Liouville first-passage percolation: subsequential scaling limit at high temperature

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Joint work with Jian Ding
First-passage percolation in $\mathbb{Z}^2$

- Nearest-neighbor graph on $\mathbb{Z}^2$ (edges go one unit north, south, east, west).
- Randomly assign a weight ("cost") to each node.
- Distance between two nodes is the total weight of the lowest-weight ("cheapest") path between them.
- This gives a *random metric space*.
- Introduced by Hammersley and Welsh (1965).
Discrete Gaussian free field

- Let $\mathcal{R} \subseteq \mathbb{Z}^2$ connected. Consider DGFF on $\mathcal{R}$ with Dirichlet boundary conditions on $\partial \mathcal{R}$: Gaussian field $Y$ on $\mathcal{R}$ with covariance function

$$\mathbb{E} Y(x) Y(y) = G_{\mathcal{R}}(x, y),$$

where $G_{\mathcal{R}}$ is the Green’s function of simple random walk on $\mathcal{R}$ killed on $\partial \mathcal{R}$.

- Log-correlated Gaussian field: $G_{\mathcal{R}}(x, y) \sim \log \frac{|\mathcal{R}|}{|x-y|}$.
- Similar correlation structure to branching random walk.
Liouville first-passage percolation

- Liouville first-passage percolation (LFPP) is first-passage percolation on $e^{\gamma Y}$, where $Y$ is a Gaussian free field and $\gamma$ is an inverse-temperature parameter:

- Duplantier–Sheffield (2008) and Rhodes–Vargas (2008), along with many others subsequently, have studied the measure arising from the exponential of log-correlated Gaussian fields. We are concerned instead with the metric.
Questions

- How does the metric (left–right crossing distance, point-to-point distance, diameter...) scale with the box size $S$ and the temperature $\gamma$?
  
  - Ding–Goswami (2016) prove that the exponent of the expectation is strictly less than 1 in the small-$\gamma$ regime via “switchings.”

- As $S \to \infty$ (with $\gamma$ held constant), does the (normalized) metric converge to some limiting metric?
  
  - We make progress in the small-$\gamma$ regime.
Main result

Let $d_s(x, y)$ be the linearly-interpolated first-passage percolation distance between $x$ and $y$ in $[0, 1]^2$, using lattice of size $S = 2^s$, normalized so that the expected left-right distance of the square is 1.

Theorem (Ding–D.)

If $\gamma$ is sufficiently small, then the sequence $\{d_s\}_{s \in \mathbb{N}}$ is tight in the Gromov–Hausdorff topology. In fact, it is tight in the uniform topology on functions $([0, 1]^2)^2 \to \mathbb{R}_{\geq 0}$.

By Prokhorov’s theorem, this implies subsequential convergence.
Small-$\gamma$ regime

- Any constant number of scales (at the top or at the bottom) are negligible—can use any fixed amount of independence needed.
  - Crossing weights are highly concentrated.
- At a constant scale, geodesics are almost straight (Hausdorff dimension close to 1).
Obtain subsequential convergence by proving tightness of the normalized metric.

Follows by Arzelà–Ascoli theorem from equicontinuity of the metric as the scale increases.

Need to get good tail bounds on the diameter of a box so we can max over many boxes.
The coefficient of variation bound

**Theorem (Ding–D.)**
*The coefficient of variation (CV = \( \sigma / \mu \)) of the crossing weights can be made arbitrarily small by making \( \gamma \) sufficiently small.*

**Corollary**
*Arbitrarily high and low crossing quantiles are multiplicatively related as long as \( \gamma \) is sufficiently small.*
Bounding $CV^2$

- Use induction from scale $s$ to scale $s + k$, where $k$ is constant but large.
- Bound the variance from above and the expectation from below.
- Without contributions from boxes between scales $s$ and $s + k$, variance and expectation “should” both go like $K = 2^k$.
  - Need to relate the coefficients (as constant multiples of the expected crossing weight at scale $s$).
- So by making $\gamma$ small and $k$ large, can make $\text{Var}/\text{E}^2$ as small as we like.
Easy and hard crossings

- Easy crossings (left) at a smaller scale are *necessary* to cross at a larger scale.
  - \( \implies \) inductive lower bounds for crossing weights
- Hard crossings (right) at a smaller scale are *sufficient* to cross at a larger scale.
  - \( \implies \) inductive upper bounds for crossing weights
- RSW result: *easy and hard crossings can be related.*
  - not obvious, and the crux of our results
Russo–Seymour–Welsh results

- Show that crossing probabilities/weights in the easy direction are related to those for the hard direction by a constant factor.
- Introduced for Bernoulli percolation by Russo, Seymour, Welsh in 1978–81.
- Try to glue together easy crossings to get a hard crossing.
RSW for Voronoi percolation (Tassion 2014)

- Tassion proved an RSW result in a weakly correlated ordinary (not first-passage) percolation setting.
- Self-dual model—go from square crossing to hard crossing rather than easy crossing to hard crossing.
- Delicate multi-scale analysis involving inductively controlling both the probability of crossings and the geometry of crossings if they do exist.
Challenges for first-passage percolation

- Goal is now to show there are good probabilities of crossings with certain *weights*.  
  - Thus need to choose these weights appropriately, and keep track of them in every construction.  
  - At the end, we need to show that the weight we get is not too big—requires our inductive hypothesis.

- We don’t have a notion of self-duality
  - Need to go from easy crossing to hard crossing rather than from square crossing to easy crossing.
Keeping track of the weights

- Tassion’s analysis is already quite delicate—having a limited “weight budget” makes things substantially trickier in many places.

- Multiscale joining procedure creates paths of weight $\sum c^n w_{s-nk}$, where $c$ is some constant and $w_n$ is the crossing weight at scale $n$.
  - Need to force $k$ to be large so that this is summable—can do this by skipping over many scales at each joining step.
  - Need the sum we get to be not too large (dominated by the largest term)—apply inductive hypothesis and our \textit{a priori} bound on easy crossings.
Future work

- Investigate properties of limit point metrics.
- Show convergence of the metrics to a limiting metric (eventual goal of current work with Jian Ding and Subhajit Goswami).
- What happens for larger $\gamma$?
Jian Ding and Alexander Dunlap, *Liouville first-passage percolation: subsequential scaling limits at high temperature*, Ann. Probab., to appear, available at http://arxiv.org/abs/1605.04011.

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Rémi Rhodes and Vincent Vargas, *KPZ formula for log-infinitely divisible multifractal random measures*, ESAIM: Probability and Statistics 15 (2011), 358–371.

Vincent Tassion, *Crossing probabilities for Voronoi percolation*, Ann. Probab. 44 (2016), no. 5, 3385–3398.