., \alpha_N)$ is a multiindex, $|\alpha| = \alpha_1 + ... + \alpha_N$, $D^\alpha_x = D^\alpha_{x_1}...D^\alpha_{x_N}$, $[l]$ is the integer part of a number $l$, $|u|^{(0)}_{\Omega_T} = \max_{\Omega_T} |u(x,t)|$, $\langle D^\alpha_x u \rangle^{(l_1-|\alpha|)}_{x,\Omega_T}$, $\langle D^\alpha_t u \rangle^{(l_2-|\alpha|)}_{t,\Omega_T}$ are the Hölder constants of the corresponding functions with respect to $x$ and $t$ correspondingly over a domain $\Omega_T$. Recall that besides quantities in (86), for functions from the space $C^{l_1,l_2}(\Omega_T)$ the Hölder constants of the derivatives $D^\alpha_x u$ with respect to $t$ are finite with some exponents. And the same is true for the Hölder constants of the derivatives $D^\alpha_t u$ with respect to $x$ and for mixed derivatives up to some order - see (7). Estimates of all these Hölder constants are obtained by interpolation with the help of (86) - see, for example, [74]. Below we will use the space $C^{l_1,l_2}(\Omega_T)$, where $l$ is a non-integer positive number and the norm in this space we will denote for simplicity by $|u|^{(l)}_{\Omega_T}$.

We will use also the spaces $C^{l_1,l_2}_0(\Omega_T)$, where zero at the bottom of the notation denotes a closed subspace of $C^{l_1,l_2}(\Omega_T)$, consisting of functions whose derivatives with respect to $t$ up to the order $[l_2]$ vanish identically at $t = 0$. The functions of these spaces can be considered to be extended identically by zero to $t \leq 0$ with the preservation of the class.

We proceed to the formulation of the problem. Denote

$$Q^{N+1}_+ = \{(x,t) \in \mathbb{R}^N \times \mathbb{R}^1 : x_N \geq 0, t \geq 0\},$$

$$Q^{N+1}_{+,T} = \{(x,t) \in \mathbb{R}^N \times \mathbb{R}^1 : x_N \geq 0, t \in [0,T]\},$$

$$Q^{N}_+ = \{(x,t) \in \mathbb{R}^N \times \mathbb{R}^1 : x_N = 0, t \geq 0\},$$

$$Q^{N}_{+,T} = \{(x,t) \in \mathbb{R}^N \times \mathbb{R}^1 : x_N = 0, t \in [0,T]\},$$

$$Q^{N} = \{(x,t) \in \mathbb{R}^N \times \mathbb{R}^1 : x_N = 0\} = \mathbb{R}^{N-1} \times \mathbb{R}^1, \quad x = (x', x_N).$$

Consider in $Q^{N+1}_+$ the following initial boundary value problem for the unknown function $u(x,t)$:

$$\frac{\partial u}{\partial t} + \Delta^2 u = f(x,t), \quad (x,t) \in Q^{N+1}_+, \quad (87)$$

$$\frac{\partial \Delta u}{\partial x_N} = g(x',t), \quad (x',t) \in Q^{N}_+, \quad (88)$$
\[ u(x,0) = 0, \quad x_N \geq 0, \quad (89) \]

\[ \frac{\partial u}{\partial t} - a \Delta_{x'} u = h_1(x',t), \quad (x',t) \in Q^N_+, \quad (90) \]

where \( \Delta \) is the Laplace operator, \( \Delta_{x'} \) is the Laplace operator with respect to the variables \( x' \), \( a \) is a positive constant, and we assume that the function \( u(x,t) \) is bounded at \( |x| \to \infty \). Together with dynamic boundary condition (90) (instead of this condition) we also consider other boundary condition

\[ \frac{\partial u}{\partial t} - a \frac{\partial u}{\partial x_N} = h_2(x',t), \quad (x',t) \in Q^N_+. \quad (91) \]

The physical meaning of the condition of the form (90) is explained, for example, in [82], and the condition (91) is explained, for example, in [67]. In this case, in [67] was considered a more general boundary condition

\[ \frac{\partial u}{\partial t} - a \Delta_{x'} u - b \frac{\partial u}{\partial x_N} = h(x',t), \quad (x',t) \in Q^N_+. \]

But (at least when considering the classes of smooth functions) the term \( b \frac{\partial u}{\partial x_N} \) is a junior one (in order) in this condition and can be omitted when considering the model problem.

We assume that the given functions \( f, g, h_1, h_2 \) have compact supports and belong to the following spaces with zero at the bottom and with some \( \gamma \in (0,1) \)

\[ f \in C^{4+\gamma,\frac{4+\gamma}{2}}_0(Q^N_+), \quad g \in C^{2+\gamma,\frac{2+\gamma}{2}}_0(Q^N_+), \quad h_1 \in C^{2+\gamma,\frac{2+\gamma}{2}}_0(Q^N_+), \quad h_2 \in C^{3+\gamma,\frac{3+\gamma}{2}}_0(Q^N_+). \quad (92) \]

We suppose the solution \( u(x,t) \in C^{4+\gamma,\frac{4+\gamma}{2}}_0(Q^N_+) \) as it is dictated by the anisotropy of equation (87). But besides we require that \( u_t(x',0,t) \in C^{2+\gamma,\frac{2+\gamma}{2}}_0(Q^N_+) \) or \( u_e(x',0,t) \in C^{3+\gamma,\frac{3+\gamma}{2}}_0(Q^N_+) \) depending on the type of dynamic boundary conditions.

In view of (89), (87) and the fact that the given functions \( f, g, h_1, h_2 \) belong to the spaces with zero at the bottom, the function \( u(x,t) \) must satisfy the condition \( \partial u/\partial t(x,0) = 0 \). Together with (89) this allows to consider the function \( u(x,t) \) and all the given functions \( f, g, h_1, h_2 \) to be extended by zero to \( t < 0 \) and consider relations (87) - (91) for all values of the time variable \( t \in \mathbb{R}_1 \).

### 3.1. Problem (87) - (90).

Consider problem (87) - (90). Denote

\[ \rho(x',t) \equiv u(x',0,t) = u(x,t)|_{x_N=0}. \quad (93) \]

Condition (90) allows to find the value of the unknown function \( u(x,t) \) at \( x_N = 0 \), that is the function \( \rho(x',t) \). Namely,

\[ \rho(x',t) = \Gamma_0(x',t) * h_1(x',t), \quad (94) \]

where \( \Gamma_0(x',t) \) is the fundamental solution of the heat operator \( L_0 \equiv \partial/\partial t - a \Delta_{x'} \). It is well known that expression (94) can be obtained from (90) by applying the Fourier transform with respect to the variables \( x' \) and \( t \). In other words, denoting for a function \( v(x',t) \)

\[ \tilde{v}(\xi_0,\xi) = \int_{\mathbb{R}^{N-1}} dt \int_{\mathbb{R}^1} e^{-i\xi_0 t-i\xi x'} v(x',t) dx' \]


and applying this transform to relation (90) (recall that all the functions are assumed to be extended by zero to \( t < 0 \)), in view of the known properties of the Fourier transform of derivatives, we find that

\[
\tilde{\rho}(\xi_0, \xi) = \frac{i\xi_0}{i\xi_0 + a\xi^2} \tilde{h}_1(\xi_0, \xi). \tag{95}
\]

Estimates for the potential in (94) are well known in the case when its density \( h_1 \in C^{k+\gamma, \frac{k+2}{2}}(Q^N_+) \). However, in our case we are dealing with another anisotropy of smoothness of the space for the density, namely \( h_1 \in C_0^{2+\gamma, \frac{2+2}{2}}(Q^N_+) \). Therefore known properties of the potential for the heat operator are inapplicable in our case. Furthermore, the results of [57] and Theorem 1.1 also are inapplicable, since the anisotropy of homogeneity of the kernel does not coincide with the anisotropy of the smoothness of the density \( h_1 \). Therefore, we obtain estimates of the Hölder constants for highest derivatives of the function \( \rho(x', t) \) in the space \( C^{4+\gamma, \frac{4+2}{2}}(Q^N_+) \) using Theorem 2.7.

Consider first the Hölder constant in the variable \( t \) of the derivative \( \rho_t(x', t) \). In view of relation (95) and known properties of the Fourier transform of derivatives

\[
\tilde{\rho}_t = \frac{i\xi_0}{i\xi_0 + a\xi^2} \tilde{h}_1(\xi_0, \xi). \tag{96}
\]

Consider the function

\[
\tilde{m}_1(\xi_0, \xi) = \frac{i\xi_0}{i\xi_0 + a\xi^2}. \tag{97}
\]

Evidently this function is homogeneous of degree zero

\[
\tilde{m}_1(\lambda^2 \xi_0, \lambda \xi) = \tilde{m}_1(\xi_0, \xi), \quad \lambda > 0. \tag{98}
\]

Besides, this function is smooth on the set \( B_1 = \{ (\xi_0, \xi) : 1/8 < |\xi_0| + \xi^2 < 8 \} \). Therefore it is trivial to verify that \( \tilde{m}_1(\xi_0, \xi) \) satisfies the conditions of Theorem 2.7. Consequently

\[
\langle \rho_t \rangle_{t, Q^N_+}^{(2a+2)} \leq C \langle h_1 \rangle_{t, Q^N_+}^{(2a+2)}. \tag{99}
\]

Further, since \( h_1 \in C_0^{2+\gamma, \frac{2+2}{2}}(Q^N_+) \), we have for a multi-index \( \alpha = (\alpha_1, ..., \alpha_{N-1}) \) with \( |\alpha| = 2 \)

\[
\widehat{D_{x'}^\alpha \rho_t} = (i\xi_{j_1})(i\xi_{j_2})\tilde{\rho}_t = \tilde{m}_1(\xi_0, \xi) \left[ (i\xi_{j_1})(i\xi_{j_2})\tilde{h}_1(\xi_0, \xi) \right] = \tilde{m}_1(\xi_0, \xi) \widehat{D_{x'}^\alpha h_1}. \]

Therefore, in view of \( \tilde{m}_1(\xi_0, 0) \equiv 1 \) and from Corollary 1,

\[
\langle D_{x'}^\alpha \rho_t \rangle_{x', Q^N_+}^{(\gamma)} \leq C \langle D_{x'}^\alpha h_1 \rangle_{x', Q^N_+}^{(\gamma)}, \quad |\alpha| = 2. \tag{100}
\]

Relations (99), (100) mean that \( \rho_t \in C_0^{2+\gamma, \frac{2+2}{2}}(Q^N_+) \) and give an estimate for \( \rho_t \) in this space.

Consider now fourth order derivatives \( D_{x'}^\alpha \rho \), \( |\alpha| = 4 \). We have with some indexes \( k_1, k_2, k_3, k_4 \)

\[
\widehat{D_{x'}^\alpha \rho} = C(i\xi_{k_1})(i\xi_{k_2})(i\xi_{k_3})(i\xi_{k_4}) \frac{1}{i\xi_0 + a\xi^2} \tilde{h}_1 =
\]

\[
= C \left( \frac{i\xi_{k_1}}{i\xi_0 + a\xi^2} \right) \left( i\xi_{k_2} \right) \left( i\xi_{k_3} \right) \left( i\xi_{k_4} \right) \tilde{h}_1 = C \frac{\xi_{k_1} \xi_{k_2}}{i\xi_0 + a\xi^2} D_{x_{k_3} x_{k_4}}^2 \tilde{h}_1 \equiv \tilde{m}_2(\xi_0, \xi) D_{x_{k_3} x_{k_4}}^2 \tilde{h}_1. \tag{101}
\]
In this case the function \( \tilde{m}_2(\xi_0, \xi) \) is homogeneous, \( \tilde{m}_2(\lambda^2 \xi_0, \lambda \xi) = \tilde{m}_2(\xi_0, \xi) \). And also \( \tilde{m}_2(\xi_0, 0) = 0 \), so it is easy to verify the conditions of Theorem 2.7. Therefore, from (101) it follows that

\[
\sum_{|\alpha|=4} (D^\alpha_x \rho)_{x,Q}^{(\gamma)} \leq C \sum_{|\beta|=2} \left\langle D^\beta_x h_1 \right\rangle_{x,Q}^{(\gamma)}.
\]

From (99), (100), (102), and from the condition \( \rho(x', 0) = 0 \) it follows that for an arbitrary \( T > 0 \)

\[
|\rho|_{C^{4+\gamma, 4+\gamma}(Q^{N+1}_{+,T})} + |\rho_t|_{C^{2+\gamma, 2+\gamma}(Q^{N}_{+,T})} \leq C_T |h_1|_{C^{2+\gamma, 2+\gamma}(Q^{N}_{+,T})}.
\]

Thus in problem (89)-(90) condition (90) can be replaced by the condition

\[ u(x', 0, t) = \rho(x', t), \]

where for the function \( \rho(x', t) \) estimate (103) is valid. Then from the results of [75] it follows that this problem has a unique solution \( u(x, t) \) and

\[
|u|_{C^{4+\gamma, 4+\gamma}(Q^{N+1}_{+,T})} \leq C_T \left( |f|_{C^{\gamma, \gamma}(Q^{N+1}_{+,T})} + |g|_{C^{2+\gamma, 2+\gamma}(Q^{N}_{+,T})} + |h_1|_{C^{2+\gamma, 2+\gamma}(Q^{N}_{+,T})} \right),
\]

and besides in view of (103)

\[
|u_t(x', 0, t)|_{C^{2+\gamma, 2+\gamma}(Q^{N}_{+,T})} \leq C_T |h_1|_{C^{2+\gamma, 2+\gamma}(Q^{N}_{+,T})}.
\]

Thus, we have proved the following assertion.

**Theorem 3.1.** Under conditions (92) and for any \( T > 0 \) problem (89)-(90) has the unique solution \( u(x, t) \) from the space \( C^{4+\gamma, 4+\gamma}(Q^{N+1}_{+,T}) \) and estimates (105), (106) are valid.

3.2. **Problem (87) - (89), (91).** As in the previous section, we reduce the problem to a problem with condition (104) instead of condition (91) after determining the function \( \rho(x', t) \equiv u(x', 0, t) \) from the conditions of the problem. However, in this case the boundary operator in the left side of (91) is not a local operator (as opposed to (90)), so its consideration requires somewhat more complex reasoning. This is due to the fact that in this case a more complex multiplier arises, which is not a homogeneous function. To study this multiplier, we extract its “main” homogeneous part. Such an approach would require to have smooth and finite solution to justify the emerging Fourier transforms. Therefore, we obtain the solvability of the problem and estimate of the solution to (87)-(89), (91) in two stages. On the first stage we assume that the data of the problem are finite and belong to the class \( C^\infty \). Here we also will obtain the solution of the problem from the class \( C^\infty \). Finally, the solvability and the estimates under assumptions (92) will be obtained by the approximation of the data by functions from \( C^\infty \) exactly as it was done at the end of the proof of Theorem 2.1. We apply such an approach to avoid consideration of known potentials connected to the problem and to have the possibility to use the well known results of [75].

So, assume first that the given functions \( f, g, h_2 \) are finite and belong to \( C^\infty \) with the supports in the sets

\[
supp(f) \subset \{(x, t) : |x| < \lambda, x_N > 0, 0 < t < 1\} \equiv G^{N+1}_\lambda, \quad \lambda \in (0, 1),
\]

\[
supp(g), \ supp(h_2) \subset \{(x', t) : |x'| < \lambda, 0 < t < 1\} \equiv G^N_\lambda.
\]

(107)
Using well-known results on the solvability of parabolic boundary value problems, we can reduce problem (87) - (89), (91) to the case when \( f \equiv 0 \) and \( g \equiv 0 \). Besides, we will denote for simplicity the function \( h_2 \) as just \( h \). Indeed, consider the problem with the unknown \( v(x, t) \)

\[
\frac{\partial v}{\partial t} + \Delta^2 v(x, t) = f(x, t), \quad (x, t) \in Q^{N+1}_+,
\]

\[
\frac{\partial \Delta v}{\partial x_N} \bigg|_{x_N=0} = g(x', t),
\]

\[
v(x', 0, t) = 0, \quad v(x, 0) = 0.
\]

From the results of \([3, 47, 48, 63, 75]\) it follows that this problem has the unique solution from the space \( C^{4+\gamma, \frac{4+\gamma}{N}}(Q^{N+1}_+) \), which obeys the estimate

\[
|v|_{Q^{N+1}_+} \leq C \left( |f|_{Q^{N+1}_+} + |g|_{Q^N_+}^{1+\gamma, \frac{4+\gamma}{N}} \right). \tag{109}
\]

Moreover, this solution is represented as

\[
v(x, t) = \int_0^t \int_{R^N_+} G_0(x, \eta, t-\tau) f(\eta, \tau) d\eta + \int_0^t \int_{R^{N-1}} G_1(x, \eta', t-\tau) g(\eta', \tau) d\eta',
\]

where \( R^N_+ = \{ x \in R^N : x_N \geq 0 \} \) and the functions \( G_0 \) and \( G_1 \) obey the estimates

\[
\left| D_\alpha^\alpha D_\beta^\beta G_0 \right| + \left| D_\alpha^\alpha D_\beta^\beta G_1 \right| \leq C_{\nu, \alpha, \beta} t^{-\frac{\nu}{N} - \frac{\nu + 1}{4} - \frac{\nu + 1}{4}} e^{-\frac{\nu + 1}{4}}, \tag{111}
\]

From (109)-(111) and from the fact that \( f \) and \( g \) are finite it follows that the function \( v(x, t) \) and its derivatives decay sufficiently rapidly at infinity, i.e. as \( |x| + |t| \to \infty \). If we make in original problem (87) - (89), (91) the change of the unknown \( u = \overline{u} + v \), it reduces to the case \( f \equiv 0, g \equiv 0 \). And the right hand side of (91) is changed

\[
\frac{\partial u}{\partial t} - a \frac{\partial u}{\partial x_N} = h \equiv h_2 + a \frac{\partial v}{\partial x_N}. \tag{112}
\]

Now from (111), (110), and (112) in view of (107) it follows that for \( |x| + |t| > 3 \)

\[
\left| D_\alpha^\alpha D_\beta^\beta \frac{\partial v}{\partial x_N} \right| \leq C_{\nu, \alpha, \beta} t^{-\frac{\nu + 1}{4} - \frac{\nu + 1}{4} - \frac{\nu + 1}{4}} e^{-\frac{\nu + 1}{4}}, \tag{113}
\]

and also

\[
\left| \frac{\partial v}{\partial x_N} \right|_{Q^N_+}^{(3+\gamma, \frac{3+\gamma}{N})} \leq C \left( |f|_{Q^{N+1}_+}^{(\gamma, \frac{3+\gamma}{N})} + |g|_{Q^N_+}^{(1+\gamma, \frac{3+\gamma}{N})} \right). \tag{114}
\]

Since the function \( h_2(x', t) \) is finite, we see that estimates (113) are valid also for the function \( h(x', t) \) in (112). Note that in view of (114)

\[
|h|_{Q^N_+}^{(3+\gamma, \frac{3+\gamma}{N})} \leq C \left( |f|_{Q^{N+1}_+}^{(\gamma, \frac{3+\gamma}{N})} + |g|_{Q^N_+}^{(1+\gamma, \frac{3+\gamma}{N})} + |h_2|_{Q^N_+}^{(3+\gamma, \frac{3+\gamma}{N})} \right). \tag{115}
\]

And also, in view of the fact that \( h_2(x', t) \) is finite together with (113), the function \( h(x', t) \) and all its derivatives belong to the space \( L^2(Q^N) \) and

\[
\sum_{|\alpha| \leq 3} \| D_\alpha^\alpha h \|_{L^2(Q^N)} \leq C \left( |f|_{Q^{N+1}_+}^{(\gamma, \frac{3+\gamma}{N})} + |g|_{Q^N_+}^{(1+\gamma, \frac{3+\gamma}{N})} + |h_2|_{Q^N_+}^{(3+\gamma, \frac{3+\gamma}{N})} \right). \tag{116}
\]
Thus, in what follows we assume that the functions $f$ and $g$ in (87), (88) are identically equal to zero and the function $h_2$ in (91) will be denoted just by $h(x', t) \equiv h_2(x', t)$ and it belongs to $C^\infty$ and satisfies the estimate

$$
\left| D_x^\alpha D_t^\beta h \right| \leq C_{\nu, \alpha, \beta} (t + 1)^{-\frac{N+1}{4} - \frac{\alpha}{4}} \frac{1}{|x'|^\frac{1}{2}} e^{-\nu \frac{|x'|}{(t + 1)^{\frac{1}{4}}}}.
$$

(117)

We are going to apply the Fourier transform to problem (87) - (89), (91). To justify our operations we need sufficiently fast decay at infinity of the function $h(x', t)$ (faster than in (117)). The reason is we are going to obtain not just estimates of the solution but to construct the solution itself without more-less explicit representation in integral form as in [47], [48], [75]. If one going to obtain the estimates only it possibly to apply the Fourier transform in variables $x'$ and $t$ and go to relation (154) directly. Besides, also for technical reasons we need sufficient smoothness of the Fourier transform of the function $h(x', t)$. At this end we differentiate relations (87) - (89), (91) with respect to $t$ and denote

$$
h^{(k)}(x', t) \equiv D_t^k h(x', t).
$$

(118)

Also we denote by $u^{(k)}(x, t)$ the solution of (87) - (89), (91) with $h^{(k)}$ instead of $h$ in the right hand side of (91). In view of the linearity of the problem we have

$$
u^{(k)}(x, t) \equiv D_t^k u(x, t).
$$

(119)

Due to (117) we have for the function $h^{(k)}$ the estimate

$$
\left| D_x^\alpha D_t^\beta h^{(k)} \right| \leq C_{\nu, \alpha, \beta} (t + 1)^{-\frac{N+1}{4} - \frac{\alpha}{4}} \frac{1}{|x'|^\frac{1}{2}} e^{-\nu \frac{|x'|}{(t + 1)^{\frac{1}{4}}}}.
$$

(120)

From this estimate it follows that

$$
\left| D_x^\alpha D_t^\beta h^{(k)} \right| \leq C_{\nu, \alpha, \beta} (1 + |t|^{\frac{1}{4}} + |x'|) - (N+1+k)|\alpha| - 4|\beta|.
$$

(121)

To obtain this estimate from (120) it is enough to consider the two cases $1 + |t|^{\frac{1}{4}} > |x'|$ and $1 + |t|^{\frac{1}{4}} < |x'|$. In the first case (121) directly follows from (120). In the second case (121) is obtained by multiplying and dividing estimate (120) by $|x'|$ at the appropriate power and applying the inequality

$$
z^A e^{-\nu z^{\frac{1}{4}}} \leq C_A e^{-\nu z^{\frac{1}{4}}}, \quad z = \frac{|x'|}{1 + t^{\frac{1}{4}}}.
$$

Recall that all the functions in (87) - (89), (91) are extended by zero to the domain $t < 0$. From (121) it follows that the function

$$
\overline{h^{(k)}}(\xi, \xi_0) = \int_{-\infty}^{+\infty} \int_{R^{N-1}} e^{-it\xi_0 - i\xi} h^{(k)}(x', t) dx' dt,
$$

belongs to the space $\overline{h^{(k)}} \in C^{4(k-1), (k-1)}(R^{N-1} \times R^1)$. Besides, the function $h^{(k)}$ itself and all its derivatives decay at infinity faster than any power and the following estimate is valid.

$$
\left| D_\xi^\alpha D_{\xi_0}^\beta h^{(k)} \right| \leq C_{m, \alpha, \beta} (1 + |\xi_0| + |\xi|)^{-m}, \quad |\alpha| + 4|\beta| \leq k - 1, m > 0.
$$

(122)
Moreover, in view of estimate (121) the function \( \hat{h}^{(k)}(\xi, \xi_0) \) can be extended to the function \( \hat{h}^{(k)}(\xi, p) \), \( p = i\xi_0 + a \), \( a > 0 \),
\[
\hat{h}^{(k)}(\xi, p) = \int_{-\infty}^{+\infty} \int_{R^{N-1}} e^{-pt - i\xi x} \hat{h}^{(k)}(x', t) dx' dt
\]
and for the extended function the following estimate is still valid
\[
\left| D_\xi^a D_p^m \hat{h}^{(k)} \right| \leq C_{m, \alpha, \beta} (1 + |p| + |\xi|)^{\alpha} \xi^{-\beta}, \ |\alpha| + 4|\beta| \leq k - 1, m > 0. \quad (123)
\]
Choosing sufficiently large natural \( k \) (the precise choice of \( k \) will be done below), make in problem (87) - (89), (91) the Fourier transform with respect to the variables \( x' \) and \( t \).
\[
\hat{u}^{(k)}(\xi, x_N, \xi_0) = \int_{-\infty}^{+\infty} \int_{R^{N-1}} e^{-i\xi_0 t - i\xi x} u^{(k)}(x', x_N, t) dx' dt. \quad (124)
\]
As a result, these relations take the form
\[
i\xi_0 \hat{u}^{(k)} + \left( -\xi^2 + \frac{d^2}{dx_N^2} \right)^2 \hat{u}^{(k)} = 0, \quad x_N > 0, \quad (125)
\]
\[
\frac{d}{dx_N} \left( -\xi^2 + \frac{d^2}{dx_N^2} \right) \hat{u}^{(k)} \bigg|_{x_N=0} = 0, \quad (126)
\]
\[
i\xi_0 \rho^{(k)} + a \left. \frac{d}{dx_N} \hat{u}^{(k)} \right|_{x_N=0} = \hat{h}^{(k)}, \quad (127)
\]
\[
\left| \hat{u}^{(k)} \right| \leq C, \quad x_N \to \infty, \quad (128)
\]
where \( \rho^{(k)}(x', t) \equiv u^{(k)}(x', 0, t) \).

Our aim now is to obtain a representation for \( \hat{\rho}^{(k)} \) and so for \( \rho^{(k)} \).

**Lemma 3.2.** For the function \( \hat{\rho}^{(k)} \) the following representation is valid
\[
\hat{\rho}^{(k)} = \frac{\hat{h}^{(k)}(\xi, \xi_0)}{i\xi_0 + 2\sqrt{\xi^2 + \xi_0} - i3\xi_0 \sqrt{\xi^2 - \xi_0}} = \frac{\hat{h}^{(k)}(\xi, \xi_0)}{M(\xi, \xi_0)}. \quad (129)
\]

**Proof.** From equation (125) in view of (128) it follows that
\[
u^{(k)}(\xi, x_N, \xi_0) = C_1(\xi, \xi_0) e^{-x_N \sqrt{\xi^2 + \xi_0}} + C_2(\xi, \xi_0) e^{-x_N \sqrt{\xi^2 - \xi_0}}. \quad (130)
\]
Here \( \sqrt{\xi} \) with the deuce in the designation means the complex function with the property
\[
\arg \sqrt{\xi} \in [0, \pi),
\]
that is the function \( \sqrt{\xi} \) maps the complex plane without the positive real half-axes to the upper half-plane. At the same time by the standard symbol \( \sqrt{\xi} \) (without the deuce) we denote the standard function with the property
\[
\arg \sqrt{\xi} \in (-\frac{\pi}{2}, \frac{\pi}{2}],
\]
that is the function \( \sqrt{\xi} \) maps the complex plane without negative real half-axes to the right half-plane. Denote
\[
\lambda_1 = \sqrt{\xi^2 + \sqrt{\xi_0}}, \quad \lambda_2 = \sqrt{\xi^2 - \sqrt{\xi_0}}. \quad (131)
\]
Condition (126) gives
\[ C_1 \lambda_1 (\xi^2 - \xi') + C_2 \lambda_2 (\xi^2 - \xi') \]
and so
\[ C_2 = C_1 \frac{\sqrt{\xi^2 + \sqrt{-i\xi_0}}}{\sqrt{\xi^2 - \sqrt{-i\xi_0}}} = C_1 \frac{\lambda_1}{\lambda_2}. \]
Consequently
\[ \widetilde{u}^{(k)}(\xi, x_N, \xi_0) = C(\xi, \xi_0) \left( e^{-\lambda_1 x_N} + \lambda_1 \frac{\lambda_2}{\lambda_1 + \lambda_2} \right) \].

Therefore,
\[ \frac{d\widetilde{\varphi}^{(k)}}{dx_N} \bigg|_{x_N=0} = C(\xi, \xi_0)(-2\lambda_1), \]
That is
\[ C(\xi, \xi_0) = \varphi^{(k)}(\xi, \xi_0) \frac{\lambda_2}{\lambda_1 + \lambda_2}, \]
\[ \frac{d\widetilde{u}^{(k)}}{dx_N} \bigg|_{x_N=0} = -\varphi^{(k)}(\xi, \xi_0) \frac{2\lambda_1 \lambda_2}{\lambda_1 + \lambda_2}. \]
Inserting relation (136) in condition (127), we obtain
\[ \varphi^{(k)}(i\xi_0 + \frac{2\sqrt{\xi^2 + \sqrt{-i\xi_0} \sqrt{\xi^2 - \sqrt{-i\xi_0}}}}{\sqrt{\xi^2 + \sqrt{-i\xi_0}} + \sqrt{\xi^2 - \sqrt{-i\xi_0}}} \equiv \tilde{M}(\xi, \xi_0) \overline{\varphi^{(k)}} = \overline{\varphi^{(k)}}(\xi, \xi_0) \]
that is
\[ \varphi^{(k)}(i\xi_0 + \frac{2\sqrt{\xi^2 + \sqrt{-i\xi_0} \sqrt{\xi^2 - \sqrt{-i\xi_0}}}}{\sqrt{\xi^2 + \sqrt{-i\xi_0}} + \sqrt{\xi^2 - \sqrt{-i\xi_0}}} = \tilde{M}(\xi, \xi_0) \overline{\varphi^{(k)}}(\xi, \xi_0). \]
This proves the lemma.

Note that the function \( \tilde{M}(\xi, \xi_0) \) can be extended in the domain \( a > 0 \) to the analytic function of the argument \( p = i\xi_0 + a \)
\[ \tilde{M}(\xi, p) = p + \frac{2\sqrt{\xi^2 + \sqrt{-p} \xi^2 - \sqrt{-p}}}{\sqrt{\xi^2 + \sqrt{-p}} + \sqrt{\xi^2 - \sqrt{-p}}}. \]

Let us show that for sufficiently large \( k \) the function \( \varphi^{(k)}(\xi, p) = \overline{\varphi^{(k)}}(\xi, p) / \tilde{M}(\xi, p) \) is sufficiently smooth, which means sufficient decay at infinity of the function \( \varphi^{(k)}(x', t) \). Let \( l \) be a natural number, \( l < k \). According to the definition of the function \( h^{(k)} \) we have \( \varphi^{(k)}(\xi, p) = p^l h^{(k-l)}(\xi, p) \) and consequently
\[ \varphi^{(k)}(\xi, p) = p^l \frac{\tilde{M}(\xi, p)}{M^{(k-l)}(\xi, p)} = \tilde{M}^{(k-l)}(\xi, p). \]
Directly from the definition of \( \tilde{M}(\xi, p) \) it follows that
\[ \tilde{N}^{(k)}(\xi, p) = \frac{p^l}{M^{(k-l)}(\xi, p)} \in C^{(k-l)}(R^{N-1} \times C_+), \quad C_+ = \{ z : \mathbb{R} z \geq 0 \}. \]
and in view of (122),
\[ |D_{\xi}^{\alpha}D_{t}^{\beta}\hat{\rho}^{(k)}(\xi,p)| \leq C_{m,\alpha,\beta}(1 + |p| + |\xi|)^{-m}, \quad |\alpha| + |\beta| \leq \frac{l-1}{2}, m > 0. \] (141)

Choose \( l = \lceil k/2 \rceil + 1 \). From the last estimate, taking into account Lemma 2.8, it follows that the function \( \rho^{(k)}(x',t) \) (that is the inverse Fourier transform of the function \( \hat{\rho}^{(k)}(\xi,\xi_{0}) \)) has the properties
\[ \rho^{(k)}(x',t) \in C^{\infty}(R^{N-1} \times R^{1}), \quad \rho^{(k)}(x',t) \equiv 0, t \leq 0, \]
\[ \left| D_{x}^{\alpha}D_{t}^{\beta}\rho^{(k)}(x',t) \right| \leq C_{k}(1 + |t| + |x'|)^{\frac{k+1}{2}}, \quad |\alpha|, \beta = 0, 1, 2, ... \] (142)

Consider now problem (87) - (89), (91) for the function \( u^{(k)}(x,t) \) and replace dynamic condition (91) by the condition
\[ u^{(k)}(x',0,t) = \rho^{(k)}(x',t), \] (143)

where \( \rho^{(k)}(x',t) \) is defined by relation (138). From [75], section 8 and from [75], theorems 4.9 it follows that there exists the unique solution of this problem from the class \( C^{\infty} \) and this solution can be represented in the form
\[ u^{(k)}(x,t) = \int_{0}^{t} d\tau \int_{R^{N-1}} G(x' - \eta, x_{N}, t - \tau)\rho^{(k)}(\eta, \tau) d\eta. \]

From estimate (142) it follows that for any \( m > 0 \) there exists \( k_{0} > 0 \) such that for \( k \geq k_{0} \) the boundary function \( \rho^{(k)}(x',t) \) in (143) belongs to the Sobolev space \( W^{4m,m}(R^{N-1} \times R^{1}) \) and satisfies the conditions of Theorem 3.4 in [75]. Therefore, the solution of this problem belongs to \( C^{\infty}(R^{N-1} \times R^{1}) \) and also it belongs to the class
\[ u^{(k)}(x,t) \in W^{4m,m}(R_{+}^{N} \times R^{1}) \cap C^{\infty}(R_{+}^{N} \times R^{1}). \] (144)

If \( m \) is sufficiently large then from (144) it follows that \( u^{(k)}(x,t) \) as a function of \( x_{N} \) with values in functional spaces defined on \( R^{N-1} \times R^{1} \) has the properties
\[ \frac{d^{l}u^{(k)}}{dx_{N}^{l}} \in C \left( [0, \infty), W^{4,l}_{2}(R^{N-1} \times R^{1}) \right), \quad l = 0, 1, 2, 3, 4. \] (145)

From the last relation and from the properties of the Fourier transform it follows that
\[ \frac{d^{l}\hat{u}^{(k)}}{dx_{N}^{l}}(\xi, x_{N}, \xi_{0}) \in C \left( [0, \infty), L^{2}(R^{N-1} \times R^{1}) \right), \quad l = 0, 1, 2, 3, 4. \] (146)

Relations (144)-(146) permit to make the conclusion that the constructed solution \( u^{(k)}(x,t) \) to the problem with condition (143) satisfies in fact boundary condition (91) with the right hand side \( h^{(k)} \). Indeed, from properties (144)-(146) of the function \( u^{(k)} \) it follows that we can make the Fourier transform with respect to the variables \( (x',t) \) in relations (87) - (89), (143). And from the way of constructing of the function \( \rho^{(k)} \) in (138) in view of bijectivity of the Fourier transform it follows that the function \( u^{(k)}(x,t) \) satisfies condition (91). Thus we proved the existence
of the solution $u^{(k)}(x, t)$ from the class $C^\infty(R^N_+ \times R^1)$ to problem (87) - (89), (91) with the right hand side $h^{(k)}$ in (91) and with $f \equiv 0$, $g \equiv 0$. Denote

$$u(x, t) = \int_0^t \frac{(t - \tau)^{k-1}}{(k - 1)!} u^{(k)}(x, \tau) d\tau$$

is the $k$-fold integral in the variable $t$ from the function $u^{(k)}(x, t)$. By virtue of the linearity of problem (87) - (89), (91) with constant coefficients the function $h^{(k)}$ for large $S$ the following estimate is valid

Lemma 3.3. The following estimate is valid

$$\sum_{|\alpha|=4} (D_{x'}^\alpha \rho^{(\gamma)})_{x',Q^N} \leq C \left( \sum_{|\alpha|=3} (D_{x'}^\alpha h^{(\gamma)})_{x',Q^N} + \sum_{|\beta|=3} \left\| D_{x'}^\beta h \right\|_{L_2(Q^N)} \right) + \varepsilon (D_t \rho)^{(3+\gamma;\frac{3+\gamma}{3})} + C_\varepsilon \| D_t \rho \|_{L_2(Q^N)} .$$

Proof. From the relations with some multi-index $\alpha, |\alpha| = 4$,

$$D_{x'}^\alpha \rho^{(k)} = \frac{\partial^4 \rho^{(k)}}{\partial x_{i_1} \partial x_{i_2} \partial x_{i_3} \partial x_{i_4}} = (i\xi_{i_1})(i\xi_{i_2})(i\xi_{i_3})(i\xi_{i_4})\rho^{(k)}(\xi, \xi_0),$$

in view of equality (138) it follows that

$$D_{x'}^\alpha \rho^{(k)} = \frac{i\xi_l}{M(\xi, \xi_0)} D_{x'}^{\beta} h^{(k)}, \ |\alpha| = 4, |\beta| = 3, l \in \{1, ..., N - 1\}.$$

And analogously for lower derivatives

$$D_{x'}^\alpha \rho^{(k)} = \frac{i\xi_l}{M(\xi, \xi_0)} D_{x'}^{\beta} h^{(k)}, \ |\alpha| \leq 4, |\beta| = |\alpha| - 1, l \in \{1, ..., N - 1\} .$$

In view of the definitions of the functions $\rho^{(k)}$ and $h^{(k)}$

$$D_{x'}^{\alpha} \rho^{(k)} = (i\xi_0)^k D_{x'}^{\alpha} \rho, \ D_{x'}^{\beta} h^{(k)} = (i\xi_0)^k D_{x'}^{\beta} h,$$

where the first equality we understand in the sense of distributions $S'(Q^N)$ as for large $k$ the functions $D_{x'}^{\alpha} \rho^{(k)}$ and, consequently, $D_{x'}^{\alpha} \rho^{(k)}$ belong to the space $L_2(Q^N) \subset S'(Q^N)$. However, we do not know for a while about properties of the function $D_{x'}^{\alpha} \rho$ besides it is an element of $S'(Q^N)$.

From (148) and (149) it follows that

$$(i\xi_0)^k \widetilde{D_{x'}^{\alpha} \rho} = (i\xi_0)^k \frac{i\xi_l}{M(\xi, \xi_0)} \widetilde{D_{x'}^{\beta} h},$$
and thus in the sense of distributions
\[
\hat{D}_x^\alpha \rho = \frac{i\xi}{M(\xi,\xi_0)} \hat{D}_x^\beta h + \hat{F}_\alpha(\xi,\xi_0),
\]
(151)
where \(\hat{F}_\alpha(\xi,\xi_0)\) is some element of \(S'(Q^N)\) with the support on the surface \(\{\xi_0 = 0\}\) and with the property
\[
(i\xi_0)^k \hat{F}_\alpha(\xi,\xi_0) = 0.
\]
(152)
Further, the function \(i\xi / M(\xi, p)\), \(p = i\xi_0 + a\), \(a \geq 0\), is bounded. Consequently, the functions \(\hat{D}_x^\beta h\) belong to the space \(L_2(R^{N-1} \times R^1)\) so that the function \(\frac{i\xi}{M(\xi,\xi_0)} \hat{D}_x^\beta h \in L_2(R^{N-1} \times R^1)\) and it satisfies the conditions of Lemma 2.8. Making in (151) the inverse Fourier transform in the sense of distributions \(S'(Q^N)\), we obtain
\[
D_x^\alpha \rho(x', t) = F^{-1} \left[ \frac{i\xi}{M(\xi,\xi_0)} \hat{D}_x^\beta h \right] + F_{\alpha}(x', t).
\]
Since \(D_x^\alpha \rho(x', t)\) and \(F^{-1} \left[ \frac{i\xi}{M(\xi,\xi_0)} \hat{D}_x^\beta h \right]\) are identically equal to zero for \(t < 0\), we see that
\[
\text{supp}(F_{\alpha}(x', t)) \subset \{(x', t) : t \geq 0)\}.
\]
(153)
Besides, from (152) it follows that \(\partial^k F_{\alpha}(x', t) = 0\) in the sense of distributions. From this and from (153) it follows in standard way that \(F_{\alpha}(x', t) = 0\). Consequently, according to (151)
\[
\hat{D}_x^\alpha \rho = \frac{i\xi}{M(\xi,\xi_0)} \hat{D}_x^\beta h, \quad 1 \leq |\alpha| \leq 4, |\beta| = |\alpha| - 1
\]
(154)
that is, in particular, the functions \(\hat{D}_x^\alpha \rho\) and, consequently, \(D_x^\alpha \rho\) belong to the space \(L_2(R^{N-1} \times R^1)\). This means that the equality (154) is valid in the usual pointwise sense as an equality of functions in \(L_2(R^{N-1} \times R^1)\). Since the function \(\frac{i\xi}{M(\xi,\xi_0)}\) is bounded, we see that from (154) it follows that
\[
\|D_x^\alpha \rho\|_{L_2(R^{N-1} \times R^1)} \leq C \|D_x^\beta h\|_{L_2(R^{N-1} \times R^1)}, \quad 1 \leq |\alpha| \leq 4, |\beta| = |\alpha| - 1.
\]
(155)
We are going to apply Theorem 2.7 to (154) to estimate the Hölder constants of higher derivatives \(D_x^\alpha \rho\), \(|\alpha| = 4\), in terms of the Hölder constants of the derivatives \(D_x^\beta h\). However, the functions \(\hat{M}(\xi,\xi_0)\) and \(\frac{i\xi}{M(\xi,\xi_0)}\) are not homogeneous and therefore it is not so easy to verify the conditions of Theorem 2.7 directly. To overcome this difficulty, we use some simple algebraic transformations to extract “main homogeneous part” of the function \(\hat{M}(\xi,\xi_0)\). Putting \(|\alpha| = 4, |\beta| = 3\), write (154) in the form
\[
\left(\hat{D}_x^\alpha \rho\right) \hat{M}(\xi,\xi_0) = i\xi \hat{D}_x^\beta h,
\]
(156)
and represent the function \(\hat{M}(\xi,\xi_0)\) as
\[
\hat{M}(\xi,\xi_0) = (i\xi_0 + |\xi|) + (i\xi_0)(C_1 \hat{D}_1 + C_2 \hat{D}_2) \hat{T}^3,
\]
(157)
where \(C_1, C_2\) are given constants,
\[
\hat{D}_1(\xi,\xi_0) \equiv \hat{d}_1 \hat{d}_2 \hat{d}_3 \hat{d}_4, \quad \hat{D}_2(\xi,\xi_0) \equiv \hat{d}_2 \hat{d}_5,
\]
(158)
\( \tilde{d}_1 = \frac{|\xi|}{\sqrt{\xi^2 + \sqrt{-i\xi_0} + \xi^2 - \sqrt{-i\xi_0}}} \), \( \tilde{d}_2 = \frac{\sqrt{\xi^2 + \sqrt{-i\xi_0} \sqrt{\xi^2 - \sqrt{-i\xi_0}}}}{\left(\sqrt{\xi^2 + \sqrt{-i\xi_0} + \xi^2 - \sqrt{-i\xi_0}}\right)^2} \),

\( \tilde{d}_3 = \frac{\sqrt{\xi^2 + \sqrt{-i\xi_0} \sqrt{\xi^2 - \sqrt{-i\xi_0}}}}{\left(\xi + \sqrt{\xi^2 + \sqrt{-i\xi_0}} + \xi^2 - \sqrt{-i\xi_0}\right)} \), \( \tilde{d}_4 = \frac{\sqrt{\xi^2 + \sqrt{-i\xi_0} \sqrt{\xi^2 - \sqrt{-i\xi_0}}}}{\left(\xi + \sqrt{\xi^2 - \sqrt{-i\xi_0}} + \xi^2 - \sqrt{-i\xi_0}\right)} \),

\( \tilde{d}_5 = \frac{\sqrt{\xi^2 + \sqrt{-i\xi_0} \sqrt{\xi^2 - \sqrt{-i\xi_0}}}}{\left(\xi^2 + \sqrt{-i\xi_0} + \xi^2 - \sqrt{-i\xi_0}\right)^2} \). \tag{159}

If we substitute (157) in (156), divide both parts by (\( i\xi_0 + |\xi| \)), and move the term with \( \tilde{T} \) in the right hand side, we obtain

\( \tilde{D}_{\rho} = \frac{i\xi_1}{\xi_0 + |\xi|} \tilde{D}_{\rho}^\rho - \frac{i\xi_0}{\xi_0 + |\xi|} (C_1 \tilde{D}_1 + C_2 \tilde{D}_2) \tilde{T}^3 \tilde{D}_{\rho}^\rho. \) \tag{162}

Taking into account that

\( \tilde{D}_{\rho} = (i\xi_m)(i\xi_m)(i\xi_m)(i\xi_m) \tilde{D}_x \tilde{D}_\rho, \quad m_1, m_2, m_3, r \in \{1, ..., N - 1\}, \)

write the relation (162) in the form

\( \tilde{D}_{\rho} = \frac{i\xi_1}{\xi_0 + |\xi|} \tilde{D}_{\rho}^\rho - \frac{i\xi_0}{\xi_0 + |\xi|} (C_1 \tilde{D}_1 + C_2 \tilde{D}_2) \prod_{k=1}^{3} (i\xi_{m_k} \tilde{T}) \tilde{D}_{\rho}, \)

or, bearing in mind that \( i\xi_0 \tilde{D}_x \tilde{\rho} = i\xi_r \tilde{D}_\rho, \)

\( \tilde{D}_{\rho}^\rho = \frac{i\xi_1}{\xi_0 + |\xi|} \tilde{D}_{\rho}^\rho - \frac{i\xi_0}{\xi_0 + |\xi|} (C_1 \tilde{D}_1 + C_2 \tilde{D}_2) \prod_{k=1}^{3} (i\xi_{m_k} \tilde{T}) \tilde{D}_{\rho}. \) \tag{163}

Note now that the multipliers \( \tilde{d}_i \) in the definitions of the functions \( \tilde{D}_1 \) and \( \tilde{D}_2 \) and also the multipliers \( (i\xi_{m_k} \tilde{T}) \) in the righthand side of (163) are homogeneous of the kind \( \tilde{m}(\lambda \xi, \lambda^4 \xi_0) = \tilde{m}(\xi, \xi_0) \), where \( \tilde{m}(\xi, \xi_0) \) means one of these multipliers. For them we can easily verify the conditions of Theorem 1.1. To do this the whole set of the variables \( (\xi, \xi_0) \) should be split into two groups \( \xi \) and \( \xi_0 \) and any \( p \in (1, 2) \) may be chosen. Besides, the differentiation with respect to \( \xi_0 \) is done only one time (up to the first order) and the considered functions are smooth with respect to \( \xi \) on the set \( B_1 = \{ (\xi, \xi_0) : 1/8 \leq |\xi_0| + |\xi|^4 \leq 8 \} \).

So for these functions \( \tilde{D}_{\xi_0} \tilde{D}_{\xi} \tilde{m}(\xi, \xi_0) \leq C_{\omega} |\xi_0|^{-1/2} \) on this set.

Further, the multipliers \( \tilde{m}_i(\xi, \xi_0) = \frac{i\xi_0}{\xi_0 + |\xi|} \) and \( \tilde{m}_r(\xi, \xi_0) = \frac{i\xi}{\xi_0 + |\xi|} \) in the right hand side of (163) are homogeneous of the kind \( \tilde{m}(\lambda \xi, \lambda^4 \xi_0) = \tilde{m}(\xi, \xi_0) \) and satisfy the conditions of Theorem 2.7. To verify the conditions of these theorem note first that the functions \( \tilde{m}_i(\xi, \xi_0) \) and \( \tilde{m}_r(\xi, \xi_0) \) are smooth with respect to the variable.
ξ_0 on the set B_2 = \{(ξ, ξ_0) : 1/8 ≤ |ξ_0| + |ξ| ≤ 8\}. And with respect to the variables ξ these functions satisfy the estimate (ω = (ω_1, ..., ω_{N-1}))

\[ |D^k_{ξ_0} D^p_ξ \tilde{m}_{L, r}(ξ, ξ_0)| ≤ C_k |ξ|^{-|ω|+2}, \quad |ω| ≥ 2. \]

Therefore we can split the set of variables (ξ, ξ_0) into two groups ξ and ξ_0 and take for the differentiation with respect to ξ the order s_1 = [(N-1)/p+γ] + 1, p ∈ (1, 2).

Then we see that for |ω| = s_1 we have

\[ |D^k_{ξ_0} D^p_ξ \tilde{m}(ξ, ξ_0)|^p ≤ C_{k, ω} |ξ|^{-((N-1)/p+γ)p+p} \]

and -[(N-1)/p+γ]p+p > -(N-1).

Thus, in view of (155) we have (|α| = 4, |β| = 3)

\[ \langle D^\alpha_{x', ρ} \rangle_{x', Q^N} \leq C \langle D^\beta_{x', h} \rangle_{x', Q^N} + C \langle D_\rho \rangle_{Q^N}. \]  (164)

To estimate the last term in the righthand side of (164) we use the following interpolation inequality

\[ \langle f \rangle_{Q^N} \leq \varepsilon \langle f \rangle_{Q^N} + C_\varepsilon \| f \|_{L^2(Q^N)}, \quad l', l, \varepsilon > 0. \]  (165)

The proof of this this inequality for l = 0, l' = γ is given in fact in [57] and for the general case the proof is analogous. Applying (165) to the function D_\rho and taking into account (155), we obtain

\[ \langle D_\rho \rangle_{Q^N} \leq \varepsilon \langle D_\rho \rangle_{Q^N}^{(3+\gamma, 14\gamma)} + C_\varepsilon \| D_\rho \|_{L^2(Q^N)} \leq \varepsilon \langle D_\rho \rangle_{Q^N}^{(3+\gamma, 14\gamma)} + C_\varepsilon \sum_{|ω|\leq 3} \| D^\omega_{x'} h \|_{L^2(Q^N)} + C_\varepsilon \| D_\rho \|_{L^2(Q^N)}. \]  (166)

From (164) and (166) it follows that

\[ \sum_{|α| = 4} \langle D^\alpha_{x', ρ} \rangle_{x', Q^N} \leq C_\varepsilon \left( \sum_{|β| = 3} \langle D^\beta_{x', h} \rangle_{x', Q^N} + \sum_{|ω|\leq 3} \| D^\omega_{x'} h \|_{L^2(Q^N)} \right) + \varepsilon \langle D_\rho \rangle_{Q^N}^{(3+\gamma, 14\gamma)} + C_\varepsilon \| D_\rho \|_{L^2(Q^N)}. \]  (167)

This proves the lemma.

Obtain now a similar estimate for \langle D_\rho \rangle_{I, Q^N}. First, in view of \tilde{D}_{x_1} \rho ∈ L^2(Q^N) from (154) and from the equality

\[ \tilde{D}_\rho = \frac{iξ_0}{iξ_t} \tilde{D}_{x_1} \rho \]

it follows that

\[ \tilde{D}_\rho = \frac{iξ_0}{M(ξ, ξ_0)} \tilde{h}. \]

Therefore \tilde{D}_\rho ∈ L^2(Q^N) since the function \frac{iξ_0}{M(ξ, ξ_0)} is bounded. Moreover,

\[ \| D_\rho \|_{L^2(Q^N)} \leq C \| h \|_{L^2(Q^N)}. \]  (168)

Further, just using (165), we obtain

\[ \langle D_\rho \rangle_{I, Q^N} \leq \varepsilon \langle D_\rho \rangle_{Q^N}^{(3+\gamma, 14\gamma)} + C_\varepsilon \| D_\rho \|_{L^2(Q^N)} \leq \varepsilon \langle D_\rho \rangle_{Q^N}^{(3+\gamma, 14\gamma)} + C_\varepsilon \| h \|_{L^2(Q^N)}. \]  (169)
From (167) and (169) it follows that
\[
\langle \rho \rangle_{Q^N}^{(4+\gamma, \frac{4+\gamma}{2})} \leq C_\varepsilon \left( \sum_{|\beta|=3} \langle D_x^\beta h \rangle_{x', Q^N}^{(\gamma)} + \sum_{|\omega| \leq 3} \|D_x^\omega h\|_{L^2(Q^N)} \right) + \varepsilon \langle D_t \rho \rangle_{Q^N}^{(3+\gamma, \frac{3+\gamma}{2})}.
\] (170)

Thus, the function \( u(x, t) \) satisfies to problem (87) - (89), (91) and condition (91) can be replaced by the condition
\[
u(x', 0) = \rho(x', t).
\] (171)

From results of [75] it follows that the solution of this problem can be represented as (formula (2.68) in [75])
\[
u(x, t) = \int_0^t d\tau \int_{R^{N-1}} K(x' - \eta, x_N, t - \tau) \rho(\eta, \tau) d\eta
\] (172)
with some kernel \( K(x, t) \). This kernel for \( Q = 0, 1, 2, \ldots \) is represented as
\[
K(x, t) = \left[ \frac{\partial}{\partial t} + \Delta^2_x \right]^Q K^{(Q)}(x, t)
\]
with some functions \( K^{(Q)}(x, t) \). Applying in (172) integrating by parts, similar to [75], we obtain the representation
\[
u(x, t) = \int_0^t d\tau \int_{R^{N-1}} K^{(1)}(x' - \eta, x_N, t - \tau) \Phi(\eta, \tau) d\eta,
\]
where
\[
\Phi(x', t) = \left[ -\frac{\partial}{\partial t} + \Delta^2_x \right] \rho(x', t).
\]

The paper [75] gives the estimate for \( \langle u \rangle_{Q^N+1}^{(4+\gamma, \frac{4+\gamma}{2})} \) on whole time interval \( t \in (-\infty, +\infty) \) in terms of \( \langle \Phi \rangle_{Q^N}^{(\gamma, \frac{\gamma}{2})} \) (estimate (3.6) in [75]) that is the estimate
\[
\langle u \rangle_{Q^N+1}^{(4+\gamma, \frac{4+\gamma}{2})} \leq C \langle \Phi \rangle_{Q^N}^{(\gamma, \frac{\gamma}{2})} \leq C \langle \rho \rangle_{Q^N}^{(4+\gamma, \frac{4+\gamma}{2})}.
\] (173)

Together with (170) this gives
\[
\langle u \rangle_{Q^N+1}^{(4+\gamma, \frac{4+\gamma}{2})} \leq C\varepsilon \left( \sum_{|\beta|=3} \langle D_x^\beta h \rangle_{x', Q^N}^{(\gamma)} + \sum_{|\omega| \leq 3} \|D_x^\omega h\|_{L^2(Q^N)} \right) + \varepsilon \langle D_t \rho \rangle_{Q^N}^{(3+\gamma, \frac{3+\gamma}{2})}.
\] (174)

But according to the construction of the functions \( u(x, t) \) and \( \rho(x', t) \) we have \( D_t \rho = D_t u(x', 0, t) \) and condition (91) is satisfied. Consequently, in view of (173) and (170)
\[
\langle D_t \rho \rangle_{Q^N}^{(3+\gamma, \frac{3+\gamma}{2})} \leq \langle h \rangle_{Q^N}^{(3+\gamma, \frac{3+\gamma}{2})} + C \left( \frac{\partial u}{\partial x_N} \right)_{Q^N}^{(3+\gamma, \frac{3+\gamma}{2})} \leq \langle h \rangle_{Q^N}^{(3+\gamma, \frac{3+\gamma}{2})} + C \langle u \rangle_{Q^N+1}^{(4+\gamma, \frac{4+\gamma}{2})}
\]
\[
\leq \langle h \rangle_{Q^N}^{(3+\gamma, \frac{3+\gamma}{2})} + C \langle \rho \rangle_{Q^N}^{(4+\gamma, \frac{4+\gamma}{2})}
\]
\[
\leq C\varepsilon \left( \langle h \rangle_{Q^N+1}^{(4+\gamma, \frac{3+\gamma}{2})} + \sum_{|\omega| \leq 3} \|D_x^\omega h\|_{L^2(Q^N)} \right) + \varepsilon \langle D_t \rho \rangle_{Q^N}^{(3+\gamma, \frac{3+\gamma}{2})}.
\] (175)
If we take in this estimate \( \varepsilon = 1/2 \) we obtain in view of estimate (174)

\[
(u)^{(4+\gamma, \frac{4+\gamma}{2})}_{Q_{+1}^N} + (D_t u(x', 0, t))^{(3+\gamma, \frac{3+\gamma}{2})}_{Q_{+1}^N} \leq C \left( \langle h \rangle_{x', Q_{+1}^N} \right)^{(3+\gamma, \frac{3+\gamma}{2})}_{Q_{+1}^N} + \sum_{|\omega|\leq 3} \| D_{x_\nu} h \|_{L_2(Q_{+1}^N)} \right).
\]

Recalling now estimates (114), (115), and (116), for original problem (87) - (89), Theorem 3.4.

Let be arbitrary and let for problem

\[
\|u\|_{X_{\gamma}^1(Q_{+1}^N)} + \|D_t u(x', 0, t)\|_{X_{\gamma}^1(Q_{+1}^N)} \leq C_T \left( |f|_{X_{\gamma}^1(Q_{+1}^N)} + |g|_{X_{\gamma}^1(Q_{+1}^N)} + |h|_{X_{\gamma}^1(Q_{+1}^N)} \right).
\]

Thus, it is proved that for finite \( f, g, h \) of the class \( C^\infty \) problem (87) - (89), (91) has the unique solution from \( C^\infty \) and the estimate (177) is valid. If now \( f, g, h \) are finite but not from \( C^\infty \) though they satisfy condition (92), we can apply smoothing these functions with subsequence passing to the limit in view of estimate (177) as it was done at the end of the proof of Theorem 2.1. Thus we have the following theorem.

**Theorem 3.4.** Let \( T > 0 \) be arbitrary and let for problem (87) - (89), (91) conditions (92) are satisfied. Then this problem has the unique solution \( u(x, t) \) from the space \( u(x, t) \in C^{4+\gamma, \frac{4+\gamma}{2}}(Q_{+1}^N) \), \( u_t(x', 0, t) \in C^{3+\gamma, \frac{3+\gamma}{2}}(Q_{+1}^N) \) and estimate (177) is valid.

Note that the uniqueness of the solution with the estimate (177) follows from this estimate itself.

4. Boundary value problems for the linearized Cahn-Hilliard equation with dynamic boundary conditions. Let \( \Omega \) be an arbitrary domain in \( R^N \) with the boundary \( \Gamma = \partial \Omega \) of the class \( C^{4+\gamma} \), \( \gamma \in (0, 1) \), \( T > 0 \), \( \Omega_T = \Omega \times [0, T] \), \( \Gamma_T = \Gamma \times [0, T] \). In the domain \( \Omega_T \) consider the following initial-boundary value problem for unknown function \( u(x, t) \)

\[
\frac{\partial u}{\partial t} + \Delta^2 u + \sum_{|\alpha|\leq 3} b_\alpha(x, t) D_{x_\alpha}^2 u = f(x, t), \quad (x, t) \in \Omega_T,
\]

\[
\frac{\partial \Delta u}{\partial \nu} \bigg|_{x \in \partial \Omega} = g(x, t), \quad (x, t) \in \Gamma_T,
\]

\[
u u(x, 0) = u_0(x), \quad t = 0, x \in \Omega,
\]

\[
\frac{\partial u}{\partial \nu} - a(x, t) \Delta u + \sum_{i=1}^N b_i(x, t) \frac{\partial u}{\partial x_i} + c(x, t) u = h_1(x, t), \quad (x, t) \in \Gamma_T.
\]

Here \( \nu \) is outer normal to the surface \( \Gamma \), \( \Delta \Gamma \) is the Laplace-Beltrami operator on the surface \( \Gamma \), \( b_\alpha(x, t), a(x, t), b_i(x, t), c(x, t), f(x, t), g(x, t) \), \( u_0(x) \), \( h_1(x, t) \) are given functions, \( a(x, t) \geq \nu > 0 \), and

\[
b_\alpha \in C^{4, \frac{4}{2}}(\Omega_T), a, b_i, c \in C^{2+\gamma, \frac{2+\gamma}{2}}(\Gamma_T),
\]

\[
f \in C^{4, \frac{4}{2}}(\Omega_T), g \in C^{1+\gamma, \frac{1+\gamma}{2}}(\Gamma_T), h_1 \in C^{2+\gamma, \frac{2+\gamma}{2}}(\Gamma_T), u_0 \in C^{4+\gamma, \frac{4+\gamma}{2}}(\Omega_T).
\]

Instead of dynamic boundary condition (181) we also consider the condition

\[
\frac{\partial u}{\partial t} + d(x, t) \frac{\partial u}{\partial \nu} + b(x, t) u = h_2(x, t), \quad (x, t) \in \Gamma_T.
\]
where \(d(x, t), b(x, t), h_2(x, t)\) are given functions and
\[
d(x, t), b(x, t), h_2(x, t) \in C^{3+\gamma, \frac{2+\gamma}{\alpha}}(\Gamma_T), \quad d(x, t) \geq \nu > 0.
\] (184)
We assume also the standard compatibility conditions up to the first order at \(t = 0, x \in \partial \Omega\). Namely, we assume that
\[
\left. \frac{\partial \Delta u_0(x)}{\partial n} \right|_{x \in \partial \Omega} = g(x, 0), \quad x \in \Gamma.
\] (185)
Besides, in the case of condition (181) we assume that for \(x \in \Gamma\)
\[
\Delta^2 u_0(x) + \sum_{|\alpha| \leq 3} b_\alpha(x, 0) D^\alpha_x u_0 - f(x, 0) =
\]
\[
= -a(x, 0) \Delta u_0(x) + \sum_{i=1}^N b_i(x, 0) \frac{\partial u_0}{\partial x_i} + c(x, 0) u_0 - h_1(x, 0),
\] (186)
and in the case of condition (183) we assume that for \(x \in \Gamma\)
\[
\Delta^2 u_0(x) + \sum_{|\alpha| \leq 3} b_\alpha(x, 0) D^\alpha_x u_0 - f(x, 0) = d(x, 0) \frac{\partial u_0}{\partial n} + b(x, 0) u_0 - h_2(x, 0).
\] (187)
The following statements are valid.

**Theorem 4.1.** Under conditions (182), (185), (186) problem (178)-(181) has the unique solution from \(C^{4+\gamma, \frac{4+\gamma}{\alpha}}(\Omega_T)\) such that \(u_t|_{\Gamma_T} \in C^{2+\gamma, \frac{2+\gamma}{\alpha}}(\Gamma_T)\) and the estimate
\[
|u|^{(4+\gamma)}_{\Omega_T} + \left| \frac{\partial u}{\partial t} \right|_{\Gamma_T}^{(2+\gamma)} \leq C_1 \left( |f|^{(\gamma)}_{\Omega_T} + |g|^{(1+\gamma)}_{\Gamma_T} + |h_1|^{(2+\gamma)}_{\Gamma_T} + |u_0|^{(4+\gamma)}_{\Gamma_T} \right)
\] (188)
is valid, where the constant \(C_1\) depends only on \(T, \nu\), and on norms of the coefficients \(b_\alpha(x, t), a(x, t), b_i(x, t), c(x, t)\) in the corresponding spaces.

**Theorem 4.2.** Under conditions (182), (184), (185), (187) problem (178)-(180), (183) has the unique solution from \(C^{4+\gamma, \frac{4+\gamma}{\alpha}}(\Omega_T)\) such that \(u_t|_{\Gamma_T} \in C^{3+\gamma, \frac{3+\gamma}{\alpha}}(\Gamma_T)\) and the following estimate
\[
|u|^{(4+\gamma)}_{\Omega_T} + \left| \frac{\partial u}{\partial t} \right|_{\Gamma_T}^{(3+\gamma)} \leq C_2 \left( |f|^{(\gamma)}_{\Omega_T} + |g|^{(1+\gamma)}_{\Gamma_T} + |h_2|^{(3+\gamma)}_{\Gamma_T} + |u_0|^{(4+\gamma)}_{\Gamma_T} \right)
\] (189)
is valid, where the constant \(C_2\) depends only on \(T, \nu\), and on the norms of the coefficients \(b_\alpha(x, t), d(x, t), b(x, t)\) in the corresponding spaces.

We do not give the proof of these theorems because it is quite standard to the present time and is identical to the proofs of the corresponding statements from [75]. The proof consists of the construction of “regularizing operator” to the problem that is construction of an operator that is near inverse operator to the operator of the problem. Such an operator is constructed with the help of a partition of unity from the solutions of simplest model problems for different points of the domain \(\Omega_T\). Now the model problems for points of the boundary \(\Gamma_T\) are studied in the previous section and the model problems for inner points of \(\Omega_T\) are studied in [75].
Remark 2. Since for the proof of theorems 4.1 and 4.2 the technique of [75] can be used, we can replace the simplest equation in (178) by more general equation, where the Laplace operator is replaced by more general operator, for example, by the operator

\[ L \equiv \sum_{i,j=1}^{N} a_{ij}(x,t) \frac{\partial^2}{\partial x_i \partial x_j}, \]

where

\[ a_{ij}(x,t) \in C^{2+\gamma, \frac{2+\gamma}{2}}(\Omega_T), \quad \nu \eta^2 \leq \sum_{i,j=1}^{N} a_{ij}(x,t) \eta_i \eta_j \leq \nu^{-1} \eta^2, \eta \in \mathbb{R}^N, \nu > 0. \]

In this case equation (178) takes the form

\[ \frac{\partial u}{\partial t} + L^2 u + \sum_{|\alpha| \leq 3} b_\alpha(x,t) D_\alpha^2 u = f(x,t). \]

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